Erratum to “On Operations and Linear Extensions of Well Partially Ordered Sets”

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In this article, we give a counter-example to Lemma 12 of the article “On Operations and Linear Extensions of Well Partially Ordered Sets” by Maciej Malicki and Aleksander Rutkowski.

Keywords: partial order

1 Introduction

In this article, we give a counter-example to Lemma 12 of the article “On Operations and Linear Extensions of Well Partially Ordered Sets” by Maciej Malicki and Aleksander Rutkowski (Malicki and Rutkowski (2004)).

2 Definitions and notations

Definition 2.1 (Rank function). Each well-founded poset \( P \) admits an ordinal valued rank function \( \text{rank}_P \), defined inductively on its elements: \( \text{rank}_P(a) = \sup_{x < P a}(\text{rank}_P(x) + 1) \)

Let \( \mathcal{P} = \{ P_t : t \in T \} \) be an ordered family of ordered sets, i.e. both \( P_t \)'s and \( T \) are partially ordered (by \( \leq_t \) and \( \leq_T \) respectively). With no loss of generality, elements of \( \mathcal{P} \) can be assumed to be pairwise disjoint. Let, for \( a \in \bigcup_{t \in T} P_t \), \( f(a) \) be that unique \( t \) such that \( a \in P_t \).

Now, assume all elements of \( \mathcal{P} \) to be well-founded and call, for \( a \in \bigcup_{t \in T} P_t \), the primitive rank of \( a \) an ordinal \( g(a) = \text{rank}_{P_{f(a)}}(a) \). Define the following ranked order \( R <_R \) on \( \bigcup_{t \in T} P_t \): \( a <_R b \) if

- either \( f(a) = f(b) \) and \( a <_{f(a)} b \),
- or \( f(a) < f(b) \) and \( g(a) \leq g(b) \).

Call the union with that order the ranked sum and denote it \( R \mathcal{P} \). Observe that \( a \leq_R b \) implies \( f(a) \leq_T f(b) \).
3 A counter example to Lemma 12

False lemma 3.1 (Lemma 12). Let both $T$ and all the components $P_i$ of $\mathcal{RP}$ be well-founded (hence $P = \mathcal{RP}$ is well-founded too). Then for each $a \in \bigcup_{i \in T} P_i$, \( \text{rank}_P(a) \leq \text{rank}_T(f(a)) + \text{rank}_{f(a)}(a) \).

Counter-example:

It is easy to construct an order $\mathcal{RP}$ with an element $a$ such that \( \text{rank}_P(a) = \text{rank}_T(f(a)) + \text{rank}_{f(a)}(a) + 1 \). Indeed consider $T = \{0, 1\}$, and $P_0 = P_1 = \omega + 1$ ($\omega$ is the first infinite ordinal). Let $a$ be the maximum of $P_1$, and $b$ be the maximum of $P_0$. Then \( \text{rank}_{\mathcal{RP}}(b) = \text{rank}_{f(a)}(b) \), hence \( \text{rank}_{\mathcal{RP}}(a) = \text{rank}_{\mathcal{RP}}(b) + 1 = \omega + 1 > \text{rank}_T(f(a)) + \text{rank}_{f(a)}(a) = \text{rank}_T(P_1) + \text{rank}_{P_1}(a) = 1 + \omega = \omega \) (ordinal sum is not commutative and $1 + \omega \neq \omega + 1$).

The problem in the proof is in the line \( \text{rank}_T(f(b)) + \text{rank}_{f(b)}(b) + 1 \leq \text{rank}_T(f(a)) + \text{rank}_{f(a)}(a) \). It should be corrected to \( \text{rank}_T(f(b)) + 1 + \text{rank}_{f(b)}(b) \leq \text{rank}_T(f(a)) + \text{rank}_{f(a)}(a) \), but then the proof by transfinite induction fails.

You cannot correct the lemma by switching both ranks, i.e. \( \text{rank}_P(a) \leq \text{rank}_{f(a)}(a) + \text{rank}_T(f(a)) \). Indeed then $T = \omega + 1$, and $P_0 = P_1 = \ldots = P_\omega = \{0, 1\}$ is a counter-example.

\[\square\]

Lemma 3.2. For any ordinal $\alpha$, there is an order $\mathcal{RP}$ with an element $a$ such that \( \text{rank}_{\mathcal{RP}}(a) = \text{rank}_T(f(a)) + \text{rank}_{f(a)}(a) + \alpha \).

Proof: Consider $T = \alpha + 1$, and $P_0 = P_1 = \ldots = P_\alpha = \beta + 1$, where $\beta$ is the first ordinal such that $\alpha + \beta = \beta$. Let $\alpha$ be the maximum of $P_\alpha$, and $b$ be the maximum of $P_0$. Then \( \text{rank}_{\mathcal{RP}}(b) = \text{rank}_{P_0}(b) \), hence \( \text{rank}_{\mathcal{RP}}(a) = \text{rank}_{\mathcal{RP}}(b) + \alpha = \beta + \alpha > \text{rank}_T(f(a)) + \text{rank}_{f(a)}(a) = \text{rank}_T(P_\alpha) + \text{rank}_{P_\alpha}(a) = \alpha + \beta = \beta. \)

\[\square\]

Lemma 3.3. For any ordinal $\alpha$, there is an order $\mathcal{RP}$ with an element $a$ such that \( \text{rank}_{\mathcal{RP}}(a) = \text{rank}_{f(a)}(a) + \text{rank}_T(f(a)) + \alpha \).

4 Conclusion

We sent an email to one of the authors on 2019/02/24 but, unfortunately, we never had an answer. We hope this erratum may be useful to the scientific community.

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References

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