Random walks on Homeo($S^1$)

Dominique MALICET

Abstract

In this paper, we study random walks $g_n = f_{n-1} \cdots f_0$ on the group $\text{Homeo}(S^1)$ of the homeomorphisms of the circle, where the homeomorphisms $f_k$ are chosen randomly, independently, with respect to a same probability measure $\nu$. We prove that under the only condition that there is no probability measure invariant by $\nu$-almost every homeomorphism, the random walk almost surely contracts small intervals. It generalizes what has been known on this subject until now, since various conditions on $\nu$ were imposed in order to get the phenomenon of contractions. Moreover, we obtain the surprising fact that the rate of contraction is exponential, even in the lack of assumptions of smoothness on the $f_k$'s. We deduce next various dynamical consequences on the random walk $(g_n)$: finiteness of ergodic stationary measures, distribution of the trajectories, asymptotic law of the evaluations, etc. The proof of the main result is based on a modification of the Ávila-Viana’s invariance principle, working for continuous cocycles on a space fibred in circles.

1 Introduction

The objective of the paper is to study properties of (left) random walks on $\text{Homeo}(S^1)$, that is to say long compositions $f_n \circ \cdots \circ f_0$ of homeomorphisms of the circle chosen randomly independently with respect to a same probability measure $\nu$. The study of independent random composition of transformations of a space $X$ is the theory of random dynamical systems (RDS). They appear naturally for example in the theory of iterated forward systems (IFS), when one wants to study the action of a finitely generated group or semigroup $G$: choosing $\nu$ uniform on the set of generators, the theory of RDS allows to study the properties of “typical” elements of $G$. The RDS also correspond to a natural family of skew-products on $X$: the ones of the form $(\omega, x) \mapsto (T\omega, f_\omega(x))$, where $T$ is a shift operator on a symbol space and $f_\omega$ only depends on the first coordinate of $\omega$.

A standard starting point in order to study a random (or deterministic) dynamical system is the question of the dependence to the initial condition. In the context of RDS of homeomorphisms of the circle, the conclusion put in evidence by various results, is that in general the following alternative holds:
either the iterated homeomorphisms preserve a common probability measure on the circle (which implies some “determinism” in the RDS)

or the RDS has the local contractions property: given any point of the circle, typical compositions of the homeomorphisms contracts some neighbourhood of the point.

In the linear case (i.e. when the homeomorphisms are the projective actions of elements of $SL_2(\mathbb{R})$), that is a well known result of H. Furstenberg [9] (and moreover, the contractions are global and exponential). In the general case, there is variations of the precise assumptions and conclusions of this fact, but we can mainly distinguish two kinds of results:

- Smooth case: In the case where the probability measure $\nu$ is supported on $\text{Diff}(S^1)$, one can use the general theory of hyperbolic dynamical systems on manifolds. If the quantity $\int \log^+ ||f'||_{\infty} d\nu(f)$ is finite, we can define Lyapunov exponents. In this context, various results of hyperbolic dynamics [2, 3, 4] imply that if there is an invariant probability measure, then one can find a negative Lyapunov exponent in the system (one can see this as a non linear analogue of the standard result of Furstenberg on random product of matrices [9]). Next, by Pesin theory (or even simpler arguments), one can deduce that the random dynamical system locally contracts, and even that the contractions are exponentially fast.

- Continuous case: In the general case of the iteration of continuous homeomorphisms, the theory of hyperbolic dynamical systems, smooth by nature, does not apply any more. Though, coupling arguments of basic theory of the homeomorphisms of the circle with probabilistic arguments, it is still possible to obtain analogue results with no regularity assumption. The most canonical result (though the older one) of this kind is probably the following theorem of Antonov:

**Theorem. (Antonov) [1]**

Let $f_1, \ldots, f_m$ a finite number of homeomorphisms of the circle preserving the orientation, such that the semi group $G_+$ generated by $f_1, \ldots, f_m$ and the semigroup $G_-$ generated by $f_1^{-1}, \ldots, f_m^{-1}$ both act minimally on $S^1$ (i.e. the orbit of every point is dense), and let $\nu$ a non degenerated probability measure on $\{1, \ldots, p\}$. Then:

- Either for any initial condition $x, y$ in $S^1$, for $\nu^N$-almost every sequence $(i_n)_{n \geq 0}$, the distance between the trajectories $f_{i_n} \circ \cdots \circ f_{i_0}(x)$ and $f_{i_n} \circ \cdots \circ f_{i_0}(y)$ goes to 0. (synchronization)

- Either there exists a probability measure invariant by all the homeomorphisms $f_i$, what implies in fact by minimality that $f_1, \ldots, f_p$ are simultaneously conjugated to rotations. (invariance)

- Or there exists $\theta$ in $\text{Homeo}_+(S^1)$ of finite order $p \geq 2$ commuting with all the $f_i$'s. (factorization)
Remark 1.1. When we are in the third case of Antonov Theorem, then one can factorize the system by identifying the points of the same orbit of $\theta$, in order to obtain a new topological circle, and homeomorphisms $f_1, \ldots, f_m$ of this circle induced by $f_1, \ldots, f_m$.

We can deduce that if $f_1, \ldots, f_m$ does not have a common invariant probability measure, then the random compositions of these homeomorphisms satisfy the property of synchronization (first point of the alternative) up to some factorization (as described below).

As a consequence of Antonov’s Theorem, it remains true that in absence of a common invariant probability measure we have the local contraction property. However, assuming no regularity for the iterated homeomorphisms has a price: additional structural assumptions are required and no speed of convergence is assured: the finiteness of the number of generators is only an assumption for convenience, and the proof of Antonov remains valid without this assumption. The minimality assumptions, though, are much deeper: the dynamics of a semigroup of Homeo(S$^1$) preserving some common interval is very different of the dynamics described in Antonov’s Theorem. And if one considers a semigroup preserving two disjoint intervals, then one can check that in general, none of the alternatives of Antonov’s Theorem are satisfied.

Variants of this theorem exist: let us cite for example [15] where the authors proved (independently of Antonov) that synchronization occur (first case of the previous theorem) under the additional assumption that $G_s$ contains a “north-south” homeomorphism, and [6] where the assumption of minimality is replaced by an assumption of symmetry ($G_s = G_-)$.

The objective of the paper is to treat the study of a general random walk on $\text{Homeo}(S^1)$. We adapt techniques coming from hyperbolic theory in this continuous context, and we show that the distinction between the regular and continuous cases described above is actually basically useless: there is no need to put additional assumptions on a random walk on $\text{Homeo}(S^1)$ to obtain the local contractions, and in fact, even the exponentially speed of contractions remains! Next we use this property of contraction to study deeply the behaviour of the random walk.

The key is to adapt the ideas of Ávila and Viana in [2] (who themselves used those of [17]) to establish that an invariance principle remains in the $C^0$-case: but instead of using the Lyapunov exponents, we will use an another analogue quantity, which measures the exponential contractions as well, but which does not require derivability to be defined. That approach allows to obtain a criterion of the existence of exponential contractions for RDS of the circle, and more generally for any cocycle on a space fibred in circles.
2 Statements of the results

2.1 The main theorem

Before stating our result, we need to formalize the notion of random walk/random dynamical systems:

Definition 2.1. Let \((G, \circ)\) a topological semigroup.

- The random walk generated by a probability measure \(\nu\) on \(G\) is the random sequence \(\omega \mapsto (f_n^\omega)_{n\in\mathbb{N}}\) of elements of \(G\) on the probability space \((\Omega, \mathbb{P}) = (G^\mathbb{N}, \nu^\mathbb{N})\), defined by: for \(\omega = (f_n)_{n\geq 0}\) in \(\Omega\) and \(n\) in \(\mathbb{N}\),
  \[f_n^\omega = f_{n-1} \circ \cdots \circ f_0.\]

- The random walk (or the probability measure \(\nu\)) is said to be non degenerated on \(G\) if the semigroup generated by the topological support of \(\nu\) is dense in \(G\). (equivalently: every open set of \(G\) has positive probability to be reached by the random walk).

- If \(G\) acts on a space \(X\) and if the probability measure \(\nu\) is non degenerated on \(G\), we say that \((G, \nu)\) is a random dynamical system (RDS) on \(X\). The skew-product associated to the RDS is the transformation \(\hat{T}\) on \(\Omega \times X\) defined by
  \[\hat{T}(\omega, x) = (T\omega, f_0(x)),\]
  where \(T\) is the shift operator on \(\Omega\) and \(f_0\) is the first coordinate of \(\omega\).

For a given random walk, we will always denote \((\Omega, \mathbb{P})\) the associated probability space.

In the case of a random walk on \(\text{Homeo}(S^1)\), there exists naturally a unique closed sub-semigroup of \(\text{Homeo}(S^1)\) on which the random walk is non degenerated (namely the closed semigroup generated by the topological support of the associated probability measure \(\nu\)). An interesting fact is that in the majority of the results that we will state, we obtain properties on the random walk depending only on assumptions on the associated semigroup.

Here is the main theorem of the paper:

Theorem A. Let \(\omega \mapsto (f_n^\omega)_{n\geq 0}\) be a non degenerated random walk on a sub-semigroup \(G\) of \(\text{Homeo}(S^1)\). Let us assume that \(G\) does not preserve any probability measure on \(S^1\). Then, for any \(x\) in \(S^1\), for \(\mathbb{P}\)-almost every \(\omega\) in \(\Omega\), there exists a neighbourhood \(I\) of \(x\) such that
  \[\forall n \in \mathbb{N}, \text{diam}(f_n^\omega(I)) \leq q^n,\]
where \(q < 1\) depends on \(\nu\) only.

We can obtain the same result for random walks on a semigroup of continuous injective transformations of a compact interval \(I\), since seeing \(I\) as a
part of \( S^1 \), such an injective map can be extended to a homeomorphism of the circle. Thus, in some sense, the surjectivity of the iterated transformations is not important. The injectivity, though, is primordial: one cannot hope to obtain a contraction phenomenon by iterating maps homotopic to \( z \mapsto z^2 \).

In the case where the semigroup \( G \) associated to a random walk on \( \text{Homeo}(S^1) \) preserves a probability measure \( \mu \), then the topological support \( K \) of \( \mu \) is a compact minimal invariant by the group \( \tilde{G} \) generated by \( G \), and hence we have the standard trichotomy: \( K \) is either \( S^1 \), a Cantor set or a finite set. It is then standard to prove that \( \tilde{G} \) is conjugated to a group of isometries if \( K = S^1 \), and conjugated to a group of isometries if \( K \) is a Cantor set. This fact allows to obtain an interesting classification of random walks on \( \text{Homeo}(S^1) \):

**Proposition 2.2.** Let \( \omega \mapsto (f^n_\omega)_{n \geq 0} \) be a non-degenerated random walk on a sub-semigroup \( G \) of \( \text{Homeo}(S^1) \). Then one (and only one) of the following possibilities occurs:

i) \( G \) does not preserve a probability measure, and the random walk is locally contracting in the sense given by Theorem A.

ii) The random walk is semiconjugated to a random walk on \( O_2(\mathbb{R}) \) (group of isometries of the circle) acting minimally on \( S^1 \).

iii) There is a finite set invariant by \( G \).

On this form, the statement is closed to Furstenberg’s one in the linear case.

### 2.2 General study of random walks acting on \( \text{Homeo}(S^1) \)

In this section, we use Theorem A as a main tool to understand the behaviour of a general random walk on \( \text{Homeo}(S^1) \).

#### 2.2.1 Distribution of the trajectories \( n \mapsto f^n_\omega(x) \)

We interest in the typical distribution of the sequence \( (f^n_\omega(x))_{n \in \mathbb{N}} \) for a given initial condition \( x \). This problem is naturally related to the study of the stationary probability measures of \( \nu \), that is the probability measures \( \mu \) on \( S^1 \) such that \( \Pi \otimes \mu \) is invariant by the skew-product \( \hat{T} \) (we refer to [8] or [14] for details). In the case where the random walk is non degenerated on a subgroup of \( \text{Homeo}(S^1) \), it has been proved that in general, the stationary probability measure is unique (see [6]). In the case of semigroups, this does not hold any more, but we prove that the number of ergodic stationary probability measures (i.e. extremal stationary probability measures) is necessarily finite, and that these probability measures give the typical distributions of the trajectories of the random walk:

**Theorem B.** Let \( \omega \mapsto (f^n_\omega)_{n \geq 0} \) be a non degenerated random walk on a sub-semigroup \( G \) of \( \text{Homeo}(S^1) \) with no finite orbit on \( S^1 \). Then:
• There is only a finite number of ergodic stationary probability measures $\mu_1, \ldots, \mu_d$. Their topological supports $F_1, \ldots, F_d$ are pairwise disjoints and are exactly the minimal invariant compacts of $G$.

• For every $x$ in $S^1$, for $\mathbb{P}$-almost every $\omega$ in $\Omega$, there exists a unique integer $i = i(\omega, x)$ in $\{1, \ldots, d\}$ such that $\text{dist}(f_n^\omega(x), F_i) \xrightarrow{n \to +\infty} 0$, and then we have

$$\frac{1}{N} \sum_{n=0}^{N-1} \delta_{f_n^\omega(x)} \xrightarrow{n \to +\infty} \mu_i$$

in the weak-$*$ topology of $C(S^1, \mathbb{R})$.

Note that in this theorem, we relaxed the condition “no invariant probability measure” to “no finite orbits”.

As a direct consequence of this theorem, we obtain that the stationary probability measure is indeed unique when the action is minimal:

**Corollary 2.3.** A non degenerated random walk on a sub-semigroup $G$ of $\text{Homeo}(S^1)$ acting minimally on $S^1$ admits a unique stationary probability measure.

At our knowledge, this fact was never proved in full generality: until now some additional assumption (smoothness, backward minimality, symmetry...) was required to obtain the unique ergodicity. And actually, we obtain a slightly stronger corollary: the action of any random walk of $\text{Homeo}(S^1)$ on a minimal invariant compact $F$ admits a unique stationary probability measure: if there is no finite orbits, that is a consequence of Theorem B and if there is a finite orbit, then $F$ is necessarily finite and the unique ergodicity is easy to prove).

### 2.2.2 Law of probability of $\omega \mapsto f_n^\omega(x)$

We focus now in the law of the random variables $X_n^g : \omega \mapsto f_n^\omega(x)$ for any given initial condition $x$ and a large integer $n$, and we ask whether for a given condition initial, the law of $X_n^g$ does not depend asymptotically on $n$.

Natural obstructions to the convergence of law of $X_n^g$ are the “ping-pong configurations” where there exists two disjoints closed sets $F_1$ and $F_2$ such that the generators of the semigroup send $F_1$ into $F_2$ and $F_2$ into $F_1$. In this case the distribution of $X_n^g$ strongly depends on the parity of $n$. And the same problem can also arise with a larger number of closed sets. That leads us to the following definition:

**Definition 2.4.** A non degenerated random walk $\omega \mapsto (f_n^\omega)_{n \geq 0}$ on a sub-semigroup $G$ of $\text{Homeo}(S^1)$ is said to be indecomposable if there does not exist a finite number $p \geq 2$ of pairwise disjoints closed subsets $F_1, \ldots, F_p$ of $S^1$ such that for $\nu$-almost every $g$ in $G$, $g(F_i) \subset F_{i+1}$ for $i = 1, \ldots, p - 1$ and $g(F_p) \subset F_1$. 
Remark 2.5. If the action of $G$ is minimal, the random walk is necessarily indecomposable since otherwise, $S^1$ would be a non trivial finite union of pairwise disjoints closed subsets.

The next theorem states that for random walks with no invariant probability measure, the only obstruction to the convergence in law of $X^n_x$ is the one described above:

**Theorem C.** Let $\omega \mapsto (f^n_\omega)_{n \geq 0}$ a non degenerated random walk on $\text{Homeo}(S^1)$ on a sub-semigroup $G$ of $\text{Homeo}(S^1)$ with no invariant probability measure on $S^1$, and such that the random walk is indecomposable. Then, for every $x$ in $S^1$, denoting by $\mu^x_n$ the law of the random variable $X^n_x : \omega \mapsto f^n_\omega(x)$, we have the convergence in law

$$\mu^x_n \xrightarrow{n \to +\infty} \mu^x,$$

where $\mu^x$ is a stationary probability measure for the random walk. Moreover, the convergence is uniform in $x$ in the sense that for any continuous test function $\varphi : S^1 \to \mathbb{R}$,

$$\sup_{x \in S^1} \left| \int_{S^1} \varphi \, d\mu^x_n - \int_{S^1} \varphi \, d\mu^x \right| \xrightarrow{n \to +\infty} 0.$$

**2.2.3 Behaviour of typical homeomorphisms $x \mapsto f^n_\omega(x)$**

Finally, we focus in the behaviour of the homeomorphism $f^n_\omega$ for $\omega$ typical and $n$ a large integer. We prove that such a homeomorphism is similar to a “staircase map” with a constant finite number of stairs. One can see this as a global version of Theorem A.

**Theorem D.** Let $\omega \mapsto (f^n_\omega)_{n \geq 0}$ be a non degenerated random walk on a sub-semigroup $G$ of $\text{Homeo}(S^1)$, such that $G$ does not preserve a common invariant probability measure on $S^1$. Then, there exists a finite number $p$ of measurable functions $\sigma_1, \ldots, \sigma_p : \Omega \to S^1$ such that: for $\mathbb{P}$-almost every $\omega$ in $\Omega$, for every closed interval $I$ included in $S^1 - \{\sigma_1(\omega), \ldots, \sigma_p(\omega)\}$, $\text{diam}(f^n_\omega(I)) \xrightarrow{n \to +\infty} 0$ exponentially fast.

This result is an analogue of Antonov’s Theorem stated in the introduction, but without minimality assumptions (and with an exponential rate of convergence). Indeed, this statement is almost a reformulation of Antonov’s Theorem, except that with his additional assumptions, Antonov obtains a more precise structure: there exists a homeomorphism $\theta$ of order $p$, commuting with all the elements of $g$ and such that

$$\{\sigma_1, \sigma_2, \ldots, \sigma_p\} = \{\sigma_1, \theta \circ \sigma_1 \circ \ldots \circ \sigma_p\}.$$

But as we said in the introduction, such a rigid conclusion cannot hold in general in a non minimal context.
2.3 Property of synchronization

In this section, we want to characterize in which situation the action of a random walk on the circle has the property of synchronization, which means that for any couple of initial conditions \(x\) and \(y\), for almost every realization of the random walk, the distance between the corresponding trajectories of \(x\) and \(y\) tends to 0. This property of synchronization has been studied in [9] in the linear case, and for example in [11, 13, 15, 10] for non linear cases.

**Definition 2.6.** If \((X, d)\) is a metric space, we say that a random walk \(\omega \mapsto (f^n_\omega)\) acting on \(X\) is synchronizing if for every \(x, y\) in \(X\), for almost every \(\omega\),

\[
d(f^n_\omega(x), f^n_\omega(y)) \longrightarrow 0.
\]

We say that it is exponentially synchronizing if the previous convergence is exponential.

What we obtain is that in the context of random walks acting on the circle, an equivalent condition for synchronization is to check that there exists a way to synchronize any pair of points:

**Theorem E.** Let \(\omega \mapsto (f^n_\omega)\) be a non degenerated random walk on a sub-semigroup \(G\) of \(\text{Homeo}(\mathbb{S}^1)\) without a common fixed point. Then the following properties are equivalent:

i) The random walk is exponentially synchronizing.

ii) The random walk is synchronizing.

iii) For every \(x, y\) in \(\mathbb{S}^1\), there exists a sequence \((g_n)\) in \(G\) such that

\[
\text{dist}(g_n(x), g_n(y)) \to 0.
\]

This allows for example to retrieve the main result of [15] in a non minimal context:

**Corollary 2.7.** \(\omega \mapsto (f^n_\omega)\) a non degenerated random walk on a sub-semigroup \(G\) of \(\text{Homeo}(\mathbb{S}^1)\) such that:

- \(G\) contains a map \(g_0\) with exactly 2 fixed points \(a\) and \(b\), one attractive, one repulsive.

- None of the sets \([a]\), \([b]\), \([a, b]\) is invariant by the semigroup \(G\).

Then the random walk is exponentially synchronizing.

That corollary follows rather easily from Theorem E for any \(x, y\) in \(\mathbb{S}^1\), one can find \(h\) in \(G\) such that \(h(x)\) and \(h(y)\) are distinct from the repulsive fixed point of \(g_0\), and then one can apply Theorem E with \(g_n = g_0^n \circ h\). The details are left to the interested reader.
We give two applications of Theorem E. The first one concerns the iterations of continuous injective transformations of an interval. The typical situation treated here is the following: consider two transformations \( f_1, f_2 : [0, 1] \rightarrow [0, 1] \), which are homeomorphisms onto their range, but which are not surjective onto \([0, 1]\) (or at least one of them). We want to know if random iterations of these transformations will contract almost surely the whole interval \([0, 1]\). This is for example the case when the ranges of \( f_1 \) and \( f_2 \) are disjoints (the “ping-pong” configuration, generating a Markov partition), or when \( f_1 \) and \( f_2 \) are Lipschitz-contracting. We give a criterion generalizing these two elementary examples:

**Corollary 2.8.** Let \( I \) be the interval \([0, 1]\) and \( \omega \mapsto (f^n_\omega) \) be a non degenerated random walk on a semi-group \( G \) of injective continuous functions from \( I \) into itself, and let us assume that

\[
\bigcap_{g \in G} g(I) = \emptyset.
\]

Then there exists \( q < 1 \) such that for \( \mathbb{P} \)-almost every \( \omega \):

\[
\forall n \in \mathbb{N}, \quad \text{diam}(f^n_\omega(I)) \leq C q^n
\]

for some constant \( C = C(\omega) \).

The assumption \( \bigcap_{g \in G} g(I) = \emptyset \) is weak (and is actually equivalent to the conclusion if \( G \) does not fix any point of \( I \)): for example, if you iterate randomly two continuous injective functions \( f_1, f_2 : I \rightarrow I \) such that \( f_1 \) has only one fixed point \( c \), and \( f_2(c) \neq c \), then the corollary applies, that is to say that random compositions of \( f_1 \) and \( f_2 \) almost surely contract the whole interval \( I = [0, 1] \) exponentially fast.

Our second application deals with the robustness of the property of synchronization (that is to say the persistence of the property to small perturbations): with Theorem E we can prove that the property of synchronization is robust among the semigroups of homeomorphisms without a common fixed point. We restrict ourselves to the case of finitely generated semigroups to avoid to manipulate intricate topologies on sets of semigroups/random walks.

**Corollary 2.9.** Consider a non degenerated random walk \( \omega \mapsto (f^n_\omega) \) on a sub-semigroup \( G \) of \( \text{Homeo}(S^1)^d \) generated by \( d \) homeomorphisms of the circle \( f_1, \ldots, f_d \) without common fixed points, and assume that \( \omega \mapsto (f^n_\omega) \) is synchronizing. Then there exists a neighbourhood \( V \) of \( (f_1, \ldots, f_d) \) in \( \text{Homeo}(S^1)^d \) such that for any \( d \)-tuple \( (\tilde{f}_1, \ldots, \tilde{f}_d) \) in \( V \), any non degenerated random walk \( \omega \mapsto (\tilde{f}^n_\omega) \) on the semigroup \( \tilde{G} \) generated by \( (\tilde{f}_1, \ldots, \tilde{f}_d) \) is (exponentially) synchronizing.

It is natural to ask whether the property of synchronization is generic, but it easy to see that it is not the case: if \( I \) is an open interval, the property

\[
\forall k \in \{1, \ldots, d\}, \quad \overline{f_k(I)} \subset I
\]
is robust, and the existence of two disjoints such intervals is an obstruction to the synchronization. However, in the case of a non degenerated random walk on subgroups of Homeo_+(S^1), Antonov’s Theorem holds (see [6]), and hence in this case, the property of synchronization is generic, because the other alternatives (existence of a common invariant probability measure or existence of a non trivial homeomorphism in the centralizer of the group) are degenerated properties. Combining this remark with Corollary 2.9 we obtain the following conclusion:

**Corollary 2.10.** Let d be an integer larger than 1. Then there exists an open dense subset \( U \) of Homeo_+(S^1)^d such that for every \((f_1, \ldots, f_d)\) in \( U \), any non degenerated random walk on the group generated by \( \{f_1, \ldots, f_d\} \) is exponentially synchronizing.

### 2.4 Random walks on Homeo_+([0, 1])

We conclude by examining the case of random walks on the group of the homeomorphisms of [0, 1] (or equivalently the group of the homeomorphisms of \( \mathbb{R} \), or still the group of the homeomorphisms of \( S^1 \) fixing a common point \( c \)). The techniques of the paper does not seem to be sufficient to treat such a random walk in a general exhaustive way. However we can still adapt our techniques to get some (partial) information. Here is a variation of our main theorem in this context:

**Theorem F.** Let \( \omega \mapsto (f^\omega_n)_{n \geq 0} \) be a non degenerated random walk on a sub-semigroup G of Homeo_+([0, 1]), let \( \mu \) be a stationary probability measure without atoms, and set \( m = \min(\text{supp}(\mu)) \) and \( M = \max(\text{supp}(\mu)) \). Then for every \( x \) in \((m, M)\), there exists a neighbourhood \( I \) of \( x \) such that

\[
\forall n \in \mathbb{N}, \quad \text{diam}(f^\omega_n(I)) \leq q^n,
\]

where \( q < 1 \) does not depend on \( x \).

This theorem gives the phenomenon of local contractions under the existence of a stationary probability measure. With some additional work, one can hope to deduce various dynamical properties from it as we do in this paper in the case of the circle. As an example, let us state the following corollary, answering by the affirmative to a question of B. Deroin in [5]: “If \( f, g \) are increasing diffeomorphisms of [0, 1], and if the measure of Lebesgue is stationary (for \( \nu = \frac{\delta f + \delta g}{2} \)), is it necessarily the only stationary probability measure?”

**Corollary 2.11.** If a random walk on Homeo_+([0, 1]) admits a stationary probability measure without atoms and with total support, then it is unique. In particular, any random walk on Homeo_+([0, 1]) acting minimally admits at most one stationary probability measure without atoms.

However, if there does not exists (non atomic) stationary probability measures, Theorem F does not give any conclusion. And when the random walk is
symmetric, in the sense that the associated probability measure \( \nu \) is invariant under the transformation \( g \mapsto g^{-1} \), it is proved in [7] that there is no stationary probability measure. However, [7] also develops techniques to obtain a good understanding of the random walk in this case. One could hope that by adapting these techniques and those of this paper it would be possible to manage the study of a general random walk on \( \text{Homeo}_+([0, 1]) \).

2.5 Scheme of the paper

The paper is organized as follows: in Section 3 we present the core argument of our results: an invariance principle for a general skew-product \( \hat{T} \) on a space \( \Omega \times S^1 \) stating that either there is a phenomenon of contractions, either “there is something invariant”. Applying the principle to the specific case where \( \hat{T} \) is associated to a random walk, we obtain Theorem A. In Section 4 we state various properties for random dynamical systems satisfying a property of contraction. In Section 5 we deduce the proofs of the other stated theorems.

3 An invariance principle

The objective of this part is to prove an invariance principle in the spirit of the works of Ledrappier [17] and Ávila-Viana [2] for one-dimensional cocycles without regularity (except the continuity). We will only use this principle for random dynamical systems, but our version is valid for a larger class of cocycles. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \( T : \Omega \rightarrow \Omega \) be a \( \mathbb{P} \)-invariant transformation. We look at the skew products on \( \Omega \times S^1 \) extending \( T \), that is the measurable transformations \( \hat{T} \) of the form \((\omega, x) \mapsto (T\omega, f_\omega(x))\), where \( f_\omega \in \text{Homeo}(S^1) \). For \( \omega \) in \( \Omega \), we will use the notation \( f_\omega^n = f_{T^{-n}\omega} \circ \cdots \circ f_{\omega} \), so that the iterates of \( \hat{T} \) are given by \( \hat{T}^n(\omega, x) = (T^n\omega, f_\omega^n(x)) \). If \( f_\omega \in \text{Diff}^1(S^1) \), the Lyapunov exponent of \( \hat{T} \) at a point \((\omega, x) \in \Omega \times S^1\) is defined as \( \lambda(\omega, x) = \lim_{n \to +\infty} \frac{\log \|f_\omega^n(x)\|}{n} \) (when the limit exists), and it measures the exponential rate of contraction of \( f_\omega^n \) at the neighbourhood of \( x \). We define a similar quantity:

**Definition 3.1.** We call exponent of contraction of \( \hat{T} \) at the point \((\omega, x)\) the non positive quantity

\[
\lambda_{\text{con}}(\omega, x) = \lim_{y \to x} \lim_{n \to +\infty} \frac{\log(\text{dist}(f_\omega^n(x), f_\omega^n(y)))}{n}.
\]

If \( \hat{\mu} \) is a \( \hat{T} \)-invariant probability measure, we define the exponent of contraction of \( \hat{\mu} \) as

\[
\lambda_{\text{con}}(\hat{\mu}) = \int_{\Omega \times S^1} \lambda_{\text{con}}(\omega, x) d\hat{\mu}(\omega, x).
\]
Remark 3.2. It is possible to prove that this quantity $\lambda_{\text{con}}(\hat{\mu})$ coincide with the Lyapunov exponent of $\hat{\mu}$ when this one is defined.

This exponent has the advantage over Lyapunov exponents that it does not need any assumption of differentiability. As a counterpart, the information provided by this exponent are slightly less precise than the one provided by the Lyapunov exponents, because it only measures the contraction of the cocycle, not the expansion, and actually the maximal contraction only, so that one cannot hope miming an Oseledec/Pesin’s theory with this naive definition in dimension larger than one. In dimension one, though, this exponent of contraction is a perfect tool. Let us illustrate this by stating our invariance principle:

**Theorem G** (Invariance principle).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a standard Borel space, with $\mathbb{P}$ probability measure, and let $T : (\omega, x) \mapsto (T\omega, f_\omega(x))$ be a measurable transformation of $\Omega \times S^1$ with $f_\omega \in \text{Homeo}(S^1)$. Then, for every $T$-invariant probability measure $\hat{\mu}$ of the form $d\hat{\mu}(\omega, x) = d\mu_\omega(x)d\mathbb{P}(\omega)$, we have the following alternative:

- either $\lambda_{\text{con}}(\hat{\mu}) < 0$,
- or for $\mathbb{P}$-almost every $\omega$, $\mu_{T\omega} = (f_\omega)_*\mu_\omega$.

When the transformation $T$ of $\Omega$ is invertible, the relation $\mu_{T\omega} = (f_\omega)_*\mu_\omega$ is only a reformulation of “$\hat{\mu}$ is stationary”, so that the invariance principle as we stated it only gives information in non-invertible context (it is possible though to get an invariance principle in an invertible context, applying the theorem to a modified system, see [17]).

### 3.1 Fibred Jacobian and fibred entropy

Following the ideas of [2][17], we define the fibred entropy of $\hat{\mu}$ as follows:

**Definition 3.3.** The fibred Jacobian $J = J(\hat{\mu}) : \Omega \times S^1 \to \mathbb{R}$ of $\hat{\mu}$ is defined by the expression

$$J(\omega, x) = \frac{d(f_\omega^{-1})_*\mu_{T\omega}}{d\mu_\omega}(x),$$

where the derivative is taken in the Radon-Nikodym sense. The fibred entropy $h(\hat{\mu})$ of $\hat{\mu}$ is defined as

$$h(\hat{\mu}) = \begin{cases} -\int_{\Omega \times S^1} \log J d\hat{\mu} & \text{if } \log J \in L^1(\Omega \times S^1, \hat{\mu}), \\ +\infty & \text{otherwise.} \end{cases}$$

By definition, the mapping $x \mapsto J(\omega, x)$ is the derivative of Radon-Nikodym of the measure $(f_\omega^{-1})_*\mu_{T\omega}$ against $\mu_\omega$, that is the $\mu_\omega$-integrable function such that we can write

$$d(f_\omega^{-1})_*\mu_{T\omega}(x) = J(\omega, x) d\mu_\omega(x) + d\tilde{\mu}_\omega(x)$$

(1)
where $\tilde{\mu}_\omega$ is singular with respect to $\mu_\omega$.

Let us state a classical general fact of geometric measure theory which allows to see a Radon-Nikodym derivative as, in some sense, a standard derivative:

**Proposition 3.4.** Let $\mu$ be a probability measure on $S^1$, and $\nu$ be any measure on $S^1$. Then:

i) For $\mu$-almost every $x$ in $S^1$,
\[
\frac{d\nu}{d\mu}(x) = \lim_{I \ni x, \text{diam}(I) \to 0} \frac{\nu(I)}{\mu(I)}
\]

(here and in the sequel, $I$ represents an interval of $S^1$).

ii) Denoting $q^*(x) = \sup_{I \ni x} \frac{\nu(I)}{\mu(I)}$,
\[
\int_{S^1} \log^+ q^*(x) d\mu(x) \leq 2\nu(S^1).
\]

This proposition is standard for $\nu$ the Lebesgue measure, as a consequence of Vitali’s covering Lemma, and as noticed in [17], the proof adapts for any measure $\nu$ if we use Besicovitch’s covering Lemma instead of Vitali.

The key of the proof of Theorem G is to see the entropy $h(\hat{\mu})$ in two different ways.

- **Firstly,** one can see $h(\hat{\mu})$ as a quantity measuring in average how much $(f_\omega^{-1})_*\mu_{\tau_\omega}$ differs from $\mu_\omega$, and obtain the following fact justifying that $h(\hat{\mu})$ deserves its appellation of entropy:

**Proposition 3.5.** We have the inequality
\[
h(\hat{\mu}) \geq 0,
\]
with equality if and only if for $\mathbb{P}$-almost every $\omega$, $\mu_{\tau_\omega} = (f_\omega)_*\mu_\omega$.

- **Secondly,** one can use Proposition 3.4 to see the Jacobian term $J(\omega, x)$ as a kind of derivative for some geometry: for $\hat{\mu}$-a.e. $(\omega, x)$ in $\Omega \times S^1$,
\[
J(\omega, x) = \lim_{y \to x} \frac{\mu_{\tau_\omega}([f_\omega(x), f_\omega(y)])}{\mu_\omega([x, y])}.
\]

It is then possible to think of $-h(\hat{\mu})$ as a kind of Lyapunov exponent, and obtain:

**Proposition 3.6.** We have the inequality
\[
\lambda_{\text{con}}(\hat{\mu}) \leq -h(\hat{\mu})
\]
Remark 3.7. With a slighter effort, we could actually prove the more precise inequality \( \lambda_{\text{con}}(\hat{\mu}) \leq -\frac{h(\hat{\mu})}{\hat{d}(\hat{\mu})} \), for a good definition of the fibred dimension \( \hat{d}(\hat{\mu}) \), which would thus belong to the big family of inequalities relating Lyapunov exponent, entropy and dimension (see for example [19][12][18]).

It is clear that Theorem [3] is a direct consequence of Propositions 3.5 and 3.6. Let us begin by proving Proposition 3.5 (the easy part):

**Proof of Proposition 3.5.** As a consequence of (1),
\[
\int_{\Omega \times S^1} J d\hat{\mu} = \int_{\Omega} \int_{S^1} J(\omega, x) d\mu_\omega(x) d\mathbb{P}(\omega) \leq \int_{\Omega} \int_{S^1} d\mu_{\omega}(x) d\mathbb{P}(\omega) = 1,
\]
hence, by Jensen inequality,
\[
-h(\hat{\mu}) = \int_{\Omega \times S^1} \log J d\hat{\mu} \leq \log \int_{\Omega \times S^1} J d\hat{\mu} \leq 0,
\]
so that \( h(\hat{\mu}) \) is non negative.

Moreover, if \( h(\hat{\mu}) = 0 \), then the Jensen inequality (2) is in fact an equality, so that \( J = 1 \hat{\mu} \)-almost everywhere. Thus, replacing it in (1), we have that for \( \mathbb{P} \)-almost every \( \omega \), \( \mu_\omega = 0 \) and \( (f^{-1}_\omega)_{\mu_{\omega}} = \mu_\omega \), hence \( \mu_{\omega} = (f^{-1}_\omega)_{\mu_{\omega}} \). \( \square \)

We focus now on the proof of Proposition 3.6. In the following subsection, we dismantle the problem and leave the core arguments for a separated treatment in the section afterwards.

### 3.2 Preliminaries: reduction of the problem

The objective of this subsection is to check that it is enough to prove Proposition 3.6 in the case where we have some useful additional properties on \( \hat{\mu} \), namely:

- \( \hat{\mu} \) is ergodic.
- None of the probability measures \( \mu_\omega \) has atoms on \( S^1 \).

The reduction of the problem to the ergodic case is done by ergodic disintegration: let us write
\[
\hat{\mu} = \int \hat{\mu}_\alpha d\alpha
\]
with \( \hat{\mu}_\alpha \) ergodic and \( d\alpha \) some probability measure on the set of ergodic probability measures. Then, writing \( d\hat{\mu}_\alpha = d\mu_{\alpha,\alpha} d\mathbb{P}_\alpha \) and setting
\[
J_\alpha(\omega, x) = \frac{d(f^{-1}_\omega)_{\mu_{\omega,\alpha}}}{d\mu_{\omega,\alpha}}(x)
\]
the Jacobian associated to \( \hat{\mu}_\alpha \), we have that \( J_\alpha = J \hat{\mu}_\alpha \)-almost everywhere, and as a consequence,
\[
\begin{align*}
    h(\hat{\mu}) &= -\iint_{\Omega \times S^1} \log f \, d\hat{\mu}_a \, d\alpha = -\iint_{\Omega \times S^1} \log J_a \, d\hat{\mu}_a \, d\alpha = \int h(\hat{\mu}_a) \, d\alpha.
\end{align*}
\]

Moreover, we also have
\[
\lambda_{\text{con}}(\hat{\mu}) = \iint_{\Omega \times S^1} \lambda_{\text{con}}(\omega, x) \, d\hat{\mu}_a(\omega, x) \, d\alpha = \int \lambda_{\text{con}}(\hat{\mu}_a) \, d\alpha,
\]

hence the inequality to prove is \( \int h(\hat{\mu}_a) \, d\alpha \leq \int \lambda_{\text{con}}(\hat{\mu}_a) \, d\alpha \), which follows from the inequalities in the reduced ergodic case \( \lambda_{\text{con}}(\hat{\mu}_a) \leq -h(\hat{\mu}_a) \).

Thus from now on, we assume that \( \hat{\mu} \) is ergodic. To treat the case where \( \mu_\omega \) has atoms, we use the following general lemma:

**Lemma 3.8.** If \( \hat{\mu} \) is ergodic, and if the set \( \{ \omega \in \Omega \mid \mu_\omega \) has atoms\( \} \) has \( \mathbb{P} \)-positive probability, then there exists a family \( (E(\omega))_{\omega \in \Omega} \) of finite subsets of \( S^1 \), all of them with same cardinal \( d \), such that for \( \mathbb{P} \)-almost every \( \omega \) in \( \Omega \), \( f_\omega(E(\omega)) = E(T\omega) \), and \( \mu_\omega = \frac{1}{d} \sum_{x \in E(\omega)} \delta_x \).

**Remark 3.9.** Actually, the proof does not use the structure of \( S^1 \) so that the statement remains valid for a cocycle \( \hat{T} \) on whatever space.

**Proof.** If \( \varphi \) is any function from \( S^1 \) into \( \mathbb{R} \), we denote \( \| \varphi \|_1 = \sum_{x \in S^1} |\varphi(x)| \), and if \( \mu \) is a measure on \( S^1 \), we denote \( \| \mu \|_\infty = \sup_{x \in S^1} \mu(\{x\}) \). With these notations, if \( \| \varphi \|_1 < +\infty \) we have
\[
\int_{S^1} \varphi d\mu \leq \| \varphi \|_1 \| \mu \|_\infty,
\]

with equality if and only if \( \varphi \) is supported on the set \( \{ x \in S^1 \mid \mu(\{x\}) = \| \mu \|_\infty \} \).

Now, in the context of the statement, let us set
\[
\mathcal{E} = \bigcup_{\omega \in \Omega} \{ \omega \} \times E(\omega) \text{ with } E(\omega) = \left\{ x \in S^1 \mid \mu_\omega(\{x\}) = \| \mu_\omega \|_\infty \right\},
\]

and let us define a function \( \varphi : (\omega, x) \mapsto \varphi_\omega(x) \) by:
\[
\varphi_\omega(x) = \begin{cases} 
    \frac{1_E(\omega)(x)}{\text{Card}(E(\omega))} & \text{if } \| \mu_\omega \|_\infty > 0 \\
    0 & \text{if } \| \mu_\omega \|_\infty = 0
\end{cases}
\]

Notice that \( \| \varphi_\omega \|_1 = 1 \) if \( \| \mu_\omega \|_\infty > 0 \) and 0 if not. Next, we have on one hand the equality
\[
\int_{\Omega \times S^1} \varphi d\hat{\mu} = \int_{\Omega} \left( \int_{S^1} \varphi_\omega d\mu_\omega \right) d\mathbb{P}(\omega) = \int_{\Omega} \| \mu_\omega \|_\infty d\mathbb{P}(\omega),
\]
and we have on the other hand the inequalities

\[
\int_{\Omega \times S^1} \varphi \circ T d\hat{\mu} = \int \left( \int_{S^1} (\varphi_{T\omega} \circ f_\omega) d\mu_\omega \right) dP(\omega) \\
\leq \int_{\Omega} ||\varphi_{T\omega} \circ f_\omega||_{l^1} ||\mu_\omega||_{l^1} dP(\omega) \\
\leq \int_{\Omega} ||\mu_\omega||_{l^1} dP(\omega)
\]

(we used the equality \(||\varphi \circ f||_{l^1} = ||\varphi||_{l^1}\), valid for \(f \in \text{Homeo}(S^1)\)).

The invariance equality \(\int \varphi d\hat{\mu} = \int \varphi \circ T d\hat{\mu}\) implies that the previous series of inequalities is in fact a series of equalities. In particular, we obtain that for \(\mathbb{P}\)-almost every \(\omega\), \(\int_{S^1} (\varphi_{T\omega} \circ f_\omega) d\mu_\omega = ||\varphi_{T\omega} \circ f_\omega||_{l^1} ||\mu_\omega||_{l^1}\), hence that \(\varphi_{T\omega} \circ f_\omega\) is supported on the set \(E(\omega)\), what means that \(f_\omega^{-1}(E(T\omega)) \subset E(\omega)\). That implies that \(\hat{T}^{-1}(E) \subset E\) (up to a set \(\hat{\mu}\)-negligible), hence, using the ergodicity of \(\hat{\mu}\) and the fact that \(E\) is not negligible by assumption, we deduce that in fact \(\hat{\mu}(E) = \hat{\mu}(\hat{T}^{-1}(E)) = 1\).

In consequence, for \(\mathbb{P}\)-almost every \(\omega, \mu_\omega(E(\omega)) = 1\) and \(f_\omega^{-1}(E(T\omega)) = E(\omega)\). The last equality implies that \(\text{Card}(E(T\omega)) \geq \text{Card}(E(\omega))\), and hence by ergodicity that \(\hat{d} = \text{Card}(E(\omega))\) does not depend on \(\omega\) (up to a negligible set). In particular, \(f_\omega : E(\omega) \to E(T\omega)\) is a bijection and \(\mu_\omega = \frac{1}{\hat{d}} \sum_{x \in E(\omega)} \delta_x\). \(\square\)

As a consequence, if the probability measures \(\mu_\omega\) have atoms for a set of \(\omega\) of \(\mathbb{P}\)-positive probability, then Lemma 3.8 implies in particular that for \(\mathbb{P}\)-almost every \(\omega\) in \(\Omega\), \(\mu_{T\omega} = (f_\omega)_* \mu_\omega\), hence \(h(\hat{\mu}) = 0\), so that the inequality \(\lambda_{\text{con}}(\hat{\mu}) \leq -h(\hat{\mu})\) is trivial.

### 3.3 Proof of Proposition 3.6

From now on, we assume that \(\hat{\mu}\) is ergodic and that the fibred probability measures \(\mu_\omega\) have no atoms.

The main idea of the proof is to use the theorem of Birkhoff to log \(I\) to see that the entropy \(h(\hat{\mu})\) represents the exponential rate of decrease of \(\frac{d\nu_{\mu_\omega}(f_\omega)}{d\mu_\omega}\), and hence of \(\frac{\mu_{\nu_{\mu_\omega}(f_\omega)}(I(\omega))}{\mu_\omega(I)}\) for \(I\) a “typical” small interval. However, it is more convenient to work with a slightly modified version of \(I\):

**Definition 3.10.** For \(\varepsilon > 0\), we define the approximated Jacobian \(I_\varepsilon = I_\varepsilon(\hat{\mu})\) as

\[
I_\varepsilon(\omega, x) = \sup \left\{ \frac{\mu_{T\omega}(f_\omega(I))}{\mu_\omega(I)} \mid x \in I, \mu_\omega(I) \leq \varepsilon \right\}.
\]
and the corresponding approximated entropy \( h \) as
\[
 h(\hat{\mu}) = \begin{cases} 
 -\int_{\Omega \times S^1} \log J_\varepsilon d\hat{\mu} & \text{if } \log J_\varepsilon \in L^1(\Omega \times S^1, \hat{\mu}) \\
 +\infty & \text{otherwise.} 
\end{cases}
\]

Notice that \( J_\varepsilon \) is well defined thanks to the fact that \( \mu_\omega \) has no atoms.

In the next lemma, we justify that the definitions of \( J_\varepsilon(\hat{\mu}) \) and \( h_\varepsilon(\hat{\mu}) \) are legitimate, in the sense that these quantities are indeed approximations of \( J(\hat{\mu}) \) and \( h(\hat{\mu}) \).

**Lemma 3.11.** We have
\[
 \lim_{\varepsilon \to 0} J_\varepsilon(\hat{\mu}) = J(\hat{\mu}) \quad \hat{\mu}\text{-almost everywhere, and}
\]
\[
 \lim_{\varepsilon \to 0} h_\varepsilon(\hat{\mu}) = h(\hat{\mu}).
\]

**Proof.** The first point is a direct consequence of Proposition 3.4 applied to \( \mu_\omega \) and \( (f_\omega^{-1})_{\mu T_\omega} \). To prove the second point, we write \( \log J_\varepsilon = u_\varepsilon - v_\varepsilon \) with \( u_\varepsilon = \sup(\log J_\varepsilon, 0) \), \( v_\varepsilon = \sup(-\log J_\varepsilon, 0) \), and we also write \( \log J = u + v \) in the same way. We have that \( u_\varepsilon \to u \) and \( v_\varepsilon \to v \) \( \hat{\mu}\text{-almost everywhere} \) by the first point. Moreover, using the second part of Proposition 3.4 we deduce that \( \sup_{\varepsilon>0} u_\varepsilon \in L^1(\hat{\mu}) \), hence by dominated convergence,
\[
 \lim_{\varepsilon \to 0} \int_{\Omega \times S^1} u_\varepsilon d\hat{\mu} = \int_{\Omega \times S^1} u d\hat{\mu}.
\]

On the other hand, \( v_\varepsilon \) is non-negative and increasing as \( \varepsilon \) decreases to 0, hence by Beppo-Levi’s Theorem,
\[
 \lim_{\varepsilon \to 0} \int_{\Omega \times S^1} v_\varepsilon d\hat{\mu} = \int_{\Omega \times S^1} v d\hat{\mu}.
\]

The claim follows directly from these two estimates. \( \square \)

The following lemma is the key part of the proof of Proposition 3.6 (and hence of Theorem G). It establishes some phenomenon of exponential local contractions under the presence of entropy:

**Lemma 3.12.** Let us assume that \( h(\hat{\mu}) \) is positive. Then, for \( \hat{\mu}\text{-almost every } (\omega, x) \in \Omega \times S^1 \), for every \( \hat{h} \in (0, h(\hat{\mu})) \), there exists \( \delta > 0 \) such that for any interval \( I \) containing \( x \) such that \( \mu_\omega(I) < \delta \),
\[
 \forall n \in \mathbb{N}, \quad \mu T_\omega(f_\omega^n(I)) \leq e^{-nh} \mu_\omega(I).
\]

**Proof.** Let \( \hat{h} \in (0, h) \) be given. By (3) one can choose \( \varepsilon > 0 \) so that \( h_\varepsilon(\hat{\mu}) > \hat{h} \). Let us take a Birkhoff point \( (\omega, x) \) of \( \log J_\varepsilon \), that is such that
\[
 \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \log J_\varepsilon \circ \hat{T}^k(\omega, x) = -h_\varepsilon(\hat{\mu}).
\]
Remark 3.13. Birkhoff’s Theorem is still valid even when $\log J_r \notin L^1(\hat{\mu})$, because one can apply Birkhoff Theorem to the function $\sup(\log J_r, -M)$ (integrable by Proposition 3.4) for $M$ arbitrarily large.

In particular there exists a constant $C_0 = C_0(\omega, x)$ such that

$$\forall n \in \mathbb{N}, \quad \prod_{k=0}^{n-1} J_r \circ \hat{T}^k(\omega, x) \leq C_0 e^{-n\tilde{h}}.$$ 

Let $I$ be an interval containing $x$ small enough so that $\mu_\omega(I) \leq \delta := \frac{\varepsilon}{1 + C_0}$, and let us set $x_n = f^n_\omega(x)$, $I_n = f^n_\omega(I)$. We claim that

$$\forall n \in \mathbb{N}, \quad \mu_{T^n \omega}(I_n) \leq e^{-n\tilde{h}} \mu_\omega(I),$$

from which the conclusion of the lemma follows. The proof of the claim is done by induction:

- For $n = 0$, the inequality is trivial.
- If the inequality is satisfied for $k = 0, \ldots, n - 1$, then, for $k = 0, \ldots, n - 1$ the interval $I_k$ contains the point $x_k$ and satisfies $\mu_{T^k \omega}(I_k) \leq \varepsilon$, hence, by definition of $J_r$,

$$\frac{\mu_{T^{k+1} \omega}(I_{k+1})}{\mu_{T^k \omega}(I_k)} = \frac{\mu_{T^{k+1} \omega}(f_{T^k \omega}(I_k))}{\mu_{T^k \omega}(I_k)} \leq J_r(T^k \omega, x_k) = J_r \circ \hat{T}^k(\omega, x),$$

and we deduce

$$\mu_{T^n \omega}(I_n) = \mu_\omega(I) \prod_{k=0}^{n-1} \frac{\mu_{T^{k+1} \omega}(I_{k+1})}{\mu_{T^k \omega}(I_k)} \leq \mu_\omega(I) \prod_{k=0}^{n-1} J_r \circ \hat{T}^k(\omega, x) \leq C_0 e^{-n\tilde{h}} \mu_\omega(I).$$

\[\square\]

Lemma 3.14. Let $\omega \mapsto \nu_{\omega}$ be any measurable function from $\Omega$ into the set of probability measures on $S^1$. Then, for $\hat{\mu}$-almost every $(\omega, x)$ in $\Omega \times S^1$, we have:

$$\lim_{y \to x} \lim_{n \to +\infty} \log(\nu_{T^n \omega}(f^n_\omega(x), f^n_\omega(y))) n \leq -h(\hat{\mu}).$$

Proof. The case where $\nu_\omega = \mu_\omega$ is a direct consequence of Lemma 3.12 that is

$$\lim_{y \to x} \lim_{n \to +\infty} \log(\mu_{T^n \omega}(f^n_\omega(x), f^n_\omega(y))) n \leq -h(\hat{\mu}). \quad (4)$$
For the general case, let us set

\[ Q^* (w, x) = \sup_{I \ni x} \frac{\nu_\omega (I)}{\mu_\omega (I)} \]

By Proposition 3.4, \( \log^+ Q^* \in L^1 (\hat{\mu}) \), hence the Birkhoff’s sums \( \frac{1}{n} \sum_{k=0}^{n-1} \log^+ Q^* \circ \hat{T}^k \) converge \( \hat{\mu} \)-almost everywhere, hence in particular \( \frac{1}{n} \log^+ Q^* \circ \hat{T}^k \) tends to 0 \( \hat{\mu} \)-almost everywhere, which implies:

\[ \lim_{y \to x} \lim_{n \to +\infty} \frac{1}{n} \log \left( \frac{v_{T_n \omega} (f^N_{T_n \omega} (x), f^N_{T_n \omega} (y))}{\mu_{T_n \omega} (f^N_{T_n \omega} (x), f^N_{T_n \omega} (y))} \right) \leq \lim_{y \to x} \lim_{n \to +\infty} \frac{1}{n} \log^+ Q^* (\hat{T}^k (\omega, x)) = 0. \quad (5) \]

The statement is then a direct consequence of (4) and (5).

The fact that for \( \mathbb{P} \)-almost every \( \omega \), \( \Pi_\omega \) is constituted of stationary probability measures is a general fact: that is the analogue of the standard Krylov-Bogolyubov Theorem for RDS, and is in fact valid for any RDS on a compact set. The proof can be found in [5] (French), or it can be seen as a consequence of Lemma 2.5 of [9].

The proof of Proposition 3.15 begins by noticing two elementary facts on the function \( (\omega, x) \mapsto \lambda_{\text{con}} (\omega, x) \).

Lemma 3.16. The function \( \lambda_{\text{con}} \) is \( \hat{T} \)-invariant \( (\lambda_{\text{con}} \circ \hat{T} = \lambda_{\text{con}}) \), and for any \( \omega \) in \( \Omega \), the function \( x \mapsto \lambda_{\text{con}} (\omega, x) \) is upper semicontinuous.

Proof. The invariance property \( \lambda_{\text{con}} \circ \hat{T} = \lambda_{\text{con}} \) comes from the fact that an interval \( I \) containing \( f_0 (x) \) is contracted by the sequence \( (f^N_{T \omega}) \) if and only if \( f_0^{-1} (I) \) is an interval containing \( x \) contracted by \( (f^N_\omega) \).

The upper semicontinuity of \( \lambda_{\text{con}} \) comes from the fact that if \( \lambda_{\text{con}} (\omega, x) < c \), then there exists an interval \( I \) containing \( x \) such that \( \text{diam}(f^N_{T \omega} (I)) = O(e^{-nc}) \) and hence \( \lambda_{\text{con}} (\omega, \cdot) < c \) on \( I \).
Then, Proposition 3.15 is actually a direct consequence of a much more general fact of random dynamical systems:

**Lemma 3.17.** Let $(G, ν)$ be a RDS on a compact space $X$, and let $φ : \Omega \times X \mapsto \mathbb{R}$ be a measurable positive function such that:

- for every $ω ∈ \Omega$, $x \mapsto φ(ω, x)$ is lower semicontinuous,
- $φ ◦ ˆT ≤ φ$ on $\Omega \times X$.

Let $x_0$ be a point of $X$ and for $ω$ in $\Omega$, let $Π_ω$ be the set of weak-* cluster values of $\big(\frac{1}{N} \sum_{n=0}^{N-1} δ_{f_n(ω)}\big)_{N ∈ \mathbb{N}}$. Then, for $ν^N$-almost every $ω$ in $\Omega = G_N$,

$$φ(ω, x_0) ≥ \sup_{µ ∈ Π_ω} \int_{Ω \times X} φ d(µ ⊗ ν),$$

Proof. Let $F_n$ the σ-algebra generated by the $n$ first canonical projections $p_k : ω = (f_j)_{j≤0} \mapsto f_k$, and set

$$\bar{φ}(x) = \mathbb{E}[φ(\cdot, x)],$$

$$Λ_n = \mathbb{E}[φ(\cdot, x_0)|F_n].$$

We know that $Λ_n \xrightarrow{n→∞} φ(\cdot, x_0)$ almost surely. On the other hand, from the inequality $φ(ω, x_0) ≥ φ ◦ ˆT^n(ω, x_0) = φ(T^nω, f_n^ω(x_0))$, we deduce by taking the conditional expectation with respect to $F_n$ that for $P$-almost every $ω$,

$$Λ_n(ω) ≥ \bar{φ}(f_n^ω(x_0)).$$

Hence, using the Cesaro theorem, for $P$-almost every $ω$,

$$φ(ω, x_0) = \lim_{N→∞} \frac{1}{N} \sum_{n=0}^{N-1} Λ_n(ω) ≥ \lim_{N→∞} \frac{1}{N} \sum_{n=0}^{N-1} \bar{φ}(f_n^ω(x_0)) = \lim_{N→∞} \int_X \bar{φ} dµ_{N,ω} \quad (6)$$

Now, we know that $\bar{φ}$ is lower semicontinuous thanks to the lower semicontinuity of $φ(ω, x)$ and Fatou’s lemma: indeed, for any $x$ in $X$,

$$\lim_{y→x} \bar{φ}(y) = \lim_{y→x} \mathbb{E}[φ(\cdot, y)] ≥ \mathbb{E}[\lim_{y→x} φ(\cdot, y)] ≥ \bar{φ}(x).$$

As a consequence, we can write:

$$\bar{φ} = \inf{ψ ∈ C(X, \mathbb{R}) | ψ ≤ \bar{φ}}$$

and for every such function $ψ ≤ \bar{φ}$, we have by (6):

$$φ(ω, x_0) ≥ \lim_{N→∞} \int_X ψ dµ_{N,ω} = \sup_{µ ∈ Π_ω} \int_X ψ dµ$$

Since $ψ$ is arbitrary, we deduce that

$$φ(ω, x_0) ≥ \sup_{µ ∈ Π_ω} \int_X φ dµ = \sup_{µ ∈ Π_ω} \int_{Ω \times X} φ dP dµ.$$

□
Let us conclude by deducing the following corollary, which is only a reformulation of Theorem [A]

**Corollary 3.18.** Let \( \omega \mapsto (f^n_\omega) \) a random walk generated by a probability measure \( \nu \) on \( \text{Homeo}(S^1) \), and let us assume that there is no probability measure on \( S^1 \) invariant by \( \nu \)-almost every homeomorphism. Then there exists \( \lambda_0 < 0 \) such that for any \( x \) in \( S^1 \), for \( \mathbb{P} \)-almost every \( \omega \),

\[
\lambda_{\text{con}}(\omega, x) \leq \lambda_0.
\]

**Proof.** By Theorem [G], \( \lambda_{\text{con}}(\mathbb{P} \otimes \mu) < 0 \) for any stationary probability measure, hence Proposition 3.15 implies immediately that for any \( x \) in \( S^1 \), \( \omega \mapsto \lambda_{\text{con}}(\omega, x) \) is negative \( \mathbb{P} \)-almost everywhere. To obtain a uniform negative bound, let us notice that this negativity implies the negativity of \( \overline{\lambda}_{\text{con}}(x) = \int_{\Omega} \lambda_{\text{con}}(\omega, x) d\mathbb{P}(\omega) \). Thus, \( x \mapsto \overline{\lambda}_{\text{con}}(x) \) is punctually negative and upper-semicontinuous, hence uniformly bounded from above by some negative number \( \lambda_0 \). Then, using Proposition 3.15 one more time, we obtain

\[
\lambda_{\text{con}}(\omega, x_0) \leq \inf_{\mu \in \Pi_{\omega}} \lambda_{\text{con}}(\mathbb{P} \otimes \mu) = \inf_{\mu \in \Pi_{\omega}} \int_{S^1} \overline{\lambda}_{\text{con}} d\mu \leq \lambda_0.
\]

for \( x \) in \( S^1 \), and \( \mathbb{P} \)-almost every \( \omega \). That achieves the proof of the corollary, and hence of Theorem [A]. \( \square \)

## 4 Locally contracting random dynamical systems

In this section, we study random dynamical systems which locally contracts, in some sense that we shall define. Throughout the section, \((X, \bar{d})\) is a metric compact space, \(G\) is a semigroup of continuous transformations of \(X\), and \(\nu\) a non degenerated probability measure on \(G\). We denote by \((\Omega, \mathbb{P}) = (G^\mathbb{N}, \nu^\mathbb{N})\) the associated probability space, \(\omega \mapsto (f^n_\omega)\) the corresponding random walk, and \(\hat{T} : \Omega \times X \rightarrow \Omega \times X\) the cocycle defined by \(\hat{T}(\omega, x) = (T^\omega f_0(x))\), where \(f_0\) is the first coordinate of \(\omega\).

We assume in all the section that the RDS satisfies the property of local contractions:

**Assumption A.** For every \( x \) in \(X\), for \( \mathbb{P} \)-almost every \( \omega \) in \( \Omega \), there exists a neighbourhood \( B \) of \( x \) such that

\[
\text{diam}(f^n_\omega(B)) \xrightarrow{n \rightarrow +\infty} 0.
\]

**Remark 4.1.** By Theorem [A] Assumption [A] is satisfied when \( G \) is a subgroup of \( \text{Homeo}_+(S^1) \) without invariant probability measure. It is also satisfied if \( X \) is a manifold and \( G \) is a semigroup of diffeomorphisms such that all the Lyapunov exponents of the random walk are negative.
4.1 Preliminaries on random sets of RDS

In this part, we state some general results on the RDS, concerning the structure of the sets invariant by $\hat{T}$. We do not use Assumption A in this part.

**Proposition 4.2.** Let $E = \cup_{\omega \in \Omega} \{\omega\} \times U(\omega)$ a subset of $\Omega \times X$ backward-invariant by $\hat{T}$ (i.e. $\hat{T}^{-1}(E) \subset E$) such that $U(\omega)$ is open in $X$ for every $\omega \in \Omega$. Let us assume that

$$P \otimes \mu(E) > 0$$

for every stationary ergodic probability measure $\mu$. Then actually,

$$P \otimes \mu(E) = 1$$

for every probability measure $\mu$ on $X$ (not necessarily stationary).

*Proof.* That is a simple application of Proposition 3.17 to $\phi = 1_E$: for $x_0$ in $X$, and with the notations of Proposition 3.17, we obtain that for almost every $\omega$, if $\mu(U(\omega)) > 0$ for some $\mu$ in $\Pi_\omega$, then $1_E(\omega, x_0) > 0$, hence $(\omega, x_0) \in E$. Thus, $P \otimes \delta_{x_0}(E) = 1$ for every $x_0$ in $X$. □

The second proposition shows that the fibres of a $\hat{T}$-invariant set cannot have many connected components (that will be the main ingredient for the proof of Theorem D).

**Proposition 4.3.** Let $E = \cup_{\omega \in \Omega} \{\omega\} \times E(\omega)$ a subset of $\Omega \times X$ totally invariant by $\hat{T}$ ($\hat{T}^{-1}(E) = E$). Then, for every stationary ergodic probability measure $\mu$, for $P$-almost every $\omega$ in $\Omega$, $E(\omega)$ has only a constant finite number $d$ of connected components of $\mu$-measure positive.

*Proof.* In order to prove this proposition, we look at the action of $\hat{T}$ in the past. To do this, we extend (canonically) the cocycle on $G \times X$, in an invertible context. This procedure is standard, we resume in the following lemma the properties of the extension we use (we refer to [16] for the details).

**Lemma 4.4.** Let $\Omega = G^Z$ and $\hat{\nu} = \nu^Z$. The transformation $\hat{T} : (\omega, x) \mapsto (T\omega, f_0(x))$ admits an invariant ergodic probability measure $\hat{\mu}$ on $\hat{\Omega} \times X$ of the form $d\hat{\mu} = d\mu_\omega(x) d\hat{\nu}(\omega)$, with:

- the function $\omega \mapsto \mu_\omega$ depending only on the negative coordinates of $\omega$,
- $\int_{\hat{\omega}} \mu_\omega d\hat{\nu}(\omega) = \mu$,
- for $\hat{\nu}$-almost every $\omega$ in $\hat{\Omega}$, $\mu_{T\omega} = (f_0)_* \mu_\omega$.

By this process, we can now prove the following general lemma:

**Lemma 4.5.** Let $(E(\omega, x))_{(\omega,x) \in \Omega \times X}$ be a family of Borelian subsets of $X$ such that

$$\forall (\omega, x) \in \Omega \times X, \quad E(T(\omega, x)) = f_0(E(\omega, x)).$$

Then the function $(\omega, x) \mapsto \mu(E(\omega, x))$ is constant $P \otimes \mu$-almost everywhere.
Proof. Let us extend canonically \((\omega, x) \mapsto E(\omega, x)\) to \(\bar{\Omega} \times X\) (by setting \(E((f_k)_{k \in \mathbb{Z}}, x) = E((f_k)_{k \in \mathbb{N}}, x)\)). For every \((\omega, x) \in \bar{\Omega} \times X, E(\omega, x) = (f_0)^{-1}(E(\bar{T}(\omega, x)))\), hence
\[
\mu_\omega(E(\omega, x)) = (f_0)_* \mu_\omega(E(\bar{T}(\omega, x))) = \mu_{\bar{T}\omega}(E(\bar{T}(\omega, x))).
\]
The function \((\omega, x) \mapsto \mu_\omega(E(\omega, x))\) is hence \(\bar{T}\)-invariant on \(\bar{\Omega} \times X\). By ergodicity of \(\bar{\mu}\), there exists a constant \(c\) such that for \(\bar{\mu}\)-almost every \((\omega, x)\) in \(\bar{\Omega} \times X\), \(\mu_\omega(E(\omega, x)) = c\). Since \(\mu_\omega\) only depends on the negative coordinates of \(\omega\) and \(E(\omega, x)\) only depends on the non-negative coordinates of \(\omega\), we deduce by integration of this equality over the negative coordinates of \(\omega\) that for \(\mathbb{P} \otimes \mu\)-almost every \((\omega, x)\) in \(\bar{\Omega} \times X\), \(\mu_\omega(E(\omega, x)) = c\).

Proposition \[\ref{prop:ergodicity_of_RDS}\] follows by choosing \(E(\omega, x)\) to be the connected component of \(x\) in \(E(\omega)\) (with the convention \(E(\omega, x) = \emptyset\) if \(x \notin E(\omega)\)), we have hence the relation \(E(\bar{T}(\omega, x)) = f_0(E(\omega, x))\). For any ergodic probability measure \(\mu\) of the RDS, by Lemma \[\ref{lem:contractibility_of_RDS}\] for \(\mathbb{P}\)-almost every \(\omega\), the function \(x \mapsto \mu(E(\omega, x))\) is equal to some positive constant \(c\) \(\mu\)-almost everywhere, which means that all the connected components of \(U(\omega)\) which are not \(\mu\)-negligible have the same \(\mu\)-measure \(c\). In particular there is only a finite number of them, namely \(\frac{1}{c}\). \(\square\)

4.2 Stationary trajectories

We prove in this part that the property of local contractions implies that the number of ergodic stationary probability measures is finite, and the trajectory of every point almost surely distribute with respect of one of them.

Let us say that a ball \(B\) is contractible if there exists a set \(\Omega' \subset \Omega\) of \(\mathbb{P}\)-positive probability such that, for \(\omega\) in \(\Omega'\), \(\text{diam}(f^n_\omega(B)) \xrightarrow{\text{n \to +\infty}} 0\). As a consequence of the assumption of local contractivity, every point contains a contractible neighbourhood.

Lemma 4.6. If \(B \subset X\) is a contractible ball, then there exists at most one ergodic stationary probability measure \(\mu\) such that \(\mu(B) > 0\).

Proof. Let \(\mu_1\) and \(\mu_2\) be two ergodic stationary measures such that \(\mu_1(B) \neq 0\) and \(\mu_2(B) \neq 0\). By Birkhoff’s theorem one can find \(x\) and \(y\) in \(B\) such that for \(\mathbb{P}\)-almost every \(\omega\) in \(\Omega\), for every continuous \(\varphi : X \to \mathbb{R}\),
\[
\frac{1}{N} \sum_{n=0}^{N-1} \varphi(f^n_\omega(x)) \xrightarrow{N \to +\infty} \int_X \varphi d\mu_1,
\]
\[
\frac{1}{N} \sum_{n=0}^{N-1} \varphi(f^n_\omega(y)) \xrightarrow{N \to +\infty} \int_X \varphi d\mu_2.
\] (7)

Since \(B\) is contractible, one can choose such an \(\omega\) for which \(\text{diam}(f^n_\omega(B))\) tends to 0. Then, for every continuous mapping \(\varphi : X \to \mathbb{R}, \varphi(f^n_\omega(x)) - \varphi(f^n_\omega(y))\) tends
to 0, hence we conclude from (7) that
\[ \int_X \varphi d\mu_1 = \int_X \varphi d\mu_2, \]
so that \( \mu_1 = \mu_2. \)

\[ \square \]

**Proposition 4.7.** The RDS has a finite number \( d \) of ergodic stationary probability measures \( \{\mu_1, \ldots, \mu_d\} \). Their respective topological supports \( F_1, \ldots, F_d \) are pairwise disjoints, and are exactly the minimal invariant compacts of \( G \).

**Proof.** Each point of \( x \) is the centre of a contractible ball, hence by compactness, we can cover \( X \) by a finite number of contractible balls \( B_1, \ldots, B_d \). By Lemma 4.6, for each \( i \), there is at most one ergodic probability measure \( \mu_i \) such that \( \mu_i(B) \neq 0 \). Hence, there are at most \( d \) stationary ergodic probability measures.

Let \( \{\mu_1, \ldots, \mu_d\} \) be the set of the ergodic probability measures and \( F_i = \text{supp}(\mu_i) \). If \( x \in F_i \cap F_j \) then if \( B \) a contractible ball centred at \( x \), we have \( \mu_i(B) \neq 0 \) and \( \mu_j(B) \neq 0 \), hence by Lemma 4.6, \( \mu_i = \mu_j \). The sets \( F_1, \ldots, F_d \) are hence pairwise disjoint.

If \( F \) is a minimal closed invariant subset of \( X \), then there exists a stationary ergodic probability measure \( \mu_i \) supported in \( F \). And since \( F_i = \text{supp}(\mu_i) \) is invariant by \( G \), we have \( F = F_i \) by minimality of \( F \) (note that we did not use the assumption of contraction to prove this inclusion).

Conversely, let \( i \) be in \( \{1, \ldots, d\} \). The closed set \( F_i = \text{supp}(\mu_i) \) is invariant, hence it contains a minimal invariant closed subset \( F \). By the previous point, \( F = F_j \) for some \( j \), but since the \( F_1, \ldots, F_d \) are pairwise disjoint, necessarily \( i = j \) and hence \( F_i = F \) is a minimal invariant subset.

\[ \square \]

**Proposition 4.8.** For every \( x \) in \( X \), for \( P \)-almost every \( \omega \) in \( \Omega \), there exists a (unique) integer \( i = i(\omega, x) \) in \( \{1, \ldots, d\} \) such that:

- The set of cluster values of the sequence \( (f_n^\omega(x))_{n \geq 0} \) is exactly \( F_i \).
- The sequence of probability measures \( \frac{1}{N} \sum_{n=0}^{N-1} \delta_{f_n^\omega(x)} \) weakly-* converges to \( \mu_i \) in \( C(X, \mathbb{R})^* \).

**Proof.** Let us consider \( E_0 \) to be the set of the points \( (\omega, x) \) such that there is a neighbourhood of \( x \) contracted by \( (f_n^\omega)_n \). We apply Proposition 4.2 to the sets

\[ \mathcal{E} = \left\{ (\omega, x) \in E_0 \text{ such that for some } i \in \{1, \ldots, d\}, \frac{1}{N} \sum_{n=0}^{N-1} \delta_{f_n^\omega(x)} \underset{n \to +\infty}{\rightarrow} \mu_i \right\} \]

and

\[ \tilde{\mathcal{E}} = \left\{ (\omega, x) \in E_0 \text{ such that for some } i \in \{1, \ldots, d\}, \lfloor f_n^\omega(x), n \in \mathbb{N} \rfloor = F_i \right\} \]
which satisfy the required assumptions, hence
\[ \forall x \in X, \quad (\mathbb{P} \times \delta_x)(\delta) = (\mathbb{P} \times \delta_x)(\tilde{\mathbb{E}}) = 1, \]
and the claimed result follows. \(\square\)

4.3 Dynamics of the transfer operator

We study in this part the sequence of the iterates of the transfer operator \(P\) of the RDS, applied to a continuous test function \(\varphi\). We prove that this sequence \((P^n \varphi)_{n \in \mathbb{N}}\) always converges uniformly in the Cesaro sense to a harmonic function, and that it actually converges uniformly in the standard sense if the RDS is indecomposable (in the sense of Definition 2.4).

The transfer operator \(P\) of the system is defined on measurable bounded functions \(\varphi : X \to \mathbb{R}\), by
\[ P \varphi = \int_G \varphi \circ f d\nu(f). \]
The iterates of \(P\) are given by
\[ P^n \varphi = \int_{\Omega} \varphi \circ f^n_\omega d\mathbb{P}(\omega), \]
so that the dynamics of \(P\) represents the evolution of the law of the random variables \(\omega \mapsto f^n_\omega(x)\).

Lemma 4.9. For every continuous \(\varphi : X \to \mathbb{R}\), the family \((P^n \varphi)_{n \in \mathbb{N}}\) is equicontinuous on \(X\).

Proof. Fix \(x\) in \(X\) and \(\varepsilon > 0\), and let \(\delta > 0\) be such that
\[ \forall x, y \in X^2, d(x, y) \leq \delta \Rightarrow |\varphi(x) - \varphi(y)| \leq \varepsilon. \]
Thanks to the local contraction assumption, we can find a ball \(B\) centred at \(x\) and a subset \(\Omega' \subset \Omega\) of probability more than \(1 - \varepsilon\) such that:
\[ \forall n \in \mathbb{N}, \forall \omega \in \Omega', \quad \text{diam}(f^n_\omega(B)) \leq \delta. \]
Then we deduce that for every integer \(n\) and every \(y\) in \(B\):
\[ |P^n \varphi(x) - P^n \varphi(y)| \leq \int_{\Omega} |\varphi(f^n_\omega(x)) - \varphi(f^n_\omega(y))|d\mathbb{P}(\omega) \leq \varepsilon\mathbb{P}(\Omega') + 2||\varphi||_\infty\mathbb{P}(\Omega - \Omega') \leq (1 + 2||\varphi||_\infty)\varepsilon. \]
Thus, \((P^n \varphi)_{n \in \mathbb{N}}\) is equicontinuous at \(x\). Since \(x\) is arbitrary and \(X\) is compact, \((P^n \varphi)_{n \in \mathbb{N}}\) is equicontinuous on \(X\). \(\square\)
We keep the notations $\mu_1, \ldots, \mu_d$ and $F_1, \ldots, F_d$ for the ergodic stationary probability measures of the RDS and their topological support.

**Proposition 4.10.** Let $E_0 = \{ \varphi \in C(X, \mathbb{R}) \mid P \varphi = \varphi \}$ be the vector space of harmonic continuous function. Then $E_0$ has finite dimension $d$, and one can find a basis $(u_1, \ldots, u_d)$ of $E_0$ where $u_i$ is valued in $[0, 1]$, $u_i = \delta_{i,j}$ on $F_j$ and $\sum_i u_i = 1$ on $X$.

For every continuous $\varphi : X \to \mathbb{R}$, we have

$$\frac{1}{N} \sum_{n=0}^{N-1} P^n \varphi \xrightarrow{\|\cdot\|_{\infty}} \psi$$

where $\psi$ is the element of $E_0$ given by

$$\psi(x) = \sum_{i=1}^{d} \left( \int_X \varphi d\mu_i \right) u_i(x).$$

**Proof.** Let $\varphi : X \to \mathbb{R}$ be a continuous function, and let $x$ be in $X$. With $i(\omega, x)$ defined as in Proposition 4.8, we have for $P$-almost every $\omega$ in $\Omega$:

$$\frac{1}{N} \sum_{n=0}^{N-1} \varphi(f^n_0(x)) \xrightarrow{\|\cdot\|_{\infty}} \int_X \varphi d\mu_i(\omega, x).$$

Integrating in $\omega$, we deduce by dominated convergence that

$$\frac{1}{N} \sum_{n=0}^{N-1} P^n \varphi(x) \xrightarrow{n \to +\infty} \sum_{i=1}^{d} u_i(x) \int_X \varphi d\mu_i,$$

where $u_i(x) = P(\omega \in \Omega \mid i(\omega, x) = i)$. Since the sequence $\left( \frac{1}{N} \sum_{n=0}^{N-1} P^n \varphi \right)_{n \in \mathbb{N}}$ is equicontinuous by Lemma 4.9, the convergence (8) is in fact uniform in $x$.

The only non trivial property to prove on the functions $u_i$ is their continuity. For a given $i$, we choose $\varphi$ such that $\varphi = \delta_{i,j}$ on $K_j$, so that (8) becomes

$$u_i = \lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} P^n \varphi$$

where the limit is uniform. The continuity of $u_i$ follows. \hfill \Box

**Proposition 4.11.** If the system is indecomposable in the sense of Definition 2.4, then for every continuous $\varphi : X \to \mathbb{R}$,

$$P^n \varphi \xrightarrow{n \to +\infty} \psi$$

where $\psi$ is defined as in the Proposition 4.10.

The fact that the system is indecomposable is used to obtain the following fact, whose proof is postpone:
Lemma 4.12. If \( \omega \mapsto (f^n_\omega)_{n \geq 0} \) acts minimally on \( X \) and is indecomposable, then for any positive integer \( p \), \( \omega \mapsto (f^{pn}_\omega)_{n \geq 0} \) also acts minimally on \( X \).

Proof of Proposition 4.11. Let \( \varphi : X \to \mathbb{R} \) be a continuous mapping. Thanks to Lemma 4.9, the only thing we need to prove is that \( \|P\varphi\|_{\infty} \) is a cluster value of \( (P^n\varphi)_{n \geq 0} \). Thus, let \( \psi = \lim_{k \to +\infty} P^n\varphi \) be a cluster value of \( (P^n\varphi)_n \).

Firstly, up to extracting a subsequence, we can assume that \( \int q d\mu_i = 0 \) for \( i = 1, \ldots, d \), so that we want to prove that \( \psi = 0 \).

Secondly, we can reduce the problem to the case where \( \varphi = \psi \); indeed, up to extracting a subsequence, we can assume that \( m_k = n_{k+1} - n_k \) tends to \( +\infty \) when \( k \) tends to \( +\infty \). Using that \( P \) is contracting for \( \| \cdot \|_\infty \), we have

\[
\|P^n\varphi - \psi\|_\infty \leq \|P^n(\varphi - P^n\varphi)\|_\infty + \|P^{n+1}\varphi - \psi\|_\infty \xrightarrow{k \to +\infty} 0
\] (9)

Thus, from now on we assume that:

- \( \int q d\mu_i = 0 \) for \( i = 1, \ldots, d \),
- \( P^n\varphi \xrightarrow{k \to +\infty} \varphi \),

and we want to prove that \( \varphi = 0 \). We begin by treating the restriction of the problem to a minimal subset \( F_i = \text{supp}(\mu_i) \). We will use the following remark:

Lemma 4.13. For any continuous \( \varphi : X \to \mathbb{R} \) and any positive integer \( k \), we have \( \|P^k\varphi\|_{L^2(\mu_i)} \leq \|\varphi\|_{L^2(\mu_i)} \) with equality if and only if for every \( P \)-almost every \( \omega, \omega' \), \( \varphi \circ f^k_\omega = \varphi \circ f^k_{\omega'} \) on \( F_i \).

Proof. The inequality is just a consequence of the Jensen inequality \( P^k(\varphi)^2 \leq P^k(\varphi^2) \) and of the \( P^k \)-invariance of \( \mu_i \), and in the equality case, for almost every \( \omega, \omega' \), \( \varphi \circ f^k_\omega = \varphi \circ f^k_{\omega'} \) \( \mu_i \)-almost everywhere, hence on \( F_i \) by continuity. \( \square \)

By the lemma, the sequence \( (\|P^n\varphi\|_{L^2(\mu_i)}) \) is non increasing. For any integer \( p \), writing that

\[
\|P^n\varphi\|_{L^2(\mu_i)} \leq \|P^{nk+p}\varphi\|_{L^2(\mu_i)} \leq \|P^{nk}\varphi\|_{L^2(\mu_i)}
\]

and passing to the limit, we obtain that \( \|P^n\varphi\|_{L^2(\mu_i)} = \|\varphi\|_{L^2(\mu_i)} \), and hence by the lemma, for \( P \)-almost every \( \omega \), \( \varphi \circ f^n_\omega = P^n\varphi \) on \( F_i \).

As a consequence, we obtain that for \( P \)-almost every \( \omega \) in \( \Omega \),

\[
\varphi \circ f^n_\omega = P^n\varphi \xrightarrow{k \to +\infty} \varphi \text{ on } F_i
\]
In particular, if \( B \) is a contractible ball, then \( \varphi \) is constant on \( B \). By compactness, \( \varphi \) only takes a finite number of values. We deduce that fixing an integer \( p = n_k \) with \( k \) large enough, we have \( \varphi \circ f^p_\omega \equiv \varphi \) on \( F_i \) for \( \mathbb{P} \)-almost every \( \omega \). Hence \( \varphi \) is constant on \( F_i \) by Lemma 4.12 and this constant is necessarily \( \int \varphi d\mu_i = 0 \).

We now go back to the whole space: we know that \( \varphi \) is identically zero on each \( F_i \). And for any \( x \) in \( X \), for almost every \( \omega \), all the cluster values of \( (f^p_n(\omega))(x) \) belong to a minimal set \( F_i \) (Proposition 4.8), hence \( \varphi(f^p_n(\omega)(x)) \to 0 \), hence by integration over \( \omega \), \( P^n\varphi(x) \to 0 \), and in particular,

\[
\varphi(x) = \lim_k P^k_n \varphi(x) = 0.
\]

Thus \( \varphi \) is identically zero on \( X \).

\( \square \)

**Proof of Lemma 4.12**  If \( F \) is a closed subset of \( X \), let us set

\[
\Theta(F) = \bigcup_{f \in \text{supp}(\nu)} f(F).
\]

We want to prove that if \( F \) is a non empty closed subset such that \( \Theta(F) \subset F \) then \( F = X \). Set

\[
\mathcal{F} = \{ F \subset X \text{ closed}, F \neq \emptyset, \Theta(F) \subset F \},
\]

and let \( F \in \mathcal{F} \) be minimal with respect to the inclusion. Then:

- for any integer \( k \), \( \Theta^k(F) \in \mathcal{F} \) (obvious);
- \( \Theta(F) = F \) by minimality of \( F \), since \( \Theta(F) \in \mathcal{F} \) and \( \Theta(F) \subset F \);
- for any integer \( k \), \( \Theta^k(F) \) is minimal with respect to the inclusion in \( \mathcal{F} \): indeed, if \( G \in \mathcal{F} \) and \( G \subset \Theta^k(F) \) with \( k < p \), then \( \Theta^{p-k}(G) \in \mathcal{F} \) and \( \Theta^{p-k}(G) \subset F \), hence \( \Theta^{p-k}(G) = F \) by minimality, and hence \( \Theta^k(F) = \Theta^p(G) \subset G \).

We conclude that the sequence \( (\Theta^k(F))_k \) is periodic, with elements that are pairwise disjoint or equal (by minimality). Let \( p' \) the smallest integer such that \( \Theta^{p'}(F) = F \). The sequence \( F, \Theta(F), \ldots, \Theta^{p'-1}(F) \) is a sequence of pairwise disjoint closed sets such that any \( f \) in \( \text{supp}(\nu) \) sends each set into the following, and the last one into the first. Because of the assumption of indecomposability, \( p' \) is necessarily equal to 1. As a consequence, \( \Theta(F) \subset F \), which means that \( F \) is invariant by any \( f \) in \( \text{supp}(\nu) \), and hence \( F = X \) by minimality of the random walk.

\( \square \)

### 4.4 Global contractions

The following theorem shows that from the local phenomenon of contractions given by Assumption A, we can obtain a phenomenon of global contractions, in the sense that almost surely, the number of domains of attraction is finite: this result is close to a result of Y. Le Jan [16].
Proposition 4.14. We do in this proposition the additional assumption that $X$ is locally connected. Then there exists a positive integer $p$, such that, for $\mathbb{P}$-almost every $\omega$ in $\Omega$, there exists $p$ connected open sets $U_1(\omega), \ldots, U_p(\omega)$, pairwise disjoints, such that:

- the union $U(\omega) = U_1(\omega) \cup \cdots \cup U_p(\omega)$ is dense in $X$,
- for every $i$ in $\{1, \ldots, p\}$, for every $x, y$ in $U_i(\omega)$,
  $$d(f^n_\omega(x), f^n_\omega(y)) \to 0.$$ 

Proof of Theorem D.

Let us consider one more time the set

$$\mathcal{E} = \{(\omega, x) \in \Omega \times S^1(f^n_\omega)\text{ contracts a neighbourhood of } x\} = \bigcup_{\omega \in \Omega} \{\omega\} \times U(\omega).$$

By Proposition 4.7, there is a finite number of stationary probability measures $\mu_1, \ldots, \mu_d$. For each $i$ in $\{1, \ldots, d\}$, let $U_i(\omega)$ be the union of the connected components of $U(\omega)$ which have a positive $\mu_i$-measure. For $\mathbb{P}$-almost every $\omega$, the set $U_i(\omega)$ is an open subset with $\mu_i$-measure 1, and has by Proposition 4.3 a finite constant number $p_i$ of connected components. Write $\tilde{U}(\omega) = U_1(\omega) \cup \cdots \cup U_d(\omega)$. As a consequence of Corollary 4.2 applied to $\tilde{\mathcal{E}} = \bigcup_{\omega \in \Omega} \{\omega\} \times \tilde{U}(\omega)$, we know that $\mathbb{P} \otimes \mu(\tilde{\mathcal{E}}) = 1$ for every probability measure $\mu$, and hence that $\tilde{U}(\omega)$ is dense for $\mathbb{P}$-almost every $\omega$. As a consequence, for $\mathbb{P}$-almost every $\omega$, $\tilde{U}(\omega)$ is a dense open subset of $X$ with a finite number $p = \sum_i p_i$ of connected components (and hence in fact, $U(\omega) = \tilde{U}(\omega)$). $\Box$

We conclude with a criterion ensuring the synchronization of the RDS.

Proposition 4.15. The following assertions are equivalent:

1. $\omega \mapsto (f^n_\omega)$ is synchronizing;
2. the random walk $\omega \mapsto (f^n_\omega, f^n_\omega)$ admits a unique stationary probability measure on $X \times X$;
3. for every $x, y$ in $X$, there exists a sequence $(g_n)_n$ of elements of $G$ such that $d(g_n(x), g_n(y)) \to 0$.

Proof. Let us notice that $\omega \mapsto (f^n_\omega, f^n_\omega)$ also satisfies the property of local contractions, so that the previous propositions of the section apply to it. We will denote by $\tilde{G}$ the semigroup associated to $\omega \mapsto (f^n_\omega, f^n_\omega)$, and by $D$ the diagonal of $X \times X$.

1 $\Rightarrow$ 3 is trivial.

3 $\Rightarrow$ 2: By Proposition 4.7, if there are two distinct stationary probability measures, then their respective topological support $F_1$ and $F_2$ would be two disjoint closed non empty subsets of $X \times X$ invariant by $\tilde{G}$. Let $(x, y)$ be any point of $F_1$. By assumption, one can find a sequence of elements $g_n$ in $G$ such
that the distance between \( g_n(x) \) and \( g_n(y) \) tends to 0. Since \( (g_n(x), g_n(y)) \in F_1 \), taking a cluster value of the sequence we deduce that \( F_1 \) intersects \( D \) at some point \((z_1, z_1)\). In the same way, \( F_2 \) intersects \( D \) at some point \((z_2, z_2)\). Choosing then a sequence \((h_n)\) in \( G \) such that \( d(h_n(z_1), h_n(z_2)) \to 0 \), any cluster value of \((h_n(z_1), h_n(z_2))\) is also a cluster value of \((h_n(z_2), h_n(z_2))\) and hence belongs to \( F_1 \cap F_2 \), which is absurd.

2 \( \Rightarrow \) 1: By Proposition 4.7, there is a unique minimal non empty closed subset \( F \) invariant by \( \tilde{G} \). Since \( D \) is \( \tilde{G} \)-invariant, \( F \subset D \). By Proposition 4.8, for every \((x, y)\) in \( X \times X \), for \( \mathbb{P} \)-almost every \( \omega \) in \( \Omega \), the set of cluster values of \((f_n^\omega(x), f_n^\omega(y))_{n \in \mathbb{N}} \) is exactly \( F \). In particular, it is included in \( D \), hence \( d(f_n^\omega(x), f_n^\omega(y)) \to 0 \) as \( n \to +\infty \). \( \square \)

5 Proof of the results

By Theorem A, a random walk on \( \text{Homeo}(S^1) \) without invariant probability measures is locally contracting, so that most of the theorems stated in the introduction becomes corollaries of the general results stated in previous part.

5.1 Behaviour of random walks on \( \text{Homeo}(S^1) \)

Proof of Theorem B

If we are in the first case of Proposition 2.2 then the result is a direct consequence of Proposition 4.7 and 4.10. If not, then we are in the second case since \( G \) has no finite orbit. That means that \( G \) is semiconjugated to a minimal semigroup of isometries, and it is classical in this case that the stationary probability measure is unique: assuming, up to a conjugation that \( G \) is a semigroup of isometries acting minimally, if \( \mu_1 \) and \( \mu_2 \) are two ergodic stationary probabilities, one can find Birkhoff’s points of \( \mu_1 \) and \( \mu_2 \) arbitrary close, and then the trajectories of these points stays close, so that \( \mu_1 \) and \( \mu_2 \) are themselves arbitrarily close, hence equal. Thus, Theorem B remains valid in this case. \( \square \)

Proof of Theorem C

This is a direct consequence of Proposition 4.11 with the notations of the statement, the distribution \( \mu_x^\omega \) is given by \( \int q \, d\mu_x^\omega = \mathbb{P}^n q(x) \), so that the proposition implies that \( (\mu_x^\omega)_n \) converges in law, uniformly in \( x \), to the stationary probability measure \( \mu_x = \sum_{i=1}^d u_i(x)\mu_i \). \( \square \)

Proof of Theorem D

As a consequence of Proposition 4.14 for \( \mathbb{P} \)-almost \( \omega \), the set \( U(\omega) \) of the points having a neighborhood contracted by \((f_\omega^n)_n \) is dense and has a finite constant number \( d \) of connected components, so that \( S^1 - U(\omega) \) is finite of cardinal \( d \). To obtain the exponential contractions, it is enough to copy the proof of Proposition 4.14 replacing \( U(\omega) \) by the set \( \tilde{U}(\omega) \) of the points having a neighborhood contracted by \((f_\omega^n)_n \). \( \square \)
5.2 Synchronization

Proof of Theorem 5.2. The only non-trivial implication is iii) \(\Rightarrow i\). Let us assume that \(G\) satisfy the property iii) is satisfied in the sense that any points \(x, y\) can be synchronized in the sense that there exists a sequence \(g_n\) such that dist\((g_n(x), g_n(y))\) \(\to 0\).

Firstly, let us justify that we are in the first case of Proposition 2.2:

If \(G\) is semi-conjugated to \(\tilde{G}\), then \(\tilde{G}\) satisfies the same property of synchronization, so that \(\tilde{G}\) is not a group of isometries, and so we are not in second case.

If \(G\) leaves a finite set invariant, with has least two distinct points, these points cannot be synchronized, which contradicts the assumptions. And \(G\) cannot fix a singleton by assumption. Hence we are not in third case.

So we are in case i), that is, the random walk satisfy the property of contractions given by Theorem A. For any \(x, y\) in the circle, one can find a sequence \(g_n\) in \(G\) such that \((f_n^a(x), f_n^b(y))\) tends to 0 exponentially fast as \(n\) tends to \(+\infty\).

Proof of Corollary 2.8. Notice that seeing \(I\) as a part of \(S^1\), we can prolong arbitrarily any injective map of \(I\) to a homeomorphisms. Thus, in order to prove the statement by using Theorem 5.2 we want to prove that

- There is no point of \(I\) fixed by every element of \(G\);
- There exist a sequence \((g_n)_n\) is \(G\) such that

\[
\text{diam}(g_n(I)) \xrightarrow{n \to +\infty} 0.
\]

The first point is straightforward, since such a point fixed by \(G\) belongs \(\bigcap_{g \in G} g(I) = \emptyset\).

Let us prove the second point. Let us denote, for \(g \in G\), \([a(g), b(g)] = g(I)\), and

\[
a = \sup_{g \in G} a(g), b = \inf_{g \in G} b(g).
\]

If \(a \leq b\), then \([a, b] \subset \bigcap_{g \in G} [a(g), b(g)] \subset \bigcap_{g \in G} g(I)\), what is a contradiction. Thus, \(a > b\), and hence one can find \(g\) and \(h\) in \(G\) such that \(a(g) > b(h)\), what implies that \(g(I) \cap h(I) = \emptyset\). Up to replacing \(g\) by \(g \circ g\), we assume that \(g\) is increasing, and then, since \(g\) has no fixed point on \(h(I)\), we deduce that the sequence \((g^n)_n\) converges on \(h(I)\) to a constant (the smaller fixed
point of $g$). In consequence, the sequence $g_n = g^n h$ satisfies the second point. That concludes the proof. □

Proof of Corollary 2.9 Let $K$ be the closed minimal invariant by $G$ (necessarily unique).

Lemma 5.1. There exists $g$ in $G$ having a robust fixed point, and $g|_K \not= Id_K$.

(We say that $g$ has a robust fixed point if every small $C^0$-perturbation of $g$ has a fixed point.)

Proof. Let $x$ be any point of $K$. By Theorem 1 one can find $\omega \in \Omega$ and a neighbourhood $I_0$ of $x$ such that $\text{diam}(f^\omega_0(I_0)) \to 0$ and $(f^\omega_n(x))$ is dense in $K$. Thus we can find some integer $n$ such that $f^\omega_0(I_0) \subset I_0 - \{x\}$. Thus $g = f^\omega_n$ satisfy $g(I_0) \subset I_0$ (which implies that $g$ has a robust fixed point ) and $g(x) \not= x$. □

Let $g$ be as in the lemma, and $I$ be an open interval intersecting $K$ such that $g$ has no fixed point on the closure of $I$. Let $x$ and $y$ be in $S^1$. For almost every $\omega$, the trajectories $(f^\omega_0(x))$ and $(f^\omega_0(y))$ are asymptotically identical and are dense in $K$. We deduce that we can find $h$ in $G$ such that $h(x), h(y) \in I$. By compactness, one can find $h_1, \ldots, h_p$ in $G$ such that for any $x, y$ in $S^1$, $h_i(x), h_i(y) \in I$ for some $i$ in $\{1, \ldots, p\}$.

Now, let $\tilde{f}_1, \ldots, \tilde{f}_d$ be small $C^0$-perturbations of the generators $f_1, \ldots, f_d$. $G$ be the semigroup generated by these new generators, and $\tilde{g}, \tilde{h}_1, \ldots, \tilde{h}_p \in G$ be the corresponding perturbations of $g, h_1, \ldots, h_p$. If the perturbations are small enough, the properties

- $\forall x \in I, \tilde{g}(x) \not= x$,
- $\tilde{g}$ has a fixed point,
- $\forall x, y \in S^1 : \exists i \in \{1, \ldots, p\} | \tilde{h}_i(x), \tilde{h}_i(y) \in I$,

are still satisfied. The two first properties imply that $(\tilde{g}^n)$ converges to a constant on $I$, and hence using the third one we deduce that for any $x, y$ in $S^1$, there exists $i$ such that $\text{dist}(\tilde{g}^n \circ \tilde{h}_i(x), \tilde{g}^n \circ \tilde{h}_i(y)) \to 0$ as $n \to +\infty$. Thus, we can use Proposition E to conclude that every random walk which is non degenerated on $G$ is synchronizing. □

5.3 Random walks on $\text{Homeo}_+(\{0, 1\})$

Proof of Theorem 1 We identify the random walk to a random walk on $\text{Homeo}(S^1)$ fixing a common point $c$, that we can assume to be unique (up to looking at an invariant subinterval). We use Proposition 3.15 let $x_0$ be between $m$ and $M$, and set $\Omega_1 = \{\omega \mid \lambda_{\omega}(0, x_0) = 0\}$. By Proposition 3.15 for almost every $\omega$ in $\Omega_1$, a weak cluster value $\mu$ of

$$\mu_{N, \omega} = \frac{1}{N} \sum_{n=0}^{N-1} \delta_{f^n(x_0)}$$
is a stationary probability measure satisfying $\lambda_{\text{con}}(P \otimes \mu) = 0$ so that by Theorem 5, $\mu$ is invariant by the random walk. But the assumptions imply that $G$ has a common fixed point $c$, hence $\mu = \delta_c$. As a consequence, $f^n_\omega(x_0) \to c$ for $n$ in a set of integers of density 1. Then, we deduce that for at least one of the two arcs $I$ connecting $c$ and $x_0$, and for a set $A$ of integers with positive density, we have $f^n_\omega(x) \to c$ for $n \in A, x \in I$. But this means that for $x$ in $I$, $(\omega, x)$ is not a Birkhoff point for $\mu$. Since $\text{supp}(\mu)$ intersects $I$, we conclude that $P(\Omega_1) = 0$. □

Proof of Corollary 2.11
This is a consequence of Proposition 4.6. Every point $x$ has a contractible neighbourhood $I$, and there exists exactly one stationary ergodic probability measure such that $\mu_x(I) > 0$ (at most one because of Proposition 4.6, and at least one because $\mu(I) > 0$). Then, $x \mapsto \mu_x$ is locally constant, hence constant, and it is the only ergodic stationary probability measure supported on $(0, 1)$, and hence is $\mu$. □

References

[1] V. A. Antonov. Modeling cyclic evolution processes: Synchronization by means of a random signal. Leningradskii Universitet Vestnik Matematika Mekhanika Astronomiia, pages 67–76, April 1984.

[2] A. Ávila and M. Viana. Extremal Lyapunov exponents: an invariance principle and applications. Inventiones mathematicae, 181(1):115–178, 2010.

[3] P.H. Baxendale. Lyapunov exponents and relative entropy for a stochastic flow of diffeomorphisms. Probability Theory and Related Fields, 81(4):521–554, 1989.

[4] H. Crauel. Extremal exponents of random dynamical systems do not vanish. Journal of Dynamics and Differential Equations, 2(3):245–291, 1990.

[5] B. Deroin. Propriétés ergodiques des groupes de difféomorphismes du cercle par rapport la mesure de lebesgue. Course, disponible at http://www.math.ens.fr/deroin/Publications/coursneuchatel.pdf.

[6] B. Deroin, V. Kleptsyn, and A. Navas. Sur la dynamique unidimensionnelle en régularité intermédiaire. Acta mathematica, 199(2):199–262, 2007.

[7] B. Deroin, V. Kleptsyn, A. Navas, and K. Parwani. Symmetric random walks on Homeo(\mathbb{R}). The Annals of Probability, 41(3B):2066–2089, 2013.

[8] Alex Furman. Random walks on groups and random transformations. Handbook of dynamical systems, 1:931–1014, 2002.

[9] H. Furstenberg. Noncommuting random products. Transactions of the American Mathematical Society, pages 377–428, 1963.
[10] Anton Gorodetski and Victor Kleptsyn. Synchronization properties of random piecewise isometries. *arXiv preprint arXiv:1408.1140*, 2014.

[11] Ale Jan Homburg. Synchronization in iterated function systems. *arXiv preprint arXiv:1303.6054*, 2013.

[12] Huyi Hu. Dimensions of invariant sets of expanding maps. *Communications in mathematical physics*, 176(2):307–320, 1996.

[13] Thomas Kaijser. On stochastic perturbations of iterations of circle maps. *Physica D: Nonlinear Phenomena*, 68(2):201–231, 1993.

[14] Yuri Kifer. *Ergodic theory of random transformations*. Springer, 1986.

[15] V.A. Kleptsyn and M.B. Nalskii. Contraction of orbits in random dynamical systems on the circle. *Functional Analysis and Its Applications*, 38(4):267–282, 2004.

[16] Y Le Jan. Équilibre statistique pour les produits de difféomorphismes aléatoires indépendants. In *Annales de l’institut Henri Poincaré (B) Probabilités et Statistiques*, volume 23, pages 111–120. Gauthier-Villars, 1987.

[17] F Ledrappier. Positivity of the exponent for stationary sequences of matrices. In *Lyapunov Exponents*, pages 56–73. Springer, 1986.

[18] François Ledrappier. Some relations between dimension and lyapounov exponents. *Communications in Mathematical Physics*, 81(2):229–238, 1981.

[19] Lai-Sang Young. Dimension, entropy and lyapunov exponents. *Ergodic theory and dynamical systems*, 2(01):109–124, 1982.