

Decay by tunneling of Bosonic and Fermionic Tonks-Girardeau Gases

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We study the tunneling dynamics of bosonic and fermionic Tonks-Girardeau gases from a hard wall trap, in which one of the walls is substituted by a delta potential. Using the Fermi-Bose map, the decay of the probability to remain in the trap is studied as a function of both the number of particles and the intensity of the end-cap delta laser. The fermionic gas is shown to be a good candidate to study deviations of the non-exponential decay of the single-particle type, whereas for the bosonic case a novel regime of non-exponential decay appears due to the contributions of different resonances of the trap.

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I. INTRODUCTION

Decay of a metastable system via tunneling is one of the most remarkable and old effects in quantum mechanics. Since Gamow’s analysis of alpha decay, resonance theory, which applies to virtually all fields from particle to molecular physics, has been motivated by this phenomenon. Simple treatments examine the escape or survival of single particle wave functions in one dimensional (1D) potentials. At this level much attention has been paid to deviations from exponential decay, and Zeno or anti-Zeno effects. Also, exact results are typically available, by means of analytical models or numerically. The more complex decay of a multiparticle unstable system is treated by more sophisticated multichannel, or reactive-scattering approaches, sometimes with statistical approximations or, depending on the system and environment, in a phenomenological way, and also using mean-field approximations. In “macroscopic quantum tunneling”, a macroscopic variable, such as the phase difference of the Cooper pair wave function across a Josephson junction obeys a simple tunneling equation for an effective particle subjected to dissipation [1, 2]. The effect of dissipation due to the perturbing environment has thus been extensively discussed and measured. Other macroscopic quantum tunneling effect much studied in recent times is the tunneling and decay of Bose-Einstein condensates; in particular the effect of effective atom-atom interaction and the non-linear term in the Gross-Pitaevskii mean-field approach [3, 4, 5].

Some works go beyond the mean field theory using simplified Hamiltonians [6], which is particularly relevant for few-body systems. These systems with not too many particles may still be amenable of exact treatments but will show differences from single and many-particle ones. The experimental study of few-body tunneling is a formidable prospect, as it requires initial preparation of a ground few-body Fock state, precise control over the tunneling time, and the ability to count single atoms with unit quantum efficiency. Until recently, such capabilities did not exist so that any experimental tests seemed unlikely. However recent developments now open the door for few body tunneling experiments and motivate the present work in anticipation of such results. The starting point was the development of a novel optical box trap that confines a degenerate Bose gas, together with single-atom counting [7, 8]. The same box trap was used to produce number squeezing of atoms by confining a degenerate Bose gas and controlled lowering of the walls until a final value [9]. The observed fluctuations in number were a factor of two below the Poissonian limit, but the residual noise can be accounted for by known sources of technical noise, so that these experiments are consistent with number-state production. This simple procedure, called “laser culling of atoms” has recently been analyzed theoretically, and is shown to produce atomic few-body Fock states for sufficiently slow ramp time and neglecting quantum tunneling through the barrier [10]. The latter effect can be highly suppressed by sculpting the shape of the barrier using the techniques described in [8]. The barrier width can then be reduced at a well defined time, allowing quantum tunneling to occur. This system should therefore enable the first experimental study of few body quantum tunneling in different regimes of interaction and with controlled number.

In this paper we study the decay from a trap by tunneling trough a delta barrier of a few-body Tonks-Girardeau gas with the aim of obtaining exact results of a few-body decay problem for which the experimental verification is

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in view \[7, 8, 9\]. In particular, we find few-body deviations from the exponential decay law. For one particle, deviations from exponential decay have been long predicted at both short and long times. Short time deviations were observed experimentally with ultracold atoms \[11\], whereas long time deviations have been observed very recently for the first time in dissolved organic materials \[12\]. A suitable system to test deviations from exponential decay at the few-body level is the bosonic Tonks-Girardeau (BTG) gas \[13\]. Such a gas actually mimics the fermionic behavior to minimize the strongly repulsive interaction. At low densities, the TG regime can be reached under a strong enough radial confinement \[14\] such that the transverse degrees of freedom are reduced to zero-point oscillations, resulting a 1D effective unit function”

\[
F(x_1, \ldots, x_N) = \prod_{i=1}^{N} \phi_0(x_i).
\]

In a similar way, one can deal with the Fermionic TG (FTG) gas \[17\], but using now the generalized FB mapping in the opposite direction. The wavefunction of the FTG gas can be written in terms of the Hartree product describing the dual (auxiliary) system, which is now the ideal Bose gas,

\[
\tilde{\psi}_B(x_1, \ldots, x_N) = \prod_{i=1}^{N} \phi_0(x_i).
\]

II. SINGLE EIGENMODE DYNAMICS

First of all we shall study the time evolution of the \(n\)-th eigenstate of a hard wall trap. As it is well-known they have the general form

\[
\phi_n(x, t = 0) = \sqrt{2/L} \sin \left( \frac{n\pi x}{L} \right) \chi_{[0,L]}(x),
\]

with \(n \in \mathbb{N}\) and \(\chi_{[0,L]}(x)\) the characteristic function in \([0, L]\).

At time equal zero the right-wall is substituted by a delta potential, \(V(x) = \eta \delta(x - L)\), which is to represent a far-detuned laser from the atomic resonance, so that atomic excitation becomes negligible. This model has been recently considered to study non-exponential decay at both short and long times at the single-particle level in \[27\]. The time evolution of a given initial state may be written in terms of the retarded Green’s function \(g(x, x'; t)\) as

\[
\phi_n(x, t) = \int_0^L g(x, x'; t) \phi_n(x', 0) dx'.
\]

Equation \[41\] may be calculated using an expansion in the eigenfunctions of the Hamiltonian \(|k^+\rangle |k \in \mathbb{R}^+\rangle\), i.e., the so called physical wave solutions,

\[
\phi_n(x, t) = \int_0^\infty dk |k^+ \rangle \langle k^+ | \phi_n \rangle e^{-ikx^2/2m},
\]

\[
\langle x | k^+ \rangle = \sqrt{\frac{2}{\pi}} \left\{ \sin(kx)/J_0(k), \quad x \leq L \right. \\
\left. (i/2) \left[ e^{-ikx} - S(k)e^{ikx} \right], \quad x \geq L, \right.
\]

In this way, one can deal with the Fermionic TG (FTG) gas \[17\], but using now the generalized FB mapping in the opposite direction. The wavefunction of the FTG gas can be written in terms of the Hartree product describing the dual (auxiliary) system, which is now the ideal Bose gas,
The decaying particle and in all figures we use dimensionless units with $L = 1$ and $2m = \hbar = 1$.

where the S-matrix $S(k) = J_-(k)/J_+(k)$, with $J_-(k) = J_+^*(k)$, and the Jost function $J_+(k)$ reads

$$J_+(k) = 1 + \frac{\eta}{2i\hbar}(e^{2ikL} - 1).$$

Above and in the rest of this work we take $k = [2mE/\hbar^2]^{1/2}$ and $\eta = [2m/\hbar^2]\eta^i$, $E$ being the energy of the decaying particle and $m$ its mass.

The position of the pole $k_j$ of the S-matrix in the $k$-plane gives us information about the lifetime $\hbar/\Gamma_j$ of the resonance and its real energy $E_j$ in the leaking trap (see Fig. 1): $E_j = k_j^2\hbar^2/(2m) = E_j - i\Gamma_j/2$. We notice that for a weaker laser (smaller $\eta$), the resonance decays faster. Besides, $j$ is chosen so that the width of the resonance increases monotonically with it, $j = 1$ corresponding to the longest lived resonance.

The limit $\eta \to 0$ is an important reference corresponding to free evolution in the presence of a hard wall at the origin, once the endcap laser at $x = L$ has been turned off. In such case, a fully analytical solution is available invoking the method of images [27], see a detailed study of the BTG expansion dynamics in the absence of the wall in [25].

The decay of the different eigenstates of the hard wall trap is shown in Fig. 2. It exhibits the characteristic transition to long times as an inverse power of time. The scale of the figure conceals an interesting behavior at short times, which is exhibited in Fig. 3. We observe, with exception of the first decaying state, that each state presents some characteristic oscillations at short times and then tends to decay with the same slope as the first state. Physical insight regarding this behavior may be obtained making use of the formalism of resonant states. In this representation one Laplace transforms $g(x, x'; t)$ in Eq. (9) into the complex $k$-plane to exploit the analytical properties of the corresponding outgoing Green’s function $G^+(x, x'; k)$. This leads to an alternative expression to Eq. (11), namely,

$$\phi_n(x, t) = \sum_{j=1}^{\infty} c_j(n)u_j(x)e^{-\hbar^i k_j^2 t/2m} +$$

$$\frac{i}{\pi} \int_{0}^{L} \phi_n(x', 0)dx' \int_{C_L} G^+(x, x'; k)e^{-i\hbar^i k_j^2 t/2m}kdk,$$

where the sum runs over the so called proper complex poles i.e., those on the fourth quadrant of the complex $k$-plane with $\text{Re}(k_j) > |\text{Im}(k_j)|$; the $u_j$’s obey outgoing boundary conditions at $x = L$ and satisfy the Schrödinger equation with complex eigenvalues $E_j = E_j - i\Gamma_j/2$; the coefficients $c_j(n)$ give the overlap of the initial state $\phi_n$ with the resonant states $u_j$ of the problem, namely, $c_j(n) = \int_{0}^{L} \phi_n(x, 0)u_j(x)dx$; the integral term involving $G^+(x, x'; k)$ stands for the non-exponential contribution, which in general may be neglected except at ultrashort or very long times. The path $C_L$ of the integral is chosen, without loss of generality, as a straight line 45 degrees off the real $k$-axis along the complex $k$-plane passing through the origin $k = 0$.

The nonescape probability is defined as

$$P_n(t) = \int_{0}^{L} |\phi_n(x, t)|^2dx,$$

(to be distinguished from the “survival probability” $|\langle \phi_n(t) | \phi_n(0) \rangle|^2$, which is harder to measure). Using the exponential contribution to Eq. (11) into Eq. (14) yields...
the nonequilibrium probability for each decaying eigenstate as

\[ P_n^{exp}(t) = \sum_{j=1}^{N} c_j(n)c_j^*(n)I_{js} e^{-i(E_j-E_s)t/\hbar} e^{-(\Gamma_j+\Gamma_s)t/2\hbar}, \]

where \( I_{js} = \int_0^L u_j(x)u_s^*(x)dx \). The exponential term in Eq. (15), with \( N = n \), reproduces very well the regime depicted in Fig. 1. It reflects a transient regime where each eigenstate \( \phi_n \) makes eventually a transition into the longest lived eigenstate (\( j = 1 \)). As shown in Fig. 1, after the exponential regime, it follows the long time \( t^{-3} \) inverse power law governed by the integral contribution to Eq. (13). Standard asymptotic analysis gives the result

\[ P_n^{long}(t) = \left( \frac{2m}{\hbar} \right)^3 \frac{L^3}{12\pi(1+\eta L)^4} \frac{C^2(n)}{t^3}, \]

where \( C(n) = \int_0^L \phi_n(x,0)xdx = L\sqrt{2L}(\frac{(-1)^n}{n\pi}). \) Clearly the transients in the transition from exponential to nonexponential behavior observed in Fig. 1 originate from the interference between the exponential and long time expressions of \( \phi_n(x,t) \). The transition may be displaced to earlier times and made more easily observable by decreasing the ratio \( R = E_1/\Gamma_1 \) of the longest lived resonance.

III. TONKS-GIRARDEAU GAS

We next generalize the notion of nonequilibrium probability to a few particle system, associating it with the average number of particles within the trap at a given time,

\[ N_T(t) = \int_0^L dx \rho(x,t), \]

where \( \rho(x,t) \) is the density normalized to the total number of particles \( N \). The nonequilibrium probability per particle thus becomes \( P(t) = N_T/N \). From this definition it is clear that for the FTG gas, \( P(t) \) is identical to the single particle nonequilibrium probability associated with the ground state of the trap. However, this means that the total signal is enhanced by a factor corresponding to the number of particles, a key advantage for the experimental study of deviations from the exponential decay law. In other words, \( \ln N_T \) for an \( N \)-particle FTG gas in its ground state, is obtained by shifting upwards the curve for \( n = 1 \) in Figs. 2 or 3 by \( \ln N \).

More remarkable yet, the BTG gas exhibits novel few-body decay features. In particular, a new regime of non-exponential decay arises from the sum of contributions of the single particle exponentials. Figure 4 shows that, for the BTG gas, the higher the number of particles the faster is the tunneling rate and, as a result of the different energy contributions, the short time dependence is enhanced by a factor corresponding to the total number of particles. From top to bottom, \( N = 1, \ldots, 10 \) with \( \eta = 5 \). The curve for the FTG gas of \( N \)-particles is obtained by shifting up the \( N = 1 \) BTG gas curve (single-particle case) by \( \ln N \).

![FIG. 3](image_url)  
**FIG. 3:** Single particle nonequilibrium probability for the first ten eigenstates of the hard wall trap, tunneling through a delta potential with \( \eta = 5 \) at short times. From top to bottom: \( n = 1, \ldots, 10 \).

![FIG. 4](image_url)  
**FIG. 4:** Logarithm of the average number of particles within the trap for a BTG gas as a function of time, for different values of the total number of particles. From top to bottom, \( N = 1, \ldots, 10 \) with \( \eta = 5 \). The curve for the FTG gas of \( N \)-particles is obtained by shifting up the \( N = 1 \) BTG gas curve (single-particle case) by \( \ln N \).

The effect of the end-cap laser intensity can be simulated by varying \( \eta \). In so doing, we have learnt that the resonance poles shift towards the real axis in the \( k \)-plane for increasing \( \eta \), see Fig. 1. For vanishing \( \eta \), even for the single particle case, and therefore the FTG gas, there is no reason to expect exponential decay since the potential is not a trap any more and there are no resonances.

This behavior is shown in Fig. 3. For a given \( \eta \), a clear difference between BTG and FTG gases is the decay rate, which is larger for the former at short times, a direct consequence of the different ground state energy for both systems, namely,

\[ E_{BTG} = \hbar^2\pi^2 N(N+1)(2N+1)/(12mL^2) \]

and

\[ E_{FTG} = \hbar^2\pi^2 N/(2mL^2). \]
important effect of the environment in organic molecules.

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This result guarantees that the generalization for our few-

body systems of the dwell time (the average time spent

Regarding the long time behavior, it follows from Eq.

10 that \(N_T(t) = NP_0(t)\) for the FTG gas exhibits the

dependence. This is also the case for BTG gas where

\[
N_T(t) = \sum_{n=1}^{N} P_n(t) \propto \sum_{n=1}^{N} \frac{C^2(n)}{t^3}.
\]

(18)

This result guarantees that the generalization for our few-

body systems of the dwell time (the average time spent

by a particle in a spatial region) is possible, being a meaningful finite quantity \([32]\).

In conclusion, we have studied exactly two related few-

body tunneling problems, namely, that of bosonic and fermionic TG gases. The FTG gas has been pointed out

as a good system to observe long time deviations from exponential decay because of the strength of the signal,

proportional to the number of particles. The recent first

measurement of long time deviations \([12]\) is based on the important effect of the environment in organic molecules

in solution, so that the deviation for “pure”, isolated systems remains to be observed \([30]\). For the bosonic case a new deviation of the exponential decay appears, which can be understood as a sum of \(N\) single-particle contributions.

It is still an experimental challenge to get to the strong TG limit in a flat box, the main limiting factor being the confinement in transverse directions. Assuming, according to Reichel and Thywissen \([33]\), a “maximal practical value” of transversal frequency of 1 MHz, \(N = 10\), a box of 10 \(\mu\)m, and the constants for rubidium 87 (scattering length \(a \approx 5\) nm), the ratio \(\alpha\) of interaction energy to the potential energy is \(\alpha \approx 440\), whereas the ratio of chemical potential to the kinetic energy is \(\gamma \approx 88\), see \([32]\). For rubidium 85 at a Feshbach resonance the scattering length, \(\alpha\) and \(\gamma\) may increase by a factor of 100 \([34]\), making optical traps a viable alternative to magnetic confinement. These values point out at a TG gas regime not too far away from current capabilities, but the actual implementation may still be difficult also because of the need for accurate single atom detection.

We may in any case expect that reaching a strict TG regime is not absolutely essential to find interesting few-body decay effects. The upshot of our analysis is that for the few-body ground state of the box, the BTG gas exhibits maximal deviations from the exponential law at short times, whereas they are completely absent in the FTG gas or ideal Bose gas. The behaviour between the ideal Bose gas and BTG gas could be smoothly extrapolated by increasing the effective interactions \([35]\). Moreover, the Bose-Fermi duality \([10]\) opens up the possibility of observing experimentally such effects in a wider class of systems.

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