Cosmological black holes: the spherical perfect fluid collapse with pressure in a FRW background

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Abstract

We have constructed a spherically symmetric structure model in a cosmological background filled with perfect fluid with non-vanishing pressure as an exact solution of the Einstein equations using the Lemaître solution. To study its local and quasi-local characteristics, including the novel features of its central black hole, we have suggested an algorithm to integrate the equations numerically. The result shows intriguing effects of the pressure inside the structure. The evolution of the central black hole within the FRW Universe, its decoupling from the expanding parts of the model, the structure of its space-like apparent horizon, the limiting case of the dynamical horizon tending to a slowly evolving horizon, and the decreasing mass infall to the black hole is also studied. The cosmological redshift of a light emitted from the cosmological structure to an observer in the FRW background is also calculated. This cosmological redshift includes local and cosmic parts that are explicitly separated. We have also formulated a modified NFW density profile for a structure to match the exact solution conditions.

Keywords: cosmological black hole, black hole apparent horizon, light redshift
1. Introduction

The term cosmological black hole (CBH) is used to describe a collapsing structure within an otherwise expanding Universe after the radiation era. This is different from the term astrophysical black holes (ABHs) coined for the applications of static or stationary asymptotically flat black holes in astrophysics (see for example the recent paper [1]). CBHs are fundamentally different from the ABHs as they are non-stationary and have a dynamical horizon, although they are just as interesting as ABHs for astrophysical applications and well suited to be used as a test arena for fundamental physics in strong gravity and quasi-local phenomena in weak gravity regimes. Despite extensive studies in relativistic structure formation and static and stationary black holes, we still have little information based on exact general relativistic studies regarding the main features of such overdense cosmological regions including its central black hole.

Since the early beginning of the discovery of the expansion of the Universe, people have been looking for models describing an overdense region in a cosmological background ([2]; see also [3]). In the matter-dominated era, the initial expansion of an overdense region within an expanding Universe will finally decouple from the background and collapse to a dynamical black hole, in contrast to the overdensity regions in the radiation-dominated era in the early Universe where the density perturbation has to be of the order of the horizon to collapse to a primordial black hole (PBH). In the matter-dominated era, we expect a CBH within the resulting structure and a very weak gravitational field outside it, differing from the familiar Schwarzschild one in that it is neither static nor asymptotically flat [4]. Therefore, such cosmological structures, if based on exact solutions of general relativity and not produced by a cut-and-paste technology, are very interesting laboratories to study not only general relativistic non-linear structures, their quasi-local features such as mass and horizon [4], black hole thermodynamics and information loss puzzle [5, 20], but the validity of the weak field approximation as well [7]. After all, the Universe is evolving and asymptotically not Minkowskian. Therefore, one needs to have a dynamical model for a black hole, to be compared with the familiar results in the literature on black holes within a static and asymptotically flat spacetime [8], where global concepts such as the event horizon are not defined. Nevertheless, depending on the specific cosmological model we may be able to define a hypersurface separating the trapped null outgoing geodesics from those approaching (future) infinity. This hypersurface effectively plays the role of the event horizon [4]. Numerically, this hypersurface is determined by tracing the outgoing null geodesics at late times. We will refer to it as the quasi event horizon.

The need for a local definition of black holes and their horizons has led us to concepts such as Hayward’s trapping horizon [9], the isolated horizon [10], Ashtekar and Krishnan’s dynamical horizon (DH) [11], and Booth and Fairhurst’s slowly evolving horizon [12]. The CBH we are going to study is an excellent example of testing these different concepts and their relationship in addition to understanding its difference to an asymptotically Minkowskian and static black hole. Now, a widely used metric to describe the gravitational collapse of a spherically symmetric dust cloud is the so-called Tolman–Bondi–Lemaître (LTB) metric [13]. Exact general relativistic models for the dynamic of an asymptotically FLRW structure leading to a central dynamical black hole based on a LTB metric with no pressure has been reported in [4, 14]. These models may be extended to a perfect fluid with a non-zero pressure, the so-called Lemaître models [17]. Our interest is now using these inhomogeneous cosmological solutions of the Einstein equations as a model for a cosmic structure with non-vanishing internal pressure leading to a dynamical CBH within an FRW matter-dominated expanding Universe. We, therefore, try an ideal fluid with an interpolating pressure function that is non-zero inside the structure and vanishes at infinity where we expect a matter-dominated
FRW Universe. To achieve this goal, we have to avoid any cut-and-paste technology when finding the solution. Any internal solution pasted to an FRW Universe does not reflect the dynamics of the inhomogeneous Universe due to the homogeneity of the FRW Universe outside the structure. In contrast, our model is just asymptotically FRW reflecting the full relativistic local and quasi-local effects due to the cosmic fluid. The ideal fluid is modeled such that the non-vanishing pressure inside the structure goes smoothly to a matter-dominated Universe far from the structure. A similar problem studied extensively in literature in the last 40 years is the issue of primordial black holes (PBHs). These are structures within the radiation-dominated phase of the Universe, usually in the late phase of inflation, due to the superhorizon perturbations [18]. PBHs are usually based on the same Lemaître cosmological solutions of Einstein equations, which are spherically symmetric and inhomogeneous. The perturbation is formalized by assuming that the horizon size $R_H$ is smaller than the structure size $R_S$, or by assuming $\epsilon \equiv \frac{R_H}{R_S} < 1$, where the relevant quantities are expanded in the powers of $\epsilon$. Using the same Lemaître solution, although not in the same coordinates, we are interested in cases where $\epsilon \gg 1$, i.e. structures are much smaller than the horizon size as expected for cosmological structures within the matter-dominated FRW Universe. Therefore, we expect to recover novel features and we also have to propose a new algorithm for how to solve the field equations numerically. In addition, we are seeking different information, such as the kind of black holes we may encounter and the characteristics of their apparent and event horizons, the effect of the pressure inside the structure on the collapse behavior and the matter flux. There are many other issues in the black hole literature to be faced in future studies, such as the definition of a non-rotating spherically symmetric dynamical black hole within an expanding Universe, its differences to the simple Schwarzschild or Kerr black hole, its event and apparent horizon and their features, the internal structure of CBHs, the rate of collapse, the effect of pressure inside the structure and its probable non-Newtonian and non-linear novel effects due to the non-vanishing matter outside, and the information puzzle. That is why the study of CBHs goes beyond the study of PBHs and is a new arena for novel black hole terminology.

Section 2 is an introduction to the spherically symmetric inhomogeneous perfect fluid cosmological models. In section 3 the result of the numerical integration is reported, expressing the main characteristics of our model assuming different pressure profiles. In section 4, the redshift of the light coming from near the structure is investigated. In section 5 we develop a modified NFW density profile for a cluster of galaxies, taking into account the requirements of our solution, and we study its evolution. We will then discuss the results in section 6. Throughout the paper we assume $8\pi G = c = 1$.

2. The general spherically symmetric solution

Consider a general inhomogeneous spherically symmetric spacetime [17] filled with a perfect fluid and a metric expressed in the comoving coordinates, $x^\mu = (t, r, \theta, \phi)$:

$$ds^2 = -e^{2\sigma} dt^2 + e^{\lambda} dr^2 + R^2 d\Omega^2,$$

where $\sigma = \sigma(t, r)$, $\lambda = \lambda(t, r)$ are functions to be determined, $R = R(t, r)$ is the physical radius, and $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ is the metric of the unit 2-sphere. The energy momentum tensor of the perfect fluid is given by

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + g^{\mu\nu} p,$$

where $\rho = \rho(t, r)$ is the mass–energy density, $p = p(t, r)$ is the pressure, and $u^\mu = (e^{-\sigma}, 0, 0, 0)$ is the perfect fluid four-velocity.
2.1. Field equations

In addition to the Einstein field equations, $G^{\mu\nu} = \kappa T^{\mu\nu} - g^{\mu\nu}\Lambda$, we will use the conservation equations in the form

$$\frac{2e^{2\sigma}}{(\rho + p)} \nabla{T}^{\mu\nu} = \lambda + \frac{2\rho}{(\rho + p)} + 4\frac{\dot{R}}{R} = 0 \quad (3)$$

$$\frac{e^{\lambda}}{(\rho + p)} \nabla{T}^{\mu\nu} = \sigma' + \frac{\dot{p}}{p + \rho} = 0, \quad (4)$$

where the dot means the derivative with respect to $t$, and the prime means the derivative with respect to $r$. The Einstein equations finally lead to following equations

$$\frac{\partial}{\partial r} \left[ R + R\dot{R}^2 e^{-2\sigma} - R\dot{R}^2 e^{-\lambda} - \frac{1}{3} \Delta R^3 \right] = \kappa \rho R^2 R', \quad (5)$$

and

$$\frac{\partial}{\partial r} \left[ R + R\dot{R}^2 e^{-2\sigma} - R\dot{R}^2 e^{-\lambda} - \frac{1}{3} \Delta R^3 \right] = -\kappa \rho R^2 \dot{R}. \quad (6)$$

The term in the brackets is related to the Misner-Sharp mass, $M$, defined by

$$\frac{2M}{R} = \dot{R}^2 e^{-2\sigma} - R\dot{R}^2 e^{-\lambda} + 1 - \frac{1}{3} \Delta R^3. \quad (7)$$

Equations (5) and (6) may now be written as

$$\kappa \rho = \frac{2M'}{R^2 R'}, \quad \kappa \rho' = -\frac{2M}{R^2 R}. \quad (8)$$

We may write equation (7) in the form of an evolution equation of the model:

$$\dot{R} = \pm e^\sigma \sqrt{\frac{2M}{R} + f + R\frac{\Lambda R^2}{3}}, \quad (9)$$

where

$$f(t, r) = R^2 e^{-\lambda} - 1 \quad (10)$$

is the curvature term, or twice the total energy of the test particle at $r$ (analogous to $f(r)$ in the LTB model). In this paper we have set $\Lambda = 0$. Note that $R(t, r)$ cannot be directly obtained from this equation because of the unknown functions $\lambda, \sigma$, and $M$.

The metric functions $g_{tt}$ and $g_{rr}$ may be obtained by integrating (4) and (3):

$$\sigma = c(t) - \int_{\rho_0}^{\rho} \frac{\dot{p}}{(\rho + p)} \left. dr \right|_{t=\text{const}} = \sigma_0 - \int_{\rho_0}^{\rho} \left. \left( \frac{\partial \rho}{\partial \rho} \right) \right|_{t=\text{const}} d\rho, \quad (11)$$

and

$$\lambda = \lambda_0(r) - 2 \int_{\rho_0}^{\rho} \frac{d\rho}{(\rho + p(\rho))} - 4 \ln \left( \frac{R}{R_0} \right) \left. \right|_{t=\text{const}}, \quad (12)$$

where $c(t)$ and $\lambda_0(r)$ are arbitrary functions of integration (see [17] for more details). In the case of $c(t)$ it is easily seen that requiring our coordinates to lead to the LTB synchronous
ones for \( p = 0 \) leads to \( c(t) = 0 \). We also note that according to (10) and the LTB coordinate conditions, the choice of \( \lambda_0(r) \) is equivalent to the choice of \( f(t_0, r) = f_0(r) \). One may prefer to choose \( f_0(r) \) and then calculate \( \lambda_0(r) \) from \( e^{\lambda_0} = R_0^{1/2}/(1 + f_0) \).

We therefore have five unknowns \( p, \rho, \sigma, \lambda, \) and \( R \), four dynamical equations \( \dot{p}, M, \dot{\lambda}, \) and \( \dot{R} \), in addition to an equation of state \( p = p(\rho) \), and the definition of the mass \( M \) (7). This defines a numerical algorithm to find solutions for the dynamics of our spherical structure after assuming the initial conditions. The algorithm for solving the coupled PDEs can be found in [23].

### 3. Equation of state and the results

We are now ready to specify the equation of state and integrate the model to see its characteristics. To have a comparative discussion of the results we consider two types of equations of state: a perfect fluid with a constant state function, \( p = w\rho \), and a more general case with the equation of state \( p = ws(r)\rho \) matching our needs for a structure with pressure inside the structure and a pressure-less matter-dominated Universe far from the structure. We may then choose the function \( s(r) \) in a way that the pressure becomes zero at infinity, i.e. at \( r \gg r_0 \). A suitable choice is \( s(r) = -e^{-r}r_0 \) where \( r_0 \) is the distance of the void (boundary of the expanding and collapsing phase) from the center of the structure. This is a more realistic model to describe a black hole collapse within the FRW Universe and to see the effect of the inside pressure while the Universe outside is matter-dominated with no pressure.

The model we envisage starts from a small inhomogeneity within an FRW Universe. The density profile should be such that the metric outside the structure tends to FRW independent of the time while the central overdensity region undergoes a collapse after some initial expansion. At the initial conditions, where the density contrast of the overdensity region is still too small, we may assume that the metric is almost FRW or LTB; the density contrast and the pressure do not play a significant role. The dynamics of the Lemaître Universe will give us the expected structure at late times. To choose the initial conditions at the time \( t_0 \), we will therefore use an LTB solution with a negative curvature function. We have in fact tried both examples of LTB and FRW initial data and found no significant difference between the final Lemaître solutions.

Now, let us choose the two initial functions \( f(r) = f(t_0, r) \) and \( M(r) = M(t_0, r) \) in the following way to achieve an asymptotically FRW final solution:

\[
\begin{align*}
    f(r) &= -\frac{1}{b}re^{-r}, \\
    M(r) &= \frac{1}{a}r^{3/2}\left(1 + r^{3/2}\right).
\end{align*}
\]

Far from the central overdensity region we have

\[
\begin{align*}
    \lim_{r \to \infty} f(r) &= 0, \\
    \lim_{r \to \infty} M(r) &= \frac{r^3}{a},
\end{align*}
\]

showing the asymptotically FRW behavior of the initial conditions. The corresponding LTB solution of the Einstein equations now gives us \( R(r) = R_0(r) \) at the initial time \( t_0 \). By choosing suitable \( a \) and \( b \) the ‘reality condition’, [21]
will be satisfied. This condition can be obtained from equation (9). In addition to the ‘reality condition’, the ‘weak energy condition’ [21] must be satisfied at every \( r \) and \( t \):

\[
\rho(r, t) \geq 0, \quad \rho(r, t) + p(r, t) \geq 0. \tag{18}
\]

It can be seen from figure 1 that our initial conditions give rise to \( \rho(r, t) \geq 0 \) at every \( t \) and \( r \).

Assuming an equation of state that satisfies the weak energy condition, \( p(r, t) \geq 0 \), is now enough to numerically calculate the necessary dynamical functions of the model. Specifically, by looking at \( \dot{R}(t, r) \) and \( \rho(t, r) \) we may extract information on how the central region starts collapsing after the initial expansion and how a black hole with distinct apparent and event horizons develops while the outer region expands as a familiar FRW Universe. We may also find out the difference in the case of the pressure-less model. It will also show if and how the very weak gravity outside the collapsed structure affects the dynamic of the central structure in comparison to the familiar Schwarzschild model. The results of the numerical calculation for both equations of state are given in the following sections.

3.1. The density behavior

The density profiles for both equations of state as a function of \( t \) and \( r \) are given in figure 1. A comparison of these figures shows the effect of the pressure on the development of the central black hole. Obviously in case of non-vanishing pressure outside the structure, the collapse is highlighted with a more steep density profile. The overdensity region in the collapsing phase is always separated from the expanding underdensity region through a void not expressible in these figures. We will consider the deepest place of the void as the boundary of the structure. This boundary is always near the boundary of the contracting and the expanding region of the model structure.

In the numerical calculation at late times, the density near the center of the black hole approaches infinity, leading to numerical errors and crashing the algorithm. We refer to this effect as the black hole singularity, as discussed in [4, 23]. The cosmological singularities (big bang and big crunch) are not included in our numerical simulation due to the initial conditions being fixed at a time much later than the big bang in the matter-dominated era. Assuming the background to be the standard flat FLRW metric, there is no big crunch singularity in our model. By selecting other backgrounds such as the closed FLRW model, the big crunch singularity will play a crucial role in the development of singularities [4].
3.2. The pressure effect

Figures 2(a) and 3(a) show the behavior of the collapsing and the expanding regions for the equation of state $p = w_0 \rho$ by depicting the corresponding Lemaître Hubble parameter $\dot{R}/R$ versus the physical radius $R$. Figures 2(b) and 3(b) show similar data for the equation of state $p = w s(r) \rho$. Note that the function $s(r)$, as defined to get the matter-dominated FRW Universe at far distances, has no significant effect, and the qualitative behavior of the dynamics of the physical radius is independent of it. Therefore, as far as we are interested in the qualitative features of the model, we will just use the simple equation of state with $s(r) = 1$.

The place of separation between the expanding and collapsing region defined by $\dot{R} > 0$ and $\dot{R} < 0$ is almost coincident with the place of the void where we have defined as the boundary of the structure. Now, from the figures we realize that the effect of the pressure in different regions of the model and its comparison to the homogeneous FRW model are intriguing. As we know already from the Friedman equations in FRW models, the pressure adds up to the density and has an attractive effect, slowing down the expansion and leading to a more negative acceleration $\ddot{a} = -\frac{1}{6}(\dot{\rho} + 3p)$.

Figure 2. The behavior of the Hubble parameter $\dot{R}/R$ in the case of $p = w_0 \rho$. Evidently the pressure slows down the collapse velocity near the center of the structure.

Figure 3. The behavior of the Hubble parameter $\dot{R}/R$ in the case of $p = w s(r) \rho$. The features are qualitatively as in figure 2.
we have a contracting overdensity region, the behavior is counter-intuitive. Except for the case of vanishing pressure, in all the other cases the pressure begins somewhere to act classically like a repulsive force opposing the collapse of the structure. To see this more clearly, we have also depicted the acceleration in Figure 4. As we approach distances near the center, the negative acceleration in the FRW limit and even inside the void gradually increases to positive values, meaning that somewhere within the structure the contraction of the structure slows down due to the pressure like a classical fluid. Therefore, the pressure effect somewhere within the structure begins to act like a repulsive force in contrast to the outer regions where its attractive nature dominates. Note that the central black hole and its horizon have a much smaller radius than the region of the repulsive pressure effect we are discussing.

3.3. The apparent and quasi event horizon

The boundary of a dynamical black hole, where the area law and the black hole temperature are defined, is a non-trivial concept (see for example [4, 19] and [20]). Our model is again a concrete example of the behavior of both the apparent and quasi event horizon (QEH) of a dynamical structure within an expanding Universe. It is easily seen that the apparent horizon of our cosmological black hole is located at $R = 2M$ [14]. We note that after the formation of an apparent horizon the central density goes to infinity, indicating the formation of the singularity at the center. Therefore, the singularity is covered by the dynamical horizon and is not naked!

This apparent horizon is calculated in $t, r$ coordinates numerically. It is always space-like, tending to be light-like at late times. This can best be seen by comparing the slope of the apparent horizon relative to the light cone at every coordinate point of it. This is in contrast to the Schwarzschild black hole horizon where it is always light-like. At late times, however, we expect the apparent horizon to become approximately light-like and approaching the event horizon. This is reflected in figure 5. It is evident that $\frac{d}{dr}|_{\text{AH}} < \frac{d}{dr}|_{\text{QEH}}$ at all times on the apparent horizon, with the difference tending to zero at late times. Therefore, the apparent horizon is always a space-like dynamical horizon leading to a slowly varying horizon at late times [4, 11]. Note that the qualitative result is independent of the equation of state.

We now show how the dynamical horizon of our cosmological black hole becomes a slowly evolving horizon at late times. Let us first define the evolution parameter $\epsilon$ such that the
The tangent vector to the dynamical horizon, $V^\mu$, is given by

$$V^\mu = \ell^\mu - cn^\mu,$$

where the two vectors $\ell^\mu$ and $n^\mu$ are normal null vectors on a space-like two surface $S$ in the $(t, r)$ plane (see [11]). We expect $c$ to go to zero at late times in order for our dynamical horizon to become a slowly evolving horizon. In the case of our Lemaître model, $c$ is calculated to be

$$c = 2 \left( \frac{M' + wM'}{M' - wM' - R^2} \right)_{AH}.$$

The result of the numerical calculation for different equations of state and different state functions is given in figure 6. The decreasing behavior of the function $c$ over the course of time independent of the equation of state is evident. We may then conclude that the dynamical horizon of the cosmological black hole tends toward a slowly evolving horizon.
3.4. The effect of $\Lambda$ on the formation of the apparent horizon

Let us now explore how the addition of a cosmological term $\Lambda$ to the Einstein equations may influence the formation of the apparent horizon. The location of the apparent horizon is now defined by

$$2M = R - \frac{\Lambda}{3}R^3. \quad (21)$$

Now, for a typical structure like a Galaxy or a cluster of galaxies we have $\frac{\Lambda}{3}R^3 \ll \frac{2M}{R}$. Therefore, the location of the apparent horizon is almost the same as in the case of the vanishing cosmological constant, i.e. $2M \approx R$. We may then refer to figure 7 for the behavior of the apparent horizon.

3.5. Mass and matter flux

Due to the expanding background, we expect the matter flux into the dynamical black hole to be decreasing and the dynamical horizon to become a slowly evolving horizon over the course of time [14]. We know already that there is no unique concept of mass in general relativity corresponding to the Newtonian concept. The question of what general relativity tells us about the mass of a cosmological structure in a dynamical setting was discussed recently [16]. It was shown [15] that the Misner-Sharp quasi-local mass, $M$, is very close to the Newtonian mass.

Let us then take the Misner-Sharp mass for this black hole and calculate the corresponding matter flux into the black hole. In the case of the Lemaître model, the matter flux is given by

$$\left. \frac{dM(r, t)}{dt} \right|_{AH} = \left. \frac{\partial M(r, t)}{\partial t} \right|_{AH} + \left. \frac{\partial M(r, t)}{\partial r} \frac{dr}{dt} \right|_{AH} = M' \left. \frac{dr}{dt} \right|_{AH}. \quad (22)$$

The result of the numerical calculation is depicted in figure 8. Note how the pressure decreases the rate of matter flux into the black hole at late times.
4. Measuring the redshift in the Lemaître metric

In cosmology we are used to interpreting the cosmological redshift according to the homogeneous FRW model. What if the Universe is inhomogeneous? In the simplest case we are ready to model a source, a cosmological structure, within an otherwise homogeneous FRW model using our Lemaître model. We assume now an observer far from the source of light near the structure. What is then the redshift measured by this observer? Given that the metric is an exact solution of the Einstein equations, we expect the redshift to include all gravitational effects including not only the cosmological redshift but also the gravitational redshift due to the overdensity of the source.

The redshift in our model can be obtained as follows. Assume the light ray coming from a source $S$ located near to the structure in the center of our inhomogeneous Lemaître model and the observer $O$ somewhere within the FRW background having corresponding 4-velocities $u^\mu_{(S)}$ and $u^\mu_{(O)}$. Let $k^\mu \equiv dx^\mu/d\beta$ be the tangent vector to the null geodesic connecting the source to the observer. The corresponding redshift $z$, i.e. the frequency shift, is then defined as [22]

$$1 + z = \left[ \begin{array}{c} k_\mu u^\mu_{(S)} \\ k_\mu u^\mu_{(O)} \end{array} \right]_{\beta_1}$$  \hspace{1cm} (23)

where $[k_\mu u^\mu_{(S)}]_{\beta_1}$ and $[k_\mu u^\mu_{(O)}]_{\beta_2}$ are evaluated at the source and observer events. Let us assume the null geodesics to be radial, i.e. $k^\mu k_\mu = 0 \equiv k_\mu k^\mu$. For the metric in equation (1) we then have

$$k^\mu = \frac{e^{-\sigma}}{e^{\sigma}} k^\mu \Rightarrow \frac{dt}{dr} = \frac{e^{\sigma}}{e^{-\sigma}}$$  \hspace{1cm} (24)

Using now the geodesic equation, it is straightforward to show that

$$\frac{dk^\mu}{d\beta} = - \left[ \sigma' + \left( - \sigma + \frac{\lambda}{2} \right) \frac{e^{\sigma}}{e^{\sigma}} \right] k^\mu k'^\mu$$  \hspace{1cm} (25)

where $\beta$ is an affine parameter. Therefore,

$$\frac{dk^\mu}{k^\mu} = - \left[ \sigma' + \left( - \sigma + \frac{\lambda}{2} \right) \frac{e^{\sigma}}{e^{\sigma}} \right] d\beta k'^\mu = - \left[ \sigma' + \left( - \sigma + \frac{\lambda}{2} \right) \frac{e^{\sigma}}{e^{\sigma}} \right] dr.$$  \hspace{1cm} (26)
Integrating equation (26) we obtain

\[ k' = c_o \exp \left( - \int \left[ \sigma' + \left( -\dot{\sigma} + \frac{\dot{\lambda}}{2} \right) \frac{e^{\sigma}}{e^{\sigma_0}} \right] \, dr \right) \]  

(27)

where \( c_o \) is a constant.

Now, the 4-velocities are given by

\[ u^\mu_{(i)} = \delta^\mu_{\nu} e^{-\sigma (r, t_i)} \]  

(28)

\[ u^\mu_{(o)} = \delta^\mu_{\nu} e^{-\sigma (r, t_o)} \]  

(29)

Using equations (23), (28), (29), and (27) we obtain the cosmological redshift in the presence of a structure (\( z_{CBH} \)):

\[ 1 + z_{CBH} = \exp \left( \int_{r_o}^{r} \left[ \sigma' + \left( -\dot{\sigma} + \frac{\dot{\lambda}}{2} \right) \frac{e^{\sigma}}{e^{\sigma_0}} \right] \, dr \right) e^{\sigma_0} \]  

(30)

\[ 1 + z_{CBH} = \exp \left( \int_{r_o}^{r} \left( -\dot{\sigma} + \frac{\dot{\lambda}}{2} \right) \frac{e^{\sigma}}{e^{\sigma_0}} \, dr \right) \]  

(31)

Note that there was no need to calculate the \( k' \) due to \( k_r u^r = 0 \).

Now, equation (31) may be integrated numerically using the equations for \( \dot{\sigma}, \sigma' \), and \( \dot{\lambda} \) from [23] for any collapsing structure in an expanding FRW Universe. The necessary initial condition may be chosen as discussed in [23].

This redshift includes the familiar cosmological FRW part, \( z_c \), as well as the gravitational redshift, \( z_G \), due to the overdensity of the structure (the cosmological black hole). In general, we then expect it to be different from that of the corresponding homogeneous FRW model. In the special case of a homogeneous Universe without a structure, it obviously reduces to the familiar FRW cosmological redshift \( z_c = \frac{a(t)}{a(t_o)} - 1 \), lacking the contribution from the local gravitational redshift of the overdensity, \( z_G \). To see the difference between the exact inhomogeneous cosmological redshift according to our model and the sum \( z_G + z_c \), let us look at some specific models.

Take a CBH model with a mass 10\(^6\)M\(_\odot\) at a distance corresponding to \( z = 0.005 \) from the observer. The gravitational redshift according to the Schwarzschild metric is given by

\[ z_G \approx = \frac{1}{1 - \frac{2M(r, t)}{R_k}} - 1 \]  

(32)

Adding to it the FRW cosmological redshift

\[ z_c = \frac{a(t_o)}{a(t)} - 1 \]  

(33)

should give us the CBH redshift we have already calculated, i.e.

\[ z_{CBH} \approx = z_G + z_c \]  

(34)

The result of the numerical calculation is given in figure 9. We see that the exact \( z_{CBH} \) is always larger than the sum of the FRW cosmological redshift and the local Schwarzschild gravitational redshift, although the difference is smaller than the observational limit of accuracy. The difference \( z_{CBH} - z_c \) is shown in figure 10. The difference goes to zero for a source at distances far from the apparent horizon. There may be, however, cases where this
difference is not to be ignored. We leave it to a more detailed study in the future to see where this difference may be of any cosmological significance.

5. Modeling the evolution of a cluster of galaxies using the Lemaître metric: the case of A2061

Now we are ready to use our exact solution and the corresponding algorithm to study the collapse of a real Galaxy cluster within an otherwise expanding Universe using a fluid model. Clusters of galaxies are modeled using a spherically symmetric (isotropic) dark matter halo assumed to dominate the dynamics of the system. The density profile of such a halo is often described by the Navarro-Frenk-White (NFW) profile [24]:

$$\rho_{\text{NFW}}(r) = \frac{\delta_c \rho_c}{4 \pi \sigma_c^3 \left(1 + \frac{r}{\sigma_c} \right)^2}.$$  \hspace{1cm} (35)
where $r_s$ is a scale radius, $\rho_c$ is the critical density, and the characteristic overdensity $\delta_c$ is a function of cluster concentration $c = r_{200}/r_s$ given by

$$
\delta_c = \frac{200}{3} \frac{c^3}{\ln(1 + c) - \frac{c}{1 + c}}.
$$

The parameter $r_{200}$ is the radius at which the average interior density is $\rho_c$, approximately equal to the virialized overdensity, $\rho_{\text{virial}} = 178 \rho_c$. The NFW profile in equation (35) has three undesirable features to be applied to our case of a cosmological structure:

1. The density tends to infinity at small $r$.
2. The NFW mass diverges at large $r$ and cannot be matched to the FRW mass.
3. Due to the lack of a void, it does not represent a density profile for a structure within an expanding Universe.

We then have to modify it to remedy these deficiencies. To resolve the first problem, we introduce a maximum density at very small $r$:

$$
\rho_{1, \text{NFW}}(r) = \frac{\delta_c \rho_c}{(\frac{c}{r} + \frac{c}{r_s})^2}.
$$

The second problem will be resolved by introducing the truncation radius $r_t$:

$$
\rho_{2, \text{NFW}}(r) = \frac{\delta_c \rho_c}{(\frac{c}{r} + \frac{c}{r_s})^2 \left(1 + \frac{c}{r_s}\right)^2} \left(\frac{r_t^2}{r^2 + r_t^2}\right)^2.
$$

In order to resolve the third problem, we add a Gaussian density profile to match the overdensity region through a void to the cosmological background, as is necessary for any exact solution of the Einstein equations (see [6] and [7]):

$$
\rho_G(r) = a \exp \left( -\frac{(r - r_1)^2}{r_0} \right).
$$

The parameters $r_0$ and $r_1$ give us the freedom to adjust the location of the void as desired. For our model cluster, A2061 [26], we choose $c = 3$, $a = 1$ Mpc, and $c = 0.1$ to have $r_{200} = 1.723$ Mpc. Our metric solution has to approach FRW at large $r$ outside the cluster. Therefore, we add a homogeneous background density such that the final density profile at $t_0$ becomes

$$
\rho_{\text{cluster}}(r) = \rho_{2, \text{NFW}}(r) - \rho_G(r) + \rho_c.
$$

| $R/R_{\text{ah}}$ | $z_G$ | $z_{\text{CBH}}$ | $(z_{\text{CBH}} - (z_G + z_c))/z_{\text{CBH}}$ |
|----------------|-------|----------------|------------------------------------------------|
| 30             | 0.01709 | 0.022173 | 0.00374 |
| 50             | 0.01015 | 0.015211 | 0.00401 |
| 70             | 0.00722 | 0.012263 | 0.00350 |
| 100            | 0.00458 | 0.00959  | 0.00101 |

Table 1. Numerical values of figure 9.
Note that the Gaussian profile must be added in such a way that
\[
\int_0^{r_1} r^2 \rho_0 \, dr = \int_0^{r_1} r^2 \rho_2(r) \, dr,
\]
where \( r_1 \) is the distance at which the density approaches the background critical density. It guarantees that the mass due to the overdensity in the structure is compensated by the underdensity region within the void (see figure 11).

Now, by choosing \( R(t_0, r) = r \) and integrating equation (8) to determine \( M(t_0, r) \) as well as having the equation of state, we can solve the coupled evolution PDEs numerically. To have zero pressure at the FRW background, a good choice for the equation of state is \( p(t, r) = w(p(t, r) - p_c) \), in which \( p_c \) is just a function of time. The other initial functions are as before. Figure 11 shows the evolution of the Galaxy cluster. By modifying the NFW density profile to match the exact general relativistic requirements, we have therefore arrived at a model cluster based on the exact solution. We may even go further and see how much we may rely on the observation of the redshift of such a cluster and compare it to the model reflected in equation (31).

Based on the redshift of clusters estimated with spectroscopic data and the redshift of the brightest cluster galaxies (BCGs) [26], we may take the cosmological redshift of the Galaxy cluster A2061 to be \( z = 0.07845 \), being located at a distance around 330 Mpc. We may therefore model our Galaxy cluster such that its cosmological redshift is \( z_c = 0.07845 \); equation (31) will then tell us if we need to add the gravitational redshift as part of the cosmological one.

Fixing the observer distance at 330 Mpc and solving equation (31) we see that the gravitational redshift is about two orders of magnitude smaller than the cosmological redshift. Therefore, we have a well-defined dynamical model of the structure including a modified density profile fitted well with the observation. The effect of the gravitational redshift may be much bigger for supermassive black holes [25]. Any other density profile may also be modified and adapted to requirements as indicated in this section.

6. Discussion

We have studied the evolution of a structure made of perfect fluid with non-vanishing pressure as an exact solution of the Einstein equations within an otherwise expanding FRW Universe. The structure boundary is separated by a void from the expanding part of the
model, which is very much like an FRW Universe already near the void. We have noticed a counter-intuitive pressure effect somewhere inside the structure where the existence of the pressure slows down the collapse like a classical fluid, in contrast to distances far from the structure. The collapsed region develops a dynamical black hole with a space-like apparent horizon, in contrast to the Schwarzschild black hole. This apparent horizon tends toward a slowly evolving horizon, becoming light-like at late times with a decreasing matter flux into the black hole. We have, therefore, to conclude that the mere existence of cosmological matter, even dust, may have a significant effect on the central black hole, differentiating it from a Schwarzschild one irrespective of how small the density is outside the structure.

The light properties of these cosmological black holes can be interesting because most of the information about the black holes, galaxies and clusters comes from their lights. We have investigated the redshift of a light emitted near a cosmological structure to a distant observer. It was shown that the exact CBH redshift of light includes both local gravitational and cosmological redshifts of the structure. Therefore, in the era of precision cosmology we may be forced to consider the effects of inhomogeneities as seen in the CBH cosmological model for high redshift surveys.

We have also seen how to generalize the existing models of the density profiles of cold dark matter within large structures using the results of our structure model. Although the gravity may be too weak near a large cosmological structure, we cannot use the Newtonian approximation due to the non-local or quasi-local cosmological effects.

References

[1] Ellis G F R arXiv:1310.4771 [gr-qc]
Berti E 2013 Braz. J. Phys. 43 341
[2] McVittie G C 1933 Mon. Not. R. Astron. Soc. 93 325
Einstein A and Straus E G 1945 Rev. Mod. Phys. 17 120
Einstein A and Straus E G 1945 Rev. Mod. Phys. 18 148
[3] Valkenburg W 2012 Gen. Rel. Grav. 44 2449–76
Chen X, Shen Y-G and Farooqi V 2011 Phys. Rev. D 84 104047
Bolejko K, Celerier M-N and Krasiński A 2011 Class. Quantum Grav. 28 164002
Abdalla E, Afshordi N, Fontanini M, Gueriente D C and Papantonopoulos E 2014 Phys. Rev. D 89 104018
[4] Firouzjaee J T and Mansouri R 2010 Gen. Rel. Grav. 42 2431
[5] Giddings S B 2012 Phys. Rev. D 85 124063
[6] Khakshournia S and Mansouri R 2002 Phys. Rev. D 65 027302
[7] Mood M P, Firouzjaee J T and Mansouri R arXiv:1304.5062 [astro-ph.CO]
Mood M P, Firouzjaee J T and Mansouri R 2013 Phys. Rev. D 88 083011
[8] Wald R M 1984 General Relativity (Chicago, IL: University of Chicago Press)
[9] Hayward S A 1994 Phys. Rev. D 49 6467
[10] Ashtekar A, Beetle C, Dreyer O, Fairhurst S, Krishnan B, Lewandowski J and Wisniewski J 2000 Phys. Rev. Lett. 85 3564–7
[11] Ashtekar A and Krishnan B 2002 Phys. Rev. Lett. 89 261101
Ashtekar A and Krishnan B 2003 Phys. Rev. D 68 104030
[12] Booth I and Fairhurst S 2004 Phys. Rev. Lett. 92 011102
[13] Tolman R C 1934 Proc. Natl. Acad. Sci. USA 20 410
Lemaître G 1933 Ann. Soc. Sci. Bruxelles I A 53 51
Bondi H 1947 Mon. Not. R. Astron. Soc. 107 343
[14] Firouzjaee J T 2012 Int. J. Mod. Phys. D 21 1250039
[15] Razbin M, Firouzjaee J T and Mansouri R 2014 Int. J. Mod. Phys. D 23 450074
[16] Firouzjaee J T, Mood M P and Mansouri R 2012 Gen. Rel. Grav. 44 639
[17] Alfedeel A A H and Hellaby C 2010 Gen. Rel. Grav. 42 1935
[18] Carr B J 1975 *Astrophys. J.* **201** I
    Shibata M and Sasaki M 1999 *Phys. Rev. D* **60** 084002
    Polnarev A G, Nakama T and Yokoyama J ’i 2012 *J. Cosmol. Astropart. Phys. JCAP09(2012)027*
[19] Faraoni V 2013 *Galaxies* **1** 113 (arXiv:1309.4915v1 [gr-qc])
[20] Firouzjaee J T and Mansouri R 2012 *Europhys. Lett.* **97** 29002
    Firouzjaee J T and Ellis G F R 2015 *Gen. Rel. Grav.* **47** 6
    Firouzjaee J T and Ellis G F R 2015 *Phys. Rev. D* **91** 103002
[21] Joshi P S and Malafarina D 2011 *Int. J. Mod. Phys. D* **20** 2641–729
[22] Dwivedi I H 1998 *Phys. Rev. D* **58** 064004
[23] Moradi R, Firouzjaee J T and Mansouri R arXiv:1301.1480 [gr-qc]
[24] Navarro J F, Frenk C S and White S D M 1995 *Mon. Not. Ry. Astron. Soc.* **275** 720
[25] Pollack J, Spergel D N and Steinhardt P J 2015 *Astrophys. J.* **804** 131
[26] Yoon J H et al 2008 *Astrophys. J. Suppl. Ser.* **176** 414