Effect of Trapped Electrons on Soliton Energy in an Inhomogeneous Magnetized Multicomponent Plasma

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Abstract. Energy carried by ion acoustic solitons are estimated in an inhomogeneous plasma containing negative ions and a group of hot and cold nonisothermal electron species in addition to positive ions in presence of external magnetic field. On the basis of Reductive Perturbation Technique (RPT), a relevant Korteweg-de Vries (KdV) equation is obtained and solved for its solitary wave solution. The profile of the propagating soliton is used to obtain an expression of the soliton energy. The characteristics of soliton and the energy carried by it are investigated under the effects of nonisothermal electrons trapped by wave potential, negative ions concentration, strength of the magnetic field and its obliqueness.

1. Introduction

Solitary waves and solitons arise in a wide range of areas such as shallow and deep water waves, optics communications, Bose-Einstein condensates, biological models, fluid dynamics, plasma physics etc. Solitary waves are the waves of permanent form which arise from a certain class of nonlinear partial differential equations and preserve their shape while traveling over large distances. Solitons are the solitary waves that interact with one another so as to keep their identity. They are such excitations in a medium which remain constant and do not dissipate their energy with time. Solitons arise due to a dynamical balance between the effect of nonlinearity and dispersion in the medium. Washimi and Taniuti [1] were the first to show that propagation of ion-acoustic solitary waves in plasma is governed by the Korteweg-de Vries (KdV) equation [2]. The first experimental observation of ion acoustic soliton was made by Ikezi et al. [3] in a Double Plasma (DP) device followed by Jeffrey and Kakutani [4]. Thereafter characteristics of ion- acoustic solitons have been studied for different plasma models ranging from ordinary homogeneous plasmas to the inhomogeneous plasmas having negative ions in addition to positive ions and electrons [5] - [17]. Moreover, there are several studies in magnetized plasmas which reflect the modifications by the magnetic field in the characteristics of solitons [8], [15], [18] - [21].

The characteristics of multicomponent plasma having positive ions, negative ions and the electrons differ from those in normal two component plasma, as the response of the plasma to disturbances is drastically modified due to the replacement of some plasma electrons by comparatively heavier negative ions. The properties of the soliton are further modified at a particular density of the negative ions which is referred to as critical density. At the critical density of the negative ions, the usual KdV
soliton picture fails and a new class of soliton termed as mKdV soliton is found to evolve and propagate. We have earlier reported the study on energies carried by KdV and mKdV soliton in an inhomogeneous magnetized plasma containing negative ions and isothermal electrons which follows the Boltzmann distribution [22].

The behavior of the electrons in plasma is found to be strongly modified by the nonlinear potential of the soliton. Usually, electrons are assumed to be isothermal during the passage of solitons but, some of the electrons may get trapped by the wave potential and interact strongly with the wave during its evolution [23]. In this case the electrons in the plasma are considered to be nonisothermal and are separated into two categories: the free electrons and the trapped electrons, thus resulting in group of electrons at two different temperatures. The Boltzmann velocity distribution considered for the isothermal electron is replaced by the superposition of two vortex like distributions for the nonisothermal electrons [24], [25]. There are several reports on the propagation of ion acoustic waves in such a two electron temperature plasma [26]–[29]. As energy is one of the important characteristics of soliton, therefore in the present paper, we extend our earlier investigation [22] on the energy carried by ion acoustic soliton in case of plasma having nonisothermal electrons.

2. Deriving the relevant KdV equation

2.1. Basic fluid equations

We consider a collisionless, weakly inhomogeneous, magnetized plasma consisting of positive and negative ions and nonisothermal electrons. A static magnetic field \( B_0 \) is applied in the z-direction and the wave propagates at an angle \( \theta \) with it in the (x, z) plane. The plasma is having a spatial density gradient along the x-direction as well as in the z-direction. Both the positive and negative ions to be singly charged . The normalized basic fluid equation governing the plasma response in present case are written as follows:

\[
\partial n_j/\partial t + \partial(n_j v_j)/\partial x + \partial(n_j v_j)/\partial z = 0, \quad (1a)
\]

\[
\partial n_e/\partial t + n_e \partial n_e/\partial x + n_e \partial n_e/\partial z + \delta \partial n_e/\partial x \pm \delta \partial n_e/\partial z = 0, \quad (1b)
\]

\[
\partial n_i/\partial t + n_i \partial n_i/\partial x + n_i \partial n_i/\partial z + \delta \partial n_i/\partial x \pm \delta \partial n_i/\partial z = 0, \quad (1c)
\]

\[
\partial \phi/\partial x^2 + \partial \phi/\partial z^2 - n_{el} - n_{eh} - n_n + n_p = 0, \quad (1d)
\]

\[
n_{el} = n_{el0} \left[ 1 + (T_{el} \phi/T_{el}) - (4/3) \phi (T_{el} \phi/T_{el})^{1/2} + (1/2) (T_{el} \phi/T_{el})^2 + \ldots \right], \quad (1e)
\]

\[
n_{eh} = n_{eh0} \left[ 1 + (T_{el} \phi/T_{eh}) - (4/3) \phi (T_{el} \phi/T_{eh})^{1/2} + (1/2) (T_{el} \phi/T_{eh})^2 + \ldots \right], \quad (1f)
\]

where for positive ions \( v = v, \delta = 1, j \rightarrow p \) and upper sign holds, whereas for negative ions \( v = u, \delta = m_p/m_n, j \rightarrow n \) and lower sign holds. Here \( m_p \) and \( m_n \) are masses of positive and negative ion, respectively and \( n_p \) and \( n_n \) are their respective densities. The densities of the electron species at lower and higher temperatures (which we call cold and hot electrons) are given by \( n_{el} \) and \( n_{eh} \), respectively together with their respective unperturbed values given by \( n_{el0} \) and \( n_{eh0} \). All the densities are normalized by the zeroth-order ion density \( n_0 \) at an arbitrary reference point, which we choose to be at \( x = z = 0 \). The temperatures of the cold and hot electrons are taken as \( T_{el} \) and \( T_{eh} \), respectively and \( T_{ef} \) is the effective temperature of the plasma defined by \( T_{ef} = (n_{el0} + n_{eh0}) T_{el} T_{eh} / (n_{el0} T_{el} + n_{eh0} T_{el}) \) [26]. [30] together with \( b_l \) and \( b_h \) taken as nonisothermal parameters defined in terms of the electron temperatures as \( b_{l,h} = (1 - \beta_{l,h}) / \pi^{1/2} \).
together with $\beta_{l,b} = T_{d,l}/T_{d}$ such that, for the case of isothermal electrons where $T_{db} = T_{d} = T_{df}$; $b_l$ and $b_n$ vanishes and the electron distributions given by Eqs. (1f) and (1g) reduces to Boltzmann distribution. Further, $(\vec{v}_l, \vec{v}_n, \vec{v}_e)$ and $(\vec{u}_l, \vec{u}_n, \vec{u}_e)$ are the x, y and z components of the positive and negative ion flow velocities respectively, normalized by the ion acoustic speed $\left( T_{p} / m_p \right)^{1/2}$ and $\phi$ is the electric potential normalized by $T_{d} / e$. The space coordinates $x$ and $z$ are normalized by the Debye length $\left( \epsilon_n k_BT_e / n_o e^2 \right)^{1/2}$ and the time $t$ is normalized by the inverse of the ion plasma frequency $\omega_p \equiv \left( n_o e^2 / \epsilon_n m_p \right)^{1/2}$. The magnetic field is expressed in term of $A \equiv \left( \epsilon_n / n_o m_p \right)^{1/2} B_0$. The specific heat ratios $\gamma_p$ for the positive ions and $\gamma_n$ for the negative ions are taken as 2 for the present case where number of degrees of freedom is 2.

2.2. Reductive Perturbation Technique (RPT)

The reductive perturbation technique envolves the expansion of dependent quantities (like densities, velocities, potential etc.) about their equilibrium position in the form of small parameter $\epsilon$, the smallness of which determines the order/strength of the perturbation. Further, we put the expanded form of these dependent quantities to the basic equations and separate the terms in different powers of smallness parameter $\epsilon$ to get set of equation in different order of perturbation. The expansion has to be done in view of dispersion, i.e. the stretching (transformation of coordinates to the wave frame of reference) and reduction of dependent quantities (type of oscillations/perturbation) lead to this balance.

In this regard, now we expand the densities, fluid velocities and electric potential in terms of a smallness parameter $\epsilon$ by taking into account the oblique incidence of the wave with respect to the magnetic field, as follows [16], [28] - [29]

\[
f = f_0(x,z) + \epsilon^{1/4} f_1(x,z,t) + \epsilon^{3/4} f_2(x,z,t) + \ldots, f \equiv n_x, n_n, n_e, \phi, v_x, u_x, \quad (2a)
\]

\[
g = g_0(x,z) + \epsilon^{1/4} g_1(x,z,t) + \epsilon^{3/4} g_2(x,z,t) + \ldots, g \equiv v_z, u_z, v_y, u_y, \quad (2b)
\]

\[
\phi = \phi_0(x,z,t) + \epsilon^{1/4} \phi_1(x,z,t) + \epsilon^{3/4} \phi_2(x,z,t) + \ldots, \quad (2c)
\]

We introduce the following stretched coordinates

\[
\xi = \epsilon^{1/4} \left[ (\hat{k} \cdot \hat{r}) / \lambda_0 - \tau \right] = \epsilon^{1/4} \left[ \frac{x \sin \theta + z \cos \theta}{\lambda_0} - \tau \right],
\]

\[
\tau = \epsilon^{3/4} \left[ \frac{\hat{k} \cdot \hat{r}}{\lambda_0} \right] = \epsilon^{3/4} \left[ x \sin \theta + z \cos \theta \right].
\]

Here $\lambda_0$ is the phase velocity of the wave and $\hat{k}$ is the unit wave vector in the $(x, z)$ plane at an angle $\theta$ with the direction of the magnetic field $\vec{B} = B_0 \hat{z}$.

2.3. Relevant KdV equation

By using Eqs. (2) and (3) into the basic equations [Eq. (1)] and collecting the terms of different powers in $\epsilon$, we obtain different sets of equations termed as zeroth, first and second order equations where zeroth order equations involves only unperturbed quantities while, first and second order equation involves quantities up to first and second order perturbation terms respectively. At the zeroth order, equation of continuity for positive and negative ion fluids and equation for charge neutrality for electrons are obtained as

\[
\sin \theta \frac{\partial \phi}{\partial n_e} |_{\lambda_0 \tau} + \cos \theta \frac{\partial \phi}{\partial n_n} |_{\lambda_0 \tau} = 0
\]

and

\[
n_{e0} + n_{e0} + n_{n0} - n_{p0} = 0,
\]

respectively. With the set of first order equations, we obtain the phase velocity relation assuming equal mass of ions ($m_p = m_n$) and hence equal x- and z-components of ion
drifts for positive and negative ions in the plasma \((v_{x0} = u_{x0} \text{ and } v_{z0} = u_{z0})\), which is given as

\[
\lambda_0 = v_0 \cos \theta + u_0 \sin \theta \pm a_0 \cos \theta.
\]  

(4)

where, \(a_0 = \left[1 + d_0 \right] / \left[1 - d_0 \right]\) \(d_0 = n_{x0} / n_{z0}\). Here the positive and negative sign correspond to propagation in fast and slow mode respectively. At the second order, we obtain the following important relation

\[
n_{p2} - n_{n2} - (n_{p0} - n_{n0}) \phi_0 + (4 / 3) \left[ \frac{\partial u_0 b_t}{\partial T_{ee}} \right] \left[ \frac{\partial u_0 b_h}{\partial T_{eh}} \right] \phi_{0 1 / 2} - n_{a0} b_0 \left[ \frac{\partial u_0 b_h}{\partial T_{eh}} \right] \phi_{0 1 / 2} + \left[ \frac{1}{\lambda_0^2} \right] \phi_{0 1 / 2} \phi_{0 1 / 2} = 0.
\]  

(5)

Equation (5) with the help of other first and second order equations (not shown in the paper) together with phase velocity relation [Eq. (4)] reduces to following relevant KdV equation with an additional term due to density inhomogeneity

\[
\frac{\partial n_{p1}}{\partial \tau} + \alpha n_{p1} \frac{\partial n_{p1}}{\partial \xi} + \beta \frac{\partial^3 n_{p1}}{\partial \xi^3} + m_{p1} \frac{\partial n_{p0}}{\partial \tau} = 0,
\]  

(6)

where,

\[
\alpha = \frac{4}{3} \left( \frac{\partial u_0 b_t}{\partial T_{ee}} \right) \left[ \frac{\partial u_0 b_h}{\partial T_{eh}} \right] \phi_{0 1 / 2} + n_{a0} b_0 \left[ \frac{\partial u_0 b_h}{\partial T_{eh}} \right] \phi_{0 1 / 2} \left( \frac{1}{\lambda_0^2} \right) \cos \theta \left( \frac{n_{p0} - n_{n0}}{n_{p0} - n_{n0}} \right) \lambda_0 - v_0)
\]  

(7a)

\[
\beta = a_0^2 \left( \lambda_0 - v_0 \right) \left( \lambda_0 - v_0 \right) \left( \frac{1}{\lambda_0^2} \right) \sin^2 \theta + A^2 \lambda_0 \cos^2 \theta / 2 \frac{A^2 \lambda_0^2}{2} \left( \frac{n_{p0} - n_{n0}}{n_{p0} - n_{n0}} \right) \cos \theta
\]  

(7b)

\[
\gamma = \frac{\lambda_0 \cos \theta}{2 \frac{n_{p0}}{\lambda_0 - v_0}} \left( \frac{\lambda_0 - v_0 - v_{z0} \cos \theta}{\lambda_0 - v_0 \sin \theta} \right) \left( \lambda_0 - v_0 \right)
\]  

(7c)

together with \(v_{z0} = v_{z0} \cos \theta + v_{z0} \sin \theta \cos \theta\).

3. Solitary wave solution of the KdV equation

To obtain the solitary wave solution of Eq. (6) We put,

\[
n_{p1} = h(\tau) n(\xi, \tau),
\]  

(8)

together with

\[
h(\tau) = \exp \left( - \int \gamma \left( \frac{\partial n_{p0}}{\partial \tau} \right) d\tau \right).
\]  

(9)

With this substitution Eq. (6) reduces to

\[
\frac{\partial n}{\partial \tau} + \alpha n \frac{\partial n}{\partial \xi} + \beta \frac{\partial^3 n}{\partial \xi^3} = 0.
\]  

(10)

To obtain the travelling wave solution of this equation, we transform the coordinates to \(\xi = w^{-1} \left( \tau - U \xi \right)\), which is a variable transformation together with \(U\) and \(w\) as the velocity shift [15] and the width of the soliton, respectively. With this transformation, Eq. (10) becomes

\[
\frac{\partial n}{\partial \xi} - \alpha U n \frac{\partial n}{\partial \xi} + \beta a^2 U^3 \frac{\partial^3 n}{\partial \xi^3} = 0.
\]  

(11)

The above equation is solved using the method suggested by Yan [31] to obtain the soliton profile as

\[
n = \frac{169}{36 \alpha^2 b U^2} \sec h^2 \left( \frac{1}{\sqrt{12 \beta U^3}} \chi \right)
\]  

(12)

where \(\chi = \tau - U \xi\), \(169 / 36 \alpha^2 b U^2 \equiv n_m\) is the peak soliton amplitude and \(\sqrt{12 \beta U^3} \equiv w\) is the soliton width.

4. Expression for soliton energy
The soliton profile [Eq. (12)] can be used to estimate the energy carried by solitons as as [22], [32] - [33]

\[ E_{\text{sol}} = \int_{-\omega}^{\infty} n^2(\chi) d\chi. \]  

(13)

Considering the unperturbed quantities as slowly varying in the present case of weekly ionized plasma, the above integration yields the expression for soliton energy in plasma having nonisothermal electrons as

\[ E_{\text{sol}} = 56.989 \rho^{1/2}/\alpha^4 f^2 U^{5/2} \]  

(14)

5. Results and Discussions

In this paper, we have studied the evolution of solitons and energy carried by them in a nonisothermal inhomogeneous plasma containing negative ions and having two group of electrons at different temperatures in presence of external magnetic field. We study the effects of concentrations of cold and hold electrons and their relative temperatures on the solitons profiles and their energies. Solitons energy is also studied under the effect of negative ion concentration and magnetic field strength and its obliqueness to the direction of wave propagation. In figure 1, we have plotted the soliton profile [eq. (12)] for two values of cold to hot electron density ratio \( n_{e0} \) (\( \equiv n_{e0}/n_{eh0} \)). Here we observe that the profiles of the solitons are significantly modified by the presence of the colder electron species and the nonisothermality of the plasma. The increase in concentration of cold electrons results in the evolution of soliton with higher amplitude. Similar results were obtained by Das and Karmakar [29] in an unmagnetized homogeneous plasma.

![Figure 1. Sketch of soliton profile based on Eq. (12) for the values of cold to hot electron density ratio](image)

In figure 2 we have plotted the energy carried by soliton as well as effective temperature of plasma under the effect of cold to hot electron density ratio. An increase in this ratio results in the reduction of the effective temperature of the plasma which eventually leads to the increase in soliton energy. Increase in the energy of soliton with increasing \( n_{e0} \) is consistent with the corresponding increase in soliton amplitude as seen in figure 1. Further, the increase of the amplitude (and hence energy) with
the decrease in the effective temperature is consistent with the result of Goswami and Buti [27] obtained in an ordinary homogeneous unmagnetized plasma having positive ions and two temperature electrons. The fact that the soliton evolving with higher amplitude contains more energy is also verified in figure 3, in which we have investigated the effect of cold and hot electron temperature ratio $T_{e_h} / T_{e_c}$ on the peak amplitude and energy of soliton. Here, we observe that both the peak amplitude and the energy of the soliton increase with $T_{e_h}$, which correspond to the increase in the nonisothermal parameter for cold electrons, $b_c$ and decrease in the nonisothermal parameter for hot electron $b_h$.

Figure 2. Plots of soliton energy and effective temperature $T_{e_h} / T_{e_c}$ w.r.t. cold to hot electron density ratio $n_{e_{cl}}$ when $T_{e_h} = 0.1$ and other parameter same as in figure 1.

Figure 3. Plots of soliton amplitude and energy w.r.t. cold to hot electrons temperature ratio $T_{e_h}$ when $n_{e_{cl}} = 0.4$ and other parameter same as figure 1.
Finally, in figure 4, we have studied the effect of negative ion concentration and strength of the magnetic field and its obliqueness on the energy of the soliton. Here, we have plotted soliton energy w.r.t. negative to positive ion density ratio $d_0$ for two values of magnetic field strength and obliqueness $\theta$. Here we observe that the presence of negative ion has a suppressing role on the soliton energy (and hence amplitude). Soliton energy gets further decreased in presence of stronger magnetic field but increases with the increase in obliqueness $\theta$. The trend of soliton energy with negative ion concentration or magnetic field strength is similar to that observed in our earlier report on soliton energy in a plasma having isothermal electron [22], but the comparison reveals that the energy of soliton propagating in a nonisothermal plasma is relatively higher than that in an isothermal plasma.

![Figure 4](image)

**Figure 4.** Plots of soliton energy w.r.t. negative to positive ion density ratio $d_0$ for two values of magnetic field strength $B_0$ and its obliqueness $\theta$ for $n_e/T_{e0}=0.4$ and $T_{e0}=0.15$. Here, solid lines are for $\theta = 30^\circ$ and dashed line is for $\theta = 10^\circ$.

Solitons are stationary, localized finite energy wave packets that arise due to a dynamical balance between the effects of nonlinearity and dispersion in the medium and once this balance is achieved, it propagates without dissipating its energy contents as long as the two properties of the medium do not change. In this theoretical study based on Reductive Perturbation Technique we ensure such a delicate balance using a smallness parameter $\varepsilon$, by appropriately choosing the stretched coordinates and properly expanding the dependent variables in terms of it. For any given medium as the order of nonlinearity increases without the corresponding increase in dispersion, the energy of wave propagating in it gets continually injected into higher frequency modes. In the time domain, it appears as the formation of shock waves. But, if the dispersion also increases to match with the nonlinearity, then in place of shock waves, solitary waves propagates with higher energy content. This accounts to observed higher energy of soliton propagating in nonisothermal plasma in comparison to the case of isothermal plasma [22]; the nonisothermal plasma offering higher nonlinearity owing to the contributions from two separate species of electrons at different temperatures. Moreover, the effect of magnetic field on energy of ion-acoustic soliton as shown in figure 4, is attributed to the nonlinearity and dispersion of the plasma getting modified due to the cyclotron motion of the positive and negative ions; at lower magnetic field the kinetic effects (like Larmor radius effects) contributes to the plasma nonlinearity resulting the higher soliton energy for lower $B_0$ values.
6. Conclusions
In this paper, we have studied the evolution of ion acoustic soliton and estimated the energy carried by it in an inhomogeneous magnetized plasma containing negative ions and nonisothermal electrons at two different temperatures in addition to the positive ions. We have obtained expressions for the profile and energy of the soliton a study on which reveals that, the characteristics of the soliton is significantly modified by the presence of nonisothermal electrons in the plasma. The presence of negative ions in the plasma or the strength of the external magnetic field has a suppressing effect on the energy carried by soliton in it. But, on contrary, presence of nonisothermal electrons increases the amplitude and energy of soliton significantly.

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