On the abc Conjecture in Algebraic Number Fields

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Abstract

In this paper we prove a weak form of the abc Conjecture generalised to algebraic number fields. Given algebraic integers $a, b, c$ in a number field $K$ satisfying $a + b = c$, we give an upper bound for the logarithm of the projective height $H_L(a, b, c)$ in terms of norms of prime ideals dividing $abcO_L$, where $L$ is the Hilbert Class Field of $K$. In many cases this allows us to give a bound in terms of the modified radical $G := G(a, b, c)$ as given by Masser in [21]. Furthermore, by employing a recent result by Le Fourn, our estimates imply the upper bound

$$\log H_L(a, b, c) < \mathcal{C} G^{\frac{1}{4} + \epsilon \log \log \log G \log \log \log G},$$

where $\mathcal{C}$ is an effectively computable constant. Further, given conditions on the largest prime ideal dividing $abcO_L$, we obtain a sub-exponential bound for $H_L(a, b, c)$ in terms of the radical. As a consequence of our results, we will give an application to the effective Skolem-Mahler-Lech problem and give an improvement to a result by Lagarias and Soundararajan on the XYZ Conjecture given in [19].

1 Introduction

Let $a, b, c := a + b$ be positive, pairwise coprime integers and define the radical

$$G(a, b, c) = G = \prod_{p | abc} p.$$

In 1988, based on a conjecture of Szpiro about elliptic curves and on the Mason-Stothers Theorem in function fields [20], Osterlék conjectured that for all positive integers as above, there exists a positive constant $C_1$ such that $c < G^{C_1}$ [20]. Further, Masser conjectured a stronger statement, that for all positive $\epsilon$ there exists a constant $C_2(\epsilon)$ such that $c < C_2(\epsilon) G^{1+\epsilon}$ [22]. While both of these conjectures are referred to as the abc conjecture, generally the second form by Masser is focused on in the literature. These conjectures have far-reaching implications across a range of topics; see [3] and the references within. We also refer the reader to Chapter 14 of [2] and Chapter 5 of [36] for a discussion of Vojta’s conjectures, a generalisation of the second formulation of the abc conjecture given above.

In [33], Stewart and Yu prove that there exists an effectively computable positive constant $C_3$ such that for all positive integers $a, b, c = a + b$ with $(a, b, c) = 1$ and $c > 2$,

$$\log c < G^{\frac{2}{3} + \frac{C_3}{\log \log c}}.$$
In [34], they were able to improve this result to
\[ \log c < C_4 G^{3/2} (\log G)^3. \]

To do this required the use of Yu’s work extending lower bounds for linear forms in logarithms to the \( p \)-adic setting [38] [39]. We note Yu further improved these bounds in a series of papers [40] [41] [42] [43]; indeed, we will use results from [43], which make use of group varieties to strengthen the relevant bounds.

Much work has also been done generalising the \( abc \) conjecture to algebraic number fields. Browkin discusses this direction of research in [4], while in [21] Masser discusses some issues regarding adapting the radical \( G \) to the case of number fields. In [10], Györy shows that given a number field \( K \) and \( a, b, c \in K^* \) with \( a + b + c = 0 \) and any \( \epsilon > 0 \), there is an effectively computable \( C_5(\epsilon) \) such that
\[ \log (H_K(a, b, c)) < C_5 N_K(a, b, c)^{1+\epsilon}, \]
where
\[ H_K(a, b, c) = \prod_{v \in M_K} \max (|a|_v, |b|_v, |c|_v), \]
and
\[ N_K(a, b, c) = \prod_{v} \text{Nm}_{Q_Q}^K(p)^{\text{ord}_p(p)}, \]
where \( p \) is a rational prime such that \( p \) lies over \( p \), \( p \) is the prime ideal of \( O_K \) corresponding to \( v \in M_K \) and \( v \) is taken over all finite places such that \( |a|_v, |b|_v, |c|_v \) are not all equal. We note that this is the same as the modified support (1.11) of [21]. Recall that the norm of a prime ideal \( p \) of the ring of integers \( O_K \) of a number field \( K \) is defined to be
\[ \text{Nm}_{Q_Q}^K(p) = p^{f_p}, \]
where \( f_p \) is the inertia degree of \( p \) over \( p \) [25]; that is
\[ f_p := [O_K/p : \mathbb{Z}/p]. \]

Initially we introduce some notation we will use throughout this article. Let \( K \) be a number field of degree \( d \) and let \( a, b, c \in O_K \setminus \{0\} \) be such that \( a + b + c = 0 \). Further, assume that \( aO_K, bO_K \) and \( cO_K \) are pairwise coprime; that is \( aO_K + bO_K = O_K \), and similarly for all other pairs. Let \( L = HCF(K) \) be the Hilbert Class Field of \( K \) (that is, the maximal abelian unramified extension of \( K \) [7]) and let
\[ G = \prod_{\mathfrak{p} \text{ prime ideal}} \text{Nm}_{Q_Q}^L(\mathfrak{p}). \]

Let \( \mathfrak{p}_a \) be the prime ideal of \( O_L \) of greatest norm dividing \( aO_L \), and similarly for \( \mathfrak{p}_b \) and \( \mathfrak{p}_c \). If \( a \) is a unit, then we write that \( \mathfrak{p}_a = 1 \) with norm 1, and similarly for \( b \) and \( c \). Write \( \mathfrak{p}_{\text{max}} \) for the prime ideal of \( O_L \) of greatest norm dividing \( G \). A priori, this is equal to one of \( \mathfrak{p}_a, \mathfrak{p}_b, \mathfrak{p}_c \).
Let
\[ h(x) = \sum_{v \in M_F} \log^+ |x|_v, \]
where \( M_F \) is the set of places of the number field \( F \) normalised so they satisfy the product formula \[2\], and \( \log^+ (\alpha) = \max(\log \alpha, 0) \). Also, let
\[ H_F(x_1, \ldots, x_n) = \prod_{v \in M_F} \max \{|x_1|_v, \ldots, |x_n|_v\}. \]

It is worth pointing out that \( H_F(x_1, \ldots, x_n) \) is the projective height, so it gives the same value for any representative of \((x_1, \ldots, x_n) \in \mathbb{P}^{n-1}(F)\). Explicitly, this means that for any \((a, b, c) \in \mathbb{P}^2(F)\) and any \( k \in F^\times\) we have that
\[ H_F(a, b, c) = H_F(ka, kb, kc). \] (1)

In particular, since in the set up of this article \( c \neq 0 \), we have that
\[ H_L(a, b, c) = H_L\left(\frac{a}{c}, \frac{b}{c}, 1\right). \]

We will generally be considering the height over the Hilbert Class Field \( L \). In this case, as \([K : \mathbb{Q}] = d\),
\[ h(x) = dh_K \log H_L(1, x), \]
where \( h_K \) is the class number of \( K \). This follows as \([L : K] = h_K [2] \) \[37\]

We note that for any \( x, y, z \in F \) where \( F \) is an algebraic number field of degree \( d \),
\[ \log H_F(x, y, z) = \log H_F\left(\frac{x}{z}, \frac{y}{z}, 1\right) \leq 2d \max \left(h\left(\frac{x}{z}\right), h\left(\frac{y}{z}\right)\right). \] (2)

This follows directly from (4.3) of \[10\]. Furthermore, we will show in Section 3 that we can write \( a = u_a a' \) where \( u_a \) is a unit such that
\[ C_6 \log |N_{L/\mathbb{Q}}(a')| \leq h\left(a'\right) \leq C_7 \log |N_{L/\mathbb{Q}}(a')|, \]
where \( C_6, C_7 \) are computable constants, and similarly for \( b \) and \( c \). We assume without loss of generality that
\[ h(a') \leq h(b') \leq h(c'). \] (3)

We initially prove the following main theorem.

**Theorem 1.** Given the set up above, relabeling \( a, b \) and \( c \) if necessary to satisfy \[3\], there exists an effectively computable constant \( C_8 \) depending only on the field \( K \) such that
\[ \log H_L(a, b, c) < \left(Nm_{\mathbb{Q}}(p_a)Nm_{\mathbb{Q}}(p_b)Nm_{\mathbb{Q}}(p_c)^2 \max \{Nm_{\mathbb{Q}}(p_b), Nm_{\mathbb{Q}}(p_c)\}\right)^{\frac{1}{3}} G^{\frac{\log \log \log G}{\log \log G}}. \] (4)
We will then give various corollaries to put the product of norms of prime ideals in terms of the radical \( G \), namely corollaries 1-7. Importantly, in Corollary 7 we will give conditions that allow us to attain a sub-exponential bound.

In later parts we will give related results that can be easier to manipulate due to fewer prime ideals on the right hand side of the inequality, attaining the following theorem.

**Theorem 2.** Given the set up above, there exists an effectively computable number \( C_9 \) depending only on the field \( K \) such that

\[
\log H_L(a, b, c) < \left( \frac{Nm_Q^L(p_b) \cdot Nm_Q^L(p_c)^2}{Nm_Q^L(p_a) \cdot Nm_Q^L(p_c)} \right)^{\frac{1}{2}} G^{C_9 \frac{\log \log \log G}{\log \log \log \log G}},
\]

where \( G = G_9 \). \( \text{(5)} \)

We will then deduce corollaries 8-13, again giving conditions in Corollary 10 that give a sub-exponential bound in terms of the radical \( G \).

We will then discuss how exploiting a method of Le Fourn [19] enables us to reduce the dependency on prime ideals to give the following result with no further conditions:

**Theorem 3.** Given the set up above, there exists an effectively computable constant \( C_{10} \) depending on \( K \) such that

\[
\log H_L(a, b, c) < \left( \frac{Nm_Q^L(p_a) \cdot Nm_Q^L(p_b) \cdot Nm_Q^L(p_c) \cdot Nm_Q^L(p'_c) \cdot Nm_Q^L(q)}{Nm_Q^L(p_b) \cdot Nm_Q^L(p_c)} \right)^{\frac{1}{2}} G^{C_{10} \frac{\log \log \log G}{\log \log \log \log G}},
\]

where \( p'_c \) is the prime ideal of third largest norm dividing \( cO_L \) and \( q \) is the prime ideal of \( O_L \) of third largest norm dividing \( bcO_L \).

We will then deduce that

\[
\log H_L(a, b, c) < G^{\frac{1}{4} + C_{11} \frac{\log \log \log G}{\log \log \log \log G}}.
\]

The results given in this paper, in particular Theorem 3, allow us to give a new method of solving the effective Skolem-Mahler-Lech problem [28] of order 3. Additionally, we use Corollary 10 to expand on results by Lagarias and Soundararajan regarding smooth solutions to the abc equation [16]. We briefly discuss both these problems here.

First we discuss the effective Skolem-Mahler-Lech problem. The problem is, given a linear recurrence sequence, to decide whether said sequence contains zeroes.

We recall that a linear recurrence sequence is a sequence \( (a_x) \) of elements of a commutative ring with 1, \( R \), satisfying a homogeneous linear recurrence relation

\[
a_{x+n} = c_1 a_{x+n-1} + \cdots + c_n a_x,
\]

where \( c_1, \ldots, c_n \in R \). We note, we will take \( R \) to be an algebraic number field.

The Skolem-Mahler-Lech Theorem is as follows.

**Theorem 4 (Skolem-Mahler-Lech).** If a sequence of numbers satisfies a linear recurrence relation over a field of characteristic zero, the zeroes of this sequence can be decomposed into the union of a finite set, and finitely many arithmetic progressions.
More comments on this theorem and a proof for sequences defined over the rationals is given in [8], along with further references.

**Remarks.** We note that there exists an algorithm to tell us if there are infinitely many zeroes, and if so to find the decomposition of these zeros into periodic sets guaranteed to exist by the Skolem–Mahler–Lech Theorem [1]. The effective Skolem-Mahler-Lech problem then is to find an algorithm to determine whether there are any non-periodic zeroes in a given linear recurrence sequence, importantly in the case where these are the only zeroes [28]. This would allow us to effectively answer whether a given linear recurrence relation contains any zeroes.

Recall, given a linear recurrence relation

\[ a_{x+n} = c_1a_{x+n-1} + \cdots + c_na_x, \]

and initial terms \( a_1, \ldots, a_n \in R \), we can find a formula for the \( m \)’th term. Given the recurrence relation, we find the characteristic polynomial

\[ f(X) = X^n - c_1X^{n-1} - \cdots - c_{n-1}X - c_n, \]

with roots \( r_1, \ldots, r_l \) with multiplicities \( m_1, \ldots, m_l \) respectively. The \( x \)'th term of the sequence then is given by

\[ a_x = g_1(x)r_1^x + \cdots + g_l(x)r_l^x, \]

where \( g_i(x) \) are polynomials with \( \text{deg} \ (g_i) \leq m_i - 1 \) which depend on the initial values \( a_1, \ldots, a_n \).

In section 8 we will show how Theorem 3 gives a new method of determining if a linear recurrence sequence of order 3 contains zeroes. For further references of preexisting methods and results on this problem and related problems we refer the reader to [14] [32] [27] and the references contained within them.

We now discuss the smooth \( abc \) Conjecture, also referred to as the \( xyz \) conjecture given by Lagarias and Soundararajan in [16]. Given a triple \( a, b, c := a + b \in \mathbb{N} \), define the smoothness of the triple

\[ S(a, b, c) := \max \{ p : p \mid abc \}. \]

In [16], Lagarias and Soundararajan give the following conjecture, which they refer to as the \( xyz \) conjecture.

**Conjecture 1** (\( xyz \) conjecture). There exists a positive constant \( \kappa \) such that the following hold.

a) For each \( \epsilon > 0 \) there are only finitely many integer solutions \( (X, Y, Z) \) to the equation \( X + Y = Z \) with \( (X, Y, Z) = 1 \) and

\[ S(X, Y, Z) < (\log H(X, Y, Z))^{\kappa-\epsilon}. \]

b) For each \( \epsilon > 0 \) there are infinitely many integer solutions \( (X, Y, Z) \) to the equation \( X + Y = Z \) with \( (X, Y, Z) = 1 \) and

\[ S(X, Y, Z) < (\log H(X, Y, Z))^{\kappa+\epsilon}. \]
When a triple \((X, Y, Z)\) satisfies \(X + Y = Z\) and \((X, Y, Z) = 1\), we will call the triple a primitive solution. Lagarias and Soundararajan go on to conjecture that \(\kappa = \frac{3}{2}\). They note however that to prove the above conjecture, one need only prove that there exists a \(\kappa_0 > 0\) satisfying part a) and a \(\kappa_1 < \infty\) satisfying part b). As a) and b) are independent, monotonicity would then imply the existence of a unique constant \(\kappa\).

Lagarias and Soundararajan prove part b) assuming the Generalised Riemann Hypothesis, and show that the \(abc\) conjecture implies part a). Further, in Corollary 1 of [15], Harper is able to show unconditionally that the \(xyz\)-smoothness exponent \(\kappa\) is finite, and showed that part b) of the conjecture holds.

Unconditionally, Lagarias and Soundararajan are able to give the following result.

**Theorem** (Theorem 2.2 of [16]). For each \(\epsilon > 0\) there are only finitely many solutions to \(X + Y = Z\) satisfying \((X, Y, Z) = 1\) and

\[
S(X, Y, Z) \leq (3 - \epsilon) \log \log H(X, Y, Z).
\]

The proof of this, and the improvement we will give below depend heavily on Northcott’s Theorem. We recall that Northcott’s Theorem tells us that for any given number field \(K\) and \(B \in \mathbb{R}, B > 0\), the set

\[
\{(X, Y, Z) \in K^3 : H(X, Y, Z) < B\}
\]

is finite [2].

Using results from this paper, we will improve this bound with the following theorem.

**Theorem 5.** Let \(\phi : \mathbb{R} \to \mathbb{R}\) be a function such that \(\phi(x) < \log \log x\) with

\[
\lim_{x \to +\infty} \phi(x) = +\infty.
\]

Then there are finitely many solutions to \(X + Y = Z\) satisfying \((X, Y, Z) = 1\) and

\[
S(X, Y, Z) \leq \log \log H(X, Y, Z) \frac{\log \log H(X, Y, Z)}{\log \log \log \log H(X, Y, Z) \phi(\log \log H(X, Y, Z))},
\]  

\[ (8) \]

We note that this result implies that there are only finitely many primitive integer triples \((X, Y, Z)\) satisfying \(X + Y = Z\) with

\[
S(X, Y, Z) < c \log \log H(X, Y, Z)
\]

for any constant \(c \in \mathbb{R}, c > 0\). This is because for any such \(c\), there is a value \(H\) such that for any \(H(X, Y, Z) > H\),

\[
\frac{\log \log H(X, Y, Z)}{\log \log \log \log H(X, Y, Z) \phi(\log \log H(X, Y, Z))} > c,
\]

and by Northcott’s Theorem, there are only finitely many triples \((X, Y, Z)\) satisfying \(H(X, Y, Z) < H\). The statement above then follows from Theorem 5 and along with the result given in Theorem 5 is also an improvement on the \(3 - \epsilon\) in Theorem 2.2 of [16].
Remarks. Independently, using different methods, Györy has been able to show a similar result to that of these results, but over the base field $K$ [12]. We will discuss this further in section 7.

When this manuscript was completed, Professor Györy informed me about a sharper abc inequality over $\mathbb{Q}$ and imaginary quadratic number fields by Mochizuki, Fesenko, Hoshi, Minamide and Porowski (submitted for publication). However, this result relies on Mochizuki’s results in Inter-universal Teichmüller Theory [23], the veracity of which is currently being debated [31].

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2 Preliminary Lemmas

In this section we state pre-existing lemmas which we will repeatedly use throughout the proof of Theorem 1. In the rest of the paper, $C_{11}, C_{12}, \ldots$ denote effectively computable constants, and we will, where relevant, state what these constants depend on. In many cases, the constants depend on properties determined by a certain field; in these cases we will sometimes explicitly give which properties of the field the constants depend on.

We note that by Lemma 1 of [6], we can find a system of fundamental units $\eta_1, \ldots, \eta_r$ where $r$ is the unit rank of $F$ such that

$$\prod_{i=1}^{r} h(\eta_i) \leq C_{11} R_F$$

where

$$C_{11} = \frac{(r-1)!}{2^{r-2} d^{r-1}}$$

and $R_F$ is the regulator of the field $F$. Throughout this paper we will use such a system of fundamental units, and often refer to them as ”the fundamental units” of the field in question.

Lemma 1. Let $F$ be a number field of degree $d$, and let $\alpha \in \mathcal{O}_F \setminus \mathcal{O}_F^*$. Then there is an effectively computable number $C_{12}(F)$, depending on the fundamental units of $\mathcal{O}_F$, and an $\epsilon \in \mathcal{O}_F^*$ such that

$$\alpha \leq C_{12} \left| N_{F/\mathbb{Q}}(\alpha) \right|^{1/d}$$

where $\alpha$ denotes the house of $\alpha$.

We recall that $\alpha$, the house of $\alpha$, is defined to be the maximal absolute value of the conjugates of $\alpha$ over $\mathbb{C}$.

Proof. This is Lemma 1.3.8 from [24].
Lemma 2. Let $F$ be an algebraic number field of degree $d$ with set of normalised places $M_F$, and let $S$ be a finite subset of $M_F$ which contains $S_\infty$, the set of infinite places. Let $s$ be the cardinality of $S$, $p_1, \ldots, p_s$ the prime ideals corresponding to finite places of $S$ and let $P = \max_i Nm_Q^F(p_i)$. Further let $R_S$ be the $S$-regulator of $F$. We note that $R_S = i_S R \prod_{i=1}^{t} \log Nm_Q^F(p_i)$, where $i_S$ is a positive divisor of the class number $h_F$ of $F$ and $R$ is the regulator of $F$ (cf. [10][9][13]). Given $\alpha, \beta$, non-zero elements of $F$, we consider the $S$-unit equation $\alpha x + \beta y = 1$ in $x, y$, where $x, y$ are $S$-units. Let $r$ denote the unit rank of $F$ and let $R = \max \{ h_F, C_{13}(r, d) R \}$, where $C_{13}(r, d)$ is given explicitly in [10]. Then, if $t = 0$, all solutions $x, y$ of the above equation satisfy
\[
\max \{ h(x), h(y) \} \leq C_{13}(r, d) \max \{ h(\alpha), h(\beta), 1 \}.
\]

If $t > 0$ then we obtain
\[
\max \{ h(x), h(y) \} \leq C_{14}(r, d) h_F R (\log^* R) R^{t+1} (\log^* R) \left( \frac{P}{\log^* P} \right) R_S \max \{ h(\alpha), h(\beta), 1 \},
\]
where $C_{14}(r, d)$ is also explicitly given in [10].

Proof. This is part of Theorem A from [10].

Lemma 3. Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers and $F$ a number field of degree $d$ containing $\alpha_1, \ldots, \alpha_n$. Let $p$ be a prime ideal of $O_F$ lying above the rational prime $p$ with ramification index $e_p$ and residue class degree $f_p$. For $\alpha \in F, \alpha \neq 0$, we write $\ord_p(\alpha)$ for the exponent to which $p$ divides the principal fractional ideal generated by $\alpha$ in $F$, and we set $\ord_p(0) = +\infty$. Let $b_1, \ldots, b_n$ be integers, and set $\Theta = \alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1$. Assume that $\Theta \neq 0$. Finally, set $h'(\alpha_j) = \max \{ h(\alpha_j), \frac{1}{\log Nm_{\infty}^F} \}$. Then
\[
\ord_p(\Theta) < (16ed)^{2(n+1)} n^{5/2} \log (2nd) \log (2d) \cdot \frac{N_{Q/F}(p)}{(\log Nm_Q^F(p))^2} \prod_{i=1}^{n} h'(\alpha_i) \log B,
\]
where $B = \max \{ \| b_1 \|, \ldots, \| b_n \|, 3 \}$.

Proof. This is a consequence of the main theorem of [14], given on page 190.

Lemma 4. Let $F$ be a number field with ring of integers $O_F$. Apply a total ordering to the prime ideals, so that if $\Nm_Q^F(\pi) > \Nm_Q^F(\eta)$, then $\pi > \eta$. Arbitrarily order ideals of the same norm. Then there is an effectively computable positive constant $C_{15}$ such that for every positive integer $r$ we have
\[
\prod_{i=1}^{r} \frac{\Nm_Q^F(p_i)}{\log \Nm_Q^F(p_i)} > \left( \frac{r}{C_{15}} \right)^r.
\]

Proof. Let $\pi_F(x)$ denote the number of prime ideals in number field $F$ of norm less than or equal to $x$. By the Landau Prime Ideal Theorem [17], we know that $\pi_F(x) \sim \frac{x}{\log x}$. Partially order the prime ideals as in the statement of the Lemma. Then by Landau,
\[
\pi_F(\Nm_Q^F(p_j)) \sim \frac{\Nm_Q^F(p_j)}{\log (\Nm_Q^F(p_j))}.
\]
Thus by Landau, there exists an effectively computable number \( C_{16} \) such that
\[
\frac{\text{Nm}_Q^F(p_j)}{\log (\text{Nm}_Q^F(p_j))} > j/C_{16}.
\]
Thus, using the inequality \( r! \geq (r/e)^r \), we see that
\[
\prod_{j=1}^r \frac{\text{Nm}_Q^F(p_j)}{\log (\text{Nm}_Q^F(p_j))} > r! \geq \left( \frac{r}{C_{17}e} \right)^r \geq \left( \frac{r}{C_{18}} \right)^r,
\]
which proves the claim in the lemma.

We finally state a lemma that we will use to tidy our end arguments.

**Lemma 5.** If \( x, a \in \mathbb{R} \) with \( \frac{a}{\log a} < x \), then \( a < \max \{ e, 2x \log x \} \).

**Proof.** If \( \frac{a}{\log a} < x \), then
\[
a < x \log a. \tag{10}
\]
As \( \frac{a}{\log a} < x \), we see that \( \log a - \log \log a < \log x \). Further, we can show that \( \frac{\log a}{2} < \log a - \log \log a \).

Combining these we get that
\[
\log a < 2 \log x. \tag{11}
\]
Multiplying together (10) and (11) we obtain that \( a < 2x \log x \). \( \square \)

## 3 Proof of the Main Theorem

Let \( K \) be a number field with ring of integers \( \mathcal{O}_K \), class number \( h_K \), and Hilbert Class Field \( L \). Let \( [K : \mathbb{Q}] = d \), so by the tower law, \( [L : \mathbb{Q}] = h_K d \).

Take \( a, b, c := -a - b \in \mathcal{O}_K \) so that
\[
a + b + c = 0, \tag{12}
\]
with the assumption that \( a\mathcal{O}_K, b\mathcal{O}_K \) and \( c\mathcal{O}_K \) are coprime. We write
\[
a\mathcal{O}_K = p_1^{e_1} \cdots p_i^{e_i}, \\
b\mathcal{O}_K = q_1^{f_1} \cdots q_u^{f_u}, \\
c\mathcal{O}_K = \tau_1^{g_1} \cdots \tau_v^{g_v},
\]
where \( p_i, q_j, \) and \( \tau_k \) are prime ideals of \( \mathcal{O}_K \) and \( e_i, f_j, g_k \) are integers.

A key property of \( L = \text{HCF}(K) \) is that every ideal \( \mathfrak{J} \) of \( \mathcal{O}_K \) is principal in \( \mathcal{O}_L \); that is \( \mathfrak{J}\mathcal{O}_L = \alpha\mathcal{O}_L \) for some \( \alpha \in \mathcal{O}_L \). By Lemma 1, we can pick the generator of each ideal so that if \( \alpha \) is the generator, then
\[
h(\alpha) \leq \log |\alpha| \leq \log \left( C_{19} (K) |N_{L/Q} (\alpha)|^{1/d} \right) = C_{20} (K) \log |N_{L/Q} (\alpha)|. \tag{14}
\]
We note that the dependence of the constants is on $K$ rather than $L$, as $L$ is uniquely determined by $K$. Further, for such algebraic $\alpha$ we have
\[
\log |N_{L/Q}(\alpha)| \leq dh(\alpha),
\]
giving us that
\[
C_{21}(K) \log |N_{L/Q}(\alpha)| \leq h(\alpha) \leq C_{22}(K) \log |N_{L/Q}(\alpha)|.
\] (15)

Recalling this, we write
\[
p_iO_L = a_iO_L
\]
\[
q_jO_L = b_jO_L
\]
\[
r_kO_L = c_kO_L,
\] (16)
where $a_i$, $b_j$, $c_k$ satisfy (14).

We can also write $aO_L$, $bO_L$ and $cO_L$ as a product of prime ideals of $O_L$. Knowing this, we will write
\[
G = \prod_{\mathfrak{P} \text{ prime ideal}} \mathfrak{P}^{\text{Nm}_{L/Q}(\mathfrak{P})}.
\] (17)
Note that by our assumptions this is equivalent to taking the field to be $L$ in equation (1.7) of [21]. Further we will denote the prime ideal of $O_L$ of largest norm dividing $aO_L$ by $p_a$, and similarly for $b$ and $c$.

From (16) we can write $a$, $b$, $c$ as follows:
\[
a = u_a a_1^{e_1} \cdots a_t^{e_t}
\]
\[
b = u_b b_1^{f_1} \cdots b_u^{f_u}
\]
\[
c = u_c c_1^{g_1} \cdots c_v^{g_v}
\] (18)
where $u_a$, $u_b$ and $u_c$ are units of $O_L$. We will also often write $u_a a' + u_b b' + u_c c' = 0$ where $a' = \prod_{i=1}^{t} a_i$ and similarly for $b'$ and $c'$. After relabeling if needed, we can assume that (3) holds. We note that if $h(c') \leq 1$ then straightforwardly the claim holds. This is because necessarily $h(a') \leq h(b') \leq h(c') \leq 1$ and Theorem 1 readily follows from (2). Similarly, after some work we do below finding a bound on $h(c')$, we see that if $h(b') \leq 1$ then again we'll find the claim straightforwardly follows, and the same logic will hold if $h(a') \leq 1$. Thus we assume in the following that
\[
1 < h(a') \leq h(b') \leq h(c').
\] (19)

Dividing through by $u_c c' = c$ in (12) we obtain that
\[
-\frac{u_a a'}{u_c c'} - \frac{u_b b'}{u_c c'} = 1,
\] (20)
so we are in a position to apply Lemma [2].

Before doing this, we note that by the remarks around (11), $\log H_L(a, b, c) = \log H_L(\frac{a}{e}, \frac{b}{f}, 1)$. This allows us to move between representatives of the projective point $[a : b : c] \in \mathbb{P}^2(L)$. 

10
First, following the notation from Lemma 2, initially take $S$ to be the set of infinite places of $L$. Applying this to (20), we obtain that

$$\max \left\{ h \left( \frac{-u_a}{u_c} \right), h \left( \frac{-u_b}{u_c} \right) \right\} = \max \left\{ h \left( \frac{u_a}{u_c} \right), h \left( \frac{u_b}{u_c} \right) \right\}$$

$$\leq C_{23} (K) \max \left\{ h \left( \frac{a'}{c'} \right), h \left( \frac{b'}{c'} \right), 1 \right\}. \quad (21)$$

We note similar bounds also hold if we divide (12) through by $u_a a'$ or $u_b b'$. Recall that $h(xy) \leq h(x) + h(y)$ [2][37]. By this fact and our assumptions on $h(a'), h(b')$ and $h(c')$, we deduce that

$$\max \left\{ h \left( \frac{a'}{c'} \right), h \left( \frac{b'}{c'} \right), 1 \right\} \leq \max \left\{ h \left( a' \right) + h \left( c' \right), h \left( b' \right) + h \left( c' \right) \right\} \leq 2 h \left( c' \right). \quad (22)$$

It thus follows that

$$h \left( \frac{u_a a'}{u_c c'} \right) \leq h \left( \frac{u_a}{u_c} \right) + h \left( a' \right) + h \left( c' \right)$$

$$\leq C_{24} h \left( c' \right) + 2 h \left( c' \right)$$

$$= C_{25} h \left( c' \right). \quad (23)$$

Similarly, we also obtain that

$$h \left( \frac{u_b b'}{u_c c'} \right) \leq C_{26} h \left( c' \right). \quad (24)$$

We again apply Lemma 2 but this time with a different set of $S$-units. Now let $S = S_\infty \cup \{ p : p \mid \mathcal{O}_L \}$ where $p$ refers both to the prime ideal $p$ and its corresponding place. Applying Lemma 2 in this case to (20), we obtain that

$$\max \left\{ h \left( \frac{u_a}{u_c} \right), h \left( \frac{u_b}{u_c} \right) \right\} \leq C_{27} (K) \left( \frac{\text{Nm}_Q^L (p_c)}{\log^* \text{Nm}_L^Q (p_c)} \right) \left( \prod_{p \mid \mathcal{O}_L \setminus \{ p \mid \mathcal{O}_L \}} \log \text{Nm}_Q^L (p) \right) \max \left\{ h \left( a' \right), h \left( b' \right), 1 \right\}$$

$$\leq C_{27} (K) \text{Nm}_Q^L (p_c) \left( \prod_{p \mid \mathcal{O}_L} \log \text{Nm}_Q^L (p) \right) h \left( b' \right). \quad (25)$$

Again, $L$ is defined by $K$ uniquely, hence the dependence of the constants here on $K$ rather than $L$. Further, by [19], we have that $\max \left\{ h \left( a' \right), h \left( b' \right), 1 \right\} = h \left( b' \right)$.  

11
From (25) we obtain that
\[ h \left( \frac{u_{a}a'}{u_{c}c'} \right) \leq h \left( \frac{u_{a}}{u_{c}c'} \right) + h \left( a' \right) \]
\[ \leq C_{27} \left( K \right) \text{Nm}_{L}^{Q} \left( p_{c} \right) \left( \prod_{\substack{p \in \mathcal{O}_{L} \\
p | bc \mathcal{O}_{L} \backslash p \neq p_{c}}} \log \text{Nm}_{L}^{Q} \left( p \right) \right) \left( h \left( b' \right) + h \left( a' \right) \right) \]
\[ \leq \left( C_{27} \left( K \right) \text{Nm}_{L}^{Q} \left( p_{c} \right) \left( \prod_{\substack{p \in \mathcal{O}_{L} \\
p | bc \mathcal{O}_{L} \backslash p \neq p_{c}}} \log \text{Nm}_{L}^{Q} \left( p \right) \right) + 1 \right) \left( h \left( b' \right) \right) \]
\[ \leq C_{28} \left( K \right) \text{Nm}_{L}^{Q} \left( p_{c} \right) \left( \prod_{\substack{p \in \mathcal{O}_{L} \\
p | bc \mathcal{O}_{L} \backslash p \neq p_{c}}} \log \text{Nm}_{L}^{Q} \left( p \right) \right) \left( h \left( b' \right) \right). \tag{26} \]

Similarly, we find that
\[ h \left( \frac{u_{b}b'}{u_{c}c'} \right) \leq C_{29} \left( K \right) \text{Nm}_{L}^{Q} \left( p_{c} \right) \left( \prod_{\substack{p \in \mathcal{O}_{L} \\
p | bc \mathcal{O}_{L} \backslash p \neq p_{c}}} \log \text{Nm}_{L}^{Q} \left( p \right) \right) \left( h \left( b' \right) \right). \tag{27} \]

We now choose another set \( S \), this time containing the infinite places and the finite places corresponding to the prime ideals dividing \( bc \mathcal{O}_{L} \); that is, \( S = S_{\infty} \cup \{ p : p | bc \mathcal{O}_{L} \} \). Applying Lemma 2 to (20) with this \( S \) we obtain that
\[ \max \left\{ h \left( \frac{u_{a}}{u_{c}c'} \right), h \left( \frac{u_{b}b'}{u_{c}c'} \right) \right\} \leq C_{30} \left( K \right) \max \left\{ \text{Nm}_{L}^{Q} \left( p_{b} \right), \text{Nm}_{L}^{Q} \left( p_{c} \right) \right\} \left( \prod_{\substack{p \in \mathcal{O}_{L} \\
p | bc \mathcal{O}_{L}}} \log \text{Nm}_{L}^{Q} \left( p \right) \right) \max \{ h \left( a' \right), 1 \}. \tag{28} \]
From (19) and (28) we obtain that

\[
\begin{align*}
    h \left( \frac{u_a u_a'}{u_c c} \right) & \leq h \left( \frac{u_a}{u_c c} \right) + h \left( a' \right) \\
    & \leq C_{30} (K) \max \left\{ \text{Nm}_{Q}^L(p_b), \text{Nm}_{Q}^L(p_c) \right\} \left( \prod_{p \in \mathcal{O}_L} \log \text{Nm}_{Q}^L(p) \right) h \left( a' \right) + h \left( a' \right) \\
    & \leq \left( C_{30} (K) \max \left\{ \text{Nm}_{Q}^L(p_b), \text{Nm}_{Q}^L(p_c) \right\} \left( \prod_{p \in \mathcal{O}_L} \log \text{Nm}_{Q}^L(p) \right) + 1 \right) h \left( a' \right) \\
    & \leq \left( C_{31} (K) \max \left\{ \text{Nm}_{Q}^L(p_b), \text{Nm}_{Q}^L(p_c) \right\} \left( \prod_{p \in \mathcal{O}_L, p \mid c} \log \text{Nm}_{Q}^L(p) \right) \right) h \left( a' \right). \quad (29)
\end{align*}
\]

Similarly, we find that

\[
\begin{align*}
    h \left( \frac{u_b u_b'}{u_c c} \right) & \leq C_{32} \max \left\{ \text{Nm}_{Q}^L(p_b), \text{Nm}_{Q}^L(p_c) \right\} \left( \prod_{p \in \mathcal{O}_L} \log \text{Nm}_{Q}^L(p) \right) h \left( a' \right). \quad (30)
\end{align*}
\]

By consideration of (23), (24), we see that

\[
\begin{align*}
    \max \left\{ h \left( \frac{u_a u_a'}{u_c c} \right), h \left( \frac{u_b u_b'}{u_c c} \right) \right\} & < \max \{ C_{25}, C_{26} \} h \left( c' \right) \\
    & = C_{33} h \left( c' \right), \quad (31)
\end{align*}
\]

while (26) and (27) show that

\[
\begin{align*}
    \max \left\{ h \left( \frac{u_a u_a'}{u_c c} \right), h \left( \frac{u_b u_b'}{u_c c} \right) \right\} & < \max \{ C_{28}, C_{29} \} \text{Nm}_{Q}^L(p_c) \left( \prod_{p \in \mathcal{O}_L, p \mid c \mathcal{O}_L} \log \text{Nm}_{Q}^L(p) \right) h\left( b' \right) \\
    & = C_{34} \text{Nm}_{Q}^L(p_c) \left( \prod_{p \in \mathcal{O}_L, p \mid c \mathcal{O}_L, p \neq p_c} \log \text{Nm}_{Q}^L(p) \right) h\left( b' \right). \quad (32)
\end{align*}
\]
In the same way, it follows from (29) and (30) that

$$\max \left\{ h \left( \frac{u_a a'}{u_c c'} \right), h \left( \frac{u_b b'}{u_c c'} \right) \right\} < \max \{ C_{31}, C_{32} \} \max \{ \text{Nm}_L^L (p_b), \text{Nm}_L^L (p_c) \} \left( \prod_{p \in \mathcal{O}_L \atop p \mid b \cap \mathcal{O}_L} \log \text{Nm}_L^L (p) \right) h(a')$$

$$= C_{35} \max \{ \text{Nm}_L^L (p_b), \text{Nm}_L^L (p_c) \} \left( \prod_{p \in \mathcal{O}_L \atop p \mid b \cap \mathcal{O}_L} \log \text{Nm}_L^L (p) \right) h(a').$$

(33)

We next prove the following lemma.

**Lemma 6.** Let $\alpha \in \{ a, b, c \}$. Then

$$h (\alpha') \leq C_{36} \left( \max_{p \mid (\alpha)_L} \text{ord}_p (\alpha) \right) \log G.$$  

(34)

**Proof.** If $\alpha$ is a unit, then we define $\max_{p \mid (\alpha)_L} \text{ord}_p (\alpha) := 1$.

By construction of $\alpha'$, we have that $h (\alpha') \leq C_{37} \log \text{Nm}_L^L (\alpha \mathcal{O}_L)$. This follows from the fact that

$$|\text{Nm}_L / Q (\alpha')| = |\text{Nm}_L / Q (\alpha)| = \text{Nm}_L^L (\alpha \mathcal{O}_L).$$

We write a factorisation of $\alpha \mathcal{O}_L = \mathfrak{p}_{1,1}^{u_1} \cdots \mathfrak{p}_{u,1}^{u_{u,1}}$ into prime ideals of $\mathcal{O}_L$. Note this may be different to the ideals given in (16), as the ideals in (16) may not be prime. Working with this prime factorisation, we obtain that

$$\log \text{Nm}_L^L (\alpha \mathcal{O}_L) = \log \left( \prod_{i=1}^{u_1} \text{Nm}_L^L (p_{1,i}) \right)$$

$$= \sum_{i=1}^{u_1} u_{i,1} \log \left( \text{Nm}_L^L (p_{i,1}) \right)$$

$$\leq \left( \max_{p \mid (\alpha)_L} \text{ord}_p (\alpha) \right) \log G.$$ 

The claim then follows. 

It follows immediately from (2), (31) and Lemma 5 that

$$\frac{\log H_L (a, b, c)}{C_{33} \log G} \leq \max_{p \mid (c)_L} \text{ord}_p (c).$$

(35)

Similarly, it follows from (2), (32) and Lemma 6 that

$$\frac{\log H_L (a, b, c)}{C_{34} \log G \text{Nm}_L^L (p_c) \left( \prod_{p \in \mathcal{O}_L \atop p \not= p_c} \log \text{Nm}_L^L (p) \right)} \leq \max_{p \mid (b)_L} \text{ord}_p (b).$$

(36)
Further, it follows from (2), (33) and Lemma 6 that

\[
\log H_L(a, b, c) \leq \max_{p \mid (b)_{L}} \text{ord}_p(a) \cdot (37)
\]

We will use Lemma 3 to establish upper bounds for the right-hand sides of (35), (36) and (37). In order to do this we need to write \(\text{ord}_p(c)\), \(\text{ord}_p(b)\) and \(\text{ord}_p(a)\) in a form where we’re able to use Lemma 3.

By the coprimeness of \(a_{O_L}, b_{O_L}\) and \(c_{O_L}\) we see that

\[
\text{ord}_p(c) = \text{ord}_p\left(\frac{c}{b}\right) = \text{ord}_p\left(\frac{-a - b}{b}\right) = \text{ord}_p\left(\frac{-a}{b} - 1\right) = \text{ord}_p\left(\frac{-u_a}{u_b} e_1^a \cdots a_{t} e_t^a b_1^f \cdots b_u^f - 1\right). (38)
\]

Similarly,

\[
\text{ord}_p(b) = \text{ord}_p\left(\frac{b}{a}\right) = \text{ord}_p\left(\frac{-a - c}{a}\right) = \text{ord}_p\left(\frac{-c}{a} - 1\right) = \text{ord}_p\left(\frac{-u_c}{u_a} c_1^g \cdots c_v^g a_{1}^{-e_1} \cdots a_{t}^{-e_t} - 1\right), (39)
\]

and

\[
\text{ord}_p(a) = \text{ord}_p\left(\frac{a}{c}\right) = \text{ord}_p\left(\frac{-b - c}{c}\right) = \text{ord}_p\left(\frac{-b}{c} - 1\right) = \text{ord}_p\left(\frac{-u_b}{u_c} b_1^f \cdots b_u^f c_1^{-g_1} \cdots c_v^{-g_v} - 1\right). (40)
\]

To apply Lemma 3 we need to bound exponents \(e_i, f_j, g_k\) and bound the heights of the units \(h(u_a), h(u_b)\) and \(h(u_c)\).

First note that

\[
\max \{\text{ord}_p(a), \text{ord}_p(b), \text{ord}_p(c)\} \leq \log H_L(a, b, c), (41)
\]
directly from the definition of $H_L$.

This follows from the definitions of projective and absolute logarithmic heights.

In order to use Yu’s bound, we need to manage the heights of $u_a$, $u_b$ and $u_c$. To do this, we will use fundamental units of $O_L^*$. By Dirichlet’s Unit Theorem, there exist fundamental units $\xi_1, \ldots, \xi_r$ of $O_L$, where $r$ is the unit rank of $O_L$ such that all units $u$ of $O_L$ can be written $u = \mu \xi_1^{\delta_1} \cdots \xi_r^{\delta_r}$, with $\mu$ a root of unity. We note again that finding a set of fundamental units is computable, for example see [5], and we can find a nice system satisfying (9). Indeed, we could use any system of fundamental units, but this choice is helpful should one wish to explicitly compute the constants. Thus once found, the product $\prod_{i=1}^r h'(\xi_i)$ we will obtain applying Lemma 3 can be upper bounded constants depending on the field. It remains to find an upper bound for $\max_i |\delta_i|$.

Note that (21) gives us that

$$h(u_a u_c) \leq C_{38} h(c') .$$

Further,

$$h(c') \leq C_{39} (K) \log |N_{L/Q}(c)| \leq C_{40} (K) h(c) \leq C_{41} (K) \log H_L(a, b, c).$$

It follows from the above comments that

$$h\left(\frac{u_a}{u_c}\right) \leq C_{42} (K) \log H_L(a, b, c). \quad (42)$$

Let $L$ have $r_1$ real embeddings $\epsilon_1, \ldots, \epsilon_{r_1}$ and $2r_2$ complex embeddings $\epsilon_{r_1+1}, \overline{\epsilon_{r_1+1}}, \ldots, \epsilon_{r_2}, \overline{\epsilon_{r_2}}$. By Dirichlet’s Unit Theorem, there are $r := r_1 + r_2 - 1$ fundamental units $\xi_1, \ldots, \xi_r$, such that for any unit $u \in O_L^*$, $u = \mu \xi_1^{\delta_1} \cdots \xi_r^{\delta_r}$ where $\mu$ is a root of unity in $L$ and $\delta_i \in \mathbb{Z}$ for all $i$.

We now prove a lemma that gives an upper bound for $\max_i |\delta_i|$.

**Lemma 7.** Given the set up above,

$$\max |\delta_i| \leq C_{43} (K) \log H_L(a, b, c). \quad (43)$$

**Proof.** Recall that there are $r + 1$ distinct embeddings of $L$ into $\mathbb{C}$. Let us denote these embeddings by $e_i$, $i = 1, \ldots, r + 1$. Furthermore for all $i = 1, \ldots, r + 1$, let us define

$$\varepsilon_i : L \rightarrow \mathbb{R}$$

$$\alpha \rightarrow \log |e_i(\alpha)|.$$

Let $u$ be an element of $L$. We see that

$$h(u) = \frac{1}{2} \sum_{i=1}^{r+1} N_i |\varepsilon_i(u)|,$$

where $N_i = 1$ if the image of $e_i$ is a subset of $\mathbb{R}$, and $N_i = 2$ in the complimentary case, when the image of $e_i$ is not a subset of $\mathbb{R}$. This is because

$$h(u) = \sum_{i=1}^{r+1} N_i \log \max (|e_i(u)|, 1) = -\sum_{i=1}^{r+1} N_i \log \min (|e_i(u)|, 1),$$

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where \( N_i \) is defined as above. Thus

\[
2h(u) = \sum_{i=1}^{r+1} N_i \log \max (|e_i(u)|, 1) - \sum_{i=1}^{r+1} N_i \log \min (|e_i(u)|, 1)
\]

\[
= \sum_{i=1}^{r+1} N_i \log |e_i(u)|
\]

\[
= \sum_{i=1}^{r+1} N_i |e_i(u)|,
\]

and thus the identity follows.

We now take advantage of some properties of \( u_a, u_b \) and \( u_c \) so we will write them explicitly. In what follows we use the case \( u = u_a u_c \), but it is true for the other relevant quotients of \( u_a, u_b \) and \( u_c \). Write \( \frac{u_a}{u_c} = \mu \xi^1 \cdots \xi^r \), where \( \mu \) is a root of unity. From the comments above it follows that

\[
h\left( \frac{u_a}{u_c} \right) = \frac{1}{2} \sum_{i=1}^{r+1} N_i \left| \varepsilon_i \left( \mu \prod_{j=1}^{r} \xi_j^{\delta_j} \right) \right|
\]

(44)

where we lose the \( \mu \) as it is a root of unity, so for all \( i, \varepsilon_i(\mu) = 0 \).

From (42), we know that \( h\left( \frac{u_a}{u_c} \right) \leq C_{42} \log H_L(a, b, c) \), giving us an upper bound for the absolute logarithmic height of the unit. Along with (44), this implies that, for all \( i = 1, \ldots, r+1 \), we have

\[
\left| \sum_{j=1}^{r} \delta_j \varepsilon_i(\xi_j) \right| < C_{44}(K) \log H_L(a, b, c).
\]

(45)

That is, (42) implies that all the exponents \( \delta_j, j = 1, \ldots, r \) satisfy (45). This holds for all \( \varepsilon_i, i = 1, \ldots, r+1 \), so (45) gives us a system of \( r+1 \) inequalities.

We pick any \( r \) inequalities of \( r+1 \) in the system (45). For the sake of concreteness, let us take the first \( r \) inequalities. We are going to deduce the upper bound for the system of inequalities (45) where \( i = 1, \ldots, r \). Note that the left-hand side of these inequalities are coordinates of the vector

\[
M \begin{pmatrix} \delta_1 \\ \vdots \\ \delta_r \end{pmatrix},
\]

where the matrix \( M \) is defined by

\[
M := (\varepsilon_i(\xi_j))_{1 \leq i, j \leq r}.
\]

By definition, the absolute value of determinant of \( M \) is equal to the regulator of the number field \( L \), and is thus non-zero. Hence \( M \) is non-degenerate. Importantly, the matrix \( M \) depends only on the number field \( L \). Further, as the value of the regulator is independent of the choice
of the \( r \) inequalities we picked, it shows that our choice of inequalities is irrelevant and we obtain the same result given a different choice of \( r \) inequalities from the \( r + 1 \) in (45).

It follows that the solutions to the system of inequalities (45) for \( i = 1, \ldots, r \) are given by \( M(B) \) where \( B \) is an \( r \)-dimensional cube \( [-C_{44}(K) \log H_L(a, b, c), C_{44}(K) \log H_L(a, b, c)]^r \). Thus these solutions form a parallelepiped, the form of which depends on \( M \) (hence eventually on \( L \) only, which is uniquely determined by \( K \)) and the linear size is given by \( C_{45}(K) \log H_L(a, b, c) \). This means that the solutions \( \delta_i \) have an upper bound of the form \( C_{46}C_{45} \log H(a, b, c) \), where the constant \( C_{46} \) depends on \( M \) (hence actually depends on \( K \)) only. We thus conclude that

\[
\max_i |\delta_i| \leq C_{47}(K) \log H_L(a, b, c),
\]
as claimed. \( \Box \)

Importantly, as commented during the proof, the method is not changed if we choose a different unit such as \( \frac{u}{t} \) and so on. Thus this lemma holds for all relevant units in this paper.

We return to considering (38). We can now use the above after writing the unit in terms of fundamental units as follows:

\[
\text{ord}_p (c) = \text{ord}_p \left( \frac{u_a}{u_b} a_1^{e_1} \cdots a_t^{e_t} b_1^{f_1} \cdots b_u^{f_u} - 1 \right)
= \text{ord}_p \left( \mu \xi_1^{d_1} \cdots \xi_r^{d_r} a_1^{e_1} \cdots a_t^{e_t} b_1^{f_1} \cdots b_u^{f_u} - 1 \right)
\]

(46)

where \( \mu \) is a root of unity. We are in a position to apply Lemma 8. Using the notation of Lemma 8 from (31) and Lemma 7 we obtain that \( \log B \leq C_{42}(K) \log \log H_L(a, b, c) \).

Applying Lemma 8 on (46), we obtain that

\[
\text{ord}_p (c) \leq C_{48}(K)^{r+t+u+2} (r + t + u + 1)^{5/2} \log (2d (r + t + u + 1)) \frac{N_{\mathbb{Q}}^L(p_\infty)}{(\log N_{\mathbb{Q}}^L(p_\infty))^2} h'(\mu) h'(\xi_1) \cdots h'(\xi_r) h'(a_1) \cdots h'(a_t) h'(b_1) \cdots h'(b_u) \log \log H_L(a, b, c).
\]

(47)

Note that \( h(\mu) = 0 \) as \( \mu \) is a root of unity so \( h'(\mu) = \frac{1}{16e^2d^2} \), which we take into the constant. Further, we recall our system of fundamental units satisfies (9) so we take \( \prod_{i=1}^t h'(\xi_i) \) into the constant.

We further note that \( |N_{\mathbb{Q}}^L(a_i)| = N_{\mathbb{Q}}^L((a_i)) = (N_{\mathbb{Q}}^L(p_i))^f_K \), and similarly for \( b_j \) and \( c_k \). We recall that by definition, \( f_K \leq d \) (29). Further, the norms of all these prime ideals are greater than 1, so for all \( x \in \{a_1, \ldots, a_s, b_1, \ldots, b_t\} \), if \( a_x \) is the prime ideal associated with \( x \), \( h'(x) \leq C_{49}(K) \log N_{\mathbb{Q}}^L(a_x) \). Putting this together with the inequality above, with other bounds used as necessary, we obtain that

\[
\text{ord}_p (c) \leq C_{50}(K)^{t+u} (r + t + u + 1)^{7/2} \log \log H_L(a, b, c)
\]

\[
N_{\mathbb{Q}}^L(p_\infty) \prod_{i=1}^t \log (N_{\mathbb{Q}}^L(p_i)) \prod_{j=1}^u \log (N_{\mathbb{Q}}^L(q_j)).
\]

(48)
Similarly, we see that by considering (39) in the same way as above, we obtain that
\[
\text{ord}_p (b) = \text{ord}_p \left( -\frac{u_c c_1^{g_1} \cdots c_v a_1^{-e_1} \cdots a_t^{-e_t}}{u_a} - 1 \right) \\
= \text{ord}_p \left( \mu' \xi_1^{g_1} \cdots \xi_v c_1^{g_1} \cdots c_v a_1^{-e_1} \cdots a_t^{-e_t} - 1 \right). \tag{49}
\]

Following the same line of reasoning as above we obtain that
\[
\text{ord}_p (b) \leq C_{51} \left( K \right)^{t+u} (r + t + v + 1)^{7/2} \log \log H_L(a, b, c) \\
Nm_{Q}^L (p_b) \prod_{i=1}^{t} \log (Nm_{Q}^K (p_i)) \cdot \prod_{j=1}^{v} \log (Nm_{Q}^K (r_j)). \tag{50}
\]

In the same way, by considering (40) we further obtain that
\[
\text{ord}_p (a) = \text{ord}_p \left( -\frac{u_c b_1^{f_1} \cdots b_{t_a} a_1^{-g_1} \cdots c_v^{g_v} - 1}{u_c} \right) \\
= \text{ord}_p \left( \mu'' \xi_1^{g_1} \cdots \xi_v b_1^{f_1} \cdots b_{t_a} a_1^{-g_1} \cdots c_v^{g_v} - 1 \right), \tag{51}
\]

and as before, applying Lemma 3 gives us that
\[
\text{ord}_p (a) \leq C_{52} \left( K \right)^{u+v} (r + u + v + 1)^{7/2} \log \log H_L(a, b, c) \\
Nm_{Q}^L (p_a) \prod_{i=1}^{u} \log (Nm_{Q}^K (q_i)) \cdot \prod_{j=1}^{v} \log (Nm_{Q}^K (q_j)). \tag{52}
\]

From this point, all constants depend on the field \( K \), in particular on the degree of the field \( d \), so we omit these dependencies.

By combining (35) and (48) we obtain that
\[
\frac{\log H_L(a, b, c)}{\log \log H_L(a, b, c)} < C_{53} \left( K \right)^{t+u} (r + t + u + 1)^{7/2} \log \log G \cdot \text{Nm}_{Q}^L (p_c) \\
\prod_{i=1}^{t} \log (Nm_{Q}^K (p_i)) \cdot \prod_{j=1}^{u} \log (Nm_{Q}^K (q_j)). \tag{53}
\]

Similarly, combining (36) and (50) gives us that
\[
\frac{\log H_L(a, b, c)}{\log \log H_L(a, b, c)} < C_{54} \left( K \right)^{t+v} (r + t + v + 1)^{7/2} \log G \cdot \text{Nm}_{Q}^L (p_b) \cdot \text{Nm}_{Q}^L (p_c) \\
\prod_{i=1}^{t} \log (Nm_{Q}^K (p_i)) \cdot \prod_{j=1}^{v} \log (Nm_{Q}^K (r_j)) \cdot \prod_{p \in \mathcal{O}_L \setminus \mathcal{P} \cap \mathcal{O}_L \setminus \mathcal{P}} \log \text{Nm}_{Q}^L (p). \tag{54}
\]
Applying the same idea, combining (37) and (52) gives us that
\[
\frac{\log H_L(a, b, c)}{\log \log H_L(a, b, c)} < C_{55}^{u+v} (r + u + v + 1)^{7/2} \log G \cdot \{\text{Nm}_Q^L(p_a) \cdot \max \{\text{Nm}_Q^L(p_b), \text{Nm}_Q^L(p_c)\}\}
\]
\[
\prod_{i=1}^{u} \log (\text{Nm}_Q^K(p_i)) \cdot \prod_{j=1}^{v} \log (\text{Nm}_Q^K(q_j)) \cdot \prod_{p \in \mathcal{O}_L \atop p | bc\mathcal{O}_L} \log \text{Nm}_Q^L(p). \tag{55}
\]

Multiplying together (53), (54) and (55) and bounding some terms for ease, we obtain that
\[
\left(\frac{\log H_L(a, b, c)}{\log \log H_L(a, b, c)}\right)^3 < C_{56}^{t+u+v} (r + t + u + v)^{21/2} (\log G)^3
\]
\[
\left(\text{Nm}_Q^L(p_a) \text{Nm}_Q^L(p_b) \text{Nm}_Q^L(p_c)\right)^2 \max \{\text{Nm}_Q^L(p_b), \text{Nm}_Q^L(p_c)\}
\]
\[
\left(\prod_{i=1}^{t} \log (\text{Nm}_Q^K(p_i)) \cdot \prod_{j=1}^{u} \log (\text{Nm}_Q^K(q_j)) \cdot \prod_{k=1}^{v} \log (\text{Nm}_Q^K(r_k))\right)^3
\]
\[
\prod_{p \in \mathcal{O}_L \atop p | c\mathcal{O}_L} \log \text{Nm}_Q^L(p) \cdot \prod_{p \in \mathcal{O}_L \atop p | bc\mathcal{O}_L} \log \text{Nm}_Q^L(p). \tag{56}
\]

We note that \(r\) depends on the field, so we can write \((r + (t + u + v))^{21/2} \leq C_{57} (t + u + v)^{21/2}\). Further, for sufficiently large \(C_{57}\) this will absorb \((t + u + v)^{21/2}\), so we can move this into the constant. We thus obtain that
\[
\left(\frac{\log H_L(a, b, c)}{\log \log H_L(a, b, c)}\right)^3 < C_{58}^{t+u+v} (\log G)^3 \left(\text{Nm}_Q^L(p_a) \text{Nm}_Q^L(p_b) \text{Nm}_Q^L(p_c)\right)^2 \max \{\text{Nm}_Q^L(p_b), \text{Nm}_Q^L(p_c)\}
\]
\[
\left(\prod_{i=1}^{t} \log (\text{Nm}_Q^K(p_i)) \cdot \prod_{j=1}^{u} \log (\text{Nm}_Q^K(q_j)) \cdot \prod_{k=1}^{v} \log (\text{Nm}_Q^K(r_k))\right)^3
\]
\[
\prod_{p \in \mathcal{O}_L \atop p | c\mathcal{O}_L} \log \text{Nm}_Q^L(p) \cdot \prod_{p \in \mathcal{O}_L \atop p | bc\mathcal{O}_L} \log \text{Nm}_Q^L(p). \tag{57}
\]

Next we aim to deal with \(\prod_{i=1}^{t} \log (\text{Nm}_Q^K(p_i)) \cdot \prod_{j=1}^{u} \log (\text{Nm}_Q^K(q_j)) \cdot \prod_{k=1}^{v} \log (\text{Nm}_Q^K(r_k))\). First note that \(\text{Nm}_Q^K(\mathfrak{p})^{h_K} = \text{Nm}_Q^L(\mathfrak{p}\mathcal{O}_L)\), where \(\mathfrak{p}\) is a prime ideal of \(\mathcal{O}_K\). We follow an idea from the first part of Section 3 of [34]. Let \(N\) be the number of prime ideals of \(\mathcal{O}_L\) such that the prime ideal \(\mathfrak{p} | (abc)\mathcal{O}_L\). By definition, these all lie above primes \(p\) of \(\mathcal{O}_K\), so \(N \geq t + u + v\). Thus from these comments and Lemma 3 we obtain that
\[
\left(\frac{t + u + v}{C_{59}}\right)^{t+u+v} \leq \left(\frac{N}{C_{60}}\right)^N < G, \tag{58}
\]

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where $C_{60}$ is the constant given by Lemma 4. It follows that
\[ t + u + v < C_{61} \frac{\log G}{\log \log G}. \] (59)

By the arithmetic-geometric mean inequality we obtain that
\[
\prod_{i=1}^{t} \log (Nm_{K}^{L}(p_{i})) \cdot \prod_{j=1}^{u} \log (Nm_{Q}^{K}(q_{j})) \cdot \prod_{k=1}^{v} \log (Nm_{Q}^{K}(r_{k})) \\
\leq \left( \frac{1}{t + u + v} \left( \sum_{i=1}^{t} \log (Nm_{K}^{L}(p_{i})) + \sum_{j=1}^{u} \log (Nm_{Q}^{K}(q_{j})) + \sum_{k=1}^{v} \log (Nm_{Q}^{K}(r_{k})) \right) \right)^{t+u+v}
\leq \left( \frac{1}{t + u + v} \sum_{\mathcal{P} \subset \mathcal{O}} \log (Nm_{Q}^{L}(\mathcal{P})) \right)
\leq \left( \frac{\log G}{t + u + v} \right)^{t+u+v}.
\] (60)

It follows from (59) and (60) that
\[
\prod_{i=1}^{t} \log (Nm_{K}^{L}(p_{i})) \cdot \prod_{j=1}^{u} \log (Nm_{Q}^{K}(q_{j})) \cdot \prod_{k=1}^{v} \log (Nm_{Q}^{K}(r_{k})) < G^{C_{62} \frac{\log \log \log G}{\log \log \log G}}. \] (61)

The same logic can be used to show that
\[
\prod_{p \subset \mathcal{O}_{L}} \log Nm_{Q}^{L}(p) < G^{C_{63} \frac{\log \log G}{\log \log G}}, \] (62)
and that
\[
\prod_{p \subset \mathcal{O}_{L}} \log Nm_{Q}^{L}(p) \left| \begin{array}{c} p \mid \mathcal{O}_{L} \\ p \neq p_{c} \end{array} \right. < G^{C_{64} \frac{\log \log G}{\log \log G}}. \] (63)

Applying these to (57), we obtain that
\[
\left( \frac{\log H_{L}(a, b, c)}{\log \log H_{L}(a, b, c)} \right)^{3} < C_{58}^{t+u+v} (\log G)^{3}
\leq \left( Nm_{Q}^{L}(p_{a}) Nm_{Q}^{L}(p_{b}) Nm_{Q}^{L}(p_{c}) \right)^{2} \max \left\{ Nm_{Q}^{L}(p_{b}), Nm_{Q}^{L}(p_{c}) \right\} \right) G^{C_{65} \frac{\log \log G}{\log \log G}}. \] (64)

Further, from Lemma 4 we obtain that
\[
C_{58}^{t+u+v} < G^{C_{66} \frac{\log \log \log G}{\log \log \log G}}. \] (65)
Further, note that \( \log G = G^{\log \log G} \). Thus we obtain that

\[
\left( \frac{\log H_L(a, b, c)}{\log \log H_L(a, b, c)} \right)^3 < \left( \text{Nm}_Q^L(p_a) \text{Nm}_Q^L(p_b) \text{Nm}_Q^L(p_c)^2 \max \{\text{Nm}_Q^L(p_b), \text{Nm}_Q^L(p_c)\} \right)
\]

\[
G^{C_{67}} \left( \frac{\log \log \log G}{\log \log G} + \frac{1}{\log \log G} + \frac{\log \log G}{\log G} \right).
\] (66)

We take the cube root of both sides, before applying Lemma \( \text{5} \), obtaining

\[
\log H_L(a, b, c) < \left( \text{Nm}_Q^L(p_a) \text{Nm}_Q^L(p_b) \text{Nm}_Q^L(p_c)^2 \max \{\text{Nm}_Q^L(p_b), \text{Nm}_Q^L(p_c)\} \right)^{\frac{1}{3}}
\]

\[
G^{C_{68}} \left( \frac{\log \log \log G}{\log \log G} + \frac{1}{\log \log G} + \frac{\log \log G}{\log G} \right).
\] (67)

Note, the dominant term in the power of \( G \) is \( \frac{\log \log \log G}{\log \log G} \). Combining this with the above proves Theorem 1.

### 3.1 Corollaries of Theorem 1

In this section we show various corollaries of Theorem 1. The first two corollaries depend on the Class Group of \( K \) and the ideals that \( p_b \) and \( p_c \) lie above.

**Corollary 1.** Assume that \( \text{Nm}_Q^L(p_b) > \text{Nm}_Q^L(p_c) \) and that \( p_b \) and \( p_c \) both lie over prime ideals of \( \mathcal{O}_K \) that do not generate the class group of \( K \). Then

\[
\log H_L(a, b, c) < G^{\frac{1}{3} + C_{69} \frac{\log \log \log G}{\log \log G}}.
\] (68)

**Proof.** By assumption,

\[
\left( \text{Nm}_Q^L(p_a) \text{Nm}_Q^L(p_b) \text{Nm}_Q^L(p_c)^2 \max \{\text{Nm}_Q^L(p_b), \text{Nm}_Q^L(p_c)\} \right)
\]

\[
= \left( \text{Nm}_Q^L(p_a) \text{Nm}_Q^L(p_b)^2 \text{Nm}_Q^L(p_c)^2 \right).
\]

Recall that in the Hilbert Class Field \( L \), a prime ideal \( p \) of \( \mathcal{O}_K \) splits into \( \frac{k}{P} \) prime ideals of \( \mathcal{O}_L \), where \( P \) is the order of \( [p] \) in the Class Group of \( K \). By assumption, there must be at least two prime ideals dividing \( b\mathcal{O}_L \) with the same norm \( \text{Nm}_Q^L(p_b) \), and similarly for \( \text{Nm}_Q^L(p_c) \).

It follows that

\[
\left( \text{Nm}_Q^L(p_a) \text{Nm}_Q^L(p_b)^2 \text{Nm}_Q^L(p_c)^2 \right) \leq G,
\]

and the claim follows.

**Corollary 2.** Assume that \( \text{Nm}_Q^L(p_b) < \text{Nm}_Q^L(p_c) \) and that \( p_c \) lies above a prime ideal of \( \mathcal{O}_K \) that has order greater than 2 in the class group of \( K \). Then

\[
\log H_L(a, b, c) < G^{\frac{1}{3} + C_{70} \frac{\log \log \log G}{\log \log G}}.
\] (69)
Thus the corollary follows.

Proof. By assumption,

\[
\left(\Nm_{\mathcal{O}_L}^L(p_a) \Nm_{\mathcal{O}_L}^L(p_b) \Nm_{\mathcal{O}_L}^L(p_c)\right)^2 \max\left\{ \Nm_{\mathcal{O}_L}^L(p_b), \Nm_{\mathcal{O}_L}^L(p_c) \right\}
\]

By the comments in the proof of the previous corollary, our assumption here gives us that there are at least 3 prime ideals of $\mathcal{O}_L$ dividing $c\mathcal{O}_L$ with the same norm, $\Nm_{\mathcal{O}_L}^L(p_c)$. It follows that

\[
\left(\Nm_{\mathcal{O}_L}^L(p_a) \Nm_{\mathcal{O}_L}^L(p_b) \Nm_{\mathcal{O}_L}^L(p_c)^3\right) \leq G,
\]

and the claim follows. \hfill \square

The following corollary holds regardless of the class field of $K$.

**Corollary 3.** Assume that $\Nm_{\mathcal{O}_L}^L(p_b) > \Nm_{\mathcal{O}_L}^L(p_c)$. Then

\[
\log H_L(a, b, c) < G^{3+\log \log \log G}. \frac{\log \log \log G}{\log \log \gamma}.
\]

Proof. Note that $\Nm_{\mathcal{O}_L}^L(p_a) \Nm_{\mathcal{O}_L}^L(p_b) \Nm_{\mathcal{O}_L}^L(p_c) \leq G$. Further, by assumption

\[
\Nm_{\mathcal{O}_L}^L(p_c) \max\left\{ \Nm_{\mathcal{O}_L}^L(p_b), \Nm_{\mathcal{O}_L}^L(p_c) \right\} = \Nm_{\mathcal{O}_L}^L(p_b) \Nm_{\mathcal{O}_L}^L(p_c)
\]

Thus the corollary follows. \hfill \square

**Corollary 4.** Assume that $\Nm_{\mathcal{O}_L}^L(p_a) > \Nm_{\mathcal{O}_L}^L(p_b) > \Nm_{\mathcal{O}_L}^L(p_c)$. Then

\[
\log H_L(a, b, c) < G^{3+\log \log \log G}. \frac{\log \log \log G}{\log \log \gamma}.
\]

If $\max\left\{ \Nm_{\mathcal{O}_L}^L(p_b), \Nm_{\mathcal{O}_L}^L(p_c) \right\} = \Nm_{\mathcal{O}_L}^L(p_c)$ then we obtain that

\[
\log H_L(a, b, c) < G^{3+\log \log \log G}. \frac{\log \log \log G}{\log \log \gamma}.
\]

Proof. By assumption, $\Nm_{\mathcal{O}_L}^L(p_b) \Nm_{\mathcal{O}_L}^L(p_c) \leq G^\frac{2}{3}$ and $\Nm_{\mathcal{O}_L}^L(p_b) \leq G^\frac{1}{3}$, $\Nm_{\mathcal{O}_L}^L(p_c) \leq G^\frac{1}{3}$. Applying this to Theorem 1 gives both parts of the corollary. \hfill \square

Remarks. If we assume that none of $a, b, c$ are units of $\mathcal{O}_K$ then the only assumption we need to obtain the first inequality in the corollary above is that $\max\left\{ \Nm_{\mathcal{O}_L}^L(p_a), \Nm_{\mathcal{O}_L}^L(p_b), \Nm_{\mathcal{O}_L}^L(p_c) \right\} = \Nm_{\mathcal{O}_L}^L(p_a)$. This follows as then by assumption, $\Nm_{\mathcal{O}_L}^L(p_a) \leq G^\frac{2}{3}$, $\Nm_{\mathcal{O}_L}^L(p_c) \leq G^\frac{1}{3}$. The argument follows.

We now present some corollaries that depend on the value of $\max\left\{ \Nm_{\mathcal{O}_L}^L(p_b), \Nm_{\mathcal{O}_L}^L(p_c) \right\}$.

**Corollary 5.** Assume that $\max\left\{ \Nm_{\mathcal{O}_L}^L(p_b), \Nm_{\mathcal{O}_L}^L(p_c) \right\} < G^{-\alpha}$ with $0 < \alpha \leq 1$. Then

\[
\log H_L(a, b, c) < G^{\frac{1+\alpha}{3}+\log \log \log G}.
\]
Proof. Again,

\[ N_{\mathbb{Q}}^L (p_a) N_{\mathbb{Q}}^L (p_b) N_{\mathbb{Q}}^L (p_c) \leq G. \]

By assumption,

\[ N_{\mathbb{Q}}^L (p_c) \max \{ N_{\mathbb{Q}}^L (p_b), N_{\mathbb{Q}}^L (p_c) \} < G^{2\alpha}. \]

Applying this to Theorem 1 gives the result.

Corollary 6. Assume that \( \max \{ N_{\mathbb{Q}}^L (p_b), N_{\mathbb{Q}}^L (p_c) \} < (\log H_L (a, b, c))^\alpha \) for \( 0 < \alpha < \frac{2}{3} \). Then

\[ \log H_L (a, b, c) < G^{\frac{1}{3-2\alpha} + C_{74} \frac{\log \log \log G}{\log \log G}}. \]

Proof. Consider (66). Applying the assumption, we can rewrite this as

\[ \left( \frac{\log H_L (a, b, c)}{\log \log H_L (a, b, c)} \right)^3 < G \left( \log H_L (a, b, c) \right)^{2\alpha} G^{C_{75} \frac{\log \log \log G}{\log \log G}}. \]

Dividing through by \( (\log H_L (a, b, c))^{2\alpha} \) we obtain that

\[ \frac{(\log H_L (a, b, c))^{3-2\alpha}}{(\log \log H_L (a, b, c))^3} < G^{1+C_{75} \frac{\log \log \log G}{\log \log G}}. \]

Taking the \( 3-2\alpha \)'th root and applying a variant of Lemma 5 gives the result.

Corollary 7. Assume that \( N_{\mathbb{Q}}^L (p_{\max}) < (\log H_L (a, b, c))^\alpha \) for \( 0 < \alpha < \frac{2}{3} \). Then

\[ \log H_L (a, b, c) < G^{\frac{C_{76} \log \log \log G}{\log \log G}} \]

\[ = G^{C_{77} \frac{\log \log \log G}{\log \log G}}. \] (70)

Remarks. We note that we can write this in the following terms. For any given \( \varepsilon > 0 \), given the assumptions of the theorem and corollary there is a computable number \( C_{78} \) such that

\[ \log H_L (a, b, c) < G^{C_{78} \varepsilon}. \]

Proof. Consider Theorem 1. By assumption,

\[ \log H_L (a, b, c) < (\log H_L (a, b, c))^{\frac{5\alpha}{3} G^{\frac{\log \log \log G}{\log \log G}}}. \]

Dividing through, we obtain that

\[ (\log H_L (a, b, c))^{1-\frac{5\alpha}{3}} < G^{\frac{\log \log \log G}{\log \log G}}. \]

Take the \( 1-\frac{5\alpha}{3} \)'th root and the result follows.
4 Method Only Using two $S$-unit bounds

Part of the difficulty in analyzing cases in the previous section comes from the number of prime ideals on the right hand side of (67). If we only use two $S$-unit bounds then, while in general the bound is worse, it is easier to analyze for corollaries. We now prove Theorem 2, as stated in the introduction.

We follow the main text until (32). We then do not use the $S$-unit bound obtained by letting $S$ be equal to the infinite places and finite places corresponding to the prime ideals of $\mathcal{O}_L$ dividing $bc\mathcal{O}_L$. Following the argument of the main text, we obtain (53) and (54). Multiplying these together we obtain that

$$
\left( \frac{\log H_L(a, b, c)}{\log \log H_L(a, b, c)} \right)^2 < c_7^{t+u+v} (r + t + u + v)^7 (\log G)^2 \left( \text{Nm}_Q^L(p_b) \text{Nm}_Q^L(p_c)^2 \right)
$$

$$
\left( \prod_{i=1}^t \log \left( \text{Nm}_Q^K(p_i) \right) \right) \left( \prod_{j=1}^u \log \left( \text{Nm}_Q^K(q_j) \right) \right) \left( \prod_{k=1}^v \log \left( \text{Nm}_Q^K(r_k) \right) \right) \cdot \prod_{p \in \mathcal{O}_L, p \mid c\mathcal{O}_L} \log \text{Nm}_Q^L(p). \quad (71)
$$

From here we follow the arguments of the proof of the main theorem, obtaining

$$
\left( \frac{\log H_L(a, b, c)}{\log \log H_L(a, b, c)} \right)^2 < \left( \text{Nm}_Q^L(p_b) \text{Nm}_Q^L(p_c)^2 \right) \cdot \prod_{p \in \mathcal{O}_L, p \mid c\mathcal{O}_L} \log \text{Nm}_Q^L(p). \quad (72)
$$

We take the square root of both sides, before applying Lemma 5 obtaining

$$
\log H_L(a, b, c) < \left( \text{Nm}_Q^L(p_b) \text{Nm}_Q^L(p_c)^2 \right)^{\frac{1}{2}} \cdot \text{Nm}_Q^L(p_b) \text{Nm}_Q^L(p_c) \cdot \prod_{p \in \mathcal{O}_L, p \mid c\mathcal{O}_L} \log \text{Nm}_Q^L(p). \quad (73)
$$

Again, $\log \log \log G$ is the dominant term in the exponent of $G$.

From this point, there are many corollaries we can find, similarly to in the previous section. However, given that there are fewer prime ideals on the right hand side of (73), they are generally easier to prove. Further, Theorem 1 and Theorem 2 are independent, so if $\text{Nm}_Q^L(p_a)$ is sufficiently large in comparison to $\text{Nm}_Q^L(p_b)$ and $\text{Nm}_Q^L(p_c)$, Theorem 2 could give a better bound.

4.1 Corollaries Of Theorem 2

This corollary relies on the class group of $\mathcal{O}_K$.

**Corollary 8.** Assume that the prime ideal $\mathfrak{r} \subset \mathcal{O}_K$ that $p_c \subset \mathcal{O}_L$ lies above does not generate the class group of $K$. Then there exists an effectively computable constant $C_{82}$ such that

$$
\log H_L(a, b, c) < G^{\frac{1}{2}} + C_{82} \frac{\log \log \log G}{\log \log G}. \quad (74)
$$
Proof. Let $\mathfrak{r}$ be a prime ideal of $\mathcal{O}_K$ dividing $c\mathcal{O}_K$ such that $p_\mathfrak{c} \subset \mathcal{O}_L$ lies above $\mathfrak{r}$. Assume that $\mathfrak{r}$ does not generate the class group of $K$. Then in $L = HCF(K)$, $\mathfrak{r}$ splits into $h_K/P$ prime ideals, where $h_K$ is the class number of $K$ and $P$ is the order of $[\mathfrak{r}]$ in $C_K$ [7] [25]. As $\mathfrak{r}$ does not generate the class group of $K$, then the order of $[\mathfrak{r}]$ is at least 2. As $\mathfrak{r}$ splits in $\mathcal{O}_L$, we know that all prime ideals lying above $\mathfrak{r}$ in $\mathcal{O}_L$ have the same norm. By assumption, we have at least two such ideals in $\mathcal{O}_L$, so $Nm_{2}^{L}(p_\mathfrak{b}) (Nm_{2}^{L}(p_\mathfrak{c}))^2 < G$. More explicitly, there is another prime ideal $p'_\mathfrak{c}$ of $\mathcal{O}_L$ lying above $\mathfrak{r}$ such that $Nm_{2}^{L}(p_\mathfrak{c}) = Nm_{2}^{L}(p'_\mathfrak{c})$. It then follows from Theorem 2 that
\[
\log H_L(a, b, c) < G^{1 + C_{82} \frac{\log \log G}{\log \log G}}. \quad (75)
\]

The following corollaries give different bounds depending on $p_{\text{max}}$ or $\max \text{ord}_p (c)$.

**Corollary 9.** Assume that $Nm_{2}^{L}(p_\mathfrak{c}) < G^\alpha$ with $0 < \alpha < 1$, or that max $\text{ord}_p (c) < G^\alpha$ with $0 < \alpha < 1$. Then there exists an effectively computable constant $C_{83}$ such that
\[
\log H_L(a, b, c) < G^{\frac{1+\alpha}{2} + C_{83} \frac{\log \log G}{\log \log G}}. \quad (76)
\]
Further, if $\max \{Nm_{2}^{L}(p_\mathfrak{b}), Nm_{2}^{L}(p_\mathfrak{c})\} < G^\alpha$ then
\[
\log H_L(a, b, c) < G^{\frac{3\alpha}{2} + C_{84} \frac{\log \log G}{\log \log G}}. \quad (77)
\]

**Remarks.** We note that $\frac{3\alpha}{2} < 1$ for $\alpha < \frac{2}{3}$, and further that $\frac{3\alpha}{2} < \frac{1+\alpha}{2}$ for $\alpha < \frac{1}{2}$. Thus, our second bound is better than our first given in this corollary for $\alpha < \frac{1}{2}$.

**Proof.** We first assume that $Nm_{2}^{L}(p_\mathfrak{c}) < G^\alpha$ where $\alpha \in (0, 1)$. Thus
\[
Nm_{2}^{L}(p_\mathfrak{b}) Nm_{2}^{L}(p_\mathfrak{c})^2 < G^{1+\alpha} < G^2.
\]
Thus from Theorem 2 we obtain that
\[
\log H_L(a, b, c) < G^{\frac{1+\alpha}{2} + C_{83} \frac{\log \log G}{\log \log G}}. \quad (78)
\]

We now assume that $\max \text{ord}_p (c) < G^\alpha$ for some $\alpha \in (0, 1)$. Then, in place of (48), we have that for all $p \mid c\mathcal{O}_L$, $\text{ord}_p (c) < G^\alpha$. It follows from (33) that
\[
\log H_L(a, b, c) < C_{84} (\log G) G^\alpha. \quad (79)
\]
We note that for $\alpha < \frac{1}{2}$, this bound is actually better than the bound that follows.

As in the proof for the main theorem, (54) still holds. Multiplying the above and (54) we obtain that
\[
\frac{(\log H_L(a, b, c))^2}{\log \log H_L(a, b, c)} < C_{85}^{t+v} (r + t + v + 1) \frac{7}{2} (\log G)^{2} G^{\alpha \text{Nm}_{2}^{L}(p_\mathfrak{b}) \cdot \text{Nm}_{2}^{L}(p_\mathfrak{c})} \prod_{i=1}^{t} \log (\text{Nm}_{K}^{L}(p_i)) \cdot \prod_{j=1}^{v} \log (\text{Nm}_{K}^{L}(r_j)) \cdot \prod_{p \in \mathcal{O}_L \setminus \{c\mathcal{O}_L\}, p \neq p_\mathfrak{c}}^{H} \log \text{Nm}_{Q}^{L}(p). \quad (80)
\]
We note that $\text{Nm}_{L}^{Q}(p_{b}) \cdot \text{Nm}_{Q}^{L}(p_{c}) \leq G$. Further, we can use the techniques from above to tidy terms in the same way as we did for the main theorem to obtain

$$\frac{(\log H_{L}(a, b, c))^{2}}{\log \log H_{L}(a, b, c)} < G^{1+\alpha}G^{C_{86} \frac{\log \log \log G}{\log \log G}}. \tag{81}$$

Taking the square root and applying a variant of Lemma 5, we obtain that

$$\log H_{L}(a, b, c) < G^{\frac{1}{2}+\alpha + C_{83} \frac{\log \log \log G}{\log \log G}}. \tag{82}$$

This proves the first part of Corollary 9. The further comments follow directly from Theorem 2 when we bound $\text{Nm}_{L}^{Q}(p_{b})$ and $\text{Nm}_{Q}^{L}(p_{c})$ above by $G^{\alpha}$. This gives us that

$$\log H_{L}(a, b, c) < G^{\frac{3\alpha}{2} + C_{84} \frac{\log \log \log G}{\log \log G}}, \tag{83}$$

as claimed, where $\frac{3\alpha}{2} < 1$ for $\alpha < \frac{2}{3}$, and is a better bound than given above for $\alpha < \frac{1}{2}$.

It also follows directly from Theorem 2 that if $\max \{\text{Nm}_{L}^{Q}(p_{b}), \text{Nm}_{Q}^{L}(p_{c})\} < G^{\alpha}$, then

$$\log H_{L}(a, b, c) < G^{\frac{3\alpha}{2} + C_{84} \frac{\log \log \log G}{\log \log G}}. \tag{84}$$

Corollary 10. Assume now that $\text{Nm}_{L}^{Q}(p_{c}) < (\log H_{L}(a, b, c))^\alpha$ with $0 < \alpha < 1$, or that $\max \text{ord}_{p}(c) < (\log H_{L}(a, b, c))^\alpha$ with $0 < \alpha < 1$. Then there exits an effectively computable constant $C_{87}$ such that

$$\log H_{L}(a, b, c) < G^{\frac{1}{2} + C_{87} \frac{\log \log \log G}{\log \log G}}. \tag{84}$$

Furthermore, if $\max \{\text{Nm}_{L}^{Q}(p_{b}), \text{Nm}_{Q}^{L}(p_{c})\} < (\log H_{L}(a, b, c))^\alpha$ for $\alpha < \frac{2}{3}$, then directly from Theorem 2 we obtain that

$$\log H_{L}(a, b, c) < G^{\frac{3\alpha}{2} + C_{84} \frac{\log \log \log G}{\log \log G}} = G^{C_{89} \frac{\log \log \log G}{\log \log G}}. \tag{85}$$

This is the best bound we achieve in this text.

Remarks. We note that the second inequality in this corollary gives a sub-exponential bound, an improvement on the bounds given in [33][34].

To more easily compare with existing results, we note we can slightly weaken this upper bound. Inequality (85) implies that given any $\varepsilon > 0$ there exists some computable $C_{90}$ such that

$$\log H_{L}(a, b, c) < G^{C_{90} - \varepsilon},$$

where importantly $C_{90}$ does not depend on $\varepsilon$.

Proof. We first assume that $\text{Nm}_{L}^{Q}(p_{c}) < (\log H_{L}(a, b, c))^\alpha$ with $\alpha \in (0, 1)$. Then from this assumption and (72), we obtain that

$$\frac{(\log H_{L}(a, b, c))^{2-\alpha}}{(\log \log H_{L}(a, b, c))^{2}} < G \cdot G^{C_{91} \frac{\log \log \log G}{\log \log G}}. \tag{86}$$
By assumption, $2 - \alpha > 1$, and we take this root to obtain that
\[
\frac{\log H_L(a, b, c)}{(\log \log H_L(a, b, c))^{2 - \alpha}} < G^{1 - \alpha + C_{92} \frac{\log \log \log G}{\log \log \log \log G}}.
\] (87)

Note that $1 > \frac{1}{2 - \alpha} > \frac{1}{2}$. Applying a variant of Lemma 5, we obtain that
\[
\log H_L(a, b, c) < G^{2 - \alpha + C_{93} \frac{\log \log \log G}{\log \log \log \log G}}.
\] (88)

Now instead assume that $\max \text{ord}_p(c) < (\log H_L(a, b, c))^{\alpha}$ for some $\alpha \in (0, 1)$. Then as above, from (35) we obtain that
\[
\log H_L(a, b, c) < C_{94} \log G \log (\log H_L(a, b, c))^{\alpha}.
\] (89)

It immediately follows that
\[
(\log H_L(a, b, c))^{1 - \alpha} < C_{94} \log G.
\] (90)

Again, we still have (54), as obtained by following the main argument. We multiply (54) by the above to obtain
\[
(\log H_L(a, b, c))^{2 - \alpha} < C_{95} \log (r + t + v + 1)^{\frac{7}{2}} (\log G)^{2} \left( \prod_{p \in \mathcal{Q}_L} \text{Nm}_{\mathcal{Q}}(p_b) \cdot \text{Nm}_{\mathcal{Q}}(p_c) \right) \prod_{i=1}^{t} \log (\text{Nm}_{\mathcal{Q}}(p_i)) \cdot \prod_{j=1}^{v} \log (\text{Nm}_{\mathcal{Q}}(r_j)) \cdot \prod_{p \in \mathcal{Q}_L} \log (\text{Nm}_{\mathcal{Q}}(p))
\] (91)

As before, we can use the same method of tidying as in the proof of the main theorem to show that
\[
(\log H_L(a, b, c))^{2 - \alpha} < G^{1 - \alpha + C_{96} \frac{\log \log \log G}{\log \log \log \log G}}.
\] (92)

Taking the $2 - \alpha$'th root and applying a variant of Lemma 5, we obtain that
\[
\log H_L(a, b, c) < G^{\frac{1}{2} - \alpha + C_{97} \frac{\log \log \log G}{\log \log \log \log G}}.
\] (93)

This concludes the proof of the first part of Corollary 10.

To see the strongest case, we appeal directly to Theorem 2. Assume that
\[
\text{Nm}(p_{\text{max}}) < (\log H_L(a, b, c))^{\alpha}
\]
with $\alpha < \frac{2}{3}$. Then we can bound $\text{Nm}_{\mathcal{Q}}(p_b)$ and $\text{Nm}_{\mathcal{Q}}(p_c)$ above by $(\log H_L(a, b, c))^{\alpha}$, obtaining
\[
\log H_L(a, b, c) < (\log H_L(a, b, c))^{\frac{2\alpha}{3}} G^{C_{98} \left( \frac{\log \log \log G}{\log \log \log \log G} + \frac{1}{\log \log \log \log \log G} + \frac{1}{\log \log \log \log \log \log G} \right)}.
\] (94)
It then follows that
\[(\log H_L(a, b, c))^{1-\frac{3\alpha}{2}} < G^{C_{98}} \left( \frac{\log \log \log G}{\log \log G} + \frac{1}{\log \log G} + \frac{\log \log G}{\log G} \right). \quad (95)\]

Taking the $\frac{1}{1-\frac{3\alpha}{2}}$-th root gives us the result, namely that
\[\log H_L(a, b, c) < G^{C_{98}} \left( \frac{\log \log \log G}{\log \log G} + \frac{1}{\log \log G} + \frac{\log \log G}{\log G} \right). \quad (96)\]

Finally, we recall that the dominant term in the exponent is $\frac{\log \log \log G}{\log \log G}$, so we obtain that
\[\log H_L(a, b, c) < G^{C_{98}} \left( \frac{\log \log \log G}{\log \log G} \right). \quad (97)\]

As commented in the statement of the theorem, this is of the form
\[\log H_L(a, b, c) < G^{C_{90} \cdot \epsilon}. \]

Remarks. While the assumptions are hard to compare due to their different natures, we can see that for all $\alpha \in (0, 1)$, \(1 + \alpha^2 \geq \frac{1}{2 - \alpha}\). Thus, generally speaking, the bound of Corollary 10 is better than that of Corollary 9. More concretely, given $(a, b, c)$ that satisfy the assumptions of both Corollary 9 and 10, Corollary 10 gives a better bound in terms of the radical $G$ than that of Corollary 9.

**Corollary 11.** Assume that \(N_{Q}^{L}(p_{\text{max}}) > G^{\alpha}\) for $\alpha > \frac{1}{3}$, and that $p_{a} = p_{\text{max}}$. Then
\[\log H_L(a, b, c) < G^{\frac{3-3\alpha}{2} + C_{99} \frac{\log \log \log G}{\log \log G}}. \quad (98)\]

If $N_{Q}^{L}(p_{\text{max}}) \leq G^{\frac{1}{2}}$ it follows directly from Theorem 2 that
\[\log H_L(a, b, c) < G^{\frac{1}{2} + C_{100} \frac{\log \log \log G}{\log \log G}}. \quad (99)\]

Remarks. Note we have the assumption that $\alpha > \frac{1}{3}$ in order to make sure that $\frac{3-3\alpha}{2} < 1$.

Further, the second inequality given is the same case as $\alpha = \frac{1}{3}$ in the last part of Corollary 9.

**Proof.** Assume that \(N_{Q}^{L}(p_{\text{max}}) > G^{\alpha}\), and assume that $p_{a} = p_{\text{max}}$. Then considering (73), we note that
\[N_{Q}^{L}(p_{b}) N_{Q}^{L}(p_{c})^{2} < (G^{1-\alpha})^{2}, \]
so
\[\left( N_{Q}^{L}(p_{b}) N_{Q}^{L}(p_{c})^{2} \right)^{\frac{1}{2}} < G^{\frac{3-3\alpha}{2}}. \]

It follows that
\[\log H_L(a, b, c) < G^{\frac{3-3\alpha}{2} + C_{99} \frac{\log \log \log G}{\log \log G}}. \quad (100)\]

We note that $\frac{3-3\alpha}{2} < 1$ for $\alpha > \frac{1}{3}$. \qed
Corollary 12. Assume $\text{ord}_{p_c}c < G^\alpha$ for $0 < \alpha \leq 1$. Then
\[
\log H_L (a, b, c) < G^{\max \left\{ \frac{3}{4} \alpha, \frac{3}{4}, \frac{1}{2} \right\} + C_{101} \frac{\log \log \log G}{\log \log G}}.
\] (101)

Proof. Assume that $\text{ord}_{p_c}c < G^\alpha$.

Note that we can write
\[
\max_{p \mid \langle c \rangle_L} \text{ord}_p (c) = \max \left\{ \max_{p \mid L} \text{ord}_p (c), \text{ord}_{p_c} (c) \right\}.
\]

By assumption, we attain the bound
\[
\max_{p \mid \langle c \rangle_L} \text{ord}_p (c) = \max \left\{ \max_{p \mid L} \text{ord}_p (c), G^\alpha \right\}.
\] (102)

From the above and (35), it follows directly that
\[
\log H_L (a, b, c) < C_{102} \log G \max \left\{ \max_{p \mid L} \text{ord}_p (c), G^\alpha \right\}.
\] (103)

We now consider cases depending on $\text{Nm}_Q^L (p_c)$.

First assume that $\text{Nm}_Q^L (p_{\text{max}}) < G^{\frac{1}{2}}$. Then we can directly use Corollary 9 to obtain the bound given there, namely
\[
\log H_L (a, b, c) < G^{\frac{3}{4} + C_{103} \frac{\log \log \log G}{\log \log G}}.
\] (104)

Thus from the above and (103), we obtain that
\[
\log H_L (a, b, c) < G^{\max \left\{ \frac{3}{4}, \alpha \right\} + C_{104} \frac{\log \log \log G}{\log \log G}}.
\] (105)

Assume now instead that $\text{Nm}_Q^L (p_{\text{max}}) \geq G^{\frac{1}{2}}$. It immediately follows that for all other prime ideals $p$ contributing to $G$, we have that $\text{Nm}_Q^L (p) < G^{\frac{1}{2}}$.

Consider now $\max_{p \mid L} \text{ord}_p (c)$. We apply Yu’s bound as before on this, taking the above comments into consideration. It follows that
\[
\max_{p \mid L} \text{ord}_p (c) < C_{105} \left( r + t + u + 1 \right)^{7/2} \log \log H_L (a, b, c) \text{Nm}_Q^L (p)
\]
\[
\prod_{i=1}^{t} \log (\text{Nm}_Q^K (p_i)) \cdot \prod_{j=1}^{u} \log (\text{Nm}_Q^K (q_j)),
\] (106)

where $\text{Nm}_Q^L (p) < G^{\frac{1}{2}}$. Following the same logic as the main text, we obtain that
\[
\frac{\log H_L(a, b, c)}{\log \log H_L(a, b, c)} < C_{106}^t (r + t + u + 1)^{7/2} \log G \cdot G^{\frac{1}{2}} \prod_{i=1}^{t} \log (Nm_Q^K (p_i)) \cdot \prod_{j=1}^{u} \log (Nm_Q^K (q_j)).
\]

(107)

Note that (54) still holds, and \(Nm_Q^K (p_b) Nm_Q^K (p_c) \leq G\). Multiplying (54) and (107), tidying terms as we do in the text, and considering (103), we obtain that

\[
\frac{\log H_L(a, b, c)}{\log \log H_L(a, b, c)} < C_{107} \log \log \log \log G + C_{109} \log \log \log G.
\]

(108)

More concisely, after applying Lemma 5, we obtain that

\[
\log H_L(a, b, c) < G^{\max \left\{ \alpha, \frac{3}{4} \right\} + C_{109} \log \log \log G}. \quad (109)
\]

Thus, in either case depending on \(Nm_Q^K (p_{\text{max}})\), we obtain that

\[
\log H_L(a, b, c) < G^{\max \left\{ \alpha, \frac{3}{4} \right\} + C_{109} \log \log \log G}. \quad (110)
\]

\textbf{Corollary 13.} Assume that \(\text{ord}_p c < (\log H_L(a, b, c))^\alpha\) for \(0 < \alpha < 1\). Then

\[
\log H_L(a, b, c) < \max \left\{ G^{\frac{3}{4} + C_{109} \log \log \log G}, G^{\alpha + C_{110} \log \log \log G} \right\}. \quad (111)
\]

\textit{Proof.} Assume that \(\text{ord}_p c < (\log H_L(a, b, c))^\alpha\) for some \(0 < \alpha < 1\). As in Corollary 12 it immediately follows that

\[
\max \text{ord}_p (c) = \max \left\{ \max \text{ord}_p (c), \text{ord}_p c (c) \right\} \leq \max \left\{ \max \text{ord}_p (c), \left( \log H_L(a, b, c) \right)^\alpha \right\}. \quad (112)
\]

This along with (35) implies that

\[
\log H_L(a, b, c) < \max \left\{ \max \text{ord}_p (c) \log G, C_{111} \left( \log H_L(a, b, c) \right)^\alpha \log G \right\}. \quad (113)
\]

If

\[
\max \left\{ \max \text{ord}_p (c) \log G, \left( \log H_L(a, b, c) \right)^\alpha \log G \right\} = C_{112} \left( \log H_L(a, b, c) \right)^\alpha \log G,
\]
then we can see that
\[ \log H_L(a, b, c) < C_{113} (\log G)^{1-\alpha}. \]

We now consider two cases.

In the first case we assume that \( Nm_Q^L(p_{\text{max}}) < G^{1/2} \). In this case we can appeal directly to Corollary 9, obtaining that
\[ \log H_L(a, b, c) < G^{1/2+C_{114}} \left( \frac{\log \log \log G}{\log \log G} \right). \] (114)

For the second case we assume that \( Nm_Q^L(p_{\text{max}}) \geq G^{1/2} \) and follow the same argument as in Case 2 in Corollary 12; see section 4.5.2.

As before, we see that for all prime ideals \( p \neq p_{\text{max}} \) contributing to \( G \), we have that \( Nm_Q^L(p) < G^{1/2} \). Applying Yu’s bound on \( \max_{p \in \mathcal{O}_L} \text{ord}_p(c) \) again, we find that
\[ \max_{p \neq p_c} \text{ord}_p(c) < C_{115}^{s+t} (r + t + u + 1)^{7/2} \log \log H_L(a, b, c) \cdot Nm_Q^L(p) \prod_{i=1}^{t} \log (Nm^K_{Q}(p_i)) \cdot \prod_{j=1}^{u} \log (Nm^K_{Q}(q_j)). \] (115)

Again, we know that \( Nm_Q^L(p) < G^{1/2} \). Following the logic of the main text and Section 4.5.2, it again follows that
\[ \frac{\log H_L(a, b, c)}{\log \log H_L(a, b, c)} < C_{116}^{s+t} (r + t + u + 1)^{7/2} G \cdot G^{1-\beta} \prod_{i=1}^{t} \log (Nm^K_{Q}(p_i)) \cdot \prod_{j=1}^{u} \log (Nm^K_{Q}(q_j)). \] (116)

After tidying as we have previously and applying Lemma 5, it follows that
\[ \log H_L(a, b, c) < G^{1/2+C_{117}} \frac{\log \log \log G}{\log \log G}. \]

Combining these results, in both cases we obtain that
\[ \log H_L(a, b, c) < \max \left\{ G^{1/2+C_{110}} \frac{\log \log \log G}{\log \log G}, C_{110} (\log G)^{1-\alpha} \right\}. \] (117)

5 Application of Le Fourn’s Method

A method of Le Fourn [19] allows us to improve the above somewhat. We will prove Theorem 3 from the introduction. The important point here is that in our \( S \)-unit bounds we can use the following lemma to replace the use of the largest prime with the third largest prime in the set \( S \). This reduces our reliance on \( p_a, p_b, p_c \).
Lemma 8. Let $K$ be a number field of degree $d$ and let $S \subset M_K$ containing all the infinite places and a finite number of finite places. Let $s = |S|$ and let $\alpha, \beta \in K^*$. Consider the $S$-unit equation

$$\alpha x + \beta y = 1$$

with $x, y \in O_S^*$.

If $S$ contains at most two finite places then all solutions of the above satisfy

$$\max \{ h(x), h(y) \} \leq C_{118}(d, s) R_S \log^+ (R_S) H,$$

where $H = \max \{ h(\alpha), h(\beta), 1, \frac{a}{d} \}$, $R_S$ is the $S$-regulator and $C_{118}(d, s)$ is given in [19]. For any general set of places $S$, all solutions of the above equation satisfy

$$\max \{ h(x), h(y) \} \leq C_{119}(d, s) P'_S R_S \left( 1 + \frac{\log^+ (R_S)}{\log^+ (P'_S)} \right) H,$$

with $P'_S$ the third largest value of the norms of ideals coming from the finite places of $S$ and $C_{119}(d, s)$ given in [19]. If there are fewer than 3 finite places in $S$ then we take $P'_S = 1$.

Proof. This is Theorem 1.4 of [19]. We note the constants are taken from [13].

When we apply Le Fourn's lemma in the places we previously applied Lemma 2 we attain (after moving things into the constant) essentially the same bounds with $p_a, p_b, p_c$ replaced by $p'_a, p'_b$ and $p'_c$ where $p'_a$ is the prime ideal of third largest norm dividing $aO_L$ and similarly for $p'_b$ and $p'_c$. If fewer than three prime ideals divide $a, b$ or $c$ then we define the corresponding norm to be 1.

We follow the proof of the main theorem, but we replace any use of Lemma 2 with Lemma 3. For the most part, all that changes is any occurrence of $p_a, p_b$ and $p_c$ arising from the use of Lemma 2 is replaced by $p'_a, p'_b$ and $p'_c$. We follow the line of reasoning from the main text up until (21). For this first application of $S$-units, where we have no finite places, we continue to use Lemma 2 as it is simpler in this case than Lemma 3. As before, we obtain (31).

As before, now let $S = S_\infty \cup \{ p : p | cO_L \}$. Applying Lemma 8 to

$$-\frac{u_a a'}{u_c c'} - \frac{u_b b'}{u_c c'} = 1,$$

we obtain that

$$\max \left\{ h \left( -\frac{u_a}{u_c c'} \right), h \left( -\frac{u_b}{u_c c'} \right) \right\} < C_{120} P'_S R_S \left( 1 + \frac{\log^+ (R_S)}{\log^+ (P'_S)} \right) \max \left\{ h \left( a' \right), h \left( b' \right), 1, \frac{\pi}{d} \right\}$$

$$< C_{121} Nm_Q^L \left( p'_a \right) R_S^2 \max \left\{ h \left( a' \right), h \left( b' \right), 1, \frac{\pi}{d} \right\}$$

$$< C_{122} Nm_Q^L \left( p'_b \right) \left( \prod_{p \in \mathbb{Q}_L, p | cO_L} \log Nm_Q^L (p) \right)^2 \max \left\{ \frac{\pi}{2} h \left( a' \right), \frac{\pi}{2} h \left( b' \right), \frac{\pi}{2} \right\}$$

$$= C_{123} Nm_Q^L \left( p'_c \right) \left( \prod_{p \in \mathbb{Q}_L, p | cO_L} \log Nm_Q^L (p) \right)^2 h \left( b' \right), \quad (118)$$

33
where the line of reasoning about \( \max \{ h(a'), h(b'), 1, \frac{r}{s} \} \) follows from assumption (19). We are thus able to replace (32) with

\[
\max \left\{ h \left( \frac{u_a d'}{u_c c'} \right), h \left( \frac{u_b b'}{u_c c'} \right) \right\} < C_{124} N \log N_{\mathbb{Q}} (p') \left( \prod_{p \subset \mathcal{O}_L} \log N_{\mathbb{Q}} (p) \right)^2 h \left( b' \right). \tag{119}
\]

As before, we now let \( S = S_{\infty} \cup \{ p : p \mid bc\mathcal{O}_L \} \). Applying Lemma 8 again, following the same method as above, in place of (33) we obtain that

\[
\max \left\{ h \left( -\frac{u_a}{u_c c'} \right), h \left( -\frac{u_b b'}{u_c c'} \right) \right\} < C_{125} N \log N_{\mathbb{Q}} (q) \left( \prod_{p \subset \mathcal{O}_L} \log N_{\mathbb{Q}} (p) \right)^2 h \left( a' \right), \tag{120}
\]

where \( q \) is the prime ideal of \( \mathcal{O}_L \) of third largest norm dividing \( bc\mathcal{O}_L \). We note that this is not necessarily \( p_b \) or \( p_c \), though it may have the same norm as one of them, and could indeed be either of them.

We now follow the argument of the main text again, using the above inequalities in place of (32) and (33) as necessary, and we end up obtaining

\[
\log H_L(a, b, c) < \left( N \log N_{\mathbb{Q}} (p_a) N \log N_{\mathbb{Q}} (p_b) N \log N_{\mathbb{Q}} (p_c) N \log N_{\mathbb{Q}} (p'_c) N \log N_{\mathbb{Q}} (q) \right)^{\frac{1}{3}} G^{C_{126} \left( \frac{\log \log \log G}{\log \log G} \right) + \frac{1}{3} G^{C_{127} \left( \frac{\log \log \log G}{\log \log G} \right)}} \tag{121}
\]

in place of (67). As before, \( \frac{\log \log \log G}{\log \log G} \) is the dominant term in the exponent of \( G \), so we can write

\[
\log H_L(a, b, c) < \left( N \log N_{\mathbb{Q}} (p_a) N \log N_{\mathbb{Q}} (p_b) N \log N_{\mathbb{Q}} (p_c) N \log N_{\mathbb{Q}} (p'_c) N \log N_{\mathbb{Q}} (q) \right)^{\frac{1}{3}} G^{C_{128} \left( \frac{\log \log \log G}{\log \log G} \right)} \tag{122}
\]

We explore some cases. If \( q = p'_b \) then

\[
N \log N_{\mathbb{Q}} (p_a) N \log N_{\mathbb{Q}} (p_b) N \log N_{\mathbb{Q}} (p_c) N \log N_{\mathbb{Q}} (p'_c) N \log N_{\mathbb{Q}} (p'_b) \leq G.
\]

If \( q = p'_c \) then there exists a prime ideal \( p''_c \), the prime of second largest norm dividing \( c\mathcal{O}_L \). Note that \( N \log N_{\mathbb{Q}} (p'_c) \leq N \log N_{\mathbb{Q}} (p'_c) \leq N \log N_{\mathbb{Q}} (p_c) \), and all these primes divide \( abc\mathcal{O}_L \) so their norms contribute to \( G \). Thus, in this case we obtain that

\[
N \log N_{\mathbb{Q}} (p_a) N \log N_{\mathbb{Q}} (p_b) N \log N_{\mathbb{Q}} (p_c) N \log N_{\mathbb{Q}} (p'_c) N \log N_{\mathbb{Q}} (q) \leq G.
\]

We have dealt with the cases where \( q = p'_b \) and \( q = p'_c \). Using the notation above, there are four further possibilities for \( q \), namely \( p_b, p_c, p''_b, p''_c \). If \( q = p''_b \) or
then substituting into the above expression, it follows from the definition of \( G \) that
\[
N_{M_L}(p_a) N_{M_L}(p_b) N_{M_L}(p_c) N_{M_L}(p'_c) N_{M_L}(q) < G.
\]

On the other hand, if \( q = p_b \) or \( p_c \), we can still upper bound this by \( G \). Assume \( q = p_b \). Then we deduce that
\[
N_{M_L}(p_b) \leq N_{M_L}(p'_c) \leq N_{M_L}(p_c).
\]
It follows then that
\[
N_{M_L}(p_a) N_{M_L}(p_b) N_{M_L}(p_c) N_{M_L}(p'_c) N_{M_L}(q) < G.
\]
The argument is symmetric so applies if \( q = p_b \). Thus in all cases we obtain that
\[
\log H_L(a, b, c) < G^{\frac{1}{3} + C_{128} \frac{1 + \log \log \log G}{\log \log G}}. \tag{123}
\]

We note that for given \( a, b, c \), once we know the prime ideals dividing \( aO_K, bO_K \) and \( cO_K \), the inequality (122) may be stronger than that given in (123).

6 Some Remarks

A combination of methods and results by Györy and Yu [13] and Györy [10, 11] with the method of Le Fourn [19] can be used directly to find results over the base field, as done by Györy in [12]. Further, in terms of \( S \), Györy improved the \( S \)-unit bound given by Le Fourn.

Györy’s result regarding the \( abc \) conjecture is as follows.

Let \( K \) be a number field and let \( a, b, c := a + b \) belong to \( K^* \). Define
\[
N_K = \prod_{\wp} N_{M_L}(p)^{\ord_p(p)},
\]
where \( \wp \) is taken from the set of finite places such that \( |a|_\wp, |b|_\wp \) and \( |c|_\wp \) are not all equal, and \( p \) is the rational prime such that \( \wp \cap \mathbb{Z} = p \). Then for all \( \varepsilon \) there is a computable constant \( C_{129} \) depending only on \( d = [K : \mathbb{Q}] \), \( \Delta_K \) and \( \varepsilon \) such that
\[
\log H_K(a, b, c) < C_{129} G_K^{\frac{1}{3} + \varepsilon}.
\]

Györy’s combination of his method with that of Le Fourn’s enables him to state his results entirely over the base field \( K \). On the other hand, the dependence on the norms of prime ideals in this paper allows us to state corollaries depending on these norms, leading to the sub-exponential bound for example. Further, we believe they may allow some attack at open problems such as the smooth \( abc \) conjecture [10].

For all these results, we have considered them in terms of \( \log H_L(a, b, c) \). We note that \( H_L(a, b, c) = H_K(a, b, c)^{h_K} \), as \( h_K = [L : K] \). Thus, as \( h_K \) depends on the field, after taking the logarithm we can incorporate the \( h_K \) into our computable constant and have the height in terms of the base field \( K \). However, so far we have been unable to do the same for the radical \( G \).
7 Application to Effective Skolem-Mahler-Lech Problem

In this section we will use our main result to allow us to determine whether a linear recurrence sequence of degree three with no repeated roots of the characteristic polynomial has zeroes. As noted in the introduction, there exists an algorithm to determine whether there are periodic zeroes, so we are concerned with the case when there are only potentially finitely many zeroes.

7.1 Case Where all Terms are Coprime

Consider a linear recurrence sequence of the following form:

\[ a_n = c_1a_{n-1} + c_2a_{n-2} + c_3a_{n-3}, \]

where the values of \( a_0, a_1 \) and \( a_2 \) are known. We form the characteristic polynomial of the sequence

\[ x^3 - c_1x^2 - c_2x - c_3 \]

and assume that this has distinct roots \( r_1, r_2, r_3 \). Let \( K = \mathbb{Q}(r_1, r_2, r_3) \). We further assume the roots are pairwise coprime when considered as principal ideals of the ring of integers \( \mathcal{O}_K \).

By our assumptions, we know that we can write

\[ a_n = k_1r_1^n + k_2r_2^n + k_3r_3^n \]

where \( k_1, k_2, k_3 \) are constants depending on \( a_0, a_1 \) and \( a_2 \). We further assume \( k_1, k_2 \) and \( k_3 \) are coprime, but we will look at ways to try and deal with this when not coprime later.

For ease, we use the result obtained by using Le Fourn’s Lemma, that is the inequality given at (123). Assume there exists an \( n \) such that \( a_n = 0 \). Explicitly,

\[ 0 = k_1r_1^n + k_2r_2^n + k_3r_3^n. \]

We are in a position to use the result. Let \( L = HCF(K) \) and define \( G \) as above. Then,

\[ \log H(k_1r_1^n, k_2r_2^n, k_3r_3^n) < G^{1 + C_{128} \log \log \log G \over \log \log G}. \]

Without loss of generality, assume that

\[ h(r_1) \leq h(r_2) \leq h(r_3). \]

Note that

\[ H(k_1r_1^n, k_2r_2^n, k_3r_3^n) = H \left( \frac{k_1}{k_3}r_1^n, \frac{k_2}{k_3}r_2^n, r_3^n \right). \]

Further, by comparing definitions,

\[ h(r_3^n) \leq \log H \left( \frac{k_1}{k_3}r_1^n, \frac{k_2}{k_3}r_2^n, r_3^n \right). \]

Moreover, \( h(r_3^n) = nh(r_3) \) [37]. Combining all this we obtain that

\[ nh(r_3) < G^{1 + C_{128} \log \log \log G \over \log \log G}. \]
It follows that
\[ n < \frac{\sqrt{G} + C_{128} \frac{\log \log \log G}{\log \log G}}{h(r_3)}, \]
giving an upper bound for \( n \).

Explicitly, given a recurrence relation satisfying the given conditions, we first check whether there are any zeroes in arithmetic progressions. If so, we are done. If not, we apply the above method, which gives an upper bound for the maximal value of \( n \) such that \( a_n = 0 \). We numerically check the values of \( a_x \) for \( x \) less than the obtained upper bound. This answers the question as to whether the recurrence sequence has a zero.

**Example 1.** Consider the linear recurrence sequence with \( a_0 = 31 \), \( a_1 = 112 \), \( a_2 = 452 \) and
\[ a_n = 10a_{n-1} - 31a_{n-2} + 30a_{n-3}. \]
This sequence has characteristic polynomial
\[ x^3 - 10x^2 + 31x - 30, \]
with roots 2, 3 and 5. Thus, \( a_n = k_12^n + k_23^n + k_35^n \), where \( k_1, k_2 \) and \( k_3 \) are to be found. They are found to be \( k_1 = 7, k_2 = 11 \) and \( k_3 = 13 \), so
\[ a_n = 7 \cdot 2^n + 11 \cdot 3^n + 13 \cdot 5^n. \]
This means \( G = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 30030 \). The largest logarithmic height of the roots is \( h(5) = \log 5 \). It follows that if \( a_n = 0 \), then
\[ n < \frac{30030^{\frac{1}{3}} + C_{128} \frac{\log \log \log 30030}{\log \log 30030}}{\log 5} \]
\[ < 20 \cdot 43^{C_{130}}. \]
In principle, \( C_{128} \) and \( C_{130} \) can be computed following the proof given in this paper. This gives an upper bound for \( n \).

In this example, once we derive \( a_n = 7 \cdot 2^n + 11 \cdot 3^n + 13 \cdot 5^n \), it is clear there are no zeroes. With more complicated examples, it may not be so obvious.

### 7.2 Case Where Terms are Not Coprime

Assume we have a linear recurrence relation as above with characteristic polynomial \( f(x) \) with roots \( r_1, r_2, r_3 \). Assume there are constants \( k_1, k_2, k_3 \) so
\[ a_n = k_1r_1^n + k_2r_2^n + k_3r_3^n. \]
We assume nothing about coprimeness. If they are all coprime, we’re done as above. We thus assume \( k_1r_1^n, k_2r_2^n, k_3r_3^n \) are not coprime.
If there exists an \( n \) such that \( a_n = 0 \), then the same prime ideal must divide all 3 terms. We can see this as if
\[
0 = k_1 r_1^n + k_2 r_2^n + k_3 r_3^n,
\]
then
\[
-k_1 r_1^n = k_2 r_2^n + k_3 r_3^n,
\]
and it follows if a prime ideal divides two of these as ideals, it has to divide the third.

More rigorously,
\[
\text{ord}_p (a + b) \geq \min \{ \text{ord}_p (a) , \text{ord}_p (b) \},
\]
with equality when \( \text{ord}_p (a) \neq \text{ord}_p (b) \). The claim directly follows from this. It also follows from this that at least two of the terms are divisible by the prime ideal to the same order.

We consider these as ideals of \( \mathcal{O}_K \). Assume that \( q \) is a prime ideal of \( \mathcal{O}_K \) dividing all three terms, and that \( \text{ord}_q (k_1 r_1^n) = \text{ord}_q (k_2 r_2^n) = l \). We write
\[
k_1 r_1^n \mathcal{O}_K = p_{k_1,1}^{e_{k_1,1}} \cdots p_{k_1,a}^{n-e_{k_1,1}} \cdots p_{r_1,b}^{e_{r_1,b}} q^l,
k_2 r_2^n \mathcal{O}_K = p_{k_2,1}^{e_{k_2,1}} \cdots p_{k_2,c}^{n-e_{k_2,1}} \cdots p_{r_2,d}^{e_{r_2,d}} q^l,
k_3 r_3^n \mathcal{O}_K = p_{k_3,1}^{e_{k_3,1}} \cdots p_{k_3,f}^{n-e_{k_3,1}} \cdots p_{r_3,g}^{e_{r_3,g}} q^m,
\]
where these ideals are prime ideals of \( \mathcal{O}_K \).

Note, it may be the case that \( l = m \). Also, if \( q \mid r_1^n \), this implies that \( l = an \) for some \( a \), but we will see this doesn’t matter for the argument. Finally, it may be that there is a further prime ideal that divides all three terms; \( p \) if so, we apply the following process iteratively on all prime ideals dividing all three terms.

We now move the the Hilbert Class Field \( L \). All the ideals above are principal as ideals of \( \mathcal{O}_L \), so we can write
\[
k_1 r_1^n = u_1 p_{k_1,1}^{e_{k_1,1}} \cdots p_{k_1,a}^{n-e_{k_1,1}} \cdots p_{r_1,b}^{e_{r_1,b}} q^l,
k_2 r_2^n = u_2 p_{k_2,1}^{e_{k_2,1}} \cdots p_{k_2,c}^{n-e_{k_2,1}} \cdots p_{r_2,d}^{e_{r_2,d}} q^l,
k_3 r_3^n = u_3 p_{k_3,1}^{e_{k_3,1}} \cdots p_{k_3,f}^{n-e_{k_3,1}} \cdots p_{r_3,g}^{e_{r_3,g}} q^m,
\]
where the terms on the right hand side are all elements of \( L \) that generate the relevant principal ideals. Thus, we can now write
\[
a_n = u_1 p_{k_1,1}^{e_{k_1,1}} \cdots p_{k_1,a}^{n-e_{k_1,1}} \cdots p_{r_1,b}^{e_{r_1,b}} q^l + u_2 p_{k_2,1}^{e_{k_2,1}} \cdots p_{k_2,c}^{n-e_{k_2,1}} \cdots p_{r_2,d}^{e_{r_2,d}} q^l + u_3 p_{k_3,1}^{e_{k_3,1}} \cdots p_{k_3,f}^{n-e_{k_3,1}} \cdots p_{r_3,g}^{e_{r_3,g}} q^m.
\]

By \((124)\), we see that \( m \geq l \). We thus divide through the above equation by \( q^l \). After repeating this for all prime ideals dividing all three terms, the remaining terms on the right hand side will all be coprime. We now assume that there exists an \( n \) such that \( a_n = 0 \), and we are in the same position as section 8.1, and the argument follows identically.

Remarks. We note that this application also follows from Győry’s result \([12]\).
8 Smooth Solutions to the \textit{abc} Conjecture

In this section we will prove Theorem 5 as given in the introduction. We will first prove the following lemma.

\textbf{Lemma 9.} Let \((X, Y, Z) \in \mathbb{Z}^3\) be a triple with smoothness \(S(X, Y, Z)\) and radical \(G(X, Y, Z)\) defined as above. Then

\[ G(X, Y, Z) \leq e^{3S(X,Y,Z)}. \]  

\textit{Proof.} From [30], we know that for \(n \geq 6,\)

\[ \frac{p_n}{n} < \log n + \log \log n, \]

where \(p_n\) denotes the \(n\)'th rational prime.

It follows that for \(n \geq 6,\)

\[ p_n \leq 2n \log n. \]

Indeed, computationally we can check primes 2, 3, 5, 7, 11 and we find that for \(n > 2\) the above inequality holds.

Further, by Rosser’s Theorem [29],

\[ p_n > n \log n. \]

Thus, it follows that the product of the first \(k\) primes satisfies the following inequality:

\[
\prod_{i=1}^{k} p_i \leq 2 \cdot 3 \cdot \prod_{i=3}^{k} 2i \log i
\]

\[
= 2 \cdot 3 \cdot 2^{k-2} (4 \cdot 5 \cdots k) \cdot \prod_{i=3}^{k} \log i
\]

\[
= 2^{k-2} \cdot k! \cdot \prod_{i=3}^{k} \log i
\]

\[
\leq 2^k \cdot k^k \cdot \prod_{i=3}^{k} \log i. \tag{126}
\]
We now take logs of each side of the inequality attaining
\[
\log \left( \prod_{i=1}^{k} p_i \right) \leq \log \left( 2^k \cdot k^k \cdot \prod_{i=3}^{k} \log i \right) \\
= k \log 2 + k \log k + \log \left( \prod_{i=3}^{k} \log i \right) \\
= k \log 2 + k \log k + \sum_{i=3}^{k} \log \log i \\
\leq k \log 2 + k \log k + k \log \log k \\
\leq 3k \log k \\
\leq 3p_k
\]}

where the last line follows from Rosser’s Theorem.

It thus follows that for a triple of pairwise coprime integers satisfying \( X + Y = Z \) with smoothness \( S(X, Y, Z) \),
\[
G(X, Y, Z) \leq \prod_{p \text{ prime}} p \\
\leq e^{3S(X, Y, Z)},
\]
by the above inequality.

We are now in a position to prove Theorem 5.

Proof. By Northcott’s Theorem, we can assume in the following that \( H(X, Y, Z) > B \) for any given bound \( B \) as we will only be excluding finitely many possible solutions \( (X, Y, Z) \) satisfying (8).

For ease of notation, write
\[
T := \log \log H(X, Y, Z). 
\]

By assumption, we have that
\[
S(X, Y, Z) < T \frac{\log T}{\log \log T \phi(T)}. 
\]

We study triples \( (X, Y, Z) \) satisfying (130), and note that for a sufficiently large \( H(X, Y, Z) \),
\[
T \frac{\log T}{\log \log T \phi(T)} < (\log H(X, Y, Z))^{\frac{1}{2}},
\]
so we can apply Corollary 10. We note the choice of the exponent to be \( \frac{1}{2} \) is incidental; indeed any exponent less than \( \frac{2}{3} \) could have been chosen. Further, the Hilbert Class Field of \( \mathbb{Q} \) is
itself $\mathbb{Q}$ [7], so the radical $G$ in this case is defined over $\mathbb{Q}$ and coincides with the radical given in [16].

By Corollary 10, we know that

$$\log H (X, Y, Z) < G^{\frac{\log \log \log G}{\log \log G}}.$$  

From the upper bound for $G$ given at (125) in Lemma 9, we obtain from the above that

$$\log H (X, Y, Z) < e^{3c_{30} S(X, Y, Z) \log \log \log e^{3S(X, Y, Z)}}$$

$$= e^{c_{131} S(X, Y, Z) \log \log 3S(X, Y, Z)}.$$  \hspace{1cm} (131)

It follows from the above that

$$\log \log H (X, Y, Z) = T < C_{131} S(X, Y, Z) \frac{\log \log 3S(X, Y, Z)}{\log 3S(X, Y, Z)}.$$  \hspace{1cm} (132)

We recall that by Northcott’s Theorem again, we can assume that $S(X, Y, Z)$ can be larger than any given constant while only dropping finitely many solutions to $X + Y = Z$. Thus, only losing finitely many solutions, for sufficiently large $S(X, Y, Z)$ it follows from (132) that

$$T < C_{132} S(X, Y, Z) \frac{\log \log S(X, Y, Z)}{\log S(X, Y, Z)}.$$  \hspace{1cm} (133)

Taking logarithms on both side of inequality (133), we can assume $S(X, Y, Z)$ is large enough to give that

$$\log T < \log \left( C_{132} S(X, Y, Z) \frac{\log \log S(X, Y, Z)}{\log S(X, Y, Z)} \right)$$

$$= \log C_{132} + \log S(X, Y, Z) + \log \log \log S(X, Y, Z) - \log \log S(X, Y, Z)$$

$$< 2 \log S(X, Y, Z).$$  \hspace{1cm} (134)

For ease later, we divide the above by 2 to give that

$$\frac{1}{2} \log T < \log S(X, Y, Z).$$  \hspace{1cm} (135)

For triples $(X, Y, Z)$ satisfying (130), taking logarithms in inequality (130) we deduce that

$$\log S(X, Y, Z) < \log \left( \frac{T \log T}{\log \log T \phi(T)} \right)$$

$$= \log T + \log \log T - \log \log \log T - \log \phi(T)$$

$$< 2 \log T.$$  \hspace{1cm} (136)

We note that this also implies that for sufficiently large $S(X, Y, Z)$,

$$\log \log S(X, Y, Z) < 2 \log \log T.$$
Substituting this and (135) into (130) we obtain that

\[
S(X, Y, Z) < T \frac{\log T}{\log \log T \phi(T)} < T \frac{2 \log S(X, Y, Z)}{\frac{1}{2} \log (S(X, Y, Z)) \phi(T)} = 4T \frac{\log S(X, Y, Z)}{\log \log (S(X, Y, Z)) \phi(T)}.
\]

(137)

Rearranging the above we obtain that

\[
\frac{1}{4} \frac{S(X, Y, Z) \log S(X, Y, Z)}{\log S(X, Y, Z)} \phi(T) < T
\]

(138)

We now have two inequalities relating \(S\) and \(T\), namely (133) and (138) given above. We compare these directly to find that

\[
\frac{1}{4} \frac{S(X, Y, Z) \log S(X, Y, Z)}{\log S(X, Y, Z)} \phi(T) < T < C_{132} \frac{S(X, Y, Z) \log S(X, Y, Z)}{\log S(X, Y, Z)}
\]

(139)

which we rewrite as

\[
\frac{S(X, Y, Z) \log S(X, Y, Z)}{\log S(X, Y, Z)} \phi(T) < T < C_{133} \frac{S(X, Y, Z) \log S(X, Y, Z)}{\log S(X, Y, Z)}
\]

(140)

Cancelling terms on both sides gives us that

\[
\phi(T) < C_{133}.
\]

(141)

However, \(\phi(T)\) tends to \(+\infty\) as \(T\) tends to 0, and as \(T = \log \log H(X, Y, Z)\), this happens as \(H(X, Y, Z)\) gets arbitrarily large. Thus, there is a value \(B\) such that if \(H(X, Y, Z) > B\), then (141) cannot hold. This gives an upper bound for values of \(H(X, Y, Z)\) such that the triple satisfies the assumptions of the theorem. It thus follows by Northcutt’s Theorem that there are only finitely many primitive triples \((X, Y, Z)\) satisfying \(X + Y = Z\) with

\[
S(X, Y, Z) \leq \log \log H(X, Y, Z) \frac{\log \log \log \log H(X, Y, Z)}{\phi(\log \log H(X, Y, Z))}
\]

We note that we could also directly prove that there are only finitely many primitive integer triples \((X, Y, Z)\) satisfying \(X + Y = Z\) with

\[
S(X, Y, Z) < c \log \log H(X, Y, Z)
\]

for any constant \(c \in \mathbb{R}, c > 0\) using the same method of proof as above, though this result follows from Theorem 5 as stated previously.
References

[1] J. Berstel and M. Mignotte. Deux propriétés décidables des suites récurrentes linéaires. *Bulletin de la Société Mathématique de France*, 104:175–184, 1976.

[2] E. Bombieri and W. Gubler. *Heights in Diophantine geometry*. Number 4. Cambridge university press, 2007.

[3] J. Browkin. The abc-conjecture. In *Number theory*, pages 75–105. Springer, 2000.

[4] J. Browkin. The abc–conjecture for algebraic numbers. *Acta Mathematica Sinica*, 22(1):211–222, 2006.

[5] J. Buchmann. On the computation of units and class numbers by a generalization of lagrange’s algorithm. *Journal of Number Theory*, 26(1):8–30, 1987.

[6] Y. Bugeaud and K. Györy. Bounds for the solutions of unit equations. *Acta Arithmetica*, 74:67–80, 1996.

[7] N. Childress. *Class field theory*. Springer Science & Business Media, 2008.

[8] G. Everest, A. J. Van Der Poorten, I. Shparlinski, T. Ward, et al. *Recurrence sequences*, volume 104. American Mathematical Society Providence, RI, 2003.

[9] J.E. Evertse and K. Györy. *Unit equations in Diophantine number theory*, volume 146. Cambridge University Press, 2015.

[10] K. Györy. On the abc conjecture in algebraic number fields. *Acta Arithmetica*, 133:281–295, 2008.

[11] K. Györy. Bounds for the solutions of s-unit equations and decomposable form equations II. *Publ.Math.Debrecen*, 94:507–526, 2019.

[12] K. Györy. *S*-unit equations and Masser’s abc-conjecture in algebraic number fields. *Submitted*, 2021.

[13] K. Györy and K. Yu. Bounds for the solutions of s-unit equations and decomposable form equations. *Acta Arithmetica*, 123:9–41, 2006.

[14] V. Halava, T. Harju, M. Hirvensalo and J. Karhumäki. Skolem’s problem—on the border between decidability and undecidability. *Technical Report 683*, Turku Centre for Computer Science, 2005.

[15] A. J. Harper. Minor arcs, mean values, and restriction theory for exponential sums over smooth numbers. *Compositio Mathematica*, 152(6):1121–1158, 2016.

[16] J. C. Lagarias and K. Soundararajan. Smooth solutions to the abc equation: the xyz conjecture. *Journal de théorie des nombres de Bordeaux*, 23(1):209–234, 2011.

[17] E. Landau. Neuer beweis des primzahlsatzes und beweis des primidealsatzes. *Mathematische Annalen*, 56(4):645–670, 1903.

[18] S. Lang. *Algebraic number theory*, volume 110. Springer Science & Business Media, 2013.
[19] S. Le Fourn. Tubular approaches to baker’s method for curves and varieties. *Algebra & Number Theory*, 14(3):763–785, 2020.

[20] R. C. Mason. *Diophantine equations over function fields*, volume 96. Cambridge University Press, 1984.

[21] D. W. Masser. On abc and discriminants. *Proceedings of the American Mathematical Society*, 130(11):3141–3150, 2002.

[22] D. W. Masser. Open problems. In *Proceedings of the symposium on Analytic Number Theory, London, 1985*. Imperial College, 1985.

[23] S. Mochizuki. Inter-universal teichmüller theory I, II, III, IV. *Publications of the Research Institute for Mathematical Sciences*, 57(1):3–723, 2021.

[24] S. Natarajan and R. Thangadurai. *Pillars of Transcendental Number Theory*. Springer-Verlag, 2020.

[25] J. Neukirch. *Algebraic number theory*, volume 322. Springer Science & Business Media, 2013.

[26] J. Oesterlé. Nouvelles approches du “théoreme” de fermat. *Astérisque*, 161(162):165–186, 1988.

[27] A. Ostafe and I. E. Shparlinski. On the Skolem problem and some related questions for parametric families of linear recurrence sequences. *Canadian Journal of Mathematics*, 1–24, 2020.

[28] J. Ouaknine and J. Worrell. Decision problems for linear recurrence sequences. In *International Workshop on Reachability Problems*, pages 21–28. Springer, 2012.

[29] B. Rosser. The n-th prime is greater than nlogn. *Proceedings of the London Mathematical Society*, 2(1):21–44, 1939.

[30] B. Rosser. Explicit bounds for some functions of prime numbers. *American Journal of Mathematics*, 63(1):211–232, 1941.

[31] P. Scholze and J. Stix. Why abc is still a conjecture, 2018.

[32] M. Sha. Effective results on the Skolem problem for linear recurrence sequences. *Journal of Number Theory*, 197:228–249, 2019.

[33] C. L. Stewart and K. Yu. On the abc conjecture. *Mathematische Annalen*, 291:225–230, 1991.

[34] C. L. Stewart and K. Yu. On the abc conjecture, II. *Duke Mathematical Journal*, 108(1):169–181, 2001.

[35] A. Surroca. Sur l’effectivite du theoreme de Siegel et la conjecture abc. *Journal of Number Theory*, 124(2):267–290, 2007.

[36] P. A. Vojta. *Diophantine approximations and value distribution theory*, volume 1239. Springer, 2006.
[37] M. Waldschmidt. *Diophantine Approximation on Linear Algebraic Groups: Transcendence Properties of the Exponential Function in Several Variables*. Springer-Verlag, 2000.

[38] K. Yu. Linear forms in $p$-adic logarithms. *Acta Arithmetica*, 53(2):107–186, 1989.

[39] K. Yu. Linear forms in $p$-adic logarithms. II. *Compositio Mathematica*, 74(1):15–113, 1990.

[40] K. Yu. Linear forms in $p$-adic logarithms. III. *Compositio Mathematica*, 91(3):241–276, 1994.

[41] K. Yu. P-adic logarithmic forms and group varieties I. 1998.

[42] K. Yu. p-adic logarithmic forms and group varieties II. *Acta Arithmetica*, 89(4):337–378, 1999.

[43] K. Yu. P-adic logarithmic forms and group varieties III. *Forum Mathematicum*, 19(2):187–280, 2007.