SOLUTIONS OF THE $\mathfrak{sl}_2$ $qKZ$ EQUATIONS MODULO AN INTEGER

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Abstract. We study the $qKZ$ difference equations with values in the $n$-th tensor power of the vector $\mathfrak{sl}_2$ representation $V$, variables $z_1, \ldots, z_n$ and integer step $\kappa$. For any integer $N$ relatively prime to the step $\kappa$, we construct a family of polynomials $f_r(z)$ in variables $z_1, \ldots, z_n$ with values in $V^\otimes n$ such that the coordinates of these polynomials with respect to the standard basis of $V^\otimes n$ are polynomials with integer coefficients. We show that the polynomials $f_r(z)$ satisfy the $qKZ$ equations modulo $N$.

Polynomials $f_r(z)$ are modulo $N$ analogs of the hypergeometric solutions of the $qKZ$ equations given in the form of multidimensional Barnes integrals.

1. Introduction

The Knizhnik-Zamolodchikov ($KZ$) differential equations are a system of linear differential equations, satisfied by conformal blocks on the sphere in the WZW model of conformal field theory. The quantum Knizhnik-Zamolodchikov ($qKZ$) equations are a difference version of the $KZ$ equations which naturally appear in the representation theory of Yangians (rational case) and quantum affine algebras (trigonometric case). As a rule one considers the $KZ$ and $qKZ$ equations over the field of complex numbers. Then these differential and difference equations are solved in multidimensional hypergeometric integrals.

In [SV] the $KZ$ differential equations were considered modulo an integer $N$. It turned out that modulo $N$ the $KZ$ equations have a family of polynomial solutions. The construction of these solutions was analogous to the construction of the multidimensional hypergeometric solutions, and these polynomial solutions were called the $N$-hypergeometric solutions.

In this paper we consider modulo $N$ the rational $\mathfrak{sl}_2$ $qKZ$ equations with values in the $n$-th tensor power of the vector representation of $V$ and with an integer step $\kappa$. The $qKZ$ equations for a function $f(z_1, \ldots, z_n)$ with values in $V^\otimes n$ have the form

$$f(z_1, \ldots, z_a - \kappa, \ldots, z_n) = K_a(z; \kappa) f(z), \quad a = 1, \ldots, n,$$

where linear operators $K_a(z; \kappa)$ are given in terms of the rational $\mathfrak{sl}_2$ $R$-matrix, see (2.3). The operators $K_a(z; \kappa)$ commute with the diagonal action of $\mathfrak{sl}_2$, and, therefore, it is sufficient to solve the $qKZ$ equations only with values in the space of singular vectors of a given weight.

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We fix an integer $N$ relatively prime to $\kappa$ and construct $V^{\otimes n}$-valued polynomials in $z_1, \ldots, z_n$ such that their coordinates in the standard basis of $V^{\otimes n}$ are polynomials with integer coefficients. Then we show that these $V^{\otimes n}$-valued polynomials satisfy the $qKZ$ equations modulo $N$.

The idea of the construction is as follows. The hypergeometric complex $V^{\otimes n}$-valued solutions of the $qKZ$ are given in [TV, MV]. The solutions with values in the subspace of singular vectors of weight $n - 2l$ are written as $l$-dimensional integrals of the form

$$f(z) = \int_{\mathbb{R}^l} \Phi(t, z) w(t, z) W(t, z) dt.$$  

Here $t = (t_1, \ldots, t_l)$, $\Phi(t, z)$ is a scalar master function given as a ratio of products of Euler gamma functions, $w(t, z)$ is the vector-valued rational weight function, and $W(t, z)$ is a scalar $\kappa$-periodic function with respect to all variables $t, z$; cf. Section A.3. To show that $f(z)$ is a solution, one observes that the difference

$$\Phi(t, z_1, \ldots, z_a - \kappa, \ldots, z_n) w(t, z_1, \ldots, z_a - \kappa, \ldots, z_n) - K_a(z, \kappa) \Phi(t, z) w(t, z)$$

is a discrete differential with respect to variables $t_1, \ldots, t_l$, that is, it can be written in the form $\sum_{s=1}^l (G_{a,s}(t) - G_{a,s}(t_1, \ldots, t_s - \kappa, \ldots, t_l))$ for appropriate functions $G_{a,s}$.

Then using Stokes’ theorem one shows that the integral of a discrete difference equals zero, and hence $f(z)$ is a solution of the $qKZ$ equations. The hypergeometric solution $f(z)$ depends on the choice of the $\kappa$-periodic function $W(t, z)$ from a suitable finite-dimensional vector space. The functions $W(t, z)$ of that space provide the convergence of the integral in (1.1) and applicability of Stokes’ theorem.

One may consider the $qKZ$ difference equations with values in a tensor product $\otimes_{i=1}^n V_i$ of finite-dimensional $\mathfrak{sl}_k$-modules and construct the corresponding hypergeometric solutions like in formula (1.1). The functions $\Phi(t, z)$ and $\bar{w}(t, z)$ used in (1.1) have meaning in that more general situation, see [TV]. They depend on the Cartan matrix and the highest weights of the modules. In our case the highest weight of the vector representation is $\lambda = 1$ and the Cartan matrix is $C = (2)$.

The main idea of the construction of polynomial solutions of the $qKZ$ equations modulo $N$, is to modify in formulas for $\Phi(t, z)$ and $w(t, z)$ the numbers $1$ coming from the highest weight and the numbers $2$ coming from the Cartan matrix in such a way, that they do not change modulo $N$ but become multiples of the step $\kappa$. So, we change in the formulas:

$$1 \to 1 + Nm = -\kappa k, \quad 2 \to 2 + Nm' = \kappa k',$$

where $m, m', k, k'$ are integers. This is possible if and only if $N$ and $\kappa$ are relatively prime.

After that change, the ratio of products of gamma functions in the master function $\Phi$ becomes a polynomial in $t, z$ due to the fundamental property of the gamma function:

$$(-\kappa)^k \frac{\Gamma\left(\frac{z-k\kappa}{-\kappa}\right)}{\Gamma\left(\frac{z}{-\kappa}\right)} = z(z-\kappa) \ldots (z-(k-1)\kappa) =: [z]_k.$$  

This modified master function (now called the master polynomial) is a product of linear factors organized into Pochhammer polynomials of the form $[x]_k$ and $[x]_{k'}$ with appropriate $x$, see Figure 1. The product $\Phi w$ also becomes a polynomial. Moreover, each of $\binom{n}{1}$ coordinates
of $\Phi w$ is a sum of $l!$ terms with each term being a product of Pochhammer polynomials of the same kind, see Figure 2.

Then we check that modulo $N$ the difference (1.2) is still a discrete differential.

It remains to find a way to integrate over $t$ which eliminates discrete differentials of polynomials with integer coefficients. Such a procedure is clearly impossible over complex numbers (since the discrete derivative map is surjective), but is well-known modulo an integer $N$. Namely, let a polynomial $G(t, z)$ be a discrete differential with respect to the $t$-variables, and let

$$G(t, z) = \sum_{r_1, \ldots, r_l} a_{r_1, \ldots, r_l}(z)[t_1]_{r_1} \cdots [t_l]_{r_l}, \quad a_{r_1, \ldots, r_l}(z) \in \mathbb{Z}[z_1, \ldots, z_n].$$

Then a coefficient $a_{r_1, \ldots, r_l}(z)$ equals zero modulo $N$ if $r_i \equiv -1 \pmod{N}$ for $i = 1, \ldots, l$.

The assignment to a polynomial $G(t, z)$ of a coefficient $a_{r_1, \ldots, r_l}(z)$ with $r_i \equiv -1 \pmod{N}$ for $i = 1, \ldots, l$, may be considered as a modulo $N$ analog of the integral over $t$ of the polynomial $G(t, z)$. We call it a difference $r$-integral. There is a generalization of the difference $r$-integral defined for all sequences of non-negative integers $r = (r_1, \ldots, r_l)$ which assigns to a polynomial $G(t, z)$ the polynomial $N_r a_{r_1, \ldots, r_l}(z)$, where $N_r$ is the smallest integer such that $N_r (r_i + 1) \equiv 0 \pmod{N}$. This assignment also gives zero in $\mathbb{Z}/N\mathbb{Z}$, if $G(t, z)$ is a discrete differential. See Section 3.2.

Thus, if

$$\Phi(t, z)w(t, z) = \sum_{r_1, \ldots, r_l} c_{r_1, \ldots, r_l}(z)[t_1]_{r_1} \cdots [t_l]_{r_l},$$

and $(r_1, \ldots, r_l)$ is a sequence of non-negative integers then $f_r = N_r c_{r_1, \ldots, r_l}(z)$ is a solution of the $qKZ$ equations modulo $N$. Keeping in mind the language of solutions over complex numbers, we call the solutions constructed in this way, the $N$-hypergeometric solutions.

Since we consider only the tensor products of $\mathfrak{sl}_2$ vector representations, and since we have no gamma functions and no analytic issues, the proofs are simpler than in [MV] or [TV] and are combinatorial in nature. Moreover, we show that our solutions satisfy the (stronger) symmetric form of the $qKZ$ equations, see (2.6). This happens due to the symmetric nature of our “integration”.

The $N$-hypergeometric solution are non-homogeneous polynomials in $z_1, \ldots, z_n$. One can consider the set of solutions of the form $\sum_r g_r(z)f_r(z)$, where $g_r(z)$ are scalar polynomials with integer coefficients satisfying the congruences

$$N_r g_r(z_1, \ldots, z_i - \kappa, \ldots, z_n) \equiv N_r g_r(z_1, \ldots, z_i, \ldots, z_n) \pmod{N}$$

for all $i$. It is an interesting problem to describe this set as a module over the ring of polynomials which are periodic modulo $N$. It is nontrivial because this module is not free in general and the answer depends on how the integers $\kappa, N, n, l$ are related to each other. We do not address this question in this first joint paper on the subject, but we show that the top degree terms of an $N$-hypergeometric solutions of the $qKZ$ equations constitute an $N$-hypergeometric solution of the $KZ$ equations constructed in [SV], see Proposition 6.2, for which some studies have been done, see [V1, V2, V3].

In contrast to the complex-valued case, the $N$-hypergeometric solutions is generically do not span the space of singular vectors. Therefore, a natural question is if there are polynomial modulo $N$ solutions of the $qKZ$ equations which are not $N$-hypergeometric.
In this paper we consider the \( qKZ \) equations with values in a tensor power of the vector representation of \( \mathfrak{sl}_2 \). In the same way we may construct \( N \)-hypergeometric solutions of the \( qKZ \) equations with values in a tensor product of finite-dimensional \( \mathfrak{sl}_k \)-modules.

The paper is organized as follows. In Section 2 we define the \( qKZ \) equations. In Section 3 we discuss the generalities of discrete differentiation and integration modulo \( N \). In Section 4 we define the main ingredients of our solutions: master polynomial and weight functions. In Section 5 we formulate and prove our main theorem which provides \( N \)-hypergeometric solutions of the \( qKZ \) equations. These are vectors of non-homogeneous polynomials in \( z \) with integer coefficients solving the \( qKZ \) equations modulo \( N \). In Section 6 we show that the sum of the top degree terms of the \( N \)-hypergeometric solutions of the \( qKZ \) equations coincide with the \( N \)-hypergeometric solutions of the \( KZ \) equations. In Appendix A we remind the constructions of the hypergeometric solutions of the \( KZ \) and \( qKZ \) equations in the special case in which the corresponding hypergeometric integrals are one-dimensional.

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2. DIFFERENCE \( qKZ \) EQUATIONS

2.1. Notations. Consider the Lie algebra \( \mathfrak{sl}_2 \) with basis \( e, f, h \) and relations \( [e, f] = h, [h, e] = 2e, [h, f] = -2f \). Let \( V \) be the two-dimensional \( \mathfrak{sl}_2 \)-module with basis \( v_1, v_2 \) and the action \( ev_1 = 0, ev_2 = v_1, fv_1 = v_2, fv_2 = 0, hv_1 = v_1, hv_2 = -v_2 \).

Fix a positive integer \( n > 1 \). The \( \mathfrak{sl}_2 \)-module \( V^\otimes n \) has weight decomposition

\[
V^\otimes n = \bigoplus_{l=0}^n V^\otimes [n - 2l],
\]

where \( V^\otimes [n - 2l] \) is the eigenspace of \( h \) with eigenvalue \( n - 2l \). Let \( \text{Sing} V^\otimes [n - 2l] \subset V^\otimes [n - 2l] \) be the subspace of singular vectors (the vectors annihilated by \( e \)).

Let \( I_l \) be the set of all \( l \)-element subsets of \( \{1, \ldots, n\} \). Denote

\[
v_I = v_{i_1} \otimes \cdots \otimes v_{i_n} \in V^\otimes n,
\]

where \( i_j = 2 \) if \( i_j \in I \) and \( i_j = 1 \) if \( i_j \notin I \). The set \( \{v_I \mid I \in I_l\} \) is a basis of \( V^\otimes [n - 2l] \).

2.2. The \( qKZ \) equations. Define the rational \( R \)-matrix acting on \( V^\otimes 2 \),

\[
R(u) = \frac{u - P}{u - 1},
\]

where \( P \) is the permutation of factors of \( V^\otimes 2 \). The \( R \)-matrix satisfies the Yang-Baxter and unitarity equations,

\[
R^{(12)}(u - v)R^{(13)}(u)R^{(23)}(v) = R^{(23)}(v)R^{(13)}(u)R^{(12)}(u - v),
\]

\[
R^{(12)}(u)R^{(21)}(-u) = 1.
\]

The first equation is an equation in \( \text{End}(V^\otimes 3) \). The superscript indicates the factors of \( V^\otimes 3 \) on which the corresponding operators act.
Let \( z = (z_1, \ldots, z_n) \). Define the qKZ operators \( K_1, \ldots, K_n \) acting on \( V^\otimes n \):

\[ K_a(z; \kappa) = R^{(a,a-1)}(z_a - z_{a-1} - \kappa) \cdots R^{(a,1)}(z_a - z_1 - \kappa) \times R^{(a,n)}(z_a - z_n) \cdots R^{(a,a+1)}(z_a - z_{a+1}), \]

where \( \kappa \) is a parameter.

The qKZ operators preserve the weight decomposition of \( V^\otimes n \), commute with the \( \mathfrak{sl}_2 \)-action, and form a discrete flat connection with step \( \kappa \),

\[ K_a(z_1, \ldots, z_b - \kappa, \ldots, z_n; \kappa) K_b(z; \kappa) = K_b(z_1, \ldots, z_a - \kappa, \ldots, z_n; \kappa) K_a(z; \kappa) \]

for \( a, b = 1, \ldots, n \), see [FR].

The system of difference equations with step \( \kappa \),

\[ f(z_1, \ldots, z_a - \kappa, \ldots, z_n) = K_a(z; \kappa) f(z), \quad a = 1, \ldots, n, \]

for a \( V^\otimes n \)-valued function \( f(z) \) is called the qKZ equations.

Since the qKZ operators commute with the action of \( \mathfrak{sl}_2 \) in \( V^\otimes n \), the qKZ operators preserve the subspace \( \text{Sing} V^\otimes n[n - 2l] \) for any integer \( l \).

2.3. The symmetric qKZ equations. Let \( P^{(a,a+1)} \) be the operator swapping the \( a \)-th and \( a + 1 \)-st tensor factors of \( V^\otimes n \).

Let \( \mu \in S_n \) be the cyclic permutation \( (1, 2, \ldots, n) \). In terms of simple transpositions we have

\[ \mu = s_{1,2}s_{2,3} \cdots s_{n-1,n}. \]

Set

\[ P(\mu) = P^{(1,2)} P^{(2,3)} \cdots P^{(n-1,n)}. \]

The following system of equations is called the symmetric qKZ equations:

\[ f(z_1, \ldots, z_{a+1}, z_a, \ldots, z_n) = P^{(a,a+1)} f(z), \quad a = 1, \ldots, n - 1, \]

\[ f(z_1 - \kappa, z_2, \ldots, z_n) = P(\mu) f(z_2, \ldots, z_n, z_1). \]

The symmetric qKZ equations imply the qKZ equations.

**Lemma 2.1.** Let \( f(z) \) satisfy (2.6). Then \( f(z) \) satisfies (2.4).

**Proof.** For (2.4) with \( a = 1 \) we have

\[
\begin{align*}
f(z_1 - \kappa, z_2, \ldots, z_n) &= P^{(1,2)} \cdots P^{(n-1,n)} f(z_2, \ldots, z_n, z_1) \\
&= P^{(1,2)} \cdots P^{(n-2,n-1)} P^{(n-1,n)} R^{(n-1,n)}(z_1 - z_n) f(z_2, \ldots, z_n, z_1, z_n) \\
&= P^{(1,2)} \cdots P^{(n-2,n-1)} R^{(n-1,n)}(z_1 - z_n) f(z_2, \ldots, z_n, z_1, z_n) \\
&= P^{(1,2)} \cdots P^{(n-3,n-2)} R^{(n-2,n-1)}(z_1 - z_n) f(z_2, \ldots, z_n, z_1, z_n) \\
&= R^{(1,n)}(z_1 - z_n) \cdots R^{(1,2)}(z_1 - z_2) f(z) = K_1(z, \kappa) f(z).
\end{align*}
\]
Now we use it to show (2.4) with $a = 2$,
\[
    f(z_1, z_2 - \kappa, \ldots, z_n) = P^{(12)}R^{(12)}(z_2 - z_1 - \kappa)f(z_2 - \kappa, z_1, \ldots, z_n)
    = P^{(12)}R^{(12)}(z_2 - z_1 - \kappa)R^{(1, n)}(z_2 - z_n) \ldots R^{(1, 2)}(z_2 - z_1)f(z_2, z_1, \ldots, z_n)
    = P^{(12)}R^{(1, 2)}(z_2 - z_1 - \kappa)R^{(1, n)}(z_2 - z_n) \ldots R^{(1, 2)}(z_2 - z_1)P^{(12)}R^{(1, 2)}(z_2 - z_1)f(z)
    = K_2(z; \kappa)f(z).
\]

The proof of equations (2.4) with $a > 2$ is similar.

In this paper we fix relatively prime integers $\kappa, N$. We consider equations (2.6) modulo $N$, and without loss of generality, we assume that $0 < \kappa < N$. We construct vectors in $V^{\otimes n}[n - 2l]$ whose coefficients in the basis $v_I$ are polynomials in variables $z$ with integer coefficients,

\[
    f(z) = \sum_{I \in I_f} f_I(z) v_I, \quad f_I(z) \in \mathbb{Z}[z],
\]

such that equations (2.6) hold modulo $N$ and such that $ef(z) = 0$ modulo $N$.

Difference equations modulo $N$ have trivial symmetries. Namely, let $f(z)$ be a polynomial solution modulo $N$ of a difference equation with step $\kappa$. Let $g(z)$ be a periodic polynomial, $g(z_1, \ldots, z_i - \kappa, \ldots, z_n) \equiv g(z)$ (mod $N$) for all $i$. Let $h(x)$ be any polynomial in variables $z$ with values in $V^{\otimes n}$ whose coordinates have integer coefficients. Then $f(z)g(z) + Nh(x)$ is also a solution of the difference equations modulo $N$. Moreover, if $N = dd', d, d' \in \mathbb{Z}$, and all coefficients of all coordinates $f_I(z)$ of $f(z)$ are divisible by $d$, then $f(z)g(z)$ is a solution for any $g(x)$ which is periodic modulo $d'$.

3. Difference differentials

Assume again that the integers $\kappa, N$ are relatively prime and $0 < \kappa < N$. Let $k, k'$ be the integers such that $0 < k < N, 0 < k' < N$,

\[
    \kappa k \equiv -1, \quad \kappa k' \equiv 2 \pmod{N}.
\]

The integers $k, k'$ exist and are unique.

3.1. Pochhammer polynomials. Let $m$ be a positive integer. Define the Pochhammer polynomial

\[
    [t]_m = \prod_{i=1}^{m} (t - (i - 1)\kappa).
\]

We have

\[
    [t - \kappa]_m = [t]_m \frac{t - \kappa m}{t}, \quad [t + \kappa]_m = [t]_m \frac{t + \kappa}{t - (m - 1)\kappa}.
\]

We also call a Pochhammer polynomial $[t]_m$ a string of length $m$ which starts at $t$ and ends at $t - (m - 1)\kappa$.

The string $[t]_m$ should be thought of as a discretized analog of the monomial $t^m$. For example, in parallel to the monomials, there is a Newton binomial formula for strings as well:

\[
    [t + z]_m = \sum_{i=1}^{m} \binom{m}{i} [t]_i [z]_{m-i}.
\]
If $N = p$ is prime, and $\kappa$ is still relatively prime to $p$, then
\[
[t]_p \equiv t^p - t \pmod{p}
\]
by the Little Fermat theorem. In this case we also have
\[
[t + z]_p \equiv (t + z)^p - (t + z) \equiv t^p - t + z^p - z \equiv [t]_p + [z]_p.
\]

For $M \in \mathbb{Z}$, we call a polynomial $f(t) \in \mathbb{Z}[t]$ an $M$-constant if $f(t - \kappa) \equiv f(t) \pmod{M}$. We note that strings, whose length is a multiple of $N$, are $N$-constants. More generally, we have
\[
b[t - \kappa]_a = b[t]_a - ab\kappa[t - \kappa]_{a-1}.
\]

Therefore, the polynomial $b[t]_a$ is an $N$-constant if and only if $ab \equiv 0 \pmod{N}$.

Let $A$ be a $\mathbb{Z}$-algebra. Our main example is $A = \mathbb{Z}[z_1, \ldots, z_n] = \mathbb{Z}[z]$. The strings $\{[t]_m, \ m \geq 0\}$ form an $A$-basis of $A[t]$ which should be thought of as a discrete analog of the monomial basis $\{t^m, \ m \geq 0\}$.

More generally, $\{\prod_{i=1}^l[t_i]_{m_i}, \ m_i \in \mathbb{Z}_{\geq 0}\}$ form an $A$-basis of $A[t_1, \ldots, t_l]$ which we will often use.

### 3.2. Difference $r$-integrals

Let $r = (r_1, \ldots, r_l)$ be a sequence of non-negative integers. Let $N_r$ be the least positive integer such that $N_r(r_i + 1) \equiv 0 \pmod{N}$ for all $i$. In terms of greatest common divisors (gcd) and least common multiples (lcm) we have

\[
N_r = \frac{N}{\gcd(r_1 + 1, \ldots, r_l + 1, N)} = \frac{N}{\gcd(N, r_1 + 1)} \cdots \frac{N}{\gcd(N, r_l + 1)}.
\]

We also set
\[
M_r = \frac{N}{N_r} = \gcd(r_1 + 1, \ldots, r_l + 1, N).
\]

For a polynomial $f(t_1, \ldots, t_l) = \sum_{m_1, \ldots, m_l} c_{m_1, \ldots, m_l} \prod_{i=1}^l[t_i]_{m_i} \in A[t_1, \ldots, t_l]$ and a sequence $r = (r_1, \ldots, r_l)$ of non-negative integers, we define the difference $r$-integral of $f$ by the formula:

\[
\{f\}_r^{t_1, \ldots, t_l} = N_r c_r.
\]

The difference $r$-integral is an $A$-linear map $\{\cdot\}_r^{t_1, \ldots, t_l} : A[t_1, \ldots, t_l] \to A$. In the next section we show that this map vanishes on discrete differentials modulo $N$.

If $r$ is such that $r_i \equiv -1 \pmod{N}$, $i = 1, \ldots, l$, then $N_r = 1$. In this case we call the sequence $r$ a maximal sequence and call the difference $r$-integral a maximal difference integral.

If $r$ is such that for some $i$ the number $r_i + 1$ is relatively prime to $N$, then $N_r = N$ and the corresponding difference $r$-integral is identically zero modulo $N$. We call such $r$ a trivial sequence and the difference $r$-integral a trivial difference integral.

If $N = p$ is prime, then all non-trivial difference integrals are maximal.
3.3. **Discrete differentials.** Define the *discrete* $t$-derivative $D_t : A[t] \to A[t]$, 
\[ D_t f(t) = f(t) - f(t - \kappa). \]

Similar to usual differentiation, the discrete derivative is an $A$ linear map which satisfy the Leibniz rule:
\[ D_t(f(t)g(t)) = (D_t f(t))g(t) + f(t - \kappa)(D_t g(t)). \]

We call polynomials in the image of the discrete $t$-derivative the *discrete differentials*. We have
\[ (3.3) \quad D_t[t]_m = m\kappa [t - \kappa]_{m-1}, \]
Modulo $N$, the discrete derivative $D_t$ has kernel given by $N$-constants, generated by 
\[ a[t]_b, \quad a, b \in \mathbb{Z}_{>0}, \quad ab \equiv 0 \pmod{N}. \]

For any $N$-constant $g(t)$ we have 
\[ D_t(g(t)f(t)) \equiv g(t)D_t f(t) \pmod{N}, \]
which explains the name “$N$-constant”.

The next proposition asserts that a difference integral computed on a discrete differential is zero.

**Proposition 3.1.** Let $r = (r_1, \ldots, r_l)$, $t = (t_1, \ldots, t_l)$, and $f \in A[t_1, \ldots, t_l]$. Then for any $j = 1, \ldots, l$, 
\[ \{D_{t_j}f\}_{t_1, \ldots, t_l} \equiv 0 \pmod{N}. \]

**Proof.** Write $f$ in the $A$-basis $\prod_{i=1}^l [t_i]_{m_i}$ and apply $D_{t_j}$ using (3.3). That gives $D_{t_j} f$ in the same basis, and the coefficient in $D_{t_j} f$ of each $[t_j]_{r_j}$ is a multiple of $r_j + 1$. Therefore, it is zero modulo $N$ when multiplied by $N_r$. \hfill \Box

3.4. **The module of $t$-periods.** In this section we discuss the set of all difference $r$-integrals of a polynomial $f(t, z) \in \mathbb{Z}[t, z]$. For simplicity, we consider the case when $t$ and $z$ are single variables. The results can be extended to the case when $t = (t_1, \ldots, t_l)$, $z = (z_1, \ldots, z_n)$. At the end of this section we formulate a remark on an analog of Fubini’s theorem for the difference $r$-integrals.

Given a polynomial $f(t, z) = \sum_{r=0}^m c_r(z)[t]_r \in \mathbb{Z}[t, z]$, we consider the following set of polynomials in $z$ with integer coefficients. The set is denoted by $\{f(t, z)\}_t$ and consists of all polynomials of the form 
\[ \sum_{r=1}^m g_r(z)N_r c_r(z) = \sum_{r=1}^m g_r(z)\{f(t, z)\}_t, \]
where each $g_r(z)$ is an arbitrary polynomial such that $N_r g_r(z - \kappa) \equiv N_r g_r(z) \pmod{N}$, in other words, each $g_r(z)$ is an arbitrary $M_r$-constant. The set $\{f(t, z)\}_t$ is called the *module of $t$-periods of the polynomial $f(t, z)$ relative to the variable $t$ and the integer $N$*. The module is the object of study in this section.
The reason for this concept is the following observation. If a polynomial \( f(t, z) \) satisfies a difference equation up to a \( t \)-derivative:

\[
f(t, z - \kappa) - A(z)f(t, z) \equiv D_t(h(t, z)) \pmod{N},
\]

then all polynomials \( u(z) \in \{ f(t, z) \}^t \) are solutions of the equation

\[
u(z - \kappa) \equiv A(z)u(z) \pmod{N}.
\]

We have \( \{ f(t, z) \}^t \subset \mathbb{Z}[z] \), but we are really interested in the projection of \( \{ f(t, z) \}^t \) to \( \mathbb{Z}/N\mathbb{Z}[z] \). The image of \( \{ f(t, z) \}^t \) in \( \mathbb{Z}/N\mathbb{Z}[z] \) is a \( \mathbb{Z}/N\mathbb{Z} \)-module.

The definition of the module of \( t \)-periods uses the \( A \)-basis \( \{ [t]_m, m \geq 0 \} \). The next proposition says that one can equivalently use the \( A \)-basis \( \{ [t + n_1 z + n_2]_m, m \geq 0 \} \) for any fixed \( n_1, n_2 \in \mathbb{Z} \) without changing the module of \( t \)-periods.

**Proposition 3.2.** Let \( n_1, n_2 \in \mathbb{Z} \) and

\[
\sum_{r=0}^{m} c_r(z)[t]_r \equiv \sum_{r=0}^{m} b_r(z)[t + n_1 z + n_2]_r \pmod{N},
\]

where \( c_r(z), b_r(z) \in \mathbb{Z}[z] \). Then the following two sets are equal:

\[
\begin{align*}
\left\{ \sum_{r=0}^{m} g_r(z)N_r c_r(z) \mid \text{each } g_r(z) \text{ is an arbitrary } M_r\text{-constant} \right\} &= \left\{ \sum_{r=0}^{m} g_r(z)N_r b_r(z) \mid \text{each } g_r(z) \text{ is an arbitrary } M_r\text{-constant} \right\}.
\end{align*}
\]

**Proof.** We give the proof for the case of \( n_2 = 0, n_1 = 1 \). The case \( n_2 = 0 \) and non-zero \( n_1 \) is the same. The case \( n_1 = 0 \) is similar (and simpler). The general case if is the combination of the previous two statements.

By (3.2),

\[
b_r(z)[t + z]_r = b_r(z) \sum_{i=0}^{r} \binom{r}{i} [t]_i [z]_{r-i}.
\]

We claim that if polynomial \( g_i(z) \in \mathbb{Z}[z] \) is an \( M_r \)-constant, then the polynomial

\[
\tilde{g}_r(z) = \binom{r}{i} \frac{N_i}{N_r} [z]_{r-i} g_i(z)
\]

has integer coefficients and, moreover, it is an \( M_r \)-constant.

It is enough to prove this for the case \( N = p^a \) where \( p \) is prime. Let \( r + 1 = p^i x, i + 1 = p^r y \), where \( x, y \) are integers not divisible by \( p \).

Let us consider \( b \leq a, c \leq a \), the other cases are similar. Then \( N_r = p^{a-b}, N_i = p^{b-c} \). If \( c \leq b \), then \( r - i \) is divisible by \( p^x \) and \([z]_{r-i} g_i N_i / N_r\) is an \( M_r \)-constant since \( i N_i / N_r \) is divisible by \( p^b \). If \( b < c \) then we use \( \binom{r}{i} \). It is well known that a binomial coefficient is divisible by \( p^d \) where \( d \) is the number of moving a digit when the difference \( r - i \) is computed in the base \( p \) system. It follows that \( \binom{r}{i} \) is divisible by \( p^{c-b} \) and the claim follows.
Thus \( g_i(z)N_i c_i(z) = g_i(z)N_i b_i(z) + \sum_{r=i+1}^n b_r(z)N_r \tilde{g}_r \) where \( \tilde{g}_r \) are \( M_r \)-constants. It follows that the left-hand side set in (3.4) is inside of the right set. Doing the change \( t \to t - z \) and repeating the argument, we obtain the other inclusion. \( \square \)

Thus, a difference \( r \)-integral of \( f(t, z) \) with respect to \( t \) may change, if the \( A \)-basis is shifted by a linear polynomial in \( z \), but the module of \( t \)-periods does not.

Another statement in the same spirit is the following proposition.

**Proposition 3.3.** For any integers \( n_1, n_2, n_3 \), with \( n_3 > 0 \), and any polynomial \( f(t, z) \in \mathbb{Z}[t, z] \), we have

\[
\{ f(t, z)[t + n_1 z + n_2] \} \equiv \{ f(t, z) \}^t \pmod{N}.
\]

**Proof.** The proof is similar to Proposition 3.2. We treat the case \( n_1 = n_3 = 1 \) and \( n_2 = 0 \).

We have modulo \( N \)

\[
[t, [t + z]_N \equiv [t, [(t - r \kappa) + z]_N = [t, \sum_{i=0}^N \left( \frac{N}{i} \right) [t - \kappa]_i [z]_{N-i} = \sum_{i=0}^N \left( \frac{N}{i} \right) [t + i [z]_{N-i}.
\]

The claim is that if \( g_r(z) \) is an \( M_r \)-constant, then

\[
\tilde{g}(z) = \left( \frac{N}{i} \right) \frac{N_{r+i}}{N_r} [z]_{N-i} g_r(x)
\]

is an \( M_{r+i} \)-constant. This is checked for \( N = p^a \) where \( p \) is prime, by using the fact that if \( i = p^a x \), where \( x \) is an integer not divisible by \( p \) and \( c < a \), then \( \left( \frac{N}{i} \right) \) is divisible by \( p^{a-c} \).

Write \( f(t, z) = \sum_{r=0}^m c_r(z)[t]_m \) and \( f(t, z)[t + z]_N = \sum_{r=0}^{N+n} b_r(z)[t]_r \). It follows from the claim that for any \( M_r \)-constant \( g_i(z) \), we have \( b_i(z)g_i(z)N_i = \sum_{r=i-N} N_r \tilde{g}_r(z) \) where we set \( c_r(z) = 0 \) for \( r < 0 \) and where \( \tilde{g}_r(z) \) are \( M_r \)-constants. Therefore, \( \{ f(t, z)[t + z]_N \} \subset \{ f(t, z) \} \) modulo \( N \).

Clearly \( N_i = N_{i+N} \) and \( g_i(z) \) is an \( M_i \)-constant if and only if \( g_i(z) \) is an \( M_{i+N} \)-constant. Therefore, if \( i \leq m \), then \( b_{i+N}[z]_m g_i(z)N_{i+N} = c_i(z)N_i g_i(z) + \sum_{r=i+1}^{i+N} c_r(z)N_r \tilde{g}_r(z) \). Thus we obtain \( c_i(z)N_i g_i(z) \in \{ f(t, z)[t + z]_N \} \) recursively starting from \( i = m \), then continuing to \( i = m - 1 \), then to \( m - 2 \), and so on. \( \square \)

We use Propositions 3.2 and 3.3 in Section 5 to explain that the family of the modulo \( N \) solutions of the \( qKZ \) equations constructed in Theorem 5.1 does not depend on various choices.

**Remark.** Consider an example. Let \( f(t_1, t_2, z) = p(z)[t_1][t_2]_2 \) and \( N = 4 \). Then

\[
\{ f \}_{t_1}^{t_2} = \{ f \}_{t_2}^{t_1} \equiv 4p(z) \equiv 0 \pmod{4}, \quad \text{but} \quad \{ f \}_{t_2}^{t_1} = 2p(z).
\]

More generally, we have an \( N \)-analog of Fubini’s theorem:

\[
\{ f(t_1, t_2, z) \}_{t_1}^{t_2} = \{ f(t_1, t_2, z) \}_{t_2}^{t_1} \equiv \frac{N_{r_1}N_{r_2}}{\gcd(N_{r_1}, N_{r_2})} \{ f(t_1, t_2, z) \}_{r_1}^{r_2} \equiv \frac{N_{r_1}N_{r_2}}{\gcd(N_{r_1}, N_{r_2})} \{ f(t_1, t_2, z) \}_{r_2}^{r_1}.
\]

Note that the integer \( N_{r_1, r_2} \) is a divisor of \( N_{r_1}N_{r_2} \) and, in general, is a proper divisor. Thus, in general, the double integration gives more difference \( r \)-integrals than the repeated integration.
Figure 1. Zeroes of the master polynomial $\Phi(t_1, t_2, z_1, \ldots, z_n)$.

4. Master function and weight functions

We define the main ingredients of the construction of the solutions of the $q$KZ equation modulo $N$: the master polynomial and weight function.

4.1. Master polynomial. Recall integers $N, \kappa, k, k'$ defined in (3.1). We also have integers $n, l$ corresponding to the choice of the space $\text{Sing} V^\otimes n[n-2l]$.

Let again $z = (z_1, \ldots, z_n), t = (t_1, \ldots, t_l)$. Define the master polynomial

$$\Phi(t, z) = \prod_{a=1}^{n} \prod_{i=1}^{l} (t_i - z_a)_k \prod_{1 \leq i < j \leq l} (t_i - t_j + 1)_{k'}.$$ 

The zeroes of the master polynomial are shown on Figure 1 for $l = 2$. The start of each string is indicated in blue. The string itself consist of $k$ or $k'$ zeroes of the master polynomial. The distance between neighboring zeroes is $-\kappa$ as shown in red in Figure 1.

Note that the master polynomial is symmetric in variables $z_1, \ldots, z_n$.

As explained in the introduction, the master polynomial is obtained from the master function used in the case of complex-valued solutions. This master function originally is a ratio of products of gamma functions, see Section A.3, which reduces to a polynomial under our assumptions.

4.2. Symmetrization modulo $N$. In the case of complex-valued solutions, one uses an action of a symmetric group $S_l$ on functions of $t_1, \ldots, t_l$ given by the formula

$$(\tau \circ f)(t_1, \ldots, t_l) = f(t_{\tau(1)}, \ldots, t_{\tau(l)}) \prod_{i<j, \tau(i)>\tau(j)} \frac{t_i - t_j - 1}{t_i - t_j + 1},$$

for any $\tau \in S_l$. We modify it as follows.
For $\tau \in S_l$ define

$$(\tau f)(t_1, \ldots, t_l) = f(t_{\tau(1)}, \ldots, t_{\tau(l)}) \prod_{i<j, \tau(i) > \tau(j)} \frac{t_i - t_j + 1 - \kappa k'}{t_i - t_j + 1}.$$ 

Note that this formula defines an action of $S_l$ modulo $N$ since $1 - \kappa k' \equiv -1 \pmod{N}$.

We also define symmetrization modulo $N$:

$$\text{Sym}_t f(t) = \sum_{\tau \in S_l} (\tau f)(t).$$

4.3. Weight function. In the case of complex-valued solutions, the coordinates of the weight function are defined as symmetrizations of the functions

$$\prod_{i=1}^l \frac{a_i - 1}{t_i - z_{a_i}} \prod_{j=1}^{a_i - 1} \frac{t_j - z - \kappa k}{t_j - z_j}.$$ 

We modify it as follows.

For $I = \{1 \leq a_1 < \cdots < a_l \leq n\} \in \mathcal{I}_l$, define the coordinates of the weight function

$$w_I(t, z) = \text{Sym}_t \left( \prod_{i=1}^l \frac{1}{t_i - z_{a_i}} \prod_{j=1}^{a_i - 1} \frac{t_j - z_j - \kappa k}{t_j - z_j} \right).$$

Since $k\kappa \equiv -1 \pmod{N}$, and $\text{Sym}_t$ coincides modulo $N$ with the previous symmetrization, this formula coincides with the formula used in the complex-valued case modulo $N$.

It is convenient to denote

$$w_I^\tau(t, z) = \tau \left( \prod_{i=1}^l \frac{1}{t_i - z_{a_i}} \prod_{j=1}^{a_i - 1} \frac{t_j - z_j - \kappa k}{t_j - z_j} \right).$$

Then $w_I(t, z) = \sum_{\tau \in S_l} w_I^\tau(t, z)$.

We also denote

$$U_I(t, z) = \Phi(t, z) w_I(t, z), \quad U_I^\tau(t, z) = \Phi(t, z) w_I^\tau(t, z).$$

Then we have a lemma saying that the summands $U_I^\tau(t, z)$ are polynomials organized in products of strings.

Lemma 4.1. For $\tau \in S_l$, the function $U_I^\tau(t, z)$ is a polynomial which is a product of $\binom{l}{2}$ strings of length $k'$, of $l(n - 1)$ strings of length $k$, and of $l$ strings of length $k - 1$,

$$U_I^\tau(t, z) = \prod_{i<j, \tau(i) < \tau(j)} [t_i - t_j + 1 - \kappa k'] \prod_{i<j, \tau(i) > \tau(j)} [t_i - t_j + 1 - \kappa k']$$

$$\times \prod_{i=1}^l \left( \prod_{s=1}^{a_i - 1} [t_{\tau(i)} - z_s - \kappa k] \prod_{s=a_i}^{a_i - 1} [t_{\tau(i)} - z_a - \kappa] k - 1 \left( \prod_{s=a_i}^{a_i + 1} [t_{\tau(i)} - z_s] k \right) \right).$$

Proof. It is easy to see that the poles of $w_I^\tau(t, z)$ cancel with the starts of the strings existing in the master polynomial, while the zeroes append those strings.
The vector $U$ the master polynomial are pictured as red circles. The new zeroes are shown in red bullets.

In the next section we show that any difference $r$-integral of $U(t, z)$ solves the symmetric $qKZ$ equations modulo $N$ and is a singular vector modulo $N$.

5. The main theorem

In this section we construct solutions modulo $N$ of the $qKZ$ equations.

5.1. The statement of the theorem.

Theorem 5.1. Let $U(t, z)$ be the polynomial in $t_1, \ldots, t_l, z_1, \ldots, z_n$ with values in $V^\otimes[n - 2l]$ defined in (4.2). Then for any $r = (r_1, \ldots, r_l)$ the difference $r$-integral $f_r(z) = \{U(t, z)\}^t_r$ is a polynomial in $z_1, \ldots, z_n$ which is a solution of the symmetric $qKZ$ equations (2.6) modulo $N$. The vector $f_r(z)$ is singular modulo $N$.

We prove the theorem in the next three sections.

If $g(z)$ is a scalar $N$-constant polynomial and $f(z)$ is a solution of the $qKZ$ modulo $N$, then $g(z)f(z)$ is a solution too. Moreover, we have a stronger statement. Namely, let $f(z) = \sum_{r \in \mathcal{I}_t} f_r(z) v_r$, $f_I(z) \in \mathbb{Z}[z]$, be a solution of the $qKZ$ equations modulo $N$, and let $a \in \mathbb{Z}$ be such that all polynomials $f_I(z)/a$ have integer coefficients. Then for any scalar $(N/\gcd(a, N))$-constant $g(z)$, the product $g(z)f(z)$ is also a solution of the $qKZ$ equations modulo $N$.
For example, if \(g_r(z) \in \mathbb{Z}[z]\) is an \(M_r\)-constant then \(g_r(z)f_r(z)\) is also a solution. Thus, by Theorem 5.1 all functions in the module of \(t\)-periods \(\{U(t, z)\}^t\) are solutions modulo \(N\) of the \(qKZ\) equations. We call all such solutions \(N\)-hypergeometric solutions. Thus, the \(N\)-hypergeometric solutions are polynomials of the form \(\sum_r g_r(z)f_r(z)\) where \(f_r(z) \in N_r\mathbb{Z}[z]\) are given in Theorem 5.1, and \(g_r(z) \in \mathbb{Z}[z]\) are \(M_r\)-constants. We give examples of \(N\)-hypergeometric solutions in Section 5.5.

Seemingly, there are several other ways to obtain solutions modulo \(N\). The first way is to choose integers \(k\) and \(k'\) as any positive integer solutions of congruences (3.1) and not as the least positive integer solutions. Such a choice will change the master and weight functions. It is clear that Theorem 5.1 still holds for the modified integrand \(U(t, z)\). However the choice of new \(k\) and \(k'\) does not produce new solutions by Proposition 3.3.

The second way is to construct solutions by integrating with respect to a different \(A\)-basis, e.g. \(\{\prod[t_i - z_i]_{m_i}, m_i \in \mathbb{Z}_{\geq 0}\}\). Then an analog of Theorem 5.1 still holds. However, a different choice of an \(A\)-basis produces no new solutions as well by Proposition 3.2.

The third way is to consider a divisor \(\tilde{N}\) of \(N\) and a solution \(\tilde{f}(z)\) of the \(qKZ\) equations modulo \(\tilde{N}\). Then \(\tilde{N}\tilde{f}(z)\) is a solution of the \(qKZ\) equations modulo \(N\). The solutions obtained in this way also are not new.

5.2. The first \(n - 1\) equations. In this section we check the first \(n - 1\) equations of (2.6). Moreover, we show that these equations are already satisfied before taking the difference \(r\)-integral.

Namely, we check modulo \(N\)

\[
g(z_1, \ldots, z_{a+1}, z_a, \ldots, z_n) \equiv P^{(a,a+1)}R^{(a,a+1)}(z_a - z_{a+1}) g(z)
\]

for \(g(z) = w(t, z) = \sum_{I \in \mathcal{I}_t} w_I v_I\).

This is sufficient because multiplication by the scalar master polynomial \(\Phi(z, t)\) preserves relation (5.1) as \(\Phi\) is symmetric in \(z_1, \ldots, z_n\). The integration preserves this relation as well since (5.1) does not depend on integration variables \(t_1, \ldots, t_I\).

We have three cases.

First, let \(I\) be such that \(a, a + 1 \not\in I\). Then \(w_I\) is symmetric in \(z_a, z_{a+1}\). The vector \(v_I\) has \(v_I \otimes v_I\) in positions \(a, a + 1\). For any \(x\), we have \(PR(x)v_1 \otimes v_1 = v_1 \otimes v_1\). Therefore, \(g = w_I v_I\) satisfies (5.1) for each \(I \in \mathcal{I}_t\).

Second, let \(I\) be such that \(a, a + 1 \in I\). The vector \(v_I\) has \(v_2 \otimes v_2\) in positions \(a, a + 1\). Again, for any \(x\), we have \(PR(x)v_2 \otimes v_2 = v_2 \otimes v_2\). So we have to check that \(w_I\) does not change when \(a\) and \(a + 1\) are swapped. We claim that for any \(\tau \in S_t\), the sum \(w_I^\tau + w_I^{(a,a+1)^\tau}\) has this property. Indeed, if \(\tau(1) = b, \tau(a + 1) = c, \) and if \(c > b\), then modulo \(N\) we have

\[
(w_I^\tau + w_I^{(a,a+1)^\tau})(t, z) = \frac{h(z, t)}{(t_b - z_a)(t_c - z_a)} \left( \frac{t_c - z_a - \kappa k}{t_c - z_{a+1}} + \frac{t_b - z_a - \kappa k}{t_b - z_{a+1}} \right) \left( \frac{t_b - t_c + 1 - \kappa k'}{t_b - t_c + 1} \right)
\]

\[
= \frac{h(z, t)}{(t_b - z_{a+1})(t_c - z_{a+1})} \left( \frac{t_c - z_{a+1} - \kappa k}{t_c - z_a} + \frac{t_b - z_{a+1} - \kappa k}{t_b - z_a} \right) \left( \frac{t_b - t_c + 1 - \kappa k'}{t_b - t_c + 1} \right)
\]

\[
= (w_I^\tau + w_I^{(a,a+1)^\tau})(t, z_1, \ldots, z_{a+1}, z_a, \ldots, z_n).
\]
Here the common factor \( h(z, t) \) does not depend on \( z_a, z_{a+1} \). The case \( b > c \) is similar. Therefore, the term \( g = w_I v_I \) satisfies (5.1).

Third, let \( I \) be such that \( a \in I \) and \( a + 1 \notin I \). We pair it up with the set \( J \) such that \( a \notin J \) and \( a + 1 \in I \) in the most natural way. Namely, let \( J = (I \setminus a) \cup \{a + 1\} \) be the set \( I \) where \( a \) is replaced with \( a + 1 \). Let \( \tau \in S_I \) and \( \tau(a) = b \). We claim that modulo \( N \),

\[
PR(z_1 - z_{a+1}))(w_I^r(t, z) v_2 \otimes v_1 + w_J^r(t, z) v_1 \otimes v_2) = w_I^r(t, z_1, \ldots, z_{a+1}, z_a, \ldots, z_n) v_2 \otimes v_1 + w_J^r(t, z_1, \ldots, z_{a+1}, z_a, \ldots, z_n) v_1 \otimes v_2.
\]

Indeed, this is equivalent to

\[
\frac{1}{z_a - z_{a+1} - 1} w_I^r(t, z) + \frac{z_a - z_{a+1}}{z_a - z_{a+1} - 1} w_J^r = h(t, z) \left( \frac{1}{z_a - z_{a+1} - 1} - \frac{z_a - z_{a+1}}{z_a - z_{a+1} - 1} \right) + \frac{1}{t_b - z_a} \frac{t_b - z_a - \kappa k'}{t_b - z_{a+1}} = \frac{h(t, z)}{t_b - z_{a+1} - 1} = w_I^r(t, z_1, \ldots, z_{a+1}, z_a, \ldots, z_n).
\]

The second equation is similar. Therefore, the sum \( g = w_I v_I + w_J v_J \) satisfies (5.1). Thus \( g = w(t, z) \) satisfies (5.1). Equation (5.1) is proved.

We note that if one replaces \( \kappa k' \) in \( w(t, z) \) by \( 2 \), and \( \kappa k \) by \( -1 \) then \( w(t, z) \) will satisfy (5.1) exactly (not only modulo \( N \)). This property of the weight function \( w(t, z) \) is known in large generality and we could derive it, for example, from [TV, Theorem 4.9] instead of checking it directly.

Also note that in the complex-valued case, there is an extra ingredient, as one multiplies by a periodic function \( W \) before integration, see Section A.3. This function is usually not symmetric in \( z \) and therefore, the integrand loses symmetry (5.1).

5.3. The \( n \)-th equation. Clearly, for any \( I \in I_t \),

\[
(P^{(\mu)})^{-1} v_I = v_{\mu^{-1} I}, \quad \mu^{-1} I = \{\mu^{-1}(a), a \in I\}.
\]

To check the \( n \)-th equation

\[
(P^{(\mu)})^{-1} f_r(z_1 - \kappa, z_2, \ldots, z_n) = f_r(z_2, \ldots, z_n, z_1),
\]

we show that if one takes \( U_I^r(t, z_1 - \kappa, z_2, \ldots, z_n) \) and changes \( t_i \to t_i - \kappa \), where \( i \) depends on \( I \) and \( \tau \), then the result is \( U_{\mu^{-1} I}^r(t, z_2, \ldots, z_n, z_1) \), where

\[
\tau' = \begin{cases} 
\tau & 1 \notin I, \\
\tau(1, 2, \ldots, l) & 1 \in I.
\end{cases}
\]

For that let us follow what happens to the strings when we shift \( z_1 \). The change \( z_1 \to z_1 - \kappa \) affects only strings of the form \( [t_i - z_1 - \kappa]_k \), and \( [t_i - z_1 - \kappa]_{k-1} \) which are shifted to \( [t_i - z_1]_k \),
and \([t_i - z_1]_{k-1}\) respectively. On Figure 2 it corresponds to the shift of the corresponding strings to the left by one position.

Let \(I\) be such that \(1 \not\in I\). Then the strings depending on \(z_1\) in \(U_I^\tau(t, z)\) are \([t_i - z_1 - \kappa]_k\), \(i = 1, \ldots, l\). Note that \(n \not\in I' = \mu^{-1}I\). Thus the strings depending on \(z_1\) \(U_{\mu^{-1}I}^\tau(t, z_2, \ldots, z_n, z_1)\) are \([t_i - z_1]_k\), \(i = 1, \ldots, l\). Therefore, the shift \(z_1 \rightarrow z_1 \kappa\) takes the strings in \(U_I^\tau(t, z)\) with the strings in \(U_{\mu^{-1}I}^\tau(t, z_2, \ldots, z_n, z_1)\). The strings which do not involve \(z_1\) match as well. Thus we have \(U_I^\tau(t, z_1 - \kappa, z_2, \ldots, z_n) = U_{\mu^{-1}I}^\tau(t, z_2, \ldots, z_n, z_1)\). Note that we did not shift \(t_i\) in this case.

Let \(I\) be such that \(1 \in I\). Let \(\tau(1) = b\). Then the strings depending on \(z_1\) in \(U_I^\tau(t, z)\) are \([t_i - z_1 - \kappa]_k\), \(i = 1, \ldots, l\), \(i \neq b\) and \([t_b - z_1 - \kappa]_{k-1}\). Note that \(\tau'(n) = \tau \mu(n) = \tau(1) = b\). Thus the strings depending on \(z_1\) \(U_{\mu^{-1}I}^\tau(t, z_2, \ldots, z_n, z_1)\) are \([t_i - z_1]_k\), \(i \neq b\) and \([t_b - z_1 - \kappa]_{k-1}\).

We shift both \(z_1 \rightarrow z_1 - \kappa\) and \(t_b \rightarrow t_b - \kappa\) in \(U_I^\tau(t, z)\). Clearly, all strings are shifted appropriately to match the strings in \(U_{\mu^{-1}I}^\tau(t, z_2, \ldots, z_n, z_1)\). Thus, in this case we have \(U_I^\tau(t_1, \ldots, t_b - \kappa, \ldots, t_l, z_1 - \kappa, z_2, \ldots, z_n) = U_{\mu^{-1}I}^\tau(t, z_2, \ldots, z_n, z_1)\).

Therefore, the \(n\)-th equation is valid for the integrand up to terms of the form \(D_{t_b} U_I^\tau(t, z_1 - \kappa, z_2, \ldots, z_n)\) which vanish after taking the difference \(r\)-integral.

5.4. \(N\)-hypergeometric solutions are singular vectors modulo \(N\). Finally, we claim that the vector \(U(t, z)\) is a singular vector modulo \(N\) to difference \(t\)-differentials, namely,

\[
eU(t, z) \equiv \sum_{J \in \mathcal{X}_{l-1}} g_J(t, z) v_J \pmod{N},
\]

where \(g_J(t, z)\) are suitable discrete \(t\)-differentials. This statement is deduced directly from [TV, Lemma 2.21]. Hence taking a difference integral of \(eU(t, z)\) we obtain the zero vector modulo \(N\).

**Example 5.2.** Let \(l = 1\). Then \(eU(t, z) = \sum_{a=1}^n U_{\{a\}}(t_1, z) v_1 \otimes \cdots \otimes v_1\), and modulo \(N\) we have

\[
\Phi(t_1 - \kappa, z) - \Phi(t_1, z) = \Phi(t, z) \left[ \prod_{a=1}^n \frac{t_1 - z_a - \kappa k}{t_1 - z_l} - 1 \right]
\]

\[
= -\Phi(t, z) \kappa k \sum_{a=1}^n \frac{1}{1 - \frac{t_1 - z_a - \kappa k}{t_1 - z_l}} = -\kappa k \sum_{a=1}^n U_{\{a\}}(t_1, z) \equiv \sum_{a=1}^n U_{\{a\}}(t_1, z).
\]

Note, that one can modify solutions modulo \(N\) by adding a vector valued polynomial \(f(z)\) of the form (2.7) such that the coordinate polynomials \(f_I(z) \in \mathbb{Z}[z]\) are all divisible by \(N\). In many cases such a modification allows us to obtain solutions which are singular vectors (not only singular vectors modulo \(N\)), see Sections 5.5.

5.5. **Example.** Let \(n = 3, l = 1, \kappa = 2, N = 3, r = 2\). Then \(k = 1\) and \(U(t_1, z_1, z_2, z_3)\) is a quadratic polynomial in \(t_1\). We have

\[
\{U(t_1, z_1, z_2, z_3)\}_2^1 = v_2 \otimes v_1 \otimes v_1 + v_1 \otimes v_2 \otimes v_1 + v_1 \otimes v_1 \otimes v_2.
\]

Note that the solution is a singular vector only modulo 3. One can subtract, for example, \(3v_2 \otimes v_1 \otimes v_1\) to make it a singular vector.
Let \( n = 3, l = 1, \kappa = 2, N = 5, r = 4 \). Then \( k = 2 \) and \( U(t_1, z_1, z_2, z_3) \) is a polynomial of degree 5 in \( t_1 \). We have

\[
\{U(t_1, z_1, z_2, z_3)\}_1^{U_1} = (14 - z_1 - 2z_2 - 2z_3)v_2 \otimes v_1 \otimes v_1 \\
+ (10 - z_2 - 2z_1 - 2z_3)v_1 \otimes v_2 \otimes v_1 + (6 - z_3 - 2z_1 - 2z_2)v_1 \otimes v_1 \otimes v_2.
\]

Again, the solution is a singular vector only modulo 5 and one can subtract a multiple of 5, for example, \((30 - 5(z_1 + z_2 + z_3))v_2 \otimes v_1 \otimes v_1\) to make the solution a singular vector.

More generally, for \( n = 3, l = 1, \kappa = 2, N = 2k + 1 \), the vector \( U(t_1, z_1, z_2, z_3) \) is a polynomial of degree \( 3k - 1 \) in \( t_1 \) and we have a difference r-integral with \( r = 2k \). The corresponding solution modulo \( N \) is a non-homogenous polynomial in \( z_1, \ldots, z_n \) of degree \( k - 1 \). If \( N = p \) is prime, then this is the only \( N \)-hypergeometric solution.

6. From difference to differential equations

In this section we show that the solutions modulo \( N \) constructed in Theorem 5.1 recover the solutions of the KZ differential equations modulo \( N \) constructed in [SV]. For simplicity and following [SV], we restrict ourselves only to maximal difference r-integrals, that is to the case of \( r \), such that \( r_i \equiv -1 \pmod{N} \).

6.1. \( N \)-hypergeometric solutions of KZ differential equations. In this section we remind the construction in [SV] of polynomial solutions modulo an integer \( N \) of the \( \mathfrak{sl}_2 \) differential KZ equations.

We consider the system of differential equations on a \( V^\otimes n \)-valued function \( f(z) \),

\[
\kappa \frac{\partial f}{\partial z_a} = \sum_{s, s \neq a} \frac{P^{(a,s)} - 1}{z_a - z_s} f, \quad a = 1, \ldots, n,
\]

where \( z = (z_1, \ldots, z_n) \) and \( \kappa \) is a parameter of the system. This system is called the KZ differential equations. This system commutes with the \( \mathfrak{sl}_2 \)-action on \( V^\otimes n \).

Let \( 0 < \kappa < N \) be the same relatively prime integers as before. Let \( l \) be a positive integer, \( 2l \leq n \). Let \( k, k' \) be the same positive integers as in (3.1). Let \( t = (t_1, \ldots, t_l) \).

Define the master polynomial

\[
\Phi^0(t, z) = \prod_{a=1}^{n} \prod_{i=1}^{l} \frac{1}{(t_i - z_a)^{k}} \prod_{1 \leq i < j \leq l} (t_i - t_j)^{k'}.
\]

For a function \( f(t) \) define

\[
\text{sym}_t f(t) = \sum_{\tau \in S_l} f(t_{\tau(1)}, \ldots, t_{\tau(l)}).
\]

For \( I = \{1 \leq a_1 < \cdots < a_l \leq n\} \in \mathcal{I}_l \), define the weight functions

\[
w^0_I(t, z) = \text{sym}_t \prod_{i=1}^{l} \frac{1}{t_i - z_{a_i}}, \quad U^0_I(t, z) = \Phi^0(t, z) w^0_I(t, z).
\]
Then $U^0_I(t, z)$ is a polynomial with integer coefficients. Expand $U^0_I(t, z)$ with respect to the $t$-variables

$$U^0_I(t, z) = \sum_{m_1, \ldots, m_l \geq 0} c^0_{I,m}(z) t_1^{m_1} \cdots t_l^{m_l}, \quad c^0_{I,m}(z) \in \mathbb{Z}[z].$$

For any $r = (r_1, \ldots, r_l) \in \mathbb{Z}_{>0}$ such that $r_i \equiv -1 \pmod{N}$, for all $i$, denote

$$f^0_r(z) = \sum_{I \in \mathcal{I}} c^0_{I,r}(z) v_I.$$  \hspace{1cm} (6.2)

**Theorem 6.1.** [SV] For any $r = (r_1, \ldots, r_l) \in \mathbb{Z}_{>0}$ such that $r_i \equiv -1 \pmod{N}$, for $i = 1, \ldots, l$, the polynomial $f^0_r(z)$ satisfies the KZ differential equations modulo $N$ and is a singular vector modulo $N$,

$$e^r f^0_r(z) \equiv 0 \pmod{N},$$  \hspace{1cm} (6.3)

$$\kappa \frac{\partial f^0_r}{\partial z_a} \equiv \sum_{s, s \neq a} \frac{P(a,s) - 1}{z_a - z_s} f^0_r \pmod{N}, \quad a = 1, \ldots, n.$$  \hspace{1cm} (6.4)

The solutions $f^0_r$ of the KZ equations modulo $N$ given by Theorem 6.1 are called the $N$-hypergeometric solutions of the KZ equations.

Note that the polynomials $U^0_I(t, z)$ are homogeneous polynomials of variables $t, z$ of degree $nk + l(l-1)k'/2 - l$. Hence the $N$-hypergeometric solution $f^0_r(z)$ is a homogeneous polynomial in $z$ of degree

$$d_r = nk + l(l-1)k'/2 - l - \sum_{i=1}^l r_i.$$  \hspace{1cm} 6.2. The top degree of the $N$-hypergeometric solutions. Unlike the solutions $f^0_r(z)$ of the KZ equations, the solutions $f_r(z)$ of the qKZ equations given by Theorem 5.1 are not homogeneous. It turns out that the top degree part of these solutions coincides with $f^0_r(z)$.

Define degree of polynomials in $\mathbb{Z}[t, z]$ by setting

$$\deg t_i = \deg z_j = 1, \quad i = 1, \ldots, l, \quad j = 1, \ldots, n.$$  

Let $r = (r_1, \ldots, r_l) \in \mathbb{Z}_{>0}$ be such that $r_i \equiv -1 \pmod{N}$, for all $i$. Let $f_r(z)$ be the $N$-hypergeometric solution of the qKZ equations defined in Theorem 5.1. Let $f^0_r(z)$ be the $N$-hypergeometric solution of the KZ equations defined in Theorem 6.1.

**Proposition 6.2.** We have

$$f_r(z) = f^0_r(z) + \ldots,$$

where the dots denote the terms of degree less than $\deg f^0_r(z)$.

**Proof.** For a polynomial $X$ in variables $t, z$ and integer coefficients, we write $X = Y + \ldots$ when $Y$ is a homogeneous polynomial in $t, z$ and all terms of $X - Y$ have degree smaller than the degree of $Y$. 
For any integer $a$, we have $t_i - t_j - a\kappa \equiv (t_i - t_j)^m + \ldots$ and $[t_i - z_j - a\kappa]_m = (t_i - z_j)^m + \ldots$. Hence $U(t, z) = U^0(t, z) + \ldots$. At the same time if

$$U(t, z) = \sum_m c_m(z) \prod_{i=1}^l [t_i]_{m_i}, \quad c_m(z) = c_m^0(z) + \ldots,$$

then $U(t, z) = \sum_m c_m^0(z) \prod_{i=1}^l t_i^{m_i} + \ldots$. Therefore, $U^0(t, z) = \sum_m c_m^0(z) \prod_{i=1}^l t_i^{m_i}$, and the proposition is proved.

Proposition 6.2 says that there are at least as many $N$-hypergeometric solutions of the $qKZ$ equations as the $N$-hypergeometric solutions of the $KZ$ equations.

It is an interesting problem to describe the module of the $N$-hypergeometric solutions in either case. The answer depends on arithmetic properties of $N, n, \kappa, l$.

**Appendix A. Hypergeometric solutions**

In this section we remind the limit which produces the $KZ$ equations from the $qKZ$ equations. We also remind the formulas for the hypergeometric complex-valued solutions of the $\mathfrak{sl}_2$ $KZ$ differential and $qKZ$ difference equations with values in $\text{Sing} V^{\otimes n}[n - 2]$, that is, in the case when $l = 1$.

**A1. KZ equations as a limit of qKZ equations.** It is well known, that the $KZ$ equations can be obtained as a limit of the $qKZ$ equations which explains the origin of Proposition 6.2. For the reader’s convenience, we recall the construction.

Let $f(z_1, \ldots, z_n)$ satisfy the $qKZ$ equations,

$$f(z_1, \ldots, z_a - \kappa, \ldots, z_n) = K_a(z; \kappa) f(z), \quad a = 1, \ldots, n. \tag{A.1}$$

Define $g(w_1, \ldots, w_n; \alpha) = f(w_1/\alpha, \ldots, w_n/\alpha)$. Then

$$g(w_1, \ldots, w_a - \alpha \kappa, \ldots, w_n) = K_a(w/\alpha; \kappa) g(w), \quad a = 1, \ldots, n, \tag{A.2}$$

where

$$K_a(w/\alpha; \kappa) = R^{(a,a-1)}(w_a - w_{a-1} - \alpha \kappa; \alpha) \ldots R^{(a,1)}(w_a - w_1 - \alpha \kappa; \alpha) \times R^{(a,n)}(w_a - w_n; \alpha) \ldots R^{(a,a+1)}(w_a - w_{a+1}; \alpha), \tag{A.3}$$

and

$$R(u; \alpha) = \frac{u - \alpha P}{u - \alpha} = 1 - \alpha \frac{P - 1}{u - \alpha}.$$

In the limit $\alpha \to 0$, the $qKZ$ difference equations (A.3) turn into the $KZ$ differential equations (6.1).
A.2. Solutions of the KZ equations. Define the master function and weight functions

\[ \Phi(t_1, z) = \prod_{a=1}^{n} \frac{(t_1 - z_a)^{-1/\kappa}}{\Gamma\left(\frac{t_1 - z_a}{\kappa}\right)}, \quad w_{(a)}(t_1, z) = \frac{1}{t - z_a}, \quad a = 1, \ldots, n. \]

Consider the vector of integrals

\[ F_{(\gamma)}^0(z) = \sum_{a=1}^{n} \int_{i\mathbb{R}} \Phi(t_1, z) w_{(a)}(t_1, z) dt_1 v_{(a)} \in V^{\otimes n}[n - 2], \]

where \( \gamma(z) \) is a flat family of 1-cycles associated with the multi-valued master function \( \Phi(t_1, z) \). Then \( F_{(\gamma)}^0(z) \) is a solution of the KZ differential equations (6.1) and \( eF_{(\gamma)}^0(z) = 0 \).

In this case the solutions are labeled by the families of cycles \( \gamma(z) \).

A.3. Solutions of the qKZ equations. Define the master function and weight functions

\[ \Phi(t_1, z) = \prod_{a=1}^{n} \frac{\Gamma\left(\frac{t_1 - z_a + 1}{\kappa}\right)}{\Gamma\left(\frac{t_1 - z_a}{\kappa}\right)}, \quad w_{(a)}(t_1, z) = \frac{1}{t_1 - z_a} \prod_{l=1}^{a-1} \frac{t_1 - z_l + 1}{t - z_l}, \quad a = 1, \ldots, n. \]

Define the trigonometric weight functions

\[ W_{(b)}(t_1, z) = e^{\pi i (t - z_b)/\kappa} \prod_{l=1}^{b-1} \frac{\sin(\pi(t - z_l + 1)/\kappa)}{\sin(\pi(t - z_l)/\kappa)}, \]

\[ W_{(b)}^{\text{sing}}(t_1, z) = W_{(b)}(t_1, z) - W_{(b+1)}(t_1, z), \quad b = 1, \ldots, n - 1. \]

Consider the vector of integrals

\[ F_{(b)}(z) = \sum_{a=1}^{n} \int_{i\mathbb{R}} \Phi(t_1, z) w_{(a)}(t_1, z) W_{(b)}^{\text{sing}}(t_1, z) dt_1 v_{(a)} \in V^{\otimes n}[n - 2], \]

\( b = 1, \ldots, n - 1 \), where \( i\mathbb{R} \) is the imaginary axis properly deformed depending on values of \( z \). Then \( F_{(b)}(z) \) is a solution of the qKZ equations (2.4) and \( eF_{(b)}^0(z) = 0 \), see [TV, Section 8].

In this case the solutions are labeled by the trigonometric weight functions \( W_{(b)}^{\text{sing}}(t_1, z) \).

One may study the functions \( F_{(b)}(z) \) under the limit \( \alpha \to 0 \) as in Section A.1 and show that the limit gives solutions \( F_{(\gamma)}^0(z) \) of Section A.2, see [TV, Section 8]. The proof of this fact is rather delicate, if compared with the trivial observation in Proposition 6.2.

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