Generalized equation of relativistic quantum mechanics in a gauge field

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We develop an unified algebraic approach to the description of gauge interactions within the framework of a new concept of quantum mechanics. The next step in generalizing the space-time and the action vector space is made. The gauge field is defined through linear mappings in the generalized space-time and the action space. Relativistic quantum mechanics equations for particles in a gauge field are derived from the structure equations for the action space expanded in the linear mappings of action vectors. In a special case, these equations are reduced to the relativistic equations for the leptons in the electroweak field. As against the standard Glashow–Weinberg–Salam model, the set of equations includes the equation for the right neutrino interacting only with the weak \( Z \)-field.

11.15.-q, 12.15.-y

I. INTRODUCTION

The new concept of quantum mechanics has been put forward in our previous works \([1,2]\). The main features of this concept are the following. We introduced the space of all contravariant tensors over the usual space-time as a \emph{generalized space-time} \( \mathbb{X} \). The space of the Clifford algebra \( C_4 \), selected from the generalized space-time \( \mathbb{X} \), was used for the description of leptons and hence was called \emph{a space of leptons}. The action was considered as a vector quantity. The action vectors formed an algebra \( \mathbb{S} \) similar to the algebra \( \mathbb{T} \). The wave function was identified with a differential of action vector. Relativistic quantum mechanics equations for free particles were derived from the structure equations for the algebra \( \mathbb{S} \).

In the present work, we develop the specified concept of quantum mechanics for the purpose of describing the interaction of particles with a gauge field. The following propositions are used as the basis for our study:

1. Linear mappings of the generalized space-time \( \mathbb{X} \) onto themselves are introduced. The linear mappings form algebra \( \mathbb{U} \). A kinematic space \( \mathbb{T} = \mathbb{X} + \mathbb{U} \), endowed with algebraic properties, is defined.

2. The action is considered as vector quantity. The action vectors form an algebra \( \mathbb{S} \) similar to the algebra \( \mathbb{T} \).

3. A gauge potential is defined as a derivative of linear mapping coordinates in the generalized space-time and the space of generalized action.

4. The partial derivation of the multiplication rule for the algebras \( \mathbb{T} \) and \( \mathbb{S} \) result in the specific differential relations called the structure equations. The structure equations for algebra \( \mathbb{S} \) are reduced to the generalized equations of quantum mechanics for particles in a gauge field.

II. LINEAR MAPPINGS OF THE GENERALIZED SPACE-TIME

A. Vector space of linear mappings

We consider \emph{linear mappings} of the generalized space-time \( \mathbb{X} \). Let us introduce a linear operator \( h( ) \) mapping \( \mathbb{X} \) to itself:

\[
x' = h(x), \quad x, x' \in \mathbb{X}.
\]

The vector of the generalized space-time can be decomposed along basis vectors \( e_I \):

\[
\]
\[ x = e_0 x^0 + e_{i_1} x^{i_1} + e_{i_2 i_3} x^{i_1 i_2} + \ldots + e_{i_n \ldots i_{2i_1}} x^{i_1 i_2 \ldots i_n} + \ldots = e_I \cdot x^I, \]

where \( e_0 \) is the unit of reals, \( \{e_{i_1}\} \) is the basis in the usual space-time, the lower collective index

\[ I = 0, i_1, (i_2 i_1), \ldots, (i_n \ldots i_{2i_1}), \ldots \]

and the upper collective index

\[ I = 0, i_1, (i_1 i_2), \ldots, (i_1 i_2 \ldots i_n), \ldots \]

are used in the last expression for compactness. Therefore, the mapped vector can be rewritten as

\[ x' = h(e_I) \cdot x^I. \]

Introduce a decomposition of vectors \( h(e_I) \) in terms of the basis vectors \( e_K \)

\[ h(e_I) = e_K \cdot h^K_I. \]

Using this relation we get

\[ x' = e_K \cdot h^K_I \cdot x^I. \quad (1) \]

Let \( \mathbb{U} \) be the set of linear mappings. We define an addition and a multiplication by a number, satisfying the distributivity rule. As a result, \( \mathbb{U} \) becomes a vector space.

Introduce basis mappings \( \mathcal{I}_K(\cdot) \) on the vector space \( \mathbb{U} \) so that

\[ h(\cdot) = h^K_I \cdot \mathcal{I}_K(\cdot), \]

where the mapping coordinates \( h^K_I \) are the decomposition coefficients of the vectors \( h(e_I) \) along the basis vectors \( e_K \). The linear mapping of the vector \( x = e_L \cdot x^L \) should have the form \( \mathbb{U} \), therefore

\[ h^K_I \cdot \mathcal{I}_K(e_L) \cdot x^L = e_K \cdot h^K_I \cdot x^I. \]

From here, a mapping rule for the basis vectors \( e_K \) of the space \( \mathbb{X} \) through the basis mappings \( \mathcal{I}_K(\cdot) \) follows

\[ \mathcal{I}_K(e_L) = e_K \cdot \delta^L_I, \quad (2) \]

where \( \delta^L_I \) is the Kronecker delta.

**B. Linear mapping group. Linear mapping algebra**

We introduce a group composition rule acting on the space \( \mathbb{U} \):

\[ h(x) = h_2(h_1(x)), \quad x \in \mathbb{X}, \quad h(\cdot), h_1(\cdot), h_2(\cdot) \in \mathbb{U}, \quad (3) \]

i.e. we require that the set of linear mappings \( \mathbb{U} \) is a group. Write the mappings involved in the composition rule through the basis mappings:

\[ h(\cdot) = h^M_L \cdot \mathcal{I}_M(\cdot), \quad h_2(\cdot) = (h_2)^M_I \cdot \mathcal{I}_M(\cdot), \quad h_1(\cdot) = (h_1)^K_L \cdot \mathcal{I}_K(\cdot). \]

From \( (3) \), we obtain the relation between the coordinates of these mappings:

\[ h^M_L = (h_2)^M_K \cdot (h_1)^K_L \]

and the composition rule for the basis mappings

\[ \mathcal{I}_M(\mathcal{I}_K(\cdot)) = \delta^L_K \cdot \mathcal{I}_M(\cdot). \]

The composition rule acting on vectors of the space \( \mathbb{U} \) can be considered as a multiplication rule and can be written in the algebraic form instead of the operational one:

\[ h = h_1 \circ h_2. \]
The composition rule for the basis mappings is rewritten as
\[ \mathcal{J}^L_K \circ \mathcal{J}^M_M = \delta^L_K \cdot \mathcal{J}^M_M. \] (4)

The linear mapping operation vector can also be considered as a multiplication
\[ x' = x \circ h. \]

The composition rule (2) for basis vectors of \( \mathbb{X} \) and \( \mathbb{U} \) is rewritten as
\[ \epsilon_L \circ \mathcal{J}^I_K = \delta^I_L \cdot \epsilon_K. \]

We suppose that the composition and addition rules for linear mappings are connected by the distributivity rule. Thereof the linear mapping vector space \( \mathbb{U} \) is an algebra.

C. Turn group of subspace of the generalized space-time

Let \( D \) be a subspace of the generalized space-time \( \mathbb{X} \). If \( D \) is an algebra, the composition rule for the basis vectors \( \epsilon_I \in D \) has the form:
\[ \epsilon_I \circ \epsilon_K = \epsilon_L \cdot C^{I}L_{JK}. \]

Here \( C^{I}L_{JK} \) are the structural constants or the parastrophic matrices of the algebra \( D \).

Let us introduce a scalar product of vectors \( x_1, x_2 \in D \):
\[ \langle x_1, x_2 \rangle = \langle \epsilon_I, \epsilon_K \rangle (x_1)^I (x_2)^K = g_{IK} \cdot (x_1)^I (x_2)^K. \]

The quantity \( g_{IK} = \epsilon_0 \cdot C^{0}I_K \) is the metric tensor. Note that for the space of leptons \( \mathbb{C}_4 \), \( g_{IK} \) represents the diagonal matrix whose the diagonal is the signature of basis vectors \( \epsilon_I \). The scalar product of vector by itself is the vector length:
\[ \langle x, x \rangle = g_{IK} \cdot x^I \cdot x^K = x^2. \]

Consider vectors \( x'_1, x'_2 \in D \) resulted from a linear mapping \( h \) of vectors \( x_1, x_2 \in D \). The linear mapping changes the scalar product of vectors in a common case. We extract from all linear mappings rotations which preserve the scalar product in \( D \):
\[ \langle x'_2, x'_1 \rangle = \langle x_2, x_1 \rangle. \] (5)

From the condition (3) it follows that the linear mapping matrix \( h^{L}I \) for rotations should satisfy
\[ g_{LM} \cdot h^{L}I \cdot h^{M}K \cdot g^{KN} = \delta^{N}I. \]

If we introduce a conjugate matrix \( \tilde{h}^{N}L \) as
\[ \tilde{h}^{N}L = g_{LM} \cdot h^{M}K \cdot g^{KN}, \]
the condition that the linear mapping is rotation takes the form
\[ \tilde{h}^{N}L = (h^{-1})^{N}L. \]

D. Parametrical representation of linear mappings

Consider vectors \( h \in \mathbb{U} \) as functions of parameters \( \varphi^\alpha \):
\[ h(\varphi^\alpha) = \mathcal{J}^I_K \cdot h^K_L(\varphi^\alpha). \]

We suppose that the group composition rule acts on parameters \( \varphi^\alpha \):
[\varphi^\alpha = \Phi(\varphi_2^\alpha, \varphi_1^\alpha),
\text{and the correspondence exists between the composition rule on } \{\varphi^\alpha\} \text{ and the multiplication rule on } U:

h(\varphi^\alpha) = h(\varphi_1^\alpha) \circ h(\varphi_2^\alpha),
\text{and the units of both groups are also in correspondence to one another}

h^{*}_I(\varphi^\alpha)|_{\varphi^\alpha=0} = \delta^{*}_I.

For the turn group, such parameters are called turn angles. Consider a differential dh for h close to the group unit:

dh(\varphi^\alpha) = \mathfrak{J}_L \frac{\partial h^{*}_I(\varphi^\alpha)}{\partial \varphi^\alpha} d\varphi^\alpha = \mathfrak{J}_L \cdot K^{*}_I \cdot d\varphi^\alpha.

Here the notation

K^{*}_I = \left. \frac{\partial h^{*}_I(\varphi^\alpha)}{\partial \varphi^\alpha} \right|_{\varphi^\alpha=0}

was introduced. The vectors

\mathfrak{J}_\alpha = \mathfrak{J}_L \cdot K^{*}_I

are the basis vectors in the space of vectors of the type dh = \mathfrak{J}_\alpha d\varphi^\alpha. The multiplication rule for these basis vectors has the form

\mathfrak{J}_\alpha \circ \mathfrak{J}_\beta = \mathfrak{J}_\gamma \cdot C^{\gamma}_{\alpha \beta}.

(7)

If we substitute (8) in (7) and take into account (8) we obtain

K^{*}_L \cdot K^{*}_I = K^{*}_I \gamma \cdot C^{\gamma}_{\alpha \beta}.

Comparing this relation with (7), we conclude that the basis vectors \mathfrak{J}_\alpha can be put into the correspondence with the matrices K^{*}_I:

\mathfrak{J}_\alpha \sim K^{*}_I.

This correspondence will be called a parametrical representation of the linear mapping algebra.

E. Turns in the Clifford algebra

For the Clifford algebra, the turn matrix around the axis passing through the origin and parallel to the vector \(\varepsilon_\alpha\), can be written as

\begin{align*}
h^{K}_I(\varphi^\alpha) = \begin{cases} 
\delta^K_I \cdot \cos \varphi^\alpha + C^K_{I\alpha} \cdot \sin \varphi^\alpha, & \text{ for } (\varepsilon_\alpha)^2 = -1; \\
\delta^K_I \cdot \cosh \varphi^\alpha + C^K_{I\alpha} \cdot \sinh \varphi^\alpha, & \text{ for } (\varepsilon_\alpha)^2 = 1,
\end{cases}
\end{align*}

where \(C^K_{I\alpha}\) are the parastrophic matrices of the regular representation of basis vectors \(\varepsilon_\alpha\) (see [2]), there is no summation over the index \(\alpha\). From here it follows that

K^{K}_I = \left. \frac{\partial h^{K}_I(\varphi^\alpha)}{\partial \varphi^\alpha} \right|_{\varphi^\alpha=0} = C^K_{I\alpha}

(8)

for turns in the Clifford algebra. If we use an inverse regular representation, i.e. the correspondence of basis vectors to parastrophic matrices, we obtain the representation of turns in the Clifford algebra through the basis vectors:

\begin{align*}
h(\varphi^\alpha) = \begin{cases} 
\varepsilon_0 \cos \varphi^\alpha + \varepsilon_\alpha \sin \varphi^\alpha, & \text{ for } (\varepsilon_\alpha)^2 = -1; \\
\varepsilon_0 \cosh \varphi^\alpha + \varepsilon_\alpha \sinh \varphi^\alpha, & \text{ for } (\varepsilon_\alpha)^2 = 1.
\end{cases}
\end{align*}

Thus

\mathfrak{J}_\alpha = \mathfrak{J}_L \cdot C^K_{I\alpha} = \varepsilon_\alpha.
F. Gauge group. Gauge field

Let us suppose that vectors \( h \in U \) are functions of vectors \( x \in X \). In this case, the linear mappings will be called *gauge transformations*. We shall assume that the gauge transformation group is responsible for interaction. The function

\[
h(x) = \mathcal{J}^{J}_K \cdot h^K_I(x)
\]

will be named a *gauge h-field*.

A differential of transformation \( h''(x) = h'(x) \circ h(x) \) is

\[
dh'' = dh' \circ h + h' \circ dh.
\]

We multiply this expression on the inverse vector \( h''^{-1} = h^{-1} \circ h^{-1} \) at the left

\[
h''^{-1} \circ dh'' = h^{-1} \circ h^{-1} \circ dh' \circ h + h^{-1} \circ dh.
\]

(9)

Introduce a function

\[
A^{KLM}(x) = (h^{-1})^{K}_{N} \frac{\partial h^{L}_{I}}{\partial x^{M}},
\]

which will be called a *gauge field potential* for an arbitrary gauge transformation. From (9) we obtain a common transformation rule for the potential

\[
A''^{KLM} = (h^{-1})^{K}_{N} \cdot A'_{NLM} \cdot h^{I}_{L} + A^{KLM}.
\]

The above definition of the potential and its transformation rule are simplified when the gauge transformations are close to the unit transformation. In this case \( dh'' = dh' + dh \),

\[
A^{KLM} = \frac{\partial h^{K}_{I}}{\partial x^{M}}, \quad A''^{KLM} = A'^{KLM} + \frac{\partial h^{K}_{I}}{\partial x^{M}},
\]

and in the parametrical representation

\[
A^{KLM} = \frac{\partial h^{K}_{I}(\varphi^\alpha)}{\partial \varphi^\alpha} \bigg|_{\varphi^\alpha=0} \frac{\partial \varphi^\alpha}{\partial x^{M}} = \mathcal{K}^{K}_{\alpha} \cdot A^{\alpha M}.
\]

G. Structure equations of the generalized space-time in a gauge field

We shall further restrict our consideration to particles, but in the conclusions we shall discuss how antiparticles can be described together with particles. Introduce a vector space \( T = X + U \) which will be called *kinematic*. Besides the multiplications \( X \circ X \rightarrow X, U \circ U \rightarrow U \) and \( X \circ U \rightarrow X \), we define the multiplication \( U \circ X \rightarrow 0 \). Thus we shall use the following multiplication rules for the basis vectors:

\[
\varepsilon^{I} \circ \varepsilon^{K} = \varepsilon_{L} \cdot C^{L}_{IK}, \quad (10a)
\]

\[
\mathcal{J}^{L}_{K} \circ \mathcal{J}^{J}_{M} = \mathcal{J}^{L}_{M} \cdot \delta_{K}^{J}, \quad (10b)
\]

\[
\varepsilon_{L} \circ \mathcal{J}^{J}_{K} = \varepsilon_{K} \cdot \delta_{L}^{J}, \quad (10c)
\]

\[
\mathcal{J}^{J}_{M} \circ \varepsilon^{K} = 0. \quad (10d)
\]

As a result, the kinematic space \( T \) becomes algebra. Note that the simplest rule (10d) is necessary for closing the kinematic algebra. We write the multiplication rule for vectors \( t, t_1, t_2 \in T \) as

\[
t = t_1 \circ t_2. \quad (11)
\]

For algebras, there are typical differential relations resulting from the derivation of the multiplication rule. These relations are called the *structure equations*. It was shown in our previous paper [3] that the quantum mechanics
equations for free particles can be derived from the structure equations for subalgebras of the generalized space-time \(X\) and those of the action space \(S X\). We shall apply this approach to the kinematic algebra \(T\).

Differential of a vector \(t\) with variation in a vector \(t_i\) will be denoted by \(\delta t_i\). The double derivation of the multiplication rule (11), at first by \(t_1\) and next by \(t_2\), gives the common structure equation of the kinematic algebra:

\[
\delta_2 \delta_1 t = \delta_1 t \circ (t)^{-1} \circ \delta_2 t.
\]

For \(t\) close to the group unit, it is reduced to

\[
\delta_2 (\delta_1 x + \delta_1 h) = (\delta_1 x + \delta_1 h) \circ (\delta_2 x + \delta_2 h),
\]

where it was taken into account that \(t = x + h\). Let us consider the projection of the last relation on the generalized space-time:

\[
\delta_2 \delta_1 x = \delta_1 x \circ \delta_2 t.
\]

This equation will be called the structure equation of the generalized space-time in a gauge field.

### III. RELATIVISTIC QUANTUM MECHANICS EQUATIONS FOR PARTICLES IN A GAUGE FIELD

#### A. Action space and its linear mappings

In our book [1], the ability of bodies to interact was associated with the presence of the action vector \(S\). Such vectors form the vector space \(S X\). We have assumed that the space \(S X\) is similar to the space \(X\), bearing in mind that basis vectors of the space \(X\) can be used as basis vectors in the space \(S X\). Therefore the vector \(S \in S X\) can be written as \(S = e_N \cdot S^N\). \(S X\) is algebra as well as \(X\) with the same multiplication rule for basis vectors.

In the present work we expand the notion of the action vector \(S\). Let us consider that the action space \(S\) is the sum \(S = S X + S U\), and the vector spaces \(S X\) and \(S U\) are similar to the spaces \(X\) and \(U\), respectively. In other words, the action vector \(S \in S\) is represented by the sum of two component

\[
S = S_x + S_h.
\]

Here \(S_x = e_N \cdot S^N \in S X\), and \(S_h = I^I_L \cdot S^I_L \in S U\).

We consider the action vector as a function of generalized space-time vector \(x\) and gauge \(h\)-field: \(S = S(x, h)\). Let us suppose that the coordinates \(S^K\) depend only on \(x\), and the coordinates \(S^N_L\) depend only on \(h(\varphi^\alpha)\), that is

\[
S(x, \varphi^\alpha) = e_N \cdot S^N_N(x) + I^I_L \cdot S^I_L(\varphi^\alpha).
\]

Consider a differential of action vector

\[
dS = \frac{\partial S}{\partial x^M} dx^M + \frac{\partial S}{\partial \varphi^\alpha} d\varphi^\alpha.
\]

Let us introduce a generalized impulse

\[
p_M \equiv - \frac{\partial S}{\partial x^M} = - \frac{\partial S_x}{\partial x^M} = - e_N \frac{\partial S^N}{\partial x^M} = e_N \cdot p^N_M,
\]

and a generalized moment

\[
m_\alpha \equiv - \frac{\partial S}{\partial \varphi^\alpha} = - \frac{\partial S_h}{\partial \varphi^\alpha} = - I^I_L \frac{\partial S^I_L}{\partial \varphi^\alpha} = I^I_L \cdot m^I_L \alpha.
\]

Thus,

\[
dS = -p_M \cdot dx^M - m_\alpha \cdot d\varphi^\alpha.
\]
We discuss the parametrical representation of vectors $S_h \in SU$. In line with the Section II D, we suppose that the vectors $S_h$ are functions of parameters $\mathfrak{s}^\alpha$ with dimensionality of action:

$$S_h(\mathfrak{s}^\alpha) = \mathcal{J}_I \cdot S^I_L(\mathfrak{s}^\alpha),$$

and the group composition rule acts on the parameters $\mathfrak{s}^\alpha$ similar to the parameters $\varphi^\alpha$:

$$\mathfrak{s}^\alpha = \Phi(\mathfrak{s}_2^\alpha, \mathfrak{s}_1^\alpha).$$

We also assume that the correspondence exists between the composition rule on $\mathfrak{s}$ and the multiplication rule on $SU$:

$$S_h(\mathfrak{s}^\alpha) = S_h(\mathfrak{s}_1^\alpha) \circ S_h(\mathfrak{s}_2^\alpha),$$

and the units of both groups are in correspondence to one another

$$S^I_L(\mathfrak{s}^\alpha)|_{\mathfrak{s}^\alpha=0} = S^0 \delta^I_L,$$

where the scalar component of action vector $S^0$ is action in a classical sense. Let us consider a differential

$$dS_h(\mathfrak{s}^\alpha) = \mathcal{J}_I \cdot \frac{\partial S^I_L(\mathfrak{s}^\alpha)}{\partial \mathfrak{s}^\alpha} \, d\mathfrak{s}^\alpha$$

for $S_h$ close to the group unit. From the similarity of the spaces $SU$ and $U$ it follows that

$$\left. \frac{\partial S^I_L(\mathfrak{s}^\alpha)}{\partial \mathfrak{s}^\alpha} \right|_{\mathfrak{s}^\alpha=0} = \mathcal{K}^I_L \cdot \mathfrak{s}^\alpha.$$

Then

$$dS_h(\mathfrak{s}^\alpha) = \mathcal{J}_I \cdot \mathcal{K}^I_L \cdot d\mathfrak{s}^\alpha = \mathcal{J}_a \cdot d\mathfrak{s}^\alpha,$$

where the relation (11) was used. The vector $\mathcal{J}_a$ can be considered as the basis one in the space of vectors of the type $dS_h = \mathcal{J}_a \cdot d\mathfrak{s}^\alpha$. In the parametrical representation, the gauge $h$-field can be expressed through angles $\varphi^\alpha$. The coordinates $S^I_L(h)$ can also be written as functions of angles $\varphi^\alpha$: $S^I_L = S^I_L(h(\varphi^\alpha))$. Therefore, one can say about the parametrical representation of action vector $S_h$. The generalized moment coordinates are written in the parametrical representation as

$$m^I_L, \beta = \frac{\partial S^I_L(h)}{\partial \varphi^\beta} = \mathcal{K}^I_L \cdot \frac{\partial \mathfrak{s}^\alpha(\varphi)}{\partial \varphi^\beta} = \mathcal{K}^I_L \cdot g^\alpha_{\beta}, \quad (12)$$

Functions $g^\alpha_{\beta} \equiv \partial \mathfrak{s}^\alpha / \partial \varphi^\beta$ form a coupling matrix.

We assume that any type of interactions can be associated with some subgroup of the gauge group. Let us consider that the subgroup of $i$-th type of interactions has a single coupling coefficient $g_i$ which will be called a gauge charge of this type of interactions. In other words, we suppose that the following relation is fulfilled

$$g^\alpha_{\beta i} = g_i \cdot \delta^\alpha_{\beta i}, \quad (13)$$

Thus the gauge charge has a meaning of the coefficient of similarity between the parameters $\mathfrak{s}^\alpha$ and $\varphi^\alpha$ used for the representation of vectors $S_h$ and $h$. Using (12) and (13) we get

$$\frac{\partial S^I_L}{\partial x^M} = \mathcal{K}^I_L \cdot \frac{\partial \mathfrak{s}^\alpha}{\partial x^M} = \mathcal{K}^I_L \cdot \frac{\partial \mathfrak{s}^\alpha(x)}{\partial x^M} = \mathcal{K}^I_L \cdot g^\alpha_{\beta} \cdot A^\beta_M = \sum_i g_i \cdot \mathcal{K}^I_L \cdot A^\alpha_M, \quad (14)$$

where $\mathcal{K}^I_{L \alpha i}$ are the parastrophic matrices of $i$-th subgroup of gauge transformations; $A^\alpha_M$ is the gauge field potential appropriate to this subgroup; the summation is over all interactions. From this relation, we obtain the potential expressed through the parameter coordinates $\mathfrak{s}^\alpha$:

$$A^I_{LM} = \frac{1}{g} \frac{\partial S^I_L}{\partial x^M} = \frac{1}{g} \mathcal{K}^I_L \cdot \frac{\partial \mathfrak{s}^\alpha}{\partial x^M}.$$
and the transformation rule for the potential:

\[ A''_{LM} = A'_{LM} + \frac{1}{g} \frac{\partial S'_L}{\partial x^M}. \]

As well as the algebra \( \mathcal{T} \), the algebra \( \mathcal{S} \) has the structure equation

\[ \delta_2 \delta_1 S = -\frac{1}{S^0} \delta_1 S \circ \delta_2 S. \]

After the projecting on the action subspace \( \mathcal{S} x \), this equation takes the form:

\[ \delta_2 \delta_1 S_x = -\frac{1}{S^0} \delta_1 S_x \circ \delta_2 S. \]

(15)

It will be called the structure equation of the action space \( \mathcal{S} x \) in a gauge field. Let us also write the structure equations in the coordinate form

\[ \delta_2 \delta_1 S^I = -\frac{1}{S^0} (C^I_{LN} \cdot \delta_2 S^N \cdot \delta_1 S^L + \delta_2 S'_L \cdot \delta_1 S^L). \]

(16)

B. Quantization equations in differentials

Hereafter, we shall use the system of natural units \((\hbar = c = 1)\).

We perform the passage to the dynamic equations of quantum mechanics by according to the new interpretation of the wave function. In [1,2], the wave function was interpreted as a differential of action vector

\[ \psi = \delta_1 S_x. \]

In the equations (15) and (16) we denote the differential \( \delta_2 \) by \( d \). Let us set \( S^0 = \hbar = 1 \), i.e. we shall consider the equations for action vector values close to the Planck constant. The structure equations with respect to the wave function

\[ d\psi + \psi \circ dS = -\psi \circ dS_x \]

or with respect to its coordinates

\[ d\psi^I + dS'_L \cdot \psi^L = -C^I_{LN} \cdot \psi^L \]

(17)

will be called quantization equations in differentials.

We shall suppose that action vectors and their linear transformations are functions of generalized space-time vectors. Then from the quantization equations in differentials (17), the relations follow

\[ \partial_M \psi^I (x) + \partial_M S'_L \cdot \psi^L = C^I_{LN} \cdot p^N_M \cdot \psi^L, \]

where \( p^L_M = -\partial_M S^L \) are the generalized impulse coordinates. These equations will be named quantum postulates for particles in a gauge field. Using the relations (1) the quantum postulates take the form

\[ \partial_M \psi^I (x) + K^I_{La} \cdot g^\alpha \cdot A^\beta_M \cdot \psi^L = C^I_{LN} \cdot p^N_M \cdot \psi^L. \]

(18)

C. Relativistic quantum mechanics equations for particles in a gauge field

Multiply the quantum postulates (18) by \( C^{MK}_I \):

\[ C^{MK}_I (\partial_M \psi^I (x) + K^I_{La} \cdot g^\alpha \cdot A^\beta_M \cdot \psi^L) = C^{MK}_I \cdot C^I_{LN} \cdot p^N_M \cdot \psi^L. \]

These equations will be called relativistic quantum mechanics equations in Dirac’s form for particles in a gauge field. We suppose that the wave function \( \psi(x) \) depends only on coordinates of the space-time \( X \). We pass from the generalized action space \( \mathcal{S} \) to the space of leptons \( \mathcal{S} C_4 \) and to the turn group in this space. Then we obtain
\[
C^{mK_I} \cdot \partial_m \psi^L (x) + C^{MK_I} \cdot C^I_{L \alpha} \cdot g^\alpha_\beta \cdot A^\beta_M \cdot \psi^L = C^{MK_I} \cdot C^I_{LN} \cdot p^N_M \cdot \psi^L, \quad (m = 1, \ldots, 4). \tag{19}
\]

Here \( \psi^L (x) \) are sixteen real functions, \( C^{mK_I} \) are the regular representation matrices of basis vectors of the space-time \( X \) in the space of leptons \( \mathbb{C}_4 \) and, in addition, the relation \( (3) \) was used. These equations will be named the relativistic quantum mechanics equations for the leptons in a gauge field.

According to \( (3) \), we assume that the generalized impulse \( p^N_M \) has only the two components
\[
p^0 = -\partial_0 S^0 = \frac{m}{2}, \quad p^{34} = -\partial_0 S^{34} = \frac{m}{2}.
\]

If we substitute these components in \( (19) \) and use the relations \( C^{mK_I} = \delta^{K_I}, C^I_{L0} = \delta^I_L \), and \( (13) \), we obtain the quantum mechanics equations for the leptons in a gauge field in the final form:
\[
C^{mK_I} \cdot \partial_m \psi^L (x) + C^{MK_I} \sum_i g_i C^I_{L \alpha_i} \cdot A^\alpha_{\alpha_M} \cdot \psi^L = \frac{m}{2} (\delta^K_L + C^K_{L34}) \psi^L. \tag{20}
\]

D. Quantum mechanics equations for the leptons in the electromagnetic field

We assume that the direct product of the turn groups about the axes \( \varepsilon_{21} \) and \( \varepsilon_{1324} \):
\[
h_1 = \varepsilon_0 \cos \varphi_{21} + \varepsilon_{21} \sin \varphi_{21}, \quad h_2 = \varepsilon_0 \cos \varphi_{1324} + \varepsilon_{1324} \sin \varphi_{1324},
\]
in the space of leptons \( \mathbb{C}_4 \) is responsible for the electromagnetic interaction of leptons. Each of the groups is isomorphic to the turn group \( U(1) \) in the complex plane. Therefore the group, responsible for the electromagnetic interaction of leptons, is isomorphic to the product \( U(1) \times U(1) \). This group will be called electrical. Thus, in order to describe the leptons in the electromagnetic field we set in the equations \( (20) \)
\[
\alpha_i = 21, 1324, \quad g_i = \frac{e}{2},
\]
where \( e \) is the elementary electric charge. Let us assume that the angles \( \varphi_{21}, \varphi_{1324} \) depend only on coordinates of the space-time \( X \). In other words, the electromagnetic field is described by the potential components
\[
A^\alpha_M = \frac{\partial \varphi^\alpha}{\partial x^M} = \{ A^{21}_m, A^{1324}_m \}.
\]

In the space of the electrical group, we select the section \( \varphi_{1324} = \varphi_{21} \equiv \varphi \). Then
\[
A^{21}_m = A^{1324}_m = A_m,
\]
where \( A_m \) are the electromagnetic potential components. As a result, the equations \( (20) \) are reduced to the quantum mechanics equations for the leptons in the electromagnetic field:
\[
C^{mK_I} \left( \partial_m \psi^L + \frac{e A_m}{2} (C^I_{L21} + C^I_{L1324}) \psi^L \right) = \frac{m}{2} (\delta^K_L + C^K_{L34}) \psi^L.
\]

Let us now pass to the quaternion representation of the parastr ophic matrices and that of the wave functions (see \( (3) \)) for which
\[
C^I_{L21} = i \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad C^I_{L1324} = i \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.
\]

For the leptons of the first generation in the electromagnetic field, we obtain
\[
||C^{mK_I}|| \left( \partial_m \begin{pmatrix} \Psi^0 \\ \Psi^{34} \\ \Psi^{123} \\ \Psi^{124} \end{pmatrix} + \frac{e A_m}{2} \begin{pmatrix} \Psi^0 + \Psi^{34} \\ \Psi^0 + \Psi^{34} \\ \Psi^{123} + \Psi^{124} \\ \Psi^{123} + \Psi^{124} \end{pmatrix} \right) = \frac{m}{2} \begin{pmatrix} \Psi^0 + \Psi^{34} \\ \Psi^0 + \Psi^{34} \\ \Psi^{123} + \Psi^{124} \\ \Psi^{123} + \Psi^{124} \end{pmatrix}.
\]

Here the matrices \( C^{mK_I} = \{ C^{aK}_I, C^{4K}_I \} \) \( (a = 1, 2, 3) \) have the form:
In order to obtain the equations with respect to the right and left components of electron and those of e-neutrino, we transform these equations by the same way as was made in \[2\]. At first we add the first equation with the second one and the third one with the fourth one. Then we subtract the third equation from the fourth one, and the first one from the second one. Taking into account that

\[
\Psi^0 + \Psi^{34} = \psi_L, \text{ the left component of electron,}
\]

\[
\Psi^{123} + \Psi^{124} = \psi_R, \text{ the right component of electron,}
\]

\[
\Psi^{123} - \Psi^{124} = \nu_L, \text{ the left component of e-neutrino,}
\]

\[
\Psi^0 - \Psi^{34} = \nu_R, \text{ the right component of e-neutrino},
\]

we obtain the quantum mechanics equations for the leptons of the first generation in the electromagnetic field

\[
i \gamma_1^m (\partial_m + i e A_m) \psi_L = m \psi_L,
\]

\[
i \gamma_2^m (\partial_m + i e A_m) \psi_R = m \psi_R,
\]

\[
i \gamma_1^m \partial_m \nu_L = 0,
\]

\[
i \gamma_2^m \partial_m \nu_R = 0.
\]

Here

\[
\gamma_1^m = \{-\sigma^a, 1\}, \quad \gamma_2^m = \{\sigma^a, 1\}.
\]

Thus, the system of four equations is transformed to the two independent systems of two equations. As one would expect, the right and left electrons interact with the electromagnetic field with the identical coupling constant, and the neutrino does not interact with the electromagnetic field.

**E. Quantum mechanics equations for the leptons in the electroweak field**

1. **The first approximation**

The interaction of leptons with the electromagnetic field is considered by according to the previous Section. We assume that the direct product of the turn groups about the axes \(\varepsilon_{1234}, \varepsilon_4, \varepsilon_{123}, \varepsilon_{34}, \varepsilon_{124}, \varepsilon_3\):

\[
h_1 = \varepsilon_0 \cos \varphi_{1324} + \varepsilon_{1324} \sin \varphi_{1324},
\]

\[
h_2 = \varepsilon_0 \cos \varphi_4 + \varepsilon_4 \sin \varphi_4,
\]

\[
h_3 = \varepsilon_0 \cos \varphi_{123} + \varepsilon_{123} \sin \varphi_{123},
\]

\[
h_4 = \varepsilon_0 \cosh \varphi_{34} + \varepsilon_{34} \sinh \varphi_{34},
\]

\[
h_5 = \varepsilon_0 \cosh \varphi_{124} + \varepsilon_{124} \sinh \varphi_{124},
\]

\[
h_6 = \varepsilon_0 \cosh \varphi_3 + \varepsilon_3 \sinh \varphi_3
\]

in the space of leptons \(C_4\) is responsible for the weak interaction of leptons. This group will be called weak. In this approximation, the turn group about the axis \(\varepsilon_{1324}\) is responsible for the mixed electroweak interaction with \(A\) and \(Z\) fields. Note that the weak group is isomorphic to the Lorentz group and can be represented as the product \(SU(2) \times SU(2)\).

Thus, in order to describe the interaction of leptons with the weak field, we should set in \(20\)

\[
\alpha_i = 1324, 4, 123, 34, 124, 3, \quad g_i = \frac{g_W}{2}.
\]

The gauge charge of the weak group is given by the constant \(g_W\). We suppose that the angles \(\varphi_{1324}, \varphi_4, \varphi_{123}\) depend on the space-time coordinates, and the angles \(\varphi_{34}, \varphi_{124}, \varphi_3\) depend on the coordinates \(x^{1324}, x^{134}, x^{124}, x^{123}\) which can be written as \(x^{m1324}\). From here follows that the weak field is described by the potential components

\[
A^\alpha_M = \frac{\partial \varphi^\alpha}{\partial x^M} = \{A^{1324}_m, A^4_m, A^{123}_m, A^{34}_{m1324}, A^{124}_{m1324}, A^3_{m1324}\}.
\]

We postulate the following correspondences:
$g_{1324} A^{1324}_{m} \equiv \frac{e}{2} A_{m} + \frac{g_{W}}{2} Z_{m}$, \hspace{1em} $A^{3}_{m1324} \equiv Z_{m}$, \hspace{1em} $A^{4}_{m} = A^{124}_{m1324} \equiv W^{1}_{m}$, \hspace{1em} $A^{123}_{m} = A^{3}_{m1324} \equiv W^{2}_{m}$,

where $Z_{m}$, $W^{1}_{m}$, $W^{2}_{m}$ are the weak field potentials.

As a result, the equations (20) is reduced to the quantum mechanics equations for the leptons in the electroweak field:

$$
C^{mK}_{l} \left( \partial_{m} \psi^{I} + \frac{e A_{m}}{2} (C^{I}_{L21} + C^{I}_{L1324}) \psi^{L} + \frac{g_{W}}{2} (Z_{m} C^{I}_{L1324} + W^{1}_{m} C^{I}_{L4} + W^{2}_{m} C^{I}_{L123}) \psi^{L} \right) + C^{1324mK}_{P} \frac{g_{W}}{2} (Z_{m} C^{P}_{L34} + W^{1}_{m} C^{P}_{L24} + W^{2}_{m} C^{P}_{L13}) \psi^{L} = \frac{m}{2} (\delta^{K}_{L} + C^{K}_{L34}) \psi^{L}.
$$

If we write the matrix $C^{1324mK}_{P}$ as the product

$$
C^{1324K}_{l} \cdot C^{mI}_{l} = -C^{mK}_{l} \cdot C^{1324I}_{l},
$$

and pass to the quaternion representation for which

$$
C^{I}_{L34} = -i C^{I}_{L1324}, \hspace{1em} C^{I}_{L124} = -i C^{I}_{L4}, \hspace{1em} C^{I}_{L3} = -i C^{I}_{L123},
$$

we obtain

$$
C^{mK}_{l} \left( \partial_{m} \psi^{I} (x) + \frac{e A_{m}}{2} (C^{I}_{L21} + C^{I}_{L1324}) \psi^{L} \right) + \frac{g_{W}}{2} (\delta^{I}_{P} - i C^{1324I}_{P}) (Z_{m} C^{P}_{L34} + W^{1}_{m} C^{P}_{L24} + W^{2}_{m} C^{P}_{L13}) \psi^{L} = \frac{m}{2} (\delta^{K}_{L} + C^{K}_{L34}) \psi^{L}.
$$

After the substitution of parastrophic matrices

$$
C^{1324I}_{P} = i, \hspace{1em} C^{I}_{L1324} = i, \hspace{1em} C^{I}_{L4} = i, \hspace{1em} C^{I}_{L123} = i,
$$

the quantum mechanics equations with respect to the quaternion components of wave function take the form

$$
\|C^{mK}_{l}\| \left\{ \begin{array}{c}
\psi^{0} \\
\psi^{34} \\
\psi^{123} \\
\psi^{124}
\end{array} \right\} + \frac{ie A_{m}}{2} \left\{ \begin{array}{c}
\psi^{0} + \psi^{34} \\
\psi^{0} + \psi^{34} \\
\psi^{123} + \psi^{124} \\
\psi^{123} + \psi^{124}
\end{array} \right\} + \frac{g_{W}}{2} (Z_{m} \left\{ \begin{array}{c}
\psi^{124} - \psi^{123} \\
\psi^{124} - \psi^{123} \\
\psi^{123} - \psi^{124} \\
\psi^{123} - \psi^{124}
\end{array} \right\} + W^{1}_{m} \left\{ \begin{array}{c}
\psi^{124} - \psi^{123} \\
\psi^{124} - \psi^{123} \\
\psi^{123} - \psi^{124} \\
\psi^{123} - \psi^{124}
\end{array} \right\} + \frac{m}{2} \left\{ \begin{array}{c}
\psi^{0} + \psi^{34} \\
\psi^{0} + \psi^{34} \\
\psi^{123} + \psi^{124} \\
\psi^{123} + \psi^{124}
\end{array} \right\} \right\} = \frac{m}{2} \left\{ \begin{array}{c}
\psi^{0} \\
\psi^{34} \\
\psi^{123} \\
\psi^{124}
\end{array} \right\}.
$$

From here the quantum mechanics equations for the leptons of the first generation in the electroweak field follow:

$$
i \gamma^{m}_{1} (\partial_{m} + ie A_{m}) e_{R} = m e_{L},
$$

$$
i \gamma^{m}_{2} (\partial_{m} e_{L} + ie A_{m} e_{L} + ig_{W} Z_{m} e_{L} - ig_{W} [W^{1}_{m} - i W^{2}_{m}] \nu_{eL}) = m e_{R},
$$

$$
i \gamma^{m}_{1} \partial_{m} \nu_{eL} = 0,
$$

$$
i \gamma^{m}_{2} (\partial_{m} \nu_{eL} - i g_{W} Z_{m} \nu_{eL} - ig_{W} [W^{1}_{m} + i W^{2}_{m}] e_{L}) = 0.
$$

The right and left electrons interact with the electromagnetic field with the identical coupling constant. In this approximation, the right electron does not interact with the weak field in contrast with the left electron and neutrino. The right neutrino does not interact with the electroweak field and therefore it cannot be revealed in the specified interactions.

2. The second approximation

In the previous Section, the variant of the electroweak theory was considered such that only the left leptons interact the weak field. In the wake of the Glashow–Weinberg–Salam theory, we shall try to take into account the interaction
of the right leptons with the weak $Z$-field. For this propose, the weak group is supplemented by the turn group about the axis $\varepsilon_{21}$ in the space of leptons $\mathbb{C}_4$. In other words, we consider that the group of turns

$$h = \varepsilon_0 \cos \varphi_{21} + \varepsilon_{21} \sin \varphi_{21}$$

is also responsible for the mixed electroweak interaction with $A$ and $Z$ fields, in addition to the turn group about the axis $\varepsilon_{1324}$. The gauge charge of this interaction with $Z$-field will be written as $g_i = -g_1/2$. As before, we assume that the angle $\varphi_{21}$ depends only on the space-time coordinates, and for this subgroup

$$g_{21} A_{21}^m \equiv \frac{e}{2} A_m - \frac{g_1}{2} Z_m.$$ 

After the calculations similar to the previous ones, we obtain

$$i \gamma_1^m (\partial_m + i e A_m - i g_1 Z_m) e_R = m e_L,$$  \hspace{1cm} (21a)

$$i \gamma_2^m (\partial_m e_L + i e A_m e_L + i [g_W - g_1] Z_m e_L - i g_W [W_{1m}^1 - i W_{2m}^2] \nu_{eL}) = m e_R,$$  \hspace{1cm} (21b)

$$i \gamma_1^m (\partial_m - i g_1 Z_m) \nu_{eR} = 0,$$  \hspace{1cm} (21c)

$$i \gamma_2^m (\partial_m \nu_{eL} - i [g_W + g_1] Z_m \nu_{eL} - i g_W [W_{1m}^1 + i W_{2m}^2] e_L) = 0.$$  \hspace{1cm} (21d)

In this approximation, the right and left electrons interact with the electromagnetic field with the identical coupling constant, the left electron and neutrino interact with the weak fields $W$ and $Z$, the right electron and neutrino interact with the weak $Z$-field.

In [3] we have shown that such a remarkable phenomenon as an $e$-$\mu$-$\tau$ universality owes its origin to the algebraic equivalency of the basis vectors $\varepsilon_{21}$, $\varepsilon_{13}$, $\varepsilon_{32}$ in the space of leptons $\mathbb{C}_4$. Recall that these basis vectors are used for describing three lepton generations. In virtue of the $e$-$\mu$-$\tau$ universality, the equations for the interaction of the muon and $\tau$-lepton with the electroweak field are similar to ones presented above.

3. **Comparison with the Glashow–Weinberg–Salam theory**

According to the Glashow–Weinberg–Salam theory, the lagrangian of electroweak interaction for the leptons of the first generation has the form (see, for example, [3]):

$$L = i \bar{e}_R \gamma_1^m (\partial_m e_R + i e A_m - i g_1 Z_m) e_R - g \sin \theta_W \bar{e}_R \gamma_1^m e_R A_m - g \sin \theta_W \bar{e}_L \gamma_2^m e_L A_m - g \sin \theta_W \bar{e}_L \gamma_2^m e_L Z_m - \frac{g}{2 \cos \theta_W} \sin^2 \theta_W \bar{e}_R \gamma_1^m e_R Z_m - \frac{\sqrt{2} g}{2 \cos \theta_W} \bar{e}_L \gamma_2^m e_L Z_m + \frac{g}{2 \cos \theta_W} \bar{\nu}_{eL} \gamma_2^m \nu_{eL} Z_m,$$

where $\theta_W$ is the Weinberg’s angle, $g$ is the standard coupling constant appropriate the subgroup $SU(2)$ of the Glashow group. The grouping of addends with identical conjugate vectors allows to write the quantum mechanics equations for the leptons of the first generation in the electroweak field in the Glashow–Weinberg–Salam model.

$$i \gamma_1^m (\partial_m + i g \sin \theta_W A_m - i g Z_m \left[ \frac{\sin^2 \theta_W}{\cos \theta_W} \right] ) e_R = m e_L,$$  \hspace{1cm} (21a)

$$i \gamma_2^m (\partial_m e_L + i g \sin \theta_W A_m e_L + i g Z_m \left[ \frac{\sin^2 \theta_W}{\cos \theta_W} \right] ) e_L - \frac{i g (W_{1m}^1 - i W_{2m}^2)}{2} \nu_{eL} = m e_R,$$  \hspace{1cm} (21b)

$$i \gamma_2^m (\partial_m \nu_{eL} - \frac{i g Z_m}{2} \left[ \frac{\sin^2 \theta_W}{\cos \theta_W} \right] ) \nu_{eL} - \frac{i g (W_{1m}^1 + i W_{2m}^2)}{2} e_L = 0.$$  \hspace{1cm} (21c)

The comparison of these equations with the system (21a-d) shows the following differences of our theory from the Glashow–Weinberg–Salam theory:

1. The system of equations is supplemented by the equation for the right neutrino.

2. The coupling constants of interaction between the leptons, the electron and the left neutrino, and the weak field differ from values obtained in the standard model. However, the concrete character of these differences depends on the chosen correspondence of our coupling coefficients $g_W$ and $g_1$ to the parameters $\theta_W$ and $g$ of the standard model. For example, if we identify the expression $(g \sin^2 \theta_W)/(2 \cos \theta_W)$ with our coupling coefficient $g_1$, we see that the coupling constant of interaction between the right electron and the weak $Z$-field in our consideration is 2 times less than one in the Glashow–Weinberg–Salam model.

3. The right neutrino interacts only with the weak $Z$-field with the same coupling constant as the right electron.
IV. CONCLUSIONS

We summarize the more important results found in the previous Sections.

1. The subgroups of turn group of the generalized action space are identified with the groups of interior symmetries responsible for interaction.

2. The direct product of turn groups about the axes $\varepsilon_{21}$ and $\varepsilon_{1324}$ in the Clifford space $\mathbb{S}C_4$ is responsible for the electromagnetic interaction of leptons.

3. The direct product of turn groups about the axes $\varepsilon_{21}$, $\varepsilon_{1324}$, $\varepsilon_4$, $\varepsilon_{123}$, $\varepsilon_{34}$, $\varepsilon_{124}$, $\varepsilon_3$ in $\mathbb{S}C_4$ is responsible for the weak interaction of leptons.

4. The gauge charge is the similarity factor between interaction subgroup parameters (turn angles) in the action space and those in the kinematic space.

5. The relativistic quantum mechanics equations for particles in a gauge field can be derived from the structure equations of the kinematic algebra. We emphasize that such a derivation have no need of the gauge principle which is usually applied for the prolongation of free wave equations onto gauge transformations.

6. The joint description of particles and antiparticles may be found only if we shall expand the kinematic algebra through the generalized conjugate space-time $\tilde{X}$ (for more details, see [2]) and the space $\tilde{U}$ of mappings in $\tilde{X}$. In doing so, the relations (10a-d) should be supplemented by multiplication rules including basis vectors of the spaces $\tilde{X}$ and $\tilde{U}$.

7. With the introduction of the generalized space-time, the space-time and interior space coordinates unite into the coordinates of single vector. This union, combined with the union of wave function components, allows to consider particles in gauge fields from common positions and to count on that the development of theory proposed by us will result in the construction of an unified theory of interactions.

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[3] L. H. Ryder, *Quantum field theory*, Cambridge University Press, s. 8.5 (1985).