Abstract

Let \((X, \|\cdot\|)\) be a real normed space of dimension \(N \in \mathbb{N}\) with a basis \((e_i)_{i=1}^N\) such that the norm is invariant under coordinate permutations. Assume for simplicity that the basis constant is at most 2. Consider any \(n \in \mathbb{N}\) and \(0 < \varepsilon < 1/4\) such that \(n \leq c(\log \varepsilon^{-1})^{-1} \log N\). We provide an explicit construction of a matrix that generates a \((1 + \varepsilon)\) embedding of \(\ell^n_2\) into \(X\).

1 Introduction

The modern formulation of Dvoretzky's theorem states that there exists a function \(\xi(\cdot) : (0, 1/2) \to (0, \infty)\) such that for all \((N, n, \varepsilon) \in \mathbb{N} \times \mathbb{N} \times (0, 1/2)\) with \(n \leq \xi(\varepsilon) \log N\), and any real normed space \((X, \|\cdot\|)\) with \(\dim(X) = N\), there exists a linear map \(T : \mathbb{R}^n \to X\) such that for all \(x \in \mathbb{R}^n\),

\[
(1 - \varepsilon)|x| \leq \|Tx\| \leq (1 + \varepsilon)|x|\]  

where \(|\cdot|\) denotes the standard Euclidean norm. The normed space \((\mathbb{R}^n, |\cdot|)\) is denoted \(\ell^n_2\).

Equation (1) expresses the fact that \(\ell^n_2\) can be \((1 + \varepsilon)\) embedded in \(X\). Geometrically, this means that any centrally symmetric convex body in high dimensional Euclidean space has cross-sections of lower dimension that are approximately ellipsoidal. The logarithmic dependence on \(N\), which is due to Milman [16], is optimal in the general setting (specifically for \(X = \ell^N_\infty\)) but can be greatly improved for other spaces such as \(\ell^n_1\). On the other hand, the optimal dependence on \(\varepsilon\) is unknown. The best current bound is \(\xi(\varepsilon) = c\varepsilon(\log \varepsilon^{-1})^{-2}\) by Schechtman [20]. If weaker forms of Knaster's problem are true [18, 12], then one could take \(\xi(\varepsilon) = c(\log \varepsilon^{-1})^{-1}\), which would be optimal since this is the correct dependence in \(\ell^N_\infty\). We refer to [17, 20, 21] for a more detailed background.

The proofs of Dvoretzky's theorem typically make use of random embeddings and are nonconstructive. A natural question (see for example Section 2.2 in [11] and Section 4 in [22]) is whether or not one can make these and other probabilistic constructions in functional analysis explicit, or at least decrease the randomness in some way.

Of course it is impossible to find an explicit Euclidean subspace of a completely general and unspecified normed space, otherwise all subspaces would be Euclidean. This follows...
from rotational invariance of the class of symmetric convex bodies in $\mathbb{R}^N$ and the fact that the orthogonal group $O(N)$ acts transitively on the Grassmannian $G_{N,n}$. The same is true even if we assume that the space has nontrivial cotype, or that the unit ball is in John’s position. One either has to construct explicit subspaces for specific spaces individually, or (most likely) we need to impose some sort of symmetry in order to get a grip on the space.

The case $X = \ell_1^N$ is particularly important from the point of view of applications, see for example [10] and the references therein. For this space, there are various algorithms to compute embeddings or decrease randomness [1 2 5 6 8 9 10 15 19], although there is still no truly explicit embedding that is as good as a random one. For $X = \ell_p^N$, where $p \in \mathbb{N}$ is an even integer and $N \geq \left( \frac{n + p - 1}{p} \right)$ which is true whenever $n \leq N^{1/p}$, the space $\ell_p^n$ embeds isometrically into $X$, [18 13]. In this case there are also various explicit embeddings. For example, the identity

$$6 \left( \sum_{i=1}^{4} x_i^2 \right)^2 = \sum_{1 \leq i < j \leq 4} (x_i + x_j)^4 + \sum_{1 \leq i < j \leq 4} (x_i - x_j)^4$$

defines an isometric embedding of $\ell_2^4$ into $\ell_4^2$. As far as we are aware, there are no known explicit embeddings (in the classical sense) or even algorithms, that apply to a wide class of spaces such as those with an unconditional or permutation invariant basis (which would be the two most natural cases to consider).

The purpose of this paper is to provide explicit $(1 + \varepsilon)$ embeddings, as in [11], of $\ell_2^n$ into a general normed space $(X, \|\cdot\|)$ with a permutation invariant basis, provided $n \leq c(\log \varepsilon^{-1})^{-1} \log N$ and we have some control over the basis constant. Our main result, Theorem 1, provides more details and complements (non-explicit) results of Bourgain and Lindenstrauss [3] and Tikhomirov [23], who studied symmetric spaces (spaces with a permutation invariant and 1-unconditional basis). The dependence on both $N$ and $\varepsilon$ that we provide is optimal.

## 2 Main result

For any $N \in \mathbb{N}$ let $S_N$ denote the permutation group on $N$ elements. Let $(X, \|\cdot\|)$ denote a real normed space of dimension $N$ with a basis $(e_i)_1^N$ such that the norm $\|\cdot\|$ is invariant under the action of $S_N$, i.e. for all $(a_i)_1^N \in \mathbb{R}^N$ and all $\sigma \in S_N$,

$$\left\| \sum_{i=1}^{N} a_{\sigma(i)} e_i \right\| = \left\| \sum_{i=1}^{N} a_i e_i \right\|$$

The basis constant of $(e_i)_1^N$ is defined as the smallest value of $K \geq 1$ such that for all $(a_i)_1^N \in \mathbb{R}^N$ and all $j \leq N$,

$$\left\| \sum_{i=1}^{j} a_i e_i \right\| \leq K \left\| \sum_{i=1}^{N} a_i e_i \right\|$$

2
Consider any \( n \in \mathbb{N} \), \( n \geq 6 \), and \( 0 < \varepsilon < (2K)^{-1} \). Set \( \delta = \varepsilon/1429 \), \( \sigma = \delta^{-4} \) and \( \alpha = 2\delta^{-4}(\log\delta^{-1})^{1/2} \). Let \( \Phi \) denote the standard normal cumulative distribution, and for each integer point \( x \in \mathbb{Z}^n \cap (\alpha\sqrt{nB_2^n}) \) define

\[
m(x) = \left[ N \prod_{i=1}^{n} \left( \Phi \left( \frac{x_i + 1/2}{\sigma} \right) - \Phi \left( \frac{x_i - 1/2}{\sigma} \right) \right) \right]
\]

\[
N' = \sum_{x \in \mathbb{Z}^n} m(x)
\]

\[
m'(x) = \begin{cases} m(x) & : x \neq 0 \\
                    m(x) + N - N' & : x = 0
\end{cases}
\]

Consider the \( N \times n \) matrix \( T \) defined as follows: For each \( x \in \mathbb{Z}^n \cap (\alpha\sqrt{nB_2^n}) \), repeat the vector \( x' = \sqrt{nx}/|x| \) a total of \( m'(x) \) times as a row of the matrix \( T \).

**Theorem 1** There exists a universal constant \( c > 0 \) with the following property. Consider any \( n, N \in \mathbb{N} \), \( K \geq 1 \), and \( 0 < \varepsilon < (2K)^{-1} \) such that

\[
6 \leq n \leq c(\log\varepsilon^{-1})^{-1}\log N
\]

Let \( (X, \|\cdot\|) \) be a real normed space of dimension \( N \) with a basis \( (e_i)_1^N \) of basis constant \( K \) that is invariant under permutations and let \( T \) be the \( N \times n \) matrix defined above. Then for all \( x \in \mathbb{R}^n \),

\[
(1 - K\varepsilon)M \left( \sum_{i=1}^{n} x_i^2 \right)^{1/2} \leq \left\| \sum_{i=1}^{N} \sum_{j=1}^{n} T_{i,j}x_je_i \right\| \leq (1 + K\varepsilon)M \left( \sum_{i=1}^{n} x_i^2 \right)^{1/2}
\]

where \( M = \|v\| \) and \( v \) is defined by (3). One can take \( c = 1/100 \), for example. To embed \( \ell_2^k \) into \( X \) for \( k < 6 \), simply take \( n = 6 \) and use the matrix \( \tilde{T} \) which consists of the first \( k \) columns of \( T \).

Let us briefly sketch the proof. Identify \( X \) with \( \mathbb{R}^N \) so that \( (e_i)_1^N \) become the standard basis vectors,

\[
e_i = (0,0\ldots,0,1,0\ldots,0)
\]

Let \( (\theta_i)_1^N \) denote the rows of \( T \). The empirical distribution of the sequence \( (\theta_i)_1^N \) is the probability measure on \( \mathbb{R}^n \) defined by

\[
\mu = \frac{1}{N} \sum_{i=1}^{N} \delta(\theta_i)
\]

where \( \delta(x) \) is the Dirac point mass at \( x \). This measure is a discrete approximation to normalized Haar measure on \( \sqrt{n}S^{n-1} \). This should be clear (at least in a rough sense) by (2) and is made precise in Lemma 8. Therefore for any \( \theta \in S^{n-1} \) the measure

\[
\mu_\theta = \text{Proj}_\theta\mu = \frac{1}{N} \sum_{i=1}^{N} \delta(\langle \theta, \theta_i \rangle)
\]
should approximate the standard normal distribution on \( \mathbb{R} \). In particular, the choice of \( \theta \) has minimal effect on \( \mu_\theta \). Since the norm \( \| \cdot \| \) is invariant under permutations, the quantity
\[
\| T \theta \| = \left\| \left( \langle \theta, \theta_i \rangle \right)_{i=1}^N \right\|
\]
depends only on \( \mu_\theta \), and does not oscillate much on \( S^{n-1} \). The result then follows by homogeneity. More details are given in Section 4.

3 Background and preliminaries

The symbols \( B_2^n \) and \( S^{n-1} \) denote the unit ball and sphere in \( \mathbb{R}^n \). Let \( \phi \) and \( \Phi \) denote the standard normal density and cumulative distribution,
\[
\phi(t) = \frac{1}{\sqrt{2\pi}} \exp \left( -t^2/2 \right)
\]
\[
\Phi(t) = \int_{-\infty}^{t} \phi(u) du
\]
Let \( X = (X_i)_{i=1}^n \) denote a random vector with the standard multivariate normal distribution and for each \( t \in \mathbb{R} \) define
\[
\Phi_n(t) = \mathbb{P} \left\{ \frac{\sqrt{n}X_1}{(\sum_1^n X_i^2)^{1/2}} \leq t \right\}
\]
Note that by uniqueness of Haar measure, \( \sqrt{n}X/|X| \) is uniformly distributed on \( \sqrt{n}S^{n-1} \), and for all \( \theta \in S^{n-1} \),
\[
\mathbb{P} \left\{ \langle \theta, \sqrt{n}X/|X| \rangle \leq t \right\} = \Phi_n(t)
\]
The corresponding density function, which is supported on \( [-\sqrt{n}, \sqrt{n}] \), can be written as
\[
\phi_n(t) = \frac{(n-1)\text{vol}_{n-1}(B_2^{n-1})}{n^{3/2}\text{vol}_n(B_2^n)} \left( 1 - \frac{t^2}{n} \right)^{(n-3)/2}
\]
Setting
\[
\lambda_n = \frac{(n-1)\text{vol}_{n-1}(B_2^{n-1})}{n^{3/2}\text{vol}_n(B_2^n)}
\]
we have the following representations
\[
\lambda_n = \frac{(n-1)\Gamma(1+n/2)}{n^{3/2}\sqrt{\pi}\Gamma(1/2+n/2)} = \left( \int_{-\sqrt{n}}^{\sqrt{n}} \left( 1 - \frac{t^2}{n} \right)^{(n-3)/2} dt \right)^{-1}
\]
The first one follows from the formula \( \text{vol}_n(B_2^n) = \pi^{n/2}/\Gamma(1+n/2) \) (see eg. [13]) while the second follows from the fact that \( \int \phi_n = 1 \). Using the inequality \( \pi - x \leq \exp(-x) \), we
see that $\lambda_n \geq 1/\sqrt{4\pi}$. By log-convexity of $\Gamma$, one can show that $\lambda_n \leq 1/\sqrt{2\pi}$. It should be clear (for at least two different reasons) that

$$\lim_{n \to \infty} \phi_n(t) = \phi(t)$$

$$\lim_{n \to \infty} \Phi_n(t) = \Phi(t)$$

This is an old result, see for example the discussion in [4], and also implies that $\lim_{n \to \infty} \lambda_n = 1/\sqrt{2\pi}$.

**Lemma 2** For all $t \geq \sqrt{n/(n-4)}$,

$$\frac{n}{2(n-3)t} \left(1 - \frac{t^2}{n}\right) \phi_n(t) \leq 1 - \Phi_n(t) \leq \frac{n}{(n-3)t} \left(1 - \frac{t^2}{n}\right) \phi_n(t)$$

**Proof.** The function $\phi_n$ is convex on $[(n/(n-4))^{1/2}, \infty)$ and log-concave on $\mathbb{R}$, i.e. log $\phi_n(x)$ is concave. The function is therefore sandwiched, on the interval $[t, \infty)$, between an affine function (below) and an exponential function (above). In particular, for all $x \geq t$,

$$\frac{-(n-3)t\phi_n(t)}{n(1-t^2/n)}(x-t) + \phi_n(t) \leq \phi_n(x) \leq \phi_n(t) \exp\left(-\frac{(n-3)(x-t)t}{n(1-t^2/n)}\right)$$

and the result follows because

$$1 - \Phi_n(t) = \int_t^\infty \phi_n(u)du$$

Taking $n \to \infty$ in Lemma 2 we recover the standard estimate for $1 - \Phi(t)$. The following lemma is well known in the more general setting of log-concave functions and we include its short proofs for completeness.

**Lemma 3** The function $\psi_n = \phi_n \circ \Phi_n^{-1} : (0, 1) \to (0, \infty)$ is concave.

**Proof.** Note that $g_n(t) = -\ln \phi_n(t)$ is convex. By the inverse function theorem and the chain rule,

$$\psi'_n(t) = \frac{\phi'_n(\Phi_n^{-1}(t))}{\phi_n(\Phi_n^{-1}(t))} = -g'_n(\Phi_n^{-1}(t))$$

Hence

$$\psi''_n(t) = -\frac{g''_n(\Phi_n^{-1}(t))}{\phi_n(\Phi_n^{-1}(t))} < 0$$

**Lemma 4** For all $0 < a < b < 1/2$,

$$|\Phi_n^{-1}(b) - \Phi_n^{-1}(a)| \leq \sqrt{\pi} \ln(b/a)$$
Proof. Consider \( \psi_n \) as in Lemma 3 and define \( h_n : (0, \infty) \to \mathbb{R} \) by \( h_n(x) = \Phi_n^{-1}(e^{-x}) \). Note that \( \psi_n \) is positive, concave, symmetric about \( x = 1/2 \), and

\[
\lim_{x \to 0} \psi_n(x) = \lim_{x \to 1} \psi_n(x) = 0
\]

Therefore \( \psi(x) \geq 2\phi_n(0)x = 2\lambda_n x \geq x/\sqrt{\pi} \) for all \( x \in (0, 1/2) \). By the inverse function theorem,

\[
h'_n(x) = \frac{-e^{-x}}{\phi_n(\Phi_n^{-1}(e^{-x}))}
\]

and \( |h'_n(x)| \leq \sqrt{\pi} \) for all \( x > \ln(2) \). Hence \( h_n \) is \( \sqrt{\pi} \)-Lipschitz on \( (\ln(2), \infty) \) and the result follows. \( \blacksquare \)

Lemma 5 There exists a sequence \( (\omega_n)_1^\infty \) with \( 0 < \omega_n < 1 \) and \( \lim_{n \to \infty} \omega_n = 1 \) such that for all \( n \in \mathbb{N} \),

\[
\text{vol}_n(B_n^2) = \left( \frac{\sqrt{2\pi e\omega_n}}{n} \right)^n
\]

Proof. This follows from the expression \( \text{vol}_n(B_n^2) = \pi^{n/2}(\Gamma(1+n/2))^{-1} \) and Sterling’s approximation. See for example Corollary 2.20 in [14]. \( \blacksquare \)

The following bound is well known, without dependence on \( n \).

Lemma 6 Let \( X \) be a random vector in \( \mathbb{R}^n \) with the standard multivariate normal distribution. Then for all \( t \geq 2\sqrt{n} \),

\[
\frac{1}{2} \leq \frac{\mathbb{P}\{|X| > t\}}{n\text{vol}_n(B_n^2)/(2\pi)^{-n/2}t^{-n} \exp(-t^2/2)} \leq \frac{4}{3}
\]

Proof. By polar integration with respect to surface area measure on \( S^{n-1} \) with \( \mu_n(S^{n-1}) = n\text{vol}_n(B_n^2) \),

\[
\mathbb{P}\{|X| > t\} = \int_{t}^{\infty} \int_{S^{n-1}} x^{n-1}(2\pi)^{-n/2} \exp(-x^2/2) d\mu_n(\omega) dx
\]

\[
= n(2\pi)^{-n/2}\text{vol}_n(B_n^2) \int_{t}^{\infty} x^{n-1} \exp(-x^2/2) dx
\]

On one hand, by the limiting case of Lemma 2

\[
\int_{t}^{\infty} x^{n-1} \exp(-x^2/2) dx \geq t^{n-1} \int_{t}^{\infty} \exp(-x^2/2) dx \geq \frac{1}{2} t^{n-2} \exp(-t^2/2)
\]

By convexity on the other hand (or basic algebra), \( tx - t^2/2 \leq x^2/2 \) for all \( x \in \mathbb{R} \). Using this together with the substitution \( u = tx - t^2 \), the inequality \( 1 + z \leq \exp(z) \) valid for all \( z \in \mathbb{R} \), and the fact that (by assumption) \( n/t^2 \leq 1/4 \),

\[
\int_{t}^{\infty} x^{n-1} \exp(-x^2/2) dx \leq \exp(t^2/2) \int_{t}^{\infty} x^{n-1} \exp(-tx) dx
\]

\[
\leq t^{n-2} \exp(-t^2/2) \int_{0}^{\infty} \exp(-3u/4) du
\]

\( \blacksquare \)

We shall also make use of the following classical result.
Theorem 7 (Hoeffding [7]) Let \((\gamma_i)_{i=1}^n\) be independent random variables with \(a_i \leq \gamma_i \leq b_i\). Then for all \(t > 0\),
\[
\mathbb{P}\left\{ \left| \frac{1}{n} \sum_{i=1}^{n} \gamma_i - \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \gamma_i \right] \right| \geq t \right\} \leq 2 \exp \left( \frac{-2t^2}{\sum_{i=1}^{n} (b_i - a_i)^2} \right)
\]

4 Main proof

Consider the cumulative distribution function
\[
F_\theta(t) = \mu_\theta ((-\infty, t]) = \frac{1}{N} |\{ i \leq N : (\theta, \theta_i) \leq t \}|
\]
and the quantile function \(F_\theta^{-1} : (0, 1) \rightarrow \mathbb{R}\) defined by
\[
F_\theta^{-1}(s) = \inf \{ t \in \mathbb{R} : F_\theta(t) \geq s \} = \sup \{ t \in \mathbb{R} : F_\theta(t) < s \}
\]
This is well defined even though \(F_\theta\) is not injective and does not have an inverse in the classical sense. Set \(a = \Phi_n (1.5)\) and \(b = \Phi_n ((1 - 17\delta)\sqrt{n})\). In Section 5 we prove the following lemma.

Lemma 8 Consider any \(\theta \in S_n^{n-1}\). Then,
\[
|F_\theta^{-1}(s) - \Phi_n^{-1}(s)| \leq \begin{cases} 
20\delta \Phi_n^{-1}(s) : & (1 - b) \leq s \leq (1 - a) \\
7\delta : & (1 - a) < s < a \\
20\delta \Phi_n^{-1}(s) : & a \leq s \leq b
\end{cases}
\]
and for \(s \in (0, 1 - b) \cup (b, 1)\), \(|F_\theta^{-1}(s) - \sqrt{n}| \leq 29\delta \sqrt{n}\).

Proof of Theorem [1] Consider any \(\theta \in S_n^{n-1}\). Let \((u_i)_{i=1}^N\) be the order statistics (non-decreasing rearrangement) of the sequence \(((\theta, \theta_i))_{i=1}^N\). Note that for all \((i - 1)/N < s \leq i/N\), \(F_\theta^{-1}(s) = u_i\). Define \(v \in \mathbb{R}^N\) by
\[
v_i = \begin{cases} 
-\sqrt{n} : & 0 \leq i - 1/2 < (1 - b)N \\
\Phi_n^{-1}((i - 1/2)/N) : & (1 - b)N \leq i - 1/2 \leq bN \\
\sqrt{n} : & bN < i - 1/2 \leq N
\end{cases}
\]
(3)

By Lemma 8
\[
|u_i - v_i| \leq \begin{cases} 
29\delta |v_i| : & 0 \leq i - 1/2 \leq (1 - a)N \\
7\delta : & (1 - a)N < i - 1/2 < aN \\
29\delta |v_i| : & aN \leq i - 1/2 \leq N
\end{cases}
\]
By the triangle inequality,
\[
|\|u\| - \|v\|| \leq \|u - v\|
\]
Define \(w, y \in \mathbb{R}^N\) by
\[
w_i = \begin{cases} 
u_i - v_i : & 0 \leq i - 1/2 < (1 - a)N \\
0 : & (1 - a)N \leq i - 1/2 \leq aN \\
u_i - v_i : & aN < i - 1/2 \leq N
\end{cases}
\]
\[ y_i = \begin{cases} 
0 & : \ 0 \leq i - 1/2 < (1-a)N \\
u_i - v_i & : (1-a)N \leq i - 1/2 \leq aN \\
0 & : aN < i - 1/2 \leq N
\end{cases} \]

It follows from this definition that \( y \) has at most \((2a-1)N + 1\) nonzero coordinates. Let \( y' \in \mathbb{R}^N \) be defined by

\[ y'_i = y_{\rho(i)} \]

where \( \rho \in S_N \) is any permutation such that if \( y'_i = 0 \) and \( |i - (N+1)/2| > |j - (N+1)/2| \), then \( y'_j = 0 \) i.e. we have moved the nonzero coordinates of \( y \) to the right and left hand tails of the vector (as a string of coordinates) and placed the zero’s in the middle. Then for all \( i \), if \( y'_i \neq 0 \) then \( |i - (N+1)/2| > (1-a)N/2 \) so \( |v_i| \geq \Phi_n^{-1} (0.5 + (1-a)/2) > 5 \times 10^{-3} \) and

\[ |y'_i| \leq 1400 \delta |v_i| \]

Here we have used the inequality \( 1 - x \geq \exp(-2e^2x/(e^2-1)) \) valid for all \( 0 \leq x \leq 1-e^{-2} \), which implies that

\[ a = 1 - \lambda_n \int_{1.5}^{\sqrt{n}} \left( 1 - \frac{t^2}{n} \right)^{(n-3)/2} dt \leq 1 - \frac{1}{\sqrt{4\pi}} \int_{1.5}^{2e^{-1}\sqrt{e^2-1}} \exp \left(- \frac{e^2t^2}{2(e^2-1)} \right) dt \leq 0.996 \]

and

\[ \Phi_n^{-1} (0.5 + (1-a)/2) \geq \Phi_n^{-1} (0.502) \geq 2 \times 10^{-3} / \lambda_n \]

This implies that

\[ \|u - v\| = \|w + y\| \leq \|w\| + \|y\| \leq 29 \delta K \|v\| + 1400 \delta K \|v\| \]

The result now follows with \( M = \|v\| \). ■

5 Analysis of the point cloud

Let \( X = (X_t)_1^n \) be a random vector in \( \mathbb{R}^n \) with the standard multivariate normal distribution (i.e. mean zero and identity covariance matrix). Let \([t]\) denote the closest integer function of \( t \in \mathbb{R} \), where we round closer to zero if \( t \) is midway between two integers. Let \( Y = (Y_t)_1^n \) where \( Y_t = [\sigma X_t] \), and let \( Z_t = Y_t - \sigma X_t \). Consider the random vector \( Q \) defined by

\[ Q = \sqrt{n}|Y|^{-1}Y \cdot 1_{\{Y \neq 0\} \cup \{|Y| \leq \alpha \sqrt{n}\}} \]

where \( 1_{\{Y \neq 0\} \cup \{|Y| \leq \alpha \sqrt{n}\}} \) is the indicator function of the event \( \{Y \neq 0\} \cup \{|Y| \leq \alpha \sqrt{n}\} \). Let \( \theta \in S^{n-1} \) denote an arbitrary unit vector.

Lemma 9 For all \( 1 \leq t \leq (1 - 3\delta)\sqrt{n} \),

\[ (1 - \delta)(1 - \Phi_n((1 + 3\delta)t)) \leq \mathbb{P} \left\{ \sum_{i=1}^{n} \theta_i Q_i > t \right\} \leq (1 + \delta)(1 - \Phi_n((1 - 2\delta)t)) \] (4)
Proof. By the union bound,
\[
\Pr \left\{ \sum_{i=1}^{n} \theta_i Q_i \leq t \right\} \leq \Pr \left\{ \sqrt{n} |Y|^{-1} \sum_{i=1}^{n} \theta_i Y_i \leq t \right\} + \Pr \{ Y = 0 \} + \Pr \{ |Y| > \alpha \sqrt{n} \}
\]

By the triangle inequality $||\sigma X| - |Y|| \leq |Z| \leq \sqrt{n}/2$ and by the definitions $\alpha = 2\delta^{-4}(\log \delta^{-1})^{1/2}$ and $\sigma = \delta^{-4}$, we have $(\alpha - 1/2)/\sigma \geq 2$. By Lemma 6 which bounds $|X|$ and Lemma 5 on vol$_{n}(B_2^n)$, we have
\[
\Pr \{ |Y| > \alpha \sqrt{n} \} \\
\leq \Pr \{ |X| > \sigma^{-1}(\alpha - 1/2)\sqrt{n} \} \\
\leq 2n\text{vol}_n(B_2^n)(2\pi)^{-n/2}(\alpha^{-1}(\alpha - 1/2)\sqrt{n})^{n-2} \exp \left( -1/2(\alpha^{-1}(\alpha - 1/2)\sqrt{n})^2 \right) \\
\leq 2e^{n/2}\sigma^{-n+2}(\alpha - 1/2)^{-n+2} \exp(-1/2\sigma^{-2}(\alpha - 1/2)^2n) \\
\leq 2^{n-1}e^{n/2} \left( \log \delta^{-1} \right)^{(n-2)/2} \delta^{10n/10} \\
\leq e^{n/2}\delta^n
\]

Since $\|\phi\|_{\infty} \leq (2\pi)^{-1/2}$, $\Pr \{ Y = 0 \} \leq (2\pi)^{-n/2}\delta^{4n}$. Using the union bound again, as well as the equations $Y = \sigma X + Z$ and $\sigma = \delta^{-4}$,
\[
\Pr \left\{ \frac{\sqrt{n}}{|Y|} \sum_{i=1}^{n} \theta_i Y_i \leq t \right\} \\
\leq \Pr \left\{ \frac{|\sigma X|}{|Y|} \frac{\sqrt{n}}{|X|} \sum_{i=1}^{n} \theta_i X_i \leq (1 + \delta)t \right\} + \Pr \left\{ \frac{|\sigma X|}{|Y|} \frac{\sqrt{n}}{|\sigma X|} \sum_{i=1}^{n} \theta_i Z_i \geq \delta t \right\} \\
\leq \Pr \left\{ \frac{\sqrt{n}}{|X|} \sum_{i=1}^{n} \theta_i X_i \leq (1 + \delta)^2 t \right\} + \Pr \left\{ \frac{|\sigma X|}{|Y|} < (1 + \delta)^{-1} \right\} + \Pr \left\{ \sum_{i=1}^{n} \theta_i Z_i > \delta^{-2}t/2 \right\} + \Pr \left\{ \frac{|\sigma X|}{|Y|} > (1 - \delta)^{-1} \right\} + \Pr \left\{ \frac{\sqrt{n}}{|X|} > \delta^{-1} \right\}
\]

As noted in Section 3
\[
\Pr \left\{ \frac{\sqrt{n}}{|X|} \sum_{i=1}^{n} \theta_i X_i \leq (1 + \delta)^2 t \right\} = \Phi_n \left( (1 + \delta)^2 t \right)
\]

By Hoeffding’s inequality,
\[
\Pr \left\{ \sum_{i=1}^{n} \theta_i Z_i \geq \delta^{-2}t/2 \right\} \leq 2 \exp(-\delta^{-4}t^2/2)
\]

and $\Pr \{ |Z| \geq \delta^{-1}\sqrt{n} \} \leq 2 \exp(-\delta^{-4}n)$. By Lemma 5 on vol$_{n}(B_2^n)$,
\[
\Pr \{ |X| \leq \delta \sqrt{n} \} \leq (2\pi)^{-n/2}\text{vol}_n (\delta \sqrt{n} B_2^n) \leq e^{n/2}\delta^n
\]
This implies, via the inequality \(|\sigma X| - |Y| \leq |Z|\), that
\[
P \{ (1 - \delta) |\sigma X| \leq |Y| \leq (1 + \delta) |\sigma X| \} \geq 1 - 2e^{n/2}\delta^n
\]

It follows that
\[
P \left\{ \sqrt{n}|Y|^{-1} \sum_{i=1}^{n} \theta_i Q_i \leq t \right\} \leq \Phi_n((1 + \delta)^2t) + 5e^{n/2}\delta^n + 2 \exp(-\delta^{-4}t^2/2)
\]

By Lemma 2
\[
\delta (1 - \Phi_n((1 + \delta)^2t)) \geq \delta (1 - \Phi_n((1 - \delta)\sqrt{n})) \geq \frac{1}{5\sqrt{2\pi}n}\delta^{(n+1)/2} \geq 10e^{n/2}\delta^n
\]

Our remaining task is to show that \(4\exp(-\delta^{-4}t^2/2) \leq \delta (1 - \Phi_n((1 + \delta)^2t))\). Using Lemma 2 again, it suffices to have
\[
4 \exp(-\delta^{-4}t^2/2) \leq \frac{\delta}{6t\sqrt{\pi}} \left( 1 - \frac{(1 + \delta)^4t^2}{n} \right)^{(n-1)/2} \tag{5}
\]

We now consider two cases. In Case 1, \(t \geq \delta \sqrt{n}\). Since \((1 + \delta)^4t^2 \leq (1 - \delta)n\), (5) is implied by
\[
\frac{1}{2}\delta^{-4}t^2 \geq \log(24\sqrt{\pi}) + \log t + \frac{n+1}{2} \log \delta^{-1}
\]
which holds by the assumption of Case 1. In Case 2, \(t < \delta \sqrt{n}\). Using the fact that \(\log(1 - x)^{-1} \leq 2x\) whenever \(0 < x < 1/2\), a sufficient condition for (5) to hold is that
\[
\delta^{-4}t^2 \geq 2\log(24\sqrt{\pi}) + 2\log t + 2\log \delta^{-1} + 3t^2
\]
which is true by the bounds imposed on \(t\) and \(\delta\). This proves the left hand inequality in (4). The right hand inequality follows similar lines. ■

**Lemma 10** For all \(0 \leq t \leq 2\),
\[
\Phi_n(t - 5\delta) \leq P \left\{ \sum_{i=1}^{n} \theta_i Q_i \leq t \right\} \leq \Phi_n(t + 5\delta)
\]
The lower bound follows by similar reasoning, and using an upper bound on the growth rate of $\Phi_n$ (via the mean value theorem),

\[
\Pr \left\{ \sum_{i=1}^{n} \theta_i Q_i \leq t \right\} 
\leq \Pr \left\{ \sqrt{n}|Y|^{-1} \sum_{i=1}^{n} \theta_i Y_i \leq t \right\} + \Pr \left\{ Y = 0 \right\} + \Pr \left\{ |Y| > \alpha \sqrt{n} \right\}
\leq \Pr \left\{ \frac{\sigma X}{|Y|} \sqrt{n} \sum_{i=1}^{n} \theta_i X_i \leq t + \delta^2 \right\} + \Pr \left\{ \frac{\sigma X}{|Y|} \left| \sum_{i=1}^{n} \theta_i Z_i \right| > \delta \right\} + 2e^{n/2}\delta^n
\leq \Pr \left\{ \frac{\sqrt{n}}{|X|} \sum_{i=1}^{n} \theta_i X_i \leq (1 + \delta)(t + \delta^2) \right\} + \Pr \left\{ \frac{\sigma X}{|Y|} < (1 + \delta)^{-1} \right\} + 2e^{n/2}\delta^n
\leq \Phi_n((1 + \delta)(t + \delta^2)) + 6e^{n/2}\delta^n + 2 \exp(-\delta^{-2}) \leq \Phi_n(t + 5\delta)
\]

The lower bound follows by similar reasoning, and using an upper bound on the growth rate of $\Phi_n$. ■

**Lemma 11** For all $0 \leq t \leq 2$,

\[
\Phi_n(t - 6\delta) \leq F_\theta(t) \leq \Phi_n(t + 6\delta)
\]

and for all $1 \leq t \leq (1 - 3\delta)\sqrt{n}$,

\[
(1 - 4\delta)(1 - \Phi_n((1 + 3\delta)t)) \leq 1 - F_\theta(t) \leq (1 + 4\delta)(1 - \Phi_n((1 - 2\delta)t))
\]

**Proof.** Consider the distribution of $Y \cdot 1_{\{|Y| \leq \alpha \sqrt{n}\}}$ and the empirical distribution of the sequence $(x_i)_1^n$, where each integer point $x$ is repeated $m'(x)$ times in the sequence,

\[
\eta = \frac{1}{N} \sum_{i=1}^{N} \delta(x_i)
\]

These two probability measures are both discrete measures supported on $\mathbb{Z}^n \cap (\alpha \sqrt{n}B_2^n)$ that we now compare. For each nonzero $x \in \mathbb{Z}^n \cap (\alpha \sqrt{n}B_2^n)$ we have, by definition of $m'(\cdot)$ and $Y$,

\[
|N \cdot \Pr \{ Y = x \} - m'(x)| \leq 1
\]

as well as,

\[
\Pr \{ Y = x \} \geq (2\pi)^{-n/2}\sigma^{-n} \exp \left( -1/2\sigma^{-2}(\alpha + 1/2)^2n \right)
= \exp \left[ -\frac{1}{2} (\sigma^{-2}(\alpha + 1/2)^2 + \log 2\pi + \log \sigma^2) n \right]
\]
By the bounds imposed on $N$, 
\[
N \geq \delta^{-1} \sigma \exp \left[ \frac{1}{2} \left( \sigma^{-2} (\alpha + 1/2)^2 + \log 2\pi + \log \sigma^2 \right) n \right]
\]  
(6)

This implies that 
\[
(1 - \delta \sigma^{-1}) \mathbb{P} \{ Y = x \} \leq \eta(\{x\}) \leq (1 + \delta \sigma^{-1}) \mathbb{P} \{ Y = x \}
\]

Consequently, for any Borel set $E \subset \mathbb{R}^n$ with $0 \notin E$, 
\[
(1 - \delta \sigma^{-1}) \mathbb{P} \{ Y \in E \cap \alpha \sqrt{n} B_2^n \} \leq \eta(E) \leq (1 + \delta \sigma^{-1}) \mathbb{P} \{ Y \in E \cap \alpha \sqrt{n} B_2^n \}
\]

Hence, the same is true when we take the radial projections of these measures onto $\sqrt{n}S_{n-1}$, i.e.  
\[
(1 - \delta \sigma^{-1}) \mathbb{P} \{ Q \in E \} \leq \mu(E) \leq (1 + \delta \sigma^{-1}) \mathbb{P} \{ Q \in E \}
\]

Lastly, 
\[
|\mu(\{0\}) - \mathbb{P} \{ Q = 0 \}| = |\mu(\mathbb{R}^n \setminus \{0\}) - \mathbb{P} \{ Q \neq 0 \}| \leq \delta \sigma^{-1} \mathbb{P} \{ Q \neq 0 \} \leq \delta \sigma^{-1}
\]

The result now follows from Lemma 10 (using properties of $\Phi_n$ as before) and Lemma 9.

**Lemma 12** For all $1 \leq t \leq (1 - 3\delta) \sqrt{n}$, 
\[
\Phi_n((1 - 10\delta)t) \leq F_{\theta}(t) \leq \Phi_n((1 + 12\delta)t)
\]

**Proof.** By Lemma 4 and Lemma 11, as well as the equation $\Phi_n(-x) = 1 - \Phi_n(x)$ and the fact that $t \geq 1$, 
\[
1 - F_{\theta}(t) \leq \Phi_n \circ \Phi_n^{-1} \left[ (1 + 4\delta)(1 - \Phi_n((1 - 2\delta)t)) \right] \\
\leq \Phi_n \left[ \sqrt{n} \ln(1 + 4\delta) - t(1 - 2\delta) \right] \\
\leq 1 - \Phi_n(t(1 - 2\delta) - 11\delta)
\]

The other side of the inequality follows similarly, using the fact that $\ln(1 - 4\delta) \geq -5\delta$.

**Proof of Lemma 8.** Without loss of generality, we may assume that $s \geq 1/2$. Consider 3 cases. In Case 1, 
\[
\Phi_n((1 - 17\delta) \sqrt{n}) < s < 1
\]

In this case, using Lemma 12, 
\[
F_{\theta} \left( (1 - 29\delta) \sqrt{n} \right) \leq \Phi_n((1 + 12\delta)(1 - 29\delta) \sqrt{n}) < s
\]

which implies that $F_{\theta}^{-1}(s) \geq (1 - 29\delta) \sqrt{n}$. On the other hand, since $\mu$ is supported on $\sqrt{n}S_{n-1}$, $F_{\theta}^{-1}(s) \leq \sqrt{n}$. In Case 2, 
\[
\Phi_n(1.5) \leq s \leq \Phi_n((1 - 17\delta) \sqrt{n})
\]

and it follows by Lemma 12 and the fact that $\Phi_n$ is strictly increasing on $[-\sqrt{n}, \sqrt{n}]$ that 
\[
F_{\theta} \left( (1 + 16\delta)^{-1} \Phi_n^{-1}(s) \right) < s \leq F_{\theta} \left( (1 - 14\delta)^{-1} \Phi_n^{-1}(s) \right)
\]

which implies $(1 - 16\delta) \Phi_n^{-1}(s) \leq F_{\theta}^{-1}(s) \leq (1 + 20\delta) \Phi_n^{-1}(s)$. In Case 3, $1/2 \leq s < \Phi_n(1.5)$ and it follows by Lemma 11 that 
\[
F_{\theta} \left( \Phi_n^{-1}(s) - 7\delta \right) < s < F_{\theta} \left( \Phi_n^{-1}(s) + 7\delta \right)
\]

and that $\Phi_n^{-1}(s) - 7\delta < F_{\theta}^{-1}(s) < \Phi_n^{-1}(s) + 7\delta$. ■
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