Differentially private graphical degree sequences
and synthetic graphs

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Abstract

Releasing the exact degree sequence of a graph for analysis may violate privacy. However, the degree sequence of a graph is an important summary statistic that is used in many statistical models. Hence a natural starting point is to release a private version of the degree sequence. A graphical degree partition is a monotonic degree sequence such that there exists a simple graph realizing the sequence. Ensuring graphicalness of the released degree partition is a desirable property for many statistical inference procedures. We present an algorithm to release a graphical degree partition of a graph under the framework of differential privacy. Unlike previous algorithms, our algorithm allows an analyst to perform meaningful statistical inference from the released degree partition. We focus on the statistical inference tasks of existence of maximum likelihood estimates, parameter estimation and goodness of fit testing for the random graph model where the degree partition is a sufficient statistic, called the beta model. We show the usefulness of our algorithm for performing statistical inference for the beta model by evaluating it empirically on simulated and real datasets. As the degree partition is graphical, our algorithm can also be used to release synthetic graphs.

1 Introduction

Privacy is a growing problem due to the large of amount of data being collected by various agencies. A lot of data is being collected in the form of graphs where the sensitive information includes not only individual records but also relationships between them. Analysis of such graph data can be very useful for advancement of research in many fields, but free access to such data must be limited due to obvious privacy concerns. One of the central goals of privacy research is to enable useful statistical analysis of such data while preserving privacy.

One property of graphs that has been given a lot of importance in the literature of random graph models is it’s degree sequence. Although there is evidence that the degree sequence alone does not capture all the structural information in a graph, see for example, [15], in many cases the only information available
is that of the degrees of a graph. Every other structural property of a graph is estimated from a random graph model. For example, in epidemiological studies of sexually transmitted disease [8], the survey collects information on the number of sexual partners of an individual. In such cases, a natural starting point is to release the degree information of a graph in a private manner.

In this paper, we study the problem of releasing a graphical degree sequence of a graph while preserving privacy of individual relations while allowing an analyst to perform standard statistical inference with the released data. Our algorithm satisfies the rigorous definition of privacy called differential privacy [3]. Our approach in releasing degree sequences can be seen in two different ways. In the context of interactive privacy scheme, our algorithm can be seen as providing a private answer to the query of degree sequence of a graph. This enables the analyst to fit all those models whose sufficient statistics are functions of the degree sequence. In the context of synthetic graphs, our algorithm can be regarded as generating synthetic graphs from the conditional (uniform) distribution of all graphs with a given degree sequence.

2 Previous Work

Considerable amount of research has been done in the area of privacy of graph data in the computer science community, for a partial survey of results on privacy techniques see [16]. Most of these techniques do not provide rigorous guarantees under arbitrary attacks which is provided by the notion of differential privacy [3]. There has been some work done in the area of protecting graphs using the notion of differential privacy. In [10], the authors show how to release the number of triangles in a graph in a private manner. In [7], the authors present algorithms to release different subgraph statistics in a private manner. However, neither of them consider degree sequences explicitly. Also, none of these works evaluate the usefulness of the output of their algorithms for performing statistical inference.

In [6], the authors present an algorithm to release the degree distribution of the graph in a differentially private manner. They do so by asking for the degree partition (an ordered degree sequence) of a graph. The degree partition has additional consistency constraints which are used to post process the answer. The authors show that one can release a very accurate estimate of the degree distribution by exploiting these constraints. However, the output from their algorithm is not directly usable for carrying out basic statistical inference tasks, as we illustrate in Section 6. Specifically, the degree partition released by their algorithm is not suitable for model testing applications and maximum likelihood estimation. The main issue is that the degree partition released by their algorithm need not be graphical, i.e there may not exist a simple graph whose degree partition corresponds to the released partition. This is a desirable property for many statistical applications where exactly specified degrees are desired, such as generating or enumerating random graphs for model-testing applications. Such applications are very common in statistical analysis of net-
works, and in fact form a central core of inference procedures. If output is not a graphical degree sequence, standard inference procedures such as conditional goodness of fit tests cannot be used and the maximum likelihood estimators may fail to exist for the private version of the released degree partition, even in the cases where the original degree partition does not suffer from these issues. For more details, see section 5.

We address these issues in our paper by presenting an algorithm to release the degree partition of a graph under differential privacy. The output from our algorithm can be used directly to perform maximum likelihood estimation and model testing of the beta model of random graphs (defined in Section 5). We built upon the work of [6] and include an additional post processing step to ensure that the released degree sequence is graphical. This work also serves to illustrate the point that, simply ensuring the closeness of L1 or L2 distance between the released and the original data may not be sufficient for statistical applications. However, this has been a common measure of utility in most work on differential privacy. Another contribution of the paper which may be of independent interest is describing a simple and efficient algorithm to test for the existence of maximum likelihood estimates of the beta model. In general, it is a difficult problem to characterize explicitly testable conditions in which the maximum likelihood estimators exist for different models. For more details on the problem of existence of mle, see [13] and references therein.

3 Preliminaries

This section introduces the preliminaries and the notation used in the paper. Let $G_n$ denote an graph on $n$ nodes and let $m$ be the number of edges in the graph. A simple graph is a graph with no self loops and multiple edges. All the graphs considered in this paper are simple. Let $G$ denote the set of all simple graphs on $n$ nodes. The distance between two graphs $G$ and $G'$ is defined as the number of edges on which the graphs differ and is denoted by $d(G, G')$. Next, we define the differential privacy for graph data.

3.1 Differential Privacy

Differential privacy for graphs is defined to protect edges in a graph (or relationships between nodes), as the following definition illustrates:

**Definition 1** (Edge Differential Privacy). Let $\epsilon > 0$. A randomized algorithm $A$ is $\epsilon$ edge differentially private if for all graphs $G$ and $G'$ such that $d(G, G') = 1$ and for all output $S$,

$$P(A(G) \in S) \leq e^\epsilon P(A(G') \in S)$$

Roughly, edge differential privacy requires that the output of the algorithm $A$ two neighboring graphs should be close to each other. A basic algorithm to release the output of any function $f$ under edge differential privacy is the
Laplace Mechanism \((\mathbb{B})\) which adds Laplace noise proportional to the global sensitivity of \(f\).

**Definition 2** (Global Sensitivity). Let \(f : G \rightarrow \mathbb{R}^k\). The global sensitivity of \(f\) is defined as

\[
GS(f) = \max_{d(G, G') = 1} ||f(G) - f(G')||_1
\]

where \(||.||_1\) is the \(L_1\) norm.

**Theorem 1** (Laplace Mechanism, \([3]\)). Let \(f : G \rightarrow \mathbb{R}^k\). Let \(Z_1, \ldots, Z_k\) be independent and identically distributed Laplace random variables with standard deviation \(\sqrt{2GS(f)}\). Then the algorithm which on input \(G\) releases \(f(G) + (Z_1, \ldots, Z_k)\) is \(\epsilon\)-differentially private.

One nice property of differential privacy is that any function of the differentially private algorithm is also differentially private as the following lemma illustrates.

**Lemma 1** (Post-processing, \([2, 11]\)). Let \(f\) be an output of a differentially private algorithm and \(g\) be any function. Then \(g(f(G))\) is also differentially private.

In the next section, we define the degree sequence and degree partition of a graph.

### 3.2 Degree sequence of a graph

Let \(G_n\) be an undirected simple graph on \(n\) nodes with \(m\) edges. The degree \(d_i\) of a node \(i\) is the number of nodes connected to it.

**Definition 3** (Degree Sequence, Degree partition and Degree distribution). The degree sequence of a graph \(d\) is defined as the sequence of degrees of each node. The ordered degree sequence, ordered in non-decreasing order is called the degree partition and is denoted by \(\bar{d}\). The degree distribution of a graph denoted by \(p\) is the sequence \(\{p_k, k = 1, \ldots, n - 1\}\) where \(p_k\) is the number of nodes of degree \(k\).

There can be more than one graph associated with the same degree sequence. Let \(G(d)\) be the set of simple graphs on \(n\) vertices with degree sequence \(d\). Also, not every integer sequence of length \(n\) is a degree sequence. Sequences that can be realized by a simple graph are called graphical degree sequences. Graphical degree sequences have been studied in depth and admit many characterizations.

One of the characterizations that is useful for our purposes is given below. The set of all degree sequences of size \(n\) is denoted by \(DS_n\). The set of all degree partitions of size \(n\) is denoted by \(DP_n\).

**Theorem 2.** [Have l- Hakimi] \([3]\) and \([7]\) Let \(d = \{d_1, \ldots, d_n\}\) be a non-decreasing sequence of integers. \(d \in DS_n\) iff \(c = \{c_1, \ldots, c_{n-1}\} \in DS_{n-1}\), where

\[
c_i = \begin{cases} 
  d_{i+1} - 1 & \text{if } 1 \leq i \leq d_1 \\
  d_{i+1} & \text{if } d_1 + 1 \leq i \leq n - 1
\end{cases}
\]
A straightforward way to release the degree sequence of a graph is to use the Laplace mechanism. Proposition 1 calculates the global sensitivity of $\bar{d}$, $d$ and $p$. Using this proposition, one can release the degree sequence by adding Laplace noise with scale parameter $b = \frac{2}{\epsilon}$. By theorem 1, this algorithm is $\epsilon$ differentially private.

**Proposition 1.** The global sensitivity of degree sequence, degree partition of a graph is 2.

It is possible to release the degree partition of a graph with smaller magnitude of noise, as illustrated by [6]. The main idea is to introduce consistency constraints in the query $q$ which hold for any graph $G$. Let the constrained query be $q_c$. The differentially private answer to the query $\hat{q}_c(G)$ need not satisfy the constraints. Hence we can post process the query $\hat{q}_c(G)$ so that it satisfies the constraints. Note that in general this approach need not improve the accuracy of the estimated answer. This is because, in general, the sensitivity of $q$ is different from sensitivity of $q_c$. However, there are many naturally occurring consistency constraints. For example, if the query asks for a degree sequence, we expect that the answer be a degree sequence. We can add more constraints to the query. For example, we can ask for the degree partition. This query has two constraints: the answer must be a set of monotonic nonnegative integers and it must also be a degree sequence. It turns out that the global sensitivity of these two queries are the same. Moreover, any kind of post processing does not violate differential privacy due to Lemma 1.

If we let $\bar{d}$ be the query that asks for the degree partition, the constraint that the differentially private answer to $\bar{d}$ needs to satisfy can be written as the geometric constraint that $\bar{d} \in DP_n$. If $z$ is the output from the Laplace mechanism, then the post processing step is equivalent to solving the following optimization problem:

$$s = \arg\min_{\bar{d} \in DP_n} ||\bar{d} - z||_1$$  \hspace{1cm} (1)

We propose a two step solution to the optimization problem 1. The first step is to compute the nearest non-decreasing integer sequence to the output of the Laplace mechanism, i.e. find the $L1$ projection of $z$ onto the set of nondecreasing
integers, denoted by \( Z \). In the second step, we find the nearest degree partition to the given nondecreasing sequence of integers. The first step of the problem is the well known case of isotonic regression, and was also the approach used by \[6\]. We present an algorithm to solve the second step of the proposed procedure. Specifically, we present an algorithm to find a degree sequence \( d \) that is closest to a given sequence of real numbers. We then show that if the given sequence is ordered, then the algorithm outputs the closest (in terms of the L1 distance) degree partition. The proposed mechanism is shown in algorithm \[1\]. Step 3 of the algorithm is the well known case of L1 isotonic regression and can be solved efficiently, see \[14\] and \[12\]. In the next section, we present an algorithm to solve step 4.

Algorithm 1

\begin{align*}
\text{Input: degree partition } & \bar{d}, \text{ privacy parameter } \epsilon \\
1: \text{ Sample } n \text{ independent Laplace random variables } & e_i \text{ with } b = 2/\epsilon \\
2: \text{ Let } z_i = d_i + e_i \text{ for } i = 1, \ldots, n \\
3: \text{ Let } c = \arg\min_{w \in Z_\leq} ||w - z||_1. \\
4: \text{ Let } s = \arg\min_{d \in DP_n} ||d - c||_1. \\
5: \text{ return } s
\end{align*}

4.1 Optimization over \( DS_n \)

In this section, we present an algorithm that finds a degree sequence closest to a given sequence of real numbers. We define “closeness” in terms of the L1 distance. The motivation for using the L1 distance is as follows. Let us assume we observe \( n \) random variables \( Z_i, i = 1 \) to \( n \) such that \( z_i = d_i + e_i \) where \( e_i \sim \text{Lap}(0, b) \), for \( i = 1 \) to \( n \) and \( d = \{d_i\} \in DS_n \) are the unknown parameters. It is very easy to see that the maximum likelihood estimates of \( d_i \) in the above estimation problem corresponds to finding an degree sequence closest to the sequence \( \{z_i\} \) in terms of the L1 distance. In essence, we are reconstructing the most likely value of the degree sequence from the observed noisy answer. The following is the main result of this section.

**Theorem 3.** Let \( z = \{z_i\} \) be a sequence of real numbers of length \( n \). The degree sequence of graph \( G \) produced by Algorithm 3 solves the optimization problem \( \arg \min_{h \in DS_n} ||h - z||_1 \).

We can obtain the following corollary which allows us to solve step 4 of algorithm 1.

**Corollary 1.** Let \( z = \{z_i\} \) be a sequence of non increasing integers of length \( n \). The degree partition of graph \( G \) output by Algorithm 3 solves the optimization problem \( \arg \min_{h \in DP_n} ||h - z||_1 \).

In the following algorithm, let \( d^* = \arg \min_{h \in DS_n} ||h - z||_1 \).
Algorithm 2  Input: A sequence $z$ of length $n$  
Output: A graph $G$ on $n$ vertices with degree sequence $d^*$

1: Let $G$ be the empty graph on $n$ vertices 
2: for $i = 1 \rightarrow n$ do 
3: Let $\text{pos} = |\{j : z_j \neq 0, i+1 \leq j \leq n\}|$ 
4: Let $h = \min(\lceil z_{(i)} \rceil, \text{pos})$ where $z_{(i)}$ is the $i^{th}$ largest element. 
5: Let $\mathcal{I} =$ indices of $h$ highest values of $\lceil z_j \rceil$ from $i+1$ to $n$ 
6: Add edge $(i, k)$ to $G$ for all $k \in \mathcal{I}$ 
7: Let $z_j = z_j - 1$ for all $j \in \mathcal{I}$ 
8: end for 
9: return $G$

Remark: Given a point $z$, algorithm[2] finds a point in $DS_n$ that is closest to $z$ in terms of L1 distance. There are many differences from the traditional projection. Firstly, the set $DS_n$ has "holes" in it, for instance, every point whose $l1$ norm is not divisible by 2 is not included in the set. Due to this reason, the closest point need not be on the boundary of the convex hull of $DS_n$. Moreover, there can be more than one degree sequence that solve the same optimization problem. Specifically, the following is true.

**Lemma 2.** Given any optimal solution $d^*$ to the optimization problem[7] we can obtain another optimal solution by increasing or decreasing the degree of a pair of nodes by adding or deleting an edge, as long as each degree remains bounded pairwise by $\lceil z \rceil$.

Using this property, we can search for an optimal degree sequence that lies inside the boundary of convex hull of $DS_n$. This is an important property for ensuring that the maximum likelihood estimates of the beta model exist, see section[3] In the next section, we present the beta model of random graphs whose sufficient statistics are the degree sequences.

5 Degree sequence and the beta model

One of the simplest model involving degree sequences of a graph is the beta model. This model admits many different characterizations, see [1] and references therein. The beta model arises as a model in the discrete exponential family of distributions on the space of graphs when the degree sequence is a sufficient statistic. We can also describe this model in terms of independent Bernoulli random variables as follows.

Let $\beta$ be a fixed point in $\mathbb{R}^n$. For a random graph on $n$ vertices, let each edge between nodes $i$ and $j$ occur independently of other edges with probability $p_{ij}$

$$p_{ij} = \frac{e^{\beta_i + \beta_j}}{1 + e^{\beta_i + \beta_j}}$$
This model is called the beta model with \( \{ \beta_i \} \) as the vector of parameters. The beta model arises as a special case of p1 models and a log linear model, see [13]. If we ignore the ordering of the nodes, then the degree partition is also a sufficient statistic for the beta model. In the next two subsections, we illustrate two common statistical inference tasks that are associated with the beta model. We will evaluate our algorithm by performing these tasks on the private version of the degree partition.

5.1 Existence of mle of the beta model

We would like to have the property that if the maximum likelihood estimates of the observed degree partition exist, then the maximum likelihood estimates of the private version of the degree partition also exist. Note that under strict implementation of differential privacy, this is not allowed, as the answer to the query “Does the mle exist” cannot be answered exactly. However, we relax this requirement, and our algorithm satisfies this property approximately. More specifically, if the mle of the observed degree partition exists, the algorithm attempts to output a degree partition whose mle also exists. This is done by making use of the property in Lemma 2.

We need an efficient way to check for the existence of mle. In [13], the authors provide conditions to check for the existence of the mle of the beta model, however their algorithm is not efficient. Here we present a simple and efficient algorithm to check for the existence of the mle for the degree partitions \( \tilde{d} \) which may be of independent interest. We conjecture that this result holds for the case of degree sequences as well. The following theorem provides conditions to check for the existence of mle for the degree partition which follows from a standard theorem of exponential families, see [9].

**Theorem 4.** Let \( d \) be a degree partition. The mle of the beta model exists iff \( d \in ri(conv(DP_n)) \) where \( conv(DP_n) \) is the convex hull of the set of degree partitions, which is true iff

1. \( d_i > 0 \) and \( d_i < n-1 \) \( \forall \ i \).

2. \( \sum_{i=1}^{k} d_i - \sum_{i=n-l+1}^{n} x_i < k(n-1-l) \) for \( 1 \leq k+1 \leq n \)

Theorem 4 shows that the mle of the beta model exists iff the degree partition lies in the relative interior of convex hull of \( DP_n \).

5.2 Conditional tests and conditionally specified models

Conditional goodness of fit tests are used to evaluate the fit of any model and are based on the space \( \mathcal{G}(d) \). To perform conditional tests, we need the released degree partition to be graphical. This is because if \( d \) is not graphical, then \( \mathcal{G}(d) \) is empty. As another example, consider conditionally specified models of random graphs. In these models, one considers the degree sequence is treated as a nuisance parameter and conditions on them. Statistical inference can be
performed by simulating from the space of all graphs given the fixed degree sequence $\mathcal{G}(d)$. But $\mathcal{G}(d)$ is empty if $d$ is not graphical. However, if $d$ is inside the convex hull of degree sequences, then one can perform tests based on the set of graphs given the expected degree sequence. However, if $d$ is outside this convex hull, then $\mathcal{G}(d)$ is empty or if $d$ is on the boundary of the convex hull, $\mathcal{G}(d)$ contains a single element. In all these cases, our algorithm outputs a degree graphical degree sequence closest to $d$.

6 Experiments

In this section, we evaluate our proposed algorithm (called isotone-hh) for releasing degree partitions (algorithm [1]) empirically and compare it with the algorithm due to [6] (called isotone). In the original algorithms, the authors use L2 minimization, but we use an L1 minimization to be consistent with our algorithm.

The main goal of these experiments is to evaluate the statistical properties of the degree partitions produced by these differentially private algorithms. There are three categories of experiments. In the first setting, we compare how close the released degree partition is to the original degree partition. In the second set of experiments, we are interested in the following basic question: If the mle exists for the original degree partition, does the mle also exist for the private version. In the last set of experiments, we evaluate the closeness of the distribution of number of triangles in the space of graphs given the original degree sequence is to the space of graphs given the private degree sequence. This distribution is important for goodness of fit tests for the beta model. Specifically, this distribution is used to compute the p-values for goodness of fit tests. We present our results for the karate dataset ([17]) obtained from the UCI network repository. This dataset is a social network of friendships between 34 members of a karate club at a US university. For the experiment related to the existence of mle, we also present our results for the family of power law graphs.

Remark: In our experiments, we only ask for the degree partition. An analyst may be interested in releasing the degree sequence when the order is set by some other requirement. In such cases, our algorithm can release a graphical degree sequence but the additional constraints of monotonicity no longer exist. In simulation experiments, we found that the degree sequence released without these additional constraints is still very noisy and not useful for statistical inference. This could be due to issues with the algorithm, but it could also be that differential privacy requires addition of a large amount of noise. Thus in cases where the ordering information is not useful, it is better to ask for the degree partition.

6.1 Existence of MLE of the beta model

As noted in section [5] the maximum likelihood estimates of the beta model exist only when the degree sequence lies in the interior of the polytope of degree
sequences. In this set of experiments, we simulate random graphs with degree sequences following the power law $p_i = P(d_i = x) \propto cx^\gamma$ for different values of $\gamma$ and different node sizes. For each simulated degree partition $(d)$, we find the degree partition released by the isotone algorithm and the isotone-hh algorithm $(d_e)$. We compute the probability that the existence of mle for the original degree partition coincides with the existence of mle exist for the released degree partition by simulating over the randomness of the Laplace noise and the random graph model. We used the conditions provided in theorem [1] to check for the existence of mle for the degree partition. The results are shown in Table 1.

Table 1: $P(\text{existence of mle of } d_e \text{ coincides with } d)$ for power law family of graphs

| n   | Isotone-hh | Isotone |
|-----|------------|---------|
|     | 1          | 1.5     | 2      | 1      | 1.5    | 2      |
| 100 | 0.983      | 0.997   | 0.910  | 0.242  | 0.240  | 0.251  |
| 200 | 0.998      | 1.000   | 0.930  | 0.240  | 0.239  | 0.241  |
| 400 | 1.000      | 1.000   | 0.956  | 0.241  | 0.240  | 0.233  |
| 500 | 1.000      | 1.000   | 0.967  | 0.243  | 0.243  | 0.232  |

From Table 1, we can see that for the isotone algorithm, the existence of mle coincides only 25 percent of the times. On the other hand, the existence of mle coincides at least 90 percent of the times for the Isotone HH algorithm. Table 2 shows the results for the Karate dataset. Again we can see that the mle exists with high probability for the isotone-hh algorithm whereas the mle exists only 50 percent of the times for the Isotone algorithm. The mean $L2$ error for both the algorithms are very close to each other.

Table 2: $P(\text{mle exists})$ and $L2$ error for Karate Dataset

| P(mle exist) | Mean L2 error |
|--------------|---------------|
| Isotone HH   | 0.998         | 52.63         |
| Isotone      | 0.499         | 56.57         |

6.2 Parameter estimates of the beta model

In the next set of experiments, we evaluate how close the maximum likelihood estimates of the beta model for the synthetic graph are to the original graph. The comparison is tricky because in more than 50 percent of the cases, the mle did not exist for the isotone algorithm. In such a case, we assumed that the parameter estimates are 0. For the degree partition of the karate dataset, we released the degree partition using the isotone and the isotone-hh algorithm 500 times. Figure 1 shows the results of the experiment; it is a plot of the estimates of the $\beta$ parameters on the y axis vs the node id on the x axis. The red, green, blue lines indicate the mean value of the parameter estimates, the maximum likelihood estimates and the 95 percent confidence intervals of the estimates respectively. We can see that the estimates for the Isotone algorithm are biased and have higher variance when compared to the isotoneHH algorithm. This
is because of the fact that the mle does not exist for many degree partitions released by the Isotone algorithm.

6.3 Empirical Null of number of triangles

In the last set of experiments, we compare $G(d)$, the space of graphs given the original degree sequence with $G(d_e)$, the space of graphs given the released degree sequence. If the released degree sequence is not graphical or is an extreme point, this set is either empty or has a single element. This set is associated with model testing applications for the beta model. For example, one common procedure for testing the fit of the beta model is to pick a statistic $T(G)$, say number of triangle and compute the sampling distribution of the number of triangles of random graphs with the fixed degree sequence. This is in line with the exact tests in the contingency table literature where one conditions on the sufficient statistics. As before, we use the Karate dataset and repeat the experiment 500 times. For each run, we release the degree partition using isotone and isotoneHH algorithms and compute the empirical null distribution of the number of triangles. Figure 2 shows the results for 10 sample runs.

For the isotone algorithm, if the released degree sequence was not graphic, we output a point mass distribution at arbitrary point, in this case -10. The blue, green and red lines in the figure show the distribution of the number of triangles obtained from the Isotone algorithm, the original degree sequence and the IsotoneHH algorithm respectively. Each panel shows the output from one random draw. We can see that in many cases, the Isotone algorithm fails to produce a valid distribution. On the other hand, the IsotoneHH algorithm produces a valid distribution which is also close to the true empirical null. However, there are cases when the empirical null is completely disjoint, for example, the
7 Conclusion and Future Work

In this paper, we presented an algorithm to release a graphical degree sequence of a graph in a differentially private manner by adding an additional step to the algorithm proposed by [6]. The main motivation for releasing a graphical degree sequence is to enable analysts to perform useful statistical inference, for example, goodness of fit tests, and maximum likelihood estimation of the beta model. We presented simpler conditions for testing the existence of mle for the beta model for a degree partition and used these conditions to empirically evaluate our algorithm and that of [6]. We found that even though the mle exists for the original degree sequence, the mle fails to exist in more than 50 percent of the cases for the sequences released by [6]. On the other hand, our proposed algorithm performs better, in that the mle exists with very high probability.

We also compared the effect of other statistical inference procedures such as parameter estimation and goodness of fit testing. Both of these are inherently tied to the nongraphical nature of the released degree sequence.

However, there are further issues that need to be addressed. For instance, to compute p-values, we need to know the observed number of triangles. Under differential privacy, the analyst obtains a private version of observed number of triangles, possibly released by using the algorithms provided in [7]. Thus, not only is the space of graphs giving rise to the empirical null distribution completely disjoint from the original space of graphs, but also the observed statistic is a noisy version of the original statistic. More work in needed to understand the behavior of p-values under such setting.
Another direction of work would be to release degree sequences for bipartite and directed graphs. The degree sequence of bipartite graphs form sufficient statistics for the so called rasch models, see for instance [13].

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Proof of theorem \[1\]

In this section, we present the proof of correctness of algorithm \[2\]. The main idea behind the proof is that degree sequences can be written as special sums, which are helpful in selecting directions for building the solution to the L1 optimization problem. We begin by some definitions.

Definition 4 (k-star degree sequence). A degree sequence $d_k$ is said to be a k-star degree sequence if there exists a graph $G \in G(d_k)$ on $n$ vertices such that $G$ is a k-star. Note that we allow the graph to have disconnected nodes, specially in case $k < n$. Let $K_n$ be the set of all k-star degree sequences of length $n$.

Lemma 3. Every degree sequence $d$ can be written as $d = \sum_i g_i$ where $g_i \in K_n$.

Proof. Let $d$ be any degree sequence. Consider repeated applications of Theorem \[2\] to $d$. Let the residue sequence obtained at each step be $r_i$. It is easy to see that this procedure terminates after atmost $n$ steps, thus it generates at most $n$ residue sequences. Moreover, $r_{i+1}$ is obtained from $r_i$ by reducing $r_i$ with a $\max_j r_i(j)$ star sequence. Let $g_i$ be the star sequence used to reduce $r_i$ to $r_{i+1}$. Since $d$ is a degree sequence, the last residue sequence will be the 0 degree sequence. It is easy to see that $d$ can be reconstructed as a sum of $g_i$, i.e. $d = \sum_i g_i$. Since each $g_i$ is a k star sequence, $g_i \in K_n$.

Definition 5 (Havel Hakimi Decomposition). The Havel Hakimi decomposition of a degree sequence $d$ is defined as the set of k-star degree sequences obtained after the application of Theroem \[4\] and is denoted by \{d\}.

Lemma \[3\] shows that every degree sequence can be written as a sum of $k$-star sequences, thus every degree sequence has a havel hakimi decomposition. Further, it is easy to see that the decomposition is unique if the order of the nodes is fixed. The next two lemmas allow us to restrict the search for optimal degree sequences in the set of degree sequences that are pointwise bounded by $z$ after eliminating the negative coordinates of $z$. 

\[14\] T. Robertson, FT Wright, R.L. Dykstra, and T. Robertson. Order restricted statistical inference, volume 229. Wiley New York, 1988.

\[15\] T.A.B. Snijders. Accounting for degree distributions in empirical analysis of network dynamics. In Dynamic Social Network Modeling and Analysis: workshop summary and papers, pages 146–161, 2003.

\[16\] X. Wu, X. Ying, K. Liu, and L. Chen. A survey of algorithms for privacy-preservation of graphs and social networks. Managing and Mining Graph Data. Kluwer Academic Publishers, Dordrecht (August 2009), 2010.

\[17\] W.W. Zachary. An information flow model for conflict and fission in small groups. Journal of anthropological research, pages 452–473, 1977.
Lemma 4. Let \((z_1, \ldots, z_n)\) be a sequence of real numbers. Let \(\mathcal{I} = \{i : z_i > 0\}\). Let \(f_z(a) = \sum_i |z_i - a_i|\). Let \(d\) be any degree sequence such that \(\text{argmin}_{a \in DS_n} f_z(a) = d\) and \(d(\mathcal{I}^c) > 0\). Then there exists a degree sequence \(d^*\) such that \(d^*(\mathcal{I}^c) = 0\) and \(f(d) = f(d^*)\).

Proof. If \(d_i = 0\forall i \in \mathcal{I}^c\), the lemma is true by letting \(d^* = d\). Hence assume that \(\exists \) at least one \(i = j \in \mathcal{I}^c\) such that \(d_j > 0\). Let \(d^*\) be the degree sequence obtained from \(d\) by reducing it with a \(d_j\)-star.

Next let us show that \(f(d^*) \leq f(d)\).

\[
f(d^*) = \sum_i |z_i - d_i^*| = \sum_{i \in \mathcal{J}} |z_i - d_i^*| + \sum_{i \in \mathcal{K}} |z_i - d_i^*| + |z_j - d_j^*|
\]

\[
= \sum_{i \in \mathcal{J}} |z_i - d_i| + \sum_{i \in \mathcal{K}} |z_i - d_i| + |z_j| \leq \sum_{i \in \mathcal{J}} |z_i - d_i| + \sum_{i \in \mathcal{K}} |z_i - d_i| + |z_j| + d_j
\]

\[
= \sum_{i \in \mathcal{J}} |z_i - d_i| + \sum_{i \in \mathcal{K}} |z_i - d_i| + |z_j - d_j|
\]

But \(d\) is such that \(\text{argmin}_{a \in DS_n} f_z(a) = d\), hence \(f(d^*) = f(d)\). If there is more than one \(j \in \mathcal{I}\) that \(d_j > 0\), we can redefine \(d^*\) iteratively until there are no such \(j\) left.

\[\square\]

Lemma 5. Let \((z_1, \ldots, z_n)\) be a sequence of non negative real numbers. Let \(f_z(a) = \sum_i |z_i - a_i|\). Let \(d\) be any degree sequence such that \(\text{argmin}_{a \in DS_n} f_z(a) = d\). Then there exists a degree sequence \(d^*\) such that \(d_i^* \leq \lfloor z_i \rfloor \forall i\) and \(f_z(d^*) = f_z(d)\).

Proof. If \(d_i \leq \lfloor z_i \rfloor \forall i\), the lemma is true by letting \(d^* = d\). Hence assume that \(\exists \) at least one \(i = j\) such that \(d_j > \lfloor z_j \rfloor\). Let \(d^*\) be defined as follows:

\[
d_i^* = \begin{cases} 
\lfloor z_i \rfloor & \text{for } i = j \\
|d_i - 1| & \text{for } i \in \mathcal{I}, j \notin \mathcal{I} \\
d_i & \text{for } i \in \mathcal{J}
\end{cases}
\]

where \(\mathcal{I}\) and \(\mathcal{J}\) are any index sets such that \(|\mathcal{I}| = d_j - \lfloor z_j \rfloor\), and \(\mathcal{I} \cup \mathcal{J} \cup \{i\} = [n]\). Clearly, \(d^*\) is a degree sequence because it is obtained by reducing \(d\) with a \(k\)-star sequence, where \(k = d_j - \lfloor z_j \rfloor\).

Next let us show that \(f(d^*) \leq f(d)\).
Lemma 6. Let $d_0$ be any degree sequence in $DS_{\leq z}$ and let $\{d_0\} = \{g_1\}$ be its haveli hakimi decomposition. Let $\{x_k\}$ be the $k$-star following sequence: $x_1 = \arg\min_{x \in \mathcal{K}_{\leq z}} \{f_z(x) : g \in \mathcal{K}_{\leq z}\}, x_{k+1} = \arg\min_{x \in \mathcal{K}_{\leq z} \setminus \{x_1, \ldots, x_k\}} \{f_z(\sum_{i=1}^k x_i + g), g \in \mathcal{K}_{\leq z} \setminus \{x_1, \ldots, x_k\}\}$. Let $d_k$ be defined as the following sequence: if $x_1 \in \{d_0\}$ then $d_1 = d_0$, else $d_1 = x_1 + \sum_{i \neq j} g_i$. Similarly, if $x_k \in \{d_0\}$, then $d_k = d_{k-1}$, else $d_k = \sum_{i=1}^k x_i + \sum_{i=k+1}^n g_i$, where $g_i \in \{d\}$. Then, $f_z(d_n) \leq f_z(d_0)$ and each $d_k \in DS_{\leq z}$.
Proof. Consider
\[
    f_z(d_k) - f_z(d_{k+1}) = \|z - \sum_{i=1}^{k} x_i - \sum_{i=k+1}^{n} g_i\| - \|z - \sum_{i=1}^{k+1} x_i - \sum_{i=k+2}^{n} g_i\|
\]

\[
= x_{k+1} - g_{k+1} = \|z - \sum_{i=1}^{k} x_i - g_{k+1}\| - \|z - \sum_{i=1}^{k} x_i - x_{k+1}\|
\]

\[
= f_z(\sum_{i=1}^{k} x_i + g_{k+1}) - f_z(\sum_{i=1}^{k} x_i + x_{k+1})
\]

\[
\geq 0
\]

Adding these inequalities for \(k = 0\) to \(k = n - 1\), we get \(f_z(d_0) - f(d_n) \geq 0\), as required. Moreover, each \(d_k\) is clearly is a degree sequence, as \(d_k\) is obtained from \(d_{k+1}\) by replacing a \(k\)-star sequence from its Havel Hakimi decomposition.

The next proposition shows how to find the best \(k\)-star sequence for the L1 optimization.

**Proposition 2.** Given a non negative sequence \(z\), the element in the set \(\mathbb{K}_{\leq z}\) that solves the following optimization problem

\[
\min_{g \in \mathbb{K}_{\leq z}} \|z - g\|_1
\]

is the following \(k\)-star sequence: if \(i^* = \{i : \lceil z_i^* \rceil = \max_i \lceil z_i \rceil\}\), then \(k = \lceil z_i^* \rceil\). Let \(\mathcal{I}\) be the index set of \(k\) largest elements of \(z\) excluding \(i^*\), then there is an edge between \(i^*\) and \(i\) for all \(i \in \mathcal{I}\).

We are now ready to present the proof of Theorem 1.

### 7.1 Proof of Theorem 1

Let \(d^*\) be the optimal degree sequence. Let \(\mathcal{I} = \{z_i : z_i \leq 0\}\). By lemma 4, we can set \(d^*(\mathcal{I}) = 0\). Thus, it is enough to find the optimal degree sequence \(d^*\) with respect to the function \(f_z(x_i)(d)\). From this point onwards, let us assume that \(\mathcal{I} = \emptyset\). This is achieved by Step 3 of algorithm 2. Moreover, from lemma 5, it is enough to consider degree sequences bounded pointwise by \(\lceil z \rceil\). Thus, we need to find the optimum over the set \(DS_{\leq z}\). By lemma 6, we can construct the optimal degree sequence over \(DS_{\leq z}\) by starting with any degree sequence \(d_0\) and replacing it by the \(k\)-star sequence defined in lemma 6. Since 0 is also a degree sequence, let the starting sequence \(d_0\) be the zero degree sequence. Then, the optimal degree sequence is \(\sum_{k=1}^{n} x_k\) where \(x_{k+1} = \arg\min \{f_z(\sum_{i=1}^{k} x_i + g), g \in \mathbb{K}_{\leq z} \setminus \{x_i\}_{i=1}^{k} \}\). It is easy to see that the optimal \(k\)-star sequence for each \(x_{k+1}\) obtained by Proposition 2 is the same sequence selected by steps 5 and 6 of algorithm 2.
Theorem 5. Given a graph $G$, we can release its degree sequence differentially privately in $O(m + n \log n)$ time. We can also release a synthetic graph corresponding to the private degree sequence in $O(m + n \log n)$ time.