Critical Phenomena and Reentrant Phase Transition of Asymptotically Reissner–Nordström Black Holes

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By considering a small correction to the Maxwell field, we show that the resultant black hole solutions (also known as the asymptotically Reissner–Nordström black holes) undergo the reentrant phase transition and can have a novel phase behavior. We also show that such a small nonlinear correction of the Reissner–Nordström black holes has high effects on the phase structure of the solutions. It leads to a new classification in the canonical ensemble of extended phase space providing the values of the nonlinearity parameter $\alpha$ being $\alpha \lesssim 4q^2/7$. We shall study these three classes and investigate deviations from those of the standard Reissner–Nordström solutions. Interestingly, we find that there is the reentrant phase transition for $\alpha < 4q^2/7$, and for the case of $\alpha = 4q^2/7$ there is no phase transition below (at) the critical point. For the last case, one finds that small and large black holes are thermodynamically distinguishable for temperatures and pressures higher than the critical ones.

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I. INTRODUCTION

It is well known that a black hole can be investigated as an ordinary thermodynamic system $[1,2]$ with typical entropy $[3]$ and temperature $[5]$ such that in most cases usually obeys the first law of thermodynamics $[6]$. It was also shown that these highly dense compact objects treat as usual thermodynamic systems enjoying the phase transition phenomenon $[7]$. More interestingly, we can see the van der Waals-like (vdW-like) phase transition including charged black hole systems by considering the correspondence $(Q, \Phi) \leftrightarrow (P, V)$ between conserved quantities and thermodynamic variables $[8,9]$. Recently, in the context of black hole thermodynamics, a possible connection between the cosmological constant as a thermodynamical pressure is proposed $[10,11]$ which has attracted much attention. This relation is defined as follows

$$P = -\frac{\Lambda}{8\pi}, \quad (1.1)$$

in which the thermodynamical volume $V$ is the conjugate quantity to pressure as

$$V = \left(\frac{\partial M}{\partial P}\right)_{{\text{rep}}}, \quad (1.2)$$

where "rep" refers to "residual extensive parameters". Indeed, the primary motivation of considering $\Lambda$ as a thermodynamical pressure comes from the fact that several physical constants, such as Yukawa coupling, gauge coupling constants, and Newton’s constant are not fixed values in some fundamental theories. In addition, in Tolman–Oppenheimer–Volkoff equation, $\Lambda$ is added to pressure that shows the cosmological constant can play the role of the thermodynamical pressure. Besides, $\Lambda$ is a slow variation parameter and has the dimension $(\text{length})^{-2}$ which is the dimension of the pressure. Usually, a vdW-like small-large black hole (SBH-LBH) phase transition can be observed in thermodynamical systems including black holes whenever $\Lambda$ behaves as a thermodynamical pressure. This type of phase transition has been studied extensively in the background spacetime of various black hole solutions (for instance, see an incomplete list $[12,23]$ and references therein written during recent years).

The reentrant phase transition (RPT) phenomenon can be observed in an ordinary thermodynamical system when a monotonic change of any thermodynamical variable provides more than one phase transition such that the final phase is macroscopically similar to the initial phase. There is a special range in temperature in the asymptotically Reissner–Nordström (ARN) black holes so that these solutions enjoy a large-small-large phase transition by a monotonic change in the pressure. This interesting phase behavior has been observed in ordinary thermodynamical systems,
such as nicotine-water mixture \[26\], liquid crystals, binary gases, multicomponent fluids, and other different typical thermodynamic systems \[27\]. In the context of black hole thermodynamics, the RPT is reported for Born-Infeld solutions \[28, 29\], rotating black holes \[30\], asymptotically dS black holes \[31\], hairy black holes \[32\], black hole solutions in massive gravity \[33\], and Born–Infeld-dilaton black holes \[34\].

In this paper, we study the thermodynamics of ARN black holes, investigate the RPT in the extended phase space, and find deviations from those of the standard Reissner–Nordström (RN) solutions. We also discuss novel phenomenon of our black hole case study and compare it with the standard RN black holes.

II. REVIEW OF SOLUTIONS AND THERMODYNAMICS

In this section, we are going to briefly mention the solutions and thermodynamics of black holes in the presence of quadratic nonlinear electrodynamics. Before proceeding, it is worthwhile to give some motivations. Nonlinear field theories are of interest in various classes of mathematical physics since most physical systems are basically nonlinear in the nature. The nonlinear electrodynamics (NED) fields are much richer than the Maxwell theory and in special cases they reduce to the linear Maxwell field. Different constraints of the Maxwell field, like the radiation propagation inside specific materials \[35–38\] and description of the self-interaction of virtual electron-positron pairs \[39, 41\], motivate one to consider NED theories as effective fields \[42, 43\]. Moreover, a well-known outstanding problem is that most gravitational theories predict a singularity in the center of black holes. It was shown that by employing the NED fields, the big bang and black hole singularities can be removed \[44–49\]. Besides, the NEDs have important effects on the structure of the superstrongly magnetized compact objects, such as pulsars and strange stars \[50–52\].

The Lagrangian of Born-Infeld-type NED theories \[53, 54, 56\], which each one was constructed based on various motivations, tends to the following form for weak nonlinearity

\[
L(F) = -F + \alpha F^2 + O(\alpha^2),
\]

where \(F = F_{\mu\nu}F^{\mu\nu}\) is the Maxwell invariant, \(F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}\) is the electromagnetic field tensor, and \(A_{\mu}\) is the gauge potential. In this equation, \(\alpha\) denotes nonlinearity parameter that is a small quantity and proportional to the inverse value of nonlinearity parameter in Born-Infeld-type NED fields. Indeed, although different models of NEDs have been constructed with various primitive aims, they contain physical and experimental importance just for the weak nonlinearity since the Maxwell field in various branches leads to near accurate or acceptable results. Thus, in transition from the Maxwell field to NEDs, considering the weak nonlinearity effects seems to be reasonable and a logical decision. In other words, we expect to obtain precise physical results with experimental agreements whenever the nonlinearity is considered as a correction to the Maxwell theory. In this context, regardless of constant parameters which are contracted in \(\alpha\), most NED Lagrangians reduce to Eq. (2.1) for weak nonlinearity and we shall consider this Lagrangian as an effective matter source coupled to gravity.

The mentioned motivations have led to publish some interesting and reasonable works by employing Eq. (2.1) as an effective Lagrangian of electrodynamics \[39, 42, 55, 67\]. Heisenberg and Euler demonstrated that quantum corrections lead to nonlinear properties of vacuum \[39, 42, 55\]. Besides, it was shown that a quartic correction of the Maxwell invariant appears in the low-energy limit of heterotic string theory \[59, 67\]. Therefore, considering a correction term to the Maxwell field and investigating Eq. (2.1) as an effective and suitable Lagrangian of electrodynamics instead of the Maxwell and other NED fields is a reasonable and logical decision.

According to the considered motivations, we consider the topological black holes in \((n+1)\)-dimensional spacetime with perturbative nonlinear electrodynamics \[57\]. The \((n+1)\)-dimensional line element reads

\[
ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{n-1}^2,
\]

where \(f(r)\) is the metric function and \(d\Omega_{n-1}^2\) represents the line element of \((n-1)\)-dimensional hypersurface with constant curvature \((n-1)(n-2)k\) and volume \(\omega_{n-1}\) with the following explicit form

\[
d\Omega_{n-1}^2 = \begin{cases}
d\theta_1^2 + \sum_{i=2}^{n-1} \prod_{j=1}^{i-1} \sin^2 \theta_j d\theta_i^2 & k = 1 \\
d\theta_1^2 + \sinh^2 \theta_1 \left( d\theta_2^2 + \sum_{i=3}^{n-1} \prod_{j=2}^{i-1} \sin^2 \theta_j d\theta_i^2 \right) & k = -1 \\
\sum_{i=1}^{n-1} d\phi_i^2 & k = 0
\end{cases}
\]
The metric function of these black holes can be obtained as
\[
f(r) = k - \frac{m}{r^{n-2}} - \frac{2Ar^2}{n(n-1)} + \frac{2q^2}{(n-1)(n-2)r^{2n-4}} - \frac{4q^4}{2(n-2)(n+2)+(n-3)(n-4)}r^{4n-6}\alpha + O(\alpha^2),
\]
(2.4)
in which \(m\) and \(q\) are two integration constants which are related to the total mass and total electric charge of the black hole, and the last term indicates the effect of nonlinearity.

The Hawking temperature of these black holes can be obtained by using the definition of the surface gravity on the outermost horizon, \(r_+\),
\[
T = \frac{1}{2\pi(n-1)} \left( \frac{(n-1)(n-2)}{2r_+} k - \Lambda r_+ - \frac{q^2}{r_+^{n-3}} + \frac{2q^4}{r_+^{4n-5}\alpha} \right) + O(\alpha^2).
\]
(2.5)

Moreover, as we are working in Einstein gravity, the entropy of the black holes can be calculated via the quarter of the event horizon area
\[
S = \frac{r_+^{n-1}}{4},
\]
(2.6)
which shows the entropy per unit volume \(\omega_{n-1}\). The electric potential \(\Phi\), measured at infinity as a reference with respect to the event horizon is given by
\[
\Phi = \frac{q}{(n-2)r_+^{n-2}} - \frac{4q^3}{(3n-4)r_+^{4n-4}\alpha} + O(\alpha^2).
\]
(2.7)

Besides, the total electric charge per unit volume \(\omega_{n-1}\), can be obtained by considering the flux of the electric field at infinity as
\[
Q = \frac{q}{4\pi}.
\]
(2.8)

At the final stage of calculating the conserved and thermodynamic quantities, one can get the total mass of obtained black holes by using the behavior of the metric at large \(r\). Therefore, the total mass per unit volume \(\omega_{n-1}\) is given by
\[
M = \frac{(n-1)m}{16\pi}.
\]
(2.9)

Considering the entropy and electric charge as a complete set of extensive parameters, one can show that these conserved and thermodynamic quantities satisfy the first law of thermodynamics
\[
dM = TdS + \Phi dQ.
\]
(2.10)

It is worthwhile to mention that all the equations (2.4)-(2.10) are representing the background geometry and thermodynamics of the higher dimensional ARN black holes and they reduce to the higher dimensional standard RN solutions in the special limit \(\alpha = 0\).

III. REENTRANT PHASE TRANSITION

It is proved that the Schwarzschild (AdS) black holes have no vdW-like phase transition, while this phenomenon has been observed in the RN black holes. Thus, the electric charge of black holes usually plays a key role in observing such a phase transition and it would be interesting to include the effects of black hole’s charge in the thermodynamic calculations. In this paper, we show how a small correction to the Maxwell field highly affects the phase transition structure of the RN solutions and extends the thermodynamical phase space into three new different regions. The mentioned correction is motivated by nonlinear properties of vacuum generated from quantum corrections, appearing a quartic correction of the Maxwell invariant in the low-energy limit of heterotic string theory, and physical and experimental importance of adding a weak nonlinearity to the Maxwell field.

In what follows, we concentrate our attention on the spherical symmetric black holes with negative cosmological constant in 4-dimensional spacetime. Calculations show that the RPT can also occur for higher dimensional solutions,
FIG. 1: $r_+ c$ and $\hat{r}_+$ in $q - \alpha$ plane. The black region on the left indicates imaginary $r_+$ and $\hat{r}_+$ whereas the colorful area represents real $r_+$ and $\hat{r}_+$. At the border between black and colorful areas, $r_+$ and $\hat{r}_+$ are equal to $2q$.

but this is not the case for positive cosmological constant and/or flat or hyperbolic solutions. The negative cosmological constant in the extended phase space plays the role of a positive thermodynamical pressure as follows [10, 11]

$$P = -\frac{\Lambda}{8\pi}. \quad (3.1)$$

In this scenario, the total mass (2.9) behaves as the enthalpy of system, and the Smarr formula and first law of thermodynamics are modified as

$$M = 2TS + \Phi Q - 2VP + 2A\alpha; \quad A = \left(\frac{\partial M}{\partial \alpha}\right)_{S,Q,P}, \quad (3.2)$$

$$dM = TdS + \Phi dQ + VdP + Ad\alpha, \quad (3.3)$$

where $A$ is a new thermodynamical variable conjugate to $\alpha$ and as mentioned before, $V$ is the thermodynamical volume conjugate to $P$ as follows

$$V = \left(\frac{\partial M}{\partial P}\right)_{S,Q,\alpha} = \frac{1}{3}r_+^3. \quad (3.4)$$

Here, we study the thermodynamics of 4-dimensional black holes in the canonical ensemble (fixed $Q$ and $\alpha$) of extended phase space. So, by using the temperature (2.5) for $n = 3$

$$T = -\frac{\Lambda r_+}{4\pi} + \frac{1}{4\pi r_+} - \frac{q^2}{4\pi r_+^2} + \frac{q^4\alpha}{2\pi r_+^4}, \quad (3.5)$$

and the relation between the cosmological constant and pressure (3.1), it is straightforward to show that the equation of state, $P = P(r_+, T)$, is given by

$$P = \frac{T}{2r_+} - \frac{1}{8\pi r_+^2} + \frac{q^2}{8\pi r_+^3} - \frac{q^4\alpha}{4\pi r_+^5}. \quad (3.6)$$

The thermodynamic behavior of the system and its global stability are governed by the free energy, and thus, we obtain the Gibbs free energy as well. We can determine the Gibbs free energy per unit volume $\omega_2$ in the extended phase space by employing the following relation

$$G = M - TS = \frac{r_+}{16\pi} - \frac{r_+^3P}{6} + \frac{3q^2}{16\pi r_+} - \frac{7q^4\alpha}{40\pi r_+^5}. \quad (3.7)$$
In the right panel, the solid lines refer to $C_P > 0$ while the dashed red lines correspond to $C_P < 0$. $C_P$ diverges at the joins of dashed and solid lines. Besides, $G - T$ curves are shifted for clarity. The vertical black line at $T_0$, $T_1 < T_0 < T_2$, shows a discontinuity in the Gibbs free energy and indicates a zeroth-order SBH-IBH phase transition.

On the other hand, the heat capacity help us to find the local thermal stability, and thus, we calculate it in extended phase space at constant pressure as

$$C_P = T \left( \frac{\partial S}{\partial T} \right)_P = \frac{r_+^2}{2} \left( 8\pi P r_+^5 + r_+^6 - q^2 r_+^4 + 2q^4 \alpha \right).$$  \hspace{1cm} (3.8)

Here, since we are working in the canonical ensemble, $C_P$ is the heat capacity at constant $P$, $Q$, and $\alpha$. The negativity of $C_P$ indicates unstable solutions while its positivity refers to local stability (or at least metastability).

In order to study the phase transition of black holes, one can use the definition of inflection point at the critical point of isothermal $P - V$ (or equivalently $P - r_+$) diagram

$$\left. \frac{\partial P(r_+, T)}{\partial r_+} \right|_T = \left. \frac{\partial^2 P(r_+, T)}{\partial r_+^2} \right|_T = 0,$$  \hspace{1cm} (3.9)
which can be used to obtain the critical horizon radius \( r_{+c} \) and critical temperature \( T_c \). One can easily show that this equation leads to the following equation for the critical horizon radius

\[
r_{+c}^6 - 6q^2r_{+c}^4 + 56q^4\alpha = 0, \quad \text{(3.10)}
\]

with at most two real positive solutions as follows

\[
r_{+c} = \sqrt{2q^2 \left( 1 + \frac{q^2}{\sqrt{\mathcal{X}}} \right) + 2\mathcal{X}}, \quad \text{(3.11)}
\]

\[
\hat{r}_{+c} = \sqrt{2q^2 \left( 1 + \frac{i(i + \sqrt{3})q^2}{2\mathcal{X}} \right) + i\left(i - \sqrt{3}\right)\mathcal{X}}, \quad \text{(3.12)}
\]

where

\[
\mathcal{X} = \left(q^6 - \frac{7}{2}q^4\alpha + \frac{1}{2}\sqrt{7aq^6(7\alpha - 4q^2)}\right)^{1/3}. \quad \text{(3.13)}
\]

From now, the thermodynamic behavior of these black holes depends on the values of (3.11) and (3.12) which are illustrated in Fig. 1 (a) when \( r_{+c} \) and \( \hat{r}_{+c} \) are imaginary/complex, there is neither vdW-like phase transition nor RPT (black region of Fig. 1). In addition, the behavior is not like the ideal gas and we shall discuss this region later. (b) in the colorful area of Fig. 1 that both \( r_{+c} \) and \( \hat{r}_{+c} \) are real, the RPT is observed which we investigate it in this section. (c) at the border of these black and colorful areas, \( r_{+c} \) is equal to \( \hat{r}_{+c} \) and is determined by \( \alpha = 4q^2/7 \). In this case, there is a critical point such that no phase transition occurs below and at this point. Besides, the SBHs and LBHs are thermodynamically distinguishable above this critical point and we will study this border in the next section. (d) for some higher values of \( q \) and \( \alpha \), \( \hat{r}_{+c} \) is imaginary/complex while \( r_{+c} \) is real. This leads to the standard vdW-like (first-order SBH-LBH) phase transition which is investigated extensively before (for instance, see [12] for the standard RN black holes and [69] for our black hole case study) and we do not consider it in this paper. The other option, means real \( \hat{r}_{+c} \) and imaginary \( r_{+c} \), is not accessible for the system.

One may note that in the absence of the nonlinearity (the case of RN black hole), \( r_{+c} \) reduces to \( \sqrt{6q} \) and \( \hat{r}_{+c} \) vanishes, as it should be. Therefore, the RN black hole can only undergo the vdW-like phase transition for nonzero values of the electric charge. This fact uncovers the significant role of the nonlinearity parameter \( \alpha \) on the phase transition structure of these black holes. However, there is a constraint on choosing the values of \( q \) and \( \alpha \). Considering the last (correction) term of Eqs. (3.6) and (3.7), we should choose some values of \( q \) and \( \alpha \) so that this term be ignorable compared with the third term, and therefore, can be considered as a perturbation. Hence, as the simplest option, we can consider some values of \( q \) and \( \alpha \) so that \( \alpha q^2 << r_{+c}^4/2 \). However, we can ignore this restriction since one can consider the last term as a nonlinear term rather than just a perturbation (correction) term.

As a typical example and without loss of generality, we consider \( q = 0.8 \) and \( \alpha = 0.1 \) that is a point in the colorful area of Fig. 1 and thus, we expect to see the RPT. Now, we can obtain the critical temperature and pressure as follows

\[
T_c = \frac{1}{2\pi r_{+c}^2} - \frac{q^2}{\pi r_{+c}^2} + \frac{4q^4\alpha}{\pi^2 r_{+c}^4}, \quad \text{(3.14)}
\]

\[
P_c = \frac{T_c}{2r_{+c}^2} - \frac{1}{8\pi r_{+c}^2} + \frac{q^2}{8\pi r_{+c}^4} - \frac{q^4\alpha}{4\pi^2 r_{+c}^6}. \quad \text{(3.15)}
\]

For the fixed \( q = 0.8 \) and \( \alpha = 0.1 \), the general behavior of the ARN black holes is shown in Fig. 2. This figure is plotted for various areas of temperature (or equivalently pressure) in \( P = r_{+} \) (or equivalently \( G - T \)) diagram. The dashed red lines show the negative heat capacity (3.8) and refer to unstable black holes while the solid lines stand for the positive heat capacity representing the stable (or metastable) black holes (for more discussion regarding the relation between Gibbs free energy and heat capacity see [22]). Considering Fig. 2, we see that a critical point located at \( P = P_c \) in \( G - T \) diagram (at \( T = T_c \) in \( P = r_{+} \) diagram with an inflection point) and demonstrates a second-order phase transition from SBHs to large ones. In \( G - T \) diagram, the curve looks like the Hawking-Page phase transition for \( P > P_c \) [11]. For \( P_1 < P < P_c \) and \( T_1 < T < T_c \), there is an area that black holes undergo the standard first-order SBH-LBH phase transition. Besides, there are three different phases including intermediate
black holes (IBHs), SBHs, and LBHs for $P \in (P_t, P_z)$. The vertical line at $T = T_0 \in (T_t, T_z)$ indicates a zeroth-order phase transition between SBHs and IBHs which is characterized by a discontinuity in the Gibbs energy. In this area of pressures and temperatures, black hole undergoes a first-order SBH-LBH phase transition as well. This behavior is known as the RPT. Note that IBHs are macroscopically similar to large ones, and thus, black holes enjoy the large-small-large phase transition in this region of pressures and temperatures. Finally, we have just LBHs for $P < P_t$ and $T < T_t$.

Figure 3 describes the coexistence lines of SBHs+LBHs (the blue line) and IBHs+SBHs (the green line) in different scales. The blue line is located between the critical point $(T_c, P_c)$ and the triple point $(T_t, P_t)$ between SBHs, IBHs, and LBHs. Similarly, the green line is bounded between this triple point and point $(T_z, P_z)$. The black holes enjoy a first (zeroth)-order phase transition from SBHs to LBHs (IBHs to SBHs) whenever they cross the blue (green) line from left to right or top to bottom. Therefore, we observe the RPT behavior of the ARN black holes for a narrow range of temperatures $T \in (T_t, T_z)$ and pressures $P \in (P_t, P_z)$.

Now, it is worthwhile to do a comparison between the ARN black holes and the RN ones to see how this small perturbation in the Maxwell field changes the thermodynamical behavior of the RN black holes significantly. Indeed, observing the RPT for this kind of black hole is very interesting since such a behavior cannot be seen for a large class of black holes even with more complicated generalizations in the matter field and/or gravitational sector of the field equation.

Figure 4 shows the differences between the ARN black holes and the RN ones. From the left panel of this figure, we find that the nonlinearity parameter reduces the pressure of SBHs significantly whereas the pressure of LBHs almost remains unchanged. Besides, the high pressure SBHs at low temperatures are not allowed to exist while this is not the case for high temperature SBHs. These facts can be seen analytically from the equation of state \[ 3.9 \] as well. For SBHs, the nonlinear term grows significantly and reduces the pressure since it has a negative sign, and finally leads to a negative pressure for these black holes that are not allowed to exist. But for high temperatures, the first term dominates the pressure and we have SBHs in this case. On the other hand, the correction term will be very small for the LBHs and does not affect the pressure of these black holes.

It is worthwhile to mention that the minimum accessible size for SBHs at (and below) the critical point is about $r_+ \sim 0.8$, and therefore, the ratio of the correction and Maxwell terms (correction/Maxwell ratio) is at most about $\sim 0.3$. Thus, the nonlinear term is small even in the worst case and never dominates the behavior of the system. In addition, from the right panel of Fig. 4, one finds that the nonlinear term creates a new region as IBHs and increases the critical temperature and pressure. This behavior can also be understood from Eqs. \[ 3.14 \] and \[ 3.15 \] by considering the fact that the last term is ignorable since $r_+^4 \gg 2\alpha q^2$ (see the left panel of Fig. 1). However, the nonlinearity parameter does not affect the SBH-LBH phase transition point significantly (the right panel of Fig. 4).
FIG. 5: $P-r_+$ and $P-T$ diagrams for the special case $\alpha = 4q^2/7$ including the RN black hole. The dotted lines in both figures show the behavior of the RN black hole. The nonlinear parameter converts the stable small RN black holes to unconditionally unstable SBHs. The nonlinear term extends the LBH region and increases the pressure of these black holes.

IV. SPECIAL CASE $\alpha = 4q^2/7$

From the previous section, we observed that the colorful region related to $\alpha < 4q^2/7$ leads to the RPT that we have studied in details. Now, we are interested to see what would happen for the other areas. Here, we are going to investigate two special cases of thermodynamical behavior related to the ARN black holes in extended phase space including a border between the black and colorful areas of Fig. 1 specified with $\alpha = 4q^2/7$, and also, the black area of this figure determined by $\alpha > 4q^2/7$. For $\alpha = 4q^2/7$, Eqs. (3.11) and (3.12) give the same results as follows

$$r_{+c} = \hat{r}_{+c} = 2q,$$

which is a critical point since the response function (3.8) diverges at this point. In this case, the critical temperature (3.14) and critical pressure (3.15) reduce to

$$T_c = \frac{1}{7\pi q}; \quad P_c = \frac{3}{256\pi q^2}. \quad (4.2)$$

Here, we fix $q$ as $q = 0.8$ to plot Fig. 5. Interestingly, this figure indicates that the small correction highly affects the thermodynamical behavior of the ARN black holes in the case of $\alpha = 4q^2/7$ as well. In this case, the correction term is very small, hence ignorable for all accessible black holes’ event horizon radius $r_+$. From the left panel, we find that the nonlinearity parameter converts the stable small RN black holes to unconditionally unstable SBHs and slightly affects the LBHs. The right panel shows the effect of $\alpha$ on LBHs so that the region of these black holes is extended and the pressure is increased. In the case of the RN black holes (and also, the critical phenomena of other black hole solutions) the SBHs and LBHs are thermodynamically indistinguishable above the critical point whereas for our black hole case study, there is always a border between (unconditionally unstable) SBHs and large ones (blue line of right panel of Fig. 5). This distinguishable property is a novel feature observed in this special type of black holes and is due to the fact that there is no SBH-LBH phase transition in this special case and SBHs are always unstable. Indeed, another interesting and new behavior is that there is no SBH-LBH phase transition at (and below) the critical point.

It is worthwhile to mention that the thermodynamic behavior of the black area of Fig. 1 determined by $\alpha > 4q^2/7$, is very similar to $\alpha = 4q^2/7$ case, but $r_{+c}$ and $\hat{r}_{+c}$ are imaginary, and thus, the critical point specified by (4.1) and (4.2) is absent. Thus, in this case, the SBHs and LBHs are always distinguishable while the SBHs are unconditionally unstable.

V. CONCLUSIONS

In this paper, we have considered the cosmological constant as thermodynamical pressure and studied the thermodynamics of 4-dimensional ARN black holes in the canonical ensemble of extended phase space and deviations from
those for the standard RN black holes were investigated. We interestingly found that by considering a small correction in the Maxwell field, the thermodynamical behavior of the RN black holes changes significantly and a novel critical phenomenon can be observed. Based on the values of the nonlinearity parameter, the phase space classified into three regions, and thus, three kinds of behaviors have been found which one of them was the RPT and the other one was a novel behavior in the extended phase space thermodynamics.

Specially, we have seen that in addition to the standard vdW-like phase transition of the black hole case study and the RN black holes, they can enjoy the RPT by considering this small correction in the Maxwell field. It was shown that this behavior happens for a narrow range of temperatures and pressures. In this range of RPT, black holes undergo a zeroth-order IBH-SBH phase transition and first-order SBH-LBH phase transition, and this behavior could be seen for special values of the nonlinearity parameter \( \alpha < 4q^2/7 \). In comparison with the RN black holes, the nonlinearity parameter highly affected the SBHs and converted them to unstable ones. This modification term created a new black hole region as IBHs and increased the critical temperature and pressure as well.

Moreover, it was shown that for special values of the nonlinearity parameter as \( \alpha \geq 4q^2/7 \), the correction term highly affects the thermodynamical behavior of the solutions as well. Specially, in the case of \( \alpha = 4q^2/7 \), we observed a novel critical point such that below and at this point, the black holes had no phase transition, and above this critical point, SBHs and LBHs were thermodynamically distinguishable. Besides, the stable small RN black holes converted to unconditionally unstable SBHs. This nonlinear term extended the area of LBHs and increased the pressure of these black holes.

As the final remark, since introducing a small correction in the Maxwell field, interestingly, had significant effects on the thermodynamical structure of the RN black holes, it would be nice to consider dynamical perturbations in the background geometry of the ARN black holes and investigate the effects of the nonlinearity parameter on the dynamical stability and quasinormal modes, and then compare them with those of the RN solutions.

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