A POSITIVE SOLUTION OF ASYMPTOTICALLY PERIODIC CHOQUARD EQUATIONS WITH LOCALLY DEFINED NONLINEARITIES

Gui-Dong Li, Yong-Yong Li, Xiao-Qi Liu and Chun-Lei Tang

School of Mathematics and Statistics
Southwest University, Chongqing 400715, China

(Communicated by Luc Nguyen)

Abstract. In this paper, we investigate the following Choquard equation

$$-\Delta u + V(x)u = \lambda (I_\alpha \ast F(u))f(u) \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3, \lambda > 0, \alpha \in (0, N)$, $V$ is an asymptotically periodic potential, $I_\alpha$ is the Riesz potential, the nonlinearity term $F(s) = \int_0^s f(t)dt$ and $f$ is only locally defined in a neighborhood of $u = 0$ and satisfies the suitable conditions. By using the Nehari manifold and the Moser iteration, we prove the existence of positive solutions for the equation with sufficiently large $\lambda$.

1. Introduction. Firstly, we introduce the space

$$E := \left\{ u \in L^2(\mathbb{R}^N) : |\nabla u| \in L^2(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} V(x)u^2dx < +\infty \right\}\,$$

equipped with the inner product

$$\langle u, v \rangle = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x)uv)dx,$$

where $V$ satisfies

$$(V_1) \ 0 \leq V(x) \leq V_\infty(x) \in L^\infty(\mathbb{R}^N) \text{ and } V - V_\infty \in F_1,$$

where

$$F_1 := \{ h(x) : \text{for any } \varepsilon > 0, \text{meas}\{x \in B_i(y) : |h(x)| \geq \varepsilon\} \rightarrow 0 \text{ as } |y| \rightarrow \infty \},$$

and $V_\infty$ satisfies $\inf_{x \in \mathbb{R}^N} V_\infty(x) > 0$ and is 1-periodic in each component $x_i (1 \leq i \leq N)$.

Then $E$ is a Hilbert space and we denote by $\| \cdot \|$ the associated norm. In this paper, we investigate the existence of positive solutions of asymptotically periodic Choquard equations with locally defined nonlinearities on $E$, that is,

$$\begin{cases}
-\Delta u + V(x)u = \lambda (I_\alpha \ast F(u))f(u), & x \in \mathbb{R}^N, \\
u \in E,
\end{cases}
\quad (P)$$

2000 Mathematics Subject Classification. Primary: 35A15, 35J60; Secondary: 35J50, 35D30.

Key words and phrases. Choquard equation, positive solution, locally defined nonlinearity, Nehari manifold, Moser iteration.

This work is partially supported by National Natural Science Foundation of China (No.11971393) and Graduate Student Scientific Research Innovation Projects in Chongqing (No. CYB19082).

* Corresponding author.
where $N \geq 3$, $\lambda > 0$, $I_\alpha$ is the Riesz potential of order $\alpha \in (0, N)$ defined for every $x \in \mathbb{R}^N \setminus \{0\}$ by

$$I_\alpha = \frac{\Gamma(\frac{N-2}{2})}{2^{\alpha} \pi^{\frac{N}{2}} \Gamma\left(\frac{\alpha}{2}\right)} |x|^{N-\alpha},$$

$F(s) = \int_0^s f(t) dt$, $V$ satisfies $(V_1)$ and $f$ satisfies the following conditions:

$(f_1)$ $f \in C(\mathbb{R}, \mathbb{R})$ and there exists $p \in [2, \frac{N+\alpha}{N-2})$ such that $\lim_{s \to 0} \frac{f(s)^p}{|s|^p} = C > 0$;  

$(f_2)$ there exist $\nu > 1, \delta > 0$ such that the function $s \mapsto \frac{f(s)}{s^\nu}$ is nondecreasing and $f(s) > 0$ on $(0, \delta)$.

In general, the hypothesis $p \in \left[\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2}\right]$ is necessary for the existence of solutions to Eq. $(P)$, see [20, 21, 23]. Since we consider the case where $p \geq 2$, the restriction $\alpha \in ((N-4)^+, N)$ on $\alpha$ imposed is natural. The exponent $\frac{N+\alpha}{N-2}$ (or $\frac{N+\alpha}{N}$) is called the upper (or lower) critical exponent with respect to the Hardy-Littlewood-Sobolev inequality.

Prior to our work, we recall some important results about Choquard equations. Especially, the case of Eq. $(P)$ is as following

$$-\Delta u + u = \left(\frac{1}{|x|} * |u|^2\right) u \quad \text{in } \mathbb{R}^3,$$

which was used to describe the quantum theory of a polaron at rest by Pekar [24] in 1954. Eq. (1) arisen in a certain approximation to Hartree-Fock theory for a one component plasma. Ph. Choquard [12] proposed it for investigation at the Symposium on Coulomb Systems, Lausanne, July, 1976. Knowledge of the solutions of Eq. (1) has a great importance for studying standing wave solutions for the Hartree-Fock equation

$$i\hbar \frac{\partial \Psi}{\partial t} = -\hbar^2 \Delta \Psi + W(x) \Psi - (\Phi(x) * \Psi^2) \Psi, \quad \text{for all } x \in \Omega,$$

(NLS)

where $\hbar > 0$, $W$ and $\Phi$ are real-valued functions and $\Omega$ is a domain in $\mathbb{R}^3$. Eq. (NLS) has many interesting applications in the quantum theory of large systems of non-relativistic bosonic atoms and molecules.

In recent years, Eq. $(P)$ has been researched intensively. When Eq. $(P)$ is autonomous, it has been studied by Lieb [12], Ma and Zhao [18], Moroz and Van Schaftingen [20, 21] and so on. When Eq. $(P)$ is non-autonomous, there are many results. Among these papers, Menzala [19] have studied the case where $V$ is spherically symmetric, decreases with $r = |x|$ and vanishes at infinity. In [5, 9], the authors have studied the case where $V$ is a periodic function. The case where $V$ is asymptotic to a positive constant has been considered in [17]. $V$ is a vanishing potential, which has received attention in the paper of Alves et al. [1]. For the case where $V$ is coercive, i.e.,

$(V_2)$ $\lim_{|x| \to +\infty} V(x) = +\infty,$

we would like to cite the paper of Van Schaftingen and Xia [27]. In [29], a more weak condition than coercivity on $V$ has been assumed, more precisely,

$(V_3)$ $V \in C(\mathbb{R}^N)$, $\inf_{x \in \mathbb{R}^N} V(x) > 0$ and there exists a constant $r > 0$ such that, for any $M > 0$, meas $\{x \in \mathbb{R}^N : |x - y| \leq r, V(x) \leq M\} \to 0$ as $|y| \to +\infty$.

A case dealing with Eq. $(P)$ with steep potential well is studied in [2, 26]. The case where $V$ is asymptotically periodic, that is,
This aim is to ensure that the associated energy functional

\[ I(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx - \frac{\lambda}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)dx \]

would be well defined and of class \( C^1 \) from the Hardy-Littlewood-Sobolev inequality and the Sobolev inequality. Classically, it follows from variational methods that critical points of \( I \) are precisely the solutions of Eq. (\( P \)) (see [28]). If we don’t make any assumptions on \( f \) at infinity, we can prove that there exists a nontrivial solution for Eq. (\( P \)). Mathematically this problem is new and interesting.

Here, the assumptions \((f_1)-(f_2)\) we make on the nonlinearity \( f \) only refer solely to its behavior in a neighborhood of \( u = 0 \), and we will show that they suffice for the existence of a positive solution of Eq. (\( P \)) when \( \lambda \) is large enough. Exactly we give our main result.

**Theorem 1.1.** Assume that \( N \geq 3, \alpha \in ((N - 4)^+, N) \), (\( V_1 \)) and \((f_1)-(f_2)\) hold. Then there exists \( \lambda_1 > 0 \) such that Eq. (\( P \)) has a positive solution for \( \lambda \geq \lambda_1 \).

**Remark 1.** For the case where the nonlinear term is only locally defined for \( |u| \) small, we should point out that we refer [6, 7, 8, 11] for references in this direction. Chu and Liu [6] investigated quasi-linear Schrödinger equations in the radial space. Costa and Wang [7] considered Schrödinger equations in bound domain. do Ó et al. [8] considered Schrödinger equations when \( V \) was coercive potential or satisfied that \( V(x)^{-1} \) belongs to \( L^1(\mathbb{R}^N) \). Li and Zhong [11] studied the Kirchhoff equation when the nonlinearity term was sub-linear growth. In these papers, the compactness is obtained obviously, then they can prove certain solutions. However, in our cases we do not have compact embedding, which is the main difficulty in this paper. And the conditions in [6, 7, 8, 11] and ours are mutually non-inclusive and the methods are different.

[30] also studied the asymptotically periodic potential. However, in our cases the nonlinear term is only locally defined for \( |u| \) small and (\( V_1 \)) is a reformatory condition about the asymptotic processes of (\( V_4 \)) (see [15]), which lead to the difference from [30]. These are a little surprising.

**Remark 2.** For instance, let \( f(s) = |s|^{p-2}s + |s|^{q-2}s, s \in \mathbb{R}, \) where \( p \in \left[ 2, \frac{N+\alpha}{N-2} \right] \) and \( p \leq q \). Then it is easy to check that \( f \) satisfies conditions (\( f_1)-(f_2) \). However, when \( \frac{N+\alpha}{N-2} < q \), this functional \( I \) takes the value \(-\infty\) for some \( u \in E \) and in
particular, it is not of class $C^1$. This implies that variational methods fail to our problem directly. Then we take a new technique to deal with our problem.

We now make some comments on the key ingredients of the analysis in this paper. Following the idea of dealing with the elliptic problem in [6, 7, 8, 11], we first extend the nonlinear term $f$ and introduce a modified Choquard equation. Next, we prove that the modified Choquard equation possesses a positive ground state solution $u$ by variational methods. Finally, inspired by the results of [7], we can show an a priori bound of the form

$$|u|_\infty < C\lambda^{-\beta}, \quad \beta > 0.$$ 

The organization of this paper is as follows. In the next section we reserve for setting the framework and establishing some preliminary results. Theorem 1.1 is proved in Section 3.

2. Preliminaries. From now on, we will use the following notations.

- $H^1(\mathbb{R}^N)$ is the usual Sobolev space endowed with the usual norm

  $$\|u\|_H^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) dx.$$ 

- $L^p(\mathbb{R}^N)$ is the usual Lebesgue space endowed with the norm

  $$|u|_p^p = \int_{\mathbb{R}^N} |u|^p dx \text{ and } |u|_\infty = \sup_{x \in \mathbb{R}^N} |u(x)| \text{ for all } p \in [1, +\infty).$$ 

- $u^\pm := \max\{\pm u, 0\}$ and $K := \{u \in E : u^+ \neq 0\}$.
- $\text{meas } \Omega$ denotes the Lebesgue measure of the set $\Omega$.
- $(\cdot, \cdot)$ denotes action of dual.
- $B_r(y) := \{x \in \mathbb{R}^N : |x - y| \leq r\}$ and $B_r := \{x \in \mathbb{R}^N : |x| \leq r\}$.
- $C$ denotes a positive constant and is possibly various in different places.

We work on the space $E$ and recall some facts that the norms $\| \cdot \|$ and $\| \cdot \|_H$ are equivalent and the embedding $E \hookrightarrow L^q(\mathbb{R}^N)$ for any $s \in [2, 2^*)$ is continuous. The proof can be done similarly to that in [15] and details are omitted here. We start by observing that $(f_1) - (f_2)$ imply that the existence of a positive constant $\delta_* \leq \min\{1, \delta\}$ such that

$$|f(s)s| \leq C|s|^p, \quad \text{for } |s| \leq \delta_*.$$ 

In order to prove our main results via variational methods, we need to modify and extend $f(u)$ for outside a neighborhood of $u = 0$ to get $\tilde{f}(u)$. We set

$$\tilde{f}(s) := \begin{cases} 0, & s \leq 0, \\ f(s), & 0 < s \leq \delta_*, \\ C_1s^{p-1}, & \delta_* < s, \end{cases}$$

and fix $C_1 > 0$ such that $\tilde{f} \in C(\mathbb{R}, \mathbb{R})$. Combining the definition of $\tilde{f}$ with the proof of Lemma 2.3 in [16] (see also [30]), one can easily complete the proof of the following lemma.

Lemma 2.1. Suppose that $(f_1) - (f_2)$ hold. Then

(i) there exists $C > 0$ such that $|\tilde{f}(s)s| \leq C|s|^p$ and $|\tilde{F}(s)| \leq C|s|^p$ for all $s \in \mathbb{R}$,

(ii) there exists $\mu \in (1, \min\{v + 1, p\})$ such that the function $s \rightarrow \frac{\tilde{f}(s)}{s^{\mu}}$ is strictly increasing on $(0, +\infty)$.


Now let us consider the modified equation of Eq. (P) given by
\[
\begin{aligned}
-\Delta u + V(x)u &= \lambda(I_\alpha * \tilde{F}(u))\tilde{f}(u), \\
u &\in E.
\end{aligned}
\]  
\text{(P)}

From (i) of Lemma 2.1, the Hardy-Littlewood-Sobolev inequality and the Sobolev inequality, we have
\[
\int_{\mathbb{R}^N} (I_\alpha * \tilde{F}(u))\tilde{F}(u)dx \leq C \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^pdx \leq C \left( \int_{\mathbb{R}^N} |u|^\frac{2Np}{N+p}dx \right)^{\frac{N+p}{N}}.
\]  
Then the corresponding functional
\[
\tilde{I}(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx - \frac{\lambda}{2} \int_{\mathbb{R}^N} (I_\alpha * \tilde{F}(u))\tilde{f}(u)dx
\]
is well defined and of class $C^1$ by a standard argument and whose derivative is given by
\[
\langle \tilde{I}'(u), v \rangle = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x)uv)dx - \lambda \int_{\mathbb{R}^N} (I_\alpha * \tilde{F}(u))\tilde{f}(u)vdx, \quad v \in E.
\]

Formally, critical points of $\tilde{I}$ are the solutions of Eq. (P). We note that nonnegative critical points of $\tilde{I}$ with $L^\infty$-norm less than or equal to $\delta_\ast$ are also solutions of the original Eq. (P). We recall the Nehari manifold
\[
\mathcal{N} := \left\{ u \in E \setminus \{0\} : \langle \tilde{I}'(u), u \rangle = 0 \right\} = \left\{ u \in K : \langle \tilde{I}'(u), u \rangle = 0 \right\}
\]
and set
\[
c := \inf_{u \in \mathcal{N}} \tilde{I}(u).
\]

**Lemma 2.2.** Suppose that $N \geq 3$, $\alpha \in ((N - 4)^+, N)$, $(V_1)$ and $(f_1) - (f_2)$ hold. Then

(a) for any $u \in K$, there exists a unique $t_u > 0$ such that $t_uu \in \mathcal{N}$. Moreover, the maximum of $\tilde{I}(tu)$ for $t > 0$ is achieved at $t_u$,

(b) there exists $\rho > 0$ such that $\|u\| \geq \rho$ for all $u \in \mathcal{N}$,

(c) $\tilde{I}$ is bounded from below on $\mathcal{N}$ by a positive constant.

Proof. (a) For any $u \in K$, we define
\[
\Psi(t) := \tilde{I}(tu) = \frac{t^2}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx - \frac{\lambda}{2} \int_{\mathbb{R}^N} (I_\alpha * \tilde{F}(tu))\tilde{f}(tu)dx,
\]
where $t \in (0, +\infty)$. It follows from (2) that
\[
\int_{\mathbb{R}^N} (I_\alpha * \tilde{F}(tu))\tilde{F}(tu)dx \leq Ct^{2p}\|u\|^{2p}.
\]
Thus one has
\[
\Psi(t) \geq \frac{t^2}{2}\|u\|^2 - \frac{\lambda C}{2}t^{2p}\|u\|^{2p}.
\]
Then there exists $t_0 > 0$ such that $\Psi(t_0) > 0$. We set $\Omega = \{ x \in \mathbb{R}^N : u(x) > 0 \}$. Combining (ii) in Lemma 2.1 with the Fatou lemma, we have
\[
\liminf_{t \to \infty} \int_{\mathbb{R}^N} \frac{(I_\alpha * \tilde{F}(tu))\tilde{F}(tu)}{t^2}dx
\]
Since \( p > 1 \), there exists \( \rho > 1 \) such that \( \Psi(t_u) = \max_{t>0} \Psi(t) \) and \( \Psi'(t_u) = 0 \), i.e., \( \tilde{I}(t_u) = \max_{t>0} \tilde{I}(t) \) and \( t_u \in \mathcal{N} \).

Suppose that there exists \( t_1 > t_2 > 0 \) such that \( t_1 u \in \mathcal{N}, i = 1, 2 \), one has

\[
\int_\Omega \int_\Omega I_u(x-y) \left[ \frac{\tilde{F}(t_1 u(x))}{t_1} \tilde{f}(t_1 u(y)) - \frac{\tilde{F}(t_2 u(x))}{t_2} \tilde{f}(t_2 u(y)) \right] dxdy = 0,
\]

which is a contradiction with (ii) of Lemma 2.1. Consequently \( t_u \) is unique.

(b) For any \( u \in \mathcal{N} \), combining (i) of Lemma 2.1, the Hardy-Littlewood-Sobolev inequality and the Sobolev embedding one obtains

\[
\|u\|^2 = \lambda \int_{\mathbb{R}^N} (I_\alpha * \tilde{F}(u)) \tilde{f}(u) dx \leq \lambda C \int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^p dx \leq \lambda C \|u\|^{2p}.
\]  

(3)

It follows from (3) that there exists \( \rho > 0 \) independent of \( u \) such that

\[
\rho \leq \|u\|.
\]

(c) Also from (2), we have

\[
\tilde{I}(u) \geq \frac{1}{2} \|u\|^2 - \frac{\lambda C}{2} \|u\|^{2p}.
\]

Since \( p > 1 \), there exists \( \sigma > 0 \) such that \( \tilde{I}(u) \geq C \sigma^2 > 0 \) for \( \|u\| = \sigma > 0 \). For any \( v \in \mathcal{N} \), there exists \( t' > 0 \) such that \( t' \|v\| = \sigma \). Combining (a) with (b), one obtains

\[
\tilde{I}(v) \geq \tilde{I}(t'v) \geq C \sigma^2.
\]

This completes the proof.

\[ \square \]

From Lemma 2.2, one can easily know (see also [28])

\[
c = \inf_{u \in \mathcal{N}} \tilde{I}(u) = \inf_{u \in \mathcal{K}} \sup_{t>0} \tilde{I}(tu) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \tilde{I}(\gamma(t)),
\]

where \( \Gamma = \{ \gamma \in C([0,1],E) : \gamma(0) = 0, ~ \tilde{I}(\gamma(t)) < 0 \} \). Notice that \( c > 0 \) from (c) of Lemma 2.2. In order to prove our results, we introduce the following equation

\[
-\Delta u + V(x)u = \lambda (I_\alpha * \tilde{F}(u)) \tilde{f}(u). \tag{P_\infty}
\]

It follows from [30] that Eq. (\( P_\infty \)) has a positive ground state solution \( \omega \). From Lemma 2.2, there exists a unique \( t_\omega > 0 \) such that \( t_\omega \omega \in \mathcal{N} \) and

\[
c \leq \tilde{I}(t_\omega) \leq \tilde{I}(t_\omega \omega) \leq \tilde{I}_\infty(\omega) := c_\infty,
\]

where \( \tilde{I}_\infty \) is the energy functional associated with Eq. (\( P_\infty \)).

**Lemma 2.3.** Suppose that \( N \geq 3, \alpha \in ((N-4)^+, N), (V_1) \) and \( (f_1) - (f_2) \) hold. If \( u \in \mathcal{N} \) and \( \tilde{I}(u) = c \), then \( u \) is a nontrivial solution of Eq. (\( \tilde{P} \)).
Proof. From [14, 28], one supposes by contradiction that \( u \) is not a nontrivial solution of Eq. (P). Then there exists \( \phi \in E \) such that

\[
\langle \tilde{T}(u), \phi \rangle = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla \phi + V(x)u\phi)dx - \lambda \int_{\mathbb{R}^N} (I_\alpha * \tilde{F}(u))\tilde{F}(u)dx < -1.
\]

Let \( \varepsilon \in (0, 1) \) be small enough such that for all \( |t - 1| \leq \varepsilon \) and \( |s| \leq \varepsilon \),

\[
\langle \tilde{T}(tu + s \phi), \phi \rangle \leq -\frac{1}{2}.
\]

We define a smooth cut-off function \( 0 \leq \tau \leq 1 \) such that \( \tau(t) = 1 \) for \( |t - 1| \leq \frac{\varepsilon}{2} \) and \( \tau(t) = 0 \) for \( |t - 1| \geq \varepsilon \). For \( t > 0 \), we set a curve \( \gamma(t) = tu + s\tau(t)\phi \). Obviously, \( \gamma \) is a continuous.

Notice that, it follows from Lemma 2.2 that \( \tilde{T}(\gamma(t)) = \tilde{T}(tu) < \tilde{T}(u) = c \) when \( |t - 1| \geq \varepsilon \). When \( |t - 1| \leq \varepsilon \), since \( \Phi(s) := \tilde{T}(tu + s\tau(t)\phi) \) is of \( C^1 \) on \( [0, \varepsilon] \), there exists \( \bar{s} \in (0, \varepsilon) \) such that

\[
\tilde{T}(tu + s\tau(t)\phi) = \tilde{T}(tu) + \tilde{T}(tu + \bar{s}\tau(t)\phi), \varepsilon\tau(t) \phi \leq \tilde{T}(tu) - \frac{1}{2}\varepsilon\tau(t) < c.
\]

Hence \( \tilde{T}(\gamma(t)) < c \) for any \( t \in (0, +\infty) \).

We define \( J(u) = \tilde{T}(u, u) \), then \( J(\gamma(1 - \varepsilon)) = J((1 - \varepsilon)u) > 0 \) and \( J(\gamma(1 + \varepsilon)) = J((1 + \varepsilon)u) < 0 \) from \( u \in N \) and Lemma 2.2. By the continuity of \( t \mapsto J(\gamma(t)) \) there exists \( \ell' \in (1 - \varepsilon, 1 + \varepsilon) \) such that \( J(\gamma(\ell')) = 0 \). Thus \( \gamma(\ell') \in N \) and \( \tilde{T}(\gamma(\ell')) < c \), which is a contradiction. This completes the proof.

Lemma 2.4. Suppose that \( N \geq 3, \alpha \in ((N - 4)^+, N) \), \( (V_1) \) and \( (f_1) - (f_2) \) hold. Then the Cerami sequence for \( \tilde{T} \) at level \( m > 0 \) (shortly: \( (Ce)_m \) sequence) is bounded in \( E \).

Proof. We recall the \( (Ce)_m \) sequence \( \{u_n\} \), that is,

\[
\tilde{T}(u_n) \rightarrow m, \quad \|\tilde{T}'(u_n)\|_*(1 + \|u_n\|) \rightarrow 0.
\]

Then

\[
o(1) = \langle \tilde{T}'(u_n), u_n^- \rangle = -\|u_n^-\|^2.
\]

Consequently we could deduce that \( \{u_n^+\} \) is also a \( (Ce)_m \) sequence. For the sake of convenience, we denote \( u_n = u_n^+ \). By contradiction, we assume that \( \|u_n\| \rightarrow +\infty \) and set \( v_n = \frac{u_n}{\|u_n\|} \). Obviously up to a subsequence, there exists a nonnegative function \( v \in E \) such that \( v_n \rightarrow v \in E, v_n \rightarrow v \in L^2_{\text{loc}}(\mathbb{R}^N) \) and \( v_n(x) \rightarrow v(x) \) a.e. in \( \mathbb{R}^N \). We define \( \Omega_1 = \{x \in \mathbb{R}^N : v(x) > 0\} \). If meas \( \Omega_1 > 0 \), the Fatou lemma and (ii) of Lemma 2.1 imply

\[
\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} \frac{(I_\alpha * \tilde{F}(u))\tilde{F}(u)}{\|u_n\|^2} dx
\]

\[
\geq \liminf_{n \rightarrow \infty} \int_{\Omega_1} \int_{\Omega_1} I_\alpha(x - y) \frac{\tilde{F}(u_n(x))\tilde{F}(u_n(y))}{u_n(x)u_n(y)} v_n(x)v_n(y)dxdy
\]

\[
= +\infty.
\]

Then

\[
0 = \limsup_{n \rightarrow \infty} \frac{\tilde{T}(u_n)}{\|u_n\|^2} = \frac{1}{2} + \frac{\lambda}{\|u_n\|^2} - \frac{1}{\|u_n\|^2} \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} \frac{(I_\alpha * \tilde{F}(u))\tilde{F}(u)}{\|u_n\|^2} dx = -\infty,
\]

CHOQUARD EQUATION 1357
which is a contradiction. Thus \( v = 0 \). We define
\[
\varrho := \lim_{n \to \infty} \sup_{z \in \mathbb{R}^N} \int_{B_1(z)} v_n^2 dx. \tag{5}
\]

If \( \varrho = 0 \), we have \( v_n \to 0 \) in \( L^{\frac{2N}{N-2}}(\mathbb{R}^N) \) from the Lions lemma \([13, 28]\). Combining with (2), we obtain
\[
\int_{\mathbb{R}^N} (I_\alpha \ast \tilde{F}(2\sqrt{mv_n})) \tilde{F}(2\sqrt{mv_n}) dx = o(1).
\]

By the continuity of \( \tilde{I} \), there exists \( t_n \in [0, 1] \) such that \( \tilde{I}(t_n u_n) = \max_{t \in [0, 1]} \tilde{I}(t u_n) \). Since \( \|u_n\| \to +\infty \), one has \( \frac{2\sqrt{m}}{|u_n|} \leq 1 \) as \( n \) large enough. We observe that
\[
\tilde{I}(t_n u_n) + o(1) \geq \tilde{I} \left( \frac{2\sqrt{m}}{|u_n|} u_n \right) + o(1)
\]
\[
= 2m \|v_n\|^2 - \frac{\lambda}{2} \int_{\mathbb{R}^N} (I_\alpha \ast \tilde{F}(2\sqrt{mv_n})) \tilde{F}(2\sqrt{mv_n}) dx + o(1)
\]
\[
= 2m + o(1).
\]

Since \( \tilde{I}(u_n) \to m \) and (a) of Lemma 2.2, we can deduce that \( t_n \in (0, 1) \) and \( \langle \tilde{I}(t_n u_n), t_n u_n \rangle = 0 \) as \( n \) large enough. Hence by Lemma 2.3 in \([16]\), one has
\[
m = \tilde{I}(u_n) + o(1)
\]
\[
= \tilde{I}(u_n) - \frac{1}{2\mu} \langle \tilde{I}(u_n), u_n \rangle + o(1)
\]
\[
= \left( \frac{1}{2} - \frac{1}{2\mu} \right) \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)u_n^2) dx
\]
\[
+ \frac{\lambda}{2} \int_{\mathbb{R}^N} (I_\alpha \ast \tilde{F}(u_n)) \left( \frac{1}{\mu} \tilde{F}(u_n)u_n - \tilde{F}(u_n) \right) dx + o(1)
\]
\[
\geq \left( \frac{1}{2} - \frac{1}{2\mu} \right) \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)u_n^2) dx
\]
\[
+ \frac{\lambda}{2} \int_{\mathbb{R}^N} (I_\alpha \ast \tilde{F}(t_n u_n)) \left( \frac{1}{\mu} \tilde{F}(t_n u_n)u_n - \tilde{F}(t_n u_n) \right) dx + o(1)
\]
\[
= \tilde{I}(t_n u_n) - \frac{1}{2\mu} \tilde{I}(t_n u_n), t_n u_n) + o(1)
\]
\[
= \tilde{I}(t_n u_n) + o(1)
\]
\[
\geq 2m + o(1),
\]
which is a contradiction.

If \( \varrho > 0 \), there exists \( \{z_n\} \subset \mathbb{R}^N \) such that
\[
\varrho \leq \int_{B_1(z_n)} v_n^2 dx.
\]

If \( \{z_n\} \) is bounded, there exists \( R > 0 \) such that
\[
\varrho \leq \int_{B_R} v_n^2 dx.
\]
which is a contradiction with \( v_n \to 0 \) in \( L^2_{\text{loc}}(\mathbb{R}^N) \). Then \( \{z_n\} \) is unbounded, up to a subsequence, \( |z_n| \to \infty \). We set \( w_n := v_n(\cdot + z_n) \), where \( w_n \) satisfies
\[
\frac{\theta}{2} \leq \int_{B_1} w_n^2 dx.
\]
(6)

Thus up to a subsequence, there exists a nonnegative function \( w \in E \) such that \( w_n \to w \) in \( E \), \( w_n \to w \) in \( L^2_{\text{loc}}(\mathbb{R}^N) \) and \( w_n(x) \to w(x) \) a.e. in \( \mathbb{R}^N \). Evidently, \( \text{meas } \Omega_2 > 0 \) where \( \Omega_2 = \{ x \in \mathbb{R}^N : w(x) > 0 \} \). In fact \( w_n(x) = \frac{u_n(x + z_n)}{\|u_n\|} \). Also from the Fatou lemma and (ii) of Lemma 2.1, one obtains
\[
\liminf_{n \to \infty} \left[ \frac{1}{\|u_n\|^2} \int_{\mathbb{R}^N} (I_\alpha * \tilde{F}(u_n)) \tilde{F}(u_n) dx \right] = \liminf_{n \to \infty} \left[ \frac{1}{\|u_n\|^2} \int_{\mathbb{R}^N} \left( I_\alpha * \tilde{F}(u_n(x + z_n)) \right) \tilde{F}(u_n(x + z_n)) dx \right] \geq \liminf_{n \to \infty} \int_{\Omega_2} \int_{\Omega_2} I_\alpha(x-y) \frac{\tilde{F}(u_n(x + z_n)) \tilde{F}(u_n(y + z_n))}{u_n(x + z_n)u_n(y + z_n)} w_n(x + z_n)w_n(y + z_n) dx dy = +\infty.
\]

Hence
\[
0 = \lim\sup_{n \to \infty} \frac{\tilde{I}(u_n)}{\|u_n\|^2} = \frac{1}{2} - \liminf_{n \to \infty} \left[ \frac{\lambda}{2\|u_n\|^2} \int_{\mathbb{R}^N} (I_\alpha * \tilde{F}(u_n)) \tilde{F}(u_n) dx \right] = -\infty,
\]
which is a contradiction. In a word, the \((Ce)_m\) sequence \( \{u_n\} \) is bounded in \( E \). \( \square \)

Now, we recall the following some facts. The proof is similar to Lemmas 2.2-2.4 in [15]. Here we omit it.

**Lemma 2.5.** Suppose that \( (V_1) \) holds. If \( \{u_n\} \) is bounded in \( E \) and \( |z_n| \to +\infty \), for any \( \phi \in C_0^\infty (\mathbb{R}^N, \mathbb{R}) \) one has
\[
\int_{\mathbb{R}^N} (V_\infty(x) - V(x)) u_n \phi(x - z_n) dx = o(1).
\]

Moreover, \( u_n \to 0 \) in \( E \), one has
\[
\int_{\mathbb{R}^N} (V_\infty(x) - V(x)) u_n^2 dx = o(1).
\]

**Proposition 1.** Suppose that \( N \geq 3, \alpha \in ((N - 4)^+, N) \), \( (V_1) \) and \( (f_1) - (f_2) \) hold. Then Eq. (\( \tilde{P} \)) has a positive ground state solution.

**Proof.** Notice that \( 0 < c \leq c_\infty \). Therefore, one of the two cases occurs:

**Case 1.** \( c = c_\infty \). It follows from (4) that
\[
c_\infty \leq \tilde{I}(t_\omega \omega) \leq \tilde{I}_\infty(t_\omega \omega) \leq \tilde{I}_\infty(\omega) = c_\infty.
\]

Then \( \omega \) is also a positive ground state solution of Eq. (\( \tilde{P} \)) from Lemma 2.3.

**Case 2.** \( 0 < c < c_\infty \). We can see that \( \tilde{I} \) satisfies the mountain pass geometry. From the mountain pass theorem [25, 28] and Lemma 2.4, there exists a nonnegative and bounded sequence \( \{u_n\} \in E \) such that
\[
\tilde{I}(u_n) \to c, \quad \|\tilde{I}'(u_n)\|_s(1 + \|u_n\|) \to 0.
\]
Then there exists a nonnegative function \( u \in E \) such that up to a subsequence, \( u_n \to u \) in \( E \), \( u_n \to u \) in \( L^2_{\text{loc}}(\mathbb{R}^N) \), and \( u_n(x) \to u(x) \) a.e. in \( \mathbb{R}^N \). For any
\( \varphi \in C_0^\infty (\mathbb{R}^N) \), one has \( 0 = \langle \mathcal{P}'(u_n), \varphi \rangle + o(1) = \langle \mathcal{P}'(u), \varphi \rangle \), i.e., \( u \) is a solution of Eq. \((\mathcal{P})\). If \( u \neq 0 \) in \( E \), combining with the Fatou lemma one obtains

\[
c = \mathcal{I}(u) + o(1) = \mathcal{I}(u) - \frac{1}{2\mu} \langle \mathcal{I}'(u_n), u_n \rangle + o(1)
\]

\[
= \left( \frac{1}{2} - \frac{1}{2\mu} \right) \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)u_n^2) \, dx
\]

\[
+ \frac{\lambda}{2} \int_{\mathbb{R}^N} \left( I_n \ast \tilde{F}(u_n) \right) \left( \frac{1}{\mu} \tilde{f}(u_n) u_n - \tilde{F}(u_n) \right) \, dx + o(1)
\]

\[
\geq \left( \frac{1}{2} - \frac{1}{2\mu} \right) \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \, dx
\]

\[
+ \frac{\lambda}{2} \int_{\mathbb{R}^N} \left( I_n \ast \tilde{F}(u) \right) \left( \frac{1}{\mu} \tilde{f}(u) u - \tilde{F}(u) \right) \, dx + o(1)
\]

\[
= \mathcal{I}(u) - \frac{1}{2\mu} \langle \mathcal{I}'(u), u \rangle + o(1)
\]

\[
= \mathcal{I}(u) + o(1).
\]

At the same time, from the definition of \( c \) and \( u \in \mathcal{N} \) one knows \( c \leq \mathcal{I}(u) \). We could deduce that \( u \) is a positive ground state solution of Eq. \((\mathcal{P})\) by the strongly maximum principle.

We assume that \( u = 0 \) (otherwise we complete the proof). Then there exists \( \varrho \geq 0 \) such that

\[
\lim_{n \to \infty} \sup_{z \in \mathbb{R}^N} \int_{B_1(z)} |u_n|^2 \, dx = \varrho.
\]

Indeed, if \( \varrho = 0 \), applying the Lions lemma \([13, 28]\), we obtain

\[
u_n \to 0 \quad \text{in} \quad L^{\frac{2N\mu}{N-2\mu}}(\mathbb{R}^N).
\]

Hence it follows from \((i)\) in Lemma 2.1 that \( \mathcal{I}(u_n) \to 0 \) as \( n \to \infty \), which is a contradiction with \( c > 0 \). Then there exists \( \{z_n\} \subset \mathbb{R}^N \) such that \( \int_{B_1(z_n)} |u_n|^2 \, dx \geq \frac{\varrho}{2} > 0 \).

If \( \{z_n\} \) is bounded, there exists \( R > 0 \) such that \( \int_{B_R(0)} |u_n|^2 \, dx \geq \frac{\varrho}{2} > 0 \), which is a contradiction with \( u_n \to 0 \) in \( L^2_{\text{loc}}(\mathbb{R}^N) \). Then \( \{z_n\} \) is unbounded. After extracting a subsequence if necessary, we have

\[
(i) \quad |z_n| \to +\infty,
\]

\[
(ii) \quad u_n(\cdot + z_n) \rightharpoonup v \neq 0 \quad \text{in} \quad E.
\]

From Lemma 2.5, we have

\[
0 = \langle \mathcal{P}'(u_n), \varphi (\cdot - z_n) \rangle + o(1) = \langle \mathcal{I}_n(v), \varphi \rangle + o(1).
\]

Then \( v \) is a nontrivial solution of Eq. \((\mathcal{P}_\infty)\). At the same time, from Lemma 2.5 we have

\[
c = \mathcal{I}(u_n) - \frac{1}{2\mu} \langle \mathcal{I}'(u_n), u_n \rangle + o(1)
\]

\[
= \left( \frac{1}{2} - \frac{1}{2\mu} \right) \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)u_n^2) \, dx
\]
Then we use the Sobolev inequality to yield
\[ \frac{\lambda}{2} \int_{\mathbb{R}^N} \left( I_\alpha * \tilde{F}(u_n) \right) \left( \frac{1}{\mu} \tilde{f}(u_n) u_n - \tilde{F}(u_n) \right) dx + o(1) \]
\[ = \left( \frac{1}{2} - \frac{1}{2\mu} \right) \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V_\infty(x) u_n^2) dx \]
\[ + \frac{\lambda}{2} \int_{\mathbb{R}^N} \left( I_\alpha * \tilde{F}(u_n(\cdot + z_n)) \right) \left( \frac{1}{\mu} \tilde{f}(u_n(\cdot + z_n)) u_n(\cdot + z_n) - \tilde{F}(u_n(\cdot + z_n)) \right) dx + o(1) \]
\[ = \tilde{I}_\infty(u_n(\cdot + z_n)) - \frac{1}{2\mu} \tilde{I}'_\infty(u_n(\cdot + z_n), u_n(\cdot + z_n)) + o(1) \]
\[ \geq \tilde{I}_\infty(v) + o(1) \]
\[ \geq c_\infty + o(1), \]
which is a contradiction.

In conclusion, whether case 1 occurs or case 2 occurs, we can prove Proposition 1. \qed

3. Proof of Theorem 1.1.

Lemma 3.1. Suppose that \( N \geq 3, \alpha \in (\mathbb{Z} - 4)^+, \alpha \), \((V_1)\) and \((f_1) - (f_2)\) hold. If \( u \) is a critical point of \( \tilde{I} \), then \( u \in L^\infty(\mathbb{R}^N) \). Furthermore, there exists a positive constant \( C \) independent of \( \lambda \) such that
\[ |u|_\infty \leq C \lambda^{\frac{N-2}{2N}} \|u\|^{\frac{N+2-N+2}{N+2-N+2}}. \]

Proof. We prove this result by using the Moser iteration. For each \( k > 0 \), we define
\[ u_k(x) = \begin{cases} u(x), & \text{if } |u(x)| \leq k, \\ \pm k, & \text{if } \pm u(x) > k. \end{cases} \]
For \( \beta > 1 \), we use \( \varphi_k = |u_k|^{2(\beta-1)} \) as a test function in \( \langle \tilde{I}'(u), \varphi_k \rangle \) to obtain
\[ \int_{\mathbb{R}^N} |u_k|^{2(\beta-1)} \nabla u^2 dx + 2(\beta - 1) \int_{\mathbb{R}^N} |u_k|^{2(\beta-2)} u u_k \nabla u \cdot \nabla u_k dx \]
\[ + \int_{\mathbb{R}^N} V(x) |u_k|^{2(\beta-1)} u^2 dx \]
\[ = \lambda \int_{\mathbb{R}^N} (I_\alpha * f(u)) |u_k|^{2(\beta-1)} u dx. \]

Then we use the Sobolev inequality to yield
\[ \beta^2 \int_{\mathbb{R}^N} \left( |u_k|^{2(\beta-1)} |\nabla u|^2 + 2(\beta - 1)|u_k|^{2(\beta-2)} u u_k \nabla u \cdot \nabla u_k \right) dx \]
\[ \geq \int_{\mathbb{R}^N} (|u_k|^{2(\beta-1)} |\nabla u|^2 + (\beta - 1)^2 |u_k|^{2(\beta-2)} |\nabla u_k|^2 \]
\[ + 2(\beta - 1)|u_k|^{2(\beta-2)} u u_k \nabla u \cdot \nabla u_k dx \]
\[ \geq \int_{\mathbb{R}^N} \left( |u_k|^{\beta-1} u \right)^2 dx \]
\[ \geq C \left( \int_{\mathbb{R}^N} |u_k|^{\beta-1} u dx \right)^\frac{2}{\beta}, \]
\[ \text{for} \ (\beta-2) + 2 = \frac{2}{\beta}. \]
from (i) in Lemma 2.1, we deduce
\[ \beta > 0, \]

and we also have used the facts that \( u^2 |\nabla u_k|^2 \leq u_k^2 |\nabla u|^2 \) and \( \beta > 1 \). If \( p > 2 \), from (i) in Lemma 2.1, we deduce
\[
\int_{\mathbb{R}^N} (I_\alpha \ast F(u)) f(u) |u_k|^{2(\beta-1)} u dx \\
\leq C \left( \int_{\mathbb{R}^N} |u| \frac{2Np}{N+\alpha} dx \right) \frac{N+\alpha}{2N} \left( \int_{\mathbb{R}^N} |u|^p |u_k|^{2(\beta-1)} \frac{2N}{N+\alpha} dx \right) \frac{N+\alpha}{N} \\
\leq C \left( \int_{\mathbb{R}^N} |u| \frac{2Np}{N+\alpha} dx \right) \frac{N+\alpha}{2N} \left( \int_{\mathbb{R}^N} |u|^p \frac{2N}{N+\alpha} dx \right) \frac{N+\alpha}{p-2} \\
\left( \int_{\mathbb{R}^N} |u_k|^{2(\beta-1)} u^2 \frac{2N}{N+\alpha-(N-2)(p-2)} dx \right) \frac{N+\alpha}{2N} \frac{p-2}{N+\alpha-(N-2)(p-2)} \\
\leq C \|u\|^{2p-2} |u_k|^{2(\beta-1)} u^2 \frac{2N}{N+\alpha-(N-2)(p-2)}.
\]

If \( p = 2 \), we also have
\[
\int_{\mathbb{R}^N} (I_\alpha \ast F(u)) f(u) |u_k|^{2(\beta-1)} u dx \\
\leq C \|u\|^{2p-2} |u_k|^{2(\beta-1)} u^2 \frac{2N}{N+\alpha-(N-2)(p-2)}.
\]

Combining (8) with (9), we obtain
\[
\left( \int_{\mathbb{R}^N} \|u_k\|_{p-2}^\beta dx \right)^{\frac{1}{\beta}} \leq C\lambda \beta^2 \|u\|^{2p-2} |u_k|^{2(\beta-1)} u^2 \frac{2N}{N+\alpha-(N-2)(p-2)}.
\]

Letting \( k \to \infty \) in (10), we have
\[
|u|_{\beta-2}^\beta \leq (C\lambda \beta^2)^{\frac{1}{\beta}} \|u\|_{p-1}^{p-1} |u| \frac{4N \beta_m}{N+\alpha-(N-2)(p-2)}.
\]

Now we carry out an iteration process. Set
\[
\beta_m = \left( \frac{N + \alpha - (N-2)(p-2)}{2(N-2)} \right)^{m+1}, \quad m = 0, 1, \ldots.
\]

Then we have
\[
\frac{4N \beta_{m+1}}{N+\alpha-(N-2)(p-2)} = 2^m \beta_m.
\]

By (11), one obtains
\[
|u|_{\beta_m}^{2^m} \leq (C\lambda \beta_m^2)^{\frac{1}{2^m}} \|u\|_{p-1}^{p-1} |u| \frac{4N \beta_m}{N+\alpha-(N-2)(p-2)}
\]
\[
= (C\lambda)^{\frac{1}{2^m}} \beta_m^{\frac{1}{2^m}} \|u\|_{p-1}^{p-1} |u|_{\beta_{m-1}}^{2^m}.
\]

By the Moser iteration, we have
\[
|u|_{\beta_m}^{2^m} \leq C\lambda^{\sum_{i=0}^m \frac{1}{2^i}} \prod_{i=0}^m \beta_i^{\frac{1}{2^i}} \|u\|_{p-1}^{(p-1)} \sum_{i=0}^m \frac{1}{2^i} |u|_{2^i}.
\]

Since \( \beta_0 = \left( \frac{N + \alpha - (N-2)(p-2)}{2(N-2)} \right) > 1 \) and \( \beta_i = \beta_{i-1}^{2^m} \), we observe that
\[
\sum_{i=0}^m \frac{1}{\beta_i} = \sum_{i=0}^m \frac{1}{\beta_{i-1}^{2^m}}, \quad \prod_{i=0}^m \beta_i^{\frac{1}{2^i}} = \prod_{i=0}^m (\beta_{i-1}^{2^m})^{\frac{1}{2^i}} = (\beta_0 \sum_{i=0}^m \frac{1}{\beta_i^{2^m}})^{\frac{1}{2^m}}.
\]
One can easily see
\[ \sum_{i=0}^{\infty} \frac{i+1}{2^{i+1}} = \beta^* < +\infty, \quad \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{2(N-2)}{N + \alpha - p(N-2)}. \]
Letting \( m \to \infty \), we would conclude that \( u \in L^\infty(\mathbb{R}^N) \) and
\[ |u|_{\infty} \leq (C\lambda)^{\frac{N-2}{4+N+\alpha-2p(N-2)}} \beta^* \|u\|_{2^*}^{\frac{(2p-2)(N-2)}{N + \alpha - p(N-2)}} |u|_{2^*} \leq C \lambda^{\frac{N-2}{4+N+\alpha-2p(N-2)}} \|u\|_{2^*}^{\frac{N+\alpha(2p-2)(N-2)}{N+\alpha-2p(N-2)}}. \]
(12)
This completes the proof.

Proof of Theorem 1.1. By proposition 1, Eq. \((\tilde{P})\) has a positive ground solution \( u \). Combining the Sobolev embedding and (i) of Lemma 2.1, one obtains
\[ c = \tilde{I}(u) - \frac{1}{2\mu} \langle \tilde{I}'(u), u \rangle \geq \left( \frac{1}{2} - \frac{1}{2\mu} \right) \|u\|^2. \]
(13)
We can see that there exists \( v \in K \cap L^\infty(\mathbb{R}^N) \) such that \( |v|_{\infty} < 1 \). Since \((f_1)\), there exists \( C > 0 \) independent of \( \lambda \) such that
\[ F(tv) \geq C |tv|^p, \quad t \in [0, 1]. \]
At the same time there exists \( \lambda_0 > 0 \) such that \( \tilde{I}(v) < 0 \) for \( \lambda \geq \lambda_0 \). Then from the definition of \( c \), we have
\[ c \leq \max_{t \in [0, 1]} \tilde{I}(tv) \]
\[ = \max_{t \in [0, 1]} \frac{t^2}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)u^2)dx - \frac{\lambda}{2} \int_{\mathbb{R}^N} (I_\alpha * F(tv)) \tilde{F}(tv)dx \]
\[ \leq \max_{t \in [0, 1]} \frac{t^2}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)u^2)dx - Ct^2 \lambda \int_{\mathbb{R}^N} (I_\alpha * |v|^p)|v|^p dx \]
\[ \leq C\lambda^{\frac{1}{2p+1}}. \]
(14)
Combining (12), (13) and (14), we have
\[ |u|_{\infty} \leq C \lambda^{\frac{N-2}{4+N+\alpha-2p(N-2)}} \lambda^{\frac{1}{2p(N+\alpha-2p(N-2))}} \leq C \lambda^{\frac{1}{2p}}. \]
Since \( 2 \leq p \), there exists \( \lambda_1 \geq \lambda_0 \) such that \( |u|_{\infty} \leq \delta_* \) for \( \lambda \geq \lambda_1 \). Therefore, from the definition of \( \tilde{f} \), we can conclude that \( u \) is also a positive solutions of Eq. \((P)\) for any \( \lambda \geq \lambda_1 \). This completes the proof of Theorem 1.1.

Under the assumption \((V_3)\), the embedding \( E \) into \( L^s(\mathbb{R}^N) \) for any \( s \in [2, 2^*) \) is compact from [4, 29]. Combining with the proof of Theorem 1.1, we can directly prove the following result.

Remark 3. Assume that \( N \geq 3, \alpha \in ((N-4)^+,N) \), \((V_3)\) and \((f_1)-(f_2)\) hold. Then there exists \( \lambda_2 > 0 \) such that Eq. \((P)\) has a positive solution for \( \lambda \geq \lambda_2 \).

Acknowledgments. The authors would like to express sincere thanks to the referees and the handling editor whose careful reading of the manuscript and valuable comments greatly improve the original manuscript.
REFERENCES

[1] C. O. Alves, G. M. Figueiredo and M. Yang, Existence of solutions for a nonlinear Choquard equation with potential vanishing at infinity, *Adv. Nonlinear Anal.*, 5 (2016), 331–345.

[2] C. O. Alves, A. B. Nóbrega and M. Yang, Multi-bump solutions for Choquard equation with deepening potential well, *Calc. Var. Partial Differential Equations*, 55 (2016), Art. 48, 28.

[3] C. O. Alves and M. Yang, Existence of semiclassical ground state solutions for a generalized Choquard equation, *J. Differential Equations*, 257 (2014), 4133–4164.

[4] T. Bartsch, Z.-Q. Wang and M. Willem, The Dirichlet problem for superlinear elliptic equations, in *Stationary Partial Differential Equations. Vol. II*, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2005, 1–55.

[5] S. Chen and L. Xiao, Existence of a nontrivial solution for a strongly indefinite periodic Choquard system, *Calc. Var. Partial Differential Equations*, 54 (2015), 599–614.

[6] C. Chu and H. Liu, Existence of positive solutions for a quasilinear Schrödinger equation, *Nonlinear Anal. Real World Appl.*, 44 (2018), 118–127.

[7] D. G. Costa and Z.-Q. Wang, Multiplicity results for a class of superlinear elliptic problems, *Proc. Amer. Math. Soc.*, 133 (2005), 787–794.

[8] J. M. do Ó, E. Medeiros and U. Severo, On the existence of signed and sign-changing solutions for a class of superlinear Schrödinger equations, *J. Math. Anal. Appl.*, 342 (2008), 432–445.

[9] F. Gao and M. Yang, A strongly indefinite Choquard equation with critical exponent due to the Hardy-Littlewood-Sobolev inequality, *Commun. Contemp. Math.*, 20 (2018), 1750037, 22.

[10] M. Ghimenti and J. Van Schaftingen, Nodal solutions for the Choquard equation, *J. Funct. Anal.*, 271 (2016), 107–135.

[11] L. Li and X. Zhong, Infinitely many small solutions for the Kirchhoff equation with local sublinear nonlinearities, *J. Math. Anal. Appl.*, 435 (2016), 955–967.

[12] E. H. Lieb, Existence and uniqueness of the minimizing solution of Choquard’s nonlinear equation, *Stud. Appl. Math.*, 57 (1976/77), 93–105.

[13] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. I, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 1 (1984), 109–145.

[14] J.-Q. Liu, Y.-Q. Wang and Z.-Q. Wang, Solutions for quasilinear Schrödinger equations via the Nehari method, *Comm. Partial Differential Equations*, 29 (2004), 879–901.

[15] J. Liu, J.-F. Liao and C.-L. Tang, A positive ground state solution for a class of asymptotically periodic Schrödinger equations, *Comput. Math. Appl.*, 71 (2016), 965–976.

[16] S. Liu, On superlinear problems without the Ambrosetti and Rabinowitz condition, *Nonlinear Anal.*, 73 (2010), 788–795.

[17] X. Liu, S. Ma and X. Zhang, Infinitely many bound state solutions of Choquard equations with potentials, *Z. Angew. Math. Phys.*, 69 (2018), Art. 118, 29.

[18] L. Ma and L. Zhao, Classification of positive solitary solutions of the nonlinear Choquard equation, *Arch. Ration. Mech. Anal.*, 195 (2010), 455–467.

[19] G. P. Menzala, On regular solutions of a nonlinear equation of Choquard’s type, *Proc. Roy. Soc. Edinburgh Sect. A.*, 86 (1980), 291–301.

[20] V. Moroz and J. Van Schaftingen, Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics, *J. Funct. Anal.*, 265 (2013), 153–184.

[21] V. Moroz and J. Van Schaftingen, Existence of groundstates for a class of nonlinear Choquard equations, *Trans. Amer. Math. Soc.*, 367 (2015), 6557–6579.

[22] V. Moroz and J. Van Schaftingen, Semi-classical states for the Choquard equation, *Calc. Var. Partial Differential Equations*, 52 (2015), 199–235.

[23] V. Moroz and J. Van Schaftingen, A guide to the Choquard equation, *J. Fixed Point Theory Appl.*, 19 (2017), 773–813.

[24] S. I. Pekar, *Untersuchungen über die Elektronentheorie der Kristalle*, Akademie-verlag, 1954.

[25] M. Schechter, *A variation of the mountain pass lemma and applications*, *J. London Math. Soc. (2)*, 44 (1991), 491–502.

[26] Z. Shen, F. Gao and M. Yang, On critical Choquard equation with potential well, *Discrete Contin. Dyn. Syst.*, 38 (2018), 3567–3593.

[27] J. Van Schaftingen and J. Xia, Choquard equations under confining external potentials, *NoDEA Nonlinear Differential Equations Appl.*, 24 (2017), Art. 1, 24.

[28] M. Willem, *Minimax Theorems*, vol. 24, Springer Science and Business Media, 1997.
[29] H. Zhang, J. Xu and F. Zhang, Bound and ground states for a concave-convex generalized Choquard equation, *Acta Appl. Math.*, **147** (2017), 81–93.

[30] H. Zhang, J. Xu and F. Zhang, Existence and multiplicity of solutions for a generalized Choquard equation, *Comput. Math. Appl.*, **73** (2017), 1803–1814.

Received March 2019; revised August 2019.

E-mail address: bestdong123@163.com
E-mail address: mathliyy518@163.com
E-mail address: 1604694612@qq.com
E-mail address: tangcl@swu.edu.cn