NORDHAUS – GADDUM TYPE RESULTS FOR WIENER LIKE INDICES OF GRAPHS

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ABSTRACT

A Nordhaus -Gaddum type result is a lower or upper bound on sum or product of a parameter of a graph and its complement. This concept was introduced in 1956 by Nordhaus E. A., Gaddum J. W. Generalized Wiener like indices such as wiener index, Detour index, Reciprocal- wiener index, Harary- wiener index, Hyper- wiener index, Reciprocal- Detour index, Harary- Detour index and Hyper- Detour index have been studied in graph theory. In this paper, Nordhaus – Gaddum type results of these indices for k-Sun graph and four regular graph are presented.

Keywords: Generalized Wiener like Polynomial, k-sun graph, Nordhaus – Gaddum results.

Introduction 1:

All graphs considered in this paper are finite, simple and connected. For a graph G = (V, E) with vertices u, v ∈ V, the distance between u and v in G, denoted by d_G(u, v), is the length of a shortest (u, v) – path in G. The Wiener index [2,3,4] of G is defined as

\[ W(G) = \frac{1}{2} \sum_{u,v \in V(G)} d_G(u, v) \]

with the summation going over all pairs of distinct vertices of G. The above definition can be further generalized in the following way:

\[ W_\lambda(G) = \frac{1}{2} \sum_{u,v \in V(G)} d_G^\lambda(u, v) \]

where \( d_G^\lambda(u, v) = (d_G(u, v))^\lambda \) and \( \lambda \) is any real number.

For particular instances of the invariant \( \lambda \), \( W_{-2} \), \( W_{-1} \) and \( \frac{1}{2} W_1 + \frac{1}{2} W_2 \) are the so called Harary index \([1]\), reciprocal Wiener index and hyper – wiener index\([5,6]\).

The detour index of G is defined as

\[ D(G) = \frac{1}{2} \sum_{u,v \in V(G)} D_G(u, v) \]

with the summation going over all pairs of distinct vertices of G. The above definition can be further generalized in the following way:

\[ D_\lambda(G) = \frac{1}{2} \sum_{u,v \in V(G)} D_G^\lambda(u, v) \]

where \( D_G(u, v) = (d_G(u, v))^\lambda \) and \( \lambda \) is any real number.

For particular instances of the invariant \( \lambda \), \( D_{-2} \), \( D_{-1} \) and \( \frac{1}{2} D_1 + \frac{1}{2} D_2 \) are the so called Harary detour index, reciprocal detour index and hyper – detour index\([W]\). The complement of a graph G, denoted by \( \bar{G} \) is the graph with the same vertex set as G, where two vertices in G are adjacent if and only if they are not adjacent in G.

Definition 1.1:

A k – Sun graph \( (k \geq 3) \) is the graph on \( 2k \) vertices obtained from a clique \( c_1, c_2, \ldots c_k \) on k vertices and an independent set on k vertices. Let \( V(G) = \{ c_1, c_2, \ldots, c_k, s_1, s_2, \ldots, s_k \} \) and \( E(G) = \{ s_i c_i, s_i c_{i+1} \mid 1 \leq i \leq k \} \cup \{ s_i c_{k}, s_k c_1 \} \cup \{ c_i c_j \mid 1 \leq i \leq k, 1 \leq j \leq k, i < j \} \) be the vertex set and edge set of G respectively.

2.2. Generalized Wiener like indices of k – sun graph and its complement graph:
Theorem 2.2.1:

Let G be a k – sun graph. Then the generalized wiener polynomial, generalized detour polynomial and the generalized circular polynomial are given by

\[ W_\lambda P(G : x) = \left( \frac{k^2 + 3k}{2} \right)x^i + (k^2 - k)x^2 + \left( \frac{k^2 - 3k}{2} \right)x^3 \]

\[ D_\lambda P(G : x) = \left( \frac{k^2 - 3k}{2} \right)x^{(2k-3)i} + (k^2 - k)x^{(2k-2)i} + \left( \frac{k^2 + 3k}{2} \right)x^{(2k-1)i}, \text{ and} \]

\[ C_\lambda P(G : x) = (\frac{k^2 - 3k}{2})x^{(2k-2)i} + kx^{(2k-1)i} + k^2x^{2k} + kx^{(2k+1)i} + \left( \frac{k^2 - 3k}{2} \right)x^{(2k+2)i}, \]

where \( k \geq 4 \) and \( \lambda \) is any real number.

Proof:

Let G be \( k \) – sun graph on \( 2k \) vertices, where \( k \geq 4 \) and \( \lambda \) is any real number.

Let \( V(G) = \{ c_1, c_2, \ldots c_k, s_1, s_2, \ldots s_k \} \) and \( E(G) = \{ s_ic_i, s_is_{i+1} ; 1 \leq i \leq k \} \cup \{ s_kc_k, s_1c_1 \} \cup \{ c_ic_j ; 1 \leq i \leq k, 1 \leq j \leq k, i < j \} \) be the vertex set and edge set of G respectively. The generalized Wiener like polynomial of G is defined as,

\[ W_\lambda P(G : x) = \sum_{u,v \in V(G)} x^{d_\lambda(u,v)}, \text{ for any real number } \lambda. \]

For the \( k \) – sun graph, the generalized wiener polynomial, the generalized detour polynomial and generalized circular polynomial of \( k \) – sun graph G are,

\[ W_\lambda P(G : x) = \sum_{1 \leq i < j \leq k} x^{d_\lambda(c_ic_j)} + \sum_{1 \leq i < j \leq k} x^{d_\lambda(c_jc_i)} + \sum_{1 \leq i < j \leq k} x^{d_\lambda(s_is_j)} \] (1)

\[ D_\lambda P(G : x) = \sum_{1 \leq i < j \leq k} x^{d_\lambda(c_ic_j)} + \sum_{1 \leq i < j \leq k} x^{d_\lambda(c_jc_i)} + \sum_{1 \leq i < j \leq k} x^{d_\lambda(s_is_j)} \] (2)

\[ C_\lambda P(G : x) = \sum_{1 \leq i < j \leq k} x^{c_\lambda(c_ic_j)} + \sum_{1 \leq i < j \leq k} x^{c_\lambda(c_jc_i)} + \sum_{1 \leq i < j \leq k} x^{c_\lambda(s_is_j)} \] (3)

The 4-sun graph and the wiener detour matrix are shown in Figure 1.1 and Figure 1.2 respectively. Figure 1.3 shows the circular matrix of the 4 – Sun graph. The wiener-detour matrix and the circular matrix gives the wiener polynomial, the detour polynomial and circular polynomial of the 4 – s-un graph.
The 5-sun graph and the wiener detour matrix are shown in Figure 1.4 and Figure 1.5 respectively. Figure 1.6 shows the circular matrix of the 5-sun graph. The wiener-detour matrix and the circular matrix gives the wiener polynomial, the detour polynomial and circular polynomial of the 5–sun graph.

$$W_x P(G : x) = 14x^7 + 12x^2 + 2x^4; D_x P(G : x) = 2x^2 + 12x^6 + 14x^7; C_x P(G : x) = 2x^6 + 4x^7 + 16x^9 + 4x^9 + 2x^{10}.$$
Let $G$ be a $k$-sun graph for $k \geq 4$. Then, the Wiener index $W_1(G) = 4k^2 - 5k$,

The Reciprocal-Wiener index $W_{-1}(G) = -[4k^2 - 5k]$,

The Harary-Wiener index $W_2(G) = -2[4k^2 - 5k]$.

The Hyper-Wiener index $WW(G) = \frac{1}{2}[12k^2 - 15k]$.

**Corollary 2.2.3:**

Let $G$ be a $k$-sun graph for $k \geq 4$. Then
The Detour index $D_{i}(G) = [4k^{3} - 6k^{2} + 5k]$. The Reciprocal Detour index $D_{r}(G) = -[4k^{3} - 6k^{2} + 5k]$.

The Harary - Detour index $D_{h}(G) = -2[4k^{3} - 6k^{2} + 5k]$. The Hyper - Detour index $DD(G) = \frac{1}{2}[12k^{3} - 18k^{2} + 15k]$.

**Theorem 2.2.4:**

Let $\bar{G}$ be the complement of $k$ – sun graph $G$. Then the generalized wiener polynomial and detour polynomial for $\bar{G}$ are respectively given by:

$$W_{\bar{G}}P(G : x) = \frac{(3k^{2} - 5k)}{2} x^{k^{2}} + \frac{(k^{2} + 3k)}{2} x^{k^{2}}; D_{\bar{G}}P(G : x) = k(2k - 1)x^{k^{2} - 1}; k \geq 5$$

**Proof:**

Let $G$ be the $k$ – sun graph on $2k$ vertices. Let $\bar{G}$ be the complement of $k$ – sun graph $G$. Figure 1.7 shows the complement graph $\bar{G}$ of $5$ – sun graph. The wiener detour matrix of the complement of $5$ – Sun graph in Figure 1.8 gives the wiener polynomial and the detour polynomial of complement of $5$ – sun graph. Figure 1.8.WDM[the complement of $5$ – Sun graph]

$$W_{\bar{G}}P(G : x) = 25x^{1^{2}} + 20x^{2^{2}}; D_{\bar{G}}P(G : x) = 45x^{9^{2}}$$

The wiener detour matrix of the complement of $6$ – Sun graph in Figure 1.9 gives the wiener polynomial and the detour polynomial of complement of $6$ – sun graph.
Figure 1.9 WDM[the complement of 6 – Sun graph]

\[ W_\lambda P(G : x) = 39x^{14} + 27x^{24}; \quad D_\lambda P(G : x) = 66x^{14} \]

For \( k = 7, 8, 9 \) the corresponding wiener polynomials and detour polynomial of the complement of \( k – \) sun graph given below,

\[ W_\lambda P(G : x) = 56x^{14} + 35x^{24}; \quad W_\lambda P(G : x) = 76x^{14} + 44x^{24}; \quad W_\lambda P(G : x) = 90x^{14} + 54x^{24} \]

\[ D_\lambda P(G : x) = 91x^{14}; \quad D_\lambda P(G : x) = 120x^{24}; \quad D_\lambda P(G : x) = 153x^{17} \]

Hence in general, the generalized wiener and detour polynomial of complement of \( k – \) sun graph are respectively given by,

\[ W_\lambda P(G : x) = \frac{(3k^2 - 5k)}{2} x^{14} + \frac{(k^2 + 3k)}{2} x^{24}; \quad D_\lambda P(G : x) = k(2k - 1)x^{(2k-1)4}, \quad k \geq 5.. \]

**Corollary 2.2.5:**

Let \( \bar{G} \) be the complement of \( k – \) sun graph \( G \), then

The Wiener index \( W_1(\bar{G}) = \frac{1}{2}[5k^2 + k] \); The Reciprocal index \( W_{-1}(\bar{G}) = -\frac{1}{2}[5k^2 + k] \);

The Harary - Wiener index \( W_{-2}(\bar{G}) = -\frac{1}{2}[10k^2 + 2k] \); The Hyper - Wiener index \( WW(\bar{G}) = \frac{1}{4}[15k^2 + 3k] \)

**Corollary 2.2.6:**

Let \( \bar{G} \) be the complement of \( k – \) sun graph \( G \), then

The Detour index \( D_1(\bar{G}) = 4k^3 - 4k^2 + k \); The Reciprocal - Detour index \( D_{-1}(\bar{G}) = -\left[4k^3 - 4k^2 + k\right] \);

The Harary - Detour index \( D_{-2}(\bar{G}) = -2\left[4k^3 - 4k^2 + k\right] \); The Hyper - Detour index \( DD(\bar{G}) = \frac{12k^3 - 12k^2 + 3k}{2} \)

**Result 2.2.7:** Nordhaus – Gaddum Equations of \( k – \) sun graph.
\[(i) W_i(G) + W_j(G) = \frac{13k^2 - 9k}{2}; (ii) W_i(T) + W_j(T) = \left[ \frac{13k^2 - 9k}{2} \right] \]

\[(iii) W_{-2}(G) + W_{-2}(\overline{G}) = -\left[ 13k^2 - 9k \right] (iv) WW(G) + WW(\overline{G}) = \frac{39k^2 - 27k}{4} \]

\[(v) D_1(G) + D_1(\overline{G}) = [8k^3 - 10k^2 + 6k]; (vi) D_{-1}(G) + D_{-1}(\overline{G}) = -[8k^3 - 10k^2 + 6k] \]

\[(vii) D_{-2}(G) + D_{-2}(\overline{G}) = -2[8k^3 - 10k^2 + 6k]; (viii) DD(G) + DD(\overline{G}) = \frac{24k^3 - 30k^2 + 18k}{2} \]

3.1. Nordhaus–Gaddum Equation for four regular graph:

In this section the generalized Wiener polynomial and generalized detour polynomial of four regular graph and complement of four regular graph are presented and also Nordhaus–Gaddum equation for four regular graph is derived.

**Algorithm for Four regular graph:**

Input: the number of vertices n of a cyclic graph.
Output: the class four regular graph with 2n vertices.

Begin
Step 1: Take a cycle C_n with vertex set V = \{v_1, v_2, \ldots v_n\} and edge set E = \{v_iv_{i+1} \cup v_nv_1: 1 \leq i \leq (n-1)\}.
Step 2: For the edge v_iv_{i+1}, 1 \leq i \leq (n-1) introduce a new vertex u_i and create new edge v_iu_i and v_{i+1}u_i.
Step 3: For the edge v_nv_1 introduce a new vertex u_n and create new edges v_nu_n and v_1u_n.
Step 4: From the set of new vertices u_i, create new edges u_iu_{i+1} for 1 \leq i \leq (n-1) and an edge u_nu_1.
Step 5: The new four regular graph G(C_n) = (V, E) has the vertex set and edge set
V_G = \{ v_1, v_2, \ldots, v_n, u_1, u_2, \ldots u_n \}
E_G = \{ u_1v_1, v_nv_1, v_iu_i, v_{i+1}u_i, v_nu_n, u_1u_n, u_{i+1}u_i, u_nu_1 / 1 \leq i \leq (n-1) \}.

**Generalized Wiener like indices of four regular graph and its complement:**

**Theorem 3.1.1:**
Let G(C_n) be a four regular graph. Then the generalized Wiener polynomial and the generalized detour polynomial are given by the following expressions:

\[ W_\Delta P(G: x) = 4nx^i + 4nx^{2i} + \ldots + 4nx^{(\frac{n-1}{2})i} + nx^{(\frac{n+1}{2})i}, \text{ when } n \text{ is odd and } n \geq 3. \]

\[ W_\Delta P(G: x) = 4nx^i + 4nx^{2i} + \ldots + 4nx^{(\frac{n-2}{2})i} + 3nx^{(\frac{n+1}{2})i}, \text{ when } n \text{ is even and } n \geq 4. \]

\[ D_\Delta P(G: x) = n(n-1)x^{(n-1)i} \]

**Proof:**
Let G = G(C_n) be a four regular graph with 2n vertices. Let V(G) = \{v_1, v_2, \ldots v_n\} and edge set
E = \{ u_1v_1, v_nv_1, v_iu_i, v_{i+1}u_i, v_nu_n, v_1u_n, u_1u_{i+1}, u_nu_1 / 1 \leq i \leq (n-1) \}.

Case(i): When n is odd.
Figure 1.10. shows the four regular graph $G(C_3)$ and Figure 1.11. gives the wiener polynomial and the detour polynomial of four regular graph $G(C_3)$.

$$\begin{align*}
W_\lambda P(G : x) &= 12x^4 + 3x^2 + 2x \\
D_\lambda P(G : x) &= 15x^4
\end{align*}$$

Figure 1.11. WDM[$G(C_3)$]

Case (i): When $n$ is odd.

Hence in general, the generalized wiener polynomial of four regular graph $G(C_n)$ is,

$$W_\lambda P(G : x) = 4nx^4 + 4nx^2 + \cdots + 4nx^{\frac{n-1}{2}} + nx^{\frac{n+1}{2}}, \text{ when } n \text{ is odd and } n \geq 3.$$
\[ W_d(G : x) = 16x^4 + 12x^2, \quad D_d(G : x) = 28x^7. \]

Hence in general, the generalized wiener polynomial of four regular graph \( G(C_n) \) is

\[ W_d(G : x) = 4nx^4 + 4nx^2 + \ldots + 4nx^{\left\lfloor \frac{n-1}{2} \right\rfloor} + nx^{\left\lfloor \frac{n-1}{2} \right\rfloor}, \quad \text{when } n \text{ is odd and } n \geq 3. \]

\[ W_d(G : x) = 4nx^4 + 4nx^2 + \ldots + 4nx^{\left\lfloor \frac{n-2}{2} \right\rfloor} + 3nx^{\left\lfloor \frac{n}{2} \right\rfloor}, \quad \text{when } n \text{ is even and } n \geq 4. \]

and the generalized detour polynomial of four regular graph \( G(C_n) \) is,

\[ D_d(G : x) = \frac{n(n-1)}{2} x^{(n-1)i} \]

**Corollary 3.1.2:**

Let \( G \) be the four regular graph. Then

The Wiener index \( W_1(G) = \frac{n^2(n+1)}{2} \); The Reciprocal - Wiener index \( W_{-1}(G) = \left[ \frac{n^2(n+1)}{2} \right] \).

The Harary - Wiener index \( W_{-2}(G) = \left[ \frac{n^2(2n+2)}{2} \right] \); The Hyper - Wiener index \( WW(G) = \left[ \frac{3n^2(n+1)}{4} \right] \).

**Corollary 3.1.3:**

Let \( G \) be the four regular graph, then

The Detour index \( D_1(G) = \frac{n(n-1)^2}{2} \); The Reciprocal index \( D_{-1}(G) = \left[ \frac{n(n-1)^2}{2} \right] \),

The Harary- Detour index \( D_{-2}(G) = -2 \left[ \frac{n(n-1)^2}{2} \right] \);
The Hyper-Detour index $DD(G) = \left\lfloor \frac{3n(n-1)^2}{4} \right\rfloor$.

**Theorem 3.1.4:**

Let $\overline{G}$ be the complement of four regular graph $G$. Then the generalized wiener polynomial and detour polynomial for $\overline{G}$ are respectively given by:

$$W_\Delta P(\overline{G} : x) = (2n^2 - 5n)x^4 + 4nx^2; D_\Delta P(\overline{G} : x) = (2n^2 - n)x^{(2n-1)^2}.$$  

**Proof:**

Let $G$ be the four regular graph and $\overline{G}$ be the complement of $G$. Figure 1.16. shows the complement graph $\overline{G}$ of four regular graph $G(C_4)$. The wiener detour matrix Figure 1.17. gives the wiener polynomial and the detour polynomial of complement of four regular graph $G(C_4)$.

$$W_\Delta P(\overline{G} : x) = 12x^4 + 16x^2; D_\Delta P(\overline{G} : x) = 28x^{7^2}.$$  

In similar manner, when $k = 5, 6, 7, 8$, the corresponding wiener polynomials of complement graph $\overline{G}$ of four regular graph given below,

$$W_\Delta P(\overline{G} : x) = 25x^4 + 20x^2; W_\Delta P(\overline{G} : x) = 42x^2 + 24x^2; W_\Delta P(\overline{G} : x) = 63x^4 + 28x^{2^2};$$  

$$W_\Delta P(\overline{G} : x) = 88x^4 + 32x^2$$

Hence in general, the generalized wiener polynomial of complement of four regular graph,

$$W_\Delta P(\overline{G} : x) = (2n^2 - 5n)x^4 + 4nx^2$$

In similar manner, when $k = 5, 6, 7, 8$, the corresponding generalized detour polynomials of complement graph $\overline{G}$ of four regular graph given below,
\[ D_4 P(G : x) = 45x^{6}; D_4 P(\tilde{G} : x) = 66x^{11}; D_4 P(G : x) = 91x^{13}; D_4 P(\tilde{G} : x) = 120x^{15} \]

Hence in general, the generalized detour polynomial of complement of four regular graph is.

\[ D_4 P(G : x) = (2n^2 - n)x^{(2n-1)^2} \]

**Corollary 3.1.5:**

Let \( \tilde{G} \) be the complement of \( G \). Then

The Wiener index \( W_1(G) = [2n^2 + 3n] \); The Reciprocal index \( W_{-1}(G) = -[2n^2 + 3n] \);

The Harary-Wiener index \( W_{-2}(G) = -2[2n^2 + 3n] \); The Hyper-Wiener index \( WW(G) = \frac{1}{2}(6n^2 + 9n) \)

**Corollary 3.1.6:**

Let \( \tilde{G} \) be the complement of \( G \). Then

The Detour index \( D_1(G) = 4n^3 - 4n^2 + n \); The Reciprocal-Detour index \( D_{-1}(G) = \left[4n^3 - 4n^2 + n\right] \)

The Harary-Detour index \( D_{-2}(G) = -2\left[4n^3 - 4n^2 + n\right] \) The Hyper-Detour index \( DD(G) = \frac{12n^3 - 12n^2 + 3n}{2} \)

**Result 3.1.7:** Nordhaus – Gaddum Equations of four regular graph.

\[(i) W_1(G) + W_1(\tilde{G}) = \frac{n^3 + 5n^2 + 6n}{2} \]; \( (ii) W_{-1}(G) + W_{-1}(\tilde{G}) = \left[\frac{n^3 + 5n^2 + 6n}{2}\right] \)

\[(iii) W_{-2}(G) + W_{-2}(\tilde{G}) = -\left[n^3 + 5n^2 + 6n\right] \]; \( (iv) WW(G) + WW(\tilde{G}) = \frac{3n^3 + 15n^2 + 18n}{4} \)

\[(v) D_1(G) + D_1(\tilde{G}) = \frac{9n^3 - 10n^2 + 3n}{2} \]; \( (vi) D_{-1}(G) + D_{-1}(\tilde{G}) = \left[\frac{9n^3 - 10n^2 + 3n}{2}\right] \)

\[(vii) D_{-2}(G) + D_{-2}(\tilde{G}) = -\left[9n^3 - 10n^2 + 3n\right] \); \( (viii) DD(G) + DD(\tilde{G}) = \frac{27n^3 - 30n^2 + 9n}{4} \).

**Conclusions:**

In 1956, Nordhaus E. A., Gaddum J. W. [7] introduced the bounds involving the chromatic number \( \chi(G) \) of a graph \( G \) and its complement. Many authors studied [8, 9] Nordhaus-Gaddum bounds for domination number, connected domination number, total domination number and also there has been many publications on Nordhaus-Gaddum type results for indices like Gutman wiener index, Steiner index, Krichhoff index. This paper deals with Nordhaus – Gaddum equations for wiener like indices to \( k \) – sun graph four regular graph.
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