Learning a local Hamiltonian from local measurements
Eyal Bairey,1 Itai Arad,1 and Netanel H. Lindner1
1Physics Department, Technion, 3200003, Haifa, Israel

Recovering an unknown Hamiltonian from measurements is an increasingly important task for certification of noisy quantum devices and simulators. Recent works have succeeded in recovering the Hamiltonian of an isolated quantum system with local interactions from long-ranged correlators of a single eigenstate. Here, we show that such Hamiltonians can be recovered from local observables alone, using computational and measurement resources scaling linearly with the system size. In fact, to recover the Hamiltonian acting on each finite spatial domain, only observables within that domain are required. The observables can be measured in a Gibbs state as well as a single eigenstate; furthermore, they can be measured in a state evolved by the Hamiltonian for a long time, allowing to recover a large family of time-dependent Hamiltonians. We derive an estimate for the statistical recovery error due to approximation of expectation values using a finite number of samples, which agrees well with numerical simulations.

Introduction. Contemporary condensed matter physics has witnessed great advancements in tools developed to obtain the state of a system given its Hamiltonian. As quantum devices are being rapidly developed, the converse task of recovering the Hamiltonian of a many-body system from measured observables is becoming increasingly important. In particular, it is a necessary step for certifying quantum simulators and devices containing many qubits. As these expand beyond the power of classical devices [1], there is a growing need to certify them using only a polynomial amount of classical computational resources as well as quantum measurements.

Various methods have been suggested for recovering a Hamiltonian based on its dynamics [2–6] or Gibbs state [7, 8]. Improvements to the system-size scaling of the efficiency of the recovery procedure can be achieved using a trusted quantum simulator [9–13], manipulations of the investigated system [14], or accurate measurements of short-time dynamics [15, 16].

Here, we suggest a framework for recovering a generic local Hamiltonian using only polynomial time and measurements. The measurements can be performed on multiple copies of a single eigenstate, a Gibbs state \( \rho = \frac{1}{Z}e^{-\beta H} \), or arbitrary initial states time-evolved by the Hamiltonian. Inspired by the recently introduced method for recovering a local Hamiltonian from a single eigenstate [17–19], our framework offers three main contributions. First, we generalize to mixed states, treating any state which commutes with the Hamiltonian at the same footing as an eigenstate. Second, we can recover a time-dependent Hamiltonian, assuming that the functional form of its time-dependence is known. Finally, we can infer the Hamiltonian of a local patch \( L \) based only on local measurements inside \( L \). This implies that the Hamiltonian of a large system can be obtained with a number of measurements and computation time linear in system size.

Problem setting. We wish to recover the Hamiltonian acting on a region \( L \) by measuring observables only in \( L \).

We would first like to make these notions precise.

We consider a Hamiltonian \( H \) on a finite lattice \( \Lambda \) in \( d \) dimensions:

\[
H = \sum_i h_i. \tag{1}
\]

We assume that \( H \) is \( k \)-local, such that each \( h_i \) acts non-trivially on no more than \( k \) spatially contiguous sites (i.e., contained within a ball of diameter \( k \)); we extend our analysis to long-ranged Hamiltonians in the discussion). We focus on a specific subset of sites \( L \subseteq \Lambda \) that defines a region whose Hamiltonian we wish to recover. We define its interior \( L_0 \subseteq L \) as the sites that interact only within \( L \), excluding the boundary sites that might be connected by \( H \) to the rest of the lattice (Fig. 1). We denote by \( H_L \) the subset of \( h_i \) terms in \( H \) that act non-trivially on \( L_0 \).

We call any state \( \rho \) that is stationary under \( H \) a steady
state (taking $\hbar = 1$):
\[ i\partial_t \rho = [H, \rho] = 0, \] (2)

In particular, $\rho$ can be any eigenstate as well as a Gibbs state. Moreover, suppose we repeatedly initialize our system in some state $\rho(0)$, then let it evolve for a random time distributed uniformly in $0 \leq t \leq T$, before measuring an operator $A$. The average outcome of these measurements is given by $\text{Tr}(\rho_{\text{avg}} A)$, where $\rho_{\text{avg}} = \frac{1}{T} \int_0^T \rho(t) \, dt$. This time-averaged density matrix approaches a steady state in trace norm, since by (2):

\[ \|\rho_{\text{avg}} - H\] (3)

Our goal is to recover $H_L$ from a steady state of $H$, based only on measurements in $L$.

Algorithm To recover $H_L$, we identify a set of local constraints on $H_L$ obeyed by any steady state $\rho$ of $H$. Since $\rho$ is stationary under $H$, so is the expectation value $\langle A \rangle \overset{\text{def}}{=} \text{Tr}(\rho A)$ of any operator $A$ under $\rho$: $\partial_t (\langle A \rangle) = -\langle [A, H] \rangle = 0$. Each operator $A$ therefore imposes the constraint $\langle i[A, H] \rangle = 0$ on $H$. If $A$ is supported only on $L_0$, this constraint becomes

\[ \langle i[A, H_L] \rangle = 0, \] (4)

since $A$ trivially commutes with $H - H_L$.

For concreteness, we assume each site is a spin $\frac{1}{2}$. We can then expand $H_L$ in a basis of products of Pauli matrices:

\[ H_L = \sum_{m=1}^M c_m S_m, \] (5)

where $c_m \in \mathbb{R}$, and $S_m$ is a product of at most $k$ Pauli operators acting on sites $\ell_1, \ell_2, \ldots, \ell_k$: $S_m \overset{\text{def}}{=} \sigma_{\ell_1}^{\alpha_{\ell_1}} \cdots \sigma_{\ell_k}^{\alpha_{\ell_k}}$. Therefore, $m$ is an index made of at most $k$ pairs $(\ell_i, \alpha_i)$, and $M$ denotes the total number of basis elements that can appear in $H_L$. The constraint (4) then becomes a linear homogeneous constraint on the vector $\vec{c} = (c_1, c_2, \ldots, c_M)$:

\[ \sum_{m=1}^M c_m \langle i[A, S_m] \rangle = 0. \] (6)

Using a set of operators $\{A_n\}_{n=1}^N$, each supported on $L_0$, we obtain a set of $N$ linear constraints:

\[ \forall n : \sum_{m=1}^M c_m \langle i[A_n, S_m] \rangle = 0, \] (7)

which is equivalent to the $N \times M$ real linear equation:

\[ K\vec{c} = 0, \] (8)

By the same observation made in Refs. [17, 18], the number of $S_m$ terms that span $H_L$ is linear in the system’s volume $|L|$. The maximal number of constraints on the other hand scales like the number of linearly independent observables $A_n$ in $L_0$, which is $4|L_0|$. Therefore, for a sufficiently large but constant region $L$ (depending on $k$ but not on $|A|$), we can always have more equations than unknowns, i.e., $N > M$. As argued in Ref. [17], we expect these equations to be generally independent, thereby providing a unique solution $\vec{c}$.

For some intuition, note that any coefficient vector $\vec{c}$ defines a local Hamiltonian and $\|K\vec{c}\|^2 = \sum_n |\langle \partial_t (A_n) \rangle|^2$ measures how fast the set of expectation values $\{\langle A_n \rangle\}$ would change under that local Hamiltonian. The true Hamiltonian, under which $\{\langle A_n \rangle\}$ are stationary, is given by a solution to $K\vec{c} = 0$. When this solution is unique, we recover the true Hamiltonian up to an overall scale.

Given a region $L$ whose Hamiltonian we wish to learn, our method is therefore as follows:

1. Identify a set of terms $\{S_m\}_{m=1}^M$ spanning the space of possible $H_L$’s.

2. Construct a constraint matrix $K_{N \times M}$ by measuring $\{\langle i[A_n, S_m] \rangle\}$ with respect to a set of constraints $\{A_n\}_{n=1}^N$ supported on $L_0$.

3. Estimate $H \propto \sum_{m=1}^M c_m S_m$ with $\vec{c}$ the lowest right-singular vector of $K$.

The lowest right-singular vector of $K$ is the numerical solution to Eq. (8), the vector that minimizes $\|K\vec{c}\|$. Namely, it is the ground-state of the correlation matrix,

\[ \mathcal{M} = K^T K. \] (9)

Finally, we can extend our method to time-dependent Hamiltonians of the form:

\[ \hat{H}(t) = \hat{H}^{(0)} + \hat{V} f(t), \] (10)

where $f(t)$ is a known function. Similarly to Eq. (3), now the time-averaged commutator $\frac{1}{T} \int_0^T [\rho(t), \hat{H}(t)] \, dt$ must decay with time; minimizing it is equivalent to finding the lowest singular vector of an extended constraint matrix $K_{N \times 2M}$ composed of time-averaged as well as time-modulated measurements (see Supplemental Materials):

\[ \forall m \leq M : K_{n,m} = \frac{1}{T} \int_0^T \langle [A_n, S_m] \rangle \, dt \]

\[ K_{n,M+m} = \frac{1}{T} \int_0^T \langle [A_n, S_m] \rangle f(t) \, dt \] (11)

Sample complexity. The complexity of our method depends on the number of observables we need to measure, and to what accuracy we need to measure each of
them. Experimentally, each observable \( \langle i|A_n, S_m \rangle \) can only be measured to finite accuracy due to statistical uncertainty in estimating it using a finite number of samples \( n_s \). We quantify the resulting error in the reconstruction process by the \( l_2 \) distance between the normalized recovered and true coefficient vectors [20],

\[
\Delta = \| \hat{c}_{\text{true}} - \hat{c}_{\text{recovered}} \|_2,
\]  

where \( \hat{c} \overset{\text{def}}{=} \frac{c}{\|c\|} \).

Following Ref. [17], we analyze the reconstruction error using a simple perturbation theory on the correlation matrix \( \mathcal{M} \). We model the error in each entry \( K_{n,m} \) obtained by \( n_s \) samples as an i.i.d Gaussian with zero mean and standard deviation \( \epsilon \approx n_s^{-\frac{1}{2}} \). To lowest order in \( \epsilon \), we estimate the expected error:

\[
\mathbb{E}(\Delta) \approx \epsilon \sqrt{\sum_{i>0} \frac{1}{\lambda_i} \Delta_{\text{ext}}^i},
\]

where \( \lambda_i \) are the eigenvalues of \( \mathcal{M} \) (see Supplemental Materials).

To open a gap in \( \mathcal{M} \) between \( \lambda_0 \) and \( \lambda_1 \) and recover a unique Hamiltonian, at least as many constraints \( N \) as unknowns \( M \) are required. This means measuring \( O(|L|) \) operators \( [A_n, S_m] \), since each constraint \( A_n \) commutes with all but a constant number of candidate Hamiltonian terms \( S_m \). Moreover, the Hamiltonian can be reconstructed in linear time in \( |L| \) and a linear number of measurements by breaking down \( L \) into smaller sub-regions and reconstructing the Hamiltonian on each of them separately. For translationally invariant Hamiltonians a single patch is sufficient, with only a constant number of operators to be measured.

**Numerical simulations.** To demonstrate our method, we simulated random one-dimensional spin chains. We considered Hamiltonians consisting of all possible 2-local terms, acting on single spins and nearest neighbors:

\[
H = \sum_{i=1}^{|\Lambda|} \sum_{\alpha=1}^3 c_{i\alpha} \sigma_i^\alpha + \sum_{i=1}^{|\Lambda|-1} \sum_{\alpha=1}^3 \sum_{\beta=1}^3 c_{i\alpha \beta} \sigma_i^\alpha \sigma_{i+1}^\beta.
\]  

In each simulation, we generated a random 2-local Hamiltonian \( H \) (Eq. (14)) on \( |\Lambda| = 12 \) sites by sampling the vector of all coefficients \( \vec{c} \) from a Gaussian distribution with zero mean and unit standard deviation, setting the energy scale for what follows. We numerically calculated the ground state of \( H \), and then recovered \( H \) from its ground-state in steps. In each step we added one row to the constraint matrix \( K \) by choosing a constraint operator \( A_n \) supported on the 6 middle sites \( L_0 \) and estimating \( \{ \langle i|A_n, S_m \rangle \}^M_{m=1} \). To measure the robustness of the reconstruction, we added to the constraint matrix \( K \) a noise matrix of i.i.d Gaussian entries with zero mean and standard deviation \( \epsilon = 10^{-12} \).

As expected, once sufficiently many constraints had been measured, our procedure recovered the Hamiltonian to high accuracy (Fig. 2). As soon as \( N = M - 1 \), the correlation matrix \( \mathcal{M} \) opened a gap, allowing to recover the coefficient vector \( \vec{c} \) given by the ground-state of \( \mathcal{M} \). As more constraints were added, the gap gradually grew. The reconstruction error decreased correspondingly, showing excellent agreement with our estimate (13).

We ran additional simulations on random \( XY \) chains to reach larger system sizes (\(|\Lambda| = 100\)). The gap of the correlation matrix seemed insensitive to the size of the sub-system for the range we examined \( 7 \leq |L| \leq 13 \) (Fig. S1).

**Reconstruction from Gibbs states.** Next, we reconstructed \( H_L \) for random spin chains from measurements of their Gibbs states. We sampled 200 random Hamiltonians (14) on \(|\Lambda| = 12 \) sites and generated Gibbs states \( \frac{1}{\beta} e^{-\beta H} \) for varying \( \beta \in [0.01, 1] \). We then measured a fixed number of observables in the \(|L| = 8 \) middle spins, corresponding to all the 4-local constraints \( A_n \) supported on the \(|L_0| = 6 \) middle spins, adding a small noise (\( \epsilon = 10^{-12} \)) to each measurement.

Our results show that the reconstruction error increased with temperature (Fig. 3). As the system approaches a fully mixed state, the commutator \([H, \rho] \) in
Eq. (2) approaches zero for every $H$, which implies that many different $H$ are becoming compatible with $\rho$. Correspondingly, the elements of the constraint matrix $K$ shrink, and so does its gap. On the other hand, at moderately small temperatures, the reconstruction quality was similar to that of ground-states.

Reconstruction from dynamics. To demonstrate Hamiltonian recovery from the dynamics of an initial state we simulated a quench protocol. We generated two random Hamiltonians $H^{(0)}, H^{(1)}$ on $|\Lambda| = 12$ sites from the ensemble (14). We initialized our system in the ground-state of $H^{(0)} + H^{(1)}$, and evolved it by $H^{(0)}$ alone. This way, the energy of the initial state with respect to the final Hamiltonian was not too high. We then attempted at different times $T$ to recover $H_L^{(0)}$ on the 8 middle spins using all the 4-local constraints $A_n$ supported on the 6 middle spins $L_0$. We did this by constructing a constraint matrix $K_T$ from time-averaged values of $\langle [A_n, S_m] \rangle$, sampled at equally spaced intervals $dt = 0.05$ up to time $T$.

After a transient period, the first excited eigenvalue of the correlation matrix $\lambda_1$ saturated (Fig. 4). The lowest eigenvalue $\lambda_0$ continued to decay, opening a gap which widened with time. This decay fits to the power law $t^{-2}$, reflecting the expected decay rate of the commutator with the true Hamiltonian from Eq. (3). Here the finite value of $\lambda_0$ plays the role of noise, leading to reconstruction error. As $\lambda_0$ decreased with time the Hamiltonian was reconstructed to better and better accuracy.

Recovery of time-dependent Hamiltonians. We repeated the quench experiments with a final Hamiltonian which depends on time, focusing on a periodic drive with a single frequency: $f(t) = \cos \omega t$. We initialized our system in the ground state of $H^{(0)} + H^{(1)}$ and evolved it in time with $H(t) = H^{(0)} + JV \cos \omega t$. $\lambda_1$ now decreases with time due to heating, leading to a larger reconstruction error compared to the time-independent case (light blue for $H^{(0)}$, green for $V$). Results are averaged over 50 randomizations.

FIG. 3. Quality of Hamiltonian reconstruction from Gibbs states $\rho = \frac{1}{Z} e^{-\beta H}$ as a function of temperature $T = \beta^{-1}$. We reconstructed $H_L$ on the $L = 8$ middle spins of random spin chains (14) of length $|\Lambda| = 12$ (see Fig. 1). The gap of the correlation matrix $M$ decreases with temperature (orange). Correspondingly, the reconstruction error due to a small measurement uncertainty ($\epsilon = 10^{-12}$) increases (light blue) according to the estimate (13) obtained from the spectrum of $M$ (dashed green). Results are averaged over 200 randomizations.

FIG. 4. Hamiltonian reconstruction from dynamics as a function of time. Left: reconstruction of a time-independent Hamiltonian. We initialize a random chain of $|\Lambda| = 12$ spins in the ground state of $H^{(0)} + H^{(1)}$, and evolve it in time with $H^{(0)}$ alone, where $H^{(0)}, H^{(1)}$ are sampled randomly from (14). We then attempt to recover $H_L$ on the 8 middle spins $L$. While the first excited eigenvalue of the correlation matrix $\lambda_1$ saturates (orange), its lowest eigenvalue $\lambda_0$ decreases with time (red), leading to a smaller reconstruction error $\Delta$ (light blue). Right: reconstruction of a time-oscillating Hamiltonian. We evolve the initial state with $H(t) = H^{(0)} + JV \cos \omega t$. $\lambda_1$ now decreases with time due to heating, leading to a larger reconstruction error compared to the time-independent case (light blue for $H^{(0)}$, green for $V$). Results are averaged over 50 randomizations.
quires a number of measurements as well as computational resources scaling linearly with system size.

Most of our formalism applies equally well to long-ranged Hamiltonians, when interactions can involve any arbitrary set of \(k\) spins. While our algorithm must then be applied to the whole system \(\Lambda\) at once rather than locally, the number of possible Hamiltonian terms still scales polynomially with system size, as \(\lambda^k\).

It is interesting to note that demanding stationarity of all the observables \(A_n\) supported on \(L_0\) is equivalent to the operator identity \(\text{Tr}_{\partial L}[\rho_L, H_L] = 0\), where \(\rho_L\) is the reduced density matrix on \(L\) and \(\text{Tr}_{\partial L}\) is a partial trace on the boundary spins included in \(L\) but not in its interior \(\partial L \overset{\text{def}}{=} L - L_0\). This identity was used in Ref. [21] to recover \(\rho_L\) given \(H_L\), whereas here we do exactly the opposite. If we enforce stationarity of all possible operators on the full system \(L = \Lambda\), our correlation matrix takes the appealing form \(\mathcal{M}_N^L = \text{Tr}(\rho L [\rho, S_i]^\dagger [\rho, S_j])\) coinciding with the correlation matrix defined in Ref. [17] (up to a scalar; see Supplemental Materials).

The correlation matrix formalism [17–19] can also be used to obtain the entanglement Hamiltonian of a given region, namely \(H_E\) such that \(\rho_L = \frac{1}{Z} e^{-H_E}\) [22]. Since we treat mixed states on the same footing as pure states, our framework applies very naturally to this task. This can be done by including boundary terms in the reconstruction and constraints, thereby searching for a local Hamiltonian which commutes with the reduced density matrix.

In contrast to the quantum case, the sample complexity of learning a classicalising model from its Gibbs state scales exponentially with the number of neighbors each spin can have [23]. For a quantum Hamiltonian however, our estimate (13) suggests that a polynomial number of measurements of each observable should suffice as long as \(\lambda_1\) does not vanish exponentially with system size. This contrast between the classical and quantum case can be formalized using a rigorous upper bound on the sample complexity of the quantum Hamiltonian learning problem [24].

We thank Miklos Santha and Anupam Prakash for illuminating discussions, and Renan Gross for critical comments on the manuscript. E. B. and N. L. acknowledge financial support from the European Research Council (ERC) under the European Union Horizon 2020 Research and Innovation Programme (Grant Agreement No. 639172). N. L. acknowledges support from the People Programme (Marie Curie Actions) of the European Union’s Seventh Framework Programme (No. FP7/2007–2013) under REA Grant Agreement No. 631096 and from the Israeli Center of Research Excellence (I-CORE) “Circle of Light.”.

[1] J. Preskill, (2018), arXiv:1801.00862.
[2] D. Burgarth, K. Maruyama, and F. Nori, Phys. Rev. A 79 (2009), 10.1103/PhysRevA.79.020305, arXiv:0810.2866.
[3] C. Di Franco, M. Paternostro, and M. S. Kim, Phys. Rev. Lett. 102 (2009), 10.1103/PhysRevLett.102.187203, arXiv:0812.3510.
[4] J. Zhang and M. Sarovar, Phys. Rev. Lett. 113, 80401 (2014).
[5] L. E. De Clercq, R. Oswald, C. Flühmann, B. Keitch, D. Kienzler, H. Y. Lo, M. Marinelli, D. Nadlinger, V. Negnevitsky, and J. F. Home, Nat. Comm. 7 (2016), 10.1038/ncomms12118, arXiv:1011.1669v3.
[6] Y. Wang, D. Dong, B. Qi, J. Zhang, I. R. Petersen, and H. Yonezawa, IEEE Transactions on Automatic Control 63, 1388 (2018), arXiv:1610.08841.
[7] K. Rudinger and R. Joynt, Phys. Rev. A 92 (2015), 10.1103/PhysRevA.92.052322, arXiv:1410.3029v1.
[8] H. J. Kappen, (2018), arXiv:1803.11278.
[9] C. E. Granade, C. Ferrie, N. Wiebe, and D. G. Cory, New Journal of Physics 14 (2012), 10.1088/1367-2630/14/10/103013, arXiv:1207.1655v1.
[10] N. Wiebe, C. Granade, C. Ferrie, and D. G. Cory, Phys. Rev. Lett. 112 (2014), 10.1103/PhysRevLett.112.190501, arXiv:1309.0876.
[11] N. Wiebe, C. Granade, C. Ferrie, and D. Cory, Phys. Rev. A 89 (2014), 10.1103/PhysRevA.89.042314, arXiv:1311.5269.
[12] N. Wiebe, C. Granade, and D. G. Cory, New Journal of Physics 17 (2015), 10.1088/1367-2630/17/2/022005, arXiv:1409.1524.
[13] J. Wang, S. Paessens, R. Santagati, S. Knauer, A. A. Gentile, N. Wiebe, M. Petruzella, J. L. O’Brien, J. G. Rarity, A. Laing, and M. G. Thompson, Nat. Phys. 13, 551 (2017), arXiv:1703.05402.
[14] S. T. Wang, D. L. Deng, and L. M. Duan, New Journal of Physics 17 (2015), 10.1088/1367-2630/17/9/093017, arXiv:1505.00665.
[15] A. Shabani, M. Mohseni, S. Lloyd, R. L. Kosut, and H. Rabitz, Phys. Rev. A 84 (2011), 10.1103/PhysRevA.84.012107, arXiv:1002.1330.
[16] M. P. Da Silva, O. Landon-Cardinal, and D. Poulin, Phys. Rev. Lett. 107 (2011), 10.1103/PhysRevLett.107.210404, arXiv:1104.3835.
[17] X.-L. Qi and D. Ranard, (2017), arXiv:1712.01850.
[18] E. Chertkov and B. K. Clark, (2018), arXiv:1802.01590.
[19] M. Greiter, V. Schnells, and R. Thomale, (2018), arXiv:1802.07827.
[20] This metric can also be written as \(\sqrt{2 - 2 \cos(\theta)}\), where \(\theta\) is the angle between the two vectors [17].
[21] A. Anshu, I. Arad, and A. Jain, Phys. Rev. B 94 (2016), 10.1103/PhysRevB.94.195143, arXiv:1603.06049.
[22] W. Zhu, Z. Huang, and Y.-c. He, (2018), arXiv:1806.08060.
[23] N. P. Santhanam and M. J. Wainwright, IEEE Transactions on Information Theory 58, 4117 (2012), arXiv:0905.2639.
[24] Future work.
[25] I. Peschel, J. Phys. A: Math. Theor. 36, L205 (2003), arXiv:0212631 [cond-mat].
[26] S. A. Cheong and C. L. Henley, Phys. Rev. B
Supplemental Materials

I: Recovering time-dependent Hamiltonians

Suppose we wish to recover a time-dependent Hamiltonian of the form:

\[ \hat{H}(t) = \hat{H}^{(0)} + \hat{V} f(t), \]

where \( f(t) \) is a known function and \( \hat{H}, \hat{V} \) are the operators we wish to learn. For any operator \( A \), Schrodinger’s equation now reads:

\[ i\partial_t (A) = \{A, \hat{H}^{(0)} \} + \{A, f(t)\hat{V} \}. \]

Integrating the above equation and expanding in local operators: \( \hat{H}^{(0)} = \sum c_m^{(0)} h_m, \ \hat{V} = \sum c_m^v h_m \), we obtain:

\[ \left| \sum c_j^{(0)} \frac{1}{T} \int_0^T (\{A, h_m\}) dt + \sum c_j^v \frac{1}{T} \int_0^T (\{A, h_m\}) f(t) dt \right| \leq \frac{2 ||A||}{T}. \]

Minimizing the LHS of (S3) with respect to a set of operators \( \{A_n\}_{n=1}^N \) amounts to finding the lowest right-singular vector of the extended constraint matrix defined in (11):

\[ \forall m \leq M : \ K_{n,m} = \frac{1}{T} \int_0^T (\{A_n, h_m\}) dt \]

\[ K_{n,m+M} = \frac{1}{T} \int_0^T (\{A_n, h_m\}) f(t) dt. \]

II: Error estimation

Experimentally, each element of the constraint matrix \( K_{n,m} = \langle [A_n, S_m] \rangle \) can only be estimated using a finite number of samples \( n_s \). Therefore, the measured empirical constraint matrix \( \hat{K} \) deviates from the true one \( K \) by a noise matrix. We would like to estimate the error in the recovered Hamiltonian due to this noise.

We study the effect of the noise by treating it as a perturbation. We assume that the correct \( \hat{K} \) has a one-dimensional kernel; namely, we were given a state \( \rho \) for which there is only one local Hamiltonian (up to an overall scalar) that satisfies Eq. (8). After many measurements, we can use the central limit theorem to model the noise as a Gaussian matrix:

\[ \hat{K} - K \approx \epsilon \hat{E}, \]

where each entry \( \hat{E}_{nm} \) of \( \hat{E} \) is an independent random variable with zero mean and unit standard deviation. It is scaled by a small parameter \( \epsilon \) which decays as \( n_s^{-1/2} \).

We wish to estimate the distance between the true and recovered Hamiltonians. This distance is given by \( ||\hat{c}_0 - c_0|| \), where \( \hat{c}_0 \) and \( c_0 \) are the ground-states of the clean \( K^T K \) and its noisy estimate \( (K + \epsilon \hat{E})^T (K + \epsilon \hat{E}) \).

We treat \( K^T K \) as an unperturbed Hamiltonian, and \( \epsilon (\hat{E}^T K + K^T \hat{E}) \) as a perturbation to first order in \( \epsilon \). We obtain:

\[ ||c_0 - \hat{c}_0|| = \epsilon \sum_{i>0} |\langle c_i | \hat{E} \rangle | c_0 \rangle \langle c_0 | c_i \rangle + O(\epsilon^2) \]

which implies that

\[ K^T K |c_0\rangle = 0 \Rightarrow ||K|c_0||^2 = \langle c_0 | K^T K |c_0\rangle = 0. \]

Similarly, \( ||K|c_i||^2 = \lambda_i \), so we can write:

\[ K |c_i\rangle = \sqrt{\lambda_i} |\hat{c}_i\rangle \]

For some unit vector \( |\hat{c}_i\rangle \). Using this we obtain:

\[ ||c_0 - \hat{c}_0|| = \epsilon \sum_{i>0} \frac{1}{\sqrt{\lambda_i}} |\langle c_i | \hat{E} \rangle | c_0 \rangle + O(\epsilon^2), \]

and therefore,

\[ ||c_0 - \hat{c}_0|| = \epsilon \sum_{i>0} \frac{1}{\lambda_i} ||(\hat{c}_i | \hat{E} | c_0 \rangle| \rangle^2 + O(\epsilon^2). \]

We can now average over the noise \( \hat{E} \) by invoking Jensen’s inequality, together with the concavity of the square root function:

\[ \mathbb{E} ||c_0 - \hat{c}_0|| = \epsilon \mathbb{E} \sqrt{\sum_{i>0} \frac{1}{\lambda_i} ||(\hat{c}_i | \hat{E} | c_0 \rangle| \rangle^2 + O(\epsilon^2) \leq \epsilon \sum_{i>0} \frac{1}{\lambda_i} \mathbb{E} (|\hat{c}_i | \hat{E} | c_0 \rangle| \rangle^2 + O(\epsilon^2) \leq \epsilon \sum_{i>0} \frac{1}{\lambda_i} + O(\epsilon^2). \]
Reconstruction gap

Λ = 100 sites. We considered sub-regions consisting of structured reduced density matrices for ground-states on long XY chains, for which the Hamiltonian is given by:

\[
H_{XY} = \frac{1}{2} \sum_{i=1}^{\Lambda} \left[ 2g_{i} \sigma_{i}^{x} + (1 + \gamma_{i}) \sigma_{i}^{z} \sigma_{i+1}^{z} + (1 - \gamma_{i}) \sigma_{i}^{y} \sigma_{i+1}^{y} \right].
\]

Using the methods described in [25–28], we constructed reduced density matrices for ground-states on Λ = 100 sites. We considered sub-regions consisting of |L| = 7, 9, 11, 13 sites, and calculated the gaps of the correlation matrices \( \mathcal{M} \) constructed with all the 4-local constraints \( A_{n} \) supported on the corresponding interior regions \( L_{0} \). The gap of the correlation matrix seems to be insensitive to sub-system size for the sizes we examined (Fig. S1, inset).

IV: Recovery for different driving parameters

Recovery of a time-dependent \( H(t) \) is only possible when heating is sufficiently slow. As time progresses, the time-averaged commutator \( [\rho(t), H(t)] \) decays, as quantified by \( \lambda_{0} \), the lowest right-singular value of the extended constraint matrix \( K_{N \times 2M} \). However, since energy is not conserved for such a system, \( \rho_{\text{avg}} \) could heat up to an infinite-temperature fully-mixed state, which trivially commutes with any Hamiltonian. One measure for this process is the next singular value \( \lambda_{1} \), quantifying how well any competing Hamiltonian would commute with \( \rho \) on average. Recovery is therefore possible whenever \( \lambda_{0} \) decays faster than \( \lambda_{1} \), such that the solution does not mix the true Hamiltonian much with any competitor.

Indeed, we find that stronger or slightly faster driving leads to a more rapid decay of the reconstruction gap (Fig. S2). This agrees with our expectation that a larger driving amplitude should lead to faster energy absorption from the drive; we expect the same from a slightly higher driving frequency within the low-frequency regime we study.

V: Relation to previously-defined correlation matrix

If we wish to recover the Hamiltonian on the full system \( \Lambda \) by enforcing stationarity of all possible constraints \( A_{n} \), the correlation matrix \( \mathcal{M} \) takes the following form, as defined in [17] (up to a multiplicative scalar):

\[
M_{ij} = \sum_{n} K_{ni} K_{nj} = \sum_{n} \langle i | h_{i}, A_{n} \rangle \langle i | h_{j}, A_{n} \rangle
\]

\[
= \sum_{n} \text{Tr} (\rho [h_{i}, A_{n}]) \text{Tr} (\rho [h_{j}, A_{n}]) = \sum_{n} \text{Tr} (A_{n} [h_{i}, \rho]) \text{Tr} (A_{n} [h_{j}, \rho])
\]

\[
= 2^{\Lambda} \text{Tr} \left( [h_{i}, \rho]^{\dagger} [h_{j}, \rho] \right)
\]

Where we used the identity \( \text{Tr} (A [B, C]) = \text{Tr} (C [A, B]) \) which follows from the cyclic property of the trace, as well as \( [A, B] = -[B, A] \). Finally, the last equality follows from the generalized Parseval identity for.
FIG. S2. Recovery of a time-dependent Hamiltonian as a function of driving amplitude and frequency. Left: right panel of 4 in main text (driving frequency $\omega = 0.05$, amplitude $J = 0.5$). Middle: double frequency ($\omega = 0.1$, $J = 0.5$). Right: double amplitude ($\omega = 0.05$, $J = 1$). The power $\alpha$ of the long-time decay of the first excited right-singular value $\lambda_1 \sim t^{-\alpha}$ is larger in the central and right panels compared to the left panel ($-1.20 \pm 0.01$ and $-1.17 \pm 0.01$ compared to $-0.946 \pm 0.003$), indicating more noisy recovery. Indeed, we expect faster heating to arise from a stronger driving amplitude, as well as from a higher driving frequency within this low-frequency regime.

\[
\langle v, w \rangle = \sum_i \langle v, e_i \rangle \langle w, e_i \rangle ,
\]

(S15) taking the operators $A_n$ to be an orthogonal basis for the operators on $\Lambda$. We normalize $A_n$ to unity in operator norm (largest eigenvalue) rather than Hilbert-Schmidt norm, since measurements in the lab yield ±1 outcomes; this is the origin of the $2^\Lambda$ factor.