On the geometric structure of certain real algebraic surfaces

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Abstract

In this paper we study the affine geometric structure of the graph of a polynomial \( f \in \mathbb{R}[x, y] \). We provide certain criteria to determine when the parabolic curve is compact and when the unbounded component of its complement is hyperbolic or elliptic. We analyse the extension to the real projective plane of both fields of asymptotic lines and the Poincaré index of its singular points when the surface is generic. Thus, we exhibit an index formula for the field of asymptotic lines involving the number of connected components of the projective Hessian curve of \( f \) and the number of the special parabolic points. As an application of this investigation, we obtain upper bounds, respectively, for the number of special parabolic points having an interior tangency and when they have an exterior tangency.

Keywords: parabolic curve, asymptotic fields of lines, real algebraic surfaces, quadratic differential forms.

MS classification: 53A15, 53A05, 14P05, 14N10, 34K32, 34G20

1 Introduction

There is a well known classification of the points of a smooth algebraic surface immersed in the three-dimensional real affine (projective or Euclidean) space. Any point belongs to one of the following types: elliptic, parabolic or hyperbolic. On generic algebraic surfaces, parabolic points appear along a smooth curve (it may be empty) called the parabolic curve of the surface.

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whose complement is constituted by the elliptic and hyperbolic domains. The configuration of these sets, invariant under the action of the affine group on 3-space, is the basic affine geometric structure of the surface. One of the goals in projective and affine differential geometry has been the study of the basic geometric structure of smooth and algebraic surfaces, see for example [2, 3, 5, 11, 15, 20]. In this paper we focus on the analysis of the basic geometric structure of algebraic surfaces in $\mathbb{R}^3$ that are the graph of a real polynomial $f \in \mathbb{R}[x, y]$.

When the parabolic curve of such a surface $S_f$ is compact, there is one unbounded component $C_u$ in the complement of this curve that plays a relevant role in the determination of this structure. If the class of this component, which can be either elliptic or hyperbolic, is known in the generic case, we can specify the class of the other connected components that are on the complement of the parabolic curve.

In section 3, we study the structure of the elliptic and hyperbolic domains: we provide generic conditions on the homogeneous part of the highest degree of $f$ that guarantees the parabolic curve is compact and indicates the class of the component $C_u$, Theorem 1. At each point on the complement of the elliptic domain there are two lines tangent to the surface, called asymptotic lines. Each one has a contact of order at least three with the surface at the point. If the point is parabolic, these two lines coincide while at a hyperbolic point, in which they become transversal.

When $f$ is a differentiable function defined on the plane $\mathbb{R}^2$, it is usual to consider a projection of the geometric structure of $S_f$ into the plane. The image under such projection of the parabolic curve is a plane curve called Hessian curve of $f$ that is defined by the equation $\text{Hess}_f(x, y) = 0$ and the images of the two fields of asymptotic lines, are described by the second fundamental form of $S_f$,

$$\Pi_f(dx, dy) = f_{xx}(x, y)dx^2 + 2f_{xy}(x, y)dxdy + f_{yy}(x, y)dy^2.$$  

In [8], V. Guínez considers positive quadratic differential equations on the plane $\mathbb{R}^2$ of the form

$$a(x, y)dx^2 + b(x, y)dxdy + c(x, y)dy^2 = 0,$$  

where $a, b, c \in \mathbb{R}[x, y]$ are polynomials of degree at most $n$, the function $b^2 - 4ac$ is nonnegative at every point of the $xy$-plane and $b^2 - 4ac$ vanish at a point $p$ if and only if $a, b, c$ vanish simultaneously at $p$. He extends the foliations determined by equation (1) to the line at infinity and proves, between other things that the topological behavior of these foliations, in a
neighborhood of an infinite singular point, is one of the types shown in figure 1 ([8], Remark 2.9).

Figure 1: Topological types of an infinite singular point.

When \( f \in \mathbb{R}[x, y] \) is a polynomial, the second fundamental form \( II_f \) is a polynomial quadratic differential form such that, in general, it is not positive: it has disjoint open sets on the plane at which the discriminant of this form is negative.

Through the projection of Poincaré from a plane into the unitary sphere, we give, in section 4, an analytic extension on the sphere of the two fields of asymptotic lines. These two fields of lines defined on the sphere have the same singular points. We prove in Theorem 2 that in fact each singular point of these fields that lie on the equator of this sphere is an infinite hyperbolic point. Therefore, its Poincaré index is well defined and it is equal to \( \frac{1}{2} \), Theorem 3. As a consequence we obtain an upper bound for the sum over all singular points lying on the equator of its Poincaré index. All this analysis allowed us to itemize, in the generic case, the Poincaré index of each singular point of the extension of each field of asymptotic lines to the real projective plane, Remark 6.

The projective Hessian curve of \( f \) is, in general, a nonsingular algebraic curve in \( \mathbb{R}P^2 \) of even degree. In this plane we define two surfaces, \( B^\pm \) whose boundary is the projective Hessian curve of \( f \). Among parabolic points of a generic surface \( S_f \) a special parabolic point is distinguished because its unique asymptotic line is tangent to the parabolic curve. The problem of determining the lowest upper bound for the number of special parabolic points of an algebraic surface in terms of the degree of the polynomial that defines it has been an interesting subject of research [12, 19, 1]. The tangency of the asymptotic direction with the parabolic curve at such a point may be interior or exterior [4]. In the last section we give a formula that relates the following three values: the Euler characteristic of the surface \( B^\pm \), the number of interior and exterior tangencies of the special parabolic points and the Poincaré index of the singular points of the fields of asymptotic lines when they are defined on \( B^\pm \), Theorem 4. Derived from this result, upper bounds for the number of interior and exterior tangencies are given, Corollary 2. Finally, in Theorem 5 we exhibit an upper bound for the number
of special parabolic points when the projective Hessian curve of \( f \) is convex and it is comprised of only exterior ovals.

2 Preliminaries

2.1 Classification of points on a generic surface

Points on a smooth surface in \( \mathbb{R}^3 \) are classified in terms of the maximum order of contact of the tangent lines at them with the surface \([19, 13, 17]\). We say that a point \( p \) is elliptic if all straight lines tangent to the surface at \( p \) have a contact of order two with the surface at that point.

An asymptotic line at a point \( p \) is a straight line tangent to the surface at \( p \) that has a contact of order greater than two with the surface. A hyperbolic point has exactly two transversal asymptotic lines while a parabolic point has one (double) asymptotic line.

The sets of elliptic and hyperbolic points are open subsets on the surface called elliptic and hyperbolic domains, respectively. On a generic smooth surface, these two domains share a common boundary called the parabolic curve which is a smooth curve constituted by the parabolic points. The unique asymptotic line at a parabolic point is transversal to the parabolic curve except at some isolated points called special parabolic points (other authors call them Gaussian cusps or godrons) and will be referred to in this article as SP points. The order of contact of the asymptotic line at each parabolic point is three while at a SP point, is four. The set of asymptotic directions makes up, globally, two continuous fields of directions tangent to the surface \([21]\) (this property is proved locally in \([6, 7]\)). Such fields are known as fields of asymptotic directions and its integral curves, as asymptotic curves.

When the surface \( S_f \) is the graph of a differentiable function \( f \) on the plane, we shall study the projection onto the plane of the geometric structure of \( S_f \) to understand its geometric behavior. In particular, consider the projection \( \pi : \mathbb{R}^3 \to \mathbb{R}^2, (x, y, z) \mapsto (x, y) \). The image of the parabolic curve on the \( xy \)-plane under \( \pi \) is the zero locus of the Hessian function \( \text{Hess} f = f_{xx}f_{yy} - f_{xy}^2 \). This curve will be called the Hessian curve of \( f \). The hyperbolic and elliptic domains are projected on subsets on the \( xy \)-plane, denoted by \( H \) and \( E \), at which the Hessian function is negative and positive, respectively. In fact, for any open set \( U \) contained in the \( xy \)-plane, we shall say that it is a hyperbolic set or an elliptic set if the Hessian function of \( f \) is negative or positive, respectively at each point on \( U \).

The projection of both fields of asymptotic directions onto the \( xy \)-plane
yields two fields of lines that are described by the quadratic differential equation:

\[ f_{xx}(x, y)\, dx^2 + 2f_{xy}(x, y) + f_{yy}(x, y)\, dy^2 = 0. \tag{2} \]

The quadratic form on the left will be referred to as the second fundamental form of \( f \) and will be denoted by \( II_f(dx, dy) \). A point \( p \) on the \( xy \)-plane is a singular point of the second fundamental form of \( f \) if the coefficients of this form, \( f_{xx}, f_{xy} \) and \( f_{yy} \) vanish at \( p \). In the rest of this work, we mainly consider the solution fields (2) that will be referred to as the fields of asymptotic directions of the surface.

We are interested in the particular case when \( f \in \mathbb{R}[x, y] \) is a polynomial. If the degree of \( f \) is \( n \), its Hessian curve is a real plane algebraic curve of degree, at most \( 2n - 4 \).

**Definition 1** The projective Hessian curve of \( f \) is the zero locus of the homogeneous polynomial \( H_f \in \mathbb{R}[x, y, z] \) which is the homogenization of the polynomial \( \text{Hess} \, f(x, y) \).

### 2.2 Real algebraic curves in \( \mathbb{R}P^2 \).

A real algebraic curve in \( \mathbb{R}P^2 \) of degree \( m \) is, up to nonzero constant factors, a homogeneous polynomial \( F \in \mathbb{R}[x, y, z] \) of degree \( m \). The polynomial equation \( F(x, y, z) = 0 \) determines the set of real points of the curve in \( \mathbb{R}P^2 \). From now on, we shall also call this set a real algebraic curve in \( \mathbb{R}P^2 \).

Each connected component of a nonsingular algebraic curve in \( \mathbb{R}P^2 \) is homeomorphic to a circle. There are two ways up to isotopy to embed a circle into the real projective plane which are called the two-sidedly and the one-sidedly. In the two-sidedly case, the complement in \( \mathbb{R}P^2 \) of the image \( L \) of the circle has two connected components, one of which is homeomorphic to an open disc and called the inside component of \( L \) while the other is homeomorphic to a Möbius strip and is known as the outside component of \( L \). Under these conditions, the image of the circle is called oval. We say that an oval is an outer oval if it is not in the inside component of any other oval. In the one-sidedly case, the complement in \( \mathbb{R}P^2 \) of the image of the circle is connected and homeomorphic to a disc. In this situation the image of the circle is called a pseudo-line. While all the connected components of a nonempty nonsingular real algebraic curve in \( \mathbb{R}P^2 \) of even degree are ovals, each nonsingular algebraic curve of odd degree is constituted by ovals (if there is any) and exactly one pseudo-line.

The complement of a nonsingular curve \( G \) of even degree in \( \mathbb{R}P^2 \) is the union of two disjoint open subsets, say \( b^+ \) and \( b^- \) (Figure 2). The set \( b^+ \) is
an orientable smooth surface at which the sign of $G$ does not change while the open set $b^-$ is a nonorientable smooth surface at which $G$ takes the other sign. The closure of $b^+$ and $b^-$ will be denoted by $B^+$ and $B^-$, respectively.

Figure 2: Open sets $b^+$ and $b^-$. 

**Definition 2** An oval of a real algebraic curve in $\mathbb{R}P^2$ of even degree $m$ is called even (odd) if it is contained in an even (odd) number of ovals of the same curve. The number of even ovals is denoted by $P$ and the number of odd ovals by $N$.

The numbers $P$ and $N$ contain information about the topology of the surfaces $B^+$ and $B^-$. Indeed, the surface $B^+$ has $P$ connected components and the surface $B^-$ has $N+1$ connected components. In 1906, Virginia Ragsdale proves that the Euler characteristics of these surfaces are $\chi(B^+) = P - N$ and $\chi(B^-) = N - P + 1$, [18]. Three decades later, I. Petrowsky shows the following

**Theorem (Petrowsky [16])** Any nonsingular real projective algebraic curve of even degree $m = 2k$ satisfies

$$-\frac{3}{2}k(k - 1) \leq P - N \leq \frac{3}{2}k(k - 1) + 1.$$

3 Determination of the elliptic and hyperbolic domains

Let $f \in \mathbb{R}[x, y]$ be a real polynomial. In this paragraph we analyse the geometric behavior of the sets $E$ and $H$, imposing some conditions to the highest degree homogeneous part of $f$. To do that, we shall consider the natural extension of this plane to the line at infinity $L_\infty$. We say that the graph $S_f$ of a polynomial $f$ of degree $n$ is *generic* if it is a generic algebraic
surface in \( \mathbb{R}^3 \) and if the projective Hessian curve of \( f \) is a nonsingular curve of degree \( 2n - 4 \). Now, we label the points that are on the line \( \mathcal{L}_\infty \).

**Definition 3** A point \([u : v : 0] \in \mathcal{L}_\infty\) is called *infinite elliptic*, *infinite parabolic* or *infinite hyperbolic* if the sign of the homogeneous polynomial \( H_f \) at this point is positive, zero or negative, respectively.

In Theorem 1 we can appreciate how in some cases the homogeneous part of highest degree of \( f \) determines the geometric structure of the surface \( S_f \) when it is generic. It is not true as shown by the following examples.

Consider the real polynomials \( f(x, y) = x^4 + 6x^2y^2 - y^4 + 3x^2y - 3xy^2 + 10y^2 - 10x^2 \) and \( g(x, y) = x^4 + 6x^2y^2 - y^4 + 3x^2y - 3xy^2 + 10y^2 + 10x^2 \). While they only differ by the quadratic homogeneous part, its geometric structure is different because in the first case the set \( b^- \cap \mathbb{R}^2 \) is hyperbolic, and in the second case, \( b^- \cap \mathbb{R}^2 \) is elliptic.

**Definition 4** A homogeneous polynomial on \( \mathbb{R}[x, y] \) is called *hyperbolic* (elliptic) if its graph is a surface with only hyperbolic (elliptic) points outside of the origin of the \( xy \)-plane.

It is worth noting that the conclusions of the next result are proved in [10] (Theorem 2), but under stronger hypothesis.

**Theorem 1** Let \( f \in \mathbb{R}[x, y] \) be a polynomial of degree \( n \geq 3 \). Suppose that \( f_n \), the homogeneous part of degree \( n \) of \( f \), is hyperbolic or elliptic. Then the Hessian curve of \( f \) is compact. Moreover, the set \( b^- \cap \mathbb{R}^2 \) is hyperbolic or elliptic providing that \( f_n \) is hyperbolic or elliptic, respectively.

**Proof.** Consider the homogeneous decomposition of the Hessian polynomial of \( f \), \( \text{Hess}_f(x, y) = \sum_{j=0}^{2n-4} h_j(x, y) \), where \( h_j(x, y) \) is a homogeneous polynomial of degree \( j \). Because the homogenization of this polynomial is \( H_f(x, y, z) = \sum_{j=0}^{2n-4} z^{2n-4-j} h_j(x, y) \), the intersection of the projective Hessian curve of \( f \) with the line at infinity \( \{z = 0\} \) is the set \( \{[x : y : 0] \in \mathbb{R}P^2 | h_{2n-4}(x, y) = 0\} \).

Suppose that \( f_n \) is a hyperbolic polynomial. The elliptic case is similar. Since the polynomial \( h_{2n-4}(x, y) = \text{Hess}_f_n(x, y) \) is nonpositive and it vanishes only at the origin, the projective Hessian curve does not intersect with the line at infinity. Accordingly, the curve \( \text{Hess}_f(x, y) = 0 \) is compact in \( \mathbb{R}^2 \) and the line at infinity is contained in \( B^- \). To show that the set \( b^- \cap \mathbb{R}^2 \) is hyperbolic, it will be enough to prove that any point of the line at infinity is an infinite hyperbolic point. By taking \( p = [1 : 0 : 0] \) we have that \( H_f(p) \) is negative because \( f_n \) is hyperbolic and \( H_f(p) = h_{2n-4}(1, 0) = \text{Hess}_f_n(1, 0) \).

\( \square \)
4 Projection into the Poincaré sphere

Let \( f \in \mathbb{R}[x, y] \) be a polynomial of degree \( n \) such that its graph \( S_f \subset \mathbb{R}^3 \) is a generic surface. Consider the two fields of asymptotic directions on the \( xy \)-plane defined by equation (2) and denote them by \( \mathbb{X}_1 \) and \( \mathbb{X}_2 \).

Now, following the ideas of Poincaré [14], we shall describe the extension of these fields to the line at infinity. To do that, we consider the central projection from a plane with coordinates \((x, y)\) into the unitary sphere \( S^2 = \{(u, v, w) \in \mathbb{R}^3 \mid u^2 + v^2 + w^2 = 1\} \subset \mathbb{R}^3 \).

This map associates to each \( x = (x, y) \) the following two points in \( S^2 \):

\[
s_1(x) = \frac{1}{\sqrt{1 + x^2 + y^2}} (x, y, 1), \quad s_2(x) = -\frac{1}{\sqrt{1 + x^2 + y^2}} (x, y, 1).
\]

**Remark 1** The images, under the central projection, of the two fields of asymptotic directions \( \mathbb{X}_1 \) and \( \mathbb{X}_2 \) into both upper and lower hemispheres are the zero loci of the induced quadratic differential forms, \( s_1^* (\mathbb{II}_f) \) and \( s_2^* (\mathbb{II}_f) \), which are defined on the complement of the equator of \( S^2 \). Moreover, the images of both fields into each open hemisphere consist of two fields of lines diffeomorphic to \( \mathbb{X}_1 \) and \( \mathbb{X}_2 \).

Similarly, as V. Guínez does in [8], we shall prove that the induced quadratic differential forms \( s_1^* (\mathbb{II}_f) \) and \( s_2^* (\mathbb{II}_f) \) can be extended to an analytical quadratic differential form defined on the sphere.

**Proposition 1** The induced differential forms \( s_1^* (\mathbb{II}_f) \) and \( s_2^* (\mathbb{II}_f) \) are extended to this analytical quadratic differential form

\[
(du \ dv \ d\omega) \begin{pmatrix}
\omega^2 F_{uu}(u, v, \omega) & \omega^2 F_{uv}(u, v, \omega) & \omega A(u, v, \omega) \\
\omega^2 F_{uv}(u, v, \omega) & \omega^2 F_{vv}(u, v, \omega) & \omega B(u, v, \omega) \\
\omega A(u, v, \omega) & \omega B(u, v, \omega) & S(u, v, \omega)
\end{pmatrix}
\begin{pmatrix}
(du) \\
(dv) \\
(d\omega)
\end{pmatrix}
\]

defined on the sphere in such a way that if a point on the equator is a point on the foliations of this form, such foliations are tangent to the equator at
In this case

\[ F(u, v, \omega) = \sum_{i=0}^{n} \omega^{n-i} f_i(u, v), \]

\[ F_{uu} = \frac{\partial^2 F}{\partial u^2}, \quad F_{uv} = \frac{\partial^2 F}{\partial u \partial v}, \quad F_{vv} = \frac{\partial^2 F}{\partial v^2}, \]

\[ A(u, v, \omega) = -uF_{uu}(u, v, \omega) - vF_{uv}(u, v, \omega), \]

\[ B(u, v, \omega) = -uF_{uv}(u, v, \omega) - vF_{vv}(u, v, \omega), \]

\[ S(u, v, \omega) = u^2F_{uu}(u, v, \omega) + 2uvF_{uv}(u, v, \omega) + v^2F_{vv}(u, v, \omega). \]

**Proof.** Consider the map \( \rho : \mathbb{R}^3 \setminus \{ \omega = 0 \} \rightarrow \mathbb{R}^2, \ (u, v, \omega) \mapsto (x, y) \) where \( x = \frac{u}{\omega}, \ y = \frac{v}{\omega}. \) This map associates each pair of antipode points on the unitary sphere \( \mathbb{S}^2 \) to the same point. We proceed to obtain the pullback \( \rho^* (II_f) \) of the second fundamental form \( II_f. \) Replacing

\[ (dx \ dy) = \left( \frac{\omega du - ud\omega}{\omega^2} \quad \frac{\omega dv - vd\omega}{\omega^2} \right) = \frac{1}{\omega^2} (du \ dv \ d\omega) \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} \]

in the expression \( II_f(dx, dy) = (dx \ dy) \begin{pmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{yx}(x, y) & f_{yy}(x, y) \end{pmatrix} (dx \ dy), \)

we have that \( \rho^* (II_f) \) is

\[ \frac{1}{\omega^4} (du \ dv \ d\omega) \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & -v \\ -u & -v & \omega \end{pmatrix} \begin{pmatrix} f_{xx}(\frac{u}{\omega}, \frac{v}{\omega}) & f_{xy}(\frac{u}{\omega}, \frac{v}{\omega}) \\ f_{yx}(\frac{u}{\omega}, \frac{v}{\omega}) & f_{yy}(\frac{u}{\omega}, \frac{v}{\omega}) \end{pmatrix} \begin{pmatrix} \omega & 0 & -u \\ 0 & \omega & -v \end{pmatrix} (du \ dv \ d\omega). \]

After multiplication by \( \omega^{n+2} \) we obtain the desired quadratic form (3). \( \square \)

We denote by \( \mathcal{Y}_1 \) and \( \mathcal{Y}_2 \) the two fields of lines tangent to the sphere that are defined by the quadratic form (3). It is worth mentioning that these fields are not defined, in general, on the whole sphere.

**Definition 5** A point \((u, v, \omega) \in \mathbb{S}^2\) is a **singular point** of \( \mathcal{Y}_1 \) and \( \mathcal{Y}_2 \) if the \( 3 \times 3 \) matrix of (3) vanishes at this point. And this point is called **infinite singular** if it lies on the equator of the unitary sphere \( \mathbb{S}^2 \).

**Remark 2** (a) If \( S_f \) is generic, all singular points of (3) (if there are any) are in the equator of the sphere \( \mathbb{S}^2 \).

(b) A point \((u_0, v_0, \omega_0) \in \mathbb{S}^2, \) with \( \omega_0 \neq 0 \) is a singular point of (3) if and only if the point \((x_0, y_0) = \left( \frac{u_0}{\omega_0}, \frac{v_0}{\omega_0} \right) \) is a singular point of the second fundamental form \( II_f. \)
(c) A point \((u_0, v_0, 0) \in \mathbb{S}^2\) is an infinite singular point of \((3)\) if and only if the polynomial 
\[
\frac{\partial^2 f_n(u, v)}{\partial u^2} u^2 + 2uv \frac{\partial^2 f_n(u, v)}{\partial u \partial v} + v^2 \frac{\partial^2 f_n(u, v)}{\partial v^2}
\]
vanishes at that point.

In the next Lemma, we prove that the number of infinite singular points of the field \(\mathbb{Y}_k, k = 1, 2\), is twice the number of distinct real linear factors of the homogeneous polynomial \(f_n\). In our case, the number of singular points of \((3)\) is always finite. This can be untrue for other binary quadratic differential equations \([8]\).

**Lemma 1** If \(f \in \mathbb{R}[x, y]\) is a polynomial of degree \(n \geq 3\), then the set of infinite singular points of \(\mathbb{Y}_i, i = 1, 2\), is

\[
\{(u, v, 0) \in \mathbb{S}^2 \mid f_n(u, v) = 0\},
\]

where \(f_n(x, y)\) is the homogeneous part of degree \(n\) of \(f(x, y)\).

**Proof.** By Remark 2 the infinite singular points \((u, v, 0) \in \mathbb{S}^2\) are given by the equation

\[
\frac{\partial^2 f_n(u, v)}{\partial u^2} u^2 + 2uv \frac{\partial^2 f_n(u, v)}{\partial u \partial v} + v^2 \frac{\partial^2 f_n(u, v)}{\partial v^2} = 0.
\]

Now, consider the well known Euler’s formula for a homogeneous polynomial \(P \in \mathbb{R}[x, y]\) of degree \(m\): \(mP(x, y) = xP_x(x, y) + yP_y(x, y)\). This formula implies the relation

\[
m(m - 1)P(x, y) = x^2P_{xx}(x, y) + 2xyP_{xy}(x, y) + y^2P_{yy}(x, y).
\]

Because the right side of the last equality coincides with the left side of (1) when considering \(P = f_n\), the equation (4) turns into the expression

\[
n(n - 1) f_n(u, v) = 0.
\]

**Remark 3** The only points on the equator of the sphere that will be considered as special parabolic points are those at which the projective Hessian curve of \(f\) is tangent to the line at infinity.

**Theorem 2** Consider a polynomial \(f \in \mathbb{R}[x, y]\) of degree \(n \geq 3\) and its homogeneous decomposition \(f = \sum_{i=1}^{n} f_i\), where \(f_i \in \mathbb{R}[x, y]\) is a homogeneous polynomial of degree \(i\). Assume that \(S_f\) is generic and that \(f_n\) and \(\text{Hess} f_n\) have no common factor. Then, all infinite singular points of the quadratic form \((3)\) are infinite hyperbolic points.
Proof. Let \( \text{Hess}f(x,y) = \sum_{j=0}^{2n-4} h_j(x,y) \) be the homogeneous decomposition of \( \text{Hess}f \). Then, its homogenization is the polynomial \( H_f(x,y,z) = \sum_{j=0}^{2n-4} z^{2n-4-j}h_j(x,y) \).

Suppose that \( \omega = (\omega_1,\omega_2,0) \) is an infinite singular point of \( (3) \). Thus, \( H_f(\omega) = h_{2n-4}(\omega_1,\omega_2) = \text{Hess}f_n(\omega_1,\omega_2) \). According to Lemma 1 and using the fact that no linear factor of \( f_n \) is a linear factor of \( \text{Hess}f_n \), we have that \( \omega \) does not belong on the projective Hessian curve of \( f \).

Now, we shall prove that \( H_f(\omega) < 0 \). Let \( l(x,y) \) be the real linear factor of \( f_n \) corresponding to \( \omega \) (Lemma 1). After an affine change of coordinates on the \( xy \)-plane we can suppose that \( l(x,y) = y \). So, \( \omega = [1:0:0] \). In this case, \( f_n(x,y) = yg(x,y) \) and \( H_f(\omega) = \text{Hess}f_n(1,0) = -(g_x(1,0))^2 < 0 \). □

Since an infinite singular point is surrounded locally by hyperbolic points (including the infinite hyperbolic points) its Poincaré index is well defined.

Theorem 3 Let \( f = \sum_{i=1}^{n} f_i \in \mathbb{R}[x,y] \) be a polynomial of degree \( n \geq 3 \). Assume that \( f_n \) has no repeated linear factor. Then, the Poincaré index of each infinite singular point of \( (3) \) is equal to \( \frac{1}{2} \). Moreover, its topological type is the one shown in figure 3.

![Figure 3: Topological type of an infinite singular point.](image)

Remark 4 If \( f \in \mathbb{R}[x,y] \) is a homogeneous polynomial of degree \( n \geq 3 \) such that its Hessian polynomial has a degree \( 2n-4 \) and has no common factor with \( f \), then all the infinite singular points of the quadratic form \( II_f \) are infinite hyperbolic and its Poincaré index is equal to \( \frac{1}{2} \) (i.e., the conclusions of Theorem 2 and Theorem 3 remain true for homogeneous polynomials).

Consider \( \mathcal{Y}_1 \) and \( \mathcal{Y}_2 \), the fields of lines tangent to the sphere \( \mathbb{S}^2 \) defined by the quadratic form \( (3) \). For \( k = 1, 2 \), the expression \( \text{Sing} (\mathcal{Y}_k) \) denotes the set of singular points of the field \( \mathcal{Y}_k \).
Corollary 1 Let \( f \in \mathbb{R}[x, y] \) be a polynomial such that its graph is generic.

i) If the projective Hessian curve of \( f \) has a non empty transversal intersection with the line at infinity, then
\[
\sum_{\xi \in \text{Sing}(\mathbb{Y}_k)} \text{Ind}(\xi) \leq n - 2, \quad \text{for } k = 1, 2.
\]

ii) If the projective Hessian curve of \( f \) does not intersect with the line at infinity, then
\[
\sum_{\xi \in \text{Sing}(\mathbb{Y}_k)} \text{Ind}(\xi) \leq n, \quad \text{for } k = 1, 2.
\]

Proof. It follows, from Proposition 1 of [9], that when the homogeneous polynomial \( f_n \) has exactly \( n \) generic real linear factors, it is hyperbolic. In such a case, the Hessian curve of \( f \) is compact by Theorem 1 and the field \( \mathbb{Y}_k \) has \( 2n \) infinite singular points. Thus, if the Hessian curve of \( f \) is unbounded, \( f_n \) has at most \( n - 2 \) real linear factors, and by Lemma 1, the maximum number of infinite singular points is \( 2(n - 2) \). The inequalities of i) and ii) follow from Theorem [3]. □

Remark 5 The fields \( \mathbb{Y}_1 \) and \( \mathbb{Y}_2 \) behave as follows:

- When \( n \) is even, if \( \mathbb{Y}_1 \) is the projection on the upper hemisphere of \( \mathbb{X}_1 \), then, on the lower hemisphere, \( \mathbb{Y}_1 \) is the projection of \( \mathbb{X}_2 \). Analogously for \( \mathbb{Y}_2 \).

- If \( n \) is odd, the field \( \mathbb{Y}_i \) is the projection into both hemispheres of either \( \mathbb{X}_1 \) or \( \mathbb{X}_2 \).

\[
\begin{array}{c}
\text{n odd} & \text{n even} \\
\text{n odd} & \text{n even}
\end{array}
\]

Figure 4: Behavior at antipodal infinite singular points on \( S^2 \).

Figure 5: Behavior at an infinite singular point on \( \mathbb{R}P^2 \).

The restriction of the fields \( \mathbb{Y}_1 \) and \( \mathbb{Y}_2 \) to the upper hemisphere or the lower hemisphere of \( S^2 \) will be called a projective extension of the two fields of
asymptotic directions $X_1$ and $X_2$. If $p$ is an infinite singular point of the field $\mathcal{Y}_i$ there exists a projective extension on which $p$ is a singular point and it is on the line at infinity. We denote by $\tilde{X}$ the field that has the point $p$ as a singular point of such a projective extension. Thus, from Remark~5 we have the following

**Remark 6** The Poincaré index of any singular point that is in the line at infinity of the field $\tilde{X}$ is equal to $\frac{1}{2}$ if $n$ is odd, and it is 1 when $n$ is even, figure 5.

In figure 6 we draw the foliation of the field $\mathcal{Y}_i$ in both hemispheres for the cubic polynomial $f(x, y) = x^2 + y^2 + y(x^2 + y^2)$, and in figure 7 that of quartic polynomial $g(x, y) = y(x + 3)(x - y)(y + x - 3)$.

![Figure 6: Foliation of asymptotic lines for a polynomial of degree 3.](image)

![Figure 7: Foliation of asymptotic lines for a polynomial of degree 4.](image)

**Proof of Theorem 3.** Let $p \in S^2 \subset \mathbb{R}^3 = \{(u, v, \omega)\}$ be an infinite singular point of the fields of directions $\mathcal{Y}_1, \mathcal{Y}_2$ defined by the quadratic form (3) on the unitary sphere $S^2$. According to Lemma 1 the homogeneous polynomial $f_n(x, y)$ has a real linear factor $l(x, y)$ that defines the point $p$. After a suitable linear change of coordinates on the $xy$-plane, we can suppose that $l(x, y) = y$. So, $p = (1, 0, 0)$ and $f_n$ can be written as

$$f_n(x, y) = y \left( \sum_{i=0}^{n-1} a_{i,n-i} x^i y^{n-1-i} \right), \text{ with } a_{n-1,1} \neq 0. \quad (5)$$

Consider the chart $u = 1$ in $\mathbb{R}^3 = \{(u, v, \omega)\}$. In this chart, the projections of $\mathcal{Y}_1$ and $\mathcal{Y}_2$ restricted to the set $\{(u, v, \omega) \in S^2 | u > 0\}$ are described by the quadratic equation

$$(dv \quad d\omega) \left( \begin{array}{cc} \omega^2 F_{vv} (1, v, \omega) & \omega B (1, v, \omega) \\ \omega B (1, v, \omega) & S (1, v, \omega) \end{array} \right) \left( \begin{array}{c} dv \\ d\omega \end{array} \right) = 0. \quad (6)$$

The two fields of directions defined by equation (6) are

$$R_k (v, \omega) dv + \tilde{S} (v, \omega) d\omega = 0, \quad k = 1, 2. \quad (7)$$
where
\[ \tilde{S}(v, \omega) = S(1, v, \omega) \]
\[ R_k(v, \omega) = -2\omega B(1, v, \omega) + 2(-1)^k \sqrt{\omega^2 HF(v, \omega)}, \]
\[ HF(v, \omega) = \left( F_{uv}^2(u, v, \omega) - F_{uu}(u, v, \omega) F_{vv}(u, v, \omega) \right) |_{u=1}. \]

The origin of the \( v\omega \)-plane is a singular point of both fields of lines and the Poincaré indexes of both fields at this point coincide. We denote by \( G_1 \) and \( G_2 \) the foliations of such fields.

The proof of this Theorem is based on the following geometric idea. Consider the sets
\[ W_U = \{ (v, \omega) \in \mathbb{R}^2 | \omega > 0 \} \quad \text{and} \quad W_L = \{ (v, \omega) \in \mathbb{R}^2 | \omega < 0 \}. \]

The key point is to prove that a neighborhood \( W \subset \mathbb{R}^2 \) exists containing the origin in which one of the two foliations, \( G_1 \) for example, is tangent to a vector field having a node at \((0, 0)\) in \( W \cap W_U \) and \( G_1 \) is tangent to a nonsingular vector field in \( W \cap W_L \). Simultaneously, we will have that the foliation \( G_2 \) is tangent to the same vector fields, but in the sets \( W \cap W_L \) and \( W \cap W_U \), respectively.

From the expression of the fields of lines described in (7) we define the following vector fields on the \( v\omega \)-plane having similar qualitative behaviors
\[ Y_k(v, \omega) = \left( 2\tilde{S}(v, \omega), \omega T_k(v, \omega) \right), \quad k = 1, 2, \]
where
\[ T_k(v, \omega) = -2B(1, v, \omega) + 2(-1)^k \sqrt{HF(v, \omega)}. \]

Remark 7 In a punctured neighborhood of the origin the foliation \( G_1 \) is tangent to the vector field \( Y_1 \) if \( \omega > 0 \), and it is tangent to the vector field \( Y_2 \) if \( \omega < 0 \). Respectively, the foliation \( G_2 \) is tangent to the vector field \( Y_2 \) when \( \omega > 0 \), and it is tangent to the vector field \( Y_1 \) for \( \omega < 0 \).

Before describing the topological type of the singular point \((0, 0)\) of the fields \( Y_1 \) we state the following

Remark 8 These expressions, \( T_1, T_2 \) and \( \tilde{S} \) satisfy
\[ T_1(v, \omega) T_2(v, \omega) = 4\tilde{S}(v, \omega) \left( \sum_{i=0}^{n} \omega^{n-i} \frac{\partial^2 f_i}{\partial v^2}(1, v) \right). \] (8)

Moreover, if \( a_{n-1,1} > 0 \), then \( T_1(0, 0) = 0 \) and \( T_2(0, 0) = 4(n-1)a_{n-1,1} \). In case \( a_{n-1,1} < 0 \), \( T_2(0, 0) = 0 \) and \( T_1(0, 0) = 4(n-1)a_{n-1,1} \).
Proof. A straightforward computation proves the equality (8). Because

\[-B(0,0) = (n-1)\frac{\partial f_n}{\partial v} |_{(1,0)} = (n-1)a_{n-1,1} \quad \text{and} \quad \]

\[HF(0,0) = Hess f_n (1,0) = -(n-1)^2 a_{n-1,1},\]

we have that \( T_k (0,0) = 2(n-1) a_{n-1,1} + 2(-1)^k (n-1) |a_{n-1,1}| \).

\[\square\]

Remark 9 If \( a_{n-1,1} \) is positive (negative), then the point \((0,0)\) is a singular point of type node of the vector field \( Y_2 \) (respectively \( Y_1 \)).

Proof. Suppose \( a_{n-1,1} > 0 \). We analyse the linear part of the vector field \( Y_k (v, \omega) \) at \((0,0)\). In a matrix form, \( DY_k (0,0) = \)

\[
= \left( \begin{array}{c}
2 \frac{\partial}{\partial v} \tilde{S}(v, \omega) \\
\frac{\partial}{\partial \omega} T_k (v, \omega) + T_k (v, \omega)
\end{array} \right) |_{(0,0)}
\]

\[
= \left( \begin{array}{c}
2(n-1) \frac{\partial}{\partial v} \sum_{i=0}^n \omega^{n-i} f_i (1, v) \\
\frac{\partial}{\partial \omega} T_k (v, \omega) + T_k (v, \omega)
\end{array} \right) |_{(0,0)}
\]

\[
= \left( \begin{array}{c}
2(n-1) a_{n-1,1} \\
2(n-1) a_{n-1,1} + 2(n-1) (-1)^k \omega |a_{n-1,1}| \end{array} \right).
\]

Therefore, the matrix \( DY_1 (0,0) \) has two nonzero real eigenvalues with the same sign if \( a_{n-1,1} < 0 \), and also in the case of \( DY_2 (0,0) \) if \( a_{n-1,1} > 0 \).

\[\square\]

Lemma 2 If \( a_{n-1,1} \) is positive (negative), then \( Y_1 \) (respectively \( Y_2 \)) is tangent to a nonsingular vector field in a punctured neighborhood of \((0,0)\).

Proof. Suppose \( a_{n-1,1} > 0 \). By Remark [8] we know that \( T_2 (0,0) \neq 0 \). We define the vector field

\[ Z_1 (v, \omega) = \left( T_2 (v, \omega), 2\omega \left( \sum_{i=2}^n \omega^{n-i} \frac{\partial^2}{\partial v^2} f_i (1, v) \right) \right). \]

The origin is a nonsingular point of this vector field. By [8] this vector field satisfies the equality:

\[ T_2 (v, \omega) Y_1 (v, \omega) = 2 \tilde{S} (v, \omega) Z_1 (v, \omega). \]

This relation implies that \( Z_1 \) is tangent to the foliation of \( Y_1 \).

\[\square\]
5 Upper bounds for the number of special parabolic points

Let $f \in \mathbb{R}[x, y]$ be a polynomial whose graph $S_f$ is generic and consider a field of asymptotic directions $X_i$ defined by the second fundamental form $\Pi_f = 0$. We denote by $\tilde{X}$ its extension to $\mathbb{R}P^2$ and by $\text{Sing}(\tilde{X})$ the set of singular points of $\tilde{X}$. In subsection 2.2, we defined on $\mathbb{R}P^2$ a nonorientable surface $B^-$ and an orientable surface $B^+$ whose boundaries are the projective Hessian curve of $f$.

In the first part of this chapter, we prove a formula that relates the Euler characteristic of the surface $B^\pm$ with the Poincaré index of the singularities of $\tilde{X}$ when it is defined on $B^\pm$, respectively. In the second part, as an application, we give an upper bound for the number of special parabolic points when the projective Hessian curve of $f$ is constituted only by exterior ovals. Before stating our results we introduce some definitions.

The tangency of the asymptotic line with the Hessian curve of $f$ at a special parabolic point is either, exterior or interior. In the first case, we say that the SP point has an interior tangency and in the second case, an exterior tangency.

**Theorem 4** Let $f \in \mathbb{R}[x, y]$ be a polynomial of degree $n$ whose graph $S_f$ is generic and its projective Hessian curve is not tangent to the line at infinity. Assume that $f_n$ and $\text{Hess}f_n$ don't have linear common factors. If $\tilde{X}$ is an extension to $\mathbb{R}P^2$ of a field of asymptotic directions, then

\[ \sum_{\xi \in \text{Sing}(\tilde{X})} \text{Ind}(\xi) = \chi(B^-) + \frac{P_i - P_e}{2} \text{ if } \tilde{X} \text{ is defined on } B^- . \]

\[ \sum_{\xi \in \text{Sing}(\tilde{X})} \text{Ind}(\xi) = \chi(B^+) + \frac{P_i - P_e}{2} \text{ if } \tilde{X} \text{ is defined on } B^+ . \]

In both cases, $P_i$ denotes the number of special parabolic points with an interior tangency and $P_e$, with an exterior tangency.

**Proof.** Because $f_n$ and $\text{Hess}f_n$ has no common factor, all infinite singular points are infinite hyperbolic, Theorem 2. Moreover, there are no SP points on the line at infinity: the projective Hessian curve of $f$ is not tangent to the line at infinity.

i) The smooth surface $B^-$ contains a finite number of orientable connected components denoted by $D_1, \ldots, D_s$ and a connected component $M$ homeomorphic to a closed Möbius strip with a finite number of open discs removed.
For \( l = 1, \ldots, s \), we denote by \( P_{Di} \) and \( P_{De} \) the number of SP points having an interior and exterior tangency on the boundary of \( D_l \), respectively.

Poincaré-Hopf’s Theorem for surfaces with boundary implies

\[
\sum_{l=1}^{s} \left( \sum_{\xi \in \text{Sing}(\tilde{X})} \text{Ind}(\xi) \right) = \sum_{l=1}^{s} \chi(D_l) + \sum_{l=1}^{s} \frac{P_{Di} - P_{De}}{2}.
\]  \( (9) \)

To conclude the proof, we shall prove a version of Poincaré-Hopf’s Theorem for the nonorientable surface \( M \).

**Lemma 3** Let \( f \in \mathbb{R}[x,y] \) be a polynomial of degree \( n \) such that its graph is generic and its projective Hessian curve is not tangent to the line at infinity. Assume that \( f_n \) and \( \text{Hess} f_n \) have no common factor. If \( \tilde{X} \) is an extension to \( \mathbb{R}P^2 \) of a field of asymptotic directions, then

\[
\sum_{\xi \in \text{Sing}(\tilde{X})} \text{Ind}(\xi) = \chi(M) + \sum_{\xi \in M} \frac{P_{Mi} - P_{Me}}{2},
\]

where the values \( P_{Mi} \) and \( P_{Me} \) denote, respectively, the numbers of special parabolic points with interior and exterior tangency on the boundary of \( M \).

**Proof.** The projective field of asymptotic directions \( \tilde{X} \) is the restriction of a field \( Y_i \) to a hemisphere of the unitary sphere. Consider the field \( Y_i \) on this sphere. It is defined on an orientable surface \( D_M \subset S^2 \) that is a double covering of \( M \), so that \( \chi(D_M) = 2\chi(M) \). By considering the Poincaré-Hopf Theorem for the field \( Y_i \),

\[
\sum_{\xi \in \text{Sing}(\tilde{X})} \text{Ind}(\xi) = \chi(D_M) + \sum_{\xi \in \tilde{X}} \frac{P_{Di} - P_{De}}{2}.
\]  \( (10) \)

Since the number of asymptotic lines tangent to the boundary of \( D_M \) is twice the number of those lines tangent to the boundary of \( M \), we have that \( P_{Di} - P_{De} = 2(P_{Mi} - P_{Me}) \). Moreover, by Remark 5

\[
\sum_{\xi \in \text{Sing}(\tilde{X})} \text{Ind}(\xi) = 2 \sum_{\xi \in \text{Sing}(\tilde{X})} \text{Ind}(\xi).
\]

By replacing these expressions in (10) we obtain the desired equality. \( \square \)
Since $B^- = \mathbb{M} \sqcup D_1 \sqcup \cdots \sqcup D_s$, the proof of Theorem 4 concludes with Lemma 3 and equality (9). □

George Salmon proves in 1927 that a generic algebraic surface of degree $n$ in $\mathbb{C}P^3$ has $2n(n-2)(11n-24)$ SP points [19]. This number is an upper bound for the number of special parabolic points on a generic algebraic surface in $\mathbb{R}P^3$. When the graph of a polynomial $f \in \mathbb{R}[x,y]$ of degree $n$ is generic, an upper bound for the number of SP points is given ([9], Theorem 5), namely,

$$\# \{\text{SP points in } S_f\} \leq (n-2)(5n-12). \tag{11}$$

**Corollary 2** Let $f \in \mathbb{R}[x,y]$ be a polynomial of degree $n$ such that $S_f$ is generic and $f_n$ has no common factor with $Hess f_n$. Suppose that the projective Hessian curve is not tangent to the line at infinity. If the polynomial $f_n$ has $k$ real linear factors, then

$$P_i \leq \frac{(n-2)(8n-21) + k}{2} \quad \text{and} \quad P_e \leq 1 + \frac{(n-2)(8n-21) - k}{2}.$$  

**Proof.** If the set $b^+ \cap \mathbb{R}^2$ is hyperbolic, then, by Theorem 4 we have that

$$\frac{P_i - P_e}{2} = \sum_{\xi \in Sing(\tilde{\times})} Ind(\xi) - \chi(B^\pm). \tag{12}$$

Since $\chi(B^+) = 1 - \chi(B^-)$, Petrowsky’s Theorem implies (subsection 2.2)

$$-\frac{3(n-2)(n-3)}{2} - 1 \leq -\chi(B^\pm) \leq \frac{3(n-2)(n-3)}{2}. \tag{13}$$

According to Lemma 1 and Theorem 3 we obtain $\sum_{\xi \in Sing(\tilde{\times})} Ind(\xi) = \frac{k}{2}$. Therefore, replacing this equality and the inequalities of (13) in (12) we get

$$-3(n-2)(n-3) - 2 + k \leq P_i - P_e \leq 3(n-2)(n-3) + k. \tag{14}$$

The proof follows from inequalities (11) and (14). □

In the case the Hessian curve of $f$ is a convex compact curve and the set $b^- \cap \mathbb{R}^2$ is hyperbolic, the second author of this paper joined L.I. Hernández-Martínez and F. Sánchez-Bringas to prove that $n(3n-14) + 18$ is an upper bound for the number of SP points lying on the boundary of the unbounded connected component $C_u$ ([10], Theorem 10). In the following result, we improve such bound under different assumptions and we analyse the unbounded case: we give an upper bound for the number of special parabolic points that are on the boundary of $\mathbb{M}$.  

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Theorem 5 Let \( f \in \mathbb{R}\{x,y\} \) be a polynomial of degree \( n \) whose graph \( S_f \) is generic and such that \( f_n \) has no common factor with \( \text{Hess} f_n \). Suppose that the projective Hessian curve of \( f \), constituted only by exterior ovals, is convex and it is not tangent to the line at infinity. If \( b^- \cap \mathbb{R}^2 \) is hyperbolic and the polynomial \( f_n \) has \( k \) real linear factors, then the maximal number of special parabolic points is \( 3(n-2)(n-3) + k \).

Proof. Since \( \tilde{X} \), the extension to \( \mathbb{R}P^2 \) of a field of asymptotic lines, is defined on \( B^- \), the expression \( P_i - P_e \) satisfies the second inequality of (14), that is, \( P_i - P_e \leq 3(n-2)(n-3) + k \). Because the projective Hessian curve of \( f \) is convex and the set \( b^- \cap \mathbb{R}^2 \) is hyperbolic, all special parabolic points have an interior tangency. Therefore, \( P_{e}^{B^-} = 0 \) and \( P_{i}^{B^-} \) equals the total number of SP points. \( \square \)

References

[1] Arnold V. I., Remarks on Parabolic Curves on Surfaces and the Higher-Dimensional Möbius-Sturm Theory, Funct. Anal. Appl. 31 No. 4 (1997), 227-239.

[2] Arnold V. I., On the problem of realization of a given Gaussian curvature function, Topol. Method Nonl. An. 11, no. 2 (1998), 199-206.

[3] Arnold V. I., Astroidal Geometry of Hypocycloides and the Hessian topology of Hyperbolic polynomials, Russ. Math. Surv. 56, no.6 (2001), 1019-1083.

[4] Banchoff T., Thom R., Sur les points paraboliques des surfaces: erratum et compléments, C. R. Acad. Sci. Paris, Série A, 291 (1980), 503-505.

[5] Bertrand B. & Brugallé E., On the number of connected components of the parabolic curve, C. R. Math. Acad. Sci. Paris 348, no. 5-6 (2010), 287–289.

[6] Dara L., Singularités génériques des équations différentielles multiformes, Bol. Soc. Brasil. Mat., 6 (1975), 95-128.

[7] Davydov A. A., Qualitative Theory of Control Systems, Translations of Math. Monographs. Amer. Math. Soc. 141, 1994.

[8] Guínez V., Nonorientable Polynomial Foliations on the Plane, J. Differ. Equations 87 (1990), 391-411.
[9] Hernández-Martínez L. I., Ortiz-Rodríguez A. & Sánchez-Bringas F., On the Affine Geometry of the Graph of a Real Polynomial, J. Dyn. Control Syst. 18, no.4 (2012), 455-465.

[10] Hernández-Martínez L. I., Ortiz-Rodríguez A. & Sánchez-Bringas F., On the Hessian geometry of a real polynomial hyperbolic near infinity. Adv. Geom. 13, no.2 (2013), 277-292.

[11] Kergosien, Y.L., Thom, R. Sur les points paraboliques des surfaces, C. R. Acad. Sci. Paris, Série A, t. 290 (1980), 705-710.

[12] Kulikov V. S., Calculation of singularities of an imbedding of a generic algebraic surface in projective sapace $P^3$, Funct. Anal. Appl. 17, no.3 (1983), 15-27.

[13] Landis E. E., Tangential singularities, Func. Anal. Appl. 15, no.2 (1981), 103-114.

[14] Poincaré H., Mémoire sur les courbes définies par une équation différentielle, J. Math. Pure Appl. 7 (1881), 375-422.

[15] Panov D. A., Especial points of surfaces in a three-dimensional projective space, Funct. Anal. Appl. 34, no.4 (2000), 276-287.

[16] Petrowsky I., On The Topology Of Real Plane Algebraic Curves, Ann. Math. 39, no. 1 (1938), 189-209.

[17] Platonova O. A., Singularities of a relative position of a surface and a line, Russ. Math. Surv. 36, no.1 (1981), 248-249.

[18] Ragsdale V., On the arrangement of the real branches of plane algebraic curves, Am. J. Math. 28, (1906), 377-404.

[19] Salmon G., A treatise in Analytic Geometry of Three Dimensions. Chelsea Publ. 1927.

[20] Segre B., THE NON-SINGULAR CUBIC SURFACES: a New Method of Investigation with Special Reference to Questions of Reality, Clarendon Press, 1942.

[21] Uribe-Vargas R. A projective invariant for swallowtails and godrons, and global theorems on the flecnodal curve, Mosc. Math. J. 6, no.4 (2006), 731-768.
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