MIXING IMPLIES EXPONENTIAL MIXING AMONG CODIMENSION ONE HYPERBOLIC ATTRACTORS AND ANOSOV FLOWS

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Abstract. On a compact manifold of any dimension $d \geq 3$, we show that joint non-integrability of the stable and unstable foliation of a hyperbolic attractor with one-dimensional expanding direction, for a vector field of class $C^2$, implies exponential mixing with respect to its physical measure.

Consequently, the set of Axiom A vector fields which mix exponentially with respect to the physical measure of its non-trivial attractors contains a $C^1$-open and $C^2$-dense subset of the set of all Axiom A vector fields.

Moreover, for codimension one $C^2$ Anosov flows in any dimension $d \geq 3$, if the flow mixes with respect to the unique physical measure, then the flow mixes exponentially, proving the Bowen-Ruelle conjecture in this setting.

Contents

1. Introduction 1
1.1. Statement of results 4
1.2. Organization of the text 5
2. Axiom A attractors and Anosov flows 6
2.1. Generalities on non-trivial hyperbolic attractors 6
2.2. Cross-sections, Poincaré maps and quotient over center-stable leaves 7
2.3. Representation as a hyperbolic skew-product suspension semiflow 8
3. Exponential mixing for generalized hyperbolic skew-product semiflows 12
3.1. Suspension semiflow over the expanding full branch Markov map 12
3.2. Hyperbolic skew-product semiflow with constant roof on stable leaves 14
3.3. Generalized hyperbolic skew-product semiflow 14
4. Joint non-integrability of stable and unstable foliation 17
References 19

1. INTRODUCTION

Anosov flows [1] and the Axiom A flows of Smale [31] have been extensively studied since the 1960s and are now relatively well understood, with the relevant exception of their rate of mixing. Exponential mixing describes the rate at which
initial information is lost under the action of the dynamics, and is crucial for establishing many other quantitative statistical properties [24] and applications of Anosov flows in other fields, e.g as models in semiclassical analysis: see [20] and references therein.

Let $M$ be a Riemannian manifold (dimension $d \geq 3$), $X$ a $C^2$ vector field on $M$ and let $\Lambda$ be an invariant, closed, topologically transitive, locally maximal hyperbolic set, that is, a basic set for the (Axiom A) flow $(X^t)_{t \in \mathbb{R}}$ of $X$. The set $\Lambda$ is an attractor if there exists a neighborhood $U$ of $\Lambda$ so that $\Lambda = \bigcap_{t \geq 0} X^t U$, and is non-trivial if it is neither an equilibrium nor a periodic solution. If such $\Lambda$ is the whole ambient manifold $M$, then the vector field is Anosov.

Here we focus on the physical/SRB measure: the unique invariant probability measure $\mu$ which is characterised by having absolutely continuous conditional measures on unstable manifolds. It is also the unique physical measure and the Gibbs measure associated to the potential chosen as minus the logarithm of the unstable Jacobian [35]. We say that $\mu$ is mixing if the correlation function $\rho_{\phi,\psi}(t) := \int_{\Lambda} \phi \circ X^t \cdot \psi \, d\mu - \int_{\Lambda} \phi \, d\mu \cdot \int_{\Lambda} \psi \, d\mu$ tends to zero for $t \to +\infty$, for all bounded measurable observables $\phi, \psi : U \to \mathbb{R}$. We say that it mixes exponentially if, for any fixed Hölder exponent $\alpha \in (0,1)$, there exist $\gamma, C > 0$ such that, for all $\phi, \psi$ which are $\alpha$-Hölder on $U$, $|\rho_{\phi,\psi}(t)| \leq C \|\phi\|_{C^\alpha} \|\psi\|_{C^\alpha} e^{-\gamma t}$ for all $t \geq 0$.

The study of the rate of mixing for hyperbolic systems goes back to the work of Sinai [30] and Ruelle [28] in the 1970s and a wealth of results were obtained for smooth maps during the subsequent years. For smooth continuous time systems the situation is more subtle.

Anosov [1] proved that geodesic flows for negatively curved compact Riemannian manifolds are always mixing and showed the Anosov alternative: a transitive volume-preserving Anosov flow is either mixing or the suspension of an Anosov diffeomorphism by a constant roof function. This was extended by Plante [25] who showed that codimension-one Anosov flows are mixing if, and only if, they are robustly mixing – among this family of flows, mixing is equivalent to the joint non-integrability of the stable and unstable foliations which is a $C^1$-open condition.

Not all Anosov nor Axiom A flows mix exponentially, as shown by Ruelle [29] and Pollicott [26], who obtained examples with arbitrarily slow mixing rates. The Bowen-Ruelle conjecture states that each mixing Anosov flow mixes exponentially [13].

Only in the final years of the XXth century a significant breakthrough was obtained in the fundamental work of Dolgopyat [18] building on the work of Chernov [16]. There exponential mixing was obtained for the first time for transitive Anosov flows with respect to physical measures under rather strong assumptions: global smoothness of stable and unstable foliations and their uniform non-integrability. Both stable and unstable foliations are always Hölder but typically no better than Hölder: generically a smooth perturbation will destroy the Lipschitz regularity of at least one of these foliations [22]. Dolgopyat conjectured [19, Conjecture 1] that the set of exponentially mixing flows contains a $C^r$-open, $C^r$-dense subset of the set of all Axiom A flows.

Field, Melbourne and Torok [21] refining Dolgopyat techniques obtained $C^2$-open, and $C^r$-dense sets of $C^r$-Axiom A flows ($r \geq 2$) for which each non-trivial basic set is rapid mixing, that is, $\rho_{\phi,\psi}(t)$ converges to zero faster than any polynomial in $t$ as $t$ grows to infinity.
Exponential mixing was proved for $C^2$ uniformly expanding surface suspension semiflows by Baladi and Vallée [10] under the assumption that the return time function is not cohomologous to a piecewise constant function. This was extended to arbitrary dimension by Avila, Gouëzel, and Yoccoz [9]. The author together with Melbourne [4] extended [10] relaxing $C^2$ to $C^{1+\alpha}$, while Butterley and Var [15] extended [9] also to $C^{1+\alpha}$ smoothness under broader geometrical conditions. For more on mixing rates of hyperbolic flows we refer the reader to the introductions of [23, 21, 15].

Recently, it was shown that exponential mixing can be obtained for Axiom A and Anosov vector fields assuming only $C^{1+\alpha}$-smoothness of the stable foliation when the unstable direction is one-dimensional. In this setting we can represent the flow as a hyperbolic skew-product suspension semiflow which admits an embedded uniformly expanding surface suspension semiflow. Since this smoothness property of the stable foliation is $C^{1}$-open under certain conditions, open families of $C^{r}$ Anosov [15] and Axiom A [2, 3] vector fields where obtained which mix robustly on its non-trivial attractors (as well as robustly mixing examples on singular attractors and attracting sets [7, 8, 6]).

Tsujii [32] proves that a $C^{r}$ open and dense subset in the space of zero divergence Anosov vector fields ($r \geq 3$), on a compact 3-dimensional manifold, mix exponentially. Recently, Tsujii and Zhang [33] claim that, among $C^{\infty}$ transitive Anosov 3-vector fields, mixing implies exponential mixing.

The purpose of this paper is to show that a $C^{1}$ open and $C^{2}$ dense subset of Axiom A vector fields mix exponentially with respect to the physical/SRB measure of its non-trivial attractors whenever the unstable bundle is one-dimensional. In particular, we obtain a $C^{1}$ open and $C^{2}$ dense subset of Axiom A vector fields of compact 3-manifolds which mix exponentially on its non-trivial attractors. This greatly extends the robustly mixing families of Axiom A vector fields obtained by the author together with Butterley and Varandas in [2, 3].

We stress that we do not require any special regularity of the invariant foliations, while Dolgopyat’s original argument [18] required $C^{1}$ regularity for both the stable and unstable foliations, and the previously cited results on robust exponential mixing [7, 2, 4, 9, 15, 6] required $C^{1+\alpha}$ regularity of the stable foliation only.

Since codimension one Anosov vector fields are transitive on higher-dimensional manifolds [34], and satisfy the Anosov alternative [25], these results provide a proof of the Bowen-Ruelle conjecture in full generality for Anosov vector fields with one-dimensional unstable direction. This extends the results of Tsujii and Zhang [32, 33]. As a consequence, a $C^{1}$ open and $C^{2}$ dense subset of Anosov vector fields $X$ on any compact 4-manifold mixes exponentially, either for the physical/SRB measure of the flow of $X$ or for the physical/SRB measure of the flow of $-X$ (these measures coincide when $X$ is volume preserving). This also extends the results of Butterley and Var [15] to larger families of Anosov vector fields.

This result paves the way to obtaining open and dense exponential mixing for all equilibrium states with respect to Hölder potentials supported on non-trivial hyperbolic attractors with one-dimensional unstable direction, extending and adapting the recent results from Daltro and Varandas [17]. For arbitrary codimension hyperbolic attractors or Anosov flows, the proof breaks down due to the crucial property that stable holonomies are $C^{1+\alpha}$ smooth, which does not hold in higher codimensions; see Subsection 2.2, [5, Section 7] and also [27] and [22].
1.1. **Statement of results.** A compact $X^t$-invariant subset $\Lambda \subset M$ is called **uniformly hyperbolic** if there exists a $DX^t$-invariant splitting $T\Lambda M = E^s \oplus E^X \oplus E^u$ (here $E^X$ denotes the one-dimensional subbundle tangent to the flow direction) and there are a Riemannian metric and a constant $\lambda > 0$ so that:

$$\|DX^t(x)v\| \leq e^{-\lambda t} \|v\|$$

for every $v \in E^s_x$, and

$$\|DX^{-t}(x)v\| \leq e^{-\lambda t} \|v\|$$

for every $v \in E^u_x$ and all $t \geq 0$. In the special case that $\Lambda = \{\sigma\}$, then $\sigma$ is a fixed point and the hyperbolic splitting can be read simply as $T_\sigma M = E^s \oplus E^u$. The vector field is **Anosov** if $M$ is a hyperbolic set. A hyperbolic set $\Lambda$ is a **basic set** if it is transitive, the periodic orbits of $X^t|_\Lambda$ are dense in $\Lambda$ and there exists an open set $U \supseteq \Lambda$ with $\Lambda = \bigcap_{t \in \mathbb{R}} X^t(U)$. A basic set $\Lambda$ is an **attractor** if there exists an open set $U \supseteq \Lambda$ so that $\Lambda = \bigcap_{t > 0} X^t(U)$. The field $X$ is **Axiom A** if its non-wandering set is a hyperbolic set and coincides with the closure of the set of critical elements, i.e. the set formed by periodic orbits and singularities. The **Spectral Decomposition Theorem** ensures that the non-wandering set of an Axiom A flow consists of a finite number of disjoint hyperbolic basic sets.

Given an Axiom A flow $X^t : M \to M$ associated to a vector field $X$ we consider the $C^r$ distance on the space of $C^r$-vector fields $\mathfrak{X}^r(M)$, that induces a natural distance on the space of flows.

**Theorem A.** Given any Riemannian manifold $M$ of dimension $d \geq 3$ and any $r \geq 2$, the $C^1$-open subset of $C^r$-vector fields $\mathcal{U} \subset \mathfrak{X}^r(M)$ such that each $X \in \mathcal{U}$ generates an Axiom A flow endowed with a non-trivial attractor with one-dimensional unstable bundle, contains a $C^1$-dense and $C^1$ open subset $\mathcal{V} \subset \mathcal{U}$ of vector fields which mix exponentially with respect to the unique physical measure on the attractor.

This is the first result on the denseness of robust exponential mixing in the Axiom A setting in the $C^2$ topology (among attractors with one-dimensional unstable bundle).

The strategy of the proof is based on a refinement of the construction of $C^2$ open sets of exponential mixing singular flows and Axiom A flows; see [2, 3, 6] and references therein. Theorem A is a consequence of the following.

**Theorem B.** Suppose that $X$ is a $C^2$ Axiom A vector field, $\Lambda$ is a non-trivial attractor whose unstable bundle has dimension one. If the stable and unstable foliations are not jointly integrable, then the flow mixes exponentially with respect to the unique physical measure supported on $\Lambda$.

Joint non-integrability of stable and unstable foliations may be interpreted in different ways. On the one hand, the stable and unstable foliations of an Axiom A flow are always transversal, consequently, if they are jointly integrable, this provides a codimension one invariant foliation which is transversal to the flow direction. On the other hand, if there exists a codimension one invariant foliation which is transversal to the flow direction, then this foliation must be subfoliated by both the stable and unstable foliations which must therefore be jointly integrable. In this case it is known [21, Proposition 3.3] that the flow is (bounded-to-one) semiconjugate to a locally constant suspension over a subshift of finite type, which need not mix, or may mix slower than exponentially.

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1 Recall that the non-wandering set of the flow consists of all points $x \in M$ so that $V \cap X^t V \neq \emptyset$ for every open neighborhood $V$ of $x$ and every large $|t|$. 
Moreover, since joint non-integrability of the stable and unstable foliations implies mixing [21, Remark 3.6(1)], the previous results can be stated informally as the Bowen-Ruelle conjecture: ”mixing implies exponentially mixing among non-trivial hyperbolic attractors of $C^2$ vector fields, with respect to the physical measure”.

The proof extends naturally to the Anosov flow setting, providing a proof of the Bowen-Ruelle conjecture for several classes of Anosov flows, since

- a codimension one\(^2\) mixing Anosov flow has stable and unstable foliations which are not jointly integrable, by Plante [25];
- the set of Anosov flows where the stable and unstable bundles are not jointly integrable is $C^1$-open and $C^r$-dense in the set of all Anosov flows, cf. Field-Melbourne-Török [21] and references therein; and
- codimension-one Anosov flows on higher dimensional manifolds (dim $M > 3$) are transitive, after Verjovsky [34].

**Theorem C.** Let $X \in X^2(M)$ be an Anosov vector field whose unstable bundle is one-dimensional. If the corresponding flow is mixing with respect to the unique physical measure, then the flow mixes exponentially.

In particular, we obtain the following corollary.

**Corollary A.**

(A) Among codimension one Anosov vector fields, on compact manifolds $M$ with dimension $d \geq 4$, there exists a $C^1$-open and $C^2$-dense subset whose vector fields $X$ mix exponentially, either for the physical/SRB measure of the flow of $X$, or for the physical/SRB measure of the flow of $-X$ (these coincide when $X$ is volume-preserving).

(B) There exists a $C^1$-open and $C^2$-dense subset of 3-dimensional transitive Anosov vector fields which mix exponentially with respect to the unique physical/SRB measure.

(C) There exists a $C^1$-open and $C^2$-dense subset of 4-dimensional Anosov vector fields $X$ which mix exponentially with respect to the unique physical/SRB measure, either for the flow of $X$, or for the flow of $-X$.

Item (B) above extends the recent result of Tsujii and Zhang [33] claimed only for $C^\infty$ Anosov vector fields (although valid for all equilibrium states with respect to Hölder continuous potentials); and also a previous result on exponential mixing for the volume measure on 3-dimensional conservative Anosov flows of class $C^r (r \geq 3)$ by Tsujii [32]. Items (A) and (C) extend the results from Butterley-War [15] from a neighborhood of conservative flows to an open and dense subset of all 4-dimensional Anosov flows and all codimension one Anosov flows.

1.2. **Organization of the text.** The proof is described in Section 2 and involves quotienting along stable manifolds of a well-chosen Poincaré section to reduce to the case of a suspension semiflow over a skew-product on an expanding Markov map. We can then apply the results of [4, 6], which implies exponential mixing for the semiflow unless the return time function is cohomologous to a piecewise constant function. This is then related to the exponential mixing for the original flow. In Section 3, we extend the main exponential mixing result from [4, 6] showing that the roof function needs not be constant on contracting fibers. This enables us to avoid the smoothness assumption of the stable foliation which was a key

\(^2\)That is, \(\dim \mathbb{E}^s = 1\) or \(\dim \mathbb{E}^u = 1\).
assumption in [2, 4, 6]. Finally, in Section 4, we relate the cohomology property mentioned above with the joint integrability of the stable and unstable foliations. Since the non-joint integrability of stable and unstable foliations is an open and dense condition (see e.g. [21, Remark 1.10] and references therein), then we obtain open and dense exponential mixing, proving Theorems A and B. Theorem C is a consequence of Theorem B with $\Lambda = M$, using the global properties of codimension one Anosov vector fields.

2. Axiom A attractors and Anosov flows

To prove Theorem B we follow the same general strategy presented at [2, Section 2], whose notation we keep for the most part.

We assume that $X$ is a $C^2$ vector field and $\Lambda \subset M$ is a hyperbolic basic set which is also an attractor. We additionally assume that the unstable bundle on $\Lambda = M$.

2.1. Generalities on non-trivial hyperbolic attractors. For $x \in \Lambda$ we denote the local (strong) stable manifold of $x$ by $W^s_\varepsilon(x) = \{ y \in M : d(X^t(y), X^t(x)) \leq \varepsilon, \forall t \geq 0 \text{ and } d(X^t(y), X^t(x)) \to 0 \text{ as } t \to +\infty \}$. This provides a collection of forward invariant embedded submanifolds, that is, $X^t(W^s_\varepsilon(x)) \subset W^s_\varepsilon(X^t(x))$ for every $x \in \Lambda$ and $t \geq 0$. The local centre-stable manifold of $x$ is given by $W^{cs}_\varepsilon(x) = \cup_{|s| \leq \varepsilon} X^t(W^s_\varepsilon(x))$. The local (strong) unstable and centre-unstable manifolds, $W^u_\varepsilon(x)$ and $W^{cu}_\varepsilon(x)$ respectively, are the corresponding local strong stable and local centre-stable manifolds for the flow $(X^{-t})$. The attractor $\Lambda$ has a local product structure, that is, there is an open neighborhood $\mathcal{J}$ of the diagonal in $M \times M$ and $\varepsilon > 0$ so that, for all $(x, y) \in \mathcal{J}_\Lambda := \mathcal{J} \cap (\Lambda \times \Lambda)$, then $W^{cu}_\varepsilon(x) \cap W^u_\varepsilon(y) \neq \emptyset$ and $W^u_\varepsilon(x) \cap W^{cx}_\varepsilon(y) \neq \emptyset$ each consist of a single point and this intersection point belongs to $\Lambda$: see e.g. [11]. We may consider the continuous maps $[\cdot, \cdot]_\Lambda : \mathcal{J}_\Lambda = \Lambda$ and $[\cdot, \cdot]_\mathcal{J} : \mathcal{J} \to \Lambda$ given by

$$ W^{cs}_\varepsilon(x) \cap W^u_\varepsilon(y) = \{ [x, y]_\Lambda \}, \quad W^u_\varepsilon(x) \cap W^{cx}_\varepsilon(y) = \{ [x, y]_\mathcal{J} \}. $$

Both the set $\mathcal{J}$ and $\varepsilon > 0$ may be chosen fixed for some $C^1$-open set $U \subset X^1(M)$ containing any given initial Axiom A flow; see e.g. [21, §4.1].

**Definition 1** ([11]). A differentiable closed $(d-1)$-dimensional disk $S \subset M$, transverse to the flow direction is called a local cross-section. A set $\mathcal{R} \subset \Lambda \cap S$ is called a rectangle if $W^{cs}_\varepsilon(x) \cap W^{cu}_\varepsilon(y) \cap \mathcal{R}$ consists of exactly one point for all $x, y \in \mathcal{R}$.

Let $\mathcal{R}_i^\dagger$ denote the interior of $\mathcal{R}_i$ as a subset of the metric space $S_i \cap \Lambda$.

**Definition 2** ([11]). A finite set of rectangles $\mathcal{R} = \{ \mathcal{R}_i \}_i$ is called a proper family (of size $\varepsilon$) if $\Lambda = \cup_{t \in [-\varepsilon, 0]} X^t(\cup_i \mathcal{R}_i) =: X^{[\varepsilon, 0]}(\cup_i \mathcal{R}_i)$, and there exist local sections $\{ S_i \}_i$ of diameter less than $\varepsilon$ such that

- $\mathcal{R}_i \subset \text{int}(S_i)$ and $\mathcal{R}_i = \mathcal{R}_i^\dagger$;
- for $i \neq j$, at least one of the sets $S_i \cap X^{[0, \varepsilon]} S_j$ and $S_j \cap X^{[0, \varepsilon]} S_i$ is empty.

Given a proper family as above, let $\Gamma := \cup_i \mathcal{R}_i$, and denote by $P$ the Poincaré return map to $\Gamma$ associated to the flow $X^t$, and by $\tau$ the return time. Although $P$ and $\tau$ are not continuous on $\Gamma$, they are continuous on

$$ \Gamma' := \{ x \in \Gamma : P^k(x) \in \cup_i \mathcal{R}_i^* \text{ for all } k \in \mathbb{Z} \}. $$
Definition 3 ([11]). A proper family \( R = \{ R_i \} \) is a Markov family if

1. \( x \in U(R_i, R_j) := \{ w \in \Gamma : w \in R_i, P(w) \in R_j \} \) implies \( S_i \cap W^c_{\varepsilon}(x) \subset U(R_i, R_j) \),

2. \( y \in V(R_k, R_i) := \{ w \in \Gamma : P^{-1}(w) \in R_k, w \in R_i \} \) implies \( S_i \cap W^u_{\varepsilon}(y) \subset V(R_k, R_i) \).

After [11, Theorem 2.4] we have that for any Markov family \( R = \{ R_i \} \) the flow is (bounded-to-one) semiconjugate to a suspension flow, with a bounded roof function also bounded away from zero, over the subshift of finite type \( \sigma_R : \Sigma_R \to \Sigma_R \) where \( \Sigma_R = \{(a_i)_i \in \mathbb{Z} : A_{a_i a_j} = 1 \ \forall i,j\} \) and \( A_{a_i a_j} = 1 \) if, and only if, there exists \( x \in \Gamma' \cap R_i \) and \( P(x) \in R_j \).

2.2. Cross-sections, Poincaré maps and quotient over center-stable leaves.

The proof of Theorem B starts by carefully choosing local cross-sections for the flow around \( \Lambda \). Let \( \rho_0 > 0 \) be the radius of injectivity of the exponential map, that is, \( \exp_{\varepsilon}^{-1} : B(x, \rho_0) \to T_x M \cap B(0, \rho_0) \) is a diffeomorphism for each \( x \) in a neighborhood \( U \) of \( \Lambda \), where the ball \( B(x, \rho_0) \) is determined by the Riemannian metric on \( M \) and the ball \( B(0, \rho_0) = \{ v \in T_x M : ||v||_x < \varepsilon \} \) by the corresponding inner product at \( T_x M \). Because \( \Lambda \) is a hyperbolic attractor, the local unstable manifold through each point of the attractor is contained within the attractor; see e.g. [12]. Hence, for any \( x \in \Lambda \) and small enough \( \varepsilon > 0 \) we have \( W^u_{\varepsilon}(x) \subset \Lambda \). The \((d-1)\)-submanifold generated by the union of images of the center-stable bundle under the exponential map through points of \( W^c_{\varepsilon}(x) \)

\[
S_{\varepsilon}(x) = \bigcup_{y \in W^c_{\varepsilon}(x)} \exp_y \left( (E^s_x \oplus \mathbb{R} \cdot X(x)) \cap B(0, \varepsilon) \right)
\]

is a local cross-section containing \( x \), for \( \varepsilon < \rho_0 \). There is no reason to expect that these local cross-sections will be foliated either by local (un)stable manifolds, even though they contain a local unstable manifold at the centre. However, since the center-stable foliation of \( X^t | U \) has \( C^2 \) leaves and \( W^c_{\varepsilon}(x) \) is a \( C^2 \)-disk in \( W^u(x) \), then \( S_{\varepsilon}(x) \) is a codimension one \( C^2 \)-hypersurface foliated by the intersection of local center-stable manifolds with \( S_{\varepsilon}(x) \), which are the \( C^2 \)-curves \( W^c_{\varepsilon}(y, S_{\varepsilon}(x)) := W^c_{\varepsilon}(y) \cap S_{\varepsilon}(x) \) for each \( y \in W^u_{\varepsilon}(x) \). We denote by \( F \) this collection of leaves that foliates \( S_{\varepsilon}(x) \), which induces a natural projection \( \pi_{\varepsilon,x} : S_{\varepsilon}(x) \to W^c_{\varepsilon}(x) \).

As a particular case of a general result [5, Section 6] for \( C^2 \) vector fields with a partially hyperbolic attracting set, we have the Hölder regularity and absolute continuity of the foliation \( F \), which enable us to obtain the following results in our low codimension setting.

Proposition 4. [5, Lemma 7.1] There exists \( 0 < \alpha < 1 \) so that the map \( \pi_{\varepsilon,x} : S_{\varepsilon}(x) \to W^c_{\varepsilon}(x) \) is \( C^{1+\alpha} \).

Remark 5. The result of Proposition 4 is essential for our argument. It is the reason for the restriction on the codimension of the stable bundle of \( \Lambda \). If \( E^s \) has codimension higher than two, the \( C^{1+\alpha} \)-smoothness of the stable holonomy \( \pi \) is expected to fail in general; see e.g. [27] and [22].

We now combine the above observations with the construction of Markov families for locally maximal hyperbolic sets to obtain.

Lemma 6. Let \( X^t : M \to M \) be an Axiom A flow and \( \Lambda \subset M \) an attractor. There exists a finite number of \( C^2 \) local cross-sections \( \mathcal{S} = \{ S_i \} \subset U \) such that the sets
$\mathcal{R}_i = S_i \cap \Lambda$ are rectangles and $\mathcal{R} = \{\mathcal{R}_i\}_i$ is a Markov family for $X^t$. Moreover, for each rectangle $\mathcal{R}_i$ there exists $x_i \in \mathcal{R}_i$ and a $C^2$-disk $\Delta_i \subset W^u_{\varepsilon}(x_i) \subset \Lambda$ such that $S_i = \cup y \Delta_i \gamma^i(y)$ where $\gamma^i(y)$ is an open subset of $W^u_{\varepsilon}(y, S_i)$ that contains $y$. In addition, the projection $\pi_i : S_i \to \Delta_i$ along the foliation $\mathcal{F}$ is $C^{1+}$ smooth.

Proof. We have shown (1) that through each point $x$ of $\Lambda$ there passes a $C^2$ codimension one disk transversal to the flow, whose diameter can be made arbitrarily small, and these disks have a $C^{1+}$ smooth projection along $\mathcal{F}$ onto the unstable disk $W^u_{\varepsilon}(x)$. This is the starting point of the proof of [11, Theorem 2.5], detailed in [11, Section 7], to show that the $X^t : \Lambda \to \Lambda$ admits a Markov family as in the statement of lemma, consisting of a finite number of rectangles of arbitrarily small size $\varepsilon$ contained in the interior of cross-sectional disks $S = \{S_i\}_i \subset U$, each endowed with an unstable disk $\Delta_i \subset S_i$ and a projection $\pi_i$, as in the statement. □

2.3. Representation as a hyperbolic skew-product suspension semiflow. Let $\mathcal{F}$ be as in Lemma 6. Now we consider the flow on $U \supset \Lambda$ as a suspension flow, Poincaré first return map $P : \mathcal{S} \to \mathcal{S}$ and Poincaré first return time $\tau : \mathcal{S} \to [\underline{\tau}, \bar{\tau}]$ for some fixed $0 < \underline{\tau} < \bar{\tau} < \infty$. The $C^2$-smooth regularity of the local cross-sections $S_i$ and the flow $X^t$ is enough to guarantee that the return map $P : \mathcal{S} \to \mathcal{S}$ is also $C^2$ on each component.

Let $\mathcal{F}_s$ denote the foliation of $U$ by local center-stable manifolds and set $\Delta = \mathcal{S} / \mathcal{F}_s$ (the quotient of $\mathcal{S}$ with respect to the local center-stable manifolds). A concrete realization of this quotient is $\Delta = \cup_i \Delta_i$. Quotienting along these manifolds, the Poincaré map $P$ induces a map $f : \Delta \to \Delta$.

Let $\mathfrak{m}$ be the normalised restriction of the Riemannian volume to the family $\Delta$ of $C^2$ curves $\{\Delta_i\}_i$. Since $\mathcal{R}$ is a Markov family, this ensures the Markov structure of $f : \Delta \to \Delta$ and hence, for each $\Delta_i$, there exists a partition $\{\Delta_{i,j}\}_j$ of a full $\mathfrak{m}$-measure subset of $\Delta_i$ such that $f : \Delta_{i,j} \to \Delta_j$ can be seen as a smooth bijection of intervals.

Because each projection $\pi_i$ is $C^{1+}$-smooth, we conclude that $f : \Delta_{i,j} \to \Delta_j$ is a $C^{1+}$ diffeomorphism for each $i$, $j$. For future convenience, let $\pi : \mathcal{S} \to \Delta$ denote the collection of the projections $\pi_i$, so that $f \circ \pi = \pi \circ P$. In fact, we may equivalently define $f := \pi \circ P$ since $P(\gamma^i(y)) \subset \gamma^i(P(y)), y \in \Delta$.

Remark 7. Let $d_s$ be the dimension of $\mathcal{F}_s$. It is not difficult to see that if the projection $\pi$ is smooth, then the foliation $\mathcal{F}$ on each element of the Markov family admits smooth charts; see e.g. the proof of [5, Theorem 6.2] in the general Hölder-continuous case. In our setting, the foliation admits $C^{1+}$ charts. In particular, this implies that we can write $P$ as a $C^{1+}$ skew-product over $f$, i.e., $P : \Delta \times \Omega \to (x, y) \mapsto (f(x), g(x, y))$, where $\Omega$ is a closed $d_s$-dimensional ball and $g : \Delta \times \Omega \to \Omega$ is also a $C^{1+}$ map. In addition, we can identify $\Delta$ with $\Delta \times \{0\} \subset \Delta \times \Omega$.

2.3.1. The piecewise expanding full branch Markov map. To relate this construction to the flows studied in [4], we need to work with a full branch expanding interval map, whereas the quotient return map $f : \cup_{i,j} \Delta_{i,j} \to \Delta$ is a transitive Markov expanding interval map, which might not be full branch. To ensure the full branch property we consider an induced map: let $F$ be the first return map to some element $\Delta_0$ of the Markov partition of $f$. Here $\Delta_0$ may be arbitrarily chosen, e.g. we take $\Delta_0$ from a dynamical refinement of the Markov partition. The induced system is a full branch Markov map $F = f^R : \cup_{i} \Delta_0^{(f)} \to \Delta_0$ where $\{\Delta_0^{(f)}\}_{r}$ is a countable partition, formed by open intervals, of a full measure subset of $\Delta_0$. In addition,
the first return time function $R : \Delta_0 \to \mathbb{N}$ is constant on each $\Delta_0^{(\ell)}$. The induced return map $\bar{F}$ over $\tilde{\Delta}_0 := \pi^{-1}\Delta_0$ in given by $\bar{F}(x) = P^R(\pi(x))(x)$ with the induced Poincaré return time $r := \sum_{j=0}^{R_\pi-1} \tau \circ P^j$ which we will treat as a roof function. Let $m$ denote the normalised restriction of the Riemannian volume to $\Delta_0$. For future convenience let $\Delta_0^{(\ell)} := \pi^{-1}\Delta_0^{(\ell)}$ for each $\ell$, so that $\bar{F}$ and $r$ become the return map and return time, respectively, to the section $\tilde{\Delta} \subset U$ under the action of the flow $X^t$, since $F$ was chosen as the first return of $f$ to $\Delta_0$. To state the following results, we define the Hölder-norm for functions $\varphi : \Delta_0 \to \mathbb{R}$:

$$\|\varphi\|_{C^\alpha} := \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^\alpha}.$$  

**Lemma 8.** Let $F : \cup_\ell \Delta_0^{(\ell)} \to \Delta_0$ be as specified above. Then there is $\alpha \in (0, 1]$ satisfying

1. for each $\ell$, $F : \Delta_0^{(\ell)} \to \Delta_0$ is a $C^{1+\alpha}$ diffeomorphism;
2. there exists $\lambda \in (0, 1)$ such that, for each inverse branch $h : \Delta_0 \to \Delta_0^{(\ell)}$ of $F$, $|h'(x)| \leq \lambda$ for all $x \in \cup_\ell \Delta_0^{(\ell)}$;
3. there exists $C_0 > 0$ such that for each inverse branch $h$ of $F$ we have $\|\log |h'|\|_{C^\alpha} \leq C_0$.

The statement of the above lemma is precisely the definition of a $C^{1+\alpha}$ uniformly-expanding full-branch Markov map of the interval studied in [4].

**Proof of Lemma 8.** Since $F$ is $C^2$, the regularity condition on $F$ and $\log J$ are satisfied. The uniform hyperbolicity of the flow ensures that there are $C > 0$, $\lambda \in (0, 1)$ such that $|Df^{-n}(x)| \leq C\lambda^n$ for all $x \in \Delta_0$, $n \in \mathbb{N}$. The construction of the induced return map to an element $\Delta_0$ of a dynamical refinement of the initial Markov partition guarantees that the least inducing time $R$ is as large as needed. Then $|h'(x)| \leq C\lambda^{R(x)} \leq \lambda < 1$ for $x \in \Delta_0$ and each inverse branch $h$ of $F$ with inducing time $R(x)$, just as in item (2), is a consequence of setting $\lambda = C \sup_x \lambda^{R(x)} \in (0, 1)$. To conclude, we note first that since $f \mid_{\Delta_j}$ is a $C^{1+\alpha}$ diffeomorphism for each element of the initial Markov partition, then $A = \sup_j \|\log |Df| \mid_{\Delta_j}\|_{C^\alpha}$ is bounded and so we obtain for each inverse branch $h$ of $F$

$$\|\log |h'|\|_{C^\alpha} = \left\| \sum_{j=0}^{R(h)-1} |\log |Df| \circ f^j \circ h| \right\|_{C^\alpha}$$

$$\leq \sum_{j=0}^{R(h)-1} \|\log |Df| \mid_{f^j(\Delta)}\|_{C^\alpha} \text{diam}(f^j h(\Delta))^\alpha$$

$$\leq A \sum_{j=0}^{R(h)-1} C\lambda^{R(h)-j} \leq AC \frac{1 - \lambda^{R(h)}}{1 - \lambda}$$

is also bounded uniformly in $h$, proving item (3).  

2.3.2. The hyperbolic skew-product. Taking advantage of the smoothness of the flow and of the projection along each center-stable leaf, as in Remark 7, we may assume without loss of generality that $\hat{\Delta} = \Delta_0 \times \Omega$ where $\Delta_0$ is an interval and $\Omega$ is a compact $d_\delta$-dimensional ball and write $\hat{F} : (x, z) \mapsto (F(x), G(x, z))$; where $F$ is the uniformly expanding $C^{1+\alpha}$ Markov interval map constructed in the previous paragraphs, and $G$ is of class $C^{1+\alpha}$ and contracting in the second coordinate, i.e.,
there are constants $C > 0$ and $\gamma_0 \in (0, 1)$ such that
\[
\|P^n(x, y) - P^n(x, z)\| \leq C\gamma_0^n\|y - z\|, \quad x \in \Delta_0 \quad \text{and} \quad y, z \in \Omega.
\]
This puts us in the setting of [14]. We observe that, by choosing first a refinement of the Markov partition, and then choosing $\Delta_0$ from the refined partition, we guarantee that the inducing time $R$ is a large as needed, so that there exists contraction at each iteration by $\hat{F}$, that is, we may take $C = 1$. For that choice of $\Delta_0$ we have the following result.

**Proposition 9.** Let $\hat{F} : \cup_i \Delta_0^{(i)} \to \hat{\Delta}_0$ be the skew-product and $F : \cup_i \Delta_0^{(i)} \to \Delta_0$ be the uniformly expanding full-brancl Markov (interval) map, as defined above. Let $\nu$ be the unique ergodic $F$-invariant absolutely continuous probability measure.

1. There exists a $C^{1+}$ map $\pi : \hat{\Delta}_0 \to \Delta_0$ such that $\hat{F} \circ \pi = \pi \circ F$ whenever both members of the equality are defined;
2. There exists an ergodic $\hat{F}$-invariant probability measure $\eta$ giving full mass to the domain of definition of $\hat{F}$ and such that $\pi_* \eta = \nu$;
3. There exists a family of probability measures $\{\eta_x\}_{x \in \Delta_0}$ on $\hat{\Delta}_0$ which is a disintegration of $\eta$ over $\nu$, that is, $x \mapsto \eta_x$ is measurable, $\eta_x$ is supported on $\pi^{-1}(x)$ and, for each measurable subset $A$ of $\hat{\Delta}_0$, we have $\eta(A) = \int \eta_x(A) d\nu(x)$. Moreover (a) this disintegration is Hölder: we can find a constant $C > 0$ such that, for any open subset $\omega \subset \cup_i \Delta_0^{(i)}$ and for each $v : \hat{\omega} = \pi^{-1}\omega \to \mathbb{R}$ such that $\|v\|_\alpha < \infty$, the function $\hat{v} : \omega \to \mathbb{R}, x \mapsto \hat{v}(x) := \int v(y) d\eta_x(y)$ is $\alpha$-Hölder and satisfies $\|\hat{v}\|_\alpha \leq C \cdot \|v\|_\alpha$.
4. There exists $\kappa \in (0, 1)$ such that, for all $w_1, w_2$ such that $\pi(w_1) = \pi(w_2)$, we have $d(\hat{F}w_1, \hat{F}w_2) \leq \kappa d(w_1, w_2)$.

In the previous item (3), $\|v\|_\alpha = |v|_\infty + \|v\|_{C^\alpha}$ with $\| \cdot \|_{C^\alpha}$ as used in Lemma 8 and we write for $v : \hat{\omega} \to \mathbb{R}$
\[
\|v\|_\alpha := |v|_\infty + \sup \left\{ \frac{|v(x, y) - v(x', y')|}{(\|x - x'\| + \|y - y'\|)^\alpha} : (x, y), (x', y') \in \hat{\omega}, (x, y) \neq (x', y') \right\}.
\]

The statement of the above proposition corresponds precisely with [4, Section 3] and says, in their terminology, that $\hat{F}$ is a hyperbolic skew-product over $F$.

**Proof of Proposition 9.** Item (1) is clear. Item (2) follows from [14, Section 3]. Item (3) follows from [14, Section 4] while item (4) is a consequence of the choice of $\Delta_0$. See e.g. [2, Section 2] and references therein for more details.\hfill $\square$

**2.3.3. The hyperbolic skew-product suspension flow.** Let $\tilde{\Delta}_0^u = \{(x, u) : x \in \Delta_0, 0 \leq u < r(x)\} \sim$ be the phase space of the suspension semiflow over $\hat{F}$ with roof $r$, where the quotient is determined by the equivalence relation $(x, r(x)) \sim (\hat{F}(x), 0)$, for all $x \in \Delta_0$. In local coordinates, the suspension semiflow $\hat{F}_t : \tilde{\Delta}_0^u \to \tilde{\Delta}_0^u$ is given by
\[
\hat{F}_t(x, s) = \left( \hat{F}^n(x), t + s - \sum_{i=0}^{n-1} r(\hat{F}^i(x)) \right) \quad \text{for all } t \geq 0,
\]
where $n = n(x, s, t)$ is a non-negative integer so that $\sum_{i=0}^{n-1} r(\hat{F}^i(x)) \leq s + t < \sum_{i=0}^{n} r(\hat{F}^i(x))$. The suspension semiflow is a factor of the original flow $X^t : U \to U$ for some neighbourhood $U$ of the attractor through
\[
\Phi : \tilde{\Delta}_0^u \to U; \quad (x, u) \mapsto X^u(x), \quad \text{since} \quad \Phi(\hat{F}_t(x, u)) = X^u(\Phi(x, u)).
\]
Moreover, $\Phi$ is invertible since $\hat{F}$ is the first return to $\hat{\Delta}_0$; and inherits the regularity of $X^t$. Up to this point, we chose the local section $\hat{\Delta} \subset U$ to represent $X^t$ as a suspension semiflow with return map $\hat{F}$ and return time given by the roof function $r$. Since the induced map $F$ is full branch Markov (countable partition), the roof function might be unbounded.

In what follows, we use this representation to obtain exponential mixing for the physical measure of the hyperbolic attractor $\Lambda$ under the condition of uniform non-integrability.

**Remark 10.** Next we crucially use the results from [4] (extended in [6]) on exponentially mixing hyperbolic suspension semiflows and so $\nu$ is assumed to be absolutely continuous with respect to the induced Lebesgue measure on $\Delta_0$ induced by the Riemannian volume. This is also the setting of [14].

Since $\eta$ is an $\hat{F}$-invariant probability measure, then $\hat{\eta} := \eta(r)^{-1} \eta \times \text{Leb}$ is an $\hat{F}_t$-invariant probability measure. Since this measure has absolutely continuous conditional measure on unstable manifolds due to the connection to the absolutely continuous $\nu$, we know that $\mu = \Phi_* \hat{\eta}$ is the unique physical/SRB measure for $X^t : U \to U$.

We observe that the skew-product $\hat{F}$ satisfies also the following

**domination condition:** there exists $c \in (0, 1)$ such that for all $(x, y) \in \hat{\Delta}_0$

$$
\|D_1 G(x, y)\| \leq c \cdot |DF(x)|.
$$

(4)

Indeed, this is a consequence of hyperbolicity in a neighborhood $U$ of $\Lambda$. Moreover, the roof function $r$ is an extension of $r \circ \pi$ on $\Delta_0$ to $\hat{\Delta}_0$ which satisfies the following

**Lipschitz condition:** there exists a constant $C > 0$ so that

$$
|r(x, y) - r(x, y')| < C \text{ dist}(y, y') \text{ for all } x \in \cup f \Delta_0^{(t)}, y, y' \in \Omega.
$$

(5)

Indeed, this follows by the $C^2$ smoothness of $\tau$ which implies that

$$
|r(x, y) - r(x, y')| \leq \sum_{i=0}^{R(x)-1} |\tau(P^i(x, y)) - \tau(P^i(x, y'))|
$$

$$
\leq |D\tau|_\infty \sum_{i=0}^{R(x)-1} d(P^i(x, y), P^i(x, y'))
$$

$$
\leq |D\tau|_\infty \sum_{i=0}^{R(x)-1} C_{\gamma_0} d((x, y), (x, y'))
$$

$$
\leq \hat{C}|D\tau|_\infty (1 - \gamma_0)^{-1} \|y - y'\|,
$$

where $\hat{C}$ is a constant taking into account the bounded distortion of distances along the contracting fibers of $F$.

Let $\mathcal{C}_{\alpha}^0(\hat{\Delta}_0)$ denote the subset of $L^\infty$ functions $\psi : \hat{\Delta}_0 \to \mathbb{R}$ such that $\|\psi\|_\alpha = |\psi|_\infty + |\psi|_{\alpha, \text{loc}}$, where

$$
|\psi|_{\alpha, \text{loc}} = \sup_{h \in H} \sup_{(x_1, y_1, u) \neq (x_2, y_2, u)} \frac{|\psi(h x_1, y_1, u) - \psi(h x_2, y_2, u)|}{|x_1 - x_2|^\alpha + \|y_1 - y_2\|};
$$

(6)

and let $\mathcal{C}_{\alpha,k}^0(\hat{\Delta}_0)$ be the subset of $\mathcal{C}_{\alpha}^0(\hat{\Delta}_0)$ given by those functions $\psi : \hat{\Delta}_0 \to \mathbb{R}$ such that $\|\psi\|_{\alpha,k} := \sum_{j=0}^k |\partial^j_\alpha \psi|_\alpha < \infty$, where $\partial_\alpha$ denotes the differentiation along the semiflow direction and $k \geq 1$ is a given integer.
We consider the cohomological property known as the uniform non integrability property of the flow [18]: recalling (cf. Remark 7) that \( \Delta_0 \) can be identified with \( \Delta_0 \times \{0\} \subset \Delta_0 \times \Omega = \hat{\Delta}_0 \), we state

(UNI): There does not exist any \( C^1 \) function \( \gamma : \Delta_0 \to \mathbb{R} \) such that \( r - \gamma \circ F + \gamma \) is constant on each \( \Delta_0^{(i)} \).

**Theorem 11.** Suppose that \( \hat{F} : \cup_t \hat{\Delta}^{(i)}_0 \to \hat{\Delta}_0 \), \( r : \cup_t \Delta_0^{(i)} \to \mathbb{R}_+ \), and \( \hat{F}_t \) are defined as before, and the suspension semiflow \( \hat{F}_t \) on \( \hat{\Delta}_0 \) preserves the measure \( \hat{\eta} \) and satisfies conditions (4) and (5). In addition, assume that (UNI) holds. Then there exist \( C, \delta > 0 \) so that for all \( \phi \in C^0_{loc}(\hat{\Delta}_0), \psi \in C^2_{loc}(\hat{\Delta}_0) \), and for all \( t \geq 0 \),

\[
\left| \int \phi \cdot \psi \circ \hat{F}_t \, d\hat{\eta} - \int \phi \, d\hat{\eta} \cdot \int \psi \, d\hat{\eta} \right| \leq C \|\phi\|_\alpha \|\psi\|_{\alpha,2} e^{-\delta t}.
\]

This ensures that the original flow \( X^t : U \to U \) also mixes exponentially for sufficiently smooth observables: if \( \phi, \psi : U \to \mathbb{R} \) are \( C^r \) for some \( r > 1 \) and \( \phi \circ \Phi, \psi \circ \Phi \) are also \( C^2_{loc}(\hat{\Delta}_0) \) observables on \( \hat{\Delta}_0 \), where \( \Phi \) is the conjugacy (3), then

\[
\int \phi \cdot (\psi \circ X^t) \, d\mu = \int (\phi \circ \Phi) \cdot (\psi \circ X^t \circ \Phi) \, d\hat{\eta} = \int (\phi \circ \Phi) \cdot (\psi \circ \Phi) \circ \hat{F}_t \, d\hat{\eta}
\]

tend to zero exponentially fast. A standard approximation argument [18, Proof of Corollary 1] provides exponential mixing for Hölder observables.

We stress that the technical novelty of Theorem 11, compared with the arguments in [2, 3], is that the roof function is allowed to be non constant on the leaves of the stable foliation \( F \) on the sections \( S \), enabling us to bypass smoothness assumptions on the stable foliation of \( \Lambda \).

3. Exponential mixing for generalized hyperbolic skew-product semiflows

Here we prove Theorem 11. We start with the exponential mixing for a suspension semiflow over the full branch Markov expanding map \( \hat{F} \), in Subsection 3.1; and then extend this result to a suspension semiflow over a hyperbolic skew-product whose roof function is constant on stable leaves; in Subsection 3.2. Up to this point we mainly review the literature. Then we obtain exponential mixing for a hyperbolic skew-product semiflow whose roof function is not necessarily constant on stable leaves, subject to some mild conditions, which enables us to obtain Theorem 11, in Subsection 3.3.

3.1. Suspension semiflow over the expanding full branch Markov map.

In the setting of Theorem 11, we define a roof function over \( \Delta_0 \) by \( r \circ \pi \) to obtain a suspension semiflow \( F_t \) over the \( C^1 \) expanding full branch Markov map \( F \). Precisely, we write \( \Delta_0^{\mathrm{r.o}} := \{(y, t) : x \in \Delta_0, 0 \leq t < r(x)\}/\sim \) with the equivalence relation \( (y, r(y)) \sim (F(y), 0), y \in \Delta_0 \) and the flow \( F_t : \Delta_0^{\mathrm{r.o}} \circ \) is given by

\[
F_t(y, s) = \left( F^k(y), t + s - \sum_{i=0}^{k-1} r(F^i(y)) \right) \quad \text{for all} \ t \geq 0,
\]

where \( k = k(y, s, t) \) is a non-negative integer so that \( \sum_{i=0}^{k-1} r(F^i(x)) \leq s + t < \sum_{i=0}^{k} r(F^i(x)) \).
Lemma 12. Let $F : \cup_l \Delta_0^{(l)} \to \Delta_0$ and $r \circ \pi : \cup_l \Delta_0^{(l)} \to \mathbb{R}_+$ be the full branch Markov map with countable partition and the roof function as defined before. There are $r_0, K, \sigma_0 > 0$ so that

1. $r \circ \pi$ is bounded from below by $r_0$;
2. $|D(r \circ \pi \circ h)|_\infty \leq K$ for every inverse branch $h$ of $F$;
3. $\int e^{\sigma_0 \tau \circ \pi} \, dm < \infty$.

Proof. Since $R(x) \underline{\leq} r(x) \leq R(x) \bar{\tau}$, items (1-2) are consequences of the boundedness of $\tau$. Indeed, for item (2) we follow the proof of Lemma 8: for each inverse branch $h$ of $F$

$$|D(r \circ \pi \circ h)|_\infty = \left| \sum_{i=0}^{R(h)} D[\tau \circ P^i \circ (\pi \circ h)] \right|_\infty \leq |D\tau|_\infty \sum_{i=0}^{R(h)} |DF^i|_{\pi h} D(\pi \circ h)|_\infty$$

$$= |D\tau|_\infty \sum_{i=0}^{R(h)} |DF^i|_h Dh|_\infty \leq |D\tau|_\infty \sum_{i=0}^{R(h)} C|DF^{i-R(h)}|_\infty$$

$$\leq |D\tau|_\infty \sum_{i=0}^{R(h)} C\lambda^{i-R(h)} \leq C|D\tau|_\infty (1-\lambda)^{-1}.$$

Item (3) follows as in [2, Lemma 10] with the same notations. \qed

Consider the uniform non integrability property of this flow

$$(\text{UNI}^0): \text{There does not exist any } C^1 \text{ function } \gamma : \Delta_0 \to \mathbb{R} \text{ such that } r \circ \pi - \gamma \circ F + \gamma$$

is constant on each $\Delta_0^{(l)}$.

The above property is also described as “$r \circ \pi$ not being cohomologous to a locally constant function”. We note that it is precisely the same UNI condition stated in the previous section.

Since $\nu$ is an $F$-invariant probability measure, then $\hat{\nu} := \frac{1}{\nu(\tau \circ \pi)} \nu \times \text{Leb}$ is an $F$-invariant probability measure on $\Delta_0^*$. We define $C_0^\alpha(\Delta_0^{\circ \pi})$ to consist of $L^\infty$ functions $\psi : \Delta_0^{\circ \pi} \to \mathbb{R}$ such that $|\psi|_\alpha = |\psi|_\infty + |\psi|_{\alpha, \text{loc}} < \infty$, where

$$|\psi|_{\alpha, \text{loc}} = \sup_{h \in \mathcal{H}} \sup_{(x_1, u) \neq (x_2, u)} \frac{|\psi(hx_1, u) - \psi(hx_2, u)|}{|x_1 - x_2|^{\alpha}}$$

and $\mathcal{H}$ is the family of inverse branches of $F$. Given an integer $k \geq 1$, we define $C_{0, k}^\alpha(\Delta_0^{\circ \pi})$ to consist of $C_0^\alpha(\Delta_0^{\circ \pi})$ functions $\psi$ with $|\psi|_{\alpha, k} = \sum_{j=0}^{k} \|\partial^j \psi\|_\alpha < \infty$, where $\partial^j$ denotes the differentiation along the semiflow direction.

Theorem (Exponential mixing for expanding semiflows). If (UNI) holds, then there are constants $c, C > 0$ so that

$$|\int (\varphi \circ F_t) \psi \, d\tilde{\nu} - \int \varphi \, d\tilde{\nu} \int \psi \, d\tilde{\nu}| \leq Ce^{-ct} \|\varphi\|_\infty \|\psi\|_{\alpha, 2}$$

for all $\varphi \in L^\infty(\Delta_0^{\circ \pi})$, $\psi \in C_{\text{loc}}^{\alpha, 2}(\Delta_0^{\circ \pi})$ and $t > 0$.

Proof. This is [6, Theorem 2.2], which is a generalization of [4, Theorem 2.1] from $\alpha$-Hölder observables to the more general class of observables presented above. \qed

Remark 14. The exponential mixing result from [4, Theorem 2.1] in turn extends [10] to $C^{1+}$ smooth from $C^2$ smooth dynamics.
We consider now the space \( \hat{\chi} \) relation \((x, r(\pi(x))) \sim (\hat{F}(x), 0) \) and the suspension semiflow over the \( \hat{F} : \hat{\Delta}_0^{r} \circ \) given by

\[
\hat{F}(x, s) = (\hat{F}^k(x), t + s - \sum_{i=0}^{k-1} (r \circ F^i)(\pi x)) \quad \text{for all } t \geq 0,
\]

where \( k = k(x, s, t) \) is a non-negative integer so that \( \sum_{i=0}^{k-1} (r \circ F^i)(\pi x) \leq s + t < \sum_{i=0}^{k} (r \circ F^i)(\pi x) \).

We stress that the roof function is constant on stable leaves by construction. In particular, condition (UNI) is unchanged for \( \hat{F} \).

We observe that Proposition 9 directly applies since the map \( \hat{F} \) is the same with respect to the construction in the previous Section 2. However we now have a potentially different probability measure \( \hat{\eta} = \eta (r \circ \pi)^{-1} \eta \times \text{Leb} \) on the suspension \( \hat{\Delta}_0^{r} \) which is \( \hat{F} \)-invariant.

Let \( C_{loc}^0(\hat{\Delta}_0^{r}) \) denote the subset of \( L^\infty \) functions \( \psi : \hat{\Delta}_0^{r} \to \mathbb{R} \) such that \( ||\psi||_\alpha = ||\psi||_\alpha + ||\psi||_{\alpha,loc} \), with the same expression (6) for the norm. Likewise we define \( C_{loc}^{0,k}(\hat{\Delta}_0^{r}) \) for each integer \( k > 1 \).

**Theorem 15.** Suppose that \( \hat{F} : \hat{\Delta}_0^{r} \to \hat{\Delta}_0^{r} \) is a \( C^{1+} \) hyperbolic skew product whose roof satisfies condition (UNI). Then there exist constants \( c, C > 0 \) such that \( |f(\varphi \circ \hat{F}) \cdot \psi d\hat{\eta} - \int \varphi d\hat{\eta} \int \psi d\hat{\eta}| \leq Ce^{-ct} \|\varphi\|_{\alpha,2} \|\psi\|_{\alpha,2} \) for all \( \varphi \in C_{loc}^{0}(\hat{\Delta}_0^{r}), \psi \in C_{loc}^{0,2}(\hat{\Delta}_0^{r}) \) and \( t > 0 \).

**Proof.** This is a extension of [4, Theorem 3.3] to this more general class of observables, obtained in [6, Theorem 2.4].  \[ \square \]

### 3.3. Generalized hyperbolic skew-product semiflow

We present here the proof of Theorem 11 by reducing its setting to the one of Theorem 15.

#### 3.3.1. The conjugation

A map \( q : \hat{\Delta}_0^{r} \to \hat{\Delta}_0^{r} \) \((w, t) \mapsto (w, t + \chi(w)) \) conjugates the suspension semiflows \( \hat{F}_t : \hat{\Delta}_0^{r} \circ \) and \( \hat{F}_t : \Delta_0^{r} \circ \), that is, \( q \circ \hat{F}_t = \hat{F}_t \circ q \) if \( \chi : \hat{\Delta}_0 \to \mathbb{R} \) satisfies the cocycle relation

\[
\chi \circ \hat{F} - \chi = r - r \circ \pi.
\]

Let \( \Xi = \hat{\Delta}_0 \setminus \cup_{i} \hat{\Delta}_0^{(i)} \) be the family of stable leaves where \( \hat{F} \) is not defined and let \( \Xi_0 = \hat{\Delta}_0 \setminus \cup_{n \geq 1} \hat{F}^{-n} \Xi \) be the subset of points of \( \hat{\Delta}_0 \) which never visit \( \Xi \) in all future iterates.

**Lemma 16.** The following defines a function on \( \Xi_0 \)

\[
\chi(w) = \sum_{m \geq 0} \left[r(\hat{F}^m \pi w) - r(\hat{F}^m w)] \right], \quad w \in \Xi_0.
\]

Moreover, the relation (9) holds and there exists \( C > 0 \) so that \( \chi | \pi^{-1}(x) \) is \( C(1 - \kappa)^{-1} \)-Lipschitz for every \( x \in \Xi_0 \).

**Proof.** Because \( w, \pi w \) belong to the same stable leaf, the skew-product structure ensures that \( \hat{F}^m \pi w, \hat{F}^m w \) also lie in one stable leaf whenever the image is defined. Hence, for \( w \in \Gamma_0 \) and \( m \geq 0 \), we get from condition (5) and contraction of stable leaves: \( |r(\hat{F}^m \pi w) - r(\hat{F}^m w)| \leq C|\hat{F}^m \pi w - \hat{F}^m w| \leq Cr^m |w - \pi w| \leq Cr^m \), and so the series defining \( \chi(w) \) is absolutely convergent. Moreover, the relation (9) is a
trivial consequence of the definition of $\chi$ and, if $\pi w' = \pi w$, then $|\chi(w) - \chi(w')|$ is bounded by $\sum_{m \geq 0} |r(\tilde{F}^m w') - r(\tilde{F}^m w)| \leq \sum_{m \geq 0} C \kappa^m |w' - w| = C \frac{|w - w'|}{1 - \kappa}$, which completes the proof of Lemma 16.

To be able to convert the relevant observables on $\hat{\Delta}_0^{\operatorname{rot}}$ to good observables on $\hat{\Delta}_0^r$ we need more regularity of $\chi$ along the unstable direction.

**Lemma 17** (Version of Livsic Theorem). Let $\chi : \Xi_0 \to \mathbb{R}$ be a function satisfying $\chi \circ F - \chi = \phi$ where $\phi \in C^0_{\operatorname{loc}}(\hat{\Delta}_0)$ for some $\theta \in (0, 1]$. Given any $\hat{F}$-invariant and transitive set $\Gamma \subset \Xi_0$, then $\chi$ admits a $\theta$-Hölder extension to the closure $\hat{\Gamma}$ of $\Gamma$.

3.3.2. **Exponential mixing.** To prove Theorem 11, we start by applying the previous lemma to $\phi = r - r \circ \tau$ with $\theta = 1$. We note that $\phi \in C^1_{\operatorname{loc}}(\hat{\Delta}_0)$ by definition of the $C^{1+}$ expanding semiflow, since $\phi$ is differentiable in each element $\pi^{-1}(\hat{\Delta}_0^{\phi})$, and $(x, y) \in \hat{\Delta}_0 \times \hat{\Omega} \mapsto \phi(x, y)$ is Lipschitz for each fixed $h \in \mathcal{H}$. More precisely, for each $\omega = \hat{\Delta}_0^{\phi}$ we have with $\hat{\omega} = \pi^{-1}(\omega)$ that $\phi \mid \hat{\omega} = (r - r \circ \tau) \mid \hat{\omega} : \hat{\omega} \to \mathbb{R}$ is a $C^{1+\alpha}$ map for some fixed $\alpha > 0$. Thus, if $\phi(hx, y) = r(hx, y) - r(\pi x, 0)$ is Lipschitz by Lemma 12(2) and assumption (5). Moreover, the subset $\Xi_0$ only depends on $\hat{F}$ and not on the suspension.

We consider the compact $\hat{F}$-transitive set $\Gamma = \operatorname{supp} \eta$ and write $\Gamma'$ and $\Gamma^{\operatorname{rot}}$ for the sets $\Gamma \times \mathbb{R}$ quotiented over the spaces $\hat{\Delta}_0^r$ and $\hat{\Delta}_0^{\operatorname{rot}}$, respectively. We note that $\hat{m}(\Gamma) = 1$, where $\hat{m} = m \times \operatorname{Leb}$ is the normalized Lebesgue volume on $\hat{\Delta}_0 \times \hat{\Omega}$.

Lemma 17 ensures that we may extend $\chi$ to a Lipschitz function first on $\Gamma$, then on $\hat{\Delta}_0$, satisfying the cocycle relation (9) on the invariant set $\Gamma$. The map $q^{-1}$ becomes an ergodic equivalence between $(\hat{\Delta}_0^r, \hat{F}, \hat{\eta})$ and $(\hat{\Delta}_0^{\operatorname{rot}}, \hat{F}, \hat{\eta})$, since $\hat{\eta}$ is $\hat{F}$-invariant and $\hat{\eta} = \hat{\Gamma}^{\operatorname{rot}}$. Moreover, if $\varphi \in C^0_{\operatorname{loc}}(\hat{\Delta}_0^{\operatorname{rot}})$, then $\varphi \circ q \in C^0_{\operatorname{loc}}(\hat{\Delta}_0^{r} \cap \Gamma^r)$, since $\partial_t (\varphi \circ q)(w, t) = (\partial_t \varphi)(q(w, t))$, $i = 1, 2$ by the expression of $q$. Similarly, if $\varphi \in C^0_{\operatorname{loc}}(\hat{\Delta}_0^r)$, then $\varphi \circ q^{-1} \in C^0_{\operatorname{loc}}(\hat{\Delta}_0^{\operatorname{rot}} \cap \Gamma^{\operatorname{rot}})$, where $q^{-1}(w, t) = (w, t - \chi(w))$. In addition, it is easy to find $C > 0$ so that

$$\|\varphi \circ q^\pm\|_{\alpha, 2} \leq \|\varphi\|_{\alpha, 2}(1 + \|\chi\|_{\alpha, \operatorname{loc}}) \quad \text{and} \quad \|\chi\|_{\alpha, \operatorname{loc}} \leq C(\|\chi\|_{\infty} + \operatorname{Lip}(\chi)).$$

Since the integrals in the statement of Theorem 15 are with respect to the measure $\hat{\eta}$, and $q^{-1} \hat{\eta} = \hat{\eta}$, then for each $\varphi \in C^0_{\operatorname{loc}}(\hat{\Delta}_0^r)$ and $\psi \in C^0_{\operatorname{loc}}(\hat{\Delta}_0^r)$ we can write

$$\left| \int (\varphi \circ q^{-1} \circ \hat{F}_t) \cdot \psi \circ q^{-1} \ d\hat{\eta} - \int \varphi \circ q^{-1} \ d\hat{\eta} \right| \psi \circ q^{-1} \ d\hat{\eta} \leq C e^{-ct} \|\varphi \circ q^{-1}\|_{\alpha} \|\psi \circ q^{-1}\|_{\alpha, 2} \leq C(1 + \|\chi\|_{\alpha}^2 e^{-ct} \|\varphi\|_{\alpha} \|\psi\|_{\alpha, 2},$$

using Theorem 15 applied to $\hat{\eta}$ together with the previous estimates. This completes the proof of Theorem 11, except for Lemma 17.

3.3.3. **Proof of a version of Livsic’s Theorem.** To prove Lemma 17 we follow the usual path of Hyperbolic Dynamics, obtaining first the following.

**Lemma 18** (Closing Lemma). Given $w \in \Xi_0, \varepsilon > 0$ and $m > 0$ such that $|w - \tilde{F}^m w| < \varepsilon$, then there exists an $m$-periodic point $p \in \Xi_0$ so that, for $g = \max\{\kappa, \lambda\}$
and a constant $K > 0$ depending only on $\hat{F}$, we get
\[ |\hat{F}^i p - \hat{F}^i w| \leq K\varepsilon^{\min\{i,m-i\}}|w - \hat{F}^m w| \leq K\varepsilon^{\min\{i,m-i\}}, \quad i = 0, \ldots, m. \]

Proof of Lemma 18. Let $\omega = \Delta_0^{(\ell)}$ and $\hat{\omega} = \pi^{-1}\omega$ be an atom of the partition of $\Delta_0$ and $m > 0$, $w \in \hat{\omega}$ such that $x_m = F^m x$ satisfies $|w - w_n| < \varepsilon$. Since $\hat{F}$ is a skew-product, we have $w = (x, y), w_n = (x_m, y_m)$ with $x, x_n \in \Delta_0$ and $y, y_n \in \Omega$.

Because $F$ is a full-branch Markov map of an interval, there exists a subinterval $\omega_m$ so that $x \in \omega_m \subset \omega$ and $F^m | \omega_m : \omega_m \to \Delta_0$ is a isomorphism.

Hence, there exists a $F^m$-fixed point $z \in \omega_m$. This implies that $\pi^{-1}z$ is a $\hat{F}^m$-invariant contracting fiber and, thus, since each fiber is the closure of an open ball in an Euclidean space, there exists a $\hat{F}^m$-fixed point $p = (z, \hat{z}) \in \pi^{-1}z \subset \hat{\omega}$.

To estimate the distance between the (periodic) orbit of $p$ and the orbit segment $w_i = \hat{F}^i w, i = 0, \ldots, m$, we consider the unstable and stable directions separately. We take the horizontal line $\xi = \omega_m \times \{y\}$ over $\omega_m$ and though $w$; and consider the images $\xi_i = \hat{F}^i \xi$, each a unstable curve through $w_i, i = 0, \ldots, m$. More precisely, from condition (4), the tangent direction at $w'_i \in \xi_i$, where $w'_i = \hat{F}^i w'$ and $w' = (x', y) \in \xi$ is given by $DF^i(w') \cdot (1, 0) = (DF^i(x'), D_1 G_1(x', y))$, where we write $\hat{F}^n(x, y) = (F^n(x), G_n(x, y)), n \geq 1$. It is straightforward to check the recurrence relation $G_{n+1}(u, v) = (F^n(u), G_n(u, v)), n \geq 1$,
\[ \frac{D_1 G_1(x', y)}{DF^i(x')} = D_1 G(\hat{F}^i(x), G_1(x, y)) < c \]
and the slope of the tangent is bounded above by $c < 1$. Thus, in particular, we obtain that $|DF^i(u)| \leq |DF^i(u, v)| < (1 + c)|DF^i(u)|$ for all $i = 1, \ldots, m$ and $(u, v) \in \Delta_0$.

![Figure 1](image.png)

**Figure 1.** Finding a periodic orbit shadowing a segment of a recurrent orbit.

Let $p' = (z, y)$ and $p'_m = \xi_m \cap \pi^{-1}z$ be points in the intersection of $\xi$ and $\xi_m$ with the $\hat{F}^m$-invariant fiber $\pi^{-1}z$; see Figure 1. We write $p'_i = \hat{F}^i p'$ for $i \geq 0$. On the one hand, the length $\ell(\xi_i)$ of $\xi_i$ satisfies
\[ |p'_{m-i} - w_{m-i}| \leq \ell(\xi_{m-i}) \leq \lambda^i \ell(\xi_m) \leq (1 + c)\lambda^i |y_m - z| \leq (1 + c)\lambda^i |w - w_m|, \]
where the last inequality follows from the bound on the slope of the tangent direction to $\xi_m$ and the definition of the norm $|\cdot|$. On the other hand, writing
for any period \( p = \lim_{i \to +\infty} p_{im}' \) is the fixed point of a \( \kappa^m \)-contraction, we have
\[
|p - p'| \leq \sum_{i \geq 0} |p_{(i+1)m}' - p_{im}'| \leq \sum_{i \geq 0} \kappa^i |p' - p'_m| \leq \frac{(1 + c)|w - w_m|}{1 - \kappa^m}
\]
where the last inequality follows by the bound on the slope of \( \xi_m \). We thus get
\[
|p_i - p_i'| \leq \kappa^i |p - p'| \leq \frac{(1 + c)\kappa^i}{1 - \kappa^m}|w - w_m|,
\]
Altogether, we obtain for \( i = 0, \ldots, m \)
\[
|p_i - w_i| \leq |p_i - p_i'| + |p_i' - w_i| \leq \frac{(1 + c)\kappa^i}{1 - \kappa^m} (1 + \kappa^m) |w - w_m|,
\]
\[
\leq K \max \{ \kappa, \lambda \}^{\min\{i, m-i\}} |w - w_m| \leq K \varepsilon^{\min\{i, m-i\}},
\]
for some constant \( K > 0 \) depending only on \( \widehat{F} \), completing the proof. \( \square \)

Now the proof of Livsic’s Theorem is straightforward.

Proof of Lemma 17. Let \( \chi, \phi \) and \( \Gamma \) be as in the statement of Lemma 17. Since, by assumption, periodic points \( p \) of \( \widehat{F} \) belong to \( \Xi_0 \), then \( S_k \phi(p) := \sum_{i=0}^{k-1} \phi(\widehat{F}^i p) = 0 \) for any period \( k \) of \( p \).

Let \( w \in \Gamma \) be such that \( \{w_i : i \geq 0\} \supset \Gamma \) where \( w_i = (x_i, y_i) = \widehat{F}^i w, i \geq 0 \). Using the cocycle relation (9) we get \( \chi(w_i) = \chi(w) + S_i \phi(w), i \geq 0 \).

Thus, whenever we have \( \varepsilon = |w - w_i| \) small for some \( i > 0 \), we have a periodic point \( p = (z, \overline{z}) \) of period \( i \) satisfying the Closing Lemma 18. We write \( p_k = (z_k, \overline{z}_k) = \widehat{F}^k p \) and let \( h_k \) be the inverse branch of \( F \) so that \( z_k \in h(\Delta_0) \), \( k = 0, \ldots, i - 1 \). Then because \( \phi \in C^\theta_{\text{loc}}(\Delta_0) \)
\[
|\chi(w_i) - \chi(w)| = |S_i \phi(w)| = |S_i \phi(w) - S_i \phi(p)| \leq \sum_{k=0}^{i-1} |\phi(w_k) - \phi(p_k)|
\]
\[
\leq \sum_{k=0}^{i-1} |\phi(h_k F x_{k+1}, y_k) - \phi(h_k F z_{k+1}, \overline{z}_k)| \leq \sum_{k=0}^{i-1} \|\phi\|_{\alpha, \text{loc}} \left( |z_{k+1} - x_{k+1}|^\theta + |y_k - \overline{z}_k| \right)
\]
\[
\leq \|\phi\|_{\alpha, \text{loc}} \left( \sum_{k=1}^{i} K^\theta \theta^{\min\{k, i-k\}} |w - w_i|^\theta + \sum_{k=0}^{i-1} K^\theta \theta^{\min\{k, i-k\}} |w - w_i|^\theta \right)
\]
\[
\leq \frac{\max\{K, K^\theta\} \|\phi\|_{\alpha, \text{loc}} |w - w_i|^\theta}{1 - \theta^\theta}.
\]
Hence, \( \chi \) is \( \theta \)-Hölder on a dense subset of \( \Gamma \). Thus, \( \chi \) admits a unique \( \theta \)-Hölder extension, with the same Hölder constant, to the closure of \( \Gamma \). This completes the proof of Lemma 17. \( \square \)

4. Joint non-integrability of stable and unstable foliation

To prove Theorem B, and then Theorems A and C, we relate (UNI) with non joint-integrability following [2] adapted to the setting where the roof function is not constant on stable leaves.
Lemma 19. Suppose that $\hat{F} : \cup_t \hat{\Delta}_0^{(t)} \to \hat{\Delta}_0$ and $r : \cup_t \hat{\Delta}_0^{(t)} \to \mathbb{R}$ are defined as in the previous section, where $r \neq r \circ \pi$ is allowed. Property (UNI) fails if, and only if, the stable and unstable foliations of the non-trivial attractor, for the underlying Axiom A vector field $X$, are jointly integrable.

This proves Theorem B. Indeed, from the representation of the flow on a neighborhood of a non-trivial hyperbolic attractor by hyperbolic skew-product semiflow presented in Section 2 together with the exponential mixing under (UNI) proved in Section 3, we have from Lemma 19 that joint non-integrability of stable and unstable foliation of a non-trivial hyperbolic attractor imply exponential mixing for the physical measure.

This also proves Theorem A, since the joint non-integrability of stable and unstable foliations is an open and dense condition [21, Remark 1.10] among hyperbolic flows.

Proof of Lemma 19. First we assume that (UNI) fails. Then there exists a $C^1$ function $\gamma : \Delta_0 \to \mathbb{R}$ such that $r \circ \pi - \gamma \circ F + \gamma$ is constant on each $\hat{\Delta}_0^{(t)}$. We also write $\gamma$ for the extension to $\hat{\Delta}_0$ which is constant along local stable manifolds. This means that $r \circ \pi - \gamma \circ F + \gamma$ is constant on each $\hat{\Delta}_0^{(t)}$. For each $t$ we can fix some $x_t \in \hat{\Delta}_0^{(t)}$ so that $\gamma(x) \leq \gamma(x_t)$ for all $x \in \hat{\Delta}_0^{(t)}$, since $\gamma : \Delta_0 \to \mathbb{R}$ is a $C^1$ function. We may assume, without loss of generality, that $\gamma(x_t) - \gamma(x) \leq \inf r$ by choosing, if needed, $\Delta_0$ from a finer refinement of the Markov partition. Since $\gamma$ is a $C^1$ function, then

$$D_t := \left\{(x, \gamma(x_t) - \gamma(x)) : x \in \hat{\Delta}_0^{(t)} \right\} \subset \hat{\Delta}_0^{(t)} \cap \hat{\Delta}_0^{(t)}.$$

is a $(d_u + d_s)$-dimensional $C^1$-hypersurface for each $t$.

The return time function to this family of local cross-sections $\cup_t D_t$ is locally constant with respect to the flow $\hat{F}_t : \hat{\Delta}_0^{(t)}$, as proved in [2, Lemma 12]. Indeed, for $t, t'$ let $(x, a) \in D_t$ such that $\hat{F}(x) \in \hat{\Delta}_0^{(t')}$ for $x$ so that $0 < t - r(\pi x) < r(Fx)$. Then $\hat{F}_t(x, \gamma(x_t) - \gamma(x)) = (\hat{F}x, t + \gamma(x_t) - \gamma(x) - r(\pi x))$ and the first $t > 0$ so that this belongs to $\cup_{t'} D_{t'}$ is

$$t_1 = (\gamma(Fx_{t'} - \gamma(x_{t'})) + (r(\pi x) - \gamma(Fx) + \gamma(x)).$$

This value is constant for all $x \in \hat{\Delta}_0^{(t)} \cap \hat{F}^{-1}\hat{\Delta}_0^{(t')}$, thus locally constant and we get $\hat{F}_t(\hat{\Delta}_0^{(t)} \cap \hat{F}^{-1}\hat{\Delta}_0^{(t')}) = D_{t'}$. Since $q^{-1} \circ \hat{F}_t = \hat{F}_t \circ q^{-1}$ we obtain

$$q^{-1}(D_{t'}) = q^{-1}(\hat{F}_t(\hat{\Delta}_0^{(t)} \cap \hat{F}^{-1}\hat{\Delta}_0^{(t')})) = \hat{F}_t(q^{-1}(\hat{\Delta}_0^{(t)} \cap \hat{F}^{-1}\hat{\Delta}_0^{(t')})).$$

This proves that

$$q^{-1}\left\{(x, a + \gamma(x) - \gamma(x)) : x \in \hat{\Delta}_0^{(t)}, a \in \mathbb{R}^+ \right\}$$

$$= \left\{(x, a + \gamma(x) - \gamma(x) - \gamma(x)) : x \in \hat{\Delta}_0^{(t)}, a \in \mathbb{R}^+ \right\}$$

defines a codimension-one invariant lamination $\hat{L}$ of $\hat{\Delta}_0$ transversal to the flow $\hat{F}_t$, whose laminas are Lipschitz graphs over $\Gamma = \text{supp } \eta$ and over each stable fiber $\pi^{-1}x, x \in \Xi_0$, according to Lemmas 16 and 17.

If $X$ is Anosov, then $\Xi = M$, so $\Gamma = \hat{\Delta}_0$ and $D_0$ is a collection of graphs of Lipschitz functions $\hat{\Delta}_0 \to \mathbb{R}$. Through the conjugation $\Phi$ given in (3), we obtain
a lamination $\mathcal{L} := \Phi(\hat{\mathcal{L}})$ by Lipschitz graphs of the ambient manifold which is transversal to $X$ and invariant under the flow $X^t$. This, in turn, implies that the stable and unstable foliations of $X$ are jointly integrable, since these are also invariant (and hyperbolic) foliations of the same flow, and thus must be contained in the leaves of $\mathcal{L}$.

If $\Lambda$ is a non-trivial proper hyperbolic attractor, again $\mathcal{L} = \Phi(\hat{\mathcal{L}})$ is a lamination of a neighborhood $U$ of $\Lambda$, transversal to $X$ and $X^t$-invariant. Then, the lamina $\mathcal{L}_x$ through $x \in \omega := \Phi(\Delta_{0}^{(t)} \cap F^{-1} \Delta_{0}^{(t)}) \subset \Delta$ must contain $W_x^u(x) \cap \omega = \omega$ together with each local stable leaf $W_x^s(y) \cap U, y \in \omega$. Thus $\omega^s := \bigcup \{ W_x^s(y) \cap U : y \in \omega \} \subset \mathcal{L}_x$ and also for each $z \in \Lambda \cap \omega^s$ we have that $W_z^u(z) \cap \omega^s$ contains a neighborhood of $z$ in $W_z^u(z)$ which is contained in $\omega^s$. This, in turn, implies that the stable and unstable foliations of $\Lambda$ are jointly integrable.

This shows that if (UNI) fails, then joint integrability of stable and unstable foliations follows. Reciprocally, it is well known [21] that joint integrability of stable and unstable foliation ensures that (UNI) fails. □

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