Accelerating Inexact Successive Quadratic Approximation for Regularized Optimization Through Manifold Identification

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Abstract For regularized optimization that minimizes the sum of a smooth term and a regularizer that promotes structured solutions, inexact proximal-Newton-type methods, or successive quadratic approximation (SQA) methods, are widely used for their superlinear convergence in terms of iterations. However, unlike the counterparts in smooth optimization, they suffer from lengthy running time in solving regularized subproblems because even approximate solutions cannot be computed easily, so their empirical time cost is not as impressive. In this work, we first show that for partly smooth regularizers, although general inexact solutions cannot identify the active manifold that makes the objective function smooth, approximate solutions generated by commonly-used subproblem solvers will identify this manifold, even with arbitrarily low solution precision. We then utilize this property to propose an improved SQA method, ISQA+, that switches to efficient smooth optimization methods after this manifold is identified. We show that for a wide class of degenerate solutions, ISQA+ possesses superlinear convergence not only in iterations, but also in running time because the cost per iteration is bounded. In particular, our superlinear convergence result holds on problems satisfying a sharpness condition that is more general than that in existing literature. We also prove iterate convergence under a sharpness condition for inexact SQA, which is novel for this family of methods that could easily violate the classical relative-error condition frequently used in proving convergence under similar conditions. Experiments on real-world problems confirm that ISQA+ greatly improves the state of the art for regularized optimization.

Keywords Variable metric · Manifold identification · Regularized optimization · Inexact method · Superlinear convergence

Mathematics Subject Classification 49M15 · 90C55 · 49K40 · 90C31 · 90C55 · 65K05

1 Introduction

Consider the following regularized optimization problem:

\[
\min_{x \in \mathcal{E}} F(x) := f(x) + \Psi(x),
\]

where \( \mathcal{E} \) is a Euclidean space with an inner product \( \langle \cdot, \cdot \rangle \) and its induced norm \( \|\cdot\| \), the regularizer \( \Psi \) is extended-valued, convex, proper, and lower-semicontinuous, \( f \) is continuously differentiable with Lipschitz-continuous gradients, and the solution set \( \Omega \) is non-empty. This type of problems is ubiquitous in applications such as machine learning and signal processing (see, for example, [5, 6, 11]). One widely-used method for (1) is inexact successive quadratic approximation (ISQA). At the \( t \)th iteration with iterate \( x^t \), ISQA obtains the update direction \( p^t \) by approximately solving

\[
p^t \approx \arg \min_{p \in \mathcal{E}} \quad Q_{H_t}(p; x^t),
\]

\[
Q_{H_t}(p; x^t) := \langle \nabla f(x^t), p \rangle + \frac{1}{2} \langle p, H_t p \rangle + \Psi(x^t + p) - \Psi(x^t),
\]

where \( H_t \) is a self-adjoint positive-semidefinite linear endomorphism of \( \mathcal{E} \). The iterate is then updated along \( p^t \) with a step size \( \alpha_t > 0 \).

ISQA is among the most efficient for (1). Its variants differ in the choice of \( H_t \) and \( \alpha_t \), and how accurately (2) is solved. In this class, proximal Newton (PN) [22, 26] and proximal quasi-Newton (PQN) [38] are popular for their fast convergence in iterations. Regrettably, their subproblem has no closed-form solution as \( H_t \) is non-diagonal, so one needs to use an iterative solver for (2) and the running time to reach the accuracy requirement...
can hence be lengthy. For example, to attain the same superlinear convergence as truncated Newton for smooth optimization, PN that takes $H_t = \nabla^2 f$ requires increasing accuracies for the subproblem solution, implying a growing and unbounded number of inner iterations of the subproblem solver. Its superlinear convergence thus gives little practical advantage in running time. On the contrary, in smooth optimization, one can solve (2) with a bounded cost by either conjugate gradient (CG) or matrix factorizations since $\Psi \equiv 0$. The advantage of second-order methods over the first-order ones in regularized optimization is therefore not as significant as that in smooth optimization.

A possible remedy when $\Psi$ is partly smooth is to switch to smooth optimization after identifying an active manifold $M$ that contains a solution $\hat{x}$ to (1) and makes $\Psi$ confined to it smooth. We say an algorithm can identify $M$ if there is a neighborhood $U$ of $\hat{x}$ such that $x' \in U$ implies $x'^{t+1} \in M$, and call it possesses the manifold identification property. Unfortunately, for ISQA, this property in general only holds when (2) is always solved exactly. Indeed, even if each $p^t$ is arbitrarily close to the corresponding exact solution, it is possible that no iterate lies in the active manifold, as shown below.

Example 1 Consider the following simple example of (1) with $\Psi(\cdot) = \| \cdot \|_1$:

$$\min_{x \in \mathbb{R}^2} (x_1 - 2.5)^2 + (x_2 - 0.3)^2 + \|x\|_1,$$

whose only solution is $\hat{x} = (2, 0)$, and $\|x\|_1$ is smooth relative to $M = \{ x | x_2 = 0 \}$ around $\hat{x}$. Consider $\{x^t\}$ with $x^t_1 = 2 + f(t), x^t_2 = f(t)$, for some $f(t) > 0$ with $f(t) \downarrow 0$, and let $H_t \equiv I, \alpha_t \equiv 1$, and $p^t = x^{t+1} - x^t$. The optimum of (2) is $p^* = \hat{x} - x^t$, so $\|x^t - \hat{x}\| = O(f(t))$ and $\|p^t - p^*\| = O(f(t))$. As $f$ is arbitrary, both the subproblem approximate solutions and their corresponding objectives converge to the optimum arbitrarily fast, but $x^t \notin M$ for all $t$.

Moreover, some versions of inexact PN generalize the stopping condition of CG for truncated Newton to require $\|r^t\| \to 0$, where

$$r^t := \arg \min_r \|r\|, \quad \text{subject to} \quad r \in \partial T_{H_t}(p^t; x^t),$$

but Example 1 gives a sequence of $\{\|r^t\|\}$ converging to 1, hinting that such a condition might have an implicit relation with manifold identification.

Interestingly, in our numerical experience in [20, 21, 27], ISQA with approximate subproblem solutions, even without an increasing solution precision and on problems that are not strongly convex, often identifies the active manifold rapidly. We thus aim to provide theoretical support for such a phenomenon and utilize it to devise more efficient and practical methods that trace the superior performance of second-order methods in smooth optimization.

In this work, we show that ISQA essentially possesses the manifold identification property, by giving a sufficient condition for inexact solutions of (2) in ISQA to identify the active manifold that is satisfied by the output of most of the widely-used subproblem solvers even if (2) is solved arbitrarily roughly. We also show that $\|r^t\| \downarrow 0$ is indeed sufficient for manifold identification, so PN can achieve superlinear convergence in a more efficient way through this process. When the iterates do not lie in a compact set, it is possible that the iterates do not converge, in which case even algorithms possessing the manifold identification property might fail to identify the active manifold because the iterates never enter a neighborhood that enables the identification. Therefore, we also show convergence of the iterates under a sharpness condition widely-seen in real-world problems that generalizes the quadratic growth condition and the weak sharp minima. Under convexity, this sharpness condition is equivalent to a type of the Kurdyka-Łojasiewicz (KL) condition [18, 29], but convergence of general ISQA methods under the KL condition is unknown since the inexactness condition can easily violate the relative-error condition needed in [24].

1.1 Related Work

ISQA for (1) for the special case of constrained optimization has been well-studied, and we refer the readers to [20] for a detailed review of related methods. We mention here in particular the works [7, 22, 26, 47] that provided superlinear convergence results. Lee et al. [22] first analyzed the superlinear convergence of PN and PQN. Their analysis considers only strongly convex $f$, so both the convergence of the iterate and the positive-definiteness...
of the Hessian are guaranteed. Their inexact version requires $\|r^t\| \downarrow 0$, which might not happen when the solutions to (2) are only approximate, as illustrated in Example 1. With the same requirement for $\|r^t\|$ as (22). Li et al. [29] showed that superlinear convergence for inexact PN can be achieved when $f$ is self-concordant. Byrd et al. [7] focused on $\mathcal{Ψ}(\cdot) = \|\cdot\|$ and showed superlinear convergence of PN under the subproblem stopping condition [10], defined in Section 2 which is achievable as long as $p^t$ is close enough to the optimum of (2). To cope with degenerate cases in which the Hessian is only positive semidefinite, Yue et al. [47] used the stopping condition of [7] to propose a damping PN for general $\mathcal{Ψ}$ and showed that its iterates converge and achieve superlinear convergence under convexity and the error-bound (EB) condition [22] even if $F$ is not coercive. A common drawback of [7, 22, 29, 47] is that they all require increasing precisions in solving the subproblem, so the superlinear rate is observed only in iterations but not in running time in their experiments. In contrast, by switching to smooth optimization after identifying the active manifold, ISQA+ achieves superlinear convergence not only in iterations but also in time, and is thus much more efficient in practice. Our superlinear convergence result also allows a broader range of degeneracy than that in [47].

Although ISQA is intensively studied, its ability for manifold identification is barely discussed because this does not in general hold, as noted in Example 1. Hare [14] showed that ISQA identifies the active manifold under the impractical assumptions that (2) is always solved exactly and the iterates converge, and his analysis cannot be extended to inexact versions. Our observation in [20, 21, 27] that ISQA identifies the active manifold empirically motivates this work to provide theoretical guarantees for this phenomenon.

Manifold identification requires the iterates, or at least a subsequence, to converge to a point of partial smoothness. In most existing analyses for (1) iterate convergence is proven under either: (i) $f$ is convex and the algorithm is a first-order one, (ii) $F$ is strongly convex, or (iii) the Kurdyka-Lojasiewicz (KL) condition holds. Analyses for the first scenario rely on the implicit regularization of first-order methods such that their iterates lie in a bounded region [28], but this is not applicable to ISQA. Under the second condition, convergence of the objective directly implies that of the iterates. For the third case, convergence of the full iterate sequence is usually proven under an assumption of a relative-error behavior, of the form

$$\min_{v \in \partial F(x^{t+1})} \|v\| \leq b \|x^{t+1} - x^t\|, \forall t$$

for some $b > 0$, as done in [2, 5, 10], but this condition can easily be violated when inexactness kicks in in ISQA, as argued by Bonettini et al. [5]. To work around this issue, [5] further assumed that the forward-backward envelope [30] of $F$ satisfies the KL condition and obtained iterate convergence under such a situation, but whether KL condition of $F$ implies that of its forward-backward envelope is unclear. The only exception to get convergence under the KL condition of $F$ for a specific type of SQA method is [47] that shows the convergence of the iterates for their specific algorithm under EB and convexity of $f$ but [47] requires $H_i$ in [2] to be the Hessian of $f$ plus a multiple of identity and their analysis cannot be extended to general $H_i$. On the other hand, our analysis for iterate convergence is novel and more general in covering a much broader algorithmic framework and requiring only a general sharpness condition for $F$ that contains both EB and the weak-sharp minima [3] as special cases.

Our design of the two-stage ISQA+ is inspired by [23, 27] in conjecturing that the active manifold is identified after the current manifold remains unchanged, but the design of the first stage is quite different and we also add in additional safeguards in the second stage. Lee and Wright [23] used dual averaging in the first stage for optimizing the expected value of an objective function involving random variables, so their algorithm is more suitable for stochastic settings. Li et al. [27] focused on distributed optimization and their usage of manifold identification is for reducing the communication cost, instead of accelerating general regularized optimization considered in this work.

1.2 Outline

This work is outlined as follows. In Section 2 we describe the algorithmic framework and give preliminary properties. Technical results in Section 3 prove the manifold identification property of ISQA and the convergence of the iterates. We then describe the proposed ISQA+ and show its superlinear convergence in running time in Section 4. The effectiveness of ISQA+ is then illustrated through extensive numerical experiments in Section 5. Section 6 finally concludes this work. Our implementation of the described algorithms is available at https://www.github.com/leepei/ISQA_plus/

2 Preliminaries

We denote the minimum of (1) by $F^*$; the domain of $\mathcal{Ψ}$ by dom($\mathcal{Ψ}$); and the set of convex, proper, and lower semicontinuous functions by $\mathcal{L}$. For any set $C$, relint($C$) denotes its relative interior. We will frequently use
the following notations.

\[ \delta_t := F(x^t) - F^*, \quad P_{\Omega}(x) := \arg \min_{y \in \Omega} \|x - y\|, \quad \text{dist}(x, \Omega) := \|x - P_\Omega(x)\|. \]

The level set \( \text{Lev} (\xi) := \{x \mid F(x) - F^* \leq \xi \} \) for any \( \xi \geq 0 \) is closed but not necessarily bounded. A function is \( L \)-smooth if it is differentiable with the gradient \( L \)-Lipschitz continuous. We denote the identity operator by \( I \). For self-adjoint linear endomorphisms \( A, B \in \mathcal{E} \), \( A \succeq B \) (\( \succeq \)) means \( A - B \) is positive definite (positive semidefinite). We abbreviate \( A \succeq \tau I \) to \( A \succeq \tau \) for \( \tau \in \mathbb{R} \). The set of \( A \) with \( A \succeq 0 \) is denoted by \( \mathcal{S}_{++} \). The subdifferential \( \partial \Psi(x) \) of \( \Psi \) at \( x \) is well-defined as \( \Psi \in \mathcal{P} \), hence so is the generalized gradient \( \partial F(x) = \nabla f(x) + \partial \Psi(x) \). For any \( g \in \mathcal{P} \), \( \tau \geq 0 \), and \( A \in \mathcal{S}_{++} \), the proximal mapping

\[ \text{prox}_t \gamma g(x) := \arg \min_{y \in \mathcal{E}} \frac{1}{2} \|x - y\|^2 + \tau \gamma (y) \] \hspace{1cm} (5)

is continuous and finite in \( \mathcal{E} \) even outside \( \text{dom}(g) \). When \( A = I \), \( [5] \) is shortened to \( \text{prox}_\tau g(x) \). For \( [2] \) we denote its optimal solution by \( p^* \). When there is no ambiguity, we abbreviate \( Q_{H_t} (\cdot ; x^t) \) to \( Q_t (\cdot) \), \( Q_t (p^t) \) to \( Q_t \), and \( Q_t (p^*) \) to \( Q_t^* \).

### 2.1 Algorithmic Framework

We give out details in defining the ISQA framework by discussing the choice of \( H_t \), the subproblem solver and its stopping condition, and how sufficient objective decrease is ensured. We consider the algorithm a two-level loop procedure, where the outer loop updates the iterate \( x^t \) and the iterations of the subproblem solver form the inner loop.

After obtaining \( p^t \) from \([2] \) we need to find a step size \( \alpha_t > 0 \) for it to ensure sufficient objective decrease. Given \( \gamma, \beta \in (0, 1) \), we take \( \alpha_t \) as the largest value in \( \{ \beta^0, \beta^1, \ldots \} \) satisfying an Armijo-like condition.

\[ F(x^t + \alpha_t p^t) \leq F(x^t) + \gamma_\alpha_t Q_t(p^t). \] \hspace{1cm} (6)

This condition is satisfied by all \( \alpha_t \) small enough as long as \( Q_t(p^t) < 0 \) and \( Q_t (\cdot) \) is strongly convex; see \([20] \), Lemma 3].

For the choice of \( H_t \), we only make the following blanket assumption without further specification to make our analysis more general.

\[ \exists M, m > 0, \quad \text{such that} \quad M \geq H_t \geq m, \forall t \geq 0. \] \hspace{1cm} (7)

For \([2] \) any suitable solver for regularized optimization, such as (accelerated) proximal gradient, (variance-reduced) stochastic gradient methods, and their variants, can be used, and the following are common for their inner loop termination:

\[ \hat{Q}_t - Q^*_t \leq \epsilon_t, \] \hspace{1cm} (8)

\[ \|x^t\| \leq \epsilon_t, \quad \text{or} \] \hspace{1cm} (9)

\[ G^*_{H_t}(p^t; x^t) := \|p^t - \hat{p}^*_t\| \leq \epsilon_t, \] \hspace{1cm} (10)

for some given \( \epsilon_t \geq 0 \) and \( \tau > 0 \), where

\[ \hat{p}^*_t := \text{prox}_{\tau \Psi} \left( (x^t + p^t) - \tau \left( \nabla f(x^t) + H_t p^t \right) \right) - x^t. \] \hspace{1cm} (11)

The point \( \hat{p}^*_t \) in \([11] \) is computed by taking a proximal gradient step of the subproblem \([2] \) from \( p^t \), and thus \( \hat{p}^* - p^t \) is the proximal gradient (with step size \( \tau \)) of \( Q_t \) at \( p^t \). Because the subproblem is strongly convex, the norm of the proximal gradient is zero at \( p^t \) if and only if \( p^t \) is the unique solution to the subproblem. We will see in the next subsection that the norm of this proximal gradient squared is also equivalent to the objective distance to the optimum of the subproblem. We summarize this framework in Algorithm 1.

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**Algorithm 1: A Framework of ISQA for \([11] \)**

```
input \( x^0 \in \mathcal{E} \), \( \gamma, \beta \in (0, 1) \)
for \( t = 0, 1, \ldots \) do
    \( \alpha_t := 1 \), pick \( \epsilon_t \geq 0 \) and \( H_t \), and solve \([2] \) for \( p^t \) satisfying \([8] \) or \([9] \) or \([10] \)
    while \([6] \) is not satisfied do \( \alpha_t := \beta \alpha_t \)
    \( x^{t+1} := x^t + \alpha_t p^t \)
```

2.2 Basic Properties

Under [7, 3] is $m$-strongly convex with respect to $p$, so the following standard results hold for any $\tau \in (0, 1/M]$ and any $p' \in E$ [12, 32].

$$\|r'\|^2 \geq 2m \left(Q_t(p') - Q_t^*\right) \geq \frac{m}{M} G_{H^t}(p'; x')^2,$$  \hfill (12)

$$G_{H^t}(p'; x')^2 \geq \tau \left(\frac{2m-1 + \tau}{1 + M\tau} - \frac{1}{2}\right) \left(Q_t(p') - Q_t^*\right).$$  \hfill (13)

Therefore, (10) and (8) are almost equivalent and implied by (9) while Example 1 has shown that (9) is a stronger condition not implied by (8). Although (13) does not show that (10) directly implies (8), once (10) is satisfied, we can use it to find $\bar{p}_t^*$ satisfying (8) from $p'$.

A central focus of this work is manifold identification, so we first formally define manifolds following [42].

A set $M \subseteq \mathbb{R}^m$ is a $C^k$ manifold around $x \in \mathbb{R}^m$ if there is a $C^k$ function $\Phi : R^m \to \mathbb{R}^{m-r}$ whose derivative at $x$ is surjective such that for $y$ close enough to $x$, $y \in M$ if and only if $\Phi(y) = 0$. Through the implicit function theorem, we can also use a $C^k$ parameterization $\phi : \mathbb{R}^p \to M$, with $\phi(y) = x$ and the derivative injective at $y$, to describe a neighborhood of $x$ on $M$. Now we are ready for the definition of partial smoothness [24] that we assume for the regularizer when discussing manifold identification.

**Definition 1 (Partly smooth)** A convex function $\Psi$ is partly smooth at a point $x^*$ relative to a set $M$ containing $x^*$ if $\partial \Psi(x^*) \neq \emptyset$ and:

1. Around $x^*$, $M$ is a $C^2$-manifold and $\Psi|_M$ is $C^2$.
2. The affine span of $\partial \Psi(x)$ is a translate of the normal space to $M$ at $x^*$.
3. $\partial \Psi$ is continuous at $x^*$ relative to $M$.

Intuitively, it means $\Psi$ is smooth around $x^*$ in $M$ but changes drastically along directions leaving the manifold. We also call this $M$ the active manifold.

As the original identification results in [15, Theorem 5.3] and [25, Theorem 4.10] require the sum $F$ to be partly smooth but our setting does not require so for $f$, we first provide a result relaxing the conditions to ensure identification in our scenario.

**Lemma 1** Consider [1] with $f \in C^1$ and $\Psi$ convex, proper, closed, and partly smooth at a point $x^*$ relative to a $C^2$-manifold $M$. If at $x^*$, the nondegenerate condition

$$0 \in \text{relint} (\partial F(x^*)) = \nabla f(x^*) + \text{relint} (\partial \Psi(x^*))$$  \hfill (14)

holds, and there is a sequence $\{x_t\}$ converging to $x^*$ with $F(x^*) \to F(x^*)$, then

$$\text{dist}(0, \partial F(x^*)) \to 0 \quad \iff \quad x_t^* \in M \quad \text{for all } t \text{ large}.$$

**Proof** We first observe that as $f \in C^1$, $x^t \to x^*$ implies $f(x^t) \to f(x^*)$, whose combination with $F(x^t) \to F(x^*)$ further implies $\Psi(x^t) \to \Psi(x^*)$. Moreover, convex functions are prox-regular everywhere. Therefore, the premises of [25, Theorem 4.10] on $\Psi$ are satisfied. We then note that

$$\text{dist}(0, \partial F(x^*)) \to 0 \quad \iff \quad \text{dist}(-\nabla f(x^*), \partial f(x^*)) \to 0.$$  \hfill (15)

Again from that $f \in C^1$, $x^t \to x^*$ implies $\nabla f(x^t) \to \nabla f(x^*)$, so by [14, 15] is further equivalent to

$$\text{dist}(-\nabla f(x^*), \partial \Psi(x^*)) \to 0,$$

which is the necessary and sufficient condition for $x^t \in M$ for all $t$ large in [25, Theorem 4.10] because (14) indicates that $-\nabla f(x^*) \in \text{relint}(\partial \Psi(x^*))$. We then apply that theorem to obtain the desired result. Here we note that for the requirements

Using [Lemma 1], we further state an identification result for (1) under our setting without the need to check whether $\{F(x^t)\}$ converges to $F(x^*)$. This will be useful in our later theoretical development.

**Lemma 2** Consider [1] with $f \in C^1$ and $\Psi \in I_0$. If $\Psi$ is partly smooth relative to a manifold $M$ at a point $x^*$ satisfying (14), and there is a sequence $\{x^t\}$ converging to $x^*$, then we have

$$\text{dist}(0, \partial F(x^*)) \to 0 \quad \Rightarrow \quad x_t^* \in M \quad \text{for all } t \text{ large}.$$

**Proof** As $\Psi \in I_0$, it is subdifferentially continuous at $x^* \in \text{dom}(\Psi)$ by [37, Example 13.30]. Thus, $\text{dist}(0, \partial F(x^*)) \to 0 \in \partial F(x^*)$ and $x^t \to x^*$ imply that $F(x^t) \to F(x^*)$. The desired result is then obtained by applying [Lemma 1].

We note that the requirement of the above two lemmas is partial smoothness of $\Psi$, instead of $F$, at $x^*$. Therefore, it is possible that $F|_M$ is not $C^2$, as we only require $\Psi|_M \in C^2$ and $f$ being $L$-smooth.
3 Manifold Identification of ISQA

Our first major result is the manifold identification property of Algorithm 1. We start with showing that the strong condition (9) with \( \epsilon_t \downarrow 0 \) is sufficient.

**Theorem 1** Consider a point \( x^s \) satisfying (14) with \( \Psi \in \Gamma_0 \) partly smooth at \( x^s \) relative to some manifold \( M \). Assume \( f \) is locally \( L \)-smooth for \( L > 0 \) around \( x^s \). If Algorithm 1 is run with the condition (9) and (7) holds, then there exist \( \epsilon, \delta > 0 \) such that \( \| x^t - x^s \| \leq \delta, \epsilon_t \leq \epsilon \), and \( \alpha_t = 1 \) imply \( x^{t+1} \in M \).

**Proof** Since each iteration of Algorithm 1 is independent of the previous ones, we abuse the notation to let \( x^t \) be the input of Algorithm 1 at the \( t \)th iteration and \( p^t \) the corresponding inexact solution to (2), but \( x^{t+1} \) is irrelevant to \( p^t \) or \( \alpha_t \). Assume for contradiction the statement is false. Then there exist a sequence \( \{x^t\} \subset E \) converging to \( x^s \), a nonnegative sequence \( \{\epsilon_t\} \) converging to 0, a sequence \( \{H_t\} \subset S_{++} \) satisfying (7) and a sequence \( \{p^t\} \subset E \) such that \( Q_{H_t}(p^t; x^t) \) in (3) satisfies (9) for all \( t \), yet \( x^t + p^t \not\in M \) for all \( t \). From (4)

\[
r^t - \nabla f(x^t) - H_t p^t \in \partial \Psi(x^t + p^t), \quad \forall t \geq 0.
\]

Therefore, we get from (16) that

\[
\text{dist}(0, \partial F(x^t + p^t)) = \text{dist}(-\nabla f(x^t + p^t), \partial \Psi(x^t + p^t)) \\
\leq \| -\nabla f(x^t + p^t) - (r^t - \nabla f(x^t) - H_t p^t) \| \\
\leq \| \nabla f(x^t + p^t) - \nabla f(x^t) \| + \| r^t \| + \| H_t \| \| p^t \| \\
\leq (L + M) \| p^t \| + \| r^t \|.
\]

From (7) and the convexity of \( \Psi, Q_{H_t}(.; x^t) \) is \( m \)-strongly convex, which implies

\[
Q_{H_t}(p; x^t) - Q_{H_t}(p^t; x^t) \geq \frac{m}{2} \| p - p^t \|^2, \quad \forall p \in E.
\]

Combining (18) and (12) shows that

\[
m^{-1} \| r^t \| + \| p^t \| \geq \| p^t \|.
\]

Since \( \{x^t\} \) converges to \( x^s \), by the argument in [13, Lemma 3.2], we get

\[
p^t = O(\| x^t - x^s \|)
\]

whenever \( x^t \) is close enough to \( x^s \). Thus (19), (20), (9), and that \( \epsilon_t \downarrow 0 \) imply

\[
\lim_{t \to \infty} \| p^t \| = 0.
\]

Substituting (9) and (21) into (17) gives \( \text{dist}(0, \partial F(x^t)) \to 0 \), and by (21) we also get \( x^t + p^t \to x^s + 0 = x^s \). Therefore, Lemma 2 implies that \( x^t + p^t \in M \) for all \( t \) large enough, proving the desired contradiction. \( \square \)

**Theorem 1** shows that if a variant of PN or PQN needs \( \| r^t \| \downarrow 0 \) to achieve superlinear convergence, \( M \) will be identified in the middle, so one can reduce the running time by switching to smooth optimization that can be conducted more efficiently while possessing the same superlinear convergence. Moreover, although Theorem 1 shows that (9) is sufficient for identifying the active manifold, it might never be satisfied as Example 1 showed. We therefore provide another sufficient condition for ISQA to identify the active manifold that is satisfied by most of the widely-used solvers for (2), showing that ISQA essentially possesses the manifold identification property. This result uses the condition (8) which is weaker than (4) and we follow [20] to define \( \{\epsilon_t\} \) using a given \( \eta \in [0, 1) \):

\[
\epsilon_t = \eta(Q_t(0) - Q^*_t) = -\eta Q^*_t.
\]

As argued in [20] and practically adopted in various implementations including [13, 19, 21], it is easy to ensure that (8) with (22) holds for some \( \eta < 1 \) under (7) (although the explicit value might be unknown) if we apply a linear-convergent subproblem solver with at least a pre-specified number of iterations to (2).

**Theorem 2** Consider the setting of Theorem 1. If Algorithm 1 is run with (8) and (22) for some \( \eta \in [0, 1) \), (7) holds, and the update direction \( p^t \) satisfies

\[
x^t + p^t = \text{prox}_{\lambda_t}^H(y^t - \Lambda_t^{-1}(\nabla f(x^t) + H_t(y^t - x^t) + s^t))
\]

where \( s^t \) satisfies \( \| s^t \| \leq R(\| y^t - (x^t + p^t) \|) \) for some \( R : [0, \infty) \to [0, \infty) \) continuous in its domain with \( R(0) = 0 \), \( \Lambda_t \in S_{++} \) with \( M_1 \geq \Lambda_t \) for \( M_1 > 0 \), and \( y^t \) satisfies

\[
\| (y^t - x^t) - p^t \| \leq \eta \| Q_t(0) - Q^*_t \| \nu
\]

for some \( \nu > 0 \) and \( \eta \geq 0 \), then there exists \( \epsilon, \delta > 0 \) such that \( \| x^t - x^* \| \leq \delta, \| Q^*_t \| \leq \epsilon \), and \( \alpha_t = 1 \) imply \( x^{t+1} \in M \).
Proof Suppose the statement is not true for contradiction. Then there exist a continuous function $R : [0, \infty) \to [0, \infty)$ with $R(0) = 0$, $\eta_1 \geq 0$, $M_1 > 0$, a sequence $\{x^t\} \subset E$ converging to $x^*$, a sequence $\{H_t\} \subset S_{++}$ satisfying (7) and

$$\lim_{t \to \infty} \min_p Q_{H_t}(p; x^t) = 0,$$  

(25)

three sequences $\{p^t\}, \{y^t\}, \{s^t\} \subset E$ and a sequence $\{A_t\} \subset S_{++}$ with $M_1 \geq A_t$ such that (8) with (22) and (23)–(24) hold, yet $x^t + p^t \notin M$ for all $t$. We abuse the notation to let $p^* \equiv Q^*_{\nu}$ and $Q^*_{\nu}$ respectively denote the optimal solution and objective value for $\min_{p, y} Q_{H_t}(p; x^*)$, but $x^{t+1} \equiv x^*$ is irrelevant to $p^*$ or $\alpha_t$.

The optimality condition of (5) applied to (23) indicates that

$$-A_t \left( y^t - y^* \right) - \left( \nabla f \left( x^t \right) + H_t \left( y^t - x^t \right) + s^t \right) \in \partial \psi \left( x^t + p^t \right).$$  

(26)

Thus, we have

$$\text{dist} \left( 0, \partial F \left( x^t + p^t \right) \right) \leq \left\| \nabla f \left( x^t + p^t \right) - A_t \left( x^t + p^t - y^t \right) - \nabla f \left( x^t \right) - H_t \left( y^t - x^t \right) - s^t \right\|$$

$$\leq \left\| \nabla f \left( x^t + p^t \right) - \nabla f \left( x^t \right) \right\| + M_1 \left( \|x^t - y^t\| + \|p^t\| \right) + M_1 \|y^t - x^t\| + \|s^t\|$$

$$\leq \left( L + M_1 \right) \|p^t\| + \left( M_1 + M_1 \right) \|y^t - x^t\| + \|s^t\|.$$  

(27)

For the first two terms in (27), the triangle inequality and (18) imply

$$\|p^t\| \leq \|p^t - p^*\| + \|p^*\| \leq \sqrt{2m^{-1}} \sqrt{Q_t(\|p^t - Q^*_t\|} + \sqrt{2m^{-1}} \sqrt{0 - Q^*_t}$$

$$\leq (\sqrt{m} + 1) \sqrt{2m^{-1}} \sqrt{Q^*_t},$$  

(28)

and

$$\|y^t - y^t\| \leq \left( \|y^t - x^t\| - p^* + \|p^*\| \right) \leq \eta_1 \left( -Q^*_t \right) + \sqrt{2m^{-1}} \sqrt{0 - Q^*_t}.$$  

(29)

For the last term in (27) we have from our definition of $s^t$ that

$$\|s^t\| \leq R \left( \|y^t - (x^t + p^*)\| \right).$$  

(30)

By substituting (28), (30) back into (27) clearly there are $C_1, C_2 > 0$ such that

$$\text{dist} \left( 0, \partial F \left( x^t + p^t \right) \right) \leq C_1 \sqrt{-Q^*_t} + C_2 (-Q^*_t)^\nu + R \left( \|y^t - (x^t + p^*)\| \right).$$  

(31)

Note that $Q_t(0; x^t) \equiv 0$, so $-Q^*_t \geq 0$ from its optimality and the right-hand side of (31) is well-defined. Next, we see from (25)–(24) and the continuity of $R$ that

$$\lim_{t \to \infty} R \left( \|y^t - x^t\| - p^* \right) = 0, \quad \Rightarrow \lim_{t \to \infty} R \left( \|y^t - x^t\| - p^* \right) = 0.$$  

(32)

Applying (25) and (32) to (31) and letting $t$ approach infinity then yield

$$\lim_{t \to \infty} \text{dist} \left( 0, \partial F \left( x^t + p^t \right) \right) = 0.$$  

(33)

Next, from (28) and (25) it is also clear that $\|p^t\| \to 0$, so from the convergence of $x^t$ to $x^*$ we have

$$x^t + p^t \to x^* + 0 = x^*.$$  

(34)

Now (34) and (33) allow us to apply Lemma 2 so $x^t + p^t \in M$ for all $t$ large enough, leading to the desired contradiction. \hfill \Box

The function $R$ can be seen as a general residual function and we just need from it that $s^t$ approaches 0 with $\|y^t - (x^t + p^*)\|$, and Theorem 2 can be used as long as we can show that such an $R$ exists, even if the exact form is unknown. Condition (24) is deliberately chosen to exclude the objective $Q_t(y^t - x^t)$ so that broader algorithmic choices like those with $y^t \notin \text{dom}(\psi)$ can be included.

One concern for Theorem 2 is the requirement of $|Q^*_t| \leq \epsilon$. Fortunately, for Algorithm 1 with (8) and (22) if $\alpha_t$ are lower-bounded by some $\alpha > 0$ (which is true under (7) by [20, Corollary 1]) and $F$ is lower-bounded, then (6) together with (8) and (22) shows that $-Q^*_t$ is summable and thus decreasing to 0.

We now provide several examples satisfying (23)–(24) to demonstrate the usefulness of Theorem 2. In our description below, $p^{i+1}$ denotes the $i$th iterate of the subproblem solver at the $t$th outer iteration and $x^{i,t} := x^t + p^{i,t}$.

- Using $x^{t+1} = x^t + p^t$ from (11). Assume we have a tentative $p^t$ that satisfies (10) for some $\epsilon_t$, and we use (11) to generate $p_t^t$ as the output satisfying (8) for a corresponding $\epsilon_t$ calculated by (13). We see that it is of the form $[23] with $s^t = 0, y^t = x^t + p^t$, and $A_t = 1/\tau$. From (13) we know that (8) is satisfied, while Corollary 3.6 and (12) guarantees (24) with $\nu = 1/2$.\footnote{\textbf{Footnote}}
– Proximal gradient (PG): These methods generate the inner iterates by
\[ x^{t,0} = x^t, \quad x^{t,i+1} = \text{prox}_{\psi}^{\lambda_{t,i}} \left( x^{t,i} - \lambda_{t,i} \left( \nabla f \left( x^t \right) + H_t \left( x^{t,i} - x^t \right) \right) \right), \quad \forall i > 0, \]
for some \( \{ \lambda_{t,i} \} \) bounded in a positive range that guarantees \( \{ Q_t(p^{t,i}) \} \) is a decreasing sequence for all \( t \) (decided through pre-specification, line search, or other mechanisms). Therefore, for any \( t \), no matter what value of \( i \) is the last inner iteration, \( (23) \) is satisfied with \( y^t = x^{t,i-1}, s^t = 0, A_{t,i} = \lambda_{t,i-1} I \). The condition \( (22) \) holds for some \( \eta < 1 \) because proximal-gradient-type methods are a descent method with \( Q \)-linear convergence on strongly convex problems, and \( (24) \) holds for \( \eta_1 \sqrt{2\eta/m} \) and \( \nu = 1/2 \) from \( (18) \).

– Accelerated proximal gradient (APG): The iterates are generated by
\[
\begin{align*}
\begin{cases}
    y^{t,i} = x^{t,0} = x^t, \\
    y^{t,i} = x^{t,i-1} + \left( 1 - \frac{2}{\sqrt{\kappa(H_t)+1}} \right) (x^{t,i-1} - x^{t,i-2}), & \forall i > 1,
\end{cases}
\end{align*}
\]
where \( \kappa(H_t) \geq 1 \) is the condition number of \( H_t \). APG satisfies \( (32) \): \( Q_t(x^{t,i} - x^t) - Q^*_t \leq -2 \left( 1 - \kappa(H_t)^{-1/2} \right)^2 Q^*_t, \quad \forall i > 0, \quad \forall t \geq 0. \)

Since \( \kappa(H_t) \geq 1 \), \( p^{t,i} \) satisfies \( (8) \) with \( (22) \) for all \( i \geq \ln 2\sqrt{\kappa(H_t)} \). If such a \( p^{t,i} \) is output as our \( p^t \), we see from \( (36) \) that \( (23) \) holds with \( s^t = 0 \) and \( A_t = \|H_t\|I \). The only condition to check is hence whether \( y^{t,i} \) satisfies \( (24) \). The case of \( i = 1 \) holds trivially with \( \eta_1 = \sqrt{2/m} \). For \( i > 1 \), \( (24) \) holds with \( \eta_1 = 3\sqrt{2\eta m^{-1}} \) and \( \nu = 1/2 \) because
\[
\begin{align*}
    \| y^{t,i} - (x^t + p^t) \| & \leq \left( 1 - \frac{2}{\sqrt{\kappa(H_t)+1}} \right) \| x^{t,i-1} - x^{t,i-2} \| + \| x^{t,i-1} - (x^t + p^t) \|
    \leq 2\sqrt{2\eta m^{-1}} - \sqrt{-Q^*_t}.
\end{align*}
\]

– Prox-SAGA/SVRG: These methods update the iterates by
\[ x^{t,i} = \text{prox}_{\psi}^{\lambda_{t,i}} \left( x^{t,i-1} - \lambda_{t,i} \left( \nabla f \left( x^t \right) + H_t \left( x^{t,i-1} - x^t \right) + s^t \right) \right), \]
with \( x^{t,0} = x^t \), \( \{ \lambda_{t,i} \} \) bounded in a positive range, and \( \{ s^{t,i} \} \) are random variables converging to 0 as \( x^{t,i} - x^t \) approaches \( p^t s^t \). (For a detailed description, see, for example, \( [36] \).) It is shown in \( [45] \) that for prox-SVRG, \( Q_t(x^{t,i} - x^t) - Q^*_t \) converges linearly to 0 with respect to \( i \) if \( \lambda_i \) is small enough, so \( (8) \) with \( (22) \) is satisfied. A similar but more involved bound for prox-SAGA can be derived from the results in \( [11] \). When \( p^t = x^{t,i} - x^t \) for some \( i > 0 \) is output as \( p^t \), we get \( y^t = x^{t,i-1} \), \( A_t = \lambda_i I \), and \( s^t = s^{t,i} \) in \( (23) \) so the requirements of \( \text{Theorem 2} \) hold.

If \( \psi \) is block-separable and \( \mathcal{M} \) decomposable into a product of submanifolds that conform with the blocks of \( \psi \), \( (23) \) can be modified easily to suit block-wise solvers like block-coordinate descent. This extension simply adapts the analysis above to the block-wise setting, so the proof is straightforward and omitted for brevity.

3.1 Iterate Convergence

\( \text{Theorem 1} \) and \( \text{Theorem 2} \) both indicate that for \( \text{Algorithm 1} \) to identify \( \mathcal{M} \), we need \( x^t \) (or at least a subsequence) to converge to a point \( x^* \) of partial smoothness. We thus complement our analysis to show the convergence of the iterates under convexity of \( f \) and a local sharpness condition, which is a special case of the KL condition and is universal in real-world problems, without any additional requirement on the algorithm such as the relative-error condition in \( [2, 3, 5] \). In particular, we assume \( F \) satisfy the following for some \( \zeta, \xi > 0 \), and \( \theta \in (0, 1] \):
\[
    \zeta \text{dist}(x, \Omega) \leq \left( F(x) - F^* \right)^\theta, \quad \forall x \in \text{Lev} (\zeta \xi) \tag{38}.
\]

This becomes the well-known quadratic growth condition when \( \theta = 1/2 \), and \( \theta = 1 \) corresponds to the weak-sharp minima \( [6] \). As discussed in \( \text{Section 1.1} \) under convexity of \( f \), \( [17] \) showed that the iterates of their PN variant converge to some \( x^* \in \Omega \) if \( F \) satisfies EB, which is equivalent to the quadratic growth condition in their setting \( [12] \). Our analysis allows broader choices of \( H_t \) and (strong) iterate convergence is proven for \( \theta \in (1/4, 1] \).
Theorem 3 Consider Algorithm 1 with any $x^0 \in E$. Assume that $\Omega \neq \emptyset$, $f$ is convex and $L$-smooth for $L > 0$, $\Psi \in \Gamma_0$, (7) holds, there is $\eta \in [0,1]$ such that $p^t$ satisfies (8) with (22) for all $t$, and that (38) holds for $\xi, \zeta > 0$ and $\theta \in (1/4, 1]$. Then $x^t \to x^*$ for some $x^* \in \Omega$.

The convergence in Theorem 3 holds true in infinite dimensional real Hilbert spaces with strong convergence (which is indistinguishable from weak convergence in the finite-dimensional case), and the proof in Appendix B is written in this general scenario. The key of its proof is our following improved convergence rate, which might have its own interest. Except for that the case of $\theta = 1/2$ has been proven by Peng et al [34] and that $\theta = 0$ reduces to the general convex case analyzed in [20], this faster convergence rate is, up to our knowledge, new for ISQA.

Theorem 4 Consider the settings of Theorem 3 but with $\theta \in [0,1]$, $E$ a real Hilbert space and $x^0 \in \text{Lev}(\xi)$. Then there is $\bar{\alpha} > 0$ such that $\alpha_t \geq \bar{\alpha}$ for all $t$ and the following hold.

1. For $\theta \in (1/2, 1]$: When $\delta_t$ is large enough such that
   \[
   \delta_t > (\zeta^2 M^{-1})^{1/\alpha_t},
   \]
   we have
   \[
   \delta_{t+1} \leq \delta_t \left(1 - \frac{(1 - \eta)\alpha_t\zeta^2 \zeta^{1-2\theta}}{2M}\right).
   \]
   Next, let $t_0$ be the first index failing (39) then for all $t \geq t_0$ we have
   \[
   \delta_{t+1} \leq \delta_t \left(1 - \frac{(1 - \eta)\alpha_t\zeta}{2}\right).
   \]

2. For $\theta = 1/2$, we have global $Q$-linear convergence of $\delta_t$ in the form
   \[
   \frac{\delta_{t+1}}{\delta_t} \leq 1 - (1 - \eta)\alpha_t \zeta. \begin{cases} \frac{\zeta^2}{2\delta_t}, & \text{if } \zeta^2 \leq \|H_t\|, \\ \frac{\|H_t\|}{2\zeta^2}, & \text{else,} \end{cases} \end{cases}
   \]
   \forall t \geq 0.

3. For $\theta \in [0, 1/2]$, (41) takes place when $\delta_t$ is large enough to satisfy (39) Let $t_0$ be the first index such that (39) fails, then
   \[
   \frac{\delta_t}{\delta_{t_0}} \leq \left(1 - 2\theta \sum_{i=t_0}^{t-1} \alpha_i\right)^{-1/2\theta} \leq ((1 - 2\theta)(t - t_0)\bar{\alpha})^{-1/2\theta}, \forall t \geq t_0.
   \]

For the range of $\theta$ in Theorem 3 convergence of the iterates generated by inexact SQA is a new result. Moreover, even if the additional conditions on the forward-backward envelope in [3] also hold, although their analysis also uses the subproblem stopping condition (8) they need a much stricter stopping tolerance of the form
\[
\epsilon_t = -\tau_t Q_t^\tau, \quad \sum \sqrt{\tau_t} \leq \infty,
\]
which clearly requires $\tau_t$ to converge to 0 fast enough and thus costs much more time in the subproblem solve. In contrast, we just need $\epsilon_t$ to be a constant factor of $Q_t^\tau$ in (22) so the number of inner iterations and thus the cost of subproblem solve can be a constant.

4 An Efficient Inexact SQA Method with Superlinear Convergence in Running Time

Now that it is clear ISQA is able to identify the active manifold, we utilize the fact that the optimization problem reduces to a smooth one after the manifold is identified to devise more efficient approaches, with safeguards to ensure that the correct manifold is really identified. The improved algorithm, ISQA*, is presented in Algorithm 2 and we explain the details below.

ISQA* has two stages, separated by the event of identifying the active manifold $\mathcal{M}$ of a cluster point $x^*$. Our analysis showed that iterates converging to $x^*$ will eventually identify $\mathcal{M}$, but since neither $x^*$ nor $\mathcal{M}$ is known a priori, the conjecture of identification can only be made when $\mathcal{M}$ remains unchanged for $S \geq 0$ iterations.

Most parts in the first stage are the same as Algorithm 1 although we have added specifications for the subproblem solver according to Theorem 2. The only major difference is that instead of linesearch, ISQA* adjusts $H_t$ and re-solve (2) whenever (16) with $\alpha_t = 1$ fails. This trust-region-like approach has guaranteed global convergence from [20] and ensures $\alpha_t = 1$ for Theorem 2 to be applicable.
preconditioned CG (PCG, see for example [33, Chapter 5]) to find an approximate solution for conducting MO at the truncated semismooth Newton (TSSN) approach. Since discussed in Section 2, we can find a φ ∇ its generalized Hessian (denoted by is preferred in practice for better running time. Although we did not assume that the exact solution to the subproblem for generating the update direction in finite time, because the subproblem manifold optimization problem, and efficient algorithms for this type is abundant (see, for example, [1] for an sophisticated when the superlinear convergence phase of the MO step is reached, using PG instead of solving (2) with a is not correctly identified, a PG step thus prevents us from sticking at a wrong manifold, while φs remain fixed, we also retain the same A iterations

\[ A^T A \] with at least \( T \) iterations

\[ \text{while} \ [6] \text{ is not satisfied with } \alpha_t = 1 \text{ do} \]

\[ \text{if } \text{Manifold optimization fails then } x^{t+1} \leftarrow x^t, \text{ Unchanged } \leftarrow 0 \]

\[ \text{else SmoothStep } \leftarrow 1 \]

\[ x^{t+1} = \text{prox}_{\rho F}(x^t - \mu_t \nabla f(x^t)) \] (44)

In the second stage, we alternate between a standard proximal gradient (PG) step and a manifold optimization (MO) step. PG is equivalent to solving (2) with \( H_t = L I \) to optimality, so Theorem 2 applies. When \( M \) is not correctly identified, a PG step thus prevents us from sticking at a wrong manifold, while when the superlinear convergence phase of the MO step is reached, using PG instead of solving (2) with a sophisticated \( H_t \) avoids redundant computation.

When the objective is partly smooth relative to a manifold \( M \), optimizing it within \( M \) can be cast as a manifold optimization problem, and efficient algorithms for this type is abundant (see, for example, [1] for an overview). The difference between applying MO methods and sticking to (2) is that in the former, we can obtain the exact solution to the subproblem for generating the update direction in finite time, because the subproblem in the MO step is simply an unconstrained quadratic optimization problem whose solution can be found by solving a linear system, while in the latter it takes indefinitely long to compute the exact solution, so the former is preferred in practice for better running time. Although we did not assume that \( f \) is twice-differentiable, its generalized Hessian (denoted by \( \nabla^2 f \)) exists everywhere since the gradient is Lipschitz-continuous [10]. As discussed in Section 2, we can find a \( C^2 \) parameterization \( \phi \) of \( M \) around \( x^* \), and we use this \( \phi \) to describe a truncated semismooth Newton (TSSN) approach. Since \( M \) might change between iterations, when we are conducting MO at the \( t \)th iteration, we find a parameterization \( \phi_t \) of the current \( M \) and a point \( y^t \) such that \( \phi_t(y^t) = x^t \). If \( M \) remains fixed, we also retain the same \( \phi_t \). The TSSN step \( q^t \) is then obtained by using preconditioned CG (PCG, see for example [33, Chapter 5]) to find an approximate solution for

\[ q^t \approx \arg \min_q \langle g^t, q \rangle + \frac{1}{2} \langle q, H_t q \rangle, \quad \text{or equivalently} \quad H_t q^t \approx -g^t, \]

\[ g^t := \nabla F(\phi_t(y^t)), \quad H_t := \nabla^2 F(\phi_t(y^t)) + \mu_t I, \quad \mu_t := c \|g^t\|^p, \] (45)

that satisfies

\[ \|H_t q^t + g^t\| \leq 0.1 \min \left\{ \|g^t\|, \|g^t\|^{1+\rho} \right\} \] (46)

with pre-specified \( c > 0 \) and \( \rho \in (0, 1) \). We then run a backtracking line search procedure to find a suitable step size \( \alpha_t > 0 \). For achieving superlinear convergence, we should accept unit step size whenever possible, so we only require the objective not to increase. If \( q^t \) is not a descent direction or \( \alpha_t \) is too small, we consider the MO step failed and go back to the first stage. If \( \alpha_t < 1 \), the superlinear convergence phase is not entered yet, and likely \( M \) has not been correctly identified, so we also switch back to the first stage. This algorithm is summarized in Algorithm 3. When products between \( \nabla^2 F(\phi_t(y^t)) \) and arbitrary points, required by PCG, cannot be done easily, one can adopt Riemannian quasi-Newton approaches like [17] instead.
**Algorithm 3:** Truncated semismooth Newton on Manifold

```
input : \( x^\prime \in \mathcal{E}, \ c > 0, \ p \in (0, 1], \ \alpha, \beta, \gamma \in (0, 1), \) and a manifold \( \mathcal{M} \nexists \)
Obtain parameterization \( \phi_t \) for \( \mathcal{M} \) and point \( y^\prime \) with \( \phi_t(y^\prime) = x^\prime \)
\( \alpha_t \leftarrow 1, \) and compute \( g^\prime \) in \( (\ref{eq:semismooth}) \nexists \) Obtain by PCG an approximation solution \( q^\prime \) to \( (\ref{eq:semismooth}) \nexists \) satisfying \( (\ref{eq:approx}) \nexists \) if \( (q^\prime, g^\prime) \nexists \gtrless 0 \nexists \) then Report MO step fails and exit

while \( F(\phi_t(y^\prime + \alpha_t q^\prime)) \nexists F(x^\prime) \nexists \) and \( \alpha_t > \alpha \nexists \) do \( \alpha_t \leftarrow \beta \alpha_t \nexists \) if \( \alpha_t \leq \alpha \nexists \) then Report MO step fails
else \( x^{t+1} \leftarrow \phi_t(y^\prime + \alpha_t q^\prime) \)
```

4.1 Global Convergence

This section provides global convergence guarantees for Algorithm 2. Because MO steps do not increase the objective value, global convergence of Algorithm 2 follows from the analysis in [20] by treating (44) as solving (2) with \( H_t = \tilde{L} I \) and noting that this update always satisfies (6) with \( \alpha_t = 1 \). For completeness, we still state these results, and provide a proof in Appendix C.

First, we restate a result in [20] to bound the number of steps spent in the while-loop for enlarging \( H_t \) in Algorithm 2.

**Lemma 3 ([20, Lemma 4]):** Given an initial choice \( H^0_t \) for \( H_t \) at the \( t \)th iteration of Algorithm 2 (so initially we start with \( H_t = H^0_t \) and modify it when (6) fails with \( \alpha_t = 1 \)) and a parameter \( \beta \in (0, 1) \). Consider the following two variants for enlarging \( H_t \), starting with \( \sigma = 1 \):

\[
\sigma \leftarrow \beta \sigma, \quad H_t \leftarrow H^0_t / \sigma, \quad (\text{Variant 1})
\]

\[
H_t \leftarrow H^0_t + \sigma^{-1} I, \quad \sigma \leftarrow \beta \sigma. \quad (\text{Variant 2})
\]

We then have the following bounds if every time the approximate solution to (2) always satisfies \( Q_{\mathcal{H}}(p^t; x^t) \leq 0 \).

1. If \( H^0_t \) satisfies \( \min_t M_t^0 \geq H^0_t \geq m_0^0 \) for some \( \min_t M_t^0 \geq m_0^0 > 0 \), then the final \( H_t \) from Variant 1 satisfies \( \|H_t\| \leq M \max \{1, L/ (3\beta m)\} \) and the while-loop terminates within \( \lceil \log_{3/2} L / m \rceil \) rounds.

2. If \( H^0_t \geq 0 \), then the final \( H_t \) from Variant 2 satisfies \( \|H_t\| \leq M + \max \{1, L/\beta\} \) and the while-loops terminates within \( 1 + \lceil \log_{3/2} L \rceil \) rounds.

Now we provide global convergence guarantees for Algorithm 2 without the need of manifold identification. From Lemma 3 we can simply assume without loss of generality that (7) holds for the final \( H_t \) that leads to the final update direction that satisfies (6).

**Theorem 5** Consider (1) with \( f \) \( L \)-smooth for \( L > 0, \ \Psi \in \mathcal{G}_0, \) and \( \Omega \neq \emptyset \). Assume Algorithm 2 is applied with an initial point \( x^0 \), the estimate \( \hat{L} \) satisfies \( \hat{L} \geq L \geq \Omega \), (7) holds for the final \( H_t \) after exiting the while-loop, and (8) is satisfied with (22) for some \( \eta \in (0, 1) \) fixed over \( t \). Let \( \{k_t\} \) be the iterations that the MO step is not attempted (so either 2 or 4 is solved approximately or (44) is conducted), then we have \( k_t \leq 2 t \) for all \( t \). By denoting \( \tilde{M} := \max \{\hat{L}, M\}, \ m := \min \{\hat{L}, m\}, \) we have the following convergence rate guarantees.

1. Let \( G_t := \arg \min_p Q_f(p; x^t), \) then \( \|G_{k_t}\| \to 0, \) and for all \( t \geq 0, \) we have

\[
\min_{0 \leq i \leq t} \|G_{k_i}\|^2 \leq \frac{F(x^0) - F^*}{\gamma(t+1)} \frac{\hat{M}^2 \left(1 + \hat{m}^{-1} + \sqrt{1 - 2\hat{M}^{-1} + \hat{m}^{-2}}\right)^2}{2(1 - \eta)\hat{m}}.
\]

Moreover, \( G_t = 0 \) if and only if \( 0 \in \partial F(x^t), \) and therefore any limit point of \( \{x^{k_t}\} \) is a stationary point of (1).

2. If in addition \( f \) is convex and there exists \( R_0 \in (0, \infty) \) such that

\[
\sup_{x : F(x) \leq F(x^0)} \|x - P_d(x)\| = R_0
\]

(in other words, (38) holds with \( \theta = 0, \ \zeta = R_0^{-1}, \) and \( \xi = F(x^0) - F^* \)), then:

2.1. When \( F(x^{k_t}) - F^* \geq M R_0^2, \) we have

\[
F \left(x^{k_t+1}\right) - F^* \leq \left(1 - \frac{\gamma(1 - \eta)}{2}\right) \left(F \left(x^{k_t}\right) - F^*\right).
\]
2.2. Let \( t_0 := \text{arg \, min}_t \{ t : F(x^{k_t}) - F^* < \bar{M} R_0^2 \} \), we have for all \( t \geq t_0 \) that

\[
F(x^{k_t}) - F^* \leq \frac{2\bar{M} R_0^2}{\gamma (1 - \eta)} (t - t_0) + 2.
\]

Moreover, we have

\[
t_0 \leq \max \left\{ 0, 1 + \frac{2}{\gamma (1 - \eta)} \log \frac{F(x^0) - F^*}{M R_0^2} \right\}.
\]

In summary, we have \( F(x^t) - F^* = O \left( t^{-1} \right) \).

3. The results of Theorem 4 hold, with \( M \) and \( \|H_t\| \) replaced by \( \bar{M} \), \( \delta_t \) by \( \delta_{k_t} \), \( \alpha_t \) and \( \bar{\alpha} \) by 1, \( \delta_t \) by \( \delta_{k_t}, \delta_{t+1} \) by \( \delta_{k_{t+1}} \), and \( \delta_{t_0} \) by \( \delta_{k_{t_0}} \).

4.2 Superlinear and Quadratic Convergence

Following the argument in the previous subsection to treat (44) as solving (2) exactly, the manifold identification property of (44) also follows from Theorem 2. We thus focus on its local convergence in this subsection. In what follows, we will show that \( x^t \) converges to a stationary point \( x^* \) satisfying (14) superlinearly or even quadratically in the second stage.

Let \( \mathcal{M} \) be the active manifold of \( x^* \) and \( \phi \) be a parameterization of \( \mathcal{M} \) with \( \phi(y^*) = x^* \) for some point \( y^* \). We can thus assume without loss of generality that \( \phi_t = \phi \) for all \( t \) that identified \( \mathcal{M} \). We denote \( F_{\phi}(y) := F(\phi(y)) \).

For simplicity, we assume that \( F_{\phi} \) is twice-differentiable with its Hessian locally Lipschitz continuous around \( y^* \). In particular, we just need the following property to hold locally in a neighborhood \( U_0 \) of \( y^* \):

\[
\nabla F_{\phi}(y_1) - \nabla F_{\phi}(y_2) - \nabla^2 F_{\phi}(y_2) = O (\|y_1 - y_2\|^2), \quad \forall y_1, y_2 \in U_0.
\]

(48)

We do not assume \( \nabla^2 F_{\phi}(y) \geq 0 \) like existing analyses for Newton’s method, but consider a degenerate case in which there is a neighborhood \( U_1 \) of \( y^* \) such that

\[
\nabla^2 F_{\phi}(y) \geq 0, \quad \forall y \in U_1.
\]

(49)

Note that (49) implies that \( F_{\phi} \) is convex within \( U_1 \). We can decompose \( \mathcal{E} \) into the direct sum of the tangent and the normal spaces of \( \mathcal{M} \) at \( x^* \), and thus its stationarity implies \( \nabla F_{\phi}(y^*) = 0 \). This and (49) mean \( y^* \) is a local optimum of \( F_{\phi} \), and hence \( x^* \) is a local minimum of \( F \) when \( f \) is \( L \)-smooth, following the argument of [43, Theorem 2.5]. We also assume that \( F_{\phi} \) satisfies a sharpness condition similar to (38) in a neighborhood \( U_2 \) of \( y^* \):

\[
\zeta^\hat{\theta} \| y - y^* \| \leq \left( F_{\phi}(y) - F(y^*) \right)^\hat{\theta}, \quad \forall y \in U_2,
\]

(50)

for some \( \zeta > 0 \) and \( \hat{\theta} \in (0, 1/2] \). By shrinking the neighborhoods if necessary, we assume without loss of generality that \( U_0 = U_1 = U_2 \) and denote it by \( U \). Note that the conventional assumption of positive-definite Hessian at \( y^* \) is a special case that satisfies (49) and (50) with \( \hat{\theta} = 1/2 \).

We define \( d_t := \| y^t - y^* \| \) and use it to bound \( y^t + q^t \) and \( \nabla F_{\phi}(y^t + q^t) \).

**Lemma 4** Consider a stationary point \( x^* \) of (11) with \( \Psi \) partly smooth at it relative to a manifold \( \mathcal{M} \) with a parameterization \( \phi \) and a point \( y^* \) such that \( \phi(y^*) = x^* \), and assume that within a neighborhood \( U \) of \( y^* \), \( F_{\phi} \) is twice-differentiable with (48) and (49) hold. Then \( y^t \in U \) implies that any \( q^t \) satisfying (46) is bounded by

\[
\| q^t \| \leq 2d_t + \mu_t^{-1} O (d_t^2) + 0.1 \mu_t^{-1} \| g^t \|^{1+\rho}.
\]

(51)

**Proof** From (46) we can find \( \psi_t \in \mathcal{E} \) such that

\[
H_t g^t + g^t = \psi_t, \quad \| \psi_t \| \leq 0.1 \| g^t \|^{1+\rho}.
\]

(52)

From (49) and (45) we have

\[
H_t \geq \mu_t > 0,
\]

(53)

\( ^1 \hat{\theta} > 1/2 \) cannot happen unless \( F_{\phi} \) is a constant in \( U_2 \).
so $H_t$ is invertible. We then get
\[
\|g^t + q^t - y^*\| \leq H_t^{-1} \|\psi_t - g^t + H_t(g^t - y^*)\| \\
\leq H_t^{-1} \left( \|\psi_t\| + \|g^t - \nabla^2 F(\phi_t(g^t))\| d_t \right) + \mu_t d_t.
\]
From the triangle inequality, we have $\|q^t\| \leq \|g^t - y^*\| + \|y^t + q^t - y^*\|$, whose combination with (54) proves (51).

**Lemma 5** Consider the setting of Lemma 4 and further assume that $\Psi \in \Gamma_0$ and $f$ is $L$-smooth. The following hold.

1. If $\rho \in (0, 1]$ and $F_\phi$ satisfies (50) with $\hat{\theta} = 1/2$ for some $\zeta > 0$, then
\[
\|y^t + q^t - y^*\| = O\left(d_t^{1+\rho}\right), \quad \|\nabla F_\phi(g^t + q^t)\| = O\left(\|g^t\|^{1+\rho}\right).
\]

2. If $\rho = 0.69$ and $F_\phi$ satisfies (50) for some $\zeta > 0$ and $\hat{\theta} \geq 3/8$, then
\[
\|y^t + q^t - y^*\| = o(d_t), \quad \|\nabla F_\phi(g^t + q^t)\| = o\left(\|g^t\|\right).
\]

**Proof** From the assumptions on $\Psi$, $\phi$, and $f$, $F_\phi$ is twice-differentiable almost everywhere, and within any compact set $K$ containing $y^*$, any $\nabla^2 F_\phi \in \partial \nabla F_\phi$ is upper-bounded by some $L_K > 0$ ($f(\phi(y))$ is differentiable with the gradient Lipschitz-continuous, and $\nabla^2(\Psi(\phi(y)))$ is upper-bounded), so $F_\phi$ is $L_K$-smooth within $K$.

Since $K$ is arbitrary, we let $K \subset U$ and obtain
\[
\|g^t\| \leq L_K d_t.
\]
(57)

Since (49) implies that $F_\phi$ is convex in $U$, (50) leads to
\[
\|\nabla F_\phi(y)\| \geq \zeta \|y - y^*\|^{(1-\hat{\theta})/\hat{\theta}}, \quad \forall y \in U.
\]
(58)

For the first case, (57) (58) show $g^t = \Theta(d_t)$, so Lemma 4 implies
\[
\|q^t\| = O(d_t) = O(g^t).
\]
(59)

Thus, by the triangle inequality, (55) is proven by
\[
\|\nabla F_\phi(g^t + q^t)\| = \|\nabla F_\phi(g^t + q^t) + \psi_t - \psi_t\| \\
\leq \|\nabla F_\phi(g^t + q^t) - g^t - H_t g^t\| + \|g^t\|^{1+\rho} \\
\leq O\left(\|g^t\|^2\right) + \mu_t \|q^t\| + 0.1 \|g^t\|^{1+\rho} O\left(\|g^t\|^{1+\rho}\right).
\]
(60)

In the second case, $\hat{\theta}/(1 - \hat{\theta}) \geq 3/5$, so (45) Lemma 4 and (57) (58) imply
\[
\|q^t\| = O\left(d_t^{0.85}\right).
\]
(61)

We then get from $\rho = 0.69$ that
\[
\|\nabla F_\phi(g^t + q^t)\| \leq O\left(\|g^t\|^2\right) + \mu_t \|q^t\| + 0.1 \|g^t\|^{1.69}
\]
\[
O\left(d_t^{1.7}\right) + 2 \mu_t d_t + O(d_t^2) + O(d_t^{1.69}) = O(d_t^{1.69}).
\]
(62)

From (58) we get that
\[
\|y^t + q^t - y^*\| = O\left(\|\nabla F_\phi(g^t + q^t)\|^{0.6}\right) \leq O\left(d_t^{1.69}\right)^{0.6} = O(d_t^{0.1014}),
\]
proving the first equation in (56). The second one is then proven by
\[
\|\nabla F_\phi(g^t + q^t)\| \leq O\left(\|d_t^{1.69}\right) \leq O\left(\|d_t^{1.014}\right).
\]
(63)

Now we are able to show two-step superlinear convergence of $\|x - x^*\|$. 

Theorem 6 Consider the setting of Lemma 5 and assume in addition that $x^*$ satisfies (14). Then there is a neighborhood $V$ of $x^*$ such that if at the $t_0$th iteration of Algorithm 2 for some $t_0 > 0$ we have that $x^{t_0} \in V$, \( \text{Unchanged} \geq S, \) $\mathcal{M}$ is correctly identified with parameterization $\phi$ and $\phi(y^*) = x^*$, and $\alpha_t = 1$ is taken in Algorithm 3 for all $t \geq t_0$, we get the following for all $t \geq t_0$.

1. For $\rho \in (0, 1]$ and $F_\phi$ satisfying (50) with $\tilde{\theta} = 1/2$ for some $\zeta > 0$:

$$
\|x^{t+2} - x^*\| = O \left( \|x^t - x^*\|^{1+\rho} \right) \|\nabla F_\phi(y^{t+2})\| = O \left( \|\nabla F_\phi(y^t)\|^{1+\rho} \right).
$$

(63)

2. For $\rho = 0.69$ and $F_\phi$ satisfying (50) for some $\zeta > 0$ and $\tilde{\theta} \geq 3/8$,

$$
\|x^{t+2} - x^*\| = o \left( \|x^t - x^*\| \right).
$$

Proof In our discussion below, $V_i$ and $U_i$ for $i \in \mathbb{N}$ are respectively neighborhoods of $x^*$ and $y^*$.

Since $\phi$ is $C^2$, there is $U_1$ of $y^*$ such that

$$
\phi(y) - \phi(y^*) = \langle \nabla \phi(y^*), y - y^* \rangle + O \left( \|y - y^*\|^2 \right), \forall y \in U_1.
$$

(64)

Because the derivative of $\phi$ at $y^*$ is injective, (64) implies

$$
\|\phi(y) - \phi(y^*)\| = \Theta \left( \|y - y^*\| \right), \forall y \in U_1.
$$

(65)

If the $t$th iteration is a TSSN step, we define $y^t$ to be the point such that $\phi(y^t) = x^t$. If either case in Lemma 5 holds and $q^t$ satisfies (46), from that $\|y^t + q^t - y^*\| = o(\|y^t - y^*\|)$ we can find $U_2 \subset U$ such that $y^t \in U_2$ implies $y^t + q^t \in U_2$. Take $U_3 := U_1 \cap U_2 \subset U$, for $y^t \in U_3$ and $x^{t+1} = \phi(y^t + q^t)$, we get $y^t + q^t \in U_3 \subset U_1$, and hence the following from (65).

$$
\|x^{t+1} - x^*\| = \Theta \left( \|y^t + q^t - y^*\| \right), \quad \|x^t - x^*\| = \Theta \left( \|y^t - y^*\| \right).
$$

(66)

On the other hand, consider the case in which the $t$th iteration is a PG step. As $\phi$ is $C^2$ and $\phi(y^*) = x^*$, we can find $V_1$ such that $\phi(U_3) \supset V_1 \cap \mathcal{M}$. From Lemma 3.2, there is $V_2$ such that $x^t \in V_2$ implies

$$
\|x^{t+1} - x^*\| = O(\|x^t - x^*\|),
$$

(67)

so there is $V_2 \subset V_3$ such that $x^{t+1} \in V_1$ if $x^t \in V_2$. Therefore, from Theorem 2 (applicable because we have assumed (14)), there is $V_3$ such that $x^t \in V_4$ implies $x^{t+1} \in \mathcal{M}$. Take $V_5 := V_1 \cap V_3$, then $x^t \in V_5$ implies $x^{t+1} \in V_1 \cap \mathcal{M}$, thus we can find $y^{t+1} \in U_3$ with $\phi(y^{t+1}) = x^{t+1}$.

Now consider the first case in the statement. If at the $t$th iteration we have $x^t \in V_5$ and have taken (44) then $x^{t+1} \in V_1 \cap \mathcal{M}$ with $x^{t+1} = \phi(y^{t+1})$ for some $y^{t+1} \in U_3$, so we can take a TSSN step at the $(t + 1)$th iteration and

$$
\|x^{t+2} - x^*\| = \Theta \left( \|y^{t+1} + q^{t+1} - y^*\| \right) = O \left( \|y^{t+1} - y^*\|^{1+\rho} \right).
$$

(68)

so there is $V_6 \subset V_5$ such that $x^t \in V_6$ implies $x^{t+2} \in V_6$ as well and the superlinear convergence in (68) propagates to $t+2, t+4, \ldots$. We therefore see that $\|x^{t+2i+1} - x^*\| = O(\|x^{t+2i-1} - x^*\|^{1+\rho})$ for $i \in \mathbb{N}$ as well, so there is $V_7 \subset V_6$ such that $x^{t+2i-1} \in V_7 \cap \mathcal{M}$ implies $x^{t+2i+1} \in V_7 \cap \mathcal{M}$ as well.

Let $V = V_7 \cap V_6$, we see that $x^t \in V \cap \mathcal{M}$ implies $x^{t+2} \in V \cap \mathcal{M}$ no matter we take PG or TSSN first, proving the first equation in (63). The convergence of $\nabla F_\phi$ then follows from (57) and (58). The superlinear convergence in the second case follows the same argument.

Note that when $\rho = 1$ in the first case, we obtain quadratic convergence.

The analysis in (47) assumed directly (58) instead of (50), together with a Lipschitzian Hessian for $f$, under the setting of regularized optimization to get a superlinear rate. In the context of smooth optimization, our analysis is more general in giving a wider range of superlinear convergence. In particular, for (58) only allowed $\tilde{\theta} = 1$, where $\tilde{\theta} := \theta/(1 - \theta)$, whereas our result extends the range of superlinear convergence to $\tilde{\theta} \geq 0.6$.

Remark 1 PCG returns the exact solution of (45) in $d$ steps, where $d$ is the dimension of $\mathcal{M}$, and each step involves only a bounded number of basic linear algebra operations, so the running time of Algorithm 3 is upper bounded. Therefore, superlinear convergence of Algorithm 2 in terms of iterations, from Theorem 6 implies that in terms of running time as well. This contrasts with existing PN approaches, as they all require applying an iterative subproblem solver to (2) with increasing precision, which also takes increasing time per iteration because (2) has no closed-form solution.
5 Numerical Results

We conduct numerical experiments on $\ell_1$-regularized logistic regression to support our theory, which is of the form in (1) with $E = \mathbb{R}^d$ for some $d \in \mathbb{N}$,

$$\Psi(x) = \lambda \|x\|_1, \quad f(x) = \sum_{i=1}^n \log (1 + \exp (-b_i (a, x))),$$

(69)

where $\lambda > 0$ decides the weight of the regularization and $(a, b_i) \in \mathbb{R}^d \times \{-1, 1\}$ for $i = 1, \ldots, n$ are the data points. Note that $\lambda \|x\|_1$ is partly smooth at every $x \in \mathbb{R}^d$ relative to $M_x := \{y \mid y_i = 0, \forall i \in J_x\}$, where $J_x := \{i \mid x_i = 0\}$. Let $I$ be the identity matrix, and $J_C x := \{1, \ldots, d\} \setminus J_x$, then viewing from the definition of $M_x$ here, the parameterization we use at each iteration is simply projecting $y_t \in \mathbb{R}^{|J_{xt}|}$ back to $\mathbb{R}^d$. Namely,

$$\phi_t(y_t) := I_{J_{xt}} y_t.$$

We use public available real-world data sets listed in Table 1. All experiments are conducted with $\lambda = 1$ in (69). All methods are implemented in C++, and we set $\gamma = 10^{-4}, \beta = 0.5, T = 5$ throughout for ISQA and ISQA$^+$.

5.1 Manifold Identification of Different Subproblem Solvers

We start with examining the ability for manifold identification of different subproblem solvers. We run both ISQA and the first stage of ISQA$^+$ (by setting $S = \infty$) and consider two settings for $H_t$. The first is the L-BFGS approximation with a safeguard in [21], and we set $m = 10$ and $\delta = 10^{-10}$ in their notation following their experimental setting. The second is a PN approach in [47] that uses $H_t = \nabla^2 f(x^t) + c \|x^t - \text{prox}_\Psi(x^t - \nabla f(x^t))\|_I$, and we set $\rho = 0.5, c = 10^{-6}$ following their suggestion. In both cases, we enlarge $H_t$ in Algorithm 2 through $H_t \leftarrow 2H_t$.

We compare the following subproblem solvers.

- SpaRSA [44]: a backtracking PG-type method with the initial step sizes estimated by the Barzilai-Borwein method.
- APG [31]: See (35)-(36). We use a simple heuristic of restarting whenever the objective is not decreasing.
- Random-permutation cyclic coordinate descent (RPCD): Cyclic proximal coordinate descent with the order of coordinates reshuffled every epoch.

The results presented in Table 2 show that all subproblem solvers for ISQA$^+$ can identify the active manifold, verifying Theorem 2. Because the step sizes are mostly one in this experiment, even solvers for ISQA$^+$ can identify the active manifold. Among the solvers, RPCD is the most efficient and stable in identifying the active manifold, so we stick to it in subsequent experiments.

5.2 Comparing ISQA$^+$ with Existing Algorithms

We proceed to compare ISQA$^+$ with the following state of the art for (1) using the relative objective value: $\left(F(x) - F^*\right)/F^*$.

- LHAC [35]: an inexact proximal L-BFGS method with RPCD for (2) and a trust-region-like approach. Identical to our L-BFGS variant, we set $m = 10$ for constructing $H_t$ in this experiment.

Table 1: Data statistics.

| Dataset    | #instances (n) | #features (d) |
|------------|----------------|---------------|
| a9a        | 32,561         | 123           |
| realsim    | 72,309         | 20,958        |
| news20     | 19,996         | 1,355,191     |
| ijcnn1     | 49,990         | 22            |
| covtype.scale | 581,012     | 54            |
| rcv1-test  | 677,399        | 47,226        |
| epsilon    | 400,000        | 2,000         |
| webspam    | 350,000        | 16,609,143    |

2 Downloaded from [http://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/](http://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/)
NewGLMNET \cite{46}: a line-search PN with an RPCD subproblem solver.

IRPN \cite{47} is another PN method that performs slightly faster than NewGLMNET, but their algorithmic frameworks are similar and the experiment in \cite{47} showed that the running time of NewGLMNET is competitive. We thus use NewGLMNET as the representative because its code is open-sourced.

For ISQA$^+$, we set $S = 10$ and use both PN and L-BFGS variants with RPCD in the first stage and Algorithm 3 with $\rho = 0.5, c = 10^{-6}$ in the second. For PCG, we use the diagonals of $H_t$ as the preconditioner. We use a heuristic to let PCG start with an iteration bound $T_0 = 5$, double it whenever $\alpha_t = 1$ until reaching the dimension of $M_t$, and reset it to 5 when $\alpha_t < 1$. For the value of $S$, although tuning it properly might lead to even better performance, we observe that the current setting already suffices to demonstrate the improved performance of the proposed algorithm.

Results in Fig. 1 show the superlinear convergence in running time of ISQA$^+$, while LHAC and NewGLMNET only exhibit linear convergence. We observe that for data with $n \gg d$, including a9a, ijcnn1, covtype.scale, and epsilon, L-BFGS approaches are faster because $H_t p$ can be evaluated cheaply (LHAC failed on covtype.scale due to implementation issues), and PN approaches are faster otherwise, so no algorithm is always superior. Nonetheless, for the same type of $H_t$, ISQA$^+$-LBFGS and ISQA$^+$-Newton respectively improve state-of-the-art algorithms LHAC and NewGLMNET greatly because of the fast local convergence, especially when the base method converges slowly.

6 Conclusions

In this paper, we showed that for regularized problems with a partly smooth regularizer, inexact successive quadratic approximation is essentially able to identify the active manifold because a mild sufficient condition is satisfied by most of commonly-used subproblem solvers. An efficient algorithm ISQA$^+$ utilizing this property is proposed to attain superlinear convergence on a wide class of degenerate problems in running time, greatly improving upon state of the art for regularized problems that only exhibit superlinear convergence in outer iterations. Numerical evidence illustrated that ISQA$^+$ significantly improves the running time of state of the art for regularized optimization.

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Table 2: Outer iterations and time (seconds) for different subproblem solvers to identify the active manifold. For each ISQA$^+$ variant, the fastest running time among all subproblem solvers is boldfaced.
Fig. 1: Comparison of Different Algorithms. For covtype.scale, LHAC is not shown because it failed to converge.

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A Proof of Theorem 4

Proof For any \( t \geq 0 \), [20] Lemma 5 gives

\[
Q^t \leq \min_{\lambda \in [0,1]} \lambda (F^* - F(x^t)) + \frac{\lambda^2}{2} \|H_t\| \text{dist}(x^t, \Omega)^2.
\]

Combining (70) and (38) then leads to

\[
Q^t \leq -\lambda \delta_t + \frac{\lambda^2 \|H_t\| \delta^2}{2\zeta^2}, \quad \forall \lambda \in [0,1].
\]

Through (6), (8), and (22), we can further deduce from (71) that

\[
\delta_{t+1} - \delta_t \leq \alpha \gamma (1 - \eta) \left( \lambda - \frac{\lambda^2 \|H_t\|}{2\zeta^2} \right), \quad \forall \lambda \in [0,1].
\]

When \( \theta = 1/2 \), clearly (71) (72) imply

\[
\delta_{t+1} \leq \min_{\lambda \in [0,1]} \delta_t \left( 1 - \alpha \gamma (1 - \eta) \left( \lambda - \frac{\lambda^2 \|H_t\|}{2\zeta^2} \right) \right).
\]

Setting \( \lambda = \min \{1, \zeta^2 \|H_t\|^{-1}\} \) in (73) then proves (42).

For \( \theta \neq 1/2 \), we will use (72) with

\[
\lambda = \min \{1, \zeta^2 \delta_t^{-1} \delta_{t+1}^{-1} M^{-1}\}. \quad (74)
\]

The case in which \( \lambda = 1 \) in (74) suggests that \( M_t^{2\theta - 1} \delta_t^{-2} \leq 1 \) and corresponds to the following linear decrease.

\[
\delta_{t+1} - \delta_t \leq \alpha \gamma (1 - \eta) \delta_t \left( 1 + \frac{M_t^{2\theta - 1}}{2\zeta^2} \right) \leq -\delta_t \frac{\alpha \gamma (1 - \eta)}{2},
\]

which is exactly (41). When \( \theta > 1/2 \), \( 2\theta - 1 > 0 \) so \( M_t^{2\theta - 1} \delta_t^{-2} \leq 1 \) if and only if (39) fails, and when \( \theta < 1/2 \), it is the opposite case because \( 2\theta - 1 < 0 \).

When \( \lambda = \zeta^2 \delta_t^{-1} \delta_{t+1}^{-1} M^{-1} \) in (74) (72) leads to

\[
\delta_{t+1} \leq \delta_t - \frac{\alpha \gamma (1 - \eta) \zeta^2}{2M} \delta_t^{-2}. \quad (75)
\]

For \( \theta > 1/2 \), this happens when (39) holds. The rate (40) is then obtained from (75) by \( -\delta_t^{-2\theta} \leq -\xi^{1-2\theta} \) for all \( t \), implied by \( x^0 \in \text{Lev} (\xi) \) and the monotonicity of \( \{\delta_t\} \). For \( \theta < 1/2 \), since \( 1 - 2\theta > 0 \), \( \lambda < 1 \) if \( \delta_t \) is small enough to fail (39). We have \( 2 - 2\theta > 1 \), so [35] and Lemma 6 in [35] Chapter 2.2 imply (43).

B Proof of Theorem 3

From Theorem 4, we have gotten a faster convergence speed of the objective. We then need to relate it to the decrease of \( Q_t \) and \( Q_t^* \) before proving Theorem 3.

Lemma 6 Consider Algorithm 1. If \( \Omega \neq \emptyset \), \( f \) is \( L \)-smooth for \( L > 0 \), \( \Psi \in \Gamma_0 \), and there are \( m > 0 \) and \( \eta \in [0,1) \) such that \( H_t \succeq m \) and \( \eta^t \) satisfies (8) with (22) for all \( t \), then

\[
Q_t - Q_t^* \leq \frac{\eta \delta_t}{\gamma \alpha (1 - \eta)}, \quad -Q_t^* \leq \frac{\delta_t}{\gamma \alpha (1 - \eta)}, \quad \forall t \geq 0,
\]

where \( \alpha > 0 \) is the constant in Theorem 4.
we have that
\[ -\dot{Q}_t \geq -(1 - \eta)Q^*_t. \]  
(77)

By [77] [6], \( \delta_t \geq 0 \), and \( \alpha_t \geq \hat{\alpha} \) from Theorem 4, the second inequality of (76) is proven by
\[ \delta_t \geq \delta_t - 0 \geq -\gamma \alpha_t (1 - \eta) Q^*_t \geq -\gamma \hat{\alpha} (1 - \eta) Q^*_t. \]  
(78)

For the first inequality, the case of \( \eta = 0 \) holds trivially. If \( \eta \in (0, 1) \), [77] implies
\[ -\eta \dot{Q}_t \geq (1 - \eta) \left( \dot{Q}_t - Q^*_t \right) \Rightarrow -\dot{Q}_t \geq \frac{1 - \eta}{\eta} \left( \dot{Q}_t - Q^*_t \right). \]  
(79)

Following the same logic for obtaining [78], we get from (6) and (79) that
\[ \eta \delta_t \geq \gamma \alpha_t (1 - \eta) \left( \dot{Q}_t - Q^*_t \right) \geq \gamma \hat{\alpha} (1 - \eta) \left( \dot{Q}_t - Q^*_t \right). \]

\[ \square \]

Proof (Proof of Theorem 3)

We first note that (44) can be seen as solving (2) exactly with \( \mathcal{H} = \hat{\mathcal{H}} \), and it is widely-known that for \( \hat{\mathcal{H}} \geq L \), (6) is always satisfied with \( \alpha_t = 1 \). Therefore, under [7] we have
\[ \hat{M} \geq H_{k_t} \geq \hat{m}, \quad \forall t \geq 0, \]  
(83)

and [44] with (22) holds for all \( k_t \). For the MO steps, we note that it does not increase the objective value, so for any computation involving the objective change, we can safely skip them. That \( k_t \leq 2t \) for any \( t \) is clear, since we do not conduct two consecutive MO steps.
1. The part of $G_t = 0$ if and only if $0 \in \partial F(x^t)$ is well-known. See, for example, [20, Lemma 7]. Thus, the claim of stationarity of the limit points follow directly from the claim that $G_{k_t} \to 0$.

Now let us prove that $G_{k_t} \to 0$ and the upper bound for $\min_{0 \leq i \leq t} \|G_{k_i}\|^2$. For the $k_t$th iteration, we get from (6) and that (22) indicates that $Q_{k_i} \leq 0$, that

$$-\gamma Q_{k_i} \leq F(x^k) - F(x^{k+1}),$$

(84)

By applying (22) and (6) to (84), we further get

$$-\gamma (1 - \eta)Q_{k_i} \leq F(x^k) - F(x^{k+1}).$$

(85)

By summing (84) over $i = 0, \ldots, t$, we get

$$-\gamma (1 - \eta) \sum_{i=0}^{t} Q_{k_i} \leq \sum_{i=0}^{t} F(x^k) - F(x^{k+1}) \leq F(x^0) - F^*, \quad (23)$$

where the penultimate inequality is from that $F(x^t)$ is nonincreasing. Following [20, Lemma 7] and [11, Lemma 3], we then have from (83) that

$$\frac{F(x^0) - F^*}{\gamma (1 - \eta)} \geq \sum_{i=0}^{t} -Q_{k_i} \geq \sum_{i=0}^{t} \frac{2\bar{m}}{\bar{M}^2 (1 + \bar{m}^{-1} + \sqrt{1 - 2\bar{M}^{-1} + \bar{m}^{-2}})} \|G_{k_i}\|^2.$$  

(86)

Since the above is true for any $t \geq 0$, it shows that $\{\|G_{k_t}\|^2\}$ is summable, so $\|G_{k_t}\|^2$ converges to 0, which implies that $G_{k_t} \to 0$. The rate is then obtained from (86) by noting that

$$\sum_{i=0}^{t} \|G_{k_i}\|^2 \geq (t + 1) \min_{0 \leq t \leq z} \|G_{k_t}\|^2.$$  

2. This part follows directly from the proof of [20, Theorem 1] by replacing $t$ by $t_k$, noting that MO steps do not increase the objective value so $\delta_{k_{t+1}} \leq \delta_{k_{t+1}}$ for all $t$, and using the bound

$$\langle x^t - P_{Q_t}(x^t), H_t(x^t - P_{Q_t}(x^t)) \rangle \leq \bar{M} \|x^t - P_{Q_t}(x^t)\|^2 \leq M R_0^2, \quad \forall t \geq 0$$

from (47)

3. This part follows from the proof of Theorem 4 again by using (83) and that $\delta_{k_{t+1}} \leq \delta_{k_{t+1}}$ for all $t$. 
