Stabilization with finite dimensional controllers for a periodic parabolic system under perturbations in the system conductivity

Ling Lei *

Department of Mathematics and Statistics, Wuhan University,
Wuhan, 430072, P.R.China

Abstract. This work studies the stabilization for a periodic parabolic system under perturbations in the system conductivity. A perturbed system does not have any periodic solution in general. However, we will prove that the perturbed system can always be pulled back to a periodic system after imposing a control from a fixed finite dimensional subspace. The paper continues the author’s previous work in [8].

Key words. approximate periodic solution, stabilization through a finite dimensional control space, parabolic system, unique continuation of elliptic equations.

AMS subject classification. 35B37, 93B99.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a $C^2$-smooth boundary $\partial \Omega$ and let $\omega \subset \Omega$ be a subdomain. Write $Q = \Omega \times (0, T)$ with $T > 0$ and write $\Sigma = \partial \Omega \times (0, T)$. Consider the following parabolic equation:

$$\begin{cases}
\frac{\partial y}{\partial t}(x,t) + L_0 y(x,t) + e(x,t)y(x,t) = f(x,t), & \text{in } Q = \Omega \times (0, T), \\
y(x,t) = 0, & \text{on } \Sigma = \partial \Omega \times (0, T),
\end{cases}$$

(1.1)

where

$$L_0 y(x,t) = - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_j} (a^{ij}(x) \frac{\partial}{\partial x_i} y(x,t)) + c(x)y(x,t)$$

*Supported by a National Science Foundation of China Research Grant (NSFC-10801108)
is considered as the system operator. Here and in all that follows, we make the following
regularity assumptions for the coefficients of $L_0$:

(I): 
$$a^{ij}(x) \in \text{Lip}(\overline{\Omega}), \quad a^{ij}(x) = a^{ji}(x), \quad \text{and} \quad \lambda^* |\xi|^2 \leq \sum_{i,j=1}^{N} a^{ij}(x) \xi_i \xi_j \leq \frac{1}{\lambda^*} |\xi|^2, \quad \text{for} \ \xi \in \mathbb{R}^N$$

(1.2)

with $\lambda^*$ a certain positive constant;

(II): 
$$c(x) \in L^\infty(\Omega), \quad e(x,t) \in L^\infty(0,T; L^q(\Omega)) \quad \text{with} \quad q > \max\{N,2\}, \quad \text{and} \quad f(x,t) \in L^2(Q).$$

(1.3)

In such a system, we regard $e(x,t)$ as a perturbation in the system conductivity. Suppose
in the ideal case, namely, in the case when the perturbation $e(x,t) \equiv 0$, (1.1) has a
periodic solution $y_0(x,t)$:

$$\begin{cases}
\frac{\partial y_0}{\partial t}(x,t) + L_0 y_0(x,t) = f(x,t), & \text{in } Q, \\
y_0(x,t) = 0, & \text{on } \Sigma, \\
y_0(x,0) = y_0(x,T), & \text{in } \Omega.
\end{cases}$$

(1.4)

Then the presence of the error term $e(x,t)$ may well destroy the periodicity of the system.
Indeed, (1.1) may no longer have any periodic solution. (See Section 3.) The problem
that we are interested in in this paper is to understand if there is a finite (constructible)
dimensional subspace $U \subset L^2(Q)$, such that, after imposing a control $u_e \in U$, we can
restore the periodic solution $y_e$. Moreover, we would like to know if $y_e$ is close to $y_0$ and if
the energy of $u_e$ is small, when $e(x,t)$ is small. Our main purpose of this paper is to show
that we can indeed achieve this goal in the small perturbation case, even if the control is
only imposed over a subregion $\omega$ of $\Omega$. The basic tool for this study is the existence and
energy estimate for the approximate periodic solutions obtained in the author’s previous
paper [8].

To state our results, we first recall the definition of approximate periodic solutions
with respect to the elliptic operator $L_0$.

Notice that $L_0$ is a symmetric operator. Consider the eigenvalue problem of $L_0$:

$$\begin{cases}
L_0 X(x) = \lambda X(x), \\
X(x)|_{\partial \Omega} = 0.
\end{cases}$$

(1.5)

Making use of the regularity assumptions of the coefficients of $L_0$, we know (see, [2] [5],
for example) that (1.5) has a complete set of eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ with the associated
eigenvectors $\{X_j(x)\}_{j=1}^{\infty}$ such that

$$L_0 X_j(x) = \lambda_j X_j(x).$$
Choose \( \{X_j(x)\}_{j=1}^\infty \) such that it forms an orthonormal basis of \( L^2(\Omega) \). Therefore, for any \( y(x,t) \in L^2(Q) \), we have \( y(x,t) = \sum_{j=1}^\infty y_j(t)X_j(x) \), where

\[
y_j(t) = \langle y(x,t), X_j(x) \rangle = \int_\Omega y(x,t)X_j(x)dx \in L^2(0,T).
\]

**Definition 1.1.** We call \( y(x,t) \) is a \( K \)-approximate periodic solution of (1.1) with respect to \( L_0 \) if

(a): \( y \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)) \) is a weak solution of (1.1);

(b): \( y \in S_K \), where \( S_K \) is the space of the following functions:

\[
S_K = \{ y(x,t) \in L^2(Q); \ y_j(0) = y_j(T), \ \text{for} \ j \geq K + 1, \ y_j(t) = \int_\Omega y(x,t)X_j(x)dx \}.
\]

When \( K = 0 \), we will always regard \( \sum_{j=1}^0 = 0 \). Hence, a 0-approximate periodic solution of (1.1) is a regular periodic solution. In what follows, we write \( \langle y(\cdot,t), y(\cdot,t) \rangle = \int_\Omega y^2(x,t)dx = \|y(\cdot,t)\|^2 \), and we denote \( y_x \) for the derivative of \( y(x,t) \) with respect to \( t \).

Our first result of this paper can be stated as follows:

**Theorem 1.1** Consider the system (1.1), where \( e(x,t) \) is regarded as a perturbation in the system conductivity. Suppose that (1.1) has a periodic solution \( y_0(x,t) \) at the ideal case with \( e(x,t) \equiv 0 \). Assume that \( \|e(x,t)\|_{L^\infty(0,T;L^2(\Omega))} = \text{ess sup}_{t \in [0,T]} \|e(\cdot,t)\|_{L^2(\Omega)} < \varepsilon \),

where \( \varepsilon < 1 \) is a small constant which depends only on \( L_0, \Omega, N, q, T \) with \( q > \max\{N,2\} \). Then there are a non-negative integer \( K_0 \), depending only on \( L_0, \Omega, N, q, T \) (but not \( f \)), and a unique outside force of the form

\[
u_e(x) := \sum_{j=1}^{K_0} u_jX_j(x) \in U = \text{span}_R \{X_1(x), X_2(x), \ldots, X_{K_0}(x)\},
\]

where \( u_j \in R \), such that the following has a unique periodic solution \( y \) satisfying:

\[
\begin{align*}
\frac{\partial y(x,t)}{\partial t} + L_0y(x,t) + e(x,t)y(x,t) &= f(x,t) + u_e(x), & \text{in} & \ Q, \\
y(x,t) &= 0, & \text{on} & \ \Sigma, \\
\langle y(x,0), X_j(x) \rangle &= \langle y_0(x,0), X_j(x) \rangle, & \text{for} & \ j \leq K_0, \\
y(x,0) &= y(x,T), & \text{in} & \ \Omega.
\end{align*}
\]

(1.6)
Moreover, we have the following energy estimate:

\[
\sup_{t \in [0,T]} \|(y - y_0)(\cdot, t)\|^2 + \int_0^T \|\nabla(y - y_0)(\cdot, t)\|^2 dt \\
\leq C(\text{system, } K_0)\|e(x, t)\|^2_{L^\infty(0,T; L^2(\Omega))}(1 + |\vec{a}|^2 + \int_Q f^2 dx dt),
\]

(1.7)

and

\[
\|u_e\|^2_{L^2(\Omega)} \leq C(\text{system, } K_0)\|e(x, t)\|^2_{L^\infty(0,T; L^2(\Omega))}(1 + |\vec{a}|^2 + \int_Q f^2 dx dt),
\]

(1.8)

where \(\vec{a} = (a_1, a_2, \cdots, a_{K_0}) = (\langle y_0(x, 0), X_1(x) \rangle, \langle y_0(x, 0), X_2(x) \rangle, \cdots, \langle y_0(x, 0), X_{K_0}(x) \rangle)\).

Here and in what follows, \(C(\text{system, } K_0)\) denotes a constant depending only on \(L_0, \Omega, N, q, T\), which may be different in different contexts.

In Section 3 of this paper, we will construct an example, showing that without outside controls, (1.1) has no periodic solutions in general. This is one of the main features in our Theorem 1.1: The control can always be taken from a certain fixed constructible finite dimensional subspace to regain the periodicity, while the perturbation space for \(e(x, t)\), which destroys the periodicity, is of infinite dimension. We also notice that our system operator \(L_0\) is not assumed to be positive.

The second part of this work is to consider the same problem as studied in the first part, but with the control only imposed over a subregion \(\omega \subset \Omega\) and time interval \(E \subset [0, T]\), \(m(E) > 0\). We will similarly obtain the following:

**Theorem 1.2.** Suppose that the system (1.1) has a periodic solution \(y_0(x, t)\) at the ideal case with \(e(x, t) \equiv 0\). Then there are a positive integer \(K_0\), a small constant \(\varepsilon > 0\), depending only on \(L_0, \Omega, N, q, T\) \((q > \max\{N, 2\})\), such that, when

\[
\|e(x, t)\|_{L^\infty(0,T; L^2(\Omega))} = \text{ess sup}_{t \in (0,T)} \|e(x, t)\|_{L^2(\Omega)} < \varepsilon,
\]

the following has a unique periodic solution:

\[
\begin{cases}
\frac{\partial y(x, t)}{\partial t} + L_0 y(x, t) + e(x, t) y(x, t) = f(x, t) + \sum_{j=1}^{K_0} \chi_{\omega}(x) \chi_{E}(t) u_j X_j(x), & \text{in } Q, \\
y(x, t) = 0, & \text{on } \Sigma, \\
\langle y(x, 0), X_j(x) \rangle = a_j, & \text{for } j \leq K_0, \\
y \in S_{K_0},
\end{cases}
\]

(1.9)

where \((a_1, a_2, \cdots, a_{K_0}) = (\langle y_0(x, 0), X_1(x) \rangle, \langle y_0(x, 0), X_2(x) \rangle, \cdots, \langle y_0(x, 0), X_{K_0}(x) \rangle) = \vec{a}\),

\((u_1, u_2, \cdots, u_{K_0}) = \vec{u} \in \mathbb{R}^{K_0}\). Moreover,

\[
|\vec{a}|^2 \leq C(\text{system, } K_0, \omega) \frac{\|e(x, t)\|^2_{L^\infty(0,T; L^2(\Omega))}}{(m(E))^2}(1 + |\vec{a}|^2 + \int_Q f^2 dx dt),
\]

(1.10)
and
\[
\sup_{t \in [0, T]} \|(y - y_0)(\cdot, t)\|^2 + \int_0^T \|\nabla(y - y_0)(\cdot, t)\|^2 dt \\
\leq C(\text{system, } K_0, \omega) \left( \frac{\|e(x, t)\|_{L^\infty([0,T];L^q(\Omega))}^2}{(m(E))^2} \right) (1 + |\vec{a}|^2 + \int_Q f^2 dx dt).
\] (1.11)

Here,
\[ \chi_\omega(x), \chi_E(t) \]
are the characteristic functions for \( \omega \) and \( E \), respectively; and \( C(\text{system, } K_0, \omega) \) is a constant depending only on \( \omega, L_0, \Omega, N, q, T \).

Theorem 1.1 and Theorem 1.2 give stabilization results for the periodic solutions of a linear parabolic system under small perturbation of the system conductivity, modifying a control from a fixed finite dimensional subspace. We do not know if similar results as in Theorem 1.1 hold under the large perturbation case.

The paper is organized as follows. In Section 2, we prove Theorem 1.1. In Section 3, we give an example to show that with a small perturbation \( e(x, t) \), (1.1) has no periodic solution in general. In section 4, we give the proof of Theorem 1.2.

2 Small perturbation

In this Section, we give a proof of Theorem 1.1, based on the author's previous paper [8]. For convenience of the reader, we first recall the following result of [8], which will be used here.

**Theorem 2.1.** Assume (1.2) and (1.3). Let \( e(x, t) \in \mathcal{M}(q, M) \), where, for any positive number \( M \) and \( q > \frac{N}{2} \),
\[ \mathcal{M}(q, M) := \{ e(x, t) \in L^\infty(0, T; L^q(\Omega)) ; \text{ess sup}_{t \in (0, T)} \| e(x, t) \|_{L^q(\Omega)} \leq M \}. \]

Then, there exists an integer \( K_0(L_0, M, \Omega, q, N, T) \geq 0 \), depending only on \( (L_0, M, \Omega, q, N, T) \) (but not \( f(x, t) \)), such that for any \( K \geq K_0(L_0, M, \Omega, q, N, T) \) and any initial value \( \vec{a} = (a_1, a_2, \ldots, a_K) \in \mathbb{R}^K \), we have a unique solution to the following equation:
\[
\begin{aligned}
&\frac{\partial y(x, t)}{\partial t} + L_0 y(x, t) + e(x, t)y(x, t) = f(x, t), &\text{in } Q, \\
y(x, t) = 0, &\text{on } \Sigma, \\
\langle y(x, 0), X_j(x) \rangle = a_j, &\text{for } j \leq K, \\
y \in S_K.
\end{aligned}
\] (2.1)

Moreover, for such a solution \( y(x, t) \), we have the following energy estimate:
\[
\sup_{t \in [0, T]} \|y(\cdot, t)\|^2 + \int_0^T \|\nabla y(\cdot, t)\|^2 dt \leq C(L_0, M, \Omega, q, N, T)(|\vec{a}|^2 + \int_Q f^2 dx dt).
\] (2.2)
Now, suppose \( y_0 \) is a periodic solution of (1.1) with \( e(x, t) = 0 \), namely,

\[
\begin{aligned}
\begin{cases}
\frac{\partial y_0}{\partial t}(x, t) + L_0y_0(x, t) = f(x, t), \
y_0(x, t) = 0, \
y_0(x, 0) = y_0(x, T),
\end{cases}
\end{aligned}
\tag{2.3}
\]

Let

\[ a_j = \langle y_0 \rangle_0(0) = \langle y_0(x, 0), X_j(x) \rangle, \quad \text{for } j = 1, 2, \ldots. \]

In all that follows, we assume that \( e(x, t) \in \mathcal{M}(q, M) \) with \( M = 1 \). By Theorem 2.1, there exists an integer \( K_0(L_0, M, \Omega, q, N, T) \geq 0 \), such that for the initial value \( \vec{a} = (a_1, a_2, \ldots, a_{K_0}) \in \mathbb{R}^{K_0} \), we have a unique solution \( y(x, t) \) satisfying the following equations:

\[
\begin{aligned}
\begin{cases}
\frac{\partial y}{\partial t}(x, t) + L_0y(x, t) + e(x, t)y(x, t) = f(x, t) + \sum_{j=1}^{K_0} u_jX_j(x), \
y(x, t) = 0, \
\langle y(x, 0), X_j(x) \rangle = a_j, \
y \in S_{K_0}.
\end{cases}
\end{aligned}
\tag{2.4}
\]

Here, \( \vec{u} = (u_1, u_2, \ldots, u_{K_0}) \in \mathbb{R}^{K_0}. \)

Subtracting (2.3) from (2.4), we get the following equation:

\[
\begin{aligned}
\begin{cases}
(y - y_0)_t + L_0(y - y_0) + e(x, t)(y - y_0) = \sum_{j=1}^{K_0} u_jX_j(x) - e(x, t)y_0, \
y(x, t) - y_0(x, t) = 0, \
\langle y(x, 0) - y_0(x, 0), X_j(x) \rangle = 0, \
(y - y_0) \in S_{K_0}.
\end{cases}
\end{aligned}
\tag{2.5}
\]

We define a map

\[ J : \mathbb{R}^{K_0} \longrightarrow \mathbb{R}^{K_0} \]

by

\[ J(u_1, u_2, \ldots, u_{K_0}) = ((y - y_0)_1(T), (y - y_0)_2(T), \ldots, (y - y_0)_{K_0}(T)). \]

Write \( v = y - y_0 = v_0 + v_u \). Here, \( v_0 \) and \( v_u \) are the solution of the following equations, respectively,

\[
\begin{aligned}
\begin{cases}
(v_0)_t + L_0v_0 + e(x, t)v_0 = -e(x, t)y_0, \
v_0 = 0, \
v_0(0) = 0, \
v_0 \in S_{K_0}.
\end{cases}
\end{aligned}
\tag{2.6}
\]
and
\[
\begin{cases}
(v_u)_t + L_0 v_u + e(x, t)v_u = \sum_{j=1}^{K_0} u_j X_j(x), & \text{in } Q, \\
v_u = 0, & \text{on } \Sigma, \\
(v_u)_j(0) = 0, & \text{for } j \leq K_0, \\
v_u \in S_{K_0}.
\end{cases}
\]
We are led to the question to find out if there is a vector \( \vec{u} = (u_1, u_2, \cdots, u_{K_0}) \in \mathbb{R}^{K_0} \) such that
\[
J(\vec{u}) = ((y - y_0)_1(T), (y - y_0)_2(T), \cdots, (y - y_0)_K(T)) = (0, 0, \cdots, 0).
\]
Indeed, if this is the case, then \( y \) is a periodic solution with the required estimate as we will see later.

For this purpose, we write \( J_0 = ((v_0)_1(T), (v_0)_2(T), \cdots, (v_0)_K(T)) \) and
\[
J^*(\vec{u}) = ((v_0)_1(T), (v_0)_2(T), \cdots, (v_0)_K(T)).
\]
Then
\[
J(\vec{u}) = J_0 + J^*(\vec{u}).
\]
Now, it is easy to see that \( J^* \) is linear in \( (u_1, u_2, \cdots, u_{K_0}) \). We next claim that \( J^* \) is invertible under the small perturbation case. If not, we can find a vector \( \vec{\xi} = (\xi_1, \xi_2, \cdots, \xi_{K_0}) \in \mathbb{R}^{K_0} \) with
\[
|\vec{\xi}| = \sqrt{\xi_1^2 + \xi_2^2 + \cdots + \xi_{K_0}^2} = 1
\]
such that \( J^*(\vec{\xi}) = 0 \). Hence, we have a unique solution to the following problem:
\[
\begin{cases}
w_t + L_0 w + e(x, t)w = \sum_{j=1}^{K_0} \xi_j X_j(x), & \text{in } Q, \\
w = 0, & \text{on } \Sigma, \\
w_j(0) = w_j(T) = 0, & \text{for } j \leq K_0, \\
w \in S_{K_0}.
\end{cases}
\]
Then by the energy estimate in Theorem 2.1, we have for \( w(x, t) \),
\[
\sup_{t \in [0, T]} \|w(\cdot, t)\|^2 + \int_0^T \|\nabla w(\cdot, t)\|^2 dt \leq C(\text{system}) \cdot T \cdot |\vec{\xi}|^2 \leq C(\text{system}, K_0).
\]
As mentioned before, we use \( C(\text{system}, K_0) \) to denote a constant depending only on \( L_0, M, \Omega, q, N, T \), which may be different in different contexts.
Next, by the Hölder inequality (see Claim 2.2 of [8]), we have
\[
\int_{\Omega} |e(x, t)w(x, t)X_j(x)|\,dx \leq C(\Omega, N, q)\|e(x, t)\|_{L^\infty(0,T;L^q(\Omega))}\|e(\cdot, t)\|_{L^2(\Omega)}^2
\]
\[+\|X_j(x)\|_{L^2(\Omega)}^2 + \|\nabla w(\cdot, t)\|_{L^2(\Omega)}^2 + \|\nabla X_j(x)\|_{L^2(\Omega)}^2\]
\[\leq C(\Omega, N, q)\|e(x, t)\|_{L^\infty(0,T;L^q(\Omega))}[1 + \lambda_j^2]
\]
\[+\|w(\cdot, t)\|_{L^2(\Omega)}^2 + \|\nabla w(\cdot, t)\|_{L^2(\Omega)}^2].
\]

By (2.9), we have
\[
\int_0^T \int_{\Omega} |e(x, t)w(x, t)X_j(x)|\,dx\,dt \leq C(\text{system, } K_0)\|e(x, t)\|_{L^\infty(0,T;L^q(\Omega))}.
\]

(2.11)

Next, from (2.10), we get
\[
(e^{\lambda_j t}w_j(t))' + \int_{\Omega} e(x, t)w(x, t)X_j(x)e^{\lambda_j t}\,dx = e^{\lambda_j t}\xi_j, \quad \text{for } j = 1, 2, \ldots, K_0.
\]

Integrating the above over [0,T], we get, for \( j = 1, 2, \ldots, K_0 \),
\[
0 + \int_0^T \int_{\Omega} e(x, t)w(x, t)X_j(x)e^{\lambda_j t}\,dx\,dt = \xi_j \int_0^T e^{\lambda_j t}\,dt.
\]

Namely,
\[
\xi_j = \begin{cases} 
\frac{\int_0^T \int_{\Omega} e(x, t)w(x, t)X_j(x)e^{\lambda_j t}\,dx\,dt}{\lambda_j^2(e^{\lambda_j T} - 1)}, & \text{for } \lambda_j \neq 0, \\
\frac{\int_0^T \int_{\Omega} e(x, t)w(x, t)X_j(x)\,dx\,dt}{T}, & \text{for } \lambda_j = 0.
\end{cases}
\]

Hence, we get, for \( j = 1, 2, \ldots, K_0 \),
\[
|\xi_j| \leq C(\text{system, } K_0)\int_0^T \int_{\Omega} |e(x, t)w(x, t)X_j(x)|\,dx\,dt
\]
\[\leq C(\text{system, } K_0)\|e(x, t)\|_{L^\infty(0,T;L^q(\Omega))}.
\]

(2.12)

We get
\[
1 = |\vec{\xi}| \leq C(\text{system, } K_0)\sqrt{K_0}\|e(x, t)\|_{L^\infty(0,T;L^q(\Omega))}.
\]

This gives a contradiction when
\[
\|e(x, t)\|_{L^\infty(0,T;L^q(\Omega))} < \frac{1}{C(\text{system, } K_0)\sqrt{K_0}}.
\]

Therefore, we showed that \( J^* \) is invertible when \( \|e(x, t)\|_{L^\infty(0,T;L^q(\Omega))} < \epsilon \) with a certain \( \epsilon \) depending only on \( L_0, \Omega, N, q, T. \)
Hence, for any given \( \tilde{b} = (b_1, b_2, \cdots, b_{K_0}) \in \mathbb{R}^{K_0} \), there exists a unique
\[
\vec{u} = (u_1, u_2, \cdots, u_{K_0}) \in \mathbb{R}^{K_0}
\]
such that
\[
J^*(\vec{u}) = J^*(u_1, u_2, \cdots, u_{K_0}) = (b_1, b_2, \cdots, b_{K_0}).
\]
Back to the equation (2.7), we have
\[
\frac{d(v_u)_{j}(t)}{dt} + \lambda_j(v_u)_{j}(t) + \int_{\Omega} e(x, t)v_u(x, t)X_j(x)\,dx = u_j, \quad \text{for } j = 1, 2, \cdots, K_0.
\]
Then
\[
\frac{d[e^{\lambda_j t}(v_u)_{j}(t)]}{dt} + \int_{\Omega} e(x, t)v_u(x, t)X_j(x)e^{\lambda_j t}\,dx = u_je^{\lambda_j t}, \quad \text{for } j = 1, 2, \cdots, K_0.
\]
Integrating the above over \([0, T]\), by the definition of \(J^*\), we have
\[
b_j e^{\lambda_j T} - 0 + \int_{0}^{T} \int_{\Omega} e(x, t)v_u(x, t)X_j(x)e^{\lambda_j t}\,dxdt = u_j \int_{0}^{T} e^{\lambda_j t}\,dt, \quad \text{for } j = 1, 2, \cdots, K_0.
\]
We then get
\[
u_j = \begin{cases} 
\frac{b_j e^{\lambda_j T} + \int_{0}^{T} \int_{\Omega} e(x, t)v_u(x, t)X_j(x)e^{\lambda_j t}\,dxdt}{\frac{1}{\lambda_j}(e^{\lambda_j T} - 1)}, & \text{for } \lambda_j \neq 0, \\
\int_{0}^{T} \int_{\Omega} e(x, t)v_u(x, t)X_j(x)\,dxdt, & \text{for } \lambda_j = 0.
\end{cases}
\]
\[
|u_j|^2 \leq 2e^{2\lambda_0 T}|b_j|^2 + 2e^{2\lambda_k T}\left[\int_{0}^{T} \int_{\Omega} e(x, t)v_u(x, t)X_j(x)\,dxdt\right]^2
\]
\[
\leq 2e^{2\lambda_0 T}|b_j|^2 + 2e^{2\lambda_k T} \cdot \sup_{\Omega} |X_j|^2 \left[\int_{0}^{T} \|e(\cdot, t)\|_{L^q(\Omega)}\|\dot{u}(\cdot, t)\|_{L^{q'}(\Omega)}\,dt\right]^2
\]
Here \(1/q + 1/q' = 1\). Since \(\Omega\) is bounded and \(q' = \frac{q}{q-1} \leq 2\), by the Hölder inequality, we have \(\|v_u\|_{L^q(\Omega)} \leq C(\Omega, q)\|v_u\|_{L^2(\Omega)}\). Hence,
\[
\left[\int_{0}^{T} \|e(\cdot, t)\|_{L^q(\Omega)}\|v_u(\cdot, t)\|_{L^{q'}(\Omega)}\,dt\right]^2 \leq C(\Omega, T, q)\|e\|_{L^\infty(0, T; L^q(\Omega))}^2 \|v_u\|_{L^2(\Omega)}^2.
\]
By the energy estimate in Theorem 2.1, we have \(\|v_u\|_{L^2(\Omega)}^2 \leq C(\text{system}, K_0)\|u\|^2\). Hence, as argument before, when \(\|e\|_{L^\infty(0, T; L^q(\Omega))}^2\) is small, we can solve the above to obtain the following:
\[
|\vec{u}|^2 \leq C(\text{system}, K_0)|\vec{b}|^2.
\]
(2.13)
Back to (2.5), we need to find \( \vec{u} = (u_1, u_2, \ldots, u_{K_0}) \) such that the solution in (2.5) has the property \( (y - y_0)_j(T) = 0 \) for \( j = 1, 2, \ldots, K_0 \). As mentioned before, \( v = y - y_0 \) is then a periodic solution. Thus \( y = v + y_0 \) is a periodic solution of (2.4) after applying the control force \( \sum_{j=1}^{K_0} u_j X_j(x) \). To this aim, we need only to find \( \vec{u} \) such that

\[
J(\vec{u}) = 0 \quad \text{or} \quad J^*(\vec{u}) = -J_0.
\]

By the definition of \( J_0 \),

\[
J_0 = \vec{b} = (-b_1, -b_2, \ldots, -b_{K_0})
\]

is given by

\[
\begin{cases}
(v_0)_t + Lv_0 + e(x, t)v_0 = -e(x, t)y_0, & \text{in } Q, \\
v_0 = 0, & \text{on } \Sigma, \\
(v_0)_j(0) = 0, (v_0)_j(T) = -b_j, & \text{for } j \leq K_0, \\
v_0 \in S_{K_0}.
\end{cases}
\]

By the energy estimate of Theorem 2.1, we have

\[
|\vec{b}|^2 \leq \|v_0(\cdot, T)\|_{L^2(\Omega)}^2 \\
\leq C(\text{system}, K_0) \int_0^T \int_{\Omega} (-ey_0)^2 dx dt \\
\leq C(\text{system}, K_0) \int_0^T \{ \|e(x, t)\|_{L^2(\Omega)}^2 \|y_0(\cdot, t)\|_{L^{2q}(\Omega)}^{2q/2} \} dt \quad (2.14) \\
\leq C(\text{system}, K_0) \|e(x, t)\|_{L^\infty(0,T;L^q(\Omega))} \|\nabla y_0\|_{L^2(Q)}^2 \\
\leq C(\text{system}, K_0) \|e(x, t)\|_{L^\infty(0,T;L^q(\Omega))} (|\vec{a}|^2 + \int_Q f^2 dx dt),
\]

where \( \vec{a} = (a_1, a_2, \ldots, a_{K_0}) = (\langle y_0(x, 0), X_1(x) \rangle, \langle y_0(x, 0), X_2(x) \rangle, \ldots, \langle y_0(x, 0), X_{K_0}(x) \rangle) \).

Thus, by (2.13), we get

\[
|\vec{a}|^2 \leq C(\text{system}, K_0) \|e(x, t)\|_{L^\infty(0,T;L^q(\Omega))}^2 (1 + |\vec{a}|^2 + \int_Q f^2 dx dt). \quad (2.15)
\]

By (2.2), (2.14) and (2.15), we obtain

\[
\sup_{t \in [0,T]} \| (y - y_0)(\cdot, t) \|^2 + \int_0^T \| \nabla (y - y_0)(\cdot, t) \|^2 dt \\
\leq C(\text{system}, K_0) \|e(x, t)\|_{L^\infty(0,T;L^q(\Omega))}^2 (1 + |\vec{a}|^2 + \int_Q f^2 dx dt),
\]

Summarizing the above, we complete the proof of Theorem 1.1. \( \blacksquare \)
3 An example

In this section, we present an example, showing that with a small perturbation \( e(x, t) \), (1.1) has no periodic solution in general. This demonstrates the importance of an outside control to gain back the periodicity as in Theorem 1.1.

We consider the following one dimensional parabolic equation:

\[
\begin{cases}
y_t - y_{xx} - y - e(x)y = f(x), & 0 \leq x \leq \pi, 0 \leq t \leq T, \\
y(0, t) = y(\pi, t) = 0, & 0 \leq t \leq T.
\end{cases}
\]

(3.1)

Let \( L_e y = -y_{xx} - y - e(x)y \) with \( e(x) \in C^0[0, \pi] \). Suppose 0 is an eigenvalue of \( L_e \) with eigenvectors \( \{X_j(x)\}_{j=1}^m \). Then (3.1) has a periodic solution if and only if

\[
\int_0^\pi f(x)X_j(x)dx = 0,
\]

for \( j = 1, 2, \cdots, m \).

Now, when \( e(x) = 0 \), then 0 is the first eigenvalue of \( L_0 \) with \( \sin x \) as a basis of the 0-eigenspace. Hence, (3.1) has a periodic solution if and only if

\[
\int_0^\pi f(x) \sin x dx = 0 \text{ or } f(x) = \sum_{j=2}^\infty a_j \sin jx, \quad \sum_{j=2}^\infty |a_j|^2 < \infty.
\]

Now suppose \( e(x) \approx 0 \). The first eigenvalue \( \lambda_e \) of \( L_e \) is given by

\[
\lambda_e = \min_{\varphi \in H^1_0(0, \pi), \|\varphi\|_{L^2(0, \pi)}=1} J_e(\varphi, \varphi),
\]

where

\[
J_e(\varphi, \varphi) = \int_0^\pi (\varphi_x^2 - \varphi^2 - e(x)\varphi^2)dx.
\]

(See [5]). Hence,

\[
\lambda_e \leq \min_{\varphi \in H^1_0(0, \pi), \|\varphi\|_{L^2(0, \pi)}=1} \int_0^\pi (\varphi_x^2 - \varphi^2)dx + \max |e(x)| \int_0^\pi \varphi^2 dx
\]

\[
\leq 0 + \max |e(x)| \int_0^\pi \varphi^2 dx
\]

\[
\leq \max |e(x)|. \quad \text{(3.2)}
\]

\[
\lambda_e = J_e(\varphi_e, \varphi_e) = \int_0^\pi (\varphi_e)^2 dx - \int_0^\pi (1 + e(x))\varphi_e^2 dx
\]

with \( \varphi_e \) the eigenvector corresponding to \( \lambda_e \) and \( \|\varphi_e\|_{L^2(0, \pi)} = 1 \).

Since 0 is the first eigenvalue of \( L_0 \), we have

\[
\lambda_e = \int_0^\pi ((\varphi_e)^2 - (\varphi_e)^2) dx - \int_0^\pi e(x)\varphi_e^2 dx
\]

\[
\geq - \max |e(x)| \quad \text{(3.3)}
\]
By (3.2) and (3.3), we get

\[ |\lambda_e| \leq \max |e(x)|, \text{ and } \lambda_e \to 0 \text{ as } e(x) \to 0. \]

Next, consider the system with \( e(x) + \lambda_e \) as the perturbation in the system conductivity:

\[
\begin{aligned}
y_t - y_{xx} - y - (e(x) + \lambda_e)y &= f(x), & 0 \leq x \leq \pi, & 0 \leq t \leq T, \\
y(0, t) = y(\pi, t) &= 0, & 0 \leq t \leq T.
\end{aligned}
\tag{3.4}
\]

Then when \( e(x) \approx 0 \), we have \( (e(x) + \lambda_e) \approx 0 \). However, if (3.4) still has a periodic solution, we have

\[ \int_0^\pi f(x)\varphi_e dx = 0. \]

If this is the case for any given \( f \), we then have

\[ \int_0^\pi \sin jx\varphi_e dx = 0, \text{ for } j = 2, 3, \ldots. \]

This implies that \( \varphi_e = C \sin x \) and thus

\[ -e(x) \sin x = \lambda_e \sin x, \text{ or } e(x) = -\lambda_e. \]

This is a contradiction unless \( e(x) \equiv \text{const.} \). This shows that for any non-constant small perturbation in \( e(x) \), for most a priori given \( f \), the periodicity of the system will get lost.

\section{Local stabilization}

In this section, we consider the same problem as studied in Section 2, but with the control only imposed over a subregion \( \omega \subset \Omega \) and time interval \( E \subset [0, T] \) with \( m(E) > 0 \).

For the proof of Theorem 1.2, we need the following lemma, whose quantitative version in the Laplacian case can be found in [6] and [7]:

**Lemma 4.1** Let \( X_{ij}(\omega) = \int_\omega X_i(x)X_j(x)dx \). Then the symmetric matrix \( X(\omega, k) = (X_{ij}(\omega))_{1 \leq i, j \leq k} \) is positive definite for any \( k \geq 1 \). In particular, it is invertible.

**Proof of Lemma 4.1:** Let \( a = (a_1, a_2, \ldots, a_k) \in \mathbb{R}^k \) and let

\[ I(a, a) = \int_\omega |\sum_{j=1}^k a_jX_j(x)|^2 dx. \]

Then

\[ I(a, a) = a \cdot X(\omega, k) \cdot a^\top, \text{ where } a^\top = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix}. \]
Apparently, $I(a,a) \geq 0$. If $X(\omega,k)$ is not positive definite, then there is a vector $a' = (a'_1, a'_2, \ldots, a'_k) \neq 0$ such that $I(a',a') = 0$. Without loss of generality, assume that $a'_k \neq 0$. Hence,

$$\sum_{j=1}^{k} a'_j X_j(x)|_{\omega} = 0. \quad (4.1)$$

We thus get over $\omega$:

$$X_k(x) = \sum_{j<k} b_j X_j(x), \quad \text{with } b_j = -\frac{a'_j}{a'_k}, \quad (4.2)$$

Applying $(L_0)^m$ to (4.2) over $\omega$, we have

$$\lambda^m_k X_k(x) = \sum_{j<k} b_j \lambda^m_j X_j(x).$$

We get

$$X_k(x) = \sum_{j<k} b_j \frac{\lambda_j}{\lambda_k}^m X_j(x) \text{ over } \omega.$$  

Letting $m \to \infty$, we get over $\omega$

$$X_k(x) = \sum_{k' \leq j < k} b_j X_j(x), \quad (4.3)$$

where

$$\left\{ \begin{array}{l}
\lambda_j = \lambda_k, \quad \text{for } j \geq k', \\
\lambda_j < \lambda_k, \quad \text{for } j < k'.
\end{array} \right. \quad (4.4)$$

By (4.4), we get over $\Omega$,

$$L_0(X_k(x) - \sum_{k' \leq j < k} b_j X_j(x)) = \lambda_k X_k(x) - \sum_{k' \leq j < l} b_j \lambda_j X_j(x) = \lambda_k [X_k(x) - \sum_{k' \leq j < k} b_j X_j(x)].$$

By (4.3) and the unique continuation for solutions of elliptic equations, we get

$$X_k(x) - \sum_{k' \leq j < k} b_j X_j(x) \equiv 0 \text{ over } \Omega.$$  

This contradicts the linear independence of the system $\{X_j\}$. □

**Proof of Theorem 1.2.**: Similar to the proof of Theorem 1.2, we need only to find a vector $\vec{u} = (u_1, u_2, \ldots, u_{K_0}) \in \mathbb{R}^{K_0}$ such that

$$J^*_\omega(\vec{u}) = -J_{0,\omega},$$

where

$$J^*_\omega(\vec{u}) = (\langle v(x,T), X_1(x) \rangle, \langle v(x,T), X_2(x) \rangle, \ldots, \langle v(x,T), X_{K_0}(x) \rangle) = (v_1(T), v_2(T), \ldots, v_{K_0}(T))$$
with \( v \) the solution of the following equation:

\[
\begin{cases}
  v_t + L_0 v + e(x, t)v = \sum_{j=1}^{K_0} \chi_\omega(x)\chi_E(t)u_j(x), & \text{in } Q, \\
  v = 0, & \text{on } \Sigma, \\
  v_j(0) = 0, & \text{for } j \leq K_0, \\
  v \in S_{K_0}.
\end{cases}
\]  

(4.5)

and

\[
J_{0,\omega} = ((v_0)_1(T), (v_0)_2(T), \cdots, (v_0)_{K_0}(T))
\]

with \( v_0 \) the solution of the following system

\[
\begin{cases}
  (v_0)_t + L_0 v_0 + e(x, t)v_0 = -e(x, t)v_0, & \text{in } Q, \\
  v_0 = 0, & \text{on } \Sigma, \\
  (v_0)_j(0) = 0, & \text{for } j \leq K_0, \\
  v_0 \in S_{K_0}.
\end{cases}
\]  

(4.6)

In the same way, if \( J^*_{\omega} \) is not invertible, then for a vector \( \vec{\xi} = (\xi_1, \xi_2, \cdots, \xi_{K_0}) \) with \( |\vec{\xi}| = 1 \), we have a solution to the following system:

\[
\begin{cases}
  v_t + L_0 v + e(x, t)v = \sum_{j=1}^{K_0} \chi_\omega(x)\chi_E(t)\xi_j(x), & \text{in } Q, \\
  v = 0, & \text{on } \Sigma, \\
  v_j(0) = 0 = v_j(T), & \text{for } j \leq K_0, \\
  v \in S_{K_0}.
\end{cases}
\]

We then get

\[
v_j(t)' + \lambda_j v_j(t) + \int_\Omega e(x, t)v(x, t)X_j(x)dx = \sum_{l=1}^{K_0} \xi_l\chi_E(t)X_l(\omega), \text{ for } j = 1, 2, \cdots, K_0.
\]

We similarly get

\[
(e^{\lambda_j t}v_j(t))' + e^{\lambda_j t}\int_\Omega e(x, t)v(x, t)X_j(x)dx = e^{\lambda_j t}\sum_{l=1}^{K_0} \xi_l\chi_E(t)X_l(\omega), \text{ for } j = 1, 2, \cdots, K_0.
\]

\[
0 + \int_0^T \int_\Omega e^{\lambda_j t}e(x, t)v(x, t)X_j(x)dxdt = \int_0^T e^{\lambda_j t}\sum_{l=1}^{K_0} \xi_l\chi_E(t)X_l(\omega)dt.
\]

We then get

\[
\begin{pmatrix}
  \int_0^T e^{\lambda_1 t}\chi_E(t)dt \\
  \int_0^T e^{\lambda_2 t}\chi_E(t)dt \\
  \vdots \\
  \int_0^T e^{\lambda_{K_0} t}\chi_E(t)dt
\end{pmatrix}
X(\omega, K_0)
\begin{pmatrix}
  \xi_1 \\
  \xi_2 \\
  \vdots \\
  \xi_{K_0}
\end{pmatrix}
\]

14
tion. Therefore, we showed that

\[
X(\omega, K_0) = \left( \begin{array}{c}
\int_0^T \int_\Omega e^{\lambda_1 t} e(x,t) v(x,t) X_1(x) dx dt \\
\int_0^T \int_\Omega e^{\lambda_2 t} e(x,t) v(x,t) X_2(x) dx dt \\
\vdots \\
\int_0^T \int_\Omega e^{\lambda_n t} e(x,t) v(x,t) X_n(x) dx dt
\end{array} \right)
\]

(4.7)

By Lemma 4.1, we know \(X(\omega, K_0)^{-1}\) is a bounded linear operator from \(R^{K_0}\) to \(R^{K_0}\).

By the energy estimate in Theorem 2.1, we have for \(v(x,t)\),

\[
\sup_{t \in [0,T]} \|v(\cdot,t)\|^2 + \int_0^T \|\nabla v(\cdot,t)\|^2 dt \leq C(\text{system}, K_0) \int_0^T \sum_{j=1}^{K_0} \chi_\omega(x) \chi_E(t) \xi_j X_j(x)^2 dx dt \\
\leq C(\text{system}, K_0) T \|\xi\|^2 \\
\leq C(\text{system}, K_0).
\]

(4.8)

By the Hölder inequality, we have,

\[
\int_\Omega |evX_j| dx \leq C(\Omega, N, q) \|e\|_{L^\infty(0,T;L^q(\Omega))}[1 + \lambda_2^2 + \|v(\cdot,t)\|^2 + \|\nabla v(\cdot,t)\|^2].
\]

(4.9)

Together with (4.8), we thus have

\[
\int_0^T \int_\Omega |evX_j| dx dt \leq C(\text{system}, K_0) \|e\|_{L^\infty(0,T;L^q(\Omega))}.
\]

(4.10)

Back to (4.7), we have

\[
\|\xi\|^2 \leq C(\text{system}, \omega, K_0) \frac{1}{(m(E))^2} \|X(\omega, K_0)^{-1}\|^2 \|e\|_{L^\infty(0,T;L^q(\Omega))}^2 \\
\leq C(\text{system}, \omega, K_0) \frac{1}{(m(E))^2} \|e\|_{L^\infty(0,T;L^q(\Omega))}^2.
\]

Hence, when \(\|e\|_{L^\infty(0,T;L^q(\Omega))}\) is sufficient small, we get \(|\xi_j|^2 < 1\). This gives a contradiction. Therefore, we showed that \(J^*_\omega\) is invertible under small perturbation. By the same arguments as those in the proof of Theorem 1.1, we can also show the energy estimates as stated in Theorem 1.2. This completes the proof of Theorem 1.2. 

\[\blacksquare\]
References

[1] V. Barbu and G. S. Wang, Feedback stabilization of periodic solutions to nonlinear parabolic-like evolution systems. Indiana Univ. Math. J. 54 (2005), no. 6, 1521–1546.

[2] L. C. Evans, Partial Differential Equations, Graduate Studies in Mathematics (Vol 19), American Mathematical Society, 1998.

[3] A. Friedman, S. Y. Huang and J. M. Yong, Optimal periodic control for the two-phase Stefan problem. SIAM J. Control Optim. 26 (1988), no. 1, 23–41.

[4] A. Friedman and L. S. Jiang, Periodic solutions for a thermostat control problem. Comm. Partial Differential Equations 13 (1988), no. 5, 515–550.

[5] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order. Grundlehren der mathematischen Wissenschaften, a Series of Comprehensive Studies in Mathematics, Vol. 224, 2. Edition, Springer 1977, 1983.

[6] G. Lebeau and L. Robbiano, Contrôle exact de l’équation de la chaleur. Comm. PDE, 20(1-2), 335-356, 1995.

[7] G. Lebeau and E. Zuazua, Null-controllability of a system of linear thermoelasticity. Arch. Rational Mech. Anal. 141, 297-329, 1998.

[8] L. Lei, Identification of Parameters through the Approximate Periodic Solutions of a Parabolic System, Journal of Optimization Theory and Applications, Vol. 137, PP: 185-204, 2008.