A class of index transforms generated by the Mellin and Laplace operators

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Abstract

Classical integral representation of the Mellin type kernel

$$x^{-z} = \frac{1}{\Gamma(z)} \int_0^{\infty} e^{-xt} t^{z-1} dt, \; x > 0, \; \text{Re } z > 0,$$

in terms of the Laplace integral gives an idea to construct a class of non-convolution (index) transforms with the kernel

$$k^\pm_z(x) = \int_0^{\infty} \frac{e^{-xt\pm 1}}{r(t)} t^{z-1} dt, \; x > 0,$$

where $r(t) \neq 0, \; t \in \mathbb{R}_+$ admits a power series expansion, which has an infinite radius of convergence and the integral converges absolutely in a half-plane of the complex plane $z$. Particular examples give the Kontorovich-Lebedev-like transformation and new transformations with hypergeometric functions as kernels. Mapping properties and inversion formulas are obtained. Finally we prove a new inversion theorem for the modified Kontorovich-Lebedev transform.

Keywords: Mellin transform, Laplace transform, Kontorovich-Lebedev transform, modified Bessel functions, hypergeometric functions

AMS subject classification: 44A15, 33C05, 33C10, 33C15

1 Introduction and preliminary results

In this paper we construct a class of integral transformations of the non-convolution type, which involves an integration with respect to parameters of hypergeometric functions. We will base on mapping and inversion properties of the Mellin and Laplace transforms [5] given, respectively, by formulas

$$(Mf)(z) = \int_0^{\infty} f(t) t^{z-1} dt, \; z \in \mathbb{C}, \quad (1.1)$$

$$(Lf)(x) = \int_0^{\infty} f(t) e^{-xt} dt, \; x \in \mathbb{R}_+, \quad (1.2)$$

where the integrals converge in an appropriate sense, which will be clarified below.

The idea to obtain such a new class of index transformations comes from classical representation of the Mellin kernel

$$x^{-z} = \frac{1}{\Gamma(z)} \int_0^{\infty} e^{-xt} t^{z-1} dt, \; x > 0, \; \text{Re } z > 0,$$

where $\Gamma(z)$ is Euler’s gamma-function. Hence doing our steps formally first, we will show how to invert using (1.3) the modified Mellin transform

$$(Ff)(z) = \int_0^{\infty} t^{-z} f(t) e^{-at} dt, \; a > 0 \quad (1.4)$$
and then will motivate it rigorously in a special class of functions. Indeed, substituting (1.3) into (1.4) and changing the order of integration, we find
\[
(Ff)(z) = \frac{1}{\Gamma(z)} \int_0^\infty x^{z-1} \int_0^\infty e^{-(x+a)t} f(t) dt dx.
\] (1.5)
Hence appealing to the inversion formula of the Mellin transform \[4\], \[5\], we have the equality
\[
\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(z)(Ff)(z)x^{-z} dz = \int_0^\infty e^{-(x+a)t} f(t) dt.
\]
Substituting again (1.3) in the left-hand side of the latter equality, we change the order of integration to get
\[
\int_0^\infty e^{-xt} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} (Ff)(z)t^{z-1} dz dt = \int_0^\infty e^{-(x+a)t} f(t) dt.
\] (1.6)
Finally canceling the Laplace transform (1.2), we come out with the inversion formula of the transformation (1.4)
\[
f(x) = \frac{e^{ax}}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} (Ff)(z)x^{z-1} dz, \ x > 0,
\] (1.7)
which coincides with the classical inversion formula for the Mellin transform up to a simple change of variables and functions.

The rigorous proof of these reciprocal formulas can be done in a special class of functions related to the Mellin transform and its inversion, which was introduced in \[7\] (see also in \[8\]). Indeed, we have

**Definition 1.** Denote by \(\mathcal{M}^{-1}(L_c)\) the space of functions \(f(x), x \in \mathbb{R}_+\), representable by inverse Mellin transform of integrable functions \(f^*(s) \in L_1(c)\) on the vertical line \(c = \{s \in \mathbb{C} : \text{Re} s = c_0\}\):
\[
f(x) = \frac{1}{2\pi i} \int_c f^*(s)x^{-s} ds.
\] (1.8)

The space \(\mathcal{M}^{-1}(L_c)\) with the usual operations of addition and multiplication by scalar is a linear vector space. If the norm in \(\mathcal{M}^{-1}(L_c)\) is introduced by the formula
\[
\|f\|_{\mathcal{M}^{-1}(L_c)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |f^*(c_0 + it)| dt,
\] (1.9)
then it becomes a Banach space.

**Definition 2** (\[7\], \[8\]). Let \(c_1, c_2 \in \mathbb{R}\) be such that \(2\text{sign } c_1 + \text{sign } c_2 \geq 0\). By \(\mathcal{M}^{-1}_{c_1,c_2}(L_c)\) we denote the space of functions \(f(x), x \in \mathbb{R}_+\), representable in the form (1.8), where \(s^{c_2}e^{\pi c_1|s|} f^*(s) \in L_1(e)\).

It is a Banach space with the norm
\[
\|f\|_{\mathcal{M}^{-1}_{c_1,c_2}(L_c)} = \frac{1}{2\pi} \int_e e^{\pi c_1|s|} |s^{c_2} f^*(s)| ds.
\]
In particular, letting \(c_1 = c_2 = 0\) we get the space \(\mathcal{M}^{-1}(L_c)\). Moreover, it is easily seen the inclusion \((c_0 \neq 0)\)
\[
\mathcal{M}^{-1}_{d_1,d_2}(L_c) \subseteq \mathcal{M}^{-1}_{c_1,c_2}(L_c)
\]
when \(2\text{sign}(d_1 - c_1) + \text{sign}(d_2 - c_2) \geq 0\).
We have

**Theorem 1.** Let \( f \in \mathcal{M}^{-1}(L_c) \), \( a > 0, c_0 < 1 \). Then transformation (1.4) is well-defined and \((Ff)(z)\) is analytic in the half-plane \( \text{Re} z < 1 - c_0 \). Moreover,

\[
(Ff)(z) = \frac{1}{2\pi i} \int_{c_0-i\infty}^{c_0+i\infty} \Gamma(1-s-z)f^*(s)a^{z+s-1}ds,
\]

(1.10)

and the operator \( F : \mathcal{M}^{-1}(L_c) \to L_1(\text{Re} - i\infty, \text{Re} + i\infty), \text{Re} < 1 - c_0 \) is bounded with the norm satisfying the estimate

\[
\|F\| \leq a^{\text{Re} z + c_0 - 1} \int_{-\infty}^{\infty} |\Gamma(1-c_0 - \text{Re}z - i\tau)|d\tau.
\]

Finally, for all \( x > 0 \) inversion formula (1.7) holds.

**Proof.** Indeed, substituting (1.3) into (1.4) we change the order of integration by Fubini’s theorem via the estimate (see (1.8))

\[
|\Gamma(z)| \leq e^{-\pi i z} z e^{-\pi |z|^2} \leq e^{-\pi z} z e^{-\pi \text{Re} z^2} \leq \frac{a^{\text{Re} z + c_0 - 1}}{2\pi} \Gamma(1-c_0 - \text{Re}z - i\tau) |f^*(s)ds| < \infty,
\]

which also guarantees the analyticity of \((Ff)(z)\) in the strip \( \text{Re} z < 1 - c_0 \). Thus calculating the inner integral with respect to \( t \) we arrive at the representation (1.10). Finally, the norm estimation is given by the inequality

\[
\|Ff\|_1 = \int_{-\infty}^{\infty} |(Ff)(\text{Re}z + i\tau)|d\tau
\]

\[
\leq a^{\text{Re} z + c_0 - 1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\Gamma(1-c_0 - \text{Re}z - i(\tau + t))f^*(c_0 + it)|dt d\tau
\]

\[
\leq a^{\text{Re} z + c_0 - 1} \|f\|_{\mathcal{M}^{-1}(L_c)} \int_{-\infty}^{\infty} |\Gamma(1-c_0 - \text{Re}z - i\tau)|d\tau.
\]

In order to prove formula (1.7), we multiply both sides of (1.10) by \( x^{z-1} \), \( x > 0 \) and integrate with respect to \( z \) over the line \( \gamma - i\infty, \gamma + i\infty \), \( \gamma < 1 - c_0 \). Hence changing the order of integration via the absolute convergence and calculating the inner integral as an inverse Mellin transform of the Gamma-function, we derive

\[
\int_{\gamma-i\infty}^{\gamma+i\infty} (Ff)(z)x^{z-1}dz = \frac{1}{2\pi i} \int_{c_0-i\infty}^{c_0+i\infty} f^*(s)a^s \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(1-s-z)(ax)^{z-1}dzds
\]

\[
= e^{-ax} \int_{c_0-i\infty}^{c_0+i\infty} f^*(s)x^{-s}ds = 2\pi i e^{-ax}f(x),
\]

which proves (1.7).

\[\square\]

**2 General non-convolution transforms**

Let us consider a general non-convolution transformation

\[
(Ff)(z) = \int_{0}^{\infty} k_z(x)f(x)dx, \quad z \in \mathbb{C},
\]

(2.1)
The operator

\[ r(t) = \sum_{k=0}^{\infty} a_k t^k \]

with an infinite radius of convergence.

**Theorem 2.** Let \( f \in \mathcal{M}^{-1}(L_c), \ c_0 < 1. \) Let \( r^{-1}(t) \in L_1(\mathbb{R}_+; t^{\gamma-c_0} dt), \gamma \in \mathbb{R}, \ \rho(s) \in L_1(1 + \gamma - c_0 - i\infty, 1 + \gamma - c_0 + i\infty), \) where \( \rho(s) \) is the Mellin transform (1.1) of the function \( r^{-1}(t) \)

\[ \rho(s) = \int_0^\infty \frac{t^{s-1}}{r(t)} dt. \]  

Then transformation (2.1) is well-defined and \( (Ff)(z), \text{Re}z = \gamma \) can be represented in the form

\[ (Ff)(z) = \frac{1}{2\pi i} \int_{c_0-i\infty}^{c_0+i\infty} \Gamma(1 - s) \rho(1 + z - s) f^*(s) ds. \]  

The operator (2.1) is bounded from \( \mathcal{M}^{-1}(L_c) \) into \( L_1(1 - c_0 + \gamma - i\infty, 1 - c_0 + \gamma + i\infty) \) and

\[ ||F|| \leq \Gamma(1 - c_0) \int_{-\infty}^{\infty} |\rho(1 - c_0 + \gamma - i\tau)| d\tau. \]  

Moreover, when \( \gamma < 0 \) and \( (Ff)(z)/\Gamma(-z) \in L_1(\gamma - i\infty, \gamma + i\infty), \) for all \( x > 0 \) the inversion formula holds

\[ f(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \hat{k}_\gamma^-(x)(Ff)(z) dz, \]  

where

\[ \hat{k}_\gamma^-(x) = \sum_{k=0}^{\infty} \frac{a_k x^{k-z-1}}{\Gamma(k-z)}. \]  

and integral (2.6) converges absolutely.

**Proof.** Since using (1.8), (2.2) and conditions of the theorem

\[ |(Ff)(z)| \leq \int_0^\infty |k_\gamma^-(x)f(x)| dx \leq \frac{1}{2\pi} \int_0^\infty |k_\gamma^-(x)| x^{-c_0} dx \int_{c_0-i\infty}^{c_0+i\infty} |f^*(s)| ds, \]

\[ \leq \frac{1}{2\pi} \int_0^\infty \int_0^\infty \frac{e^{-x/t} x^{-c_0}}{|r(t)|} t^{Rez-1} dt dx \int_{c_0-i\infty}^{c_0+i\infty} |f^*(s)| ds, \]

\[ = \frac{\Gamma(1 - c_0)}{2\pi} \int_0^\infty \int_{-\infty}^{\infty} \frac{|f^*(s)| ds}{|r(t)|} dt \int_{c_0-i\infty}^{c_0+i\infty} |f^*(s)| ds < \infty, \]

one can substitute (1.8) into (2.1) and change the order of integration via the absolute convergence. After calculation of the inner integral employing the convolution property of the Mellin transform [5] and minding (2.3), we come out with representation (2.4). Hence

\[ ||Ff||_1 \leq \frac{\Gamma(1 - c_0)}{2\pi} \int_{c_0-i\infty}^{c_0+i\infty} \int_{-\infty}^{\infty} |\rho(1 + \gamma - c_0 - i\tau)| f^*(s) ds d\tau \]

\[ \leq \Gamma(1 - c_0)||f||_{\mathcal{M}^{-1}(L_c)} \int_{-\infty}^{\infty} |\rho(1 + \gamma - c_0 - i\tau)| d\tau, \]
which yields (2.5). Returning to (2.4) we multiply both sides of this equality by \(x^z\), \(x > 0\) and integrate with respect to \(z\) over the line \((\gamma - i\infty, \gamma + i\infty)\). Changing the order of integration by Fubini’s theorem, which is applicable by virtue of the absolute convergence of the corresponding integral, we calculate the inner integral via the inversion theorem for the Mellin transform [5], since the original function and its image are integrable. Therefore reciprocally from (2.3) for all \(x > 0\) we have

\[
\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \rho(1 + z - s)x^z\,dz = x^{s-1}|r(1/x)|^{-1}
\]

and

\[
\int_{\gamma - i\infty}^{\gamma + i\infty} (Ff)(z)x^z\,dz = [r(1/x)]^{-1} \int_{c_0 + i\infty}^{c_0 - i\infty} \Gamma(1 - s)f^*(s)x^{s-1}\,ds.
\]

Hence appealing in the right-hand side of the latter equality to the Parseval identity for the Mellin transform, we derive

\[
\frac{r(1/x)}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} (Ff)(z)x^z\,dz = \int_{0}^{\infty} e^{-xt}f(t)\,dt.
\]  

(2.8)

Meanwhile, bearing in mind the series representation of the function \(r(1/x)\) and its infinite radius of convergence, we observe that the series of coefficients \(\sum_{k=0}^{\infty} a_k\) converges absolutely. So, the left-hand side of (2.8) can be treated as follows

\[
\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} (Ff)(z) \sum_{k=0}^{\infty} a_k x^{-k}\,dz = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} (Ff)(z) \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k - z)} \int_{0}^{\infty} e^{-xt} t^{k-1}\,dt\,dz
\]

\[
= \int_{0}^{\infty} e^{-xt} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} (Ff)(z) \sum_{k=0}^{\infty} \frac{a_k t^{k-1}}{\Gamma(k - z)} \,dz\,dt,
\]

where the change of the order of integration is possible via the integrability of the function \((Ff)(z)/\Gamma(-z)\) and the estimate \((\gamma < 0)\)

\[
\int_{\gamma - i\infty}^{\gamma + i\infty} \left| \frac{(Ff)(z)}{\Gamma(-z)} \right| \sum_{k=0}^{\infty} \frac{|a_k| \Gamma(k - \gamma)}{(-\gamma)_k} \,dz \leq \int_{\gamma - i\infty}^{\gamma + i\infty} \left| \frac{(Ff)(z)}{\Gamma(-z)} \right| \sum_{k=0}^{\infty} \frac{|a_k| \Gamma(k - \gamma)}{(-\gamma)_k} \,dz
\]

\[
= \Gamma(-\gamma) \int_{\gamma - i\infty}^{\gamma + i\infty} \left| \frac{(Ff)(z)}{\Gamma(-z)} \right| \sum_{k=0}^{\infty} |a_k| < \infty,
\]

where \((-\gamma)_k\) is the Pochhammer symbol [1], Vol. I.

Thus returning to (2.8), we get the equality

\[
\int_{0}^{\infty} e^{-xt} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} (Ff)(z) \sum_{k=0}^{\infty} \frac{a_k t^{k-1}}{\Gamma(k - z)} \,dz\,dt = \int_{0}^{\infty} e^{-xt} f(t)\,dt, \quad x > 0.
\]  

(2.9)

As we see, equality (2.9) is true for all \(x > 0\), where functions under the convergent Laplace integrals in its both sides are continuous on \(\mathbb{R}_+\) owing to the condition \(f \in \mathcal{M}^{-1}(L_c)\) and assumptions of the theorem. Therefore one can cancel the Laplace transform (1.2) in (2.9) by virtue of the uniqueness theorem (see in [2]). Consequently, we established the inversion formula (2.6) and completed the proof of Theorem 2. \(\square\)
We note, that if \( r(t) \equiv P_n(t) \) is a polynomial with no zeros on the line \( \mathbb{R}_+ \), then Theorem 2 is true, where instead of the series final sums are involved. But in this case we can also prove similarly the theorem about mapping properties and inversion formula of the following transformation

\[
(Ff)_n(z) = \int_0^\infty k_n(z,x) f(x) dx, \quad z \in \mathbb{C}, \ n \in \mathbb{N},
\]

(2.10)

where

\[
k_n(z,x) = \int_0^\infty \frac{e^{-zt}}{P_n(t)} t^{z-1} dt, \ x > 0
\]

(2.11)

and \( P_n(t) = \sum_{k=0}^n a_k t^k \). Precisely, we have the following result.

**Theorem 3.** Let \( n \in \mathbb{N}, f \in \mathcal{M}^{-1}(L_c), c_0 < 1. \) Let \( \gamma \in \mathbb{R}, -c_0 < \gamma < n - c_0, \ \rho_n(s) \in L_1(\gamma + c_0 - i\infty, \gamma + c_0 + i\infty), \) where \( \rho_n(s) \) is the Mellin transform (1.1) of \( P_n^{-1}(t) \)

\[
\rho_n(s) = \int_0^\infty \frac{t^{s-1}}{P_n(t)} dt.
\]

(2.12)

Then transformation (2.10) is well-defined and \( (Ff)_n(z) \) has the representation

\[
(Ff)_n(z) = \frac{1}{2\pi i} \int_{c_0-i\infty}^{c_0+i\infty} \Gamma(1-s) \rho_n(s+z)f^*(s) ds.
\]

(2.13)

The operator (2.10) is bounded from \( \mathcal{M}^{-1}(L_c) \) into \( L_1(c_0 + \gamma - i\infty, c_0 + \gamma + i\infty) \) and

\[
||F|| \leq \Gamma(1-c_0) \int_{-\infty}^\infty |\rho_n(c_0+\gamma-i\tau)| d\tau.
\]

Moreover, when \( \max (-c_0, n-1) < \gamma < n - c_0 \) and \( (Ff)_n(z)/\Gamma(1+z) \in L_1(\gamma - i\infty, \gamma + i\infty) \), for all \( x > 0 \) the inversion formula holds

\[
f(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \hat{k}_n(z,x)(Ff)(z) dz,
\]

(2.14)

where

\[
\hat{k}_n(z,x) = \sum_{k=0}^n \frac{a_k x^{-k}}{\Gamma(1+z-k)}
\]

(2.15)

and integral (2.14) converges absolutely.

### 3 Examples of new index transforms

We start this section showing an interesting example of the transform (2.1) recently discovered by the author (see in [10]). In fact, let \( r(t) = e^{-t} \). Then calculating the integral (2.2), we get \( k_\gamma(x) = 2x^{\gamma/2}K_\gamma(2\sqrt{x}) \), where \( K_\gamma(2\sqrt{x}) \) is the modified Bessel function [1], Vol. II. As it is easily seen, integral (2.1) converges absolutely for any \( x \in \mathbb{R}_+, z \in \mathbb{C} \) and represents an entire function of \( z \). On the other hand, the kernel \( k_\gamma(x) \) can be written with the use of the Parseval relation for the Mellin transform, which leads to the representation

\[
2x^{\gamma/2}K_\gamma(2\sqrt{x}) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(s+z)\Gamma(s)x^{-s} ds, \ x > 0.
\]

(3.1)
Thus the transformation (2.1) in this case has the form

$$(Ff)(z) = 2 \int_{0}^{\infty} x^{z/2} K_{z}(2\sqrt{x}) f(x) dx.$$ (3.2)

This transformation looks like the Kontorovich-Lebedev transform [4], [8], [9]. However, it is a completely different operator and cannot be reduced to the Kontorovich-Lebedev integral by any change of variables and functions. As far as the author is aware, the transform (3.2) was not studied yet, taking into account his mapping properties and inversion formula in an appropriate class of functions. An analog of Theorem 2 for this case is

**Theorem 4** [10]. Let $f \in \mathcal{M}^{-1}(L_{c})$ and $c_{0} < 1$. Then transformation (3.2) is well-defined and $(Ff)(z)$ is analytic in the half-plane $\text{Re} z > c_{0} - 1$. Further,

$$(Ff)(z) = \frac{1}{2\pi i} \int_{c_{0}-i\infty}^{c_{0}+i\infty} \Gamma(1-s) \Gamma(1-s) f^{*}(s) ds,$$ (3.3)

and the operator $F : \mathcal{M}^{-1}(L_{c}) \rightarrow L_{1}(\gamma - i\infty, \gamma + i\infty)$, $\gamma > c_{0} - 1$ is bounded with the norm satisfying the estimate

$$||F|| \leq \Gamma(1-c_{0}) \int_{-\infty}^{\infty} |\Gamma(1-c_{0} + \gamma + i\tau)| d\tau.$$ 

Moreover, when $c_{0} - 1 < \gamma < 0$ and $(Ff)(z)/\Gamma(-z) \in L_{1}(\gamma - i\infty, \gamma + i\infty)$, for all $x > 0$ the inversion formula holds

$$f(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} L_{-1}(1+z) \left(2\sqrt{t}\right) t^{-(1+i)/2} (Ff)(z) dz,$$ (3.4)

where $I_{\nu}(w)$ is the modified Bessel function of the third kind [2], Vol. II and the integral (3.4) converges absolutely.

The Kontorovich-Lebedev-like transformation (3.2) can be generalized considering the following kernel

$$S_{m}(x, z) = \int_{0}^{\infty} e^{-\frac{x}{2} - t^{m} z^{-1}} dt, \quad x > 0, \quad m \in \mathbb{N}.$$ (3.5)

We calculate integral (3.5) in terms of the Meijer $G$-function [1], Vol. I. Precisely, we derive

$$\int_{0}^{\infty} e^{-\frac{x}{2} - t^{m} z^{-1}} dt = \frac{1}{2\pi i m} \int_{\nu-i\infty}^{\nu+i\infty} \Gamma \left( \frac{s+z}{m} \right) \Gamma(s) x^{-s} ds.$$ 

Appealing to the Gauss-Legendre multiplication formula for gamma-function [1], Vol.I

$$\Gamma(ms) = m^{ms-1/2} (2\pi)^{(1-m)/2} \prod_{k=0}^{m-1} \Gamma \left( s + \frac{k}{m} \right), \quad m \in \mathbb{N},$$

the latter Mellin-Barnes integral becomes the following Meijer $G$-function

$$\frac{1}{2\pi i m^{1/2}} \int_{\nu-i\infty}^{\nu+i\infty} \Gamma \left( \frac{s+z}{m} \right) \Gamma(s) x^{-s} ds = \frac{(2\pi)^{(1-m)/2}}{2\pi i m^{1/2}} \int_{m-i\infty}^{m+i\infty} \Gamma \left( s + \frac{z}{m} \right) \prod_{k=0}^{m-1} \Gamma \left( s + \frac{k}{m} \right) \left( \frac{x}{m} \right)^{-ms} ds.$$
The inversion formula (2.6) for this case is given accordingly

\[
(Ff)(z) = \frac{(2\pi)^{(1-m)/2}}{m^{1/2}} \int_0^\infty G^{m+1,0} \left( \frac{x}{m} \right) \left| \begin{array}{c} 0, \frac{1}{m}, \ldots, \frac{m-1}{m}, \frac{z}{m} \\ 0, \frac{1}{m}, \ldots, \frac{m-1}{m} \end{array} \right| f(x)dx, \quad z \in \mathbb{C}. \tag{3.6}
\]

The inversion formula (2.6) for this case is given accordingly

\[
f(x) = \frac{1}{2\pi i} \int_{c_0-i\infty}^{c_0+i\infty} \hat{S}_m(z,x)(Ff)(z)dz, \quad x > 0, \tag{3.7}
\]

where the kernel \(\hat{S}_m(z,x)\) can be expressed in terms of the hyper-Bessel functions. In fact, the corresponding kernel (2.7) is written as the generalized hypergeometric series

\[
\hat{S}_m(z,x) = \sum_{n=0}^\infty \frac{x^{mn-z}}{n! \Gamma(mn-z)} = (2\pi)^{(m-1)/2} m^{1/2} x^{-1} \sum_{n=0}^\infty \frac{(xm)^{mn-z}}{n! \prod_{k=0}^{m-1} \Gamma(n+\frac{k-z}{m})}.
\]

We have

\textbf{Theorem 5.} Let \(m \in \mathbb{N}, f \in \mathcal{M}^{-1}(L_c)\) and \(c_0 < 1\). Then transformation (3.6) is well-defined and \((Ff)(z)\) is analytic in the half-plane \(\text{Re} z > c_0 - 1\). Further,

\[
(Ff)(z) = \frac{1}{2\pi im} \int_{c_0-i\infty}^{c_0+i\infty} \Gamma \left( \frac{1-s+z}{m} \right) \Gamma(1-s)f^*(s)ds,
\]

and the operator \(F : \mathcal{M}^{-1}(L_c) \rightarrow L_1(\gamma-i\infty, \gamma+i\infty), \gamma > c_0 - 1\) is bounded with the norm satisfying the estimate

\[
||F|| \leq \Gamma(1-c_0) \int_{-\infty}^\infty \left| \Gamma \left( \frac{1}{m}(1-c_0 + \gamma + i\tau) \right) \right| d\tau.
\]

Moreover, when \(c_0 - 1 < \gamma < 0\) and \((Ff)(z)/\prod_{k=0}^{m-1} \Gamma(\frac{k-z}{m}) \in L_1(\gamma-i\infty, \gamma+i\infty),\) for all \(x > 0\) the inversion formula (3.7) holds with the absolutely convergent integral.

Next, calling relation (2.3.6.9) in [3], Vol. 1

\[
\int_0^\infty e^{-xt}(t+1)^n t^{z-1} dt = \Gamma(z)\Psi(z, z+1-n; x), \quad x > 0, \quad \text{Re} z > 0, \quad n \in \mathbb{N}, \tag{3.8}
\]

where \(\Psi(a,b;w)\) is Tricomi’s function [1], Vol. 1, we are ready to introduce the corresponding index transform (2.10) in the form

\[
(Ff)(z) = \Gamma(z) \int_0^\infty \Psi(z, z+1-n; x) f(x)dx. \tag{3.9}
\]

According to Theorem 3 and equalities (2.14), (2.15) its inversion formula is given by the following integral

\[
f(x) = \frac{1}{2\pi i} \int_{c_0-i\infty}^{c_0+i\infty} \left( \sum_{k=0}^n \frac{n}{k} \frac{x^{1-z}}{\Gamma(1+z-k)} \right) (Ff)(z)dz, \quad x > 0. \tag{3.10}
\]
But the finite sum inside (3.10) can be expressed in terms of the generalized Laguerre polynomials. In fact, appealing to relation (7.17.1.1) in [3], Vol. 3 we obtain
\[ \sum_{k=0}^{n} \left( \begin{array} {c} n \\ k \end{array} \right) \frac{x^{z-k}}{\Gamma(1+z-k)} = \frac{x^{z}}{\Gamma(1+z)} \, {}_{2}F_{0} \left( -n, -\frac{1}{x}; 1 \right) = \frac{x^{z-n}n!}{\Gamma(1+z)} L_{n}^{z-n}(-x). \]

Hence an analog of Theorem 3 is

**Theorem 6.** Let \( n \in \mathbb{N}, f \in \mathcal{M}^{-1}(L_{c}), c_{0} < 1 \). Let \( \gamma \in \mathbb{R}, 0 < \gamma < n - c_{0} \). Then transformation (3.9) is well-defined and \((Ff)(z)\) has the representation
\[ (Ff)(z) = \frac{1}{2\pi i(n-1)!} \int_{c_{0}-i\infty}^{c_{0}+i\infty} \Gamma(s+z)\Gamma(n-s-z)\Gamma(1-s)f(s)ds. \]

The operator (3.9) is bounded from \( \mathcal{M}^{-1}(L_{c}) \) into \( L_{1}(c_{0} + \gamma - i\infty, c_{0} + \gamma + i\infty) \) and
\[ ||F|| \leq \frac{\Gamma(1-c_{0})}{(n-1)!} \int_{-\infty}^{\infty} |\Gamma(c_{0} + \gamma + i\tau)\Gamma(n-c_{0} - \gamma - i\tau)|d\tau. \]
Moreover, if \( n-1 < \gamma < n-c_{0} \) and \((Ff)(z)/\Gamma(1+z) \in L_{1}(\gamma - i\infty, \gamma + i\infty)\), for all \( x > 0 \) the inversion formula (3.10) holds
\[ f(x) = \frac{n!}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{x^{z-n}}{\Gamma(1+z)} L_{n}^{z-n}(-x)(Ff)(z)dz, \ x > 0, \]
where the latter integral is absolutely convergent.

In particular, letting \( n = 1 \) in (3.8), we use relation (2.3.6.13) in [3], Vol. 1 to get
\[ \int_{0}^{\infty} \frac{e^{-xt}}{t+1} t^{z-1}dt = \Gamma(z)e^{x} \Gamma(1-z,x), \ x > 0, \ Rez > 0, \quad (3.11) \]
where \( \Gamma(w,a) \) is incomplete gamma-function [1], Vol. 1. Consequently, we have the reciprocal pair of index transforms
\[ (Ff)(z) = \Gamma(z) \int_{0}^{\infty} \Gamma(1-z,x)e^{x} f(x)dx, \quad (3.12) \]
\[ f(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{x^{z}}{\Gamma(z)} \left[ \frac{1}{z} + \frac{1}{x} \right] (Ff)(z)dz, \ x > 0. \quad (3.13) \]

**Theorem 7.** Let \( f \in \mathcal{M}^{-1}(L_{c}), c_{0} < 1 \). Let \( \gamma \in \mathbb{R}, 0 < \gamma < 1 - c_{0} \). Then transformation (3.12) is well-defined and \((Ff)(z)\) has the representation
\[ (Ff)(z) = \frac{1}{2\pi i} \int_{c_{0}-i\infty}^{c_{0}+i\infty} \frac{\Gamma(1-s)\Gamma(1-s)(c_{0} + \gamma + i\tau)}{\sin(\pi(c_{0} + \gamma + i\tau))} f(s)ds. \]

The operator (3.12) is bounded from \( \mathcal{M}^{-1}(L_{c}) \) into \( L_{1}(c_{0} + \gamma - i\infty, c_{0} + \gamma + i\infty) \) and
\[ ||F|| \leq \pi \Gamma(1-c_{0}) \int_{-\infty}^{\infty} |\sin(\pi(c_{0} + \gamma + i\tau))|d\tau. \]
Moreover, if \( 0 < \gamma < 1 - c_{0} \) and \((Ff)(z)/\Gamma(1+z) \in L_{1}(\gamma - i\infty, \gamma + i\infty)\), for all \( x > 0 \) the inversion formula (3.13) holds, where the integral is absolutely convergent.
A final example, which we are going to consider is generated by relation (2.3.7.8) in [3], Vol. 1. It involves a combination of hypergeometric functions. Let

\[ H(z, x) = \int_0^\infty \frac{e^{-xt}}{(t^2 + 1)^n} t^{z-1} dt = x^{2n-z}\Gamma(z-2n) \, _1F_2 \left( n; 1 + n - \frac{z}{2}, n + \frac{1 - z}{2}; -\frac{x^2}{4} \right) \]

moreover, if \( n \in \mathbb{N}, x > 0, \Re z > 0 \).

Therefore we arrive at the following result.

**Theorem 8.** Let \( n \in \mathbb{N}, f \in M^{-1}(L_c), c_0 < 1 \). Let \( \gamma \in \mathbb{R}, 0 < \gamma < n - c_0 \). Then transformation (3.15) is well-defined and \((F f)(z)\) has the representation

\[ (F f)(z) = \int_0^\infty H(z, x) f(x) dx \]

admits the following inversion formula

\[ f(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{x^{z-2k}}{\Gamma(1+z-2k)} (F f)(z) dz, \quad x > 0, \]

In the meantime,

\[ \sum_{k=0}^{n} \binom{n}{k} \frac{x^{z-2k}}{\Gamma(1+z-2k)} = \frac{x^z}{\Gamma(1+z)} \, _3F_0 \left( -n, -\frac{z}{2}, -\frac{1 - z}{2}; -\frac{x^2}{4} \right). \]

Therefore we arrive at the following result.

**Theorem 8.** Let \( n \in \mathbb{N}, f \in M^{-1}(L_c), c_0 < 1 \). Let \( \gamma \in \mathbb{R}, 0 < \gamma < n - c_0 \). Then transformation (3.15) is well-defined and \((F f)(z)\) has the representation

\[ (F f)(z) = \frac{1}{4\pi i(n-1)!} \int_{c_0 + i\infty}^{c_0 + i\infty} \Gamma \left( \frac{s + z}{2} \right) \Gamma \left( n - \frac{s + z}{2} \right) \Gamma(1-s) f^*(s) ds. \]

The operator (3.15) is bounded from \( M^{-1}(L_c) \) into \( L_1(c_0 + \gamma - i\infty, c_0 + \gamma + i\infty) \) and

\[ ||F|| \leq \frac{\Gamma(1-c_0)}{2(n-1)!} \int_{-\infty}^{\infty} \left| \Gamma \left( \frac{c_0 + \gamma + i\tau}{2} \right) \Gamma \left( n - \frac{c_0 + \gamma + i\tau}{2} \right) \right| d\tau. \]

Moreover, if \( n - 1 < \gamma < n - c_0 \) and \((F f)(z)/\Gamma(1+z) \in L_1(\gamma-i\infty, \gamma+i\infty)\), for all \( x > 0 \) the inversion formula (3.16) holds

\[ f(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{x^z}{\Gamma(1+z)} \, _3F_0 \left( -n, -\frac{z}{2}, -\frac{1 - z}{2}; -\frac{x^2}{4} \right) (F f)(z) dz, \quad x > 0, \]

where the latter integral is absolutely convergent.
4 A new inversion theorem for the modified Kontorovich-Lebedev transform

In this final section we will prove a new inversion theorem for the modified Kontorovich-Lebedev transform

\[ g(x) = \int_0^\infty e^{-x/2} K_{i\tau} \left( \frac{x}{2} \right) f(x) dx, \quad \tau \in \mathbb{R}. \quad (4.1) \]

It is easily seen that making elementary changes of variables and functions we come out with the classical Kontorovich-Lebedev transform (cf. [4], [8], [9]). We will prove that an arbitrary function \( f \in M^{-1/2}_{1/4}(\mathcal{L}_c) \), \( \nu \in \mathbb{R} \) (see Definition 2) can be expanded for all \( x > 0 \) in terms of the following iterated Kontorovich-Lebedev integral

\[ f(x) = e^{x/2} \int_{-\infty}^{\infty} \tau \sinh \pi \tau K_{i\tau} \left( \frac{x}{2} \right) \int_0^\infty e^{-y/2} K_{i\tau} \left( \frac{y}{2} \right) f(y) dy. \quad (4.2) \]

We note, that similar expansion in the space \( M^{-1/4}_{1/4}(\mathcal{L}) \) was studied in [6].

**Theorem 9.** Let \( f \in M^{-1/2}_{1/4}(\mathcal{L}_c) \), \( \nu > c_0 - 1 \), \( 0 < c_0 < 1 \). The expansion (4.2) holds for all \( x > 0 \), where the integral with respect to \( y \) is absolutely convergent and the integral with respect to \( \tau \) exists in the Riemann improper sense.

**Proof.** Calling relation (8.4.23.3) in [3], Vol. 3, we use the Parseval equality for the Mellin transform [5] and Definition 2 to write integral (4.1) in the form

\[ \int_0^\infty e^{-y/2} K_{i\tau} \left( \frac{y}{2} \right) f(y) dy = \frac{\sqrt{\pi}}{2 \pi i} \int_{c_0-i\infty}^{c_0+i\infty} \frac{\Gamma(1-s-i\tau)\Gamma(1-s+i\tau)}{\Gamma(3/2-s)} f^*(s) ds. \quad (4.3) \]

But due to Stirling’s formula for gamma-functions [1], Vol. I, we have \( \Gamma(3/2-s) = O(|s|^{1-c_0}e^{-\pi|s|/2}) \), \( |s| \to \infty \). Therefore via conditions of the theorem the function

\[ \frac{f^*(s)}{\Gamma(3/2-s)} \]

is Lebesgue integrable over the line \( (c_0 - i\infty, c_0 + i\infty) \). Hence denoting by

\[ h(x) = \frac{1}{2\pi i} \int_{c_0-i\infty}^{c_0+i\infty} \frac{f^*(s)}{\Gamma(3/2-s)} x^{-s} ds \quad (4.4) \]

and applying again the Parseval equality for the Mellin transform together with representation (3.1) for the modified Bessel function, the right-hand side of (4.3) becomes

\[ \frac{\sqrt{\pi}}{2\pi i} \int_{c_0-i\infty}^{c_0+i\infty} \frac{\Gamma(1-s-i\tau)\Gamma(1-s+i\tau)}{\Gamma(3/2-s)} f^*(s) ds = 2\sqrt{\pi} \int_0^\infty K_{2\nu}(2\sqrt{y}) h(y) dy. \quad (4.5) \]

Substituting the right-hand side of (4.5) into (4.2), we change the order of integration by Fubini’s theorem, which is applicable owing to the absolute convergence of the iterated integral. Indeed, fixing a positive \( x \), we appeal to the uniform inequality for the modified Bessel function [9]

\[ |K_{i\tau}(x)| \leq e^{-\delta|\tau|} K_0(x \cos \delta), \quad \delta \in \left[ 0, \frac{\pi}{2} \right], \]


definition (4.4) of \( h(x) \), and asymptotic behavior of the modified Bessel function \([1]\), Vol. II to have the estimate

\[
\int_{-\infty}^{\infty} \tau \sinh \pi \tau K_{i\tau} \left( \frac{x}{2} \right) \left| \int_0^\infty |K_{2i\tau}(2\sqrt{y})h(y)| dyd\tau \leq \frac{1}{2\pi} K_0 \left( \frac{x\cos \delta}{2} \right) \int_{-\infty}^{\infty} |\tau \sinh \pi \tau| e^{-3\delta/|\tau|}d\tau \times \int_0^\infty K_0(2\cos \delta \sqrt{y})y^{-c_0}dy \int_{c_0-i\infty}^{c_0+i\infty} \left| \frac{f^*(s)}{\Gamma(3/2-s)} \right| ds < \infty,
\]

since one can choose \( \delta \in ]\pi/3, \pi/2[ \). Calculating the inner index integral via relation (2.16.52.9) in \([3]\), Vol. 2, which is slightly corrected by the author

\[
\int_{-\infty}^{\infty} \tau \sinh \pi \tau K_{i\tau} \left( \frac{x}{2} \right) K_{2i\tau}(2\sqrt{y})d\tau = \frac{1}{2} \sqrt{\frac{\pi y}{x}} e^{-x/2-y/x},
\]

we take the result, writing the right-hand side of (4.2) in the form

\[
\frac{1}{x^{3/2}} \int_0^\infty e^{-y/x}h(y)\sqrt{y}dy.
\]

Meanwhile, using expression (4.4) for \( h(x) \) and changing the order of integration after its substitution in the latter integral, we easily deduce the equalities

\[
\frac{1}{x^{3/2}} \int_0^\infty e^{-y/x}h(y)\sqrt{y}dy = \frac{1}{2\pi i} \int_{c_0-i\infty}^{c_0+i\infty} \frac{f^*(s)}{\Gamma(3/2-s)} \int_0^\infty e^{-y/x}y^{1/2-s}dyds
\]

\[
= \frac{1}{2\pi i} \int_{c_0-i\infty}^{c_0+i\infty} f^*(s)x^{-s}ds = f(x)
\]

via Definition 2. Thus we proved (4.2) and completed the proof of the theorem.

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