Multilayer heat equations and their solutions via oscillating integral transforms.

Andrey Itkin 1 Alexander Lipton 2 and Dmitry Muravey 3

1 Tandon School of Engineering, New York University, New York, USA
2 The Jerusalem School of Business Administration, The Hebrew University of Jerusalem, Jerusalem, Israel; Connection Science and Engineering, Massachusetts Institute of Technology, Cambridge, MA, USA
3 Moscow State University, Moscow, Russia

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By expanding the Dirac delta function in terms of the eigenfunctions of the corresponding Sturm-Liouville problem, we construct some new (oscillating) integral transforms. These transforms are then used to solve various finance, physics, and mathematics problems, which could be characterized by the existence of a multilayer spatial structure and moving (time-dependent) boundaries (internal interfaces) between the layers. Thus, constructed solutions are semi-analytical and extend the authors’ previous work (Itkin, Lipton, Muravey, Multilayer heat equations: application to finance, FMF, 1, 2021). However, our new method doesn’t duplicate the previous one but provides alternative representations of the solution which have different properties and serve other purposes.

Introduction

Undoubtedly, integral transforms are one of the most beautiful developments of classical mathematics. They build a robust and unified basis for solving various linear differential and integral equations of mathematical physics, applied mathematics, and engineering and can be utilized for both the initial and initial-boundary problems, usually with constant coefficients. Moreover, with ever greater modern demand for mathematical methods in science and engineering, the utility of and interest in the integral transforms increase with time. Thus, even though integral transforms already have many mathematical and physical applications, they constantly find new applications in advanced study and research.

As mentioned in (Stetz), the idea that beauty is "right" and "true" has been around since humankind developed abstract thought. The Latin phrase *Pulchritudo splendor varitatis* - "beauty is the splendor of truth" - is thousands of years old and suggests that beauty and truth are interrelated. Indeed, it seems appropriate that something beautiful would be true, but it is more realistic to think that something hideous could also be true.
This thought immediately comes to mind when looking at some modern problems in physics, biology, finance, etc. Among other essential problems, let us mention just a few: (a) the transdermal drug release from an iontophoretic system, (Pontrelli et al., 2016); (b) the transmembrane diffusion in cells; (c) the growth of diffusive brain tumors, which considers the brain tissue’s heterogeneity, (Asvestas et al., 2014); (d) distribution of organisms in rivers - an advection-diffusion problem, (Ramirez et al., 2013); (e) pricing and calibration with piecewise constant, linear and quadratic volatility, (Lipton and Sepp, 2011; Itkin and Lipton, 2018; Lipton, 2018; Lipton et al.); (f) the oscillating/skew Brownian Motion, the local time diffusion, (Lejay, 2006; Lejay and Pigato, 2018); (g) the occupation times, optimal stopping/first passage time problems for the processes with broken drift, (Salminen and Stenlund, 2021; Mordecki and Salminen, 2019; Ankirchner et al., 2021); (h) parabolic equations and diffusion processes on graphs, (Lejay, 2006).

Solving these and similar problems, characterized by multiple spatial layers (with different physical properties) and moving boundaries between these layers, is very important, even though it is impossible to find their solutions by using the standard operational calculus of classical integral transforms. Therefore, to proceed, we need to sacrifice the beauty of elegant and straightforward classical transforms and instead build a new class of integral transforms. Of course, these transforms have to be adapted to the specific structure at hand; see, e.g., (Itkin et al., 2021a) where various time-dependent problems of mathematical finance and physics with moving boundaries are solved by doing precisely that. Usually, those transforms are constructed by using some basis in eigenfunctions found by solving the corresponding Sturm-Liouville (SL) problem, (Antimirov et al.).

In this paper, we extend the approach of (Itkin et al., 2021a) by applying its main idea to solving various multilayer problems for the one-dimensional heat equation. Let us introduce the time $t \in [0, \infty)$, the space coordinate $x \in \mathbb{R}$, and a function $u(t, x) : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$, $u \in C^2$. Suppose that the whole domain $\Omega = \mathbb{R}$ could be split into $N + 2$ non-overlapping layers $-\infty < y_0(t) < \ldots < y_N(t) < \infty$, $\forall t \geq 0$, i.e. $\Omega = \bigcup_{i=0}^{N} \Omega_i$, where each layer is a curvilinear strip

\[
\Omega_i = [y_{i-1}(t), y_i(t)] \times \mathbb{R}^+, \quad y_i(t) < y_{i+1}(t), \quad \forall t \geq 0, \quad \forall i = 1, \ldots, N, \tag{1}
\]

and $y_i(t) \in C^1$, $i = 0, \ldots, N$. Also, let us consider a diffusion (thermal conductivity) process in $\mathbb{R}$ such that the diffusion coefficient $\sigma$ is a piecewise constant function of $x$, i.e.

\[
\sigma(x) = \begin{cases} 
\sigma_i, & x \in [y_{i-1}(t), y_i(t)], \\
\sigma_-, & x \in (-\infty, y_0(t)], \\
\sigma_+, & x \in [y_N(t), \infty), 
\end{cases} \tag{2}
\]

where $\sigma_i > 0, \sigma_- > 0, \sigma_+ > 0$ are constants. This assumption can be relaxed, so that $\sigma_\pm = \sigma_p m(t), \sigma_i = \sigma_i(t)$, however, for simplicity we assume them to be constant. This assumption doesn’t impact the main idea of the proposed approach and the construction of oscillating integral transforms but makes many formulae look lighter.

Suppose the evolution of this diffusion process is described by the heat equation

\[
u_t = \sigma^2(x)u_{xx}, \tag{3}
\]

which should be solved subject to the initial condition

\[
u(0, x) = \delta(x - x_0), \tag{4}
\]

where $\delta(x - x_0)$ is the Dirac delta function, and also the matching conditions (see the corresponding discussion in (Itkin et al., 2021b) and references therein)

\[
\lim_{x \to y_i(t) - 0} u(t, x) = \lim_{x \to y_i(t) + 0} u(t, x), \tag{5}
\]
Multilayer heat equations and their solutions via oscillating integral transforms.

\[
\lim_{x \to y_i(t) - 0} \sigma_i^2 u(x, t) = \lim_{x \to y_{i+1}(t) + 0} \sigma_{i+1}^2 u(x, t), \quad i = 0, \ldots, N.
\]

These conditions mean the continuity of the function \( u \) and its flux over each boundary in the multilayer media. Note that such conditions could vary depending on the problem; see (Carr and March, 2018).

The known methods of solving Eq. (3), Eq. (4), Eq. (5) can be conventionally classified as follows

(a) eigenfunction expansions, (Carr and March, 2018);
(b) the Laplace Transform, (Lipton and Sepp, 2011; Itkin and Lipton, 2018);
(c) the method of heat potentials (HP) and generalized integral transform (GIT), (Itkin et al., 2021b,a);
(d) the Fokas method, (Asvestas et al., 2014; Deconinck et al., 2014, 2016);
(e) probabilistic methods, (Lejay, 2006).

The methods (a), (b) (d) can be used only for problems with straight boundaries; method (c) works even if the boundaries are curvilinear. Therefore, we want to treat multilayer problems by using a flavor of the framework (c).

To illustrate the main idea, we recall a recent paper (Itkin et al., 2021b) where the multilayer method has been introduced in detail. Let us consider the \( i \)-th layer in Eq. (3) and the corresponding integral transform

\[
\bar{u}(t, \lambda) = \int_{y_{i-1}(t)}^{y_i(t)} \sin(\lambda(x - y_i(t))) u(t, x) dx.
\]

Applying this transform to Eq. (3) for the \( i \)-th layer and then constructing the inverse transform allows a closed form representation of the solution \( u(t, x) \), \( x \in [y_{i-1}(t), y_i(t)] \) which, however, also depends on four yet unknown functions

\[
\psi_i^-(t) = \lim_{x \to y_i(t) - 0} u(t, x), \quad \psi_i^+(t) = \lim_{x \to y_i(t) + 0} u(t, x),
\]

\[
\Psi_i^-(t) = \lim_{x \to y_i(t) - 0} u_x(t, x), \quad \Psi_i^+(t) = \lim_{x \to y_i(t) + 0} u_x(t, x).
\]

It turns out that with allowance for the matching Eq. (5) and boundary Eq. (3) conditions the unknown functions \( \psi_i^\pm(t), \Psi_i^\pm(t) \) solve a system of Volterra equations of the second kind. Once this system is solved (numerically), the solution of the problem Eq. (3) is obtained. It is important to emphasize that this procedure relies on the first step, which consists in using an integral transform to obtain a formal solution at every layer, and the second step utilizing the matching conditions to derive the Volterra equations.

Although it is shown in (Itkin et al., 2021b) that this approach works well, here we slightly modify it by changing the order of the corresponding steps. It means that for each problem under consideration, we want to design a special integral transform with the kernel satisfying the matching conditions Eq. (5). Once this is done, the problem Eq. (5) can be solved by using a direct/inverse integral transform in line of (Itkin et al., 2021a). Surprisingly, in many cases, the corresponding integral transforms can be explicitly obtained. By analogy to the oscillating Brownian Motion, (Lejay, 2006), we call them "oscillating integral transforms" (OIT).

The novelty of this paper is twofold. First, we derive some integral transforms which, to the best of our knowledge, yet have never been reported in the literature. Therefore, this is a contribution in a purely mathematical context. Second, we develop a new method of solving the multilayer heat equation (MHE) problem in Eq. (3), Eq. (4), Eq. (5) with curvilinear boundaries. Moreover, since the Green’s function of this problem is constructed in a semi-explicit form, several problems, such as determining the first passage time, optimal stopping, etc. for processes associated with Eq. (3) with discontinuous coefficients, can be solved by either the HP or GIT methods, (Itkin et al., 2021a).

It is worth discussing the second point in more detail. In (Itkin et al., 2021b) we developed a multilayer method that makes it possible to find semi-analytical solutions of some one-dimensional PDEs
that otherwise cannot be solved analytically. Our approach works because the corresponding PDEs cannot be reduced to the heat or Bessel equation in the whole spatial domain but are reducible in each layer separately. As a result, the layer boundaries can be chosen arbitrarily (to some extent) based on the argument of accuracy and convenience. In particular, we constructed these boundaries so that the corresponding Volterra equations could be solved by using the Laplace transform. However, suppose we want to use this method to solve physical interlayer (interface) boundaries, which are defined by the essence of the problem and cannot be chosen at will. In this case, we have to solve the Volterra equations the way they are, so no further simplifications are possible.

In this case, the gradients of the solution could potentially misbehave close to each internal boundary in a sense that our series representation of the solution could converge very slowly; see (Itkin et al., 2021b). Therefore, the numerical solution for the corresponding Volterra equations could be unstable. However, by adding artificial boundaries to make the Volterra integral a convolution, the Volterra equation can be solved using the Laplace transform, thus eliminating potential problems. As a result, the corresponding solution is well-behaved.

To address the issues mentioned above, we develop another flavor of the multilayer method, which gives rise to different are more numerically stable Volterra equations. To proceed, we need the boundary points to be internal for an appropriate basis, rather than the boundary points for another basis. If this choice is possible, the described problem of potential instability in the numerical solution of the Volterra equations, which occurs close to the layer boundaries, disappears. Therefore, our new method doesn’t duplicate that one in (Itkin et al., 2021b) because now we obtain alternative representations of the solution with better properties.

The paper is organized as follows. Section 1 presents a construction of the OIT by using series expansions of the Dirac delta function and building an orthonormal basis of eigenfunctions for the corresponding Sturm-Liouville problem. We consider various cases characterized by a spectrum of eigenvalues: continuous, discrete, or mixed. In Section 2 some examples of this technique are presented in more detail. We describe a model with two non-zero layers and a discrete spectrum and another model with three layers and a continuous spectrum. Next, section 3.2 is devoted to problems that give rise to multilayer heat equations with curvilinear boundaries. We discuss in detail the time-dependent oscillating Brownian motion and various applications from finance and physics. For a two-layer problem with moving (time-dependent) boundary (the interlayer interface), we derive a system of linear Volterra integral equations for the values of the solution and its gradient at the interface. By solving this system, we can express the solution of the whole problem in the closed-form. Thus, our method is semi-analytical. To the best of our knowledge, in the existing literature, the problem at hand was solved just numerically (for the time-dependent boundary) or analytically (for the constant boundary). Thus, our approach is new and contributes to the existing literature. In the next section, we demonstrate the power of our method as applied to solving a practical problem - solidification (or melting) of a liquid-solid system at the time-dependent phase interface. Again, so far in the literature, this problem was solved only numerically, e.g., by using finite differences (FD), which brings some issues when constructing an efficient FD scheme at the domain with the interphase moving boundary. Then, we introduce and solve a multilayer problem with the number of layers $N > 2$ and piecewise constant coefficients and time-dependent interfaces in Section 3.2. We summarize our conclusions in the final section.

We tried doing our best to simplify the exposition of the method and make it transparent to all readers regardless of their background (finance, physics, or mathematics). For this reason, we moved a significant part of derivations into appendices. Section 1.2 is a bit technical; however, it would be difficult to understand our approach without it. If necessary, it could be skipped at first reading so that after Introduction, the reader can immediately proceed to Section 2.
1 Building integral transforms for multilayer problems

For the reader’s convenience and to eliminate overcomplicated formulae, we begin this section by introducing a special notation used across the whole paper.

1.1 Notation

By \( y \) we denote the following vector
\[
y = (y_0, y_1, \ldots, y_N)^	op, \quad y_{i-1} < y_i, \quad i = 1 \ldots N.
\]
(7)
The vector \( \sigma \) is the \( N + 2 \)-dimensional vector with positive entries
\[
\sigma = (\sigma_-, \sigma_1, \ldots, \sigma_N, \sigma_+)^\top,
\]
(8)
and by \( \sigma^2 \) we mean a component-wise square of \( \sigma \). We also introduce the vector indicator function \( \mathbb{1}_{x|y} \) associated with \( y \)
\[
\mathbb{1}_{x|y} = (\mathbb{1}_{x<y_0}, \mathbb{1}_{y_0<x<y_1}, \ldots, \mathbb{1}_{y_{N-1}<x<y_N}, \mathbb{1}_{x>y_N})^\top.
\]
(9)
If \( y \) is a scalar, i.e. \( y = y \), the indicator vector \( \mathbb{1}_{x|y} \) is defined as
\[
\mathbb{1}_{x|y} = (\mathbb{1}_{x<y}, \mathbb{1}_{x>y})^\top.
\]
(10)
Let the operator \( \langle a, b \rangle \) denote a scalar product
\[
\langle a, b \rangle = a^\top b = b^\top a = \sum_{i=1}^N a_i b_i.
\]
(11)
Let us also introduce \( R(\varphi), H(\varphi) \) and \( S(\alpha, \beta) \)
\[
R(\varphi) = \begin{bmatrix}
\cos \varphi & -\sin \varphi \\
\sin \varphi & \cos \varphi
\end{bmatrix}, \quad H(\varphi) = \begin{bmatrix}
cosh \varphi & \sinh \varphi \\
\sinh \varphi & \cosh \varphi
\end{bmatrix}, \quad S_{\alpha,\beta} = \begin{bmatrix}
\alpha & 0 \\
0 & \beta
\end{bmatrix},
\]
(12)
which are matrices of rotation, hyperbolic rotation and stretch, respectively. The special case \( S_{1,\beta} \) is denoted as \( S_{\beta} = S_{1,\beta} \).

Suppose we are given a set of squared matrices \( A_k, \quad k = 1, \ldots, K \). A product over a set \( k \in [a,b] \) is defined as
\[
\prod_{k=a}^b A_k = A_b A_{b-1} \ldots A_{a+1} A_a.
\]
(13)

1.2 Construction of transforms using series expansion of the Dirac delta function

Our focus in this paper is on solving various multilayer problems by constructing an appropriate integral transform. In doing so, we begin with an observation that the Dirac delta function can be represented as a series expansion in eigenfunctions of some Sturm-Liouville (SL) problem, (Titchmarsh). We discuss various approaches to obtaining this representation in Section 1.3. Then an integral transform associated with a given expansion can be constructed as follows.

Let \( y \) and \( \sigma \) be the vectors defined in Eq. (7) and Eq. (8) respectively. Let us introduce a function \( u_\lambda(x|x_0) : \mathbb{R} \mapsto \mathbb{R}^+, \quad u \in C^2 \) parameterized by \( \lambda \in \mathbb{R} \) and the initial condition \( u(x_0) = \delta(x - x_0) \). We consider the following SL problem for the function \( u_\lambda(x|x_0) \)
\[
\frac{\partial}{\partial x} \left( \mathbb{1}_{x|y} \sigma^2 \frac{\partial}{\partial x} u_\lambda(x|x_0) \right) - \lambda u_\lambda(x|x_0) = -\delta(x - x_0),
\]
(14)
which should be solved subject to the vanishing boundary conditions

\[ \lim_{x \to \pm \infty} u_\lambda(x|x_0) = 0, \quad (15) \]

and also the matching conditions

\[ \lim_{x \to y_i^-} u_\lambda(x|x_0) = \lim_{x \to y_i^+} u_\lambda(x|x_0), \quad (16) \]

\[ \lim_{x \to y_i^-} \sigma_i^2 u'_\lambda(x|x_0) = \lim_{x \to y_i^+} \sigma_i^2 u'_\lambda(x|x_0), \quad i = 0, \ldots, N, \]

\[ \lim_{x \to y_0^-} \sigma_0^2 u'_\lambda(x|x_0) = \lim_{x \to y_0^+} \sigma_0^2 u'_\lambda(x|x_0), \]

\[ \lim_{x \to y_N^-} \sigma_N^2 u'_\lambda(x|x_0) = \lim_{x \to y_N^+} \sigma_N^2 u'_\lambda(x|x_0). \]

Once the function \( u_\lambda(x|x_0) \) is known, the following representations of the Dirac delta function holds

\[ \delta(x - x_0) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} u_\lambda(x|x_0) d\lambda. \quad (17) \]

Indeed, suppose that the function \( u(t, x) \) solves the heat equation at \( (t, x) \in \mathbb{R}^+ \times \mathbb{R} \) with the initial condition \( u(0, x) = \delta(x - x_0) \). Taking the Laplace transform, one can see that the image \( u_\lambda(x|x_0) \) solves Eq. (14). Therefore, given \( u_\lambda(x|x_0), u(t, x) \) can be restored via the inverse Laplace transform (the Bromwich integral). Then setting \( t = 0 \) we obtain Eq. (17).

In Eq. (17) \( \gamma \) is a real-valued constant chosen in such a way that all poles of the function under the integral are located in a complex plane to the left of a straight line parallel to the imaginary axis \( x \in (\gamma, -i\infty), (\gamma, i\infty) \). As per (Titchmarsh), this gives rise to the delta function expansion in non-normalized eigenfunctions of the SL problem of the form

\[ \delta(x - x_0) = \int_{-\infty}^{+\infty} F_\omega(x) B_\omega(x_0) d\omega, \quad (18) \]

where \( F_\omega(x) \) is some function of \( x \) and \( B_\omega(x_0) \) - some function of \( x_0 \), so the integrand in Eq. (18) factorizes into a product. Then, using the filtering property of delta function

\[ f(x) = \int_{-\infty}^{+\infty} \delta(x - x_0) f(x_0) dx_0, \quad (19) \]

we arrive at the following construction

\[ \tilde{f}(\omega) = \int_{-\infty}^{+\infty} F_\omega(x) f(x) dx, \quad f(x) = \int_{-\infty}^{+\infty} B_\omega(x) \tilde{f}(\omega) d\omega, \]

which is an integral transform built based on the eigenfunctions of the corresponding SL problem. Thus, constructing the necessary integral transform is almost straightforward once the corresponding expansion of the Dirac delta function is derived. In the next section, we consider the latter problem in more detail.

1.3 Expansion of the Dirac delta function into series of eigenfunctions

Since solutions to this problem could have different properties depending on the explicit form of \( y, \sigma \), below, we develop a more granular classification.

**Problems with continuous spectrum.** This corresponds to the special case \( y = y \), so Eq. (15), and Eq. (16) take the form

\[ \lim_{x \to \pm \infty} u_\lambda(x|x_0) = 0, \quad \lim_{x \to y^-} u_\lambda(x|x_0) = \lim_{x \to y^+} u_\lambda(x|x_0), \quad \lim_{x \to y^-} \sigma_i^2 u'_\lambda(x|x_0) = \lim_{x \to y^+} \sigma_i^2 u'_\lambda(x|x_0). \quad (20) \]
Problems with discrete spectrum. This occurs when the vector \( \sigma \) is of a special form
\[
\sigma = (0, \sigma_1, \sigma_2, \ldots, \sigma_{N-1}, \sigma_N, 0)^\top,
\]
(21)
or when the absorbing boundary conditions at \( y_0 \) and \( y_N \) are used. Then Eq. (15), Eq. (16) read
\[
\lim_{x \to y_0^+} u_\lambda(x|x_0) = 0, \quad \lim_{x \to y_N^-} u_\lambda(x|x_0) = 0, \\
\lim_{x \to y_i^-} u_\lambda(x|x_0) = \lim_{x \to y_i^+} u_\lambda(x|x_0), \quad \lim_{x \to y_i^+} \sigma_i^2 u_\lambda'(x|x_0) = \lim_{x \to y_{i+1}^-} \sigma_{i+1}^2 u_\lambda'(x|x_0), \quad i = 1, \ldots, N - 1.
\]
(22)
Problems with mixed spectrum. In the general case the problem Eq. (14), Eq. (15) Eq. (16), as well as the problems corresponding to the vector \( \sigma \) with \( \sigma_- = 0 \) or \( \sigma_+ = 0 \) have mixed spectra.

Below we analyze each that case in more detail, find an explicit solution for \( u_\lambda(x|x_0) \) and obtain the corresponding representation for the Dirac delta function \( \delta(x - x_0) \).

1.3.1 Continuous spectrum. The oscillating Fourier transform

In this section, we derive a new (to the best of our knowledge) integral transform, which can be viewed as a generalization of the Fourier transform. We call it the oscillating Fourier transform (OIT).

It is well-known that the classical Fourier transform can be easily derived from the following identity
\[
\delta(x - x_0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega(x-x_0)} d\omega.
\]
(23)
The expansion Eq. (23) is associated with the exponential function
\[
f_\omega(x) = e^{i\omega x}, \quad x \in (-\infty, \infty).
\]
(24)
By analogy, for our problem as the basis function we consider a piecewise exponential function
\[
g_\omega(x) = \begin{cases} e^{i\omega(x-y)/\sigma_-}, & x < y, \\ e^{i\omega(x-y)/\sigma_+}, & x > y. \end{cases}
\]
(25)
In case \( \sigma_- = \sigma_+ \) the function \( g \) up to a constant multiplier coincides with the function \( f \).

Given the vector \( \sigma^2 = (\sigma_1^2, \sigma_2^2, \ldots)^\top \) let us define the function \( u_\lambda(x|x_0) \) as the solution of Eq. (14), Eq. (20). Our goal is to find an explicit expression for \( u_\lambda(x|x_0) \). Once this is done, we use Eq. (17) and get the necessary representation of \( \delta(x-x_0) \).

The solution of this problem is given in Appendix A. The result reads
\[
\delta(x - x_0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathbf{1}_{x\geq0}^\top F_{x-y,\sigma}(\omega) B_{x_0-y,\sigma}(\omega) \mathbf{1}_{x_0\geq0} d\omega,
\]
(26)
where the definitions of matrices \( F_{x-y,\sigma}(\omega), B_{x_0-y,\sigma}(\omega) \) are given in Eq. (A.7).

Since the Forward \( F \) and Backward \( B \) matrices in Eq. (26) depend only on \( x \) and \( x_0 \), respectively, \( F \) can be interpreted as a kernel of some integral transform, and \( B \) - as an inversion kernel. Indeed, using Eq. (19) together with Eq. (26) yields
\[
f(x_0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathbf{1}_{x|y}^\top F_{x-y,\sigma}(\omega) B_{x_0-y,\sigma}(\omega) \mathbf{1}_{x_0|y} f(x) dx d\omega.
\]
(27)
Splitting Eq. (27) into two integrals we arrive at the following integral transform

\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{f}(\omega) B_{x \rightarrow y, \sigma}(\omega) \mathbf{1}_{x \mid y} d\omega. \]  

(28)

The two-component image \( \tilde{f}(\omega) \) is defined as follows

\[ \tilde{f}(\omega) = \begin{bmatrix} \tilde{f}_-(\omega) \\ \tilde{f}_+(\omega) \end{bmatrix} = \int_{-\infty}^{+\infty} F_{x \rightarrow y, \sigma}(\omega) \mathbf{1}_{x \mid y} f(x) dx. \]  

(29)

We call it the Oscillating Fourier transform. Contrary to the original Fourier transform, the oscillating Fourier transform in Eq. (29), Eq. (28) has a two-component image.

1.3.2 Discrete spectrum. Problem in a strip

In this case it is possible to directly derive an orthogonal basis associated with the solution of the corresponding problem in a strip with matching conditions in Eq. (22).

But first, let us consider a slightly modified problem

\[ \frac{\partial}{\partial x} \left( \langle \sigma^2, \mathbf{1}_{x \mid y} \rangle \frac{\partial}{\partial x} Z(x) \right) = \lambda^2 Z(x), \quad Z(x \leq y_0) = Z(x \geq y_N) = 0. \]  

(30)

Here the function \( Z(x) = Z_{\lambda, \gamma}(x) \in C^1 \forall x \in [y_0, y_N] \), and so is the flux

\[ \sigma_i^2 Z'_x |_{x \rightarrow y_i - 0} = \sigma_{i+1}^2 Z'_x |_{x \rightarrow y_i + 0}, \quad i = 1, \ldots, N - 1. \]  

(31)

The functions \( Z_{\lambda, \gamma}(x) \) form an orthogonal basis in \([y_0, y_N] \). Indeed, let us introduce two functions \( Z_{\lambda, \gamma}(x) \) and \( Y_{\mu, \gamma}(x) \) associated with two different parameters \( \lambda, \mu, \lambda \neq \mu \) in Eq. (30), and consider the following identity

\[ \int_{y_0}^{y_N} \left[ \frac{\partial}{\partial x} \left( \langle \sigma^2, \mathbf{1}_{x \mid y} \rangle \frac{\partial}{\partial x} Z(x) \right) Y(x) - \frac{\partial}{\partial x} \left( \langle \sigma^2, \mathbf{1}_{x \mid y} \rangle \frac{\partial}{\partial x} Y(x) \right) Z(x) \right] dx = (\lambda^2 - \mu^2) \int_{y_0}^{y_N} Z(x) Y(x) dx. \]  

(32)

The integral in the LHS of Eq. (32) vanishes. Indeed, integrating by parts, we obtain

\[ \int_{y_0}^{y_N} \left[ \langle \sigma^2, \mathbf{1}_{x \mid y} \rangle \frac{\partial}{\partial x} Z(x) \right] Y(x) - \frac{\partial}{\partial x} \left( \langle \sigma^2, \mathbf{1}_{x \mid y} \rangle \frac{\partial}{\partial x} Y(x) \right) Z(x) \right] dx \]

\[ = \int_{y_0}^{y_N} \left[ \langle \sigma^2, \mathbf{1}_{x \mid y} \rangle \left( Z'(x) Y'(x) - Y'(x) Z'(x) \right) \right] dx \]

\[ + \sum_{i=1}^{N-1} \langle \sigma^2, \mathbf{1}_{x \mid y} \rangle \left[ Z'(y_i) Y(y_i) - Y'(y_i) Z(y_i) \right] \bigg|_{y_i = y_i + 0} - \sum_{i=0}^{N-1} \langle \sigma^2, \mathbf{1}_{x \mid y} \rangle \left[ Z'(y_i) Y(y_i) - Y'(y_i) Z(y_i) \right] \bigg|_{y_i = y_i - 0} \]

\[ = \sum_{i=1}^{N-1} \left( \sigma_i^2 \left[ Z'(y_i + 0) Y(y_i) - Y'(y_i + 0) Z(y_i) \right] - \sigma_{i-1}^2 \left[ Z'(y_i - 0) Y(y_i) - Y'(y_i - 0) Z(y_i) \right] \right) \]

\[ = \sum_{i=1}^{N} \left( Y(y_i) \left[ \sigma_i^2 Z'(y_i + 0) - \sigma_{i-1}^2 Z'(y_i - 0) \right] - Z(y_i) \left[ \sigma_i^2 Y'(y_i + 0) - \sigma_{i-1}^2 Y'(y_i - 0) \right] \right) = 0. \]

Since \( \lambda \neq \mu \), that means that the functions \( Z(x) \) and \( Y(x) \) are orthogonal.
We proceed with an explicit construction of the orthogonal basis associated with the problem Eq. (14), Eq. (15), Eq. (22). Let us define the function \( \Theta_{y, \lambda}(x) \) as follows
\[
\Theta_{y, \lambda}(x) = [0, \Theta_{y, \lambda, 1}(x), \ldots, \Theta_{y, \lambda, N(x)}, 0]^T.
\]
In turn, each component of \( \Theta_{y, \lambda}(x) \) can be represented as
\[
\Theta_{y, \lambda, i}(x) = C_i(\lambda, y) \cos(\bar{\lambda}_i x) + D_i(\lambda, y) \sin(\bar{\lambda}_i x), \quad \bar{\lambda}_i = \lambda / \sigma_i.
\]
The unknown coefficients \( C_i, D_i, \ i = 1, \ldots, N \) can be found via recursion
\[
C_1(\lambda, y) = -\sin(\bar{\lambda}_i y_0), \quad D_1(\lambda, y) = \cos(\bar{\lambda}_i y_0),
\]
\[
\begin{bmatrix}
\cos(\bar{\lambda}_{i+1} y_i) & \sin(\bar{\lambda}_{i+1} y_i) \\
-\sin(\bar{\lambda}_{i+1} y_i) & \cos(\bar{\lambda}_{i+1} y_i)
\end{bmatrix}
\begin{bmatrix}
C_{i+1}(\lambda, y) \\
D_{i+1}(\lambda, y)
\end{bmatrix}
= \begin{bmatrix}
\cos(\bar{\lambda}_i y_i) & \sin(\bar{\lambda}_i y_i) \\
-si \sin(\bar{\lambda}_i y_i) & si \cos(\bar{\lambda}_i y_i)
\end{bmatrix}
\begin{bmatrix}
C_i(\lambda, y) \\
D_i(\lambda, y)
\end{bmatrix}, \quad s_i = \frac{\sigma_i}{\sigma_{i+1}}.
\]
An explicit solution of Eq. (36) can be expressed by using the stretch and rotation matrices introduced in Eq. (12)
\[
\begin{bmatrix}
C_{i+1}(\lambda, y) \\
D_{i+1}(\lambda, y)
\end{bmatrix}
= (R_{\bar{\lambda}_{i+1} y_i})^{-1} S_{s_i} R_{\bar{\lambda}_i y_i} \begin{bmatrix}
C_i(\lambda, y) \\
D_i(\lambda, y)
\end{bmatrix}.
\]

Defining the length \( l_i = y_i - y_{i-1} \) of each segment and using the property \( R_{\alpha}^{-1} = R_{-\alpha}, R_{\alpha + \beta} = R_{\alpha} R_{\beta} \) this recurrent formula can be written as
\[
\begin{bmatrix}
C_i(\lambda, y) \\
D_i(\lambda, y)
\end{bmatrix}
= R_{\bar{\lambda}_{i-1} y_i} \prod_{j=1}^{i-1} \Lambda_j R_{\bar{\lambda}_{i+1} y_i} \begin{bmatrix}
C_1(\lambda, y) \\
D_1(\lambda, y)
\end{bmatrix}, \quad \Lambda_1 = S_{s_1}, \quad \Lambda_i = S_{s_i} R_{\bar{\lambda}_{i-1} l_{i-1}}, \quad i > 1.
\]

The eigenvalues \( \lambda_n \) solve the transcendental equation
\[
\cos(\bar{\lambda}_N y_N) C_N(\lambda, y) + \sin(\bar{\lambda}_N y_N) D_N(\lambda, y) = 0, \quad \lambda = \lambda_n.
\]
Taking into account the following identities
\[
R_{\bar{\lambda}_i y_i} \begin{bmatrix}
C_1(\lambda, y) \\
D_1(\lambda, y)
\end{bmatrix}
= R_{\bar{\lambda}_i y_i} \begin{bmatrix}
-\sin(\bar{\lambda}_1 y_0) \\
\cos(\bar{\lambda}_1 y_0)
\end{bmatrix}
= \begin{bmatrix}
\sin(\bar{\lambda}_1 l_1) \\
\cos(\bar{\lambda}_1 l_1)
\end{bmatrix},
\]
\[
\begin{bmatrix}
\cos(\bar{\lambda}_N y_N) \\
\sin(\bar{\lambda}_N y_N)
\end{bmatrix}^T R_{-\bar{\lambda}_N y_{N-1}} = \begin{bmatrix}
\cos(\bar{\lambda}_N l_N) \\
\sin(\bar{\lambda}_N l_N)
\end{bmatrix}^T,
\]
the Eq. (39) can also be re-written in a more explicit form
\[
\begin{bmatrix}
\cos(\bar{\lambda}_N l_N) \\
\sin(\bar{\lambda}_N l_N)
\end{bmatrix}^T \left( \prod_{i=1}^{N-1} \Lambda_i \right) \begin{bmatrix}
\sin(\bar{\lambda}_1 l_1) \\
\cos(\bar{\lambda}_1 l_1)
\end{bmatrix} = 0. \quad \lambda = \lambda_n.
\]

Thus, we arrive at the following series expansion for the delta function
\[
\delta(x - x_0) = \sum_{n=1}^{\infty} \frac{\langle \Theta_{\lambda_n, y}(x_0), 1_{x \mid y} \rangle \langle \Theta_{\lambda_n, y}(x), 1_{x \mid y} \rangle}{\int_{y_0}^{y_N} \langle \Theta_{\lambda_n, y}(\xi), 1_{\xi \mid y} \rangle d\xi}.
\]
Oscillating Fourier Series. Using the representation Eq. (41) we obtain the following Oscillating Fourier series

$$f(x) = \sum_{n=1}^{\infty} \frac{\langle \Theta_{\lambda_n,y}(x), 1_{x|y} \rangle}{\int_{y_0}^{y_N} \langle \Theta_{\lambda_n,y}(\xi), 1_{\xi|y} \rangle^2 d\xi} \bar{f}(\lambda_n), \quad \bar{f}(\lambda) = \int_{y_0}^{y_N} f(\xi) \langle \Theta_{\lambda,y}(\xi), 1_{\xi|y} \rangle d\xi.$$ (42)

The eigenvalues $\lambda_n$ are defined as an increasing sequence of the solutions of Eq. (40).

The function $\Theta_{\lambda_n,y}(x)$ can be considered as a generalization of the sine function in

$$\sin \left( \frac{\pi x}{y} \right).$$

Indeed, if $\sigma$ has the form

$$\sigma = [0, \sigma, \sigma, \ldots, \sigma, 0]^T,$$

all stretch matrices $S$ in Eq. (38) and Eq. (40) are equal to the identity matrix. Using a semi-group property of the rotation matrices yields the following equation for the eigenvalues $\lambda_n$

$$\begin{bmatrix} \cos \left( \frac{\lambda}{\sigma} l_N \right) \\ \sin \left( \frac{\lambda}{\sigma} l_N \right) \end{bmatrix}^T R = \sum_{k=2}^{N-1} \frac{\lambda_n l_k}{\sigma} \begin{bmatrix} \sin \left( \frac{\lambda}{\sigma} l_1 \right) \\ \cos \left( \frac{\lambda}{\sigma} l_1 \right) \end{bmatrix} = 0,$$ (43)

from which it follows that $\sin \left( \frac{\lambda_n}{\sigma} l_1 \right) = 0$ and $\lambda_n = \frac{\pi n \sigma}{l}$, $l = \sum_{k=1}^{N-1} l_k = y_N - y_0$. Computing the constants $C_i$ and $D_i$ and substituting the values of $\lambda_n$ gives rise to the following formula for $\Theta_{y,\lambda_n}(x)$

$$\Theta_{y,\lambda_n}(x) = \left[ 0, \sin \left( \frac{\pi n x - y_0}{l} \right), \sin \left( \frac{\pi n x - y_0}{l} \right) + \frac{1}{\sqrt{\lambda_1}}, \frac{1}{\sqrt{\lambda_1}} \right]^T.$$ (44)

Hence, when $\sigma_i = \sigma = \text{const.}$, $\forall i = 1, \ldots, N$ the eigenfunctions coincide with the sine functions.

1.3.3 Mixed spectrum

As compared with the previous cases this is the most complicated problem. Let us slightly change the notation to $\sigma_- = \sigma_0$, $\sigma_+ = \sigma_{N+1}$, $l_i = y_i - y_{i-1}$, $l_0 = 0$. Let us look for the solution $u_\lambda(x|x_0)$ in the form

$$u_\lambda(x|x_0) = \langle u, 1_{x|y} \rangle, \quad u = (u_0, u_1, \ldots, u_N, u_{N+1}),$$ (45)

where

$$u_i = (1 - \delta_i,N+1) C_i \cosh (\tilde{\lambda}_i x) + (1 + \delta_i,0) D_i \sinh (\tilde{\lambda}_i x) + \frac{1}{2\sqrt{\lambda_1}} e^{-\tilde{\lambda}_i|x-x_0|},$$ (46)

and $\delta_{i,b}$ is the Kronecker delta. Let us define the auxiliary matrix $G_\lambda(\alpha, \beta|z)$

$$G_\lambda(\alpha, \beta|z) = \begin{bmatrix} e_\lambda(z, \beta, \alpha) - e_\lambda(z, z, \beta) \\ e_\lambda(z, \alpha, \beta) - e_\lambda(z, \alpha, \alpha) \\ e_\lambda(z, \alpha, \beta) - e_\lambda(z, \beta, \beta) \\ e_\lambda(z, \beta, \alpha) - e_\lambda(z, \beta, \beta) \end{bmatrix}, \quad e_\lambda(z, a, b) = \frac{1}{2\sqrt{\lambda b}} e^{-|z-b|\sqrt{\lambda}}.$$ (47)

Using the matching conditions at $x = y_0$ for the functions $u_0$ and $u_1$, we obtain the equation

$$\begin{bmatrix} C_1 \\ D_1 \end{bmatrix} = \mathcal{H}_{-\tilde{\lambda}_1 y_0} S_{y_0} \begin{bmatrix} 1 \\ C_- \end{bmatrix} C_- + \mathcal{H}_{-\tilde{\lambda}_1 y_0} G_\lambda(\sigma_0, \sigma_1|x_0 - y_0) 1_{x_0|y_0}, \quad s_+ = \frac{\sigma_+}{\sigma_0}.$$ (48)

By analogy with Section 1.3.2, it is possible to derive recurrent relationships between the coefficients $C_i$, $D_i$ and $C_{i+1}$, $D_{i+1}$ which read

$$\begin{bmatrix} C_{i+1} \\ D_{i+1} \end{bmatrix} = \mathcal{H}_{-\tilde{\lambda}_{i+1} y_0} S_{y_i} \mathcal{H}_{\tilde{\lambda}_i y_i} \begin{bmatrix} C_i \\ D_i \end{bmatrix} + \mathcal{H}_{-\tilde{\lambda}_{i+1} y_i} G_\lambda(\sigma_i, \sigma_{i+1}|x_0 - y_i) 1_{x_0|y_i}.$$ (49)
And at the last point \( x = y_N \) the matching conditions yield the equation
\[
\begin{bmatrix}
1 \\
-1
\end{bmatrix}
D_+ = S_N H_{\lambda N y N} \begin{bmatrix}
C_n \\
D_n
\end{bmatrix} + G_{\lambda}(\sigma_N, \sigma_{N+1} | x_0 - y_N) \mathbb{1}_{x_0|y_N}.
\] (50)

Combining Eq. (48), Eq. (49), Eq. (50) together we arrive at the system of equations to determine \( C_- \) and \( D_+ \)
\[
\begin{bmatrix}
1 \\
-1
\end{bmatrix}
D_+ = \prod_{j=0}^{N} S_{s_j} H_{\lambda j j} \begin{bmatrix}
1 \\
1
\end{bmatrix} C_- + \sum_{k=0}^{N} \left( \prod_{p=k+1}^{N} S_{s_p} H_{\lambda p p} \right) G_{\lambda}(\sigma_k, \sigma_{k+1} | x_0 - y_k) \mathbb{1}_{x_0|y_k}.
\] (51)
This system can also be re-written in the matrix form if we define the matrix \( \Lambda(\lambda) \)
\[
\Lambda(\lambda) = \prod_{j=0}^{N} S_{s_j} H_{\lambda j j},
\] (52)
with elements \( \Lambda_{i,j} \), \( i, j = 1, \ldots, N + 1 \), so
\[
\begin{bmatrix}
\Lambda_{11}(\lambda) + \Lambda_{12}(\lambda) & -1 \\
\Lambda_{21}(\lambda) + \Lambda_{22}(\lambda) & 1
\end{bmatrix}
\begin{bmatrix}
C_- \\
D_+
\end{bmatrix} = - \frac{\Lambda^*(\lambda)}{\det \Lambda^*(\lambda)} \sum_{k=0}^{N} \left( \prod_{p=k+1}^{N} S_{s_p} H_{\lambda p p} \right) G_{\lambda}(\sigma_k, \sigma_{k+1} | x_0 - y_k) \mathbb{1}_{x_0|y_k},
\] (53)
Solving for the coefficients \( C_- \) and \( D_+ \) yields
\[
\begin{bmatrix}
C_- \\
D_+
\end{bmatrix} = - \frac{\Lambda^*(\lambda)}{\det \Lambda^*(\lambda)} \sum_{k=0}^{N} \left( \prod_{p=k+1}^{N} S_{s_p} H_{\lambda p p} \right) G_{\lambda}(\sigma_k, \sigma_{k+1} | x_0 - y_k) \mathbb{1}_{x_0|y_k},
\] (54)
where \( \Lambda^* \) is an adjacent matrix
\[
\Lambda^*(\lambda) = \begin{bmatrix}
1 & 1 \\
-\Lambda_{21}(\lambda) - \Lambda_{22}(\lambda) & \Lambda_{11}(\lambda) + \Lambda_{12}(\lambda)
\end{bmatrix}.
\] (55)
In a similar manner let us define a sequence of matrices
\[
\Lambda_k^p(\lambda) = \prod_{j=k}^{p-1} S_{s_j} H_{\lambda j j}, \quad \Lambda(\lambda) = \Lambda_0^{N+1}(\lambda), \quad \Lambda_k^p(\lambda) = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.
\] (56)
Re-writing the recursion in Eq. (49) by using these matrices and solving for the coefficients \( C_i \) and \( D_i \) yields
\[
\begin{bmatrix}
C_i \\
D_i
\end{bmatrix} = C_- H_{-\lambda_i y_{i-1}} \Lambda_i^0(\lambda) \begin{bmatrix}
1 \\
1
\end{bmatrix} + H_{-\lambda_i y_{i-1}} \sum_{k=0}^{i} \Lambda_{k+1}^{i+1}(\lambda) G_{\lambda}(\sigma_k, \sigma_{k+1} | x_0 - y_k) \mathbb{1}_{x_0|y_k}.
\] (57)
Therefore, the \( i \)-th component of \( u \) can be expressed as
\[
u_i(x) = \begin{bmatrix}
\cosh(\tilde{\lambda}_i(x - y_{i-1})) \\
\sinh(\tilde{\lambda}_i(x - y_{i-1}))
\end{bmatrix}^\top \left( C_- \Lambda_i^0(\lambda) \begin{bmatrix}
1 \\
1
\end{bmatrix} + \sum_{k=0}^{i} \Lambda_{k+1}^{i+1}(\lambda) G_{\lambda}(\sigma_k, \sigma_{k+1} | x_0 - y_k) \mathbb{1}_{x_0|y_k} \right) + \frac{e^{-\tilde{\lambda}_i |x-x_0|}}{2 \sqrt{\lambda} \sigma_i}.
\] (58)
Having Eq. (57) we can substitute it into Eq. (45) and then into the Bromwich integral in Eq. (17). It turns out, however, that the integrands in Eq. (58) have a branching point at \( \lambda = 0 \) and poles at \( \lambda = \lambda_n \), \( n = 1 \ldots \infty \) where \( \lambda_n \) are the roots of the equation \( \det \Lambda^*(\lambda) = 0 \). By doing contour integration, see Appendix B, we obtain the necessary representation of the Dirac delta function
\[
\delta(x - x_0) = \left\langle \frac{1}{\pi i} \int_0^\infty \left[ u_{e^{-i \omega \eta}}(x | x_0) - u_{e^{i \omega \eta}}(x | x_0) \right] \omega \, d\omega + \sum_{k=1}^\infty \text{Res} \left( u_k(x | x_0) \right), \mathbb{1}_{x_0|y} \right\rangle.
\] (59)
2 Representative examples

As mentioned already, the series expansion of the Dirac delta function in the basis of eigenfunctions of the corresponding SL problem provides a mechanism of building integral transforms with necessary properties suitable for solving a given multilayer problem. This section offers examples showing how this relatively abstract concept can be implemented in two and three-dimensional cases.

2.1 Model with two non-zero layers

Let us consider the problem Eq. (14), Eq. (22) with \( N = 2 \), so the corresponding vectors \( \mathbf{y} \) and \( \boldsymbol{\sigma} \) take the form

\[
\mathbf{y} = (y_0, y_1, y_2)^\top, \quad \boldsymbol{\sigma} = (0, \sigma_1, \sigma_2, 0)^\top,
\]

and the basis function \( \Theta_{\mathbf{y}, \lambda}(x) \) reads

\[
\Theta_{\mathbf{y}, \lambda}(x) = \begin{cases}
0, & x \in (-\infty, y_0], \\
C_1(\lambda, \mathbf{y}) \cos(\lambda_1 x) + D_1(\lambda, \mathbf{y}) \sin(\lambda_1 x), & x \in [y_0, y_1], \\
C_2(\lambda, \mathbf{y}) \cos(\lambda_2 x) + D_2(\lambda, \mathbf{y}) \sin(\lambda_2 x), & x \in [y_1, y_2], \\
0, & x \in [y_2, \infty). 
\end{cases}
\]

The coefficients \( C_1(\lambda, \mathbf{y}) \) and \( D_1(\lambda, \mathbf{y}) \) found by using the corresponding procedure described in Section 1.3.2 read

\[
C_1(\lambda, \mathbf{y}) = -\sin(\lambda_1 y_0), \quad D_1(\lambda, \mathbf{y}) = \cos(\lambda_1 y_0),
\]

and so \( C_2(\lambda, \mathbf{y}) \) and \( D_2(\lambda, \mathbf{y}) \) follow as

\[
\begin{bmatrix} C_2(\lambda, \mathbf{y}) \\ D_2(\lambda, \mathbf{y}) \end{bmatrix} = \mathcal{R}_{\lambda_2 y_1} \mathcal{S}_{s_1} \mathcal{R}_{\lambda_1 y_1} \begin{bmatrix} C_1(\lambda, \mathbf{y}) \\ D_1(\lambda, \mathbf{y}) \end{bmatrix}.
\]

Thus, the function \( \Theta_{\lambda, \mathbf{y}}(x) \) in the explicit form reads

\[
\Theta_{\lambda, \mathbf{y}}(x) = \begin{cases}
0, & x \in (-\infty, y_0], \\
\sin(\lambda_1(x - y_0)), & x \in [y_0, y_1], \\
\cos(\lambda_2(x - y_1)) \sin(\lambda_1 l_1) + s_1 \sin(\lambda_2(x - y_1)) \cos(\lambda_1 l_1), & x \in [y_1, y_2], \\
0, & x \in [y_2, \infty). 
\end{cases}
\]

In turn, the eigenvalues \( \{\lambda_n\}_{n=1}^{\infty} \) solve the equation

\[
\begin{bmatrix} \cos(\lambda_2(y_2 - y_1)) & \sin(\lambda_2(y_2 - y_1)) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & s_1 \end{bmatrix} \begin{bmatrix} \sin(\lambda_1(y_1 - y_0)) \\ \cos(\lambda_1(y_1 - y_0)) \end{bmatrix} = 0,
\]

or, alternatively

\[
(\sigma_2 - \sigma_1) \sin(\lambda_2 l_2 - \lambda_1 l_1) = (\sigma_2 + \sigma_1) \sin(\lambda_2 l_2 + \lambda_1 l_1).
\]

Therefore, we define the eigenvalues \( \lambda_1, \lambda_2, \cdots \) as the ordered sequence of positive roots of this equation. With allowance for Eq. (65), the function in Eq. (63) can be simplified

\[
\Theta_{\mathbf{y}, \lambda_n}(x) = \Theta_{\mathbf{y}, \lambda_n}(x) = \begin{cases}
0, & x \in (-\infty, y_0], \\
\sin(\lambda_1(x - y_0)), & x \in [y_0, y_1], \\
\frac{\sin(\lambda_1 l_1)}{\sin(\lambda_2 l_2)} \sin(\lambda_2(y_2 - x)), & x \in [y_1, y_2], \\
0, & x \in [y_2, \infty),
\end{cases}
\]

with \( \lambda = \lambda_n \).
**Flat $\sigma$.** If $\sigma_1$ is close to $\sigma_2$, so the function $\sigma$ is almost flat, the following approximation holds

$$\lambda_n \approx \lambda_n^0 = \frac{\pi n}{l_1/\sigma_1 + l_2/\sigma_2} \quad (67)$$

Using the Taylor series expansion, the next approximation $\lambda_n^1$ can also be obtained by solving the equation

$$\alpha \sin \left( \beta \left( \lambda_n^0 + \lambda_n^1 \right) \right) = \sin \left( \left( \lambda_n^0 + \lambda_n^1 \right) \left( \frac{l_2}{\sigma_2} + \frac{l_1}{\sigma_1} \right) \right), \quad \alpha = \frac{\sigma_2 - \sigma_1}{\sigma_2 + \sigma_1}, \quad \beta = \frac{l_2}{\sigma_2} - \frac{l_1}{\sigma_1},$$

or

$$\alpha \left[ \sin \left( \lambda_n^0 \beta \right) + \lambda_n^1 \beta \cos \left( \lambda_n^0 \beta \right) \right] \approx (-1)^n \lambda_n^1 \left[ \frac{l_2}{\sigma_2} + \frac{l_1}{\sigma_1} \right],$$

which yields

$$\lambda_n^1 \approx \frac{\alpha \sin \left( \beta \lambda_n^0 \right)}{(-1)^n \left[ \frac{l_2}{\sigma_2} + \frac{l_1}{\sigma_1} \right] - \alpha \cos \left( \beta \lambda_n^0 \right)}.$$

Therefore, in the next order of approximation we get $\lambda_n \approx \lambda_n^0 + \lambda_n^1$. Surprisingly, this formula works well even if $|\sigma_1 - \sigma_2|$ is sufficiently large. In Fig.1 the error in computing the first 30 eigenvalues using the approximations with one and two terms in the Taylor series expansion of $\sigma$ is presented using $\sigma_1 = 7$, $\sigma_2 = 0.7$, $l_1 = 1.2$ and $l_2 = 1$. The figure shows that the zero-order approximation $\lambda_n^0$ has poor quality, and the error in computing the first five eigenvalues is about 5-10 percent. However, the first-order approximation is much more accurate and provides about one percent difference in the worst case (the 3rd and 6th eigenvalues, respectively). In any case, those approximations can be used as a good initial guess when solving Eq. (65) numerically. Note, that it is possible to derive similar approximations for the general multi-layer case in Eq. (40).

![Figure 1: The error of zero and first order approximations in solving Eq. (65) as compared with the numerical solution of Eq. (65) when $\sigma_1 = 7$, $\sigma_2 = 0.7$, $l_1 = 1.2$, $l_2 = 1$.](image-url)
2.2 Model with three layers and continuous spectrum

Here we consider the problem Eq. (14), Eq. (20) with $N = 2$, so the corresponding vectors $\mathbf{y}$ and $\mathbf{\sigma}$ are

$$\mathbf{y} = (y_0, y_1)^T, \quad \mathbf{\sigma} = (\sigma_-, \sigma_1, \sigma_+)^T.$$  

Using the result in Eq. (46), the function $u_\lambda(x|x_0)$ can be represented as

$$u_\lambda(x|x_0) = \frac{1}{2\lambda} \begin{bmatrix} \lambda_- e^{-\lambda_- |x-x_0|} \\ \lambda_1 e^{-\lambda_1 |x-x_0|} \\ \lambda_+ e^{-\lambda_+ |x-x_0|} \end{bmatrix}^T \mathbf{1}_{x|y} + \mathbf{1}_{y_0<x<y_1} \begin{bmatrix} \cosh(\lambda_1(x-y_0)) \\ \sinh(\lambda_1(x-y_0)) \end{bmatrix}^T \mathbf{G}_\lambda(\mathbf{\sigma_-}, \mathbf{\sigma_1}|x_0-y_0) \mathbf{1}_{x|y_0} \quad (69)$$

$$+ \begin{bmatrix} \mathbf{1}_{x<y_0} \\ \mathbf{1}_{x>y_1} \end{bmatrix} \begin{bmatrix} e^{-\lambda_- x} \\ 0 \end{bmatrix} + \mathbf{1}_{y_0<x<y_1} \begin{bmatrix} \cosh(\lambda_1(x-y_0)) \\ \sinh(\lambda_1(x-y_0)) \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ -s_- & 0 \end{bmatrix} \begin{bmatrix} C_- \\ D_+ \end{bmatrix},$$

while the constants $C_-$ and $D_+$ read

$$\begin{bmatrix} C_- \\ D_+ \end{bmatrix}(\lambda) = -\frac{\mathbf{\Lambda}^*(\lambda)}{\det \mathbf{\Lambda}^*(\lambda)} \left( \mathbf{S}_{s_1} \mathbf{H}_{\lambda_1 l_1} \mathbf{G}_\lambda(\mathbf{\sigma_-}, \mathbf{\sigma_1}|x_0-y_0) \mathbf{1}_{x_0|y_0} + \mathbf{G}_\lambda(\mathbf{\sigma_1}, \mathbf{\sigma_+}|x_0-y_1) \mathbf{1}_{x_0|y_1} \right),$$  

$$\mathbf{\Lambda}(\lambda) = \mathbf{S}_{s_1} \mathbf{H}_{\lambda_1 l_1} \mathbf{S}_{s_-} = \begin{bmatrix} \cosh(\tilde{\lambda}_1 l_1) & s_- \sinh(\tilde{\lambda}_1 l_1) \\ s_1 \sinh(\tilde{\lambda}_1 l_1) & \frac{s_-}{s_1} \cosh(\tilde{\lambda}_1 l_1) \end{bmatrix}.$$  

The determinant $\det \mathbf{\Lambda}^*(\lambda)$ can be found explicitly

$$\det \mathbf{\Lambda}^*(\lambda) = (s_1 + s_-) \sinh(\tilde{\lambda}_1 l_1) + (1 + s_-) \cosh(\tilde{\lambda}_1 l_1).$$  

Moreover, Eq. (39) can also be solved in closed form which yields

$$\tanh(\tilde{\lambda}_1 l_1) = \frac{\sigma_1(\sigma_- + \sigma_+)}{\sigma_1^2 + \sigma_- \sigma_+} = \frac{s_- + 1/s_1}{1 + s_-/s_1}.$$  

From here we find the explicit representation for the eigenvalues $\lambda_n^\pm$

$$\lambda_n^\pm = \frac{\sigma_1^2}{l_1^2} \left( \chi^2 - \pi^2 n^2 \pm 2i\pi \chi \right), \quad \chi = -\arctan \left( \frac{s_- + 1/s_1}{1 + s_-/s_1} \right).$$  

Thus, we have managed to explicitly find all the components in Eq. (59), which makes it possible to construct an oscillating integral transform in this case. However, since the final result looks bulky, and we don’t use it in this paper, it will be published elsewhere.

3 MHE for problems with curvilinear boundaries

The previous two sections cover the construction of a particular type of integral transform - the OIT. Although these sections are a bit technical, we cannot omit them since OIT is new and has never been reported in the literature. However, once these results have been obtained and described, we can concentrate on solving some more practical problems which regularly occur in various areas of science and engineering. In particular, in this section, we solve MHE defined at the space domain with curvilinear (time-dependent) boundaries by using the corresponding OIT, constructed in Eq. (27) and Eq. (42). A similar OIT can also be derived for problems with mixed spectrum by using a general representation Eq. (59), but we don’t consider it in this paper.
3.1 Time-dependent oscillating Brownian motion

The oscillating Brownian motion (OBM) is described in detail in (Keilson and Wellner). Let \( \sigma(x) = \sigma_+, x \geq 0, \sigma(x) = \sigma_-, x < 0 \) and let \( m(dx) = 2\sigma^{-2}(x)dx \) be speed measure. In (Ito and McKeen) the authors construct a diffusion process \( Y_t \) with an arbitrary speed measure \( m \) (which has support at an interval \( Q \subseteq \mathbb{R} \)) from Brownian motion and show that \( Y_t \) is a strong Markov, conservative diffusion process on \( Q \). They call it the OBM with the desired speed measure \( m \). The process \( Y_t \) 'oscillates' in a sense that it behaves like a Brownian motion which changes variance parameter each time it crosses zero. In the well-known special case \( \sigma_+ = \sigma_\alpha = \sigma, Y_t \) is an ordinary Brownian motion with variance \( \sigma^2 \). Another known case is \( \sigma_+ = \sigma, \sigma_- = \infty \), and then \( Y_t \) is a reflecting Brownian motion.

Applications of the OBM in finance could be found in (Lejay and Pigato; Gairat and Shcherbakov) who proposed a threshold local volatility model with piecewise constant volatility and drift (the geometric oscillating Brownian motion (GOBM)) \(^1\). This model is an instance of the tiled volatility model considered in (Lipton and Sepp, 2011) and later extended in (Itkin and Lipton, 2018). A detailed description of the mathematical aspects of this problem and solutions can also be found in (Lipton, 2018).

In the GOBM model, a fixed threshold separates two regimes for the prices. The volatility and the drift parameter can assume two possible values, according to the stock price position, above or below the threshold. Thus, \( \sigma_- \) is the volatility below the threshold, \( \sigma_+ \) - the volatility above the threshold, and similarly \( b_-, b_+ \) for the drift. It turns out that such model accounts for the leverage effect when \( \sigma_- > \sigma_+ \).

In this case, when prices are low, volatility increases, consistently with what is observed on empirical financial data. As mentioned in (Lejay and Pigato), a motivation for considering such price dynamics coming from a different viewpoint is given in (Ankirchner et al.). It is shown in (Ankirchner et al.) that the GOBM describes the price dynamics corresponding to the optimal strategy for a manager who can control, in a stylized setting, the volatility of the value of a firm, getting bonus payments when the value process performs better than a reference index.

In terms of financial mathematics, the GOBM model is a local volatility model of the form

\[
\frac{dS_t}{S_t} = \mu(S_t)dt + \sigma(S_t)S_t dW_t, \tag{74}
\]

where \( S_t \) is the stock price, \( T \) is the time, \( \mu(S_t) \) is the drift, \( \sigma(S_t) \) is the local volatility function, \( W_t \) is the standard Brownian motion, and the drift and local volatility are defined as

\[
\sigma(x) = \begin{cases} 
\sigma_-, & x < m, \\
\sigma_+, & x \geq m.
\end{cases} \quad \text{and} \quad \mu(x) = \begin{cases} 
\mu_-, & x < m, \\
\mu_+, & x \geq m.
\end{cases} \tag{75}
\]

In the GOBM model, the threshold \( m \) is constant, yet the time-dependent version of the model has not been considered in the literature.

Another step in this direction has been done in (Friz et al.) who proposed the Step Stochastic Volatility Model (SSVM). The authors were looking for a (possible) simple (without adding jumps or non-Markovian rough fractional volatility dynamics) modification of a class of stochastic volatility models to be capable of producing extreme short-dated implied volatility skew\(^2\). They found that introducing a leverage effect by making volatility discontinuous at the money (by multiplying the (backbone) stochastic volatility with distinct factors, say \( \sigma_- \) and \( \sigma_+ \) depending on when the considered option is out-of-the-money or in-the-money) achieves this goal, and the implied skew generated by such model explodes as \( T^{-1/2} \), (Pigato).

---

\(^1\)Note that in (Goovaerts et al.) the authors study a class of tractable diffusions suitable for the model’s primitives of interest rates. They consider scalar diffusions with scale \( s(x) \) and speed \( m(x) \) densities discontinuous at the level \( x^* \), which they call a family of Self Exciting Threshold diffusions. Those processes can also be considered as examples of the OBM.

\(^2\)As the authors mentioned, much of the recent success of rough SVMs is due to the (desired) implied skew blow up at rate \( T^{-H-1/2} \), \( T \) is the time to maturity, and the Hurst parameter \( H \in (0,1/2] \) quantifies the roughness of the volatility process. The blowup can be at most of the order \( T^{-1/2} \) which is a model-free consequence of no-arbitrage, (Lee, 2002).
Going from finance to physics, diffusion in nonuniform media is another example of the OBM or skew Brownian motion playing an essential role in practice; see, e.g., in (Sattin; Andreucci et al.) and references therein. In the presence of substantial inhomogeneities, even the hydrodynamic Fokker-Planck equation limit may become inaccurate and mask some features of the real solution, as computed from the Master Equation. For example, the author considers an experimental test - a study of tracer diffusion between gelatin solutions with different viscosity or different effective diffusivity. The width of the interface between the two solutions is minimal and can be put to zero. Hence, the whole system may be modeled as two regions with different jumping lengths - the same setting as regions with different variance in the mathematical finance context. Again, the relevant domain is split by a constant interface $x = 0$. However, this problem can be further generalized by considering time-dependent boundaries.

For the financial models mentioned above, it seems more plausible that since the boundary (threshold) for the oscillating volatility should be close to the at-the-money (ATM) level, it would be beneficial to consider it a function of time. In this paper, we concentrate on MHEs. However, our method can be further generalized to the problems that can be reduced to the MHE by transformations. In many cases, even the time-dependent models can be reduced to the MHE (or to the Multilayer Bessel equations, (Decamps et al.)). However, the boundaries inherent to those problems and constant in the original variables become time-dependent in the new variables; see various examples in (Itkin et al., 2021a). Therefore, our framework covers both problems with time-dependent interfaces, as well as models with time-dependent parameters.

With allowance for all that, we consider a problem with two layers and time-dependent interface $y(\tau)$ where $\tau = T - t$ is the backward time

$$
\frac{\partial u}{\partial \tau} = \frac{\partial}{\partial x} \left( \langle 1 \rangle_{x|y(\tau)} \sigma^2 \frac{\partial u}{\partial x} \right), \quad \sigma = (\sigma_-, \sigma_+) \tag{76}
$$

$$
\lim_{x \to \pm \infty} u(\tau, x) = 0, \quad u(0, x) = \delta(x - x_0),
$$

$$
\lim_{x \to y(\tau)-0} u(\tau, x) = \lim_{x \to y(\tau)+0} u(\tau, x), \quad \lim_{x \to y(\tau)-0} \sigma^- u_x'(\tau, x) = \lim_{x \to y(\tau)+0} \sigma^+ u_x'(\tau, x).
$$

The solution of this problem is given in Appendix C and reads

$$
u(\tau, x) = \left\langle 1 \right|_{x|y(0)} P_\sigma(x - y(\tau), \tau, |x_0 - y(\tau), 0) \right\rangle + \int_0^\tau \psi(s) \left\langle 1 \right|_{-1} P_\sigma(x - y(\tau), \tau|y(s) - y(\tau), s) \right\rangle ds \tag{77}
$$

$$
+ \int_0^\tau \varphi(s) \left\langle 1 \right|_{-1} \eta_\sigma(x - y(\tau), \tau|y(s) - y(\tau), s) \right\rangle ds,
$$

where functions $P_\sigma(z, \tau|\zeta, s)$, $\eta_\sigma(z, \tau|\zeta, s)$ are defined in Eq. (C.10), and $\psi(\tau) = \Phi(s) + y'(s)\varphi(s)$. The functions $\varphi(\tau)$ and $\Phi(\tau)$ are introduced in Eq. (C.4) and are yet unknown values of the solution and its spatial gradient at the interface boundary $y(\tau)$. However, differentiating Eq. (77) on $x$ and substituting $x \to y(\tau) - 0$ gives rise to a system of linear Volterra integral equations of the second kind for $\varphi(\tau)$ and $\Phi(\tau)$

$$
\varphi(\tau) = \langle 1 \rangle_{x|y(0)} P_\sigma^1(0, \tau, |x_0 - y(\tau), 0) \rangle + \int_0^\tau \psi(s) \left\langle 1 \right|_{-1} P_\sigma^1(0, \tau|y(s) - y(\tau), s) \right\rangle ds \tag{78}
$$

$$
+ \int_0^\tau \varphi(s) \left\langle 1 \right|_{-1} \eta_\sigma^1(0, \tau|y(s) - y(\tau), s) \right\rangle ds,
$$

$$
\frac{\Phi(\tau)}{\sigma^2} = \langle 1 \rangle_{x|y(0)} \frac{\partial P_\sigma^1}{\partial x}(0, \tau, |x_0 - y(\tau), 0) \rangle + \int_0^\tau \psi(s) \left\langle 1 \right|_{-1} \frac{\partial P_\sigma^1}{\partial x}(0, \tau|y(s) - y(\tau), s) \right\rangle ds
$$

3Alternatively, due to the last boundary condition in Eq. (76) we can use $x \to y(\tau) + 0$. 

\[ + \int_0^\tau \varphi(s) \left[ \begin{bmatrix} -1 \end{bmatrix}, \frac{\partial \eta_1^i}{\partial x}(0, \tau|y(s) - y(\tau), s) \right] ds. \]

where the superscript in \( P^i \), \( \eta^i \) denotes a transposed \( i \)-th column, e.g., \( P^i = \begin{bmatrix} P_{1i}^i \end{bmatrix}^\top \). This system can be solved numerically; see a detailed discussion in (Itkin et al., 2021a). Therefore, the proposed method is semi-analytical in a sense that the solution Eq. (77) is explicit but depends on two functions that should be found numerically.

Note, that in the case \( y(\tau) = \text{const} \), from Eq. (C.10) we have \( \eta_0^i(0, \tau|0, s) = 0 \),

\[ P_\sigma(0, \tau|0, s) = \frac{1}{2\sqrt{\pi}(\tau - s)} \begin{bmatrix} \frac{1}{\sigma_-} & 0 & 1 + \Sigma \ 0 & \frac{1}{\sigma_+} & 1 + \Sigma \end{bmatrix}, \]

and, therefore,

\[ 2\sqrt{\pi}(\tau - s) \begin{bmatrix} 1 & 1 + \Sigma & 1 - \Sigma \ 1 & 1 - \Sigma & 1 - \Sigma \end{bmatrix} = \frac{1}{\sigma_-} - \frac{1}{\sigma_+} = 0. \]

Hence, \( \varphi(\tau) \) in Eq. (78) becomes an explicit function (since two integrals in the RHS vanish). In a similar way it can be shown that \( \frac{\partial P_\sigma^1}{\partial x}(0, \tau|0, s) = 0 \). Since \( \varphi(\tau) \) is already known, the function \( \Phi(\tau) \) also becomes known from the second line of Eq. (78). Thus, in this case, we don’t need to solve the Volterra equations, and the solution is expressed in a semi-closed form.

Also, note that from computational point of view, it is convenient to re-write Eq. (78) in variables \( \phi(\tau), \psi(\tau) \) to get

\[ \varphi(\tau) = \langle 1_{x_0|y(0)}, P_\sigma^1(0, \tau|x_0 - y(\tau), 0) \rangle + \int_0^\tau \psi(s) K_p(s, \tau) ds + \int_0^\tau \varphi(s) K_p(s, \tau) ds, \]

\[ \psi(\tau) - y'(\tau) \varphi(\tau) \quad \text{or} \quad \psi(\tau) = \langle 1_{x_0|y(0)}, \frac{\partial P_\sigma^1}{\partial x}(0, \tau|x_0 - y(\tau), 0) \rangle + \int_0^\tau \psi(s) C_p(s, \tau) ds + \int_0^\tau \psi(s) C_p(s, \tau) ds, \]

\[ K_p(s, \tau) = \begin{bmatrix} -1 \end{bmatrix}, P_\sigma^1(0, \tau|y(s) - y(\tau), s) \right] \]

\[ K_p(s, \tau) = \begin{bmatrix} -1 \end{bmatrix}, \frac{\partial \eta_1^i}{\partial x}(0, \tau|y(s) - y(\tau), s) \right] \]

\[ C_p(s, \tau) = \begin{bmatrix} -1 \end{bmatrix}, \frac{\partial \eta_1^i}{\partial x}(0, \tau|y(s) - y(\tau), s) \right] \]

When solving this system of equations using quadratures, e.g., the Simpson rule, the matrix in the RHS becomes block-triangular. Therefore, this system can be solved with complexity \( O(M^2) \) where \( M \) is the number of computational nodes in the time-space \( \tau \in [0, T] \), while providing the fourth-order of approximation \( O(M^{-4}) \).

The representation of the Volterra equations in the form of Eq. (79) immediately reveals the fact that for the linear boundary \( y(\tau) = a + b\tau, a, b - \text{const} \), Eq. (79) can be solved by using the Laplace transform. Indeed, taking the Laplace transform of both parts of Eq. (79) and using the convolution theorem yields a new system of algebraic equations

\[ \tilde{\varphi}(p) = A(p) + \tilde{\psi}(p) \tilde{K}_p(p) + \tilde{\varphi}(p) \tilde{K}_p(p) \]

\[ \tilde{\psi}(p) - b\tilde{\varphi}(p) \quad \text{or} \quad \tilde{\psi}(p) = B(p) + \tilde{\psi}(p) \tilde{C}_p(p) + \tilde{\psi}(p) \tilde{C}_p(p) \]

where the bar over the function \( f(\tau) \) means its Laplace transform \( \tilde{f}(p) = \int_0^\infty f(\tau)e^{-\tau p} d\tau \), and

\[ A(p) = \int_0^\infty e^{-\tau p} \langle 1_{x_0|y(0)}, P_\sigma^1(0, \tau|x_0 - y(\tau), 0) \rangle d\tau, \]

\[ B(p) = \int_0^\infty e^{-\tau p} \langle 1_{x_0|y(0)}, \frac{\partial P_\sigma^1}{\partial x}(0, \tau|x_0 - y(\tau), 0) \rangle d\tau, \]

\[ \tilde{\psi}(p) = \begin{bmatrix} -1 \end{bmatrix}, P_\sigma(0, \tau|0, s) \right] \]

\[ \tilde{\varphi}(p) = \begin{bmatrix} -1 \end{bmatrix}, \frac{\partial \eta_1^i}{\partial x}(0, \tau|0, s) \right] \]

\[ \tilde{\psi}(p) = \begin{bmatrix} -1 \end{bmatrix}, \frac{\partial \eta_1^i}{\partial x}(0, \tau|0, s) \right] \]

\[ \tilde{\varphi}(p) = \begin{bmatrix} -1 \end{bmatrix}, \frac{\partial \eta_1^i}{\partial x}(0, \tau|0, s) \right] \]
This problem is a generalization of the one considered in Section 3.2 for the case when the number of layers \( N > 2 \). When the boundaries between all layers (the interfaces) are constant, this problem was considered, e.g., in (Hickson et al.) and solved numerically. As the authors mention, diffusion processes through a multilayered material are of interest for various applications, including industrial, biological, electrical, and environmental areas. Some industrial applications include annealing steel coils, the performance of semiconductors and electrodes, geological profiles, and measuring greenhouse gas emissions from soil surfaces. Biological applications include determining the effectiveness of drug carriers inserted into living tissue, the probing of biological tissue with infrared light, and analyzing the heat production of muscle, see the corresponding references in (Hickson et al.) and the discussion.

When the interfaces are moving, a general semi-analytical method for solving these problems was presented in (Itkin et al., 2021b). Here, as explained in the Introduction, we develop an alternative (but similar) approach.

The problem under consideration is as follows. Given the time \( t \in [0, \infty) \), the space coordinate \( x \in \mathbb{R} \), and a function \( u(t, x) : \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}^+ \), \( u \in C^2 \). Suppose that the whole domain \( \Omega = \mathbb{R} \) could be split into \( N + 2 \) non-overlapping layers \(-\infty < y_0(t) < \ldots < y_N(t) < \infty, \forall t \geq 0 \), i.e. \( \Omega = \bigcup_{i=0}^{N} \Omega_i \), where each layer is a curvilinear strip as in Eq. (1). Let us consider a diffusion (thermal conductivity) process in \( \mathbb{R} \) such that the diffusion coefficient \( \sigma \) is a piecewise constant function of \( x \), i.e.

\[
\sigma(x) = \begin{cases} 
\sigma_i, & \forall x \in [y_{i-1}(t), y_i(t)], \quad i = 1, \ldots, N, \\
0, & x \in (-\infty, y_0(t)], \\
0, & x \in [y_N(t), \infty),
\end{cases}
\]

where \( \sigma_i > 0, \forall i = 1, \ldots, N \) are constants. Suppose the evolution of this diffusion process is described by the heat equation

\[
\frac{\partial u}{\partial \tau} = \frac{\partial}{\partial x} \left( \sigma^2 \frac{\partial u}{\partial x} \right) + g(\tau, x), 
\]

\[
u(\tau, y_0(\tau)) = u(\tau, y_N(\tau)) = 0, \]

\[
u(\tau, x) = f(x), \quad \sigma = [0, \sigma_0, \ldots, \sigma_N, 0]^T
\]

where \( f(x), g(\tau, x) \in \mathbb{R} \) are given functions.

\[
\mathcal{K}_\psi(s, \tau) = \int_0^\infty e^{-pt} \langle [\psi(s, \eta)], \mathcal{P}_s^1(0, \tau | -\tau, 0) \rangle \, d\tau, \quad \mathcal{K}\psi(s, \tau) = \int_0^\infty e^{-pt} \langle [\psi(s, \eta)], \mathcal{P}_s^1(0, \tau | -\tau, 0) \rangle \, d\tau,
\]

\[
\mathcal{C}_\psi(s, \tau) = \int_0^\infty e^{-pt} \langle [\psi(s, \eta)], \mathcal{P}_s^1(0, \tau | -\tau, 0) \rangle \, d\tau, \quad \mathcal{C}_\psi(s, \tau) = \int_0^\infty e^{-pt} \langle [\psi(s, \eta)], \mathcal{P}_s^1(0, \tau | -\tau, 0) \rangle \, d\tau.
\]

Solving Eq. (80) we obtain

\[
\varphi(p) = \frac{\sigma^2 (A(p)\mathcal{C}_\psi(p) - B(p)\mathcal{K}_\psi(p)) - A(p)}{Z(p)}, \quad \psi(p) = \frac{B(p)(\mathcal{K}_\psi(p) - 1)\sigma^2 - A(p)(b + \mathcal{C}_\psi(p))\sigma^2}{Z(p)}.
\]

Then application of the inverse Laplace transform (taken numerically) solves the problem.

### 3.2 MHE with piecewise constant coefficients and time-dependent interfaces

This problem is a generalization of the one considered in Section 3.2 for the case when the number of layers \( N > 2 \). When the boundaries between all layers (the interfaces) are constant, this problem was considered, e.g., in (Hickson et al.) and solved numerically. As the authors mention, diffusion processes through a multilayered material are of interest for various applications, including industrial, biological, electrical, and environmental areas. Some industrial applications include annealing steel coils, the performance of semiconductors and electrodes, geological profiles, and measuring greenhouse gas emissions from soil surfaces. Biological applications include determining the effectiveness of drug carriers inserted into living tissue, the probing of biological tissue with infrared light, and analyzing the heat production of muscle, see the corresponding references in (Hickson et al.) and the discussion.

When the interfaces are moving, a general semi-analytical method for solving these problems was presented in (Itkin et al., 2021b). Here, as explained in the Introduction, we develop an alternative (but similar) approach.

The problem under consideration is as follows. Given the time \( t \in [0, \infty) \), the space coordinate \( x \in \mathbb{R} \), and a function \( u(t, x) : \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}^+ \), \( u \in C^2 \). Suppose that the whole domain \( \Omega = \mathbb{R} \) could be split into \( N + 2 \) non-overlapping layers \(-\infty < y_0(t) < \ldots < y_N(t) < \infty, \forall t \geq 0 \), i.e. \( \Omega = \bigcup_{i=0}^{N} \Omega_i \), where each layer is a curvilinear strip as in Eq. (1). Let us consider a diffusion (thermal conductivity) process in \( \mathbb{R} \) such that the diffusion coefficient \( \sigma \) is a piecewise constant function of \( x \), i.e.

\[
\sigma(x) = \begin{cases} 
\sigma_i, & \forall x \in [y_{i-1}(t), y_i(t)], \quad i = 1, \ldots, N, \\
0, & x \in (-\infty, y_0(t)], \\
0, & x \in [y_N(t), \infty),
\end{cases}
\]

where \( \sigma_i > 0, \forall i = 1, \ldots, N \) are constants. Suppose the evolution of this diffusion process is described by the heat equation

\[
\frac{\partial u}{\partial \tau} = \frac{\partial}{\partial x} \left( \sigma^2 \frac{\partial u}{\partial x} \right) + g(\tau, x), 
\]

\[
u(\tau, y_0(\tau)) = u(\tau, y_N(\tau)) = 0, \]

\[
u(\tau, x) = f(x), \quad \sigma = [0, \sigma_0, \ldots, \sigma_N, 0]^T
\]

where \( f(x), g(\tau, x) \in \mathbb{R} \) are given functions.
To solve this problem we define the integral transform

$$\tilde{u}(\tau, \lambda) = \int_{-\infty}^{\infty} u(\tau, x) (1_{x|y(\tau)}, \Theta_{y(\tau), \lambda}(x)) dx,$$

(83)

where the basis function $\Theta_{y(\tau), \lambda}(x)$ is defined by Eq. (34) via Eq. (35). It is clear that

$$\tilde{u}(\tau, \lambda) = \sum_{i=1}^{N} \int_{y_{i-1}(\tau)}^{y_{i}(\tau)} u(\tau, x) \left[C_i(\lambda, y(\tau)) \cos \left(\frac{\lambda x}{\sigma_i}\right) + D_i(\lambda, y(\tau)) \sin \left(\frac{\lambda x}{\sigma_i}\right)\right] dx$$

(84)

$$= \sum_{i=1}^{N} \left\{C_i(\lambda, y(\tau)) \tilde{u}_i^C(\tau, \lambda) + D_i(\lambda, y(\tau)) \tilde{u}_i^S(\tau, \lambda)\right\},$$

$$\tilde{u}_i^C(\tau, \lambda) = \int_{y_{i-1}(\tau)}^{y_{i}(\tau)} u(\tau, x) \cos \left(\frac{\lambda x}{\sigma_i}\right) dx, \quad \tilde{u}_i^S(\tau, \lambda) = \int_{y_{i-1}(\tau)}^{y_{i}(\tau)} u(\tau, x) \sin \left(\frac{\lambda x}{\sigma_i}\right) dx.$$
where the authors also show that their approach favorably compares with the finite-difference method.

There exist various engineering and scientific problems involving heat conduction with moving boundaries schematically presented in Fig. 2. Among many others, let us consider a typical example of the freezing problem, *(Han; Jiji)*, which is such as freezing or melting for solar storage systems, high-temperature droplet evaporation, ablation, etc.

Again, this system can be solved numerically, as discussed in *(Itkin et al., 2021a)*. Thus, our method can be conventionally classified as semi-analytical in the sense that the solution in Eq. (88) is explicit but depends on two functions that should be computed numerically by solving the linear Volterra equations in Eq. (89). The corresponding solution, however, can be found very efficiently; see *(Itkin et al., 2021a)*, where the authors also show that their approach favorably compares with the finite-difference method.

### 3.3 Freezing and solidification

There exist various engineering and scientific problems involving heat conduction with moving boundaries such as freezing or melting for solar storage systems, high-temperature droplet evaporation, ablation, among many others. Let us consider a typical example of the freezing problem, *(Han; Jiji)*, which is schematically presented in Fig. 2.

![Figure 2: Heat conduction with moving boundary - a freezing problem: blue - the temperature profile, solid red - the boundaries, dashed red - the interphase (moving) boundary.](image)

We consider a rectangular region which is split into two subregions \( y(\tau) < x < y_+ \) and \( y_- < x < y(\tau) \) by an interphase boundary \( x = y(\tau) \). For example, suppose that we have liquid water to the right of the boundary and solid ice to the left. The boundary \( x = y(\tau) \) is a phase transition boundary which position...
is unknown yet but could be found based on some physics consideration. The temperature distribution in this two-phase region is governed by two heat equations, one for the solid phase and the other for the liquid phase. Our problem boils down to determining the moving boundary \( y(\tau) \) and the temperature distribution in both phases. However, first, we need to describe the physics of the process in more detail. Real geometries are much more complex than the one depicted in Fig. 2, which is produced based on the following simplifications:

- properties of each phase are uniform and remain constant;
- the effect of liquid phase motion due to changes in density is negligible;
- we consider just one-dimensional heat conduction;
- we neglect by any energy generation.

In addition, we assume that the water temperature at \( x \to y_+ \) is constant and equal to \( T_l \), and the ice temperature at \( x \to y_- \) is also constant and equal to \( T_s \). Thus, our system at both ends is connected to thermostats which support those constant temperatures. Also, we assume that water undergoes a phase change at a fixed temperature at the phase interface (the melting or freezing temperature) \( T_m \). However, the physical properties of both media are different. Therefore, the system with constant temperatures at both ends and phase transition at the interface can only exist if the interface moves in time.

Based on our assumptions, we can write two heat equations for the temperature \( T(\tau, x) \)

\[
\frac{\partial T}{\partial \tau} = \kappa_W \frac{\partial^2 T}{\partial x^2}, \quad y(\tau) < x < y_+, \quad (90)
\]

\[
\frac{\partial T}{\partial \tau} = \kappa_I \frac{\partial^2 T}{\partial x^2}, \quad y_- < x < y(\tau),
\]

where \( \kappa_W, \kappa_I \) are the corresponding thermal diffusivities. These equations have to be solved subject to the boundary conditions

\[
T(\tau, y_+) = T_l, \quad T(\tau, y_-) = T_s, \quad T(\tau, y(\tau) - 0) = T(\tau, y(\tau) + 0) = T_m. \quad (91)
\]

The last condition means continuity of temperature at the phase interface. As shown in [Jiji], a conservation of energy argument gives rise to one more boundary condition at the interface

\[
k_I \frac{\partial T}{\partial x} \bigg|_{x=y(\tau)-0} - k_W \frac{\partial T}{\partial x} \bigg|_{x=y(\tau)+0} = \rho_I L_h y'(\tau). \quad (92)
\]

Here \( L_h \) is the latent heat of fusion, \( \rho_I, \rho_W \) are the corresponding densities in two phases and \( k_I, k_W \) - their thermal conductivities. They are connected to \( \kappa_I, \kappa_W \) via the relationship \( \kappa_j = k_j/(\rho_j C_j) \), \( j = I, W \) where \( C_I, C_W \) are specific heat capacities. The Eq. (92) is the interface energy equation and is valid for both solidification and melting. However, for melting \( \rho_I \) in the RHS of Eq. (93) should be replaced with \( \rho_W \). Thus, in contrast to temperature, the temperature flux experiences a jump when crossing the phase interface, and the value of this jump is proportional to \( y'(\tau) \).

As discussed in more detail in Section 4, in some papers an alternative form of Eq. (93) is used, which can be obtained by assuming \( \rho_W C_W \approx \rho_I C_I \). Taking the values at temperature \( T = 0^\circ C \) and atmospheric pressure \( \rho_W = 997 \) kg/m\(^3\), \( \rho_I = 917 \) kg/m\(^3\), \( C_W = 4.22 \) kJ/kg/K, \( C_I = 2.09 \) kJ/kg/K, (Vargaftik; Keenan), one can see that this assumption is not very accurate. However, without loss of generality, below instead of Eq. (92) we use its simplified form

\[
\kappa_I \frac{\partial T}{\partial x} \bigg|_{x=y(\tau)-0} - \kappa_W \frac{\partial T}{\partial x} \bigg|_{x=y(\tau)+0} = \rho_I L y'(\tau), \quad (93)
\]

where \( L = L_h/C_a \), and \( C_a \) is an average value

\[
C_a = (\rho_W C_W + \rho_I C_I)/2. \quad (94)
\]
Finally, we need to set the initial state of the system. For instance, one can assume that at \( \tau = 0 \) the interface boundary is at \( x = y_\tau \), and, hence, \( T(0,x) = T_l \). Again, it is worth emphasizing that the law of motion \( y(\tau) \) is unknown in advance and should be determined together with the temperature profile by solving the described problem. However, this is standard for the so-called Stefan problems, see (Jiji) and references therein.

Thus, the stated problem can be solved by using the approach developed in (Itkin and Muravey, 2021). Indeed, in the left region (ice), temperature evolution is determined by the heat equation with given temperatures at the edges \( T_s, T_m \). Solving this problem and finding the temperature distribution is exactly equivalent (in financial terms) to pricing double barrier option (with barriers at \( x = y_- \) and \( x = y(\tau) \) and the payoff at \( \tau = 0 \) equal to \( T_l \) which pays rebates at hit equal to \( T_s \) and \( T_m \), respectively. Given the law \( y(\tau) \), this problem can be solved by the method developed in (Itkin and Muravey, 2021). Similarly, for the right region same problem is defined at the domain \( y(\tau) \leq x \leq y_+ \). Therefore, the whole algorithm could be as follows. We start with some initial guess for \( y(\tau) = y_0(\tau) \) and solve the above problems for the left and right regions. As a part of this solution we also obtain its gradients at \( x = y_0(\tau) - 0 \) and \( x = y_0(\tau) + 0 \). Then, substituting them into Eq. (93) and solving for \( y(\tau) \) yields its next approximation \( y_1(\tau) \). We proceed in the same way until the solution converges up to a given tolerance. This method is advantageous as compared with sophisticated numerical methods, e.g., described in (Hickson et al.), because it is uniform (no need for a complex construction of the finite-difference scheme at the moving interface) and semi-analytical, so fast.

However, in this paper, for the reasons explained in the Introduction, we solve this problem by using a slight modification of the method described in the previous section. First, here the problem has inhomogeneous boundary conditions, Eq. (91). Second, the matching condition for the first derivative contains a jump. Fortunately, the problem in Eq. (90), Eq. (91) can be reduced to that in Eq. (82), and the latter can be solved in a semi-explicit form via the OIT, see Appendix E.

We construct the solution of Eq. (90) as oscillating Fourier series with the terms

\[
1_{x<y(\tau)} \Theta_{y(\tau),\lambda_n(\tau),1}(x) + 1_{x>y(\tau)} \Theta_{y(\tau),\lambda_n(\tau),2}(x) = 1_{x<y(\tau)} \sin(x_n^-(\tau,x)) + 1_{x>y(\tau)} K_n(\tau) \sin(x_n^+(\tau,x)),
\]

\[
x_n^-(\tau,x) = \lambda_n(\tau)(x-y_-)/\sqrt{\kappa_I}, \quad l_-(\tau) = (y(\tau) - y_-)/\sqrt{\kappa_I}, \quad K_n(\tau) = \frac{\sin[\lambda_n(\tau)l_-(\tau)]}{\sin[\lambda_n(\tau)l_+(\tau)]}, (95)
\]

\[
x_n^+(\tau,x) = \lambda_n(\tau)(y_+ - x)/\sqrt{\kappa_W}, \quad l_+(\tau) = (y_+ - y(\tau))/\sqrt{\kappa_W}.
\]

The spectrum \( \lambda_n(\tau), \ n = 1,\ldots, \) is defined as an ordered sequence of positive roots of the equation

\[
(\sqrt{\kappa_W} - \sqrt{\kappa_I}) \sin[\lambda_n(\tau)(l_+(\tau) - l_-(\tau))] = (\sqrt{\kappa_W} + \sqrt{\kappa_I}) \sin[\lambda_n(\tau)(l_+(\tau) + l_-(\tau))]. (96)
\]

This equation has to be solved numerically. However, in some cases an analytic or asymptotic solution is also possible. For instance, at \( \tau = 0 \) we have \( y(0) = y_- \) and, hence

\[
\lambda_n(0) = 2\pi n/l_+(0), \quad n \in \mathbb{Z}. (97)
\]

The factor of 2 is required in order to satisfy the boundary conditions.

It can be observed that with this value of \( \lambda_n(0) \) the value of \( K_n(0) \) becomes an indeterminate form since it is 0/0. By using L’Hospital’s rule and the fact that \( \lambda'_n(0) = 0 \) (which can be proved by differentiating Eq. (96) with respect to \( \tau \) and taking the limit \( \tau \to 0 \)), we have

\[
K_n(0) = -\sqrt{\kappa_W}/\sqrt{\kappa_I} \neq 0. (98)
\]

An asymptotic solution can also be constructed by having in mind that, e.g., at atmospheric pressure and at \( T = 0^\circ C \) we have \( \kappa_W \approx 0.13 \text{ mm}^2/\text{s}, \kappa_I \approx 1.02 \text{ mm}^2/\text{s}, \) (Vargaftik; Keenan), so \( \kappa_W/\kappa_I \ll 1 \). If \( l_+(\tau) \) and \( l_-(\tau) \) are of the same order of magnitude, then Eq. (96) can be transformed to

\[
0 = \sin[\lambda_n(\tau)(l_+(\tau) + l_-(\tau))] + \sin[\lambda_n(\tau)(l_+(\tau) - l_-(\tau))] = 2\sin[\lambda_n(\tau)l_+(\tau)]\cos[\lambda_n(\tau)l_-(\tau)],
\]
with the obvious solutions\footnote{Again, we take period $2\pi n$ rather than $\pi n$ to correctly obey the boundary conditions.}

$$
\lambda_{n,1}(\tau) = \pi n/l_+(\tau), \quad \lambda_{n,2}(\tau) = (\pi/2 + 2\pi n)/l_-(\tau), \quad \tau > 0, \ n \in \mathbb{Z}.
$$

(99)

Substitution of $\lambda_{n,1}(\tau)$ into the definition of $K_n(\tau)$ makes $K_n(\tau)$ singular, so $\lambda_{n,1}(\tau)$ has to be excluded. Therefore, in this case the asymptotic solution is given by $\lambda_{n,2}(\tau)$. If, however, $l_-(\tau) \ll l_+(\tau)$, then the asymptotic solution is given by $\lambda_{n,1}(\tau)$.

Once the eigenvalues $\lambda_n(\tau)$ are found, the temperature $T(\tau, x)$ can be represented as

$$
T(\tau, x) = \eta(\tau, x) + \sum_{n=1}^{\infty} \frac{S_n(\tau) + R_n(\tau)}{N_n(\tau)} \mathcal{M}(\tau, x),
$$

(100)

$$
\mathcal{M}(\tau, x) = \begin{cases}
1_{x<y(\tau)} \sin[x_n^-(\tau, x)] + 1_{x>y(\tau)} K_n(\tau) \sin[x_n^+(\tau, x)], & x \neq y(\tau), \\
\frac{1}{2} \left[ s^+(y(\tau)) \sin[x_n^-(\tau, y(\tau))] + s^-(y(\tau)) K_n(\tau) \sin[x_n^+(\tau, y(\tau))] \right], & x = y(\tau),
\end{cases}
$$

where $s^\pm(y(\tau)) = 1 + \delta_{y(\tau)} \mp y(\tau)$.

After some tedious algebra (which we partly present in Appendix E, but omit other details) the functions $R_n(\tau)$, $S_n(\tau)$ and $N_n(\tau)$ can be represented in the form (the terms in the below definitions slightly differ from those in Eq. (E.14) since for convenience we regrouped them a bit)

$$
N_n(\tau) = \int_{y_-}^{y(\tau)} \Theta^2_{y(\tau), n} (\xi) d\xi + \int_{y(\tau)}^{y_+} \Theta^2_{y(\tau), n} (\xi) d\xi = \frac{1}{2} l_-(\tau) \sqrt{\kappa_I} + \frac{1}{2} K_2^2(\tau) l_+(\tau) \sqrt{\kappa_W},
$$

$$
S_n(\tau) = e^{-\tau \lambda_n^2(\tau)} \lambda_n(\tau) \sqrt{\kappa_I} (T_I - T_s) - \frac{1}{\lambda_n^2(\tau)} \left[ B_-(\tau) \kappa_I \Theta y(\tau), n(1)(y(\tau)) - B_+(\tau) \kappa_W \Theta y(\tau), n(2)(y(\tau)) \right]
$$

$$
+ (T_m - T_I) e^{-\tau \lambda_n^2(\tau)} \left[ \sqrt{\kappa_I} \cos (x^- (\tau, y(0))) + \sqrt{\kappa_W} K_n(\tau) \cos (x^+ (\tau, y(0))) \right]
$$

$$
R_n(\tau) = \int_0^\tau e^{-(\tau - s) \lambda_n^2(\tau)} \left[ \Phi(s) \Theta y(\tau), n(1)(s) + B_-(s) \kappa_I \Theta y(\tau), n(1)(y(s)) - B_+(s) \kappa_W \Theta y(\tau), n(2)(y(s)) \right] ds.
$$

To improve their convergence, we add and subtract $B_-(\tau) \kappa_I \sin[x_n^- (\tau, y(\tau))]$ and $B_+(\tau) \kappa_W K_n(\tau) \sin[x_n^+ (\tau, y(\tau))]$ under the last integral to obtain

$$
R_n(\tau) = \int_0^\tau e^{-(\tau - s) \lambda_n^2(\tau)} \left\{ \Phi(s) \left[ \sin[x_n^- (\tau, y(s))] - K_n(\tau) \sin[x_n^+ (\tau, y(s))] \right]
$$

$$
+ B_-(s) \kappa_I \sin[x_n^- (\tau, y(s))] - B_-(\tau) \kappa_I \sin[x_n^- (\tau, y(\tau))] - B_+(s) \kappa_W K_n(\tau) \sin[x_n^+ (\tau, y(s))] + B_+(\tau) \kappa_W K_n(\tau) \sin[x_n^+ (\tau, y(\tau))] \right\} ds
$$

$$
+ \int_0^\tau e^{-(\tau - s) \lambda_n^2(\tau)} \left[ B_-(\tau) \kappa_I \sin[x_n^- (\tau, y(\tau))] - B_+(\tau) \kappa_W K_n(\tau) \sin[x_n^+ (\tau, y(\tau))] \right] ds.
$$

The last integral can be computed analytically while we move the result to the definition of $S_n$. Hence, finally this yields

$$
S_n(\tau) = e^{-\tau \lambda_n^2(\tau)} \lambda_n(\tau) \left\{ \sqrt{\kappa_I} (T_I - T_s) + (T_m - T_I) \left[ \sqrt{\kappa_I} \cos (x^- (\tau, y(0))) + \sqrt{\kappa_W} K_n(\tau) \cos (x^+ (\tau, y(0))) \right] \right\}
$$

(102)
Thus, from Eq. (100) we obtain
\[
\tau \text{ equality follows from Eq. (97)). Also, at }
\tau \text{ Behavior of the solution at small }
\tau \Phi(\tau) = \frac{1}{\lambda_n(\tau)} \left[ \frac{B^+}{\kappa_I} \sin(x_n^-) - B^- \kappa W K_n \sin(x_n^+) \right] \left\{ \Phi(s) \left[ \sin(x_n^-) - K_n \sin(x_n^+) \right] \right\}.
\]

It can be seen, that at \( \tau = 0, y(0) = y_- \) and from Eq. (102) we have \( R_n(0) = 0 \). It turns out that \( S_n(0) = 0 \) as well. Indeed, by definition \( x_n^-(0, y(0)) = 0, x_n^+(0, y(0)) = \lambda_n(0) l_+ = 2\pi n, n \in \mathbb{Z} \) (the last equality follows from Eq. (97)). Also, at \( \tau = 0 \) we have \( T_s = T_1, B_-(0) = B_+(0) = 0 \) and \( K_n(0) \) is given by Eq. (98). Then a simple algebra yields the result. However, \( N_n(0) \neq 0 \). Indeed, Eq. (96) implies that at \( \tau = 0 \) the value of \( \lambda_n(0) \) is given by Eq. (97). Therefore, we have \( l_-(0) = 0, \lambda_{W,n}(0) = 2\pi n \) and thus,
\[
N_n(0) = \frac{1}{2} K_n^2(0) l_+(0) = \frac{1}{2} \frac{\kappa W}{\kappa_I} l_+(0) \neq 0.
\]

Thus, from Eq. (100) we obtain \( T(0,x) = \eta(0,x) \) and, in particular, \( T(0,y(0)) = T_1 \). This is the correct value since at \( \tau = 0, y(0) = y_- \) we have a uniform media and, hence, \( T_s = T_m = T_1 \).

Based on Eq. (E.12) one can see that the free boundary \( y(\tau) \) and the function \( \Phi(\tau) \) - the flux\(^6\) of the modified temperature \( T \) at the boundary \( x = y(\tau) \) (see the definition in Eq. (E.4)), can be found by solving the system of nonlinear Volterra integral equations of the second kind
\[
T_m = \eta(\tau, y(\tau)) + \frac{1}{2} \sum_{n=1}^{\infty} \frac{S_n(\tau) + R_n(\tau)}{N_n(\tau)} \left[ s^+(y(\tau)) \sin(x_n^-) + s^-(y(\tau)) K_n \sin(x_n^+) \right],
\]
\[
\Phi(\tau) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{S_n(\tau) + R_n(\tau)}{N_n(\tau)} \lambda_n(\tau) \left[ s^+(y(\tau)) \sqrt{\kappa_I} \cos(x_n^-) - s^-(y(\tau)) \sqrt{\kappa W} K_n \cos(x_n^+) \right].
\]

**Behavior of the solution at small \( \tau \).** It is worth noting that based on the definitions in Eq. (102), at small \( \tau \) we have
\[
S_n(\tau) \approx e^{-\tau^2(\tau)} \frac{\sqrt{\kappa_I}}{\lambda_n(\tau)} (T_1 - T_s), \quad N_n(\tau) \approx N_n(0), \quad R_n(\tau) \approx 0,
\]
while \( \lambda_n(\tau) \) is given by Eq. (97). Assuming \( x \neq y(\tau) \), overall, we obtain
\[
\Phi(\tau) = \frac{\sqrt{\kappa_I}(T_1 - T_s)}{2 N_n(0)} \sum_{n=1}^{\infty} e^{-\tau^2(\tau)} \left[ s^+(y(\tau)) \sqrt{\kappa_I} \cos(x_n^-) - s^-(y(\tau)) \sqrt{\kappa W} K_n \cos(x_n^+) \right],
\]
\[
x_n^-(\tau, y(\tau)) \approx \frac{2\pi n (x - y_-)}{(y(\tau) - y_-)}, \quad x_n^+ (\tau, x) \approx \frac{2\pi n (y_+ - x)}{(y_+ - y(\tau))}.
\]

\(^6\)In this section we denote the flux as \( \Phi \) while in previous ones that was the gradient. They differ just by a constant multiplier \( \kappa_W \) or \( \kappa_I \), and hopefully this change doesn’t bring any confusion.
We proceed with the observation that the sums in Eq. (104) could be expressed via Jacobi theta functions of the third kind, (Mumford et al., 1983). By definition
\[
\theta_3(z, \omega) = 1 + 2 \sum_{n=1}^{\infty} \omega^2 \cos(2nz).
\]

Using these definitions Eq. (104) can be re-written in the form
\[
\Phi(\tau) = \frac{\sqrt{R_I}(T_i - T_s)}{4N_n(0)} \left[ s^+(y(\tau)) \sqrt{R_I} \theta_3(z^-(x), \omega(\tau)) - s^-(y(\tau)) \sqrt{\kappa_W} K_n(0) \theta_3(z^-(x), \omega(\tau)) \right],
\]
\[
\omega(\tau) = e^{-4\pi^2 \tau / \ell^2(\tau)}, \quad z^-(x) = \pi \frac{x - y^-}{y(\tau) - y^-}, \quad z^+(x) = \pi \frac{y^+ - x}{y^+ - y^-}.
\]

A well-behaved theta function must have parameter \(|\omega| < 1\), (Mumford et al., 1983). In our case this condition holds for any \(\tau > 0\).

The RHS of Eq. (106) depends on \(x\) via functions \(z_{\pm}(x)\). It is easy to check that a similar representation holds for \(T(\tau, x)\) in Eq. (100), but now instead of \(\theta_3(z_{\pm}(x), \omega(\tau))\) we have to use \(\Theta_3(z_{\pm}(x), \omega(\tau))\), where
\[
\Theta_3(z_{\pm}(x), \omega(\tau)) = \int_0^x \theta_3(z_{\pm}(\xi), \omega(\tau)) d\xi.
\]

Since function \(\Theta_3(z, \omega)\) vanishes at \(z = 0\), the boundary conditions at \(x = y_{\pm}\) are satisfied. The result in Eq. (106) to some extent is not a surprise since it is well-known that the Jacobi theta function is the fundamental solution of the one-dimensional heat equation with spatially periodic boundary conditions, (Ohyama, 1995), and so is its integral in \(z\). The representation of the solution via the Jacobi theta function has been used in a series of authors’ papers devoted to pricing exotic financial derivatives (the barrier options), see, e.g., (Itkin et al., 2021a) and references therein. However, when the clock \(\tau\) runs ahead from \(\tau = 0\), the form of the solution in Eq. (100) becomes more complex.

### 3.3.1 Solution of Volterra equation

As shown in the previous section, the temperature \(T(\tau, x)\) across the whole two-phase domain (see Fig. 2) is determined by Eq. (100) if we know the trajectory \(y(\tau)\) of the interphase (moving) boundary as well as the flux \(\Phi(\tau)\) at this boundary. These two functions solve Eq. (103) - the system of two Volterra integral equation of the second kind. This system is linear in \(\Phi(\tau)\) and nonlinear in \(y(\tau)\).

(Itkin et al., 2021a) present various methods of solving Volterra integral equations and the corresponding references. Here we suggest solving this system sequentially in time. For doing so we first choose the upper bound \(\bar{\tau}\) for the time \(\tau\) and consider all moments of time \(\tau \in [0, \bar{\tau}]\). Then we discretize the time by creating a uniform grid in \(\tau\) with step \(h\), so our discrete time is defined at the points \([0, h, 2h, \ldots, \bar{\tau}]\). We solve Eq. (103) first for \(\tau = h\), the for \(\tau = 2h\) and so on.

To make the structure of the Volterra equations Eq. (103) more transparent, we re-write it in the form
\[
f_1(\tau) = \int_0^\tau \Phi(s)K_1(\tau, s)ds - \Phi(\tau), \quad f_2(\tau) = \int_0^\tau \Phi(s)K_2(\tau, s)ds,
\]
where
\[
f_1(\tau) = -\sum_{n=1}^{\infty} \frac{S_n(\tau) + P_n(\tau)}{2N_n(\tau)} \lambda_n(\tau) \left[ s^+(y(\tau)) \sqrt{R_I} \cos[x_n^-(\tau, y(\tau))] - s^-(y(\tau)) \sqrt{\kappa_W} K_n(\tau) \cos[x_n^+(\tau, y(\tau))] \right],
\]
\[
f_2(\tau) = T_m - \eta(\tau, y(\tau)) - \sum_{n=1}^{\infty} \frac{S_n(\tau) + P_n(\tau)}{2N_n(\tau)} \left[ s^+(y(\tau)) \sin[x_n^-(\tau, y(\tau))] + s^-(y(\tau)) K_n(\tau) \sin[x_n^+(\tau, y(\tau))] \right],
\]

Multilayer heat equations and their solutions via oscillating integral transforms.

Page 25 of 43
Multilayer heat equations and their solutions via oscillating integral transforms.

\[ K_1(\tau, s) = \sum_{n=1}^{\infty} \frac{\lambda_n(\tau)}{2N_n(\tau)} e^{-(\tau-s)\lambda_n^2(\tau)} \left[ \sin[x_n^-(\tau, y(s))] - K_n(\tau) \sin[x_n^+(\tau, y(s))] \right] \]
\[ \times \left[ s^+(y(\tau))\sqrt{\kappa_f} \cos[x_n^-(\tau, y(\tau))] - s^-(y(\tau))\sqrt{\kappa_f} K_n(\tau) \cos[x_n^+(\tau, y(\tau))] \right], \]  
(108)

\[ K_2(\tau, s) = \sum_{n=1}^{\infty} \frac{1}{2N_n(\tau)} e^{-(\tau-s)\lambda_n^2(\tau)} \left[ \sin[x_n^-(\tau, y(s))] - K_n(\tau) \sin[x_n^+(\tau, y(s))] \right] \]
\[ \times \left[ s^+(y(\tau)) \sin[x_n^-(\tau, y(\tau))] + s^-(y(\tau)) K_n(\tau) \sin[x_n^+(\tau, y(\tau))] \right], \]

\[ P_n(\tau) = \int_{0}^{\tau} e^{-(\tau-s)\lambda_n^2(\tau)} \left\{ \kappa_I [B_-(s) \sin[x_n^-(\tau, y(s))] - B_-(\tau) \sin[x_n^-(\tau, y(\tau))]] \right. \]
\[ - K_n(\tau) \kappa_W \left[ B_+(s) \sin[x_n^+(\tau, y(s))] - B_+(\tau) \sin[x_n^+(\tau, y(\tau))] \right] \right\} ds. \]

As \( S_n(\tau) \) and \( P_n(\tau) \) are proportional to \( e^{-\tau\lambda_n^2(\tau)} \) all sums in Eq. (108) converge to a finite limit. One can check that \( f_1(0) = P_n(0) = K_1(0, 0) = K_2(0, 0) = 0, f_2(0) = T_m - T_i \), and thus \( \Phi(0) = 0 \). Also, \( K_1(\tau, \tau) = K_2(\tau, \tau) = 0 \). Therefore, Eq. (107) has an explicit solution for \( \Phi(\tau) \)

\[ \Phi(\tau) = \int_{0}^{\tau} \Phi(s)K_1(\tau, s)ds - f_1(\tau). \]  
(109)

Numerically this solution can be computed recursively. One can observe that \( f_1(\tau) \neq 0, f_2(\tau) \neq 0 \). Therefore, at the first step \( \tau = h \) the second equation in Eq. (107) implies \( f_2(h) = 0 \). This is a nonlinear algebraic equation with respect to \( y(h) \). Solving it numerically by using some initial guess, e.g., \( y(h) = y(0) = y_- \), we obtain \( y(h) \), and then from Eq. (109) get \( \Phi(h) = -f_1(h) \). Here the function \( f_1(h) \) is computed by using the already found value of \( y(h) \).

At the next step \( \tau = 2h \) the values of \( y(0), y(h), \Phi(0), \Phi(h) \) are already known. Hence, \( y(2h) \) solves the nonlinear equation

\[ f_2(2h) = h\Phi(h)K_2(2h, h) = -hf_1(h)K_2(2h, h), \]

while \( \Phi(2h) \) immediately follows from Eq. (109) and reads\(^7\)

\[ \Phi(2h) = h\Phi(h)K_1(2h, h) - f_1(2h) = -hf_1(h)K_1(2h, h) - f_1(2h). \]

And so on.

In our numerical experiments we choose constant values of the model parameters which are given in Tab. 1. Note, \( y'(\tau) \) is measured in (mm/s), and \( \Phi(\tau) \) in (K-mm/s).

| \( y_- \) | \( y_+ \) | \( T_m \) | \( T_i \) | \( \kappa_I \) | \( \kappa_W \) | \( \rho_I \) | \( \rho_W \) | \( L \) | \( \rho_I L \) |
|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| mm     | mm     | K      | K      | mm\(^2\)/s | mm\(^2\)/s | kg/m\(^3\) | kg/m\(^3\) | K m\(^3\)/kg | K        |
| 1      | 50     | 270    | 273    | 290     | 1.02   | 0.13   | 917    | 997    | 0.054   | 49.86   |

Table 1: Parameters of the model used in the Example. For those that depend on pressure it is assumed to be 1 atm.

Our results are presented in Fig. 3: a) the temperature profile; b) the law of the interphase boundary movement \( y(\tau) \); the jump in gradients at the interphase boundary; all as a function of time \( \tau \). In our calculations we took \( n = 50 \), because further increase of \( n \) doesn’t affects the results. Since our initial condition

\(^7\)Again, we use trapezoidal quadratures, but at \( s = 0 \) the gradient \( \Phi(s) \) vanishes, and at \( s = \tau \) so does the kernel \( K_1(\tau, s) \).
is singular (that is because $y(0) = y_-, T(y_-) = T(y) = T_i$, and right at the first step in time the temperatures jump to their constant values $T(y_-) = T_s, T(y) = T_m$) we construct a nonuniform grid in time starting with small steps $h = 0.01$ sec and then increasing the step up to $h = 15$ sec. Also, an important point is to compute the correct values of $\lambda_n(\tau)$ by solving Eq. (96) (see comments after Eq. (99)). We do it by using the chebfun package in Matlab, then $\text{lambd}a = \text{roots}(\text{chebfun}(\text{fun1}, [1.e-6, \text{upLimit}]))$ and taking only odd roots. A typical elapsed time for one step in time is 0.14 sec on a standard PC with 3.2 Ghz CPU running under Windows and Matlab 2020.

\begin{figure}[h]
\centering
\begin{subfigure}{0.4\textwidth}
\centering
\includegraphics[width=\textwidth]{a.png}
\caption{(a)}
\end{subfigure}
\begin{subfigure}{0.4\textwidth}
\centering
\includegraphics[width=\textwidth]{b.png}
\caption{(b)}
\end{subfigure}
\begin{subfigure}{0.4\textwidth}
\centering
\includegraphics[width=\textwidth]{c.png}
\caption{(c)}
\end{subfigure}
\caption{Results on calculations: a) The temperature profile as a function of the time $\tau$ and distance $x$, the dashed line corresponds to $T(\tau, x) = T_m$; (b) The law of the interphase boundary movement $y(\tau)$; c) Jump in the gradient of temperature at the interphase boundary.}
\end{figure}

4 Conclusions

In this paper, we present another approach to solving various MHE problems, which can be viewed as an alternative to the method described in (Itkin et al., 2021b). This approach exploits the series expansion of
the Dirac delta function into eigenfunctions of the corresponding Sturm-Liouville problem. We construct some new (oscillating) integral transforms and use them to solve several multilayer problems occurring in physics, finance, and biology. The number of such problems in science and engineering is vast.

The advantage of our method lies in the fact that our solutions are semi-analytical. This means that the answer is represented explicitly (perhaps, via some integrals). These integrals, however, depend on few yet unknown functions that solve a system of linear Volterra integral equations of the second kind. We derive this system in the paper and emphasize that, in general, it has to be solved numerically. However, as is shown in detail in (Itkin et al., 2021a), the corresponding numerical procedures are very efficient. In particular, they provide better speed and accuracy than the standard finite-difference (FD) methods.

Our main contribution to the existing literature lies in the fact that we consider moving boundaries, depending on time, while still constructing a semi-analytical solution. All previous approaches to these problems are purely numerical and rely on finite differences, finite elements, and similar methods. But, again, the semi-analytical solution has the advantage that the most challenging part of the problem is solved analytically, and the remaining part is solved numerically but with high efficiency.

As we mentioned earlier in the paper, various time-dependent problems with constant boundaries can be reduced to the MHE (or even to the Multilayer Bessel equations, (Decamps et al.)) by a set of transformations. However, as a result, the boundaries become time-dependent in the new variables. Therefore, our setting covers a broader class of problems than what the title claims.

For financial applications, this semi-analytical form is also very convenient to compute sensitivities of asset prices to various parameters of the model; see a thorough discussion in (Itkin et al., 2021a).

For physical and biological problems, the following comment is in order. This paper considers matching conditions at the layer boundaries (internal interfaces), which are the continuity of the solution and its flux over the boundary. As discussed in the literature, see, e.g., in (Hickson et al.), depending on what kind of physics problem is considered, various conditions at the internal interface can be set. It is possible to distinguish at least three different sets of matching conditions of particular interest. The first one assumes the temperature and flux continuity at the interface; we use it throughout this paper. The second matching condition deals with the conductivities, $k_i$, as opposed to the diffusivities, and reads

$$ k_i \frac{\partial u_i}{\partial x} \bigg|_{x_i} = k_{i+1} \frac{\partial u_{i+1}}{\partial x} \bigg|_{x_i}, $$

where $k_i = \rho_i c_i \kappa_i$. This equation is more general than the continuity of the flux. It is applicable when the density, $\rho_i$, and specific heat capacity $c_i$ are possibly different in each layer. The continuities of the flux and conductivities highlight the key difference between heat and mass transfer problems, as the former expresses the equality of the mass flux and the latter of the heat flux.

The third matching condition assumes a jump either in the solution or its flux (or both) at the interface (especially if this is a phase interface). Accordingly, it can be written in the form

$$ \kappa_i \frac{\partial u_i}{\partial x} = H_i (u_{i+1} - u_i), $$
$$ \kappa_{i+1} \frac{\partial u_{i+1}}{\partial x} = H_{i+1} (u_{i+1} - u_i), $$

where $H_i$ is the contact transfer coefficient. This matching condition is more general than the previous ones as it models the roughness of the contact between the layers and contact resistance at the interfaces, see (Hickson et al.; Illingworth and Golosnoy). If $H_i \to \infty$, then the contact is perfect, and hence this limit represents the equivalent matching conditions from the previous cases. In a similar way the jump in the function value $u$ at the interface is often modeled by the condition $u_i(\tau, y_i(\tau)) = P_i u_{i+1}(\tau, y_i(\tau))$ where $P_i$ is the partitioning coefficient which could be set constant or, in a more general case, is a function of time.
As far as our approach is concerned, it can deal with the second and third kinds of interface conditions. The integral transforms are the same, but solutions for the direct and inverse transforms are different. They rely on expanding discontinues functions into the Fourier series, which is possible and works fine. A detailed description of our method will be published elsewhere.

Another extension of the problem under consideration stems from choosing the boundary conditions at the external boundary. Usually, various analytical and semi-analytical solutions are constructed for the infinite or semi-infinite spatial domain. However, in real problems, e.g., that one discussed in (Illingworth and Golosnoy) and dedicated to finding transient solutions to diffusion problems in two distinct phases separated by a moving boundary, such an assumption is not valid. Indeed, in addition to ignoring the actual boundary conditions at the boundary walls, analytical methods have to assume that one of the phases has zero initial size. Such highly restrictive conditions mean that it is not possible to construct an analytical model for many situations, which are of considerable practical or industrial importance. However, for our method, these conditions are not as restrictive. Again, a detailed consideration will be presented elsewhere.

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### Appendices

#### A The expansion of the Dirac delta function for the continuous spectrum

Let us seek for the unknown function $u_\lambda(x|x_0)$ to be in the form

$$u_\lambda(x|x_0) = \mathbb{1}_{x \mid y} \left[ \frac{A_- e^{-\tilde{\lambda}_-(x-y)} + \frac{1}{2\sigma_- \sqrt{\lambda}} e^{-\tilde{\lambda}_-(|x-x_0|)}}{A_+ e^{-\tilde{\lambda}_+(x-y)} + \frac{1}{2\sigma_+ \sqrt{\lambda}} e^{-\tilde{\lambda}_+|x-x_0|}} \right] \tag{A.1}$$

where functions $A_\pm$ are chosen to satisfy the matching conditions Eq. (20). In more detail, we need

$$\lim_{x \to y^\pm 0} u_\lambda = A_\pm + \frac{1}{2\sigma_\pm \sqrt{\lambda}} e^{-\tilde{\lambda}_\pm|y-x_0|},$$
Combining these two equations yields a linear system of equations for $A_-$ and $A_+$

$$
\begin{bmatrix}
1 & -1 \\
\sigma_- & \sigma_+
\end{bmatrix}
\begin{bmatrix}
A_- \\
A_+
\end{bmatrix}
= \frac{1}{2\sqrt{\lambda}} \begin{bmatrix}
e^{-\lambda_+|y-x_0|}(1_{x_0>y} - 1_{x_0<y}) - e^{-\lambda_-|y-x_0|}(1_{x_0>y} - 1_{x_0<y}) \\
e^{-\lambda_+|y-x_0|}(1_{x_0>y} - 1_{x_0<y}) - e^{-\lambda_-|y-x_0|}(1_{x_0>y} - 1_{x_0<y})\end{bmatrix},
$$

with the following solution

$$
\begin{bmatrix}
A_- \\
A_+
\end{bmatrix}
= \frac{\Sigma e^{-\lambda_-|x_0-y|} + e^{-\lambda_+|x_0-y|}}{2\sigma_-\sqrt{\lambda} + 2\sigma_+\sqrt{\lambda}} \begin{bmatrix}
\frac{2\sigma_+}{\sigma_- + \sigma_+} e^{-\lambda_+|x_0-y|} - \frac{2\sigma_-}{\sigma_- + \sigma_+} e^{-\lambda_-|x_0-y|} \\
\frac{2\sigma_-}{\sigma_- + \sigma_+} e^{-\lambda_+|x_0-y|} - \frac{2\sigma_+}{\sigma_- + \sigma_+} e^{-\lambda_-|x_0-y|}\end{bmatrix}
\begin{bmatrix}1_{x_0>y} \\ 1_{x_0<y}\end{bmatrix},
$$

$$\Sigma = \frac{\sigma_- - \sigma_+}{\sigma_- + \sigma_+}.
$$

Accordingly, the function $u_\lambda(x|x_0)$ can be represented as

$$
u_\lambda(x|x_0) = \begin{bmatrix}1_{x<y}e^{-\lambda_-|x-y|} \\ 1_{x>y}e^{-\lambda_+|x-y|} \end{bmatrix}^\top \begin{bmatrix}A_- \\
A_+\end{bmatrix} + \begin{bmatrix}1_{x>y} \\ 1_{x<y}\end{bmatrix} \begin{bmatrix}e^{-\lambda_-|x-x_0|} e^{-\lambda_-|x-x_0|} \\
e^{-\lambda_-|x-x_0|} e^{-\lambda_-|x-x_0|}\end{bmatrix} \begin{bmatrix}1_{x_0>y} \\ 1_{x_0<y}\end{bmatrix},
$$

Figure 4: Contours of integration in a complex plane in case of the continuous spectrum.

Now we are can compute the integral in Eq. (17). Let us extend the original contour $(\gamma, -i\infty) - (\gamma, i\infty)$ to the loop contour $L$ in Fig. 4 which can be described as follows. It starts with a parallel line $(\gamma, -iR) - (\gamma, iR)$, extending to a big symmetric arcs $\Gamma_-$ and $\Gamma_+$ around the point $(\gamma, 0)$ with the radius
To solve Eq. (76) we use the oscillating transform

\[ \mathbf{u}(\tau, \omega) = \begin{bmatrix} \bar{u}_-(\tau, \omega) \\ \bar{u}_+(\tau, \omega) \end{bmatrix} = \int_{-\infty}^{\infty} \mathcal{F}(\tau, \sigma) \mathbb{1}_{x|y}(\tau) u(\tau) \, dx, \]  

(C.1)

\[ R \] with the two horizontal line segments \( l_1, l_2 \) connecting to a small circle \( \gamma_\varepsilon \) around the origin with the radius \( \varepsilon \).

Using a standard technique, we take a limit \( \varepsilon \to 0, R \to \infty \), apply the Cauchy Residue theorem and obtain

\[ \oint_L u_\lambda(x|x_0) \, d\lambda = 0. \]  

(A.4)

Therefore,

\[ \delta(x - x_0) = \frac{1}{2\pi i} \int_{\gamma_{-\infty}}^{\gamma_{+\infty}} u_\lambda(x|x_0) \, d\lambda = \int_0^\infty \frac{u_{\varepsilon-i\omega}^2(x|x_0) - u_{\varepsilon+i\omega}^2(x|x_0)}{\pi i} \omega \, d\omega. \]  

(A.5)

Omitting tedious algebra, we find the following representation of the Dirac delta function

\[ \delta(x - x_0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \begin{bmatrix} e^{i\omega x} & e^{i\omega y} \\ x^y & x^y \end{bmatrix} \begin{bmatrix} \frac{1}{1 + \Sigma e^{2i\omega x - y}} \\ \frac{1}{1 - \Sigma e^{2i\omega x - y}} \end{bmatrix} \begin{bmatrix} e^{-i\omega x} & e^{-i\omega y} \\ x^y & x^y \end{bmatrix} \, d\omega. \]  

(A.6)

We proceed with separation of the variables \( x \) and \( x_0 \) in Eq. (A.6). Introducing the matrices (i.e. Forward and Backward)

\[ \mathcal{F}_{x,\sigma}(\omega) = S \begin{bmatrix} e^{i\omega x} & 0 \\ 0 & e^{i\omega y} \end{bmatrix}, \]  

(A.7)

\[ \mathcal{B}_{x,\sigma}(\omega) = \begin{bmatrix} \frac{1}{\sigma_-} (1 + \Sigma) e^{i\omega x} & \frac{1}{\sigma_-} (1 + \Sigma) e^{i\omega y} \\ \frac{1}{\sigma_+} (1 - \Sigma) e^{-i\omega x} & \frac{1}{\sigma_+} (1 - \Sigma) e^{-i\omega y} \end{bmatrix}, \]  

the Eq. (A.6) can be re-written as

\[ \delta(x - x_0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathcal{F}_{x-y,\sigma}(\omega) \mathcal{B}_{x_0-y,\sigma}(\omega) \mathbb{1}_{x|y} \, d\omega. \]  

(A.8)

It is worth mentioning that the matrix \( \mathcal{F}_{x,y,\sigma} \) has a semigroup property with respect to the matrix multiplication, i.e.

\[ \mathcal{F}_{x,\sigma}(\omega) \mathcal{F}_{z,\sigma}(\omega) = \mathcal{F}_{x+z,\sigma}(\omega). \]  

(A.9)

### B Contour integration

By analogy with computation of the integral in Eq. (17), we extend the contour \((\gamma, -i\infty) - (\gamma, i\infty)\) to the loop contour \( L \) shown in Fig. 5. However, in this case the integration domain contains simple poles which are determined by the equation det \( \Lambda^\ast(\lambda) = 0 \). Again, we proceed in a way similar to that for Eq. (17), i.e., by taking a limit \( \varepsilon \to 0, R \to \infty \) and applying the Cauchy Residue theorem. Omitting an intermediate algebra, we provide just the final result which is the representation in Eq. (59).

### C Solution of Eq. (76)

To solve Eq. (76) we use the oscillating transform
Multilayer heat equations and their solutions via oscillating integral transforms.

Figure 5: Contours of integration in a complex plane in case of the mixed spectrum.

where the matrix $\mathcal{F}_{x,\sigma}(\omega)$ was defined in Eq. (A.7). Using the semigroup property Eq. (A.9) in the inversion formula (28) yields

$$u(\tau,x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{u}(\tau,\omega)^{\top} \mathcal{F}_{-\sigma,y(\tau),\sigma}(\omega) B_{x,y(\tau)}(\omega) \mathbb{1}_{x,y(\tau)} d\omega.$$  (C.2)

The images $\bar{u}_-$ and $\bar{u}_+$ can be easily found in the explicit form and read

$$\bar{u}_-(\tau,\omega) = \int_{-\infty}^{y(\tau)} e^{-\sigma_- u(\tau,x)} dx, \quad \bar{u}_+(\tau,\omega) = \int_{y(\tau)}^{\infty} e^{\sigma_+ u(\tau,x)} dx$$  (C.3)

To make these integrals well-behaved we need function $u(\tau,x)$ to vanish at plus and minus infinity faster than the corresponding exponent\(^9\). Now we define new functions $\Phi(\tau), \varphi(\tau)$ as

$$\sigma_-^2 \frac{\partial u(\tau,x)}{\partial x} \bigg|_{x=y(\tau)-0} = \sigma_+^2 \frac{\partial u(\tau,x)}{\partial x} \bigg|_{x=y(\tau)+0} = \Phi(\tau), \quad u(\tau,y(\tau)) = \varphi(\tau),$$  (C.4)

and apply the transform Eq. (C.1) directly to the both sides of the last line in Eq. (76)

\begin{align*}
\int_{-\infty}^{y(\tau)} \sigma_-^2 u_{xx} e^{\sigma_-} dx &= e^{-\sigma_-} \Phi(\tau) - i\omega \sigma_- \int_{-\infty}^{y(\tau)} u_x e^{\sigma_-} dx \\
&= e^{-\sigma_-} \Phi(\tau) - i\omega \sigma_- e^{\sigma_-} \varphi(\tau) - \omega^2 \bar{u}_-(\tau,\omega), \\
\int_{-\infty}^{y(\tau)} u_{\tau} e^{\sigma_-} dx &= \frac{\partial}{\partial \tau} \int_{-\infty}^{y(\tau)} e^{\sigma_-} u(\tau,x) dx - y'(\tau) e^{\sigma_-} u(\tau,y(\tau)) = \frac{\partial \bar{u}_-}{\partial \tau} - y'(\tau) e^{\sigma_-} \varphi(\tau),
\end{align*}

\(^9\)Actually, as shown below integrals on $\omega$ in Eq. (C.9) are regular, so there is no problem with convergence of these transforms.
where \( \Psi \).

This result can be also expressed in the matrix form

\[
\begin{align*}
\int_{y(\tau)}^{+\infty} \sigma_+^2 u_{xx} e^{\sigma_+ x} dx &= -e^{\sigma_+ y(\tau)} \Phi(\tau) - i \omega \sigma_+ \int_{y(\tau)}^{+\infty} u_x e^{\sigma_+ y} dx \\
&= -e^{\sigma_+ y(\tau)} \Phi(\tau) + i \omega \sigma_+ e^{\sigma_+ y(\tau)} \varphi(\tau) - \omega^2 \bar{u}_+(\tau, \omega),
\end{align*}
\]

\[
\int_{y(\tau)}^{+\infty} u_{r} e^{\sigma_+ r} dx = \frac{\partial}{\partial \tau} \int_{y(\tau)}^{+\infty} e^{\sigma_+ u(x, \tau) dx + y'(\tau) e^{\sigma_+ y(\tau)}} = \frac{\partial \bar{u}_+}{\partial \tau} + y'(\tau) e^{\sigma_+ \phi(\tau)}.
\]

Therefore, the matching conditions in Eq. (76) give rise to two independent ordinary differential equations (ODE) (with respect to \( \tau \))

\[
\frac{\partial \bar{u}_-}{\partial \tau} + \omega^2 \bar{u}_- = e^{\sigma_-} \Phi(\tau) - i \omega \sigma_- e^{\sigma_-} \varphi(\tau) + y'(\tau) e^{\sigma_-} \varphi(s), \quad \bar{u}_-(0, \omega) = 1_{x_0 < y(0)} e^{\sigma_-} ,
\]

\[
\frac{\partial \bar{u}_+}{\partial \tau} + \omega^2 \bar{u}_+ = -e^{\sigma_+} \Phi(\tau) + i \omega \sigma_+ e^{\sigma_+} \varphi(\tau) - y'(\tau) e^{\sigma_+} \varphi(s), \quad \bar{u}_+(0, \omega) = 1_{x_0 > y(0)} e^{\sigma_+} .
\]

These equations can be solved analytically via a standard technique

\[
\bar{u}_-(\tau, \omega) = 1_{x_0 < y(0)} \int_{0}^{\tau} e^{-\omega^2(\tau-s)} \left[ e^{\sigma_- s} \Phi(s) - i \omega \sigma_- e^{\sigma_-} \varphi(s) + y'(s) e^{\sigma_-} \varphi(s) \right] ds,
\]

\[
\bar{u}_+(\tau, \omega) = 1_{x_0 > y(0)} \int_{0}^{\tau} e^{-\omega^2(\tau-s)} \left[ e^{\sigma_+ s} \Phi(s) - i \omega \sigma_+ e^{\sigma_+} \varphi(s) + y'(s) e^{\sigma_+} \varphi(s) \right] ds.
\]

This result can be also expressed in the matrix form

\[
\bar{u}(\tau, \omega) = e^{-\omega^2 \mathcal{F}_{x_0, \sigma}(\omega) \mathbbm{1}_{x_0 < y(0)}} + \int_{0}^{\tau} e^{-\omega^2(\tau-s)} \mathcal{F}(\sigma)(0, \omega) \left[ \Phi(s) - i \omega \sigma_- \varphi(s) + y'(s) \varphi(s) \\
- \Phi(s) + i \omega \sigma_+ \varphi(s) - y'(s) \varphi(s) \right] ds.
\]

Applying the inversion formula Eq. (C.2) we arrive at the following representation

\[
u(\tau, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\omega^2(\tau-s)} \mathcal{F}_{x_0}(\omega) \mathcal{B}_{x_0}(\sigma)(0) \mathbbm{1}_{y(\tau)} d\omega
\]

\[
+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{0}^{\tau} e^{-\omega^2(\tau-s)} \mathbf{\Psi}(s, \omega) \mathcal{F}(\sigma)(y(\tau), \omega) \mathcal{B}_{x_0}(\sigma)(0) \mathbbm{1}_{y(\tau)} d\omega d\omega,
\]

where \( \mathbf{\Psi}(s, \omega) \) is defined as

\[
\mathbf{\Psi}(s, \omega) = \left[ \Phi(s) + y'(s) \varphi(s) \right] \left[ \begin{array}{c} 1 \\
-1 \end{array} \right] - i \omega \varphi(s) \left[ \begin{array}{c} \sigma_- \\
0 \end{array} \right] \left[ \begin{array}{c} 1 \\
-1 \end{array} \right].
\]

The integrals with respect to \( \omega \) can be computed analytically. Indeed, using the identities, (Gradshteyn and Ryzhik, 2007)

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\omega^2(\tau-s)+i\omega} d\omega = \frac{e^{-\frac{\sigma^2}{4(\tau-s)}}}{2\sqrt{\pi(\tau-s)}}, \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} i\omega e^{-\omega^2(\tau-s)+i\omega} d\omega = -\frac{\alpha e^{-\frac{\sigma^2}{4(\tau-s)}}}{4\sqrt{\pi(\tau-s)}},
\]

and the explicit formula for the matrix product \( \mathcal{F}_{\xi, \sigma}(\omega) \mathcal{B}_{\xi, \sigma}(\omega) \)

\[
\mathcal{F}_{\xi, \sigma}(\omega) \mathcal{B}_{\xi, \sigma}(\omega) = \left[ \begin{array}{c} \frac{1}{\sigma_-} e^{\sigma_-(\Sigma + \Sigma e^{\sigma_-})} \\
\frac{1}{\sigma_+} e^{\sigma_+(1 + \Sigma)} e^{-\frac{\sigma_-}{\sigma_+}} \left( \Sigma - \sigma_- \right) \end{array} \right], \quad \Sigma = \frac{\sigma_- - \sigma_+}{\sigma_- + \sigma_+}.
\]
An explicit representation of the inverse transform in Eq. (83) can be obtained if we know all its components \( \tilde{u}^C_i(\tau, \lambda) \) to the initial ODE problems for functions \( \tilde{u}^C_i(\tau, \lambda) \) constructing the whole image \( \tilde{u} \) as a weighted sum. For doing so, let us denote the values of \( u(\tau, x) \) and their gradients at all internal interfaces between the layers as

\[
\frac{\partial u(\tau, x)}{\partial x} \bigg|_{x=y_i(\tau)-0} = \phi_i^+(\tau), \quad \frac{\partial u(\tau, x)}{\partial x} \bigg|_{x=y_i(\tau)+0} = \phi_i^-(\tau),
\]

Applying the transform in Eq. (83) to Eq. (82), integrating by parts and collecting terms gives rise to the initial ODE problems for functions \( \tilde{u}^C_i(\tau, \lambda) \), \( \tilde{u}^S_i(\tau, \lambda) \):

\[
\frac{d\tilde{u}^C_i(\tau, \lambda)}{d\tau} = -\lambda^2 \tilde{u}^C_i(\tau, \lambda) + \int_{y_i(\tau)-0}^{y_i(\tau)} g(\tau, \xi) \cos \left( \lambda_i^2 \xi \right) d\xi + \sigma_i^2 \left[ \cos \left( \lambda_i y_i(\tau) \right) \Psi_i^+(\tau) - \cos \left( \lambda_i^{-1} y_i(\tau) \right) \Psi_i^-(\tau) \right]
\]

\[
+ \sigma_i \lambda \left[ \sin \left( \lambda_i y_i(\tau) \right) \phi_i^+(\tau) - \sin \left( \lambda_i^{-1} y_i(\tau) \right) \phi_i^-(\tau) \right]
\]

\[
+ y_i'(\tau) \cos \left( \lambda_i y_i(\tau) \right) \phi_i^+(\tau) - y_i'(\tau) \cos \left( \lambda_i^{-1} y_i(\tau) \right) \phi_i^-(\tau)
\]

\[
\tilde{u}^C_i(0, \lambda) = \int_{y_i(0)-0}^{y_i(0)} f(\xi) \cos \left( \lambda_i \xi \right) d\xi
\]
Multilayer heat equations and their solutions via oscillating integral transforms.

\[
\frac{\tilde{u}_i^S(\tau, \lambda)}{d\tau} = -\lambda^2 \tilde{u}_i^S(\tau, \lambda) + \int_{y_i(\tau)}^{y_i(\tau)} g(\tau, \xi) \sin \left( \xi \right) d\xi + \int_0^\tau \int_{y_{i-1}(\tau)}^{y_i(\tau)} e^{-\lambda^2(\tau-s)} g(s, \xi) \cos \left( \xi \right) dsd\xi
\]

\[
- \sigma_i \lambda \left[ \cos \left( \lambda_i y_i(\tau) \right) \phi_i^+(\tau) - \cos \left( \lambda_i y_{i-1}(\tau) \right) \phi_i^-(\tau) \right],
\]

\[
+ y_i'(\tau) \sin \left( \lambda_i y_i(\tau) \right) \phi_i^+(\tau) - y_{i-1}'(\tau) \sin \left( \lambda_i y_{i-1}(\tau) \right) \phi_i^-(\tau)
\]

\[
\tilde{u}_i^S(0, \lambda) = \int_{y_i(0)}^{y_i(0)} f(\xi) \sin \left( \lambda_i x \right) d\xi.
\]

Since each ODE is linear, these problems can be easily solved to get

\[
u_i^C(\tau, \lambda) = e^{-\lambda^2\tau} \int_{y_i(0)}^{y_i(0)} f(\xi) \sin \left( \lambda_i \xi \right) d\xi + \int_0^\tau e^{-\lambda^2(\tau-s)} \int_{y_i(\tau)}^{y_i(s)} g(s, \xi) \cos \left( \lambda_i \xi \right) dsd\xi
\]

\[
+ \sigma_i^2 \int_0^\tau e^{-\lambda^2(\tau-s)} \left[ \cos \left( \lambda_i y_i(s) \right) \Psi_i^+(s) - \cos \left( \lambda_i y_{i-1}(s) \right) \Psi_i^-(s) \right] ds
\]

\[
+ \sigma_i \lambda \int_0^\tau e^{-\lambda^2(\tau-s)} \left[ \sin \left( \lambda_i y_i(s) \right) \phi_i^+(s) - \sin \left( \lambda_i y_{i-1}(s) \right) \phi_i^-(s) \right] ds
\]

\[
- \int_0^\tau e^{-\lambda^2(\tau-s)} \left[ \phi_i'(s) \sin \left( \lambda_i y_i(s) \right) \phi_i^+(s) - \phi_{i-1}'(s) \sin \left( \lambda_i y_{i-1}(s) \right) \phi_i^-(s) \right] ds.
\]

To proceed, it is convenient to define the auxiliary function \( \theta_{y(\tau), \lambda}(x) \)

\[
\theta_{y(\tau), \lambda, k}(x) = -\sigma_i \lambda \left( C_k(\tau, \lambda) \sin \left( \lambda_i x \right) + D_k(\tau, \lambda) \cos \left( \lambda_i x \right) \right).
\]

With this new notation Eq. (84) reads

\[
\tilde{u}(\tau, \lambda) = e^{-\lambda^2\tau} \int_{y_0(0)}^{y_N(0)} (1_{[y_0(\tau), \lambda]}(y_N(\tau), \lambda)) f(\xi) d\xi + \int_0^\tau \int_{y_0(\tau)}^{y_N(\tau)} e^{-\lambda^2(\tau-s)} (1_{[y_0(\tau), \lambda}(y_N(\tau), \lambda)) g(s, \xi) dsd\xi
\]

\[
+ \int_0^\tau e^{-\lambda^2(\tau-s)} \sigma_i^2 \theta_{y(\tau), \lambda, N}(y_N(\tau), \lambda)) \Psi_1^+(s) ds - \int_0^\tau e^{-\lambda^2(\tau-s)} \sigma_1^2 \theta_{y(\tau), \lambda, N}(y_{N}(\tau), \lambda)) \Psi_1^- (s) ds
\]

\[
+ \sum_{i=1}^{N-1} \int_0^\tau e^{-\lambda^2(\tau-s)} \left[ \sigma_i^2 \theta_{y(\tau), \lambda, i}(y_i(s)) \Psi_i^+(s) - \sigma_{i+1}^2 \theta_{y(\tau), \lambda, i+1}(y_{i+1}(s)) \Psi_{i+1}^-(s) \right] ds
\]

\[
- \int_0^\tau e^{-\lambda^2(\tau-s)} \theta_{y(\tau), \lambda, N}(y_N(s)) \phi_N^+(s) ds + \int_0^\tau e^{-\lambda^2(\tau-s)} \theta_{y(\tau), \lambda, N}(y_0(s)) \phi_N^-(s) ds
\]

\[
- \sum_{i=1}^{N-1} \int_0^\tau e^{-\lambda^2(\tau-s)} \left[ \theta_{y(\tau), \lambda, i}(y_i(s)) \phi_i^+(s) - \theta_{y(\tau), \lambda, i+1}(y_{i+1}(s)) \phi_{i+1}^- (s) \right] ds
\]

\[
+ \int_0^\tau e^{-\lambda^2(\tau-s)} y_N'(s) \theta_{y(\tau), \lambda, N}(y_N(s)) \psi_N^+(s) ds - \int_0^\tau e^{-\lambda^2(\tau-s)} y_0'(s) \theta_{y(\tau), \lambda, N}(y_0(s)) \psi_N^- (s) ds
\]
allows one to re-write Eq. (D.3) in the form
\[
y \bar{\bar{u}}(s) = \Theta_{y(\tau),\lambda,i}(y_i(s)) \psi_0^+(s) - \Theta_{y(\tau),\lambda,i+1}(y_{i+1}(s)) \psi_{i+1}^+(s)
\]

This representation can be further simplified. Taking into account the following identities
\[
\phi_0^-(\tau) = \phi_0^+(\tau) = 0, \quad \sigma_i^2 \Psi_i^+ = \sigma_i^2 \Psi_i^-, \quad \phi_i^+(\tau) = \phi_i^-(\tau), \quad i = 1, \ldots, N - 1,
\]
and making change of variables
\[
\sigma_i^2 \Psi_i^-(\tau) = \Phi_i(\tau), \quad \sigma_i^2 \Psi_i^+(\tau) = \Phi_N(\tau), \quad \phi_i^+(\tau) = \phi_{i+1}^-(\tau) = \phi_1(\tau)
\]
allows one to re-write Eq. (D.3) in the form
\[
\bar{u}(\tau, \lambda) = e^{-\lambda^2 T} \int_{y_0(0)}^{y_N(0)} \langle \mathbf{1}_{y(\xi)}, \Theta_{y(\tau),\lambda}(\xi) \rangle f(\xi) d\xi + \int_0^T \int_{y_0(s)}^{y_N(s)} e^{-\lambda^2 (T-s)} \langle \mathbf{1}_{y(\xi)}, \Theta_{y(\tau),\lambda}(\xi) \rangle g(s, \xi) ds d\xi
\]
\[
+ \int_0^T e^{-\lambda^2 (T-s)} \Phi(s) \Theta_{y(\tau),\lambda,N}(y_N(s)) ds - \int_0^T e^{-\lambda^2 (T-s)} \Phi_s(\tau) \Theta_{y(\tau),\lambda,1}(y_0(s)) ds + \sum_{i=1}^{N-1} \int_0^T e^{-\lambda^2 (T-s)} \Phi_i(s) \left[ \Theta_{y(\tau),\lambda,i}(y_i(s)) - \Theta_{y(\tau),\lambda,i+1}(y_{i+1}(s)) \right] ds
\]
\[
- \sum_{i=1}^{N-1} \int_0^T e^{-\lambda^2 (T-s)} \phi_i(s) \left[ \theta_{y(\tau),\lambda,i}(y_i(s)) - \theta_{y(\tau),\lambda,i+1}(y_{i+1}(s)) \right] ds + \sum_{i=1}^{N-1} \int_0^T e^{-\lambda^2 (T-s)} y_i(s) \phi_i(s) \left[ \Theta_{y(\tau),\lambda,i}(y_i(s)) - \Theta_{y(\tau),\lambda,i+1}(y_{i+1}(s)) \right] ds.
\]

Since by the definition of \( \theta \) and \( \phi_1(\tau) \) we have
\[
\Theta_{y(\tau),\lambda,0}(x) = 0, \quad \Theta_{y(\tau),\lambda,N+1}(x) = 0, \quad \phi_0(\tau) = 0, \quad \phi_N(\tau) = 0,
\]
the Eq. (D.4) can be transformed to
\[
\bar{u}(\tau, \lambda) = e^{-\lambda^2 T} \int_{y_0(0)}^{y_N(0)} \langle \mathbf{1}_{y(\xi)}, \Theta_{y(\tau),\lambda}(\xi) \rangle f(\xi) d\xi + \int_0^T \int_{y_0(s)}^{y_N(s)} e^{-\lambda^2 (T-s)} \langle \mathbf{1}_{y(\xi)}, \Theta_{y(\tau),\lambda}(\xi) \rangle g(s, \xi) ds d\xi
\]
\[
+ \sum_{i=0}^{N-1} \int_0^T e^{-\lambda^2 (T-s)} \Phi_i(s) \left[ \Theta_{y(\tau),\lambda,i}(y_i(s)) - \Theta_{y(\tau),\lambda,i+1}(y_{i+1}(s)) \right] ds
\]
\[
- \sum_{i=0}^{N-1} \int_0^T e^{-\lambda^2 (T-s)} \phi_i(s) \left[ \theta_{y(\tau),\lambda,i}(y_i(s)) - \theta_{y(\tau),\lambda,i+1}(y_{i+1}(s)) \right] ds + \sum_{i=0}^{N-1} \int_0^T e^{-\lambda^2 (T-s)} y_i(s) \phi_i(s) \left[ \Theta_{y(\tau),\lambda,i}(y_i(s)) - \Theta_{y(\tau),\lambda,i+1}(y_{i+1}(s)) \right] ds.
\]

Finally, we define \( N + 1 \)-dimensional vector functions
\[
\Phi(s) = [\Phi_0(s), \Phi_1(s), \ldots, \Phi_N(s)]^T, \quad \phi(s) = [0 = \phi_0(s), \phi_1(s), \ldots, \phi_N(s) = 0]^T, \quad \text{D.6}
\]
\[
\Omega_{y(\tau),\lambda}(s) = \left[ 0 - \Theta_{y(\tau),\lambda,1}(y_0(s)), \ldots, \Theta_{y(\tau),\lambda,i}(y_i(s)) - \Theta_{y(\tau),\lambda,i+1}(y_{i+1}(s)), \ldots, \Theta_{y(\tau),\lambda,N}(y_N(s)) - 0 \right]^T,
\]
\[
\omega_{y(\tau),\lambda}(s) = \left[ 0 - \theta_{y(\tau),\lambda,1}(y_0(s)), \ldots, \theta_{y(\tau),\lambda,i}(y_i(s)) - \theta_{y(\tau),\lambda,i+1}(y_{i+1}(s)), \ldots, \theta_{y(\tau),\lambda,N}(y_N(s)) - 0 \right]^T,
\]
and $N + 1 \times N + 1$ diagonal matrix $\mathcal{Y}(\tau)$

$$\mathcal{Y}(\tau) = \text{diag} [y_0(\tau), y_1(\tau), y_2(\tau), \ldots y_N(\tau)]^\top, \quad \mathcal{Y}'(\tau) = \text{diag} [y'_0(\tau), y'_1(\tau), y'_2(\tau), \ldots y'_N(\tau)]^\top. \quad \text{(D.7)}$$

and arrive at a very compact representation of $\tilde{u}(\tau, \lambda)$

$$\tilde{u}(\tau, \lambda) = e^{-\lambda^2 \tau} \int_{-\infty}^{+\infty} \langle 1_{x', y(\tau)}, \Theta_{y(\tau), \lambda}(\xi) \rangle f(\xi) d\xi + \int_{0}^{\tau} \int_{-\infty}^{+\infty} e^{-\lambda^2 (\tau - \xi)} \langle 1_{x', y(\xi)}, \Theta_{y(\xi), \lambda}(\xi) \rangle g(s, \xi) ds \xi \quad \text{(D.8)}$$

$\text{E} \quad \text{The solution of the problem Eq. (90)}$

The problem in Eq. (90) has inhomogeneous boundary conditions, Eq. (91). Therefore, first we make a change of variables to transform Eq. (90) to the problem with homogeneous boundary conditions by setting

$$T(\tau, x) = \mathcal{T}(\tau, x) + \eta(\tau, x), \quad \eta(\tau, x) = \theta(y(\tau) - x) F_-(\tau, x) + \theta(x - y(\tau)) F_+(\tau, x), \quad \text{(E.1)}$$

where $\theta(x)$ is the Heaviside theta function with $\theta(0) = 1/2$. Functions $F_\pm(\tau, x)$ should be such to provide the correct values of the temperature at the lower and upper boundaries, and also a jump of the gradient and continuity of the temperature at the interface boundary. Since this gives four conditions, we proceed by setting

$$F_-(\tau, x) = A_-(\tau) + B_-(\tau) x, \quad F_+(\tau, x) = A_+(\tau) + B_+(\tau) x. \quad \text{(E.2)}$$

Four yet unknown functions $A_-(\tau), B_-(\tau), A_+(\tau), B_+(\tau)$ solve the linear system

$$\begin{bmatrix} 1 & y_- & 0 & 0 \\ 0 & 0 & 1 & y_+ \\ 1 & y(\tau) & -1 & y(\tau) \\ 0 & \kappa_1 & 0 & -\kappa_W \end{bmatrix} \begin{bmatrix} A_-(\tau) \\ B_-(\tau) \\ A_+(\tau) \\ B_+(\tau) \end{bmatrix} = \begin{bmatrix} T_s \\ T_0 \\ 0 \\ \rho_1 L y'(\tau) \end{bmatrix}. \quad \text{(E.3)}$$

With these definitions the problem in Eq. (90) transforms to

$$\frac{\partial \mathcal{T}}{\partial \tau} = \kappa_W \frac{\partial^2 \mathcal{T}}{\partial x^2} + g_+(\tau, x), \quad y(\tau) < x < y_+, \quad \text{(E.4)}$$

$$\frac{\partial \mathcal{T}}{\partial \tau} = \kappa_1 \frac{\partial^2 \mathcal{T}}{\partial x^2} + g_-(\tau, x), \quad y_- < x < y(\tau),$$

$$\mathcal{T}(\tau, y_-) = \mathcal{T}(\tau, y_+) = 0, \quad \mathcal{T}(0, x) = T_0 - \eta(0, x),$$

$$\mathcal{T}(\tau, y(\tau) - 0) = \mathcal{T}(\tau, y(\tau) + 0) = T_m - \eta(\tau, y(\tau)) \equiv \phi(\tau),$$

$$\kappa_1 \frac{\partial \mathcal{T}}{\partial x} \bigg|_{x=y(\tau)-0} = \kappa_W \frac{\partial \mathcal{T}}{\partial x} \bigg|_{x=y(\tau)+0} \equiv \Phi(\tau).$$

Here

$$g_-(\tau, x) = -A'_-(\tau) - B'_-(\tau) x, \quad g_+(\tau, x) = -A'_+(\tau) - B'_+(\tau) x.$$ 

Thus, we obtained a similar problem for $\mathcal{T}(\tau, x)$, but now with homogeneous boundary conditions at both ends, while the heat equations acquire new source terms $g_\pm(x)$.

Solving Eq. (E.3) yields

$$A_-(\tau) = \frac{-y_- T_1 \kappa_W + T_s (y_+ \kappa_1 + y(\tau)(\kappa_W - \kappa_1)) + y_- [y(\tau) - y_+] \rho_1 L y'(\tau)}{D(\tau)}, \quad \text{(E.5)}$$
Together they allow simplification of Eq. (E.7) to the form

\[ B_{-}(\tau) = \frac{[T_1 - T_s] k_W + [y_+ - y(\tau)] \rho I L y'_{\tau}(\tau)}{D(\tau)}, \]

\[ A_+(\tau) = \frac{-T_l [y(\tau)(k_I - k_W) + y_- k_W] + y_+ [T_s k_I + (y(\tau) - y_-) \rho I L y'_{\tau}(\tau)]}{D(\tau)}, \]

\[ B_+(\tau) = \frac{[T_1 - T_s] k_I + [y_- - y(\tau)] \rho I L y'_{\tau}(\tau)}{D(\tau)}, \]

where

\[ D(\tau) = \det \begin{bmatrix} 1 & y_- & 0 & 0 \\ 0 & 0 & 1 & y_+ \\ 1 & y(\tau) & -1 & -y(\tau) \\ 0 & k_I & 0 & -k_W \end{bmatrix} = k_I (y_+ - y(\tau)) + k_W (y(\tau) - y_-) > 0. \] (E.6)

Note, that at \( \tau = 0 \) we have \( y(0) = y_- \), \( T_s = T_m = T_l \) and \( y'(0) = 0 \). Then, Eq. (E.5) implies \( B_{-}(0) = B_{+}(0) = 0 \), \( A_{-}(0) = A_{+}(0) = T_l \), i.e. Eq. (E.5) replicates the correct boundary condition at \( x = y_- \).

The problem in Eq. (E.4) can be solved by directly applying the OIT in Eq. (83), which immediately gives rise to an explicit representation of \( \mathcal{T} \)

\[ \mathcal{T}(\tau, x) = \left\{ \mathbb{1}_{x \leq y(\tau)}, \sum_{n=1}^{\infty} \frac{\Theta_{y(\tau), \lambda_n(\tau)}(x)}{N_n(\tau)} \left[ e^{-\tau \lambda_n^2(\tau)} \int_{-\infty}^{+\infty} \left( 1_{\xi < y(0)} \Theta_{y(\tau), \lambda_n(\tau)}(\xi) \right) \left[ T_l + \eta(0, \xi) \right] d\xi \right] + \right. \]

\[ + \int_{0}^{\tau} \int_{-\infty}^{+\infty} e^{-(\tau-s)\lambda_n^2(\tau)} \left( 1_{\xi \geq y(s)} \Theta_{y(\tau), \lambda_n(\tau)}(\xi) \right) \left[ 1_{\xi < y(s)} g_- (s, \xi) + 1_{\xi > y(s)} g_+ (s, \xi) \right] d\xi dsd\xi \]

\[ + \left. \int_{0}^{\tau} e^{-(\tau-s)\lambda_n^2(\tau)} \left[ \left( \Phi(s), \Omega_{y(\tau), \lambda_n(\tau)}(s) \right) + \left( \phi(s), \mathcal{Y}^\prime(s) \Omega_{y(\tau), \lambda_n(\tau)}(s) - \omega_{y(\tau), \lambda_n(\tau)}(s) \right) \right] ds \right\}. \] (E.7)

with

\[ \Omega_{y(\tau), \lambda}(s) = \left[ 0 - \Theta_{y(\tau), \lambda, 1}(y_-), \Theta_{y(\tau), \lambda, 1}(y(s)) - \Theta_{y(\tau), \lambda, 2}(y(s)), \Theta_{y(\tau), \lambda, 2}(y_+) - 0 \right]^T, \]

\[ \omega_{y(\tau), \lambda}(s) = \left[ 0 - \Theta_{y(\tau), \lambda, 1}(y_-), \Theta_{y(\tau), \lambda, 1}(y(s)) - \Theta_{y(\tau), \lambda, 2}(y(s)), \ldots, \Theta_{y(\tau), \lambda, 2}(y_+) - 0 \right]^T, \]

\[ \theta_{y(\tau), \lambda_n(\tau), 1}(x) = \lambda_n(\tau) \sqrt{\frac{\kappa_I}{2}} \cos \left( \frac{\lambda_n(\tau)(x - y_-)}{\sqrt{\kappa_I}} \right), \]

\[ \theta_{y(\tau), \lambda_n(\tau), 2}(x) = -\lambda_n(\tau) \sqrt{\frac{\kappa_W}{2}} K_n(\tau) \cos \left( \frac{\lambda_n(\tau)(y_+ - x)}{\sqrt{\kappa_W}} \right). \]

Since \( \Theta_{y(\tau), \lambda, 1}(y_-) = \Theta_{y(\tau), \lambda, 2}(y_+) = 0 \), each scalar product in the final formula contains just one summand. Therefore, in the RHS of Eq. (E.7) there are only yet two unknown functions \( \Phi(s) \) and \( \phi(s) \), and we can set

\[ \Phi(s) = [0, \Phi(s), 0]^T, \quad \phi(s) = [0, \phi(s), 0]^T, \]

Also, in our case

\[ \mathcal{Y}^\prime(\tau) = \text{diag} \left[ 0, y(\tau), 0 \right]^T. \]

Together they allow simplification of Eq. (E.7) to the form

\[ \mathcal{T}(\tau, x) = \left\{ \mathbb{1}_{x \leq y(\tau)}, \sum_{n=1}^{\infty} \frac{\Theta_{y(\tau), \lambda_n(\tau)}(x)}{N_n(\tau)} \left[ e^{-\tau \lambda_n^2(\tau)} \int_{-\infty}^{+\infty} \left( 1_{\xi < y(0)} \Theta_{y(\tau), \lambda_n(\tau)}(\xi) \right) \left[ T_l - \eta(0, \xi) \right] d\xi \right] \right. \]

\[ + \int_{0}^{\tau} \int_{-\infty}^{+\infty} e^{-(\tau-s)\lambda_n^2(\tau)} \left( 1_{\xi \geq y(s)} \Theta_{y(\tau), \lambda_n(\tau)}(\xi) \right) \left[ 1_{\xi < y(s)} g_- (s, \xi) + 1_{\xi > y(s)} g_+ (s, \xi) \right] d\xi dsd\xi \]

\[ + \left. \int_{0}^{\tau} e^{-(\tau-s)\lambda_n^2(\tau)} \left[ \left( \Phi(s), \Omega_{y(\tau), \lambda_n(\tau)}(s) \right) + \left( \phi(s), \mathcal{Y}^\prime(s) \Omega_{y(\tau), \lambda_n(\tau)}(s) - \omega_{y(\tau), \lambda_n(\tau)}(s) \right) \right] ds \right\}. \] (E.8)
where

\[ F_n(\tau) = e^{-\tau \lambda_n^2(\tau)} \left( \int_{y(-)}^{y(0)} \Theta_{y(\tau), \lambda_n(\tau), 1}(\xi) [T_1 - A_{-}(0) - B_{-}(0)\xi] d\xi \right) \]  

or

\[ T(\tau, x) = \sum_{n=1}^{\infty} \frac{F_n(\tau) + G_n(\tau) + H_n(\tau)}{N_n(\tau)} \left[ \mathbb{1}_{x<y(\tau)} \Theta_{y(\tau), \lambda_n(\tau), 1}(x) + \mathbb{1}_{x>y(\tau)} \Theta_{y(\tau), \lambda_n(\tau), 2}(x) \right] . \]  

(E.9)

where

\begin{align*}
F_n(\tau) &= e^{-\tau \lambda_n^2(\tau)} \left( \int_{y(-)}^{y(0)} \Theta_{y(\tau), \lambda_n(\tau), 1}(\xi) [T_1 - A_{-}(0) - B_{-}(0)\xi] d\xi \right) \\
G_n(\tau) &= -\int_{0}^{\tau} e^{-(\tau-s) \lambda_n^2(\tau)} \left( \int_{y(-)}^{y(s)} \Theta_{y(\tau), \lambda_n(\tau), 1}(\xi) [A'_{-}(s) + B'_{-}(s)\xi] d\xi \right) ds + \int_{y(0)}^{y(\tau)} \Theta_{y(\tau), \lambda_n(\tau), 2}(\xi) d\xi, \\
H_n(\tau) &= \int_{0}^{\tau} e^{-(\tau-s) \lambda_n^2(\tau)} \left[ \phi(s) \Omega_{y(\tau), \lambda_n(\tau)}(s) + \phi(s) \left( y'(s) \Omega_{y(\tau), \lambda_n(\tau)}(s) - \omega_{y(\tau), \lambda_n(\tau)}(s) \right) \right] ds, \\
N_n(\tau) &= \int_{y(-)}^{y(\tau)} \Theta_{y(\tau), \lambda_n(\tau), 1}(\xi) d\xi + \int_{y(\tau)}^{y(\tau)} \Theta_{y(\tau), \lambda_n(\tau), 2}(\xi) d\xi,
\end{align*}

and

\begin{align*}
\Omega_{y(\tau), \lambda_n(\tau)}(s) &= \Theta_{y(\tau), \lambda_n(\tau), 1}(y(s)) - \Theta_{y(\tau), \lambda_n(\tau), 2}(y(s)), \\
\omega_{y(\tau), \lambda_n(\tau)}(s) &= \theta_{y(\tau), \lambda_n(\tau), 1}(y(s)) - \theta_{y(\tau), \lambda_n(\tau), 2}(y(s)).
\end{align*}

(E.11)

As now follows from Eq. (E.9) and the boundary conditions, given \( y(\tau) \) the unknown functions \( \Phi(\tau) \) and \( \phi(\tau) \) solve a system of linear Volterra integral equations of the second kind

\begin{align*}
\phi(\tau) &= \sum_{n=1}^{\infty} \frac{F_n(\tau) + G_n(\tau) + H_n(\tau)}{N_n(\tau)} \left[ s^+(y(\tau)) \Theta_{y(\tau), \lambda_n(\tau), 1}(y(\tau)) + s^-(y(\tau)) \Theta_{y(\tau), \lambda_n(\tau), 2}(y(\tau)) \right], \\
\Phi(\tau) &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{F_n(\tau) + G_n(\tau) + H_n(\tau)}{N_n(\tau)} \left[ s^+(y(\tau)) \theta_{y(\tau), \lambda_n(\tau), 1}(y(\tau)) + s^-(y(\tau)) \theta_{y(\tau), \lambda_n(\tau), 2}(y(\tau)) \right],
\end{align*}

(E.12)

where \( s^\pm(y(\tau)) = 1 + \mathbb{1}_{y=0} \).

**Further simplifications.** The Eq. (E.9) can be further simplified and rewritten in terms of the original function \( T(\tau, x) \). Integrating by parts the formula for \( G_n(\tau) \) in Eq. (E.10) yields

\begin{align*}
G_n(\tau) &= e^{-\tau \lambda_n^2(\tau)} \left[ \int_{y(-)}^{y(0)} \Theta_{y(\tau), \lambda_n(\tau), 1}(\xi) [A_{-}(0) + B_{-}(0)\xi] d\xi + \int_{y(0)}^{y(\tau)} \Theta_{y(\tau), \lambda_n(\tau), 2}(\xi) [A_{+}(0) + B_{+}(0)\xi] d\xi \right] \\
&\quad - \int_{y(-)}^{y(\tau)} \Theta_{y(\tau), \lambda_n(\tau), 1}(\xi) [A_{-}(\tau) + B_{-}(\tau)\xi] d\xi - \int_{y(\tau)}^{y(\tau)} \Theta_{y(\tau), \lambda_n(\tau), 2}(\xi) [A_{+}(\tau) + B_{+}(\tau)\xi] d\xi \\
&\quad + \int_{0}^{\tau} \frac{d}{ds} \left\{ e^{-(\tau-s) \lambda_n^2(\tau)} \int_{y(-)}^{y(s)} \Theta_{y(\tau), \lambda_n(\tau), 1}(\xi) \right\} [A_{-}(s) + B_{-}(s)\xi] d\xi ds.
\end{align*}
Recall that $B_-(0) = B_+(0) = 0$, $A_-(0) = A_+(0) = T_l$. Using the identities

$$T_l \oint_{y(0)}^{y(s)} \Theta_{y(\tau), \lambda_n(\tau), 1}(\xi) d\xi = T_l \frac{\sqrt{K_I}}{\lambda_n(\tau)} \left\{ 1 - \cos[x_n^-(\tau, y(0))] \right\} , \quad x_n^-(\tau, x) = \lambda_n(\tau)(x - y_-) / \sqrt{K_I},$$

$$T_l \oint_{y(0)}^{y(s)} \Theta_{y(\tau), \lambda_n(\tau), 2}(\xi) d\xi = T_l K_n(\tau) \frac{\sqrt{K_W}}{\lambda_n(\tau)} \left\{ 1 - \cos[x_n^+(\tau, y(0))] \right\} , \quad x_n^+(\tau, x) = \lambda_n(\tau)(y_+ - x) / \sqrt{K_W},$$

$$\int_{-}^{y(s)} \left[ A_-(s) + B_-(s) \xi \right] \Theta_{y(\tau), \lambda_n(\tau), 1}(\xi) d\xi = \frac{\sqrt{K_I} T_l}{\lambda_n(\tau)} + \frac{\kappa_l B_-(s)}{\lambda_n^2(\tau)} \Theta_{y(\tau), \lambda_n(\tau), 1}(y(s)) - \frac{A_-(s) + B_-(s) y(s)}{\lambda_n(\tau)} \Theta_{y(\tau), \lambda_n(\tau), 1}(y(s)),$$

$$\int_{y(s)}^{y(s)} \left[ A_+(s) + B_+(s) \xi \right] \Theta_{y(\tau), \lambda_n(\tau), 2}(\xi) d\xi = K_n(\tau) \frac{\sqrt{K_W}}{\lambda_n(\tau)} - \frac{B_+(s) \kappa_W}{\lambda_n^2(\tau)} \Theta_{y(\tau), \lambda_n(\tau), 2}(y(s)) + \frac{A_+(s) + B_+(s) y(s)}{\lambda_n^2(\tau)} \Theta_{y(\tau), \lambda_n(\tau), 2}(y(s)),$$

$$\int_{0}^{\tau} \frac{d}{ds} \left\{ e^{-(\tau-s) \lambda_n^2(\tau)} \int_{-s}^{s} \Theta_{y(\tau), \lambda_n(\tau), 1}(\xi) d\xi \right\} [A_-(s) + B_-(s) \xi] d\xi ds = \int_{0}^{\tau} e^{-(\tau-s) \lambda_n^2(\tau)} \left\{ \lambda_n^2(\tau) \int_{-s}^{s} \Theta_{y(\tau), \lambda_n(\tau), 1}(\xi) \left[ A_-(s) + B_-(s) \xi \right] d\xi + \Theta_{y(\tau), \lambda_n(\tau), 1}(\xi) \left[ A_-(s) + B_-(s) \xi \right] y'(s) \right\} ds,$$

taking into account that $A_-(s) + B_-(s) y(s) = A_+(s) + B_+(s) y(s)$, $\omega(y(\tau), \lambda_n(\tau) = 0$ and collecting all terms, we arrive at the following representations

$$T(\tau, x) = \frac{\sum_{n=1}^{\infty} S_n(\tau) + R_n(\tau)}{N_n(\tau)} \mathcal{M}(\tau, x),$$

$$\mathcal{M}(\tau, x) = \begin{cases} 1_{x < y(\tau)} \Theta_{y(\tau), \lambda_n(\tau), 1}(x), & x \neq y(\tau), \\
\frac{1}{2} \left[ s^+(y(\tau)) \Theta_{y(\tau), \lambda_n(\tau), 1}(y(\tau)) + s^-(y(\tau)) \Theta_{y(\tau), \lambda_n(\tau), 2}(y(\tau)) \right], & x = y(\tau), \end{cases}$$

$$\Phi(\tau) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{S_n(\tau) + R_n(\tau)}{N_n(\tau)} \left[ s^+(y(\tau)) \Theta_{y(\tau), \lambda_n(\tau), 1}(y(\tau)) + s^-(y(\tau)) \Theta_{y(\tau), \lambda_n(\tau), 2}(y(\tau)) \right].$$

Here $s^+(y(\tau)) = 1 + 1_{y_+ = y(\tau)}$ and

$$S_n(\tau) = \frac{T_l e^{-\tau \lambda_n^2(\tau)}}{\lambda_n(\tau)} \left[ \sqrt{K_I} \left( 1 - \cos[x_n^-(\tau, y(0))] \right) + K_n(\tau) \sqrt{K_W} \left( 1 - \cos[x_n^+(\tau, y(0))] \right) \right] - \frac{1}{\lambda_n(\tau)} \left[ \sqrt{K_I} T_s + \sqrt{K_W} K_n(\tau) T_l \right] \frac{\kappa_l}{\lambda_n^2(\tau)} \Theta_{y(\tau), \lambda_n(\tau), 1}(y(\tau)) - \frac{B_-(\tau) \kappa_I \Theta_{y(\tau), \lambda_n(\tau), 1}(y(\tau)) - B_+(\tau) \kappa_W \Theta_{y(\tau), \lambda_n(\tau), 2}(y(\tau))}{\lambda_n^2(\tau)},$$

$$R_n(\tau) = \int_{0}^{\tau} e^{-(\tau-s) \lambda_n^2(\tau)} \left\{ \Phi(s) \Omega_y(\tau, \lambda_n(\tau))(s) + [B_-(s) \kappa_I + y'(s) T_m] \Theta_{y(\tau), \lambda_n(\tau), 1}(y(s)) - [B_+(s) \kappa_W + y'(s) T_m] \Theta_{y(\tau), \lambda_n(\tau), 2}(y(s)) - T_m \omega_y(\tau, \lambda_n(\tau))(s) \right\} ds$$
\[ + \frac{1 - e^{-\tau \lambda_n^2(\tau)}}{\lambda_n(\tau)} \left[ \sqrt{\kappa_I T_s} + \sqrt{\kappa_W K_n(\tau) T_I} \right]. \]

The terms containing \( y'(\tau) \) can be integrated by parts to yield
\[
\int_0^\tau e^{-(\tau-s)\lambda_n^2(\tau)} y'(s) \Theta_y(\tau),\lambda_n(\tau),1(y(s)) ds = \int_0^\tau e^{-(\tau-s)\lambda_n^2(\tau)} \sin (x^- (\tau, y(s))) \, dy(s)
\]
\[
= \frac{\sqrt{\kappa_I}}{\lambda_n(\tau)} e^{-\tau \lambda_n^2(\tau)} \cos (x^- (\tau, y(0))) - \frac{\sqrt{\kappa_I}}{\lambda_n(\tau)} \cos (x^- (\tau, y(\tau)))
\]
\[
+ \sqrt{\kappa_I} \lambda_n(\tau) \int_0^\tau e^{-(\tau-s)\lambda_n^2(\tau)} \cos (x^- (\tau, y(s))) \, ds,
\]
\[
\int_0^\tau e^{-(\tau-s)\lambda_n^2(\tau)} y'(s) \Theta_y(\tau),\lambda_n(\tau),2(y(s)) ds = K_n(\tau) \int_0^\tau e^{-(\tau-s)\lambda_n^2(\tau)} \sin (x^+ (\tau, y(s))) \, dy(s)
\]
\[
= -K_n(\tau) \frac{\sqrt{\kappa_W}}{\lambda_n(\tau)} e^{-\tau \lambda_n^2(\tau)} \cos (x^+ (\tau, y(0))) + K_n(\tau) \frac{\sqrt{\kappa_W}}{\lambda_n(\tau)} \cos (x^+ (\tau, y(\tau)))
\]
\[
+ K_n(\tau) \sqrt{\kappa_W} \lambda_n(\tau) \int_0^\tau e^{-(\tau-s)\lambda_n^2(\tau)} \cos (x^+ (\tau, y(s))) \, ds.
\]

Taking into account that from Eq. (96)
\[
\sqrt{\kappa_I} \cos [x^- (\tau, y(\tau))] + \sqrt{\kappa_W} K_n(\tau) \cos [x^+ (\tau, y(\tau))] = 0, \quad (E.15)
\]
and slightly regrouping the terms between \( S_n \) and \( R_n \) we get another representation of these functions which now doesn’t contain \( y'(\tau) \)

\[
S_n(\tau) = \frac{e^{-\tau \lambda_n^2(\tau)}}{\lambda_n(\tau)} \left[ \sqrt{\kappa_I} (T_I - T_s) - \frac{1}{\lambda_n^2(\tau)} \left[ B_-(\tau) \kappa_I \Theta_y(\tau),\lambda_n(\tau),1(y(\tau)) - B_+(\tau) \kappa_W \Theta_y(\tau),\lambda_n(\tau),2(y(\tau)) \right] \right]
\]
\[
+ \left( T_m - T_s \right) \frac{e^{-\tau \lambda_n^2(\tau)}}{\lambda_n(\tau)} \left[ \sqrt{\kappa_I} \cos (x^- (\tau, y(0))) + \sqrt{\kappa_W} K_n(\tau) \cos (x^+ (\tau, y(0))) \right]
\]
\[
R_n(\tau) = \int_0^\tau e^{-(\tau-s)\lambda_n^2(\tau)} \left[ \Phi(s) \Omega_y(\tau),\lambda_n(\tau)(s) + B_- (s) \kappa_I \Theta_y(\tau),\lambda_n(\tau),1(y(s)) - B_+ (s) \kappa_W \Theta_y(\tau),\lambda_n(\tau),2(y(s)) \right] ds.
\]