ON THE SLOPE OF RELATIVELY MINIMAL FIBRATIONS ON RATIONAL COMPLEX SURFACES

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Abstract. Given a relatively minimal fibration $f : S \rightarrow \mathbb{P}^1$, defined on a rational surface $S$, with a general fiber $C$ of genus $g$, we investigate under what conditions the inequality $6(g - 1) \leq K_f^2$ occurs, where $K_f$ is the canonical relative sheaf of $f$. We give sufficient conditions for having such inequality, depending on the genus and gonality of $C$ and the number of certain exceptional curves on $S$. We illustrate how these results can be used for constructing fibrations with the desired property. For fibrations of genus $11 \leq g \leq 49$ we prove the inequality: $6(g - 1) + 4 - 4\sqrt{g} \leq K_f^2$.

1. Introduction

Let $S$ be a projective complex nonsingular surface. A fibration $f : S \rightarrow X$ on $S$ is a morphism onto a projective curve $X$ with connected fibers. Throughout this paper $X$ will be equal to $\mathbb{P}^1$. We use the usual identification of divisors and their associated sheaves and write, for example, $H^i(L)$ instead of $H^i(S, \mathcal{O}_S(L))$.

Given a fibration $f$, the relative canonical sheaf of $f$ is defined as $K_f = K_S \otimes f^*K_X^{-1}$ if $X$ is rational then $K_f = K_S(2C)$, with $C$ a general fiber (which is assumed to have genus $g \geq 2$). This sheaf is known to be big and nef for non-isotrivial and relatively minimal fibrations, a result proved in the foundational papers by Arakelov and Parshin (see [1] and [9]). In the case when $S$ is rational and $g > 0$, $K_f$ is nef even if we drop the hypothesis on isotriviality. This is due to the fact that in this case $K_f$ is an effective divisor, thus $K_f \cdot E < 0$ would imply that $E$ is a vertical $(-1)$-curve which gives a contradiction with the relatively minimal hypothesis (see proof of 2.1).

The basic numerical invariant associated to a non-isotrivial fibration is the so called slope of $f$ defined as:

$$\lambda_f = \frac{K_f^2}{\deg(f_*K_f)}.$$

In the case of our interest, when $S$ is a rational surface, $\deg(f_*K_f) = g$ (see Lemma 2.2) for any fibration $f$ and $\lambda_f = K_f^2/g$.

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The study of the restrictions that the slope of a fibration must satisfy in relation to the genus \( g \) is a central issue in the theory. As noted before, in the case of rational surfaces the study of \( \lambda_f \) is equivalent to that of \( K_f^2 \).

This paper is devoted to the relation between \( K_f^2 \) and \( g \) in the case of rational surfaces. Beside its importance for the study of the slope, this relation is relevant for another problem, the bounding of the minimal number \( \sigma \) of singular fibers that a semi-stable non-isotrivial fibration must have.

The strict canonical inequality (see [11] and [13]) states that:

\[
K_f^2 < (\sigma - 2)(2g - 2),
\]

for any semi-stable, non-isotrivial fibration \( f : S \to \mathbb{P}^1 \) of genus \( g \geq 2 \).

In this way inequalities of the sort of \( n(g - 1) \leq K_f^2 \), for some integer \( n \), lead to lower bounds for the number \( \sigma \) (see [11] and [12]). For instance, the inequality

\[
6(g - 1) \leq K_f^2,
\]

implies, for semistable and non-isotrivial fibrations, that \( \sigma \geq 6 \).

For the case of surfaces of non-negative Kodaira dimension it is known that in fact the inequality \( 6(g - 1) \leq K_f^2 \) (and in consequence \( \sigma \geq 6 \)) holds for any semi-stable and non-isotrivial fibration (see [6] and [12]).

However, it is a hard problem to determine for which fibrations on a rational or ruled surface the inequality \( 6(g - 1) \leq K_f^2 \) is valid. Simple examples of rational surface admitting a fibration for which \( K_f^2 < 6(g - 1) \) are shown after the statement of Theorem 3.7. In this paper we obtain several general conditions to guarantee the validity of this inequality for rational surfaces.

Our method is based on the study of the linear systems \(|C + nK_S|\) with \( n = 2, 3 \). If any of these linear systems is non-empty, then it is possible to compute its Zariski-Fujita decomposition \( P + N \), with \( P \) a nef divisor. The resulting inequality \( P^2 \geq 0 \) gives inequalities involving \( K_f^2 \), \( g \) and some auto-intersection numbers of exceptional divisors on \( S \). If moreover, \( P \) is big then \( \chi(P + K_S) \geq 0 \) also gives useful inequalities. Extra hypotheses on the genus \( g \) and the gonality of \( C \) allows us to guarantee the non-emptiness of these linear systems. These hypotheses are imposed in order to be able to apply Reider’s method to the study of the linear systems. These results are summarized in Theorems 3.3 and 3.7.

First of all we can compute the negative part \( N_1 \) of the Zariski-Fujita decomposition of \( C + 2K_S \) and the auto-intersection \( N_1^2 = -l \). \( N_1 \) is given by the expression:

\[
N_1 = \sum_{i=1}^s \left[ (l(G_i) + 1)G_i + \sum_{j=1}^{l(G_i)} (l(G_i) - j + 1)E_{ij} \right],
\]

where \( \{G_i\} \) is the set of \((-1)\)–sections of \( f \) and \( \sum_{j=1}^{l(G_i)} E_{ij} \) is a maximal chain (possibly empty) of vertical \((-2)\) curves satisfying:
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$$\Gamma_i \cdot E_{i1} = 1, \quad E_{ik} \cdot E_{im} = \begin{cases} 1 & \text{if } |k - m| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, in this notation $l(\Gamma_i)$ is the length of the chain of $(-2)$-curves attached to $\Gamma_i$, and $s$ the number of $-1$ sections of $f$. The resulting auto-intersection number $l = -N_1^2$ is $l = \sum_i(l(\Gamma_i)+1)$. We call such a chain of $(-2)$-curves a $(-2)$-divisor.

**Theorem 3.3** Let $f : S \to \mathbb{P}^1$ be a non-isotrivial relatively minimal fibration on a rational surface $S$, with general fiber $C$ of genus $g$. Then the following statements hold:

i) If $C + 2K_S$ is effective, then $C + 2K_S - N_1$ is nef, and

$$0 \leq 4K_f^2 - 24(g - 1) + l.$$  

ii) If $g \geq 7$ and the gonality of $C$ is at least 4, then $C + 2K_S$ is effective.

iii) If $g \geq 11$ and the gonality of $C$ is at least 5, then $C + 2K_S - N_1$ is also big and

$$0 \leq 3K_f^2 - 19(g - 1) + l + 1.$$  

In particular if $l + 1 \leq g - 1$ then $6(g - 1) \leq K_f^2$.

We remark that if we contract the $(-1)$-sections we obtain a new rational surface and an associated pencil having its base locus just on the image of the $\Gamma_i$. After this contraction the images of $E_{i1}$ are $(-1)$ curves and can be contracted again to a non-singular rational surface if we continue this procedure we can finally contract the divisor $N_1$ and obtain a new rational nonsingular surface and a pencil on it with its base locus in the image of the connected components of $N_1$. Conversely, resolving the base locus of the resulting pencil we obtain the original fibration on $S$.

In order to explain the content of Theorem 3.3 we introduce a basic example. Start with a pencil generated by two irreducible, nonsingular plane curves of degree $d$ intersecting each other transversally and consider the fibration $f : S \to \mathbb{P}^1$ obtained by blowing up the $d^2$ base points. The invariants associated are:

$$g - 1 = \frac{d(d - 3)}{2}, \quad K_f^2 = K_S(2C)^2 = 3d^2 - 12d + 9,$$

and finally, $\mathbb{P}^2$ being a minimal surface, $l$ coincides in this case with the number of $(-1)$ sections of $f$, therefore $l = d^2$.

In this way,

$$K_f^2 - 6(g - 1) = 9 - 3d,$$

which is negative for $d \geq 3$. On the other hand, $C + 2K_S$ is effective for $d \geq 6$ (see the computations in Section 4, Example (1)).
Now, if we add the prescribed term in Theorem 3.3 i), namely \( \frac{l}{4} \) (we are dividing the inequality by 4), then we obtain:

\[
\frac{d^2}{4} - 3d + 9,
\]

that is in fact positive for all values of \( d \). Note also that in these examples the gonality of the general fiber is \( d - 1 \).

Returning to the general situation, a similar analysis can be made for \( C + 3K_S \). The negative part of \( C + 3K_S - N_1 \) in the Zariski-Fujita decomposition is of the form \( N_1 + N'_1 + N_2 \). Explicitly the divisors \( N'_1 \) and \( N_2 \) are given by:

\[
N'_1 = \sum_{j=1}^{t'} (l'(\Gamma_i) - j + 1)E'_{ij},
\]

where \( E'_i = \sum_j E'_{ij} \) are maximal \((-2)\) divisors such that \( E'_i \cdot C = 1 \) and \( E'_i \cdot \Gamma_i = 1 \), and

\[
N_2 = \sum_{i=1}^{t} [(m(\Delta_i) + 1)\Delta_i + \sum_{j=1}^{m(\Delta_i)} (m(\Delta_i) - j + 1)F_{ij}],
\]

with \( \{\Delta_1, \ldots, \Delta_t\} \) the set of \((-1)\) curves on \( S \) satisfying \( \Delta_i \cdot C = 2 \), and \( F_i = \sum F_{ij} \) are vertical maximal \((-2)\)-divisors such that \( F_i \cdot \Delta_i = 1 \).

Denote \( l' = \sum_{i=1}^{t'} (l'(\Gamma_i) + 1) \) and \( m = \sum_{i=1}^{t} (m(\Delta_i) + 1) \). Then \( N'^2 = -l' \) and \( N_2^2 = -m \).

With this notation in mind we have:

**Theorem 3.7** Let \( f : S \to \mathbb{P}^1 \) a non-isotrivial relatively minimal fibration on a rational surface \( S \), with general fiber \( C \) of genus \( g \). Then the following statements hold:

i) If \( C + 3K_S - N_1 \) is effective, then \( C + 3K_S - 2N_1 - N'_1 - N_2 \) is nef, and

\[
0 \leq 9K_f^2 - 60(g - 1) + 4l + l' + m.
\]

ii) If the gonality of \( C \) is at least 6 and \( g \geq 23 \), then \( C + 3K_S - N_1 \) is effective.

Once again, the divisors \( N'_1 \) and \( N_2 \) can be contracted to obtain a new non-singular surface with a pencil associated to \( f \). If \( N_2 \) is non-empty the generic element of this pencil will have in general singularities. These singularities are of nodal type if the chains \( F_{ij} \) are empty.

Thus, in order to produce examples of fibrations satisfying the hypothesis of Theorems 3.3 or 3.7, we must start with pencils of curves with a certain bounded kind of singularities. This is the task in section 4, where we explain how the previous theorems can be used in order to obtain fibrations satisfying \( 6(g - 1) \leq K_f^2 \). The examples are given by blowing up the base locus of pencils of nodal curves on
minimal rational surfaces.

Before section 4, as a part of a preparatory discussion for Theorem 3.7, we obtain Theorem 3.5, which gives a general and uniform bound for the slope of a relatively minimal fibration.

**Theorem 3.5** Let \( f : S \to \mathbb{P}^1 \) be a relatively minimal fibration on a rational surface \( S \). If the genus \( g \) of the fiber is greater than 11 and the gonality is at least 5, then

\[
(5 + \frac{1}{2})(g - 1) - 5 \leq K_f^2
\]

holds.

Finally, returning to the example on pencils of non-singular plane curves, note that in order to obtain a positive quantity it is sufficient to add \( 3d - 9 \) to \( K_f^2 - 6(g - 1) \). The number \( 3d - 9 \) is approximately equal to \( \sqrt{g} \). In section 5 we find another class of fibrations satisfying an inequality of the type:

\[
6(g - 1) \leq K_f^2 + O(\sqrt{g}),
\]

(denoting by \( O(\sqrt{g}) \) a quantity comparable to \( \sqrt{g} \)). More precisely we prove:

**Theorem 5.2** If \( f : S \to \mathbb{P}^1 \), is a relatively minimal fibration of genus \( 11 \leq g \leq 49 \) on a rational surface \( S \) such that the gonality of the general fiber \( C \) is at least 5 and the surface \( T \) obtained by blowing-down the divisor \( N_1 \) satisfies that \( K_T^2 < 0 \), then:

\[
6(g - 1) + 4 - 4\sqrt{g} \leq K_f^2.
\]

Theorem 5.2, together with the previous example gives some evidence in the sense that probably any relatively minimal fibration on a rational surface satisfies an inequality of the type:

\[
6(g - 1) \leq K_f^2 + O(\sqrt{g}).
\]

It should be noticed that Theorem 3.3 is valid on any algebraic surface satisfying \( h^1(S, \mathcal{O}_S) = h^2(S, \mathcal{O}_S) = 0 \), and all the results in our paper are still valid if moreover we assume that \( K_T^2 \leq 9 \), with \( K_T \) standing for the surface obtained after contracting the support of the divisor \( N_1 \). However, as remarked before, the inequality \( 6(g - 1) \leq K_f^2 \) is true for non-isotrivial fibrations on surfaces of non-negative Kodaira dimension. Obtaining analogous results in the case of non-rational surfaces of negative Kodaira dimension seems to be of natural interest.

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2. Preliminaries and notation

We always denote by \( f : S \to \mathbb{P}^1 \) a relatively minimal fibration on a rational surface \( S \). We will denote by \( K_S \) a canonical divisor of \( S \).
$C$ will denote a general fibre of $f$ which is assumed to have genus $g$, and $K_f = K_S(2C)$ will be the relative canonical sheaf of $f$.

In order to simplify the notation we shall write:

$$a = K_f^2 \quad \text{and} \quad b = g - 1.$$  

Some standard equalities are used systematically:

$$C \cdot K_f = 2b, \quad K_f^2 = a - 8b.$$  

The following Lemma, a result taken from [10], will be invoked several times ($S$ is here an arbitrary surface):

**Lemma 2.1.** Let $L$ be a nef divisor on $S$.

i) If $L^2 \geq 5$ and $|L + K_S| = \emptyset$ then there exists a base point free pencil $|E|$ on $S$ such that either $E \cdot L = 0$ or 1.

ii) If $L^2 \geq 10$ and $|L + K_S|$ does not define a birational map then there exists a base point free pencil $|E|$ on $S$ such that either $E \cdot L = 1$ or 2.

**Proof.** ii) is just Corollary 2 in [10]. Even when i) is not explicitly stated in [10] it follows just by the same argument used in the proof of Corollary 2 in [10]. □

Now, in part for the sake of completeness, in part for illustrating the kind of argument that will be used in the sequel, we state and prove a Lemma that is by now well known (see [6] and [12]).

**Lemma 2.2.** Let $f$ be, as before, a relatively minimal fibration on a rational surface $S$. Suppose that $g > 0$, then $|C + K_S|$ is effective and nef.

**Proof.** We can be more specific about the dimension of the space of sections of $C + K_S$.

Indeed, the surface $S$ is rational, thus $h^0(K_S) = h^1(K_S) = 0$. By Leray’s spectral sequence we have also $h^0(f_*K_S) = h^1(f_*K_S) = 0$. Thus in the decomposition of $f_*K_S$ like a sum of invertible sheaves in $\mathbb{P}^1$ we must have:

$$f_*K_S = \bigoplus \mathcal{O}_{\mathbb{P}^1}(-1),$$

therefore,

$$h^0(C + K_S) = h^0(f_*K_S \otimes \mathcal{O}_{\mathbb{P}^1}(1)) = g.$$  

This proves the first assertion. Now, if $(C + K_S) \cdot E < 0$ for some irreducible curve $E$, then, being $C$ nef, $E$ must be a vertical ($-1$)–curve (see [12], proof of Proposition 4.1) and in consequence $C \cdot E = 0$. But this is impossible because $f$ is relatively minimal. This proves the Lemma. □
3. Adjoint systems and the slope of \( f \)

In this section we investigate the properties of the linear systems \( |C + mK_S| \) for \( m = 1, 2, 3 \). We apply their properties to the study of the slope of \( f \).

**Lemma 3.1.** If \( b \geq 6 \) and the gonality of \( C \) is at least 4, then \( h^0(C + 2K_S) > 0 \) and \( a \geq 5b \).

**Proof.** This follows easily from Corollary 4.4 of [7]. \( \square \)

If \( C + 2K_S \) is effective, it admits a Zariski-Fujita decomposition as the sum of a nef divisor and a negative part, let us compute this negative part.

Let

\[
N_1 = \sum_{i=1}^{s} [(l(\Gamma_i) + 1)\Gamma_i + \sum_{j=1}^{l(\Gamma_i)} (l(\Gamma_i) - j + 1)E_{ij}],
\]

where \( \{\Gamma_1, ..., \Gamma_s\} \) is the set of \((-1)\)-sections of \( f \) and \( \sum E_{ij} \) is a maximal chain (possibly empty) of vertical \((-2)\) curves satisfying:

\[
\Gamma_i \cdot E_{i1} = 1, \quad E_{ik} \cdot E_{im} = \begin{cases} 
1 & \text{if } |k - m| = 1, \\
0 & \text{otherwise},
\end{cases}
\]

and \( l(\Gamma_i) \) will denote the length of the maximal chain \( \sum E_{ij} \). If the chain is empty, then we convey that \( l(\Gamma_i) = 0 \). In the language of [3], this is expressed by saying that the divisor \( E_i = \sum E_{ij} \) is a \((-2)\)-curve and \( \Gamma_i \cdot E_i = 1 \). We prefer to call such a chain a \((-2)\)-divisor, in order to distinguish it from the irreducible case. It should be noticed that there is only one maximal connected \((-2)\)-divisor (possibly empty) attached to each \((-1)\)-section.

We will also use the notation \( \Gamma = \sum_{i=1}^{s} \Gamma_i \).

**Lemma 3.2.** The negative part of \( C + 2K_S \) in the Zariski-Fujita decomposition is \( N_1 \) and \( N_1^2 = -l \), where \( l = \sum_{i=1}^{s} (l(\Gamma_i) + 1) \).

**Proof.** Although our calculus relies on the argument used in [5] Proposition 4.1, it is slightly different and uses the Zariski-Fujita algorithm in order to obtain a more explicit expression.

If \( (C + 2K_S) \cdot D < 0 \) for some irreducible curve \( D \), then being \( C \) nef and \( D^2 < 0 \) we must have that \( D \) is a \((-1)\)-curve such that \( C \cdot D \leq 1 \). Since \( f \) is relatively minimal we conclude that \( D \) is a \((-1)\)-section.

Next we need to find the irreducible curves \( D \) such that:

\[(1) \quad (C + 2K_S - \Gamma) \cdot D < 0.\]

Since \( D \neq \Gamma \), we have that it is not a \((-1)\)-curve, therefore \( D \cdot K_S \geq 0 \) and \( D \cdot \Gamma > 0 \).
Let $T$ be the surface obtained by contracting $\Gamma$. Then in $T$, (11) becomes:

$$(C_0 + K_T) \cdot \pi_* D < 0,$$

with $\pi$ standing for the contraction and $C_0 \in |\pi(C)|$.

Just by the same argument as before we get that $\pi_* D$ is a $(-1)$-curve with $0 \leq C_0 \cdot \pi_* D \leq 1$. In particular $\Gamma \cdot D \leq 1$. Thus we obtain that $D \cdot \Gamma = 1$, $K_S \cdot \Gamma = 0$ and $C \cdot D = 0$. In this way $D = E_{i1}$ for some $i$.

Now, the Zariski-Fujita algorithm commands solving the system of equation in $\alpha_j, \beta_k$:

$$(C + 2K_S - \Gamma) \cdot \Gamma_i = (\sum \alpha_j \Gamma_j + \sum \beta_k E_{k1}) \cdot \Gamma_i,$$

$$(C + 2K_S - \Gamma) \cdot E_{i1} = (\sum \alpha_j \Gamma_j + \sum \beta_k E_{k1}) \cdot E_{i1},$$

for $i = 1, \ldots, s$. It is easy to see that the solution is $\alpha_j = \beta_k = 1$ for all $j$ and $k$. So in this step we need to subtract $\Gamma + \sum E_{i1}$ to $C + 2K_S - \Gamma$.

The rest of the computation is iterative. Assume that on some step of the algorithm we have obtained the divisor:

$$C + 2K_S - N,$$

with $N = \sum_i [(l_i + 1)\Gamma_1 + \sum_{j=1}^{l_i} (l_i - j + 1)E_{ij}]$.

Let $T$ now be the surface obtained by contracting the support of $N$. We obtain, with the analogous notation:

$$(C_0 + K_T) \cdot \pi_* D < 0,$$

therefore, as before, $C \cdot D = 0$ and $N \cdot D = 1$. So, $D$ must be equal to $E_{i, i+1}$ for some $i$. The Zariski-Fujita algorithm leads again to subtract $\sum [\Gamma_i + \sum_{j=1}^{l_i+1} E_{ij}]$. Therefore, the negative part is:

$$N_1 = \sum_{i=1}^s [(l(\Gamma_i) + 1)\Gamma_i + \sum_{j=1}^{l(\Gamma_i)} (l(\Gamma_i) - j + 1)E_{ij}].$$

Now we will calculate $N_1^2$:
Theorem 3.3. Let $S$ be a relatively minimal fibration on a rational surface $S$, with general fiber $C$ of genus $g$. Then the following statements hold:

i) If $C + 2K_S$ is effective, then $C + 2K_S - N_1$ is nef, and

$$0 \leq 4K_f^2 - 24(g - 1) + l.$$

ii) If $g \geq 7$ and the gonality of $C$ is at least 4, then $C + 2K_S$ is effective.

iii) If $g \geq 11$ and the gonality of $C$ is at least 5, then $C + 2K_S - N_1$ is also big and

$$0 \leq 3K_f^2 - 19(g - 1) + l + 1.$$

In particular if $l + 1 \leq g - 1$ then $a \geq 6b$.

Proof. Parts i) and ii) follow from Lemma 3.2, the inequality in i) merely expresses the fact that $(C + 2K_S - N_1)^2 \geq 0$.

We proceed with the proof of iii). We will prove that $|C + 2K_S|$ defines a birational map. This would imply that the nef part of the divisor is big (see [2], 14.18).

Assume, for contradiction, that $|C + 2K_S|$ does not define a birational map. We know, by Lemma [3, 1] that $a \geq 5b$, thus, $(C + K_S)^2 = a - 4b > b > 10$, since $g \geq 11$. By Lemma [2, 2]ii), $S$ admits a base point free pencil $|E|$ with $E \cdot (C + K_S) = 1$ or 2. If, for instance, $E \cdot (C + K_S) = 1$, then

$$N_1^2 = \left( \sum_{i=1}^{s} [(l(G_i) + 1)G_i + \sum_{j=1}^{l(G_i)} (l(G_i) - j + 1)E_{ij}] \right)^2$$

$$= \sum_{i=1}^{s} [(l(G_i) + 1)G_i + \sum_{j=1}^{l(G_i)} (l(G_i) - j + 1)E_{ij}]^2$$

(because $\Gamma_i \cdot \Gamma_k = \Gamma_k \cdot E_{ij} = E_{kj} \cdot E_{ij} = 0$ if $k \neq i$)

$$= \sum_{j=1}^{l(G_i)} [- (l(G_i) + 1)^2 + 2(l(G_i) + 1)(l(G_i) + \sum_{j=1}^{l(G_i)} l(G_i) - j + 1)E_{ij}]^2$$

(because $\Gamma_i \cdot E_{ij} = 0$ if $j > 1$)

$$= \sum_{j=1}^{l(G_i)} [- (l(G_i) + 1)^2 + 2(l(G_i) + 1)(l(G_i) - l(G_i) - j + 1) - 2]$$

$$= \sum_{j=1}^{l(G_i)} \frac{(l(G_i) - j + 1)E_{ij})^2}{2} = \sum_{j=1}^{l(G_i)} (-2(l(G_i) - j + 1)^2 + 2 \sum_{j=1}^{l(G_i)} (l(G_i) - j + 1)(l(G_i) - j))$$

$$= \sum_{j=1}^{l(G_i)} [- (l(G_i) + 1)(l(G_i) + 1 - 2l(G_i) + l(G_i))] = -l.$$

□
1 = K_S \cdot E + C \cdot E = 2g_E - 2 + C \cdot E.

The only possibility for this equality holding is \( g_E = 0 \) and \( C \cdot E = 3 \), but then \( C \) must be trigonal. Similarly, \( E \cdot (K_S + C) = 2 \) implies that \( C \) is tetragonal or hyperelliptic. The last assertion follows from Mumford’s vanishing theorem:

\[
0 \leq \chi(C + 3K_S - N_1) = h^0(C + 3K_S - N_1) = \frac{(C + 2K_S - N_1) \cdot (C + 3K_S - N_1)}{2} + 1 = 3K_f^2 - 19(g - 1) + l + 1.
\]

□

Now, we can make a similar analysis for the linear system \(|C + 3K_S|\). We start by proving the following bound for \( l \):

**Lemma 3.4.** Assume \( g \geq 11 \) and the gonality of \( C \) is at least 5, then

\[
l \leq \frac{5}{2}b + 14.
\]

**Proof.** Let \( \pi : S \to T \) be the contraction of the divisor \((N_1)_{red}\). After contracting \( \Gamma_i, E_i \) is contracted to a \((-1)\) curve, and the same is valid for \( E_{i,j+1} \) after contracting \( E_{i,j} \), so we see that \( T \) is a nonsingular surface.

Moreover,

\[K_S = \pi^* K_T + N_1,\]

and,

\[K_T^2 = (\pi^* K_T + N_1)^2 = a - 8b.\]

Thus, \( \pi^* K_T^2 = a - 8b + l \). Combining this with part iii) of Theorem 3.3 we obtain the equality:

\[
h^0(C + 3K_S - N_1) - 1 + 2l - 3\pi^* K_T^2 = 5b.
\]

Using \( K_T^2 \leq 9 \) we obtain the desired inequality. □

So far we have obtained, under the hypothesis of Theorem 3.3 a bound for the slope of \( f \):

**Theorem 3.5.** Let \( f : S \to \mathbb{P}^1 \) be a relatively minimal fibration on a rational surface \( S \). If the genus \( g \) of the fiber is greater than or equal to 11 and the gonality of \( C \) is at least 5 then

\[
(5 + \frac{1}{2})(g - 1) - 5 \leq K_f^2.
\]

**Proof.** The statement is just a combination of Theorem 3.3 and Lemma 3.4. □

We return to the analysis of the linear system \(|C + 3K_S|\). The following is analogous to Lemma 3.1.

**Proposition 3.6.** If \( b \geq 22 \) and the gonality of \( C \) is at least 6, then \(|C + 3K_S - N_1| \neq \emptyset\).
Proof. The hypotheses of Theorem 3.3, iii) are satisfied, therefore \( C + 2K_S - N_1 \) is nef. Note that, by Lemma 3.4 and part iii) of Theorem 3.3

\[
(C + 2K_S - N_1)^2 = 4a - 24b + l
\]

\[
\geq \frac{4}{3}(19b - l - 1) - 24b + l
\]

\[
= \frac{1}{3}(4b - l - 4) \geq \frac{4}{3}b - \frac{5}{6}b - 6 = \frac{1}{2}b - 6.
\]

Thus, if \( b \geq 22 \) then \( (C + 2K_S - N_1)^2 \geq 5 \). By Lemma 2.1 i) if \( |C + 3K_S - N_1| = \emptyset \), then there exists a base point free pencil \( |E| \) such that:

\[
E \cdot (C + 2K_S - N_1) = 1 \text{ or } 0.
\]

We have either that \( E \) is contracted by \( |C + 2K_S - N_1| \) or \( E \) is a rational curve with \( E^2 = 0 \) and \( E \cdot (C + 2K_S - N_1) = 1 \).

In the first case \( E \) is contracted as well by \( C + 2K_S \), but \( C + K_S \) is nef and \( (C + K_S)^2 = a - 4b \geq 10 \). Thus, by part ii) of Lemma 2.1 there exists a pencil \( |E'| \), such that:

\[
(C + K_S) \cdot E' = 2 \text{ or } 3.
\]

But then:

\[
C \cdot E' - 2 \leq 2 \text{ or } 3,
\]

which gives a contradiction with the assumption on the gonality of \( C \).

On the other hand, if \( E \) is rational then \( E \cdot K_S = -2 \) and:

\[
E \cdot C - 4 - N_1 \cdot E = 1.
\]

Consider the \( \mathbb{P}^1 \)- fibre bundle associated with \( E \) (see [3], V 4.3):

\[
\begin{array}{ccc}
S & \xrightarrow{\phi} & R \\
\downarrow & & \downarrow \\
\mathbb{P}^1 & & \mathbb{P}^1.
\end{array}
\]

\( R \) is a Hirzebruch surface \( \mathbb{F}_n \) and, if \( n \neq 1 \) then \( \mathbb{F}_n \) is minimal and \( N_1 \) must be contracted by \( \phi \). We conclude that \( N_1 \) is a sum of vertical curves with respect to \( |E| \), and \( N_1 \cdot E = 0 \).

If \( R = \mathbb{F}_1 \) then we could have \( N_1 \cdot E = 1 \), but in this case \( C \cdot E = 6 \) and the image of \( C \) in \( \mathbb{F}_1 \) becomes equivalent to \( \Gamma_0 + 6E \), with \( \Gamma_0 \) denoting the \((-1)\)-section, that is, the image of \( N_1 \) under \( \phi \). A simple calculus using the adjunction formula on \( \mathbb{F}_1 \) shows that \( b = 17 \). Therefore, the only possibility is \( N_1 \cdot E = 0 \) and \( E \cdot C \leq 5 \). This final contradiction proves the Proposition.

Next we need to obtain the negative part of \( C + 3K_S - N_1 \), the computation is quite analogous to that of the negative part of \( C + 2K_S \), using in this case that \( C + 2K_S - N_1 \) is nef. This negative part is \( N_1 + N'_1 + N_2 \) with
\[ N'_i = \sum_{j=1}^{l'(\Gamma_i)} (l'(\Gamma_i) - j + 1)E'_{ij}, \]

where \( E'_i = \sum_i E'_{ij} \) are maximal \((-2)\) divisors such that \( E'_i \cdot C = 1 \) and \( E'_i \cdot \Gamma_i = 1 \), of length \( l'(\Gamma_i) \); and

\[ N_2 = \sum_{i=1}^{t} [(m(\Delta_i) + 1)\Delta_i + \sum_{j=1}^{m(\Delta_i)} (m(\Delta_i) - j + 1)F_{ij}], \]

with \( \{\Delta_1, ..., \Delta_t\} \) the set of \((-1)\) curves on \( S \) satisfying \( \Delta_i \cdot C = 2 \), and \( F_i = \sum F_{ij} \) are vertical maximal \((-2)\) divisors such that \( F_i \cdot \Delta_i = 1 \) and of length \( m(\Delta_i) \). To each \((-1)\)-section \( \Gamma_i \) there is associated a unique (possible empty) divisor \( E'_i \) and the same is true for divisors \( \Delta_i \) with respect to the chains \( F_i \).

Denote \( l'' = \sum_{i=1}^{s} (l'(\Gamma_i) + 1) \) and \( m = \sum_{i=1}^{t} (m(\Delta_i) + 1) \).

As in the case of \( N'_1 \), a similar calculus shows that \( N'_1^2 = -l'' \) and \( N'_2 = -m \).

We have obtained, in analogy with Theorem 3.3:

**Theorem 3.7.** Let \( f : S \to \mathbb{P}^1 \) be a relatively minimal fibration on a rational surface \( S \), with general fiber \( C \) of genus \( g \). Then the following statements hold:

i) If \( C + 3K_S - N_1 \) is effective, then \( C + 3K_S - 2N_1 - N'_1 - N_2 \) is nef, and

\[ 0 \leq 9K_f^2 - 60(g - 1) + 4l + l'' + m. \]

ii) If the gonality of \( C \) is at least 6 and \( g \geq 23 \), then \( C + 3K_S - N_1 \) is effective.

The proof of i) follows, as in the analogous case in Theorem 3.2, from the fact that \((C + 3K_S - 2N_1 - N'_1 - N_2)^2 \geq 0\), since the divisor is nef. Part ii) follows from Proposition 3.6.

In particular, under the hypothesis of Theorem 3.7(i), if \( 4l + l'' + m \leq 6(g - 1) \), then \( 6(g - 1) \leq K_f^2 \).

4. Examples

In the following examples we start with a surface \( T \) and a pencil \( W \) of nodal curves on \( T \). The fibration will be obtained in a surface \( S \) by blowing-up the base locus of the pencil. The general fiber of the fibration will be denoted by \( C \) and will correspond with the proper transform of the general element \( C_0 \in W \).

In general it is not true that blowing up the base locus of a pencil on a minimal surface \( T \) gives rise to a relatively minimal fibration. The simplest example comes from considering the pencil generated by an irreducible conic \( Q \) in \( \mathbb{P}^2 \) and the product of two lines \( L_1, L_2 \) that intersects in a point not contained in \( Q \). After blowing up the base locus of this pencil the proper transform of both, \( L_1 \) and \( L_2 \) become vertical \((-1)\)-curves. More complicated examples can be constructed. Fortunately if we limit the singularities of the general element of the pencil we can gain control of the situation.
Thus, we start by computing the conditions to have in the following examples relatively minimal fibrations. Consider a curve $C_0 = D_0 + D_1$ in the pencil $W$ and letting $p_1, ..., p_l$ be the nonsingular points in the base locus of $W$ and letting $q_1, ..., q_m$ be the nodal points in the base locus of $W$, we will use the following notation:

\[
\begin{align*}
  l_{01} &= \#\{p_k : p_k \in D_0\} \\
  l_{02} &= \#\{q_k : q_k \text{ is simple for } D_0\} \\
  m_0 &= \#\{q_k : q_k \text{ is node for } D_0\} \\
  l_0 &= l_{01} + l_{02}.
\end{align*}
\]

Suppose that the proper transform $\tilde{D}_0 \subset S$ is a vertical (-1)-curve. We have the following equations:

(i) $\tilde{D}_0^2 = -1 = D_0^2 - l_{01} - l_{02} - 4m_0$, 
(ii) $g\tilde{D}_0 = 0$, which implies that $\frac{D_0(D_0 + K_T)}{2} = m_0 = -1$,
(iii) $\tilde{D}_0 \cdot C_0 = 0$, that is, $D_0 \cdot C_0 - l_{01} - 2l_{02} - 4m_0 = 0$.

From (i) and (ii) we have that $D_0^2 - l_{01} - l_{02} + 1 = 2D_0^2 + 2D_0 \cdot K_T + 4$, and $D_0^2 + 2D_0 \cdot K_T + 3 = -(l_{01} + l_{02})$, that is, if $D_0^2 + 2D_0 \cdot K_T + 3 > 0$ curve $D_0$ cannot exist. On the other hand we have from (i) and (iii) that $D_0 \cdot C_0 - l_{01} - 2l_{02} = D_0^2 - l_{01} - l_{02} + 1$, then $D_0 \cdot C_0 - D_0^2 = l_{02} + 1 > 0$. This implies that if there exists curve $D_0$ the following two conditions hold simultaneously:

\[
D_0^2 + 2D_0 \cdot K_T < 0, \quad D_0 \cdot C_0 - D_0^2 > 0.
\]

For the case $T = \mathbb{P}^2$ we have that:

\[
\begin{align*}
  \tilde{D}_0^2 &= d_0^2 - l_0 - 4m_0 = -1 \\
  g\tilde{D}_0 - 1 &= \frac{d_0(d_0 - 3)}{2} - m_0 = -1 \\
  C_0 \cdot \tilde{D}_0 &= d_0d - l_01 - 2l_{02} - 4m_0 = 0.
\end{align*}
\]

where $d_0$ is the degree of $D_0$. From the above equations we can conclude that $d_0(d_0 - 6) + 3 = -l_0$ and $d_0(2d_0 - d) > 0$. If $d_0 \geq 6$ then we have a contradiction with the first equation and if $d_0 < 6$ then $d < 12$. In conclusion a nodal pencil of degree $d \geq 12$ in $\mathbb{P}^2$ gives place to a relatively minimal fibration.

Similar computations show that if $T = \mathbb{P}^2$ and the class of $C_0$ is $(\alpha, \beta)$ with $\alpha, \beta \geq 8$ then the associated fibration is relatively minimal.

(1) We want to use Theorem 3.7(i) in order to construct a pencil of plane curves such that the associated fibration satisfies $6(g - 1) \leq K_T^2$.

Consider positive integers $l, m, d$ satisfying the conditions $l + 4m = d^2$ and $l = 2m$, i.e., $6m = d^2$. Fix $m$ points $q_1, ..., q_m \in \mathbb{P}^2$, if $V(q_1, ..., q_m)$ denotes the linear system of plane curves of degree $d$ having nodes at $q_i$, then:
Consider the Hirzebruch surface $T$. In the previous section are given again in this example by $N$ the fibration satisfying $6\beta_0$ obtain a curve of genus $82$, so with this numerical conditions we have a exceptional divisors associated with $q_4$. Since $h_1T$ on $N_\alpha\beta$ only if $\Gamma_i$ the arithmetic genus of $C_0$ is $g_0 = 1 + \frac{C_0^2}{2} + \frac{K_T^2}{2}$, i.e., $g_0 - 1 = \alpha\beta - \alpha - \beta$. Suppose also that $C_0$ has $m$ nodes. Then its geometric genus $g$ is $\alpha\beta - \alpha - \beta + 1 - m$. The classes of $K_S$ and $C$ in $Pic(S)$ are given by:

$$K_S = [(-2, -2), 1, 1, ..., 1]$$
$$C = [(\alpha, \beta), -1, ..., -1, -2, ..., -2].$$

Here we use again the standard ordered basis for $Pic(S) \cong Pic(F_0) \oplus_i \Gamma_i, Z \oplus_j \Delta_j, Z$ with $\Gamma_i$ the exceptional divisors associated with $p_i$ and $\Delta_j$ the exceptional divisors associated with $q_j$. The divisors $N_1$ and $N_2$ as defined in the previous section are given again in this example by $N_1 = \sum \Gamma_i$ and $N_2 = \sum \Delta_j$. Since $h^0(T, O_T(C_0)) - 1 \geq \frac{C_0^2 - C_0 - K_T^2}{2}$, we have that $\dim |C_0| > 0$ if and only if $\alpha\beta + \alpha + \beta > 3m$. In order to apply Theorem $3.7$ we need $|C + 3K_S - N_1| \neq \emptyset$, for this, it is enough to have $\alpha \geq 7$ and $\beta \geq 7$, because $3K_S + C = [(\alpha - 6, \beta - 6), -2, -2, -1, ..., -1]$. In order to guarantee that

$$\dim V(q_1, ..., q_m) \geq \frac{d(d + 3)}{2} - 3m = \frac{l - 2m + 3d}{2} = \frac{3d}{2},$$

which is positive. Taking a general pencil contained in $V(q_1, ..., q_k)$, and blowing up its base locus, we will obtain a rational fibered surface $S$. For $d > 9$ the condition $|C + 3K_S| \neq \emptyset$ is satisfied because:

$$K_S = [-3, 1, 1, ..., 1]$$
$$C = [d, -1, ..., -1, -2, ..., -2].$$

The above equalities mean equality of classes in the Picard group of $S$, where we use the standard ordered basis for $Pic(S) \cong Pic(F^2) \oplus_i \Gamma_i, Z \oplus_j \Delta_j, Z$ with $\Gamma_i$ the exceptional divisors associated with $p_i$ and $\Delta_j$ the exceptional divisors associated with $q_j$, and denote by $H$ the hyperplane class in $F^2$. Therefore

$$C + 3K_S = [d - 9, 2, ..., 2, 1, ..., 1].$$

Note that the decomposition of the divisor $C + 3K_S = (d - 9)H + (\sum 2\Gamma_i + \sum \Delta_j)$ is the Zariski-Fujita decomposition of $C + 3K_S$. In this case, being $F^2$ a minimal surface the divisor $N_1$ defined in the previous section is just $\sum \Gamma_i$ and $N_2 = \sum \Delta_j$. The numbers $l$ and $m$ coincide with the previously defined in section 3. By the genus formula $6b = 3d^2 - 9d - 6m$, in order to be in the hypothesis of the remark after Theorem $3.7$ we need the inequality $4l + m \leq 6b$ which is equivalent to $3d \leq m$. If we take $d = 18, m = 54$ we obtain a curve of genus $82$, so with this numerical conditions we have a fibration satisfying $6b \leq a$. Moreover the gonality of $C$ is $d - 2 = 16$ (see [4]).

(2) Consider the Hirzebruch surface $T = F_0$ and let $C_0$ be an effective divisor on $T$ of class $\langle \alpha, \beta \rangle$. The arithmetic genus of $C_0$ is $g_0 = 1 + \frac{C_0^2}{2} + \frac{K_T^2}{2}$, i.e., $g_0 - 1 = \alpha\beta - \alpha - \beta$. Suppose also that $C_0$ has $m$ nodes. Then its geometric genus $g$ is $\alpha\beta - \alpha - \beta + 1 - m$. The classes of $K_S$ and $C$ in $Pic(S)$ are given by:

$$K_S = [(-2, -2), 1, 1, ..., 1]$$
$$C = [(\alpha, \beta), -1, ..., -1, -2, ..., -2].$$

Here we use again the standard ordered basis for $Pic(S) \cong Pic(F_0) \oplus_i \Gamma_i, Z \oplus_j \Delta_j, Z$ with $\Gamma_i$ the exceptional divisors associated with $p_i$ and $\Delta_j$ the exceptional divisors associated with $q_j$. The divisors $N_1$ and $N_2$ as defined in the previous section are given again in this example by $N_1 = \sum \Gamma_i$ and $N_2 = \sum \Delta_j$. Since $h^0(T, O_T(C_0)) - 1 \geq \frac{C_0^2 - C_0 - K_T^2}{2}$, we have that $\dim |C_0| > 0$ if and only if $\alpha\beta + \alpha + \beta > 3m$. In order to apply Theorem $3.7$ we need $|C + 3K_S - N_1| \neq \emptyset$, for this, it is enough to have $\alpha \geq 7$ and $\beta \geq 7$, because $3K_S + C = [(\alpha - 6, \beta - 6), -2, -2, -1, ..., -1]$. In order to guarantee that
the resulting fibration is relatively minimal we must take both \( \alpha, \beta \geq 8 \).

We also need to have \( 4l + m \leq 6b \). Since \( 2\alpha\beta = C_0^2 = l + 4m \), then
\( 8\alpha\beta - 15m \leq 6\alpha\beta - 6\alpha - 6\beta - 6m \), therefore
\( 3m \geq \frac{2}{3} \alpha\beta + 2\alpha + 2\beta \). Then
the condition \( 3m \in \left[ \frac{2}{3} \alpha\beta + 2\alpha + 2\beta, \alpha\beta + \alpha + \beta \right] \) is satisfied taking for
example \( \alpha = \beta = 8 \) and \( m = 26 \). Then if we blow-up the nodes in the
pencil given by \( C_0 \) we obtain the following diagram:

\[
\begin{array}{ccc}
S & \rightarrow & T \\
\downarrow f & & \downarrow \\
\mathbb{P}^1, & & \\
\end{array}
\]

where \( f \) is a fibration of genus 23 that satisfies \( 6b \leq a \).

(3) In this example we exhibit a fibration for which the equality \( K_f^2 = 6(g - 1) \)
holds. Following the notation of the above example consider the numbers
\( \alpha, \beta, l \) and \( m \) satisfying \( 3m < \alpha\beta + \alpha + \beta, l + 4m = 2\alpha\beta \), since
\( a = 8(\alpha - 1)(\beta - 1) - l - 9m \) and \( b = \alpha\beta - \alpha - \beta - m \), the condition \( a = 6b \) is
equivalent to having \( 2\alpha + 2\beta = m + 8 \). For example the numbers \( \alpha = \beta = 8 \),
\( m = 24 \) satisfy the above conditions. In this case we obtain a fibration of
genus 25. We must observe that the minimal degree of a map \( C_0 \to \mathbb{P}^1 \) is
\( \alpha \) but we can not guarantee that \( \alpha \) is the gonality of its normalization \( C \),
it could be in fact lower (see [8]).

5. Slope of fibration with \( 11 \leq g \leq 49 \)

Let as before \( f : S \to \mathbb{P}^1 \) be a relatively minimal fibration on a rational surface.
Through this section assume moreover that \( g \geq 11 \) and the gonality of \( C \) is at least 5.

Using Lemma 2.2 and Lemma 3.1 we have that \( K_f \) is big and nef. Thus:
\[
h^0(nK_f) = \frac{nK_f((n-1)K_f + 2C)}{2} + 1 \\
= \frac{n(n-1)a + 2bn}{2} + 1.
\]

On the other hand, consider the direct image sheaves:

\[
f_*nK_f = \bigoplus_{i=1}^{b(2n-1)} \mathcal{O}(a_i).
\]

Note that by Mumford’s vanishing \( h^1(nK_f - (n+1)C) = h^1((n-1)(C + K_S) + K_S) = 0 \) for \( n \geq 2 \), as \( C + K_S \) is nef. Thus, by the projection formula

\[
a_i - (n + 1) \geq -1, \\
a_i \geq n.
\]

In this way \( 2(n + 1)(2n - 1)b \leq 2h^0(f_*nK_f) = 2h^0(nK_f) = n(n-1)a + 4bn + 2 \).
Subtraction of both terms leads to the conclusion that \( q(x) = (a - 4b)x^2 - (a - \)
$2b)x + 2b + 2$ is positive when evaluated in any integer $n \geq 2$.

Some properties of this polynomial are summarized in:

**Proposition 5.1.** Let $f : S \to \mathbb{P}^1$ be a semistable, non isotrivial fibration on a rational surface $S$. Then:

i) $q$ is positive when evaluated in any integer $n$.

ii) If $q$ has real roots then they are located in $(0, 1) \cup (1, 2)$.

iii) The discriminant of $q$ is $\Delta_q = (a - 6b)^2 - 8(a - 4b)$.

iv) If $\Delta_q \leq 0$ then:

$$6(g - 1) + 4 - 4\sqrt{g} \leq K_f^2.$$  

**Proof.** i) By construction $q(n) > 0$ for $n \geq 2$. It is easy to verify that $q(1) = 2$ and $q(0) = 2b + 2$. Moreover, the critical value of $q$ is $\frac{2(a - 4b)}{a - 2b}$ which turns out to satisfy

$$0 < \frac{2(a - 4b)}{a - 2b} < 2.$$  

This proves i) and ii). Part iii) is just a direct computation.

For iv), consider the discriminant $\Delta_q$ as a polynomial in $a$:

$$\Delta_q(a) = a^2 - (12b + 8)a + 36b^2 + 32b.$$  

If $\Delta_q(a) \leq 0$ then $a$ is in the “negative region” of this quadratic function. Thus,

$$\frac{12b + 8 - \sqrt{(12b + 8)^2 - 4(36b^2 + 32b)}}{2} \leq a.$$  

This proves the Proposition. 

The next Theorem gives an inequality for the slope of fibrations of genus $11 \leq g \leq 49$, with an extra assumption on the surface $T$ obtained after blowing down the negative part of $C + 2K_S$.

**Theorem 5.2.** If $f : S \to \mathbb{P}^1$ is a relatively minimal fibration of genus $11 \leq g \leq 49$ on a rational surface $S$ such that the gonality of the general fiber $C$ is at least 5 and the surface $T$ obtained by blowing-down the divisor $N_1$ satisfies that $K_T^2 < 0$, then:

$$6(g - 1) + 4 - 4\sqrt{g} \leq K_f^2.$$  

**Proof.** We can assume $a \leq 6b$. Following the proof of Lemma 3.4 and by Theorem 3.3 (iii) we have $2\ell \leq 5b + 1 + 3\pi_*(K_f^2)$, the condition $K_T^2 < 0$ implies $\ell + 1 \leq \frac{5b}{2}$, since $3a \geq 19b - \ell - 1$ we obtain $\frac{11}{2}b \leq a$. Hence we have:

$$\Delta_q = (a - 6b)^2 - 8(a - 4b) \leq (1/2)^2b^2 - 12b = \frac{b(b - 48)}{4}.$$  

Combining with part iv) of Proposition 5.1 we get the theorem. 

\qed
ON THE SLOPE OF FIBRATIONS

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