A numeric-analytical method for solving the Cauchy problem for ordinary differential equations.

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Abstract

In the paper we offer a functional-discrete method for solving the Cauchy problem for the first order ordinary differential equations (ODEs). This method (FD-method) is in some sense similar to the Adomian Decomposition Method. But it is shown that for some problems FD-method is convergent whereas ADM is divergent. The results presented in the paper can be easily generalized on the case of systems of ODEs.

Key words: FD-method, Adomian decomposition method, generating function method, ordinary differential equation, Cauchy problem, exponential convergence rate.

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1. Introduction.

Many scientific papers devoted to the Adomian decomposition method (ADM) have been published during last two decades. For the first time this method was proposed by the American physicist G. Adomian as a method for solving an operator equations (see Adomian [1984]-Adomian [1994]). The method is based on the specific analytical representation of the exact solution.

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More precisely, the solution is represented in terms of a rapidly convergent infinite series with easily computable terms (see Cherruault, 1989, El-Kalla 2007 and the references therein). In spite of the considerable scientific activity, the necessary and sufficient conditions that provide a convergence of these series are unknown. Partially, this topic was discussed in Cherruault, 1989 — Hosseini, Nasabzadeh 2006, where authors study the question about convergence of ADM applied to the nonlinear operator equation $y - N (y) = f$, where $N (\cdot)$ is a nonlinear operator in a Hilbert space $H$. The paper Himoun, Abbaoui and Cherruault 2003 offers sufficient conditions that provide a convergence of ADM applied to the Cauchy problem on a finite segment. In Hashim, Noorani, Ahmad, Bakar, Ismail and Zakaria 2006 authors apply ADM (in a rather modified form) to the Lorenz system. The paper Inc and Cherruault 2005 is devoted to the application of ADM to the solution of the nonlinear Volterra-Fredholm integro-differential equation.

The author of El-Kalla 2007 proposed a modification of the ADM. Using several numerical examples, he has shown that this modified method converges faster than the ordinary one. Nevertheless, one crucial fact was overlooked in his work, as it turns out, this modified ADM coincides with the fixed point iteration method (c.f. Gavrilyuk, Lazurchak, Makarov and Sytnyk 2009).

The idea of ADM is similar to the idea of a functional-discrete method, called FD-method, which was firstly introduced in Makarov 1991. In this paper the FD-method was applied to the solution of the Sturm-Liouville problem and the remarkable convergence results were obtained. Then, in Gavrilyuk, Klymenko, Makarov and Rossokhata 2007 – Gavrilyuk et al. 2009, FD-method was successfully applied to several operator equations and, in particular, to the boundary value problems.

The essential difference between ADM and FD-method is expressed by the fact that the last one has a built-in adjustable parameter, by varying which we can provide the convergence of FD-method even, when ADM is found to be divergent.

Let us outline the general idea of FD-method by applying it to the following Cauchy problem:

$$L_r \left( u \left( x \right) \right) - N \left( x, u \left( x \right) \right) u \left( x \right) = \phi \left( x \right), x \in \left[ x_0, +\infty \right),$$

$$u \left( x_0 \right) = u_0, \quad \frac{d}{dx} u \left( x_0 \right) = u_0^{(1)}, \ldots, \quad \frac{d^{r-1}}{dx^{r-1}} u \left( x_0 \right) = u_0^{(r-1)},$$

(1)
where \( \mathbb{L}_r (\cdot) = \frac{d^r}{dx^r} (\cdot) + \sum_{k=1}^{r-1} p_k (x) \frac{d^{(r-k)}}{dx^{(r-k)}} (\cdot) \) is a linear differential operator of the \( r \)-th order, \( r \in \mathbb{N}, p_k (x) \in C ([x_0, +\infty)) \); \( N (x, u): [x_0, +\infty) \times \mathbb{R} \to \mathbb{R} \) is a given nonlinear function, \( \phi (x) \in C ([x_0, +\infty)) \). From now on we make the assumption that \( N (x, u) \) is continuous in \( x \) on \([x_0, +\infty)\) and infinitely differentiable with respect to \( u \) on \( \mathbb{R} \), i.e., \( N (x, u) \in C_{x,u}^{0,\infty} ([x_0, +\infty) \times \mathbb{R}) \).

The main idea of FD-method is to find an approximation to the exact solution \( u (x) \) of problem (1) in the form of the finite subsum \( \bar{u} (x) = \sum_{i=0}^{\infty} u^{(i)} (x) \) of the infinite series representation

\[
u (x) = \sum_{i=0}^{\infty} u^{(i)} (x).
\]

To define each term of series (2) we need to introduce a grid:
\[
\hat{\omega} = \{ x_0 < x_1 < x_2 < \ldots, \ x_n \to +\infty, \ n \to +\infty \},
\]

\[
h = \sup_{i \in \mathbb{N}} \{ h_i = x_i - x_{i-1} \}.
\]

We define the functions \( u^{(i)} (x) \in C^{r-1} ([x_0, +\infty)) \) to be the solutions of the following linear Cauchy problems

\[
\mathbb{L}_r (u^{(0)} (x)) = N (x, u^{(0)} (x_{i-1})) u^{(0)} (x) = \phi (x),
\]

\[
u^{(0)} (x_0) = u^{(0)}_0, \ \frac{d}{dx} u^{(0)} (x_0) = u^{(1)}_0, \ldots, \ \frac{d^{r-1}}{dx^{r-1}} u^{(0)} (x_0) = u^{(r-1)}_0,
\]

\[
x \in [x_{i-1}, x_i], \ i = 1, 2, \ldots,
\]

\[
\mathbb{L}_r (u^{(j+1)} (x)) = N (x, u^{(0)} (x_{i-1})) u^{(j+1)} (x) =
\]

\[
= N' (x, u^{(0)} (x_{i-1})) u^{(0)} (x) u^{(j+1)} (x_{i-1}) + F^{(j+1)} (x),
\]

\[
u^{(j+1)} (x_0) = 0, \ \frac{d}{dx} u^{(j+1)} (x_0) = 0, \ldots, \ \frac{d^{r-1}}{dx^{r-1}} u^{(j+1)} (x_0) = 0,
\]

\[
x \in [x_{i-1}, x_i], \ i = 1, 2, \ldots
\]

with the matching conditions

\[
\left[ \frac{d^k}{dx^k} u^{(j)} (x) \right]_{x=x_i} = \lim_{x \to x_i +0} \frac{d^k}{dx^k} u^{(j)} (x) - \lim_{x \to x_i -0} \frac{d^k}{dx^k} u^{(j)} (x) = 0,
\]

\[
k = 0, 1, \ldots, r - 1; \ i = 1, 2, \ldots; j = 0, 1, \ldots,
\]
where
\[
F^{(j+1)}(x) = \sum_{p=1}^{j} A_{j+1-p}(N(x,(\cdot)); u^{(0)}(x_{i-1}), \ldots, u^{(j+1-p)}(x_{i-1})) u^{(p)}(x) + \\
+ \sum_{p=0}^{j} \left[ A_{j-p}(N(x,(\cdot)); u^{(0)}(x), u^{(1)}(x), \ldots, u^{(j-p)}(x)) - \\
-A_{j-p}(N(x,(\cdot)); u^{(0)}(x_{i-1}), u^{(1)}(x_{i-1}), \ldots, u^{(j-p)}(x_{i-1})) \right] u^{(p)}(x) + \\
+ A_{j+1}(N(x,(\cdot)); u^{(0)}(x_{i-1}), \ldots, u^{(j)}(x_{i-1}), 0) u^{(0)}(x), j = 0, 1, \ldots
\] (7)

And
\[
A_k(N(x,(\cdot)); u^{(0)}, u^{(1)}, \ldots, u^{(k)}) = \frac{1}{k!} \frac{d^k}{dt^k} \left[ N\left(x, \sum_{i=0}^{\infty} t^i u^{(i)}(x)\right) \right]_{t=0}
\] (8)

are well-known Adomian’s polynomials (see Seng, Abbaoui and Cherruault [1996], Abbaoui, Cherruault and Seng [1995]).

Definition 1. We say that the FD-method for the Cauchy problem (1) converges (to the exact solution of problem (1)) on \([x_0, x_0 + H]\), 0 < H ≤ +∞, if there exists a real number \(\tilde{H} > 0\) such that for any grid \(\omega(3)\) with \(h \leq \tilde{H}\), series (2), with the terms computed from (4)–(7), converges uniformly (to the exact solution of problem (1)) on \([x_0, x_0 + H]\).

In the present paper we consider a particular case of the Cauchy problem (1) when \(r = 1, L_1(\cdot) = \frac{d}{dx}(\cdot)\), more precisely
\[
\frac{d}{dx} u(x) - N(x, u(x)) u(x) = \phi(x),
\]
\[
u(x_0) = u_0, \quad x \in [x_0, +\infty).
\] (9)

The main result of the paper is presented by the following theorem:

Theorem 2. Let the Cauchy problem (9) satisfies the following conditions:

1. \(N(x, u) = \sum_{i=0}^{\infty} a_i(x) u^i\), where \(a_i(x) \in C([x_0, +\infty))\). And there exists a sequence of real numbers \(B_i > 0, \ldots i = 0, 1, \ldots\), such that
\[
\sup_{x \in [x_0, +\infty)} |a_i(x)| \leq B_i, \quad 0 \leq B_i \in \mathbb{R}, \quad i = 0, 1, \ldots,
\]
and the series \(\sum_{i=0}^{\infty} B_i u^i\) is convergent \(\forall u \in \mathbb{R}\);
2. \( \phi(x) \) is a continuous and bounded function on \([x_0, +\infty)\), with
\[
\sup_{x \in [x_0, +\infty)} |\phi(x)| = k < +\infty;
\]

3. \( (N(x, u)u)'_u = N(x, u) + uN'_u(x, u) < -\alpha, < -\infty, \forall x \in [x_0, +\infty), \forall u \in \mathbb{R}. \)

Then for any initial condition \( u_0 \in \mathbb{R} \) the solution \( u(x) \) of problem (9) exists on \([x_0, +\infty)\). The FD-method converges on \([x_0, +\infty)\) to the exact solution of problem (9). Moreover, the following error estimations hold true
\[
\sup_{x \in [0, +\infty)} |u(x) - \bar{u}(x)| \leq \frac{C}{(1 + m)^{1+\varepsilon}} \frac{(h/R)^{m+1}}{1 - h/R} \text{ if } h < R, \tag{10}
\]
\[
\sup_{x \in [0, +\infty)} |u(x) - \bar{u}(x)| \leq C \sum_{j=m+1}^{\infty} \frac{1}{(j + 1)^{1+\varepsilon}} \text{ if } h = R, \tag{11}
\]
were \( \varepsilon, R, C \) are positive real numbers that depend on the input data of the problem only.

2. Justification of the FD-method algorithm for solving the Cauchy problem on the infinite interval.

To begin with, let us introduce some useful notations. For each real-valued function \( u(x) \) defined on \( (a, b) \), \( a < b \leq +\infty \) we denote
\[
\|u(x)\|_{0, \infty, (a, b)} = \sup_{x \in (a, b)} |u(x)|,
\]
and \( \forall u(x) \in C((a, b)) \) we denote
\[
\|u(x)\|_{1, \infty, (a, b)} = \max \left\{ \|u(x)\|_{0, \infty, (a, b)}, \left\| \frac{d}{dx} u(x) \right\|_{0, \infty, (a, b)} \right\}.
\]
For the future convenience we would like to define such notations
\[
\|u(x)\|_{i, (a, b)} \overset{\text{def}}{=} \|u(x)\|_{i, \infty, (a, b)}, \|u(x)\|_{i, [x_0, +\infty)} \overset{\text{def}}{=} \|u(x)\|_{i, \infty, [x_0, +\infty)}, \quad i = 1, 2. \tag{12}
\]
Before proceed to the proof of theorem 2 we need to prove several auxiliary statements, presented below.
Lemma 3. Let conditions 2) and 3) of theorem 2 be fulfilled, and \( N(x, u) \in C^{1,0}_{x,u}([x_0, +\infty) \times \mathbb{R}) \). Then for any \( h_i > 0, i = 1, 2, \ldots \) the solution \( u^{(0)}(x) \) of the following Cauchy problem with piecewise constant argument (see. Akhmet 2007)

\[
\frac{d}{dx} u^{(0)}(x) - N(x, u^{(0)}(x)) u^{(0)}(x) = \phi(x), \quad x \in [x_{i-1}, x_i], \quad x = x_{i-1} + h_i, \tag{13}
\]

\[
u^{(0)}(x_0) = u_0 \in \mathbb{R}, \quad [u^{(0)}(x)]_{x=x_i} = 0, \quad i = 1, 2, \ldots \tag{14}
\]

exists on \([x_0, +\infty)\) and \(|u^{(0)}(x)| \leq \mu = \max \{|u_0|, \frac{k}{\alpha}\}, \forall x \in [x_0, +\infty)\).

Proof. Since equation (13) is linear on each segment \([x_{i-1}, x_i]\) and \( N(x, u) \) is defined on the whole real line as a function of \( u \), the solution of problem (13), (14) on the interval \([x_0, +\infty)\) exists obviously:

\[
u^{(0)}(x) = \exp \left\{ \int_{x_{i-1}}^{x} N(\xi, u^{(0)}(x_{i-1})) d\xi \right\} u^{(0)}(x_{i-1}) + \]

\[
+ \int_{x_{i-1}}^{x} \exp \left\{ \int_{\xi}^{x} N(\tau, u^{(0)}(x_{i-1})) d\tau \right\} \phi(\xi) d\xi, \quad x \in [x_{i-1}, x_i]. \tag{15}
\]

Thus, it remains only to prove that this solution is bounded by some positive constant \( \mu \). The conditions of this lemma together with the mean value theorem provide us with the following inequality

\[
N(x, u) u^2 = u \left[ N(x, u) u - N(x, 0) \cdot 0 \right] =
\]

\[
= u \left[ (N'_u(x, \tilde{u}) \tilde{u} + N(x, \tilde{u})) \right] \leq -\alpha u^2, \quad \tilde{u} \in [0, u].
\]

If we, additionally, take into account that the function \( N(x, u) \), is continuous, we will obtain the inequality \( N(x, u) \leq -\alpha, \forall u \in \mathbb{R} \). Then multiplying both sides of equation (13) (for \( i = 1 \)) by \( u^{(0)}(x) \) and using the above estimate for \( N(x, u) \) we get the following

\[
\frac{1}{2} \frac{d}{dx} (u^{(0)}(x))^2 = N(x, u^{(0)}(x_0)) (u^{(0)}(x))^2 + \phi(x) u^{(0)}(x),
\]

\[
\frac{1}{2} \frac{d}{dx} (u^{(0)}(x))^2 \leq -\alpha (u^{(0)}(x))^2 + k |u^{(0)}(x)|. \tag{16}
\]

Inequality (16) states that solution \( u^{(0)}(x) \) of problem (13), (14) on \([x_0, x_1]\) can’t leave a segment \([-\mu, \mu] \).
As a matter of fact, if we suppose that there exist \( x_*, x^* \in [x_0, x_1] \) such that \( x_* < x^* \) and
\[
\begin{align*}
|u^{(0)}(x_*)| &= \mu \geq \frac{k}{\alpha}, \\
|u^{(0)}(x)| &> \mu, \quad \forall x \in (x_*, x^*],
\end{align*}
\]
then we immediately obtain from (16)
\[
\left. \frac{d}{dx} \left( \left( u^{(0)}(x) \right)^2 \right) \right|_{x=x^*} < 0.
\]
This contradicts our assumption (17). So, \( u^{(0)}(x) \in [-\mu, \mu], \quad \forall x \in [x_0, x_1] \).
In a similar way we can prove that the solution of the Cauchy problem (13), (14) on \([x_1, x_2]\) belongs to the segment \([-\mu, \mu]\) too and so on. This completes the proof of the lemma.

The following lemma can be considered as a generalization of lemma 2.1 from Gavrilyuk et al. [2009].

**Lemma 4.** Let \( N(x,u) \in C_{x,u}^{0,\infty}([x_0, +\infty) \times \mathbb{R}) \) and \( u^{(j)}(x) \in C^1([x_0, +\infty)) \), \( j = 0, 1, 2, \ldots \). Furthermore, let there exists a function \( \tilde{N}(u) \in C^\infty(\mathbb{R}) \) such that
\[
\sup_{x \in [x_0, +\infty)} \left| \frac{\partial^p}{\partial u^p} N(x,u) \right| \leq \frac{d^p}{du^p} \tilde{N}(|u|) \quad \forall u \in \mathbb{R}, \quad \forall p \in \mathbb{N}\cup\{0\}.
\]

Then the following inequalities hold true
\[
\| A_k (N(x, (\cdot)); u^{(0)}(x), \ldots, u^{(k)}(x)) - \\
- A_k \left( N(x, (\cdot)); u^{(0)}(x_{i-1}), \ldots, u^{(k)}(x_{i-1}) \right) \|_{0,[x_{i-1}, x_i]} \leq \\
\leq h_i A_k \left( \tilde{N}'(u) \left( \begin{array}{c} u^{(0)}(x) \\ u^{(0)}(x) \\ \vdots \\ u^{(k)}(x) \end{array} \right) \right) \leq \\
\leq h_i A_k \left( \tilde{N}'(u) \left( \begin{array}{c} u^{(0)}(x) \\ u^{(0)}(x) \\ \vdots \\ u^{(k)}(x) \end{array} \right) \right),
\]
\( \forall x \in [x_{i-1}, x_i], \quad k = 0, 1, 2, \ldots, i = 1, 2, \ldots \).
Proof. Throughout the proof we use the notation

\[ N^{(p)}(u) = \frac{\partial^p}{\partial u^p} N(x, u), \quad u^{(j)} = u^{(j)}(x), \quad u^{(j)}_{i-1} = u^{(j)}(x_{i-1}). \]

Let us fix any arbitrary \( x \in [x_{i-1}, x_i] \). It is well known that the following representation for the Adomian’s polynomial holds (see Abbaoui, Pujol, Cherruault, Himoun and Grimalt 2001, Seng et al. 1996):

\[ A_k \left( N(\cdot); u^{(0)}, \ldots, u^{(k)} \right) = \sum_{\alpha_1 + \ldots + \alpha_k = k} \sum_{\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_k, \alpha_i \in \mathbb{N}[\cup \{0\}] \}

Therefore

\[ \left\| A_k \left( N(\cdot); u^{(0)}, \ldots, u^{(k)} \right) - A_k \left( N(\cdot); u^{(0)}_{i-1}, \ldots, u^{(k)}_{i-1} \right) \right\|_{0, [x_{i-1}, x_i]} \leq \]

\[ \leq \left\| \sum_{\alpha_1 + \ldots + \alpha_k = k} \left\{ N^{(\alpha_1)} \left( u^{(0)} \right) \frac{(u^{(1)})^{(\alpha_1-\alpha_2)}}{(\alpha_1-\alpha_2)!} \ldots \frac{(u^{(k)})^{(\alpha_k)}}{(\alpha_k)!} \right\} \right\|_{0, [x_{i-1}, x_i]} \]

\[ = \left\| h^* \sum_{\alpha_1 + \ldots + \alpha_k = k} \left\{ N^{(\alpha_1+1)} \left( \tilde{u}^{(0)} \right) \frac{du^{(0)}(x)}{dx} \bigg|_{x=x_i} \right\} \right\|_{0, [x_{i-1}, x_i]} \]

\[ + \ldots + N^{(\alpha_1)} \left( \tilde{u}^{(0)} \right) \frac{(\tilde{u}^{(1)})^{(\alpha_1-\alpha_2)}}{(\alpha_1-\alpha_2)!} \ldots \frac{(\tilde{u}^{(k)})^{(\alpha_k-1)}}{(\alpha_k-1)!} \frac{du^{(k)}(x)}{dx} \bigg|_{x=x_i} \right\} \right\|_{0, [x_{i-1}, x_i]} \]

\[ \leq h_i \sum_{\alpha_1 + \ldots + \alpha_k = k} \frac{\|u^{(1)}\|_{1,[x_{i-1}, x_i]}}{(\alpha_1-\alpha_2)!} \ldots \frac{\|u^{(k-1)}\|_{1,[x_{i-1}, x_i]}}{(\alpha_{k-1}-\alpha_k)!} \frac{\|u^{(k)}\|_{1,[x_{i-1}, x_i]}}{(\alpha_k)!} \times \]

\times \left\{ \|N^{(\alpha_1+1)} \left( u^{(0)} \right) \|_{0,[x_{i-1}, x_i]} + \alpha_1 \|N^{(\alpha_1)} \left( u^{(0)} \right) \|_{0,[x_{i-1}, x_i]} \right\} \leq \]

8
\[ \sum_{\alpha_1 + \ldots + \alpha_k = k} \frac{\|u^{(1)}\|^\alpha_1 \alpha_2}{(\alpha_1 - \alpha_2)!} \ldots \frac{\|u^{(k-1)}\|^\alpha_{k-1} \alpha_k}{(\alpha_{k-1} - \alpha_k)!} \frac{\|u^{(k)}\|^\alpha_k}{(\alpha_k)!} \times \]
\[ \times \left\{ \tilde{N}^{(\alpha_1+1)} \left( \|u^{(0)}\|_{1,[x_{i-1}, x_i]} \right) \|u^{(0)}\|_{1,[x_{i-1}, x_i]} + \alpha_1 \tilde{N}^{(\alpha_1)} \left( \|u^{(0)}\|_{1,[x_{i-1}, x_i]} \right) \right\} \]
\[ = h_i A_k \left( \tilde{N}' (u) \ ; \ \|u^{(0)}\|_{1,[x_{i-1}, x_i]} , \ldots , \|u^{(k)}\|_{1,[x_{i-1}, x_i]} \right) \leq \]
\[ \leq h_i A_k \left( \tilde{N}' (u) \ ; \ \|u^{(0)}\|_{1, [x_0, +\infty)} , \ldots , \|u^{(k)}\|_{1, [x_0, +\infty)} \right) , \]

where \( \tilde{u}^{(j)} = u^{(j)} (\overline{x}) \), \( \overline{x} \in [x_{i-1}, x_i] \), \( h^* = x - x_{i-1} \). And this is precisely the assertion of the lemma.

The following lemma can be considered as a generalization of lemma 2.2 from Gavrilyuk et al. [2009].

**Lemma 5.** For any scalar function \( \tilde{N} (u) \in C^\infty (\mathbb{R}) \) the following equality holds true

\[ A_{j+1} \left( \tilde{N} (x) ; V_0, V_1, \ldots, V_j, 0 \right) = \]
\[ = \frac{1}{(j + 1)!} \left\{ \frac{d^{j+1}}{dz^{j+1}} \left( \tilde{N} (f (z)) - (f (z) - V_0) \tilde{N}' (V_0) \right) \right\}_{z=0} , \]

where \( j = 0, 1, \ldots, f (z) = \sum_{i=0}^{\infty} z^i V_i. \)

The proof of lemma 5 is trivial.

**Proof. (Of theorem 2)** Let the conditions of theorem 2 be fulfilled. It is easy to seen that for any initial condition \( u_0 \in \mathbb{R} \), the solution of the Cauchy problem [7] exists and is unique on \([0, +\infty)\). Moreover, it belongs to the segment \([-\mu, \mu] \), \( \mu = \max \left\{ \|u_0\|, \frac{\epsilon}{\alpha} \right\} \) (see. Demidivich [1967], p.278). So, we are going to prove that this solution can be found by the FD-method.

Let us fix any arbitrary infinite grid

\[ \hat{\omega} = \{ x_i = x_{i-1} + h_i, \ h_i > 0, \ i = 1, 2, \ldots ; x_i \to +\infty, \ i \to +\infty \}, \quad (18) \]
\[ h = \sup_{i \in \mathbb{N}} h_i \]
and embed the Cauchy problem (9) into the more general problem
\[
\frac{d}{dx}u(x,t) - N(x, u(x_{i-1}, t)) u(x, t) - \\
t \{N(x, u(x, t)) - N(x, u(x_{i}, t))\} u(x, t) = \phi(x), \ x \in [x_{i-1}, x_{i}], \ (19)
\]
\[u(x, t)\big|_{x=x_{i}} = 0, \ i = 1, 2, \ldots, \ u(x_{0}, t) = u_{0}, \ \forall t \in [0, 1].\]

Apparently, if we set \(t = 1\) then the solution of (19) coincides with the solution of problem (9):
\[u(x, 1) = u(x).\]

If we set \(t = 0\), on the other hand, we will get the base problem
\[
\frac{d}{dx}u^{(0)}(x) - N(x, u^{(0)}(x_{i-1})) u^{(0)}(x) = \phi(x), \ x \in [x_{i-1}, x_{i}], \ i = 1, 2, \ldots,
\]
\[u^{(0)}(x_{0}) = u_{0}, \ u^{(0)}(x)\big|_{x=x_{i}} = 0, \ i = 1, 2, \ldots, \ (20)\]

which is analogous to problem (4). Then, we assume that the solution of problem (19) can be found in the form of series
\[u(x, t) = \sum_{j=0}^{\infty} t^{j} u^{(j)}(x), \ \forall t \in [0, 1], \ \forall x \in [x_{0}, +\infty), \ (21)\]

\[
\frac{d}{dx}u(x, t) = \sum_{j=0}^{\infty} t^{j} \frac{d}{dx} u^{(j)}(x), \ \forall t \in [0, 1], \ \forall x \in [x_{i-1}, x_{i}], \ i = 1, 2, \ldots, \ (22)\]

where coefficients \(u^{(j)}(x)\) depend on \(x\) only. By substituting (21) and (22) into (19) and comparing the coefficients in front of the powers of \(t\), we obtain the following recurrence sequence of linear problems for \(u^{(j+1)}(x)\) (with a piecewise constant coefficient):
\[
\frac{d}{dx}u^{(j+1)}(x) - N(x, u^{(0)}(x_{i-1})) u^{(j+1)}(x) = \\
N_{u}^{i}(x, u^{(0)}(x_{i-1})) u^{(0)}(x) u^{(j+1)}(x_{i-1}) + F^{j+1}(x), \ (23)
\]
\[x \in [x_{i-1}, x_{i}], \ i = 1, 2, \ldots, \]
where

\[ F^{(j+1)}(x) = \sum_{p=1}^{j} A_{j+1-p} \left( N(x, (\cdot)); u^{(0)}(x_{i-1}), \ldots, u^{(j+1-p)}(x_{i-1}) \right) u^{(p)}(x) + \]
\[ + \sum_{p=0}^{j-1} \left[ A_{j-p} \left( N(x, (\cdot)); u^{(0)}(x), u^{(1)}(x), \ldots, u^{(j-p)}(x) \right) \right. \]
\[ - A_{j-p} \left( N(x, (\cdot)); u^{(0)}(x_{i-1}), u^{(1)}(x_{i-1}), \ldots, u^{(j-p)}(x_{i-1}) \right) \] \]
\[ + A_{j+1} \left( N(x, (\cdot)); u^{(0)}(x_{i-1}), u^{(1)}(x_{i-1}), \ldots, u^{(j)}(x_{i-1}), 0 \right) u^{(0)}(x), \]
\[ \left[ u^{(j+1)}(x) \right]_{x=x_i} = 0, \quad i = 1, 2, \ldots, \quad u^{(j+1)}(x_0) = 0, \quad j = 0, 1, 2, \ldots. \]

It is easy to see, that problems (23) are analogous to problems (5). Let us express equation (23) in the equivalent form

\[ \frac{du^{(j+1)}(x)}{dx} - q_i(x) u^{(j+1)}(x) = \]
\[ = N_u'(x, u^{(0)}(x_{i-1})) \left[ u^{(j+1)}(x_{i-1}) \int_{x_{i-1}}^{x} \frac{d}{d\xi} u^{(0)}(\xi) \, d\xi \right. \]
\[ + u^{(0)}(x_{i-1}) \int_{x_{i-1}}^{x} \frac{d}{d\xi} u^{(j+1)}(\xi) \, d\xi \] \]
\[ + F^{(j+1)}(x), \quad x \in [x_{i-1}, x_i], \]
\[ \left[ u^{(j+1)}(x) \right]_{x=x_i} = 0, \quad i = 1, 2, \ldots, \quad u^{(j+1)}(x_0) = 0, \]

where

\[ q_i(x) = N(x, u^{(0)}(x_{i-1})) + N'_u(x, u^{(0)}(x_{i-1})) u^{(0)}(x_{i-1}). \]

The continuous solution of the Cauchy problem (24) can be represented in
the following form

\[ u^{(j+1)}(x) = \exp \left\{ \int_{x_{i-1}}^{x} q_i(\xi) \, d\xi \right\} + \]

\[ + \int_{x_{i-1}}^{x} \exp \left\{ \int_{\xi}^{x} q_i(\tau) \, d\tau \right\} N'_u(\xi, u^{(0)}(x_{i-1})) \times \]

\[ \times \int_{x_{i-1}}^{\xi} \frac{d}{d\eta} u^{(0)}(\eta) \, d\eta \, d\xi \] \[ u^{(j+1)}(x_{i-1}) + \]

\[ + u^{(0)}(x_{i-1}) \int_{x_{i-1}}^{x} \exp \left\{ \int_{\xi}^{x} q_i(\tau) \, d\tau \right\} N'_u(\xi, u^{(0)}(x_{i-1})) \int_{x_{i-1}}^{\xi} \frac{d}{d\eta} u^{(j+1)}(\eta) \, d\eta \, d\xi + \]

\[ + \int_{x_{i-1}}^{x} \exp \left\{ \int_{\xi}^{x} q_i(\tau) \, d\tau \right\} F^{(j+1)}(\xi) \, d\xi, \quad x \in [x_{i-1}, x_i]. \] \tag{25}

On the other hand, from (23) we will obtain

\[ u^{(j+1)}(x) = \exp \left\{ \int_{x_{i-1}}^{x} n_i(\xi) \, d\xi \right\} + \]

\[ + \int_{x_{i-1}}^{x} \exp \left\{ \int_{\xi}^{x} n_i(\eta) \, d\eta \right\} N'_u(\xi, u^{(0)}(x_{i-1})) u^{(0)}(\xi) \, d\xi \] \[ u^{(j+1)}(x_{i-1}) + \]

\[ + \int_{x_{i-1}}^{x} \exp \left\{ \int_{\xi}^{x} n_i(\eta) \, d\eta \right\} F^{(j+1)}(\xi) \, d\xi, \] \tag{26}

where

\[ n_i(x) = N(x, u^{(0)}(x_{i-1})) \quad x \in [x_{i-1}, x_i]. \]

If we differentiate (26) with respect to \( x \) we will obtain
\[
\frac{d}{dx} u^{(j+1)}(x) = \left[ n_i(x) \exp \left\{ \int_{x_{i-1}}^{x} n_i(\xi) \, d\xi \right\} + N'_u(x, u^{(0)}(x_{i-1})) u^{(0)}(x) + \\
+ n_i(x) \int_{x_{i-1}}^{x} \exp \left\{ \int_{\xi}^{x} n_i(\tau) \, d\tau \right\} N'_u(\xi, u^{(0)}(x_{i-1})) u^{(0)}(\xi) \, d\xi \right] u^{(j+1)}(x_{i-1}) + \\
+ n_i(x) \int_{x_{i-1}}^{x} \exp \left\{ \int_{\xi}^{x} n_i(\tau) \, d\tau \right\} F^{(j+1)}(\xi) \, d\xi + F^{(j+1)}(x), \quad x \in [x_{i-1}, x_i].
\]

Equation (27) could be expressed in the another form
\[
\frac{d}{dx} u^{(j+1)}(x) = p_i(x) u^{(j+1)}(x_{i-1}) + \\
+ n_i(x) \int_{x_{i-1}}^{x} \exp \left\{ \int_{\xi}^{x} n_i(\tau) \, d\tau \right\} F^{(j+1)}(\xi) \, d\xi + F^{(j+1)}(x), \quad x \in [x_{i-1}, x_i].
\]

where
\[
p_i(x) = n_i(x) \exp \left\{ \int_{x_{i-1}}^{x} n_i(\xi) \, d\xi \right\} + N'_u(x, u^{(0)}(x_{i-1})) u^{(0)}(x) + \\
+ n_i(x) \int_{x_{i-1}}^{x} \exp \left\{ \int_{\xi}^{x} n_i(\tau) \, d\tau \right\} N'_u(\xi, u^{(0)}(x_{i-1})) u^{(0)}(\xi) \, d\xi, \quad x \in [x_{i-1}, x_i].
\]

From (28), (29) and lemma 3 we obtain the following estimates
\[
\left\| \frac{d}{dx} u^{(j+1)}(x) \right\|_{0,[x_{i-1}, x_i]} \leq \left\| p_i(x) \right\|_{0,[x_{i-1}, x_i]} \left\| u^{(j+1)}(x_{i-1}) \right\| + \\
+ (1 + Nhe^{Nh}) \left\| F^{(j+1)}(x) \right\|_{0,[x_{i-1}, x_i]}, \quad N = \max_{\left\| u \right\| \leq \mu, \quad x \in [x_0, +\infty)} |N(x,u)|
\]

for \(i = 1, 2, \ldots\). For abbreviation, we denote (using the result of lemma 3)
\[
B = \max_{\left\| u \right\| \leq \mu, \quad x \in [x_0, +\infty)} |N'_u(x,u)| \left\{ |N(x,u)| \mu + k \right\} < +\infty,
\]
\[
C = \max_{\left\| u \right\| \leq \mu, \quad x \in [x_0, +\infty)} |N'_u(x,u)| \mu < +\infty.
\]
Then, from (25) we obtain the estimates

$$
\| u^{(j+1)}(x) \|_{0,[x_{i-1}, x_i]} \leq \left[ 1 + \frac{Bh_i^2}{2} \right] u^{(j+1)}(x_{i-1}) + \frac{Ch_i^2}{2} \| F^{(j+1)}(x) \|_{0,[x_0, +\infty)} ,
$$

$$
\left| u^{(j+1)}(x_i) \right| \leq \left[ e^{-\alpha h_i} + \frac{Bh_i^2}{2} \right] u^{(j+1)}(x_{i-1}) + \frac{Ch_i^2}{2} \| F^{(j+1)}(x) \|_{0,[x_0, +\infty)}, \quad \alpha = 1, 2, \ldots .
$$

Combining the last two inequalities, the result of lemma 3 and (30), we get

$$
\| u^{(j+1)}(x_i) \|_{0,[x_{i-1}, x_i]} \leq \left[ 1 + \mathcal{B} h_i^2 \right] u^{(j+1)}(x_{i-1}) + h_i \| F^{(j+1)}(x) \|_{0,[x_0, +\infty)},
$$

$$
\left| u^{(j+1)}(x_i) \right| \leq \left[ e^{-\alpha h_i} + \mathcal{B} h_i^2 \right] u^{(j+1)}(x_{i-1}) + h_i \| F^{(j+1)}(x) \|_{0,[x_0, +\infty)}, \quad (31)
$$

were

$$
\mathcal{B} = \frac{B}{2} + \frac{Cp}{2}, \quad D = 1 + h_i \frac{C}{2} (1 + Ne^{Nh}) ,
$$

$$
P = Ne^{Nh} + C + NhC = C + Ne^{Nh} (1 + hC) \geq \| p(x) \|_{0,[x_0, +\infty)} .
$$

Without loss of generality, we can assume that \( \| F^{j+1}(x) \|_{0,[x_0, +\infty)} > 0 \).

Since that we denote

$$
y_i = \| u^{(j+1)}(x_i) \|_{0,[x_{i-1}, x_i]}^{-1} \| F^{(j+1)}(x) \|_{0,[x_0, +\infty)} ^{-1} \quad (32)
$$

and rewrite (32) in the following form

$$
y_i \leq \left[ e^{-\alpha h_i} + \mathcal{B} h_i^2 \right] y_{i-1} + h_i D, \quad y_0 = 0, \quad i = 1, 2, \ldots . \quad (33)
$$

It is easy to see that for all \( h_i > 0 \)

$$
e^{-\alpha h_i} + \mathcal{B} h_i^2 \leq 1 - \alpha h_i + h_i^2 \left( \mathcal{B} + \frac{\alpha^2}{2} \right) = 1 - h_i \left( \alpha - h_i \left( \mathcal{B} + \frac{\alpha^2}{2} \right) \right).
$$

Thus the inequality

$$
h_i \leq h \leq \frac{\alpha}{2\mathcal{B} + \alpha^2}, \quad i = 1, 2, \ldots , \quad (34)
$$
implies
\[ e^{-\alpha h_i} + Bh_i^2 \leq 1 - \frac{h_i \alpha}{2}. \]

The last inequality, under conditions (34), guarantee that the recurrence sequence
\[ Y_i = \left( 1 - h_i \frac{\alpha}{2} \right) Y_{i-1} + h_i D, \quad i = 1, 2, \ldots, \quad Y_0 = 0, \quad (35) \]
is a majorant for the sequence \( y_i \) (33).

It is easy to seen that (35) can be reduced to the form
\[ Z_i = (1 - \bar{h}_i) Z_{i-1}, \quad i = 1, 2, \ldots, \quad Z_0 = 1, \]
where
\[ \bar{h}_i = h_i \frac{\alpha}{2}, \quad Z_i = 1 - \bar{V}_i \bar{V}_i = \frac{\alpha Y_i}{2D}, \quad i = 1, 2, \ldots, \]
hence that
\[ Z_i = \prod_{p=1}^{i} (1 - h_p), \quad i = 1, 2, \ldots, \quad Z_0 = 1. \]

We conclude from the last expression that the inequalities
\[ 0 < h_i \leq \mu_1 = \min \left\{ \frac{4}{\alpha}, \frac{\alpha}{2B + \alpha^2} \right\}, \quad \forall i \in \mathbb{N} \quad (36) \]
implies the estimates
\[ y_i \leq \bar{V}_i \leq \frac{4}{\alpha} D, \quad \forall i \in \mathbb{N}. \]

From now on we orient the grid \( \hat{\omega} \) (18) by requirement that (36) is satisfied. Thus the inequalities (30) and (31) yield
\[
\left\| u^{(j+1)}(x) \right\|_{0, [x_{i-1}, x_i]} \leq \left( 1 + B \mu_1^2 \frac{2}{\alpha} D + \mu_1 D \right) \left\| F^{(j+1)}(x) \right\|_{0, [x_0, +\infty)},
\]
\[
\left\| \frac{d}{dx} u^{(j+1)}(x) \right\|_{0, [x_{i-1}, x_i]} \leq \left( pD \frac{2}{\alpha} + \frac{Q}{\alpha} + 1 \right) \exp \left( \frac{C}{\alpha} \right) \left\| F^{(j+1)}(x) \right\|_{0, [x_0, +\infty)}.
\]

Combining the last two inequalities, we obtain the estimate
\[
\left\| u^{(j+1)}(x) \right\|_{1, [x_0, +\infty]} \leq \sigma \left\| F^{(j+1)}(x) \right\|_{0, [x_0, +\infty)} \quad (37),
\]
where
\[
\sigma = \max \left\{ \left(1 + B\mu_1^2 \right) \frac{2}{\alpha} D + \mu_1 D, \left(\frac{\mu_1^2}{\alpha} + \frac{2}{\alpha} + 1 \right) \exp \left(\frac{C}{\alpha}\right) \right\}.
\]

To make the following estimations we need to use the results of lemmas 4 and 5. The requirements of these lemmas are satisfied if we set (in the notations of lemma 4)
\[
\tilde{N}(u) = \sum_{i=0}^{\infty} \|a_i(x)\|_0 u^i \leq \sum_{i=0}^{\infty} B_i u^i, \forall u \in \mathbb{R}.
\] (38)

(37) is equivalent to
\[
\|u^{(j+1)}(x)\|_1 \leq \sigma \left\{ \sum_{p=0}^{j} A_{j+1-p} \left(\tilde{N}(u); \|u^{(0)}(x)\|_1, \ldots, \|u^{(j+1-p)}(x)\|_1\right) \|u^{(p)}(x)\|_1 + \\
+h \sum_{p=0}^{j} A_{j-p} \left(\tilde{N}'(u) u; \|u^{(0)}(x)\|_1, \ldots, \|u^{(j-p)}(x)\|_1\right) \|u^{(p)}(x)\|_1 + \\
+ \frac{\|u^{(0)}(x)\|_1}{(j+1)!} \left[ \frac{d^{j+1}}{dz^{j+1}} \left(\tilde{N} \left(\sum_{s=0}^{\infty} z^s \|u^{(s)}(x)\|_1\right)\right)_{z=0} \right], j = 0, 1, 2, \ldots.
\] (39)

Denote
\[
u_j = \frac{\|u^{(j)}(x)\|_1}{h^j}, j = 0, 1, \ldots.
\] (40)

Let us define a sequence \(\{V_i\}_{i=1}^{\infty}\) by the following recursive formula
\[
V_{j+1} = \sigma \left\{ \sum_{p=1}^{j} A_{j+1-p} \left(\tilde{N}(u); V_0, \ldots, V_{j+1-p}\right) V_p + \\
+ \sum_{p=0}^{j} A_{j-p} \left(\tilde{N}'(u) u; V_0, \ldots, V_{j-p}\right) V_p + \right\}
\] (41)
\[ + \frac{V_0}{(j + 1)!} \frac{d^{j+1}}{dz^{j+1}} \left( \tilde{N} \left( \sum_{s=0}^{\infty} z^s V_s \right) \right) \bigg|_{z=0} - V_{j+1} V_0 \tilde{N}'(V_0) \right \}, \quad j = 0, 1, \ldots \]

or, in more convenient form,

\[ V_{j+1} = \frac{\sigma}{1 + \sigma V_0 \tilde{N}'(V_0)} \left\{ \sum_{p=0}^{j} A_{j+1-p} \left( \tilde{N} (u; V_0, \ldots, V_{j+1-p}) V_p + \right. \right. \]

\[ + \sum_{p=0}^{j} A_{j-p} \left( \tilde{N}' (u; V_0, \ldots, V_{j-p}) V_p \right) \right\}, \quad (42) \]

where \( V_0 = \mu \).

It is easy to seen that \( \nu_i \leq V_i, \forall i \in \mathbb{N} \bigcup \{0\} \).

We are now in a position to prove that for \( h \) sufficiently small the assumptions (21) and (22) hold. In order to do that, it is sufficient to show that the function

\[ g(z) = \sum_{j=0}^{\infty} z^j V_j \quad \text{(43)} \]

has a nonempty open domain.

From (42) we obtain

\[ g(z) - V_0 = \frac{\sigma}{1 + \sigma V_0 \tilde{N}'(V_0)} \left\{ g(z) \left[ \tilde{N} (g(z)) - \tilde{N} (V_0) \right] + z g^2 (z) \tilde{N}' (g(z)) \right\}. \quad (44) \]

Let us express \( z \) from (44) as a function of \( g \):

\[ z(g) = \frac{1}{g^2 \tilde{N}'(g)} \left\{ \frac{1}{\Sigma} (g - V_0) - \left( \tilde{N} (g) - \tilde{N} (V_0) \right) g \right\}, \]

\[ V_0 \leq g, \quad \Sigma = \frac{\sigma}{1 + \sigma V_0 \tilde{N}'(V_0)}. \quad (45) \]

Evidently, the function \( z(g) \) \( (45) \) is defined and continuously differentiable in some open neighbourhood of the point \( g = V_0 \). To prove the existence of the inverse function \( g = g(z) \) defined in some open neighbourhood of the point \( z = 0 \) it is sufficient to show that \( z'(V_0) > 0 \). The last fact immediately follows from formula \( (45) \):

\[ z'(V_0) = \lim_{g \to V_0} \frac{z(g) - z(V_0)}{g - V_0} = \]
\[
\text{lim}_{g \to V_0} \frac{1}{V_0^2 N'(V_0)} \left( \frac{1}{\Sigma} \frac{\tilde{N}(g) - \tilde{N}(V_0)}{g - V_0} \right) = \frac{1}{\sigma V_0^2 N'(V_0)} > 0.
\]

From (38) it follows that \(\tilde{N}(u) \to +\infty\) as \(u \to +\infty\). Hence, from equation (45) we have:

\[
\text{lim}_{g \to +\infty} z(g) \leq 0.
\]

The last inequality together with (46) provides the existence of a point \(g_{\text{max}}\) such that \(g_{\text{max}} > V_0\), \(z'(g_{\text{max}}) = 0\) and \(\forall g \in (V_0, g_{\text{max}}) \ z'(g) > 0\). \(R = z_{\text{max}} = z(g_{\text{max}})\) is the radius of convergence of the power series (43). In other words,

\[
R^j V_j \leq C \frac{1}{(j + 1)^{1+\varepsilon}},
\]

for sufficiently small \(\varepsilon > 0\), where \(C\) is some constant independent of \(h_i\), \(i \in \mathbb{N}\).

Returning to notation (40), we obtain

\[
\|u^{(j)}(x)\|_{1,[x_0, +\infty)} \leq \frac{C}{(j + 1)^{1+\varepsilon}} \left( \frac{h}{R} \right)^j, \ j = 0, 1, \ldots
\]

This estimate gives us the following sufficient condition for series (43) to be convergent:

\[
\frac{h}{R} \leq 1.
\]

Finally, we have proved that for \(h > 0\) sufficiently small (to be exact, \(h < \min \{\mu_1, R\}\)) assumptions (21) and (22) are fulfilled. Therefore, it is remain to prove that the sum of the uniformly convergent series (2) is a solution to problem (9). To do that we need to add up the base problem (20) and equations (23) for \(j = 0, 1, \ldots\). As a result we obtain

\[
\sum_{j=0}^\infty \frac{d}{dx} u^{(j)}(x) - \sum_{j=0}^\infty \sum_{p=0}^j A_{j-p} \left( N(x, (\cdot)); u^{(0)}(x), \ldots, u^{(j-p)}(x) \right) u^{(p)}(x) = \phi(x),
\]

\[
(48)
\]
Using the theorem about the composition of two power series (see Fihtenholts [1966], p. 485), it is easy to prove the equality

\[
N \left( x, \sum_{i=0}^{\infty} t^i u^{(i)}(x) \right) \sum_{i=0}^{\infty} t^i u^{(i)}(x) = \sum_{j=0}^{\infty} \sum_{p=0}^{j} A_{j-p} \left( N(x, \cdot), u^{(0)}(x), \ldots, u^{(j-p)}(x) \right) u^{(p)}(x),
\]

\[\forall t \in [0, 1], \forall x \in [x_0, +\infty).\] Thus, taking into account the uniformly convergence of the series \[\sum_{i=0}^{\infty} \frac{d}{dx} u^{(i)}(x)\] on each interval \((x_{i-1}, x_i), i = 1, 2, \ldots,\) we can rewrite (48) in the following form

\[
\frac{d}{dx} \tilde{u}(x) - N(\tilde{u}(x)) \tilde{u}(x) = \phi(x), \forall x \in \bigcup_{i=1}^{\infty} (x_{i-1}, x_i), \tilde{u}(x) = \sum_{j=0}^{\infty} u^{(j)}(x),
\]

(49)

Recall that \(\tilde{u}(x_0) = u_0.\) Then the continuity of \(\tilde{u}(x),\) the existence and uniqueness of the solution \(u(x)\) of Cauchy problem (9) on \([x_0, +\infty)\) along with (49) imply our final goal \(\tilde{u}(x) \equiv u(x), \forall x \in [x_0, +\infty).\) This means that the FD-method for the Cauchy problem (9) converges to the exact solution of the problem in the sense of definition 1.

We leave it to the reader to verify the error estimates (10), (11). They could be easily derived from (47).

3. Examples.

**Example 1.** As an example let us consider the following Cauchy problem

\[
\frac{d}{dx} u(x) = -u^3(x) - u(x) + \cos(x) + \sin(x) + \sin^3(x), \quad u(0) = 0.
\]

(50)

It easy to seen that the exact solution of problem (50) is \(u^*(x) = \sin(x).\)

To solve problem (50) numerically, we, first of all, may try to apply the ADM. We approximate the solution by the \(m-\)th partial sum \(u_A^m\) of the series \(\sum_{i=0}^{\infty} u_A^{(i)}(x),\) where \(u_A^{(i)}(x)\) can be found from the sequence of Cauchy problems

\[
\frac{d}{dx} u_A^{(0)}(x) = -u_A^{(0)}(x) + \cos(x) + \sin(x) + \sin^3(x), \quad u_A^{(0)}(0) = 0,
\]
\[
\frac{d}{dx} u_{A}^{(i)}(x) = -\frac{d}{dt} \left( \sum_{i=0}^{\infty} t^{i} u_{A}^{(i)}(x) \right)^{3} \Bigg|_{t=0} - u_{A}^{(i)}(x), \quad u_{A}^{(i)}(0) = 0, \quad i = 1, 2, \ldots.
\]

The results are presented on the Fig. 1. The graphs on Fig. 1 show that the

ADM for the Cauchy problem (50) is divergent on \([0, 6]\).

The application of FD-method to problem (50) with the grid

\[\omega = \left\{ x_0 = 0, \quad x_i = \frac{1}{3}i, \quad i = 1 \ldots, 144 \right\}\]

yields results which are presented on Fig.2, Fig.3. Here we use the notations:

\[\delta_{i}(x) = \sin(x) - \hat{u}_{i}(x), \quad i = 0, 1, \ldots.\]

The graphs on Fig. 2 and Fig. 3 confirm the exponential convergence rate of series (2) to the exact solution of problem (50).

Example 2.

Let us consider the Cauchy problem

\[
\frac{d}{dx} u(x) + \left( \frac{1}{\sqrt{x}} + 1 \right) u^{3}(x) = \left( \frac{1}{\sqrt{x}} + 1 \right) \sin(2\sqrt{x} + x),
\]

(51)
Figure 2: Example 1. FD-method. The graphs of absolute errors. a) $\delta_0(x)$; b) $\delta_1(x)$.

Figure 3: Example 1. FD-method. The graphs of absolute errors. a) $\delta_2(x)$; b) $\delta_3(x)$; $x \in [0, 48]$.

$x \in [0, 1]$, $u(0) = 1$.

Problem (51) has a singularity at the point $x = 0$ : $\lim_{x \to 0} \frac{d}{dx} u(x) = +\infty$.  

21
Using routines from the computer algebra system Maple 12, we tried to find the solution of problem (51) either analytically or numerically, but all that was in vain. In the present version Maple is unable to solve such problems. The conditions of theorem 2 are not fulfilled for this problem, either. Regardless of that, the ideas of ADM and FD-method are naturally applicable.
to this problem. However, as it turned out, the ADM is divergent on \([0, 1]\) in this case.

To apply the FD-method we need to introduce a grid on the segment \([0, 1]\)

\[\hat{\omega} = \{0 = x_0, x_i = 0.05i, i = 1, 2, \ldots, 20\}.\]

The base problem is stated as follows

\[
\frac{d}{dx} u^{(0)}(x) + \left( \frac{1}{\sqrt{x}} + 1 \right) \left( u^{(0)}(x_{i-1}) \right)^2 u^{(0)}(x) = \left( \frac{1}{\sqrt{x}} + 1 \right) \sin \left( 2\sqrt{x} + x \right),
\]

\[x \in [x_{i-1}, x_i] \ u(0) = 1.\]

It admits the analytical solution

\[
u^{(0)}(x) = \exp \left( - \left( 2\sqrt{\xi} + \xi \right) \right) \left|_{\xi = x_{i-1}}^{\xi = x} \right. \left( u^{(0)}(x_{i-1}) \right)^2 u^{(0)}(x_{i-1}) + \left. \right.

\[
+ \int_{x_{i-1}}^{x} \left( \frac{1}{\sqrt{\xi}} + 1 \right) \exp \left( (2\sqrt{\tau} + \tau) \right) \left|_{\tau = x_{i-1}}^{\tau = \xi} \left( u^{(0)}(x_{i-1}) \right)^2 \right) \sin \left( 2\sqrt{\xi} + \xi \right) d\xi = \exp \left( - \left( 2\sqrt{\xi} + \xi \right) \right) \left|_{\xi = x_{i-1}}^{\xi = x} \right. \left( u^{(0)}(x_{i-1}) \right)^2 u^{(0)}(x_{i-1}) + \right.

\]

Figure 6: Example 2. FD-method. Graphs of discrepancy. \(v_2(x)\).
\[
+ \frac{(u^{(0)}(x_{i-1}))^2}{(u^{(0)}(x_{i-1}))^4 + 1} \left[ \exp \left( - (2\sqrt{\tau} + \tau) \bigg|_{\tau=\xi} (u^{(0)}(x_{i-1}))^2 \right) \times \\
\times \left\{ \sin \left( 2\sqrt{\xi} + \xi \right) - \frac{1}{(u^{(0)}(x_{i-1}))^2} \cos \left( 2\sqrt{\xi} + \xi \right) \right\} \bigg|_{\xi=x}^{\xi=x_{i-1}} \right].
\]

Similar analytical formulas were obtained for \( u^{(1)}(x) \) and \( u^{(2)}(x) \). For error control we use the discrepancy

\[
\nu_n(x) = \sqrt{x} \left( \frac{d}{dx} u^n(x) \right) + (1 + \sqrt{x}) \left( \left( \frac{d}{dx} u^n(x) \right)^3 - \sin \left( 2\sqrt{x} + x \right) \right).
\]

The results are presented on Fig. 4 – Fig. 6. As above we use the notation

\[
u_n(x) = \sum_{i=0}^{m} u^{(i)}(x).
\]

It is easy to seen that the discrepancy is decreasing exponentially even behind the point of singularity \( x = 0 \).

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