On entropic uncertainty relations for measurements of energy and its “complement”

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Reformulation of Heisenberg’s uncertainty principle in application to energy and time is a powerful heuristic principle. In a qualitative form, this statement plays the important role in foundations of quantum theory and statistical physics. A typical meaning of energy-time uncertainties is as follows. If some state exists for a finite interval of time, then it cannot have a completely definite value of energy. It is also well known that the case of energy and time principally differs from more familiar examples of two non-commuting observables. Since quantum theory was originating, many approaches to energy-time uncertainties have been proposed. Entropic approach to formulating the uncertainty principle is currently the subject of active researches. Using the Pegg concept of complementarity of the Hamiltonian, we obtain uncertainty relations of the “energy-time” type in terms of the Rényi and Tsallis entropies. Although this concept is somehow restricted in scope, derived relations can be applied to systems typically used in quantum information processing. Both the state-dependent and state-independent formulations are of interest. Some of the derived state-independent bounds are similar to the results obtained within a more general approach on the base of sandwiched relative entropies. In this regard, our relations provide an alternative viewpoint on such bounds. The developed approach also allows us to address the case of detection inefficiencies. It is worth for several reasons including possible information-processing applications.

Keywords: energy-time uncertainty principle, complement of the Hamiltonian, canonical conjugacy, Rényi entropy, Tsallis entropy

I. INTRODUCTION

The uncertainty principle is widely known among scientific achievements inspired by quantum physics. The Heisenberg’s thought experiment with microscope was first analyzed in qualitative sense [1]. As a formal statement, it was explicitly derived by Kennard [2]. Thus, the product of the standard deviations of position and momentum in the same state cannot be less than $\hbar/2$. This formulation can be extended to arbitrary pairs of observables [3]. Robertson’s formulation has later been criticized for several reasons [4, 5]. There is no general consensus concerning a proper formulation of the uncertainty principle [6]. Entropic functions provide a powerful tool to characterize uncertainties in quantum measurements [7–10]. Other approaches to express uncertainties in quantum measurements are currently the subject of interest. In particular, modern investigations concern fine-grained uncertainty relations [11, 14], the sum of variances [13, 16], majorization relations [17, 20], and effective anticommutators [21]. Uncertainty relations can be distinguished not only with respect to the way to measure uncertainties. In general, the case of energy and time cannot be treated similarly to the case of usual observables. Another important question concerns the role of order in which measurements have been performed. Traditional formulations are served as preparation uncertainty relations [22], since repeated trials with the same quantum state are assumed. In contrast to the traditional preparation scenario, uncertainty relations for successive measurements were introduced for several reasons.

Considering Heisenberg’s thought experiment with microscope, we conclude that definite momentum cannot be localized in space. In a similar manner, completely definite value of energy cannot be localized in time. The role of time in quantum theory has many facets [23, 26]. Moreover, there is a principal reason against the existence of universal form of a self-adjoint time operator conjugate to the Hamiltonian. In many basic models, the Hamiltonian spectrum is discrete and bounded from below. This fact is very important from the physical viewpoint. It is one of well known Pauli’s remarks that a time operator would imply that the Hamiltonian has the entire real line as its spectrum (see, e.g., footnote 2 in Sec. 8 of English translation [27]). Various approaches to energy-time uncertainty relations are reviewed in [28]. First formal derivations were explicitly given by Mandelstam and Tamm [29] and by Fock and Krylov [30]. Further development of this direction was given in [31, 32]. Such results were also interpreted as setting a fundamental bound on how fast any quantum system can evolve [33]. Despite of many known attempts, the authors of [34] expressed the opinion that the proper interpretation of energy-time uncertainty relations remains to be given. Entropic uncertainty relations for energy and time can be approached by constructing an almost-periodic time observable [35]. The latter is rather inadequate clock for aperiodic systems. The quantum-clock view on time uncertainty was recently developed in [36].

The author of [37] introduced the concept of complement of the Hamiltonian. This notion is treated as some quantity that is complementary to the Hamiltonian. The question has been resolved for a system with discrete energy levels, for which the ratios of the energy differences are rational exactly or approximately [37]. It can be represented by a non-orthogonal resolution of the identity as well as by an Hermitian operator acting in a suitably extended space.
This approach leads to the commutation relation that is formally equivalent to the phase-number commutator within the Pegg–Barnett formalism [38–40]. Using the introduced quantities, Pegg [37] derived uncertainty relations of the Robertson type. Although the scope of Pegg’s approach is somehow restricted, it is suitable for many interesting models. Moreover, it approaches the problem along a direction that is typically used to motivate impossibility of a Hermitian time operator. Thus, the notion of energy complement provides an alternate way to understand “energy-time” issue. It seems that this approach to energy-time uncertainty relations has received less attention than it deserves.

The aim of this work is to formulate entropic uncertainty relations for energy and its complement taken within the Pegg approach [37]. Our consideration is rather complementary to entropic uncertainty relations obtained recently in [36]. We present entropic uncertainty relations that are immediately related to measurement statistics. Both the state-dependent and state-independent formulations will be addressed. In addition, the developed approach allows one to take into account the case of detection inefficiencies. On the other hand, the scope of our results is more restricted as related to the special case of systems with discrete energy levels. This paper is organized as follows. The preliminary material is reviewed in Section II. Here, we recall the definition of used entropic functions and describe some details of the Pegg approach to the problem of energy-time uncertainties. Main results are presented in Section III. We derive uncertainty relations of the Maassen–Uffink type as well as relation with the same parameter in the corresponding entropies. In Section IV we conclude the paper with a summary of results obtained.

II. PRELIMINARIES

In this section, we review the required material concerning generalized entropies. The Pegg concept of complement of the Hamiltonian will be recalled as well. Let \( p = \{ p_i \} \) be a discrete probability distribution. For \( 0 < \alpha \neq 1 \), the Rényi \( \alpha \)-entropy is defined as [41]

\[
R_\alpha(p) := \frac{1}{1 - \alpha} \ln \left( \sum_i p_i^\alpha \right).
\]

(1)

It is known that this entropy does not increase with growth of \( \alpha \). The Rényi \( \alpha \)-entropy is certainly concave for \( \alpha \in (0; 1) \). For \( \alpha > 1 \), it is neither purely convex nor purely concave [42]. Here, the situation actually depends on the dimensionality of probabilistic vectors. For a discussion of basic properties of (1), see section 2.7 of [43]. In the limit \( \alpha \to 1 \), we have the Shannon entropy

\[
H_1(p) := - \sum_i p_i \ln p_i.
\]

(2)

The limit \( \alpha \to \infty \) leads to the so-called min-entropy

\[
R_\infty(p) = - \ln(\max p_i).
\]

(3)

Tsallis entropies form another important family of generalized entropies. For \( 0 < \alpha \neq 1 \), the Tsallis \( \alpha \)-entropy is defined as [44]

\[
H_\alpha(p) := \frac{1}{1 - \alpha} \left( \sum_i p_i^\alpha - 1 \right) = - \sum_i p_i^\alpha \ln_\alpha(p_i).
\]

(4)

Here, the \( \alpha \)-logarithm of positive \( x \) is given as \( \ln_\alpha(x) = (x^{1-\alpha} - 1)/(1 - \alpha) \). Substituting \( \alpha = 1 \), the right-hand side of (4) gives (2). It will be convenient to introduce norm-like functionals of discrete probabilistic vectors. For \( \beta > 0 \), we define

\[
\|p\|_\beta := \left( \sum_i p_i^\beta \right)^{1/\beta}.
\]

(5)

The right-hand side of (5) gives a legitimate norm only for \( \beta \geq 1 \). Hence, we can write

\[
R_\alpha(p) := \frac{\alpha}{1 - \alpha} \ln \|p\|_\alpha, \quad H_\alpha(p) := \frac{\|p\|_\alpha^\alpha - 1}{1 - \alpha}.
\]

(6)

For \( \alpha > 1 > \beta > 0 \), we obviously have

\[
\|p\|_\alpha \leq 1, \quad \|p\|_\beta \geq 1.
\]

(7)
We will also concern differential entropies assigned to a continuously changed variable. In principle, the formula can be rewritten immediately. When the variable of interest is distributed according to the probability density function \( w(\tau) \), then

\[
R_\alpha(w) := \frac{1}{1-\alpha} \ln \left( \int w(\tau)^\alpha d\tau \right),
\]

where \( 0 < \alpha \neq 1 \). The integral is assumed to be taken over the interval of values, for which \( w(\tau) \) is defined. The corresponding interval will follow from the context. In the limit \( \alpha \to 1 \), the expression leads to the differential Shannon entropy

\[
H_1(w) := - \int w(\tau) \ln w(\tau) d\tau.
\]

It is convenient to extend the notion to the case of probability density functions. For the given density function \( w(\tau) \) and \( \beta > 0 \), we write

\[
\|w\|_\beta := \left( \int w(\tau)^\beta d\tau \right)^{1/\beta}.
\]

In the case of discrete probability distributions, for \( \alpha > 1 > \beta > 0 \) we have. It is provided by the normalization \( |p||1 = 1 \). On the other hand, for probability density functions the normalization \( \|w\|_1 = 1 \) does not provide restrictions analogous to (7). One of corollaries of this fact is that differential entropies are not positive definite in general. Formulating the Tsallis version of uncertainty relations with continuous time, we will use entropies taken with binning only.

There are several possible ways to fit a quantum counterpart of generalized entropic functions. The main point is that we deal here with the case of non-commuting variables. One of existing approaches is based on the concept of the so-called “sandwiched” divergences. In general, the concept of relative entropy, or divergence, plays a key role in quantum information theory [45, 46]. Quantum relative entropies of the Rényi type are considered as a generalization of this concept. To resolve the non-commutative case, sandwiched entropies have found to be useful [17]. Another approach to parameterized quantum entropies was thoroughly examined in [45]. The sandwiched Rényi relative entropies allow one to define the corresponding conditional entropies. Using such entropies, the authors of [30] formulated entropic energy-time uncertainty relations with a quantum memory. In the following, we consider an alternative approach based on the concept of complement of the Hamiltonian.

Let us proceed to the problem of energy-time uncertainty relations. Following Einstein, the authors of [49] emphasized that “nature provides its own way to localize a point in spacetime”. That is, coordinates are only convenient but not preexisting tools. Concrete values of coordinates have no significance unless the used reference frame is somehow anchored to certain events. Without a further clarification, our everyday understanding of the word “time” cannot be applied in quantum scales. Of course, this question is typically asked within the context of quantum gravity [50, 51]. On the other hand, limitations on the accuracy of a quantum clock are closely related to Heisenberg’s uncertainty principle [50]. To simplify formulas, we will further deal with the units in which \( \hbar = 1 \). Then the energy scale is inverse to the time scale. The problem of existence of a proper time operator has found a certain attention (see, e.g. section III.8 of [52]). For a free non-relativistic particle with the standard Hamiltonian of kinetic energy, time representation is built by means of the Fourier transform. The corresponding entropic uncertainty relation merely repeats the relation of Becker [53] which was originally derived by Hirschman [50]. For discrete semi-bounded Hamiltonians, the problem was formally analyzed in [57]. The result of [57] has been criticized in [53]. We will use the approach of Pegg [37] who proposed explicit constructions for discrete systems with levels of a certain structure.

Let us consider the system with \( d + 1 \) energy levels \( \varepsilon_n \). It is convenient to choose the lowest level \( \varepsilon_0 = 0 \) [37]. We will also assume that energy values are non-degenerate and numbered in increasing order. The Hamiltonian is accordingly represented as

\[
E = \sum_{n=0}^{d} \varepsilon_n \langle \varepsilon_n | \varepsilon_n \rangle,
\]

where \( |\varepsilon_n\rangle \) denotes \( n \)-th energy eigenstate. In the case of unitary evolution, a pure state changes in time according to

\[
\exp(-iE t) |\psi\rangle = \sum_{n=0}^{d} \exp(-i\varepsilon_n t) c_n |\varepsilon_n\rangle, \quad c_n = \langle \varepsilon_n | \psi \rangle.
\]
The author of \[37\] asked a quantity conjugate to the Hamiltonian in the sense that \( E \) is the generator of shifts. So, one seeks states of the form \(|\tau\rangle\), for which

\[
\exp(-iE \Delta \tau)|\tau\rangle = |\tau + \Delta \tau\rangle.
\]

(13)

It is not difficult to get the final expression \[37\]

\[
|\tau\rangle = \frac{1}{\sqrt{d+1}} \sum_{n=0}^{d} \exp(-i\varepsilon_n \tau)|\varepsilon_n\rangle.
\]

(14)

Such expressions are typical in considering eigenstates of complementary observables in finite dimensions. For equidistant levels, we will deal just with two complementary observables. In the context of uncertainty relations, this question was analyzed in \[58\]. The main question is how to treat the case of unequally spaced energy levels \[37\]. Nevertheless, one is able to get a non-orthogonal resolution of the identity \[58\]. The main question is how to treat the case of unequally spaced energy levels \[37\].

In principle, the parameter \( \tau \) can be varied continuously. In this way, we have arrived at an over-complete set of kets of the form \(14\). It is generally impossible to pick out a subset of \( d + 1 \) states \(|\tau\rangle\) that are orthogonal \[37\]. Nevertheless, one is able to get a non-orthogonal resolution of the identity on \(\mathcal{H}_{d+1}\). Suppose that the ratios \(\varepsilon_n/\varepsilon_1\) are rational numbers or can be sufficiently closely approximated by them. For the former, we write

\[
\frac{\varepsilon_n}{\varepsilon_1} = \frac{B_n}{A_n},
\]

(15)

where integers \(B_n\) and \(A_n\) are mutually prime. By \(r_1\), one denotes the lowest common multiple of the values of \(A_n\) for \(n > 1\). Defining \(r_0 = 0\) and \(r_n = r_1 B_n/A_n\) for \(n > 1\), we deal with integer numbers \(r_n\). This results in the formula

\[
\varepsilon_n = \frac{2\pi r_n}{T_c},
\]

(16)

where \(T_c = 2\pi r_1/\varepsilon_1\). Following \[37\], we take \(s + 1\) states of the form \(13\) for the values

\[
\tau_m = \tau_0 + m \frac{T_c}{s + 1} \quad (m = 0, 1, \ldots, s).
\]

(17)

The intermediate values \(\tau_1, \ldots, \tau_s\) are uniformly distributed between the points \(\tau_0\) and \(\tau_0 + T_c\). As was shown in \[37\], one finally gets

\[
\frac{d + 1}{s + 1} \sum_{m=0}^{s} |\tau_m\rangle\langle\tau_m| = \mathbb{1}_{d+1},
\]

(18)

where \(\mathbb{1}_{d+1}\) is the identity operator on \(\mathcal{H}_{d+1}\). Therefore, we have arrived at a non-orthogonal resolution of the identity for measuring an energy complement. The relation \(15\) is satisfied exactly when the ratios \(\varepsilon_n/\varepsilon_1\) are rational and the differences \(r_m - r_n\) are not multiples of \(s + 1\). One can ensure the latter by choosing \(s + 1 > \max r_n\). In other respects, we have a freedom in the choice of \(s \geq d\). If these ratios are irrational but sufficiently well approximated by rational numbers, the relation \(15\) holds up to a negligible additive term \[37\]. To each energy level \(\varepsilon_n\), we can assign the natural period \(2\pi/\varepsilon_n\). When \(\varepsilon_n/\varepsilon_1\) are exact rational numbers, the characteristic time \(T_c\) has a simple physical interpretation. It represents the smallest non-zero time taken for the system to return to its initial state \[37\]. Hence, the state \(|\tau + T_c\rangle\) will coincide with \(|\tau\rangle\). Focusing on the corresponding range in \(17\) prevents us from including the same state twice or more.

Dealing with a positive operator-values measure (POVM), we still not reach an observable represented by a Hermitian operator. On the other hand, uncertainties themselves are rather connected with a spread of probability distribution. In this regard, the entropic way to formulate uncertainty relations is quite sufficient since entropies are immediately calculated for concrete values of probabilities. Although the question of building the complement observable can be resolved within Naimark’s extension \[37\], we can express entropic uncertainty relations without it.

Measuring the energy, we use projective-valued measure \(\mathcal{E} = \{|\varepsilon_n\rangle\langle\varepsilon_n|\}\). To the given state \(\rho\), we assign the probabilities \(\langle\varepsilon_n|\rho|\varepsilon_n\rangle\). By \(R_{\alpha}(\mathcal{E}; \rho)\) and \(H_{\alpha}(\mathcal{E}; \rho)\), we denote the \(\alpha\)-entropies \(11\) and \(41\) calculated with these probabilities. The complement of energy is described by rank-one POVM \(\mathcal{T} = \{|	heta_m\rangle\langle\theta_m|\}\), where

\[
|\theta_m\rangle = \sqrt{\frac{d + 1}{s + 1}} |\tau_m\rangle.
\]

(19)

To the prepared state \(\rho\), we assign the entropies \(R_{\alpha}(\mathcal{T}; \rho)\) and \(H_{\alpha}(\mathcal{T}; \rho)\) calculated according to the probabilities \(\langle\theta_m|\rho|\theta_m\rangle\). In many respects, the above construction is similar to the Pegg–Barnett formalism \[38\,14\]. This formalism
allows us to fit a Hermitian operator to represent quantum phase. Since Dirac’s famous work [59] on quantum electrodynamics had appeared, the quantum phase problem has been studied from different viewpoints [34, 60]. An intuitive assumption is that the operators of optical phase and photon number are canonically conjugate. Instead of using the infinite Hilbert space from the begin, the Pegg–Barnett formalism deals with a finite but arbitrarily large state space [38, 39]. The final step is to find the limit of desired quantities as the dimensionality tends to infinity. The authors of [61] have developed this approach with respect to the concept of canonical conjugacy.

### III. MAIN RESULTS

In this section, we derive various forms of entropic uncertainty relations for energy and its complement. Let us begin with entropic uncertainty relations of the Maassen–Uffink type. Following [62], we introduce the quantity

$$g(\mathcal{E}, T; \rho) := \max \left| \frac{\langle \varepsilon_n | \theta_m \rangle \langle \theta_m | \rho | \varepsilon_n \rangle}{\langle \varepsilon_n | \rho | \varepsilon_n \rangle^{1/2} \langle \theta_m | \rho | \theta_m \rangle^{1/2}} \right|,$$

where the maximization is performed under the conditions $\langle \varepsilon_n | \rho | \varepsilon_n \rangle \neq 0$ and $\langle \theta_m | \rho | \theta_m \rangle \neq 0$. In general, the quantity $\mu$ depends on the used construction of states $|\theta_m\rangle$. To get uncertainty relations in terms of Rényi entropies, purely algebraic operations are required. The case of Tsallis entropies is not so immediate. We will use the method of [63], where the minimization problem was examined. Entropic uncertainty relations for energy and its complement are posed as follows. For any prepared state $\rho$, we have

$$R_\alpha(\mathcal{E}; \rho) + R_\beta(T; \rho) \geq -2 \ln g(\mathcal{E}, T; \rho),$$

$$H_\alpha(\mathcal{E}; \rho) + H_\beta(T; \rho) \geq \ln \mu \{ g(\mathcal{E}, T; \rho)^{-2} \},$$

where positive entropic parameters obey $1/\alpha + 1/\beta = 1$ and $\mu = \max\{\alpha, \beta\}$. The condition $1/\alpha + 1/\beta = 2$ reflects the fact that the Maassen–Uffink result is based on Riesz’s theorem [64]. An alternative viewpoint is that the above uncertainty relations follow from the monotonicity of the quantum relative entropy [65]. For an arbitrary choice of $\alpha$ and $\beta$, the problem of obtaining general entropic bounds was examined in [66].

The inequalities (21) and (22) are preparation uncertainty relations formulated in terms of both the Rényi and Tsallis entropies. Due to (20), these entropic bounds depend on the way in which we have built the POVM $T = \{ |\theta_m\rangle \langle \theta_m| \}$. This POVM is constructed of kets that are mutually unbiased with the eigenstates of the Hamiltonian. A certain freedom takes place in the choice of actual referent values $\tau_n$ of time. Thus, we obtained a kind on entropic “energy-time” uncertainty relations. It is not insignificant that our relations directly connect to measurement statistics. In this regard, they differ from entropic uncertainty relations derived in [36]. Another distinction is that both the bounds (21) and (22) explicitly depend on the measured state $\rho$.

As was explained in [66], the above state-dependent uncertainty bounds can be converted into a state-independent form. It turned out that state-independent entropic bounds are expressed in terms of $s$ solely. To do so, we merely write

$$g(\mathcal{E}, T; \rho) \leq f(\mathcal{E}, T) := \max \left| \langle \varepsilon_n | \theta_m \rangle \right|,$$

This inequality follows from combining (20) with the Cauchy–Schwarz inequality. It is easy to check that $f(\mathcal{E}, T) = (s + 1)^{-1/2}$ according to the chosen $s$. As a result, we obtain

$$R_\alpha(\mathcal{E}; \rho) + R_\beta(T; \rho) \geq \ln(s + 1),$$

$$H_\alpha(\mathcal{E}; \rho) + H_\beta(T; \rho) \geq \ln \mu(s + 1),$$

where $1/\alpha + 1/\beta = 1$ and $\mu = \max\{\alpha, \beta\}$. In other words, the entropic bounds (23) and (24) are expressed in terms of the number $s$ of the reference instants of time. Here, we see a similarity to the entropic bound in energy-time uncertainty relations given in [36]. The following fact should be pointed out. Since the states $|\theta_m\rangle$ are sub-normalized, the corresponding probabilities cannot reach 1. Hence, the entropies $R_\beta(T; \rho)$ and $H_\beta(T; \rho)$ in the above relations are certainly non-zero. Using $\langle \theta_m | \rho | \theta_m \rangle \leq (d + 1)/(s + 1)$, we easily obtain

$$R_\beta(T; \rho) \geq \ln \left( \frac{s + 1}{d + 1} \right) =: \Gamma.$$

Of course, this estimation from below is only approximate. Nevertheless, it can be used to characterize an unavoidable amount of uncertainty in the measurement $T$. Subtracting $\Gamma$ from both the sides of (21), we have

$$R_\alpha(\mathcal{E}; \rho) + R_\beta(T; \rho) - \Gamma \geq \ln(d + 1).$$
After subtracting, the entropic lower bound is determined by the logarithm of dimensionality for every choice of time moments.

Using the entropic approach, we can take into account possible inefficiencies of the detectors used. Since measurement devices inevitably suffer from losses, the “no-click” probability is non-zero in practice. Here, we consider the following model. Let the parameter \( \eta \in [0; 1] \) characterize a detector efficiency. To the given value \( \eta \) and probability distribution \( p = \{p_i\} \), we assign a “distorted” distribution \( \tilde{p}^{(\eta)} \) such that

\[
p_i^{(\eta)} = \eta p_i, \quad p_\emptyset^{(\eta)} = 1 - \eta.
\]

The probability \( p_\emptyset^{(\eta)} \) corresponds to the no-click event. The above formulation is inspired by the first model of detection inefficiencies used by the authors of [67] for cycle scenarios of the Bell type. For the sake of simplicity, we restrict a consideration to the Shannon entropies. It was mentioned in [62] that

\[
H_\alpha(p^{(\eta)} = \eta H_\alpha(p) + h_\alpha(\eta),
\]

where the binary Tsallis entropy \( h_\alpha(\eta) \) reads as

\[
h_\alpha(\eta) = - \eta^\alpha \ln \eta - (1 - \eta)^\alpha \ln (1 - \eta).
\]

For the Shannon entropies, one gives

\[
H_1(p^{(\eta)}) = \eta H_1(p) + h_1(\eta).
\]

In the case considered, we have

\[
H_1(E^{(\eta)}; \rho) + H_1(T^{(\eta)}; \rho) \geq -2 \eta \ln g(E, T; \rho) + 2h_1(\eta) \geq \eta \ln(s + 1) + 2h_1(\eta).
\]

By \( \eta = \min\{\eta_E, \eta_T\} \geq 1/2, \) we mean the minimum of the two efficiencies corresponding respectively to measurements of energy and its complement. We see that detector inefficiencies will produce additional uncertainties in the entropies of actually measured data. For low values of the efficiency, measurement statistics will mainly reflect detector-generated uncertainties.

Since the states \( |\theta_m\rangle \) lead to a non-orthogonal resolution of the identity in \( \mathcal{H}_{d+1} \), they cannot be eigenstates of a Hermitian operator acting in this space. On the other hand, any POVM-measurement can be realized as a projective one in suitably extended space. In principle, this possibility is established by the Naimark theorem. Its general discussion with applications can be found in [52, 68]. It is sufficient for our aims to focus on the case of rank-one POVMs. Then the corresponding projective measurement may be constructed in a simplified manner as follows (see, e.g., section 3.1 of [43]). Components of \( s + 1 \) kets \( |\theta_m\rangle \) are treated as elements of certain \( (d + 1) \times (s + 1) \) matrix. Adding this matrix by suitable number of rows, we can obtain a unitary matrix of size \( s + 1 \). Each Hermitian operator acting in the extended space \( \mathcal{H}_{s+1} = \mathcal{H}_{d+1} \oplus \mathcal{K} \) should have \( s + 1 \) eigenstates. The energy basis will include extra states, so that we obtain an orthogonal resolution \( \tilde{\mathcal{E}}_{s+1} = \{ |\tilde{\ell}\rangle \langle \tilde{\ell}| \}_{\tilde{\ell}=0}^{s} \). Following [37], we consider kets of the form

\[
|\tilde{\theta}_m\rangle = \frac{1}{\sqrt{s + 1}} \sum_{\ell=0}^{s} \exp(-i\ell \theta_m) |\tilde{\ell}\rangle.
\]

In this way, we obtain another orthogonal resolution \( \tilde{T}_{s+1} = \{ |\tilde{\theta}_m\rangle \langle \tilde{\theta}_m| \}_{m=0}^{s} \). It must be stressed that the original energy eigenstates \( |\varepsilon_n\rangle \) with \( n = 0, 1, \ldots, d \) are rearranged so that

\[
|\tilde{\ell}\rangle = |\varepsilon_{\ell-n}\rangle \oplus 0,
\]

whenever \( \ell = r_n \). The latter is possible due to \( s + 1 > \max r_n \). If \( \ell \neq r_n \) for all \( n = 0, 1, \ldots, d \), then the ket \( |\tilde{\ell}\rangle \) has non-zero components only in \( \mathcal{K} \). In [33], the numbers \( \theta_m \) are defined as

\[
\theta_m = \frac{2 \pi \tau_m}{T_c}.
\]

These numbers lie in a range of length \( 2\pi \) between \( \theta_0 \) and \( \theta_0 + 2\pi \), where \( \theta_0 = 2 \pi \tau_0/T_c \). To each density matrix \( \rho \) on \( \mathcal{H}_{d+1} \), we assign the matrix \( \tilde{\rho} \) of size \( s + 1 \) by adding zero rows and columns. Obviously, we have

\[
\langle \tilde{\theta}_m|\tilde{\rho}|\tilde{\theta}_m\rangle = \langle \theta_m|\rho|\theta_m\rangle.
\]
for all $m$, \( \langle \bar{\ell} | \bar{\rho} | \bar{\ell} \rangle = \langle \varepsilon_n | \rho | \varepsilon_n \rangle \) for $\ell = r_n$, and \( \langle \bar{\ell} | \bar{\rho} | \bar{\ell} \rangle = 0 \) for $\ell \neq r_n$. In the case considered, we introduce the following two operators,

\[
\sum_{\ell=0}^{s} \ell |\bar{\ell}\rangle \langle \bar{\ell}|, \quad \sum_{m=0}^{s} \theta_m |\bar{\theta}_m\rangle \langle \bar{\theta}_m|.
\]

(37)

Up to a factor, the former operator gives a Hamiltonian acting in the extended space. The second one is formally equivalent to the operator of optical phase due to Pegg and Barnett. For the above operators, one can easily obtain uncertainty relations of the Robertson type. Their discussion together with the limiting case $s \to \infty$ can be found in [37]. On the other hand, entropic uncertainty relations are rather connected with resolutions of the identity. In this sense, we will mainly focus on probability distributions and, after taking the limit, probability density functions.

The author of [37] also mentioned how to unify the approach for all systems of the type considered. We can examine basic quantities in the limit $s \to \infty$. As the difference between successive values of $\tau_m$ tends to zero, the probability to lie in the small range between $\tau$ and $\tau + \Delta \tau$ is assumed to be obtained from $\rho$ by adding zero rows and columns. According to (8), we introduce differential Rényi $\alpha$-entropies $R_\alpha(w_\rho)$ and $R_\alpha(U_\rho)$. In contrast to entropies of discrete probability distributions, differential entropies satisfy the condition $1/\alpha + 1/\beta = 2$. To consider the limit $s \to \infty$, we treat probability distributions as related to the extended space $\mathcal{H}_{s+1}$. As was mentioned above, the observables (37) are canonically conjugate in the sense of the Pegg–Barnett formalism. For probabilistic vectors $p = \{p_n\}$ with $p_n = \langle \varepsilon_n | \rho | \varepsilon_n \rangle$ and $q = \{q_m\}$ with $q_m = \langle \theta_m | \rho | \theta_m \rangle$, we have

\[
\|p\|_\alpha \leq \left( \frac{1}{s+1} \right)^{(1-\beta)/\beta} \|q\|_\beta, \quad \|q\|_\alpha \leq \left( \frac{1}{s+1} \right)^{(1-\beta)/\beta} \|p\|_\beta,
\]

(40)

where $1/2 < \beta < 1 < \alpha$. The formulas (40) follow from the Riesz theorem. The limiting procedure results in the probability density function, so that $q_m$ is finally replaced with $U_\rho(\theta_m) d\theta$. Here, we can write

\[
\|p\|_\alpha \leq \left( \frac{1}{2\pi} \right)^{(1-\beta)/\beta} \|U_\rho\|_\beta, \quad \|U_\rho\|_\alpha \leq \left( \frac{1}{2\pi} \right)^{(1-\beta)/\beta} \|p\|_\beta,
\]

(41)

where $1/\alpha + 1/\beta = 2$ and $1/2 < \beta < 1 < \alpha$. These relations can be derived similarly to the method of the paper [62]. The latter is devoted to number-phase uncertainty relations in terms of generalized entropies. Differential entropies are calculated with probability density functions that depend on rescaling of the random variable. It is better to do this step in terms of norm-like functionals. Combining $U_\rho(\theta) d\theta = w_\rho(\tau) d\tau$ with $\theta = 2\pi\tau/T_c$, we also obtain

\[
\|U_\rho\|_\beta = \left( \frac{2\pi}{T_c} \right)^{(1-\beta)/\beta} \|w_\rho\|_\beta.
\]

(42)

Hence, the “twin” relations (41) are rewritten as

\[
\|p\|_\alpha \leq \left( \frac{1}{T_c} \right)^{(1-\beta)/\beta} \|w_\rho\|_\beta, \quad \|w_\rho\|_\alpha \leq \left( \frac{1}{T_c} \right)^{(1-\beta)/\beta} \|p\|_\beta,
\]

(43)

under the same conditions on $\alpha$ and $\beta$. Using simple algebraic operations, we convert (43) into entropic uncertainty relations with continuous time, viz.

\[
R_\alpha(\mathcal{E}; \rho) + R_\beta(w_\rho) \geq \ln T_c,
\]

(44)
where \( 1/\alpha + 1/\beta = 2 \). The obtained entropic bound is very similar to the bound given in [36]. It seems that entropic bounds of such a kind are different manifestations of the same fundamental restriction. Note that our relation deals with entropic functions directly related to measurement statistics. In this sense, one characterizes energy-time uncertainties in a very traditional style. Thus, we have obtained an old-fashioned counterpart of entropic energy-time relations proposed in [36].

Since the right-hand side of (44) involves a dimensional parameter, there is a dependence on the chosen unit of time. On the other hand, differential entropy \( R_\delta(w_P) \) also depends on the time unit. The mentioned dependence is such that rescaling time will contribute the additive term to both the sides of the relation (44). In this sense, our entropic uncertainty relations with continuous time are independent of the time unit. To get a dimensionless formulation explicitly, we can consider entropic uncertainty relations with time binning. The interval \([\tau_0; \tau_0 + T_c]\) is divided into the set of bins between some ordered marks \( \tau_j \). In contrast to the case (17), these values can generally be chosen in arbitrary way. By \( \delta \tau \), we mean the maximum of the differences \( \tau_{j+1} - \tau_j \). Instead of \( w_P(\tau) \), we now deal with probabilities of the form

\[
q_j^{(\delta)} := \int_{\tau_j}^{\tau_{j+1}} w_\rho(\tau) \, d\tau, \tag{45}
\]

resulting in the discrete distribution \( q_j^{(\delta)} \). Due to (43), we obtain the inequalities

\[
\|p\|_\alpha \leq \left( \frac{\delta\tau}{T_c} \right)^{(1-\beta)/\beta} \|q_j^{(\delta)}\|_\beta, \quad \|q_j^{(\delta)}\|_\alpha \leq \left( \frac{\delta\tau}{T_c} \right)^{(1-\beta)/\beta} \|p\|_\beta, \tag{46}
\]

where \( 1/\alpha + 1/\beta = 2 \) and \( 1/2 < \beta < 1 < \alpha \). Details of deriving (46) from (43) are quite similar to that was given in section 3.3 of [70]. Using (46), we finally obtain

\[
R_\alpha(E; \rho) + R_\beta(q_j^{(\delta)}; \rho) \geq \ln \left( \frac{T_c}{\delta\tau} \right), \tag{47}
\]

\[
H_\alpha(E; \rho) + H_\beta(q_j^{(\delta)}; \rho) \geq \ln_\mu \left( \frac{T_c}{\delta\tau} \right), \tag{48}
\]

where \( 1/\alpha + 1/\beta = 2 \) and \( \mu = \max\{\alpha, \beta\} \). The inequalities (47) and (48) give entropic uncertainty relations with time binning. As was already mentioned, the actual bins can be chosen irrespectively to (17). In this sense, uncertainty relations of “energy-time” kind are written in unifying way, when the system considered is characterized by the single parameter \( T_c \). Of course, the above results are derived under assumptions used initially in building the POVM \( \mathcal{T} = \{\theta_m\}^{m=1}_{m=0} \).

Let us consider an example of preparation uncertainty relations for energy and its complement. The simplest case deals with repeated measurements on a single qubit. It can be meant as a spin-1/2 particle in an external magnetic field. The Hamiltonian is proportional to the \( z \)-Pauli matrix. However, we recall that the energy scale should be shifted to provide \( \varepsilon_0 = 0 \). The latter is required to construct POVMs \( \mathcal{T} = \{\theta_m\}^{m=1}_{m=0} \). The number \( s + 1 \) of referent moments changes from 2 up to infinity. In this example, we may simply put \( \tau_0 = 0 \). It is usual to represent qubit states by vectors of the Bloch ball. In Fig. 1 we plot the left-hand side of (24) together with lower bound \( \ln(s + 1) \) for several values of \( \beta \). The Bloch vector \( r \) points out along the \( x \)-axis, whereas its modulus is taken to be \( r = 1 \) and \( r = 0.75 \). For equatorial qubit states, the entropy \( R_\alpha(E; \rho) \) is constant. Thus, the curves mainly reflect changes in \( R_\beta(T; \rho) \). The abscissa includes values of \( s + 1 \) between 2 and 1000, whence a pass to the case of continuous time with binning seems to be clear. All the curves lie near \( \ln(s + 1) \), so the state-independent lower bound is sufficiently tight. When \( r \) decreases, the curves become more closely to each other, though they slightly shift upward. To take into account this small increase, we can consider the state-dependent relation (24).

Using the treatment of measurements in \( H_{s+1} \), we can obtain entropic uncertainty relations of another type. By construction, the two bases \( \{\bar{\theta}_m\}^{s}_{m=0} \) and \( \{\bar{\theta}_m\}^{s}_{m=0} \) are mutually unbiased. Hence, we can write entropic uncertainty relations for MUBs derived in [71] and later extended [62]. If the density matrix \( \rho \) is obtained from \( \bar{\rho} \) by adding zero components, then

\[
R_\alpha(E; \rho) = R_\alpha(\bar{E}; \bar{\rho}), \quad R_\alpha(T; \rho) = R_\alpha(\bar{T}; \bar{\rho}), \tag{49}
\]

and similarly for the Tsallis entropies. By suitable substitutions into formulas (17) and (18) of [62], for \( \alpha \in (0; 2] \) one gets

\[
H_\alpha(E; \rho) + H_\alpha(T; \rho) \geq 2 \ln_\alpha \left( \frac{2s + 2}{(s + 1) \text{tr}(\bar{\rho}^2) + 1} \right) \geq 2 \ln_\alpha \left( \frac{2s + 2}{s + 2} \right). \tag{50}
\]
Uncertainty relations for MUBs in terms of Rényi entropies were presented in [62] and later improved in [72]. Applying the results of [62, 72] to the case considered, for \( \alpha \geq 2 \) we obtain
\[
R_\alpha(E; \rho) + R_\alpha(T; \rho) \geq 2 \alpha - 1 \ln \left( \frac{2s + 2}{(s + 1) \text{tr}(\rho^2) + 1} \right) + 2 \alpha - 4 \frac{\ln \left( \sqrt{2s + \sqrt{2}} \right)}{\alpha - 1} \ln \left( \frac{\sqrt{2s + \sqrt{2}}}{\sqrt{s(s + 1) \text{tr}(\rho^2) - s + \sqrt{2}}} \right).
\]
(51)

In particular, the corresponding min-entropies obey
\[
R_\infty(E; \rho) + R_\infty(T; \rho) \geq 2 \ln \left( \frac{\sqrt{2s + \sqrt{2}}}{\sqrt{s(s + 1) \text{tr}(\rho^2) - s + \sqrt{2}}} \right).
\]
(52)

Thus, we have obtained state-dependent uncertainty relation in terms of both the Tsallis and Rényi entropies. The derived bounds are expressed in terms of purity \( \text{tr}(\rho^2) \). The above expressions are especially useful, when purity of the measured state is sufficiently far from 1. For the case of pure states, the results (51) and (52) are used with \( \text{tr}(\rho^2) = 1 \). For instance, the min-entropies satisfy
\[
R_\infty(E; |\psi\rangle\langle\psi|) + R_\infty(T; |\psi\rangle\langle\psi|) \geq 2 \ln \left( \frac{\sqrt{2s + \sqrt{2}}}{s + \sqrt{2}} \right).
\]
(53)

Of course, the latter remains valid for arbitrary state. Using the results of [72], we can improve (53). By \( \|X\|_\infty \), we mean the spectral norm of the operator \( X \). It is defined as the maximal singular value of \( X \). Let \( L \) and \( N \) be positive operators that satisfy \( L \leq 1_{d+1} \) and \( N \leq 1_{d+1} \); then [73]
\[
\text{tr}(L\rho) + \text{tr}(N\rho) \leq 1 + \|\sqrt{L} \sqrt{N}\|_\infty.
\]
(54)

This results generalizes an inequality mentioned in [74] for measurements in two orthonormal bases. The authors of [73] used (5.1) to derive generalized uncertainty relations of the Landau–Pollak type. Substituting \( L = |\theta_m\rangle\langle\theta_n| \) and \( N = |\varepsilon_n\rangle\langle\varepsilon_n| \) gives
\[
\|\sqrt{L} \sqrt{N}\|_\infty = \frac{1}{\sqrt{s+1}}.
\]
(55)

We now combine (54) with (55) and also take into account \( q_m(T; \rho) \leq (d + 1)/(s + 1) \). Together, these observations leads to
\[
\max p_n(E; \rho) + \max q_m(T; \rho) \leq 1 + \Upsilon, \quad \Upsilon := \min \left\{ \frac{1}{\sqrt{s+1}}, \frac{d + 1}{s + 1} \right\}.
\]
(56)
When \( s + 1 > (d + 1)^2 \), we actually have \( T = (d + 1)/(s + 1) \). We also note that the function \( x \mapsto -\ln x \) is convex and decreasing. Combining these points with (54) and (55), one gets

\[
R_\infty(\mathcal{E}; \rho) + R_\infty(\mathcal{T}; \rho) \geq 2 \ln \left( \frac{2}{1 + \eta} \right).
\] (57)

For large \( s \), the right-hand side of (57) is approximately equal to \( \ln 4 \). In the same limit, the right-hand side of (52) becomes \( \ln 2 - \ln(\text{tr}(\rho^2)) \). Applying the latter to the completely mixed state, we obtain the lower bound \( \ln(2d + 2) \).

When we consider low-purity states of a system with several energy levels, the formula (52) is better than (57). In other cases, the result (57) seems to be preferable.

Using Tsallis entropies with the same parameter \( \alpha \), we can again address the case of detection inefficiencies. It is natural to suppose that both the efficiencies \( \eta_\mathcal{E} \) and \( \eta_\mathcal{T} \) are not less than 1/2. Due to (39) and (40), one gets

\[
H_\alpha(\mathcal{E}(\eta_\mathcal{E}); \rho) + H_\alpha(\mathcal{T}(\eta_\mathcal{T}); \rho) \geq 2\eta_0 \ln \left( \frac{2s + 2}{(s + 1) \text{tr}(\rho^2) + 1} \right) + 2h_\alpha(\eta).
\] (58)

where \( \alpha \in (0; 2] \) and \( \eta = \min\{\eta_\mathcal{E}, \eta_\mathcal{T}\} \). The result (58) is an entropic uncertainty relation in the model of detection inefficiencies. Entropies of actual probability distributions take into account not only quantum uncertainties. In the case \( \alpha = 1 \), the inefficiency-free lower bound is multiplied by \( \eta \) and added by \( 2h_1(\eta) \). Observations of similar kind were already reported in [62].

**IV. CONCLUSIONS**

We have addressed the problem of formulating entropic uncertainty relations of the “energy-time” type for a system with discrete levels of a certain structure. It was emphasized by founders that a universal form of time operator hardly exists. The derived results are based on Pegg’s concept of complement of the Hamiltonian. When ratios of energy values are rational exactly or approximately, we can construct measurements with the required properties. The Pegg concept allows us to treat the energy-time uncertainty principle similarly to other cases. It also reflects adequately features of time measurements, including possibly arbitrary choice of reference moments. To express quantum uncertainties, generalized entropies of the Rényi and Tsallis types were utilized. The derived uncertainty relations are immediately related to actual measurement statistics. Since our relations characterize energy-time uncertainties in more traditional style, they differ from recent results reported in [30]. Note also that state-dependent uncertainty relations were formulated. On the other hand, obtained entropic bounds of the Maassen–Uffink type turned out to be very similar. In this regard, Pegg’s concept of the Hamiltonian complement leads to a supplementary treatment of the bounds (24) and (44) within the preparation scenario. Of course, this treatment is restricted to discrete systems of the considered type.

In suitably extended space, the measurement of energy and its complement can be treated as mutually unbiased. Hence, we derived state-dependent entropic relations beyond the Maassen–Uffink approach. The corresponding bounds are expressed in terms of purity of the measured quantum state. Another type of entropic uncertainty relations follows from inequalities of the Landau–Pollak type. Entropic uncertainty relations provide not only another way to express some incompatibility of certain physical quantities. Such relations may be of practical interest as imposing some restrictions on probabilities of corresponding measurements. In this regard, the question of detection inefficiencies was incorporated into a consideration. Basic findings are similar to that was described previously. In reality, inefficiency-free entropic bounds will be multiplied by some factor depending on the efficiency parameter. In addition, there are additive entropic terms related purely to the employed detectors. It is known that entropic uncertainty relations can be useful for information-processing applications. Although the presented relations are restricted in their scope, they are applicable to typical systems used for information processing. Of course, many additional aspects should be taken into account in practice.

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