$E_{7(7)}$ symmetry in perturbatively quantised $\mathcal{N} = 8$ supergravity

Guillaume Bossard

AEI, Max-Planck-Institut für Gravitationsphysik

Penn State
September 2010
Outline

- Duality invariant action
- Quantum equivalence
- Regularisation
- $\mathfrak{su}(8)$ current anomaly
- $\mathfrak{e}_7(7)$ current anomaly
- Instanton effects
- Conclusion and outlook

[ G. Bossard, C. Hillmann and H. Nicolai, 1007.5472 ]
Self-dual 4-form in type IIB

Using ADM decomposition [M. Henneaux and C. Teitelboim]

\[ g_{\mu\nu} dx^{\mu} dx^{\nu} = - N^2 dt^2 + h_{ij} (dx^i + N^i dt)(dx^j + N^j dt) \]

one can write a local action for type IIB such that

\[ \mathcal{L}_{\text{4-form}} = \frac{1}{2 \cdot 5!} \left( \frac{1}{4!} \varepsilon^{ijklmnopq} (\partial_0 G_{ijkl} - N^r G_{ijklr}) G_{mnopq} ight. \\
\left. - N \sqrt{h} h^i h^j h^k h^l h^m h^r \hat{G}^i_{ijklm} \hat{G}_{nopqr} \right) \]

Indeed, the equations of motion are

\[ \partial_0 G_{ijklp} - 5 \partial_i \left( N^q G_{ijklp} q + \frac{N}{5! \sqrt{h}} h^i m h^k n h^l o h^p q \varepsilon^{mnopqrstuv} \hat{G}_{rstuv} \right) \approx 0 \]
Self-dual 4-form in type IIB

Using ADM decomposition [M. Henneaux and C. Teitelboim]

\[ g_{\mu\nu}dx^\mu dx^\nu = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \]

one can write a local action for type IIB such that

\[ \mathcal{L}_{4\text{-form}} = \frac{1}{2 \cdot 5!} \left( \frac{1}{4!} \epsilon^{ijklmnopq} (\partial_0 C_{ijkl} - N^r G_{ijklr}) G_{mnopq} \right) \]

\[ -N \sqrt{h} h^{in} h^{jp} h^{kr} h^{mq} \hat{G}_{ijklm} \hat{G}_{nopqr} \]

Indeed, the equations of motion are

\[ 5 \partial_{[i} \left( \partial_0 C_{jklp]} - N^q G_{jklp]}q \right) - \frac{N}{5! \sqrt{h}} h_{j[m} h_{k|n} h_{l|opo} h_{p]q} \epsilon^{mnoprstuv} \hat{G}_{rstuv} \approx 0 \]
Self-dual 4-form in type IIB

Using ADM decomposition [M. Henneaux and C. Teitelboim]

\[ g_{\mu\nu}dx^\mu dx^\nu = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \]

one can write a local action for type IIB such that

\[
\mathcal{L}_{4\text{-form}} = \frac{1}{2 \cdot 5!} \left( \frac{1}{4!} \varepsilon_{ijklmnopq} (\partial_0 C_{ijkl} - N^r G_{ijklr}) G_{mnopq} - N \sqrt{h} h^{in} h^{jp} h^{kp} h^{rq} \hat{G}_{ijklm} \hat{G}_{nopqr} \right)
\]

Indeed, the equations of motion are

\[
\partial_0 C_{ijkl} - N^p G_{ijklp} - \frac{N}{5! \sqrt{h}} h_{im} h_{jn} h_{ko} h_{lp} \varepsilon_{mnopqrstuv} \hat{G}_{qrstuv} \approx 4 \partial_{[i} C_{ijkl]} \]

\[ \Rightarrow \hat{G} \approx \star \hat{G} \]
Self-dual 4-form in type IIB

Using ADM decomposition [M. Henneaux and C. Teitelboim]

\[ g_{\mu\nu}dx^\mu dx^\nu = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \]

one can write a local action for type IIB such that

\[ \mathcal{L}_{4\text{-form}} = \frac{1}{2 \cdot 5!} \left( \frac{1}{4!} \varepsilon^{ijklmnopq} (\partial_0 C_{ijkl} - N^r G_{ijklr}) G_{mnopq} \right. \]

\[ \left. - N \sqrt{h} h^{in} h^{jo} h^{kp} h^{lq} h^{mr} \hat{G}_{ijklm} \hat{G}_{nopqr} \right) \]

Diffeomorphism invariance is realised such that

\[ \delta C_{ijkl} = (\xi^p + \xi^0 N^p) G_{ijklp} + \xi^0 \frac{N}{5! \sqrt{h}} h_{im} h_{jn} h_{ko} h_{lp} \varepsilon^{mnopqrstuv} \hat{G}_{qrstuv} \]

\[ \approx \xi^\mu G_{ijkl\mu} \]
$E_7(7)$-invariant action

Defining $G_{mn} \equiv [\mathcal{V}^t \mathcal{V}]_{mn}$ and the symplectic form $\Omega_{mn}$

$$G^{mn} = \Omega^{mp} \Omega^{nq} G_{pq} \quad (G^{mp} G_{pn} = \delta^m_n)$$

and the ‘complex structure’

$$J^m_n \equiv G_{mp} \Omega^{pn} \implies J^m_p J^p_n = -\delta^m_n$$

one defines for 56 vectors $A^m_i$ [ C. Hillmann ]

$$- \mathcal{L}_{vec} = \frac{1}{4} \Omega_{mn} \varepsilon^{ijk} (\partial_0 A^m_i + N^l F^m_{il}) F^n_{jk} + \frac{1}{4} N \sqrt{h} G_{mn} h^{ik} h^{jl} \hat{F}^m_{ij} \hat{F}^n_{kl}$$

$$\implies 2 \partial[i] \left( \partial_0 A^m_j + N^l F^m_{jl} \right]_l - \frac{N}{2 \sqrt{h}} h_{lj} \varepsilon^{lpq} J^m_n \hat{F}^n_{pq} \right) \approx 0$$
The action is supersymmetric.
Equivalence with Cremmer–Julia

Coulomb Gauge

\[-L_{\text{vec}} = \frac{1}{4} \Omega_{mn} \varepsilon^{ijk} (\partial_0 A_i^m + N^l F^m_{il}) F^m_{jk} + \frac{1}{4} N \sqrt{\hbar} G_{mn} h^{ik} h^{jl} \hat{F}^m_{ij} \hat{F}^m_{kl} + b_m \partial_i A_i^m\]

Darboux basis \( \Omega_{mn} = \Omega_{\bar{m}\bar{n}} = 0, \ \Omega_{m\bar{n}} = -\Omega_{\bar{n}m} = \delta_{m\bar{n}} \)

\[-L_{\text{vec}} = \frac{1}{2} \left( \delta_{m\bar{n}} \varepsilon^{ijk} (\partial_0 A_i^m + N^l F^m_{il}) + N \sqrt{\hbar} G_{m\bar{n}} h^{ik} h^{jl} F^m_{ij} \right) F^\bar{n}_{kl} + \frac{1}{4} N \sqrt{\hbar} G_{mn} h^{ik} h^{jl} F^m_{ij} F^m_{kl} + \frac{1}{4} N \sqrt{\hbar} G_{mn} h^{ik} h^{jl} F^m_{ij} F^m_{kl} + b_m \partial_i A_i^m + b_m \partial_i A_i^\bar{m}\]

Guillaume Bossard (AEI)

\( E_7(7) \) symmetry in perturbatively quantised \( \mathcal{N} = 8 \) supergravity
Equivalence with Cremmer–Julia

Darboux basis \( \Omega_{mn} = \Omega_{\overline{m}\overline{n}} = 0, \Omega_{m\overline{n}} = -\Omega_{\overline{n}m} = \delta_{m\overline{n}} \)

\[
- \mathcal{L}_{\text{vec}} = \frac{1}{2} \left( \delta_{\overline{m}n} \varepsilon^{ijkl} (\partial_0 A^m_i + N^l F^m_{il}) + N \sqrt{h} G_{\overline{m}n} h^{ik} h^{jl} F^m_{ij} \right) F^\overline{n}_{kl} \\
+ \frac{1}{4} N \sqrt{h} G_{mn} h^{ik} h^{jl} F^m_{ij} F^\overline{n}_{kl} + \frac{1}{4} N \sqrt{h} G_{mn} h^{ik} h^{jl} F^m_{ij} F^n_{kl} + b_m \partial_i A^m_i + b_{\overline{m}} \partial_i A^{\overline{m}}_i
\]

Integrating \( b_{\overline{m}} \) implements strongly \( \partial_i A^{\overline{m}}_i = 0 \)

\[
\Pi^{i\overline{m}} = \varepsilon^{ijkl} \partial_j A^{\overline{m}}_k \quad A^{\overline{m}}_i = -[\partial_i \partial_l]^{-1} \varepsilon_{ijkl} \partial_j \Pi^k \overline{m} 
\]

with \( \partial_i \Pi^{i\overline{m}} = 0 \).
Equivalence with Cremmer–Julia

Darboux basis $\Omega_{mn} = \Omega_{\bar{m}\bar{n}} = 0$, $\Omega_{m\bar{n}} = -\Omega_{\bar{n}m} = \delta_{m\bar{n}}$

$$-\mathcal{L}_{\text{vec}} = \frac{1}{2} \left( \delta_{m\bar{n}} \varepsilon^{ijk} \left( \partial_0 A^m_i + N^l F^m_{i\bar{l}} \right) + N \sqrt{h} G_{m\bar{n}} h^{ik} h^{jl} F^m_{ij} \right) F^\bar{n}_{kl}$$

$$+ \frac{1}{4} N \sqrt{h} G_{m\bar{n}} h^{ik} h^{jl} F^\bar{m}_{ij} F^\bar{n}_{kl} + \frac{1}{4} N \sqrt{h} G_{mn} h^{ik} h^{jl} F^m_{ij} F^n_{kl} + b_m \partial_i A^m_i + b_{\bar{m}} \partial_i A_{\bar{m}}^i$$

Integrating $b_{\bar{m}}$ implements strongly $\partial_i A_{\bar{m}}^i = 0$

$$\Pi^{i\bar{m}} = \varepsilon^{ijk} \partial_j A_{k}^{\bar{m}} \quad A_{\bar{m}}^i = -[\partial_i \partial_l]^{-1} \varepsilon_{ijkl} \partial_j \Pi^{k\bar{m}}$$

with $\partial_i \Pi^{i\bar{m}} = 0$. 

Guillaume Bossard (AEI)

$E_7(7)$ symmetry in perturbatively quantised $\mathcal{N} = 8$ supergravity
**Equivalence with Cremmer–Julia**

Darboux basis $\Omega_{mn} = \Omega_{\bar{m}\bar{n}} = 0, \, \Omega_{m\bar{n}} = -\Omega_{\bar{n}m} = \delta_{m\bar{n}}$

$$-\mathcal{L}_{\text{vec}} = \frac{1}{2} \left( 2\delta_{m\bar{n}} \left( \partial_0 A_i^m - \partial_i A_0^m + N^l F_{il}^m \right) + N \sqrt{\hbar} \varepsilon_{ilh} G_{m\bar{n}} h^{lj} h^{lk} F_{jk}^m \right) \Pi^i \bar{n}$$

$$+ \frac{1}{2} \frac{N}{\sqrt{h}} G_{m\bar{n}} h_{ij} \Pi^i \bar{m} \Pi^j \bar{n} + \frac{1}{4} N \sqrt{h} G_{mn} h^{ik} h^{jl} F_{ij}^m F_{kl}^n + b_m \partial_i A_i^m$$

Integrating $b_m$ implements strongly $\partial_i A_i^\bar{m} = 0$

$$\Pi^i \bar{m} = \varepsilon^{ijk} \partial_j A_k^\bar{m} \quad A_i^\bar{m} = - \left[ \partial_l \partial_i \right]^{-1} \varepsilon_{ijk} \partial_j \Pi^k \bar{m}$$

with $\partial_i \Pi^i \bar{m} = 0$. 

---

Guillaume Bossard (AEI)

**E$_7$(7) symmetry in perturbativelyquantised $N = 8$ supergravity**
Equivalence with Cremmer–Julia

Gaussian integration of the momentum variables $\Pi^i \bar{m}$

$$-\mathcal{L}_{\text{vec}} = \frac{1}{2} \left( 2\delta_{m\bar{n}} (F_{0i} + N^l F_{i\bar{l}}^m) + N \sqrt{h} \varepsilon_{ilh} G_{m\bar{n}} h^{lj} h^{hk} F_{jk}^m \right) \Pi^i \bar{n}$$

$$+ \frac{1}{2} N \sqrt{h} G_{m\bar{n}} h_{ij} \Pi^{i \bar{m}} \Pi^{j \bar{n}} + \frac{1}{4} N \sqrt{h} G_{mn} h^{ik} h^{jl} F_{ij}^m F_{kl}^n + b_m \partial_i A_i^m$$

gives rise to

$$-\mathcal{L}_{\text{vec}} = \frac{1}{4} \left( \sqrt{-g} \delta_{m\bar{m}} \delta_{n\bar{n}} H^{m\bar{n}} g^{\mu\sigma} g^{\nu\rho} F_{\mu\nu}^m F_{\sigma\rho} + \epsilon^{\mu\nu\sigma\rho} \delta_{m\bar{m}} H^{m\bar{n}} G_{n\bar{n}}^m F_{\mu\nu}^m F_{\sigma\rho} \right)$$

where $H^{m\bar{n}}$ is the inverse of $G_{m\bar{n}}$ which satisfies

$$G_{mn} - G_{m\bar{m}} H^{m\bar{n}} G_{n\bar{n}} = \delta_{m\bar{m}} \delta_{n\bar{n}} H^{m\bar{n}} \quad \delta_{m\bar{m}} H^{m\bar{n}} G_{n\bar{n}} = \delta_{n\bar{n}} H^{m\bar{n}} G_{m\bar{m}}$$
**$E_7(7)$-invariant gauge-fixing**

di Vecchia–Ferrara–Girardello computed that $\mathfrak{su}(8)$ gauge invariance is **anomalous** at one-loop.

→ fix coordinates for $E_7(7)/SU_c(8)$

Equivalence ensured [ B. de Wit and M. T. Grisaru ].

In the symmetric gauge $\mathcal{V} = \exp(\Phi)$, $G_{mn} = [\exp(2\Phi)]_{mn}$ and with $f(\Phi) \ast X \equiv f(\text{ad}_\Phi)X$

$$\delta^{E_7(7)} \Phi = [C_\xi, \Phi] - \frac{\Phi}{\tanh \Phi} \ast C_p, \quad \delta^{E_7(7)} \psi = \delta^{\mathfrak{su}(8)} \left( C_\xi + \tanh(\Phi/2) \ast C_p \right) \psi$$

and the supersymmetry variation is

$$\delta^{\text{Susy}} \Phi = \frac{\Phi}{\sinh \Phi} \ast [\bar{\epsilon} \chi]$$
\[ E_{7(7)} \text{ current} \]

The current 3-form \[ j \text{ current} \] [ C. Hillmann ] ( “≈” [ M. K. Gaillard and B. Zumino ] )

\[
J(X) = X_m \epsilon^n \left[ J_{\text{inv}} \epsilon^m + CS(A)^{mp} \Omega_p n \right]
\]

with

\[
CS(A)^{mn} \equiv \frac{1}{2} A^m \wedge dA^n + A^m_i \left( \partial_0 A^n_j + N^k F^m_{jk} - \frac{N}{2\sqrt{h}} h_{jl} \delta^{lpq} J^n \hat{F}_{pq} \right) dt \wedge dx^i \wedge dx^j
\]

and

\[
J_{\text{inv}} = \nu^{-1} \left( e^a \psi^i \gamma_a \psi^j + \frac{1}{6} \epsilon_{abcd} e^b \wedge e^c \wedge e^d \chi^{ikl} \gamma^a \chi_{jkl} \right) \nu
\]

\[
* \hat{A}_{ijkl} - e^a \wedge \chi_{ijkl} \gamma_a \psi^j
\]

Guillaume Bossard (AEI)
Feynman rules

\[-L_0 = \frac{1}{2} \Omega_{mn} \varepsilon^{ijk} \partial_0 A^m_i \partial_j A^n_k + \frac{1}{2} G_{mn} (\delta^{ik} \delta^{jl} - \delta^{il} \delta^{jk}) \partial_i A^m_j \partial_k A^n_l + b_m \partial_i A^m_i\]

defines the propagator

\[\Delta(p) = \frac{1}{p^2} \begin{pmatrix} \Omega_{mn} \varepsilon^{ijk} p_0 p^k - G_{mn} (\delta_{ij} p^2 - p_i p_j) & ip_i \delta^m_n \\ p_0^2 - p^2 + i\varepsilon & 0 \end{pmatrix} \]

with 56 poles

\[2 |p| \text{ res } (\Delta) \bigg|_{p_0 = \pm |p|} = \Omega_{mn} \varepsilon^{ijk} \hat{p}^k \mp \hat{G}_{mn} (\delta_{ij} - \hat{p}_i \hat{p}_j)\]
Feynman rules

\[-L_0 = \frac{1}{2} \Omega_{mn} \varepsilon^{ijk} \partial_0 A^m_i \partial_j A^n_k + \frac{1}{2} G_{mn} (\delta^{ik} \delta^{jl} - \delta^{il} \delta^{jk}) \partial_i A^m_j \partial_k A^n_l + b_m \partial_i A^m_i \]

defines the propagator

\[\Delta(p) = \frac{1}{p^2} \begin{pmatrix} \Omega^{mn} \varepsilon_{ijk} p_0 p_k - G^{mn} (\delta_{ij} p^2 - p_i p_j) & ip_i \delta^m_n \\ \frac{p_0^2 - p^2 + i \varepsilon}{p^2} & 0 \end{pmatrix} \]

with 56 poles

\[J^m_n e^n_i(\sigma, p) = \varepsilon_i^{jk} \hat{p}_k e^m_j(\sigma, p) = i\hbar(\sigma) e^m_i(\sigma, p)\]
Pauli–Villars regularisation

\[- \mathcal{L}_0(M) = \frac{1}{2} \Omega_{mn} \varepsilon^{ijk} \partial_0 A^m_i \partial_j A^n_k + \frac{i}{2} \Gamma_{mn} \varepsilon^{ijk} MA^m_i \partial_j A^n_k + \frac{1}{2} G_{mn} \left( \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} \right) \partial_i A^m_j \partial_k A^n_l + b_m \partial_i A^m_i \]

defines the propagator

\[\Delta(p, M) = \frac{1}{p^2} \left( \begin{array}{ccc}
\frac{\Omega^{mn} \varepsilon_{ijk} p_0 p^k + \Gamma^{mn}_{ij} M p^k - G^{mn}(\delta_{ij} p^2 - p_i p_j)}{p_0^2 - p^2 - M^2 + i\varepsilon} & ip_i \delta^m_n \\
- ip_j \delta^n_m & 0
\end{array} \right)\]

with 56 poles.

\[\Gamma_{m \bar{n}} = \Gamma_{\bar{n}m} = \delta_{m \bar{n}} \] breaks $\text{su}(8)$ to $\text{so}(8)$.
Pauli–Villars regularisation

\[ \Delta(p, M) = \frac{1}{p^2} \left( \begin{array}{cc} \Omega^{mn} \varepsilon_{ijk} p^i p^k + \Gamma^{mn} \varepsilon_{ijk} M p^k - G^{mn} (\delta_{ij} p^2 - p_i p_j) & ip_i \delta^m_n \\ p_o^2 - p^2 - M^2 + i \varepsilon & -ip_j \delta^m_n \\ -ip_j \delta^m_n & 0 \end{array} \right) \]

With the massive complex structure \( J(p, M)^2 = -1 \)

\[ J^m_n(p, M) \equiv \frac{1}{|p|} \left( \sqrt{p^2 + M^2} J^m_n + MG^{mp} \Gamma_{pn} \right) \]

the 56 poles

\[ J^m_n(p, M) e_i^n(\sigma, p) = \varepsilon_i^{jk} \hat{p}_k e_j^m(\sigma, p) = ih(\sigma) e_i^m(\sigma, p) \]
1-loop anomaly

\[- \mathcal{L}_0[B] = \frac{1}{2} \Omega_{mn} \varepsilon^{ijk} (\partial_0 A_i^m + B_0^m p A_i^p) (\partial_j A_k^n + B_j^n q A_k^q) \]

\[+ \frac{1}{2} G_{mn} (\delta^{ik} \delta^j l - \delta^{il} \delta^{jk}) (\partial_i A_j^m + B_i^m p A_j^p) (\partial_k A_l^n + B_k^n q A_l^q) + b_m (\partial_i A_i^m + B_i^m n A_i^n) \]

defines the vertex

\[\Upsilon^0(k + p, k) = \begin{pmatrix} i \Omega_{mn} \varepsilon^{ijk} (k_k + \frac{1}{2} p_k) & 0 \\ 0 & 0 \end{pmatrix}, \]

\[\Upsilon^k(k + p, k) = \begin{pmatrix} i \Omega_{mn} \varepsilon^{ijk} (k_0 + \frac{1}{2} p_0) + i G_{mn} \left( (2k^k + p^k) \delta^{ij} - \delta^{ki} (k^j + p^j) - \delta^{kj} k^i \right) & -\delta^{ki} \delta^m_n \\ \delta^{kj} \delta^m_n & 0 \end{pmatrix} \]

and

\[R^{0k} = R^{k0} = \begin{pmatrix} \frac{1}{2} \Omega_{mn} \varepsilon^{ijk} & 0 \\ 0 & 0 \end{pmatrix}, \quad R^{kl} = \begin{pmatrix} G_{mn} (\delta^{kl} \delta^{ij} - \delta^{kj} \delta^{li}) & 0 \\ 0 & 0 \end{pmatrix} \]
1-loop anomaly

Which satisfy

\[-i p_\mu \gamma_\mu(k + p, k) = \Delta^{-1}(k + p) - \Delta^{-1}(k)\]

and

\[i p_\nu R^{\mu\nu} = \gamma_\mu(k + q, k + p) - \gamma_\mu(k + q, k)\]

So the anomaly

\[A^\mu_\nu(p_1, p_2) \text{Tr } J X_1 X_2 X_3 \equiv i(p_1 \sigma + p_2 \sigma)J^\mu(X_1, p_1)J^\nu(X_2, p_2)J^\sigma(X_3, -p_1 - p_2)\]

\[- \left\langle J^\mu(X_1, p_1)J^\nu([X_2, X_3], -p_1)\right\rangle_{\text{vec}} - \left\langle J^\mu([X_1, X_3], -p_2)J^\nu(X_2, p_2)\right\rangle_{\text{vec}}\]

is the difference of two linearly divergent integrals.
1-loop anomaly

The Pauli-Villars vertex satisfies

\[ -i p_\mu \Upsilon^\mu(k + p, k) = \Delta^{-1}(k + p, M) - \Delta^{-1}(k, -M) - M \Upsilon_5(2k + p) \]

So the regularised anomaly

\[
A_{vec}^{\mu\nu}(p_1, p_2) \text{Tr } J X_1 X_2 X_3 \equiv i(p_1 \sigma + p_2 \sigma) \left\langle J^\mu(X_1, p_1) J^\nu(X_2, p_2) J^\sigma(X_3, -p_1 - p_2) \right\rangle_{vec+PV} - \left\langle J^\mu([X_1, X_2], -p_1) J^\nu(X_2, p_2) \right\rangle_{vec+PV} - \left\langle J^\mu([X_1, X_3], -p_2) J^\nu(X_2, p_2) \right\rangle_{vec+PV}
\]

is the vanishing difference of two linearly divergent integrals, plus a finite integral involving \( \Upsilon_5 \).
1-loop anomaly

The ‘axial-axial-axial’ anomaly is

\[
A^{\mu\nu}_{\text{vec}}(p_1, p_2) = -\frac{1}{28} \lim_{M \to \infty} \left[ M \int \frac{dk^4}{(2\pi)^4} \text{Tr} \left( \Delta(k+p_1, M)\gamma^\mu(k+p_1, k)\Delta(k, -M)\gamma^\nu(k, k+p_1)\Delta(k+p_1, M)\gamma_5(2k+p_1-p_2) \right. \\
\left. + \Delta(k+p_1, M)R^{\mu\nu} \Delta(k-p_2, M)\gamma_5(2k+p_1-p_2) \right) \right]
\]

and one computes

\[
A^{0i}_{\text{vec}} = 0 = A^{i0}_{\text{vec}}
\]

\[
A^{ij}_{\text{vec}} = \frac{1}{6\pi^2} \varepsilon^{ijk}(p_{10}p_{2k} - p_{20}p_{1k}) - \frac{1}{6\pi^2} \varepsilon^{ijk}(p_{10} + p_{20})(p_{1k} - p_{2k})
\]
1-loop anomaly

With the finite renormalisation

\[- \mathcal{L}_0[B] = \frac{1}{2} \Omega_{mn} \varepsilon^{ijk} (\partial_0 A_i^m + B_0^m p A_i^p) (\partial_j A_k^n + B_j^n q A_k^q) \]

\[+ \frac{1}{2} G_{mn} (\delta^k l \delta^j l - \delta^i l \delta^j k) (\partial_i A_j^m + B_i^m p A_j^p) (\partial_k A_l^n + B_k^n q A_l^q) + b_m (\partial_i A_i^m + B_i^m n A_i^n) \]

\[+ \frac{1}{6\pi^2} \varepsilon^{ijk} J_{mn} B_0^n p B_i^p q \partial_j B_k^q m \]

one gets

\[A^\mu^\nu_{\text{vec}}(p_1, p_2) = \frac{1}{6\pi^2} \varepsilon^{\mu^\nu^\sigma^\rho} p_1^\sigma p_2^\rho . \]
1-loop anomaly

This confirms that the anomaly coefficient is defined by the family’s index.

\[ \text{O. Alvarez, I. M. Singer and B. Zumino} \]

\[
\text{ch}[\text{Ind}(D)] = \int_{S^2 \times M} \hat{A}(Z) \wedge \text{ch} \left( F + \delta A + \Box^{-1}_A [\delta A^\mu, \delta A_\mu] \right)
\]

Marcus computation establishes the absence of \( su(8) \) anomaly.
1-loop anomaly

This confirms that the anomaly coefficient is defined by the family’s index.

\[
\text{ch}[\text{Ind}(D)] = -\frac{3}{(2\pi)^3} \int_{S^2 \times M} \hat{A}_0 \text{Tr} \left( \Box^{-1}_A [\delta A^\mu, \delta A_\mu] F \wedge F + \delta A \wedge \delta A \wedge F \right)
\]

Marcus computation establishes the absence of \( su(8) \) anomaly.

\[
(-3) \times 1 + 2 \times 4 + (-1) \times 5 = 0
\]
1-loop anomaly

Taking one $u(1)$ generator $X$ such that

\[
8 \cong 1^{(7)} \oplus 7^{(-1)} , \quad 28 \cong 7^{(6)} \oplus 21^{(-2)} , \quad 56 \cong 21^{(5)} \oplus 35^{(-3)}
\]

one computes $\text{Tr} \ X^3$

\[
7^3 - 7 \times 1^3 = 1 \times 336 , \quad 7 \times 6^3 - 21 \times 2^3 = 4 \times 336 , \quad 21 \times 5^3 - 35 \times 2^3 = 5 \times 336
\]

Marcus computation establishes the absence of $su(8)$ anomaly.

\[
(-3) \times 1 + 2 \times 4 + (-1) \times 5 = 0
\]
Wess–Zumino consistency condition

The horizontal Russian formula [ R. Stora ]

\[(d + \delta^\xi)(B_\xi + C_\xi) + (B_\xi + C_\xi)^2 = F_\xi\]

permits to derive the algebraic anomaly

\[\delta^\xi \int \text{Tr} \ dC_\xi \left( B_\xi F_\xi - \frac{1}{2} B_\xi^3 \right) = 0\]

from the Cartan homotopy formula

\[(d + \delta^\xi) \text{Tr} \left( \tilde{B}_\xi F_\xi^2 - \frac{1}{2} \tilde{B}_\xi^3 F_\xi + \frac{1}{10} \tilde{B}_\xi^5 \right) = \text{Tr} \ F_\xi^3 = 0\]
The $SU(8)$-equivariant cohomology

The $K$-equivariant cohomology of $\mathfrak{g}$

\[
\delta_{C_t}^{\mathfrak{g}} \Phi = - \frac{\Phi}{\tanh(\Phi)} \ast C_p = -C_p - \frac{1}{3} [\Phi, [\Phi, C_p]] + \mathcal{O}(\Phi^4)
\]

\[
\delta_{C_t}^{\mathfrak{g}} C_p = 0
\]

is trivial

\[
\mathcal{H}^n_K(\delta_{C_t}^{\mathfrak{g}}) \cong \emptyset \quad \text{for } n \geq 1
\]

\[
\mathcal{H}^0_K(\delta_{C_t}^{\mathfrak{g}}) \cong \left\{ E_7(7) \right\} \text{-invariants}
\]
The $SU(8)$-equivariant cohomology

Using a spectral decomposition of $\delta^{g}_{C_t} = \sum_n \delta^{g(n)}_{C_t}$

$$\delta^{g(0)}_{C_t} \Phi = -C_p$$
$$\delta^{g(0)}_{C_t} C_p = 0$$

$$\delta^{g(1)}_{C_t} \Phi = -\frac{1}{3} [\Phi, [\Phi, C_p]]$$
$$\delta^{g(1)}_{C_t} C_p = 0$$

One trivialises a cocycle $\mathcal{F} = \sum_n \mathcal{F}^{(n)}$

$$\delta^{g} \mathcal{F} = 0 \implies \delta^{g(0)} \mathcal{F}^{(0)} = 0$$

$$\mathcal{F}^{(0)} = \delta^{g(0)} \mathcal{G}^{(0)} \implies \delta^{g(0)} \left( \mathcal{F}^{(1)} - \delta^{g(1)} \mathcal{G}^{(0)} \right) = 0$$

$$\left( \mathcal{F}^{(1)} - \delta^{g(1)} \mathcal{G}^{(0)} \right) = \delta^{g(0)} \mathcal{G}^{(1)} \implies \delta^{g(0)} \left( \mathcal{F}^{(2)} - \delta^{g(1)} \mathcal{G}^{(1)} - \delta^{g(2)} \mathcal{G}^{(0)} \right) = 0$$

$$\cdots$$
The $SU(8)$-equivariant cohomology

One computes the exact sequence

$$0 \rightarrow \mathcal{H}_K^1(\delta_{C_t}^g) \xrightarrow{\iota} \mathcal{H}^1(\delta^g) \xrightarrow{\pi} \mathcal{H}^1(\delta_t^g)$$

Indeed

$$\delta^g(\mathcal{F} \cdot C_t + G \cdot C_p) = \delta^t(\mathcal{F} \cdot C_t) + \delta_{C_t}^g \mathcal{F} \cdot C_t + \delta^t(G \cdot C_p) - \mathcal{F} \cdot C_p^2 + \delta_{C_t}^g G \cdot C_p$$

If

$$\mathcal{F} \cdot C_t = \delta^t \mathcal{K} \implies G \cdot C_p - \delta_{C_t}^g \mathcal{K} \in \mathcal{H}_K^1(\delta_{C_t}^g)$$

Therefore

$$\text{Ker}(\pi) = 0$$

Guillaume Bossard (AEI)
The $SU(8)$-equivariant cohomology

Moreover

$$\mathcal{H}_K^n(\delta^g_{C_8} | d) \cong \emptyset \quad \text{for } n \geq 1$$

and one computes the exact sequence

$$0 \rightarrow \mathcal{H}_K^1(\delta^g_{C_8} | d) \xrightarrow{\iota} \mathcal{H}^1(\delta^g | d) \xrightarrow{\pi} \mathcal{H}^1(\delta^\xi | d) \rightarrow \mathcal{H}_K^2(\delta^g_{C_8} | d)$$

which implies then

$$\mathcal{H}^1(\delta^g | d) \cong \mathcal{H}^1(\delta^\xi | d)$$
Wess–Zumino consistency condition

The semi-horizontal Russian formula

\[(d + \delta^g)(B_\ell + C_\ell + \tanh(\Phi/2) \ast (B_p + C_p)) + (B_\ell + C_\ell + \tanh(\Phi/2) \ast (B_p + C_p))^2 = F_\ell + \tanh(\Phi/2) \ast F_p + d_B(\tanh(\Phi/2)) \ast (B_p + C_p)\]

The Cartan homotopy formula

\[(d + \delta^g)\text{Tr} \left( \tilde{B}_g \tilde{F}_g^2 - \frac{1}{2} \tilde{B}_g^3 \tilde{F}_g + \frac{1}{10} \tilde{B}_g^5 \right) = \text{Tr} \tilde{F}_g^3\]

has a right-hand-side, but

\[(d + \delta^g_{C_\ell}) \text{Tr} \tilde{F}_g^3 = 0\]
Wess–Zumino consistency condition

\[(d + \delta^g) \text{Tr} \tilde{F}_g^3 = 0\]

and

\[\text{Tr} \tilde{F}_g^3 = (d + \delta^g) \tilde{M}\]

such that

\[(d + \delta^g) \left[ \text{Tr} \left( \tilde{B}_g \tilde{F}_g^2 - \frac{1}{2} \tilde{B}_g^3 \tilde{F}_g + \frac{1}{10} \tilde{B}_g^5 \right) - \tilde{M} \right] = 0\]

permits to derive the anomaly

\[\delta^g \int \left[ \text{Tr} \left( \tilde{B}_g \tilde{F}_g^2 - \frac{1}{2} \tilde{B}_g^3 \tilde{F}_g + \frac{1}{10} \tilde{B}_g^5 \right) - \tilde{M} \right]_{(4,1)} = 0\]
Potential anomalies

\[ \langle J^\mu_1(x_1) J^\nu_2(x_2) J^\sigma_3(x_3) \rangle \]

\[ \langle J^\mu_1(x_1) J^\nu_2(x_2) J^\sigma_3(x_3) \prod_{n=1}^{N+1} \Phi(y_n) \rangle \]

\[ \langle J^\mu_1(x_1) J^\nu_2(x_2) J^\sigma_3(x_3) \prod_{n=1}^{N+2} \Phi(y_n) \rangle \]

\[ \langle J^\mu_1(x_1) J^\nu_2(x_2) J^\sigma_3(x_3) \prod_{n=1}^{N+3} \Phi(y_n) \rangle \]

\( E_7(7) \) symmetry in perturbatively quantised \( \mathcal{N} = 8 \) supergravity
Potential anomalies

For example

\[ \langle J^\mu_1(x_1) J^\nu_2(x_2) J^\sigma_p(x_3) \Phi(x_4) \rangle \]

would gives for the \( J^\mu_1 \) conservation

\[
- ip_2 \sigma \left\langle J^\mu (X_1, p_1) J^\sigma (X_2, p_2) J^\nu (Y_1, p_3) \Phi(Y_2, -p_1 - p_2 - p_3) \right\rangle_{\text{vec} + \text{PV}} \\
= \left\langle J^\mu ([X_1, X_2], p_1 + p_2) J^\sigma (Y_1, p_3) \Phi(Y_2, -p_1 - p_2 - p_3) \right\rangle_{\text{vec} + \text{PV}} \\
+ \left\langle J^\mu (X_1, p_1) J^\sigma ([Y_1, X_2], p_2 + p_3) \Phi(Y_2, -p_1 - p_2 - p_3) \right\rangle_{\text{vec} + \text{PV}} \\
+ \left\langle J^\mu (X_1, p_1) J^\sigma (Y_1, p_3) \Phi([Y_2, X_2], -p_1 - p_3) \right\rangle_{\text{vec} + \text{PV}} \\
+ \frac{1}{6\pi^2} \varepsilon^{\mu \nu \sigma \rho} p_1 \sigma p_2 \rho \text{Tr} J X_1 X_2 [Y_1, Y_2] 
\]
Potential anomalies

For example

\[
\left\langle J_\mu^\mu(x_1) J_\nu^\nu(x_2) J_\sigma^\sigma(x_3) \Phi(x_4) \right\rangle
\]

and for the \( J_p \) conservation

\[
- i p_3 \sigma \left\langle J^\mu(X_1, p_1) J^\nu(X_2, p_2) J^\sigma(Y_1, p_3) \Phi(Y_2, -p_1 - p_2 - p_3) \right\rangle_{\text{vec+PV}} \\
= \left\langle J^\mu([X_1, Y_1], p_1 + p_3) J^\nu(X_2, p_2) \Phi(Y_2, -p_1 - p_2 - p_3) \right\rangle_{\text{vec+PV}} \\
+ \left\langle J^\mu(X_1, p_1) J^\nu([X_2, Y_1], p_2 + p_3) \Phi(Y_2, -p_1 - p_2 - p_3) \right\rangle_{\text{vec+PV}} \\
+ \left\langle J^\mu(X_1, p_1) J^\nu(X_2, p_2) \Phi(Y_1, p_3) \Phi(Y_2, -p_1 - p_2 - p_3) \right\rangle_{\text{vec+PV}} \\
+ \frac{1}{6\pi^2} \varepsilon^{\mu\nu\sigma\rho} (3p_1 \sigma p_2 \rho + (p_1 \sigma - p_2 \sigma) p_3 \rho) \text{Tr} JX_1X_2[Y_1, Y_2]
\]
**$E_{7(7)}$ Ward identities**

$E_{7(7)}$ Ward identities for the 1PI correlation functions at all orders in perturbation theory. The linear $\mathfrak{su}(8)$ Ward identities

\[
\sum_{i \in I} \delta^{(4)}(x - x_i) \left\langle [X, \Phi(x_i)] \prod_{j \neq i} \Phi(x_j) \right\rangle = \partial_{\mu} \left\langle J_{\mu}^I(X, x) \prod_{i \in I} \Phi(x_i) \right\rangle
\]

and the $\mathfrak{e}_{7(7)} \oplus \mathfrak{su}(8)$ Slavnov–Taylor identities

\[
\sum_{J \subset I} \left\langle \phi_A(x) \prod_{i \in J} \Phi(x_i) \right\rangle \left\langle \left[ \frac{\Phi}{\tanh(\Phi)}(x) * X \right]^A \prod_{j \in I \setminus J} \Phi(x_j) \right\rangle
\]

\[
= \sum_{J \subset I} \partial_{\mu} \left\langle J_{\mu}^I(X, x) \prod_{i \in J} \Phi(x_i) \right\rangle \left\langle \prod_{j \in I \setminus J} \Phi(x_j) \right\rangle
\]
Instanton background

The globally defined Lagrangian

\[-\mathcal{L}_{\text{vec}} = \frac{1}{4} \Omega_{mn} \varepsilon^{ijk} \partial_0 (A^m_i - \hat{A}^m_i) (F^n_{jk} + \hat{F}^n_{jk}) + \frac{1}{4} \Omega_{mn} \varepsilon^{ijk} N^l F^m_{il} F^n_{jk} + \frac{1}{4} N \sqrt{h} G_{mn} h^{ik} h^{jl} \hat{F}^m_{ij} \hat{F}^n_{kl}\]

Then the Chern–Simons term in the current

\[CS(A)^{mn} \equiv \frac{1}{2} (A^m - \hat{A}^m) \wedge (F^n + \hat{F}^n)\]

\[+ (A^m_i - \hat{A}^m_i) \left( \partial_0 A^n_j + N^k F^n_{jk} - \frac{N}{2 \sqrt{h}} h_{jl} \varepsilon^{lpq} J^n_{o \hat{F}^o_{pq}} \right) dt \wedge dx^i \wedge dx^j\]

So the current conservation breaks to

\[dJ(X) \approx \frac{1}{2} X^m \Omega_{pn} \hat{F}^m \wedge \hat{F}^n\]

Guillaume Bossard (AEI)
Instanton background

Classical breaking of the Ward identity

\[ \Gamma[g] = \Gamma[1] + \Omega_{mp} g^p_n \frac{1}{2\pi} \int \dot{F}^m \wedge \dot{F}^n \]

Ricci flat \( M \) with non-trivial cohomology group

\[ \mathcal{H}^2_+ (M, \mathbb{Z}) \times \mathcal{H}^2_+ (M, \mathbb{Z}) \rightarrow \mathcal{H}^4 (M, \mathbb{Z}) \]

one has

\[ \Gamma[g] = \Gamma[1] + 2\pi \Omega_{mp} g^p_n q^m q^n \]

and

\[ E_7(\mathbb{C}) \rightarrow E_7(\mathbb{Z}) \]
• Conclusion

★ Manifestly $E_{7(7)}$ invariant perturbation theory
  ➡ Manifestly gauged fixed invariant action
  ➡ consistent Pauli–Villars regularisation
★ Quantum equivalence with conventional $\mathcal{N} = 8$ formulation.
★ One single algebraic anomaly with zero coefficient
  ➡ Explicit verification of the family’s index
★ Non-perturbative $E_{7(7)}(\mathbb{Z})$
Outlook

* Implications for logarithmic divergences
  
  ➡️ First divergence (not before) 7-loop in $\mathcal{N} = 8$.
  
  [H. Elvang and M. Kiermaier]
  [BHS][BEFKMS]
  
  ➡️ 5-loop for $\mathcal{N} = 6$, 4-loop for $\mathcal{N} = 5$.

* Constraints on on-shell amplitudes?
  
  ➡️ multi-soft limits at higher orders

* Instanton corrections
  
  ➡️ $E_{7(7)}(\mathbb{Z})$ effective action in field theory

Guillaume Bossard (AEI)