Moving vortices in anisotropic superconductors

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The magnetic field of moving vortices in anisotropic superconductors is considered in the framework of time-dependent London approach. It is found that at distances large relative to the core size, the field may change sign that alludes to a non-trivial intervortex interaction which depends on the crystal anisotropy and on the speed and direction of motion. These effects are caused to the electric fields and corresponding normal currents which appear due to the moving vortex magnetic structure. We find that the motion related part of the magnetic field attenuates at large distances as $1/r^3$ unlike the exponential decay of the static vortex field. The electric field induced by the vortex motion decreases as $1/r^2$.

I. INTRODUCTION

The problem of interaction of vortices in anisotropic superconductors has been studied extensively in early 90s both theoretically [1–3] and experimentally [4]. For vortices parallel to one of the principal crystal directions the problem is solved just by rescaling the isotropic results. In particular the interaction is repulsive for any position of the second vortex relative to the first. However, the force direction in general is not along the vector $\mathbf{R}$ connecting the vortices, in other words, for an arbitrary positions of the pair there is a torque, unless $\mathbf{R}$ is directed along principal directions [5].

The situation is different if parallel vortices are tilted out of principal directions [1–3]. Then, at distances of the order of London penetration depth $\lambda$, the magnetic field $h(R)$ of a single tilted vortex may change sign and approach zero for $R \to \infty$ being negative. In other words, the vortex-vortex interaction being repulsive at short distances may turn attractive at large distances. This leads to formation of chains of vortices in tilted fields [4].

In this paper we consider the magnetic field and current distributions of moving anisotropic vortices. Commonly, moving vortices are considered as static but displaced as a whole. It was argued, however, that out-of-core moving vortex structure differs from the static case due to out-of-core dissipation [6, 7]. The moving vortex magnetic field $h(r,t)$ generates the electric field and currents of normal excitations, which in turn distort the field $h$. We show that at large distances the distortion is not small and even able to change the field sign. Unexpectedly, this distortion attenuates with distance as a power law $1/R^3$, i.e. much slower than the standard decay of undistorted field $\sim e^{-R/\lambda}$.

At distances large in comparison to the core size of interest in this work, one can use the time-dependent London approach based on the assumption that the current consists of the normal and superconducting parts:

$$J = \sigma \mathbf{E} - \frac{2e^2|\Psi|^2}{mc} \left( A + \frac{\phi_0}{2\pi} \mathbf{\nabla} \chi \right),$$

where $A$ is the vector potential, $\Psi$ is the order parameter, $\chi$ is the phase, $\phi_0$ is the flux quantum, $\mathbf{E}$ is the electric field, and $\sigma$ is the conductivity associated with normal excitations.

The conductivity $\sigma$ approaches the normal state value $\sigma_n$ when the temperature $T$ approaches $T_c$; in s-wave superconductors it vanishes with decreasing temperature along with the density of normal excitations. This is not the case, however, for strong pair-breaking when superconductivity is gapless while the density of states approaches the normal state value at all temperatures. Unfortunately, not much experimental information about the $T$ dependence of $\sigma$ is available. Theoretically, this question is still debated, e.g. Ref. [8] discusses possible enhancement of $\sigma$ due to inelastic scattering. Experimentally, interpretation of the microwave absorption data is not yet settled either [9].

At distances large in comparison with the vortex core size, $|\Psi|$ is a constant $\Psi_0$ and Eq. (1) becomes:

$$\frac{4\pi}{c} J = 4\pi \sigma \mathbf{E} - \frac{1}{\lambda^2} \left( A + \frac{\phi_0}{2\pi} \mathbf{\nabla} \chi \right),$$

where $\lambda^2 = mc^2/8\pi e^2 |\Psi_0|^2$ is the London penetration depth. Acting on this by curl one obtains:

$$\mathbf{h} - \lambda^2 \nabla^2 h + \tau \frac{\partial h}{\partial t} = \phi_0 z \sum_\nu \delta(\mathbf{r} - \mathbf{r}_\nu),$$

where $\mathbf{r}_\nu(t)$ is the position of the $\nu$-th vortex which may depend on time $t$, $z$ is the direction of vortices, and the relaxation time

$$\tau = 4\pi \sigma \lambda^2 / c^2.$$

Equation (3) can be considered as a general form of the time dependent London equation (TDL). The anisotropic generalization of this equation was given in [11] and reproduced here in Section III.
II. VORTEX AT REST IN ANISOTROPIC CASE

For an arbitrary oriented vortex in anisotropic material this problem have been considered in [1, 10]. In general, results are cumbersome, so here we consider a simple situation of an orthorhombic superconductor in field along the c axis. The London equation in this case is:

$$h_z(x, y) - \lambda_1^2 \frac{\partial^2 h_z}{\partial y^2} - \lambda_2^2 \frac{\partial^2 h_z}{\partial x^2} = \phi_0 \delta(r),$$  \hspace{1cm} (5)

Here, the frame $x, y, z$ is chosen to coincide with $a, b, c$ of the crystal, $r = (x, y)$, $\lambda_{xx}^2 = \lambda_1^2$ and $\lambda_{yy}^2 = \lambda_2^2$ are the diagonal components of the tensor $(\lambda^2)_{ik}$. The solution of this equation is

$$h_z(x, y) = \frac{\phi_0}{2\pi \lambda_1 \lambda_2} K_0(\rho), \quad \rho^2 = \frac{x^2}{\lambda_1^2} + \frac{y^2}{\lambda_2^2}.$$  \hspace{1cm} (6)

Current densities follow:

$$J_x = - \frac{c \phi_0}{8\pi^2 \lambda_1 \lambda_2} y K_1(\rho), \quad J_y = \frac{c \phi_0}{8\pi^2 \lambda_1 \lambda_2} x K_1(\rho),$$  \hspace{1cm} (7)

where $K_{0,1}$ are Modified Bessel functions.

It is easy to see that the contours $h_z(x, y) = \text{const}$ coincide with the stream lines of the current, an example is shown in Fig. 1. The current lines have the expected ellipse-like shape.

This is, however, not the case for the distribution of the current values $J(x, y) = \sqrt{J_x^2 + J_y^2}$. An example is shown in Fig. 2. Hence, the geometry of the streamlines of the vector $J$ differs from that of contours $|J(x, y)| = \text{const}$, unlike the isotropic case where they are in fact the same.

FIG. 1. The stream lines of the current for $\gamma = \lambda_3/\lambda_1 = 3$ or, which is the same, contours of constant $h_z(x, y)$. $\lambda_1$ is taken as unit length.

FIG. 2. The contours of constant current values $J(x, y) = \sqrt{J_x^2 + J_y^2}$ for $\lambda_3/\lambda_1 = 3$. $x$ and $y$ are in units of $\lambda_1$.

III. MOVING VORTEX

The anisotropic generalization of the isotropic Eq. (2) for the current is straightforward:

$$J_k = \sigma_{kl} E_l - \frac{c}{4\pi} (\lambda^{-2})_{kl} \left( A_l + \frac{\phi_0}{2\pi} \frac{\partial \chi}{\partial x_l} \right).$$  \hspace{1cm} (8)

Here, $\sigma_{kl}$ and $(\lambda^{-2})_{kl}$ are tensors of the conductivity due to normal excitations and of the inverse square of the penetration depth.

Having in mind to derive an equation for the magnetic field $h$ we first have to get rid of the vector potential. To this end, multiply both sides by $4\pi (\lambda^2)_{k\mu} / c$ where $(\lambda^2)_{k\mu}$ is the tensor inverse to $(\lambda^{-2})_{k\mu}$ and sum up over $k$. Then apply curl to both sides and use the relation

$$\text{curl}(A + \phi_0 \nabla \chi/2\pi) = h - \phi_0 \hat{z} \delta(r - r_v),$$  \hspace{1cm} (9)

where $r_v$ is the vortex core position.

It is convenient to use in the following the notation\(\text{curl}_\nu V = \epsilon_{\nu \mu \sigma} \partial V_\mu / \partial x_\sigma\) where $\epsilon_{\nu \mu \sigma}$ is Levi-Chivita unit and symmetric tensor: $\epsilon_{xzy} = 1$ and so do all components with even number of transpositions of indices, it is $-1$ for odd numbers, and zero otherwise.

Hence, applying $\epsilon_{\nu \mu \sigma} \partial / \partial x_\sigma$ to Eq. (8), one obtains the anisotropic version of TDL [11]:

$$h_\nu + \frac{4\pi}{c} \epsilon_{\nu \mu \sigma} \lambda_{k\mu}^2 \frac{\partial J_k}{\partial x_\sigma} - \frac{4\pi}{c} \epsilon_{\nu \mu \sigma} \lambda_{k\mu}^2 \frac{\partial E_l}{\partial x_\sigma} \delta(\rho - r_v).$$  \hspace{1cm} (10)

In this form, the equation is valid for an arbitrary oriented vortex and any crystal anisotropy.

For an orthorhombic crystal in which the vortex and its field are along one of the principal directions (call it
for small $S$ can take larger values, we restrict this discussion by sound presently attainable [14, 15]. Although in principle stationary motion with a constant velocity, we can set here

$$E_{kx} = -\frac{k_y}{k_x} F_{ky} = -\frac{ik_y}{ck^2} \frac{\partial h_{kz}}{\partial t},$$

so that we obtain the 2D Fourier transform of Eq. (11):

$$h_k (1 + k_x^2 \lambda_x^2 + k_y^2 \lambda_y^2) + \frac{4\pi\sigma}{c^2} \frac{\lambda_x^2 \partial h_k}{k^2} + \frac{\lambda_x^2 \partial h_k}{k^2} = \phi_0 e^{-ikt},$$

where $h_k$ is the Fourier transform of $h_z(r)$. In isotropic case we obtain the equation studied in [7]. We further denote $\lambda_y^2 = \lambda_2^2$, $\lambda_x^2 = \lambda_1^2$ and $\lambda = \sqrt{\lambda_1 \lambda_2}$. The anisotropy parameter is defined as $\gamma = \lambda_2/\lambda_1$. Then, we obtain:

$$h_k (1 + k_x^2 \lambda_x^2 + k_y^2 \lambda_y^2) + \tau \frac{\lambda_x^2 k_x^2 + \lambda_y^2 k_y^2}{\lambda^2 k^2} \frac{\partial h_k}{\partial t} = \phi_0 e^{-ikt}.$$ 

with $\tau = 4\pi\sigma \lambda^2/c^2$. This is a linear differential equation for $h_k(t)$ with the solution

$$h_k = \frac{\phi_0 e^{-ikt}}{C - iDK \cdot s}, \quad s = \nu t,$$

$$C = 1 + k_x^2 \lambda_x^2 + k_y^2 \lambda_y^2, \quad D = \frac{\lambda_x^2 k_x^2 + \lambda_y^2 k_y^2}{\lambda^2 k^2}.$$ 

Since we are interested in stationary motion with a constant velocity, we can set here $t = 0$.

The dimensionless parameter

$$S = \frac{s}{\lambda} = \frac{4\pi\nu\sigma \lambda}{c^2}$$

is small even for vortex velocities exceeding the speed of sound presently attainable [14, 15]. Although in principle $S$ can take larger values, we restrict this discussion by small $S$ and call this case a “slow motion”.

### IV. SLOW MOTION

For $s \to 0$ one can expand $h(k, s)$ in powers of small $s$ up to $O(s)$:

$$h_k = \frac{\phi_0}{C} + i \frac{\phi_0 D}{C^2} k \cdot s,$$ 

The first term corresponds to the static solution discussed above:

$$h_0(x, y) = \frac{\phi_0}{2\pi \lambda^2} K_0(\rho), \quad \rho^2 = \frac{x^2}{\lambda_2^2} + \frac{y^2}{\lambda_1^2}.$$ 

The correction due to motion is given by

$$\frac{\delta h_k \lambda^2}{\phi_0} = i \frac{(\lambda_x^2 k_x^2 + \lambda_y^2 k_y^2)}{k^2(1 + \lambda_x^2 k_x^2 + \lambda_y^2 k_y^2)^2} k \cdot s,$$ 

To separate the part that does not disappear when $\lambda_1 = \lambda_2$, one can use the identity

$$\frac{\lambda_x^2 k_x^2 + \lambda_y^2 k_y^2}{k_x^2 + k_y^2} = \lambda_x^2 + \frac{k_y^2(\lambda_1^2 - \lambda_2^2)}{k_x^2 + k_y^2}$$

to obtain:

$$\frac{4\pi^2 \lambda^2 \delta h(r)}{\phi_0} = \lambda_1^2 \int \frac{d^2 k (k \cdot s) e^{ikr}}{(1 + \lambda_x^2 k_x^2 + \lambda_y^2 k_y^2)^2} + (\lambda_2^2 - \lambda_1^2) \int \frac{d^2 k (k \cdot s) e^{ikr}}{(1 + \lambda_x^2 k_x^2 + \lambda_y^2 k_y^2)^2}.$$ 

Evaluation of the first contribution is outlined in Appendix A:

$$h_1 = -\frac{\phi_0}{2\pi \lambda^2} \frac{S_x X + S_y Y \gamma^2}{2} K_0 \left( \sqrt{\frac{X^2}{\gamma} + Y^2 \gamma} \right),$$

where

$$S = \frac{s}{\lambda}, \quad X = \frac{x}{\lambda}, \quad Y = \frac{y}{\lambda}, \quad \lambda = \sqrt{\lambda_1 \lambda_2}, \quad \gamma = \frac{\lambda_2}{\lambda_1}.$$ 

It is shown in [17] that in the isotropic case for a vortex moving along $x$:

$$h(r) = \frac{\phi_0}{2\pi \lambda^2} e^{-s^2/2\lambda^2} K_0 \left( \frac{r}{2\lambda} \sqrt{4 + s^2/\lambda^2} \right)$$

in common units. Expanding this in small $s$ one obtains for a slow motion:

$$\delta h(r) = -\frac{\phi_0}{4\pi \lambda^4} s x K_0 \left( \frac{r}{\lambda} \right).$$ 

Hence, $h_1$ of Eq. (22) has the correct isotropic limit.

The second integral over two components of $k$ in Eq. (21) can be reduced to integrals over a single variable which are easy to deal with numerically, see Appendix B:
Thus, the vortex field can be calculated as $h = h_0 + h_1 + h_2$ with $h_0$ given in Eq. (18), $h_1$ in Eq. (22), and $h_2$ in Eq. (26). The results obtained with the help of Wolfram Mathematica package are shown below.

One can see in Fig. 3 that the current stream-lines (or, what is the same, contours $h(x, y) = \text{const}$) in the vicinity of the moving vortex core are only weakly distorted relative to the static elliptic shape. The most interesting feature of this distribution is that at large distances $h(x, y)$ changes sign in some parts of the $(x, y)$ plane. Since the interaction energy of the vortex at the origin with another one at $(x, y)$ is proportional to $h(x, y)$, the presence of domains with $h < 0$ means that for the second vortex in these domains the intervortex interaction is attractive.

The field distribution is different for the motion along $y$ axis shown in Fig. 4.

It is seen that the flux in front of the moving vortex is depleted whereas behind it is enhanced, the feature first discussed in [16] for the isotropic case. This feature remains also for a general direction of motion; an example of motion along the line $x = y$ is shown in Fig. 5. Moreover, Fig. 3–5 show that this depletion may even change sign of the field.

It is worth mentioning that the London theory is reliable in the region $r \gg \xi$, $\xi$ being the core size, and so are our predictions of a non-trivial behavior of $h(x, y)$ at large distances.

It is instructive to see how the interaction changes along certain directions. E.g., for $S_x = 0, S_y = 0.1$, the motion along $y$-axis, $h(0, Y)$ is positive if $0 < Y \lesssim 2.5$ (so that the second vortex at $(0, Y)$ in this region is repelled by the vortex at the origin). If the second vortex is at $2.5 \lesssim Y < \infty$ the interaction is attractive. This is illustrated in Fig. 6.

### A. Asymptotic behavior of $h(0, Y)$ for $Y \to \infty$

For $X = 0$, Eq. (26) yields

$$
2\pi \lambda^2 \phi_0 h_2 = \frac{(\gamma^2 - 1)}{4\gamma} S_y Y \int_0^\infty d\zeta \frac{[3K_0(\eta) - \eta K_1(\eta)]}{(\zeta + \gamma)^{1/2}(\zeta + 1/\gamma)^{5/2}},
$$

$$
\eta = \frac{|Y|}{\sqrt{\zeta + 1/\gamma}}.
$$

Going to the integration variable $\eta$, we get

$$
2\pi \lambda^2 \phi_0 h_2 = \frac{(\gamma^2 - 1)}{2\gamma} S_y Y^2 \int_0^\gamma d\eta \eta^{3/2} [3K_0(\eta) - \eta K_1(\eta)] \frac{1}{\sqrt{Y^2 + \eta^2(\gamma - 1/\gamma)}}.
$$

Fig. 7 shows that the integrand here is substantial only in the finite region $0 < \eta \lesssim 10$. Therefore being interested in the asymptotic behavior for $|Y| \to \infty$, one can replace
the denominator by \(|Y|\) and the upper limit of integration by \(\infty\):

\[
\frac{2\pi\lambda^2}{\phi_0}h_2(0,Y) = \frac{\gamma^2 - 1}{2\gamma} \frac{S_y}{Y^3} \int_0^\infty d\eta \eta^3 \left[3K_0 - \eta K_1\right] e^{-\gamma^2 - 1 \lambda S_y Y^3}.
\]

Thus, \(h_2(0,Y)\) is negative when \(Y \to \infty\) and positive for \(Y \to -\infty\). It decays as \(1/Y^3\), therefore, the total field \(h_0 + h_1 + h_2\) attenuates as a power law as well, since \(h_0\) and \(h_1\) decay exponentially and at large distances can be disregarded. Hence, \(h_2\) can be replaced with \(h\) in this region. This conclusion agrees with direct numerical evaluation of \(h(0,Y)\) shown in Fig. 6.

In the same way one can obtain the leading term in the asymptotic behavior for \(Y = S_y = 0\) for the motion along the \(x\) axis:

\[
h(X,0) \sim \frac{\phi_0}{2\pi\lambda^2} \frac{\gamma^2 - 1}{2\gamma} \frac{2S_x}{X^3}.
\]

For the sake of brevity we do not provide other terms in the asymptotic series.

The power-law decay of the field \(h(x,y)\) for vortices moving in anisotropic superconductors is a surprising feature. Clearly, this feature disappears for vortices at rest as well as for vortices moving in isotropic materials. Formally, the power-law behavior in real space originates in the factor \(1/k^2\) in Fourier transforms, see e.g. Eq. (21), which, however, cancels out for \(\gamma = 1\).

V. ELECTRIC FIELD FOR SLOW MOTION

In the approximation linear in velocity, we have according to Eq. (15)

\[
\frac{\partial h_k}{\partial t} = -i \frac{\phi_0 (k \cdot v)}{C} \cdot \frac{C_1 + k_x^2 \lambda_2^2 + k_y^2 \lambda_1^2}{k^2 C}, \quad C = 1 + \frac{k_x^2 \lambda_2^2 + k_y^2 \lambda_1^2}{k^2 C}.
\]

This yields the electric field

\[
E_{k_x} = -\frac{k_y}{k_x} \frac{\phi_0 k_y (k \cdot s)}{C} = -\frac{\phi_0 k_y (k \cdot s)}{2\pi c_0}, \quad s = v \tau.
\]

see Eqs. (12). Hence, we have in real space

\[
E_x = -\frac{\phi_0}{4\pi^2 c_0 \lambda} \int \frac{d^2 k k_y (k \cdot s)}{k^2 C} e^{i k \tau},
\]

or, using \(\lambda = \sqrt{\lambda_1 \lambda_2}\) as the unit length,

\[
E_x = -\frac{\phi_0}{4\pi^2 c_0 \lambda} \int \frac{d^2 q q_y (q \cdot S)}{q^2 C} e^{i q \tau}.
\]

Here, \(q = k \lambda\), \(R = (X,Y) = r / \lambda\) (see definitions (23), and

\[
C = 1 + q_x^2 \gamma + q_y^2 \gamma, \quad C = \lambda_2 / \lambda_1.
\]

In the same way we obtain

\[
E_y = \frac{\phi_0}{4\pi^2 c_0 \lambda} \int \frac{d^2 q q_x (q \cdot S)}{q^2 C} e^{i q \tau}.
\]

The integrals in Eqs. (34) and (36) are dimensionless.
It is of interest to see the streamlines of \( \mathbf{E} \) (or, that is the same, of the normal current \( \mathbf{J}_n = \sigma \mathbf{E} \)). To this end, we calculate the stream function \( G(x, y) \) such that \( E_x = \partial_y G \) and \( E_y = -\partial_x G \); the streamlines then are given by contours \( G(x, y) = \text{const} \). In Fourier space we have \( E_{xk} = i k_y G_k \) so that

\[
G_k = \frac{i \phi_0}{c \epsilon_0} \frac{k \cdot s}{k^2 C}, \quad G(r) = \frac{i \phi_0}{4 \pi^2 \epsilon_0} \int \frac{d^2 q (q \cdot S) e^{i q R}}{q^2 C}. \tag{37}
\]

The formal procedure of reducing the double to single integration in Eq. (37) is similar to that used for \( h(r) \) and is outlined in Appendix C. The result is:

\[
G(r) = -\frac{\phi_0}{4 \pi \epsilon_0} \int_0^\infty \frac{d \eta K_0(R \sqrt{\eta})}{\sqrt{\mu \nu}} \left( \frac{S_x X}{\mu} + \frac{S_y Y}{\nu} \right), \quad \mu = 1 + \eta \gamma, \quad \nu = 1 + \eta / \gamma, \quad R = \sqrt{\frac{X^2}{\mu} + \frac{Y^2}{\nu}}. \tag{38}
\]

Figs. 8 and 9 show two examples of \( J_x \)-streamlines (or contours \( G(X, Y) = \text{const} \)) obtained by numerical integration of Eq. (38).

![FIG. 8. Streamlines of the field \( \mathbf{E} \) (or of the normal current \( \mathbf{J}_n \)) for the vortex moving along \( X = Y \) \((S_x = 0.1, S_y = 0)\). \( \gamma = \lambda_2 / \lambda_1 = 3 \). \( X, Y \) are in units of \( \lambda = \sqrt{\lambda_1 \lambda_2} \). Positive constants by the contours correspond to the clockwise current direction, negative otherwise.](image)

The electric field is now readily obtained by differentiation of \( G \). We will not write down these cumbersome expressions. Instead we consider the asymptotic behavior of electric fields at large distances in two relatively simple cases using the method employed above for asymptotic behavior of \( h(0,y) \) and \( h(x,0) \). Omitting formalities, we give the results:

\[
G(X,0) \sim -\frac{\phi_0 S_x}{2 \pi \epsilon_0 X}, \quad |X| \to \infty, \tag{39}
\]

that yields

\[
E_x(X,0) = 0, \quad E_y(X,0) \sim \frac{\phi_0 S_x}{2 \pi \epsilon_0 X^2}. \tag{40}
\]

Similarly, for the motion along \( Y \) axis

\[
E_y(0,Y) = 0, \quad E_x(0,Y) \sim \frac{\phi_0 S_y}{2 \pi \epsilon_0 Y^2}. \tag{41}
\]

Interestingly, the material anisotropy does not enter these results at all. This means that the power-law decay of the electric field should exists also in the isotropic case. In fact, for \( \gamma = 1 \) one has from Eq. (37)

\[
G(X,0) = \frac{i \phi_0 S_x}{4 \pi^2 \epsilon_0} \int \frac{d^2 q q_x e^{i q X}}{q^2 (1 + q^2)}, \tag{42}
\]

which is readily done integrating first over the angle between \( q \) and \( X \). We obtain:

\[
G(X,0) = \frac{\phi_0 S_x}{2 \pi \epsilon_0} \left[ K_1(X) - \frac{1}{X} \right], \tag{43}
\]

that gives

\[
E_y(X,0) = -\frac{\phi_0 S_x}{2 \pi \epsilon_0} \left[ K'_1(X) + \frac{1}{X^2} \right]. \tag{44}
\]

Figure 10 shows that the field \( E_y(X,0) \) changes sign at \( x / \lambda \approx 1 \), reaches maximum near 2, and slowly decays as a power law \( \lambda^2 / x^2 \). This is quite surprising since the electric field power-law decay means that no screening of \( \mathbf{E} \) is involved, in other words, that there is no Meissner-type effect for the field \( \mathbf{E} \).

VI. DISCUSSION

We have studied effects of vortex motion within time dependent London theory, which is based on the assump-
The dashed line shows the power-law term $1/X^2$. $X$ are in units of $\lambda$.

The physical reason for this change is the induced electric field $E$ and along with it the currents of normal excitations $\sigma E$. This field is obtained by solving quasi-stationary Maxwell equations $\text{curl}E = \partial_t h/c$, the condition of quasi-neutrality $\text{div}E = 0$, coupled with the time-dependent London equation (basically, the same procedure as in deriving time-dependent Ginzburg-Landau equations [13]). Unlike $h$, the field $E$ cannot be screened in the bulk of the material, so that one may say that there is no “Meissner effect” for the electric field per se.

It turns out that in anisotropic case the magnetic field of moving vortex has a power law dependence on distances $r >> \lambda$: $h \propto (\gamma^2 - 1)v/r^3$ ($\gamma$ is the anisotropy parameter, $v$ is the vortex velocity). The exponentially decaying part of $h$ is still present, but at large distances it is irrelevant in comparison with the power-law part. In isotropic case, the power law gives way to the standard exponential decay. The electric field, however, goes as $1/r^2$ in both cases.

Most of our calculation were done for orthorhombic materials with the in-plane anisotropy parameter $\gamma = 3$ and the vortex along $c$. Such materials in fact exist, examples are NiBi films [19], or Ta$_3$Pd$_3$Te$_16$ [20].

### Appendix A

Consider the integral

$$\int \frac{d^2k e^{ikr}}{(1 + k^2)^2} = \int_0^\infty \frac{dk}{(1 + k^2)^2} \int_0^{2\pi} d\varphi e^{ikr \cos(\alpha - \varphi)} = 2\pi \int_0^\infty \frac{dk}{(1 + k^2)^2} J_0(kr) = \pi r K_1(r), \quad (A1)$$

$k$ and $r$ are at angles $\varphi$ and $\alpha$ relative to $x$. Apply $\partial_x$ to both sides:

$$\int d^2k_x e^{ikr} = i\pi x K_0(r), \quad (A2)$$

The evaluation of the first integral in Eq. (21) is now straightforward.
Appendix B

The second contribution in Eq. (21) consists of parts related to x and y projection of the velocity. With the help of identities
\[ \frac{1}{f} = \int_0^\infty du e^{-fu}, \quad \frac{1}{f^2} = \int_0^\infty du u e^{-fu}, \] (B1)
one recasts the x-part:
\[ I_x = \frac{S_x(1 - \gamma^2)}{\gamma} \int \frac{d^2q}{q^2(1 + \gamma q_x^2 + q_y^2)^2} = \int_0^\infty d\xi \int_0^\infty du u e^{-u} \int d^2q q_y^2 q_x e^{i q R - \xi q_x^2 - \xi q_y^2}. \] (B2)
Here we use \( \lambda \) as a unit length: \( q = k\lambda, R = r/\lambda, \lambda_3^2 = \lambda^2, \lambda_1^2 = \lambda^2/\gamma, \) and \( S = s/\lambda. \) The anisotropy parameter is \( \gamma = \lambda_2/\lambda_1. \) We now introduce a new integration variable \( \zeta \) via \( \xi = \zeta u: \)
\[ I_x = \frac{S_x(1 - \gamma^2)}{\gamma} \int_0^\infty d\zeta \int_0^\infty du u^2 e^{-u} \int d^2q q_y^2 q_x e^{i q R - u(\zeta + \gamma)q_x^2 - u(\zeta + \gamma)q_y^2}. \] (B3)
Integrals over \( q_x, q_y \) are evaluated with the help of the known Fourier transform of a Gaussian:
\[ \int d\xi e^{i q_x x - a q_x^2} = \sqrt{\frac{\pi a}{4u^2}} e^{-x^2/4a}. \] (B4)
Integration over \( u \) can be done utilizing relations
\[ \int_0^\infty \frac{du}{u} \exp \left( -u - \frac{u^2}{4} \right) = 2 K_0(w), \quad \int_0^\infty \frac{du}{u^2} \exp \left( -u - \frac{u^2}{4} \right) = \frac{4}{w} K_1(w). \] (B5)
We obtain after straightforward algebra:
\[ I_x = \frac{i\pi(1 - \gamma^2)}{2\gamma} S_x X \int_0^\infty \frac{d\zeta}{(\zeta + \gamma)^{3/2}(\zeta + 1/\gamma)^{3/2}} \left[ K_0(R_\zeta) - \frac{Y^2}{(\zeta + 1/\gamma)R_\zeta} K_1(R_\zeta) \right], \quad R_\zeta = \frac{X^2}{\zeta + \gamma} + \frac{Y^2}{\zeta + 1/\gamma}. \] (B6)
In a similar fashion one obtains the part proportional to \( S_y \) and Eq. (26).

Appendix C: Electric field and normal currents

We evaluate here the stream function of Eq. (37):
\[ G = \frac{i q_0 C}{4\pi^2 e^2} \hat{G} = \int \frac{d^2q(q \cdot S)}{q^2 C} e^{i q R}. \] (C1)
The following manipulation is similar to that outlined in Appendix B for \( h(X,Y): \)
\[ \hat{G} = \int d^2q(q \cdot S)e^{i q R} \int_0^\infty du e^{-uq_2} \int_0^\infty d\xi e^{-\xi} \int d^2q(q \cdot S)e^{i q R - uq_2 - \xi(q_x^2 + q_y^2/\gamma)}. \] (C2)
Further, we write the last integral \( \int d^2q \ldots = S_x I_x + S_y I_y \) with
\[ I_x = \int_{-\infty}^\infty dq_x q_x e^{i q_x x - q_x^2(u + \xi \gamma)} \int_{-\infty}^\infty dq_y e^{i q_y Y - q_y^2(u + \xi \gamma)} \] (C3)
and \( I_y \) which is obtained from \( I_x \) by replacing \( x \leftrightarrow y \) and \( \gamma \leftrightarrow 1/\gamma. \) The integral over \( q_x \) and \( q_y \) are related to the known Fourier transform of a Gaussian and we obtain:
\[ I_x = \frac{i\pi X}{2(u + \xi \gamma)^{3/2}(u + \xi \gamma)^{1/2}} \exp \left( -\frac{X^2}{4(u + \xi \gamma)} - \frac{Y^2}{4(u + \xi \gamma)} \right) \] (C4)
and the part \( \hat{G} \) proportional to \( S_x \) takes the form
\[ \hat{G}_x = \frac{i\pi X S_x}{2} \int_0^\infty du \int_0^\infty \frac{d\xi e^{-\xi}}{(u + \xi \gamma)^{3/2}(u + \xi \gamma)^{1/2}} \exp \left( -\frac{X^2}{4(u + \xi \gamma)} - \frac{Y^2}{4(u + \xi \gamma)} \right). \] (C5)
To integrate over $u$ we can use Eq. (B5). To this end we introduce a new integration variable $\eta$ instead of $\xi$ via $\xi = u\eta$. Then the integral over $\xi$ becomes

$$
\frac{1}{u} \int_{0}^{\infty} d\eta \frac{e^{-\eta u}}{(1 + \eta \gamma)^{3/2}(1 + \eta / \gamma)^{1/2}} \exp \left( -\frac{R_{\eta}^2}{4u} \right), \quad R_{\eta}^2 = \frac{X^2}{1 + \eta \gamma} + \frac{Y^2}{1 + \eta / \gamma}.
$$

(C6)

Now, the integration over $u$ is done with the help of Eq. (B5) and we obtain:

$$
\hat{G}_x = i \pi S_x X \int_{0}^{\infty} d\eta \frac{(1 + \eta \gamma)^{1/2}(1 + \eta / \gamma)^{3/2}}{(1 + \eta \gamma)^{1/2}(1 + \eta / \gamma)^{3/2}} K_0(R_{\eta} \sqrt{\eta}).
$$

(C7)

The part $G_y$ follows immediately after the replacements $x \leftrightarrow y$ and $(1 + \eta \gamma) \leftrightarrow (1 + \eta / \gamma)$:

$$
\hat{G}_y = i \pi S_y Y \int_{0}^{\infty} d\eta \frac{(1 + \eta \gamma)^{3/2}(1 + \eta / \gamma)^{1/2}}{(1 + \eta \gamma)^{3/2}(1 + \eta / \gamma)^{1/2}} K_0(R_{\eta} \sqrt{\eta}).
$$

(C8)