The Lamb System with a Nonuniform String

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Abstract

The propagator method is applied to investigate the scattering problem in a taut string with discontinues mass density interacting with a harmonic oscillator. The impedance of string is obtained in terms of its Green’s function and then is used to explain its effect on the time evolution of oscillator. The Green’s function of whole system (string + oscillator) is obtained to investigate the response of system to an impulse.

Keywords: Lamb model; nonuniform string; Green’s function; input impedance.

1 Introduction

The Lamb system, originally introduced by H. Lamb in 1900, is a simple setup made of an oscillator coupled to an infinite taut string [1]. It is a simple and useful model to realize the radiation damping. When the oscillator begins to oscillate, its energy diminishes by the waves emitted along the string, which in turn, results in a dissipative force on the motion of oscillator, similar to the phenomenon that takes place in classical electrodynamics when a charged oscillator loses energy by emitting the electromagnetic waves.

The original Lamb model is linear, i.e. the oscillator is harmonic. A nonlinear oscillator leads to chaotic reflected and transmitted waves [2]. In a model with two nonlinear oscillators, the reciprocity is lost, which means that the same wave transmits differently in two opposite directions [3]. The long time asymptotics for an anharmonic oscillator are obtained in [4] where the finite energy solution decays to a sum of dispersive and stationary waves. A similar setup with massless case is studied in [5]. The lamb system with finite and infinite domains and dispersive medium is investigated in [6]. The main achievement of [6] is that in a dispersive medium with periodic boundary condition, the Lamb oscillator generates fractal solution provided that the medium posses an asymptotically sub-linear dispersion relation. In an unforced medium fractalization takes place only if the dispersion relation grows super-linearly. The problem of instability in Lamb system and its gyroscopic variant in n-dimensions is studied in [7]. It is found that the transfer of energy from the gyroscopic Lamb oscillator to the wave field leads to an instability. Then, inclusion of the dispersion term results in a stabilization due to a shift in the potential energy of the oscillator.
The Lamb problem, indeed, could be considered as a prototype for the general problem of wave-particle interaction. The dynamics of a particle coupled to a general wave field is investigated from a wider perspective in [8]. The scattering problem in a Lamb system with a uniform string and a harmonic oscillator is presented by some textbooks, where the equation of motion of the oscillator, i.e. equation (50), is employed to achieve the scattering coefficients [9, 10].

The emphasis put on the use of Green’s function together with a nonuniform model for the string, are main aspects of the present work. This paper consists of 6 sections. In second section, the equation of motion for a string with an arbitrary distribution of mass interacting with an oscillator is obtained, starting from the Lagrangian density of the problem. In section 3, we establish the appropriate Green’s function for a string made of two semi-infinite strings with different densities \( \rho_+ \). We then apply the propagator method to obtain the reflection and transmission of an incoming wave scattered off the oscillator. In section 4, it is demonstrated how the dynamics of oscillator is affected by the impedance of string due to presence of a dissipation term. The Green’s function of string obtained in the section 3, is used to calculate the impedance. It is found that the total impedance is a sum of the impedances of the two semi-infinite strings joined together at \( x = 0 \). In section 5, response of the system to an impulse delivered to the oscillator, is obtained by evaluating the Green’s function of the whole system. Finally, a short conclusion and a future research direction is included in section 6.

2 A string interacting with an oscillator

We consider an infinite taut string with a constant tension \( T \) and a nonuniform mass density \( \mu(x) \). The field \( \phi(x,t) \) denotes the displacement of string from the equilibrium. The Lagrangian density for the displacement field is given by

\[
\mathcal{L} = \frac{1}{2} \mu(x) \left( \frac{\partial \phi(x,t)}{\partial t} \right)^2 - \frac{1}{2} T \left( \frac{\partial \phi(x,t)}{\partial x} \right)^2 + \phi(x,t) F(x,t) + \mathcal{L}_{\text{int}}. \tag{1}
\]

Here \( F(x,t) \) is an external force acting on the string and \( \mathcal{L}_{\text{int}} \) denotes interaction of string with its environment. Inserting the above Lagrangian into the Euler-Lagrange equation

\[
\frac{\partial}{\partial x} \left( \frac{\partial \mathcal{L}}{\partial \phi} \right) - \frac{\partial \mathcal{L}}{\partial t} \frac{\partial \phi}{\partial t} - \frac{\partial \mathcal{L}_{\text{int}}}{\partial \phi} = 0, \tag{2}
\]

yields the equation of motion

\[
\mu(x) \frac{\partial^2 \phi(x,t)}{\partial t^2} - T \frac{\partial^2 \phi(x,t)}{\partial x^2} - \frac{\partial \mathcal{L}_{\text{int}}}{\partial \phi} = F(x,t). \tag{3}
\]

For a harmonic oscillator with mass \( m \) attached to string at \( x = 0 \), the mass density and interaction Lagrangian become

\[
\mu(x) = \rho(x) + m \delta(x), \tag{4}
\]

\[
\mathcal{L}_{\text{int}} = -\frac{1}{2} \kappa \phi^2(x,t) \delta(x). \tag{5}
\]
where $\kappa = m\omega_0^2$ is stiffness of the spring and $\rho(x)$ is mass density of the free string. Thus, (4) and (5) modifies (3) to [5, 6]

$$G^{-1}\phi(x, t) = \frac{1}{T}F(x, t),$$

with

$$G^{-1} = \left(\frac{1}{\mu^2(x)}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) + \frac{1}{T}\delta(x)\left(\frac{m}{\mu^2} + \kappa^2\right).$$

where the speed of wave is denoted by $v = \left(\frac{T}{\rho}\right)^{1/2}$. A formal solution of equation (6) is given by

$$\phi(x, t) = \frac{1}{T} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dt' G(x; x', t')F(x', t'), \quad t' < t.$$  \hspace{1cm} (8)

where $G(x; x', t')$ is the retarded Green’s function or propagator of the system. For this propagator we have

$$G^{-1}G(x; x', t') = \delta(t - t')\delta(x - x').$$  \hspace{1cm} (9)

and

$$G(x; x', t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} g(x, x'; \omega).$$  \hspace{1cm} (10)

with $g(x, x'; \omega)$ as its Fourier transform. Then, inserting (10) into (9) reveals that

$$\left(\frac{\partial^2}{\partial x^2} + k^2(x) + V(x)\right)g(x, x'; k) = -\delta(x - x'), \quad k(x) = \frac{\omega}{v(x)},$$  \hspace{1cm} (11)

where the potential is

$$V(x) = \frac{\lambda}{T}\delta(x), \quad \lambda(\omega) = m(\omega^2 - \omega_0^2).$$  \hspace{1cm} (12)

In absence of the potential term, the Green’s function $g(x, x'; k)$ reduces to the Green’s function of the free string $g_0(x, x'; k)$ for which we have

$$\left(\frac{\partial^2}{\partial x^2} + k^2(x)\right)g_0(x, x'; k) = -\delta(x - x').$$  \hspace{1cm} (13)

3 Propagator approach to scattering problem

The scattering problem deals with determination of the reflection and transmission coefficients of an incoming wave scattered from the oscillator [9, 10]. We derive these coefficients with an approach based on Green’s function [11]. An elementary, yet straightforward method is provided in appendix A. For a time-harmonic solution $\phi(x, t) = \varphi(x)e^{-i\omega t}$, equation (6) reduces to

$$\left(\frac{d^2}{dx^2} + k^2(x) + V(x)\right)\varphi(x) = 0.$$  \hspace{1cm} (14)

where we have discarded the external force. In propagator method, for any localized potential $V(x)$, the solution of equation (14) is expressed in terms of the Green’s function of string as [11]

$$\varphi(x) = \int_{-\infty}^{\infty} dx' g_0(x, x')V(x') + \left[ g_0(x, x') \frac{\partial \varphi(x')}{\partial x'} - \varphi(x') \frac{\partial g_0(x, x')}{\partial x'} \right]_{x' = -\infty}^{x' = \infty}.$$  \hspace{1cm} (15)
Figure 1: An infinite string consisting of two different semi-infinite strings connected at \( x = 0 \). An oscillator made of a mass and a spring (dashed line) is coupled to the string at the joining point \( x = 0 \).

Now, we assume an infinite string made of two semi-infinite strings joined together at \( x = 0 \) (Fig.1). The semi-infinite strings have the mass densities \( \rho_{\pm} \). For such a string with discontinuous density, one can write

\[
\rho(x) = \rho_- \Theta(-x) + \rho_+ \Theta(x),
\]

which, by virtue of \( k \sim \rho \), gives rise to

\[
k^2(x) = k_-^2 \Theta(-x) + k_+^2 \Theta(x), \quad k_{\pm} = \frac{\omega}{v_{\pm}},
\]

where the Heaviside step function is defined as

\[
\Theta(x) = \begin{cases} 
1, & 0 < x, \\
\frac{1}{2}, & x = 0, \\
0, & x < 0.
\end{cases}
\]

Therefore, on substituting (17) into (14) and (13), we get

\[
\left( \frac{d^2}{dx^2} + k_-^2 \Theta(-x) + k_+^2 \Theta(x) \right) \varphi(x) + V(x)\varphi(x) = 0,
\]

and

\[
\left( \frac{\partial^2}{\partial x^2} + k_-^2 \Theta(-x) + k_+^2 \Theta(x) \right) g_0(x; x') = -\delta(x - x').
\]

The next step is to construct the Green’s function. The form of Green’s function depends on the position of source. Therefore, we write

\[
g_0(x, x'; k) = g_-(x, x'; k)\Theta(-x') + g_+(x, x'; k)\Theta(x').
\]

We denote \( g_- \) and \( g_+ \) as the corresponding Green’s functions for a source located in negative \((x' < 0)\) and positive \((0 < x')\) domains of \( X\)-axis, respectively. To derive the explicit form of \( g_{\pm}(x, x'; k) \), we begin with a source positioned in the negative region. Hence, we can have

\[
g_-(x, x'; k) = C_-(x, x'; k)\Theta(-x) + C_+(x, x'; k)\Theta(x).
\]
On substituting (22) into (20), we find
\begin{equation}
\left( \frac{\partial^2}{\partial x^2} + k^2 \right) C_+(x, x'; k) = -\delta(x - x'), \quad 0 < x \\
\quad = 0,
\end{equation}
\begin{equation}
\left( \frac{\partial^2}{\partial x^2} + k^2 \right) C_-(x, x'; k) = -\delta(x - x'), \quad x < 0
\end{equation}
\begin{equation}
\frac{\partial}{\partial x} C_+(0, x'; k) = \frac{\partial}{\partial x} C_-(0, x'; k).
\end{equation}

The set of equations (A.3)-(A.6) are invoked to obtain the above three equations. The Dirac delta in (23) is set to zero because for this term we have always \( x \neq x' \). We add the continuity of Green’s function at \( x = 0 \), to the set of above formulas
\begin{equation}
C_-(0, x'; k) = C_+(0, x'; k).
\end{equation}

Now, we follow the well-known formula to construct the Green’s function [11]
\begin{equation}
g(x, x'; k) = \frac{\varphi_>(x)\varphi_<(x')}{W(x)}\Theta(x - x') + \frac{\varphi_>(x')\varphi_<(x)}{W(x)}\Theta(x' - x),
\end{equation}
where the Wronskian is given by
\begin{equation}
W(x) = \varphi_>(x)\frac{d\varphi_<(x)}{dx} - \varphi_<(x)\frac{d\varphi_>(x)}{dx}.
\end{equation}
The functions \( \varphi_<(x) \) and \( \varphi_>(x) \) are linearly independent solutions of the corresponding wave equation. For \( C_-(x, x'; k) \), two such a solutions are
\begin{equation}
\varphi_<(x) = e^{-ik_-x} + A_Re^{-ik_-x}, \quad x < x' < 0,
\end{equation}
\begin{equation}
\varphi_>(x) = e^{ik_-x} + A_Re^{-ik_-x}, \quad x' < x < 0.
\end{equation}
These terms yield the corresponding Wronskian
\begin{equation}
W = -2ik_-(1 + A_R),
\end{equation}
and the Green’s function
\begin{equation}
C_-(x, x'; k) = \frac{i}{2k_-} \left( e^{ik_-|x-x'|} + A_Re^{-ik_-x+x'} \right),
\end{equation}
according to (27). To obtain \( g + (x, x'; k) \), taking into account (23), we must have
\begin{equation}
C_+(x, x'; k) \sim e^{ik+x}.
\end{equation}
Then, upon applying the continuity condition (26), we arrive at
\begin{equation}
C_+(x, x'; k) = \frac{i}{2k_-} A_Re^{ik_x-x_-ix}.
\end{equation}
Therefore, from (34), (32) and (22) we find the expression
\begin{equation}
g_-(x, x'; k) = \frac{i}{2k_-} \left( e^{ik_-|x-x'|} + \frac{k_- - k_+}{k_+ + k_-} e^{-ik_-x+x'} \right) \Theta(-x) + \left( \frac{i}{k_- + k_+} e^{ik_+x-x_-ix} \right) \Theta(x).
\end{equation}
Similarly, by putting the source in region $0 < x'$, writing down the homogenous solutions and using (27), one obtains
\[
g_+ (x, x'; k) = \left( \frac{i}{k_- + k_+} e^{ik_+ x' - ik_- x} \right) \Theta(-x) + \frac{i}{2k_+} \left( e^{ik_+ |x-x'|} - \frac{k_- - k_+}{k_+ + k_-} e^{ik_+ (x+x')} \right) \Theta(x). \tag{36}
\]
These two expressions together with the formula (21), determine the Green’s function of the string, from which we immediately obtain
\[
g_0 (x, 0; k) = \frac{1}{2} g_- (x, 0; k) + \frac{1}{2} g_+ (x, 0; k) = \frac{i}{k_+ + k_-} \left[ e^{-ik_- x} \Theta(-x) + e^{ik_+ x} \Theta(x) \right]. \tag{37}
\]
Having determined the Green’s function, by means of (15), we arrive at
\[
\varphi(x) = \frac{\lambda}{T} g_0 (x, 0) \varphi(0) + A_T e^{ik_+ x} \Theta(x) + \left( e^{ik_- x} + A_R e^{-ik_- x} \right) \Theta(-x). \tag{38}
\]
The second and third terms arise from the boundary term and can be realized in the following way: at distances far from the origin, the effect of oscillator can be ignored and the wave function obeys
\[
\lim_{x \to \infty} \varphi(x) \approx A_T e^{ik_+ x}, \tag{39}
\]
\[
\lim_{x \to -\infty} \varphi(x) \approx e^{ik_- x} + A_R e^{-ik_- x}. \tag{40}
\]
A little algebra reveals that here is no contribution from the first term (39), but (40) leads to an expression for the boundary term of the form
\[
\left[ g_0 (x, x') \frac{\partial \varphi(x')}{\partial x'} - \varphi(x') \frac{\partial g_0 (x, x')}{\partial x'} \right]_{x'=\infty} - \infty = A_T e^{ik_+ x} \Theta(x) + \left( e^{ik_- x} + A_R e^{-ik_- x} \right) \Theta(-x). \tag{41}
\]
To get the final form of the wave function, we set $x = 0$ on the right-hand side of (38) and use (37). We obtain
\[
\varphi(0) = \frac{2k_-}{k_+ + k_- + i\frac{\lambda}{T}}, \tag{42}
\]
which when inserted into (38), yields
\[
\varphi(x) = A_T e^{ik_+ x} \Theta(x) + \left( e^{ik_- x} + A_R e^{-ik_- x} \right) \Theta(-x), \tag{43}
\]
with the scattering coefficients given by
\[
A_R = \frac{k_- - k_+ + i\frac{\lambda}{T}}{k_+ + k_- - i\frac{\lambda}{T}}, \tag{44}
\]
\[
A_T = \frac{2k_-}{k_+ + k_- - i\frac{\lambda}{T}}. \tag{45}
\]
These relations indicate the effects of oscillator and density discontinuity on the scattering of the incoming wave, simultaneously. For a uniform string with $k_- = k_+ = k$, there is only a contribution from the oscillator and we recover the textbook result [9, 10]
\[
A_R = \frac{i\frac{\lambda}{T}}{2k - i\frac{\lambda}{T}}, \tag{46}
\]
\[
A_T = \frac{2k}{2k - i\frac{\lambda}{T}}. \tag{47}
\]
When the oscillator is absent, or the frequency of the incoming wave equals the natural frequency of the oscillator $\omega_0$, we have $\lambda = 0$. In these cases, equations (44) and (45) reduce to

$$A_R = \frac{k_- - k_+}{k_+ + k_-}, \quad (48)$$
$$A_T = \frac{2k_-}{k_+ + k_-}. \quad (49)$$

implying that there is only a scattering due to the discontinuity of density at $x = 0$.

4 Radiation impedance

The back-reaction of string on the motion of oscillator arises as an impedance-related dissipative term. To see how this happens, let us integrate both sides of (6) over an infinitesimal interval including the oscillator. So, we obtain [1-7, 9, 10]

$$\left( \frac{m}{2}\frac{d^2}{dt^2} + \kappa \right) \xi(t) = T \left. \frac{\partial \phi(x,t)}{\partial x} \right|_{x=0^+}^{x=0^-}, \quad \xi(t) \equiv \phi(0,t). \quad (50)$$

where $\xi(t)$ denotes the displacement of oscillator. With the aid of equation (B.4) we arrive at the formula, which exhibits the role of impedance of medium (string) on the dynamics of oscillator

$$\ddot{\xi} + \frac{1}{m} Z(0) \dot{\xi} + \omega_0^2 \xi = 0. \quad (51)$$

The equation (51) yields the rate of energy loss of the oscillator as

$$\frac{dE}{dt} = -Z(0) \dot{\xi}^2, \quad (52)$$

where the mechanical energy of oscillator is

$$E = \frac{1}{2} m \dot{\xi}^2 + \frac{1}{2} m \omega_0^2 \xi^2. \quad (53)$$

Thus, the dissipative nature of impedance in equation (52) allows one to interpret the input impedance of string as the "radiation impedance" of the oscillator. The equation (51), for an under-damped oscillator ($Z < \omega_0$), has a general solution of the form

$$\xi(t) = e^{-\frac{1}{2m}Z(0)t} \left( a_+ e^{i\Omega t} + a_- e^{-i\Omega t} \right). \quad (54)$$

where $\Omega = \sqrt{\omega_0^2 - Z^2/4m^2}$. With the aid of the equations (37) and (B.6) the impedance at $x_0 = 0$ is readily found to be

$$Z(0) = -\frac{T}{i\omega g(0,0;k)} = \rho_- v_- + \rho_+ v_+. \quad (55)$$

which is the sum of impedances of the two strings connected at origin. So, for an oscillator with initial condition $\xi(0) = 0$, equations (54) and (55) yield the time evolution of the form

$$\xi(t) = \xi_0 e^{-\frac{1}{2m}(\rho_- v_- + \rho_+ v_+)^t} \sin(\Omega t). \quad (56)$$
For a uniform string, the Green’s function can be obtained from (21), (35) and (36) upon setting \( k_- = k_+ \), to give [11]

\[
g(x, x'; k) = \frac{i}{2k} e^{ik|x-x'|},
\]

from which, by means of (B.6), the impedance is found to be

\[
Z(0) = 2\rho v,
\]

yielding [6]

\[
\xi(t) = \xi_0 e^{-\frac{1}{m}\rho v t} \sin(\Omega t).
\]

by virtue of (54). Needless to say, the relations (58) and (59) also can be obtained by letting \( \rho_- = \rho_+ = \rho \) in (55) and (56). We skip the cases \( \omega_0 = Z \) and \( \omega_0 < Z \), as they lead to the critically damped and over-damped motions, respectively.

### 5 Response of system to an impulse

The response of system to an impulse of the from \( F(x, t) = J_0 \delta(x) \delta(t) \) exerted at \( x = 0 \), is attainable from (8). We find immediately

\[
\phi(x, t) = \frac{J_0}{T} G(xt; 00) = \frac{J_0}{T} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} g(x, 0; \omega).
\]

On the other hand, the equation (11) admits a perturbative solution for the Green’s function as

\[
g(x, x'; k) = g_0(x, x'; k) + \int_{-\infty}^{\infty} dx'' g_0(x, x''; k)V(x'')(x'', x'; k).
\]

For the localized potential (12), this equation can be solved exactly to give

\[
g(x, x'; k) = g_0(x, x'; k) + \frac{\lambda}{T} g_0(x, 0; k)g(0, x'; k).
\]

From this expression one can obtain \( g(0, x'; k) \) by letting \( x = 0 \), on both sides of (62). Substituting the resultant term in (62), then recasts it into

\[
g(x, x'; k) = g_0(x, x'; k) + \frac{\lambda}{T} g_0(x, 0; k)g(0, x'; k)\frac{1}{1 - \frac{\lambda}{T} g_0(0, 0; k)}.
\]

In particular, from (63) we have

\[
g(x, 0; k) = \frac{g_0(x, 0; k)}{1 - \frac{\lambda}{T} g_0(0, 0; k)},
\]

which, by means of (12) and (37) takes the form

\[
g(x, 0; \omega) = \frac{-Te^{-i\omega \frac{x}{v}}}{m} \Theta(-x) + \frac{e^{i\omega \frac{x}{v}}}{m} \Theta(x).
\]
with the impedance $Z(0)$ given by (55). On inserting (65) into (60), one encounters with an integral of the form
\[ \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega\tau_{\pm}}}{\omega^2 + \frac{1}{m} Z(0)\omega - \omega_0^2}, \quad \tau_{\pm} = t \pm \frac{x}{v_{\pm}}. \] (66)

This integral has two singular points in the lower half-plane of the complex plane, given by
\[ \omega_{\pm} = -\frac{i}{2m} Z(0) \pm \Omega. \] (67)

Thus, by employing the residue theorem and enclosing the integration contour in the lower (upper) half-plane for $0 < \tau_{\pm}$ ($\tau_{\pm} < 0$), the integral is found to be
\[ \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega\tau_{\pm}}}{\omega^2 + \frac{1}{m} Z(0)\omega - \omega_0^2} = \begin{cases} \frac{1}{\pi} e^{-\frac{1}{2m} Z(0)\tau_{\pm}} \sin(\Omega\tau_{\pm}), & 0 < \tau_{\pm}, \\ 0, & \tau_{\pm} < 0. \end{cases} \] (68)

So, for those points that holds in causality condition, $0 < \tau_{\pm}$, the solution becomes
\[ \phi(x, t) = \frac{J_0}{m\Omega} e^{-\frac{1}{2m} Z(0)(t + \frac{x}{v_{\pm}})} \sin \Omega \left( t + \frac{x}{v_{\pm}} \right) \Theta(-x) + \frac{J_0}{m\Omega} e^{-\frac{1}{2m} Z(0)(t - \frac{x}{v_{\pm}})} \sin \Omega \left( t - \frac{x}{v_{\pm}} \right) \Theta(x). \] (69)

In obtaining the above relation we have supposed an under-damped oscillator ($Z < \omega_0$) as in previous section. This equation represents a sinusoidal wave originating from the oscillator and traveling to the left and right sides of the oscillator with different speeds $v_{\pm}$ and a decaying amplitude. For the motion of oscillator, after setting $x = 0$ in (69), we recover the solution (56)
\[ \phi(0, t) = \frac{J_0}{m\Omega} e^{-\frac{1}{2m}(\rho - v_{\pm})^2 t} \sin(\Omega t). \] (70)

with $\xi_0 = \frac{J_0}{m\Omega}$. For a uniform string, after letting $v_{\pm} = v$, in (69), we obtain [6]
\[ \phi(x, t) = \frac{J_0}{m\Omega} e^{-\frac{1}{2m} v(t - |x|)} \sin \Omega \left( t - \frac{|x|}{v} \right), \quad |x| < vt. \] (71)

6 Conclusion

The Green’s function method is an elegant yet unifying mathematical toolkit. In this paper, we investigated some problems related with a nonuniform Lamb model in a unified manner thanks to the unifying characteristic of the Green’s function technique. We used the Green’s function of string to obtain the scattering coefficients and to describe the effective dynamics of the oscillator. We showed that the Green’s function describing the string and oscillator as a whole, accurately explains the response of system to an impulse.

As a future research plan, it is of interest to look at the effective dynamics of a particle coupled to a dispersive or nondispersive, finite or infinite media, through the lens of the Green’s function theory.
An elementary approach to scattering problem

Equation (19) admits a solution of the form

\[ \varphi(x) = f_-(x)\Theta(-x) + f_+(x)\Theta(x). \quad (A.1) \]

Plugging this into (19) gives rise to

\[ \Theta(-x) \left( \frac{d^2}{dx^2} + k_-^2 \right) f_-(x) + \Theta(x) \left( \frac{d^2}{dx^2} + k_+^2 \right) f_+(x) + \left( \frac{df_+(0)}{dx} - \frac{df_-(0)}{dx} + \frac{\lambda}{T} \varphi(0) \right) \delta(x) = 0, \quad (A.2) \]

where we have gained the well-known properties associated with the Dirac-delta and step functions

\[ h(x)\delta(x) = h(0)\delta(x), \quad (A.3) \]
\[ h(x)\partial_x\delta(x) = -\partial_x h(0)\delta(x), \quad (A.4) \]
\[ \delta(x) = \delta(-x), \quad (A.5) \]
\[ \partial_x\Theta(x) = \delta(x). \quad (A.6) \]

The first two terms of (A.2) yield

\[ \left( \frac{d^2}{dx^2} + k_-^2 \right) f_-(x) = 0, \quad (A.7) \]
\[ \left( \frac{d^2}{dx^2} + k_+^2 \right) f_+(x) = 0, \quad (A.8) \]

while the third term gives

\[ \frac{df_+(0)}{dx} - \frac{df_-(0)}{dx} = -\frac{\lambda}{T} \varphi(0). \quad (A.9) \]

This equation is indeed the equation of motion of the oscillator. It can be also achieved by analyzing the forces acting on the mass [1, 2, 9, 10]. However, expressing the solution in terms of the step function as in (A.1), leads to an automatic derivation. The continuity of string at \( x = 0 \) demands

\[ f_+(0) = f_-(0). \quad (A.10) \]

With an incoming wave from \(-\infty\) of the form \( e^{ikx} \), we assume the ansatz

\[ f_-(x) = e^{ik_-x} + A_R e^{-ik_-x}, \quad (A.11) \]
\[ f_+(x) = A_T e^{ik_+x}. \quad (A.12) \]

where \( A_R \) and \( A_T \) represent reflection and transmission coefficients, respectively. These coefficients can be obtained straightforwardly, by means of (A.9) and (A.10). Then, we obtain

\[ A_R = \frac{k_- - k_+ + i\lambda}{k_- + k_+ - i\lambda}, \quad (A.13) \]
\[ A_T = \frac{2k_-}{k_- + k_+ - i\lambda}. \quad (A.14) \]

in accordance with (44) and (45).
B Impedance in terms of Green’s function

The point impedance, or input impedance, of a string with an arbitrary mass distribution at a given point \( x_0 \), is defined as the ratio of the transverse speed to the force acting on, namely

\[
Z(x_0) = \frac{f(t)}{\partial_t \phi(x_0, t)}, \tag{B.1}
\]

Thus, with the external force \( F(x, t) = f(t) \delta(x - x_0) \) by means of (6), the wave equation takes the form

\[
\left( \frac{1}{v^2(x)} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \phi(x, t) = \frac{f(t)}{T} \delta(x - x_0). \tag{B.2}
\]

Integrating the both sides over an infinitesimal region around the point \( x_0 \), then yields

\[
f(t) = -T \frac{\partial \phi(x_0, t)}{\partial x} \bigg|_{x_0-\epsilon}^{x_0+\epsilon}, \tag{B.3}
\]

which allows one to re-write the impedance as

\[
Z(x_0) = -T \frac{\partial \phi(x_0, t)}{\partial x} \bigg|_{x_0-\epsilon}^{x_0+\epsilon}. \tag{B.4}
\]

For a time-harmonic solution, the above expression can be re-arranged to give

\[
Z(x_0) = \frac{T}{i\omega} \frac{\varphi_<(x_0)\partial_x \varphi_>(x_0) - \varphi_>(x_0)\partial_x \varphi_<(x_0)}{\varphi_<(x_0)\varphi_>(x_0)} \tag{B.5}
\]

since \( \varphi_<(x_0) = \varphi_>(x_0) \). Comparing (B.5) to the formula (27) then yields

\[
Z(x_0) = -\frac{T}{i\omega g_0(x_0, x_0; k)}. \tag{B.6}
\]

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