Universal fast expansion for solving nonlinear problems

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Abstract. Formulas are given for the fast expansion of a smooth function on a segment, which allows one to increase the rate of convergence of the Fourier series unlimitedly depending on the order of expansion. The theorems on the convergence of a series in fast expansion, the possibility of its multiple termwise differentiation, the uniqueness of fast expansion, are proved, and the error in the residual series in fast expansion is estimated. Algorithms for applying fast expansion are given, the concept of the fast expansion operator, which is necessary for solving nonlinear integro-differential problems, is introduced.

1. Introduction

The theory of Fourier series for non-periodic functions $f(x)$ in the general case given on a segment is significantly different from the case when a periodic function $f(x)$ is given on an unbounded axis. Classical Fourier series on a segment are rarely used in solving engineering problems because of their slow convergence in the general case. Numerous literatures are devoted to the problem of improving the convergence of Fourier series. The beginning of such studies was laid in the works of Academician A. N. Krylova [1]. In [2], a fundamental theory of spectral expansion is presented, conditions for convergence are obtained, and many key theorems are proved. Then appeared the works [3-6] and others, where Padé approximants are used to improve convergence. Here, discontinuous functions are mainly considered, during differentiation of which it is necessary to take into account the consequences of discontinuities, called the Gibbs effect. Such series converge slowly and various formulas for calculating the Fourier coefficients are proposed to increase their convergence rate. The accuracy can be increased by about 4 to 10 times [6]. For comparison, we note that if the fast expansion in the cosine and sine Fourier series proposed here is limited to only three terms with $p = 4$, then the discarded fourth term will be less than the first one approximately in $7 \cdot 10^{-4}$ times, which ensures high accuracy and minimal computational costs.

In [7], to improve the convergence of the series, the correction functions are used when considering the problems of the natural vibrations of elastic structures by the Bubnov-Galerkin method. Other methods for improving the convergence of series by the method of synthesis of dynamic characteristics were proposed in studies [8-12]. Some improvements are given, but then the computational complexity significantly increases. Exact formulas for estimating the rate of convergence of Fourier series in orthogonal polynomials on some classes of functions in space $L_p$ are given in [13], where cases of direct expansion in Laguerre, Hermite, and Jacobi polynomials without using boundary functions are considered.
2. Materials and methods

The application of expansions of discontinuous functions in Fourier series in continuum mechanics is very difficult, especially when considering multidimensional problems, since finding the position of the curved surface of discontinuities is an additional difficult task and therefore the use of Fourier series becomes very problematic. In such cases, the area of the body under consideration is divided into parts so that in each part all functions are sufficiently smooth. At the conjugate boundaries of the parts of the body, special conditions are used, called conditions for discontinuities. A similar approach is convenient and allows you to get a solution with high accuracy.

In this regard, in contrast to the known methods, another approach to the problem of improving the convergence of Fourier series is proposed below, called the fast expansion method in this paper, applicable to nonlinear problems for a curvilinear region with different boundary conditions.

Let us make the analysis of the properties of the Fourier coefficients. In fast expansions, a certain sufficiently smooth function \( f(x) \) given on a segment will be represented as the sum of a specially constructed boundary function \( M_p(x) \) and the Fourier series for the difference \( f(x) - M_p(x) \). The boundary function \( M_p(x) \) strongly affects the properties of the difference \( f(x) - M_p(x) \). Fast sine and cosine expansions are constructed on this principle [14-16], which have shown their effectiveness in solving applied problems. Such expansions are especially convenient if the statement of the problem contains only even, or only odd derivatives. For cases when derivatives of any integer orders are used in the problem, this method is proposed – the universal method of fast expansion.

Let some smooth function be given on the segment \([-a, a]\), represented by the classical Fourier series uniformly converging in the metric \( L^2 \):

\[
f(x) = c_0 + \sum_{n=1}^{\infty} c_n \cos \frac{n\pi x}{a} + \sum_{m=1}^{\infty} b_m \sin \frac{m\pi x}{a}, \quad f(x) \in \left\{ C^{(p+2)}, L^2_p (-a \le x \le a) \right\},
\]  

where \( L^2_p \) are the classes of Sobolev-Liouville functions, so far we will not specify the value of \( p \).

The convergence of series (1) and subsequent series at the ends of the segment \([-a, a]\) is given significant importance, since the proposed method is designed to solve applied problems when information at the boundaries of the body is no less important than information at points inside the body. The Fourier coefficients in (1) are determined by the classical formulas:

\[
c_0 = \frac{1}{2a} \int_{-a}^{a} f(x) \, dx, \quad c_n = \frac{1}{a} \int_{-a}^{a} f(x) \cos \frac{n\pi x}{a} \, dx, \quad b_m = \frac{1}{a} \int_{-a}^{a} f(x) \sin \frac{m\pi x}{a} \, dx.
\]  

We offer the following transformation of expressions (2). In the formula for sine coefficients \( b_m \), we integrate twice by parts:

\[
b_m = \frac{1}{a} \int_{-a}^{a} f(x) \sin \frac{m\pi x}{a} \, dx = -\left(\frac{1}{m\pi}\right)^{\frac{m}{2}} \left( f(a) - f(-a) \right) - \left(\frac{a}{m\pi}\right)^{\frac{m}{2}} \int_{-a}^{a} f''(x) \sin \frac{m\pi x}{a} \, dx.
\]

Let the function \( f(x) \) satisfy the additional condition

\[
f(a) = f(-a).
\]

Then the expression (3) for \( b_m \) will take the form

\[
b_m = -\left(\frac{a}{m\pi}\right)^{\frac{m}{2}} \int_{-a}^{a} f''(x) \sin \frac{m\pi x}{a} \, dx.
\]
When the condition (4) is met the rate of decrease $b_m$ with growth $m$ substantially increases. To obtain conditions for an even greater increase in the rate of decrease $b_m$, equality (5) is again integrated twice in parts:

$$b_m = \frac{1}{a} \left( \frac{a}{m\pi} \right)^3 \int_{-a}^{a} f^*(x) d\cos m\pi \frac{x}{a} = \frac{(-1)^n}{a} \left( \frac{a}{m\pi} \right)^3 (f^*(a) - f^*(-a)) + \frac{1}{a} \left( \frac{a}{m\pi} \right)^4 \int_{-a}^{a} f^{(4)}(x) \sin m\pi \frac{x}{a} dx.$$  \hspace{1cm} (6)

We require that in addition to (4) the additional condition to be met

$$f^*(a) = f^*(-a).$$  \hspace{1cm} (7)

Then the rate of decrease $b_m$ from (6) increases even more

$$b_m = \frac{1}{a} \left( \frac{a}{m\pi} \right)^4 \int_{-a}^{a} f^{(4)}(x) \sin m\pi \frac{x}{a} dx.$$  \hspace{1cm} (8)

But in order to use the formula (8) now it is necessary to fulfill two equalities at once – (4) and (7). When (5) was obtained, the integration for parts was applied twice to the formula for $b_m$ from (2), when (8) was obtained, a total of 4 times were used. If integration by parts in (2) is performed $2M_0$ times, then, similarly to (8), we will have

$$b_m = \frac{(-1)^{M_0}}{a} \left( \frac{a}{m\pi} \right)^{2M_0} \int_{-a}^{a} f^{(2M_0)}(x) \sin m\pi \frac{x}{a} dx, \quad m=1,2,...$$  \hspace{1cm} (9)

By analogy with (4) and (7), to fulfill formula (9), the function $f(x)$ must satisfy the following set of additional equalities

$$f^{(2P_0)}(a) = f^{(2P_0)}(-a), \quad P_0 = 0 \div (M_0 - 1).$$  \hspace{1cm} (10)

Similarly, we transform the expression for cosine coefficients $c_n$, i.e. integrate by parts the expression for $c_n$ from (2):

$$c_n = \frac{1}{a} \int_{-a}^{a} f(x) \cos n\pi \frac{x}{a} dx = -\frac{1}{a} \int_{-a}^{a} f'(x) \sin n\pi \frac{x}{a} dx.$$  \hspace{1cm} (11)

We conclude from this: if $f(x) \in C^{(1)}(-a \leq x \leq a)$, then the integrals $\int_{-a}^{a} f'(x) \sin n\pi \frac{x}{a} dx$ and $\int_{-a}^{a} f(x) \sin n\pi \frac{x}{a} dx$ will have the same rate of decrease with increasing $m$ and $n$ (only limited functions $f(x)$ and $f'(x)$ will be different in them), and therefore the coefficients $c_n$ from (11) in comparison with $b_m$ from (2) decrease faster in the classical Fourier series. Those Fourier series converge slowly, mainly due to the slow decrease of its sine coefficients $b_m$. To obtain new conditions for increasing the convergence rate of a series in formula (11) for $c_n$, we perform integration by parts twice:
\[ c_n = -\frac{1}{a} \int_a^0 f'(x) \sin n\pi \frac{x}{a} \, dx = \frac{1}{a} \left( \frac{a}{n\pi} \right)^2 (-1)^n \left( f'(a) - f'(-a) \right) + \frac{1}{a} \left( \frac{a}{n\pi} \right)^3 \int_a^0 f''(x) \sin n\pi \frac{x}{a} \, dx. \]  

(12)

We require that the equality be fulfilled in (12)

\[ f'(a) = f'(-a). \]  

(13)

Then from (12) we get

\[ \tilde{c}_n = \frac{1}{a} \left( \frac{a}{n\pi} \right)^3 \int_a^0 f''(x) \sin n\pi \frac{x}{a} \, dx. \]  

(14)

From (14) it can be seen that if \( f(x) \) has property (13), then the rate of decrease \( c_n \) with growth of \( n \) is much higher compared to the general case when \( c_n \) is calculated by formula (11). The convergence rate of the series (1) with respect to the cosine itself can be further increased. To do this, in the integral (14) we apply integration by parts two more times:

\[ c_n = \frac{1}{a} \left( \frac{a}{n\pi} \right)^3 \int_a^0 f''(x) \sin n\pi \frac{x}{a} \, dx = \frac{1}{a} \left( \frac{a}{n\pi} \right)^4 \left( f''(a) - f''(-a) \right) \]

\[ -\frac{1}{a} \left( \frac{a}{n\pi} \right)^5 \int_a^0 f^{(5)}(x) \sin n\pi \frac{x}{a} \, dx. \]  

(15)

We require that in addition to (13) the additional equality have to be fulfilled

\[ f''(a) = f''(-a). \]  

(16)

Then formula (15) for \( c_n \) takes the form

\[ c_n = \frac{1}{a} \left( \frac{a}{n\pi} \right)^5 \int_a^0 f^{(5)}(x) \sin n\pi \frac{x}{a} \, dx. \]  

(17)

That is, with the simultaneous fulfillment of conditions (13) and (16), the rate of decrease of the cosine coefficients \( c_n \) increases significantly. Applying integration by parts \( 2N_0+1 \) times to the formula for \( c_n \) from (2), and fulfilling additional conditions

\[ f^{(2N_0+1)}(a) = f^{(2N_0+1)}(-a), \quad Q_0 = 1 \div N_0 \]  

(18)

each time, we come to the following formula for calculating the coefficients \( c_n \):

\[ c_n = \frac{(-1)^{N_0+1}}{a} \left( \frac{a}{n\pi} \right)^{2N_0+1} \int_a^0 f^{(2N_0+1)}(x) \sin n\pi \frac{x}{a} \, dx. \]  

(19)

Similarly, we transform the expression for cosine coefficients \( c_n \), i.e. integrate by parts the expression for \( c_n \) from (2):

\[ c_n = \frac{1}{a} \int_a^0 f(x) \cos n\pi \frac{x}{a} \, dx = -\frac{1}{a} \frac{a}{n\pi} \int_a^0 f'(x) \sin n\pi \frac{x}{a} \, dx. \]  

(20)

Under conditions (10) and (18), the rate of decrease of the Fourier coefficients \( b_n \) in (9) and \( c_n \) in (19) increases rapidly, but the class of functions for which equalities (10) + (18) are fulfilled is
essentially limited. This disadvantage is compensated as follows.

Let us consider the boundary function. To fulfill the conditions (10), (18) we represent \( f(x) \) as the sum of some boundary function \( M_p(x) \) and the function \( \psi_p(x) \), which we will expand in a rapidly converging Fourier series. In (10), it should be considered \( 2P_0 = p \), in (1.18) \( 2N_0 + 1 = p \), i.e. \( p \) - order of highest derivative.

Expression

\[
f(x) = M_p(x) + \psi_p(x), \quad p = 0,1,\ldots
\]

we will call a fast expansion \( f(x) \) of the \( p \) – order if the construction \( M_p(x) \) is such that the difference \( f(x) - M_p(x) = \psi_p(x) \) satisfies the additional conditions (10), (18) and then \( \psi_p(x) \) can be written down as quickly converging Fourier series.

It was shown above that in the classical Fourier series when \( m = n \) the cosine-coefficient decreases faster than the sine coefficient. Applying the fast expansion of the zeroth order (21) if \( p = 0 \), when only one additional condition (4) is satisfied, the coefficients \( c_n \) will decrease faster than \( b_n \). If two conditions (4) + (7) are satisfied and \( M_p(x) = M_1(x) \) are used in (21), again the coefficients \( c_n \) will decrease faster as compared to \( b_n \). With a monotonous increase in the number of additional conditions (10), (18), alternation occurs: either \( c_n \) or \( b_n \) decreases faster. When an odd amount of these additional conditions (10) + (18) is fulfilled, the decrease rate of the Fourier sine coefficients is higher; when an even number of additional conditions are fulfilled, the decrease rate of the Fourier cosine coefficients is higher. The order of the rate of decrease of the Fourier coefficients in the series for \( \psi_p(x) \) is obtained due to a special construction \( M_p(x) \) called the boundary one, since all its coefficients will be determined through the values \( f(x) \) and its derivatives up to the order \( p \) inclusively at the boundaries of the segment \([-a,a]\). For construction of \( M_p(x) \), we will use special polynomials \( P_q(x) \) of even and odd degrees of the variable \( x \), called fast polynomials. We introduce recurrence formulas for fast polynomials using definite integrals

\[
P_0(x) = \frac{x}{2a}, P_{2q+1}(x) = \int_0^x P_{2q}(t) dt, P_{2q+2}(x) = \int_0^x P_{2q+1}(t) dt - \frac{x}{a} \int_0^x P_{2q+1}(t) dt, \quad q = 0,1,\ldots
\]

A polynomial of the first degree \( P_0(x) \) is initial, the rest are obtained by formulas (22). We write the first four polynomials up as examples

\[
P_0(x) = \frac{x}{2a}, \quad P_1(x) = \frac{x^2}{4a}, \quad P_2(x) = \frac{1}{12} \left( \frac{x^4}{a} - ax \right), \quad P_3(x) = \frac{1}{12} \left( \frac{x^4}{4a} - ax^2 \right).
\]

From (22) and (23) we can establish the following properties of polynomials \( P_q(x) \) at the ends of the segment \([-a,a]\):

\[
P_{2q}(a) = P_{2q}(-a) = 0, \quad P_{2q+1}(a) = P_{2q+1}(-a), \quad q = 1,2,\ldots
\]

These polynomials have the property of evenness and oddness: polynomials with even numbers are odd functions of \( x \), with odd numbers are even functions of \( x \):

\[
P_{2q}(x) = -P_{2q}(-x), \quad P_{2q+1}(x) = P_{2q+1}(-x), \quad q = 0,1,\ldots
\]
In addition to (24), polynomials \( P_p(x) \) have the following differential properties

\[
P'_{2q+1}(x) = P_{2q}(x), \quad P'_{2q+2}(x) = P_{2q+1}(x) - \frac{1}{a_0} \int_0^a P_{2q+1}(t) \, dt, \quad q = 0, 1, \ldots
\] (26)

In the simplest case, when it is only necessary to satisfy condition (4) and obtain a fast expansion of zeroth order, we represent \( f(x) \) by the sum

\[
f(x) = M_0(x) + \psi_0(x), \quad M_0(x) = P_0(x) \frac{f(a) - f(-a)}{a}
\]

\[
\psi_0(x) = f(x) - M_0(x) = f(x) - P_0(x) \frac{f(a) - f(-a)}{a}.
\] (27)

In (27) only a polynomial \( P_0(x) \) was used for which \( \psi_0(x) \) satisfies one additional condition (4):

\[
\psi_0(a) = f(a) - \frac{1}{2} \left( f(a) - f(-a) \right) = \frac{1}{2} \left( f(a) + f(-a) \right),
\]

\[
\psi_0(-a) = f(-a) + \frac{1}{2} \left( f(a) - f(-a) \right) = \frac{1}{2} \left( f(a) + f(-a) \right).
\]

That is, at the ends of the segment \([-a, a]\) the function \( \psi_0(x) \) takes equal values and therefore satisfies condition (4). As a result, we have for obtained Fourier series \( \psi_0(x) \) the convergence rate is higher than the classical Fourier series (1) for \( f(x) \). This is the fast expansion of zeroth order. In order to organize the fast expansion of the first order

\[
f(x) = M_1(x) + \psi_1(x)
\] (28)

we construct the boundary function \( M_1(x) \) so that \( \psi_1(x) \) satisfies both conditions (4) and (13). For this we write \( M_1(x) \) using the polynomials in (22)

\[
M_1(x) = M_0(x) + P_1(x) \frac{f'(a) - f'(-a)}{a} = \frac{x}{2a} \left( f(a) - f(-a) \right) + \frac{x^2}{4a} \left( f'(a) - f'(-a) \right).
\] (29)

Using (28) and (29) we find the expression for \( \psi_1(x) \):

\[
\psi_1(x) = f(x) - M_1(x) = f(x) - P_0(x) \frac{f(a) - f(-a)}{a} - P_1(x) \left( f'(a) - f'(-a) \right).
\] (30)

For the function \( \psi_1(x) \) obtained in (30), the conditions (4) and (13) are satisfied:

\[
\psi_1(a) = \psi_1(-a) = \frac{1}{2} \left( f(a) + f(-a) \right) - \frac{a}{4} \left( f'(a) - f'(-a) \right),
\]

\[
\psi_1'(a) = \psi_1'(-a) = -\frac{1}{2a} \left( f(a) - f(-a) \right) + \frac{1}{2} \left( f'(a) + f'(-a) \right).
\]

Like (29), the second-order function \( M_2(x) \) is represented by the sum

\[
M_2(x) = M_1(x) + P_2(x) \frac{f''(a) - f''(-a)}{a} = \frac{x}{2a} \left( f(a) - f(-a) \right) + \frac{x^2}{4a} \left( f'(a) - f'(-a) \right)
\]

\[
+ \frac{1}{12} \left( \frac{x^3}{a} - ax \right) \left( f''(a) - f''(-a) \right).
\] (31)
So \( \psi_2(x) \) is determined by the equality

\[
\psi_2(x) = f(x) - M_2(x) = f(x) - \sum_{q=0}^{\infty} P_q(x) \left( f^{(q)}(a) - f^{(q)}(-a) \right).
\] (32)

It can be verified directly that \( \psi_2(x) \) from (32) satisfies all three conditions (4), (13) and (7) simultaneously.

From the three examples (27), (30), (32) it can be seen that the boundary function \( M_p(x) \) for arbitrary \( p \) can be written as

\[
M_p(x) = \sum_{q=0}^{p} A_q(f) P_q(x), \quad A_q(f) = f^{(q)}(a) - f^{(q)}(-a).
\] (33)

And fast expansion (21) is represented by the equality

\[
f(x) = M_p(x) + c_0^{(p)} + \sum_{n=1}^{\infty} c_n^{(p)} \cos \frac{n \pi x}{a} + \sum_{m=1}^{\infty} b_m^{(p)} \sin \frac{m \pi x}{a}, \quad p = 0, 1, \ldots
\] (34)

The definition of the boundary function \( M_p(x) \) by expression (33) and the fast expansion by formula (34) allows us to prove several key theorems.

Theorem 1. If the boundary function \( M_p(x) \) is defined by equality (33), then the difference

\[
f(x) - M_p(x) = \psi_p(x), \quad \psi_p(x) = f(x) - \sum_{q=0}^{p} P_q(x) \left( f^{(q)}(a) - f^{(q)}(-a) \right)
\] (35)

will satisfy the additional conditions (10) and (18), i.e. all its derivatives from zero to the \( p \) order inclusive at the ends of the segment \([-a, a]\) are equal to each other

\[
\psi^{(q)}_p(a) = \psi^{(q)}_p(-a), \quad \forall q = 0 \div p.
\] (36)

Proof. It has already been proved that three functions \( \psi_0(x), \psi_2(x) \) satisfy conditions (10) and (18). We assume that the difference \( f(x) - M_{p-1}(x) = \psi_{p-1}(x) \) satisfies conditions (10) and (18). Let us prove that \( \psi_p(x) \) from (35) will also satisfy these conditions.

For this, we rewrite the expression \( M_p(x) \) from (33) in the form

\[
M_p(x) = \sum_{q=0}^{p} P_q(x) \left( f^{(q)}(a) - f^{(q)}(-a) \right) = M_{p-1}(x) + P_p(x) \left( f^{(p)}(a) - f^{(p)}(-a) \right).
\] (37)

Since, by assumption, \( \psi_{p-1}(x) \) satisfies conditions (10) and (18), it remains to prove that the term

\[
P_p(x) \left( f^{(p)}(a) - f^{(p)}(-a) \right)
\] in (37) also satisfies conditions (10) and (18).

For an even or odd index \( p \), the polynomial \( P_p(x) \) at the ends of the segment by property (24) satisfies conditions (10) and (18), i.e. \( P_p(a) = P_p(-a) \). All subsequent derivatives of \( P_p(x) \) translated \( P_p(x) \) into lower polynomials used in \( \psi_{p-1}(x) \), which satisfy conditions (10) and (18). This means that \( P_p(x) \) in (37) satisfies conditions (10) and (18). The theorem is proved.

We obtain from theorem 1 that, using the polynomials \( P_q(x) \) from (22), the fast expansion (34) for
\( f(x) \) should be defined by the equalities:

\[
\begin{aligned}
 f(x) &= \sum_{q=0}^{p} A_q(f) P_q(x) + c_0^{(p)} + \sum_{n=1}^{\infty} c_n^{(p)} \cos n\pi \frac{x}{a} + \sum_{m=1}^{\infty} b_m^{(p)} \sin m\pi \frac{x}{a}, \\
 A_q(f) &= f^{(q)}(a) - f^{(q)}(-a), \quad q = 0 \div p, \quad f(x) \in \left\{ C^{(p+1)}, L_2^{p+1} \mid -a \leq x \leq a \right\}.
\end{aligned}
\]  

(38)

Here the constants

\[
A_q(f) = A_p(f), \quad c_0^{(p)}, \quad c_n^{(p)}, \quad b_m^{(p)}
\]

are called the fast expansion coefficients (38). If \( f(x) \) is known, then the coefficients \( A_q(f) \) are found through the derivatives \( f^{(q)}(a), f^{(q)}(-a) \) in the second line of expression (38). The remaining coefficients from (39), i.e. Fourier coefficients, are calculated by the formulas:

\[
\begin{aligned}
 c_0^{(p)} &= \frac{1}{2a} \int_{-a}^{a} f(x) \, dx, \\
 c_n^{(p)} &= \frac{1}{a} \int_{-a}^{a} f(x) \cos n\pi \frac{x}{a} \, dx, \\
 b_m^{(p)} &= \frac{1}{a} \int_{-a}^{a} f(x) \sin m\pi \frac{x}{a} \, dx, \\
 (m,n) &= 1,2,\ldots
\end{aligned}
\]

(40)

If \( f(x) \) is unknown, then to determine it using formula (38), it suffices to find the coefficients (39) from the solution of some given problem. The classical Fourier series for the smooth \( f(x) \) on a segment \([-a,a] \) in the general case has the properties of all types of convergence, but has two drawbacks: such a series converges slowly and cannot be differentiated termwise, since in the general case a divergent series can be obtained.

In contrast to the classical series, the fast expansion (38) has remarkable properties, which we state in the following lemma.

Lemma 1 on the convergence of a fast expansion. Let \( f(x) \) be continuous on the segment \([-a,a] \) and satisfies the condition

\[
\begin{aligned}
 f(x) \in \left\{ C^{(p+1)}, L_2^{p+1} \mid \forall x \in [-a,a] \right\}.
\end{aligned}
\]

(41)

Then the Fourier series in the fast expansion (38) has all kinds of convergence of the classical Fourier series for \( \forall x \in [-a,a] \).

The proof of this lemma follows from the fact that in the fast expansion (38) the boundary function \( M_p(x) \) is finite and polynomial. Therefore, all types of convergence that are satisfied for \( f(x) \) occur for the Fourier series constructed for difference \( f(x) - M_p(x) \), since \( M_p(x) \) this property does not deteriorate.

The term \( M_p(x) \) increases the rate of decrease of the Fourier coefficients depending on the order \( p \) of the boundary function. This can be verified by performing multiple integration by parts in formulas (40) for the coefficients \( c_n^{(p)} \) and \( b_m^{(p)} \).

For even \( p = 2s \), \( s = 0,1,\ldots \) from (40), after \( (p+2) \) multiple integration by parts in the integral for \( c_n^{(p)} \) and \( (p+3) \) multiple integration by parts in the integral for \( b_m^{(p)} \), we obtain
\[ c^{(p)}_n = \frac{1}{a} \int_{-a}^{a} \left[ f(x) - M_{2s}(x) \right] \cos n\pi \frac{x}{a} \, dx \]
\[ = \frac{1}{a} (-1)^s \left( \frac{a}{n\pi} \right)^{p+2} \int_{-a}^{a} f^{(p+2)}(x) \cos n\pi \frac{x}{a} \, dx \] \tag{42}

\[ b^{(p)}_n = \frac{1}{a} \int_{-a}^{a} \left[ f(x) - M_{2s} \right] \sin m\pi \frac{x}{a} \, dx = (-1)^s \frac{1}{a} \left( \frac{a}{m\pi} \right)^{p+3} \int_{-a}^{a} f^{(p+3)}(x) \cos m\pi \frac{x}{a} \, dx - (-1)^{n} A_{p+2}(f) \] \tag{43}

We obtain from (42): for even \( p = 2s \) sine coefficients \( b^{(p)}_n \) in (42) decrease faster compared to cosine coefficients \( c^{(p)}_n \).

For odd \( p = 2s + 1 \), s = 0, 1, ... from (40), after \((p+3)\) multiple integration by parts in the integral for \( c^{(p)}_n \) and \((p+2)\) multiple integration by parts in the integral for \( b^{(p)}_n \), we obtain

\[ c^{(p)}_n = \frac{1}{a} (-1)^s \left( \frac{a}{n\pi} \right)^{p+2} \int_{-a}^{a} f^{(p+2)}(x) \cos n\pi \frac{x}{a} \, dx - (-1)^{n} A_{p+2}(f) \] \tag{43}

When deriving dependencies (42) - (43), we used expressions (22) - (26). It can be seen from (42) - (43) that with an increase the order \( p \) of the used boundary function, the value of which is selected in accordance with the applied problem under consideration, the rate of decrease of the Fourier coefficients \( b^{(p)}_n \) and \( c^{(p)}_n \) also increases significantly. In view of (42) - (43), the fast expansion (34) can be represented by the expressions with even \( p = 2s \), fast expansion is represented by the formula

\[ f(x) \in \left\{ C^{(p+3)}_1, L^{p+3}_2 (-a \leq x \leq a) \right\}, \ s = 0, 1, ... ; \]

\[ f(x) = \sum_{q=0}^{2s} A_q(f) P_q(x) + (-1)^s \frac{1}{a} \sum_{n=1}^{\infty} \left( \frac{a}{n\pi} \right)^{p+2} \int_{-a}^{a} f^{(p+2)}(x) \cos n\pi \frac{x}{a} \, dx \cos n\pi \frac{x}{a} \]
\[ + \frac{1}{2a} \int_{-a}^{a} f(x) - \sum_{q=0}^{p} A_q(f) P_q(x) \, dx \]
\[ + (-1)^s \frac{1}{a} \sum_{n=1}^{\infty} \left( \frac{a}{m\pi} \right)^{p+3} \int_{-a}^{a} f^{(p+3)}(x) \cos m\pi \frac{x}{a} \, dx - (-1)^{n} A_{p+2}(f) \sin m\pi \frac{x}{a} \] \tag{44}

In the case of odd \( p = 2s + 1 \), fast expansion has the following form

\[ f(x) \in \left\{ C^{(p+3)}_1, L^{p+3}_2 (-a \leq x \leq a) \right\}, \]

\[ f(x) = \sum_{q=0}^{2s+1} A_q(f) P_q(x) + (-1)^s \frac{1}{a} \sum_{n=1}^{\infty} \left( \frac{a}{n\pi} \right)^{p+3} \int_{-a}^{a} f^{(p+3)}(x) \cos n\pi \frac{x}{a} \, dx - (-1)^{n} A_{p+2}(f) \cos n\pi \frac{x}{a} \]
\[ + \frac{1}{2a} \int_{-a}^{a} \left( f(t) - \sum_{q=0}^{p} A_q(f) P_q(t) \right) dt \] \tag{45}
The Fourier series in (44) and (45) admits termwise differentiation $p$ times. The speed of convergence of the series decreases with the growth of the order of the derivative. Therefore, in consideration of a certain differential problem raises the question of the legitimacy of using fast expansion in (38). In this context, we prove the following theorem.

Theorem 2 on the differentiability of fast expansion. Let $f(x)$ on the segment $[-a, a]$ satisfies the conditions (41). Then fast expansion (44) or (45) admits $r$-times termwise differentiation, remaining fast expansion of $f(x)$ derived from the boundary function $M_{p-r}$, where $r = 1/p$, in addition, for derivatives of Fourier series up to order $p$ inclusive, we will have the formula:

$$f'(x) = \sum_{q=0}^{p} A_q(f) P_q(x) + c^{(p)}_0 + \sum_{n=1}^{\infty} c^{(p)}_n \cos n\pi \frac{x}{a} + \sum_{m=1}^{\infty} b^{(p)}_m \sin m\pi \frac{x}{a}.$$  (46)

As an example, when $p = 3$ for $f(x)$ using (38) we write the fast expansion

$$f(x) = \sum_{q=0}^{3} A_q(f) P_q(x) + c^{(3)}_0 + \sum_{n=1}^{\infty} c^{(3)}_n \cos n\pi \frac{x}{a} + \sum_{m=1}^{\infty} b^{(3)}_m \sin m\pi \frac{x}{a};$$  (47)

$$c^{(3)}_n = \frac{1}{a} \int_{-a}^{a} f(x) - \sum_{q=0}^{3} A_q(f) P_q(x) \cos n\pi \frac{x}{a} \, dx = -\frac{1}{a} \int_{-a}^{a} f'(x) - \sum_{q=0}^{2} A_q(f) P_q(x) \sin n\pi \frac{x}{a} \, dx;$$

$$b^{(3)}_m = \frac{1}{a} \int_{-a}^{a} f(x) - \sum_{q=0}^{3} A_q(f) P_q(x) \sin m\pi \frac{x}{a} \, dx = -\frac{1}{a} \left( \frac{1}{m\pi} \right) \int_{-a}^{a} f'(x) - \sum_{q=0}^{2} A_q(f) P_q(x) \cos m\pi \frac{x}{a} \, dx.$$  (48)

Let us differentiate the left and right side of (47) once:

$$f''(x) \sim \sum_{q=0}^{2} A_q(f) P_q(x) - \sum_{n=1}^{\infty} n\pi \frac{c^{(3)}_n}{a} \sin n\pi \frac{x}{a} + \sum_{m=1}^{\infty} m\pi \frac{b^{(3)}_m}{a} \cos m\pi \frac{x}{a} + A_0(f) \frac{1}{2a} - A_1(f) \frac{a}{12}.  \quad (49)$$

As in (47) equality is not yet proven, then the equal sign substituted for the mark of conformity $\sim$.

Now by analogy with (47) we write down the fast expansion for the derivative $f'(x)$ for $p = 2$ with boundary function $M_{2}$, formally substituting $f(x)$ by $f'(x)$:

$$f'(x) = \sum_{q=0}^{2} A_q(f') P_q(x) + c^{(2)}_0 + \sum_{n=1}^{\infty} c^{(2)}_n \cos n\pi \frac{x}{a} + \sum_{m=1}^{\infty} b^{(2)}_m \sin m\pi \frac{x}{a}.$$  (49)

Let us consider the following equalities:

$$A_q(f') = A_{q+1}(f).$$
\[ c^{(2)}_0 = \frac{1}{2a} \int_{-a}^{a} \left( f(x) - \sum_{q=0}^{2} A_q (f') P_q(x) \right) dx = \frac{1}{2a} \left( f(a) - f(-a) \right) - A_1 (f') \frac{x^3}{24a^3} \bigg|_{-a}^{a} = \frac{1}{2a} A_0 (f) - A_2 (f) \frac{a}{12}. \]

Simple calculations can show that the right-hand sides in (48) and (49) are the same, that is what we wanted to prove.

Fast expansions are applicable for the solution of complex nonlinear integro-differential problems [14-16]. To ensure that the obtained solution is the solution of the problem under consideration, we prove the following lemma.

**Lemma 2 on the uniqueness of fast expansion.** If two functions \( f_1(x), f_2(x) \in C^{[p+3]}(L^2_{-a}(-a \leq x \leq a)) \) and their respective coefficients of fast expansion of the \( p \)-order on the segment \([-a,a]\) are equal

\[ \left( A_{1,1}(f) \mp A_{p,1}(f) \right) = \left( A_{1,2}(f) \mp A_{p,2}(f) \right), \quad p=1,2,\ldots, \quad (50) \]

\[ \left( c_{n,1}^{(p)}, c_{n,2}^{(p)}, b_{n,1}^{(p)}, b_{n,2}^{(p)} \right) = \left( c_{n,1}^{(p)}, c_{n,2}^{(p)}, b_{n,1}^{(p)}, b_{n,2}^{(p)} \right), \quad p=1,2,\ldots, \quad (51) \]

then these functions are equal

\[ f_1(x) = f_2(x), \quad \forall x \in [-a,a]. \quad (52) \]

**Proof.** Fast expansions for these functions in accordance with (38) consist of the sum of two parts:

\[ f_i(x) = M_{i,p}(x) + \psi_{i,p}, \quad p=1,2,\ldots, i-1,2, \quad \psi_{i,p} = c_{i,0}^{(p)} + \sum_{n=1}^{\infty} c_{n,i}^{(p)} \cos \frac{n \pi x}{a} + \sum_{m=1}^{\infty} b_{n,m}^{(p)} \sin \frac{m \pi x}{a}. \quad (53) \]

From the equality (50) we have \( M_{p,1}(x) = M_{p,2}(x) \), then from (51) follows

\[ \psi_{p,1} = \psi_{p,2}, \quad p=1,2,\ldots, \quad \forall x \in [-a,a]. \]

The lemma is proved.

Let us consider an error estimate for the partial sums. When examining applied problems the sum in the Fourier series (44) or (45) are restricted to a finite number of terms, which introduces an error. To estimate the error in the even-numbered \( p = 2s \) let us present the Fourier series in the form:

\[ p = 2s, \quad s=0,1,\ldots; \quad f(x) = \sum_{q=0}^{p} A_q P_q(x) + \left( -1 \right)^s \frac{1}{a} \sum_{n=1}^{N} \left( \frac{a}{n \pi} \right)^{p+2} \right) [n,p] \cos \frac{n \pi x}{a} \]

\[ + \frac{1}{2a} \left[ \int_{-a}^{a} \left( f(x) - \sum_{q=0}^{p} A_q P_q(x) \right) dx + \left( -1 \right)^s \frac{1}{a} \sum_{m=1}^{M} \left( \frac{a}{m \pi} \right)^{p+3} [m,p] \sin \frac{m \pi x}{a}, \quad (54) \]

\[ \left[ n,p \right] = \left( -1 \right)^s \left( f^{(p+1)}(a) - f^{(p+1)}(-a) \right) - \int_{-a}^{a} f^{(p+2)}(x) \cos \frac{n \pi x}{a} dx, \]

\[ \left[ m,p \right] = \int_{-a}^{a} f^{(p+3)}(x) \cos \frac{m \pi x}{a} dx - \left( -1 \right)^s \left( f^{(p+2)}(a) - f^{(p+2)}(-a) \right). \]

Here the order \( p \) specifies an even number \( p = 2s \), the series for cosines and sines of (44) are recorded by partial sums with \( N \) and \( M \) summands accordingly. For residual series can be offered following assessment.

Let the function have the property \( f^{(r)}(x) \in C^{[p-r+3]}(x \in [-a,a]), \quad r = 0 \div p \). Then there exists the
upper bound $R$ of values $[n, p], [m, p]$ indicated in (54):

$$\sup\{[n, p], [m, p]\} = R, \quad \forall (n, m) = 1, 2, \ldots$$  \hspace{1cm} (55)

For the residual cosine series in (54) can be written the inequality

$$\sum_{n=N+1}^{\infty} \left(\frac{1}{n} \right)^{p+2} \left\lbrack n, p \right\rbrack \cos n\pi \frac{x}{a} \leq R \left\{ \left(\frac{1}{N+1} \right)^{p+2} + \sum_{n=N+1}^{\infty} \left(\frac{1}{n} \right)^{p+2} \right\} \leq R \left\{ \left(\frac{1}{N+1} \right)^{p+2} + \int_{N+1}^{\infty} \left(\frac{1}{x} \right)^{p+2} dx \right\}$$  \hspace{1cm} (56)

Similarly, we get an estimate for the residual sine series:

$$\sum_{m=M+1}^{\infty} \left(\frac{1}{m} \right)^{p+3} \left\lbrack m, p \right\rbrack \sin m\pi \frac{x}{a} \leq R \left\{ \left(\frac{1}{M+1} \right)^{p+3} + \sum_{m=M+1}^{\infty} \left(\frac{1}{m} \right)^{p+3} \right\} \leq R \left\{ \left(\frac{1}{M+1} \right)^{p+3} + \int_{M+1}^{\infty} \left(\frac{1}{x} \right)^{p+3} dx \right\}$$  \hspace{1cm} (57)

Estimates (56), (57) remain valid when $p = 2s - 1$ is odd as well.

3. Results and discussion

At first glance, the obtained expressions (44) and (45) are complicated. They were only needed temporarily to show the fast convergence of the series in (38). In fact, we need to use simple classical formulas (38), (40).

To solve boundary-value or other integro-differential problems, it is necessary to use the fast expansion operator $Ch_p(f(x))$ defined by the equality:

$$f(x) = Ch_p(f(x)) = M_p(x) + \psi_p(x), \quad f(x) \in C^{(p+3)}(x \in [-a, a]).$$  \hspace{1cm} (58)

Here, $M_p(x)$ – the boundary function from (33), the $\psi_p(x)$ – the complete fast converging Fourier series for the difference $f(x) - M_p(x)$ from (38) for a given $f(x)$.

Definition. By operator $Ch_p(f(x))$ of the fast expansion (38) we mean the set of operations on $f(x)$, the application of which allows us to determine the expansion coefficients of (39) and thereby to represent $f(x)$ as the sum (58). The set of operations is represented by algorithms 1 and 2.

Algorithm 1 operator definition $Ch_p(f(x))$, if $f(x)$ is given:

1. Determine the size $a$ of the segment $[-a, a]$, choose the order $p$ of the boundary function and the number of terms $N$ and $M$ taken into account in the partial sums of the cosine and sine Fourier series for $\psi_p(x)$ from (38). When choosing $p, N, M$, one must take into account that with an increase the order $p$, the rate of convergence of the Fourier series grows rapidly, then the number of terms $N$ and $M$ in sums can be reduced, but the number of unknowns increases in $M_p(x)$.

2. Calculate the derivatives of the $q$ order $f^{(q)}(-a), f^{(q)}(a), q = 0 \div p$ at the ends of the segment $[-a, a]$. 

3. Using the formulas (38), calculate the coefficients $A_q$, $q = 0 \div p$. The boundary function $M_\phi(x)$ is determined by these actions.

4. Now we must calculate the Fourier coefficients $c_0^{(p)}$, $c_n^{(p)}$, $b_m^{(p)}$ using formulas (40) and write down the fast expansion (38) in explicit form.

Let us define algorithm 2 of the definition of the operator $Ch_p(f(x))$, if $f(x)$ is unknown.

Let $f(x)$ be determined from the solution of some nonlinear integro-differential problem with boundary conditions:

$$D_{2r}(f(x)) = 0, \quad L_i(f(x)) = 0, \quad i = 1 \div r, \quad p - 1 \geq 2r.$$  \hfill (59)

Here $2r$ is the order of the highest derivative. We impose a restriction on the functionals $D_{2r}(f(x))$, $L_i(f(x))$, $i = 1 \div r$ so that, when substituting the expansions $Ch_p(f(x))$ from (38) into them, they satisfy the smoothness conditions:

$$D_{2r}(Ch_p(f(x))) \in C^{(p+2-2r)}(x \in [-a, a]), \quad L_i(f(x)) \in C^{(p+2)}(x \in [-a, a]), \quad i = 1 \div r.$$  \hfill (60)

To find the coefficients (39), we perform the following steps:

1. We represent $f(x)$ by the fast expansion (38), where the coefficients (39) are now unknown.

2. Substitute $Ch_p(f(x))$ from (58) into the given boundary conditions

$$L_i(Ch_p(f(x))) = 0, \quad i = 1 \div r.$$  \hfill (61)

In (61), we have $2r$ equations for the coefficients (39).

3. Now we substitute $Ch_p(f(x))$ from (58) into the integro-differential equation from (59):

$$D_{2r}(Ch_p(f(x))) = 0.$$  \hfill (62)

By construction, the Fourier series in $Ch_p(f(x))$ admits $p$-multiple termwise differentiation, therefore, operation (62) is legal.

4. In equation (62), the Fourier series in $Ch_p(f(x))$ is differentiated $2r$ times, and therefore, equation (62) can be additionally differentiated termwise by $(p - 2r)$ times:

$$D_{2r}(Ch_p(f(x)))^{(i)} \in C^{(p+1-2r)}(x \in [-a, a]), \quad i = 1 \div p - 2r.$$  \hfill (63)

5. After that, in the differentiated expressions (63) for each value of $i$ we put $x = \pm a$. The difference between the obtained equations with $x = a$ and $x = -a$ allows you to get the following $p - 2r$ equations:

$$\left. D_{2r}(Ch_p(f(x)))^{(i)} \right|_{x = a} - \left. D_{2r}(Ch_p(f(x)))^{(i)} \right|_{x = -a} = 0, \quad i = 1 \div (2p - 2r).$$  \hfill (64)

This operation corresponds to the formula for $A_q(D_{2r})$, $q = 1 \div p - 2r$ from (38) applied to the functional $D_{2r}$. As a whole, system (61) + (64) will consist of $p$ algebraic equations. Moreover, the number $p$ is equal to the number of $A_q$ unknowns in the boundary function $M_\phi(x)$.

6. Now, from the left and right sides of equation (62), we calculate the trigonometric Fourier
coefficients by the formulas:

\[ \int_{-a}^{a} D_{2r} \left( Ch_p \left( f(x) \right) \right) dx = 0, \quad \int_{-a}^{a} D_{2r} \left( Ch_p \left( f(x) \right) \right) \cos \frac{n\pi x}{a} dx = 0, \]

\[ \int_{-a}^{a} D_{2r} \left( Ch_p \left( f(x) \right) \right) \sin \frac{m\pi x}{a} dx = 0, \quad (n,m) = 1 \div (N,M). \]  

(65)

The resulting system (61) + (64) + (65) is closed with respect to the unknowns (39). It follows from theorem 2 that the obtained solution for \( (N,M) \to \infty \) satisfies the boundary conditions and differential equation (59).

Since all operations in algorithm 2 can also be performed when considering nonlinear problems with nonlinear boundary conditions

\[ L_i \left( Ch_p \left( f(x) \right) \right) \bigg|_{i=1}^{i=r} = 0, \quad i = 1 \div r \quad \text{and} \quad D_{2r} \left( Ch_p \left( f(x) \right) \right) = 0, \]

this method of fast expansions is also applicable to nonlinear problems.

Practical examples of the application of this method have shown excellent results in accuracy and saving time spent on computers.

4. Conclusion
Distinctive features of this universal method of fast expansions are:

1. The method allows to obtain an approximate solution of nonlinear problems for a curvilinear region in an analytical form with high accuracy and to carry out further studies of the solution of the problem.

2. The method comes down to a minimal algebraic system, the solution of which requires minimal time in computer calculations.

3. The convergence of the method to the exact solution is proved if the number of terms in the used series is unlimited.

4. The error estimate is given in explicit form.

According to the listed signs, this method is significantly superior to all known analytical methods. The above error estimates (56), (57) show that even if we take into account a very small number of terms in the Fourier series \( M = N = 2 \) and \( p = 4 \), the error will be less \( 7 \times 10^{-4} \), the number of unknowns from (39) will be only 10. In the classical definition of the boundary conditions (Dirichlet, or Neumann), some of the unknowns are calculated directly and the algebraic system is further reduced. Such accuracy is quite acceptable for engineering practice. These estimates allow you to select the values \( p, N, M \) when solving a particular problem. It must be borne in mind that the order \( p \) of magnitude must be no lower than the order of the differential equation under consideration. The choice \( M, N \) is influenced by the oscillations of the functions when defining the internal source and boundary conditions, the oscillations of the shape of the region for which a solution to a certain problem of continuum mechanics is constructed, if any.

The above example shows that the method of fast expansions proposed here is quite effective, since \( M, N \) can be small. Therefore, at low time costs for computers, the method allows you to solve complex problems with high accuracy.

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