Two-dimensional dilaton gravity coupled to massless spinors

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Abstract

We derive exact solutions of two-dimensional dilaton gravity coupled to massless spinors for some particular choices of the dilatonic potential. For constant dilatonic potential the model turns out to be completely solvable and the general solution is found. For linear and exponential dilatonic potentials we present the class of exact solutions with a Killing vector.

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1 Introduction

Recently, two-dimensional dilaton-matter-gravity (DMG) theories have attracted much attention because of their connection to string theory and also, for some choices of the dilatonic potential and matter fields, to dimensionally reduced models of gravity in $N \geq 3$ dimensions (see, for instance, [1]-[3] and references quoted therein.) Furthermore, DMG models can be seen as simplified models which are useful to clarify some conceptual problems of quantum gravity. So several 1+1 models of DMG have been discussed in the literature, both from the classical and quantum points of view [4]-[8]. In this context, attention has been focused essentially on dilaton-gravity (DG) models coupled to gauge and scalar fields. (For short reviews, see e.g. [9, 10].) Not surprisingly, we are not aware of papers dealing with models of two-dimensional DG coupled to fermionic fields. The lack of discussion about two-dimensional spinor-dilaton-gravity (SDG) models is indeed somehow related to the absence of a general formalism able to describe gravity dynamically coupled to spinorial fields. The aim of this paper is to fill this gap partially and present some exact solutions of two-dimensional SDG. We shall use a general formalism for spinor-gravity theories that is well posed from a geometrical point of view and is potentially able to deal with interactions between gravity and spinors without any restriction on dimension or signature (see [11, 12]). The formalism also describes a truly relativistic field theory, in the sense that it does not make use of any background field (such as fixed background metrics) in contrast to what is often done when dealing with spinors. As a consequence, we are able to describe both the effect of the gravitational field on spinors (as in the case of ‘background theories’ on curved spaces) as well as the effect of spinors on the gravitational field itself.

The framework recalls gauge theories in their geometrical formulation where one starts from a principal bundle on spacetime $M$, the so-called structure bundle $\Sigma$, that encodes the symmetry structure of the theory. The configuration bundle $C$ is then associated to the structure bundle, i.e. the principal automorphisms of the structure bundle are represented on the configuration bundle by means of a canonical action. Automorphisms of the structure bundle are requested to be symmetries of the Lagrangian $L$, namely to act on the configuration bundle leaving the scalar Lagrangian unchanged.

Gravity is described by new variables, called spin frames, suitably related to spin structures on spacetime and defined without any reference to any preferred background metric. So at a fundamental level no metric appears at all. The metric is canonically induced by the spin frame (see equation (A.4)) once the field equations have been solved:

\[
(M, \Sigma) \leadsto (C, L) \leadsto \text{dilaton+Einstein+Dirac equations} \leadsto \text{spin frames} \leadsto \text{a geometry } g \text{ on } M \quad (1.1)
\]

Locally, spin frames resemble vielbein because both assign a $GL(N)$ matrix to each point of spacetime $M$ ($\dim(M) = N$). However, spin frames and vielbein transform differently.
Vielbein are natural objects, i.e. they transform naturally under both diffeomorphisms of $M$ and the appropriate representation of the orthogonal group, and these two actions are completely unrelated; spin frames transform just under automorphisms of the structure bundle $\Sigma$ (see equation (A.6) in the appendix), as a kind of gauge fields.

Let us mention here that the spin frame formalism allows to overrun an obstacle that is usually encountered in the discussion of spinor-gravity theories. In the standard approach one needs vielbein to write the Dirac Lagrangian because the world indices of the covariant derivatives must be transformed into vielbein indices to be contracted with Dirac matrices. This prevents the theory from being geometrically well defined since in this case vielbein are dynamical fields and have to be varied to obtain the field equations. However, vielbein are intrinsically local objects, i.e. they are local sections of the frame bundle $L(M)$. On reasonably general manifolds, i.e. non-parallelizable ones, there is no global section of $L(M)$ at all.

In this paper we are just interested in field equations and exact solutions so local variational techniques provide (local) field equations which are invariant under gauge transformations. Due to this invariance, local solutions fit together smoothly on patches and we can always glue them together to obtain a global solution. However, we prefer to introduce the spin frame formalism in view of forthcoming investigations such as, for instance, the study of conserved quantities (that are not local quantities) and higher-dimensional models. In this perspective, the standard formalism is not adequate since one needs a global frame to define a global Lagrangian and spinor theories make sense just on parallelizable manifolds. Unlike ordinary frames, spin frames are geometrically well defined in any dimension and for any signature. Moreover, they are guaranteed to be continuous and globally defined on every spin manifold, i.e. on a much wider class of manifolds than the class of parallelizable ones. (Of course, in two dimensions any spin manifold is parallelizable so, for the purposes of this paper, we do not really need spin frames.)

Since here we simply derive exact solutions, we relegate to the appendix the discussion of the spin frame formalism and technical details.

## 2 Action and field equations

From now on, we shall work locally. The global framework (see the appendix) may be recovered at the end, when both the manifold structure of $M$ and the structure bundle $\Sigma$ over it can be obtained by gluing local solutions.

Let us consider massless spinors. (The massive case will be discussed elsewhere.) The SDG Lagrangian density is (we follow the conventions of [15])

$$L = \sqrt{-g} \left[ -\phi R + U(\phi) + i \alpha \left( \bar{\psi} \gamma^a \nabla_a \psi - \nabla_a \bar{\psi} \gamma^a \psi \right) \right], \quad (2.1)$$

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where $R$ is the scalar curvature of the metric $g_{\mu\nu}$ induced by a spin frame $\Lambda$ on $\Sigma$, $\sqrt{-g}$ is the square root of (the absolute value) of its determinant, $\gamma^a$ are the two-dimensional Dirac matrices defined in the appendix, and $\alpha$ is a (real) coupling constant. We have denoted the conjugate spinors by $\bar{\psi} = \psi^\dagger \cdot \gamma_0$ and the covariant derivative of spinors by

$$\nabla_a \psi = e^\mu_a \left( d_\mu \psi + \frac{1}{8}[\gamma_b, \gamma_c] \psi \Gamma^b_\mu \right), \quad \nabla_a \bar{\psi} = e^\mu_a \left( d_\mu \bar{\psi} - \frac{1}{8} \bar{\psi} [\gamma_b, \gamma_c] \Gamma^b_\mu \right), \quad \text{(2.2)}$$

where $\Gamma^{ab}_\mu = e^a_\rho \left( \Gamma^\rho_{\sigma\mu} e^b_\sigma + d_\mu e^b_\rho \right)$ are the coefficients of the spin connection, and $\Gamma^e_{\sigma\mu}$ are the Christoffel symbols of the induced metric $g_{\mu\nu}$. The function $U(\phi)$ is the dilatonic potential.

Equation (2.1) is the local expression of a global Aut($\Sigma$)-covariant Lagrangian and defines the action

$$S_D = \int_D L \, d^2x, \quad \text{(2.3)}$$

where $d^2x$ is the local volume element and $D \subset M$ is a compact two-dimensional submanifold with a compact one-dimensional boundary $\partial D$. Varying the action (2.3) we obtain the field equations

$$\gamma^a \nabla_a \psi = 0, \quad \text{(2.4)}$$

$$-R + \frac{dU}{d\phi} = 0, \quad \text{(2.5)}$$

$$(\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla_\sigma \nabla^\sigma) \phi + \frac{1}{2} U g_{\mu\nu} = H_{\mu\nu}, \quad \text{(2.6)}$$

where $H_{\mu\nu}$ is the Hilbert stress tensor evaluated on-shell, i.e. modulo terms that vanish on solutions:

$$H_{\mu\nu} = \frac{i\alpha}{2} \left( \nabla_a \bar{\psi} \gamma_b \psi - \bar{\psi} \gamma_b \nabla_a \psi \right) e^a_\mu e^b_\nu. \quad \text{(2.7)}$$

Since the Lagrangian density $L$ is covariant with respect to automorphisms of $\Sigma$, these transformations map solutions into solutions, leaving the field equations invariant. If $(e^\mu_a, \phi, \psi)$ is a solution of the field equations over the open set $U_\alpha$, then

$$(e^\mu_a, \phi, \psi) \sim (J^\nu_{\mu} e^\nu_b e^b_a(\varphi^{-1}), \phi, \varphi \cdot \psi) \quad \text{(2.8)}$$

is also a solution over the open set $f(U_\alpha)$ whatever automorphism $\Phi : \Sigma \to \Sigma$ projecting onto the diffeomorphism $f : M \to M$ is considered.

### 3 Exact solutions

Since the manifold $M$ is two-dimensional any metric $g$ can be locally cast into the conformal form

$$g = 4\rho^2(u, v) \cdot \frac{1}{2} (du \otimes dv + dv \otimes du), \quad \text{(3.1)}$$
where \( u \) and \( v \) are related to the timelike and spacelike coordinates by the relations 
\[
\begin{aligned}
u &= (t - x)/2 \\
v &= (t + x)/2.
\end{aligned}
\]

Relying on the covariance properties of the solutions we may choose a standard representative 
\( e^\mu_a \) in the class of spin frames inducing \( g \). Given the representative, a generic spin 
frame is of the form 
\[
\begin{aligned}
e^\prime_\mu_a &= e^\mu_a A^b_a \\
A^b_a &\in SO(1,1).
\end{aligned}
\]

The choice of a particular representative is a sort of gauge fixing that simplifies the field equations. A 
convenient choice is 
\[
\|e^\mu_a\| = e[\rho] = \begin{pmatrix} \rho & -\rho \\ \rho & \rho \end{pmatrix}, \quad \rho > 0.
\]

The general solution of the Dirac equation (2.4) can be written in the form (from now on 
we will consider \( \alpha > 0 \); the generalization to the case \( \alpha < 0 \) is straightforward and will be 
omitted)
\[
\psi[\rho, \psi_u, \psi_v] = \frac{1}{(\alpha \rho)^{1/2}} \left( \begin{array}{c} \psi_v \\ \psi_u \end{array} \right)
\]

where \( \psi_u(u) \) and \( \psi_v(v) \) are arbitrary functions of \( u \) and \( v \), respectively.

Using equation (3.3) in the field equations (2.5) and (2.6) we obtain
\[
\begin{aligned}
\partial_u \partial_v (\ln \rho^2) - \rho^2 \frac{dU}{d\phi} &= 0, \\
\partial_u \partial_v \phi - \rho^2 U(\phi) &= 0, \\
\rho^2 \partial_u \left( \frac{\partial_u \phi}{\rho^2} \right) &= F[\psi_u], \\
\rho^2 \partial_v \left( \frac{\partial_v \phi}{\rho^2} \right) &= G[\psi_v],
\end{aligned}
\]

where we have set
\[
\begin{aligned}
F[\psi_u] &= i |\psi_u|^2 \partial_u \left[ \ln \left( \frac{\psi^*_v}{\psi_u} \right) \right], \\
G[\psi_v] &= i |\psi_v|^2 \partial_v \left[ \ln \left( \frac{\psi^*_u}{\psi_v} \right) \right].
\end{aligned}
\]

Let us remark that the gauge fixing (3.2) is not covariant, so it destroys the explicit 
covariance of the field equations (2.4)-(2.6). However, the covariance of equations (2.4)-(2.6) 
can be restored \textit{a posteriori} by noticing that any solution of equations (3.4)-(3.7) 
identifies a whole class of physically equivalent solutions of equations (2.4)-(2.6). Indeed, let 
us consider the transformations induced by a vertical automorphism of \( \Sigma \) on 
\( (\epsilon[\rho], \psi[\rho, \psi_u, \psi_v], \phi) \), where \( \epsilon \) and \( \psi \) are defined in equations (3.2) and (3.3)
\[
(\epsilon, \psi, \phi) \mapsto (\epsilon', \psi', \phi).
\]
Here \( \epsilon' = \ell(S(u, v)) \cdot \epsilon, \psi' = S(u, v) \cdot \psi \) and

\[
S(u, v) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}
\]
(3.12)
is a generic matrix in \( \text{Spin}(1, 1) \). It is straightforward to see that \((\epsilon', \psi', \phi)\) is again a solution of equations (2.4)-(2.6) corresponding to the same metric \( g \) induced by \( \epsilon \) (see equation (A.4)).

An explicit invariance of equations (3.4)-(3.7) is given by the coordinate transformations

\[
u = u(\tilde{u}), \quad v = v(\tilde{v}).
\]
(3.13)

Under these transformations the conformal factor \( \rho^2(u, v) \) of the metric and the dilaton transform as (we consider without loss of generality \( (d\tilde{u}/du) > 0 \) and \( (d\tilde{v}/dv) > 0 \))

\[
\rho^2(u, v) = \rho^2(\tilde{u}, \tilde{v}) \frac{d\tilde{u}}{du} \frac{d\tilde{v}}{dv},
\]
(3.14)

\[
\psi(u, v) = S(\tilde{u}, \tilde{v}) \psi_u(\tilde{u}, \tilde{v}),
\]
(3.15)

\[
\phi(u, v) = \tilde{\phi}(\tilde{u}, \tilde{v}),
\]
(3.16)

where \( S \) is the two-dimensional \( \text{Spin}(1, 1) \) matrix

\[
S(\tilde{u}, \tilde{v}) = \begin{pmatrix} \left[ \frac{d\tilde{u}}{du} \right]^{-1/4} & \left[ \frac{d\tilde{v}}{dv} \right]^{1/4} \\ 0 & \left[ \frac{d\tilde{v}}{dv} \right]^{-1/4} \left[ \frac{d\tilde{u}}{du} \right]^{1/4} \end{pmatrix}.
\]
(3.17)
The field equations are explicitly conformally invariant under the transformations (3.14)-(3.16). Hence, the rotation of the spinor by the angle

\[
\theta = -\frac{1}{4} \ln \left( \frac{d\tilde{u}}{du} \right) + \frac{1}{4} \ln \left( \frac{d\tilde{v}}{dv} \right)
\]
(3.18)
preserves explicitly the conformal invariance of the theory. Finally, the functionals \( F[\psi_u] \) and \( G[\psi_v] \) transform as

\[
F[\psi_u] = \tilde{F}[\psi_\tilde{u}] \left( \frac{d\tilde{u}}{du} \right)^2, \quad G[\psi_v] = \tilde{G}[\psi_\tilde{v}] \left( \frac{d\tilde{v}}{dv} \right)^2,
\]
(3.19)

under the transformation rule (3.13).

In the following sections we will find and discuss several exact solutions for our model. The strategy is the following: starting from equations (3.4)-(3.7) we choose the functionals \( F[\psi_u] \) and \( G[\psi_v] \) in equations (3.6)-(3.7). (This is essentially equivalent to choosing a particular class of spinors for the model.) Then we solve the field equations (3.4)-(3.7) for a given choice of the potential \( U(\phi) \). This completes the integration of the model.
3.1 Solutions with constant-phase spinors

The simplest family of exact solutions is obtained when the functionals $F[\psi_u]$ and $G[\psi_v]$ are identically zero. In this case the metric and the dilaton decouple from spinors and the system formally reduces to pure DG. The most general class of spinors satisfying this condition is given by

$$
\psi_u(u) = \psi^*_u(u) e^{-i\chi_u}, \quad \psi_v(v) = \psi^*_v(v) e^{-i\chi_v},
$$

(3.20)

where $\chi_u$ and $\chi_v$ are real constant parameters.

Two-dimensional pure DG has been discussed extensively in the literature. So, without entering details, let us recall some interesting properties of the model that will be useful later. It is well known that two-dimensional pure DG can be completely integrated. A straightforward and powerful way to prove this can be found in [10]: the field equations can be transformed into linear equations via a Bäcklund-like transformation and the general solution can be written as a function of (two) free fields. A remarkable feature of this model is that the metric tensor and the dilaton $\phi$ are actually depending only on one of the two original free fields. As a consequence, the general solution is ‘static’, i.e. the metric and the dilaton can be cast into the form

$$
ds^2 = 4 f(u,v) du dv, \quad f(u,v) = h(\chi) \partial_u \chi \partial_v \chi, \quad \phi \equiv \phi(\chi),
$$

(3.21) (3.22)

where $\chi$ is a solution of the two-dimensional D’Alembert equation $\partial_u \partial_v \chi = 0$. (This property is nothing other that the generalized Birkhoff theorem for this class of models.) Let us stress that our definition of ‘staticity’ is slightly different form the usual definition that can be found in the literature (see, for instance, [16]). Here the concept of staticity refers to the fields, i.e. to the metric and the dilaton, and not to the geometry. In this context ‘staticity’ simply means that a Killing vector, not necessarily timelike and hypersurface orthogonal, does exist. Hence, for solutions of the form (3.21) and (3.22) we can always choose a system of coordinates such that the solution depends on a single coordinate, either the spacelike or the timelike one. For instance, the two-dimensional Schwarzschild metric $ds^2 = (1 - 2M/x) dt^2 - (1 - 2M/x)^{-1} dx^2$ is ‘static’ in the sense of (3.21) and (3.22) even though in the region $x < 2M$ the metric tensor depends on the timelike variable.

Let us make a short digression and discuss how the existence of solutions of the form (3.21) and (3.22) is related to the structure of the field equations. Let us consider equations (3.4)-(3.7). It is easy to prove the following theorem:

• Theorem. For any static solution (in the sense defined above) there is a set of coordinates $(\tilde{u}, \tilde{v})$ such that $\tilde{F}[\psi_\tilde{u}] = \tilde{G}[\psi_\tilde{v}] = constant$. In the coordinates $(\tilde{u}, \tilde{v})$ the metric reads $ds^2 = 4h(\tilde{u} + \tilde{v}) d\tilde{u} d\tilde{v}$.

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The proof is very simple. Let us consider a static solution. The latter can be cast into the form (3.21) and (3.22). The general solution of the D’Alembert equation in the lightcone coordinates can be written as a sum of an arbitrary function of $u$ and an arbitrary function of $v$. Setting $\chi = \tilde{u}(u) + \tilde{v}(v)$ and using the definition of staticity (3.21) and (3.22) with $f = \rho^2$ it is straightforward to verify that in the $(\tilde{u}, \tilde{v})$ coordinates equations (3.6) and (3.7) reduce to the form

$$h\left(\frac{\phi'}{h}\right)' = \tilde{F}[\psi_\tilde{u}], \quad h\left(\frac{\phi'}{h}\right)' = \tilde{G}[\psi_\tilde{v}],$$

(3.23)

where primes denote derivatives with respect to $\chi$. Finally, from equations (3.23) it follows immediately $\tilde{F} = \tilde{G} = constant$.

Hence, for static solutions we can always find a set of coordinates such that the right-hand side of the two constraints (3.6) and (3.7) are identical. In this case the field equations (3.4-3.7) can be reduced to a system of ordinary differential equations. Since for static solutions $F[\psi_\tilde{u}] = G[\psi_\tilde{v}] = constant$ is a necessary condition, the general static solution of the model can be found by imposing directly $F[\psi_\tilde{u}] = G[\psi_\tilde{v}] = constant$ in the field equations. Clearly, the above condition is satisfied in the case of pure dilaton-gravity when $F \neq 0$, $G \neq 0$ the Bäcklund-like transformation cannot be implemented and non-static solutions exist.

In conclusion, the model with constant-phase spinors behaves essentially like a pure DG system. The Hilbert tensor is identically zero and the spinor field is decoupled both from the dilaton and the gravitational field. Geometrically, for suitable choices of the dilaton potential $U(\phi)$ black-hole-like solutions can be found. For instance, when $U(\phi) \propto 1/\sqrt{\phi}$ the model can be interpreted as the effective two-dimensional reduction of vacuum spherically symmetric Einstein gravity. (The dilaton plays the role of the scale factor of the sphere.) As a consequence, the model can be physically interpreted as a black hole in a bath of massless non-interacting (ghost) spinors.

### 3.2 Solutions with constant dilatonic potential

Since constant-phase spinors do not interact with either the dilaton or with the gravitational field, let us look for a model in which the spinors have a non-vanishing Hilbert tensor. Let us change our strategy and choose a particular dilatonic potential, leaving undetermined the functionals $F[\psi_\tilde{u}]$ and $G[\psi_\tilde{v}]$, i.e. the form of the spinor (3.3). The simplest choice is the constant dilaton potential $U(\phi) = -\Lambda$. In this case the spacetime is flat and the system is completely solvable.

The field equation for the dilaton (3.4) and the dynamical Einstein equation (3.5) read

$$\partial_u \partial_v (\ln \rho) = 0,$$

(3.24)

$$\partial_u \partial_v \phi + \Lambda \rho^2 = 0.$$  

(3.25)
Since equation (3.24) does not involve the dilaton $\phi$ the geometry of spacetime is decoupled both from the dilaton and the spinorial field. So the first equation can be immediately solved and the general solution can be written in the convenient form

$$\rho^2(u, v) = \frac{d\tilde{u} \, d\tilde{v}}{du \, dv},$$

(3.26)

where $\tilde{u}$ and $\tilde{v}$ are arbitrary functions of $u$ and $v$, respectively. From equation (3.26) it is easy to see that $\tilde{u}$ and $\tilde{v}$ are the lightcone coordinates of the two-dimensional Minkowski spacetime. In the lightcone coordinates the remaining equations become

$$\partial_\tilde{u} \partial_\tilde{v} \phi + \Lambda = 0,$$

(3.27)

$$\partial^2_\tilde{u} \phi = \tilde{F}[\psi_\tilde{u}],$$

(3.28)

$$\partial^2_\tilde{v} \phi = \tilde{G}[\psi_\tilde{v}],$$

(3.29)

where $\tilde{F}$ and $\tilde{G}$ are related to $F$ and $G$ in the original $(u, v)$ variables by equations (3.19).

The metric and the spinor read

$$ds^2 = 4d\tilde{u}d\tilde{v}, \quad \psi(\tilde{u}, \tilde{v}) = \frac{1}{\sqrt{\alpha}} \left( \begin{array}{c} \psi_\tilde{v} \\ \psi_\tilde{u} \end{array} \right).$$

(3.30)

(This corresponds to set directly $\rho = 1$ in the field equations and use flat coordinates; the functions $\psi_\tilde{u}$ and $\psi_\tilde{v}$ define the spinor in the Minkowskian system of coordinates.)

Equations (3.27)-(3.29) can be easily solved. The general solution for arbitrary $\tilde{F}$ and $\tilde{G}$ is

$$\phi(\tilde{u}, \tilde{v}) = \phi_0 + \phi_\tilde{u} \tilde{u} + \phi_\tilde{v} \tilde{v} - \Lambda \tilde{u} \tilde{v} + U(\tilde{u}) + V(\tilde{v}),$$

(3.31)

where $\phi_0$, $\phi_\tilde{u}$, and $\phi_\tilde{v}$ are constants, and

$$U(\tilde{u}) = \int_{\tilde{u}_1}^{\tilde{u}} d\tilde{u}' \int_{\tilde{u}_2}^{\tilde{u}'} d\tilde{u}'' \tilde{F}[\psi_\tilde{u}''], \quad V(\tilde{v}) = \int_{\tilde{v}_1}^{\tilde{v}} d\tilde{v}' \int_{\tilde{v}_2}^{\tilde{v}'} d\tilde{v}'' \tilde{G}[\psi_\tilde{v}''].$$

(3.32)

Even though the general solution is known, it is interesting to derive the static solutions directly from the field equations. From the theorem of the previous section it follows that for static solutions there exists a set of coordinates $(\bar{u}, \bar{v})$ such that

$$F[\psi_{\bar{u}}] = G[\psi_{\bar{v}}] = \epsilon,$$

(3.33)

where $\epsilon$ is a real constant. In this system of coordinates the fields can be chosen to depend only on $x = \bar{v} - \bar{u}$. (Alternatively, on $t = \bar{v} + \bar{u}$.) So it is easy to integrate the field equations. We have two families of distinct solutions:

- **Solution (3.2a):**

$$ds^2 = 4\rho_0^2 e^{k(\bar{v} - \bar{u})} d\bar{u}d\bar{v},$$

(3.34)

$$\phi = \phi_0 - \frac{\epsilon}{k}(\bar{v} - \bar{u}) \pm \frac{\Lambda \rho_0^2}{k^2} e^{k(\bar{v} + \bar{u})}, \quad k \neq 0,$$

(3.35)

$$\psi = \frac{1}{(\alpha \rho_0)^{1/2}} e^{-k(\bar{v} + \bar{u})/4} \left( \begin{array}{c} \psi_{\bar{v}} \\ \psi_{\bar{u}} \end{array} \right);$$

(3.36)
• Solution (3.2b) (degenerate case, $\Lambda \neq 0$, $\epsilon \neq 0$):

$$ds^2 = \pm 4 \frac{\epsilon}{\Lambda} d\bar{u} d\bar{v}, \quad (3.37)$$

$$\phi = \phi_0 + \phi_1 (\bar{v} \mp \bar{u}) + \frac{\epsilon}{2} (\bar{v} \mp \bar{u})^2, \quad (3.38)$$

$$\psi = \left[ \pm \frac{(\Lambda/\epsilon)^{1/4}}{\sqrt{\alpha}} \right] \begin{pmatrix} \psi_{\bar{v}} \\ \psi_{\bar{u}} \end{pmatrix}, \quad (3.39)$$

where $\rho_0 > 0$, $k$, $\phi_0$, and $\phi_1$ are integration constants. Note that both solutions are describing a flat two-dimensional manifold, as expected for a constant dilaton potential (see equation (2.3)). However, the two solutions correspond to different choices of the spinor. This can be easily seen using Minkowskian coordinates and comparing the two solutions. In the Minkowskian coordinates $(\tilde{u}, \tilde{v})$ the dilaton and the spinor are

• Solution (3.2b):

$$\phi = \tilde{\phi}_0 - \Lambda \tilde{u} \tilde{v} - \tilde{\epsilon} \ln |\tilde{u} \tilde{v}|, \quad (3.40)$$

$$\psi = \frac{1}{\sqrt{\alpha}} \begin{pmatrix} \psi_{\tilde{v}} \\ \psi_{\tilde{u}} \end{pmatrix}, \quad (3.41)$$

where $\tilde{\phi}_0$ and $\tilde{\epsilon} = \epsilon / k^2$ are constants and $\psi_{\tilde{u}}$, $\psi_{\tilde{v}}$ are defined by the relations

$$\tilde{F}[\psi_{\tilde{u}}] = \frac{\tilde{\epsilon}}{\tilde{u}^2}, \quad \tilde{G}[\psi_{\tilde{v}}] = \frac{\tilde{\epsilon}}{\tilde{v}^2}; \quad (3.42)$$

• Solution (3.2b):

$$\phi = \phi_0 + \tilde{\phi}_1 (\tilde{v} \mp \tilde{u}) \pm \frac{\Lambda}{2} (\tilde{v} \mp \tilde{u})^2, \quad (3.43)$$

$$\psi = \frac{1}{\sqrt{\alpha}} \begin{pmatrix} \psi_{\tilde{v}} \\ \psi_{\tilde{u}} \end{pmatrix} \equiv \left[ \pm \frac{(\Lambda/\epsilon)^{1/4}}{\sqrt{\alpha}} \right] \begin{pmatrix} \psi_{\tilde{v}} \\ \psi_{\tilde{u}} \end{pmatrix}, \quad (3.44)$$

where $\tilde{\phi}_1$ is a constant and

$$\tilde{F}[\psi_{\tilde{u}}] = \tilde{G}[\psi_{\tilde{v}}] = \pm \Lambda. \quad (3.45)$$

In the case (3.2b) the spinor is chosen so that the two functionals $\tilde{F}$ and $\tilde{G}$ are equal to $\pm \Lambda$ in the flat system of coordinates; in the first case the spinors are chosen so that the functionals $\tilde{F}$ and $\tilde{G}$ are not constant in the Minkowskian coordinates. Note that the solution (3.2b) is singular for $\epsilon = 0$ and has no counterpart in the pure DG theory unless $\Lambda = 0$. (This is evident in the Minkowskian coordinates because $\tilde{F}$ and $\tilde{G}$ cannot be zero for $\Lambda \neq 0$, see equations (3.45).) Solution (3.2a) reduces to the pure DG solution when $\tilde{\epsilon} = 0$. Finally, both solutions can be obtained directly from (3.31) using (3.42) and (3.45).
To conclude the discussion of the solutions with constant dilatonic potential let us find a representation for the spinor. Let us consider solution (3.2b). Since the metric is flat plane-wave spinors are the natural choice. In the \((\tilde{u}, \tilde{v})\) coordinates plane-wave spinors are

\[
\psi = \frac{1}{2\sqrt{2\pi}} \left( e^{ip\tilde{u}} e^{ip\tilde{v}} \right),
\]

(3.46)

where \(p\alpha/(4\pi) = \pm\Lambda\). Now, let us consider solution (3.2a). In this case plane-wave spinors (3.46) are not compatible with equations (3.40) and (3.41) and equation (3.42) because the solution is singular for \(\tilde{u} = 0\) and \(\tilde{v} = 0\). We can choose instead

\[
\psi = \frac{1}{2\sqrt{2\pi}} \left( |\tilde{v}|^{\pm ip - 1/2} \right),
\]

(3.47)

where \(p\alpha/(4\pi) = \tilde{\epsilon}\) and signs correspond to \(\tilde{v} > 0, \tilde{v} < 0\) and \(\tilde{u} > 0, \tilde{u} < 0\), respectively. Spinors in equation (3.47) satisfy the conditions (3.42) and correspond (apart from a constant phase factor) to plane waves in the \((\tilde{u}, \tilde{v})\) system of coordinates.

3.3 Solutions with linear dilatonic potential

We have seen that for a constant dilatonic potential the model is completely solvable. However, the gravitational degree of freedom represented by the conformal factor \(\rho\) decouples dynamically both from the dilaton and the spinor field. As a consequence, the two-dimensional spacetime is simply the Minkowski spacetime and the model can be interpreted as a bath of interacting spinor and dilaton fields in a flat background. This is not surprising. Indeed, the curvature of the spacetime is proportional to the derivative of the dilatonic potential, as one can easily verify from the field equation (2.5).

So it is interesting to explore models with curved geometry and see how the presence of spinors modifies the structure of the spacetime. The simplest non-flat model is identified by the linear dilatonic potential \(U(\phi) = -\lambda\phi\). (In pure DG the model is known as ‘pure string-inspired dilaton-gravity’ or the ‘Jackiw-Teitelboim model’ and has been investigated extensively in the literature; see, for instance, [4]-[8] and references quoted therein.) Even though the dynamics of the spacetime remains decoupled from the dilaton and the spinor field (the manifold has constant curvature \(\lambda\)) the structure of the model is richer with respect to the flat case. For instance, pure string-inspired two-dimensional DG can be interpreted as the effective reduced theory of vacuum axisymmetric gravity plus cosmological constant in \(2 + 1\) dimensions (see, for example, [17]). It is well known that axisymmetric black holes of constant curvature do exist in \(2 + 1\) dimensions [18]. So the model with linear dilatonic potential can be interpreted as describing a bath of spinors in a black hole background and is worthwhile investigating.
Let us consider a linear dilatonic potential and restrict attention on static solutions. Solving the field equations we find three families of solutions (we consider for simplicity the fields depending on $x$. Static solutions depending on $t$ can be obtained by the substitution $x \to t$, $\lambda \to -\lambda$)

- Solution (3.3a):
  \[
  \rho^2(x) = \frac{2}{\lambda a^2} \frac{1}{P(x)^2}, \\
  \phi(x) = \phi_0 Q(x) + \frac{\epsilon a^2}{2} \left[ \frac{x}{a} Q(x) - 1 \right],
  \]
  where
  \[
  P(x) = \sinh \left( \frac{x - x_0}{a} \right), \quad Q(x) = \coth \left( \frac{x - x_0}{a} \right), \quad \text{for } \sigma = 1, \\
  P(x) = \cosh \left( \frac{x - x_0}{a} \right), \quad Q(x) = \tanh \left( \frac{x - x_0}{a} \right), \quad \text{for } \sigma = -1.
  \]

- Solution (3.3b):
  \[
  \rho^2(x) = \frac{2}{\lambda a^2} \frac{1}{\sin^2 \left( \frac{x - x_0}{a} \right)}, \\
  \phi(x) = \phi_0 \cot \left( \frac{x - x_0}{a} \right) - \frac{\epsilon a^2}{2} \left[ \frac{x}{a} \cot \left( \frac{x - x_0}{a} \right) - 1 \right].
  \]

- Solution (3.3c):
  \[
  \rho^2(x) = \frac{2}{\lambda (x - x_0)^2}, \\
  \phi(x) = \frac{\phi_0}{x - x_0} + \frac{\epsilon}{6} (x - x_0)^2;
  \]
  where $x_0$, $\phi_0$ and $a \neq 0$ are constants of integration and $\epsilon$ is defined as in the previous section.

It is straightforward to verify that the three solutions describe a two-dimensional space-time with constant Ricci scalar $R = -\lambda$, as expected from the dilaton field equation. However, the presence of matter fields, i.e. the dilaton and the spinor, breaks the isometry group of the de Sitter metric [19]. As a consequence, the three solutions are diffeomorphism inequivalent solutions of the field equations and correspond to different physical descriptions. Let us see this in detail and define the spacelike coordinate
  \[
  y = -\int \rho^2(x) dx.
  \]
In the \((t, y)\) coordinates the line element has the standard ‘Schwarzschild-like’ form

\[
    ds^2 = f(y)dt^2 - f(y)^{-1}dy^2, \quad f(y) = \frac{\lambda}{2} \left[ y^2 + \gamma \right],
\]

where

\[
    y = \frac{2}{\lambda a} \coth \frac{x - x_0}{a}, \quad \gamma = -\frac{4}{\lambda^2 a^2}, \quad (3.58)
\]

\[
    y = \frac{2}{\lambda a} \tanh \frac{x - x_0}{a}, \quad \gamma = -\frac{4}{\lambda^2 a^2}, \quad (3.59)
\]

for solutions (3.3a) and

\[
    y = \frac{2}{\lambda a} \cot \frac{x - x_0}{a}, \quad \gamma = \frac{4}{\lambda^2 a^2}, \quad (3.60)
\]

\[
    y = \frac{2}{\lambda (x - x_0)}, \quad \gamma = 0, \quad (3.61)
\]

for solutions (3.3b) and (3.3c), respectively. The Penrose diagrams for the above solutions can be found in [19]. The physical meaning of the solutions (3.3a)-(3.3c) is obvious: the three solutions describe the dilaton and the spinor field in different regions of the de Sitter spacetime separated by Killing horizons.

Solutions (3.3a) can be cast in a simpler and unique form using reparametrizations in \(u\) and \(v\). Setting for simplicity \(x_0 = 0\) we have

\[
    ds^2 = \frac{8}{\lambda (\tilde{v} - \tilde{u})^2} \tilde{d}u \tilde{d}v, \quad (3.62)
\]

\[
    \phi = \phi_0 \frac{\tilde{v} + \tilde{u}}{\tilde{v} - \tilde{u}} + \tilde{\epsilon} \left[ \frac{\tilde{v} + \tilde{u}}{\tilde{v} - \tilde{u}} \ln \left| \frac{\tilde{v}}{\tilde{u}} \right| - 2 \right], \quad (3.63)
\]

\[
    \psi = \left| \frac{\lambda}{2} \right|^{1/4} \sqrt{\frac{|\tilde{v} - \tilde{u}|}{\alpha}} \left( \psi_{\tilde{v}} \psi_{\tilde{u}} \right), \quad (3.64)
\]

where

\[
    u = \frac{a}{2} \ln |\tilde{u}|, \quad v = \frac{a}{2} \ln |\tilde{v}|, \quad (3.65)
\]

and \(\tilde{\epsilon} = \epsilon a^2/4\). The two choices (3.50) and (3.51) correspond to the regions \(\tilde{v} \tilde{u} > 0\) and \(\tilde{u} \tilde{v} < 0\), respectively. In the \((\tilde{u}, \tilde{v})\) coordinates plane-wave spinors read

\[
    \psi = \sqrt{\frac{|\tilde{v} - \tilde{u}|}{8\pi}} \left( \frac{\lambda}{2} \right)^{\pm i p - 1/2} \left( |\tilde{v}|^{\pm i p - 1/2} \right), \quad (3.66)
\]

where \(\tilde{\epsilon}\) is defined as a function of \(p\) as below equation (3.47) and signs correspond to \(\tilde{v} > 0\), \(\tilde{v} < 0\) and \(\tilde{u} > 0\), \(\tilde{u} < 0\), respectively.
In conclusion, the linear-potential model describes a spinor field interacting with the dilaton in a fixed background with constant curvature. The geometry of spacetime is dynamically decoupled both from the spinor and the dilaton. This happens both for solutions with a Killing vector and for general solutions because the curvature of spacetime is completely determined by the dilatonic potential. In the next section we will discuss a more interesting case: the exponential dilatonic potential.

### 3.4 Solutions with exponential dilatonic potential

Let us consider a potential of the form

$$U(\phi) = \lambda e^{k\phi}, \quad (3.67)$$

where $\lambda \neq 0$ and $k \neq 0$ are constants. The field equations can be solved exactly in the static sector. We find four distinct families of solutions (again we consider the fields depending on $x$):

- **Solution (3.4a):**
  $$\rho^2(x) = \frac{\gamma^2}{k\lambda \sinh[\gamma(x-x_0)]} e^{-k(\phi_0+ax)} ,$$
  $$\phi(x) = \phi_0 + ax - \frac{1}{k} \ln \{\sinh[\gamma(x-x_0)]\} ,$$
  $$\epsilon = ka^2 - \gamma^2/k ; \quad (3.70)$$

- **Solution (3.4b):**
  $$\rho^2(x) = -\frac{\gamma^2}{k\lambda \sinh[\gamma(x-x_0)]} e^{-k(\phi_0+ax)} ,$$
  $$\phi(x) = \phi_0 + ax - \frac{1}{k} \ln \{\sinh[\gamma(x-x_0)]\} ,$$
  $$\epsilon = ka^2 - \gamma^2/k ; \quad (3.73)$$

- **Solution (3.4c):**
  $$\rho^2(x) = -\frac{\gamma^2}{k\lambda \sin [\gamma(x-x_0)]} e^{-k(\phi_0+ax)} ,$$
  $$\phi(x) = \phi_0 + ax - \frac{1}{k} \ln \{\sin [\gamma(x-x_0)]\} ,$$
  $$\epsilon = ka^2 + \gamma^2/k ; \quad (3.76)$$
• Solution (3.4d):

\[
\rho^2(x) = \frac{1}{k} e^{-k\sqrt{\epsilon}(x-x_0)} \frac{1}{x-x_0},
\]

(3.77)

\[
\phi(x) = \frac{\sqrt{\epsilon}}{k}(x-x_0) - \frac{1}{k} \ln (x-x_0);
\]

(3.78)

where \(a, \gamma \neq 0\), and \(x_0\) are constants of integration.

It is straightforward to prove that the four families of solutions are actually distinct. This can be done by calculating the Ricci scalar. From the dilaton field equation (2.5) we have

\[
R = \frac{\lambda}{k} e^{k\phi}.
\]

(3.79)

So the Ricci scalar is proportional to the inverse of \(\cosh(\gamma(x-x_0)), \sinh(\gamma(x-x_0)), \sin(\gamma(x-x_0))\) and \((x-x_0)\) for the four solutions, respectively. In the first case the manifold is regular for any value of the spacelike coordinate \(x\). Conversely, for the solutions (3.4b)-(3.4d) the manifolds have singular points: \(x = x_0\) for solutions (3.4b) and (3.4d), and \(x = x_0 = n\pi/\gamma\) for the case (3.4c), where the dilaton field blows up. The discussion of the geometrical structure and of the global properties of these solutions is beyond the purpose of this paper and will be considered elsewhere. However, let us spend a few words on some particular cases. Solutions (3.4c) have no limit for \(\epsilon = 0\), i.e. they are not solutions of the pure DG theory without spinors. Conversely, solutions (3.4a), (3.4b), and (3.4d) admit a pure DG limit. Solutions (3.4a) and (3.4b) can be rewritten in a more interesting form using reparametrizations in \(u\) and \(v\). Using the coordinates

\[
u = \frac{1}{\gamma} \ln |\bar{u}|, \quad v = \frac{1}{\gamma} \ln |\bar{v}|,
\]

(3.80)

solutions (3.4a) and (3.4b) read

• Solution (3.4a):

\[
\begin{align*}
ds^2 &= \pm \frac{8}{\lambda k} e^{-k\phi_0} |\bar{u}|^\kappa \frac{d\bar{u}d\bar{v}}{|\bar{v}^2 + \bar{u}^2|}, \\
\phi &= \phi_0 + \frac{1}{k} \ln \left[ \frac{|\bar{v}|^\kappa}{\bar{v}^2 + \bar{u}^2} \right].
\end{align*}
\]

(3.81)

(3.82)

• Solution (3.4b):

\[
\begin{align*}
ds^2 &= \pm \frac{8}{\lambda k} e^{-k\phi_0} |\bar{u}|^\kappa \frac{d\bar{u}d\bar{v}}{|\bar{v}^2 - \bar{u}^2|}, \\
\phi &= \phi_0 + \frac{1}{k} \ln \left[ \frac{|\bar{v}|^\kappa}{\bar{v}^2 - \bar{u}^2} \right].
\end{align*}
\]

(3.83)

(3.84)
where the upper signs refer to the I and IV quadrants of the $(\bar{u}, \bar{v})$ plane, the lower signs refer to the II and III quadrants, and $\kappa = \pm \sqrt{1 + \epsilon k/\gamma^2}$. Note that $\kappa = \pm 1$ for the pure DG case. The Ricci scalar in the new coordinates is (the signs refer to solutions (3.4a) and (3.4b), respectively):

$$R = \lambda k e^{k\phi_0} |\bar{v}|^{\kappa} \frac{2|\bar{u}\bar{v}|}{\bar{v}^2 \pm \bar{u}^2}.$$  (3.85)

From equation (3.85) we see that the geometrical structure of the solutions depends on the value of the parameters $\kappa$ and $k$. First of all, the sign of the curvature is constant for fixed $\lambda$ and $k$, as expected from equation (3.79). Furthermore, when $\kappa = \pm 1$, i.e. when there are no spinors, the spacetime (3.4a) is never singular and the spacetime (3.4b) is singular only for $\bar{v}^2 - \bar{u}^2 = 0$. This picture changes drastically if $\kappa \neq 1$. In this case the spacetime is singular on $\bar{u} = 0 (\kappa > 1)$ and $\bar{v} = 0 (\kappa < -1)$. When spinors are present the dilaton field blows up for $\bar{u} = 0$ and $\bar{v} = 0$. In pure DG this happens only for $\bar{u} = 0$ or $\bar{v} = 0$, depending on the $\kappa$-branch chosen.

### 4 Conclusions and perspectives

In this paper we have presented and discussed some exact solutions of two-dimensional DG coupled to massless spinors. In our formalism the gravitational field is described by new variables, the so-called spin frames, related to spin structures on the spacetime manifold and defined without any reference to any preferred background metric. The theory is geometrically well defined on spin manifolds and describes gravity dynamically coupled to spinorial fields. Hence, the formalism describes both the effect of gravity on spinors and the effect of spinors on the gravitational field for a large class of physical manifolds.

We have focused attention on three physically interesting choices of the dilatonic potential: constant, linear and exponential potentials. In the first case the theory is completely solvable and represents a remarkable new example in the family of two-dimensional DMG integrable systems. We have integrated locally the field equations and discussed briefly the properties of the solutions. Even though in this simple model the spacetime is flat and there are no black hole solutions, several interesting conclusions can be drawn. Indeed, the presence of the spinor field changes dramatically the structure of the solutions with respect to the pure DG case. For instance, the Birkhoff theorem is no longer valid and solutions without any symmetries appear. This important property seems not to depend on the particular model chosen, i.e. on the dimension of the spacetime and on the form of the dilatonic potential. Since two-dimensional DG theories can be interpreted, for suitable choices of the dilatonic potential, as effective dimensionally reduced models of gravity in $N > 2$ dimensions, the presence of spinors may have interesting consequences on the physics of black holes in higher dimensions. So the investigation of effective models describing three- or four-dimensional black holes interacting with the spinor field is worthwhile investigating and may constitute a fruitful subject for future research. A small step in this direction has been done in section 16.
where we have discussed the linear dilatonic potential. Indeed, the Jackiw-Teitelboim model can be interpreted as the effective reduced theory of vacuum axisymmetric gravity plus cosmological constant in $2 + 1$ dimensions. Since axisymmetric black holes of constant curvature do exist in $2 + 1$ dimensions \cite{18}, the model describes a bath of spinors in a black hole background.

Furthermore, the Hamiltonian formulation and quantization of the theory are subjects worth to be explored. In particular, the quantization of pure DG with constant dilatonic potential has been studied in detail in the literature \cite{4}-\cite{8}. (The main results of these investigations are summarized in the report \cite{9}.) Since the presence of spinors does not destroy the complete solvability of the system, the quantization of SDG with constant potential might not be very different from the quantization of the CGHS model \cite{20} and the well known techniques developed for the CGHS model might be successfully used for SDG.

Finally, let us mention that an alternative viewpoint on topological two-dimensional gravity has been recently suggested by one of the authors (in collaboration with Volovich, see \cite{21, 22}) on the basis of a metric-affine formalism (or first-order formalism à la Palatini). It is well known that in two dimensions the metric and the metric-affine formalisms are not equivalent (see also \cite{23, 24}). In the future we aim to discuss the application of spin frames to this alternative formalism.

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Appendix: The Formalism of Spin Frames

In this appendix, we briefly outline the spin frame formalism (see also \cite{11, 12}) and set up notations.

Let us consider a two-dimensional, connected, orientable manifold $M$ that allows metrics of signature $(1, 1)$. Since $\dim(M) = N = 2$, the relevant groups are $O(1) \simeq \mathbb{Z}_2$, $SO(1, 1) \simeq \mathbb{R}$, and $\text{Spin}(1, 1) \simeq \mathbb{R}$ which is embedded into $\mathbb{C}^+(1, 1) \simeq \mathbb{R}^2$ as a branch of hyperbola. Due to the choice $\dim(M) = N = 2$, the group covering $\ell : \text{Spin}(1, 1) \to SO(1, 1)$ is trivial and one-to-one when we restrict ourselves to the connected components of unity, as we have always understood. (For the general formalism in $N > 2$ dimensions, see e.g. \cite{12}.)

This topological requirement ensures that we may consider reductions of the tangent bundle $TM$ to the structure group $O(1) \times O(1) \subset \text{GL}(2)$ or, equivalently, that there exists
a splitting sequence of the form
\[ 0 \longrightarrow T \overset{i_+}{\longrightarrow} TM \overset{i_-}{\longrightarrow} S \longrightarrow 0, \]  
(A.1)
where \( T \) and \( S \) are two line bundles over \( M \).

The manifold \( M \) is interpreted as the spacetime. One should also require \( M \) to be a spin manifold, i.e. to have vanishing second Stiefel-Whitney class \( w_2(M, \mathbb{Z}_2) = 0 \). This is again a topological condition on \( M \). Of course, since in two dimensions any orientable manifold is a spin manifold, we do not impose any further restriction on \( M \).

Let us consider a Spin\((1, 1)\)-principal bundle \( \Sigma \) over \( M \), i.e. a principal \( \mathbb{R} \)-bundle. A spin frame on \( \Sigma \) is a global vertical principal morphism \( \Lambda : \Sigma \rightarrow L(M) \) of \( \Sigma \) in the frame bundle \( L(M) \); namely, the following diagrams commute:
\[
\begin{array}{c}
\Sigma \xrightarrow{\Lambda} L(M) \\
\downarrow \ \\
M \\
\end{array}
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
\] (A.2)

where \( R \) denotes the canonical right action defined on any principal bundles. Here \( \hat{\ell} = i \circ \ell \), \( i : SO(1, 1) \rightarrow GL(2) \) is the canonical immersion and \( \ell : Spin(1, 1) \rightarrow SO(1, 1) \) is the covering map.

A spin principal bundle \( \Sigma \) is called structure bundle for \( M \) if at least one spin frame on \( \Sigma \) does exist. Of course, there are spin principal bundles that are not structure bundles for \( M \); however, since \( M \) is a spin manifold there is at least one structure bundle for \( M \).

Let us consider now a structure bundle \( \Sigma \) for \( M \) and a spin frame \( \Lambda \) on it. A SO\((1, 1)\)-principal subbundle \( \text{Im}(\Lambda) \) is induced in \( L(M) \) and, according to [12], uniquely identifies a metric \( g(\Lambda) \) of signature \((1, 1)\) called the metric induced by \( \Lambda \) (see below equation (A.4)). Let us remark that \( \Lambda \) is onto the special orthonormal frame bundle \( SO(M, g(\Lambda)) \equiv \text{Im}(\Lambda) \) and is a covering. Thus \( (\Sigma, \Lambda) \) is a spin structure on \( (M, g(\Lambda)) \) in the sense of [13, 14]. Conversely, if \( (\Sigma, \tilde{\Lambda}) \) is a spin structure on \( (M, g) \), by composing \( \tilde{\Lambda} : \Sigma \rightarrow SO(M, g) \) with the canonical immersion \( i_g : SO(M, g) \rightarrow L(M) \) we obtain \( \Lambda_g = i_g \circ \tilde{\Lambda} \), which is a spin frame on \( \Sigma \).

Since spin frames induce a metric uniquely (but not vice versa) they represent natural candidates as dynamical fields to describe gravity. However, in order to be interpreted as fields, they should have some value at any point of \( M \). In other words, spin frames on \( \Sigma \) must be sections of some bundle over \( M \). This can be easily achieved by considering the associated bundle \( \Sigma_\rho = (\Sigma \times_M L(M)) \times_\rho GL(2) \) through the action
\[
\rho : \text{Spin}(1, 1) \times GL(2) \times GL(2) \rightarrow GL(2) : (S, J, e) \mapsto J \cdot e \cdot \ell(S^{-1}). \]  
(A.3)

Finally, one may also show that there is a one-to-one correspondence between spin frames on \( \Sigma \) and sections of the spin frame bundle \( \Sigma_\rho \).
Points in $\Sigma_\rho$ are represented by $[\sigma, \partial, e^\mu_a]_\rho$, where $[.]_\rho$ denotes the orbit under the action, $\sigma \in \Sigma$, $\partial \in L(M)$, and $e_\rho^a \in \text{GL}(2)$; thus $(x^\mu, e_\rho^a)$ are local coordinates on $\Sigma_\rho$. Since there is no preferred choice of a unit point, $\Sigma_\rho$ is neither a principal bundle nor a group bundle, even though it has fibres that are diffeomorphic to the group $\text{GL}(2)$.

The construction of the induced metric presented above provides a canonical global epimorphism $g_\Sigma : \Sigma_\rho \to \text{Met}(M; 1, 1)$, where $\text{Met}(M; 1, 1)$ is the bundle of all metrics of signature $(1, 1)$ on $M$; this morphism is called inducing metric morphism and is locally given by
\[
g_\Sigma(x^\mu, e_\mu^a) = \left(x^\mu, g^a_{\mu \nu} \right),
\]
where $e^a_\mu$ is the inverse matrix of $e_\rho^a$ and $\eta_{ab}$ is the canonical diagonal matrix of signature $(1, 1)$. Latin indices are lowered and raised by $\eta_{ab}$ and Greek indices by the induced metric $g_{\mu \nu}$; thus one can write simply $e_\mu^a$ regardless if the latter is given by $\eta_{ab} e_\mu^b$ or $e_\nu^b g_{\mu \nu}$ since these two expressions coincide.

Of course, metric actions written on $\text{Met}(M; 1, 1)$ can be canonically pulled-back on $\Sigma_\rho$ along the inducing metric morphism. If the metric Lagrangian is generally covariant then the pull-back Lagrangian on $\Sigma_\rho$ is covariant with respect to the group $\text{Aut}(\Sigma)$ of any (not necessarily vertical) automorphism of $\Sigma$. This group acts on $\Sigma_\rho$ as
\[
\Phi \in \text{Aut}(\Sigma) \quad \mapsto \quad \Phi_\rho : \Sigma_\rho \to \Sigma_\rho : [\sigma, \partial, e_\rho^a]_\rho \mapsto [\Phi(\sigma), L_f(\partial), e_\rho^a]_\rho,
\]
where $L_f : L(M) \to L(M)$ is the natural lift to the frame bundle of the diffeomorphism $f : M \to M$ induced on $M$ by $\Phi$. Locally we have
\[
\Phi(x, S) = (f(x), \varphi(x) \cdot S) \quad \mapsto \quad \Phi_\rho(x, e_\rho^a) = (f(x), J_\rho^\mu(x) e_\rho^a J_\rho^{\nu \ell}(\varphi^{-1}(x)))
\]
\[
\varphi : U \subset M \to \text{Spin}(1, 1), \quad J_\rho^\mu(x) = \frac{\partial f^\mu}{\partial x^\rho}(x).
\]

Equation (A.6) shows that spin frames behave in a very different way from standard frames with respect to symmetries (i.e. transformation laws). Indeed, the transformation laws of a frame with respect to a diffeomorphism $f : M \to M$ is
\[
e_\rho^\mu_a(x') = J_\rho^\nu e_\rho^a(x), \quad x' = f(x), \quad J_\rho^\mu = \partial_{\nu} f^\mu.
\]
These two different behaviours account for different Lie derivatives used in literature and then for different conserved quantities. Furthermore, they also account for different ways of gluing patches together, so they may be relevant for exact solutions too.

In order to consider interactions with spinors we choose a Spin$(1, 1)$-representation $\lambda$ on a suitable vector space $V$. For instance, in two dimensions we can choose $V = \mathcal{F}^2$ and the representation induced by the two-dimensional Dirac matrices
\[
\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]
Let us consider the associated spinor bundle $\Sigma_\lambda = \Sigma \times_\lambda V$ whose sections are, by definition, spinor fields on $M$. A point of $\Sigma_\lambda$ is given by $[\sigma, \psi]$, where $[\cdot]_\lambda$ denotes the orbit, $\sigma \in \Sigma$, and $\psi \in V$; the local coordinates are $(x^\mu, \psi^i)$. Again we have an action of $\text{Aut}(\Sigma)$ on $\Sigma_\lambda$ given by

$$\Phi_\lambda[\sigma, \phi]_\lambda = [\Phi(\sigma), \psi]_\lambda, \quad \Phi_\lambda(x^\mu, \psi^i) = (f^\mu(x), \lambda^i(\phi(x))\psi^i).$$  \hspace{1cm} (A.9)

Finally, dilatons (scalar fields) are sections of the trivial bundle $\Delta = M \times \mathcal{C}$. On $\Delta$ we have a (trivial) action of $\text{Aut}(\Sigma)$ given by

$$\Phi_\Delta(x, \phi) = (f(x), \phi).$$  \hspace{1cm} (A.10)

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