Additional symmetries of the Zakharov-Shabat hierarchy, String equation and Isomonodromy

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We are going to explain and to prove the following statement: Isomonodromic deformations are nothing but symmetries of the Zakharov-Shabat (isospectral) hierarchy, both the basic ones (belonging to the hierarchy) and additional, restricted to the submanifold of solutions to the string equation. All the symmetries commute with each other, so the equations can be integrated together.

A more precise formulation will follow (Theorem below). All necessary definitions will be given. We also construct quantities which are first integrals both for isospectral and for isomonodromic deformations.

Apparently, many formulas we are writing here can be extracted in one or another form from the literature ([1-4]). What especially interested us was a context and interpretation. Our goal is to express the theory in hierarchy terms. It is common nowadays to study integrable systems not individually but in big collections called hierarchies. A hierarchy is always a set of commuting vector fields on an infinite dimensional manifold.

A Zakharov-Shabat equation has a form \([\partial_{t_1} - A_1(z), \partial_{t_2} - A_2(z)] = 0\) where \(A_1(z)\) and \(A_1(z)\) are rational functions of \(z\), and \(\partial_{t_1}\) and \(\partial_{t_2}\) are derivatives with respect to some variables. There are very many such equations. How to consider them as a hierarchy, that was discussed in [5]. However, there is one more activity where operators of type \(\partial_{t_i} - A_i(z)\) are involved. These are the so-called monodromy preserving deformations, see [1-4]. Incorporating them into the ZS hierarchy, that is what we are doing here.

1. Definition of the ZS hierarchy.

A definition was presented in [5], we repeat it here for convenience.
Let \(a_k, k = 1, ..., N\) be a given set of complex numbers. Let, for every \(k\),
\[
\hat{w}_k = \sum_{0}^{\infty} w_{ki}(z - a_k)^i,
\]
be a formal series. The entries of \(n \times n\) matrices \(w_{ki} = \{w_{ki,\alpha\beta}\}\) belong to an algebra where \(\det w_{k0}\) is invertible. The formal series \(\hat{w}_k\) can be inverted. Define
\[
R_{k\alpha} = \hat{w}_k E_\alpha(z - a_k)^{-i} \hat{w}_k^{-1}
\]
where \(E_\alpha\) is a matrix with only one non-vanishing element, equal 1, on the \((\alpha, \alpha)\) place.

We have two kinds of objects. Such quantities as \(\hat{w}_k\) and \(R_{k\alpha}\) are formal series, or jets, at the points \(a_k\). If \(j_k\) is a jet, then \(j_k\) symbolizes its principal part, i.e., a sum of negative
powers of $z - a_k$, and $j_k^+$ the rest of the series. If the principal part contains finite number of terms it can be considered as a global meromorphic function. Global functions are objects of the second kind. A global function gives rise to a jet at every $a_k$, as its Laurent expansion there. In particular, $j_k^-$ can be considered as a jet at a point $a_{k_1}$, different from $a_k$.

**Definitions.** (i) A hierarchy corresponding to a fixed set $\{a_k\}$ is the totality of equations

$$\partial_{kal} \hat{w}_{k_1} = \begin{cases} -R_{kal}^+ \hat{w}_{k_1}, & k = k_1 \\ R_{kal}^- \hat{w}_{k_1}, & \text{otherwise} \end{cases}, \quad \partial_{kal} = \partial/\partial t_{kal}. \quad (1)$$

In the second case $R_{kal}^-$ is considered as a jet at $a_{k_1}$; $t_{kal}$ are some variables.

(ii) A ZS hierarchy is an inductive limit of hierarchies with fixed sets $\{a_k\}$, with respect to a natural embedding of a hierarchy corresponding to a subset in to a hierarchy corresponding to a larger set, as a subhierarchy.

Note that $R_{ka0} = \hat{w}_k E_\alpha \hat{w}_k^{-1} = R_{ka0}^+$ and $\partial_{kal} \hat{w}_{k_1} = \hat{w}_{k_1} E_\alpha$; any linear combination of derivatives $\partial_{kal}$ acting on $\hat{w}$ multiplies it on the right by a constant diagonal matrix. Nothing depends on this transformation, and we will not use the variables $t_{ka0}$ at all.

Two jets $R_{kal}$ and $R_{k_1a_1}$ with the same $k$ commute.

**Lemma.** Equalities

$$\partial_{kal} R_{k_1a_1} = \begin{cases} -[R_{kal}^+, R_{k_1a_1}], & k = k_1 \\ [R_{kal}, R_{k_1a_1}], & \text{otherwise} = [R_{kal}^-, R_{k_1a_1}] \end{cases}$$
hold.

**Proof.** It easily can be obtained from the definition of $R_{kal}$. □

**Proposition 1.** All operators $\partial_{kal}$ commute.

For a proof see [1]. This proposition entitles us to call the above defined object a hierarchy. The following proposition readily can be proven by a simple straightforward computation:

**Proposition 2.** A dressing formula

$$\hat{w}_{k_1}(\partial_{kal} - E_\alpha (z - a_k)^{-l} \delta_{kk_1}) \hat{w}_{k_1}^{-1} = \partial_{kal} - B_{kal}, \quad B_{kal} = R_{kal}^-$$

is equivalent to Eq.(1).

In the above equality, $B_{kal}$ is assumed to be a jet at $a_{k_1}$. However, it does not depend on $k_1$ at all and can be considered as a global function of $z$ with the only pole of $l$th order at $a_k$. Let

$$w_k = \hat{w}_k \exp \xi_k \quad \text{where} \quad \xi_k = \sum_{l=1}^{\infty} \sum_{\alpha=1}^{n} t_{kal} E_\alpha (z - a_k)^{-l}.$$  

**Definition.** The collection $w = \{w_k\}$ is the formal Baker function of the hierarchy.
The statement of the Proposition 2 can be rewritten in terms of the Baker function as

$$w_{k_1} \partial_{kal} w_{k_1}^{-1} = \partial_{kal} - B_{kal}$$  \hspace{1cm} (2)

and the equations (1) as

$$\partial_{kal} w_{k_1} = B_{kal} w_{k_1}.$$  \hspace{1cm} (3)

**Proposition 3.** All operators $\partial_{kal} - B_{kal}$ commute.

This is an immediate corollary of the Proposition 1 and Eq. (2).

**Remark.** Without difficulty we could include a case when a pole is at infinity. Then $\hat{w}_\infty = \sum_0^\infty w_\infty z^{-i}$. All formulas stay the same with $(z - a_k)$ being replaced by $z^{-1}$ and + and −, as superscripts, being swapped. For simplicity of writing, we skip this case.

One can consider arbitrary linear combinations of the above constructed operators,

$$L = \sum_{kal} \lambda_{kal}(\partial_{kal} - B_{kal}) = \partial + U$$

where $\partial = \sum_{kal} \lambda_{kal}\partial_{kal}$ and $U = -\sum_{kal} \lambda_{kal}B_{kal}$. Only finite number of $\lambda_{kal}$ are assumed to be non-zero. Two such operators commute which yields equations of the Zakharov-Shabat type

$$\partial U_1 - \partial_1 U = [U_1, U].$$

Functions $U$ and $U_1$ are rational functions of the parameter $z$.

### 2. Additional symmetry and string equation.

In the case of the KP hierarchy, it is well-known that there are additional symmetries ([6],[7],[8], see also [9]) which do not belong to the hierarchy, and do not commute between themselves. They often are called “Virasoro symmetries”, according to their rules of commutation. Especially graphic and convenient is the way they are presented in [7]. A similar construction can be performed for the ZS hierarchy ([5]). Dressing an obvious relation $[\partial_z, \partial_{kal}] = 0$ with the help of $w_i$ at the point $a_i$ we have $0 = w_i[\partial_z, \partial_{kal}]w_i^{-1} = [\partial_z - M_i, \partial_{kal} - B_{kal}]$, i.e.,

$$\partial_{kal} M_i = \partial_z B_{kal} - [M_i, B_{kal}]$$  \hspace{1cm} (5)

where

$$\partial_z - M_i = w_i \partial_z w_i^{-1} = \partial_z - \partial_z w_i \cdot w_i^{-1} = \partial_z - \partial_z \hat{w}_i \cdot \hat{w}_i^{-1} - \hat{w}_i \xi_{iz} \hat{w}_i^{-1}$$

and $\xi_{iz} = \partial \xi_i / \partial z = t_i \partial \Lambda_i$. The quantity

$$M_i = \partial_z \hat{w}_i \cdot \hat{w}_i^{-1} + \hat{w}_i \xi_{iz} \hat{w}_i^{-1}$$  \hspace{1cm} (6)

is a jet at the point $a_i$. Notice that

$$M_i^- = (\hat{w}_i \xi_{iz} \hat{w}_i^{-1})^- = (w_i \xi_{iz} w_i^{-1})^-$$
while $B_{\text{kal}}$ can be written as

$$B_{\text{kal}} = (w_k(\partial_{\text{kal}}\xi_k)w_k^{-1})^{-1}. $$

Taking negative and positive parts of (5), we get at the point $a_i$

$$\begin{align*} 
(1) & \quad i = k \quad \partial_{\text{kal}}M_k^- = \partial_z B_{\text{kal}} - [M_k, B_{\text{kal}}]^{-}, \\
(2) & \quad i \neq k \quad \partial_{\text{kal}}M_i^- = -[M_i, B_{\text{kal}}]^{-}, \\
\end{align*}$$

Definition. The additional symmetry is given by the system of differential equations

$$\partial^* w_j = (M_j^+ - \sum_{i \neq j} M_i^-)w_j$$

where $\partial^*$ is a derivative with respect to a parameter $t^*$ of the symmetry.

As it is easy to see, the equation of the additional symmetry implies

$$\partial^* R_{\text{kal}} = [M_k^+ - \sum_{i \neq k} M_i^-, R_{\text{kal}}] \quad \text{and} \quad \partial^* B_{\text{kal}} = [M_k^+ - \sum_{i \neq k} M_i^-, B_{\text{kal}}].$$

This has the meaning of the equality of two jets at $a_k$.

**Proposition 4.** The additional symmetry commutes with operators of the hierarchy, $[\partial^*, \partial_{\text{kal}}] = 0$, i.e., it is a symmetry, indeed.

A proof is below. In the theory of the KP hierarchy there is the celebrated string equation. It is nothing but condition of invariance of field variables with respect to a definite additional symmetry (see, e.g., [10], [11]). In a more general sense, “a string equation” is the invariance with respect to any additional symmetry.

**Definition.** A string equation is condition that $w_i$ do not depend on $t^*$.

This condition is compatible with the hierarchy, by virtue of the Proposition 4, and yields

$$M_k^+ - \sum_{i \neq k} M_i^- = 0, \quad k = 1, ..., N. \quad (9)$$

The jets $w_k$ are defined up to multiplication on the right by constant diagonal matrices that may be a series in powers of $(z - a_k)^{-1}$. This does not change $R_{\text{kal}}$ and the equations of the hierarchy. However, $M_k$ will modify. One can use that possibility to incorporate terms of degree $-1$ into $\partial_z \xi_k$, so put $\xi_k = \sum_{l=1}^{\infty} \sum_{\alpha=1}^{n} t_{k\alpha} E_{\alpha}(z - a_k)^{-l} + \sum_{\alpha=1}^{n} \lambda_{\alpha} \log(z - a_k)$ to have

$$\partial_z \xi_k = -\sum_{l=1}^{\infty} \sum_{\alpha=1}^{n} t_{k\alpha} E_{\alpha} l(z - a_k)^{-l-1} + \sum_{\alpha=1}^{n} \lambda_{\alpha} (z - a_k)^{-1}.$$  

3. Toward the isomonodromy equations.
Let us introduce new (“additional”) variables $t^*_i$ and write differential equations
\[ \partial^*_i w_j = \begin{cases} -M^*_i w_j, & i \neq j \\ M^*_i w_j, & i = j \end{cases}, \quad \partial^*_i = \partial / \partial t^*_i, \]
the left- and the right-hand sides are jets at $a_i$. A sum of these vector fields equals the vector field $\partial^*$, the additional symmetry,
\[ \partial^* w_j = \sum_i \partial^*_i w_j. \]
Is each of them a symmetry? Alas, this is not the case. Nevertheless, they also have an advantage: they commute between themselves.

**Proposition 5.** The following commutation rules hold:
\[ [\partial^*_i, \partial_{k\alpha l}] w_j = (\delta_{ik} - \delta_{ij})(\partial_z B_{k\alpha l}) w_j, \quad [\partial^*_i, \partial^*_k] w_j = 0. \]

**Proof.** In order to prove the first relation, we consider the cases when all three indices $i, k$ and $j$ are distinct, when a pair of them coincide, and when all three are equal. Different letters will always symbolize different numbers here.

1) \[ \partial^*_i \partial_{k\alpha l} w_j = \partial^*_i (B_{k\alpha l} w_j) = -[M^-_i, B_{k\alpha l}]^- w_j - B_{k\alpha l} M^-_i w_j. \]
Let us explain. $M^-_i$ is a rational function with a pole at $a_i$, we consider it as a global function generating a (positive) Laurent series at $a_k$. Then we take its commutator with $B_{k\alpha l}$ which is a negative jet at this point and take the negative part of the commutator. It is a global function having a purely positive expansion at $a_j$. A superscript $-$ accompanied by a subscript $k$ denotes taking the negative part of an expansion in powers of $z - a_k$.

How we got the formula? Compute:
\[ \partial^*_i B_{k\alpha l} = \partial^*_i (w_k (\partial_{k\alpha l} \xi_k) w_k^{-1})^{-1}_k \]
\[ = -(M^-_i w_k (\partial_{k\alpha l} \xi_k) w_k^{-1} - w_k (\partial_{k\alpha l} \xi_k) w_k^{-1} M^-_i)_i = -[M^-_i, R_{k\alpha l}]^- = -[M^-_i, B_{k\alpha l}]^- \]  \quad (10)
The matrix $M^-_i$ has a purely positive expansion at $a_k$, therefore the positive part of $R_{k\alpha l}$ does not make any contribution to the negative part of the commutator, and one can replace $R_{k\alpha l}$ by $B_{k\alpha l}$. Now,
\[ \partial_{k\alpha l} \partial^*_i w_j = -\partial_{k\alpha l} (M^-_i w_j) = -[B_{k\alpha l}, M^-_i]^- w_j - M^-_i B_{k\alpha l} w_j. \]
The formula
\[ \partial_{k\alpha l} M^-_i = [B_{k\alpha l}, M^-_i]^- \]  \quad (11)
used here immediately follows from (7). Finally,
\[ [\partial^*_i, \partial_{k\alpha l}] w_j = -([M^-_i, B_{k\alpha l}]^- + [M^-_i, B_{k\alpha l}]^-) w_j - [B_{k\alpha l}, M^-_i]^- w_j = 0 \]
since a sum of all the principal parts of a rational function equals the function itself (generally, up to a constant, the latter is zero here because the function decays at infinity).
2) It is easy to see that in the previous case it was not important that \( k \neq j \), i.e.,
\[
[\partial^*_k, \partial_{j\alpha l}]w_j = 0.
\]

3) \( \partial^*_k \partial_{kal} w_j = \partial^*_k (B_{kal} w_j) = [M^+_k, B_{kal}]^- w_j - B_{kal} M^- w_j \)
and (see the first of eqs. (7))
\[
\partial_{kal} \partial^*_k w_j = -\partial_{kal} (M^- w_j) = -(\partial_z B_{kal} - [M_k, B_{kal}]^-) w_j - M^- B_{kal} w_l
\]
whence
\[
[\partial^*_k, \partial_{kal}]w_j = -([M^-_k, B_{kal}]^- + [B_{kal}, M^-_k] - \partial_z B_{kal}) w_j = (\partial_z B_{kal}) w_j.
\]

4) \( \partial^*_j \partial_{kal} w_j = \partial^*_j (B_{kal} w_j) = -[M^-_j, B_{kal}]^- w_j + B_{kal} M^+_j w_j \)
\[= -(M^-_j, B_{kal}) + [M^-_j, B_{kal}]^- w_j - B_{kal} M^+_j w_j = ([M^-_j, B_{kal}] - [M_j, B_{kal}]^-) w_j - B_{kal} M^+_j w_j \]
and (see the last of the eqs. (7))
\[
\partial_{kal} \partial^*_j w_j = \partial_{kal} (M^+_j w_j) = -(\partial_z B_{kal} - [M_j, B_{kal}]^+) w_j - M^+_j B_{kal} w_j.
\]

Therefore,
\[
[\partial^*_j, \partial_{kal}] w_j = -([M^-_j, B_{kal}] - [M_j, B_{kal}] + [M^+_j, B_{kal}] + \partial_z B_{kal}) w_j = -(\partial_z B_{kal}) w_j.
\]

5) \( \partial^*_{j\alpha l} \partial_{j\alpha l} w_j = \partial^*_j (B_{j\alpha l} w_j) = [M^+_j, B_{j\alpha l}]^- w_j + B_{j\alpha l} M^+_j w_j \)
and (see the second of the eqs. (7))
\[
\partial_{j\alpha l} \partial^*_j w_j = \partial_{j\alpha l} (M^+_j w_j) = -[M_j, B_{j\alpha l}]^+ w_j + M^+_j B_{j\alpha l} w_j = -[M^+_j, B_{j\alpha l}]^+ w_j + M^+_j B_{j\alpha l} w_j
\]
whence
\[
[\partial^*_j, \partial_{j\alpha l}] w_j = [M^+_j, B_{j\alpha l}] w_j - [M_j^+, B_{j\alpha l}] w_j = 0.
\]
The first assertion is proven. The second one requires the following computations.

6) \( \partial^*_k \partial^*_i w_j = -\partial^*_k (M^-_i w_j) = [M^-_k, M^-_i]^- w_j + M^-_i M^- w_j = [M^-_k, M^-_i]^- w_j + M^-_i M^- w_j \)
and
\[
\partial^*_i \partial^*_k w_j = [M^-_i, M^-_k]^- w_j + M^-_k M^- w_j
\]
which implies
\[
[\partial^*_k, \partial^*_i] w_j = ([M^-_k, M^-_i]^- + [M^-_i, M^-_k]^- - [M^-_k, M^-_i]) w_j = 0.
\]

7) It is easy to see that nothing changes in the preceding proof if \( k = j \). Thus,
\[
[\partial^*_k, \partial^*_i] w_j = 0
\]
for all \( k, i \) and \( j \). The proposition is proven.

**Corollary.** The sum \( \partial^* = \sum \partial_i^* \) commutes with all \( \partial_{kal} \). This is the above additional symmetry.

Indeed,
\[
[\sum \partial_i^*, \partial_{kal}]w_j = (\partial_z B_{kal} - \partial_z B_{kal})w_j = 0.
\]

The Baker function \( \{w_j\} \) depends on \( \{a_i\} \) as parameters. The next important step is to make the parameters variable, namely, put \( a_i = t_i^* \). The total derivative with respect to \( a_i \) is \( D_i = \partial_i^* + \partial/\partial a_i \) where \( \partial/\partial a_i = \partial_{a_i} \) is a partial derivative with respect to \( a_i \) which enters \( w_j \) explicitly in the form of \( z - a_i \). Evidently, \( (\partial/\partial a_j)w_j = -\partial_z w_j \) and \( (\partial/\partial a_i)w_j = 0 \) when \( i \neq j \), hence
\[
D_iw_j = -M_i^- w_j, \quad i \neq j; \quad D_jw_j = (M_j^+ - M_j^-)w_j = -M_j^- w_j,
\]
i.e., in all cases
\[
D_iw_j = -M_i^- w_j.
\]

**Proposition 6.** Vector fields (12) commute with each other and with \( \partial_{kal} \)’s.

**Proof.** We have,
\[
D_kD_iw_j = -D_k(M_i^- w_j) = [M_k^-, M_i^-]w_j + M_k^- M_i^- w_j = [M_k^-, M_i^-]_i w_j + M_i^- M_k^- w_j
\]
and
\[
D_iD_kw_j = -\partial_i^*(M_k^- w_j) = [M_i^-, M_k^-]_k w_j + M_k^- M_i^- w_j,
\]
hence
\[
[D_k, D_i]w_j = ([M_k^-, M_i^-]_i + [M_k^-, M_i^-]_k - [M_k^-, M_i^-]_i)w_j = 0.
\]
Since the commutators \( [\partial_i^*, \partial_{kal}] \) are already found (proposition 5), it remains to find additional terms \( [\partial_{a_i}, \partial_{kal}]w_j \). It is easy to see that this is zero when all three indices are distinct. Now,
\[
\partial_{a_k}\partial_{kal}w_j = \partial_{a_k}(B_{kal}w_l) = -(\partial_z B_{kal})w_j, \quad \partial_{kal}\partial_{a_k}w_j = 0,
\]
Thus, \( [\partial_{a_k}, \partial_{kal}]w_j = -(\partial_z B_{kal})w_j \). This additional term exactly cancels with the value of the commutator in the proposition 5.

Similarly,
\[
\partial_{a_j}\partial_{kal}w_j = \partial_{a_j}(B_{kal}w_j) = B_{kal}(-\partial_z w_j) = -B_{kal}M_j^- w_j,
\]
and
\[
\partial_{kal}\partial_{a_j}w_j = -\partial_{kal}(M_j^- w_j) = -(\partial_z B_{kal} - [M_j, B_{kal}])w_j - M_j B_{kal}w_j
\]
(see (5)). This yields that
\[
[\partial_{a_j}, \partial_{kal}]w_j = (\partial_z B_{kal})w_j
\]
which also cancels. Finally,
\[
\partial_{a_j}\partial_{j}w_j = \partial_{a_j}(B_{jkal}w_j) = -\partial_z(B_{jkal}w_j)
\]
and

\[ \partial_{j\alpha l} \partial_{a_j} w_j = \partial_{j\alpha l} (-\partial_z w_j) = -\partial_z (\partial_{j\alpha l} w_j) = -\partial_z (B_{j\alpha l} w_j) \]

and \([\partial_{a_j}, \partial_j] w_j = 0\). This completes the proof.

**Corollary.** If (12) is considered as a system of differential equations where \(D_i\) are total derivatives with respect to \(a_i\) then this system is consistent and also compatible with the hierarchy equations \(\partial_{k\alpha l} w_j = B_{k\alpha l} w_j\).

Thus, \(D_i\) are true symmetries of the ZS hierarchy, in contrast to \(\partial^*_i\). We call them additional symmetries\(^2\).

Now, we turn to the additional symmetry \(\partial^* = \sum \partial^*_i\). In terms of \(D_i\) that is

\[ \partial^* = \partial_z + \sum D_i. \]

This operator commutes with all \(D_k\)'s since \(\partial_z\) commutes with all other involved vector fields. The string equation \((\partial_z + \sum D_i) w_j = 0\) is, therefore, invariant with respect to the equations (12). In more detail, the string equation looks like

\[ (\partial_z - \sum M_i^-) w_j = 0, \quad j = 1, \ldots, N. \quad (13) \]

Summarizing everything said before, we formulate the following

**Theorem.** Let

\[ \xi_k = \sum_{l=1}^{\infty} \sum_{\alpha=1}^{n} t_{k\alpha} E_\alpha (z - a_k)^{-l} + \sum_{\alpha=1}^{n} \lambda_{k\alpha} \log(z - a_k) \]

where \(\{t_{k\alpha}\}, \{a_k\}\) and \(z\) are independent complex variables. Let \(\partial_{k\alpha l} = \partial/\partial t_{k\alpha l}\) and \(D_k = \partial/\partial a_k\) be “total partial derivatives” taking into account variables involved both explicitly and implicitly through the unknowns \(w_k\). Let

\[ w_k = \sum_{m=0}^{\infty} w_{km} (z - a_k)^m \exp \xi_k, \quad B_{k\alpha l} = (w_k (\partial_{k\alpha l} \xi_k) w_k^{-1})^-, \quad M_k^- = (w_k (\partial_z \xi_k) w_k^{-1})^- \]

where the superscript \(^-\) means the negative (principal) part of the formal Laurent series in powers of \(z - a_k\). Then all equations:

\[ (\partial_{k\alpha l} - B_{k\alpha l}) w_j = 0, \quad k, j = 1, \ldots, N; \quad \alpha = 1, \ldots, n, \quad l = 1, \ldots \quad (14) \]

\[ (D_k + M_k^-) w_j = 0, \quad k, j = 1, \ldots, N, \quad (15) \]

and

\[ (\partial_z - \sum_{k} M_k^-) w_j = 0, \quad j = 1, \ldots, N \quad (16) \]

\(^2\)Commutativity of additional symmetries seemingly contradicts the statement that they are analogues of the noncommutative Orlov-Shulman additional symmetries. In fact, the whole collection of \(D_i\) is an analogue of only one O-S symmetry which splits into \(N\) commuting symmetries, accordingly to the number of poles.
are compatible. Thus, they can be solved all together and a solution \( w_j(t; a; z) \) found.

The whole system is called isomonodromic equations, Eqs. (14) describing monodromy preserving deformations of diagonal elements \( t_{kal} \) and Eqs. (15) those of poles. The equations (14) (without restriction (16)) are known as the ZS isospectral hierarchy.

4. First integrals.

Let

\[
R_i^C = w_i C w_i^{-1}
\]

where \( C \) is a constant diagonal matrix.

**Proposition 7.** The jets at the point \( a_i \):

\[
J_{ikal}(z) = \int \text{tr} R_i^C \partial_z B_{kal} dt_{kal} \quad \text{and} \quad J_{ik}^* = \int \text{tr} R_i^C \partial_z M_k^- da_k, \quad \forall k
\]

are generators of first integrals of the equations (14) and (15), i.e., coefficients of their expansions in powers of \( z - a_i \) are first integrals.

The integral is understood in an algebraic sense: if \( f(t) \) is a differential polynomial in elements \( \{w_{jm, \alpha \beta}\} \) then \( f(t) dt_{kal} \) is the class of equivalence of \( f(t) \) modulo exact derivatives \( \partial_k a(t) \) where \( g(t) \) are also differential polynomials.

**Proof.** (1) For brevity, we write \( B_k \) instead of \( B_{kal}, \partial_k \) for \( p_{kal} \), and so on.

We have

\[
\partial_t \text{tr} R_i^C \partial_z B_k = \text{tr} ( [B_t, R_i^C] \partial_z B_k + R_i^C \partial_z [B_t, B_k]^- ).
\]

This is an equality of jets at \( a_i \). Now we use the following transformations:

\[
\text{tr} [B_t, R_i^C] \partial_z B_k = - \text{tr} R_i^C [B_t, B_k, z] \quad \text{where} \quad B_{k,z} = \partial_z B_k, \quad [B_t, B_k] = [B_t, B_k]^- + [B_t, B_k]_l^-
\]

and \( [B_t, B_k]_l^- = - \partial_z B_l \) whence

\[
\partial_t \text{tr} R_i^C \partial_z B_k = \text{tr} ( - R_i^C [B_t, B_k, z] + R_i^C \partial_z [B_t, B_k] + R_i^C \partial_k B_{t,z} ) = \text{tr} ( R_i^C [B_{t,z}, B_k] + R_i^C \partial_k B_{t,z} )
\]

\[
= \text{tr} ( [B_k, R_i^C] B_{t,z} + R_i^C \partial_k B_{t,z} ) = \text{tr} ( (\partial_k R_i^C) B_{t,z} + R_i^C \partial_k B_{t,z} ) = \partial_k \text{tr} R_i^C B_{t,z}.
\]

A nice equality is obtained:

\[
\partial_t \text{tr} R_i^C \partial_z B_k = \partial_k \text{tr} R_i^C \partial_z B_l,
\]

It implies that \( \partial_t \text{tr} R_i^C \partial_z B_k dt_k = 0 \).

(2) Now, we compute \( D_l \text{tr} R_i^C \partial_z B_k \). First, let \( l \neq k \).

\[
D_l \text{tr} R_i^C \partial_z B_k
\]

\[
= \text{tr} ( - [M_l^-, R_i^C] B_{k,z} - R_i^C [M_l^-, B_k]^- ) = \text{tr} ( - [M_l^-, R_i^C] B_{k,z} - R_i^C [M_l^-, B_k]_z - R_i^C [M_l^-, B_k]_{l,z} )
\]

\(^3\)About the isomonodromic deformations see [1], [2], [3], [4].
Transform the last term using (7):

\[ R_i^C[M_i^-, B_{k_l}^i] = R_i^C[M_i B_{k_l}^i] = R_i^C(-\partial_k M_i^-,B_{k_l}^i). \]

We have

\[
D_l \text{tr } R_i^C \partial_z B_l = \text{tr } (-R_i^C[M_i^-, B_{k_l}^i] + R_i^C \partial_k M_i^-, B_{k_l}^i = \text{tr } (-M_i^-, \partial_k R_i^C, -R_i^C \partial_k M_i^-, B_{k_l}^i) = -\partial_k \text{tr } (R_i^C M_i^-, B_{k_l}^i).
\]

Secondly, let \( l = k \). We have

\[
D_l B_l = (\partial_l^* + \partial_a) B_l = [M_i^+, B_l] - \partial_z B_l = [M_i^+, B_l] - (\partial_l M_i^- + [M_l, B_l^-])
\]

(see (7)), therefore,

\[
D_l \text{tr } R_i^C \partial_z B_l = \text{tr } (-R_i^C[M_i^+, B_l] - R_i^C \partial_l M_i^-, B_{k_l}^i) = \text{tr } (-M_i^-, \partial_l R_i^C) - R_i^C \partial_l M_i^-, B_{k_l}^i)
\]

Thus, we obtained two formulas

\[ \partial_l \text{tr } R_i^C \partial_z B_l = \partial_k \text{tr } R_i^C \partial_z B_l, \quad \text{and} \quad D_l \text{tr } R_i^C \partial_z B_k = -\partial_k \text{tr } R_i^C \partial_z M_i^-, \quad (17). \]

One easily can prove one more equation, in addition to Eqs. (17),

\[ D_k \text{tr } R_i^C M_i^-, B_{k_l}^i = D_l \text{tr } R_i^C M_k^-, \quad (18) \]

which completes the proof.

Together, (17) and (18) are equivalent to the statement that the form

\[ \omega = \text{tr } \sum_{k_l} R_i^C B_{k_l}^i dt_{k_l} - \text{tr } \sum_k R_i^C M_{k_l}^- da_k \]

is closed.

**Remark.** The relations (17) and (18) hold always, whenever \( l \) equals \( i \) or not. However, if one wishes to obtain conservation laws expanding (17) and (18) in powers of \( z - a_i \), the case \( l = i \) will be exceptional, one fails to get a conservation law. It is better to eliminate this case demanding that \( a_i \) be an additional point that is not variable. For example, it is convenient to take this point at infinity.

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