New integral representations of Whittaker functions for classical Lie groups

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Abstract. The present paper proposes new integral representations of \( g \)-Whittaker functions corresponding to an arbitrary semisimple Lie algebra \( g \) with the integrand expressed in terms of matrix elements of the fundamental representations of \( g \). For the classical Lie algebras \( \mathfrak{sp}_{2\ell} \), \( \mathfrak{so}_{2\ell} \), and \( \mathfrak{so}_{2\ell+1} \) a modification of this construction is proposed, providing a direct generalization of the integral representation of \( \mathfrak{gl}_{\ell+1} \)-Whittaker functions first introduced by Givental. The Givental representation has a recursive structure with respect to the rank \( \ell + 1 \) of the Lie algebra \( \mathfrak{gl}_{\ell+1} \), and the proposed generalization to all classical Lie algebras retains this property. It was observed elsewhere that an integral recursion operator for the \( \mathfrak{gl}_{\ell+1} \)-Whittaker function in the Givental representation coincides with a degeneration of the Baxter \( \mathcal{D} \)-operator for \( \widehat{\mathfrak{gl}}_{\ell+1} \)-Toda chains. In this paper \( \mathcal{D} \)-operators for the affine Lie algebras \( \widehat{\mathfrak{sp}}_{2\ell} \), \( \widehat{\mathfrak{so}}_{2\ell+1} \) and a twisted form of \( \widehat{\mathfrak{gl}}_{\ell} \) are constructed. It is then demonstrated that the relation between integral recursion operators for the generalized Givental representations and degenerate \( \mathcal{D} \)-operators remains valid for all classical Lie algebras.

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Introduction

A remarkable integral representation for common eigenfunctions of $\mathfrak{gl}_{\ell+1}$-Toda chain Hamiltonian operators was proposed by Givental [1] (see also [2]). This integral representation arises naturally in a construction of a mirror dual of Type A topological closed strings on $\mathfrak{gl}_{\ell+1}$-flag manifolds. The Givental integral representation has many interesting properties. For example, it has an explicit recursive structure with respect to the rank $\ell + 1$ of the corresponding Lie algebra $\mathfrak{gl}_{\ell+1}$. The integrand in the integral representation admits a purely combinatorial description in terms of a simple graph which can be identified with the Gelfand–Zetlin graph [3], originally introduced for the description of finite-dimensional representations of $\mathfrak{gl}_{\ell+1}$. The same graphs also arise in the study of flat degenerations of flag manifolds to the Gorenstein toric Fano varieties [4], [5].
In [6], the Givental integral representation was reconsidered in the framework of the representation-theory approach to quantum integrable systems. According to Kostant [7], [8] the common eigenfunctions of $g$-Toda chain Hamiltonian operators are given by generalizations of the classical Whittaker functions [9] and can be expressed in terms of matrix elements of infinite-dimensional representations of the universal enveloping algebra $U(g)$. It was demonstrated in [6] that the Givental representation of $gl_{\ell+1}$-Toda chain eigenfunctions coincides with an integral representation of the corresponding matrix element obtained using a particular parametrization of an open part of the $gl_{\ell+1}$-flag manifold.

A conceptual explanation for the special choice of coordinates on flag manifolds can be given using a relation with the Baxter $Q$-operator formalism. In [6] it was noted that the Givental integral representation has a recursive structure connecting the $gl_\ell$- and $gl_{\ell+1}$-Whittaker functions by simple integral transformations. The corresponding integral operator coincides with a particular degeneration of the Baxter $Q$-operator for the $\hat{gl}_{\ell+1}$-Toda chain [10]. It is well known that $Q$-operators realize the quantum Bäcklund transformations in quantum integrable systems [11]. In the classical limit the $Q$-operator formalism enables one to define a special coordinate system on the phase space of the $\hat{gl}_{\ell+1}$-Toda chain. Thus, degenerate $Q$-operators determine particular coordinates on an open part of the flag manifolds $GL_{\ell+1}/B$ and lead directly to the Givental integral representation of a $gl_{\ell+1}$-Whittaker function.

Until now no generalization of the Givental integral representation of $gl_{\ell+1}$-Whittaker functions to Lie algebras other then $gl_{\ell+1}$ has been known. The only known generalization [5], [12] of the Givental construction is an integral representation for common eigenfunctions of certain degenerations of $gl_{\ell+1}$-Toda chains [13]. It is based on flat degenerations of partial flag manifolds $G/P$ for $G = GL(\ell+1, \mathbb{C})$, $P$ being a parabolic subgroup of $G$ [5].

In this paper we propose a universal construction of an integral representation of a $g$-Whittaker function for an arbitrary semisimple Lie algebra $g$. For classical Lie algebras $sp_{2\ell}$, $so_{2\ell}$, and $so_{2\ell+1}$ we are able to modify this construction to obtain a generalization of the Givental construction for classical Lie algebras. The integral representations for classical Lie algebras obtained in this way possess all the characteristic properties of the original Givental integral representation. The integral representations for the classical Lie algebras have a recursive structure. The integrands of the integral representations have combinatorial descriptions in terms of graphs. The proposed generalization to the classical Lie algebras is based on a modification of a factorized representation of generic elements of maximal unipotent subgroups of the corresponding Lie groups [14] (see also [15] and [16]). The construction of the modified factorized representation essentially uses a realization of maximal unipotent subgroups of classical Lie groups as explicitly described subgroups of upper triangular matrices (see for instance, [17]). Below we introduce the Baxter $Q$-operators associated with the classical affine Lie algebras $\hat{so}_{2\ell}$, $\hat{so}_{2\ell+1}$ and a twisted form of $\hat{gl}_{2\ell}$. We demonstrate that the relation between recursion integral operators of the generalized Givental representation and degenerate $Q$-operators remains valid for all classical Lie algebras.

A novel feature of the given integral representation is that, in contrast with the case of $gl_{\ell+1}$ (where the kernel of the recursion operator is given by a simple func-
tion), the integral kernels of the recursion operators for all other classical Lie groups are given by non-trivial integrals. This suggests that the recursion operators can be obtained as a composition of elementary operators. Indeed, for zero eigenvalues, recursion operators connecting the Toda chain eigenfunctions of Lie algebras with adjacent ranks can be represented as compositions of elementary recursion operators connecting the Toda chain eigenfunctions of different classical series.

We stress that the construction of the integral representations of $\mathfrak{g}$-Whittaker functions presented in this paper has a natural interpretation in terms of the torification of flag manifolds associated with the classical Lie groups. The graph encoding the integrand of the Givental representation for a classical Lie group can be identified with the Gelfand–Zeitlin graph for classical groups (see for instance, [18]). These graphs also explicitly describe toric degenerations of the corresponding flag manifolds (thus generalizing the results of [5] to all classical Lie groups).

The Givental-type integral representation of $\mathfrak{g}$-Whittaker functions for classical Lie algebras has a direct application to the construction of a mirror dual to Type A closed strings on flag manifolds associated with classical Lie groups $G$, $\mathfrak{g} = \text{Lie}(G)$. According to Givental [1] the mirror dual to Type A topological string theories on flag manifolds associated with Lie groups $G$ should be the Landau–Ginzburg models associated with Langlands dual Lie groups $G^\vee$ such that the generating function of the genus-zero correlators is a $\mathfrak{g}^\vee$-Whittaker function, $\mathfrak{g}^\vee = \text{Lie}(G^\vee)$. In the case $\mathfrak{g} = \mathfrak{gl}_{\ell+1}$ Givental provides a description of the superpotential of the dual Landau–Ginzburg model in terms of the integrand of the integral representation of the corresponding Whittaker function. Moreover, interpretation of the integrands of the integral representations of a $\mathfrak{gl}_{\ell+1}$-Whittaker function in terms of a torification of flag manifolds [4], [5], [19] allows a direct construction of the mirror map. We conjecture that the integrands of the integral representations for classical Lie algebra Whittaker functions have similar relations to the superpotentials of the dual Landau–Ginzburg models.

The plan of this paper is as follows. In Part 1 we propose new integral representations of $\mathfrak{g}$-Whittaker functions corresponding to an arbitrary semisimple Lie algebra $\mathfrak{g}$ with the integrands expressed in terms of the matrix elements of the fundamental representations of $\mathfrak{g}$. The corresponding assertion is given in Proposition 1.1. For the classical Lie algebras $\mathfrak{so}_{2\ell+1}$, $\mathfrak{sp}_{2\ell}$, and $\mathfrak{sp}_{2\ell}$ we then propose a modification of this construction leading to a direct generalization of the integral representation of $\mathfrak{gl}_{\ell+1}$-Whittaker functions first introduced by Givental. The main results are formulated in Theorems 1.3, 1.6, 1.10, and 1.14, respectively. In Part 2 we collect the proofs of the results presented in Part 1.

The results of this paper were first announced in [20] (see also [21] and [22]) and were presented at various conferences during the last four years.

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1. Part 1. Results

1.1. Toda chain eigenfunctions as matrix elements. The eigenfunctions of
\( g \)-Toda chains are given by particular matrix elements of infinite-dimensional rep-
resentations of a Lie algebra \( g \) [7], [8] (for details, see for instance, [23]). In this
subsection we provide integral representations of these matrix elements with inte-
grands being expressed in terms of matrix elements of finite-dimensional represen-
tations of \( g \). In the following subsections we derive explicit expressions for the
Corresponding matrix elements of finite-dimensional representations using a spe-
cial parametrization of maximal unipotent subgroups, and obtain integral repre-
sentations of \( g \)-Toda chain eigenfunctions generalizing the results of Givental for
\( g = gl_{\ell+1} \). The construction will be given for all classical Lie algebras. We start
with standard definitions in the theory of Lie algebras, mainly following [24] (for
a discussion of root data for reductive groups see, for instance, [25]).

1.1.1. Root data for reductive groups. The root data are given by a quadruple
\((X, \Phi, X^\vee, \Phi^\vee)\), where \( X \) is a lattice of a finite rank, \( X^\vee \) is the dual lattice, \( \Phi \) and
\( \Phi^\vee \) are subsets of \( X \) and \( X^\vee \) supplied with a bijection \( \alpha \mapsto \alpha^\vee \) of \( \Phi \) onto \( \Phi^\vee \), with
the following conditions. First, \( \langle \alpha, \alpha^\vee \rangle = 2 \) for any \( \alpha \in \Phi \). Second, the subsets \( \Phi \) and
\( \Phi^\vee \) should be invariant under all automorphisms \( s_\alpha, s_\alpha^\vee \),
\( s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha, \quad s_\alpha^\vee(y) = y - \langle y, \alpha \rangle \alpha^\vee, \quad x \in X, \quad y \in X^\vee, \quad \alpha \in \Phi \).

Let \( Q \subset X \) be the sublattice generated by the elements of \( \Phi \), and \( P \) the lattice
defined by
\[ P = \{ x \in X \otimes \mathbb{Q} \mid \langle x, \alpha^\vee \rangle \in \mathbb{Z}, \alpha \in \Phi \} . \]
Then \( Q \subset P \) and \( P/Q \) is a finite group. Let \( X_0 \subset X \) be the sublattice defined by
\[ X_0 = \{ x \in X \mid \langle x, y \rangle = 0, y \in \Phi^\vee \} . \]

With any reductive Lie group one can associate root data. Let \( G \) be a connected
complex reductive Lie group and \( H \subset G \) a maximal torus (Cartan subgroup).
We associate with the pair \((G, H)\) the root data \((X, \Phi, X^\vee, \Phi^\vee)\) as follows. Here
\( X \) is the free Abelian finite-rank group of characters of \( H \), and \( X^\vee = \text{Hom}(\mathbb{C}^\ast, H) \)
is the dual group of one-parameter multiplicative subgroups of \( H \). The pairing
\( \langle \cdot, \cdot \rangle: X \times X^\vee \rightarrow \mathbb{Z} \) is defined by
\[ \lambda(u(t)) = t^{\langle \lambda, u \rangle}, \quad \lambda \in X, \quad u \in X^\vee, \quad t \in \mathbb{C}^\ast. \]
Then \( \Phi \) and \( \Phi^\vee \) are finite subsets of \( X \) and \( X^\vee \), respectively, and there is a bijection
\( \alpha \mapsto \alpha^\vee \) of \( \Phi \) onto \( \Phi^\vee \).

The adjoint action of \( H \) on the Lie algebra \( g = \text{Lie}(G) \) defines the following
decomposition (root decomposition)
\[ g = h \oplus \sum_{\alpha \in \Phi} \mathbb{C}e_\alpha, \quad h = \text{Lie}(H), \]
and thus defines a subset \( \Phi \subset X \). Let \( B \) be the Borel subgroup containing \( H \). There
is a unique ordering \( > \) on \( \Phi \) such that \( b = \text{Lie}(B) \) is generated by \( h = \text{Lie}(H) \) and
\( e_\alpha \) with \( \alpha > 0 \). One fixes a basis \( \Pi = \{\alpha_i, i \in \Gamma\} \) in \( \Phi \) compatible with this ordering and associated with \( B \), where \( \Gamma \) is some set.

There is a decomposition \( G = Z_0 \cdot G' \) where \( Z_0 \) is the identity component of the centre \( Z \) of \( G \) and \( G' \) is a semisimple group (the derived group of \( G \)). We have \( H = Z_0 \cdot H' \), where \( H' \) is a maximal torus of \( G' \). The root data associated with \( (G', H') \) are \((X/X_0, \Phi, Q^\vee, \Phi^\vee)\), with \( Q \subset X/X_0 \). Given a basis \( \{\alpha_i, i \in \Gamma\} \) in \( Q^\vee \) and a basis \( \{\omega_j, j \in \widetilde{\Gamma}\} \) in \( X \), one can choose a set of representatives of the form \( \{\omega_i' = \omega_i + X_0, i \in \Gamma \subset \widetilde{\Gamma}\} \) in \( X/X_0 \) such that \( \{\omega_i', i \in \Gamma\} \) forms a basis dual to \( \{\alpha_i', i \in \Gamma\} \).

From now on, if not explicitly mentioned, let \( g \) be a semisimple Lie algebra of rank \( \ell \). Let \( \mathfrak{h} \subset g \) be a Cartan subalgebra and \( \mathfrak{b}_\pm \) a pair of opposite Borel subalgebras of \( g \) compatible with the ordering \( > \) on \( \Phi \) and such that \( \mathfrak{h} \subset \mathfrak{b}_\pm \). Then the root decomposition is given by \( g = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \), where \( \mathfrak{n}_\pm \subset \mathfrak{b}_\pm \) are the nilpotent radicals. Denote by \( \Gamma \) the set of vertices of the Dynkin diagram associated with the root system of \( g \), and let \( |\Gamma| = \dim \mathfrak{h} = \ell \). Let \( \Pi = \{\alpha_i \in \mathfrak{h}^*, i \in \Gamma\} \) be the set of simple roots, \( \{\omega_i \in \mathfrak{h}^*, i \in \Gamma\} \) the set of fundamental weights, and \( \Pi^\vee = \{\alpha_i^\vee \in \mathfrak{h}, i \in \Gamma\} \) the set of coroots defined by \( \langle \alpha_i^\vee, \omega_j \rangle = \delta_{ij} \). Let \( A = [a_{ij}], i, j = 1, \ldots, \ell \), be the Cartan matrix of \( g \) defined by \( a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle \). Denote by \( R_+ \) the set of positive roots of \( g \) and let \( \rho \) be one half of the sum of the positive roots, \( \rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha \). There are positive rational numbers \( d_1, \ldots, d_\ell \) such that the matrix \( [b_{ij}] = [d_i a_{ij}] \) is symmetric. Define a symmetric bilinear form on \( \mathfrak{h}^* \) by \( \langle \alpha_i, \alpha_j \rangle = b_{ij} \). This form defines a non-degenerate pairing \( \nu: \mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^* \), given by \( \nu(\alpha_i^\vee) = d_i^{-1} \alpha_i \).

Denote by \( e_i, f_i, h_i, i = 1, \ldots, \ell \), the set of standard generators (the Chevalley generators) of the semisimple Lie algebra \( g \): \( e_i = e_{-\alpha_i}, f_i = e_{\alpha_i}, h_i = \alpha^\vee, i = 1, \ldots, \ell \). They satisfy the following relations:

\[
[h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j, \quad [e_i, f_j] = \delta_{ij} h_i, \quad (\text{ad } e_i)^{1-a_{ij}} e_j = 0, \quad (\text{ad } f_i)^{1-a_{ij}} f_j = 0, \quad i \neq j.
\]

An invariant symmetric bilinear form on \( g \) is given by

\[
(h_i, h_j) = b_{ij} d_i^{-1} d_j^{-1}, \quad (e_i, f_j) = \delta_{ij} d_i^{-1}, \quad (e_i, h_j) = (f_i, h_j) = 0.
\]

The only example of a non-semisimple reductive Lie algebra that will be considered in this paper is the reductive Lie algebra \( \mathfrak{gl}_{\ell+1} \). In this case we explicitly define the Lie algebra by generators and relations as follows. We introduce the set of generators

\[
\{e_{i,i+1}, i = 1, \ldots, \ell; \quad e_{k,k}, \quad k = 1, \ldots, \ell + 1\}
\]

of \( \mathfrak{gl}_{\ell+1} \). They satisfy the relations

\[
[e_{i,i}, e_{j,j}] = 0, \quad [e_{i,i+1}, e_{i+1,i}] = e_{i,i} - e_{i+1,i+1},
\]

\[
[e_{i,i}, e_{i,i+1}] = e_{i,i+1}, \quad [e_{i+1,i+1}, e_{i,i+1}] = -e_{i+1,i+1},
\]

\[
[e_{i,i}, e_{i+1,i}] = -e_{i+1,i}, \quad [e_{i+1,i+1}, e_{i+1,i}] = e_{i+1,i},
\]

\[
(\text{ad } e_{i,i+1})^2 e_{j,j+1} = 0, \quad (\text{ad } e_{i+1,i})^2 e_{j+1,j} = 0, \quad |i - j| = 1.
\]
Let \( W = W(\mathfrak{g}) \) be the Weyl group of the root system associated with the Lie algebra \( \mathfrak{g} \). It is generated by simple reflections \( s_1, \ldots, s_\ell \) acting by linear transformations on \( \mathfrak{h}^* \):

\[
s_i(\lambda) = \lambda - \langle \lambda, a_i^\vee \rangle a_i, \quad \lambda \in \mathfrak{h}^*.
\]

The defining relations can be represented as

\[
s_i^2 = 1, \quad (s_is_j)^{m_{ij}} = 1, \quad i, j = 1, \ldots, \ell,
\]

where

\[
m_{ij} = 2, 3, 4, 6, \infty
\]

for

\[
a_{ij}a_{ji} = 0, 1, 2, 3, 4,
\]

respectively. For any \( w \in W \) a reduced word is a sequence of indices \( I_w = (i_1, \ldots, i_{l(w)}) \), \( i_k \in \Gamma \), of shortest possible length such that \( w = s_{i_1}s_{i_2} \cdots s_{i_{l(w)}} \). The integer \( l(w) \) is called the length of \( w \). Denote by \( w_0 \) the unique element of maximal length in the Weyl group and let \( m = l(w_0) \). In what follows we fix a lift \( \hat{w} \in G, \mathfrak{g} = \text{Lie}(G) \) of an element \( w \in W \) such that \( w(u) = \text{ad}_u w, u \in \mathfrak{g} \). For simple reflections \( s_i \) we define

\[
\hat{s}_i = e^{f_i}e^{-e_i}e^{f_i},
\]

and for \( w = s_{i_1} \cdots s_{i_{l(w)}} \) we take \( \hat{w} = \hat{s}_{i_1} \cdots \hat{s}_{i_{l(w)}} \). Thus defined, \( \hat{w} \) does not depend on the decomposition of \( w \) into a product of simple reflections (see for instance, \cite{24}, Lemma 3.8).

1.1.2. Whittaker model of principal series representations. Let \( \mathcal{U}(\mathfrak{g}) \) be a universal enveloping of \( \mathfrak{g} \) and let \( V, V' \) be \( \mathcal{U}(\mathfrak{g}) \)-modules. These modules are said to be dual if there is a non-degenerate pairing \( \langle \cdot, \cdot \rangle : V' \times V \to \mathbb{C} \) such that \( \langle v', Xv \rangle = -\langle Xv', v \rangle \) for all \( v \in V, v' \in V' \), and \( X \in \mathfrak{g} \). We will assume that the action of the Cartan subalgebra on \( V \) and \( V' \) can be integrated to an action of the maximal torus (the Cartan torus).

Let \( B_- = N_- H \) and \( B_+ = HN_+ \) be a pair of opposite Borel subgroups, where \( H \) is a maximal torus, and \( \mathfrak{n}_\pm \) are opposite maximal unipotent subgroups of \( G \). Let \( \mathfrak{h} = \text{Lie}(H) \) and \( \mathfrak{n}_\pm = \text{Lie}(N_\pm) \). The characters of \( \mathfrak{n}_\pm = \text{Lie}(N_\pm) \) are defined by their values on simple-root generators. Let \( \chi_\pm : \mathfrak{n}_\pm \to \mathbb{C} \) be the characters of \( \mathfrak{n}_\pm \) defined by \( \chi_+(f_i) := -1 \) and \( \chi_-(e_i) := -1 \) for all \( i = 1, \ldots, \ell \). A vector \( \psi_R \in V \) is called the Whittaker vector with respect to \( \chi_+ \) if

\[
f_i\psi_R = \chi_+(f_i)\psi_R = -\psi_R, \quad i = 1, \ldots, \ell,
\]

and a vector \( \psi_L \in V' \) is called the Whittaker vector with respect to \( \chi_- \) if

\[
e_i\psi_L = \chi_-(e_i)\psi_L = -\psi_L, \quad i = 1, \ldots, \ell.
\]
Consider the principal series representation \( \text{Ind}_{B_-}^G \chi_\mu \) of \( G \) induced by a character \( \chi_\mu \) of \( B_- = H N_- \) that is trivial on \( N_- \). It is realized in the space of functions \( f \in L^2(G) \) satisfying the \( B_- \)-equivariance condition
\[
f(bg) = \chi_\mu(b) f(g), \tag{1.9}
\]
where the action of \( G \) is given by the right action \((g_1 \cdot f)(g_2) = f(g_2 g_1)\). We shall be interested in the infinitesimal form \( \text{Ind}_{\mathcal{U}(g)}^{\mathfrak{g}}(g) \chi_\mu \) of this representation. The action of the Lie algebra \( \mathfrak{g} = \text{Lie}(G) \) is given by the infinitesimal form of the right action
\[
(Xf)(g) = \frac{d}{d\epsilon} f(g e^{\epsilon X}) \bigg|_{\epsilon \to 0}, \quad X \in \mathfrak{g}. \tag{1.10}
\]
Denote by \( V_\mu \) the corresponding \( \mathcal{U}(\mathfrak{g}) \)-module.

Let \( G(\mathbb{R}) \) be a totally split real form of a reductive Lie group \( G \), \( \mathfrak{g}_\mathbb{R} = \text{Lie}(G(\mathbb{R})) \) the corresponding Lie algebra, and \( N_+ (\mathbb{R}) \subset N_+ \cap G(\mathbb{R}) \) a unipotent subgroup of \( G(\mathbb{R}) \). Let \( \text{d} \mu_{G(\mathbb{R})} \) be a bi-invariant (Haar) measure on \( G(\mathbb{R}) \). We have the Bruhat decomposition \( G(\mathbb{R}) = \bigcup_{w \in W} B_- (\mathbb{R}) w B_+ (\mathbb{R}) \). Let \( G_0 (\mathbb{R}) = B_- (\mathbb{R}) N_+ (\mathbb{R}) \) be a \( w = 1 \) component in this decomposition. Up to normalization the restriction of the measure \( \text{d} \mu_{G(\mathbb{R})} \) to \( G_0 (\mathbb{R}) \) has the form [26]
\[
\text{d} \mu_{G(\mathbb{R})}(g) = \delta_{B_+ (\mathbb{R})}(b) \text{d} \mu_{B_+ (\mathbb{R})}(b) \wedge \text{d} \mu_{N_+ (\mathbb{R})}(x). \tag{1.11}
\]
Here \( \delta_{B_+ (\mathbb{R})} \) is the modular function on \( B_+ (\mathbb{R}) \). For any \( b = ng_0 \in N_+ (\mathbb{R}) H \) it is equal to \( \delta_{B_+ (\mathbb{R})}(b) = \exp \{ 2 \langle \rho, \log g_0 \rangle \} \).

Let \( \mu = i \lambda - \rho \). Consider the following non-degenerate pairing \( V_\mu \times V_\mu \to \mathbb{C} \):
\[
\langle f_1, f_2 \rangle = \int \text{d} \mu_{N_+ (\mathbb{R})}(x) f_1(x) \overline{f_2(x)},
\]
where \( \text{d} \mu_{N_+ (\mathbb{R})} \) is the restriction of \( (1.11) \) to \( N_+ (\mathbb{R}) \). On \( V_\mu \) it defines the structure of a unitary representation \( \pi_\mu \) of \( \mathcal{U}(\mathfrak{g}_\mathbb{R}) \), and we have \( \langle f_1, X f_2 \rangle = - \langle X f_1, f_2 \rangle \) for any \( X \in \mathfrak{g}_\mathbb{R} \).

We shall consider a slightly more general pairing defined as follows. Note that \( N_+ (\mathbb{R}) \subset N_+ \) is a real non-compact middle-dimensional subgroup. There is a natural complex structure on \( \mathfrak{g} = \mathfrak{g}_\mathbb{R} \otimes \mathbb{C} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1} \) which induces the complex structure on \( N_+ \). Consider the space of holomorphic functions on \( N_+ \). It is a module with respect to the action of the corresponding subalgebra of \( \mathfrak{g}^{1,0} \subset \mathfrak{g} \). The right-invariant measure \( \text{d} \mu_{N_+ (\mathbb{R})} \) can be extended to a holomorphic top-dimensional form \( \text{d} \mu_{N_+}^{\text{hol}} \) on \( N_+ \). Let \( C \subset N_+ \) be an arbitrary non-compact middle-dimensional submanifold. Consider the pairing
\[
\langle f_1, f_2 \rangle_C = \int_C \text{d} \mu_{N_+}^{\text{hol}}(x) f_1(x) \overline{f_2(x)}
\]
on the space \( \mathcal{S}^{\text{hol}}_C(N_+) \) of holomorphic functions on \( N_+ \) exponentially decreasing together with all their derivatives when restricted to \( C \). This pairing satisfies the conditions \( \langle f_1, X f_2 \rangle_C = - \langle X f_1, f_2 \rangle_C \) for any \( X \in \mathfrak{g}^{1,0} \).
1.1.3. Whittaker functions as Toda chain wave functions. According to Kostant [7], [8] the eigenfunctions of a $\mathfrak{g}$-Toda chain can be written in terms of the invariant pairing on the Whittaker modules as follows:

$$\Psi^\lambda(x) = e^{-(\rho,x)}\langle \psi_L, \pi\mu(e^{-h_x})\psi_R \rangle, \quad x \in \mathfrak{h},$$  

(1.12)

where $h_x := \sum_{i=1}^\ell (\omega_i, x)h_i$. In the special case $\mathfrak{g} = \mathfrak{sl}_2$ the function $\Psi^\lambda(x)$, $x \in \mathbb{R}$, coincides with the classical modified Bessel function of the second kind [9] (sometimes called the McDonald function). In what follows we will use the term ‘$\mathfrak{g}$-Whittaker function’ for (1.12) (see, for instance, [23]). A slightly different notion of the Whittaker functions was used in [27], [28].

One can introduce a set of commuting differential operators $\mathcal{H}_k \in \text{Diff}(\mathfrak{h})$, $k = 1, \ldots, \ell$, corresponding to a set $\{c_k, k = 1, \ldots, \ell\}$ of generators of the centre $\mathfrak{Z}(\mathfrak{g}) \subset \mathcal{W}(\mathfrak{g})$ as follows:

$$\mathcal{H}_k \Psi^\lambda(x) = e^{-(\rho,x)}\langle \psi_L, \pi\mu(e^{-h_x})c_k\psi_R \rangle.$$

(1.13)

The quadratic generator of the centre $\mathfrak{Z}(\mathfrak{g})$ (Casimir element) is given by

$$c_2 = \frac{1}{2} \sum_{i,j=1}^\ell c_{ij}h_i h_j + \frac{1}{2} \sum_{\alpha \in R^+} (e_\alpha f_\alpha + f_\alpha e_\alpha),$$

(1.14)

where the matrix $\|c_{ij}\| = \|d_i d_j (b^{-1})_{ij}\|$ is inverse to the matrix $\|(\alpha_i, \alpha_j)\|$.

Let $\{\epsilon_i, i = 1, \ldots, \ell\}$ be an orthonormal basis $((\epsilon_i, \epsilon_j) = \delta_{ij})$ in $\mathfrak{h}$ and let $x = \sum_{i=1}^\ell x_i \epsilon_i$ be a decomposition of $x \in \mathfrak{h}$ in this basis. Then the projection (1.13) of (1.14) gives the well-known $\mathfrak{g}$-Toda chain Hamiltonian operator (see, for example, [13])

$$\mathcal{H}^\mathfrak{g}_2 = -\frac{1}{2} \sum_{i=1}^\ell \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^\ell d_i e^{(\alpha_i, x)}.$$  

(1.15)

The operators $\mathcal{H}_k$ provide a complete set of commuting $\mathfrak{g}$-Toda chain Hamiltonians [7].

The $\mathfrak{g}$-Toda chain eigenfunctions (1.12) are in an abstract form. To get explicit integral representations we write the matrix elements (1.12) of infinite-dimensional representations in terms of matrix elements of finite-dimensional representations of $\mathcal{W}(\mathfrak{g})$. Let $\pi_i, i = 1, \ldots, \ell$, be the set of fundamental representations corresponding to all the fundamental weights $\omega_i, i = 1, \ldots, \ell$, of $\mathfrak{g}$, and let $\xi^+_i, i = 1, \ldots, \ell$, be the highest/lowest vectors in these representations determined by the conditions $n_\pm \xi_i^\pm = 0, i = i, \ldots, \ell$, and normalized so that $\langle \xi^-_i | \xi^+_i \rangle = 1$. For the highest weight vector $\xi^+_i$ in a fundamental representation on $V_{\omega_i}$ we have $s^{-1}_i \xi^+_i = e_i \xi^+_i$. Let us consider the following matrix elements in fundamental (finite-dimensional) representations:

$$\Delta_{\omega_i, w}(g) = \langle \xi^-_i | \pi_i(g) \pi_i(\hat{w}) | \xi^+_i \rangle, \quad w \in W, \quad g \in G,$$

(1.16)

for $i = 1, \ldots, \ell$. 




Lemma 1.1. The left/right Whittaker vectors defined by (1.8) and (1.7) are given by:

$$
\psi_R(v) = \exp\left\{ t \sum_{i=1}^{\ell} \frac{\Delta_{\omega_i, s_i^{-1}}(v)}{\Delta_{\omega_i, 1}(v)} \right\}, \tag{1.17}
$$

$$
\psi_L(v) = \prod_{i=1}^{\ell} \left( \Delta_{\omega_i, w_0^{-1}}(v) \right)^{i(\lambda, \alpha_i^\vee) - 1} \cdot \exp\left\{ \sum_{i=1}^{\ell} \frac{\Delta_{\omega_i, w_0^{-1}}(v)}{\Delta_{\omega_i, 1}(v)} \right\}. \tag{1.18}
$$

The proof is given in Part 2, §2.2.

Proposition 1.1. Let $\mathfrak{g}$ be any semisimple Lie algebra. The common $\mathfrak{g}$-Toda chain eigenfunctions (1.12) can be represented in the following integral form:

$$
\Psi^g_{\lambda}(x) = e^{t(\lambda, x)} \int_{C} d\mu_{N_+}^{\text{hol}}(v) \prod_{i=1}^{\ell} \left( \Delta_{\omega_i, w_0^{-1}}(v) \right)^{i(\lambda, \alpha_i^\vee) - 1} \times \exp\left\{ \sum_{i=1}^{\ell} \left( \frac{\Delta_{\omega_i, w_0^{-1}}(v)}{\Delta_{\omega_i, 1}(v)} - e^{(\alpha_i, x)} \frac{\Delta_{\omega_i, s_i^{-1}}(v)}{\Delta_{\omega_i, 1}(v)} \right) \right\}, \tag{1.19}
$$

where $\Delta_{\omega_i, w}(g)$ is given by (1.16) and $C \subset N_+$ is a middle-dimensional non-compact cycle such that the integrand decreases exponentially at the boundary and at infinity. The measure for the integration is the restriction to $C$ of the holomorphic continuation $d\mu_{N_+}^{\text{hol}}$ of the right-invariant measure $d\mu_{N_+}(\mathbb{R})$ on $N_+(\mathbb{R})$.

The first example of this type of integral representation for a $\mathfrak{gl}_n$-Whittaker function was considered in [29]. Its generalization given above is straightforward. The proof of Proposition 1.1 is given in Part 2, §2.2.

The expression (1.19) for the Whittaker function is much more detailed than (1.12) but does not yet provide an explicit integral representation. To obtain explicit integral representations of the Whittaker functions one needs to choose a parametrization of $N_+$ (or an open part of it) and express the measure $d\mu_{N_+}^{\text{hol}}$ and the various matrix elements in (1.19) in terms of the coordinates on $N_+$. A natural choice would be a factorized representation of the elements of an open part of a maximal unipotent subgroup of an arbitrary Lie group [14] (see also [16], [15]). For each $i \in \Gamma$ let $X_i(t) = \exp\{tf_i\}$ be a one-parameter subgroup in $N_+$. For the longest element $w_0$ in the Weyl group $W$ we pick a decomposition corresponding to a reduced word $I_{m_0} = (i_1, \ldots, i_m)$, $l(w_0) = m = \dim N_+$. Then the map

$$
\mathbb{C}^m \rightarrow N_+^{(0)}, \quad (t_1, \ldots, t_m) \mapsto v(t_1, \ldots, t_m) = X_{i_1}(t_1) \cdots X_{i_m}(t_m), \tag{1.20}
$$

is a birational isomorphism. This provides a so-called factorized parametrization of an open part $N_+^{(0)}$ of $N_+$. Parameterizations corresponding to different choices of the reduced word $I_{m_0}$ are related by birational transformations described explicitly by Lusztig [14]. The right-invariant measure has the following description in the factorized representation.
Lemma 1.2. The right-invariant measure $d\mu_{N_+}^{\text{hol}}$ in the factorized parametrization is given by

$$d\mu_{N_+}^{\text{hol}}(v) = \prod_{i=1}^\ell \Delta_{\omega_i,w_0^{-1}}(v) \prod_{k=1}^m \frac{dt_k}{t_k}. \tag{1.21}$$

The proof is given in Part 2, §2.1.

Thus, the problem of finding explicit integral representations of Whittaker functions in the factorized parametrization (1.20) is reduced to a calculation of the matrix elements of finite-dimensional representations of $\mathfrak{g}$ in this parametrization. In what follows we provide explicit expressions for finite-dimensional matrix elements for classical Lie groups (2.2) and give corresponding integral representations of the Whittaker functions (see Theorems 1.1, 1.4, 1.8, and 1.12). We stress, however, that the integral representation for $\mathfrak{g} = \mathfrak{gl}_{\ell+1}$ obtained in this way does not coincide with the Givental representation [1]. Note that for classical series of Lie algebras the factorized parametrization (1.20) has a recursive structure with respect to the rank $\ell$, reflecting the recursive structure of the reduced decomposition of $w_0 \in W$. This recursive structure does not, however, translate into a simple recursive structure for the matrix elements of the infinite-dimensional representations in the factorized parametrization and does not reproduce the recursive structure of the Givental integral representation.

In [6] a modification of the factorized parametrization (1.20) for $\mathfrak{g} = \mathfrak{gl}_{\ell+1}$ was proposed, and it was shown that the integral representation (1.19) in this parametrization reproduces the Givental integral representation of $\mathfrak{gl}_{\ell+1}$-Whittaker functions exactly. In particular, for this parametrization the recursive structure of the reduced decomposition of $w_0 \in W(\mathfrak{g})$ translates directly into the recursive structure of the integral representation of the corresponding Whittaker function.

Below we generalize the results of [6] to all classical series of Lie algebras. We propose a modification of the factorized parametrization (1.20) based on a particular realization of maximal unipotent subgroups $N_+ \subset G$ of classical Lie groups as explicitly defined subgroups of the maximal unipotent subgroups of general linear groups. For any classical simple Lie group, the maximal unipotent subgroup can be realized as a subgroup of a group of upper triangular matrices of appropriate size with 1s on the diagonal (see, for instance, [17]). The corresponding subset of upper triangular matrices for a classical Lie group can be described explicitly. We define a parametrization of maximal unipotent subgroups of classical Lie groups by constructing a special form of parametrization of the corresponding subset of upper triangular matrices. Using this parametrization, we derive explicit integral representations of the Whittaker functions associated with all classical groups and demonstrate that these integral representations have all the characteristic properties of the Givental integral representation for $\mathfrak{gl}_{\ell+1}$-Whittaker functions. In particular, the recursive structure of the Whittaker functions is explicit in this new parametrization.

1.2. Integral representations of $\mathfrak{gl}_{\ell+1}$- and $\mathfrak{sl}_{\ell+1}$-Toda chain eigenfunctions. In this subsection we recall a construction of integral representations of $\mathfrak{gl}_{\ell+1}$- and $\mathfrak{sl}_{\ell+1}$-Toda chain eigenfunctions using a factorized parametrization (1.20) of an open part $N_+^{(0)}$ of the maximal unipotent subgroup $N_+ \subset \text{GL}(\ell + 1)$ and its
and the eigenfunction must satisfy the equations of quantum integrable system having a set of orthonormal basis in \( R \). The coroots \( \alpha_i^\vee \) can be identified with the corresponding roots \( \alpha_i \) with respect to the pairing in \( R^{\ell+1} \). With this root/weight system one associates a \( gl_{\ell+1} \)-Toda quantum integrable system having a set of \( \ell+1 \) pairwise commuting functionally independent quantum Hamiltonians \( H^{\ell+1}_k \), \( k = 1, \ldots, \ell+1 \). We are interested in the explicit integral representations for common eigenfunctions of the complete set of quantum Hamiltonian operators for \( gl_{\ell+1} \). For instance, the linear and quadratic quantum Hamiltonians of a \( gl_{\ell+1} \)-Toda chain are given by
\[
H^{\ell+1}_1 = -i \sum_{i=1}^{\ell+1} \frac{\partial}{\partial x_i},
\]
\[
H^{\ell+1}_2 = -\frac{1}{2} \sum_{i=1}^{\ell+1} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{\ell+1} e^{x_i-1-x_i},
\]
and the eigenfunction must satisfy the equations
\[
H^{\ell+1}_1 (x) \Psi^{\ell+1}_{x,\ldots,x}(x_1, \ldots, x_{\ell+1}) = \sum_{i=1}^{\ell+1} \lambda_i \Psi^{\ell+1}_{x,\ldots,x}(x_1, \ldots, x_{\ell+1}),
\]
\[
H^{\ell+1}_2 (x) \Psi^{\ell+1}_{x,\ldots,x}(x_1, \ldots, x_{\ell+1}) = \frac{1}{2} \sum_{i=1}^{\ell+1} \lambda_i^2 \Psi^{\ell+1}_{x,\ldots,x}(x_1, \ldots, x_{\ell+1}).
\]
A common eigenfunction of the quantum Hamiltonians has the following representation as a matrix element:
\[
\Psi^{\ell+1}_{x}(x) = e^{\sum x_i \rho_i} \langle \psi_L, \pi_\mu (e^{\sum x_i E_i}) \psi_R \rangle,
\]
where \( \rho_i = -(\ell - 2i + 2)/2, i = 1, \ldots, \ell+1 \), are the components of \( \rho \) in the standard basis \( \{ \epsilon_i, i = 1, \ldots, \ell+1 \} \) in \( R^{\ell+1} \).

The construction for the semisimple Lie algebra \( sl_{\ell+1} \) is quite similar to that for the reductive Lie algebra \( gl_{\ell+1} \). The roots and fundamental weights for the semisimple Lie algebra \( sl_{\ell+1} \) can be written in the form (see, for instance, [30])
\[
\alpha_i = \epsilon_i+1 - \epsilon_i, \quad \omega_i = - (\epsilon_1 + \cdots + \epsilon_i) + \frac{i}{\ell+1} (\epsilon_1 + \cdots + \epsilon_{\ell+1})
\]
for \( i = 1, \ldots, \ell \). This representation of the \( A_\ell \) root/weight system can be obtained from the root/weight system of the reductive Lie algebra \( gl_{\ell+1} \) as follows. Let us pick an orthogonal basis of fundamental weights for \( gl_{\ell+1} \),
\[
\omega'_i = -\epsilon_1 - \cdots - \epsilon_i,
\]
such that $\langle \omega'_i, \alpha^\vee_j \rangle = \delta_{i,j}$ for $i, j = 1, \ldots, \ell$ and $\langle \omega'_{\ell+1}, \alpha^\vee_j \rangle = 0$ for $j = 1, \ldots, \ell$. Then $\omega'_{\ell+1}$ can be considered as a generator of $X_0$. Introducing

$$\omega_i = \omega'_i - \frac{i}{\ell+1} \omega'_{\ell+1},$$

one readily obtains the set (1.28) of fundamental weights for $\mathfrak{sl}_{\ell+1}$.

With this root data one associates an $\mathfrak{sl}_{\ell+1}$-Toda quantum integrable system possessing a set of $\ell$ pairwise commuting functionally independent Hamiltonians $\mathcal{H}_k^{\mathfrak{sl}_{\ell+1}}, k = 1, \ldots, \ell$. However, it is convenient to consider $\mathfrak{sl}_{\ell+1}$-Toda chain Hamiltonians as a subset $\mathcal{H}_k^{\mathfrak{gl}_{\ell+1}}, k = 2, \ldots, (\ell+1)$, of $\mathfrak{gl}_{\ell+1}$-Toda chain Hamiltonians acting on the kernel of the linear Hamiltonian $\mathcal{H}_1^{\mathfrak{gl}_{\ell+1}}$. For instance, an eigenfunction of a quadratic quantum Hamiltonian of a $\mathfrak{sl}_{\ell+1}$-Toda chain must satisfy the equation

$$\mathcal{H}_2^{\mathfrak{sl}_{\ell+1}} \Psi^\mathfrak{sl}_{\ell+1}(x_1, \ldots, x_{\ell+1}) = \left\{ -\frac{1}{2} \sum_{i=1}^{\ell+1} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{\ell} e^{x_{i+1}-x_i} \right\} \Psi^\mathfrak{sl}_{\ell+1}(x_1, \ldots, x_{\ell+1})$$

$$= \frac{1}{2} \sum_{i=1}^{\ell+1} \lambda_i^2 \Psi^\mathfrak{sl}_{\ell+1}(x_1, \ldots, x_{\ell+1})$$

(1.29)

with the additional constraint $\lambda_1 + \cdots + \lambda_{\ell+1} = 0$. The eigenfunctions for an $\mathfrak{sl}_{\ell+1}$-Toda chain can also be written using the following set of variables

$$\Psi^\mathfrak{sl}_{\ell+1}(y_1, \ldots, y_{\ell}) = \Psi^\mathfrak{sl}_{\ell+1}(x_1, \ldots, x_{\ell+1}), \quad \nu_i = \lambda_{i+1} - \lambda_i, \quad y_i = x_{i+1} - x_i.$$  

(1.30)

Note that without imposing the constraint $\lambda_1 + \cdots + \lambda_{\ell+1} = 0$ the eigenfunctions of the $\mathfrak{sl}_{\ell+1}$-Toda chain can be expressed in terms of the eigenfunctions of a $\mathfrak{gl}_{\ell+1}$-Toda chain as

$$\Psi_{\nu_1, \ldots, \nu_{\ell}}(y_1, \ldots, y_{\ell}) = \exp \left\{ -\frac{i}{\ell+1} \sum_{i=1}^{\ell+1} \nu_i \sum_{i=1}^{\ell+1} x_i \right\} \Psi^\mathfrak{gl}_{\ell+1}(x_1, \ldots, x_{\ell+1}),$$

(1.31)

where we use the notation of (1.30). In what follows we will mainly consider $\mathfrak{gl}_{\ell+1}$-Toda chain eigenfunctions, making comments on the corresponding modifications for the $\mathfrak{sl}_{\ell+1}$ case (and mainly using the non-reduced form $\Psi^\mathfrak{gl}_{\ell+1}(x)$).

1.2.1. $\mathfrak{gl}_{\ell+1}$-Whittaker function: factorized parametrization. To make the integral representation (1.19) for $\mathfrak{gl}_{\ell+1}$-Whittaker functions explicit one needs to pick a particular parametrization of $N_+ \subset \text{GL}(\ell+1)$. Let $w_0$ be the element of maximal length in the Weyl group $W = S_{\ell+1}$ of $\mathfrak{gl}_{\ell+1}$. Consider a reduced decomposition of $w_0$ corresponding to the following reduced word $I_\ell$:

$$I_\ell = (i_1, i_2, i_3, \ldots, i_{m_\ell}) := (1, 21, 321, \ldots, (\ell \ldots 21)), \quad m_\ell = \frac{\ell(\ell+1)}{2}.$$
Note that the reduced word $I_\ell$ has an obvious recursive structure: $I_\ell = I_{\ell-1} \sqcup (\ell \ldots 21)$. Thus, the corresponding parametrization of unipotent elements $v^{(\ell)}$ in an open part $N_+^{(0)}$ of $N_+$ can be written in a recursive form:

$$v^{(\ell)} = v^{(\ell-1)} x^{\ell}_{\ell-1},$$

where

$$x^{\ell}_{\ell-1} = X_{\ell}(y_{\ell,1}) \cdots X_2(y_{2,\ell-1})X_1(y_{1,\ell})$$

and $X_i(y) = \exp(y f_i)$. The parameters $y_{ik}$ of one-parameter subgroups will be called factorization parameters. The action of $\mathfrak{gl}_{\ell+1}$ on $G/B_-$ considered in §1.1.2 defines an action of the Lie algebra on the space $V_\mu$ of functions restricted to $N_+^{(0)}$.

**Proposition 1.2.** The following differential operators define a realization of the representation $\pi_\mu$ of $\mathcal{U}(\mathfrak{gl}_{\ell+1})$ on $V_\mu$ in terms of the factorized parametrization (1.32), (1.33):

$$E_{i,i+1} = \frac{\partial}{\partial y_{i,\ell+1-i}} + \sum_{k=1}^\ell \sum_{s=0}^{k-1} y_{i-1-s,\ell+2-i} \frac{\partial}{\partial y_{i-1-k,\ell+1-i}} - \frac{\partial}{\partial y_{i-1-k,\ell+2-i}} \},$$

$$E_{i,i} = \mu_i - \sum_{k=1}^{\ell+1-i} y_{i,k} \frac{\partial}{\partial y_{i,k}} + \sum_{k=1}^{\ell-2-i} y_{i-1,k} \frac{\partial}{\partial y_{i-1,k}},$$

$$E_{i+1,i} = \sum_{n=1}^{\ell+1-i} y_{i,n} \left\{ \mu_i - \mu_{i+1} - y_{i,n} \frac{\partial}{\partial y_{i,n}} - \sum_{j=1}^\ell \sum_{k=1}^{n-1+\delta_{i,j+1}} a_{ij} y_{j,k} \frac{\partial}{\partial y_{j,k}} \right\},$$

where $E_{i,j} = \pi_\mu(e_{i,j})$, $\mu_k = i\lambda_k - \rho_k$ and $\rho_k = -(\ell - 2k + 2)/2$ for $k = 1, \ldots, \ell + 1$.

The proof is given in Part 2, §2.4.1.

The calculation of the matrix elements appearing in the integral (1.19) in the factorized parametrization (1.32), (1.33) can be done following [16] and [15] (see section 2.3 for details). Another, more straightforward, approach is to find left and right Whittaker vectors by solving the equations (1.7), (1.8) directly. In what follows we will use the conventions that $\sum_{i=k}^j = 0$ when $k > j$ and $\prod_{i=k}^j = 1$ when $k > j$.

Recall that according to (1.7), (1.8) the Whittaker vectors are the solutions to the equations

$$E_{i,i+1} \psi_R = -\psi_R, \quad E_{i+1,i} \psi_L = -\psi_L, \quad i = 1, \ldots, \ell.$$
Lemma 1.3. The following expressions hold for the left/right Whittaker vectors in terms of the factorization parameters:

\[
\psi_R(y) = \exp \left\{ -\sum_{i=1}^{\ell} \sum_{n=1}^{\ell+1-i} y_{i,n} \right\},
\]

(1.35)

\[
\psi_L(y) = \prod_{i=1}^{\ell} \left( \prod_{k=1}^{\ell} \prod_{n=1}^{i} y_{k+1-k,n} \right)^{\mu_{i+1} - \mu_i} \times \exp \left\{ -\sum_{k=1}^{\ell} \frac{1}{y_{\ell+1-k,k}} \left( 1 + \sum_{n=1}^{\ell} \prod_{i=1}^{\ell} y_{\ell+1-k-i,k+1} \right) \right\}.
\]

(1.36)

Taking into account (1.21), we arrive at the following expression for the \( gl_{\ell+1} \)-Whittaker function in the factorized parametrization.

Theorem 1.1. Eigenfunctions of the \( gl_{\ell+1} \)-Toda chain (1.27) admit the integral representation

\[
\Psi_{\lambda_1, \ldots, \lambda_{\ell+1}}^{gl_{\ell+1}}(x_1, \ldots, x_{\ell+1}) = e^{i \sum_{k=1}^{\ell+1} \lambda_k x_k} \int_{C} \prod_{i=1}^{\ell} \prod_{n=1}^{\ell+1-i} \frac{dy_{i,n}}{y_{i,n}} \prod_{i=1}^{\ell} \left( \prod_{k=1}^{\ell} \prod_{n=1}^{i} y_{k+1-k,n} \right)^{i(\lambda_{i+1} - \lambda_i)} \times \exp \left\{ -\sum_{k=1}^{\ell} \frac{1}{y_{\ell+1-k,k}} \left( 1 + \sum_{n=1}^{\ell} \prod_{i=1}^{\ell} y_{\ell+1-k-i,k+1} \right) \right\}.
\]

(1.37)

Here \( C \subset \mathbb{N}_+ \) is a middle-dimensional non-compact submanifold such that the integrand decreases exponentially at the boundary and at infinity. In particular, one can take for \( C \) a slightly deformed subspace \( \mathbb{R}_{+}^{\ell(\ell+1)/2} \) or \( \mathbb{C}_{+}^{\ell(\ell+1)/2} \) making the integral (1.37) convergent.

The proof is given in Part 2, §2.3.1.

1.2.2. \( gl_{\ell+1} \)-Whittaker function: modified factorized parametrization. Now we consider a modification of the factorized parametrization (1.32), (1.33) leading to the Givental integral representation of a \( gl_{\ell+1} \)-Whittaker function. This modified factorized parametrization was first introduced in [6]. We stress that there is an important difference between the factorized and the modified factorized parameterizations. Note that the parametrization (1.32), (1.33) is defined in terms of elements of the subgroup \( N_+ \). To define a modified factorized parametrization of \( N_+ \) we shall consider the image of a group element in a faithful finite-dimensional representation of the group. In the case of \( gl_{\ell+1} \) and \( sl_{\ell+1} \) we use the tautological representation \( \pi : gl_{\ell+1} \rightarrow \text{End}(\mathbb{C}^{\ell+1}) \). Let \( \epsilon_{i,j} \) be the elementary \( (\ell + 1) \times (\ell + 1) \) matrix with 1 at the \((i, j)\) position and zeros elsewhere. We consider the diagonal matrices

\[
U_k = \sum_{i=1}^{k} e^{-x_{k,i}} \epsilon_{i,i} + \sum_{i=k+1}^{\ell+1} \epsilon_{i,i}
\]

(1.38)
and introduce the following upper triangular deformation of $U_k$:

$$
\tilde{U}_k = \sum_{i=1}^{k} e^{-x_{k,i}} \epsilon_{i,i} + \sum_{i=k+1}^{\ell+1} \epsilon_{i,i} + \sum_{i=1}^{k-1} e^{-x_{k-1,i}} \epsilon_{i,i+1}.
$$

The modified factorized parametrization of $N_+$ is then defined as follows.

**Theorem 1.2.** (i) The image of a generic unipotent element $v \in N_+$ in the tautological representation $\pi : \mathfrak{gl}_{\ell+1} \to \text{End}(\mathbb{C}^{\ell+1})$ can be represented in the form

$$
\pi(v) = \tilde{U}_2 U_2^{-1} \tilde{U}_3 U_3^{-1} \cdots \tilde{U}_\ell U_\ell^{-1} \tilde{U}_{\ell+1},
$$

where it is assumed that $x_{\ell+1,i} = 0$, $i = 1, \ldots, \ell + 1$.

(ii) The expression (1.40) defines a parametrization of the totally non-negative part $N_+^{(0)} \subset N_+$.

**Proof.** Let $v(y)$ be elements of $N_+$ parameterized according to (1.32), (1.33). Let us now change the variables in the following way:

$$
y_{i,n} = e^{x_{n+i,i+1} - x_{n+i-1,i}}, \quad 1 \leq n \leq i \leq \ell,
$$

where it is assumed that $x_{\ell+1,n} = 0$, $n = 1, \ldots, \ell + 1$. One can easily check that after the change of variables the image $\pi(v)$ of $v$ defined by (1.32), (1.33) transforms into (1.40). Taking into account that the change of variables (1.41) is invertible, we obtain a parametrization of $N_+^{(0)} \subset N_+$. □

Applying the change of variables (1.41) to the expressions in Proposition 1.2, one obtains the realization in the modified factorized parametrization.

**Proposition 1.3.** The following differential operators define a realization of the representation $\pi_\mu$ of $\mathfrak{gl}_{\ell+1}$ on $V_\mu$ in terms of the modified factorized parametrization (1.38)–(1.40) of $N_+$:

$$
E_{i,i} = \mu_i - \sum_{k=1}^{i-1} \frac{\partial}{\partial x_{i+1+k-i,k}} + \sum_{k=i}^{\ell} \frac{\partial}{\partial x_{k,i}},
$$

$$
E_{i,i+1} = -\sum_{n=1}^{i} e^{x_{\ell-i+n,n} - x_{\ell-i+1,n,n}} \sum_{k=1}^{n} \left\{ \frac{\partial}{\partial x_{\ell-k+k,k}} - \frac{\partial}{\partial x_{\ell-k+k,k-1}} \right\},
$$

$$
E_{i+1,i} = \sum_{n=1}^{\ell+1-i} e^{x_{n+i,i+1} - x_{k+i-1,i}} \left[ \mu_i - \mu_{i+1} + \sum_{k=1}^{n} \left\{ \frac{\partial}{\partial x_{i+k-1,i}} - \frac{\partial}{\partial x_{i+k-1,i+1}} \right\} \right],
$$

where $E_{i,j} = \pi_\mu(e_{i,j})$, and it is assumed that $x_{\ell+1,i} = 0$, $i = 1, \ldots, \ell + 1$.

This realization of the principal series representation of $\mathfrak{gl}_{\ell+1}$ by differential operators is based on a particular parametrization of the maximal unipotent subgroup $N_+$ appearing in the Gauss decomposition of the group $\text{GL}(\ell + 1)$ and was inspired by the Givental integral formula. In [6] we coined the term ‘Gauss–Givental representation’ for this realization of the principal series representation. Applying the change of variables (1.41) to the expressions in Lemma 1.3, one obtains Whittaker vectors in the modified factorized parametrization.
Lemma 1.4. The following expressions hold for the left/right Whittaker vectors:

\[
\psi_R(x) = \exp \left\{ - \sum_{i=1}^{\ell} \sum_{n=1}^{\ell+i-1} e^{x_{n+i,i+1} - x_{n+i-1,i}} \right\},
\]

\[
\psi_L(x) = \exp \left\{ \sum_{k=1}^{\ell} \sum_{i=1}^{k} (\mu_k - \mu_{k+1}) x_{k,i} \right\} \exp \left\{ - \sum_{i=1}^{\ell} \sum_{k=1}^{\ell+i-1} e^{x_{k+i-1,k} - x_{k+i,k}} \right\},
\]

where \( x_{\ell+1,i} = 0 \) for \( i = 1, \ldots, \ell + 1 \).

Now we are ready to write the integral representation of the pairing (1.27) using the modified factorized parametrization. Going from (1.27) to (1.19), and then to (1.37), we act by the Cartan torus element \( e^{\sum x_i E_{ii}} \) to the right in (1.27). A different choice (for example, action to the left) leads to a transformation of the integrand by adding a total derivative to it. The choice of the diagonal group element made in (1.19), (1.37) is not the most symmetric one. One of the special features of the Gauss–Givental representation is that up to a simple exponential factor in \( \psi_L(x) \) the left and right Whittaker vectors (1.43) are very similar (unlike in the case of the factorized parametrization (1.17), (1.18)). We would like to retain this symmetry in the integrand of the integral representation. Let us represent the Cartan subgroup element \( h_x \) in the following way:

\[
e^{h_x} = e^{H_L} e^{H_R},
\]

where

\[
e^{h_x} = e^{\sum x_i E_{ii}} = \exp \left\{ \sum_{i=1}^{\ell} x_{\ell+1,i} \left( \mu_i - \sum_{k=1}^{i-1} \frac{\partial}{\partial x_{\ell+1+k-i,k}} + \sum_{k=i}^{\ell} \frac{\partial}{\partial x_{k,i}} \right) \right\},
\]

\[
e^{H_L} = \exp \left\{ \sum_{i=1}^{\ell+1} x_{\ell+1,i} \sum_{k=i}^{\ell} \frac{\partial}{\partial x_{k,i}} \right\},
\]

\[
e^{H_R} = \exp \left\{ \sum_{i=1}^{\ell+1} x_{\ell+1,i} \mu_i - \sum_{i=1}^{\ell} x_{\ell+1,i} \sum_{k=1}^{\ell} \frac{\partial}{\partial x_{\ell+1+k-i,k}} \right\}.
\]

In the calculation of the matrix element we will choose the differential operator \( H_L \) acting on the left vector and \( H_R \) acting on the right Whittaker vector in (1.27). In this way we obtain the following integral formula for eigenfunctions of the \( \mathfrak{gl}_{\ell+1} \)-Toda chain.

Theorem 1.3. Eigenfunctions of the \( \mathfrak{gl}_{\ell+1} \)-Toda chain (1.27) admit the integral representation

\[
\Psi_{\lambda_1, \ldots, \lambda_{\ell+1}}(x_1, \ldots, x_{\ell+1}) = \int_C \prod_{k=1}^{\ell} dx_k e^{\mathcal{F}_{\ell+1}(x)},
\]

where \( x_k = (x_{k,1}, \ldots, x_{k,k}) \), \( k = 1, \ldots, \ell \), and the function \( \mathcal{F}_{\ell+1}(x) \) is given by

\[
\mathcal{F}_{\ell+1}(x) = \ell \sum_{n=1}^{\ell+1} \lambda_n \left( \sum_{i=1}^{n-1} x_{n,i} - \sum_{i=1}^{n-1} x_{n-1,i} \right) - \sum_{k=1}^{\ell} \sum_{i=1}^{k} (ex_{k+1,i} - ex_{k+1,i} + ex_{k+1,i} - ex_{k+1,i}).
\]
Here $x_i = x_{\ell+1,i}$, $i = 1, \ldots, \ell+1$, and $C \subset N_+$ is a middle-dimensional non-compact submanifold such that the integrand decreases exponentially at the boundary and at infinity. In particular, $C$ can be chosen to be a slightly deformed subspace $\mathbb{R}(\ell+1)/2 \subset \mathbb{C}(\ell+1)/2$ making the integral (1.47) convergent.

Note that the above integral formula (1.47), (1.48) can also be obtained directly from (1.37) by the substitution

$$y_{i,n} = e^{x_{\ell+1,i} - x_{\ell+1,i+1}} e^{x_{n+i+1} - x_{n+i-1,i}}, \quad i = 1, \ldots, \ell, \quad n = 1, \ldots, \ell + 1 - i.$$  

(1.49)

**Corollary 1.1.** The $\mathfrak{sl}_{\ell+1}$-Whittaker function has the following integral representation:

$$
\Psi_{\lambda_2 - \lambda_1, \ldots, \lambda_{\ell+1} - \lambda_{\ell}}(x_2 - x_1, \ldots, x_{\ell+1} - x_{\ell}) = \exp \left\{ \frac{\ell}{\ell + 1} \sum_{i=1}^{\ell+1} x_i \left( \ell \lambda_i - \sum_{j \neq i} \lambda_j \right) \right\} \int_C \prod_{k=1}^{\ell} dx_k e^{x_{k+1} - x_k}(x),
$$

where

$$
\mathcal{F}_{\mathfrak{sl}_{\ell+1}}(x) = \sum_{k=1}^{\ell} \left( \lambda_{k+1} - \lambda_k \right) \left( x_{k+1,k} - x_{k,i} \right) - \sum_{k=1}^{\ell} \sum_{i=1}^{k+1} \left( e^{x_{k,i} - x_{k+1,i}} + e^{x_{k+1,i} - x_{k,i}} \right)
$$

and it is assumed that $x_k = x_{\ell+1,k}$, $k = 1, \ldots, \ell + 1$.

**Example 1.1.** Let $x_1 = x_{2,1}$, $x_2 = x_{2,2}$. In the case $\ell = 1$ the expression (1.47), (1.48) becomes

$$
\Psi_{\lambda_1, \lambda_2}(x_1, x_2) = \int_{\mathbb{R}} dx_{1,1} \exp \left\{ i \lambda_2(x_1 + x_2 - x_{1,1}) + i \lambda_1 x_{1,1} - e^{x_{1,1} - x_1} - e^{x_2 - x_{1,1}} \right\}.
$$

Here the $\mathfrak{sl}_2$-Whittaker function (1.50) coincides with the modified (or hyperbolic) Bessel function of the second kind [9]:

$$
\Psi_{\lambda_2 - \lambda_1}(x_2 - x_1) = e^{\frac{\ell}{2}(\lambda_2 - \lambda_1)(x_1 + x_2)} \int_{\mathbb{R}} dx_{1,1} e^{i(\lambda_1 - \lambda_2)x_{1,1} - e^{x_{1,1} - x_1} - e^{x_2 - x_{1,1}}} = \int_{\mathbb{R}} dx_{1,1} \exp \left\{ i(\lambda_1 - \lambda_2)x_{1,1} - e^{\frac{x_2 - x_1}{2}}(e^{x_{1,1}} + e^{-x_{1,1}}) \right\} = K_{i(\lambda_1 - \lambda_2)}(2e^{\frac{x_2 - x_1}{2}}).$

(1.50)

The integral representation (1.47), (1.48) of the $\mathfrak{g}_{\ell+1}$-Toda chain eigenfunctions was first obtained by Givental in his study of quantum cohomology of the $\mathfrak{g}_{\ell+1}$-flag manifold [1] (see also [2]). The description of the Givental integral formula in terms of the matrix element (1.27) was first obtained in [6].
According to Givental the function $F^{gl_{\ell+1}}(x)$ for $\lambda_k = 0, k = 1, \ldots, \ell+1$, admits a simple description in terms of the following diagram:

![Diagram](image)

(1.51)

We assign variables $x_{k,i}$ to vertices $(k,i)$ and functions $e^{y-x}$ to arrows $(x \rightarrow y)$ of the diagram (1.51). Then the potential function $F^{gl_{\ell+1}}(x)$ (1.48) at the zero spectrum $\lambda_i = 0$ is given by the sum of the functions assigned to all the arrows.

As demonstrated in Theorem 1.2, the variables $\{x_{k,i}, 1 \leq i \leq k \leq \ell\}$ provide a parametrization of an open part $N^{(0)}_+$ of the flag manifold $X = SL(\ell+1,\mathbb{C})/B$. The non-compact manifold $N^{(0)}_+$ has a natural action of the torus $T^{(\omega_0)}$ and can be compactified to a Gorenstein toric variety. The set of the relations defining this compactification can be described as follows. We introduce the new variables

$$a_{k,i} = e^{x_{k,i}-x_{k+1,i}}, \quad b_{k,i} = e^{x_{k+1,i+1}-x_{k,i}}, \quad 1 \leq i \leq k \leq \ell,$$

assigned to arrows of the diagram (1.51). Then the following defining relations hold:

$$a_{k,i}b_{k,i} = b_{k+1,i}a_{k+1,i+1}, \quad 1 \leq i \leq k < \ell,$$

$$a_{\ell,i}b_{\ell,i} = e^{x_{\ell+1,i+1}-x_{\ell+1,i}}, \quad i = 1, \ldots, \ell.$$  

(1.52)

They can be interpreted as relations between various compositions of elementary paths having the same initial and final vertexes. The set of relations between more general paths (following from (1.52)) provides a toric embedding of the flag manifold degeneration (see [5] and [19] for details). We also note that the diagram (1.51) coincides with the Gelfand–Zetlin diagram for $gl_{\ell+1}$.

1.2.3. Relations with $\hat{gl}_{\ell+1}$-Toda chain Baxter $\mathcal{B}$-operator. The integral representation (1.47), (1.48) of the $gl_{\ell+1}$-Whittaker function has a recursive structure with respect to the rank $\ell +1$ of the Lie algebra. Indeed, this integral representation can be rewritten in the following form:

$$\Psi^{\hat{gl}_{\ell+1}}_{\lambda_1, \ldots, \lambda_{\ell+1}}(x_{\ell+1}) = \int_C \prod_{k=1}^{\ell} dx_k \prod_{k=1}^{\ell+1} Q^{\hat{gl}_k}_{\lambda_k}(x_k, x_{k-1}; \lambda_k),$$

(1.53)
where

\[
Q_{gl_k}^{\ell+1}(x_{k+1}; x_k; \lambda_{k+1}) = \exp \left\{ i\lambda_{k+1} \left( \sum_{i=1}^{k+1} x_{k+1,i} - \sum_{i=1}^{k} x_{k,i} \right) - \sum_{i=1}^{k} (e^{x_{k,i} - x_{k+1,i}} + e^{x_{k+1,i+1} - x_{k,i}}) \right\}.
\]  

Here we use the notation \( x_k = (x_{k,1}, \ldots, x_{k,k}) \) and assume that \( Q_{gl_0}^{l+1} = e^{i\lambda_1 x_{1,1}} \).

Let us choose linear coordinates \( x_k = (x_{k,1}, \ldots, x_{k,k}) \) in \( C_k \). Let \( C_k \) be a non-compact middle-dimensional submanifold in \( C_k \) such that (1.54) as a function of \( x_k \) decreases exponentially at possible boundaries and at infinity in \( C_k \). Consider the integral operator

\[
(Q_{gl_k}^{l+1} f)(x_{k+1}) = \int_{C_k} Q_{gl_k}^{l+1}(x_{k+1}; x_k; \lambda_{k+1}) f(x_k) \, dx_k
\]

acting on functions which do not grow too rapidly at possible boundaries and at infinity in \( C_k \). The integral operators \( Q_{gl_k}^{l+1} \) provide a recursive construction of \( gl_{\ell+1} \)-Whittaker functions:

\[
\Psi_{\lambda_1,\ldots,\lambda_{\ell+1}}^{gl_{\ell+1}}(x_{\ell+1}) = \int_{\mathcal{C}_{\ell}} \, dx_{\ell} \, Q_{gl_{\ell}}^{l+1}(x_{\ell+1}; x_{\ell}; \lambda_{\ell+1}) \Psi_{\lambda_1,\ldots,\lambda_{\ell}}^{gl_{\ell}}(x_{\ell}).
\]  

There is a natural oriented path in the diagram (1.51) which can be associated with the recursion operator \( Q_{gl_{\ell}}^{l+1} \):

\[
\begin{align*}
x_{\ell+1,1} & \quad x_{\ell+1,2} & \quad \ldots & \quad x_{\ell+1,\ell+1} \\
x_{\ell,1} & \quad \ldots & \quad x_{\ell,\ell}
\end{align*}
\]  

(1.56)

The diagram (1.51) can be considered as a collection of the oriented paths (1.56), and thus the recursive structure of the integral representation (1.47) is encoded in (1.51) in an obvious way.

Following (1.55) the integral operators \( Q_{gl_k}^{l+1} \) with kernels \( Q_{gl_k}^{l+1}(x_{k+1}; x_k; \lambda_{k+1}) \) satisfy braiding relations with the quantum Toda chain Hamiltonians. For example, the following relation holds between the quadratic Hamiltonians \( \mathcal{H}_2^{gl_{k+1}}(x_{k+1}) \) and \( \mathcal{H}_2^{gl_k}(x_k) \):

\[
\mathcal{H}_2^{gl_{k+1}}(x_{k+1}) Q_{gl_k}^{l+1}(x_{k+1}; x_k; \lambda_{k+1}) = Q_{gl_k}^{l+1}(x_{k+1}; x_k; \lambda_{k+1}) \mathcal{H}_2^{gl_k}(x_k) + \frac{1}{2} \lambda_{k+1}^2.
\]  

In the relation above and similar ones we shall assume that Hamiltonian operators on the left-hand side act to the right and Hamiltonians on right-hand side act to
the left. Similar braiding relations hold for higher quantum Hamiltonian operators (see [6] for details).

The recursion operators $Q_{\mathfrak{gl}_k}^{\mathfrak{gl}_{k+1}}$ appear to be related to an important object in the theory of quantum integrable systems, the $\mathcal{Q}$-operator. A particular instance of the $\mathcal{Q}$-operator was introduced by Baxter [31] to solve a class of quantum integrable models. In the case of a (periodic) $\mathfrak{gl}_{\ell+1}$-Toda chain with the quadratic Hamiltonian

$$\mathcal{H}_{2}^{\mathfrak{gl}_{\ell+1}} = -\frac{1}{2} \sum_{i=1}^{\ell+1} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{\ell} e^{x_{i+1} - x_i} + g e^{x_{1} - x_{\ell+1}}, \quad (1.58)$$

where $g$ is an arbitrary coupling constant, the $\mathcal{Q}$-operator has the following integral kernel:

$$\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(x^{(\ell+1)}, y^{(\ell+1)}; \lambda) = \exp \left\{ i \lambda \sum_{i=1}^{\ell+1} (x_i - y_i) \right\} - \left( \sum_{i=1}^{\ell} \left( e^{y_i - x_i} + e^{x_{i+1} - y_i} \right) + e^{y_{\ell+1} - x_{\ell+1}} + g e^{x_{1} - y_{\ell+1}} \right), \quad (1.59)$$

Here we use the notation $x^{(\ell+1)} = (x_1, \ldots, x_{\ell+1})$ and $y^{(\ell+1)} = (y_1, \ldots, y_{\ell+1})$. This $\mathcal{Q}$-operator was first constructed in [10]. It commutes with all the $\mathfrak{gl}_{\ell+1}$-operators and generates quantum Bäcklund transformations [10]. For instance, for the quadratic Hamiltonians we have:

$$\mathcal{H}_{2}^{\mathfrak{gl}_{\ell+1}}(x^{(\ell+1)}, \mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(x^{(\ell+1)}, y^{(\ell+1)}; \lambda) = \mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(x^{(\ell+1)}, y^{(\ell+1)}; \lambda) \mathcal{H}_{2}^{\mathfrak{gl}_{\ell+1}}(y^{(\ell+1)}). \quad (1.60)$$

To establish a relation between the Baxter $\mathcal{Q}$-operator for the $\mathfrak{gl}_{k+1}$-Toda theory and a recursion operator for the $\mathfrak{gl}_{k+1}$-Toda theory it is useful to introduce a slightly modified recursion operator $Q_{\mathfrak{gl}_{k} \oplus \mathfrak{gl}_{1}}^{\mathfrak{gl}_{k+1}}$ with the kernel

$$Q_{\mathfrak{gl}_{k} \oplus \mathfrak{gl}_{1}}^{\mathfrak{gl}_{k+1}}(x^{(k+1)}, y^{(k+1)}; \lambda) = e^{-i \lambda y_{k+1}} Q_{\mathfrak{gl}_{k}}^{\mathfrak{gl}_{k+1}}(x^{(k+1)}, y^{(k)}; \lambda)$$

$$= \exp \left\{ i \lambda \left( \sum_{i=1}^{k+1} x_i - \sum_{i=1}^{k+1} y_i \right) - \sum_{i=1}^{k} \left( e^{y_i - x_i} + e^{x_{i+1} - y_i} \right) \right\}, \quad (1.61)$$

where $x^{(k+1)} = (x_1, \ldots, x_{k+1})$, $y^{(k)} = (y_1, \ldots, y_k)$, and $y^{(k+1)} = (y_1, \ldots, y_k, y_{k+1})$. This modified operator intertwines the Hamiltonian operators of $\mathfrak{gl}_{k+1}$- and $\mathfrak{gl}_{k} \oplus \mathfrak{gl}_{1}$-Toda chains (the new variable $y_{k+1}$ appears only in a description of the $\mathfrak{gl}_{1}$-Toda chain). Thus, for the quadratic Hamiltonian operators we have

$$\mathcal{H}_{2}^{\mathfrak{gl}_{k+1}}(x^{(k+1)}) Q_{\mathfrak{gl}_{k} \oplus \mathfrak{gl}_{1}}^{\mathfrak{gl}_{k+1}}(x^{(k+1)}, y^{(k+1)}; \lambda)$$

$$= Q_{\mathfrak{gl}_{k} \oplus \mathfrak{gl}_{1}}^{\mathfrak{gl}_{k+1}}(x^{(k+1)}, y^{(k+1)}; \lambda) \left( \mathcal{H}_{2}^{\mathfrak{gl}_{k}}(y^{(k)}) + \mathcal{H}_{2}^{\mathfrak{gl}_{1}}(y_{k+1}) \right),$$

where $\mathcal{H}_{2}^{\mathfrak{gl}_{1}}(y_{k+1}) = -\frac{1}{2} \frac{\partial^2}{\partial y_{k+1}^2}$. Obviously, the projection of the above relation on the subspace of functions $F(y^{(k)}, y_{k+1}) = e^{-i \lambda y_{k+1}} f(y^{(k)})$ leads to (1.57).
Now consider the one-parameter family of integral operators
\[
\mathcal{Q}^\mathfrak{gl}_{\ell+1} \left( x^{(k+1)}, y^{(k+1)}; \lambda, \varepsilon \right) = \varepsilon^{-1} \lambda \exp \left\{ \sum_{i=1}^{\ell+1} (x_i - y_i) \right. \\
- \left( \sum_{i=1}^{\ell} (e^{y_i-x_i} + e^{x_{i+1}-y_i}) + \varepsilon e^{y_{i+1}-x_{i+1}} + \varepsilon^{-1} g e^{x_i-y_{i+1}} \right) \right\}, \quad (1.62)
\]

obtained from (1.59) by the shift of variable \( y_{\ell+1} \rightarrow y_{\ell+1} + \log \varepsilon \). The limiting behaviour of (1.60) when \( \varepsilon \rightarrow 0, g \varepsilon^{-1} \rightarrow 0 \) can be described as follows:
\[
Q^\mathfrak{gl}_{\ell+1} \left( x^{(k+1)}, y^{(k+1)}; \lambda \right) = \lim_{\varepsilon \rightarrow 0, \quad g \varepsilon^{-1} \rightarrow 0} \varepsilon^{1/2} \mathcal{Q}^\mathfrak{gl}_{\ell+1} \left( x^{(k+1)}, y^{(k+1)}; \lambda; \varepsilon \right). \quad (1.63)
\]

This provides a relation between the Baxter \( \mathcal{Q} \)-operator and the (modified) recursion operator.

### 1.3. Integral representations of \( \mathfrak{so}_{2\ell+1} \)-Toda chain eigenfunctions

Now consider the one-parameter family of integral operators
\[
\mathcal{Q}^\mathfrak{gl}_{\ell+1} \left( x^{(k+1)}, y^{(k+1)}; \lambda, \varepsilon \right) = \varepsilon^{-1} \lambda \exp \left\{ \sum_{i=1}^{\ell+1} (x_i - y_i) \right. \\
- \left( \sum_{i=1}^{\ell} (e^{y_i-x_i} + e^{x_{i+1}-y_i}) + \varepsilon e^{y_{i+1}-x_{i+1}} + \varepsilon^{-1} g e^{x_i-y_{i+1}} \right) \right\}, \quad (1.62)
\]

This provides a relation between the Baxter \( \mathcal{Q} \)-operator and the (modified) recursion operator.

Consider a \( B_\ell \)-type root system corresponding to the Lie algebra \( \mathfrak{so}_{2\ell+1} \). Let \( \{\epsilon_1, \ldots, \epsilon_\ell\} \) be an orthonormal basis in \( \mathbb{R}^{\ell} \). We realize \( B_\ell \) roots, coroots, and fundamental weights as vectors in \( \mathbb{R}^{\ell} \) in the following way:
\[
\alpha_1 = \epsilon_1, \quad \alpha_1^\vee = 2 \epsilon_1, \quad \omega_1 = \frac{1}{2} (\epsilon_1 + \cdots + \epsilon_\ell), \\
\alpha_2 = \epsilon_2 - \epsilon_1, \quad \alpha_2^\vee = \epsilon_2 - \epsilon_1, \quad \omega_2 = \epsilon_2 + \cdots + \epsilon_\ell, \quad (1.64)
\]
\[
\ldots \\
\alpha_\ell = \epsilon_\ell - \epsilon_{\ell-1}, \quad \alpha_\ell^\vee = \epsilon_\ell - \epsilon_{\ell-1}, \quad \omega_\ell = \epsilon_\ell.
\]
The Cartan matrix is then given by \( a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle \), and the positive rational numbers \( d_1 = 1/2, d_2 = 1, \ldots, d_\ell = 1 \) are such that the matrix \( \| b_{ij} \| = \| d_i a_{ij} \| \) is symmetric. One associates with these data a quantum Toda chain with a quadratic Hamiltonian
\[
\mathcal{H}^B \left( x^{(k+1)}, y^{(k+1)}; \lambda, \varepsilon \right) = \varepsilon^{-1} \lambda \exp \left\{ \sum_{i=1}^{\ell+1} (x_i - y_i) \right. \\
- \left( \sum_{i=1}^{\ell} (e^{y_i-x_i} + e^{x_{i+1}-y_i}) + \varepsilon e^{y_{i+1}-x_{i+1}} + \varepsilon^{-1} g e^{x_i-y_{i+1}} \right) \right\}, \quad (1.62)
\]

One can complete (1.65) to a complete set of \( \ell \) pairwise commuting functionally independent Hamiltonians \( \mathcal{H}^B_k, k = 1, \ldots, \ell \), for a \( \mathfrak{so}_{2\ell+1} \)-Toda chain. We are looking for an integral representation of a common eigenfunction of \( \mathcal{H}^B_k \). The eigenfunction problem for the quadratic Hamiltonian can be written in the following form:
\[
\mathcal{H}^B \left( x^{(k+1)}, y^{(k+1)}; \lambda, \varepsilon \right) = \varepsilon^{-1} \lambda \exp \left\{ \sum_{i=1}^{\ell+1} (x_i - y_i) \right. \\
- \left( \sum_{i=1}^{\ell} (e^{y_i-x_i} + e^{x_{i+1}-y_i}) + \varepsilon e^{y_{i+1}-x_{i+1}} + \varepsilon^{-1} g e^{x_i-y_{i+1}} \right) \right\}, \quad (1.62)
\]
1.3.1. so\(_{2\ell+1}\)-Whittaker function: factorized parametrization. A reduced word for the element \(w_0\) of maximal length in the Weyl group of \(B_\ell\) type can be represented in the recursive form

\[
I_\ell = (i_1, i_2, i_3, \ldots, i_{m_\ell}) := (1, 212, 32123, \ldots, (\ell \ldots 212 \ldots \ell)), \quad m_\ell = \ell^2,
\]

where the indices \(i_k \in \Gamma = \{1, \ldots, \ell\}\) correspond to elementary reflections with respect to simple roots \(\alpha_{i_k}\). Let \(N_+ \subset G\) be a maximal unipotent subgroup of \(G = SO(2\ell + 1)\). One associates with the reduced word \(I_\ell\) the following recursive parametrization of a generic unipotent element \(v^{B_\ell} \in N_+\):

\[
v^{B_\ell} = v^{B_{\ell-1}} \mathcal{X}^{B_\ell}_{B_{\ell-1}},
\]

where

\[
\mathcal{X}^{B_\ell}_{B_{\ell-1}} = X_\ell(y_{\ell,1}) \cdots X_k (y_{k,2(\ell+1-k)-1}) \cdots X_2 (y_{2,2\ell-3}) \times X_1(y_{1,\ell}) X_2(y_{2,2\ell-2}) \cdots X_k(y_{k,2(\ell+1-k)}) X_\ell(y_{\ell,2}).
\]

Here \(X_i(y) = e^{y f_i}\) and \(f_i \equiv f_{\alpha_i}\) are simple root generators. The subset \(N_+^{(0)}\) of elements admitting such a representation is an open part of \(N_+\). The action of the Lie algebra \(so_{2\ell+1}\) on \(N_+^{(1.10)}\) considered at the beginning of \(\S 1.1.2\) defines an action of the Lie algebra on \(N_+^{(0)}\). The following proposition explicitly describes this action on the space \(V_\mu\) considered as a space of equivariant functions on \(N_+^{(0)}\).

**Proposition 1.4.** The following differential operators determine a realization of a principal series representation \(\pi_\mu\) of \(\mathcal{W}(so_{2\ell+1})\) in terms of a factorized parametrization of \(N_+^{(0)}\):

\[
F_1 = \frac{\partial}{\partial y_{1,\ell}} + \sum_{n=1}^{\ell-1} \left\{ \prod_{j=n}^{\ell-1} \frac{y_{2,2i}}{y_{2,2i-1}} \left( \frac{\partial}{\partial y_{1,n}} - \frac{\partial}{\partial y_{1,n+1}} \right) + 2 \frac{y_{2,2(n-1)}}{y_{1,n}} \prod_{i=n+1}^{\ell-1} \frac{y_{2,2i}}{y_{2,2i-1}} \left( \frac{\partial}{\partial y_{2,2n-1}} - \frac{\partial}{\partial y_{2,2n}} \right) \right\},
\]

\[F_k = \frac{\partial}{\partial y_{k,2(\ell+1-k)}} + (1 - \delta_{k,\ell}) \sum_{n=1}^{\ell-k} \left\{ \prod_{i=n}^{\ell-k} \frac{y_{k+1,2i}}{y_{k+1,2i-1}} \frac{y_{k,2(i+1)-1}}{y_{k,2(i+1)}} \left( \frac{\partial}{\partial y_{k,2n}} - \frac{\partial}{\partial y_{k,2n+1}} \right) + \frac{y_{k+1,2n}}{y_{k,2(n+1)}} \prod_{i=n+1}^{\ell-k} \frac{y_{k+1,2i}}{y_{k+1,2i-1}} \frac{y_{k,2(i+1)-1}}{y_{k,2(i+1)}} \right\}, \quad k = 2, \ldots, \ell,
\]

\[H_k = \langle \mu, \alpha_k^{\vee} \rangle + \sum_{i=1}^{\ell} a_{k,i} \sum_{j=1}^{n_i} y_{i,j} \frac{\partial}{\partial y_{i,j}}, \quad 1 \leq k \leq \ell,
\]
\[ E_1 = \sum_{n=1}^{\ell} y_{1,n} \left\{ -\langle \mu, \alpha_1^\vee \rangle + \sum_{j=1}^{2n-3} 2y_{2,j} \frac{\partial}{\partial y_{2,j}} - 2 \sum_{j=1}^{n-1} y_{1,j} \frac{\partial}{\partial y_{1,j}} - y_{1,n} \frac{\partial}{\partial y_{1,n}} \right\}, \]

\[ E_2 = \sum_{n=1}^{2(\ell-1)} y_{2,n} \left\{ -\langle \mu, \alpha_2^\vee \rangle + \sum_{j=1}^{\alpha(n)+1} y_{1,j} \frac{\partial}{\partial y_{1,j}} - 2 \sum_{j=1}^{n-1} y_{2,j} \frac{\partial}{\partial y_{2,j}} \right\} + \sum_{j=1}^{2(\beta(n)-3)} y_{3,j} \frac{\partial}{\partial y_{3,j}} - y_{2,n} \frac{\partial}{\partial y_{2,n}} \right\}, \]

\[ E_k = \sum_{n=1}^{2(\ell+1-k)} y_{k,n} \left\{ -\langle \mu, \alpha_k^\vee \rangle + \sum_{j=1}^{2(\alpha(n)+2)} y_{k-1,j} \frac{\partial}{\partial y_{k-1,j}} - 2 \sum_{j=1}^{n-1} y_{k,j} \frac{\partial}{\partial y_{k,j}} \right\} + \sum_{j=1}^{2(\beta(n)-3)} y_{k+1,j} \frac{\partial}{\partial y_{k+1,j}} - y_{k,n} \frac{\partial}{\partial y_{k,n}} \right\}, \]

where \( \pi_\mu(e_i) = E_i, \pi_\mu(f_i) = F_i, \pi_\mu(h_i) = H_i, \) \( i = 1, \ldots, \ell, \) for \( 1 \leq k \leq \ell, \) \( a_{ij} \) is a Cartan matrix, and the terms containing \( y_{i,j} \) with the indexes not in the set \( \{1 \leq i, j \leq \ell\} \) are assumed to be omitted. The following notation is also used: \( \alpha(n) := [n/2], \beta(n) := [(n + 1)/2]. \)

For the proof see Part 2, § 2.4.2.

The left/right Whittaker vectors are given by (1.7), (1.8), and in the factorized parametrization they have the following expressions.

**Lemma 1.5.** The left/right Whittaker vectors are given by

\[ \psi_R(y) = \exp \left\{ -\left( \sum_{n=1}^{\ell} y_{1,n} + \sum_{k=1}^{n_k} \sum_{n=1}^{\ell} y_{k,n} \right) \right\}; \]

\[ \psi_L(y) = \left( \prod_{n=1}^{\ell} y_{1,n} \prod_{i=2}^{n_k} y_{i,2n-1} \right)^{\langle \mu, \alpha_1^\vee \rangle} \]

\[ \times \left( \prod_{k=2}^{\ell} \prod_{n=2}^{k} y_{2,n} \prod_{i=k+1}^{n_k} y_{i,2n-1} \prod_{i=2}^{n_k} y_{i,2n-1} y_{i,2n} \right)^{\langle \mu, \alpha_k^\vee \rangle} \]

\[ \times \exp \left\{ -\left( \sum_{n=1}^{\ell} \frac{1}{y_{1,n}} \left( 1 + \frac{y_{2,2(n-1)}}{y_{2,2(n-1)}-1} \right) \prod_{i=n+1}^{\ell} \frac{y_{2,2(i-1)}}{y_{2,2(i-1)}-1} \right) \right\} \]

\[ + \sum_{k=2}^{n_k/2} \sum_{n=1}^{\ell} \frac{1}{y_{k,2n}} \left( 1 + \frac{y_{k+1,2(n-1)}}{y_{k+1,2(n-1)}-1} \right) \prod_{i=n+1}^{\ell} \frac{y_{k+1,2(i-1)}}{y_{k+1,2(i-1)}-1} \right\}. \]

where \( n_1 = \ell \) and \( n_k = 2(\ell + 1 - k), \) \( k = 2, \ldots, \ell. \)

For the proof see Part 2, § 2.3.2.

Using (1.12) and (1.21), we obtain an integral representation of \( \mathfrak{so}_{2\ell+1} \)-Whittaker functions in the factorized parametrization.
Theorem 1.4. The eigenfunctions (1.12) of the $\mathfrak{so}_{2\ell+1}$-Toda chain admit the following integral representation:

\[
\Psi_{\lambda_1, \ldots, \lambda_\ell}(x_1, \ldots, x_\ell) = e^{\lambda_1 x_1 + \cdots + \lambda_\ell x_\ell} \int C \prod_{i=1}^\ell \prod_{k=1}^{n_i} dy_{i,k} \left( \prod_{n=1}^\ell y_{1,n} \prod_{i=2}^{\ell} \prod_{n=1}^{n_i/2} y_{i,2n-1} \prod_{i=2}^{\ell} \prod_{n=1}^{n_i/2} y_{i,2n-1} y_{i,2n} \right)^{2\lambda_1} \\
\times \prod_{k=2}^\ell \left( \prod_{n=2}^\ell y_{1,n}^2 \prod_{i=k+1}^{n_i/2} y_{i,2n-1} \prod_{i=2}^{\ell} \prod_{n=1}^{n_i/2} y_{i,2n-1} y_{i,2n} \right)^{i(\lambda_k - \lambda_{k-1})} \\
\times \exp \left\{ - \left( \sum_{n=1}^{\ell} \frac{1}{y_{1,n}} \left( 1 + \frac{y_{2,2(n-1)}}{y_{2,2(n-1)} - 1} \right) \prod_{i=n+1}^{\ell} \frac{y_{2,2(i-1)}}{y_{2,2(i-1)} - 1} \right) \sum_{k=2}^{\ell} \sum_{n=1}^{n_k/2} \frac{1}{y_{k,2n}} \left( 1 + \frac{y_{k+1,2(n-1)}}{y_{k+1,2(n-1)} - 1} \right) \prod_{i=n+1}^{n_k/2} \frac{y_{k+1,2(i-1)}}{y_{k+1,2(i-1)} - 1} \frac{y_{k,2i-1}}{y_{k,2i}} \right\} \\
+ e^{x_1} \sum_{n=1}^{\ell} y_{1,n} + \sum_{k=2}^{\ell} e^{x_{k-x_{k-1}}} \sum_{n=1}^{n_k} y_{k,n} \right\},
\]

(1.74)

where $n_1 = \ell$, $n_k = 2(\ell + 1 - k)$, $k = 2, \ldots, \ell$, and $C \subset N_+$ is a middle-dimensional non-compact submanifold such that the integrand decays exponentially at the boundary and at infinity. In particular, one can choose $C$ to be a slightly deformed subspace $\mathbb{R}^{4\ell}_+$ of $\mathbb{C}^{4\ell}$ making the integral convergent.

The proof is given in Part 2, § 2.3.2.

Example 1.2. Let $\ell = 2$. In this case, the general formula (1.74) acquires the form

\[
\Psi_{\lambda_1, \lambda_2}(x_1, x_2) = e^{\lambda_1 x_1 + \lambda_2 x_2} \int C \prod_{i,k=1}^2 dy_{i,k} (y_{1,1} y_{2,1} y_{1,2})^{2\lambda_1} (y_{2,1} y_{1,2} y_{2,2})^{\lambda_2 - \lambda_1} \\
\times \exp \left\{ - \left( \frac{1}{y_{1,1}} + \frac{1}{y_{2,2}} \left( \frac{1}{y_{1,1}} + \frac{1}{y_{1,2}} \right) - \frac{1}{y_{2,2}} \right) \right\} \\
- e^{x_1} (y_{1,1} + y_{1,2}) - e^{x_2 - x_1} (y_{2,1} + y_{2,2}) \right\},
\]

(1.75)

where one can choose $C$ to be a slightly deformed subspace $\mathbb{R}^4_+$ of $\mathbb{C}^4$ making the integral convergent.

1.3.2. $\mathfrak{so}_{2\ell+1}$-Whittaker function: modified factorized parametrization. In this part we introduce a modified factorized parametrization of $N_+ \subset \mathrm{SO}(2\ell + 1)$. We use this parametrization to construct the integral representations for $\mathfrak{so}_{2\ell+1}$-Whittaker functions. In contrast to the integral representations described above, these integral representations have a simple recursive structure with respect to the rank $\ell$ and can be described in purely combinatorial terms using suitable graphs. Thus, these representations can be considered as a generalization of the Givental integral representation to the case $\mathfrak{so}_{2\ell+1}$.

There is a realization of the tautological representation $\pi: \mathfrak{so}_{2\ell+1} \to \mathrm{End}(\mathbb{C}^{2\ell+1})$ such that the Chevalley generators corresponding to the Borel (Cartan) subalgebra
of $\mathfrak{so}_{2\ell+1}$ are realized by upper triangular (respectively, diagonal) matrices. This defines an embedding $\mathfrak{so}_{2\ell+1} \subset \mathfrak{gl}_{2\ell+1}$ such that the Borel (Cartan) subalgebra is mapped into the Borel (Cartan) subalgebra (see, for instance, [17]). To define the corresponding embedding of the groups, consider the following involution on $GL(2\ell + 1)$:

$$g \mapsto g^* := \hat{W}_0^{-1}(g^{-1})^t \hat{W}_0,$$

where $a^t$ is induced by the standard transposition of a matrix $a$ and $\hat{W}_0$ is a lift of the maximal length element $w_0$ in the Weyl group $S_{2\ell+1}$ of $\mathfrak{gl}_{2\ell+1}$. In matrix form this can be written as

$$\pi(\hat{W}_0) = SJ,$$

where $S = \text{diag}(1, -1, \ldots, 1, -1, 1)$ and $J = ||J_{i,j}|| = ||\delta_{i,j,2\ell+2}||$. The orthogonal group $G = \text{SO}(2\ell + 1)$ can then be defined as the following subgroup of $GL(2\ell + 1)$:

$$\text{SO}(2\ell + 1) = \{ g \in GL(2\ell + 1) : g^* = g \}.$$

Let $\epsilon_{i,j}$ be the elementary $(2\ell + 1) \times (2\ell + 1)$ matrix with 1 at the $(i, j)$ position and zeros elsewhere. We introduce the following matrices:

$$U_n = \sum_{i=1}^{\ell-n}(\epsilon_{i,i} + \epsilon_{2\ell+2-i,2\ell+2-i}) + \sum_{i=1}^{n}\epsilon_{\ell-n+i,\ell-n+i} + e^{-z_{n,1}}\epsilon_{\ell+1,\ell+1}$$

$$+ e^{-z_{n,1}}\epsilon_{\ell+2,\ell+2} + \sum_{i=1}^{n-1}e^{-z_{n,i+1}}\epsilon_{\ell+1+i,\ell+1+i},$$

(1.77)

$$\tilde{U}_n = U_n + \sum_{i=1}^{n-1}(1 + \delta_{i,1})e^{-x_{n-1,i}}\epsilon_{\ell+1+i,\ell+2+i}$$

(1.78)

and

$$V_n = \sum_{i=1}^{\ell-n}(\epsilon_{i,i} + \epsilon_{2\ell+2-i,2\ell+2-i}) + \sum_{i=1}^{n}e^{x_{n,i}}\epsilon_{\ell-1+i,\ell-1+i} + e^{-z_{n,1}}\epsilon_{\ell+1,\ell+1}$$

$$+ \sum_{i=1}^{n}\epsilon_{\ell+i+1,\ell+i+1},$$

(1.79)

$$\tilde{V}_n = V_n + 2e^{-z_{n,1}}\epsilon_{\ell,\ell+1} + \sum_{i=1}^{n-1}e^{z_{n,i+1}}\epsilon_{\ell-i,\ell+1+i}$$

(1.80)

for $n = 2, \ldots, \ell$, and

$$Z_k = \sum_{i=1}^{\ell-1}(\epsilon_{i,i} + \epsilon_{2\ell+2-i,2\ell+2-i}) + \epsilon_{\ell+1,\ell+1} + e^{-z_{n,1}}\epsilon_{\ell,\ell} + e^{-z_{n,1}}\epsilon_{\ell+2,\ell+2},$$

(1.81)

$$R_k = \sum_{i=1}^{2\ell+1}\epsilon_{i,i} + 2e^{2z_{n,1}}\epsilon_{\ell,\ell+2}$$

(1.82)
for $k = 1, \ldots, \ell$, and
\[
U_1 = \sum_{i=1}^{\ell-1} \epsilon_{i,i} + e^{-z_{1,1}} \epsilon_{\ell,\ell} + e^{z_{1,1}} \epsilon_{\ell+1,\ell+1} + \sum_{i=\ell+2}^{2\ell+1} \epsilon_{i,i},
\]
(1.83)
\[
\tilde{U}_1 = \sum_{i=1}^{\ell-1} \epsilon_{i,i} + e^{-z_{1,1}} \epsilon_{\ell,\ell} + e^{z_{1,1}} \epsilon_{\ell+1,\ell+1} + \sum_{i=\ell+2}^{2\ell+1} \epsilon_{i,i} + 2e^{x_{1,1}} \epsilon_{\ell,\ell+1},
\]
(1.84)
\[
V_1 = \sum_{i=1}^{\ell} \epsilon_{i,i} + e^{-z_{1,1}} \epsilon_{\ell+1,\ell+1} + e^{z_{1,1}} \epsilon_{\ell+2,\ell+2} + \sum_{i=\ell+3}^{2\ell+1} \epsilon_{i,i},
\]
(1.85)
\[
\tilde{V}_1 = U_1^* = \sum_{i=1}^{\ell} \epsilon_{i,i} + e^{-z_{1,1}} \epsilon_{\ell+1,\ell+1} + e^{z_{1,1}} \epsilon_{\ell+2,\ell+2} + \sum_{i=\ell+3}^{2\ell+1} \epsilon_{i,i} + 2e^{x_{1,1}} \epsilon_{\ell+1,\ell+2},
\]
(1.86)

where it is assumed that $x_{\ell,k} = 0$, $k = 1, \ldots, \ell$. Note that $\tilde{V}_i$, $\tilde{U}_i$ can be considered as off-diagonal deformations of $V_i$, $U_i$. Now we can define a modified factorized representation for $N_+ \subset \text{SO}(2\ell + 1)$.

**Theorem 1.5.** (i) The image of a generic unipotent element $v^{B_\ell} \in N_+$ in the tautological representation $\pi: \mathfrak{so}_{2\ell+1} \rightarrow \text{End}(\mathbb{C}^{2\ell+1})$ can be represented in the form
\[
\pi(v^{B_\ell}) = \mathfrak{X}_1 \mathfrak{X}_2 \cdots \mathfrak{X}_\ell, \quad \mathfrak{X}_n = \pi(\mathfrak{X}_{B_{n-1}}^{B_n}), \quad n = 1, \ldots, \ell,
\]
(1.87)

where
\[
\mathfrak{X}_1 = \tilde{U}_1 U_1^{-1} Z_1 R_1 Z_1^{-1}\tilde{V}_1 V_1^{-1},
\]
\[
\mathfrak{X}_n = \tilde{U}_n U_n^{-1} Z_n R_{n-1} Z_n^{-1}[\tilde{U}_n U_n^{-1}]^* \cdot [\tilde{V}_n V_n^{-1}]^* Z_n R_n Z_n^{-1}\tilde{V}_n V_n^{-1} \quad \text{for } n = 2, \ldots, \ell,
\]
(1.88)

where it is assumed that $x_{\ell,k} = 0$, $k = 1, \ldots, \ell$.

(ii) The expressions (1.87) and (1.88) define a parametrization of the totally non-negative part $N^{(0)}_+ \subset N_+$.

**Proof.** Let $v^{B_\ell}(y)$ be a parametrization of an open part of $N_+$ according to (1.67), (1.68). Let $\tilde{X}_k(y) = e^{y_{\ell+1-k, \ell+2-k}}$, $k = 2, \ldots, \ell$, be unipotent elements in $\text{GL}(2\ell + 1)$ corresponding to simple roots $\alpha_1, \ldots, \alpha_\ell$. The action of the involution $^*$ is given by $\tilde{X}_k(y)^* = \tilde{X}_{\ell-k}(y)$. We then embed an elementary unipotent element $X_i(y)$ of $\text{SO}(2\ell + 1)$ into $\text{GL}(2\ell + 1)$ as follows:
\[
X_1(y) = e^{2y_{\ell,\ell+1}} e^{2y_{\ell+1,\ell+2}}, \quad X_k(y) = \tilde{X}_i(y)^* \tilde{X}_i(y), \quad k = 2, \ldots, \ell.
\]

This maps an arbitrary regular unipotent element $v^{B_\ell}$ into a unipotent subgroup of $\text{GL}(2\ell + 1)$. Let us now change the variables in the following way:
\[
y_{1,1} = e^{x_{1,1}}, \quad y_{1,k} = e^{x_{k-1,1}} e^{x_{k,1}}, \quad k = 2, \ldots, \ell,
\]
\[
y_{k,2r-1} = e^{x_{k+r-1,k}-x_{k+r-2,k}-1}, \quad r = 1, \ldots, \ell + 1 - k,
\]
\[
y_{k,2r} = e^{x_{k+r-1,k}-x_{k+r-2,k}-1},
\]
(1.89)
where the conditions $x_{\ell,k} = 0$, $k = 1, \ldots, \ell$, are assumed. By using elementary operations it is easy to check that after the change of variables (1.89) the image $\pi(v^{B_1})$ of $v^{B_1}$ defined by (1.67), (1.68) transforms into (1.87), (1.88), and this leads to a parametrization of $N_{\mu}^{(0)} \subset N_{\mu}$. □

The modified factorized parametrization of a unipotent subgroup $N_{\mu} \subset SO(2\ell+1)$ defines a particular realization of a principal series representation of $\mathcal{W}(\mathfrak{so}_{2\ell+1})$ by differential operators. This can be obtained using the change of variables (1.89) applied to the representation given in Proposition 1.4. We shall use the term Gauss–Givental representation for this realization.

**Proposition 1.5.** The following differential operators define a representation $\pi_{\mu}$ of $\mathfrak{so}_{2\ell+1}$ on $V_{\mu}$ in terms of the modified factorized parametrization:

\[
F_1 = -e^{z_{1,1}} \frac{\partial}{\partial z_{1,1}} - 2 \sum_{n=2}^{\ell} e^{z_{n,1}} \left\{ \frac{1}{2} \frac{\partial}{\partial z_{n,1}} + \frac{e^{x_{n,1}}}{e^{x_{n-1,1}} + e^{x_{n,1}}} \frac{\partial}{\partial z_{n,2}} \right. \\
+ \left. \sum_{k=1}^{n-1} \left( \frac{\partial}{\partial x_{k,1}} + \frac{\partial}{\partial x_{k,2}} + \frac{\partial}{\partial x_{k,1}} \right) \right\},
\]

(1.90)

\[
F_2 = \sum_{n=1}^{2\ell-3} e^{x_{\beta(n)+1,2} - z_{\alpha(n)+2,2}} \left\{ \frac{\partial}{\partial z_{1,1}} + \frac{\partial}{\partial x_{1,1}} + \sum_{k=1}^{\beta(n)} \left( \frac{\partial}{\partial z_{k+1,1}} + \frac{\partial}{\partial z_{k+1,2}} - \frac{\partial}{\partial z_{k+1,3}} \right) \right. \\
+ \left. \sum_{k=1}^{\alpha(n)+1} \left( \frac{\partial}{\partial x_{k,1}} - \frac{\partial}{\partial x_{k,2}} \right) + (-1)^n \frac{e^{x_{\beta(n)+1,1}}}{e^{x_{\alpha(n)+1,1}} + e^{x_{\alpha(n)+2,1}}} \frac{\partial}{\partial z_{\alpha(n)+2,1}} \right\},
\]

(1.91)

\[
F_i = \sum_{n=1}^{2(\ell-i)+1} e^{x_{\beta(n)+1,i} - z_{\alpha(n)+1,i}} \left\{ \frac{\partial}{\partial z_{i-1,i-1}} + \frac{\partial}{\partial x_{i-1,i-1}} \\
+ \sum_{k=1}^{\alpha(n)} \left( \frac{\partial}{\partial x_{i+k-1,i-1}} - \frac{\partial}{\partial x_{i+k-1,i}} \right) \\
+ \sum_{k=1}^{\beta(n)} \left( \frac{\partial}{\partial z_{i+k-1,i}} - \frac{\partial}{\partial z_{i+k-1,i+1}} \right) \right\} \quad \text{for } i = 3, \ldots, \ell,
\]

(1.92)

\[
H_k = \langle \mu, \alpha_k^\vee \rangle + \sum_{n=1}^{\ell} (-1)^{\delta_{n,1}} a_{k,n} \sum_{i=n}^{\ell} \frac{\partial}{\partial z_{i,n}}, \quad k = 1, \ldots, \ell,
\]

(1.93)

\[
E_1 = -\sum_{n=1}^{2\ell-1} e^{x_{\beta(n)+1,i} - z_{\alpha(n)+1,i}} \left\{ 2\mu_1 + 2 \sum_{k=1}^{\alpha(n)} \frac{\partial}{\partial x_{k,1}} + (-1)^n \frac{\partial}{\partial z_{\alpha(n)+1,1}} \right\},
\]

(1.94)

\[
E_2 = \sum_{n=1}^{2(\ell-i)+1} e^{x_{\beta(n)+1,i} - z_{\alpha(n)+1,i}} \left\{ \mu_1 - \mu_2 + \sum_{k=1}^{\beta(n)} \left( \frac{\partial}{\partial x_{k,1}} - \frac{\partial}{\partial x_{k,2}} \right) - \sum_{k=1}^{\alpha(n)} \frac{\partial}{\partial z_{k+1,2}} \right\},
\]

(1.95)
New integral representations of Whittaker functions

where \( E_i = \sum_{n=1}^{2(\ell+1-i)} e^{\frac{i}{2}(n+i-1,i-1)} \left\{ \mu_{i-1} - \mu_i + \sum_{k=0}^{\alpha(n)} \left( \frac{\partial}{\partial z_{k+i-1,i-1}} - \frac{\partial}{\partial z_{k+i-1,i}} \right) \right\} \)

for \( i = 3, \ldots, \ell \). (1.96)

Here \( E_i = \pi_\lambda(e_i), F_i = \pi_\lambda(f_i), H_i = \pi_\lambda(h_i), x_{\ell,k} = 0, k = 1, \ldots, \ell, \) are assumed, and the derivatives with respect to \( x_{i,k}, z_{i,k}, i < k, \) and \( x_{\ell,n}, n = 1, \ldots, \ell, \) are omitted. The following notation is also used: \( \alpha(n) := [n/2], \beta(n) := [(n+1)/2] \).

We shall write the matrix element (1.12) explicitly in the Gauss–Givental representation. The Whittaker vectors \( \psi_R \) and \( \psi_L \) in this representation satisfy the systems of differential equations (1.7) and (1.8):

\[
F_i \psi_R(x) = -\psi_R(x), \quad E_i \psi_L(x) = -\psi_L(x), \quad i = 1, \ldots, \ell.
\]

Their solutions have the following form.

**Lemma 1.6.** The functions

\[
\psi_L(x, z) = e^{2\mu_1 x_{1,1}} \prod_{n=2}^{\ell} (e^{x_{n,1} + x_{n-1,1}}) e^x \times \prod_{n=1}^{\ell} \exp \left\{ -\mu_n \left( \sum_{i=1}^{n} x_{n,i} + 2z_{n,1} - 2 \sum_{i=2}^{n} z_{n,i} + \sum_{i=1}^{n-1} x_{n-1,i} \right) \right\} \\
\times \exp \left\{ -\left( \sum_{k=1}^{\ell} e^{x_{k,1}} + \sum_{k=2}^{\ell} e^{x_{k,k-2,k}} + \sum_{k=2}^{\ell} \sum_{n=k+1}^{\ell} (e^{x_{n-1,k-2,k,n}} + e^{x_{n,k-2,k,n}}) \right) \right\},
\]

\[ (1.97) \]

\[
\psi_R(x, z) = \exp \left\{ -\left( e^{x_{1,1} - z_{1,1}} + \sum_{n=2}^{\ell} (e^{x_{n-1,1} - z_{n,1}} + e^{x_{n,1} - z_{n,1}}) \right) \\
+ \sum_{k=2}^{\ell} \sum_{n=k}^{\ell} (e^{x_{n,k-2,k,n-1}} + e^{x_{n,k-2,k,n-1}}) \right\} \]

(1.98)

are solutions of the linear differential equations (1.7), (1.8). Here \( x_{\ell,k} = 0 \) for \( k = 1, \ldots, \ell \) and \( \mu_k = i\lambda_k - \rho_k, \) where \( \rho_k = (2k - 1)/2 \).

Now we are ready to find the integral representation for the pairing (1.12) in terms of modified factorization parameters. To obtain an explicit expression for the integrand, one uses the same type of decomposition of a Cartan subgroup element as in the case of \( g_{\ell+1} \):

\[
e^{-h_x} = e^{H_L e^{H_R}},
\]

where

\[
-h_x = H_L + H_R = -\sum_{i=1}^{\ell} \langle \omega_i, x \rangle \langle \mu, \alpha_i \rangle.
\]
\[ + x_{\ell,1} \sum_{n=1}^{\ell} \frac{\partial}{\partial z_{n,1}} + \sum_{k=1}^{\ell-1} (x_{\ell,i} - x_{\ell,i+1}) \sum_{n=k}^{\ell} \frac{\partial}{\partial z_{n,k}} \]  

(1.99)

with

\[ H_L = \sum_{k=1}^{\ell-1} \sum_{n=1}^{\ell} x_{\ell,n} \frac{\partial}{\partial x_{k,n}} + \sum_{k=2}^{\ell} \sum_{n=2}^{k} x_{\ell,n} \frac{\partial}{\partial z_{k,n}}, \]  

(1.100)

\[ H_R = -h_x - H_L. \]  

(1.101)

We recall that the differential operator \( H_L \) acts on the left vector and \( H_R \) acts on the right vector in (1.12). Taking into account the results of Lemma 1.6, one obtains the following theorem.

**Theorem 1.6.** The eigenfunctions (1.12) of the \( \mathfrak{so}_{2\ell+1} \)-Toda chain admit the integral representation

\[ \Psi_{\lambda_1, \ldots, \lambda_\ell}(x_{\ell,1}, \ldots, x_{\ell,\ell}) = \int_C \prod_{k=1}^{\ell-1} dx_k \prod_{k=1}^{\ell} dz_k e^{\mathcal{J}_{B_\ell}}, \]  

(1.102)

where

\[ \mathcal{J}_{B_\ell} = -i \lambda_1 (-x_{1,1} + 2z_{1,1}) \]

\[ - i \sum_{n=2}^{\ell} \lambda_n \left( \sum_{i=1}^{n} x_{n,i} + 2z_{n,1} - 2 \sum_{i=1}^{n} z_{n,i} + \sum_{i=1}^{n-1} x_{n-1,i} - 2 \log(e^{x_{n,1}} + e^{x_{n-1,1}}) \right) \]

\[ - \left\{ \sum_{n=1}^{\ell} e^{z_{n,1}} + \sum_{k=2}^{\ell} \sum_{n=k+1}^{\ell} (e^{x_{n-1,k}} - z_{n,k} + e^{x_{n,k}} - z_{n,k}) \right\} \]

\[ + \sum_{n=k}^{\ell} (e^{x_{n,k}} - x_{n-1,k-1} - e^{z_{n,k}} - x_{n,k-1}) + \sum_{n=1}^{\ell} e^{x_{n,n}} - z_{n,n} \]  

with \( x_i := x_{\ell,i}, i = 1, \ldots, \ell \). Here \( C \subset N_+ \) is a middle-dimensional non-compact submanifold such that the integrand decreases exponentially at the boundary and at infinity. In particular, the domain of integration can be chosen to be a slightly deformed subspace \( \mathbb{R}^m \subset \mathbb{C}^m, m = l(w_0) \), making the integral convergent.

We note that the integral (1.102) can be obtained directly from (1.74) using the following change of variables:

\[ y_{1,1} = e^{-x_{\ell,1}} e^{x_{1,1} - z_{1,1}}, \]

\[ y_{1,k} = e^{-x_{\ell,1}} (e^{x_{k-1,1} - z_{1,k}} + e^{x_{k,1} - z_{1,k}}), \quad k = 2, \ldots, \ell, \]

\[ y_{k,2r-1} = e^{-x_{\ell,k-1} - x_{\ell,k}} e^{x_{k+r-1,k} - x_{k+r-2,k-1}}, \quad r = 1, \ldots, \ell + 1 - k, \]

\[ y_{k,2r} = e^{-x_{\ell,k-1} - x_{\ell,k}} e^{x_{k+r-1,k} - x_{k+r-1,k-1}}. \]  

(1.103)
Example 1.3. For \( \ell = 2 \) the general formula (1.102) has the following form:

\[
\Psi_{\lambda_1, \lambda_2}^B(x_{2,1}, x_{2,2}) = \int_C dz_{1,1} dx_{1,1} dz_{2,1} dz_{2,2}
\times e^{-\lambda_1(-x_{1,1}+2z_{1,1})-\lambda_2(2z_{2,1}-2z_{2,2}+x_{1,1}+x_{2,1}+x_{2,2})}(e^{x_{2,1}} + e^{x_{1,1}})^{2\lambda_2}
\times \exp\{-e^{2z_{1,1}} + e^{z_{2,1}} + e^{x_{2,2}-z_{2,2}} + e^{x_{1,1}-z_{1,1}} + e^{x_{1,1}-z_{2,1}}
+ e^{z_{2,1}-x_{1,1}} + e^{z_{2,2}-x_{2,1}}\}.
\] (1.104)

There is a simple combinatorial description of the potential \( \mathcal{F}^B \) for the zero spectrum \( \{\lambda_i = 0\} \). Namely, it can be represented as the sum over all the arrows in the following diagram for \( B_\ell \):

Here we use the same rules for assigning variables to the arrows of the diagram as in the \( A_\ell \) case. In addition we assign the functions \( e^x/2 \) to the arrows \( \circ \rightarrow x \).

Note that the diagram for \( B_\ell \) can be obtained by a factorization of the diagram (1.51) for \( A_{2\ell} \). Namely, consider the involution

\[
*: X \mapsto -\hat{W}_0^{-1}X^t\hat{W}_0,
\] (1.105)

where \( \hat{W}_0 \) is the longest element in the Weyl group \( W(A_{2\ell}) = S_{2\ell+1} \) and \( X^t \) denotes the standard transposition. The corresponding action on the modified factorization parameters is given by

\[
*: x_{k,i} \mapsto -x_{k,k+1-i}.
\] (1.106)

This defines a factorization of the \( A_{2\ell} \)-diagram that gives the diagram for \( B_\ell \).

An analogue of the toric relations (1.52) can be obtained as follows. We associate the variables \( a_{k,i}, b_{k,i}, c_{k,i}, d_{k,i} \) with the arrows of the Givental diagram:

\[
a_{k,1} = b_{k,1} = \frac{1}{2}e^{2z_{k,1}}, \quad k = 1, \ldots, \ell;
\]

\[
a_{k,i} = e^{z_{k,i}-x_{k-1,i-1}}, \quad b_{k,i} = e^{z_{k,i}-x_{k,i-1}}, \quad 1 < i \leq k \leq \ell;
\]

\[
c_{k,a} = e^{x_{k,a}-2z_{k,a}}, \quad 1 \leq a \leq k \leq \ell;
\]

\[
d_{m,j} = e^{x_{m,j}-z_{m+1,j}}, \quad 1 \leq j \leq m < \ell.
\] (1.107)
Then the following relations hold:

\[ a_{k,1} = b_{k,1}, \quad 1 \leq k \leq \ell, \]
\[ d_{k,i} a_{k+1,i+1} = c_{k+1,i} b_{k+1,i+1}, \quad 1 \leq i \leq k < \ell - 1, \]
\[ b_{k,i} c_{k,i} = a_{k+1,i} d_{k,i}, \quad 1 \leq i \leq k < \ell - 1, \]
\[ b_{\ell,i} c_{\ell,i} = e^{x_{\ell,i}}, \quad b_{\ell,i} c_{\ell,i} = e^{x_{\ell,i}-x_{\ell,i-1}}, \quad i = 2, \ldots, \ell. \] (1.108)

These equalities can be considered as relations between elementary paths on the Givental diagram. Using relations following from (1.108) for more general paths, one can define a toric degeneration of the \( \mathfrak{so}_{2\ell+1} \) flag manifolds that generalizes the results in [5]. Note that the above diagram coincides with the Gelfand–Zetlin diagram for \( \mathfrak{so}_{2\ell+1} \).

1.3.3. Recursion for \( \mathfrak{so}_{2\ell+1} \)-Whittaker functions and the 2-operator for a \( B^{(1)}_{\ell} \)-Toda chain. The integral representation (1.102) in Theorem 1.6 for \( \mathfrak{so}_{2\ell+1} \)-Whittaker functions possesses a remarkable recursive structure with respect to the rank \( \ell \). For convenience we adopt the following notation: \( a_n = (a_{n,1}, \ldots, a_{n,n}) \) for \( n = 1, \ldots, \ell \). Let us introduce the integral operators \( Q^{B_n}_{B_{n-1}} \), \( n = 2, \ldots, \ell \), with kernels

\[ Q^{B_n}_{B_{n-1}}(x_n, x_{n-1}; \lambda_n), \]
defined as follows:

\[ Q^{B_n}_{B_{n-1}}(x_n, x_{n-1}; \lambda_n) = \int d\zeta_n Q^{B_n}_{BC_n}(x_n, \zeta_n) Q^{B_{n-1}}_{B_{n-1}}(\zeta_n, x_{n-1})(e^{x_{n,1}} + e^{x_{n-1,1}})^{2i\lambda_n} \]
\[ \times \exp \left\{ -i \lambda_n \left( \sum_{i=1}^{n} x_{n,i} + 2 \zeta_{n,1} - 2 \sum_{i=2}^{n} \zeta_{n,i} + \sum_{i=1}^{n-1} x_{n-1,i} \right) \right\}, \] (1.109)

where

\[ Q^{B_n}_{BC_n}(x_n, \zeta_n) = \exp \left\{ -\frac{1}{2} e^{\zeta_{n,1}} - \sum_{i=1}^{n-1} (e^{x_{n,i} - \zeta_{n,i}} + e^{x_{n,i+1} - x_{n,i+1}}) - e^{x_{n,n} - \zeta_{n,n}} \right\}, \] (1.110)
\[ Q^{B_{n-1}}_{B_{n-1}}(\zeta_n, x_{n-1}) = \exp \left\{ -\frac{1}{2} e^{\zeta_{n,1}} - \sum_{i=1}^{n-1} (e^{x_{n-1,i} - \zeta_{n,i}} + e^{x_{n,i+1} - x_{n-1,i}}) \right\}. \] (1.111)

We set for \( n = 1 \)

\[ Q^{B_1}_{B_0}(x_{1,1}; \lambda_1) = \Psi_{A_1}(x_{1,1}) = \int dz_{1,1} e^{i\lambda_1(x_{1,1} - 2z_{1,1})} \exp \left\{ -(e^{z_{1,1}} + e^{x_{1,1} - z_{1,1}}) \right\}. \]

Remark 1.1. We change the variables \( \zeta_n \) under the integral in (1.109) as follows:

\[ z_{n,1} \rightarrow -z_{n,1} + \log(e^{x_{n-1,1}} + e^{x_{n,1}}), \]
\[ z_{n,k} \rightarrow -z_{n,k} + x_{n-1,k-1} + x_{n,k-1} + \log(e^{x_{n-1,k}} + e^{x_{n,k}}) - \log(e^{x_{n-1,k-1}} + e^{x_{n,k-1}}) \quad \text{for} \quad k = 2, \ldots, n - 1, \]
\[ z_{n,n} \rightarrow -z_{n,n} + x_{n-1,n-1} + x_{n,n-1} + x_{n,n} - \log(e^{x_{n-1,n-1}} + e^{x_{n,n}}). \]
Then we arrive at a simpler integral representation of the recursion operator:

\[
Q_{B_{n-1}}^B(x_n, x_{n-1}; \lambda_n) = \int d\lambda_n Q_{BC_n}^B(x_n, z_n)Q_{B_{n-1}}^{BC_n}(z_n, x_{n-1})
\times \exp \left \{ \imath \lambda_n \left ( \sum_{i=1}^n x_{n,i} + 2z_{n,1} - 2 \sum_{i=2}^n z_{n,i} + \sum_{i=1}^{n-1} x_{n-1,i} \right ) \right \}
\]

(1.112)

and of the \(so_{2\ell+1}\)-Toda wave function.

By using the integral operators \(Q_{B_{n-1}}^B\) the integral representation (1.102) can be written in a recursive form.

**Theorem 1.7.** An eigenfunction of the \(B_\ell\)-Toda chain can be written as

\[
\Psi_{\alpha_1, \ldots, \alpha_\ell}^B(x_1, \ldots, x_\ell) = \int d\lambda_1 \prod_{k=1}^{\ell-1} dx_k \prod_{k=1}^\ell Q_{B_{k-1}}^{B_k}(x_k, x_{k-1}; \lambda_k),
\]

or equivalently,

\[
\Psi_{\alpha_1, \ldots, \alpha_\ell}^B(x_1, \ldots, x_\ell) = \int d\lambda_1 \prod_{k=1}^{\ell-1} dx_k Q_{B_{k-1}}^{B_k}(x_k, x_{k-1}; \lambda_k) \Psi_{\alpha_1, \ldots, \alpha_\ell}^{B_{\ell-1}}(x_1, \ldots, x_{\ell-1}),
\]

(1.113)

where \(x_i := x_{\ell,i}, \ i = 1, \ldots, \ell\), and \(C \subset \mathbb{N}_+\) is a middle-dimensional non-compact submanifold such that the integrand decreases exponentially at possible boundaries and at infinity.

We note that in contrast to the case of \(gl_{\ell+1}\) integral representations, the kernels of \(Q_{B_{n-1}}^B, n = 1, \ldots, \ell\), have a more complicated form. A curious new structure appears if we consider the Whittaker functions for the zero spectrum \(\{\lambda_i = 0\}\). As is clear from (1.109), the kernel of \(Q_{B_{n-1}}^B\) is given by the convolution of two kernels \(Q_{BC_n}^B(x_n, z_n)\) and \(Q_{B_{n-1}}^{BC_n}(z_n, x_{n-1})\). The corresponding integral operators \(Q_{BC_n}^B\) and \(Q_{B_{n-1}}^{BC_n}\) can be regarded as elementary intertwiners connecting the Toda chains for root systems \(B_n, BC_n\), and \(BC_n, B_{n-1}\). The \(B_\ell\)-Toda chain\(^1\) is defined in terms of the non-reduced root system \(BC_\ell\) in a standard fashion. We recall the construction of the non-reduced root system \(BC_\ell\). A root system of \(BC_\ell\) type can be realized in terms of an orthogonal basis \(\{\epsilon_i\}\) in \(\mathbb{R}_\ell\) as

\[
\alpha_0 = 2\epsilon_1, \quad \alpha_1 = \epsilon_1, \quad \alpha_{i+1} = \epsilon_{i+1} - \epsilon_i, \quad 1 \leq i \leq \ell - 1,
\]

(1.114)

and the corresponding Dynkin diagram is

\[
\alpha_0 \quad || \quad \alpha_1 \quad || \quad \alpha_2 \quad \cdots \quad \alpha_{n-1} \quad || \quad \alpha_n,
\]

(1.115)

---

\(^1\)The \(BC_\ell\)-Toda chain can also be considered as a most general form of the \(C_\ell\)-Toda chain (see, for example, [32], Remark on p. 61). In what follows we will use the term ‘\(BC_\ell\)-Toda chain’ to distinguish it from the more standard \(C_\ell\)-Toda chain that will be consider below.
where the first vertex from the left corresponds to a reduced root \( \alpha_1 = \epsilon_1 \) and a non-reduced root \( \alpha_0 = 2\epsilon_1 \). Then, for example, the quadratic Hamiltonian operator for the \( BC_\ell \)-Toda chain is given by

\[
\mathcal{H}_{2BC}^{(\ell)}(x^{(\ell)}) = -\frac{1}{2} \sum_{i=1}^{\ell} \frac{\partial^2}{\partial x_i^2} + \frac{1}{4} \left( e^{x_1} + \frac{1}{2} e^{2x_1} \right) + \sum_{i=1}^{\ell-1} e^{x_{i+1} - x_i}. \tag{1.116}
\]

The integral operators \( Q_{BC_n}^B_n \) and \( Q_{B_{n-1}}^{BC_n} \) intertwine the Hamiltonian operators of Toda chains of different types. Thus, for quadratic Hamiltonians one can check the following relations directly.

**Proposition 1.6.** 1. The operators \( Q_{BC_n}^B_n \) and \( Q_{B_{n-1}}^{BC_n} \) defined by the kernels (1.110) and (1.111) intertwine the quadratic Hamiltonians of the \( B \)- and \( BC \)-Toda chains:

\[
\mathcal{H}_{2BC}^{B_n}(x_n)Q_{BC_n}^{B_{n-1}}(x_{n-1}; z_n) = Q_{BC_n}^{B_{n-1}}(x_{n-1}; z_n)\mathcal{H}_{2B_{n-1}}^{B_n}(x_n), \tag{1.117}
\]

\[
\mathcal{H}_{2B_n}(z_n)Q_{BC_n}^{B_{n-1}}(x_{n-1}; z_n) = Q_{BC_n}^{B_{n-1}}(z_n; x_{n-1})\mathcal{H}_{2B_{n-1}}^{B_n}(z_n). \tag{1.118}
\]

2. The integral operator \( Q_{B_{n-1}}^{B_n} \) at \( \lambda_n = 0 \) intertwines the Hamiltonians \( \mathcal{H}_{2B_n}^{B_{n-1}} \) and \( \mathcal{H}_{2B_n}^{B_{n-1}} \):

\[
\mathcal{H}_{2B_n}(x_n)Q_{B_{n-1}}^{B_n}(x_{n-1}; z_{n-1}; \lambda_n = 0) = Q_{B_{n-1}}^{B_n}(x_{n-1}; x_n; \lambda_n = 0)\mathcal{H}_{2B_{n-1}}^{B_n}(x_{n-1}). \tag{1.119}
\]

The kernel \( Q_{B_{n-1}}^{B_n}(x_{n-1}; x_n; \lambda_n = 0) \) can be succinctly encoded into the following subdiagram of the \( \mathfrak{so}_{2\ell+1} \) Givental diagram:

\[
\begin{array}{c}
\circ \\
\circ \rightarrow z_{n,1} \rightarrow x_{n,1} \\
\mid \hspace{1cm} \mid \\
x_{n-1,1} \rightarrow z_{n,2} \rightarrow \cdots \\
\mid \hspace{1cm} \\
\cdots \rightarrow x_{n,n-1} \\
\mid \hspace{1cm} \\
x_{n-1,n-1} \rightarrow z_{n,n} \rightarrow x_{n,n}
\end{array} \tag{1.120}
\]

Here the upper and lower descending paths in the oriented diagram correspond to the kernels of the elementary intertwiners \( Q_{BC_n}^B \) and \( Q_{B_{n-1}}^{BC_n} \), respectively. The convolution of the kernels \( Q_{BC_n}^B \) and \( Q_{B_{n-1}}^{BC_n} \) in (1.109) at \( \lambda_n = 0 \) corresponds to the integration with respect to the variables \( z_{n,1} \) associated with the inner vertices of the subdiagram (1.120).

As in the case of \( \mathfrak{gl}_{\ell+1} \), the recursion operators \( Q_{B_{n-1}}^B \) can be considered as particular degenerations of the Baxter \( \mathcal{Q} \)-operators for affine \( B^{(1)}_\ell \)-Toda chains.
Below we provide the integral representations for these $\mathcal{Q}$-operators. We stress that up to now $\mathcal{Q}$-operators were known only for the $\mathfrak{g}l_{\ell+1}$ case. We do not present here the complete set of properties characterizing the $\mathcal{Q}$-operators introduced, but only consider the commutation relations with quadratic Hamiltonians of affine Toda chains. A detailed account will be given elsewhere.

We start with a description of the $B^{(1)}_\ell$-Toda chain. The set of simple roots of the affine root system $B^{(1)}_\ell$ can be represented in the following form:

$$\alpha_1 = \epsilon_1, \quad \alpha_{i+1} = \epsilon_{i+1} - \epsilon_i, \quad 1 \leq i \leq \ell - 1, \quad \alpha_{\ell+1} = -\epsilon_\ell - \epsilon_{\ell-1}. \quad (1.121)$$

The corresponding Dynkin diagram is

$$\begin{align*}
\alpha_1 & \leftrightarrow \alpha_2 \cdots \alpha_{\ell-1} \\
\alpha_\ell & \downarrow \\
\alpha_{\ell+1} &
\end{align*}$$

These root data allow us to define an affine $B^{(1)}_\ell$-Toda chain with a quadratic Hamiltonian given by

$$\mathcal{H}_2^{B^{(1)}_\ell} = -\frac{1}{2} \sum_{i=1}^{\ell+1} \frac{\partial^2}{\partial x_i^2} + \frac{1}{2} e^{x_1} + \sum_{i=1}^{\ell-1} e^{x_{i+1} - x_i} + ge^{-x_\ell - x_{\ell-1}}. \quad (1.122)$$

Here $g$ is an arbitrary coupling constant.

We define the Baxter $\mathcal{Q}$-operator for the $B^{(1)}_\ell$-Toda chain as an integral operator with the kernel

$$\mathcal{Q}^{B^{(1)}_\ell}(x, z; \lambda) = \int d\bar{z}^{(\ell)} (e^{x_1} + e^{y_1})^{2\lambda}(e^{-x_\ell} + e^{-y_\ell})^{-2\lambda}$$

$$\times \exp\left\{ -\lambda \left( \sum_{i=1}^{\ell} x_i + 2z_1 - 2 \sum_{i=2}^{\ell} z_i + \sum_{i=1}^{\ell} y_i \right) \right\} \times Q^{B^{(1)}_\ell}_{BC^{(1)}_\ell}(\bar{x}^{(\ell)}, \bar{z}^{(\ell)}) Q^{B^{(1)}_\ell}_{BC^{(1)}_\ell}(\bar{z}^{(\ell)}, \bar{y}^{(\ell)}), \quad (1.123)$$

where

$$Q^{B^{(1)}_\ell}_{BC^{(1)}_\ell}(\bar{x}^{(\ell)}, \bar{y}^{(\ell)}) = \exp\left\{ -\left( \frac{1}{2} e^{z_1} + \sum_{i=1}^{\ell-1} (e^{y_i - z_i} + e^{z_{i+1} - y_i}) \right) \right\} + e^{y_\ell - z_\ell} + ge^{-y_\ell - z_\ell} \quad (1.124)$$

and

$$Q^{B^{(1)}_n}_{BC^{(1)}_n}(\bar{x}^{(\ell)}, \bar{z}^{(\ell)}) = Q^{B^{(1)}_n}_{B^{(1)}_n}(\bar{z}^{(\ell)}, \bar{x}^{(\ell)}). \quad (1.125)$$

Here $x^{(\ell)} = (x_1, \ldots, x_\ell), \ z^{(\ell)} = (z_1, \ldots, z_\ell), \text{ and } y^{(\ell)} = (y_1, \ldots, y_\ell)$. The following proposition can be proved by a direct check.
Proposition 1.7. The $\mathcal{Q}$-operator (1.123) commutes with the quadratic Hamiltonian of the $B^{(1)}_\ell$-Toda chain, that is, the kernel intertwines the Hamiltonians $\mathcal{H}^{B^{(1)}_\ell}_2$:

$$\mathcal{H}^{B^{(1)}_\ell}_2(x^{(\ell)}), \mathcal{Q}^{B^{(1)}_\ell}_2(x^{(\ell)}, y^{(\ell)}; \gamma) = \mathcal{Q}^{B^{(1)}_\ell}_2(x^{(\ell)}, y^{(\ell)}; \gamma), \mathcal{H}^{B^{(1)}_\ell}_2(y^{(\ell)}).$$

(1.126)

Now we will demonstrate that the recursion operator $Q^{B^{(1)}_\ell}_{B^{(1)}_{\ell-1}}$ can be considered as a degeneration of the Baxter $\mathcal{Q}$-operators for $B^{(1)}_\ell$. Let us introduce a slightly modified recursion operator $Q^{B^{(1)}_\ell}_{B^{(1)}_{\ell-1} \oplus B_1}$ with the kernel

$$Q^{B^{(1)}_\ell}_{B^{(1)}_{\ell-1} \oplus B_1}(x^{(\ell)}, y^{(\ell)}; \gamma) := e^{i\lambda y_\ell} Q^{B^{(1)}_\ell}_{B^{(1)}_{\ell-1}}(x^{(\ell)}, y^{(\ell-1)}; \gamma),$$

(1.127)

where $y^{(\ell-1)} = (y_1, \ldots, y_{\ell-1})$. The operator (1.127) intertwines the Hamiltonians of the $so_{2\ell+1}$ and $so_{2\ell-1} \oplus so_2$-Toda chains. Thus, for the quadratic Hamiltonians we have

$$\mathcal{H}^{B^{(1)}_\ell}_2(x^{(\ell)}) Q^{B^{(1)}_\ell}_{B^{(1)}_{\ell-1} \oplus B_1}(x^{(\ell)}, y^{(\ell)}; \gamma) = Q^{B^{(1)}_\ell}_{B^{(1)}_{\ell-1} \oplus B_1}(x^{(\ell)}, y^{(\ell)}; \gamma) \times \left( \mathcal{H}^{B^{(1)}_{\ell-1}}_2(y^{(\ell-1)}) + \mathcal{H}^{B^{(1)}_1}(y_\ell) \right),$$

(1.128)

where $\mathcal{H}^{B^{(1)}_1}(y_\ell) = -\frac{1}{2} \frac{\partial^2}{\partial y_\ell^2}$. Obviously, the projection of the above relation on the subspace of functions $F(y^{(\ell)}) = \exp(i\lambda y_\ell)f(y^{(\ell-1)})$ recovers the initial recursion operator satisfying the relation

$$\mathcal{H}^{B^{(1)}_\ell}_2(x^{(\ell)}) Q^{B^{(1)}_\ell}_{B^{(1)}_{\ell-1}}(x^{(\ell)}, y^{(\ell)}; \gamma) = Q^{B^{(1)}_\ell}_{B^{(1)}_{\ell-1}}(x^{(\ell)}, y^{(\ell-1)}; \gamma) \left( \mathcal{H}^{B^{(1)}_{\ell-1}}_2(y^{(\ell-1)}) + \frac{1}{2} \lambda^2 \right).$$

(1.129)

We consider the one-parameter family of kernels

$$\mathcal{Q}^{B^{(1)}_\ell}_{B^{(1)}_{\ell-1}}(x^{(\ell)}, y^{(\ell)}; \gamma, \epsilon) = \epsilon^{i\lambda} e^{i\lambda y_\ell} \int dz^{(\ell)} (e^{x_1 + e^{y_1}})^{2i\lambda} (\epsilon e^{y_\ell - x_\ell} + 1)^{-2i\lambda} \times \exp \left\{ -i\lambda \left( \sum_{i=1}^{\ell} z_i + 2z_1 - 2 \sum_{i=2}^{\ell} z_i + \sum_{i=1}^{\ell-1} y_i \right) \right\} Q^{B^{(1)}_{BC^{(1)}_\ell}}_{B^{(1)}_{\ell}}(x^{(\ell)}, z^{(\ell)}; \gamma),$$

(1.129)

where

$$Q^{B^{(1)}_{BC^{(1)}_\ell}}_{B^{(1)}_{\ell}}(z^{(\ell)}, y^{(\ell)}; \gamma) = \exp \left\{ - \left( \frac{1}{2} e^{z_1} + \sum_{i=1}^{\ell-1} \left( e^{y_i - z_i + e^{z_{i+1}} - y_i} + \epsilon e^{y_\ell - z_\ell} + \epsilon^{-1} e^{-y_\ell - z_\ell} \right) + 1 \right) e^{-y_\ell - z_\ell} \right\},$$

(1.130)

which are obtained from the kernel of the operator $\mathcal{Q}^{B^{(1)}_\ell}$ by the change of variable $y_\ell \rightarrow y_\ell + \log \epsilon$. Let us consider the limiting behaviour of (1.129), (1.130) as $\epsilon \to 0$, $\epsilon \to -1 \to 0$. Then the following relation holds between the $\mathcal{Q}$-operator for the $B^{(1)}_\ell$-Toda chain and the (modified) recursion operator for the $so_{2\ell+1}$-Whittaker function:

$$Q^{B^{(1)}_\ell}_{B^{(1)}_{\ell-1} \oplus B_1}(x^{(\ell)}, y^{(\ell)}; \gamma) = \lim_{\epsilon \to 0} \lim_{\epsilon \to -1} Q^{B^{(1)}_\ell}(x^{(\ell)}, y^{(\ell)}; \gamma, \epsilon).$$
1.4. Integral representations of \( \mathfrak{sp}_{2\ell} \)-Toda chain eigenfunctions. In this subsection we provide an analogue of the Givental integral representation of the Whittaker functions for the Lie algebras \( \mathfrak{sp}_{2\ell} \). As in the case of \( \mathfrak{so}_{2\ell+1} \), we start with a derivation of the integral representation of \( \mathfrak{sp}_{2\ell} \)-Whittaker functions using the factorized parametrization. Then we consider a modification of the factorized parametrization leading to a Givental-type integral representation of \( \mathfrak{sp}_{2\ell} \)-Whittaker functions.

Consider a \( C_\ell \)-type root system corresponding to the Lie algebra \( \mathfrak{sp}_{2\ell} \). Let \( \{\epsilon_1, \ldots, \epsilon_\ell\} \) be an orthonormal basis in \( \mathbb{R}^\ell \). We use the following realization of simple roots, coroots, and fundamental weights as vectors in \( \mathbb{R}^\ell \):

\[
\begin{align*}
\alpha_1 &= 2\epsilon_1, \\
\alpha_2 &= \epsilon_2 - \epsilon_1, \\
\vdots \\
\alpha_\ell &= \epsilon_\ell - \epsilon_{\ell-1}, \\
\alpha^\vee_1 &= \epsilon_1, \\
\alpha^\vee_2 &= \epsilon_2 - \epsilon_1, \\
\vdots \\
\alpha^\vee_\ell &= \epsilon_\ell - \epsilon_{\ell-1}, \\
\omega_1 &= \epsilon_1 + \ldots + \epsilon_\ell, \\
\omega_2 &= \epsilon_2 + \ldots + \epsilon_\ell, \\
\ldots \\
\omega_\ell &= \epsilon_\ell.
\end{align*}
\] (1.131)

The Cartan matrix \( \|a_{ij}\| = \|\langle \alpha^\vee_j, \alpha_i \rangle\| \) becomes a symmetric matrix \( \|b_{ij}\| = \|d_i a_{ij}\| \) for \( d_1 = 2, d_i = 1, i = 2, \ldots, \ell \). One associates with these root data a \( \mathfrak{sp}_{2\ell} \)-Toda chain with the quadratic Hamiltonian given by

\[
H^{C_\ell}_2 = -\frac{1}{2} \sum_{i=1}^\ell \frac{\partial^2}{\partial z_i^2} + 2e^{2z_1} + \sum_{i=1}^{\ell-1} e^{z_{i+1} - z_i}.
\] (1.132)

One can supplement (1.132) to get a complete set of \( \ell \) pairwise commuting operators \( H^{C_\ell}_2 \) for the \( C_\ell \)-Toda chain. We are looking for integral representations of common eigenfunctions of the complete set of commuting Hamiltonians. The corresponding eigenfunction problem for the quadratic Hamiltonian can be written in the following form:

\[
H^{C_\ell}_2 \Psi_{\lambda_1, \ldots, \lambda_\ell}^{C_\ell}(z_1, \ldots, z_\ell) = \frac{1}{2} \sum_{i=1}^\ell \lambda_i^2 \Psi_{\lambda_1, \ldots, \lambda_\ell}^{C_\ell}(z_1, \ldots, z_\ell).
\] (1.133)

1.4.1. \( \mathfrak{sp}_{2\ell} \)-Whittaker function: factorized parametrization. The reduced word for the maximal length element \( w_0 \) in the Weyl group of \( \mathfrak{sp}_{2\ell} \) can be represented in the recursive form:

\[
I_\ell = (i_1, i_2, i_3, \ldots, i_\ell) := (1, 212, 32123, \ldots, (\ell \ldots 212 \ldots \ell)), \quad m_\ell = \ell^2,
\]

where the indices \( i_k \in \Gamma = \{1, \ldots, \ell\} \) correspond to elementary reflections with respect to the simple roots \( \alpha_{i_k} \). Let \( N_+ \subset G \) be a maximal unipotent subgroup of \( G = \text{Sp}(2\ell) \). One associates with the reduced word \( I_\ell \) the following recursive parametrization of a generic element \( v^{C_\ell} \in N_+ ^{C_\ell} \):

\[
v^{C_\ell} = v^{C_{\ell-1}} X^{C_\ell}_{C_{\ell-1}},
\] (1.134)

where

\[
X^{C_\ell}_{C_{\ell-1}} = X_\ell(y_\ell, 1) \cdots X_k(y_{k,2(\ell+1-k)-1}) \cdots X_2(y_{2,2\ell-3}) \times X_1(y_1, \ell) X_2(y_{2,2\ell-2}) \cdots X_k(y_{k,2(\ell+1-k)}) \cdots X_\ell(y_\ell, 2)
\] (1.135)
Here $X_i(y) = e^{y f_i}$ and $f_i \equiv f_{\alpha_i}$ are simple root generators. The subset $N_+^{(0)}$ admitting the representation (1.134), (1.135) is an open part of $N_+$. The action of the Lie algebra $\mathfrak{sp}_{2\ell}$ on $N_+$ given by (1.10) defines an action on the space of functions on $N_+^{(0)}$. The following proposition explicitly describes this action on the space $V_\mu$ of (twisted) functions on $N_+^{(0)}$.

**Proposition 1.8.** The following differential operators define a realization of the representation $\pi_\mu$ of $\mathcal{U}(\mathfrak{sp}_{2\ell})$ on $V_\mu$ in terms of the factorized parametrization of $N_+^{(0)}$:

$$
F_1 = \frac{\partial}{\partial y_{1,\ell}} + \sum_{n=1}^{\ell-1} \sum_{j=1}^{\ell-1} \frac{1}{2} \left( \frac{\partial}{\partial y_{1,n}} \right)^2 - \frac{\partial}{\partial y_{1,n+1}} \left( \frac{\partial}{\partial y_{1,n}} \right)^2 \left\{ \frac{\partial}{\partial y_{2,j}} - \frac{\partial}{\partial y_{2,j-1}} \right\},
$$

$$
F_k = \frac{\partial}{\partial y_{k,2(\ell-1)+k}} + \sum_{n=1}^{\ell-k} \sum_{j=1}^{\ell-k} \frac{1}{2} \left( \frac{\partial}{\partial y_{k,n+1}} \right)^2 - \frac{\partial}{\partial y_{k,n}} \left( \frac{\partial}{\partial y_{k,n}} \right)^2 \left\{ \frac{\partial}{\partial y_{k,n+1}} - \frac{\partial}{\partial y_{k,n}} \right\},
$$

for $k = 2, \ldots, \ell$, and

$$
H_k = \langle \mu, \alpha_k^\vee \rangle + \sum_{i=1}^{\ell} a_{k,i} \sum_{j=1}^{n_1} y_{i,j} \frac{\partial}{\partial y_{i,j}}, \quad k = 1, \ldots, \ell,
$$

$$
E_1 = \sum_{n=1}^{\ell} y_{1,n} \left\{ -\langle \mu, \alpha_1^\vee \rangle + \sum_{j=1}^{2n-1} y_{2,j} \frac{\partial}{\partial y_{2,j}} - \sum_{j=1}^{n-1} y_{1,j} \frac{\partial}{\partial y_{1,j}} - y_{1,n} \frac{\partial}{\partial y_{1,n}} \right\},
$$

$$
E_2 = \sum_{n=1}^{2(\ell-1)} y_{2,n} \left\{ -\langle \mu, \alpha_2^\vee \rangle + \frac{\alpha(n+1)}{2} \sum_{j=1}^{n-1} y_{1,j} \frac{\partial}{\partial y_{1,j}} - \sum_{j=1}^{n-1} y_{2,j} \frac{\partial}{\partial y_{2,j}} \right\},
$$

$$
E_k = \sum_{n=1}^{2(\ell+k-1)} y_{k,n} \left\{ -\langle \mu, \alpha_k^\vee \rangle + \frac{\alpha(n+1)}{2} \sum_{j=1}^{n-1} y_{k-1,j} \frac{\partial}{\partial y_{k-1,j}} - \sum_{j=1}^{n-1} y_{k,j} \frac{\partial}{\partial y_{k,j}} \right\}, \quad k = 2, \ldots, \ell,
$$

where $E_i = \pi_\mu(e_i)$, $H_i = \pi_\mu(h_i)$, $F_i = \pi_\mu(f_i)$, $i = 1, \ldots, \ell$, $n_1 = \ell$, $n_k = 2(\ell+1-k)$ for $1 < k \leq \ell$, and $\alpha(n) := \lfloor n/2 \rfloor$, $\beta(n) := \lceil (n+1)/2 \rceil$.

The proof is given in Part 2, §2.4.3.
The left/right Whittaker vectors defined by (1.7) and (1.8) have the following expressions in the factorized parametrization.

**Lemma 1.7.** The left/right Whittaker vectors in the factorized parametrization are given by:

\[
\psi_R(y) = \exp \left\{ - \left( \sum_{n_1}^\ell y_{1,n} + \sum_{k=2}^{n_k} \sum_{n=1}^\ell y_{k,n} \right) \right\},
\]

\[
\psi_L(y) = \prod_{i=1}^\ell \left( \prod_{n=1}^\ell y_{1,n} \prod_{k=2}^\ell y_{k,n} \right) \prod_{k=i+1}^\ell y_{k,2n-1} \langle \mu, \alpha \rangle
\]

\[
\times \exp \left\{ - \left( \sum_{n_1}^\ell \frac{1}{y_{1,n}} \left( 1 + \frac{y_{2,2(n-1)}}{y_{2,2(n-1)-1}} \right) \right) \prod_{i=n_1+1}^\ell \left( \frac{y_{2,2(i-1)}}{y_{2,2(i-1)-1}} \right)^2
\]

\[
+ \sum_{k=2}^\ell \sum_{n=1}^{n_k/2} \frac{1}{y_{k,2n}} \left( 1 + \frac{y_{k+1,2(n-1)}}{y_{k+1,2(n-1)-1}} \right) \prod_{i=n_1+1}^\ell \frac{y_{k+1,2(i-1)-1}}{y_{k+1,2(i-1)-1}} \frac{y_{k,2i-1}}{y_{k,2i}} \right\},
\]

where \( n_1 = \ell \) and \( n_k = 2(\ell + 1 - k) \) for \( k = 2, \ldots, \ell \).

The proof is given in Part 2, §2.3.3.

Using the expressions (1.140) for the left/right Whittaker vectors, we obtain the integral representation of the \( \mathfrak{sp}_{2\ell} \)-Whittaker function in terms of the factorized parametrization.

**Theorem 1.8.** The eigenfunctions of the \( \mathfrak{sp}_{2\ell} \)-Toda chain admit the integral representation

\[
\Psi_{\lambda_1, \ldots, \lambda_\ell}^{C_\ell} (z_1, \ldots, z_\ell) = e^{i\lambda_1 z_1 + \cdots + i\lambda_\ell z_\ell} \int_C \prod_{i=1}^\ell \prod_{k=1}^\ell \frac{dy_{i,k}}{y_{i,k}} \left( \prod_{n=1}^\ell y_{1,n} \prod_{k=2}^\ell y_{k,n} \prod_{k=i+1}^\ell y_{k,2n-1} \right)^{i\lambda_1}
\]

\[
\times \prod_{i=2}^\ell \left( \prod_{n=1}^\ell y_{1,n} \prod_{k=2}^\ell y_{k,n} \prod_{k=i+1}^\ell y_{k,2n-1} \right)^{i(\lambda_i - \lambda_{i-1})}
\]

\[
\times \exp \left\{ - \left( \sum_{n_1}^\ell \frac{1}{y_{1,n}} \left( 1 + \frac{y_{2,2(n-1)}}{y_{2,2(n-1)-1}} \right) \right) \prod_{i=n_1+1}^\ell \left( \frac{y_{2,2(i-1)}}{y_{2,2(i-1)-1}} \right)^2
\]

\[
+ \sum_{k=2}^\ell \sum_{n=1}^{n_k/2} \frac{1}{y_{k,2n}} \left( 1 + \frac{y_{k+1,2(n-1)}}{y_{k+1,2(n-1)-1}} \right) \prod_{i=n_1+1}^\ell \frac{y_{k+1,2(i-1)-1}}{y_{k+1,2(i-1)-1}} \frac{y_{k,2i-1}}{y_{k,2i}} \right\}
\]

where \( n_1 = \ell \) and \( n_k = 2(\ell + 1 - k) \) for \( k = 2, \ldots, \ell \). The domain of integration \( C \subset N_+ \) is a middle-dimensional non-compact submanifold such that the integrand decreases exponentially at the boundary and at infinity. In particular, one can choose \( C \) to be a slightly deformed subspace \( \mathbb{R}^2_+ \) of \( \mathbb{C}^2 \) ensuring convergence of the integral.
The proof is given in Part 2, §2.3.3.

**Example 1.4.** For $\ell = 2$ the general formula (1.141) acquires the form

\[
\Psi_{\lambda_1, \lambda_2}^{C_2}(z_1, z_2) = e^{i\lambda_1 z_1 + i\lambda_2 z_2} \int_C \prod_{i,k=1}^2 \frac{dy_{i,k}}{y_{i,k}} (y_{1,1} y_{2,1} y_{1,2})^{1\lambda_1} (y_{2,1} y_{1,2} y_{2,2})^{1\lambda_2 - i\lambda_1} \times \exp\left\{ -\left( \frac{1}{y_{1,1}} \left( \frac{y_{2,2}}{y_{2,1}} \right)^2 + \frac{1}{y_{1,2}} \left( \frac{y_{2,1}}{y_{2,2}} + 1 \right)^2 + \frac{1}{y_{2,2}} \right) + e^{2z_1} (y_{1,1} + y_{1,2}) + e^{2z_2 - z_1} (y_{2,1} + y_{2,2}) \right\}.
\]

(1.142)

1.4.2. $\mathfrak{sp}_{2\ell}$-Whittaker function: modified factorized parametrization. In this part we introduce a modified factorized parametrization of an open part of $N_+ \subset \text{Sp}(2\ell)$. We use this parametrization to construct integral representations for $\mathfrak{sp}_{2\ell}$-Whittaker functions. In analogy to the integral representation of $\mathfrak{so}_{2\ell+1}$-Whittaker functions considered above, these integral representations have a simple recursive structure with respect to the rank $\ell$ and can be described in purely combinatorial terms using suitable graphs. These representations can be considered as a generalization of the Givental integral representations to the case $\mathfrak{g} = \mathfrak{sp}_{2\ell}$.

We follow the same approach that was used in the description of the modified factorized representation for $\mathfrak{so}_{2\ell+1}$. There is a realization of the tautological representation $\pi: \mathfrak{sp}_{2\ell} \to \text{End}(\mathbb{C}^{2\ell})$ such that the Chevalley generators corresponding to the Borel (Cartan) subalgebra of $\mathfrak{sp}_{2\ell}$ are realized by upper triangular (respectively, diagonal) matrices. This defines an embedding $\mathfrak{sp}_{2\ell} \subset \mathfrak{gl}_{2\ell}$ such that the Borel (Cartan) subalgebra maps into the Borel (Cartan) subalgebra (see, for example, [17]). To define the corresponding embedding of the groups we consider the following involution on $\text{GL}(2\ell)$:

\[
g \mapsto g^* := \tilde{W}_0^{-1} (g^{-1})' \tilde{W}_0,
\]

(1.143)

where $a \to a'$ is induced by the standard transposition of matrices and $\tilde{W}_0$ is a lift of the longest element in the Weyl group $S_{2\ell}$ for $\mathfrak{gl}_{2\ell}$. In matrix form it can be written as

\[
\pi(\tilde{W}_0) = SJ,
\]

where $S = \text{diag}(1, -1, \ldots, 1, -1)$ and $J = \|J_{i,j}\| = \|\delta_{i+j, 2\ell+1}\|$. The symplectic group $G = \text{Sp}(2\ell)$ can then be identified with the following subgroup of $\text{GL}(2\ell)$ (see, for instance, [17]):

\[
\text{Sp}(2\ell) = \{ g \in \text{GL}(2\ell) : g^* = g \}.
\]

Let $\epsilon_{i,j}$ stand for the elementary $2\ell \times 2\ell$ matrix with 1 at the $(i, j)$ position and zeros elsewhere. We introduce the following $2\ell \times 2\ell$ matrices:

\[
U_n = \sum_{i=1}^{\ell} \epsilon_{i,i} + \sum_{i=1}^{n} e^{-x_{i,1}} \epsilon_{\ell+i, \ell+i} + \sum_{i=n+1}^{\ell} \epsilon_{\ell+i, \ell+i},
\]

(1.144)

\[
\tilde{U}_n = U_n + e^{z_{n,1}} \epsilon_{\ell, \ell+1} + \sum_{i=1}^{n-1} e^{-z_{n,1}} \epsilon_{\ell+i, \ell+i+1},
\]

(1.145)
Here we assume that $z$ of $GL(2)$ and $z_\ell$.

We can define a modified factorized parametrization as follows.

**Proof.** Let $v^{C_\ell}$ be parametrization of an open part of $N_+^{(0)}$ in the tautological representation $\pi: sp_{2\ell} \to \text{End}(C^{2\ell})$ can be represented in the form

$$\pi(v^{C_\ell}) = x_1 x_2 \cdots x_\ell, \quad x_n = \pi(x_{C_n}^{C_{n-1}}) \quad \text{for} \quad n = 1, \ldots, \ell,$$

where

$$x_1 = 1 + e^{x_{1.1} + z_{1.1}} \epsilon_{-1,1},$$

$$x_n = [U_n^* U_n^{-1}] \cdot V_n V_n^{-1}[V_n^* V_n^{-1}], \quad n = 2, \ldots, \ell,$$

and $z_{\ell,k} = 0, k = 1, \ldots, \ell$.

(ii) The expressions (1.150), (1.151) define a parametrization of the totally non-negative open part $N_+^{(0)} \subset N_+$.

**Proof.** Let $v^{C_\ell}(y)$ be parametrization of an open part of $N_+$ according to (1.134), (1.135). Let $\tilde{X}_k(y) = e^{y\epsilon_{1-k,1-k+2-k}}, k = 1, \ldots, \ell$, be unipotent elements in GL$(2\ell)$, and suppose that the action of the involution $*$ is given by

$$\tilde{X}_1(y)^* = \tilde{X}_1(y), \quad \tilde{X}_i(y)^* = \tilde{X}_{\ell+i}(y), \quad i = 2, \ldots, \ell.$$

Then we embed the elementary unipotent subgroups $X_i(y)$ of $Sp(2\ell)$ into GL$(2\ell)$ as follows:

$$X_1(y) = \tilde{X}_1(y), \quad X_i(y) = \tilde{X}_i(y)^* \tilde{X}_i(y), \quad i = 2, \ldots, \ell.$$

This maps an arbitrary regular unipotent element $v^{C_\ell}$ into a unipotent subgroup of GL$(2\ell)$. Let us now change the variables in the following way:

$$y_{1,1} = e^{x_{1.1} + z_{1.1}},$$

$$y_{1,k} = e^{z_{k-1.1} + x_{k.1}} + e^{z_{k.1} + x_{k.1}}, \quad k = 2, \ldots, \ell,$$

$$y_{k,2r-1} = e^{x_{k+r-1.1} - z_{k+r-2.1}}, \quad r = 1, \ldots, \ell + 1 - k,$$

$$y_{k,2r} = e^{x_{k+r-1.1} - z_{k+r-1.1}}.$$

Here we assume that $z_{\ell,k} = 0$ for $k = 1, \ldots, \ell$. 

\[\tilde{U}'_n = U_n + \sum_{i=1}^{n-1} e^{-z_{n-i-1,1}} \epsilon_{\ell+i,\ell+1+i}, \quad (1.146)\]

\[V_n = \sum_{i=1}^{\ell-n} \epsilon_{i,1} + e^{-z_{n,1}} \epsilon_{\ell+1,\ell+1} + \sum_{i=1}^{n} e^{-z_{n-1,i}} \epsilon_{\ell+1-i,\ell+1-i}, \quad (1.147)\]

\[\tilde{V}_n = V_n + e^{x_{n,1}} \epsilon_{\ell,\ell+1} + \sum_{i=1}^{n} e^{x_{n-i,1}} \epsilon_{\ell+1-i,\ell+2-i}, \quad (1.148)\]

\[\tilde{V}'_n = V_n + \sum_{i=1}^{n} e^{x_{n-i,1}} \epsilon_{\ell+1-i,\ell+2-i}. \quad (1.149)\]
By elementary manipulations it is easy to check that after the change of variables (1.152) the image $\pi(v^{\text{ce}})$ of $v^{\text{ce}}$ defined by (1.134), (1.135) transforms into (1.150), (1.151). Taking into account that this change of variables is invertible, we obtain a parametrization of $N_+^{(0)} \subset N_+$.

The modified factorized parametrization of a unipotent group $N_+$ defines a particular realization of a principal series representation of $\mathcal{U}(\mathfrak{sp}_{2\ell})$ by differential operators. It can be obtained using the change of variables (1.152) applied to the realization given in Proposition 1.8. We shall use the term Gauss–Givental representation for this realization of the representation of $\mathcal{U}(\mathfrak{sp}_{2\ell})$.

**Proposition 1.9.** The following differential operators define a representation $\pi_{\mu}$ of $\mathcal{U}(\mathfrak{sp}_{2\ell})$ on $V_{\mu}$ in terms of the modified factorized parametrization:

\[
F_1 = \sum_{n=1}^{2\ell-1} e^{z_{n,1-2} \alpha(n)+1,1} \left\{ \frac{\partial}{\partial x_{1,1}} + \sum_{k=2}^{\beta(n)} \left( \frac{\partial}{\partial x_{k,1}} - \frac{\partial}{\partial x_{k,2}} \right) - \sum_{k=1}^{\alpha(n)} \frac{\partial}{\partial z_{k,1}} \right\}, \quad (1.153)
\]

\[
F_2 = -\sum_{n=1}^{2\ell-3} e^{z_{n,1-2} \alpha(n)+2,2} \left\{ \frac{\partial}{\partial x_{1,1}} - \frac{\partial}{\partial z_{1,1}} + \sum_{k=1}^{\beta(n)} \left( \frac{\partial}{\partial x_{k+1,1}} - \frac{\partial}{\partial x_{k+1,2}} + (1 - \delta_{n,1}) \frac{\partial}{\partial x_{k+1,3}} \right) \right. \\
- \left. \sum_{k=1}^{\alpha(n)} \left( \frac{\partial}{\partial z_{k+1,1}} - \frac{\partial}{\partial z_{k+1,2}} \right) + (-1)^n \frac{e^{z_{n,1-2} \alpha(n)+1,1}}{e^{z_{\alpha(n)+1,1} + e^{z_{2\ell}+2,1}} \frac{\partial}{\partial x_{\alpha(n)+2,1}}} \right\}, \quad (1.154)
\]

\[
F_k = \sum_{2(\ell-k)+1} e^{z_{n,1-2} \alpha(n)+k-1,1-k} \alpha(n) \left\{ \frac{\partial}{\partial x_{k-1,k-1}} + \frac{\partial}{\partial x_{k,k}} + \sum_{m=1}^{\beta(n)} \left( \frac{\partial}{\partial z_{k-1+m,k-1}} - \frac{\partial}{\partial z_{k-1+m,k}} \right) \right. \\
+ \left. \sum_{m=1}^{\alpha(n)} \left( \frac{\partial}{\partial x_{k-1+m,k}} - \frac{\partial}{\partial x_{k-1+m,k+1}} \right) \right\} \quad \text{for} \quad k = 3, \ldots, \ell, \quad (1.155)
\]

\[
H_k = \langle \mu, \alpha_k \rangle + \sum_{n=1}^{\ell} a_{k,n} \sum_{i=n}^{\ell} \frac{\partial}{\partial x_{i,n}}, \quad k = 1, \ldots, \ell, \quad (1.156)
\]

\[
E_1 = -\sum_{n=1}^{2\ell-1} e^{z_{n,1-2} \alpha(n)+1,1} \left\{ \mu_1 + \sum_{k=1}^{\beta(n)} \frac{\partial}{\partial x_{k,1}} + \sum_{k=1}^{\alpha(n)} \frac{\partial}{\partial z_{k,1}} \right\}, \quad (1.157)
\]

\[
E_2 = \sum_{n=1}^{2\ell-2} e^{z_{n,1-2} \alpha(n)+1,1} \left\{ \mu_1 - \mu_2 + \frac{\partial}{\partial x_{1,1}} + \frac{\partial}{\partial z_{1,1}} + \sum_{k=1}^{\alpha(n)} \left( \frac{\partial}{\partial x_{k+1,1}} - \frac{\partial}{\partial x_{k+1,2}} \right) + \sum_{k=1}^{\beta(n)-1} \left( \frac{\partial}{\partial z_{k+1,1}} - \frac{\partial}{\partial z_{k+1,2}} \right) \right\}, \quad (1.158)
\]


\[ E_k = \sum_{n=1}^{2(\ell+1-k)} e^{x_{\beta(n)+k-1,k}-z_{\beta(n)+k-1,k}} \left\{ \langle \mu, \alpha_k \rangle + \frac{\partial}{\partial x_{k-1,k-1}} + \frac{\partial}{\partial z_{k-1,k-1}} \right\} \]

\[ + \sum_{m=1}^{\alpha(n)} \left( \frac{\partial}{\partial x_{k-1+m,k-1}} - \frac{\partial}{\partial x_{k-1+m,k}} \right) \]

\[ + \sum_{m=2}^{\beta(n)} \left( \frac{\partial}{\partial z_{k-2+m,k-1}} - \frac{\partial}{\partial z_{k-2+m,k}} \right) \}

for \( k = 3, \ldots, \ell \). \tag{1.159}

Here \( z_{\ell,i} = 0, i = 1, \ldots, \ell \), and the derivatives with respect to \( x_{i,k}, z_{i,k}, i < k \), are omitted. Also, \( E_i = \pi_\mu(e_i), F_i = \pi_\mu(f_i), H_i = \pi_\mu(h_i), i = 1, \ldots, \ell \), and \( \alpha(n) := [n/2], \beta(n) := [(n+1)/2] \).

We shall write the matrix element (1.12) explicitly using the Gauss–Givental representation defined above. The Whittaker vectors \( \psi_R \) and \( \psi_L \) in this representation satisfy the system of differential equations (1.7) and (1.8). The solutions have the following general form. Using the explicit change of variables (1.152), we obtain expressions for the Whittaker vectors in the modified factorized parametrization.

**Lemma 1.8.** The following expressions hold for the left/right Whittaker vectors:

\[ \psi_R = \exp \left\{ -e^{x_{1,1}+z_{1,1}} - \sum_{n=2}^{\ell} (e^{x_{n-1,1}+x_{n,1}} + e^{x_{n,1}+x_{n,1}}) \right\} \]

\[ - \sum_{k=2}^{\ell} \sum_{n=k}^{\ell} (e^{x_{n,k}-z_{n-1,k-1}} + e^{x_{n,k}-z_{n,k-1}}) \right\}, \tag{1.160} \]

\[ \psi_L = e^{\mu_1 z_{1,1}} \prod_{n=2}^{\ell} (e^{z_{n,1}} + e^{z_{n-1,1}})^{\mu_n} \]

\[ \times \prod_{n=1}^{\ell} \exp \left\{ -\mu_n \left( \sum_{i=1}^{n} z_{n,i} - x_{n,1} - 2 \sum_{i=2}^{n} x_{n,i} + \sum_{i=1}^{n-1} z_{n-1,i} \right) \right\} \]

\[ \times \exp \left\{ -\sum_{k=1}^{\ell} (e^{z_{k,k}-x_{k,k}} + \sum_{n=k+1}^{\ell} e^{x_{n-1,k}-x_{n,k}} + e^{x_{n,k}-x_{n,k}}) \right\}, \tag{1.161} \]

where \( z_{\ell,k} = 0 \) and \( \mu_k = i\lambda_k - \rho_k, \rho_k = k \) for \( k = 1, \ldots, \ell \).

Now we are ready to find an integral representation of the pairing (1.12) for \( g = \mathfrak{sp}_{2\ell} \). To obtain an explicit expression for the integrand, one uses the same type of decomposition for the Cartan element as previously for \( \mathfrak{g}l_{\ell+1} \) and \( \mathfrak{sp}_{2\ell} \):

\[ e^{-h_z} = \pi_\lambda \left( \exp \left( -\sum_{i=1}^{\ell} (\omega_i, z) h_i \right) \right) = e^{HL} e^{HR}, \]

where

\[ -h_z = H_L + H_R = -\langle \mu, z_\ell \rangle - 2z_{\ell,1} \sum_{n=1}^{\ell} \frac{\partial}{\partial x_{n,1}} + \sum_{k=1}^{\ell-1} (z_{\ell,i} - z_{\ell,i+1}) \sum_{n=k}^{\ell} \frac{\partial}{\partial x_{n,k}} \tag{1.162} \]
with
\[ H_L = \sum_{k=1}^{\ell} z_{\ell,k} \left( \sum_{n=k}^{\ell-1} \frac{\partial}{\partial z_{n,k}} + \sum_{n=k}^{\ell} \frac{\partial}{\partial x_{n,k}} \right), \] (1.163)
\[ H_R = -\langle \mu, z_{\ell} \rangle - \sum_{k=1}^{\ell-1} z_{\ell,k} \left( \sum_{n=k}^{\ell} \frac{\partial}{\partial x_{n,k}} - \sum_{n=k}^{\ell-1} \frac{\partial}{\partial z_{n,k}} - \sum_{n=k+1}^{\ell} \frac{\partial}{\partial x_{n,k+1}} \right). \] (1.164)

We assume that \( H_L \) acts on the left vector and \( H_R \) acts on the right vector in (1.12). Taking into account Lemma 1.8, one obtains the following theorem.

**Theorem 1.10.** The eigenfunctions (1.12) of the \( sp_{2\ell} \)-Toda chain admit the integral representation
\[ \Psi_{\lambda_1, \ldots, \lambda_\ell}^{C_\ell}(z_1, \ldots, z_\ell) = \int_C \prod_{k=1}^{\ell-1} dz_k \prod_{k=1}^{\ell} dx_k e^{F^{C_\ell}}, \] (1.165)
where
\[ F^{C_\ell} = i\lambda_1 z_{1,1} - \sum_{n=2}^{\ell} i\lambda_n \left( \sum_{i=1}^{n} z_{n,i} - x_{n,1} - 2 \sum_{i=2}^{n} x_{n,i} \right) \\
+ \sum_{i=1}^{n-1} z_{n-1,i} - \log(e^{z_{n,1}} + e^{z_{n-1,1}}) \\
- \left\{ \sum_{k=1}^{\ell} (e^{z_{k,k}} - x_{k,k}) + \sum_{n=k+1}^{\ell} e^{z_{n-1,k}} - x_{n,k} + e^{z_{n,k}} - x_{n,k} \right\} + e^{x_{1,1} + z_{1,1}} \\
+ \sum_{n=2}^{\ell} (e^{z_{n-1,1} + x_{n,1}} + e^{z_{n,1} + x_{n,1}}) + \sum_{k=2}^{\ell} \sum_{n=k}^{\ell} \left( e^{x_{n,k} - z_{n-1,k-1}} + e^{x_{n,k} - z_{n,k-1}} \right) \right\} \] (1.166)

with \( z_i := z_{\ell,i}, 1 \leq i \leq \ell \). Here \( C \subset N_+ \) is a middle-dimensional non-compact submanifold such that the integrand decays exponentially at the boundary and at infinity. In particular, the domain of integration can be chosen to be a slightly deformed subspace \( \mathbb{R}^{2\ell} \subset \mathbb{C}^{2\ell} \) making the integral convergent.

Note that the integral (1.165), (1.166) can be obtained from (1.141) by the following direct substitution:
\[ y_{1,1} = e^{-2z_{1,1}} e^{x_{1,1} + z_{1,1}}, \]
\[ y_{1,k} = e^{-2z_{1,1}} (e^{z_{k-1,1} + x_{k,1}} + e^{z_{k,1} + x_{k,1}}), \quad k = 2, \ldots, \ell, \]
\[ y_{k,2r-1} = e^{-2z_{1,1}} e^{x_{k+r-1,1} - z_{k+r-1,1}} e^{x_{k+r-2,k-1}}, \quad r = 1, \ldots, \ell + 1 - k, \]
\[ y_{k,2r} = e^{-2z_{1,1}} e^{x_{k+r-1,1} - z_{k+r-1,1}} e^{x_{k+r-2,k-1}}. \]

**Example 1.5.** For \( \ell = 2 \) the general expression (1.165) acquires the form
\[ \Psi_{\lambda_1, \lambda_2}^{C_2}(z_1, z_2) = \int_C dx_{1,1} dx_{2,1} dx_{2,2} dz_{1,1} (e^{z_{1,1}} + e^{z_{2,1}})^{i\lambda_2} \]
× \exp \left\{ t\lambda_1 x_{1,1} - t\lambda_2 (z_{2,1} + z_{2,2} - (x_{2,1} + 2x_{2,2}) + z_{1,1}) 
- (e^{z_{1,1} - x_{1,1}} + e^{x_{1,1} + z_{1,1}} + e^{x_{2,1} - x_{2,2}} + e^{x_{2,1} + x_{2,2}} 
+ e^{x_{2,1} - x_{2,2}} + e^{x_{2,1} + x_{2,2}} + e^{z_{2,2} - z_{2,1}} + e^{z_{2,2} - z_{2,1}}) \right\}, \quad (1.168)

where \( z_1 = z_{2,1}, z_2 = z_{2,2} \).

There is a simple combinatorial description of the potential \( \mathcal{F}^{C_\ell} \) for the zero spectrum \( \{ \lambda_i = 0 \} \). Namely, it can be represented as the sum over all arrows in the diagram

We use the same rule to assign variables to the arrows of the diagram as for \( A_\ell \). In addition we assign to the symbol \( z \rightarrow x \) the exponential \( e^{z+x} \).

Note that the diagram for \( C_\ell \) can be obtained by a factorization of the diagram for \( A_{2\ell-1} \). Consider the involution

\[ \ast : X \mapsto -\hat{W}_0^{-1} X^t \hat{W}_0, \quad (1.169) \]

where \( \hat{W}_0 \) is a lift of the longest element in the Weyl group \( W(A_{2\ell-1}) = S_{2\ell} \) and \( X^t \) denotes the standard transposition. The corresponding action on the modified factorization parameters is given by

\[ \ast : x_{k,i} \mapsto -x_{k,k+1-i}. \quad (1.170) \]

This defines a factorization of the \( A_{2\ell-1} \)-diagram that produces the diagram for \( C_\ell \).

One can easily write a \( C_\ell \)-analogue of the toric \( A_\ell \)-relations (1.52). Let us introduce the variables

\[
\begin{align*}
a_{n,1} &= e^{x_{n,1} + z_{n-1,1}}, & a_{n,i} &= e^{x_{n,i} - z_{n-1,i-1}}, & 2 \leq i \leq n \leq \ell; \\
b_{k,1} &= e^{x_{k,1} + z_{k,1}}, & k &= 1, \ldots, \ell; & b_{n,i} &= e^{x_{n,i} - x_{n,i-1}}, & 2 \leq i \leq n \leq \ell; \\
c_{k,i} &= e^{z_{k,i} - x_{k,i}}, & 1 \leq i \leq k \leq \ell; \\
d_{m,j} &= e^{z_{m,j} - x_{m+1,j}}, & 1 \leq j \leq m < \ell.
\end{align*}
\]

(1.171)
Then the following relations hold:

\[ c_{k,i}b_{k,i} = d_{k,i}a_{k+1,i}, \quad a_{k,i}d_{k-1,i-1} = b_{k,i}c_{k,i-1}, \quad 1 \leq i \leq k < \ell; \]
\[ b_{\ell,1}c_{\ell,1} = e^{2z_{\ell,1}}, \quad c_{\ell,i}b_{\ell,i} = e^{z_{\ell,i-1}z_{\ell,i-1}}, \quad i = 2, \ldots, \ell. \] (1.172)

The above relations can be considered as relations between elementary paths on the Givental diagram. Using relations following from (1.172) for more general paths, one can define a toric degeneration of the \( C_\ell \)-flag manifolds, thus generalizing results of [5] and [19]. Note that the above diagram coincides with the Gelfand–Zetlin diagram for \( \mathfrak{sp}_{2\ell} \).

1.4.3. Recursion for \( \mathfrak{sp}_{2\ell} \)-Whittaker functions and the \( \mathcal{Q} \)-operator for an \( A^{(2)}_{2\ell-1} \)-Toda chain. The integral representation (1.165), (1.166) of \( \mathfrak{sp}_{2\ell} \)-Whittaker functions possesses a recursive structure with respect to the rank \( \ell \). For any \( n = 2, \ldots, \ell \) let us introduce integral operators \( Q^c_{n-1} \) with the integral kernels

\[
Q^c_{n-1}(z_n, z_{n-1}; \lambda_n) = \int dx_n Q^c_n(z_n, x_n)Q^D_{n-1}(x_n, z_{n-1})(e^{zn,1} + e^{zn-1,1})^{i\lambda_n}
\times \exp \left\{-i\lambda_n \left( \sum_{i=1}^{n} z_{n,i} - x_{n,1} - 2 \sum_{i=2}^{n} x_{n,i} + \sum_{i=1}^{n-1} z_{n-1,i} \right) \right\},
\] (1.173)

where

\[
Q^D_{n-1}(x_n, z_{n-1}) = \exp \left\{-e^{x_{n,1}+zn-1,1} - \sum_{i=1}^{n-1} \left( e^{zn-1,i-x_{n,i}} + e^{x_{n,i+1}-zn-1,i} \right) \right\},
\] (1.174)
\[
Q^c_{n}(z_n, x_n) = \exp \left\{-e^{x_{n,1}+zn,1} - \sum_{i=1}^{n-1} \left( e^{zn,i-x_{n,i}} + e^{x_{n,i+1}-zn,i} - e^{zn,n-x_{n,n}} \right) \right\}.
\] (1.175)

For \( n = 1 \) we define

\[
Q^c_0 = \int dx_{1,1} e^{1\lambda_1 x_{1,1}} \exp \left\{- (e^{x_{1,1}+z_{1,1}} + e^{z_{1,1}-x_{1,1}}) \right\}.
\]

By using the integral operators \( Q^c_{n-1} \) the integral representation for \( \mathfrak{sp}_{2\ell} \)-Whittaker functions can be written in recursive form.

**Theorem 1.11.** The integral representations of \( \mathfrak{sp}_{2\ell} \)-Toda chain eigenfunctions (1.165) can be written as

\[
\Psi^c_{\lambda_1, \ldots, \lambda_\ell}(z_1, \ldots, z_\ell) = \int_{C} \prod_{k=1}^{\ell-1} d\tilde{z}_k \prod_{n=1}^{\ell} Q^c_{n-1}(z_n, z_n; \lambda_n),
\] (1.176)

or equivalently,

\[
\Psi^c_{\lambda_1, \ldots, \lambda_\ell}(z_\ell) = \int_{\mathcal{G}_\ell} d\tilde{z}_{\ell-1} Q^c_{\ell-1}(\tilde{z}_{\ell-1}; \tilde{z}_\ell; \lambda_\ell) \Psi^c_{\lambda_1, \ldots, \lambda_{\ell-1}}(\tilde{z}_{\ell-1}).
\] (1.177)
Here $z_i := z_{\ell,i}$, $i = 1, \ldots, \ell$, and $C \subset \mathbb{N}_+$ is a middle-dimensional non-compact submanifold such that the integrand decreases exponentially at the boundary and at infinity.

This recursive form of the integral representation is similar to the case of $\mathfrak{so}_{2\ell+1}$. Its recursive kernel $Q_{C_{n-1}}^{C_n}$ is given by a non-trivial integral in contrast to the $\mathfrak{gl}_{\ell+1}$ case (1.54). As in the case of the $\mathfrak{so}_{2\ell+1}$-Whittaker functions, a new structure appears if we consider the Whittaker function for the zero spectrum $\{\lambda_i = 0\}$. It is clear from (1.173) that the kernels $Q_{C_{n-1}}^{C_n}$ at $\lambda_n = 0$ are given by convolutions of the kernels $Q_{D_n}^{D_n}(z_n, x_n)$ and $Q_{C_{n-1}}^{D_n}(x_n, z_{n-1})$. The corresponding integral operators $Q_{D_n}^{C_n}, Q_{C_{n-1}}^{C_n}$ can be regarded as elementary intertwiners connecting Hamiltonians of Toda chains for $C_n, D_n$ and $D_n, C_{n-1}$ root systems, respectively. For example, it is easy to check the intertwining relations for the quadratic Hamiltonians directly. Indeed, the $D_{\ell}$-Toda chain has the following quadratic Hamiltonians (for a more detailed discussion see §2.4.3):

$$H_{2\ell}^D(z^{(\ell)}) = -\frac{1}{2} \sum_{i=1}^{\ell} \frac{\partial^2}{\partial x_i^2} + e^{x_1+x_2} + \sum_{i=1}^{\ell-1} e^{x_{i+1}-x_i}. \quad (1.178)$$

**Proposition 1.10.** The integral operators $Q_{C_{n-1}}^{C_n}, Q_{D_n}^{C_n}$, and $Q_{C_{n-1}}^{D_n}$ satisfy the following relations.

1. The operators $Q_{D_n}^{C_n}$ and $Q_{C_{n-1}}^{D_n}$ intertwine the quadratic Hamiltonians for $C$- and $D$-Toda chains:

$$H_{2\ell}^D(x_n)Q_{C_{n-1}}^{D_n}(x_n, z_{n-1}) = Q_{D_n}^{D_n}(z_n, x_{n-1})H_{2\ell}^{C_{n-1}}(z_{n-1}), \quad (1.179)$$

$$H_{2\ell}^C(z_n)Q_{C_{n-1}}^{C_n}(z_n, x_n) = Q_{C_{n-1}}^{D_n}(z_n, x_n)H_{2\ell}^{D_n}(z_n). \quad (1.180)$$

2. The operator $Q_{C_{n-1}}^{C_n}$ at $\lambda_n=0$ intertwines the Hamiltonians $H_{2\ell}^C$ and $H_{2\ell}^{C_{n-1}}$:

$$H_{2\ell}^C(z_n)Q_{C_{n-1}}^{C_n}(z_n, z_{n-1}) = Q_{C_{n-1}}^{C_n}(z_n, z_{n-1})H_{2\ell}^{C_{n-1}}(z_{n-1}). \quad (1.181)$$

The integral kernel of $Q_{C_{n-1}}^{C_n}$ can be succinctly encoded into the following diagram:
Here the upper (lower) boundary of the oriented diagram corresponds to the kernels of the elementary intertwiners $Q^C_n (Q^D_n)$ and the convolution of the two kernels is given by integration with respect to the variables $x_{n,1}, \ldots, x_{n,n}$ on the diagonal of the diagram.

As in the cases of $\mathfrak{gl}_{\ell+1}$ and $\mathfrak{sp}_{2\ell}$, the recursion operators $Q^C_n$ can be considered as degenerations of the Baxter $\mathcal{Q}$-operators for the twisted affine $A^{(2)}_{2\ell-1}$-Toda chain introduced below. We stress that up to now $\mathcal{Q}$-operators for $A^{(2)}_{2\ell-1}$ were not known. We will not present here a complete set of the characteristic properties of the $\mathcal{Q}$-operators introduced and will only consider commutation relations for the quadratic affine Toda chain Hamiltonians. A detailed account will be given elsewhere.

We start with a description of $A^{(2)}_{2\ell-1}$-Toda chains. The set of simple roots of the affine root system $A^{(2)}_{2\ell-1}$ can be represented in terms of the orthogonal basis \{\(\epsilon_i, i = 1, \ldots, \ell\)\} in \(\mathbb{R}^\ell\) as follows:

\[
\alpha_1 = 2\epsilon_1, \quad \alpha_{i+1} = \epsilon_{i+1} - \epsilon_i, \quad 1 \leq i \leq \ell - 1, \quad \alpha_{\ell+1} = -\epsilon_\ell - \epsilon_{\ell-1}, \quad (1.182)
\]

and the corresponding Dynkin diagram is given by

\[
\alpha_1 \rightarrow \alpha_2 \rightarrow \cdots \rightarrow \alpha_{\ell-1} \rightarrow \alpha_\ell
\]

These root data enable us to define the affine $A^{(2)}_{2\ell-1}$-Toda chain with quadratic Hamiltonian given by

\[
\mathcal{H}^{A^{(2)}_{2\ell-1}}(z^{(\ell)}; \lambda) = -\frac{1}{2} \sum_{i=1}^{\ell} \frac{\partial^2}{\partial z_i^2} + 2e^{2z_1} + \sum_{i=1}^{\ell-1} e^{z_{i+1} - z_i} + ge^{-z_{\ell-1} - z_\ell}, \quad (1.183)
\]

where \(g\) is an arbitrary parameter.

We define the Baxter $\mathcal{Q}$-operator for the $A^{(2)}_{2\ell-1}$-Toda chain as an integral operator with the integral kernel

\[
\mathcal{Q}^{A^{(2)}_{2\ell-1}}(z^{(\ell)}; y^{(\ell)}; \lambda) = \int dx^{(\ell)} Q^{A^{(2)}_{2\ell-1}}(z^{(\ell)}, x^{(\ell)}) Q^{A^{(2)}_{2\ell-1}}(x^{(\ell)}, y^{(\ell)})
\]

\[
\times \exp\left\{-\lambda \left(\sum_{i=1}^{\ell} z_i - x_1 - 2 \sum_{i=2}^{\ell} x_i + \sum_{i=1}^{\ell} y_i\right)\right\}(e^{z_1} + e^{y_1})^{\lambda}(e^{-z_\ell} + e^{-y_\ell})^{-2\lambda}, \quad (1.184)
\]
where

\[ Q_{A_{2\ell-1}^{(2)}}^{(2)}(z^{(\ell)}, \bar{z}^{(\ell)}) = \exp \left\{ -e^{z_1+x_1} - \sum_{i=1}^{\ell-1} \left( e^{z_i-x_i} + e^{x_i+1-z_i} - e^{z_{i+1}-x_i} - e^{z_i-x_{i+1}} - e^{y_{i+1}-y_i} - e^{y_i-x_{i+1}} - ge^{-z_{i+1}-x_i} \right) \right\}, \]

(1.185)

\[ Q_{A_{2\ell-1}^{(2)}}^{(2)}(x^{(\ell)}, y^{(\ell)}) = \exp \left\{ -e^{y_1+x_1} - \sum_{i=1}^{\ell-1} \left( e^{y_i-x_i} + e^{x_i+1-y_i} - e^{y_{i+1}-x_i} - e^{y_i-x_{i+1}} - e^{y_{i+1}+x_i} - e^{y_i-x_{i+1}} - ge^{-y_{i+1}-x_i} \right) \right\}, \]

(1.186)

Here we use the notation \( z^{(\ell)} = (z_1, \ldots, z_{\ell}) \), \( y^{(\ell)} = (y_1, \ldots, y_{\ell}) \).

The following statement can be verified in a straightforward manner.

**Proposition 1.11.** The \( \mathcal{Q} \)-operator (1.184) commutes with the quadratic Hamiltonian of the \( A_{2\ell-1}^{(2)} \)-Toda chain:

\[ \mathcal{H} A_{2\ell-1}^{(2)}(z^{(\ell)}) Q_{A_{2\ell-1}^{(2)}}^{(2)}(z^{(\ell)}, \bar{z}^{(\ell)}) = Q_{A_{2\ell-1}^{(2)}}^{(2)}(z^{(\ell)}, \bar{z}^{(\ell)}) \mathcal{H} A_{2\ell-1}^{(2)}(y^{(\ell)}). \]

(1.187)

Now we will demonstrate that the recursion operator \( Q_{C_{\ell-1}}^{\mathcal{C}_\ell} \) can be obtained by a degeneration of the Baxter \( \mathcal{Q} \)-operator for \( A_{2\ell-1}^{(2)} \). We consider a slightly modified recursion operator \( Q_{C_{\ell-1}^{\mathcal{C}_\ell}}^{\mathcal{C}_\ell} \) with kernel given by

\[ Q_{C_{\ell-1}^{\mathcal{C}_\ell}}^{\mathcal{C}_\ell}(z^{(\ell)}, y^{(\ell)}; \lambda) := e^{i\lambda y_{\ell}} Q_{C_{\ell-1}}^{\mathcal{C}_\ell}(z^{(\ell)}, y^{(\ell-1)}; \lambda), \]

where \( y^{(\ell-1)} = (y_1, \ldots, y_{\ell-1}) \). The operator thus defined intertwines the Hamiltonians of the \( \mathfrak{sp}_{2\ell} \) and \( \mathfrak{sp}_{2\ell-2} \oplus \mathfrak{sp}_{2} \)-Toda chains. Thus, for the quadratic Hamiltonians we have

\[ \mathcal{H}^{\mathcal{C}_\ell}_{2}(z^{(\ell)}) Q_{C_{\ell-1}^{\mathcal{C}_\ell}}^{\mathcal{C}_\ell}(z^{(\ell)}, y^{(\ell)}; \lambda) = Q_{C_{\ell-1}^{\mathcal{C}_\ell}}^{\mathcal{C}_\ell}(z^{(\ell)}, y^{(\ell)}; \lambda) \times \left( \mathcal{H}^{\mathcal{C}_\ell}_{2}(y^{(\ell-1)}) + \mathcal{H}^{\mathcal{C}_\ell}_{2}(y_{\ell}) \right), \]

where \( \mathcal{H}^{\mathcal{C}_\ell}_{2}(y_{\ell}) = -\frac{1}{2} \frac{\partial^2}{\partial y_{\ell}^2} \). Obviously, the projection of above equation on the subspace of functions \( F(y^{(\ell)}) = \exp(i\lambda y_{\ell}) f(y^{(\ell-1)}) \) recovers the initial recursion operator satisfying the conditions

\[ \mathcal{H}^{\mathcal{C}_\ell}_{2}(z^{(\ell)}) Q_{C_{\ell-1}^{\mathcal{C}_\ell}}^{\mathcal{C}_\ell}(z^{(\ell)}, y^{(\ell-1)}; \lambda) = Q_{C_{\ell-1}^{\mathcal{C}_\ell}}^{\mathcal{C}_\ell}(z^{(\ell)}, y^{(\ell-1)}; \lambda) \left( \mathcal{H}^{\mathcal{C}_\ell}_{2}(y^{(\ell-1)}) + \frac{1}{2} \lambda^2 \right). \]

(1.188)

Consider the one-parameter family of kernels

\[ Q_{A_{2\ell-1}^{(2)}}^{(2)}(z^{(\ell)}, y^{(\ell)}; \lambda, \varepsilon) \]

\[ = \varepsilon^{i\lambda} e^{i\lambda y_{\ell}} \int d\tilde{z}^{(\ell+1)} Q_{A_{2\ell-1}^{(2)}}^{(2)}(\tilde{z}^{(\ell+1)}, \bar{z}^{(\ell+1)}) Q_{A_{2\ell-1}^{(2)}}^{(2)}(\tilde{z}^{(\ell+1)}, y^{(\ell)}; \varepsilon) \times \exp \left\{ -i\lambda \left( \sum_{i=1}^{\ell} z_i - x_1 - 2 \sum_{i=2}^{\ell} x_i + \sum_{i=1}^{\ell-1} y_i \right) \right\} (e^{z_1} + e^{y_1})^{i\lambda} (\varepsilon e^{y_{\ell}-z_{\ell}+1} + 1)^{-2\lambda}, \]

(1.189)
where
\[
Q^{A_{2\ell-1}^{(2)}}_{A_{2\ell-1}^{(2)}}(z^{(\ell+1)}, y^{(\ell)}; \varepsilon) = \exp \left\{ -e^{y_1+x_1} - \sum_{i=1}^{\ell-1}(e^{y_i-x_i} + e^{x_{i+1}-y_i}) - \varepsilon(e^{y_\ell-x_\ell} - 1) + \varepsilon^{-1}ge^{-y_\ell-x_\ell} \right\}, \tag{1.190}
\]
is obtained by the shift of variable \(y_\ell \to y_\ell + \log \varepsilon\) in (1.184)-(1.186). Then the following relation holds between the \(Q\)-operator for the \(A_{2\ell-1}^{(2)}\)-Toda chain and the recursion operator for the \(sp_{2\ell}\)-Whittaker function:
\[
Q^{C_{\ell-1}}_{C_{\ell-1} \oplus C_1}(z^{(\ell)}, y^{(\ell)}; \lambda) = \lim_{\varepsilon \to 0, \varepsilon^{-1}y \to 0} \varepsilon^{-1} \mathcal{Q}^{A_{2\ell-1}^{(2)}}(z^{(\ell)}, y^{(\ell)}; \lambda; \varepsilon). \tag{1.191}
\]

1.5. Integral representations of \(so_{2\ell}\)-Toda chain eigenfunctions. In this subsection we provide an analogue of the Givental integral representation of the Whittaker functions for the Lie algebras \(so_{2\ell}\). As in the previously considered cases, we start with a derivation of an integral representation of the \(so_{2\ell}\)-Whittaker functions using a factorized parametrization. Then we consider a modification of the factorized parametrization leading to a Givental-type integral representations of the \(so_{2\ell}\)-Whittaker functions.

Consider a \(D_\ell\) root system corresponding to the Lie algebra \(so_{2\ell}\). Let \(\{\epsilon_1, \ldots, \epsilon_\ell\}\) be an orthonormal basis in \(\mathbb{R}^\ell\). We realize \(D_\ell\) simple roots and fundamental weights as the following vectors in \(\mathbb{R}^\ell\):
\[
\begin{align*}
\alpha_1 &= \epsilon_2 - \epsilon_1, & \omega_1 &= \frac{1}{2}(-\epsilon_1 + \epsilon_2 + \cdots + \epsilon_\ell), \\
\alpha_2 &= \epsilon_2 + \epsilon_1, & \omega_2 &= \frac{1}{2}(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_\ell), \\
\alpha_3 &= \epsilon_3 - \epsilon_2, & \omega_3 &= \epsilon_3 + \cdots + \epsilon_\ell, \\
\vdots & & \vdots \\
\alpha_n &= \epsilon_\ell - \epsilon_{\ell-1}, & \omega_\ell &= \epsilon_\ell.
\end{align*}
\tag{1.192}
\]
The coroots \(\alpha_i^\vee\) can be identified with the corresponding roots \(\alpha_i\) using the standard scalar product in \(\mathbb{R}^\ell\). One associates with these root data the \(so_{2\ell}\)-Toda chain with the quadratic Hamiltonian given by
\[
\mathcal{H}_2^{D_\ell} = -\frac{1}{2} \sum_{i=1}^\ell \frac{\partial^2}{\partial x_i^2} + e^{x_1+x_2} + \sum_{i=1}^{\ell-1} e^{x_{i+1}-x_i}. \tag{1.193}
\]
We can supplement (1.193) to get a complete set of \(\ell\) pairwise commuting functionally independent Hamiltonians \(\mathcal{H}_k^{D_\ell}\) for the \(so_{2\ell}\)-Toda chain. We are looking for integral representations of common eigenfunctions of this complete set of Hamiltonians. The corresponding eigenfunction problem for the quadratic Hamiltonian can be written in the form
\[
\mathcal{H}_2^{D_\ell} \Psi^{D_\ell}_{\lambda_1, \ldots, \lambda_\ell}(x_1, \ldots, x_\ell) = \frac{1}{2} \sum_{i=1}^\ell \lambda_i^2 \Psi^{D_\ell}_{\lambda_1, \ldots, \lambda_\ell}(x_1, \ldots, x_\ell). \tag{1.194}
\]
1.5.1. \( \mathfrak{so}_{2\ell}\)-Whittaker function: factorized parametrization. The reduced word for the maximal length element \( w_0 \) in the Weyl group of \( \mathfrak{so}_{2\ell} \) can be represented in the following recursive way:

\[
I_\ell = (i_1, i_2, \ldots, i_m) := (12, 3123, \ldots, (\ell \ldots 3123 \ldots \ell)), \quad m_\ell = \ell(\ell - 1),
\]

where the index \( i_k \in \Gamma = \{1, \ldots, \ell\} \) corresponds to an elementary reflection with respect to the simple root \( \alpha_{i_k} \). Let \( N_+ \subset G \) be a maximal unipotent subgroup of \( G = \text{SO}(2\ell) \). One associates with the reduced word \( I_\ell \) the following recursive parametrization of a generic element \( v^{D_\ell} \in N_+ \):

\[
v^{D_\ell} = v^{D_{\ell - 1}} \mathfrak{X}^{D_\ell}_{D_{\ell - 1}}, \quad (1.195)
\]

where

\[
\mathfrak{X}^{D_\ell}_{D_{\ell - 1}} = X_\ell(y_{\ell,1}) \cdots X_k(y_{k,2(\ell + 1 - k)-1}) \cdots X_3(y_{3,2\ell - 5})X_1(y_{1,\ell - 1}) \cdots \times X_2(y_{2,\ell - 1})X_3(y_{3,2\ell - 4}) \cdots X_k(y_{k,2(\ell + 1 - k)}) \cdots X_\ell(y_{\ell,2}). \quad (1.196)
\]

Here \( X_i(y) = e^{y f_i} \) and the \( f_i \equiv f_{\alpha_i} \) are simple root generators. The subset \( N_+^{(0)} \) of elements admitting a representation \((1.195), (1.196)\) is an open part of \( N_+ \). The action of the Lie algebra \( \mathfrak{so}_{2\ell} \) on \( N_+ \) (1.10) defines an action on the space of functions on \( N_+^{(0)} \). The explicit description of the action on the space \( V_\mu \) of (twisted) functions on \( N_+^{(0)} \) is given below.

**Proposition 1.12.** The following differential operators define a realization of a principal series representation \( \pi_\mu \) of \( \mathcal{U}(\mathfrak{so}_{2\ell}) \) on \( V_\mu \) in terms of a factorized parametrization of \( N_+^{(0)} \):

\[
F_i = \frac{\partial}{\partial y_{i, \ell - 1}} + \sum_{n = 1}^{\alpha(\ell) - 1} \prod_{k = 2n - 1}^{\ell - 2} \frac{y_{3,2k}}{y_{3,2k-1}} \prod_{j = n}^{\alpha(\ell) - 1} \frac{y_{i,2j+\omega(\ell)}}{y_{i,j+1+\omega(\ell)}} \frac{y_{i,j+1+\omega(\ell)}}{y_{i,2j+\omega(\ell)}} \times \left\{ \frac{\partial}{\partial y_{i,2n-1+\omega(\ell)}} - \frac{\partial}{\partial y_{i,2n+\omega(\ell)}} \right\} + \sum_{n = 1}^{\beta(\ell) - 1} \prod_{k = 2n - 1}^{\ell - 2} \frac{y_{3,2k}}{y_{3,2k-1}} \frac{y_{i,\ell - 1}}{y_{i,\ell - 1}} \times \prod_{j = n + 1}^{\beta(\ell) - 1} \frac{y_{i,2j-\omega(\ell)}}{y_{i,2j-1-\omega(\ell)}} \frac{y_{i,2j-1-\omega(\ell)}}{y_{i,2j-\omega(\ell)}} \left\{ \frac{\partial}{\partial y_{i,2n-1+\omega(\ell)}} - \frac{\partial}{\partial y_{i,2n+1+\omega(\ell)}} \right\} \\
+ \sum_{n = 1}^{\ell - 3} \prod_{k = n}^{\ell - 2} \frac{y_{3,2k}}{y_{3,2k-1}} \prod_{j = \alpha(n+\omega(\ell))}^{\alpha(\ell) - 1} \frac{y_{i,\ell - 1}}{y_{i,\ell - 1}} \frac{y_{i,2j+1+\omega(\ell)}}{y_{i,2j+\omega(\ell)}} \times \prod_{m = \beta(n+\omega(\ell))}^{\beta(\ell) - 2} \frac{y_{i,2m+2-\omega(\ell)}}{y_{i,2m+1-\omega(\ell)}} \left\{ \frac{\partial}{\partial y_{i,2n-1}} - \frac{\partial}{\partial y_{3,2n}} \right\} \quad (1.197)
\]

for \( i = 1, 2 \) and \( *: 1 \leftrightarrow 2 \).
\[ F_k = \frac{\partial}{\partial y_{k,2}(\ell+1-k)} \]
\[
+ (1 - \delta_{k,\ell}) \sum_{n=1}^{\ell-k} \prod_{j=n}^{\ell-k} \frac{y_{k+1,2j} y_{k,2j+1}}{y_{k+1,2j-1} y_{k,2j+2}} \left( \frac{\partial}{\partial y_{k,2n}} - \frac{\partial}{\partial y_{k,2n+1}} \right) 
+ \frac{y_{k+1,2}(\ell-k)}{y_{k,2}(\ell+1-k)} \prod_{j=n}^{\ell-1-k} \frac{y_{k+1,2j} y_{k,2j+3}}{y_{k+1,2j+1} y_{k,2j+2}} \left( \frac{\partial}{\partial y_{k+1,2n-1}} - \frac{\partial}{\partial k+1,2n} \right) \right) \]
\]
\[ (1.198) \]

for \( k = 3, \ldots, \ell, \)
\[ H_k = \langle \mu, \alpha_k^\vee \rangle + \sum_{i=1}^{\ell} a_{k,i} \sum_{j=1}^{n_i} y_{i,j} \frac{\partial}{\partial y_{i,j}} \]
\[ (1.199) \]

for \( k = 1, \ldots, \ell, \)
\[ E_i = \sum_{n=1}^{\ell-1} y_{i,n} \left\{ -\langle \mu, \alpha_i^\vee \rangle - 2 \sum_{k=1}^{n-1} y_{i,k} \frac{\partial}{\partial y_{i,k}} + 2 \sum_{k=1}^{2n-3} y_{3,k} \frac{\partial}{\partial y_{3,k}} - y_{i,n} \frac{\partial}{\partial y_{i,n}} \right\} \]
\[ (1.200) \]

for \( i = 1, 2, \)
\[ E_3 = \sum_{n=1}^{2(\ell-2)} y_{3,n} \left\{ \mu_2 - \mu_3 + \sum_{k=1}^{\alpha(n)+1} \left( y_{1,k} \frac{\partial}{\partial y_{1,k}} + y_{2,k} \frac{\partial}{\partial y_{2,k}} \right) \right. 
+ \sum_{k=1}^{2\beta(n)-3} y_{4,k} \frac{\partial}{\partial y_{4,k}} - 2 \sum_{k=1}^{n-1} y_{3,k} \frac{\partial}{\partial y_{3,k}} - y_{3,n} \frac{\partial}{\partial y_{3,n}} \right\} \]
\[ (1.201) \]
\[ E_k^{(\ell)} = \sum_{n=1}^{2(\ell+1-k)} y_{k,n} \left\{ \mu_{k-1} - \mu_k + \sum_{j=1}^{2\alpha(n)+2} y_{k-1,j} \frac{\partial}{\partial y_{k-1,j}} + \sum_{j=1}^{2\beta(n)-3} y_{k+1,j} \frac{\partial}{\partial y_{k+1,j}} 
- 2 \sum_{j=1}^{n-1} y_{k,j} \frac{\partial}{\partial y_{k,j}} - y_{k,n} \frac{\partial}{\partial y_{k,n}} \right\}, \quad k = 4, \ldots, \ell, \]
\[ (1.202) \]

where \( \alpha(n) := [n/2], \beta(n) := [(n+1)/2], \) and \( \varpi(n) = (1 - (-1)^n)/2. \)

The proof is given in Part 2, §2.4.4.

The left/right Whittaker vectors are defined by (1.7) and (1.8) and can be found explicitly in the factorized parametrization.
Lemma 1.9. The following expressions hold for the left/right Whittaker vectors:

\[
\psi_R(y) = \exp \left\{ - \left( \sum_{n=1}^{\ell-1} y_{1,n} + \sum_{n=1}^{\ell-1} y_{2,n} + \sum_{k=3}^{\ell} \sum_{n=1}^{2(\ell+1-k)} y_{k,n} \right) \right\},
\]

\[
\psi_L(y) = \left( \prod_{k=1}^{\ell-1} y_{1,k} \right)^{1-\varpi(k)} \left( \prod_{n=3}^{\ell} y_{1,2(n+1-i)-1} \right)^{\mu,\alpha_1^\gamma} \left( \prod_{i=k+1}^{\ell} y_{1,2n-1} \right)^{\mu,\alpha_1^\nu}
\]

\[
\times \exp \left\{ - \left( \sum_{n=1}^{\ell-1} \left[ \frac{1}{y_{1,\ell-1}} \prod_{k=1}^{n-1} \left( \frac{y_{1,\ell-k}}{y_{1,\ell-k-1}} \right)^{1-\varpi(k)} \left( \frac{y_{2,\ell-k}}{y_{2,\ell-k-1}} \right)^{\varpi(k)} \right] \left( 1 + \frac{y_{3,2(\ell-n-1)}}{y_{3,2(\ell-n-1)-1}} \right) \right) \right\},
\]

where \( \varpi(n) = \frac{(1 - (-1)^n)}{2} \) is the parity of \( n \).

The proof is given in Part 2, § 2.3.4.

By using (1.17) and (1.18) it is easy to obtain the integral representations of an \( \mathfrak{so}_{2\ell} \)-Whittaker function in the factorized parametrization.

Theorem 1.12. The eigenfunctions of the \( \mathfrak{so}_{2\ell} \)-Toda chain admit the following integral representation:

\[
\Psi_{\lambda_1, \ldots, \lambda_\ell}^D(x_1, \ldots, x_\ell) = e^{i\lambda_1 x_1 + \cdots + i\lambda_\ell x_\ell} \int_C \prod_{i=1}^{\ell} y_{i,k} dy_{i,k}
\]

\[
\times \prod_{k=1}^{\ell-1} \left( \frac{y_{1,k}}{y_{2,k}} \right)^{i\lambda_1(1)^k} \left( \prod_{n=3}^{\ell} y_{1,2(n+1-i)-1} \right)^{2i\lambda_2}
\]

\[
\times \prod_{k=3}^{\ell} \left( \prod_{i=1}^{n_k} y_{i,n} \prod_{i=k+1}^{n_k} y_{i,2n-1} \right)^{i(\lambda_k - \lambda_{k-1})}
\]

\[
\times \exp \left\{ - \left( \sum_{n=1}^{\ell-1} \left[ \frac{1}{y_{1,\ell-1}} \prod_{k=1}^{n-1} \left( \frac{y_{1,\ell-k}}{y_{1,\ell-k-1}} \right)^{1-\varpi(k)} \left( \frac{y_{2,\ell-k}}{y_{2,\ell-k-1}} \right)^{\varpi(k)} \right] \left( 1 + \frac{y_{3,2(\ell-n-1)}}{y_{3,2(\ell-n-1)-1}} \right) \right) \right\},
\]
Example 1.6. For \( \ell = 3 \) the general formula (1.205) acquires the following form

\[
\Psi_{D_3}^{\mu_1,\mu_2,\mu_3}(x_1, x_2, x_3) = e^{\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3} \int_C \prod_{i=1}^{3} \prod_{k=1}^{2} \frac{dy_{i,k}}{y_{i,k}} \times (y_{1,1} y_{3,1} y_{2,2})^{(\lambda_2 - \lambda_1)} (y_{2,1} y_{3,1} y_{1,2})^{(\lambda_2 + \lambda_1)} (y_{3,1} y_{1,2} y_{2,2} y_{3,2})^{(\lambda_3 - \lambda_2)} 
\times \exp \left\{ \frac{1}{y_{1,2}} \left( 1 + \frac{y_{3,2}}{y_{3,1}} \right) + \frac{1}{y_{1,2}} \frac{y_{2,2}}{y_{3,1}} \right\} + \frac{1}{y_{3,2}} + e^{x_2 - x_1}(y_{1,1} + y_{1,2}) + e^{x_2 + x_1}(y_{1,2} + y_{2,2}) + e^{x_3 - x_2}(y_{3,1} + y_{3,2}) \right\}.
\]

(1.206)

1.5.2. \( \mathfrak{so}_{2\ell} \)-Whittaker function: modified factorized parametrization. In this part we introduce a modified factorized parametrization of an open part \( N_+^{(0)} \) of a maximal unipotent subgroup \( N_+ \subset \mathfrak{so}(2\ell) \). We use this parametrization to construct integral representations for \( \mathfrak{so}_{2\ell} \)-Whittaker functions. Like other series of classical Lie algebras, these integral representations for \( \mathfrak{so}_{2\ell} \)-Whittaker functions have a simple recursive structure with respect to the rank \( \ell \) and can be described in purely combinatorial terms using suitable graphs. These representations can be considered as a generalization of Givental integral representations to the case \( \mathfrak{g} = \mathfrak{so}_{2\ell} \).

We follow the same approach that was used in the description of the modified factorized representation for other classical groups. There is a realization of the tautological representation \( \pi: \mathfrak{so}_{2\ell} \to \text{End}(\mathbb{C}^{2\ell}) \) such that the Chevalley generators corresponding to the Borel (Cartan) subalgebra of \( \mathfrak{so}_{2\ell} \) are realized by upper triangular (respectively, diagonal) matrices. This defines an embedding \( \mathfrak{so}_{2\ell} \subset \mathfrak{gl}_{2\ell} \) such that the Borel (Cartan) subalgebra maps into the Borel (Cartan) subalgebra (see, for instance, [17]). To define the corresponding embedding of the groups, consider the following involution on \( \text{GL}(2\ell) \):

\[
g \mapsto g^* := \hat{W}_0^{-1}(g^{-1})^t \hat{W}_0,
\]

(1.207)
where \( a' \) is induced by the standard transposition of a matrix \( a \) and \( \tilde{W}_0 \) is a lift of the longest element in the Weyl group \( \tilde{W}(\mathfrak{gl}_{2\ell}) = S_{2\ell} \). In matrix form it can be written as

\[
\pi(\tilde{W}_0) = SJ,
\]

for \( S = \text{diag}(1, -1, \ldots, (-1)^{\ell-1}, (-1)^{\ell-1}, \ldots, 1) \) and \( J = \|J_{i,j}\| = \|\delta_{i+j, 2\ell+1}\| \). The orthogonal group \( G = \text{SO}(2\ell) \) can then be identified with the following subgroup of \( \text{GL}(2\ell) \) (see, for example, [17]):

\[
\text{SO}(2\ell) = \{ g \in \text{GL}(2\ell) : g^* = g \}.
\]

Let \( \epsilon_{i,j} \) be the elementary \( 2\ell \times 2\ell \) matrix with 1 at the \((i,j)\) position and zeros elsewhere. We introduce the following matrices:

\[
U_2 = \sum_{i=1}^{\ell} \epsilon_{i,i} + e^{-x_{1,1}} \epsilon_{\ell+1,\ell+1} + e^{x_{1,1}} \epsilon_{\ell+2,\ell+2} + \sum_{i=\ell+3}^{2\ell} \epsilon_{i,i},
\]

\[ (1.208) \]

\[
\tilde{U}_2 = U_2 + e^{x_{1,1}} (2\epsilon_{\ell,\ell+1} + \epsilon_{\ell+1,\ell+2});
\]

\[
U_n = \sum_{i=1}^{\ell} \epsilon_{i,i} + e^{-x_{n-1,1}} \epsilon_{\ell+1,\ell+1} + \sum_{k=1}^{n-1} e^{x_{n-1,k}} \epsilon_{\ell+1+k,\ell+1+k} + \sum_{i=\ell+1+n}^{2\ell} \epsilon_{i,i},
\]

\[ (1.209) \]

\[
\tilde{U}_n = U_n + 2e^{x_{n-1,1}} \epsilon_{\ell,\ell+1} + \sum_{k=1}^{n-1} e^{x_{n-1,k}} \epsilon_{\ell+k,\ell+1+k},
\]

\[
\tilde{U}'_n = U_n + 2e^{x_{n-1,1}} \epsilon_{\ell,\ell+1} + e^{x_{n-1,1}} \epsilon_{\ell+1,\ell+2},
\]

\[
\tilde{U}''_n = U_n + \sum_{k=2}^{n-1} e^{x_{n-1,k}} \epsilon_{\ell+k,\ell+1+k}
\]

for \( n = 3, \ldots, \ell, \) and

\[
V_2 = \sum_{i=1}^{\ell} \epsilon_{i,i} + \epsilon_{\ell+1+i,\ell+1+i} + e^{-x_{1,1}} \epsilon_{\ell,\ell} + e^{x_{1,1}} \epsilon_{\ell+1,\ell+1},
\]

\[ (1.210) \]

\[
\tilde{V}_2 = V_2 + e^{x_{n-1,1}} (2\epsilon_{\ell,\ell+1} + \epsilon_{\ell+1,\ell+2}),
\]

\[
V_n = \sum_{i=1}^{\ell-n} \epsilon_{i,i} + e^{-x_{n,1}} \epsilon_{\ell,\ell} + e^{x_{n,1}} \epsilon_{\ell+1,\ell+1}
\]

\[
+ \sum_{k=2}^{n} e^{x_{n,k}} \epsilon_{\ell+1-k,\ell+1-k} + \sum_{i=\ell+2}^{2\ell} \epsilon_{i,i},
\]

\[ (1.211) \]

\[
\tilde{V}_n = V_n + e^{x_{n-1,1}} \epsilon_{\ell,\ell+1} + \sum_{k=1}^{n-1} (1 + \delta_{k,1}) e^{x_{n-1,k}} \epsilon_{\ell-k,\ell+1-k},
\]

\[
\tilde{V}'_n = V_n + e^{x_{n-1,1}} (2\epsilon_{\ell-1,\ell} + \epsilon_{\ell,\ell+1}),
\]

\[
\tilde{V}''_n = V_n + \sum_{k=2}^{n-1} e^{x_{n-1,k}} \epsilon_{\ell-k,\ell+1-k}
\]
for \( n = 3, \ldots, \ell \). The matrices \( U_n, \tilde{U}_n, V_n, \tilde{V}_n \) satisfy the relations
\[
\tilde{U}_n = \tilde{U}_n'' U_n^{-1} \tilde{U}_n', \quad \tilde{V}_n = \tilde{V}_n'' V_n^{-1} \tilde{V}_n', \quad n = 3, \ldots, \ell.
\] (1.212)

**Theorem 1.13.** i) The image of a generic element \( v^{D \ell} \in N_+^{(0)} \) in the tautological representation \( \pi: \mathfrak{so}_{2\ell} \rightarrow \text{End}(\mathbb{C}^{2\ell}) \) can be represented in the form
\[
\pi(v^{D \ell}) = \mathcal{X}_2 \mathcal{X}_3 \cdots \mathcal{X}_\ell, \quad \mathcal{X}_n = \pi(\mathcal{X}_{D_{n-1}}),
\] (1.213)
with
\[
\mathcal{X}_2 = R_{\ell+1} [\tilde{U}_2 U_2^{-1}]^* R_{\ell+1} R_{\ell-1} \tilde{U}_2 U_2^{-1} R_{\ell-1} R_{\ell-1}
\times [\tilde{V}_2 V_2^{-1}]^* R_{\ell-1} R_{\ell+1} \tilde{V}_2 V_2^{-1} R_{\ell+1} R_{\ell},
\]
(1.214)
\[
\mathcal{X}_n = [\tilde{U}_n U_n^{-1}]^* R_{\ell-1} \tilde{U}_n U_n^{-1} R_{\ell-1} R_{\ell+1} \tilde{U}_n U_n^{-1} R_{\ell+1} S_{\ell} R_{\ell-1}
\times [\tilde{V}_n V_n^{-1}]^* R_{\ell-1} R_{\ell+1} \tilde{V}_n V_n^{-1} R_{\ell+1} R_{\ell} \tilde{V}_n V_n^{-1}
\] (1.215)
for \( n = 3, \ldots, \ell \), where it is assumed that \( x_{\ell,i} = 0 \) for \( i = 1, \ldots, \ell \), and \( R_i \) is defined as follows:
\[
R_i = \sum_{k=1}^{i-1} \epsilon_{k,k} + \epsilon_{i,i+1} + \epsilon_{i+1,i} + \sum_{k=i+2}^{2\ell} \epsilon_{k,k}.
\] (1.216)

ii) The expressions (1.213)–(1.216) define a parametrization of a totally non-negative part \( N_+^{(0)} \subset N_+ \) of \( N_+ \).

**Proof.** Let \( v^{D \ell}(y) \) be the parametrization of \( N_+ \) according to (1.195), (1.196). Let
\[
\tilde{X}_1(y) = e^{y_{\ell+1,1}}, \quad \tilde{X}_2(y) = e^{2y_{\ell+1,1}}, \\
\tilde{X}_k(y) = e^{y_{\ell+1-k,1,1}} y_{\ell+1-k,1,1}, \quad k = 3, \ldots, \ell.
\]
Then the action of the involution \( * \) has the form
\[
\tilde{X}_1(y)^* = e^{y_{\ell+1,1} + 2}, \quad \tilde{X}_2(y)^* = e^{2y_{\ell+1,1} + 1}, \\
\tilde{X}_k(y)^* = \tilde{X}_{\ell+1-k}(y), \quad k = 3, \ldots, \ell.
\]
Let us embed the elementary unipotent subgroups \( X_i(y) \) of \( \text{SO}(2\ell) \) into \( \text{GL}(2\ell) \) as follows:
\[
X_i(y) = \tilde{X}_i(y)^* \tilde{X}_i(y), \quad i = 1, \ldots, \ell.
\]
This maps arbitrary regular unipotent elements \( v^{D \ell} \) into a unipotent subgroup of \( \text{GL}(2\ell) \). Let us change the variables in the following way:
\[
y_{1,n} = e^{x_{n,1} - x_{n,1}}, \quad n = 1, \ldots, \ell - 1, \\
y_{2,n} = e^{x_{n,1} + x_{n,1}}, \quad n = 1, \ldots, \ell - 1, \\
y_{k,2r-1} = e^{x_{k+r-2,1+k-1}} x_{k+r-2,1+k-1}, \quad k = 3, \ldots, \ell, \\
y_{k,2r} = e^{x_{k+r-2,1+k-1}} x_{k+r-2,1+k-1}, \quad r = 1, \ldots, \ell + 1 - k.
\] (1.217)
Here \( x_{\ell,k} = 0, \ k = 1, \ldots, \ell \). It is easy to check that after the change of variables (1.217) the image \( \pi(v^{D \ell}) \) of \( v^{D \ell} \) defined by (1.195), (1.196) transforms into (1.213)–(1.215). Taking into account that this change of variables is invertible, we obtain a parametrization of \( N_+^{(0)} \subset N_+ \). \( \square \)
The modified factorized parametrization of a unipotent group $N_+$ defines a particular realization of a principal series representation of $\mathcal{U}(\mathfrak{so}_2\ell)$ by differential operators. It can be obtained using the change of variables (1.217) applied to the realization given in Proposition 1.12. We shall use the term Gauss–Giventhal representation for this realization of a representation of $\mathcal{U}(\mathfrak{so}_2\ell)$.

**Proposition 1.13.** The following differential operators define a representation $\pi_\mu$ of $\mathcal{U}(\mathfrak{so}_2\ell)$ on $V_\mu$ in terms of a modified factorized parametrization of $N_+^{(0)}$:

\[
F_i = \sum_{n=1}^{\ell-1} e^{x_{1}(n),1-z_{n,1}} e^{x_{\gamma_{i}(n),1+x_{n+1},1}} \left[ \frac{e^{x_{\gamma_{i}(n),1+x_{n+1},1} - e^{x_{\gamma_{i}(n),1+x_{n+1},1}}}}{(e^{x_{n,1}} + e^{x_{n+1,1}})^2} \frac{\partial}{\partial z_{n,1}} + (\delta_{n,1} - 1) \frac{\partial}{\partial z_{n,2}} \right] + (\delta_{n,1} - 1) \frac{\partial}{\partial x_{n,1}}
\]

\[
+ \left( -1 \right)^{\ell+i-k} \frac{e^{x_{1}(n),1 - e_{x_{1}(n),1} - e_{x_{k,1}+1,1}}}{e^{x_{k,1}+1,1}} \frac{\partial}{\partial z_{k,1}} \right], \quad i = 1, 2, \quad (1.218)
\]

where $*: 1 \leftrightarrow 2$, $\varpi(n) = (1 - (-1)^n)/2$ is the parity of $n$, and $\gamma_i(n) = 2[(n + \varpi(\ell + 1 - i))/2] + \varpi(\ell - i)$,

\[
F_k = \sum_{n=1}^{2(\ell-k)+1} e^{x_{\beta(n),k-1,k-z_{\alpha(n)}+k-1,k-1}} \left[ \frac{\partial}{\partial x_{\alpha(n)+k-1,k-1}} + (\delta_{n,1} - 1) \frac{\partial}{\partial x_{\alpha(n)+k-1,k-1}} \right]
\]

\[
+ \varpi(n) \left[ \frac{\partial}{\partial z_{\alpha(n)+k-1,k-1}} + (\delta_{n,1} - 1) \frac{\partial}{\partial z_{\alpha(n)+k-1,k-1}} \right] \quad (1.219)
\]

for $k = 3, \ldots, \ell$, and

\[
H_i = \langle \mu, \alpha_i^\vee \rangle + 2 \left\{ \sum_{n=1}^{\ell-1} \varpi(n + i^*) e^{x_{n,1}} + \varpi(n + i) e^{x_{n+1,1}} \right\}
\]

\[
+ (-1)^i \left( \frac{\partial}{\partial x_{1,1}} + \frac{\partial}{\partial x_{\ell-1,1}} \right) \right\} - \sum_{n=2}^{\ell-1} \frac{\partial}{\partial z_{n,2}} \quad (1.220)
\]

for $i = 1, 2$, where $\varpi(k) = (1 - (-1)^k)/2$,

\[
H_k = \langle \mu, \alpha_k^\vee \rangle + \sum_{i=2}^{\ell} a_{k,i} \sum_{j=1}^{\ell} \frac{\partial}{\partial z_{j-1,i-1}} \quad \text{for} \quad k = 3, \ldots, \ell, \quad (1.221)
\]

\[
E_i = \sum_{n=1}^{2\ell-2} e^{x_{\beta(n),1} + (-1)^i x_{\alpha(n)+1,1}} \left[ \langle \mu, \alpha_i^\vee \rangle + (-1)^{i+1} \right]
\]

\[
\times \sum_{k=1}^{\beta(n)} \left\{ \frac{\partial}{\partial x_{k,1}} + (1 - \delta_{k,1}) \delta_{k,1} \frac{\partial}{\partial x_{k,2}} \right\}
\]

\[
- \sum_{k=1}^{\alpha(n)} \left\{ \frac{\partial}{\partial z_{k,1}} + \delta_{k,1} (1 - \delta_{k,1}) \frac{\partial}{\partial z_{k,2}} \right\} \quad (1.222)
\]
for $i = 1, 2$, and

$$
E_k = \sum_{n=1}^{2(\ell+1-k)} e^{z\beta(n)+k-2} x_{\alpha(n)+k-1,k-1} \left( \langle \mu, \alpha_k \rangle + \frac{\partial}{\partial x_{\beta(n)+k-2,1}} + (\delta \beta(n), 1 - 1) \frac{\partial}{\partial x_{\beta(n)+k-2,2}} \right)
+ \sum_{m=1}^{\alpha(n)} \left\{ \frac{\partial}{\partial z_{k-2+m,k-2}} - 2 \frac{\partial}{\partial z_{k-2+m,k-1}} + (1 - \delta m, 1) \frac{\partial}{\partial z_{k-2+m,k}} \right\}
+ (1 - \varpi(n)) \left\{ \frac{\partial}{\partial z_{k-2+m,k-1}} + (\delta n, 1) \frac{\partial}{\partial z_{k-2+m,k}} \right\},
\right.

(1.223)

where $3 \leq k \leq \ell$ and it is assumed that $x_{\ell,k} = 0$.

We shall write the matrix element (1.12) for $g = so_{2\ell}$ explicitly using the Gauss–Givental representation defined above. The Whittaker vectors $\psi_R$ and $\psi_L$ in this representation should satisfy the systems of differential equations (1.7) and (1.8). The solutions have the following form.

Lemma 1.10. The following expressions hold for the left/right Whittaker vectors:

\[
\begin{align*}
\psi_R &= \exp \left\{ - \sum_{n=1}^{\ell-1} \left( e^{z_{n,1} - x_{n,1}} + e^{z_{n,1} - x_{n+1,1}} + e^{z_{n,1} + x_{n+1,1}} + e^{z_{n,1} + x_{n+1,1}} \right) 
- \sum_{k=3}^{\ell} \sum_{n=1}^{\ell+1-k} \left( e^{z_{k+n-2,1} - x_{k+n-2,1}} + e^{z_{k+n-2,1} - x_{k+n-1,1}} \right) \right\},
\end{align*}
\]

(1.224)

\[
\begin{align*}
\psi_L &= e^{2\mu x_{1,1}} \prod_{n=2}^{\ell} \left( e^{x_{n,1}} + e^{x_{n-1,1}} \right)^{2\mu_n} 
&\times \prod_{n=1}^{\ell} \exp \left\{ -\mu_n \left( \sum_{i=1}^{n} x_{n,i} - 2 \sum_{i=1}^{n-1} z_{n-1,i} + \sum_{i=1}^{n-1} x_{n-1,i} \right) \right\} 
&\times \exp \left\{ -\sum_{k=1}^{\ell-1} \left( e^{x_{k+1,k+1} - z_{k,k}} + \sum_{i=k+1}^{\ell-1} \left( e^{x_{i,k+1} - z_{i,k}} + e^{x_{i+1,k+1} - z_{i,k}} \right) \right) \right\}.
\end{align*}
\]

(1.225)

where $x_{k,k} = 0, k = 1, \ldots, \ell$, $\mu_n = i\lambda_n - \rho_n$, $\rho_1 = 0$, and $\rho_n = n - 1$ for $1 < n \leq \ell$ ($\sum_{i}^{j} = 0$ when $j < i$).

Now we are ready to find the integral representation of the pairing (1.12) for $g = so_{2\ell}$. To obtain an explicit expression for the integrand, one uses the same type of decomposition of the Cartan element as for other classical groups in the previous subsections:

\[
e^{-h_x} = \pi_\lambda \left( \exp \left( - \sum_{i=1}^{\ell} (\omega_i, x) h_{\bar{i}} \right) \right) = e^{H_L} e^{H_R},
\]

where $h_x = \pi_\lambda \left( \exp \left( - \sum_{i=1}^{\ell} (\omega_i, x) h_{\bar{i}} \right) \right)$.
where
\[-h_x = H_L + H_R = \sum_{i=1}^{\ell} \mu_i x_{\ell,i} + \sum_{k=3}^{\ell} (x_{\ell,k} - x_{\ell,k-1}) \sum_{i=k-1}^{\ell-1} \frac{\partial}{\partial z_{i,k-1}} + x_{\ell,2} \sum_{i=1}^{\ell-1} \frac{\partial}{\partial z_{i,1}} + x_{\ell,1} \left( \frac{\partial}{\partial x_{\ell,-1,1}} + \frac{\partial}{\partial x_{1,1}} - \sum_{k=1}^{\ell-1} (-1)^k e^{x_{k+1,1}} - e^{x_{k,1}} \right) \frac{\partial}{\partial z_{k,1}} \]

with
\[H_L = \sum_{k=1}^{\ell} x_{\ell,k} \left( \sum_{i=k}^{\ell-1} \frac{\partial}{\partial x_{i,k}} + \sum_{i=k-1}^{\ell-1} \frac{\partial}{\partial z_{i,k-1}} \right), \quad H_R = -h_x - H_L. \]

We assume that \( H_L \) acts on the left vector and \( H_R \) acts on the right vector in (1.12). Taking into account Proposition 1.10, one obtains the following theorem.

**Theorem 1.14.** The eigenfunctions (1.12) of the \( so_{2\ell} \)-Toda chain admit the integral representation
\[
\Psi_{\lambda_1, \ldots, \lambda_\ell}^D(x_{\ell,1}, \ldots, x_{\ell,\ell}) = \int_C \prod_{k=1}^{\ell-1} dx_k \, dz_k \, e^{D}, \quad (1.229)
\]
where
\[
D = -\nu \lambda_1 x_{1,1} - \sum_{n=2}^{\ell} n \lambda_n \left( \sum_{i=1}^n x_{n,i} - 2 \sum_{i=1}^{n-1} z_{n-1,i} + \sum_{i=1}^{n-1} x_{n-1,i} - 2 \log(e^{x_{n,1}} + e^{x_{n-1,1}}) \right)
\]
\[\quad - \sum_{k=1}^{\ell-1} \left( e^{x_{k+1,1} - z_{k,k}} + \sum_{i=k+1}^{\ell-1} \left( e^{x_{i,k+1} - z_{i,k}} + e^{x_{i+1,k+1} - z_{i,k}} \right) \right) \]
\[\quad - \sum_{n=1}^{\ell+1-k} \sum_{k=3}^{\ell} \left( e^{z_{n,1} - x_{n,1} + e^{z_{n,1} - x_{n+1,1}} + e^{z_{n,1} + x_{n,1}} + e^{z_{n,1} + x_{n+1,1}}} \right) \]
\[\quad - \sum_{k=3}^{\ell} \sum_{n=1}^{\ell-k} \left( e^{z_{n-2,k-1} - x_{n-2,k-1} - e^{z_{n-2,k-1} - x_{n-2,k-1}}} \right),
\]

with \( x_i := x_{i,i}, 1 \leq i \leq \ell \), and where \( C \subset N_+ \) is a middle-dimensional non-compact submanifold such that the integrand decreases exponentially at the boundary and at infinity. In particular, one can take \( C \) to be a slightly deformed subset \( \mathbb{R}^{\ell(\ell-1)} \subset \mathbb{R}^{\ell(\ell-1)} \) making the integral convergent.

Note that the integral (1.229) can be obtained directly from (1.205) using the following change of variables:
\[
y_{1,n} = e^{x_{n,1} - x_{n,2}} \left( e^{z_{n,1} - z_{n,1}} + e^{z_{n,1} - z_{n,1}} \right), \quad n = 1, \ldots, \ell - 1,
\]
\[
y_{2,n} = e^{-x_{n,1} - x_{n,2}} \left( e^{z_{n,1} + x_{n,1}} + e^{z_{n,1} + x_{n,1}} \right), \quad n = 1, \ldots, \ell - 1,
\]
\[
y_{k,2r-1} = e^{x_{k,1} - x_{k,r-1}} \left( e^{z_{k+r-2,k-1} - e^{z_{k+r-2,k-1}} \right), \quad k = 3, \ldots, \ell,
\]
\[
y_{k,2r} = e^{x_{k,1} - x_{k,r-1}} \left( e^{z_{k+r-2,k-1} - x_{k+r-2,k-1}} \right), \quad r = 1, \ldots, \ell + 1 - k.
\]

(1.230)
Example 1.7. For $\ell = 2$ the general expression (1.229) acquires the form

$$
\Psi_{\lambda_1, \lambda_2}^{D_2}(x_{2,1}, x_{2,2}) = \int_C dx_{1,1} dz_{1,1} \left(e^{x_{1,1}} + e^{x_{2,1}}\right)^{2t\lambda_2} \\
\times \exp\left\{\imath \lambda_1 x_{1,1} - \imath \lambda_2 \left(x_{2,1} + x_{2,2} - 2z_{1,1} + x_{1,1}\right) - (e^{z_{1,1}} + e^{z_{1,1}+x_{1,1}} + e^{z_{1,1}-x_{2,1}} + e^{z_{1,1}+x_{2,1}} + e^{x_{2,2}-z_{1,1}})\right\}. \tag{1.231}
$$

Let us consider the following transformation of the integration variables:

$$
z_{1,1} \mapsto -z_{1,1} + x_{2,2} - \log(e^{x_{1,1}} + e^{x_{2,1}}), \quad x_{1,1} \mapsto x_{1,1} - z_{1,1} - x_{2,1}, \quad \tag{1.232}
$$

which implies that

$$
\Psi_{\lambda_1, \lambda_2}^{D_2}(x_{2,1}, x_{2,2}) = e^{-\imath \lambda_1 x_{2,1} + \imath \lambda_2 x_{2,2}} \int_C dz_{1,1} dx_{1,1} \exp\left\{\imath (\lambda_1 - \lambda_2) x_{1,1} - (e^{z_{1,1}} + e^{z_{1,1}+x_{1,1}} + e^{z_{1,1}-x_{2,1}} + e^{z_{1,1}+x_{2,1}} + e^{x_{2,2}-z_{1,1}})\right\}.
$$

Then we change the integration variables as follows

$$
z_{1,1} \mapsto z_{1,1} + \frac{x_{2,2} - x_{2,1}}{2}, \quad x_{1,1} \mapsto x_{1,1} + \frac{x_{2,1} + x_{2,2}}{2},
$$

and arrive at

$$
\Psi_{\lambda_1, \lambda_2}^{D_2}(x_{2,1}, x_{2,2}) = K_{\imath (\lambda_1 - \lambda_2)} \left(2e^{\frac{x_{2,2} - x_{2,1}}{2}}\right) K_{-\imath (\lambda_1 + \lambda_2)} \left(2e^{\frac{x_{2,1} + x_{2,2}}{2}}\right), \tag{1.233}
$$

where by (1.50)

$$
K_{\gamma}(2e^y) = \int_{\mathbb{R}} dt e^{\imath \gamma t - y(e^t + e^{-t})} = \Psi_{\gamma}^{sl_2}(y).
$$

The Whittaker function identity (1.233) is provided by the algebra isomorphism $so_4 \simeq sl_2 \oplus sl_2$. 

There is a simple combinatorial description of the potential $\mathcal{F}^{D_\ell}$ for the zero spectrum $\{\lambda_i = 0\}$. Namely, it can be represented as the sum over the arrows in the following diagram:

```
         z_{\ell-1,1} \rightarrow x_{2,1} \rightarrow x_{\ell,1}
          \downarrow \quad \quad \downarrow
          x_{\ell-1,1} \rightarrow z_{\ell-1,1} \rightarrow x_{\ell,2}

          \downarrow \quad \quad \downarrow
          x_{\ell-1} \rightarrow z_{\ell,1} \rightarrow \cdots \rightarrow x_{\ell-1,\ell-1} \rightarrow z_{\ell-1,\ell-1} \rightarrow x_{\ell,\ell}
```

We use the same rule to assign variables to the arrows of the diagram as for $A_\ell$. In addition we assign to the symbol $z \rightarrow x$ the exponential $e^{z+x}$. 


Note that the diagram for $D_\ell$ can be obtained by a factorization of the diagram for $A_{2\ell-1}$. Consider the involution
\[ * : X \mapsto -\hat{W}_0^{-1}X^t\hat{W}_0, \tag{1.234} \]

where $\hat{W}_0$ is a lift of the longest element in the Weyl group $W(A_{2\ell-1}) = S_{2\ell}$ and $X^t$ denotes the standard transposition. The corresponding action on the modified factorization parameters is given by
\[ * : x_{k,i} \mapsto -x_{k,k+1-i}. \tag{1.235} \]

This defines a factorization of the $A_{2\ell-1}$-diagram that produces the diagram for $D_\ell$. Note that diagram for $D_\ell$ can be also obtained by erasing the last series of vertices and arrows on the right-hand slope of the diagram for $C_\ell$.

An analogue of the toric relations (1.52) is as follows. We introduce variables $a_{i,k}, b_{i,k}, c_{i,k}, d_{i,k}$ associated with the arrows of the diagram:
\[
\begin{align*}
  a_{n,1} &= e^{x_{n,1} + z_{n-1,1}}, & a_{n,i} &= e^{x_{n,i} - z_{n-1,i-1}}, & 2 \leq i \leq n \leq \ell; \\
  b_{k,1} &= e^{x_{k,1} + z_{k,1}}, & k &= 1, \ldots, \ell - 1; \\
  b_{n,i} &= e^{z_{n,i} - x_{n,i-1}}, & 2 \leq i \leq n < \ell; \\
  c_{m,j} &= e^{z_{m,j} - x_{m,j}}, & d_{m,j} &= e^{z_{m,j} - x_{m+1,j}}, & 1 \leq j \leq m < \ell.
\end{align*}
\]

Then the following relations hold:
\[
\begin{align*}
  b_{k,i}c_{k,i} &= a_{k+1,i}d_{k,i}, & a_{k+1,i+1}d_{k,i} &= b_{k+1,i+1}c_{k+1,i}, & 1 \leq i \leq k < \ell, \\
  a_{\ell,1}a_{\ell,2} &= e^{x_{\ell,1} + x_{\ell,2}}, & a_{\ell,i}d_{\ell-1,i-1} &= e^{x_{\ell,i} - x_{\ell,i-1}}, & i = 2, \ldots, \ell. \tag{1.237}
\end{align*}
\]

1.5.3. Recursion for $\mathfrak{so}_{2\ell}$-Whittaker functions and the $\mathcal{Q}$-operator for a $D_\ell^{(1)}$-Toda chain. The integral representation (1.229) of $\mathfrak{so}_{2\ell}$-Whittaker functions possesses a recursive structure with respect to the rank $\ell$. For any $n = 2, \ldots, \ell$ let us introduce integral operators $Q^{C_n}_{D_{n-1}}$ with the kernels $Q^{C_n}_{D_{n-1}}(x_n, z_{n-1}; \lambda_n)$ defined as follows:
\[
Q^{D_n}_{D_{n-1}}(x_n, z_{n-1}; \lambda_n) = \int d\lambda_n \left( \lambda_n \right) Q^{C_n}_{C_{n-1}}(x_n, z_{n-1})Q^{C_{n-1}}_{D_{n-1}}(z_{n-1}, x_{n-1}) \times \exp \left\{ -\lambda_n \left( \sum_{i=1}^{n} x_{n,i} - 2 \sum_{i=1}^{n-1} z_{n-1,i} + \sum_{i=1}^{n-1} x_{n-1,i} \right) \right\} (e^{x_{n-1,1}} + e^{x_{n,1}})^{2n}\lambda_n,
\]

where
\[
\begin{align*}
  Q^{C_n}_{C_{n-1}}(x_n, z_{n-1}) &= \exp \left\{ -\left( e^{x_{n,1} + z_{n-1,1}} + \sum_{i=1}^{n-1} \left( e^{z_{n-1,i} - x_{n,i}} + e^{x_{n+1,i} - z_{n-1,i}} \right) \right) \right\}, \tag{1.238} \\
  Q^{C_{n-1}}_{D_{n-1}}(z_{n-1}, x_{n-1}) &= \exp \left\{ -\left( e^{x_{n-1,1} + z_{n-1,1}} + \sum_{i=1}^{n-2} \left( e^{z_{n-1,i} - x_{n-1,i}} + e^{x_{n-1,i+1} - z_{n-1,i}} + e^{z_{n-1,n-1} - x_{n-1,n-1}} \right) \right) \right\},
\end{align*}
\]
and for \( n = 1 \) we define
\[
Q^{D_1}_{D_0}(x_{1,1}; \lambda_1) = e^{\lambda_1 x_{1,1}}.
\]

By using \( Q^{D_n}_{D_{n-1}} \), \( n = 1, \ldots, \ell \), the integral representation \((1.229)\) can be written in a recursive form.

**Theorem 1.15.** The eigenfunctions for the \( \mathfrak{so}_{2\ell} \)-Toda chain can be written in the following recursive form:

\[
\Psi^{D_\ell}_{\lambda_1, \ldots, \lambda_\ell}(x_1, \ldots, x_\ell) = \int_C \prod_{k=1}^{\ell-1} dx_k \prod_{k=1}^{\ell} Q^{D_k}_{D_{k-1}}(x_k, x_{k-1}; \lambda_k), \quad (1.239)
\]

or equivalently,

\[
\Psi^{D_\ell}_{\lambda_1, \ldots, \lambda_\ell}(x_\ell) = \int_{C_{\ell-1}} dx_{\ell-1} Q^{D_\ell}_{D_{\ell-1}}(x_\ell, x_{\ell-1}; \lambda_\ell) \Psi^{D_{\ell-1}}_{\lambda_1, \ldots, \lambda_{\ell-1}}(x_{\ell-1}),
\]

where \( x_n := x_{n,n}, 1 \leq n \leq \ell \). Here \( C \subset N_+ \) is a middle-dimensional non-compact submanifold such that the integrand decreases exponentially at the boundary and at infinity.

As for other classical Lie algebras different from \( \mathfrak{gl}_{\ell+1} \), the specialization to the zero spectrum \( \{ \lambda_n = 0 \} \) reveals a more refined recursive structure. In this case the kernel of the operator \( Q^{D_n}_{D_{n-1}} \) is reduced to a convolution of two kernels \( Q^{C_{n-1}}_{C_{n-1}}(x_n, z_{n-1}) \) and \( Q^{C_{n-1}}_{D_{n-1}}(z_{n-1}, x_{n-1}) \). The corresponding integral operators \( Q^{D_n}_{C_{n-1}}, Q^{C_{n-1}}_{D_{n-1}} \) can be regarded as elementary intertwiners connecting the Toda chain Hamiltonians for the \( D_n \), \( C_{n-1} \), \( C_{n-1} \), \( D_{n-1} \) root systems. Thus, for quadratic Hamiltonians one can easily verify the following relations directly.

**Lemma 1.11.** The operators \( Q^{D_n}_{D_{n-1}}, Q^{D_n}_{C_{n-1}}, \) and \( Q^{C_{n-1}}_{D_{n-1}} \) satisfy the following intertwining relations with the quadratic Toda Hamiltonians.

1. The operators \( Q^{D_n}_{C_{n-1}} \) and \( Q^{C_{n-1}}_{D_{n-1}} \) intertwine the quadratic Hamiltonians of \( C \) - and \( D \) - Toda chains:

\[
\mathcal{H}^{D_n}_{2}(x_n) Q^{D_{n-1}}_{C_{n-1}}(x_n, z_{n-1}) = Q^{D_{n-1}}_{C_{n-1}}(x_n, z_{n-1}) \mathcal{H}^{C_{n-1}}_{2}(x_{n-1}), \quad (1.240)
\]

\[
\mathcal{H}^{C_{n}}_{2}(z_n) Q^{C_{n}}_{D_{n-1}}(z_n, x_n) = Q^{D_{n}}_{D_{n-1}}(z_n, x_n) \mathcal{H}^{D_{n}}_{2}(x_n), \quad (1.241)
\]

2. The operator \( Q^{D_{n-1}}_{D_n} \) for \( \lambda_n = 0 \) intertwines the Hamiltonians \( \mathcal{H}^{D_n}_{2} \) and \( \mathcal{H}^{D_{n-1}}_{2} \):

\[
\mathcal{H}^{D_n}_{2}(x_n) Q^{D_{n-1}}_{D_n}(x_n, x_{n-1}; \lambda_n = 0) = Q^{D_{n-1}}_{D_n}(x_n, x_{n-1}; \lambda_n = 0) \mathcal{H}^{D_{n-1}}_{2}(x_{n-1}), \quad (1.242)
\]

where

\[
\mathcal{H}^{C_n}_{2} = -\frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2}{\partial z_i^2} + 2e^{2z_1} + \sum_{i=1}^{n-2} e^{z_{i+1} - z_i}, \quad (1.243)
\]

\[
\mathcal{H}^{D_n}_{2} = -\frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + e^{x_1 + x_2} + \sum_{i=1}^{n-1} e^{x_{i+1} - x_i}. \quad (1.244)
\]
The integral kernel of the operator \( Q_{D_{n-1}}^{D_n} (x_n, x_{n-1}) \) at \( \lambda_n = 0 \) can be succinctly encoded into the following diagram:

\[
\begin{array}{c}
z_{n-1,1} \quad \times \quad x_{n,1} \\
\downarrow \\
x_{n-1,1} \rightarrow z_{n-1,1} \rightarrow x_{n,2} \\
\downarrow \\
x_{n-1,2} \rightarrow z_{n-1,2} \rightarrow \cdots \\
\downarrow \\
\cdots \rightarrow x_{n,n-1} \\
\downarrow \\
x_{n-1,n-1} \rightarrow z_{n-1,n-1} \rightarrow x_{n,n} \\
\end{array}
\]  

(1.245)

Here the upper and lower descending paths of the oriented diagram correspond to the kernels of the elementary intertwiners \( Q_{C_{n-1}}^{D_n} \) and \( Q_{D_{n-1}}^{C_{n-1}} \), respectively. The convolution of the kernels \( Q_{C_{n-1}}^{D_n} \) and \( Q_{D_{n-1}}^{C_{n-1}} \) corresponds to integration with respect to the variables \( z_{n-1,i} \) associated with the inner vertices of the subdiagram (1.245).

As with other classical series of Lie algebras, the recursion operators \( Q_{D_{n-1}}^{D_n} \) can be considered as degenerations of Baxter \( \mathcal{Q} \)-operators for affine \( D^{(1)}_\ell \)-Toda chains. We recall the root data for \( D^{(1)}_\ell \). Simple roots of the affine root system \( D^{(1)}_\ell \) can be represented as vectors in \( \mathbb{R}^\ell \) in the following way:

\[
\alpha_1 = \epsilon_1 + \epsilon_2, \quad \alpha_{i+1} = \epsilon_{i+1} - \epsilon_i, \quad 1 \leq i \leq \ell - 1, \quad \alpha_{\ell+1} = -\epsilon_{\ell} - \epsilon_{\ell-1},
\]

and the Dynkin diagram is given by

\[
\begin{array}{c}
\alpha_1 \\
\downarrow \\
\alpha_3 \quad \cdots \quad \alpha_{\ell-1} \\
\downarrow \\
\alpha_2 \\
\downarrow \\
\alpha_{\ell} \\
\alpha_{\ell+1}
\end{array}
\]

The corresponding \( D^{(1)}_\ell \)-Toda chain quadratic Hamiltonian is defined by

\[
\mathcal{H}_2^{D^{(1)}_\ell} = -\frac{1}{2} \sum_{i=1}^{\ell} \frac{\partial^2}{\partial x_i^2} + e^{x_1+x_2} + \sum_{i=1}^{\ell-2} e^{x_{i+1}-x_i} + ge^{x_\ell-x_{\ell-1}} + ge^{-x_\ell-x_{\ell-1}}. \quad (1.246)
\]
We define the Baxter $\mathcal{Q}$-operator of the $D^{(1)}_\ell$-Toda chain as an integral operator with the integral kernel

$$\mathcal{Q}^{(1)}_{D^{(1)}_\ell}(x^{(\ell)}, y^{(\ell)}; \lambda) = \int d\tilde{x}^{(\ell-1)} Q^{(1)}_{D^{(1)}_\ell}(\tilde{x}^{(\ell)}, \tilde{x}^{(\ell-1)}) Q^{(1)}_{D^{(1)}_\ell}(\tilde{x}^{(\ell-1)}, y^{(\ell)})$$

$$\times \exp \left\{ -\lambda \left( \sum_{i=1}^{\ell} x_i - 2 \sum_{i=1}^{\ell-1} z_i + \sum_{i=1}^{\ell} y_i \right) \right\} (e^{x_1} + e^{y_1})^{2\lambda}(e^{-x_\ell} + e^{-y_\ell})^{-2\lambda},$$

where

$$Q^{(1)}_{C^{(1)}_{\ell-1}}(x^{(\ell)}, z^{(\ell-1)}) = \exp \left\{ -e^{z_1+x_1} - \sum_{i=1}^{\ell-1} (e^{z_i-x_i} + e^{x_{i+1}-z_i}) - ge^{-x_\ell-z_{\ell-1}} \right\}$$

and

$$Q^{(1)}_{C^{(1)}_{\ell-1}}(x_1, \ldots, x_\ell; z_1, \ldots, z_\ell) = Q^{(1)}_{D^{(1)}_\ell}(z_1, \ldots, z_\ell; x_1, \ldots, x_\ell).$$

Here we use the notation $x^{(\ell)} = (x_1, \ldots, x_\ell)$, $y^{(\ell)} = (y_1, \ldots, y_\ell)$.

**Proposition 1.14.** The $\mathcal{Q}$-operator (1.247) commutes with the quadratic Hamiltonian of the $D^{(1)}_\ell$-Toda chain:

$$\mathcal{H}^{(1)}_{D^{(1)}_\ell}(x^{(\ell)}) \mathcal{Q}^{(1)}_{D^{(1)}_\ell}(x^{(\ell)}, y^{(\ell)}) = \mathcal{Q}^{(1)}_{D^{(1)}_\ell}(x^{(\ell)}, y^{(\ell)}) \mathcal{H}^{(1)}_{D^{(1)}_\ell}(y^{(\ell)}).$$

Now we show that the recursion operator $Q^{(1)}_{D^{(1)}_{\ell-1}}$ can be considered as a degeneration of the Baxter $\mathcal{Q}$-operators for $D^{(1)}_\ell$. Let us introduce a slightly modified recursion operator with the kernel $Q^{(1)}_{D^{(1)}_{\ell-1} \oplus D_1}$:

$$Q^{(1)}_{D^{(1)}_{\ell-1} \oplus D_1}(x^{(\ell)}, y^{(\ell)}; \lambda) = e^{\lambda y_\ell} Q^{(1)}_{D^{(1)}_{\ell-1}}(x^{(\ell)}, y^{(\ell)-1}; \lambda),$$

where $y^{(\ell-1)} = (y_1, \ldots, y_{\ell-1})$. This operator intertwines the Hamiltonians of the $\mathfrak{so}_{2\ell}$- and $\mathfrak{so}_{2\ell-2} \oplus \mathfrak{so}_2$-Toda chains. For instance, for the quadratic Hamiltonians we have

$$\mathcal{H}^{(1)}_{\mathfrak{so}_{2\ell}}(y^{(\ell)}) = -\frac{1}{2} \frac{\partial^2}{\partial y_\ell^2}.\]$$

Obviously, the projection of the above relation on the subspace of functions $F(y^{(\ell)}) = \exp(\lambda y_\ell)f(y^{(\ell)-1})$ recovers the initial recursion operator satisfying the relation

$$\mathcal{H}^{(1)}_{\mathfrak{so}_{2\ell}}(y^{(\ell)}) Q^{(1)}_{D^{(1)}_{\ell-1}}(x^{(\ell)}, y^{(\ell)-1}; \lambda) = Q^{(1)}_{D^{(1)}_{\ell-1}}(x^{(\ell)}, y^{(\ell)-1}; \lambda) \left( \mathcal{H}^{(1)}_{\mathfrak{so}_{2\ell}}(y^{(\ell)-1}) + \frac{1}{2} \lambda^2 \right).$$

(1.252)
Let us introduce the one-parameter family of operators with the kernels

\[Q^{D_{\ell}}_1 \left( x^{(\ell)}, y^{(\ell)}; \lambda; \varepsilon \right) := \varepsilon^\lambda e^{i\lambda y_\ell} \int d\bar{z}^{(\ell-1)} Q^{D_{\ell-1}}_{C_{\ell-1}} \left( x^{(\ell)}, z^{(\ell-1)} \right) Q^{C_{\ell-1}}_{D_{\ell-1}} \left( z^{(\ell-1)}, y^{(\ell)}; \varepsilon \right) \times \exp \left\{ -i\lambda \left( \sum_{i=1}^\ell x_i - 2 \sum_{i=1}^{\ell-1} z_i + \sum_{i=1}^{\ell-1} y_i \right) \right\} \left( e^{x_1} + e^{y_1} \right)^{2\lambda} (\varepsilon e^{-x_\ell+y_\ell} + 1)^{-2\lambda},\]

where

\[Q^{D_{\ell-1}}_{C_{\ell-1}} \left( x^{(\ell)}, z^{(\ell-1)} \right) = \exp \left\{ -e^{z_1+x_1} - \sum_{i=1}^{\ell-1} (e^{z_i-x_{i+1}} + e^{x_{i+1}-z_i}) - ge^{-x_{\ell-1}z_{\ell-1}} \right\} \]

and

\[Q^{C_{\ell-1}}_{D_{\ell-1}} \left( z_1, \ldots, z_{\ell-1}, y_1, \ldots, y_{\ell}; \varepsilon \right) = \exp \left\{ -e^{z_1+y_1} - \sum_{i=1}^{\ell-2} (e^{z_i-y_i} + e^{y_{i+1}-z_i}) - e^{z_{\ell-1}-y_{\ell-1}} - \varepsilon e^{y_{\ell}-z_{\ell-1}} - \varepsilon^{-1} ge^{-x_{\ell-1}z_{\ell-1}} \right\}.\]  

These operators are obtained by the shift of variable \(y_\ell \rightarrow y_\ell + \log \varepsilon\) in (1.247)–(1.249). Then the following relation holds between the \(Q\)-operator for the \(D_{\ell}^{(1)}\)-Toda chain and the (modified) recursion operator for the \(so_{2\ell}\)-Whittaker function:

\[Q^{D_{\ell}}_{D_{\ell-1} \oplus D_1} \left( x^{(\ell)}, y^{(\ell)}; \lambda \right) = \lim_{\varepsilon, g \to 0} \varepsilon^{-1} Q^{D_1^{(1)}} \left( x^{(1)}, y^{(1)}; \lambda; \varepsilon \right).\]  

2. Part 2. Proofs

Let \(G\) be a complex connected simply-connected semisimple Lie group of rank \(\ell\) and \(g = \text{Lie}(G)\) the corresponding semisimple Lie algebra. Let \(\Gamma = \{1, \ldots, \ell\}\) be the set of vertices of the Dynkin graph of \(g\), and denote the Chevalley generators by \(f_i, h_i, e_i, i \in \Gamma\). Let us fix a Borel subgroup \(B_+\) and let \(T\) be the maximal torus \(T \subset B_+\). This defines a pair \(N_+, N_-\) of opposite unipotent subgroups of \(G\), \(N_+ \subset B_+\). Let \(\{\alpha_i, i \in \Gamma\}\) be the set of simple roots, \(\{\gamma_k, k = 1, \ldots, m\}\), \(m = \frac{1}{2} \text{dim} g/\mathfrak{h}\), the set of all positive roots, and \(\{\alpha_i^\vee, i \in \Gamma\}\) the set of simple coroots. For every \(i \in \Gamma\) there is a group homomorphism

\[\varphi_i : SL(2) \longrightarrow G,\]

which is defined as follows. We introduce a set of one-parameter subgroups \(e^{tf_i} = X_i(t) \subset N_+, e^{te_i} = Y_i(t) \subset N_-\), and \(e^{th_i} = \alpha_i^\vee(t) \subset T\). The homomorphisms (2.1) are defined as

\[\varphi_i(e^{te}) = e^{te_i}, \quad \varphi_i(e^{tf}) = e^{tf_i}, \quad \varphi_i(e^{th}) = \alpha_i^\vee(t),\]
Remark 2.1. For classical Lie groups one has given by

\[ \hat{s} = e^f e^{-e} e^f, \quad \hat{s}_i = e^f_i e^{-e_i} e^{f_i}. \]  

(2.3)

The lifts of Weyl group generators defined in this way are obviously compatible with the homomorphisms (2.1), that is, \( \varphi_i(\hat{s}) = \hat{s}_i \). We have the relations

\[ \hat{s}^{-1} e \hat{s} = -f, \quad \hat{s}_i^{-1} e_i \hat{s}_i = -f_i. \]  

(2.4)

The action \( w_0(\alpha_i) = -\alpha_i^\ast \) of the maximal length element \( w_0 \) in the Weyl group on simple roots defines an involution \( i \mapsto i^\ast \). The corresponding action of \( \hat{w}_0 \) is given by

\[ \hat{w}_0^{-1} e_i \hat{w}_0 = -f_i^\ast. \]  

(2.5)

**Remark 2.1.** For classical Lie groups one has \( i^\ast = \ell + 1 - i \) for \( G = \text{SL}(\ell + 1) \), and \( i^\ast = i \) for \( G = \text{SO}(2\ell + 1) \) and for \( G = \text{Sp}(2\ell) \). In the case \( G = \text{SO}(2\ell) \) (for \( \ell \geq 2 \)) the action of the involution \( * \) is as follows:

\[ *: \quad 1 \mapsto \begin{cases} 1, & \ell \text{ even}, \\ 2, & \ell \text{ odd}, \end{cases} \quad 2 \mapsto \begin{cases} 2, & \ell \text{ even}, \\ 1, & \ell \text{ odd}, \end{cases} \]  

(2.6)

where the enumeration of roots of \( \text{SO}(2\ell) \) is given by (1.192).

In what follows we will consider matrix elements of the finite-dimensional representations on \( V_{\omega_i} \) of \( g \) corresponding to the fundamental weights \( \omega_i, i \in \Gamma \). Let \( \xi_{\omega_i}^+ \) and \( \xi_{\omega_i}^- \) be the highest and lowest vectors in \( V_{\omega_i} \) determined by the condition \( n_\pm \xi_{\omega_i}^\pm = 0 \) and normalized so that \( \langle \xi_{\omega_i}^{-}|\xi_{\omega_i}^{+}\rangle = 1 \). For the lift (2.3) of the elements of the Weyl group we have (see, for instance, [24], Lemma 3.8, or [15], (2.29))

\[ \hat{w}_0^{-1} \xi_{\omega_i}^+ = \xi_{\omega_i}^-, \quad \hat{s}_i^{-1} \xi_{\omega_i}^+ = e_i \xi_{\omega_i}^+. \]  

(2.7)

We consider the following parametrization of a generic group element \( g \in G \)

\[ g = g^{(-)} g^{(0)} g^{(+)} = \exp \left( \sum_{\alpha \in \Phi_+} u_{-\alpha} e_\alpha \right) \exp \left( \sum_{i=1}^{\ell} u_i h_i \right) \exp \left( \sum_{\alpha \in \Phi_+} u_{\alpha} f_\alpha \right). \]  

(2.8)

For the coordinates \( u_i \) corresponding to the Cartan generators \( h_i \) and for the coordinates \( u_{\pm \alpha} \), corresponding to the simple root generators \( e_\alpha, f_\alpha \), there are simple expressions in terms of matrix elements of the fundamental representations on \( V_{\omega_i}, i = 1, \ldots, \ell \):

\[ u_{\alpha_i}(g) = \frac{\langle \xi_{\omega_i}^-|\pi_i(g)\pi_i(e_i)|\xi_{\omega_i}^+\rangle}{\langle \xi_{\omega_i}^-|\pi_i(g)|\xi_{\omega_i}^+\rangle}, \quad u_{-\alpha_i}(g) = \frac{\langle \xi_{\omega_i}^-|\pi_i(f_i)\pi_i(g)|\xi_{\omega_i}^+\rangle}{\langle \xi_{\omega_i}^-|\pi_i(g)|\xi_{\omega_i}^+\rangle}, \]  

(2.9)

\[ u_i(g) = \langle \xi_{\omega_i}^-|\pi_i(g)|\xi_{\omega_i}^+\rangle, \]  

where \( \pi_i \equiv \pi_{\omega_i} \) is a fundamental representation on \( V_{\omega_i} \). We define the generalized twisted minors as

\[ \Delta_{\omega_1, \omega_2}(g) = \langle \xi_{\omega_1}^-|\pi_{\omega_1}(g)\pi_{\omega_1}(\hat{w})|\xi_{\omega_1}^+\rangle, \quad g \in G. \]  

(2.10)
Then the coordinates $u_i$ and $u_{\alpha_i}$ of a twisted unipotent element $v\dot{w}^{-1}_0 \in G$ (where $v \in N_+$) can be expressed in terms of the twisted minors (2.10) as follows:

$$e^{u_i(v\dot{w}^{-1}_0)} = \Delta_{\omega_i,\dot{w}^{-1}_0}(v),$$

$$u_{\alpha_i}(v\dot{w}^{-1}_0) = \frac{\langle \xi_i^- | \pi_i(v\dot{w}^{-1}_0)|\xi_i^+ \rangle}{\langle \xi_i^- | \pi_i(v\dot{w}^{-1}_0)|\xi_i^+ \rangle} = \frac{\langle \xi_i^- | \pi_i(v)|\pi_i(\dot{w}^{-1}_0)|\pi_i(\dot{s}_i^{-1})|\xi_i^+ \rangle}{\langle \xi_i^- | \pi_i(v\dot{w}^{-1}_0)|\xi_i^+ \rangle}.$$  \hspace{1cm} (2.11)

In what follows we use the shorthand notation

$$\Delta'_i(v) := \langle \xi_i^- | \pi_i(vf_i,\dot{w}^{-1}_0)|\xi_i^+ \rangle = -\Delta_{\omega_i,\dot{w}^{-1}_0}(v),$$

$$\Delta_i(v) := \Delta_{\omega_i,\dot{w}^{-1}_0}(v).$$ \hspace{1cm} (2.12)

2.1. Measure in $N_+$: proof of Lemma 1.2. In this part we derive an explicit expression (1.21) for a measure $d\mu_{N_+}(x)$ on a unipotent subgroup $N_+ \subset G$ of any classical Lie group, using the factorized parametrization (1.20) of $N_+$. The case of a general semisimple Lie group follows from the results in [33].

Recall that for a reduced word $I_\ell = (i_1, \ldots, i_{m_\ell})$ of $w_0$ there is a birational isomorphism $\mathbb{C}^{m_\ell} \rightarrow N_+$. In particular, given a unipotent element $v \in N_+(0)$, we have the factorized representation

$$v(t) = X_{i_1}(t_1)X_{i_2}(t_2) \cdots X_{i_{m_\ell}}(t_{m_\ell}),$$ \hspace{1cm} (2.13)

where $X_i(t) = e^{tf_i}$. The variables $t_i$ are called the factorization parameters of $v$.

**Proposition 2.1.** Let $v(t) \in N_+^{(0)}$ be the factorized parametrization (2.13) corresponding to a reduced word $I = (i_1, \ldots, i_{m_\ell})$. Then the measure

$$d\mu_{N_+}(v(t)) = \prod_{k=1}^{\ell} \prod_{i=1}^{m_\ell} t_i^{(\omega_k,i)} \cdot \prod_{i=1}^{m_\ell} \frac{dt_i}{t_i}.$$ \hspace{1cm} (2.14)

is the restriction of the right-invariant measure $d\mu_{N_+}$ to $N_+^{(0)}$, that is,

$$d\mu_{N_+}(v(t)) = d\mu_{N_+}(v(t)X_j(\tau)), \hspace{1cm} j = 1, \ldots, \ell.$$ \hspace{1cm} (2.15)

**Proof.** To prove the proposition we consider the specific dependence on the choice of a reduced word $I = (i_1, \ldots, i_{m_\ell})$. Let $t^I = (t_1^I, \ldots, t_{m_\ell}^I)$ be the factorization parameters corresponding to a reduced word $I$. According to [16] (Theorem 4.3) one has the following expressions for the matrix elements:

$$\Delta_k(t^I) := \Delta_k(x(t^I)w_0^{-1}) = \prod_{i=1}^{m_\ell} (t_i^I)^{(\omega_k,i)}.$$ \hspace{1cm} (2.16)

Two parameterizations $x(t^I)$ and $x(t'^I)$ of $N_+^{(0)}$ corresponding to reduced words $I$ and $I'$ are related by a birational transformation.
\textbf{Lemma 2.1.} For any reduced decompositions of $w_0$ corresponding to reduced words $I$ and $I'$ the following relations hold.
\begin{enumerate}
  \item \[ \Delta_k(t^I) = \Delta_k(t^{I'}), \quad 1 \leq k \leq \ell. \] (2.17)
  \item \[ \bigwedge_{j=1}^{m_i} \frac{dt^I_j}{t^I_j} = \bigwedge_{j=1}^{m_i} \frac{dt^{I'}_j}{t^{I'}_j}. \] (2.18)
\end{enumerate}

Proof of the lemma. It is shown in [14] that birational transformations $R^{I'}_I$ of $N_+$ corresponding to any two reduced words $I$ and $I'$ can be represented as a composition of elementary transformations (so-called 3-, 4- and 6-moves). Therefore, to prove (2.17), (2.18) one has to check these identities for the elementary moves only in the following cases. Namely, in the case of the classical Lie groups it is enough to consider the following two birational transformations $R^{I'}_I: t^I \to t^{I'}$:
\begin{enumerate}
  \item $X_i(t_1)X_j(t_2)X_i(t_3) = X_j(t'_1)X_i(t'_2)X_j(t'_3)$ for $a_{ij} = a_{ji} = -1$,
  \item $X_j(t_1)X_i(t_2)X_j(t_3)X_i(t_4) = X_i(t'_1)X_j(t'_2)X_i(t'_3)X_j(t'_4)$ for $a_{ij} = -1$ and $a_{ji} = -2$, where we denote $t = t^I$ and $t' = t^{I'}$.
\end{enumerate}

The proof of the identity (2.17) for elementary 3-, 4- and 6-moves follows in a straightforward manner from results in [16]. For completeness we check (2.18) for 3- and 4-moves only.

1) In the case $a_{ij} = a_{ji} = -1$ we have to consider the birational transformation between the parameterizations associated with the reduced words $I = (\ldots iji \ldots)$ and $I' = (\ldots ji j \ldots)$. We have the following relation between the parameters:

\[ v = X_i(t_1)X_j(t_2)X_i(t_3) = X_j(t'_1)X_i(t'_2)X_j(t'_3), \]

where

\[ t'_1 = \frac{t_2t_3}{t_1 + t_3}, \quad t'_2 = t_1 + t_3, \quad t'_3 = \frac{t_1t_2}{t_1 + t_3}. \]

A direct check gives

\[ d \log t'_1 \wedge d \log t'_2 \wedge d \log t'_3 = d \log t_1 \wedge d \log t_2 \wedge d \log t_3. \] (2.19)

2) In the case $a_{ij} = -1$, $a_{ji} = -2$ we have to consider the birational transformation between the parameterizations associated with the reduced words $I = (\ldots ji ji \ldots)$ and $I' = (\ldots i ji j \ldots)$. Thus, we have the following relation between the parameters

\[ X = X_j(t_1)X_i(t_2)X_j(t_3)X_i(t_4) = X_i(t'_1)X_j(t'_2)X_i(t'_3)X_j(t'_4), \]

with

\[ t'_1 = \frac{t_2^2t_3^2t_4}{t_1^2t_2 + (t_1 + t_3)^2t_4}, \quad t'_2 = \frac{t_1^2t_2 + (t_1 + t_3)^2t_4}{t_1t_2 + (t_1 + t_3)t_4}, \]

\[ t'_3 = \frac{(t_1t_2 + (t_1 + t_3)t_4)^2}{t_1^2t_2 + (t_1 + t_3)^2t_4}, \quad t'_4 = \frac{t_1t_2t_3}{t_1t_2 + (t_1 + t_3)t_4}. \] (2.20)

One can readily verify the following identity:

\[ d \log t'_1 \wedge d \log t'_2 \wedge d \log t'_3 \wedge d \log t'_4 = d \log t_1 \wedge d \log t_2 \wedge d \log t_3 \wedge d \log t_4. \]

This completes the proof of the lemma.
Now we can complete the proof of Proposition 2.1. To establish the right-invariance of the measure $d\mu_{N_+}(v)$ we use (2.17), (2.18). For any simple root $\alpha_i$ one can find a reduced word $I(\alpha_i) = (j_1, \ldots, j_m)$ for $w_0$ with $m = m_\ell$ such that $j_m = i$. Then the identities (2.17), (2.18) imply that 

$$d\mu_{N_+}(v^{I(\alpha_i)}) = d\mu_{N_+}(v^{tI(\alpha_i)}).$$

In this way we obtain 

$$v^{I(\alpha_i)} \cdot X_i(\tau) = X_{j_1}(t_1)X_{j_{m-1}}(t_{m-1})X_i(t_m + \tau). \quad (2.21)$$

By construction the factorization parameter $t_m$ appears only in the (monomial) expression for $\Delta_j(v(t))$ as a homogeneous factor of degree one. In this way, the factorization parameter $t_m$ appears in the measure $d\mu_{N_+}$ only in the $\alpha_j$-component $\Delta_j(v(t))d\log t_m$, and hence $d\mu_{N_+}$ is invariant under the shift $t_m \to t_m + \tau$. Thus, the measure is right-invariant with respect to the action of $X_j(\tau)$ for any $j = 1, \ldots, \ell$, and so it is right-invariant with respect to the whole of $N_+$. This completes the proof of Proposition 2.1, and using (2.16) we complete the proof of Lemma 2.1.

2.2. Whittaker vectors for classical Lie groups: proofs of Lemma 1.1 and Proposition 1.1. In this subsection we derive expressions for the left and right $g$-Whittaker vectors in terms of the matrix elements of finite-dimensional representations of $g$. The Whittaker vectors satisfy the following equations:

$$f_i\psi_R = -\psi_R, \quad e_i\psi_L = -\psi_L, \quad i = 1, \ldots, \ell. \quad (2.22)$$

Integrating the actions of the nilpotent Lie subalgebras $n_\pm \subset g$ to actions of the nilpotent Lie subgroups $N_\pm \subset G$, we can write the equations for the $g$-Whittaker functions in terms of one-parameter subgroups $X_i(t) \subset N_+$ and $Y_i(t) \subset N_-$ as follows:

$$\pi_\mu(X_i(t))\psi_R(v) = e^{-t}\psi_R(v), \quad \pi_\mu(Y_i(t))\psi_L(v) = e^{-t}\psi_L(v),$$

where $i = 1, \ldots, \ell, v \in N_+$. Equivalently, one has for any $z_\pm \in N_\pm$

$$\pi_\mu(z_+)_i\psi_R(v) = \exp\left\{-\sum_{i=1}^{\ell}(z_+_i)\right\}\psi_R(v), \quad (2.23)$$

$$\pi_\lambda(z_-)_i\psi_L(v) = \exp\left\{-\sum_{i=1}^{\ell}(z_-)_i\right\}\psi_L(v),$$

where $(z_\pm)_i := u_{\pm\alpha_i}(z_\pm)$. The construction of the right Whittaker vector is straightforward. Note that we have a simple identity

$$u_{\alpha_i}(v_1v_2) = u_{\alpha_i}(v_1) + u_{\alpha_i}(v_2), \quad v_1, v_2 \in N_+.$$
Then from (2.23) we infer that the right Whittaker vector is given by a multiplicative character of the maximal unipotent subgroup $N_+$

$$\psi_R(v) = \exp\left\{ - \sum_{i=1}^{\ell} v_i \right\} = \exp\left\{ - \frac{\Delta_{\omega_i, s_{i-1}^{-1}}(v)}{\Delta_{\omega_i, 1}(v)} \right\}, \quad v \in N_+, \quad (2.24)$$

where $v_i := u_{\alpha_i}(v)$ and we use (2.9) to express $v_i$ in terms of matrix elements.

To construct the left Whittaker vector in terms of matrix elements we use an inner automorphism of $G$, acting on $z \in G$ as $z^\tau = \dot{w}^{-1}_0 z \dot{w}_0$. Taking into account that $\dot{w}^{-1}_0 \mathcal{X}_*(t) \dot{w}_0 = Y_i(t)$, we have $\dot{w}^{-1}_0 N_+ \dot{w}_0 = N_-$. The equation for the left Whittaker vector

$$\pi_\mu(Y_i(t))\psi_L(v) = e^{-t}\psi_L(v), \quad i = 1, \ldots, \ell,$$

can now be written in the form

$$\pi_\mu(z^\tau)\psi_L(v) = \exp\left\{ - \sum_{i=1}^{\ell} z_i \right\}\psi_L(v), \quad z \in N_+. \quad (2.25)$$

The left Whittaker vector can be obtained by a twist of the right vector

$$\psi_L(v) = \psi_R(v\dot{w}^{-1}_0)^{-1},$$

where the function $\psi_R$ is considered as a $B_-$-equivariant function on $G$ (see (1.9) for the precise definition). Using Gauss decomposition and the parametrization (2.8), (2.9), we obtain for the left Whittaker vector

$$\psi_L(v) = \exp\left\{ i\lambda - \rho, \sum_{i=1}^{\ell} u_i(v\dot{w}^{-1}_0)h_i \right\} \exp\left\{ \sum_{i=1}^{\ell} u_{\alpha_i}(v\dot{w}^{-1}_0) \right\}.$$

In terms of the matrix elements of finite-dimensional representations we have the representation

$$\psi_L(v) = \prod_{i=1}^{\ell} \Delta_{\omega_i, \dot{w}^{-1}_0}(v)^{(i\lambda - \rho, \alpha_i^\vee)} \exp\left\{ \frac{\Delta_{\omega_i, \dot{w}^{-1}_0}(v)}{\Delta_{\omega_i, 1}(v)} \right\}$$

$$= \prod_{i=1}^{\ell} \Delta_i(v)^{(i\lambda - \rho, \alpha_i^\vee)} \exp\left\{ - \frac{\Delta_i'(v)}{\Delta_i(v)} \right\}. \quad (2.26)$$

This completes the proof of Lemma 1.1. The proof of Proposition 1.1 is then obtained by combining the expressions for the right Whittaker vector and the left Whittaker vector twisted by the action of the factors $e^{H_L}$ and $e^{H_R}$ of the Cartan element $e^{H_L + H_R} = \exp(-h_x) = \exp\left\{ - \sum_{i=1}^{\ell} \langle \omega_i, x \rangle h_i \right\}$.

### 2.3. Explicit evaluation of matrix elements.

To construct integral representations of the Whittaker functions one has to express various matrix elements in the integral formulae (1.19) by using factorized and modified factorized parameterizations of group elements. This can be done in a rather straightforward manner using
the results of \cite{16}, \cite{15}. Below we shall use a recursive structure of the reduced word $I_\ell$ corresponding to the maximal length element $w_0$ in the Weyl group of classical Lie algebras. This recursive structure translates into recursion formulae for the relevant ratios of matrix elements. Solving the recursion equations, we find explicit expressions for $\psi_L$ and $\psi_R$ in a (modified) factorized parametrization. This provides corresponding integral representations for the Whittaker functions of classical Lie groups. In the case of the modified factorized parametrization we obtain a generalization of the Givental integral representation for $g$ Lie groups. In the case of the modified factorized parametrization we obtain a generalization of the Givental integral representation for $g = \mathfrak{gl}_{\ell+1}$ to other classical Lie algebras.

\subsection{Expressions for $\mathfrak{gl}_{\ell+1}$ matrix elements: proofs of Theorems 1.1 and 1.3} In this subsection we introduce expressions for matrix elements relevant to the construction of integral representations of $\mathfrak{gl}_{\ell+1}$-Whittaker functions, using a factorized parametrization of an open part of $N_+ \subset \text{GL}(\ell + 1)$. This provides a proof of the integral representations of $\mathfrak{gl}_{\ell+1}$-Whittaker functions presented in Part 1.

The eigenfunctions of the $\mathfrak{gl}_{\ell+1}$- and $\mathfrak{sl}_{\ell+1}$-Toda chains differ by a simple factor \eqref{eq:factorized}, and the Whittaker vectors $\psi_L$, $\psi_R$ are the same for both Lie algebras. Thus, we use the $\mathfrak{sl}_{\ell+1}$ root data for calculations of the matrix elements $\Delta_{\omega_i, \omega_0^{-1}}(v)$, $\Delta_{\omega_i, \omega_0^{-1} s_i^{-1}}(v)$, $i = 1, \ldots, \ell$, in the fundamental representations of $\mathfrak{gl}_{\ell+1}$ and in addition let $\Delta_{\omega_{\ell+1}, \omega_0^{-1}}(v) = 1$. The $\mathfrak{sl}_{\ell+1}$ root data are given by \eqref{eq:rootdata}. The reduced decomposition $w_0 = s_{i_1} s_{i_2} \cdots s_{i_m}$ of the maximal length element $w_0 \in W$ corresponding to a reduced word $I_\ell = (i_1, \ldots, i_m)$ with $m = m_\ell = \ell(\ell + 1)/2$ provides a total ordering of the positive coroots by $R_+^\vee = \{ \gamma_k^\vee = s_{i_1} \cdots s_{i_{k-1}} \alpha_k^\vee, k = 1, \ldots, m_\ell \}$. We consider a decomposition of $w_0$ described by the reduced word

$$I_\ell = (1, 21, \ldots, (\ell \ldots 21)).$$

The corresponding ordering of the positive coroots is given by

$$\gamma_1^\vee = \alpha_1^\vee, \quad \gamma_2^\vee = \alpha_1^\vee + \alpha_2^\vee, \quad \gamma_3^\vee = \alpha_2^\vee, \quad \ldots \quad \gamma_{m_\ell}^\vee = \alpha_{m_\ell}^\vee.$$

\begin{equation}
\gamma_{m_{\ell-1}+1}^\vee = \alpha_1^\vee + \cdots + \alpha_\ell^\vee,
\end{equation}

The recursive parametrization of an open part $N_+^{(0)}$ of $N_+$ corresponding to the reduced word $I_\ell$ is as follows. Given $v^{A_\ell} \in N_+^{(0)}$, we have

$$v^{A_\ell}(y) = \mathfrak{X}_1(y) \mathfrak{X}_2(y) \cdots \mathfrak{X}_\ell(y),$$

where

$$\mathfrak{X}_k(y) = X_k(y_{k,n_k,k}) \cdots X_2(y_{2,n_{k-1},2}) X_1(y_{1,n_{k-1},1})$$

and $\mathfrak{X}_1 = X_1(y_{1,1})$. Here we adopt the following notation. Let $|I_\ell| = m_\ell$ be the length of $w_0$. For the root system of type $A_\ell$ one has $m_\ell = \ell(\ell + 1)/2$. Then for any $k \in \{1, \ldots, \ell\}$ consider a subword

$$I_k = (i_1, \ldots, i_{m_k}) \subset I_\ell = (i_1, \ldots, i_{m_k}, i_{m_k+1}, \ldots, i_{m_\ell}),$$

with $|I_k| = m_k = k(k + 1)/2$. Let $A_k$ be a corresponding root subsystem in $R_+$ and let $v^{A_k} = \mathfrak{X}_1 \cdots \mathfrak{X}_k$ be a factorized parametrization of the corresponding
Lemma 2.3. Let the following relations hold for the matrix elements of $v$:

$$n_{k,i} = k + 1 - i, \quad 1 \leq i \leq \ell. \tag{2.29}$$

We are interested in explicit expressions for the following matrix elements in terms of the factorization parameters $\{y_{i,n}\}$:

$$\Delta_i(v) := \langle \xi_{\omega_i}^- | \pi_i(v \tilde{w}_0^{-1}) | \xi_{\omega_i}^+ \rangle, \quad \Delta'_i(v) := \langle \xi_{\omega_i}^- | \pi_i(v f_i \tilde{w}_0^{-1}) | \xi_{\omega_i}^+ \rangle,$$

where $\pi_i = \pi_{\omega_i}$, $i = 1, \ldots, \ell$, is the fundamental representation with highest weight $\omega_i$, and $\xi_{\omega_i}^+$ and $\xi_{\omega_i}^-$ are the highest and the lowest weight vectors in the representations $\pi_i$, $i = 1, \ldots, \ell$, determined by the conditions $n \pm \xi_i^\pm = 0$ and normalized so that $\langle \xi_{\omega_i}^+ | \xi_{\omega_i}^- \rangle = 1$. Note that according to (2.5), for the Lie algebra $\mathfrak{sl}_{\ell+1}$ we have $i^* = \ell + 1 - i$. The proof of the following statement is obtained by an iterative evaluation of the matrix elements, taking into account the Serre relations and the defining ideals of the fundamental representations, and using the technique of [16].

Lemma 2.2. For $i = 1, \ldots, \ell$ the following recursion relations hold:

$$\Delta_i(v)^{A_{\ell}} = \left( \prod_{k=1}^{i} y_{\ell+1-k,i} \right) \Delta_i(v)^{A_{\ell-1}},$$

$$\left( \frac{\Delta'_i(v)}{\Delta_i(v)} \right)^{A_{\ell}} = \frac{1}{y_{\ell+1-i,i}} + (1 - \delta_{i,\ell}) \frac{y_{\ell-i,i+1}}{y_{\ell+1-i,i}} \left( \frac{\Delta'_i(v)}{\Delta_i(v)} \right)^{A_{\ell-1}}. \tag{2.30}$$

The matrix elements can then be found by solving the recursion relations (2.30).

Lemma 2.3. Let $v$ be defined by (1.32) and (1.33). Then for $i = 1, \ldots, \ell$ the following relations hold for the matrix elements of $v$ in terms of the variables $y_{i,k}$:

$$\Delta_{\omega_i, s_i}(v)^{A_{\ell}} = \sum_{n=1}^{\ell} y_{i,n},$$

$$\Delta_i(v)^{A_{\ell}} = \prod_{k=1}^{i} \prod_{n=1}^{\ell} y_{k+1-n,n}, \tag{2.31}$$

$$\left( \frac{\Delta'_k(v)}{\Delta_i(v)} \right)^{A_{\ell}} = \frac{1}{y_{\ell+1-i,i}} \left[ 1 + (1 - \delta_{i,\ell}) \sum_{n=1}^{\ell-i} \prod_{k=1}^{n} y_{\ell+1-k,i+1} \right].$$

Combining these expressions with the expression (1.21) for the invariant measure on $N_+$ and substituting into (1.17), (1.18), and (1.19), one obtains the integral formula (1.37) and completes the proof of Theorem 1.1.

Now consider the integral representation for the $\mathfrak{gl}_{\ell+1}$-Whittaker function in the modified factorized parametrization (1.41). We start with an analogue of the recursion relations (2.30) for matrix elements in the modified factorized parametrization. To simplify the formulation of the recursion relations it turns out to be useful to consider a twisted version

$$y_{i,n} = e^{x_{\ell+1,i} - x_{\ell+1,i+1}} e^{x_{n,i+1} - x_{n+i-1,i}} \tag{2.32}$$
of the modified parametrization (1.41) by taking into account the action of the part \( e^{HR} \) (1.46) of the Cartan generators \( e^{-h\ell} \). The simple change of variables (2.32) applied to (2.30) gives the following.

**Lemma 2.4.** 1) In the modified factorized parametrization (2.32) the recursion relations (2.30) are given by

\[
\left( \frac{\Delta'_i(v)}{\Delta_i(v)} \right)^{A_{\ell}} = e^{x_{\ell,\ell-i+1} - x_{\ell,\ell-i} + 1} \frac{e^{x_{\ell,\ell-i+1} - x_{\ell,\ell-i}} \left( \frac{\Delta'_i(v)}{\Delta_i(v)} \right)^{A_{\ell-1}}}{e^{x_{\ell,\ell-i+1} - x_{\ell,\ell-i}}}. \tag{2.33}
\]

2) The recursion equations have the solution

\[
\left( \frac{\Delta'_i(v)}{\Delta_i(v)} \right)^{A_{\ell}} = \sum_{n=1}^{\ell+1-i} e^{x_{n+i-1,n} - x_{n+i,n}}, \tag{2.34}
\]

\[
\Delta_{\omega_i,w_{0}^{-1}}(v)^{A_{\ell}} = \exp\left\{ \sum_{n=1}^{i} (x_{\ell+1,n} - x_{i,n}) \right\}, \tag{2.35}
\]

where \( i = 1, \ldots, \ell \) and it is assumed that \( \Delta_{\ell+1}(v) = 1 \).

Substituting (2.34), (2.35) into (1.17), (1.18), we now obtain the Whittaker vectors in the parametrization (2.32). Taking \( x_{\ell+1,k} = 0, k = 1, \ldots, \ell + 1 \), we recover the expressions for the Whittaker vectors given in Lemma 1.4. To prove Theorem 1.3 it remains to take into account the form of the measure \( d\mu_{N_+} \) in the modified factorized parametrization. This completes the proof of the theorem.

2.3.2. Expressions for \( \mathfrak{so}_{2\ell+1} \) matrix elements: proofs of Theorems 1.4 and 1.6.

In this subsection we introduce expressions for matrix elements relevant to the construction of integral representations of \( \mathfrak{so}_{2\ell+1} \)-Whittaker functions, using a factorized parametrization of an open part of \( N_+ \subset \text{SO}(2\ell+1) \). This provides a proof of the integral representations of \( \mathfrak{so}_{2\ell+1} \)-Whittaker functions presented in Part 1.

We are using the root data given by (1.64). The reduced decomposition \( w_0 = s_{i_1} s_{i_2} \cdots s_{i_{m_\ell}} \) of the maximal length element \( w_0 \in W(\mathfrak{so}_{2\ell+1}) \) corresponding to a reduced word \( I_\ell = (i_1, \ldots, i_{m_\ell}) \) with \( m_\ell = \ell^2 \) provides a total ordering of the positive coroots by \( R_+^\vee = \{ \gamma_k^\vee = s_{i_1} \cdots s_{i_{k-1}} \alpha_k^\vee, k = 1, \ldots, m_\ell \} \) for \( \mathfrak{so}_{2\ell+1} \). We consider a decomposition of \( w_0 \) described by the following reduced word

\[ I_\ell = (1, 212, \ldots, (\ell \ldots 212 \ldots \ell)). \]
The corresponding ordering of the positive coroots is given by:

\[
\begin{align*}
\gamma_2^\vee &= \alpha_1^\vee + \alpha_2^\vee, \\
\gamma_1^\vee &= \alpha_1^\vee, \\
\gamma_3^\vee &= \alpha_1^\vee + 2\alpha_2^\vee, \\
\gamma_4^\vee &= \alpha_2^\vee,
\end{align*}
\]

\[
\gamma_{(\ell-1)^2+1}^\vee = \alpha_1^\vee + 2(\alpha_2^\vee + \cdots + \alpha_{\ell-1}^\vee) + \alpha_\ell^\vee,
\]

\[
\gamma_{(\ell-1)^2+2}^\vee = \alpha_1^\vee + 2(\alpha_2^\vee + \cdots + \alpha_{\ell-2}^\vee) + \alpha_{\ell-1}^\vee + \alpha_\ell^\vee,
\]

\[
\begin{align*}
\cdots \cdots \cdots & \\
\gamma_{(\ell-1)}^\vee &= \alpha_1^\vee + \alpha_2^\vee + \cdots + \alpha_\ell^\vee, \\
\gamma_{(\ell-1)+1}^\vee &= \alpha_1^\vee + 2(\alpha_2^\vee + \cdots + \alpha_\ell^\vee), \\
\gamma_{(\ell-1)+2}^\vee &= \alpha_2^\vee + \cdots + \alpha_\ell^\vee, \\
\cdots \cdots \cdots & \\
\gamma_{\ell^2}^\vee &= \alpha_\ell^\vee.
\end{align*}
\]

The recursive parametrization of an open part \(N_+^{(0)}\) of \(N_+\) corresponding to the reduced word \(I_\ell\) is as follows. Given \(v^{B_\ell} \in N_+^{(0)}\), we have

\[
v^{B_\ell}(y) = \mathcal{X}_1(y)\mathcal{X}_2(y) \cdots \mathcal{X}_\ell(y),
\]

where

\[
\mathcal{X}_k(y) = X_k(y_{k,n_k,k-1}) \cdots X_2(y_{2,n_2,1})X_1(y_{1,n_1,1})X_2(y_{2,n_2,2}) \cdots X_k(y_{k,n_k,k})
\]

and \(\mathcal{X}_1 = X_1(y_{1,1})\). For any \(k \in \{1, \ldots, \ell\}\) we consider a subword

\[
I_k = (i_1, \ldots, i_{m_k}) \subset I_\ell = (i_1, \ldots, i_{m_1}, i_{m_1+1}, \ldots, i_{m_\ell})
\]

with \(|I_k| = m_k = k^2\). Let \(B_\ell\) be a corresponding root subsystem in \(R_+\) and \(v^{B_\ell} = \mathcal{X}_1 \cdots \mathcal{X}_\ell\) a factorized parametrization of the corresponding subgroup. The factorization parameters for \(v^{B_k}(y)\) can be naturally enumerated as \(\{y_{i,n}; 1 \leq i \leq k, 1 \leq n \leq n_{k,i}\}\), where

\[
n_{k,1} = k, \quad n_{k,i} = 2(k+1-i), \quad 1 < i \leq \ell.
\]

We also use the notation \(n_i := n_{\ell,i}\).

We are interested in explicit expressions for the following matrix elements in terms of the factorization parameters \(\{y_{i,n}\}\):

\[
\Delta_i(v) := \langle \xi_{\omega_i}^- | \pi_i (v \bar{w}_0^{-1}) | \xi_{\omega_i}^+ \rangle, \quad \Delta'_i(v) := \langle \xi_{\omega_i}^- | \pi_i (v f_i \bar{w}_0^{-1}) | \xi_{\omega_i}^+ \rangle,
\]

where \(\pi_i = \pi_{\omega_i}\) is the fundamental representation with highest weight \(\omega_i\), \(\xi_{\omega_i}^+\) and \(\xi_{\omega_i}^-\) are the highest and lowest weight vectors in the representation \(\pi_i\) such that \(\langle \xi_{\omega_i}^+ | \xi_{\omega_i}^- \rangle = 1\). Note that for the Lie algebra \(\mathfrak{so}_{2\ell+1}\) we have \(i \mapsto i^*\) for the involution defined by (2.5). The proof of the following statement is obtained by an iterative evaluation of the matrix elements, taking into account the Serre relations and the defining ideals of the fundamental representations, and using the technique of [16].
Lemma 2.5. Let \( v := v^{B_\ell} \) be defined by (2.37). Then the following recursion equations hold:

\[
\Delta_1(v)^{B_\ell} = \left(y_{1,\ell} \prod_{k=2}^{\ell} y_{k, n_k-1}\right) \cdot \Delta_1(v)^{B_{\ell-1}},
\]

\[
\Delta_i(v)^{B_\ell} = \left(y_{1,\ell}^2 \prod_{k=2}^{i} y_{k, n_k-1} y_{k, n_k} \prod_{k=i+1}^{\ell} y_{k, n_k-1}\right) \cdot \Delta_i(v)^{B_{\ell-1}}, \quad 1 < i \leq \ell,
\]

\[
\left(\frac{\Delta'_i(v)}{\Delta_i(v)}\right)^{B_\ell} = \frac{1}{y_{1,\ell}} \left(1 + \frac{y_{2,2(\ell-1)}}{y_{2,2(\ell-1)-1}}\right) + \frac{y_{2,2(\ell-1)}}{y_{2,2(\ell-1)-1}} \left(\frac{\Delta'_i(v)}{\Delta_i(v)}\right)^{B_{\ell-1}}, \tag{2.39}
\]

\[
\left(\frac{\Delta'_k(v)}{\Delta_k(v)}\right)^{B_\ell} = \frac{1}{y_{k,2(\ell+1-k)}} \left(1 + \frac{y_{k+1,2(\ell-k)}}{y_{k+1,2(\ell-k)-1}}\right) + \frac{y_{k,2(\ell+1-k)-1}}{y_{k,2(\ell+1-k)}} \left(\frac{\Delta'_k(v)}{\Delta_k(v)}\right)^{B_{\ell-1}},
\]

where \( n_k = 2(\ell + 1 - k), \ k = 2, \ldots, \ell, \) and \( \prod_{n=M}^{N} a_n = 1 \) for \( N < M. \)

Now the matrix elements can be found by solving the recursion relations given above.

Lemma 2.6. Let \( v \) be defined by (1.67) and (1.68). The following expressions hold for the matrix elements of \( v \) in terms of the variables \( y_{i,k}: \)

\[
\Delta_{\omega_i, s_i}(v)^{B_\ell} = \sum_{n=i}^{i+n_1} y_{i,n}, \quad i = 1, \ldots, \ell, \tag{2.40}
\]

\[
\Delta_1(v)^{B_\ell} = \prod_{n=1}^{\ell} y_{1,n} \cdot \prod_{k=2}^{n} \prod_{n=1}^{n_k/2} y_{k, 2n-1}, \tag{2.41}
\]

\[
\Delta_k(v)^{B_\ell} = \prod_{n=2}^{n_1} y_{1,n}^2 \cdot \prod_{i=k+1}^{\ell} \prod_{n=1}^{n_i/2} y_{i, 2n-1} \cdot \prod_{i=2}^{k} \prod_{n=1}^{n_i/2} y_{i, 2n-1} y_{i, 2n}, \tag{2.42}
\]

\[
\left(\frac{\Delta'_1(v)}{\Delta_1(v)}\right)^{B_\ell} = \sum_{n=1}^{\ell} \frac{1}{y_{1,n}} \left(1 + \frac{y_{2,2(n-1)}}{y_{2,2(n-1)-1}}\right) \prod_{i=n+1}^{\ell} \frac{y_{2,2(i-1)}}{y_{2,2(i-1)-1}}, \tag{2.43}
\]

\[
\left(\frac{\Delta'_k(v)}{\Delta_k(v)}\right)^{B_\ell} = \frac{1}{y_{k,2(\ell+1-k)}} \left(1 + \frac{y_{k+1,2(\ell-k)}}{y_{k+1,2(\ell-k)-1}}\right) + \frac{y_{k,2(\ell+1-k)-1}}{y_{k,2(\ell+1-k)}} \left(\frac{\Delta'_k(v)}{\Delta_k(v)}\right)^{B_{\ell-1}},
\]
for the matrix elements of $v$

Lemma 2.7. Choose a unipotent element $H$ taking into account the action of the part $k$ to be useful to consider a twisted version of the recursion relations (2.39) for the matrix elements in the modified factorized integral formula (1.74), which completes the proof of Theorem 1.4.

We now consider the integral representation for the $\mathfrak{so}_{2\ell+1}$-Whittaker function in the modified factorized parametrization (1.89), (1.90). We start with an analogue of the recursion relations (2.39) for the matrix elements in the modified factorized parametrization. To simplify the formulation of the recursion relations, it turns out to be useful to consider a twisted version

$$y_{1,1} = e^{-x_{\ell,1}}e^{x_{\ell,1}-z_{1,1}}, \quad y_{1,k} = e^{-x_{\ell,1}}(e^{x_{k-1,1}-z_{k,1}} + e^{x_{k,1}-z_{k,1}}),$$

$$y_{k,2r-1} = e^{x_{\ell,k-1}-x_{\ell,k}}e^{z_{k+r-1,k}-z_{k+r-2,k-1}},$$

$$y_{k,2r} = e^{x_{\ell,k-1}-x_{\ell,k}}e^{z_{k+r-1,k}-z_{k+r-1,k-1}}$$

for $k = 2, \ldots, \ell$ and $r = 1, \ldots, \ell + 1 - k$, of the modified parametrization (1.89) by taking into account the action of the part $H_R$ of the Cartan generators (1.99).

Lemma 2.7. Choose a unipotent element $v \in N_+^{(0)}$. The following expressions hold for the matrix elements of $v$ in terms of the variables $x_{k,i}$, $z_{k,i}$ defined in (2.45):

$$\frac{\Delta_1^2(v)}{\Delta_2(v)} = e^{x_{\ell,1} - 2z_{1,1}},$$

$$\frac{\Delta_k(v)}{\Delta_{k+1}(v)} = \exp\left\{-\sum_{i=1}^k x_{k,i} - 2z_{k,1} + 2\sum_{i=2}^k z_{k,i} - \sum_{i=1}^{k-1} x_{k-1,i}\right\} (e^{x_{k-1,1}} + e^{x_{k,1}})^2, \quad k = 2, \ldots, \ell, \quad \delta_{\ell+1}(v) = 1 \quad is \ assumed;$$

$$\frac{\Delta'_1(v)}{\Delta_1(v)} = \sum_{k=1}^{\ell} e^{z_{k,1}},$$

$$\frac{\Delta'_k(v)}{\Delta_k(v)} = e^{x_{\ell,k} - z_{k,k}} + \sum_{n=k+1}^{\ell} (e^{x_{n-1,k} - z_{n,k}} + e^{x_{n,k} - z_{n,k}}), \quad k = 2, \ldots, \ell.$$

Here $x_{\ell,k} = 0$, $k = 1, \ldots, \ell$. Terms like $e^{z_{\ell+1,i}}$ in (2.47) are omitted, and as usual, $\sum_{n=i}^{j}$ is taken to be 0 whenever $i > j$.

Substituting (2.46), (2.47) into (1.17), (1.18), we obtain the Whittaker vectors in the parametrization (2.45). Taking $x_{\ell,k} = 0$, $k = 1, \ldots, \ell$, we recover the expressions for the Whittaker vectors given in Lemma 1.6. To prove Theorem 1.6 it remains to take into account the form of the measure $d\mu_{N_+}$ in the modified factorized parametrization. This completes the proof of the theorem.
2.3.3. Expressions for $\mathfrak{sp}_{2\ell}$ matrix elements: proofs of Theorem 1.8 and 1.10. In this subsection we introduce expressions for matrix elements relevant to the construction of integral representations of $\mathfrak{sp}_{2\ell}$-Whittaker functions, using a factorized parametrization of an open part of $N_+ \subset \text{Sp}(2\ell)$. This provides a proof of the integral representations of $\mathfrak{sp}_{2\ell}$-Whittaker functions presented in Part 1.

We are using the root data for $\mathfrak{g} = \mathfrak{sp}_{2\ell}$ given by (1.131). The reduced decomposition $w_0 = s_{i_1} s_{i_2} \cdots s_{i_m}$, $m = m_\ell$, of the maximal length element $w_0 \in W(C_\ell)$ corresponding to a reduced word $I_\ell = (i_1, \ldots, i_{m_\ell})$ provides a total ordering of the positive coroots by $R_\vee = \{ \gamma_1^\vee = s_{i_1} \cdots s_{i_{k-1}} \alpha_k^\vee, \ k = 1, \ldots, m_\ell \}$. For the root system of type $C_\ell$ one has $m_\ell = \ell^2$. We consider the decomposition of $w_0$ described by the following reduced word:

$$I_\ell = (1, 212, \ldots, (\ell \cdots 212 \cdots \ell)).$$

The corresponding ordering of the positive coroots is given by

$$\gamma_1^\vee = \alpha_1^\vee, \quad \gamma_2^\vee = 2\alpha_1^\vee + \alpha_2^\vee, \quad \gamma_3^\vee = \alpha_1^\vee + \alpha_2^\vee, \quad \gamma_4^\vee = \alpha_2^\vee, \quad \cdots \quad \gamma_\ell^\vee = \alpha_\ell^\vee,$$

$$\gamma_1^\vee = \alpha_1^\vee, \quad \gamma_2^\vee = 2\alpha_1^\vee + \alpha_2^\vee + \cdots + \alpha_\ell^\vee,$$

$$\gamma_1^\vee = \alpha_1^\vee,$$

$$\gamma_2^\vee = \alpha_1^\vee + \cdots + \alpha_\ell^\vee.$$

The recursive parametrization of an open part $N_+^{(0)}$ of $N_+$ defined by the reduced word $I_\ell$ is as follows. Given $v^{C_\ell} \in N_+^{(0)}$, we have

$$v^{C_\ell}(y) = \mathfrak{X}_1(y) \mathfrak{X}_2(y) \cdots \mathfrak{X}_\ell(y),$$

where

$$\mathfrak{X}_k(y) = X_k(y_{k,n_{k,k-1}}) \cdots X_2(y_{2,n_{k,2}-1}) X_1(y_{1,n_{k,1}}) X_2(y_{2,n_{k,2}}) \cdots X_k(y_{k,n_{k,k}})$$

and $\mathfrak{X}_1 = X_1(y_{1,1})$. For any $k \in \{1, \ldots, \ell\}$ we consider a subword

$$I_k = (i_1, \ldots, i_{m_k}) \subset I_\ell = (i_1, \ldots, i_{m_\ell}, i_{m_\ell+1}, \ldots, i_{m_\ell})$$

with $|I_k| = m_k = k^2$. Let $C_k$ be the corresponding root subsystem in $R_\vee$ and let $v^{C_k} = \mathfrak{X}_1 \cdots \mathfrak{X}_k$ be a factorized parametrization of the corresponding subgroup.
The factorization parameters for \( v^C_k(y) \) can be naturally enumerated as \( \{ y_{i,n}; 1 \leq i \leq k, 1 \leq n \leq n_{k,i} \} \), where

\[
 n_{k,1} = k, \quad n_{k,i} = 2(k + 1 - i), \quad 1 < i \leq \ell. \tag{2.50}
\]

Also, \( n_i := n_{\ell,i} \).

We are interested in explicit expressions for the following matrix elements in terms of the factorization parameters \( \{ y_{i,n} \} \):

\[
\Delta_i(v) := \langle \xi^{-}_{\omega_i} | \pi_i(v^0_{\ell-1}) | \xi^{+}_{\omega_i} \rangle, \\
\Delta_i'(v) := \langle \xi^{-}_{\omega_i} | \pi_i(v^f i_{\ell-1}) | \xi^{+}_{\omega_i} \rangle,
\]

where \( \pi_i = \pi_{\omega_i} \) is the fundamental representation with highest weight \( \omega_i \), and \( \xi^{+}_{\omega_i} \) and \( \xi^{-}_{\omega_i} \) are the highest and lowest weight vectors in the representation \( \pi_i \) such that \( \langle \xi^{+}_{\omega_i} | \xi^{-}_{\omega_i} \rangle = 1 \). Note that for the Lie algebra \( sp_{2\ell} \) the involution \( i \to i^* \) defined by (2.5) is trivial: \( i^* = i \). The proof of the following statement is obtained by an iterative evaluation of the matrix elements, taking into account the Serre relations and the defining ideals of the fundamental representations, and using the technique of [16].

**Lemma 2.8.** The matrix elements \( \Delta_i(v)^{C\ell} \) and \( \Delta_i'(v)^{C\ell} \) satisfy the following recursion relations:

\[
\Delta_i(v)^{C\ell} = \left( y_{1,n_1} \prod_{k=2}^{i} y_{k,n_{k-1},k} \prod_{k=1+1}^{\ell} y_{k,n_{k-1},k} \right) \Delta_i(v)^{C_{\ell-1}}, \quad i = 1, \ldots, \ell,
\]

\[
\left( \frac{\Delta_i'(v)}{\Delta_1(v)} \right)^{C\ell} = \frac{1}{y_{1,1}} \left( 1 + \frac{y_{2,2(\ell-1)}}{y_{2,2(\ell-1)-1}} \right)^2 + \frac{y_{2,2(\ell-1)}}{y_{2,2(\ell-1)-1}} \left( \frac{\Delta_1(v)}{\Delta_1(v)} \right)^{C_{\ell-1}},
\]

\[
\left( \frac{\Delta_k(v)}{\Delta_1(v)} \right)^{C\ell} = \frac{1}{y_{k,2(\ell+1-k)}} \left( 1 + \frac{y_{k+1,2(\ell-k)}}{y_{k+1,2(\ell-k)-1}} \right) + \frac{y_{k,2(\ell+1-k)-1}}{y_{k+1,2(\ell-k)-1}} \frac{y_{k+1,2(\ell-k)}}{y_{k+1,2(\ell-k)-1}} \left( \frac{\Delta_k(v)}{\Delta_k(v)} \right)^{C_{\ell-1}}, \quad \text{for } k = 2, \ldots, \ell - 1,
\]

\[
\left( \frac{\Delta_{\ell}(v)}{\Delta_1(v)} \right)^{C\ell} = \frac{1}{y_{1,2}}.
\]

The matrix elements can now be found by solving the recursion relations (2.51).
Lemma 2.9. Let $v$ be defined by (1.134) and (1.135). The following relations hold for the matrix elements of $v$ in terms of the variables $y_{i,k}$:

$$
\Delta_{\omega_i,\bar{s}_i}(v)^{C_\ell} = \sum_{n=1}^{\ell} y_{i,n}, \quad i = 1, \ldots, \ell, \\
$$

$$
\Delta_{\omega_i,\bar{w}_0}(v)^{C_\ell} = \prod_{n=1}^{n_1} y_{1,n} \prod_{k=2}^{n_i} y_{k,n} \prod_{k=i+1}^{\ell} y_{k,2n-1}, \\
$$

$$
\left( \frac{\Delta'_1(v)}{\Delta_1(v)} \right)^{C_\ell} = \sum_{n=1}^{n_1} \frac{1}{y_{1,n}} \left( 1 + \frac{y_{2,2(n-1)}}{y_{2,2(n-1)} - 1} \right)^{2\ell-1} \prod_{i=n}^{\ell} \frac{y_{2,2i}}{y_{2,2i-1}}, \\
$$

$$
\left( \frac{\Delta'_k(v)}{\Delta_k(v)} \right)^{C_\ell} = \sum_{n=1}^{n_k/2} \frac{1}{y_{k,2n}} \left( 1 + \frac{y_{k+1,2(n-1)}}{y_{k+1,2(n-1)} - 1} \right) \times \prod_{i=n+1}^{n_k/2} \frac{y_{k+1,2(i-1)}}{y_{k+1,2(i-1)} - 1} \frac{y_{k,2i-1}}{y_{k,2i}} \quad \text{for} \quad k = 2, \ldots, \ell - 1, \\
$$

$$
\left( \frac{\Delta'_\ell(v)}{\Delta_\ell(v)} \right)^{C_\ell} = \frac{1}{y_{\ell,2}}.
$$

Combining these expressions with the expression (1.21) for the invariant measure on $N_+$ and substituting into (1.17), (1.18), and (1.19), one completes the proof of Theorem 1.8.

We now consider the integral representation for the $\mathfrak{sp}_{2\ell}$-Whittaker function in the modified factorized parametrization (1.150), (1.151). We start with an analogue of the recursion relations (2.51) for the matrix elements in the modified factorized parametrization. To simplify the formulation of the recursion relations it turns out to be useful to consider a twisted version of the modified parametrization (1.152) by taking into account the action of the part $e^{H_R}$ (1.162) of the Cartan element $e^{-h_z}$. Thus, we consider the following change of variables:

$$
y_{1,1} = e^{-2\varepsilon_1} e^{x_{1,1} + z_{1,1}}, \\
y_{1,k} = e^{-2\varepsilon_1} \left( e^{z_{k-1,1} + x_{k,1}} + e^{z_{k,1} + x_{k,1}} \right), \\
y_{k,2r-1} = e^{z_{k-1,k-1} - z_{k,k}} e^{x_{k,r-1,k} - z_{k+r-2,k-1}}, \\
y_{k,2r} = e^{z_{k-1,k-1} - z_{k,k}} e^{x_{k,r-1,k} - z_{k+r-1,k-1}},
$$

here $k = 2, \ldots, \ell$, $r = 1, \ldots, \ell - k$. 
Lemma 2.10. 1) In the modified factorized parametrization the recursion relations (2.51) are given by

\[
\left( \frac{\Delta_k'}{\Delta_k} \right)^{C_n} = e^{z_{n-1},k-x_{n,k}} + e^{z_{n,k}} + e^{(\alpha_k, z_{n-1})} \left( \frac{\Delta_k'}{\Delta_k} \right)^{C_{n-1}}, \quad 1 \leq k < n < \ell,
\]

with the solution

\[
\left( \frac{\Delta_k'(v)}{\Delta_k(v)} \right)^{C_\ell} = e^{z_{\ell,k}-x_{\ell,k}} + \sum_{n=k+1}^\ell (e^{z_{n-1},k-x_{n,k}} + e^{z_{n,k}-x_{n,k}}), \quad k = 1, \ldots, \ell,
\]

where \( z_n = (z_{n,1}, \ldots, z_{n,n}) \) and \( (\alpha_k, z_n) = z_{n,k+1} - z_{n,k}, (\alpha_k, z_{n-1}) = z_{n-1,k+1} - z_{n-1,k} \).

2) The following expressions hold for \( \Delta_k(v) \) in terms of the variables \( x_{k,i}, z_{k,i} \):

\[
\left( \frac{\Delta_1(v)}{\Delta_2(v)} \right)^{C_\ell} = e^{-z_{\ell,1}x_{1,1}}
\]

\[
\left( \frac{\Delta_k(v)}{\Delta_{k+1}(v)} \right)^{C_\ell} = e^{-z_{\ell,k}}(e^{z_{k,1}} + e^{z_{k-1,1}})
\]

\[
\times \exp \left\{ - \sum_{i=1}^k z_{k,i} + x_{k,1} + 2 \sum_{i=2}^k x_{k,i} - \sum_{i=1}^{k-1} z_{k-1,i} \right\},
\]

where \( k = 2, \ldots, \ell \) and it is assumed that \( \Delta_{\ell+1} = 1 \).

Substituting (2.53), (2.54) into (1.17), (1.18), we obtain the left/right Whittaker vectors in the twisted parametrization (2.53). Taking \( z_{\ell,k} = 0, k = 1, \ldots, \ell \), we recover the formulae for the Whittaker vectors given in Lemma 1.6. To prove Theorem 1.10 it remains to take into account the measure \( d\mu_{N^+} \) in the modified factorized parametrization. This completes the proof of the theorem.

2.3.4. Expressions for \( \mathfrak{so}_{2\ell} \) matrix elements: proofs of Theorem 1.12 and 1.14. In this subsection we introduce expressions for matrix elements relevant to the construction of integral representations of \( \mathfrak{so}_{2\ell} \)-Whittaker functions, using a factorized parametrization of an open part of \( N^+ \subset \text{SO}(2\ell) \). This provides a proof of the integral representations of \( \mathfrak{so}_{2\ell} \)-Whittaker functions presented in Part 1.

We are using the root data given by (1.192). The reduced decomposition \( w_0 = s_{i_1}s_{i_2} \cdots s_{i_{m_\ell}} \) of the maximal length element \( w_0 \in W \) corresponding to a reduced word \( I_\ell = (i_{1}, \ldots, i_{m_\ell}) \) with \( m_\ell = \ell(\ell - 1) \) provides a total ordering of the positive coroots by \( R^+ = \{ \gamma_k^\vee = s_{i_1} \cdots s_{i_{k-1}} \alpha_k^\vee, k = 1, \ldots, m_\ell \} \) for \( \mathfrak{sp}_{2\ell} \). We consider a decomposition of \( w_0 \) described by the reduced word

\[
I_\ell = (12, 3123, \ldots, (\ell \ldots 3123 \ldots \ell))
\]
The corresponding ordering of the positive coroots is given by

\[
\begin{align*}
\gamma_1^\vee &= \alpha_1^\vee, \\
\gamma_2^\vee &= \alpha_2^\vee, \\
\gamma_3^\vee &= \alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee, \\
\gamma_4^\vee &= \alpha_2^\vee + \alpha_3^\vee, \\
\gamma_5^\vee &= \alpha_1^\vee + \alpha_3^\vee, \\
\gamma_6^\vee &= \alpha_3^\vee, \\
\gamma_{m_{\ell-1}+1}^\vee &= \alpha_1^\vee + \alpha_2^\vee + 2(\alpha_3^\vee + \cdots + \alpha_{\ell-1}^\vee) + \alpha_\ell^\vee, \\
\gamma_{m_{\ell-1}+2}^\vee &= \alpha_1^\vee + \alpha_2^\vee + 2(\alpha_3^\vee + \cdots + \alpha_{\ell-2}^\vee) + \alpha_{\ell-1}^\vee + \alpha_\ell^\vee, \\
\vdots & \quad (2.55) \\
\gamma_{m_{\ell-1}+\ell-2}^\vee &= \alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee + \cdots + \alpha_\ell^\vee, \\
\gamma_{m_{\ell-1}+\ell-1}^\vee &= p_{\ell-1}\alpha_1^\vee + p_\ell\alpha_2^\vee + \alpha_3^\vee + \cdots + \alpha_\ell^\vee, \\
\gamma_{m_{\ell-1}+\ell}^\vee &= p_\ell\alpha_1^\vee + p_{\ell+1}\alpha_2^\vee + \alpha_3^\vee + \cdots + \alpha_\ell^\vee, \\
\gamma_{m_{\ell-1}+\ell+1}^\vee &= \alpha_3^\vee + \cdots + \alpha_\ell^\vee, \\
\vdots & \\
\gamma_{m_{\ell}}^\vee &= \alpha_\ell^\vee.
\end{align*}
\]

The recursive parametrization of an open part \(N_+^{(0)}\) of \(N_+\) corresponding to a reduced word \(I_\ell\) is as follows. Given \(v^{D_\ell} \in N_+^{(0)}\), we have

\[
v^{D_\ell}(y) = \mathcal{X}_2(y)\mathcal{X}_2(y)\cdots \mathcal{X}_\ell(y),
\]

where

\[
\mathcal{X}_k(y) = X_k(y_{k,n_{k,k-1}})\cdots X_3(y_{2,n_{k,3}-1})X_1(y_{1,n_{k,1}})X_2(y_{1,n_{k,2}}) \\
\times X_3(y_{2,n_{k,3}})\cdots X_k(y_{k,n_{k,k}})
\]

and \(\mathcal{X}_2 = X_1(y_{1,1})X_2(y_{2,1})\). For any \(k \in \{1, \ldots, \ell\}\) consider a subword

\[
I_k = (i_1, \ldots, i_{m_k}) \subset I_\ell = (i_1, \ldots, i_{m_k}, i_{m_k+1}, \ldots, i_{m_\ell})
\]

with \(|I_k| = m_k = k(k-1)\). Let \(D_k\) be a corresponding root subsystem in \(R_+\) and let \(v^{D_k} = \mathcal{X}_2\cdots \mathcal{X}_k\) be a factorized parametrization of the corresponding subgroup. The factorization parameters for \(v^{D_k}(y)\) can be naturally enumerated as \(\{y_{i,n}; 1 \leq i \leq k, 1 \leq n \leq n_{k,i}\}\), where

\[
n_{k,1} = n_{k,2} = k-1, \quad n_{k,i} = 2(k+1-i), \quad 2 < i \leq \ell. \quad (2.57)
\]

Also, \(n_i = n_{\ell,i}\).

We are interested in explicit expressions for the following matrix elements in terms of factorization parameters \(\{y_{i,n}\}\):

\[
\Delta_i(v) := \langle \xi^-_{\omega_i} | \pi_i(v\hat{w}_0^{-1})| \xi^+_{\omega_i} \rangle, \quad \Delta'_i(v) := \langle \xi^-_{\omega_i} | \pi_i(vf_i\hat{w}_0^{-1})| \xi^+_{\omega_i} \rangle,
\]

where \(\pi_i = \pi_{\omega_i}\) is the fundamental representation with highest weight \(\omega_i\), \(\xi^+_{\omega_i}\), and \(\xi^-_{\omega_i}\) are the highest and lowest weight vectors in the representation \(\pi_i\) such that
Note that for the Lie algebra \( \mathfrak{so}_{2\ell} \) we have \( i \to i^* \) for the involution defined by (2.5). The proof of the following statement is obtained by an iterative evaluation of the matrix elements, taking into account the Serre relations and the defining ideals of the fundamental representations, and using the technique of [16].

**Lemma 2.11.** The following recursion relations hold:

\[
\Delta_1(v)^{D_\ell} = \left( (y_{1,n_1})^{P_{\ell-1}}(y_{2,n_2})^{P_{\ell}} \prod_{k=3}^{\ell} y_{k,n_k-1} \right) \Delta_1(v)^{D_{\ell-1}}, \\
\Delta_2(v)^{D_\ell} = \left( (y_{1,n_1})^{P_{\ell}}(y_{2,n_2})^{P_{\ell+1}} \prod_{k=3}^{\ell} y_{k,n_k-1} \right) \Delta_2(v)^{D_{\ell-1}}, \\
\Delta_i(v)^{D_\ell} = \left( y_{1,n_1} y_{2,n_2} \prod_{k=3}^{i} y_{k,n_k-1} \prod_{k=i+1}^{\ell} y_{k,n_k-1}^2 \right) \Delta_i(v)^{D_{\ell-1}}, \quad 2 < i < \ell, \\
\left( \frac{\Delta_1'}{\Delta_1} \right)^{D_{2r}} = \frac{1}{y_{1,2r-1}} \left( 1 + \frac{y_{3,2(2r-2)}}{y_{3,2(2r-2)-1}} \right) + \frac{y_{2,2r-1}}{y_{1,2r-1}} \frac{y_{3,2(2r-2)}}{y_{3,2(2r-2)-1}} \left( \frac{\Delta_1'}{\Delta_1} \right)^{D_{2r-1}}, \\
\left( \frac{\Delta_1'}{\Delta_1} \right)^{D_{2r+1}} = \frac{1}{y_{2,2r}} \left( 1 + \frac{y_{3,2(2r-1)}}{y_{3,2(2r-1)-1}} \right) + \frac{y_{1,2r}}{y_{2,2r}} \frac{y_{3,2(2r-1)}}{y_{3,2(2r-1)-1}} \left( \frac{\Delta_1'}{\Delta_1} \right)^{D_{2r}}, \tag{2.58} \\
\left( \frac{\Delta_2'}{\Delta_2} \right)^{D_{2r}} = \frac{1}{y_{2,2r-1}} \left( 1 + \frac{y_{3,4(r-1)}}{y_{3,4(r-1)-1}} \right) + \frac{y_{1,2r-1}}{y_{2,2r-1}} \frac{y_{3,4(r-1)}}{y_{3,4(r-1)-1}} \left( \frac{\Delta_2'}{\Delta_2} \right)^{D_{2r-1}}, \\
\left( \frac{\Delta_2'}{\Delta_2} \right)^{D_{2r+1}} = \frac{1}{y_{1,2r-1}} \left( 1 + \frac{y_{3,2(2r-1)}}{y_{3,2(2r-1)-1}} \right) + \frac{y_{2,2r}}{y_{1,2r}} \frac{y_{3,2(2r-1)}}{y_{3,2(2r-1)-1}} \left( \frac{\Delta_2'}{\Delta_2} \right)^{D_{2r}}.
\]

The matrix elements can be evaluated by solving the recursion equations, and the results of the calculations are presented in the following lemma.

**Lemma 2.12.** Let \( v \) be defined by (2.56). The following expressions hold for the matrix elements of \( v \) in terms of the variables \( y_{i,k} \):

\[
\Delta_i(v) = \prod_{n=1}^{\ell-1} y_{1,n}^{\varpi(n)} y_{1,i,n}^{1-\varpi(n)} \prod_{n=1}^{(\ell-1)/2} y_{2,2n} \prod_{n=3}^{\ell} y_{i,2,(n+1-i)-1}, \quad i = 1, 2; \\
\Delta_k(v) = \prod_{n=k}^{\ell} \left( y_{1,n-1} y_{2,n-1} \prod_{j=3}^{k} y_{j,2,(n-j)+1} y_{j,2,(n-j)+2} \prod_{m=k+1}^{n} y_{m,2,(n-m)-1} \right) \tag{2.59}
\]

for \( k = 3, \ldots, \ell \), where \( \varpi(n) = \left( 1 - (-1)^n \right) / 2 \) is the parity of \( n \) and \( n_k = 2(\ell+1-k) \), \( 2 < k \leq \ell; \)

\[
\left( \frac{\Delta_1'}{\Delta_1} + \frac{\Delta_2'}{\Delta_2} \right)^{D_\ell} = \sum_{n=1}^{\ell-1} \left[ \frac{1}{y_{1,\ell-1}} \prod_{k=1}^{n-1} \left( \frac{y_{1,\ell-k}}{y_{1,\ell-k-1}} \right)^{1-\varpi(k)} \left( \frac{y_{2,\ell-k}}{y_{2,\ell-k-1}} \right)^{\varpi(k)} \right]
\]
Lemma 2.13. Thus, we consider the following change of variables: taking into account the action of the part to be useful to consider a twisted version of the modified parametrization (1.217) by parametrization. To simplify the formulation of the recursion relations it turns out of the recursion relations (2.58) for the matrix elements in the modified factorized in the modified factorized parametrization (1.217). We start with an analogue for the elements obtained in Lemma 2.12 into the formula (1.19) in Proposition 1.1. (the integral formula (1.205)) follows by a direct substitution of the expressions in elements obtained in Lemma 2.12 into (2.24) and (2.26). The result of Theorem 1.12

We now consider the integral representation for the

Solving the recursion equations, one easily obtains the following result. for $k = 3, \ldots, \ell$.

For the proof of Lemma 1.9 one must substitute the expressions for the matrix elements obtained in Lemma 2.12 into (2.24) and (2.26). The result of Theorem 1.12 (the integral formula (1.205)) follows by a direct substitution of the expressions in Lemma 2.12 into the formula (1.19) in Proposition 1.1.

We now consider the integral representation for the $\mathfrak{so}_{2\ell}$-Whittaker function in the modified factorized parametrization (1.217). We start with an analogue of the recursion relations (2.58) for the matrix elements in the modified factorized parametrization. To simplify the formulation of the recursion relations it turns out to be useful to consider a twisted version of the modified parametrization (1.217) by taking into account the action of the part $e^{H_{R}} (1.226)$ of the Cartan element $e^{-h_{x}}$. Thus, we consider the following change of variables:

\[
y_{1,n} = e^{x_{\ell,1}-x_{\ell,2}}(e^{x_{n,1}+x_{n,1}} + e^{x_{n,1}+x_{n,1}}), \quad n = 1, \ldots, \ell - 1, \\
y_{2,n} = e^{-x_{\ell,1}-x_{\ell,2}}(e^{x_{n,1}+x_{n,1}} + e^{x_{n,1}+x_{n,1}}), \quad n = 1, \ldots, \ell - 1, \\
y_{k,2r-1} = e^{x_{\ell,k-1}-x_{\ell,k}}e^{x_{k+r-2,k-1}+x_{k+r-2,k-1}}, \quad k = 3, \ldots, \ell, \\
y_{k,2r} = e^{x_{\ell,k-1}-x_{\ell,k}}e^{x_{k+r-2,k-1}+x_{k+r-2,k-1}}, \quad r = 1, \ldots, \ell + 1 - k.
\]  

Lemma 2.13. The following recursion relation holds in the variables (2.62):

\[
\left( \frac{\Delta'_{1}}{\Delta_{1}} \right)^{D_{n}} + \left( \frac{\Delta'_{2}}{\Delta_{2}} \right)^{D_{n}} = e^{x_{n-1,2}+x_{n-1,1}} + e^{x_{n,2}+x_{n-1,1}} \\
+ e^{x_{n-1,2}+(-1)^{\varpi(n)}x_{n-1,1}} \left( \frac{\Delta'_{1}}{\Delta_{1}} \right)^{D_{n-1}} \\
+ e^{x_{n-1,1}+(-1)^{\varpi(n)}x_{n-1,2}} \left( \frac{\Delta'_{2}}{\Delta_{2}} \right)^{D_{n-1}},
\]  

where $\varpi(n) = (1 - (-1)^{n})/2$ is the parity of $n = 3, \ldots, \ell$.

Solving the recursion equations, one easily obtains the following result.
Lemma 2.14. For a given unipotent element \( v \in N_+ \) the following expressions hold for the matrix elements of \( v \) in the modified parametrization:

\[
\frac{\Delta_1'}{\Delta_1} + \frac{\Delta_2'}{\Delta_2} = \sum_{k=1}^{\ell-1} (e^{z_{k,k} - x_{k,k-1}} + e^{z_{k,k} - x_{k+1,k}}),
\]
\[
\frac{\Delta_1'}{\Delta_1} = e^{z_{\ell-1,\ell+1-k} - x_{\ell,\ell+1-k}}, \quad k = 3, \ldots, \ell,
\]
\[
\frac{\Delta_2}{\Delta_1} = e^{-x_{\ell,1}} e^{x_{1,1}},
\]
\[
- \frac{\Delta_1}{\Delta_3} e^{-x_{\ell,2}} \exp \left\{ -(x_{2,1} + x_{2,2}) + 2z_{1,1} - x_{1,1} \right\} (e^{x_{1,1}} + e^{x_{2,1}})^2,
\]
\[
- \frac{\Delta_k}{\Delta_{k+1}} e^{-x_{\ell,k}} \exp \left\{ -\sum_{i=1}^{k} x_{k,i} + 2\sum_{i=1}^{k-1} z_{k-1,i} - \sum_{i=1}^{k-1} x_{k-1,i} \right\} \times (e^{x_{k-1,1}} + e^{x_{k,1}})^2,
\]

and it is assumed that \( \Delta_{\ell+1} = 1 \).

Substituting (2.64) into (1.17), (1.18), we obtain the Whittaker vectors in a parametrization (2.62). Taking \( x_{\ell,k} = 0, k = 1, \ldots, \ell \), we recover the expressions for the Whittaker vectors given in Lemma 1.10. To prove Theorem 1.14 it remains to take into account the measure \( d\mu_{N_+} \) in the modified factorized parametrization. This completes the proof of the theorem.

2.4. Realization of \( \mathcal{U}(g) \) by differential operators. In this part we prove formulae for the realization of classical Lie algebra generators of types \( X_\ell \) for \( X = A, B, C, D \) by differential operators acting in the space of equivariant (twisted) functions on \( N_+ \) equipped with the factorized parametrization. The analogous formulae for the realization of Lie algebra generators in the modified factorized parametrization (Gauss–Givental representation) can be obtained in a straightforward manner by a simple change of variables discussed in Part 1, and will not be considered in this section.

We outline the general strategy used below for obtaining the realization of the Lie algebra by the differential operators. Let \( V_\mu \) be the space of equivariant functions on \( B_- \setminus G \):

\[
f(bg) = \chi_\mu(b)f(g), \quad b \in B_-,
\]

where \( \chi_\mu \) is a character of the Borel subgroup \( B_- \subset G \). The principal series representation of \( \mathcal{U}(g) \) on \( V_\mu \) is defined as

\[
(Xf)(v) = \frac{\partial}{\partial \varepsilon} f(ve^{\varepsilon X})|_{\varepsilon \to 0}, \quad X \in g.
\]

Let \( I_\ell = (i_1, \ldots, i_{m_\ell}) \) be the reduced word corresponding to the reduced decomposition \( w_0 = s_{i_1} \cdots s_{i_{m_\ell}} \) of the longest Weyl group element \( w_0 \in W(X_\ell) \) (here \( X_\ell \) belongs to one of the classical series \( A, B, C, \) or \( D \) and has rank \( \ell \)). One can choose \( I = I_\ell \) with a recursive structure with respect to the rank \( \ell \) of the Lie algebra. We consider the recursive factorized parametrization of unipotent elements of
the corresponding classical Lie group $G$ of type $X_\ell$:

$$v^{X_\ell} = \mathfrak{X}_1^{X_\ell} \cdots \mathfrak{X}_j^{X_\ell} = v^{X_{\ell-1}} \mathfrak{X}_j^{X_{\ell-1}}. \quad (2.67)$$

For a Lie algebra $\mathfrak{g}$ of classical type $X_\ell$ we will derive explicit formulae defining representations of $\mathfrak{U}(\mathfrak{g})$ on $V_\mu$ in two steps. In the first step we use the recursive structure $(2.67)$ to construct recursion relations between the classical Lie algebra generators for Lie algebras of the same type with ranks differing by 1. In the second step we solve the recursion relations between the images $\pi_\mu(e_i), \pi_\mu(f_i), \pi_\mu(h_i)$ of the Chevalley generators of algebras of types $X_\ell$ and $X_{\ell \pm 1}$ to obtain explicit formulae for the action of the generators on the space $V_\mu$.

We start with a list of relevant relations between one-parameter subgroups of $G$ (see, for example, [14] and [16]). Let $e_i, h_i, f_i, i = 1, \ldots, \ell$, be a Chevalley basis of elements of a Lie algebra $\mathfrak{g}$ of classical type $X_\ell$, and let $A = \|a_{ij}\|$ be the Cartan matrix. Let us introduce the one-parameter subgroups

$$X_i(y) = e^{yf_i}, \quad \alpha_i^\vee(1 + \varepsilon) = e^{y h_i}, \quad Y_i(y) = e^{y e_i}, \quad i = 1, \ldots, \ell. \quad (2.68)$$

Then the following relations hold:

$$X_i (y) \alpha_j^\vee (1 + \varepsilon) = \alpha_j^\vee (1 + \varepsilon) X_i (y + a_{j i} \varepsilon y) \mod \varepsilon^2, \quad (2.69)$$

$$X_i (y) Y_i (\varepsilon) = Y_i (\varepsilon) \alpha_i^\vee (1 - \varepsilon y) X_i (y - \varepsilon y^2) \mod \varepsilon^2. \quad (2.70)$$

For $a_{i j} = a_{j i} = -1$ we have

$$X_i (y_1) X_j (y_2) X_i (\varepsilon) = X_j \left( \frac{y_2}{y_1} \right) X_i (y_1 + \varepsilon) X_j \left( y_2 - \varepsilon \frac{y_2}{y_1} \right) \mod \varepsilon^2. \quad (2.71)$$

For $a_{i j} = -2$ and $a_{j i} = -1$ we have

$$X_j (y_1) X_i (y_2) X_j (y_3) X_i (\varepsilon) = X_i \left( \frac{y_3}{y_1} \right) X_j \left( y_1 + 2 \varepsilon \frac{y_3}{y_2} \right) X_i \left( y_2 + \varepsilon - \varepsilon \frac{y_3}{y_1} \right) \times X_j \left( y_3 - 2 \varepsilon \frac{y_3}{y_2} \right) \mod \varepsilon^2. \quad (2.72)$$

For $a_{j i} = -2$ and $a_{i j} = -1$ we have

$$X_j (y_1) X_i (y_2) X_j (y_3) X_i (\varepsilon) = X_i \left( \frac{y_3}{y_1} \right) X_j \left( y_1 + \varepsilon \frac{y_1 y_3 + y_3^2}{y_1 y_2} \right) X_i \left( y_2 + \varepsilon \frac{y_1^2 - y_3^2}{y_1^2} \right) \times X_j \left( y_3 - \varepsilon \frac{y_1 y_3 + y_3^2}{y_1 y_2} \right) \mod \varepsilon^2. \quad (2.73)$$

The derivation of the recursion relation for the generators of the Lie algebra is as follows. Consider the right action of the one-parameter subgroups $(2.68)$ on the recursive factorized parametrization $(2.67)$ of an element $v^{X_\ell} \subset N_+(0)$. One uses the relations $(2.69)$–$(2.73)$ to move the generators one step to the left. For example, in the case of the one-parameter subgroup generated by $f_i$ we have

$$v^{(\ell)} X_i (\varepsilon) = v^{(\ell-1)} X_{\ell-1} (y) X_i (\varepsilon) = v^{(\ell-1)} X_i (c_i (y) \varepsilon) X_{\ell-1} (y' (y)) \mod \varepsilon^2. \quad (2.74)$$
This leads to recursion relations expressing classical Lie algebra generators of rank \( \ell \) in terms of classical Lie algebra generators of rank \( \ell - 1 \) and differential operators with respect to the variables \( y_{i,n} \) parametrizing \( \mathcal{X}^\ell_{\ell-1} \). In the final step of the reduction we use (2.65). In the following subsections we provide recursion relations and explicit formulae for the generators of all classical Lie algebras without further comments.

### 2.4.1. Generators for \( \mathfrak{gl}_{\ell+1} \): proof of Proposition 1.2.

Let \( E_{i,i+1}^{(\ell+1)}, E_{i,i}^{(\ell+1)}, E_{i+1,i}^{(\ell+1)} \) be the images of the Chevalley generators \( e_{i,j} \) of \( \mathfrak{gl}_{\ell+1} \) acting in the principal series representation \( (\pi_\mu, V_\mu) \). Below we present the recursion relations and explicit expressions for these generators.

The recursion relations are given by

\[
E_{i,i+1}^{(\ell+1)} = \frac{\partial}{\partial y_{i,\ell+1-i}} + \frac{y_{i-1,\ell+2-i}}{y_{i,\ell+1-i}} \left( E_{i-1,i}^{(\ell)} - \frac{\partial}{\partial y_{i-1,\ell+2-i}} \right),
\]

\[
E_{i,i}^{(\ell+1)} = \mu_i^{(\ell+1)} - \mu_i^{(\ell)} + E_{i,i}^{(\ell)} + y_{i-1,\ell+2-i} \frac{\partial}{\partial y_{i-1,\ell+2-i}}
- y_{i,\ell+1-i} \frac{\partial}{\partial y_{i,\ell+1-i}}, \quad i \neq \ell + 1,
\]

\[
E_{i+1,i}^{(\ell+1)} = \mu_{\ell+1}^{(\ell+1)} + y_{\ell+1,\ell+1} \frac{\partial}{\partial y_{\ell+1,1}},
\]

(2.75)

\[
E_{i+1,i}^{(\ell+1)} = E_{i+1,i}^{(\ell)} + y_{i,\ell+1-i} \left( E_{i,i}^{(\ell)} - E_{i+1,i}^{(\ell)} \right)
- y_{i,\ell+1-i} \left( y_{i,\ell+1-i} \frac{\partial}{\partial y_{i,\ell+1-i}} - y_{i+1,\ell-i} \frac{\partial}{\partial y_{i+1,\ell-i}} \right).
\]

Solving the recursion relations, we arrive at (1.34), and this completes the proof of Proposition 1.2.

### 2.4.2. Generators for \( \mathfrak{so}_{2\ell+1} \): proof of Proposition 1.4.

Let \( E_i^{(\ell)}, H_i^{(\ell)}, F_i^{(\ell)} \) be the images of the Chevalley generators \( e_{i,j} \) of \( \mathfrak{so}_{2\ell+1} \) acting in the principal series representation \( (\pi_\mu, V_\mu) \). Below we present the recursion relations and explicit expressions for these generators.

The recursion relations are given by:

\[
F_1^{(\ell)} = \frac{\partial}{\partial y_{1,\ell}} + \frac{y_{2,2(\ell-1)}}{y_{2,2(\ell-1)-1}} \left( F_1^{(\ell-1)} - \frac{\partial}{\partial y_{1,\ell}} \right)
+ 2 \frac{y_{2,2(\ell-1)}}{y_{1,\ell}} \left\{ \frac{\partial}{\partial y_{2,2(\ell-1)-1}} - \frac{\partial}{\partial y_{2,2(\ell-1)}} \right\},
\]

(2.76)

\[
F_k^{(\ell)} = \frac{\partial}{\partial y_{k,2(\ell+1-k)}} + \frac{y_{k,2(\ell+1-k)-1}}{y_{k,2(\ell+1-k)}} \frac{y_{k+1,2(\ell-k)}}{y_{k+1,2(\ell-k)-1}} \left( F_k^{(\ell-1)} - \frac{\partial}{\partial y_{k,2(\ell+1-k)-1}} \right)
+ \frac{y_{k+1,2(\ell-k)}}{y_{k,2(\ell+1-k)}} \left\{ \frac{\partial}{\partial y_{k+1,2(\ell-k)-1}} - \frac{\partial}{\partial y_{k+1,2(\ell-k)}} \right\}, \quad k = 2, \ldots, \ell - 1,
\]

\[
E_1^{(\ell)} = E_1^{(\ell-1)} - y_{1,\ell} H_1^{(\ell-1)} + 2y_{1,\ell} y_{2,2(\ell-1)-1} \frac{\partial}{\partial y_{2,2(\ell-1)-1}} - y_{1,\ell}^2 \frac{\partial}{\partial y_{1,\ell}},
\]

(2.77)
\[ E_2^{(\ell)} = E_2^{(\ell-1)} - \left( y_{2,2(\ell-1)-1} + y_{2,2(\ell-1)} \right) H_2^{(\ell-1)} \]
\[ + \left( y_{2,2(\ell-1)-1} + y_{2,2(\ell-1)} \right) y_{3,2(\ell-2)-1} \frac{\partial}{\partial y_{3,2(\ell-2)-1}} + 2y_{2,2(\ell-1)} y_{1,\ell} \frac{\partial}{\partial y_{1,\ell}} \]
\[ - \left\{ y_{2,2(\ell-1)-1}^2 \frac{\partial}{\partial y_{2,2(\ell-1)-1}} + 2y_{2,2(\ell-1)} y_{2,2(\ell-1)-1} \frac{\partial}{\partial y_{2,2(\ell-1)-1}} \right\}, \]
\[ E_k^{(\ell)} = E_k^{(\ell-1)} - \left( y_{k,2(\ell+1-k)-1} + y_{k,2(\ell+1-k)} \right) H_k^{(\ell-1)} \]
\[ + \left( y_{k,2(\ell+1-k)-1} + y_{k,2(\ell+1-k)} \right) y_{k+1,2(\ell-k)-1} \frac{\partial}{\partial y_{k+1,2(\ell-k)-1}} \]
\[ + y_{k,2(\ell+1-k)} \left\{ y_{k-1,2(\ell+2-k)-1} \frac{\partial}{\partial y_{k-1,2(\ell+2-k)-1}} + 2y_{k,2(\ell+1-k)} y_{k,2(\ell+1-k)-1} \frac{\partial}{\partial y_{k,2(\ell+1-k)-1}} \right\}, \]
\[ (2.78) \]
\[ (2.79) \]

For the generators of the Cartan subalgebra we have
\[ H_k^{(\ell)} = \langle \mu^{(\ell)}, \alpha_k^\vee \rangle + \sum_{i=1}^{\ell} a_{k,i} \sum_{j=1}^{n_i} y_{i,j} \frac{\partial}{\partial y_{i,j}}, \quad k = 1, \ldots, \ell, \]
\[ (2.80) \]

where \( n_1 = \ell, n_k = 2(\ell + 1 - k) \) for \( 1 < k \leq \ell \).

Solving the recursion relations, one obtains (1.69), (1.70), and (1.71). This completes the proof of Proposition 1.4.

2.4.3. Generators for \( \mathfrak{sp}_{2\ell} \): proof of Proposition 1.8. Let \( E_i^{(\ell)}, H_i^{(\ell)}, F_i^{(\ell)}, i = 1, \ldots, \ell, \) be the images of the Chevalley generators of \( \mathfrak{sp}_{2\ell} \) acting in the principal series representation \( (\pi_\mu, V_\mu) \). Below we present the recursion relations and explicit expressions for these generators.

The recursion relations are given by:
\[ F_1^{(\ell)} = \frac{\partial}{\partial y_{1,\ell}} + \left( \frac{y_{2,2(\ell-1)} y_{2,2(\ell-1)} - y_{2,2(\ell-1)}}{y_{2,2(\ell-1)-1}} \right)^2 \left( F_1^{(\ell-1)} - \frac{\partial}{\partial y_{1,\ell}} \right) \]
\[ + \frac{y_{2,2(\ell-1)-1} y_{2,2(\ell-1)} + y_{2,2(\ell-1)}^2}{y_{1,\ell} y_{2,2(\ell-1)-1}} \left\{ \frac{\partial}{\partial y_{2,2(\ell-1)-1}} \right\}, \]
\[ (2.81) \]
\[ F_k^{(\ell)} = \frac{\partial}{\partial y_{k,2(\ell+1-k)}} + \frac{y_{k,2(\ell+1-k)-1} + y_{k+1,2(\ell-k)} y_{2,2(\ell-1)-1}}{y_{k,2(\ell+1-k)}} \left( F_k^{(\ell-1)} - \frac{\partial}{\partial y_{k,2(\ell+1-k)-1}} \right) \]
\[ + \frac{y_{k+1,2(\ell-k)-1}^2}{y_{k,2(\ell+1-k)}} \left\{ \frac{\partial}{\partial y_{k+1,2(\ell-k)-1}} \right\}, \quad 1 < k < \ell, \]
\[ (2.82) \]
Generators for \( \mathfrak{so}_{2\ell} \): proof of Proposition 1.12. Let \( E_i^{(\ell)}, H_i^{(\ell)}, F_i^{(\ell)}, i = 1, \ldots, \ell \) be the images of the Chevalley generators of \( \mathfrak{so}_{2\ell} \) acting in the principal
The recursion relations are given by

\[
F_1^{(\ell)} = \frac{\partial}{\partial y_1,\ell-1} + \frac{y_2,\ell-1}{y_1,\ell-1} \frac{y_3,2(\ell-2)}{y_1,\ell-1} \left( F_2^{(\ell-1)} - \frac{\partial}{\partial y_2,\ell-1} \right) \\
+ \frac{y_3,2(\ell-2)}{y_1,\ell-1} \left\{ \frac{\partial}{\partial y_3,2(\ell-2)} - \frac{\partial}{\partial y_3,2(\ell-2)} \right\},
\]

\[
F_2^{(\ell)} = \frac{\partial}{\partial y_2,\ell-1} + \frac{y_1,\ell-1}{y_2,\ell-1} \frac{y_3,2(\ell-2)}{y_2,\ell-1} \left( F_1^{(\ell-1)} - \frac{\partial}{\partial y_1,\ell-1} \right) \\
+ \frac{y_3,2(\ell-2)}{y_2,\ell-1} \left\{ \frac{\partial}{\partial y_3,2(\ell-2)} - \frac{\partial}{\partial y_3,2(\ell-2)} \right\},
\]

\[
F_k^{(\ell)} = \frac{\partial}{\partial y_k,2(\ell+1-k)} + \frac{y_k,2(\ell+1-k)-1}{y_k,2(\ell+1-k)} \frac{y_{k+1,2(\ell-k)}}{y_k,2(\ell+1-k)-1} \left( F_k^{(\ell-1)} - \frac{\partial}{\partial y_{k+2(\ell+1-k)-1}} \right) \\
+ \frac{y_{k+1,2(\ell-k)}}{y_k,2(\ell+1-k)} \left\{ \frac{\partial}{\partial y_{k+2(\ell+1-k)-1}} - \frac{\partial}{\partial y_{k+2(\ell+1-k)-1}} \right\}, \quad k = 3, \ldots, \ell - 1,
\]

\[
E_i^{(\ell)} = E_i^{(\ell-1)} - \frac{y_i,\ell-1}{y_i,\ell-1} \frac{\partial}{\partial y_i,\ell-1} \\
+ y_i,\ell-1 \left\{ -H_i^{(\ell-1)} + \frac{y_3,2(\ell-2)}{y_3,2(\ell-2)} \frac{\partial}{\partial y_3,2(\ell-2)} \right\}, \quad i = 1, 2,
\]

\[
E_3^{(\ell)} = E_3^{(\ell-1)} - \left\{ \frac{y_3,2(\ell-2)}{y_3,2(\ell-2)} \frac{\partial}{\partial y_3,2(\ell-2)} - \frac{y_3,2(\ell-2)}{y_3,2(\ell-2)} \frac{\partial}{\partial y_3,2(\ell-2)} \right\} \\
+ \left( y_3,2(\ell-2)-1 + y_3,2(\ell-2) \right) \left\{ -H_3^{(\ell-1)} + \frac{y_4,2(\ell-3)}{y_4,2(\ell-3)} \frac{\partial}{\partial y_4,2(\ell-3)} \right\} \\
+ y_3,2(\ell-2) \left\{ y_1,\ell-1 \frac{\partial}{\partial y_1,\ell-1} + y_2,\ell-1 \frac{\partial}{\partial y_2,\ell-1} \right\},
\]

\[
E_k^{(\ell)} = E_k^{(\ell-1)} - \left\{ \frac{y_k,2(\ell+1-k)-1}{y_k,2(\ell+1-k)-1} \frac{\partial}{\partial y_k,2(\ell+1-k)-1} - \frac{y_k,2(\ell+1-k)}{y_k,2(\ell+1-k)} \frac{\partial}{\partial y_k,2(\ell+1-k)} \right\} \\
+ \left( y_k,2(\ell+1-k)-1 + y_k,2(\ell+1-k) \right) \times \left\{ -H_k^{(\ell-1)} + \frac{y_{k+1,2(\ell-k)}}{y_{k+1,2(\ell-k)}} \frac{\partial}{\partial y_{k+1,2(\ell-k)}} \right\} \\
+ y_k,2(\ell+1-k) \left\{ y_{k-1,2(\ell+2-k)-1} \frac{\partial}{\partial y_{k-1,2(\ell+2-k)-1}} \right\} \quad 3 < k \leq \ell.
\]
The action of the Cartan subalgebra is given by

\[ H_i^{(\ell)} = \langle \mu, \alpha_i^\vee \rangle + \sum_{k=1}^{\ell} a_{i,k} \sum_{j=1}^{n_k} y_{k,j} \frac{\partial}{\partial y_{k,j}}, \quad k = 1, \ldots, \ell, \tag{2.83} \]

where \( n_1 = n_2 = \ell - 1 \), \( n_k = 2(\ell + 1 - k) \) for \( 2 < k \leq \ell \).

Solving the above recursion relations, we obtain the expressions (1.197)–(1.202). This completes the proof of Proposition 1.12.

Bibliography

[1] A. Givental, “Stationary phase integrals, quantum Toda lattices, flag manifolds and the mirror conjecture”, Topics in singularity theory, Amer. Math. Soc. Transl. Ser. 2, vol. 180, Amer. Math. Soc., Providence, RI 1997, pp. 103–115; arXiv: alg-geom/9612001.

[2] D. Joe and B. Kim, “Equivariant mirrors and the Virasoro conjecture for flag manifolds”, Int. Math. Res. Not. 2003:15 (2003), 859–882; arXiv: math.AG/0210377.

[3] И. М. Гельфанд, М. С. Цейтлин, “Конечномерные представления группы унимодулярных матриц”, Докл. АН СССР 71:5 (1950), 825–828. [I. M. Gelfand and M. S. Tsetlin, “Finite-dimensional representations of groups of unimodular matrices”, Dokl. Akad. Nauk SSSR 71:5 (1950), 825–828.]

[4] V. V. Batyrev, “Toric degenerations of Fano varieties and constructing mirror manifolds”, The Fano Conference, Univ. Torino, Turin 2004, pp. 109–122; arXiv: alg-geom/9712034.

[5] V. V. Batyrev, I. Ciocan-Fontanine, B. Kim, and D. van Straten, “Mirror symmetry and toric degenerations of partial flag manifolds”, Acta Math. 184:1 (2000), 1–39; arXiv: math.AG/9803108.

[6] A. Gerasimov, S. Kharchev, D. Lebedev, and S. Oblezin, “On a Gauss–Givental representation of quantum Toda chain wave function”, Int. Math. Res. Not. 2006 (2006), Art. ID 96489, 23 pp.; arXiv: math.RT/0505310.

[7] B. Kostant, “Quantization and representation theory”, Representation theory of Lie groups: from classical to quantum (Montréal, QC, 1999), CRM Proc. Lecture Notes, vol. 26, Amer. Math. Soc., Providence, RI 2000, pp. 227–250.

[8] B. Kostant, “On Whittaker vectors and representation theory”, Invent. Math., 1978, no. 2, 48, 101–184.

[9] E. T. Whittaker and G. N. Watson, A course of modern analysis, 4th ed., Cambridge Univ. Press, Cambridge 1996, 608 pp.

[10] V. Pasquier and M. Gaudin, “The periodic Toda chain and a matrix generalization of the Bessel function recursion relation”, J. Phys. A 25:20 (1992), 5243–5252.

[11] E. K. Sklyanin, “Bäcklund transformations and Baxter’s Q-operator”, Integrable systems: from classical to quantum (Montréal, QC, 1999), CRM Proc. Lecture Notes, vol. 26, Amer. Math. Soc., Providence, RI 2000, pp. 227–250.

[12] K. Rietsch, “A mirror construction for the totally nonnegative part of the Peterson variety”, Nagoya Math. J. 183 (2006), 105–142; arXiv: math.AG/0604170.

[13] М. А. Ольшанецкий, А. М. Переломов, А. Г. Рейман, М. А. Семенов-Тян-Шанский, “Интегрируемые системы. II”, Динамические системы – 7, Итоги науки и техники. Соврем. пробл. матем. Фундам. напр., 16, ВИНТИ,
New integral representations of Whittaker functions

M. 1987, c. 86–226; English transl., M. A. Ol’shanetskij, M. A. Perelomov, A. G. Reyman, and M. A. Semenov-Tian-Shansky, “Integrable systems. II”, Dynamical systems. VII. Integrable systems, nonholonomic dynamical systems, Encyclopaedia Math. Sci., vol. 16, Springer-Verlag, Berlin 1994, pp. 83–259.

[14] G. Lusztig, “Total positivity in reductive groups”, Lie theory and geometry, Progr. Math., vol. 123, Birkhäuser Boston, Boston, MA 1994, pp. 531–568.

[15] S. Fomin and A. Zelevinsky, “Double Bruhat cells and total positivity”, J. Amer. Math. Soc. 12:2 (1999), 335–380; arXiv:math.RT/9802056.

[16] A. Berenstein and A. Zelevinsky, “Total positivity in Schubert varieties”, Comment. Math. Helv. 72:1 (1997), 128–166.

[17] В.Г. Дринфельд, В.В. Соколов, “Алгебры Ли и уравнения типа Кортевег–де Фриза”, Итоги науки и техники. Соврем. пробл. матем. Нов. достиж., 24, ВИНИТИ, М. 1984, с. 81–180; English transl., V.G. Drinfeld and V. V. Sokolov, “Lie algebras and equations of Kortweg–de Vries type”, J. Soviet Math. 30:2 (1985), 1975–2036.

[18] A.D. Berenstein and A.V. Zelevinsky, “Tensor product multiplicities and convex polytopes in partition space”, J. Geom. Phys. 5:3 (1989), 453–472.

[19] V. Lakshmibai, “Degeneration of flag varieties to toric varieties”, C. R. Acad. Sci. Paris Sér. I Math. 321:9 (1995), 1229–1234.

[20] A. Gerasimov, D. Lebedev, and S. Oblezin, On a Gauss–Givental representation for classical groups, arXiv:math.RT/0608152.

[21] A. Gerasimov, D. Lebedev, and S. Oblezin, Baxter Q-operator and Givental integral representation for $C_n$ and $D_n$, arXiv:math.RT/0609082.

[22] A. Gerasimov, D. Lebedev, S. Oblezin, “Quantum Toda chains intertwined”, Algebra and Analysis 22:3 (2010), 107–141; English edition, A. Gerasimov, D. Lebedev, and S. Oblezin, “Quantum Toda chains intertwined”, St. Petersburg Math. J. 22:3 (2011), 411–435.

[23] P. Etingof, Differential topology, infinite-dimensional Lie algebras, and applications, Amer. Math. Soc. Transl. Ser. 2, vol. 194, Amer. Math. Soc., Providence, RI 1999, pp. 9–25; arXiv:math.QA/9901053.

[24] V.G. Kac, Infinite-dimensional Lie algebras, 3rd ed., Cambridge University Press, Cambridge 1990, xxii+400 pp.

[25] T.A. Springer, “Reductive groups”, Automorphic forms, representations and L-functions, Part 1 (Oregon State Univ., Corvallis, OR 1977), Proc. Sympos. Pure Math., vol. 33 (A. Borel, W. Casselman, eds.), Amer. Math. Soc., Providence, RI 1979, pp. 3–27.

[26] S. Helgason, Differential geometry, Lie groups, and symmetric spaces, Pure Appl. Math., vol. 34, Academic Press, New York–London 1978, xv+628 pp.

[27] H. Jacquet, “ Fonctions de Whittaker associées aux groupes de Chevalley”, Bull. Soc. Math. France 95 (1967), 243–309.

[28] M. Hashizume, “Whittaker functions on semisimple Lie groups”, Hiroshima Math. J. 12:2 (1982), 259–293.

[29] A. Gerasimov, S. Kharchev, A. Morozov, M. Olshanetsky, A. Marshakov, and A. Mironov, “Liouville type models in the group theory framework. I. Finite-dimensional algebras”, Internat. J. Modern Phys. A 12:14 (1997), 2523–2583.

[30] N. Bourbaki, Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie: Chapitre VI: Systèmes des racines, Hermann, Paris 1968.

[31] R. J. Baxter, Exactly solved models in statistical mechanics, Academic Press, London 1982, xii+486 pp.
A. A. Gerasimov, D. R. Lebedev, and S. V. Oblezin

[32] A. Г. Рейман, М. А. Семенов-Тян-Шанский, “Интегрируемые системы”, *Теоретико-групповой подход*, Современная математика, Институт компьютерных исследований, М.–Ижевск 2003, 352 с. [A.G. Reyman and M.A. Semenov-Tian-Shansky, *Integrable Systems. Group theory approach*, Modern Mathematics, Institute for Computer Sciences, Moscow–Izhevsk 2003.]

[33] K. Rietsch, “A mirror symmetric construction of $qH^*_T(G/P)(q)$”, *Adv. Math.* 217:6 (2008), 2401–2442; arXiv: math.AG/0511124v2.

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