We investigate the ground-state phases of a mixture of spin-1 and spin-2 Bose-Einstein condensates at zero magnetic field. In addition to the intra-spin interactions, two spin-dependent interaction coefficients are introduced to describe the inter-spin interaction. We systematically explore the wide parameter space, and obtain phase diagrams containing a rich variety of phases. For example, there exists a phase in which the spin-1 and spin-2 vectors are tilted relative to each other breaking the axial symmetry.

I. INTRODUCTION

There is a wide variety of quantum fluids with internal degrees of freedom, such as superfluid $^3$He [1], $p$-wave and $d$-wave superconductors [2], possible superfluids in neutron stars [3, 4], and spinor Bose-Einstein condensates (BECs) of atomic gases [5, 6]. In these systems, the order parameters have spin or angular momentum degrees of freedom, and their ground-state phases, dynamics, and topological excitations are richer than those of single-component superfluids. If two or more quantum fluids with internal degrees of freedom are mixed, the order-parameter space is greatly extended and the physics is further enriched.

Spinor BECs of ultracold atoms are suitable systems for realizing such a mixture of quantum fluids due to their high controllability. However, in most previous experiments, spinor BECs of spin-1, spin-2, and spin-3 atoms have been realized only individually [7–11]. The ground state of a spin-1 BEC can be ferromagnetic or antiferromagnetic, and topological excitations, such as monopoles [12], skyrmions [13], half-quantum vortices [14], and knots [15], are possible. A spin-2 BEC is more intriguing because of the presence of the cyclic phase and non-Abelian vortices [16, 17]. We expect that a mixture of such spinor BECs will exhibit novel quantum phases and topological excitations. A spin-1/spin-1 mixture has been studied theoretically and phase diagrams and many-body properties have been determined [18, 19]. The spin dynamics in a mixture of a spin-1 $^{23}$Na BEC and a spin-1 $^{87}$Rb thermal gas have been observed experimentally [20].

Recently, a mixture of spin-1 and spin-2 $^{87}$Rb BECs was realized experimentally, and the spin dynamics were observed [31]. Motivated by this experiment, in the present paper we theoretically investigate the ground-state phase diagrams of the mixture of spin-1 and spin-2 BECs at zero magnetic field. The spin-1 and spin-2 BECs have one and two spin-dependent interaction coefficients, respectively. In addition to these intra-spin interactions, in a spinor mixture we must consider the inter-spin interaction, which is described by two spin-dependent interaction coefficients for the spin-1/spin-2 mixture. This gives a total of five spin-dependent interaction coefficients. We therefore study the ground-state phase diagrams by varying these five interaction coefficients. Using the Monte Carlo method, we determine the phase diagrams for various sets of the parameters. Unlike for the phase diagrams of the individual spin-1 and spin-2 BECs, the spinor mixture has phases that continuously change with respect to the interaction coefficients, including phases in which the spin-1 and spin-2 vectors are tilted from each other, breaking the axial symmetry. According to the interaction coefficients measured in Ref. [31], the ground state of the mixture of spin-1 and spin-2 $^{87}$Rb BECs is different from that of the individual spin-1 and spin-2 BECs.

This paper is organized as follows. Section III states the problem and reviews the ground states of spin-1 and spin-2 BECs. Section IV describes the numerical calculations and the various phase diagrams of the spinor mixture. Section V concludes the study.

II. FORMULATION OF THE PROBLEM

The spin state of spin-1 and spin-2 atoms are denoted by $|f,m\rangle$, where $f = 1, 2$ and $m = -f, -f + 1, \cdots, f$. We consider BECs with spin-1 and spin-2 atoms at zero temperature and zero magnetic field in the mean-field approximation. The macroscopic wave function for the BEC of spin state $|f,m\rangle$ is expressed as $\psi_m(r) = \sqrt{\rho_f(r)} \zeta_m^{(f)}(r)$, where $\rho_f(r)$ is the density and $\zeta_m^{(f)}(r)$ is the complex spin vector normalized as $\sum_m |\zeta_m^{(f)}(r)|^2 = 1$. The energy $E_f$ of a spin-$f$ BEC with atomic mass $M_f$ confined in a trap potential $V_f(r)$ is given by [3, 5, 16, 17]

$$E_f = \int dr \sum_{m=-1}^1 \psi_m(r)^* \left[ -\frac{\hbar^2}{2M_f} \nabla^2 + V_f(r) \right] \psi_m(r) + \frac{1}{2} \int dr \left[ g_0^{(1)} + g_1^{(1)} \mathbf{F}^{(1)}(r) \cdot \mathbf{F}^{(1)}(r) \right] \rho_f^2(r)$$
for a spin-1 BEC and
\[
E_2 = \int dr \sum_{m=-2}^{2} \psi_m^{(2)*}(r) \left[ -\frac{\hbar^2}{2M_2} \nabla^2 + V_2(r) \right] \psi_m^{(2)}(r) + \frac{1}{2} \int dr \left[ g_0^{(2)} + g_1^{(2)} F^{(2)}(r) \cdot F^{(2)}(r) + g_2^{(2)} |A_0^{(2)}(r)|^2 \right] \rho^2(r)
\]
for a spin-2 BEC, where
\[
F^{(f)}(r) = \sum_{mm'} c^{(f)*}_{mm'}(r) S^{(f)}_{mm'} \zeta^{(f)}_{mm'}(r)
\]
is the mean spin vector, with \( S^{(f)} \) being the vector of \((2f+1) \times (2f+1)\) matrices for spin \(f\), and
\[
A_0^{(2)} = \frac{1}{\sqrt{3}} \left( 2 \zeta_2^{(2)} \zeta_{-2}^{(2)} - 2 \zeta_1^{(2)} \zeta_{-1}^{(2)} + \zeta_0^{(2)} \zeta^{(2)} \right)
\]
is the spin-singlet scalar for spin 2. The interaction coefficients in Eqs. (1) and (2) have the forms
\[
\begin{align*}
g_0^{(1)} &= \frac{4 \pi \hbar^2 a_0^{(1)} + 2 a_0^{(1)}}{M_1}, \\
g_1^{(1)} &= \frac{4 \pi \hbar^2 a_2^{(1)} - a_0^{(1)}}{M_1}, \\
g_0^{(2)} &= \frac{4 \pi \hbar^2 4 a_2^{(2)} + 3 a_0^{(2)}}{M_2}, \\
g_1^{(2)} &= \frac{4 \pi \hbar^2 a_4^{(2)} - a_2^{(2)}}{M_2}, \\
g_2^{(2)} &= \frac{4 \pi \hbar^2 7 a_0^{(2)} - 10 a_2^{(2)} + 3 a_4^{(2)}}{M_2}
\end{align*}
\]
where \(a_f^{(f)}\) is the \(s\)-wave scattering length between spin-\(f\) atoms with colliding channel of total spin \(F\). We denote the spin vectors as \(\zeta_1^{(1)} = (\zeta_1^{(1)}, \zeta_0^{(1)}, \zeta_{-1}^{(1)})\) and \(\zeta_1^{(2)} = (\zeta_1^{(2)}, \zeta_0^{(2)}, \zeta_{-1}^{(2)}, \zeta_{-2}^{(2)}).\)

Before considering the mixture of spinor BECs, we summarize the ground-state phases for individual spin-1 and spin-2 BECs in a uniform system. The ground state of a spin-1 BEC depends on the sign of \(g_1^{(1)}\). When \(g_1^{(1)} < 0\), the ground state is the fully-polarized ferromagnetic state
\[
\zeta_1^{(1)} = e^{i \chi} \hat{R}(1,0,0),
\]
where \(\chi\) is an arbitrary phase and \(\hat{R}\) is an arbitrary SO(3) rotation in the spin space. When \(g_1^{(1)} > 0\), the ground state is the polar state
\[
\zeta_0^{(1)} = e^{i \chi} \hat{R}(0,1,0).
\]

The spin-2 BEC has more variety of ground states. When \(g_1^{(2)} < 0\) and \(g_2^{(2)} > 20g_1^{(2)}\), the ground state is the ferromagnetic state
\[
\zeta_0^{(2)} = e^{i \chi} \hat{R}(1,0,0,0,0).
\]
where $P^{(12)}$ is defined in Eq. (A7). The interaction coefficients in Eq. (18) are given by

\begin{align}
    g_0^{(12)} &= \frac{2\pi\hbar^2}{M_{12}} \left(2a_{2}^{(12)} + a_{3}^{(12)}\right), \\
    g_1^{(12)} &= \frac{2\pi\hbar^2}{M_{12}} \left(a_{3}^{(12)} - a_{2}^{(12)}\right), \\
    g_2^{(12)} &= \frac{2\pi\hbar^2}{M_{12}} \left(3a_{1}^{(12)} - 5a_{2}^{(12)} + 2a_{3}^{(12)}\right),
\end{align}

where $M_{12} = (M_{1}^{-1} + M_{2}^{-1})^{-1}$ is the reduced mass and $a_{F}^{(12)}$ is the s-wave scattering length between spin-1 and spin-2 atoms with colliding channel of total spin $F$.

In the following analysis, we assume that the spin healing lengths are much larger than the size of the atomic cloud and we neglect the spatial variation of the spin states $\zeta^{(j)}$. The kinetic and potential energy terms in $E_1$ and $E_2$ in Eqs. (1) and (2) then become independent of the spin states $\zeta^{(j)}$. The spin-dependent part of the total energy $E = E_1 + E_2 + E_{12}$ thus reduces to

\[
    E_{\text{spin}} = \frac{1}{2} \left( c_1^{(1)} F^{(1)} \cdot F^{(1)} + c_1^{(2)} F^{(2)} \cdot F^{(2)} + c_2^{(12)} |A_0^{(2)}|^2 \right) + c_1^{(12)} F^{(1)} \cdot F^{(1)} + c_2^{(12)} F^{(2)} \cdot F^{(2)},
\]

where

\[
    c_n^{(f)} = g_n^{(f)} \int \rho^2(r) dr, \\
    c_n^{(12)} = g_n^{(12)} \int \rho_1(r) \rho_2(r) dr
\]

with $n = 1, 2$. In the rest of this paper, we normalize the interaction coefficients $c_1^{(1)}, c_1^{(2)}, c_2^{(1)}, c_{12}^{(1)},$ and $c_{12}^{(2)}$ by $4\pi\hbar^2 a_B \int \rho^2 dr / M_1$, where $a_B$ is the Bohr radius, and therefore these interaction coefficients are dimensionless.

Our purpose is to find the spin states $\zeta^{(1)}$ and $\zeta^{(2)}$ that minimize the energy $E_{\text{spin}}$. We numerically obtain the ground state as follows. First we set complex random numbers to $c_n^{(f)}$ and minimize the energy in a stochastic manner, that is, we try a small random change to the spin state $c_n^{(f)} + \delta c_n^{(f)}$ and adopt the change if the energy is lowered. After sufficiently many steps in this random walk in the spin space, we obtain a metastable state or the ground state. Repeating this procedure many times with different initial random states, we can exclude metastable states and determine the true ground state.

III. GROUND STATES OF A SPIN-1/SPIN-2 MIXTURE

To see the effect of the interaction between the spin-1 and spin-2 BECs, we first consider the case without the intra-spin interactions, $c_1^{(1)} = c_1^{(2)} = c_2^{(2)} = 0$. The spin-dependent energy then reduces to $E_{\text{spin}} = c_1^{(12)} F^{(1)}$.
The subscripts indicate the intermediate states as defined in Table I. The letters indicate the sign of the trivial group. The subscripts + or − is added to a–d to indicate the sign of $\mathbf{F}^{(1)} \cdot \mathbf{F}^{(2)}$.

Next we consider the cases of nonzero intra-spin interaction coefficients of $^{87}$Rb for $\rho_1 = \rho_2$ in Eq. (21). There is a remarkable number of phases with complicated structures. If the inter-spin interaction is absent, i.e., at the origin of the phase diagram, the ground state for spin 1 is the ferromagnetic state and that for spin 2 is the nematic state. Comparing Fig. 3 with Fig. 2, we find that the four phases in Fig. 2 FF$_+^-$, FF$_+^+$, PU, and PB, also appear in Fig. 3 where the continuous degeneracy in Fig. 2 is removed and the PF state disappears in Fig. 3. There are many intermediate states, labeled by lower-case letters classified in Table I. In the regions of these intermediate states, either or both of the spin-1 and spin-2 states continuously change with respect to $c_{1(2)}$ and $c_{2(2)}$. We now consider the phases along the dotted line in Fig. 3. When $c_{1(2)}$ is large and negative, the ground state is the FF$_+$ state. When $c_{1(2)}$ crosses the phase boundary between FF$_+$ and $a_+$, the lengths of the spin-1 and spin-2 vectors begin to decrease, as shown in Fig. 4(a). In this $a_+$ phase, the spin vectors $\mathbf{F}^{(1)}$ and $\mathbf{F}^{(2)}$ remain in the same direction. In contrast, in the $b_+$ phase, the directions of the spin vectors $\mathbf{F}^{(1)}$ and $\mathbf{F}^{(2)}$ become different. This can be regarded as axisymmetry breaking of the magnetization, that is, if we fix the vector $\mathbf{F}^{(1)}$ to the $z$ direction, the vector $\mathbf{F}^{(2)}$ has a component $F_{2\perp}$ perpendicular to the $z$ axis. Examples of such axisymmetry breaking states are shown in Fig. 4(b). Axisymmetry breaking has been found in a spin-1/spin-1 mixture in Ref. [24]. In the FF$_+^+$ phase, the directions of the spin vectors $\mathbf{F}^{(1)}$ and $\mathbf{F}^{(2)}$ become the same again. In this phase, the spin-1 state returns to $\zeta_{1(2)} = \zeta_{F}^{(1)}$ and the spin-2 state is $\zeta_{2(2)} = \zeta_{F}^{(2)}$, which does not depend on $c_{1(2)}$ and $c_{2(2)}$ within the phase, as seen from the plateau in Fig. 4(a). In the $a_+$, $b_+$, and $a_-$ phases, the spin states continuously change again; the spin vectors $\mathbf{F}^{(1)}$ and $\mathbf{F}^{(2)}$ are in the same direction in the $a_+$ phase while they take different directions in the $b_+$ phase. The FU state is connected to the origin of the phase diagram. The phases on the right-hand side of the phase diagram, $a_-$, $b_+$, · · · are similar to the corresponding phases $a_+$, $b_+$, · · · where the spin vector $\mathbf{F}^{(1)}$ or $\mathbf{F}^{(2)}$ is flipped, i.e., the time-reversal transformation is applied to the spin-1 or spin-2 state. For example, in the FF$_-$ phase, when the spin-1 state

| $F^{(1)}$ | $F^{(2)}$ | $A^{(2)}$ | $F^{(1)} \times F^{(2)}$ | isotropy group |
|---|---|---|---|---|
| a | nonzero | nonzero | nonzero | 0 | $Z_2$ |
| b | nonzero | nonzero | nonzero | 0 | 0 | $Z_4$ |
| c | 1 | nonzero | nonzero | nonzero | 0 | $Z_3$ |
| d | 1 | nonzero | 0 | 0 | 0 | $Z_2 \times Z_2$ |

TABLE I: Classification of the intermediate states that change continuously in the phase diagram. “nonzero” indicates that the value depends on $c_{1(2)}$ and $c_{2(2)}$. E indicates the trivial group. The subscript + or − is added to a–d to indicate the sign of $\mathbf{F}^{(1)} \cdot \mathbf{F}^{(2)}$. 3$c_{2(2)}^{(12)}$/10, the ground state is continuously degenerate: the linear combination of the “PF” (polar-ferromagnetic) and “PB” (polar-biaxial nematic) states is the ground state with an energy $E_{\text{spin}} = 3c_{2(2)}^{(12)}/10$.

Next we consider the cases of nonzero intra-spin interaction coefficients $c_{1(2)}^{(1)}$, $c_{1(2)}^{(2)}$, and $c_{2(2)}^{(2)}$. Figure 3 shows the ground-state phase diagram for $c_{1(2)}^{(1)} = -0.46$, $c_{1(2)}^{(2)} = 1.1$, and $c_{2(2)}^{(2)} = -0.05$, which correspond to the interaction

![Ground-state phase diagram](image-url)
is \( \zeta^{(1)} = (1, 0, 0) \), the spin-2 state is \( \zeta^{(2)} = (0, 0, 0, 1, 0) \), which is the time-reversal state of \( \zeta^{(2)} = (0, 1, 0, 0, 0) \) in the FF\( _c \) phase. For \( \zeta^{(12)} < 0 \), the phase structures are simpler. In the \( c_{\pm} \) phases, the spin-1 state is fixed to the ferromagnetic state, while the spin-2 state continuously changes with \( F^{(1)} \) and \( F^{(2)} \) being kept in the same direction. A typical \( c \) state is shown in Fig. 4(b).

In the experiment in Ref. 31, the values of the inter-spin scattering lengths of \(^{87}\)Rb were measured, which correspond to \( c^{(12)} \simeq 0.83 \) and \( c^{(12)} \simeq 4.8 \) in the present case, if \( \rho_1 = \rho_2 \) in Eq. (21), i.e., an almost 1:1 mixture of spin-1 and spin-2 atoms. In the phase diagram in Fig. 3 these values correspond to the PB state, namely, the polar state for spin 1 and the biaxial nematic state for spin 2. The ground state phase of the spin-1 \(^{87}\)Rb BEC alone is the ferromagnetic state and that for spin-2 is the biaxial or uniaxial nematic state. Thus, the ground state of the 1:1 mixture of spin-1 and spin-2 \(^{87}\)Rb BECs is different from those of the individual BECs due to the inter-spin interaction.

Figure 4 shows the ground-state phase diagram for \( c^{(1)} = -0.46, c^{(2)} = -1.1, \) and \( c^{(2)} = 1.5 \). If the inter-spin interaction is absent, the ground state is the ferromagnetic state both for spin 1 and spin 2 for these parameters. The phase diagram is much simpler than Fig. 3. Comparing Fig. 5 with Fig. 2 we find that the PB and PU states disappear in Fig. 5. Between the PF and FF\( _c \) phases, there exists the region of the b state, in which the axisymmetry is broken. For the present parameters, the spin 2 state is almost the ferromagnetic state in the b phase. The angle between the two spin vectors changes from 0 to \( \pi \) across the region of the b state.

Figure 5 shows the ground-state phase diagram for \( c^{(1)} = -0.46, c^{(2)} = 1.1, \) and \( c^{(2)} = 1.5 \). If the inter-spin interaction is absent, the phase diagram, the ground state of the spin-1 BEC is the ferromagnetic state and that of the spin-2 BEC is the cyclic state for these parameters. The phase diagram is again very complicated. Let us examine the phases along the dotted line. As \( c^{(12)} \) is increased from a large negative value, the ground state changes from the FF\( _c \) state to the \( a_+ \), \( b_+ \), and FF\( _c' \) states, which is similar to the case in Fig. 3. After that, a new phase appears, labeled by d\( _+ \). In this phase, the value of \( |A_{0}^{(2)}| \) in the spin-2 state vanishes, as in the cyclic state, whereas \( |F^{(2)}| \) is finite, as shown in Fig. 7(a). The spin-1 state is in the ferromagnetic state \( \zeta^{(1)} = \zeta^{(1)} \). From the shape of the spherical harmonic representation in Fig. 7(b), we find that this state may be regarded as an intermediate state between the FC and FF\( _c \) states. The d\( _\pm \) states also exist in the region \( c^{(12)} < 0 \). The structures of the a\( _\pm \), b\( _\pm \), and c\( _\pm \) regions in Fig. 6 appear to be different from those in
FIG. 6: (color online) Ground-state phase diagram for $c_{1}^{(1)} = -0.46$, $c_{1}^{(2)} = 1.1$, and $c_{2}^{(2)} = 1.5$. The ground state for $c_{1}^{(1)} = 0$ is the ferromagnetic state for spin 1 and the cyclic state for spin 2. The region of many phases in (a) is magnified in (b). The physical quantities along the dotted line are shown in Fig. 7(a). The spin states at the black dots are shown in Fig. 7(b).

Figure 8 shows the ground-state phase diagram for $c_{1}^{(1)} = 0.46$, $c_{1}^{(2)} = 1.1$, and $c_{2}^{(2)} = 1.5$. If the inter-spin interaction is absent, the ground state of the spin-1 BEC is the polar state and that of the spin-2 BEC is the cyclic state for these parameters. In this phase diagram, a new state appears, labeled e. The e state has no magnetization for both spin 1 and spin 2, $F_{1} = F_{2} = 0$, as shown in Fig. 10(a). From the shapes of the spherical harmonic representation in Fig. 10(b), the e state is an intermediate state between the cyclic and nematic states. In the phase diagram, the regions of the e state are located at the heads of the PB and PU regions. For the parameters in Fig. 10, interestingly, the two regions of the e state are detached from each other near the origin, where the $a_{\pm}$ states fill in. Although in Fig. 10 the quantities $F_{1}$, $F_{2}$, and $A_{0}^{(2)}$ seem to jump at the boundary of the e region, they continuously change across the very narrow regions of the $a_{\pm}$ states. In all of the phase diagrams presented above, these quantities continuously change at the phase boundaries of the intermediate (a, b, c, d, and e) regions.

FIG. 7: (color online) (a) Dependence of $F_{1}, F_{2}, F_{\perp}$, and $|A_{0}^{(2)}|$ on $c_{1}^{(12)}$ along the dotted line in Fig. 6. Here, the spin-1 and spin-2 states are rotated so that $F_{1}$ is in the $z$ direction, and hence $F_{\perp}$ is always zero. The small changes in $F_{1}$ and $|A_{0}^{(2)}|$ are magnified in the insets. (b) Spherical-harmonic representations of the spin states marked by the black dots in Fig. 6, where the left- and right-hand figures are the spin-1 and spin-2 states, respectively.

Figure 9 shows the ground-state phase diagram for $c_{1}^{(1)} = 0.46$, $c_{1}^{(2)} = 1.1$, and $c_{2}^{(2)} = 1.5$. If the inter-spin interaction is absent, the ground state of the spin-1 BEC is the polar state and that of the spin-2 BEC is the cyclic state for these parameters. In this phase diagram, a new state appears, labeled e. The e state has no magnetization for both spin 1 and spin 2, $F_{1} = F_{2} = 0$, as shown in Fig. 10(a). From the shapes of the spherical harmonic representation in Fig. 10(b), the e state is an intermediate state between the cyclic and nematic states. In the phase diagram, the regions of the e state are located at the heads of the PB and PU regions. For the parameters in Fig. 10, interestingly, the two regions of the e state are detached from each other near the origin, where the $a_{\pm}$ states fill in. Although in Fig. 10 the quantities $F_{1}, F_{2}$, and $A_{0}^{(2)}$ seem to jump at the boundary of the e region, they continuously change across the very narrow regions of the $a_{\pm}$ states. In all of the phase diagrams presented above, these quantities continuously change at the phase boundaries of the intermediate (a, b, c, d, and e) regions.

Figure 11 shows the ground-state phase diagram for $c_{1}^{(1)} = 0.46$, $c_{1}^{(2)} = -0.0005$, and $c_{2}^{(2)} = 1$. If the inter-spin interaction is absent, the ground state of the spin-1 BEC is the polar state and that of the spin-2 BEC is the ferromagnetic state for these parameters. We take the
FIG. 8: (color online) Ground-state phase diagram for $c_1^{(1)} = 0.46$, $c_1^{(2)} = 1.1$, and $c_2^{(2)} = -1$. The ground state for $c_1^{(12)} = c_2^{(12)} = 0$ is the polar state for spin 1 and the nematic state for spin 2.

small value of $c_1^{(2)}$, because the PU region is far from the origin for a larger value of $c_1^{(2)}$. The $a_\pm$ states occupy the region near the origin instead of the PU state. Compared with Fig. 2 the degeneracy is removed and the PF state remains in the upper region of Fig. 8.

Finally, we mention the order-parameter manifold of the ground-state. In the case of individual spin-1 and spin-2 BECs, the Hamiltonian is invariant with respect to changes in the global phase, $U(1)$, and the rotation in the spin space, $SO(3)$. The ground state therefore has continuous degeneracy, with a manifold represented by $U(1) \times SO(3)$. However, for example, the spin-1 ferromagnetic state in Fig. 1(a) is invariant with respect to rotation around the symmetry axis (with a global phase shift due to the spin-gauge symmetry). In other words, the isotropy group of the spin-1 ferromagnetic state is $SO(2)$. The order-parameter manifold of the spin-1 ferromagnetic state is thus $U(1) \times SO(3) / SO(2) \simeq SO(3)$.

The isotropy group of the spin-1 polar state is $SO(2) \times \mathbb{Z}_2$. Since Fig. 1(b) is invariant with respect to rotation around the symmetry axis and upside-down rotation with global phase $\pi$.

In the case of the spin-1/spin-2 mixture, the Hamiltonian is invariant with respect to changes in the global phase for each of the spin-1 and spin-2 states, in addition to the spin rotation of both spin-1 and spin-2 states, and then the symmetry group of the Hamiltonian is $U(1) \times U(1) \times SO(3)$. For example, the isotropy group of the FF state is $SO(2)$, and therefore the order-parameter manifold of the FF state is $U(1) \times SO(3)$. Similarly, the FF and PU states have this manifold. The isotropy groups of the intermediate states are summarized in Table I, whose symmetries are lower than those of individual spin states. For example, the symmetry-broken state $b$ in Table I only has the trivial isotropy group (only the identity element).

FIG. 9: (color online) Ground-state phase diagram for $c_1^{(1)} = 0.46$, $c_1^{(2)} = 1.1$, and $c_2^{(2)} = 1.5$. The ground state for $c_1^{(12)} = c_2^{(12)} = 0$ is the polar state for spin 1 and the cyclic state for spin 2. The region of many phases in (a) is magnified in (b). The physical quantities along the dotted line in (a) are shown in Fig. 10(a). The spin states at the black dots are shown in Fig. 10(b).

IV. CONCLUSIONS

We have investigated the ground-state phase diagrams of a mixture of spin-1 and spin-2 BECs in the mean-field approximation. We obtained two types of ground
states. One is a pair of known stationary states in spin-1 and spin-2 BECs, such as the FF and PB states. In the other type of ground state, either or both of the spin states continuously change with respect to the interaction coefficients. The latter type of ground state is classified in Table I.

For the various choices of the intra-spin interaction coefficients, \( c_1^{(1)}, c_1^{(2)}, c_2^{(2)} \), we obtained the phase diagrams with respect to the inter-spin interaction coefficients, \( c_1^{(12)} \) and \( c_2^{(12)} \). These phase diagrams have remarkably rich structures. In all the phase diagrams, the FF\(_{+}\) and FF\(_{-}\) phases occupy the regions of large negative and positive \( c_1^{(12)} \), respectively. Also, the PF, or the PB and FF\(_{\pm}\) phases are located in the \( c_2^{(12)} > 0 \) region, and the PU phase is located in the \( c_2^{(12)} < 0 \) region (except Fig. 6). Between these phases, there exist various intermediate phases with interesting phase structures. Among them, we found the axisymmetry broken phase (b in Table I), in which the spin-1 and spin-2 vectors are tilted from each other.

We have also determined the ground-state phase of a mixture of spin-1 and spin-2 \(^{87}\)Rb BECs, using the measured interaction coefficients \([31]\). It has been known that the ground state of the spin-1 \(^{87}\)Rb BEC alone is the ferromagnetic state and that of spin-2 BEC is a linear combination of the uniaxial and biaxial nematic states at zero magnetic field. By contrast, for an almost 1:1 mixture, the ground state is the polar state for spin 1 and the biaxial-nematic state for spin 2. The ground state of the spinor mixture of \(^{87}\)Rb BECs is thus changed by the interaction between spin-1 and spin-2 BECs.

The present study can be extended in various directions. For example, the magnetic field dependence (linear and quadratic) of the phase diagrams is the next planned extension of this work. Since the ground-state manifolds of the spinor mixture are different from those of single BECs, novel topological excitations will be possible. If phase separation occurs in the spinor mixture, we expect that the interface between domains will create interesting problems.

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Appendix A: Derivation of interaction energy between spin-1 and spin-2 atoms

The spin state of colliding spin-1 and spin-2 atoms can be represented by the bases as

\[ |\mathcal{F}, \mathcal{M}\rangle = \sum_{mn'} C_{mn'}^{\mathcal{F}, \mathcal{M}} |1, m\rangle |2, m'\rangle, \quad (A1) \]

where \( C_{mn'}^{\mathcal{F}, \mathcal{M}} \) is the Clebsch-Gordan coefficient, \( \mathcal{F} = 1, 2, \) and \( 3 \) are total spin, and \( \mathcal{M} = -\mathcal{F}, -\mathcal{F} + 1, \cdots, \mathcal{F}. \) The projection operator for the colliding channel of total spin \( \mathcal{F} \) is defined by

\[ \hat{P}_{\mathcal{F}} = \sum_{\mathcal{M} = -\mathcal{F}}^\mathcal{F} |\mathcal{F}, \mathcal{M}\rangle \langle \mathcal{F}, \mathcal{M}|, \quad (A2) \]

which is rotation invariant. In the present Hilbert space, the identity operator \( \hat{I} \) is given by

\[ \hat{P}_1 + \hat{P}_2 + \hat{P}_3 = \hat{I}. \quad (A3) \]

We define the spin operators acting on the spin-1 and spin-2 states as \( \hat{f}_1 \) and \( \hat{f}_2, \) respectively. We find

\[ \hat{f}_1 \cdot \hat{f}_2 = \frac{1}{2} (\hat{f}_1 + \hat{f}_2)^2 - \frac{1}{2} \sum_{f=1,2} f (f + 1) \hat{I} \]

\[ = \frac{1}{2} \sum_{\mathcal{F} = 1, 2, 3} \mathcal{F} (\mathcal{F} + 1) \hat{P}_{\mathcal{F}} - 4 \hat{I}. \quad (A4) \]

Since the Hamiltonian must be rotation invariant, the two-body interaction Hamiltonian between spin-1 and spin-2 atoms is written as

\[ \hat{H}_{12} = \frac{2\pi \hbar^2}{M_{12}} \sum_{\mathcal{F} = 1, 2, 3} a_{\mathcal{F}} \hat{P}_{\mathcal{F}} \delta(r_1 - r_2), \quad (A5) \]

where \( M_{12} = (M_1^{-1} + M_2^{-1})^{-1} \) is the reduced mass. Using Eqs. (A3) and (A4), the interaction Hamiltonian can be rewritten as

\[ \hat{H}_{12} = (g_{0}^{(12)} \hat{I} + g_{1}^{(12)} \hat{f}_1 \cdot \hat{f}_2 + g_{2}^{(12)} \hat{P}_1) \delta(r_1 - r_2), \quad (A6) \]

where \( g_{0}^{(12)}, g_{1}^{(12)}, \) and \( g_{2}^{(12)} \) are defined in Eq. (19). The mean-field energy is thus given by Eq. (13), where \( P_1^{(12)} = |A_{1,1}|^2 + |A_{1,0}|^2 + |A_{1,-1}|^2 \) with

\[ A_{1,1} = \frac{1}{\sqrt{10}} \zeta_1^{(1)} \zeta_0^{(2)} - \frac{3}{10} \zeta_0^{(1)} \zeta_1^{(2)} + \frac{3}{5} \zeta_1^{(1)} \zeta_2^{(2)}, \quad (A7a) \]

\[ A_{1,0} = \sqrt{\frac{3}{10}} \zeta_1^{(1)} \zeta_0^{(2)} - \frac{2}{5} \zeta_0^{(1)} \zeta_1^{(2)} + \sqrt{\frac{3}{10}} \zeta_2^{(1)} \zeta_1^{(2)}, \quad (A7b) \]

\[ A_{1,-1} = \sqrt{\frac{3}{5}} \zeta_1^{(1)} \zeta_0^{(2)} - \sqrt{\frac{3}{10}} \zeta_0^{(1)} \zeta_1^{(2)} + \frac{1}{\sqrt{10}} \zeta_1^{(1)} \zeta_0^{(2)}. \quad (A7c) \]

Appendix B: Linear stability analysis and phase boundaries

We perform a linear stability analysis of a stationary state to obtain the phase boundaries analytically. The total energy is given by

\[ E = \frac{c_0^{(1)}}{2} \left( \sum_{m=-1}^1 |\zeta_m^{(1)}|^2 \right)^2 + \frac{c_0^{(2)}}{2} \left( \sum_{m=-2}^2 |\zeta_m^{(2)}|^2 \right)^2 + \frac{1}{2} \left( c_1^{(1)} \mathbf{F}^{(1)} \cdot \mathbf{F}^{(1)} + c_1^{(2)} \mathbf{F}^{(2)} \cdot \mathbf{F}^{(2)} + c_2^{(2)} |A_0^{(2)}|^2 \right) + c_1^{(12)} \mathbf{F}^{(1)} \cdot \mathbf{F}^{(2)} + c_2^{(12)} P_1^{(12)}. \quad (B1) \]

Using this energy, the Gross-Pitaevskii (GP) equation is written as

\[ i\hbar \frac{\partial \zeta_m^{(f)}}{\partial t} = \frac{\partial E}{\partial \zeta_m^{(f)}}. \quad (B2) \]

All of the ground states in the phase diagrams are stationary solutions of the GP equation. We write a stationary solution as

\[ \zeta_m^{(f)}(t) = e^{-i\mu_f t/\hbar} Z_m^{(f)}, \quad (B3) \]

where \( \mu_f \) is the chemical potential for spin \( f. \) We consider a small deviation from the stationary solution as

\[ \zeta_m^{(f)}(t) = e^{-i\mu_f t/\hbar} \left( Z_m^{(f)} + u_m^{(f)} e^{-i\omega t} + v_m^{(f)} e^{i\omega t} \right). \quad (B4) \]

Substituting this into Eq. (B2) and taking the first-order terms of \( u_m^{(f)} \) and \( v_m^{(f)} \), we obtain an \( 8 \times 8 \) eigenvalue equation with respect to \( \omega. \) If one or more eigenvalues are negative or complex, the stationary state \( Z_m^{(f)} \) is not the ground state.

For example, we take the stationary state \( Z_m^{(f)} \) as the ferromagnetic state \( \zeta^{(1)} = (1, 0, 0) \) and \( \zeta^{(2)} = (1, 0, 0, 0, 0), \) which corresponds to the FF+ state in the phase diagrams. Diagonalizing the eigenvalue equation, we obtain

\[ \omega = -3c_1^{(12)}, \quad (B5a) \]

\[ \omega = -6c_1^{(2)} - 3c_1^{(12)} + \frac{3}{10} c_2^{(12)}, \quad (B5b) \]

\[ \omega = -8c_1^{(2)} + \frac{2}{5} c_2^{(2)} - 4c_1^{(12)} + \frac{3}{5} c_2^{(12)}, \quad (B5c) \]

\[ \omega = -c_1^{(1)} - 2c_1^{(2)} - 3c_1^{(12)} + \frac{7}{20} c_2^{(12)} \]

\[ \pm \left[ A^2 - \frac{1}{2} A c_1^{(12)} + \left( \frac{7}{20} c_1^{(12)} \right)^2 \right]^{1/2}. \quad (B5d) \]

and \( \omega = 0, \) where \( A = c_1^{(1)} - 2c_1^{(2)} + c_1^{(12)}. \) In the case of Fig. 2, for example, the condition \( \omega > 0 \) for Eqs. (B5a) and (B5c) gives \( c_1^{(12)} < 0 \) and \( c_2^{(12)} > 10c_2^{(12)}, \) which agree with the phase boundary of the FF+ phase in Fig. 2. On
the other hand, for the phase diagram in Fig. 3 the phase boundary of the FF+ phase is determined by Eqs. (B5c) and (B5d) for $c_2^{(12)} < 0$ and $c_2^{(12)} > 0$, respectively.

Taking the stationary state $Z_n^{(f)}$ as $\zeta^{(1)} = (1, 0, 0)$ and $\zeta^{(2)} = (0, 0, 0, 0, 1)$, i.e., the FF− state, we obtain

\[
\omega = -6c_1^{(2)} + 5c_2^{(12)} - \frac{3}{5}c_2^{(12)},
\]

(B6a)

\[
\omega = -8c_1^{(2)} + 2c_2^{(12)} + 4c_1^{(12)} - \frac{3}{5}c_2^{(12)},
\]

(B6b)

\[
\omega = \pm \left( -c_1^{(1)} + 2c_1^{(2)} + c_1^{(12)} - \frac{1}{20}c_2^{(12)} \right)
\]

\[+ \left[ B^2 - \frac{11}{10}c_2^{(12)} + \frac{97}{400}c_2^{(12)} \right]^{1/2},
\]

(B6c)

\[
\omega = \frac{c_1^{(12)} - \frac{3}{10}c_2^{(12)}},
\]

(B6d)

and $\omega = 0$, where $B = c_1^{(1)} + 2c_2^{(2)} - 3c_1^{(12)}$. For example, for the phase diagram in Fig. 3 the phase boundary of the FF− phase is determined by Eqs. (B6b) and (B6c) for $c_2^{(12)} < 0$ and $c_2^{(12)} > 0$, respectively.

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