On mix-norms and the rate of decay of correlations

Bryan W Oakley, Jean-Luc Thiffeault and Charles R Doering

1 Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706, United States of America
2 Center for the Study of Complex Systems, Department of Mathematics and Department of Physics, University of Michigan, Ann Arbor, MI 48109, United States of America

E-mail: boakley@wisc.edu, jeanjuc@math.wisc.edu and doering@umich.edu

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Abstract

Two quantitative notions of mixing are the decay of correlations and the decay of a mix-norm—a negative Sobolev norm—and the intensity of mixing can be measured by the rates of decay of these quantities. From duality, correlations are uniformly dominated by a mix-norm; but can they decay asymptotically faster than the mix-norm? We answer this question by constructing an observable with correlation that comes arbitrarily close to achieving the decay rate of the mix-norm. Therefore the mix-norm is the sharpest rate of decay of correlations in both the uniform sense and the asymptotic sense. Moreover, there exists an observable with correlation that decays at the same rate as the mix-norm if and only if the rate of decay of the mix-norm is achieved by its projection onto low-frequency Fourier modes. In this case, the function being mixed is called \( q \)-recurrent; otherwise it is \( q \)-transient. We use this classification to study several examples and raise questions for future investigations.

Keywords: mix-norms, decay of correlations, mixing, dynamical systems

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(Some figures may appear in colour only in the online journal)
1. Introduction

Consider a spatially-periodic mean-zero function \( f'(x) = f(t, x) \) bounded uniformly in \( L^2(\mathbb{T}^d) \) for all \( t > 0 \). For example, \( f(t, x) \) might be a solution to the advection-diffusion equation

\[
\frac{\partial f}{\partial t} + u \cdot \nabla f = D\Delta f,
\]

with \( f^0 \in L^2(\mathbb{T}^d) \) and smooth divergence-free velocity field \( u(t, x) \). We may also consider \( D = 0 \) in equation (1.1), in which case it is the transport equation. Another example, in the context of dynamical systems, is when an initial condition \( f^0 \in L^2 \) is transported by an area-preserving map \( M \) via the transfer operator \( f^{n+1} = f^n \circ M^{-1} \).

Decay of the correlation function \( C_t(g) = \langle f'(t), g \rangle \) as \( t \to \infty \) for observables \( g \) in \( L^2(\mathbb{T}^d) \) corresponds to mixing of \( f' \) as \( t \to \infty \) [32]. Mathew et al [25] introduced the \( H^{-1/2} \) norm as another criterion to quantify mixing, and Lin et al [19] extended this to any negative Sobolev (e.g., \( H^{-q} \)) norm and showed that correlations decay to zero if and only if any such ‘mix-norm’ decays to zero. That is,

\[
\lim_{t \to \infty} \langle f'(t), g \rangle = 0 \quad \forall g \in L^2 \iff \lim_{t \to \infty} \| f'(t) \|_{H^{-q}} = 0, \quad \text{for any } q > 0.
\]

Mix-norms are well-suited to quantification of mixing efficiencies [9, 18, 23, 31, 33–36], to lower bounds on the rate of mixing in general [15, 21, 22], and to analyzing mixing [24, 26, 27, 37]. Moreover, such negative Sobolev spaces provide a natural setting for a discussion of enhanced dissipation and relaxation [1, 4, 6–8, 12, 16, 17]. Mathew et al [25] introduced the mix-norm in the context of spatial averages over strips, and made the connection to weak convergence (see also [39]).

While mix-norms are well-adapted to the PDE context, correlations and weak convergence are more commonly studied in the context of ergodic theory. A central question, then, is the quantitative relationship between decay rate of correlations and decay of mix-norms. This is the central focus of this paper where we will work in a setting where the evolution of a function \( f'(x) \) is given, arising from the continuous-time solution of a PDE or in discrete times from an iterated map.

When studying a collection of functions converging to zero as \( t \to \infty \), such as \( \langle f'(t), g \rangle \) for \( g \in X \) where \( X \subset L^2 \) is some Banach space, there are several reasonable ways to define a rate of decay:

(a) We can consider a uniform upper bound [2, 5, 10, 20, 30].
(b) We can say that each function is \( O(t^\varphi) \) where \( \varphi(t) \) is some rate function [38]. This lifts the tail of the rate function by multiplying by a constant that depends on \( g \in X \).
(c) We can instead lift the tail of the rate function by translation and say that each function is bounded above by a translation of some rate function [11].

We summarize as follows (for concreteness, fix some \( q > 0 \) and consider \( X = H^q(\mathbb{T}^d) \)):

(a) Correlations decay at the uniform rate \( r(t) \) for \( g \in H^q \) if

\[
\langle f'(t), g \rangle \leq r(t) \| g \|_{H^q} \quad \forall g \in H^q.
\]

(b) Another criterion to quantify mixing, and Lin et al [25] extended this to any negative Sobolev (e.g., \( H^{-q} \)) norm and showed that correlations decay to zero if and only if any such ‘mix-norm’ decays to zero. That is

\[
\lim_{t \to \infty} \langle f'(t), g \rangle = 0 \quad \forall g \in L^2 \iff \lim_{t \to \infty} \| f'(t) \|_{H^{-q}} = 0, \quad \text{for any } q > 0.
\]

We say that \( \varphi(t) = O(t^\varphi) \) as \( t \to \infty \) if there are \( T, M \) so that \( \varphi(t) \leq Mt^\varphi \) for \( t > T \). For \( b(t) > 0 \), this is equivalent to \( \limsup_{t \to \infty} |a(t)|/b(t) = C \in [0, \infty) \). Moreover, we say \( \varphi(t) = o(b(t)) \) if \( C = 0 \).

We summarize as follows (for concreteness, fix some \( q > 0 \) and consider \( X = H^q(\mathbb{T}^d) \)):

(a) Correlations decay at the uniform rate \( r(t) \) for \( g \in H^q \) if

\[
\langle f'(t), g \rangle \leq r(t) \| g \|_{H^q} \quad \forall g \in H^q.
\]
(b) Correlations decay at the asymptotic rate \( g(t) \) for \( g \in H^q \) if

\[
|\langle f', g \rangle| = O(g), \text{ for each } g \in H^q.
\]  

That is,

\[
\limsup_{t \to \infty} \frac{|\langle f', g \rangle|}{g(t)} = C_g \in [0, \infty).
\]  

(c) Correlations decay at the translational rate \( \lambda(t) \) for \( g \in H^q \) if for each \( g \in H^q \) there exists \( \tau_\lambda \in \mathbb{R} \) such that for all \( t > \tau_\lambda \) we have

\[
|\langle f', g \rangle| \leq \lambda(t - \tau_\lambda) \|g\|_{H^q}.
\]  

From considering duality in section 2, we find that the smallest uniform rate is the mix-norm \( \|f'||_{H^q} \). Since any uniform rate trivially satisfies the definitions of asymptotic rate and translational rate, the question is whether there is a \( g \) (or \( \lambda \)) that decays faster than \( \|f'||_{H^q} \). We answer this question by showing that we cannot have \( g = o(\|f'||_{H^q}) \). Similarly, given the additional assumption that \( \limsup_{t \to \infty} \lambda(t - \tau)/\lambda(t) \) is finite, we cannot have \( \lambda = o(\|f'||_{H^q}) \).

We remark that this growth condition on \( \lambda \) is satisfied by power law and exponential functions.

We prove the above facts by constructing an observable \( g \in H^q \) such that \( |\langle f', g \rangle| \) decays arbitrarily closely to the mix-norm. Namely, for any positive \( h(t) = o(\|f'||_{H^q}) \) there is a \( g \in H^q \) such that \( |\langle f', g \rangle| \) is big-O but not little-O of \( h \). Note that this is not the same as asymptotic equivalence because the correlation may be much smaller than \( h \) at certain times.

Let \( P_I f' \) denote the projection of \( f' \) onto the Fourier modes \( I \). We say \( f' \) is \( q \)-recurrent (otherwise it is \( q \)-transient) if there is a finite set \( I \) where \( \|P_I f'||_{H^q} \) is big-O but not little-O of \( \|f'||_{H^q} \). Heuristically, in this case, the decay of the mix-norm is characterized by \( P_I f' \). We prove \( f' \) is \( q \)-recurrent if and only if there is a \( g \in H^q \) such that \( |\langle f', g \rangle| \) is big-O but not little-O of \( \|f'||_{H^q} \). Therefore, \( q \)-recurrence is the criterion for the existence of a correlation that obtains the decay rate of the mix-norm.

In section 2 we introduce the key definitions and main theorems. Section 3 contains examples, and sections 4 and 5 contains the full proofs of the theorems.

2. Overview

Throughout, it will be more convenient to work with the homogeneous Sobolev spaces \( \dot{H}^\alpha \) for \( \alpha \in \mathbb{R} \). Since the torus \( \mathbb{T}^d \) is a compact manifold, Poincaré’s inequality applies [14] so that the \( H^\alpha \) norm and \( \dot{H}^\alpha \) norm are equivalent for mean-zero functions. For \( \alpha > 0 \), the \( \dot{H}^{-\alpha} \) norm is typically defined via the duality equation

\[
\|f\|_{\dot{H}^{-\alpha}} = \sup_{g \in H^{\alpha}} \frac{|\langle f, g \rangle|}{\|g\|_{H^\alpha}}.
\]

However, there is an equivalent definition [13] for all \( \alpha \in \mathbb{R} \). Let \( f_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-2\pi i x k} \, dx \) denote the Fourier coefficients of \( f(x) \). Then

\[4\] The growth condition is not satisfied by functions that decay faster than exponentially, such as \( \lambda(t) = e^{-t^2} \). In this case, \( \lambda \) is not asymptotically equivalent to its own translation: for large \( t \), we see \( e^{-t^2} = e^{2t^2} e^{-t^2} \gg e^{-t^2} \). For \( \lambda \) not satisfying the growth condition, a translation of \( \lambda \) is much larger than a constant multiple of \( \lambda \). If there is a correlation bounded by a translation of \( \lambda \) but not a constant multiple of \( \lambda \), then \( \lambda = o(\rho) \).
Figure 1. Plotted above are the mix-norm and correlations with different choices of $g$, demonstrating that the mix-norm is the envelope over $|\langle f^t, g \rangle|$ with $\|g\|_{H^q} = 1$.

$$\|f\|_{H^p} = \left( \sum_{k \neq 0} k^{2p} |f_k|^2 \right)^{1/2},$$

where $k^2 = |k|^2 = |k_1|^2 + \ldots + |k_d|^2$. We will typically omit the $k \neq 0$ under the sum since $f_0 = 0$ for mean-zero functions.

Similarly, correlations have a simple expression. Since the trigonometric functions $\{e^{2\pi i k \cdot x}\}$ provide an orthonormal basis for $L^2(\mathbb{T}^d)$, the Fourier transform is a unitary map to $L^2(\mathbb{Z}^d)$. Therefore the Fourier transform preserves the inner product \cite{13} and we have Plancherel’s theorem:

$$\langle f, g \rangle = \sum_k f_k \bar{g}_k \quad \forall f, g \in L^2(\mathbb{T}^d).$$

Now say $q > 0$ and consider $g \in \dot{H}^q$ for the rest of this paper. For time-dependent $f^t(x)$, the duality equation implies $|\langle f^t, g \rangle| \leq \|f^t\|_{H^{-q}} \|g\|_{H^q}$ for all $t$. Moreover, fix $t = t_0$ and take $g$ with Fourier coefficients

$$g_k = f_{k_0} k^{-2q} |f_{k_0}|^{-1}.$$

Then $\|g\|_{H^q} = 1$ and Plancherel’s theorem gives $\langle f_{k_0}, g \rangle = \|f_{k_0}\|_{H^{-q}}$. The correlation achieves the mix-norm at the time $t_0$. Since $t_0$ is arbitrary, we see that $\|f^t\|_{H^{-q}}$ is the envelope of the set of functions $|\langle f^t, g \rangle|$ with $\|g\|_{H^q} = 1$, as in figure 1. This shows the point-wise smallest uniform rate of decay of correlations, in the sense of equation (1.2), is the mix-norm $\|f^t\|_{H^{-q}}$.

Using only duality, the most that can be said about the relationship between the rate of decay of a correlation and the rate of decay of the mix-norm is that

$$|\langle f^t, g \rangle| = O\left(\|f^t\|_{H^{-q}}\right) \quad \text{for each } g \in \dot{H}^q.$$

However, each correlation could decay strictly faster than the mix-norm as illustrated in figure 1. We are then led to ask if such a situation is possible.

When is $|\langle f^t, g \rangle| = o\left(\|f^t\|_{H^{-q}}\right)$ for each $g \in \dot{H}^q$? To answer this question, we must construct functions $g \in \dot{H}^q$ such that the correlations $|\langle f^t, g \rangle|$ decay as slowly as possible. To do this, we first classify $f^t$ as either $q$-recurrent or $q$-transient as follows.
For a set $I \subset \mathbb{Z}^d$ let

$$P_I f' = \sum_{k \in I} f_k e^{2\pi i \mathbf{x} \cdot \mathbf{k}}$$

denote the projection of $f'$ onto the Fourier modes $\mathbf{k} \in I$. Then

$$\|P_I f'\|_{H^{-q}}^2 = \sum_{k \in I} k^{-2q} |f_k|^2$$

measures the amount of mix-norm supported on $I$. We often refer to this as the Fourier energy contained in $I$. This notion of energy is $q$-dependent, though the $q$ will usually be clear from the context.

**Definition 1.** We say $f'$ is $q$-recurrent if there exists a finite set $I \subset \mathbb{Z}^d$ such that

$$\limsup_{t \to \infty} \frac{\|P_I f'_m\|_{H^{-q}}}{\|f'_m\|_{H^{-q}}} > 0.$$  \hspace{1cm} (2.2)

Functions that are not $q$-recurrent will be called $q$-transient.

**Remark.** We emphasize that $q$-recurrence is a property of $f'$ which encompasses both the stirring action and the initial condition coupled together. To clarify, in the context of the advection-diffusion equation (1.1), $q$-recurrence is a property of a particular realization of $\mathbf{u}$ and $f^0$ taken together—it is not just a property of the vector field $\mathbf{u}$. Flows which don’t mix, like $\mathbf{u} = \text{const.}$, will trivially induce a $q$-recurrent trajectory $f'$ for any initial condition and any $q$. However, this stability does not hold in general. In example 2 of section 3, we will see that a given $f'$ may be $q$-recurrent for some (larger) values of $q$ and $q$-transient for other (smaller) $q$. Additionally, we will find that a given flow $\mathbf{u}$ may induce a $q$-recurrent evolution $f'$ for some initial conditions and $q$-transient for others.

From inequality (2.2) and the trivial bound $\|P_I f'_m\|_{H^{-q}} \leq \|f'_m\|_{H^{-q}}$ we see that $\|P_I f'_m\|_{H^{-q}}$ is big-O but not little-O of $\|f'_m\|_{H^{-q}}$. Unpacking the definition of limit supremum offers another interpretation: there exists $\delta > 0$ and a sequence $t_m \to \infty$ where

$$\|P_I f'_m\|_{H^{-q}} \geq \delta \|f'_m\|_{H^{-q}}.$$  \hspace{1cm} (2.3)

This means that there is at least a $\delta$ fraction of the mix-norm supported on $I$ at arbitrarily large times. As time progresses the Fourier energy could move off of $I$, but we can always find a future time $t_{m+1}$ where a proportion $\delta$ of the mix-norm is again on $I$. In other words, some Fourier energy always returns to populate the spatial scales in $I$. In this case, test functions $g$ with coefficients for $\mathbf{k} \in I$ similar to that in equation (2.1) will match well with $f'$ at times $t_m$ (after possibly taking a subsequence) so that in section 4 we can prove the following theorem, a central result of our paper.

**Remark.** The case $\|f^0\|_{H^{-q}} = 0$ is degenerate in the context of the advection-diffusion equation and dynamical systems. In those settings, if the mix-norm is zero at any finite time it will remain zero for future times. In such a case where $f'$ is eventually zero, the mix-norm and correlations will trivially decay at the same rate, the zero function. Otherwise, if the mix-norm is nonzero for a sequence $t_m \to \infty$, we may drop the times where the mix-norm is zero so that $\|f'_m\|_{H^{-q}} > 0$ for all $t > 0$. Without loss of generality, to simplify the presentation of our results, we will take this as an assumption in the theorems below.

Theorem 1. Let $f^t$ be a mean-zero function in $L^2(\mathbb{T}^d)$ with $\|f^t\|_{H^{-q}} > 0$ for all $t > 0$. Then $f^t$ is $q$-recurrent if and only if there is a function $g \in \dot{H}^q$ such that

$$\limsup_{t \to \infty} \frac{|\langle f^t, g \rangle|}{\|f^t\|_{H^{-q}}} > 0.$$  

Equivalently, there is a function $g \in \dot{H}^q$, a constant $c > 0$, and a sequence $t_m \to \infty$ where

$$|\langle f^{t_m}, g \rangle| \geq c \|f^{t_m}\|_{H^{-q}}.$$  (2.4)

Remark. As demonstrated in figure 2, it is possible that $|\langle f^t, g \rangle|$ is small at times $t \neq t_m$ and so we do not show asymptotic equivalence. We interpret our result as demonstrating that the correlation does not decay asymptotically faster than the mix-norm in the sense that $|\langle f^t, g \rangle|$ is big-O but not little-O of $\|f^t\|_{H^{-q}}$. From this theorem, the answer to our previously posed question ‘when is $|\langle f^t, g \rangle| = o(\|f^t\|_{H^{-q}})$ for each $g \in \dot{H}^q$?’ is exactly when $f^t$ is $q$-transient.

This naturally prompts us to ask if $f^t$ is $q$-transient and we carefully choose $g \in \dot{H}^q$, how slowly can we make $|\langle f^t, g \rangle|$ decay? The following theorem answers this question.

Theorem 2. Let $f^t$ be a mean-zero function in $L^2(\mathbb{T}^d)$ with $\|f^t\|_{H^{-q}} > 0$ for all $t > 0$. For any positive function $h(t)$ such that $h(t) = o(\|f^t\|_{H^{-q}})$, there is a function $g \in \dot{H}^q$ such that

$$\limsup_{t \to \infty} \frac{|\langle f^t, g \rangle|}{h(t)} > 0.$$  

Remark. Theorems 1 and 2 do not require $f^t$ to be bounded uniformly in $L^2(\mathbb{T}^d)$ in time, nor for the mix-norm to decay to zero. Additionally, theorem 2 is valid whether $f^t$ is $q$-recurrent or $q$-transient.

The proof of theorem 2 is deferred until section 5, but we present the idea behind the proof now. If $f^t$ is $q$-recurrent, then the proof is accomplished by a result similar to theorem 1. For $q$-transient functions, the proof relies on the construction of sets $I_m$ and times $t_m$ satisfying certain properties, the first being that we want the finite disjoint sets $I_m \subset \mathbb{Z}^d$ to capture a large amount of the Fourier energy at time $t_m$. We can do this since $q$-transience ensures that we can wait for the next time $t_{m+1}$ where a proportion of the Fourier energy moves off of $I_m$ and never comes back. Then by choosing the Fourier coefficients of $g$ on $I_m$ to agree with $f^t$ at time $t_m$, we can guarantee that $|\langle f^t, g \rangle|$ will be large at time $t_m$. Hence, the function $g$ in theorem 2 which gives the slowly decaying $|\langle f^t, g \rangle|$ has Fourier coefficients.
\[
g_k = \begin{cases} 
 f_k^* k^{-2q} \| f_m \|_{H^{\tau_q}}^{-2} h(t_m), & k \in I_m; \\
 0, & \text{otherwise};
\end{cases}
\]  
(2.5)

where \( I_m \) are disjoint and \( \| P_m f_m \|_{H^{\tau_q}} \) captures a nonzero proportion of \( h(t_m) \), similar to inequality (2.3). These coefficients are similar to those of equation (2.1) except with an extra factor of \( h/\| f \|_{H^{\tau_q}} \). This factor is needed so that we can satisfy the second property we require from the sets \( I_m \) and times \( t_m \): by taking a subsequence, we can use the fact that \( h(t) = o \left( \| f \|_{H^{\tau_q}} \right) \) to make the \( g_k \) decay fast enough as \( k \to \infty \) to have \( g \in \dot{H}^q \). Hence, although correlations may not achieve the decay rate of the mix-norm, they may achieve the decay rate of \( h \).

These theorems allow us to show the result outlined in the introduction. The following corollary reveals it is not possible to find a \( \varrho \) or \( \lambda \) (under a given growth condition) that is little-O of the mix-norm.

**Corollary 1.**

(a) For any \( \varrho \) satisfying equation (1.3), we have
\[
\limsup_{t \to \infty} \frac{\varrho(t)}{\| f \|_{H^{\tau_q}}} > 0.
\]

(b) For \( \lambda \) satisfying equation (1.5) and \( \limsup_{t \to \infty} \lambda(t - \tau)/\lambda(t) \) finite for any \( \tau \in \mathbb{R} \), we have
\[
\limsup_{t \to \infty} \frac{\lambda(t)}{\| f \|_{H^{\tau_q}}} > 0.
\]

**Proof of corollary 1.** Seeking contradiction we suppose there is a \( \varrho(t) \) satisfying equation (1.3) such that \( \varrho(t) = o \left( \| f \|_{H^{\tau_q}} \right) \). Choosing \( h(t) = \sqrt{\varrho(t)} \| f \|_{H^{\tau_q}} \), the geometric mean of \( \varrho \) and the mix-norm, we see
\[
\limsup_{t \to \infty} \frac{h(t)}{\| f \|_{H^{\tau_q}}} = \limsup_{t \to \infty} \sqrt{\frac{\varrho(t)}{\| f \|_{H^{\tau_q}}}} = 0.
\]

Then theorem 2 assures there is a \( g \in \dot{H}^q \) with
\[
\limsup_{t \to \infty} \frac{\| f \cdot g \|}{h} > 0.
\]

Then we have arrived at a contradiction:
\[
\limsup_{t \to \infty} \frac{\| f \cdot g \|}{h} \leq \limsup_{t \to \infty} \frac{\| f \cdot g \|}{\varrho} \cdot \limsup_{t \to \infty} \frac{\varrho}{h} = 0
\]
since \( \limsup_{t \to \infty} \| f \cdot g \| / \varrho \) is finite by equation (1.3) and \( \limsup_{t \to \infty} \varrho/h = 0 \) as in equation (2.6).

A similar argument gives us the second half of the corollary. In this case choose \( h(t) = \sqrt{\lambda(t)} \| f \|_{H^{\tau_q}} \) and apply theorem 2 to produce a test function \( g \) which comes with a \( \tau_g \) from equation (1.5). Then we have another contradiction:
\[
\limsup_{t \to \infty} \frac{\| f \cdot g \|}{h(t)} \leq \limsup_{t \to \infty} \frac{\| f \cdot g \|}{\lambda(t - \tau_g)} \cdot \limsup_{t \to \infty} \frac{\lambda(t - \tau_g)}{\lambda(t)} \cdot \limsup_{t \to \infty} \frac{\lambda(t)}{h(t)} = 0
\]
since \( \limsup_{t \to \infty} \lambda(t - \tau_g)/\lambda(t) \) is finite by hypothesis. \( \square \)
In section 3 we present an example of a $q$-transient function and alter it to send energy down the spectrum less efficiently, resulting in $q$-recurrence for a range of $q$. We then include diffusion at every time step, demonstrating a transition to $q$-recurrence for all $q > 0$. We include a numerical example and provide intuition about how to recognize when $f^t$ is $q$-recurrent. Finally, we prove the theorems in generality in sections 4 and 5.

3. Examples

Example 1 (baker’s map and $q$-transience). Let $B$ be the baker’s map, the area preserving transformation of the domain $[0,1]^2$ as pictured in figure 3. For the $y$-independent initial function $f^0(x,y) = 2 \cos(2\pi x)$, applying the baker’s map simply doubles the frequency in the $x$ direction. After $n$ applications of the baker’s map we have $f^n = f^0 \circ B^{-n} = 2 \cos(2\pi 2^n x)$. As a result, the Fourier coefficients have the simple expression

$$f^n_k = \begin{cases} 1 & k_1 = 2^n, k_2 = 0; \\ 0 & \text{otherwise.} \end{cases}$$

This is a one dimensional action on Fourier coefficients $f^n_k = f_{k_1,0}$ via an infinite dimensional matrix $A_{k\ell}$ as

$$f^{n+1}_k = \sum_{\ell} A_{k\ell} f^n_{\ell} \quad (3.1)$$

where

$$A_{k\ell} = \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots \\ 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & & 1 \\ & & & & \ddots \end{pmatrix}$$

is populated by 1’s along a subdiagonal of slope $-2$ and 0’s everywhere else.
Table 1. Nonvanishing Fourier coefficients of \( f^n \) defined by equation (3.1), for \( f^0(x) = 2 \cos(2\pi x) \).

| \( k \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ... |
|-------|---|---|---|---|---|---|---|---|-----|
| \( f_0^k \) | 1 |   |   |   |   |   |   |   |     |
| \( f_1^k \) | a | b |   |   |   |   |   |   |     |
| \( f_2^k \) | a^2 | a \cdot b | b |   |   |   |   |   |     |
| \( f_3^k \) | a^3 | a^2 \cdot b | a \cdot b | b |   |   |   |   |     |

The entire mix-norm is supported on just one Fourier mode and given any finite set \( I \in \mathbb{Z}^d \), it is clear that, as \( n \) increases, the Fourier energy will move off of \( I \) and never return. Therefore \( f^n \) is \( q \)-transient \( \forall q > 0 \).

Example 2 (baker-like action and \( q \)-recurrence). We now alter the previous example so that the energy of \( f^n \) is sent down the spectrum less effectively, the result being a \( q \)-recurrent function (if \( q \) is large enough). This time, consider the action on the Fourier coefficients of \( f^n(x) \) as in equation (3.1) via the infinite dimensional matrix

\[
\mathbf{\tilde{A}}_{kl} = \begin{pmatrix}
1 & 2 & 3 & 4 & \ldots \\
(\begin{array}{cccc}
a & b & \ldots & 1 \\
b & 1 & \ldots & 2 \\
1 & \ldots & \ldots & \ldots \\
\end{array}) & 1 & 2 & 3 & 4 & \ldots \\
\end{pmatrix}
\]

where \( a, b > 0 \) are constants such that \( a^2 + b^2 = 1 \).

Remark. It is not evident that the current example is still a dynamical systems example. That is, we do not know that there is a map \( T : [0,1]^d \rightarrow [0,1]^d \) so that \( f^{n+1} = f^n \circ T \) and \( f_k^{n+1} = \sum A_k f_k^n \). Moreover, such a map might not be injective, surjective, or unique. For example, \( B^{-1} \) from the previous example is not injective when thought of as a map on \([0,1]\), but it is when we allow it to move through another dimension \((d = 2)\). For the present example, taking \( a, b = \sqrt{2}/2 \), the initial function \( f^0(x) = 2 \cos(2\pi x) \) is transformed to \( f^1(x) = \sqrt{2}(a \cos(2\pi x) + b \cos(4\pi x)) \) after one time step. Since the range of \( f^0 \) and \( f^1 \) are not the same set, any such \( T \) cannot be surjective. In this example, we see the map is also not measure preserving. Lastly, if \( f^n \) is constant, then \( T \) can be any map. These caveats notwithstanding, we consider the current example as a study of possible ways for energy to move down the spectrum and proceed with an analysis.

We show the coefficients of \( f^n \) for initial function \( f^0(x) = 2 \cos(2\pi x) \) in table 1. Heuristically, the energy starts concentrated on the \( k = 1 \) mode and subsequently splits between modes \( k = 1, 2 \) so that \( L^2 \) norm is preserved. After that the \( k = 1 \) mode continues to donate a proportion \( b \) of its energy to \( k = 2 \) and the energy on \( k = 2 \) is transported down the spectrum at the same rate as the baker’s map \((k = 2^n)\).
Notice that \( f^n \) is mean-zero because it is the finite sum of cosine functions with full period. We can compute the \( L^2 \) norm and find \( f^n _{L^2} = 1 \forall n \). Hence \( \Delta \tilde{A}_M \) is a unitary map on \( \ell^2 \) by the polarization identity. Therefore \( f^n \) is bounded uniformly in \( L^2 \) and so \( f^n \) satisfies the hypothesis of the theorems in section 2.

Consider the contribution to the mix-norm from mode \( k = 1 \)
\[
\mathcal{E}_1^n = \| P_{k=1} f^n \|_{H^{-q}}^2 = \sum_{k=1} k^{-2q} |f^n_k|^2 = |f^n_1|^2 = a^{2n},
\]
and compare it to the contributions from modes \( k > 1 \) (a geometric sum):
\[
\mathcal{E}_{k>1}^n = \| P_{k>1} f^n \|_{H^{-q}}^2 = \sum_{k=1} k^{-2q} |f^n_k|^2
\]
\[
= \begin{cases} 
    c_{q,a} \left(a^{2n} - 2^{-2q} \right), & \text{for } a \neq 2^{-q}; \\
    b^2 a^{2n} n, & \text{for } a = 2^{-q}; 
\end{cases}
\]
where
\[
c_{q,a} = \frac{b^2}{aq^{2q} - 1}.
\]
For \( a > 2^{-q} \), we see that \( \mathcal{E}_{k>1}^n \sim \mathcal{E}_1^n \) as \( n \to \infty \) and the mode \( k = 1 \) captures a non-zero proportion of the mix-norm for arbitrarily large \( n \). Therefore \( f^n \) is \( q \)-recurrent for \( q > \log_2(1/a) \).

For \( a \leq 2^{-q} \), \( \mathcal{E}_{k>1}^n = o(\mathcal{E}_1^n) \) so the mode \( k = 1 \) does not capture a non-zero proportion of the mix-norm for arbitrarily large \( n \). This suggests that \( f^n \) is \( q \)-transient for \( q \leq \log_2(1/a) \). To prove \( q \)-transience, we need to show the same holds for an arbitrary finite set \( I \). Take \( I = [0, 2^R] \) for some \( R \in \mathbb{N} \). For \( n > R \),
\[
\| P_I f^n \|_{H^{-q}}^2 = c_{R,a} a^{2n},
\]
where
\[
c_{R,a} = \begin{cases} 
    1 + c_{q,a} \left(1 - a^{-2R} 2^{-2qR} \right), & \text{for } a \neq 2^{-q}; \\
    1 + b^2 R, & \text{for } a = 2^{-q}; 
\end{cases}
\]
and we conclude that
\[
\lim_{n \to \infty} \frac{\| P_I f^n \|_{H^{-q}}^2}{\| f^n \|_{H^{-q}}^2} = \begin{cases} 
    c_{R,a} \left(1 + c_{q,a} \right)^{-1}, & \text{for } a > 2^{-q}; \\
    0, & \text{for } a \leq 2^{-q}. 
\end{cases}
\]
Therefore \( f^n \) is \( q \)-transient for \( q \leq \log_2(1/a) \) and \( q \)-recurrent for \( q > \log_2(1/a) \).

**Remark.** In the above example, an initial function with \( f^0 = 0 \) has a trajectory \( f^n \) that is \( q \)-transient. This is because the coefficients do not see the alterations we have made to the baker’s map and are sent down the spectrum at the same rate as the baker’s map \( (k = 2^n) \). We emphasize that \( q \)-recurrence is a property of a particular realization of the flow and initial condition taken together—a given flow may induce a \( q \)-recurrent evolution \( f^n \) for some initial conditions and \( q \)-transient for others.

**Example 3 (baker-like action with diffusion).** We use the same matrix \( \tilde{A}_M \) and initial condition as in example 2 but now we also include diffusion. Without diffusion the conclusion
from the previous section was that \( f^n \) is \( q \)-recurrent if \( q \) was large enough. We now show that with diffusion, \( f^n \) is \( q \)-recurrent for all \( q \). Along the way, we show the rate of decay of the Sobolev norm \( \|f^n\|_{H^\beta} \) is independent of \( \beta \in \mathbb{R} \).

Let

\[
\gamma_k = \exp\left(-\kappa(2\pi k)^2\right) \tag{3.2}
\]

and update the Fourier coefficients according to

\[
f_{k}^{n+1} = \sum_{\ell} \gamma_k \tilde{A}_{k\ell} f_{\ell}^n \tag{3.3}
\]

where \( \tilde{A}_{k\ell} \) is the matrix defined in example 2. This matrix multiplication will result in pulsed diffusion with diffusion constant \( \kappa \). We display the coefficients of \( f^n \) in table 2.

The amount of mix-norm found on mode \( k = 1 \),

\[
E^n_1 = \|P_{k=1}f^n\|_{H^\beta}^2 = \sum_{k=1} k^{2\beta} |f_k^n|^2 = (a\gamma_1)^{2n}, \tag{3.4}
\]

is asymptotically equivalent to the amount found on modes \( k > 1 \) because

\[
E^n_{k>1} = \|P_{k>1}f^n\|_{H^\beta}^2 = \sum_{k>1} k^{2\beta} |f_k^n|^2
\]

\[
= \sum_{\ell=1}^{n} (2\ell)^{2\beta} \big((a\gamma_1)^{-\ell} b \prod_{s=1}^{\ell} \gamma_s\big)^2
\]

\[
= (a\gamma_1)^{2n} \sum_{\ell=1}^{n} (2\ell)^{2\beta} \big((a\gamma_1)^{-\ell} b \prod_{s=1}^{\ell} \gamma_s\big)^2,
\]

where the factor

\[
c_{n,\beta,a,\kappa} := \sum_{\ell=1}^{n} (2\ell)^{2\beta} \big((a\gamma_1)^{-\ell} b \prod_{s=1}^{\ell} \gamma_s\big)^2 \tag{3.5}
\]

limits to a finite constant \( c_{\beta,a,\kappa} \) as \( n \to \infty \). We see that \( c_{n,\beta,a,\kappa} \) converges since the factor

\[
\gamma_{2\ell} = \exp\left(-\kappa(2\pi 2\ell)^2\right) \tag{3.6}
\]

dominates the terms in the sum to render the sum convergent. Lastly, notice that \( c_{\beta,a,\kappa} \to \infty \) as \( \beta \to \infty \) or \( a \to 0 \). We conclude that \( f^n \) is \( q \)-recurrent for all \( q \in \mathbb{R} \) and, moreover, that all of the Sobolev norms decay at the same rate.

**Example 4 (sine flow).** Lastly we consider a computational example, the random sine flow, which is a simple model flow that is empirically quite efficient at mixing [29, 35]. The sine flow is a two-dimensional time-periodic flow with a full period consisting of the shear flow

\[
u_1(t, x) = \sqrt{2} (0, \sin(2\pi x + \psi_1)), \quad 0 \leq t < 1/2, \tag{3.7a}
\]

followed by

\[
u_2(t, y) = \sqrt{2} (\sin(2\pi y + \psi_2), 0), \quad 1/2 \leq t < 1, \tag{3.7b}
\]
Table 2. Nonvanishing Fourier coefficients of $f^n$ defined by equation (3.3), for $f^0(x) = 2 \cos(2\pi x)$.

| $k$ | $f^0_1$ | $f^1_1$ | $f^2_1$ | $f^3_1$ |
|-----|---------|---------|---------|---------|
| 1   | 1       | $a \gamma_1$ | $a^2 \gamma_1$ | $a^3 \gamma_1$ |
| 2   | $b \gamma_2$ | $a^2 \gamma_1 b \gamma_2$ | $a^3 \gamma_1 b \gamma_2$ | $a^3 \gamma_1 b \gamma_2 \gamma_4$ |
| 3   |         |         |         |         |
| 4   |         |         |         |         |
| 5   |         |         |         |         |
| 6   |         |         |         |         |
| 7   |         |         |         |         |
| 8   |         |         |         |         |

Figure 4. For the advection-diffusion equation (1.1) with $u$ given by the random sine flow (3.7), the rate of decay of the mix-norms is independent of $q$. The initial condition is $f^0(x) = \sqrt{2} \cos(2\pi x)$, and the diffusivity is $D = 10^{-5}$.

In general, if $f^t$ is $q$-recurrent then the decay rate of the mix-norm is independent of $q$ in the following sense:

**Theorem 3.** If $f^t$ is $q$-recurrent, then it is also $q'$-recurrent for any $q' > q$. Moreover, we have

$$\limsup_{t \to \infty} \frac{\|f^t\|_{H^{-q'}}}{\|f^t\|_{H^{-q}}} > 0.$$  

Then together with the trivial estimate

$$\|f^t\|_{H^{-q'}} \leq \|f^t\|_{H^{-q}}$$  \hspace{1cm} (3.8)

we conclude that $\|f^t\|_{H^{-q'}}$ is big-$O$ but not little-$O$ of $\|f^t\|_{H^{-q}}$. 

With $(x,y) \in [0,1]^2$ and periodic spatial boundary conditions. Here $\psi_1$ and $\psi_2$ are random phases, uniformly distributed in $[0,2\pi]$, chosen independently at every period. Unlike the pulsed diffusion in example 3, diffusion acts continuously by solving the advection–diffusion equation (1.1) with diffusivity $D = 10^{-5}$. We display $\|f^t\|_{H^{-q}}$ for various $q$ in figure 4, for initial condition $f^0(x) = \sqrt{2} \cos(2\pi x)$, and observe that the mix-norms all decay at the same rate, at least within numerical fluctuations.
Proof. Since $f'$ is $q$-recurrent, there is a finite set $I$ such that
\[ 0 < \limsup_{t \to \infty} \frac{\| P_I f' \|_{H^{-q}}}{\| f' \|_{H^{-q}}} \quad (3.9) \]

Say $I \subset [-R, R]$ for some $R \in \mathbb{N}$. Then
\[ \| P_I f' \|_{H^{-q}} \leq R^{q' - q} \| f' \|_{H^{-q'}}. \quad (3.10) \]

Putting together equations (3.8)–(3.10) we obtain
\[ 0 < R^{q' - q} \limsup_{t \to \infty} \frac{\| P_I f' \|_{H^{-q'}}}{\| f' \|_{H^{-q'}}}. \]

We conclude $f'$ is $q'$-recurrent. Moreover, the trivial estimate $\| P_I f' \|_{H^{-q'}} \leq \| f' \|_{H^{-q'}}$ together with equations (3.9) and (3.10) imply
\[ 0 < R^{q' - q} \limsup_{t \to \infty} \frac{\| f' \|_{H^{-q'}}}{\| f' \|_{H^{-q'}}}. \quad (3.11) \]

One question for further investigation is whether a converse to the above theorem exists. That is, can we conclude $f'$ is $q$-recurrent for a range of $q$ if the mix-norms decay at the same rate for the those $q$? Another question concerns the transition from $q$-transient to $q$-recurrent when including pulsed diffusion. Does $q$-recurrence imply an introduction of the Batchelor scale and anomalous dissipation [3, 27, 28]?}

4. Proof of theorem 1

We begin by generalizing the definition of $q$-recurrent functions to the notion of (q,h)-recurrent functions.

Definition 2. For positive functions $h(t)$, we say $f'$ is (q,h)-recurrent if there exists a finite set $I \subset \mathbb{Z}^d$ such that
\[ \limsup_{t \to \infty} \frac{\| P_I f' \|_{H^{-q}}}{h} > 0. \quad (4.1) \]

Functions that are not (q,h)-recurrent are called (q,h)-transient.

Lemma 1. If $f'$ is (q,h)-recurrent, then there is a function $g \in \dot{H}^\beta$ such that
\[ \limsup_{t \to \infty} \frac{|\langle f', g \rangle|}{h} > 0. \]

Moreover, $g \in \dot{H}^\beta$ for any $\beta \in \mathbb{R}$ with
\[ \| g \|_{\dot{H}^\beta}^2 = 2 \sum_{k \in I} k^{2(q' - q)}. \quad (4.2) \]

Proof. There exists a constant $c > 0$ and a sequence of times $t_m \to \infty$ where
\[ \sum_{k \in I} |f'_k|^2 k^{-2q} \geq c^2 h^2(t_m). \quad (4.3) \]
Recall the signum function
\[
\text{sgn } x = \begin{cases} 
1, & x \geq 0; \\
-1, & x < 0.
\end{cases}
\] (4.4)

Now notice that for each fixed time \( t_m \), \( \{ {f^m}_k \}_{k \in I} \) is a list of \(|I|\) numbers in \( \mathbb{C} \). Write \( {f^m}_k = a_k^m + ib_k^m \) where \( a_k^m \) and \( b_k^m \) are real. Then \( {f^m}_k \) is found in one of the four quadrants of the complex plane, depending on the two possibilities for \( \text{sgn } a_k^m \) and two possibilities for \( \text{sgn } b_k^m \). Thus, \( \{ {f^m}_k \}_{k \in I} \) has \( 4^{|I|} \) possible states. Since we have an infinite sequence of times \( t_m \), one of these states must occur infinitely many times. By taking a subsequence \( t_{m\ell} \), we can ensure \( \{ {f^m}_k \}_{k \in I} \) is the same state for all \( \ell \). Let \( \{(c_k, d_k)\}_{k \in I} \) encode this state, meaning that
\[
c_k = \text{sgn } a_k^m \quad \text{and} \quad d_k = \text{sgn } b_k^m \quad \text{for all } \ell.
\] Let
\[
g_k = \begin{cases} 
(c_k + id_k) k^{-q}, & k \in I; \\
0, & \text{otherwise.}
\end{cases}
\] (4.5)

Notice that \( g \in \mathcal{H}^3 \) because
\[
\|g\|_{\mathcal{H}^3}^2 = \sum_{k \in I} |g_k|^2 = \sum_{k \in I} (|c_k|^2 + |d_k|^2) = 2 \sum_{k \in I} k^{-2} < \infty
\] (4.6)
since \( I \) is a finite set. We have
\[
\sum_{k \in I} {f^m}_k g_k = \sum_{k \in I} (a_k^m + ib_k^m) (c_k - id_k) k^{-q}
\]
\[
= \sum_{k \in I} \left( a_k^m c_k + b_k^m d_k + i \left( b_k^m c_k - a_k^m d_k \right) \right) k^{-q}
\]
\[
= \sum_{k \in I} \left( |a_k^m|^2 + |b_k^m|^2 \right) k^{-q}.
\] We conclude that
\[
|\langle f^m , g \rangle| \geq \Re \sum_{k \in I} {f^m}_k \bar{g}_k
\]
\[
= \sum_{k \in I} \left( |a_k^m|^2 + |b_k^m|^2 \right) k^{-q}
\]
\[
\geq \sum_{k \in I} \sqrt{|a_k^m|^2 + |b_k^m|^2} k^{-q}
\]
\[
= \sum_{k \in I} |{f^m}_k| k^{-q},
\]
as desired. Lastly, we use dominance of \( \ell^2 \) by \( \ell^1 \) together with equation (4.3) to conclude
\[
|\langle f^m , g \rangle| \geq c \, h(t_m), \forall m,\ell.
\] □
Lemma 1 characterizes the behavior of (q,h)-recurrent functions. We now develop the tool we need to further analyze (q,h)-transient functions.

**Lemma 2.** Let \( f \) be (q,h)-transient for some positive \( h = O(\|f\|_{H^{-q}}) \). For any \( \delta \) with \( 0 < \delta < 1 \), there exist sets \( I_i = \{k|I_{i-1} < |k| \leq J_i\} \) with \( J_0 = -1 \) and a sequence of times \( T_i \to \infty \) satisfying the following.

(a) The set \( I_i \) captures a significant proportion of the Fourier energy at time \( T_i \), so that

\[
\sum_{k \in I_i} |f^T_k|^2 k^{-2q} \geq (1 - \delta) \|f^T\|^2_{H^{-q}}. \tag{4.7}
\]

(b) Enough of the Fourier energy does not return to lower frequency modes, so that

\[
\sum_{|k| \leq J_{i-1}} |f^T_k|^2 k^{-2q} \leq \delta h^2(t) \quad \text{for all } t \geq T_i. \tag{4.8}
\]

**Proof.** Base case: \( i = 1 \). Since (by definition of absolute convergence)

\[
\lim_{j \to \infty} \sum_{|k| \leq J_i} |f^0_k|^2 k^{-2q} = \|f^0\|^2_{H^{-q}} \tag{4.9}
\]

there is a \( J_1 \) such that

\[
\sum_{|k| \leq J_1} |f^0_k|^2 k^{-2q} \geq (1 - \delta) \|f^0\|^2_{H^{-q}}. \tag{4.10}
\]

We see \( I_1 = \{k|J_0 < |k| \leq J_1\} \) where \( J_0 = -1 \) and \( T_1 = 0 \) therefore trivially satisfy equations (4.7) and (4.8).

Induction step. Suppose that we are given \( J_{i-2}, J_{i-1} \) and \( T_{i-1} \) that satisfy equations (4.7) and (4.8). Since \( h = O(\|f^0\|_{H^{-q}}) \), there is a \( c > 0 \) and a time \( T \) so that \( h(t) \leq c \|f^0\|_{H^{-q}} \) \forall t \geq T.

Recall that \( f^T \) is (q,h)-transient and so, for any finite set \( I \),

\[
\lim_{t \to \infty} \sup \frac{\|P_I f^T\|_{H^{-q}}}{h} = 0. \tag{4.11}
\]

That is, given finite set \( I \), then for any constant \( \epsilon \) there is \( T_\epsilon \) so that

\[
\sum_{k \in I} |f^T_k|^2 k^{-2q} \leq \epsilon h^2(t) \quad \text{for all } t \geq T_\epsilon. \tag{4.12}
\]

Take \( I = \{k \text{ s.t. } |k| \leq J_{i-1}\} \) and \( \epsilon = \min(\delta, \delta/c^2) \). We conclude that there exists \( T_i \) with \( T_i \geq T_{i-1} + 1 \) and \( T_i \geq T \) such that equation (4.8) is satisfied. Moreover, we have

\[
\sum_{|k| \leq J_{i-1}} |f^T_k|^2 k^{-2q} \leq \delta \|f^T\|^2_{H^{-q}}. \tag{4.13}
\]

Hence,

\[
\sum_{|k| > J_{i-1}} |f^T_k|^2 k^{-2q} \geq (1 - \delta) \|f^T\|^2_{H^{-q}}. \tag{4.14}
\]
That is,
\[
\lim_{J \to \infty} \sum_{J_{i-1} < |k| < J_i} |f_{Ti}^k|^2 k^{-2q} = C \|f_{Ti}^k\|_{H^{-q}}^2,
\]
(4.15)
where \(C > (1 - \delta)\). From the definition of the limit, it follows that there is a \(J_i\) large enough so that for \(I_i = \{k | J_{i-1} < |k| < J_i\}\) we have
\[
\sum_{k \in I_i} |f_{Ti}^k|^2 k^{-2q} \geq (1 - \delta) \|f_{Ti}^k\|_{H^{-q}}^2
\]
which is equation (4.7).

Having developed all of the tools we will need, we now prove theorem 1 and, in the next section, theorem 2.

**Theorem 1.** Let \(f^t\) be a mean-zero function in \(L^2(\mathbb{T}^d)\) with \(\|f^t\|_{H^{-q}} > 0\) for all \(t > 0\). Then \(f^t\) is \(q\)-recurrent if and only if there is a function \(g \in \dot{H}^q\) such that
\[
\limsup_{t \to \infty} \frac{|\langle f^t, g \rangle|}{\|f^t\|_{H^{-q}}} > 0.
\]

**Proof.** The forward direction is a special case of lemma 1 with \(h(t) = \|f^t\|_{H^{-q}}\). We assume \(f^t\) is \(q\)-transient and show \(\langle f^t, g \rangle = o(\|f^t\|_{H^{-q}})\) for all \(g \in \dot{H}^q\), so
\[
\limsup_{t \to \infty} \frac{|\langle f^t, g \rangle|}{\|f^t\|_{H^{-q}}} < \infty, \quad \forall g \in \dot{H}^q.
\]
(4.17)
Seeking a contradiction, we suppose there exists a \(g \in \dot{H}^q\) such that
\[
\limsup_{t \to \infty} \frac{|\langle f^t, g \rangle|}{\|f^t\|_{H^{-q}}} = C > 0.
\]
(4.18)
There is a sequence \(t_n \to \infty\) such that
\[
|\langle f^{t_n}, g \rangle| \geq \frac{C}{2} \|f^{t_n}\|_{H^{-q}}, \quad \forall n.
\]
(4.19)
We will show that equation (4.19) implies that the Fourier coefficients of \(g\) decay too slowly for \(g\) to be in \(\dot{H}^q\). Since \(g \in \dot{H}^q\), we can choose \(\delta\) small enough that \(\frac{2}{\delta} - 2 \sqrt{\delta} \|g\|_{L^2} = C_0 > 0\.

Applying lemma 2 with \(h(t) = \|f^t\|_{H^{-q}}\), there exist sets \(I_i = \{k | J_{i-1} < |k| \leq J_i\}\) and a sequence of times \(T_i \to \infty\) (without loss of generality, say that \(T_i\) is a subsequence of \(\{t_n\}\) above) such that we have equations (4.7) and (4.8). Equation (4.7) implies
\[
\sum_{|k| > J_i} |f_{Ti}^k|^2 k^{-2q} \leq \delta \|f_{Ti}^k\|_{H^{-q}}^2.
\]
(4.20)
Note that
\[
\langle f_{Ti}^k, g \rangle = \sum_{j \geq 1} \sum_{k \in I_j} \hat{f}_{Ti}^k \hat{g}_k
= \sum_{k \in I_j} \hat{f}_{Ti}^k \hat{g}_k + E
\]
(4.21)
and we can bound the error using the Cauchy–Schwarz inequality

\[ E = \sum_{k \notin I_i} f_{T_k} \bar{g}_k \leq \left( \sum_{k \notin I_i} \left| f_{T_k} \right|^2 k^{-2q} \right)^{1/2} \left( \sum_{k \notin I_i} |g_k|^2 k^{2q} \right)^{1/2}. \] (4.22)

Applying equation (4.7) of lemma 2, we have

\[ |E| \leq \delta \| f_{T_i} \|_{H^{-q}} \| g \|_{H^q}. \] (4.23)

and therefore

\[ \left| \langle f_{T_i}, g \rangle \right| \leq \left| \sum_{k \in I_i} f_{T_k} \bar{g}_k \right| + \sqrt{\delta} \| f_{T_i} \|_{H^{-q}} \| g \|_{H^q}. \] (4.24)

Putting together equations (4.19) and (4.24), we have

\[ \left( C_0 - \sqrt{\delta} \| g \|_{H^q} \right) \| f_{T_i} \|_{H^{-q}} \leq \left| \sum_{k \in I_i} f_{T_k} \bar{g}_k \right|. \] (4.25)

Since \( g \in \dot{H}^q \), we can choose \( \delta \) small enough that

\[ C_0 := \left( C_2 - \sqrt{\delta} \| g \|_{H^q} \right) > 0. \] (4.26)

Applying Cauchy–Schwarz to the right-hand side of equation (4.25) we have

\[ C_0 \| f_{T_i} \|_{H^{-q}} \leq \left( \sum_{k \in I_i} \left| f_{T_k} \right|^2 k^{-2q} \right)^{1/2} \left( \sum_{k \in I_i} |g_k|^2 k^{2q} \right)^{1/2} \] (4.27)

\[ \leq \| f_{T_i} \|_{H^{-q}} \left( \sum_{k \in I_i} |g_k|^2 k^{2q} \right)^{1/2}. \] (4.28)

Therefore

\[ \sum_{k \in I_i} |g_k|^2 k^{2q} \geq C_0^2, \quad \forall i. \] (4.29)

This shows that the coefficients of \( g \) are large on sets \( I_i \) and we have

\[ \| g \|_{H^q}^2 = \sum_i \sum_{k \in I_i} \left| g_k \right|^2 k^{2q} \geq \sum_i C_0^2 = \infty. \] (4.30)

We conclude that \( g \) is not in \( H^q \)—a contradiction. \[ \square \]
5. Proof of theorem 2

**Theorem 2.** Let $f^t$ be a mean-zero function in $L^2(\mathbb{T}^d)$ with $\|f^t\|_{H^{-q}} > 0$ for all $t > 0$. For any positive function $h(t)$ such that $h(t) = o \left( \|f^t\|_{H^{-q}} \right)$, there is a function $g \in \dot{H}^q$ such that

$$\limsup_{t \to \infty} \frac{|\langle f^t, g \rangle|}{h(t)} > 0.$$ 

**Proof of theorem 2.** If $f^t$ is $(q, h)$-recurrent, then we are done by lemma 1, so say $f^t$ is $(q, h)$-transient. Take some $\delta < 1/3$ and apply lemma 2 to construct sets $\{I_i\}_{i=1}^\infty$ and a sequence $\{T_i\}_{i=1}^\infty$. Let $\{T_{i\ell}\}_{\ell=1}^\infty$ be a subsequence of $\{T_i\}_{i=1}^\infty$ satisfying

$$\sum_{\ell > L} \left( \frac{h(T_{i\ell})}{\|f^{T_{i\ell}}\|_{H^{-q}}} \right)^2 \leq \delta^2 \left( \frac{h(T_i)}{\|f^{T_i}\|_{H^{-q}}} \right)^2, \quad (5.1)$$

and

$$\sum \left( \frac{h(T_{i\ell})}{\|f^{T_{i\ell}}\|_{H^{-q}}} \right)^2 \leq \delta. \quad (5.2)$$

This can be done since $h(t) = o(\|f^t\|_{H^{-q}})$. Let $g$ be the function with Fourier coefficients given by

$$g_k = \begin{cases} \int f_{T_{i\ell}}^T k^{-2q} \|f_{T_{i\ell}}\|_{H^{-q}}^2 h(T_{i\ell}) \quad k \in I_i; \\ 0 \quad \text{otherwise}. \end{cases} \quad (5.3)$$

Equation (5.2) allows us to conclude that $g \in \dot{H}^q$:

$$\|g\|_{\dot{H}^q}^2 = \sum |g_k|^2 k^{2q} \leq \sum_{i_j} \sum_{k: k \in I_{i_j}} \left| \int f_{T_{i\ell}}^T k^{-2q} \|f_{T_{i\ell}}\|_{H^{-q}}^2 h(T_{i\ell}) \right|^2 k^{2q} \leq \sum_{i_j} \left( \sum_{k: k \in I_{i_j}} \left| \int f_{T_{i\ell}}^T k^{-2q} \right|^2 k^{-2q} \right) \leq \sum_{i_j} \left( \sum_{k: k \in I_{i_j}} \left| \int f_{T_{i\ell}}^T k^{-2q} \right|^2 k^{2q} \right).$$

We will now finish the proof by showing $|\langle f^T_{T_{i\ell}}, g \rangle| \geq (1 - 3\delta) h(T_{i\ell})$. We begin with some notation. Split the following sum into two parts:

$$\langle f^T_{T_{i\ell}}, g \rangle = \sum_{i_j} \sum_{k: k \in I_{i_j}} \left| \int f_{T_{i\ell}}^T k^{-2q} \|f_{T_{i\ell}}\|_{H^{-q}}^2 h(T_{i\ell}) \right|^2 \leq \sum_{i_j} \sum_{k: k \in I_{i_j}} \left( \int f_{T_{i\ell}}^T k^{-2q} \|f_{T_{i\ell}}\|_{H^{-q}}^2 h(T_{i\ell}) \right)^2 = S_{i\ell} T_{i\ell} + E_{i\ell},$$

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where $S^{T_i}$ is the sum when $j_i = i_i$: 

$$S^{T_i} = \sum_{j_i = i_i} \sum_{k \in I_{j_i}} f^T_{k_i} \frac{f_{k_i} T_i k^{-2q}}{f^T_{k_i} k^{-2q} \lVert f^T_i \rVert_{H^{-q}}^2 h(T_{j_i})}$$

$$= \sum_{k \in I_{j_i}} \lVert f_{k_i} \rVert^2 k^{-2q} \lVert f^T_i \rVert_{H^{-q}}^2 h(T_{j_i}),$$

and $E^{T_i}$ is the sum over $j_i \neq i_i$: 

$$E^{T_i} = \sum_{j_i \neq i_i} \sum_{k \in I_{j_i}} f^T_{k_i} \frac{f_{k_i} T_i k^{-2q}}{f^T_{k_i} k^{-2q} \lVert f^T_i \rVert_{H^{-q}}^2 h(T_{j_i})}.$$ 

(5.4)

The idea is that $g_k$ is constructed to agree well with $f^T_{k_i}$ when $k \in I_{j_i}$. We will show that $S^{T_i}$ dominates the error $E^{T_i}$. Consider $j_i < i_i$ and $j_i > i_i$ separately in equation (5.4); taking absolute value, we have 

$$|E^{T_i}| \leq \sum_{j_i \neq i_i} \sum_{k \in I_{j_i}} \lVert f_{k_i} \rVert \lVert f^T_{k_i} \rVert k^{-2q} \lVert f^T_i \rVert_{H^{-q}}^2 h(T_{j_i})$$ 

(5.5)

and let $E_1^{T_i}$ be the sum over $j_i < i_i$: 

$$E_1^{T_i} := \sum_{j_i < i_i} \sum_{k \in I_{j_i}} f^T_{k_i} \frac{f_{k_i} T_i k^{-2q}}{f^T_{k_i} k^{-2q} \lVert f^T_i \rVert_{H^{-q}}^2 h(T_{j_i})}.$$ 

(5.6)

Similarly define $E_2^{T_i}$ to be the sum over $j_i > i_i$: 

$$E_2^{T_i} := \sum_{j_i > i_i} \sum_{k \in I_{j_i}} f^T_{k_i} \frac{f_{k_i} T_i k^{-2q}}{f^T_{k_i} k^{-2q} \lVert f^T_i \rVert_{H^{-q}}^2 h(T_{j_i})}.$$ 

(5.7)

We now bound $E_1^{T_i}$ using the Cauchy–Schwarz inequality: 

$$E_1^{T_i} \leq \left( \sum_{j_i < i_i} \sum_{k \in I_{j_i}} \lVert f^T_{k_i} \rVert^2 k^{-2q} \right)^{1/2} \left( \sum_{j_i < i_i} \sum_{k \in I_{j_i}} \lVert f_{k_i} T_i k^{-2q} \lVert f^T_i \rVert_{H^{-q}}^{-4} h^2(T_{j_i}) \right)^{1/2}$$

$$\leq \left( \sum_{j_i < i_i} \sum_{k \in I_{j_i}} \lVert f^T_{k_i} \rVert^2 k^{-2q} \right)^{1/2} \left( \sum_{j_i < i_i} \lVert f^T_i \rVert_{H^{-q}}^{-2} h^2(T_{j_i}) \right)^{1/2}.$$ 

We use equation (4.8) from lemma 2 to bound the first factor and equation (5.2) to bound the second factor: 

$$E_1^{T_i} \leq (\delta h^2(T_{i_i}))^{1/2} (\delta)^{1/2} = \delta h(T_{i_i}).$$ 

(5.8)

We similarly bound $E_2^{T_i}$.
Using equation (5.1), we find
\[ E_{2}^{T_{i}} \leq \left\| f_{T_{i}} \right\|_{H^{-q}} \left( \sum_{j>\ell} \left\| f_{T_{j}} \right\|_{H^{-q}}^{-2} h^2(T_{j}) \right)^{1/2}. \]

Using equation (5.1), we find
\[ E_{2}^{T_{i}} \leq \left\| f_{T_{i}} \right\|_{H^{-q}} \left( \sum_{j>\ell} \left\| f_{T_{j}} \right\|_{H^{-q}}^{-2} h^2(T_{j}) \right)^{1/2}. \]

and therefore
\[ \langle f_{T_{i}}, g \rangle = S_{1}^{T_{i}} + E_{2}^{T_{i}} - E_{1}^{T_{i}} \geq S_{1}^{T_{i}} - \delta h(T_{i}) - \delta h(T_{i}). \]

Again using equation (4.7) from lemma 2 that the set \( I_{i} \) captures a large proportion of the Sobolev norm, we conclude
\[ \langle f_{T_{i}}, g \rangle \geq (1 - \delta) \left\| f_{T_{i}} \right\|_{H^{-q}}^{2} \left\| f_{T_{i}} \right\|_{H^{-q}}^{-2} h(T_{i}) - 2\delta h(T_{i}) \]
\[ \geq (1 - 3\delta) h(T_{i}) \]
\[ \Box \]

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ORCID iDs

Jean-Luc Thiffeault © https://orcid.org/0000-0001-7724-7966

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