Tsallis, Rényi, and Shannon entropies for time-dependent mesoscopic RLC circuits

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We calculate information measures using the Tsallis, Rényi, and Shannon entropies for two classes (Lane–Emden and Caldirola–Kanai) of time-dependent mesoscopic RLC circuits. To determine the expressions for the entropies, we used the dynamical invariant method to obtain the exact Schrödinger wave function \( \psi_n(x, t) \). For the state \( n = 0 \), all the expressions found are given in terms of \( \rho \), a c-number quantity satisfying a nonlinear differential equation. For the Caldirola–Kanai system, the entropies do not vary with time, but decrease with increasing damping factor. For the Lane–Emden system, the entropies decrease with increasing time. We also analyze the behavior of the Shannon and Rényi lengths in the charge and magnetic flux spaces and compare them with the respective standard deviations.

Subject Index A60, A61, A64

1. Introduction

Owing to the advance of mesoscopic physics and nanotechnology [1–3], the miniaturization of electronic devices and integrated circuits to nanoscale is now possible [4–7]. For instance, García [4] described a process of writing/reading in nanoscale devices. Cleland et al. [5] studied the \( I \times V \) curves in nanoscale resistors. They observed that the fluctuation of the charge obeys an equation of motion identical to that of a damped harmonic oscillator with an external force. Dobisz et al. [6] demonstrated that it is possible to use a polydiacetylene with urethane substituents as a high-resolution negative resist. Gabelli et al. [7] constructed a nanoscale fully coherent quantum resistance–capacitance (RC) circuit to study transport properties.

Despite the fact that there has been no realization of an experimental nanoscale RLC circuit, different quantization schemes for RLC circuits have already been reported in the literature. When devices or circuits are so small that the inelastic coherence of the charge carrier approaches the Fermi wavelength, the application of classical mechanics fails and quantum effects have to be considered. In 1998, Zhang et al. [8] presented a quantization scheme for an RLC (R, L, and C are constants) circuit with a source and studied the fluctuations of the charge and magnetic flux of the circuit in several quantum states. Wei and Lei [9] showed that the algebraic method solves the dynamical problem of a mesoscopic RLC circuit with a source ruled by a time-dependent Hermitian Hamiltonian. Xu [10] calculated the Hannay angle in a time-dependent RLC circuit. Choi [11] took advantage of an invariant operator and successive unitary transformations to derive the solution of
the Schrödinger equation of a time-independent RLC electric circuit with a power source. In a series of papers, Pedrosa and coworkers [12–15] employed the Lewis–Riesenfeld method [16] to obtain the solution of the Schrödinger equation for time-dependent RLC circuits and constructed their coherent and squeezed states.

According to Rajagopal [17], recent investigations of nanoscale quantum devices and small clusters of atoms and molecules have shown new features of both non-equilibrium dynamics and non-extensivity. Nanoscale devices are restricted to certain size and shape uncertainties, which manifest in the system properties as fluctuations. These fluctuations can be calculated by using the Tsallis and Rényi entropies [18,19], which are one-parameter generalizations of the Shannon entropy [20] used to quantify the loss of information of a system. The Heisenberg uncertainty relation imposes a lower limit for the product of the position and momentum variances. The use of the Tsallis, Rényi, and Shannon entropies to calculate the variances of canonical variables may lead to values of the uncertainty product beyond the limit determined by the Heisenberg uncertainty relation.

Many papers have reported on the study of information measures in position and momentum spaces using Tsallis and Rényi entropies for quantum systems described by time-independent Hamiltonians. For instance, Ghosh and Chaudhuri [21] considered the hydrogen atom and the D-dimensional time-independent harmonic oscillator to calculate the position and momentum entropies analytically for ground and excited states in the light of Tsallis entropy. They verified that, in both cases, the generalized entropic uncertainty relation and pseudo-additivity relation hold. On the other hand, Bouvrie et al. [22] presented a detailed study of the information-theoretic lengths of Rényi, Fisher, and Shannon, in both position and momentum spaces, of the lowest-lying states of one-delta and twin-delta time-independent potentials. They showed that these quantities reveal several individual and combined spreading features of the position and momentum of the particle that cannot be described by the usual standard deviation.

Here we calculate the information measures using Tsallis, Rényi, and Shannon entropies for the lowest-lying states of quantum systems described by time-dependent Hamiltonians. To do so, we consider an RLC circuit without a source, where R, L, and C depend on time. We follow the procedure described in Ref. [13] to obtain the solution of the Schrödinger equation for time-dependent Hamiltonians. We investigate two classes of systems: Caldirola–Kanai and Lane–Emden oscillators. In Sect. 2, we briefly outline the definitions needed for the calculations. In Sect. 3, we present and discuss the results. Section 4 contains a summary of the work.

2. Theory

For RLC circuits, the two canonically conjugated variables are the charge \((Q)\) and magnetic flux \((\Phi)\). The 1D Tsallis entropy in the charge \((T_{q,Q}(t))\) and magnetic flux \((T_{q,\Phi}(t))\) spaces for continuous probability densities, \(\varrho(Q,t)\) and \(\gamma(\Phi,t)\), reads [18]

\[
T_{q,Q}(t) = \frac{1}{q-1} \left( 1 - \int [\varrho(Q,t)]^q dQ \right),
\]

\[
T_{q,\Phi}(t) = \frac{1}{q-1} \left( 1 - \int [\gamma(\Phi,t)]^q d\Phi \right),
\]

and the Tsallis total (joint) entropy, \(T_{q,Q+\Phi}(t)\), satisfies the pseudo-additivity relation [23]:

\[
T_{q,Q+\Phi}(t) = T_{q,Q}(t) + T_{q,\Phi}(t) + (1 - q) T_{q,Q}(t) T_{q,\Phi}(t).
\]
The probability density in the charge (magnetic flux) $\varrho(Q,t)$ ($\gamma(\Phi,t)$) space is given by $|\psi_n(Q,t)|^2$ ($|\phi_n(\Phi,t)|^2$), where $\psi_n(Q,t)$ satisfies the time-dependent Schrödinger equation:

$$i\hbar \frac{\partial \psi_n(Q,t)}{\partial t} = H(t) \psi_n(Q,t),$$

where the canonical magnetic flux $\Phi$ is defined as $\Phi = -i\hbar \frac{\partial}{\partial Q}$, and $\phi_n(\Phi,t)$ is given by

$$\phi_n(\Phi,t) = \int \frac{dQ e^{-i\Phi Q}}{(2\pi\hbar)^{1/2}} \psi_n(Q,t).$$

To find $\psi_n(Q,t)$ for a time-dependent Hamiltonian, $H = H(Q,\Phi,t)$, we use the results shown in Ref. [13], where the Lewis–Riesenfeld method [16] was employed to obtain the solution of the time-dependent Schrödinger equation of a time-dependent RLC circuit without a source. When $R$, $L$, and $C$ depend on time, the Hamiltonian reads

$$H(t) = e^{-\Lambda(t)} \frac{\Phi^2}{2L(t)} + e^{\Lambda(t)} \frac{L(t) \omega^2(t)}{2} \frac{Q^2}{2},$$

where $\omega^2(t) = (1/L(t) C(t))$ is the frequency of the circuit in the absence of resistance, $Q(t)$ is the charge, $\Phi(t)$ is the magnetic flux, and $\Lambda(t)$ is given by

$$\Lambda(t) = \int^t \frac{R(\tau)}{L(\tau)} d\tau.$$

According to Ref. [13], the solution of the time-dependent Schrödinger equation for the Hamiltonian given by Eq. (6) reads

$$\psi_n(Q,t) = e^{i\theta_n(t)} \left[ \frac{1}{\pi^{1/2} n!2^n \rho} \right]^{1/2} \exp \left[ \frac{iL(t)e^{\Lambda(t)}}{2\hbar} \left( \frac{\dot{\rho}}{\rho} + \frac{i}{L(t)e^{\Lambda(t)}\rho^2(t)} \right) \right] Q^2
\times H_n \left[ \left( \frac{1}{\hbar} \right)^{1/2} \left( \frac{Q}{\rho} \right) \right],$$

where

$$\theta_n(t) = -\left( n + \frac{1}{2} \right) \int^t \frac{1}{L(\tau) e^{\Lambda(\tau)} \rho^2} d\tau,$$

and $\rho(t)$ satisfies the generalized Milne–Pinney [24,25] equation:

$$\dot{\rho} + \gamma(t) \dot{\rho} + \omega^2(t) \rho = \frac{1}{L(t) e^{\Lambda(t)} \rho^3},$$

where $\gamma(t) = \left( \frac{L+R}{L} \right)$ is a time-dependent damping factor.

From Eq. (8), we write the following relations (in units of $\hbar = 1$):

$$\langle Q \rangle = \langle \Phi \rangle = 0,$$

$$\langle Q^2 \rangle = \rho^2 \left( n + \frac{1}{2} \right),$$

$$\langle \Phi^2 \rangle = \frac{1}{\rho^2} \left( 1 + L(t)^2 e^{2\Lambda(t)} \rho^2 \rho^2 \right) \left( n + \frac{1}{2} \right),$$

$$\Delta Q \Delta \Phi = \left( 1 + L(t)^2 e^{2\Lambda(t)} \rho^2 \rho^2 \right)^{1/2} \left( n + \frac{1}{2} \right).$$
The uncertainty relation $\Delta Q \Delta \Phi$ is minimized for $\rho = \text{constant}$ and $n = 0$. For $n = 0$, $\psi_0 (x, t)$ and $\phi_0 (p, t)$ are, respectively, given by

$$\psi_0 (Q, t) = \left[ \frac{1}{\pi \rho^2} \right]^{1/4} e^{i \theta_0 (t)} \exp \left[ \frac{i L (t) e^{\lambda (t)}}{2 \hbar} \left( \frac{\dot{\rho}}{\rho} + \frac{i}{L (t) e^{\lambda (t)}} \rho \frac{d}{dt} \right) \right] Q^2, \quad (15)$$

$$\phi_0 (\Phi, t) = \left[ \frac{1}{\pi \rho^2} \right]^{1/4} e^{i \theta_0 (t)} \left[ \rho^2 \Theta (t) \right]^{1/2} \exp \left[ -\Phi^2 \rho^2 \frac{d}{2 \Theta (t)} \right]. \quad (16)$$

where $\Theta (t) = \left( \frac{1+L(t)e^{\lambda (t)} + \rho \frac{d}{dt}}{1+L(t)^2 e^{2\lambda (t)}} \right)$. 

Therefore, $T_{q, Q} (t)$, $T_{q, \Phi} (t)$, and $T_{q, Q+\Phi} (t)$ in the state $n = 0$ read

$$T_{q, Q} (t) = \frac{1}{q-1} \left[ 1 - \frac{(\pi \rho^2)^{1/2(1-q)}}{q^{1/2}} \right], \quad (17)$$

$$T_{q, \Phi} (t) = \frac{1}{q-1} \left[ 1 - \frac{1}{q^{1/2}} \left( \pi \left( \frac{1+L(t)^2 e^{2\lambda (t)}}{\rho^2} \right) \right)^{1/2(1-q)} \right], \quad (18)$$

$$T_{q, Q+\Phi} (t) = \frac{1}{q-1} \left[ 1 - \frac{(\pi)^{1-q}}{q} \left( 1+L(t)^2 e^{2\lambda (t)} \rho^2 \right)^{1/2(1-q)} \right]. \quad (19)$$

For constant values of $L (t)$ and $C (t)$ and $R (t) = 0$, $\rho$ is a constant and Eq. (19) becomes

$$T_{q, Q+\Phi} (t) = \frac{1}{q-1} \left[ 1 - \frac{(\pi)^{1-q}}{q} \right], \quad (20)$$

which is the Tsallis joint entropy for the $n = 0$ state of the time-independent LC circuit without a source. Since the time-independent LC circuit is analogous to the time-independent harmonic oscillator, Eq. (20) coincides with that found by Portesi and Plastino [23].

Another class of generalized entropy was introduced by Rényi [19] and the 1D Rényi entropy in the charge ($R_Q$) and magnetic flux ($R_\Phi$) spaces reads

$$R_{q, Q} (t) = \frac{1}{1-q} \ln \left\{ \int \left[ |\psi_n (Q, t)|^2 \right]^q dQ \right\}, \quad (21)$$

$$R_{q, \Phi} (t) = \frac{1}{1-q} \ln \left\{ \int \left[ |\phi_n (\Phi, t)|^2 \right]^q d\Phi \right\}. \quad (22)$$

By using Eqs. (15) and (16), we find

$$R_{q, Q} (t) = \frac{1}{2} \ln \left( \pi \rho^2 \right) - \frac{\ln q}{2 (1-q)}, \quad (23)$$

$$R_{q, \Phi} (t) = \frac{1}{2} \left[ \ln \left( \frac{\pi}{\rho^2} \right) \left( 1+L(t)^2 e^{2\lambda (t)} \rho^2 \right) \right] - \frac{\ln q}{2 (1-q)}, \quad (24)$$

$$R_{q, Q+\Phi} (t) = \ln \left( \pi \left( 1+L(t)^2 e^{2\lambda (t)} \rho^2 \right)^{1/2} \right) - \frac{\ln q}{1-q}. \quad (25)$$

Recall that the Rényi joint entropy is additive, i.e., $R_{q, Q+\Phi} (t) = R_{q, Q} (t) + R_{q, \Phi} (t)$, and Eq. (25) is related to Eq. (19) by the expression

$$R_{q, Q+\Phi} (t) = \frac{1}{1-q} \ln \left[ 1 + (1-q) T_{q, Q+\Phi} (t) \right]. \quad (26)$$
In the $q \to 1$ limit, Eqs. (19) and (25) reduce to the Shannon joint entropy [26] ($S_{Q+\Phi}(t)$) of a time-dependent RLC circuit

$$S_{Q+\Phi}(t) = \ln \left[ \pi e \left( 1 + L(t)^2 e^{2\lambda(t)} \rho^2 \dot{\rho}^2 \right)^{1/2} \right],$$  \hspace{1cm} (27)

where

$$S_Q(t) = \frac{1}{2} \ln \left( e\pi \rho^2 \right),$$  \hspace{1cm} (28)

$$S_\Phi(t) = \frac{1}{2} \ln \left( \left( \frac{\pi e}{\rho^2} \right) \left( 1 + m^2 \rho^2 \dot{\rho}^2 \right) \right).$$  \hspace{1cm} (29)

For given $L(t)$, $C(t)$, and $R(t)$, one has to solve Eq. (10) to determine the Tsallis, Rényi, and Shannon entropies. According to Lewis and Riesenfeld [16], only real solutions of $\rho(t)$ are acceptable.

3. Results and discussion

The well known Caldirola–Kanai oscillator whose Hamiltonian is $H(t) = e^{-\gamma t} p^2 + e^{\gamma t} \frac{\omega_0^2}{2} x^2$ [27,28] was introduced in the literature as a prototype to describe dissipation in quantum mechanics. The reason is that the equation of motion obtained from the Heisenberg equations reads

$$\ddot{x} + \gamma \dot{x} + \omega^2(t)x = 0,$$  \hspace{1cm} (30)

which is analogous to that of the damped harmonic oscillator with a constant damping factor, $\gamma$.

For a Caldirola–Kanai RLC circuit, $R = R_0$, $L = L_0$, and $C = C_0$ are all constants, and Eq. (6) reads

$$H(t) = e^{-\gamma t} \frac{\Phi^2}{2L_0} + e^{\gamma t} L_0 \omega_0^2 \frac{Q^2}{2},$$  \hspace{1cm} (31)

where $\omega_0^2 = 1/L_0 C_0$ and $\gamma = R_0/L_0$.

The solution of the respective Milne–Pinney equation (see Eq. (10)) is given by [12]

$$\rho(t) = \frac{e^{-\gamma t/2}}{(L_0 \Omega)^{1/2}},$$  \hspace{1cm} (32)

where $\Omega = \left( \omega_0^2 - \frac{\gamma^2}{4} \right)^{1/2} > 0$.

The expressions for $T_{q,Q}(t)$, $T_{q,\Phi}(t)$, $T_{q,Q+\Phi}(t)$, $R_{q,Q}(t)$, $R_{q,\Phi}(t)$, $R_{q,Q+\Phi}(t)$, and $S_{Q+\Phi}(t)$ read

$$T_{q,Q}(t) = \frac{1}{q - 1} \left[ 1 - \frac{1}{q^{1/2}} \left( \frac{\pi e^{-\gamma t}}{L_0 \Omega} \right)^{\frac{1}{2}(1-q)} \right],$$  \hspace{1cm} (33)

$$T_{q,\Phi}(t) = \frac{1}{q - 1} \left( 1 - \frac{1}{q^{1/2}} \left[ \pi \left( 1 + \frac{\gamma^2}{4\Omega^2} \right) \frac{L_0 \Omega}{e^{-\gamma t}} \right]^{\frac{1}{2}(1-q)} \right),$$  \hspace{1cm} (34)
Fig. 1. Variation of $T_{q,Q+\phi}$, $R_{q,Q+\phi}$, and $S_{Q+\phi}$ with $\gamma$. In the plots, we used $L_0 = 1$ and $\omega_0^2 = 25$.

$$T_{q,Q+\phi}(t) = \frac{1}{q - 1} \left[ 1 - \frac{(\pi)^{1-q}}{q \left( 1 + \frac{\gamma^2}{4\Omega^2} \right)^{\frac{1}{2}(1-q)}} \right],$$  \hspace{1cm} (35)

$$R_{q,Q}(t) = \frac{1}{2} \ln \left( \frac{e^{-\gamma t}}{L_0\Omega} \right) - \frac{\ln q}{2(1-q)},$$ \hspace{1cm} (36)

$$R_{q,\phi}(t) = \frac{1}{2} \left[ \ln \left( \pi \left( 1 + \frac{\gamma^2}{4\Omega^2} \right) \frac{L_0\Omega}{e^{-\gamma t}} \right) \right] - \frac{\ln q}{2(1-q)},$$ \hspace{1cm} (37)

$$R_{q,Q+\phi}(t) = \ln \left[ \pi e \left( 1 + \frac{\gamma^2}{4\Omega^2} \right)^{1/2} \right] - \frac{\ln q}{1-q},$$ \hspace{1cm} (38)

$$S_{Q+\phi}(t) = \ln \left[ \pi e \left( 1 + \frac{\gamma^2}{4\Omega^2} \right)^{1/2} \right].$$ \hspace{1cm} (39)

Although Eqs. (33), (34), (36), and (37) depend on time, Eqs. (35), (38), and (39) do not. Both $T_{q,Q+\phi}$ and $R_{q,Q+\phi}$ decrease with increasing $q$. Since $T_{q,Q+\phi}$ and $R_{q,Q+\phi}$ tend to $S_{Q+\phi}$ in the limit $q \to 1$, we consider $q = 1.1$ and $q = 0.9$ as upper and lower limits, respectively. Figure 1 shows the variation of $T_{q,Q+\phi}$, $R_{q,Q+\phi}$, and $S_{Q+\phi}$ with $\gamma$. As expected, the entropies increase with increasing damping factor $\gamma$. In other words, the loss of information increases with increasing damping. We also observe that $R_{q,Q+\phi}$ is closer to $S_{Q+\phi}$ than $T_{q,Q+\phi}$ for the two values of $q$ (0.9 and 1.1) considered. This is owing to the fact that $R_{q,Q+\phi}$ differs from $S_{Q+\phi}$ by $\ln q^{1-q}$.

We observe from Eqs. (17), (18), (23), (24), (28), and (29) that, depending on the parameters, the values of $T_q(t)$, $T_{q,\phi}(t)$, $T_{q,Q+\phi}(t)$, $R_q(t)$, $R_{q,\phi}(t)$, $S_Q(t)$, and $S_{Q}(t)$ may become negative. For instance, from Eq. (36) we observe that, for asymptotic times ($t \to \infty$), $R_q(t) \to -\infty$. According to Ref. [23], at this limit no information is lost. However, to compare the Rényi and Shannon entropy measures with the standard deviations $\Delta Q$ and $\Delta \Phi$, which are always positive quantities, it is suitable to use the Rényi and Shannon lengths defined as $L_{q,t}^J(t) = e^{H_q(t)}$ [22].
where $j = R, S$ and $i = q, \Phi$. In this case, we have

$$L^R_{q, Q}(t) = \sqrt{\frac{\pi}{L_0\Omega}} e^{-\gamma t/2} q^{\frac{1}{2q(q-1)}}, \quad (40)$$

$$L^R_{q, \Phi}(t) = \sqrt{\frac{\pi L_0\Omega}{1 + \frac{\gamma^2}{4\Omega^2}}} e^{\gamma t/2} q^{\frac{1}{2q(q-1)}}, \quad (41)$$

$$L^S_Q(t) = \sqrt{\frac{\pi e}{L_0\Omega}} e^{-\gamma t/2}, \quad (42)$$

$$L^S_{\Phi}(t) = \sqrt{\pi e L_0\Omega \left(1 + \frac{\gamma^2}{4\Omega^2}\right)} e^{\gamma t/2}, \quad (43)$$

which are all positive definite quantities.

The Tsallis length is undefined (see the properties of a length in Ref. [29]). From Eqs. (11)-(13), we have

$$\Delta Q = \frac{1}{\sqrt{2}} \frac{e^{-\gamma t/2}}{(L_0\Omega)^{1/2}}, \quad (44)$$

$$\Delta \Phi = \frac{1}{\sqrt{2}} \frac{(L_0\Omega)^{1/2}}{e^{-\gamma t/2}} \left(1 + \frac{\gamma^2}{4\Omega^2}\right). \quad (45)$$

Plots of $\Delta Q(t)$ and $L^j_{q, Q}(t)$, and $\Delta \Phi$ and $L^j_{q, \Phi}(t)$, are shown in Figs. 2(a) and (b), respectively.

We observe that $\Delta Q(t)$ and $L^j_{q, Q}(t)$ decrease while $\Delta \Phi$ and $L^j_{q, \Phi}(t)$ increase with increasing time. The decrease (increase) observed in Fig. 2(a) (2(b)) indicates that the accuracy in predicting the charge (magnetic flux) of the particle increases (decreases) with increasing time. For asymptotic times ($t \to \infty$), $\Delta Q, L^R_{q, Q}(t)$, and $L^S_Q(t)$ tend to zero.

The Lane–Emden oscillator [30–33] is another class of damped systems where the damping factor scales as $t^{-1}$. In this way, we consider $L(t) = L_0 t^\alpha$, $R(t) = R_0 t^{\alpha-1}$, and $C(t) = C_0 t^{-\alpha}$, where $\alpha > 0$ is a positive parameter. The Hamiltonian, the equation of motion, and the Milne–Pinney equation become, respectively,

$$H(t) = \frac{t^{-(\alpha + \gamma)}}{2 L_0^2} + t^{(\alpha + \gamma)} \frac{L_0 \omega_0^2}{2} Q^2, \quad (46)$$

$$\frac{\dot{Q}}{t} + \frac{\alpha + \gamma}{t} Q + \omega^2(t) Q = 0, \quad (47)$$

$$\frac{\dot{\rho}}{t} + \frac{\alpha + \gamma}{t} \dot{\rho} + \omega_0^2 \rho = \frac{1}{L_0^2(\alpha + \gamma) \omega_0^4}, \quad (48)$$

where, again, $\omega_0^2 = 1/L_0 C_0$ and $\gamma = R_0/L_0$.

It is worth mentioning that one can solve Eq. (48) for any positive value of $\alpha + \gamma$. For $\alpha + \gamma = 2$, the solution of Eq. (48) reads

$$\rho(t) = \frac{t^{-1}}{\sqrt{L_0 \omega_0^4}}, \quad (49)$$

and the expressions for $T_{q, Q}(t)$, $T_{q, \Phi}(t)$, $T_{q, Q+\Phi}(t)$, $R_{q, Q}(t)$, $R_{q, \Phi}(t)$, $R_{q, Q+\Phi}(t)$, and $S_{Q+\Phi}(t)$ are given by

$$T_{q, Q}(t) = \frac{1}{q - 1} \left[ 1 - \frac{1}{q^{1/2}} \left( \frac{\pi}{L_0 \omega_0^2 t^2} \right)^{1/(1-q)} \right], \quad (50)$$
Fig. 2. Variations of (a) $\Delta Q(t)$ and $L_{q,Q}^{j}(t)$, and (b) $\Delta \Phi$ and $L_{q,\Phi}^{j}(t)$ for the Caldirola–Kanai RLC circuit with $t$. In the plots, we used $L_0 = R_0 = 1$.

\[
T_{q,\Phi}(t) = \frac{1}{q-1} \left\{ 1 - \frac{1}{q^{1/2}} \left[ \pi \left( 1 + \frac{1}{\omega_0^2 t^2} \right) L_0 \omega_0 t^2 \right]^{1/(1-q)} \right\}, \tag{51}
\]

\[
T_{q,Q+\Phi}(t) = \frac{1}{q-1} \left[ 1 - \frac{(\pi)^{1-q}}{q} \left( 1 + \frac{1}{\omega_0^2 t^2} \right)^{1/(1-q)} \right], \tag{52}
\]

\[
R_{q,Q}(t) = \frac{1}{2} \ln \left( \frac{\pi}{L_0 \omega_0 t^2} \right) - \frac{\ln q}{2(1-q)}, \tag{53}
\]

\[
R_{q,\Phi}(t) = \frac{1}{2} \left\{ \ln \left[ \pi L_0 \omega_0 t^2 \left( 1 + \frac{1}{\omega_0^2 t^2} \right) \right] \right\} - \frac{\ln q}{2(1-q)}, \tag{54}
\]

\[
R_{q,Q+\Phi}(t) = \ln \left[ \pi \left( 1 + \frac{1}{\omega_0^2 t^2} \right)^{1/2} \right] - \frac{\ln q}{1-q}. \tag{55}
\]
Fig. 3. Variation of $S_{q+\Phi, t}$, $R_{q+\Phi, t}$, and $S_{q+\Phi}$ with $t$ for the Lane–Emden RLC circuit. In the plots, we used $\omega_0^2 = 1$.

\[
S_{q+\Phi} (t) = \ln \left[ \pi e \left( 1 + \frac{1}{\omega_0^2} \right)^{1/2} \right]. \quad (56)
\]

For the Lane–Emden oscillators, the expression for the joint entropies $T_{q+\Phi}(t)$, $R_{q+\Phi}(t)$, and $S_{q+\Phi}(t)$ depend on time. In Fig. 3 we show the time evolution of the joint entropies. We observe that, when $t \to \infty$, $T_{q+\Phi}(t)$, $R_{q+\Phi}(t)$, and $S_{q+\Phi}(t)$ tend to values corresponding to those obtained for the usual time-independent LC circuit. This is due to the fact that the damping factor goes to zero for asymptotic times (see Eq. (47)).

The standard deviations $\Delta Q$ and $\Delta \Phi$ for the Lane–Emden RLC circuit are given by

\[
\Delta Q = \frac{1}{\sqrt{2\omega_0 t}}, \quad (57)
\]

\[
\Delta \Phi = \frac{1}{\sqrt{2}} \sqrt{\left( 1 + \frac{1}{\omega_0^2 t^2} \right) L_0 \omega_0 t}, \quad (58)
\]

and the Rényi and Shannon lengths for the Lane–Emden RLC circuit read

\[
L_R^{q, Q} (t) = \sqrt{\frac{\pi}{L_0 \omega_0 t^2}} q^{\frac{1}{2(q-1)}}, \quad (59)
\]

\[
L_R^{q, \Phi} (t) = \sqrt{\frac{\pi L_0 \omega_0 t^2}{\omega_0^2 t^2}} \left( 1 + \frac{1}{\omega_0^2 t^2} \right) q^{\frac{1}{2(q-1)}}, \quad (60)
\]

\[
L_S^{Q} (t) = \frac{\pi e}{L_0 \omega_0 t^2}, \quad (61)
\]

\[
L_S^{\Phi} (t) = \frac{\pi e L_0 \omega_0 t^2}{\omega_0^2 t^2} \left( 1 + \frac{1}{\omega_0^2 t^2} \right). \quad (62)
\]

Figures 4(a) and (b) depict the time evolution of $\Delta Q (t)$ and $L_{q, Q}^j (t)$, and $\Delta \Phi$ and $L_{q, \Phi}^j (t)$, respectively. Again, we observe that $\Delta Q (t)$ and $L_{q, Q}^j (t)$ decrease while $\Delta \Phi$ and $L_{q, \Phi}^j (t)$ increase with
Fig. 4. Variations of (a) $\Delta Q(t)$ and $L_{q, Q}(t)$, and (b) $\Delta \Phi$ and $L_{q, \Phi}(t)$ for the Lane–Emden RLC circuit with $t$. In the plots, we used $L_0 = R_0 = 1$.

increasing time. However, the decrease (increase) observed in Fig. 4(a) (4(b)) is slower than that shown in Fig. 2(a) (2(b)). This is owing to the fact that in the Lane–Emden RLC circuit the damping factor goes to zero with increasing time, while in the Caldirola–Kanai RLC circuit it remains constant.

4. Summary

In this work, we have calculated the Tsallis, Rényi, and Shannon joint entropies for two classes of damped systems: Caldirola–Kanai and Lane–Emden mesoscopic RLC circuits. The damping is constant for the former, while it decreases with increasing time for the latter. Using the Lewis–Riesenfeld method [16] to obtain the solution of the time-dependent Schrödinger equation, we were able to write analytical expressions for the entropies in the lowest-lying states of the systems (see Eqs. (33)–(39) and (50)–(56)). For the Caldirola–Kanai mesoscopic RLC circuit, the joint entropies $T_{q, Q+\Phi}(t)$, $R_{q, Q+\Phi}(t)$, and $S_{Q+\Phi}(t)$ are constants and increase with increasing damping factor, $\gamma$. On the
other hand, for the Lane–Emden mesoscopic RLC circuit, $T_{q,\Phi}(t)$, $R_{q,\Phi}(t)$, and $S_{q,\Phi}(t)$ (see Fig. 3) depend on time and decrease with increasing time. This is owing to the fact that the damping goes to zero for asymptotic times, and, therefore, the Lane–Emden mesoscopic RLC circuits become the usual time-independent mesoscopic LC circuit.

We have also obtained the behavior of the Rényi and Shannon lengths and the standard deviations with time for the Caldirola–Kanai and Lane–Emden mesoscopic RLC circuits (see Figs. 2 and 4, respectively). For both oscillators, $\Delta Q(t)$ and $L_{q,\Phi}^j(t)$ decrease while $\Delta \Phi$ and $L_{q,\Phi}^j(t)$ increase with increasing time. However, the decrease/increase observed in Fig. 4 is less pronounced than those observed in Fig. 2. Again, this results from the time dependence of the damping factor in the case of the Lane–Emden mesoscopic RLC circuit.

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