ON DETERMINANTAL EQUATIONS FOR CURVES AND FROBENIUS SPLIT HYPERSURFACES

KIRTI JOSHI

ABSTRACT. I consider the problem of existence of intrinsic determinantal equations for plane projective curves and hypersurfaces in projective space and prove that in many cases of interest there exist intrinsic determinantal equations. In particular I prove (1) in characteristic two any ordinary, plane projective curve of genus at least one is given by an intrinsic determinantal equation (2) in characteristic three any plane projective curve is an intrinsic Pfaffian (3) in any positive characteristic any plane projective curve is set theoretically the determinant of an intrinsic matrix (4) in any positive characteristic, any Frobenius split hypersurface in \( \mathbb{P}^n \) is given by set theoretically as the determinant of an intrinsic matrix with homogeneous entries of degree between 1 and \( n - 1 \). In particular this implies that any smooth, Fano hypersurface is set theoretically given by an intrinsic determinantal equation and the same is also true for any Frobenius split Calabi-Yau hypersurface.

Ryōkan! How nice to be like a fool
for then one’s Way is grand beyond measure

(Master) Tainin Kokusen (to Ryōkan Taigu)

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1. INTRODUCTION

The problem of finding determinantal equations for varieties goes back to the grandmasters of our subject. Readers should consult the two surveys ([2, 3]) for an excellent introduction to this beautiful but difficult subject.

The classical problem alluded to here is the following. Let \( k \) be an algebraically closed field. Let \( X \hookrightarrow \mathbb{P}^n \) be a smooth projective hypersurface of degree \( d \) given by an equation \( G = 0 \). One says that \( X \) is a determinantal hypersurface if there exists a square matrix \( M \), with at least two rows, whose entries are homogeneous polynomials in coordinates of \( \mathbb{P}^n \) such that that \( \det(M) = G \) (the

\[ Key \text{ words and phrases.} \text{ determinantal equations, frobenius splitting, ordinary curves, calabi-yau varieties.} \]
entries of $M$ may not be linear in coordinates of $\mathbb{P}^n$). Note that this condition is invariant under automorphisms of $\mathbb{P}^n$.

As one learns from ([2, 3]), in modern parlance, the question of whether $X$ is or is not a determinantal hypersurface reduces to finding a coherent sheaf $\mathcal{E}$ on the ambient projective space with a certain type of a (minimal) resolution; the matrix $M = M_\mathcal{E}$ is a part of this resolution datum. More precisely the question reduces to finding an arithmetically Cohen-Macaulay coherent sheaf $\mathcal{E}$ on $\mathbb{P}^n$ (with some additional properties). The matrix $M_\mathcal{E}$ is well-defined up to a choice of coordinates and generators and up to these choices $M$ is an invariant of the isomorphism class of $\mathcal{E}$. However the coherent sheaf $\mathcal{E}$ which provides determinantal equations is seldom unique so there are many such (inequivalent) determinantal representations of $X$ each depending on the isomorphism class of the coherent sheaf $\mathcal{E}$ giving rise to it.

There are weaker variants of this question: when does there exist a matrix $M$ of homogeneous polynomials in the coordinates of $\mathbb{P}^n$ such that $\det(M) = G^m$ for some integer $m \geq 1$. If this weaker condition holds then I will say that $X$ is a set theoretic determinantal hypersurface. The problem of whether or not a given hypersurface is a set theoretic determinantal hypersurface also has a translation in terms of finding a coherent sheaf with a resolution of a suitable sort and again the coherent sheaf which provides such a representation is not unique.

If the matrix $M$ providing a (set theoretic) determinantal equation for $X$ consists of linear (resp. quadratic, cubic, etc.) homogeneous polynomials then one says that $X$ is a linear (resp. quadratic, cubic, etc.) (set theoretic) determinantal hypersurface. If the matrix $M$ has entries of bounded degrees (independently of degree of $X$) then one says that $X$ is a bounded (set theoretic) determinantal hypersurface. If the matrix $M$ is skew-symmetric and if $\text{Pfaff}(M) = G$ then one says that $X$ is a Pfaffian determinantal hypersurface (recall that the Pfaffian $\text{Pfaff}(M)$ of a skew-symmetric matrix satisfies $\det(M) = \text{Pfaff}(M)^2$).

As is explained in ([2]) the problem of finding linear determinantal equations is equivalent to finding an Ulrich bundle on $X$. Existence of such a bundle is known in a few cases including complete intersections and is a difficult conjecture of ([5]) in general ([3]) provides an excellent introduction to Ulrich bundles. As was proved in ([5]), the existence of an Ulrich bundle on any variety (i.e. not necessarily a complete intersection) implies, remarkably, that this projective variety is still determined by a single linear determinantal equation in the sense that its Chow form (see [5]) is given by a single (set-theoretic) linear determinantal equation in a suitable Grassmannian coordinate system, and as most varieties are not complete intersections, this is the best one can hope to achieve.

The main question which this note concerns itself with the question of provenance of determinantal equations (resp. theoretic determinantal equations) when they exist: when does there exist an intrinsic (set theoretic) determinantal equation for a hypersurface? Equivalently: when does there exist an intrinsic coherent sheaf $\mathcal{E}$ on $X$ which provides a (set theoretic) determinant equation for $X$?

I do not attempt to make a precise definition of intrinsic (as such an exercise might be too restrictive). However it will be clear from the proofs given below that the sheaves I provide here are intrinsically defined on $X$ (i.e. independent of any choices whatsoever including the embedding of $X \hookrightarrow \mathbb{P}^n$), and these sheaves behave well whenever $X$ moves in a flat family and their formation commutes with arbitrary base change. In characteristic zero I do not know how to find intrinsic equations for hypersurfaces. But in characteristic $p > 0$ this problem sometimes lends itself a natural solution in some cases under arithmetic assumptions on $X$. To explain the arithmetic assumptions on $X$ let me introduce additional notions.
Suppose $k$ has characteristic $p > 0$. Let $X$ be a smooth, projective variety of dimension $n$. Let $d : \mathcal{O}_X \to \Omega^1_X$ be the differential. The image $B^1_X = d(\mathcal{O}_X)$ is a subsheaf of $\Omega^1_X$ consisting of locally exact differentials. As $d(f^d g) = f^d dg$ one sees that $B^1_X$ is a locally free subsheaf $B^1_X \subset F_s(\Omega^1_X)$ and the differential $d : \mathcal{O}_X \to B^1_X$ provides the fundamental exact sequence

\begin{equation}
0 \to \mathcal{O}_X \to F_s(\mathcal{O}_X) \to B^1_X \to 0.
\end{equation}

Note that if $X$ is a smooth projective curve then $B^1_X$ is locally free of rank $p - 1$ and degree $(p - 1)(g - 1)$. A smooth, projective variety over $k$ is said to be Frobenius split if and only if $B^1_X$ splits as an exact sequence of $\mathcal{O}_X$-modules. By (1.1) it is immediate from Frobenius splitting hypothesis that one has

\begin{equation}
H^i(X, B^1_X) = 0 \text{ for all } i \geq 0.
\end{equation}

One says that a smooth projective curve $X$ is ordinary if $H^i(X, B^1_X) = 0$ for all $i \geq 0$ (equivalently Frobenius morphism $H^i(X, \mathcal{O}_X) \to H^i(X, \mathcal{O}_X)$ is an isomorphism for all $i \geq 0$. This is equivalent to the condition that the Jacobian of $X$ is an ordinary abelian variety.

If $X \to S$ is a flat family of smooth, proper varieties then there is an open subset of $S$ over which geometric fibers are Frobenius split, similarly if $X \to S$ is a smooth, proper family of curves then there is an open subset of $S$ over which the fibers are ordinary. In many interesting situations for example if $S$ is, say, the moduli of space curves, or scheme parameterizing smooth hypersurfaces in $\mathbb{P}^n$ of degree at most $n + 1$ then this open subset of $S$ is also non-empty (see [7]). So Frobenius splitting (resp. ordinarity) can be viewed as an arithmetic condition as well as a geometric genericity condition in moduli. However it is important to note that Frobenius splitting (resp. ordinarity) is a condition which can often be tested for a given curve (for example the Fermat curve $x^n + y^n + z^n = 0$ is ordinary if $p \equiv 1 \mod n$) whereas most genericity conditions are often difficult to check for a given curve.

It is a pleasure to thank N. Mohan Kumar for comments and corrections.

2. DETERMINANTAL EQUATIONS

A vector bundle $\mathcal{E}$ on $\mathbb{P}^n$ will called a lineal bundle if $\mathcal{E}$ is a direct sum of line bundles on $\mathbb{P}^n$.

Let $X \hookrightarrow \mathbb{P}^n$ be a smooth, projective hypersurface of dimension $\geq 1$ (so $n \geq 2$). If $n = 2$ then $X$ is a plane curve. The following result is a restatement of the fundamental results of ([2], [3]), which establish a correspondence between sheaves of appropriate sort on $\mathbb{P}^n$ and (set theoretic) (linear) determinantal equations for $X$, in a form convenient for my purposes.

**Proposition 2.1.** Let $X \hookrightarrow \mathbb{P}^n$ be a smooth, projective hypersurface given by an equation $G = 0$ of degree $d$ and suppose $\mathcal{E}$ is a coherent sheaf on $\mathbb{P}^n$ supported on $X$.

1. The following are equivalent:
   (a) $\mathcal{E}$ is an ACM bundle on $X$.
   (b) $\mathcal{E}$ admits a minimal resolution of the form
   \[ 0 \to \mathcal{F}_1 \xrightarrow{M} \mathcal{F}_0 \to \mathcal{E} \to 0, \]
   where $\mathcal{F}_1, \mathcal{F}_0$ are lineal bundles (of rank $\text{rk}(\mathcal{F}_1) = \text{rk}(\mathcal{F}_0)$ on $\mathbb{P}^n$).
   (c) $\det(M) = G^n$ in other words $X$ is a set theoretic determinantal hypersurface.

2. The following are equivalent:
   (a) $\mathcal{E}$ is an Ulrich bundle.
(b) $E$ admits a minimal linear resolution of the form
\[ 0 \to F_1 \xrightarrow{M} F_0 \to E \to 0, \]
with $F_0, F_1$ lineal vector bundles and the entries of $M$ are homogeneous linear forms in the coordinates of $\mathbb{P}^n$.

(c) $\det(M) = G^r$, and $M$ has linear entries in other words $X$ is a set theoretic linear determinantal hypersurface.

This proposition reduces the task of finding set theoretic determinantal equations to finding vector bundles with appropriate kind of resolutions.

3. INTRINSIC DETERMINANTAL EQUATIONS FOR PLAIN CURVES

**Theorem 3.1.** Let $X$ be a smooth, projective, ordinary curve $X \subset \mathbb{P}^2$ over an algebraically closed field $k$ of characteristic $p > 0$.

1. Suppose $p = 2$ then $X$ is an intrinsically linear, symmetric determinantal curve.
2. Suppose $p = 3$ then $X$ is an intrinsically linear Pfaffian determinantal curve.
3. Suppose $p > 3$ then $X$ is an intrinsically set theoretic linear determinantal curve.

**Proof.** Let $\omega_X = \Omega^1_X$ be the canonical divisor of $X$. Before proceeding recall that $B^1_X$ carries a natural, perfect skew-symmetric pairing $B^1_X \otimes B^1_X \to \omega_X$ (see [11]). To prove (1) it suffices to observe that for $p = 2$, $B^1_X$ is a line bundle of degree $\deg(B^1_X) = g - 1$ and the pairing $B^1_X \otimes B^1_X \to \omega_X$ gives $(B^1_X)^2 = \omega_X$ so $B^1_X$ is a natural theta divisor on $X$. The assumption that $X$ is ordinary says that $H^i(X, B^1_X) = 0$ for $i \geq 0$ and hence one simply appeals to Proposition 2.1 and ([2, Proposition 4.2]).

Now suppose $p = 3$ and consider then $B^1_X$ is a locally free sheaf of rank two equipped with a natural symplectic pairing which also implies that for $p = 3$ one has a canonical isomorphism $\det(B^1_X) = \wedge^2 B^1_X = \omega_X$. Now assertion (2) follows from Proposition 2.1 and ([2, Proposition 5.1]).

Finally note that (3) follows from the observation of ([8]) that if $X$ is ordinary, then $B^1_X(1)$ is an Ulrich bundle i.e. a bundle with linear syzygies (see [2] for the definition of an Ulrich bundle). So under the hypothesis that $X$ is a plane curve one has, by Proposition 2.1 a resolution
\[ 0 \to F_1 \xrightarrow{M} F_0 \to B^1_X(1) \to 0, \]
where $M$ is a linear matrix. By twisting by $O_X(-1)$ one has
\[ 0 \to F_1(-1) \xrightarrow{M} F_0(-1) \to B^1_X(-1) \to 0, \]
Since $B^1_X$ is an intrinsically defined vector bundle on $X$, one sees that $X$ is intrinsically a set theoretic linear determinantal curve in $\mathbb{P}^2$. This completes the proof. 

4. INTRINSIC CHOW FORM FOR ORDINARY CURVES IN PROJECTIVE SPACE

Suppose $X \hookrightarrow \mathbb{P}^n$ is a smooth, projective curve embedded in $\mathbb{P}^n$ and equipped with $O_X(1)$. Let Grass$(2, n)$ be the Grassmannian of lines in $\mathbb{P}^n$. Then the image of the incidence scheme
\[ \text{Inc}(X) = \{(x, \ell) \in X \times \text{Grass}(2, n) : x \in \ell\} \]
under the projection to Grass$(2, n)$ is a divisor, denoted Chow$(X)$, and called the Chow divisor or the Chow form of $X$ in $\mathbb{P}^n$. To put it differently, the Chow form of $X$ is the scheme of lines in
\( \mathbb{P}^n \) which meet \( X \). More generally, as is shown in ([5] Theorem 1.4), if \( \mathcal{E} \) is a vector bundle on \( X \) then \( \mathcal{E} \) has a Chow divisor, denoted \( \text{Chow}(\mathcal{E}) \) in \( \text{Grass}(2, n) \) which satisfies
\[
\text{Chow}(\mathcal{E}) = \text{rk}(\mathcal{E}) \cdot \text{Chow}(X).
\]
Combining this result with the ([8, Theorem 2.1]) one has the following

**Theorem 4.1.** Let \( X \hookrightarrow \mathbb{P}^n \) be a smooth, projective ordinary curve then \( X \) has an intrinsic set-theoretic linear determinantal Chow form. More precisely there exists a divisor \( D \hookrightarrow \text{Grass}(2, n) \) which is given globally by a single intrinsic linear determinantal equation in coordinates of \( \text{Grass}(2, n) \) and whose support is the Chow form of \( X \).

5. **Regularity of \( B^1_X \) for Frobenius split varieties**

The main theorem of this section (Theorem[5,1]) provides a bound on the Mumford-Castelnuovo regularity of \( B^1_X \) for any smooth, projective Frobenius split variety over an algebraically closed field in characteristic \( p > 0 \). This result will be needed in the next section. Recall that if \( \mathcal{F} \) is a coherent sheaf on \( \mathbb{P}^n \) then its Mumford-Castelnuovo regularity, denoted \( \text{reg}(\mathcal{F}) \), is the smallest integer \( d \) such that
\[
H^i(\mathbb{P}^n, \mathcal{F}(d - i)) = 0 \quad \forall i > 0.
\]

**Theorem 5.1.** Let \( X \hookrightarrow \mathbb{P}^n \) be a smooth projective variety over an algebraically closed field characteristic \( p > 0 \) equipped with \( \mathcal{O}_X(1) \) provided by this embedding. Assume that \( X \) is Frobenius split. Then \( \text{reg}(B^1_X) \leq \dim(X) \).

**Proof.** So one has to prove that \( H^i(\mathbb{P}^n, B^1_X(d - i)) = 0 \) for all \( i > 0 \) and \( d = \dim(X) \), or equivalently (by Leray spectral sequence for \( X \hookrightarrow \mathbb{P}^n \)) one has to prove that \( H^i(X, B^1_X(d - i)) = 0 \) for \( i > 0 \). This is proved as follows. First note that if \( i = d \) then \( H^d(X, B^1_X) = 0 \) by our assumption that \( X \) is Frobenius split. So assume \( 1 \leq i \leq d - 1 \). Twisting the split exact sequence (1.1) (this is where Frobenius splitting is used) by \( \mathcal{O}_X(d - i) \) one has a split exact sequence
\[
0 \to \mathcal{O}_X(d - i) \to F_*(\mathcal{O}_X)(d - i) \to B^1_X(d - i) \to 0,
\]
where \( 1 \leq i \leq d - 1 \). So \( H^i(B^1_X(d - i)) \) is a direct summand of \( H^i(F_*(\mathcal{O}_X)(d - i)) \) and so it is enough to prove that the latter is zero for \( 1 \leq i \leq d - 1 \). For \( i \) in this range, \( \mathcal{O}_X(d - i) \) is ample and
\[
H^i(X, F_*(\mathcal{O}_X)(d - i)) = H^i(X, \mathcal{O}_X(p(d - i)))
\]
by the projection formula. By ([10]) as \( X \) is Frobenius split one sees that
\[
H^i(X, \mathcal{O}_X(p(d - i))) = 0
\]
and hence its direct summand \( H^i(X, B^1_X(d - i)) = 0 \) for \( 1 \leq i \leq d - 1 \). Thus the result is established.

6. **Determinantal Equations for Frobenius split Hypersurfaces**

Let me now prove the following theorem. I do not know if there is any classical analogue of this theorem which shows that any Frobenius split hypersurface (of any dimension) is necessarily a set-theoretic determinantal variety. Note that while the next theorem makes no assumption on the degree of \( X \) let me point out that it is immediate from ([10]) that any Frobenius split hypersurface \( X \subset \mathbb{P}^n \) has \( \deg(X) \leq n + 1 \) (i.e. \( X \) is Fano or a Calabi-Yau hypersurface) and by ([6]) one sees that any smooth Fano hypersurface in \( \mathbb{P}^n \) is Frobenius split.
**Theorem 6.1.** Let $X \hookrightarrow \mathbb{P}^n$ be a smooth, projective, Frobenius split hypersurface given by some equation $G = 0$. Then there exists an intrinsic square matrix $M$ whose entries are homogeneous polynomials bounded between 1 and $n-1$ and such that $\det(M) = G^r$ for some $r \geq 0$. In other words every smooth, projective Frobenius split hypersurface is always an intrinsic, bounded set theoretic determinantal hypersurface. In particular every smooth Fano hypersurface is a bounded set theoretic determinantal hypersurface.

The following Lemma will be used in the proof.

**Lemma 6.2.** Let $X \hookrightarrow \mathbb{P}^n$ be a smooth projective variety over an algebraically closed field characteristic $p > 0$ equipped with $\mathcal{O}_X(1)$ provided by this embedding. Assume that $X$ is Frobenius split with $\dim(X) \geq 2$. Then

$$\text{Hom}(\mathcal{O}_X(m), B^1_X) = 0 \text{ for all } m \geq 0.$$  

**Proof.** First observe that if $m = 0$ then $\text{Hom}(\mathcal{O}_X, B^1_X) = 0$ is equivalent to $H^0(X, B^1_X) = 0$ which is immediate from (1,1) as $X$ is Frobenius split. Next assume $m \geq 1$ then (using the fact that $X$ is Frobenius split)

$$\text{Hom}(\mathcal{O}_X(m), B^1_X) = H^0(X, B^1_X(-m)) \subseteq H^0(X, F_s(\mathcal{O}_X)(-m)) = H^0(\mathcal{O}_X(-pm)) = 0$$

as $\mathcal{O}_X(-pm)$ is anti-ample for $m \geq 1$. This proves the lemma. □

**Proof.** The regularity bound of the previous section, while elementary, is the main reason why one gets bounded degrees as will be shown here. Let $X \in \mathbb{P}^n$ be a smooth, projective hypersurface. Then by Theorem 5.1 one has $\text{reg}(B^1_X) \leq \dim(X) = n - 1$. Let

$$0 \to \mathcal{G}_1 = \oplus_i \mathcal{O}_{\mathbb{P}^n}(-b_i) \xrightarrow{M} \mathcal{G}_0 \to B^1_X \to 0$$

be a minimal resolution of $B^1_X$. By (8) $B^1_X$ is arithmetically Cohen-Macaulay coherent sheaf on $\mathbb{P}^n$ one can read off the regularity of $B^1_X$ from the $b_i$. Specifically one has (see [4, Exercise 4, Page 85])

$$\text{reg}(B^1_X) = \max(b_i) - 1.$$

As $\text{reg}(B^1_X) \leq n - 1$ by Theorem 5.1 one sees that $\max(b_i) - 1 \leq n - 1$ (by Theorem 5.1). Hence $\max(b_i) \leq n - 1 + 1 = n$. Hence $b_i \leq n$ for all $i$.

I claim that $\text{Hom}(\mathcal{O}_{\mathbb{P}^n}(m), \mathcal{G}_0) = 0$ for any $m \geq 0$. Indeed if this is non-zero then one has a non-zero morphism $\mathcal{O}_X(m) \to B^1_X$ with $m \geq 0$ which is impossible by Lemma 6.2. Hence $\mathcal{O}_X(m) \subseteq \text{ker}(\mathcal{G}_0, B^1_X) = \mathcal{G}_1$ and so one has a trivial subcomplex $0 \to \mathcal{O}_X(m) = \mathcal{O}_X(m) \to 0$ of our minimal resolution which contradicts minimality of the resolution. Moreover one also has from this that $\text{Hom}(\mathcal{O}_X(m), \mathcal{G}_1) = 0$ for any $m \geq 0$. Thus any line bundle which is a free direct summand of $\mathcal{G}_0$ or $\mathcal{G}_1$ is of the form $\mathcal{O}_{\mathbb{P}^n}(-m)$ with $m \geq 1$. Thus $B^1_X$ has a minimal resolution of the asserted type where the entries of the matrix $M$ for the morphism $\mathcal{G}_1 \to \mathcal{G}_0$ are homogeneous polynomials in coordinates of $\mathbb{P}^n$ of degrees bounded by 1 and $n - 1$.

□

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**Math. department, University of Arizona, 617 N Santa Rita, Tucson 85721-0089, USA.**

**E-mail address:** kirti@math.arizona.edu