FREQUENCY CONTROL OF SINGULARLY PERTURBED
FORCED DUFFING’S OSCILLATOR

ROBERT VRABEL AND MARCEL ABAS

Abstract. We analyze the dynamics of the forced singularly perturbed differential equation of Duffing’s type. We explain the appearance of the large frequency nonlinear oscillations of the solutions. It is shown that the frequency can be controlled by a small parameter at the highest derivative. We give some generalizations of results obtained recently by B.S. Wu, W.P. Sun and C.W. Lim, Analytical approximations to the double-well Duffing oscillator in large amplitude oscillations, Journal of Sound and Vibration, Volume 307, Issues 3-5, (2007), pp. 953-960. The new method for an analysis of the nonlinear oscillations which is based on the dynamic change of coordinates is proposed.

1. INTRODUCTION

Nonlinear oscillations comprise a large class of nonlinear dynamical systems, and arise naturally from many physical systems such as mechanics, chemistry, and engineering. Also a variety of biological phenomena can be characterized as nonlinear oscillations, ranging from heartbeat, neuronal activity, to population cycles (9).

The forced Duffing oscillator exhibits behavior, from limit cycles to chaos due to its nonlinear dynamics. When the periodic force that drives the system is large, chaotic behavior emerges and the phase space diagram is a strange attractor. In that case the behavior of the system is sensitive to the initial condition (11).

In this work we focus our attention to the nonlinear oscillations in the context of the singularly perturbed forced oscillator of Duffing’s type with a nonlinear restoring force

\[ \epsilon^2 \left( a^2(t) y' \right)' + f(y) = m(t), \quad 0 < \epsilon << 1 \]

or rewriting to the autonomous system form

\[ \epsilon y' = \frac{w}{a} \]
\[ \epsilon w' = \frac{m(t)}{a} - \frac{f(y)}{a} - \epsilon \frac{a'}{a} w \]
\[ t' = 1 \]

where \( a, m \) are the \( C^1 \) functions on the interval \( \langle t_B, t_E \rangle \), \( a \) is positive and \( f \) is a \( C^1 \) function on \( \mathbb{R} \).

System (2), (3), (4) is an example of a singularly perturbed system, because in the limit \( \epsilon \to 0^+ \), it does not reduce to a differential equation of the same type, but...
to an algebraic-differential reduced system

\[
0 = \frac{w}{a} \\
0 = \frac{m(t) - f(y)}{a} \\
t' = 1.
\]

Another way to study the singular limit \( \epsilon \to 0^+ \) is by introducing the new independent variable \( \tau = \frac{t}{\epsilon} \) which transforms (2), (3), (4) to the system

\[
\frac{dy}{d\tau} = \frac{w}{a} \\
\frac{dw}{d\tau} = \frac{m(t) - f(y)}{a} - \frac{\epsilon}{a} w \\
\frac{dt}{d\tau} = \epsilon.
\]

Taking the limit \( \epsilon \to 0^+ \), we obtain the so-called associated system (11)

\[
\frac{dy}{d\tau} = \frac{w}{a} \\
\frac{dw}{d\tau} = \frac{m(t) - f(y)}{a} \\
\frac{dt}{d\tau} = 0 \quad \text{i.e. } \ t = t^* = \text{const.}
\]

The critical manifold \( S \) is defined as a solution of the reduced system i.e.

\[
S := \{(t,y,w) : \ t \in (t_B,t_E), f(y) = m(t), w = 0\}
\]

which corresponds to a set of equilibria for the associated system (5), (6), (7).

We assume that

1. The critical manifold is S-shaped curve with two folds, i.e. it can be written in the form \( t = \varphi(y) \), \( t \in (t_B,t_E) \) and the function \( \varphi \) has precisely two critical points, one non-degenerate minimum \( y_{\text{min}} \) and one non-degenerate maximum \( y_{\text{max}} \) and let \( y_{\text{min}} < y_{\text{max}} \). Thus, the critical manifold can be broken up into three pieces \( S_b, S_m, S_a \), separated by the minimum and maximum (Fig. 1). These three pieces are defined as follows

   \[
   S_b = \{(y,\varphi(y)) : \ y < y_{\text{min}}\} \\
   S_m = \{(y,\varphi(y)) : \ y_{\text{min}} < y < y_{\text{max}}\} \\
   S_a = \{(y,\varphi(y)) : \ y_{\text{max}} < y\}
   \]

2. \( \varphi'(y) \neq 0 \) for \( y \neq y_{\text{min}}, y_{\text{max}} \)

3. \( \frac{df}{dy}(y) < 0 \) for every \( (t,y,0) \in S_m \) and \( \frac{df}{dy}(y) > 0 \) for every \( (t,y,0) \in S_a \cup S_b \).

Let \( t_{\text{min}} = \varphi(y_{\text{min}}), \ t_{\text{max}} = \varphi(y_{\text{max}}) \). Denote by

\[
\begin{align*}
   u_1(t) &= \varphi^{-1}(t) : \ t \in (t_B,t_{\text{max}}), \ y_{\text{max}} \leq u_1(t) \\
   u_2(t) &= \varphi^{-1}(t) : \ t \in (t_{\text{min}},t_{\text{max}}), \ y_{\text{min}} \leq u_2(t) \leq y_{\text{max}} \\
   u_3(t) &= \varphi^{-1}(t) : \ t \in (t_{\text{min}},t_E), \ u_3(t) \leq y_{\text{min}}.
\end{align*}
\]
We divide the phase diagram of (2), (3), (4) into three charts, for $K_1, K_2, K_3$, where

$$
K_1 \subset (t_B, t_{\min})
$$

$$
K_2 \subset (t_{\min}, t_{\max})
$$

$$
K_3 \subset (t_{\max}, t_E)
$$

are the compact sets.

The situation considered here is principally different from the one in [5], where two pieces of critical manifold are attracting and one is repelling. In this paper, two pieces $S_a$ and $S_b$ of the critical manifold are not normally hyperbolic ([1]) and consequently the geometric singular perturbation theory developed by N. Fenichel ([1]) is not applicable to our case. Indeed, all of the characteristic roots of associated system (5), (6), (7), $\lambda_{1,2}(t, y, w) = \pm a^{-1}(t)\sqrt{-\frac{df}{dy}(y)}$, $\lambda_3 = 0$, $(t, y, w) \in S_a \cup S_b$ lie on the imaginary axis. The piece $S_m$ is a normally hyperbolic manifold.

We generalize the results presented in [10], where unforced and undamped double-well Duffing oscillator with $\epsilon = 1$ was examined. Moreover, the considerations below can be useful in the design of the high-frequency circuits (see e.g. [3, 7], and the references therein) and we introduce the parameter $\epsilon$ as a modeling tool for the frequency control of the oscillations.

Our considerations relies on a suitable combination the phase-space analysis and the generalized polar coordinate transformations.

Consider the function

$$
H(t, y, w) = \frac{1}{2}w^2 + V(t, y), \quad V(t, y) = \int_0^y f(s)ds - m(t)y.
$$

Let

$$
H^0(t) = \begin{cases} 
V(t, u_1(t)) & \text{for } t \in (t_B, t_{\min}) \\
V(t, u_2(t)) & \text{for } t \in (t_{\min}, t_{\max}) \\
V(t, u_3(t)) & \text{for } t \in (t_{\max}, t_E).
\end{cases}
$$
We use the level surfaces $H(t, y, w) = H^\epsilon$ of $H$ with

$$H^\epsilon(t, y, w) = H^0(t) + \Delta(t) + h^\epsilon(t, y, w)$$

to characterize the trajectories of (2), (3), (4), where $h^\epsilon = O(\epsilon^\nu), \nu > 0$ for $t \in \langle t_B, t_E \rangle$ and $y, w$ bounded is a positive function such that $H^\epsilon(t, y, w)$ is continuous; $\Delta(t) \geq \Delta > 0$ on $\langle t_B, t_E \rangle$ where $\Delta$ is an arbitrarily small constant. These surfaces in $(t, y, w)$-space are defined by

$$w = \pm\left(2(H^\epsilon(t, y, w) - V(t, y))\right)^{\frac{1}{2}}$$

extending it as long as $w$ remains real. In our case such trajectories, lying on the surface $w = w(t, y, \epsilon)$, are bounded for every small $\epsilon$ (Fig. 2). On the charts for $K_1$ and $K_3$ there is a motion in a single potential well and on the chart for $K_2$, double well with a barrier in between.

Let $H^\epsilon(t) = H^\epsilon(t, y^\epsilon(t), w^\epsilon(t))$, where $(y^\epsilon, w^\epsilon)$ is a solution of (2), (3) on $\langle t_B, t_E \rangle$ and let $y^\epsilon_L(t), y^\epsilon_R(t)$ are the roots of equation

$$H^\epsilon(t) = V(t, y)$$
on $\langle t_B, t_E \rangle$. Obviously,

$$y^\epsilon_L(t) < u_1(t) < y^\epsilon_R(t) \text{ on } K_1$$
$$y^\epsilon_L(t) < u_1(t) < u_3(t) < y^\epsilon_R(t) \text{ on } K_2$$
$$y^\epsilon_L(t) < u_3(t) < y^\epsilon_R(t) \text{ on } K_3$$

Further, denote $y^\epsilon_L(t), y^\epsilon_R(t)$ the roots of equation

$$H^0(t) + \Delta(t) = V(t, y).$$

Hence, $y^\epsilon_L(t) < u_2(t) < y^\epsilon_R(t)$ on $K_2$ and $y^\epsilon_L(t) \to y^\epsilon_R(t)$ from left side and $y^\epsilon_R(t) \to y^\epsilon_R(t)$ from right side on $K_1 \cup K_2 \cup K_3$ for $\epsilon \to 0^+$.

The derivative of $H^\epsilon(t)$ along any solution path of (2), (3), (4) is

$$H^\epsilon'(t) = w^\epsilon w^\epsilon' + f(y^\epsilon)y^\epsilon' - [m(t)y^\epsilon]'$$

$$= w^\epsilon \left[ -\frac{f(y^\epsilon)}{ea} + \frac{m(t)}{ea} - \frac{a'}{a} w^\epsilon \right] + f(y^\epsilon)y^\epsilon' - [m(t)y^\epsilon]'$$

$$= -\frac{a'(t)}{a(t)}(w^\epsilon)^2 - m'(t)y^\epsilon.$$
Fig. 2a: The function $f(y) - m(t^*)$ and its corresponding $V(t^*, y)$ and phase-diagram $(t^*, y, w)$ for fixed $t^* \in (t_B, t_{\text{min}})$.

Fig. 2b: The function $f(y) - m(t^*)$ and its corresponding $V(t^*, y)$ and phase-diagram $(t^*, y, w)$ for $t^* = t_{\text{min}}$. 
Fig. 2c: The function $f(y) - m(t^*)$ and its corresponding $V(t^*, y)$ and phase-diagram $(t^*, y, w)$ for fixed $t^* \in (t_{\text{min}}, t_{\text{max}})$

Fig. 2d: The function $f(y) - m(t^*)$ and its corresponding $V(t^*, y)$ and phase-diagram $(t^*, y, w)$ for $t^* = t_{\text{max}}$
Let
\[ \chi(t, y) = (y - u_2(t)) \frac{\partial}{\partial y} \left[ \frac{\int_{u_2(t)}^{y} (f(s) - m(t))ds}{(y-u_2(t))^2/2} \right]. \]

We make the following assumption.

(A4) The function
\[ \chi(t, y) > -\frac{4\Delta}{(y-u_2(t))^2} \]
for \( y \in (u_1(t), u_2(t)) \cup (u_2(t), u_3(t)) \) and for every fixed \( t \in K_2 \).

In this work we show that under the assumptions (A1) – (A4) the Diff. Eq. (1) admits the nonlinear oscillations of solution \( y^\epsilon \) with a frequency tending to infinity for \( \epsilon \to 0^+ \).

2. Generalized polar coordinate transformation

We introduce the variable \( v = \epsilon a^2 y' \) and write (1) in the following system
\[ y' = \frac{v}{\epsilon a^2} \]
\[ v' = \frac{m(t)}{\epsilon} - \frac{f(y)}{\epsilon}. \]

Then, we put \( y = u_i(t) + r \cos \gamma, i = 1, 2, 3 \) and \( v = -r \sin \gamma \) on the charts \( K_1, K_2, K_3 \), respectively. We obtain the following differential equation for \( \gamma \)
\[ \gamma' = \frac{1}{\epsilon} \left[ \frac{1}{a^2(t)} \sin^2 \gamma + f_i(t, y) \cos^2 \gamma + \frac{\epsilon u_i'(t)}{r} \sin \gamma \right] \]
or by using identity $\sin^2 \alpha + \cos^2 \alpha = 1$

\[(8) \quad \gamma' = \frac{1}{\epsilon} \left[ \frac{1}{a^2(t)} + \cos^2 \gamma \left( f_i(t, y) - \frac{1}{a^2(t)} \right) + \frac{\epsilon u_i'(t)}{r} \sin \gamma \right]
\]

where

\[r = \sqrt{(y - u_i)^2 + w^2},
\]

\[f_i(t, y) = \frac{f(y) - m(t)}{y - u_i(t)}, \quad f_i'(t, u_i(t)) = \frac{df}{dy}(u_i(t)) \quad i = 1, 2, 3.
\]

3. ANÁLISIS DE GRÁFICO PARA $K_2$

En esta sección probamos que bajo la suposición (A1) – (A4) es

\[(9) \quad \gamma' \geq \frac{1}{\epsilon} c_{K_2}
\]

en $K_2$, donde $c_{K_2}$ es una constante positiva. Primero estimamos $r = r'(t)$

\[r_{\min}(K_i) \overset{\text{def}}{=} \min_{K_i} r'(t)
\]

\[= \min_{K_i} \left\{ u_i(t) - y_{L_i}(t), y_{R_i}(t) - u_i(t), \sqrt{2}a(t) \sqrt{H'(t) - V(t, u_i(t))} \right\}.
\]

Porque

\[u_i(t) - y_{L_i}(t) > u_i(t) - y_{R_i}(t) > 0,
\]

\[y_{R_i}(t) - u_i(t) > y_{R_i}(t) - u_i(t) > 0\]

y

\[H'(t) - V(t, u_i) = \Delta(t) + h'(t) > \Delta > 0
\]

es $r_{\min}(K_i) > 0$ para todo pequeño $\epsilon$ y $i = 1, 2, 3$. Así, tercera expresión en [8]

\[\left| \frac{\epsilon u_i'(t)}{r} \sin \gamma \right| \leq \frac{\epsilon |u_i'(t)|}{r_{\min}(K_2)} = O(\epsilon)
\]

en $K_2$. Para existencia oscilaciones en gráfico $K_2$ es fundamental el análisis de la expresión

\[(10) \quad \cos^2 \gamma \left( f_2(t, y) - \frac{1}{a^2(t)} \right).
\]

Claramente,

\[\left| \cos^2 \gamma \left( f_2(t, y) - \frac{1}{a^2(t)} \right) \right| = \left| \frac{(y - u_2)^2 \left( f_2(t, y) - \frac{1}{a^2(t)} \right)}{(y - u_2)^2 + a^2w^2} \right|.
\]

Para $H'$ es $(y - u_2)^2 + a^2w^2 \neq 0$ en el camino de solución existe independiente de $\epsilon$ constante $\delta_1 = \delta_1(\eta) > 0$, tal que para todo $\eta$, $0 < \eta < \frac{1}{a^2(t)}

\left| \cos^2 \gamma \left( f_2(t, y) - \frac{1}{a^2(t)} \right) \right| \leq \frac{1}{a^2(t)} - \eta
\]

para $y \in (u_2(t) - \delta_1, u_2(t) + \delta_1)$. Ahora analice la expresión (10) en el intervalo

\[(11) \quad (u_1(t) - \delta_2, u_2(t) - \delta_1) \cup (u_2(t) + \delta_1, u_3(t) + \delta_2),
\]

donde $\delta_2$ es la constante positiva elegida adecuada tal que $u_1(t) - \delta_2 \geq y_{L_2}(t)$ y

$u_3(t) + \delta_2 \leq y_{R_2}(t)$. 
Moreover, for $\delta$ exists $> 0$ on $(\epsilon, \epsilon')$. Hence taking into consideration that $c_2(t)\leq \min_{t\in K_2} c_2 \in R$ we obtain that

\[
\int_{t_1}^{t_2} (f(s) - m(t)) ds < 0.
\]

Thus, there exists $\delta_2 > 0$ such that

\[
\cos^2 \gamma \left( \overline{f_2}(t, y) - \frac{1}{a^2(t)} \right) < 0
\]

on $[11]$. Now, let $c_{K_2}$ from $[9]$ be $c_{K_2} = \min \{c_{K_2,1}, c_{K_2,2}, c_{K_2,3}\}$ where

\[
c_{K_2,1} = \min \left\{ \eta - \frac{\epsilon |u_2'(t)|}{r_{\min}(K_2)}, t \in K_2 \right\}
\]

\[
c_{K_2,2} = \min \left\{ \frac{1}{a^2(t)} - \frac{1}{a^2(t)} \left[ \frac{1}{a^2(t)} - \overline{f_2}(t, y) \overline{f_2}(t, y) \right] \left[ \frac{1}{a^2(t)} - \frac{1}{a^2(t)} \right] \right\} - \frac{\epsilon |u_2'(t)|}{r_{\min}(K_2)},
\]

$t \in K_2, y \in [11]$

\[
c_{K_2,3} = \min \left\{ \frac{1}{a^2(t)} \sin^2 \gamma + \overline{f_2}(t, y) \cos^2 \gamma - \frac{\epsilon |u_2'(t)|}{r_{\min}(K_2)},
\]

$t \in K_2, y \in \langle y_1^0(t), y_0^1(t) \rangle \setminus [11], \gamma \in R$.

Taking into consideration that $\overline{f}_2 > 0$ for $t \in K_2$ and $y \in \langle y_1^0(t), y_0^1(t) \rangle \setminus [11]$, we conclude that $c_{K_2} > 0$ for every sufficiently small $\epsilon, \epsilon \in (c, c_0)$.
4. Analysis of the charts for $K_1$ and $K_3$

On the difference of $K_2$, the analysis in the charts for $K_i, i = 1, 3$ is easy in comparison with $K_2$ one. The function $f_i(t,y) > 0$ for $t \in K_i$ and $y \in \langle y^i_L(t), y^i_R(t) \rangle, i = 1, 3$. Let

$$c_{K_i} = \min \left\{ \frac{1}{a^2(t)} \sin^2 \gamma + f_i(t,y) \cos^2 \gamma - \frac{\epsilon |u'_i(t)|}{r_{\min}(K_i)}, \right. $$

$$t \in K_i, y \in \langle y^i_L(t), y^i_R(t) \rangle, \gamma \in \mathbb{R}, i = 1, 3.$$ 

The constants $c_{K_i}, i = 1, 3$ are positive for every sufficiently small $\epsilon$, $\epsilon \in (\epsilon, \epsilon_0)$. Thus, $\gamma = \gamma'(t)$ is increasing on $K_i, i = 1, 2, 3$

$$\gamma' \geq \frac{1}{\epsilon} c_{K_i} \tag{12}$$

5. Frequency control of nonlinear oscillations

In this section we show that the parameter $\epsilon$ play role modeling tool for the frequency control of the nonlinear oscillations. Let us denote by $s_i$ the spacing between two successive zeros of $y_i - u_i$ and by $z_i(y)$ the number of zeros of $y_i - u_i$ on $K_i, i = 1, 2, 3$, where $y = y'(t)$ is a solution of [1], then integrating the inequality [12] with respect to the variable $t$ between two successive zeros of $y - u_i$ we obtain immediately

$$\int_{\text{zero } (j)}^{\text{zero } (j+1)} \gamma' dt \geq \frac{c_{K_i}}{\epsilon} \int_{\text{zero } (j)}^{\text{zero } (j+1)} dt$$

$$\pi \geq \frac{c_{K_i}}{\epsilon} s_i.$$  

Hence,

$$s_i \leq \frac{\epsilon \pi}{c_{K_i}}$$

and

$$\lim_{\epsilon \to 0^+} z_i(y') = \infty, i = 1, 2, 3.$$  

Now we summarize the results of this article (pictorially, see Fig. 3).

6. Statement of main result

**Theorem 6.1.** Under the assumptions (A1)–(A4) there exists solution $y'$ of [1] for $\epsilon \in (0, \epsilon_0)$ such that $z_i(y') \to \infty$ with amplitude $y^i_R(t) - u_i(t)$ tendings to $y^i_R(t) - u_i(t)$ for subintervals of $K_i$ where $y' - u_i \geq 0$ and with amplitude $u_i(t) - y^i_L(t)$ tendings to $u_i(t) - y^i_R(t)$ for subintervals of $K_i$ where $y' - u_i \leq 0, i = 1, 2, 3.$
Finally, we remark that the proposed technique is an appropriate tool for detection and detailed analysis of the nonlinear oscillations in the dynamical systems but there is another powerful way to analyse the systems under consideration. Indeed, after selecting the new time $\tau = t/\epsilon$, system (2), (3), (4) becomes a particular case of more general system of type

$$
\begin{align*}
\frac{dx}{d\tau} &= \frac{\partial H}{\partial y}(t,x,y) + \epsilon f_1(t,x,y), \\
\frac{dy}{d\tau} &= -\frac{\partial H}{\partial x}(t,x,y) + \epsilon f_2(t,x,y), \\
\frac{dt}{d\tau} &= \epsilon.
\end{align*}
$$

Assuming under the study of system (13) that for our values $t$ there exists a family of closed trajectories inside the levels $\{(x,y) : H(t,x,y) = \text{const}\}$, one can introduce new variables $(I, \phi)$ corresponding to these trajectories, in which the subsystem

$$
\begin{align*}
\dot{I} &= 0, \\
\dot{\phi} &= \omega(I,t),
\end{align*}
$$

where $\omega(I,t) > 0$. In new variables system (13) takes the form

$$
\begin{align*}
\frac{dI}{d\tau} &= \epsilon \Delta_1(I,\phi,t,\epsilon), \\
\frac{d\phi}{d\tau} &= \omega(I,t) + \epsilon \Delta_2(I,\phi,t,\epsilon),
\end{align*}
$$

Now one needs to add only that to system (15) the standard averaging techniques with respect $\phi$ could be applied (see e.g. [2], [3], [6]).

**Acknowledgments**

We would like to express our gratitude for all the valuable and constructive comments we have received from the referee.

This research was supported by Slovak Grant Agency, Ministry of Education of Slovak Republic under grant number 1/0068/08.
References

[1] N. Fenichel: Geometric singular perturbation theory for ordinary differential equations, J. Differential Equations 31, (1979), pp. 53-98.
[2] V. Gaitgory and G. Grammel: On the construction of asymptotically optimal controls for singularly perturbed systems, Systems & Control Letters, Volume 30, Issues 2-3, (1997), pp. 139-147.
[3] F. Herzel and B. Heinemann: High-frequency noise of bipolar devices in consideration of carrier heating and low temperature effects, Solid-State Electronics, Volume 38, Issue 11, (1995), pp. 1905-1909.
[4] Christopher K.R.T. Jones: Geometric Singular Perturbation Theory, C.I.M.E. Lectures, Montecatini Terme, June 1994, Lecture Notes in Mathematics 1609, Springer-Verlag, Heidelberg, (1995).
[5] M. Krupa and P. Szmolyan: Relaxation Oscillation and Canard Explosion, Journal of Differential Equations, Volume 174, Issue 2, (2001), pp. 312-368.
[6] N. M. Krylov and N. N. Bogoliubov: Introduction to Nonlinear Mechanics, Princeton: Princeton University Press, (1947).
[7] P. Mei, Ch. Cai and Y. Zou: A Generalized KYP Lemma-Based Approach for $H_\infty$ Control of Singularly Perturbed Systems, Circuits, Systems, and Signal Processing, Volume 28, no.6, (2009), pp. 945-957.
[8] J. Sanders, F. Verhulst, and J. Murdock: Averaging Methods in Nonlinear Dynamical Systems, Springer, New York, (2007).
[9] R. Srebro: The Duffing oscillator: a model for the dynamics of the neuronal groups comprising the transient evoked potential, Electroencephalography and clinical neurophysiology. Evoked potentials, Volume 96, no.6, (1995), pp. 561-573.
[10] B. S. Wua, W. P. Suna and C. W. Lim: Analytical approximations to the double-well Duffing oscillator in large amplitude oscillations, Journal of Sound and Vibration, Volume 307, Issues 3-5, (2007), pp. 953-960.
[11] Y. Ye, L. Yue, D. P. Mandic and Y. Bao-Jun: Regular nonlinear response of the driven Duffing oscillator to chaotic time series, Chinese Phys. B 18 958, (2009). doi: 10.1088/1674-1056/18/3/020

Robert Vrabel, Institute of Applied Informatics, Automation and Mathematics, Faculty of Materials Science and Technology, Hajdoczyho 1, 917 01 Trnava, Slovakia
E-mail address: robert.vrabel@stuba.sk

Marcel Abas, Institute of Applied Informatics, Automation and Mathematics, Faculty of Materials Science and Technology, Hajdoczyho 1, 917 01 Trnava, Slovakia
E-mail address: abas@stuba.sk