Einstein-Hilbert Path Integrals in $\mathbb{R}^4$

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Abstract

A hyperlink is a finite set of non-intersecting simple closed curves in $\mathbb{R} \times \mathbb{R}^3$. The dynamical variables in General Relativity are the vierbein $e$ and a $\text{su}(2) \times \text{su}(2)$-valued connection $\omega$. Together with Minkowski metric, $e$ will define a metric $g$ on the manifold.

The Einstein-Hilbert action $S(e, \omega)$ is defined using $e$ and $\omega$. We will define a path integral $I$ by integrating a functional $H(e, \omega)$ against a holonomy operator of a hyperlink $L$, and the exponential of the Einstein-Hilbert action, over the space of vierbeins $e$ and $\text{su}(2) \times \text{su}(2)$-valued connections $\omega$.

Three different types of functional will be considered for $H$, namely area of a surface, volume of a region and the curvature of a surface $S$. Using our earlier work done on Chern-Simons path integrals in $\mathbb{R}^3$, we will derive and write these infinite dimensional path integrals $I$ as the limit of a sequence of Chern-Simons integrals.

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1 Einstein-Hilbert Action

Consider a 3-manifold $M$, hence a 4-manifold $\mathbb{R} \times M$, and a principal bundle $V$ over $\mathbb{R} \times M$, with structure group $G$. Let $\mathfrak{g}$ be the Lie Algebra of $G$. The vector space of all smooth $\mathfrak{g}$-valued one forms on the manifold $\mathbb{R} \times M$ will be denoted by $\mathcal{A}_{\mathbb{R} \times M, \mathfrak{g}}$. Denote the group of all smooth $G$-valued mappings on $\mathbb{R} \times M$ by $\mathcal{G}$, called the gauge group. The gauge group induces a gauge transformation on $\mathcal{A}_{\mathbb{R} \times M, \mathfrak{g}}$, $\mathcal{A}_{\mathbb{R} \times M, \mathfrak{g}} \times \mathcal{G} \to \mathcal{A}_{\mathbb{R} \times M, \mathfrak{g}}$, given by

$$A \cdot \Omega \equiv A^\Omega := \Omega^{-1}d\Omega + \Omega^{-1}A\Omega$$

for $A \in \mathcal{A}_{\mathbb{R} \times M, \mathfrak{g}}$, $\Omega \in \mathcal{G}$. The orbit of an element $A \in \mathcal{A}_{\mathbb{R} \times M, \mathfrak{g}}$ under this operation will be denoted by $[A]$ and the set of all orbits by $\mathcal{A}/\mathcal{G}$.

The 4-manifold we will consider in this article is $\mathbb{R} \times \mathbb{R}^3 \equiv \mathbb{R}^4$, with tangent bundle $T\mathbb{R}^4$. The tangent-space indices are denoted by $a, b, c, d$ and ‘Lorentz’ indices as $\mu, \gamma, \alpha, \beta$, both taking values from $\{0, 1, 2, 3\}$.

Let $V \to \mathbb{R} \times \mathbb{R}^3$ be a 4-dimensional vector bundle, with structure group $SO(3, 1)$. This implies that $V$ is endowed with a metric, $\eta^{\mu\nu}$, of signature $(-, +, +, +)$, and a volume form $\epsilon_{abcd}$.

**Notation 1.1** Fix the standard coordinates on $\mathbb{R}^4 \equiv \mathbb{R} \times \mathbb{R}^3$, with time coordinate $x_0$ and spatial coordinates $(x_1, x_2, x_3)$. 

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Let $Λ^q(\mathbb{R}^4)$ denote the $q$-th exterior power of $\mathbb{R}^4$ and we choose the canonical basis $(dx_0, dx_1, dx_2, dx_3)$ for $Λ^1(\mathbb{R}^4)$. Using the standard coordinates on $\mathbb{R}^4$, let $Λ^1(\mathbb{R}^3)$ denote the subspace in $Λ^1(\mathbb{R}^4)$ spanned by $(dx_1, dx_2, dx_3)$. Finally, a basis for $Λ^2(\mathbb{R}^4)$ is given by

$$\{dx_0 \wedge dx_1, dx_0 \wedge dx_2, dx_0 \wedge dx_3, dx_2 \wedge dx_3, dx_3 \wedge dx_1, dx_1 \wedge dx_2\}.$$  

We adopt Einstein’s summation convention, i.e. we sum over repeated superscripts and subscripts.

Suppose $V$ has the same topological type as $T\mathbb{R}^4$, so that isomorphisms between $V$ and $T\mathbb{R}^4$ exist. Hence we may assume that $V$ is a trivial bundle over $\mathbb{R}^4$. Without loss of generality, we will assume the Minkowski metric $η_{ab}$ is given by

$$η = -dx_0 \otimes dx_0 + 3 \sum_{i=1}^{3} dx_i \otimes dx_i.$$  

And $ε^{μγαβ} = ε^{μγαβ}$ is equal to 1 if the number of transpositions required to permute (0123) to $(μγαβ)$ is even; otherwise it takes the value -1.

However, there is no natural choice of an isomorphism. A vierbein $e$ is a choice of isomorphism between $T\mathbb{R}^4$ and $V$. It may be regarded as a $V$-valued one form, obeying a certain condition of invertibility. A spin connection $ω^a_{μγ}$ on $V$, is anti-symmetric in its indices $μ, γ$. It takes values in $Λ^2(V)$, whereby $Λ^k(V)$ denotes the $k$-th antisymmetric tensor power or exterior power of $V$. The isomorphism $e$ and the connection $ω$ can be regarded as the dynamical variables of General Relativity.

The curvature tensor is defined as

$$R^b_{aμγ} = \partial_a ω^b_{μγ} - \partial_μ ω^b_{aγ} + [ω^a_{μ}, ω^b_{γ}]|_a, \quad \partial_a \equiv ∂/∂x_a,$$

or as $R = dω + ω ∧ ω$. It can be regarded as a two form with values in $Λ^2(V)$.

Using the above notations, the Einstein-Hilbert action is written as

$$S_{EH}(e, ω) = \frac{1}{8} \int_{\mathbb{R}^4} ε^{μγαβ} ε_{abcd} e^a_{μ} e^b_{γ} R_{αβ}^{cd}. \quad (1.1)$$

The expression $e ∧ e ∧ R$ is a four form on $\mathbb{R}^4$ taking values in $V ⊗ V ⊗ Λ^2(V)$ which maps to $Λ^4(V)$. But $V$ with the structure group $SO(3,1)$ has a natural volume form, so a section of $Λ^4(V)$ may be canonically regarded as a function. Thus Equation (1.1) is an invariantly defined integral. By varying Equation (1.1) with respect to $e$, we will obtain the Einstein equations in vacuum. See [14].

The metric $η^{ab}$ on $V$, together with the isomorphism $e$ between $T\mathbb{R}^4$ and $V$, gives a (non-degenerate) metric $g^{ab} = e^a_{μ} e^b_{γ} η^{μγ}$ on $T\mathbb{R}^4$. By varying Equation (1.1) with respect to the connection $ω$, we will obtain an equation that identifies $ω$ as the Levi-Civita connection associated with the metric $g^{ab}$.

2 Notations

Throughout this article, $√-1$ will be denoted by $i$. 

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Notation 2.1 (Subspaces in \( \mathbb{R}^4 \))

In this article, we will write \( \mathbb{R}^4 \equiv \mathbb{R} \times \mathbb{R}^3 \), whereby \( \mathbb{R} \) will be referred to as the time-axis and \( \mathbb{R}^3 \) is the spatial 3-dimensional Euclidean space. In future, when we write \( \mathbb{R}^3 \), we refer to the spatial subspace in \( \mathbb{R}^4 \). Let \( \pi_0 : \mathbb{R}^4 \to \mathbb{R}^3 \) denote this projection.

Let \( \{e_i\}_{i=1}^3 \) be the standard basis in \( \mathbb{R}^3 \). And \( \Sigma_i \) is the plane in \( \mathbb{R}^3 \), containing the origin, whose normal is given by \( e_i \). So, \( \Sigma_1 \) is the \( x_2 - x_3 \) plane, \( \Sigma_2 \) is the \( x_3 - x_1 \) plane and finally \( \Sigma_3 \) is the \( x_1 - x_2 \) plane.

Note that \( \mathbb{R} \times \Sigma_i \cong \mathbb{R}^3 \) is a 3-dimensional subspace in \( \mathbb{R}^4 \). Here, we replace one of the axis in the spatial 3-dimensional Euclidean space with the time-axis. Let \( \pi_1 : \mathbb{R}^4 \to \mathbb{R} \times \Sigma_i \) denote this projection.

Notation 2.2 (Indices)

In this article, the symbols are indexed by several indices. To make it easier for the reader to follow, we will reserve certain symbols for specific indices.

In the rest of the article, indices labeled \( i, j, k, \tilde{i}, \tilde{j}, \tilde{k} \) will only take values from 1 to 3. These indices will keep track of the spatial coordinate \( x_i \).

Indices such as \( a, b, c, d \) and greek indices such as \( \mu, \gamma, \alpha, \beta \) will take values from 0 to 3. We will use the greek indices to index the basis in \( \Lambda^2(V) \).

We will let \( I = [0, 1] \) be the unit interval, and

\[
I^2 \equiv I \times I, \quad I^3 = I \times I \times I, \quad I^4 = I \times I \times I \times I.
\]

We will let \( s, \bar{s}, t, \bar{t} \) denote real numbers in \( I \) and \( \bar{s} = (s, \bar{s}), \bar{t} = (t, \bar{t}) \). And \( ds d\bar{s}, dt d\bar{t} \). Typically, \( s, \bar{s}, t, \bar{t} \) will be reserved as the variable for some parametrization, i.e. \( \bar{p} : s \in I \mapsto \mathbb{R}^4 \).

Notation 2.3 (Symmetric group \( S_3 \))

Let \( S_3 \) denote the symmetric group on the set \( \{1, 2, 3\} \). In this group, there is a cyclic subgroup, \( C_3 \) given by the set \( C_3 = \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\} \). And \( \Upsilon \) denote the set \( \{(2, 3), (3, 1), (1, 2)\} \).

Let \( \tau : \{1, 2, 3\} \to \Upsilon \), by

\[
\tau : 1 \mapsto (2, 3), \quad 2 \mapsto (3, 1), \quad 3 \mapsto (1, 2).
\]

Let \( \epsilon^{ijk} \equiv \epsilon_{ijk} \) be defined on the set \( \{1, 2, 3\} \), by

\[
\epsilon^{123} = \epsilon^{231} = \epsilon^{312} = 1, \quad \epsilon^{213} = \epsilon^{321} = \epsilon^{132} = -1,
\]

if \( i, j, k \) are all distinct; 0 otherwise.

Notation 2.4 (Vectors in \( \mathbb{R}^4 \))

More often than not, given a symbol \( p \), we will use \( \bar{p} \equiv (p_0, p_1, p_2, p_3) \) to denote a 4-vector, \( p \equiv (p_1, p_2, p_3) \) to denote a 3-vector and \( \bar{p} \) to denote a 2-vector.

Suppose we have a vector \( \sigma \equiv (\sigma_1, \sigma_2, \sigma_3) \in \mathbb{R}^3 \). We will write \( \sigma : (0, \sigma) \equiv (0, \sigma_1, \sigma_2, \sigma_3) \).

Write \( x = (x_1, x_2, x_3) \). For \( i = 1, 2, 3 \), we will write

\[
\hat{x}_i = \begin{cases} (x_2, x_3), & i = 1; \\ (x_1, x_3), & i = 2; \\ (x_1, x_2), & i = 3. \end{cases}
\]
Notation 2.5 (Representation of $\mathfrak{su}(2) \times \mathfrak{su}(2)$)

Let $V$ be a vector space of dimension 4. In the rest of this article, we take our principal bundle over $\mathbb{R}^4$ to be trivial, i.e. $\mathbb{R}^4 \times V \to \mathbb{R}^4$ will be our trivial bundle in consideration. Fix a basis $\{E^\gamma\}$ in $V$. Write $E^{\gamma \mu} = E^\gamma \wedge E^\mu \in \Lambda^2(V)$, thus $\{E^{\gamma \mu}\}_{0 \leq \gamma, \mu \leq 3}$ is a basis for $\Lambda^2(V)$.

Let $\mathfrak{su}(2)$ be the Lie Algebra of $SU(2)$. We can map $\Lambda^2(V)$ to the Lie Algebra $\mathfrak{su}(2) \times \mathfrak{su}(2)$ via a linear map. Let $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ be any basis for the first copy of $\mathfrak{su}(2)$ and $\{\check{e}_1, \check{e}_2, \check{e}_3\}$ be any basis for the second copy of $\mathfrak{su}(2)$, satisfying the conditions

$$[\check{e}_1, \check{e}_2] = \check{e}_3, \quad [\check{e}_2, \check{e}_3] = \check{e}_1, \quad [\check{e}_3, \check{e}_1] = \check{e}_2.$$ $\check{e}_1, \check{e}_2, \check{e}_3$ will be fixed throughout this article. Using the above basis, define

$$E^{01} \mapsto (\hat{e}_1, 0) \equiv \hat{E}^{01}, \quad E^{02} \mapsto (\hat{e}_2, 0) \equiv \hat{E}^{02}, \quad E^{03} \mapsto (\hat{e}_3, 0) \equiv \hat{E}^{03}$$

and

$$E^{23} \mapsto (0, \check{e}_1) \equiv \check{E}^{23}, \quad E^{31} \mapsto (0, \check{e}_2) \equiv \check{E}^{31}, \quad E^{12} \mapsto (0, \check{e}_3) \equiv \check{E}^{12}. $$

Do note that $\hat{E}^{\alpha \beta} = -\check{E}^{\beta \alpha} \in \mathfrak{su}(2) \times \mathfrak{su}(2)$. Refer to Notation 2.3. Now $\hat{E}^{\tau(1)} = \check{E}^{23}$, $\hat{E}^{\tau(2)} = \check{E}^{31}$ and $\hat{E}^{\tau(3)} = \check{E}^{12}$.

This isomorphism that sends $E^{\alpha \beta} \mapsto \check{E}^{\alpha \beta}$ will be fixed throughout this article. Using the above basis, define

$$\mathcal{E}^+ = \sum_{i=1}^{3} \hat{e}_i, \quad \mathcal{E}^- = \sum_{i=1}^{3} \check{e}_i,$$

and a $4 \times 4$ complex matrix

$$\mathcal{E} = \begin{pmatrix} -\mathcal{E}^+ & 0 \\ 0 & \mathcal{E}^- \end{pmatrix}. $$

Write

$$\mathcal{F}^+ = \sum_{j=1}^{3} \check{E}^{0j} \equiv (\mathcal{E}^+, 0), \quad \text{and} \quad \mathcal{F}^- = \sum_{j=1}^{3} \hat{E}^{\tau(j)} \equiv (0, \mathcal{E}^-).$$

For $A, B, C, D \in \mathfrak{su}(2)$, we define the Lie bracket on $\mathfrak{su}(2) \times \mathfrak{su}(2)$ as

$$[(A, B), (C, D)] = ([A, C], [B, D]) \in \mathfrak{su}(2) \times \mathfrak{su}(2).$$

Let $\rho^\pm : \mathfrak{su}(2) \to \text{End}(V^\pm)$ be an irreducible finite dimensional representation, indexed by half-integer and integer values $j, z \geq 0$. The representation $\rho : \mathfrak{su}(2) \times \mathfrak{su}(2) \to \text{End}(V^+) \times \text{End}(V^-)$ will be given by $\rho = (\rho^+, \rho^-)$, with

$$\rho : \alpha_i \hat{E}^{0i} + \beta_j \hat{E}^{\tau(j)} \mapsto \left( \sum_{i=1}^{3} \alpha_i \rho^+(\hat{e}_i), \sum_{j=1}^{3} \beta_j \rho^-(\check{e}_j) \right).$$

By abuse of notation, we will now write $\rho^+ \equiv (\rho^+, 0)$ and $\rho^- \equiv (0, \rho^-)$ in future and thus $\rho^+(\hat{E}^{0i}) \equiv \rho^+(\hat{e}_i)$, $\rho^-(\hat{E}^{\tau(j)}) \equiv \rho^-(\check{e}_j)$. Also write

$$\rho^+(\mathcal{F}^+) = \sum_{j=1}^{3} \rho^+(\check{E}^{0j}), \quad \text{and} \quad \rho^-(\mathcal{F}^-) = \sum_{j=1}^{3} \rho^-(\hat{E}^{\tau(j)}).$$

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Note that the dimension of $V^\pm$ is given by $2j_{\rho^\pm} + 1$. Then it is known that the Casimir operator is

$$\sum_{i=1}^{3} \rho^+(\hat{E}^{\rho_0^i})\rho^+(\hat{E}^{\rho_0^i}) \equiv \sum_{i=1}^{3} \rho^+(\hat{e}_i)\rho^+(\hat{e}_i) = -\xi_{\rho^+}I_{\rho^+},$$

$$\sum_{i=1}^{3} \rho^-(\hat{E}^{\rho_0^e(i)})\rho^-(\hat{E}^{\rho_0^e(i)}) \equiv \sum_{i=1}^{3} \rho^-(\hat{e}_i)\rho^-(\hat{e}_i) = -\xi_{\rho^-}I_{\rho^-},$$

$I_{\rho^\pm}$ is the $2j_{\rho^\pm} + 1$ identity operator for $V^\pm$ and $\xi_{\rho^\pm} := j_{\rho^\pm}(j_{\rho^\pm} + 1)$.

Without loss of generality, we assume that $\rho^+(\hat{E})$ is skew-Hermitian for any $\hat{E} \in \mathfrak{su}(2)$, so by choosing a suitable basis in $V^+$, we will always assume that $\rho^+(i\hat{e}^+)$ is diagonal, with the real eigenvalues given by the set

$$\begin{cases} \{\pm\hat{\lambda}_1, \pm\hat{\lambda}_2, \ldots, \pm\hat{\lambda}_{(2j_{\rho^+}+1)/2}\}, & 2j_{\rho^+} + 1 \text{ is even;} \\ \{\pm\hat{\lambda}_1, \pm\hat{\lambda}_2, \ldots, \pm\hat{\lambda}_{j_{\rho^+}}, 0\}, & 2j_{\rho^+} + 1 \text{ is odd.} \end{cases}$$

Similarly, by choosing another suitable basis in $V^-$, we will always assume that $\rho^-(i\hat{e}^-)$ is diagonal, with the real eigenvalues given by the set

$$\begin{cases} \{\pm\hat{\lambda}_1, \pm\hat{\lambda}_2, \ldots, \pm\hat{\lambda}_{(2j_{\rho^-}+1)/2}\}, & 2j_{\rho^-} + 1 \text{ is even;} \\ \{\pm\hat{\lambda}_1, \pm\hat{\lambda}_2, \ldots, \pm\hat{\lambda}_{j_{\rho^-}}, 0\}, & 2j_{\rho^-} + 1 \text{ is odd.} \end{cases}$$

Then, we have

$$\text{Tr} \rho^+(e^{i\hat{e}^+}) = \begin{cases} \sum_{n=0}^{(2j_{\rho^+}+1)/2} \text{cosh}(\hat{\lambda}_n), & 2j_{\rho^+} + 1 \text{ is even;} \\ 1 + \sum_{n=0}^{j_{\rho^+}} \text{cosh}(\hat{\lambda}_n), & 2j_{\rho^+} + 1 \text{ is odd,} \end{cases}$$

and

$$\text{Tr} \rho^-(e^{i\hat{e}^-}) = \begin{cases} \sum_{n=0}^{(2j_{\rho^-}+1)/2} \text{cosh}(\hat{\lambda}_n), & 2j_{\rho^-} + 1 \text{ is even;} \\ 1 + \sum_{n=0}^{j_{\rho^-}} \text{cosh}(\hat{\lambda}_n), & 2j_{\rho^-} + 1 \text{ is odd.} \end{cases}$$

In either case, we have $\text{Tr} \rho^\pm(e^{i\hat{e}^\pm}) \geq 1$ and hence $\log \text{Tr} \rho^\pm(e^{i\hat{e}^\pm}) \geq 0$, for any irreducible representation. Finally, note that $\text{Tr} \rho^\pm(e^{i\hat{e}^\pm})$ is well-defined, even though $\hat{E}^\pm$ is not in general.

**Remark 2.6** By choosing the group $SU(2) \times SU(2)$, we actually define a spin structure on $\mathbb{R}^4$.

**Notation 2.7** (On hyperlinks in $\mathbb{R}^4$)

For a finite set of non-intersecting simple closed curves in $\mathbb{R}^3$ or in $\mathbb{R} \times \Sigma_i$, we will refer to it as a link. If it has only one component, then this link will be referred to as a knot. A simple closed curve in $\mathbb{R}^4$ will be referred to as a loop. A finite set of non-intersecting loops in $\mathbb{R}^4$ will be referred to as a hyperlink in this article. We say a link or hyperlink is oriented if we assign an orientation to its components.

Let $L$ be a hyperlink. We say $L$ is a time-like hyperlink, if given any 2 distinct points $p \equiv (x_0, x_1, x_2, x_3), q \equiv (y_0, y_1, y_2, y_3) \in L$, $p \neq q$, we have

- $\sum_{i=1}^{3} (x_i - y_i)^2 > 0$;
• if there exists \( i, j, i \neq j \) such that \( x_i = y_i \) and \( x_j = y_j \), then \( x_0 - y_0 \neq 0 \).

Throughout this article, all our hyperlinks in consideration will be time-like. We refer the reader to [10] as to why the term time-like was used.

We will have 2 different hyperlinks, \( T = \{ t^u : u = 1, \ldots, \pi \} \) and \( L = \{ l^v : v = 1, \ldots, n \} \). The former will be called a matter hyperlink; the latter will be referred to as a geometric hyperlink.

The symbols \( u, \bar{u}, v, \bar{v} \) will be indices, taking values in \( \mathbb{N} \). They will keep track of the loops in our hyperlinks \( T \) and \( L \). The symbols \( \pi \) and \( \bar{n} \) will always refer to the number of components in \( T \) and \( L \) respectively.

Given a hyperlink \( T \) and a hyperlink \( L \), we also assume that together (by using ambient isotopy if necessary), they form another hyperlink with \( \pi + \bar{n} \) components. Denote this new hyperlink by \( \chi(T, L) \equiv \chi((T^u)_{u=1}^{\pi}, (L^v)_{v=1}^{\bar{n}}) \).

Color the matter hyperlink \( T \), which means we choose a representation \( \rho_{\bar{u}} : s\text{u}(2) \times s\text{u}(2) \rightarrow \text{End}(V_{\bar{u}}^+) \times \text{End}(V_{\bar{u}}^-) \) for each component \( T^u, u = 1, \ldots, \pi \), in the hyperlink \( T \). Note that we do not color \( L \), i.e. we do not choose a representation for \( L \). Finally, we will also refer \( \chi(T, L) \) as a colored hyperlink.

**Notation 2.8** (Parametrization of curves)
Let \( \vec{y}^u \equiv (y_0^u, y_1^u, y_2^u) : [0, 1] \rightarrow \mathbb{R} \times \mathbb{R}^3 \) be a parametrization of a loop \( T^u, u = 1, \ldots, \pi \). We will write \( y^u(s) = (y_0^u(s), y_1^u(s), y_2^u(s)) \) and \( \vec{y}^u(s) \equiv \vec{y}_v \). We will also write \( \vec{y} = (y_0, y_1, y_2) \). Similar notation for \( \vec{y} \), \( v = 1, \ldots, \bar{n} \), which is a parametrization of a loop \( L^v \). When the loop is oriented, we will choose a parametrization which is consistent with the assigned orientation.

Let \( \rho : S \rightarrow \mathbb{R}^3 \) for some set \( S \). Typically, \( S = I, \dot{I}^2 \) or \( \dot{I}^3 \). We can write \( \rho(s) \equiv (\rho_1(s), \rho_2(s), \rho_3(s)) \). Refer to Notation 2.4. We will write

\[
\hat{\rho}_i(s) = \begin{cases} 
(\rho_2(s), \rho_3(s)), & i = 1; \\
(\rho_1(s), \rho_3(s)), & i = 2; \\
(\rho_1(s), \rho_2(s)), & i = 3.
\end{cases}
\]

**Notation 2.9** (Parametrization of surfaces) Choose an orientable, closed and bounded surface \( S \subset \mathbb{R}^3 \), with or without boundary. If it has a boundary \( \partial S \), then \( \partial S \) is assumed to be a time-like hyperlink. Do note that we allow \( S \) to be disconnected, with finite number of components. Parametrize it using

\[
\tilde{\sigma} : (t, \bar{t}) \in I^2 \mapsto (\sigma_0(t, \bar{t}), \sigma_1(t, \bar{t}), \sigma_2(t, \bar{t}), \sigma_3(t, \bar{t})) \in \mathbb{R}^4
\]

and let

\[
J_{\alpha\beta} = \frac{\partial \sigma_\alpha}{\partial t} \frac{\partial \sigma_\beta}{\partial \bar{t}} - \frac{\partial \sigma_\alpha}{\partial \bar{t}} \frac{\partial \sigma_\beta}{\partial t}.
\]

When we project \( S \) inside \( \mathbb{R}^3 \) as \( \pi_0(S) \), we can parametrize it using \( \sigma \equiv (\sigma_1, \sigma_2, \sigma_3) \). Let

\[
K_{\sigma}(t, \bar{t}) \equiv K_{\sigma}(t, \bar{t}) := (J_{01}(t, \bar{t}), J_{02}(t, \bar{t}), J_{03}(t, \bar{t})),
\]

\[
J_{\sigma}(t, \bar{t}) \equiv J_{\sigma}(t, \bar{t}) := \frac{\partial}{\partial t} \sigma(t, \bar{t}) \times \frac{\partial}{\partial \bar{t}} \sigma(t, \bar{t}) \equiv (J_{23}(t, \bar{t}), J_{31}(t, \bar{t}), J_{12}(t, \bar{t})).
\]

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3 Einstein-Hilbert path integral

Any spin connection $\omega$ described in Section II can be written as $\omega \equiv A_{\alpha\beta}^a \otimes dx_a \otimes \dot{E}^{\alpha\beta}$, whereby $A_{\alpha\beta}^a : \mathbb{R}^4 \rightarrow \mathbb{R}$ is smooth and we identify $\Lambda^2(V)$ with $\mathfrak{su}(2) \times \mathfrak{su}(2)$. See Notation 2.5. Considering that this space of smooth spin connections is too big for our purpose, we need to ‘trim’ down this space.

If we are considering $A_{\mathbb{R}^4, g}$, the standard approach would be to consider $A_{\mathbb{R}^4, g}$, modulo gauge transformations. Let $\{F_a\}$ be any basis in $\mathfrak{g}$. Under axial gauge fixing, every $A \in A_{\mathbb{R}^4, g}$ can be gauge transformed into $A_{\alpha}^a \otimes dx_i \otimes F^\alpha$, $A_{\alpha}^a : \mathbb{R}^3 \rightarrow \mathbb{R}$ smooth, subject to the conditions

$$A_{\alpha}^1(0, x^1, 0) = 0, \ A_{\alpha}^2(0, x^1, x^2, 0) = 0, \ A_{\alpha}^3(0, x^1, x^2, x^3) = 0. \quad (3.1)$$

Now, 3+1 gravity is not a gauge theory, in the sense that if we interpret $\epsilon$ and $\omega$ as gauge fields, then the Einstein-Hilbert action should be invariant under gauge transformation. But there is no such action in gauge theory. Furthermore spin connection is $\Lambda^2(V)$-valued one form, not exactly a gauge field. However, we still can apply axial gauge fixing and by making the identification $\Lambda^2(V) \equiv \mathfrak{su}(2) \times \mathfrak{su}(2)$, we now consider

$$\omega = A_{\alpha\beta}^a \otimes dx_i \otimes \dot{E}^{\alpha\beta} \in \mathcal{S}_\kappa(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3) \otimes \mathfrak{su}(2) \times \mathfrak{su}(2) =: L_\omega. \quad (3.2)$$

Observe that $A_{\alpha\beta}^1 = -A_{\beta\alpha}^2 \in \mathcal{S}_\kappa(\mathbb{R}^4)$ and $\mathcal{S}_\kappa(\mathbb{R}^4)$ is the Schwartz space discussed in Section II.

Remark 3.1. Note that the restrictions given by Equation (3.1) will not be imposed on $\omega$.

Recall $\epsilon$ is $V$-valued one form. Even though $\epsilon$ is not a gauge, we will still apply axial gauge fixing argument as above. As a consequence, we will have to drop the invertibility condition, and consider all non-invertible transformations $\epsilon : T\mathbb{R}^4 \rightarrow V$, written as

$$\epsilon = B_{\gamma}^i \otimes dx_i \otimes E^\gamma \in \mathcal{S}_\kappa(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3) \otimes V =: L_\epsilon. \quad (3.3)$$

Remark 3.2. 1. Note that $\epsilon(\partial / \partial x_0) = 0$, so after applying axial gauge fixing, we consider non-invertible transformations $\epsilon$.

2. The restrictions given by Equation (3.3) will not be imposed on $\epsilon$.

3. The reader may object to apply axial gauge fixing to $\epsilon$; after all $\epsilon$ is not a gauge and in General Relativity, $\epsilon$ defines a metric, which is non-degenerate in classical General Relativity. However, as discussed in [13], we must consider $\epsilon$ to be non-invertible to make sense of or develop 2+1 quantum gravity. Likewise here, to develop a 3+1 quantum gravity, we have to consider non-invertible $\epsilon$.

At this point, it is good to discuss the significance of $\omega$ and $\epsilon$. By identifying $\Lambda^2(V)$ with $\mathfrak{su}(2) \times \mathfrak{su}(2)$, we interpret $\omega$ as a connection with values in $\mathfrak{su}(2) \times \mathfrak{su}(2)$. In Notation 2.5, the first copy of $\mathfrak{su}(2)$ is generated by $\{E^{0i}\}_{i=1}^3$, which corresponds to boost in the $x_i$ direction in the Lorentz group; the second copy of $\mathfrak{su}(2)$ is generated by $\{E^{1\tau(i)}\}_{i=1}^3$, which corresponds to rotation about the $x_i$-axis in the Lorentz group. When we give a representation $\rho^\tau$ to a colored loop $\tilde{l}$, which we interpret as representing a particle, we are effectively assigning values to the translational and angular momentum of this particle.
The vierbein $e$ can be interpreted as translating $V$, a 4-dimensional vector space. By choosing an orthonormal basis $\{f_\alpha\}_{\alpha=0}^3$ using the Minkowski metric, we may interpret $f_\alpha$ as generator for translation in the $f_\alpha$ direction, which corresponds to translation in the Poincaré group. Note that the Lorentz group is a Lie subgroup of the Poincaré Lie group. This means that we may think of $\{\omega, e\}$ as a connection with values in the Poincaré Lie Algebra.

**Definition 3.3 (Time ordering operator)**
For any permutation $\sigma \in S_r$,

$$\mathcal{T}(A(s_{\sigma(1)}) \cdots A(s_{\sigma(r)})) = A(s_1) \cdots A(s_r), \quad s_1 > s_2 > \ldots > s_r.$$

Suppose now our matrices $A^u(s)$ are indexed by the curves $u$ and time $s$. Extend the definition of the time ordering operator, first ordering in decreasing values of $u$, followed by the time $s$.

Consider two oriented hyperlinks, $L = \{(l^u)_{u=1}^n\}$ in $\mathbb{R} \times \mathbb{R}^3$. Color each component of $L$ with representation $\rho_u$. The hyperlinks $L$ and $L'$ are entangled together to form an oriented colored hyperlink, denoted by $\chi(L, L')$. Let $q \in \mathbb{R}$ be known as a charge.

Define

$$V((l^u)_{u=1}^n)(e) := \exp \left[ \sum_{u=1}^n \sum_{\gamma=0}^3 B^u_\gamma \otimes dx_\gamma \right],$$

$$W(q; (l^u, \rho_u)_{u=1}^n)(\omega) := \prod_{u=1}^n \text{Tr}_{\rho_u} \mathcal{T} \exp \left[ q \int_{l^u} A^u_{\alpha\beta} \otimes dx_\alpha \otimes \hat{E}^\alpha_\beta \right].$$

Here, $\mathcal{T}$ is the time-ordering operator defined in Definition 3.3.

**Remark 3.4** The term $\mathcal{T} \exp [q \int_{l^u} \omega]$ needs some further explanation. If we regard $\Lambda^2(V) \cong su(2) \times su(2)$, then the former is known as a holonomy operator along a loop $l^u$, for a spin connection $\omega$.

Our aim in this article is to give a plausible definition for an Einstein-Hilbert path integral, of the form

$$\frac{1}{Z} \int_{\omega \in L_\omega, \ e \in L_e} H(e, \omega)V((l^u)_{u=1}^n)(e)W(q; (l^u, \rho_u)_{u=1}^n)(\omega) \ e^{i S_{EH}(e, \omega)} \ DeD\omega, \quad (3.4)$$

whereby $De$ and $D\omega$ are Lebesgue measures on $L_e$ and $L_\omega$ respectively and

$$Z = \int_{\omega \in L_\omega, \ e \in L_e} e^{i S_{EH}(e, \omega)} \ DeD\omega.$$

Here, $H$ is some continuous function, possibly taking values in $\mathbb{R}$ or in some Lie Algebra. In this article, we will consider 3 possible functions, namely

- the area of some surface $S \subset \mathbb{R}^3$,
- the volume of a region $R \subset \mathbb{R}^3$,
- and finally the curvature, integrated over some surface $S \subset \mathbb{R}^4$. 

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These 3 functions will be dealt with separately, in Sections 8, 9 and 10 respectively.

To define these three path integrals, we will show that one can write the path integral in the form of a Chern-Simons path integral, which was studied in [7] and [9]. See Section 7. Using similar arguments in our previous work, we will write down a set of Chern-Simons rules, given by Definition 6.14, to define the path integral in all three cases. Our main result in this article is to derive these definitions, which are respectively given by Definitions 8.21, 9.12 and 10.20 respectively. In a sequel to this article, we will compute explicitly these path integrals, using topological invariants. See [4], [5] and [6].

The idea of using hyperlinks in $\mathbb{R} \times \mathbb{R}^3$ in the quantization of gravity is not new. The idea of using loops to describe quantum gravity appeared in [11]. In that article, the authors wrote down a path or functional integral using a suitable (infinite dimensional) measure. Unfortunately, they did not state the choice of this measure or even give a plausible definition for such an ill-defined integral.

4 Schwartz space

**Notation 4.1** For a vector space $V$, $V^{\otimes n} \equiv V \otimes \cdots \otimes V$ will mean the $n$-th tensor product of $V$. The notation $V^{\times n} \equiv V \times \cdots \times V$ means the $n$-th direct product of $V$. If $V$ is an inner product space, then $V^{\otimes n}$ inherits the tensor inner product.

**Notation 4.2** In this article, $\vec{y} \equiv (y_0, y)$, whereby $y \equiv (y_1, y_2, y_3) \in \mathbb{R}^3$. If $x \in \mathbb{R}^n$, we will write $(p_{\kappa}^x)^2$ to denote the $n$-dimensional Gaussian function, center at $x$, variance $1/\kappa^2$. For example,

$$p_{\kappa}^x(\cdot) = \frac{\kappa^2}{2\pi} e^{-\kappa^2|\cdot-x|^2/4}, \ x \in \mathbb{R}^4.$$

We will also write $(q_{\kappa}^x)^2$ to denote the 1-dimensional Gaussian function, i.e.

$$q_{\kappa}^x(\cdot) = \frac{\sqrt{\kappa}}{(2\pi)^{1/4}} e^{-\kappa^2(\cdot-x)^2/4}.$$

**Notation 4.3** Later on, we will approximate the Dirac-delta function with $p_\kappa$. The bigger the $\kappa$, the better is the approximation. In the end, we will let $\kappa$ go to infinity.

Let $\tilde{\kappa} := \kappa/2\sqrt{4\pi}$. This is an important factor, which we need to use throughout this article. As we will see later, we need the correct powers of $\tilde{\kappa}$, to ensure that our path integrals will converge to something meaningful.

Consider the inner product space $S_\kappa(\mathbb{R})$ which is contained inside the Schwartz space. The space $S_\kappa(\mathbb{R})$ consists of functions of the form $f \sqrt{\phi_\kappa}$, whereby $\phi_\kappa$ is the Gaussian function $\phi_\kappa(x) = ke^{-\kappa^2|x|^2/2}/(2\pi)^{1/2}$ and $f$ is a polynomial.

Let $f, g$ be polynomials in $\mathbb{R}$. The inner product $\langle \cdot, \cdot \rangle$ is given by

$$\langle f \sqrt{\phi_\kappa}, g \sqrt{\phi_\kappa} \rangle = \int_{\mathbb{R}} f \cdot g \cdot \phi_\kappa \ d\lambda,$$

$\lambda$ is Lebesgue measure on $\mathbb{R}$. Let $S_\kappa(\mathbb{R})$ be the smallest Hilbert space containing $S_\kappa(\mathbb{R})$, using this inner product. Let $\overline{S}_\kappa(\mathbb{R}^4)$ be the smallest Hilbert space containing $S_\kappa(\mathbb{R})^{\otimes 4}$. 
Suppose \( f \in S_c(\mathbb{R}^4) \) and \( g \notin \mathcal{S}_c(\mathbb{R}^4) \), but \( g \) is bounded and continuous. If further \( f \) is \( L^1 \) integrable, by abuse of notation, we will write

\[
\langle f, g \rangle := \int_{\mathbb{R}^4} f \cdot g \, d\lambda,
\]

integrating using Lebesgue measure on \( \mathbb{R}^4 \).

Suppose \( V \) is some vector space and consider the tensor product \( C^\infty(\mathbb{R}^4) \otimes V \). Let \( \sum_u \alpha_u \otimes \beta_u \in C^\infty(\mathbb{R}^4) \otimes V \). We will abuse notation and write for \( \gamma \in C^\infty(\mathbb{R}^4) \),

\[
\bigg\langle \gamma, \sum_u \alpha_u \otimes \beta_u \bigg\rangle := \sum_u \int_{\mathbb{R}^4} \gamma \cdot \alpha_u \, d\lambda \otimes \beta_u,
\]

provided the integral \( \int_{\mathbb{R}^4} \gamma \cdot \alpha_u \, d\lambda \) converges.

In summary, we wish to highlight to the reader, that in the rest of this article, when we write \( \langle \cdot, \cdot \rangle \), it means integrate using Lebesgue measure, for a given product of \( C^\infty \) functions. If \( f, g \in C^\infty(\mathbb{R}^n) \), then

\[
\langle f, g \rangle \equiv \int_{\mathbb{R}^n} f \cdot g \, d\lambda,
\]

whereby we integrate the product using Lebesgue measure over \( \mathbb{R}^n \).

5 Important Linear operators

See Notation 4.1. We will often write \( V^{\times 3} \) to mean the vector space consisting of 3-vectors, whose components take values in \( V \). For example, \( \mathbb{R}^3 \) will be the usual vector space consisting of real 3-vectors. The vector space \( V^{\times 3} \) will refer to the space containing vectors with 9 components, whose components take values in \( V \).

Given a 3-vector \( u = (u_1, u_2, u_3), u_i \in C^\infty(\mathbb{R}^4) \), we may identify \( C^\infty(\mathbb{R}^4)^{\times 3} \) with a subspace inside \( (C^\infty(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3))^{\times 3} \), by

\[
u \in C^\infty(\mathbb{R}^4)^{\times 3} \iff (u_1 \otimes dx_1, u_2 \otimes dx_2, u_3 \otimes dx_3) \in (C^\infty(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3))^{\times 3}.
\]

We will write a 9-vector in \( V^{\times 9} \) as \( u = (u_1, u_2, u_3) \), whereby each \( u_i \in V^{\times 3} \). In the case when \( V = C^\infty(\mathbb{R}^4) \), we may identify \( C^\infty(\mathbb{R}^4)^{\times 9} \) with a subspace inside \( (C^\infty(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3))^{\times 9} \), by

\[
u \iff (u_1 \otimes dx_1, u_2 \otimes dx_2, u_3 \otimes dx_3) \in (C^\infty(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3))^{\times 9}.
\]

So, each \( u_i \otimes dx_i \in (C^\infty(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3))^{\times 3} \).

The Hodge star operator \( * \) is a linear isomorphism between \( \Lambda^1(\mathbb{R}^3) \) and \( \Lambda^2(\mathbb{R}^3) \) using the volume form \( dx_1 \wedge dx_2 \wedge dx_3 \), i.e.

\[
* (dx_1) = dx_2 \wedge dx_3, \quad *(dx_2) = dx_3 \wedge dx_1, \quad *(dx_3) = dx_1 \wedge dx_2.
\]

We define another linear isomorphism \( \frac{\partial}{\partial x} \) between \( \Lambda^1(\mathbb{R}^3) \) and \( \Lambda^2(\mathbb{R}^3) \), by

\[
\frac{\partial}{\partial x_i} (dx_j \wedge dx_k + dx_k \wedge dx_j) = dx_i,
\]

for \( (i, j, k) \in C_3 \).

Certain operators arise during the analysis of the Chern-Simons path integrals in [7], [8] and [9]. The following linear operators act on dense subsets in \( \mathcal{F}_c(\mathbb{R}) \) and \( \mathcal{F}_c(\mathbb{R}^4) \).
Definition 5.1 (Integral operators)

1. For \( x = (x_0, x_1, x_2, x_3) \), write

\[
x(s) := \begin{cases} 
(s_0, x_1, x_2, x_3), & a = 0; \\
(x_0, s_1, x_2, x_3), & a = 1; \\
(x_0, x_1, s_2, x_3), & a = 2; \\
(x_0, x_1, x_2, s_3), & a = 3.
\end{cases}
\]

2. Let \( \partial_a \equiv \partial/\partial x_a \) be a differential operator. There is an operator \( \partial_a^{-1} \) acting on a dense subset in \( \mathbb{S}_\kappa(\mathbb{R}^4) \),

\[
(\partial_a^{-1} f)(x) := \frac{1}{2} \int_{-\infty}^{x_a} f(x(s_a)) \, ds_a - \frac{1}{2} \int_{x_a}^{\infty} f(x(s_a)) \, ds_a, \quad f \in \mathbb{S}_\kappa(\mathbb{R}^4).
\]

Here, \( x_a \in \mathbb{R} \). Notice that \( \partial_a \partial_a^{-1} f \equiv f \) and \( \partial_a^{-1} f \) is well-defined provided \( f \) is in \( L^1 \), but it is not inside \( \mathbb{S}_\kappa(\mathbb{R}^4) \).

3. Let \( \lambda \in \mathbb{S}_\kappa(\mathbb{R}^4) \). We define an operator

\[
m_i^\kappa(\lambda) := \frac{1}{\kappa} \partial_i + \lambda, \quad i = 1, 2, 3,
\]

\( \lambda \) acts on \( f \in \mathbb{S}_\kappa(\mathbb{R}^4) \) by multiplication. Its inverse, \( m_i^\kappa(\lambda)^{-1} \), acts on a dense subset in \( \mathbb{S}_\kappa(\mathbb{R}^4) \) by

\[
(m_i^\kappa(\lambda)^{-1} h)(x) := \frac{\kappa}{2} \left[ \int_{-\infty}^{x_i} - \int_{x_i}^{\infty} \right] e^{(s_i - x_i)\lambda(x(s_i))} h(x(s_i)) \, ds_i, \quad h \in \mathbb{S}_\kappa(\mathbb{R}^4).
\]

Remark 5.2 When \( \lambda \equiv 0 \), then \( m_i^\kappa(0)^{-1} = \kappa \partial_i^{-1} \).

Notation 5.3 Refer to Notations \ref{notations4} and \ref{notations2}. For each \( i = 1, 2, 3 \), write

\[
\langle \hat{p}_\kappa^x, p_i^\kappa \rangle_i = \langle \hat{p}_\kappa^x, p_i^\kappa \rangle \langle \hat{q}_\kappa^{\hat{x}_i}, \kappa \partial_0^{-1} q_i^\kappa \rangle \langle \partial_0^{-1} q_i^\kappa, q_0^\kappa \rangle.
\]

Here,

\[
\partial_0^{-1} q_i^\kappa(t) \equiv \frac{1}{2} \int_{-\infty}^{t} q_i^\kappa(\tau) \, d\tau - \frac{1}{2} \int_{t}^{\infty} q_i^\kappa(\tau) \, d\tau.
\]

Note that \( \langle \partial_0^{-1} q_i^\kappa, q_0^\kappa \rangle \) means we integrate \( \partial_0^{-1} q_i^\kappa \cdot q_0^\kappa \) over \( \mathbb{R} \), using Lebesgue measure. It is well-defined because \( q_i^\kappa \) is in \( L^1 \). Refer to Section \ref{sections4}.

To define our Einstein-Hilbert path integrals later on, we need to introduce the following differential operators.

Definition 5.4 (Differential operators)
1. We will consider real 3-vectors, each component taking values in the real line. However, we may consider a 3-vector whose components are in $C^\infty(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3)$. Given a differential operator $\partial_j$, it acts on $u^i \otimes dx_i \in C^\infty(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3)$, $j = 1, 2, 3$ by

$$\partial_j(u^i \otimes dx_i) = \sum_{i \neq j} \frac{\partial u_i}{\partial x_j} \otimes dx_j \wedge dx_i \in C^\infty(\mathbb{R}^4) \otimes \Lambda^2(\mathbb{R}^3).$$

Here, there is no implied sum over the $j$.

2. Consider a 3-vector $u = (u_1, u_2, u_3) \in (C^\infty(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3))^3$. Given $v = (v_1, v_2, v_3) \in (C^\infty(\mathbb{R}^4))^3$, we can define the cross product $v \times u$ by

$$v \times u = (v_2u_3 - v_3u_2, v_3u_1 - v_1u_3, v_1u_2 - v_2u_1).$$

3. Let $u = (u_1, u_2, u_3) \in (C^\infty(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3))^3$. By abuse of notation, we define

$$\nabla \times (u_1, u_2, u_3) := (\partial_2u_3 - \partial_3u_2, \partial_3u_1 - \partial_1u_3, \partial_1u_2 - \partial_2u_1).$$

Note that each component lies in $C^\infty(\mathbb{R}^4) \otimes \Lambda^2(\mathbb{R}^3)$.

6 Chern-Simons Integrals

Suppose we have a (real) Hilbert space $H$, with inner product $\langle \cdot, \cdot \rangle$. Write $H_C \equiv C \otimes_{\mathbb{R}} H$ to be the complexification of $H$. The Chern-Simons integral is typically an infinite dimensional integral over $H^{\times 2} \equiv H \times H$.

In [7] and [9], we defined the Chern-Simons path integrals over in $\mathbb{R}^3$ and $S^2 \times S^1$ respectively. The definition of such ill-defined path integrals is via constructing an Abstract Wiener space using the Segal Bargmann Transform, followed by defining a path integral of the form

$$\frac{1}{Z} \int_{(w_+, w_-) \in H^{\times 2}} e^{i\langle w_+, \alpha_+ \rangle} e^{i\langle w_-, \alpha_- \rangle} e^{i\langle w_+, w_- \rangle} \, Dw_+ Dw_-,$$

$\alpha_+, \alpha_- \in H$, with

$$Z = \int_{(w_+, w_-) \in H^{\times 2}} e^{i\langle w_+, w_- \rangle} \, Dw_+ Dw_-,$$

over the Abstract Wiener space. Note that $Dw_\pm$ is some Lebesgue measure over $H$, which does not exist. See [3].

Using Fourier transform and analytic continuation (See Proposition 3.3 in [7]), we define the above integral as $e^{i\langle \alpha_+, \alpha_- \rangle}$, $i = \sqrt{-1}$. From this definition, we can easily extend to more general type of path integrals.

**Definition 6.1** An integral written on $H^{\times 2}$ is said to be a Chern-Simons integral if it is of the form

$$\frac{1}{Z} \int_{(w_+, w_-) \in H^{\times 2}} F(w_+, w_-) e^{i\langle w_+, w_- \rangle} \, Dw_+ Dw_-,$$

for some continuous function $F$ and

$$Z = \int_{(w_+, w_-) \in H^{\times 2}} e^{i\langle w_+, w_- \rangle} \, Dw_+ Dw_-$$

is some normalization constant.
Let $T_u : \mathbb{C} \to \text{End}(H\mathbb{C})$ be a linear operator. And let $\beta_u, \bar{\beta}_u, \alpha_\pm \in H$ be fixed, for $u = 1, \ldots, n$ and $\bar{u} = 1, \ldots, d$. Typically, the Chern-Simons integral we want to compute is of the form
\[
\frac{1}{Z} \int_{(w_-, \bar{w}_-) \in H^{\times 2}} F((w_-, \beta_1), \ldots, (w_-, \beta_n)) e^{(w_-, \sum_{u=1}^d T_u((w_-, \bar{\beta}_u)) \alpha_u)} e^{i(w_-, \beta_+)} Dw_+ Dw_-. \tag{6.1}
\]
Here, $F$ is some continuous function on $\mathbb{R}^n$, which admits an analytic continuation on $\mathbb{C}^n$.

**Remark 6.2** Expression (6.1) is not the most general form whereby we can make a sensible definition for it, but for those Chern-Simons integrals that we are going to consider, this will suffice.

The following definition is taken from Definition 8 in [7], which can be thought of as a generalization of Proposition 3.3 in [7].

**Definition 6.3** We define the integral in Expression (6.1) as
\[
F(i(\alpha_+, \beta_1), \ldots, i(\alpha_+, \beta_n)) e^{i(\alpha_+, \sum_{u=1}^d T_u(\alpha_+ \beta_u)) \alpha_u}. \tag{6.2}
\]

**Remark 6.4** From $e^{(w_+, \alpha_+)}$, we replace $w_-$ with $i\alpha_+$ in the expression
\[
F((w_-, \beta_1), \ldots, (w_-, \beta_n)) e^{(w_-, \sum_{u=1}^d T_u((w_-, \bar{\beta}_u)) \alpha_u)} \tag{6.3}
\]

This will give us the expression in Definition (6.2).

Unfortunately, the Chern-Simons path integrals that one is interested in is not exactly in the above form. (See [7] and [9].) Typically, we need to consider
\[
\frac{1}{Z} \int_{(u_+, u_-) \in H^{\times 2}} F(u_+, u_-) e^{i(u_+, T u_-)} Du_+ Du_-, \tag{6.4}
\]
with
\[
Z = \int_{(u_+, u_-) \in H^{\times 2}} e^{i(u_+, T u_-)} Du_+ Du_.
\]
Here, $T$ is some unbounded operator acting on $H$.

To define the path integral in Expression (6.4) the conventional approach would be to write it in the form
\[
\frac{1}{Z} \int_{(u_+, u_-) \in H^{\times 2}} F(u_+, T^{-1} T u_-) e^{i(u_+, T u_-)} \det[T]^{-1} Du_+ D[T u_-],
\]
with
\[
Z = \int_{(u_+, u_-) \in H^{\times 2}} e^{i(u_+, T u_-)} \det[T]^{-1} Du_+ D[T u_-].
\]
Now $\det[T]^{-1}$ is typically some undetermined constant. It will be factored out and canceled with another copy in $Z$. Finally, we replace $T u_- \mapsto u_-$ and we interpret Expression (6.4) as
\[
\frac{1}{Z} \int_{(u_+, u_-) \in H^{\times 2}} F(u_+, u_-) e^{i(u_+, u_-)} Du_+ Du_-, \tag{6.5}
\]
with
\[
\tilde{Z} = \int_{(u_+, u_-) \in H^{\times 2}} e^{i(u_+, u_-)} Du_+ Du_-
\]
Because it is not possible to define the determinant of $T$, the above heuristic change of variables argument is not mathematically rigorous. However, it does serve as a starting point on how to define the Chern-Simons integrals that we are really interested in and give us a plausible definition for such an integral.
6.1 Chern-Simons Path Integral in $\mathbb{R}^4$

We now need to describe our infinite dimensional Hilbert space, defined over a 4-manifold $\tilde{M} = \mathbb{R} \times \mathbb{R}^3$.

The infinite dimensional Hilbert space we will consider is $H = L^2(\tilde{M}) \otimes W$, $W$ is some finite dimensional inner product space. The inner product on $H$ is the tensor inner product from both $L^2(\tilde{M})$ and $W$. We will denote the inner products from $L^2(\tilde{M})$, $W$ and their tensor inner product by the same symbol $\langle \cdot, \cdot \rangle$.

Let $\{e^i_u\}$ be some orthonormal basis in $W$, indexed by $i$ and $u$. The index $u$ can take values from 1 to some whole number, but $i$ will take values 1, 2 and 3. Hence, the dimension of $W$ must be a multiple of 3.

Any $w \in H$ can be written in the form $w \equiv w^i_u \otimes e^i_u$. Suppose $L : W \to \text{End}(W)$. We also assume that for $x \in \tilde{M}$, $L(x)$ is a skew-symmetric operator, i.e. $\langle L(x)z, y \rangle = -\langle z, L(x)y \rangle$. For $w^i_u, w^j_v \in L^2(\tilde{M})$, define a linear operator $L(w^i_u \otimes e^i_v) : L^2(\tilde{M}) \otimes W \to L^2(\tilde{M}) \otimes W$ by

$$[L(w^i_u \otimes e^i_v)(w^j_v \otimes e^j_w)](m) := L(w^i_u(m)e^i_v)(w^j_v(m)e^j_w) \in W.$$ 

Note that here, we evaluate at the point $m \in \tilde{M}$, so $w^i_u(m)e^i_v \in W$, similarly for $w^i_u(m)e^i_v$. Hence the operator $L(w^i_u \otimes e^i_v)$ acts pointwise for each $m \in \tilde{M}$. Typically, $L(w^i_u \otimes e^i_v)$ is some multiplication operator.

Finally, let $D : L^2(\tilde{M}) \otimes W \to L^2(\tilde{M}) \otimes W$ be some skew-symmetric differential operator, acting on $L^2(\tilde{M}) \otimes W$. So $D + L(w_-)$ is skew-symmetric, i.e.

$$\langle [D + L(w_-)]w_+, w_- \rangle = -\langle w_+, [D + L(w_-)]w_- \rangle, \quad w_\pm \in L^2(\tilde{M}) \otimes W.$$ 

At the point $\bar{x} \in \tilde{M}$,

$$[D + L(w_-)]w_+(\bar{x}) = (Dw_+) (\bar{x}) + L(w_-(\bar{x}))w_+(\bar{x}) = [D + L(w_-(\bar{x}))](w_+(\bar{x}).$$

Hence,

$$[D + L(w_-)]^{-1} w_+(\bar{x}) \equiv [D + L(w_-(\bar{x}))]^{-1} w_+(\bar{x}). \quad (6.3)$$

**Definition 6.5** Let $H = L^2(\tilde{M}) \otimes W$, whereby $W$ is a finite dimensional inner product space and $\{x_0, x_1, x_2, x_3\}$ be coordinates for $\mathbb{R} \times \mathbb{R}^3$.

Fix a basis $\{e^i_u : i = 1, 2, 3, u = 1, \ldots, d\}$ for $W$ and thus any $w_u \in H$ can be written in the form $w_u = w^i_u \otimes e^i_u$ in the latter. We denote all inner products with $\langle \cdot, \cdot \rangle$. Introduce a Lie Algebra, $V$ and suppose $\{E^a\}_{a=1}^d \subset V$ is a basis.

Define

$$\Omega_1 := \left\{ \int_{L} w^i_{\pm, u} \otimes dx_i \otimes E^u \mid L \text{ is a } 1 \text{- dim submanifold in } \mathbb{R} \times \mathbb{R}^3 \right\},$$

$$\Omega_2 := \left\{ \int_{S} g^{ab}(w_-) \otimes dx_a \wedge dx_b \mid S \text{ is a } 2 \text{- dim submanifold in } \mathbb{R} \times \mathbb{R}^3 \right\},$$

$$\Omega_3 := \left\{ \int_{R} g^{abc}(w_-) \otimes dx_a \wedge dx_b \wedge dx_c \mid R \text{ is a } 3 \text{- dim submanifold in } \mathbb{R} \times \mathbb{R}^3 \right\}.$$ 

In the above, $g^{ab}$ and $g^{abc}$ are continuous functions on $W$.
Remark 6.6  1. Note that the functions \( g^{ab} \) need not be real-valued. To keep things in general, one can assume that it takes values in some finite dimensional vector space.

2. For a 1-submanifold \( M^1 \) to be considered in \( \Omega_1 \), we typically consider a hyperlink. However, we may in certain occasions consider an open curve. Note that an integral in \( \Omega_1 \) is actually \( V \)-valued.

3. For a 2-submanifold \( M^2 \) to be considered in \( \Omega_2 \), henceforth referred to as a surface, we do allow the surface to have a boundary. Typically, we consider closed and bounded surfaces, with or without boundary. We can consider a finite union of such surfaces, with empty intersection.

4. For a 3-submanifold \( M^3 \) to be considered in \( \Omega_3 \), it should be closed and bounded, henceforth referred to as a compact region. We allow it to be disconnected, with finitely many components.

Example 6.7 Assume that \( L \equiv \{ l^v \}_{v=1}^\infty \) is a hyperlink. Now, to compute a typical integral from \( \Omega_1 \), for each loop \( l^v \), we will choose a parametrization \( \bar{y}^v \equiv (y_0^v, y_1^v, y_2^v, y_3^v) : I \rightarrow \bar{M} \) such that \( \bar{y}^v(I) \) describes the loop \( l^v \subset \bar{M} \). Similarly, for each loop \( l^w \) in \( L \equiv \{ l^w \}_{w=1}^\infty \), let \( \bar{y}^w \equiv (y_0^w, y_1^w, y_2^w, y_3^w) : I \rightarrow \bar{M} \) such that \( \bar{y}^w(I) \) describes the loop \( l^w \subset \bar{M} \).

Explicitly, the integral in \( \Omega_1 \) is computed as

\[
\int_{L} \sum_{v=1}^\infty \int_{I} w^v_{+,u}(\bar{y}^v(s))y^v_i(s) \otimes E^u \, ds =: \int_{\bar{L}} G^1 \{ w^v_{+,u}y^v_i \otimes E^u \}. \tag{6.4}
\]

Here, \( E^u \equiv E^u \) and the subscript \( s \) is to keep track of the ordering of the matrices \( E^u \). This is necessary because when we take tensor products of these terms, we need to time order them according to the time ordering operator, given in Definition 3.3.

We will also need the following integral,

\[
\int_{L} \sum_{u=1}^d \sum_{v=1}^\infty \int_{I} w^v_{-,u}(\bar{y}^v(s))g^v_i(s) \, ds =: \int_{\bar{L}} G^1 \{ w^v_{-,u}g^v_i \}. \tag{6.5}
\]

Note that this integral is real-valued.

Example 6.8 (Surface Integral)
Assume that \( M^2 = S \) is a connected surface and \( \{ g^{ab} \} \) is a set of continuous real-valued functions on \( W \). Now, to compute a typical integral from \( \Omega_2 \), we will choose a parametrization \( \bar{\sigma} \equiv (\sigma_0, \sigma_1, \sigma_2, \sigma_3) : I^2 \rightarrow \bar{M} = \mathbb{R} \times \mathbb{R}^3 \) such that \( \bar{\sigma}(I^2) \) describes the surface \( S \subset \bar{M} \). See Notation 5.55.

Recall \( w_- = w_{-\,u} \otimes e^u_i \) and \( w_-(\bar{x}) \equiv w_{-\,u}(\bar{x})e^u_i \). Explicitly, the integral becomes \( (\hat{s} = (\hat{s}, \hat{t})) \)

\[
\int_{S} g^{ab}(w_-) \, dx_a \wedge dx_b = \int_{I^2} g^{\hat{b}i}(w_-(\bar{\sigma}))(J_{0\hat{i}}(\hat{s}) \, d\hat{s} + \int_{I^2} g^{\hat{b}j}(w_-(\bar{\sigma})) \, J_{\hat{r}\hat{j}}(\hat{s}) \, d\hat{s} =: \int_{M^2} G^2 \{ w_{-\,u}(\bar{x}) \}.
\]
Example 6.9 (Volume Integral)
Assume that $M^3 \equiv R$ is a compact region and \{g^{abc}\} is a set of real-valued continuous functions on $W$. For simplicity, we assume that $\Omega^3 \subset \{0\} \times \mathbb{R}^3$. Now, to compute a typical integral from $\Omega_3$, we will choose a parametrization $\vec{\rho} \equiv (0, \rho_1, \rho_2, \rho_3) \equiv (0, \rho) : I^3 \to M$ such that $\vec{\rho}(I^3)$ describes the compact region $R \subset \mathbb{R} \times \mathbb{R}^3$. Now, $w_- = w_{\rho}^\pm \otimes e^u_\mp$ and $w_- (\vec{\omega}) \equiv w_{\rho}^\pm (\vec{x}) e^u_\mp$.

Explicitly, the integral becomes ($\tau \equiv (\tau_1, \tau_2, \tau_3)$)

$$
\int_R g^{abc}(w_-) dx_a \wedge dx_b \wedge dx_c \equiv \int_{I^3} g^{abc}(w_-(\vec{\rho}(\tau))) \left| \frac{\partial \vec{\rho}}{\partial \tau_1} \right| \left| \frac{\partial \vec{\rho}}{\partial \tau_2} \right| \left| \frac{\partial \vec{\rho}}{\partial \tau_3} \right| (\tau) \, d\tau_1 d\tau_2 d\tau_3
$$

Recall the time ordering operator $\mathcal{T}$ defined in Definition 3.3. Note that

$$
\mathcal{T} \left[ \exp \left[ \sum_{i=1}^7 \int_I w_{\rho}^\pm (\vec{\rho}(s)) y_i^v(s) \otimes E^u_\pm ds \right] \right] = \bigoplus_{n=0}^\infty V_{\mathbb{C}}^\otimes n,
$$

where $\vec{\rho}^\tau = (\vec{x}, \vec{\rho}, \tau)$, we will choose a parametrization $\vec{\rho}$. For simplicity, we assume that $\vec{\rho}$.

**Remark 6.10** We refer the reader to [8] on how does one compute

$$
\mathcal{T} \left[ \exp \left( \sum_{i=1}^7 \int_I w_{\rho}^\pm (\vec{\rho}(s)) y_i^v(s) \otimes E^u_\pm ds \right) \right].
$$

However, as shown in our calculations later, it is not necessary to keep track of the subscript $s$ and we can effectively drop the time ordering operator for simplicity. Thus, the reader can safely ignore the subscript $s$.

Recall $w_\pm = w_{\rho}^\pm \otimes e^u_\mp$ and let

$$
\exp \left[ i (w_{\rho}^\pm \otimes e^u_\mp, [D + L(w_-)] w_{\rho}^\pm \otimes e^u_\mp) \right] \, D w_\pm \equiv D \Lambda.
$$

Let $Z = \int_{H \subset H} D \Lambda$. As it turns out, certain functional integrals can be written similarly to a Chern-Simons integral.

Suppose we want to make sense of the following path integral

$$
\frac{1}{Z} \int_{(w_+, w_-) \in H^2} \int_{M^p} G^p (w_{\rho}^\pm (\vec{\omega})) \otimes \left[ e^I \wedge w_{\rho}^\pm \otimes dx_\pm \otimes E^u_\mp \right] \sum_{i} w_{\rho}^\pm \otimes dx_i \right] D \Lambda, \quad (6.6)
$$

for $p = 0, 2, 3$. See Examples 6.1, 6.8 and 6.9. When $p = 0$, we define $\int_{M^p} G^p (w_- (\vec{\omega})) := 1$.

**Remark 6.11** 1. The above path integral is not the most general form which we can give a definition.
2. If we apply the time ordering operator, then the path integral will take values in

\[ \bigoplus_{n=0}^{\infty} V^{\otimes n}_C. \]

Let \( \delta^2 \) be the Dirac-delta function, i.e. \( w^i_{\pm,u}(x) \equiv \langle w^i_{\pm,u}, \delta^2 \rangle \). Of course, \( \delta^2 \) does not lie in the Hilbert space \( L^2(\hat{M}) \). So we have to approximate it using \( L^1 \) functions on \( \hat{M} \), i.e. \( p^2_\kappa \rightarrow \delta^2 \) in the distribution sense, as \( \kappa \rightarrow \infty \).

**Example 6.12** Refer to Example 6.7. We can write Equations (6.4) and (6.5) as

\[ \sum_{i=1}^{\infty} \int_I ds \langle w^i_{+,u}, \delta^0(s) \rangle y_i^{v',i}(s) \otimes E^u =: \int_I C^i \{ \langle w^i_{+,u}, \delta^0 \rangle y_i^{v',i} \otimes E^u \}, \tag{6.7} \]

respectively. Replace \( \delta \) with \( p_\kappa \) in Equation (6.7), then we have

\[ \sum_{i=1}^{\infty} \int_I ds \sum_{d=1}^{d} \langle w^i_{+,u}, p^0_\kappa(s) \rangle y_i^{v',i}(s) \otimes E^u =: \int_I G^i \{ \langle w^i_{+,u}, p^0_\kappa \rangle y_i^{v',i} \otimes E^u \}. \tag{6.8} \]

Write

\[ \pi_+ \{ \bar{g}^v \} := \sum_{v=1}^{\infty} \sum_{i=1}^{3} \sum_{u=1}^{d} \int_I ds \ p^0_\kappa(s) y_i^{v',i}(s) \otimes e_i^u \otimes E^u. \tag{6.9} \]

**Remark 6.13** Note that \( \pi_+ \{ \bar{g}^v \} \) is in \( L^2(\hat{M}) \otimes W \otimes V \). And

\[ \left( \int_I ds \ p^0_\kappa(s) y_j^{v',j}(s) \otimes e_j^0 \otimes E^u \right) = \int_I ds \left( \int_I \int ds \ p^0_\kappa(s) y_j^{v',j}(s) \otimes E^u \right), \]

which takes values in \( V \). Note that on both the LHS and RHS, there is no sum over the index \( j \) and \( \bar{u} \).

To write down the definition for the path integral in Expression (6.6) we have to go through the long and tedious process of constructing an Abstract Wiener space using the Segal Bargmann Transform, applying Definition 6.3 and furthermore, approximating the Dirac-delta function using a Gaussian function. This was the approach used in \( [8] \) and \( [9] \) to define the Chern-Simons path integrals, given respectively by Equation (2.6) in \( [8] \) and Equation (24) in \( [9] \). The same approach can be adapted to give a definition for Expression (6.6).

In anticipation of future work, we propose to write down a set of rules, so that in future, one can write down a definition for the path integral in Expression (6.6) without going through the full derivation using the Abstract Wiener space approach. These rules are derived from Equation (2.6) in \( [8] \) and Equation (24) in \( [9] \).

**Definition 6.14** The following are the Chern-Simons rules for evaluating an integral given by Expression (6.6).
1. Replace \( w_{i,u}^i(\vec{y}) \) with \( \langle w_{i,u}^i, \delta \bar{y} \rangle \).

2. Write \( T_{w_{-}} = D + L(w_{-}) \) and note that \( T_{w_{-}}^{-1} \equiv [D + L(w_{-})]^{-1} \) is skew-symmetric, i.e.

\[
\langle T_{w_{-}}^{-1}w_{+}, w_{-} \rangle = -\langle w_{+}, T_{w_{-}}^{-1}w_{-} \rangle, \quad w_{\pm} \in L^2(M) \otimes W.
\]

See Equation (6.3) for the definition of \( T_{w_{-}}^{-1} \).

Now

\[
w_{i,u}^j(\vec{x}) = \langle w_{i,u}^j, \vec{y} \rangle e_{i,j}^u = \left\langle [T_{w_{-}}^{-1}w_{-}^j, \delta \bar{y}^u](\vec{x}), e_{i,j}^u \right\rangle = \left\langle T_{w_{-}}^{-1}(x)[T_{w_{-}}w_{-}](\vec{x}), e_{i,j}^u \right\rangle.
\]

Here, \( T_{w_{-}}^{-1} = [D + L(w_{-}(\vec{x}))]^{-1} \).

So we can write

\[
\langle T_{w_{-}}^{-1}(x)[T_{w_{-}}w_{-}](\vec{x}), e_{i,j}^u \rangle = \langle T_{w_{-}}^{-1}(x)T_{w_{-}}w_{-}, \delta \bar{y}^u \rangle = -\langle T_{w_{-}}w_{-}, T_{w_{-}}^{-1}(x)\delta \bar{y}^u \rangle.
\]

Hence the integral can be written in the form

\[
\frac{1}{Z} \int_{(w_{+}, w_{-}) \in H \times 2} D\bar{w}_{+}Dw_{-}e^{i(w_{+}, T_{w_{-}}w_{-})} \int_{M^p} G^p \left\{ \left\langle T_{w_{-}}^{-1}(x)T_{w_{-}}w_{-}, \delta \bar{y}^u \right\rangle \right\}
\]
\[
\otimes \exp \left( \int_{\mathcal{T}} G^i \left\{ (w_{i,u}^j, \delta \bar{y}^u) y_{i,j}^u \otimes E^u \right\} \right)
\]
\[
\exp \left( \int_{L} G^1 \left\{ \left\langle T_{w_{-}}^{-1}(x)T_{w_{-}}w_{-}, \delta \bar{y}^u \right\rangle \otimes \sum_{i=1}^{3} g_{i,j}^u e_{i}^u \right\} \right)
\]

which is equal to

\[
\frac{1}{Z} \int_{(w_{+}, w_{-}) \in H \times 2} D\bar{w}_{+}Dw_{-}e^{i(w_{+}, T_{w_{-}}w_{-})} \int_{M^p} G^p \left\{ -\left\langle T_{w_{-}}w_{-}, T_{w_{-}}^{-1}(x)\delta \bar{y}^u \right\rangle \right\}
\]
\[
\otimes \exp \left( \int_{\mathcal{T}} G^i \left\{ (w_{i,u}^j, \delta \bar{y}^u) y_{i,j}^u \otimes E^u \right\} \right)
\]
\[
\exp \left( \int_{L} G^1 \left\{ -\left\langle T_{w_{-}}w_{-}, T_{w_{-}}^{-1}(x)\delta \bar{y}^u \right\rangle \otimes \sum_{i=1}^{3} g_{i,j}^u e_{i}^u \right\} \right)
\]

Replace \( T_{w_{-}}w_{-} \) by \( w_{-} \) and write \( w_{-}(\vec{x}) = \langle w_{i,u}^i, \delta \bar{y}^u \rangle e_{i}^u \in W \). Hence

\[
T_{w_{-}}^{-1}(x) = T_{w_{-}}^{-1}(x)\delta \bar{y}^u = T_{w_{-}}^{-1}(x)\delta \bar{y}^u = T_{w_{-}}^{-1}(x)\delta \bar{y}^u = T_{w_{-}}^{-1}(x)\delta \bar{y}^u = T_{w_{-}}^{-1}(x)\delta \bar{y}^u = T_{w_{-}}^{-1}(x)\delta \bar{y}^u.
\]

So we will now focus on the following path integral of the form

\[
\frac{1}{Z} \int_{(w_{+}, w_{-}) \in H \times 2} D\bar{w}_{+}Dw_{-}e^{i(w_{+}, w_{-})} \int_{M^p} G^p \left\{ -\left\langle w_{-}, T \left[ w_{i,u}^i, \delta \bar{y}^u \right] e_{u}^j \right\rangle \otimes \sum_{i=1}^{3} g_{i,j}^u e_{i}^u \right\}
\]
\[
\otimes \exp \left( \int_{\mathcal{T}} G^i \left\{ (w_{i,u}^j, \delta \bar{y}^u) y_{i,j}^u \otimes E^u \right\} \right)
\]
\[
\exp \left( \int_{L} G^1 \left\{ -\left\langle w_{-}, T \left[ w_{i,u}^i, \delta \bar{y}^u \right] e_{u}^j \right\rangle \otimes \sum_{i=1}^{3} g_{i,j}^u e_{i}^u \right\} \right),
\]

(6.10)
which resembles that of a Chern-Simons integral and
\[ Z = \int_{(w_+, w_-) \in H^2} Dw_+ Dw_- e^{i(w_+, w_-)}. \]

**Remark 6.15** Instead of giving a plausible meaning to Expression (6.10), we will now define Expression (6.10) by treating it as a Chern-Simons integral.

3. Because \( \delta^2 \) is not inside \( L^2(\tilde{M}) \), we replace it with a Gaussian-like function supported in \( \tilde{M} \), \( p^\kappa_\pi \), such that \( p^\kappa_\pi \to \delta^2 \) in distribution. The above integral now becomes
\[
\frac{1}{Z} \int_{(w_+, w_-) \in H^2} Dw_+ Dw_- e^{i(w_+, w_-)} \int_{M^p} G^p \left\{ - \left\langle w_-, T \left( (w^\pi_+, p^\pi_\kappa) e^u_j \right) \right\rangle^{-1} p^\pi_\kappa \otimes e^u_i \right\}
\]
\[
\otimes \exp \left( \int_{\mathcal{T}} C_{\mathcal{T}}^1 \left\{ \left\langle \left( w^\pi_+, p^\pi_\kappa \right) y^\kappa_{\pi'} \otimes E^u \right\rangle \right\} \right)
\]
\[
\exp \left( \int_{\mathcal{L}} G_{\mathcal{L}}^1 \left\{ - \left\langle w_-, T \left[ \left( w^\pi_-, p^\pi_\kappa \right) e^u_j \right] \right\rangle^{-1} p^\pi_\kappa \otimes \sum_{i=1}^3 \delta_i e^u_i \right\} \right) \right) \].

Note that the operator \( T \left( (w^\pi_-, p^\pi_\kappa) e^u_j \right) \) acts on \( p^\pi_\kappa \otimes \sum_{i=1}^3 \delta_i e^u_i \in L^2(\tilde{M}) \otimes W \).

4. We need to put in factors of \( \bar{\kappa} = \kappa/2\sqrt{\pi} \). For each integral from \( \Omega_{\pi'} \), we scale it by factors of \( \bar{\kappa} \) as follows:
\[
\int_{M^p} G^p \mapsto \bar{\kappa} \int_{M^p} G^p, \quad \int_{\mathcal{L}} G_{\mathcal{L}}^1 \mapsto \bar{\kappa} \int_{\mathcal{L}} G_{\mathcal{L}}^1.
\]
However, for \( \int_{\mathcal{T}} C_{\mathcal{T}}^1 \), we scale it by a factor of \( \kappa \bar{\kappa} \), i.e.
\[
\int_{\mathcal{T}} C_{\mathcal{T}}^1 \mapsto \kappa \bar{\kappa} \int_{\mathcal{T}} C_{\mathcal{T}}^1.
\]

5. Refer to Equation (6.9) for the definition of \( \pi_+ \{ \bar{y}^\pi \} \). From Equations (6.7) and (6.8),
\[
\kappa \bar{\kappa} \int_{\mathcal{T}} C_{\mathcal{T}}^1 \left\{ \left\langle \left( w^\pi_+, p^\pi_\kappa \right) y^\kappa_{\pi'} \otimes E^u \right\rangle \right\} = \left\langle w_+, \kappa \bar{\kappa} \pi_+ \{ \bar{y}^\pi \} \right\rangle.
\]
Finally apply Definition (6.9), i.e. replace \( w^\pi_+ \) and \( w_+ \) respectively with
\[
w^\pi_-, u \mapsto \sqrt{-1} \kappa \bar{\kappa} \pi^\dagger_{+, u} \{ \bar{y}^\pi \} := \sqrt{-1} \kappa \bar{\kappa} \sum_{v=1}^\pi \int I p^\pi_\kappa(s) y^v_{\pi'}(s) \, ds,
\]
\[
w_-, u \mapsto \sqrt{-1} \kappa \bar{\kappa} \pi_+ \{ \bar{y}^\pi \} = \sqrt{-1} \kappa \bar{\kappa} \sum_{v=1}^\pi \sum_{i=1}^d \sum_{u=1}^d \int I p^\pi_\kappa(s) y^v_{\pi'}(s) \otimes e^u_i \otimes E^u_i,
\]
and the path integral is defined as
\[
\bar{\kappa} \int_{M^p} G^p \left\{ - \left\langle i \kappa \bar{\kappa} \pi_+ \{ \bar{y}^\pi \}, T \left[ \left( i \kappa \bar{\kappa} \pi^\dagger_{+, u} \{ \bar{y}^\pi \}, p^\pi_\kappa \right) e^u_j \right] \right\rangle^{-1} p^\pi_\kappa \otimes e^u_i \right\}
\]
\[
\otimes \exp \left( \bar{\kappa} \int_{\mathcal{L}} G_{\mathcal{L}}^1 \left\{ - \left\langle i \kappa \bar{\kappa} \pi_+ \{ \bar{y}^\pi \}, T \left[ \left( i \kappa \bar{\kappa} \pi^\dagger_{+, u} \{ \bar{y}^\pi \}, p^\pi_\kappa \right) e^u_j \right] \right\rangle^{-1} p^\pi_\kappa \otimes \sum_{i=1}^3 \delta_i e^u_i \right\} \right) .
\]
Remark 6.16

1. In Item 4, we scaled by factors

\[ \kappa \]

and

\[ \nu_\alpha(1; \bar{v}, \kappa) = \sqrt{-1\kappa \pi \bar{u}} \{ \bar{y}^\nu \}, \nu_\alpha(s) \] := \sqrt{-1\kappa \pi} \int_I \langle p_{\kappa}^{\bar{v}}(s), p_{\kappa}^{\bar{u}} \rangle \ y_i^{\nu}(s) \ ds \]

and

\[ \nu_\alpha(p; \kappa) = \sqrt{-1\kappa \pi \bar{u}} \{ \bar{y}^\nu \}, \nu_\alpha(s) \] := \sqrt{-1\kappa \pi} \sum_{i=1}^{\pi} \int_I \langle p_{\kappa}^{\bar{v}}(s), p_{\kappa}^{\bar{u}} \rangle \ y_i^{\nu}(s) \ ds. \]

Note that \( \nu_\alpha(1; \bar{v}, \kappa) \in \mathbb{C} \) and \( \nu_\alpha(1; \bar{v}, \kappa)e_i^u, \nu_\alpha(p, \kappa)e_i^u \in \mathbb{C} \otimes W \). Refer to Examples 6.8 and 6.9. Then we can rewrite our expression for the path integral as

\[ \hat{p} \int_{M_p} G^p \left\{ -i \left( \kappa\pi \bar{v}_+ \{ \bar{y}^\nu \}, T \left[ \nu_\alpha(p; \kappa)e_i^u \right]^{-1} p_{\kappa}^{\bar{v}} \otimes e_i^u \right) \right\} \]

\[ \otimes \exp \left( -ik \int_{L} \mathcal{L} \left\{ \left( \kappa\pi \bar{v}_+ \{ \bar{y}^\nu \}, T \left[ \nu_\alpha(1; \bar{v}, \kappa)e_i^u \right]^{-1} p_{\kappa}^{\bar{v}} \otimes \sum_{i=1}^{3} \theta_i^{\bar{v}} e_i^u \right) \right\} \right) \]

(6.11)

with

\[ \bar{v}_+ \{ \bar{y}^\nu \} := \sum_{i=1}^{\pi} \sum_{u=1}^{d} \sum_{a=1}^{3} \int_I \ ds \ p_{\kappa}^{\bar{v}}(s) \ y_i^{\nu}(s) \otimes e_i^u \otimes E_s^u. \]

Remark 6.16

1. In Item [4] we scaled by factors \( \kappa \) for each integral over a submanifold. The reason for this scaling originated from our work in [7] and [9], whereby we scaled each line integral by a factor \( \kappa/2 \), instead of \( \kappa \). But for the line integral involving \( L \), we need an extra \( \kappa \). The reason is because we need this extra factor to obtain non-trivial limits when we take \( \kappa \) going to infinity. Such an extra factor is also used in [7] and [9].

2. Note that \( T \left[ \nu_\alpha(1; \bar{v}, \kappa)e_i^u \right]^{-1} \) can be defined, depending on how we define \( D \) and \( L \). And it acts on \( p_{\kappa}^{\bar{v}} \otimes e_i^u \in L^2(\hat{M}) \otimes W \). Typically,

\[ T \left[ \nu_\alpha(1; \bar{v}, \kappa)e_i^u \right]^{-1} p_{\kappa}^{\bar{v}} \otimes \sum_{a=1}^{d} \sum_{i=1}^{3} \theta_i^{\bar{v} \bar{v}} e_i^u = f_a^{\bar{v}}(\kappa; \bar{y}^\nu) \otimes e_i^u, \]

whereby \( f_a^{\bar{v}}(\kappa; \bar{y}^\nu) \) is a \( C^\infty \) bounded function on \( \hat{M} \).

Thus,

\[ \langle \kappa\pi \bar{v}_+ \{ \bar{y}^\nu \}, T \left[ \nu_\alpha(1; \bar{v}, \kappa)e_i^u \right]^{-1} p_{\kappa}^{\bar{v}} \otimes \sum_{a=1}^{d} \sum_{j=1}^{3} \theta_i^{\bar{v} \bar{v} \bar{v}} e_j^u \rangle \]

\[ = \kappa \pi \sum_{i=1}^{\pi} \int_I \ ds \ \langle p_{\kappa}^{\bar{v}}(s), f_a^{\bar{v}}(\kappa; \bar{y}^\nu) \rangle \ y_i^{\nu}(s) \otimes E_s^u \]

and

\[ \langle p_{\kappa}^{\bar{v}}(s), f_a^{\bar{v}}(\kappa; \bar{y}^\nu) \rangle = \int_{\mathbb{R}^4} p_{\kappa}^{\bar{v}}(s) \cdot f_a^{\bar{v}}(\kappa; \bar{y}^\nu) \ d\lambda \]

is dependent on \( s \). (See Section 6.4)
3. The term
\[
\frac{1}{k^3} \left\{ \kappa \kappa \pi \{ \bar{y} \}, T [\nu^\prime_\alpha(1; \bar{v}, \kappa) e^\prime_\alpha]^{-1} p^\kappa \otimes \sum_{i=1}^{3} g^\kappa e^\alpha_i \right\}
\]
is expressed explicitly as
\[
\sum_{v=1}^{d} \sum_{v=1}^{d} \int \mathcal{D} s \ k \kappa \left\langle p^\kappa \{ \bar{y} \}, \mathcal{F}^\kappa(s) \right\rangle \ y^\nu(s) \times E^\nu.
\]

4. The term
\[
\int_{M^p} G^p \left\{ -i \left( \kappa \kappa \pi \{ \bar{y} \}, T [\nu^\prime_\alpha(p; \kappa) e^\alpha]^{-1} (p^\kappa \otimes e^\alpha) \right) \right\}
\]
\[
\equiv \int_{M^p} G^p \left\{ i \left( \kappa \kappa T \nu^\prime_\alpha(p; \kappa) e^\alpha \right]^{-1} \pi \{ \bar{y} \}, p^\kappa \otimes e^\alpha \right\}.
\]
(6.12)
deserves some explanation.

Now,
\[
T [\nu^\prime_\alpha(p; \kappa) e^\alpha]^{-1} \pi \{ \bar{y} \} \equiv \sum_{v=1}^{d} \sum_{v=1}^{d} \sum_{i=1}^{3} \int T [\nu^\prime_\alpha(p; \kappa) e^\alpha]^{-1} \left( p^\kappa \otimes y^\nu(s) e^\alpha \right) \times E^\nu
\]
lies inside \( W \otimes V \). This linear operator \( T [\nu^\prime_\alpha(p; \kappa) e^\alpha]^{-1} \) acts on \( p^\kappa \otimes y^\nu(s) e^\alpha \).

Recall that \( V \) is some Lie Algebra. Hence the RHS of Equation (6.12) will involve \( g(E) \), whereby \( g \) is some continuous function and \( E \) will be a matrix. There will be some issues on how to define \( g(E) \), as it is not true that \( g(E) \) can make sense of. In certain cases, we will show that \( g(E) \) can be rigorously defined. So, we will leave the matter as it is and address it later for specific examples to come.

5. Later, we will show that the terms \( k^\nu \rightarrow 0 \) and \( k^\nu(1; \bar{v}, \kappa) \rightarrow 0 \) as \( k \) goes to 0. To compute the limit of Expression (6.12) as \( k \) goes to infinity, it suffices to compute the limit of
\[
\bar{k}^p \int_{M^p} G^p \left\{ i \left( \kappa \kappa D^{-1} \pi \{ \bar{y} \}, p^\kappa \otimes e^\alpha \right) \right\}
\]
\[
\otimes \exp \left[ -i \bar{k} \int \frac{1}{k^3} \left\{ \kappa \kappa \pi \{ \bar{y} \}, D^{-1} \left( p^\kappa \otimes \sum_{i=1}^{3} g^\kappa e^\alpha_i \right) \right\} \right].
\]

7 Einstein-Hilbert Path Integral

We can now begin to define the path integral in Expression (3.4). First, we need to rewrite it to make it look like the Chern-Simons path integral, as defined in Expression (6.6).

**Notation 7.1** Recall we defined \( A^\alpha_{\alpha' \beta} \) and \( B^\beta \) in Equations (3.2) and (3.3) respectively. Define the following 3-vectors,
\[
B^\beta = (B^\beta_1, B^\beta_2, B^\beta_3), \quad B_0 = (B^\beta_0, B^\beta_0, B^\beta_0), \quad B_i = (B^\beta_1, B^\beta_i, B^\beta_i),
\]
\[
A^\alpha_0 = (A^\alpha_{01}, A^\alpha_{02}, A^\alpha_{03}), \quad A^\alpha = (A^\alpha_{21}, A^\alpha_{31}, A^\alpha_{12}),
\]
\[
A^\alpha = (A^\alpha_{21}, A^\alpha_{23}, A^\alpha_{23}), \quad A_{31} = (A^\alpha_{31}, A^\alpha_{31}, A^\alpha_{31}), \quad A_{12} = (A^\alpha_{12}, A^\alpha_{12}, A^\alpha_{12}).
\]
For 3-vectors \( \mathbf{x}, \mathbf{y} \) will denote the usual dot product and \( \mathbf{x} \times \mathbf{y} \) will denote the cross product.

Write \( B = (B^1, B^2, B^3) \) and \( A_0 = (A_{0\alpha}, A_{0\beta}, A_{0\gamma}) \). Let \( \tilde{B} = B_1 \times B_0 \) and hence write \( \tilde{B} = (\tilde{B}^1, \tilde{B}^2, \tilde{B}^3) \), \( A = (A_{23}, A_{31}, A_{12}) \). These vectors thus defined are 9-vectors. By abuse of notation, we write for a 9-vector \( F = (F^1, F^2, F^3) \), each \( F^i \) is a 3-vector,

\[
B \times F = \frac{1}{2} \left( B^2 \times F^3 - B^3 \times F^2, B^3 \times F^1 - B^1 \times F^3, B^1 \times F^2 - B^2 \times F^1 \right).
\]

In particular,

\[
B \times B = \frac{1}{2} \left( B^2 \times B^1 - B^3 \times B^2, B^3 \times B^1 - B^1 \times B^3, B^1 \times B^2 - B^2 \times B^1 \right)
= (B^2 \times B^1, B^3 \times B^1, B^1 \times B^2) = ([B \times B]_1, [B \times B]_2, [B \times B]_3),
\]

which is a 9-vector. Finally, we write

\[
\partial_0 A_0 \cdot B \times B := \partial_0 A_0 \cdot [B \times B]_1.
\]

We will also use the dot to denote the usual scalar product for 9-vectors.

With Notation \( 7.1 \) we have the following proposition.

**Proposition 7.2** Using Equations \( (3.2) \) and \( (3.3) \), the Einstein-Hilbert action given in Equation \( (1.1) \) can be written as

\[
\int_{\mathbb{R}^4} \partial_0 A_0 \cdot B \times B - \int_{\mathbb{R}^4} \partial_0 A \cdot \tilde{B},
\]

whereby it is understood we integrate over Lebesgue measure on \( \mathbb{R}^4 \).

**Proof.** Refer to Notation \( 2.5 \). Suppose that \( \omega \) and \( e \) are given by Equations \( (3.2) \) and \( (3.3) \). We have

\[
e \wedge e = B^i \gamma \wedge E^\gamma \wedge dx_i \wedge B^j_\mu \wedge E^\mu \wedge dx_j
= \frac{1}{2} B^i_\mu B^j_\nu \wedge (E^\gamma \wedge E^\mu \wedge E^\nu) \wedge dx_i \wedge dx_j
= B^i_\mu B^j_\nu \wedge E^\gamma \wedge dx_i \wedge dx_j + B^i_\mu B^j_\nu \wedge E^\mu \wedge dx_2 \wedge dx_3 + B^i_\mu B^j_\nu \wedge E^\nu \wedge dx_3 \wedge dx_1.
\]

And

\[
d\omega + \omega \wedge \omega = \partial_0 A^i_\alpha_\beta \wedge E^{i\alpha} \wedge dx_0 \wedge dx_i + \frac{1}{2} A^{i_1}_{\alpha_\beta} A^{i_2}_{\mu_\nu} \wedge [E^{i_1\alpha}, E^{i_2\beta}] \wedge dx_i \wedge dx_j
+ \frac{\partial}{\partial x_i} A^i_\alpha_\beta \wedge E^{i\alpha} \wedge dx_i \wedge dx_j, \quad \partial_0 \equiv \frac{\partial}{\partial x_0}.
\]

Thus, the Einstein-Hilbert action becomes

\[
\frac{1}{8} \int_{\mathbb{R}^4} e^{abcd} [e \wedge e]_{cd} \wedge [R]_{ab}
= \frac{1}{8} \int_{\mathbb{R}^4} e^{abcd} B^{i_1}_\gamma B^{i_2}_\mu [E^{i_1\alpha}]_{ab} \cdot \partial_0 A^{i_1}_{\alpha_\beta} [E^{i_2\beta}]_{cd} dx_1 \wedge dx_2 \wedge dx_0 \wedge dx_3
+ \frac{1}{8} \int_{\mathbb{R}^4} e^{abcd} B^{i_3}_\mu [E^{i_1\alpha}]_{ab} \cdot \partial_0 A^{i_3}_{\alpha_\beta} [E^{i_2\beta}]_{cd} dx_2 \wedge dx_3 \wedge dx_0 \wedge dx_1
+ \frac{1}{8} \int_{\mathbb{R}^4} e^{abcd} B^{i_4}_\mu [E^{i_1\alpha}]_{ab} \cdot \partial_0 A^{i_4}_{\alpha_\beta} [E^{i_2\beta}]_{cd} dx_3 \wedge dx_1 \wedge dx_0 \wedge dx_2.
\]

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The integral can be written as
\[
\frac{1}{2} \int_{\mathbb{R}^4} B^1 \times B^2 \cdot \partial_0 A^0_0 + B^2 \times B^3 \cdot \partial_0 A^0_1 + B^3 \times B^1 \cdot \partial_0 A^0_3
\]
\[
- \frac{1}{2} \int_{\mathbb{R}^4} B^2 \times B^1 \cdot \partial_0 A^0_0 + B^3 \times B^2 \cdot \partial_0 A^0_1 + B^1 \times B^3 \cdot \partial_0 A^0_3
\]
\[
- \int_{\mathbb{R}^4} B^1 B^3_0 \cdot \partial_0 A^3 + B^2 B^3_0 \cdot \partial_0 A^1 + B^3 B^3_0 \cdot \partial_0 A^2
\]
\[
+ \int_{\mathbb{R}^4} B^2 B^3_0 \cdot \partial_0 A^3 + B^3 B^3_0 \cdot \partial_0 A^1 + B^3 B^3_0 \cdot \partial_0 A^2,
\]
which can be further simplified into
\[
\frac{1}{2} \int_{\mathbb{R}^4} B^1 \times B^2 \cdot \partial_0 A^0_0 + B^2 \times B^3 \cdot \partial_0 A^0_1 + B^3 \times B^1 \cdot \partial_0 A^0_3
\]
\[
- \frac{1}{2} \int_{\mathbb{R}^4} B^2 \times B^1 \cdot \partial_0 A^0_0 + B^3 \times B^2 \cdot \partial_0 A^0_1 + B^1 \times B^3 \cdot \partial_0 A^0_3
\]
\[
- \int_{\mathbb{R}^4} \partial_0 A_{23} \cdot [B_1 \times B_0] + \partial_0 A_{31} \cdot [B_2 \times B_0] + \partial_0 A_{12} \cdot [B_3 \times B_0].
\]

Using Notation 7.1 the Einstein-Hilbert action can be written as
\[
\int_{\mathbb{R}^4} \partial_0 A_0 \cdot B \times B - \int_{\mathbb{R}^4} \partial_0 A \cdot \hat{B}.
\]

If we treat \( \hat{B} \) as an independent variable, then we are lead to define a path integral with product
\[
\exp \left[ i \int_{\mathbb{R}^4} \partial_0 A_0 \cdot B \times B - \partial_0 A \cdot \hat{B} \right] D[B] D[A] D[\hat{B}] D[A],
\]
with all \( A_0, A, B \) and \( \hat{B} \) being independent variables. Write
\[
DA = D[B] D[A_0] D[\hat{B}] D[A].
\]

Notice that there are 2 measures, namely
\[
\exp \left[ i \int_{\mathbb{R}^4} \partial_0 A_0 \cdot B \times B \right] D[B] D[A_0] \text{ and } \exp \left[ -i \int_{\mathbb{R}^4} \partial_0 A \cdot \hat{B} \right] D[\hat{B}] D[A].
\]

To reconcile with the model in Subsection 7.1 we see that \( W = \mathbb{R}^9 \). The Lie Algebra we have in mind will be \( V = su(2) \times su(2) \). The Hilbert space we need to consider will hence be
\[
H = \mathbb{S}_c(\mathbb{R}^4) \otimes \mathbb{R}^9 \otimes su(2) \times su(2).
\]

**Notation 7.3** Let \( \hat{E} = (1, 1, 1) \) be a 3-vector.

**Notation 7.4** Refer to Notations 2.2 and 2.5. Consider two oriented hyperlinks, \( \mathcal{L} = \{ \mathcal{L}^i \}_{i=1}^{\dim}, \)
\( \mathcal{L} = \{ \mathcal{L}^\alpha \}_{\alpha=1}^{2n} \) in \( \mathbb{R}^4 \), which together form an oriented hyperlink \( \chi(\mathcal{L}, \mathcal{L}) \). Color the former \( \mathcal{L} \) with a representation for each component. Let \( q \in \mathbb{R} \) be known as a charge. Define
\[
V(\{ \mathcal{L}^\alpha \}_{\alpha=1}^{2n})((B^i)) := \exp \left[ \sum_{i=1}^{n} \int_{\mathcal{L}^i} \hat{E} \cdot B^i \otimes dx_i + B^\alpha_0 \otimes dx_i \right],
\]
\[
W(q; \{ \mathcal{L}^i \}, \rho_{\alpha})((A_{\alpha,\beta}^k)) := \prod_{\alpha=1}^{2n} \text{Tr} \rho_{\alpha} \mathcal{J} \exp \left[ q \int_{\mathcal{L}^i} A^\alpha_{\alpha,\beta} \otimes dx_i \otimes \hat{E}^{\alpha,\beta} + A^k_{\alpha,\beta} \otimes dx_k \otimes \hat{E}(j) \right].
\]
Remark 7.5  
1. Here, $T$ is the time-ordering operator defined in Definition 3.3.
2. The functional $W$ is to compute the holonomy of a connection, along a hyperlink $\bar{L}$.
3. The functionals $V$ and $W$ will appear in Sections 8, 9 and 10.

8 Area Path Integral

Refer to Notation 2.9. Fix a closed and bounded orientable surface, with or without boundary, denoted by $S$, inside $\mathbb{R}^3$. We allow $S$ to be disconnected, with finite number of components. Parametrize it using $\sigma: \hat{t} \equiv (t, \tilde{t}) \in I^2 \rightarrow \sigma(\hat{t}) \in \mathbb{R}^3$ and let $J_\sigma$ denote the Jacobian of $\sigma$. We will write $J_\sigma = (J_{23}, J_{31}, J_{12})$.

We can always write $S$ as a finite disjoint union of surfaces because we allow ambient isotopy of $S$, so without any loss of generality, we assume that the surface lies in the $x_2 - x_3$ plane. Hence, $S \subset \mathbb{R}^3$ with normal given by $e_1 = (1, 0, 0)$. Write $\vec{\sigma} = (0, \sigma)$. We will also assume that the projected link $\pi_0(L)$ intersects $S$ at finitely many points.

Using the dynamical variables $\{B_i^\alpha\}$ and the Minkowski metric $\eta^{ab}$, we see that the metric $g^{ab} \equiv B^a_\mu \eta^{\mu\gamma} B^b_\gamma$ and the area is given by

$$\text{Area of } S(\{B_i^\mu\}) \equiv A_S(\{B_i^\mu\}) := \int_S \sqrt{g^{22}g^{33} - (g^{23})^2} \ dA.$$  

Explicitly,

$$A_S(\{B_i^\mu\}) = \int_{I^2} \sqrt{g^{22}g^{33} - (g^{23})^2} (\tilde{\sigma}(\hat{t})) J_{23}(\hat{t}) \ d\hat{t}.$$  

Refer to Notation 7.4. We want to define the following area path integral,

$$\frac{1}{Z_{EH}} \int V((\hat{L})^n_{\rho=1})(\{B_i^\mu\})W(q; \{\vec{\tau}_\rho\}_{\rho=1})\left(\{A_{\alpha\beta}^k\}\right)$$

$$\times \ A_S(\{B_i^\mu\}) \exp \left[ i \int_{\mathbb{R}^4} \partial_0 A_0 \cdot B \times B - \partial_0 A \cdot \hat{B} \right] \ D\Lambda, \quad (8.1)$$

with

$$Z_{EH} = \int \exp \left[ i \int_{\mathbb{R}^4} \partial_0 A_0 \cdot B \times B - \partial_0 A \cdot \hat{B} \right] \ D\Lambda.$$  

Remark 8.1  
1. When $S$ is the empty set, we define $A_\emptyset \equiv 1$, so we write Expression 8.1 as $Z(q; \chi(T, L))$, which in future be termed as the Wilson Loop observable of the colored hyperlink $\chi(T, L)$.

2. We will write Expression 8.1 as $\hat{A}_S[Z(q; \chi(T, L))]$.

We will now make use of Definition 6.14 to make sense of the path integral in Expression 8.1. We will go through the steps in Definition 6.14 in detail.
Let $e_1 = (1, 0, 0)$ be a 3-vector. Firstly, note that $g^{ij} = B^i \cdot B^j - B_0^i \cdot B_0^j$ and all other entries are 0. Secondly, it is straightforward to see that

$$g^{22}g^{33} - (g^{33})^2 = |B^2 \times B^3|^2 - |B_1 \times B_0 \cdot e_1|^2 - |B_2 \times B_0 \cdot e_1|^2 - |B_3 \times B_0 \cdot e_1|^2.$$ 

Refer to Notation 6.14

Observe that $B^2 \times B^3 - B^3 \times B^2 = 2B^2 \times B^3$. Thus, we can also write

$$g^{22}g^{33} - (g^{33})^2 = |[B \times B_1]|^2 - |B_1 \times B_0 \cdot e_1|^2 - |B_2 \times B_0 \cdot e_1|^2 - |B_3 \times B_0 \cdot e_1|^2. \quad (8.2)$$

**Notation 8.2** Write $m(B) := B \times$, $m(B^i) := B^i \times$ and $m(B_0) := B_0 \times$ as operators, so we have

$$m(B^i)B^j \equiv B^i \times B^j, \quad m(B_0)B_0 \equiv B_1 \times B_0 \times$$

and $m(B)B \equiv B \times B$. There is no sum over the index $i$.

For a 3-vector $v \equiv (v_1, v_2, v_3)$, we write $m(B)^{-1}v$ to mean $m(B)^{-1}$ acting on the 9-vector $(v_1 \hat{E}, v_2 \hat{E}, v_3 \hat{E})$, $\hat{E} = (1, 1, 1)$. And $[m(B)^{-1}]^i_3$ will also refer to the $j$-th spatial component, which is a 3-vector, with $[m(B)^{-1}v]_i$ referring to its $i$-th component. We will also write $m(B)^{-1}(v \otimes \hat{E}) \equiv m(B)^{-1}v$.

**Definition 8.3** Let $\delta^2$ be the Dirac-delta function, i.e. for any function $f$, $(f, \delta^2) = f(\vec{r})$. Also refer to Notation 7.3

Define

$$\tilde{V}([B^i]) := \exp \left\{ \sum_{j=1}^{3} [B^i] - \sum_{v=1}^{n} \int_{0}^{1} \left[ m(B, \delta^2) \right]^{-1} (\delta^2_x \delta^2_y) \right\} d\delta$$

$$+ \left\{ \tilde{B}^i - \sum_{v=1}^{n} \int_{0}^{1} m(B, \delta^2) \right\}^{-1} (\delta^2_x \delta^2_y) d\delta, \quad (8.3)$$

$$\tilde{W}([A^i_{\alpha \beta}]) := \prod_{u=1}^{3} \text{Tr}_{\rho_u} \mathcal{T} \exp \left[ \left( A^0_{0}, -q \int_{0}^{1} d\delta \delta^0 \delta^0 \cdot y_{u,s} \otimes \hat{E}_{s} \right) + \left( A^k_{s}, -q \int_{0}^{1} d\delta \delta^0 \delta^k \cdot y_{u,s} \otimes \hat{E}_{s} \right) \right] \quad (8.4)$$

**Lemma 8.4** Refer to the parametrizations $\vec{y}^u$ and $\vec{e}^u$ defined in Notation 8.8

After doing a change of variables given in Notation 8.9 the path integral in Expression (8.4) becomes

$$\frac{1}{Z_{EH}} \int \tilde{V}([B^i]) \tilde{W}([A^i_{\alpha \beta}]) \tilde{A}_S(B, [\tilde{B}^i]) e^{i f_{\mathbb{A}} A^0 \cdot B - A \cdot \tilde{B}} D\Lambda \quad (8.5)$$

with

$$\tilde{Z}_{EH} = \int \exp \left[ i \int_{\mathbb{R}^4} A^0 \cdot B - A \cdot \tilde{B} \right] D\Lambda,$$

after applying Steps 1 and 2 in Definition 6.14

Here, $\tilde{V}$ and $\tilde{W}$ are defined by Equations (8.3) and (8.4) and

$$\tilde{A}_S(B, [\tilde{B}^i]) := \int_{\mathbb{R}^2} \left[ |(B^1, \delta^2(i))|^2 - \sum_{j=1}^{3} (\delta^2(i) \cdot e_1)^2 \right] J_{23}(\vec{f}) d\vec{f}.$$
Here, $\hat{A}$ with skew-symmetric operators. Hence computed as $W(\{T\}_{u=1}^\infty)(\{A_{\alpha\beta}^k\}) = \prod_{u=1}^\infty \text{Tr}_{\rho_u} \mathcal{T} \exp \left[ \int_0^1 ds \left( \langle \hat{E} \cdot B^i, \delta \hat{\rho}^i \rangle \varphi_{j,s}^{\nu,i} + \langle [m(B_u)]^{-1} \hat{B}^i, \delta \hat{\rho}^i \rangle \varphi_{j,s}^{\nu,i} \right) \right]$, and $s$ keeps track of the ordering. This is Step $\text{III}$ in Definition $6.14$.

Refer to Notation $8.2$. We want to replace $B \hat{x} B \equiv m(B)B$ with $B$, $\partial_0 A_0$ with $A_0$ and $\partial_0 A$ with $A$, so that the action becomes $\int_S A_0 \cdot B - A \cdot \hat{B}$.

This means we need to replace $B_0$ with $m(B)^{-1} \hat{B}^i$, $B$ with $\hat{B} = m(B)^{-1} B$ and $A_{\alpha\beta}^k$ with $\partial_0^{-1} A_{\alpha\beta}^k$, hence

$$V(\{L^i\}_{v=1}^\infty)(\{B_0^i\}) \longrightarrow \text{exp} \left[ \sum_{v=1}^\infty \int_0^1 ds \left( \langle \hat{E} \cdot B^i, \delta \hat{\rho}^i \rangle \varphi_{j,s}^{\nu,i} + \langle [m(B_v)]^{-1} \hat{B}^i, \delta \hat{\rho}^i \rangle \varphi_{j,s}^{\nu,i} \right) \right] = \text{exp} \left[ \sum_{v=1}^\infty \int_0^1 ds \left( \langle \hat{E} \cdot B^i, \delta \hat{\rho}^i \rangle \varphi_{j,s}^{\nu,i} + \langle [m(B_v)(\hat{\rho}_v)]^{-1} \hat{B}^i, \delta \hat{\rho}^i \rangle \varphi_{j,s}^{\nu,i} \right) \right],$$

$W(q; \{T\}_{u=1}^\infty)(\{A_{\alpha\beta}^k\}) \longrightarrow W(q; \{T\}_{u=1}^\infty)(\{\partial_0^{-1} A_{\alpha\beta}^k\})$.

Note that

$$\langle \hat{E} \cdot B^i, \delta \hat{\rho}^i \rangle \equiv \langle \hat{E} \cdot [m(B(\hat{\rho}_v))]^{-1} B^i, \delta \hat{\rho}^i \rangle.$$ 

Because $m(B_i)$ and $m(B)$ are skew-symmetric operators, therefore $m(B_i)^{-1}$ and $m(B)^{-1}$ are skew-symmetric operators. Hence

$$\exp \left[ \sum_{v=1}^\infty \int_0^1 ds \left( \langle \hat{E} \cdot B^i, \delta \hat{\rho}^i \rangle \varphi_{j,s}^{\nu,i} + \langle [m(B_v)(\hat{\rho}_v)]^{-1} \hat{B}^i, \delta \hat{\rho}^i \rangle \varphi_{j,s}^{\nu,i} \right) \right] \equiv \tilde{V}(\{B_0^i\})$$

$$:= \exp \left\{ \sum_{v=1}^\infty \int_0^1 \left[ [m(B(\hat{\rho}_v))]^{-1} (\varphi_{j,s}^{\nu,i} \delta \hat{\rho}^i) \right] d\bar{s} \right\} \right\} \quad \text{and} \quad \langle \hat{B}^i, \delta \hat{\rho}^i \rangle \equiv \langle \hat{B}^i, \delta \hat{\rho}^i \rangle \equiv \langle \hat{B}^i, \delta \hat{\rho}^i \rangle.$$ 

Replaced $B(\hat{\rho}_v)$ and $B_i(\hat{\rho}_v)$ with $\langle B, \delta \hat{\rho}^i \rangle$ and $\langle B_i, \delta \hat{\rho}^i \rangle$ and we will obtain the term $\tilde{V}$.

And making the same substitutions inside Equation $8.2$, we will have the area integrand computed as

$$\int_S \sqrt{|B^i|^2 - \sum_{j=1}^3 \| \hat{B}^j \cdot e_1 \|^2} dA = \int_{J_2} \sqrt{|B^i(\hat{\sigma}(\tilde{i}))|^2 - \sum_{j=1}^3 |\hat{B}^j(\hat{\sigma}(\tilde{i})) \cdot e_1|^2} J_{23}(\tilde{i}) \, d\tilde{i}$$

$$= \int_{J_2} \sqrt{|B^i(\hat{\sigma}(\tilde{i}))|^2 - \sum_{j=1}^3 |\hat{B}^j(\hat{\sigma}(\tilde{i})) \cdot e_1|^2} J_{23}(\tilde{i}) \, d\tilde{i}.$$

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Note that $\partial_0^{-1}$ is skew-symmetric because $\partial_0$ is skew-symmetric by doing a simple integration by parts, i.e. $(\partial_0^{-1} A_{\alpha \beta}, B^\mu) = (A_{\alpha \beta}^\mu, -\partial_0^{-1} B^\mu)$. Hence

$$W(q; \{ A^k_{\alpha \beta} \}_{k=1}^{N})(\{ \partial_0^{-1} A^k_{\alpha \beta} \}) = \tilde{W}(\{ A^k_{\alpha \beta} \})$$

whereby for a 9-vector $m$

$$W(q; \{ A^k_{\alpha \beta} \}_{k=1}^{N})(\{ \partial_0^{-1} A^k_{\alpha \beta} \}) = \tilde{W}(\{ A^k_{\alpha \beta} \})$$

whereby for a 9-vector $m$

$$W(q; \{ A^k_{\alpha \beta} \}_{k=1}^{N})(\{ \partial_0^{-1} A^k_{\alpha \beta} \}) = \tilde{W}(\{ A^k_{\alpha \beta} \})$$

With

$$W(q; \{ A^k_{\alpha \beta} \}_{k=1}^{N})(\{ \partial_0^{-1} A^k_{\alpha \beta} \}) = \tilde{W}(\{ A^k_{\alpha \beta} \})$$

We refer the reader to Section 5. Also recall Definition 5.4. We refer the reader to Section 5. Also recall Definition 5.4.

Before we proceed to the rest of the steps in Definition 6.14, we first point out that $m(B(\vec{x}))-1$, $m(B(\vec{x}))-1$ cannot be defined. So, we need to make an approximation to these operators, which is the next definition. The reader may wish to compare with the one given in Definition 5.1.

**Definition 8.5** We refer the reader to Section 5. Also recall $x(s_i)$ defined in Definition 5.1.

For $i = 1, 2, 3$, define $\partial_i^c := \frac{1}{4}\partial_i$, which maps $C^\infty(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3) \rightarrow C^\infty(\mathbb{R}^4) \otimes \Lambda^2(\mathbb{R}^3)$ and define for a 9-vector $F \equiv (F^1, F^2, F^3)$, $F^i \in (C^\infty(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3))^3$,

$$m_\kappa(B)F := (\partial_1 F^1, \partial_2 F^2, \partial_3 F^3) \tilde{\times} F$$

$$\equiv ([m_\kappa(B)F]^1, [m_\kappa(B)F]^2, [m_\kappa(B)F]^3),$$

whereby for $(i, j, k) \in C_3$,

$$[m_\kappa(B)F]^i = (\partial_i^c F^k + \frac{1}{2} dx_j \wedge B^j \times F^k) + (\partial_k^c F^j + \frac{1}{2} dx_k \wedge B^k \times F^j).$$

Here, it is understood that $[m_\kappa(B)F]^i \in (C^\infty(\mathbb{R}^4) \otimes \Lambda^2(\mathbb{R}^3) \equiv (C^\infty(\mathbb{R}^4) \otimes \Lambda^2(\mathbb{R}^3))^3$ and

$$\partial_i^c F^j \equiv (\partial_1 F^1, \partial_2 F^2, \partial_3 F^3), \quad F^i \in (C^\infty(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3),$$

$$B^i \times F^j \equiv (B^1 F^2_1 - B^2 F^3_1, B^2 F^3_1 - B^3 F^1_1, B^3 F^1_1 - B^1 F^2_1), \quad F^i \in C^\infty(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3).$$

See Items 2 and 3 in Definition 5.1.

Let $F = (F^1, F^2, F^3) \equiv (F^i \otimes dx_1, F^2 \otimes dx_2, F^3 \otimes dx_3)$, $F^i \in C^\infty(\mathbb{R}^4)^3 \times 3$. Write $v_1 = \frac{1}{2} dx_2 \wedge dx_3$, $v_2 = \frac{1}{2} dx_3 \wedge dx_1$, and $v_3 = \frac{1}{2} dx_1 \wedge dx_2$. Observe that as $\kappa \rightarrow \infty$,

$$m_\kappa(B)F \rightarrow \left\{ (B^2 \times F^3 - B^3 \times F^2) \otimes v_1, (B^3 \times F^1 - B^1 \times F^3) \otimes v_2, (B^1 \times F^2 - B^2 \times F^1) \otimes v_3 \right\},$$

which can be identified with $m(B)F \equiv B \tilde{\times} F$ using the Hodge star operator.

The inverse is given by

$$m_\kappa(B)^{-1}F = ([m_\kappa(B)^{-1}F]^1, [m_\kappa(B)^{-1}F]^2, [m_\kappa(B)^{-1}F]^3),$$

whereby for $(i, j, k) \in C_3$,

$$[m_\kappa(B)^{-1}F]^i = m_\kappa(B)^{-1}F^k \otimes dx_k - m_\kappa(B)^{-1}F^j \otimes dx_j,$$

with

$$m_\kappa(B)^{-1}F^j(x) := \kappa \left[ \int_{-\infty}^{x_i} - \int_{x_i}^{\infty} \right] e^{\kappa(s_1-x_1)}(\frac{1}{2} B^i \times F^j)(x(s_1))dx_i.$$
Here, \([B^i \times] : C^\infty(\mathbb{R}^4)^3 \to C^\infty(\mathbb{R}^4)^3\) is viewed as a linear operator acting on 3-vectors, hence represented by a 3 \times 3 matrix. Note that \([m_\kappa(B)^{-1}F^i_j] \in [C^\infty(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3)]^3 \times 3\). In other words, \(m_\kappa(B)^{-1}F\) is a 9-vector, each component taking values in \(C^\infty(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3)\).

A direct computation will show that
\[
m_\kappa(B)m_\kappa(B)^{-1}F = \left( F^1 \otimes (dx_1 \land dx_2 + dx_3 \land dx_4), F^2 \otimes (dx_2 \land dx_3 + dx_1 \land dx_4), F^3 \otimes (dx_3 \land dx_1 + dx_2 \land dx_4) \right).
\]

Each component in the RHS of the preceding equation is in \([C(\mathbb{R}^4) \otimes \Lambda^2(\mathbb{R}^3)]^3\). Using \(\bar{\iota}\) defined in Section 5 will identify the RHS of the preceding equation with \(F\).

**Remark 8.6**

1. Explicitly, if \(B^i = (b_1, b_2, b_3)\), then the matrix representing \([B^i \times]\) is given by
\[
\begin{pmatrix}
0 & -b_3 & b_2 \\
b_3 & 0 & -b_1 \\
-b_2 & b_1 & 0
\end{pmatrix}.
\]

2. Refer to Remark 5.2. When \(B \equiv 0\), we have for \((i, j, k) \in C_3\),
\[
[m_\kappa(0)^{-1}F^i_j \equiv \kappa(\partial_j^i F^k \otimes dx_k - \partial^i_k F^j \otimes dx_j).
\]

Here, it is understood that \(F^i \in C^\infty(\mathbb{R}^4)^3\) and the integral operator \(\partial_i^{-1}\) acts on the 3-vector componentwise.

Let \(\nabla \equiv (\partial_1, \partial_2, \partial_3)\). For \(i = 1, 2, 3\), we will also define \(m_\kappa(B_i)\), by
\[
m_\kappa(B_i)F = \left[ \frac{1}{\kappa} \nabla + B_i \right] \times F, \quad (8.9)
\]
with \(F = (F^1, F^2, F^3) \in (C^\infty(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3))^3\).

The components take values in \(C^\infty(\mathbb{R}^4) \otimes \Lambda^2(\mathbb{R}^3)\) and is given by
\[
[m_\kappa(B_i)F^i_j = \left( \partial_j^k F^k + dx_j \land B_i^j F^k \right) + \left( \partial^i_k F^j + dx_k \land B_i^k F^j \right), \quad i = 1, 2, 3,
\]
for \((i, j, k) \in C_3\) and \(\partial_i^k \equiv \frac{1}{\kappa} \partial_i\).

Let \(F = (F^1, F^2, F^3) \equiv (F^1 \otimes dx_1, F^2 \otimes dx_2, F^3 \otimes dx_3)\), \(F^i \in C^\infty(\mathbb{R}^4)\). Write \(w_1 = dx_2 \land dx_3, w_2 = dx_3 \land dx_1, w_3 = dx_1 \land dx_2\). Observe that as \(\kappa \to \infty\),
\[
m_\kappa(B_i)F \longrightarrow \left( (B_i^2 F^3 - B_i^3 F^2) \otimes w_1, (B_i^3 F^1 - B_i^1 F^3) \otimes w_2, (B_i^1 F^2 - B_i^2 F^1) \otimes w_3 \right),
\]
which can be identified with \(m(B_i)F \equiv B_i \times F\) using the Hodge star operator.

Its inverse is defined by, for \((i, j, k) \in C_3\),
\[
[m_\kappa(B_j)^{-1}F^i_j = m_\kappa(B_j)^{-1}F^k \otimes dx_k - m_\kappa(B_j^k)^{-1}F^j \otimes dx_j, \quad j = 1, 2, 3, \quad (8.10)
\]
with
\[
m_\kappa(B_j^i)^{-1}G(x) := \frac{\kappa}{2} \left[ \int_{-\infty}^{s_j} - \int_{x_j}^{\infty} \right] e^{\kappa(s_j - x_j)} B_i^j G(x(s_j)) ds_j. \quad (8.11)
\]
Note that each component of the 3-vector $m\kappa(B_i)^{-1}F$ is in $C^\infty(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3)$.

A direct computation will show that

$m\kappa(B_i)m\kappa(B_i)^{-1}F$

$= (F^1 \otimes (dx_1 \wedge dx_2 + dx_3 \wedge dx_4), F^2 \otimes (dx_2 \wedge dx_3 + dx_1 \wedge dx_4), F^3 \otimes (dx_3 \wedge dx_1 + dx_2 \wedge dx_3))$.

Using $\dagger$ will identify the RHS of the preceding equation with $F$.

Remark 8.7 It is interesting to consider $B_i = 0$. Then from Equation 8.4, we see that $m\kappa(0) \equiv \frac{1}{\kappa} \nabla \times \kappa$. Thus, we have

$\nabla \times (m\kappa(0))^{-1} F = \kappa F$,

after we make the identification between $\Lambda^1(\mathbb{R}^3)$ and $\Lambda^2(\mathbb{R}^3)$ using $\dagger$.

Furthermore, for $(i, j, k) \in C_3$,

$[m\kappa(0)^{-1}F] = \kappa [\partial_j^{-1}F^k \otimes dx_k - \partial_k^{-1}F^j \otimes dx_j]$.

Refer to Notation 2.8. To make sense of the path integral given by Expression 8.5, we will make the following approximations.

1. We will approximate the Dirac-delta function with a Gaussian function $p_\kappa$, defined in Notation 4.3.

2. We will approximate the operators $m(B)$ and $m(B_i)$ with the operators $m\kappa(B)$ and $m\kappa(B_i)$ respectively.

This completes Step 3 in Definition 6.14.

Definition 8.8 Define

$\hat{V}_\kappa(\{B^i_j\}) := \exp \left\{ \sum_{j=1}^{3} \int_{0}^{1} \left[ \left[ m\kappa(\{B, p^v_\kappa\}) \right]^{-1} (\bar{g}_s^{\nu} p^{\nu}_v) \right]^j ds \right\}$

and

$\hat{W}_\kappa(\{A^{k}_{\alpha\beta}\}) = \exp \left[ q \left\{ A^{i}_{\alpha j}, -\kappa \bar{\kappa} \sum_{u=1}^{\pi} \int_{0}^{1} ds \, \partial_0^{-1} p^v_\kappa \right\} y_{u, s}^{\nu} \otimes \hat{E}_s^{\nu j} \right.$

$\left. + q \left\{ A^{i}_{\alpha j}, -\kappa \bar{\kappa} \sum_{u=1}^{\pi} \int_{0}^{1} ds \, \partial_0^{-1} p^v_\kappa \right\} y_{k, s}^{\nu} \otimes \hat{E}_s^{\nu j} \right]$. (8.12)

Remark 8.9

1. Note that $\hat{W}_\kappa(\{A^{k}_{\alpha\beta}\})$ will take values inside $\bigoplus_{n=0}^{\infty} (\mathfrak{su}(2) \times \mathfrak{su}(2)) \otimes^n$. 

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2. Write \( \nu = (\nu^1, \nu^2, \nu^3) \), a 9-vector by

\[
\nu^i = \delta_{i,3} p_\kappa^3 E, \quad E = (1, 1, 1).
\]

Note that \( \langle B, p_\kappa^3 \rangle \) is a 9-vector, so

\[
\left[ m_\kappa(\langle B, p_\kappa^3 \rangle) \right]^{-1} \left( \delta_{\kappa,\rho} p_\rho^3 \right) = \left[ m_\kappa(\langle B, p_\kappa^3 \rangle) \right]^{-1} \nu \equiv (F^1, F^2, F^3),
\]

and its component, a 3-vector written as

\[
F^j = \left[ m_\kappa(\langle B, p_\kappa^3 \rangle) \right]^{-1} \left( \delta_{\kappa,\rho} p_\rho^3 \right)^j.
\]

Recall in Step 4 of Definition 6.14 we need to do some scaling. The line integrals are respectively

\[
\int_{I_2} \left| \left\langle B^1, p_\kappa^3 \right\rangle \right|^2 - \sum_{j=1}^3 \left| \left\langle B^j, p_\kappa^3 \right\rangle \cdot e_j \right|^2 J_{23}(l) \, dl.
\]

Definition 8.10 \((\tilde{\text{Tr}})\)

Define a linear functional \((\tilde{\text{Tr}})\) as follows. Suppose a matrix \( A \) is indexed by time \( s \) and representation \( \rho(i), i = 1, \ldots, r \). In other words, \( A \equiv A(\rho(i), s) \). Let \( \{A(\pi_1, s_1), \ldots, A(\pi_n, s_n)\} \) be a finite set of matrices. Let \( S_i = \{ j \in \{1, \ldots, n \} : \pi_j = \rho(i) \} \) and write \( m_i := |S_i| \). For any \( n \geq 1 \), define a linear operator,

\[
\tilde{\text{Tr}} : A(\pi_1, s_1) \otimes \cdots \otimes A(\pi_n, s_n)
\]

\[
\rightarrow \text{Tr}_{\rho(1)}[A(\rho(1), s_{\beta_1(1)}) \cdots \otimes A(\rho(1), s_{\beta_1(m_1)})] \cdots \text{Tr}_{\rho(r)}[A(\rho(r), s_{\beta_r(1)}) \cdots A(\rho(r), s_{\beta_r(m_r)})],
\]

such that for each \( i = 1, \ldots, r \), \( s_{\beta_i(1)} > s_{\beta_i(2)} > \ldots > s_{\beta_i(m_i)} \) and \( \beta_i(j) \in S_i \) for \( j = 1, \ldots, m_i \).

Together with the preceding approximation, we replace \( \tilde{V} \) and \( \tilde{V} \) with \( \tilde{W}_\kappa \) and \( \tilde{\nu}_\kappa \) respectively in Definition 8.8. Because \( \tilde{W}_\kappa(\{B^j_\kappa\}) \) and \( \tilde{\nu}_\kappa(\{B^j_\kappa\}) \) are scalars, we can bring them inside the time ordering operator \( \mathcal{T} \) and trace, which is the linear functional \( \tilde{\text{Tr}} \). Thus we approximate our path integral in Equation (8.13) with

\[
\frac{1}{Z_{EH}} \tilde{\text{Tr}} \int \tilde{W}_\kappa(\{B^j_\kappa\}) \tilde{V}_\kappa(\{A^k_{\beta,\beta} \}) e^{\int_{\mathcal{T}} A^0 B - A \tilde{B} DA}.
\]

This completes Steps 4 and 8 in Definition 6.14.

Definition 8.11 Given a colored oriented hyperlink \( \mathcal{L} \equiv (\mathcal{T}, \rho_u)_{u=1}^{n_1} \) and another oriented hyperlink \( \mathcal{L} \equiv (\mathcal{T}, \rho_u)_{u=1}^{n_1} \) as in Notation 2.7, we obtain a new colored oriented hyperlink \( \chi(\mathcal{L}, \mathcal{L}) \). Recall we parametrize \( \mathcal{T} \) using \( \rho_u \) and \( \mathcal{L} \) using \( \rho_u \) respectively from Notation 2.8.
Define
\[ \tilde{E} = (\tilde{E}^{01}, \tilde{E}^{02}, \tilde{E}^{03}), \rho_s^+(\tilde{E}_s) = (\rho_s^+(\tilde{E}_s^{01}), \rho_s^+(\tilde{E}_s^{02}), \rho_s^+(\tilde{E}_s^{03})) \]
and
\[ \lambda_s^e = (\lambda_s^{e,1}, \lambda_s^{e,2}, \lambda_s^{e,3}), \quad \tilde{\lambda}_s^e = (\tilde{\lambda}_s^{e,1}, \tilde{\lambda}_s^{e,2}, \tilde{\lambda}_s^{e,3}), \]
which is a 9-vector and 3-vector respectively, with
\[
\lambda_{s}^{e,j}(s) = -\frac{ikq^j}{2\sqrt{4\pi}} \sum_{u=1}^{n} \int_{0}^{1} ds \left\langle p_{k, s}^{u}, \partial_{0}^{-1} p_{k}^{u} \right\rangle y_{j,s}^{u} \otimes \tilde{E} \in C^{\alpha, 3}, \quad \tilde{E} = (1, 1, 1),
\]
and
\[
\tilde{\lambda}_{s}^{e,j}(s) = -\frac{ikq^j}{2\sqrt{4\pi}} \sum_{u=1}^{n} \int_{0}^{1} ds \left\langle p_{k, s}^{u}, \partial_{0}^{-1} p_{k}^{u} \right\rangle y_{j,s}^{u} \in C,
\]
for \( s \in I = [0, 1] \).

Remark 8.12 1. We can also identify \( \tilde{\lambda}_s^e(s) \) with \( \lambda_s^e(s) \).

2. Recall from Section 4 \( \left\langle p_{k}^{u}, \partial_{0}^{-1} p_{k}^{u} \right\rangle \) means \( \int_{\mathbb{R}} p_{k}^{u} \cdot \left[ \partial_{0}^{-1} p_{k}^{u} \right] \) d\( \lambda \) and \( \partial_{0}^{-1} p_{k}^{u} \) is defined using Equation (5.12) in Definition 5.7.

Definition 8.13 Refer to Definition 8.1. Recall that we have 2 copies of \( \mathfrak{su}(2) \) and \( \rho_u \equiv (\rho_u^+, \rho_u^-) \), \( u = 1, \ldots, n \), from Notation 2.3. Using Notation 2.3, we will write
\[
W^+(q; \tilde{T}, \tilde{L}) = \exp\left\{ \frac{ikq^3}{4\pi} \sum_{u=1}^{n} \int_{\mathbb{R}} ds \left\langle y_{j,s}^{u}, \partial_{0}^{-1} p_{k}^{u} \right\rangle \rho_s^+(\tilde{E}_s), \left\{ [m_s(\lambda_s^e(s))]^{-1} (\tilde{g}_s^+, \tilde{p}_s^+) \right\} \right\}
\]
\[
W^-(q; \tilde{T}, \tilde{L}) = \exp\left\{ -\frac{ikq^3}{4\pi} \sum_{u=1}^{n} \int_{\mathbb{R}} ds \left\langle y_{j,s}^{u}, \partial_{0}^{-1} p_{k}^{u} \right\rangle \left\{ [m_s(\lambda_s^e(s))]^{-1} (\tilde{g}_s^+, \tilde{p}_s^+) \right\} \right\} \otimes \sum_{j=1}^{3} (\tilde{E}_s^{(j)})^{-1}.
\]
Note that \( W^\pm \in \bigoplus_{n=0}^{\infty} (\mathfrak{su}(2))^\otimes_n \), but \( T W^\pm \equiv W^\pm \). If we multiply the matrices after applying the time ordering operator, then \( W^\pm \in SU(2) \).

We will define
\[
Z(\kappa, \chi; \tilde{T}, \tilde{L}) := \prod_{u=1}^{n} \left[ \text{Tr}_{\rho_s^+} W_u^+(q; \tilde{T}, \tilde{L}) + \text{Tr}_{\rho_s^-} W_u^-(q; \tilde{T}, \tilde{L}) \right].
\]
See also Remark 5.14.

Remark 8.14 1. Note that \( [m_s(\lambda_s^e(s))]^{-1} (\tilde{g}_s^+, \tilde{p}_s^+) \) is a 9-vector, i.e.
\[
[m_s(\lambda_s^e(s))]^{-1} (\tilde{g}_s^+, \tilde{p}_s^+) \equiv m_s(\lambda_s^e(s))^{-1} \left( \tilde{g}_s^+ \tilde{E}, \tilde{g}_s^+ \tilde{E}, \tilde{g}_s^+ \tilde{E} \tilde{p}_s^+ \right), \quad \tilde{E} = (1, 1, 1),
\]
and its components given explicitly, for \( (i, j, k) \in C_3 \),
\[
\left\{ [m_s(\lambda_s^e(s))]^{-1} (\tilde{g}_s^+, \tilde{p}_s^+) \right\}^i \equiv m_s(\lambda_s^{e,i}(s))^{-1} \left( \tilde{p}_s^+, \tilde{E} \right) g_{k,s}^i - m_s(\lambda_s^{e,k}(s))^{-1} \left( \tilde{p}_s^+, \tilde{E} \right) g_{j,s}^k,
\]
which are defined by Equations (8.7) and (8.8). Hence we notice that \( m_s(\lambda_s^e(s))^{-1} (\tilde{g}_s^+, \tilde{p}_s^+) \equiv m_s(0)^{-1} (\tilde{g}_s^+, \tilde{p}_s^+) \).
2. Similarly, \( \left( m_\kappa(\vec{\lambda}_\kappa(s)) \right)^{-1} (g_{s}^{v}, p_{k}^{v}) \) is a 3-vector and \( \left\{ \left( m_\kappa(\vec{\lambda}_\kappa(s)) \right)^{-1} (g_{s}^{v}, p_{k}^{v}) \right\} \) refers to its \( j \)-th component and explicitly given, for \((i, j, k) \in C_3\),

\[
\left\{ \left( m_\kappa(\vec{\lambda}_\kappa(s)) \right)^{-1} (g_{s}^{v}, p_{k}^{v}) \right\} _j \equiv m_\kappa(\vec{\lambda}_\kappa(s))^{-1} \left( g_{k}^{v}, p_{k}^{v} \right) - m_\kappa(\vec{\lambda}_\kappa(s))^{-1} \left( g_{k}^{v}, p_{k}^{v} \right) \]

which are defined by Equations (8.14) and (8.11).

3. For a given 3-vector \( y = (y_1, y_2, y_3) \in \mathbb{C}^{x^3} \), the term \( \langle \tilde{E}, y \rangle \) means \( y_j \otimes \tilde{E}^{0j} \). Thus if we write

\[
\left\{ \left( m_\kappa(\lambda_\kappa(s)) \right)^{-1} (g_{s}^{v}, p_{k}^{v}) \right\} \equiv (x_1^j, x_2^j, x_3^j) \in (C^\infty(\mathbb{R}^4))^{x^3},
\]

then

\[
\left\{ \left( m_\kappa(\lambda_\kappa(s)) \right)^{-1} (g_{s}^{v}, p_{k}^{v}) \right\} \equiv y_j^u \left( \partial_0^{-1} p_{k}^{u} - x_i^j \right) \otimes \rho^+_u (\tilde{E}_u^0).
\]

Using Remarks [8.6] and [8.12] we leave to the reader to show that it is equal to

\[- \epsilon^{ijk} \left\langle p_{k}^v, p_{k}^v \right\rangle_k y_j^u \left( \partial_0^{-1} p_{k}^{u} - x_i^j \right) \otimes \sum_{v=1}^{3} \rho^+_u (\tilde{E}_u^0).\]

Refer to Expression (8.24).

**Notation 8.15** Refer to Notation [2.2] Write

\[
a_\kappa = k^3 \int_{P^k} \left\{ - \sum_{i=1}^{3} \left( \frac{\pi}{12} \sum_{u=1}^{\kappa} \int_{0}^{1} y_{j,s}^u \partial_0^{-1} p_{k}^{u} ds, p_{k}^{u} (i) \right) \otimes \rho^+_u (\tilde{E}_u^0) \right\}^{1/2} J_{23}(i) \, d\nu.
\]

\[
b_\kappa = k^3 \int_{P^k} \left\{ \frac{\pi}{12} \sum_{u=1}^{\kappa} \int_{0}^{1} y_{j,s}^u \partial_0^{-1} p_{k}^{u} ds, p_{k}^{u} (i) \right\} \otimes \rho^-_u (\tilde{E}_u^{*}(i)) \right\}^{1/2} J_{23}(i) \, d\nu.
\]

And

\[
\left\langle \int_{0}^{1} y_{j,s}^u \partial_0^{-1} p_{k}^{u} ds, p_{k}^{u} (i) \right\rangle \equiv \int_{0}^{1} ds \, y_{j,s}^u \left( \partial_0^{-1} p_{k}^{u} - p_{k}^{u} (i) \right).
\]

See also Section [4].

**Lemma 8.16** Recall \( \mathcal{W}_k^+ \) were defined in Definition [8.13] and also refer to Notation [8.15] Apply Step [5] in Definition [6.14] the path integral in Expression (8.15) is hence computed as

\[
\bar{\text{Tr}} \left( a_\kappa \otimes \mathcal{W}_k^+ (q; T^u, L) \right)
\]

and also refer to Notation [8.15] Apply Step [5] in Definition [6.14] the path integral in Expression (8.15) is hence computed as

\[
\bar{\text{Tr}} \left( a_\kappa \otimes \mathcal{W}_k^+ (q; T^u, L) \right).
\]

**Proof.** From Equation (8.12) and according to Step [5] in Definition [6.14] we replace \( B^j \) and \( \tilde{B}^i \) inside the function \( \bar{V}_k (\{ \tilde{B}_k^j \}) \) with

\[
B^j \longmapsto - i \kappa \tilde{k} \sum_{u=1}^{\tilde{u}} \int_{0}^{1} ds \, y_{j,s}^u \partial_0^{-1} p_{k}^{u} \otimes (\rho^+_u (\tilde{E}_u^{01}), \rho^+_u (\tilde{E}_u^{02}), \rho^+_u (\tilde{E}_u^{03})),
\]

\[
\tilde{B}^i \longmapsto i \kappa \tilde{k} \sum_{u=1}^{\tilde{u}} \int_{0}^{1} ds \, y_{j,s}^u \partial_0^{-1} p_{k}^{u} \otimes \rho^-_u (\tilde{E}_u^{*}(i)).
\]

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Expression (8.18) will give us
\[ m_\kappa\left(\langle B, p_\kappa^\ell \rangle\right)^{-1} \mapsto m_\kappa(\lambda_\kappa^\ell (\hat{s}))^{-1} \]
and it also means that
\[ B_i \mapsto -iq\kappa \sum_{u=1}^{\mathcal{F}} \int_0^1 ds \ y_u^{j,s} \partial_0^{-1} p_\kappa^\ell \otimes \rho_\kappa^+(\check{E}_s^q). \]

Hence it means we replace
\[ m_\kappa\left(\langle B_i, p_\kappa^\ell \rangle\right)^{-1} \mapsto m_\kappa(\hat{\lambda}_\kappa^\ell (\hat{s}))^{-1}. \]

Note that \( \lambda_\kappa^\ell (\hat{s}) \) and \( \hat{\lambda}_\kappa^\ell (\hat{s}) \) are given by Equations (8.14) and (8.15) respectively. Substitute all these inside \( V_\kappa \) as defined in Definition 8.8 and we will obtain
\[ \left( \otimes_{u=1}^{\mathcal{F}} W_\kappa^+ (q_i; \hat{\tau}^b, \check{\mathcal{L}}) \right) \left( \otimes_{u=1}^{\mathcal{F}} W_\kappa^- (q_i; \hat{\tau}^b, \check{\mathcal{L}}) \right). \]

Making the substitution given by Equation (8.18), we see that
\[ \left| \langle B^i, p_\kappa^{j(t)} \rangle \right|^2 \mapsto - \sum_{i=1}^{\mathcal{F}} \left| \left\langle \left\langle q \kappa \sum_{u=1}^{\mathcal{F}} \int_0^1 ds \ y_u^{j,s} \partial_0^{-1} p_\kappa^{j(t)} \otimes \rho_\kappa^+(\check{E}_s^q) \right| \right. \right|^2 \]
\[ = -q^2 \kappa^2 \sum_{u, u=1}^{\mathcal{F}} \int_{\mathcal{J}_u} ds \ y_u^{j,s} \partial_0^{-1} p_\kappa^{j(t)} \left. \right\rangle \left\langle y_u^{j,s} \partial_0^{-1} p_\kappa^{j(t)} \otimes \rho_\kappa^+(\check{E}_s^q) \right. \right\rangle \otimes A(u, s; \check{u}, \check{s}), \]

where \( A(u, s; \check{u}, \check{s}) := \sum_{i=1}^{\mathcal{F}} \rho_\kappa^+(\check{E}_s^q) \otimes \rho_\kappa^+(\check{E}_s^q). \)

Making the substitution given by Equation (8.19), we see that
\[ - \left| \langle \tilde{B}^i, p_\kappa^{j(t)} \rangle \cdot e_1 \right|^2 \mapsto \left| \left\langle \left\langle q \kappa \sum_{u=1}^{\mathcal{F}} \int_0^1 ds \ y_u^{j,s} \partial_0^{-1} p_\kappa^{j(t)} \otimes \rho_\kappa^+(\check{E}_s^q) \right| \right. \right|^2 \]
\[ = q^2 \kappa^2 \sum_{u, u=1}^{\mathcal{F}} \int_{\mathcal{J}_u} ds \ y_u^{j,s} \partial_0^{-1} p_\kappa^{j(t)} \left. \right\rangle \left\langle y_u^{j,s} \partial_0^{-1} p_\kappa^{j(t)} \otimes \rho_\kappa^+(\check{E}_s^q) \right. \right\rangle \otimes B^i(u, s; \check{u}, \check{s}), \]

where \( B^i(u, s; \check{u}, \check{s}) := \rho_\kappa^+(\check{E}_s^q) \otimes \rho_\kappa^+(\check{E}_s^q). \)

Suppose our representation is given by \( \rho_\kappa \equiv (\rho_\kappa^+, 0) \). Apply Step 5 in Definition 6.3, substitute inside Expression (8.13) and with Notation (8.15) we will obtain
\[ \overline{\text{Tr}} \left. a_\kappa \left( \otimes_{u=1}^{\mathcal{F}} \exp \left\{ \frac{iq \kappa^3}{4 \pi} \sum_{u=1}^{\mathcal{F}} \int_{\mathcal{J}_u} ds \ y_u^{j,s} \partial_0^{-1} p_\kappa^{j(t)} \otimes \rho_\kappa^+(\check{E}_s), \{ m_\kappa(\lambda_\kappa^\ell (\hat{s}))^{-1} (\check{E}_s^q \check{E}_s^{q'}) \} i \right\} \right. \right\} \]
\[ \equiv \overline{\text{Tr}} \left. \left. a_\kappa \left( \otimes_{u=1}^{\mathcal{F}} W_\kappa^+ (q_i; \hat{\tau}^b, \check{\mathcal{L}}) \right) \right. \right\} \]

which follows from Definition 8.13.
Similarly, suppose our representation is given by $\rho_u \equiv (0, \rho^-_u)$. Apply Step 10 in Definition 8.3 substitute inside Expression 8.13 and with Notation 8.15 we will obtain

$$\tilde{\text{Tr}} \left[ b_n \prod_{u=1}^{n} \exp \left\{ -i g \kappa_0^3 \sum_{v=1}^{n} \int_{I^v} ds \left\{ y_{0,s}^{-1} \rho_0^- p^-_{\kappa(v)} \left\{ m_\kappa(x^\kappa_0(s)) \right\}^{-1} \left( \rho_0^+ p^+_{\kappa(v)} \right) \right\} \right\} \right] = \tilde{\text{Tr}} \left[ b_n \prod_{u=1}^{n} W_n(q; \tilde{T}, \tilde{L}) \right],$$

which follows from Definition 8.13. 

In the general case $\rho_u \equiv (\rho_u^+, \rho_u^-)$, Expressions 8.20 and 8.21 will give us our desired result.

\[\blacksquare\]

**Remark 8.17**

1. When $S = \emptyset$, then Expression 8.17 reduces to $Z(\kappa, q; \chi(T, L))$ as defined in Equation 8.16.

2. The term

$$Z(q; \chi(T, L)) := \lim_{\kappa \to \infty} Z(\kappa, q; \chi(T, L))$$

will be referred to as the Wilson Loop observable for a colored hyperlink $\chi(T, L)$.

Unfortunately, $a_\kappa$ and $b_\kappa$ are not defined, if the components of the hyperlink are colored differently. The reason is that we do not know how to take the square root of $\sum_{i=1}^{3} \rho_u^+ (\tilde{E}^0(i)) \otimes \rho_u^- (\tilde{E}^0(i))$ and $\sum_{i=1}^{3} \rho_u^+ (\tilde{E}^\kappa(v)) \otimes \rho_u^- (\tilde{E}^{\kappa(v)}),$, if $\rho_u^+ \neq \rho_u^-$. 

What if all the representations are the same? Unfortunately, we will still have problems defining $a_\kappa$ and $b_\kappa$. The reason is the square root function is not analytic at 0, so we do not know how to apply the time ordering operator to the square root.

In a sequel [4] to this article, we will show that the area path integral can be computed from the intersection points between the link $\pi_0(T)$ and the surface $S$. At any such intersection point, and termed as piercing, we see that it involves only one component $\tilde{T}$ inside $T$ and only one point in $T'$, so the sum of the tensor products inside $a_\kappa$ and $b_\kappa$ reduce down to

$$\sum_{i=1}^{3} \rho_u^+ (E^{0\kappa(v)}(i)) \rho_u^- (E^{0\kappa(v)}(i)) \equiv -\xi_{\rho_u^+} I_{\rho_u^+}, \text{ and } \sum_{i=1}^{3} \rho_u^+ (E^{\kappa(v)}(i)) \rho_u^- (E^{\kappa(v)}(i)) \equiv -\xi_{\rho_u^-} I_{\rho_u^-},$$

$I_{\rho_u^\pm}$ are respectively identity matrices. It is now clear how to take the square root.

If the representations are all the same, i.e. $\rho_u^\pm \equiv \rho^\pm$, then we replace both $a_\kappa$ and $b_\kappa$ with

$$\bar{a}_\kappa = k^3 \int_{I^2} \left\{ \left[ q \sum_{u=1}^{n} \int_{0}^{1} y_{1,s}^{-1} \rho_0^- p^-_{\kappa(u)} ds, p^-_{\kappa(u)} \right] \right\}^2 \otimes \xi_{\rho_u^+} I_{\rho_u^+} \right\}^{1/2} \right|_{J_{23}(i)d\tau},$$

$$\bar{b}_\kappa = k^3 \int_{I^2} \left\{ - \left[ q \sum_{u=1}^{n} \int_{0}^{1} y_{1,s}^{-1} \rho_0^- p^-_{\kappa(u)} ds, p^-_{\kappa(u)} \right] \right\}^2 \otimes \xi_{\rho_u^-} I_{\rho_u^-} \right\}^{1/2} \right|_{J_{23}(i)d\tau},$$

respectively, in Expression 8.17.
Hence Expression 8.24 upon further simplification, gives us
\[
\prod_{u=1}^{\pi} \left\{ \kappa^3 \int_{I} \left\langle \sum_{u=1}^{\pi} \kappa \int_{0}^{1} y_{1,3}^{u, l} \partial_{0}^{-1} \psi_{k}^{u, l} ds, \psi_{k}^{u, l}(t) \right\rangle \sqrt{\xi_{p_k}} J_{23}(t) dt \right\}^{1/\pi} \right\}
\]
\[
\left\{ \kappa^3 \int_{I} \left\langle \sum_{u=1}^{\pi} \kappa \int_{0}^{1} y_{1,3}^{u, l} \partial_{0}^{-1} \psi_{k}^{u, l} ds, \psi_{k}^{u, l}(t) \right\rangle \sqrt{\xi_{p_k}} J_{23}(t) dt \right\}^{1/\pi} \right\}
\] (8.22)

We will now define the path integral in Expression 8.24 as the limit of Expression 8.22 as \( \kappa \) goes to infinity. We can further simplify this expression using the following lemma.

**Lemma 8.18** We have \( \kappa \lambda^{\nu}_{L}(s) \to 0 \) and \( \kappa \lambda^{\nu}_{L}(s) \to 0 \) as \( \kappa \to \infty \).

**Proof.** Note that
\[
\kappa^3 \left\langle \psi_{k}^{u, l}, \partial_{0}^{-1}(\psi_{k}^{u, l}) \right\rangle = \kappa^2 \left\langle \psi_{k}^{u, l}, \psi_{k}^{u, l} \right\rangle \cdot \kappa \left\langle \psi_{k}^{u, l}, \partial_{0}^{-1} \psi_{k}^{u, l} \right\rangle.
\]

The proof now follows directly from Lemma A.1, the details to be left to the reader. ■

**Corollary 8.19** Refer to Definition 5.7. Define \( \hat{W}^\pm_k(q; T^0, L) \) as
\[
\hat{W}^\pm_k(q; T^0, L) := \exp \left\{ \frac{iq}{4\pi} \kappa^3 \sum_{u=1}^{\pi} \int_{I} ds \ e^{ijk} \left\langle \psi_{k}^{u, l, \nu}, \psi_{k}^{u, l} \right\rangle \kappa y_{1,3}^{u, l} \hat{g}_{j, s} \otimes \mathcal{F}^\pm \right\},
\] (8.23)

whereby \( \mathcal{F}^\pm \) was defined in Notation 2.7. We thus have
\[
\lim_{\kappa \to \infty} \kappa \hat{W}^\pm_k(q; T^0, L) = \lim_{\kappa \to \infty} \kappa \hat{W}^\pm_k(q; T^0, L).
\]

**Proof.** From Lemma 8.18, together with Remarks 8.6 and 8.7, we see that to compute the limit as \( \kappa \) goes to infinity, for the exponent in \( \hat{W}^\pm_k(q; T^0, L) \), it suffices to compute the limit as \( \kappa \) goes to infinity for
\[
\pm \frac{iq}{4\pi} \kappa^3 \sum_{u=1}^{\pi} \int_{I} ds \ \left\{ \left\langle y_{1,3}^{u, l} \partial_{0}^{-1} \psi_{k}^{u, l}, \kappa \left[ \hat{g}_{j, s} \partial_{0}^{-1} - \hat{g}_{j, s} \partial_{0}^{-1} \right] \psi_{k}^{u, l} \right\rangle \right\}
\]
\[
+ \left\langle y_{2,3}^{u, l} \partial_{0}^{-1} \psi_{k}^{u, l}, \kappa \left[ \hat{g}_{j, s} \partial_{0}^{-1} - \hat{g}_{j, s} \partial_{0}^{-1} \right] \psi_{k}^{u, l} \right\rangle
\]
\[
+ \left\langle y_{3,3}^{u, l} \partial_{0}^{-1} \psi_{k}^{u, l}, \kappa \left[ \hat{g}_{j, s} \partial_{0}^{-1} - \hat{g}_{j, s} \partial_{0}^{-1} \right] \psi_{k}^{u, l} \right\rangle \otimes \mathcal{F}^\pm.
\] (8.24)

Using Notations 2.8 and 5.3 applied to Expression 8.24, after some simple manipulation, Expression 8.24 can be written compactly as
\[
\pm \frac{iq}{4\pi} \kappa^3 \sum_{u=1}^{\pi} \int_{I} ds \ e^{ijk} \left\langle \psi_{k}^{u, l, \nu}, \psi_{k}^{u, l} \right\rangle \kappa y_{1,3}^{u, l} \hat{g}_{j, s} \otimes \mathcal{F}^\pm.
\]

This completes the proof. ■

With this corollary, we can define the area path integral by replacing \( W^\pm_k \) with \( \hat{W}^\pm_k \) in Expression 8.22 and taking the limit of this new expression as \( \kappa \) goes to infinity. This is for the case when the representations are all the same.
Notation 8.20 Suppose we have a list of irreducible representations of $\mathfrak{su}(2) \times \mathfrak{su}(2)$, $\{\rho_1, \ldots, \rho_m\}$ and there are $\tilde{m}$ distinct representations, labeled as $\{\gamma_1, \ldots, \gamma_{\tilde{m}}\}$, arranged in any order. For representation $\gamma_u$, let $\Gamma_u$ denote the set of integers in $\{1, 2, \ldots, \tilde{m}\}$ such that $\rho_v = \gamma_u$, $v \in \Gamma_u$.

If the representations are not the same, we can define the area path integral in the following manner.

Definition 8.21 (Area Path Integral)
Write the surface $S$ into a disjoint union of smaller surfaces $S_1, S_2, \ldots, S_{\tilde{m}}$.

$m$ as defined in Notation 8.20. In other words, $S_v$ will be the (possibly disconnected) surface whereby those hyperlinks colored with the same representation $\gamma_v$ pierce it. Let $I^2 = \bigcup_{v=1}^{\tilde{m}} I^v_2$ such that $\sigma : I^v_2 \to S_v$ is a parametrization of $S_v$.

Hence we can write the path integral in Expression 8.13 as

$$\frac{1}{Z_{EH}} \overline{\text{Tr}} \int \hat{V}_u(B_{\mu}) \hat{W}_u(A_{\kappa}) \bar{A}_{\kappa} S(B, \{\hat{B}^i\}) \cdot e^{\int I_{\kappa} A_0 B - A \cdot \hat{B}} \mathrm{d}\Lambda = \sum_{m=1}^{\tilde{m}} \frac{1}{Z_{EH}} \overline{\text{Tr}} \int \hat{V}_u(B_{\mu}) \hat{W}_u(A_{\kappa}) \bar{A}_{\kappa} S(B, \{\hat{B}^i\}) \cdot e^{\int I_{\kappa} A_0 B - A \cdot \hat{B}} \mathrm{d}\Lambda.$$

The path integral in Expression 8.1 is now defined as the limit as $\kappa$ goes to infinity, of the expression

$$\prod_{a=1}^{\tilde{m}} \left\{ \sum_{v=1}^{\tilde{m}} \kappa^3 \int _{I^v_2} \left< q \sum_{u \in \Gamma_v} \kappa \int_0^1 y_{1,4}^{-1} p_{\kappa}^0 \mathrm{d}s, p_{\kappa}^0(0, \hat{t}) \right> \sqrt{\xi_{\rho_v}} J_{23}(\hat{t}) \mathrm{d}\hat{t} \right\}^{1/\tilde{m}} \overline{\text{Tr}} \hat{W}_{\kappa}^+ (q; \hat{t}^0, L) + \int _{I^v_2} \left< q \sum_{u \in \Gamma_v} \kappa \int_0^1 y_{1,4}^{-1} p_{\kappa}^0 \mathrm{d}s, p_{\kappa}^0(0, \hat{t}) \right> \sqrt{\xi_{\rho_v}} J_{23}(\hat{t}) \mathrm{d}\hat{t} \right\}^{1/\tilde{m}} \overline{\text{Tr}} \hat{W}_{\kappa}^- (q; \hat{t}^0, L).$$

Remark 8.22 The above expression is dependent on how we partition the surface $S$ into $\bigcup_{v=1}^{\tilde{m}} S_v$. But its limit as $\kappa$ goes to infinity will be shown to be independent of this partition in a sequel to this article.

9 Volume Path Integral

Fix a closed and bounded 3-manifold $R \subset \mathbb{R}^3$, referred as a compact region from now on, possibly disconnected with a finite number of components. We identify it as $\{0\} \times R \subset \{0\} \times \mathbb{R}^3$ inside $\mathbb{R} \times \mathbb{R}^3$. Furthermore, $L$ is disjoint from $R$.

Notation 9.1 Refer to Notation 2.8. Let $\rho : I^3 \to \mathbb{R}^3$ be any parametrization of $R$. Let $|J_{\rho}|(r)$ denote the determinant of the Jacobian of $\rho$, $r = (r_1, r_2, r_3)$. And write $\mathrm{d}r = dr_1 dr_2 dr_3$. We will also write $\bar{\rho}(r) \equiv \bar{r}, \equiv (0, \rho(r)) \in \mathbb{R}^4$. 

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Using the dynamical variables \( \{B^i_\mu\} \) and the Minkowski metric \( \eta^{ab} \), we see that the metric \( g^{ab} \equiv B^a_i \eta^{ij} B^b_j \) and the volume \( V_R \) is given by

\[
V_R(\{B^i_\mu\}) = \int_R \sqrt{- \epsilon_{ijk} g^{ij} g^{jk}}.
\]

Refer to Notation 7.3. It is not difficult to see that the volume is indeed given by

\[
V_R(\{B^i_\mu\}) = \int_R \left\{ \frac{B}{3} \cdot \bar{B} - \sum_{(i,j) \in \mathcal{Y}} |B_i \times B_j| + \sum_{i=1}^3 \frac{1}{3} B_i \times B_0 \right\}^{1/2} dV.
\]

**Remark 9.2** Note that \( B_i \times B_0 \cdot B_0 \equiv 0 \). However, we include this term inside this formula, as when we do the substitution \( B_i \times B_0 \to \bar{B}^i \), this term will give us a non-trivial contribution.

Refer to Notation 7.4. We want to define a volume path integral

\[
\frac{1}{Z_{EH}} \int \tilde{V}(\{B^i_\mu\}) W(\{\chi, \rho\}) \tilde{W}(\{A^k_{i\alpha\beta}\}) \times V_R(\{B^i_\mu\}) \exp \left[ i \int_{\mathbb{R}^4} \bar{\partial}_0 A_0 \cdot B \bar{\xx} B - \partial_0 A \cdot \bar{\xx} B \right] d\Lambda. \tag{9.1}
\]

**Remark 9.3** 1. When \( R \) is the empty set, we define \( V_\emptyset \equiv 1 \), so we write Expression 9.7 as \( \hat{V}_R[Z(q; \chi(\bar{\mathcal{T}}, \mathcal{L}))] \), which was termed as the Wilson Loop observable of the colored hyperlink \( \chi(\bar{\mathcal{T}}, \mathcal{L}) \) in Remark 8.1.

2. We will write Expression 9.7 as \( \hat{V}_R[Z(q; \chi(\bar{\mathcal{T}}, \mathcal{L}))] \).

We will now make use of Definition 6.14 to make sense of the path integral in Expression 9.1.

**Lemma 9.4** Recall we defined \( \hat{V}(\{B^i_\mu\}) \) and \( \tilde{W}(\{A^k_{i\alpha\beta}\}) \) in Equations 8.3 and 8.4 respectively. Refer to the parametrizations \( \tilde{\rho}^a \) and \( \tilde{\rho}^a \) defined in Notation 2.8. After doing a change of variables given in Notation 8.3 the path integral in Expression 9.1 is defined as

\[
\frac{1}{Z_{EH}} \int \tilde{V}(\{B^i_\mu\}) \tilde{W}(\{A^k_{i\alpha\beta}\}) \tilde{V}_R(\{B^i_\mu\}) e^{i \int_{\mathbb{R}^4} A_0 \cdot B - A \cdot \bar{\xx} B} d\Lambda. \tag{9.2}
\]

with

\[
\tilde{Z}_{EH} = \int \exp \left[ i \int_{\mathbb{R}^4} A_0 \cdot B - A \cdot \bar{\xx} B \right] d\Lambda,
\]

after applying Steps 7 and 2 in Definition 6.14.

Here,

\[
\tilde{V}_R(\{B^i_\mu\}) = \int_{\mathcal{I}} \left\{ \left[ \frac{\langle [m((B_i, \delta \bar{\xx} (r)))]^{-1} B_i, \delta \bar{\xx} (r) \rangle}{3} - \sum_{(i,j) \in \mathcal{Y}} \langle \bar{B}_i, \delta \bar{\xx} (r) \rangle \cdot \langle B_j, \delta \bar{\xx} (r) \rangle \right]^2 \right. \\
+ \left. \sum_{i=1}^3 \frac{1}{3} (\bar{B}_i, \delta \bar{\xx} (r)) \cdot \langle \bar{B}_i, [m((B_i, \delta \bar{\xx} (r)))]^{-1} \delta \bar{\xx} (r) \rangle \right\}^{1/2} |J_\mu| \, dr.
\]

Note that we can also write

\[
\langle \bar{B}_j, \delta \bar{\xx} (r) \rangle = \left[ \langle [m((B, \delta \bar{\xx} (r))]^{-1} B_j, \delta \bar{\xx} (r) \rangle \right].
\]

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Proof. We have shown in Lemma 8.4 how to obtain \( \tilde{V} \) and \( \tilde{W} \) respectively. We will only show how to make the substitution inside the volume integrand \( V_R \), which we will now give the details.

Apply Step 2 in Definition 6.14 and using the notations and substitutions as discussed in Lemma 8.4 make the following change of variables,

\[
B \times B \mapsto B, \quad B \mapsto \tilde{B} \equiv m(B)^{-1} B, \quad B_i \times B_0 \mapsto \tilde{B}_i, \quad B_j \mapsto \tilde{B}_j,
\]

and

\[
B_0 \mapsto m(B_1)^{-1} \tilde{B}_i, \quad B_i \times B_0 \mapsto \tilde{B}_i m(B_1)^{-1} \tilde{B}_i.
\]

Therefore, the volume integrand becomes

\[
\tilde{V}_R(\{B_\mu^i\}) = \int_R \left\{ \left| \frac{\tilde{B}}{3} \cdot B \right|^2 - \sum_{(i,j) \in \tau} \left| \tilde{B}_i \cdot B_j \right|^2 + \frac{3}{3} \left| \tilde{B}_i \cdot m(B_1)^{-1} \tilde{B}_i \right|^2 \right\}^{1/2} dV.
\]

In terms of the parametrization \( \rho \),

\[
\tilde{V}_R(\{B_\mu^i\}) = \int_R \left\{ \left| \frac{\tilde{B}}{3} \cdot B \right|^2 - \sum_{(i,j) \in \tau} \left| \tilde{B}_i \cdot B_j \right|^2 + \frac{3}{3} \left| \tilde{B}_i \cdot m(B_1)^{-1} \tilde{B}_i \right|^2 \right\}^{1/2} (\tilde{\rho}(r)) |J_\rho|(r) dr.
\]

Note that \( B(\tilde{x}) \equiv m(B(\tilde{x}))^{-1} B(\tilde{x}) \) and \([m(B_1)^{-1} \tilde{B}_i](\tilde{x}) \equiv m(B_1(\tilde{x}))^{-1} \tilde{B}_i(\tilde{x})\).

Apply Step 1 in Definition 6.14 and write

\[
\tilde{B}(\tilde{\rho}(r)) = \langle [m(B_1, \delta \tilde{\rho}(r))]^{-1} B, \delta \tilde{\rho}(r) \rangle,
\]

\[
B(\tilde{\rho}(r)) = \langle B, \delta \tilde{\rho}(r) \rangle, \quad \tilde{B}_i(\tilde{\rho}(r)) = \langle \tilde{B}_i, \delta \tilde{\rho}(r) \rangle, \quad B_i(\tilde{\rho}(r)) = \langle B_i, \delta \tilde{\rho}(r) \rangle,
\]

\[
[m(B_1)^{-1} \tilde{B}_i](\tilde{\rho}(r)) \equiv m(B_1(\tilde{\rho}(r)))^{-1} \tilde{B}_i(\tilde{\rho}(r)) = \langle m(B_1(\tilde{\rho}(r)))^{-1} \tilde{B}_i, \delta \tilde{\rho}(r) \rangle
\]

\[
= - \langle \tilde{B}_i, m(B_1(\tilde{\rho}(r)))^{-1} \delta \tilde{\rho}(r) \rangle = - \langle \tilde{B}_i, [m((B_1, \delta \tilde{\rho}(r)))^{-1} \delta \tilde{\rho}(r)] \rangle,
\]

because \( m(B)^{-1} \) and \( m(B_1)^{-1} \) are skew-symmetric. Substitute into the above volume integrand and we will obtain our result.

As stated in Section 8, both \( m(B(\tilde{x}))^{-1} \), \( m(B_i(\tilde{x}))^{-1} \) cannot be defined. Recall in Section 8, we approximate the Dirac-delta function with a Gaussian function \( \rho_\kappa \) with \( m(B) \) and \( m(B_i) \) with \( m_\kappa(B_i) \). And we need to add in factors of \( \kappa \). See Definition 8.8.

Because \( R \) is 3-dimensional submanifold, we need to include a factor of \( \kappa^3 \) to the volume integral. Hence we replace \( \tilde{V}_R(\{B_\mu^i\}) \) with

\[
\tilde{V}_{\kappa,R}(\{B_\mu^i\}) = \kappa^3 \int_R \left\{ \frac{[m_\kappa((B, \rho_\kappa^{(r)}))]^{-1} B, \rho_\kappa^{(r)}}{3} \cdot (B, \rho_\kappa^{(r)}) \right\}^2
\]

\[
- \sum_{(i,j) \in \tau} \left| \langle \tilde{B}_i, \rho_\kappa^{(r)} \rangle \cdot \langle B_j, \rho_\kappa^{(r)} \rangle \right|^2
\]

\[
+ \frac{3}{3} \left| \langle \tilde{B}_i, \rho_\kappa^{(r)} \rangle \cdot \tilde{B}_i, [m_\kappa((B, \rho_\kappa^{(r)}))]^{-1} \rho_\kappa^{(r)} \rangle \right|^2 \right\}^{1/2} |J_\rho|(r) dr. \tag{9.3}
\]

Note that we write

\[
\langle \tilde{B}_j, \rho_\kappa^{(r)} \rangle := \langle m_\kappa((B, \rho_\kappa^{(r)}))^{-1} B_{\kappa,j}, \rho_\kappa^{(r)} \rangle,
\]

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after making the approximations.

Together with the preceding approximation, we replace \( \tilde{W} \) and \( \tilde{V} \) with \( \tilde{W}_\alpha \) and \( \tilde{V}_\alpha \) respectively, defined earlier in Definition 8.8. Because \( \tilde{V}_\alpha (\{B_{\mu}^0\}) \) and \( \tilde{V}_{\alpha,R}(\{\{B_{\mu}^0\}\}) \) are scalars, we can bring them inside the time ordering operator \( \mathcal{T} \) and trace, which is the linear functional \( \tilde{\mathcal{T}} \). Thus we approximate our path integral in Expression 9.2 with

\[
\frac{1}{Z_{EH}} \tilde{\mathcal{T}} \int \tilde{V}_\alpha (\{B_{\mu}^0\}) \tilde{W}_\alpha (\{A_{\alpha \beta}^0\}) \tilde{V}_{\alpha,R}(\{\{B_{\mu}^0\}\}) e^{i \int_{\mathcal{R}} A_{\alpha} B - A \tilde{B} D \lambda}. \tag{9.4}
\]

This completes Steps 3 and 4 in Definition 6.14.

**Definition 9.5** Define the following 9-vector \( \lambda_\kappa = (\lambda^1_\kappa, \lambda^2_\kappa, \lambda^3_\kappa) \), whereby for \( r \in T^3 \),

\[
\lambda^i_\kappa (r) = \frac{-i q \kappa^2}{2 \sqrt{4 \pi}} \sum_{u=1}^{n} \int_0^1 ds \, \left( p^i_{\kappa} - \partial_0^{-1} p^i_{\kappa} \right) y^u_{i,s} \otimes \bar{E}. \quad \kappa \in \mathbb{C}^3.
\]

Also define a 3-vector \( \tilde{\lambda}_\kappa (r) \) by

\[
\tilde{\lambda}_\kappa (r) = \frac{-i q \kappa^2}{2 \sqrt{4 \pi}} \sum_{u=1}^{n} \int_0^1 ds \, \left( p^i_{\kappa} - \partial_0^{-1} p^i_{\kappa} \right) y^u_{s,i}.
\]

**Remark 9.6**

1. We can also identify \( \tilde{\lambda}_\kappa (r) \otimes \bar{E} \) with \( \lambda_\kappa (r) \).

2. Now, \( \left( p^i_{\kappa}, \partial_0^{-1} p^i_{\kappa} \right) \) means \( \int_{\mathcal{R}} p^i_{\kappa} - \partial_0^{-1} p^i_{\kappa} \, d\lambda \) and \( \partial_0^{-1} p^i_{\kappa} \) was defined using Equation 5.1.

**Lemma 9.7** Recall \( W_{\kappa, \lambda}^0 \) were defined in Definition 8.13. Apply Step 5 in Definition 6.14, the path integral in Expression 9.4 is hence computed as

\[
\tilde{\mathcal{T}} \left( A_{\kappa} \bigotimes_{u=1}^{\pi} W_{\kappa, \lambda}^0 (q_i, T^u_i, L) \right)
\]

whereby

\[
A_{\kappa} := \kappa^3 \int_0^1 \sqrt{\alpha_\kappa (\bar{p}(r)) |J_\rho|(r)} \, dr, \quad B_{\kappa} := \kappa^3 \int_0^1 \sqrt{\beta_\kappa (\bar{p}(r)) |J_\rho|(r)} \, dr,
\]

with \( \alpha_\kappa \) and \( \beta_\kappa \) both defined in Equations 9.6 and 9.8 respectively.

**Proof.** In the proof of Lemma 8.16 we obtain the function \( \tilde{V}_\alpha (\{B_{\mu}^0\}) \) using the substitutions given by Expressions 8.18 and 8.19.

Equation 8.18 will give us

\[
m_\kappa (\{B, p^i_{\lambda_\kappa} \})^{-1} \rightarrow m_\kappa (\lambda_\kappa (s))^{-1}, \quad m_\kappa (\{B, p^i_{\lambda_\kappa} \})^{-1} \rightarrow m_\kappa (\lambda_\kappa (r))^{-1}
\]

and it also means that

\[
B_i \rightarrow -i q \kappa \kappa \sum_{u=1}^{\pi} \int_0^1 ds \, y^u_{s,i} \partial_0^{-1} p^i_{\kappa} \otimes \rho^u_{\alpha} (\bar{E}_{\kappa}).
\]

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Hence it means we replace

\[ m_\alpha(\langle B, p_\alpha^r \rangle)^{-1} \rightarrow m_\alpha(\hat{\lambda}_\alpha^r(s))^{-1} \text{ and } [m_\alpha(\langle B, p_\alpha^r \rangle)]^{-1} \rightarrow m_\alpha(\hat{\lambda}_\alpha(r))^{-1}. \]

Equation 8.18 will also imply that

\[ [m_\alpha(\langle B, p_\alpha^r \rangle)]^{-1} B \]

\[ \rightarrow m_\alpha(\hat{\lambda}_\alpha(r))^{-1} \left\{ -i\kappa \sum_{u=1}^{\pi} \int_0^1 ds \ y_s^u \partial_0^{-1} p_\kappa^r \otimes (\rho_\kappa^+ (\hat{E}_u^0), \rho_\kappa^+ (\hat{E}_u^2), \rho_\kappa^+ (\hat{E}_u^3)) \right\} \]

\[ \equiv -i\kappa \sum_{u=1}^{\pi} \int_0^1 ds \ m_\alpha(\hat{\lambda}_\alpha(r))^{-1} [y_s^u \partial_0^{-1} p_\kappa^r] \otimes (\rho_\kappa^+ (\hat{E}_u^0), \rho_\kappa^+ (\hat{E}_u^2), \rho_\kappa^+ (\hat{E}_u^3)). \]

Equation 8.19 will lead us to replace

\[ m_\alpha(\langle B, p_\alpha^r \rangle)^{-1} \rightarrow m_\alpha(\hat{\lambda}_\alpha(r))^{-1}. \]

Note that \( \hat{\lambda}_\alpha^r(s) \) and \( \hat{\lambda}_\alpha^r(\hat{s}) \) were given by Equations (8.14) and (8.15) respectively; \( \lambda_\alpha(r) \) and \( \hat{\lambda}_\alpha(r) \) were defined in Definition 9.5.

Suppose \( \rho_\alpha \equiv (\rho_\alpha^+, 0) \). Now the above substitutions, when applied to \( \tilde{V}_\alpha \) as defined in Definition 8.8, will yield \( W_\alpha^+ \) and when applied to the term

\[ \left| \left( m_\alpha(\langle B, p_\alpha^r \rangle)^{-1} B, p_\alpha^r \right) \right|^2, \]

will yield (after some simplification)

\[ \alpha_\alpha(\rho_\alpha) := q^4 \kappa^4 \left\{ \sum_{u,a=1}^{\pi} \kappa \int_0^1 ds \ m_\alpha(\hat{\lambda}_\alpha(r))^{-1} [y_s^u \partial_0^{-1} p_\kappa^r, \partial_0^{-1} p_\kappa^r] \cdot \kappa \int_0^1 ds \ \partial_0^{-1} p_\kappa^r \otimes \rho_\kappa^+ (\hat{E}_u^0) \right\}^2. \]

Note that we made use of \( \langle \partial_0^{-1} f, g \rangle = -\langle f, \partial_0^{-1} g \rangle \) to obtain the above formula. Refer to Remark 9.8 for the meaning of the term \( \left\langle \partial_0^{-1} p_\kappa^r, m_\alpha(\hat{\lambda}_\alpha(r))^{-1} [y_s^u \partial_0^{-1} p_\kappa^r] \right\rangle \).

Apply Step 5 in Definition 6.3, the path integral in Expression 9.4 is now define as

\[ \text{Tr} A_\alpha \bigotimes_{u=1}^n \exp \left\{ \frac{iq \kappa^3}{4\pi} \sum_{v=1}^n \int_{I^2} ds \ \left\{ y_s^v \partial_0^{-1} p_\kappa^r \otimes \rho_\kappa^+ (\hat{E}_u^0), \left[ m_\alpha(\lambda_\alpha^r(s))^{-1} (y_s^v \partial_0^{-1} p_\kappa^r) \right] \right\} \right\} \]

\[ \equiv \text{Tr} \left[ \kappa^3 \int_{I^3} \sqrt{a_\alpha(\rho_\alpha)} |d\rho_\alpha| \right. \left. \bigotimes_{u=1}^n W_\alpha^+(q; \tilde{T}_u^+, \tilde{L}) \right], \]

which follows from Definition 8.13.

Suppose \( \rho_\alpha \equiv (0, \rho_\alpha^+). \) Now the above substitutions, when applied to \( \tilde{V}_\alpha \) as defined in Definition 8.8, will yield \( W_\alpha^- \).
From Notation 2.5, note that $\rho^+(E_0^u) \otimes \rho^-(\hat{E}^r(i)) \equiv 0$ for any $i$ and $j$. Hence, after the substitution, the term $\sum_{(i,j) \in T} |\hat{B}_i \cdot B_j|^2$ will not contribute to anything, but

$$\sum_{i=1}^3 \frac{1}{3} |\langle \hat{B}_i, p_{\hat{\kappa}}(r) \rangle \cdot \langle \hat{B}_i, [m_\kappa(\langle B_i, p_{\kappa}^c(r) \rangle)]^{-1} p_{\kappa}(r) \rangle |^2$$

will give us the term

$$\beta_\kappa(p(r)) := q^4 \kappa^4 \sum_{u, \bar{u}=1}^n \int \frac{1}{4\pi} \int ds \left( \delta^{\rho^{-1} p_{\kappa}^c(r), p_{\kappa}^c(r)} \cdot \delta^{[m_\kappa(\lambda_\kappa(s))]^{-1} (\hat{\delta}_s^r p_{\kappa}^c(r))} \right) \otimes \sum_{j=1}^3 \rho^-(\hat{E}_s^r(i))$$

Apply Step 5 in Definition 6.3 the path integral in Expression 9.4 is now define as

$$\tilde{\text{Tr}} B_\kappa \prod_{u=1}^n \exp \left\{ -\frac{iq^3}{4} \sum_{u=1}^n \int ds \left( \delta^{\rho^{-1} p_{\kappa}^c(r), p_{\kappa}^c(r)} \cdot \delta^{[m_\kappa(\lambda_\kappa(s))]^{-1} (\hat{\delta}_s^r p_{\kappa}^c(r))} \right) \otimes \sum_{j=1}^3 \rho^-(\hat{E}_s^r(i)) \right\}$$

which follows from Definition 8.13.

In the general case $\rho_u \equiv (\rho_u^+, \rho_u^-)$, Expressions 9.7 and 9.9 will give us our desired result. This completes Step 5 in Definition 6.14.

**Remark 9.8**

1. Note that $m_\kappa(\lambda_\kappa(r))^{-1}[y_{s^r}^u p_{\kappa}^c(r)] \equiv (f_1, f_2, f_3)$ is a 3-vector, each $f_i$ is a function on $\mathbb{R}^4$. And $\langle m_\kappa(\lambda_\kappa(r))^{-1}[y_{s^r}^u p_{\kappa}^c(r)], \rho_\kappa^+(\hat{E}_s^r(i)) \rangle$ means

$$\langle \hat{\delta}_s^r p_{\kappa}^c(r), (f_1, f_2, f_3) \rangle = \left( \langle \hat{\delta}_s^r p_{\kappa}^c(r), f_1 \rangle, \langle \hat{\delta}_s^r p_{\kappa}^c(r), f_2 \rangle, \langle \hat{\delta}_s^r p_{\kappa}^c(r), f_3 \rangle \right).$$

2. When $V = \emptyset$, then Expression 9.5 reduces to $Z(\kappa, \rho^-(\hat{E}_s^r(i)))$ as defined in Equation 8.10.

There is a problem with Expression 9.5 as the square root of Expressions 9.6 and 9.8 do not make sense. The problem lies with

$$\sum_{i=1}^3 \rho^+_u(\hat{E}_s^0) \otimes \rho^-_u(\hat{E}_s^0)$$

and

$$\sum_{i=1}^3 \rho^-_u(\hat{E}_s^r(i)) \otimes \rho^-_u(\hat{E}_s^r(i))$$

which lies in $\text{End}(V_+^u)$ and $\text{End}(V_+^-)$ respectively.

Of course, the above sum of tensor products is not the problem. The problem will come later, when we have to take the absolute value. We do not know how to take the modulus of

$$\sum_{i=1}^3 \rho^+_u(\hat{E}_s^0) \otimes \rho^+_u(\hat{E}_s^0)$$

and

$$\sum_{i=1}^3 \rho^-_u(\hat{E}_s^r(i)) \otimes \rho^-_u(\hat{E}_s^r(i))$$

when $\rho^+_u \neq \rho^-_u$.

Now, the reader may think that if we make the representation to be the same throughout, i.e. all the component in the matter hyperlink have the same color, then we will resolve the problem.
Unfortunately, Expression 9.9 is still not defined. The problem lies with the square root and the time ordering operator. The square root function or the modulus function is not analytic at 0, so it is not clear how to apply the time ordering operator.

In a sequel to this article, we will show that when we take the limit, the path integral is computed using the crossings of a link diagram. In fact, there is another problem that is lurking, which is the self-linking problem (for links) first pointed out by Witten in [13]. We will not give the details here; we refer the reader to [8] on an explanation of the self-linking problem. A quick answer is that we need to consider a ribbon or a framed link instead, not a link. A ribbon is a link, equipped with a non-tangential normal, also known as a frame, to the link. This normal, will give rise to half-twists on the link diagram.

Project the matter hyperlink in $\mathbb{R}^3$ and we will refer to it as a link. Let $l^1, \ldots, l^\pi$ be the components in the link, with each $l^u$ being a knot in $\mathbb{R}^3$. For each knot $l^u$, let $N(l^u)$ be a tubular neighborhood of $l^u$, homeomorphic to the torus, containing $l^u$ as a longitude curve in it, such that $N(l^u) \cap N(l^\bar{u}) = \emptyset$ if $u \neq \bar{u}$.

Now, we can always write the compact region $R$ as a disjoint union $\bigcup_{v=1}^{m} R_v$, such that either

i. $R_v \subset N(l^u)$ for some $u$ and $R_v$ contains an arc $\bar{\ell} \subset l^u$, such that for every $k = 1, 2, 3$, $\bar{\ell}$ projects onto the plane $\Sigma_k$ to form a link diagram as defined in [8] with no crossing;

ii. $R_v \cap N(l^u) = \emptyset$ for every $u = 1, \ldots, \pi$.

So, it suffices to consider the path integral over a compact region $R \subset \mathbb{R}^3$ which either is inside the tubular neighborhood for some knot $l^u$ or it does not intersect any tubular neighborhood at all. By considering such a compact region $R$, we will see in a sequel to this article, the crossings in a link diagram no longer contribute to the path integral.

From [8], we see that each half-twist on a link diagram of a knot $l^u$ gives rise to an operator

$$\sum_{i=1}^{3} \rho^{\alpha\beta}_u(E^{(i)}) \rho^{\beta\alpha}_u(E^{(i)})$$

and $\sum_{i=1}^{3} \rho^{\alpha\beta}_u(E^{(i)}) \rho^{\beta\alpha}_u(E^{(i)})$, which are $-\xi_{\rho^\alpha u} I_{\rho^\alpha u}$ and $-\xi_{\rho^\alpha u} I_{\rho^\alpha u}$ respectively. Then, it is clear how to take the absolute value, i.e.

$$\left| -\xi_{\rho^\alpha u} I_{\rho^\alpha u} \right| = \xi_{\rho^\alpha u} I_{\rho^\alpha u}, \quad \left| -\xi_{\rho^\alpha u} I_{\rho^\alpha u} \right| = \xi_{\rho^\alpha u} I_{\rho^\alpha u}.$$

Do note that the volume functional commutes with the matrices, so indeed, we really need to get some scalar multiple of the identity.

Notation 9.9 Let $L = \{l^1, \ldots, l^\pi\}$ be a matter hyperlink. Project each $l^u$ into $\mathbb{R}^3$ to form a knot $l^u$ and let $N(l^u)$ be a tubular neighborhood of $l^u$. Given a compact region $R \subset \mathbb{R}^3$, write $R = \bigcup_{v=1}^{m} R_v$ as a disjoint union, such that either $R_v \subset N(l^u)$ for some $u$ or $R_v \cap N(l^\bar{u}) = \emptyset$ for every $u = 1, \ldots, \pi$. Let $I^3_v \subset I^3$ such that $\rho: I^3_v \to R_v \subset R$ be a parametrization of $R_v$.

Definition 9.10 From Notation 8.9, we write $R = \bigcup_{v=1}^{m} R_v$ as a disjoint union of regions. Hence we can write the path integral in Expression 9.4 as

$$\frac{1}{Z_{EH}} \text{Tr} \int \tilde{V}_k([B^\mu_k]) \tilde{W}_k([A^k_{\alpha\beta}]) \tilde{V}_{k,R}([B^\mu_k]) \cdot e^{i \int_{R} A_0 B - A \bar{B} D A}$$

$$= \sum_{v=1}^{m} \frac{1}{Z_{EH}} \text{Tr} \int \tilde{V}_k([B^\mu_k]) \tilde{W}_k([A^k_{\alpha\beta}]) \tilde{V}_{k,R_v}([B^\mu_k]) \cdot e^{i \int_{R_v} A_0 B - A \bar{B} D A}.$$
We now define the path integral
\[
\frac{1}{Z_{EH}} \int \tilde{V}_\kappa(\{B^i_\mu\}) \tilde{W}_\kappa(\{A^k_{\alpha_\beta}\}) \tilde{V}_\kappa, R(\{B^i_\mu\}) \cdot e^{i \int_{\mu} A_0 \cdot B - A \cdot \tilde{B} \cdot DA},
\]
as
\[
q^2 \prod_{u=1}^{\bar{n}} \left\{ \left[ \sum_{v=1}^{m} \kappa^2 \kappa^3 \sum_{u=1}^{\bar{n}} \int_{r \in \mathbb{I}^2} dr |J_\rho| \right] \int_{\mathbb{I}^2} d\bar{s} \left\langle \partial_0^{-1} p_{\kappa}^{\mu}, p_{\kappa}^{\rho(r)} \right\rangle y_{s, u, j, k} \cdot \frac{\kappa}{3} \right\frac{1}{3} \TR_{\rho_0} W_\kappa^+(\{\kappa, T^0\}, L)
\]
\[
+ \left[ \sum_{v=1}^{m} \kappa^2 \kappa^3 \sum_{u=1}^{\bar{n}} \int_{r \in \mathbb{I}^2} dr |J_\rho| \right] \int_{\mathbb{I}^2} d\bar{s} \left\langle \partial_0^{-1} p_{\kappa}^{\mu}, m_\kappa(\tilde{\lambda}_\kappa(r))^{-1} y_{s, u, j, k} \right\rangle \cdot \frac{\kappa}{3} \right\frac{1}{3} \TR_{\rho_0} W_\kappa^-(\{\kappa, T^0\}, L)\right\}, (9.10)
\]
from Expression (9.10). Notice that it is no longer necessary to keep track of the matrices, so we remove the time ordering operator.

**Lemma 9.11** Refer to Notation (5.3) and Equation (8.23). The limit of Expression (9.10) as \( \kappa \) goes to infinity, is given by computing the limit of
\[
q^2 \prod_{u=1}^{\bar{n}} \left\{ \left[ \sum_{v=1}^{m} \kappa^2 \kappa^3 \sum_{u=1}^{\bar{n}} \int_{r \in \mathbb{I}^2} dr |J_\rho| \right] \int_{\mathbb{I}^2} d\bar{s} \left\langle \partial_0^{-1} p_{\kappa}^{\mu}, p_{\kappa}^{\rho(r)} \right\rangle y_{s, u, j, k} \cdot \frac{\kappa}{3} \right\frac{1}{3} \TR_{\rho_0} W_\kappa^+(\{\kappa, T^0\}, L)
\]
\[
+ \left[ \sum_{v=1}^{m} \kappa^2 \kappa^3 \sum_{u=1}^{\bar{n}} \int_{r \in \mathbb{I}^2} dr |J_\rho| \right] \int_{\mathbb{I}^2} d\bar{s} \left\langle \partial_0^{-1} p_{\kappa}^{\mu}, m_\kappa(\tilde{\lambda}_\kappa(r))^{-1} y_{s, u, j, k} \right\rangle \cdot \frac{\kappa}{3} \right\frac{1}{3} \TR_{\rho_0} W_\kappa^-(\{\kappa, T^0\}, L)\right\}, (9.11)
\]
as \( \kappa \) goes to infinity. Here,
\[
\bar{\kappa} = \sqrt{\frac{\pi}{2}} \kappa \left( \frac{\kappa}{2} \pi \right)^{1/2}.
\]

**Proof.** From Lemma (8.15) we have seen that \( \kappa \lambda^0_\kappa(s) \to 0 \) and \( \tilde{\lambda}^0_\kappa(s) \to 0 \) as \( \kappa \to \infty \).

Now,
\[
\kappa^2 \left\langle p_{\kappa}^{\rho}, \partial_0^{-1} p_{\kappa}^{\rho} \right\rangle = \kappa^2 \left\langle q_{\kappa}^{\rho}, q_{\kappa}^{\rho} \right\rangle \cdot \kappa \left\langle q_{\kappa}^{\rho}, q_{\kappa}^{\rho} \right\rangle\]
and
\[
\kappa^3 \left\langle \partial_0^{-1} p_{\kappa}^{\rho}, p_{\kappa}^{\rho(r)} \right\rangle = \kappa^2 \left\langle p_{\kappa}^{\rho}, p_{\kappa}^{\rho(r)} \right\rangle \cdot \kappa \left\langle q_{\kappa}^{\rho}, q_{\kappa}^{\rho} \right\rangle = \kappa^2 e^{-\kappa^2|\kappa^2 - \rho(r)|^2/8} \cdot \kappa \left\langle \partial_0^{-1} q_{\kappa}^{\rho}, q_{\kappa}^{\rho} \right\rangle,
\]
and both converge to 0, using Lemma (A.1).
Refer to Definition 8.1. From Item 3 in Lemma 8.1.1, we see that
\[
\left\langle \partial_0^{-1} p_K^{\bar{\nu}^r} \cdot p_K^{\bar{\rho}(r)} \right\rangle = \left\langle \partial_0^{-1} q_K^{\nu_s^r} \cdot q_K^0 \right\rangle \cdot \left\langle p_K^{\nu_s^r} \cdot p_K^{\bar{\rho}(r)} \right\rangle = e^{-\kappa^2 [\nu_s^r - \rho(r)]^2 / 8} \left\langle \partial_0^{-1} q_K^{\nu_s^r} \cdot q_K^0 \right\rangle.
\]

From Remark 9.7, we see that
\[
m_{\nu}(0)^{-1} \left\langle y_{\nu}^{u,i} \cdot p_K^{\bar{\rho}(r)} \right\rangle \cdot y_{\nu}^u = \left\langle p_K^{\bar{\rho}(r)} \cdot p_K^{\bar{\rho}(r)} \right\rangle \left[ y_{\nu}^{u,i} \cdot y_{\nu}^{u,i} \right]^k \equiv \left\langle p_K^{\bar{\rho}(r)} \cdot p_K^{\bar{\rho}(r)} \right\rangle \left[ y_{\nu}^{u,i} \cdot y_{\nu}^{u,i} \right]^k,
\]
using Notation 5.3.

Since \(\kappa\lambda_\nu, \kappa\lambda_\nu \to 0\), the limit of Expressions 9.6 and 9.8 are equivalent to compute the limits of
\[
q^4 \kappa^4 k^4 \sum_{u=1}^{\pi} \int_{l_2} d\bar{s} \left\langle p_K^{\bar{\rho}(r)} \cdot p_K^{\bar{\rho}(r)} \right\rangle \left[ y_{\nu}^{u,i} \cdot y_{\nu}^{u,i} \right]^k e^{-\kappa^2 [\nu_s^r - \rho(r)]^2 / 8} \left\langle \partial_0^{-1} q_K^{\nu_s^r} \cdot q_K^0 \right\rangle \otimes \frac{\xi_{\nu}^{u,i}}{3} \right]^2 \]
\[
=q^4 \kappa^4 k^4 \sum_{u=1}^{\pi} \int_{l_2} d\bar{s} \epsilon^{ijk} \left\langle p_K^{\bar{\rho}(r)} \cdot p_K^{\bar{\rho}(r)} \right\rangle \left[ y_{\nu}^{u,i} \cdot y_{\nu}^{u,i} \right]^k e^{-\kappa^2 [\nu_s^r - \rho(r)]^2 / 8} \left\langle \partial_0^{-1} q_K^{\nu_s^r} \cdot q_K^0 \right\rangle \otimes \frac{\xi_{\nu}^{u,i}}{3} \right]^2 \]
respectively as \(\kappa\) goes to infinity.

Replace Expressions 9.6 and 9.8 in Expression 9.10 with the above expressions and using the proof in Corollary 8.4.9, we will obtain our result.

**Definition 9.12 (Volume Path Integral)**

We define Expression 9.1 as the limit as \(\kappa\) goes to infinity, of Expression 9.11.

**Remark 9.13** The path integral in Expression 9.11 will of course depend on the choice of partition \(\{R_i\}_{i=1}^m\). But its limit as \(\kappa\) goes to infinity will be shown to be independent of this partition in a sequel.

## 10 Curvature Path Integral

Because curvature is a two form, we need to choose a surface \(S\) and integrate curvature over it. Now, \(S\) should be orientable, closed and bounded, with or without boundary, and disjoint from \(L\).

We allow \(S\) to be disconnected, with finite number of components. Furthermore, we insist the link \(\pi_0(L)\) intersects \(\pi_0(S)\) at most finitely many number of points inside \(\pi_0(S)\).

**Notation 10.1** Let
\[
\partial_0 A^i_{\alpha\beta} \equiv \frac{\partial A^i_{\alpha\beta}}{\partial x_0}, \quad \partial_i A^j_{\alpha\beta} \equiv \frac{\partial A^j_{\alpha\beta}}{\partial x_i}.
\]
In terms of \( \{ A_{\alpha\beta}^k \} \), we define
\[
F_S(\{ A_{\alpha\beta}^k \}) := \frac{1}{2} \int_S \partial_0 A_{\alpha\beta}^k \otimes dx_0 \otimes \hat{E}^{\alpha\beta} + \frac{1}{2} \sum_{i=1}^3 \int_S \partial_i A_{\alpha\beta}^k \otimes dx_i \otimes \hat{E}^{\alpha\beta} \\
+ \frac{1}{2} \int_S A_{\alpha\beta} A_{\gamma\mu}^i \otimes dx_i \otimes \partial_0 [\hat{E}^{\alpha\beta}, \hat{E}^{\gamma\mu}] \\
:= F_1^S(\{ A_{\alpha\beta}^k \}) + F_2^S(\{ A_{\alpha\beta}^k \}) + F_3^S(\{ A_{\alpha\beta}^k \}).
\]

Note that we do not specify any representation for \( \hat{E}^{\alpha\beta} \).

Refer to Notation 7.4. We are going to define the following curvature path integral, given by
\[
1 \int Z_{EH} F_S(\{ A_{\alpha\beta}^k \}) V(\{ B_{\mu}^i \}) W(\{ A_{\alpha\beta}^k \}) e^{i \int_{\mathcal{R}^4} A_0 \cdot B \times B - A \cdot \tilde{B} \} D\Lambda}, \tag{10.1}
\]
where
\[
Z_{EH} = \int \exp \left[ i \int_{\mathcal{R}^4} \partial_0 A_0 \cdot B \times B - \partial_0 A \cdot \tilde{B} \right] D\Lambda.
\]

We will now make use of Chern-Simon rules given in Definition 6.14 to make sense of the path integral in Expression 10.1 as before.

**Remark 10.2**

1. When \( S \) is the empty set, we define \( F_0 \equiv 1 \), so we write Expression [10.1] as \( Z(q; \chi(\mathcal{T}, \mathcal{L})) \), which was termed as the Wilson Loop observable of the colored hyperlink \( \chi(\mathcal{T}, \mathcal{L}) \) in Remark 8.1.

2. The path integral will take values in \( su(2) \times su(2) \).

## 10.1 \( F_1^S \) Path Integral

We will first define the path integral
\[
\frac{1}{Z_{EH}} \int F_S(\{ A_{\alpha\beta}^k \}) V(\{ B_{\mu}^i \}) W(\{ A_{\alpha\beta}^k \}) e^{i \int_{\mathcal{R}^4} A_0 \cdot B \times B - A \cdot \tilde{B} \} D\Lambda}, \tag{10.2}
\]
which we will denote it as \( \hat{F}_S[Z(q; \chi(\mathcal{T}, \mathcal{L}))] \).

Recall we defined \( \tilde{V}(\{ B_{\mu}^i \}) \) and \( \tilde{W}(\{ A_{\alpha\beta}^k \}) \) respectively in Equations (8.3) and (8.4).

**Lemma 10.3** After doing a change of variables given in Notation 8.2, the path integral in Expression [10.2] is defined as
\[
\frac{1}{Z_{EH}} \int \tilde{F}_S(\{ A_{\alpha\beta}^k \}) \tilde{V}(\{ B_{\mu}^i \}) \tilde{W}(\{ A_{\alpha\beta}^k \}) e^{i \int_{\mathcal{R}^4} A_0 \cdot B \times B - A \cdot \tilde{B} \} D\Lambda} \tag{10.3}
\]
with
\[
\tilde{Z}_{EH} = \int \exp \left[ i \int_{\mathcal{R}^4} A_0 \cdot B - A \cdot \tilde{B} \right] D\Lambda,
\]
after applying Steps 7 and 8 in Definition 6.14. Here, \( \tilde{F}_S(\{ A_{\alpha\beta}^k \}) \) is defined in Equation [10.4].
Proof. In Lemma \ref{Lemma8.4}, we replace $\partial_0 A$ with $A_0$ and $\partial_0 A$ with $A$ respectively. Recall we define $J_{\alpha\beta}$ in Notation \ref{Notation2.9}. Thus define

$$F_2^k(\{A_{\alpha\beta}^k\}) \mapsto \int_{\mathbb{R}^2} d\vec{r} \langle A_{ij}^{k}, \delta^{g(\vec{r})} \rangle J_{0i}(\vec{r}) \otimes \tilde{E}^{ij} = \int_{\mathbb{R}^2} d\vec{r} \langle A_{ij}^{k}, \delta^{g(\vec{r})} \rangle J_{0i}(\vec{r}) \otimes \tilde{E}^{ij} \equiv F_2^k(\{A_{\alpha\beta}^k\}).$$

The rest of the proof is exactly the same as the proof in Lemma \ref{Lemma8.4}, which will give us Expression \ref{Expression10.3}.

As explained in the earlier sections, $m(B(\vec{x}))^{-1}$, $m(B_{i}(\vec{x}))^{-1}$ cannot be defined. To make sense of the path integral given by Expression \ref{Expression10.3}, we will make the following approximations as done in Sections \ref{Section8} and \ref{Section9}. We approximate the Dirac-delta function with $p_\kappa$, $m(B)$ with $m_\kappa(B)$ and $m(B_i)$ with $m_\kappa(B_i)$.

With the above approximations, the path integral that was obtained previously is of the form

$$\frac{1}{Z_{EH}} \int F(\{B_i\}, \{\tilde{B}_i\}) e^{(A_{ij}^{\alpha_i}, \alpha_i^\prime)} e^{(A_{ij}^{\beta_i}, \beta_i^\prime)} e^j_{\lambda^j} B^j \otimes A_0 B^A \tilde{A}^B D\Lambda,$$  \hspace{1cm} \text{(10.5)}$$

whereby $F$ is some continuous function dependent on the variables $B_i$ and $\tilde{B}_i$, and $\alpha_i^\prime$, $\beta_i^\prime$ are in $C^\infty(\mathbb{R}^4)$.

According to the rules of the Chern-Simons integral given in Definition \ref{Definition6.14}, we substitute $B_i^j$ with $\alpha_i^\prime$ and $\tilde{B}_i^j$ with $\beta_i^\prime$ respectively inside the function $F$, and we will obtain the formula for the path integral. More importantly, in Sections \ref{Section8} and \ref{Section9}, we replace

$$\left[ m_\kappa((B, \delta^{\vec{e}_i})) \right]^{-1} \longrightarrow m_\kappa(\hat{\lambda}^{\kappa}(\hat{s}))^{-1},$$  \hspace{1cm} \text{(10.6)}$$

$$\left[ m_\kappa((B, \delta^{\vec{e}_i})) \right]^{-1} \longrightarrow m_\kappa(\hat{\lambda}^{\kappa}(\hat{s}))^{-1}.$$  \hspace{1cm} \text{(10.7)}$$

However, if one look at Expression \ref{Expression10.3}, it is not in the form given by Expression \ref{Expression10.5} even after we replace $m(B(\vec{g}_k))^{-1}$ and $m(B_i(\vec{g}_k))^{-1}$ with $m_\kappa(B(\vec{g}_k))^{-1}$ and $m_\kappa(B_i(\vec{g}_k))^{-1}$ respectively.

Refer to Notation \ref{Notation2.8}. To define the above path integral, we make an approximation to $m(B(\vec{g}_k))$ and $m(B_i(\vec{g}_k))$. However, unlike the approximation we made in Sections \ref{Section8} and \ref{Section9}, to simplify the integral, we approximate it with $m_\kappa(\hat{\lambda}^{\kappa}(\hat{s}))$ and $m_\kappa(\hat{\lambda}^{\kappa}(\hat{s}))$ respectively. Specifically, we will replace $m(B(\vec{g}_k))^{-1}$ and $m(B_i(\vec{g}_k))^{-1}$ with the substitutions given by Equations \ref{Expression10.6} and \ref{Expression10.7} respectively. As mentioned earlier, the reason is Expression \ref{Expression10.3} is not a Chern-Simons integral. To facilitate computations, we will first make this approximation.

With these approximations, we replace $\hat{W}$ and $\hat{V}$ with $\hat{W}_\kappa$ and $\hat{V}_\kappa$ respectively in Definition \ref{Definition10.4}.
Definition 10.4 Define

\[ \tilde{V}_\kappa(\{B_\mu\}) := \exp \left\{ \sum_{j=1}^{3} \left( B_j, -\tilde{\kappa} \sum_{v=1}^{2} \int_0^1 ds \left[ m_\kappa(\lambda_\kappa^v(\tilde{s}))^{-1} \left( \theta_\kappa^v \phi_\kappa^v \right) \right] \right) \right\} \]

\[ + \sum_{i=1}^{3} \left( B_i, -\tilde{\kappa} \sum_{v=1}^{2} \int_0^1 ds \left[ m_\kappa(\tilde{\lambda}_\kappa^v(\tilde{s}))^{-1} \left( \theta_\kappa^v \phi_\kappa^v \right) \right] \right) \}, \quad (10.8) \]

\[ W_\kappa(\{A_{\alpha_0}\}) := \prod_{u=1}^{n} Tr_{p_u} \exp \left\{ \left[ A_{k_0}^{(1)} - q \tilde{\kappa} \sum_{j=1}^{3} \int_0^1 ds \left[ \partial_0^{-1} \phi_\kappa^v \gamma_{u_j} \right] \right] \otimes \hat{E}_0 \right\} \]

\[ + \left[ A_{r(j)}^{(1)} - q \tilde{\kappa} \sum_{j=1}^{3} \int_0^1 ds \left[ \partial_0^{-1} \phi_\kappa^v \gamma_{u_j} \right] \right] \otimes \hat{E}_{r(j)} \].

Remark 10.5

1. We add in factors of \( \tilde{\kappa} \) as before.

2. Also refer to Remark 8.3.

Recall from Step [\( \text{def} \)] in Definition 6.14, we need to scale the surface integral with a factor of \( \tilde{\kappa}^2 \). Thus we replace \( \tilde{F}_1(\{A_{\alpha_0}\}) \) with

\[ \tilde{F}_{1,S}(\{A_{\alpha_0}\}) := \tilde{\kappa}^2 \int I^2 \left[ \langle A_{0j}, \bar{p}_\kappa^{(i)} \rangle J_0(\hat{t}) d\hat{t} \otimes \hat{E}_{(j)} \right] + \left[ A_{r(j)}^{(i)}, \bar{p}_\kappa^{(i)} \right] J_0(\hat{t}) d\hat{t} \otimes \hat{E}_{r(j)} \]

\[ \equiv \left[ A_{0j}, \int I^2 \tilde{\kappa}^2 p_\kappa^{(i)} J_0(\hat{t}) d\hat{t} \otimes \hat{E} \right] + \left[ A_{r(j)}^{(i)}, \int I^2 \tilde{\kappa}^2 p_\kappa^{(i)} J_0(\hat{t}) d\hat{t} \otimes \hat{E}_{r(j)} \right]. \]

Hence we approximate our path integral in Expression 10.3 with

\[ \frac{1}{Z_{EH}} \int \tilde{F}_{1,S}(\{A_{\alpha_0}\}) \tilde{V}_\kappa(\{B_\mu\}) W_\kappa(\{A_{\alpha_0}\}) e^{i \frac{1}{\tilde{\kappa}} A_0 - B \cdot \bar{B} - D \Lambda}. \]

This completes Steps [3] and [4] in Definition 6.14.

Lemma 10.6

Recall \( \tilde{E} = (1, 1, 1) \) and \( \tilde{E} \) was defined in Definition 8.11. Refer to Definition 8.13 where \( W_{\kappa}^{(1)}(q; T, L, L') \) was defined. Apply Step [3] in Definition 6.14, the path integral in Expression 10.9 is hence computed as

\[ \left\{ -i \tilde{\kappa} \sum_{v=1}^{2} \int I^2 d\hat{t} \left[ m_\kappa(\lambda_\kappa^v(\tilde{s}))^{-1} \left( \theta_\kappa^v \phi_\kappa^v \right) \right] \right\} \kappa \int I^2 \tilde{\kappa}^2 p_\kappa^{(i)} J_0(\hat{t}) d\hat{t} \otimes \hat{E} \]

\[ + i \tilde{\kappa} \sum_{v=1}^{2} \int I^2 d\hat{t} \left[ m_\kappa(\lambda_\kappa^v(\tilde{s}))^{-1} \left( \theta_\kappa^v \phi_\kappa^v \right) \right] \kappa \int I^2 \tilde{\kappa}^2 p_\kappa^{(i)} J_0(\hat{t}) d\hat{t} \otimes \sum_{j=1}^{3} \hat{E}_{r(j)} \right\} Z(\kappa, q; \chi(T, L)). \]

(10.10)

Proof. From Equation 10.8, according to the rules of Definition 6.14 of the Chern-Simons path integral, we now replace

\[ (A_{01}, A_{02}, A_{03}) \rightarrow - i \tilde{\kappa} \sum_{v=1}^{2} \int I d\hat{t} \left[ m_\kappa(\lambda_\kappa^v(\tilde{s}))^{-1} \left( \theta_\kappa^v \phi_\kappa^v \right) \right] \]

\[ A_{r(j)} \rightarrow i \tilde{\kappa} \sum_{v=1}^{2} \int I d\hat{t} \left[ m_\kappa(\lambda_\kappa^v(\tilde{s}))^{-1} \left( \theta_\kappa^v \phi_\kappa^v \right) \right] \]

(10.11)
Remark 10.7 Inside the functionals $\bar{W}_\kappa$ and $F^1_{\kappa,S}$. Note that $\int_I ds \left\{ m_\kappa(\lambda_\kappa^v(s))^{-1} (\theta_s^v \theta_s^v) \right\}^k$ is a 3-vector.

It can be shown that the substitution inside $\bar{W}_\kappa$ will yield $Z(\kappa, q; \chi(L, L))$, by using a similar argument in Lemma 8.16. Notice that it is now no longer necessary to have the time ordering operator. So, we will focus on making the substitution inside $F^1_{\kappa,S}$. After the substitution, we obtain the term

$$\int_F ds \left\{ m_\kappa(\lambda_\kappa^v(s))^{-1} (\theta_s^v \theta_s^v) \right\}^k, \int_I \kappa^2 p^\rho_\kappa (J_0(t)) dt \otimes \hat{E}$$

This completes the proof. □

**Remark 10.7** Refer to Remark 8.14 for an explanation of Expression 10.12.

Using Lemma 8.18, we see that to compute the limit of the first and second term in Expression 10.12 it suffices to compute the limit of

$$A^\pm_\kappa := \frac{i e^3}{32\pi \sqrt{4\pi}} \sum_{n=1}^{\infty} \int_I \left\{ \left( \theta_{k_s}^v \theta_{k_s}^v - \theta_{k_s}^v \theta_{k_s}^v \right) p^v_{k_s} \right\} J_{01}(t)$$

respectively, as $\kappa$ goes to infinity.

From the proof of Lemma 8.19 we see that

$$\lim_{\kappa \to \infty} Z(\kappa, q; \chi(L, L)) = \lim_{\kappa \to \infty} \prod_{u=1}^\pi \left[ Tr_{\rho_u^+} \bar{W}_\kappa(q; T_u^+, L) + Tr_{\rho_u^-} \bar{W}_\kappa(q; T_u^-, L) \right].$$

**Definition 10.8** We define Expression 10.13 as

$$\lim_{\kappa \to \infty} \left[ A^+_{\kappa} + A^-_{\kappa} \right] \prod_{u=1}^\pi \left[ Tr_{\rho_u^+} \bar{W}_\kappa(q; T_u^+, L) + Tr_{\rho_u^-} \bar{W}_\kappa(q; T_u^-, L) \right].$$

Here, $A^\pm_\kappa$ were defined by Expression 10.12.

### 10.2 $F^2_S$ Path Integral

We will now define the path integral

$$\frac{1}{Z_{EH}} \int F^2_S([A_{\alpha_{ij}}]) V([L^v_{\kappa}]) \sum_{v=1}^{\kappa} (B^v_{\mu_{ij}}) W(q; \{ T^v_{\mu}, \rho_u \}^\pi_{u=1}) \left( \{ A_{k_{ij}} \} e^{ik f_{k_{ij}} \partial_{\kappa_{ij}} B A \times B - A \times B} \right),$$

which we will denote it as $\hat{F}^2_S[Z(q; \chi(L, L))]$.
Refer to Notation 2.8. Consider the term $\frac{1}{2} \sum_{i=1}^{3} \int_{S} \partial_i A_{\alpha\beta}^i \otimes dx_i \wedge dx_j \otimes \tilde{E}^{\alpha\beta}$. Observe that we can write this term as

$$
\frac{1}{2} \sum_{i=1}^{3} \int_{S} \partial_i A_{\alpha\beta}^i \otimes dx_i \wedge dx_j \otimes \tilde{E}^{\alpha\beta} = \frac{1}{2} \int_{J^2} \nabla \times (A_{\alpha\beta}^1, A_{\alpha\beta}^2, A_{\alpha\beta}^3)(\sigma(\tilde{t})) \cdot (J_{23}, J_{31}, J_{12})(\tilde{t}) d\tilde{t} \otimes \tilde{E}^{\alpha\beta}.
$$

**Remark 10.9** Note that $\nabla \times (A_{\alpha\beta}^1, A_{\alpha\beta}^2, A_{\alpha\beta}^3)$ means take the curl of the 3-vector $(A_{\alpha\beta}^1, A_{\alpha\beta}^2, A_{\alpha\beta}^3)$.

**Lemma 10.10** Recall $\hat{E} = (1, 1, 1)$. Let $\delta^2$ be the Dirac-delta function, i.e., for any function $f$, $(f, \delta^2) = f(\hat{E})$. Refer to the parametrizations $\hat{\vartheta}$ and $\hat{\vartheta}$ defined in Notation 2.8. After doing a change of variables given in Notation 2.8, the path integral in Expression (10.16) is defined as

$$
\frac{1}{Z_{EH}} \int \tilde{F}^2_{\alpha\beta}(A_{\alpha\beta}) \tilde{V}(\{B_{\mu}\}) \tilde{W}(\{A_{\alpha\beta}\}) e^{i \int_{S} \tilde{A}_0 \cdot B - A \cdot \hat{B}} d\Lambda,
$$

with

$$
\tilde{Z}_{EH} = \int \exp \left[ i \int_{S} A_0 \cdot B - A \cdot \hat{B} \right] d\Lambda,
$$

after applying Steps 1 and 2 in Definition 6.14.

Here, $\tilde{V}(\{B_{\mu}\})$ and $\tilde{W}(\{A_{\alpha\beta}\})$ were defined in Equations (8.13) and (8.14) respectively and

$$
\tilde{F}^2_{\alpha\beta}(A_{\alpha\beta})
= \int_{J^2} d\tilde{t} \left< A_{0j}, \partial_{0j}^{-1} \nabla \times (\delta^2(\tilde{t}), J_{\sigma}(\tilde{t})) \right> \otimes \tilde{E}^{0j} + \int_{J^2} d\tilde{t} \left< A_{\tau(j)}, \partial_{0\tau}^{-1} \nabla \times (\delta^2(\tilde{t}), J_{\sigma}(\tilde{t})) \right> \otimes \tilde{E}^{\tau(j)}.
$$

**Proof.** By writing $A_{\alpha\beta} = (A_{\alpha\beta}^1, A_{\alpha\beta}^2, A_{\alpha\beta}^3)$ and $J_{\sigma} = (J_{23}, J_{31}, J_{12})$, we see that

$$
(\nabla \times A_{\alpha\beta}(\hat{E})) \cdot J_{\sigma}(\tilde{t}) = (\nabla \times A_{\alpha\beta}^1, \delta^2 J_{\sigma}(\tilde{t})) \equiv (\nabla \times A_{\alpha\beta}, \delta^2 J_{\sigma}(\tilde{t}))
= (A_{\alpha\beta}, -\nabla \times \delta^2 J_{\sigma}(\tilde{t})).
$$

In the last equality, we make use of the fact that $\nabla \times$ is an skew-symmetric operator.

Note that $A_{\alpha\beta} \mapsto \partial_{0\tau}^{-1} A_{\alpha\beta}$ and $\partial_{0\tau}^{-1} f, g = -(f, \partial_{0\tau}^{-1} g)$. The rest of the proof is similar to the proof in Lemma 10.3 hence omitted.

As explained in Subsection 10.4, Expression (10.16) is not a Chern-Simons integral, therefore we need to make some approximations, i.e., approximate $m(B)$ and $m(B_{\mu})$ with $m_k(B)$ and $m_{k_{\mu}}(B_{\mu})$, and $\delta^2$ with $p_k^2$. After making these approximations, it is still not a Chern-Simons integral. So, we need to make use of the substitutions given by Equations (10.6) and (10.7). Finally we need to add in factors of $k$.

Thus we approximate our path integral in Expression (10.16) with

$$
\frac{1}{Z_{EH}} \int \tilde{F}^2_{\alpha\beta}(A_{\alpha\beta}) \tilde{V}_{\kappa}(\{B_{\mu}\}) \tilde{W}_{\kappa}(\{A_{\alpha\beta}\}) e^{i \int_{S} \tilde{A}_0 \cdot B - A \cdot \hat{B}} d\Lambda.
$$

Here, $\tilde{V}_{\kappa}(\{B_{\mu}\})$ and $\tilde{W}_{\kappa}(\{A_{\alpha\beta}\})$ were defined in Definition 10.4 and

$$
\tilde{F}^2_{\kappa, S}(A_{\alpha\beta})
:= k^2 \int_{J^2} d\tilde{t} \left< A_{0j}, \partial_{0j}^{-1} \nabla \times (p_k^2(\tilde{t}), J_{\sigma}(\tilde{t})) \right> \otimes \tilde{E}^{0j} + k^2 \int_{J^2} d\tilde{t} \left< A_{\tau(j)}, \partial_{0\tau}^{-1} \nabla \times (p_k^2(\tilde{t}), J_{\sigma}(\tilde{t})) \right> \otimes \tilde{E}^{\tau(j)}
\equiv \left< A_{0j}, k^2 \int_{J^2} d\tilde{t} \partial_{0j}^{-1} \nabla \times (p_k^2(\tilde{t}), J_{\sigma}(\tilde{t})) \right> \otimes \tilde{E}^{0j} + \left< A_{\tau(j)}, k^2 \int_{J^2} d\tilde{t} \partial_{0\tau}^{-1} \nabla \times (p_k^2(\tilde{t}), J_{\sigma}(\tilde{t})) \right> \otimes \tilde{E}^{\tau(j)}.
$$

(10.18)
This completes Steps 3 and 4 in Definition 6.14.

**Notation 10.11** Write \( \hat{J} (i) = (\hat{J}^1 (i), \hat{J}^2 (i), \hat{J}^3 (i)) \), whereby

\[
\hat{J}^i = J_{\tau (i)} \otimes (\hat{E}^{01}, \hat{E}^{02}, \hat{E}^{03}).
\]

**Lemma 10.12** Refer to Definition 8.13 where \( W_\kappa (q; \overline{L}, \overline{l}^w) \) was defined. Apply Step 3 in Definition 6.14 the path integral in Expression (10.19) is hence computed as \( \tilde{A}_\kappa Z(\kappa, q; \chi (\overline{L}, \overline{l})) \), whereby

\[
\tilde{A}_\kappa := -i \kappa \sum_{k=1}^3 \left[ \sum_{s=1}^{\infty} \int_I ds \left\{ \left[ m_\kappa (\lambda^\kappa (s)) \right]^{-1} \left( \varphi_{\kappa, s} (p_{\kappa} \otimes \hat{J} (i)) \right) \right\} \right]
\]

\[
\times \left[ \sum_{i=1}^3 \int_I \partial_0^{-1} \nabla \times \left( \varphi_{\kappa} (p_{\kappa} \otimes \hat{J} (i)) \right) \right] \otimes \mathbb{F}^-.
\]

(10.19)

Here, both \( \lambda^\kappa \) and \( \tilde{\lambda}^\kappa \) were defined in Equations (8.14) and (8.15) respectively. And note that \( Z(\kappa, q; \chi (\overline{L}, \overline{l})) \) was defined in Equation (8.16).

**Proof.** According to the rules of Definition 6.14 of the Chern-Simons integral and from Equation (10.8), we use the Substitution given by Equations (10.11) inside the functionals \( W_\kappa \) and \( \tilde{F}^2 \) given by Equations (10.8) and (10.18) respectively.

One can show that the substitution inside \( W_\kappa \) will yield \( Z(\kappa, q; \chi (\overline{L}, \overline{l})) \). Now refer to Notation 10.11 Making the above substitution inside \( \tilde{F}^2 \) will yield the term \( \tilde{A}_\kappa \).

**Remark 10.13**

1. See Remark 8.14 for an explanation of RHS of Equation (10.19).

2. The term

\[
\partial_0^{-1} \nabla \times \left( \varphi_{\kappa} (p_{\kappa} \otimes \hat{J} (i)) \right) = \partial_0^{-1} \nabla \times \left( \varphi_{\kappa} \otimes J_{\sigma} (i) \right) \otimes (\hat{E}^{01}, \hat{E}^{02}, \hat{E}^{03})
\]

can be written as \( (a^1, a^2, a^3) \otimes (\hat{E}^{01}, \hat{E}^{02}, \hat{E}^{03}) \), with each \( a^i \) given by for \( (i, j, k) \in C_3 \),

\[
a^i = \partial_0^{-1} (\partial_0 p_{\kappa} (\hat{J} (i))) \otimes J_{\tau (k) (i)} - \partial_0^{-1} (\partial_0 p_{\kappa} (\hat{J} (i))) \otimes J_{\tau (k) (i)}.
\]

Note that \( \partial_0^{-1} (\partial_0 p_{\kappa} (\hat{J} (i))) \) is defined using Equation (5.1).

Now \( \langle \nabla \times f, g \rangle = \langle f, -\nabla \times g \rangle \) and from Remark 8.4

\[
- \nabla \times m_\kappa (0)^{-1} = -\kappa.
\]

Hence \( m_\kappa (0)^{-1} = \kappa (\nabla \times )^{-1} \). Since \( \kappa \lambda_\kappa \to 0 \) and \( \kappa \tilde{\lambda}_\kappa \to 0 \) as \( \kappa \to \infty \) by Lemma 8.18 therefore to compute the limit of Expression (10.19) as \( \kappa \) goes to infinity is equivalent to compute the limit of

\[
\tilde{B}_\kappa := \frac{i \kappa^3}{32 \pi \sqrt{4 \pi}} \int_{I^3} \kappa \left( \partial_0^{-1} p_{\kappa} (\hat{J} (i)) \otimes \varphi_{\kappa} \otimes J_{\sigma} (i) \right) \otimes \sum_{i=1}^3 \hat{E}^{0i} ds,\]

\[
- \frac{i \kappa^3}{32 \pi \sqrt{4 \pi}} \int_{I^3} \kappa \left( \partial_0^{-1} p_{\kappa} (\hat{J} (i)) \otimes \varphi_{\kappa} \otimes J_{\sigma} (i) \right) \otimes \sum_{j=1}^3 \hat{E}^{0(j)} ds,
\]

as \( \kappa \) goes to infinity.
Definition 10.14 Refer to Equation [10.14] We define Expression [10.15] by
\[
\lim_{\kappa \to \infty} \hat{B}_\kappa \prod_{u=1}^n \left[ \text{Tr}_{\rho_u} \mathcal{W}_\kappa^+(q; T^u, L) + \text{Tr}_{\rho_u} \mathcal{W}_\kappa^-(q; T^u, L) \right],
\]
whereby \( \hat{B}_\kappa \) was defined in the preceding paragraph.

10.3 \( F_3^3 \) Path Integral

Refer to Notations [2.2] and [2.9] We have finally come to the last term
\[
F_3^3(\{A_{\alpha\beta}\}) = \frac{1}{2} \int_{\mathcal{S}} A_{\alpha\beta}^{i} A_{\gamma\mu}^{j} \odot dx_i \wedge dx_j \odot [\mathcal{E}^{\alpha\beta}, \mathcal{E}^{\gamma\mu}].
\]

We will now define the path integral
\[
\frac{1}{Z_{EH}} \int F_3^3(\{A_{\alpha\beta}\})V(\{\Sigma^\alpha\}_{\alpha=1}^n)(\{B^\mu\}_{\mu=1}^n)W(q; \{\bar{T}^n\}_{\mu=1}^n)(\{A_{\alpha\beta}\})e^{\int_{\mathcal{S}} A_{\alpha\beta} \cdot \bar{E} \cdot (B^\beta - A^\beta B - A \cdot \bar{B}) DA}, \tag{10.20}
\]
which we will denote it as \( F_3^2[Z(q; \chi(L, L))] \).

Lemma 10.15 Recall \( \mathcal{E} = (1, 1, 1) \). Let \( \delta^2 \) be the Dirac-delta function, i.e. for any function \( f \), \( \langle f, \delta^2 \rangle = f(\bar{x}) \). Refer to the parametrizations \( \bar{\sigma}^\alpha \) and \( \bar{\sigma}^\alpha \) defined in Notation [2.8] After doing a change of variables given in Notation [8.2], the path integral in Expression [10.20] is defined as
\[
\frac{1}{Z_{EH}} \int F_3^3(\{A_{\alpha\beta}\})\bar{V}(\{B^\mu_{\alpha}\})\bar{W}(\{A_{\alpha\beta}\})e^{\int_{\mathcal{S}} A_{\alpha\beta} \cdot B - A \cdot \bar{B} DA} \tag{10.21}
\]
with
\[
\bar{Z}_{EH} = \int \exp \left[ i \int_{\mathbb{R}^4} A_{\alpha\beta} \cdot B - A \cdot \bar{B} \right] DA,
\]
after applying Steps [7] and [8] in Definition [6.14].

Here, \( \bar{V}(\{B^\mu_{\alpha}\}) \) and \( \bar{W}(\{A_{\alpha\beta}\}) \) were defined in Equations [8.3] and [8.4] respectively and \( F_3^2(\{A_{\alpha\beta}\}) \) is defined by Equation [10.22]. See Equations [10.23] and [10.24].

Proof. Now we can write
\[
A_{\alpha\beta}(\bar{x})A_{\gamma\mu}(\bar{x}) \equiv \langle A_{\alpha\beta}^i, \delta^2 \rangle \langle A_{\gamma\mu}^j, \delta^2 \rangle =: \langle A_{\alpha\beta}^i \odot A_{\gamma\mu}^j, \delta^2 \odot \delta^2 \rangle,
\]
using the tensor inner product.

Refer to Notations [2.2], [2.3] and [2.9] Then we can write
\[
F_3^3(\{A_{\alpha\beta}\})
= 8 \sum_{(i,j) \in T} \sum_{(i,j) \in T} \int_{\mathbb{R}^4} d\bar{t} A_{0i}^{\alpha}(\bar{\sigma}(\bar{t}))A_{0j}^{\beta}(\bar{\sigma}(\bar{t}))J_{ij}(\bar{t}) \odot [\mathcal{E}^{0\alpha}, \mathcal{E}^{0\beta}]
+ 8 \sum_{(i,j) \in T} \sum_{(i,j) \in T} \int_{\mathbb{R}^4} d\bar{t} A_{i\gamma}^{\alpha}(\bar{\sigma}(\bar{t}))A_{j\mu}^{\beta}(\bar{\sigma}(\bar{t}))J_{ij}(\bar{t}) \odot [\mathcal{E}^{\gamma\alpha}, \mathcal{E}^{\mu\beta}]
= 8 \sum_{(i,j) \in T} \sum_{(i,j) \in T} \int_{\mathbb{R}^4} d\bar{t} \langle A_{0i}^{\alpha} \odot A_{0j}^{\beta}, \delta(\bar{t}) \odot \delta(\bar{t}) \rangle J_{ij}(\bar{t}) \odot [\mathcal{E}^{0\alpha}, \mathcal{E}^{0\beta}]
+ 8 \sum_{(i,j) \in T} \sum_{(i,j) \in T} \int_{\mathbb{R}^4} d\bar{t} \langle A_{i\gamma}^{\alpha} \odot A_{j\mu}^{\beta}, \delta(\bar{t}) \odot \delta(\bar{t}) \rangle J_{ij}(\bar{t}) \odot [\mathcal{E}^{\gamma\alpha}, \mathcal{E}^{\mu\beta}].
\]

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Note that \( A_{\alpha\beta} \mapsto \partial^{-1}_0 A_{\alpha\beta} \) and \( \langle \partial^{-1}_0 f, g \rangle = -\langle f, \partial^{-1}_0 g \rangle \). So we define
\[
F^3_S(\{A^k_{\alpha\beta}\}) := 8 \sum_{(i,j) \in Y} \sum_{(i,j) \in Y} \int_{I^2} d\hat{t} \left\langle A_{0i} \otimes A_{0j}, \partial_0^{-1} \delta^\beta(\hat{t}) \otimes \partial_0^{-1} \delta^\beta(\hat{t}) \right\rangle J_{ij}(\hat{t}) \otimes [\hat{E}^{0i}, \hat{E}^{0j}]
+ 8 \sum_{(i,j) \in Y} \sum_{(i,j) \in Y} \int_{I^2} d\hat{t} \left\langle A_{0i} \otimes A_{0j}, \partial_0^{-1} \delta^\beta(\hat{t}) \otimes \partial_0^{-1} \delta^\beta(\hat{t}) \right\rangle J_{ij}(\hat{t}) \otimes [\hat{E}^{\tau(i)}, \hat{E}^{\tau(j)}]
\equiv 8 \sum_{(i,j) \in Y} \int_{I^2} d\hat{t} \left\langle A_{\tau(i)} \otimes A_{\tau(j)}, \partial_0^{-1} \delta^\beta(\hat{t}) \otimes \partial_0^{-1} \delta^\beta(\hat{t}) J_{\sigma}(\hat{t}) \right\rangle \otimes [\hat{E}^{\tau(i)}, \hat{E}^{\tau(j)}].
\tag{10.22}
\]

Here,
\[
A_{0i} \otimes A_{0j} \equiv \left( A_{0i}^2 \otimes A_{0j}^3, A_{0i}^3 \otimes A_{0j}^1, A_{0i}^1 \otimes A_{0j}^3 \right),
\tag{10.23}
A_{\tau(i)} \otimes A_{\tau(j)} \equiv \left( A_{\tau(i)}^2 \otimes A_{\tau(j)}^3, A_{\tau(i)}^3 \otimes A_{\tau(j)}^1, A_{\tau(i)}^1 \otimes A_{\tau(j)}^3 \right),
\tag{10.24}
\]
are respectively 3-vectors, whose components are in \( C^\infty(\mathbb{R}) \otimes C^\infty(\mathbb{R}) \). The rest of the proof is similar to the proof in Lemma 10.3 hence omitted.

As explained in Subsection 10.1 Expression (10.21) is not a Chern-Simons integral. To make it into a Chern-Simons integral, we make use of the substitutions given by Equations (10.6) and (10.7). We also approximate \( \delta^2 \) with \( p^2_0 \). And we need to add in factors of \( \hat{k} \).

Thus we approximate our path integral in Expression (10.16) with
\[
\frac{1}{Z_{EH}} \int \hat{F}^3_{\kappa,S}(\{A^k_{\alpha\beta}\}) \hat{V}_\kappa(\{B^k_\mu\}) \hat{W}_\kappa(\{A^k_{\alpha\beta}\}) e^{t \int_{A^0 - A^0} B - A^0 \hat{B} DA}.
\tag{10.25}
\]

Here, \( \hat{V}_\kappa(\{B^k_\mu\}) \) and \( \hat{W}_\kappa(\{A^k_{\alpha\beta}\}) \) were defined in Definition (10.3) and
\[
\hat{F}^3_{\kappa,S}(\{A^k_{\alpha\beta}\})
:= 8k^2 \sum_{(i,j) \in Y} \int_{I^2} d\hat{t} \left\langle A_{0i} \otimes A_{0j}, \partial_0^{-1} p^\kappa(\hat{t}) \otimes \partial_0^{-1} p^\kappa(\hat{t}) J_{\sigma}(\hat{t}) \right\rangle \otimes [\hat{E}^{0i}, \hat{E}^{0j}]
+ 8k^2 \sum_{(i,j) \in Y} \int_{I^2} d\hat{t} \left\langle A_{\tau(i)} \otimes A_{\tau(j)}, \partial_0^{-1} p^\kappa(\hat{t}) \otimes \partial_0^{-1} p^\kappa(\hat{t}) J_{\sigma}(\hat{t}) \right\rangle \otimes [\hat{E}^{\tau(i)}, \hat{E}^{\tau(j)}]
\equiv \sum_{(i,j) \in Y} \left\langle A_{0i} \otimes A_{0j}, 8k^2 \int_{I^2} d\hat{t} \partial_0^{-1} p^\kappa(\hat{t}) \otimes \partial_0^{-1} p^\kappa(\hat{t}) J_{\sigma}(\hat{t}) \right\rangle \otimes [\hat{E}^{0i}, \hat{E}^{0j}]
+ \sum_{(i,j) \in Y} \left\langle A_{\tau(i)} \otimes A_{\tau(j)}, 8k^2 \int_{I^2} d\hat{t} \partial_0^{-1} p^\kappa(\hat{t}) \otimes \partial_0^{-1} p^\kappa(\hat{t}) J_{\sigma}(\hat{t}) \right\rangle \otimes [\hat{E}^{\tau(i)}, \hat{E}^{\tau(j)}].
\tag{10.26}
\]

This completes Steps 3 and 4 in Definition 6.14.
Notation 10.16 Refer to Notation 8.22. Observe that
\[
\sum_{(i,j)\in \Upsilon} [\hat{E}^{ij}, E^{ij}] = (\mathcal{E}^+, 0) = \mathcal{F}^+, \quad \sum_{(i,j)\in \Upsilon} [\hat{E}^{\tau(i)}, \hat{E}^{\tau(j)}] = (0, \mathcal{E}^-) = \mathcal{F}^-.
\]

Lemma 10.17 Refer to Definition 8.14 where \( W^\pm_\kappa(q; \bar{L}, \bar{L}^v) \) was defined. Apply Step 6 in Definition 6.14, the path integral in Expression 10.26 is hence computed as \( (A_\kappa + B_\kappa)Z(\kappa, q; \chi(\bar{L}, \bar{L})) \), whereby \( A_\kappa \) is given by Expression 10.27 and \( B_\kappa \) is given by Expression 10.28.

Proof. According to the rules of Definition 6.14 of the Chern-Simons integral and from Equation (10.8), we apply the substitution given in Equation (10.11) inside the functionals \( \hat{W}_\kappa \) and \( \hat{F}^2_\kappa \) given by Equations (10.25) and (10.26) respectively. Here, both \( \lambda^v_\kappa \) and \( \lambda^v_\tilde{\kappa} \) were defined by Equations (8.14) and (8.15) respectively.

The 3-vector
\[
\sum_{v = 1}^n \int_I ds \left( \left\{ m_\kappa(\lambda^v_\kappa(s)) \right\}^{-1} (\hat{\varepsilon}_s^v, \hat{\varepsilon}_s^\tilde{v}) \right)_i \equiv (C^i_1, C^i_2, C^i_3),
\]
will have each component
\[
C^i_j := \sum_{v = 1}^n \int_I ds \left( \left\{ m_\kappa(\lambda^v_\kappa(s)) \right\}^{-1} (\hat{\varepsilon}_s^v, \hat{\varepsilon}_s^\tilde{v}) \right)_j.
\]

It was shown that the substitution inside \( \hat{W}_\kappa \) will yield \( Z(\kappa, q; \chi(\bar{L}, \bar{L})) \). Thus after making the above substitution into \( \hat{F}^2_\kappa(S(A_\kappa)) \), should give us the sum of 2 terms, \( A_\kappa + B_\kappa \), the first term \( A_\kappa \) being (after some simplification)
\[
8 (i\bar{\kappa})^2 \sum_{(i,j)\in \Upsilon} \sum_{(i,j)\in \Upsilon} \int_{I^2} d\bar{t} J_{ij}(\bar{t}) \left\{ \hat{\kappa} \sum_{v = 1}^n \int_I ds \left( \left\{ m_\kappa(\lambda^v_\kappa(s)) \right\}^{-1} (\hat{\varepsilon}_s^v, \hat{\varepsilon}_s^\tilde{v}) \right)_i \cdot \partial_0^{-1} \hat{p}_\kappa^{\tilde{\kappa}(\bar{t})} \right\}
\times \left\{ \hat{\kappa} \sum_{v = 1}^n \int_I ds \left( \left\{ m_\kappa(\lambda^v_\kappa(s)) \right\}^{-1} (\hat{\varepsilon}_s^v, \hat{\varepsilon}_s^\tilde{v}) \right)_j \cdot \partial_0^{-1} \hat{p}_\kappa^{\tilde{\kappa}(\bar{t})} \right\} \rangle \langle \hat{E}^{\tilde{\kappa}}, \hat{E}^{\tilde{\kappa}}] \rangle,
\]
and the second term \( B_\kappa \) being
\[
8 (i\bar{\kappa})^2 \sum_{(i,j)\in \Upsilon} \int_{I^2} \left\{ \hat{\kappa} \sum_{v = 1}^n \int_I ds \left( \left\{ m_\kappa(\lambda^v_\kappa(s)) \right\}^{-1} (\hat{\varepsilon}_s^v, \hat{\varepsilon}_s^\tilde{v}) \right)_i \cdot \partial_0^{-1} \hat{p}_\kappa^{\tilde{\kappa}(\bar{t})} \right\}
\times \left\{ \hat{\kappa} \sum_{v = 1}^n \int_I ds \left( \left\{ m_\kappa(\lambda^v_\kappa(s)) \right\}^{-1} (\hat{\varepsilon}_s^v, \hat{\varepsilon}_s^\tilde{v}) \right)_j \cdot \partial_0^{-1} \hat{p}_\kappa^{\tilde{\kappa}(\bar{t})} \right\} J_{ij}(\bar{t}) d\bar{t} \otimes (0, \mathcal{E}^-).
\]

Remark 10.18 Refer to Remark 8.14 for a detailed explanation of Expressions (10.27) and (10.28).
Refer to Notation 2.3. Since \( \kappa \lambda_\nu^v \to 0 \) and \( \kappa \lambda_\nu^v \to 0 \) as \( \kappa \) goes to infinity in Lemma 8.18, to compute the limit for the terms \( A_\kappa \) and \( B_\kappa \) as \( \kappa \) goes to infinity, it suffices to compute the limit of

\[
\bar{C}_\kappa^\pm := \frac{-\kappa^4}{32\pi^2} \sum_{(i,j,k) \in C_3} \sum_{v,v'=1}^n \int I_4 \left\{ \left\langle \partial_0^{-1} p_{\kappa}^{\pm}(i), \kappa \left[ \frac{\nu'}{\partial_{k,v}} \partial_{j,v}^{-1} - \frac{\nu'}{\partial_{j,v}} \partial_{k,v}^{-1} \right] p_{\kappa}^{\pm} \right\rangle \right. \\
\times \left\langle \partial_0^{-1} p_{\kappa}^{\pm}(j), \kappa \left[ \frac{\nu'}{\partial_{k,v}} \partial_{i,v}^{-1} - \frac{\nu'}{\partial_{i,v}} \partial_{k,v}^{-1} \right] p_{\kappa}^{\pm} \right\rangle \right\} J_{ij}(\hat{t}) \, d\hat{s}d\hat{t} \otimes \mathcal{F}^\pm,
\]

respectively as \( \kappa \) goes to infinity. Note that \( \hat{s} = (s, \bar{s}) \) and \( \hat{t} = (t, \bar{t}) \).

**Definition 10.19** Refer to Equation 10.14. We define Expression 10.20 by

\[
\lim_{\kappa \to \infty} \left[ \bar{C}_\kappa^+ + \bar{A}_\kappa + \bar{B}_\kappa + \bar{C}_\kappa^- \right] \prod_{u=1}^{\pi} \left[ \text{Tr}_{\rho_{\kappa}^u} \hat{W}_{\kappa}^+(q; \hat{T}^u, \mathcal{L}) + \text{Tr}_{\rho_{\kappa}^u} \hat{W}_{\kappa}^-(q; \hat{T}^u, \mathcal{L}) \right],
\]

whereby \( \bar{C}_\kappa^\pm \) were defined in the preceding paragraph.

Putting all together, we can now make the final definition.

**Definition 10.20** (Curvature Path Integral)

Refer to Definitions 10.8, 10.14 and 10.19. We define the curvature path integral given by Expression 10.1 as

\[
\lim_{\kappa \to \infty} \left[ (\bar{A}_\kappa^+ + \bar{A}_\kappa^-) + \bar{B}_\kappa + (\bar{C}_\kappa^+ + \bar{C}_\kappa^-) \right] \prod_{u=1}^{\pi} \left[ \text{Tr}_{\rho_{\kappa}^u} \hat{W}_{\kappa}^+(q; \hat{T}^u, \mathcal{L}) + \text{Tr}_{\rho_{\kappa}^u} \hat{W}_{\kappa}^-(q; \hat{T}^u, \mathcal{L}) \right].
\]

### A Important Lemmas

**Lemma A.1** Refer to Notation 4.2.

1. Let \( s, t \in \mathbb{R}, s \neq t \). Then

\[
\lim_{\kappa \to \infty} \kappa \sqrt{2\pi} \langle q^s_\kappa, \partial_0^{-1} q^t_\kappa \rangle = \begin{cases} 1, & s > t; \\
-1, & s < t. \end{cases}
\]

2. Let \( a, b \in \mathbb{R}^2 \). Then,

\[
\langle p^a_{\kappa}, p^b_{\kappa} \rangle = e^{-\kappa^2 |a-b|^2/8}.
\]

3. Let \( a, b \in \mathbb{R}^3 \). Then,

\[
\langle p^a_{\kappa}, p^b_{\kappa} \rangle = e^{-\kappa^2 |a-b|^2/8}.
\]

**Proof.** The proof of Item 1 can be found in Lemma 4.5 in [7]. We reproduce it here for the convenience of the reader.
By definition of $\partial_0^{-1}$, we have

$$
\frac{\kappa}{\sqrt{2\pi}} \langle q_\kappa, \partial_0^{-1} q_\kappa \rangle
= \frac{\kappa}{\sqrt{2\pi}} \int_{\mathbb{R}} \sqrt{\kappa} \sqrt{2\pi} e^{-\kappa \frac{(y-t)^2}{4}} \frac{1}{2} \left[ \int_{0}^{\infty} \sqrt{\kappa} e^{-\kappa \frac{y^2}{4}} dy - \int_{-\infty}^{0} \sqrt{\kappa} e^{-\kappa \frac{y^2}{4}} dy \right] dx
$$

$$
\to \begin{cases} 
1, & s > t; \\
-1, & s < t,
\end{cases}
as \kappa \to \infty.
$$

Let $x = (x_+, x_-)$. Item (2) follows from direct integration,

$$
\int_{\mathbb{R}^2} \frac{\kappa}{\sqrt{2\pi}} e^{-\kappa \frac{|x-a|^2}{4}} \frac{\kappa}{\sqrt{2\pi}} e^{-\kappa \frac{|x-b|^2}{4}} dx_+ dx_- = e^{-\kappa \frac{|a-b|^2}{8}}.
$$

Note that $\langle \mu^\kappa_a, \mu^\kappa_b \rangle$ means integrate the product over $\mathbb{R}^3$ using Lebesgue measure. The proof of Item 3 is similar to the above item, so omitted.

References

[1] R. W. R. Darling. *Differential forms and connections*. Cambridge University Press, Cambridge, 1994.

[2] Brian C. Hall. *Lie groups, Lie algebras, and representations*, volume 222 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2003. An elementary introduction.

[3] Hui Hsiung Kuo. *Gaussian measures in Banach spaces*. Lecture Notes in Mathematics, Vol. 463. Springer-Verlag, Berlin, 1975.

[4] Adrian P. C. Lim. Area operator in loop quantum gravity. *Preprint*.

[5] Adrian P. C. Lim. Path integral quantization of volume. *Preprint*.

[6] Adrian P. C. Lim. Quantized curvature in loop quantum gravity. *Preprint*.

[7] Adrian P. C. Lim. Chern-Simons path integral on $\mathbb{R}^3$ using abstract Wiener measure. *Commun. Math. Anal.*, 11(2):1–22, 2011.

[8] Adrian P. C. Lim. Non-abelian gauge theory for Chern-Simons path integral on $R^3$. *Journal of Knot Theory and its Ramifications*, 21(4), 2012.

[9] Adrian P. C. Lim. Chern-Simons path integrals in $S^2 \times S^1$. *Mathematics*, 3:843–879, 2015.

[10] Adrian P. C. Lim. Invariants in quantum geometry. *Preprint*, 2017.

[11] Carlo Rovelli and Lee Smolin. Knot theory and quantum gravity. *Phys. Rev. Lett.*, 61:1155–1158, Sep 1988.
[12] Carlo Rovelli and Lee Smolin. Discreteness of area and volume in quantum gravity. *Nuclear Physics B*, 442(3):593–619, 1995.

[13] Thomas Thiemann. Lectures on loop quantum gravity. *Lect. Notes Phys.*, 631:41–135, 2003. [41(2002)].

[14] Edward Witten. (2+1)-Dimensional Gravity as an Exactly Soluble System. *Nucl. Phys.*, B311:46, 1988.

[15] Edward Witten. Quantum field theory and the Jones polynomial. *Comm. Math. Phys.*, 121(3):351–399, 1989.