Maximal Unitarity at Two Loops

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(with S. Caron-Huot, H. Johansson and D. Kosower)
Part 1: Introduction

- motivations for studying amplitudes
- modern methods for computation at one loop
The searches at LHC for physics beyond the Standard Model require a detailed understanding of background, especially QCD, processes.
Examples of signals and QCD backgrounds

**Signal:** An example of a Higgs boson process:

\[ \begin{array}{c}
g \\
H \\
g 
\end{array} \rightarrow \begin{array}{c}
W^- \\
\mu^- \\
\bar{\nu}_\mu \\
W^+ \\
q \\
\bar{q} \} 2 \text{ jets}
\end{array} \]

**Background:** An example of a QCD background process:

\[ \begin{array}{c}
g \\
W^- \\
g 
\end{array} \rightarrow \begin{array}{c}
\bar{d} \\
\mu^- \\
\bar{\nu}_\mu \\
u \} 2 \text{ jets}
\end{array} \]
In fact, there are two important motivations:

- **LHC phenomenology**
  Quantitative estimates of QCD background: needed for precision measurements, uncertainty estimates of NLO calculations, and reducing renormalization scale dependence.

- **Reveal fascinating structure in QFT**
  For $\mathcal{N} = 4$ SYM: hidden symmetries (integrability $\rightarrow$ non-perturbative solution) and new dualities (to Wilson loops and correlators).
  For $\mathcal{N} \leq 4$: connection to multivariate complex analysis and algebraic geometry.
In practice, the Feynman diagram prescription produces a very large number of terms: e.g. for the five-gluon tree-level amplitude

\[ k_1 \cdot k_4 \varepsilon_2 \cdot k_1 \varepsilon_1 \cdot \varepsilon_3 \varepsilon_4 \cdot \varepsilon_5 \]
Feynman diagrams hide simplicity

Yet, the final result for five-gluon tree-level amplitude is simple,

\[ A_{5}^{\text{tree}}(1^\pm, 2^+, 3^+, 4^+, 5^+) = 0 \]

\[ A_{5}^{\text{tree}}(1^-, 2^-, 3^+, 4^+, 5^+) = \frac{i\langle 1 2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 5 \rangle \langle 5 1 \rangle}. \]

This strongly suggests there should exist better methods for computing amplitudes.

At one-loop level, unitarity has proven very successful, allowing e.g. the calculation of \( qg \rightarrow W + \) multi-jets.

This talk is about extending generalized unitarity (systematically) to two loops.
Integral reductions and integral basis

Feynman rules $\rightarrow$ numerator powers in integrals

At one loop, all such integrals can be expanded in a basis.

For example, consider the box insertion

\[ \ell \cdot k_4 = \frac{1}{2} \left( (\ell + k_4)^2 - \ell^2 \right) \]

This can be reduced to

\[ \begin{array}{c}
2 \\
\ell \\
3 \\
4
\end{array} = \begin{array}{c}
2 \\
\ell \\
3 \\
4
\end{array} \]
Integral reductions and integral basis

Feynman rules \( \rightarrow \) \textit{numerator powers in integrals}

At one loop, all such integrals can be expanded in a \textit{basis}.

For example, consider the box insertion

\[
\begin{align*}
\ell \cdot k_4 &= \frac{1}{2} \left( (\ell + k_4)^2 - \ell^2 \right),
\end{align*}
\]

By using the identity \( \ell \cdot k_4 = \frac{1}{2} \left( (\ell + k_4)^2 - \ell^2 \right) \), this can be reduced to
Use integral reductions to write the one-loop amplitude as a linear combination of *basis integrals*

\[ A^{(1)} = c_1 + c_2 + c_3 + c_4 + \text{rational terms} \]
The modern unitarity approach (1/2)

Use integral reductions to write the one-loop amplitude as a linear combination of \textit{basis integrals}

\[ A^{(1)} = c_1 + c_2 + c_3 + c_4 \]

\[ \text{+ rational terms} \]

To determine \( c_i \), apply cuts \( \frac{1}{(\ell - K)^2} \longrightarrow \delta((\ell - K)^2) \) to both sides.
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Applying a quadruple cut [Britto, Cachazo, Feng] isolates a single box integral:
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To determine \( c_i \), apply cuts \( \frac{1}{(\ell - K)^2} \rightarrow \delta((\ell - K)^2) \) to both sides. Applying a quadruple cut [Britto, Cachazo, Feng] isolates a single box integral:

\[ \frac{1}{2} \sum_{\text{kin sols } j=1}^{4} A_j^{\text{tree}} \]
A triple cut will leave $4 - 3 = 1$ free complex parameter $z$. Parametrizing the loop momentum,

$$\ell^\mu = \alpha_1 K_1^{b\mu} + \alpha_2 K_2^{b\mu} + \frac{z}{2} \langle K_1^{-} | \gamma^\mu | K_2^{-} \rangle + \frac{\alpha_4(z)}{2} \langle K_2^{-} | \gamma^\mu | K_1^{-} \rangle$$

one obtains an explicit formula for the triangle coefficient [Forde]
Part 2: From trees to two loops

- maximal cuts at two loops
- constructing two-loop amplitudes out of tree-level data
- elliptic integrals in $\mathcal{N} = 4$ SYM amplitudes
Expand the massless 4-point two-loop amplitude in a basis, e.g.

\[ A_{4}^{2-\text{loop}} = c_{1}(\epsilon) + c_{2}(\epsilon) + \text{ints with fewer props} + \text{rational terms} \]
Expand the massless 4-point two-loop amplitude in a basis, e.g.

\[ A^{2-\text{loop}}_4 = c_1(\epsilon) + c_2(\epsilon) \]

Compute \( c_1(\epsilon) \) and \( c_2(\epsilon) \) according to

\[ \prod_j A^\text{tree}_j \rightarrow \text{MACHINE} \rightarrow c_1(\epsilon) \text{ and } c_2(\epsilon) \]
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Compute \( c_{1}(\epsilon) \) and \( c_{2}(\epsilon) \) according to

The machinery: \textit{contour integrals} \( \oint_{\Gamma_{j}}(\cdots) \)

The philosophy: basis integral \( I_{j} \leftrightarrow \text{unique } \Gamma_{j} \text{ producing } c_{j} \)
The anatomy of two-loop maximal cuts

Cutting all seven visible propagators in the double-box integral,

\[
\int d^4 p \, d^4 q \prod_{i=1}^{7} \frac{1}{\ell_i^2} \quad \rightarrow \quad \int d^4 p \, d^4 q \prod_{i=1}^{7} \delta^\mathbb{C}(\ell_i^2) = \oint_\Gamma \frac{dz}{z(z + \chi)},
\]

produces (cf. [Buchbinder, Cachazo]), setting \( \chi \equiv \frac{t}{s} \),

\[
\int \frac{dz}{z(z + \chi)}.
\]

a contour integral in the complex plane.
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\]

a contour integral in the complex plane.

Jacobian poles \( z = 0 \) and \( z = -\chi \): composite leading singularities encircle \( z = 0 \) and \( z = -\chi \) with \( \Gamma = \omega_1 C_\epsilon(0) + \omega_2 C_\epsilon(-\chi) \)

\( \rightarrow \) freeze \( z \) ("8th cut")
Choosing contours: *die Qual der Wahl*

Six inequivalent classes of solutions to on-shell constraints

4 massless external states $\longrightarrow$ 8 independent leading singularities
Choosing contours: *die Qual der Wahl*

Six inequivalent classes of solutions to on-shell constraints

4 massless external states $\rightarrow$ 8 independent leading singularities

*How do we select contours within this variety of possibilities?*
Principle for selecting contours

To fix the contours, insist that

vanishing Feynman integrals must have vanishing heptacuts.

This ensures that

\[ I_1 = I_2 \implies \text{cut}(I_1) = \text{cut}(I_2). \]
Principle for selecting contours

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Origin of terms with vanishing $\mathbb{R}^D \times \mathbb{R}^D$ integration:

reduction of Feynman diagram expansion to a \textit{basis of integrals}

(including use of integration-by-parts identities).

Remarkable simplification:

- 4 massless external states: \[ 22 \longrightarrow 2 \text{ double-box integrals} \]
- 5 massless external states: \[ 160 \longrightarrow 2 \text{ “turtle-box” integrals} \]
- 5 massless external states: \[ 76 \longrightarrow 1 \text{ pentagon-box integral} \]
There are two classes of constraints on $\Gamma$'s:

1) Levi-Civita integrals. For example,

\[
\varepsilon(p, 1, 2, 4) = 0 \quad \implies \quad \varepsilon(p, 1, 2, 4) = 0
\]
There are two classes of constraints on $\Gamma$’s:

1) Levi-Civita integrals. For example,

\[
\varepsilon(p, 1, 2, 4) = 0 \quad \Rightarrow \quad \varepsilon(p, 1, 2, 4) = 0
\]

2) Integration by parts (IBP) identities must be preserved. For example,

\[
\frac{\chi}{8s_{12}} - \frac{3}{4}s_{12} + \cdots \quad \Rightarrow \quad \frac{\chi}{8s_{12}} - \frac{3}{4}s_{12} + \cdots
\]
The constraints in the case of four massless external momenta:

\[
\begin{align*}
\omega_1 - \omega_2 &= 0 \\
\omega_3 - \omega_4 &= 0 \\
\omega_5 - \omega_6 &= 0 \\
\omega_7 - \omega_8 &= 0 \\
\omega_3 + \omega_4 - \omega_5 - \omega_6 &= 0 \\
\omega_1 + \omega_2 - \omega_5 - \omega_6 + \omega_7 + \omega_8 &= 0
\end{align*}
\]

leaving \(8 - 4 - 2 = 2\) free winding numbers.
Master contours: the concept

Going back to the two-loop basis expansion

\[ A_{4}^{2\text{-loop}} = c_{1}(\epsilon) + c_{2}(\epsilon) \]

and applying a heptacut one finds

\[ \prod_{j=1}^{6} A_{j}^{\text{tree}} = c_{1}(\epsilon) + c_{2}(\epsilon) \]

+ ints with fewer props
+ rational terms
Master contours: the concept

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Exploit free parameters \( \rightarrow \exists \) contours with

\[ P_1 : (\text{cut}(I_1), \text{cut}(I_2)) = (1, 0) \]
\[ P_2 : (\text{cut}(I_1), \text{cut}(I_2)) = (0, 1) . \]

We call such \( P_i \) master contours.
Master contours: results

With four massless external states,

\[
c_1 = \frac{i\chi}{8} \int_{P_1} \frac{dz}{z(z + \chi)} \prod_{j=1}^{6} A_{j}^{\text{tree}}(z)
\]

\[
c_2 = -\frac{i}{4s_{12}} \int_{P_2} \frac{dz}{z(z + \chi)} \prod_{j=1}^{6} A_{j}^{\text{tree}}(z)
\]

With our choice of basis integrals, the \( P_i \) are

\[ n = \text{winding number} \]
Characterizing the on-shell solutions

There are six solutions for the heptacut loop momenta

Set \( k_i^\mu = \lambda_i \sigma^\mu \tilde{\lambda}_i \) and classify each vertex according to

\[
\lambda_a \propto \lambda_b \propto \lambda_c \quad (\text{MHV}) \quad \rightarrow \quad \bullet
\]

\[
\tilde{\lambda}_a \propto \tilde{\lambda}_b \propto \tilde{\lambda}_c \quad (\text{MHV}) \quad \rightarrow \quad \circ
\]
heptacut solutions $\rightarrow$ Riemann spheres

\[ c_\triangle = \oint_{C_\epsilon(\infty)} \frac{dz}{z} \prod_{j=1}^{3} A^{\text{tree}}_j(z) \]

points $\in S_i \cap S_j \rightarrow$ no notion of $\bullet$ or $\circ \rightarrow$ resp. prop. is soft
also: $S_i \cap S_j \subset \{\text{leading singularities}\}$

two-loop leading singularities $\rightarrow$ IR singularities of integral
Observation: leading-singularity residues cancel between virtual (a) and real (b) contributions to cross section in complete analogy with the KLN theorem on IR cancelations.
Classification of heptacut solutions

Arbitrary # of external states. Define

\[
\mu_i \equiv \begin{cases} 
  m & \text{if } i^{th} \text{ vertical prop. } \in 3\text{-pt. vertex} \\
  M & \text{if } i^{th} \text{ vertical prop. } \notin 3\text{-pt. vertex}
\end{cases}
\]

The solution to \( \ell_i^2 = 0, \ i = 1, \ldots, 7 \) is

- case 1 (M,M,M,M): 1 torus
- case 2 (M,M,m,m) etc.: 2 \( \mathbb{CP}^1 \) with \( S_i \leftrightarrow \) distrib. of \( \bullet, \bigcirc \)
- case 3 (M,m,m,m) etc.: 4 \( \mathbb{CP}^1 \) with \( S_i \leftrightarrow \) distrib. of \( \bullet, \bigcirc \)
- case 4 (m,m,m,m): 6 \( \mathbb{CP}^1 \) with \( S_i \leftrightarrow \) distrib. of \( \bullet, \bigcirc \)
Uniqueness of master contours

Limits $\mu_i \to m \implies$ chiral branchings: torus $\mu_3 \to m$

Each torus-pinching: new IR-pole + new residue thm
\[ \implies \# \text{ of lead. sing. same in all cases} \]

In all cases:
\[ \# \text{ of master } \Gamma\text{'s} = \# \text{ of basis integrals} \]
\[ \implies \text{all linear relations are preserved} \]
\[ \implies \text{perfect analogy with one-loop generalized unitarity} \]
Symmetries and systematics of IBP constraints

The IBP constraints are invariant under flips.
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Reverse logic $\rightarrow$ demand constraints to be invariant under flips and $\pi$-rotations.

$\{M,m,m\}$ case: choose basis, e.g. $\omega_{1,2,5,6} = 0$

$r_1^{(b)}(\omega_3 + \omega_4 + \omega_7 + \omega_8) + r_2^{(b)}(\omega_9 + \omega_{10} - \omega_{11} - \omega_{12}) = 0$

where, in fact, $r_1^{(b)} = r_2^{(b)} \neq 0$.
Symmetries and systematics of IBP constraints

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\[ r_1^{(b)}(\omega_3 + \omega_4 + \omega_7 + \omega_8) + r_2^{(b)}(\omega_9 + \omega_{10} - \omega_{11} - \omega_{12}) = 0 \]

where, in fact, \( r_1^{(b)} = r_2^{(b)} \neq 0 \).

\{m,m,m\} case:

1) constraint from \{M,m,m\} case inherited.

2) new flip symmetry $\rightarrow$ new constraint:

\[ r_1^{(c)}(\omega_3 + \omega_4) + r_2^{(c)}(\omega_{11} + \omega_{12} - \omega_{13} - \omega_{14}) = 0 \]

as expressed in the basis $\omega_{1,2,5,6,7,8} = 0$.

In fact, \( r_1^{(c)} = -r_2^{(c)} \neq 0 \).
Integrals with fewer propagators

Solution to slashed-box on-shell constraints:

On-shell constraints leave $8 - 5 = 3$ free complex parameters.

Multivariate residues depend on the order of integration.

**Example:** $f(z_i) = \frac{z_1}{z_2(a_1z_1 + a_2z_2)(b_1z_1 + b_2z_2)}$. Residues at $(z_1, z_2) = (0, 0)$:

$$
\frac{1}{(2\pi i)^2} \int_{C_\epsilon(0) \times C_\epsilon'(0)} dz_1 \, dz_2 \, f(z_i) = \frac{1}{a_1 b_1}
$$

$$
\frac{1}{(2\pi i)^2} \int_{C_\epsilon(0) \times C_\epsilon'(0)} dz_1 \, dz_2 \, f(z_i) = \frac{a_2}{a_1(a_1 b_2 - a_2 b_1)}
$$

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Elliptic curves vs. polylogs

\[
\begin{align*}
\text{sunrise integral not expressible through polylogs} & \quad \rightarrow \quad \text{neither should 10-point integral be} \\
\text{Analytic expression} & \quad \leftrightarrow \quad \text{maximal cut?} \\
\text{Wilson-loop amplitude correspondence} & \quad \rightarrow \quad \text{Maximal Unitarity at Two Loops}
\end{align*}
\]

\[\mathcal{N} = 4 \ \text{SYM:} \quad A^{(2)}(10-\text{scalar } N^3\text{MHV}) \quad \infty\]
Conclusions and outlook

- First steps towards fully automatized two-loop amplitudes
- Integration-by-parts identities $\rightarrow$ reduce \# of Feynman integrals by factor of 10-100

- Two-loop master contours are unique
  $\rightarrow$ perfect analogy with one-loop unitarity

- Classification of maximal-cut solutions

- Maximal cuts contain vital information:
  pinches/punctures $\rightarrow$ IR/UV divergences
  branch cuts $\rightarrow$ non-polylogs in uncut integral

- Underlying algebraic geometry $\rightarrow$ deeper understanding of maximal cuts (i.e., contour constraints)
Integrals and integral bases

- ideal two-loop basis: chiral integrals
- evaluate 4-point chiral integrals analytically
The two-loop integral coefficients $c_i$ have $O(\epsilon)$ corrections. Important to know, as the integrals have poles in $\epsilon$. 
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IR-finite integrals $\rightarrow \mathcal{O}(\epsilon)$ corrections not needed for amplitude

Candidates: num. insertions $\rightarrow 0$ in collinear int. regions, e.g.

\[
I_{++} \equiv I\left[1\ell_1|2\rangle\langle 3|\ell_2|4\right] \times [23\langle 14\rangle
\]

\[
I_{+-} \equiv I\left[1\ell_1|2\rangle\langle 4|\ell_2|3\right] \times [24\langle 13\rangle
\]

Essentially the chiral integrals of [Arkani-Hamed et al.]

$I_{++}$ and $I_{+-}$ lin. independent $\rightarrow$ use in any gauge theory

Philosophy: maximally IR-finite basis

$\rightarrow$ minimize need for cuts in $D = 4 - 2\epsilon$
$I_{+\pm}$ are finite $\longrightarrow$ can be computed in $D = 4$

1) Feynman parametrize

$$I_{++} = -\chi^2 \left( 1 + (1 + \chi) \frac{\partial}{\partial \chi} \right) I_1(\chi) \quad \text{and} \quad I_{+-} = -(1 + \chi)^2 \left( 1 + \chi \frac{\partial}{\partial \chi} \right) I_1(\chi)$$

where

$$I_1(\chi) = \int d^3a \, d^3b \, dc \, c \, \delta(1 - c - \sum_i a_i - \sum b_i) \left( \sum_i a_i \sum b_i + c(\sum_i a_i + \sum b_i) \right)^{-1}$$

$$\frac{1}{\left( a_1 a_3 (c + \sum_i b_i) + (a_1 b_4 + a_3 b_6 + a_2 b_5 \chi) c + b_4 b_6 (c + \sum_i a_i) \right)^2}$$

2) “Projectivize”

$$I_1(\chi) = 6 \int_1^\infty dc \, \int_0^\infty d^7(a_1 a_2 a_3 a_4 b_1 b_2 b_3 b_4) \frac{1}{\text{vol(GL(1))}} \frac{1}{(cA^2 + A.B + B^2)^4}$$
3) Obtain symbol

Integrate projective form one variable at the time, at the level of the symbol.

\[ S[l_1(\chi)] = \frac{2}{\chi} [\chi \otimes \chi \otimes (1 + \chi) \otimes (1 + \chi)] - \frac{2}{1 + \chi} [\chi \otimes \chi \otimes (1 + \chi) \otimes \chi] \]

4) “Integrate” symbol, using

a) \( l_1 \) has transcendentality 4 (fact, not a conjecture)
b) \( l_1 \) has no \( u \)-channel discontinuity
c) Regge limits:

\[
l_1(\chi) \to \frac{\pi^2}{6} \log^2 \chi + \left(4\zeta(3) - \frac{\pi^2}{3}\right) \log \chi + O(1) \quad \text{as} \quad \chi \to 0
\]

\[
l_1(\chi) \to 6\zeta(3) \frac{\log \chi}{\chi} + O(\chi^{-1}) \quad \text{as} \quad \chi \to \infty
\]
In conclusion, for the “chiral” integrals

\[ I_{++} \equiv I \left[ [1|\ell_1|2]\langle 3|\ell_2|4] \right] \times [2\ 3]\langle 1\ 4] \]
\[ I_{+-} \equiv I \left[ [1|\ell_1|2]\langle 4|\ell_2|3] \right] \times [2\ 4]\langle 1\ 3] \]

we find the results

\[ I_{++}(\chi) = 2H_{-1,-1,0,0}(\chi) - \frac{\pi^2}{3} \text{Li}_2(-\chi) \]
\[ + \left( \frac{\pi^2}{2} \log(1+\chi) - \frac{\pi^2}{3} \log \chi + 2\zeta(3) \right) \log(1+\chi) - 6\chi\zeta(3) \]

\[ I_{+-}(\chi) = 2H_{0,-1,0,0}(\chi) - \pi^2 \text{Li}_2(-\chi) - \frac{\pi^2}{6} \log^2 \chi - 4\zeta(3) \log \chi - \frac{\pi^4}{10} - 6(1+\chi)\zeta(3) \]

Actual chiral integrals: transcendentality-breaking terms cancel.