Interaction of solitary waves in longitudinal magnetic field in two-fluid MHD

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Abstract. The interaction of solitary waves in a model of two-fluid MHD is studied analytically and numerically in the most general case of waves in cold plasma in longitudinal magnetic field. The distinctive feature of this work is the use of "exact" equations rather than an approximate approach (a model equation). Numerical analysis of the solutions of this system of eight partial differential equations shows that the the interaction of solitary waves found in this case is the same (with great accuracy) as that of solitons, i.e., solitary waves that are solutions of various model equations. The solitary waves considered here transport plasmoids with velocities of the order of the Alfven velocity. The main finite-difference method used here for solving the said equations is a natural generalization of the classical two-step Lax-Wendorff scheme.

1. Introduction

The study of solitons is traditionally based on certain model equations [1], starting with the Korteweg-de Vries equation, the nonlinear Schrödinger equation, the sine-Gordon equation. For plasma environments, this list has been considerably extended due to Zakharov equations, Kadomtsev-Petviashvili equation, the equations of nonlinear Alfven waves and magnetic-sound waves [2], etc.

In this paper, we study numerically the interaction of solitary waves of a special type: the so-called wave packets, each having the form of nonlinear vibrations whose amplitude is modulated by a solitary wave. It is the longitudinal magnetic field that (being responsible for the rotation of plasma particles in the transverse plane) makes the transverse velocity components, as well as the magnetic and the electric fields, experience nonlinear vibrations which, under certain conditions, form a wave packet running along the magnetic field. In the case of cold plasma, these waves are presented in [3]. The principal distinction of the present work from other similar studies consists in avoiding the ideology of model equations, thereby making it possible to use the exact equations of two-fluid plasma hydrodynamics [4] (i.e., the equations expressing the fundamental laws of conservation of mass, energy, and momentum of electrons and ions, as well as the laws of electrodynamics) for both finding the solitary waves and studying their interaction.

It is shown here that, when colliding, the wave packets are similar to material particles: they preserve their shape, velocity, amplitude, etc., and the collision process has a finite duration.
2. Basic Equations

In the absence of dissipation, for fully ionized two-component quasi-neutral cold plasma the equations of two-fluid hydrodynamics of plasma [4] can be written in the following one-fluid form [5]:

\[
\frac{\partial \rho}{\partial t} + \text{div}\rho \mathbf{U} = 0, \quad \frac{\partial \rho \mathbf{U}}{\partial t} + \text{Div}(\rho \mathbf{UU} + \Pi^\rho + \Pi^\epsilon) = 0 \tag{1}
\]

\[
\mathbf{E} + \frac{c^2 \lambda_i \lambda_e}{4\pi \rho} \text{rot}\mathbf{E} = -\frac{1}{c}\left[\mathbf{U}, \mathbf{H}\right] + \frac{1}{\rho} \text{DivW} \tag{1}
\]

\[
\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} + \text{rot}\mathbf{E} = 0, \quad \text{div}\mathbf{H} = 0, \quad \text{rot}\mathbf{H} = \frac{4\pi}{c} \mathbf{j}
\]

where the tensors \(\Pi^\rho\), \(\Pi^\epsilon\), \(W\) have the form

\[
\Pi^\rho = \frac{H^2}{8\pi} I_3 - \frac{HH}{4\pi}, \quad \Pi^\epsilon = \lambda_i \lambda_e \frac{\mathbf{j} j}{\rho}, \quad W = (\lambda_i - \lambda_e)(\Pi^\rho + \Pi^\epsilon) + \lambda_i \lambda_e (\mathbf{U} j + j \mathbf{U}) \tag{2}
\]

In (1)-(2) \(I_3\) is the three-dimensional identity tensor, \(\rho = \rho_i + \rho_e\) is the total density, \(\mathbf{U} = (\rho_i \mathbf{v}_i + \rho_e \mathbf{v}_e)/\rho\) is the mass hydrodynamic plasma velocity, \(\lambda_i = m_i / e\), \(\lambda_e = m_e / (Ze)\), where \(Z\) is the ion charge multiplicity. Thus, (1)-(2) is a closed system of equations for the unknown quantities \(\rho\), \(\mathbf{U}\), \(\mathbf{H}\), \(\mathbf{E}\).

Consider the solutions of equations (1)-(2) as functions of \(t\), \(r\) in terms of the combined argument \(\theta = rk - at\), where \(k\) is a vector of unit length and \(a\) is a constant. Such solutions are usually referred to as planar traveling waves, \(a\) is the phase velocity of the wave, \(k\) is the wave propagation vector. We are interested in the interaction of solitary traveling waves such that all their respective parameters have finite and equal limit values as \(\theta \to \pm \infty\). The parameters of an arbitrary traveling wave \(U(\theta), \rho(\theta), \mathbf{H}(\theta), \mathbf{E}(\theta)\), for fixed \(k\) and \(a\), are found by substituting these functions into system (1)-(2). After simple transformations, we obtain the following equations for the transverse magnetic field \(H_\perp(\theta)\) and the longitudinal velocity \(u(\theta) = U_\parallel(\theta) - a\) (in the reference frame of the traveling wave):

\[
J_u + \frac{H^2_\perp}{8\pi} = D
\]

\[
\left( u - \frac{H^2}{4\pi J} \right) H_\perp - \frac{c^2 \lambda_i \lambda_e}{4\pi J} \frac{d}{d\theta} \left( u \frac{d H_\perp}{d \theta} \right) + \frac{cH_\parallel}{4\pi J} (\lambda_i - \lambda_e) u \left[ \mathbf{k}, \frac{d H_\perp}{d \theta} \right] + q = 0 \tag{3}
\]

Where each vector is expanded along \(||\) and across \(\perp\) the direction of wave vector \(k\), and \(J \neq 0\), \(D > 0\), \(q \perp k\) are arbitrary constants of integration. Moreover, we always have \(H_\parallel(\theta) = \text{const}\) in the traveling wave. In a special case, the two-fluid form of system (3) is obtained in [6], and in the general case, in [3].

In [7], the authors study collisions of solitary waves represented by the solutions of system (3) of the form \(H_\perp(\theta) = H(\theta)e_0\) with \(H_\parallel = 0\), where \(e_0 \perp k\) is an arbitrary unit vector, \(q = q e_0\), in which case the magnetic field \(H\) varies only in magnitude and has a constant direction in the transverse plane. The case \(q = 0\) corresponds to system (3) being invariant with respect to rotations in the transverse plane.

2. Solving Equations (3) for Solitary Waves in the Case \(q = 0\)

System (3) with \(q = 0\) is completely integrable. Let us write this system in the dimensionless form
With the dimensionless parameters \( \xi, b, \Lambda \) expressed by \( \xi^2 = c^2 \lambda_c / (4\pi \rho_0 L_0^2), \ b = H_\parallel / H_0, \ \Lambda = (\lambda_\parallel / \lambda_c)^{1/2} - (\lambda_c / \lambda_\parallel)^{1/2} \)

Here, \( L_0, H_0, V_0, \rho_0 \) are the characteristic scales of length, magnetic field strength, velocity, and density, respectively; and \( H_0, V_0, \rho_0 \) are chosen related to the traveling wave constants \( J \) and \( D \) by \( H_0 = \sqrt{4\pi D}, \ \rho_0 = 4J^2 / D, \ V_0 = D / (2J), \ V_0 = H_0 / (4\pi \rho_0) \). Thus, we seek a solution of system (5) in the fixed domain \( |H_\perp| < \sqrt{2} \) and this solution depends on the parameters \( \xi, b \).

In order to find a solution of system (4), we set \( H_\perp = 0, H_\parallel, H_z \) and pass to the polar coordinates \( y = H_\parallel \cos \varphi, \ z = H_\parallel \sin \varphi \) on the transverse plane. From system (4), we obtain the solution in explicit form

\[
\theta(H) = \pm \xi \int (2 - H^2) \frac{dH}{\sqrt{2(E - \Phi(H))}}, \ \varphi(H) = \pm \int \frac{2\Omega + \Lambda bH^2}{2H^2 \sqrt{2(E - \Phi(H))}} dH
\]

Here, \( E, \Omega \) are arbitrary constants, and the potential function \( \Phi(H) \) has the form

\[
\Phi(H) = \frac{H^2}{8} - \frac{1}{2} (1 - \eta^2 b^2) H^2 + \frac{\Omega^2}{2H^2}, \ \eta^2 = 1 + \frac{\Lambda^2}{4}
\]

The desired solutions of the solitary wave type correspond to the degenerate cases and exist only if

\[
\Omega = 0, \ E = 0, \ 0 < \alpha^2 = 1 - \eta^2 b^2 < 1/2.
\]

Then formulas (5) define two semibounded arcs whose analytic expressions are obtained by calculating the integrals \( 0 < H < H_m = 2\alpha \)

\[
\theta(H) = \pm \xi \int \frac{(2 - H^2)}{\sqrt{-2\Phi(H)}} dH = \mp \xi \left(1 - \ln \left(\frac{H_m + \sqrt{H_m^2 - H^2}}{H_m - \sqrt{H_m^2 - H^2}}\right)\right)
\]

\[
\varphi(H) = \pm \frac{\Lambda b}{2} \int \frac{dH}{\sqrt{-2\Phi(H)}} = \mp \frac{\Lambda b}{2} \ln \left(\frac{H_m + \sqrt{H_m^2 - H^2}}{H_m - \sqrt{H_m^2 - H^2}}\right)
\]

Consider the boundary value problem of exciting a solitary wave (in resting plasma of density \( \rho_\infty \) with the magnetic field strength \( H_\infty = H_\parallel \)) traveling along the magnetic field. For the characteristic scales we then have

\[ u_\infty = -a, \ J = \rho_\infty u_\infty = -a \rho_\infty, \ D = \rho_\infty u_\infty^2 = \rho_\infty a^2, \ V_0 = |a| / 2, \ \rho_0 = 4\rho_\infty, \ H_0 = \sqrt{4\pi \rho_\infty} |a| \]

Hence, we get \( b = H_\perp / H_0 = V_\perp / |a| \), where \( V_\perp = H_\perp / \sqrt{4\pi \rho_\infty} \). Then, from (6) we obtain the following constraints on the plasma velocity \( a \)

\[
|V_\perp| \eta < |a| < \sqrt{2} |V_\perp| \eta
\]

Moreover, it is not difficult to obtain formulas for the relative amplitude \( A(a) \) of the solitary wave

\[
A(a) = \frac{H_m H_0}{|H_\parallel|} = 2 \left|\frac{a}{V_\perp}\right| \sqrt{1 - \eta^2 (V_\perp / a)^2} = 2 \left(\frac{a}{V_\perp}\right)^2 - \eta^2
\]
The function \( A(a) \) is monotonically increasing in \( |a| \) from 0 to \( 2\eta^2 \). In particular, for electron-ion plasma, the maximal amplitude is \( \approx \Lambda \), which, for instance, in the case of hydrogen plasma is \( \approx 42 \).

3. Numerical Modeling Method

Consider three problems regarding the interaction of the above waves. We are interested in three cases: A) motion of two solitary waves of the same amplitude towards each other; B) motion of two solitary waves of different amplitudes towards each other; C) a solitary wave of greater amplitude catching up with a solitary wave of a smaller amplitude moving in the same direction.

In this connection, we are going to obtain numerical solutions of problem (1)-(2) in the plane case, \( \partial / \partial y = \partial / \partial z = 0 \), on the line \( -\infty < x < +\infty \) for \( t \geq 0 \), with the initial conditions at \( t = 0 \) (to be specified below) and the boundary conditions at infinity

\[
\rho(\pm\infty) = \rho_0, \quad U_x(\pm\infty) = 0, \quad U(\pm\infty) = 0, \quad H(\pm\infty) = 0, \quad E(\pm\infty) = 0
\]

Where the \( x \) axis is directed along the vector \( k \) and complex notation is used for the components of the vectors \( U, H, E \):

\[
U = U_y + iU_z, \quad H = H_y + iH_z, \quad E = E_y + iE_z.
\]

In dimensionless form, system (1)-(2) can be writes as

\[
\frac{\partial u}{\partial t} + \frac{\partial f(u,H)}{\partial x} = 0, \quad \frac{\partial H}{\partial t} + i\frac{\partial E}{\partial x} = 0
\]

\[
E - g(u)\frac{\partial^2 E}{\partial x^2} = D(u,H)
\]

With the boundary conditions

\[
u_1(\pm\infty) = 1, \quad u_2(\pm\infty) = 0, \quad u_3(\pm\infty) = 0, \quad H(\pm\infty) = 0, \quad E(\pm\infty) = 0
\]

Here

\[
u = (u_1, u_2, u_3) = (\rho, \rho U_x, \rho U_y)
\]

\[
f(u,H) = \left( \frac{u_2^2}{u_1}, \frac{u_2 u_3}{u_1}, \frac{H}{2}, \frac{u_3^2}{u_1}, \frac{u_2^2}{u_1}, \frac{H}{2}, \rho U_x, \rho U_y, \rho U_z - H_\parallel H \right)
\]

\[
g(u) = \frac{\xi^2}{u_1}, \quad D(u,H) = \frac{i}{u_1} \left( H_\parallel u_3 - H_\parallel u_2 - i\Lambda \xi H_\parallel \frac{\partial H}{\partial x} + \xi^2 \frac{\partial}{\partial x} \left( \frac{u_3 u_2}{u_1} \frac{\partial H}{\partial x} \right) \right)
\]

Where \( i \) is the imaginary unit, \( \xi \) is the similarity number from Sect. 2, and the characteristic scales satisfy the relations

\[
E_0 = V_0 H_0 / c, \quad V_0 = H_0 / \sqrt{4\pi \rho_0}, \quad t_0 = L_0 / V_0, \quad \rho_0 = \rho_0
\]

Our numerical method for solving system (11)-(12) is based on the two-step Lax-Wendroff \([8]\) scheme for hydrodynamic equations.

4. Calculation Results

We start with rewriting (7) for the solitary wave in dimensionless variables adopted in (11):

\[
\vartheta(H) = \pm \frac{\xi}{a} \left[ \sqrt{H_m^2 - H^2} - \frac{a^2}{H_m} \ln \frac{H_m + \sqrt{H_m^2 - H^2}}{H_m - \sqrt{H_m^2 - H^2}} \right]
\]

\[
\phi(H) = \pm \frac{\Lambda}{2H_m} \ln \frac{H_m + \sqrt{H_m^2 - H^2}}{H_m - \sqrt{H_m^2 - H^2}}, \quad u = -\frac{2a^2}{2a}, \quad \rho = \frac{2a^2}{2a^2 - H^2}, \quad E = iaHe^{i\omega}
\]
Here, $0 < H \leq H_m = (4a^2 - 4 - \Lambda^2)^{1/2}$, $\eta < |a| < \sqrt{2}\eta$, $a$ is the dimensionless phase velocity (the characteristic velocity scale is $V_A = H_\parallel / \sqrt{4\pi \rho_\infty}$). The function $H(\theta) = H_a(\theta)$ is obtained by inverting $\varphi(H)$ in (13).

Consider the calculation results in the case of electron-positron plasma, for $\Lambda = 0$, $\varphi(H) = \text{const}$. Assuming this constant to be equal to zero, we get $H_y = H_a(\theta)$, $H_z = 0$, $E_y = 0$, $E_z = aH_y$, while the remaining parameters are determined by (13). Thus, in the case of electron-positron plasma, the wave packet degenerates into a linearly polarized solitary wave. It is assumed below that $\xi = 1$, $H_\perp = 1$.

A) Consider the evolution of two waves of the same amplitude $A = 0.9$ moving towards each other with phase velocity $a = 1.1$. In this case, system (11)-(12) is solved numerically with the initial condition for the magnetic field

$$H(0,x) = H_{1,1}(x-25), \quad 0 \leq x \leq 50$$

$$H(0,x) = H_{-1,1}(x-75), \quad 50 \leq x \leq 100$$
on the interval $[0,100]$. The initial conditions for the other plasma parameters are calculated from $H(0,x)$ with the help of (13). Figure 1 represents the calculation results for the profiles $H(t,x)$ at different time instants.

B) Consider two solitary waves of different amplitudes moving towards each other with different phase velocities: the wave moving from left to right has amplitude $A = 0.9$ and phase velocity $a = 1.1$, while $A = 0.7$ and $a = -1.05$ for the wave moving from right to left. In this case, the initial magnetic field has the form

$$H(0,x) = H_{1,1}(x-25), \quad 0 \leq x \leq 50$$

$$H(0,x) = H_{-1,05}(x-75), \quad 50 \leq x \leq 100$$

Figure 2 shows the calculated profiles of $H(t,x)$ at different time instants.

C) Finally, consider the case of a wave with greater amplitude $A = 1.3$ and a greater phase velocity $a = 1.2$ catching up with a wave with smaller amplitude $A = 0.7$ and a smaller phase velocity $a = 1.05$. The initial condition for the magnetic field has the form

$$H(0,x) = H_{1,2}(x-60), \quad 0 \leq x \leq 80$$

$$H(0,x) = H_{1,05}(x-115), \quad 80 \leq x \leq 140$$

The solution is sought on the interval $[0,700]$ and $H(0,x) = 0$ for $140 \leq x \leq 700$. Figure 3 shows the profiles $H(t,x)$ at different time instants. Time goes on all the figures from top to bottom.

5. Concluding Remarks

The general qualitative conclusion of this investigation is that the solitary waves in plasma, considered here in the framework of the two-fluid MHD, exhibit the behavior of genuine solitons, in the sense that after their interaction they preserve their characteristics (velocity, amplitude, etc). Our calculations show that in the case of waves moving towards each other, the tail of each wave after the interaction exhibits oscillations of small amplitude filling the space between the diverging waves. These oscillations can be interpreted as radiation produced upon the collision of waves. This question, however, requires further investigation.

All profiles shown above were obtained for the magnetic field, but similar distributions along the $x$ axis take place for the other plasma parameters. The properties of interacting solitary waves described in this paper are typical mainly for the solutions of model equations and are usually explained by the complete integrability of the corresponding equations [1,2] and an infinite number of their first integrals.
In this paper, these effects have been discovered numerically on the basis of a complete system of
equations of two-fluid hydrodynamics of cold plasma, i.e., the laws of conservation of mass and
momentum, together with the Maxwell equations. It should be mentioned that some mathematical
properties (the Painleve property) of traveling wave equations (8) were considered in [9].

The present work continues the attempts to construct new models of plasma for the investigation of
processes in plasma accelerators and new types of magnetic traps.

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