The Asymptotic Form of the Energy Density of
Weakly Interacting Particles
in a Static, Spherical Geometry

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Abstract

The asymptotic form of the energy density for a gas of particles surrounding
a sphere of mass $M$ and radius $R$ is studied using Einstein’s equations. It
is shown that if the pressure of the gas $p$ varies linearly with the energy
density $\rho$ for small $\rho$, then $\rho \sim 1/r^2$ for large $r$.

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In the Schwarzschild solution of Einstein’s equations for a static, spheri-
cal geometry [1] the spacetime surrounding the sphere of mass $M$ and radius
$R$ is taken to be empty and free of particles. The real world is certainly not
so clean and simple, however and a more realistic model would also include
a gas of particles which surround the sphere. With the addition of these
particles the question then becomes what affect, if any, they will have on
the geometry of the spacetime. At first glance this would seem to be an
impossible question to answer. Not only must an equation of state for the
particles be known, this equation of state must also have been calculated in
the presence of the very gravitational field that it itself determines. What
we shall find, however, is that due to the restrictions that Einstein’s equa-
tions themselves put on the energy density of the particles, the asymptotic
behavior of this spacetime will be surprisingly simple to determine.

In this paper we shall study some of the asymptotic properties of a
static, spherical geometry filled with a gas of particles. To be specific, the
system we shall be considering consists of a sphere with mass $M$ and radius
$R$ which is surrounded by an arbitrary gas of particles. We shall not need
to specify the type of particles surrounding the sphere, nor shall we need
to fix their temperature or density. All that we shall require is that they
have an equation of state which has certain properties. For convenience we
have also assumed that the sphere has not undergone complete gravitational
collapse into a black hole and that the system as a whole is in thermodynamic
equilibrium. Moreover, unlike the usual treatment of matter in general
relativity, we shall not a priori confine the particles to be within a sphere
of any fixed radius, but will instead let the system itself determine how far
into the spacetime the particles will spread.

Although we have used the word “gas” to describe the particles, this is
only a matter of convenience. The particles are not only allowed to interact
with gravity, thereby determining the geometry of the spacetime, but also with each other. Since the most stringent constraint on the energy density, and thus the geometry of the spacetime, comes not from the equation of state of the particles, but rather from Einstein’s equations themselves, all that we need require is that the equation of state satisfy a few physically reasonable requirements. In the end we shall find that if the particles interact among themselves to any significant extent, then the spacetime will be asymptotically flat. The energy density of the particles will die off faster than $1/r^2$ and we may characterize the particles as being confined within a sphere of a certain radius which will require a specific choice of the equation of state to determine precisely. If the particles are very weakly interacting, on the other hand, then we shall find that the spacetime is not asymptotically flat. The energy density will die off as $1/r^2$ for large $r$ and we cannot characterize the particles as being confined within a sphere of any fixed radius. The affect of the particles on the geometry of the spacetime is instead all pervasive.

We begin with the most general form of the metric for a static, spherical geometry \[ ds^2 = -fdt^2 + hdr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 , \]

where $f$ and $h$ are unknown functions of $r$ only. We are only interested in the structure of the spacetime for $r > R$ and shall always assume that this holds. As usual, we shall take the average energy momentum tensor for the particles to be of the form $\langle T_{\mu\nu} \rangle = \rho u_\mu u_\nu + p(g_{\mu\nu} + u_\mu u_\nu)$ where $u_\mu$ is a unit velocity vector which lies in the direction of the timelike Killing vector for the system, while $\rho$ and $p$ are the energy density and pressure, respectively, of the particles surrounding the sphere. The two Einstein’s equations we
shall find of use are \[2\]

\[
8\pi \rho = \frac{h'}{rh^2} + \frac{1}{r^2} \left( 1 - \frac{1}{h} \right), \quad (1a)
\]

\[
8\pi p = \frac{f'}{rfh} - \frac{1}{r^2} \left( 1 - \frac{1}{h} \right), \quad (1b)
\]

where the primes denote derivatives with respect to \( r \) and we are using units in which \( G = c = 1 \). From the conservation equation \( \nabla^\mu \langle T_{\mu\nu} \rangle = 0 \) we also have the Tolman-Oppenheimer-Volkoff (TOV) equation \[2\] for hydrostatic equilibrium

\[
\frac{dp}{dr} + \frac{1}{2} (p + \rho) \frac{1}{f} \frac{df}{dr} = 0. \quad (2)
\]

At this point we shall assume that we are given an equation of state for the particles which we shall write as

\[
p = w(\rho), \quad (3)
\]

where \( w(\rho) \) is a functional of \( \rho \). It is further assumed that this equation of state includes the rest mass of the particles and was calculated in the presence of the gravitational field. As such both \( p \) and \( \rho \) must also be functions of \( r \). We next make the anzatz that \( w(\rho) \) itself has no explicit \( r \) dependence, but is instead dependent on \( r \) only through its dependence on \( \rho \). We justify this anzatz with the following. First, we know that in Minkowski spacetime the equation of state may always be written in the form given in eq. (3). As the system is in thermodynamic equilibrium, and as there are no external fields present, any \( r \) dependence in \( w(\rho) \) must then be due to the curvature of the spacetime. Second, \( \log f \) may, under certain circumstances, be interpreted as being twice the gravitation potential and is therefore a measure of the “gravitational force” on the particles. Third, we note that all explicit \( r \) dependence in the TOV equation can be canceled and one can instead consider it as a differential equation determining \( p \) in...
terms of $f$. In fact, any dependence of $p$ on $r$ is due to $f$. Finally, the geometry of the spacetime, namely $f$ and $h$, is ultimately determined by the energy density and pressure of the particles. Since the equation of state on curved spacetimes must also reduce to the equation of state calculated in Minkowski spacetimes in the limit where $f \to 1$ and $h \to 1$, we would therefore expect $w(\rho)$ to depend on $r$ only implicitly through $\rho$.

The equation of state will in general be very complicated, if it can be calculated at all. Fortunately, we shall not need any specific form of eq. (3), but rather that it have a couple of physically reasonable properties which we would expect from any physical system. First, the pressure $p$ must be finite as $\rho \to 0$, meaning that the gas of particles cannot become infinitely stiff as the energy density of the particles is reduced to zero. Actually, we shall need the more stringent requirement that

$$\lim_{\rho \to 0} w(\rho) \equiv w_0,$$

exists and is a finite. This means, in particular, that for small $\rho$, $p \sim \rho^{1+n}$ for $n > 0$. We justify this requirement with the observation that if the particles were consisted only of photons, then the pressure must be one third the energy density. This holds even in curved spacetimes. Since the photons do not interact among themselves, we have at least one example of a gas of non-interacting particles for which the pressure varies linearly with the energy density. We would therefore expect that if the particles do interact with one another, then the pressure should vary at least linearly with the energy density, if not stronger. We would not expect the pressure to have a weaker power law dependence on the energy density than a linear one.

With the Einstein’s equations, and the TOV equation, we have now three differential equations determining $\rho$, $f$, and $h$. (Since the equation of state is given, $p$ is determined as soon as $\rho$ is.) We shall then, presumably,
need a set of three initial conditions to determine the system completely. The initial condition for $\rho$ is straightforward; we need only take its value at the surface of the sphere $\rho_R \equiv \rho(R)$. The initial conditions for $f$ and $h$, on the other hand, are much more difficult to determine. Fortunately, we are only interested in the asymptotic nature of the spacetime and we shall later find that the asymptotic forms of both $\rho$ and $h$ are independent of their initial conditions. Whatever initial condition we choose for $h$ is thus immaterial for our purposes. $f$, on the other hand, does depend on initial conditions; not only on its own, but also on $\rho_R$ as well. Since, however, we may always rescale the time coordinate, its dependence on its own initial condition is not extremely relevant and we may formally take it to be $f(R)$.

Because the equation of state may be quite complicated, the solution of eqs. (1) for all $r$ is not trivial to find. In, however, the asymptotic limit where $r \to \infty$ the situation simplifies dramatically. To see how this happens, we integrate eq. (1a) to formally give

$$\frac{1}{h} = 1 - \frac{2m_0}{r} - \frac{8\pi}{r^2} \int_R^r \rho(\bar{r})\bar{r}^2 d\bar{r},$$

for $r > R$ where $m_0$ is an integration constant. Since $\rho$ is not known explicitly, this would seem to be of little use. Note, however, that $h > 0$ for all $r > R$. Consequently, $\rho \to 0$ as $r \to \infty$. In fact, $\rho$ must die off at least as fast as $1/r^2$. Thus, because we are only interested in the asymptotic behavior of the spacetime, only the behavior of the equation of state when $\rho \to 0$, namely $w_0$, will be relevant in our analysis. Keeping this in mind, we next make use of the equation of state in eq. (2) to obtain

$$\frac{1}{f} \frac{df}{dr} = -\frac{2}{1 + w(\rho)} \left( \frac{dw}{d\rho} + w(\rho) \right) \frac{1}{\rho} \frac{d\rho}{dr}. \quad (4)$$

We then define

$$\rho = \frac{\Delta}{4\pi r^2},$$
and note that $\Delta$ either vanishes as $r \to \infty$ or else approaches a constant value. The important point is that $\Delta$ must be finite as $r \to \infty$. Then defining $h^{-1} = 1 - 2K$, eqs. (1) become

$$\Delta = \frac{dK}{dy} + K, \quad (5a)$$
$$\Delta w_0 = -\frac{w_0(1 - 2K)}{1 + w_0} \left( \frac{1}{\Delta} \frac{d\Delta}{dy} - 2 \right) - K, \quad (5b)$$

for large $y$ where $y \equiv \log(r/r_0)$ for some $r_0$. Solutions to these equations have two different behaviors depending upon whether or not $w_0$ vanishes.

**Case 1: $w_0 = 0$**

From eq. (5b), $K \to 0$ as $r \to \infty$. Consequently from eq. (5a), $\Delta \to 0$ as well. This means that the energy density, and consequently the pressure, must decrease faster than $1/r^2$ for large $r$. Then from eq. (4), $f \to c_k$ where $c_k$ is a constant which can be set to 1 by suitably rescaling the time coordinate. Consequently, if $w_0 = 0$, then $f \to 1$ and $h \to 1$ as $r \to \infty$ and the spacetime is asymptotically flat.

**Case 2: $w_0 \neq 0$**

This case is much more interesting. We write eqs. (5) as a set of two coupled, non-linear differential equations

$$\frac{dK}{dy} = \Delta - K, \quad (6a)$$
$$\frac{d\Delta}{dy} = -\frac{1 + w_0}{w_0} \frac{\Delta}{1 - 2K} \left\{ \Delta w_0 + \left( \frac{1 + 5w_0}{1 + w_0} \right) K - \frac{2w_0}{1 + w_0} \right\}. \quad (6b)$$

Fortunately, these two equations are autonomous, meaning they have no explicit $y$ dependence. Obtaining asymptotic solutions to eqs. (6) is then straightforward and involves looking for fixed points $(\Delta_a, K_a)$ of eqs. (6) where the derivatives of both $\Delta$ and $K$ vanish. (See [3]. We caution the reader that, depending on the literature, the term “critical point” is often used instead of “fixed point”.) From eqs. (6) one of these fixed points occurs
at $\Delta_a = 0 = K_a$, which is the $w_0 = 0$ case once again. A second fixed point occurs at

$$\Delta_a = K_a = \frac{2w_0}{(1 + w_0)^2 + 4w_0}.$$ 

If we next expand eqs. (6) about this fixed point, then

$$\frac{d}{dy} \left( \frac{\delta \Delta}{\delta K} \right) = \left( -\frac{2w_0}{1 + w_0} - \frac{2(1+5w_0)}{(1+w_0)^2} \right) \left( \frac{\delta \Delta}{\delta K} \right), \tag{7}$$

where $\delta \Delta = \Delta - \Delta_a$ and $\delta K = K - K_a$. The asymptotic behavior of the solutions to eqs. (6) depends on the eigenvalues $\lambda_{\pm}$ of this matrix. Writing $\lambda_{\pm} = -\eta \pm i\varphi$, we find that

$$\eta = \frac{1}{2} \left( 1 + \frac{2w_0}{1 + w_0} \right), \quad \varphi = \frac{1}{2} \left( \frac{7 + 42w_0 - w_0^2}{1 + w_0} \right)^{1/2}.$$ 

Since $\rho \geq 0$ for all $r$, $\Delta_a$ must be positive. This condition, combined with the requirement that $h^{-1} = 1 - 2K_a > 0$ for all $r$, means that $w_0 > 0$. Moreover, because $p \leq \rho/3$, $w_0 \leq 1/3$, and it is then straightforward to see that $\varphi$ will always be a real number. Consequently, this fixed point is stable and all solutions to eqs. (6) will eventually spiral counterclockwise into this fixed point in the $\Delta$-$K$ plane no matter what their initial conditions were originally. We can see this explicitly by solving eq. (7) to give

$$K(r) \approx \Delta_a \left\{ 1 + A \left( \frac{r}{r_0} \right)^{-\eta} \sin \left( \varphi \log \frac{r}{r_0} \right) \right\},$$

$$\Delta(r) \approx \Delta_a \left\{ 1 + A \left( \frac{r}{r_0} \right)^{-\eta} \left[ \varphi \cos \left( \varphi \log \frac{r}{r_0} \right) \right. \right.$$ 

$$+ (1 - \eta) \sin \left( \varphi \log \frac{r}{r_0} \right) \right\},$$

for large $r$. $A$, and $r_0$ are constants which depend on the specific equation of state and initial condition for $\rho$ and $p$. Notice also that because $0 < w_0 \leq 1/3$, $0 < \Delta_a \leq 3/14$, and $1/2 < \eta \leq 3/4$ while $\sqrt{7}/2 < \varphi \leq \sqrt{47}/4$. Solutions to eqs. (6) therefore approaches this fixed point very slowly; the fastest being $r^{-3/4}$. 

8
From eq. (4) we can solve for $f$ in terms of $\rho$. In the asymptotic limit we find that $f \approx kr^q$ where $q = 4w_0/(1 + w_0)$,

$$k = f(R) \left( \frac{4\pi \rho R}{\Delta_a} \right)^{q/2} \exp \left\{ 2 \left( \frac{\rho R \frac{dw}{d\rho} |_{\rho R} + w_R}{1 + w_R} \right) - \frac{q}{2} \right\} + 2 \int_0^1 x \log x \frac{d^2}{dx^2} \left( \frac{xw'(x) + w(x)}{1 + w(x)} \right) dx$$

and $w_R \equiv w(\rho_R)$. Since $0 < w_0 \leq 1/3$, $0 < q \leq 1$. Consequently, for large $r$ the metric becomes

$$ds^2 = -kr^q dt^2 + \frac{1}{1 - 2K_a} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2,$$

and we can see explicitly that this spacetime is not asymptotically flat.

To summarize, we have shown that if the particles interact among themselves to such an extent that $w_0 = 0$, then a static, spherical spacetime containing these particles is asymptotically flat, as expected. Moreover, we see that the energy density of these particles must necessarily decrease faster than $1/r^2$; most probably very much faster. As the Schwarzchild geometry is also asymptotically flat and involves a mass density which is confined within a sphere of definite radius, we may therefore characterize these particles as effectively being confined within a sphere of a certain radius which will require a specific equation of state to determine. The affect of these particles on the curvature of the spacetime, like the affect of the mass $M$ in the Schwarzchild solution, eventually dies away and the spacetime asymptotically approaches the Minkowski spacetime. When, on the other hand, the particles interact with each other so weakly that $w_0 \neq 0$, then their energy density decreases as $1/r^2$ and their affect on the curvature of the spacetime is correspondingly long range. They cannot be charactorized as being contained within a sphere of any definite radius and are instead spread throughout the spacetime. In fact, we find that $f \sim r^q$ while $h$ approaches
a constant as $r \to \infty$ and a spacetime filled with these weakly interacting particles is not asymptotically flat.

What is of even greater interest is the universal nature of the energy density in the asymptotic limit

$$\rho_a = \frac{\Delta_a c^4}{4\pi r^2 G},$$

where we have replaced the correct factors of $c$ and $G$. (The subscript $a$ denotes the asymptotic limit.) Notice that the form of $\rho_a$ is the same irrespective of the mass and radius of the sphere, irrespective of any “initial condition” for $\rho$ at $r = R$, and irrespective of any form that the equation of state may take so long as $w_0 \neq 0$. In fact, the only property of the particles that it does depend upon is $w_0$, a dimensionless number. From dimensional arguments, if the particles surrounding the sphere are massless, then $w_0$ must simply be a number. It cannot even depend on the temperature of the gas, since, aside from the Planck mass, there is no other length scale one can use to construct a $w_0$. If, on the other hand, the particles have a mass $m$, then $w_0$ is either once again a number, or else is a function of the ratio $T/m$, where $T$ is the temperature of the gas.

Notice that although this universality extends to $h$, it does not extend to include $f$. For large $r$, $f \approx kr^q$ and although $q$ depends only on $w_0$, $k$ depends not only on $f(R)$ but also on $\rho_R$. Consequently, the motion of test particles in this spacetime will always be dependent the specific choice of initial conditions for $f$ and $\rho$, and thus on the detail properties of the particles and of the spherical mass.

The question remains as to how one may go about calculating $w_0$. For certain cases this is trivial. Suppose $T_{\mu\nu}$ is the energy momentum operator for the particles such that its equilibrium average is $\langle T_{\mu\nu} \rangle = \rho u_\mu u_\nu + p(g_{\mu\nu} + u_\mu u_\nu)$. If this energy momentum operator is traceless,
as it is for a gas of pure photons, then \( \langle T^\mu_\mu \rangle = 0 \) and \( p = \rho/3 \). Consequently, any system which has a traceless energy momentum operator has a \( w(\rho) = w_0 = 1/3 \). Calculating \( w_0 \) for other systems, on the other hand, is a much more formidable task, although at first glance not an impossible one. Because \( w_0 \) is determined when the energy density of the particles is vanishingly small, this suggests that when calculating \( w_0 \) one may, as a first approximation, neglect the effect of the particles on the curvature of the spacetime and may instead treat them as test particles. One can then in principle use the vacuum solutions of Einstein’s equations as a background field and calculate the equation of state for the system using methods described in [4] .

We end this paper with a couple of observations. First, the \( 1/r^2 \) behavior in eq. (8) is precisely what one would expect for the energy density of “dark matter” based on Newtonian gravity [5] – [7]. Unfortunately, when one has gone to such a large \( r \) such that eq. (8) holds, one also finds that the spacetime no longer close to being Minkowskian. It is instead extremely curved (see [8]). Second, the form of \( \rho_a \) is quite peculiar in that it contains no explicit length scale for \( r \). The only length scale which can be constructed solely from universal constants, however, is the Planck length \( l^2_{pl} = \hbar G/c^3 \). If we use it, then we can write eq. (8) as

\[
\rho_a = \rho_{pl} \frac{\Delta a}{4\pi} \left( \frac{l_{pl}}{r} \right)^2
\]

where \( \rho_{pl} = c^7/(\hbar G^2) \) is the Planck energy density. Although \( \rho_{pl} \) is very large, eq. (9) is valid only at very large \( r \). Since \( l_{pl} \sim 10^{-33}\text{cm} \), this ensures that the size of \( \rho \) will always be physically reasonable.

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