We analyze the following general version of the deterministic Hats game. Several sages wearing colored hats occupy the vertices of a graph. Each sage can have a hat of one of $k$ colors. Each sage tries to guess the color of his own hat merely on the basis of observing the hats of his neighbors without exchanging any information. A predetermined guessing strategy is winning if it guarantees at least one correct individual guess for every assignment of colors.

We present an example of a planar graph for which the sages win for $k = 14$. We also give an easy proof of the theorem about the Hats game on “windmill” graphs. Bibliography: 7 titles.

1. Introduction

The hat guessing game goes back to an old popular Olympiad problem. Its generalization to arbitrary graphs has recently attracted the interest of mathematicians (see, e.g., [1–3]). The theory of the game is based on methods of combinatorial graph theory.

In this paper, we consider the version of the hat guessing game where sages located at graph vertices try to guess the colors of their own hats while they can see the colors of hats on the sages at the adjacent vertices only. The sages act as a team, using a deterministic strategy fixed at the beginning of the game. If at least one of them guesses the color of his own hat correctly, we say that the sages win.

Most studies of this game consider the version where each sage gets a hat of one of $k$ colors. The maximum number $k$ for which the sages can ensure winning on a graph $G$ is called the hat guessing number of the graph $G$ and denoted by $HG(G)$. The computation of the hat guessing number for an arbitrary graph is a hard problem. Currently, it is solved only for few classes of graphs: for complete graphs, trees (folklore), cycles [4], and pseudotrees [5]. Also, there are some results for “book” and “windmill” graphs, see [2].

N. Alon et al. [3] studied the relation between the hat guessing number and other graph parameters. The problem of bounds on the hat guessing numbers of planar graphs was mentioned in [1, Conjecture 4] and [2, Question 5.2]. At the moment, the maximum known hat guessing number for planar graphs is 12 (see [2]).

In the previous papers [6] and [7] (joint with V. Retinsky), the authors considered the version of the game with a variable number of hats (i.e., when the number of possible colors can differ from sage to sage). This version is not only of independent interest, but opens a more flexible approach to the analysis of the classical hat guessing game, because it has nontrivial techniques for building strategies.

In this paper, we continue to study the hat guessing game with a variable number of colors. We show, using the constructors machinery, how to build a planar graph with hat guessing number at least 14. Also, we give a quite simple proof of the windmills theorem from [2].

This paper is structured as follows.

In the second section, we give necessary definitions and notation, recall several constructor theorems from [6], and give a simple proof of the windmills theorem from [2] as an example of using these constructors.

In the third section, we build an outerplanar graph with hat guessing number at least 6 (Example 3.4) and a planar graph with hat guessing number at least 14 (Theorem 3.6).
2. Constructors

2.1. Definitions and notation. We use the following notation.

- \( G = (V, E) \) is a visibility graph, i.e., a graph with sages at its vertices; we identify a sage with the corresponding vertex.
- \( h: V \to \mathbb{N} \) is a “hatness” function, which indicates the number of different hat colors that a sage can get. For a sage \( A \in V \), we call the number \( h(A) \) the hatness of \( A \). We may assume that the hat color of the sage \( A \) is a number from 0 to \( h(A) - 1 \), or a residue modulo \( h(A) \).

Definition. The hat guessing game, or HATS for short, is a pair \( G = (G, h) \) where \( G \) is a visibility graph and \( h \) is a hatness function. So, sages are located at the vertices of the visibility graph \( G \) and participate in a test. During the test, each sage \( v \) gets a hat of one of \( h(v) \) colors. The sages do not communicate with each other and try to guess the colors of their own hats. If at least one of their guesses is correct, the sages win, or the game is winning. In this case, we say that the graph is winning too, keeping in mind that this property depends also on the hatness function. Games in which the sages have no winning strategy are said to be losing.

Denote by \( (G, \ast m) \) the classical hat guessing game where the hatness function has constant value \( m \). We say that a game \( G_1 = (G_1, h_1) \) majorizes a game \( G_2 = (G_2, h_2) \) if \( G_1 = G_2 \) and \( h_1(v) \geq h_2(v) \) for every \( v \in V \). Obviously, if a winning game \( G_1 \) majorizes a game \( G_2 \), then \( G_2 \) is winning too. And vice versa: if a losing game \( G_2 \) is majorized by a game \( G_1 \), then \( G_2 \) is losing too.

Let \( m = \min_{A \in V(G)} h(A) \). Then the game \( (G, h) \) majorizes the game \( (G, \ast m) \). In this case, if the game \( G = (G, h) \) is winning, then \( HG(G) \geq m \).

2.2. Constructors. By constructors we mean theorems that allow us to build new winning graphs by combining graphs for which the winning property is already proved. Here are several constructors from the papers [6, 7].

Definition. Let \( G_1 = (V_1, E_1), G_2 = (V_2, E_2) \) be two graphs sharing a common vertex \( A \). The sum of the graphs \( G_1, G_2 \) with respect to the vertex \( A \) is the graph \( (V_1 \cup V_2, E_1 \cup E_2) \). We denote this sum by \( G_1 +_A G_2 \).

Let \( G_1 = (G_1, h_1), G_2 = (G_2, h_2) \) be two games such that \( V_1 \cap V_2 = \{A\} \). The game \( G = (G_1 +_A G_2, h) \), where \( h(v) \) equals \( h_1(v) \) for \( v \in V(G_1) \setminus \{A\} \) and \( h(A) = h_1(A) \cdot h_2(A) \) (Fig. 1), is called the product of the games \( G_1, G_2 \) with respect to the vertex \( A \). We denote the product by \( G_1 \times_A G_2 \).

\[
\begin{align*}
\text{Fig. 1. The product of games.}
\end{align*}
\]

Theorem 2.1 (on the product of games, [6, Theorem 3.1]). Let \( G_1 = (G_1, h_1) \) and \( G_2 = (G_2, h_2) \) be two games such that \( V(G_1) \cap V(G_2) = \{A\} \). If the sages win in the games \( G_1 \) and \( G_2 \), then they win also in the game \( G = G_1 \times_A G_2 \).
Theorem 2.2 ([7, Theorem 4.1]). Let $G_1$ and $G_2$ be graphs such that $V(G_1) \cap V(G_2) = \{A\}$, $G = G_1 +_A G_2$. Let $G_1 = \langle G_1, h_1 \rangle$ and $G_2 = \langle G_2, h_2 \rangle$ be losing games, $h_1(A) \geq h_2(A) = 2$. Then the game $\mathcal{G} = \langle G_1 +_A G_2, h \rangle$ is losing, where

$$h(x) = \begin{cases} h_1(x) & \text{if } x \in V(G_1), \\ h_2(x) & \text{if } x \in V(G_2) \setminus A. \end{cases}$$

Theorem 2.3 (on the “cone” over a graph, [7, Theorem 4.5]). Assume that a game $\mathcal{G} = \langle G, h \rangle$, where $V(G) = \{A_1, A_2, \ldots, A_k\}$, and $k$ graphs $G_i = \langle G_i, h_i \rangle$, $1 \leq i \leq k$, are winning and the sets $V(G_i)$ are disjoint. In each graph $G_i$, one vertex is labeled $O$ and one of its neighbors is labeled $A_i$ in such a way that $h_1(O) = h_2(O) = \ldots = h_k(O)$. Consider a new graph $G' = \langle V(G'), E(G') \rangle$, where

$$V(G') = V(G_1) \cup \ldots \cup V(G_k), \quad E(G') = E(G_1) \cup \ldots \cup E(G_k) \cup E(G).$$

Then the game $\langle G', h' \rangle$ is winning, where

$$h'(x) = \begin{cases} h_i(x) & \text{if } x \text{ belongs to one of the sets } V(G_i) \setminus \{A_i\}, \\ h_i(A_i)h(A_i) & \text{if } x \text{ coincides with } A_i. \end{cases}$$

To apply these constructor theorems, we need “bricks”, i.e., examples of winning (or losing) graphs. The following theorem gives us a whole class of such examples.

Theorem 2.4 ([6, Theorem 2.1]). Let $a_1, a_2, \ldots, a_n$ be the hatnesses of $n$ sages located at the vertices of the complete graph. Then the sages win if and only if

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} \geq 1.$$  (1)

2.3. Application of constructors: windmills. Let us show how constructors can be applied to study games with constant hatness function. The following theorem was proved in [2]. Using constructors makes it almost obvious.

Let $k$ and $n$ be arbitrary positive integers. Let $G_1, G_2, \ldots, G_n$ be $n$ copies of the complete graph $K_k$ in each of which a vertex $A$ is marked. A windmill is a graph of the form $W_{k,n} = G_1 +_A G_2 +_A \ldots +_A G_n$. The vertex $A$ in this graph is called the axis of the windmill.

Theorem 2.5 ([2, Theorem 1.4]). Let $k \geq 2$ and $n \geq \log_2(2k-2)$. Then $\HG(W_{k,n}) = 2k-2$.

Proof. First, we prove that the game $\mathcal{G}_1 = \langle W_{k,n}, h \rangle$ is winning. Consider the complete graph $K_k$ in which one of the vertices is labeled $A$. Let $h$ be the hatness function such that $h(A) = 2$ and the value of $h$ for the other vertices is equal to $2k-2$. The game $G = \langle K_k, h \rangle$ is winning by Theorem 2.4, because \(\frac{1}{k-1} + \frac{1}{2} = 1\). Using Theorem 2.1, multiply $n$ copies of $G$ with respect to the vertex $A$. We obtain a winning game $\mathcal{G}_2 = \langle W_{k,n}, h \rangle$ on the windmill $W_{k,n}$ with axis $A$, where $h_2(A) = 2^n \geq 2k-2$ and $h_2(v)$ is equal to $2k-2$ for vertices $v \neq A$. The game $\mathcal{G}_2$ majorizes the game $\mathcal{G}_1$, so the game $\mathcal{G}_1$ is winning.

Now, we prove that the game $\mathcal{G}'_1 = \langle W_{k,n}, h' \rangle$ is losing. Label two vertices in the complete graph $K_k$ by $A$ and $B$ and consider the game $\mathcal{G}' = \langle K_k, h' \rangle$ where $h'(A) = 2$, $h'(B) = 2k-1$, and the values of $h'$ at the other vertices are equal to $2k-2$. By Theorem 2.4, the game $\mathcal{G}'$ is winning. Using Theorem 2.2, multiply $n$ copies of the game $\mathcal{G}'$ with respect to the vertex $A$. We obtain a losing game $\mathcal{G}'_2 = \langle W_{k,n}, h'_2 \rangle$, where $h'_2(A) = 2$ and the values of $h'_2$ are equal to $2k-2$ or $2k-1$ at the other vertices. The game $\mathcal{G}'_2$ is majorized by $\mathcal{G}'_1$, so the game $\mathcal{G}'_1$ is losing.

The winning property of the game $\mathcal{G}_1$ and the losing property of the game $\mathcal{G}'_1$ together mean exactly that $\HG(W_{k,n}) = 2k-2$. \qed
In a similar way we can prove the lower bound on $HG$ from Theorem 1.5 in [2]. Unfortunately, we do not have suitable constructors for the upper bound. This demonstrates once again that to prove that a game is losing is much harder than to prove that a game is winning.

3. Planarity

Recall that a graph is said to be planar if it can be drawn on the plane in such a way that its edges intersect only at their endpoints. If there is an embedding of the graph in the plane such that all vertices belong to the unbounded face of the embedding, then the graph is said to be outerplanar.

The relation between the number $HG(G)$ and the planarity of a graph $G$ is one of the open problems. Namely, does there exist a planar graph with an arbitrarily large hat guessing number? In this section, we build an outerplanar graph $G$ with $HG(G) \geq 6$ and a planar graph $G$ with $HG(G) \geq 14$. Currently, we see no approaches that could allow one to increase these numbers.

3.1. Planarity and the product and cone constructors. The product constructor (Theorem 2.1, Fig. 1), obviously, preserves planarity: by “multiplying” winning planar graphs we obtain a winning planar graph.

**Definition.** The list of values of a hatness function $h$ is the list $L(h)$ of all values of $h$ arranged in nondecreasing order. For example, the graph $G$ in Fig. 3 has the list of values $L(G) = (6, 6, \ldots, 6, 8)$.

Hereafter, we do not consider trivial games for which $\min h = 1$.

**Lemma 3.1** (on the second minimum). Let $G = (G, h)$ be a winning game on a planar graph $G$ and $L(h) = (a_1, a_2, \ldots)$. Then there exists a planar graph $G'$ with $HG(G') \geq a_2$.

**Proof.** Let $A$ be the sage with hatness $a_1$. Pick $k$ such that $a_1^k \geq a_2$ (this can be done, because $a_1 > 1$). Multiply $k$ copies of the game $G$ with respect to the vertex $A$ using Theorem 2.1. We obtain a winning game $G' = (G', h')$, where $h'(A) = a_1^k \geq a_2$. The game $G'$ majorizes the game $(G', \ast a_2)$. So, the latter is winning too, and $HG(G') \geq \min h' = a_2$. The graph $G'$ is planar as a sum of planar graphs. \hspace{1cm} \Box

Informally, this lemma means that searching for planar graphs with large hat guessing number, we can “allow” one vertex to have a small hatness (it is reasonable to assign this vertex the smallest possible hatness, i.e., 2). Similar statements are true for every graph property preserved under the multiplication of graphs with respect to a vertex (e.g., for outerplanarity).

The cone theorem, like the product theorem, can be used to build planar graphs with large hat guessing number due to the following observation.

**Lemma 3.2.** Let $G = (G, \ast x)$ be a winning game on an outerplanar graph $G$, let $G_1 = (G_1, h_1)$ be a winning game on a planar graph $G_1$, and let $L(h_1) = (a_1, a_2, a_3, \ldots)$. Then there exists a graph $G'$ with $HG(G') \geq \min(a_2 \cdot x, a_3)$.

**Proof.** Apply Theorem 2.3 for the game $G$ and the collection of games $G_i$, $1 \leq i \leq |V(G)|$, where for every $i$ the game $G_i$ is a copy of the game $G_1$. In each graph $G_i$, denote by $O$ and $A_i$ the vertices where the sages with hatnesses $a_1$ and $a_2$, respectively, are located (see Fig. 2, Example 3.3, and Sec. 3.3 below). As a result, we obtain a winning game $G' = (G', h')$. The graph $G'$ is planar, because in this construction we can draw all graphs $G_i$ in the outer face of the graph $G$.

The list $L(h')$ contains the values $a_1, a_3, a_4, \ldots$ and $a_2 \cdot x$. The second minimum in this list is $a_3$ or $a_2 \cdot x$. So, by Lemma 3.1, there exists a planar graph with hat guessing number at least $\min(a_2 \cdot x, a_3)$. \hspace{1cm} \Box
This lemma shows that when building planar graphs with large hat guessing number, the hatness function can have a relatively small second minimum. However, when applying the cone theorem, we have to compensate this deficiency using outerplanar graphs with relatively large hat guessing number.

**Example 3.3.** The game “26666” in Fig. 2 is winning (the values of the hatness function are indicated near each vertex). It is obtained in the spirit of Lemma 3.2. For this, we apply the cone theorem for the outerplanar graph \( G \): we glue the graph \( G_1 \) and its copy \( G_2 \) together at the vertex \( O \) and construct a copy of the graph \( G \) on the vertices \( A_1 \) and \( A_2 \).

\[
\text{The game } G_1 \\
\begin{array}{c}
2 \quad A_1 \\
2 \quad O \\
2 \quad A_2 \\
\end{array}
\]

\[
\text{The game } G \\
\begin{array}{c}
2 \quad O \\
6 \quad A_1 \\
3 \quad A_2 \\
6 \\
\end{array}
\]

\[
\text{The game } G_2 \\
\begin{array}{c}
2 \quad O \\
6 \quad A_1 \\
3 \quad A_2 \\
6 \\
\end{array}
\]

\[
\text{The game “26666”}
\]

Fig. 2. An application of the cone theorem.

**Example 3.4.** Multiplying three copies of the game “26666” (Fig. 2) by Theorem 2.1, we obtain a game “Trefoil” shown in Fig. 3. This is an example of a winning outerplanar graph with hat guessing number at least 6. Due to computer experiments, we are sure that this hat guessing number is exactly 6, but we will not prove this here.

\[
\text{Fig. 3. The game “Trefoil.”}
\]

3.2. An example of an arithmetic strategy on an almost complete graph

**Definition.** An almost complete graph is a complete graph without one edge; we denote it by \( K_n^- \). If the vertices are numbered, we assume that the edge between the last two vertices is removed.
In [7], the authors proved Theorem 2.4 by demonstrating strategies based on arithmetic considerations. We use this approach to prove that the following game on an almost complete graph is winning.

**Lemma 3.5.** The game $G = (K_5^-, [2, 3, 14, 14, 14])$ is winning.

**Proof.** Denote the sages by $A_2, A_3, A_{14}, B_{14}, C_{14}$; the subscripts show the hatness, the edge $B_{14}C_{14}$ is absent. Denote the colors of the sages’ hats (given or assumed) by $a_2, a_5, a_{14}, b_{14}, c_{14}$. Below, the calculations are modulo 42. For every hat arrangement, consider the sum

$$S = 21a_2 + 14a_3 + 3a_{14} \mod 42.$$ 

Let $M$ be the set of residues modulo 42. For every $x \in M$, the 2-orbit of $x$ is the set $\{x, x + 21\} \subset M$, the 3-orbit is the set $\{x, x + 14, x + 28\} \subset M$, and the 14-orbit is the set $\{x + 3i : i = 0, 1, 2, \ldots, 14\} \subset M$. The sets $B = \{0, 1, 2\} \subset M$ and $C = \{0, 4, 8\} \subset M$ are called traps. It is obvious that these sets meet every 14-orbit exactly at one point.

Now we describe a winning strategy for the sages. The sages $B_{14}$ and $C_{14}$ see the hats of the sages $A_2, A_3,$ and $A_{14}$ and can compute $S$. Let the sage $B$ check the hypothesis $S + 3b_{14} \in B$, and the sage $C_{14}$ check the hypothesis $S + 3c_{14} \in C$.

The sages $A_2, A_3,$ and $A_{14}$ see the hat colors of $B_{14}$ and $C_{14}$ and draw the conclusion that if

$$S \in (B - 3b_{14}) \quad \text{or} \quad S \in (C - 3c_{14})$$

(the subtractions are modulo 42), then $B_{14}$ or $C_{14}$ guess their own color correctly. Thus, if $S \notin (B - 3b_{14}) \cup (C - 3c_{14})$, then the sages $A_2, A_3,$ or $A_{14}$ must guess correctly. For this, construct disjoint sets $A_2, A_3, A_{14}$ such that

$$A_2 \cup A_3 \cup A_{14} \cup (B - 3b_{14}) \cup (C - 3c_{14}) = M.$$ 

Let every sage $A_i$, where $i = 2, 3, 14$, choose an assumed color of his hat so that $S \in A_i$ under this assumption. This can certainly be done if the set $A_i$ meets each $i$-orbit at exactly one point. For this, let $A_2$ be an interval of the form $[x, x + 20]$ consisting of 21 consecutive residues and $A_3$ be an interval of the form $[x, x + 13]$. The set $A_{14}$ must consist of three numbers with different residues modulo 3. In this way, if for the given hat arrangement the sum $S$ belongs to the set $A_i$, then the sage $A_i$ does indeed guess his hat color correctly.

Up to a cyclic permutation, there are 14 cases of mutual position of the shifted traps $B - 3b_{14}$ and $C - 3c_{14}$. Without loss of generality, we may assume that the trap $C = \{0, 4, 8\}$ is not shifted and the trap $B$ has one of the positions $\{3i, 3i + 1, 3i + 2\}, i = 0, 1, 2, \ldots, 13$. In each of these cases, define $A_2, A_3, A_{14}$ as specified in the table.

| $C$  | $B$  | $A_2$ | $A_3$ | $A_{14}$ | Superpositions |
|------|------|-------|-------|---------|----------------|
| 0, 4, 8 | 0, 1, 2 | [5, 25] | [26, 39] | 40, 41, 3 | 0, 8 |
| 0, 4, 8 | 3, 4, 5 | [7, 27] | [28, 41] | 1, 2, 6 | 4, 8 |
| 0, 4, 8 | 6, 7, 8 | [11, 31] | [32, 3] | 5, 9, 10 | 0, 8 |
| 0, 4, 8 | 9, 10, 11 | [15, 35] | [36, 7] | 12, 13, 14 | 0, 4 |
| 0, 4, 8 | 12, 13, 14 | [15, 35] | [36, 7] | 9, 10, 11 | 0, 4 |
| 0, 4, 8 | 15, 16, 17 | [21, 41] | [1, 14] | 18, 19, 20 | 4, 8 |
| 0, 4, 8 | 18, 19, 20 | [21, 41] | [1, 14] | 15, 16, 17 | 4, 8 |
| 0, 4, 8 | 21, 22, 23 | [25, 3] | [5, 18] | 19, 20, 24 | 0, 8 |
| 0, 4, 8 | 24, 25, 26 | [29, 7] | [9, 22] | 23, 27, 28 | 0, 4 |
| 0, 4, 8 | 27, 28, 29 | [5, 25] | [32, 3] | 26, 30, 31 | 0, 8 |
| 0, 4, 8 | 30, 31, 32 | [9, 29] | [33, 4] | 5, 6, 7 | 0, 4 |
| 0, 4, 8 | 33, 34, 35 | [12, 32] | [36, 7] | 9, 10, 11 | 0, 4 |
| 0, 4, 8 | 36, 37, 38 | [15, 35] | [1, 14] | 39, 40, 41 | 4, 8 |
| 0, 4, 8 | 39, 40, 41 | [15, 35] | [1, 14] | 36, 37, 38 | 4, 8 |

$\square$
3.3. A planar graph with hat guessing number at least 14

**Theorem 3.6.** There exists a planar graph $G''$ with hat guessing number at least 14.

The proof immediately follows from Lemma 3.2 applied to $G_1 = \langle K_5^-, [2, 3, 14, 14, 14] \rangle$ and the game $\mathcal{G} = \text{“Trefoil”}$ on an outerplanar graph (Fig. 3).

**Proof.** Let $\mathcal{G} = \text{“Trefoil”}$ (see Fig. 3); this is a game on an outerplanar graph $G$ with 13 vertices, denote them by $A_1, A_2, \ldots, A_{13}$. Consider 13 copies $G_i, 1 \leq i \leq 13$, of the game $\langle K_5^-, [2, 3, 14, 14, 14] \rangle$. In each of these games, denote by $O$ the vertex of hatness 2 and denote by $A_i$ the vertex of hatness 3. It is easy to see that the graphs $G_i$ are planar. Applying the cone theorem to these graphs (see Fig. 4), we obtain a game $\langle G', h' \rangle$ with one vertex $O$ of hatness 2 and several vertices of hatnesses 14, 18, or 24. Finally, multiply four copies of the game $\langle G', h' \rangle$ with respect to the vertex $O$.

The carefully crafted game $\mathcal{G}'' = \langle G'', h'' \rangle$ is played on a planar graph $G''$ where

$$G'' = G' +_O G' +_O G' +_O G'$$ and $\text{HG}(G'') \geq \min h'' = 14$. \qed

![Fig. 4. The game $\langle G', h' \rangle$. Each petal stands for a copy of the graph $K_5^-$. Gluing four copies of this game together, we obtain a planar graph with $\text{HG}(G' +_2 G' +_2 G' +_2 G') \geq 14$.](image)

4. Conclusion

The version of the HATS game with nonconstant hatness function and the theory of constructors have proved to be a fruitful approach to the study of the classical hat guessing game. They provide natural formulations, structurally visible examples, visual and at the same time meaningful proofs. But the computational complexity of the game prevents from putting forward hasty conjectures and effectively protects the game from a complete analysis.

Translated by the authors.
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