Nonlocality, nonlinearity, and time inconsistency in stochastic differential games

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Abstract
This paper studies the well-posedness of a class of nonlocal fully nonlinear parabolic systems, which nest the equilibrium Hamilton–Jacobi–Bellman (HJB) systems that characterize the time-consistent Nash equilibrium point of a stochastic differential game (SDG) with time-inconsistent (TIC) preferences. The nonlocality of the parabolic systems stems from the flow feature (controlled by an external temporal parameter) of the systems. This paper proves the existence and uniqueness results as well as the stability analysis for the solutions to such systems. We first obtain the results for the linear cases for an arbitrary time horizon and then extend them to the quasilinear and fully nonlinear cases under some suitable conditions. Two examples of TIC SDG are provided to illustrate financial applications with global solvability. Moreover, with the well-posedness results, we establish a general multidimensional Feynman–Kac formula in the presence of nonlocality (time inconsistency).

Keywords
Existence and uniqueness, Feynman–Kac formula, Mathematics of behavioral economics, Nonlocal nonlinear parabolic systems, Stochastic differential games, Time inconsistency
1 | INTRODUCTION

The aim of this paper is to study the well-posedness of nonlocal fully nonlinear higher-order systems for the unknown $u$ of the form

$$
\begin{cases}
  u_s(t, s, y) = F\left(t, s, y, (\partial_I u)_{|I| \leq 2r}(t, s, y), (\partial_I u)_{|I| \leq 2r}(s, s, y)\right), \\
  u(t, 0, y) = g(t, y), \quad 0 \leq s \leq t \leq T, \quad y \in \mathbb{R}^d,
\end{cases}
$$

where $u, F, g$ are $m$-dimensional real-valued, $r$ is a positive integer, the nonlinearity $F$ could be nonlinear with respect to its all arguments, and both $s$ and $y$ are dynamical variables while $t$ should be considered as an external parameter. Here, $I = (i_1, \ldots, i_j)$ is a multi-index with $j = |I|$, and $\partial_I u := \frac{\partial^{j}}{\partial y_{i_1} \cdots \partial y_{i_j}}$. To clarify, the system (1) consists of $m$ coupled $\mathbb{R}$-valued nonlocal fully nonlinear equations $u^a$ ($a = 1, \ldots, m$):

$$
\begin{cases}
  u^a_s(t, s, y) = F^a\left(t, s, y, u^1(t, s, y), \ldots, u^m(t, s, y), \frac{\partial}{\partial y_{i_1}} u^1(t, s, y), \ldots, \frac{\partial^2}{\partial y_{i_d} \cdots \partial y_{i_d}} u^m(t, s, y)\right), \\
  u^a(t, 0, y) = g^a(t, y), \quad 0 \leq s \leq t \leq T, \quad y \in \mathbb{R}^d, \quad a = 1, \ldots, m,
\end{cases}
$$

where the superscripts $a$ of $u, F, g$ represent the $a$th entry of the corresponding vector functions. The systems above characterize a series of systems indexed by $t$, which are connected via their dependence on $(\partial_I u)_{|I| \leq 2r}(s, s, y)$. The diagonal dependence, referring to that $(\partial_I u)_{|I| \leq 2r}$ of $F$ are evaluated not only at $(t, s, y)$ but also at $(s, s, y)$, directly results in nonlocality. When $r = 1$ and $m = 1$, the system (1) or (2) is reduced to nonlocal fully nonlinear parabolic partial differential equations (PDEs) studied in Lei and Pun (2023).

The diagonal dependence is an inevitable consequence when we look for an equilibrium solution for a time-inconsistent (TIC) dynamic choice problem. The TIC problem was first seriously studied in Strotz (1955) from an economic perspective, in which a consistent planning of control policies, characterized by a Nash equilibrium (NE), is considered as a suitable solution; see Pollak (1968). With the popularity of behavioral economics, it has been recognized that the time inconsistency (also abbreviated as TIC) or dynamic inconsistency is prevalent when we consider behavioral factors in dynamic choice problems; see Thaler (1981) for some empirical evidence, Laibson (1997) and Frederick et al. (2002) for nonexponential discounting, Kahneman and Tversky (1979) and Tversky and Kahneman (1992) for (cumulative) prospect theory. However, a rigorous treatment in continuous-time settings was only available a decade ago; see He and Zhou (2022) for a review. Basak and Chabakauri (2010) and Pun (2018b) adopt a recursive approach, originally suggested in Strotz (1955), to derive a time-consistent (TC) portfolio strategy for the mean–variance (MV) investor in continuous time. The MV analysis, pioneered by Markowitz (1952), is free of TIC in the static setting (single period) but the dynamic choice problem under the MV criterion is TIC due to the nonlinearity of the variance operator. Subsequently, Björk and Murgoci (2014) and Björk et al. (2017) establish heuristic analytical frameworks for discrete- and continuous-time TIC stochastic control problems, which can successfully address the state-dependence issue (a type of TIC) of risk aversion in portfolio selection (Björk et al., 2014), but they leave behind many open problems, including the existence and uniqueness of the solution to their deduced Hamilton–Jacobi–Bellman (HJB) equation system.
Using a discretization approach, Yong (2012) and Wei et al. (2017) derive a so-called equilibrium HJB equation, which accords with the HJB system in Björk et al. (2017), to characterize the equilibrium solutions to TIC stochastic control problems. The equilibrium HJB equation is a nonlocal PDE, whose nonlocality comes from the diagonal dependence, and is a special case of Equation (1). Wei et al. (2017) established the existence and uniqueness results for the equation in a quasi-linear setting, which requires the linear dependence on the second-order derivative at local point \((t,s,y)\) and the removal of the second-order derivative at diagonal \((s,s,y)\), such that the diffusion term of the state process was restricted to be uncontrolled. From the perspective of stochastic differential equations (SDEs), Wang and Yong (2019), Hamaguchi (2021b), Wang (2020), Hamaguchi (2021a), Wang and Yong (2021) made attempts to the TIC problem with a flow of forward-backward SDEs (FBSDEs) or backward stochastic Volterra integral equation (BSVIE) but their results are still subject to the same restriction as in the PDE theory or limitation to the first-order dependence. However, their work has converted the key open problem in Björk et al. (2017) to another open problem in PDE or SDE theory. Recently, Lei and Pun (2023) has shown the existence and uniqueness of a nonlocal fully nonlinear parabolic PDE in a small-time setting.

Following Lei and Pun (2023), this paper extends the well-posedness results from a nonlocal fully nonlinear second-order PDE to a system of coupled nonlocal fully nonlinear higher-order PDEs, from small-time (in the sense of maximally defined regularity) to global settings, and from the conventional space of bounded functions to a weighted space with exponential growth functions. Similarly, the essential difficulty for constructing a desired contraction (to use fixed-point arguments) is the presence of the highest-order diagonal term \((\partial^2_{t} u)_{|I|=2}(t,s,y)\). To see this, we discuss an intuitive attempt heuristically, from which we outline the distinct feature of our problem. To show the existence and uniqueness of Equation (1), it is intuitive to consider a mapping from \(u\) to \(w\) that satisfies

\[
\begin{align*}
\begin{cases}
 w(t,s,y) &= F\left(t,s,y,(\partial^2_{t} w)_{|I|=2}(t,s,y),(\partial^2_{t} u)_{|I|=R}(s,s,y)\right), \\
 w(t,0,y) &= g(t,y), \quad 0 \leq s \leq t \leq T, \quad y \in \mathbb{R}^d.
\end{cases}
\end{align*}
\]

(3)

Thanks to the classical theory of parabolic systems (Solonnikov, 1965; Ladyženskaja et al., 1968; Èidel’man, 1969), the mapping is well-defined. By replacing the intractable diagonal term with a known vector-valued function \(u\), the well-posedness of the higher-order system (3) parameterized by \(t\) promises the existence and uniqueness of the solution \(w\). Moreover, if \(R = 2r\), it is clear that the fixed point solves the original system (1). However, for the case of \(R = 2r\), since the input \(u\) is of the same order of the output \(w\), it is not immediate to show the contraction. The curse has limited all aforementioned works to a restricted case of \(R = 1\) and \(r = 1\) except for Lei and Pun (2023) that extends the study to the case of \(R = 2\) and \(r = 1\). This paper leverages on the techniques developed in Lei and Pun (2023) to further extend the study to \(R = 2r\) with an arbitrary positive integer \(r\).

The mathematical extension achieved in this paper has two immediate implications, namely the establishments of mathematical foundation of TIC stochastic differential games (SDGs) and a general multidimensional Feynman–Kac formula under the framework of nonlocality.

1. Some specific types of TIC SDGs are studied in Wei et al. (2018), Lazrak et al. (2023) for zero-sum games and Wang and Wu (2021) for nonzero-sum games but they still...
left behind the existence and uniqueness results. In this paper, we will show the relevance of the system (1) by introducing a general formulation of nonzero-sum TIC SDGs. Under some regularity assumptions as in Friedman (1972), Friedman (1976), Bensoussan and Frehse (2000), we yield the parabolic systems for the TIC SDGs as a special case of Equation (1).

2. The extended version of Feynman–Kac formula paves a new path for the SDE theory to study a flow of the multidimensional second-order FBSDEs (or 2FBSDEs), which is also called the multidimensional second-order BSVIEs (or 2BSVIEs), where both forward and backward SDEs are multidimensional. The 2BSDEs were first introduced in Cheridito et al. (2007) to provide a probabilistic interpretation of a fully nonlinear parabolic PDE.

To clarify, our paper considers only the TIC caused by the initial-time-dependence in the control/game problems, and thus the parabolic systems of our interest (1) only involve the nonlocality with a two-time-variable structure. It is noteworthy that the initial-state-dependence and nonlinearity of conditional expectations also form the sources of TIC and there exist similar arguments to convert the control/game problem into a parabolic PDE systems with nonlocality in state; see Björk et al. (2017), Landriault et al. (2018), Hu et al. (2012), Hu et al. (2017), Hernández and Possamaï (2023), Hernández and Possamaï (2021), Yan and Yong (2019). We do not attempt the initial-state-dependence in this paper as it poses technical challenges. Its key difference from our consideration is that the state variable is multidimensional and unrestricted, whereas the time variable is naturally bounded especially in a finite-time framework.

This paper contributes to the theories of PDE, SDE, and SDG, especially for the treatment of nonlocality in the multidimensional setting. Specifically,

Section 2 (SDG aspect) formulates TIC SDGs that incorporate with TIC behavioral factors, which facilitate developments of many studies in financial economics including robust stochastic controls and games under relative performance concerns. We heuristically derive the associate equilibrium HJB systems and reveal its relation with the TIC SDGs. Our focus is then placed on the well-posedness of such nonlocal systems as it serves as the prerequisite of using its solution to characterize the solution to the TIC SDGs. Noteworthy is that our study allows the diffusion of the state process to be controllable, which breaks through the existing bottleneck of TIC stochastic control problems.

Section 3 (PDE aspect) presents our main results of well-posedness of nonlocal higher-order systems in linear, quasilinear, and fully nonlinear settings individually. Our results generalize the existing studies while potential extensions are discussed. To our best knowledge, our well-posedness results in a larger function space that accommodate more complex research objects over a longer time horizon open the frontier of the existing literature on nonlocal PDEs/systems.

Section 4 (SDG and PDE) analyzes the solvability of the equilibrium HJB systems in Section 2 with the well-posedness results in Section 3. Moreover, we illustrate two financial examples of TIC SDG that are globally solvable.

Section 5 (SDE and PDE) provides a nonlocal Feynman–Kac formula linking the solution to a flow of multidimensional 2FBSDEs to that of a nonlocal fully nonlinear parabolic system.

Section 6 concludes.
2 NONZERO-SUM TIME-INCONSISTENT STOCHASTIC DIFFERENTIAL GAMES

In this section, we follow the frameworks of Friedman (1972, 1976); Bensoussan and Frehse (2000) to formulate general \( m \)-player nonzero-sum TIC SDGs, where preferences and utility functions for each player are time-varying.

Let \( (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}) \) be a completed filtered probability space on which a \( k \)-dimensional standard Brownian motion \( \{W(\tau)\}_{\tau \geq 0} \) with the natural filtration \( \mathbb{F} = \{\mathcal{F}_\tau\}_{\tau \geq 0} \) augmented by all the \( \mathbb{P} \)-null sets in \( \mathcal{F} \) is well-defined. Let \( \{X(\tau)\}_{\tau \in \mathbb{R}^d} \) be the controlled \( d \)-dimensional state process driven by the forward SDE (FSDE):

\[
\begin{align*}
dX(\tau) &= b(\tau, X(\tau), \alpha(\tau))d\tau + \sigma(\tau, X(\tau), \alpha(\tau))dW(\tau), \quad \tau \in [s, T], \\
X(s) &= \xi, \quad \xi \in L^2_{\mathcal{F}_s}(\Omega; \mathbb{R}^d),
\end{align*}
\]

where \( L^2_{\mathcal{F}_s}(\Omega; \mathbb{R}^d) \) is the set of \( \mathbb{R}^d \)-valued, \( \mathcal{F}_s \)-measurable, and square-integrable random variables and \( \alpha(\cdot) : [s, T] \times \Omega \to U \) with the aggregated control process that consists of all \( m \) players’ controls characterized by \( \{\alpha_a\}_{a=1}^m \), that is, \( \alpha = ((\alpha_1)^T, \ldots, (\alpha_m)^T)^T \), with \( \alpha_a(\cdot) : [s, T] \times \Omega \to U^a \subseteq \mathbb{R}^{p_a} \) and \( \sum_{a=1}^m p_a = p \). Here, \( k, d, m, p_a \) are arbitrary positive integers. Hereafter, we follow the notations in Han et al. (2021) for the control policies that the left subscript and superscript denote the time bounds of truncated control policies, that is, \( \{\alpha^{ub}_\tau\}_{\tau \in [lb, ub]} \) (they are suppressed when they are 0 and \( T \), respectively), while the right subscript indicates the control at specific time point. Denote by \( \alpha_a^{−a} \) the aggregated controls except for \( \alpha_a \) such that \( \alpha \) consists of \( \alpha^a \) and \( \alpha^{−a} \) for any \( a \), denoted by \( \alpha = \alpha^a \oplus \alpha^{−a} \). Moreover, it is useful to introduce \( \mathcal{X}^{s, \xi, \alpha}_\tau \) (or \( \mathcal{X}_\tau \) for short) the set of reachable states at time \( \tau \) from the time-state \((s, \xi)\) with the strategy \( \alpha \), which is defined by

\[
\mathcal{X}^{s, \xi, \alpha}_\tau := \text{Int} \mathcal{X}^{s, \xi, \alpha}_\tau \cup \left\{ \gamma \in \partial \mathcal{X}^{s, \xi, \alpha}_\tau : \mathbb{P}(X(\tau) \in \partial \mathcal{X}^{s, \xi, \alpha}_\tau \cap B(\gamma, \delta)) > 0 \forall \delta > 0 \right\},
\]

where \( \mathcal{X}^{s, \xi, \alpha}_\tau \) is the support of the distribution of \( X(\tau) \) of Equation (4), the interior and the boundary of which in \( \mathbb{R}^d \) are denoted by \( \text{Int} \mathcal{X}^{s, \xi, \alpha}_\tau \) and \( \partial \mathcal{X}^{s, \xi, \alpha}_\tau \), respectively, and \( B(y, \delta) \) denotes the ball centered at \( y \) with radius \( \delta \). We refer the readers to Section 3 of He and Jiang (2021) for more details. Next, let \( \{(Y(\tau), Z(\tau))\}_{\tau \in [s, T]} \equiv \{(Y(\tau; s, \xi, \alpha), Z(\tau; s, \xi, \alpha))\}_{\tau \in [s, T]} \) be the adapted solution (see Ma & Yong (1999, Proposition 3.3) for the solvability) to the following backward SDE (BSDE):

\[
\begin{align*}
dY(\tau) &= -h(s, \tau, X(\tau), \alpha(\tau), Y(\tau), Z(\tau))d\tau + Z(\tau)dW(\tau), \quad \tau \in [s, T], \\
Y(T) &= g(s, X(T)),
\end{align*}
\]

where \( X \) satisfies Equation (4) and for \( \Psi = Y, h, \) or \( g, \) \( \Psi = (\Psi^1, \ldots, \Psi^m)^T \) and \( Z \) is \( \mathbb{R}^{m \times k} \)-valued. Equations (4) and (5) jointly form forward–backward SDEs (FBSDEs).

We presume that each Player \( a \) \((a = 1, \ldots, m)\) aims to choose her control \( \alpha_a \) to minimize the following cost functional:

\[
J^a(s, \xi, \alpha_a \oplus \alpha^{−a}) := Y^a(s; s, \xi, \alpha).
\]
With the similar arguments in Karoui et al. (1997), it turns out that under some mild conditions, the cost functional $J^a$ may be expressed as

$$J^a(s, \xi; s^a \alpha^a \bigoplus s^a \alpha^-a) = \mathbb{E} \left[ \int_s^T h^a(s, \tau, X(\tau), \alpha(\tau), Y(\tau), Z(\tau)) d\tau + g^a(s, X(T)) \bigg| F_s \right].$$

Note that when $m = 1$, the problem is reduced to the TIC problem with recursive cost functional considered in Wei et al. (2017), Yan and Yong (2019). Moreover, if both $h$ and $g$ are independent of the initial time $s$, then it is further reduced to a TC problem with a recursive utility, considered in Karoui et al. (2001). Furthermore, when $h$ depends neither on $s$ nor $(Y(\cdot), Z(\cdot))$, the cost functional reduces to the classical one; see Yong and Zhou (1999).

A typical example of such initial-time-dependent cost functionals adopts nonexponential or hyperbolic discounting factors; see Laibson (1997), Frederick et al. (2002). For illustration, we assume a Markovian framework and that all the coefficient and objective functions, $b : [0, T] \times \mathbb{R}^d \times U \rightarrow \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^d \times U \rightarrow \mathbb{R}^d \times \mathbb{R}^m \times (\mathbb{R}^d)^m$, $h^a : \nabla [0, T] \times \mathbb{R}^d \times U \times \mathbb{R}^m \times (\mathbb{R}^d)^m \rightarrow \mathbb{R}$, and $g^a : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ are deterministic, where $\nabla[0, T] := \{ (\tau_1, \tau_2) \in [0, T]^2 : 0 \leq \tau_1 \leq \tau_2 \leq T \}$. Moreover, we define the set of all admissible control processes on $[s, T]$ as follows:

$$\mathcal{A}^a := \left\{ \alpha : [s, s'] \times \Omega \rightarrow U : \alpha(\cdot) \text{ is } \mathcal{F}-\text{progressively measurable with } \mathbb{E} \int_s^{s'} |\alpha(\tau)|^2 d\tau < \infty \right\}.$$

Similarly, we define the admissible set $\mathcal{A}^a$ for each player $a$ by replacing $U$ with $U^a$.

Under some mild conditions (see Ma & Yong (1999, Proposition 3.3)), for any $(s, \xi) \in [0, T] \times \mathcal{L}^2 F_s \Omega$, $\mathbb{R}^d$ and $s^a \alpha \in \mathcal{A}^a$, the controlled FBSDEs (4)–(5) admit a unique $\mathcal{F}$-adapted solution $\{X(\tau), Y(\tau), Z(\tau)\}_{\tau \in [s, T]}$.

The $m$-player game is formed, attributed to the common state processes and the recursion of the cost functionals on the aggregated $(Y(\tau), Z(\tau))$. Each player wants to minimize her own cost functional, naturally resulting in a NE point. However, since the cost functions $h^a$ and $g^a$ in Equation (6) are dependent on the initial time $s$, we will observe TIC of the decision-making. In other words, the NE point found at time $t$ may not be the NE point when we evaluate again the SDG (4) with Equation (6) at time $s > t$. To deal with the TIC, we introduce the concept of time-consistent NE (TC-NE) point below, in line with the initiative of Strotz (1955).

## 2.1 Time-consistent Nash equilibrium point

Heuristically, we are treating the TIC SDGs as “games in subgames” while the similar concept is first proposed in Pun (2018a) for robust TIC stochastic controls, where the problem is recast as a (two-player) nonzero-sum TIC SDG played by the agent and the nature. A TC-NE point of TIC SDG (4)–(5) with Equation (6) finds the NE point over $[s, T]$ given that the players adopt the predetermined NE points over $[s + \varepsilon, T]$ for a small time elapse $\varepsilon > 0$ and $s \in [0, T]$. In light of this search, the NE points identified backwardly are subgame perfect equilibrium (SPE). Note that the SPE concept is concerned about the (aggregated) controls across time and it implies the so-called time consistency of the NE points. We give the formal definition of TC-NE point as follows.
Definition 2.1 (Time-consistent Nash equilibrium (TC-NE) point). Let $U$ be a nonempty set of $\mathbb{R}^p$ and $U^a \subseteq \mathbb{R}^{d_a}$ for $a = 1, \ldots, m$ be the control set for the player $a$. A continuous map $\overline{\alpha} : [0, T] \times \mathbb{R}^d \rightarrow U$ is called a closed-loop TC-NE point of the nonzero-sum TIC SDG (4)–(5) with Equation (6) if the following two conditions hold:

1. For any $(t, y) \in [0, T] \times \mathbb{R}^d$, the state equation
   \[
   \begin{cases}
   d\overline{X}^i, y(t) = b\left(\tau, \overline{X}^i, y(\tau), \overline{X}^i, y(\tau)\right) d\tau + \sigma\left(\tau, \overline{X}^i, y(\tau), \overline{X}^i, y(\tau)\right) dW(\tau), \\
   \overline{X}^i, y(t) = y, \quad y \in \mathbb{R}^d,
   \end{cases}
   \]
   admits a unique solution $\{\overline{X}^i, y(\tau)\}_{\tau \in [t, T]}$;

2. For any $(a, s, \alpha^a, x) \in \{1, \ldots, m\} \times [t, T) \times U^a \times \mathbb{R}_+$, let $\{\overline{X}^{s, x}(\tau)\}_{\tau \in [s, T]}$ solves
   \[
   \begin{cases}
   d\overline{X}^{s, x}(\tau) = b\left(\tau, \overline{X}^{s, x}(\tau), \alpha^a + \overline{\alpha}^{-a}(\tau, \overline{X}^{s, x}(\tau))\right) d\tau + \sigma\left(\tau, \overline{X}^{s, x}(\tau), \alpha^a + \overline{\alpha}^{-a}(\tau, \overline{X}^{s, x}(\tau))\right) dW(\tau), \\
   \overline{X}^{s, x}(s) = x,
   \end{cases}
   \]
   then the following inequality holds:
   \[
   \lim_{\epsilon \downarrow 0} \frac{J^a\left(s, x, s, \overline{\alpha}^{a, e, \alpha^a} \oplus \{\overline{\alpha}^{-a}(\tau, \overline{X}^{s, x}(\tau))\}_{\tau \in [s, T]}\right) - J^a\left(s, x; \{\overline{\alpha}(\tau, \overline{X}^{s, x}(\tau))\}_{\tau \in [s, T]}\right)}{\epsilon} \geq 0, \quad (7)
   \]
   where
   \[
   s, \overline{\alpha}^{a, e, \alpha^a}(\tau) := \alpha^a \cdot 1_{[s, s+\epsilon)}(\tau) \otimes \alpha^a(\tau, \overline{X}^{s, x}(\tau)) \cdot 1_{[s+\epsilon, T)}(\tau) = \begin{cases} 
   \alpha^a, & \tau \in [s, s+\epsilon), \\
   \alpha^a(\tau, \overline{X}^{s, x}(\tau)), & \tau \in [s+\epsilon, T].
   \end{cases} \quad (8)
   \]
   Furthermore, $\{\overline{X}^{i, y}(\tau)\}_{\tau \in [t, T]}$ and $V^a(t, y) \equiv J^a(t, y; \{\overline{\alpha}(\tau, \overline{X}^{i, y}(\tau))\}_{\tau \in [t, T]})$ for $a = 1, \ldots, m$ and $t \in [0, T]$ are called the TC-NE state process and the TC-NE value functions, respectively.

Remark 2.2. For the local optimality condition (7) and the piecewise-defined strategy (8) in Definition 2.1, He and Jiang (2021) conduct in-depth studies on the choice of reference points $x$ and perturbations $\alpha^a$ in a small time period of the length $\epsilon$. It summarizes a variety of similar but different concepts of closed-loop equilibrium strategies in the existing literature (see, for instance, Ekeland & Lazrak, 2006, 2010; Basak & Chabakauri, 2010; Björk & Murgoci, 2010; Björk et al., 2014; Dai et al., 2021 where the perturbations of Equation (8) are chosen from a set of all constant strategies; Ekeland et al. (2012), Ekeland and Pirvu (2008), Björk et al. (2017) where the perturbed strategies of Equation (8) are constructed by pasting two feasible deterministic feedback strategies). One main result of He and Jiang (2021) is to show the equilibrium strategy is independent of whether the alternative strategies are constant or deterministic strategies. In other words, $\alpha^a$ can be taken as a deterministic feedback strategy $\alpha^d = \alpha^d(\cdot, \cdot)$, which would facilitate later
analyses in Section 2.2. Another key contribution of He and Jiang (2021) is to elaborate the set of $x$ and to show the advantage of replacing the whole set $\mathbb{R}^d$ with the set $\mathcal{X}_s$ of reachable states in Equation (7). We assume that $\mathcal{X}_s = \mathbb{R}^d$ for all $s$ throughout our paper, except for Section 4.1.2 where we consider the power-utility model with $\mathcal{X}_s = (0, \infty)$. In addition to the closed-loop strategies, the existing literature also define so-called open-loop equilibrium policies (see Hu et al., 2012, 2017; Yan and Yong, 2019). In this paper, we handle TIC problem by the means of closed-loop strategies within a game-theoretical framework. When $m = 1$, the TC-NE point of the nonzerosum TIC SDG is reduced to the SPE of the corresponding TIC control problem; see Wei et al. (2017), Yong (2012), Yan and Yong (2019). When the TIC sources are eliminated, it is clear that the local optimality described by Equation (7) agrees with the conventional dynamic optimality.

The inequality (7) implies that each player is locally optimal in minimizing the cost functional $J^a$ over $[s, s + \varepsilon)$ in a proper sense and no player can do better by unilaterally changing their strategy. The basic idea is illustrated in Figure 1, which also clarifies the notations we used.

In the next subsection, we shall characterize the TC-NE point as well as the TC-NE value function with a differential equation approach. While a single equilibrium HJB equation is used to characterize the SPE of the TIC control problem in Björk et al. (2017), Wei et al. (2017), it can be imagined that a system of equilibrium HJB equations is needed for our case. Prior to its derivation, we first introduce some notations and make an assumption as with Friedman (1972), Friedman (1976), Bensoussan and Frehse (2000).

For $(t, s, y, \alpha^a \oplus \alpha^{-a}, u, p, q^a) \in \nabla [0, T] \times \mathbb{R}^d \times U \times \mathbb{R}^m \times (\mathbb{R}^d)^m \times \mathcal{S}^d$, $a = 1, \ldots, m$, where $\mathcal{S}^d$ is the set of all $d \times d$ symmetric matrices, we denote the Hamiltonian by a $\mathbb{R}^m$-valued function $\mathcal{H}(t, s, y, \alpha, u, p, q) = (\mathcal{H}^1(t, s, y, \alpha^1, \alpha^{-1}, u, p, q^1), \ldots, \mathcal{H}^m(t, s, y, \alpha^m, \alpha^{-m}, u, p, q^m))^T$ with $q = \{q^a\}^m_{a=1}$ and $p = \{p^a\}^m_{a=1}$, and $\mathcal{H}^a$ defined by

\[
\mathcal{H}^a(t, s, y, \alpha^a, \alpha^{-a}, u, p, q^a) = \frac{1}{2} \text{tr}[q^a \cdot (\sigma^\top)(s, y, \alpha^a \oplus \alpha^{-a})] + (p^a)^\top b(s, y, \alpha^a \oplus \alpha^{-a})
+ h^a(t, s, y, \alpha^a \oplus \alpha^{-a}, u, \{(p^a)^\top \sigma(s, y, \alpha^a \oplus \alpha^{-a})\}^m_{a=1}). \tag{9}
\]
Assumption 2.3 (Generalized minimax condition). There exist functions $\phi^a(t, s, y, u, p, q) : \nabla[0, T] \times \mathbb{R}^d \times \mathbb{R}^m \times (\mathbb{R}^d)^m \times (\mathbb{S}^d)^m \rightarrow \mathbb{R}^p$ for $a = 1, \ldots, m$ with needed regularity such that

1. for any $(t, s, y, u, p, q) \in \nabla[0, T] \times \mathbb{R}^d \times \mathbb{R}^m \times (\mathbb{R}^d)^m \times (\mathbb{S}^d)^m$, $\phi^a(t, s, y, u, p, q) \in \mathcal{U}^a$;
2. for any $(t, s, y, u, p, q) \in \nabla[0, T] \times \mathbb{R}^d \times \mathbb{R}^m \times (\mathbb{R}^d)^m \times (\mathbb{S}^d)^m$, 

$$
\min_{\alpha^a \in \mathcal{U}^a} \mathcal{H}^a(t, s, y, \alpha^a, \phi^{-a}(t, s, y, u, p, q), u, p, q^a) = \mathcal{H}^a(t, s, y, \phi^a(t, s, y, u, p, q), u, p, q^a).
$$

This generalized minimax condition has implied the existence of the NE point at each $(t, s) \in \nabla[0, T]$ in the sense that all $\phi^a$ are found simultaneously. It is desirable as we are discussing about a general setting and it is normally equivalent to model assumptions on $b, \sigma$, and $h^a$.

2.2 | Heuristic derivation of equilibrium HJB system

In this subsection, we derive the system of equilibrium HJB equations, characterizing the TC-NE point in Definition 2.1, from which we reveal that it is a special case of our nonlocal parabolic system (1). Since the focus of our paper is on the well-posedness of Equation (1) and the nested HJB system rather than the latter’s origination, a heuristic derivation in the similar fashion of Björk and Murgoci (2014), Björk et al. (2017), Björk and Murgoci (2010) will be in place. For simplicity, we show only where the nonlocal terms $(\partial_t u)|_{t \leq s}(s, s, y)$ come from and the linkage with the classical HJB equations. For a rigorous derivation, one can follow the discretization approach in Yong (2012), Wei et al. (2017) or a rigorous argument in He and Jiang (2021) to derive the nonlocal parabolic system but it is too lengthy and thus not adopted here.

In light of the methodology in Björk and Murgoci (2014), Björk et al. (2017), Björk and Murgoci (2010), there are three main steps (Step 1-Step 3) to obtain the equilibrium HJB system of a multiplayer nonzero-sum TIC SDGs. For the sake of simplification of the heuristic derivation, we assume that $\xi = y \in \mathbb{R}^d$ and $h = 0$ in Equations (4)–(6), and adopt the deterministic feedback-type controls throughout this subsection. Next, let us consider

$$J^a(s, y; \alpha(\cdot, \cdot)) = \mathbb{E}_{s,y}[g^a(s, X(T; s, y, \alpha(\cdot, \cdot)))], \quad a = 1, \ldots, m,$$

where $\mathbb{E}_{s,y}$ is the conditional expectation under $X(s) = y$ and $X(\cdot; s, y, \alpha(\cdot, \cdot))$ (or $X^\alpha(\cdot)$ for short) is the unique adapted solution to Equation (4) on $[s, T]$ with $\alpha(\cdot, \cdot) : [0, T] \times \mathbb{R}^d \rightarrow \mathcal{U}$ and $X(s) = y$. The set of feasible feedback strategies is denoted by $\mathcal{U}$, which can be roughly understood as the class of deterministic functions that are regular enough to promise the well-posedness of the state process (4) and the cost functional (6). We refer the readers to Definition 2.2 of Björk et al. (2017) and Definition 2.1 of He and Jiang (2021) for more details.

**Definition 2.4.** Given a feasible feedback strategy $\alpha \in \mathcal{U}$, we define $u(t, s, y; \alpha) : \nabla[0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^m$ by

$$u(t, s, y; \alpha) = \mathbb{E}_{s,y}[g(t, X^\alpha(T))], \quad (10)$$
that is, its component $u^a(t, s, y; \alpha) = E_{s,y}[g^a(t, X^\alpha(T))]$ for $a = 1, \ldots, m$.

For any $t \in [0, T]$ and $\alpha \in \mathbb{U}$, the process $u^a(t, s, X^\alpha(s); \alpha)$ is a martingale and Equation (10) satisfies

$$
\begin{align*}
A^\alpha u^a(t, s, y; \alpha) &= 0, \quad t \leq s \leq T, \\
u(t, T, y; \alpha) &= g(t, y), \quad y \in \mathbb{R}^d.
\end{align*}
$$

where $A^\alpha$ is the controlled infinitesimal generator of the FSDE (4):

$$
A^\alpha = \frac{\partial}{\partial s} + \frac{1}{2} \sum_{i,j=1}^{d} (\sigma \sigma^T)_{ij}(s, y, \alpha) \frac{\partial^2}{\partial y_i \partial y_j} + \sum_{i=1}^{d} b_i(s, y, \alpha) \frac{\partial}{\partial y_i}.
$$

Similar to the classical dynamic programming principle in Yong and Zhou (1999), we need to first derive a recursive relation between cost functionals/value functions evaluated at two different initial points $(s, y)$ and $(s + \epsilon, X^\alpha(s + \epsilon))$. Then, by sending the mesh size of the time interval partition $\epsilon$ to zero, a nonlocal system of parabolic type is derived, through which a closed-loop TC-NE point can be identified and the TC-NE value function can be obtained.

**Step 1: The recursion for cost functionals $J(s, y; \alpha)$:** From the Markovian structure and Definition 2.4, we have $E_{s+\epsilon, X^\alpha(s+\epsilon)}[g^a(s+\epsilon, X^\alpha(T))] = u^a(s + \epsilon, s + \epsilon, X^\alpha(s + \epsilon); \alpha)$, which yields $J^a(s + \epsilon, X^\alpha(s + \epsilon); \alpha) = u^a(s + \epsilon, s + \epsilon, X^\alpha(s + \epsilon); \alpha)$. Taking conditional expectation at $(s, y)$ on both sides of the latter equation, we have

$$
E_{s,y}[J^a(s + \epsilon, X^\alpha(s + \epsilon); \alpha)] = J^a(s, y; \alpha) + E_{s,y}[u^a(s + \epsilon, s + \epsilon, X^\alpha(s + \epsilon); \alpha)]
$$

Moreover, by the tower rule of conditional expectations in the last term, we obtain the recursive equation for $J^a(s, y; \alpha)$ as follows:

$$
E_{s,y}[J^a(s + \epsilon, X^\alpha(s + \epsilon); \alpha)] = J^a(s, y; \alpha) + E_{s,y}[u^a(s + \epsilon, s + \epsilon, X^\alpha(s + \epsilon); \alpha)]
$$

Moreover, by the tower rule of conditional expectations in the last term, we obtain the recursive equation for $J^a(s, y; \alpha)$ as follows:

$$
E_{s,y}[J^a(s + \epsilon, X^\alpha(s + \epsilon); \alpha)] = J^a(s, y; \alpha) + E_{s,y}[u^a(s + \epsilon, s + \epsilon, X^\alpha(s + \epsilon); \alpha)]
$$

Moreover, by the tower rule of conditional expectations in the last term, we obtain the recursive equation for $J^a(s, y; \alpha)$ as follows:

$$
E_{s,y}[J^a(s + \epsilon, X^\alpha(s + \epsilon); \alpha)] = J^a(s, y; \alpha) + E_{s,y}[u^a(s + \epsilon, s + \epsilon, X^\alpha(s + \epsilon); \alpha)]
$$

Moreover, by the tower rule of conditional expectations in the last term, we obtain the recursive equation for $J^a(s, y; \alpha)$ as follows:

$$
E_{s,y}[J^a(s + \epsilon, X^\alpha(s + \epsilon); \alpha)] = J^a(s, y; \alpha) + E_{s,y}[u^a(s + \epsilon, s + \epsilon, X^\alpha(s + \epsilon); \alpha)]
$$

Moreover, by the tower rule of conditional expectations in the last term, we obtain the recursive equation for $J^a(s, y; \alpha)$ as follows:

$$
E_{s,y}[J^a(s + \epsilon, X^\alpha(s + \epsilon); \alpha)] = J^a(s, y; \alpha) + E_{s,y}[u^a(s + \epsilon, s + \epsilon, X^\alpha(s + \epsilon); \alpha)]
$$

Moreover, by the tower rule of conditional expectations in the last term, we obtain the recursive equation for $J^a(s, y; \alpha)$ as follows:

$$
E_{s,y}[J^a(s + \epsilon, X^\alpha(s + \epsilon); \alpha)] = J^a(s, y; \alpha) + E_{s,y}[u^a(s + \epsilon, s + \epsilon, X^\alpha(s + \epsilon); \alpha)]
$$

Moreover, by the tower rule of conditional expectations in the last term, we obtain the recursive equation for $J^a(s, y; \alpha)$ as follows:

$$
E_{s,y}[J^a(s + \epsilon, X^\alpha(s + \epsilon); \alpha)] = J^a(s, y; \alpha) + E_{s,y}[u^a(s + \epsilon, s + \epsilon, X^\alpha(s + \epsilon); \alpha)]
$$

**Step 2: The recursion for TC-NE value functions $V(s, y)$:** Based on Equation (12), we aim to derive a recursive equation for $V^a$. We first define a perturbed feedback strategy $\alpha^a, s, \epsilon, \alpha^a(\tau, y)$ such that $\alpha^a, s, \epsilon, \alpha^a(\tau, y) := \alpha^a(\tau, y)$ for $\tau \in [s, s + \epsilon)$ and $\alpha^a, s, \epsilon, \alpha^a(\tau, y) := \alpha^a(\tau, y)$ for $\tau \in [s + \epsilon, T]$, where $\alpha^a$ is an arbitrary element in $U^a$ that consists of the $a$th component of feasible controls in $U$ and $\alpha$ can be viewed as a candidate equilibrium strategy. Note that $\alpha^a, s, \epsilon, \alpha^a(\tau, y)$ is a function rather than a process $\alpha^a, s, \epsilon, \alpha^a$ in Definition 2.1 while they have similar roles. Noteworthy is that the perturbed strategy $\alpha^a, s, \epsilon, \alpha^a(\tau, y)$ is constructed with two feedback strategies $\alpha^a \in U^a$ and $\alpha^a$ rather than by pasting a constant strategy $\alpha^a \in U^a$ and a feedback one $\alpha^a$ (as in Equation 8). However, Remark 2.2 illustrates that the slight difference does not affect our characterization of the equilibrium point and
its associated HJB equations/systems. Then, Definition 2.1 implies that for \( a = 1, \ldots, m \),

\[
J^a(s + \epsilon, X^a(s + \epsilon); \tilde{\alpha}^{a,s,\epsilon} \oplus \tilde{\alpha}^{-a}) = V^a(s + \epsilon, X^a(s + \epsilon)),
\]

(13)

\[
u^a(t, s + \epsilon, X^a(s + \epsilon); \tilde{\alpha}^{a,s,\epsilon} \oplus \tilde{\alpha}^{-a}) = u^a(t, s + \epsilon, X^a(s + \epsilon)),
\]

(14)

where \( X^a(\cdot) \) represents \( X(\cdot; s, y, \tilde{\alpha}^{a,s,\epsilon} \oplus \tilde{\alpha}^{-a}) \) and the function \( u(t, \cdot, \cdot) \) is defined by Equation (10) with \( \alpha \) replaced by \( \tilde{\alpha} \). Next, inspired by the discrete setting of TIC stochastic control problem in Björk and Murgoci (2014), Björk and Murgoci (2010), it is anticipated from Equation (7) that

\[
“J^a(s, y; \tilde{\alpha}^{a,s,\epsilon} \oplus \tilde{\alpha}^{-a}) \geq V^a(s, y) \text{ for } \forall \alpha^a \in \cup^a \text{ with the equality holds when } \alpha^a(s, y) = \tilde{\alpha}^a(s, y).”
\]

(15)

However, for a continuous-time model, this statement is not always true since it is still possible that \( J^a(s, y; \tilde{\alpha}^{a,s,\epsilon} \oplus \tilde{\alpha}^{-a}) < V^a(s, y) \) for sufficiently small \( \epsilon \) and certain \( \alpha^a \in \cup^a \); see Björk et al. (2017), He and Jiang (2021). Hence, the following analyses of Equations (16) and (17) are rather heuristic, while they are included in our derivation of equilibrium HJB systems as they inspire us on how to investigate continuous-time TIC problems via the lens of discrete-time setting; see Björk et al. (2017), Björk and Murgoci (2010) for similar heuristic arguments. For a formal and rigorous proof, readers are suggested to refer to Theorem 3.3 of He and Jiang (2021) and Section 4 of Wei et al. (2017). Note that no matter whether the argument is formal or not, one always obtains the same HJB equations.

Next, we find that Equations (15) and (12) indicate that

\[
\inf_{\alpha^a \in \cup^a} \{ \mathbb{E}_{s,y} \left[ J^a(s + \epsilon, X^a(s + \epsilon); \tilde{\alpha}^{a,s,\epsilon} \oplus \tilde{\alpha}^{-a}) \right] - V^a(s, y) \\
- \mathbb{E}_{s,y} \left[ u^a(s + \epsilon, s + \epsilon, X^a(s + \epsilon); \tilde{\alpha}^{a,s,\epsilon} \oplus \tilde{\alpha}^{-a}) \right] \\
+ \mathbb{E}_{s,y} \left[ u^a(s, s + \epsilon, X^a(s + \epsilon); \tilde{\alpha}^{a,s,\epsilon} \oplus \tilde{\alpha}^{-a}) \right] \} = 0,
\]

(16)

By the expressions of Equations (13) and (14), we obtain the following recursion for \( V \):

\[
\inf_{\alpha^a \in \cup^a} \{ \mathbb{E}_{s,y} [ V^a(s + \epsilon, X^a(s + \epsilon))] - V^a(s, y) \\
- (\mathbb{E}_{s,y} [ u^a(s + \epsilon, s + \epsilon, X^a(s + \epsilon))] - \mathbb{E}_{s,y} [ u^a(s, s + \epsilon, X^a(s + \epsilon))]) \} = 0,
\]

(17)
the first line of which can be approximated by \( \mathbb{E}_{s,y}[V^a(s + \epsilon, X^{a\bar{a}}(s + \epsilon))] - V^a(s, y) \approx \mathcal{A}^{\bar{a}}V^a(s, y)\epsilon \) and the second line of which can be expressed as

\[
\mathbb{E}_{s,y}[u^a(s + \epsilon, s + \epsilon, X^{\bar{a}}(s + \epsilon))] - u^a(s, s, y) \approx \left[ \mathcal{A}^{\bar{a}}u^a(s, s, y) - (\mathcal{A}^{\bar{a}}u^a(t, s, y))_{|t=s} \right] \epsilon
\]

**Step 3: Equilibrium HJB system:** Letting \( \epsilon \to 0 \) in Equation (17) gives a deterministic system:

\[
\inf_{\alpha^a \in U^a} \left\{ A^{a}V^a(s, y) - A^{a}u^a(s, s, y) + (A^{a}u^a(t, s, y))_{|t=s} \right\} = 0, \quad a = 1, 2, \ldots, m,
\]

with boundary conditions \( V(T, y) = g(T, y) \), where

\[
A^{a} = \frac{\partial}{\partial s} + \frac{1}{2} \sum_{i,j=1}^{d} (\sigma\sigma^\top)_{ij}(s, y, \alpha) \frac{\partial^2}{\partial y_j \partial y_i} + \sum_{i=1}^{d} b_i(s, y, \alpha) \frac{\partial}{\partial y_i}
\]

for any \( \alpha \in U \). Note that the key difference between the operators \( \mathcal{A}^{a} \) and \( A^{a} \) is that the former corresponds to a function \( \alpha \in U \) while the latter corresponds to a point \( \alpha \in U \). By the generalized minimax condition in Assumption 2.3 and noting that \( V^a(s, \cdot) = u^a(s, s, \cdot) \), we know that the infimum above is achievable and the minimum is expressed by

\[
\alpha^{*a}(s, y) = \phi^a(s, s, y, u(s, s, y), u_y(s, s, y), u_{yy}(s, s, y)), \quad a = 1, \ldots, m. \quad (18)
\]

By the earlier discussion in Step 2, we must have \( \bar{\alpha}^a(\cdot, \cdot) = \alpha^{*a}(\cdot, \cdot) \) to form a closed-loop TC-NE point. With the representation of \( \bar{\alpha} \), we can then solve for \( u^a(\cdot, \cdot, \cdot) \) from Equation (11) with \( \alpha \) replaced by \( \bar{\alpha} \), which is the equilibrium HJB system we look for, but we present only the system for the general case below to save space. It is noteworthy that even we can focus on solving \( u \) hereafter, the derivation uses the definition of \( V \) and thus we require \( u(t, s, y) \) to be first-order differentiable in \( t \).

**General case:** Even if the running cost functional \( h \) is nonzero and \( \xi \) is a random variable, the heuristic derivation above is almost identical except for more tedious expressions, that is, we can obtain a generalized HJB system:

\[
\inf_{\alpha^a \in U^a} \left\{ A^{a}V^a(s, y) - A^{a}u^a(s, s, y) + (A^{a}u^a(t, s, y))_{|t=s} + h^a(s, s, y, \alpha, u(s, s, y), u_y(s, s, y)\sigma(s, y, \alpha)) \right\} = 0 \quad (19)
\]
and the same expression of $\bar{\alpha}^a$ in Equation (18). Plugging Equation (18) into a modified version of Equation (11) (with $h$ as a nonhomogeneous term), we obtain

$$
\begin{align*}
\begin{cases}
\frac{d}{dt}(u(t,s,y)) + \frac{1}{2} \text{tr}\left[\sigma \sigma^T(s,y,u(s,y))\right] u_{yy}(t,s,y) \\
+ b(s,y,u(s,y)) \cdot u_y(s,y) + h(s,y,u(s,y),u_y(s,y)) + \sigma(s,y,u(s,y),u_y(s,y)) \cdot u_{yy}(t,s,y) = 0,
\end{cases}
\end{align*}
$$

Similar to the notations $\mathfrak{a}$ and $\mathfrak{a}^a$, for $(t,s,y,u,p,q,l,m,n) \in \mathbb{V}[0,T] \times \mathbb{R}^d \times \mathbb{R}^m \times (\mathbb{R}^d)^m \times \mathbb{R}^m \times (\mathbb{R}^d)^m \times (\mathbb{S}^d)^m$, we denote the nonlinearity $\mathcal{H}$ by a $\mathbb{R}^m$-valued function $\mathcal{H}(t,s,y,u,p,q,l,m,n) = (\mathcal{H}^1(t,s,y,u,p,q,l,m,n), ..., \mathcal{H}^m(t,s,y,u,p,q,l,m,n))$ with $\mathcal{H}^a(t,s,y,u,p,q,l,m,n) = \mathcal{H}^a(t,s,y,u(p,q,l,m,n), u, p, q^a)$ for $a = 1, 2, ..., m$. Then, we can simplify the system above as a nonlocal parabolic system for $u$ of the form

$$
\begin{align*}
\begin{cases}
u(t,s,y) + \mathcal{H}(t,s,y,u(t,s,y),u_y(t,s,y),u_{yy}(t,s,y), u(s,y), u_y(s,y), u_{yy}(s,y)) = 0, \\
u(t,T,y) = g(t,y), & 0 \leq t \leq s \leq T, \quad y \in \mathbb{R}^d, \quad a = 1, ..., m.
\end{cases}
\end{align*}
$$

**Remark 2.5.** Though the derivations above are heuristic, we can similarly develop the system-version results of Yong (2012), Wei et al. (2017), Yan and Yong (2019), following their discretization approach or the argument in Theorem 3.3 of He and Jiang (2021), and we will also end up with the equilibrium HJB equation/system (20) as well as a closed-loop TC-NE point (18) and the associated TC-NE value functions $V(s,y) := u(s,y)$ in the sense of Definition 2.1. The relationship between Equations (19) and (20) is discussed in Hernández and Possamaï (2021). While it is not the focus of this paper, we summarize the mathematical claims/conjectures about the connection between solutions to Equation (20) and TIC SDGs in Section 2.3. Moreover, an interesting fact is that we only need to study the HJB system (20) in the set of reachable states rather than in $\mathbb{R}^d$; see Section 2.3 of He and Jiang (2021) and the example in Section 4.1.2.

From the derivations in this subsection, the inclusion of the nonlocal terms $(\partial_{y} u)|_{y \leq \bar{\alpha}(s,y)}$ is rationalized: the characterizations of $\alpha^*$ (or $\bar{\alpha}$) and $u$ are coupled. It is also easy to see that in the TC case (independent of the initial time point), that is, $u(t,y) = u(s,y) = V(s,y)$, the HJB system (19) or equilibrium HJB system (20) reduce to the classical ones (see Yong & Zhou, 1999). Proposition 3.2 below tells us that Equation (20) is a special case of Equation (1) with $r = 1$. Moreover, a closely related topic is robust TIC stochastic controls via the formulation of nonzero-sum TIC SDGs. From which, we can observe many solvable examples in finance and insurance; see Pun (2018a), Han et al. (2021), Han et al. (2022), Yan et al. (2020), Lei and Pun (2020). By carefully choosing cost functionals of nonzero-sum SDGs, we can model the relative performance concerns among multiple agents in decision-making and thus our theory can extend the related works of Espinosa and Touzi (2015), Pun and Wong (2016), Pun et al. (2016), Lacker and Zariphopoulou (2019) by introducing TIC or behavioral factors.
2.3 The relation between the equilibrium HJB system and TIC SDGs

The derivation of the equilibrium HJB system alone does not justify its mathematical connection with the stochastic control/game problem (4)–(6). We need to show two aspects of the connection, namely sufficiency and necessity, constituting two conjectures below:

**Sufficiency/Verification theorem:** The solutions to Equations (18) and (20) indeed give a TC-NE point and a TC-NE value function. Mathematically, we assume that \( u(t, s, y) \in C^{1,1,2} \) solves Equation (20) and that the infimum of Equation (19) is attained for every \((s, y)\) in the sense of NE. Then, the minimizer of \( H \) (under Assumption 2.3), given in Equation (18), is a closed-loop TC-NE point and the function \( V(s, y) := u(s, s, y) \) is the TC-NE value function as in Definition 2.1;

**Necessity:** Every TC-NE point must minimize the Hamiltonian associated to TIC problem (4)–(6) and the corresponding value function solves the HJB system (19). Mathematically, we assume that there exist a closed-loop TC-NE point \( \bar{\alpha}(s, y) \) and the corresponding value function \( V(s, y) \in C^{1,2} \) and we define \( u(t, s, y) := \bar{V}(s, s, y, \bar{\alpha}) \), where \( \bar{V} \) comes from \( \{ (\bar{X}(\tau; s, y, \bar{\alpha}), \bar{V}(\tau; s, y, \bar{\alpha}), \bar{Z}(\tau; s, y, \bar{\alpha}))_{\tau \in [s, T]} \} \) being the adapted solution of the family of FBSDEs parameterized by \( t \):

\[
\begin{align*}
    d\bar{X}(\tau) &= b\left( \tau, \bar{X}(\tau), \bar{\alpha}(\tau, \bar{X}(\tau)) \right) d\tau + \sigma\left( \tau, \bar{X}(\tau), \bar{\alpha}(\tau, \bar{X}(\tau)) \right) dW(\tau), \quad \tau \in [s, T], \\
    d\bar{V}(\tau) &= -h\left( t, \tau, \bar{X}(\tau), \bar{\alpha}(\tau, \bar{X}(\tau)), \bar{V}(\tau), \bar{Z}(\tau) \right) d\tau + \bar{Z}^T(\tau) dW(\tau), \quad \tau \in [s, T], \\
    \bar{X}(s) &= y, \quad \bar{V}(T) = g\left( t, \bar{X}(T) \right), \quad (t, s) \in \mathbb{V}[0, T], \quad y \in \mathbb{R}^d.
\end{align*}
\]

Then, \((V(s, y), u(t, s, y))\) solves Equations (19) and (20) while \( \bar{\alpha}(s, y) \) realizes the infimum of Equation (19).

By the derivations in the previous subsection and using the similar arguments in Björk et al. (2017), it is easy to establish the verification theorem for the Markovian setting while we omit the straightforward proof. It should be noted that the equilibrium HJB equation and the verification theorem for the non-Markovian setting are attempted in Hernández and Possamaï (2023). The necessity issue is a difficult problem, while we refer the readers to the latest progress along this line, such as Lindensjö (2019), Hernández and Possamaï (2023), He and Jiang (2021), Hamaguchi (2021a) and a comprehensive literature review of the field He and Zhou (2022). Specifically, Hernández and Possamaï (2023) prove the necessity for the scalar case (stochastic control problem) in a general non-Markovian setting. With a more mathematically rigorous definition of the SPE/TC solution (similar to Definition 2.1) and a discretization approach, Wei et al. (2017), Yan and Yong (2019), Wang et al. (2022), Wang and Yong (2021) show intuitively the desired mathematical connection between TIC stochastic control problem and the associated equilibrium HJB equation, given that the latter is well-posed. Even we consider TIC SDGs with higher dimensions, one could expect that the sufficiency and the necessity above are provable.

However, they are not the focus of this paper while the well-posedness of the equilibrium HJB system appears to be the core of the concerns above. It is noteworthy that the solvability of TIC SDGs has not been provided by the works on the sufficiency and necessity. The desired mathematical connection between the equilibrium HJB equations/systems and TIC stochastic
controls/SDGs is meaningless if the equilibrium HJB equations/systems are not well-posed; see the assumptions in the two conjectures above. Moreover, though some works have proved the sufficiency and necessity, they uniformly assumed that the volatility $\sigma$ of Equation (4) is free of control such that the nonlocal second-order term in Equation (20) vanishes; see Wei et al. (2017), Hernández and Possamaï (2023). Hence, our well-posedness results, which get rid of this bottleneck, benefit the studies on more general problems of TIC SDGs as well as the previous studies on TIC stochastic controls. While there may not be a specific order of studying the sufficiency, the necessity, and the well-posedness of the nonlocal parabolic system, this paper addresses the last one and based on which, the other two are relatively simple given the extensive related studies in the literature. By establishing the existence, uniqueness, stability, and regularities of solutions of Equation (20), our PDE results directly imply the existence and uniqueness of TIC problems where both the drift and the volatility are controlled.

3 WELL-POSEDNESS OF NONLOCAL PARABOLIC SYSTEM

In this section, we present our main results about the well-posedness issues of the nonlocal higher-order systems (1). The overall idea is to first study the case with a linear operator, which will then be used to infer the well-posedness results for the system (1) with a general nonlinear operator, together with the linearization method. All the proofs are deferred to Appendix A.

While the SDG and the Feynman–Kac formula are usually formulated in a backward setting, we first show the equivalence between the solvabilities of the nonlocal backward (terminal-value) problems,

$$\begin{align*}
\{ u_t(t, s, y) + F (t, s, y, (\partial^j u)_{|j| \leq 2r}(t, s, y), (\partial^j u)_{|j| \leq 2r}(s, s, y)) & = 0, \\
u(t, T, y) & = g(t, y), 0 \leq t \leq s \leq T, y \in \mathbb{R}^d,
\end{align*}$$

and the nonlocal forward (initial-value) problems (1). There are a few noteworthy differences between the two systems: first, for the backward problem (21), if we move the $F$ to the right-hand side, we will have a negative sign $-F$, compared to the forward problem (1); second, the ordering between $t$ and $s$ is the opposite of one another. The symmetry between Equations (1) and (21) is shown in Figure 2. Notation-wise, we use the time region $\Delta[0, T] := \{(\tau_1, \tau_2) \in [0, T]^2 : 0 \leq \tau_2 \leq \tau_1 \leq T\}$ for forward problems to distinguish from $\nabla[0, T]$ for backward problems.

**Proposition 3.1.** The solvabilities of Problems (1) and (21) are equivalent.

Given Proposition 3.1, we only study the forward problem (1) in this section as it can simplify the notations. To this end, we introduce some norms and the induced Banach spaces for the problems of our interest. The solvability of Equation (1) will be first investigated in the usual space of bounded and continuous functions in Sections 3.1–3.3. Subsequently, in order to meet practical needs for more financial applications, it is necessary to extend the well-posedness results in an exponentially weighted space of growth functions in Section 3.4.
3.1 Norms and Banach spaces

For a \(m\)-dimensional real-valued array \(x = (x^1, x^2, \ldots, x^m)\), \(|x| := \left(\sum_{i=1}^m (x^i)^2\right)^{1/2}\). Given \(0 \leq a \leq b \leq T\), we denote by \(C([a, b] \times \mathbb{R}^d; \mathbb{R}^m)\) the set of all the continuous and bounded \(\mathbb{R}^m\)-valued functions in \([a, b] \times \mathbb{R}^d\) endowed with the supremum norm \(|\cdot|_\infty := \sup_{[a, b] \times \mathbb{R}^d} |\cdot|\). Whenever no confusion arises, we write \(|\cdot|_\infty\) instead of \(|\cdot|_{[a, b] \times \mathbb{R}^d}^\infty\).

Then, we revisit the definition of “parabolic” Hölder spaces, which is commonly adopted in the studies of local parabolic equations, including Eidel’man (1969), Lei and Pun (2023). Let \(C^{l-\frac{r}{2}}([a, b] \times \mathbb{R}^d; \mathbb{R})\) be the Banach space of the functions \(\varphi(s, y)\) such that \(\varphi(s, y)\) is continuous in \([a, b] \times \mathbb{R}^d\), its derivatives of the form \(\partial^h_s \partial^j_y \varphi\) for \(2rh + j < l\) exist, and it has a finite norm defined by

\[
|\varphi|^{(l)}_{[a,b] \times \mathbb{R}^d} = \sum_{k \leq |l|} \sum_{2rh+j=k} \left| \partial^h_s \partial^j_y \varphi \right|_\infty + \sum_{2rh+j=|l|} \left\langle \partial^h_s \partial^j_y \varphi \right\rangle_y^{(l-|l|)} + \sum_{0 < l - 2rh - j < 2r} \left\langle \partial^h_s \partial^j_y \varphi \right\rangle_{y}^{(l-2rh-j)/2r},
\]

where \(r\) is always a positive integer, \(l\) is a noninteger positive number and \([\cdot]\) is the floor function, \(\partial^h_s \partial^j_y \varphi\) represents the \(d^j\)-dimensional array, the entries of which are the \(j\)th-order mixed partial derivatives of \(\frac{\partial^h \varphi}{\partial s \cdots \partial y}\) in \(y\), that is, \(\frac{\partial^{h+j} \varphi}{\partial s \cdots \partial y_1 \cdots \partial y_j}\). Moreover, for \(0 < \alpha < 1\) and \(\rho_0 > 0\),

\[
\langle \varphi \rangle_y^{(\alpha)} := \sup_{s \in [a, b], y, y' \in \mathbb{R}^d} \left\{ \begin{array}{l}
\frac{|\varphi(s, y) - \varphi(s, y')|}{|y - y'|^{\alpha}}, \quad \langle \varphi \rangle_s^{(\alpha)} \\
\frac{|\varphi(s, y) - \varphi(s', y)|}{|s - s'|^{\alpha}}.
\end{array}\right.
\]

The defined norms depend on \(\rho_0\) but indeed for different \(\rho_0 > 0\), they are equivalent. Hence, we suppress the dependence on \(\rho_0\) will not be noted unless otherwise specified. Moreover,
wherever no confusion arises, we do not distinguish between \(|\varphi|^{(l)}_{[a,b] \times \mathbb{R}^d}\) and \(|\varphi|^{(l)}_{\mathbb{R}^d}\) for functions \(\varphi(y)\) independent of \(s\).

Now, we are ready to define the norms and Banach spaces for nonlocal systems of unknown vector-valued functions \(u(t,s,y) = (u^1(t,s,y), u^2(t,s,y), \ldots, u^m(t,s,y))^\top\). For any \(t\) and \(\delta\) such that \(0 \leq t \leq \delta \leq T\), we introduce the following norms:

\[
[u]^{(l)}_{[0,\delta]} \ := \ \sup_{t \in [0,\delta]} \{ |u(t,\cdot,\cdot)|^{(l)}_{[0,t] \times \mathbb{R}^d} \},
\]

\[
\|u\|^{(l)}_{[0,\delta]} \ := \ \sup_{t \in [0,\delta]} \{ |u(t,\cdot,\cdot)|^{(l)}_{[0,t] \times \mathbb{R}^d} + |u_t(t,\cdot,\cdot)|^{(l)}_{[0,t] \times \mathbb{R}^d} \},
\]

where \(|u(t,\cdot,\cdot)|^{(l)}_{[0,t] \times \mathbb{R}^d} := \sum_{a \leq m} |u^a(t,\cdot,\cdot)|^{(l)}_{[0,t] \times \mathbb{R}^d}|. Then, these norms induce the following spaces, respectively,

\[
\Theta^{(l)}_{[0,\delta]} := \{ u(\cdot,\cdot,\cdot) \in C(\Delta[0,\delta] \times \mathbb{R}^d; \mathbb{R}^m) : [u]^{(l)}_{[0,\delta]} < \infty \},
\]

\[
\Omega^{(l)}_{[0,\delta]} := \{ u(\cdot,\cdot,\cdot) \in C(\Delta[0,\delta] \times \mathbb{R}^d; \mathbb{R}^m) : \|u\|^{(l)}_{[0,\delta]} < \infty \},
\]

where \(C(\Delta[0,\delta] \times \mathbb{R}^d; \mathbb{R}^m)\) is the set of all continuous and bounded \(\mathbb{R}^m\)-valued functions defined in \(\{0 \leq s \leq t \leq \delta\} \times \mathbb{R}^d\). It is easy to see that both \(\Theta^{(l)}_{[0,\delta]}\) and \(\Omega^{(l)}_{[0,\delta]}\) are Banach spaces. The definitions above leverage not only the order relation between \(t\) and \(s\) but also the sufficient regularities in all arguments.

### 3.2 Nonlocal linear higher-order parabolic systems

Let \(L\) be a family of nonlocal, linear, and strongly elliptic operator of order \(2r\), whose \(a\)th entry, \((Lu)^a\), \(a = 1, \ldots, m\), takes the form

\[
(Lu)^a(t,s,y) := \sum_{|l| \leq 2r, b \leq m} A^l_{ab}(t,s,y) \partial_l u^b(t,s,y) + \sum_{|l| \leq 2r, b \leq m} B^l_{ab}(t,s,y) \partial_l u^b(s,s,y),
\]

where the nonlocality stems from the presence of \(\partial_l u(s,s,y)\) and the strong ellipticity condition implies that there exists some \(\lambda > 0\) such that

\[
(-1)^r \sum_{a,b,|l|=2r} A^l_{ab}(t,s,y) \xi_{i_1} \ldots \xi_{i_{2r}} v^a v^b \geq \lambda |\xi|^{2r} |v|^2,
\]

\[
(-1)^r \sum_{a,b,|l|=2r} (A^l_{ab} + B^l_{ab})(t,s,y) \xi_{i_1} \ldots \xi_{i_{2r}} v^a v^b \geq \lambda |\xi|^{2r} |v|^2
\]
uniformly for any \((t, s) \in \Delta[0, T]\), \(y, \xi \in \mathbb{R}^d\), and \(v \in \mathbb{R}^m\). Next we consider a nonlocal linear system:

\[
\begin{cases}
    u_s(t, s, y) = (Lu)(t, s, y) + f(t, s, y), \\
u(t, 0, y) = g(t, y),
\end{cases}
\]

where all coefficients \(A_{b}^{al}\) and \(B_{b}^{al}\) belong to \(\Omega^{(\alpha)}_{[0,T]}\). Moreover, the inhomogeneous term \(f \in \Omega^{(\alpha)}_{[0,T]}\) and the initial condition \(g \in \Omega^{(2r+\alpha)}_{[0,T]}\).

Suppose that \(u(t, s, y)\) is differentiable with respect to \(t\), then by differentiating (25) with respect to \(t\), the derivative \(\frac{\partial u}{\partial t}\) satisfies

\[
\begin{cases}
    \left(\frac{\partial u}{\partial t}\right)^a(t, s, y) = \sum_{|I| \leq 2r, b \leq m} A_{b}^{al}(\cdot)\partial_t\left(\frac{\partial u}{\partial t}\right)^b(t, s, y) - \sum_{|I| \leq 2r, b \leq m} B_{b}^{al}(\cdot)\int_s^t \partial_t\left(\frac{\partial u}{\partial t}\right)^b(\theta, s, y)d\theta + f^a(\cdot), & a = 1, \ldots, m, \\
    \left(\frac{\partial u}{\partial t}\right)(t, 0, y) = g_i(t, y), & 0 \leq s \leq t \leq T, \ y \in \mathbb{R}^d.
\end{cases}
\]

By taking advantage of the integral representations,

\[
\partial_t u^b(t, s, y) - \partial_t u^b(s, s, y) = \int_s^t \partial_t\left(\frac{\partial u}{\partial t}\right)^b(\theta, s, y)d\theta, \quad \text{for } |I| \leq 2r, \ b \leq m,
\]

it is clear that \((u, \frac{\partial u}{\partial t})\), denoted by \((u, v)\), satisfies the following system of \(2m\) equations:

\[
\begin{align*}
u^i_s(t, s, y) &= \sum_{|I| \leq 2r, b \leq m} (A + B)^{al}_b(\cdot)\partial_t u^b(t, s, y) - \sum_{|I| \leq 2r, b \leq m} B^{al}_b(\cdot)\int_s^t \partial_t u^b(\theta, s, y)d\theta + f^a(\cdot), \quad a = 1, \ldots, m, \\
\psi^i_s(t, s, y) &= \sum_{|I| \leq 2r, b \leq m} A^{al}_b(\cdot)\partial_t \psi^b(t, s, y) + \sum_{|I| \leq 2r, b \leq m} \left(\frac{\partial A}{\partial t} + \frac{\partial B}{\partial t}\right)^a_b(\cdot)\partial_t u^b(t, s, y) \\
&- \sum_{|I| \leq 2r, b \leq m} \left(\frac{\partial B}{\partial t}\right)^a_b(\cdot)\int_s^t \partial_t \psi^b(\theta, s, y)d\theta + f^a(\cdot), \quad a = 1, \ldots, m,
\end{align*}
\]

\[
(u, v)(t, 0, y) = (g, g_i)(t, y), \quad 0 \leq s \leq t \leq T, \ y \in \mathbb{R}^d.
\]

The following lemma reveals that problems (25) and (27) are equivalent.

**Lemma 3.2.**

1. If \(u\) is a solution of Equation (25), then \((u, u_i)\) solves Equation (27).
2. Conversely, if Equation (27) admits a solution pair \((u, v)\), then \(u\) solves Equation (25).

With Lemma 3.2, it makes sense for us to shift our focus to the well-posedness of Equation (27).
Theorem 3.3. Let $L$ be the nonlocal, linear, and strongly elliptic operator of order $2r$ defined in Equation (22) with all coefficients belonging to $\Omega^{(\alpha)}_{[0,T]}$. If $f \in \Omega^{(\alpha)}_{[0,T]}$ and $g \in \Omega^{(2r+\alpha)}_{[0,T]}$, then Equation (27) admits a unique solution pair $(u, v) \in \Theta^{(2r+\alpha)}_{[0,T]} \times \Theta^{(2r+\alpha)}_{[0,T]}$ in $\Delta[0, T] \times \mathbb{R}^d$.

Next, thanks to the equivalence between Equation (25) and Equation (27), we will establish the global well-posedness of nonlocal linear systems. Moreover, we will derive a Schauder-type estimate of solutions of Equation (25). It not only justifies the stability of the solutions to Equation (25) with respect to the data $(f, g)$, but also establishes a foundation for the further analysis of nonlocal fully nonlinear systems (1) in the next section.

Theorem 3.4. Suppose that all coefficient functions and $f$ of Equation (25) belong to $\Omega^{(\alpha)}_{[0,T]}$ and assume that $g \in \Omega^{(2r+\alpha)}_{[0,T]}$. Then the nonlocal linear system (25) admits a unique solution $u \in \Omega^{(2r+\alpha)}_{[0,T]}$ in $\Delta[0, T] \times \mathbb{R}^d$. Furthermore, we obtain the following Schauder estimate:

$$\|u\|^{(2r+\alpha)}_{[0,T]} \leq C \left( \|f\|^{(\alpha)}_{[0,T]} + \|g\|^{(2r+\alpha)}_{[0,T]} \right).$$

(28)

Consequently, let $u$ and $\hat{u}$ be solutions to Equation (25) corresponding to $(f, g)$ and $(\hat{f}, \hat{g})$, respectively, then

$$\|u - \hat{u}\|^{(2r+\alpha)}_{[0,T]} \leq C \left( \|f - \hat{f}\|^{(\alpha)}_{[0,T]} + \|g - \hat{g}\|^{(2r+\alpha)}_{[0,T]} \right).$$

(29)

3.3 Nonlocal fully nonlinear higher-order parabolic systems

After studying the solvability of nonlocal linear system, we will adopt the method of linearization to prove the well-posedness of nonlocal fully nonlinear system of the form (1).

To take advantage of the results of nonlocal linear systems in Section 3.2, we require certain regularity assumptions on $F$ and $g$. Generally speaking, we require the initial condition $g \in \Omega^{(2r+\alpha)}_{[0,T]}$. The nonlinearity $F$ is a vector-valued function $(t, s, y, z) \mapsto F(t, s, y, z)$ defined in $\Pi = \Delta[0, T] \times \mathbb{R}^d \times B(\bar{z}, R_0)$, where $\bar{z} \in \mathbb{R}^m \times (\mathbb{R}^d)^m \times \cdots \times (\mathbb{R}^d)^m \times \mathbb{R}^m \times (\mathbb{R}^d)^m \times \cdots \times (\mathbb{R}^d)^m$ and $B(\bar{z}, R_0)$ is an open ball centered at $\bar{z}$ with a positive radius $R_0$. Denoting by $O$ the open set (ball) in $\Omega^{(2r+\alpha)}_{[0,T]}$ consisting of all the functions $u$ such that the range of $(|\partial_1 u|_{|\partial_1 u| \leq 2r}(t, s, y), (|\partial_1 u|_{|\partial_1 u| \leq 2r}(s, s, y))$ is contained in the open ball, then the nonlinearity $F$ can be regarded as a mapping from $O \rightarrow \Omega^{(\alpha)}_{[0,T]}$. We require $F$ satisfies that

(i) **Ellipticity condition**: For any $\xi = (\xi_1, \ldots, \xi_d)^T \in \mathbb{R}^d$ and $v = (v^1, \ldots, v^m)^T \in \mathbb{R}^m$, there exists a $\lambda > 0$ such that

$$(-1)^{r-1} \sum_{a,b,|\mu|=2r} \partial_1 F^a_b(t, s, y, z) \xi_{i_1} \cdots \xi_{i_2r} v^a v^b \geq \lambda |\xi|^{2r} |v|^2,$$

(30)

$$(-1)^{r-1} \sum_{a,b,|\mu|=2r} \left( \partial_1 F^a_b + \partial_1 F^d_b \right)(t, s, y, z) \xi_{i_1} \cdots \xi_{i_2r} v^a v^b \geq \lambda |\xi|^{2r} |v|^2,$$

(31)
TABLE 1  First-order derivatives of $F$ required to be Hölder and Lipschitz continuous.

| $\mathcal{X}$ | $t$ | $s$ | $y$ | $\partial_t u^b(t, s, y)$ | $\partial_s u^b(s, s, y)$ |
|---|---|---|---|---|---|
| $F_x$ | $\sqrt{}$ | | | $\sqrt{}$ | $\sqrt{}$ |

TABLE 2  Second-order derivatives of $F$ required to be Hölder and Lipschitz continuous.

| $\mathcal{X}$ | $t$ | $s$ | $y$ | $\partial_t u^b(t, s, y)$ | $\partial_s u^b(s, s, y)$ |
|---|---|---|---|---|---|
| $F_y$ | | | | $\sqrt{}$ | $\sqrt{}$ |
| $\partial_t u^b(t, s, y)$ | $\sqrt{}$ | | | $\sqrt{}$ | $\sqrt{}$ |
| $\partial_s u^b(s, s, y)$ | $\sqrt{}$ | | | $\sqrt{}$ | $\sqrt{}$ |

hold uniformly with respect to $(t, s, y, z) \in \Pi$;

(ii) **Hölder continuity:** There exists a positive constant $K$ such that

$$K := \sup_{t \in [0, \delta], z \in B(\bar{z}, R_0)} |F(t, \cdot, \cdot, z)|^{(\alpha)}_{[0, t] \times \mathbb{R}^d} < \infty;$$  \hspace{1cm} (32)

(iii) **Lipschitz continuity:** There exists a $L > 0$ such that for any $(t, s, y, z_1), (t, s, y, z_2) \in \Pi$,

$$|F(t, s, y, z_1) - F(t, s, y, z_2)| \leq L|z_1 - z_2|,$$  \hspace{1cm} (33)

where $\partial_t F^a_b$ denotes the derivative of $F^a$ with respect to its argument $\partial_t u^b(t, s, y)$ while $\partial_t \tilde{F}^a_b$ denotes the derivative of $F^a$ with respect to its argument $\partial_t u^b(s, s, y)$ and the generic notation $F$ represents $F$ itself and some of its first- and second-order derivatives, whose variables to be differentiated are indicated by “$\sqrt{}/$” in Tables 1 and 2. Hereafter, we also adopt the similar notations for second-order derivatives of $F$: $\partial_t^2 F^a_{bc}$ denotes the derivative of $\partial_t F^a_b$ with respect to its argument $\partial_t u^c(t, s, y)$ and $\partial_t^2 \tilde{F}^a_{bc}$ denotes the derivative of $\partial_t \tilde{F}^a_b$ with respect to its argument $\partial_t u^c(t, s, y)$. In fact, for a simple check, the assumptions above over $(t, s)$ and $z$ are satisfied if $F$ is thrice continuously differentiable with respect to its corresponding arguments.

3.3.1  Small-time well-posedness of nonlocal fully nonlinear systems

Before we present our main result, we stress that the standard linearization methods are not applicable for the nonlocal case. In the setting of local parabolic systems, Éidel’man (1969) introduced a so-called “quasi-linearization method” and studied local existence for fully nonlinear parabolic problems by transforming fully nonlinear systems into quasi-linear systems. Noteworthy, Khudyaev (1963), Šopolov (1970) utilized a variant of this method to investigate fully nonlinear PDEs or systems. The linearization method, which we propose to prove for Theorem 3.5 below, is substantially inspired by Kruzhkov et al. (1975), Lunardi (1995). Although there are some previous works on how to linearize nonlinear equations or systems, it is still difficult to extend the existing methods from a local setting to a nonlocal setting. In fact, even for a nonlocal linear
Theorem 3.5 below is our main innovative result, which shows the (small-time) well-posedness of nonlocal fully nonlinear systems (1).

**Theorem 3.5.** Let $F$ satisfies the conditions (30)–(33). Suppose that $g \in \Omega_{[0,T]}^{(2r+\alpha)}$ and that the range of $((\partial_t g)_{|t| \leq 2r}(t, y), (\partial_g g)_{|t| \leq 2r}(s, y))$ is contained in the ball centered at $z$ with radius $R_0/2$. Then, there exist $\delta > 0$ and a unique $u \in \Omega_{[0,\delta]}^{(2r+\alpha)}$ satisfying Equation (1) in $\Delta[0, \delta] \times \mathbb{R}^d$.

It should be noted that in the small-time setting, we only require the conditions of Equations (30)–(33) of $\mathcal{F}$ in an open ball $B(z, R_0)$ while the range of $((\partial_t g)_{|t| \leq 2r}(t, y), (\partial_g g)_{|t| \leq 2r}(s, y))$ is contained in $\mathcal{O}$ with a smaller ball $B(z, R_0/2)$. For a pair $(F, g)$ that satisfies their coupled assumptions, Theorem 3.5 provides the local/maximally defined (see Remark 3.6) well-posedness of Equation (1). The (relaxed) local assumptions on $F$ facilitates a larger class of Equation (1). From the proof of Theorem 3.5 (see Equation A.20) and the example provided below, we find that the local solution always exists if $g \in \mathcal{O}$ since $\mathcal{O}$ is an open set in $\Omega_{[0,T]}^{(2r+\alpha)}$. Hence, to check if Equation (1) exists a local solution, it is more convenient to check if the conditions (30)–(33) of $F$ can be satisfied in a small open ball centered at the range of $g$, instead of in $B(z, R_0)$. For the global solvability (well-posedness), we will discuss it in Section 3.3.2.

**Remark 3.6 (Maximally defined solutions).** We have proven the local well-posedness of Equation (1) in $\Delta[0, \delta] \times \mathbb{R}^d$ and thus the diagonal condition can be determined for $s \in [0, \delta]$. After which, the nonlocal fully nonlinear system (1) is reduced to a classical local fully nonlinear systems parameterized by $t$. Then we take $\delta$ as initial time and $u(t, \delta, y)$ as initial datum, we can extend the solution to a larger time interval up to the maximal interval. It is analogous to the process of identifying the global solution of nonlocal linear systems in the proof of Theorem 3.3. The procedure could be repeated up to a maximally defined solution $u : \Delta[0, \sigma] \times \mathbb{R}^d \rightarrow \mathbb{R}^m$, belonging to $\Omega_{[0,\sigma]}^{(2+\alpha)}$ for any $\sigma < \tau$. The time region $\Delta[0, \tau]$ is maximal in the sense that if $\tau < \infty$, then there does not exist any solution of Equation (1) belonging to $\Omega_{[0,\tau]}^{(2+\alpha)}$; see Figure A.1. An example of $\tau < T$ can be proposed similarly as in the local case; see Lieberman (1996, pp. 203). It is noteworthy that the problem of existence at large for arbitrary initial data is a difficult task even in the classical fully nonlinear case. The difficulty is caused by the fact that a priori estimate in a very high norm $| \cdot |_{[a,b] \times \mathbb{R}^d}^{(2+\alpha)}$ is needed to establish the existence at large. To this end, there will be severe restrictions on the nonlinearities. More details are discussed in Krylov (1987), Lieberman (1996).

**Remark 3.7 (Stability analysis).** Consider a family of nonlinearities $F(t, s, y, z; \lambda)$ parameterized by a parameter $\lambda \in \Lambda$, where $\Lambda$ is a Banach space under $\| \cdot \|_{\Lambda}$. For any $(t, s, y, u) \in \Delta[0, \delta] \times \mathbb{R}^d \times \Omega_{[0,\delta]}^{(2+\alpha)}$, it is assumed that

$$
\| F(\cdot, \cdot, \cdot, (\partial_t u)_{|t| \leq 2r}(\cdot, \cdot, \cdot), (\partial_s u)_{|t| \leq 2r}(\cdot, \cdot, \cdot); \lambda) 
- F(\cdot, \cdot, \cdot, (\partial_t u)_{|t| \leq 2r}(\cdot, \cdot, \cdot), (\partial_s u)_{|t| \leq 2r}(\cdot, \cdot, \cdot); \hat{\lambda}) \|_{[0,\delta]}^{(\alpha)} \leq \gamma \| \lambda - \hat{\lambda} \|_{\Lambda}.
$$
Suppose $u$ and $\hat{u}$ correspond to $(\lambda, g)$ and $(\hat{\lambda}, \hat{g})$, respectively. We have the following estimate:

$$
\|u - \hat{u}\|_{[0, \delta]}^{(2\alpha + \beta)} \leq C\left(\|\lambda - \hat{\lambda}\|_A + \|g - \hat{g}\|_{[0, \delta]}^{(2\alpha + \beta)}\right),
$$

which follows directly the proofs of our Theorems 3.5 and 8.3.2 in Lunardi (1995).

Before we study the global well-posedness, we provide an example to understand the assumptions on $F$ and $g$ in Theorem 3.5. Without loss of generality, we assume that $r = m = d = 1$ and the coefficients of Equations (4) and (6) are

$$
\begin{align*}
 h(t, s, y, \alpha) &= C^{(1)}(t, s, y) + \frac{1}{2}C^{(2)}(t, s, y)\alpha^2, \\
 b(s, y, \alpha) &= B^{(1)}(s, y) + B^{(2)}(s, y)\alpha, \\
 A(s, y, \alpha) &= \frac{1}{2}\sigma(s, y, \alpha)\sigma(s, y, \alpha) = A^{(1)}(s, y) + \frac{1}{2}A^{(2)}(s, y)\alpha^2.
\end{align*}
$$

Then, it is clear that the optimum of the Hamiltonian is attained by $\phi(t, s, y, u, p, q) = -B^{(2)}(s, y)p/(A^{(2)}(s, y)q + C^{(2)}(s, s, y))$. Moreover, according to Equation (18), the equilibrium control is given by

$$
\bar{\alpha}(s, y) = \frac{-B^{(2)}(s, y)u_y(s, s, y)}{A^{(2)}(s, y)u_{yy}(s, s, y) + C^{(2)}(s, s, y)}.
$$

Consequently, with a variable substitution, the (backward) equilibrium HJB equation can be reformulated forwardly as

$$
\begin{align*}
 u_t(t, s, y) &= \left(\dot{A}^{(1)}(s, y) + \frac{1}{2}\dot{A}^{(2)}(s, y)\left(\frac{B^{(2)}(s, y)u_y(s, s, y)}{A^{(2)}(s, y)u_{yy}(s, s, y) + C^{(2)}(s, s, y)}\right)^2\right)u_{yy}(t, s, y) \\
 &\quad + \left(\dot{B}^{(1)}(s, y) - \frac{B^{(2)}(s, y)u_y(s, s, y)}{A^{(2)}(s, y)u_{yy}(s, s, y) + C^{(2)}(s, s, y)}\right)u_y(t, s, y) \\
 &\quad + \dot{C}^{(1)}(s, y) + \frac{1}{2}\dot{C}^{(2)}(s, s, y)\left(\frac{B^{(2)}(s, y)u_y(s, s, y)}{A^{(2)}(s, y)u_{yy}(s, s, y) + C^{(2)}(s, s, y)}\right)^2,
\end{align*}
$$

where $\dot{G}(t, s, y) = G(T - t, T - s, y)$. Finally, by our local well-posedness results (Theorem 3.5), there exists $\delta > 0$ such that Equation (34) is solvable in $\Delta[0, \delta]$, if there exists a constant $\epsilon > 0$ such that

1. $\dot{A}^{(2)}(s, y)\dot{g}_{yy}(s, s, y) + \dot{C}^{(2)}(s, s, y) \geq \epsilon$
2. $\frac{\partial \bar{F}}{\partial q} = \dot{A}^{(1)}(s, y) + \frac{1}{2}\dot{A}^{(2)}(s, y)\left(\frac{B^{(2)}(s, y)\dot{g}_y(s, s, y)}{A^{(2)}(s, y)\dot{g}_{yy}(s, s, y) + C^{(2)}(s, s, y)}\right)^2 \geq \epsilon$
3. $\frac{\partial \bar{F}}{\partial q} + \frac{\partial \bar{F}}{\partial n} = \dot{A}^{(1)}(s, y) + \dot{f}(t, s, y, \dot{g}_y(t, s, y), \dot{g}_{yy}(t, s, y), \dot{g}_y(s, s, y), \dot{g}_{yy}(s, s, y)) \geq \epsilon$

where $\dot{F}$ is the nonlinearity of Equation (34) and $\dot{f}$ represents the remaining terms of $\frac{\partial \bar{F}}{\partial q} + \frac{\partial \bar{F}}{\partial n}$ excluding $\dot{A}^{(1)}$. In general, it is not necessary to identify the properties of nonlinearity $\dot{F}$ in a large
ball $B(\bar{z}, R_0)$. According to the proof of Theorem 3.5, the local solution of Equation (1) can arbitrarily approach to the initial data $g$ by choosing a small enough $\delta$; see Equation (A.20). Consequently, if the domain, where nonlinearity $\dot{F}$ satisfies these requirements in Theorem 3.5, contains an open ball centered at $g$ (i.e., $g \in \mathcal{O}$), then there exists a local solution for the nonlocal systems. It is clear that $\dot{F}$ is locally Lipschitz and H"older continuous. Moreover, for a large enough $\dot{A}(1)$ and $\dot{C}(2)$, the three inequalities above hold such that Equation (34) is solvable at least in a small time interval.

3.3.2 On the global well-posedness of nonlocal nonlinear systems

In this subsection, we show that Equation (1) is well-posed globally, that is, $\tau = T$, if a very sharp a priori estimate is available. Moreover, we introduce a class of nonlocal nonlinear system called nonlocal quasilinear system of the form (36) and we establish its global solvability under a growth condition.

In contrast with the nonlocal linear systems (25), where the (small-time) solution can be extended arbitrarily many times to a global solution over $\Delta[0,T]$ for any $T < \infty$, it is possible for the nonlinear case (1) that the extension procedure is terminated at some $\tau < T$. The dissatisfying result is caused mainly by the fact in the proof of Theorem 3.5 that in order to obtain a $\frac{1}{2}$-contraction from $u$ to $U$ defined by $U_s = L_0U + F(u) - L_0u$, we need to strike a balance between $R$ and $\delta$ such that $C(R)\delta^{\frac{\alpha}{2r}} < \frac{1}{2}$. In the extension procedure in view of Remark 3.6, it is possible that the solution $u$ blows up near $\tau < T$. In this case, both $R$ and $C(R)$ tend to infinity under the norm $\|\cdot\|_{(2r+\alpha)(0,\tau)}$. From this perspective, it becomes clear that the inequality $C(R)\delta^{\frac{\alpha}{2r}} < \frac{1}{2}$ has restricted $\delta$ to be infinitely small. Consequently, the extension procedure is forced to stop to generate a maximally defined solution over $[0, \tau)$ instead of a global solution over the whole interval $[0, T]$.

In fact, it has been an unavoidable problem in the study of differential equations. To extend the maximally defined solution from $[0, \tau)$ to $[0, T]$, the key step is to show that the mapping $s \mapsto u(\cdot, s, \cdot)$ is uniformly continuous in some sense such that an analytic continuation argument works. Next, inspired by Lunardi (1989), Prato and Tubaro (1996), we show that it is possible to have $\tau = T$ if a very sharp a priori estimate is available.

**Theorem 3.8.** Let $F$ and $g$ satisfy the assumptions of Theorem 3.5 with $\alpha$ replaced by $\alpha' > \alpha$. For a fixed $g \in \Omega^{(2r+\alpha')}_{[0,T]}$, let $u$ be the maximally defined solution of problem (1) over $[0, \tau)$. Assume further that there exists a finite constant $M > 0$ such that

$$\|u\|^{(2r+\alpha')}_{[0,\sigma]} \leq M \text{ for all } \sigma \in [0, \tau),$$

(35)

then we have either $\lim_{s \to \tau} u(\cdot, s, \cdot) \in \partial \mathcal{O}$ or $\tau = T$.

Generally speaking, in the proof of Theorem 3.5, the $R$ in the $C(R)\delta^{\frac{\alpha}{2r}}$ depends on $\|u\|^{(2r+\alpha)}_{[0,\delta]}$ since the $\frac{1}{2}$-contraction operator $U = \Lambda(u)$ via $U_s = L_0U + F(u) - L_0u$ defined in a closed set $U$ of $\Omega^{(2r+\alpha)}_{[0,\delta]}$. Hence, the prior estimate of $\|u\|^{(2r+\alpha)}_{[0,\sigma]} \leq M$ for all $\sigma \in [0, \tau)$ is not enough for the existence in the large. Instead, we need an estimate on the modulus of continuity of $s \mapsto u(\cdot, s, \cdot)$. Similar sufficient conditions to obtain a priori estimates like Equation (35) for the classical
PDEs/systems can be found in Krylov (1987), Lieberman (1996). However, it is not straightforward to express such conditions in terms of coefficients and data of the local and nonlocal fully nonlinear system.

Next, we show that the desired sharp a priori estimate is available for a class of nonlocal nonlinear systems, namely nonlocal quasilinear systems, of the form:

\[
\begin{cases}
  u^a(t,s,y) = \sum_{|I|=2r,b \leq m} A^a_{I,b}(s,y) \partial_I u^b(t,s,y) + Q^a(t,s,y,\partial_I u^b(t,s,y),\partial_I u^b(s,s,y)), \\
  u(t,0,y) = g(t,y), & 0 \leq s \leq t \leq T, \quad y \in \mathbb{R}^d, \quad a = 1, \ldots, m.
\end{cases}
\]  

(36)

Compared with Equation (1), Equation (36) is free of the highest order nonlocal term \((\partial_I u^b(t,s,y))_{|I|=2r}\) and is linear in the highest order local term \((\partial_I u^b(t,s,y))_{|I|=2r}\). It is clear that Equation (36) is a special case of Equation (1). The nonlocal quasilinear systems are relevant from both theoretical and practical viewpoints, since they cover the equilibrium HJB systems of TIC SDG problems, where the diffusion of Equation (4) is uncontrolled, that is, \(\sigma(s,y,\alpha) = \sigma(s,y)\).

By leveraging Theorems 3.5 and 3.8 of nonlocal fully nonlinear systems (1), we aim to show that Equation (36) is solvable globally under some technical conditions in the theorem below. In fact, our results have been the best in the existing literature on nonlocal PDEs/systems in terms of the global well-posedness issues.

**Theorem 3.9.** Suppose that all coefficient functions \(A^a_I\) of Equation (36) belong to \(\Omega^{(\alpha)}\) and satisfy Equation (23), \(g \in \Omega^{(2r+\alpha)}\), and the nonlinearity \(Q^a(t,s,y,w,\overline{w})\) has enough regularities required in Equations (32) and (33), satisfies a linear growth condition: \(|Q^a| \leq K(1 + |w|)\), and has its bounded first-order partial derivatives with respect to \(t\) and \(w\). Then, the nonlocal quasilinear system (36) admits a unique solution \(u \in \Omega^{(2r+\alpha)}\) in \(\Delta[0,T] \times \mathbb{R}^d\).

As closing remarks of this section, we review our studies on nonlocal linear (25), quasilinear (36), and fully nonlinear systems (1) in parallel. Our analyses are based on the Banach fixed point arguments to first establish their small-time solvability and then extend the results to a longer time horizon, while the later extension faces different situations for different systems. In the case of nonlocal linear systems (25), a contractive mapping can be constructed by choosing a suitably small \(\delta\) such that \(C^{1-\frac{\alpha}{2r}} \leq \frac{1}{2}\) with a constant \(C\) depending only on the data of Equation (25). Hence, there is no issue as discussed at the beginning of Section 3.3.2 and thus the global well-posedness of Equation (25) can be obtained without extra conditions. In contrast, the nonlocal nonlinear systems, including the quasilinear (36) and the fully nonlinear (1) systems, need to balance \(R\) and \(\delta\) such that \(C(R)\delta^{\frac{\alpha}{2}} \leq \frac{1}{2}\). Through mathematical analyses, the \(R\) for the quasilinear system (36) is quantified by \(\|\cdot\|_{|a,b|}^{(2r-1+\alpha)}\) while the fully nonlinear one is quantified by \(\|\cdot\|_{|a,b|}^{(2r+\alpha)}\). This observation explains the different levels of difficulty when we establish their global existence; see Theorems 3.8 and 3.9. It is an important and promising research direction to rewrite the condition (35) in terms of the model coefficients of the original problem (1).
3.4 | Well-posedness in a weighted space

In this subsection, we extend the main results in the previous subsections to a weighted space, which allows its elements (functions) as well as their partial derivatives grow exponentially in the spatial variable \( y \). Throughout this subsection, we consider the exponential weights, defined by 
\[
\varphi(y) = \exp\{1 + \langle Sy, y \rangle^{1/2}\}
\]
for any \( y \in \mathbb{R}^d \), \( S \) being any symmetric positive-definite matrix with eigenvalues in \( [\lambda, \lambda] \) and \( \lambda > 0 \). First of all, we introduce the following weighted norms:

\[
|\varphi|_{\varphi, [a, b] \times \mathbb{R}^d}^{1,(l)} = |\varphi/\varphi|_{[a, b] \times \mathbb{R}^d},
\]

\[
|\varphi|_{\varphi, [a, b] \times \mathbb{R}^d}^{2,(l)} = \sum_{k \leq \lfloor l \rfloor} \sum_{2r + j = k} \left| \frac{\partial^h \partial^l_f \varphi}{\varphi} \right|_{0}^{\infty} + \sum_{0 < 2r - j < 2r} \left\langle \frac{\partial^h \partial^l_f \varphi}{\varphi} \right\rangle_y^{(l - \lfloor l \rfloor)} + \sum_{0 < 2r - j < 2r} \left\langle \frac{\partial^h \partial^l_f \varphi}{\varphi} \right\rangle_y^{(l - \lfloor l \rfloor)} ,
\]

\[
|\varphi|_{\varphi, [a, b] \times \mathbb{R}^d}^{3,(l)} = \sum_{k \leq \lfloor l \rfloor} \sum_{2r + j = k} \left| \frac{\partial^h \partial^l_f \varphi}{\varphi} \right|_{0}^{\infty} + \sum_{0 < 2r - j < 2r} \left\langle \frac{\partial^h \partial^l_f \varphi}{\varphi} \right\rangle_y^{(l - \lfloor l \rfloor)}
\]

\[
+ \sup_{s \in [a, b], y, y' \in \mathbb{R}^d, 0 < |y - y'| \leq \rho_0} \frac{|\partial^h \partial^l_f \varphi(s,y) - \partial^h \partial^l_f \varphi(s,y')|}{|y - y'|^{(l - \lfloor l \rfloor)}} \min \{ \varphi^{-1}(y), \varphi^{-1}(y') \}.
\]

Next, before defining weighted spaces, we illustrate the following equivalence property.

**Lemma 3.10.** The three norms defined in Equations (37)–(39) are equivalent.

By Lemma 3.10, the norms defined in Equations (37)–(39) can be all denoted by a unified notation \( |\varphi |_{\varphi, [a, b] \times \mathbb{R}^d}^{(l)} \). Similar to Section 3.1, we can then define weighted norms \( |u|_{\varphi, [0, \delta]}^{(l)} \) and \( \|u\|_{\varphi, [0, \delta]}^{(l)} \) and weighted spaces \( C_{\varphi}^{1/2r, l}, \Theta_{\varphi}^{(l)} \) and \( \Omega_{\varphi}^{(l)} \). Although the norms defined in Equations (37)–(39) are equivalent, it is useful to distinguish them since they have their own advantages. The first one (37) presents an intuitive understanding of functions with an exponential growth in the spatial argument, while the other two weighted norms (38) and (39) are convenient in showing some related conclusions in Theorem 3.12 below.

Next, we introduce a class of nonlinearities that extend the nonlinearity \( F \) in Section 3.3 for the study of well-posedness in the weighted spaces.

**Definition 3.11.** A pair of \( (F, g) \) is appropriate if there exist \( \delta, R > 0 \) such that for any \( u \in \Omega_{\varphi}^{(2r+\alpha)} : u(t, 0, y) = g(t, y), \|u - g\|_{\varphi, [0, \delta]}^{(2r+\alpha)} \leq R \),

(a) \( F(t, s, y, (\partial_t u)_{|t| \leq 2r}(t, s, y), (\partial_s u)_{|t| \leq 2r}(s, s, y)) \in \Omega_{\varphi, [0, \delta]}^{(\alpha)} \) while \( F_t \) at \( u \) belongs to \( \Theta_{\varphi, [0, \delta]}^{(\alpha)} \);

(b) both \( \partial_t F \) and \( \partial_s F \) at \( u \) belong to \( \Omega_{[0, \delta]}^{(\alpha)} \).
\[
\begin{align*}
(\text{c}) & \quad \left\{ \begin{array}{l}
(-1)^{-r} \sum_{a,b,|l|=2r} \partial_j F_b(t, s, y, (\partial_i g)|_{l|\leq r}(t, y), (\partial_i g)|_{l|\leq r}(0, y)) \xi_{j1} \cdots \xi_{l1} v^a v^b \geq \lambda |\xi|^2 |u|^2,
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
(\text{d}) & \quad \left\{ \begin{array}{l}
|\Delta_{y} \mathbf{F}(t, s, y, (\partial_j u)|_{l|\leq r}(t, s, y), (\partial_j u)|_{l|\leq r}(s, s, y))| \leq C(R)(|s - s'|^{\alpha} + |y - y'|^{\alpha}),
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
(\text{e}) & \quad \left\{ \begin{array}{l}
|\Delta_{y} \partial_j F_{bc}(t, s, y, (\partial_j u)|_{l|\leq r}(t, s, y), (\partial_j u)|_{l|\leq r}(s, s, y)) \cdot \partial_j u^i(t, s, y)|
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
& \leq C(R)(|s - s'|^{\alpha} + |y - y'|^{\alpha}),
\end{align*}
\]

where \(\Delta_{y} \varphi(s, y) := \varphi(s', y') - \varphi(s, y)\).

The conditions (a)–(e) in Definition 3.11 allow us to utilize the methodologies in Sections 3.2 and 3.3, including the linearization method and the fixed-point argument, to study nonlocal fully nonlinear systems in a weighted space. The first three conditions (a)–(c) guarantee that the mapping \(u \mapsto U\), defined by \(U_s = L_0 u + F(u) - L_0 u\), is well-defined. Moreover, with (d) and (e), we can prove that it is contractive.

Now, we are ready to show the extension of the well-posedness results for nonlocal systems in a weighted space.

**Theorem 3.12.** All well-posedness results for nonlocal systems in Sections 3.2 and 3.3 can be extended to the setting with weighted spaces defined in this subsection. Specifically, we have

1. If all coefficients of \(L\) defined in Equation (22) belong to \(\Omega_{[0, T]}^{(2+\alpha)}\), \(f \in \Omega_{[\tau, T]}^{(2+\alpha)}\) and \(g \in \Omega_{[\tau, T]}^{(2+\alpha)}\), then the nonlocal linear system (25) admits a unique solution \(u \in \Omega_{[\tau, T]}^{(2+\alpha)}\) in \(\Delta[0, T] \times \mathbb{R}^d\). Moreover,

\[
\|u\|_{[0, T]}^{(2+\alpha)} \leq C \left( \|f\|_{[\tau, T]}^{(2+\alpha)} + \|g\|_{[\tau, T]}^{(2+\alpha)} \right).
\]

2. Suppose that the pair of \((F, g)\) is appropriate in the sense of Definition 3.11. Then, there exist \(\tau > 0\) and a unique maximally defined solution \(u \in \Omega_{[\tau, T]}^{(2+\alpha)}\) satisfying Equation (1) in \(\Delta[0, T] \times \mathbb{R}^d\).

3. Assume further that \(\|u\|_{[\tau, T]}^{(2+\alpha)} \leq M\) for some finite constant \(M > 0\) across all \(\sigma \in [0, \tau]\), then either the pair of \((F, \lim_{\tau \to T} u(\cdot, s, \cdot))\) is not appropriate or \(\tau = T\). Consequently, the nonlocal quasilinear system (36) is globally solvable.

In this refined framework, although we allow the nonhomogeneous term \(f\) and the initial data \(g\) to increase exponentially in the spatial variable (more specifically, \(f \in \Omega_{[\tau, T]}^{(2+\alpha)}\) and \(g \in \Omega_{[\tau, T]}^{(2+\alpha)}\)), it is still required that all coefficients of \(L\) defined in Equation (22) belong to \(\Omega_{[0, T]}^{(2+\alpha)}\). This also induces the condition (b) of Definition 3.11 that requires that \(\partial_j F\) and \(\partial_j \overline{F}\) at \(u\) belong to ordinary normed spaces instead of the weighted ones. Nevertheless, these conditions are necessary for our analyses, including the linearization method adopted in the later analysis of solvability of nonlocal nonlinear systems in the weighted spaces, and satisfied by our first financial example in Section 4.
Remark 3.13. [Potential relaxations] Echoing our discussion at the end of Section 2.3, we establish the first analytical framework for such emerging type of nonlocal PDEs/systems with a two-time-variable structure. Hence, we have to require some additional conditions and regularities for the nonlocal setting to support our proofs. From this perspective, one may work towards lifting the restrictions so as to embrace a larger class of nonlocal systems. It should be noticed that the main restrictions originate from the limitations on coefficients of the nonlocal linear operator \( L \) defined in Equation (22). Here, we list some promising future extensions. We first note that this paper adopts a concept of strongly ellipticity conditions (23)–(24), (30)–(31), and (c) in Definition 3.11. Following Ėidel’man (1969), Ladyženskaja et al. (1968), Friedman (1964), the conditions can be substituted so as to contain a larger class of parabolic systems (in the sense of Petrovsky-type). Moreover, we may substantially improve the results by modifying the underlying norms and spaces to obtain a more general PDE theory, which allows degenerate coefficients.

4 WELL-POSEDNESS OF EQUILIBRIUM HJB SYSTEMS AND EXAMPLES

The previous two sections have established the linkage between the TIC SDGs and equilibrium HJB systems and the well-posedness of nonlocal parabolic systems that nest the equilibrium HJB systems. This section intends to summarize our results in the context of TIC SDGs and provide two examples (in finance), the induced nonlocal fully nonlinear systems of which are globally solvable under some technical conditions in Propositions 4.2 and 4.4.

Let us denote \( \dot{H}(t,s,y,z) = H(T-t, T-s, y, z) \) defined in Equation (20), where \( z = (u, p, q, l, m, n) \), and \( \dot{g}(t,y) = g(T-t, y) \). Then we have the following theorem, which follows directly Proposition 3.1 and Theorems 3.5 and 3.9.

Theorem 4.1. For any fixed \( T > 0 \), suppose that \( \dot{g} \in \Omega^{2+\alpha} [0,T] \) and assume that \( \dot{H} \) and \( \dot{g} \) are regular enough in the sense that \( \dot{H} \) satisfies the conditions of Equations (30)–(33) with the open ball \( B(\dot{g}, R_0) \) containing the range of \( ((\dot{g}_{|l| \leq 2}(t,y)), (\dot{g}_{|l| \leq 2}(s,y))) \) for any \( (t, s) \in \Delta [0,T] \) and some radius \( R_0 > 0 \). Then, for such a TIC SDG problem (4)–(5) with Equation (6), we have that

1. in the case of that both the drift and the diffusion of (4) are controlled, there exist \( \tau \in (0, T] \) and a unique maximally defined solution \( u \in \Omega^{2+\alpha} [T-\tau, T] \) satisfying the equilibrium HJB system (20) in \( \nabla [T-\tau, T] \times \mathbb{R}^d \). Moreover, whenever the domain of \( \dot{H} \) is large enough and Equation (35) holds, we have \( \tau = T \);
2. in the case of that only the drift is controlled while the diffusion of Equation (4) is uncontrolled, that is, \( \sigma(s,y,a) = \sigma(s,y) \), the equilibrium HJB system (20) admits a unique global solution \( u \in \Omega^{2+\alpha} [0,T] \) in \( \nabla [0, T] \times \mathbb{R}^d \).

The regularity requirements of \( \dot{H} \) and \( \dot{g} \) in Theorem 4.1 characterize the “needed regularity” of \( \phi \) in Assumption 2.3. By the implied existence of the NE point at each time point from Assumption 2.3, it is clear that the Hamiltonian \( \dot{H} \) satisfies Equations (30), (32), and (33). We only need to check if \( \dot{H} \) and \( \dot{g} \) jointly satisfy the condition (31). Especially for a nonlocal quasilinear system (i.e., a TIC SDG with controls on drift only), it is easy to see that conditions (30) and (31) hold. Hence, with smooth enough coefficients in Equations (4)–(6), the corresponding equilibrium HJB system satisfies the requirements in Theorem 4.1. Finally, based on the conjectures in Section 2.3, we may
conclude that the associated TIC SDG admits a TC-NE point \( \overline{\alpha} = \phi(s, s, y, (u)_{\|u\| \leq 2}(s, s, y)) \) and the TC-NE value function \( V(s, y) = u(s, s, y) \) for \( (s, y) \in [T - \tau, T] \times \mathbb{R}^d \). Analogously, Theorem 3.12 supports that all results in Theorem 4.1 still hold in a weighted space.

### 4.1 Financial examples

In this subsection, we provide two examples of TIC SDGs among \( m \) players on \([0, T]\) for an arbitrary positive integer \( m \geq 1 \) and an arbitrary large time \( T > 0 \), the first one of which studies how the investors (players) choose their own optimal investment strategies to increase their own exponential utility and the second one of which studies the optimal investment and consumption strategy pairs of investors that optimize their own power utility. These two examples showcase two different uses of our well-posedness results in Section 3.4. The first example satisfies all the conditions in our refined (weighted-norm) framework and thus the corresponding TIC problem is globally solvable over the whole time horizon \( V[0, T] \). The second example is, however, not covered by our framework due to its degeneracy property, but this interesting and relevant example is still considered here while it shows the necessity of extending our analytic framework from the nondegenerate setting to a degenerate one; see also the discussion in Remark 3.13. Though the second example is beyond our general framework, we make another problem-specific attempt in the similar spirit of the proof of Theorem 3.8. Specifically, with some suitable ansatzs of solutions, both examples admit explicit expressions of Equations (18) and (20) while the latter can be further reduced to ordinary differential equation (ODE) systems. We can then show the global solvability of these ODE systems and thus we obtain the global solvability of the two listed examples.

We first introduce the general setup for the two examples. Suppose that the \( m \) players have similar interests, for example, a risk investment fund managed by \( m \) investors (managers). To maintain and increase their own utility, each investor needs to study their own optimal investment and consumption strategy. Consider a market model in which there are one bond with the riskless interest rate \( r > 0 \) and some risky assets while the \( a \)th investor has estimated the appreciation rate of his/her return on investment (ROI) by \( \mu_a > r \) and its volatility by \( \sigma_a > 0 \). Further assume that the \( m \) investors’ ROIs are uncorrelated, the yield rate vector \( P \in \mathbb{R}^m \) of the \( m \) investors is characterized by

\[
dP(s) = \text{diag}(P(s))(\mu ds + \sigma dW(s))
\]

where \( \text{diag}(P(s)) \) is a diagonal matrix with main diagonal elements of \( P(s) \), \( \mu = (\mu_1, \mu_2, \ldots, \mu_m)^\top \), and \( \sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_m) \). Denoted by \( \alpha_a(\cdot) \) the dollar amounts managed by the \( a \)th investor and \( \{X(t)\}_{t \in [0, T]} \) the aggregated wealth process (from this perspective, we are similarly considering a problem by a fund of funds), we can obtain the following FSDE for \( \{X(t)\}_{t \in [0, T]} \):

\[
\begin{aligned}
\left\{ 
\begin{array}{l}
dX(s) = \left[rX(s) + (\mu - r1)^\top \alpha(s) - 1^\top c(s)\right] ds + \alpha^\top(s) \sigma dW(s), \\
X(t) = y, \quad 0 \leq t \leq T, \quad y \in \mathbb{X},
\end{array}
\right.
\end{aligned}
\]

where \( 1 = (1, \ldots, 1)^\top \in \mathbb{R}^m \) and \( c = (c^1, \ldots, c^m)^\top \) with \( c^a(\cdot) \) the consumption rate of the \( a \)th investor valued in \( C \), \( \alpha^a \) is valued in \( A_a \), and \((A_a, C, \mathbb{X})\) will be specified for our examples. For any fixed \( t \in [0, T] \), the admissible set of investment-consumption strategy pairs is then defined as the set of progressively measurable processes \( \{(\alpha(s), c(s))\}_{s \geq t} \) such that \( \alpha^a(s) \in A_a \) and \( c^a(s) \in C \) for \( a = 1, \ldots, m \) and that the FSDE (41) has a strong solution \( \{X(s)\}_{s \geq t} \) with \( X(s) \in \mathbb{X}, \mathbb{P}\text{-a.s.}, \) for \( s \geq t \).
Next, to characterize the investors’ preferences, let \((Y(\cdot), Z(\cdot))\) be the adapted solution to the following BSDE:

\[
\begin{aligned}
    \begin{cases}
        dY(s) = -h(t, s, X(s), \alpha(s), c(s), Y(s))ds + Z(s)dW(s), & t \leq s \leq T, \\
        Y(T) = g(t, X(T)), & 0 \leq t \leq T,
    \end{cases}
\end{aligned}
\]  

(42)

where the generator \(h\) and terminal condition \(g\) are both deterministic \(\mathbb{R}^m\)-valued functions, and they will be specified in the study of different utility problems. Then, we define the recursive utility functional of the \(a\)th investor for \(a = 1, \ldots, m\) as follows:

\[
    J^a(t, y; \alpha, c) := Y^a(t; t, y, \alpha, c).
\]

Consequently, the problem of maximizing \(J^a(t, y; \alpha, c)\) for \(a = 1, \ldots, m\) is a TIC SDG since \(h\) and \(g\) both depend on the initial time point \(t\). Note that in Section 2, we illustrate with an SDG with minimization while it is equivalent to considering maximization. It is noteworthy that the \(a\)th functional \(J^a\) is a Uzawa-type differential utility being not only recursive (in the sense that it depends on \(Y^a\) itself) but also dependent on other investors’ utility functionals \(Y^{\neq a}\). It is sensible because enormous experiments in behavioral economics/finance show that people’s assessment on their wellbeing is relative rather than absolute. Moreover, the controlled FBSDEs of the \(m\) investors are coupled together through \((\alpha, c)\) and \(Y\) in the BSDE and \((\alpha, c)\) in the FSDE. Furthermore, compared to the existing literature on the well-posedness results, we allow the diffusion of the wealth process \(X\) to be controlled.

4.1.1 TIC Merton problem with exponential utility and zero consumption

In the first example, we assume that

\[
\begin{aligned}
    \begin{cases}
        \mathbb{A} = \mathbb{R}, & \mathbb{C} = \{0\}, & \mathbb{X} = \mathbb{R}, \\
        h(t, s, X(s), \alpha(s), c(s), Y(s)) = -R(t, s)Y(s), \\
        g(t, X(T)) = -T(t) \exp\{-\eta X(T)\}, & \eta > 0,
    \end{cases}
\end{aligned}
\]

(43)

where \(R\) and \(T\) are \(\mathbb{R}^{m \times m}\)- and \(\mathbb{R}^m\)-valued continuous and positive functions, respectively. Next, we will show that this example can be analyzed within our framework in Section 3.4. Specifically, in order to show the well-posedness of solutions to the TIC SDG (41)–(43), the main steps are listed as follows:

1. embed the original TIC SDG problem (41)–(43) \((P)\) into a family of problems \(P_\gamma\) parameterized by \(\gamma \geq 0\) such that \(P_0 = P\);
2. prove the global well-posedness of solutions of \(P_\gamma\) in the case of \(\gamma > 0\) such that the mapping from \(\gamma \in (0, \infty)\) to the solution \(U_\gamma(\cdot, \cdot, \cdot) \in \Omega^{2+\alpha}_{\gamma}[0, T]\) is well-defined;
3. show that \(\gamma \mapsto U_\gamma(\cdot, \cdot, \cdot)\) admits a unique analytic continuation at \(\gamma = 0\) such that the problem \(P\) (i.e., \(P_0\)) has global existence and uniqueness of solutions as well.

These three steps not only show the global well-posedness of solutions of nonlocal HJB system and the TIC SDG but also give explicit representations for equilibrium strategies (18) and equilibrium value functions of Equation (20). We cannot directly analyze \(P_0\) as its nonlinearity is not regular.
enough and thus we parametrize the problem such that the nonlinearity of $P_\gamma$ with $\gamma > 0$ satisfies the regularity conditions in our framework.

First of all, it is more convenient to consider a transformed state $X(s) \exp\{r(T-s)\}$ for the dynamics $X(s)$ of Equation (41) before our analyses. By Corollary 5.6 of Yong and Zhou (1999), it is clear that

$$
\begin{aligned}
\left\{ \begin{array}{l}
\frac{dX(s)}{ds} = (\mu-rI)^T \exp\{r(T-s)\} \alpha(s) ds + \alpha^T(s) \sigma \exp\{r(T-s)\} dW(s), \quad t \leq s \leq T, \\
X(t) = y \exp\{r(T-t)\}, \quad 0 \leq t \leq T, \quad y \in \mathbb{R}.
\end{array} \right.
\end{aligned}
$$

Consequently, without loss of generality, let us consider a modified version of Equation (41) with the following form:

$$
\begin{aligned}
\left\{ \begin{array}{l}
\frac{dX(s)}{ds} = \hat{\mu}^T(s) \alpha(s) ds + \alpha^T(s) \hat{\sigma}(s) dW(s), \quad t \leq s \leq T, \\
X(t) = y, \quad 0 \leq t \leq T, \quad y \in \mathbb{R},
\end{array} \right.
\end{aligned}
$$

(44)

where $\hat{\mu}(s) = (\mu-rI)^T \exp\{r(T-s)\}$ and $\hat{\sigma}(s) = \sigma \exp\{r(T-s)\}$.

**Step 1: A family of parameterized problems $P_\gamma$.** Next, let us consider a family of problems (BSDEs) parameterized by an external parameter $\gamma \geq 0$,

$$
\begin{aligned}
&c(s) = 0, \\
h_\gamma(t, s, X(s), \alpha(s), c(s), Y(s)) = \gamma \left( \begin{array}{l}
\omega_1^T(t, s, X(s)) \otimes 1 \\
\omega_2(t, s, X(s)) \otimes 1
\end{array} \right) (\alpha(s) \otimes \alpha(s)) \\
g_\gamma(t, X(T)) = \gamma g_1(t) \exp(\eta X(T)) - g_2(t) \exp(-\eta X(T)), \quad \eta > 0,
\end{aligned}
$$

(45)

where $\otimes$ denotes the Kronecker product, $\odot$ denotes the Hadamard product, $\omega_3(t, s) = R(t, s)$, $g_2(t) = T(t)$, and $\omega_1(t, s, y)$, $\omega_2(t, s, y)$, $g_1(t)$, and $g_2(t)$ are all $\mathbb{R}^m$-valued continuous functions that will be specified later. It is clear that Equation (45) reduces to Equation (43) when $\gamma = 0$ and $\omega_2 = 0$. Our later specification will also parametrize $\omega_1$ and $\omega_2$ with $\gamma$ and $\omega_2 \equiv 0$ when $\gamma = 0$. Thus in this case, Equation (45) is actually parameterized by a single parameter $\gamma$.

It is noteworthy that there are multiple embedding schemes while any of them can work out the well-posedness of solutions of the problem (41)–(43) as long as the mapping from $\gamma$ to the solution of $P_\gamma$ is well-defined and is at least Cauchy-continuous at the point that reduces the parametrized problem to $P$. Moreover, the fact about whether the problem is well-posed is free of the choice of the embedding scheme. We will show that the embedding Equation (45) with Equation (50) facilitate Steps 2 and 3. The relationship between parameterized data and solutions was discussed in the earlier stability analysis of nonlocal systems; see Remark 3.7.

**Step 2. The well-definedness of $\gamma \mapsto U_\gamma(\cdot, \cdot, \cdot)$ with $\gamma > 0$.** According to the definitions of $(X(\cdot), Y(\cdot), Z(\cdot))$ formulated by controlled FBSDEs (44)–(45), the Hamiltonian system of the $m$ players has the form: for $a = 1, \ldots, m$,

$$
\mathcal{H}_\gamma^a(t, s, y, \alpha, u, p, q) = \frac{1}{2} \left( \sum_{1 \leq b \leq m} (\hat{\sigma}_b(s) \alpha^b)^2 \right) q^a + \left( \sum_{1 \leq b \leq m} \hat{\mu}_b(s) \alpha^b \right) p^a + \sum_{1 \leq b \leq m} \gamma \omega_1^b(t, s, y) \alpha^b - \sum_{1 \leq b \leq m} \omega_2^b(t, s, y) (\alpha^b)^2 - \sum_{1 \leq b \leq m} \omega_3^a(t, s) u^b,
$$
Maximizing the above with respect to $\alpha^a$ with fixed $\alpha^{-a}$, $p > 0$, and $q < 0$ yields
\[
\alpha^a = \frac{\gamma w_1^a(t, s, y) + \hat{\mu}_a(s)p^a}{2w_2^a(t, s, y) - \hat{\sigma}^a(s)q^a} = -\frac{\hat{w}_1^a(t, s, y) + (\mu_a - r)p^a}{\hat{w}_2^a(t, s, y) - \hat{\sigma}^a S^a \exp\{-r(T - s)}\right\}, \quad a = 1, \ldots, m.
\]
where $\hat{w}_1^a(t, s, y) := \gamma w_1^a(t, s, y)\exp\{-r(T - s)\}$ and $\hat{w}_2^a(t, s, y) := 2w_2^a(t, s, y)\exp\{-2r(T - s)\}$. Thus, eventually, the equilibrium strategy will be given by
\[
\alpha^a(s, y) = \frac{\hat{w}_1^a(s, y) + (\mu_a - r)U^a_y(s, y)}{\hat{w}_2^a(s, y) - \hat{\sigma}^a U^a_yy(s, y) \exp\{-r(T - s)}\right\} (46)
\]
with $U(t, s, y) = (U^1(t, s, y), \ldots, U^m(t, s, y))$ ($y$ is suppressed) being the solution to an equilibrium HJB system of the form:
\[
\begin{align*}
U^a_s(t, s, y) + \frac{1}{2} \sum_{1 \leq b \leq m} \left( \frac{\sigma_b U^b_y(s, y) + \sigma_b(\mu_b - r)U^b_y(s, y)}{\hat{w}_2^b(s, y) - \hat{\sigma}^b U^b_yy(s, y)} \right)^2 U^a_yy(t, s, y) + \sum_{1 \leq b \leq m} \frac{\gamma w_1^b(s, y) + (\mu_b - r)U^b_y(s, y)}{\hat{w}_2^b(s, y) - \hat{\sigma}^b U^b_yy(s, y)} U^a_y(t, s, y) + \exp\{-r(T - s)\} - \exp\{-2r(T - s)\} - \sum_{1 \leq b \leq m} \frac{w_2^b(t, s, y)}{\hat{w}_2^b(s, y) - \hat{\sigma}^b U^b_yy(s, y)} \right) - \sum_{1 \leq b \leq m} w_3^{ab}(t, s)U^b(t, s, y) = 0,
\end{align*}
\]
$U(t, T, y) = \gamma g_1(t)\exp\{\eta y\} - g_2(t)\exp\{-\eta y\}$, $0 \leq t \leq s \leq T$, $y \in \mathbb{R}$, $a = 1, \ldots, m$.

It is equivalent to solving the following forward problem:
\[
\begin{align*}
U^a_s(t, s, y) &= \frac{1}{2} \sum_{1 \leq b \leq m} \left( \frac{\sigma_b U^b_y(s, T - s, y) + \sigma_b(\mu_b - r)U^b_y(s, T - s, y)}{\hat{w}_2^b(T - s, s, y) - \hat{\sigma}^b U^b_yy(s, s, y)} \right)^2 U^a_yy(t, s, y) + \sum_{1 \leq b \leq m} \frac{\gamma w_1^b(s, s, y) + (\mu_b - r)U^b_y(s, s, y)}{\hat{w}_2^b(T - s, s, y) - \hat{\sigma}^b U^b_yy(s, s, y)} U^a_y(t, s, y) + \exp\{-rs\} - \exp\{-2rs\} - \sum_{1 \leq b \leq m} w_3^{ab}(T - s, s, y)U^b(t, s, y) = 0,
\end{align*}
\]
$U(t, 0, y) = \gamma g_1(T - t)\exp\{\eta y\} - g_2(T - t)\exp\{-\eta y\}$, $0 \leq s \leq t \leq T$, $y \in \mathbb{R}$, $a = 1, \ldots, m$.

Next, let us consider the partial derivatives of the nonlinearity of Equation (48) with respect to its arguments in order to verify its regularities. The nonlinearity of Equation (48) is denoted by $\mathbb{H} := \mathbb{H}_j(t, s, y, z)$. After a rather lengthy but straightforward calculation, we can obtain all partial derivatives in Tables 1 and 2, which are all listed in Appendix B. Consequently, for the initial condition $U(t, 0, y) = \gamma g_1(T - t)\exp\{\eta y\} - g_2(T - t)\exp\{-\eta y\}$, one can verify that the pair of $(\mathbb{H}, U(t, 0, y))$ is appropriate in the sense of Definition 3.11 for some suitable $\hat{w}_1$ and $\hat{w}_2$; see Equation (50). Hence, our well-posedness results in Section 3.4 promise that there exist $\delta \in (0, T]$ and a unique solution satisfying (48) in $\Delta[0, \delta] \times \mathbb{R}$. Equivalently, the backward problem (47) is solvable as well in $\mathbb{V}[T - \delta, T] \times \mathbb{R}$.
In order to find an explicit solution to Equation (47) and show its global solvability in the whole time horizon $\nabla[0,T]$, we consider the following ansatz:

$$U(t, s, y) = \varphi_1(t, s) \exp\{\gamma y\} - \varphi_2(t, s) \exp\{-\gamma y\}, \quad (t, s) \in \nabla[T - \delta, T],$$

(49)

for some suitable $\varphi_1(\cdot, \cdot)$, and $\varphi_2(\cdot, \cdot)$. Then we have $\varphi_1(t, T) = \gamma g_1(t)$ and $\varphi_2(t, T) = g_2(t)$. Furthermore, let us assume that

$$\tilde{\omega}^a(t, s, y) = \gamma \omega^a(t, s, y) \exp\{-r(T - s)\} = \gamma W^a(t, s) \exp\{\gamma y\} \exp\{-r(T - s)\},$$

$$\tilde{\omega}^b(t, s, y) = 2 \omega^b(t, s, y) \exp\{-2r(T - s)\} = \frac{\sigma^2_{\gamma}}{\mu_a - r} \omega^a(t, s, y) + 2\sigma^2_{\gamma} \eta^2 \varphi_1^a(t, s) \exp[\gamma y],$$

(50)

where $W^a_1(t, s)$ is a given continuously differentiable and positive function. Under the assumptions of Equation (50), we have $\tilde{\omega}^a(s, y) = \frac{1}{\eta \sigma^2_a} \exp(-r(T - s))$ for $a = 1, 2, \ldots, m$. Subsequently, by simple calculation, $\varphi_1(\cdot, \cdot)$ and $\varphi_2(\cdot, \cdot)$ solve the following ODE systems, respectively,

$$\begin{cases}
\varphi_1(t, s) = \gamma \varphi_1(t, s) + M_1(t, s) = 0, \\
\varphi_1(t, T) = \gamma g_1(t), \quad (t, s) \in \nabla[T - \delta, T],
\end{cases}$$

$$\begin{cases}
\varphi_2(t, s) = \gamma \varphi_2(t, s) + M_2(t, s) = 0, \\
\varphi_2(t, T) = g_2(t), \quad (t, s) \in \nabla[T - \delta, T],
\end{cases}$$

(51)

where

$$N_1(t, s) = \operatorname{diag}\left\{ \sum_b \frac{3(\mu_b - r)^2}{2\sigma_b^2}, \ldots, \sum_b \frac{3(\mu_b - r)^2}{2\sigma_b^2} \right\} - \left( \frac{\mu_1 - r}{\sigma_1^2}, \ldots, \frac{\mu_m - r}{\sigma_m^2} \right) \otimes 1 - \omega_3(t, s)$$

and

$$M_1(t, s) := \gamma \overline{M}_1(t, s) = \gamma \left( \frac{(\mu_1 - r) \exp\{-r(T - s)\}}{2\sigma_1^2 \eta}, \ldots, \frac{(\mu_m - r) \exp\{-r(T - s)\}}{2\sigma_m^2 \eta} \right) \otimes 1 \right| W_1.$$ Build.

Moreover, $N_2(t, s) = - \operatorname{diag}\left\{ \sum_b \frac{(\mu_b - r)^2}{2\sigma_b^2}, \ldots, \sum_b \frac{(\mu_b - r)^2}{2\sigma_b^2} \right\} - \omega_3(t, s)$. By the classical theory of ODE systems, systems (51) admit a unique solution for $(t, s) \in \nabla[T - \delta, T]$ since $\sup_{t,s} |N_i(t, s)| (i = 1, 2)$ are both bounded. Furthermore, the ansatz solution (49) of Equation (47) can be represented by

$$U(t, s, y) = \left[ \gamma f_1(t, s) f_1^{-1}(t, T) g_1(t) + \gamma \int_s^T f_1(t, \tau) f_1^{-1}(t, \tau) \overline{M}_1(t, \tau) d\tau \right] \exp\{\gamma y\} - f_2(t, s) f_2^{-1}(t, T) g_2(t) \exp\{-\gamma y\},$$

(52)

where $f_i(t, s)$ is the fundamental matrix of the $i$th ODE system of Equation (51), $f_i^{-1}(t, s)$ the associated inverse matrix, and

$$f_1(t, s) = I + \int_s^T N_1(t, \tau) d\tau + \int_s^T N_1(t, \tau) \int_\tau^T N_1(t, \sigma) d\sigma d\tau + \cdots, \quad i = 1, 2,$$

(53)

in which $I$ is $m \times m$ identity matrix. Note that Equation (53) converges absolutely for every $s \in [T - \delta, T]$ and uniformly on every compact interval in $[T - \delta, T]$. In particular for $i = 1, 2$, if the matrix $N_i(t, s)$ satisfies the Lappo–Danilevskii condition (see Remark 4.3), then $f_i(t, s) = \exp\{\int_s^T N_i(t, \tau) d\tau\}$.

Note that $U(t, s, y)$ of Equation (52) does not blow-up at $s = T - \delta$ for any $t \in [0, T - \delta]$ such that we can update a new terminal condition at $s = T - \delta$. Furthermore, thanks to Equation (50), the uniformly elliptic conditions, the locally Lipschitz and Hölder continuity still hold within a small open ball centered at the range of the updated data. Consequently, one can repeat indefinitely the solving procedure up to a global solution for Equation (47) over $\nabla[0,T]$. With our well-posedness results of nonlocal systems in Section 3.4, we show that the mapping from the parameter $\gamma > 0$ into the solution of Equation (47), that is, $U_{\gamma}(t, s, y) := U(t, s, y)$, is well-defined.
Step 3. Analytic continuation of $\gamma \mapsto U_\gamma(\cdot,\cdot,\cdot)$ at $\gamma = 0$. For the original problem $P_0$, that is, $\gamma = 0$, our well-posedness results of nonlocal systems are not feasible even in a small time interval since the locally Lipschitz and Hölder continuity conditions are violated for some derivatives of the nonlinearity $H$. However, for any fixed $\gamma > 0$, we have shown that the mapping $\gamma \mapsto U_\gamma(\cdot,\cdot,\cdot)$ is well-defined and has an explicit formula (52). From which, we can easily see that the mapping $\gamma \mapsto U_\gamma(\cdot,\cdot,\cdot)$ is at least uniformly continuous in $\gamma$ and thus we can extend it at $\gamma = 0$ uniquely. Consequently, we can obtain the unique solution for $P_0$ in $V[0,T]$, $U(t,s,y) = -f_2(t,s)g_2(t)\exp(-\eta y)$. Furthermore, the closed-loop TC-NE point and the corresponding TC-NE value function of the TIC SDG (41)–(43) have the following explicit representations:

$$
\tilde{\alpha}_a(s,y) = \frac{1}{\eta} \frac{\mu_a - r}{\sigma^2_a} \exp\{-r(T - s)\},
$$
$$
V(s,y) = -f_2(s,s)T(s)\exp\{-\eta y \exp\{r(T - s)\}\}, \quad (s,y) \in [0,T] \times \mathbb{R},
$$

by noting also the relationship between Equation (41) and Equation (44).

Indeed, by directly making an ansatz $U(t,s,y) = -\varphi_2(t,s)\exp(-\eta y)$ for the solution of $P_0$ (i.e., Equations 43 and 44), we can still obtain the same explicit solution (54). However, as we stressed before, our well-posedness results do not cover the problem $P_0$ since Equations (50) and (51) induce $\tilde{w}_1 = \tilde{w}_2 = \varphi_1 = 0$ when $\gamma = 0$. Hence, it is necessary to embed $P_0$ into a family of problem $P_\gamma$ ($\gamma > 0$).

Finally, let us summarize our results in the following proposition.

Proposition 4.2. Suppose that $R$ and $T$ are continuously differentiable, then the TIC SDG (41)–(43) admits a unique solution in $V[0,T]$, and the closed-loop TC-NE point and the TC-NE value function are given in Equation (54).

Remark 4.3. The matrix $N_i(t,s)$ satisfies the Lappo-Danilevskii condition, which means that it commutes with its integral, that is, $N_i(t,s) \cdot \int_s^T N_i(t,\tau) d\tau = \int_s^T N_i(t,\tau) d\tau \cdot N_i(t,s)$. Let us list four cases in which the condition holds, (1) $m = 1$; (2) $w(t,s) = w(t)$; (3) $w(t,s)$ is a diagonal matrix; (4) $N(t,s)$ and $N(t,\tau)$ commute for all $s, \tau$, and $t$.

4.1.2 TIC Merton investment-consumption problem with power utility

In our second example, we assume that

$$
\begin{cases}
A_1 = \mathbb{R}, \quad C = [0,\infty), \quad X = (0,\infty),
\h(t,s,X(s),\alpha(s),c(s),Y(s)) = v(t,s)c^\beta(s) - w(t,s)Y(s),
\text{g}(t,X(T)) = g(t)X^\beta(T), \quad \beta > 0,
\end{cases}
$$

where $v$ and $w$ are both $\mathbb{R}^{m \times m}$-valued functions and $g$ is $\mathbb{R}^m$-valued continuous and positive function. In this case, each player $a$ ($a = 1, 2, \ldots, m$) needs to choose an investment and consumption strategy pair $(\alpha^a(s), c^a(s))$ valued in $\mathbb{R} \times [0,\infty)$ to optimize their own power utility. With a specific model, we can obtain explicit expressions of Equations (18) and (20) while the latter can be further reduced to an ODE system with an ansatz. In the similar spirit of the proof of Theorem 3.8, we can show the global solvability of the ODE system and thus we obtain the global
well-posedness of Equation (20). However, although our results are applicable to the state process with controlled drift and volatility, which is the case of this example, they do not cover this example due to its degeneracy property. Moreover, since the power utility function is defined over \((0, \infty)\), we also need the constraint that the solution \(\{X(s)\}_{t,T}\) of Equation (41) is almost surely nonnegative, that is, \(X_s = (0, \infty)\). Such a constraint is not necessary for our first example since the domain of exponential utility function is \(\mathbb{R}\).

According to the definitions of \((X(\cdot), Y(\cdot), Z(\cdot))\) formulated by controlled FBSDEs (41)–(42) with Equation (55), the Hamiltonian system of the \(m\) players has the form: for \(a = 1, \ldots, m\),

\[
\mathcal{H}^a(t, s, y, \alpha, c, u, p, q) = \frac{1}{2} \left( \sum_{1 \leq b \leq m} (\sigma_b \alpha^b)^2 \right) q^a + \left[ ry + \sum_{1 \leq b \leq m} ((\mu_b - r)\alpha^b - c^b) \right] p^a + \sum_{1 \leq b \leq m} (v^{ab}(t, s)(c^b)^\beta - w^{ab}(t, s)u^b),
\]

where \(v^{ab}\) and \(w^{ab}\) represent the \((a, b)\)-entry of matrices \(v\) and \(w\), respectively. Maximizing the above with respect to \(\alpha^a\) and \(c^a\) with fixed \(\alpha^{-a}, c^{-a}, p > 0,\) and \(q < 0\) yields

\[
\bar{\alpha}^a = -\frac{(\mu_a - r)p^a}{\sigma_a^2 q^a}, \quad \bar{c}^a = \left( \frac{p^a}{\beta v^{aa}(t, s)} \right)^{\frac{1}{\beta - 1}}, \quad a = 1, \ldots, m.
\]

Thus, eventually, the equilibrium strategy will be given by

\[
\bar{\alpha}^a(s, y) = -\frac{(\mu_a - r)U^a_y(s, s, y)}{\sigma_a^2 U^a_{yy}(s, s, y)}, \quad \bar{c}^a(s, y) = \left( \frac{U^a_y(s, s, y)}{\beta v^{aa}(s, s)} \right)^{\frac{1}{\beta - 1}},
\]

for \((s, y) \in [0, T] \times (0, \infty)\) with \(U(t, s, y) = (U^1(t, s, y), \ldots, U^m(t, s, y))\) being the solution to an equilibrium HJB system:

\[
\begin{cases}
U^a_s(t, s, y) + \mathcal{H}^a(t, s, y, \bar{\alpha}(s, y), \bar{c}(s, y), U(t, s, y), U^a_y(t, s, y), U^a_{yy}(t, s, y)) = 0, \\
U(t, T, y) = g(t)y^\beta, \quad 0 \leq t \leq s \leq T, \quad y \in (0, \infty), \quad a = 1, \ldots, m.
\end{cases}
\]

The \(m\) HJB equations system of Equation (57) are coupled with each other via the equilibrium strategy \((\bar{\alpha}, \bar{c})\) and the recursive dependence on \(Y\) in the generators of Equation (55), that is, the terms of \(U(t, s, y)\) of Equation (57). By substituting Equation (56) into the Hamiltonian, Equation (57) becomes

\[
\begin{cases}
U^a_s(t, s, y) + \frac{1}{2} \left( \sum_{1 \leq b \leq m} \left( \frac{(\mu_b - r)U^b_y(s, s, y)}{\sigma_b U^b_{yy}(s, s, y)} \right)^2 \right) U^a_{yy}(t, s, y) \\
+ ry - \sum_{1 \leq b \leq m} \left( \frac{(\mu_b - r)^2 U^b_y(s, s, y)}{2 \sigma_b U^b_{yy}(s, s, y)} - \left( \frac{U^b_y(s, s, y)}{2 \beta v^{bb}(s, s)} \right)^{\frac{1}{\beta - 1}} \right) U^a_y(t, s, y) \\
+ \sum_{1 \leq b \leq m} \left( v^{ab}(t, s) \left( \frac{U^b_y(s, s, y)}{\beta v^{bb}(s, s)} \right)^{\frac{\beta}{\beta - 1}} - w^{ab}(t, s)U^b(t, s, y) \right) = 0, \\
U(t, T, y) = g(t)y^\beta, \quad 0 \leq t \leq s \leq T, \quad y \in (0, \infty), \quad a = 1, \ldots, m.
\end{cases}
\]
It is clear that the first-order derivative of the nonlinearity of Equation (58) with respect to 
\( U^a_{yy}(t, s, y) \) at \( U(t, T, y) \) would be degenerate. Consequently, our well-posedness results are not 
applicable to analyze its solvability. Thus, this second example is more of an inspiration and serves 
as an indication of the validity of the general (degenerate) case. To facilitate our analysis of Equation (57), we 
need a more explicit form of \( U^a_{yy}(t, s, y) \) and thus inspired by its terminal condition, we consider the following ansatz:

\[
U^a(t, s, y) = \varphi^a(t, s)y^\beta, \quad 0 \leq t \leq s \leq T, 
\]

for some suitable \( \varphi^a(\cdot, \cdot), a = 1, \ldots, m \). Then we have \( \varphi^a(t, T) = g^a(t) \) and by simple calculation, \( \varphi(t, s) \) satisfies the following system of ODEs:

\[
\begin{align*}
\varphi^a_s(t, s) + & \left[ r \beta + \sum_{1 \leq b \leq m} \left( \frac{(\mu_b - r)^2}{2\sigma^2_b(1-\beta)} \right) - \sum_{1 \leq b \leq m} \beta \left( \frac{\varphi^b(s, s)}{\nu^{bb}(s, s)} \right)^\frac{1}{\beta-1} \right] \varphi^a(t, s) \\
+ & \sum_{1 \leq b \leq m} v^{ab}(t, s) \left( \frac{\varphi^b(s, s)}{\nu^{bb}(s, s)} \right)^\frac{\beta}{\beta-1} - \sum_{1 \leq b \leq m} w^{ab}(t, s) \varphi^b(t, s) = 0, \\
\varphi(t, T) = g(t), \quad 0 \leq t \leq s \leq T, \quad a = 1, \ldots, m.
\end{align*}
\]

Denoting by \( k = r \beta + \sum_{1 \leq b \leq m} \left( \frac{(\mu_b - r)^2}{2\sigma^2_b(1-\beta)} \right) \), the ODE system above becomes

\[
\begin{align*}
\varphi^a_s(t, s) + & A(t, s, \varphi(s, s)) \cdot \varphi(t, s) + f(t, s, \varphi(s, s)) = 0, \\
\varphi(t, T) = g(t), \quad 0 \leq t \leq s \leq T.
\end{align*}
\]

where

\[
A(t, s, \varphi(s, s)) = -w(t, s) + \text{diag} \left\{ k - \sum_b \beta \left( \frac{\varphi^b(s, s)}{\nu^{bb}(s, s)} \right)^\frac{1}{\beta-1}, \ldots, k - \sum_b \beta \left( \frac{\varphi^b(s, s)}{\nu^{bb}(s, s)} \right)^\frac{1}{\beta-1} \right\}
\]

\[
f(t, s, \varphi(s, s)) = \left[ \sum_b v^{1b}(t, s) \left( \frac{\varphi^b(s, s)}{\nu^{bb}(s, s)} \right)^\frac{\beta}{\beta-1}, \ldots, \sum_b v^{mb}(t, s) \left( \frac{\varphi^b(s, s)}{\nu^{bb}(s, s)} \right)^\frac{\beta}{\beta-1} \right]^T.
\]

According to the classical theory of system of ODEs, the fundamental matrix \( \chi \) makes it possible to write every solution of the inhomogeneous system (60) in the form of Cauchy’s formula

\[
\psi(t, s) = \chi(t, s, \psi(s, s)) \chi^{-1}(t, T, \psi(T, T)) g(t) + \int_s^T \chi(t, s, \psi(s, s)) \chi^{-1}(t, \tau, \psi(\tau, \tau)) f(t, \tau, \psi(\tau, \tau)) d\tau, 
\]

where \( \chi^{-1} \) is the inverse matrix of \( \chi \), and

\[
\chi(t, s, \psi(s, s)) = I + \int_s^T A(t, \tau, \psi(\tau, \tau)) d\tau + \int_s^T A(t, \tau, \psi(\tau, \tau)) \int_\tau^T A(t, \sigma, \psi(\sigma, \sigma)) d\sigma d\tau + \cdots,
\]
in which $I$ is $m \times m$ identity matrix. Note that Equation (62) converges absolutely for every $s \in [0, t]$ and uniformly on every compact interval in $[0, t]$. Taking $t = s$ gives us that

$$
\psi(s, s) = \chi(s, s, \psi(s))\chi^{-1}(s, T, \psi(T, T))g(t) + \int_s^T \chi(s, s, \psi(s))\chi^{-1}(s, \tau, \psi(\tau, \tau))f(s, \tau, \psi(\tau, \tau))d\tau,
$$

By introducing $\bar{\psi}(s) = \psi(s, s)$, we have

$$
\bar{\psi}(s) = \chi(s, s, \bar{\psi}(s))\chi^{-1}(s, T, \bar{\psi}(T))g(s) + \int_s^T \chi(s, s, \bar{\psi}(s))\chi^{-1}(s, \tau, \bar{\psi}(\tau))f(s, \tau, \bar{\psi}(\tau))d\tau,
$$

which is a nonlinear integral system for the unknown function $s \mapsto \bar{\psi}(s)$. Once the diagonal value $\bar{\psi}(s) = \psi(s, s)$ can be determined uniquely, there exists a unique solution $\psi(t, s)$ from the integral equation (61) of $\psi(t, s)$. By Equations (56) and (59), the equilibrium investment-consumption strategy and equilibrium value functions can be represented with $\bar{\psi}(s)$ in Equation (63) as follows:

$$
\alpha^a(s, y) = \frac{(\mu_a - r)}{\sigma_a^2(1 - \beta)} y, \quad \xi^a(s, y) = \left(\frac{v^{aa}(s, s)}{\bar{\psi}(s)}\right)^{\frac{1}{1 - \beta}} y, \quad V(s, y) = \bar{\psi}(s)y^\beta, \quad (s, y) \in [0, T] \times (0, \infty),
$$

The preliminary analyses above provide us the analytical form of the TC-NE value function, while the key is to make use of the ansatz (59) to transform the nonlocal PDE system (57) into a classical (local) ODE system (60) (and equivalently, an conventional integral system (63)). We can then again use the contraction mapping arguments to establish the local well-posedness of Equation (63). Moreover, to prove its solvability in an arbitrary large time interval, we shall show the boundedness of the solution of Equation (63) such that the extension procedure can be completed. The following proposition supplements the mathematical details of the above.

**Proposition 4.4.** Suppose that $v$, $w$, and $g$ are continuously differentiable, then there exists $\delta \in (0, T]$ such that the TIC SDG problem (41)–(42) with Equation (55) admits a closed-loop TC-NE point $(\bar{\alpha}, \bar{\xi})(s, y)$ given by Equation (56) and the corresponding TC-NE value function $V(s, y) = \bar{\psi}(s)y^\beta$ over $s \in [T - \delta, T]$. Moreover, if $w$ is a diagonal matrix and $v$ and $g$ satisfy Equations (A.56) and (A.57), then $\delta = T$, which implies that the TIC SDG problem (41)–(42) with Equation (55) is globally solvable.

**Remark 4.5.** For the condition (A.57), Yong (2012), Wei et al. (2017) have investigated the case where $v^{ab}(t, s) = v^{aa}(t, s)$ for $b \neq a$ and $a = 1, \ldots, m$. As they showed, the continuous differentiability of $v$ in $s$ can guarantee Equation (A.57) for this special case. Moreover, in contrast to Proposition 4.2 that provides the existence and uniqueness of solutions of equilibrium HJB system (47), Proposition 4.4 only promises the existence of solutions for the TIC SDG problem (41)–(42) with Equation (55). Since the equilibrium HJB system (58) cannot be covered by the current framework, we merely constructed one solution for the power utility model via the ansatz (59).

Our TIC SDG examples and results generalize the ones in the existing literature. Specifically, in the case of $m = 1$, the TIC SDG is reduced to the TIC stochastic control problem in Wei et al. (2017) with recursive utility functional and in Yong (2012) with nonrecursive one. Noteworthy is that the well-posedness results in Yong (2012), Wei et al. (2017) do not allow the diffusion to be controlled.
Moreover, when \( m = 1, \mathbf{v}, \mathbf{w}, \) and \( \mathbf{g} \) are all independent of \( t \), the problem is reduced to the TC case with recursive utility functional studied in Karoui et al. (2001). Based on these restrictions, if \( \mathbf{v} \) is constant and \( \mathbf{w} = 0 \), the examples are further reduced to the classical Merton problem in Merton (1971).

5 | FEYNMAN–KAC FORMULA FOR NONLOCAL PARABOLIC SYSTEMS

In this section, we provide a nonlocal version of the Feynman–Kac formula, which establishes a closed link between the solutions to a flow of FBSDEs in the multidimensional case and nonlocal second-order parabolic systems. All the proofs are deferred to Appendix A.

Before we present the formula, we reveal more properties of the solution to Equation (1). Like the classical theory of parabolic systems, the stronger conditions imposed to the nonlinearity \( \mathbf{F} \) and the given data \( \mathbf{g} \) suggest the higher regularity of the corresponding solutions of nonlocal higher-order systems.

**Lemma 5.1.** Let \( k \) and \( K \) be both non-negative integers satisfying \( k \leq K \). Suppose that \( \mathbf{F} \) is smooth and regular enough and \( \mathbf{g} \in \Omega_{[0,T]}^{2r+K+\alpha} \), then there exist \( \delta > 0 \) and a unique \( \mathbf{u} \) in \( \Delta[0,\delta] \times \mathbb{R}^d \) satisfying Equation (1) with \( \partial^k_y \mathbf{u} \in \Omega_{[0,\delta]}^{2r+\alpha} \) for all \( k \leq K \).

Next, to connect parabolic systems with the theory of FBSDEs, we consider a second-order backward nonlocal fully nonlinear system with \( r = 1 \) of the form:

\[
\begin{align*}
\mathbf{u}_s(t,s,y) + \mathbf{F}(t,s,y,\mathbf{u}(t,s,y),\mathbf{u}_y(t,s,y),\mathbf{u}_y(t,s,y),\mathbf{u}_yy(t,s,y),\mathbf{u}(s,s,y),\mathbf{u}_y(s,s,y),\mathbf{u}_yy(s,s,y)) &= 0, \\
\mathbf{u}(t,T,y) &= \mathbf{g}(t,y), \quad t_0 \leq t \leq s \leq T, \quad y \in \mathbb{R}^d,
\end{align*}
\]

where \( \mathbf{F} \) has enough regularities and \( t_0 \) is suitable in the sense that \( [t_0,T] \) is a subset of the time interval for the maximally defined solution of Equation (65). The following theorem reveals the relationship between the solutions to a nonlocal fully nonlinear second-order system and to a flow of 2FBSDEs (67).

**Theorem 5.2.** Suppose that \( \mathbf{F} \) has enough regularities, \( \sigma(s,y) \in C^{1,2}([t_0,T] \times \mathbb{R}^d) \), and \( \mathbf{g} \in \Omega_{[t_0,T]}^{3+\alpha} \). Then, Equation (65) admits a unique solution \( \mathbf{u}(t,s,y) \) that is first-order continuously differentiable in \( s \) and third-order continuously differentiable with respect to \( y \) in \( \nabla [t_0,T] \times \mathbb{R}^d \). Moreover, for any \( a = 1, \ldots, m \), let

\[
\begin{align*}
\mathbf{Y}^a(t,s) &:= \mathbf{u}^a(t,s,X(s)), \\
\mathbf{Z}^a(t,s) &:= (\sigma^\top \mathbf{u}^a_y)(t,s,X(s)), \\
\mathbf{Y}^a(t,s) &:= \left( \sigma^\top (\sigma^\top \mathbf{u}^a_y)_y \right)(t,s,X(s)), \\
\mathbf{A}^a(t,s) &:= D(\sigma^\top \mathbf{u}^a_y)(t,s,X(s)),
\end{align*}
\]

where \( (\sigma^\top \mathbf{u}^a_y)(t,s,y) = \sigma^\top (s,y)\mathbf{u}^a_y(t,s,y) \) and the operator \( \mathbf{D} \) is defined by

\[
\mathbf{D} \varphi^a = \varphi_s + \frac{1}{2} \sum_{i,j=1}^{d} (\sigma \sigma^\top)_{ij} \frac{\partial^2 \varphi^a}{\partial y_i \partial y_j} + \sum_{i=1}^{d} b_i \frac{\partial \varphi^a}{\partial y_i},
\]
then the family of random fields \((X(\cdot), Y(\cdot, \cdot), Z(\cdot, \cdot), \Gamma(\cdot, \cdot), A(\cdot, \cdot))\) is an adapted solution of the following flow of 2FBSDEs:

\[
X(s) = y + \int_{t_0}^{s} b(\tau, X(\tau))d\tau + \int_{t_0}^{s} \sigma(\tau, X(\tau))dW(\tau),
\]

\[
Y^a(t, s) = g^a(t, X(T)) + \int_{s}^{T} F^a(t, \tau, X(\tau), Y(t, \tau), Y(\tau, \tau), Z(t, \tau), Z(\tau, \tau), \Gamma(t, \tau), \Gamma(\tau, \tau))d\tau
\]

\[- \int_{s}^{T} (Z^a)^\top (t, \tau)dW(\tau),
\]

\[
Z^a(t, s) = Z^a(t, t_0) + \int_{t_0}^{s} A^a(t, \tau)d\tau + \int_{t_0}^{s} \Gamma^a(t, \tau)dW(\tau), \quad t_0 \leq t \leq s \leq T, \quad y \in \mathbb{R}^d,
\]

where \(F^a\) is defined by

\[
F^a(t, \tau, X(\tau), Y(t, \tau), Y(\tau, \tau), Z(t, \tau), Z(\tau, \tau), \Gamma(t, \tau), \Gamma(\tau, \tau)) = \overline{F}^a(t, \tau, X(\tau), u(t, \tau, X(\tau)), u_y(t, \tau, X(\tau)), u_{yy}(t, \tau, X(\tau)))
\]

\[
(68)
\]

with the definition of \(\overline{F}^a\)

\[
\overline{F}^a(t, \tau, y, u(t, \tau, y), u_y(t, \tau, y), u_{yy}(t, \tau, y), u_y(t, \tau, y), u_{yy}(t, \tau, y))
\]

\[
:= F^a(t, \tau, y, u(t, \tau, y), u_y(t, \tau, y), u_{yy}(t, \tau, y), u_y(t, \tau, y), u_{yy}(t, \tau, y))
\]

\[- \frac{1}{2} \sum_{i,j=1}^{d} \sigma \sigma^\top (\tau, y) \frac{\partial^2 u^a}{\partial y_i \partial y_j}(t, \tau, y) - \sum_{i=1}^{d} b_i(\tau, y) \frac{\partial u^a}{\partial y_i}(t, \tau, y).
\]

We make three important observations about the stochastic system \((67)\): (I) When the generator \(F\) is independent of diagonal terms, that is, \(Y(\tau, \tau), Z(\tau, \tau), \Gamma(\tau, \tau)\), the flow of FBSDEs \((67)\) is reduced to a family of 2FBSDEs parameterized by \(t\), which is exactly the 2FBSDEs in Kong et al. (2015) and equivalent to the ones in Cheridito et al. (2007) for any fixed \(t\); (II) Equation \((67)\) is more general than the systems in Wang and Yong (2019), Wang (2020), Hamaguchi (2021b), Lei and Pun (2023) since it allows for a nonlinearity of \((Y(t, \tau), Z(t, \tau), \Gamma(t, \tau))\) by increasing the dimensions and/or introducing an additional SDE of \((\Gamma, A)\) as well as diagonal terms \((Y(\tau, \tau), Z(\tau, \tau), \Gamma(\tau, \tau))\) in almost arbitrary way; (III) Theorem 5.2 shows how to solve the flow of multidimensional 2FBSDEs \((67)\) from the perspective of nonlocal systems. Inspired by Cheridito et al. (2007), Soner et al. (2011), the opposite implication of solutions (from 2FBSDEs to PDE) is likely valid by establishing the well-posedness of Equation \((67)\) in the theoretical framework of SDEs. However, it is beyond the scope of this paper, while we will prove the existence and uniqueness of Equation \((67)\) in our future works.

6 CONCLUSIONS

We provided the conditions on the nonlocal higher-order systems, under which the global well-posedness of the linear, quasilinear, fully nonlinear systems can be proved. The results are
significant for a general class of nonzero-sum TIC SDGs that we formulated and discussed in Section 2. Moreover, we present a nonlocal multidimensional version of a Feynman–Kac formula. It provides new insights into the studies of a flow of 2FBSDEs or 2BSVIEs.

We presented two immediate applications (SDG and SDE) drawing upon our main results from the PDE perspective. In fact, the study of systems of differential equations is crucial for developing other mathematical tools (in PDE), such as quasilinearization among many others, in a new environment (here, with nonlocality). The quasilinearization is a common technique in the classical (fully nonlinear) PDE problems. Specifically, under suitable regularity assumptions, we can differentiate the fully nonlinear equation of an unknown function $\varphi$ with respect to each state variable $y_i$ and yield an induced quasilinear system for $(\frac{\partial \varphi}{\partial y_1}, \ldots, \frac{\partial \varphi}{\partial y_d})$. Before taking advantage of the mathematical results of quasilinear systems, it is crucial to verify the equivalence between the original fully nonlinear equation for $\varphi$ and the induced quasilinear system for $(\varphi, \frac{\partial \varphi}{\partial y_1}, \ldots, \frac{\partial \varphi}{\partial y_d})$. The verification, however, requires the existence and uniqueness of (nonlocal) linear systems. Hence, our study for nonlocal linear systems can serve as a prerequisite for one to develop quasilinearization methods for nonlocal differential equations.

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DATA AVAILABILITY STATEMENT
Data sharing is not applicable to this paper as no dataset was generated or analyzed during this study.

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**APPENDIX A: PROOFS OF STATEMENTS**

*Proof of Proposition 3.1.* Let \( \mathbf{u}(t, s, y) = \mathbf{v}(T - t, T - s, y) \), then Equation (21) can be written as

\[
\begin{aligned}
&\mathbf{v}_{T-s}(T - t, T - s, y) + \mathbf{F}\left(t, s, y, (\partial\mathbf{v})_{|\theta| \leq 2 r}(T - t, T - s, y), (\partial_{\mathbf{v}})_{|\mathbf{v}| \leq 2 r}(T - s, T - s, y)\right) = 0, \\
&\mathbf{v}(T - t, 0, y) = \mathbf{g}(t, y), \quad 0 \leq t \leq s \leq T, \quad y \in \mathbb{R}^d.
\end{aligned}
\]

(A.1)

Next, we introduce \( t' = T - t \), \( s' = T - s \), and \( y' = y \). Then Equation (A.1) is equivalent to

\[
\begin{aligned}
&\mathbf{v}(t', s', y') = \mathbf{F}\left(T - t', T - s', y', (\partial_{\mathbf{v}})_{|\theta| \leq 2 r}(t', s', y'), (\partial_{\mathbf{v}})_{|\mathbf{v}| \leq 2 r}(s', s', y')\right), \\
&\mathbf{v}(t', 0, y') = \mathbf{g}(t' - s', y'), \quad 0 \leq s' \leq t' \leq T, \quad y' \in \mathbb{R}^d.
\end{aligned}
\]

By modifying the nonlinearity \( \mathbf{F} \) and the data \( \mathbf{g} \) but not really affecting their properties with respect to the time variables, the problem above can be further reformulated as a forward problem:

\[
\begin{aligned}
&\mathbf{v}_{t'}(t', s', y') = \mathbf{F}'\left(t', s', y', (\partial_{\mathbf{v}})_{|\theta| \leq 2 r}(t', s', y'), (\partial_{\mathbf{v}})_{|\mathbf{v}| \leq 2 r}(s', s', y')\right), \\
&\mathbf{v}(t', 0, y') = \mathbf{g}'(t', y'), \quad 0 \leq s' \leq t' \leq T, \quad y' \in \mathbb{R}^d,
\end{aligned}
\]

which completes the proof. \( \square \)
Proof of Lemma 3.2. The first claim is obvious. We prove the second one below. According to Equation (27), it is obvious that \( \frac{\partial u}{\partial t} \) satisfies

\[
\begin{align*}
\left( \frac{\partial u}{\partial t} \right)_s (t, s, y) &= \sum_{|l| \leq 2r, b \leq m} (A + B)_b^{al} (\cdot) \partial_j \left( \frac{\partial u}{\partial t} \right)_b (t, s, y) + \sum_{|l| \leq 2r, b \leq m} \left( \frac{\partial A}{\partial t} + \frac{\partial B}{\partial t} \right)_b^{al} (\cdot) \partial_j u^b (t, s, y), \\
- \sum_{|l| \leq 2r, b \leq m} \left( \frac{\partial B}{\partial t} \right)_b^{al} (\cdot) \int_s^t \partial_j v^b (\theta, s, y) d\theta - \sum_{|l| \leq 2r, b \leq m} B_b^{al} (\cdot) \partial_j v^b (t, s, y) + f_t^{al} (\cdot), \quad a = 1, \ldots, m,
\end{align*}
\]

which completes the proof.

By replacing \( v \) with \( \frac{\partial u}{\partial t} \) in the first equation of Equation (27), we have

\[
\begin{align*}
\left( \frac{\partial u}{\partial t} \right)_s (t, s, y) &= \sum_{|l| \leq 2r, b \leq m} (A + B)_b^{al} (\cdot) \partial_j u^b (t, s, y) + \sum_{|l| \leq 2r, b \leq m} B_b^{al} (\cdot) \int_s^t \partial_j \left( \frac{\partial u}{\partial t} \right)_b (\theta, s, y) d\theta + f_t^{al} (\cdot), \\
\left( \frac{\partial u}{\partial t} \right)_s (t, 0, y) &= g_t (y), \quad 0 \leq s \leq t \leq \delta, \quad y \in \mathbb{R}^d.
\end{align*}
\]

Proof of Theorem 3.3. We first adopt the Banach fixed point arguments to prove the local well-posedness of Equation (27) and then extend the local solution to the whole triangular time region \( \Delta[0, T] \). Method of contraction mapping. According to Equation (27), we first construct a mapping \( \Gamma \) from \( v \) to \( V \), where \( V \) is part of the solution \( (u, V) \) to

\[
\begin{align*}
\left( \frac{\partial u}{\partial t} \right)_s (t, s, y) &= \sum_{|l| \leq 2r, b \leq m} (A + B)_b^{al} (\cdot) \partial_j u^b (t, s, y) - \sum_{|l| \leq 2r, b \leq m} B_b^{al} (\cdot) \int_s^t \partial_j \left( \frac{\partial u}{\partial t} \right)_b (\theta, s, y) d\theta + f_t^{al} (\cdot), \\
\left( \frac{\partial v}{\partial t} \right)_s (t, s, y) &= \sum_{|l| \leq 2r, b \leq m} A_b^{al} (\cdot) \partial_j v^b (t, s, y) + \left( \frac{\partial A}{\partial t} + \frac{\partial B}{\partial t} \right)_b^{al} (\cdot) \partial_j u^b (t, s, y), \\
\left( u, v \right)_s (t, 0, y) &= (g_t, g_t) (y), \quad a = 1, \ldots, m, \quad 0 \leq s \leq t \leq \delta, \quad y \in \mathbb{R}^d.
\end{align*}
\]

The operator \( \Gamma (v) = V \) is defined in the set

\[
\mathcal{V} = \left\{ v (\cdot, \cdot, \cdot) \in C(\Delta[0, \delta] \times \mathbb{R}^d; \mathbb{R}^m) : \left\| v \right\|_{[0, \delta]}^{(2r+\alpha)} < \infty \right\}.
\]
Thanks to the theory of classical parabolic system parameterized by $t$, the operator $\mathbf{V} = \Gamma(\mathbf{v})$ is well-defined. Next, we are to prove that this mapping is a contraction, that is, for any $\mathbf{v}, \hat{\mathbf{v}} \in \mathcal{V}$, it holds that

$$[\Gamma(\mathbf{v}) - \Gamma(\hat{\mathbf{v}})]_{(2r+\alpha)}^{(0,\delta)} \leq \frac{1}{2}[\mathbf{v} - \hat{\mathbf{v}}]_{(2r+\alpha)}^{(0,\delta)}. \quad (A.4)$$

It is clear that

$$\begin{align*}
\left\{ \begin{array}{l}
(\mathbf{u} - \hat{\mathbf{u}})_s^a (t, s, y) = \sum_{|I| \leq 2r, b \leq m} (A + B)_b^a (\partial_t (\mathbf{u} - \hat{\mathbf{u}})^b (t, s, y) - B_b^a (\cdot) \int_s^t \partial_t (\mathbf{v} - \hat{\mathbf{v}})^b (\cdot, s, y) \, d\theta), \\
(\mathbf{V} - \hat{\mathbf{V}})_s^a (t, s, y) = \sum_{|I| \leq 2r, b \leq m} A_b^a (\cdot) \partial_t (\mathbf{V} - \hat{\mathbf{V}})^b (t, s, y) - \left( \frac{\partial b}{\partial t} \right)_b^a (\cdot) \int_s^t \partial_t (\mathbf{v} - \hat{\mathbf{v}})^b (\cdot, s, y) \, d\theta \\
+ (\frac{\partial A}{\partial t} + \frac{\partial B}{\partial t})_b^a (\cdot) \partial_t (\mathbf{u} - \hat{\mathbf{u}})^b (t, s, y), \\
(\mathbf{u} - \hat{\mathbf{u}}, \mathbf{V} - \hat{\mathbf{V}}) (t, 0, y) = (0, 0), \quad a = 1, \ldots, m, \quad 0 \leq s \leq t \leq \delta, \quad y \in \mathbb{R}^d.
\end{array} \right.
\end{align*}$$

(A.5)

By the classical theory of parabolic system (see Solonnikov, 1965, Theorem 4.10 or Ladyženskaja et al., 1968, Theorem 10.2), we have that for any fixed $t \in [0, \delta]$,

$$\left| (\mathbf{u} - \hat{\mathbf{u}})(t, \cdot, \cdot) \right|_{[0, \delta] \times \mathbb{R}^d}^{(2r+\alpha)} \leq C \sum_{|I| \leq 2r, b \leq m} \left| \int_s^t \partial_t (\mathbf{v} - \hat{\mathbf{v}})^b (\cdot, \cdot, \cdot) d\theta \right|_{[0, \delta] \times \mathbb{R}^d}^{(\alpha)}. \quad (A.6)$$

for some constant $C$, which is generic in this paper and its value could vary in different inequalities below. Then, similar to (Lei and Pun, 2023, Section 2), it holds that

$$\left| (\mathbf{u} - \hat{\mathbf{u}})(t, \cdot, \cdot) \right|_{[0, \delta] \times \mathbb{R}^d}^{(2r+\alpha)} \leq C \delta^{1 - \alpha} \cdot \sup_{t \in [0, \delta]} \left\{ \left| (\mathbf{v} - \hat{\mathbf{v}})(t, \cdot, \cdot) \right|_{[0, \delta] \times \mathbb{R}^d}^{(2r+\alpha)} \right\}. \quad (A.7)$$

Similarly, we also have a priori estimate

$$\left| (\mathbf{V} - \hat{\mathbf{V}})(t, \cdot, \cdot) \right|_{[0, \delta] \times \mathbb{R}^d}^{(2r+\alpha)} \leq C \delta^{1 - \alpha} \cdot \sup_{t \in [0, \delta]} \left\{ \left| (\mathbf{v} - \hat{\mathbf{v}})(t, \cdot, \cdot) \right|_{[0, \delta] \times \mathbb{R}^d}^{(2r+\alpha)} \right\}. \quad (A.8)$$

Here, the constant $C$ is independent of $\delta, \mathbf{g}, \mathbf{v},$ and $\hat{\mathbf{v}}$. Consequently, under the norm $[\cdot]_{[0, \delta]}^{(2r+\alpha)} := \sup_{t \in [0, \delta]} \left| \cdot \right|_{[0, \delta] \times \mathbb{R}^d}^{(2r+\alpha)}$, we achieve a contraction

$$[\Gamma(\mathbf{v}) - \Gamma(\hat{\mathbf{v}})]_{[0, \delta]}^{(2r+\alpha)} \leq \frac{1}{2}[\mathbf{v} - \hat{\mathbf{v}}]_{[0, \delta]}^{(2r+\alpha)}. \quad (A.9)$$

with a suitably small $\delta$.

**A contraction $\Gamma$ mapping $\mathcal{V}$ into itself.** On the other hand, we also have

$$[\mathbf{V}]_{[0, \delta]}^{(2r+\alpha)} = [\Gamma(\mathbf{v})]_{[0, \delta]}^{(2r+\alpha)} \leq C \left( [\mathbf{v}]_{[0, \delta]}^{(2r+\alpha)} + [\mathbf{f}]_{[0, \delta]}^{(\alpha)} + [\mathbf{g}]_{[0, \delta]}^{(2r+\alpha)} \right) < \infty. \quad (A.10)$$

Here, $C$ is a constant depending on $A$ and $B$. Therefore, it follows that $\mathbf{V} \in \Theta_{[0, \delta]}^{(2r+\alpha)}$. Consequently, $\Gamma$ is a contraction, mapping $\mathcal{V}$ into itself, and thus it has a unique fixed point $\mathbf{v}$ in $\Theta_{[0, \delta]}^{(2r+\alpha)}$ such that $\Gamma(\mathbf{v}) = \mathbf{v}$. Finally, the unique fixed point $\mathbf{v}(t, s, y)$ uniquely determines a function $\mathbf{u}(t, s, y)$.
via Equation (A.3). Moreover, given \( v \in \Theta_{[0,\delta]}^{(2r+\alpha)} \), it is clear that \([u]_{[0,\delta]}^{(2r+\alpha)} \leq \infty\) as well. Therefore, there exists a unique solution pair \((u,v) \in \Theta_{[0,\delta]}^{(2r+\alpha)} \times \Theta_{[0,\delta]}^{(2r+\alpha)}\) to Equation (25) in \(\Delta[0,\delta] \times \mathbb{R}^d\).

**Extension of solutions to the whole time (triangular) region.** We have proven that there exists a time horizon \(\delta_1 \in (0, T]\) such that the local well-posedness of \((u,v)\) holds in \(R_1 = \{ 0 \leq s \leq t \leq \delta_1 \}\) in Figure A.1. If \(\delta_1 = T\), the proof is completed. Otherwise, we begin the extension procedure of solutions of Equation (27). Due to the local well-posedness of Equation (27) over \(R_1\), we can determine the diagonal condition for \(s \in [0, \delta_1]\). Then the nonlocal Equations (25) and (26) reduce to classical systems with a parameter \(t\). Therefore, we can extend uniquely our solution \((u,v)\) from \(R_1\) to \(R_1 \cup R_2 = \{ 0 \leq s \leq \delta_1, s \leq t \leq T \}\) in Figure A.1. Subsequently, we acquire a new initial condition at \(s = \delta_1\) for \(t \in [\delta_1, T]\). Taking \(\delta_1\) as an initial time and \((u(t, \delta_1, y), v(t, \delta_1, y))\) as initial datum, one can extend the solution to a larger time intervals \(R_1 \cup R_2 \cup R_3\) as illustrated in Figure A.1. Hence, we can extend uniquely the solution from \(\Delta[0, \delta_1]\) to \(\Delta[0, \delta_2]\), and then \(R_1 \cup R_2 \cup R_3 \cup R_4\). Considering the facts that \(\Gamma\) is defined in the whole space \(\Theta_{[a,b]}^{(2r+\alpha)}\) and the constant \(C\) in front of Equations (A.6) and (A.7) only depend on \(A\) and \(B\) instead of the norms of \(g, v\), and \(\hat{v}\), we can always construct a \(\frac{1}{2}\)-contraction mapping to find the solution in a larger time region. Consequently, the extension procedure could be repeated up to a global solution pair \((u,v) \in \Theta_{[0,T]}^{(2r+\alpha)} \times \Theta_{[0,T]}^{(2r+\alpha)}\).

**Proof of Theorem 3.4.** As Lemma 3.2 shows, the first component \(u\) of the unique solution \((u,v)\) of Equation (27) solves the nonlocal linear equation (25) in \(\Delta[0,T] \times \mathbb{R}^d\). By noting of \(v = u_t\) in Equation (27), it is clear that \(u \in \Omega_{[0,T]}^{(2r+\alpha)}\) thanks to \((u,u_t) \in \Theta_{[0,T]}^{(2r+\alpha)} \times \Theta_{[0,T]}^{(2r+\alpha)}\). Moreover, by Equation (27), we have

\[
\sum_{a \leq m} |u^a(t, \cdot)|_{[0,T] \times \mathbb{R}^d}^{(2r+\alpha)} \leq C \left( \delta^{1-\frac{\alpha}{2r}} \sup_{t \in [0,\delta]} \sum_{a \leq m} |u^a(t, \cdot)|_{[0,T] \times \mathbb{R}^d}^{(2r+\alpha)} + \sup_{t \in [0,\delta]} \sum_{a \leq m} |f^a(t, \cdot)|_{[0,T] \times \mathbb{R}^d}^{(\alpha)} \right) + \sup_{t \in [0,\delta]} \sum_{a \leq m} |g^a(t, \cdot)|_{[0,T] \times \mathbb{R}^d}^{(2r+\alpha)}
\]

(A.11)
and

\[ \sum_{a \leq m} |u^a(t, \cdot, \cdot)|^{(2r+\alpha)}_{[0, t] \times \mathbb{R}^d} \leq C \left( \sum_{a \leq m} |u^a(t, \cdot, \cdot)|^{(2r+\alpha)}_{[0, t] \times \mathbb{R}^d} + \delta^{1 - \frac{\alpha}{2r}} \sup_{t \in [0, \delta]} \sum_{a \leq m} |u^a(t, \cdot, \cdot)|^{(2r+\alpha)}_{[0, t] \times \mathbb{R}^d} \right) + \sup_{t \in [0, \delta]} \sum_{a \leq m} |f^a(t, \cdot, \cdot)|^{(\alpha)}_{[0, t] \times \mathbb{R}^d} + \sup_{t \in [0, \delta]} \sum_{a \leq m} g^a(t, \cdot, \cdot)|^{(2r+\alpha)}_{\mathbb{R}^d} \right). \]  \tag{A.12}

Consequently, for a small enough $\delta$, it holds

\[ \sum_{a \leq m} \left\{ |u^a(t, \cdot, \cdot)|^{(2r+\alpha)}_{[0, t] \times \mathbb{R}^d} + |u^a(t, \cdot, \cdot)|^{(2r+\alpha)}_{[0, t] \times \mathbb{R}^d} \right\} \leq \sup_{t \in [0, \delta]} \sum_{a \leq m} |u^a(t, \cdot, \cdot)|^{(2r+\alpha)}_{[0, t] \times \mathbb{R}^d} + \sup_{t \in [0, \delta]} \sum_{a \leq m} |f^a(t, \cdot, \cdot)|^{(\alpha)}_{[0, t] \times \mathbb{R}^d} \right) + \sup_{t \in [0, \delta]} \sum_{a \leq m} \left( |g^a(t, \cdot, \cdot)|^{(2r+\alpha)}_{\mathbb{R}^d} + |g^a(t, \cdot, \cdot)|^{(2r+\alpha)}_{\mathbb{R}^d} \right), \]

which leads to a Schauder prior estimate of local solutions of Equation (25) in $\Delta[0, \delta] \times \mathbb{R}^d$

\[ \|u\|^{(2r+\alpha)}_{[0, \delta]} \leq C \left( \|f\|^{(\alpha)}_{[0, t]} + \|g\|^{(2r+\alpha)}_{[0, \delta]} \right). \]  \tag{A.13}

Now, we have established a Schauder prior estimate (A.13) of solutions of Equation (25) in a short-time region $\Delta[0, \delta]$. Next, we will show that it still holds for the case $\delta = T$.

First, given the well-posedness of Equation (25) over $R_1$, we consider a classical PDE

\[ \left\{ \begin{array}{l}
 u^a_\tau(t, s, y) = \sum_{|l| \leq 2r, b \leq m} A^a(t, s, y) \partial_l u^b(t, s, y) + \eta^a(t, s, y), \\
 u(t, 0, y) = g(t, y),
 \end{array} \right. \]  \tag{A.14}

where $\eta^a(t, s, y) := \sum_{|l| \leq 2r, b \leq m} B^a(t, s, y) \partial_l u^b(s, s, y) + f(t, s, y)$. By the classical theory of parabolic system (see Solonnikov, 1965, Theorem 4.10 or Ladyženskaja et al., 1968, Theorem 10.2), Equation (A.14) admits a unique solution in $(t, s, y) \in R_2 \times \mathbb{R}^d$. Moreover, for any $t \in [\delta_1, T]$, we have

\[ |u(t, s, \cdot)|^{(2r+\alpha)}_{[0, \delta_1] \times \mathbb{R}^d} \leq C \left( \sum_{|l| \leq 2r, b \leq m} B^a(t, s, y) \partial_l u^b(s, s, y) + f(t, s, y) \right)^{(2r+\alpha)}_{\mathbb{R}^d} \leq C \left( \sum_{|l| \leq 2r, b \leq m} B^a(t, s, y) \partial_l u^b(s, s, y) + f(t, s, y) \right)^{(2r+\alpha)}_{\mathbb{R}^d} \right) + \sup_{t \in [0, \delta]} \sum_{a \leq m} \left( |g^a(t, \cdot, \cdot)|^{(2r+\alpha)}_{\mathbb{R}^d} + |g^a(t, \cdot, \cdot)|^{(2r+\alpha)}_{\mathbb{R}^d} \right), \]

the second inequality of which comes from Equation (A.13). After taking supremum with respect to $t$, it implies directly that $\|u\|^{(2r+\alpha)}_{R_1 \cup R_2}$ (i.e., $\|u\|^{(2r+\alpha)}_{[0, T] \times [0, t \wedge \delta_1]}$) is bounded by $\|f\|^{(\alpha)}_{[0, T]}$ and $\|g\|^{(2r+\alpha)}_{[0, T]}$.

The notation of $\| \cdot \|^{(2r+\alpha)}_{R_1 \cup R_2}$ means that the supremum with respect to time pair $(t, s)$ is taken over $R_1 \cup R_2$.

Next, if we update the initial condition with $g'(t, y) := u(t, \delta_1, y)$, then it is clear that

\[ \|g'\|^{(2r+\alpha)}_{[\delta_1, T]} \leq C \left( \sum_{|l| \leq 2r, b \leq m} B^a(t, s, y) \partial_l u^b(s, s, y) + f(t, s, y) \right)^{(2r+\alpha)}_{\mathbb{R}^d} \leq C \left( \sum_{|l| \leq 2r, b \leq m} B^a(t, s, y) \partial_l u^b(s, s, y) + f(t, s, y) \right)^{(2r+\alpha)}_{\mathbb{R}^d} \right) + \sup_{t \in [0, \delta]} \sum_{a \leq m} \left( |g^a(t, \cdot, \cdot)|^{(2r+\alpha)}_{\mathbb{R}^d} + |g^a(t, \cdot, \cdot)|^{(2r+\alpha)}_{\mathbb{R}^d} \right), \]
Hence, Theorem 3.3 tells us that the problem

\[
\begin{align*}
\mathbf{u}_s(t,s,y) &= (L\mathbf{u})(t,s,y) + f(t,s,y), \\
\mathbf{u}(t,\delta, y) &= g'(t,y), \\
\delta_1 &\leq s \leq t \leq T, \quad y \in \mathbb{R}^d.
\end{align*}
\] (A.15)

admits a unique solution \( \mathbf{u} \in \Omega^{(2r+\alpha)}_{[\delta_1, \delta_2]} \) and there exists a constant \( \delta_2 \in (\delta_1, T] \) such that

\[
\|\mathbf{u}\|^{(2r+\alpha)}_{[\delta_1, \delta_2]} \leq C \left( \|f\|^{(\alpha)}_{[\delta_1, \delta_2]} + \|g'\|^{(2r+\alpha)}_{[\delta_1, \delta_2]} \right) \leq C \left( \|f\|^{(\alpha)}_{[0,T]} + \|g\|^{(2r+\alpha)}_{[0,T]} \right). \] (A.16)

Similarly, we can argue that \( \|\mathbf{u}\|^{(2r+\alpha)}_{[\delta_n, T] \times [\delta_n, T]} \leq C \left( \|f\|^{(\alpha)}_{[0,T]} + \|g\|^{(2r+\alpha)}_{[0,T]} \right) \), hence, for each extension from \((t, s) \in [\delta_n, T] \times [\delta_n, \tau] \) into \((t, s) \in [\delta_{n+1}, T] \times [\delta_{n+1}, \tau] \), it holds that

\[
\|\mathbf{u}\|^{(2r+\alpha)}_{[\delta_{n+1}, T] \times [\delta_{n+1}, \tau]} \leq C \left( \|f\|^{(\alpha)}_{[0,T]} + \|g\|^{(2r+\alpha)}_{[0,T]} \right), \quad n = 0, 1, 2, \ldots, N, \] (A.17)

where \( \delta_0 = 0 \) and \( \delta_{N+1} = T \). Furthermore, it is clear that \( N \) is finite and determined only by \( A \) and \( B \) according to Equations (A.6), (A.7), (A.11), and (A.12).

Subsequently, for any \( t \in [0, T] \), \( 0 \leq s \leq s' \leq t \leq T \), and \( y \in \mathbb{R}^d \), we assume that \( s \in [\delta_n, \delta_{n+1}] \) and \( s' \in [\delta_m, \delta_{m+1}] \) for \( 0 \leq n < m \leq N \) without loss of generality. Then, it follows that

\[
\sup_{0 \leq s < s' \leq t} \frac{|D_i^s D_j^s \mathbf{u}(t,s,y) - D_i^s D_j^{s'} \mathbf{u}(t,s',y)|}{|s-s'|^{l-2r-i-j}} \leq \sum_{i<k<j} \frac{|D_i^s D_j^s \mathbf{u}(t,\delta_k,y) - D_i^s D_j^{s'} \mathbf{u}(t,\delta_{k+1},y)|}{|\delta_k - \delta_{k+1}|^{l-2r-i-j}} + 
\sum_{i<k<j} \frac{|D_i^s D_j^s \mathbf{u}(t,\delta_k,y) - D_i^s D_j^{s'} \mathbf{u}(t,\delta_{k+1},y)|}{|\delta_k - \delta_{k+1}|^{l-2r-i-j}} \leq C \left( \|f\|^{(\alpha)}_{[0,T]} + \|g\|^{(2r+\alpha)}_{[0,T]} \right) \]

holds for \( 0 < l - 2r - i - j < 2r \). Here, the constant \( C \) only depends on \( \alpha \), \( T \), \( A \), and \( B \). In addition, whenever \((t, s), (t, s') \in [\delta_n, T] \times [\delta_n, \tau] \), the inequality is obvious from Equation (A.17). Consequently, according to the definition of \( \|\cdot\|^{(l)}_{[a,b]} \), we have

\[
\|\mathbf{u}\|^{(2r+\alpha)}_{[0,T]} \leq C \left( \|f\|^{(\alpha)}_{[0,T]} + \|g\|^{(2r+\alpha)}_{[0,T]} \right).
\]

Finally, by considering the nonlocal system satisfied by the difference between \( \mathbf{u} \) and \( \hat{\mathbf{u}} \), we can similarly derive the stability analysis (29). \( \square \)
Proof of Theorem 3.5. Overall speaking, we search for the solution of nonlocal fully nonlinear system as a fixed point of the operator $\Lambda$, defined in

$$
\mathcal{U} = \left\{ u \in \Omega^{(2r+\alpha)}_{[0,\delta]} : u(t, 0, y) = g(t, y), \| u - g \|^{(2r+\alpha)}_{[0,\delta]} \leq R \right\}
$$

for a constant $R$, by $\Lambda(u) = U$, where $U$ is the solution of

$$
\begin{align*}
U_s(t, s, y) &= L_0 U + F(t, s, y, (\partial_I u)|_{\| \cdot \| \leq 2r}(t, y), (\partial_I u)|_{\| \cdot \| \leq 2r}(s, y)) - L_0 u, \\
U(t, 0, y) &= g(t, y), \quad 0 \leq s \leq t \leq \delta, \quad y \in \mathbb{R}^d
\end{align*}
$$

(A.18)

in which

$$
(L_0 u)^a(t, s, y) = \sum_{|l| \leq 2r, b \leq m} \partial_l F^a_b(t, 0, y, \theta_0(t, y)) \cdot \partial_l u^b(t, s, y)
$$

$$
+ \sum_{|l| \leq 2r, b \leq m} \partial_l F^a_b(t, 0, y, \theta_0(t, y)) \cdot \partial_l g^b(s, s, y)
$$

(A.19)

with $\theta_0(t, y) := ((\partial_I g)|_{\| \cdot \| \leq 2r}(t, y), (\partial_I g)|_{\| \cdot \| \leq 2r}(0, y))$. Note that $\partial_l F^a_b(t, 0, y, \theta_0(t, y))$ is meant to be evaluated at $(t, 0, y, \theta_0(t, y))$, that is, $(t, 0, y, (\partial_I g)|_{\| \cdot \| \leq 2r}(t, y), (\partial_I g)|_{\| \cdot \| \leq 2r}(0, y))$. Similarly, the same convention applies to $\partial_l F^a_b(t, 0, y, \theta_0(t, y))$. Generally speaking, there are three conditions imposed to $\delta$ and $R$:

1. To validate the form $F(t, s, y, (\partial_I u)|_{\| \cdot \| \leq 2r}(t, y), (\partial_I u)|_{\| \cdot \| \leq 2r}(s, s, y))$, it requires that the range of various derivatives of $u$ in $\mathcal{U}$ is contained in $B(\mathcal{Z}, R_0)$. Specifically, since

$$
\sup_{\Delta[0,\delta] \times \mathbb{R}^d} \sum_{|l| \leq 2r, b \leq m} \left( |\partial_l u^b(t, s, y) - \partial_l g^b(t, y)| + |\partial_l u^b(s, s, y) - \partial_l g^b(s, y)| \right) \leq C\delta^\alpha R, \quad (A.20)
$$

it should hold that $C\delta^\alpha R \leq R_0/2$;

2. To ensure that $\Lambda$ is a $\frac{1}{2}$-contraction, we need to show that

$$
||\Lambda(u) - \Lambda(\hat{u})||^{(2r+\alpha)}_{[0,\delta]} \leq C(R)\delta^\alpha ||u - \hat{u}||^{(2r+\alpha)}_{[0,\delta]}, \quad (A.21)
$$

which requires $C(R)\delta^\alpha \leq \frac{1}{2}$;

3. Before applying the Banach fixed point theorem, we need to prove that $\Lambda$ maps $\mathcal{U}$ into itself, that is, $||\Lambda(u) - g||^{(2r+\alpha)}_{[0,\delta]} \leq R$. Hence, $R$ should be suitably large such that $||\Lambda(g) - g||^{(2r+\alpha)}_{[0,\delta]} \leq R/2$.

First, it is clear that the range of the derivatives of $u$ in $\mathcal{U}$ is contained in $B(\mathcal{Z}, R_0)$ because of Equation (A.20) as well as the fact that the range of $g$ is contained in $B(\mathcal{Z}, R_0/2)$.

Next, we are to show that the operator $\Lambda(u)$ is a contraction defined in $\mathcal{U}$, that is, for any $u$, $\hat{u} \in \mathcal{U}$, Equation (A.21) holds. Let us consider the equation for $U - \hat{U} := \Lambda(u) - \Lambda(\hat{u})$, satisfying
\[
(U - \hat{U})_s(t, s, y) = L_0(U - \hat{U}) + F(t, s, y, (\partial_t u)_{|t| \leq \sigma}(t, s, y), (\partial_t u)_{|t| \leq \sigma}(s, s, y)) - F(t, s, y, (\partial_t \hat{u})_{|t| \leq \sigma}(t, s, y), (\partial_t \hat{u})_{|t| \leq \sigma}(s, s, y)) - L_0(u - \hat{u}), \quad (A.22)
\]

\[
(U - \hat{U})(t, 0, y) = 0, \quad 0 \leq s \leq t \leq \delta, \quad y \in \mathbb{R}^d.
\]

According to the prior estimates (28) and (29) of nonlocal linear system, we have

\[
\|U - \hat{U}\|^{(2r + \alpha)}_{[0, \delta]} \leq C\|\varphi\|^{(\alpha)}_{[0, \delta]}, \quad (A.23)
\]

where the constant \(C\) is independent of \(\delta\) and the inhomogeneous term \(\varphi\) is given by

\[
\varphi(t, s, y) = F(t, s, y, (\partial_t u)_{|t| \leq \sigma}(t, s, y), (\partial_t u)_{|t| \leq \sigma}(s, s, y)) - F(t, s, y, (\partial_t \hat{u})_{|t| \leq \sigma}(t, s, y), (\partial_t \hat{u})_{|t| \leq \sigma}(s, s, y)) - L_0(u - \hat{u}),
\]

whose \(a\)th entry, \(\varphi^a(t, s, y)\) for \(a = 1, \ldots, m\), admits an integral representation:

\[
\int_0^1 \frac{d}{d\sigma} F^a(t, s, y, \theta_a(t, s, y))d\sigma - (L_0(u - \hat{u}))^a
\]

\[
= \int_0^1 \sum_{|l| \leq 2r, b \leq m} \partial_l F^a_b(t, s, y, \theta_a(t, s, y)) \cdot (\partial_l u^b(t, s, y) - \partial_l \hat{u}^b(t, s, y))d\sigma
\]

\[
+ \int_0^1 \sum_{|l| \leq 2r, b \leq m} \partial_l \bar{F}^a_b(t, s, y, \theta_a(t, s, y)) \cdot (\partial_l u^b(s, s, y) - \partial_l \hat{u}^b(s, s, y))d\sigma - (L_0(u - \hat{u}))^a
\]

\[
= \int_0^1 \sum_{|l| \leq 2r, b \leq m} (\partial_l F^a_b(t, s, y, \theta_a(t, s, y)) - \partial_l \bar{F}^a_b(t, 0, y, \theta_0(t, y))) \cdot \partial_l (u - \hat{u})^b(t, s, y)d\sigma
\]

\[
+ \int_0^1 \sum_{|l| \leq 2r, b \leq m} (\partial_l \bar{F}^a_b(t, s, y, \theta_a(t, s, y)) - \partial_l \bar{F}^a_b(t, 0, y, \theta_0(t, y))) \cdot \partial_l (u - \hat{u})^b(s, s, y)d\sigma,
\]

(A.24)

in which \(\theta_a(t, s, y) := \sigma((\partial_t u)_{|t| \leq \sigma}(t, s, y), (\partial_t u)_{|t| \leq \sigma}(s, s, y)) + (1 - \sigma)((\partial_t \hat{u})_{|t| \leq \sigma}(t, s, y), (\partial_t \hat{u})_{|t| \leq \sigma}(s, s, y)).\)

In order to obtain \(\|\varphi\|^{(\alpha)}_{[0, \delta]}\) we need to estimate \(|\varphi^a(t, \cdot, \cdot)|^{(\alpha)}_{[0, t] \times \mathbb{R}^d}\) and \(|\varphi^a(t, \cdot, \cdot)|^{(\alpha)}_{[0, t] \times \mathbb{R}^d}\) for any \(t \in [0, \delta]\) and \(a = 1, \ldots, m\).

Estimates of \(|\varphi^a(t, \cdot, \cdot)|^{(\alpha)}_{[0, t] \times \mathbb{R}^d}\). In the investigation of the difference \(|\varphi^a(t, s, y) - \varphi^a(t, s', y)|\) for any \(0 \leq s \leq s' \leq t \leq \delta \leq T\) and \(y \in \mathbb{R}^d\), it is convenient to add and subtract

\[
= \int_0^1 \sum_{|l| \leq 2r, b \leq m} \partial_l F^a_b(t, s', y, \theta_a(t, s', y)) \cdot \partial_l (u - \hat{u})^b(t, s, y)d\sigma
\]

\[
+ \int_0^1 \sum_{|l| \leq 2r, b \leq m} \partial_l \bar{F}^a_b(t, s', y, \theta_a(t, s', y)) \cdot \partial_l (u - \hat{u})^b(s, s, y)d\sigma.
\]
Subsequently, we ought to estimate

\[
|\partial_t F_b^a(t, s, y, \theta(t, s, y)) - \partial_t F_b^a(t, s', y, \theta(t, s', y))|,
|\partial_s F_b^a(t, s, y, \theta(t, s, y)) - \partial_s F_b^a(t, s', y, \theta(t, s', y))|,
|\partial_s F_b^a(t, s', y, \theta(t, s', y)) - \partial_t F_b^a(t, 0, y, \theta_0(t, y))|,
\]

where \(|l| \leq 2r\) and \(a, b = 1, \ldots, m\). Note that

\[
|\partial_t F_b^a(t, s, y, \theta(t, s, y)) - \partial_t F_b^a(t, s', y, \theta(t, s', y))| \leq K(s' - s)^{\frac{\alpha}{2}} + \sum_{b \leq m} \left(|u^b(t, \cdot, \cdot)|_{[0,t] \times \mathbb{R}^d}^{(2r+\alpha)} (s'' - s)^{\frac{\alpha}{2}} + \sup_{\mathbb{R}^d} |u^b(t, \cdot, \cdot)|_{[0,t] \times \mathbb{R}^d}^{(2r+\alpha)} (s'' - s)^{\frac{\alpha}{2}}
\]

and

\[
|\partial_s F_b^a(t, s', y, \theta(t, s', y)) - \partial_t F_b^a(t, 0, y, \theta_0(t, y))| \leq K(s' - 0)^{\frac{\alpha}{2}} + \sum_{b \leq m} \left(|u^b(t, \cdot, \cdot)|_{[0,t] \times \mathbb{R}^d}^{(2r+\alpha)} (s'' - s)^{\frac{\alpha}{2}} + \sup_{\mathbb{R}^d} |g^b(t, \cdot, \cdot)|_{[2r+\alpha]} (s'' - s)^{\frac{\alpha}{2}}
\]

where \(L > 0\) is a constant, which can be different from line to line and the subscripts of \(C\) are to represent different constant values within the derivation. In a similar manner, we can obtain

\[
|\partial_s F_b^a(t, s, y, \theta(t, s, y)) - \partial_s F_b^a(t, s', y, \theta(t, s', y))| \leq C_3(R)(s' - s)^{\frac{\alpha}{2}},
|\partial_s F_b^a(t, s', y, \theta(t, s', y)) - \partial_t F_b^a(t, 0, y, \theta_0(t, y))| \leq C_4(R)\delta^{\frac{\alpha}{2}}.
\]
Consequently, it holds that

$$\left| \Phi^a(t, s, y) - \Phi^a(t, s', y) \right|$$

\[ \leq \int_0^1 \left[ \sum_{|l| \leq 2r, b \leq m} \left| \partial_l F_b^a(t, s, y, \Theta_\sigma(t, s, y)) - \partial_l F_b^a(t, s', y, \Theta_\sigma(t, s', y)) \right| \cdot \left| \partial_l (u - \hat{u})^b(t, s, y) \right| d\sigma \]

\[ + \int_0^1 \left[ \sum_{|l| \leq 2r, b \leq m} \left| \partial_l \bar{F}_b^a(t, s, y, \Theta_\sigma(t, s, y)) - \partial_l \bar{F}_b^a(t, s', y, \Theta_\sigma(t, s', y)) \right| \cdot \left| \partial_l (u - \hat{u})^b(s, s, y) \right| d\sigma \]

\[ + \int_0^1 \left[ \sum_{|l| \leq 2r, b \leq m} \left| \partial_l F_b^a(t, s', y, \Theta_\sigma(t, s', y)) - \partial_l F_b^a(t, 0, y, \Theta_0(t, y)) \right| \left| \partial_l (u - \hat{u})^b(t, s, y) \right| \]

\[ - \partial_l (u - \hat{u})^b(t, s', y) \right| d\sigma \]

\[ + \int_0^1 \left[ \sum_{|l| \leq 2r, b \leq m} \left| \partial_l \bar{F}_b^a(t, s', y, \Theta_\sigma(t, s', y)) - \partial_l \bar{F}_b^a(t, 0, y, \Theta_0(t, y)) \right| \left| \partial_l (u - \hat{u})^b(s, s, y) \right| d\sigma \]

\[ - \partial_l (u - \hat{u})^b(s', s', y) \right| d\sigma \]

\[ \leq C_1(R)(s' - s)^\alpha \frac{a}{\delta_\sigma^\alpha} \sum_{b \leq m} \left| \partial_l (u - \hat{u})^b(t, \cdot, \cdot) \right|^{(\alpha)}_{[0, t] \times \mathbb{R}^d} \]

\[ + C_2(R)\delta_\sigma^\alpha (s' - s)^\alpha \sum_{b \leq m} \left| \partial_l (u - \hat{u})^b(t, \cdot, \cdot) \right|^{(\alpha)}_{[0, t] \times \mathbb{R}^d} \]

\[ + C_3(R)(s' - s)^\alpha \delta_\sigma^\alpha \sum_{b \leq m} \left| \partial_l (u - \hat{u})^b(s, \cdot, \cdot) \right|^{(\alpha)}_{[0, s] \times \mathbb{R}^d} \]

\[ + C_4(R)\delta_\sigma^\alpha (s' - s)^\alpha \sum_{b \leq m} \left( \sup_{\sigma \in (s, s')} \left| \partial_l (u - \hat{u})^b(\sigma, \cdot, \cdot) \right|^{(\alpha)}_{[0, \inf] \times \mathbb{R}^d} + \left| \partial_l (u - \hat{u})^b(\cdot, \cdot, \cdot) \right|^{(\alpha)}_{[0, s'] \times \mathbb{R}^d} \right) \]

\[ \leq C_5(R)\delta_\sigma^\alpha (s' - s)^\alpha \left\| u - \hat{u} \right\|^{(2r + \alpha)}_{[0, \delta]}, \]

which implies the following by noting that $\Phi(t, 0, y) \equiv 0$,

$$\left| \Phi^a(t, \cdot, \cdot) \right|_{[0, t] \times \mathbb{R}^d} \leq C_5(R)\delta_\sigma^\alpha \left\| u - \hat{u} \right\|^{(2r + \alpha)}_{[0, \delta]}.$$  \hspace{1cm} (A.26)

To estimate $|\Phi^a(t, s, y) - \Phi^a(t, s', y)|$, it is convenient to add and subtract

$$\int_0^1 \left[ \sum_{|l| \leq 2r, b \leq m} \left( \partial_l F_b^a(t, s, y', \Theta_\sigma(t, s, y')) - \partial_l F_b^a(t, 0, y', \Theta_0(t, y')) \right) \cdot \partial_l (u - \hat{u})^b(t, s, y) d\sigma \right.$$

$$\left. + \int_0^1 \left[ \sum_{|l| \leq 2r, b \leq m} \left( \partial_l \bar{F}_b^a(t, s, y', \Theta_\sigma(t, s, y')) - \partial_l \bar{F}_b^a(t, 0, y', \Theta_0(t, y')) \right) \cdot \partial_l (u - \hat{u})^b(s, s, y) d\sigma \right.$$
Note that

\[
\left| \partial_t F^a_b(t, s, y, \Theta_c(t, s, y)) - \partial_t F^a_b(t, s, y', \Theta_c(t, s, y')) \right| + \left| \partial_F^{(a)}(t, 0, y, \Theta_0(t, y)) - \partial_F^{(a)}(t, 0, y', \Theta_0(t, y')) \right| \\
\leq 2K|y - y'|^{\alpha} + L|\partial_t \Theta_c(t, s, y) - \Theta_c(t, s, y')| + L|\partial_t \Theta(t, y) - \Theta_0(t, y')|
\]

\[
\leq 2K|y - y'|^{\alpha} + L|y - y'|^{\alpha} \sum_{b \leq m} \left( |u^b(t, \cdot, \cdot)|^{(2r+\alpha)}_{[0,t] \times \mathbb{R}^d} + |\hat{u}^b(t, \cdot, \cdot)|^{(2r+\alpha)}_{[0,t] \times \mathbb{R}^d} + \left| g^b(t, \cdot) \right|^{(2r+\alpha)}_{[0,t] \times \mathbb{R}^d} \right) \right)
\]

\[
\left( K + L \left( \left\| u \right\|^{(2r+\alpha)}_{[0,t]} + \left\| \hat{u} \right\|^{(2r+\alpha)}_{[0,t]} + \left\| g \right\|^{(2r+\alpha)}_{[0,t]} \right) \right) |y - y'|^{\alpha} \leq C_6(R) |y - y'|^{\alpha},
\]

and for every \( y \in \mathbb{R}^d \),

\[
\left| \partial_t F^a_b(t, s, y, \Theta_c(t, s, y)) - \partial_t F^a_b(t, 0, y, \Theta_0(t, y)) \right| \leq K(s - 0)^{\frac{\alpha}{2}} + L|\partial_t \Theta_c(t, s, y) - \Theta_0(t, y)|
\]

\[
\leq K(s - 0)^{\frac{\alpha}{2}} + L \sum_{b \leq m} \left( |(u - g)^b(t, \cdot, \cdot)|^{(2r+\alpha)}_{[0,t] \times \mathbb{R}^d} + \sup_{\tilde{s} \in (0,s)} \left| g^b(\tilde{s}, \cdot) \right|^{(2r+\alpha)}_{\mathbb{R}^d} \right) \right)
\]

\[
\leq \left( K + L \left( \left\| u - g \right\|^{(2r+\alpha)}_{[0,t]} + \left\| \hat{u} - g \right\|^{(2r+\alpha)}_{[0,t]} + \left\| g \right\|^{(2r+\alpha)}_{[0,t]} \right) \right) (s - 0)^{\frac{\alpha}{2}} \leq C_7(R) \delta^{\frac{\alpha}{2}}.
\]

Similarly, we also have

\[
\left| \partial_F^{(a)}(t, 0, y, \Theta_0(t, y)) - \partial_F^{(a)}(t, 0, y', \Theta_0(t, y')) \right| \leq C_8(R) |y - y'|^{\alpha},
\]

\[
\left| \partial_t \overline{F}_b(t, s, y, \Theta_c(t, s, y)) - \partial_t \overline{F}_b(t, 0, y, \Theta_0(t, y)) \right| \leq C_9(R) \delta^{\frac{\alpha}{2}}.
\]

Hence, we have

\[
\left| \varphi^a(t, s, y) - \varphi^a(t, s, y') \right|
\]

\[
\leq C_6(R) |y - y'|^{\alpha} \delta^{\frac{\alpha}{2}} \sum_{b \leq m} \left| \partial_t (u - \hat{u})^b(t, \cdot, \cdot) \right|^{(a)}_{[0,t] \times \mathbb{R}^d} + C_7(R) \delta^{\frac{\alpha}{2}} |y - y'|^{\alpha} \sum_{b \leq m} \left| \partial_t (u - \hat{u})^b(t, \cdot, \cdot) \right|^{(a)}_{[0,t] \times \mathbb{R}^d}
\]

\[
+ C_8(R) |y - y'|^{\alpha} \delta^{\frac{\alpha}{2}} \sum_{b \leq m} \left| \partial_t (u - \hat{u})^b(s, \cdot, \cdot) \right|^{(a)}_{[0,t] \times \mathbb{R}^d} + C_9(R) \delta^{\frac{\alpha}{2}} |y - y'|^{\alpha} \sum_{b \leq m} \left| \partial_t (u - \hat{u})^b(s, \cdot, \cdot) \right|^{(a)}_{[0,t] \times \mathbb{R}^d}
\]

\[
\leq C_{10}(R) \delta^{\frac{\alpha}{2}} |y - y'|^{\alpha} \left\| u - \hat{u} \right\|^{(2r+\alpha)}_{[0,t]}.
\]
From Equations (A.25)-(A.27), for any \( t \in [0, \delta] \) and \( a = 1, \ldots, m \), we obtain

\[
|\varphi^a(t, \cdot, \cdot)|_{\[0, t\] \times \mathbb{R}^d}^{(\alpha)} \leq C_{11}(R)\delta^{\frac{\alpha}{2r}} \|u - \hat{u}\|_{\{2r+\alpha\}}^{\{2r+\alpha\}}. \tag{A.28}
\]

**Estimates of \( |\varphi^a(t, \cdot, \cdot)|_{\[0, t\] \times \mathbb{R}^d}^{(\alpha)} \).** We now analyze the Hölder continuity of \( \varphi^a(t, \cdot, \cdot) \) with respect to \( s \) and \( y \) in \([0, t] \times \mathbb{R}^d\). According to the integral representation of \( \varphi^a(t, s, y) \) (A.24), we have

\[
\varphi^a(t, s, y) = \int_0^1 \sum_{|l| \leq 2r, b \leq m} \left[ \frac{\partial (\hat{\varphi}^a(t, t, s, \cdot, \cdot)) - \partial \varphi^a(t, 0, y, \theta_0(y))}{\partial t} \cdot \partial_t (u - \hat{u}) (t, s, y) \right. \\
+ (\partial F^a_b(t, s, y, \theta_\varsigma(t, s, y)) - \partial F^a_b(t, 0, y, \theta_0(t, y))) \cdot \partial_t (u - \hat{u}) (t, s, y) \bigg] d\sigma \\
+ \int_0^1 \sum_{|l| \leq 2r, b \leq m} \left[ \frac{\partial (\hat{\varphi}^a(t, t, s, \cdot, \cdot)) - \partial \varphi^a(t, 0, y, \theta_0(y))}{\partial t} \cdot \partial_t (u - \hat{u}) (s, s, y) \right] d\sigma \\
=: \{M_1 + M_2 + M_3\} + \{M_4 + M_5\},
\]

where

\[
M_1 = \int_0^1 \sum_{|l| \leq 2r, b \leq m} \left( \partial^2_{ll} F^a_b(t, s, y, \theta_\varsigma(t, s, y)) - \partial^2_{ll} F^a_b(t, 0, y, \theta_0(t, y)) \right) \cdot \partial_t (u - \hat{u}) (t, s, y) d\sigma
\]

\[
M_2 = \int_0^1 \sum_{|l| \leq 2r, b \leq m} \left[ \sum_{|l| \leq 2r, c \leq m} \left( \partial^2_{ll} F^a_{bc}(t, s, y, \theta_\varsigma(t, s, y)) \cdot \left( \sigma \partial_j u^c(t, s, y) + (1 - \sigma) \partial_j \hat{u}^c(t, s, y) \right) \\
- \partial^2_{ll} F^a_{bc}(t, 0, y, \theta_0(t, y)) \cdot \partial_j g^c(t, y) \right) \right] \cdot \partial_t (u - \hat{u}) (t, s, y) d\sigma
\]

\[
M_3 = \int_0^1 \sum_{|l| \leq 2r, b \leq m} \left( \partial F^a_b(t, s, y, \theta_\varsigma(t, s, y)) - \partial F^a_b(t, 0, y, \theta_0(t, y)) \right) \cdot \partial_t (u - \hat{u}) (t, s, y) d\sigma
\]

\[
M_4 = \int_0^1 \sum_{|l| \leq 2r, b \leq m} \left( \partial^2_{ll} F^a_b(t, s, y, \theta_\varsigma(t, s, y)) - \partial^2_{ll} F^a_b(t, 0, y, \theta_0(t, y)) \right) \cdot \partial_t (u - \hat{u}) (s, s, y) d\sigma
\]

\[
M_5 = \int_0^1 \sum_{|l| \leq 2r, b \leq m} \left[ \sum_{|l| \leq 2r, c \leq m} \left( \partial^2_{ll} F^a_{bc}(t, s, y, \theta_\varsigma(t, s, y)) \cdot \left( \sigma \partial_j u^c(t, s, y) + (1 - \sigma) \partial_j \hat{u}^c(t, s, y) \right) \\
- \partial^2_{ll} F^a_{bc}(t, 0, y, \theta_0(t, y)) \cdot \partial_j g^c(t, y) \right) \right] \cdot \partial_t (u - \hat{u}) (s, s, y) d\sigma.
\]

It is easy to see that the estimates of \( M_1, M_3, \) and \( M_4 \) are similar to the terms of \( |\varphi^a(t, \cdot, \cdot)|_{\[0, t\] \times \mathbb{R}^d}^{(\alpha)} \). Hence, we focus on the remaining two terms \( M_2 \) and \( M_5 \). We denote \( \eta^a(t, s, y) = M_2 + M_5 \).
In order to estimate \(|\eta^a(t, s, y) - \eta^a(t', s', y)|\) for \(0 \leq s \leq s' \leq t \leq \delta \leq T\) and any \(y \in \mathbb{R}^d\), it is convenient to add and subtract

\[
\int_0^1 \sum_{|I| \leq 2r, b \leq m} \left[ \sum_{|I| \leq 2r, c \leq m} \left( \partial_{ij}^2 F_{bc}^a(t, s', y, \theta_\sigma(t, t', y)) \cdot (\sigma \partial_j u_i^c(t, s', y) + (1 - \sigma) \partial_j \hat{u}_i^c(t, s', y)) \right) \right] \\
\times \partial_j (u - \hat{u})^b(t, s, y) d\sigma \\
+ \int_0^1 \sum_{|I| \leq 2r, b \leq m} \left[ \sum_{|I| \leq 2r, c \leq m} \left( \partial_{ij}^2 \bar{F}_{bc}^a(t, s', y, \theta_\sigma(t, t', y)) \cdot (\sigma \partial_j u_i^c(t, s', y) + (1 - \sigma) \partial_j \hat{u}_i^c(t, s', y)) \right) \right] \\
\times \partial_j (u - \hat{u})^b(s, s, y) d\sigma.
\]

Subsequently, we need to estimate

\[
\left| \partial_{ij}^2 F_{bc}^a(t, s, y, \theta_\sigma(t, s, y)) \cdot (\sigma \partial_j u_i^c(t, s, y) + (1 - \sigma) \partial_j \hat{u}_i^c(t, s, y)) \right| \leq C_{12}(R)(s' - s)^{\frac{\alpha}{2}}, \tag{A.29}
\]

\[
\left| \partial_{ij}^2 F_{bc}^a(t, s', y, \theta_\sigma(t, t', y)) \cdot (\sigma \partial_j u_i^c(t, s', y) + (1 - \sigma) \partial_j \hat{u}_i^c(t, s', y)) \right| \leq C_{13}(R)\delta^{\frac{\alpha}{2}}, \tag{A.30}
\]

\[
\left| \partial_{ij}^2 \bar{F}_{bc}^a(t, s', y, \theta_\sigma(t, t', y)) \cdot (\sigma \partial_j u_i^c(t, s', y) + (1 - \sigma) \partial_j \hat{u}_i^c(t, s', y)) \right| \leq C_{14}(R)\delta^\alpha, \tag{A.31}
\]

\[
\left| \partial_{ij}^2 \bar{F}_{bc}^a(t, s', y, \theta_\sigma(t, t', y)) \cdot (\sigma \partial_j u_i^c(t, s', y) + (1 - \sigma) \partial_j \hat{u}_i^c(t, s', y)) \right| \leq \left| \partial_{ij}^2 \bar{F}_{bc}^a(t, 0, y, \theta_0(t, y)) \cdot \partial_j g_i^c(t, y) \right| \leq C_{15}(R)\delta^\alpha. \tag{A.32}
\]

Note that
Similarly, we also have Equation (A.30) \( \leq C_{14}(R)(s' - s)^{\frac{\alpha}{2\gamma}} \) and Equation (A.32) \( \leq C_{15}(R)\delta^{\frac{\alpha}{2\gamma}} \). Hence, we obtain that

\[
|\eta'(t, s, y) - \eta''(t, s', y)| \leq C_{12}(R)(s' - s)^{\frac{\alpha}{2\gamma}} \sum_{b \leq m} |\partial_j (u - \hat{u})^b_b (t, \cdot, \cdot)\left|_{[0, t] \times \mathbb{R}^d} + C_{14}(R)(s' - s)^{\frac{\alpha}{2\gamma}} \sum_{b \leq m} |\partial_j (u - \hat{u})^b_b (s, \cdot, \cdot)\left|_{[0, t] \times \mathbb{R}^d}
\]

\[
\times \sum_{b \leq m} |\partial_j (u - \hat{u})^b_b (t, \cdot, \cdot)\left|_{[0, t] \times \mathbb{R}^d} + C_{14}(R)(s' - s)^{\frac{\alpha}{2\gamma}} \sum_{b \leq m} \left( \sup_{R \subseteq (s, s')} |\partial_j (u - \hat{u})^b_b (\tilde{s}, \cdot, \cdot)\right\left|_{[0, t] \times \mathbb{R}^d} + |\partial_j (u - \hat{u})^b_b (s', \cdot, \cdot)\right|_{[0, t] \times \mathbb{R}^d}
\]

\[
\leq C_{16}(R)\delta^{\frac{\alpha}{2\gamma}} (s' - s)^{\frac{\alpha}{2\gamma}} \|u - \hat{u}\|^{(2r + \alpha)}_{[0, \delta]},
\]

which implies the following by noting that \( \eta(t, 0, y) \equiv 0 \),

\[
|\eta''(t, \cdot, \cdot)|^{\infty}_{[0, t] \times \mathbb{R}^d} \leq C_{16}(R)\delta^{\frac{\alpha}{2\gamma}} \|u - \hat{u}\|^{(2r + \alpha)}_{[0, \delta]}.
\]

(A.33)

In order to estimate \( |\eta'(t, s, y) - \eta''(t, s, y')| \) for \( 0 \leq s \leq t \leq \delta \leq T \) and any \( y, y' \in \mathbb{R}^d \), it is convenient to add and subtract

\[
\int_0^1 \sum_{|\ell| \leq 2r, b \leq m} \left[ \sum_{|\ell| \leq 2r, c \leq m} \right] \left( \partial_{ij} F^{bc}_{bc} (t, s, y', \theta_\sigma(t, s, y')) - (\sigma \partial_j u^c_i(t, s, y') + (1 - \sigma)\partial_j \hat{u}_i^c(t, s, y')) \right) \cdot \partial_j (u - \hat{u})^b_b (t, s, y)d\sigma
\]

\[
+ \int_0^1 \sum_{|\ell| \leq 2r, b \leq m} \left[ \sum_{|\ell| \leq 2r, c \leq m} \right] \left( \partial_{ij} F^{bc}_{bc} (t, s, y', \theta_\sigma(t, s, y')) - (\sigma \partial_j u^c_i(t, s, y') + (1 - \sigma)\partial_j \hat{u}_i^c(t, s, y')) \right) \cdot \partial_j (u - \hat{u})^b_b (t, s, y)d\sigma.
\]

Then we need to estimate the error (for \( F \))

\[
\left| \partial_{ij} F^{bc}_{bc} (t, s, y, \theta_\sigma(t, s, y)) \cdot (\sigma \partial_j u^c_i(t, s, y') + (1 - \sigma)\partial_j \hat{u}_i^c(t, s, y')) - \partial_{ij} F^{bc}_{bc} (t, 0, y, \theta_\sigma(t, y)) \cdot \partial_j g^c_i(t, y) \right.
\]

\[
\left. - \partial_{ij} F^{bc}_{bc} (t, s, y', \theta_\sigma(t, s, y')) \cdot (\sigma \partial_j u^c_i(t, s, y') + (1 - \sigma)\partial_j \hat{u}_i^c(t, s, y')) \right.
\]

\[
+ \partial_{ij} F^{bc}_{bc} (t, 0, y, \theta_\sigma(t, y')) \cdot \partial_j g^c_i(t, y') \right|
\]

(A.35)
as well as the error (for $\overline{F}$)

$$
\begin{aligned}
&\left| \partial_{ij}^2 \overline{F}_{bc}^a(t, s, y, \theta_c(t, s, y)) \cdot \left( \sigma \partial_j \hat{u}^c_i(t, s, y) + (1 - \sigma) \partial_j \hat{u}^c_i(t, s, y) \right) - \partial_{ij}^2 \overline{F}_{bc}^a(t, 0, y, \theta_0(t, y)) \cdot \partial_j g^c_i(t, y) \\
&\quad - \partial_{ij}^2 \overline{F}_{bc}^a(t, s, y', \theta_c(t, s, y')) \cdot \left( \sigma \partial_j \hat{u}^c_i(t, s, y') + (1 - \sigma) \partial_j \hat{u}^c_i(t, s, y') \right) \\
&\quad + \partial_{ij}^2 \overline{F}_{bc}^a(t, 0, y', \theta_0(t, y')) \cdot \partial_j g^c_i(t, y') \right|.
\end{aligned}
$$

(A.36)

Moreover, we also need to estimate

$$
\begin{aligned}
&\left| \partial_{ij}^2 F_{bc}^a(t, s, y, \theta_c(t, s, y)) \cdot \left( \sigma \partial_j u^c_i(t, s, y) + (1 - \sigma) \partial_j \hat{u}^c_i(t, s, y) \right) - \partial_{ij}^2 F_{bc}^a(t, 0, y, \theta_0(t, y)) \cdot \partial_j g^c_i(t, y) \right|
\end{aligned}
$$

(A.37)

and

$$
\begin{aligned}
&\left| \partial_{ij}^2 F_{bc}^a(t, s, y, \theta_c(t, s, y)) \cdot \left( \sigma \partial_j u^c_i(t, s, y) + (1 - \sigma) \partial_j \hat{u}^c_i(t, s, y) \right) - \partial_{ij}^2 \overline{F}_{bc}^a(t, 0, y, \theta_0(t, y)) \cdot \partial_j g^c_i(t, y) \right|
\end{aligned}
$$

(A.38)

Note that

(A.35) $\leq \left| \partial_{ij}^2 F_{bc}^a(t, s, y, \theta_c(t, s, y)) \cdot \left( \sigma \partial_j u^c_i(t, s, y) + (1 - \sigma) \partial_j \hat{u}^c_i(t, s, y) \right) \\
\quad - \partial_{ij}^2 F_{bc}^a(t, s, y', \theta_c(t, s, y')) \cdot \left( \sigma \partial_j u^c_i(t, s, y') + (1 - \sigma) \partial_j \hat{u}^c_i(t, s, y') \right) \\
\quad + \partial_{ij}^2 F_{bc}^a(t, 0, y', \theta_0(t, y')) \cdot \partial_j g^c_i(t, y') \right| =: N_1 + N_2.$

For $N_1$, it holds that

$$
\begin{aligned}
N_1 \leq \left| \partial_{ij}^2 F_{bc}^a(t, s, y, \theta_c(t, s, y)) \cdot \left( \sigma \partial_j u^c_i(t, s, y) + (1 - \sigma) \partial_j \hat{u}^c_i(t, s, y) \right) \\
\quad - \partial_{ij}^2 F_{bc}^a(t, s, y, \theta_c(t, s, y)) \cdot \left( \sigma \partial_j u^c_i(t, s, y') + (1 - \sigma) \partial_j \hat{u}^c_i(t, s, y') \right) \\
\quad + \partial_{ij}^2 F_{bc}^a(t, s, y', \theta_c(t, s, y')) \cdot \left( \sigma \partial_j u^c_i(t, s, y') + (1 - \sigma) \partial_j \hat{u}^c_i(t, s, y') \right) \\
\quad - \partial_{ij}^2 F_{bc}^a(t, s, y', \theta_c(t, s, y')) \cdot \left( \sigma \partial_j u^c_i(t, s, y') + (1 - \sigma) \partial_j \hat{u}^c_i(t, s, y') \right) \right| \leq C_1 r |y - y'|^\alpha.
\end{aligned}
$$

For $N_2$,

$$
\begin{aligned}
N_2 \leq \left| \partial_{ij}^2 F_{bc}^a(t, 0, y', \theta_0(t, y')) \cdot \partial_j g^c_i(t, y') - \partial_{ij}^2 F_{bc}^a(t, 0, y, \theta_0(t, y)) \cdot \partial_j g^c_i(t, y') \right| \\
\quad + \partial_{ij}^2 F_{bc}^a(t, 0, y, \theta_0(t, y)) \cdot \partial_j g^c_i(t, y') \right| \leq C_2 |y - y'|^\alpha.
\end{aligned}
$$
From the estimates of $N_1$ and $N_2$, we have Equation (A.35) $\leq C_{19}(R)|y - y'|^\alpha$. Moreover, we have

\[
(A.37) \leq \left| \partial^2_{ij} F^a_{bc}(t, s, y, \theta_c(t, s, y)) \cdot \partial_j \psi^c(t, s, y) + (1 - \sigma) \partial_j \hat{\psi}^c(t, s, y) \right|
\]

\[
- \partial^2_{ij} F^a_{bc}(t, s, y, \theta_c(t, s, y)) \cdot \partial_j g^c(t, y)
\]

\[
+ \left| \partial^2_{ij} F^a_{bc}(t, s, y, \theta_c(t, s, y)) \cdot \partial_j g^c(t, y) - \partial^2_{ij} F^a_{bc}(t, 0, y, \theta_0(t, y)) \cdot \partial_j g^c(t, y) \right| \leq C_{20}(R) \delta \frac{\alpha}{\alpha z}
\]

Similarly, for $\overline{F}$, we have Equation (A.36) $\leq C_{21}(R)|y - y'|^\alpha$ and Equation (A.38) $\leq C_{22}(R)\delta \frac{\alpha}{\alpha z}$. Hence, we have

\[
|\eta^a(t, s, y) - \eta^a(t, s, y')| \leq C_{19}(R)|y - y'|^\alpha \sum_{b \leq m} |\partial_j \psi(a - \hat{\psi})^b(t, s, \cdot, \cdot)|^{(\alpha)}_{[0, t] \times \mathbb{R}^d}
\]

\[
+ C_{20}(R)\delta \frac{\alpha}{\alpha z} |y - y'|^\alpha \sum_{b \leq m} |\partial_j \psi(a - \hat{\psi})^b(t, s, \cdot, \cdot)|^{(\alpha)}_{[0, t] \times \mathbb{R}^d}
\]

\[
+ C_{21}(R)|y - y'|^\alpha \delta \frac{\alpha}{\alpha z} \sum_{b \leq m} |\partial_j \psi(a - \hat{\psi})^b(s, \cdot, \cdot)|^{(\alpha)}_{[0, t] \times \mathbb{R}^d}
\]

\[
+ C_{22}(R)\delta \frac{\alpha}{\alpha z} |y - y'|^\alpha \sum_{b \leq m} |\partial_j \psi(a - \hat{\psi})^b(s, \cdot, \cdot)|^{(\alpha)}_{[0, t] \times \mathbb{R}^d}
\]

\[
\leq C_{23}(R)\delta \frac{\alpha}{\alpha z} |y - y'|^\alpha \|u - \widehat{u}_z\|^{(2\alpha + \alpha)}_{[0, \delta]}
\]

Therefore, together with Equation (A.34), we have

\[
|\eta^a(t, \cdot, \cdot)|^{(\alpha)}_{[0, t] \times \mathbb{R}^d} \leq C_{24}(R)\delta \frac{\alpha}{\alpha z} \|u - \widehat{u}_z\|^{(2\alpha + \alpha)}_{[0, \delta]}
\]

(A.40)

Since $M_1, M_3$, and $M_4$ satisfy the same estimates, it holds that

\[
|\phi^a(t, \cdot, \cdot)|^{(\alpha)}_{[0, t] \times \mathbb{R}^d} \leq C_{25}(R)\delta \frac{\alpha}{\alpha z} \|u - \widehat{u}_z\|^{(2\alpha + \alpha)}_{[0, \delta]}
\]

(A.41)

Finally, we have a contraction

\[
\|\Lambda(u) - \Lambda(\widehat{u})\|^{(2\alpha + \alpha)}_{[0, \delta]} \leq C \|\phi\|^{(\alpha)}_{[0, \delta]} = C \sup_{t \in [0, \delta]} \sum_{a \leq m} \left\{ \phi^a(t, \cdot, \cdot)|^{(\alpha)}_{[0, t] \times \mathbb{R}^d} + \phi^a(t, \cdot, \cdot)|^{(\alpha)}_{[0, t] \times \mathbb{R}^d} \right\}
\]

\[
\leq C(R)\delta \frac{\alpha}{\alpha z} \|u - \widehat{u}_z\|^{(2\alpha + \alpha)}_{[0, \delta]}
\]

(A.42)

A contraction $\Lambda$ mapping $U$ into itself. To show the contraction, we need to choose a suitably large $R$ such that $\Lambda$ maps $U$ into itself. If $\delta$ and $R$ satisfy

\[
C(R)\delta \frac{\alpha}{\alpha z} \leq \frac{1}{2},
\]
then $\Lambda$ is a $\frac{1}{2}$-contraction and for any $u \in U$, we have

$$\|\Lambda(u) - g\|_{[0,\delta]}^{(2r+\alpha)} \leq \frac{R}{2} + \|\Lambda(g) - g\|_{[0,\delta]}^{(2r+\alpha)}.$$  

Define the function $G := \Lambda(g) - g$ as the solution of the equation

$$G_s(t, s, y) = L_0 G + F \left(t, s, y, (\partial_t g)|_{t \leq 2r}(t, y), (\partial_t g)|_{t \leq 2r}(s, y)\right),$$

$$G(t, 0, y) = 0, \quad 0 \leq s \leq t \leq \delta, \quad y \in \mathbb{R}^d.$$  

By Equation (28), there is $C > 0$, independent of $\delta$, such that

$$\|G\|_{[0,\delta]}^{(2r+\alpha)} \leq C \sup_{t \in [0,\delta]} \sum_{a \leq m} \left\{ \left|\psi^a(t, \cdot, \cdot)\right|_{[0,\delta] \times \mathbb{R}^d}^{(\alpha)} + \left|\psi^a(t, \cdot, \cdot)\right|_{[0,\delta] \times \mathbb{R}^d}^{(\alpha)} \right\} = : C',$$

where $\psi^a(t, s, y) = F^a(t, s, y, (\partial_t g)|_{t \leq 2r}(t, y), (\partial_t g)|_{t \leq 2r}(s, y)).$

To conclude, we have

$$\|\Lambda(u) - g\|_{[0,\delta]}^{(2r+\alpha)} \leq \frac{R}{2} + C'.$$

Therefore, for a suitably large $R$, $\Lambda$ is a contraction mapping $U$ into itself and it has a unique fixed point $u$ in $U$ satisfying

$$\left\{ \begin{array}{l}
\text{Definethefunction } G := \Lambda(g) - g \text{ as the solution of the equation} \\
G_s(t, s, y) = L_0 G + F \left(t, s, y, (\partial_t g)|_{t \leq 2r}(t, y), (\partial_t g)|_{t \leq 2r}(s, y)\right), \\
G(t, 0, y) = 0, \quad 0 \leq s \leq t \leq \delta, \quad y \in \mathbb{R}^d.
\end{array} \right.$$  

(A.43)

Uniqueness. To complete the proof, we have to show that $u$ is the unique solution of Equation (1) in $\Omega_{[0,\delta]}^{(2r+\alpha)}$. It follows the Schauder-type estimate (Theorem 3.4) for the nonlocal, homogeneous, linear, and strongly parabolic system with initial value zero, which is satisfied by the difference of any two solutions $u$, and $\overline{u}$ in $\Omega_{[0,\delta]}^{(2r+\alpha)}$ to the system (A.43). On the other hand, since $\Lambda$ is a contraction, it can be done with some standard arguments.

If Equation (A.43) admits two fixed points $u$ and $\overline{u}$, let

$$\overline{t} = \sup \left\{ t \in [0, \delta] : u(t, s, y) = \overline{u}(t, s, y), \quad (t, s, y) \in \Delta[0, t] \times \mathbb{R}^d \right\}.$$  

We shall focus only on the case when $\overline{t} < \delta$ because if $\overline{t} = \delta$, then $u = \overline{u}$ in the whole $\Delta[0, \delta] \times \mathbb{R}^d$ and the proof is completed. According to the definition of $\overline{t}(< \delta)$, we know that $u(t, s, y) = \overline{u}(t, s, y)$ in $R_{1} \times \mathbb{R}^d$ in Figure A.2. Hence, we obtain diagonal conditions, namely $(\partial_t u)|_{t \leq 2r}(s, s, y) = (\partial_t \overline{u})|_{t \leq 2r}(s, s, y)$ for any $s \in [0, \overline{t}]$ and $y \in \mathbb{R}^d$. By observing Equation (A.43) provided that the same initial and diagonal conditions (i.e., the initial condition 1 and the diagonal condition in Figure A.2), the classical PDE theory promises that $u$ and $\overline{u}$ coincide in $R_{1} \cup R_{2} \times \mathbb{R}^d$.

Next, let $u(t, \overline{t}, y) = \overline{u}(t, \overline{t}, y) = \overline{g}(t, y)$ for $(t, y) \in [\overline{t}, T] \times \mathbb{R}^d$. Based on the new initial condition (i.e., the initial condition 2 in Figure A.2), we consider the following initial value problem:

$$\left\{ \begin{array}{l}
\text{We shall focus only on the case when } \overline{t} < \delta \text{ because if } \overline{t} = \delta, \text{ then } u = \overline{u} \text{ in the whole } \\
\Delta[0, \delta] \times \mathbb{R}^d \text{ and the proof is completed. According to the definition of } \overline{t}(< \delta), \text{ we know that } \\
u(t, s, y) = \overline{u}(t, s, y) \text{ in } R_{1} \times \mathbb{R}^d \text{ in Figure A.2. Hence, we obtain diagonal conditions, namely } \\
(\partial_t u)|_{t \leq 2r}(s, s, y) = (\partial_t \overline{u})|_{t \leq 2r}(s, s, y) \text{ for any } s \in [0, \overline{t}] \text{ and } y \in \mathbb{R}^d. \text{ By observing Equation (A.43) provided that the same initial and diagonal conditions (i.e., the initial condition 1 and the diagonal condition in Figure A.2), the classical PDE theory promises that } u \text{ and } \overline{u} \text{ coincide in } \\
(R_{1} \cup R_{2}) \times \mathbb{R}^d. \\
\text{Next, let } u(t, \overline{t}, y) = \overline{u}(t, \overline{t}, y) = \overline{g}(t, y) \text{ for } (t, y) \in [\overline{t}, T] \times \mathbb{R}^d. \text{ Based on the new initial condition (i.e., the initial condition 2 in Figure A.2), we consider the following initial value problem:} \\
\left\{ \begin{array}{l}
u(t, s, y) = F \left(t, s, y, (\partial_t u)|_{t \leq 2r}(t, s, y), (\partial_t u)|_{t \leq 2r}(s, s, y)\right), \\
u(t, \overline{t}, y) = \overline{g}(t, y), \quad \overline{t} \leq s \leq t \leq T, \quad y \in \mathbb{R}^d.
\end{array} \right. \\
(A.44)
\right.$$
Our previous proof shows that Equation (A.44) admits a unique solution in the set

$$\overline{U} = \left\{ u \in \Omega_{[\overline{t}, \overline{t}+\delta]}^{(2r+\alpha)} : u(t, \overline{t}, y) = \overline{g}(t, y), \|u - \overline{g}\|_{[\overline{t}, \overline{t}+\delta]}^{(2r+\alpha)} \leq \overline{R} \right\}$$

provided that $\overline{R}$ is large enough and $\delta$ is small enough. Considering $\overline{R}$ larger than $\|u - \overline{g}\|_{[\overline{t}, \overline{t}+\delta]}^{(2r+\alpha)}$ and $\|\overline{u} - \overline{g}\|_{[\overline{t}, \overline{t}+\delta]}^{(2r+\alpha)}$, we have $u = \overline{u}$ in $R_3 \times \mathbb{R}^d$. Hence, for any $y \in \mathbb{R}^d$, $u$ equals to $\overline{u}$ in $\{(t, s) : \overline{t} \leq t \leq \overline{t} + \delta, 0 \leq s \leq \overline{t}\} \cup R_3$, which contradicts the definition of $\overline{t}$. Consequently, $\overline{t} = \delta$ and $u = \overline{u}$. This completes the proof. $\Box$

**Proof of Theorem 3.8.** Assume that $\lim_{s \to \tau} u(t, s, y) \in \mathcal{O}$. To obtain the global solvability, the maximally defined solution in $[0, \tau)$ has to be extended continuously to a closed interval $[0, \tau]$ such that we can update the initial data with $u(\cdot, \tau, \cdot) \in \Omega_{[\tau, T]}^{(2r+\alpha)}$.

According to the definition of $\Omega_{[a, b] \times [c, d]}^{(2r+\alpha)}$ and the extension procedure in Figure A.1, for each fixed $t \in [\tau, T]$, it requires that the mapping $u : s \mapsto u(t, s, y)$ from $[0, \tau)$ to $C^{2r+\alpha}(\mathbb{R}^d; \mathbb{R}^m)$ is at least uniformly continuous such that the limit value $\lim_{s \to \tau} u(t, s, y)$ exists. Similar to the proof of Theorem 3.4, by the estimate (35) and extension arguments for the triangle area $(t, s) \in \Delta[0, \sigma]$ to the corresponding trapezoid region $(t, s) \in [0, T] \times [0, t \wedge \sigma]$ for any fixed $t \in [\tau, T]$, we have

$$u(t, \cdot, \cdot) \in B\left([0, \tau); C^{2r+\alpha'}(\mathbb{R}^d; \mathbb{R}^m)\right), \quad u_3(t, \cdot, \cdot) \in B\left([0, \tau); C^{\alpha'}(\mathbb{R}^d; \mathbb{R}^m)\right),$$

where $B([a, b); X)$ denotes the space of bounded function defined in $[a, b)$ and valued in the Banach space $X$. By an interpolation result (see Sinestrari (1985, Proposition 2.7) or Lunardi (1995, Chapter 1)), it follows that $u(t, \cdot, \cdot) \in C^{1-\theta}([0, \sigma]; C^{\alpha' + 2r\sigma}(\mathbb{R}^d; \mathbb{R}^m))$ for every $\sigma \in (0, \tau)$ with Hölder constant independent of $\sigma$. By choosing $\theta = 1 - \frac{\sigma - \alpha}{2r}$, it follows

$$u(t, \cdot, \cdot) \in C^{-\frac{\sigma - \alpha}{2r}}\left([0, \sigma]; C^{2r+\alpha}(\mathbb{R}^d; \mathbb{R}^m)\right).$$
Consequently, for each $t \in [\tau, T]$, $u(t, \cdot, \cdot)$ can be continued at $s = \tau$ in such a way that the extension $u(\cdot, \tau, \cdot)$ belongs to $\Omega^{(2r+\alpha)}_{[\tau, T]}$. After updating with $u(\cdot, \tau, \cdot)$ as a new initial condition at $s = \tau$, by Theorem 3.5, the nonlocal system (1) restricted in $(t, s, y) \in [\tau, T] \times [\tau, t] \times \mathbb{R}^d$ admits a unique solution $u \in \Omega^{(2r+\alpha)}_{[\tau, T]}$ for some $\tau_1 > 0$, which contradicts the definition of $\tau$. Therefore, we have $\tau = T$ or $\lim_{s \to \tau} u(\cdot, s, \cdot) \in \partial \mathcal{O}$. 

**Proof of Theorem 3.9.** Thanks to Theorem 3.5 and Remark 3.6, Equation (36) admits a unique maximally defined solution $u \in \Omega^{(2r+\alpha)}_{[0, \tau)}$ in the maximal interval $\Delta[0, \tau]$. We need to prove that the solution can be extended uniquely into $\Delta[0, T]$. 

According to the proof of Theorem 3.5 and the formulation of $L_0$, we replace the nonlinearity $F$ of Equation (A.18) with the right side of Equation (36), that is, $\sum A\partial_t u + Q(u)$. In the quasilinear case of Equation (36), it is clear that the radius $R$ of $C(R) \delta \mathcal{F}$ in the proof of Theorem 3.5 only depends on $\|g\|_{[0, \delta]}$ and $\|u\|_{[0, \delta]}$ instead of $\|g\|_{[2r+\alpha]}$ and $\|u\|_{[2r+\alpha]}$ as in the fully nonlinear case. Consequently, in order to establish the existence in the large time interval, we only need to investigate and control the solutions of Equation (36) under the norm $\| \cdot \|_{[2r+\alpha]}$. It suffices to show that the mapping $u : s \mapsto u(t, s, y)$ from $[0, \tau]$ to $C^{2r-1+\alpha}(\mathbb{R}^d; \mathbb{R}^m)$ is uniformly continuous under the conditions in Theorem 3.9.

To this end, we note from the nonlocal quasilinear system of Equation (36) that

\[
\begin{cases}
\left( \frac{\partial u}{\partial t} \right)^a(t, s, y) = \sum_{y=2r-1}^m A_n^b(s, y) \partial_t \left( \frac{\partial u}{\partial t} \right)^b(t, s, y) + \sum_{|l| \leq 2r-1, b \leq m} \partial_t Q_l^a(u) \partial_l \left( \frac{\partial u}{\partial t} \right)^b(t, s, y) + Q_l^a(u), \\
\left( \frac{\partial u}{\partial t} \right)^a(t, 0, y) = g_a(t, y), \quad 0 \leq s \leq t \leq \tau, \quad y \in \mathbb{R}^d, \quad a = 1, \ldots, m.
\end{cases}
\]  

(A.45)

where $\partial_t Q_l^a(u)$ and $Q_l^a(u)$ represent the first-order partial derivatives of the nonlinearity $Q^a$ with respect to its argument $\partial_t u^b(t, s, y)$ and $t$, respectively, while both of them are evaluated at $(t, s, y, (\partial_t u)_{|l| \leq 2r-1}(t, s, y), (\partial_t u)_{|l| \leq 2r-1}(s, s, y))$. According to the linear growth condition and bounds of $Q$ and the Grönwall–Bellman inequality, it is clear from Equations (36) and (A.45) that there exists a constant $K$ depending only on the given coefficients and data of Equation (36) such that $\| u \|_{[2r+\alpha]} \leq K$. By the classical theory of PDEs, it further implies that $u \in \Omega^{(2r-1+\alpha)}_{[0, \tau)}$ and $\| u \|_{[2r-1+\alpha]} \leq K$. Consequently, the nonlocal and nonlinear term $Q^a(u)$ of Equation (36) belongs to $\Omega^{(\alpha)}_{[0, \tau)}$. Thanks to Theorem 3.4, the nonlocal quasilinear system (36) admits a unique solution $u \in \Omega^{(2r+\alpha)}_{[0, T]}$ in $\Delta[0, T] \times \mathbb{R}^d$ and $\| u \|_{[0, \tau]} \leq K$, where $K$ could vary from line to line. With similar arguments as the proof of Theorem 3.8, for each $t \in [\tau, T]$, we can extend $u : s \mapsto u(t, s, y)$ from $[0, \tau]$ to $C^{2r-1+\alpha}(\mathbb{R}^d; \mathbb{R}^m)$ at $s = \tau$. With the achieved limit point $u(\cdot, \tau, \cdot) \in \Omega^{(2r-1+\alpha)}_{[\tau, T]}$, by Lemma 8.5.5 in Lunardi (1995), it follows that $u(\cdot, \tau, \cdot) \in \Omega^{(2r+\alpha)}_{[\tau, T]}$ and $\| u \|_{[0, T]} \leq K$. Hence, by updating the initial condition, we can extend the maximally defined solution up to the whole time region $\Delta[0, T]$.

**Proof of Lemma 3.10.** For the equivalence between Equation (37) and Equation (38), we refer the readers to Lemma 2.5 of Lorenzi (2000). It comes directly from the basic properties of the exponential weight $\varphi(y)$. Next, we will show the equivalence between Equations (38) and (39). Let us consider $f(y) \in C^\alpha(\mathbb{R}^d; \mathbb{R})$ and $0 < |y - y'| \leq 1$. Without loss of generality, we assume that
\( (Sy, y)^{1/2} < (Sy', y')^{1/2} \). Then,

\[
\left| \frac{f(y) - f(y')}{\varphi(y')} - \frac{f(y) - f(y')}{\varphi(y)} \right| = \left| \frac{f(y)}{\varphi(y)} - \frac{f(y)}{\varphi(y')} \right| + \left| \frac{f(y')}{\varphi(y')} - \frac{f(y')}{\varphi(y)} \right| \leq \left| \frac{f(y)}{\varphi(y)} \right| |y - y'|^{\alpha} + \left| \frac{f(y')}{\varphi(y')} \right| |y - y'|^{\alpha},
\]

where \( C \) depends on the maximum eigenvalue \( \lambda \) of \( S \). Similarly, it holds that

\[
\left| \frac{f(y) - f(y')}{\varphi(y')} - \frac{f(y) - f(y')}{\varphi(y)} \right| \leq C \left| \frac{f(y)}{\varphi(y)} \right| |y - y'|^{\alpha} + \left( \frac{f(y')}{\varphi(y')} \right)_{\psi, [0, T]} |y - y'|^{\alpha} \min \{ \varphi^{-1}(y), \varphi^{-1}(y') \}. \tag{A.47}
\]

Consequently, thanks to Equations (A.46) and (A.47), it is easy to see the equivalence between Equation (38) and Equation (39).

**Proof of Theorem 3.12.** In order to show the well-posedness result and the estimate (40) of solutions of the nonlocal linear systems (25) in the weighted space \( \Omega^{(i)}_{\varphi, [0, T]} \), we firstly consider a simplified case of Equation (25) where \( A^{ai}_{b}(t, s, y) = A^{al}_{b}(s, y) \), \( B^{ai}_{b}(t, s, y) = 0 \), \( f(t, s, y) = f(s, y) \), and \( g(t, s, y) = 0 \). Then, according to the classical theory of PDE systems (see Friedman (1964, Chapter 9) or Éidel’man (1969, Chapter 1.3)), Equation (25) admits a unique solution \( u(s, y) \) of the form

\[
u(s, y) = \int_{0}^{s} \int_{\mathbb{R}^d} Z(s, \tau, y, \xi) f(\tau, \xi) d\xi
\]

and for \( |I| \leq 2r \), its derivatives are expressed as

\[
\partial_{I} u(s, y) = \int_{0}^{s} d\tau \int_{\mathbb{R}^d} \partial_{I} Z(s, \tau, y, \xi) [f(\tau, \xi) - f(\tau, y)] d\xi + \int_{0}^{s} \left( \partial_{I} \int_{\mathbb{R}^d} Z(s, \tau, y, \xi) d\xi \right) f(\tau, y) d\xi,
\]

where \( Z(s, \tau, y, \xi) \) is the fundamental solution of Equation (25) with \( A^{al}_{b}(t, s, y) = A^{al}_{b}(s, y) \) and \( B^{ai}_{b}(t, s, y) = 0 \). With the upper bound of \( Z \) (see Friedman (1964, Chapter 9) or Éidel’man (1969, Chapter 1.3)), it is clear that

\[
\left| \frac{u(s, y)}{\varphi(y)} \right| \leq \int_{0}^{s} d\tau \int_{\mathbb{R}^d} \left| Z(s, \tau, y, \xi) \right| \left| \frac{f(\tau, \xi)}{\varphi(\xi)} \right| \left| \frac{\varphi(\xi)}{\varphi(y)} \right| d\xi \leq Cs \left| f \right|^{(\alpha)}_{\varphi, [0, T] \times \mathbb{R}^d}, \tag{A.48}
\]

where \( C \) only depends on the coefficients \( A^{ai}_{b} \) and \( S \). The inequality (A.48) makes full use of the facts that the upper bound of \( |Z| \) contains \( \exp\{-c|y - \xi|^{\frac{2r}{2r-1}}\} \) while \( |\varphi(\xi)/\varphi(y)| \) is bounded by \( \exp|\lambda|y - \xi| \). Similarly, we also have

\[
|\partial_{I} u(s, y)| \leq \int_{0}^{s} d\tau \int_{\mathbb{R}^d} \left| \partial_{I} Z(s, \tau, y, \xi) \right| \left| f(\tau, \xi) - f(\tau, y) \right| d\xi + \int_{0}^{s} \left| \partial_{I} \int_{\mathbb{R}^d} Z(s, \tau, y, \xi) d\xi \right| \left| f(\tau, y) \right| d\xi
\]

\[
\leq \int_{0}^{s} d\tau \int_{\mathbb{R}^d} \left( \left| \partial_{I} Z(s, \tau, y, \xi) \right| \left| f^{(\alpha)}_{\varphi, [0, T] \times \mathbb{R}^d}(\varphi(\xi) + \varphi(y)) \right| y - \xi|^{\alpha} \right) d\xi
\]

\[
+ \int_{0}^{s} \left| \partial_{I} \int_{\mathbb{R}^d} Z(s, \tau, y, \xi) d\xi \right| \left| f^{(\alpha)}_{\varphi, [0, T] \times \mathbb{R}^d}(\varphi(y)) \right| y - \xi|^{\alpha} d\xi
\]

\[
\leq Cs^{-\frac{2r-1}{2r}} \left| f^{(\alpha)}_{\varphi, [0, T] \times \mathbb{R}^d}(\varphi(y)), \tag{A.49}\right|
\]
the second inequality of which holds thanks to \( f \in \mathcal{C}^{\frac{\alpha}{2\gamma},\alpha}([0,T] \times \mathbb{R}^d,\mathbb{R}^m) \) and the last one is shown by a similar argument of Equation (A.48) as well as the upper bound of \( \partial_t Z \) illustrated in Friedman (1964) and Eidel’man (1969).

Furthermore, in order to estimate the Hölder continuity of \( \partial_z u \) in \( y \), we need to estimate the difference between \( \partial_z u(s, y) \) and \( \partial_z u(s, y') \), denoted by \( \Delta \partial_z u(s, y) \). First, we consider the case where \( s \leq |y - y'|^{2r} \). From Equation (A.49), we have

\[
|\Delta \partial_z u(s, y)| \leq C|y - y'|^{2r - |I| + \alpha} |f|_{\mathcal{F},[0,T] \times \mathbb{R}^d} \left( \varphi(y) + \varphi(y') \right).
\]

In particular for \( |I| = 2r \), we obtain

\[
|\Delta \partial_z u(s, y)| \leq C|y - y'|^{2r - |I| + \alpha} |f|_{\mathcal{F},[0,T] \times \mathbb{R}^d} \varphi(y) + \varphi(y')
\]

for \( |I| = 2r \). Consequently, we have a priori estimate of the solution of the simplified system

\[
\sup_{s \in [0,T]} |u(s, \cdot)|_{\mathcal{F},\mathbb{R}^d}^{(2r+\alpha)} \leq C |f|_{\mathcal{F},[0,T] \times \mathbb{R}^d}^{(\alpha)}.
\]

Furthermore, thanks to the regularities of \( A^{aI}_b \) and \( f \), we have

\[
\sup_{s \in [0,T]} |u(s, \cdot)|_{\mathcal{F},\mathbb{R}^d}^{(\alpha)} \leq C |f|_{\mathcal{F},[0,T] \times \mathbb{R}^d}^{(\alpha)}
\]

as well. According to the interpolation theory (see Lunardi (1995, Proposition 1.1.4 or Lemma 5.1.1) or Sinestrari (1985, Proposition 2.7)), it holds that

\[
|u|_{\mathcal{F},[0,T] \times \mathbb{R}^d}^{(2r+\alpha)} \leq C |f|_{\mathcal{F},[0,T] \times \mathbb{R}^d}^{(\alpha)}.
\]

For the general setting of nonlocal linear system (25), its global well-posedness and the Schauder’s estimate (40) can be both proven with the same arguments in Theorem 3.3 and Theorem 3.4 by replacing the estimate used in Equation (A.6) with the weighted one (A.51). After verifying the claims for the nonlocal linear system, it is clear that the conditions of Definition 3.11 and the updated Schauder’s estimate (40) suffice to guarantee the local solvability of fully nonlinear systems in the weighted space. The proof is the same as that of Theorem 3.5. In the same spirit of Section 3.3.2, the last claim of Theorem 3.12 can be proven as well.
Proof of Lemma 5.1. It is clear that the claim holds when \( k = 0 \). Let us consider \( k = 1 \). Suppose that \( u \) is the solution of Equation (1) in \( \Omega^{(2r+\alpha)}_{[0,\delta]} \), then the family of partial derivatives \( \frac{\partial u}{\partial y_i} (i = 1, \ldots, d) \) satisfy (for \( a = 1, \ldots, m \))

\[
\begin{align*}
\left\{ \frac{\partial u}{\partial y_i} & (t, s, y) = \sum_{|I|\leq 2r, |b|\leq m} \left( \partial F^a_{b}(u) \cdot \partial I \left( \frac{\partial u}{\partial y_i} \right) (t, s, y) + \partial \partial F^a_{b}(u) \cdot \partial I \left( \frac{\partial u}{\partial y_i} \right) (s, s, y) \right) + \partial y_i F^a(u), \\
\left( \frac{\partial u}{\partial y_i} \right) (t, 0, y) &= g_{y_i}(t, y), \quad 0 \leq s \leq t \leq \delta, \quad y \in \mathbb{R}^d,
\end{align*}
\]

where \( \partial F^a_{b}(u) \), \( \partial \partial F^a_{b}(u) \), and \( \partial y_i F^a(u) \) are the derivatives of \( F \) all evaluated at

\[
\left\{ t, s, y, (\partial_f u)_{|I|\leq 2r} (t, s, y), (\partial_f u)_{|I|\leq 2r} (s, s, y) \right\}.
\]

Given \( u \in \Omega^{(2r+\alpha)}_{[0,\delta]} \), all coefficients (\( \partial F^a_{b}(u) \) and \( \partial \partial F^a_{b}(u) \)) and the inhomogeneous term \( \partial y_i F^a(u) \) are all in \( \Omega^{(2r+\alpha)}_{[0,\delta]} \). Moreover, the regularity of \( g \in \Omega^{(2r+K+\alpha)}_{[0,T]} \) ensures \( g_{y_i} \in \Omega^{(2r+\alpha)}_{[0,\delta]} \). Therefore, the well-posedness of nonlocal linear higher-order systems promises \( D_{\partial_f u} \in \Omega^{(2r+\alpha)}_{[0,\delta]} \). Similarly, we can also show iteratively the cases for \( k \leq K \).

\[ \square \]

Proof of Theorem 5.2. First, under the regularity assumptions of \( F \) and \( g \), Proposition 3.1 and Lemma 5.1 guarantee that there exists a unique solution \( u(t, s, y) \) of Equation (65), which is first-order continuously differentiable in \( s \) and third-order continuously differentiable with respect to \( y \). Consequently, the family of random fields \( (X, Y, Z, \Gamma, A) \) defined by Equation (66) is well-defined (adapted).

Next, we show that the random field solves the flow of 2FBSDEs, that is, Equation (67). For any fixed \( (t, s) \in \mathcal{V}[t_0, T] \), we apply the Itô’s formula to the map \( \tau \rightarrow u^a(t, \tau, X(\tau)) \) on \([s, T]\). Then we have

\[
du^a(t, \tau, X(\tau)) = \left[ u^a_t(t, \tau, X(\tau)) + \sum_{i=1}^d b_i(t, \tau, X(\tau)) \left( \frac{\partial u}{\partial y_i} \right)^a (t, \tau, X(\tau)) \right] d\tau
\]

\[
+ \frac{1}{2} \sum_{i,j=1}^d \sigma \sigma^\top (\tau, X(\tau)) \left( \frac{\partial^2 u}{\partial y_i \partial y_j} \right)^a (t, \tau, X(\tau)) d\tau
\]

\[
+ \left( u^a_t \right)^\top (t, \tau, X(\tau)) \sigma (\tau, X(\tau)) dW(\tau)
\]

\[
= \left[ -F^a(t, \tau, X(\tau), u(t, \tau, X(\tau)), u_y(t, \tau, X(\tau)), u_{yy}(t, \tau, X(\tau)), u_{yyy}(t, \tau, X(\tau)), u_y(t, \tau, X(\tau)), u_{yy}(t, \tau, X(\tau)), u_{yyy}(t, \tau, X(\tau)) \right]
\]

\[
+ \frac{1}{2} \sum_{i,j=1}^d \sigma \sigma^\top (\tau, X(\tau)) \left( \frac{\partial^2 u}{\partial y_i \partial y_j} \right)^a (t, \tau, X(\tau)) - \sum_{i=1}^d b_i(t, \tau, X(\tau)) \left( \frac{\partial u}{\partial y_i} \right)^a (t, \tau, X(\tau)) d\tau
\]

\[
+ \left( u^a_y \right)^\top (t, \tau, X(\tau)) \sigma (\tau, X(\tau)) dW(\tau)
\]

\[
= -F^a(t, \tau, X(\tau), Y(t, \tau), Y(t, \tau), Z(t, \tau), Z(t, \tau), \Gamma(t, \tau), \Gamma(t, \tau)) d\tau + (Z^a)^\top (t, \tau) dW(\tau),
\]

which indicates \( dY(t, \tau) = -Fd\tau + Z^a(t, \tau) dW(\tau) \). Similarly, for any fixed \( (t, s) \in \mathcal{V}[t_0, T] \), by applying the Itô’s formula to \( \tau \rightarrow (\sigma \sigma^\top u^a) (t, \tau, X(\tau)) \) on \([t_0, s]\), we can also verify that \( dZ^a(t, \tau) = A^a(t, \tau) d\tau + F^a(t, \tau) dW(\tau) \) for \( a = 1, \ldots, m \). Hence, Equation (66) is an adapted solution of Equation (67).

\[ \square \]
Proof of Proposition 4.4. By the classical theory of ODE systems, we could use the conventional contraction mapping arguments to obtain the local well-posedness of Equation (63). Next, we consider a special case of Equation (60), where \( \mathbf{w} \) is a diagonal matrix, that is, \( \mathbf{w} = \text{diag}\{w^{11}, w^{22}, \ldots, w^{mm}\} \). Under this condition, the system of ODEs (60) has the following form:

\[
\begin{align*}
\dot{\varphi}^a(t, s) + \sum_{1 \leq b \leq m} \beta \left( \varphi^b(s, s) \frac{\varphi^b(t, t)}{\varphi^b(s, s)} \right)^{\frac{1}{\beta - 1}} \varphi^a(t, s) + \sum_{1 \leq b \leq m} \mathbf{v}^{ab}(t, s) \left( \frac{\varphi^b(s, s)}{\varphi^b(s, s)} \right)^{\frac{1}{\beta - 1}} = 0, \\
\varphi(t, T) = \mathbf{g}(t), \quad 0 \leq t \leq s \leq T, \quad a = 1, \ldots, m,
\end{align*}
\]

(A.53)

where \( k^a(t, s) = k - w^{aa}(t, s) \). In this special case, for \( a = 1, \ldots, m \), we have

\[
\begin{align*}
\varphi^a(t, s) &= \exp \left\{ \int_s^T \left[ k^a(t, \tau) - \sum_{1 \leq b \leq m} \beta \left( \varphi^b(\tau, \tau) \frac{\varphi^b(t, t)}{\varphi^b(\tau, \tau)} \right)^{\frac{1}{\beta - 1}} \right] d\tau \right\} g^a(t) \\
&+ \int_s^T \exp \left\{ \int_s^{\tau} \left[ k^a(\tau, \sigma) - \sum_{1 \leq b \leq m} \beta \left( \varphi^b(\sigma, \sigma) \frac{\varphi^b(t, t)}{\varphi^b(\sigma, \sigma)} \right)^{\frac{1}{\beta - 1}} \right] d\sigma \right\} \left( \sum_{1 \leq b \leq m} \mathbf{v}^{ab}(\tau, \sigma) \frac{\varphi^b(\sigma, \sigma)}{\varphi^b(\sigma, \sigma)} \right)^{\frac{1}{\beta - 1}} d\tau.
\end{align*}
\]

Denoting by \( \tilde{\varphi}^a(s) = \frac{\varphi^a(s, s)}{\varphi^a(t, t)} \), \( \tilde{g}^a(t) = \frac{g^a(t)}{g^a(t, t)} \), and \( \tilde{\mathbf{v}}^{ab}(t, s) = \frac{\mathbf{v}^{ab}(t, s)}{\mathbf{v}^{aa}(t, t)} \), we obtain

\[
\begin{align*}
\tilde{\varphi}^a(s) &= \exp \left\{ \int_s^T \left[ k^a(s, \tau) - \sum_{1 \leq b \leq m} \beta \tilde{\varphi}^b(\tau, \tau) \frac{1}{\beta - 1} \right] d\tau \right\} \tilde{g}^a(s) \\
&+ \int_s^T \exp \left\{ \int_s^{\tau} \left[ k^a(\tau, \sigma) - \sum_{1 \leq b \leq m} \beta \tilde{\varphi}^b(\sigma, \sigma) \frac{1}{\beta - 1} \right] d\sigma \right\} \left( \sum_{1 \leq b \leq m} \tilde{\mathbf{v}}^{ab}(\tau, \sigma) \frac{\varphi^b(\sigma, \sigma) \varphi^b(\tau, \tau)}{\varphi^b(\sigma, \sigma)} \right)^{\frac{1}{\beta - 1}} d\tau. \quad (A.54)
\end{align*}
\]

Let

\[
\tilde{\varphi}^a(s) = \tilde{\varphi}^a(s) \prod_{1 \leq b \leq m} \exp \left\{ \beta \int_s^T \tilde{\varphi}^b(\tau, \tau) \frac{1}{\beta - 1} d\tau \right\}, \quad \tilde{g}^a(s) = \tilde{g}^a(s) \exp \left\{ \int_s^T k^a(s, \tau) d\tau \right\}, \quad \tilde{\mathbf{v}}^{ab}(s, \sigma) = \tilde{\mathbf{v}}^{ab}(s, \sigma) \exp \left\{ \int_s^\sigma k^a(s, \tau) d\tau \right\}.
\]

Then, we have

\[
\tilde{\varphi}^a(s) = \tilde{g}^a(s) + \int_s^T \left( \sum_{1 \leq b \leq m} \tilde{\mathbf{v}}^{ab}(s, \sigma) \varphi^b(\sigma) \frac{1}{\beta - 1} \right) d\sigma. \quad (A.55)
\]

We impose the following conditions: there exist some constants \( g_0 > 0 \) and \( \gamma > 0 \) such that

\[
\tilde{g}^a(s) \geq \frac{g^a(s)}{\varphi^a(s, s)} \exp \left\{ \int_s^T k^a(s, \tau) d\tau \right\} \geq g_0 \quad (A.56)
\]
and
\[ \hat{v}^{ab}(s, \sigma) = \hat{v}^{ab}(s, \sigma) \exp \left\{ \int_s^\sigma k^a(s, \tau) d\tau \right\} = \frac{v^{ab}(s, \sigma)}{v^{ab}(s, s)} \exp \left\{ \int_s^\sigma k^a(s, \tau) d\tau \right\} \geq e^{-\gamma(\sigma-s)} \] (A.57)

hold for \( a, b = 1, \ldots, m \). With conditions (A.56) and (A.57), we have
\[ \hat{\phi}^a(s)e^{-\gamma s} \geq g_0e^{-\gamma s} + \sum_{1 \leq b \leq m} \int_s^T [\hat{\phi}^b(\sigma)e^{-\gamma \sigma}] \hat{\phi}^b(\sigma)^{\frac{1}{\beta-1}} d\sigma =: \omega(s), \quad a = 1, \ldots, m. \]

Note that
\[ \frac{d\omega(s)}{ds} = -\gamma g_0e^{-\gamma s} - \sum_{1 \leq b \leq m} [\hat{\phi}^b(s)e^{-\gamma s}] \hat{\phi}^b(s)^{\frac{1}{\beta-1}} \leq -\gamma g_0e^{-\gamma s} - \sum_{1 \leq b \leq m} \xi^b(s)\hat{\phi}^b(s)^{\frac{1}{\beta-1}}, \] (A.58)

\[ d\left( \omega(s) \prod_{1 \leq b \leq m} \exp \left\{ -\int_s^T \hat{\phi}^b(\tau)^{\frac{1}{\beta-1}} d\tau \right\} \right) \leq -\gamma g_0e^{-\gamma s} \prod_{1 \leq b \leq m} \exp \left\{ -\int_s^T \hat{\phi}^b(\tau)^{\frac{1}{\beta-1}} d\tau \right\} ds. \]

By integrating both sides above over \([s, T]\), it follows that
\[ g_0e^{-\gamma T} - \omega(s) \prod_{1 \leq b \leq m} \exp \left\{ -\int_s^T \hat{\phi}^b(\tau)^{\frac{1}{\beta-1}} d\tau \right\} \leq g_0 \int_s^T e^{-\gamma \tau} \prod_{1 \leq b \leq m} \exp \left\{ -\int_s^T \hat{\phi}^b(\tau)^{\frac{1}{\beta-1}} d\tau \right\} d\sigma. \]

Hence, for \( a = 1, \ldots, m \),
\[ \omega(s) \geq \prod_{1 \leq b \leq m} \exp \left\{ \int_s^T \hat{\phi}^b(\tau)^{\frac{1}{\beta-1}} d\tau \right\} \left( g_0e^{-\gamma T} + g_0 \int_s^T e^{-\gamma \tau} \prod_{1 \leq b \leq m} \exp \left\{ -\int_s^T \hat{\phi}^b(\tau)^{\frac{1}{\beta-1}} d\tau \right\} d\sigma \right). \]

Thus, for \( a = 1, \ldots, m \), we have
\[ \hat{\phi}^a(s) = \hat{\phi}^a(s) \prod_{1 \leq b \leq m} \exp \left\{ -\beta \int_s^T \hat{\phi}^b(\tau)^{\frac{1}{\beta-1}} d\tau \right\} \geq e^{\beta\gamma} \omega(s) \prod_{1 \leq b \leq m} \exp \left\{ -\beta \int_s^T \hat{\phi}^b(\tau)^{\frac{1}{\beta-1}} d\tau \right\} \]
\[ \geq e^{\gamma s} \prod_{1 \leq b \leq m} \exp \left\{ (1-\beta) \int_s^T \hat{\phi}^b(\tau)^{\frac{1}{\beta-1}} d\tau \right\} \left( g_0e^{-\gamma T} + g_0 \int_s^T e^{-\gamma \tau} \prod_{1 \leq b \leq m} \exp \left\{ -\int_s^T \hat{\phi}^b(\tau)^{\frac{1}{\beta-1}} d\tau \right\} d\sigma \right), \]

Moreover, we can also obtain the upper bounds:
\[ \hat{v}^a(s) = \exp \left\{ \int_s^T \left[ k^a(s, \tau) - \sum_{1 \leq b \leq m} \beta \hat{\phi}^b(\tau)^{\frac{1}{\beta-1}} \right] d\tau \right\} \hat{g}^a(s) \]
\[ + \int_s^T \exp \left\{ \int_s^\sigma \left[ k^a(s, \tau) - \sum_{1 \leq b \leq m} \beta \hat{\phi}^b(\tau)^{\frac{1}{\beta-1}} \right] d\tau \right\} \left( \sum_{1 \leq b \leq m} \hat{v}^{ab}(s, \sigma)\hat{\phi}^b(\sigma)^{\frac{1}{\beta-1}} \right) d\sigma \]
\[ \leq \exp \left\{ \int_s^T k^a(s, \tau) d\tau \right\} \hat{g}^a(s) + \int_s^T \exp \left\{ \int_s^\sigma k^a(s, \tau) d\tau \right\} \left( \sum_{1 \leq b \leq m} \hat{v}^{ab}(s, \sigma)\hat{\phi}^b(\sigma)^{\frac{1}{\beta-1}} \right) d\sigma \leq C. \]

After showing \( \hat{\phi}^a(s) \in [c, C] \) for any \( s \in [0, T] \) and \( a = 1, \ldots, m \), we are ready to prove the global well-posedness of Equation (A.58). Specifically, by choosing a suitably small \( s \in [0, T] \), we can first obtain a small-time solvability of Equation (A.58) with the Banach fixed-point arguments.
This appendix presents the partial derivatives of the nonlinearity of Equation (48) $\mathbb{H} := \mathbb{H}_\gamma(t, s, y, z)$ with respect to its arguments. For its first-order derivative, we have

$$\partial_t \mathbb{H}^a_b = \begin{cases} 0, & \text{if } |I| = 0, \\ \sigma_j^a(\mu_b - r)\bar{w}_b(T - t, T - s) + \sigma_j^a(\mu_b - r)^2 \bar{w}_b^2(T - t, T - s), & \text{if } a = b, |I| = 1, \\ \frac{1}{2} \sum_{1 \leq b \leq m} \left( \frac{\sigma_j^a(\mu_b - r)\bar{w}_b(T - t, T - s) + \sigma_j^a(\mu_b - r)\bar{w}_b^2(T - t, T - s)}{\bar{w}_b^2(T - t, T - s) - \sigma_j^a \bar{u}_b^b(T - t, T - s)} \right)^2, & \text{if } a = b, |I| = 2, \\ 0, & \text{if } a \neq b, |I| = 1,2, \end{cases}$$

(B.1)

Furthermore, the second-order derivatives can be derived from Equations (B.1) and (B.2):

$$\partial_{tt}^2 \mathbb{H}^a_b = \begin{cases} 0, & \text{if } |I| = 0, \\ \frac{-\gamma \exp[-rs]\bar{w}_b(T - t, T - s)(\mu_b - r)}{\bar{w}_b^2(T - t, T - s) - \sigma_j^a \bar{u}_b^b(T - t, T - s)} + \frac{\exp[-2rs]\bar{w}_b(T - t, T - s)(2(\mu_b - r)\bar{w}_b(T - t, T - s) + 2(\mu_b - r)^2 \bar{w}_b^2(T - t, T - s))}{(\bar{w}_b^2(T - t, T - s) - \sigma_j^a \bar{u}_b^b(T - t, T - s))^2}, & \text{if } |I| = 1, \\ \frac{-\gamma \exp[-rs]\bar{w}_b(T - t, T - s)(\mu_b - r)}{\bar{w}_b^2(T - t, T - s) - \sigma_j^a \bar{u}_b^b(T - t, T - s)} + \frac{\exp[-2rs]\bar{w}_b(T - t, T - s)(\sigma_j^a \bar{w}_b^2(T - t, T - s) + \sigma_j^a(\mu_b - r)\bar{w}_b^2(T - t, T - s))}{(\bar{w}_b^2(T - t, T - s) - \sigma_j^a \bar{u}_b^b(T - t, T - s))^3}, & \text{if } |I| = 2, \end{cases}$$

where $\bar{w}_b(T - t, T - s)$, $\bar{u}_b^b(T - t, T - s)$, and $\bar{u}_b^b(T - t, T - s)$ are partial derivatives of $\mathbb{H}$ with respect to $t$, $s$, and $y$, respectively.

Finally, for all $|I| = 0, 1, 2$, $\partial_{tt}^2 \mathbb{H}^a_b = 0$.

Next, the bounds of $\mathbb{H}(s)$ guarantee the extension from the local solution to an arbitrary large time interval by standard continuation arguments.
\[
\frac{\partial^2}{\partial t^2} \mathbb{I}_{ij} = \begin{cases} 
0, & \text{if } |I| = |J| = 0 \text{ or } a \neq c, \\
\frac{\hat{w}_b(T-s,T-s,y) - \sigma^2 b U_{yy}(s,s,y)}{\sigma^2_b(\mu_b - r) \hat{w}_b(T-s,T-s,y) + \sigma^2_b(\mu_b - r)^2 U_b(s,s,y)} & \text{if } |I| = |J| = 2, a = c, \\
\frac{(\hat{w}_b(T-s,T-s,y) - \sigma^2 b U_{yy}(s,s,y))^2}{\sigma^2_b(\mu_b - r) \hat{w}_b(T-s,T-s,y) + \sigma^2_b(\mu_b - r)^2 U_b(s,s,y)} & \text{if } |I| = 1, |J| = 2, a = c, \\
\frac{(\sigma^2_b \hat{w}_b(T-s,T-s,y) + \sigma^2_b(\mu_b - r) U_b(s,s,y))^2}{(\hat{w}_b(T-s,T-s,y) - \sigma^2 b U_{yy}(s,s,y))^3} & \text{if } |I| = |J| = 2, a = c.
\end{cases}
\]