Criticality of two-dimensional disordered Dirac fermions in the unitary class and universality of the integer quantum Hall transition

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Two-dimensional (2D) Dirac fermions are a central paradigm of modern condensed matter physics, describing low-energy excitations in graphene, in certain classes of superconductors, and on surfaces of 3D topological insulators. At zero energy $E = 0$, Dirac fermions with mass $m$ are band insulators, with the Chern number jumping by unity at $m = 0$. This observation motivated Ludwig et al. [Phys. Rev. B 50, 7326 (1994)] to suggest a relation between 2D disordered Dirac fermions (DDF) and the integer quantum Hall transition (IQHT). They conjectured that the transitions in both systems are controlled by the same fixed point and possess the same universal critical properties. Given the far reaching implications for modern condensed matter physics and our understanding of disordered critical points in general, it is surprising that the above conjecture has never been tested numerically. Here, we report the results of extensive numerics on criticality and energy-mass phase diagram of 2D-DDF in the unitary symmetry class. We find a critical line at $m = 0$, with energy dependent localization length exponent. At large energies, our results for the DDF are consistent with state-of-the-art numerical results $\nu_{\text{IQH}} = 2.56 \pm 0.62$ from models of the IQHT. At $E = 0$ however, we obtain $\nu_0 = 2.30 \pm 0.36$ incompatible with $v_{\text{IQH}}$. Our result has practical importance for a variety of experimental systems but also challenges conjectured relations between models of the IQHT.

Introduction. The integer quantum Hall effect appears when a two-dimensional (2D) electron gas is placed in a strong perpendicular magnetic field. Without disorder, the electron eigenstates form Landau levels, and each filled level contributes unity to the total Chern number $C$. Disorder is essential for experimental observation of the (dimensionless) quantized Hall conductivity $\sigma_{xy} = C$. Disorder broadens the Landau levels into bands and localizes eigenstates on a scale $\xi(E)$ that diverges in a power law fashion at a critical energy $E_c$ [1]:

$$\xi(E) \sim |E - E_c|^{-\nu_{\text{IQH}}}.$$  

For Fermi energies $E \neq E_c$ and system sizes $L \gg \xi(E)$ the Hall conductivity is quantized. The integer quantum Hall transition (IQHT) at $E = E_c$ is the most studied Anderson transition [2] because of its conceptual simplicity, low dimensionality, and relevance to experiments. However, critical properties at the IQHT are notoriously difficult to compute analytically; they are mostly known from numerical studies which employed the Chalker-Coddington (CC) network model [3–10], microscopic continuous [11–13], lattice [14–17], and Floquet Hamiltonians [18]. In recent works, the critical properties agree between models, indicating universality of the IQHT. They include the localization length exponent $\nu_{\text{IQH}} = 2.56 \pm 2.62$ and the leading irrelevant exponent $y \simeq 0.4$ (with large error bars). At criticality, $y$ describes the approach of the dimensionless quasi-1D Lyapunov exponent $\Gamma$ to its limiting value at infinite system size $\Gamma_0^{\text{IQH}} = 0.77 \pm 0.3$ [9]. A similar exponent $y$ was found for the average conductance $\overline{\gamma}$ of a square sample with limiting value $\overline{\gamma}_{\text{IQH}} = 0.58 \pm 0.62$ [19–20]. For ongoing analytical work on the IQHT, see [21] and the discussion below. The IQHT has also been discussed recently in the context of exotic topological superconductor surface states [22].

A longstanding conjecture by Ludwig et al. [23] states that the IQHT fixed point also controls the criticality of 2D disordered Dirac fermions (DDF) in the unitary symmetry class [24]. The clean Dirac Hamiltonian is

$$H_0 = \hbar v (-i\sigma_x \partial_x - i\sigma_y \partial_y) + m\sigma_z,$$

with $2 \times 2$ Pauli matrices $\sigma_\mu$, mass $m$, and velocity $v$. The spectrum of $H_0$ has a gap of size $2|m|$ symmetric around $E = 0$. For Fermi energies $E$ within the gap, the system is a band insulator with a half-integer quantized $\sigma_{xy} = C(m) = -\frac{1}{2}\text{sgn}(m)$ [25], see Fig. 1(a). If the

![FIG. 1. Schematic phase diagram for massive 2D Dirac fermions. In the clean case (a), a metal intervenes between two band insulators with different Chern numbers $C$ at $|m| > |E|$. With disorder in the unitary class (b), the critical line $m = 0$ separates Anderson insulators. The localization length exponent $\nu$ at $E = 0$ is significantly different from the one at $E > 0$ where it is compatible with the value established for the IQHT. The dashed lines indicate the values $E = 0.3, 0.7$ which we used in the simulations of the model.](https://arxiv.org/abs/2008.09025v1)
Dirac fermion is regularized on a lattice as in the Haldane model \cite{28} or Eq. \cite{7} below, the Hamiltonian \cite{2} only describes the low-energy excitations near a certain point in the Brillouin zone. Bloch states elsewhere contribute another 1/2 to C, such that $|\sigma_{xy}|$ jumps between zero and one as $m$ changes sign.

With $m$ taking the role of energy, the superficial similarity of this transition to the IQHT motivated Ludwig et al. \cite{25} to study effects of disorder on the Dirac Hamiltonian, placing the model in the unitary symmetry class, 

$$H = H_0 + \sum_{\mu=0,x,y,z} U_\mu(x,y) \sigma_\mu.$$  

The random scalar ($U_0$) and vector ($U_{x,y}$) potentials, and the random part of the mass ($U_z$) are taken to be independent Gaussian fields with the correlators $U_\mu(r)U_\nu(r') = \delta_{\mu\nu}K_\mu(|r'-r|)$ and zero mean. We expect that for $m \neq 0$, the eigenstates or $L$ are localized with the localization length $\xi(m) \sim |m|^{-\nu_E}$, c.f. Eq. \cite{1}, with a possibly $E$-dependent critical exponent.

Although model \text{(3)} is not solvable analytically, the conjecture $\nu_E = \nu_{\text{IQH}}$ is consistent with several plausible arguments which we discuss below. However, despite the importance of 2D Dirac Hamiltonians in modern condensed matter physics, the conjectured emergence of IQHT criticality in DDF was never checked numerically. In what follows, we address this issue with extensive numerical simulations employing different microscopic models and scaling observables. We start with the continuum model \text{(3)} and use the transfer matrix (TM) approach in quasi-1D (q1D) geometry to find the critical behavior near the line $m = 0$ in the $m$-$E$ plane, see Fig. \text{1(b)}. At large $E$ our results are consistent with $\nu_E = \nu_{\text{IQH}}$, but the critical exponent at zero energy $\nu_{E=0} = 2.30\pm 0.36$ is close to, but strikingly incompatible with $\nu_{\text{IQH}}$. We corroborate our results in a lattice model of DDF, employing an alternative 2D scaling observable \cite{29}.

**Continuum model and disorder induced length scale.**

We start with Hamiltonian \text{(3)} at $E = 0$ and smooth disorder with correlation length $\alpha$,

$$K_\mu(r) = W^2 e^{-r^2/2\alpha^2}/2\pi.$$  

We use $a$ and $\hbar v/a$ as units of length and energy. Then the disorder strength $W$, taken to be the same for all four disorder fields, is dimensionless and is the bare energy scale in the model. The mean free path $l_W$ equals the quasiparticle decay time, $l_W \equiv -1/Im \Sigma_{\uparrow\uparrow}(0,0)$ defined in terms of the disorder-averaged Green function $G(k,\omega) = [\omega - H_0(k) - \Sigma(k,\omega)]^{-1}$. For weak disorder $W \ll 1$, a perturbative renormalization group (RG) \cite{25,30} gives, for $m = 0$, $l_W \propto e^{-W^2}$, with $c = O(1)$. To ensure that our system sizes $L \gg l_W$, we work with strong disorder $W \geq 1.5$ where a numerically exact method \cite{31} yields $l_W^{\text{exact}} = 1.5$. We also observe that for $kl_W > 1$, the peaks in the spectral function $A(k,\omega) = -\frac{1}{\pi}Im G(k,\omega)$ occur at frequencies $\omega \simeq \pm hvk$, i.e. the velocity $v$ is almost un-renormalized. We conclude that for $W = 1.5$, system sizes $L \gtrsim O(10)$ are large enough to exhibit disorder-dominated physics.

Lyapunov exponent (LE). A common method to analyze critical behavior in disordered systems employs the self-averaging of the LEs $\gamma_i$ of q1D samples of width $L_y$ and length $L_x \to \infty$ \cite{32}. The smallest $\gamma_i > 0$ (the inverse of the 1D localization length) gives the scaling variable $\Gamma = \gamma L_y$, which increases (decreases) with $L_y$ in a localized (extended) phase and is scale-invariant at a critical point. Following Ref. \cite{33}, we use finite $L_x = O(10^3)$ and find $\Gamma$ as the average over $\gtrsim 200$ disorder realizations, see the supplemental material (SM) for details.

The eigenvalue problem for the DDF \text{(3)} can be rewritten as $\partial_x \psi(x, k_y) = f(\psi(x, k_y))$. The right hand side contains scattering between transversal wavevectors $k_y$ but is local in $x$, which allows us to express the TM in exponential form. We impose periodic boundary conditions (BC) in the $y$ direction. We discretize the $x$ direction and stabilize the TM multiplication by repeated QR-decompositions \cite{11} (to obtain $\Gamma$) or in a scattering matrix formalism \cite{34} (for the conductance of moderately sized systems). Both methods are numerically exact and faithfully treat a single Dirac node without band bending or node doubling, and maintain the statistical symmetry $m \to -m$. The only approximations are related to the cutoff $|k_y| \leq k_{\text{max}}$ and the $x$-discretization. The associated length scales (taken equal) were chosen much smaller than $\alpha$, and the results are converged with respect to these parameters.

Results for the dimensionless LE $\Gamma$ at $E = 0$, $W = 1.5$, various masses $m$ and system widths $L_y$ are presented in Fig. \text{2}. The solid lines are fits to the scaling function

$$\Gamma(m, L_y) = \Gamma_0 + \alpha_1 L_y^{-\nu} + \alpha_2 m^2 L_y^{\nu'/2},$$  

which is the lowest-order polynomial ansatz allowed by symmetry, including an irrelevant contribution. The fit gives the following critical properties:

$$\nu_{E=0} = 2.32(1), \quad y = 0.51(3), \quad \Gamma_0 = 0.84(1),$$  

the number in parentheses denotes one standard deviation. In the SM, we give a detailed account for the fitting procedure and show its stability with respect to higher order terms in Eq. \text{5} and a removal of data points of large $m$ and small $L_y$. There, we also present data for an increased disorder strength $W = 2.0$, which yields $\nu_{E=0} = 2.31(2), y = 0.51(3)$ and $\Gamma_0 = 0.84(1)$ compatible with anticipated disorder-independent critical properties.

**Lattice model and alternative scaling observable.**

We now confirm the value of $\nu_{E=0}$ using a square-lattice regularization of the DDF allowing access to an alternative scaling observable introduced by Fulga et al. \cite{35}. In momentum space, the clean model reads \cite{36}

$$H_0^L = \sigma_x \sin k_x + \sigma_y \sin k_y + \sigma_z (m - 2 + \cos k_x + \cos k_y),$$

(7)
where the lattice spacing and the energy scale have been set to unity. For $|k| \ll 1$, this model reduces to the Dirac Hamiltonian \( H \), with a topological transition at \( m = 0 \) where the Chern number \( c \) changes by 1. We add on-site disorder potentials, \( V = \sum_{x,\mu} U_{\mu}(r_x) \sigma_{\mu} \) with \( U_{\mu}(r_x) \) uniformly drawn from the interval \([-w/2, w/2]\) independently for each lattice site \( r_x \) and \( \mu = 0, x, y, z \). Transport calculations use the Kwant software package \(^{37}\) and employ two extended leads in the \( x \)-direction, described by the Hamiltonian

\[
H_{\text{Lead}}(k_x, k_y) = \sigma_x \sin k_x + \sigma_z (1 + \cos k_x) .
\] (8)

The lattice model \(^{2}\) has no symmetry that ensures \( m_c = 0 \) in the presence of disorder. However, the Dirac node energy is not renormalized away from \( E = 0 \) due to the sum of non-trivial Pauli matrices in Eq. (7). Band bending effects are important for momenta and energies above unity. Transport simulations of a \( 200 \times 200 \) square system with periodic BC in the \( y \) direction and disorder strength \( w = 2.5 \) reveal a well defined conductance peak in a mass range of width \( \simeq 0.3 \) centered around \( m \simeq 0.1 \) (data not shown). Likewise, the conductance for open BC in the \( y \) direction shows a smooth transition from 0 to 1 within the same mass range, revealing the integrity of chiral edge states. This shows that the localization lengths for \( w = 2.5 \) are compatible with achievable system sizes for a numerically convenient range of \( m \). Thus, we use \( w = 2.5 \) for further analysis.

To determine the exponent \( \nu_{E=0} \), we consider the reflection matrix \( r(\phi) \) of the left lead as a function of the phase \( \phi \) of twisted BC in the \( y \) direction. For a given disorder realization, the critical value \( m = m_c \) occurs when there exist a \( \phi \) such that \( r(\phi) \) has a zero eigenvalue and \( \det r(\phi) = 0 \). Fulga et al. \(^{35}\) showed that a scaling observable \( \Lambda \) can be obtained by working with generalized twisted BC \( \psi_{x,y=L-1} = z \psi_{x,y=0} \) for all \( x = 0, 1, ..., L - 1 \), and \( z \in \mathbb{C} \). Now, \( \det r(z) \) has zeros \( z_0 \) even for \( m \neq m_c \) but with \( |z_0| \neq 1 \). Then for the \( z_0 \) closest to the unit circle we define \( \Lambda = \log|z_0| \) as a measure of distance to criticality \( \Lambda = 0 \). For the CC model, Fulga et al. demonstrated scaling of \( \Lambda \) with system size \( L \), giving \( \nu = 2.56(3) \) \(^{35}\) compatible with results from the TM method.

We computed \( \Lambda \) for the lattice DDF \( H_0^L + V \) for \( m \) around 0.1 and system sizes between \( L = 60 \) and 200, see Fig. 3 for the results and the SM for details of the fit. We find \( \nu_{E=0} = 2.33(3) \) in agreement with the result for the continuum model. Notably, the observable \( \Lambda \) shows no discernible corrections to scaling, which allows us to omit the irrelevant terms in the scaling function for \( \Lambda \). Repeating the analysis for \( w = 2.25 \) and 2.75 (not shown) yields compatible \( \nu \) within the given error bars.

**Results for finite energy \( (E > 0) \).** We now consider the continuum model \(^{3}\) with smooth disorder \(^{4}\) at finite energy \( E > 0 \) \((E < 0 \) is related by the statistical particle-hole symmetry). In the SM, we present scaling results for the LE \( \Gamma \) for \( E = 0.3 \) and \( E = 0.7 \) at disorder strength \( W = 2 \). As for the \( E = 0 \) case we find localizing behavior for any \( m \neq 0 \). The exponent \( \nu_{E=0} = 2.34(2) \) is slightly higher than \( \nu_{E=0} \). However, \( \nu_{E=0.7} = 2.53(2) \) differs significantly from \( \nu_{E=0} \) and is much closer to \( \nu_{\text{QH}} \). Other critical properties \( (\Gamma_0 \) and \( y ) \) for \( E > 0 \) are compatible with \( E = 0 \) results.

To further probe the critical line \( m = 0 \), we computed the critical distributions of the Landauer conductance \( g \) of \( L \times L \) systems with periodic BC in the \( y \) direction, and metallic leads modeled as highly doped Dirac nodes \(^{38}\), see SM for details at \( E = 0 \). Such distributions and their moments are expected to be scale-invariant and universal \(^{2,4}\). In Fig. 4 we present results for the

**FIG. 2.** Top: LEs \( \Gamma \) for \( E = 0 \) and \( W = 1.5 \) as functions of \( m^2 \). System widths are \( L_y = 40, 50, 60, 90, 102, 120, 136, 160 \) from small to large slope. The relative error is \( \lesssim 0.2\% \), the error bars are smaller than the dots. Solid lines denote the best fit [Eq. (5)] with fit parameters as given in the panels. Bottom: Closeup at criticality \((m = 0)\) with extrapolation to infinite system size determining \( \Gamma_0 \) (cross).

**FIG. 3.** Scaling plot of the variable \( \Lambda \) for the lattice model at energy \( E = 0 \) and disorder strength \( w = 2.5 \). Dots represent averages over at least \( 10^4 \) disorder realizations for system sizes \( L = 60, 80, 110, 150, 200 \) and the solid curves are results of the fitting procedure described in the SM.
average \( \bar{g} \). We observe that for \( E \lesssim 0.3, \bar{g} \approx 0.5 \) is almost independent of the disorder strength and \( E \), which we interpret as evidence of proximity to the underlying fixed point. With increasing \( L, \bar{g} \) slightly increases, consistent with decreasing \( \Gamma(m = 0) \) in Fig. 2 (bottom).

For \( 0.3 \lesssim E, \bar{g} \) begins to depend on \( W \), and varies with \( E \) by \( \sim 50\% \) for \( W = 1.5 \) but only by \( \sim 10\% \) for \( W = 2.5 \). For \( W = 1.5 \) and \( E > 0.6, \bar{g} \) slightly decreases when \( L \) grows form 100 to 200. We interpret this as a remnant of the crossover from the diffusive to the critical behavior. It is consistent that LEs obtained in this regime (not shown) cease to obey critical scaling.

**Discussion.** In summary, our numerical results for DDF are consistent with localized behavior anywhere in the mass-energy plane except on a critical line \( m = 0 \), see Fig. 1(b). At \( m = 0 \), both the dimensionless LE extrapolated to infinite system size \( \Gamma_0 = 0.82 - 0.85 \) and the irrelevant exponent \( y \) do not vary significantly with energy or disorder strength below \( E \approx 1 \), while the average conductance \( \bar{g} \) of fixed-size square samples at stronger disorder varies at most by \( \sim 10\% \).

In contrast, the localization length exponent \( \nu_E \) significantly depends on energy. While \( \nu_{E=0.7} = 2.53(2) \) is more or less consistent with the established value for the IQHT \( \nu_{\text{IQH}} = 2.56 - 2.62 \), the value \( \nu_{E=0} \) is significantly smaller. Taking a union over error bars for the two models and two scaling methods that we used, we obtain the following conservative estimate:

\[
\nu_{E=0} = 2.30 - 2.36. \quad (9)
\]

The value \( \nu_{E=0.3} = 2.34(2) \) is consistent with Eq. (9).

Let us now put our findings in the context of existing arguments and first discuss the case of large \( E \) and low \( W \) characterized by a large Drude conductivity \( \sigma_{xx}^D \gg 1 \). In the SM, we numerically confirm that this regime is achievable in the DDF, albeit not for the parameters used for the scaling analysis above. Large \( \sigma_{xx}^D \) controls the derivation of an effective field theory for the DDF with short-range disorder \( \sigma_{xx} \) as it justifies the required saddle point approximation. The resulting non-linear sigma model with \( \theta \)-term can also be derived for other models of the IQHT: the Schrödinger equation with short-range disorder and strong magnetic field \cite{10, 11}, and the CC network model \cite{42, 43}. These relations rationalize our finding of IQHT-like criticality in the DDF at \( E = 0.7 \).

Note, however, that the CC model lacks the large parameter analogous to \( \sigma_{xx}^D \) and the derivation of the sigma model for it is uncontrolled, as well as for the DDF at \( E \approx 0 \), where \( \sigma_{xx} < 1 \).

We now focus on \( E = 0 \) and discuss possible reasons for the inequality \( \nu_{E=0} \neq \nu_{\text{IQH}} \) in two scenarios.

(a) **Insufficient system size.** Given the long and tortuous history of numerical work on the IQHT, where refined fitting functions and the ability to study larger systems shifted the value of \( \nu \) considerably over time, we cannot exclude the scenario that \( \nu_{E=0} \) in Eq. (9) is not the true asymptotic value, and further increase in system sizes would lead to a crossover to \( \nu_{\text{IQH}} \). Indeed, taking the difference \( \Gamma(m = 0, L_{E, \text{max}}) - \Gamma_0 \) as a proxy for the distance to the critical point, its value at \( E = 0 \) is roughly three times larger than at \( E = 0.7 \).

However, our system sizes, quality of numerical data, and the data analysis are comparable to the ones in recent works on the IQHT. Also, we do not see a tendency for a drift in \( \nu_{E=0} \) if the minimal \( L_y \) involved in the fit is increased from 40 to 68, see SM. Finally, we corroborated our result \( (9) \) at a different disorder strength and using an alternative scaling observable in a lattice model regularization of the DDF. Our finding for \( \nu_{E=0} \) is also supported by numerical results on a massless DDF in a magnetic field \cite{44}. At strong enough potential disorder, only the critical state deriving from the Landau level at \( E = 0 \) persists, separating localized states at \( E \lesssim 0 \). The scaling of \( d\sigma_{xy}/dE|_{E=0} \) and the width of the conductance peak around \( E = 0 \) with system size \( L \) gave \( \nu \approx 2.3 \), but no error bars were provided.

(b) **Different fixed points.** In a more intriguing scenario our results can be explained if we conjecture the existence of two different fixed points. One of them is the conventional IQHT fixed point that controls the critical behavior of DDF at \( E > 0 \), while the other fixed point controls the system at \( m = 0, E = 0 \). We conjecture that this fixed point is multicritical, where both \( m \) and \( E \) are relevant, with the RG eigenvalues \( y_m = 1/\nu_{E=0} \) and \( y_E \). The RG flow near this point would resemble that near the tricritical point in the Ising model with vacancies \cite{45} or near the point \( \sigma_{xx} = 0, \sigma_{xy} = 1/2 \) in the Khmelnitskii-Pruisken two-parameter flow diagram \cite{46, 47}. In this scenario the critical behavior at any \( E > 0 \) should be the same, and coincide with that for the IQHT.

Our inability to see this at \( E = 0.3 \) may stem from the small (or even zero, if \( E \) is marginally relevant) value of the crossover exponent \( \phi = y_E/y_m \) at the multicritical point, resulting in the cusp-like shape of the crossover line in Fig. 1(b). However, this scenario cannot explain the apparent energy independence of \( \Gamma_0 \).

Finally, we review the standard (non-rigorous) argu-
ments for $\nu_E = 0 = \nu_{\text{QH}}$, and identify their possible flaws.

(i) The original argument of Ludwig et al. [25] assumes sufficiently smooth disorder in the DDF model. Then the low energy states are chiral Jackiw-Rebbi fermions moving along lines of zero mass $m + U_z(x, y) = 0$. Other disorders lead to random phases accumulated between saddle points in $U_z$, where scattering controlled by the value of $m$ occurs. This argument parallels the one that leads to the CC network model [19]. A possible flaw is that unlike in the large $B$-field case of the IQHT, non-chiral higher energy states might be important in DDF. Indeed, solving the Dirac equation for a linear domain wall $U_z(x) = cx$ arising from our smooth disorder potential with $c \sim W/a$, the first pair of counterpropagating edge modes appears at $E_1 \sim \pm \sqrt{W}$. Naively, it takes a disorder strength $W \gtrsim 1$ to occupy these modes by $U_0$ fluctuations, but their relevance for the network model is presently unclear. In addition, the effects of geometric disorder might further modify the above argument, see the discussion below.

(ii) Another argument in favor for the equivalence of the CC network and DDF was given by Ho and Chalker [50]. The authors showed that the spectrum of quasienergies of the clean CC model (viewed as a Floquet system) has a Dirac point with mass $m$ proportional to the deviation from the critical point. Then they argued that disorder in the CC model leads to all possible types of disorder in the Dirac model. However, the mapping assumes smooth disorder in the network model, and may be invalid for strong disorder.

In either case, our results raise the issue of universality of the IQHT, which was also challenged by two recent results. Refs. [51, 52] numerically studied the role of structural (geometric) disorder in the CC model and reported $\nu \approx 2.37(2)$. The drop in $\nu$ seems to indicate the relevance of geometric disorder, and is consistently reproduced by the alternative scaling method of Ref. [53]. In ongoing work [53]. In a different development, Zirnbauer [23] proposed a solvable conformal field theory for the IQHT, predicting $\nu = \infty$ and $y = 0$. If true, this would imply apparent critical exponents that show logarithmic flow with $L$, invalidate all existing numerical studies of the exponent $\nu$ at the IQHT, and exclude any possibility to find universal critical behavior numerically.

Outlook. We hope that our findings will prompt a careful re-examination of criticality at the IQHT and other Anderson transitions. For future work on critical DDF, it is interesting to numerically study the multifractal properties of wavefunctions and compare them to established results for the IQHT [2]. Further, extension of our methods to DDF in the symmetry classes of the spin and thermal quantum Hall effects is worthwhile.

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Supplemental material for “Criticality of two-dimensional disordered Dirac fermions in the unitary class and universality of the integer quantum Hall transition”

Quasi-1D Lyapunov exponents: Additional data, fitting procedure

Here, we give a detailed description of the ensemble of data sets, fitting procedure and results for the quasi-1D Lyapunov exponents Γ of the disordered Dirac fermion. Table I summarizes all data sets used in this study. The first two columns denote energy E and disorder strength W, respectively. The third column gives the length $L_x$ of the quasi-1D slabs, the widths are $L_y = 40, 50, 68, 90, 102, 120, 136, 160$ for all data sets. We use a linearly spaced grid of squared Dirac masses $m^2$ as given in the respective column. The minimal number of disorder realizations per tuple $(m, L_y)$ is shown in the next column. In Fig. S1 we show a typical histogram of the ensemble of finite-length Γ approximants, for $m^2 = 0.05, L_y = 102$ and $E = 0, W = 1.5$. Following Ref. [32], we confirm their gaussian distribution (red line) and approximate Γ as the mean, the error $\sigma_\Gamma$ is the standard deviation of this ensemble divided by the square root of the number of disorder realizations. The maximal error so obtained in given in the respective column of Table I. For a few representative data points we have increased $L_x$ by a factor of five and checked the Γ only vary within error bars, moreover note the factor of three between the two values for $L_x$ used for the two data sets at $E = 0$. The so obtained dimensionless Lyapunov exponents Γ are shown as dots in Fig. 2 of the main text ($E = 0, W = 1.5$) and in Figs. S2-S4 for the remaining $(E, W)$ parameter pairs of table I.

![Fig. S1. Histogram of finite-length Lyapunov exponents for $E = 0, W = 1.5$ at $L_y = 102$ and $m^2 = 0.05$, involving 1000 disorder realizations. The red line is a gaussian fit.](image)

We now consider the scaling ansatz for fitting the $\Gamma(m, L_y)$ data. Following Ref. [12], we take into account the relevant and the leading irrelevant scaling variables, $x_R(m)$ and $x_I(m)$, respectively:

$$\Gamma(m, L_y) = F \left( X_R = x_R(m) L_y^{1/\nu}, X_I = x_I(m) L_y^{-y} \right). \quad (S1)$$

The relevant variable vanishes at the critical point, $x_R(0) = 0$. To satisfy the (statistical) symmetry $m \to -m$, we follow the customary choice [12, 33] an expand $x_R(m)$ as an odd-power polynomial of order $R$, and $x_I(m)$ as an even-power polynomial of order $I$. Then the series expansion of Eq. (S1) must contain only even powers of $X_R$:

$$\Gamma(m, L_y) = \Gamma_0 + \alpha_{01} X_I + \alpha_{20} X_R^2 + \alpha_{02} X_I^2 + \alpha_{21} X_R^2 X_I + \ldots \quad (S2)$$

We fix the expansion orders $R, I$ and the truncation of the series in Eq. (S2) and find the parameters $\nu, y, \Gamma_0, \alpha_{ij}$ using non-linear least squares fitting. The quality of a fit is reported by the value of $\chi^2 = \chi^2/(N - N_c)$ where $N$ is the number of data points $\Gamma_j$ ($j$ stands for the tuple $(m, L_y)$), $N_c$ is the number of fitting parameters in the chosen scaling function, and $\chi^2 = \sum_{j=1}^N (\Gamma_j - \Gamma)^2 / \sigma_j^2$ is the sum of squared residues weighted by the variances $\sigma_j^2$. The errors of the fit parameters (one standard deviation) are obtained from repeated fitting of ~100 synthetic data sets generated from a normal distribution of mean $\Gamma_j$ and variance $\sigma_j^2$. The fit results, i.e. the optimized universal parameters of the ansatz with the overall lowest $\chi^2$ and all relative errors of fit parameters below 30%, are given in the last three columns of Table I and also in Fig. 2 and S2-S4. It turns out that for all data sets considered, the minimal ansatz which allows for an irrelevant contribution (first three terms on the rhs of Eq. S2 with $R = 1, I = 0$ and $N_c = 5$, given explicitly in Eq. (5) of the main text) is chosen by our fitting algorithm. As shown in the stability plots of Figs. S2 to S4 while a higher order fitting function can give a slightly smaller value of $\chi^2$, this always comes at the cost of a relative error exceeding our stability bound above. As a further stability criterion, we request that with one or two small system widths $L_y$ or a few largest masses $m$ removed from the fit, critical properties still overlap in error bars. This is also confirmed in Table I, see column titled “restriction”.
TABLE I. Overview of quasi-1D Lyapunov exponent data sets and fitting results for the best fitting function, Eq. (5) of the main text. The set of system widths is $L_y = 40, 50, 68, 90, 102, 120, 136, 160$ for all data sets unless restricted for a stability check.

| $E$ | $W$ | $L_y$ ($10^4$) | $m^2$ | min realizations | max $\sigma_l / \Gamma$ | restriction | $\nu$ | $\gamma$ | $\Gamma_0$ | $\chi^2$ | $N$ |
|-----|-----|----------------|------|------------------|----------------------|------------|------|-------|---------|-------|-----|
| 0.0 | 1.5 | 1              | 0.00000625,...,0.00625 | 400 | 0.2\% | none | 2.32(1) | 0.51(3) | 0.84(1) | 1.060 | 88 |
|     |     |                | $m^2 \leq 0.005$ | 2.32(1) | 0.54(2) | 0.85(1) | 0.996 | 72 |
|     |     |                | $L_y \geq 68$ | 2.32(1) | 0.39(9) | 0.82(4) | 0.978 | 66 |
| 0.0 | 2.0 | 3              | 0.00005,...,0.005 | 200 | 0.2\% | none | 2.31(2) | 0.51(3) | 0.84(1) | 0.960 | 88 |
|     |     |                | $m^2 \leq 0.004$ | 2.32(3) | 0.51(5) | 0.84(1) | 1.163 | 72 |
|     |     |                | $L_y \geq 68$ | 2.30(3) | 0.47(10) | 0.84(2) | 0.919 | 66 |
| 0.3 | 2.0 | 1              | 0.0000625,...,0.005 | 200 | 0.25\% | none | 2.32(2) | 0.50(5) | 0.84(1) | 0.940 | 72 |
|     |     |                | $m^2 \leq 0.004375$ | 2.34(3) | 0.42(6) | 0.82(2) | 0.943 | 66 |
| 0.7 | 2.0 | 1              | 0.000125,...,0.01 | 550 | 0.2\% | none | 2.34(2) | 0.51(5) | 0.85(1) | 0.833 | 56 |
|     |     |                | $m^2 \leq 0.0075$ | 2.35(4) | 0.57(10) | 0.83(1) | 1.001 | 56 |

FIG. S2. Quasi-1D Lyapunov exponents as in Fig. 2 of the main text, but for $E = 0$ and $W = 2.0$. Solid lines denote the best fit with the ansatz in Eq. (2) of the main text.

FIG. S3. Quasi-1D Lyapunov exponents as in Fig. 2 of the main text, but for $E = 0.3$ and $W = 2.0$. Solid lines denote the best fit with the ansatz in Eq. (2) of the main text.

Details for calculation and fitting of the alternative scaling observable

We provide details here of calculating the alternative scaling observable $\Lambda$ proposed by Fulga et al. [35]. For tight-binding models, the concept of virtual leads extending in $y$-direction is used to apply the generalized twisted transversal boundary conditions mentioned in the main text [35]. When extending the method to the DDF lattice model, and more generally any model with multiple orbitals per site, it is crucial to match the propagating modes in the virtual leads on top and bottom. With the kwant software package [37] this is not automatically ensured, but can be corrected through the following procedure.

With the PropagatingModes method of kwant, the propagating modes in the top and bottom leads can be compared. They are beyond the user’s control and are found to be different generically. We construct unitary matrices that rotate the, say, ingoing top lead modes to match the outgoing modes of the bottom lead, $\psi_{in}^{m} = \sum_{k} U_{nk} \tilde{\psi}_{k}$ and likewise for the outgoing modes of the top lead. Applying these transformations in the Schrödinger equation defining the scattering matrix $S$ returned by
FIG. S4. Quasi-1D Lyapunov exponents as in Fig. 2 of the main text, but for \(E = 0.7\) and \(W = 2.0\). Solid lines denote the best fit with the ansatz in Eq. (2) of the main text.

KWANT, we can transform to a scattering matrix \(\tilde{S} = (U_{\text{out}})^T S(U_{\text{in}})^\ast\) defined in a matching mode basis for top and bottom leads which can now be safely short-circuited including the \(z\)-factor as detailed in Ref. [35].

As explained in the main text, we find the \(\nu = \nu_E = 0.0, W = 1.5, L_y = 40 - 160, m^2 = 0.00625\)

FIG. S5. Stability plot showing the performance of different fitting functions for the quasi-1D Lyapunov exponents obtained for \(E = 0\) and \(W = 1.5\) with all available \((m, L_y)\) data points. The labels on the \(x\)-axis encode the chosen fitting function, c.f. Eq. (S2). The green color indicates the best fit for which the maximum relative error of all fit parameters is below 30%, see dashed line in the second panel. In the third panel, horizontal lines denote \(\nu = 2.32\) as obtained recently for the CC network model in Ref. [52] and our estimate \(\nu_{E=0} = 2.32\).

We finally remark that Fulga et al. [35] claim that their scaling observable requires samples of large aspect ratio. However, we find that calculations with square samples yield converging numerics and save considerable computational resources. We discern no underlying argument in favor of particular aspect ratios.
Landauer conductance for square system at $m = 0, E = 0$

Critical conductance distributions depend on the sample shape and boundary conditions, and are typically very broad [2, 3]. In addition, unlike the self-averaging Lyapunov exponents, critical transport observables exhibit multifractality: their different moments are described by different scaling dimensions [54–56].

In the main text, we have presented disorder averaged square system Landauer conductance $\overline{g}$ for $m = 0$ and a range of energies. Here, focusing on the case $E = 0$, we present the histogram of $\log g$, see Fig. S10 (top) for a range of system sizes $L = 28–280$. It is qualitatively similar to the IQHT case [20, 57].

In Fig. S10 (middle), we attempt a power law fit of the average conductance, using the ansatz $\overline{g}(L) = g - \alpha L^{-\overline{y}}$. It yields $\overline{y} = 0.18(6)$ and $\overline{g} = 0.60(4)$. We emphasize that despite the large number of disorder realizations, the fit is very instable and we were not able to find meaningful error bars for the data points given the broad and unknown probability distribution of $g$. The bottom panel presents a logarithmic fit $\overline{g}(L) = \overline{g}(\infty) - b \log(\lambda L)$ as has been suggested in literature as well [16].

Notice that due to the above-mentioned multifractality of conductances at criticality, we do not expect that the irrelevant exponent $\overline{y}$ describing the approach of $\overline{g}(L)$ to its limiting value is the same as the exponent $y$ for the self-averaging Lyapunov exponent $\Gamma$, which is related to the conductance of a strongly localized quasi-1D system.
In Ref. [39], Ostrovsky et al. derived Pruisken’s nonlinear sigma model as an effective long-range theory for the DDF. As emphasized in the main text, this derivation is only controlled if the longitudinal Drude conductivity $\sigma_{xx}^{\text{Drude}}$ is much larger than unity. In Fig. S11, we plot the numerically obtained Drude conductivity for the continuum DDF with smooth disorder at $m = 0$ for a range of energies $E$ and disorder strengths $W$, see caption for details of the procedure. In the metallic limit of large $E$ and small $W$, the data obeys $\sigma_{xx}^{\text{Drude}} \sim 1/W^2$ [39]. For the parameter combinations $(E, W)$ used for the scaling analysis above, the Drude conductivity in Fig. S11 is of order unity such that the derivation of Ref. [39] is not controlled. Note that this does not rule out the suggested phase diagram in Fig. 1(b) of the main text.
FIG. S10. Landauer conductance of square continuum Dirac systems, Eq. (3) of the main text, with $m = 0$, $E = 0$ and smooth disorder, Eq. (4), of strength $W = 2.5$. The top panel shows the histogram of $\log g$ for four system sizes between $L = 28$ and 280 based on between 80000 and 20000 disorder realizations. The middle panel shows the mean square conductance (with additional intermediate system sizes) with a fit (solid line) to the power-law $\overline{g}(L) = \overline{g}(\infty) - aL^{-y}$ with fit parameters given in the panel. The bottom panel shows the same data with a logarithmic fit $\overline{g}(L) = \overline{g}(\infty) - b\log(\lambda L)$. The solid line indicates a power law fit as described in the main text.
FIG. S11. Drude conductivity $\sigma_{xx}^\text{Drude}$ as a function of energy $E$ and disorder strength $W$ for the DDF continuum model with smooth disorder. The data is obtained from conductance simulations for samples of large aspect ratio $L_y \gg L_x$ as the inverse slope of resistance curves $\mathcal{R}(L_x) = \frac{1}{\mathcal{G}(L_x)}$ for $L_x$ in the range of a few mean free paths. The mean free path is estimated by comparison to the conductance data of the clean ballistic case $W = 0$. 

\hspace{1cm}