Abstract: The main purpose of this work is to identify the general quadratic operator which is invariant under the action of Linear Canonical Transformations (LCTs). LCTs are known in signal processing and optics as the transformations which generalize certain useful integral transforms. In quantum theory, they can be identified as the linear transformations which keep invariant the canonical commutation relations characterizing the coordinates and momenta operators. In this paper, LCTs corresponding to a general pseudo-Euclidian space are considered. Explicit calculations are performed for the monodimensional case to identify the corresponding LCT invariant operator then multidimensional generalizations of the obtained results are deduced. It was noticed that the introduction of a variance-covariance matrix, of coordinate and momenta operators, and a pseudo-orthogonal representation of LCTs facilitate the identification of the invariant quadratic operator. According to the calculations carried out, the LCT invariant operator is a second order polynomial of the coordinates and momenta operators. The coefficients of this polynomial depend on the mean values and the statistical variances-covariances of the coordinates and momenta operators themselves. The eigenstates of the LCT invariant operator are also identified with it and some of the main possible applications of the obtained results are discussed.

Keywords: Linear Canonical Transformation, Quantum theory, Invariant, Quadratic operator, Eigenstates

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1- Introduction

Linear Canonical Transformations (LCTs) are studied and used in several areas like signal processing, optics, and quantum physics [1-9]. In the fields of signal processing and optics, they are known to be the generalization of some useful integrals transforms such as Fourier and Fractional Fourier Transforms. In quantum theory, they can be identified as the linear transformations which keep invariant the canonical commutation relations characterizing the coordinates and momenta operators: their set can be considered as a symmetry group of these canonical commutation relations [10-11]. Some operators related to LCTs and their representations were already considered by various authors [5-6, 9]. However no study has been done before regarding the identification of the invariant quadratic operator associated with LCTs, with a view to an application in relativistic quantum physics, as it is envisaged in the present work. It can be remarked that from a physical point of view, the introduction of relativistic canonical commutations relations which is needed in order to define the associated LCTs rise the problem of the existence of a time operator. However, this problem was already tackled by various authors as can be seen in the refs. [12-16]. The results established in the present work also show that the introduction of this operator could have interesting consequences.

In this work, the general case of LCTs corresponding to an N-dimensional pseudo-euclidian space is considered. Most of the LCTs currently used in the fields of signal processing and optics can be viewed as LCTs associated to an euclidian space with a dimension equal to 1 (monodimensional case). This monodimensional case is particularly studied in the section 2.

An example of a well-known quadratic operator that can be considered as invariant under the action of some particular LCT's, in the framework of non-relativistic quantum theory, is the Hamiltonian operator of a harmonic oscillator. In fact, the Hamiltonian operator of a harmonic oscillator of mass $m$ and angular frequency $\omega$ is

$$H = \frac{(p)^2}{2m} + \frac{1}{2}m\omega^2(x)^2$$

we may consider the family of Linear Canonical Transformations of the form

$$\begin{pmatrix} p' \\ x' \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} p \\ x \end{pmatrix}$$

with $$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -m\omega\sin\theta & \cos\theta \end{pmatrix} \Rightarrow ad - bc = 1 \iff [p', x'] = [p, x] = -i\hbar$$

it is easy to verify that the Hamiltonian operator $H$ is invariant under the action of this particular LCTs

$$H' = \frac{(p')^2}{2m} + \frac{1}{2}m\omega^2(x')^2 = \frac{(p)^2}{2m} + \frac{1}{2}m\omega^2(x)^2 = H$$

It can be noticed through the next sections that the general invariant quadratic operator identified through this work has a similarity with this Hamiltonian operator. Because of this similarity, the formalism that is considered in the identification of the eigenstates of this general invariant quadratic operator is analogous to the well-known formalism associated with the theory of harmonic oscillator. These eigenstates share themselves some similarities with what are called coherent states, generalized coherent states and squeezed states in the literature [17-20]. It follows that the formalism considered in the present work can also be considered as an extension of the theory of quantum harmonic oscillator in a more general relativistic framework with an establishment of a link between it and a general theory of linear canonical transformations.

The LCT invariant operator identified through this work can also be considered as a generalization of a linear combination of the reduced dispersion-codispersion operators, denoted $\delta(x, \omega)^{\nu}_{\mu}$, introduced in our previous work [9]. The results that are established here generalize and bring also more clarifications to the formulation developed and used in the references [9-11].

In order to simplify the presentation of the calculations and results associated with the identification of the LCT invariant operator in the next sections, we recall in this introduction section the definition and some of the main properties of the reduced dispersion operator $\delta(x, \omega)^{\nu}_{\mu}$, established in [9]. The notations that are used are not exactly the same as those utilized in [9] (especially for the mean values) but this change in notations is more suited to the current purpose. Some natural extensions which can be very easily deduced from the results obtained in [9] are also considered.

Let us consider a multidimensional theory corresponding to a pseudo-Euclidian space with a dimension equal to $N$ (\(\mu, \nu = 0, 1, \ldots N - 1\)). It was established in [9] that some equivalent expressions of a reduced dispersion-codispersion operator $\delta(x, \omega)^{\nu}_{\mu}$ are (natural unit system, with the reduced Planck’s constant $\hbar = 1$ and speed of light $c = 1$, is used)
\[
\begin{align*}
\mathbf{z}^\mu_{\nu} &= \frac{1}{2} a^\mu_a a^\nu_b [ (p_a - \langle p_a \rangle)(p_\beta - \langle p_\beta \rangle) + 4 b_{\alpha \rho}b_{\beta \gamma}(x^\rho - \langle x^\rho \rangle)(x^\gamma - \langle x^\gamma \rangle) ] \\
&= \frac{1}{2} \delta^\mu_\alpha \delta^\nu_\beta [4 a A_{\alpha \rho} A_{\beta \gamma}(p^\rho - \langle p^\rho \rangle)(p^\gamma - \langle p^\gamma \rangle) + (x_\alpha - \langle x_\alpha \rangle)(x_\beta - \langle x_\beta \rangle)] \\
&= \frac{1}{2} [a^\mu_a a^\nu_b (p_\rho - \langle p_\rho \rangle)(p_\beta - \langle p_\beta \rangle) + \delta^\mu_\alpha \delta^\nu_\beta (x_\rho - \langle x_\rho \rangle)(x_\beta - \langle x_\beta \rangle)] \\
&= \frac{1}{4} (p_\mu p_\nu + x_\mu x_\nu) \\
&= \frac{1}{4} (z^\mu_\mu z^\nu_\nu + z^\mu_\nu z^\nu_\mu)
\end{align*}
\]

in which:

- \( p_\mu \) and \( x^\nu \) are the momenta and coordinates operators. They satisfy the canonical commutation relations

\[
\begin{align*}
[p_\mu, x^\nu] &= i \delta^\nu_\mu \\
[p_\mu, p_\nu] &= 0 \\
[x_\mu, x_\nu] &= 0
\end{align*}
\]
the corresponding eigenvalue equation is

$$z_{\mu}[(p_\mu, (x_\mu), \mathcal{A}_{\mu
u} B_{\nu\mu})] = (z_\mu)[(p_\mu, (x_\mu), \mathcal{A}_{\mu
u} B_{\nu\mu})]$$

(9)

with $(z_\mu) = (p_\mu) + 2iB_{\mu\nu}(x^\nu)$. Because of the relation (9) and for the sake of simplicity, the notation $[(z_\mu)]$ may be used instead of $[(p_\mu, (x_\mu), \mathcal{A}_{\mu
u} B_{\nu\mu})]$ for this eigenstate in (9).

The operators $\hat{z}_\mu$ and $\hat{z}_\mu$ in (7) are related with the operator $z_\mu$ in (8) by the relation

$$\begin{cases}
\hat{z}_\mu = a_\mu^+(z_v - (z_\mu)) \\
\hat{z}_\mu = a_\mu^-(z_v - (z_\mu))
\end{cases}$$

(10)

The last relation in (1) and the commutations relations of the operators $z_\mu$ and $z_\mu^+$ in (7) permit to verify that the general eigenstates of the operators $\hat{z}_\mu^+$, denoted $[(n_\mu, (z_\mu))]$ satisfy the eigenvalue equations (without a summation on $\mu$)

$$\hat{z}_\mu^+[(n_\mu, (z_\mu))] = \frac{1}{4} (\hat{z}_\mu^+ z_\mu + z_\mu \hat{z}_\mu^+) [(n_\mu, (z_\mu))] = \frac{1}{4} (2n_\mu + 1) [(n_\mu, (z_\mu))]$$

(11)

with $n_\mu$ integer numbers. The case $n_\mu = 0$ for all $\mu$ corresponds to the state in the relations (3) and (9)

$$|(0, (z_\mu))] = [(z_\mu)] = [(p_\mu, (x_\mu), \mathcal{A}_{\mu
u} B_{\nu\mu})]$$

(12)

These states are analogs of what are called coherent states and squeezed coherent states in the literature [19-22]. The operators $z_\mu$ and $\hat{z}_\mu^+$ acts as the ladder operators on the states $[(n_\mu, (z_\mu))]$: they permits the increasing and decreasing of the values of $n_\mu$.

For the case of a monodimensional space ($N = 1$) with the metric $\eta_{00} = 1$. Index $\mu$ takes the unique value 0 and we have the relations

$$\begin{cases}
\hat{z}_0^+ = \hat{z}_0 = \frac{1}{2} \left[ \mathcal{A}_{00}(p_0 - (p_0))^2 + B_{00}(x_0 - (x_0))^2 \right] \\
= \frac{1}{4} \left[ (p_0)^2 + (x_0)^2 \right] \\
= \frac{1}{4} (\hat{z}_0^+ z_0 + z_0 \hat{z}_0^+) = \frac{1}{4} (2\hat{z}_0^+ z_0 + 1) = \frac{1}{4} (\hat{z}_0^+ z_0 + 1)
\end{cases}$$

(13)

$$\begin{cases}
\left( a_0^0 \right)^2 = \mathcal{A}_{00} \\
\left( \beta_0^0 \right)^2 = B_{00} \\
\mathcal{A}_{00} = \frac{1}{4} \left[ a_0^0 \beta_0^0 \right] = \frac{1}{2} \\
(p_0 + 2iB_{00} x_0)
\end{cases}$$

(14)

For the case of a monodimensional space ($N = 1$) with the metric $\eta_{00} = 1$. Index $\mu$ takes the unique value 0 and we have the relations

$$\begin{cases}
\hat{z}_0^+ = 1 \left( \beta_0 \right) = \frac{1}{\sqrt{2}} p_0 + i x_0 \\
\left[ \mathcal{A}_{00}, \beta_0 \right] = i
\end{cases}$$

(16)

The wavefunctions, in coordinate representation, corresponding to the states $|n_0, (z_0)\rangle$ are the harmonic Hermite-Gaussian functions that were introduced and studied in our previous works [9, 21],

$$\langle x^0 | n_0, (z_0) \rangle = \frac{H_n(\sqrt{z_0^0}(x^0 - \langle z_0 \rangle))}{\sqrt{2^n n_0! \sqrt{2\pi a_{00}}}} e^{-\frac{a_{00}(x^0 - \langle z_0 \rangle)^2}{2}}$$

(18)
It can be verified that the operator \( \sum_{k=0}^{N} \) in (13) is invariant under the action of a set of particular LCTs which include the fractional Fourier transforms. In the present work, these particular LCTs are explicitly defined by the relation (36) in the section 2 below. All the relations associated to this operator are then covariant under the action of these particular LCTs. Through the next sections, the generalization of this operator which is invariant under the action of any LCT will be identified. The main relations, analog of (14), (15), (16) and (17), which are associated with this general invariant operator and are covariant under the action of any LCT will also be established. The multidimensional generalization of the obtained results will also be considered. Some main possible applications of these results are discussed in the conclusion section.

Through this work, boldface type is used for the quantum operators.

2- LCT invariant quadratic operator and its eigenstates: monodimensional case

2.1 Definition of the LCTs in quantum theory

In quantum theory, Linear Canonical Transformations can be defined as the linear transformations of coordinates and momenta operators which keep invariant the canonical commutations in the relation (2). In the case of a general pseudo-Euclidean space as considered in the previous section, they can be defined through the following relations.

\[
\begin{align*}
\{p'_\mu, x'_\nu\} &= \{p_\mu, x_\nu\} = \eta_{\mu\nu} \\
\{p'_\mu, p'_\nu\} &= \{p_\mu, p_\nu\} = 0 \\
\{x'_\mu, x'_\nu\} &= \{x_\mu, x_\nu\} = 0
\end{align*}
\]

(19)

If we denote respectively \( a, b, c, d \) and \( \eta \) the \( N \times N \) matrices corresponding respectively to the parameters \( a_\mu, b_\nu, c_\nu, d_\mu \) and \( \eta_{\mu\nu} \), the relation (20a) is equivalent to the matrix relation

\[
\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix}
\]

(20b)

The relation (20b) means that the \( 2N \times 2N \) matrix \( \begin{pmatrix} a & c \\ b & d \end{pmatrix} \) belongs to the pseudo-symplectic group \( Sp(2N,\mathbb{R}) \).

For the case of a monodimensional space \( (N = 1) \) with the metric \( \eta_{00} = 1 \) (which means in particular that we have \( p_0 = p^0 \) and \( x_0 = x^0 \)) the relations (19) and (20a) are reduced to

\[
\begin{align*}
p'_0 &= a_0^0 p_0 + b_0^0 x_0 \\
x'_0 &= c_0^0 p_0 + d_0^0 x_0 \quad a_0^0 d_0^0 - b_0^0 c_0^0 = 1
\end{align*}
\]

(21)

In view of the multidimensional generalization, the index 0 is kept through all calculation that are performed for this monodimensional case.

2.2 Laws of transformation of wavefunctions and relation with integral transforms

Let \( |\psi\rangle \) be a random general state and let us denote \( (x^0)|\psi\rangle \) and \( (x^0)|\psi\rangle \) the wavefunctions respectively in the \( x^0 \)-representation and \( x^0 \)-representation. It can be established from the relation (21) that these two wavefunctions are linked by the following integral transform

\[
(x^0)|\psi\rangle = \int (x^0)|x^0\rangle(x^0)|\psi\rangle \ dx^0 = K \int (x^0)|\psi\rangle e^{i\int x^0 a^0_{0}x^0 - 0_{0}^2(x^0)^2} dx^0
\]

(22)

The relations (22) show explicitly the equivalence between the operator transformation (21) and the integral transforms which are currently considered in the fields of signal processing and optics (\( K \) being a constant) [1-8].

In our framework, i.e. quantum theory, these integral transforms can be identified as the laws of transformation of wavefunctions. In fact, the relations (22) show that under the action of an LCT, the state \( |\psi\rangle \) itself can be considered as an invariant but it is the wavefunction which changes. Explicitly, we may write

\[
|\psi\rangle = \int |x^0\rangle(x^0)|\psi\rangle dx^0 = \int |x^0\rangle(x^0)|\psi\rangle dx^0
\]

(23)
The relation (22) and (23) show explicitly that the LCT corresponds to a basis change between the basis \(|x^0\rangle\) and \(|x^0_0\rangle\). The wavefunctions related by the relation (22) being the components of the state vector \(|\psi\rangle\) respectively in each of these bases. This basis change can be defined explicitly by the relation

\[
|x^0\rangle = \int |x^0\rangle(x^0|x^0\rangle) \, dx^0 = K \int |x^0\rangle e^{-\frac{1}{2}\left(a^{(x^0_x)^2} + a^{(x^0)^2}\right)} \, dx^0
\]  

(24)

### 2.3 Laws of transformation of mean values and statistical variance-covariance

Let be respectively \((x_0)_{\psi} = \langle \psi | x_0 | \psi \rangle\) and \((p_0)_{\psi} = \langle \psi | p_0 | \psi \rangle\) the mean values of the coordinate operator \(x_0\) and momentum operator \(p_0\) corresponding to a general state \(|\psi\rangle\). Their laws of transformations can be deduced easily from the relation (21)

\[
\begin{align*}
(p'_0)_{\psi} &= a^0_0(p_0)_{\psi} + b^0_0(x_0)_{\psi} \\
(x'_0)_{\psi} &= c^0_0(p_0)_{\psi} + d^0_0(x_0)_{\psi}
\end{align*}
\]  

(25)

We have then also the relation

\[
\begin{align*}
(p'_0 - (p'_0))_{\psi} &= a^0_0(p_0 - (p_0))_{\psi} + b^0_0(x_0 - (x_0))_{\psi} \\
(x'_0 - (x'_0))_{\psi} &= c^0_0(p_0 - (p_0))_{\psi} + d^0_0(x_0 - (x_0))_{\psi}
\end{align*}
\]  

(26)

It can be remarked that the relation (26) remains the same even if instead of the Linear Transformation (21) an Affine Transformation is considered i.e. a Linear Transformation combined with translations:

\[
\begin{align*}
(p'_0) &= a^0_0p_0 + b^0_0x_0 + e \\
x'_0 &= c^0_0p_0 + d^0_0x_0 + f
\end{align*}
\]  

(27)

in which \(e\) and \(f\) are constant parameters defining the translations. Let us define and consider the following operators

\[
\begin{align*}
p_{00} &= (p_0 - (p_0)) \\
x_{00} &= (x_0 - (x_0)) \\
c_{00} &= (x_0 - (x_0)) \\
d_{00} &= (x_0 - (x_0))
\end{align*}
\]  

(28)

The mean values of \(\langle p_{00} \rangle, \langle x_{00} \rangle, \langle c_{00} \rangle, \langle d_{00} \rangle\) in the state \(|\psi\rangle\) corresponds respectively to the momentum statistical dispersion, the coordinate statistical dispersion and the momentum-coordinate codispersion (statistical variance-covariances). They will be respectively denoted \(\langle p_{00} \rangle_{\psi}, \langle x_{00} \rangle_{\psi}, \langle c_{00} \rangle_{\psi}, \langle d_{00} \rangle_{\psi}\) and \(\langle c_{00} \rangle_{\psi}\)

\[
\begin{align*}
\langle p_{00} \rangle_{\psi} &= \langle p_0 - (p_0) \rangle_{\psi} \\
\langle x_{00} \rangle_{\psi} &= \langle x_0 - (x_0) \rangle_{\psi} \\
\langle c_{00} \rangle_{\psi} &= \langle x_0 - (x_0) \rangle_{\psi} \\
\langle d_{00} \rangle_{\psi} &= \langle x_0 - (x_0) \rangle_{\psi}
\end{align*}
\]  

(29)

The law of transformations of the operators in (28) can be deduced from the relation (26), we obtain

\[
\begin{align*}
p_{00}' &= (a^0_0)^2p_{00} + (b^0_0)^2x_{00} + 2a^0_0b^0_0c^0_0 \psi_{00} \\
x_{00}' &= (c^0_0)^2x_{00} + (d^0_0)^2x_{00} + 2c^0_0d^0_0 \psi_{00} \\
c_{00}' &= a^0_0c^0_0p_{00} + b^0_0d^0_0x_{00} + a^0_0c^0_0 \psi_{00} + b^0_0d^0_0 \psi_{00} \\
d_{00}' &= a^0_0d^0_0p_{00} + b^0_0c^0_0x_{00} + b^0_0c^0_0 \psi_{00} + a^0_0d^0_0 \psi_{00} \\
\psi_{00}' &= a^0_0c^0_0p_{00} + b^0_0d^0_0x_{00} + (a^0_0c^0_0 + b^0_0d^0_0) \psi_{00}
\end{align*}
\]  

(30)
The laws of transformation of the statistical variances-covariances considered in (29) can be deduced easily from (30)

\[
\begin{align*}
\langle \psi | p_{00} | \psi \rangle &= (a_0^2)^2 \langle \psi | p_{00} | \psi \rangle + (b_0^2)^2 \langle \psi | x_{00} | \psi \rangle + 2a_0b_0 \langle \psi | e_{00}^* | \psi \rangle \\
\langle \psi | x_{00} | \psi \rangle &= (c_0^2)^2 \langle \psi | p_{00} | \psi \rangle + (d_0^2)^2 \langle \psi | x_{00} | \psi \rangle + 2c_0d_0 \langle \psi | e_{00}^* | \psi \rangle \\
\langle \psi | e_{00}^* | \psi \rangle &= a_0^2 c_0^2 \langle \psi | p_{00} | \psi \rangle + b_0^2 d_0^2 \langle \psi | x_{00} | \psi \rangle + a_0d_0 \langle \psi | e_{00}^* | \psi \rangle \\
\langle \psi | e_{00}^* | \psi \rangle &= a_0^2 c_0^2 \langle \psi | p_{00} | \psi \rangle + b_0^2 d_0^2 \langle \psi | x_{00} | \psi \rangle + (a_0d_0 + b_0c_0) \langle \psi | e_{00}^* | \psi \rangle
\end{align*}
\] (31)

2.4 LCT invariant quadratic operator and invariant scalar

Taking into account the relation \(a_0^2 d_0^2 - b_0^2 c_0^2 = 1\) which define the LCT in (26), it can be deduced from the relations (30) and (31) that we have, for any state \(|\psi\rangle\), the relations

\[
\frac{1}{2} [\langle \psi | x_{00} | \psi \rangle p_{00} + \langle \psi | p_{00} | \psi \rangle x_{00}] - \langle \psi | e_{00}^* | \psi \rangle e_{00}^* = \frac{1}{2} [\langle \psi | x_{00} | \psi \rangle p_{00} + \langle \psi | p_{00} | \psi \rangle x_{00}] - \langle \psi | e_{00}^* | \psi \rangle e_{00}^* \\
\langle \psi | p_{00} | \psi \rangle \langle \psi | x_{00} | \psi \rangle - [\langle \psi | e_{00}^* | \psi \rangle]^2 = \langle \psi | x_{00} | \psi \rangle \langle \psi | p_{00} | \psi \rangle - [\langle \psi | e_{00}^* | \psi \rangle]^2
\]

(32)

(33)

The relations (32) and (33) mean that the quadratic operator

\[
\Xi_0^+ = \Xi_0^- = \frac{1}{2} [\langle \psi | x_{00} | \psi \rangle p_{00} + \langle \psi | p_{00} | \psi \rangle x_{00}] - \langle \psi | e_{00}^* | \psi \rangle e_{00}^*
\]

is an LCT invariant operator and the quantity \(\langle \psi | x_{00} | \psi \rangle \langle \psi | p_{00} | \psi \rangle - [\langle \psi | e_{00}^* | \psi \rangle]^2\) is an LCT invariant scalar. The notation \(\Xi_0^+\) is chosen for the invariant operator in (34) because, as it will be seen, it can be considered as a generalization of the reduced dispersion operator in the relation (13). In fact, if we consider for instance the particular case \(|\psi\rangle = |\langle z_0 |\rangle\) being the state in the relation (17) we have

\[
\begin{align*}
\langle z_0 | p_{00} | z_0 \rangle &= B_{00} \\
\langle z_0 | x_{00} | z_0 \rangle &= A_{00} \\
\langle z_0 | e_{00}^* | z_0 \rangle &= \frac{i}{2} \\
\langle z_0 | e_{00}^* | z_0 \rangle &= -\frac{i}{2} \\
\langle z_0 | e_{00}^* | z_0 \rangle &= 0
\end{align*}
\] (35)

The operator in (34) is equal to the operator in (13) in this particular case. It means as said that the LCT invariant operator in (34) is a generalization of the operator in (13). This generalization can be also highlighted regarding the following fact: if the transformations (31) and the relation (35) are considered, it can be remarked that the operator in (13) is explicitly invariant if and only if the following relations hold:

\[
\begin{align*}
\langle \psi | p_{00} | \psi \rangle \langle \psi | x_{00} | \psi \rangle &= A_{00} B_{00} = \frac{1}{4} \quad \iff \quad a_0^2 = d_0^2 = \cos \theta \\
|b_0^2 - 2B_{00} \sin \theta| &= c_0^2 = -2A_{00} \sin \theta
\end{align*}
\] (36)

The conditions and parameterizations in (36) corresponds to a fractional Fourier-like transforms. The operator in (13) is invariant under the action of particular LCTs corresponding to (36) while its generalization (34) is invariant under the action of any LCT.

The main difference in the expressions of the operators in (13) and in (34) is the presence of the term \(\langle \psi | e_{00}^* | \psi \rangle e_{00}^*\) in (34). The relation (59) in the paragraph 2.7 below show also, explicitly, that the operator in (34) is the LCT invariant generalization of the operator in (13).

2.5 Eigenstate of the LCT- invariant operator corresponding to its lowest eigenvalue

Let \(|\chi\rangle\) be the particular eigenstate of \(\Xi_0^+\) associated with its lowest eigenvalue. For sake of simplicity, let us denote \((p_{00}, x_{00}, (\langle p_{00} \rangle, \langle x_{00} \rangle), (\langle e_{00}^* \rangle, \langle e_{00}^* \rangle)\) and \(\langle e_{00}^* \rangle\) the mean values and statistical variances-covariances corresponding to \(|\chi\rangle\).
\begin{align}
(p_0) &= \langle x \mid p_0 \rangle \quad (x_0) = \langle x \mid x_0 \rangle \\
(p_{00}) &= \langle x \mid p_{00} \rangle \quad (x_{00}) = \langle x \mid x_{00} \rangle = \langle x \mid (x_0 - x_0)^2 \rangle \\
(q_{00}) &= \langle \chi \mid q_{00} \rangle \quad (\chi_{00}) = \langle \chi \mid (x_0 - x_0)(p_0 - p_0) \rangle \\
(q_{00}^* &= \frac{1}{2}((q_{00}^*)^2) = \frac{1}{2}((x_0 - x_0)(p_0 - p_0) \rangle \
(\chi_{00}) &= \langle \chi \mid (x_0 - x_0)(p_0 - p_0) \rangle 
\end{align}

The expression of the LCT invariant operator (34) then becomes (for \( |\psi \rangle = |\chi \rangle \))

\[ \hat{\mathbf{z}}^{0+}_{00} = \hat{z}^+_{00} = \frac{1}{2}([x_{00} p_{00} + (p_{00}) x_{00}] - (\psi_{00}) q_{00}^* \]

in the \( x^0 \) coordinate representation, we have for the operators \( p_{00}, x_{00} \) and \( q_{00}^* \)

\begin{align}
\hat{p}_{00} &= (\frac{\partial}{\partial x^0} - (p_0))^2 \\
\hat{x}_{00} &= (x_0 - x_0)^2 \\
\hat{q}_{00}^* &= \frac{1}{2}([\frac{\partial}{\partial x^0} - (p_0))(x_0 - x_0) + (x_0 - x_0)(\frac{\partial}{\partial x^0} - (p_0)]
\end{align}

So the coordinate representation \( \hat{\mathbf{z}}^+_{00} \) of \( \mathbf{z}^{0+}_{00} = \mathbf{z}^+_{00} \) is a second order differential operator. Its form suggest us to suppose that the wavefunction, in coordinate representation, corresponding to \( |\chi \rangle \) is of the form:

\[ \langle x^0 \mid \chi \rangle = Ce^{-\alpha(x^0 - \beta)^2 + \gamma x^0} \]

(40)

The expression of the parameters \( C, \alpha, \beta \) and \( \gamma \) are to be determined. If we denote \( \lambda \) the eigenvalue of \( \mathbf{z}^{0+}_{00} = \mathbf{z}^+_{00} \) corresponding to \( |\chi \rangle \), we have the eigenvalues equations

\[ \hat{z}^+_{00} |\chi \rangle = \lambda |\chi \rangle \iff \hat{z}^+_{00} [Ce^{-\alpha(x^0 - \beta)^2 + \gamma x^0}] = \lambda Ce^{-\alpha(x^0 - \beta)^2 + \gamma x^0} \]

(41)

It can be established that to satisfy the relations (37) and (41), taking into account the relations (38) and (39), the following relations must hold:

\begin{align}
\alpha &= \frac{1 + 2i(q_{00}^*)}{4(x_{00})} = i \frac{2(q_{00}^*) - i}{4(x_{00})^2} = i \frac{(q_{00}^*)}{2(x_{00})} \\
\beta &= \langle x_0 \rangle \quad \gamma = -i(p_0) \quad \lambda = \frac{1}{4} \sqrt{(x_{00})(p_{00}) - (q_{00}^*)^2} = \frac{1}{4}
\end{align}

(42)

according to the last line in (42), the LCT-invariant scalar, as defined in the relation (33), corresponding to the state \( |\chi \rangle \) is equal to

\[ \langle x_{00}(p_{00}) - (q_{00}^*)^2 = 4 \lambda^2 = \frac{1}{4} \]

(43)

the expression of the normalization constant \( C \) in (40) can be deduced from the normalization relation

\[ \langle \chi \mid \chi \rangle = \int |\chi^0 \rangle |\chi^0 \rangle^* d^4 x = 1 \Rightarrow C = \frac{e^{if}}{(2\pi(x_{00}))^{1/4}} \]

(44)

\( f \) is a real number (it doesn’t depend on the variable \( x^0 \) but may depend on the mean values \( (x_0) \) and \( (p_0) \). It follows that the most explicit general expression of the coordinate wavefunction \( \langle x^0 \mid \chi \rangle \) associated to a state \( |\chi \rangle \) is

\[ \langle x^0 \mid \chi \rangle = \frac{e^{-\frac{i(q_{00}^*)}{24(x_{00})(p_{00})}(x^0 - (x_0)^2 + i(p_0)x^0 + if}}{(2\pi(x_{00}))^{1/4}} \]

(45)

It can be shown that the corresponding momentum wavefunction \( \langle p^0 \mid \chi \rangle \) which is the Fourier transform of \( \langle x^0 \mid \chi \rangle \) is

\[ \langle p^0 \mid \chi \rangle = \frac{1}{\sqrt{2\pi}} \int \langle x^0 \mid \chi \rangle e^{-ip_0 x^0} dx^0 = \frac{e^{-\frac{i(q_{00}^*)}{24(p_{00})(p_{00})(x^0)^2 + i(p_0)(p_0) + if}}}{(2\pi(p_{00}))^{1/4}} \]

(46)
Remark: It may be noted that the codispersion (i.e. statistical covariance) \( \langle q_{00}^2 \rangle \) and \( \langle p_{00}^2 \rangle \) are complex numbers while \( \langle q_{00}^2 \rangle \) is a real number. Their explicit expressions and the relations between them are

\[
\begin{align*}
\langle q_{00} \rangle &= \langle \chi |(p_0 - \langle p_0 \rangle)(x_0 - \langle x_0 \rangle)\rangle \chi = \frac{2\langle q_{00}^2 \rangle + i}{2} = \langle q_{00} \rangle^* \\
\langle q_{00}^2 \rangle &= \langle \chi |(x_0 - \langle x_0 \rangle)(p_0 - \langle p_0 \rangle)\rangle \chi = \frac{2\langle q_{00}^2 \rangle - i}{2} = \langle q_{00}^\ast \rangle^* \\
\langle q_{00}^\ast \rangle - \langle q_{00} \rangle = i \quad \langle q_{00} \rangle + \langle q_{00}^\ast \rangle = 2\langle q_{00}^\ast \rangle
\end{align*}
\]

The wavefunctions \( \langle x^0 | \chi \rangle \) and \( \langle p^0 | \chi \rangle \) can be put in the form

\[
\begin{align*}
\langle x^0 | \chi \rangle &= e^{-\frac{1}{2}(x_0^2 - \langle x_0 \rangle^2) + i(x_0 \langle p_0 \rangle + \langle x_0 \rangle p_0)} \frac{1}{\sqrt{2\pi \langle x_0 \rangle}} \\
\langle p^0 | \chi \rangle &= e^{-\frac{1}{2}(p_0^2 - \langle p_0 \rangle^2) + i(x_0 \langle p_0 \rangle + \langle x_0 \rangle p_0)} \frac{1}{\sqrt{2\pi \langle p_0 \rangle}}
\end{align*}
\]

But unlike the case in the relations (14), (15) and (18), the parameters \( \mathcal{A}_{00} \) and \( \mathcal{B}_{00} \) in (48) are not real numbers but complex numbers. However, their product is equal to a real number. We have the following explicit relations:

\[
\begin{align*}
\mathcal{B}_{00} &= i \frac{\langle q_{00} \rangle}{2\langle x_0 \rangle} = \frac{1 + 2i\langle q_{00}^\ast \rangle}{4\langle x_0 \rangle} \quad \mathcal{A}_{00} = \frac{1 + 4\langle q_{00}^\ast \rangle^2}{16\langle x_0 \rangle \langle p_0 \rangle} = \frac{1}{4} \\
\mathcal{A}_{00} \mathcal{B}_{00} &= \frac{1 + 4\langle q_{00}^\ast \rangle^2}{16\langle x_0 \rangle \langle p_0 \rangle}
\end{align*}
\]

in the limit \( q_{00}^\ast = 0 \), \( \mathcal{A}_{00} \) and \( \mathcal{B}_{00} \) become real numbers and the particular case corresponding to the relations (14) is obtained.

2.6 Ladder operators and general eigenstates of the LCT invariant operator

As the LCT invariant operator \( \mathbf{z}^0+ \mathbf{z}^0_0 \) in (37) is a generalization of the operator in (13), the state \( | \chi \rangle \) can be seen as a generalization of the state \( |x_0 \rangle \) in (17). To make it more explicit and to obtain the eigenstates of the LCT invariant operator \( \mathbf{z}^0+ \), corresponding to higher eigenvalues, the generalization of the operator \( \mathbf{z}_0 \) in (15) and the ladder operators \( \mathbf{z}_0, \mathbf{z}^+_0 \) in (16) are expected to be introduced.

Let us consider the expression of \( \langle x^0 | \chi \rangle \) in (48), it can be deduced from this expression that we have the relation

\[
(i \frac{\partial}{\partial x^0} + 2i \mathcal{B}_{00} x^0) \langle x^0 | \chi \rangle = (\mathcal{B}_{00} + 2i \mathcal{B}_{00} x^0) \langle x^0 | \chi \rangle
\]

the relation (50) is equivalent to the eigenvalue equation

\[
\mathbf{z}_0 | \chi \rangle = \langle \mathbf{z}_0 | | \chi \rangle \quad \text{with} \quad \left\{ \begin{array}{c} \mathbf{z}_0 = p_0 + 2i \mathcal{B}_{00} x^0 \\
\langle \mathbf{z}_0 | = \langle p_0 + 2i \mathcal{B}_{00} (x^0) \end{array} \right.
\]

The \( \mathcal{B}_{00} \) in this expression is given in the relation (49) such that we have more explicitly:

\[
\mathbf{z}_0 = p_0 + 2i \mathcal{B}_{00} x^0 = p_0 + i - 2 \frac{\langle q_{00}^\ast \rangle}{\langle x_0 \rangle} x^0 = p_0 - \frac{\langle q_{00}^\ast \rangle}{\langle x_0 \rangle} x^0
\]

It can then be shown that the following commutation relation holds

\[
[\mathbf{z}_0, x^+_0] = \frac{1}{\langle x_0 \rangle}
\]

The relation (53) suggest us to suppose that the definition of the ladder operators should be
\[
\begin{align*}
&\bar{z}_0 = \sqrt{(x_{oo})(z_0 - \langle z_0 \rangle)} = \sqrt{(x_{oo})}[(p_0 - \langle p_0 \rangle) - \frac{(q_{oo}^\nu)}{(x_{oo})}(x_0 - \langle x_0 \rangle)] \\
&\bar{z}_0^+ = \sqrt{(x_{oo})}(z_0^+ - \langle z_0 \rangle^*) = \sqrt{(x_{oo})}[(p_0 - \langle p_0 \rangle) - \frac{(q_{oo}^\nu)}{(x_{oo})}(x_0 - \langle x_0 \rangle)]
\end{align*}
\]

(54)

Taking into account the relation (53), we have as expected the commutation relation

\[ [\bar{z}_0, \bar{z}_0^+] = [\bar{z}_0, \bar{z}_0^+] = 1 \]

(55)

Using the relation (54) and taking into account the relations (43) and (47), it can be deduced from the relation (38) that the expression of the invariant quadratic operator in terms of the ladder operators \( \bar{z}_0 \) and \( \bar{z}_0^+ \) is

\[ \mathcal{Z}_0^+ = \mathcal{Z}_0 = \frac{1}{2} [(x_{oo})\mathbf{p}_{oo} + (p_{oo})x_{oo}] - (q_{oo}^\nu) = \frac{1}{4} (\bar{z}_0^+ \bar{z}_0 + \bar{z}_0 \bar{z}_0^+) = \frac{1}{4} (2\bar{z}_0 \bar{z}_0 + 1) \]

(56)

The relations (55) and (56) justify the choice in the relation (54).

The eigenvalue equation in (51) suggest the use of the notation \( |(z_0)\rangle \) for the eigenstate \( |\lambda\rangle \). This state \( |\lambda\rangle \) is a common eigenstate of the operators \( \mathcal{Z}_0 \) and \( \mathcal{Z}_0^+ \): according to (51), it is an eigenstate of \( \mathcal{Z}_0 \) with the eigenvalue equal to \( \langle z_0 \rangle \) and according to (41) and (42) it is an eigenstate of \( \mathcal{Z}_0^+ \) with its lowest eigenvalue equal to \( \frac{1}{4} \).

The relations (55) and (56) which are analogs to the relations characterizing a linear harmonic oscillator suggest us to denote \( |n_0, (z_0)\rangle \) a general eigenstates of \( \mathcal{Z}_0^+ = \mathcal{Z}_0^+ \) with \( n_0 \) a positive integer. Explicitly, we have the relations

\[
\begin{align*}
&\mathcal{Z}_0 |(z_0)\rangle = \langle z_0 |(z_0)\rangle \quad \mathcal{Z}_0^+ |(z_0)\rangle = \frac{1}{4} |(z_0)\rangle \\
&|n_0, (z_0)\rangle = \frac{(\mathcal{Z}_0^+)^{n_0}}{\sqrt{n_0!}} |(z_0)\rangle \\
&\mathcal{Z}_0^+ |n_0, (z_0)\rangle = \frac{1}{4} (2n_0 + 1) |n_0, (z_0)\rangle \quad n_0 \in \mathbb{N}
\end{align*}
\]

(57)

The relations in (57) are exactly the generalization of the relations in (17). The particular case (17) corresponds to the limit \( q_{oo}^\nu = 0 \). On one hand, the reduced dispersion operator in (17) is invariant under the action of the particular LCTs corresponding to relation (36) then the relation (17) is covariant under the action of these particular LCTs. On the other hand, the operator \( \mathcal{Z}_0^+ = \mathcal{Z}_0^+ \) in (56) and (57) is invariant under the action of any LCT: it follows that the relation (57) which generalizes (17) is also covariant under the action of any LCT. In other words, the LCT group is a symmetry group for the relations (56) and (57).

In our previous work [9], the particular reduced dispersion operator corresponding to the relations (17) and its eigenstates were implicitly used to define the phase space representation. Following the above results, it can be remarked that a more general phase space representation which is explicitly LCT-covariant can be obtained if this particular reduced dispersion operator and its eigenstates used in these formulations are replaced by their generalizations which correspond to the relations (56) and (57).

2.7 Laws of transformations of the reduced and ladder operators

The adequate generalization of the reduced operators \( \mathbf{p}_0 \) and \( \mathbf{x}_0 \) can be identified from the relation (54) taking into account (49)

\[
\begin{align*}
&\mathbf{z}_0 = \mathbf{p}_0 + i\mathbf{x}_0 \\
&\mathbf{z}_0^+ = \mathbf{p}_0 - i\mathbf{x}_0
\end{align*}
\]

\[
\begin{align*}
&\mathcal{Z}_0 = \sqrt{2} (x_{oo}) \mathcal{Z}_0 = \sqrt{2} (x_{oo}) (p_0 - \langle p_0 \rangle) - \sqrt{2} (q_{oo}^\nu) (x_0 - \langle x_0 \rangle) \\
&\mathcal{Z}_0^+ = \sqrt{2} (x_{oo}) (\mathcal{Z}_0^+ + \mathbf{z}_0^+ - \mathbf{z}_0 - \langle \mathbf{z}_0 \rangle^*) = \frac{1}{\sqrt{2}} (\mathbf{z}_0^+ - \mathbf{z}_0 - \langle \mathbf{z}_0 \rangle^*) (x_0 - \langle x_0 \rangle)
\end{align*}
\]

(58)

As expected, the commutation relation is \( [\mathbf{p}_0, \mathbf{x}_0] = i \) and we have for the invariant operator

\[ \mathcal{Z}_0 = \frac{1}{2} [(x_{oo})\mathbf{p}_{oo} + (p_{oo})x_{oo}] - (q_{oo}^\nu) = \frac{1}{4} (\mathbf{z}_0^+ \mathbf{z}_0 + \mathbf{z}_0 \mathbf{z}_0^+) = \frac{1}{4} [(\mathbf{p}_0)^2 + (\mathbf{x}_0)^2] \]

(59)
Let us consider the LCT (26) in which the state \( |\psi\rangle \) is taken to be the common eigenstate \( |z_0\rangle \) of \( z_0 \) and \( z_{200} \) as defined by the relation (57) \( |\psi\rangle = |z_0\rangle \). We have then:

\[
\begin{align*}
(p'_0, p_0) &= a_0^b (p_0 - p_0) + b_0^a (x_0 - x_0) \\
(x'_0, x_0) &= \frac{c_0^b}{d_0^a} (p_0 - p_0) + d_0^a (x_0 - x_0) \\
\end{align*}
\]

The laws of transformation of the statistical dispersion-codispersion can be deduced from the relations (31) and (37). We obtain

\[
\begin{align*}
(p'_0) &= (a_0^b)^2 (p_0) + 2a_0^b b_0^a x_0 + (b_0^a)^2 (x_0) \\
(x'_0) &= (c_0^b)^2 (p_0) + 2c_0^b d_0^a x_0 + (d_0^a)^2 (x_0) \\
\end{align*}
\]

After a lengthy but straightforward calculations, it can be deduced from the relations (58), (60), (61) and (43) that the laws of transformation of the reduced operators are

\[
\begin{align*}
(p'_0) &= \Pi_0^a x_0 + \Theta_0^b x_0 = (\cos \theta) p_0 + (\sin \theta) x_0 \\
(x'_0) &= \Xi_0^a x_0 + \Lambda_0^b x_0 = -(\sin \theta) p_0 + (\cos \theta) x_0 \\
\end{align*}
\]

with

\[
\begin{align*}
\Pi_0^a &= \Lambda_0^b = \cos \theta = \frac{2(c_0^b (e_0^b) + d_0^a (x_0))}{\sqrt{(c_0^b)^2 + 4(c_0^b (e_0^b) + d_0^a (x_0))^2}} \\
\Xi_0^a &= \Xi_0^b = -\sin \theta = \frac{c_0^b}{\sqrt{(c_0^b)^2 + 4(c_0^b (e_0^b) + d_0^a (x_0))^2}} \\
\end{align*}
\]

A proof of the relation (62) from a matrix formulation perspective is given in the section 3.

It follows from the relation (62) and (63) that the matrix \( \begin{pmatrix} \Pi_0^a & \Xi_0^a \\ \Theta_0^b & \Lambda_0^b \end{pmatrix} \) which corresponds to the laws of transformation of the reduced operator is an orthogonal matrix belonging to the group \( SO(2) \) while the original matrix \( \begin{pmatrix} a_0^b & c_0^b \\ b_0^a & d_0^a \end{pmatrix} \) defining the LCT (60) is an element of the special linear group \( SL(2) \). It follows that the laws of transformation of the reduced operator define an orthogonal representation of the LCT. A spinorial representation can also be deduced from this orthogonal representation.

From the relation between the reduced operator and the ladder operators \( z_0 \) and \( z_{200} \) in (58) and the laws of transformation (62). It can be deduced that the law of transformation of the ladder operators are

\[
\begin{align*}
(z'_0) &= e^{-i\theta} z_0 \\
(z'_{200}) &= e^{i\theta} z_{200} \\
\end{align*}
\]

We may remark that the relation (64) describes a transformation which has a similarity with Bogolioubov transformations [22-23]. It can be deduced from (64) that we have for the transformation of the operator \( z_0 \) and its mean value (and eigenvalue) \( \langle z_0 \rangle \)

\[
\begin{align*}
[z_0 - \langle z_0 \rangle] &= e^{-i\theta} [z_0 - \langle z_0 \rangle] \\
\end{align*}
\]

with

\[
e^{-i\theta} = \frac{\sqrt{\langle x_{200} \rangle}}{\sqrt{\langle x_{00} \rangle}} = \frac{2 \langle x_{00} \rangle}{\sqrt{(c_0^b)^2 + 4(c_0^b (e_0^b) + d_0^a (x_0))^2}} \\
\]

2.8 Laws of transformations of the eigenstates of the invariant operator

It follows from the relation (65) that the respective eigenstates \( |z_0 \rangle \) and \( |z_{200} \rangle \) of \( z_0 \) and \( z_{200} \) respectively with the eigenvalue \( \langle z_0 \rangle \) and \( \langle z_{200} \rangle \) are equals or more generally just related by a factor which is an unitary complex number

\[
|z_0 (z'_0)\rangle = |z_{200} (\langle z_{200} \rangle)\rangle \\
\{z_0 (z_{200})\rangle = \langle z_{200} \rangle \} = e^{i\xi} |\langle z_0 \rangle\rangle \quad \xi \in \mathbb{R} \\
\]
The relation (64) is an unitary transformation and it is, as expected, compatible with the fact that the operator $\mathfrak{Z}_{00}$ in (57) is LCT invariant

$$\mathfrak{Z}_{00}^+ = \mathfrak{Z}_{00}^{0+} = \frac{1}{4} (2\mathfrak{Z}_{00}^+ \mathfrak{Z}_{00} + 1) = \frac{1}{4} (2\mathfrak{Z}_{00}^+ \mathfrak{Z}_{00} + 1) = \mathfrak{Z}_{00}^{0+} = \mathfrak{Z}_{00}^+$$

The law of transformations of the eigenstates $|n_\nu, (z_0)\rangle$ of $\mathfrak{Z}_{00}^+$ can be deduced from the relations (57), (64) and (67), we have

$$|n_\nu, (z_0')\rangle = \frac{(z_0')^n_0}{\sqrt{n_0!}}|n_0\rangle = e^{i(n_0\zeta \theta)} \frac{(z_0)^n_0}{\sqrt{n_0!}}|n_0\rangle = e^{i(n_0\zeta \theta)} |n_\nu, (z_0)\rangle$$

The state $|n_\nu, (z_0)\rangle$ and $|n_\nu, (z_0)\rangle$, which are both eigenstates of the LCT invariant operators $\mathfrak{Z}_{00}^+$ are, as it may be expected, related by the factor $e^{i(n_0\zeta \theta)}$ which is a unitary complex number. It follows from the relation (69) that for any operator $A$, we have the relation

$$\langle n_\nu, (z_0')|A|n_\nu, (z_0)\rangle = \langle n_\nu, (z_0)|A|n_\nu, (z_0)\rangle$$

3-Matrix formalism

Let us introduce three parameters $a$, $b$ and $c$ defined from the statistical variance-covariance of coordinate and momentum through the following relations

$$\begin{cases} a = \sqrt{|x_{00}|} \Rightarrow (a)^2 = \langle x_{00} \rangle \\ \theta = \frac{1}{2} \frac{a}{z(x_{00})} = \frac{1}{2} \frac{\theta}{z(x_{00})} \Leftrightarrow a \theta = \frac{1}{2} \\ c = \frac{a(z_{00})}{\langle x_{00} \rangle} = \frac{\theta (z_{00})}{\sqrt{|x_{00}|}} \end{cases}$$

Using the relation (71), the relations between the momentum and coordinate operators $p_0 \cdot x_0$ and their reduced forms $p_0$ and $x_0$ can be written in the matrix form

$$\begin{bmatrix} P_0 & x_0 \end{bmatrix} = \sqrt{2} (p_0 - \langle p_0 \rangle \ x_0 - \langle x_0 \rangle) \begin{bmatrix} a & 0 \\ -c & \theta \end{bmatrix}$$

Inversely

$$\begin{bmatrix} p_0 - \langle p_0 \rangle & x_0 - \langle x_0 \rangle \end{bmatrix} = \sqrt{2} \begin{bmatrix} p_0 & x_0 \end{bmatrix} \begin{bmatrix} \theta & 0 \\ c & a \end{bmatrix}$$

The matrix $\sqrt{2} \begin{bmatrix} \theta & 0 \\ c & a \end{bmatrix}$ is the inverse of the matrix $\sqrt{2} \begin{bmatrix} a & 0 \\ -c & \theta \end{bmatrix}$. Explicitly, we have the relation

$$2 \begin{bmatrix} \theta & 0 \\ c & a \end{bmatrix} \begin{bmatrix} a & 0 \\ -c & \theta \end{bmatrix} = 2 \begin{bmatrix} a & 0 \\ -c & \theta \end{bmatrix} \begin{bmatrix} \theta & 0 \\ c & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

From the relation (71) and (43), it can be deduced that we have the relation

$$\begin{cases} \langle x_{00} \rangle = (a)^2 \\ \langle z_{00}^2 \rangle = ac \\ \langle p_{00} \rangle = \frac{1 + 4\langle z_{00}^2 \rangle}{\langle x_{00} \rangle} = (\theta)^2 + (c)^2 \end{cases}$$

The relations in (75) can be written in the matrix form

$$\begin{bmatrix} \langle p_{00} \rangle & \langle z_{00}^2 \rangle \\ \langle z_{00}^2 \rangle & (x_{00}) \end{bmatrix} = \begin{bmatrix} (\theta)^2 + (c)^2 & ac \\ ac & (a)^2 \end{bmatrix} = \begin{bmatrix} \theta & 0 \\ c & a \end{bmatrix} \begin{bmatrix} \theta & 0 \\ c & a \end{bmatrix} \begin{bmatrix} \theta & 0 \\ c & a \end{bmatrix}$$

The matrix $\begin{bmatrix} \langle p_{00} \rangle & \langle z_{00}^2 \rangle \\ \langle z_{00}^2 \rangle & (x_{00}) \end{bmatrix}$ is a momentum-coordinate statistical variance-covariance matrix. The law of transformation in (61) can be written in the matrix form
\[ \left( \begin{array}{c} \langle p_{00} \rangle \\ \langle p'_{00} \rangle \\ \langle x_{00} \rangle \end{array} \right) = \left( \begin{array}{ccc} a_0^0 & c_0^0 \\ b_0^0 & d_0^0 \end{array} \right)^T \left( \begin{array}{c} \langle p_{00} \rangle \\ \langle q_{00} \rangle \\ \langle x_{00} \rangle \end{array} \right) \left( \begin{array}{ccc} a_0^0 & c_0^0 \\ b_0^0 & d_0^0 \end{array} \right) \] (77)

From the relation (77), the invariance of the LCT scalar \( \langle x_{00} \rangle (p_{00}) - \langle q_{00} \rangle^2 \) can be interpreted as the invariance of the determinant of the statistical variance-covariance matrix \( \Sigma_0^0 \). The relation (77) show that this invariance is a direct consequence of the relation \( a_0^0 d_0^0 - b_0^0 c_0^0 = 1 \).

The matrix formalism considered here can also be used to provide a proof for the relation (62). Let us write the transformation of the reduced operator in the form

\[ \left( \begin{array}{c} p'_{0} \\ x'_{0} \end{array} \right) = \left( \begin{array}{c} p_{0} \\ x_{0} \end{array} \right) \left( \begin{array}{cc} \Pi_0^0 & \Xi_0^0 \\ \Theta_0^0 & \Lambda_0^0 \end{array} \right) \] (78)

On one hand, taking into account the relations (72), (73) and (74), it can be deduced that we have the relation

\[ \left( \begin{array}{cc} \Pi_0^0 & \Xi_0^0 \\ \Theta_0^0 & \Lambda_0^0 \end{array} \right) = 2 \left( \begin{array}{cc} \phi & 0 \\ -c & \phi \end{array} \right) \left( \begin{array}{cc} a & c' \\ -c' & a \end{array} \right) \] (79)

Inversely

\[ \left( \begin{array}{cc} a & c \\ -c & a \end{array} \right) \left( \begin{array}{cc} a' & c' \\ -c' & a' \end{array} \right) = 2 \left( \begin{array}{cc} \phi & 0 \\ -c & \phi \end{array} \right) \left( \begin{array}{cc} \Pi_0^0 & \Xi_0^0 \\ \Theta_0^0 & \Lambda_0^0 \end{array} \right) \] (80)

On the other hand, it can be deduced from the relations (76) and (77) that we have

\[ \left( \begin{array}{cc} \phi' & 0 \\ c' & a \end{array} \right)^T \left( \begin{array}{cc} a' & c' \\ -c' & a' \end{array} \right) = \left( \begin{array}{cc} a & c \\ -c & a \end{array} \right) \left( \begin{array}{cc} \phi & 0 \\ -c & \phi \end{array} \right) \left( \begin{array}{cc} \Pi_0^0 & \Xi_0^0 \\ \Theta_0^0 & \Lambda_0^0 \end{array} \right) \] (81)

A combination of the relations (79), (81) and (74) leads to the relation

\[ \left( \begin{array}{cc} \Pi_0^0 & \Xi_0^0 \\ \Theta_0^0 & \Lambda_0^0 \end{array} \right)^T \left( \begin{array}{cc} \Pi_0^0 & \Xi_0^0 \\ \Theta_0^0 & \Lambda_0^0 \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \iff \left( \begin{array}{cc} \Pi_0^0 & \Xi_0^0 \\ \Theta_0^0 & \Lambda_0^0 \end{array} \right) = \left( \begin{array}{cc} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{array} \right) \] (82)

The matrix relation in (82) is equivalent to the relations in (62).

4- Multidimensional generalization

4.1 Momenta-coordinates statistical variance-covariances and matrix formalism

Let us now consider the multidimensional case. Let \( \{ (x_{\mu}) \} \) be the state corresponding to the following wavefunction (in coordinate representation)

\[ \{ (x_{\mu}) \} = e^{-i \nu (x_{\mu}^0 + (x_{\mu})^0) - (x_{\mu}^2) - (x_{\mu})^2 + if} \] (83)

The relation (83) is a multidimensional generalization of (45). \( f \) is a real number which may depend on the mean values \( \langle x_{\mu} \rangle \) and \( \langle p_{\mu} \rangle \). The following relations which are associated to this wavefunction can be verified

\[
\begin{align*}
\left\langle (x_{\mu}) \right| (x_{\nu}) & = \delta_{\mu}^{\nu} \\
\left\langle (p_{\mu}) \right| (p_{\nu}) & = \delta_{\mu}^{\nu} \\
\left\langle (x_{\mu}) \right| (x_{\nu} - x_{\mu}) (x_{\nu} - x_{\nu}) & = \langle x_{\mu \nu} \\n\left\langle (p_{\mu}) \right| (p_{\nu} - p_{\mu}) (p_{\nu} - p_{\nu}) & = \langle q_{\mu \nu} \\
\left\langle \xi_{\mu}^\nu \right| (\xi_{\nu}^\nu) & = \delta_{\nu}^{\mu} \text{ with } \langle \xi_{\mu}^\nu \rangle = \eta^{\mu \nu} (x_{\mu}) \\
B_{\mu \nu} & = \frac{1}{2} \langle \xi_{\mu}^\nu \rangle \end{align*}
\] (84)

we have also the following eigenvalue equations and relations which justify the notation \( \{ (x_{\mu}) \} \) for these state itself.
As multidimensional generalization of the relations (29) and (37), the following full set of statistical dispersion-differences (variance–covariances) can be considered together

(85)

Taking into account the commutation relation [(\(p_\mu - p_\nu\), \((x_\nu - (x_\nu)\)] = \(i\eta_{\mu\nu}\), the following relation can be deduced from the three last relations in (86)

(87)

Let us also introduce the parameters \(a_{\mu}^\mu\), \(\theta_{\mu}^\nu\) and \(c_{\mu}^\nu\) defined through the relations

(88)

(88) is the multidimensional generalization of the relation (71). Let us denote respectively \(a\), \(\theta\), \(c\), \(\eta\), \(P\), \(X\), and \(\phi\) the \(N \times N\) matrices corresponding respectively to the parameters \(a_{\mu}^\mu\), \(\theta_{\mu}^\nu\), \(c_{\mu}^\nu\), \(\eta_{\mu\nu}\), \((p_\mu, x_\mu)\) and \((\phi_{\mu\nu}^\nu)\). It can be deduced from the relations (84), (85), (86), (87) and (88) that we have the matrix relation

(89)

(89) is the multidimensional generalization of the relation (76).

Let us now consider the LCT defined through the relations (19), (20a) and (20b). Taking into account the relations (84) and (86), we obtain the laws of transformations

(90)

The matrix form of the relation (90) is
It can be deduced, using the relation (89) that the relation (91) is equivalent to the following relation

\[
\left( \begin{array}{c}
\delta' \\
2ad'c'd'
da'
\end{array} \right)^T \left( \begin{array}{c}
\eta \\
\eta
\end{array} \right) = \left( \begin{array}{c}
a \\
-b
\end{array} \right)^T \left( \begin{array}{c}
P' \Theta x' \\
\chi'
\end{array} \right) \left( \begin{array}{c}
a \\
-b
\end{array} \right)
\]

(91)

4.2 Reduced operators

Taking into account of the relations (84), (87) and (88), the following operators may be defined and deduced from the operators \( z_\mu \) in the relation (85)

\[
\begin{align*}
z'_\mu &= a\mu(x_\sigma) = a\mu(p_\sigma - p_\nu) + c\mu(x_\nu - x_\sigma) + i\delta_\mu^\nu(x_\nu - x_\sigma) \\
\mathbf{z}'_\mu &= d\mu(z_\sigma - (x_\sigma)) = a\mu(p_\sigma - p_\nu) + c\mu(x_\nu - x_\sigma) - i\delta_\mu^\nu(x_\nu - x_\sigma)
\end{align*}
\]

(93)

According to the commutation relations in (93), the operators \( z_\mu \) and \( z'^\mu \) have the properties of ladder operators. The following reduced momenta and coordinates operators can be identified from (93)

\[
\begin{align*}
p_\mu &= \frac{1}{\sqrt{2}}(z_\mu^+ + z_\mu) = \sqrt{2}a\mu(p_\nu - p_\nu) - \sqrt{2}c\mu(x_\nu - x_\nu) \\
\mathbf{x}_\mu &= \frac{i}{\sqrt{2}}(z_\mu^+ - z_\mu) = \sqrt{2}\delta_\mu^\nu(x_\nu - x_\nu)
\end{align*}
\]

(94)

If we denote respectively \((p - \mathbf{x})\) and \((p - \mathbf{x})\) the \(1 \times 2N\) matrices corresponding respectively to \((p_\mu, \mathbf{x}_\mu)\) and \((p_\nu - p_\nu, x_\nu - x_\nu)\), we can deduce the matrix form of the relation (94)

\[
(p - x) = \sqrt{2}(p - (p - x))(\begin{array}{c} \alpha \\ -c \\ \begin{array}{c} 0 \\ \delta 
\end{array} \end{array})
\]

(95)

Inversely

\[
(p - (p - x))(p - x) = \sqrt{2}(p - (p - x))(\begin{array}{c} \alpha \\ -c \\ \begin{array}{c} 0 \\ \delta 
\end{array} \end{array})
\]

(96)

and we have also the relation

\[
2\left( \begin{array}{c}
\delta \\
2ac\delta \\
\begin{array}{c} a \\ -c \\ \delta 
\end{array} \\
\begin{array}{c} 0 \\ 0 \\ 1 
\end{array}
\end{array} \right) = 2\left( \begin{array}{c}
\alpha \\
-\frac{1}{\delta}c \\
\begin{array}{c} 0 \\ \delta \\
\begin{array}{c} 0 \\ 0 
\end{array} \\
\end{array} \\
\begin{array}{c} 1 \\ 0 
\end{array}
\end{array} \right)
\]

(97)

(95), (96) and (97) are the multidimensional generalization of the relations (72), (73) and (74).

The law of transformation of the reduced operators defined in (94) can be written in the form

\[
\begin{align*}
p'_\mu &= \Pi_\mu^x p_\mu + \Theta_{\mu x} x_\mu \\
\mathbf{x}'_\mu &= \Xi_{\mu x} p_\mu + \Lambda^\mu x_\mu
\end{align*}
\]

(98)

In which \(\Pi, \Xi, \Theta, \Lambda\) are the \(N \times N\) matrices corresponding to the parameters \(\Pi_{\mu \nu}, \Theta^\nu_{\mu}, \Xi^\nu_{\mu x}\) and \(\Lambda^\nu_{\mu x}\). It can be deduced from the relation (92), (95), (97) and (98) that we have the relation

\[
\begin{align*}
\left( \begin{array}{c}
\Pi \\
\Theta \\
\Xi \\
\Lambda
\end{array} \right) \left( \begin{array}{c}
\eta \\
0
\end{array} \right) &= \left( \begin{array}{c}
\Pi \\
\Theta \\
\Xi \\
\Lambda
\end{array} \right) \left( \begin{array}{c}
\eta \\
0
\end{array} \right)
\end{align*}
\]

(99)

The relation (99) is the multidimensional generalization of the relations (62) and (82). It means that the \(2N \times 2N\) matrix \(\left( \begin{array}{c}
\Pi \\
\Theta \\
\Xi \\
\Lambda
\end{array} \right)\) corresponding to the transformation (98) of the reduced operators in (94) belongs to the indefinite orthogonal group \(O(2N_+, 2N_-)\) while the \(2N \times 2N\) matrix \(\left( \begin{array}{c}
a \\
b \\
c \\
d
\end{array} \right)\) corresponding to the LCT defined in (19), (20a) and (20b) itself is an element of the pseudo-symplectic group \(\hat{Sp}(2N_+, 2N_-)\). It follows that the transformation (98) define a pseudo-orthogonal representation of the LCT defined through the relations (19), (20a) and (20b). As we also have the relation \(\left[ p'_\mu, x'_\mu \right] = [p_\mu, x_\mu] = i\eta_{\mu \nu}\), according to (94), it follows that the \(2N \times 2N\)
matrix $\left( \frac{\Pi}{\Theta} \frac{\Xi}{\Lambda} \right)$ also belongs to the pseudo-symplectic group $Sp(2N_+, 2N_-)$ i.e. defines an LCT. In other words this matrix belongs to the intersection of the group $Sp(2N_+, 2N_-)$ and $O(2N_+, 2N_-)$. Explicitly, we have then the following relations

$$
\left( \frac{\Pi}{\Theta} \frac{\Xi}{\Lambda} \right) T \left( \begin{array}{cc} \eta & 0 \\ 0 & \eta \end{array} \right) \left( \frac{\Pi}{\Theta} \frac{\Xi}{\Lambda} \right) = \left( \begin{array}{cc} \eta & 0 \\ 0 & \eta \end{array} \right) \Leftrightarrow \left( \frac{\Pi}{\Theta} \frac{\Xi}{\Lambda} \right) \xi = \xi \Leftrightarrow \left( \frac{\Pi}{\Theta} \frac{\Xi}{\Lambda} \right) \eta = \eta
$$

(100)

it can be deduced from the relation (100) that we can have the parameterization

$$
\left( \frac{\Pi}{\Theta} \frac{\Xi}{\Lambda} \right) = e^{\left( \begin{array}{c} \xi \\ \eta \end{array} \right)}
$$

(101)

The law of transformations of the ladder operators $z^\mu_\nu$ and $z_\mu$ can be deduced from the relations (94), (98) and (100). We obtain

$$
\left\{ \begin{align*}
z^\mu_\nu &= (\Pi^\nu - i\Theta^\nu)z_\nu \\
z^\mu_\nu &= (\Pi^\mu + i\Theta^\nu)z^\nu
\end{align*} \right.
$$

(102)

The transformations in (102), which are the multidimensional generalizations of (64) share also some similarities with the Bogolioubov transformations [22-23].

4.3 Invariant quadratic operator

It can be deduced from the relations (98) and (100) that we have the quadratic invariant operator

$$
\mathcal{Z}^+ = \eta^{\mu\nu}z^\mu_\nu = \frac{1}{4} \eta^{\mu\nu}(p_\mu^\dagger p_\nu^\dagger + \Phi^\mu_\nu + \Phi^\nu_\mu) = \frac{1}{4} \eta^{\mu\nu}(p_\mu^\dagger p_\nu^\dagger + \Phi^\mu_\nu + \Phi^\nu_\mu) = \eta^{\mu\nu}z^\mu_\nu = \mathcal{Z}^+
$$

(103)

In term of the operators $z^\mu_\nu$ and $z_\mu$, we have (with a summation on the index $\mu$ and $\nu$)

$$
\mathcal{Z}^+ = \eta^{\mu\nu}z^\mu_\nu = \frac{1}{4} \eta^{\mu\nu}(z^\mu_\nu + z^\nu_\mu) = \frac{1}{4} \eta^{\mu\nu}(2z^\mu_\nu + \eta_{\mu\nu}) = \frac{1}{4}(2z^\mu_\nu + \eta_{\mu\nu}) = \delta^\mu_\nu z^\mu_\nu
$$

(104)

It follows from (104) that the eigenstates of the LCT invariant operator $\mathcal{Z}^+$ are the common eigenstates of the operators $\mathcal{Z}^\mu_\nu$. The canonical commutations relation in (93) and the relation (104) implies that the operators $z^\mu_\nu$ and $z^\mu_\nu$ are acting as ladder operators. This fact permit to deduce any eigenstate, denoted $|(z_\mu)|$ of the operators $\mathcal{Z}^\mu_\nu$ and $\mathcal{Z}^\mu_\nu$ from the common eigenstate $|(z_\mu)|$ of the operators $z_\mu$ and $z^\mu_\nu$ defined through (85). Explicitly, it can be deduced from the relations (93) and (104) that we have the relations

$$
\left\{ \begin{align*}
|z_\mu||(z_\mu)| &= \frac{1}{4} \frac{1}{n} \sum_{n=0}^{N-1} |(z_\mu)| \\
|n_\mu (z_\mu)| &= \frac{1}{4} \frac{1}{N!} |(z_\mu)| \\
\mathcal{Z}^+ |(z_\mu)| &= \frac{1}{4} (2n + 1) |(z_\mu)| \\
\mathcal{Z}^+ |n_\mu (z_\mu)| &= \frac{1}{4} (2n + N) |n_\mu (z_\mu)|
\end{align*} \right.
$$

(105)

with $n = \sum_{\mu=0}^{N-1} n_\mu$.
4.4 Invariant quadratic operator and spinorial representation of LCTs

A spinorial representation of LCTs can be established from the pseudo-orthogonal representation defined through the relations (98), (100) and (101). An explicit study of this spinorial representation and its applications in particle physics is considered in [11]. The establishment of this spinorial representation is based on the introduction of the operator

\[ \mathfrak{p} = \alpha^\mu \mathbf{p}_\mu + \beta^\mu \mathbf{x}_\mu \]

(106)

in which the \( \alpha^\mu \) and \( \beta^\mu \) are the generators of the Clifford algebra \( \mathbb{C\ell}(2N_+, 2N_-) \). They verify the following anticommutation relations

\[
\begin{align*}
\{\alpha^\mu \alpha^\nu + \alpha^\nu \alpha^\mu\} &= 2\eta^\mu\nu \\
\{\beta^\mu \beta^\nu + \beta^\nu \beta^\mu\} &= 2\eta^\mu\nu \\
\{\alpha^\mu \beta^\nu + \beta^\nu \alpha^\mu\} &= 0
\end{align*}
\]

(107)

From the relations (103), (106) and (107) and the commutation relations \( [\mathbf{p}_\mu, \mathbf{x}_\nu] = i\eta_{\mu\nu} \), it can be established that we have between the invariant quadratic operator \( \mathfrak{z}^+ \) and the operator \( \mathfrak{p} \) the relation

\[ (\mathfrak{p})^2 = 4\mathfrak{z}^+ + i\eta_{\mu\nu} \alpha^\mu \beta^\nu \iff \mathfrak{z}^+ = \frac{1}{4}[(\mathfrak{p})^2 - i\eta_{\mu\nu} \alpha^\mu \beta^\nu] \]

(108)

5. Discussions and conclusions

The expression of the general invariant quadratic operator associated with any LCT that we are looking for is given by the relations (34) or (59) for the monodimensional case and by the relations (103) or (104) for the multidimensional one. Their eigenstates and corresponding eigenvalues are respectively given by the relations (57) and (105). These LCT invariant operators are second order polynomials of the momenta and coordinates operators. The coefficients of the polynomials depend on the mean values and statistical variances-covariances of these momenta and coordinates operators.

It can be remarked that these LCT invariant operators are generalizations of the operator \( \mathfrak{z}^+ = \eta^{\mu\nu} \mathfrak{z}^{+\mu\nu} \) associated respectively with the reduced dispersion operators in the relations (13) for the monodimensional case and in (1) for the multidimensional one.

The main difference between the reduced dispersion operators (1) and their generalization in (103) and (104) can be seen if one compares the expressions of the reduced and ladders operators (93) and (94) with (6) and (7): there is in (93) and (94) an extra term with the factor \( c^\mu_\nu \) which is related, according to (88), with the fact that we may have in general

\[ \langle \phi^\nu_\mu \rangle = \frac{1}{2} \langle \lbrace \langle \mathbf{p}_\mu - \langle \mathbf{p}_\mu \rangle \rangle \langle \mathbf{x}_\nu - \langle \mathbf{x}_\nu \rangle \rangle + \langle \mathbf{x}_\nu - \langle \mathbf{x}_\nu \rangle \rangle \langle \mathbf{p}_\mu - \langle \mathbf{p}_\mu \rangle \rangle \rangle \rangle \neq 0 \]

(109)

\[ \langle \lbrace \mathbf{p}_\mu \rangle \rangle \] being the eigenstate corresponding to the lowest eigenvalue of the LCT invariant operator as shown by the relation (105).

The operator \( \mathfrak{z}^+ = \eta^{\mu\nu} \mathfrak{z}^{+\mu\nu} \) associated with the particular reduced dispersion operators defined in (1) is invariant under the action of a set of particular LCTs which include Lorentz transformations and fractional Fourier transforms while its generalization given through the relations (103) and (104) is invariant under the action of any LCT.

Through our previous works (in [9] for instance), the eigenstates of the operator associated with the particular reduced dispersion operators in (1) were implicitly used to construct the called phase space representation of quantum theory. Taking into account the results established through this paper, it is clear that these states are to be replaced by their generalization in (105) which are the eigenstates of the general LCT -invariant operator (103). In this case, we obtain a fully and explicitly LCT covariant phase space representation. This LCT covariant phase space representation may be used to formulate an LCT covariant relativistic quantum thermodynamic. In fact, it is known, from the kinetic theory of gases for instance, that thermodynamic variables can be linked with the statistical variables of particles speeds and then with the statistical variances of their momenta. And it can be remarked, in the relations (86), (88) and (94) for instance, that the statistical variances-covariances of momenta and coordinates is expected to be at the core of the formulation of this LCT covariant phase space representation. It is also known, as considered in [24] for instance, that the obtention of phase space distribution may be exploited to establish a link between quantum theory and thermodynamics. This phase space representation can also possibly be used in the study of the relation between quantum and classical theories.

The possibility of considering an LCT group as a symmetry group in relativistic quantum physics is considered in [11]. The results established in the present work have important implications in this framework given the well-
known importance of symmetry and invariance in physics [25-27]. An LCT- covariant wave equation corresponding to a scalar wavefunction or an equation for a scalar field can be for instance obtained using the LCT invariant operator $\mathbf{\hat{s}}^*$ itself. An LCT covariant spinorial wave equation or an equation for a spinor field may also be established using the spinorial representation of LCTs that can be deduced from the pseudo-orthogonal representation defined through the relations (98), (100) and (101). This spinorial representation can also lead to a natural classification of the elementary fermions of the Standard Model of particle physics as it is shown in [11].

The results established through this work can also be used and applied in all areas concerned by the LCTs.

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