Quantum Phase Transition between $Z_2$ spin liquid and columnar Valence Bond Crystals on a Triangular lattice

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Tremendous progress has been made in the last two years in searching for spin liquid states in quantum frustrated spin models by various numerical methods [1–5]. Using the topological entanglement entropy [6, 7], $Z_2$ topological liquid phase was identified in the phase diagram of frustrated spin models and quantum boson models on the square lattice [2] and Kagome lattice [1, 3]. Although the $Z_2$ liquid phase itself does not break any symmetry, it was confirmed numerically that a $Z_2$ spin liquid phase can be very close to certain valence bond solid (VBS) state which breaks the translation symmetry [1]. Thus it is conceivable that under weak perturbations the $Z_2$ spin liquid can be driven into a VBS phase. This liquid-VBS quantum phase transition is what we will discuss in the current work. In previous works, this liquid-VBS transition was thoroughly studied on the square lattice [5, 9], honeycomb lattice [4], and also Kagome lattice [10]. The universality class of the liquid-VBS transition in general depends on the nature of the VBS pattern. On the square lattice and honeycomb lattice, quantum phase transitions between $Z_2$ liquid and simple VBS phases such as columnar, resonating-plaquette, and staggered VBS phases have all been well-understood [3, 11]. However, on the triangular lattice, starting from a $Z_2$ liquid phase, previous studies only obtained the transition into a $\sqrt{12} \times \sqrt{12}$ VBS pattern with a large unit cell [12–15] [36], while a direct transition between the $Z_2$ liquid and the simple columnar or resonating-plaquette VBS patterns was never understood in previous theoretical analysis. This is precisely the gap that we will fill in this paper (Fig. 1).

We choose to study spin systems on the triangular lattice, because the spin-1/2 organic materials with triangular lattice are the best experimental candidates for spin liquid [17–25], and some of the organic materials in the same family indeed have the columnar VBS order [24]. Our analysis predicts that on an isotropic triangular lattice, the $Z_2$ liquid to the 12-fold degenerate columnar/plaquette VBS order may belong to a 3d $O(6)^*\$ universality class (Fig. 1); while on a distorted triangular lattice, the VBS pattern becomes either 2-fold or 4-fold degenerate (depending on the details of the distortion), and this liquid-VBS transition can reduce to a 3d $XY^*$ transition. All these transitions have very large anomalous dimension of the VBS order parameter, and these anomalous dimensions can be tested by future numerical simulations.

Motivated by the recent discovery of $Z_2$ spin liquid on the square and Kagome lattice, we expect that the same $Z_2$ spin liquid can be realized with a certain spin-1/2 Hamiltonian on the triangular lattice as well. In order to analyze a spin liquid, it is standard to introduce the slave particles: $\vec{S}_i = \frac{1}{2} f^\dagger_{i,\alpha} \vec{\sigma}_{\alpha\beta} f_{i,\beta}$. $f_{i,\alpha}$ can be either bosonic or fermionic spin-1/2 excitations, but either choice is subject to a local constraint in order to match the slave particle Hilbert space with the spin Hilbert space: $\sum_{\alpha} f^\dagger_{i,\alpha} f_{i,\alpha} = n_i = 1$. This local constraint introduces a continuous gauge symmetry (U(1) for bosonic spinons and SU(2) for fermionic spinons), which at low energy can be broken down to a $Z_2$ gauge symmetry by the mean field state of $f_{i,\alpha}$. Then the low energy physics of this state is described by a pure $Z_2$ gauge field. In this paper we assume that the $Z_2$ spin liquid itself respects all the symmetry of the lattice, which is possible on the triangular lattice [27].

In both the $Z_2$ spin liquid and VBS phase, the spin excitation $f_{i,\alpha}$ is fully gapped. Therefore we can
excitation of $\sigma$ over three or five bonds. The gauge constraint and $h > 0$ in addition, forces each site to only share an odd number of bonds. Physically, this constraint is therefore the operator that creates/annihilates a bond. The first term in Eq. 1 is a ring product of $\sigma^z$ on every triangle plaquette on the lattice; we will keep $K > 0$ so that the Hamiltonian Eq. 1 favors $\sigma^z = 1$ on all the solid links, while $\eta = -1$ on the dotted links. Every unit cell has four sites $0 \cdots 3$. In order to obtain the quantum VBS order, we consider the quantum Ising model defined on this honeycomb lattice with Ising couplings between sites up to 7th nearest-neighbor (One pair of 7th neighbor sites is shown on the figure).

Here $p$ and $q$ denote the sites of the dual honeycomb lattice (plaquettes of the triangular lattice). In this dual representation, $\tau^a$ are operators, while $\eta_{pq} = \pm 1$ are pure numbers. $\tau^x_p$ is the vison density ($\tau^x_p = -1$ means there is one vison at dual site $p$), and $\tau^z_p$ creates/annihilates a vison on dual site $p$.

Because of the constraint Eq. 2 $\eta_{pq}$ must also satisfy a constraint: $\prod_{pq \in \text{hexagon}} \eta_{pq} = -1$, i.e. the dual quantum Ising model is fully frustrated. Here we choose $\eta_{pq} = -1$ on the dotted links in Fig. 2, while $\eta_{pq} = +1$ otherwise. In this dual Hamiltonian, $\tau^x$ itself is not gauge invariant because it is only defined as a bilinear. The choice of $\eta_{pq}$ we have made on the dual lattice is just a gauge choice, which has to apparently break the lattice symmetry, hence each unit cell on the dual lattice contains four sites $(0 \cdots 3)$. The correct lattice symmetry transformation for the dual vison $\tau^x$ must be combined with a nontrivial $Z_2$ gauge transformation; this combined transformation is called the projective symmetry group (PSG). The dual quantum Ising model has to be invariant under PSG, and the ellipse in Eq. 4 can contain arbitrary further neighbor Ising couplings allowed by the PSG.

The liquid-VBS phase transition corresponds to the Ising disorder-order phase transition in the dual Hamiltonian (Eq. 4), which is driven by the condensation of $\tau^x_p$. Because Eq. 4 is a fully frustrated quantum Ising model, $\tau^x$ can condense at nonzero momenta in the dual Brillouin zone (BZ). If Eq. 4 only has nearest neighbor hopping (which is the case studied in Ref. [13]), then there are four different minima in the BZ (Fig 2a), and the vison condensate corresponds to a $\sqrt{12} \times \sqrt{12}$ VBS pattern with a large unit cell $[12] [12] [12]$. The PSG also guarantees that this liquid-VBS transition belongs to a
3d $O(4)^*$ universality class.

Our goal is to study the quantum phase transition between the $Z_2$ liquid and the simple columnar/plaquette VBS order on the triangular lattice (Fig. 4). With nearest neighbor Ising couplings only, the dual Hamiltonian (Eq. 4) will not produce the columnar/plaquette VBS order. We have to turn on further neighbor couplings in Eq. 4 that are allowed by the PSG. We have thoroughly explored the possible phases of Eq. 4. A negative 2ndneighbor Ising coupling on the dual lattice will destabilize the original order in Fig. 4 at the cost of a ring degeneracy. This ring degeneracy is not broken until seventh neighbor couplings are added. With the seventh neighbor couplings, the minima of Eq. 4 are then stabilized at six different momenta (Fig. 4):

\[
\begin{align*}
\vec{Q}_1 &= (0, 0), \quad \vec{Q}_2 = (\frac{\pi}{\sqrt{3}}, \frac{\pi}{3}), \\
\vec{Q}_3 &= (\frac{\pi}{\sqrt{3}}, 0), \quad \vec{Q}_4 = (0, \frac{\pi}{3}), \\
\vec{Q}_5 &= -\vec{Q}_6 = (\frac{\pi}{2\sqrt{3}}, \frac{\pi}{6}). 
\end{align*}
\]

(5)

To analyze the low energy physics, we expand the Ising operator $\tau^2$ at these six minima:

\[
\tau^2_{r,\alpha} = \sum_{a=1}^{4} \psi_{a,r} \psi_{a,\alpha} e^{i\vec{Q}_a \cdot \vec{r}} + \sqrt{2} \psi_5(r) \text{Re} [e^{i\vec{Q}_5 \cdot \vec{r}}] + \sqrt{2} \psi_6(r) \text{Im} [e^{i\vec{Q}_6 \cdot \vec{r}}].
\]

(6)

Here $\alpha = 0 \cdots 3$ denotes the four sites in every unit cell; $\psi_r$ with $a = 1 \cdots 6$ are the six low energy modes at the BZ minima; and $\psi_r$ are all real fields. The four component vectors $\psi_{a,r}$ (given in the appendix) are the wave functions of $\tau^2_{a,r}$ at each momentum $\vec{Q}_a$. The action of the PSG on $\tau^2_{a,r}$ will induce the action of the PSG on $\psi_{a,r}$.

The PSG group is generated by the following transformations (assume the lattice constant of the original triangular lattice is $\sqrt{3}$):

\[
\begin{align*}
T_1 : \quad x &\to x + \sqrt{3}, \quad \psi_r \to T_{1} \psi_{r,\alpha}; \\
T_2 : \quad x &\to x + \frac{\sqrt{3}}{2}, \quad y \to y + \frac{3}{2}, \quad \psi_r \to T_{2} \psi_{r,\alpha}; \\
P_z : \quad y &\to -y, \quad \psi_r \to \psi_{r,\alpha}; \\
R_{\pi/3} : \quad \psi_r \to R_{\pi/3} \psi_{r,\alpha}; \\
TR : \quad t &\to -t, \quad \psi_r \to \psi_r.
\end{align*}
\]

(7)

Here $R_{\pi/3}$ is the rotation by $\pi/3$ around a hexagon center. $\psi_r$ carries a six dimensional representation of the PSG, and the PSG representation matrices are given in the appendix.

The low energy physics of the dual Hamiltonian (Eq. 4) can be completely described by $\psi_r$ and its effective Lagrangian, which must be invariant under the PSG. The PSG allowed Lagrangian reads:

\[
\mathcal{L} = \sum_{a=1}^{6} (\partial_{\mu} \psi_a)^2 + r(\psi_a)^2 + g \sum_{a=1}^{6} (\psi_a^2)^2 + \sum_{a=1}^{3} v_4 |\Psi_a|^4 + \sum_{a=1}^{3} v_4^* |\Psi_{a+1}|^2 \cos(2\theta_a) \sin(2\theta_{a+1}),
\]

(8)

where $\Psi_a = |\Psi_a| e^{i\theta_a}$. The complex fields $\Psi_a$ are defined as $\Psi_1 = \psi_1 + i\psi_2$, $\Psi_2 = (\psi_3 + i\psi_4)e^{i\pi/4}$, $\Psi_3 = \psi_5 + i\psi_6$, and $\Psi_{a+3} = \Psi_a$. The first three terms in Eq. 8 are invariant under an enlarged $O(6)$ rotation of $\psi_a$, while the $v_4$ and $v_4^*$ terms break this $O(6)$ symmetry. We always assume that $g$ is the dominant 4th order term in Eq. 8.

Under this assumption, there is a competition between $v_4$ and $v_4^*$, and when $v_4 < -|v_4^*|/2$, only one of the three $\langle \Psi_a \rangle$ is nonzero.

In this paper we will focus on the case with $v_4 < -|v_4^*|/2$, where only one of the three $\langle \Psi_a \rangle$ is nonzero. In this case $v_4^*$ is unimportant in the vison condensate. At the 8th order, there is a $Z_8$ anisotropy term allowed by the PSG:

\[
\mathcal{L}_8 = v_8 \sum_{a=1}^{3} |\Psi_a|^8 \cos(8\theta_a).
\]

(9)
In this case, with either sign of \( v \) in the horizontal direction (Fig. 4a), patterns with bonds (links with \( n \) paper. The VBS order depends on the sign of \( v \) at the point and the ordered phases we are considering in this PSG, but they are unimportant to both the critical transition. The first three terms in Eq. 8 are invariant of \( \Psi_4 \) and can therefore be safely ignored at this transition.

Some other 6th and 8th order terms are also allowed by the PSG, but they are unimportant to both the critical point and the ordered phases we are considering in this paper. The VBS order depends on the sign of \( v_4 \) and \( v_8 \). In this case, with either sign of \( v_8 \), the VBS orders are 12 fold degenerate

\[
\begin{align*}
\Psi_4 &= 0, \quad \Psi_8 < 0, \quad \Psi_4 \sim \exp(i\theta_a), \quad \theta_a = \frac{n\pi}{4}, \\
\Psi_4 &= 0, \quad \Psi_8 > 0, \quad \Psi_4 \sim \exp(i\theta_a), \quad \theta_a = \frac{\pi}{8} + \frac{n\pi}{4}, \\
\Psi_4 &\neq 0, \quad \Psi_8 = 0, \quad \Psi_4 \sim \exp(i\theta_a), \quad \theta_a = \frac{n\pi}{8} + \frac{n\pi}{4}, \\
\Psi_4 &\neq 0, \quad \Psi_8 \neq 0, \quad \Psi_4 \sim \exp(i\theta_a), \quad \theta_a = \frac{n\pi}{8} + \frac{3n\pi}{4}.
\end{align*}
\]

The negative and positive \( v_8 \) correspond to the columnar and plaquette VBS orders on the triangular lattice respectively. For example, when \( \langle \Psi_3 \rangle \neq 0 \) and \( \langle \Psi_1 \rangle = \langle \Psi_2 \rangle = 0 \), \( \theta_3 = \frac{n\pi}{4} \) corresponds to four VBS patterns with bonds (links with \( \langle \sigma^z \rangle \sim -1 \) aligning in the horizontal direction (Fig. 4b, 4f). In Eq. 10 \( n \leq 3 \) (instead of 7) because \( \theta_a \) and \( \theta_a + \pi \) are physically equivalent since the vison fields \( \tau^z \) and \( \Psi_a \) are only defined up to an arbitrary \( Z_2 \) gauge transformation \( \Psi_a \rightarrow -\Psi_a \). The bond density with different condensates of \( \Psi_a \) is plotted in Fig. 4a.

Now let us study the nature of the quantum phase transition. The first three terms in Eq. 8 are invariant under an \( O(6) \) rotation of \( \psi_a \). All of the 6th and 8th order terms are clearly irrelevant at the 3d \( O(6) \) fixed point and can therefore be safely ignored at this transition. The only potentially relevant perturbations are \( v_4 \) and \( v'_4 \), whose scaling dimensions are unclear, but they can be determined by straightforward high order \( \epsilon = 4 - d \) expansion. If \( v_4 \) and \( v'_4 \) are irrelevant, then the transition of the dual Lagrangian Eq. 8 should belong to the 3d \( O(6) \) universality class. However, any physical order parameter will always be a bilinear of \( \psi_a \). For example, the four columnar VBS order parameters with horizontal valence bonds is \( V_3 \sim \Psi_3^2 \). Thus this order parameter has an enormous anomalous dimension \( \eta \) where

\[
\langle V_3(x) V_3(x') \rangle \sim \frac{1}{|x-x'|^{1+\eta}}.
\]

Note that \( \eta \) must be greater than 1, which is much larger than any ordinary Wilson-Fisher fixed point. Thus we call the transition in the original Hamiltonian (Eq. 1) a 3d \( O(6)^* \) universality class, which should have dynamical exponent \( z = 1 \), and the same critical exponent \( \nu \) as the ordinary 3d \( O(6) \) transition, but a much larger anomalous dimension \( \eta \).

All of the previous discussions were under the assumption that the triangular lattice is fully isotropic. In real materials, a triangular lattice is usually distorted. For example, in the triangular lattice spin-1/2 material \( \text{Cs}_2\text{CuCl}_4 \) [29, 30], the Heisenberg coupling is much stronger along one of the three directions. Now let us break the \( \pi/3 \) rotation symmetry but keep the translation \( (T_1, T_2) \), reflection \( y \rightarrow -y (P_x) \), and inversion \( r \rightarrow -r ((R_{\pi/3})^3) \) symmetries. This is precisely the symmetry of the material \( \text{Cs}_2\text{CuCl}_4 \). The locations of the minima in the BZ are stable against this symmetry reduction. However, although \( \Psi_1 \) and \( \Psi_2 \) are still degenerate, they are no longer degenerate with \( \Psi_3 \); namely the 12 fold degeneracy between different columnar VBS orders will be lifted.

If \( \Psi_3 \) is the lowest energy mode (i.e. the spin coupling along the horizontal links on the triangular lattice is stronger than the other two directions), then the PSG allowed Lagrangian reads

\[
\begin{align*}
\mathcal{L} &= |\partial_x \Psi_3|^2 + r|\Psi_3|^2 + g|\Psi_3|^4 + g_6|\Psi_3|^6 \\
&+ v_8|\Psi_3|^8 \cos(8\theta_3) + \cdots
\end{align*}
\]

Again, depending on the sign of \( v_8 \), the ground state of the VBS phase (the phase with \( r < 0 \)) is either a columnar (Fig. 4a, 4b) or plaquette (Fig. 4c) VBS, both four-fold degenerate. The first three terms of Eq. 12 describe a 3d XY transition. In this case, because \( g_6 \) and \( v_8 \) are clearly irrelevant at the 3d XY fixed point, and because the VBS order parameter is still a bilinear of \( \psi_a \), the liquid-VBS phase transition belongs to the well-studied 3d XY* transition with anomalous dimension \( \eta \sim 1.49 \) [3] for the VBS order parameter.

If \( \Psi_1 \) and \( \Psi_2 \) are the lowest energy modes, then the low energy effective Lagrangian reads

\[
\mathcal{L} = \sum_{a=1,2} |\partial_x \Psi_a|^2 + r|\Psi_a|^2 + g \left( \sum_{a=1,2} |\Psi_a|^2 \right)^2
\]
Again we focus on the case with $v_4 < -|v'_4|/2$ when exactly one of $\langle \Psi_a \rangle$ is nonzero. (This eliminates the role of the $v'_4$ term in the ordered phase.) In this case, for either sign of $v'_4$, the vison condensate is a four-fold degenerate columnar VBS state (Fig. [3], [4]).

Now let us further reduce the symmetry. For example, if the $P_x$ ($y \to -y$) symmetry is broken while inversion is still preserved (this is the symmetry of most organic spin liquid materials), then the columnar VBS order has only a two-fold degeneracy which only breaks translation symmetry. Now the PSG allowed Lagrangian reads

$$L = |\partial_j \Psi_3|^2 + r |\Psi_3|^2 + g |\Psi_3|^4 + v_4 |\Psi_3|^4 \cos(4\theta_3). \quad (14)$$

For either sign of $v_4$, there is a two-fold degenerate columnar VBS order. In this case the liquid-VBS transition is still the 3d XY* transition because it is well-known that the $Z_4$ anisotropy $\cos(4\theta)$ on a 3d XY fixed point is irrelevant [31, 32].

Close to the liquid-VBS critical point, since $v_8$ in Eq. 12 and $v_4$ in Eq. 13 are both irrelevant, Eq. 12 and Eq. 13 have an emergent $U(1)$ global symmetry. Thus we can view the VBS order as a superfluid phase that spontaneously breaks this $U(1)$ symmetry; therefore the liquid-VBS transition can also be viewed as a liquid-superfluid phase transition. If we approach this critical point from the superfluid (VBS) side, then this transition is driven by the proliferation of vortex excitations of the superfluid phase. In 2+1d space-time, a superfluid phase is dual to a 2+1d U(1) gauge field. Therefore this liquid-VBS phase transition is dual to a Higgs transition:

$$L_{dual} = (|\partial_\mu - i2\alpha_\mu|^2 \Phi| + r'|\Phi|^2 + g'|\Phi|^4 + \frac{1}{e^2} f_{\mu\nu}, \quad (15)$$

where $\Phi$ is a complex field that annihilates a pair of vortices. After condensation ($r' < 0$) $\Phi$ breaks the U(1) gauge field to a $Z_2$ gauge field. Hence the condensate of vortex pairs has a $Z_2$ topological order, which is precisely the topological order of the $Z_2$ spin liquid state we started with.

In summary, in this work we studied the quantum phase transition between $Z_2$ liquid and columnar VBS orders on both the isotropic and distorted triangular lattices. The critical theories proposed in this work can be checked by future numerical simulations once a $Z_2$ spin liquid phase is identified on the triangular lattice. It would also be interesting to study the direct quantum phase transition from the magnetic order to the columnar VBS orders on the triangular lattice, which can be viewed as a triangular lattice generalization of the deconfined quantum critical point [33, 34]. This transition would be driven by condensation of Skyrmions or vortices of the magnetic order. Eventually we also plan to understand the global phase diagram around the $Z_2$ spin liquid, which probably involves a noncollinear spiral spin order, the columnar/plaquette VBS order discussed in this current paper, and a collinear spin order. A similar global phase diagram was studied in Ref. [32] for the case with four vison minima in the BZ, and we plan to generalize this to our current case with columnar VBS order. We expect to understand this global phase diagram for both spin-1/2 and spin-1 systems on the triangular lattice. We will leave these subjects to future study.

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[36] This VBS pattern has 12 sites in one unit cell on the triangular lattice [12]. However, in terms of the dual quantum Ising model on the honeycomb lattice, this pattern has 48 sites per unit cell [13].

Appendix

In this appendix we present more details about the dual frustrated quantum Ising model. With a 7th neighbor Ising coupling, in a finite region of the phase diagram, the minima of the vison band structure are stabilized at six different minima (Fig. 3):

\[
\tilde{Q}_1 = (0, 0), \quad \tilde{Q}_2 = \left(\frac{\pi}{\sqrt{3}}, \frac{\pi}{3}\right), \\
\tilde{Q}_3 = \left(\frac{\pi}{\sqrt{3}}, 0\right), \quad \tilde{Q}_4 = (0, \frac{\pi}{3}), \\
\tilde{Q}_5 = -\tilde{Q}_6 = \left(\frac{\pi}{2\sqrt{3}}, \frac{\pi}{6}\right).
\]

(16)

To analyze the low energy physics, we can expand the Ising operator \(\tau^z\) at these six minima:

\[
\tau^z_{r,\alpha} = \sum_{\alpha=1}^{4} \psi_{\alpha}(r) v_{\alpha,\alpha} e^{i\tilde{Q}_{\alpha}\cdot r}
\]

In this equation \(v_{\alpha,\alpha}\) are six four-component vectors:

\[
v_1 = \begin{pmatrix} 1 & -\sqrt{2} \\ 1 & \sqrt{2} \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 & 1 \\ -\sqrt{2} & 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 & 1 \\ 1 & -\sqrt{2} \end{pmatrix}, \quad v_4 = \begin{pmatrix} 1 & 1 \\ -\sqrt{2} & 1 \end{pmatrix}, \\
v_5 = \begin{pmatrix} e^{-\frac{4\pi}{3}} \\ e^{\frac{4\pi}{3}} \end{pmatrix}, \quad v_6 = v_5^*.
\]

(17)

The low energy modes \(\psi_{\alpha}\) carry a six dimensional representation of the PSG of the system. The entire PSG of the system is generated by the elements in Eq. 7. The representation matrices are:

\[
T_1 = \begin{pmatrix} +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},
\]

(18)

\[
T_2 = \begin{pmatrix} +1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},
\]

\[
P_x = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -\sqrt{2} & \frac{\sqrt{2}}{\sqrt{2}} \\ 0 & 0 & 0 & -\sqrt{2} & -\frac{\sqrt{2}}{2} & \sqrt{2} \end{pmatrix},
\]

(19)

\[
R_{\pi/3} = \begin{pmatrix} 0 & 0 & -1 & -\sqrt{2} & +\sqrt{2} & 0 \\ 0 & 0 & 0 & +\sqrt{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & -1 & 0 & -\sqrt{2} \\ 0 & 0 & 0 & -\sqrt{2} & +\sqrt{2} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.
\]