PISOT FAMILY SELF-AFFINE TILINGS, DISCRETE SPECTRUM, AND THE MEYER PROPERTY

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ABSTRACT. We consider self-affine tilings in the Euclidean space and the associated tiling dynamical systems, namely, the translation action on the orbit closure of the given tiling. We investigate the spectral properties of the system. It turns out that the presence of the discrete component depends on the algebraic properties of the eigenvalues of the expansion matrix $\phi$ for the tiling. Assuming that $\phi$ is diagonalizable over $\mathbb{C}$ and all its eigenvalues are algebraic conjugates of the same multiplicity, we show that the dynamical system has a relatively dense discrete spectrum if and only if it is not weakly mixing, and if and only if the spectrum of $\phi$ is a “Pisot family.” Moreover, this is equivalent to the Meyer property of the associated discrete set of “control points” for the tiling.

1. Introduction

Given a self-affine tiling $T$ of $\mathbb{R}^d$, we consider the tiling space, or “hull” $X_T$, defined as the orbit closure of $T$ in the “local” topology (please see the next section for precise definitions and statements). The translation action by $\mathbb{R}^d$ is uniquely ergodic, so we get a measure-preserving tiling dynamical system $(X_T, \mathbb{R}^d, \mu)$. We are interested in its spectral properties, specifically, in the discrete component of the spectrum which may be defined as the closed linear span of the eigenfunctions in $L^2(X_T, \mu)$. In particular, we would like to know when the tiling system is weakly mixing, which means absence of non-trivial eigenfunctions.

Our results give a complete answer to these questions in terms of the expansion matrix $\phi$ of the tiling, under the assumption that it is diagonalizable over $\mathbb{C}$ and its eigenvalues are algebraic conjugates of the same multiplicity. Let $\Lambda = \{\lambda_1, \ldots, \lambda_d\} = \text{Spec}(\phi)$ be the set of (real and complex) eigenvalues of $\phi$. It is known [19, 26] that all $\lambda_i$ are algebraic integers. Following Mauduit [29], we say that they form a Pisot family if for every $\lambda \in \Lambda$ and every Galois conjugate $\lambda'$ of $\lambda$, if $\lambda' \not\in \Lambda$, then $|\lambda'| < 1$. We prove that $(X_T, \mathbb{R}^d, \mu)$ has a relatively dense set of eigenvalues (equivalently, the set of eigenvalues of full rank $d$) if and only if $\Lambda$ is a Pisot family, and this is also equivalent to $(X_T, \mathbb{R}^d, \mu)$ being not weakly mixing. An example shows that if the multiplicities of the eigenvalues of $\phi$ are not equal, even if $\text{Spec}(\phi)$
is a Pisot family, the set of eigenvalues of the tiling dynamical system (not to be confused with \( \text{Spec}(\phi) \)) may fail to be relatively dense in \( \mathbb{R}^d \).

Special cases of our theorem were established earlier: for self-similar tilings of \( \mathbb{R}^d \), with \( d \leq 2 \), in [35], and for self-similar tilings of \( \mathbb{R}^d \) with a pure dilation expanding matrix \( \theta I \), in [37]. The present paper covers a much more general self-affine case.

Additional motivation for our work comes from the theory of aperiodic order and mathematics of quasicrystals. Considering specially chosen “control points” in the tiles, we obtain a Delone set \( C \), that is, a uniformly discrete and relatively dense subset of \( \mathbb{R}^d \), which is a substitution Delone set, see [22]. (We should note that in geometric analysis Delone sets are usually called separated nets.) It can be viewed as an atomic configuration, and it turns out that its diffraction spectrum is, in a certain precise sense, a “part” of the dynamical spectrum of the system \((X_T, \mathbb{R}^d, \mu)\), with the Bragg peaks (sharp bright spots on the diffraction picture) coming from the eigenvalues, see [9, 25, 13, 4]. Thus, for instance, weak mixing implies that there are no Bragg peaks, which indicates a certain level of “disorder.”

A Delone set \( Y \subset \mathbb{R}^d \) is Meyer if it is relatively dense and \( Y - Y \) is uniformly discrete. Answering a question of Lagarias [21], we showed in [26] that having a relatively dense set of Bragg peaks is equivalent to \( Y \) being Meyer, for a primitive substitution Delone set associated with a self-affine tiling. Our results in this paper imply that this is also characterized by the Pisot family condition (under our assumptions on \( \phi \)). The notion of Meyer set proved to be important in the study of aperiodic order, see e.g. [2, 30, 27, 24, 5]. In [2], under the Meyer set assumption for a substitution tiling, a computational algorithm is developed to decide whether the dynamical system has pure discrete spectrum. It should be noted that we do not address the question of pure discrete spectrum in the present paper but focus on the discrete spectral component.

1.1. Structure of the proof. A criterion for \( x \in \mathbb{R}^d \) to be an eigenvalue of the system \((X_T, \mathbb{R}^d, \mu)\) was obtained in [35, 37]. From it, the necessity of the Pisot family condition follows rather easily, see [35, 34]. For the converse, we need information on the location of control points, which is a manifestation of certain “rigidity” of self-affine tilings. For a self-similar tiling of the plane \( C \approx \mathbb{R}^2 \) with a complex expansion constant \( \lambda \), a result of Kenyon [17] says that the control points are contained in \( \mathbb{Z}[\lambda]a \), for some \( a \in \mathbb{C} \). We need an extension of this statement to the higher-dimensional self-affine case, which is a key result for us (see Theorem [3.1] below). This result does not depend on the Pisot family condition. We prove it using the techniques developed by Thurston [40] in the 2-dimensional self-similar case and extended to the \( d \)-dimensional self-affine case in [13, 20], where a necessary condition (which may be called the “Perron family condition”) for \( \phi \) to be an expansion map was obtained. It is here that we use the assumption that \( \phi \) is diagonalizable: the analog of the main theorem in [20] is open even for a \( 2 \times 2 \) Jordan block.
2. Definitions and statement of results

We briefly review the basic definitions of tilings and substitution tilings (see [27, 34] for more details). We begin with a set of types (or colors) \( \{1, \ldots, \kappa\} \), which we fix once and for all. A tile in \( \mathbb{R}^d \) is defined as a pair \( T = (A, i) \) where \( A = \text{supp}(T) \) (the support of \( T \)) is a compact set in \( \mathbb{R}^d \) which is the closure of its interior, and \( i = l(T) \in \{1, \ldots, \kappa\} \) is the type of \( T \). We let \( g + T = (g + A, i) \) for \( g \in \mathbb{R}^d \). We say that a set \( P \) of tiles is a patch if the number of tiles in \( P \) is finite and the tiles of \( P \) have mutually disjoint interiors. A tiling of \( \mathbb{R}^d \) is a set \( T \) of tiles such that \( \mathbb{R}^d = \bigcup \{ \text{supp}(T) : T \in T \} \) and distinct tiles have disjoint interiors. Given a tiling \( T \), finite sets of tiles of \( T \) are called \( T \)-patches. For \( A \subset \mathbb{R}^d \), let \( [A]^T = \{ T \in T : \text{supp}(T) \cap A \neq \emptyset \} \).

We always assume that any two \( T \)-tiles with the same color are translationally equivalent. (Hence there are finitely many \( T \)-tiles up to translation.)

We say that a tiling \( T \) has finite local complexity (FLC) if for each radius \( R > 0 \) there are only finitely many translational classes of patches whose supports lies in some ball of radius \( R \).

A tiling \( T \) is said to be repetitive if translations of any given patch occur uniformly dense in \( \mathbb{R}^d \); more precisely, for any \( T \)-patch \( P \), there exists \( R > 0 \) such that every ball of radius \( R \) contains a translated copy of \( P \).

Given a tiling \( T \), we define the tiling space as the orbit closure of \( T \) under the translation action: \( X_T = \{-g + T : g \in \mathbb{R}^d \} \), in the well-known “local topology”: for a small \( \epsilon > 0 \) two tilings \( S_1, S_2 \) are \( \epsilon \)-close if \( S_1 \) and \( S_2 \) agree on the ball of radius \( \epsilon^{-1} \) around the origin, after a translation of size less than \( \epsilon \). It is known that \( X_T \) is compact whenever \( T \) has FLC. Thus we get a topological dynamical system \( (X_T, \mathbb{R}^d) \) where \( \mathbb{R}^d \) acts by translations. This system is minimal (i.e. every orbit is dense) whenever \( T \) is repetitive. Let \( \mu \) be an invariant Borel probability measure for the action; then we get a measure-preserving system \( (X_T, \mathbb{R}^d, \mu) \).

Such a measure always exists; under the natural assumption of uniform patch frequencies, it is unique, see [25]. Tiling dynamical system have been investigated in a large number of papers; we do not provide an exhaustive bibliography, but mention a few: [32, 7, 15, 16]. They have also been studied as translation surfaces or \( \mathbb{R}^d \)-solenoids [6, 12].

**Definition 2.1.** A vector \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d \) is said to be an eigenvalue for the \( \mathbb{R}^d \)-action if there exists an eigenfunction \( f \in L^2(X_T, \mu) \), that is, \( f \not\equiv 0 \) and for all \( g \in \mathbb{R}^d \) and \( \mu \)-almost all \( S \in X_T \),

\[
(2.1) \quad f(S - g) = e^{2\pi i \langle g, \alpha \rangle} f(S).
\]

Here \( \langle \cdot, \cdot \rangle \) denotes the standard scalar product in \( \mathbb{R}^d \).

Note that this “eigenvalue” is actually a vector. In physics it might be called a “wave vector.” We can also speak about eigenvalues for the topological dynamical system \( (X_T, \mathbb{R}^d) \); then the eigenfunction should be in \( C(X_T) \) and the equation (2.1) should hold everywhere.
Next we define substitution tilings. Let $\phi$ be an expanding linear map in $\mathbb{R}^d$, which means that all its eigenvalues are greater than one in modulus. The following definition is essentially due to Thurston [40].

**Definition 2.2.** Let $A = \{T_1, \ldots, T_\kappa\}$ be a finite set of tiles in $\mathbb{R}^d$ such that $T_i = (A_i, i)$; we will call them prototiles. Denote by $A^+$ the set of patches made of tiles each of which is a translate of one of $T_i$’s. We say that $\omega : A \to A^+$ is a tile-substitution (or simply substitution) with expansion map $\phi$ if there exist finite sets $D_{ij} \subset \mathbb{R}^d$ for $i, j \leq \kappa$, such that

$$\omega(T_j) = \{u + T_i : u \in D_{ij}, i = 1, \ldots, \kappa\} \quad \text{for} \quad j \leq \kappa,$$

with

$$\phi A_j = \bigcup_{i=1}^{\kappa} (D_{ij} + A_i).$$

Here all sets in the right-hand side must have disjoint interiors; it is possible for some of the $D_{ij}$ to be empty.

The substitution (2.2) is extended to all translates of prototiles by $\omega(x + T_j) = \phi x + \omega(T_j)$, and to patches and tilings by $\omega(P) = \bigcup \{\omega(T) : T \in P\}$. The substitution $\omega$ can be iterated, producing larger and larger patches $\omega^k(T_j)$. To the substitution $\omega$ we associate its $\kappa \times \kappa$ substitution matrix with the entries $\sharp(D_{ij})$. The substitution $\omega$ is called primitive if the substitution matrix is primitive. We say that $\mathcal{T}$ is a fixed point of a substitution if $\omega(\mathcal{T}) = \mathcal{T}$.

**Definition 2.3.** A repetitive fixed point of a primitive tile-substitution with FLC is called a self-affine tiling. It is called self-similar if the expansion map is a similitude, that is, $|\phi(x)| = \theta |x|$ for all $x \in \mathbb{R}^d$, with some $\theta > 1$.

**Remark 2.4.** 1. A fixed point of a primitive tile-substitution is not necessarily of finite local complexity, see [8, 11]. Thus we have to assume FLC explicitly.

2. It is well-known (and easy to see, e.g. in the one-dimensional case) that the fixed point may be repetitive even for a non-primitive substitution. Conversely, the fixed point of a primitive substitution need not be repetitive. (However, if the tile-substitution is primitive and the fixed point tiling $\mathcal{T}$ has a tile which contains the origin in its interior, then $\mathcal{T}$ is repetitive [33].)

3. For a self-similar tiling of $\mathbb{R}^d$, with $d \leq 2$, we can speak of an expansion factor; it is a real number if $d = 1$ and a complex number if $d = 2$ (we then view the plane as a complex plane).

An important question, first raised by Thurston [40], is to characterize which expanding linear maps may occur as expansion maps for self-affine (self-similar) tilings. It is pointed out in [40] that in one dimension, $\lambda$ is an expansion factor if and only if $\theta = |\lambda|$ is a Perron number, that is, an algebraic integer greater than one whose Galois conjugates are all strictly
less than $\theta$ in modulus (necessity follows from the Perron-Frobenius theorem and sufficiency follows from a result of Lind [28]). In two dimensions, Thurston [40] proved that if $\lambda$ is a complex expansion factor of a self-similar tiling, then $\lambda$ is a complex Perron number, that is, an algebraic integer whose Galois conjugates, other than $\lambda$, are all less than $|\lambda|$ in modulus. The following theorem was stated in [19], but complete proof was not available until recently.

**Theorem 2.5.** [19, 20] Let $\phi$ be a diagonalizable (over $\mathbb{C}$) expansion map on $\mathbb{R}^d$, and let $T$ be a self-affine tiling of $\mathbb{R}^d$ with expansion $\phi$. Then

(i) every eigenvalue of $\phi$ is an algebraic integer;

(ii) if $\lambda$ is an eigenvalue of $\phi$ of multiplicity $k$ and $\gamma$ is an algebraic conjugate of $\lambda$, then either $|\gamma| < |\lambda|$, or $\gamma$ is also an eigenvalue of $\phi$ of multiplicity greater or equal to $k$.

**Remark 2.6.** 1. Note that if $|\gamma| = |\lambda|$ in part (ii) of the theorem, then the multiplicities of $\gamma$ and $\lambda$ are the same.

2. It is conjectured that the condition on $\phi$ in the theorem is also sufficient. There are partial results in this direction [18]; see [20] for a discussion.

For a self-affine tiling $T$, the corresponding tiling dynamical system $(X_T, \mathbb{R}^d)$ is uniquely ergodic, see [27, 34]. Denote by $\mu$ the unique invariant probability measure. There is a rich structure associated with self-affine tiling dynamical systems. As a side remark, we mention that the substitution map $\omega$ extends to an endomorphism of the tiling space, which is hyperbolic in a certain sense, see [3]. The partition of the tiling space according to the type of the tile containing the origin provides a Markov partition for $\omega$. The situation is especially nice when $T$ is non-periodic, which is equivalent to $\omega$ being invertible [36]. In order to state our results we need the following.

**Definition 2.7.** [29] A set $\Lambda$ of algebraic integers is called a Pisot family if for every $\lambda \in \Lambda$, if $\gamma$ is an algebraic conjugate of $\lambda$ and $\gamma \not\in \Lambda$, then $|\gamma| < 1$. Otherwise $\Lambda$ is called non-Pisot.

In this paper we assume that:

- all the eigenvalues of $\phi$ are algebraic conjugates with the same multiplicity.

Let $\text{Spec}(\phi)$ be the set of all eigenvalues of $\phi$ (the spectrum of $\phi$). By assumption, there exists a monic irreducible polynomial $p(t) \in \mathbb{Z}[t]$ (the minimal polynomial) such that $p(\lambda) = 0$ for all $\lambda \in \text{Spec}(\phi)$.

**Theorem 2.8.** Let $T$ be a self-affine tiling of $\mathbb{R}^d$ with a diagonalizable expansion map $\phi$. Suppose that all the eigenvalues of $\phi$ are algebraic conjugates with the same multiplicity. Then the following are equivalent:

(i) The set of eigenvalues of $(X_T, \mathbb{R}^d, \mu)$ is relatively dense in $\mathbb{R}^d$.

(ii) $\text{Spec}(\phi)$ is a Pisot family.

(iii) The system $(X_T, \mathbb{R}^d, \mu)$ is not weakly mixing (i.e., it has eigenvalues other than 0).
Remark 2.9. 1. In part (i) we could equally well talk about the topological dynamical system $(X_T, \mathbb{R}^d)$ since every eigenfunction may be chosen to be continuous \[37\].

2. The necessity of the Pisot family condition for self-affine tiling systems that are not weakly mixing was proved by Robinson \[34\] in a more general case; it is a consequence of \[35\].

Example 2.10. (i) In Fig. 2 and Fig. 3 of \[20\] a self-affine tiling $T_1$ is given, with the diagonal expansion matrix $\text{Diag}[\lambda_1, \lambda_2]$ where $\lambda_1 \approx 2.19869$ and $\lambda_2 \approx -1.91223$ are roots of the polynomial $x^3 - x^2 - 4x + 3$. Observe that $\{\lambda_1, \lambda_2\}$ is a Pisot family, hence the set of eigenvalues for the associated dynamical system is relatively dense in $\mathbb{R}^2$.

(ii) The assumption of equal multiplicity cannot be dropped from Theorem 2.8. Indeed, consider the tiling $T$ which is a “direct product” of $T_1$ defined in (i) and a self-similar tiling $T_2$ of $\mathbb{R}$ with expansion $\lambda_1$. Such a tiling $T_2$ exists by \[28\] (see \[38\] for more details) since $\lambda_1$ is a Perron number. Direct product substitution tilings have been studied by S. Mozes \[31\] and N. P. Frank \[10\]. It is easy to see that the set of eigenvalues for the dynamical system $(X_T, \mathbb{R}^3)$ is obtained as a direct sum of those which correspond to the systems $(X_{T_1}, \mathbb{R}^2)$ and $(X_{T_2}, \mathbb{R})$. By \[35\], the system $(X_{T_2}, \mathbb{R})$ is weakly mixing, because $\lambda_1$ is not a Pisot number. Thus, the tiling $T$ has expansion map $\phi = \text{Diag}[\lambda_1, \lambda_2, \lambda_1]$ for which $\text{Spec}(\phi)$ is a Pisot family, but the associated dynamical system does not have a relatively dense set of eigenvalues.

Next we state our result on Meyer sets. Recall that a Delone set is a relatively dense and uniformly discrete subset of $\mathbb{R}^d$.

Definition 2.11. A Delone set $Y$ is called a Meyer set if $Y - Y$ is uniformly discrete.

There is a standard way to choose distinguished points in the tiles of a self-affine tiling so that they form a $\phi$-invariant Delone set. They are called control points.

Definition 2.12. \[40, 33\] Let $T$ be a fixed point of a primitive substitution with expansion map $\phi$. For each $T$-tile $T$, fix a tile $\gamma T$ in the patch $\omega(T)$; choose $\gamma T$ with the same relative position for all tiles of the same type. This defines a map $\gamma : T \to T$ called the tile map. Then define the control point for a tile $T \in T$ by

$$\{c(T)\} = \bigcap_{n=0}^{\infty} \phi^{-n}(\gamma^n T).$$

The control points have the following properties:

(a) $T' = T + c(T') - c(T)$, for any tiles $T, T'$ of the same type;

(b) $\phi(c(T)) = c(\gamma T)$, for $T \in T$.

Control points are also fixed for tiles of any tiling $S \in X_T$: they have the same relative position as in $T$-tiles. Note that the choice of control points is non-unique, but there are only finitely many possibilities, determined by the choice of the tile map.
Let
\[ C := C(T) = \{ c(T) : T \in \mathcal{T} \} \]
be a set of control points of the tiling \( T \) in \( \mathbb{R}^d \). Let
\[ \Xi := \Xi(T) = \bigcup_{i=1}^{\kappa} (C_i - C_i), \]
where \( C_i \) is the set of control points of tiles of type \( i \). Equivalently, \( \Xi \) is the set of translation vectors between two \( T \)-tiles of the same type.

**Corollary 2.13.** Let \( T \) be a self-affine tiling of \( \mathbb{R}^d \) with a diagonalizable expansion map \( \phi \). Suppose that all the eigenvalues of \( \phi \) are algebraic conjugates with the same multiplicity. Then the set of control points \( C \) is Meyer if and only if \( \text{Spec}(\phi) \) is a Pisot family.

This is an immediate consequence of Theorem 2.8 and [26, Th. 4.14].

### 3. Preliminaries

Recall that \( \phi \) is assumed to be diagonalizable over \( \mathbb{C} \). For a complex eigenvalue \( \lambda \) of \( \phi \), the 2 \( \times \) 2 diagonal block
\[
\begin{pmatrix}
\lambda & 0 \\
0 & \overline{\lambda}
\end{pmatrix}
\]
is similar to a real 2 \( \times \) 2 matrix
\[
\begin{pmatrix}
a & -b \\
b & a
\end{pmatrix}
= S^{-1}
\begin{pmatrix}
\lambda & 0 \\
0 & \overline{\lambda}
\end{pmatrix}
S,
\]
where \( \lambda = a + ib, a, b \in \mathbb{R} \), and \( S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \). Since \( \phi \) is diagonalizable over \( \mathbb{C} \), we can assume, by appropriate choice of basis, that \( \phi \) is in the real canonical form of the linear map, see [14, Th. 6.4.2]. This means that \( \phi \) is block-diagonal, with the diagonal entries equal to \( \lambda \) corresponding to real eigenvalues, and diagonal 2 \( \times \) 2 blocks of the form
\[
\begin{pmatrix}
a_j & -b_j \\
b_j & a_j
\end{pmatrix}
\]
corresponding to complex eigenvalues \( a_j + ib_j \).

Let \( J \) be the multiplicity of each eigenvalue of \( \phi \). We can write
\[
\phi = \begin{bmatrix}
\psi_1 & \cdots & O \\
\vdots & \ddots & \vdots \\
O & \cdots & \psi_J
\end{bmatrix}
\quad \text{and} \quad
\psi_j = \psi := \begin{bmatrix}
A_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & A_{s+t}
\end{bmatrix}
\quad \text{for any } 1 \leq j \leq J
\]
where \( A_k \) is a real 1 \( \times \) 1 matrix for \( 1 \leq k \leq s \), a real 2 \( \times \) 2 matrix of the form
\[
\begin{pmatrix}
a_k & -b_k \\
b_k & a_k
\end{pmatrix}
\]
for \( s + 1 \leq k \leq s + t \), and \( O \) is the \((s + 2t) \times (s + 2t) \) zero matrix. Then the eigenvalues of \( \psi \) are
\[
\lambda_1, \ldots, \lambda_s, \lambda_{s+1}, \overline{\lambda_{s+1}}, \ldots, \lambda_{s+t}, \overline{\lambda_{s+t}}.
\]
Let \( m := s + 2t \); this is the size of the matrix \( \psi \). For each \( 1 \leq j \leq J \), let
\[
H_j = \{0\}^{(j-1)m} \times \mathbb{R}^m \times \{0\}^{d-jm}.
\]
Further, for each $H_j$ we have the direct sum decomposition

$$H_j = \bigoplus_{k=1}^{s+t} E_{jk},$$

such that each $E_{jk}$ is $\phi, \phi^{-1}$-invariant and $\phi|_{E_{jk}} \approx A_k$, identifying $E_{jk}$ with $\mathbb{R}$ or $\mathbb{R}^2$. Define a norm on $\mathbb{R}^d$ by

$$\|x\| = \max_{j,k} \|x_{jk}\| \text{ for } x = \sum_{j=1}^{J} \sum_{k=1}^{s+t} x_{jk}, \ x_{jk} \in E_{jk},$$

where $\|x_{jk}\|$ is the Euclidean norm on $E_{jk}$, so that $\|\phi x_{jk}\| = |\lambda_{jk}| \|x_{jk}\|$. Let $Q[\phi] := \{ p(\phi) : p \in \mathbb{Q}[x] \}$, $\mathbb{Z}[\phi] := \{ p(\phi) : p \in \mathbb{Z}[x] \}$.

Let $P_j$ be the canonical projection of $\mathbb{R}^d$ onto $H_j$ such that

$$P_j(x) = x_j,$$

where $x = x_1 + \cdots + x_J$ and $x_j \in H_j$ with $1 \leq j \leq J$. Let $\phi_j = \phi|_{H_j}$.

We define $\alpha_j \in H_j$ such that for each $1 \leq n \leq d$,

$$\alpha_j)_n = \begin{cases} 
1 & \text{if } (j-1)m+1 \leq n \leq jm; \\
0 & \text{else}.
\end{cases}$$

The next theorem is a key result of the paper; it is the manifestation of rigidity alluded in the Introduction.

**Theorem 3.1.** Let $T$ be a self-affine tiling of $\mathbb{R}^d$ with a diagonalizable expansion map $\phi$. Suppose that all the eigenvalues of $\phi$ are algebraic conjugates with the same multiplicity. Then there exists an isomorphism $\rho : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\rho \phi = \phi \rho \quad \text{and} \quad C \subset \rho(\mathbb{Z}[\phi] \alpha_1 + \cdots + \mathbb{Z}[\phi] \alpha_J),$$

where $\alpha_j, 1 \leq j \leq J$, are given as above.

The reason we call this “rigidity” is by analogy with [17, Th. 9] (see the discussion at the beginning of the proof in [17]).

We give a proof of Theorem 3.1 in Section 5 below and make use of it in proving the main theorem in Section 4. Note that the choice of $\alpha_j$ is rather arbitrary; it is “hidden” in the linear isomorphism $\rho$.

Now we continue with the preliminaries; we need to handle the real and complex eigenvalues a little bit differently. Consider the linear injective map $\mathcal{F} : \mathbb{R}^m \to \mathbb{R}^s \oplus \mathbb{C}^{2t}$ given by

$$\mathcal{F}(x_1, \ldots, x_s, x_{s+1}, \ldots, x_{s+2t}) =$$

$$= \left( x_1, \ldots, x_s, \frac{x_{s+1} + ix_{s+2}}{\sqrt{2}}, \frac{x_{s+1} - ix_{s+2}}{\sqrt{2}}, \ldots, \frac{x_{s+2t-1} + ix_{s+2t}}{\sqrt{2}}, \frac{x_{s+2t-1} - ix_{s+2t}}{\sqrt{2}} \right).$$
In other words, identifying $H_j$ with $\mathbb{R}^m$, we apply the transformation $S$ from (3.1) in every subspace $E_{jk}$, $k = s + 1, \ldots, s + t$. In view of (3.1), we have

$$\mathcal{F}(\psi x) = D\mathcal{F}(x) \quad \text{and} \quad \mathcal{F}(\psi^T x) = \overline{D\mathcal{F}(x)},$$

where

$$D = \text{Diag}[\lambda_1, \ldots, \lambda_s, \lambda_{s+1}, \ldots, \lambda_{s+t}, \lambda_{s+t}]$$

is a diagonal matrix.

The following lemma is well-known and easy to prove using the Vandermonde matrix.

**Lemma 3.2.** Let $D$ be a diagonal matrix on $\mathbb{C}^m$ with distinct complex eigenvalues. Let \( z = [z_1, \ldots, z_m]^T \in \mathbb{C}^m \) be such that $z_k \neq 0$ for all $1 \leq k \leq m$. Then \( \{z, Dz, \ldots, D^{m-1}z\} \) is linearly independent over $\mathbb{C}$.

**Corollary 3.3.** Suppose that $x \in \mathbb{R}^m$ is such that $z = \mathcal{F}(x)$ has all $(m)$ non-zero coordinates. Then both \( \{x, \psi x, \ldots, \psi^{m-1}x\} \) and \( \{x, \psi^T x, \ldots, (\psi^T)^{m-1}x\} \) are linearly independent over $\mathbb{R}$.

**Proof.** We have

$$\mathcal{F}(\{x, \ldots, \psi^{m-1}x\}) = \{z, \ldots, D^{m-1}z\}$$

by (3.6). By Lemma 3.2, the set \( \{z, \ldots, D^{m-1}z\} \) is independent over $\mathbb{C}$ and hence \( \{x, \psi x, \ldots, \psi^{m-1}x\} \) is independent over $\mathbb{R}$, using the fact that $\mathcal{F}$ is injective. The proof for the second set (with transpose matrices) is exactly the same. \(\square\)

**Corollary 3.4.** The set \( W := \{\alpha_1, \ldots, \phi^{m-1}\alpha_1, \ldots, \alpha_J, \ldots, \phi^{m-1}\alpha_J\} \) forms a basis of $\mathbb{R}^d$.

**Proof.** Identifying $H_j$ with $\mathbb{R}^m$, we have $\phi_j = \phi|_{H_j} \approx \psi$ and use the isomorphism $\mathcal{F}$ defined above. In view of (3.3), all the components of $z_j = \mathcal{F}(\alpha_j)$ are non-zero, so the claim follows from Corollary 3.3. \(\square\)

For $x, y \in \mathbb{R}^m$ we use the standard scalar product $\langle x, y \rangle = \sum_{k=1}^m x_k y_k$, and for $z, u \in \mathbb{R}^s \oplus \mathbb{C}^{2t}$ the scalar product is given by

$$\langle z, u \rangle_c = \sum_{k=1}^{s+2t} z_k \overline{u_k}.$$

Observe that

$$\langle x, y \rangle = \langle \mathcal{F}(x), \mathcal{F}(y) \rangle_c \quad \text{for all} \quad x, y \in \mathbb{R}^m.$$

Recall also that for any $m \times m$ matrix $A$,

$$\langle Ax, y \rangle = \langle x, A^T y \rangle \quad \text{for all} \quad x, y \in \mathbb{R}^m.$$
4. Proof of the main theorem (proof of Theorem 2.8)

Here we deduce Theorem 2.8 from Theorem 3.1. Recall that a set of algebraic integers \( \Theta = \{\theta_1, \ldots, \theta_r\} \) is a Pisot family if for any \( 1 \leq j \leq r \), every Galois conjugate \( \gamma \) of \( \theta_j \) with \( |\gamma| \geq 1 \) is contained in \( \Theta \). We denote by \( \text{dist}(x, \mathbb{Z}) \) the distance from a real number \( x \) to the nearest integer.

**Proposition 4.1.** Let \( T \) be a self-affine tiling of \( \mathbb{R}^d \) with a diagonalizable expansion map \( \phi \). Suppose that all the eigenvalues of \( \phi \) are algebraic conjugates with the same multiplicity. If \( \text{Spec}(\phi) \) is a Pisot family, then the set of eigenvalues of \((X_T, \mathbb{R}^d, \mu)\) is relatively dense.

**Proof.** Recall that \( \Xi = \{x \in \mathbb{R}^d : \exists T \in T, T + x \in T\} \) is the set of "return vectors" for the tiling \( T \), and let \( K = \{x \in \mathbb{R}^d : T - x = T\} \) be the set of translational periods. Clearly, \( K \subset \Xi \subset \mathbb{C} - \mathbb{C} \). We know from [37, Th. 3.13] that \( \gamma \) is an eigenvalue for \((X_T, \mathbb{R}^d, \mu)\) if and only if

\[
\lim_{n \to \infty} e^{2\pi i \langle \phi^n(x), \gamma \rangle} = 1 \quad \text{for all} \quad x \in \Xi \quad \text{and} \quad e^{2\pi i \langle x, \gamma \rangle} = 1 \quad \text{for all} \quad x \in K.
\]

Let \( \alpha_j \in H_j \) be the vectors from (3.4). Consider them as vectors in \( \mathbb{R}^m \), and let \( F \) be the linear map \( \mathbb{R}^m \to \mathbb{R}^s \oplus \mathbb{C}^{2t} \) given by (3.5). Recall that \( \phi_j = \phi|_{H_j} \) has \( s \) real and \( 2t \) complex eigenvalues, and \( m = s + 2t \). Define \( \beta_j \in H_j \cong \mathbb{R}^m \) so that

\[
(F(\beta_j))_k = (F(\alpha_j))^{-1}_k \quad \text{for} \quad 1 \leq k \leq m.
\]

More explicitly,

\[
(\beta_j)_k = (\alpha_j)_k^{-1} \quad \text{for} \quad 1 \leq k \leq s
\]

and

\[
(\beta_j)_{s+2k-1} \pm i(\beta_j)_{s+2k} = \frac{2}{(\alpha_j)_{s+2k-1} \mp i(\alpha_j)_{s+2k}} \quad \text{for} \quad 1 \leq k \leq t.
\]

Note that \( \beta_j \in H_j \) are well-defined, and \( F(\beta_j) \) have all non-zero coordinates in \( H_j \). Thus,

\[
B_j := \{\beta_j, \ldots, (\phi^T)^{m-1}\beta_j\}
\]

is a basis of \( H_j \) by Corollary 3.3 (note that \( H_j \) is also \( \phi^T \)-invariant and \( \phi^T|_{H_j} \) is isomorphic to \( \psi^T \)). It follows that \( B := \bigcup_{j=1}^l B_j \) is a basis of \( \mathbb{R}^d \). We will show that all elements of the set \((\lambda^T)^{-1}(\phi^T)^KB\) are eigenvalues for the tiling dynamical system, for \( K \) sufficiently large.

By the definition of \( \beta_j \), in view of (3.8) and (3.6), for any \( n \in \mathbb{Z}_{\geq 0} \) and \( 0 \leq l < m \),

\[
\langle \phi^n\alpha_j, (\phi^T)^l\beta_j \rangle = \langle \phi^{n+l}\alpha_j, \beta_j \rangle = \langle F(\phi^{n+l}\alpha_j), F(\beta_j) \rangle_C = \langle D^{n+l}F(\alpha_j), F(\beta_j) \rangle_C = \sum_{k=1}^s \lambda_k^{n+l} + \sum_{k=s+1}^{s+t} (\lambda_k^{n+l} + \bar{\lambda}_k^{n+l}).
\]
Here $D$ is the diagonal matrix from (3.7). Since $\text{Spec}(\phi)$ is a Pisot family, it follows that 
$$\text{dist}((\phi^n \alpha_j, (\phi^T)^l \beta_j), \mathbb{Z}) \to 0, \text{ as } n \to \infty.$$ (This is a standard argument: the sum of $(n+l)$-th powers of all zeros of a polynomial in $\mathbb{Z}[x]$ is an integer, hence the distance from the sum in (4.2) to $\mathbb{Z}$ is bounded by the sum of the moduli of $(n+l)$-th powers of their remaining conjugates, which are all less than one in modulus. Thus, this distance tends to zero exponentially fast.) Observe also that $\langle \phi^n \alpha_u, \beta_j \rangle = 0$ if $u \neq j$, hence
$$\lim_{n \to \infty} e^{2\pi i \langle \phi^n y, (\phi^T)^l \beta_j \rangle} = 1 \quad \text{for all } y \in \mathbb{Z}[\phi] \alpha_1 + \cdots + \mathbb{Z}[\phi] \alpha_j.$$ Therefore, by Theorem 3.11 using that $\Xi \subset C - C$, we obtain
$$\lim_{n \to \infty} e^{2\pi i \langle \phi^n x, (\rho^T)^{-1} (\phi^T)^l \beta_j \rangle} = 1 \quad \text{for all } x \in \Xi.$$ Furthermore, by [37, Cor. 4.4], the convergence is uniform in $x \in \Xi$, that is,
$$\lim_{n \to \infty} \sup_{x \in \Xi} |e^{2\pi i \langle \phi^n x, (\rho^T)^{-1} (\phi^T)^l \beta_j \rangle} - 1| = 0.$$ Recall that $\mathcal{K} \subset \Xi$, and $\mathcal{K}$ is a discrete subgroup in $\mathbb{R}^d$. So for every $x \in K$,
$$\lim_{n \to \infty} \sup_{k \in \mathbb{Z}_+, x \in \mathcal{K}} |e^{2\pi i \langle \phi^n(kx), (\rho^T)^{-1} (\phi^T)^l \beta_j \rangle} - 1| = \lim_{n \to \infty} \sup_{k \in \mathbb{Z}_+, x \in \mathcal{K}} |e^{2\pi ik \langle \phi^n x, (\rho^T)^{-1} (\phi^T)^l \beta_j \rangle} - 1| = 0.$$ It follows that there exists $K_l \in \mathbb{Z}_+$ such that for any $n \geq K_l$, for all $x \in K$,
$$\sup_{k \in \mathbb{Z}_+, x \in \mathcal{K}} |e^{2\pi ik \langle \phi^n x, (\rho^T)^{-1} (\phi^T)^l \beta_j \rangle} - 1| < 1/2.$$ However, unless $\langle \phi^n x, (\rho^T)^{-1} (\phi^T)^l \beta_j \rangle \in \mathbb{Z}$ for all $x \in \mathcal{K}$, (4.4) does not hold. Thus
$$e^{2\pi i \langle \phi^n x, (\rho^T)^{-1} (\phi^T)^l \beta_j \rangle} = e^{2\pi i \langle x, (\rho^T)^{-1} (\phi^T)^l \beta_j \rangle} = 1 \quad \text{for all } x \in \mathcal{K} \text{ and all } n \geq K_l.$$ Let $K = \max\{K_l : 0 \leq l < m\}$. Then
$$e^{2\pi i \langle x, (\rho^T)^{-1} (\phi^T)^{K+l} \beta_j \rangle} = 1 \quad \text{for all } x \in \mathcal{K}.$$ So from (4.3) and (4.5) it follows that $(\rho^T)^{-1} (\phi^T)^{K+l} \beta_j$ is an eigenvalue of $(X_T, \mathbb{R}^d, \mu)$ for $l = 0, \ldots, m-1$. We have shown that all vectors of the set $(\rho^T)^{-1} (\phi^T)^K B$, where $B = \bigcup_j B_j$ and $B_j$ are given by (1.1), are eigenvalues of $(X_T, \mathbb{R}^d, \mu)$. We know $\phi^T$ is invertible (it is expanding), $\rho$ is a linear isomorphism, and $B$ is a basis of $\mathbb{R}^d$, hence we obtain a basis of $\mathbb{R}^d$ consisting of eigenvalues. Integer linear combinations of eigenvalues are eigenvalues as well, so the set of eigenvalues of $(X_T, \mathbb{R}^d, \mu)$ is relatively dense in $\mathbb{R}^d$. \hfill \square

The next lemma is essentially due to Robinson [34] in a more general case; we provide a proof for completeness.

**Lemma 4.2.** If $\gamma$ is a non-zero eigenvalue of $(X_T, \mathbb{R}^d, \mu)$, then $\text{Spec}(\phi)$ is a Pisot family.
Proof. Let \( x \in \Xi \). By Theorem 3.1 we have \( x = \rho(\sum_{j=1}^J p_j(\phi)\alpha_j) \) for some polynomials \( p_j \in \mathbb{Z}[x] \). Let \( (\rho^T \gamma)_j = p_j(\rho^T \gamma) \). We again use the linear injective map \( F : H_j \approx \mathbb{R}^m \to \mathbb{R}^s \oplus \mathbb{C}^2t \) defined by (3.5) and obtain, using (3.8) and (3.6),

\[
\langle \phi^n x, \gamma \rangle = \sum_{j=1}^J \langle \phi^n p_j(\phi)\alpha_j, (\rho^T \gamma)_j \rangle
\]

\[
= \sum_{j=1}^J \langle F(\phi^n p_j(\phi)\alpha_j), F((\rho^T \gamma)_j) \rangle
\]

\[
= \sum_{j=1}^J \left( \sum_{k=1}^s \lambda_k^n p_j(\lambda_k)z_{jk}\zeta_{jk} + \sum_{k=s+1}^{s+t} 2\text{Re}\left[\lambda_k^n p_j(\lambda_k)z_{jk,2k-s-1}\zeta_{jk,2k-s-1}\right] \right)
\]

\[
= \sum_{k=1}^s c_k \lambda_k^n + \sum_{k=s+1}^{s+t} (c_k \lambda_k^n + \overline{c_k} \lambda_k^n),
\]

where \( (z_{jk})_{k=1}^{s+2t} = F(\alpha_j), (\zeta_{jk})_{k=1}^{s+2t} = F((\rho^T \gamma)_j) \), and \( c_k \) are some complex numbers. By the assumption that \( \gamma \) is an eigenvalue and [35, Th. 4.3] we have

\[
(4.6) \quad \text{dist}\left( \sum_{k=1}^s c_k \lambda_k^n + \sum_{k=s+1}^{s+t} (c_k \lambda_k^n + \overline{c_k} \lambda_k^n), \mathbb{Z} \right) = \text{dist}(\langle \phi^n x, \gamma \rangle, \mathbb{Z}) \xrightarrow{n \to \infty} 0.
\]

Since \( \Xi \) is relatively dense in \( \mathbb{R}^d \) and \( \gamma \neq 0 \), we can easily make sure that \( \langle x, \gamma \rangle \neq 0 \), and hence not all coefficients \( c_k \) in (4.6) are equal to zero. Then we can apply a theorem of Vijayaraghavan [41, Th. 4] (a generalization of the classical result of Pisot and Vijayaraghavan) to conclude that \( \text{Spec}(\phi) \) is a Pisot family. (More precisely, we obtain that a subset of \( \text{Spec}(\phi) \) is a Pisot family, but since all elements of \( \text{Spec}(\phi) \) are conjugates and have modulus greater than one, we get the claim.) \( \square \)

**Theorem 4.3.** Let \( T \) be a self-affine tiling of \( \mathbb{R}^d \) with a diagonalizable expansion map \( \phi \). Suppose that all the eigenvalues of \( \phi \) are algebraic conjugates with the same multiplicity. Then the following are equivalent:

(i) \( \text{Spec}(\phi) \) forms a Pisot family;

(ii) the set of eigenvalues of \( (X_T, \mathbb{R}^d, \mu) \) is relatively dense;

(iii) \( (X_T, \mathbb{R}^d, \mu) \) is not weakly mixing;

(iv) \( \mathcal{C} = \{ c(T) : T \in T \} \) is a Meyer set.

**Proof.** (i) \( \Rightarrow \) (ii) by Prop. 4.1

(ii) \( \Rightarrow \) (iii) is obvious.

(iii) \( \Rightarrow \) (i) by Lemma 4.2

(ii) \( \Leftrightarrow \) (iv) by [26, Th. 4.14]. \( \square \)

Theorem 2.8 is contained in Theorem 4.3 so it is proved as well.
5. Structure of the control point set (proof of Theorem 3.1)

Now we make an isomorphic transformation $\tau$ of the tiling $T$ into another tiling whose control point set contains $\alpha_1, \ldots, \alpha_J$ such that $\tau$ commutes with $\phi$. This gives the structure of the control point set of $T$ that we use in proving the main theorem in Section 4.

In Corollary 3.4 we showed that $W = \{\alpha_1, \ldots, \phi^{m-1} \alpha_1, \ldots, \alpha_J, \ldots, \phi^{m-1} \alpha_J\}$ is a basis for $\mathbb{R}^d$. Since $C$ is relatively dense in $\mathbb{R}^d$, for any $\epsilon > 0$, for every $j = 1, \ldots, J$, there exists $y_j \in C$ such that

$$\frac{\|y_j\|}{\|\alpha_j\|} - \epsilon < \epsilon.$$  

For $\epsilon > 0$ sufficiently small, $(j - 1)m + 1, \ldots, (jm)$-th entries of $y_j$ are all non-zero for any $1 \leq j \leq J$, and the set

$$Y := \{y_1, \ldots, \phi^{m-1} y_1, \ldots, y_J, \ldots, \phi^{m-1} y_J\} \subset C$$

is a basis of $\mathbb{R}^d$. We fix such a basis $Y$.

**Lemma 5.1.** Let $\tau : \mathbb{R}^d \to \mathbb{R}^d$ be a linear map such that for each $1 \leq j \leq J$ and $0 \leq k < m$,

$$\tau(\phi^k y_j) = \phi^k \alpha_j.$$  

Then $\tau$ is an isomorphism of $\mathbb{R}^d$ such that $\tau \phi = \phi \tau$.

**Proof.** We first notice that $\tau$ is an isomorphism of $\mathbb{R}^d$, since $Y$ and $W$ are bases of $\mathbb{R}^d$. In order to show that $\phi \tau(x) = \tau \phi(x)$, $x \in \mathbb{R}^d$, it is enough to check this on the basis $Y$. For the vectors $\phi^k y_j$, $0 \leq k < m - 1$, this holds by definition, so we only need to consider $\phi^{m-1} y_j$. Let $p(t)$ be the characteristic polynomial of $\psi$. Then $p(\psi) = 0$ by Cayley-Hamilton, and also $p(\phi) = 0$, since $\phi$ is a direct sum of $J$ copies of $\psi$. Thus

$$\phi^m = a_0 I + \cdots + a_{m-1} \phi^{m-1}$$

for some $a_0, \ldots, a_{m-1} \in \mathbb{R}$, hence

$$\phi \tau(\phi^{m-1} y_j) = \phi^m \alpha_j = a_0 \alpha_j + \cdots + a_{m-1} \phi^{m-1} \alpha_j = a_0 \tau(y_j) + \cdots + a_{m-1} \tau(\phi^{m-1} y_j) = \tau(\phi^m y_j),$$

as desired. \hfill \Box

Let

$$\tau(T) := \{\tau(T) : \tau(T) = (\tau(A), i), \text{ where } T \in \mathcal{T} \text{ and } T = (A,i)\}.$$  

Note that $\tau(T)$ is a fixed point of a primitive substitution with the expansion map $\phi$. Indeed, $\omega'(\tau(T)) = \tau(T)$ where $\omega'$ is defined by

$$\omega'(\tau(T_j)) = \{u + \tau(T_i) : u \in \tau(\mathcal{D}_{ij}), i = 1, \ldots, \kappa\} \text{ for } j \leq \kappa,$$
where
\[
\phi \tau(A_j) = \bigcup_{i=1}^{\kappa} (\tau(D_{ij}) + \tau(A_i)).
\]
We define the tile map \( \gamma' : \tau(T) \to \tau(T) \) so that for each \( T \)-tile \( T \),
\[
\gamma'(\tau(T)) = \tau(\gamma(T)).
\]
We define the control point for a tile \( \tau(T) \in \tau(T) \) by
\[
\{ c(\tau(T)) \} = \bigcap_{n=0}^{\infty} \phi^{-n}(\gamma'^n \tau(T)).
\]
Then
\[
c(\tau(T)) = \tau c(T).
\]
Applying the isomorphism \( \tau \) commuting with \( \phi \), we can reduce our problem to the case when the control point set of the tiling contains \( \alpha_1, \ldots, \alpha_J \). Thus, in the rest of this section (except the last paragraph which proves Theorem 3.1), we assume that \( C \) contains \( \alpha_1, \ldots, \alpha_J \).

The following two propositions were obtained in [20] in a special case. They are needed to get the structure of control point set which we use in Section 4. In the appendix, we provide the proof, which is similar to that in [20], for completeness.

In the next two propositions we do not assume that all the eigenvalues of \( \phi \) are conjugates and have the same multiplicity. Let \( G_\lambda \) be the real \( \phi \)-invariant subspace of \( \mathbb{R}^d \) corresponding to an eigenvalue \( \lambda \in \text{Spec}(\phi) \).

**Proposition 5.2.** Let \( C \) be a set of control points for a self-affine tiling \( T \) of \( \mathbb{R}^d \) with an expansion map \( \phi : \mathbb{R}^d \to \mathbb{R}^d \) which is diagonalizable over \( \mathbb{C} \). Let \( C_\infty = \bigcup_{k=0}^{\infty} \phi^{-k} C \) and let \( D \) be a finitely generated \( \mathbb{Q}[\phi] \)-module containing \( C_\infty \). Let \( H \) be a vector space over \( \mathbb{R} \) and \( A : H \to H \) be an expanding linear map, diagonalizable over \( \mathbb{C} \). Let \( g : D \to H \) be such that \( g = A^{-1} \circ g \circ \phi \) and
\[
g(y_1) - g(y_2) = g(y_1 - y_2) \quad \text{for any} \ y_1, y_2 \in D.
\]
Let \( f := g|_{C_\infty} : C_\infty \to H \) be such that for any \( y_1, y_2 \in C \),
\[
||f(y_1) - f(y_2)|| \leq C ||y_1 - y_2|| \quad \text{for some} \ C > 0.
\]
Moreover, suppose that there exist \( \gamma > 1 \) and a norm \( || \cdot || \) in \( H \) such that
\[
||Ay|| \geq \gamma ||y|| \quad \text{for any} \ y \in H.
\]
Then the following hold:

(i) The map \( f \) is uniformly continuous on \( C_\infty \), and hence extends by continuity to a map \( f : \mathbb{R}^d \to H \) satisfying \( f \circ \phi = A \circ f \).
For any $\lambda \in \text{Spec}(\phi)$ such that $|\lambda| = \gamma$, and any $a \in \mathbb{R}^d$,

$$f|_{a+G_\lambda} \text{ is affine linear.}$$

Let $P_\lambda$ be the canonical projection of $\mathbb{R}^d$ to $G_\lambda$ commuting with $\phi$, which exists by the diagonalizability assumption on $\phi$. Denote by $G_\lambda^\perp = (I - P_\lambda)\mathbb{R}^d$ the complementary $\phi$-invariant subspace. We consider the set $(I - P_\lambda)\Xi$, that is, the projection of $\Xi$ to $G_\lambda^\perp$ (recall that $\Xi$ is the set of translation vectors between two $T$-tiles of the same type). In some directions this projection may look like a lattice, i.e. be discrete. We consider the directions in which this set is not discrete, and denote the span of these directions by $G'$. We will prove that $f$ is affine linear on all $G'$ slices. More precisely, for any $\epsilon > 0$, define

$$G_\epsilon := \text{Span}_\mathbb{R} (B_\epsilon \cap (I - P_\lambda)\Xi) \quad \text{and} \quad G' := \bigcap_{\epsilon > 0} G_\epsilon.\quad (5.5)$$

Now let

$$G := G' + G_\lambda.\quad (5.6)$$

Note that $G$ is a subspace of $\mathbb{R}^d$ which is $\phi$-invariant, because

$$\phi\Xi \subset \Xi \quad \text{and} \quad \phi P_\lambda = P_\lambda \phi.$$

**Proposition 5.3.** Under the assumptions of Proposition 5.2, $f|_{a+G}$ is affine linear for any $a \in \mathbb{R}^d$. \qed

**Lemma 5.4.** Let all eigenvalues of $\phi$ be algebraic conjugates with the same multiplicity. If $\lambda$ is the smallest in modulus eigenvalue of $\phi$, then

$$G = G' + G_\lambda = \mathbb{R}^d.$$  

**Proof.** This is proved in [20] (although not stated there explicitly). Indeed, in the last part of [20], labeled Conclusion of the proof of Theorem 3.1, it is proved that the subspace $G$ (denoted $E$ there) contains, for each conjugate of $\lambda$ greater or equal than $\lambda$ in modulus, an eigenspace of dimension at least $\dim(G_\lambda)$. Note that in [20] the setting is more general, of an arbitrary diagonalizable over $\mathbb{C}$ matrix $\phi$. In our case all eigenvalues are conjugates of the same multiplicity, and $\lambda$ is the smallest in modulus, hence $G$ contains the entire $\mathbb{R}^d$. \qed

Since $\mathcal{T}$ has FLC, the $\mathbb{Z}$-module generated by $\mathcal{C}$, denoted by $\langle \mathcal{C} \rangle_{\mathbb{Z}}$, is finitely generated. Let $\{v_1, \ldots, v_N\}$ be a generating set in $\mathcal{C}$. For each $v_n$ with $1 \leq n \leq N$,

$$v_n = a_{11}^{(n)} \alpha_1 + \cdots + a_{1m}^{(n)} \phi^{m-1} \alpha_1 + \cdots + a_{J1}^{(n)} \alpha_J + \cdots + a_{jm}^{(n)} \phi^{m-1} \alpha_j$$

where $a_{jk}^{(n)} \in \mathbb{R}$, $1 \leq j \leq J$, and $1 \leq k \leq m$. Thus

$$\mathcal{C} \subset \mathcal{D} := \sum_{j=1}^J \sum_{k=1}^m \sum_{n=1}^N \mathbb{Q}[\phi](a_{jk}^{(n)} \alpha_j) = \sum_{j=1}^J \sum_{k=1}^m \sum_{n=1}^N \mathbb{Q}[\phi_j](a_{jk}^{(n)} \alpha_j).$$
Lemma 5.5. $\mathbb{Q}[\phi]_j$ is a field.

Proof. This is clearly a ring, so we just need to show that $p(\phi_j)$ has an inverse for $p \in \mathbb{Q}[x]$, if it is a non-zero matrix. We need to use that all the eigenvalues of $\phi_j$ are conjugates so they have the same irreducible polynomial $p(x)$. If $q(x) \in \mathbb{Z}[x]$ is monic, such that $p(x)$ does not divide $q(x)$, then we can find monic polynomials $h_1(x), h_2(x) \in \mathbb{Z}[x]$ such that $h_1(x)p(x) + h_2(x)q(x) = 1$, which means that $h_2(\lambda)$ is the inverse of $q(\lambda)$ for any eigenvalue $\lambda$ of $\phi_j$. \hfill $\Box$

Let $D_j = P_j(D)$, where $P_j$ is the canonical projection of $\mathbb{R}^d$ onto $H_j$ as defined in (3.3). Observe that $D_j$ is a vector space over the field $\mathbb{Q}[\phi_j]$, so we can write

$$D_j = \bigoplus_{t=1}^{r_j} \mathbb{Q}[\phi_j](a_j t \alpha_j) = \bigoplus_{t=1}^{r_j} \mathbb{Q}[\phi](a_j t \alpha_j),$$

where $a_j 1 = 1, a_j t \in \mathbb{R}$ with $1 \leq t \leq r_j$, and $\{a_j 1, \ldots, a_j r_j\}$ is linearly independent over $\mathbb{Q}$. Note that

$$D = D_1 \bigoplus \cdots \bigoplus D_j.$$

We define $\mathbb{Q}[\phi]$-module homomorphisms

$$\sigma_j : D \to \mathbb{Q}[\phi] \alpha_j$$

such that

$$\begin{align*}
\sigma_j(a_j t \phi^n \alpha_j) &= \phi^n \alpha_j \quad \text{for any } 1 \leq t \leq r_j, n \in \mathbb{Z}_{\geq 0} \\
\sigma_j(a_u u \phi^n \alpha_u) &= 0 \quad \text{for any } u \neq j, n \in \mathbb{Z}_{\geq 0}.
\end{align*}$$

Recall that $C_\infty := \bigcup_{k=0}^{\infty} \phi^{-k} C$. Observe that $D \supset C_\infty$, since $\phi^{-1}$ is a rational linear combination of $\{I, \phi, \ldots, \phi^{m-1}\}$ by (5.1). We define $\sigma'_j : C_\infty \to \mathbb{Q}[\phi] \alpha_j$ to be the restriction of $\sigma_j$, that is, $\sigma'_j := \sigma_j|_{C_\infty}$.

Using the same arguments as in [37, Lemma 5.3] (which followed [40]), we obtain the next lemma.

Lemma 5.6. For any $\xi, \xi' \in C$,

$$\|\sigma'_j(\xi) - \sigma'_j(\xi')\| \leq C \|\xi - \xi'\| \quad \text{for some } C > 0.$$

Now we use Prop.5.2, Prop.5.3 and Lemma5.4 to prove Theorem3.1 and assume that all the assumptions of the latter hold. In addition, suppose that the set of control points contains $\alpha_1, \ldots, \alpha_J$. Fix $1 \leq j \leq J$. We consider the maps $g = \sigma_j : D \to H_j$ and $f = \sigma'_j : C_\infty \to H_j$, and let $A = \phi_j = \phi|_{H_j}$. Note that (5.4) holds with $\gamma$ equal to the smallest absolute value of eigenvalues of $\phi_j$ (or $\phi$) and the norm defined as in (3.2). Thus, all the hypotheses of Prop.5.2, Prop.5.3 and Lemma5.4 are satisfied, and we obtain that for each $1 \leq j \leq J$, the (extended) map $\sigma'_j$ is linear on $\mathbb{R}^d$ and commutes with $\phi$.

Lemma 5.7. The map $\rho' := \sigma'_1 + \cdots + \sigma'_J$ is the identity map on $\mathbb{R}^d$. 


Proof. Note that
\[ \rho' : \mathbb{R}^d \to \mathbb{R}^d \text{ is linear and } \rho' \phi = \phi \rho' \]
and for any \( 1 \leq j \leq J \),
\[ \rho' (\alpha_j) = (\sigma_1' + \cdots + \sigma_j' ) (\alpha_j) = 0 + \cdots + 0 + \sigma_j' (\alpha_j) + 0 + \cdots + 0 = \alpha_j. \]
Since \( W = \{ \alpha_1, \ldots, \phi^{-1} \alpha_1, \ldots, \alpha_J, \ldots, \phi^{-1} \alpha_J \} \) is a basis of \( \mathbb{R}^d \), \( \rho' \) is the identity map on \( \mathbb{R}^d \).

Now we do not assume that the control point set of \( T \) contains \( \alpha_1, \ldots, \alpha_J \) in order to prove Theorem 3.1. Instead, we apply the above propositions and lemmas to \( \tau(\mathcal{C}) \).

Proof of Theorem 3.1. By Lemma 5.7 for each \( \xi \in \mathcal{C} \),
\[ \tau(\xi) = \rho' (\tau(\xi)) = (\sigma_1' + \cdots + \sigma_j' ) (\tau(\xi)) = (\sigma_1 + \cdots + \sigma_j ) (\tau(\xi)) \]
(5.7)
\[ \in \mathbb{Q}[\phi] \alpha_1 + \cdots + \mathbb{Q}[\phi] \alpha_J. \]
Since \( \mathcal{C} \) is finitely generated, we multiply (5.7) by a common denominator \( b \in \mathbb{Z}_+ \) to get
\[ b \cdot \tau(\mathcal{C}) \subset \mathbb{Z}[\phi] \alpha_1 + \cdots + \mathbb{Z}[\phi] \alpha_J. \]
Let \( \rho := \frac{1}{b} \cdot \tau^{-1} \). Then
\[ \mathcal{C} \subset \rho(\mathbb{Z}[\phi] \alpha_1 + \cdots + \mathbb{Z}[\phi] \alpha_J) \]
where \( \rho \) is an isomorphism of \( \mathbb{R}^d \) which commutes with \( \phi \).

6. Appendix

We give the proofs of Prop. 5.2 and Prop. 5.3 after a sequence of auxiliary lemmas. The arguments are similar to those in [20], but we present them in a more general form for our purposes.

Denote by \( B_R(a) \) the open ball of radius \( R \) centered at \( a \) and let \( B_R := B_R(0) \). We will also write \( \overline{B_R}(a) \) for the closure of \( B_R(a) \). Let \( r = r(T) > 0 \) be such that for every \( a \in \mathbb{R}^d \) the ball \( \overline{B_r}(a) \) is covered by a tile containing \( a \) (which need not be unique) and its neighbors. Let \( \lambda_{\max} \) be the largest eigenvalue of \( \phi \).

Lemma 6.1. The function \( f \) is uniformly continuous on \( \mathcal{C}_\infty \).

Proof. This is very similar to [20, Lem. 3.4]. It is enough to show that
\[ \xi_1, \xi_2 \in \mathcal{C}_\infty, \ |\xi_1 - \xi_2| \leq r \implies \| f(\xi_1) - f(\xi_2) \| \leq L |\xi_1 - \xi_2|^{\alpha}, \]
for \( \alpha = \frac{\log \gamma}{\log \lambda_{\max}} \) and some \( L > 0 \) (that is, \( f \) is Hölder continuous on \( \mathcal{C}_\infty \)).

Let \( \xi_1, \xi_2 \in \mathcal{C}_\infty \) satisfy \( |\xi_1 - \xi_2| = \delta \leq r \). Then there exist \( y_1, y_2 \in \mathcal{C} \) such that \( \phi^{-s} y_1 = \xi_1 \) and \( \phi^{-s} y_2 = \xi_2 \) for some \( s \in \mathbb{Z}_{\geq 0} \). We choose the smallest \( l \in \mathbb{Z}_{\geq 0} \) such that
\[ \phi^s B_\delta(\phi^{-s} y_1) \subset \phi^l B_r(\phi^{-l} y_1), \]
which is equivalent to \( \phi^{-l}B_\delta(\phi^{-s}y_1) \subset B_r(\phi^{-l}y_1) \). Since \( \delta \leq r \), we have \( l \leq s \) and hence \( l \) is the smallest integer satisfying

\[
|\lambda_{\text{max}}|^{s-l}\delta \leq r.
\]

Thus,

\[
|\lambda_{\text{max}}|^{s-l} > \frac{r}{\delta}|\lambda_{\text{max}}|^{-1}.
\]  

Observe that \( y_2 \in \phi^sB_\delta(\phi^{-s}y_1) \subset \phi^lB_r(\phi^{-l}y_1) \), hence \( \phi^{-l}y_1 \) and \( \phi^{-l}y_2 \) are in the same or in the neighboring tiles of \( T \) by the choice of \( r \). It is shown in the course of the proof of [20, Lem. 3.4] that we can write \( y_1 - y_2 = \sum_{h=1}^{l} \phi^hw_h \), where \( w_h \in W \) for some finite set \( W \subset \phi^{-1}\Xi \) which depends only on the tiling \( T \) (a similar statement, but without precise value of \( l \) is proved in [26, Lemma 4.5]). So

\[
||f(\xi_1) - f(\xi_2)|| \leq L'\gamma^{l-s},
\]

for some \( L' > 0 \) independent of \( l \). Notice that \( \gamma^{l-s} = (|\lambda_{\text{max}}|^{l-s})^\alpha \), where \( \alpha = \frac{\log \gamma}{\log |\lambda_{\text{max}}|} \). Thus

\[
||f(\xi_1) - f(\xi_2)|| \leq L'(|\lambda_{\text{max}}|^{l-s})\alpha
\]

\[
< L'\left(\frac{|\lambda_{\text{max}}|}{r}\right)^\alpha \quad \text{by (6.2)}
\]

\[
= L||\xi_1 - \xi_2||^\alpha \quad \text{where } L := L'\left(\frac{|\lambda_{\text{max}}|}{r}\right)^\alpha,
\]

and (6.1) is proved. \( \square \)

Since \( C_\infty \) is dense in \( \mathbb{R}^d \), we can extend \( f \) to a map \( f : \mathbb{R}^d \to H \) by continuity, and moreover,

\[
f \circ \phi = A \circ f
\]

(we denote the extended function by the same symbol \( f \)). This proves part (i) of Prop. 5.2.

**Lemma 6.2.** Let \( T \) and \( T + z \) be tiles in \( T \). Then

\[
f(\xi + z) = f(\xi) + g(z) \quad \text{for any } \xi \in \text{supp}(T).
\]  

(6.3)
Proof. It is enough to show that (6.3) holds for a dense subset of \( \text{supp}(T) \), namely, \( C_\infty \cap \text{supp}(T) \). Suppose that \( \xi = \phi^{-k}c(S) \), where \( S \in \omega^k(T) \). Note that

\[
S + \phi^kz \in \omega^k(T + z) \subset T.
\]

Then

\[
f(\xi + z) = f(\phi^{-k}c(S) + z) = f(\phi^{-k}(c(S) + \phi^kz))
\]

\[
= f\phi^{-k}(c(S) + \phi^kz) = A^{-k}g(c(S) + \phi^kz)
\]

\[
= A^{-k}(g(c(S)) + g(\phi^kz)) = A^{-k}g(c(S)) + A^{-k}g\phi^k(z)
\]

\[
= f(\phi^{-k}c(S)) + g(z) = f(\xi) + g(z).
\]

\( \square \)

Recall that \( A \) is diagonalizable over \( \mathbb{C} \). For \( \theta \in \text{Spec}(A) \) let \( p_\theta : H \to H \) be the canonical projection onto the real \( A \)-invariant subspace for \( A \) corresponding to \( \theta \), so that we have

\[
I_H = \sum_{\theta \in \text{Spec}(A)} p_\theta.
\]

Define

\[
f_\theta = p_\theta \circ f, \quad \theta \in \text{Spec}(A).
\]

Note that

\[
f = \sum_{\theta \in \text{Spec}(A)} f_\theta.
\]

Suppose that \( \lambda \in \text{Spec}(\phi) \) satisfies \( |\lambda| = \gamma \).

Lemma 6.3. For \( \theta \in \text{Spec}(A) \) and \( a \in \mathbb{R}^d \),

\[
\begin{cases}
  f_\theta|_{a + G_\lambda} \text{ is Lipschitz} & \text{if } |\theta| = |\lambda|; \\
  f_\theta|_{a + G_\lambda} \text{ is constant} & \text{if } |\theta| > |\lambda|.
\end{cases}
\]

Moreover, the Lipschitz constant is uniform in \( a \in \mathbb{R}^d \) (equal to \( C \) from (6.3)).

Proof. Let \( \xi_1, \xi_2 \in a + G_\lambda \) for some \( a \in \mathbb{R}^d \). For any \( l \in \mathbb{Z}_+ \), using the norm in \( H \) analogous to that in (3.2), so that \( \|A \circ p_\theta(x)\| = |\theta| \|p_\theta(x)\| \), we obtain

\[
\|f_\theta(\xi_1) - f_\theta(\xi_2)\| = \|(p_\theta \circ f)\phi^{-l}(\phi^l\xi_1) - (p_\theta \circ f)\phi^{-l}(\phi^l\xi_2)\|
\]

\[
= \|p_\theta(A^{-l}f(\phi^l\xi_1) - A^{-l}f(\phi^l\xi_2))\|
\]

\[
= |\theta|^{-l}\|p_\theta(f(\phi^l\xi_1) - f(\phi^l\xi_2))\|
\]

\[
\leq |\theta|^{-l}\|f(\phi^l\xi_1) - f(\phi^l\xi_2)\|.
\]

(6.6)

Note that there exist \( y_1, y_2 \in C \) such that

\[
\|\phi^l\xi_1 - y_1\| < \delta_1 \text{ and } \|\phi^l\xi_2 - y_2\| < \delta_1
\]
for some fixed $\delta_1 > 0$. Since $f$ is uniformly continuous,
\[
\|f(\phi^1_1) - f(y_1)\| < \delta_2 \quad \text{and} \quad \|f(\phi^1_2) - f(y_2)\| < \delta_2
\]
for some fixed $\delta_2 > 0$. By the assumption on $f$, we have $\|f(y_1) - f(y_2)\| \leq C\|y_1 - y_2\|$. Thus
\[
\|f(\phi^1_1) - f(\phi^1_2)\| < \|f(y_1) - f(y_2)\| + 2\delta_2 \\
\leq C\|y_1 - y_2\| + 2\delta_2 \\
< C((|\phi^1_1 - \phi^1_2| + 2\delta_1) + 2\delta_2).
\]
(6.7)

Applying (6.7) to (6.3), we obtain that for any $l \in \mathbb{Z}_+$
\[
\|f_\theta(\xi_1) - f_\theta(\xi_2)\| \\
< |\theta|^{-l} \left( C\|\phi^1_1 - \phi^1_2\| + 2C\delta_1 + 2\delta_2 \right) \\
= C|\lambda|^l |\theta|^{-l} \|\xi_1 - \xi_2\| + \frac{1}{|\theta|} (2C\delta_1 + 2\delta_2).
\]
Thus if $|\theta| = |\lambda|$, we have that $f_\theta|_{a+G_\lambda}$ is Lipschitz with a uniform Lipschitz constant $C$, and if $|\theta| > |\lambda|$, we have that $f_\theta|_{a+G_\lambda}$ is constant.

Remark 6.4. First note that $|\lambda| = \gamma \leq \min\{|\theta| : \theta \in \text{Spec}(A)\}$ by (5.4). The last lemma implies that for any $\xi \in \mathbb{R}^d$ and $w \in G_\lambda$, the vector $f(\xi + w) - f(\xi)$ is in the subspace generated by eigenspaces of $A$ corresponding to eigenvalues $\theta$ for which $|\theta| = |\lambda|$. We make use of this observation to show (6.10) in Lemma 6.6 below.

From Lemma 6.3 and (6.5), we get the following corollary.

Corollary 6.5. $f|_{a+G_\lambda}$ is Lipschitz for any $a \in \mathbb{R}^d$.

We now prove furthermore that $f$ is affine linear on $G_\lambda$ slices of $\mathbb{R}^d$.

Lemma 6.6. $f|_{a+G_\lambda}$ is affine linear for any $a \in \mathbb{R}^d$.

Proof. This is analogous to [20] Lem. 3.7, but in some places the presentation is sketchy, so we provide complete details for the readers’ convenience.

Since $f|_{a+G_\lambda}$ is Lipschitz for any $a \in \mathbb{R}^d$, it is a.e. differentiable by Rademacher’s theorem, and hence $f$ is differentiable in the direction of $G_\lambda$ a.e. in $\mathbb{R}^d$, by Fubini’s theorem. Let
\[
D(z)u = \lim_{t \to 0} \frac{f(z + tu) - f(z)}{t} \quad \text{for} \ u \in G_\lambda \ \text{and} \ z \in \mathbb{R}^d.
\]
The limit exists a.e. $z \in \mathbb{R}^d$ and for all $u \in G_\lambda$, and $D(z)u$ is a linear transformation in $u$ (from $G_\lambda$ to $H$). Moreover $D(z)$ is a measurable function of $z$, being a limit of continuous functions. By the definition of total derivative,
\[
\lim_{n \to \infty} F_n(z) = 0 \quad \text{for a.e.} \ z \in \mathbb{R}^d, \ \text{where} \ F_n(z) = \sup_{u \in G_\lambda} \frac{\|f(z + u) - f(z) - D(z)u\|}{\|u\|}.
\]
By Egorov’s theorem, \( \{F_n\} \) converges uniformly on a set of positive measure. This implies that there exists a sequence of positive integers \( N_l \uparrow \infty \) such that

\[
(6.8) \quad \Omega := \left\{ \xi \in \mathbb{R}^d : \forall l \geq 1, \forall u \in B_{1/N_l} \cap G_l, \frac{|f(\xi + u) - f(\xi) - D(\xi)u|}{\|u\|} < \frac{1}{l} \right\}
\]

has positive Lebesgue measure.

Our goal is proving that \( \Omega \) has full Lebesgue measure. The argument is based on a kind of “ergodicity”. First observe from Lemma 6.2 that \( \Omega \) is “piecewise translation-invariant” in the following sense:

\[
(6.9) \quad (T \in \mathcal{T}, T + x \in \mathcal{T}, \xi \in \Omega \cap \text{supp}(T)) \implies \xi + x \in \Omega.
\]

Second, \( \Omega \) is forward invariant under the expansion map \( \phi \). Indeed, let \( \xi \in \Omega \) and \( u \in \phi(B_{1/N_l}) \cap G_\lambda \). Then

\[
\|f(\phi^l \xi + u) - f(\phi^l \xi) - AD(\xi)\phi^{-1}u\|
= \|A(f(\phi^l \xi + \phi^{-1}u) - f(\phi^l \xi) - D(\xi)\phi^{-1}u)\|
= \|\lambda\|f(\phi^l \xi + \phi^{-1}u) - f(\phi^l \xi) - D(\xi)\phi^{-1}u\|
\]

by Remark 6.4.

This implies that \( D(\phi^l \xi) \) exists and equals \( AD(\xi)\phi^{-1} \), and since \( \phi(B_{1/N_l}) \supset B_{1/N_l} \supset B_{1/N_l} \), we also obtain that

\[
\phi(\Omega) \subset \Omega.
\]

We will need a version of the Lebesgue-Vitali density theorem where the differentiation basis is the collection of sets of the form \( \phi^{-l}B_1 \), \( l \geq 0 \), and their translates. It is well-known that such sets form a density basis, see [39, pp. 8–13]. Let \( y \) be a density point of \( \Omega \) with respect to this density basis. Then

\[
m(\Omega \cap \phi^{-l}B_1(\phi^ly)) \geq (1 - \epsilon_l)m(\phi^{-l}B_1) \quad \text{for some } \epsilon_l \to 0,
\]

where \( m \) denotes the Lebesgue measure. Note that

\[
m(\Omega \cap B_1(\phi^ly)) \geq m(\phi^l \Omega \cap B_1(\phi^ly))
= |\det \phi|^l m(\Omega \cap \phi^{-l}B_1(\phi^ly))
\geq |\det \phi|^l (1 - \epsilon_l)m(\phi^{-l}B_1)
= (1 - \epsilon_l)m(B_1).
\]

By FLC and repetitivity, there exists \( R > 0 \) such that \( B_R \) contains equivalence classes of all the patches \( [B_1(\phi^ly)]^T \). Then for any \( l \in \mathbb{Z}_+ \), there exists \( y_l \in B_R \) such that

\[
[B_1(y_l)]^T = [B_1(\phi^ly)]^T + (y_l - \phi^ly).
\]

By (6.9), we have

\[
m(\Omega \cap B_1(y_l)) \geq (1 - \epsilon_l)m(B_1),
\]
This is similar to [20, Lem. 3.8], but again, there are some differences.

Choose \( n_l \in \mathbb{Z}_+ \) so that \( |\lambda|^n_l > N_l \). Repeating the argument of (6.10) we obtain

\[
\xi \in \phi^n \Omega \quad \Rightarrow \quad \|f(\xi + v) - f(\xi) - D(\xi)v\| \leq \frac{|v|}{l}
\]

for all \( v \in \phi^n \left( B_{1/N_l} \cap G_\lambda \right) \supset B_{|\lambda|^n_l/N_l} \cap G_\lambda \supset B_1 \cap G_\lambda \).

Thus \( f(\xi + v) = f(\xi) + D(\xi)v \) for any \( \xi \in \bigcap_{i=1}^\infty \phi^n \Omega \) and \( v \in B_1 \cap G_\lambda \). Note that \( \bigcap_{i=1}^\infty \phi^n \Omega \) has full measure, hence it is dense in \( \mathbb{R}^d \). So for any \( \xi \in \mathbb{R}^d \), we can find a sequence \( \{\xi_j\} \subset \bigcap_{i=1}^\infty \phi^n \Omega \) such that \( \xi_j \to \xi \). Since \( f|_{\xi_j + G_\lambda} \) is Lipschitz with a uniform Lipschitz constant \( C \), the derivatives \( D(\xi_j) \) are uniformly bounded, and we can assume that \( D(\xi_j) \) converges to some linear transformation \( D_\xi \) by passing to a subsequence. Then we can let \( j \to \infty \) to obtain

\[
f(\xi + v) = f(\xi) + D_\xi v \quad \text{for all } v \in B_1 \cap G_\lambda.
\]

Since this holds for every point in \( \mathbb{R}^d \), we obtain that \( D_\xi = D_{\xi'} \) for any \( \xi, \xi' \in \mathbb{R}^d \) with \( \xi - \xi' \in G_\lambda \), and \( f|_{\xi + G_\lambda} \) is affine linear for any \( \xi \in \mathbb{R}^d \).

This concludes the proof of Proposition 5.2.

Recall [5.5] that \( G' = \bigcap_{\epsilon > 0} G_\epsilon \) and \( G_\epsilon = \text{Span}_{\mathbb{R}}(B_\epsilon \cap (I - P_\lambda)\Xi) \).

**Lemma 6.7.** There exists \( \epsilon > 0 \) such that \( G' = G_{\epsilon'} \) for every \( 0 < \epsilon' \leq \epsilon \), and moreover,

\[
G' = G'' := \text{Span}_{\mathbb{R}}(B_{\epsilon'} \cap (I - P_\lambda)(C_1 - C_1)) \quad \text{for all } 0 < \epsilon' \leq \epsilon.
\]

**Proof.** Observe that \( G_{\epsilon'} \subset G_\epsilon \) for \( \epsilon' < \epsilon \). These are finite-dimensional subspaces over \( \mathbb{R} \), hence they must stabilize, which yields the first claim.

To prove the second claim, we just need to show \( G' \subset G'' \) since \( C_1 - C_1 \subset \Xi \). There exists \( k \in \mathbb{Z}_+ \) such that \( \phi^k \Xi \subset C_1 - C_1 \) (just choose \( k \) such that \( \omega^k(T_1) \) contains tiles of all types). Then

\[
G' = \phi^k G' = \phi^k G_{\epsilon'/\|\phi^k\|} \subset \text{Span}_{\mathbb{R}}(B_{\epsilon'} \cap (I - P_\lambda)\phi^k \Xi) \subset G''.
\]

as desired. \( \Box \)

**Proof of Proposition 5.2.** This is similar to [20, Lem. 3.8], but again, there are some differences, and we provide more details here.

For any \( z \in \mathbb{R}^d \) and \( \xi \in G \), we have \( f(z + P_\lambda \xi) = f(z) + E(z)P_\lambda \xi \) by Lemma 6.6. Since \( f \) is uniformly continuous, \( E(z) \) is independent of the choice of \( z \). So for \( \xi \in G \),

\[
f(a + \xi) = f(a + (I - P_\lambda)\xi + P_\lambda \xi) = f(a + (I - P_\lambda)\xi) + E P_\lambda \xi ,
\]

(6.11)
for some fixed linear transformation \( E : G_\lambda \rightarrow H \). Thus we only need to prove that \( f \) is affine linear on all \( G' \)-slices. Let \( T \) be a tile of type 1 in \( T \). For any \( a \in (\text{supp}(T))^\circ \), choose \( r > 0 \) such that \( B_r(a) \subset (\text{supp}(T))^\circ \). We will show that

\[
(6.12) \quad f \left( \frac{\zeta_1 + \zeta_2}{2} \right) = \frac{f(\zeta_1) + f(\zeta_2)}{2} \quad \text{for all } \zeta_1, \zeta_2 \in B_r(a) \cap (a + G').
\]

In other words, \( f|_{a + G'} \) satisfies the so-called Jensen functional equation, and since \( f \) is continuous, this will imply that \( f|_{a + G'} \) is locally affine linear, see [1] 2.1.4. By expanding (using \( \phi \)-invariance) and translating (using \( (6.9) \)), we will then conclude that \( f|_{a + G'} \) is affine linear for all \( a \in \mathbb{R}^d \).

Now we show (6.12). By Lemma 6.7, for any \( \epsilon' > 0 \) with \( \epsilon' \leq \epsilon \), there exists a basis \( \{y_1, \cdots, y_s\} \) of \( G' \) such that for each \( 1 \leq j \leq s \), \( y_j \in B_{\epsilon'}(I - P_\lambda)z_j \) for some \( z_j \in C_1 - C_1 \). Let \( \zeta_1, \zeta_2 \in B_r(a) \cap (a + G') \) and fix small \( \epsilon' > 0 \) such that \( \epsilon' \leq \epsilon \) and

\[
\epsilon' < \left( r - \max\{||\zeta_1||, ||\zeta_2||\} \right)/4s.
\]

We consider the lattice generated by the \( y_j \)'s in \( G' \). It defines a grid with grid cells of diameter less than \( s \max_j ||y_j|| < s\epsilon' \). Thus there exist \( b_j \in \mathbb{Z} \), \( 1 \leq j \leq s \), such that

\[
|| \sum_{j=1}^s b_j y_j - \frac{(\zeta_2 - \zeta_1)}{2} || < s\epsilon'.
\]

Let \( \tilde{\zeta} := \zeta_1 + \sum_{j=1}^s b_j y_j \), so that

\[
|| \frac{\zeta_1 + \zeta_2}{2} - \tilde{\zeta} || < s\epsilon'.
\]

Translate our grid in such a way that \( \zeta_1 \) is the origin and consider a ‘grid geodesic’ connecting \( \zeta_1 \) to \( \tilde{\zeta} = \zeta_1 + \sum_{j=1}^s b_j y_j \) in the \( s\epsilon' \)-tube around the line segment \([\zeta_1, \tilde{\zeta}]\). By the choice of \( \epsilon' \), this ‘grid geodesic’ is contained in \( B_r(a) \cap (a + G') \). It is a sequence of points \( \xi_1 = \zeta_1, \xi_2, \cdots, \xi_L = \tilde{\zeta}, \) where \( \xi_{i+1} - \xi_i = y_{t(i)}, \ y_{t(i)} = (I - P_\lambda)z_{t(i)}, \ z_{t(i)} \in C_1 - C_1, \) and \( L = \sum_{j=1}^s |b_j| \). For each \( z_{t(i)} \in C_1 - C_1 \), there exists a tile \( S \) of type 1 such that \( S + z_{t(i)} \in T \). By Lemma 6.2 we have \( f(\eta + z_{t(i)}) = f(\eta) + g(z_{t(i)}) \) for any \( \eta \in \text{supp}(S) \). In view of (6.11),

\[
(6.13) \quad f(\eta + y_{t(i)}) = f(\eta) + g(z_{t(i)}) - E P_\lambda(z_{t(i)}).
\]

Since \( \xi_t, \xi_{t+1} \in \text{supp}(T) \) which is of type 1, using Lemma 6.2 again we obtain

\[
\begin{align*}
 f(\xi_{t+1}) - f(\xi_t) &= f(\xi_t + y_{t(i)}) - f(\xi_t) \\
 &= f(\eta + y_{t(i)}) - f(\eta) = g(z_{t(i)}) - E P_\lambda(z_{t(i)}).
\end{align*}
\]

Then

\[
(6.14) \quad f(\tilde{\zeta}) - f(\zeta_1) = \sum_{i=1}^L (g(z_{t(i)}) - E P_\lambda(z_{t(i)})).
\]

Note that \( \zeta_2 - (\zeta_2 + \zeta_1 - \tilde{\zeta}) = \sum_{j=1}^s b_j y_j \). The point \( \zeta_2 + \zeta_1 - \tilde{\zeta} \) is symmetric to \( \tilde{\zeta} \) with respect to \( \frac{\zeta_1 + \zeta_2}{2} \), so it is also within \( s\epsilon' \) of \( \frac{\zeta_1 + \zeta_2}{2} \). The grid geodesic which connected \( \zeta_1 \) to
\( \tilde{\zeta} \), translated by \( \zeta_2 - \tilde{\zeta} \), connects \( \zeta_2 + \zeta_1 - \tilde{\zeta} \) to \( \zeta_2 \) inside \( B_r(a) \cap (a + G') \). Thus, we obtain, repeating the argument above, that

\[
(6.15) \quad f(\zeta_2) - f(\zeta_2 + \zeta_1 - \tilde{\zeta}) = \sum_{l=1}^{L} (g(z_{t(l)}) - E P_{\lambda} (z_{t(l)})).
\]

Since \( \|\tilde{\zeta} - \frac{\zeta_2 + \zeta_1}{2}\| < s\epsilon' \), by uniform continuity

\[
\max\{\|f(\tilde{\zeta}) - f(\frac{\zeta_2 + \zeta_1}{2})\|, \|f(\zeta_2 + \zeta_1 - \tilde{\zeta}) - f(\frac{\zeta_2 + \zeta_1}{2})\|\} < \delta(\epsilon')
\]

where \( \delta(\epsilon') \to 0 \) as \( \epsilon' \to 0 \). Combining this with (6.14) and (6.15) yields (6.12), as desired.

This completes the proof of Theorem 3.1.

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