Static spherically symmetric constant density relativistic and Newtonian stars in the Lobachevskyan geometry.

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Abstract

The present paper has the purpose to illustrate the importance of the ideas and constructions of the Non-Euclidean (Lobachevsky) Geometry, which can be applied even today for solving some conceptually important problems. We study the static and spherically symmetric solutions to the Einstein field equations under the assumption that the space-time may possess an arbitrary number of spatial dimensions. A new exact solution of a perfect fluid sphere of constant (homogeneous) energy-density which agrees with interior Lobachevsky geometry for 3D and 4D spaces are found. We discuss the property of temporal scalar field arise in lower-dimensional theories as the reduction of extra dimension.

1 Introduction

Since the pioneering work of Lobachevsky [1] the aim of solving the problems on the astronomical verification of the geometry of our visible world and on the kinds of changes which will occur in mechanics after introducing in it the new geometry has been pursued. In the framework of Lobachevsky hypothesis the forces produce all by themselves the motion, velocity, time, mass and even distances and angle [2] . The lower-dimensional theories of gravity are a good candidate for analyze this problems while is that it is simple enough to be soluble but yet contains non-trivial features. The lower-dimensional theories of gravity have been intensively investigated, primarily because of
they can assume as a theoretical laboratories for applying techniques that appear intractable or awkward in the in four-dimensional general relativity. There are other reasons for studying dynamical gravity in three dimensions as membrane models of extended relativistic systems. The 3-membrane is a three-dimensional immersion in some higher-dimensional space. Its dynamics can be formulated in terms of a three-dimensional field theory and its internal stresses described in terms of an intrinsic geometry induced by the immersion.

It is surprising that in 2+1 dimensions, Einstein general theory of relativity has very little in common with gravitation in a spacetime vacuum having four dimensions. The Riemann-Christoffel tensor is uniquely determined by the Ricci tensor, which vanishes outside the sources. Hence, the gravitational dynamics of test particles must then be induced by topological effects in flat space do not feel any gravitational field. Thus vacuum Einstein gravity in three dimensions is devoid of any limit that may be identified with an analogue of Newtonian gravity [3], [4]. A proper relativistic theory of gravitation in 3 dimensions needs some additional ingredient besides the metric tensor. It is well known, in dimension lower than four, this ingredient present in Jordan, Brans-Dicke theory - the scalar field is essential to have a Newtonian regime at low energies. Furthermore, the metric in 2+1 dimensions in which the curvature tensor vanishes and gravity is due to torsion (teleparallel theory) gives the correct Newtonian limit [5].

A simple and elegant idea in this connection employ the equivalence between the (D + 1) Kaluza-Klein theories with D-dimensional Einstein theory with source [6], [7]. Furthermore, this correspondence gives a mechanism for formulation of a 3-dimensional theory containing temporal scalar field. In this scheme space and time are treated in completely different ways. This might seem to be a retrogressive step, but actually the approach achieves everything that Einstein did by presupposing a four-dimensional unity of space and time.

This paper details the new approach to the problem of separate the temporal scalar field from the four-dimensional metric context of general relativity which where outlined in previous letter [8]. Section 2 develops the formalism related to the correspondence between the ordinary 4 - dimensional gravity with 3 - dimensional theories with temporal scalar field. In Section 3 the exact solutions for the static constant density homogeneous sphere has been obtained. Finally, Sec. 4 briefly discussed some aspects of
these theories. We choose units such that $G_N = c = 1$, and let Latin indices run 0-3 and Greek indices run 0-2.

2 Equivalence between Einstein and 3 dimensional theories with temporal scalar field.

The investigation of a lower dimension theory of gravity with additional scalar field appears to be the way to construct a model with properties more akin to those of general relativity. The most natural way to introduce the scalar field is a setting up a correspondence between the ordinary 4 - dimensional gravity with 3 - dimensional theories with source. We do not claim that this is the only possibility but this would bring some more clarification on the role of the scalar fields in physics. The procedure to go from D+1 to D is by now well known (see [6], [7]). Consider the case of 4 dimensional vacuum field equations

\[ 4R_{ab} = 0, \tag{1} \]

where $R_{ab}$ is a Ricci tensor in the 4 - dimensional space. It is well known that equation (1) gives rise to 3 dimensional Einstein equations with source in the form [7]:

\[ 3R_{\alpha\beta} = 3T^\phi_{\alpha\beta}, \tag{2} \]

where $T^\phi_{\alpha\beta}$ is the induced energy-momentum tensor.

Following earlier work, it is inquired how far the 4 - dimensional equations without sources may be reduced to the Einstein equations with sources. It is shown by algebraic means that this can be done, provided the extra part of the 4 - dimensional geometry is used appropriately to define an effective 3-D energy-momentum tensor of a scalar field. We start with the 4 dimensional source-free Einstein field equations (1) and take the metric to be in the form:

\[ 4g_{ab} = \begin{pmatrix} 3g_{\alpha\beta} & 0 \\ 0 & g_{dd} \end{pmatrix}, \tag{3} \]
where both the 4 and 3 dimensional metrics $g_{ab}$ and $g_{a\beta}$ are in general dependent on the reducible coordinate $x^d$ and we write the $g_{dd}$ as

$$g_{dd} = \phi^2, \quad g^{dd} = \frac{1}{\phi^2}. \quad (4)$$

Now the source free field equations in 4 dimensions are given by

$$4R_{ab} = 0 \Rightarrow 4R_{dd} = 0 \quad (5)$$

and

$$4R_{\alpha\beta} = 3R_{\alpha\beta} - \frac{\phi_{\alpha\beta}}{\phi} \frac{1}{2 \phi^2} \left( \frac{\phi_{d\beta}}{\phi} \phi_{\alpha\beta,d} - g_{\alpha\beta,dd} + g^{\lambda\mu} g_{\alpha\lambda,d} g_{\beta\mu,d} - \frac{1}{2} g^{\mu
u} g_{\mu\nu,d} g_{\alpha\beta,d} \right), \quad (6)$$

In this way the 4-dimensional source free Einstein type equations are related to a 3 dimensional theory with sources, from the (5) we obtain

$$3R_{\alpha\beta} - \frac{1}{2} 3Rg_{\alpha\beta} = 3T_{\alpha\beta}^\phi, \quad (7)$$

$$\Box\phi = -\frac{1}{4\phi} g^{\mu\nu} g_{\mu\nu,d} - \frac{1}{2\phi} g^{\lambda\mu} g_{\lambda\mu,d} + \frac{\phi_{d\beta}}{2\phi^2} g^{\lambda\mu} g_{\lambda\mu,d}, \quad (8)$$

$$\left[ \frac{1}{2 \sqrt{g_{dd}}} \left( g^{\beta\mu} g_{\mu\alpha,d} - \delta_{\alpha}^{\beta} g^{\nu\mu} g_{\nu\mu,d} \right) \right]_{;\beta} = 0. \quad (9)$$

These give the components of the induced energy-momentum tensor since Einstein’s equations (7) hold.

$$3T_{\alpha\beta} = \frac{\phi_{\alpha\beta}}{\phi} - \frac{1}{2 \phi^2} \left[ \frac{\phi_{d\beta}}{\phi} \phi_{\alpha\beta,d} - g_{\alpha\beta,dd} + g^{\lambda\mu} g_{\alpha\lambda,d} g_{\beta\mu,d} - \frac{1}{2} g^{\mu\nu} g_{\mu\nu,d} g_{\alpha\beta,d} \right] + \frac{g_{d\alpha\beta}}{8\phi^2} \left[ g^{\mu\nu} g_{\mu\nu,d} + (g^{\mu\nu} g_{\mu\nu,d})^2 \right]. \quad (10)$$

The generally covariant d’Alembertian $\Box$ is defined to be covariant divergence of $\phi\rho$: 

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\[ \Box \phi = \phi^{\rho} ; \rho = \sqrt{-g} \left( \sqrt{-g} \phi^{\rho} \right) ; \rho . \]

From the above it is seen that, the physical metric obtained in this way can then be used to evaluate possible effects from the extra dimension. This means that it can employ the equivalence between spacelike or timelike dimensions with the scalar field, respectively. In the first case we find the field equations are identical to the Jordan, Brans-Dicke field equations in 2+1 dimensions with the free parameter \( \omega = 0 \). However, timelike extra dimensions reduction procedure from 4 dimensions leads to a spacetime containing temporal scalar field. The addition of the temporal scalar field to the Einstein tensor gives a unique theory.

Note that in more general case stress-energy tensor of matter is assumed to have been derived from the Lagrangian in the usual way.

3 Spacetime geometry of static constant density homogeneous stars.

The apparently simple problem of the static spherically symmetric perfect fluid has by now generated hundreds of scientific papers. The first two exact solution of Einstein’s field equations were obtained by Schwarzschild, soon after Einstein advent of general relativity. The first solution describes the geometry of the space-time exterior to a prefect fluid sphere in hydrostatic equilibrium. While the other, known as interior Schwarzschild solution, corresponds to the interior geometry of a fluid sphere of constant (homogeneous) energy-density. The importance of these two solutions is well known. Granted, a completely general method to solve Einstein’s field equations does not exist and therefore in this paper we focus on spherically symmetric system. To this end, let us consider the field produced by a static and isotropic source in regions devoid of sources in \( D = 3 \). Taking into account that the field equations of this theory was obtained using the reduction procedure from usual 4-dimensional general relativity we can find the corresponding solution of Einstein fields equations using the line element in the form [9]:

\[ ds^2 = e^\alpha dr^2 + e^\beta \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) . \] (11)
where $\alpha$ and $\beta$ are functions of $r$ alone. We assume that spacetime is static; the metric and the scalar field can be chosen such that

$$g_{ab,t} = \frac{\partial g_{ab}}{\partial t} = \phi_t = 0$$  \hfill (12)

The corresponding energy-momentum tensor of matter is specialized to that of perfect fluid

$$T^M_{\alpha\beta} = p\, g_{\alpha\beta} + (p + \mu)\, U_\alpha U_\beta$$  \hfill (13)

where $p$ and $\mu$ are the proper pressure and energy density, respectively. $U_\alpha$ are the components of the fluid velocity, which verifies for the static case $g_{\mu\nu} U_\alpha U_\beta = -1$, it follows that

$$T^M_{tt} = -\mu\, g_{tt},\ T^M_{ab} = p\, g_{ab}$$  \hfill (14)

Having added (17) and (14) into (7) and (8), we obtain the following equations for the metric (11):

$$\frac{1}{2}\alpha' \beta' - \frac{1}{2}\beta'^2 - \frac{\alpha' \phi'}{2\phi} = 4e^{\alpha}\pi (p - \mu)$$

$$e^{\alpha - \beta} + \frac{1}{4}\alpha' \beta' - \frac{1}{2}\beta'^2 - \frac{3\alpha'' \phi'}{2\phi} = 4e^{\alpha}\pi (p - \mu)$$

$$- (\alpha' - 2\beta') \frac{\phi'}{2\phi} + \frac{3\alpha'' \phi''}{2\phi} = -4e^{\alpha}\pi (\mu + 3p)$$  \hfill (15)

For the further specification of the problem an equation of state is needed. We use the function of it a constant density. For this case the solutions of equations (15) can be found analytically, by among the others there is
\[ g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \left(\kappa \sinh \left(\frac{r}{\kappa}\right)\right)^2 & 0 & 0 \\ 0 & 0 & \left(\kappa \sinh \left(\frac{r}{\kappa}\right) \sin \theta\right)^2 & 0 \\ 0 & 0 & 0 & c_2 \left(2c_1 + \kappa \cosh \left(\frac{r}{\kappa}\right)\right)^2 \end{pmatrix} \]

\[ \phi = c_2 + \kappa c_1 \cosh \left(\frac{r}{\kappa}\right) \]

\[ p = -\frac{c_2 + 3\kappa c_1 \cosh \left(\frac{r}{\kappa}\right)}{8\pi \kappa^2 \left(2c_1 + \kappa \cosh \left(\frac{r}{\kappa}\right)\right)} \]

where \( c_1 \) and \( c_2 \) arbitrary constant and \( \kappa = \sqrt{\frac{3}{8\pi \mu}} \). We have seen that this solution provides the Lobachevsky geometry in 3 dimensional spaces. The curvature of this space depends from the energy density of matter. Thus in this sense the forth dimension generates additional effective sources and these in turn act to curve the 3-dimensional spacetime too. Moreover the solution (16) should correspond to an interior solution for the constant density stars of the Einstein theory. In a suitable reference system the spherically symmetric static metric can be written in the standard form [9]:

\[ ds^2 = -e^\gamma dt^2 + e^\alpha dr^2 + e^\beta \left(d\theta^2 + \sin^2 \theta d\varphi^2\right) , \]

In the case \( \mu = \text{const} \) standard Einstein fields equations yield

\[ g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \left(\kappa \sinh \left(\frac{r}{\kappa}\right)\right)^2 & 0 & 0 \\ 0 & 0 & \left(\kappa \sinh \left(\frac{r}{\kappa}\right) \sin \theta\right)^2 & 0 \\ 0 & 0 & 0 & c_2 \left(2c_1 + \kappa \cosh \left(\frac{r}{\kappa}\right)\right)^2 \end{pmatrix} \]

\[ p = -\frac{2c_1 + 3\kappa \cosh \left(\frac{r}{\kappa}\right)}{8\pi \kappa^2 \left(2c_1 + \kappa \cosh \left(\frac{r}{\kappa}\right)\right)} \]

where \( c_1 \) and \( c_2 \) arbitrary constant and \( \kappa = \sqrt{\frac{3}{8\pi \mu}} \). There is formal connection between the lower-dimensional theory with a temporal scalar field and that of Einstein, but there are differences and the physical interpretation is
quite different. We should note that despite the mathematical equivalence of the two solutions, that fact that they come from different physical theories makes them conceptually distinct. To conclude our discussion of the Lobachevsky geometry in theories of gravity we make some remarks about Newtonian physics. Newtonian gravity in Lobachevskyan space is defined by the fundamental solution of the Poisson equation

\[ \Delta \Phi = 4\pi G_N \mu \]  

where \( \Phi \) is a gravitation potential acting on a test body and produced by the mass \( \mu \). In this note we focus on static spherically symmetric constant density stars with metric

\[ ds^2 = d\rho^2 + r^2 \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right), \]

then the Laplace operator in the Lobachevskyan space is defined as:

\[ \Delta = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right), \]

The difference between this and Euclidean geometries consists in the dependence of \( r \) on \( \rho \): in the Euclidean geometry \( r = \rho \) while in the Lobachevskyan one [11]

\[ r = \kappa \sinh \left( \frac{\rho}{\kappa} \right) \]

After simple manipulations, one can obtain the solution of equation (19)

\[ \Phi = c_2 + \kappa \left( 2\pi G_N \rho \mu + c_1 \right) \coth \left( \frac{\rho}{\kappa} \right) \]

with \( c_1 \) and \( c_2 \) constants arising from integration.

4 Conclusion

During the recent years, there has been a lot of activity on theories with extra dimension. The correspondence between extra and lower dimensions theories
is of potential importance in view of great deal of effort that is currently going into study of lower dimensions theories, and particularly in view of radical differences between such theories and 4-dimensional gravity. Its interesting that the dependence of the metric on the extra coordinates leads to, a new force term. Moreover, the particle will move under influence of tow forces; the gravitational one and extra force which does depend on the velocity [10].

In the spirit of Newtonian theory, we separate the temporal scalar field from the four-dimensional metric context of general relativity. In this scheme space and time are treated in completely different ways. In contrast with 2+1 dimension picture dynamics induced by the new temporal field contains all features that one expects in the Einstein theory of gravitation.

The influence of the gravitational field upon the properties of solutions of material field equations depends essentially on the matter distribution. The simplest function of it used in gravitational theories is a constant density. This approach gives interesting results, we obtain the solution of field equations to be described by Lobachevsky geometry.

Static spherically symmetric perfect fluid models are also interest in comparison with Newtonian theory. The replacement of Euclidian by Lobachevsky geometry in Newtonian dynamics allows one the possibility to explain the form of the rotation velocities of galaxies without the dark matter hypothesis [11]. Comparing Newtons expression of the gravitational force to expression of relativistic theory in Lobachevsky space we find that the curvature of space depends from the energy density of matter. Thus in this sense the temporal scalar field generates additional effective sources and these in turn act to curve the 3-dimensional spacetime too.

It is useful to illustrate the difference between the two theories, the Newtonian and Relativistic ones, by using a Lobachevsky space as example. This avenue has already proved fruitful in the case of low dimensional configuration.

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