On the Ulam–Hyers–Rassias stability for nonlinear fractional differential equations using the $\psi$-Hilfer operator

J. Vanterler da C. Sousa and E. Capelas de Oliveira

Abstract. We study the existence and uniqueness of solution of a nonlinear Cauchy problem involving the $\psi$-Hilfer fractional derivative. In addition, we discuss the Ulam–Hyers and Ulam–Hyers–Rassias stabilities of its solutions. A few examples are presented to illustrate the possible applications of our main results.

Mathematics Subject Classification. 26A33, 34A08, 34A34, 34D20.

Keywords. Existence and uniqueness, nonlinear Cauchy problem, Ulam–Hyers stability, Ulam–Hyers–Rassias stability.

1. Introduction

Fractional calculus, or fractional analysis, a branch of mathematical analysis, is a generalization of classical, integer order differentiation and integration to arbitrary, and non-integer order [1–4]. The discovery of new physical phenomena and the study of chaotic systems have given rise to the proposition of new fractional differential and integral operators that would allow a better description of such systems [5–9]. In this context, Sousa and Oliveira [2] have recently proposed a fractional differentiation operator, which they called $\psi$-Hilfer operator that has the special property of unifying several different fractional operators, that is, of generalizing those fractional operators.

Fractional differential equations arise naturally in different fields such as biology, engineering, medicine, physics, and mathematics [1,4–9]. Using fractional derivatives to model phenomena has proved an excellent tool. On the other hand, this also intensifies the studies about the existence and uniqueness of solutions of fractional differential equations with fractional differentiation operators, either with time delay, linear or nonlinear. We refer the reader, for example, to references [10–15].

This work was completed with the support of our T\TeX-pert.
The stability problem of differential equations was formulated and discussed by Ulam and Hyers [16–19]. Between 1978 and 1988, Rassias established the Ulam–Hyers stability of linear and nonlinear mappings [20,21]. The study of Ulam stability and data dependence of fractional differential equations was initiated by Wang et al. [22]. Studies about Ulam–Hyers and Ulam–Hyers–Rassias stability for fractional differential equations can be found in [22–25]. On the other hand, the study of the existence and stability of solutions of implicit fractional differential equations are being object of research in the fractional calculus area, being approached by a rigorous mathematics [26]. We can also mention other recent papers related the existence, uniqueness and stability of solutions of the implicit fractional differential equations [27–35].

In this paper, we consider nonlinear fractional differential equations:

\[
\begin{align*}
H_{\alpha,\beta}^{\alpha,\beta;\psi} y(t) &= f\left(t, y(t), H_{\alpha,\beta}^{\alpha,\beta;\psi} y(t)\right) \\
I_{a+}^{1-\gamma;\psi} y(a) &= y_a,
\end{align*}
\]

where \(H_{\alpha,\beta}^{\alpha,\beta;\psi}(\cdot)\) is the \(\psi\)-Hilfer fractional derivative [2] of order \(0 < \alpha \leq 1\) and type \(0 \leq \beta \leq 1\), \(I_{a+}^{1-\gamma;\psi}(\cdot)\) is the Riemann–Liouville fractional integral of order \(1-\gamma\), \(\gamma = \alpha + \beta(1-\alpha)\), with respect to function \(\psi\) [1,4], \(f: J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) is a given function space, \(t \in J = [a, T]\) with \(T > a\) and \(y_a \in \mathbb{R}\).

The main purpose of this paper is to study the existence and uniqueness of solutions for the nonlinear Cauchy problem, Eq. (1.1), by means of Banach’s contraction principle. In addition, we demonstrate four types of stability, Ulam–Hyers, generalized Ulam–Hyers, Ulam–Hyers–Rassias, and generalized Ulam–Hyers–Rassias stabilities for the fractional differential equation (1.1) in the case \(0 < \alpha \leq 1\).

This paper is organized as follows: in Sect. 2, we present the one parameter Mittag–Leffler function and some particular cases. We present the definitions of the \(\psi\)-Hilfer fractional derivative and the Riemann–Liouville fractional integral with respect to a function \(\psi\), the spaces in which these fractional operators are defined and Gronwall’s inequality, among other results [36]. We also present four definitions of stability Ulam–Hyers, generalized Ulam–Hyers, Ulam–Hyers–Rassias, and generalized Ulam–Hyers–Rassias using the \(\psi\)-Hilfer fractional derivative. In Sect. 3, we study the existence and uniqueness of the solutions of the proposed nonlinear Cauchy problem, using Banach’s contraction principle. In Sect. 4, we study the stabilities presented in Sect. 2 and make some important observations about them. In Sect. 5, we present some particular cases of the nonlinear implicit fractional differential equations we have been studying and discuss their stability according to the criteria of Ulam–Hyers and Ulam–Hyers–Rassias. We also discuss the importance and advantages of using the \(\psi\)-Hilfer fractional derivative. Concluding remarks close this paper.
2. Preliminaries

In this section, we present the definition of the one parameter Mittag–Leffler function and some particular cases. We introduce the definitions of the \(\psi\)-Hilfer fractional derivative and the Riemann–Liouville fractional integral with respect to a function, the spaces in which they are defined and theorems involving these operators, in particular, Gronwall’s inequality. One of the main results of this paper is the study of stability of the solutions Cauchy problem, Eq. (1.1). We did that for the definitions of Ulam–Hyers, generalized Ulam–Hyers, Ulam–Hyers–Rassias and generalized Ulam–Hyers–Rassias stabilities using the \(\psi\)-Hilfer fractional derivative.

The classical Mittag–Leffler function is the most important function of fractional calculus, specially in the study of linear fractional differential equations with constant coefficients. It also plays an important role in the study of the stability of the solutions of linear and nonlinear differential equations. We will deal only with the one parameter Mittag–Leffler function; for Mittag–Leffler functions with two, three, and more parameters, we suggest the book [37].

In 1903, Mittag–Leffler [38] introduced the classic Mittag–Leffler function with only one complex parameter.

**Definition 2.1.** [37] (One parameter Mittag–Leffler function). The Mittag–Leffler function is given by the series:

\[
E_{\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu k + 1)},
\]

where \(\mu \in \mathbb{C}, \Re(\mu) > 0\) and \(\Gamma(z)\) is a gamma function, given by

\[
\Gamma(z) = \int_{0}^{\infty} e^{-t} t^{z-1} dt,
\]

\(\Re(z) > 0\).

The error function is defined by means of the following integral:

\[
\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_{0}^{z} e^{-t^2} dt.
\]

In particular, if \(\mu = 1/2\) in Eq. (2.1), we have

\[
E_{1/2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma\left(\frac{k}{2} + 1\right)} = \frac{1}{\Gamma(1)} + \frac{z}{\Gamma\left(\frac{3}{2}\right)} + \frac{z^2}{\Gamma(2)} + \frac{z^3}{\Gamma\left(\frac{5}{2}\right)} + \frac{z^4}{\Gamma(3)} + \frac{z^5}{\Gamma\left(\frac{7}{2}\right)} + \cdots + \frac{(z^2)^n}{n!} + \frac{z^{2n+1}}{\Gamma\left(n + \frac{3}{2}\right)} + \cdots
\]

(2.2)

Note that, for \(k = 2n, n \in \mathbb{N}\), we have

\[
A(z) = \sum_{k=0}^{\infty} \frac{(z^2)^k}{k!} = \exp(z^2).
\]

(2.3)
On the other hand, for $k = 2n + 1$, $n \in \mathbb{N}$, we have
\[
B(z) = \sum_{k=0}^{\infty} \frac{z^{2n+1}}{\Gamma\left(n + \frac{3}{2}\right)} = \exp(z^2) \text{erf}(z).
\]

Then, substituting Eqs. (2.3) and (2.4) into Eq. (2.2), we have
\[
\mathbb{E}_{1/2}(z) = \sum_{k=0}^{\infty} \frac{(z^2)^k}{k!} + \sum_{k=0}^{\infty} \frac{z^{2n+1}}{\Gamma\left(n + \frac{3}{2}\right)} = \exp(z^2) [1 + \text{erf}(z)].
\]

For $z = \lambda^\beta$, in Eq. (2.5), we have
\[
\mathbb{E}_{1/2}(\lambda^\beta) = \exp(\lambda^{2\beta}) [1 + \text{erf}(\lambda^\beta)].
\]

Taking the limits $\beta \to 1$ and $\beta \to 0$, on both sides of Eq. (2.6), we have
\[
\mathbb{E}_{1/2}(\lambda) = \exp(\lambda^2) [1 + \text{erf}(\lambda)]
\]
and
\[
\mathbb{E}_{1/2}(1) = \exp(1) [1 + \text{erf}(1)] \simeq 5.002,
\]
respectively.

Let $[a, b] \ (0 < a < b < \infty)$ be a finite interval on the half-axis $\mathbb{R}^+$ and let $C[a, b]$ be the space of continuous functions $f$ on $[a, b]$ with the norm defined by [1]
\[
\|f\|_{C[a, b]} = \max_{t \in [a, b]} |f(t)|.
\]

The weighted space $C_{1-\gamma;\psi}([a, b], \mathbb{R})$ of functions $f$ on $(a, b)$ is defined by
\[
C_{1-\gamma;\psi} [a, b] = \{ f : (a, b) \to \mathbb{R}; (\psi(t) - \psi(a))^{1-\gamma} f(t) \in C[a, b] \}, \quad 0 \leq \gamma < 1,
\]
with the norm
\[
\|f\|_{C_{1-\gamma;\psi} [a, b]} = \|(\psi(t) - \psi(a))^{1-\gamma} f(t)\|_{C[a, b]} = \max_{t \in [a, b]} |(\psi(t) - \psi(a))^{1-\gamma} f(t)|.
\]

The weighted space $C^\alpha_{\gamma;\psi} [a, b]$ of functions $f$ on $(a, b)$ is defined by
\[
C^\alpha_{\gamma;\psi} [a, b] = \{ f : (a, b) \to \mathbb{R}; \ f(t) \in C^{\alpha-1} [a, b] ; \ f^{(n)}(t) \in C_{\gamma;\psi} [a, b] \}, \quad 0 \leq \gamma < 1,
\]
with the norm:
\[
\|f\|_{C^\alpha_{\gamma;\psi} [a, b]} = \sum_{k=0}^{n-1} \|f^{(k)}\|_{C[a, b]} + \|f^{(n)}\|_{C_{\gamma;\psi} [a, b]}.
\]

For $n = 0$, we have, $C^0_{\gamma;\psi} [a, b] = C_{\gamma;\psi} [a, b]$.

The weighted space $C^{\alpha,\beta}_{\gamma;\psi} [a, b]$ is defined by
\[
C^{\alpha,\beta}_{\gamma;\psi} [a, b] = \{ f \in C_{\gamma;\psi} [a, b]; \ H^D_{a+} f \in C_{\gamma;\psi} [a, b] \}, \quad \gamma = \alpha + \beta (1 - \alpha).
\]

We now present the Riemann–Liouville fractional integral with respect to a function $\psi$ and the $\psi$-Hilfer fractional derivative, recently introduced by Sousa and Oliveira [2].
Definition 2.2. Let \((a, b) (−∞ ≤ a < b ≤ ∞)\) be a finite or infinite interval of the real line \(\mathbb{R}\) and let \(α > 0\). In addition, let \(ψ(x)\) be an increasing and positive monotone function on \((a, b)\), having a continuous derivative \(ψ'(t)\) on \((a, b)\). The left-sided fractional integral of a function \(f\) with respect to function \(ψ\) on \([a, b]\) is defined by

\[
I_{a+}^{α;ψ} f(t) = \frac{1}{Γ(α)} \int_{a}^{t} \psi'(t) (ψ(t) − ψ(s))^{α−1} f(s) \, ds. \tag{2.8}
\]

The right-sided fractional integral is defined in an analogous form [1].

Definition 2.3. Let \(n − 1 < α < n\) with \(n ∈ \mathbb{N}\); let \(I = [a, b]\) be an interval, such that \(−∞ ≤ a < b ≤ ∞\) and let \(f, ψ ∈ C^n([a, b], \mathbb{R})\) be two functions, such that \(ψ\) is increasing and \(ψ'(t) \neq 0\), for all \(t ∈ I\). The left-sided \(ψ\)-Hilfer fractional derivative \(\mathbf{H}D_{a+}^{α, β; ψ} (\cdot)\) of function \(f\), of order \(α\) and type \(0 ≤ β ≤ 1\), is defined by

\[
\mathbf{H}D_{a+}^{α, β; ψ} f(t) = I_{a+}^{β(n−α); ψ} \left( \frac{1}{ψ'(t)} \frac{d}{dt} \right)^n I_{a+}^{(1−β)(n−α); ψ} f(t). \tag{2.9}
\]

The right-sided \(ψ\)-Hilfer fractional derivative is defined in an analogous form [2].

The \(ψ\)-Hilfer fractional derivative, as above defined, can be written in the form:

\[
\mathbf{H}D_{a+}^{α, β; ψ} f(t) = I_{a+}^{−α; ψ} D_{a+}^{γ; ψ} f(t), \tag{2.10}
\]

with \(γ = α + β (n − α)\) and where \(I_{a+}^{−α; ψ} (\cdot)\) is the \(ψ\)-Riemann–Liouville fractional integral and \(D_{a+}^{γ; ψ} (\cdot)\) is the \(ψ\)-Riemann–Liouville fractional derivative [1,4]. Here we consider the \(ψ\)-Hilfer fractional derivative for \(n = 1\).

The following two theorems will be important throughout this paper.

**Theorem 2.4.** If \(f ∈ C_1^{α, ψ}[a, b]\), \(0 < α < 1\) and \(0 ≤ β ≤ 1\), then

\[
I_{a+}^{α;ψ} \mathbf{H}D_{a+}^{α, β; ψ} f(t) = f(t) − \left( \frac{ψ(x) − ψ(a)}{Γ(γ)} \right)^{γ−1} I_{a+}^{(1−β)(1−α); ψ} f(a). \]

**Proof.** See [2]. □

**Theorem 2.5.** Let \(f ∈ C_1^{α, ψ}[a, b]\), \(α > 0\) and \(0 ≤ β ≤ 1\); then, we have

\[
\mathbf{H}D_{a+}^{α, β; ψ} I_{a+}^{α; ψ} f(t) = f(t). \]

**Proof.** See [2]. □

Recently, Sousa and Oliveira [2] introduced a new class of fractional derivatives and integrals and, using a result on Gronwall’s inequality [36], obtained the corresponding class of Gronwall’s inequalities.

Their results allow us to state the following Gronwall’s inequality involving the \(ψ\)-Riemann–Liouville fractional integral.

**Theorem 2.6.** Let \(u, v\) be two integrable functions and \(g\) a continuous function, with domain \([a, b]\). Let \(ψ ∈ C^1 [a, b]\) an increasing function, such that \(ψ'(t) \neq 0, ∀t ∈ [a, b]\). Assume that
1. \( u \) and \( v \) are nonnegative.
2. \( g \) is nonnegative and nondecreasing.

If

\[
u(t) \leq v(t) + g(t) \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} u(s) \, ds,
\]

then

\[
u(t) \leq v(t) + \int_a^t \sum_{k=1}^{\infty} \frac{[g(t) \Gamma(\alpha)]^k}{\Gamma(\alpha k)} \psi'(s) [\psi(t) - \psi(s)]^{\alpha k-1} v(s) \, ds,
\]

\( \forall t \in [a, b] \).

**Proof.** See [36]. \( \square \)

**Corollary 2.7.** [36] Let \( \alpha > 0 \), \( I = [a, b] \) and \( \psi \in C^1([a, b], \mathbb{R}) \) a function, such that \( \psi \) is increasing and \( \psi'(t) \neq 0 \) for all \( t \in I \). Suppose that \( b \geq 0 \), \( v \) is a nonnegative function locally integrable on \([a, b]\) and \( u \) is nonnegative and locally integrable on \([a, b]\) with

\[
u(t) \leq v(t) + b \int_a^t \psi'(s) [\psi(t) - \psi(s)]^{\alpha-1} u(s) \, ds, \quad \forall t \in [a, b].
\]

Then, we can write

\[
u(t) \leq v(t) + \int_a^t \sum_{k=1}^{\infty} \frac{[b \Gamma(\alpha)]^k}{\Gamma(\alpha k)} \psi'(s) [\psi(t) - \psi(s)]^{\alpha k-1} v(s) \, ds, \quad \forall t \in [a, b].
\]

**Corollary 2.8.** Under the hypotheses of Theorem 2.6, let \( v \) be a nondecreasing function on \([a, b]\). Then, we have

\[
u(t) \leq v(t) E_{\alpha} \left( g(t) \Gamma(\alpha) [\psi(t) - \psi(a)]^\alpha \right), \quad \forall t \in [a, b],
\]

where \( E_{\alpha}(\cdot) \) is the Mittag–Leffler function defined by Eq. (2.1).

**Proof.** See [36]. \( \square \)

In the theory of fractional differential equations, there are several types of Ulam stability. We present here the Ulam–Hyers stability, the Ulam–Hyers–Rassias stability and their respective generalizations using the \( \psi \)-Hilfer fractional derivative.

**Definition 2.9.** Equation (1.1) is Ulam–Hyers stable if there exists a real number \( c_f > 0 \), such that for each \( \varepsilon > 0 \) and for each solution \( z \in C^1_{1-\gamma,\psi}(J, \mathbb{R}) \) of the inequality

\[
\left| H D^{\alpha,\beta;\psi}_{a+} z(t) - f \left( t, z(t), H D^{\alpha,\beta;\psi}_{a+} z(t) \right) \right| \leq \varepsilon, \quad t \in J
\]

there exists a solution \( y \in C^1_{1-\gamma,\psi}(J, \mathbb{R}) \) of Eq. (1.1), such that

\[
|z(t) - y(t)| \leq c_f \varepsilon, \quad t \in J.
\]
Volterra integral equation and the nonlinear Cauchy problem, Eq. (1.1). In we present and prove a lemma which guarantees the equivalence between the obtaining this result, namely, through the integral equation. In this section, uniqueness of solutions of Cauchy-type problem has an interesting way of mathematicians, but also researchers in other fields. The study of the existence and shown to be of great interest to the academic community, not only mathe-

\[ \psi_3. \] Existence and uniqueness of solutions to \( \psi \)-Hilfer nonlinear fractional differential equations

\[ \text{Definition 2.10.} \] Equation (1.1) is generalized Ulam–Hyers stable if there exists \( \Phi_f \in C(\mathbb{R}_+, \mathbb{R}_+) \), \( \Phi_f(0) = 0 \), such that for each solution \( z \in C^1_{1-\gamma, \psi}(J, \mathbb{R}) \) of the inequality equation (2.12), there exists a solution \( y \in C^1_{1-\gamma, \psi}(J, \mathbb{R}) \) of Eq. (1.1), such that

\[ |z(t) - y(t)| \leq \Phi_f \varepsilon, \quad t \in J. \]

\[ \text{Definition 2.11.} \] Equation (1.1) is Ulam–Hyers–Rassias stable with respect to \( \varphi \in C(J, \mathbb{R}) \) if there exists a real number \( c_f > 0 \), such that for each \( \varepsilon > 0 \) and for each solution \( z \in C^1_{1-\gamma, \psi}(J, \mathbb{R}) \) of the inequality

\[ \left| \int_0^t \frac{d^\alpha \beta \psi}{d t^\alpha d z^\beta} z(t) - f \left( t, z(t), \int_0^t \frac{d^\alpha \beta \psi}{d t^\alpha d z^\beta} z(t) \right) \right| \leq \varepsilon \varphi(t), \quad t \in J \]

there exists a solution \( y \in C^1_{1-\gamma, \psi}(J, \mathbb{R}) \) of Eq. (1.1), such that

\[ |z(t) - y(t)| \leq c_f \varepsilon \varphi(t), \quad t \in J. \]

\[ \text{Definition 2.12.} \] Equation (1.1) is generalized Ulam–Hyers–Rassias stable with respect to \( \varphi \in C(J, \mathbb{R}_+) \) if there exists a real number \( c_{f, \varphi} > 0 \), such that for each solution \( z \in C^1_{1-\gamma, \psi}(J, \mathbb{R}) \) of the inequality:

\[ \left| \int_0^t \frac{d^\alpha \beta \psi}{d t^\alpha d z^\beta} z(t) - f \left( t, z(t), \int_0^t \frac{d^\alpha \beta \psi}{d t^\alpha d z^\beta} z(t) \right) \right| \leq \varphi(t), \quad t \in J, \]

there exists a solution \( y \in C^1_{1-\gamma, \psi}(J, \mathbb{R}) \) of Eq. (1.1), such that

\[ |z(t) - y(t)| \leq c_{f, \varphi} \varphi(t), \quad t \in J. \]

\[ \text{Remark 2.13.} \] A function \( z \in C^1_{1-\gamma, \psi}(J, \mathbb{R}) \) is a solution of the inequality equation (2.12) if and only if there exists a function \( g \in C(J, \mathbb{R}) \), such that

- \( |g(t)| \leq \varepsilon. \)
- \( \int_0^t \frac{d^\alpha \beta \psi}{d t^\alpha d z^\beta} z(t) = f \left( t, z(t), \int_0^t \frac{d^\alpha \beta \psi}{d t^\alpha d z^\beta} z(t) \right) + g(t), \quad t \in J. \)

3. Existence and uniqueness of solutions to \( \psi \)-Hilfer nonlinear fractional differential equations

The nonlinear Cauchy-type problem using fractional derivatives has been shown to be of great interest to the academic community, not only mathematicians, but also researchers in other fields. The study of the existence and uniqueness of solutions of Cauchy-type problem has an interesting way of obtaining this result, namely, through the integral equation. In this section, we present and prove a lemma which guarantees the equivalence between the Volterra integral equation and the nonlinear Cauchy problem, Eq. (1.1). In the sequence, using Banach’s contraction principle, we present the proof of the first main result of this paper, the existence and uniqueness of solutions to the nonlinear Cauchy problem.

\[ \text{Lemma 3.1.} \] Let a function \( f(t, u, v) : J \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) be continuous. Then, the problem, Eq. (1.1), is equivalent to the problem:

\[ y(t) = \frac{(\psi(t) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} y_a + I^\alpha_{a+} y(t), \quad (3.1) \]
where \( g \in C(J, \mathbb{R}) \) satisfies the functional equation:

\[
g(t) = f \left( t, \frac{(\psi(t) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} y_a + \int_{a+}^{\psi(t)} g \left( \psi(s) \right) ds \right) .
\]

**Proof.** Applying the fractional integral operator \( I_{a+}^{\alpha;\psi} (\cdot) \) on both sides of the fractional equation, Eq. (1.1), and using Theorem 2.4, we get

\[
y(t) = \frac{(\psi(t) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} y_a + \int_{a+}^{\psi(t)} g \left( \psi(s) \right) ds .
\]

Therefore

\[
y(t) = \frac{(\psi(t) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} y_a + I_{a+}^{\alpha;\psi} g(t) .
\]

(3.2)

On the other hand, if \( y \) satisfies Eq. (3.2), then it satisfies Eq. (1.1). However, applying the fractional derivative \( H_{a+}^{\alpha,\beta;\psi} (\cdot) \) on both sides of Eq. (3.2) and using Theorem 2.5, we have

\[
H_{a+}^{\alpha,\beta;\psi} y(t) = H_{a+}^{\alpha,\beta;\psi} \left( \frac{(\psi(t) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} y_a + \int_{a+}^{\psi(t)} g \left( \psi(s) \right) ds \right) = g(t) ,
\]

where, for \( 0 < \gamma < 1 \), we have \( H_{a+}^{\alpha,\beta;\psi} (\psi(t) - \psi(a))^{\gamma-1} = 0 \) [2,10]. □

**Theorem 3.2.** We assume the following hypotheses:

1. (H1) The function \( f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is continuous.
2. (H2) There exist constants \( k > 0 \) and \( l > 0 \), such that

\[
| f(t, u, v) - f(t, \bar{u}, \bar{v}) | \leq k |u - \bar{u}| + l |v - \bar{v}|
\]

for any \( u, v, \bar{u}, \bar{v} \in \mathbb{R} \) and \( t \in J \).

If

\[
\left( \frac{k}{\Gamma(\alpha + 1)} (\psi(t) - \psi(a))^{\alpha} + l \right) < 1
\]

then there exists a unique solution for the Cauchy problem, Eq. (1.1) on \( J \).

**Proof.** Define the operator \( M : C_{1-\gamma,\psi}(J, \mathbb{R}) \to C_{1-\gamma,\psi}(J, \mathbb{R}) \) by

\[
Mz(t) = f \left( t, \frac{(\psi(t) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} y_a + \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} z(s) ds, z(s) \right)
\]

(3.4)

for each \( t \in J \).
Let \( u, w \in C_{1-\gamma, \psi} (J, \mathbb{R}) \). Then, for \( t \in J = [a, T] \) and using hypothesis (H1), we have

\[
\left| (\psi(t) - \psi(a))^{1-\gamma} (Mu(t) - Mw(t)) \right| \\
\leq \frac{k}{\Gamma(\alpha)} \int_{a}^{t} \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} \left| (\psi(t) - \psi(a))^{1-\gamma} (u(s) - w(s)) \right| ds \\
+ l \left| (\psi(t) - \psi(a))^{1-\gamma} (u(t) - w(t)) \right| \\
\leq \frac{k}{\Gamma(\alpha)} \int_{a}^{t} \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} ds + l \left\| u - w \right\|_{C_{1-\gamma, \psi}} \\
\leq \left[ \frac{k}{\Gamma(\alpha + 1)} (\psi(T) - \psi(a))^\alpha + l \right] \left\| u - w \right\|_{C_{1-\gamma, \psi}}.
\]

(3.5)

Evaluating the maximum values, for \( t \in [a, b] \), of both sides of Eq. (3.5) and using the definition of the norm in the weighted space, we get

\[
\left\| Mu - Mw \right\|_{C_{1-\gamma, \psi}} \leq \left[ \frac{k}{\Gamma(\alpha + 1)} (\psi(T) - \psi(a))^\alpha + l \right] \left\| u - w \right\|_{C_{1-\gamma, \psi}}.
\]

It follows from this result and Eq. (3.3) that operator \( M \) is a contraction, then, we conclude that operator \( M \) has a unique fixed point \( z \in C_{1-\gamma, \psi} (J, \mathbb{R}) \), given by Banach’s contraction principle.

Therefore

\[
z(t) = f(t, y(t), z(s)),
\]

for each \( t \in J \), where

\[
y(t) = \frac{(\psi(t) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} y_a + \frac{1}{\Gamma(\alpha)} \int_{a}^{t} \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} z(s) ds.
\]

This implies that \( H^\alpha D_{a+}^{\alpha, \beta; \psi} y(t) = z(t) \). Consequently

\[
H^\alpha D_{a+}^{\alpha, \beta; \psi} y(t) = f \left( t, y(t), H^\alpha D_{a+}^{\alpha, \beta; \psi} y(t) \right).
\]

□

4. Ulam–Hyers and Ulam–Hyers–Rassias stabilities

In this section, we present and prove two theorems showing that the nonlinear Cauchy problem, Eq. (1.1), using \( \psi \)-Hilfer fractional derivative admits Ulam–Hyers and Ulam–Hyers–Rassias stabilities.

The first theorem is about Ulam–Hyers stability.

**Theorem 4.1.** Suppose the validity of (H1), (H2) and Eq. (3.3). Then, Eq. (1.1) is Ulam–Hyers stable.

**Proof.** Let \( z \in C_{1-\gamma, \psi} (J, \mathbb{R}) \) be a solution of inequality equation (2.12), that is

\[
\left| H^\alpha D_{a+}^{\alpha, \beta; \psi} z(t) - f \left( t, z(t), H^\alpha D_{a+}^{\alpha, \beta; \psi} z(t) \right) \right| \leq \varepsilon, \quad t \in J.
\]

(4.1)
Let us denote by \( y \in C_{1−\gamma,\psi} (J, \mathbb{R}) \) the unique solution of the Cauchy problem, so that
\[
H \mathbb{D}_{a+}^{\alpha,\beta;\psi} y (t) = \mathbb{D}_{a+}^{\alpha,\beta;\psi} y (t)
\]
for each \( t \in J \), \( 0 < \alpha \leq 1 \) and \( 0 \leq \beta \leq 1 \); \( I_{a+}^{(1−\beta)(1−\alpha);\psi} z (a) = I_{a+}^{(1−\beta)(1−\alpha);\psi} z (a) \).

Using Lemma 3.1, we have
\[
y (t) = \frac{\left( \psi (t) - \psi (a) \right) \gamma^{-1}}{\Gamma (\gamma)} y_a + \frac{1}{\Gamma (\alpha)} \int_a^t \psi' (s) (\psi (t) - \psi (s))^{\alpha-1} g_y (s) \, ds,
\]
where \( g_y \in C_{1−\gamma,\psi} (J, \mathbb{R}) \) satisfies the functional equation:
\[
g_y (t) = f \left( t, \frac{\left( \psi (t) - \psi (a) \right) \gamma^{-1}}{\Gamma (\gamma)} y_a + I_{a+}^{\alpha;\psi} g_y (t), g_y (t) \right).
\]

Applying operator \( I_{a+}^{\alpha;\psi} (\cdot) \) on both sides of Eq. (4.1) and using Theorem 2.4, we have
\[
\left| I_{a+}^{\alpha;\psi} H \mathbb{D}_{a+}^{\alpha,\beta;\psi} z (t) - I_{a+}^{\alpha;\psi} H f \left( t, z (t), H \mathbb{D}_{a+}^{\alpha,\beta;\psi} z (t) \right) \right| \leq I_{a+}^{\alpha;\psi} \varepsilon;
\]
this implies that
\[
\left| z (t) - \frac{\left( \psi (t) - \psi (a) \right) \gamma^{-1}}{\Gamma (\gamma)} z_a - \frac{1}{\Gamma (\alpha)} \int_a^t \psi' (s) (\psi (t) - \psi (s))^{\alpha-1} g_z (s) \, ds \right|
\]
\[
\leq \varepsilon I_{a+}^{\alpha;\psi} 1.
\]

Hence, we obtain
\[
\left| z (t) - \frac{\left( \psi (t) - \psi (a) \right) \gamma^{-1}}{\Gamma (\gamma)} z_a - \frac{1}{\Gamma (\alpha)} \int_a^t \psi' (s) (\psi (t) - \psi (s))^{\alpha-1} g_z (s) \, ds \right|
\]
\[
\leq \frac{\varepsilon (\psi (T) - \psi (a)) \alpha}{\Gamma (\alpha + 1)}, \quad (4.2)
\]
where \( g_z \in C_{1−\gamma,\psi} (J, \mathbb{R}) \) satisfies the functional equation:
\[
g_z (t) = f \left( t, \frac{\left( \psi (t) - \psi (a) \right) \gamma^{-1}}{\Gamma (\gamma)} z_a + I_{a+}^{\alpha;\psi} g_z (t), g_z (t) \right).
\]

On the other hand, we have, for each \( t \in J \)
\[
|z (t) - y (t)|
\]
\[
\leq \left| z (t) - \frac{\left( \psi (t) - \psi (a) \right) \gamma^{-1}}{\Gamma (\gamma)} z_a - \frac{1}{\Gamma (\alpha)} \int_a^t \psi' (s) (\psi (t) - \psi (s))^{\alpha-1} g_z (s) \, ds \right|
\]
\[
+ \frac{1}{\Gamma (\alpha)} \int_a^t \psi' (s) (\psi (t) - \psi (s))^{\alpha-1} |g_z (s) - g_y (s)| \, ds,
\]
where
\[
g_y (t) = f (t, y (t), g_y (t))
\]
and
\[
g_z (t) = f (t, z (t), g_z (t)).
\]
Using hypothesis (H2) of Theorem 3.2 and the two equalities above for \(g_z(\cdot)\) and \(g_y(\cdot)\), we have, for each \(t \in J\)
\[
|g_z(t) - g_y(t)| \leq k |z(t) - y(t)| + l |g_z(t) - g_y(t)|,
\]
which can be written as
\[
|g_z(t) - g_y(t)| \leq \frac{k}{1-l} |z(t) - y(t)|. \tag{4.4}
\]

Using Eqs. (4.2), (4.3), (4.4), and Theorem 2.6, we have
\[
|z(t) - y(t)| \leq \frac{\varepsilon \psi(T)}{\Gamma(\alpha + 1)} + \frac{k}{(1-l)\Gamma(\alpha)} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} |z(s) - y(s)| \, ds
\]
\[
\leq \frac{\varepsilon \psi(T)}{\Gamma(\alpha + 1)} \left[ 1 + \int_a^t \sum_{n=1}^{\infty} \left( \frac{k}{1-l} \right)^n \frac{1}{\Gamma(n\alpha)} \psi'(s) (\psi(t) - \psi(s))^{n\alpha-1} \, ds \right]
\]
\[
\leq \frac{\varepsilon \psi(T)}{\Gamma(\alpha + 1)} \frac{1}{1 - \frac{k}{\Gamma(\alpha + 1)}} \left[ 1 + \sum_{n=1}^{\infty} \frac{1}{\Gamma(n\alpha + 1)} \left( \frac{k}{1-l} \right)^{n\alpha} \right]
\]
\[
= \frac{\varepsilon \psi(T) - \psi(a)}{\Gamma(\alpha + 1)} \bigg( \frac{k}{1-l} \bigg) (\psi(T) - \psi(a))^\alpha \tag{4.5}
\]
Then, for \(c_f := \frac{\psi(T) - \psi(a)}{\Gamma(\alpha + 1)} \bigg( \frac{k}{1-l} \bigg) (\psi(T) - \psi(a))^\alpha\) with \(t \in J = [a, T], T > a\), we conclude from Eq. (4.5) that Eq. (1.1) is Ulam–Hyers stable. On the other hand, choosing \(\Phi(\varepsilon) = c \varepsilon, \Phi(0) = 0\), we obtain that Eq. (1.1) is generalized Ulam–Hyers stable.  

\[\square\]

**Theorem 4.2.** Assume the validity of (H1), (H2) and Eq. (3.3). Assume also the validity of hypothesis (H3), i.e., that function \(\varphi \in C(J, \mathbb{R}_+)\) is increasing and there exists \(\lambda_\varphi > 0\), such that for each \(t \in J,\) we have
\[
I_{a+}^{\alpha;\varphi}(t) \leq \lambda_\varphi \varphi(t).
\]
Then, Eq. (1.1) is Ulam–Hyers–Rassias stable with respect to \(\varphi\).

**Proof.** Let \(z \in C_{1-\gamma,\psi}(J, \mathbb{R})\) be a solution of the inequality:
\[
\left| H D_{a+}^{\alpha;\psi} z(t) - f(t, z(t), H D_{a+}^{\alpha;\psi} z(t)) \right| \leq \varepsilon \varphi(t), \quad t \in J, \quad \varepsilon > 0. \tag{4.6}
\]
On the other hand, let us denote by \(y \in C_{1-\gamma,\psi}(J, \mathbb{R})\) the unique solution of the Cauchy problem
\[
H D_{a+}^{\alpha;\psi} y(t) = f(t, y(t), H D_{a+}^{\alpha;\psi} y(t)), \quad t \in J, \quad 0 < \alpha \leq 1, \quad 0 \leq \beta \leq 1,
\]
Using Lemma 3.1, we have
\[
y(t) = \frac{(\psi(t) - \psi(a))^{-1}}{\Gamma(\gamma)} y_a + \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} g_y(s) \, ds,
\]
where \( g_z \in C_{1-\gamma,\psi}(J,\mathbb{R}) \) satisfies the functional equation

\[
g_y(t) = f \left(t, \frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)} y_a + I_{a+}^{\alpha;\psi} g_y(t), g_y(t)\right).
\]

Applying operator \( I_{a+}^{\alpha;\psi} \cdot \) on both sides of Eq. (4.6) and using Theorem 2.4, we obtain

\[
\left| z(t) - \frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)} z_a - I_{a+}^{\alpha;\psi} f(t, z(t), H D_{\alpha;\psi} z(t)) \right| \\
\leq \varepsilon \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) (\psi(t)-\psi(s))^{\alpha-1} \varphi(s) \, ds \leq \varepsilon \lambda \varphi(t),
\]

where \( g_z \in C_{1-\gamma,\psi}(J,\mathbb{R}) \) satisfies the functional equation:

\[
g_z(t) = f \left(t, \frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)} z_a + I_{a+}^{\alpha;\psi} g_z(t), g_z(t)\right).
\]

Then, performing the same steps of Eq. (4.3), we have, for each \( t \in J \)

\[
|z(t) - y(t)| \\
= \left| z(t) - \frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)} z_a - \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) (\psi(t)-\psi(s))^{\alpha-1} g_y(s) \, ds \right| \\
\leq \left| z(t) - \frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)} z_a - \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) (\psi(t)-\psi(s))^{\alpha-1} g_z(s) \, ds \right| \\
+ \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) (\psi(t)-\psi(s))^{\alpha-1} |g_z(s) - g_y(s)| \, ds,
\]

where

\[
g_y(t) = f(t, y(t), g_y(t))
\]

and

\[
g_z(t) = f(t, z(t), g_z(t)).
\]

Then, using Eqs. (4.4), (4.7) and (4.8), we have

\[
|z(t) - y(t)| \\
\leq \varepsilon \lambda \varphi(t) + \frac{k}{(1-l) \Gamma(\alpha)} \int_a^t \psi'(s) (\psi(t)-\psi(s))^{\alpha-1} |z(s) - y(s)| \, ds \\
\leq \varepsilon \lambda \varphi(t) + \frac{k \|z - y\|_\infty}{(1-l) \Gamma(\alpha)} \int_a^t \psi'(s) (\psi(t)-\psi(s))^{\alpha-1} \, ds \\
\leq \varepsilon \lambda \varphi(t) + \frac{k \|z - y\|_\infty}{(1-l) \Gamma(\alpha + 1)} (\psi(T)-\psi(s))^\alpha.
\]

Thus, we have

\[
\|z - y\|_\infty \left[ 1 - \frac{k (\psi(t)-\psi(a))^{\alpha}}{(1-l) \Gamma(\alpha + 1)} \right] \leq \varepsilon \lambda \varphi(t).
\]
Using Eq. (3.3), we can rewrite Eq. (4.10) in the following form

\[ \|z - y\|_\infty \leq \left[ 1 - \frac{k(\psi(t) - \psi(a))}{(1-\lambda_l) \Gamma(\alpha + 1)} \right] \lambda_{\varphi} \psi(t). \]

Then, for \( c_f := \left[ 1 - \frac{k(\psi(t) - \psi(a))}{(1-\lambda_l) \Gamma(\alpha + 1)} \right] \lambda_{\varphi} \) with \( t \in J = [a, T) \), \( T > a \), we conclude from Eq. (4.9) that Eq. (1.1) is Ulam–Hyers–Rassias stable.

Proposing a fractional operator of differentiation or integration is important and motivating to researchers working in the field of fractional calculus. However, proposing a fractional operator that unifies a vast number of definitions is not a simple task, let alone an easy one. The \( \psi \)-Hilfer fractional derivative defined in Eq. (2.8) contains, as special cases, several different classes (definitions) of fractional derivatives. Thus, the nonlinear Cauchy problem we have just proposed does also contain, as special cases, corresponding Cauchy problems for those classes of fractional derivatives. Moreover, the results we have just proved about the uniqueness of its solution and its stabilities also apply to the special cases.

These classes of fractional derivatives are obtained by means of an adequate choice of \( \psi(\cdot) \) and by considering the limits \( \beta \to 1 \) or \( \beta \to 0 \).

The choice of parameter \( a \) also determines the particular cases of the operators. For instance, to recover our results for the Hadamard fractional derivative, we choose \( \psi(t) = \ln t \), take the limit \( \beta \to 0 \) and put \( a = 1 \), because \( \ln t \) is not defined for \( a = 0 \).

We now present some of these particular cases of Ulam–Hyers and Ulam–Hyers–Rassias stabilities. In the next section, we present some examples and discuss some particular derived from them:

- Choosing \( \psi(t) = \ln t \), \( a = 1 \) and applying the limit \( \beta \to 0 \) on both sides of Eq. (1.1) and applying Theorems 4.1 and 4.2, we obtain the stability of Ulam–Hyers and Ulam–Hyers–Rassias for the Cauchy problem with Hadamard fractional derivative [39].
- For \( \psi(t) = t \), taking the limit \( \beta \to 1 \) on both sides of Eq. (1.1) and applying Theorems 4.1 and 4.2, we obtain the stability of Ulam–Hyers and Ulam–Hyers–Rassias for the Cauchy problem with the Caputo fractional derivative [40].
- For \( \psi(t) = t \), taking the limit \( \beta \to 0 \) on both sides of Eq. (1.1) and applying Theorems 4.1 and 4.2, we obtain the stability of Ulam–Hyers and Ulam–Hyers–Rassias for the Cauchy problem with the Riemann–Liouville fractional derivative.
- Choosing \( \psi(t) = t^\rho \), taking the limit \( \beta \to 0 \) on both sides of Eq. (1.1) and applying Theorems 4.1 and 4.2, we obtain the stability of Ulam–Hyers and Ulam–Hyers–Rassias for the Cauchy problem with the Katugampola fractional derivative.
- For \( \psi(t) = \ln t \), \( a = 1 \) and applying the limit \( \beta \to 0 \) on both sides of Eq. (1.1) and applying Theorems 4.1 and 4.2, we obtain the stability of
Ulam–Hyers and Ulam–Hyers–Rassias for the Cauchy problem with the Caputo–Hadamard fractional derivative.

We thus see that the Cauchy problem proposed for the \(\psi\)-Hilfer fractional derivative is in fact general, and with this it is possible to deduce the existence and uniqueness, as well as the stability of Ulam–Hyers and Ulam–Hyers–Rassias, for the nonlinear Cauchy problem, Eq. (1.1).

5. Examples

In this section, we consider some particular cases of the nonlinear implicit fractional differential equations to apply our results in the study of stabilities, specifically, generalized Ulam–Hyers and Ulam–Hyers–Rassias.

We believe that the best way to understand the results obtained here is through examples. Then, we use similar ideas to those used by several researchers in recent studies, involving stability and uniqueness of solutions of implicit fractional differential equations, among them, Benchohra, Graef, Nieto, Wang, Abbas and Henderson [29–31,33]. Furthermore, for other examples of implicit fractional differential equations, we suggest [27,28,32,34,35].

Consider the nonlinear implicit fractional differential equations (NIFDEs) of the form:

\[
\begin{aligned}
H_{a+}^{\alpha,\beta;\psi} y(t) &= f \left( t, y(t), H_{a+}^{\alpha,\beta;\psi} y(t) \right), \ t \in [a, T] \\
I_{a+}^{\alpha;\psi} y(a) &= 1,
\end{aligned}
\]  
(5.1)

where \(H_{a+}^{\alpha,\beta;\psi}(\cdot)\) is the \(\psi\)-Hilfer fractional derivative and \(I_{a+}^{\alpha;\psi}(\cdot)\) is \(\psi\)-Riemann–Liouville fractional integral. The following examples are particular cases of the NIFDEs given by Eq. (5.1).

**Example.** Consider the NIFDEs, Eq. (5.1). Taking \(\psi(t) = t\), \(a = 0\), \(T = 1\), \(\alpha = 1/2\), the limit \(\beta \to 0\), \(y : [0, 1] \to \mathbb{R}\) and \(f : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) a nonlinear function defined by

\[
f \left( t, y(t), RL_D_{0+}^{1/2,0;\psi} y(t) \right) = \frac{\lambda}{20} E_{1/2} \left( t^{1/2} \right) y(t) + \frac{\lambda}{10} RL_D_{0+}^{1/2,0;\psi} y(t),
\]

for all \(t \in [0, 1]\) and with \(0 < \lambda < \frac{13}{5}\), we get a particular case of the NIFDEs, Eq. (5.1), involving the Riemann–Liouville fractional derivative.

For all \(u, v, \bar{u}, \bar{v} \in \mathbb{R}\) and \(t \in [0, 1]\), we have

\[
|f(t, u, v) - f(t, \bar{u}, \bar{v})| = \left| \frac{\lambda}{20} E_{1/2} \left( t^{1/2} \right) u + \frac{\lambda}{10} v - \frac{\lambda}{20} E_{1/2} \left( t^{1/2} \right) \bar{u} - \frac{\lambda}{10} \bar{v} \right| \\
\leq \frac{\lambda}{20} E_{1/2} \left( t^{1/2} \right) |u - \bar{u}| + \frac{\lambda}{10} |v - \bar{v}|.
\]
Thus, condition (H2) of Theorem 3.2 is satisfied with \( k = \frac{\lambda}{20} \mathbb{E}_{1/2} (t^{1/2}) \) and \( l = \frac{\lambda}{10} \). On the other hand, using Eq. (2.7), the condition
\[
\frac{k (\psi (t) - \psi (a))^\alpha}{\Gamma (\alpha + 1) (1 - l)} = \frac{\lambda \mathbb{E}_{1/2} (t^{1/2}) t^{1/2}}{\Gamma (3/2) (1 - \frac{\lambda}{10})} = \frac{\lambda \mathbb{E}_{1/2} (t^{1/2}) t^{1/2}}{\sqrt{\pi} (10 - \lambda)} \leq \frac{\lambda \mathbb{E}_{1/2} (1)}{\sqrt{\pi} (10 - \lambda)} < 1,
\]
for \( 0 < \lambda < \frac{13}{5} \), is satisfied.

Therefore, by Theorem 3.2, the problem has a unique solution. Consequently, by Theorem 4.1, Eq. (5.2) is Ulam–Hyers stable.

Example. Consider the NIFDEs, Eq. (5.1). For \( \lambda < 0 \) and \( \alpha = 1/2 \), taking the limit \( \beta \rightarrow 1 \), we get a particular case of the NIFDEs, Eq. (5.1), involving the Caputo fractional derivative.

For all \( t \in [0, 1] \) and with \( 0 < \lambda < \frac{13}{10} \), we get a particular case of the NIFDEs, Eq. (5.1), involving the Caputo fractional derivative.

Thus, condition (H2) of Theorem 3.2 is satisfied with \( k = \frac{\lambda}{20} \mathbb{E}_{1/2} (t^{1/2}) \) and \( l = \frac{\lambda}{10} \). On the other hand, the condition:
\[
\frac{k \psi (t) - \psi (a))^\alpha}{\Gamma (\alpha + 1) (1 - l)} = \frac{\lambda \mathbb{E}_{1/2} (t^{1/2}) t^{1/2}}{\sqrt{\pi} (10 - \lambda)} \leq \frac{\lambda \mathbb{E}_{1/2} (1)}{\sqrt{\pi} (10 - \lambda)} < 1,
\]
for \( 0 < \lambda < \frac{13}{10} \), is satisfied.

Therefore, by Theorem 3.2, the problem has a unique solution. Consequently, by Theorem 4.1, Eq. (5.3) is Ulam–Hyers stable.

Example. Consider the NIFDEs, Eq. (5.1). For \( \psi (t) = \ln t \), taking the limit \( \beta \rightarrow 0 \), we get a particular case of the NIFDEs, Eq. (5.1), involving the Hadamard fractional derivative.
For all $u, v, \pi, \nu \in \mathbb{R}$ and $t \in [1, e]$, we have
\[
|f(t, u, v) - f(t, \pi, \nu)| = \left| \frac{\lambda}{20} E_{1/2} \left( \ln t^{1/2} \right) u + \frac{\lambda}{10} v - \frac{\lambda}{20} E_{1/2} \left( \ln t^{1/2} \right) \pi - \frac{\lambda}{10} \nu \right|
\leq \frac{\lambda}{20} E_{1/2} \left( \ln t^{1/2} \right) |u - \pi| + \frac{\lambda}{10} |v - \nu|.
\]

Thus, condition (H2) of Theorem 3.2 is satisfied with $k = \frac{\lambda}{20} E_{1/2} \left( \ln t^{1/2} \right)$ and $l = \frac{\lambda}{10}$. On the other hand, using Eq. (2.7), the condition
\[
k \left( \psi(t) - \psi(a) \right)^{\alpha} \frac{\lambda}{\Gamma(\alpha + 1) (1 - l)} \leq \frac{\lambda E_{1/2} (1)}{\sqrt{\pi} (10 - \lambda)} < 1,
\]
for $0 < \lambda < \frac{13}{5}$, is satisfied.

Therefore, by Theorem 3.2, the problem has a unique solution. Consequently, by Theorem 4.1 the Eq. (5.4), has Ulam–Hyers stability.

**Example.** Consider the NIFDEs, Eq. (5.1). For $\psi(t) = t$, $a = 1$, $T = 1$, $\alpha = 1/2$, taking the limit $\beta \rightarrow 1/2$, $y : [0, 1] \rightarrow \mathbb{R}$ and $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ a nonlinear function defined by
\[
f \left( t, y(t), H D_{0+}^{1/2, 1/2}, t \right) = \frac{\lambda}{20} E_{1/2} \left( \lambda^{1/2} t^{1/2} \right) y(t) + \frac{\lambda}{10} H D_{0+}^{1/2, 1/2} y(t),
\]
(5.5)
for all $t \in [0, 1]$ and with $0 < \lambda < \frac{13}{10}$, we get a particular case of the NIFDEs, Eq. (5.1), involving the $\psi$-Hilfer fractional derivative.

For all $u, v, \pi, \nu \in \mathbb{R}$ and $t \in [0, 1]$, we get
\[
|f(t, u, v) - f(t, \pi, \nu)| = \left| \frac{\lambda}{20} E_{1/2} \left( \lambda^{1/2} t^{1/2} \right) u + \frac{\lambda}{10} v - \frac{\lambda}{20} E_{1/2} \left( \lambda^{1/2} t^{1/2} \right) \pi - \frac{\lambda}{10} \nu \right|
\leq \frac{\lambda}{20} E_{1/2} \left( \lambda^{1/2} t^{1/2} \right) |u - \pi| + \frac{\lambda}{10} |v - \nu|.
\]

Thus, condition (H2) of Theorem 3.2 is satisfied with $k = \frac{\lambda}{20} E_{1/2} \left( \lambda^{1/2} t^{1/2} \right)$ and $l = \frac{\lambda}{10}$. On the other hand, the condition
\[
k \left( \psi(t) - \psi(a) \right)^{\alpha} \frac{\lambda}{\Gamma(\alpha + 1) (1 - l)} \leq \frac{\lambda E_{1/2} (\lambda^{1/2})}{\sqrt{\pi} (10 - \lambda)} < 1,
\]
for $0 < \lambda < \frac{13}{5}$, is satisfied.

Therefore, by Theorem 3.2, the problem has a unique solution. Consequently, by Theorem 4.1, Eq. (5.5) is Ulam–Hyers stable.

**Remark 5.1.** We presented four particular cases for the study of Ulam–Hyers stability. We have seen that for the Cauchy problems using the Riemann–Liouville and Hadamard fractional derivatives, the stability range $0 < \lambda < \frac{13}{5}$ is the same. On the other hand, for the Cauchy problem involving the
Caputo fractional derivative, the interval is $0 < \lambda < \frac{13}{10}$. Thus, we can say that the stability interval for parameter $\lambda$ is larger when we use Riemann–Liouville and Hadamard fractional derivatives than when we considered the Caputo fractional derivative. Therefore, we conclude that, for the Cauchy problem, Eq. (5.1), its particular cases have different stability intervals for each particular fractional derivative.

The following examples are associated with the stability of Ulam–Hyers–Rassias and there are particular cases of the NIFDEs, Eq. (5.1).

**Example.** Consider the NIFDEs, Eq. (5.1). Taking $\psi(t) = \ln t$, $a = 1$, $T = e$, $\alpha = 1/2$, the limit $\beta \to 0$, $y : [1, e] \to \mathbb{R}$ and $f : [1, e] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a nonlinear function defined by

$$f \left( t, y(t), \mathcal{H}^{1/2,0;\ln t} y(t) \right) = \frac{1}{20} \ln t^{1/2} \cos t \ y(t) + \frac{1}{20} \mathcal{H}^{1/2,0;\ln t} y(t),$$

for all $t \in [1, e]$, we get a particular case of the NIFDEs, Eq. (5.1), involving the Hadamard fractional derivative.

Note that $f$ is a set of continuous functions. Then, for all $u, v, \overline{u}, \overline{v} \in \mathbb{R}$ and $t \in [1, e]$, we have

$$|f(t, u, v) - f(t, \overline{u}, \overline{v})| = \left| \frac{\ln t^{1/2}}{20} \cos t \ u + \frac{v}{20} - \frac{\ln t^{1/2}}{20} \cos t \ \overline{u} - \frac{\overline{v}}{20} \right| \leq \frac{1}{20} \left( |u - \overline{u}| + |v - \overline{v}| \right).$$

Thus, condition (H2) of Theorem 3.2 is satisfied with $k = l = 1/20$. Therefore, we get

$$I_{1+}^{1/2;\ln t} \varphi (t) = \frac{1}{\Gamma(1/2)} \int_1^t \left( \ln \frac{s}{t} \right)^{-1/2} (\ln s)^{1/2} \frac{ds}{s} \leq \frac{1}{\sqrt{\pi}} \int_1^t \left( \ln \frac{s}{t} \right)^{-1/2} \frac{ds}{s} = \frac{2}{\sqrt{\pi}} (\ln t)^{1/2}.$$ 

Then, for $\lambda_\varphi := \frac{2}{\sqrt{\pi}}$ and $\varphi (t) = (\ln t)^{1/2}$, condition (H3) of Theorem 4.2 is satisfied. Therefore, by Theorem 3.2 the problem has a unique solution in $J$. Consequently, from Theorem 4.2, Eq. (5.6) is Ulam–Hyers–Rassias stable.

**Example.** Consider the NIFDEs, Eq. (5.1). Taking $\psi(t) = \ln t$, $a = 1$, $T = e$, $\alpha = 1/2$, the limit $\beta \to 1$, $y : [1, e] \to \mathbb{R}$ and $f : [1, e] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a nonlinear function defined by

$$f \left( t, y(t), \mathcal{C}^{1/2,1;\ln t} y(t) \right) = \frac{\lambda}{20} \ln t^{1/2} \cos t \ y(t) + \frac{\lambda}{20} \mathcal{C}^{1/2,1;\ln t} y(t),$$

for all $t \in [1, e]$, we get a particular case of the NIFDEs, Eq. (5.1), involving the Caputo fractional derivative.

Note that $f$ is a set of continuous functions. Then, for all $u, v, \overline{u}, \overline{v} \in \mathbb{R}$ and $t \in [1, e]$, we have

$$|f(t, u, v) - f(t, \overline{u}, \overline{v})| = \left| \frac{\lambda}{20} \ln t^{1/2} \cos t \ u + \frac{v}{20} - \frac{\lambda}{20} \ln t^{1/2} \cos t \ \overline{u} - \frac{\overline{v}}{20} \right| \leq \frac{\lambda}{20} \left( |u - \overline{u}| + |v - \overline{v}| \right).$$

Thus, condition (H2) of Theorem 3.2 is satisfied with $k = l = 1/20$. Therefore, we get

$$I_{1+}^{1/2;\ln t} \varphi (t) = \frac{1}{\Gamma(1/2)} \int_1^t \left( \ln \frac{s}{t} \right)^{-1/2} (\ln s)^{1/2} \frac{ds}{s} \leq \frac{1}{\sqrt{\pi}} \int_1^t \left( \ln \frac{s}{t} \right)^{-1/2} \frac{ds}{s} = \frac{2}{\sqrt{\pi}} (\ln t)^{1/2}.$$ 

Then, for $\lambda_\varphi := \frac{2}{\sqrt{\pi}}$ and $\varphi (t) = (\ln t)^{1/2}$, condition (H3) of Theorem 4.2 is satisfied. Therefore, by Theorem 3.2 the problem has a unique solution in $J$. Consequently, from Theorem 4.2, Eq. (5.6) is Ulam–Hyers–Rassias stable.
for all \( t \in [1, e] \), we get a particular case of the NIFDEs, Eq. (5.1), involving the Caputo–Hadamard fractional derivative.

Note that \( f \) is a set of continuous functions. Then, for all \( u, v, \bar{u}, \bar{v} \in \mathbb{R} \) and \( t \in [1, e] \), we get

\[
| f(t, u, v) - f(t, \bar{u}, \bar{v}) | = \left| \frac{\lambda \ln t^{1/2}}{20} \cos t \ u + \frac{\lambda v}{20} - \frac{\lambda \ln t^{1/2}}{20} \cos t \ \bar{u} - \frac{\lambda \bar{v}}{20} \right|
\]

\[
\leq \frac{\lambda}{20} (|u - \bar{u}| + |v - \bar{v}|).
\]

Then, condition (H2) of Theorem 3.2 is satisfied with \( k = l = \lambda /20 \).

Therefore, we have

\[
\int_{1/2}^{1/2; \ln t} \varphi(t) = \frac{1}{\Gamma(1/2)} \int_{1}^{t} \left( \frac{\ln s}{s} \right)^{-1/2} \left( \ln s \right)^{1/2} \frac{ds}{s}
\]

\[
\leq \frac{1}{\Gamma(1/2)} \int_{1}^{t} \left( \frac{\ln s}{s} \right)^{-1/2} \frac{ds}{s}
\]

\[
= \frac{2}{\sqrt{\pi}} (\ln t)^{1/2}.
\]

Then, for \( \lambda \varphi := \frac{2}{\sqrt{\pi}} \) and \( \varphi(t) = (\ln t)^{1/2} \), condition (H3) of Theorem 4.2 is satisfied. Therefore, by Theorem 3.2 the problem has a unique solution in \( J \). Consequently, from Theorem 4.2, Eq. (5.7) is Ulam–Hyers–Rassias stable.

**Example.** Consider the NIFDEs, Eq. (5.1). Taking \( \psi(t) = t^\rho \), \( 0 \leq \rho \leq 4 \), \( a = 0 \), \( T = 1 \), \( \alpha = 1/2 \), the limit \( \beta \to 1 \), \( \gamma : [0, 1] \to \mathbb{R} \) and \( f : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) a nonlinear function defined by

\[
f(t, y(t), C_{K}D_{0+}^{1/2, 1; \rho} y(t)) = \frac{\lambda}{20} t^{\rho/2} \cos t \ y(t) + \frac{\lambda}{20} C_{K}D_{0+}^{1/2, 1; \rho} y(t),
\]

(5.8)

for all \( t \in [0, 1] \), we get a particular case of the NIFDEs, Eq. (5.1), involving the Caputo–Katugampola fractional derivative.

Note that \( f \) is a set of continuous functions. Then, for all \( u, v, \bar{u}, \bar{v} \in \mathbb{R} \) and \( t \in [0, 1] \), we have

\[
| f(t, u, v) - f(t, \bar{u}, \bar{v}) | = \left| \frac{\lambda t^{\rho/2}}{20} \cos t \ u + \frac{\lambda v}{20} - \frac{\lambda t^{\rho/2}}{20} \cos t \ \bar{u} - \frac{\lambda \bar{v}}{20} \right|
\]

\[
\leq \frac{\lambda}{20} (|u - \bar{u}| + |v - \bar{v}|).
\]

Then, condition (H2) of Theorem 3.2 is satisfied with \( k = l = \lambda /20 \).

Therefore, we get

\[
\rho I_{0+}^{1/2; \rho} \varphi(t) = \frac{\rho^{1/2}}{\Gamma(1/2)} \int_{0}^{t} (t^{\rho} - s^{\rho})^{-1/2} s^{\rho+1/2} ds
\]

\[
\leq \frac{\rho^{1/2}}{\Gamma(1/2)} \int_{0}^{t} (t^{\rho} - s^{\rho})^{-1/2} ds
\]

\[
= \frac{2\rho^{1/2}}{\sqrt{\pi} (\rho + 1)} t^{\rho/2}.
\]
Then, for $\lambda_{\varphi} := \frac{2\rho^{1/2}}{\sqrt{\pi} (\rho + 1)}$ and $\varphi(t) = t^{\rho/2}$, condition (H3) of Theorem 4.2 is satisfied. Therefore, by Theorem 3.2, the problem has a unique solution in $J$. Consequently, from Theorem 4.2, Eq. (5.1) is Ulam–Hyers–Rassias stable.

6. Concluding remarks

In the first part of this paper, we studied the existence and uniqueness of solution of a nonlinear Cauchy problem. In the second part, we investigated the Ulam–Hyers and Ulam–Hyers–Rassias stabilities of such solutions and discussed some particular cases. Some important points were presented in the body of the article, in particular, the big gain obtained by studying the Cauchy problem using the $\psi$-Hilfer fractional derivative. We discussed some examples of stability, presenting the stability interval of each Cauchy problem and comparing them.

An interesting extension of our studies would be to discuss global attractivity for the nonlinear Cauchy problem using the $\psi$-Hilfer fractional derivative [41–43].

Acknowledgements

We are grateful to the anonymous referees for the suggestions that improved the manuscript.

References

[1] Kilbas, A.A., Srivastava, H.M., Trujillo, J.: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, p. 207. Elsevier, Amsterdam (2006)
[2] Sousa, J., da Vanterler, C., de Oliveira, E.C.: On the $\psi$-Hilfer fractional derivative. Commun. Nonlinear Sci. Numer. Simul. 60, 72–91 (2018)
[3] Sousa, J., da Vanterler, C., de Oliveira, E.C.: On two new operators in fractional calculus and application (2017). arXiv:1710.0371
[4] Samko, S., Kilbas, A., Marichev, O.: Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach, London (1993)
[5] Rosa, E.C.F.A., de Oliveira, E.C.: Relaxation equations: fractional models. J. Phys. Math. https://doi.org/10.4172/2090-0902.1000146
[6] Silva Costa, F., Contharteze Grigoletto, E., Vaz Jr., J., Capelas de Oliveira, E.: Slowing-down of neutrons: a fractional model. Commun. Appl. Ind. Math. 6(2), e-538 (2015)
[7] Sousa, J.Vanterler da C., de Oliveira, E.C., Magna, L.A.: Fractional calculus and the ESR test. AIMS Math. 2(4), 692–705 (2017)
[8] Herrmann, R.: Fractional Calculus: An Introduction for Physicists. World Scientific Publishing Company, Singapore (2011)
[9] Atanackovic, T.M., Pilipovic, S., Stankovic, B., Zorica, D.: Fractional Calculus with Applications in Mechanics: Vibrations and Diffusion Processes. Wiley-ISTE, London (2014)
[10] Furati, K.M., Kassim, M.D.: Existence and uniqueness for a problem involving Hilfer fractional derivative. Comput. Math. Appl. 64(6), 1616–1626 (2012)
[11] Benchohra, M., Slimani, B.A.: Existence and uniqueness of solutions to impulsive fractional differential equations. Electron. J. Differ. Equ. 2009, 1–11 (2009)
[12] Benchohra, M., Lazreg, J.E.: Nonlinear fractional implicit differential equations. Commun. Appl. Anal. 17, 471–482 (2013)
[13] Benchohra, M., Lazreg, J.E.: Existence and uniqueness results for nonlinear implicit fractional differential equations with boundary conditions. Rom. J. Math. Comput. Sci. 4, 60–72 (2014)
[14] Khan, R.A., Rehman, M., Henderson, J.: Existence and uniqueness of solutions for nonlinear fractional differential equations with integral boundary conditions. Fract. Differ. Calc. 1(1), 29–43 (2011)
[15] Zhou, Y.: Existence and uniqueness of solutions for a system of fractional differential equations. J. Fract. Calc. Appl. Anal. 12, 195–204 (2009)
[16] Hyers, D.H.: On the stability of the linear functional equation. Proc. Natl. Acad. Sci. 27(4), 222–224 (1941)
[17] Ulam, S.M.: Problems in Modern Mathematics, Science edn. Wiley, New York (1940)
[18] Ulam, S.M.: A Collection of Mathematical Problems, vol. 8. Interscience Publishers Inc, New York (1960)
[19] Aoki, T.: On the stability of the linear transformation in Banach spaces. J. Math. Soc. Jpn. 2(1), 64066 (1950)
[20] Park, C., Rassias, T.M.: Homomorphisms and derivations in proper JCQ-triples. J. Math. Anal. Appl. 337(2), 1404–1414 (2008)
[21] Rassias, T.M.: On the stability of the linear mapping in Banach spaces. Proc. Am. Math. Soc. 72(2), 297–300 (1978)
[22] Wang, J., Lin, Z.: Ulam’s type stability of Hadamard type fractional integral equations. Filomat 28(7), 1323–1331 (2014)
[23] Huang, J., Li, Y.: Hyers–Ulam stability of delay differential equations of first order. Math. Nachr. 289(1), 60–66 (2016)
[24] Wang, J., Lv, L., Zhou, Y.: Ulam stability and data dependence for fractional differential equations with Caputo derivative. Electron. J. Qual. Theory Differ. Equ. 2011(63), 1–10 (2011)
[25] Ibrahim, R.W.: Generalized Ulam–Hyers stability for fractional differential equations. Int. J. Math. 23(5), 1250056 (2012)
[26] Abbas, S., Benchohra, M., Graef, J.R., Henderson, J.: Implicit Fractional Differential and Integral Equations: Existence and Stability, vol. 26. Walter de Gruyter GmbH & Co KG, London (2018)
[27] Vivek, D., Kanagarajan, K., Elsayed, E.M.: Some existence and stability results for Hilfer-fractional implicit differential equations with nonlocal conditions. Mediterr. J. Math. 15, 15 (2018)
[28] Yang, D., Wang, J.: Non-instantaneous impulsive fractional-order implicit differential equations with random effects. Stoch. Anal. Appl. 35, 719–741 (2017)
[29] Kucche, K.D., Sutar, S.T.: Stability via successive approximation for nonlinear implicit fractional differential equations. Moroc. J. Pure Appl. Anal. 3, 36–54 (2017)
[30] Benchohra, M., Bouriah, S., Graef, J.R.: Nonlinear implicit differential equations of fractional order at resonance. Electron. J. Differ. Equ. 2016, 1–10 (2016)
[31] Benchohra, M., Bouriah, S., Nieto, J.J.: Existence of periodic solutions for nonlinear implicit Hadamard’s fractional differential equations, Revista de la Real Academia de Ciencias Exactas. Físicas y Naturales Serie A Matemáticas 112, 25–35 (2018)
[32] Karthikeyanl, P., Arul, R.: Stability for impulsive implicit Hadamard fractional differential equations. Malaya J. Matematik 6, 28–33 (2017)
[33] Abbas, S., Benchohra, M., Henderson, J.: Weak solutions for implicit fractional differential equations of hadamard type. Adv. Dyn. Syst. Appl. 11, 1–13 (2016)
[34] Bhairat, S.P., Dhaigude, D.B.: Ulam stability for system of nonlinear implicit fractional differential equations. Prog. Nonlinear Dyn. Chaos 6, 29–38 (2018)
[35] Sutar, S.T., Kucche, K.D.: Global existence and uniqueness for implicit differential equation of arbitrary order. Fract. Differ. Calc. 5, 199–208 (2015)
[36] Sousa, J., da Vanterler, C., de Oliveira, E.C.: A Gronwall inequality and the Cauchy-type problem by means of $\psi$-Hilfer operator (2017). arXiv:1709.03634
[37] Gorenflo, R., Kilbas, A.A., Mainardi, F., Rogosin, S.V.: Mittag-Leffler Functions, Related Topics and Applications. Springer, Berlin (2014)
[38] Mittag-Leffler, G.M.: Sur la nouvelle fonction $E_{\alpha}(x)$. C. R. Acad. Sci. Paris 137, 554–558 (1903)
[39] Benchohra, M., Lazreg, J.E.: Existence and Ulam stability for nonlinear implicit fractional differential equations with Hadamard derivative. Stud. Univ. Babes Bolyai Math. 62(1), 27–38 (2017)
[40] Benchohra, M., Lazreg, J.E.: On stability for nonlinear implicit fractional differential equations. Le Matematiche 70(2), 49–61 (2015)
[41] Dhage, B.C.: Global attractivity results for comparable solutions of nonlinear hybrid fractional integral equations. Differ. Equ. Appl. 6, 165–186 (2014)
[42] Nieto, J.J., Chen, F., Zhou, Y.: Global attractivity for nonlinear fractional differential equations. Nonlinear Anal. Real World Appl. 13(1), 287–298 (2012)
[43] Abbas, S., Benchohra, M., Nieto, J.: Global attractivity of solutions for nonlinear fractional order Riemann–Liouville Volterra–Stieltjes partial integral equations. Electron. J. Qual. Theory Differ. Equ. 2012(81), 1–15 (2012)

J. Vanterler da C. Sousa and E. Capelas de Oliveira
Department of Applied Mathematics
Imecc-Unicamp
Campinas SP13083-859
Brazil
e-mail: ra160908@ime.unicamp.br; capelas@ime.unicamp.br