A quantum cosmology and discontinuous signature changing classical solutions

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Abstract

We revisit the classical and quantum cosmology of a universe in which a self interacting scalar field is coupled to gravity with a flat FRW type metric undergoing continuous signature transition. We arrange for quantum cosmologically allowed discontinuity in the classical solutions at the signature changing hypersurface, provided these solutions be dual in some respects. This may be of some importance in the study of early universe within the signature changing scenarios.

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1 Introduction

The initial idea of signature change is due to Hartle and Hawking [1]. This idea makes it possible to have both Euclidean and Lorentzian metrics in the path integral approach to quantum gravity. It was later shown that signature change may happen even in classical general relativity [2]-[5]. More recently, the people have studied this issue in the Brane-World scenario, as well [6]. From a classical point of view, the signature change may prevent the occurrence of singularities in general relativity, such as the Big Bang, which may be replaced by a compact Euclidean domain prior to the birth of time in the Lorentzian domain, the so-called no-boundary proposal [1]. Alternatively, the classical signature change scenario may be an effective classical description of the quantum tunnelling approach for the creation of Lorentzian universe [7].

In general, there are two different approaches to the issue of classical signature change: continuous and discontinuous. In the continuous approach, passing from Euclidean to Lorentzian domain, the signature of metric changes continuously, hence the metric becomes degenerate at the transition hypersurface. In the discontinuous approach however, the metric is non-degenerate everywhere and discontinuous at the transition hypersurface. In both approaches the dynamical fields and their first derivatives satisfy specific junction conditions. In Ellis et al point of view [2], both the fields and their first derivatives are continuous, while in that of Hayward [3] the fields are continuous but their derivatives are zero at the transition hypersurface.

In this paper, we first review the model adopted by Dereli and Tucker [4] in which a self interacting scalar field is coupled to gravity. In the classical version of this model, Einstein field equations are solved such that the scalar field and the scale factor are considered as dynamical variables, giving rise to cosmological solutions with degenerate metrics describing continuous transition from Euclidean to Lorentzian domains. The quantum cosmological version of this model is also used to derive the wavefunction of the universe by solving the corresponding Wheeler-DeWitt equation [8]. The Wheeler-DeWitt equation is obtained by adopting a new choice of variables, through which Einstein classical equations of motion arise from an anisotropic constrained oscillator-ghost-oscillator Hamiltonian. A family of Hilbert subspaces are then derived in which states are identified with non dispersive wave packets which remarkably peak in the vicinity of classical loci with parameterizations corresponding to metric solutions of Einstein equations that admit a continuous signature transition [8].

Then, we pay more attention to the previously obtained duality transformations, on the parameters of the scalar field potential, by which the classical cosmology transforms to its dual, but the quantum cosmology remains unchanged [9]. Since these duality transformations do not affect the results in [8], then a remarkable correspondence can exist between the self dual quantum cosmology and the dual classical cosmologies. This motivates the definition of a potential distribution over the entire manifold such that Euclidean and Lorentzian parts are endowed by dual potentials. Therefore, according to the above correspondence, the dual classical cosmologies are indistinguishable at the quantum level, and the jump from one solution in the Euclidean part to its dual in the Lorentzian part is consistent with this quantum cosmology.

Therefore, in spite of the common continuity requirement on the signature changing clas-
classical solutions [2]-[5], the quantum cosmology discussed here allows for discontinuity in the corresponding classical solutions passing through the hypersurface of signature change. Of course, this is not so surprising because the quantum mechanics always changes a continuous picture to a discontinuous one.

We shall discuss on the possible interpretation of these allowed jumps (supported by quantum cosmology) as sudden birth of a Lorentzian universe from a rather small size Euclidean region. It is to be noted that this interpretation is in no way equal to an inflationary model. But it deserves to be studied to see how the sudden creation of a Lorentzian universe may be accommodated in a simple quantum cosmology admitting classical signature transition.

2 Dual classical solutions

We consider the Einstein-Hilbert action

$$ I = \int \sqrt{|g|} \left[ \frac{1}{16\pi G} \mathcal{R} + \mathcal{L}_M \right] d^4x, \quad (1) $$

where $\mathcal{R}$ is the scalar curvature, $\mathcal{L}_M = \frac{1}{2} \partial_\phi \phi \partial^\phi \phi - U(\phi)$ is the real scalar field Lagrangian and $\phi$ is assumed to be a homogeneous field which depends merely on the time parameter. We take the chart $\{ \beta, x_1, x_2, x_3 \}$ and parameterize the metric as FRW type [4]

$$ g = -\beta d\beta \otimes d\beta + R(\beta)^2 \left[ 1 + \frac{k}{4} r^2 \right]^2 \sum_i dx_i \otimes dx_i, \quad (2) $$

where $r^2 = \sum x_i x_i$, $R(\beta)$ is the scale factor with $k = \{-1, 0, 1\}$ representing open, flat or closed universes and $\beta$ is the lapse function producing the hypersurface of signature change at $\beta = 0$. For $\beta > 0$, the cosmic time can be written as $t = \frac{2}{3} \beta^{3/2}$. By calculating the scalar curvature $\mathcal{R}$ and using the transformations [4]

$$ X = R^{3/2} \cosh(\alpha \phi), \quad (3) $$

$$ Y = R^{3/2} \sinh(\alpha \phi), \quad (4) $$

the corresponding effective Lagrangian is obtained for a flat universe $k = 0$

$$ \mathcal{L} = \frac{1}{2 \alpha^2} (-X^2 + Y^2) - (X^2 - Y^2)U(\phi(X,Y)). \quad (5) $$

We know from the (3+1) decomposition of the Einstein theory of gravity, that the Hamiltonian $\mathcal{H}$ corresponding to $\mathcal{L}$, must vanish identically leading to a zero energy condition

$$ \mathcal{H} = \frac{1}{2 \alpha^2} (-\dot{X}^2 + \dot{Y}^2) + (X^2 - Y^2)U(\phi(X,Y)) = 0. \quad (6) $$

Now, we take the potential as [4], [9]

$$ U(\phi) = \lambda + \frac{1}{2 \alpha^2} m^2 \sinh^2(\alpha \phi) + \frac{1}{2 \alpha^2} b \sinh(2\alpha \phi), \quad (7) $$

The models with $k \neq 0$ was also studied in [12].
where $\lambda = U \mid_{\phi=0}$, $m^2 = \frac{\partial^2 U}{\partial \phi^2} \mid_{\phi=0}$ and $b$ are the bare cosmological constant, positive mass square and coupling constant, respectively. The minimum of the potential occurs at $\phi = -\frac{1}{2\alpha} \tanh^{-1}(\frac{2b}{m^2})$ when $|2b/m^2| < 1$ and we will interpret this minimum as the effective cosmological constant

$$\Lambda_{\text{eff}} = \lambda + \frac{m^2}{4\alpha^2} \left( \sqrt{1 - \frac{4b^2}{m^4}} - 1 \right), \quad (8)$$

where the second term is interpreted as the contribution of the field excitation. Variation of the action with respect to the dynamical variables $X$ and $Y$ gives the dynamical equations

$$\ddot{\xi} = M \xi, \quad (9)$$

subjected to the zero energy condition

$$\dot{\xi}^T J \dot{\xi} = \xi^T J M \xi, \quad (10)$$

where $\xi = \begin{pmatrix} X \\ Y \end{pmatrix}$, $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and

$$M = \begin{pmatrix} 2\alpha^2 \lambda & b \\ b & 2\alpha^2 \lambda - m^2 \end{pmatrix}.$$ 

One may then define the normal modes $\alpha = S^{-1} \xi$ by

$$\alpha = \begin{pmatrix} u \\ v \end{pmatrix}, \quad S = \begin{pmatrix} \frac{m^2 - \sqrt{m^4 - 4b^2}}{2b} & -\frac{m^2 + \sqrt{m^4 - 4b^2}}{2b} \\ 1 & 1 \end{pmatrix}, \quad (11)$$

which diagonalize the matrix $M$ as follows

$$S^{-1} M S = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}.$$ 

The zero energy condition then reads as

$$\alpha^T \mathcal{J} \dot{\alpha} = \alpha^T \mathcal{I} \alpha, \quad (12)$$

where $\mathcal{J} = S^T J S$ and $\mathcal{I} = S^T J M S$. The normal modes $\begin{pmatrix} u \\ v \end{pmatrix}$, with zero energy condition (12) evaluated at $t = 0$, for $\lambda_+ , \lambda_- < 0$ lead to the following classical loci [8]

$$v = 2A_0 \cos \left[ \frac{1}{r} \cos^{-1} \left( \frac{ru}{2A_0} \right) \right], \quad |u| \leq \frac{2A_0}{r}, \quad (13)$$

$$v = 2A_0 \cosh \left[ \frac{1}{r} \cosh^{-1} \left( \frac{ru}{2A_0} \right) \right], \quad |u| > \frac{2A_0}{r}.$$
where \( r = \sqrt{\frac{\lambda_+}{\lambda_-}} \), \( 0 < r < 1 \), \( \epsilon = \pm 1 \) indicates two ways of satisfying the constraint \( \mathcal{H} = 0 \), and \( \lambda_\pm \) are the eigenvalues of the decoupling matrix given by

\[
\lambda_\pm = \frac{3\lambda}{4} - \frac{m^2}{2} \pm \frac{1}{2}\sqrt{m^4 - 4b^2}. \tag{14}
\]

An interesting feature of this model is that one can find a class of transformations on the space of parameters \( \{\lambda, m^2, b\} \) leaving the eigenvalues \( \lambda_\pm \) of the decoupling matrix invariant [9]. These transformations can be written as

\[
\lambda \to \tilde{\lambda} \equiv \frac{1}{4\alpha^4} \lambda^{-1}, \\
m^2 \to \tilde{m}^2 \equiv m^2 - \frac{4\alpha^4 \lambda^2 - 1}{\alpha^2 \lambda}, \\
b^2 \to \tilde{b}^2 \equiv b^2 + m^2[(2\alpha^2 \lambda)^{-1} - 2\alpha^2 \lambda] + [(2\alpha^2 \lambda)^{-1} - 2\alpha^2 \lambda]^2. \tag{15}
\]

It is seen that although the classical loci (13) do not change under (15), the corresponding solutions \( R(\beta) \) and \( \phi(\beta) \) change, since \( X(\beta) \) and \( Y(\beta) \) are related to \( u(\beta) \) and \( v(\beta) \) by the decoupling matrix which is changed under (15).

Therefore, if we define (15) as \textit{duality} transformations, then we have two sets of solutions for \( R(\beta) \) and \( \phi(\beta) \) corresponding to dual sets of physical parameters. We interpret the new parameters as dual bare cosmological constant, dual mass square and dual coupling constant, respectively.

As is discussed in [4], in order to have signature transition from Euclidean to Lorentzian, both eigenvalues \( \lambda_\pm \) must be negative, so equation (14) gives

\[
\lambda < \frac{4}{3}\left[\frac{m^2}{2} - \frac{1}{2}\sqrt{m^4 - 4b^2}\right], \tag{16}
\]

but it does not guarantee that the dual potential has also a minimum and a positive mass square. In order the dual potential has these features, as well, we take

\[
\frac{m^2}{2} - \frac{1}{2}\sqrt{m^4 - 4b^2} \leq 1. \tag{17}
\]

If we now choose the set of suitably small couplings \( \{\lambda, m^2, b\} \)

\[
\lambda \simeq 0 \quad m^2 \ll 1 \quad b \simeq 0, \tag{18}
\]

with \( b \ll m^2 \), then \( |2b/m^2| < 1 \) and the potential \( U \) will have a minimum. The dual transformations map the small values of \( \lambda, m^2 \) and \( b \) to very large values of the corresponding dual parameters, \( \tilde{\lambda}, \tilde{m}^2 \) and \( \tilde{b} \). Fortunately, the large values for \( \tilde{b} \) and \( \tilde{m}^2 \) satisfy \( |2\tilde{b}/\tilde{m}^2| < 1 \) so that we have a minimum for \( \tilde{U} \), as well [9]. It then follows that two different classical cosmologies, one with very small values for the bare cosmological constant, mass scale and coupling constant and the other with large ones, exhibit the same signature dynamics on the configuration space \((u, v)\).
3 A self-dual quantum cosmology

We have shown, in this model, that it is possible to find dual classical cosmologies \((R, \phi)\) and \((\tilde{R}, \tilde{\phi})\) corresponding to the same classical cosmology defined on the \((u, v)\) configuration space. On the other hand, it was shown that the Wheeler-Dewitt equation for this model in the mini-superspace \((u, v)\) is

\[
\left\{ \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} - \omega_1^2 u^2 + \omega_2^2 v^2 \right\} \Psi(u, v) = 0,
\]

(19)

where \(\omega_1^2 = -\lambda_+\), \(\omega_2^2 = -\lambda_-\). To obtain the solutions \(\Psi^{(m_1, m_2)}(u, v)\) a quantization condition was required

\[
\frac{\lambda_+}{\lambda_-} = \left( \frac{2m_1 + 1}{2m_2 + 1} \right)^2,
\]

(20)

which was imposed on the parameters of the scalar field potential. For a given pair of \((m_1, m_2)\) it was shown by graphical analysis that the absolute value of these solutions have maxima in the vicinity of classical loci (13) on the \((u, v)\) configuration space which can exhibit a signature transition [8]. This correspondence between classical loci and quantum cosmological prediction is considerably interesting. In the previous section, we have shown that there are duality type transformations on the parameters of a given scalar field potential, giving rise to a dual potential, such that the solutions of the field equations on the \((R, \phi)\) configuration space transform to the dual solutions \((\tilde{R}, \tilde{\phi})\), while the solutions on the \((u, v)\) configuration space are self dual. Applying the quantum cosmology discussed above to this picture turns out that for any pair of \((m_1, m_2)\) defining a distinct quantum cosmology in terms of the variables \((u, v)\), we may correspond dual classical solutions \((R, \phi)\) and \((\tilde{R}, \tilde{\phi})\), admitting signature transition from Euclidean to Lorentzian spacetime. This is because, the two sets of dual classical solutions \((R, \phi)\) and \((\tilde{R}, \tilde{\phi})\) have the same quantum cosmology concentrated on both of them over the configuration space \((u, v)\).

4 Distributional classical solutions

Now, we consider a manifold with a distribution of the dual potentials \(U\) and \(\tilde{U}\). Suppose we have a distribution [13]

\[
U(\phi) = \Theta^+ U^+(\phi) + \Theta^- U^-(\phi),
\]

(21)

where \(U^+ \equiv U, U^- \equiv \tilde{U}\) and \(\Theta^+, \Theta^-\) are Heaviside distributions with support in regions \(\beta > 0\) (Lorentzian), and \(\beta < 0\) (Euclidean) respectively, such that

\[
d\Theta^\pm = \pm \delta,
\]

(22)

where \(\delta\) is the hypersurface Dirac distribution with support on hypersurface \(\beta = 0\). At the transition hypersurface \(\beta = 0\) we assume that the potential is regularly discontinuous

\[
[U(\phi)]_{\beta=0} = U^+(\phi) \mid_{\beta=0} - U^-(\phi) \mid_{\beta=0},
\]

(23)
where \( U^+(\phi) \big|_{\beta=0} = \lim_{\beta \to 0^+} U(\phi) \) and \( U^-(\phi) \big|_{\beta=0} = \lim_{\beta \to 0^-} U(\phi) \). This assumption is valid if \( \lambda \neq 0 \), otherwise we will have a divergent \( \tilde{U} \). In general, such a manifold is not expected to yield continuous solutions for the scale factor and the scalar field passing through the transition hypersurface \( \beta = 0 \), and we obtain distributional solutions\(^2\)

\[
\Re(\beta) = \Theta^+ R^+(\beta) + \Theta^- R^-(\beta),
\]

\[
\Phi(\beta) = \Theta^+ \phi^+(\beta) + \Theta^- \phi^-(\beta),
\]

with regularly discontinuous scale factor and scalar field

\[
[R(\beta)]_{\beta=0} \neq 0 \quad , \quad [\phi(\beta)]_{\beta=0} \neq 0.
\]

But, considering Eq.(13) and self-duality of the parameter \( r = \sqrt{\frac{\lambda}{\lambda}} \) the solutions on the \((u,v)\) configuration space are smooth and continuous passing through the hypersurface \( \beta = 0 \), namely

\[
[u(\beta)]_{\beta=0} = [v(\beta)]_{\beta=0} = 0.
\]

The relevance of this classical model manifests when quantum cosmology comes in to play the role. We have already seen that given two disjoint Lorentzian and Euclidean regions with potentials \( U \) and \( \tilde{U} \) defined over them, respectively, there are classical solutions \((R(\beta), \phi(\beta))\) on the former and \((\tilde{R}(\beta), \tilde{\phi}(\beta))\) on the later one, both of them corresponding to the same classical cosmology in \((u,v)\) space. On the other hand, as in [8] in which a good correspondence was shown between the classical loci in \((u,v)\) space and the solutions of the Wheeler-DeWitt equation, one obtains the same correspondence here. However, unlike [8] where there is a one to one relation between \((R, \phi)\) and \((u,v)\), we have here a two to one relation between \\{(\(R, \phi\), (\(\tilde{R}, \tilde{\phi}\))\} and \((u,v)\) classical cosmologies. Therefore, the solutions of Wheeler-DeWitt equation in \((u,v)\) mini-superspace will support not only \((R, \phi)\) cosmology but also its dual, namely \((\tilde{R}, \tilde{\phi})\). Therefore, following the idea in regarding the Wheeler-Dewitt solutions more primitive than the classical solutions of Einstein equations\(^3\), we may suggest that to the extent, we are concerned with quantum cosmology considered here, the distributions (24), (25) as combinations of dual classical cosmologies are also expected to be the allowed classical predictions. Hence, the strict continuity condition of the classical fields \( R(\beta) \) and \( \phi(\beta) \) in passing through the transition hypersurface \( \beta = 0 \) may be relaxed if one takes quantum mechanics into account and concentrate on the good correspondence between quantum cosmology and dual classical solutions in this model. In this regard, the jumps of \( R(\beta) \) and \( \phi(\beta) \) are quantum cosmologically allowed effects which should be interpreted in an appropriate way.

To the authors knowledge, this is a novel idea in the cosmological context which may deserve further scrutiny. In particular, it is of value to know whether this kind of relation between classical distinguishable and quantum indistinguishable states exists in non-cosmological contexts. To this end, one may hope to seek such a relation in any model in which there are some duality transformations that affect the classical state but leave the quantum state unchanged.

\(\text{Note that, } \dot{R}, \dot{\tilde{R}}, \dot{\phi} \text{ and } \dot{\tilde{\phi}} \text{ vanish approaching the hypersurface } \beta = 0 \text{ due to } \dot{\alpha}(0) = 0.\)

\(\text{A similar idea was followed by Hawking and Page, investigating the wormholes [11]}

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Discussion

Following the model adopted by Dereli and Tucker, we have shown that in a FRW type manifold endowed completely by $U$ or $\tilde{U}$, we have signature changing classical solutions $(R, \phi)$ or $(\tilde{R}, \tilde{\phi})$, corresponding to a same quantum cosmology. However, the FRW type manifold which we have assumed here is endowed by a distributional potential (21) and correspondingly has distributional solutions with regular jumps at the hypersurface of signature change. These distributional solutions are also capable of being the classical predictions of the quantum cosmology under consideration. One should then interpret these jumps of classical solutions as a quantum cosmologically allowed effect. By this effect, we do not mean at all a quantum transition in the signature of the metric. Wheeler-DeWitt equation is time ($\beta$) independent so is signature independent, as well. Therefore, both classical limits in Euclidean and Lorentzian regions are described by a self dual quantum state $|\psi >_E = |\tilde{\psi} >_L$, and so we have a trivial quantum transition, namely $L < \psi |\psi >_E = 1$. Of course this is not surprising because the model is so chosen to represent this feature. The main point of this model is not to concentrate on the above (trivial) quantum transition from Euclidean to Lorentzian state, rather the interesting idea is that this self dual quantum state allows for a reasonable discontinuous jump from one classical limit in the Euclidean region to the dual one in the Lorentzian region.

If we regard the configuration space $(u, v)$ as a more primitive concept than spacetime, we must look for a correspondence between the properties of the quantum states $\Psi^{(m_1, m_2)}(u, v)$ and an evolving classical cosmology admitting signature transition. As is discussed in [8], a change of coordinate $\beta \rightarrow \beta' = F(\beta)$ for the spacetime patch induces a change of parametrization for the classical loci (13) and this corresponds to an alternative choice of the classical time. To the extent that the classical loci are delineated by a particular state $\Psi^{(m_1, m_2)}(u, v)$ we may assert that the family of classical times, regarded as alternative parametrization of such loci, arise dynamically from this quantum state. From this point of view, we expect that classical signature change ($\beta < 0 \rightarrow \beta > 0$) is already included in the information within $\Psi^{(m_1, m_2)}(u, v)$ [4].

According to the scenario for creation of the universe from nothing, we know that the universe with a finite size of Planck length $l_P$ is emerged suddenly, through a quantum tunnelling effect, from a universe with zero size (nothing) [7]. This sudden creation of the finite size from zero size universe is typically an example of jump in the classical solutions of the Einstein equations, namely from $R = 0$ to $R \sim l_P$, which is allowed by quantum tunnelling. In the same way, one may interpret the jumps in the classical solutions (24) and (25) as sudden birth of a rather large size Lorentzian universe from a small size Euclidean region, which is allowed by quantum cosmology under consideration.

To this end, considering the transformations (15) together with $\xi = S\alpha$ and the inverse transformations

$$
R = (X^2 - Y^2)^{1/3},
$$

$$
\phi = \frac{1}{\alpha} \tanh^{-1} \left( \frac{Y}{X} \right),
$$

and

$$
\tilde{R} = (\tilde{X}^2 - \tilde{Y}^2)^{1/3},
$$
\[ \tilde{\phi} = \frac{1}{\alpha} \tanh^{-1}\left( \frac{\tilde{Y}}{\tilde{X}} \right), \]

one finds that the set of parameters

\[ \lambda \simeq 0, \quad m^2 \ll 1, \quad b \approx 0, \]

leads to an infinitesimal scale factor in the Euclidean domain and a finite scale factor in the Lorentzian one [9]. In the same way, these parameters map a rather large \(|\tilde{\phi}|\) in the Euclidean domain to the small \(|\phi|\) in the Lorentzian domain\(^4\). Exactly, like the quantum tunnelling scenario [7] in which the Lorentzian universe pops out from zero size (nothing) to the Planck size, through a Euclidean instanton solution, the universe in this simple model is born at zero size in the Euclidean domain (instanton solution) and evolves to a very small size (presumably the Planck size), approaching the end of Euclidean domain at \(\beta \to 0^-\). Up to this point, the scenario is in complete similarity with quantum tunnelling one, because the Euclidean solutions as instantons (or real tunnelling solutions) links the zero size to the Planck size universe. However, this Planck size new born universe at the end of Euclidean domain jumps to a finite size at the beginning of the Lorentzian domain at \(\beta \to 0^+\). This is the main feature of the model in that the planck size universe experience a considerable expansion (jump) in size in a very short period of time. This is reminiscent of the inflationary scenario where the universe (after quantum tunnelling) is exponentially expanded from the Planck size to a finite size in a tiny fraction of a second. Therefore, if we compare these models and consider the Euclidean instanton solution in the present model as the classical description of quantum tunnelling, then we may interpret the immediate jump in the scale factor as an inflation-like behavior. It is worth noticing that this sudden expansion coincides with the signature transition in the very short interval \(\Delta \beta = [\beta \to 0^+ - \beta \to 0^-]\) and ends up at the beginning of the Lorentzian domain after which the standard model is applied.

The extent of jumps in the scale factor and the scalar field depends on the extent of jumps from the parameters \(\{\tilde{\lambda}, \tilde{m}^2, \tilde{b}\}\) to their dual values \(\{\lambda, m^2, b\}\). On the other hand, due to the duality relations between the two sets of parameters the extent of jumps in these parameters depend on the initial values \(\{\tilde{\lambda}, \tilde{m}^2, \tilde{b}\}\). Therefore, the extent of jump in the scale factor or the scalar field depends on \(\{\lambda, m^2, b\}\). In this regard, the large values for these parameters, leading to a small effective cosmological constant \(\Lambda_{\text{eff}}, (= \Lambda_{\text{eff}})\) [9], can arrange for a big jump in the scale factor at the beginning of the Lorentzian region. An effective small cosmological constant is then playing an important role in the large expansion of the scale factor at early universe. It is interesting that, unlike some standard inflationary models with large cosmological constant, this model predicts an inflation-like behavior for the scale factor \(R\) with a small effective cosmological constant.

The jump in the scalar field also deserves further scrutiny. The Euclidean set of large parameters \(\{\tilde{\lambda}, \tilde{m}^2, \tilde{b}\}\) give rise to a large potential \(\tilde{U}(\tilde{\phi})\) in comparison to \(U(\phi)\) in terms of the Lorentzian set of small parameters \(\{\lambda, m^2, b\}\). Therefore, the jump in the scalar field at the signature transition hypersurface \(\beta = 0\) means a jump from the huge potential \(\tilde{U}(\tilde{\phi})\) to the small one \(U(\phi)\). The latter jump may be regarded as a phase transition which \(\text{coincides}\)

\(^4\)Considering \(\xi = S\alpha\) and (11), one may find that \(|X| \gg |\tilde{X}|\) and \(Y = \tilde{Y}\); hence we obtain \(|\phi| \ll |\tilde{\phi}|\).
with the signature transition. The huge energy which is released during this phase transition may be converted into the creation of particles in the Lorentzian region to start the hot big bang. The small parameters \(\{\lambda, m^2, b\}\) in the Lorentzian region results in a flat potential. On the other hand, the minimum of the potential \(U(\phi)\) occurs at the negative value \(\phi = -\frac{1}{2\alpha}\tanh^{-1}\left(\frac{2b}{m^2}\right) \ll 0\). According to the signature dynamics however, the negative \(\phi \ll 0\) is realized in the Lorentzian region at \(\beta \gg 0\), which means that the minimum of the potential is realized at \(\beta \gg 0\). Therefore, after phase transition at \(\beta = 0\) the scalar field in the Lorentzian side rolls down to its minimum at \(\beta \gg 0\) very slowly due to the flat potential and releases its remnant energy into the universe at a small rate.

In conclusion, it is worth noting that the correspondence between the quantum states and the classical loci is independent of the following scale transformations

\[
\lambda \rightarrow \xi \lambda, \quad m^2 \rightarrow \xi m^2, \quad b \rightarrow \xi b,
\]

on the parameter space, leading to \(\lambda_{\pm} \rightarrow \xi \lambda_{\pm}\) and \(r \rightarrow r\). These leave the classical loci Eq.(13) and the quantization condition (20) unchanged and do not affect the general patterns of the quantum amplitude \(|\Psi^{(m_1,m_2)}(u,v)|^2\) [8]. This indicates that a class of universes may be defined by the equivalency relation

\[
U(\phi) \rightarrow \xi U(\phi),
\]

in which a good correspondence exists between the quantum states and the classical loci, for each universe. This, on the other hand, shows that such a correspondence exists for an equivalency class of universes with the effective cosmological constants related to each other by

\[
\Lambda_{\text{eff}} \rightarrow \xi \Lambda_{\text{eff}}.
\]

Note that, although the above scale transformations do not affect the classical loci, but change the domain of the Euclidean region [8], in a parametric way, as

\[-(3\pi/2r\omega_2)^{2/3} \leq \beta < 0 \longrightarrow -(3\pi/2r\omega_2)^{2/3}\xi^{-1/3} \leq \beta < 0\]

while the domain of the Lorentzian region \(0 < \beta \leq \infty\) remains unchanged. Therefore, each universe in the above equivalency class has its own Euclidean region whose domain is different from other one.

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