NOTE ON FOURIER EXPANSIONS ATcusps

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Abstract. This was originally an appendix to our paper ‘Fourier expansions at cusps’ [1]. The purpose of this note is to give a proof of a theorem of Shimura on the action of \(\text{Aut}(\mathbb{C})\) on modular forms for \(\Gamma(N)\) from the perspective of algebraic modular forms. As the theorem is well-known, we do not intend to publish this note but want to keep it available as a preprint.

In this note we give a new proof of the following theorem which is originally due to Shimura, see [5, Theorem 8] and [6, Lemma 10.5]. It gives the interaction between the \(\text{SL}_2(\mathbb{Z})\)-action and the \(\text{Aut}(\mathbb{C})\)-action on spaces of modular forms on the group \(\Gamma(N)\). These actions on a modular form \(f(\tau) = \sum_n a_n e^{2\pi i n \tau/w}\) of weight \(k \geq 1\) are defined as follows:

\[
(f|g)(\tau) = \frac{1}{(C\tau + D)^k} f\left(\frac{A\tau + B}{C\tau + D}\right), \quad f^\sigma(\tau) = \sum_n \sigma(a_n)e^{2\pi i n \tau/w}
\]

for \(g = (\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}) \in \text{SL}_2(\mathbb{Z})\) and \(\sigma \in \text{Aut}(\mathbb{C})\). For any integer \(N \geq 1\) we denote \(\zeta_N = e^{2\pi i /N}\).

Theorem 1. [5, 6] Let \(f \in M_k(\Gamma(N))\) be a modular form of weight \(k \geq 1\) on \(\Gamma(N)\) and \(g\) and \(\sigma\) be as above such that \(\sigma(\zeta_N) = \zeta_N^\lambda\) with \(\lambda \in (\mathbb{Z}/N\mathbb{Z})^\times\). Then

\[(f|g)^\sigma = f^\sigma|g_\lambda,\]

where \(g_\lambda\) is any lift in \(\text{SL}_2(\mathbb{Z})\) of the matrix \(\left(\begin{smallmatrix} A & \lambda B \\ C & D \end{smallmatrix}\right) \in \text{SL}_2(\mathbb{Z}/N\mathbb{Z})\).

This theorem immediately implies that if a modular form \(f\) of level \(N\) has Fourier coefficients in a field \(K_f\), then the Fourier coefficients of \(f|\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)\) for any \(\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \text{SL}_2(\mathbb{Z})\) will lie in \(K_f(\zeta_N)\). In [1] we obtain improved results if \(f\) is a modular form for \(\Gamma_0(N)\) or \(\Gamma_1(N)\) and in the case of newforms on \(\Gamma_0(N)\), we determine the number field generated by the Fourier coefficients of \(f|\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)\) explicitly.

We recall the theory of algebraic modular forms, in order to give a new proof of Theorem 1. For more details on this theory, see [3, Chap. II] and the references therein.

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Definition 2. Let $R$ be an arbitrary commutative ring, and let $N \geq 1$ be an integer. A test object of level $N$ over $R$ is a triple $T = (E, \omega, \beta)$ where $E/R$ is an elliptic curve, $\omega \in \Omega^1(E/R)$ is a nowhere vanishing invariant differential, and $\beta$ is a level $N$ structure on $E/R$, that is an isomorphism of $R$-group schemes

$$\beta : (\mu_N)_R \times (\mathbb{Z}/N\mathbb{Z})_R \xrightarrow{\cong} E[N]$$

satisfying $e_N(\beta(\zeta, 0), (1, 1)) = \zeta$ for every $\zeta \in (\mu_N)_R$. Here $(\mu_N)_R = \text{Spec} \mathbb{Z}[t]/(t^N - 1)$ is the scheme of $N$-th roots of unity, and $e_N$ is the Weil pairing on $E[N]$.

If $\phi : R \to R'$ is a ring morphism, we denote by $T_{R'} = (E_{R'}, \omega_{R'}, \beta_{R'})$ the base change of $T$ to $R'$ along $\phi$.

The isomorphism classes of test objects over $\mathbb{C}$ are in bijection with the set of lattices $L$ in $\mathbb{C}$ endowed with a symplectic basis of $\frac{1}{N}L/L$ [3, §8]. Another example is given by the Tate curve $Tate(q) = \mathbb{G}_m/q^Z$ [2, §8]. It is an elliptic curve over $\mathbb{Z}((q))$ endowed with the canonical differential $\omega_{\text{can}} = dx/x$ and the level $N$ structure $\beta_{\text{can}}(\zeta, n) = \zeta q^{n/N} \mod q^Z$. The test object $(Tate(q), \omega_{\text{can}}, \beta_{\text{can}})$ is defined over $\mathbb{Z}((q^{1/N}))$.

Definition 3. An algebraic modular form of weight $k \in \mathbb{Z}$ and level $N$ over $R$ is the data, for each $R$-algebra $R'$, of a function

$$F = F_{R'} : \{\text{isomorphism classes of test objects of level } N \text{ over } R'\} \to R'$$

satisfying the following properties:

1. $F(E, \lambda^{-1}\omega, \beta) = \lambda^k F(E, \omega, \beta)$ for every $\lambda \in (R')^\times$;
2. $F$ is compatible with base change: for every morphism of $R$-algebras $\psi : R' \to R''$ and for every test object $T$ of level $N$ over $R'$, we have $F_{R''}(T_{R''}) = \psi(F_{R'}(T))$.

We denote by $M_k^{alg}(\Gamma(N); R)$ the $R$-module of algebraic modular forms of weight $k$ and level $N$ over $R$.

Evaluating at the Tate curve provides an injective $R$-linear map

$$M_k^{alg}(\Gamma(N); R) \hookrightarrow \mathbb{Z}((q^{1/N})) \otimes_\mathbb{Z} R$$

called the $q$-expansion map. The $q$-expansion principle states that if $R'$ is a subring of $R$, then an algebraic modular form $F \in M_k^{alg}(\Gamma(N); R)$ belongs to $M_k^{alg}(\Gamma(N); R')$ if and only if the $q$-expansion of $F$ has coefficients in $R'$.

Algebraic modular forms are related to classical modular forms as follows. To any algebraic modular form $F \in M_k^{alg}(\Gamma(N); \mathbb{C})$, we associate the function $F^{an} : \mathcal{H} \to \mathbb{C}$ defined by

$$F^{an}(\tau) = F\left(\frac{\mathbb{C}}{2\pi i\mathbb{Z} + 2\pi i\tau\mathbb{Z}}, dz, \beta_{\tau}\right)$$

with $\beta_{\tau}(\zeta^n, n) := [2\pi i(m + n\tau)/N]$.

Proposition 4. The map $F \mapsto F^{an}$ induces an isomorphism between $M_k^{alg}(\Gamma(N); \mathbb{C})$ and the space $M'_k(\Gamma(N))$ of weakly holomorphic modular forms on $\Gamma(N)$ (that is, holomorphic on $\mathcal{H}$ and meromorphic at the cusps). Moreover, the $q$-expansion of $F$ coincides with that of $F^{an}$.

\footnote{Our definition of the Weil pairing is the reciprocal of Silverman’s definition [7, III.8]. With our definition, we have $e_N(1/N, \tau/N) = e^{2\pi i/N}$ on the elliptic curve $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ with $\text{Im}(\tau) > 0$.}
We now interpret the action of $\text{SL}_2(\mathbb{Z})$ on modular forms in algebraic terms. Let $F \in M_k^{\text{alg}}(\Gamma(N); \mathbb{C})$ with $f = F^{\text{an}}$, and let $g = (a/b, c/d) \in \text{SL}_2(\mathbb{Z})$. A simple computation shows that

$$
(f|kg)(\tau) = F\left(\frac{c}{2\pi i(\tau + \tau z)}, dz, \beta'_r\right)
$$

where the level $N$ structure $\beta'_r$ is given by

$$
\beta'_r(\zeta^m, n) = \beta_r(\zeta^{md+nb}, mc + na).
$$

Let $\psi : (\mathbb{Z}/N\mathbb{Z})^2 \rightarrow \mu_N(\mathbb{C}) \times \mathbb{Z}/N\mathbb{Z}$ be the isomorphism defined by $\psi(a, b) = (\zeta_N^a, a)$. Let us identify the level structure $\beta_r$ (resp. $\beta'_r$) with the map $\alpha_r = \beta_r \circ \psi$ (resp. $\alpha'_r = \beta'_r \circ \psi$). Then (4) shows that

$$
\alpha'_r(a, b) = \alpha_r((a, b)g).
$$

What we have here is the right action of $\text{SL}_2(\mathbb{Z})$ on the row space $(\mathbb{Z}/N\mathbb{Z})^2$, which induces a left action on the set of level $N$ structures. As we will see, all this makes sense algebraically.

For any $\mathbb{Z}[\zeta_N]$-algebra $R$, we denote by $\zeta_{N,R}$ the image of $\zeta_N = e^{2\pi i/N}$ under the structural morphism $\mathbb{Z}[\zeta_N] \rightarrow R$.

**Lemma 5.** If $R$ is a $\mathbb{Z}[\zeta_N, 1/N]$-algebra, then there is an isomorphism of $R$-group schemes $(\mathbb{Z}/N\mathbb{Z})_R \cong (\mu_N)_R$ sending $1$ to $\zeta_{N,R}$.

**Proof.** Note that $(\mu_N)_R = \text{Spec } R[t]/(t^N - 1) = \text{Spec } R[\mathbb{Z}/N\mathbb{Z}]$ and $(\mathbb{Z}/N\mathbb{Z})_R = \text{Spec } R[\mathbb{Z}/N\mathbb{Z}]$. If $R = \mathbb{C}$, then $\mathbb{C}[\mathbb{Z}/N\mathbb{Z}] \cong \mathbb{C}[\mathbb{Z}/N\mathbb{Z}]$ because all irreducible representations of $\mathbb{Z}/N\mathbb{Z}$ have dimension 1. This isomorphism $\mathcal{F}_C$ is given by the Fourier transform, and both $\mathcal{F}_C$ and $\mathcal{F}_C^{-1}$ have coefficients in $\mathbb{Z}[\zeta_N, 1/N]$ with respect to the natural bases. It follows that in general $R[\mathbb{Z}/N\mathbb{Z}] \cong R[\mathbb{Z}/N\mathbb{Z}]$ and this isomorphism sends $[1]$ to $(\zeta^a_{N,R})_{a \in \mathbb{Z}/N\mathbb{Z}}$. \hfill \square

Let $R$ be a $\mathbb{Z}[\zeta_N, 1/N]$-algebra. We have an isomorphism of $R$-group schemes

$$
\psi_R : (\mathbb{Z}/N\mathbb{Z})_R^2 \rightarrow (\mu_N)_R \times (\mathbb{Z}/N\mathbb{Z})_R
$$

given by $\psi_R(a, b) = (\zeta_N^a, a)$. The group $\text{SL}_2(\mathbb{Z})$ acts from the right on the row space $(\mathbb{Z}/N\mathbb{Z})_R^2$ by $R$-automorphisms, and for $\alpha : (\mathbb{Z}/N\mathbb{Z})_R^2 \rightarrow E[N]$ we define

$$
(\alpha \cdot g)(a, b) = \alpha((a, b)g) \quad ((a, b) \in (\mathbb{Z}/N\mathbb{Z})^2).
$$

Using $\psi_R$, we transport this to a left action of $\text{SL}_2(\mathbb{Z})$ on the set of level $N$ structures of an elliptic curve over $R$. Given a test object $T = (E, \omega, \beta)$ over $R$, we define $g \cdot T := (E, \omega, g \cdot \beta)$. For any $F \in M_k^{\text{alg}}(\Gamma(N); R)$, we define $F|g \in M_k^{\text{alg}}(\Gamma(N); R)$ by the rule $(F|g)(T) = F(g \cdot T)$ for any test object $T$ over any $R$-algebra $R'$. The computation (3) then shows that the right action of $\text{SL}_2(\mathbb{Z})$ on $M_k^{\text{alg}}(\Gamma(N); \mathbb{C})$ corresponds to the usual slash action on $M_k^{\text{alg}}(\Gamma(N))$.

**Remark 6.** The action of $\text{SL}_2(\mathbb{Z})$ on algebraic modular forms over $\mathbb{Z}[\zeta_N, 1/N]$-algebras has the following consequence: if a classical modular form $f \in M_k(\Gamma(N))$ has Fourier coefficients in some subring $A$ of $\mathbb{C}$, then for any $g \in \text{SL}_2(\mathbb{Z})$, the Fourier expansion of $f|g$ lies in $\mathbb{Z}[|q^{1/N}|] \otimes A[\zeta_N, 1/N]$. We now interpret the action of $\text{Aut}(\mathbb{C})$ in algebraic terms (see [4, p. 88]). Let $\sigma \in \text{Aut}(\mathbb{C})$. For any $\mathbb{C}$-algebra $R$, we define $R^{\sigma} := R \otimes_{\mathbb{C}, \sigma^{-1}} \mathbb{C}$, which means that $(ax) \otimes 1 = x \otimes \sigma^{-1}(a)$ for all $a \in \mathbb{C}$, $x \in R$. We endow $R^{\sigma}$ with the structure of a $\mathbb{C}$-algebra using the map

$$
(\sigma \cdot g)(a, b) = \sigma((a, b)g).
$$
\(a \in C \mapsto 1 \otimes a \in R^\sigma\). We denote by \(\phi_\sigma : R \to R^\sigma\) the map defined by \(\phi_\sigma(x) = x \otimes 1\). The map \(\phi_\sigma\) is a ring isomorphism, but one should be careful that \(\phi_\sigma\) is not a morphism of \(\mathbb{C}\)-algebras, as it is only \(\sigma^{-1}\)-linear. For any test object \(T\) over \(R\), we denote by \(T^\sigma\) its base change to \(R^\sigma\) using the ring morphism \(\phi_\sigma\).

Let \(F \in M_k^{\text{alg}}(\Gamma(N); \mathbb{C})\) be an algebraic modular form. For any \(\mathbb{C}\)-algebra \(R\), we define

\[
F^\sigma_R : \{\text{isomorphism classes of test objects of level } N \text{ over } R\} \to R
\]

\[
T \mapsto \phi_\sigma^{-1}(F_{R^\sigma}(T^\sigma)).
\]

One may check that the collection of functions \(F^\sigma_R\) satisfies the conditions (1) and (2) above, hence defines an algebraic modular form \(F^\sigma \in M_k^{\text{alg}}(\Gamma(N); \mathbb{C})\). Moreover, since the Tate curve is defined over \(\mathbb{Z}((q))\), one may check that the map \(F \mapsto F^\sigma\) corresponds to the usual action of \(\text{Aut}(\mathbb{C})\) on the Fourier expansions of modular forms: for every \(F \in M_k^{\text{alg}}(\Gamma(N); \mathbb{C})\) and every \(\sigma \in \text{Aut}(\mathbb{C})\), we have \((F^\sigma)^{\text{an}} = (F^{\text{an}})^\sigma\).

We finally come to the proof of Theorem 1.

**Proof.** Let \(f \in M_k(\Gamma(N))\) with corresponding algebraic modular form \(F \in M_k^{\text{alg}}(\Gamma(N); \mathbb{C})\). Let \(g \in \text{SL}_2(\mathbb{Z})\) and \(\sigma \in \text{Aut}(\mathbb{C})\). We take as test object \(T = (\text{Tate}(q), \omega_{\text{can}}, \beta_{\text{can}})\) over \(R = \mathbb{Z}((q^{1/N})) \otimes \mathbb{C}\). Since a modular form is determined by its Fourier expansion, and unravelling the definitions of \(F\) and \(F^\sigma\), it suffices to check that the test objects \(g \cdot T^\sigma\) and \((g_\lambda \cdot T)^\sigma\) over \(R^\sigma\) are isomorphic. Since \(\text{SL}_2(\mathbb{Z}/N\mathbb{Z})\) acts only on the level structures of the test objects, we have to show that

\[
\phi_\sigma^{-1}(F_{R^\sigma}(T^\sigma)) = (g \cdot \beta_{\text{can}})^\sigma = (g_\lambda \cdot \beta_{\text{can}})^\sigma.
\]

For any scheme \(X\) over \(R\), let \(X^\sigma\) denote its base change to \(R^\sigma\) along \(\phi_\sigma\). Since \(\phi_\sigma\) is a ring isomorphism, the canonical projection map \(X^\sigma \to X\) is an isomorphism of schemes, and we also denote by \(\phi_\sigma : X \to X^\sigma\) the inverse map.

Put \(E = \text{Tate}(q)\) and \(\beta = \beta_{\text{can}}\). Let \(\alpha = \beta \circ \psi_R : (\mathbb{Z}/N\mathbb{Z})^2_R \xrightarrow{\cong} E[N]\). By functoriality, the level structure \(\beta^\sigma\) is given by the following commutative diagram

\[
\begin{array}{ccc}
(Z/N\mathbb{Z})^2_R & \xrightarrow{\psi_R} & (\mu_N)_R \times (Z/N\mathbb{Z})_R \\
\gamma \downarrow \quad & \quad & \quad \downarrow \phi_\sigma \\
(Z/N\mathbb{Z})^2_{R^\sigma} & \xrightarrow{\psi_{R^\sigma}} & (\mu_N)_{R^\sigma} \times (Z/N\mathbb{Z})_{R^\sigma} \\
\end{array}
\]

Let us compute the dotted arrow \(\gamma\). Since \(\phi_\sigma\) is \(\sigma^{-1}\)-linear, we have \(\phi_\sigma(\zeta_{N,R}) = \zeta_{N,R^\sigma}^{\lambda^{-1}}\). It follows that

\[
\phi_\sigma(\psi_R(a,b)) = \phi_\sigma(\zeta_{N,R}^{\lambda^{-1}} a) = (\zeta_{N,R^\sigma}^{\lambda^{-1}} b) = \psi_{R^\sigma}(a, \lambda^{-1}b)
\]

so that \(\gamma(a,b) = (a, \lambda^{-1}b)\). We may thus express \(a^\sigma\) in terms of \(a\) by

\[
(10) \quad a^\sigma(a,b) = \phi_\sigma \circ \alpha \circ \gamma^{-1}(a,b) = \phi_\sigma \circ \alpha(a, \lambda b) = \phi_\sigma \circ \alpha \left(\begin{array}{c} a \\ \lambda b \end{array} \right).
\]
Let us make explicit both sides of (7). By (6) and (10), the left hand side is given by

\[(g \cdot \alpha^\sigma)(a, b) = \alpha^\sigma((a, b)g) = \phi_\sigma \circ \alpha \left( (a, b)g \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \right).\]

Let us now turn to the right hand side of (7). By (6), we have \((g_\lambda \cdot \alpha)(a, b) = \alpha((a, b)g_\lambda)\). Applying the commutative diagram (8) with \(\alpha\) replaced by \(g_\lambda \cdot \alpha\), we get

\[(g_\lambda \cdot \alpha)^\sigma(a, b) = \phi_\sigma \circ (g_\lambda \cdot \alpha) \left( (a, b) \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \right) = \phi_\sigma \circ \alpha \left( (a, b) \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} g_\lambda \right).\]

Finally, we note that \(g \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} g_\lambda\).

\[\square\]

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