Nonlinear integral equations for the SL(2, $\mathbb{R}$)/U(1) black hole sigma model

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Abstract

It was previously established that the critical staggered XXZ spin chain provides a lattice regularization of the black hole conformal field theory (CFT). We reconsider the continuum limit of this spin chain with the exact method of nonlinear integral equations (NLIEs), paying particular attention to the effects of a singular integration kernel. With the help of the NLIEs, we rederive the continuous black hole spectrum, but also numerically match the density of states of the spin chain with that of the CFT, which is a new result. Finally, we briefly discuss the integrable structure of the black hole CFT and the identification of its massive integrable perturbation on the lattice.

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1. Introduction

The relationship between integrable spin chains and integrable quantum field theories has been a long and fruitful one. In this respect, the sine–Gordon (SG) model is an illustrative example. Undoubtedly, its large volume physics can be entirely understood without any reference to a spin chain by means of the asymptotic Bethe ansatz, which is based on the exact (anti)soliton $S$-matrix of Zamolodchikov and Zamolodchikov [1]. And although the vacuum energy in finite size can be computed from the IR data via the thermodynamic Bethe ansatz (TBA) [2, 3], excited states are inaccessible this way$^3$. A different approach is based on the observation that the XXZ spin chain provides a lattice discretization for the SG model in the sense that low-energy excitations of the former scatter with the $S$-matrix of the latter [5]. One can then solve the XXZ spectrum problem in finite size by elementary means [6, 7], take the continuum limit and produce a single nonlinear integral equation (NLIE) solving the entire finite size spectrum problem for the SG quantum field theory [8–11]. The ultimate check of this NLIE is that it produces in the IR the required asymptotic multiparticle spectrum and in the UV the expected spectrum of a free massless compact boson. Similar results were subsequently

$^3$ The generalization in [4] of the SG TBA to multiparticle states uses the Destri–de Vega equation derived from the lattice and, hence, it is not based on the IR data only.
derived for the affine Toda theories with imaginary coupling and their unitary restrictions
[12], equivalent to massive perturbations of rational conformal field theories (CFTs), from the
analysis of integrable spin chains of higher rank quantum groups [13].

In contrast, our current understanding of integrable massive perturbations of non-rational
CFTs, starting from first principles and including excited states, is limited to the IR region
only, the only notable exception being the sinh–Gordon (ShG) model [14]. To understand why
this is uncomforting, take for instance the real affine-Toda theories. These are characterized by
a solitonless massive spectrum and diagonal scattering [15, 16]. Naively, it seems rather
surprising that a theory defined by such simple IR data can develop in the UV all the
complicated features characteristic of a non-rational CFT, i.e. continuous spectrum, non-
normalizable vacuum, etc. For the ShG model the mechanism of this process was understood
in [14], building on earlier results from [17]. It is worth noticing that, again, the main tool was
a NLIE for the finite size spectrum derived from a (tailor made) lattice ShG discretization [18].
The remarkable observation of [14, 19] was that the NLIE could encode both the scattering
data in the IR and such fine non-rational CFT structures as reflection amplitudes in the UV.
Clearly, it would be nice to have more examples of perturbed non-rational CFTs for which one
can study the evolution from IR to UV explicitly by means of NLIEs derived from a lattice
discretization.

Motivated by these considerations, we study the integrable structure of the SL(2, ℤ)/U(1)
Euclidean black hole sigma model CFT\(^4\) defined by the (one loop) metric [21]
\[
ds^2 = \frac{k}{2} \left( d\rho^2 + \tanh^2 \rho \, d\phi^2 \right),
\]
(1.1)
starting from its lattice discretization as a staggered XXZ spin chain. For this chain the
emergence of a continuous spectrum in the continuum limit was first noticed in [22], further
studied in [23] and finally identified with the black hole spectrum in [24]. Our main result is a
set of two NLIE for the black hole CFT, which we derive from the lattice. The NLIEs reproduce
the conformal dimensions of all primary states in the continuous component of the black hole
spectrum [25], but also allow to compute the eigenvalues of all mutually commuting local
conserved charges of the CFT on these primary states. In this sense, our NLIEs characterize
the quantum integrable structure of the black hole CFT, which is related to quantization of
the second Poisson structure of the nonlinear Schrödinger hierarchy [26]. Coming back to
our initial discussion of IR to UV flows, one expects that there is an integrable massive
deforation of the black hole CFT which preserves this integrable structure and which can
be realized straightforwardly on the lattice following the standard recipe of [27, 28]. We shall
discuss this point further in the concluding section.

Our NLIEs have some unusual features, some of which were expected [22]. Firstly, the
integral kernels defining them are ‘singular’, i.e. do not decay at infinity. Closely related to this
fact is the appearance of unusual source terms and non-monotonic counting functions (even in
the absence of holes). To make sure that the NLIEs make sense, we have performed a very non-
trivial check on them by matching numerically the density of states in the spin chain with the
density of states in the CFT. More precisely, one can imagine the target space of the black hole
CFT as a semi-infinite cigar degenerating into a cylinder of radius \(\sqrt{k/2}\) at asymptotic infinity.
Now if the continuous black hole spectrum is regularized by cutting off the infinite tail of
the cigar with a Liouville wall as in [25], then we find that the resulting density of states agrees
precisely with the density of spin chain states for which the discrete quantum numbers are kept
fixed in the continuum limit. Equivalently, the difference between the reflection amplitude at

\(^4\) The black hole CFT is equivalent to the sine–Liouville model via the strong–weak coupling duality of Fateev et al.;
see [20] for a proof.
the tip of the cigar and off the Liouville wall determines (a subleading term in) the asymptotic of the NLIE in the region where the Bethe roots condense.

The paper is structured as follows. In section 2 we recall the staggered XXZ spin chain discretization of the black hole CFT, its Bethe ansatz solution and specify the class of excited states to which we restrict our subsequent analysis. In section 3 we derive the NLIEs for the finite size spin chain spectrum and then take their continuum limit in section 4. We then put the latter NLIEs to work in section 5, where we compute the conformal spectrum of the spin chain and match it with the spectrum of the black hole. We also give an integral representation for the generating functions of the local integrals of motion of the CFT. Finally, in section 6 we explain the numerical algorithm used to compute the density of states in the spin chain and compare the results to CFT predictions. There is also a short appendix collecting some of the more technical calculations.

2. Staggered six-vertex model

2.1. Transfer matrix and conserved quantities

We consider the six-vertex (6V) model on the square lattice, with the Boltzmann weights given by the $R$-matrix (see figure 1):

$$R(u) = \frac{1}{a(u)} \begin{pmatrix} a(u) & 0 & 0 & 0 \\ 0 & b(u) & e^{-iu}c & 0 \\ 0 & e^{iu}c & b(u) & 0 \\ 0 & 0 & 0 & a(u) \end{pmatrix}, a(u) = \sin(\gamma - u), b(u) = \sin u, c = \sin \gamma,$$

where $u$ is the spectral parameter, and $\gamma$ defines the Baxter’s ‘anisotropy parameter’ $\Delta = (a^2 + b^2 - c^2)/2ab = -\cos \gamma$. In this paper, we consider the regime:

$$0 < \gamma < \frac{\pi}{2}.$$  \hspace{1cm} (2.1)

The additional exponentials $e^{\pm iu}$ appearing in the off-diagonal terms of the $R$-matrix can be removed by a U(1)-gauge transformation; keeping them has the advantage of making the $R$-matrix $\pi$-periodic.

Spectral parameters are carried by the lines of the lattice, and the weights for a vertex with spectral parameters $u$ and $v$ are given by $R(u - v)$. We introduce a staggering of the horizontal and vertical spectral parameters, as shown in figure 2. For a row of $2L$ sites, the one-row transfer matrix with twisted periodic boundary conditions is:

$$t(u) = \text{Tr}_0 \left[ \exp(i\varphi \sigma_0^z) R_{0,2L} \left( u - \frac{\pi}{2} \right) R_{0,2L-1}(u) \cdots R_{02} \left( u - \frac{\pi}{2} \right) R_{01}(u) \right],$$

where we take $-\pi < \varphi < \pi$ and, for simplicity, restrict to $L$ even. The quantum Hamiltonian is defined by

$$H := \frac{1}{2} \sin 2\gamma \left[ t^{-1}(0) \frac{dt}{du}(0) + t^{-1}(\pi/2) \frac{dt}{du}(\pi/2) \right].$$  \hspace{1cm} (2.2)
and has the explicit form

\[
H = \sum_{j=1}^{2L} \left[ -\frac{1}{2} \sigma_j^x \sigma_{j+2}^x + \sin^2 \gamma \sigma_j^z \sigma_{j+2}^z - \frac{i}{2} \sin \gamma (\sigma_{j-1}^z - \sigma_{j+2}^z) (\sigma_j^y \sigma_{j+1}^y + \sigma_{j+2}^y \sigma_{j+3}^y) \right] + L \cos 2\gamma,
\]

(2.3)

where \( \sigma_j^a \) are the Pauli matrices at site \( j \). In terms of the conserved U(1) charge

\[
S^z = \frac{1}{2} \sum_{j=1}^{2L} \sigma_j^z
\]

the twisted periodic boundary conditions for the local spin operators are given by

\[
\sigma_{2L+j}^a = e^{2i\phi S^z} \sigma_j^a e^{-2i\phi S^z}.
\]

The momentum operator is determined by the two-row transfer matrix at \( u = 0 \) by

\[
e^{\mathbf{P}} := t(\pi/2) t(0) = \exp \left[ i\phi (\sigma_1^z + \sigma_2^z) \right] \mathbf{T}, \quad e^{i\mathbf{PL}} = e^{2i\phi S^z},
\]

(2.4)

where \( \mathbf{T} \) is the two-site translation operator. Similarly, we define the quasi-shift operator

\[
\tilde{T} := t(\pi/2) \mathbf{T}^{-1}(0),
\]

(2.5)

which will play an important role in the following. A little algebra shows that \( \tilde{T} \) has the form of a diagonal-to-diagonal (or ‘light-cone’) transfer matrix, as depicted in figure 3:

\[
\tilde{T} = \left[ \prod_{j=1}^L \tilde{R}_{2j,2j+1}(\pi/2) \right] \times e^{i\phi \sigma^1} \times \left[ \prod_{j=1}^L \tilde{R}_{2j-1,2j}(\pi/2) \right] \times e^{-i\phi \sigma^1},
\]

(2.6)

where \( \tilde{R}_{12}(u) := P_{12} R_{12}(u) \), and \( P_{12} \) is the permutation operator. Let us notice the relation \( R_{ij}(\pi/2) R_{ji}(\pi/2) = I \), which is useful for deriving equations (2.4) and (2.6).

As usual, higher order (local) conserved quantities can be generated by expanding the logarithms of the transfer matrices \( t(u) \) and \( t(u + \pi/2) \) around \( u = 0 \).
The function $\phi_\alpha$ we shall restrict in the following for definiteness to configurations of Bethe roots with

$Q$ and we have defined $t$

The eigenvalues of $2.2$. Bethe ansatz solution

The quasi-momentum $\tau$ (2.5) and (2.6) for $2L = 8$ sites with periodic boundary conditions. At each vertex, sits an $R(\pi/2)$ matrix acting in the vertical direction. The dotted line represents the twist.

For small system sizes one can check numerically that the vacuum of the Hamiltonian (2.3) is antiferromagnetic and the corresponding Bethe roots lie on the lines $\text{Im} \lambda = \pm \pi/2$. In the following we shall consider low-energy solutions of the form

$\{\lambda_j\}_{j=1,...,r} = \{\lambda_{aj} - i\sigma/2\}_{j=1,...,n_0} \cup \{\lambda_{aj} + i\sigma/2\}_{j=1,...,n_1}$, with $\lambda_{aj}$ real. The logarithmic form of the BAE for this type of solutions is

$L_p(\lambda_{aj}) = 2\pi l_{aj} - 2\varphi - \sum_{b=0,1}^{n_0} \sum_{l=1}^{n_0} \theta_{a-b}(\lambda_{aj} - \lambda_{bl})$, for $a \in \{0, 1\}$,

where $L_{aj}$ are the Bethe integers with $L_{aj} \in (r_a - 1)/2 + \mathbb{Z}$. The momentum and scattering phases are

$p(\lambda) = \phi_{\pi/2 - \gamma}(\lambda)$, $\theta_0(\lambda) = \phi_{\gamma}(\lambda/2)$, $\theta_{\pm 1}(\lambda) = -\phi_{\pi/2 - \gamma}(\lambda/2)$

and we have defined

$\phi_0(\lambda) := 2\text{Arctan}(\tanh \lambda \cotan \alpha) = -i \log \frac{\sinh(i\alpha - \lambda)}{\sinh(i\alpha + \lambda)}$.

The function $\phi_0$ is analytic on the strip $|\text{Im} \lambda| < \alpha$, and the properties of $\phi_0$ that we shall need in the subsequent calculations are given in the appendix.

The total momentum and energy can be written as:

$P = 2\varphi + \sum_{a,j} \phi_{\pi/2 - \gamma}(\lambda_{aj})$, $E = -\sin 2\gamma \sum_{a,j} \phi_{\pi/2 - \gamma}(\lambda_{aj})$.

The quasi-momentum associated to the quasi-shift $\tilde{\tau}$ is defined as:

$K := \frac{\Lambda(\pi/2)}{\Lambda(0)} = \sum_{a,j} (-1)^a k(\lambda_{aj})$, $k(\lambda) := \log \frac{\cosh \lambda + \sin \gamma}{\cosh \lambda - \sin \gamma}$.

Note that, under the exchange of $\{\lambda_{0j}\}$ and $\{\lambda_{1j}\}$, $P$ and $E$ are even, whereas $K$ is odd. Therefore, we shall restrict in the following for definiteness to configurations of Bethe roots with $r_0 \leq r_1$. 

2.3. Bethe integers

Our aim in this paper is to study the solutions of the BAE with real roots \( \lambda_{aj} \), corresponding to a vacuum with holes located immediately above (resp. below) the ‘Fermi point’ \( \max(\lambda_{aj}) \) (resp. \( \min(\lambda_{aj}) \)). In analogy to the XXZ case, we expect (at least for certain values of \( \gamma \)) precisely these solutions to carry the energy and momentum quanta, whereas the complex solutions usually carry the spin quanta. Also, solutions with more general distributions of holes do exist and contribute to the low-energy spectrum, but here we wish to concentrate on the simplest interesting case, namely the analogue of electro-magnetic excitations of the XXZ chain. The comprehensive description of low-energy (possibly complex) solutions is of course an interesting project, but is certainly a more involved task and is left for future work.

The ground state solution is fixed by the following configuration of Bethe integers:

\[
I_{a_1}, \ldots, I_{a_{ra}} = -\frac{r_0 - 1}{2}, -\frac{r_0 - 3}{2}, \ldots, \frac{r_0 - 1}{2},
\]

where \( r_0 = r_1 = L/2 \). In analogy to the XXZ case [30], we restrict to hole configurations obtained by two procedures.

First, one can remove (add) some roots from (to) the ground state:

\[
r_a = L/2 - m_a.
\]

Since the total magnetization is \( S^z = m_0 + m_1 \), these are called magnetic excitations. We introduce the even and odd ‘magnetic charges’

\[
m := m_0 + m_1 \geq 0, \quad \tilde{m} := m_0 - m_1 \geq 0,
\]

which we require to be non-negative. It is important to realize that \( \tilde{m} \) is not the eigenvalue of a conserved charge. Rather, we shall derive the relation \( \tilde{m} \propto K \log L \) which holds only in the continuum limit.

Secondly, one can shift all the \( I_{aj} \) by an integer \( e \in \mathbb{Z} \); these are called electric excitations. A combined electro-magnetic excitation corresponds to the configuration:

\[
I_{a_1}, \ldots, I_{a_{ra}} = -\frac{r_0 - 1}{2} + e, -\frac{r_0 - 3}{2} + e, \ldots, \frac{r_0 - 1}{2} + e.
\]

Thus, the Bethe integer configuration is determined by three integer numbers \( (m, \tilde{m}, e) \), with \( m \equiv \tilde{m} \mod 2 \). Note that real-valued solutions \( \{\lambda_{aj}\} \) only exist under some additional conditions on \( (m, e) \), which we shall give explicitly in section 3.1. In the following, we shall always assume that these conditions are satisfied.

3. Finite size NLIE with singular kernels

3.1. Counting functions

The BAE (2.9) and the conserved quantities (2.12) and (2.13) can be re-expressed in terms of the counting functions. For \( |\text{Im} \lambda| < \min(2\gamma, \pi/2 - \gamma) \), we define:

\[
Z_a(\lambda) := Lp(\lambda) + 2\varphi + \sum_{b,\ell} a_{a-b}(\lambda - \lambda_{b\ell}),
\]

so that the BAE (2.9) simply read

\[
Z_a(\lambda_{aj}) = 2\pi I_{aj}.
\]

The limiting values of \( Z_a \) are given by (A.4):

\[
Z_a(\pm \infty) = \pm \left\lfloor \frac{\pi}{2} [L - (-1)^a \tilde{m}] + \left( 2\gamma - \frac{\pi}{2} \right) m \right\rfloor + 2\varphi.
\]
Note that the condition $2\pi \max(I_{aj}) < Z_a(\infty)$ restricts the values of $(m, \epsilon)$:

$$\epsilon < \frac{\gamma m + \varphi}{\pi} + \frac{1}{2},$$

and similarly at $\lambda \to -\infty$. In the following, we shall always assume that (i) $\epsilon$ is small enough so that a real solution to the BAE exists and (ii) larger values of $\epsilon$ are present in the spectrum with the same expression for the energy, but correspond to complex solutions. This second assumption is merely based on analogy to the XXZ case [30], but may be verified in future studies of complex solutions of the staggered 6V model BAE.

The function $[1 + (-1)^{\epsilon} \exp(iZ_0)]$ vanishes at $\lambda_{aj}$, but it can have additional real roots: these are called ‘holes’, and denoted $\eta_{aj}$. The corresponding Bethe integers are denoted $I_{h,aj}$, and we have BAE for holes:

$$Z_a(\eta_{aj}) = 2\pi I_{h,aj}, \quad \text{with} \quad I_{h,aj} \in \left\lfloor \frac{r_a}{2} + Z \right\rfloor + Z.$$  \hspace{1cm} (3.4)

Let us now do a numerical experiment: we take $L = 100$, fix the integers $(m, \tilde{m}, \epsilon)$, and solve the BAE (2.9). Using the numerical values of the $\lambda_{aj}$, we can compute the functions $Z_a(\lambda)$ (3.1): see figures 4 and 5. We then observe the following facts.

(i) The function $Z_0$ has two extrema, whereas $Z_1$ is increasing.
(ii) For each Bethe integer in the ranges $2\pi \max(I_{aj}) < 2\pi I_{h,aj} < Z_a(+\infty)$ and $Z_a(-\infty) < 2\pi I_{h,aj} < 2\pi \min(I_{aj})$, we have an ordinary hole, in the region where $Z_0'>0$.
(iii) For each Bethe integer in the ranges $Z_0(+\infty) < 2\pi I_{h,aj} < \max Z_0$ and $\min Z_0 < I_{h,aj} < Z_0(-\infty)$, we have a pair of extraordinary holes, with both signs of $Z_0'$.

Thus, the number of positive/negative ordinary holes for $Z_0$ are, respectively,

$$N_{h,aj}^\pm = \left[ \frac{\pm Z_a(\pm\infty)}{2\pi} - \max(\pm I_{aj}) \right] = \left[ \frac{\gamma m + \varphi}{\pi} \mp \epsilon + \frac{1}{2} \right] := N_{h}^\pm,$$  \hspace{1cm} (3.5)

and the number of pairs of positive/negative extraordinary holes for $Z_0$ is, respectively,

$$\tilde{N}_{h}^\pm = \left[ \frac{\max(\pm Z_a)}{2\pi} - \max(\pm I_{aj}) \right].$$

We denote by $N_h$ (resp. $\tilde{N}_h$) the total number of ordinary (resp. extraordinary) holes.

Figure 4. The counting functions $Z_0$ and $Z_1$ at finite size $L = 100$, for $\gamma = 1.24$, $\varphi = 0$, $\epsilon = 0$, $m = 2$, $\tilde{m} = 12$. 

Note that $2\pi \max(I_{aj}) < Z_a(\infty)$ restricts the values of $(m, \epsilon)$: 

$$\epsilon < \frac{\gamma m + \varphi}{\pi} + \frac{1}{2},$$

and similarly at $\lambda \to -\infty$. In the following, we shall always assume that (i) $\epsilon$ is small enough so that a real solution to the BAE exists and (ii) larger values of $\epsilon$ are present in the spectrum with the same expression for the energy, but correspond to complex solutions. This second assumption is merely based on analogy to the XXZ case [30], but may be verified in future studies of complex solutions of the staggered 6V model BAE.

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$$Z_a(\eta_{aj}) = 2\pi I_{h,aj}, \quad \text{with} \quad I_{h,aj} \in \left\lfloor \frac{r_a}{2} + Z \right\rfloor + Z.$$  \hspace{1cm} (3.4)

Let us now do a numerical experiment: we take $L = 100$, fix the integers $(m, \tilde{m}, \epsilon)$, and solve the BAE (2.9). Using the numerical values of the $\lambda_{aj}$, we can compute the functions $Z_a(\lambda)$ (3.1): see figures 4 and 5. We then observe the following facts.

(i) The function $Z_0$ has two extrema, whereas $Z_1$ is increasing.
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and the number of pairs of positive/negative extraordinary holes for $Z_0$ is, respectively,

$$\tilde{N}_{h}^\pm = \left[ \frac{\max(\pm Z_a)}{2\pi} - \max(\pm I_{aj}) \right].$$

We denote by $N_h$ (resp. $\tilde{N}_h$) the total number of ordinary (resp. extraordinary) holes.
3.2. Nonlinear integral equations

Following [9], we shall reformulate the BAE (2.9) as NLIE for the counting functions $Z_a$. This reformulation is most suited for taking the scaling limit. The nonlinear part of the equations involves the functions

$$U_a(\lambda) := \log[1 + (-1)^{\gamma} e^{i Z_a(\lambda)}], \quad \overline{U}_a(\lambda) := \log[1 + (-1)^{\gamma} e^{-i Z_a(\lambda)}],$$

where we have used the principal determination of the logarithm, the cut-line being the negative axis. The functions $U_a(\lambda)$ and $\overline{U}_a(\lambda)$ will stay away from the branch cut in the domains where $\text{Im}[Z_a(\lambda)] > 0$ and, respectively $\text{Im}[Z_a(\lambda)] < 0$, since the arguments of the logarithms have positive real part. Hence, their imaginary parts are restricted to the domain $-\pi/2 < \text{Im} U_a(\lambda), \text{Im} \overline{U}_a(\lambda) < \pi/2$.

We can define some integration paths $\Gamma_a$ in the complex plane, on which the $U_a$ are well defined. First, since $Z'_1(\lambda) > 0$ for real $\lambda$, we can take $\Gamma_1 = i\delta + \mathbb{R}$, with $0 < \delta < \min(\gamma, \pi - 2\gamma)$ finite, but small enough so that $\text{Im}[Z_1(\lambda)] > 0$ on $\Gamma_1$. For $Z_0$, the path has to be in the upper half-plane in the vicinity of $Z'_0 > 0$, and in the lower half-plane for $Z'_0 < 0$. The two paths $\Gamma_0$ and $\Gamma_1$ are depicted in figure 6. Similarly, $\overline{U}_a$ is well defined on the contour $\Gamma_a$ conjugate to $\Gamma_a$.

We can now state the basic identity expressing sums over the Bethe roots in terms of the $\eta_{aj}$ and $U_a$ (see the appendix for a proof):

$$\sum_j f(\lambda_{aj}) = \int \frac{d\mu}{2\pi} f(\mu) Z_a(\mu) - \sum_j \nu_{aj} f(\eta_{aj})$$

$$- \frac{1}{2\pi} \left[ \int_{\Gamma_0} d\mu \ f(\mu) U'_a(\mu) - \int_{\Gamma_1} d\mu \ f(\mu) \overline{U}_a(\mu) \right],$$

which holds if $f$ is a smooth function for real $\lambda$ that increases slow enough at infinity. The numbers $\nu_{aj}$ are signs, defined as

$$\nu_{aj} := \text{sign} Z_a(\eta_{aj}).$$
They appear in (3.8) because the integration contour $C_a = (-\Gamma_a) \cup \Gamma_a$ encloses some holes in the clockwise direction, and other holes in the anti-clockwise direction.

Applying (3.8) to $Z'_a$, we get:

$$
\sum_b \left[ \left( \delta_{ab} - \frac{K_{a-b}}{2\pi} \right) \ast Z'_b \right](\lambda) = Lp'(\lambda) - \sum_{b,h} v_{bh} K_{a-b}(\lambda - \eta_{bh}) \\
- \frac{1}{2i\pi} \sum_b \left[ \int_{\Gamma_b} d\mu K_{a-b}(\lambda - \mu) U'_b(\mu) - \int_{\Gamma_b} d\mu K_{a-b}(\lambda - \mu) U'_b(\mu) \right],
$$

(3.10)

where $\ast$ denotes the convolution product (see (A.2)), and we have introduced the kernels

$$
K_{a-b} := \theta'_{a-b}.
$$

(3.11)

From (2.10), (A.1) we compute the Fourier transforms

$$
\hat{K}_0(\omega) = \frac{2\pi \sinh(\pi - 2\gamma)\omega}{\sinh \pi \omega}, \quad \hat{K}_{-1}(\omega) = -\frac{2\pi \sinh 2\gamma \omega}{\sinh \pi \omega}.
$$

(3.12)

Let us first deal with the even part of equation (3.10). The even kernel and its inverse read:

$$
\hat{K}_{ev}(\omega) := (\hat{K}_0 + \hat{K}_{-1})(\omega) = \frac{2\pi \sinh(\pi/2 - 2\gamma)\omega}{\sinh \pi \omega/2},
$$

(3.13)

$$
1 + \hat{J}_{ev}(\omega) := \frac{2\pi}{2\pi - \hat{K}_{ev}(\omega)} = \frac{\sinh \pi \omega/2}{2\sinh \gamma \omega \cosh(\pi/2 - \gamma)\omega}.
$$

(3.14)

Summing (3.30) over $a$, convolving with $(1 + J_{ev})$, and then integrating with respect to $\lambda$, we get

$$
Z_{ev}(\lambda) = 2L\sigma(\lambda) + 2C - \sum_{b,j} v_{bj} H_{ev}(\lambda - \eta_{bj}) \\
- \frac{1}{2i\pi} \sum_b \left[ \int_{\Gamma_b} d\mu H_{ev}(\lambda - \mu) U'_b(\mu) - \int_{\Gamma_b} d\mu H_{ev}(\lambda - \mu) U'_b(\mu) \right],
$$

(3.15)

where $Z_{ev} := Z_0 + Z_1$, the bulk source term is

$$
\sigma(\lambda) := [(1 + J_{ev}) \ast p](\lambda) = 2\arctan \left[ \frac{\pi \lambda}{2(\pi - 2\gamma)} \right],
$$

(3.16)

and the integrated kernel is the odd function

$$
H_{ev}(\lambda) := 2\pi \int_0^\lambda d\mu J_{ev}(\mu).
$$

(3.17)
The Fourier transform of the singular kernel is determined by the pole at \( \omega \). The principal value can be computed by averaging the integrals over \( \omega \). Using a similar argument for \( Z_0(-\infty) \), we get

\[
\text{Im} U_0(\pm \infty) = \pm (\gamma m - \pi N_B^o) + \varphi - \pi \epsilon,
\]

which, as it should, lies in the interval (3.7). Plugging these values in (3.15), we obtain

\[ C = 2\varphi. \]  

We now turn to the odd part of the NLIE (3.10). This involves the odd kernels:

\[
\hat{K}_{\text{odd}}(\omega) := (\hat{K}_0 - \hat{K}_1)(\omega) = \frac{2\pi \cosh(\pi/2 - 2\gamma)\omega}{\cosh \pi \omega/2},
\]

\[
1 + \hat{I}_{\text{odd}}(\omega) := \frac{2\pi}{2\pi - \hat{K}_{\text{odd}}(\omega)} = \frac{\cosh \pi \omega/2}{2 \sinh \gamma \omega \sinh(\pi/2 - \gamma)\omega}.
\]

Its treatment will be very different, due to the singularity of \( \hat{I}_{\text{odd}} \) at \( \omega = 0 \). Fourier transforming, we can write:

\[
\hat{Z}_{\text{odd}}(\omega) = \hat{I}_{\text{odd}}(\omega) \times \left\{-2\pi \sum_{b,j} (-1)^b v_{b,j} e^{i\omega b}\right\}
\]

\[
-\frac{1}{i} \sum_{b} (-1)^b \left[ \int_{\Gamma_b} d\mu \ e^{i\mu} U'_b(\mu) - \int_{\Gamma_b} d\mu \ e^{-i\mu} U'_b(\mu) \right],
\]

where \( Z_{\text{odd}} := Z_0 - Z_1 \). In the limit \( \omega \to 0 \), we have

\[
\hat{Z}_{\text{odd}}(0) = Z_{\text{odd}}(+\infty) - Z_{\text{odd}}(-\infty) = -2\pi \tilde{m}, \quad \hat{I}_{\text{odd}}(\omega) \sim \frac{\omega^{-2}}{\alpha(\pi - 2\gamma)}. \]

Taking this limit in (3.22) yields the consistency conditions for \( n = 0, 1, 2 \):

\[
\sum_{b,j} (-1)^b v_{b,j} \eta_{b,j} + \frac{1}{i} \sum_{b} (-1)^b \int_{\Gamma_b} d\mu \ \text{Im}[\mu^n U'_b(\mu)] = \begin{cases} 0 & n = 0, 1 \\ -\pi \tilde{m}/\alpha & n = 2, \end{cases}
\]

where we have defined

\[ \alpha := \frac{\pi}{2\gamma(\pi - 2\gamma)}. \]

These can be gathered into a single equation for any \( \lambda \):

\[
\sum_{b,j} (-1)^b v_{b,j} (\lambda - \eta_{b,j})^2 + \frac{1}{2\pi} \sum_{b} (-1)^b \left[ \int_{\Gamma_b} d\mu \ (\lambda - \mu)^2 U'_b(\mu) - \int_{\Gamma_b} d\mu \ (\lambda - \mu)^2 U'_b(\mu) \right]
\]

\[ = -\frac{\pi \tilde{m}}{\alpha}. \]

The Fourier transform of the singular kernel \( \hat{I}_{\text{odd}} \) can be defined in various ways. If we use the principal value prescription:

\[
J_{\text{odd}}(\lambda) := \int \frac{d\omega}{2\pi} e^{-i\omega \hat{I}_{\text{odd}}(\omega)}, \quad H_{\text{odd}}(\lambda) := 2\pi \int_0^\lambda d\mu \ J_{\text{odd}}(\mu),
\]

then \( J_{\text{odd}} \) is even and \( H_{\text{odd}} \) is odd. Their asymptotic for large \( \lambda \), up to exponentially small terms, is determined by the pole at \( \omega = 0 \)

\[
J_{\text{odd}}(\lambda) \sim \mp \alpha \lambda /\pi, \quad H_{\text{odd}}(\lambda) \sim \mp (\alpha \lambda^2 - \beta), \quad \lambda \to \pm \infty.
\]

The principal value can be computed by averaging the integrals over \( \mathbb{R} \pm i\epsilon \).
where, although irrelevant in the following, \( \beta = (\pi^2 \alpha - 5\pi)/6 \). After inverse Fourier transforming and integrating equation (3.22), we get

\[
Z_{\text{odd}}(\lambda) = 2\tilde{C} - \sum_{b,j} (-1)^b v_{bj} H_{\text{odd}}(\lambda - \eta_{bj})
\]

\[
- \frac{1}{2\pi} \sum_{b} (-1)^b \left[ \int_{\Gamma_b} d\mu \ H_{\text{odd}}(\lambda - \mu) U'_{\mu}(\mu) - \int_{\Gamma_b} d\mu \ H_{\text{odd}}(\lambda - \mu) \overline{U}'_{\mu}(\mu) \right],
\]

where \( \tilde{C} \) is an integration constant. The consistency condition (3.26) actually ensures that the solution \( Z_{\text{odd}}(\lambda) \) has finite limits at \( \pm\infty \). Taking \( \lambda \to \pm\infty \) and using (3.26), we find that \( \tilde{C} = 0 \).

Finally, we can recombine (3.15) and (3.22). Introducing

\[
J_{a-b} := \frac{1}{4}[V_{\text{ev}} + (-1)^{a-b} H_{\text{odd}}], \quad H_{a-b} := \frac{1}{4}[H_{\text{ev}} + (-1)^{a-b} H_{\text{odd}}],
\]

and recalling the BAE for holes, we have the following system of equations:

\[
Z_a(\lambda) = L \sigma(\lambda) + 2\phi - \sum_{b,j} v_{bj} H_{a-b}(\lambda - \eta_{bj})
\]

\[
- \frac{1}{2\pi} \sum_{b} \left[ \int_{\Gamma_b} d\mu \ H_{a-b}(\lambda - \mu) U'_{\mu}(\mu) - \int_{\Gamma_b} d\mu \ H_{a-b}(\lambda - \mu) \overline{U}'_{\mu}(\mu) \right],
\]

\[
Z_a(\eta_{aj}) = 2\pi I_{a,aj}.
\]

The NLIE (3.30) are exact for finite \( L \) and they hold for \( |\text{Im}\lambda| < \min(2\gamma, \pi/2 - \gamma) \). It is important to realize that, at this stage, integration by parts in the first equation of (3.30) is not possible, because the kernels \( H_{a-b}(\lambda) \) diverge at \( \pm\infty \); see (3.28).

### 3.3. Equations for conserved charges

In this section, shall express conserved quantities which are defined as sums over roots

\[
V_{\text{ev}} := \sum_{a,j} v(\lambda_{aj}) \quad \text{and} \quad W_{\text{odd}} := \sum_{a,j} (-1)^a w(\lambda_{aj}),
\]

where \( v(\lambda) \) and \( w(\lambda) \) are smooth functions, in terms of \( Z_a \) and sums over holes. Using equation (3.8) and the NLIE (3.30), we get

\[
V_{\text{ev}} = 2L v_\infty + \sum_{b,\ell} v_{b\ell} v_h(\eta_{b\ell}) + \frac{1}{2\pi} \sum_{b} \left[ \int_{\Gamma_b} d\lambda \ v_h(\lambda) U'_{\lambda}(\lambda) - \int_{\Gamma_b} d\lambda \ v_h(\lambda) U'_h(\lambda) \right],
\]

where

\[
v_\infty := \int \frac{d\lambda}{2\pi} \sigma'(\lambda) v(\lambda), \quad v_h := -(1 + J_{\text{ev}}) \star v.
\]

The first term in (3.31) is the bulk value, and the next terms give the finite-size corrections to \( V_{\text{ev}} \). Since the energy and momentum of a Bethe root are given by

\[
p(\lambda) = \phi_{\gamma/2-\gamma}(\lambda), \quad \epsilon(\lambda) = -\sin2\gamma \phi'_{\gamma/2-\gamma}(\lambda),
\]

the total energy and momentum read

\[
P = 2\phi + \sum_{b,\ell} v_{b\ell} p_h(\eta_{b\ell}) + \frac{1}{2\pi} \sum_{b} \int_{\Gamma_b} d\lambda \ \text{Im}[p_h(\lambda) U'_h(\lambda)],
\]

(3.33)
\[ E = 2Le_{\infty} + \sum_{b,\ell} v_{\ell b} \epsilon_b(\eta_{b\ell}) + \frac{1}{\pi} \sum_b \int_{\Gamma_b} \mathrm{d}\lambda \, \text{Im} \epsilon_b(\lambda) U'_b(\lambda), \]  
where the energy and momentum of a hole (in the density approximation) are 
\[ p_h(\lambda) = -\sigma(\lambda), \quad \epsilon_h(\lambda) = v_F / \cosh \frac{\pi \lambda}{\pi - 2\gamma}, \]  
and we have introduced the ‘Fermi velocity’ \( v_F := \pi \sin 2\gamma / (\pi - 2\gamma). \) Note that we have the exact relation \( \epsilon_h = v_F \cos p_h \) and hence, at \( \lambda \rightarrow \pm \infty, \) since \( p_h \rightarrow \mp \pi / 2, \) the holes have a linear dispersion relation.

For the odd conserved quantities, we can derive an expression similar to (3.31) 
\[ W_{\text{odd}} = \sum_{b,\ell} (-1)^h v_{\ell b} w_b(\eta_{b\ell}) + \frac{1}{2i\pi} \sum_b (-1)^h \left[ \int_{\Gamma_b} \mathrm{d}\lambda \, w_b(\lambda) U'_b(\lambda) - \int_{\Gamma_b} \mathrm{d}\lambda \, w_b(\lambda) \overline{U}'_b(\lambda) \right], \]  
where \( w_b := -(1 + J_{\text{odd}}) \star w. \) Hence, we can write the quasi-momentum (2.13) as 
\[ K = \sum_{b,\ell} (-1)^h v_{\ell b} k_b(\eta_{b\ell}) + \frac{1}{\pi} \sum_b (-1)^h \int_{\Gamma_b} \mathrm{d}\lambda \, \text{Im} \epsilon_b(\lambda) U'_b(\lambda), \]  
where the quasi-momentum of a hole, computed in the appendix, reads 
\[ k_b(\lambda) = \log \left( 2 \cosh \frac{\pi \lambda}{\pi - 2\gamma} \right). \]  

### 3.4. Higher-order conserved charges

At the value of the spectral parameter \( u = \gamma / 2 - \pi / 4 =: u_0, \) the vertex model is invariant under a \( \pi / 2 \) rotation: this is usually referred to as the ‘isotropic value’ of \( u. \) From (2.7), we can write 
\[ \Lambda(u) = e^{i\varphi} \prod_{a,j} e^{-i\phi_{aj}/2} \left[ \frac{1}{2} (\lambda_{aj} + i\pi a - i\pi / 2 + 2iu) \right] \times \left[ 1 + (-1)^{e_0} e^{-iz_0(2iu_0 - 2iu)} \right], \]  
\[ \Lambda(u + \pi / 2) = e^{i\varphi} \prod_{a,j} e^{-i\phi_{aj}/2} \left[ \frac{1}{2} (\lambda_{aj} + i\pi a + i\pi / 2 + 2iu) \right] \times \left[ 1 + (-1)^{e_0} e^{-iz_0(2iu_0 - 2iu)} \right]. \]  

Taking the logarithm we get 
\[ \log \Lambda(u) = i\varphi - i \sum_{a,j} \phi_{aj} / 2 \left[ \frac{1}{2} (\lambda_{aj} + i\pi a - i\pi / 2 + 2iu) \right] + U_0(2iu_0 - 2iu), \]  
\[ \log \Lambda(u + \pi / 2) = i\varphi - i \sum_{a,j} \phi_{aj} / 2 \left[ \frac{1}{2} (\lambda_{aj} + i\pi a + i\pi / 2 + 2iu) \right] + \overline{U}_1(2iu_0 - 2iu). \]  

For the even (free energy) and odd combinations: 
\[ F(u) := -\log[\Lambda(u)\Lambda(u + \pi / 2)], \quad G(u) := \log[\Lambda(u + \pi / 2)/\Lambda(u)], \]  
one arrives with the help of equation (A.3) at 
\[ F(u) = -2i\varphi - i \sum_{a,j} \phi_{aj} / 2 - (\overline{U}_0 + \overline{U}_1)(2iu_0 - 2iu), \]  
\[ G(u) = \sum_{a,j} (-1)^e k(\lambda_{aj} + 2iu) + (\overline{U}_0 - \overline{U}_1)(2iu_0 - 2iu). \]  

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If we now restrict to\(^6\)
\[
\pi/6 < \gamma < \pi/2, \quad u_0 < \text{Re} u < 0,
\]
then the summands in equation (3.42) are analytic functions and \(\gamma - \pi/2 < \text{Im} \, 2i(u_0 - u) < 0\). Hence, we can use equations (3.31) and (3.36) to express \(F(u)\) and \(G(u)\) in terms of the solution of the NLIE (3.30) as follows:
\[
\begin{align*}
F(u) &= -2i\varphi + 2L f_\infty (u) - i \sum_{b, \ell} v_{bd} \, p_b(2i\upsilon_{bd} + 2iu) - (U_0 + U_1)(2iu_0 - 2iu) \\
&\quad - \frac{1}{2\pi} \sum_b \left[ \int_{\Gamma_1} d\lambda \, p_b(\lambda + 2iu)U'_b(\lambda) - \int_{\Gamma_b} d\lambda \, p_b(\lambda + 2iu)U'_b(\lambda) \right], \\
G(u) &= \sum_{b, \ell} (-1)^b v_{bd} \, k_b(\eta_{bd} + 2iu) + (U_0 - U_1)(2iu_0 - 2iu) \\
&\quad + \frac{1}{2\pi} \sum_b (-1)^{b} \left[ \int_{\Gamma_b} d\lambda \, k_b(\lambda + 2iu)U'_b(\lambda) - \int_{\Gamma_b} d\lambda \, k_b(\lambda + 2iu)U'_b(\lambda) \right],
\end{align*}
\]  
(3.44)
(3.45)
where \(f_\infty (u) := -i(\sigma^r \cdot p)(2iu)\) is the bulk free energy density. Unless there are complex solutions of the BAE in the strip (3.43), the functions \(F(u)\) and \(G(u)\) will be analytic there. From \(F(0), F'(0)\) and \(G(0)\), one recovers \(P, E\) and \(K\). Higher derivatives of \(F\) and \(G\) give the eigenvalues of all the (local) conserved charges which commute with \(H\).

4. Scaling limit of the NLIE

4.1. Definitions

We shall now focus on the excitations concentrated at the boundaries of the Fermi sphere of roots where the dispersion relation for the holes linearizes and the gap closes as \(O(1/L)\) in the \(L \to \infty\) limit. This is the ‘conformal regime’ described by a CFT.\(^7\) The latter was identified with the SL(2, \(\mathbb{R}\))/U(1) black hole sigma model; \(O(1/L)\) corrections to the energy (3.34) then give the central charge and spectrum of this CFT [24]. Before we are able to reproduce these corrections we need to take the scaling limit of the NLIE (3.30).

We define the scaling limit as
\[
\begin{align*}
L &\to \infty, \quad (m, e, K) \text{ fixed.}
\end{align*}
\]
(4.1)
Numerical inspection shows that \(K \propto \tilde{m}/\log L\) for large \(L\). Hence, the scaling limit requires taking \(\tilde{m} \to \infty\). We shall confirm this behaviour in section 4.3.

Like in the XXZ case [6, 7], from the asymptotic behaviour
\[
\sigma(\lambda) \sim \pm[\pi/2 - 2e(\pm\lambda)], \quad e(\lambda) := \exp\left( -\frac{\pi \lambda}{\pi - 2\gamma} \right)
\]
(4.2)
we find that the \(\lambda\) dependent part of the source term \(L\sigma(\lambda)\) of (3.30) will be of order one if \(\lambda\) remains near the ‘Fermi levels’ \(\pm \Lambda\) with
\[
\Lambda := \frac{\pi - 2\gamma}{\pi} \log L.
\]
(4.3)
In the limit (4.1), the Bethe roots and holes arrange as follows (see figure 7). For \(|\lambda| \ll \Lambda\), the spacing between roots is of order \(1/L\), and the corresponding contributions to conserved

\(^{6}\) The values \(0 < \gamma < \pi/6\) can be treated in a similar way by changing variable \(u \to \gamma - u\).

\(^{7}\) Instead, one could engineer a gap in the dispersion relation of the holes with \(|\lambda| < \Theta\) by adding an additional \textit{imaginary} staggering \(2i\Theta\) to the spectral parameters in figure 2, see [27, 28]. This is the ‘massive regime’ described by a massive integrable quantum field theory.
quantities are contained in the bulk term $2L v_\infty$ of (3.31). Actually, this regime is well described by the linear approximation and the Wiener–Hopf approach of Yang and Yang [32]. For $|\lambda| \approx \Lambda$, the roots are of the form $\lambda_{aj} = \pm (\Lambda + \lambda_{aj}^\pm)$, where $\lambda_{aj}^\pm$ remains finite as $L$ becomes large, and similarly for holes. This is the scaling regime which determines the finite-size corrections (shaded regions in figure 7). Also, as we shall see in the subsequent calculations, the NLIE for the positive and negatives holes (right and left movers) decouple in the scaling limit.

Let us then define the shifted counting functions:

$$ Z_\pm^a(\lambda) := \pm Z_a(\pm (\Lambda + \lambda)) - \pi r_a. \quad (4.4) $$

Notice that they have a finite value at $\lambda \to +\infty$ in the limit (4.1):

$$ Z_a^\pm(+\infty) = 2\gamma_m \pm 2\varphi. \quad (4.5) $$

Correspondingly, we write

$$ U_\pm^a(\lambda) := U_a(\lambda + \lambda) = \log[1 + e^{iZ_a^\pm(\lambda)}], $$

$$ U_\pm^a(\lambda) := \overline{U_a}(\lambda - \lambda) = \log[1 + e^{iZ_a^\pm(\lambda)}], \quad (4.6) $$

and define $\overline{U}_\pm^a(\lambda)^* := U_\pm^a(\lambda)^*$, where the operation $(\cdot)^*$ denotes complex conjugation. The scaled integration paths $\Omega_\pm^a = \Omega_a$ compatible with equation (4.6) are shown in figure 8. Their complex conjugate is denoted by $\overline{\Omega}_a$. From equation (3.18) we get the boundary conditions

$$ \text{Im} \ U_\pm^a(+\infty) = \gamma m - \pi N_\pm^a \pm (\varphi - \pi e), \quad (4.7) $$

while from the behaviour of the source term $L_o(\lambda)$ at $L \to \infty$ and $\lambda$ of order one we expect $\text{Im} \ Z_a^\pm(i\delta - \infty) = +\infty$, and thus:

$$ U_\pm^a(i\delta - \infty) = 0. \quad (4.8) $$

The shifted Bethe holes and the associated signs are defined as

$$ \{\eta_{aj}\} = \{\Lambda + \eta_{aj}^+\} \cup \{-\Lambda - \eta_{aj}^-\}, \quad \nu_{aj}^\pm = \text{sign} \ Z_a^\pm(\eta_{aj}^\pm), \quad (4.9) $$
and the shifted Bethe integers for roots are
\[ I_{ij}^+ := \pm I_{ij} - r_i / 2 \in \pm 1/2 - \mathbb{N}. \] (4.10)
The Bethe integers for holes take values in the complementary interval in \( \mathbb{Z} + 1/2. \)

### 4.2. Nonlinear integral equations

For the sake of clarity, we first derive the NLIE for the \( Z_+ \), and state the analogous results for the \( Z_- \). Furthermore, like in section 3.2, we separate the discussion of the even and odd (under the exchange of indices \( a = 0 \leftrightarrow 1 \)) parts of the NLIE.

In the even NLIE (3.15), we perform the change of variables \( \lambda \rightarrow \Lambda + \lambda \), and then integrate by parts. In the scaling limit (4.1), the negative holes and the negative part of the integral contribute an additive constant, and we obtain:

\[ Z_+^0(\lambda) = -4e(\lambda) + 2C^+ - \sum_{b, \ell} \nu_b^\ell H_{\ell}(\lambda - \eta_b^\ell) \]
\[ - \frac{1}{4} \sum_b \left[ \int_{\Omega_b} d\mu J_{\ell}(\lambda - \mu) U_b^+(\mu) - \int_{\Pi_b} d\mu J_{\ell}(\lambda - \mu) U_b^+(\mu) \right], \] (4.11)

where, taking into account equations (4.7) and (4.8), we get

\[ C^+ = \frac{\pi}{2} \left( m + \frac{\psi}{\gamma} \right) + \pi \left( 1 - \frac{\pi}{4\gamma} \right) (N_h^+ + 2e). \] (4.12)

Before we deal with the odd part it is useful to define the modified kernels

\[ \tilde{H}_{\text{odd}}(\lambda) := H_{\text{odd}}(\lambda) - (\alpha\lambda^2 - \beta), \quad \tilde{H}_{\text{odd}}' = 2\pi \tilde{J}_{\text{odd}}, \] (4.13)

which decay exponentially at \( \lambda \rightarrow -\infty \). Now, notice that \( Z_{\text{odd}}^0(\lambda) = Z_{\text{odd}}(\Lambda + \lambda) + \pi \tilde{m} \) and write the consistency conditions (3.24)-(3.26) as

\[ \pi \tilde{m} = -\sum_{b, \ell} (-1)^b \nu_b^\ell [\alpha(\lambda - \eta_b^\ell)^2 - \beta] - \frac{1}{2\pi} \sum_{b} (-1)^b \left[ \int_{\Gamma_b} d\mu [\alpha(\lambda - \mu)^2 - \beta] U_b'(\mu) \right. \]
\[ \left. - \int_{\Pi_b} d\mu \left[ \alpha(\lambda - \mu)^2 - \beta \right] U_b'(\mu) \right], \]

Adding this equation to equation (3.29) we get:

\[ Z_{\text{odd}}(\lambda) = \sum_{b, \ell} (-1)^b \nu_b^\ell \tilde{H}_{\text{odd}}(-\lambda + \eta_b^\ell) + \frac{1}{2\pi} \sum_{b} (-1)^b \left[ \int_{\Gamma_b} d\mu \tilde{H}_{\text{odd}}(-\lambda + \mu) (U_b^+)' \right. \]
\[ \left. - \int_{\Pi_b} d\mu \tilde{H}_{\text{odd}}(-\lambda + \mu) (U_b^+)' \right], \] (4.14)

where, because the kernels \( -\tilde{H}_{\text{odd}}(-\lambda) = H_{\text{odd}}(\lambda) + (\alpha\lambda^2 - \beta) \) decay exponentially at \( \lambda \rightarrow +\infty \), the left movers have decoupled. The message conveyed by this equation is that \( Z_{\text{odd}}^+(\lambda) \) decays exponentially at \( \lambda \rightarrow +\infty \). In order to understand the asymptotic behaviour of \( Z_{\text{odd}}(\lambda) \) at \( \lambda \rightarrow -\infty \) we perform the replacement

\[ \tilde{H}_{\text{odd}}(-\lambda) = -\tilde{H}_{\text{odd}}(\lambda) - 2(\alpha\lambda^2 - \beta) \]
in equation (4.14), which, after integration by parts, brings it to the form

\[ Z_{\text{odd}}^+(\lambda) = 4A_+^+ \lambda - 2A_2^+ - \sum_{b, \ell} (-1)^b \nu_b^\ell \tilde{H}_{\text{odd}}(\lambda - \eta_b^\ell) \]
\[ - \frac{1}{4} \sum_b (-1)^b \left[ \int_{\Omega_b} d\mu \tilde{J}_{\text{odd}}(\lambda - \mu) U_b^+(\mu) - \int_{\Pi_b} d\mu \tilde{J}_{\text{odd}}(\lambda - \mu) U_b^+(\mu) \right]. \] (4.15)
Here $A^+_a$ are constants defined as
\[
A^+_a := \alpha \left\{ \sum_{b,t} (-1)^b v^+_{bt} (\eta_{bt}^+)^p + \frac{1}{\pi} \sum_b (-1)^b \int_{\Omega_b} d\mu \ \text{Im} [\mu^a (U_b^+)'] (\mu) \right\}. \tag{4.16}
\]

The first two terms on the rhs of equation (4.15) give the asymptotic behaviour of $Z^+_a(\lambda)$ at $\lambda \to -\infty$. Notice that the integration by parts was possible because $\tilde{H}_{\text{odd}}(\lambda) U^+_a(\lambda)$ decays (exponentially) at $\lambda \to \pm \infty$.

We can now recombine the NLIE (4.11) and (4.15), and, using the kernels
\[
\tilde{J}_{a-b} := \frac{1}{2} [U_{ev} + (-1)^{a-b} \tilde{H}_{\text{odd}}] , \quad \tilde{H}_{a-b} := \frac{1}{2} [H_{ev} + (-1)^{a-b} \tilde{H}_{\text{odd}}].
\]
cast the result in its final form\(^8\).

\[
Z^+_a(\lambda) = 2e(\lambda) + C^+ + (-1)^a (2A^+_a - A^+_a) - \sum_{b,t} v^+_{bt} \tilde{H}_{a-b}(\lambda - \eta_{bt}^+) - \frac{1}{2} \sum_{b} \left[ \int_{\Omega_b} d\mu \ \tilde{J}_{a-b}(\lambda - \mu) U^+_b(\mu) - \int_{\Omega_b} d\mu \ \tilde{J}_{a-b}(\lambda - \mu) \bar{U}^+_b(\mu) \right],
\]

\[
C^+ = \frac{\pi}{2} \left( m + \frac{\varphi}{\gamma} \right) + \pi \left( 1 - \frac{\pi}{4\gamma} \right) (N^+_h + 2\epsilon),
\]

\[
A^+_a = \alpha \left\{ \sum_{b,t} (-1)^b v^+_{bt} (\eta_{bt}^+)^p + \frac{1}{\pi} \sum_b (-1)^b \int_{\Omega_b} d\mu \ \text{Im} [\mu^a (U_b^+)'] (\mu) \right\},
\]

\[
Z^+_a(\eta^+_{a}) = 2 \pi \Lambda^+_h. \tag{4.17}
\]

As it is apparent in (4.17), the most notable effect of the pole at $\omega = 0$ of $\tilde{H}_{\text{odd}}$, see equation (3.21), is to create an additional source term proportional to $\lambda$ in the NLIE. Moreover, for one value of $a$, the source term $-2e(\lambda) + 2\Lambda^+_a |\lambda|$ is increasing, whereas for the other value, the source term $-2e(\lambda) - 2\Lambda^+_a |\lambda|$ has one local maximum: this is reminiscent of the behaviour of $Z_0$ and $Z_1$ observed numerically for finite system sizes (see figures 4 and 5).

Finally, the minus version of all equations in this section, i.e. for the left movers, can be obtained by replacing all $+$-superscripts with $-$-superscripts and flipping the sign of $e$ and $\varphi$.

4.3. Constraints and source terms

Notice that for $L$ large and $\lambda$ of order one, the increasing bulk term $L \sigma(\lambda)$ dominates on the rhs of $Z_a(\lambda)$ in equation (3.30). Hence, in this regime $U_a$ and $\bar{U}_a$ get exponentially suppressed with the system size $L$
\[
|U_a(\lambda + i\delta) = |\bar{U}_a(\lambda - i\delta)| \approx e^{-L \Lambda(\lambda)} \ll 1 , \quad \lambda \in \mathbb{R}. \tag{4.18}
\]

These functions will be of order one precisely in the scaling regime, i.e. for $|\lambda|$ of order $\Lambda$. Hence, the following approximation is exact up to exponentially small terms in $L$
\[
\int_{\Gamma_a} d\mu \ f(\mu) U_a'(\mu) - \int_{\Gamma_a} d\mu \ f(\mu) \bar{U}_a'(\mu) \approx \int_{\Omega_a} d\mu [f^+(\mu)(U_a^+)'(\mu) + f^-(\mu)(\bar{U}_a^+)'(\mu)]
\]

\[
- \int_{\Omega_a} d\mu [f^+(\mu)(\bar{U}_a^+)'(\mu) + f^-(\mu)(U_a^+)')(\mu)], \tag{4.19}
\]

where $f^\pm(\mu) := f(\pm \Lambda \pm \mu)$, provided the integral on the lhs exists.

\(^8\) Similar NLIEs appear in the unpublished notes of Lukyanov on the ordinary differential equation/integrable model (ODE/IM) correspondence for the complex SG model [31].
Let us now consider the scaling limit of the three consistency conditions (3.24). The \( n = 0 \) constraint gives, after taking into account the boundary conditions (4.7, 4.8), the trivial relation \( \sum_{a} (-1)^{a}(N_{h,a}^{+} + N_{h,a}^{-}) = 0 \), which holds because \( N_{h,a}^{\pm} = N_{h}^{\pm} \), see section 3.1. The \( n = 1 \) constraint yields, after applying equation (4.19), the non-trivial identity \( A_{1}^{+} = A_{1}^{-} \). Also, using the asymptotic behaviour

\[
\kappa_{h}(\lambda) \sim \pm \pi \lambda / (\pi - 2\gamma), \quad \lambda \rightarrow \pm \infty
\]

(4.20)

we get from (3.37):

\[
K = 2\gamma (A_{1}^{+} + A_{1}^{-}) = 4\gamma A_{1}^{\pm}.
\]

(4.21)

Moreover, after substituting \( \eta_{b} \rightarrow \pm (\Lambda + \eta_{b}^{\pm}) \) in the consistency condition (3.24) for \( n = 2 \), applying (4.19) and expanding in powers of \( \Lambda \), we obtain the relation

\[
2\Lambda (A_{2}^{+} + A_{2}^{-}) = (A_{2}^{+} + A_{2}^{-}) = -\pi \tilde{m}.
\]

Inserting (4.21) and using the definition of \( \Lambda \), we get

\[
\tilde{m} = -\frac{\pi - 2\gamma}{\pi^{2}\gamma} K \log L - \frac{1}{\pi} (A_{1}^{+} + A_{1}^{-}).
\]

(4.22)

This equation gives the precise scaling of the integer \( \tilde{m} \) as \( L \rightarrow \infty \) with \( K \) fixed. Moreover, as it was previously argued in [24], the constant term \( (A_{2}^{+} + A_{2}^{-}) \) is related to the finite part of the density of states in the energy spectrum: see section 6.1.

Let us comment on the nature of the source terms proportional to \((-1)^{n}\) in the NLIE (4.17). The first term equals \( K\lambda / 2\gamma \). Recall that, in our setting, \( K \) is a fixed parameter of the problem, exactly like \( m, e \) and \( \varphi \), and so this term should be viewed as an input data of the NLIE. In contrast, the value of the second term \(-A_{2}^{\pm}\) is fixed by the boundary condition (4.5) at \( \lambda \rightarrow +\infty \). Our approach does not give an explicit expression of \( A_{2}^{\pm} \) in terms of \( (m, e, K) \), but rather in terms of the solution of the NLIE.

5. Energy spectrum and higher spin charges

5.1. Exact energy spectrum

Using the asymptotic behaviour

\[
\epsilon_{\nu}(\lambda) \sim 2\nu\epsilon_{\nu}(\pm \lambda), \quad \lambda \rightarrow \pm \infty
\]

(5.1)

and equation (4.19), the total energy (3.34) takes the form \( E = 2Le_{\infty} + E^{+} + E^{-} \), with

\[
E^{\pm} = \frac{2\nu\epsilon_{\nu}}{L} \left\{ \sum_{b,j} \nu_{b}^{\pm} e(\eta_{b}^{\pm}) - \frac{1}{\pi} \sum_{b} \int_{\Omega_{b}} d\lambda \cdot \text{Im}[e'(\lambda)U_{b}^{\pm}(\lambda)] \right\}.
\]

(5.2)

For the sake of clarity, we shall first explain the calculation of \( E^{+} \) and in the end list the modification required to compute \( E^{-} \).

First, we shall restrict the NLIE (4.17) to the real axis. Let us define the functions:

\[
Q_{\nu}^{+}(\text{Re}\lambda) := \lim_{\delta \rightarrow 0} 2\text{Im} U_{\nu}^{+}(\lambda), \quad \lambda \in \Omega_{\nu},
\]

(5.3)

where the dependence on \( \delta \) is contained in \( \Omega_{\nu} \), and the even and odd combinations:

\[
Q_{\nu}^{+} := Q_{\nu}^{e} + Q_{\nu}^{o}, \quad Q_{\nu}^{o} := Q_{\nu}^{e} - Q_{\nu}^{o}.
\]

Then the NLIE equations (4.11) and (4.15) take the compact form

\[
Z_{\nu}^{+}(\lambda) = -4\epsilon(\lambda) + 2C^{+} - \sum_{b,l} \nu_{b}^{\nu} H_{\nu}(\lambda - \eta_{b}^{\nu}) - J_{\nu} \bullet Q_{\nu}^{+}(\lambda),
\]

\[
Z_{\nu}^{+}(\lambda) = 4A_{1}^{+} \lambda - 2A_{2}^{+} - \sum_{b,l} (-1)^{b} \nu_{b}^{\nu} H_{\nu}(\lambda - \eta_{b}^{\nu}) - \tilde{J}_{\nu} \bullet Q_{\nu}^{+}(\lambda).
\]
Notice that the derivative \((Q_+^+)')\) is not well defined, and so it was important to integrate by parts before taking the limit. In the following it will be convenient to work with a more symmetric form of the above equations

\[
Z_c^+(\lambda) = -4e(\lambda) + 2C^+ - \sum_{b,i} v_{b,i}^+ H_{ci}(\lambda - \eta_{bi}^+) - J_{ci} \ast Q_{ci}^+(\lambda),
\]

\[
Z_{odd}^+(\lambda) = 2A_1^+ \lambda - A_1^+ - \sum_{b,i} (-1)^i v_{b,i}^+ H_{odi}(\lambda - \eta_{bi}^+) - J_{odi} \ast Q_{odi}^+(\lambda),
\]

which is well defined because \(Q_{odi}^+\) decays exponentially at \(+\infty\) (see equation (4.7)), and where

\[
A_n^+ = \alpha \left[ \sum_{b,i} (-1)^i v_{b,i}^+ - \frac{1}{2\pi} \int d\mu \eta_{n}^{\mu - 1} Q_{odi}^+(\mu) \right].
\]

Next, we manipulate in the standard way the NLIE equation (5.4) in order to compute

\[
E^+ = \frac{2\pi J}{L} \left[ \sum_{b,i} v_{b,i}^+ e(\eta_{bi}^+) - \frac{1}{2\pi} \int d\lambda e'(\lambda) Q_{odi}^+(\lambda) \right].
\]

First step is to use the last equation in (4.17)

\[
2\pi I_n^+ := 2\pi \sum_{a,j} v_{a,j}^+ f_{b,a}^+ = \sum_{a,j} v_{a,j}^+ Z_{odi}^+(\eta_{aj}^+),
\]

in order to get after some simple algebra

\[
2\pi I_n^+ = -2 \sum_{b,i} v_{b,i}^+ e(\eta_{bi}^+) + 2N_+^+ C^+ + A_1^+ \sum_{a,j} (-1)^a v_{a,j}^+ \eta_{aj}^+
\]

\[
- \frac{1}{2} \sum_{a,j} v_{a,j}^+ \int d\lambda [J_{ci}(\lambda - \eta_{aj}^+) Q_{odi}^+(\lambda) + (-1)^a J_{odi}(\lambda - \eta_{aj}^+) Q_{odi}^+(\lambda)].
\]

If we now use this expression to compute the sum over \(v_{b,i}^+ e(\eta_{bi}^+)\) and inject the result into (5.5) then we get

\[
\frac{LE^+}{v_F} = -2\pi I_n^+ + 2N_+^+ C^+ + \frac{(A_1^+)^2}{\alpha} + \int \frac{d\lambda}{4\pi} Q_{odi}^+(\lambda) \left[ -4e'(\lambda) - \sum_{a,j} v_{a,j}^+ H_{ci}(\lambda - \eta_{aj}^+) \right]
\]

\[
+ \int \frac{d\lambda}{4\pi} Q_{odi}^+(\lambda) \left[ 2A_1^+ - \sum_{a,j} (-1)^a v_{a,j}^+ H_{odi}(\lambda - \eta_{aj}^+) \right].
\]

In the integrals, the factors in the brackets are the derivatives of the source terms of the NLIE (5.4). Hence, we can write:

\[
\frac{LE^+}{v_F} = -2\pi I_n^+ + 2N_+^+ C^+ + \frac{(A_1^+)^2}{\alpha} + \sum a \int \frac{d\lambda}{2\pi} (Z_n^+)'(\lambda) Q_{odi}^+(\lambda)
\]

\[
+ \sum a \int \int \frac{d\lambda \alpha}{4\pi} [Q_{odi}^+(\lambda) J_{odi}(\lambda - \mu) Q_{odi}^+(\mu) + Q_{odi}^+(\lambda) J_{odi}(\lambda - \mu) Q_{odi}^+(\mu)].\]

After a change of variables \(u = \exp(iz_\alpha^+),\) we obtain

\[
\int \frac{d\lambda}{2\pi} (Z_n^+)'(\lambda) Q_{odi}^+(\lambda) = -\frac{1}{\pi} \Re \left[ \int \frac{du}{z_\alpha^+} \frac{\log(1 + u)}{u} \right].
\]
where \( \gamma_0^+ \) is any path enclosed in the unit disk, going from 0 to \( \exp[iZ_0^+(\infty)] = \exp[iQ_0^+(\infty)] \).

We choose the particular form of the integration path:

\[
\int_{\gamma_1^+} du \log(1 + u) = \int_0^1 du \log(1 + u) + \int_0^{Q_1^+(\infty)} d(e^{i\theta}) \log(1 + e^{i\theta}).
\]

Using formulas (A.8) and (A.9), we get

\[
\int \frac{d\lambda}{2\pi} (Z_1^+)^{(\lambda,0)}(\lambda) \cdot Q^+_c(\lambda) = -\frac{\pi}{12} + \frac{|Q_1^+(\infty)|^2}{4\pi}.
\]

In the double integral of (5.7) only the even part contributes, because \( J_{\text{odd}} \) is odd and \( Q_{\text{odd}}^+ \) decays exponentially at \( \pm \infty \). Applying the ‘lemma 1’ of [9] we then get:

\[
\int \int \frac{d\lambda}{4\pi} Q^+_c(\lambda)J^+_c(\lambda - \mu)Q^{-}_c(\mu) = \frac{\pi}{32\gamma} [Q^+_c(\infty)]^2.
\]

Gathering the terms from (5.8) and (5.9), we obtain

\[
\frac{\text{LE}^+}{\nu_F} = -\frac{2\pi I^+_h}{6} + 2N^+_h C^+ + \left(\frac{A^+_h}{\alpha}\right)^2 - \frac{\pi}{6} + \frac{|Q^+_c(\infty)|^2}{32\gamma}.
\]

Finally, from equation (4.7) we get

\[
Q^+_c(\infty) = 4(y m - \varphi - 4\pi (e + N^+_h)),
\]

from equation (4.10) we can compute \( I^+_h = N^+_h (N^+_h + 2e) \), and from equation (3.37) we have \( K = A^+_h/4\gamma \). Combining everything together \( N^+_h \) drops out of equation (5.10) and we get:

\[
\frac{\text{LE}^+}{\nu_F} = -\frac{\pi}{6} + \frac{\varphi - 2\gamma}{8\pi \gamma} K^2 + \frac{1}{2\gamma} (y m - \pi e + \varphi)^2.
\]

The calculation of \( E^- \) is exactly analogous with the only difference being that all + - superscripts get replaced by − - superscripts and \( e, \varphi \) flip signs

\[
\frac{\text{LE}^-}{\nu_F} = -\frac{\pi}{6} + \frac{\varphi - 2\gamma}{8\pi \gamma} K^2 + \frac{1}{2\gamma} (y m + \pi e - \varphi)^2.
\]

5.2. Identification to the \( \text{SL}(2,\mathbb{R})/\text{U}(1) \) sigma model

From conformal invariance, we expect the following form for the scaling corrections to the energy and momentum:

\[
E^+ + E^- = 2\pi \nu_F \left( h + \tilde{h} - \frac{c}{12} \right), \quad P = \frac{2\pi}{L} (h - \tilde{h}) \quad \text{mod} \ 2\pi,
\]

where \( c \) is the central charge of the CFT and \( h, \tilde{h} \) are the conformal dimension of its primaries.

The total momentum \( P \) is easily computed from equation (2.12) by summing the BAE (2.9):

\[
P = \frac{2\pi}{L} m \left( \frac{\varphi}{\pi} - e \right) \quad \text{mod} \ 2\pi,
\]

which agrees with equation (2.4). Comparing equation (5.13) with equations (5.11), (5.12) and (5.14), we get:

\[
h - \frac{c}{24} = -\frac{1}{12} + \frac{\pi - 2\gamma}{16\pi^2 \gamma} K^2 + \frac{1}{4\pi \gamma} (y m - \pi e + \varphi)^2.
\]

\[
\tilde{h} - \frac{c}{24} = -\frac{1}{12} + \frac{\pi - 2\gamma}{16\pi^2 \gamma} K^2 + \frac{1}{4\pi \gamma} (y m + \pi e - \varphi)^2.
\]
We emphasize the fact that this conformal spectrum was obtained analytically from the scaled NLIE of the lattice model\(^9\).

Notice that the spectrum (5.15) has a continuous quantum number—the quasi-momentum \(K\)—corresponding to the quasi-shift conserved charge (2.6) and its eigenvalue (2.13). Therefore, our spin chain spectrum must correspond to a non-rational CFT with a non-normalizable Virasoro vacuum, i.e. we do not expect a state with \(h = \tilde{h} = 0\) to be part of its spectrum. Instead, the spin chain vacuum, characterized by \(e = m = K = 0\), should be identified with the primary state of the CFT with the lowest possible conformal dimension.

If we now identify
\[
 s := \frac{\pi - 2\gamma}{4\pi\gamma} K, \quad k := \frac{\pi}{\gamma},
\]
where \(k \in [2, \infty]\), and set \(\phi = 0\) then the spectrum (5.15) coincides exactly with the continuous component of the SL(2, \(\mathbb{R}\))/U(1) sigma model spectrum at level \(k\) given by
\[
 c = \frac{3k}{k - 2} - 1, \quad h = \frac{(m - ke)^2}{4k} + \frac{s^2 + 1/4}{k - 2}, \quad \tilde{h} = \frac{(m + ke)^2}{4k} + \frac{s^2 + 1/4}{k - 2},
\]
where we recall that primaries of the SL(2, \(\mathbb{R}\))/U(1) coset algebra are labelled by pairs of SL(2, \(\mathbb{R}\)) affine primaries of spin \(j = -1/2 + is\) from the continuous series and U(1) vertex operators with 'winding number' \(e\) and 'momentum' \(m\). Geometrically, \(2s\) is interpreted as the momentum carried by the string in the non-compact axial direction of SL(2, \(\mathbb{R}\))/U(1), which is shaped as an infinite cigar, while \(e\) and \(m\) are the winding and momentum numbers in the compact direction.

Notice that the state with the lowest conformal dimension \(h_0 = \tilde{h}_0 = 1/4(k - 2)\) corresponds, as expected, to the ground state of the spin chain. The effective central charge observed in the spin chain (within the pure hole sector) is then \(c_{\text{eff}} = c - 24h_0 = 2\), which is illustrative of the fact that there are two hole species.

5.3. Scaling limit of transfer matrices

In this section we shall take the scaling limit of the generating functions for the higher order conserved charges on the lattice using the expressions obtained in section 3.4. The goal is to derive some equations that allow, at least in principle, to compute the mutually commuting local integrals of motion of the scaling theory in terms of the solution to the NLIE. This type of equations serve to pin down the integrable structure of the scaling CFT, i.e. of the SL(2, \(\mathbb{R}\))/U(1) sigma model.

5.3.1. Free energy. First, let us show that if we scale the lattice transfer matrices according to equation (4.1) while keeping the spectral parameter \(u\) finite, then the result can be expressed in terms of \(E^\pm\) and \(K\), i.e. one does not produce a generating function for the higher order conserved charges.

\(^9\) Note that the dimensions (5.15) are obtained under the assumption that all Bethe roots \(\lambda_{ij}\) are real. However, for large enough values of \(\varphi\), a solution with complex roots exists, and becomes the lowest energy state in the sector with \((e, m, \varphi)\) fixed [22]. Although this type of state plays an important role in the statistical model, its study is beyond the scope of the present paper.
Let us now take a different scaling limit of \( L \) up to exponentially small terms in \( F \). Then, a straightforward calculation leads to

\[
F(u) \approx L_f \left( 1 - i e(2u)E^+ - e(-2iu)E^- \right) / v_F \quad \text{mod } \pi i,
\]

up to exponentially small terms in \( L \). The first term is the bulk free energy that we can simply subtract. The scaling correction term becomes, after using equations (5.11) and (5.12),

\[
\frac{1}{L} \left[ \pi e \sin 2\pi u / (\pi - 2\gamma) - 2i\pi [e(2u)\hat{h} - e(-2iu)\hat{h}] \right].
\]

At the isotropic point \( u \to u_0 \), this coincides with the asymptotic behaviour predicted by CFT. Similarly, taking the scaling limit of \( G(u) \) in equation (3.45) using equation (4.20) we get

\[
G(u) \approx K,
\]

up to exponentially small terms in \( L \).

### 5.3.2. Higher spin charges

Let us now take a different scaling limit of \( F(u), G(u) \)

\[
F^\pm(u) := \lim_{L \to \infty} \pm [F(\pm u \pm i\Lambda/2) - 2L_f \nu(\pm u \pm i\Lambda/2)]
\]

\[
G^\pm(u) := \lim_{L \to \infty} G(\pm u \pm i\Lambda/2).
\] (5.20)

Then, a straightforward calculation leads to

\[
F^+(u) = -i \sum_{b,\ell} v_{b,\ell}^+ \tilde{p}_b(\eta_{b,\ell}^+ + 2iu) - (\bar{U}_0^+ + \bar{U}_1^+)(2iu_0 - 2iu)
\]

\[
- \frac{1}{2\pi} \sum_b \left[ \int_{\Omega_b} d\lambda \tilde{p}_b(\lambda + 2iu)(U_b^+)'(\lambda) - \int_{\Omega_b^*} d\lambda \tilde{p}_b(\lambda + 2iu)(\bar{U}_b^+)'(\lambda) \right],
\]

\[
F^-(u) = -i \sum_{b,\ell} v_{b,\ell}^- \tilde{p}_b(\eta_{b,\ell}^- + 2iu) + (U_0^- + U_1^-)(-2iu - 2iu_0)
\]

\[
- \frac{1}{2\pi} \sum_b \left[ \int_{\Omega_b} d\lambda \tilde{p}_b(\lambda + 2iu)(U_b^-)'(\lambda) - \int_{\Omega_b^*} d\lambda \tilde{p}_b(\lambda + 2iu)(\bar{U}_b^-)'(\lambda) \right] \hspace{1cm} \text{(5.21)}
\]

modulo 2\(\pi\) and, similarly,

\[
G^+(u) = K + \sum_{b,\ell} (-1)^b v_{b,\ell}^+ \tilde{k}_b(\eta_{b,\ell}^+ + 2iu) + (\bar{U}_0^+ - \bar{U}_1^+)(2iu_0 - 2iu)
\]

\[
+ \frac{1}{2\pi} \sum_b (-1)^b \left[ \int_{\Omega_b} d\lambda \tilde{k}_b(\lambda + 2iu)(U_b^+)'(\lambda) - \int_{\Omega_b^*} d\lambda \tilde{k}_b(\lambda + 2iu)(\bar{U}_b^+)'(\lambda) \right].
\]

\[
G^-(u) = K + \sum_{b,\ell} (-1)^b v_{b,\ell}^- \tilde{k}_b(\eta_{b,\ell}^- + 2iu) + (U_0^- - U_1^-)(-2iu - 2iu_0)
\]

\[
+ \frac{1}{2\pi} \sum_b (-1)^b \left[ \int_{\Omega_b} d\lambda \tilde{k}_b(\lambda + 2iu)(U_b^-)'(\lambda) - \int_{\Omega_b^*} d\lambda \tilde{k}_b(\lambda + 2iu)(\bar{U}_b^-)'(\lambda) \right].
\] (5.22)
Here we have defined the functions
\[ \tilde{p}_h(\lambda) = p_h(\lambda) + \pi/2, \]
\[ \tilde{k}_h(\lambda) = k_h(\lambda) - \pi\lambda/(\pi - 2\gamma) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} e(\lambda)^{2n} \]
(5.23)
decaying exponentially at \( \lambda \to +\infty \).

First, notice that \( F^\pm \) and \( G^\pm \) have a form very similar to their lattice counterparts (3.44) and (3.45). Therefore, it is reasonable to expect that they satisfy a Baxter type equation in analogy with equation (2.7).

Secondly, expanding \( F^\pm(u), G^\pm(u) \) around \( 2i u \to +\infty \pm 2i u_0 \pm i\delta \) we get an asymptotic expansion in powers of \( e(2i u) \) (not to be confused with a power series expansion)
\[ F^\pm_+(u) \sim \sum_{n=0}^{\infty} e(2i u)^{2n+1} F^\pm_{2n+1}, \quad G^\pm_+(u) \sim \sum_{n=0}^{\infty} e(2i u)^{2n} G^\pm_{2n}, \]
(5.24)

where we have used equation (5.23); the contribution of the terms \( U^+_a \) and, respectively, \( U^-_a \) was neglected because they have a faster then exponential decay, see equation (4.8) and the discussion of the asymptotic of \( Z^\pm_\alpha(\lambda) \) at \( \text{Re}\lambda \to -\infty \) in section 4.2. The dominant terms of the asymptotic expansion (5.24) are given by
\[ F^+_1 = -2\pi i(h - c/24), \quad F^-_1 = -2\pi i(\tilde{h} - c/24), \quad G^+_0 = K. \]

Equations (5.24) generate the entire hierarchy of mutually commuting local conserved charges of the scaling CFT, i.e. the \( \text{SL}(2,\mathbb{R})/U(1) \) sigma model. With this interpretation \( e(2i u) \) has the natural scale of \([\text{energy}]^{-1}\), while \( F^\pm_{2n-1} \) are the conserved charges of holomorphic currents of spin \( 2n \) and, finally, \( G^\pm_{2n} \) are the conserved charges of holomorphic currents of spin \( 2n + 1 \). This is in agreement with the expectation [26] that the integrable structure of the black hole CFT is described by the nonlinear Schrödinger equation. We leave the comparison of the higher spin charges following from equations (5.22) and (5.22) with the CFT predictions of [26, 33–35] to future work.

6. Density of states

6.1. Spin chain and CFT definitions

The matching of spectra suggests that the staggered 6V model has a conformal scaling limit described by the \( \text{SL}(2,\mathbb{R})/U(1) \) sigma model. However, since the spectra are continuous, the identification between the string axial momentum \( 2s \) and the quasi-momentum \( K \) given in equation (5.16) is not very convincing—the matching would work for any positive constant multiplying \( K^2 \) in equation (5.15). To eliminate this ambiguity, the identification (5.16) must be accompanied by a comparison of the densities of states.

The density of states in the spin chain is a quantity diverging with the system size \( L \) and defined as follows. First, let us fix the quantum numbers \((m, e)\). Then, from (4.22) we have
\[ \tilde{m} = -\frac{4\pi}{\pi} \log \frac{L}{L_0} + B, \]
(6.1)
where
\[ B(s) := \frac{1}{4s} (A^+_s + A^-_s) + \log L_0, \]
(6.2)
and \( L_0 \) is an arbitrary constant, depending only on \( \gamma \), which explicitly takes into account the ambiguity of separating the logarithmically divergent term from the finite part \( B(s) \). The allowed values of \( \tilde{m} \) are all the positive integers with the same parity as \( m \). Hence, they differ by steps of \( \delta \tilde{m} = 2 \). If one increases \( \tilde{m} \) by a finite amount, one gets

\[
\delta \tilde{m} = \frac{4\delta s}{\pi} \left[ \log \frac{L}{L_0} + \partial_s (sB) \right],
\]

and any sum over \( \tilde{m} \) becomes, in the scaling limit:

\[
\sum_{\tilde{m}} (\cdots) \longrightarrow \int ds \rho(s) (\cdots),
\]

where the density of states is

\[
\rho(s) := \frac{\delta \tilde{m}}{2\delta s} = \frac{2}{\pi} \left[ \log \frac{L}{L_0} + \partial_s (sB) \right].
\]

Thus, we have found that the source terms \( A^\pm_2 \) are directly related to the finite part of the density of states for the scaling limit of the staggered 6V model.

There is a heuristic way to compute the large-\( s \) behaviour of \( B(s) \). The source term of the NLIE (4.17) for \( Z_0^± \) is up to exponentially small terms given by \( \sigma_0(s) := -2e(\lambda) + C^± + 2\pi s \lambda / (\pi - 2\gamma) - A^±_2 \). It has a maximum at \( \lambda^* = -(1 - 2\gamma / \pi) \log(-s) \), with \( \sigma_0(\lambda^*) = -2s[\log(-s) - 1] + C^± - A^±_2 \). From equations (4.22) and (5.16) it follows that \( s \to 0 \) since \( \tilde{m} > 0 \). For large \( -s \), this maximum has a large value, and the number of extraordinary pairs of holes becomes large, too. If we approximate \( Z_0^±(\lambda) \) by \( \sigma_0(\lambda) \) then it is easy to see that for large \( s \) the maximum becomes highly peaked and, thus, one can assume that the majority of holes are closely and centrally distributed around \( \lambda^* \)

\[
\{ \eta_{0j}^+ \} \simeq \{ \lambda^* \pm \delta \eta_{0j}^+ \}, \quad \{ \eta_{0j}^- \} \simeq \{ \lambda^* \pm \delta \eta_{0j}^- \}.
\]

In this approximation, we have, from (4.16):

\[
A^±_1 \simeq -2\alpha \sum_j \delta \eta_{0j}, \quad A^±_2 \simeq -4\alpha \lambda^* \sum_j \delta \eta_{0j},
\]

where the contribution of integrals and ordinary holes is subdominant. Thus we get \( A^\pm_2 \simeq -2s \log(-s) \), which then gives from equation (6.2) the asymptotic

\[
B(s) \simeq -\log(-s) \quad \text{for } s \to -\infty.
\]

In comparison, the density of states in the \( SL(2, \mathbb{R})/U(1) \) sigma model has the form

\[
\rho_{BH}(s) = \frac{1}{\pi} \left[ \log \epsilon + \partial_s (sB_{BH}) \right],
\]

\[
B_{BH}(s) = \frac{1}{2\pi} \Im \log \left[ \Gamma \left( 1 - m + ek \frac{i}{2} \right) \Gamma \left( 1 - m - ek \frac{i}{2} \right) \right],
\]

see [25]. It is useful to recall that this density of states was also computed by discretizing the spectrum of the axial string momentum 2\( s \) by adding a Liouville wall to the action, which confines the movement of the centre of the string in the axial direction to a region of length \( \log \epsilon \). The finite part of the density of states can then be extracted from the reflection amplitudes of the \( SL(2, \mathbb{R})/U(1) \) sigma model at the tip of the cigar and of the Liouville theory off of the Liouville wall. More precisely, \( 8sB_{BH} \) is the difference of the two reflection amplitudes, respectively.

Notice that our function \( B(s) \) has the correct behaviour at \( s \to -\infty \). For finite \( s \), the expression (4.16) for \( A^±_2 \) does not lend itself to an analytical computation like in section 5.1, essentially because \( A^±_2 \) are not local conserved quantities of the Hamiltonian. Thus, in section 6.2, we shall study the NLIE numerically to obtain the full function \( B(s) \).
6.2. Numerical comparison

6.2.1. Numerical algorithm. The main purposes of our algorithm are (i) to check the validity of the scaling regime by providing a numerical solution of the NLIE (4.17) and (ii) to compute the integration constants $A^i_x$, which give access to the finite part of the density of states.

In this section, although we work with the scaled NLIE (4.17), we omit the ± indices to lighten the notation. The system we have to solve is

$$Z_a(\lambda) = \sigma_a(\lambda) - \sum_{b,\ell} v_{b\ell} \widetilde{H}_{a-b}(\lambda - \eta_{b\ell})$$

$$- \frac{1}{i} \sum_{b} \left[ \int_{\Omega_b} d\mu \ \widetilde{J}_{a-b}(\lambda - \mu) U_b(\mu) - \int_{\overline{\Omega}_b} d\mu \ \widetilde{J}_{a-b}(\lambda - \mu) \overline{U}_b(\mu) \right],$$

(6.10)

$$Z_a(\eta_{aj}) = 2\pi i_{b,aj},$$

(6.11)

where the full source term is given by

$$\sigma_a(\lambda) := -2e(\lambda) + C + (-1)^a \left( \frac{2\pi s}{\pi - 2\nu} \lambda - A_2 \right).$$

(6.12)

In (6.10)–(6.12), the number of ordinary holes $N_0$ is given by (3.5), and the integration constant $C$ by (4.17). In contrast, the constant $A_2$ is not known, and it determines, among other things, the number of extraordinary holes $N_h$. As explained in section 4.3, the correct value of $A_2$ is the one for which both $Z_0(\lambda)$ and $Z_1(\lambda)$ converge to their expected limit (4.5) at $\lambda \to +\infty$. Thus, our algorithm uses a trial value for $A_2$, and evolves it to get the $Z_0(+\infty)$ as close as possible to their expected values.

Let us now explain how the algorithm solves the system (6.10) and (6.11) for a given value of $A_2$. First, we discretize the paths $\Omega_{aj}$, and write the integrals as

$$\int_{\Omega_{aj}} f(\mu) \ d\mu \longrightarrow \sum_{j=1}^{N} w_{aj} f(\mu_{aj}),$$

(6.13)

where $\{\mu_{aj}, w_{aj}\}$ are suitable points and weights for the approximation of integrals over $\Omega_{aj}$. Equation (6.10) is then used in two different ways. First, equations (6.10) for $\lambda = \mu_{aj}, \ldots, \mu_{aN}$ form a closed system for the unknowns $\{\eta_{aj} := Z_a(\mu_{aj})\}$. Second, for $\lambda = \eta_{aj}$, the left-hand side of (6.10) is replaced by $2\pi i_{b,aj}$, and we have a system of BAE for the unknowns $\{\eta_{aj}\}$. We denote by $\{\varepsilon_{aj}^{(1)}, \varepsilon_{aj}^{(2)} \ldots\}$ the sequence of numerical estimates for $\eta_{aj}$, and

$$u_{aj}^{(n)} := \log[1 + \exp(\varepsilon_{aj}^{(n)})].$$

Also, we have the sequence of estimates $\{\eta_{aj}^{(0)}, \eta_{aj}^{(1)}, \eta_{aj}^{(2)} \ldots\}$ for $\eta_{aj}$. We define the basic iteration, giving $\varepsilon_{aj}^{(n+1)}$ in terms of $u_{aj}^{(n)}$ and $\eta_{aj}^{(n)}$:

$$\varepsilon_{aj}^{(n+1)} = \sigma_a(\mu_{aj}) - \sum_{b,\ell} v_{b\ell} \widetilde{H}_{a-b}(\mu_{aj} - \eta_{b\ell}^{(n)})$$

$$- \frac{1}{i} \sum_{b} \sum_{\ell=1}^{N} \left[ w_{b\ell} \widetilde{J}_{a-b}(\mu_{aj} - \mu_{b\ell}) u_{b\ell}^{(n)} - \overline{w}_{b\ell} \widetilde{J}_{a-b}(\mu_{aj} - \overline{\mu}_{b\ell}) \overline{u}_{b\ell}^{(n)} \right].$$

(6.14)

Note that the integration path $\Omega_0$ is not a simple straight line, but rather a broken line which goes under the real axis at some value $\lambda$ (see figure 8). This value is determined at each iteration, using the criterion that all the $\varepsilon_{aj}^{(n+1)}$ should have positive imaginary parts.

The $\{\eta_{aj}^{(n+1)}\}$ are found by solving the following nonlinear system by the multivariate Newton–Raphson method:

$$2\pi i_{b,aj} = \sigma_a(\eta_{aj}^{(n+1)}) - \sum_{b,\ell} v_{b\ell} \widetilde{H}_{a-b}(\eta_{aj}^{(n+1)} - \eta_{b\ell}^{(n)})$$

$$-2 \sum_{b} \sum_{\ell=1}^{N} \text{Im}[w_{b\ell} \widetilde{J}_{a-b}(\eta_{aj}^{(n+1)} - \mu_{b\ell}) \overline{u}_{b\ell}^{(n)}].$$

(6.15)
The initial values in the algorithm are given by the source terms only
\[ z_{0,j}^{(0)} = \sigma(\mu_{0,j}), \quad 2\pi I_{h,j} = \sigma(\eta_{0,j}) - \sum_{b,\ell} v_{b,\ell} \bar{H}_{a-b}(\eta_{0,j} - \eta_{b,\ell}). \]

Moreover, at any point in the algorithm, we can evaluate \( Z_a \) for real \( \lambda \) by the extrapolation formula:
\[ Z_a^{(n)}(\lambda) := \sigma_0(\lambda) - \sum_{b,\ell} v_{b,\ell} \bar{H}_{a-b}(\lambda - \eta_{b,\ell}^{(n)}) - 2 \sum_{b} \sum_{\ell=1}^{N} \text{Im} \left[ w_{b,\ell} \tilde{J}_{a-b}(\lambda - \mu_{b,\ell}) u_{b,\ell}^{(n)} \right]. \]

After enough iterations, we have reached, up to machine precision, a fixed point for the counting functions
\[ Z_a^{(\infty)} := \lim_{n \to \infty} Z_a^{(n)} \].

6.2.2. Numerical results. For small values of \( A_2 \), one has \((Z_0^{(\infty)}(+\infty), Z_1^{(\infty)}(+\infty)) = (-\infty, +\infty)\), whereas for large values of \( A_2 \), it is \((+\infty, -\infty)\). These two regimes are separated by a value \( A_2^* \), for which we observe \( Z_0^{(\infty)}(+\infty) = Z_1^{(\infty)}(+\infty) = 2\gamma m - 2\psi \). We can follow a simple dichotomy procedure to find \( A_2^* \).

Figures 9–11 show the behaviour of \( Z_0^{(n)} \) for various values of \( A_2 \). Figure 12 shows the finite part of the density of states obtained from the numerical solution of the NLIE, which displays excellent agreement with the density of states in the SL(2, \( \mathbb{R} \))/U(1) sigma model.

7. Discussion

In this work we have considered the continuum limit of the critical staggered XXZ spin chain defined in [22] and further studied in [23, 24]. Using the method of NLIEs to compute scaling corrections we have recovered the continuous spectrum computed in [24] in the Wiener–Hopf approximation, which coincides with the continuous spectrum of the SL(2, \( \mathbb{R} \))/U(1) Euclidean black hole CFT\(^{10}\). Additionally, we have numerically computed with the NLIEs the density of states in the spin chain and found perfect agreement with the density of states in the black hole CFT, see [25].
hole CFT. The NLIEs that we derived from the lattice displayed essentially new features such as: integral kernels that do not decay at infinity, non-monotonic solutions (i.e. counting functions) even in the absence of holes and unusual source terms/asymptotic behaviour in the region where the Bethe roots condense. Our analysis shows that these are closely related to the non-rational nature of the CFT.

We have only very briefly discussed the integrable structure of the black hole CFT and found that there is one conserved charge at every integer non-negative spin. This is in agreement with the expectation of [26] that the higher spin integrals of motions belong to the nonlinear Schrödinger hierarchy. It would be very interesting to study in more detail the integrable structure of the black hole CFT along the lines of [36–38] (see also [19]), i.e. construct the Q-functions, the Baxter equation, the T-system, compute a few local and non-local higher spin integrals of motion, compare them to the CFT predictions of [26, 33–35] and find some ODE reproducing these quantities via the ODE/IM correspondence of [39]. Such a correspondence

Figure 10. The iterates $Z^{(n)}_0$ for $\gamma = 1.24, \varphi = 0, m = e = 0, s = -0.555$, when $A_2 > A_2^*$.  

Figure 11. The iterates $Z^{(n)}_0$ for $\gamma = 1.24, \varphi = 0, m = e = 0, s = -0.555$, when $A_2 = A_2^*$.  

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should make it possible to compute analytically the constants $A^\pm_2$ which appear in the NLIEs (4.17) and which determine the density of states, see [14, 19, 40] for examples. A good starting point is the ODE/IM correspondence of [41] for the Fateev SS model, which in a certain limit gives the black hole CFT.

Another interesting direction of research is to engineer a gap in the critical staggered XXZ spin chain following the standard recipe of [27, 28] and take the continuum limit in such a way that the resulting NLIE describe an integrable massive perturbation of the black hole CFT. The integrable structure should remain invariant under the perturbation, which is clear on the lattice. We notice that there are at least two different integrable massive perturbations of the black hole CFT, known as complex sinh–Gordon (CShG) models [42–45], but only one of them has a spin 2 integral of motion [26, 33, 46] required by the lattice discretization11. The respective CShG model is classically defined by the action

$$\mathcal{A} = \int k \frac{1}{2} \left[ \frac{\partial_\rho \chi \partial_\rho \overline{\chi}}{1 + \chi \overline{\chi}} - m^2 \chi \overline{\chi} \right] d^2 x,$$

where we can recognize in the first term the original cigar metric (1.1) in the complex coordinates $\chi = e^{i\phi} \sinh \rho$ and $m^2$ is the coupling to the massive perturbation. This model is classically integrable [42, 43] and there are strong perturbative [47–49] and non-perturbative arguments [46, 50, 51] that it is also quantum integrable. Its particle spectrum and exact $S$-matrix have been conjectured in [46, 50, 52]. It would be very interesting to make sense of the scattering theory for the massive deformation of the staggered XXZ spin chain and see how it compares with the CShG model. Partial results in this direction were obtained in [53].

Finally, let us mention possible extensions of our work to supergroup spin chains, which arise naturally in the study of two-dimensional disordered quantum phase transitions, like the integer quantum Hall effect [54] or spin quantum Hall effect [55]. Under some specific conditions, some of these supergroup spin chains are suspected [56–59] to have a continuous conformal spectrum in the scaling limit. In the density approximation, they are characterized

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11 We thank Hubert Saleur for pointing out this fact to us.
by the appearance of singular kernels in the linear integral form of the BAE, just like for the staggered XXZ spin chain, which produces a strongly degenerate spectrum. It would be interesting to generalize the method of NLIE to these chains as well, prove in this way the emergence of a continuous spectrum and compute its form together with the density of states. These data should allow to unambiguously identify the scaling CFTs, and ultimately to describe the non-rational CFTs associated to some disordered quantum phase transitions.

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Appendix. Useful formulas

Proof of the summation formula (3.8). The contour $C_a := (-\Gamma_a) \cup \Gamma_a$ encloses all Bethe roots $\lambda$ counter-clockwise, and holes with $\nu_{aj} = +1$ (resp. $\nu_{aj} = -1$) counter-clockwise (resp. clockwise). Since we chose $\delta$ in such a way that the roots and holes are the only solutions to $1 + (-1)^{\nu_{aj}} e^{iZ_a} = 0$ in the strip $|\text{Im}\lambda| < \delta$, we can write

$$\frac{1}{2\pi} \int_{C_a} \frac{(-1)^{\nu_{aj}} e^{iZ_a(\lambda)} e^{iZ_{a}(\lambda)}}{1 + (-1)^{\nu_{aj}} e^{iZ_{a}(\lambda)}} f(\lambda) \, d\lambda = \sum_j f(\lambda_{aj}) + \sum_j \nu_{aj} f(\eta_{aj}).$$

We then substitute under the integral:

$$\frac{(-1)^{\nu_{aj}} iZ_a(\lambda) e^{iZ_{a}(\lambda)}}{1 + (-1)^{\nu_{aj}} e^{iZ_{a}(\lambda)}} = \left\{ \frac{d}{d\lambda} \log [1 + (-1)^{\nu_{aj}} e^{iZ_{a}(\lambda)}] \right\}_{\lambda \in \Gamma_a},$$

which gives the relation (3.8). Finally, from the asymptotic of $U_a'(\lambda) \sim e^{-|\lambda|}$ at $\lambda \to \pm \infty$, which follows directly from equations (3.1) and (3.3), we see that the integral in equation (3.8) is well defined if $f(\lambda)$ grows slower then $e^{a|\lambda|}$ with $a < 1$ when $\lambda \to \pm \infty$.

Fourier transforms and convolution products:

$$\hat{f}(\omega) := \int d\lambda \, f(\lambda) e^{i\omega \lambda}, \quad f(\lambda) = \frac{1}{2\pi} \int d\omega \, \hat{f}(\omega) e^{-i\omega \lambda}. \quad (A.1)$$

$$(f \star g)(\lambda) := \int d\mu \, f(\mu) g(\lambda - \mu), \quad \hat{f} \hat{g} = \hat{f} \star \hat{g}. \quad (A.2)$$

Properties of the functions $\phi_\alpha$ for $0 < \alpha < \pi/2$ and $|\text{Im}\lambda| < \alpha$:

$$\exp[i\phi_{2\alpha}(2\lambda)] = -\exp[i\phi_\alpha(\lambda) + i\phi_\alpha(\lambda + i\pi/2)]. \quad (A.3)$$

$$\phi_\alpha(\pm \infty) = \pm (\pi - 2\alpha). \quad (A.4)$$

$$\phi_\alpha'(\lambda) = \frac{2 \sin 2\alpha}{\cosh 2\lambda - \cos 2\alpha}. \quad (A.5)$$

$$\hat{\phi}_\alpha'(\omega) = \frac{2\pi \sinh(\pi/2 - \alpha) \omega}{\sinh \pi \omega/2}. \quad (A.6)$$
Computation of the quasi-momentum of a hole. Using the $2\pi$-periodicity of $k(\lambda)$ one can easily compute its Fourier transform by deforming contours

$$k(\lambda) := \log \frac{\cosh \lambda + \sin \gamma}{\cosh \lambda - \sin \gamma} = \int \frac{dz}{\omega} \frac{e^{-iz\lambda} \sinh \omega \gamma}{\omega \cosh \pi \omega/2}.$$

Notice that the large $\lambda$ behaviour of the two hand sides agree. With this one gets

$$k_h(\lambda) := -[(1 + J_{\text{odd}}) \star k](\lambda) = \int \frac{d\omega}{2\omega} \frac{e^{-iz\lambda}}{\sinh(\pi \omega/2 - \gamma)} \omega.$$  \hspace{1cm} (A.7)

To arrive at (3.38) we use the periodicity of $\sinh(\pi \omega/2 - \gamma)$ to first compute $k_h'(\lambda) = \pi \tanh(\pi \lambda/(\pi - 2\gamma))/(\pi - 2\gamma)$ and then integrate the result. The integration constant can be fixed by comparing with the asymptotic of (A.7) at $\lambda \to \pm \infty$.

Dilogarithm integrals:

$$\int_0^1 \frac{du}{u} \log(1 + u) = \frac{\pi^2}{12}$$  \hspace{1cm} (A.8)

$$\Re \int_0^\alpha \frac{d(e^{i\theta})}{e^{i\theta}} \log(1 + e^{i\theta}) \frac{e^{i\theta}}{e^{i\theta}} = -\frac{\alpha^2}{4}, \quad \text{for } -\pi < \alpha < \pi.$$  \hspace{1cm} (A.9)

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