DIVISIBILITY OF SOME BINOMIAL SUMS

HE-XIA NI AND HAO PAN

Abstract. With help of \( q \)-congruence, we prove the divisibility of some binomial sums. For example, for any integers \( \rho, n \geq 2 \),

\[
\sum_{k=0}^{n-1} (4k + 1) \binom{2k}{k}^\rho \cdot (-4)^{\rho(n-1-k)} \equiv 0 \pmod{2^{\rho-2}n\binom{2n}{n}}.
\]

1. Introduction

In [5], Ramanujan listed 17 curious convergent series concerning \( 1/\pi \). For example, Ramanujan found that

\[
\sum_{k=0}^{\infty} \frac{6k + 1}{256^k} \cdot \binom{2k}{k}^3 = \frac{4}{\pi}.
\]

(1.1)

Nowadays, the theory of Ramanujan-type series has been greatly developed. In [1], Guillera gave a summary for the methods to deal with Ramanujan-type series.

In the recent years, the arithmetic properties of truncated Ramanujan-type series also be investigated. In [3], van Hamme proposed 13 conjectured congruences concerning truncated Ramanujan-type series. For example,

\[
\sum_{k=0}^{n-1} \frac{6k + 1}{256^k} \cdot \binom{2k}{k}^3 \equiv (-1)^{n-2}p \pmod{p^4},
\]

(1.2)

where \( p > 3 \) is a prime. Now all conjectures of van Hamme have been confirmed. The reader may refer to [7, 4] for the history of the proofs of van Hamme’s conjectures.

On the other hand, Sun [6] discovered that the convergent series concerning \( \pi \) often corresponds to the divisibility of some binomial sums. For example, Sun conjectured that for each integer \( n \geq 2 \)

\[
\sum_{k=0}^{n-1} (5k + 1) \binom{2k}{k}^2 \binom{3k}{k} \cdot (-192)^{n-1-k} \equiv 0 \pmod{n\binom{2n}{n}},
\]

(1.3)

2010 Mathematics Subject Classification. Primary 11B65; Secondary 05A10, 05A30, 11A07.

Key words and phrases. congruence; \( q \)-binomial coefficient.
which corresponds to the identity of Ramanujan

\[
\sum_{k=0}^{\infty} \frac{5k + 1}{(-192)^k} \cdot \binom{2k}{k}^2 \binom{3k}{k} = \frac{4\sqrt{3}}{\pi}.
\]  \hfill (1.4)

In this paper, we shall consider the divisibility of some binomial sums similar as (1.3). For \( \alpha \in \mathbb{Q} \setminus \mathbb{Z} \) and \( n \in \mathbb{Z}^+ \), define

\[ N_{\alpha,n} := \text{the numerator of } n \cdot \left| \binom{-\alpha}{n} \right| . \]

It is easy to see that \( N_{1/2,n} \) coincides with the odd part of \( n\binom{2n}{n} \).

**Theorem 1.1.** Suppose that \( \rho \) is a positive integer and \( \alpha \) is a non-integral rational number. Then for each integer \( n \geq 1 \),

\[
\sum_{k=0}^{n-1} (2k + \alpha) \binom{-\alpha}{k}^\rho \equiv 0 \pmod{N_{\alpha,n}}.  \hfill (1.5)
\]

In particular, substituting \( \alpha = 1/2 \) in (1.5), we may obtain that

**Corollary 1.1.** Suppose that \( \rho \geq 2 \) is an integer. Then for each integer \( n \geq 2 \),

\[
\sum_{k=0}^{n-1} (4k + 1) \binom{2k}{k}^\rho \cdot (-4)^{\rho(n-1-k)} \equiv 0 \pmod{2^{\rho-2}n \binom{2n}{n}}.  \hfill (1.6)
\]

2. \( q \)-CONGRUENCE

First, let us introduce the notion of \( q \)-congruence. For any \( x \in \mathbb{Q} \), define

\[ [x]_q := \frac{1 - q^x}{1 - q}. \]

Clearly if \( n \in \mathbb{N} = \{0, 1, 2, \ldots\} \), then \([n]_q = 1 + q + \cdots + q^{n-1}\) is a polynomial in \( q \).

For \( a, b \in \mathbb{N} \) and \( n \in \mathbb{Z}^+ \), if \( a \equiv b \pmod{n} \), then letting \( m = (a - b)/n \),

\[ [a]_q - [b]_q = \frac{q^b - q^a}{1 - q} = q^b \cdot \frac{1 - q^{nm}}{1 - q} = q^b \cdot \frac{[m]_q}{[n]_q} \equiv 0 \pmod{[n]_q}, \]

where the above congruence is considered over the polynomial ring \( \mathbb{Z}[q] \). Furthermore, we also have

\[
\frac{[nm]_q}{[n]_q} = \frac{1 - q^{nm}}{1 - q^n} = 1 + q^n + q^{2n} + \cdots + q^{(m-1)n} \equiv 1 + 1 + \cdots + 1 = m \pmod{[n]_q}.  \hfill (2.1)
\]

Note that (2.1) is still valid when \( m \) is a negative integer, since \([nm]_q = -q^{nm}[ -nm]_q\).

For \( d \geq 2 \), let \( \Phi_d(q) \) denote the \( d \)-th cyclotomic polynomial, i.e.,

\[
\Phi_d(q) = \prod_{1 \leq k \leq d \atop (d,k) = 1} (q - e^{2\pi \sqrt{-1} \frac{k}{d}}).
\]
It is well-known that $\Phi_d(q)$ is an irreducible polynomial with integral coefficients. Also, we have

$$[n]_q = \prod_{d \geq 2, d \mid n} \Phi_d(q).$$

So $\Phi_d(q)$ divides $[n]_q$ if and only if $d$ divides $n$. Furthermore,

$$\Phi_d(1) = \begin{cases} p, & \text{if } d = p^k \text{ for some prime } p, \\ 1, & \text{otherwise}. \end{cases} \quad (2.2)$$

For $n \in \mathbb{N}$, define

$$(x; q)_n := \begin{cases} (1 - x)(1 - xq) \cdots (1 - xq^{n-1}), & \text{if } n \geq 1, \\ 1, & \text{if } n = 0. \end{cases}$$

Also, define the $q$-binomial coefficient

$$\begin{bmatrix} x \\ n \end{bmatrix}_q := \frac{(q^{x-n+1}; q)_n}{(q; q)_n}.$$

Clearly

$$\lim_{q \to 1} \begin{bmatrix} x \\ n \end{bmatrix}_q = \binom{x}{n}.$$

Furthermore, it is easy to see that

$$\begin{bmatrix} -\frac{r}{m} \\ n \end{bmatrix}_{q^m} = (-1)^n q^{-nr - m(n-1)} \cdot \frac{(q^r; q^m)_n}{(q^m; q^m)_n}.$$

Suppose that $r \in \mathbb{Z}$, $m \in \mathbb{Z}^+$ and $(r, m) = 1$. For each positive integer $d$ with $(d, m) = 1$, let $\lambda_{r,m}(d)$ be the integer lying in $\{0, 1, \ldots, d-1\}$ such that

$$r + \lambda_{r,m}(d)m \equiv 0 \pmod{d}. \quad (2.3)$$

Let

$$S_{r,m}(n) = \left\{ d \geq 2 : \left\lfloor \frac{n - 1 - \lambda_{r,m}(d)}{d} \right\rfloor = \left\lfloor \frac{n}{d} \right\rfloor \right\}.$$

Evidently for each $d > \max_{0 \leq j \leq n-1} |r + jm|$, we must have $\lambda_{r,m}(d) > n - 1$, whence $d \not\in S_{r,m}(n)$. So $S_{r,m}(n)$ is always a finite set. Let

$$A_{r,m,n}(q) = \prod_{d \in S_{r,m}(n)} \Phi_d(q) \quad (2.4)$$

and

$$C_{m,n}(q) = \prod_{d \mid n, (d,m)=1} \Phi_d(q). \quad (2.5)$$
Clearly, if \( d \mid n \), then we can’t have \( d \in \mathcal{S}_{r,m}(n) \). So \( A_{r,m,n}(q) \) and \( C_{m,n}(q) \) are co-prime. Furthermore, as we shall see in the next section,

\[
A_{r,m,n}(1)C_{m,n}(1) = N_{\frac{r}{m},n}.
\]

(2.6)

The following theorem is the key ingredient of this paper.

**Theorem 2.1.** Suppose that \( r \in \mathbb{Z} \) and \( m \in \mathbb{Z}^+ \). Assume that \( \mu_0(q), \mu_1(q), \ldots \) is a sequence of rational functions in \( q \) such that for any \( d \in \mathbb{Z}^+ \) with \( (m,d) = 1 \),

(i) \( \nu_k(q) \) is \( \Phi_d(q) \)-integral for each \( k \geq 0 \), i.e., the denominator of \( \nu_k(q) \) is not divisible by \( \Phi(q) \);

(ii) for any \( s, t \in \mathbb{N} \) with \( 0 \leq t \leq d - 1 \),

\[
\nu_{sd+t}(q) \equiv \mu_s(q)\nu_t(q) \pmod{\Phi_d(q)},
\]

where \( \mu_s(q) \) is a \( \Phi_d(q) \)-integral rational function only depending on \( s \);

(iii) \[
\sum_{k=0}^{d-1} \frac{(q^r; q^m)_k}{(q^m; q^m)_k} \cdot \nu_k(q) \equiv 0 \pmod{\Phi_d(q)}.
\]

Then

\[
\sum_{k=0}^{n-1} \frac{(q^r; q^m)_k}{(q^m; q^m)_k} \cdot \nu_k(q) \equiv 0 \pmod{A_{r,m,n}(q)C_{m,n}(q)}.
\]

(2.7)

Before we give the proof of Theorem 2.1, which will occupy the subsequence section, let us see an immediate consequence of Theorem 2.1.

**Corollary 2.1.** Under the Proposition, additionally assume that for each positive integer \( n \), there exists a polynomial \( B_n(q) \) with integral coefficients such that

(i)

\[
B_n(q) \sum_{k=0}^{n-1} \frac{(q^r; q^d)_k}{(q^d; q^d)_k} \cdot \nu_k(q)
\]

is a polynomial with integral coefficients.

(ii) \( B_n(1) \) is not divisible by any prime \( p \) with \( p \nmid m \);

Then for any \( n \geq 1 \), we have

\[
\sum_{k=0}^{n-1} (-1)^k \left( \frac{r}{k} \right) \cdot \nu_k(1) \equiv 0 \pmod{N_{\frac{r}{m},n}}.
\]

(2.8)
Proof. By Theorem 2.1, we have

\[ B_n(q) \sum_{k=0}^{n-1} (q^r; q^m)_k \cdot \nu_k(q) = A_{r,m,n}(q) C_{m,n}(q) \cdot H(q), \]

where \( H(q) \) is a polynomial in \( q \). Notice that the greatest common divisor of all coefficients of \( A_{r,m,n}(q) C_{m,n}(q) \) is just 1. According to a well-known result of Gauss, we know that the coefficients of \( H(q) \) must be all integers. Hence substituting \( q = 1 \) in (2.7), we get

\[ B_n(1) \sum_{k=0}^{n-1} (-1)^k \left( \frac{r}{m} \right)_k \cdot \nu_k(1) = N_{r,m,n} \cdot H(1) \equiv 0 \pmod{N_{r,m,n}}. \]

Since \( N_{r,m,n} \) is prime to \( B_n(1) \), (2.8) is concluded. \( \square \)

3. Proof of Theorem 2.1

In this section, we shall complete the proof of Theorem 2.1. First, we need several auxiliary lemmas.

Lemma 3.1. Let \( r \in \mathbb{Z} \) and \( m, d \in \mathbb{Z}^+ \) with \( (m, d) = 1 \). Then

\[ \frac{(q^r; q^m)_d}{1 - q^d} \equiv r + \lambda_{r,m}(d)m \pmod{\Phi_d(q)}, \] (3.1)

where \( \lambda_{r,m} \) is the one defined by (2.3).

Proof. Clearly

\[ \frac{(q^r; q^m)_d}{1 - q^d} = \frac{1 - q^{r+\lambda_{r,m}(d)m}}{1 - q^d} \prod_{0 \leq j \leq d-1 \atop r+jm \neq 0 \pmod{d}} (1 - q^{r+jm}) \]
\[ \equiv \frac{1 - q^{d \lambda_{r,m}(d)m}}{1 - q^d} \prod_{j=1}^{d-1} (1 - q^j) \equiv \frac{r + \lambda_{r,m}(d)m}{d} \cdot (q; q)_{d-1} \pmod{\Phi_d(q)}. \]

Now for every primitive \( d \)-th root of unity \( \xi \), we have

\[ (q; q)_{d-1} \big|_{q=\xi} = \prod_{j=1}^{d-1} (1 - \xi^j) = \lim_{x \to 1} \prod_{j=1}^{d-1} (x - \xi^j) = \lim_{x \to 1} \frac{x^d - 1}{x - 1} = d. \]

So

\[ (q; q)_{d-1} \equiv d \pmod{\Phi_d(q)}. \] \( \square \)
Lemma 3.2. Under the assumptions of Lemma 3.1 for any $s, t \in \mathbb{N}$ with $0 \leq t \leq d - 1,$

\[
\frac{(q^r; q^m)_{sd+t}}{(q^m; q^m)_{sd+t}} = \frac{(r+\lambda_{r,m}(d)m)}{md} \frac{(q^r; q^m)_t}{(q^m; q^m)_t} \pmod{\Phi_d(q)}. \tag{3.2}
\]

Proof. By Lemma 3.1, we have

\[
\frac{(q^r; q^m)_{sd+t}}{(1 - q^d)^s} = (q^{r+smd}; q^m) \prod_{j=0}^{s-1} (q^{r+jmd}; q^m) \equiv (q^r; q^m) \prod_{j=0}^{s-1} (r + \lambda_{r,m}(d)m + jmd) \pmod{\Phi_d(q)}.
\]

Similarly,

\[
\frac{(q^m; q^m)_{sd+t}}{(1 - q^d)^s} \equiv (q^m; q^m) \prod_{j=0}^{s-1} (m + (d-1)m + jmd) \pmod{\Phi_d(q)}.
\]

Clearly

\[
\prod_{j=0}^{s-1} \frac{r + \lambda_{r,m}(d)m + jmd}{md + jmd} = \frac{(r+\lambda_{r,m}(d)m)}{md} (1)^s.
\]

Thus we get (3.2), since $(q^m; q^m)_t$ is prime to $\Phi_d(q)$ for each $0 \leq t \leq d - 1.$ \hfill \qed

Let $\lfloor \cdot \rfloor$ denote the floor function, i.e., $\lfloor x \rfloor = \max \{ k \in \mathbb{N} : k \leq x \}$ for every $x \in \mathbb{R}.$

Lemma 3.3. Suppose that $r \in \mathbb{Z}, m \in \mathbb{N}$ and $(r, m) = 1.$ Then

\[
\frac{(q^r; q^m)_n}{(q^m; q^m)_n} \prod_{(d, m) > 1} \Phi_d(q)^{\lfloor \frac{n-md}{d} \rfloor} = (-1)^\delta q^\Delta \prod_{d \in S_{r,m}(n)} \Phi_d(q), \tag{3.3}
\]

where $\delta = |\{0 \leq j \leq n-1 : r + jm < 0\}|$ and

\[
\Delta = \sum_{0 \leq j \leq n-1 \atop r+jm < 0} (r+jm).
\]

Proof. Note that for any $h \in \mathbb{N}$

\[
1 - q^h = \prod_{d \nmid h} \Phi_d(q).
\]

So

\[
(q^r; q^m)_n = (-1)^\delta q^\Delta \prod_{(d, m) = 1} \Phi_d(q)^{|\{0 \leq j \leq n-1 : r+jm \equiv 0 \pmod{d}\}|}.
\]
It is easy to check that
\[ |\{0 \leq j \leq n-1 : r + jm \equiv 0 \pmod{d}\}| = 1 + \left\lfloor \frac{n-1 - \lambda_{r,m}(d)}{d} \right\rfloor. \]

Similarly,
\[ (q^m; q^m)_n = \prod_{d \geq 1} \Phi_d(q)^{|\{1 \leq j \leq n : jm \equiv 0 \pmod{d}\}|}, \]
and
\[ |\{1 \leq j \leq n : jm \equiv 0 \pmod{d}\}| = \left\lfloor \frac{n(m,d)}{d} \right\rfloor. \]

Hence \( d \in S_{r,m}(n) \) if and only if \((d, m) = 1 \) and
\[ |\{0 \leq j \leq n-1 : r + jm \equiv 0 \pmod{d}\}| = |\{1 \leq j \leq n : jm \equiv 0 \pmod{d}\}| + 1. \]

We immediately get (3.3). \[\square\]

Let
\[ B_{r,m,n}(q) = \prod_{\substack{|l_n|_{l \geq 2} \\ (d,m)=l}} \Phi_d(q)^{\frac{\alpha m}{\pi l}}, \tag{3.4} \]

Then (3.3) is equivalent to
\[ \frac{(q^r; q^m)_n}{(q^m; q^m)_n} = (-1)^{\delta q} \cdot \frac{A_{r,m,n}(q)}{B_{r,m,n}(q)}. \]

According to the definitions, clearly \( B_{r,m,n}(q) \) is prime to \( A_{r,m,n}(q)C_{m,n}(q) \). Also, \( A_{r,m,n}(1)C_{m,n}(1) \) and \( B_{r,m,n}(1) \) are co-prime integers. Moreover, \( B_{r,m,n}(q) \) is divisible by
\[ \frac{[n]_q}{C_{m,n}(q)} = \prod_{\substack{d|n \\ (d,m)>1}} \Phi_d(q). \]

So we must have \( A_{r,m,n}(1)C_{m,n}(1) \) coincides with the numerator of \( n \cdot \left\lfloor \left( -\frac{r}{m} \right) \right\rfloor \), i.e., (2.6) is valid.

Now we are ready to prove Theorem 2.1.

**Proof of Theorem 2.1.** It suffices to show that the left side of (2.7) is divisible by \( \Phi_d(q) \) for those \( d \in S_{r,m}(n) \) and \( d \mid n \) with \((m,d) = 1.\)

Suppose that \( d \in S_{r,m}(n) \). Write \( n = ud + v \) where \( 0 \leq v \leq d-1 \). Let
\[ h = \lambda_{r,m}(d), \quad w = \frac{r + \lambda_{r,m}(d)m}{d}. \]

Note that \( d \in S_{r,m}(n) \) implies that \( v \geq 1 + h. \) Hence for any \( v \leq t \leq d-1 \), we have
\[ (q^r; q^m)_t = \prod_{0 \leq j \leq t-1 \atop j \neq h} (1 - q^{r+jm}) \equiv 0 \pmod{\Phi_d(q)}. \]
Thus the requirement (iii) of Theorem 2.1 is satisfied.

Note that applying Lemma 3.2, we get

\[ \frac{(q^r; q^m)_{ud+t}}{(q^m; q^m)_{ud+t}} \equiv 0 \pmod{\Phi_d(q)}. \]

Thus applying Lemma 3.2, we get

\[ \sum_{k=0}^{n-1} \frac{(q^r; q^m)_k}{(q^m; q^m)_k} \cdot \nu_k(q) = \sum_{k=0}^{ud-d-1} \frac{(q^r; q^m)_k}{(q^m; q^m)_k} \cdot \nu_k(q) = \sum_{s=0}^{u} \sum_{t=0}^{d-1} \frac{(q^r; q^m)_t}{(q^m; q^m)_t} \cdot \nu_t(q) \equiv 0 \pmod{\Phi_d(q)}. \]

Furthermore, assume that \( d \mid n \) and \( (m, d) = 1 \). Let \( u = n/d \). Then in view of (3.2), we also have

\[ \sum_{k=0}^{n-1} \frac{(q^r; q^m)_k}{(q^m; q^m)_k} \cdot \nu_k(q) = \sum_{s=0}^{u-1} \frac{(w)_s}{(1)_s} \cdot \mu_s(q) \sum_{t=0}^{d-1} \frac{(q^r; q^m)_t}{(q^m; q^m)_t} \cdot \nu_t(q) \equiv 0 \pmod{\Phi_d(q)}. \]

(3.5)

4. Proofs of Theorem 1.1 and Corollary 1.1

Proof Theorem 1.1 Write \( \alpha = r/m \), where \( r \in \mathbb{Z}, m \in \mathbb{Z}^+ \) and \( (r, m) = 1 \). Assume that \( d \geq 1 \) and \( (m, d) = 1 \). Let \( h = \lambda_{r,m}(d) \). Clearly \( r \equiv -hm \pmod{d} \). Then

\[ \frac{(q^r; q^m)_k}{(q^m; q^m)_k} = \frac{(q^{-hm}; q^m)_k}{(q^m; q^m)_k} = (-1)^k q^{m(k^2 - mhk)} [h]_q \left( \frac{h}{k} \right)_q \pmod{\Phi_d(q)}. \]

Note that

\[ \sum_{k=0}^{d-1} q^{mk}[2mk - hm]_q \cdot \left[ \frac{h}{k} \right]_q^\rho = \sum_{k=0}^{h} q^{m(h-k)}[2m(h-k) - hm]_q \cdot \left[ \frac{h}{k} \right]_q^\rho = - \sum_{k=0}^{h} q^{mk}[2mk - hm]_q \cdot \left[ \frac{h}{k} \right]_q^\rho. \]

We must have

\[ \sum_{k=0}^{d-1} q^{mk}[2mk + r]_q \cdot (-1)^k q^{\rho(mh(k^2 - mhk))} \cdot \frac{(q^r; q^m)_k}{(q^m; q^m)_k} \]

\[ \equiv \sum_{k=0}^{d-1} q^{mk}[2mk - hm]_q \cdot \left[ \frac{h}{k} \right]_q^\rho \equiv 0 \pmod{\Phi_d(q)}. \]

(4.1)

Thus the requirement (iii) of Theorem 2.1 is satisfied.
We still need to verify the requirement (ii) of Theorem 2.1. By Lemma 3.2, for each \( s, t \in \mathbb{N} \) with \( 0 \leq t \leq d - 1 \),

\[
q^{m(s^d+t)}[2m(sd + t) + r]q \cdot \frac{(q^r; q^m)^{\rho - 1}_t}{(q^m; q^m)^{\rho - 1}_t} = \frac{N}{(1)^{\rho - 1}_s} \cdot q^{mt}[2mt + r]q \cdot \frac{(q^r; q^m)^{\rho - 1}_t}{(q^m; q^m)^{\rho - 1}_t} \pmod{\Phi_d(q)}.
\]

And

\[
(-1)^{sd+t}q^{mh(sd+t)-m(s^d+1)} = (-1)^{sd+t}q^{mh(sd+t)-msdt- m(s^d+1)} \equiv (-1)^{sd}q^{-m(s^d+1)} \equiv (-1)^s \pmod{\Phi_d(q)}.
\]

If \( d \) is odd, then clearly

\[
(-1)^{sd}q^{-m(s^d+1)} \equiv (-1)^s \pmod{\Phi_d(q)}.
\]

If \( d \) is even, then

\[
1 + q^{d/2} = \frac{1 - q^d}{1 - q^{d/2}} \equiv 0 \pmod{\Phi_d(q)},
\]

i.e., \( q^{d/2} \equiv -1 \pmod{\Phi_d(q)} \). So

\[
(-1)^{sd}q^{-m(s^d+1)} = (q^{d/2})^{-ms(dt)} \equiv (-1)^s \pmod{\Phi_d(q)},
\]

by noting that \( m \) is odd since \((m,d) = 1\). That is, we always have

\[
(-1)^{sd+t}q^{mh(sd+t)-msdt- m(s^d+1)} \equiv (-1)^s \cdot (-1)^tq^{-m(s^d+1)} \pmod{\Phi_d(q)}.
\]

Thus applying Theorem 2.1, we obtain that

\[
\sum_{k=0}^{n-1} q^{mk}[2mk + r]q \cdot (-1)^{ok}q^{\rho(mkh-m(s^d+1))} \cdot \frac{(q^r; q^m)^{\rho}_k}{(q^m; q^m)^{\rho}_k} \equiv 0 \pmod{\Phi_d(q)} \pmod{A_{r,m,n}(q)C_{m,n}(q)}.
\]

(4.2)

On the other hand, clearly \( B_{r,m,n}(q) \) is divisible by \( B_{r,m,k}(q) \) provided \( 0 \leq k \leq n - 1 \). It follows from Lemma 3.3 that

\[
B_{r,m,n}(q)^\rho \sum_{k=0}^{n-1} q^{mk}[2mk + r]q \cdot \frac{(q^r; q^m)^{\rho}_k}{(q^m; q^m)^{\rho}_k} \equiv 0 \pmod{N_{r,m,n}(q)}.
\]

is a polynomial with integral coefficients. And by (2.2), each prime factor of \( B_{r,m,n}(1) \) must divide \( m \). In view of Corollary 2.1, we have

\[
\sum_{k=0}^{n-1} (2mk + r) \cdot \left( \frac{-r}{m} \right)^\rho \equiv 0 \pmod{N_{r,m,n}(q)}.
\]

So (1.5) is valid since \( N_{r,m,n} \) and \( m \) are co-prime. \( \square \)
Proof of Corollary 1.1. As we have mentioned, $N_{2n}$ coincides with the odd part of $n \binom{2n}{n}$. So by substituting $\alpha = 1/2$ in Theorem 1.1, we only need to compute the 2-adic of the left side of (1.6). For a positive integer $a$, let $\text{ord}_2(a)$ denote the 2-adic order of $a$, i.e., $2^{\text{ord}_2(a)} | a$ but $2^{\text{ord}_2(a)+1} \nmid a$. For each $0 \leq k \leq n - 1$, since

$$n \binom{2n}{n} = \binom{2k}{k} \cdot \frac{2^{n-k} \cdot (2n-1)(2n-3) \cdots (2k+1)}{(n-1)(n-2) \cdots (k+1)},$$

we have

$$\text{ord}_2 \left( n \binom{2n}{n} \right) \leq n - k + \text{ord}_2 \left( \binom{2k}{k} \right).$$

Also, $\binom{2k}{k}$ is even for each $k \geq 1$, since

$$\binom{2k}{k} + 2 \sum_{j=0}^{k-1} \binom{2k}{j} = 2^{2k}.$$

Hence for each $0 \leq k \leq n - 1$,

$$\text{ord}_2 \left( \binom{2k}{k}^\rho \cdot 4^{\rho(n-1-k)} \right) \geq (\rho - 1) + 2(n - 1 - k) + \text{ord}_2 \left( \binom{2k}{k} \right)$$

$$\geq (\rho - 2) + \text{ord}_2 \left( n \binom{2n}{n} \right).$$

□

References

[1] J. Guillera, Kind of proofs of ramanujan-like series, preprint, [arXiv:1203.1255](https://arxiv.org/abs/1203.1255).
[2] Victor J. W. Guo and J.-C. Liu, $q$-Analogues of two Ramanujan-type formulas for $1/\pi$, preprint, [arXiv:1802.01944](https://arxiv.org/abs/1802.01944).
[3] L. van Hamme, Some conjectures concerning partial sums of generalized hypergeometric series, in: p-Adic Functional Analysis, Nijmegen, 1996, in: Lecture Notes in Pure and Appl. Math., vol. 192, Dekker, New York, 1997, pp. 223-236.
[4] R. Osburn and W. Zudilin, On the (K.2) supercongruence of Van Hamme, J. Math. Anal. Appl., 433(2016), 706-711.
[5] S. Ramanujan, Modular equations and approximations to $\pi$, Quart. J. Math., 45(1914), 350-372.
[6] Z.-W. Sun, Open conjectures on congruences, preprint, [arXiv:0911.5665](https://arxiv.org/abs/0911.5665).
[7] H. Swisher, On the supercongruence conjectures of van Hamme, Res. Math. Sci., 2(2015), Art. 18, 21 pp.

Department of Mathematics, Nanjing University, Nanjing 210093, People’s Republic of China

E-mail address: nihexia@yeah.net

Department of Mathematics, Nanjing University, Nanjing 210093, People’s Republic of China

E-mail address: haopan79@zoho.com