Mirror-time phase transition induced by non-Hermitian skin effect under magnetic field

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The discovery of non-Hermitian skin effect (NHSE) is one of the main recent advances in non-Hermitian physics. Yet, its interplay with natural or artificial gauge fields remains elusive, which is significant for its control and applications. Here, we uncover the intriguing effects induced by the competition between the NHSE and magnetic field. We find that a magnetic field can strongly suppress and thus effectively control the NHSE. Moreover, the Onsager-Lifshitz quantization rule persists against the NHSE in the long-wavelength limit, which preserves real Landau levels. A real-to-complex transition of the entire energy spectra can be induced by the spontaneous breaking of the underlying mirror-time reversal (MT) symmetry. We propose an order parameter to characterize the phase transition, which not only gives the phase boundary but also quantifies the MT-symmetry breaking. Our work paves the way towards the effective control of the NHSE and the exploration of the novel MT phase transition in non-Hermitian magnetic systems.

Non-Hermitian physics \[1-3\] has attracted growing research interest recently for its intriguing properties and potential applications that can be implemented in various physical systems, including photonic systems \[4-9\], open quantum systems coupled to the environment \[10-13\], and the engineered Laplacian in electrical circuits \[14\]. Quasiparticles with finite lifetimes in condensed matter offer another interesting platform for the exploration of non-Hermitian physics \[15, 16\]. Moreover, the interplay between non-Hermiticity and band topology has been studied extensively and achieved plentiful interesting results \[17\], such as the anomalous edge modes \[18, 19\], the enriched classification of the topological insulating and superconducting phases \[20, 21\], exotic semimetal phases \[22, 24\], and the topological lasing \[24, 26\].

It is now well accepted that the conventional bulk-boundary correspondence principle for the Hermitian system fails \[18, 19\] due to the so-called non-Hermitian skin effect (NHSE) \[20, 28\]. Instead, the band topology should be described by the non-Bloch band theory \[27\].

The NHSE is a unique phenomenon due to non-Hermiticity \[27, 34\], where all the bulk states are driven to the edge of the system under the open boundary condition (OBC). It originates from the point gap topology of the energy spectra under the periodic boundary condition (PBC) that is not respected by the OBC \[30, 32\]. In the simple case as discussed in this work, a persistent current flows inside the system under the PBC as shown in Figs. 1(a) and 1(b). Therefore, the NHSE can be regarded as a particular type of delocalization effect that can even drive the system from the localized phase to the delocalized one \[33, 37\]. The NHSE has been confirmed in recent experiments in a variety of physical systems \[38-41\]. Very recently, an intriguing interplay has been predicted between the NHSE and the parity-time (PT)-symmetric transition \[35, 47\], which opens up the possibility for manipulating the PT transition by the NHSE.

In contrast to the physical picture of the NHSE, a magnetic field gives rise to the opposite effect. The motion of free charged particles in a magnetic field forms cyclotron orbits with the guiding centers localized in space; see Fig. 1(c). The quantization of these cyclotron orbits results in dispersionless Landau levels with vanishing mobility. The high degeneracy of the Landau levels also indicates that the corresponding eigenstates distribute uniformly in space. Therefore, the physical effects caused by the magnetic field seem completely incompatible with those due to the NHSE. Given that magnetic fields and the NHSE may coexist in a variety of natural and artificial systems \[48-51\], it is of importance to study the fascinating interplay and competition between these two effects.

In this work, we investigate the combined effects of NHSE and magnetic field on the nonreciprocal square lattice and honeycomb lattice with unequal hopping in the x-direction, as shown in Figs. 1(a) and 1(b), whose low-energy effective models correspond to the normal particle [Fig. 1(c)] and massless Dirac particle [Fig. 1(d)], respectively. Several interesting physical results have been obtained. We show that a small magnetic field is sufficient to drive the skin modes from the boundary into the bulk, leading to a suppression of the NHSE [cf. Fig. 2]. The inverse action of the NHSE on the whole magnetic energy spectra strongly relies on the boundary condition in the x-direction, making them remain entirely real under the OBC but become partially complex under the PBC. Interestingly, the low-energy spectra are always real, regardless of the boundary conditions, and the Onsager-Lifshitz quantization rule hold true in this regime in spite of the NHSE. More importantly, we show that the real-to-complex phase transition of the entire energy spectra stems from the spontaneous breaking of the inherent mirror-time reversal (MT) symmetry; see
FIG. 1. Schematic illustration of (a) nonreciprocal Harper-Hofstadter model and (b) nonreciprocal honeycomb lattice model with unequal hopping strengths \( t \pm \delta_x \) in the \( x \)-direction and equal hopping \( t \) in others. The gray dashed box in (b) denotes the unit cell composed of two sites A, B and \( a_{1,2} \) are the unit vectors. (c) Low-energy parabolic dispersion with linear imaginary part corresponding to the square lattice model in (a). (d) Dirac cone dispersion and its imaginary part corresponding to the honeycomb lattice in (b). The signs of the imaginary part of the energy coincide with those of the velocity. The contours in (c) and (d) are the closed orbits that satisfy the Onsager-Lifshitz quantization rule. (e) Semiclassical cyclotron motion of charged particles in a magnetic field. (f) Semiclassical picture of the \( \mathcal{MT} \) symmetry, in which the successive actions of the \( \mathcal{MT} \) operation and the time evolution \( U(t) \) leave the state unchanged.

An order parameter is proposed to accurately describe the \( \mathcal{MT} \)-symmetry breaking, which is defined by the average quantum distance of the eigenstates generated by the \( \mathcal{MT} \) operation. The \( \mathcal{MT} \) phase transition is an important extension of the familiar \( \mathcal{PT} \) transition \([1,2]\), which generally exist in a class of magnetic systems and may lead to interesting observations and applications \([7,9]\).

Model.-To be specific, we first study the nonreciprocal Harper-Hofstadter model \([30,32]\) on a square lattice [cf. Fig. 1(a)] under a magnetic field \( B \) in the \( z \)-direction as

\[
H = - \sum_{m,n} \left( t_x c_{m+1,n}^\dagger c_{m,n} + t_x c_{m,n}^\dagger c_{m+1,n} \right) + te^{i2\pi m\phi} c_{m+1,n+1}^\dagger c_{m,n+1} + te^{-i2\pi m\phi} c_{m,n+1}^\dagger c_{m,n} + \frac{t\delta_x}{2} c_{m,n+1}^\dagger c_{m,n+1} + \frac{t\delta_x}{2} c_{m,n}^\dagger c_{m,n},
\]

where \( c_{m,n}^\dagger(c_{m,n}) \) are the creation (annihilation) operator on the site \((m,n)\), \( t_{\pm} = t \pm \delta_x \) describe the nonreciprocal hopping in the \( x \)-direction with \( \delta_x \) the strength of nonreciprocity or non-Hermiticity. The phase factor \( \phi = \Phi/\Phi_0 \) is defined by the magnetic flux \( \Phi = Ba^2 \) through a lattice cell (lattice constant \( a \)) divided by the flux quantum \( \Phi_0 = h/q \) with \( q \) the charge of the particle.

Here, the Landau gauge \( A = (0, Bx) \) has been adopted. In the rest of the paper, we set \( h = q = a = 1 \) in all numerical calculations for simplicity. The boundary condition in the \( x \)-direction plays an essential role in the presence of the NHSE. We denote the OBC and PBC in the \( \alpha \)-direction (\( \alpha = x, y \)) as \( \alpha \)-OBC and \( \alpha \)-PBC for brevity.

At \( B = 0 \), the energy spectrum under the \( x, y \)-PBC is \( E = -2t(\cos k_x + \cos k_y) + 2i\delta_x \sin k_x \) with \( k_{x,y} \) the wave vectors in the \( x, y \)-directions, which exhibits a point gap topology for a given \( k_y \) \([30,32]\); see Fig. 2(b). The low-energy expansion at the band bottom yields the parabolic dispersion of a normal particle plus a non-vanishing imaginary component as \( E \sim t(k_x^2 + k_y^2) + 2i\delta_x k_x \); see Fig. 1(c). Due to the point gap topology, the system exhibits NHSE in the \( x \)-direction under the \( x \)-OBC \([30,32]\). This can be seen from the eigenfunctions under the \( x, y \)-OBC as \( \psi(x, y) = \psi_{m,n} = (t_x^+/t_x^0)^{m/2} \sin^2(mk_x) \sin^2(nk_y) \) where \( k_x = \pi l_x/(M+1), l_x = 1, \ldots, M \) and \( k_y = \pi l_y/(N+1), l_y = 1, \ldots, N \), and \( M(N) \) is the system size in the \( x(y) \)-direction. For \( \delta_x = 0 \), the wave functions reduce to the standing waves; The nonreciprocal hopping with \( \delta_x > 0 \)
results in an envelope function modulation \((t_x^+/t_x^-)^{m/2}\) on top of the standing waves, which is just the NHSE. As a result, all the wave functions are localized at the right boundary; see Fig. 2(a). In the opposite limit with \(\delta_x = 0\) and finite \(B\), Eq. (1) reduces to the conventional Harper-Hofstadter model [52–53], which yields the familiar butterfly energy spectra; see Fig. 1(a). The Landau fan structure near the band edges for small \(y\) represents the highly degenerate Landau levels that yields the familiar butterfly energy spectra; see Fig. 4(a).

**Suppression of the NHSE.** It is of particular interest to investigate the interplay between the NHSE and the magnetic field. The boundary condition in the \(y\)-direction is unimportant to the main results, which is set to the \(y\)-PBC for simplicity. The Hamiltonian can then be Fourier transformed into \(\tilde{H} = -\sum_{m,k_y} t_x^+ c_{m+1,k_y} c_{m,k_y} + t_x^- c_{m,k_y} c_{m+1,k_y} + 2t \cos (k_y + 2\pi m\phi) c_{m,k_y}^\dagger c_{m,k_y}\). Taking the \(x\)-OBC, we plot in Fig. 2(a) the spatial distribution function \(W(x,k_y) = \sum_i |\psi_i^R(m,k_y)|^2 / M\) defined by all the right eigenstates \(\psi_i^R(m,k_y)\) (labeled by \(i\)) of \(\tilde{H}\) for a given \(k_y\). One can see that a small \(B\) is sufficient to drive the skin modes to penetrate deeply into the bulk, showing a considerable suppression of the NHSE. This result generally holds for all transverse wave vectors \(k_y\).

The above results can be understood by the following arguments. At \(B = 0\), the energy spectra \(E_{k_y}(k_x)\) for a given \(k_y\) forms a closed loop with a point gap topology in its complex plane under the \(x\)-PBC; see Fig. 2(b), indicating the presence of the NHSE under the \(x\)-OBC [36,37]. For finite \(B\), real energy spectra form and develop from band edges to the center with \(B\) increased, accompanied by a shrinkage of the complex loop; see Fig. 2(b). According to the correspondence between the spectra configuration under the \(x\)-PBC and the occurrence of the NHSE under the \(x\)-OBC, this indicates a tendency to suppress the NHSE. One can also analyze the results from the perspective of real space under the \(x\), \(y\)-OBC and start with the opposite limit of \(\delta_x = 0\). With increasing \(\delta_x\) from zero, the wave functions under magnetic field are modulated by the exponential envelope function introduced by the NHSE; see Figs. 2(c1), 2(c4). Then the results in Fig. 2(a) can be understood as the superposition of all the broken loops in real space.

**NHSE on the magnetic energy spectra.** The non-Hermitian systems can have completely different energy spectra under the PBC and OBC [18–19]. Next, we study the NHSE on the magnetic energy spectra under both the \(x\)-OBC and the \(x\)-PBC, while the \(y\)-PBC is adopted for both cases with the transverse modes labeled by \(k_y\). The energy spectra under the \(x\)-OBC are shown in Fig. 3(a), where the in-gap streaks are the edge states. It turns out that the butterfly diagram depends on \(\delta_x\) rather weakly for \(\delta_x < 2t/5\) (see Supplementary Information for details). Moreover, the energy spectra are entirely real. In fact, under the \(x\)-OBC, by use of similarity transformation \(S^{-1}c_{m,n}^\dagger S = r^{-m}c_{m,n}^\dagger\) with \(r = \sqrt{t_x^+ / t_x^-}\), Hamiltonian (1) can be transformed into a Hermitian Hamiltonian as \(\tilde{H} = \tilde{H}^\dagger S^{-1}\) with the hopping parameters in Eq. (1) replaced by \(t_x^+ \rightarrow \sqrt{t_x^+ t_x^-}\), which gives real energy spectra [35]. We also investigate more general cases with nonreciprocal hopping in both the \(x\)- and \(y\)-directions and the next-nearest neighbor hopping (see Supplementary Information for details). The main results remain unchanged while a simple similarity transformation is not suitable in those cases.

In contrast to the stationary energy spectra under...
FIG. 3. (a) Energy spectra under the x-OBC and y-PBC with $\delta_x = 0.1$ and $M = 50$. (b) Zoom of the Landau fan. (c) Energy levels labeled by i under the x-OBC and y-OBC with $\delta_x = 0.15$ and $M = N = 50$. The effect of $\delta_x$ on the magnetic orbits of the (d1,d2) 8th and (d3,d4) 618th eigenstates labeled by the dots “a” and “b” in (c) with $B = 0.02$. In all figures, $t = 0.5$.

FIG. 4. Energy spectra under the x-PBC and y-PBC calculated in the momentum space ($k_x, k_y$) for (a) $\delta_x = 0$ and (b) $\delta_x = 0.2$. (c) The complex energy spectra corresponding to the dash lines in (a,b). (d) The $\delta_x$ dependence of the real and imaginary parts of the energy spectra under the x-PBC and y-PBC with $B = 0.05$ and $k_x = k_y = 0$. In all figures, $t = 0.5$.

The x-OBC where the boundaries play an essential, the spectra under the x-OBC is related to the wave packet dynamics inside the bulk without touching the boundaries. Under the x-PBC, no similarity transformation can be employed even in the simplest case of Eq. (1) and the complex energy spectra start to form from the band center due to the nonreciprocal hopping $\delta_x$; see Figs. 3(a), 3(c). As a result, the self-similar fractal patterns merge into continuous pieces along with multiple gap closing. Interestingly, some of the energy levels coalesce in pairs that creates multiple exceptional points; see Fig. 4(d). Given that the change of $k_y$ has little effect on the magnetic spectra, which means that a large number of exceptional points can be implemented because the number of $k_y$ is proportional to the system size $N$. At the same time, the energies near the band top and bottom remain real for $k \ll 1/l_B$ with $l_B = \sqrt{\hbar/cB}$ the magnetic length, immune to the NHSE; see Figs. 4(b) and 4(c). It means that the magnetic field prevents the system from the real-to-complex spectral transition induced by the NHSE, which again reflects their incompatible nature. It then follows that the combined action of the NHSE and strong lattice modulation at short wavelengths leads to the spectral phase transition of the system.

It is worth noting that under both x-OBC and x-PBC, the Landau fan in the long-wavelength limit exhibits equal level spacing and linear dependence on $B$; see Fig. 8(b) and Supplementary Information for more details, which reduces to the behavior of free particles with a quadratic dispersion [cf. Fig. 1(c)]. It indicates that the Oonsager-Lifshitz quantization rule remains valid even in the presence of the NHSE. For clarity, we plot in Figs. 3(d1)-3(d4) the local density of states (LDOS) obtained from $\rho_E(x,y) = -\text{Im}[G_E^r(x,y)]/\pi$ with $G_E^r = (E - H + i0^+)^{-1}$ as the retarded Green’s function. $G_E^r(x,y)$ is defined by both the left and right eigenstates as $G_E^r(x,y) = \sum_i \psi_i^0(x,y)\psi_i^0(x,y)^*$, which determines the dynamics of the particle. One can see that the magnetic orbit corresponding to the degenerate Landau level [labeled as “a” in Fig. 3(c)] remains unchanged for finite $\delta_x$. As a result, the magnetic flux encircled by the orbit does not change, which can explain the Oonsager-Lifshitz quantization survives, despite the NHSE. On the contrary, the magnetic orbit changes drastically by $\delta_x$ for the unquantized states [labeled as “b” in Fig. 3(c)]. Accordingly, the encircled flux and the energy are both modified by $\delta_x$.

$MT$ symmetry and $M\bar{T}$ transition.- It can be verified that the Hamiltonian (1) possesses the combined $M\bar{T}$ symmetry, i.e.,

$$MTHT(M\bar{T})^{-1} = H,$$

(2)
with the mirror reflection ($\mathcal{M}$) about the $x$-axis and time reversal ($\mathcal{T}$) operation defined by
\[ \mathcal{M}c_{m,n}\mathcal{M}^{-1} = c_{m,-n}, \mathcal{T}c_{m,n}\mathcal{T}^{-1} = c_{m,n}, \mathcal{T}\psi\mathcal{T}^{-1} = -\psi. \] (3)

A quasi-classical picture of the $\mathcal{MT}$ symmetry is illustrated in Fig. [1](f). Any quantum state under the successive actions of the $\mathcal{MT}$ operation and the time evolution $U(t)$ remains the same, that is, $U(t)\mathcal{M}TU(t)\mathcal{M}T = 1$. The constraint by the $\mathcal{MT}$ symmetry can be rewritten in another standard form as $\mathcal{MOH}^{-1}(\mathcal{MO})^{-1} = H$, with $O$ the transpose operation, and then $H$ is said to be $\mathcal{MO}$-pseudo-Hermition $[9]$. As a result, the energy spectra can be either entirely real or composed of complex conjugate pairs. For a specific state, the real (complex) nature of the energy corresponds to its wave function with (without) the $\mathcal{MT}$ symmetry $[50]$.

The breaking of the $\mathcal{MT}$ symmetry for the $i$th normalized eigenstate $\psi_i^R$ can be measured by the Hilbert-Schmidt quantum distance $d_{\text{HS}}$ $[37]$, which is given by
\[ d_{\text{HS}}^i = \sqrt{1 - |\langle \mathcal{MT}_i \rangle|^2}, \langle \mathcal{MT}_i \rangle = \langle \psi_i^R|\mathcal{MT}|\psi_i^R \rangle. \] (4)

It characterizes the quantum mechanical distance between the wave functions before and after the $\mathcal{MT}$ operation. For the state that satisfies the $\mathcal{MT}$ symmetry, the $\mathcal{MT}$ operation yields only an overall phase factor, i.e., $\mathcal{MT}|\psi_i^R \rangle = e^{i\theta}|\psi_i^R\rangle$, so that $d_{\text{HS}}^i = 0$, meaning that the state under the $\mathcal{MT}$ action remains the same. In contrast, if the state breaks the $\mathcal{MT}$ symmetry, one has $0 < d_{\text{HS}}^i \leq 1$.

From what has been discussed above, the system will be in the $\mathcal{MT}$-symmetric phase if the energy spectra are entirely real with $d_{\text{HS}} = 0$ for all the eigenstates; Otherwise, the $\mathcal{MT}$ symmetry is spontaneously broken if some energy values become complex with the corresponding distance $d_{\text{HS}} > 0$ $[9]$. It is then convenient to introduce an order parameter to quantify the spontaneous $\mathcal{MT}$-symmetry breaking. An insightful choice of the order parameter can be the average quantum distance of all $N$ eigenstates defined as
\[ d_{\text{HS}} = \frac{1}{N} \sum_{i=1}^{N} d_{\text{HS}}^i. \] (5)

Whether the system is in the $\mathcal{MT}$-symmetric phase or $\mathcal{MT}$-broken phase can be distinguished by $d_{\text{HS}} = 0$ and $d_{\text{HS}} > 0$, similar to the spontaneous symmetry breaking in continuous phase transitions. Importantly, in addition to being a criterion of the $\mathcal{MT}$ transition, the magnitude of $d_{\text{HS}}$ can tell to what extent the $\mathcal{MT}$ symmetry is broken.

As shown in Figs. [3](a) and [3](b), the energy spectra under the $x$-OBC and $x$-PBC are in different $\mathcal{MT}$ phases. Thus, a tunable boundary condition should be able to drive a continuous $\mathcal{MT}$ transition. In what follows we employ the generalized boundary conditions $[19] [58]$ defined by the hopping between the boundary sites $(1, n)$ and $(M, n)$ as $-\gamma_B(t_x^e c_{1,n}^\dagger + t_x^e c_{M,n}^\dagger)$. The parameter $\gamma_B \in [0, 1]$ and its two limits $\gamma_B = 0$ and $\gamma_B = 1$ correspond to the $x$-OBC and $x$-PBC, respectively. We plot in Fig. 5 the average quantum distance $d_{\text{HS}}$ as a function of $\gamma_B$ and $\delta_x$ for zero and finite $B$. The states number $N$ in Eq. (5) counts all states $\psi_i^R(m, k_y)$ labeled by $i$ and $k_y$, and the $\mathcal{MT}$ operator acts on the wave function as $\mathcal{MT}\psi_i^R(m, k_y) = \psi_i^{R*}(m, k_y)$. One can see that there is a clear phase boundary formed between the $\mathcal{MT}$-symmetric ($d_{\text{HS}} = 0$) and $\mathcal{MT}$-broken ($d_{\text{HS}} > 0$) region. Moreover, such a phase boundary can also be obtained by the critical points of the real-to-complex spectral transition, in good coincidence with that given by the $d_{\text{HS}}$ contour; see Fig. [5](a). This is assured by the theorem associated with the $\mathcal{MT}$ antiunitary symmetry $[56]$. Another interesting observation is that the critical phase boundary can be well fitted with an exponential function; see Figs. [5](a) and [5](b). In particular, such an exponential phase boundary can be mathematically proved for $B = 0$; see Supplementary Information for details.

In the absence of a magnetic field, the system is in the $\mathcal{MT}$-symmetric and $\mathcal{MT}$-broken phase for the $x$-OBC ($\gamma_B = 0$) and $x$-PBC ($\gamma_B = 1$), respectively; see Fig. [5](a). By tuning the boundary parameter $\gamma_B$, a continuous $\mathcal{MT}$ transition connecting two limiting cases can be implemented. However, varying with $\delta_x$, there is no phase transition happening in either the $x$-OBC or the $x$-PBC. Remarkably, a finite magnetic field can effectively suppress the $\mathcal{MT}$-symmetry breaking; see Fig. [5](b), which is reflected in two aspects. First, the order parameter $d_{\text{HS}}$ diminishes in the $\mathcal{MT}$-broken region so that the symmetry breaking becomes weaker, meaning that a stronger nonreciprocal hopping $\delta_x$ is required to break the $\mathcal{MT}$ symmetry compared with that

![FIG. 5](image_url)

The order parameter $d_{\text{HS}}$ as a function of $\delta_x$ and $\gamma_B$ with (a) $B = 0$ and (b) $B = 0.02$. Critical points in (a) mark the real-to-complex spectral transition. The phase boundaries defined by the $d_{\text{HS}}$ contours are fitted by the exponential functions. Other parameters are set as $M = 50$ and $t = 0.5$. 51 equally spaced values of $k_y$ within $[-\pi, \pi]$ are used in the calculation.
at $B = 0$. Second, the $\mathcal{MT}$-symmetric region with large $\gamma_B$ expands with $B$ increased. This fact reflects once more the incompatible nature between the NHSE and the magnetic field. A direct result of such magnetic suppression is that a $\mathcal{MT}$ transition can be driven by either $\delta_x$ or $B$ for a finite system (see Supplementary Information for more details).

**Results for nonreciprocal honeycomb lattice.** To verify the universality of our results, in what follows we investigate the model of nonreciprocal honeycomb lattice in Fig. 1(b). With the same Landau gauge $A = (0, Bx)$ adopted and the zigzag edges oriented along the $y$-direction, the Hamiltonian for the nonreciprocal honeycomb lattice reads

$$H' = \sum_{mn} \left( t^+_x b^\dagger_{m+1,n} a_{m,n} + t_x a^\dagger_{m,n} b_{m+1,n} \right) + t \sum_{mn} \left( e^{i2\pi\phi'} m b^\dagger_{m,n} a_{m,n} + e^{-i2\pi\phi'} m a^\dagger_{m,n+1} a_{m,n} + \text{H.c.} \right),$$

where $a^\dagger_{m,n}, b^\dagger_{m,n}$ $(a_{m,n}, b_{m,n})$ are the creation (annihilation) operators for the $A, B$ sublattices, respectively, and $\mathcal{R}_{m,n} = m a_1 + n a_2$ denote the lattice sites with $(a_1, a_2)$ the unit vectors in Fig. 1(b). The phase factor is defined by $\phi' = \Phi'/(2\Phi_0)$ with $\Phi = 3\sqrt{3}B a^2/2$ the flux through a unit cell and $\phi'$ the bond length that is set to $\phi' = 1$ henceforth.

At $B = 0$, the low-energy dispersion reduces to $E = \pm[3tk_x/2 + isgn(k_x)\delta_x]$ with $sgn$ the signum function. Without nonreciprocal hopping, it corresponds to the massless Dirac particle, as shown in Fig. 1(d). The sign of the imaginary part of the energy coincides with that of the group velocity. It indicates a nonreciprocal propagation of the wave packet that grows (decays) towards the $x$ (−$x$) direction. Accordingly, the NHSE should emerge under the $x$-OBC. Taking the $y$-PBC, we perform the calculation on the zigzag ribbon with its boundaries aligned in the $y$-direction as shown in Fig. 1(a). The Fourier transformed Hamiltonian is $H' = \sum_{m,k_y} \left[ \Delta a^\dagger_{m,k_y} b_{m,k_y} + \Delta^* b^\dagger_{m,k_y} a_{m,k_y} + t^+_x b^\dagger_{m,k_y} a_{m,k_y} + t_x a^\dagger_{m,k_y} b_{m,k_y} \right]$ with $\Delta = 2t\cos[\sqrt{3}k_y/2 + \pi\phi(m - 5/6)]$. The spatial distribution function is calculated by $W'(x, k_y) = \sum_{m}(\psi^R_{a,b}(m, k_y)^2 + \psi^R_{b,a}(m, k_y)^2)/(2M)$ with $\psi^R_{a,b}(m, k_y)$ and $\psi^R_{b,a}(m, k_y)$ as the A and B components of the right eigenstates (labeled by $i$), respectively. The spatial distribution of $W'(x, k_y)$ is plotted in Fig. 6(a). Similar to the results of the square lattice, the skin modes are strongly suppressed by just a small $B$. It is closely related to the shrinkage of two complex loops of the energy spectra under the $x$-PBC (for an arbitrary $k_y$); see Fig. 6(b).

We plot the magnetic energy spectra in Fig. 7 under both the $x$-OBC and $x$-PBC. The spectra under the $x$-OBC are entirely real despite the NHSE. In contrast, complex spectra are induced by the NHSE under the $x$-PBC, where the fractal patterns merge into continuous pieces in the parametric regions far away from the long-wavelength limit. In the vicinity of the Dirac points, the same Landau fan structures arise for both the $x$-OBC and $x$-PBC; see Figs. 7(a) and 7(d). In particular, the quantized energy levels satisfy $E_n \propto \pm\sqrt{n}E$ with $n = 0, 1, \cdots$; see Figs. 7(d) and 7(b), manifesting the massless Dirac particle. Therefore, the Onsager-Lifshitz quantization rule persists against the NHSE for the massless Dirac particle as well.

A real-to-complex spectral transition can also be implemented for the honeycomb lattice, which is attributed to the spontaneous breaking of the $\mathcal{MT}$ symmetry. The nonreciprocal honeycomb lattice under a magnetic field satisfies $\mathcal{MT}H'(\mathcal{MT})^{-1} = H'$ as well [cf. Fig. 3(b)]. In order to implement a continuous phase transition, we introduce a tunable boundary hopping, $\gamma_B(t^+_x b^\dagger_{1,n} a_{M,n} + t_x a^\dagger_{1,n} b_{1,n})$, between the outmost sites $(1,n)_B$ and $(M,n)_B$, where the hopping strength satisfies $\gamma_B \in [0,1]$ with the two limits $\gamma_B = 0$ and $\gamma_B = 1$ corresponding to the $x$-OBC and $x$-PBC, respectively. We calculate the order parameter $d_{\text{HS}}$ in Eq. 5 using $\mathcal{MT}\psi^R_{a,b}(m, k_y) = \psi^R_{a,b}(m, k_y)$ and plot the phase diagrams in Fig. 8. The phase diagrams resemble Fig. 5 for the square lattice quite well, which reveals the universality of the spectral phase transition induced by spontaneously $\mathcal{MT}$-symmetry breaking. Specifically, a magnetic field can effectively suppress the $\mathcal{MT}$-symmetry breaking, which is reflected in two aspects, suppression of the the order parameter and expansion of the $\mathcal{MT}$-symmetric region with large $\gamma_B$. Moreover, the $\mathcal{MT}$ phase transition can be implemented by tuning either $B$ or $\delta_x$.

From the discussion above, one can see that the main results in this section are in full agreement with those of the square lattice. Therefore, the interplay between the
FIG. 7. (a) Energy spectra under the $x$-OBC and $y$-PBC with $\delta_x = 0.1$ and $M = 100$. (b) Zoom of the Landau fan with the rescaled horizontal ordinate $\sqrt{B}$. Energy spectra under the $x$-PBC and $y$-PBC calculated in the momentum space $(k_x, k_y)$ for (c) $\delta_x = 0$ and (d) $\delta_x = 0.2$. In all figures, $t = 1$.

NHSE and magnetic field obeys general laws of physics, which reflect their incompatible and competitive nature and give rise to rich physical phenomena.

FIG. 8. The order parameter $d_{HS}$ as a function of $\delta_x$ and $\gamma_B$ with (a) $B = 0$ and (b) $B = 0.04$. Critical points in (a) mark the real-to-complex spectral transition. The phase boundaries defined by the $d_{HS}$ contours are fitted by the exponential functions. Other parameters are set as $M = 25$ and $t = 1$.

Discussion and outlook.- We have shown that the fascinating interplay between NHSE and magnetic field can result in many novel effects. (i) It provides an effective way to control the NHSE by a magnetic field. (ii) The $MT$ phase transition is an important generalization of the celebrated $\mathcal{PT}$ transition, which can be implemented in a large class of natural and artificial magnetic systems. (iii) The order parameter proposed here can be applied to general types of non-Hermitian phase transition, which can not only give definite phase boundaries but also provide a quantitative description of the symmetry breaking. It is also of great interest to explore possible physical meaning of the order parameter in open quantum systems. (iv) It has been shown that the Onsager-Lifshitz quantization rule survives from the NHSE, which protects the low-energy Landau levels from becoming complex. A natural question is whether more general Bohr-Sommerfeld quantization rule will persist against the NHSE in other problems. (v) Exceptional points are created during the $MT$ phase transition, the same as that in the $\mathcal{PT}$ transition. The advantage of the current system is that a large number of exceptional points can be generated due to the high degeneracy of the energy spectra. Potential applications of this property require further exploration.

The two ingredients here are the NHSE and magnetic field, which can be implemented in various physical systems, such as cold atoms [44, 48, 49], photonic [59, 60] and acoustic [61] systems, condensed matter physics [15, 16, 62, 63] and electric circuits [38, 64–67]. Therefore, our proposal can be achieved in experiments by state-of-the-art techniques.

Note added.- During the preparation of the manuscript, we became aware of a related work posted on arXiv [68]. In addition to the magnetic suppression of the NHSE studied in both works from different perspectives, our work on the NHSE mainly focuses on the unique $MT$ phase transition, the properties of the magnetic energy spectra, and the Onsager-Lifshitz quantization rule.

Data availability.- All relevant data are available from the corresponding author upon reasonable request.

Code availability.- The codes that support the findings of this study are available from the corresponding author upon reasonable request.

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Additional information

Supplementary information is available for this paper at XXXXXXXXXX.

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