Gravitational lensing by magnetized compact object in the presence of plasma

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Abstract. We study the gravitational lensing in the weak field approximation assuming the presence of a plasma and of a magnetic field around a compact gravitational source. The external magnetic field causes the split of the image, as the counterpart of the Zeeman effect. The magnetic field affects the magnification of images, creating additional components. We also study the time delay of an electromagnetic signal due to the geometry and the gravitational field around the source. We show that the time delay strongly depends on the plasma parameters. Lastly, we consider the effects of the presence of an inhomogeneous plasma on the gravitational lensing.
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1 Introduction

Light deflection and lensing in curved spacetime due to the presence of matter and/or energy density is one of distinguished features of general relativity. The gravitational lensing in the weak field approximation has been studied by many authors. The first studies on microlensing are in Refs. [1–5]. A review article on this subject is Ref. [6]. Strong gravitational lensing and the angular sizes and magnification factors for relativistic rings formed by photons were studied in [7]. The gravitational lensing by wormholes was considered in [8]. The authors of Ref. [9] studied the gravitational lensing and the ghost images in the regular Bardeen no-hair spacetimes.

The interaction of the photons with the plasma surrounding the gravitational lens in the presence of a strong magnetic field is particularly interesting from both the theoretical and the observational point of view. According to the no-hair theorem, astrophysical black holes do not possess their own magnetic field. Nevertheless, an external magnetic field can be generated by the current of the surrounding plasma or (in binary systems) by the companion star, if the latter is a neutron star or a magnetar with a strong magnetic field. The electromagnetic field configuration in the vicinity of a black hole immersed in an external magnetic field was first considered by Wald [10]. After that seminal work, many other authors have studied the properties of the spacetime around a black hole immersed in an external magnetic field [11–25]. The magnetic field of a current loop around a black hole was considered in [26]. The charged-fluid toroidal structures surrounding a static charged black hole in an asymptotically uniform magnetic field was studied in [27]. The role of gravitational lensing on the study of the distribution of stars in the Milky Way and on the study of dark matter and dark energy on very large scales is discussed in [28]. An overview of the problems of the plasma influence on the effects of gravitational lensing is reported in [29]. Optical effects related to Keplerian discs orbiting Kehagias-Sfetsos naked singularities were discussed in [30]. In Ref. [31], the optical phenomena in the field of braneworld Kerr black holes were studied in detail.

Gravitational lensing by a rotating massive object in a plasma has been considered by different authors [6, 32–40]. The recent study in Ref. [41] is devoted to gravitational lensing by
different types of regular black holes in the presence of plasma. Various optical properties of black holes in the presence of plasma, like the black hole shadow, were studied in Refs. [42–50].

The aim of the present work is to study the gravitational lensing in a weak gravitational field in the presence of plasma and magnetic field. The paper is organized as follows. In Sect. 2, we present the master equations for the description of the plasma around the black hole in the presence of a magnetic field and for the description of the photon motion. Sect. 3 is devoted to the study of the observational consequences of the gravitational lensing in the presence of plasma and magnetic field. The summary of the results are reported in Sect. 4.

Throughout the paper we employ the convention of a metric with signature (−, +, +, +). We use units in which $G = c = 1$, but we restore $G$ and $c$ when we have to compare our findings with observational data. Greek indices run from 0 to 3, while Latin indices run from 1 to 3.

2 Theoretical framework

2.1 Uniform magnetic field in the vicinity of a black hole

In this subsection, we will briefly discuss the electromagnetic field in the vicinity of a static compact object. We will also estimate the value of magnetic fields for supermassive and stellar-mass black holes, respectively. In order to study the propagation of the photons (electromagnetic waves) through a magnetized plasma in some arbitrary space-time we will consider magneto-hydrodynamic equations which is written as [51–53]

\[
\nabla_a F_{\mu \nu} = 0, \tag{2.1}
\]

\[
\nabla_a F^{\alpha \beta} = J^\beta, \tag{2.2}
\]

\[
V^\alpha \nabla_\alpha V^\beta = \frac{q}{m} F^\beta_\alpha V^\alpha, \tag{2.3}
\]

\[
\nabla_\alpha (N_q V^\alpha) = 0, \tag{2.4}
\]

\[
V^\alpha V^\alpha = -1, \tag{2.5}
\]

where $F_{\alpha \beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the electromagnetic field tensor with the vector potential $A_\alpha$ of the electromagnetic field, $J^\alpha$ is the electric current of the electrons and ions and, $q$ and $m$ are their charge and mass, respectively. $N_q$ and $V^\alpha$ are the number density and four-velocity of the charged particles.

The fundamental equation for the small perturbation of the vector potential $\hat{A}_\alpha$ is given by [51–53]

\[
D^{\alpha \beta} \hat{A}_\beta = \left[ h^{\alpha \mu} V^\nu \nabla_\nu (\nabla_\mu - \delta^\mu_\mu \nabla_\lambda) \right], \tag{2.6}
\]

\[
+ (\omega^{\alpha \mu} + \omega^{\alpha \mu}_L + \Theta^{\alpha \mu} + \Theta h^{\alpha \mu} + \frac{q}{m} E^\alpha V^\mu)(\nabla_\mu - \delta^\mu_\mu \nabla_\lambda) \]

\[
+ \omega^\mu (h^{\alpha \beta} \nabla_\lambda + \Theta^{\alpha \beta} - \omega^{\alpha \beta}) \right] \hat{A}_\beta = 0, \tag{2.7}
\]

where $\omega_\mu = (4 \pi N_q q^2 / m)^{1/2}$ is the plasma frequency and $\omega_L$ is Larmor frequency which defined as

\[
\omega_L = \sqrt{\omega_\mu \omega^\mu_L}, \quad \omega^\mu_L = \frac{q}{2m} \eta^{\mu \nu \alpha \beta} V_\nu B_{\alpha \beta}, \tag{2.7}
\]

with $B_{\alpha \beta} = h^{\alpha \mu} h^{\beta \nu} F_{\mu \nu}, h^\alpha_\beta = \delta^\alpha_\beta + V^\alpha V_\beta$ and $\eta^{\mu \nu \alpha \beta}$ is the Levi-Civita symbol in four dimensional curved space and other definitions are

\[
E^\alpha = F^\alpha_\beta V^\beta, \quad \nabla_\beta V^\alpha = \omega^\alpha_\beta + \Theta^{\alpha \beta} - V_\beta V^\gamma \nabla_\gamma V^\alpha ,
\]

\[
\text{– 2 –}
\]
\[ \omega_{\alpha\beta} = -\omega_{\beta\alpha} = -\frac{q}{m} B_{\alpha\beta}, \quad \Theta_{\alpha\beta} = \Theta_{\beta\alpha}, \quad \Theta = \nabla_\alpha V^\alpha. \]

Let us consider the spacetime of a static black hole with total mass \( M \). In spherical coordinate \((t, r, \theta, \phi)\), the line element is

\[ ds^2 = -N^2 dt^2 + \frac{1}{N^2} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \tag{2.8} \]

where the lapse function has a form \( N^2 = 1 - 2M/r \).

In Ref. [26], the author considered a current loop around a static black hole. He solved the general relativistic Maxwell equations (2.1) and (2.2) for the vector potential \( A_\alpha \) in the case of a current loop surrounding a black hole at the distance \( R \). It was shown that the vector potential in both interior and exterior regions of the current can be expressed in the following dipolar form [26, 54]

\[ A_\alpha = -\frac{3}{8} \delta_\alpha^\phi \mu r^2 \sin^2 \theta \begin{cases} g(R), & 2M \leq r \leq R, \\ g(r), & r \geq R, \end{cases} \tag{2.9} \]

with

\[ g(r) = \ln \left(1 - \frac{2M}{r}\right) + \frac{2M}{r} \left(1 + \frac{M}{r}\right), \]

where \( \mu \) is the magnetic dipole moment and it can be written in terms of the electric current \( I \) as

\[ \mu = \pi R^2 N_R I, \tag{2.10} \]

where subscript \( R \) denotes the function at \( r = R \).

**Exterior magnetic field outside loop** – The expression (2.9) for the vector potential of the electromagnetic field outside the current loop \((r > R)\) allows one to find the components of the exterior magnetic field [26]

\[ B^\phi(r, \theta) = -\frac{3\mu}{4M^2r} \left[ \ln N^2 + \frac{2M}{r} \left(1 + \frac{M}{r}\right) \right] \cos \theta, \tag{2.11} \]

\[ B^\theta(r, \theta) = \frac{3\mu N}{4M^2r} \left[ \ln N^2 + \frac{1}{N^2} + 1 \right] \sin \theta. \tag{2.12} \]

Neglecting higher order terms in \( M/r \) and \( M/R \) one can estimate the magnetic field as

\[ \lim_{M/r \to 0, M/R \to 0} B^\phi(r, \theta) = \frac{2\mu_0}{r^3} \cos \theta, \tag{2.13} \]

\[ \lim_{M/r \to 0, M/R \to 0} B^\theta(r, \theta) = \frac{\mu_0}{r^3} \sin \theta, \tag{2.14} \]

where \( \mu_0(\mu_0 = \pi R^2 I) \) is the Newtonian value of the magnetic dipole moment.

**Exterior uniform magnetic field within loop** – From Eq. (2.9), we can easily see that inside the current loop \((2M < r < R)\) the vector potential of the electromagnetic field can be expressed in terms of a uniform magnetic field \( B \) as

\[ A_\phi = \frac{1}{2} B r^2 \sin^2 \theta, \tag{2.15} \]

---
where the uniform magnetic field has the form

\[ B = -\frac{3\mu}{4M^3} \left[ \ln N_R^2 + \frac{2M}{R} \left( 1 + \frac{M}{R} \right) \right], \tag{2.16} \]

and is oriented along the z-axes. If we use the expression for the vector potential in Eq. (2.15), we find the components of the interior magnetic field

\[ B^\hat{r} = B \cos \theta, \tag{2.17} \]
\[ B^\hat{\theta} = BN \sin \theta. \tag{2.18} \]

The total magnetic field is \( B_T = (B^\hat{r}^2 + B^\hat{\theta}^2)^{1/2} \). In the limit of weak gravitational field, we can estimate the magnetic field as \( B_T \simeq B \) and we get

\[ B_T \simeq B = -\frac{3\mu}{4M^3} \left[ \ln N_R^2 + \frac{2M}{R} \left( 1 + \frac{M}{R} \right) \right]. \tag{2.19} \]

**Quasi-uniform magnetic field case** – Let us now assume that the plasma is axially symmetrically distributed around a Schwarzschild black hole. An electromagnetic wave (photon) does not interact with the magnetic field, because in the vacuum a magnetic field does not affect the propagation of light rays. This means that it does not matter whether the magnetic field has a dipolar (or multipolar) structure outside the plasma. However, when the light ray propagates in the magnetized plasma (medium) it can be in a resonance state due to the cyclotron frequency of charged particles. We thus use the expression in Eq. (2.19) for the magnetic field in the interior region of the current loop. For simplicity, we assume that the magnetic field is uniform in the vicinity of the black hole, in particular near the equatorial plane (\( \theta \simeq \pi/2 \)).

In order to estimate the magnetic field strength in the interior region of the loop, we consider the zone near the horizon of the black hole (\( 2M < r < R \)). Let us assume that the electric current loop is located at the radial coordinate \( R = 6M \), corresponding to the radius of the innermost stable circular orbit (ISCO) for a test-particle around a Schwarzschild black hole. We can now evaluate the magnetic field in the expression in Eq. (2.19) as

\[ B \simeq \frac{I}{M} \rightarrow \frac{I}{GM/c^2}, \tag{2.20} \]

which depends on the value of the current \( I \) and mass of the black hole \( M \). We can define the electric current \( I \) as

\[ I = e n_j v_j S, \tag{2.21} \]

where \( e \) is the electric charge of an electron, and \( n_j \) and \( v_j \) are, respectively, the density and the velocity of the charged particles in plasma. The subscript “\( j \)” in Eq. (2.21) refers to the type of particle. \( S = \pi hl \) is the elliptic cross section of the accretion disc with the height \( h \) and width \( l \).

Using Eqs. (2.20) and (2.21) and the following input values

\[ n_e \sim 10^5 \text{ cm}^{-3}, \quad h \sim 10^4 \text{ cm}, \quad l \sim 4M, \]

we get an estimate for the typical magnetic field strength around stellar-mass black holes

\[ B \simeq 6.5 \left( \frac{n_e}{10^5 \text{ cm}^{-3}} \right) \left( \frac{h}{10^4 \text{ cm}} \right) \left( \frac{l}{4M} \right) \left( \frac{M}{M_{\odot}} \right)^{-1} 10^8 \text{ G}. \tag{2.22} \]
Similarly, using the input values
\[ n_e \sim 10^4 \text{cm}^{-3}, \ h \sim 10^5 \text{cm}, \ l \sim 4 \times 10^{-5} M, \]
we get an estimation for the magnetic field strength around supermassive black holes
\[ B \simeq 4.3 \left( \frac{n_e}{10^4 \text{cm}^{-3}} \right) \left( \frac{h}{10^5 \text{cm}} \right) \left( \frac{l}{4 \cdot 10^{-5} M} \right) \left( \frac{M}{10^6 M_\odot} \right)^{-1} 10^4 \text{G} . \]  
(2.23)

Note that in both cases the particle is mildly relativistic \( v \simeq 0.4c \).

### 2.2 Photon motion in the plasma surrounding a black hole

In this subsection, we will consider the photon motion around a static black hole taking into account that the compact object is surrounded by a plasma and there is a non-vanishing magnetic field. At large distances from the black hole, the spacetime geometry tends to be flat and thus photons move along straight lines. The photons approaching the central object deviate from a straight line path. In order to study the photon trajectory, we consider the following set of the differential equations [55]:
\[ \frac{dx^\mu}{d\lambda} = \frac{\partial H}{\partial p^\mu}, \quad \frac{dp^\mu}{d\lambda} = -\frac{\partial H}{\partial x^\mu} , \]  
(2.24)

where \( \lambda \) is an affine parameter depending on proper time \( \tau \). \( H \) is the Hamiltonian of the photon and can be written as [55]
\[ H(x^\mu, p^\mu) = \frac{1}{2} \left[ g^{\mu\nu} p^\mu p^\nu - (n^2 - 1) (p_\mu V^\mu)^2 \right] = 0 . \]  
(2.25)

In Eq. (2.25), \( n \) is the refractive index of the medium, \( p^\mu \) is the 4-momentum of the photon, and \( V^\mu \) is the 4-velocity of the medium. According to Ref. [55], we have to take into consideration the following relation between the momentum and the 4-velocity of the photon in the medium:
\[ p_\mu V^\mu = -\frac{\hbar \omega(x^i)}{c} , \]  
(2.26)

were \( \omega(x^i) \) is the photon frequency in the medium, and \( \hbar \) and \( c \) are, respectively, the Planck constant and the speed of light in the vacuum.

Assuming that the photon is moving along the \( z \)-axis in flat spacetime, the 4-momentum \( p^\mu \) can be written as [56]
\[ p^\mu = \frac{\hbar \omega}{c} (1, 0, 0, n) , \quad p_\mu = \frac{\hbar \omega}{c} (-1, 0, 0, n) . \]  
(2.27)

Let us now consider the weak field limit. The covariant and contravariant components of the metric tensor can be written as
\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{and} \quad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} , \]  
(2.28)

where \( \eta_{\mu\nu} \) is the metric tensor in Minkowski space, \( h_{\mu\nu} \) is a small perturbation, and the following conditions hold
\[ \eta_{\mu\nu} = \eta^{\mu\nu} , \quad h_{\mu\nu} = h^{\mu\nu} \quad \text{and} \quad h_{\mu\nu} h^{\mu\nu} \rightarrow 0 . \]  
(2.29)
In the presence of a weak inhomogeneous plasma and a weak gravitational field, the photon equations of motion can be rewritten as
\[
\frac{dz}{d\lambda} = \frac{\hbar \omega}{c}, \tag{2.30}
\]
\[
\frac{dp_{\mu}}{dz} = \frac{1}{2} \frac{\hbar \omega}{c} \left( h_{zz,\mu} + \frac{1}{n^2} h_{tt,\mu} - \frac{\omega_e^2}{n^2 \omega^2} \right), \tag{2.31}
\]
where \(\omega_e\) is the frequency of the electron plasma and is defined as
\[
\omega_e^2(x) = \frac{4 \pi e^2 N_e(x)}{m_e}, \tag{2.32}
\]
\(N_e = N_e(x^i)\) is the electron concentration with respect to the coordinate in the plasma, and \(m_e\) is the electron mass.

The deflection angle is defined as the difference between the directions of the incoming and outgoing light rays. Following Ref. [56], we can write the expression of the deflection angle of the light ray in the plane perpendicular to the z-axis as
\[
\hat{\alpha}_i = e_i \text{ in} - e_i \text{ out}, \tag{2.33}
\]
where \(e_i\) is the unit vector along the vector \(p_i\), i.e. \(e_i = p_i/p\), and \(p = \sqrt{p_x^2 + p_y^2 + p_z^2} = p_z = \hbar \omega/c\). Employing Eqs. (2.30) and (2.31), we obtain the formula for the absolute value of the deflection angle
\[
\alpha = |\hat{\alpha}_k| = \frac{1}{2} \left| \int_{-\infty}^{\infty} \frac{\partial}{\partial x^k} \left( h_{zz} + \frac{1}{n^2} h_{tt} - \frac{\omega_e^2}{n^2 \omega^2} \right) dz \right|, \tag{2.34}
\]
For an inhomogeneous plasma, the refraction index \(n\) depends not only on the frequency of the electron plasma \(\omega_e\) but also on the magnetic field generated by the accretion disk of the black hole. From the fundamental equation (2.6) one can obtain dispersion relation as follows (See e.g., [52, 53])
\[
(\omega^2 - k^2) \left[ \omega^2 \omega_L^2 (\omega^2 - \omega_e^2 - k^2) + \omega_e^2 (\omega_L \cdot k)^2 \right]
- \omega^2 (\omega^2 - \omega_e^2)(\omega^2 - \omega_e^2 - k^2)^2 = 0, \tag{2.35}
\]
where \(k\) is the electromagnetic wave vector. After considering \(\omega_L \cdot k = \omega_L k \cos \psi\). Taking the inhomogeneity of the plasma and the presence of a magnetic field into account, the refraction index can be written as
\[
n^2 = n^2 \pm = 1 - \frac{\omega_e^2}{\omega^2} - \frac{\omega_e^2 \omega_L}{\omega^2} f_\pm (\omega_L, \omega_e), \tag{2.36}
\]
where \(\psi\) is the angle between the magnetic field relative to the direction of the photon, and the unknown function \(f_\pm (\omega_L, \omega_e)\) is defined as
\[
f_\pm (\omega_L, \omega_e) = \frac{1}{2} \frac{\omega \omega_L (\omega^2 + (\omega^2 - 2 \omega_e^2) \cos^2 \psi) \pm \omega^2 \sqrt{4(\omega^2 - \omega_e^2)^2 \cos^2 \psi + \omega^2 \omega_L^2 \sin^4 \psi}}{\omega^2 (\omega^2 - \omega_e^2 - \omega_L^2) + \omega_e^2 \omega_L^2 \cos^2 \psi}, \tag{2.37}
\]
In the case when $\psi = 0$ the expression (2.36) for the refractive index takes the following form

$$n_+^2 = 1 - \frac{\omega^2}{\omega^2} + \frac{\omega_L^2}{\omega^2(\omega + \omega_L)} ,$$

(2.38)

which is responsible for the case when photon comes to parallel to the magnetic field line, while $\psi = \pi/2$ case when magnetic field line perpendicular to the direction of the photons and the refractive index can take a form

$$n_+^2 = 1 - \frac{\omega^2}{\omega^2} + \frac{\omega^2 - \omega_L^2}{\omega^2 - \omega_L^2} , \quad n_-^2 = 1 - \frac{\omega^2}{\omega^2} ,$$

(2.39)

From the equation (2.36) one can easily see that absence of the magnetic field ($\omega_L = 0$) the refractive index takes simple form $n^2 = 1 - \omega^2/\omega^2$ as in [56]. Using the Eq.(2.7) and after doing some algebraic calculation one can obtain the explicit form of the Larmor frequency for the electron in the following form

$$\omega_L(r, \theta) = \frac{e}{m_e} \sqrt{\frac{1}{2} F_{\mu\nu} F^{\mu\nu}} = \omega_c \sqrt{1 - \frac{2M}{r} \sin^2 \theta} ,$$

(2.40)

where $\omega_c$ is the cyclotron frequency due to the uniform magnetic field $B$ and can be written as

$$\omega_c = \frac{eB}{m_e c} ,$$

(2.41)

which is same quantity as Larmor frequency in the external uniform magnetic field.

We can estimate the typical cyclotron frequency for supermassive and stellar-mass black holes:

- Supermassive black holes are located at the centre of galaxies and they have a mass in the range $M \sim 10^6 - 10^{10} M_\odot$. Typical values of $B$, $\omega_c$, and $\lambda$ are [57–60]:

$$B \sim 10^4 \text{ G}, \quad \omega_c \sim 30 \text{ GHz}, \quad \lambda \sim 6.3 \text{ cm}.$$  

Here $\omega_c$ is at super high radio frequencies.

- Stellar-mass black holes in the known X-ray binaries have a mass in the range $M \sim 3 - 20 M_\odot$. Typical values of $B$, $\omega_c$, and $\lambda$ are [61]:

$$B \sim 10^8 \text{ G}, \quad \omega_c \sim 300 \text{ THz}, \quad \lambda \sim 6.3 \times 10^{-4} \text{ cm}.$$  

Now $\omega_c$ is in the infrared spectrum.

Since very-long baseline interferometry observations are supposed to detect the radiation emitted by the accreting gas around the event horizon of the supermassive black holes Sgr A* and M87, it is important to study these effects in view of their near-future detectability.
2.3 Polarization angle of the light in the medium

In this subsection, we will consider the polarization angle of the light due to Faraday rotation in the presence of a magnetized plasma in the background of a static compact object. In Refs. [62] and [63], the rotation angle of the polarization plane during the propagation of the light ray in the plasma is considered at the leading order of the magnetic field. Here we will use the following more general form of the polarization angle

\[ 
\Delta \phi = \int ds k (n_+ - n_-) \simeq \int ds k \frac{\omega^2 \omega_L}{2\omega^4} (f_+ - f_-) 
\]

(2.42)

where \( k \) is the absolute value of the wave vector and can be written in terms of the frequency \( k = |\mathbf{k}| = 2\pi/\lambda = \omega \). Then polarization angle in the expression (2.42) will take a form for the different values of the inclination angle

\[ 
\Delta \phi = \int ds \frac{\omega^2 \omega_L}{2\omega} \begin{cases} 
\frac{1}{\omega^2 - \omega_L^2}, & \psi = 0 \\
\frac{2\omega_0}{\omega^2 - \omega_L^2 + \omega_0^2}, & \psi = \pi/2 
\end{cases} 
\]

(2.43)

By using the expression (2.42) for the polarization angle the rotation measure \( RM \) can be calculated as

\[ 
RM = \frac{\Delta \phi}{\lambda^2} = \frac{\omega^2}{4\pi^2} \Delta \phi 
\]

(2.44)

The scattering cross-sections of right-hand “+” and left-hand “−” polarized photons are different and can be calculated in the following way

\[ 
\sigma_\pm = \sigma_{\text{Th}} \left( 1 \pm \frac{\omega_L}{\omega} \cos \psi \right) 
\]

(2.45)

where \( \sigma_{\text{Th}} = (8\pi/3)(e^2/m_e c^2)^2 \) is the classical Thomson cross section. According to Eq. (2.45), the light ray from a magnetized plasma around a compact object is circularly polarized.

2.4 Deflection angle around a static black hole

This subsection is devoted to find the deflection angle of a photon moving in an inhomogeneous magnetized plasma in the spacetime of a static compact object. In the weak field approximation, the metric in Eq. (2.8) can be rewritten in following form

\[ 
ds^2 = ds_0^2 + \frac{2M}{r} dt^2 + \frac{2M}{r} dr^2 
\]

(2.46)

where \( ds_0^2 = -dt^2 + dx^2 + dy^2 + dz^2 \) is the line element in flat space. The components of small perturbations of the metric tensor are \( h_{\alpha\beta} \) and in the Cartesian frame are [64]

\[ 
h_{tt} = \frac{2M}{r}, \quad h_{ij} = \frac{2M}{r} \hat{n}_i \hat{n}_j, \quad h_{zz} = \frac{2M}{r} \cos^2 \theta, 
\]
where \( \hat{n}_i \) is the component of the unit vector with the same direction as the radius vector \( r_i = (x, y, z) \) and has the form \( \hat{n}_i = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta) \). Before calculating the deflection angle, it is useful to introduce the form of the plasma frequency as \( \omega_e(r) = \omega_0(R_0/r)^h \), where \( h \) and \( R_0 \) are constants and \( \omega_0 \) is the plasma frequency of the plasma at infinity. Using Eq. (2.34), we get the expression for the deflection angle of a light ray passing near a magnetized static compact object:

\[
\alpha^\pm = \frac{2M}{b} \left[ 1 + \left( 1 - \frac{\omega_0^2}{\omega^2} - \frac{\omega_0^2 \omega_e}{\omega^2} f_\pm (\omega_c, \omega_0) \right)^{-1} \right]
- \frac{\omega_0^2}{\omega^2} \frac{\sqrt{\pi} \Gamma((h + 1)/2)}{\Gamma(h/2)} \left( \frac{R_0}{b} \right)^h \left( 1 - \frac{\omega_0^2}{\omega^2} - \frac{\omega_0^2 \omega_e}{\omega^2} f_\pm (\omega_c, \omega_0) \right)^{-1} + O(M^2/b^2),
\]

where \( \Gamma(x) \) is the gamma function

\[
\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.
\]

In the case when \( \omega \gg \omega_c \), we have

\[
\alpha^\pm \simeq \frac{2M}{b} \left( 1 + \frac{\omega^2}{\omega^2 - \omega_0^2} \pm \frac{\omega^3 \omega_0^2 \omega_e \cos \psi}{(\omega^2 - \omega_0^2)^2} \right)
- \frac{\omega_0^2}{\omega^2} \frac{\sqrt{\pi} \Gamma((h + 1)/2)}{\Gamma(h/2)} \left( \frac{R_0}{b} \right)^h \left( \frac{\omega^2}{\omega^2 - \omega_0^2} \pm \frac{\omega_0^2 \omega_e \cos \psi}{(\omega^2 - \omega_0^2)^2} \right) + O(M^2/b^2).
\]

It is worth noting that in Eq. (2.47) there are four resonance states corresponding to the solutions of the equation \( \omega^3 - 2 \omega_0^2 \omega_e f_\pm (\omega_c, \omega_0) - \omega_0^2 = 0 \).

### 3 Observational effects

We can now study the observational consequences of gravitational lensing, namely the magnification of image sources, Einstein rings, etc. For this purpose, we can use the lens equation, which relates the angle \( \beta \) of the real object from the observer-lens axis, the angle \( \theta \) of the apparent image of the object from the observer-lens axis, and the deflection angle \( \alpha \):

\[
D_s \theta = D_s \beta + D_{ls} \alpha,
\]

where \( D_s \) is the distance between the observer and the source and \( D_{ls} \) is the distance between the lens and the source (see Fig. 1). In the weak field approximation, we can use the relation \( \alpha \sim 1/b \) in order to express the small angle \( \theta = b/D_l \), where \( D_l \) is the distance between the observer and the lens, as shown in Fig 1. Hereafter, we will consider two cases: \( i) \) homogeneous and \( ii) \) inhomogeneous plasma around a gravitational source. We will always take the effects of magnetic field into account.

#### 3.1 Homogeneous plasma

In the case of a homogeneous magnetized plasma, the plasma frequency in Eq. (2.32) is constant and in the weak field approximation the lens equation reduces to

\[
\beta = \theta - \frac{\Theta^\pm}{\theta},
\]

where
Figure 1. The black hole BH is between the source and the observer. The light emitted by the source and detected by the observer is affected by the gravitational lensing of the BH.

where

$$\Theta_\pm = \Theta_E, \sqrt{\frac{1}{2} \left[ 1 + \left( 1 - \frac{\omega_c^2}{\omega^2} \frac{\omega_0^2}{\omega^2} f_\pm (\omega_c, \omega_0) \right)^{-1} \right]}, \quad (3.3)$$

and $\Theta_E$ is defined as

$$\Theta_E = \sqrt{\frac{4M D_{\text{bh}}}{D_l D_s}}. \quad (3.4)$$

here $\Theta_\pm$ is the Einstein ring splitted into two rings due to the magnetic Zeeman effect. In the absence of magnetic fields, corresponding to the case $\omega_c = 0$, we can obtain the value of the unsplitted $\Theta$ in the plasma, as was done in [37, 41, 56]. In vacuum ($\omega_0 = 0$) $\Theta = \Theta_E$.

The split of the Einstein ring due to the magnetic field, or “Zeeman effect”, expressed in Eq. (3.3) is schematically shown in Fig. 3. The upper plot of Fig. 3 illustrates how the rings split into two. The lower plot shows how the angle changes due to the presence of the plasma as well as of the magnetic field.

Figure 4 shows the angular size of the Einstein ring as a function of the cyclotron frequency. From Fig. 4, we can see that in the resonance state, when $\omega_c \sim 0.84 \omega$, the size of the Einstein ring increases. An observation of the ring in the corresponding range of frequency would detect the change of the size and the form of the Einstein ring due to the existence of a magnetic field.

Let us now consider the image magnification due to lensing. First, we write $\theta$ in terms of $\beta$. The solution of Eq. (3.2) is

$$\theta = \frac{1}{2} \left( \beta \pm \sqrt{\beta^2 + 4\Theta_\pm^2} \right). \quad (3.5)$$

We define the image magnification as

$$\mu = \left| \frac{\theta d\theta}{\beta d\beta} \right|. \quad (3.6)$$
Figure 2. Einstein ring for the plasma frequency $\omega_0 = 0.4 \omega$ and the cyclotron frequency $\omega_c = 0.6 \omega$. The black line corresponds to the vacuum $\Theta_E$, the blue line is for the plasma $\Theta$, and the dashed red and green lines are for $\Theta_+$ and $\Theta_-$ at $\psi = 0$.

Figure 3. Einstein ring for the plasma frequency $\omega_0 = 0.4 \omega$ and the cyclotron frequency $\omega_c = 0.6 \omega$. The black line corresponds to the vacuum $\Theta_E$, the blue line is for the plasma $\Theta$, and the dashed red and green lines are for $\Theta_+$ and $\Theta_-$ at $\psi = \pi/2$.

Using Eq. (3.5), we can easily find the following expressions

$$
\mu_1^\pm = \frac{1}{4} \left[ \frac{\beta}{\sqrt{\beta^2 + 4\Theta_\pm^2}} + \frac{\sqrt{\beta^2 + 4\Theta_\pm^2}}{\beta} \pm 2 \right],
$$

$$
\mu_2^\pm = \frac{1}{4} \left[ \frac{\beta}{\sqrt{\beta^2 + 4\Theta_\pm^2}} - \frac{\sqrt{\beta^2 + 4\Theta_\pm^2}}{\beta} \pm 2 \right].
$$

(3.7)  
(3.8)
The total magnification is

$$\mu_1^\pm = \mu_1^+ + \mu_2^+ = \frac{\beta^2 + 2\Theta_\pm^2}{\beta \sqrt{\beta^2 + 4\Theta_\pm^2}}. \quad (3.9)$$

The ratio of the two magnifications is

$$\nu^\pm = \frac{\mu_1^\pm}{\mu_2^\pm} = \left(\frac{\theta_1^\pm}{\theta_2^\pm}\right)^2 = \left[\frac{\sqrt{\beta^2 + 4\Theta_\pm^2} + \beta}{\sqrt{\beta^2 + 4\Theta_\pm^2} - \beta}\right]^2, \quad (3.10)$$

where we have used following notations

$$\theta_1^\pm \theta_2^\pm = -\Theta_\pm^2, \quad \theta_1^\pm + \theta_2^\pm = \beta, \quad (3.11)$$

$$\theta_1^\pm - \theta_2^\pm = \sqrt{\beta^2 + 4\Theta_\pm^2}.$$

Figures 5 and 6 show the magnification of the first and of the second images due to weak lensing in the presence of a homogeneous plasma and a magnetic field. The upper and lower plots correspond, respectively, to the first and to the second solution for the magnification presented in Eqs. (3.7) and (3.8). From these plots we can easily see that due to the magnetic “Zeeman effect” the magnification plots split into two lines (dashed and dotted lines) with respect to the unmagnetized plasma case (solid line). Magnetic fields, in principle, cause the amplification of the magnification of image sources (see Fig. 5 and Fig. 6).

Fig. 7 shows the ratio of different magnifications of the image source; the upper and the lower “solid” lines correspond to the case of a plasma without magnetic field, which is the
situation shown in [56]; the dashed and dotted-dashed lines show the split of the line due to the presence of a homogeneous plasma and to the magnetic “Zeemann” effect.

A variable source behind a lensing object produces an observable variable image. However, the source and the image will not necessarily vary simultaneously: in general, there will be a time delay between the two events and there are two contributions. First, there is a purely geometrical time delay. Second, there is a delay due to the potential of the lensing object, the so-called Shapiro time delay.

If the set-up is that illustrated in Fig. 1, we have the following relation

$$D_1 + D_{ls} - D_s = \frac{D_1 D_s}{2D_{ls}} (\theta - \beta)^2 = c \Delta t_g , \quad (3.12)$$

where $\Delta t_g$ is the time delay caused by the spacetime geometry. There are two values of $\theta$ corresponding to the two values of the geometrical time delay. The time delay of one of the images with respect to the other one is

$$\Delta t_g^\pm = \frac{D_1 D_s}{2D_{ls}} [ (\theta_1^\pm - \beta)^2 - (\theta_2^\pm - \beta)^2 ] \quad (3.13)$$

Recalling the lens equation in Eq. (3.2) and the notations in Eq. (3.11), we can rewrite Eq. (3.13) as

$$\Delta t_g^\pm = \frac{D_1 D_s}{2D_{ls}} \Theta^\pm \left[ \frac{1}{\theta_1^{\pm \pm}} - \frac{1}{\theta_2^{\pm \pm}} \right]$$

$$= -2M \frac{\beta \sqrt{\beta^2 + 4\Theta^2}}{\Theta_E^2} .$$

Figure 6. Magnification for the first image $\mu_1$ as a function of $\beta/\Theta_E$. All lines are plotted for $\omega_0 = 0.4\omega$ and $\omega_c = 0.6\omega$ when $\psi = \pi/2$.

Figure 7. Ratio of the magnification $\mu_{\pm,i}/\mu_0$, ($i = 1, 2$) as a function of $\beta/\Theta_E$. All lines are plotted for $\omega_0 = 0.4\omega$ and $\omega_c = 0.6\omega$ when $\psi = 0$ (left panel) and $\psi = \pi/2$ (right panel).
The time delay caused by the gravitational potential (Shapiro time delay) is

\[ \Delta t_{ls} = 2M \ln \left( \frac{b}{2D_{ls}} \right), \quad (3.14) \]

\[ \Delta t_{l} = 2M \ln \left( \frac{b}{2D_{l}} \right). \quad (3.15) \]

The total Shapiro time delay for the gravitational potential can be written as the sum of these two components

\[ \Delta t_{p} = \Delta t_{ls} + \Delta t_{l} = 2M \ln \left( \frac{b}{2D_{ls}} \right) + 2M \ln \left( \frac{b}{2D_{l}} \right). \quad (3.16) \]

Considering that at very large distances from the lens the potential delay is negligible, we can easily calculate the difference in the time delay from one of the images to the other one. We consider the distance \( D \) such that \( D \gg b \) but \( D \ll (D_{ls}, D_{l}) \). We can rewrite Eq. (3.16) as

\[ \Delta t_{p} = 2M \left[ \ln \left( \frac{b}{2D_{ls}} \right) + \ln \left( \frac{b}{2D_{l}} \right) + \ln \left( \frac{D_{ls}^{2}}{D_{ls}D_{l}} \right) \right] \]

\[ = 4M \ln \left( \frac{b}{2D} \right) + 2M \ln \left( \frac{D_{ls}^{2}}{D_{ls}D_{l}} \right). \quad (3.17) \]

If we write the impact parameter as \( b = \theta D \), we can find the difference of two time delays

\[ \Delta t_{p}^{\pm} = 4M \left[ \ln \left( \frac{b_{1}}{2D} \right) - \ln \left( \frac{b_{2}}{2D} \right) \right] \]

\[ = 4M \ln \left( \frac{\theta_{1}}{\theta_{2}} \right). \quad (3.18) \]

Lastly, the total time delay that arises from both the geometry and the gravitational potential turns out to be

\[ \Delta T_{\pm} = 4M \ln \left[ \frac{\sqrt{\beta^{2} + 4\Theta_{\pm}^{2}} + \beta}{\sqrt{\beta^{2} + 4\Theta_{\pm}^{2}} - \beta} \right] - 2M \frac{\beta \sqrt{\beta^{2} + 4\Theta_{\pm}^{2}}}{\Theta_{\pm}^{2}}. \quad (3.19) \]

Fig. 8 shows the dependence of the time delay on the angle \( \beta \). As we can see from this plot, the presence of the plasma and of the magnetic field causes a shift of the peak of the time delay. Moreover, the positive and negative time delays due to the magnetic split (Zeeman effect) have their maximum at different values of \( \beta \). Fig. 8 also shows the time delay in the case of vacuum (dot-dashed line in Fig. 8).

### 3.2 Inhomogeneous plasma

Let us now study the effects of an inhomogeneous magnetized plasma. To do this, we assume that \( h = 1 \) in the expression for the plasma frequency. This can be regarded as a toy model for a preliminary study. Substituting Eq. (2.47) into Eq. (3.1), we obtain the following expression for \( \beta \)

\[ \beta = \theta - \frac{\Theta_{\pm}^{2}}{\theta} - \frac{\Phi_{\pm}}{\theta^{2}}, \quad (3.20) \]
Figure 8. Time delay $\Delta T^\pm$ as a function of $\beta/\Theta_E$ when $\psi = 0$ (left panel) and $\psi = \pi/2$ (right panel).

where

$$\Phi_\pm = \frac{\pi \Theta_E^2}{4} \frac{R_0^2}{2M D_1} \frac{\omega_0^2}{\omega^2} \left(1 - \frac{\omega_0^2}{\omega^2} - \frac{\omega_0^2 \omega_c}{\omega^2} f_\pm(\omega_c, \omega_0)\right)^{-1}.$$  

The brightness magnification of the source can be calculated through the formula

$$\mu = \sum_k \left| \frac{\theta_k d\theta_k}{\beta d\beta} \right|,$$

where $N$ is the number of images of the source (star). In this case, the lens equation can be written as

$$\theta^3 - \beta \theta^2 - \Theta_\pm^2 \theta - \Phi_\pm = 0.$$  

In order to solve the equation, we introduce the new variable $x = \theta - \beta \frac{2}{3}$, which we plug into Eq. (3.22). We get

$$x^3 + px^2 + q = 0,$$  

where

$$p = -\frac{1}{3} \beta^2 - \Theta_\pm^2,$$  

$$q = -\frac{2}{27} \beta^3 - \frac{1}{3} \beta \Theta_\pm^2 - \Phi_\pm.$$  

Note that Eq. (3.23) has three real solutions only in case the following condition holds

$$\frac{q^2}{4} + \frac{p^3}{27} = 0,$$

and those solutions have the form

$$x_k = 2 \sqrt[3]{s} \cos \left(\frac{\gamma + 2\pi k}{3}\right), \quad k = 0, 1, 2,$$

where

$$s = \sqrt[3]{\frac{p^3}{27}}, \quad \cos \gamma = -\frac{q}{2r}.$$
The magnification for the gravitational lens surrounded by an inhomogeneous magnetized plasma assumes the form

\[
\mu = \sum_{k} \left| \frac{\theta_k}{\beta} \frac{d\theta_k}{d\beta} \right| = \sum_{k} \left| \left( \frac{x_k}{\beta} + \frac{1}{3} \right) \left( \frac{dx_k}{d\beta} + \frac{1}{3} \right) \right| = \sum_{k} \frac{1}{3\beta} \left( 2\sqrt{3} \cos \gamma + 2\pi k + \frac{\beta}{3} \right) \times \left( 1 - 2\sqrt{3} \frac{d\gamma}{d\beta} \sin \gamma + 2\pi k + \frac{2}{3} \frac{ds}{\sqrt{s^2} d\beta} \cos \gamma + \frac{2\pi k}{3} \right). \tag{3.27}
\]

It is now easy to calculate the Einstein ring in the case of an inhomogeneous plasma by setting \(\beta = 0\) in Eq. (3.26). The three different values of the Einstein ring are

\[
\Theta_{1,\pm}^p = \frac{2}{\sqrt{3}} \Theta_\pm \cos \left[ \frac{1}{3} \arccos \left( \frac{3\sqrt{3} \Phi_\pm}{2 \Theta_\pm^3} \right) \right], \quad \tag{3.28}
\]

\[
\Theta_{2,\pm}^p = \frac{2}{\sqrt{3}} \Theta_\pm \cos \left[ \frac{1}{3} \arccos \left( \frac{3\sqrt{3} \Phi_\pm}{2 \Theta_\pm^3} + \frac{2\pi}{3} \right) \right], \quad \tag{3.29}
\]

\[
\Theta_{3,\pm}^p = \frac{2}{\sqrt{3}} \Theta_\pm \cos \left[ \frac{1}{3} \arccos \left( \frac{3\sqrt{3} \Phi_\pm}{2 \Theta_\pm^3} + \frac{4\pi}{3} \right) \right]. \tag{3.30}
\]

4 Conclusions

In this paper we have studied the gravitational lensing in the weak field approximation, extending previous work in the literature. We have considered a plasma and a magnetic field around a gravitational source. Our results can be summarized as follows:

- In the presence of a magnetic field, we may observe the split of the Einstein ring, as the counterpart of the Zeeman effect. When the cyclotron frequency approaches the plasma frequency, the size and the form of the ring change because of the presence of a resonance state. This is a pure magnetic effect and can potentially help to study magnetic fields through gravitational lensing effects.

- We have studied the magnification of the image source due to weak lensing in the presence of a homogeneous plasma and of a magnetic field. Due to the magnetic “Zeeman effect”, the magnification splits into two additional components with respect to the unmagnetized plasma case.

- We have also studied the time delay due to the geometry and the gravitational field around a gravitational source. We found that the presence of a plasma and of a magnetic field sufficiently changes the time delay depending on the angle \(\beta\).

- As a toy model, we have considered a power law density plasma. Inhomogeneities in the plasma also lead to image source magnifications. We found that an inhomogeneous plasma increases the source image magnification.
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References

[1] B. Paczynski, Gravitational microlensing by the galactic halo, Astrophys J. 304 (May, 1986) 1–5.
[2] C. Alcock, C. W. Akerlof, R. A. Allsman, T. S. Axelrod, D. P. Bennett, S. Chan et al., Possible gravitational microlensing of a star in the Large Magellanic Cloud, Nature 365 (Oct., 1993) 621–623, [astro-ph/9309052].
[3] E. Aubourg, P. Barayre, S. Bréhin, M. Gros, M. Lachièze-Rey, B. Laurent et al., Evidence for gravitational microlensing by dark objects in the Galactic halo, Nature 365 (Oct., 1993) 623–625.
[4] A. Udalski, M. Szymanski, J. Kubiak, M. Kubiak, W. Krzeminski, M. Mateo et al., The optical gravitational lensing experiment. Discovery of the first candidate microlensing event in the direction of the Galactic Bulge, Acta Astronomica 43 (July, 1993) 289–294.
[5] B. Paczynski, Gravitational Microlensing in the Local Group, Annual Review of Astronomy and Astrophysics 34 (1996) 419–460, [astro-ph/9604011].
[6] V. Perlick, Gravitational Lensing from a Spacetime Perspective, Living Reviews in Relativity 7 (Sept., 2004) 9.
[7] O. Y. Tsupko and G. S. Bisnovatyi-Kogan, Relativistic Rings due to Schwarzschild Gravitational Lensing, Gravitation and Cosmology 15 (June, 2009) 184–187.
[8] K. K. Nandi, Y.-Z. Zhang and A. V. Zakharov, Gravitational lensing by wormholes, Phys. Rev. D 74 (July, 2006) 024020, [gr-qc/0602062].
[9] J. Schee and Z. Stuchlík, Gravitational lensing and ghost images in the regular Bardeen no-horizon spacetimes, JCAP 6 (June, 2015) 48, [1501.00835].
[10] R. M. Wald, Black hole in a uniform magnetic field, Phys. Rev. D. 10 (Sept., 1974) 1680–1685.
[11] A. N. Aliev, D. V. Galtsov and V. I. Petukhov, Negative absorption near a magnetized black hole - Black hole masers, Astrophys. Space Sci. 124 (July, 1986) 137–157.
[12] A. N. Aliev and D. V. Gal’tsov, REVIEWS OF TOPICAL PROBLEMS: “Magnetized” black holes, Soviet Physics Uspekhi 32 (Jan., 1989) 75–92.
[13] A. N. Aliev and N. Özdemir, Motion of charged particles around a rotating black hole in a magnetic field, Mon. Not. R. Astron. Soc. 336 (Oct., 2002) 241–248, [gr-qc/0208025].
[14] A. N. Aliev and V. P. Frolov, Five-dimensional rotating black hole in a uniform magnetic field: The gyromagnetic ratio, Phys. Rev. D 69 (Apr., 2004) 084022, [arXiv:hep-th/0401095].
[15] V. P. Frolov, Weakly magnetized black holes as particle accelerators, Phys. Rev. D. 85 (Jan., 2012) 024020, [1110.6274].
[16] V. P. Frolov and A. A. Shoom, *Motion of charged particles near a weakly magnetized Schwarzschild black hole*, *Phys. Rev. D.* **82** (Oct., 2010) 084034, [1008.2988].

[17] B. Toshmatov, A. Abdujabbarov, B. Ahmedov and Z. Stuchlík, *Motion and high energy collision of magnetized particles around a Hořava-Lifshitz black hole*, *Astrophys Space Sci* **360** (Nov., 2015) 19.

[18] Z. Stuchlík, J. Schee and A. Abdujabbarov, *Ultra-high-energy collisions of particles in the field of near-extreme Kehagias-Sfetsos naked singularities and their appearance to distant observers*, *Phys. Rev. D* **89** (May, 2014) 104048.

[19] A. A. Abdujabbarov, B. J. Ahmedov and V. G. Kagramanova, *Particle motion and electromagnetic fields of rotating compact gravitating objects with gravitomagnetic charge*, *General Relativity and Gravitation* **40** (Dec., 2008) 2515–2532, [0802.4349].

[20] A. Abdujabbarov, B. Ahmedov, O. Rahimov and U. Salikhbaev, *Magnetized particle motion and acceleration around a Schwarzschild black hole in a magnetic field*, *Physica Scripta* **89** (Aug., 2014) 084008.

[21] A. Abdujabbarov and B. Ahmedov, *Test particle motion around a black hole in a braneworld*, *Phys. Rev. D* **81** (Feb., 2010) 044022, [0905.2730].

[22] A. Abdujabbarov, B. Ahmedov and A. Hakimov, *Particle motion around black hole in Hořava-Lifshitz gravity*, *Phys. Rev. D* **83** (Feb., 2011) 044053, [1101.4741].

[23] A. A. Abdujabbarov, B. J. Ahmedov, S. R. Shaymatov and A. S. Rakhmatov, *Penrose process in Kerr-Taub-NUT spacetime*, *Astrophys Space Sci* **334** (Aug., 2011) 237–241, [1105.1910].

[24] A. A. Abdujabbarov, A. A. Tursunov, B. J. Ahmedov and A. Kuvatov, *Acceleration of particles by black hole with gravitomagnetic charge immersed in magnetic field*, *Astrophys Space Sci* **343** (Jan., 2013) 173–179, [1209.2680].

[25] A. A. Abdujabbarov, B. J. Ahmedov and N. B. Jurayeva, *Charged-particle motion around a rotating non-Kerr black hole immersed in a uniform magnetic field*, *Phys. Rev. D.* **87** (Mar., 2013) 064042.

[26] J. A. Petterson, *Magnetic field of a current loop around a Schwarzschild black hole*, *Phys. Rev. D* **10** (Nov., 1974) 3166–3170.

[27] J. Kovář, P. Slaný, C. Cremaschini, Z. Stuchlík, V. Karas and A. Trova, *Electrically charged matter in rigid rotation around magnetized black hole*, *Phys. Rev. D* **90** (Aug., 2014) 044029, [1409.0418].

[28] F. De Paolis, M. Giordano, G. Ingrosso, L. Manni, A. Nucita and F. Strafella, *The Scales of Gravitational Lensing*, *Universe* **2** (Mar., 2016) 6, [1604.06601].

[29] G. Bisnovatyi-Kogan and O. Tsupko, *Gravitational Lensing in Presence of Plasma: Strong Lens Systems, Black Hole Lensing and Shadow*, *Universe* **3** (July, 2017) 57.

[30] Z. Stuchlík and J. Schee, *Optical effects related to Keplerian discs orbiting Kehagias-Sfetsos naked singularities*, *Classical and Quantum Gravity* **31** (Oct., 2014) 195013, [1402.2891].

[31] J. Schee and Z. Stuchlík, *Optical Phenomena in the Field of Braneworld Kerr Black Holes*, *International Journal of Modern Physics D* **18** (2009) 983–1024, [0810.4445].

[32] V. Perlick, *Exact gravitational lens equation in spherically symmetric and static spacetimes*, *Phys. Rev. D* **69** (Mar., 2004) 064017, [gr-qc/0307072].

[33] A. Grenzebach, V. Perlick and C. Lämmerzahl, *Photon regions and shadows of Kerr-Newman-NUT black holes with a cosmological constant*, *Phys. Rev. D* **89** (June, 2014) 124004, [1403.5234].

[34] A. Rogers, *Frequency-dependent effects of gravitational lensing within plasma*, *Mon. Not. R. Astron. Soc.* **451** (July, 2015) 17–25, [1505.06790].
A. Grenzebach, V. Perlick and C. Lämmerzahl, *Photon Regions and Shadows of Accelerated Black Holes*, arXiv:1503.03036 (Mar., 2015), [1503.03036].

O. Y. Tsupko and G. S. Bisnovatyi-Kogan, *Gravitational bending of light rays in plasma*, in *American Institute of Physics Conference Series* (S. K. Chakrabarti, A. I. Zhuk and G. S. Bisnovatyi-Kogan, eds.), vol. 1206 of *American Institute of Physics Conference Series*, pp. 180–187, Jan., 2010, DOI.

V. S. Morozova, B. J. Ahmedov and A. A. Tursunov, *Gravitational lensing by a rotating massive object in a plasma*, *Astrophys Space Sci* 346 (Aug., 2013) 513–520.

O. Y. Tsupko and G. S. Bisnovatyi-Kogan, *Gravitational lensing in the presence of plasmas and strong gravitational fields*, *Gravitation and Cosmology* 20 (July, 2014) 220–225.

V. Perlick, O. Y. Tsupko and G. S. Bisnovatyi-Kogan, *Influence of a plasma on the shadow of a spherically symmetric black hole*, *Phys. Rev. D* 92 (Nov., 2015) 104031, [1507.04217].

V. Perlick and O. Y. Tsupko, *Light propagation in a plasma on Kerr spacetime: Separation of the Hamilton-Jacobi equation and calculation of the shadow*, ArXiv e-prints (Feb., 2017), [1702.08768].

A. Abdujabbarov, B. Toshmatov, J. Schee, Z. Stuchlík and B. Ahmedov, *Gravitational lensing by regular black holes surrounded by plasma*, *International Journal of Modern Physics D* 26 (2017) 1741011–187.

A. Abdujabbarov, F. Atamurotov, Y. Kucukakca, B. Ahmedov and U. Camci, *Shadow of Kerr-Taub-NUT black hole*, *Astrophys. Space Sci.* 344 (Apr., 2013) 429–435.

F. Atamurotov, A. Abdujabbarov and B. Ahmedov, *Shadow of rotating non-Kerr black hole*, *Phys. Rev. D* 88 (Sept., 2013) 064004.

F. Atamurotov, A. Abdujabbarov and B. Ahmedov, *Shadow of rotating Hořava-Lifshitz black hole*, *Astrophys. Space Sci.* 348 (Nov., 2013) 179–188.

F. Atamurotov, B. Ahmedov and A. Abdujabbarov, *Optical properties of black holes in the presence of a plasma: The shadow*, *Phys. Rev. D* 92 (2015) 084005, [1507.08131].

A. A. Abdujabbarov, L. Rezzolla and B. J. Ahmedov, *A coordinate-independent characterization of a black hole shadow*, *Mon. Not. R. Astron. Soc.* 454 (Dec., 2015) 2423–2435, [1503.09054].

A. Abdujabbarov, M. Amir, B. Ahmedov and S. G. Ghosh, *Shadow of rotating regular black holes*, *Phys. Rev. D* 93 (May, 2016) 104004, [1604.03809].

A. Abdujabbarov, B. Juraev, B. Ahmedov and Z. Stuchlík, *Shadow of rotating wormhole in plasma environment*, *Astrophys Space Sci.* 361 (2016) 226.

A. Hakimov and F. Atamurotov, *Gravitational lensing by a non-Schwarzschild black hole in a plasma*, *Astrophys Space Sci.* 361 (Mar., 2016) 112.

A. Abdujabbarov, B. Toshmatov, Z. Stuchlík and B. Ahmedov, *Shadow of the rotating black hole with quintessential energy in the presence of plasma*, *International Journal of Modern Physics D* 26 (2017) 1750051–239.

R. A. Breuer and J. Ehlers, *Propagation of high-frequency electromagnetic waves through a magnetized plasma in curved space-time. I*, *Proceedings of the Royal Society of London Series A* 370 (Mar., 1980) 389–406.

R. A. Breuer and J. Ehlers, *Propagation of high-frequency electromagnetic waves through a magnetized plasma in curved space-time. II - Application of the asymptotic approximation*, *Proceedings of the Royal Society of London Series A* 374 (Jan., 1981) 65–86.

R. A. Breuer and J. Ehlers, *Propagation of electromagnetic waves through magnetized plasmas in arbitrary gravitational fields*, *Astronomy and Astrophysics* 96 (Mar., 1981) 293–295.
[54] G. Preti, *On charged particle orbits in dipole magnetic fields around Schwarzschild black holes*, *Classical and Quantum Gravity* 21 (July, 2004) 3433–3445.

[55] J. L. Synge, *Relativity: The General Theory*. North-Holland, Amsterdam, 1960.

[56] G. S. Bisnovatyi-Kogan and O. Y. Tsupko, *Gravitational lensing in a non-uniform plasma*, *Monthly Notices of the Royal Astronomical Society* 404 (June, 2010) 1790–1800, [1006.2321].

[57] M. Y. Piotrovich, N. A. Silant’ev, Y. N. Gnedin and T. M. Natsvlishvili, *Magnetic Fields of Black Holes and the Variability Plane*, ArXiv e-prints (Feb., 2010) , [1002.4948].

[58] A.-K. Baczko, R. Schulz, M. Kadler, E. Ros, M. Peruchò, T. P. Krichbaum et al., *A highly magnetized twin-jet base pinpoints a supermassive black hole*, *Astronomy and Astrophysics* 593 (Sept., 2016) A47, [1605.07100].

[59] V. L. Fish, S. S. Doeleman, C. Beaudoin, R. Blundell, D. E. Bolin, G. C. Bower et al., *1.3 mm Wavelength VLBI of Sagittarius A*: Detection of Time-variable Emission on Event Horizon Scales, *Astrophys. J. Lett.* 727 (Feb., 2011) L36, [1011.2472].

[60] M. D. Johnson, V. L. Fish, S. S. Doeleman, D. P. Marrone, R. L. Plambeck, J. F. C. Wardle et al., *Resolved magnetic-field structure and variability near the event horizon of Sagittarius A*, *Science* 350 (Dec., 2015) 1242–1245, [1512.01220].

[61] J.-F. Donati, F. Paletou, J. Bouvier and J. Ferreira, *Direct detection of a magnetic field in the innermost regions of an accretion disk*, *Nature* 438 (Nov., 2005) 466–469, [astro-ph/0511695].

[62] D. Narasimha and S. M. Chitre, *Gravitational Lens Systems to probe Extragalactic Magnetic Fields*, ArXiv e-prints (Feb., 2008) , [0802.4044].

[63] M. Sereno, *Gravitational Faraday rotation in a weak gravitational field*, *Phys. Rev. D* 69 (Apr., 2004) 087501, [astro-ph/0401295].

[64] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, Course of Theoretical Physics, Volume 2. Elsevier Butterworth-Heinemann, Oxford, 2004.