Symmetries of Everything

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October 28, 2018

Abstract

I argue that string theory can not be a serious candidate for the Theory of Everything, not because it lacks experimental support, but because of its algebraic shallowness. I describe two classes of algebraic structures which are deeper and more general than anything seen in string theory:

1. The multi-dimensional Virasoro algebras, i.e. the abelian but non-central extension of the algebra of vector fields in $N$ dimensions by its module of closed dual one-forms.

2. The exceptional simple Lie superalgebra $\mathfrak{mb}(3|8)$, which is the deepest possible symmetry (depth 3 in its consistent Weisfeiler grading). The grade zero subalgebra, which largely governs the representation theory, is the standard model algebra $sl(3) \oplus sl(2) \oplus gl(1)$. Some general features can be extracted from an $\mathfrak{mb}(3|8)$ gauge theory even before its detailed construction: several generations of fermions, absence of proton decay, no additional gauge bosons, manifest CP violation, and particle/anti-particle asymmetry.

I discuss classifications supporting the claim that every conceivable symmetry is known.

1 What’s wrong with string theory?

Recently it has become clear that not all mathematical physicists are entirely convinced that string theory is the ultimate Theory of Everything.

\footnote{The bulk text was written in spring 2001 and is identical to version 2, except for the corrected definition of $\mathfrak{mb}(3|8)$, which has been available in \cite{27}. The footnotes contain some observations added after the appearance of D. Friedan: \textit{A tentative theory of large-distance physics}, hep-th/0204131 (2002).}
According to my own prejudices, there are only two valid reasons to pursue a branch of theoretical physics: experimental support and intrinsic mathematical beauty. String theory has always disagreed with experiments, to the extent that it makes any falsifiable predictions whatsoever (extra dimensions, supersymmetric partners, 496 gauge bosons, ...). Of course, one may always argue that these features will show up at higher energies, not accessible to experimental falsification within the foreseeable future. However, my objection to string theory is not the fact that it is experimentally wrong (or void, according to taste), but rather that it is not based on the deepest and most general algebraic structures conceivable.

The algebraic structures of string theory (finite-dimensional semisimple Lie algebras such as $\text{so}(32)$ and $E_8 \times E_8$, and central extensions of infinite-dimensional Lie algebras such as the Virasoro and superconformal algebras) were probably the deepest structures known in 1984, at the time of the first string revolution. Actually, several of these algebras were discovered by string theorists in an earlier era. However, the algebra community has not been lazy after 1984. During the 1990s, there has been progress along at least two lines, which make the symmetries of string theory appear shallow in comparison.

1. String theory relies on the distinguished status of central extensions of infinite-dimensional Lie (super)algebras of linear growth. While it is true that only a few algebras admit central extensions, the algebra $\text{vect}(N)$ of vector fields acting on $N$-dimensional spacetime admits abelian but non-central extensions, which naturally generalize the Virasoro algebra to $N > 1$ dimensions. Geometrically, the multi-dimensional Virasoro algebra is an extension by the module of closed dual one-forms, i.e. closed $(N-1)$-forms. In particular, in $N = 1$ dimension, a closed dual one-form is a closed zero-form is a constant function, so the extension is central in this case, but not otherwise. However, compared to other abelian extensions, this cocycle is very close to central, as discussed in the next section.

Modulo technicalities, $\text{vect}(N)$ is the Lie algebra of the diffeomorphism group in $N$ dimensions. Other names for this algebra are diffeomorphism algebra and generalized Witt algebra. Clearly, abelian extensions of algebras...
of polynomial growth are more general than central extensions of algebras of linear growth, since central is a special case of abelian and linear is a special case of polynomial.

The point that the existence of multi-dimensional Virasoro algebras is a problem for string theory’s credibility was made already in [22], in the following form: There are no obstructions to superization, so the algebra vect(n|m) acting on n|m-dimensional superspace also admits two Virasoro-like extensions. Naïvely, one would expect to obtain such extensions of every subalgebra of vect(n|m) by restriction, but in some boring cases, the generically non-central extension reduces to a central one. And in some trivial cases, it vanishes completely, apart from cohomologically trivial terms. The superconformal algebra is such a boring but non-trivial case, rather than being algebraically distinguished.

These algebras ought to be relevant to quantum gravity, because

- The symmetry group of classical gravity is the full space-time diffeomorphism group.
- In quantum theory, symmetries are only represented projectively. On the Lie algebra level, this means that the symmetry algebra acquires an extension (provided that the algebra is big enough).

If we put these two observations together, we see that the symmetry algebra of quantum gravity should be an extension of the diffeomorphism algebra vect(4). Phrased differently: I suggest that quantum gravity should be quantum general covariant.

vect(N) has no central extensions if N > 1 [9], but it has many abelian extensions by irreducible modules [11, 25]. However, most (possibly all) extensions by tensor modules are limiting cases of trivial extensions, in the sense that one can construct a one-parameter family of trivial cocycles reducing to the non-trivial cocycle for a critical value of the parameter (= the conformal weight). In contrast, the Virasoro-like cocycles are not limits of trivial cocycles, because the module of closed dual forms does not depend on any continuous parameter. Moreover, these are the kinds of cocycles that arise in Fock representations. There are indications that an interesting class of lowest-energy modules can only be constructed in four spacetime dimensions [26].

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6Note added in version 3: Diffeomorphism symmetry (without compensating background fields) is the key lesson from general relativity. Projectivity is the key lesson from quantum theory. To ignore the key lessons from the two main discoveries in twentieth century physics does not seem to be the right approach to quantum gravity.
2. The classification of simple infinite-dimensional Lie superalgebras of vector fields was recently completed by Kac, Leites, Shchepochkina, and Cheng. In particular, the exception $\mathfrak{mb}(3|8) (= E(3|8))$ is the deepest possible symmetry, at least in a technical sense (depth 3 in its consistent Weisfeiler grading). Moreover, its grade zero subalgebra, which largely governs the representation theory, is $\mathfrak{sl}(3) \oplus \mathfrak{sl}(2) \oplus \mathfrak{gl}(1)$, i.e. the non-compact form of the symmetries of the standard model. The irreps of an algebra of vector fields is 1-1 with the irreps of its grade zero subalgebra. E.g., the $\mathfrak{vect}(N)$ irreps (tensor densities and closed forms) are 1-1 with the irreps of its grade zero subalgebra $\mathfrak{gl}(N)$. One may therefore speculate that an $\mathfrak{mb}(3|8)$ symmetry could be mistaken experimentally for an $\mathfrak{sl}(3) \oplus \mathfrak{sl}(2) \oplus \mathfrak{gl}(1)$ symmetry.

It is remarkable that the sole requirements of simplicity and maximal depth immediately lead to the symmetries of the standard model, without any arbitrary symmetry breaking, compactification, magic, or mystery. Of course, this observation does not prove that $\mathfrak{mb}(3|8)$ is relevant to physics, but I believe that this is line of research worth pursuing. Although $\mathfrak{mb}(3|8)$ acts on 3|8-dimensional superspace, it is not technically a supersymmetry, which would require that the fermionic coordinates carry Lorentz spin. Supersymmetry has attracted much interest because it is the only known way to combine the Poincaré algebra with internal symmetries in a non-trivial way, thus circumventing the Coleman-Mandula theorem. It also makes it possible to unite bosons and fermions into the same multiplet. On the other hand, the assumption that such unification is desirable remains experimentally unproven, despite 30 years of effort. In contrast, in conformal field theory (CFT), several Virasoro multiplets (both bosons and fermions) are combined into a single algebraic entity united by fusion rules. E.g., the Ising model consists of three Virasoro modules (unity, spin, and energy), and the situation is similar for all other CFTs. Avoiding supersymmetry has the additional benefit that the theory will not be plagued by supersymmetric partners, none of which has been experimentally observed.

Kac has apparently been talking about the possible connection between the exceptional superalgebras and the standard model for well over a year.

\footnote{When an algebra is first mentioned, I give both the names used by Shchepochkina and Leites (lowercase fraktur) and those used by Kac and Cheng (uppercase roman). Subsequently, I prefer the former notation, both because it was Shchepochkina who discovered the exceptions, and because Kac’ notation is unsuitable to exhibit the family structure of the regradings.}

\footnote{Kac has mainly worked with another exception $\mathfrak{vi}(3|6) (= E(3|6))$, which has the same grade zero subalgebra but is not quite as deep as $\mathfrak{mb}(3|8)$ (depth 2 versus depth 3).}
Although R. Kaluza did not pursue the idea to its logical conclusion, but so far has the physics community avoided to notice this development. Even though Kac’s suggestion is not at all convincing in its details, the observation that the standard model algebra arises naturally in a mathematically deep context should evoke interest. To my knowledge, the exceptional Lie superalgebras is the only place were this happens in an unambiguous way. In section 3 I discuss some immediate consequences of a tentative \( \mathfrak{mb}(3|8) \) gauge theory: several generations of fermions, absence of proton decay, no additional gauge bosons, manifest CP violation, and particle/anti-particle asymmetry. Further progress must await the development of \( \mathfrak{mb}(3|8) \) representation theory.

It is an undeniable fact that some of the methods of string theory have found very fruitful applications in other places. In particular, CFT is an extremely successful theory of two-dimensional critical phenomena \([2, 11]\), and many string theorists have made important contributions to this area. However, that CFT is useful in two-dimensional statistical physics does not mean that it has anything to do with four-dimensional high-energy physics. Moreover, CFT is not the ToE even in statistical physics, because it does not apply to the experimentally more interesting case of three-dimensional phase transitions. It should be noted that anomaly cancellation is not an issue in this context, because every interesting statistical model corresponds to a CFT with non-zero central charge.

Mirror symmetry also led to progress in algebraic geometry quite a few years back, although I know very little about this subject. However, it should be noted that the structures described in the present paper are local. Interesting new local structures are not so easy to come across, and local is a prerequisite for global.

Several string theorists have informed me that symmetry considerations not apply to M-theory, because nobody knows the algebraic structures behind it, and symmetries are not important in M-theory anyway. In all established and successful physical theories, such as special and general relativity, Maxwell/Yang-Mills theory, Dirac equation, and the standard model, symmetries are absolutely fundamental. And the arguably most successful theory of all in its domain of validity, namely CFT applied to two-dimensional critical phenomena, is nothing but representation theory thinly veiled by physics formalism. That symmetries should be unimportant in M-theory, the alleged mother of all theories, does not seem plausible to me. It is also notable that the importance of symmetries was down-played during the 1990s, at the same time as it became obvious that string theorists are no longer in touch with the research frontier on infinite-dimensional Lie (super)algebras.
The multi-dimensional Virasoro algebra and its Fock modules are described in section 2, and the simple Lie superalgebras in section 3. In the last section I discuss to what extent every conceivable symmetry is known. The title of this paper is thus not only a travesty of string theory’s claim to be the ToE, but it also carries some algebraic substance.

2 Multi-dimensional Virasoro algebra

To make the connection to the Virasoro algebra very explicit, I write down the brackets in a Fourier basis. Start with the Virasoro algebra $Vir$:

$$[L_m, L_n] = (n - m)L_{m+n} - \frac{c}{12}(m^3 - m)\delta_{m+n},$$

where $\delta_m$ is the Kronecker delta. When $c = 0$, $L_m = -i \exp(iax)\partial_x$, $m \in \mathbb{Z}$. The element $c$ is central, meaning that it commutes with all of $Vir$; by Schur’s lemma, it can therefore be considered as a $c$-number. Now rewrite $Vir$ as

$$[L_m, L_n] = (n - m)L_{m+n} - cm^2nS_{m+n},$$
$$[L_m, S_n] = (n + m)S_{m+n},$$
$$[S_m, S_n] = 0,$$
$$nS_m = 0.$$

It is easy to see that the two formulations of $Vir$ are equivalent (I have absorbed the linear cocycle into a redefinition of $L_0$). The second formulation immediately generalizes to $N$ dimensions. The generators are $L_\mu(m) = -i \exp(iax)\partial_\mu$ and $S_\mu(m)$, where $x = (x^\mu)$, $\mu = 1, 2, ..., N$ is a point in $N$-dimensional space and $m = (m_\mu)$. The Einstein convention is used (repeated indices, one up and one down, are implicitly summed over). The defining relations are

$$[L_\mu(m), L_\nu(n)] = n_\mu L_\nu(m + n) - m_\nu L_\mu(m + n)$$
$$+ (c_1m_\nu n_\mu + c_2m_\mu n_\nu)m_\rho S^\rho(m + n),$$
$$[L_\mu(m), S_\nu(n)] = n_\mu S_\nu(m + n) + \delta_\nu^\mu m_\rho S^\rho(m + n),$$
$$[S_\mu(m), S_\nu(n)] = 0,$$
$$m_\mu S_\nu(m) = 0.$$

This is an extension of $\mathfrak{Vect}(N)$ by the abelian ideal with basis $S_\mu(m)$. This algebra is even valid globally on the $N$-dimensional torus $\mathbb{T}^N$. Geometrically,
we can think of \( L_\mu(m) \) as a vector field and \( S^\mu(m) \) as a dual one-form; the last condition expresses closedness. The cocycle proportional to \( c_1 \) was discovered by Rao and Moody \([35]\), and the one proportional to \( c_2 \) by myself \([20]\). There is also a similar multi-dimensional generalization of affine Kac-Moody algebras. The relevant cocycle was presumably first written down by Kassel \([17]\), and its modules were studied in \([33, 34, 3, 3, 4]\). In the mathematics literature, the multi-dimensional Virasoro and affine algebras are often referred to as "Toroidal Lie algebras".

The \( \text{vect}(N) \) module spanned by \( S^\nu(n) \) contains the trivial submodule \( \mathbb{C}^N = H^1_{dR}(\mathbb{T}^N) \) (de Rham homology) spanned by \( S^\nu(0) \):

\[
[L_\mu(m), S^\nu(0)] = \delta^\nu_\mu m_\rho S^\rho(m) \equiv 0.
\]

Since \( S^\nu(0) \) is central, it can be considered as a c-number. If we replace \( S^\nu(n) \mapsto S^\nu(n) + n_\rho F^{\nu\rho}(n) \), where \( F^{\nu\rho}(n) = -F^{\rho\nu}(n) \) and

\[
[L_\mu(m), F^{\nu\rho}(n)] = n_\mu F^{\nu\rho}(m + n) + \delta^\nu_\mu m_\sigma F^{\nu\sigma}(m + n) + \delta^\rho_\mu m_\sigma F^{\nu\sigma}(m + n),
\]

the \( LS \) bracket is unchanged, whereas the cocycle becomes

\[
(c_1 m_\rho n_\mu + c_2 m_\mu n_\rho)(m_\rho S^\rho(m + n) + m_\rho n_\sigma F^{\rho\sigma}(m + n)).
\]

We can use this freedom to set \( S^\nu(n) = 0 \) for all \( n \neq 0 \), so we have almost a central extension by the \( N \)-dimensional module \( H^1_{dR}(\mathbb{T}^N) \). However, not quite, because the condition is not preserved:

\[
[L_\mu(-n), S^\nu(n)] = -\delta^\nu_\mu n_\rho S^\rho(0),
\]

which is non-zero unless all \( S^\rho(0) = 0 \). Nevertheless, this argument shows that the multi-dimensional Virasoro cocycle is close to central. Similarly, I expect that \( \text{vect}(M_N) \), where \( M_N \) is an \( N \)-dimensional manifold, has Virasoro-like extensions labelled by \( H^1_{dR}(M_N) \).

The theory of Fock modules was constructed in \([23]\). \( \text{vect}(N) \) is generated by Lie derivatives \( L_\xi \), where \( \xi = \xi^\mu(x) \partial_\mu \) is a vector field; \( L_\mu(m) \) is the Lie derivative corresponding to \( \xi = -i \exp(\text{i} m_\rho x^\rho) \partial_\mu \). The classical modules are tensor densities \([36]\) (primary fields in CFT parlance), which transform as

\[
[L_\xi, \phi(x)] = -\xi^\mu(x) \partial_\mu \phi(x) - \partial_\nu \xi^\nu(x) T^\mu_\nu \phi(x),
\]

where \( T^\mu_\nu \) satisfies \( gl(N) \):

\[
[T^\mu_\nu, T^\rho_\sigma] = \delta^\nu_\sigma T^\mu_\rho - \delta^\mu_\rho T^\nu_\sigma.
\]
Naïvely, one would start from a classical field and introduce canonical momenta $\pi(x)$ satisfying $[\pi(x), \phi(y)] = \delta^N(x - y)$. This gives the following expression for $L_\xi$:

$$L_\xi = \int d^N x \, \xi^\mu(x) \partial_\mu \phi(x) + \partial_\nu \xi^\mu(x) T_{\mu\nu}^\nu \phi(x).$$

To remove an infinite vacuum energy, we must normal order. However, this approach only works when $N = 1$, because in higher dimensions infinities are encountered. This is in accordance with the well-known fact the $\text{vect}(N)$ only admits central extensions when $N = 1$.

Instead, the crucial idea is to first expand all fields in a multi-dimensional Taylor series around the points along a one-dimensional curve ("the observer’s trajectory"), and then to truncate at some finite order $p$. Let $m = (m_1, m_2, ..., m_N)$, all $m_\mu \geq 0$, be a multi-index of length $|m| = \sum_{\mu=1}^N m_\mu$. Denote by $\mu$ a unit vector in the $\mu$:th direction, so that $m + \mu = (m_1, ..., m_\mu + 1, ..., m_N)$, and let

$$\phi_{m}(t) = \partial_{m_1} \cdots \partial_{m_N} \phi(q(t))$$

be the $|m|$:th order derivative of $\phi(x)$ on the observer’s trajectory $q^\mu(t)$. Such objects transform as

$$[L_\xi, \phi_{m}(t)] = \partial_{m_1} \cdots \partial_{m_N} \phi(q(t))$$

where explicit expressions for the matrices $T_m(\xi)$ are given in [23]. We thus obtain a realization of $\text{vect}(N)$ on the space of trajectories in the space of tensor-valued $p$-jets. This space consists of finitely many functions of a single variable, which is precisely the situation where the normal ordering prescription works. After normal ordering, we obtain a Fock representation of the multi-dimensional Virasoro algebra described above. The expression for $L_\xi$ reads

$$L_\xi = \int dt \, \{ :\xi^\mu(q(t)) p_\mu(t) : + \sum_{|n| \leq |m| \leq p} :\pi^m(t) T_m(\xi(q(t))) \phi_n(t) : \},$$

where $[p_\mu(s), q^\nu(t)] = \delta^\nu_\mu \delta(s - t)$ and $[\pi^m(s), \phi_n(t)] = \delta^m_n \delta(s - t)$. Some observations are in order:
1. The action on jet space is non-linear; the observer’s trajectory transforms non-linearly, and although vector fields act linearly on the Taylor coefficients, they act with matrices depending non-linearly on the base point. Hence the resulting extension is non-central.

2. Classically, \( \text{vect}(N) \) acts in a highly reducible fashion. In fact, the realization is an infinite direct sum because neighboring points on the trajectory transform independently of each other. To lift this degeneracy, I introduced an additional \( \text{vect}(1) \) factor, describing reparametrizations. The relevant algebra is thus the DRO (Diffeomorphism, Reparametrization, Observer) algebra \( DRO(N) \), which is the extension of \( \text{vect}(N) \oplus \text{vect}(1) \) by its four Virasoro-like cocycles.

3. The reparametrization symmetry can be eliminated with a constraint, but then one of the spacetime direction (“time”) is singled out. Two of the four Virasoro-like cocycles of \( DRO(N) \) transmute into the complicated anisotropic cocycles found in [2]; these are colloquially known as the “messy cocycles”. By further specialization to scalar-valued zero-jets on the torus, the results of Rao and Moody are recovered [3].

After the Fock modules were constructed, more interesting lowest-energy modules were considered in [24]. This unpublished paper is far too long and contains some flaws, mainly because I didn’t have the right expressions for the abelian charges at the time, but I think that the main idea is sound.

In classical physics one wants to find the stationary surface \( \Sigma \), i.e. the set of solutions to the Euler-Lagrange (EL) equations, viewed as a submanifold embedded in configuration space \( Q \). Dually, one wants to construct the function algebra \( C(\Sigma) = C(Q)/I \), where \( I \) is the ideal generated by the EL equations. For each field \( \phi_\alpha \) and EL equation \( \mathcal{E}_\alpha = 0 \), introduce an anti-field \( \phi^{*\alpha} \) of opposite Grassmann parity. The extended configuration space \( C(Q^*) = C[\phi, \phi^*] \) can be decomposed into subspaces \( C^g(Q^*) \) of fixed antifield number \( g \), where \( \text{afn} \phi_\alpha = 0 \), \( \text{afn} \phi^{*\alpha} = 1 \). The Koszul-Tate (KT) complex

\[
0 \xleftarrow{\delta} C^0(Q^*) \xrightarrow{\delta} C^1(Q^*) \xrightarrow{\delta} C^2(Q^*) \xrightarrow{\delta} \ldots ,
\]

where \( \delta \phi_\alpha = 0 \) and \( \delta \phi^{*\alpha} = \mathcal{E}_\alpha \), yields a resolution of \( C(\Sigma) \); the cohomology groups \( H^g(\delta) = 0 \) unless \( g = 0 \), and \( H^0(\delta) = C(Q)/I \) [12].

\(^9\) I refer to the parameters multiplying the cocycles as abelian charges, in analogy with the central charge of the Virasoro algebra.
My idea was to consider not just functions on the stationary surface, but all differential operators on it. The KT differential \( \delta \) can then be written as a bracket: \( \delta F = [Q, F] \), where the KT charge \( Q = \int \mathcal{E}^\alpha \pi^*_\alpha \) and \( \pi^*_\alpha \) is the canonical momentum corresponding to \( \phi^\alpha \). If we pass to the space of \( p \)-jets before momenta are introduced, the construction of Fock modules above applies. Since the KT charge consists of commuting operators, it does not need to be normal ordered, and the cohomology groups are well-defined \( DRO(N) \) modules of lowest-energy type.

I think that this construction can be viewed as a novel method for quantization, although the relation to other methods is not clear. However, it was never my intention to invent a new quantization scheme, but rather to construct interesting \( DRO(N) \) modules. An outstanding problem is to take the maximal jet order \( p \) to infinity, because infinite jets essentially contain the same information as the original fields. This limit is problematic, because the abelian charges diverge with \( p \), but it seems that this difficulty may be bypassed in four dimensions, as announced in \[26\].

3 Classification of simple infinite-dimensional Lie superalgebras of vector fields and the exceptional Lie superalgebra \( mb(3|8) \)

Technically, the classification deals with polynomial vector fields acting on a superspace of dimension \( n|m \) (\( n \) bosonic and \( m \) fermionic directions), i.e. of simple subalgebras of \( vect(n|m) \). However, the restriction to polynomials is not philosophically essential, because the results also apply to functions that can be approximated by polynomials, e.g. analytic functions. So it is really a classification of simple algebras of local vector fields.

Let \( \mathfrak{g} \subset vect(n|m) \) be such an algebra. It has a Weisfeiler grading of depth \( d \) if it can be written as

\[
\mathfrak{g} = \mathfrak{g}_{-d} + \cdots + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \cdots,
\]

where \( \mathfrak{g}_{-1} \) is an irreducible \( \mathfrak{g}_0 \) module and \( \mathfrak{g}_k \) consists of vector fields that are homogeneous of degree \( k \). However, it is not the usual kind of homogeneity, because we do not assume that all directions are equivalent. Denote the coordinates of \( n|m \)-dimensional superspace by \( x^i \) and let \( \partial_i \) be the corresponding derivatives. Then we define the grading by introducing positive integers \( w_i \) such that \( \deg x^i = w_i \) and \( \deg \partial_i = -w_i \). The operator which computes the Weisfeiler grading is \( Z = \sum_i w_i x^i \partial_i \), and \( \mathfrak{g}_k \) is the subspace
of vector fields $X = X^i(x)\partial_i$ satisfying $[Z, X] = kX$. If we only considered $\mathfrak{g}$ as a graded vector space, we could of course make any choice of integers $w_i$, but we also want $\mathfrak{g}$ to be graded as a Lie algebra: $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. The depth $d$ is identified with the maximal $w_i$. Denote the negative part by $\mathfrak{g}_- = \mathfrak{g}_{-d} + \ldots + \mathfrak{g}_{-1}$; it is a nilpotent algebra and a $\mathfrak{g}_0$ module.

The main tool for constructing algebras of vector fields is Cartan prolongation. In the mathematics literature, it is defined recursively in a way which is not so easy for a physicist to understand. Therefore, I propose the following alternative definition of Cartan prolongation:

1. Start with a realization for the non-positive part $\mathfrak{g}_0 \ltimes \mathfrak{g}_- \subset \mathfrak{g}$ in $n|m$-dimensional superspace.

2. Determine the most general set of structures preserved by $\mathfrak{g}_0 \ltimes \mathfrak{g}_-$. Such a structure is either some differential form, or an equation satisfied by forms (Pfaff equation), or a system of Pfaff equations.

3. Define the Cartan prolong $\mathfrak{g} = (\mathfrak{g}_{-d}, \ldots, \mathfrak{g}_{-1}, \mathfrak{g}_0)_*$ as the full subalgebra of $\mathfrak{vect}(n|m)$ preserving the same structures.

Clearly, the set of vector fields that preserve some structure automatically define a subalgebra of $\mathfrak{vect}(n|m)$. By choosing the maximal set of equations, this subalgebra must be $\mathfrak{g}$ itself. If one is lucky, $\mathfrak{g}$ is now simple and infinite-dimensional. The mathematicians have determined when this happens.

The simplest example is the prolong $(\mathfrak{n}, sl(n))_*$, where $\mathfrak{n}$ stands for the $n$-dimensional $sl(n)$ module with basis $\partial_i$. The vector fields are, at non-positive degrees

| deg | vector field       |
|-----|--------------------|
| -1  | $\partial_i$       |
| 0   | $x^i\partial_j - \frac{1}{n}\delta^i_j x^k \partial_k$ |

These vector fields $X = X^i(x)\partial_i$ preserve the volume form $vol$. The prolong is the algebra $\mathfrak{svect}(n)$ of all divergence-free vector fields, all of which preserve $vol$. 

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The classification \([15, 16]\) consists of a list of ten series

| Series | Description |
|--------|-------------|
| \(\text{vect}(n|m) = W(n|m)\) | arbitrary v.f. in \(n|m\) dimensions, |
| \(\text{svect}(n|m) = S(n|m)\) | divergence-free v.f., |
| \(\mathfrak{h}(n|m) = H(n|m)\) | Hamiltonian v.f. (\(n\) even), |
| \(\mathfrak{l}(n) = \mathfrak{HO}(n|n)\) | odd Hamiltonian or Leitesian v.f. \(\subset \text{vect}(n|n)\), |
| \(\mathfrak{sle}(n) = \mathfrak{SHO}(n|n)\) | divergence-free Leitesian v.f., |
| \(\mathfrak{f}(n|m) = K(n|m)\) | contact v.f. (\(n\) odd), |
| \(\mathfrak{m}(n) = \mathfrak{KO}(n|n+1)\) | odd contact v.f. \(\subset \text{vect}(n|n+1)\), |
| \(\mathfrak{sm}_\beta(n) = \mathfrak{SKO}(n|n+1; \beta)\) | a deformation of div-free odd contact v.f., |
| \(\tilde{\mathfrak{sle}}(n) = \mathfrak{SHO}^-(n|n)\) | a deformation of \(\mathfrak{sle}(n)\), |
| \(\tilde{\mathfrak{sm}}(n) = \mathfrak{SKO}^-(n|n+1)\) | a deformation of \(\mathfrak{sm}(n)\). |

Moreover, there are five exceptions, described as Cartan prolongs

\[
\begin{align*}
\text{vas}(4|4) &= E(4|4) = (\text{spin, as})_*, \\
\text{vic}(3|6) &= E(3|6) = (2, 3^* \otimes 2, sl(3) \oplus sl(2) \oplus gl(1))_*, \\
\text{fsle}(5|10) &= E(5|10) = (5, 5^* \wedge 5^*, sl(5))_*, \\
\text{mb}(3|8) &= E(3|8) = (2, 2, 3^* \otimes 2, sl(3) \oplus sl(2) \oplus gl(1))_*, \\
\text{fas}(1|6) &= E(1|6) \subset \mathfrak{f}(1|6) = (1, 6, so(6) \oplus gl(1)).
\end{align*}
\]

The finite-dimensional superalgebra \(\text{as}\) is the central extension of the special periplectic algebra \(\mathfrak{sp}(4)\) discovered by A. Sergeev; \(\mathfrak{sp}(n)\) has no central extensions for other values of \(n\). The construction of \(\text{fas}(1|6)\) is slightly more complicated than Cartan prolongation; it is a subalgebra of \(\mathfrak{f}(1|6)\).

A more geometric way to describe these algebras is by stating what structures they preserve, or what other conditions the vector fields obey.

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\[10\] Leites and Shchepochkina use different names for \(\mathfrak{sm}_\beta(n), \tilde{\mathfrak{sle}}(n)\) and \(\tilde{\mathfrak{sm}}(n)\), but unfortunately I do not quite understand their notation.
This is a very new result for the exceptions $[27]$. 

| Algebra | Basis | Description/structure preserved |
|---------|-------|---------------------------------|
| $\text{vect}(n|m)$ | $u^i, \theta^a$ | $-$ |
| $\text{svect}(n|m)$ | $u^i, \theta^a$ | $\text{vol}$ |
| $h(n|m)$ | $u^i, \theta^a$ | $\omega_{ij} du^i du^j + g_{ab} \theta^a \theta^b$ |
| $\text{le}(n)$ | $u^i, \theta_i$ | $du^i d\theta_i$ |
| $\text{sl}(n)$ | $u^i, \theta_i$ | $du^i d\theta_i, \text{vol}$ |
| $\mathfrak{t}(n+1|m)$ | $t, u^i, \theta^a$ | $dt + \omega_{ij} u^i du^j + g_{ab} \theta^a \theta^b = 0$ |
| $\mathfrak{m}(n)$ | $\tau, u^i, \theta_i$ | $d\tau + u^i d\theta_i + \theta_i du^i = 0$ |
| $\mathfrak{sm}_\beta(n)$ | $\tau, u^i, \theta_i$ | $M_f \in \mathfrak{m}(n): \text{div}_\beta M_f = 0$ |
| $\sim \text{sl}(n)$ | $u^i, \theta_i$ | $(1 + \theta_1...\theta_n) \xi, \xi \in \text{sl}(n)$ |
| $\sim \mathfrak{sm}(n)$ | $\tau, u^i, \theta_i$ | $(1 + \theta_1...\theta_n) \xi, \xi \in \mathfrak{sm}_{n+2}(n)$ |
| $\mathfrak{fsl}(5|10)$ | $u^i, \theta_{ij}$ | $du^i + \frac{1}{2} \epsilon^{ijklm} \theta_{jk} d\theta_{tm} = 0, \text{vol}$ |
| $\mathfrak{fle}(3|6)$ | $u^i, \theta_{ia}$ | $du^i + \epsilon^{ijk} e^{ab} \theta_{ja} d\theta_{kb} = 0, \text{vol}$ |
| $\mathfrak{fas}(1|6)$ | $t, \theta^a$ | $K_f \in \mathfrak{t}(1|6): \frac{\partial^2 f}{\partial \tau \partial \theta_i} = \epsilon_{abcde} f \frac{\partial^2 f}{\partial \theta_i \partial \theta_j}$ (?)* |

In this table, $u^i$ denotes bosonic variables, $\theta^a$, $\theta_i$ and $\theta_{ij}$ fermionic variables, and $t$ ($\tau$) is an extra bosonic (fermionic) variable. The indices range over the dimensions indicated: $i = 1, ..., n$ and $a = 1, ..., m$, except for $\mathfrak{fle}(3|6)$ where $a = 1, 2$ only. $\theta_{ij} = -\theta_{ji}$ in $\mathfrak{fsl}(5|10)$ so there are only ten independent fermions. $\omega_{ij} = -\omega_{ji}$ and $g_{ab} = g_{ba}$ are structure constants, and $\epsilon^{ab}$, $\epsilon^{ijk}$ and $\epsilon^{ijklm}$ are the totally anti-symmetric constant tensors in the appropriate dimensions. The notation $\alpha = 0$ (or $\alpha^i = 0$) implies that it is this Pfaff equation that is preserved, not the form $\alpha$ itself; a vector field $\xi$ acts on $\alpha$ as $\mathcal{L}_\xi \alpha = f_\xi \alpha$, $f_\xi$ some polynomial function. $\text{vol}$ denotes the volume form; vector fields preserving $\text{vol}$ satisfy $\text{div} \xi = (-)^{\xi+\mu} \partial_{\mu} \xi^\mu = 0$.

$$\text{div}_\beta M_f = 2(-)^f \left( \frac{\partial^2 f}{\partial u^i \partial \theta_i} + (u^i \frac{\partial}{\partial u^i} + \theta_i \frac{\partial}{\partial \theta_i} - n\beta) \frac{\partial f}{\partial \tau} \right)$$

is a deformed divergence. In the conjectured description of $\mathfrak{fas}(1|6)$, indices are lowered by means of the metric $g_{ab}$. The geometrical meaning of $\mathfrak{fas}(4|4)$ is not clear to me, but the differential equations that it satisfies are written down in $[41]$. 

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Finally, let us describe the exceptional algebra $\mathfrak{mb}(3|8) = (\mathfrak{2,3,3}^* \otimes \mathfrak{2}, \mathfrak{sl}(3) \oplus \mathfrak{sl}(2) \oplus \mathfrak{gl}(1))_*$. A basis for $3|8$-dimensional space is given by

$$
\begin{align*}
\theta_{ia}, & \quad (\text{degree } 1, 2 \times 3 = 6 \text{ fermions}), \\
u^i, & \quad (\text{degree } 2, 3 \text{ bosons}), \\
\vartheta^a, & \quad (\text{degree } 3, 2 \text{ fermions}),
\end{align*}
$$

where $i = 1, 2, 3$ is a three-dimensional index and $a = 1, 2$ a two-dimensional index. Denote the corresponding derivatives by $d^{ia}, \partial_i$ and $\partial_a$, respectively. We can explicitly describe the vector fields at non-positive degree$^\text{[11]}$.

| deg | vector field |
|-----|--------------|
| -3  | $F_a = \partial_a$ |
| -2  | $E_i = \partial_i + \theta_i^a \partial_a$ |
| -1  | $D^{ia} = d^{ia} + 3\epsilon^{ijk} \theta^a_j \partial_k + \epsilon^{ijk} \theta^a_j \theta^b_k \partial_b + u^i \partial^a$ |
| 0   | $I_i^k = u^k \partial_i - \theta_{ia} \partial^a - \frac{1}{3}\delta^k_i (u^i \partial_i - \theta_{ia} d^{ia})$ ($\mathfrak{sl}(3)$) |
| 0   | $J_a^i = \vartheta^a_\partial \partial_i - \theta_{ia} d^{ia} - \frac{1}{2} \delta^a_i (\vartheta^a_\partial \partial_a - \theta_{ia} d^{ia})$ ($\mathfrak{sl}(2)$) |
| 0   | $Z = 3 \vartheta^a \partial_a + 2 u^i \partial_i + \theta_{ia} d^{ia}$ ($\mathfrak{gl}(1)$) |

Geometrically, $\mathfrak{mb}(3|8)$ preserves the system of dual Pfaff equations $\tilde{D}^{ia} = 0$, where

$$
\tilde{D}^{ia} = d^{ia} - 3\epsilon^{ijk} \theta^a_j \partial_k + \epsilon^{ijk} \theta^a_j \theta^b_k \partial_b - u^i \partial^a.
$$

In other words, $\mathfrak{mb}(3|8)$ consists of vector fields $X$ such that $[\tilde{D}^{ia}, X] = f_j^{ia}(X)\tilde{D}^{jb}$ for some polynomial functions $f_j^{ia}(X)$.

Given an algebra $\mathfrak{g}$, a subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and an $\mathfrak{h}$ representation $\varphi$, one can always construct the induced $\mathfrak{g}$ representation $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \varphi$, where $\mathcal{U}(\cdot)$ denotes the universal enveloping algebra. For finite-dimensional algebras, the induced representation is usually too big to be of interest, but for Cartan prolongs the situation is different. There is a 1-1 correspondence between $\mathfrak{g}$ irreps and irreps of its grade zero subalgebra $\mathfrak{g}_0$, as follows. Start from a $\mathfrak{g}_0$ irrep and construct the corresponding induced $\mathfrak{g}$ representation. If this is irreducible, which is often the case, we are done. Otherwise, it contains an irreducible subrepresentation. A well-known example is the Cartan prolong $\mathfrak{vect}(n) = (\mathfrak{n, gl(n)})_*$. A $\mathfrak{gl}(n)$ module is a tensor with certain symmetries.

Note added in version 3: The following two formulas were incorrect in version 2.
and the induced $\text{vect}(n)$ module is the corresponding tensor density. It is irreducible unless it is totally anti-symmetric and has weight zero, i.e. it is a differential form. Then it contains the submodule of closed forms \[33\].

It sometimes happens that a Lie superalgebra can be realized in more than one way as vector fields acting on superspaces (of different dimensions). One then says that the algebra has a Weisfeiler regrading. No proper Lie algebra has a regrading. The five exceptional Lie superalgebras have fifteen regradings altogether, listed in the format \(name(n|m):depth\).

- $\text{vect}(5|4):2$, $\text{vect}(3|6):2$, $\text{vas}(4|4):1$, $\text{kas}(1|6):2$, $\text{vas}(5|5):2$, $\text{vas}(4|4):1$, $\text{vas}(4|3):1$, $\text{mb}(4|5):2$, $\text{mb}(5|6):2$, $\text{mb}(3|8):3$, $\text{ksle}(9|6):2$, $\text{ksle}(11|9):2$, $\text{ksle}(5|10):2$, $\text{ksle}(11|9;CK):3$. Note that $\text{ksle}(5|10)$ has two inequivalent realizations on 119-dimensional superspace.

Among all gradings, the consistent ones play a distinguished role. A grading is consistent if the even subspaces are purely bosonic and the odd ones are purely fermionic. Most gradings are inconsistent, and no superalgebra has more than one consistent grading.

**Theorem [15]:** The only simple Lie superalgebras with consistent gradings are the contact algebras $\mathfrak{k}(1|m)$ and the exceptions $\mathfrak{ksle}(5|10)$, $\mathfrak{vect}(3|6)$, $\mathfrak{mb}(3|8)$ and $\mathfrak{kas}(1|6)$.

If we change the function class from polynomials to Laurent polynomials, the contact algebras $\mathfrak{k}(1|m)$ for $m \leq 4$ have central extensions, known in physics as the $N = m$ superconformal algebra (the case $m = 0$ is the Virasoro algebra). Conjecture: these central extensions are obtained by restriction from the abelian Virasoro-like extensions of $\text{vect}(1|m)$. This statement has been proven in the case $m = 1$ [22]. $\mathfrak{k}(1|m)$ does not have central extensions for $m > 4$, but clearly it has abelian Virasoro-like extensions, which could be called the $N = m$ superconformal algebra also for $m > 4$. Although the superconformal algebras play an important role in string theory, their description as the vector fields that preserve the Pfaff equation $\alpha = dt + g_{ab}\theta^a d\theta^b = 0$ might be unknown to some physicists. Nor is it common knowledge that the restriction to $m \leq 4$ is unnecessary in the centerless case.

If $\mathfrak{mb}(3|8)$ is to replace the standard model algebra, it must be gauged, i.e. one must pass to the algebra $\text{map}(4,\mathfrak{mb}(3|8))$ of maps from four-dimensional spacetime to $\mathfrak{mb}(3|8)$. After inclusion of gravity, the full symmetry algebra becomes $\text{vect}(4) \ltimes \text{map}(4,\mathfrak{mb}(3|8))$. Obviously,

$$\text{vect}(4) \ltimes \text{map}(4,\mathfrak{mb}(3|8)) \subset \text{vect}(4) \ltimes \text{map}(4,\text{vect}(3|8)) \subset \text{vect}(7|8).$$

The middle algebra consists of those vector fields in 7|8-dimensional space.
that preserve the splitting between horizontal and vertical directions. This chain of inclusions proves that $\text{vect}(4) \times \text{map}(4,\text{mb}(3|8))$ has well-defined abelian extensions and Fock modules; consider the restriction from $\text{vect}(7|8)$.

The algebra $\text{map}(4,\mathfrak{g})$ encodes a very natural generalization of the gauge principle. Gauging a rigid symmetry $\mathfrak{g}_0$ makes it local in spacetime, but it is still rigid in the fiber directions; the finite-dimensional algebra $\mathfrak{g}_0$ acts on each fiber. The replacement of $\mathfrak{g}_0$ by its infinite-dimensional prolong $\mathfrak{g} = (\mathfrak{g}_-,\mathfrak{g}_0)$, makes the symmetry local in the fibers as well, while maintaining the essential features of representations and physical predictions. The only freedom lies in the nilpotent algebra $\mathfrak{g}_-$; it is natural to choose it such that $\mathfrak{g}$ is simple.

Unfortunately, the details of an $\text{mb}(3|8)$ gauge theory must await the development of the $\text{mb}(3|8)$ representation theory, in particular the list of degenerate irreducible modules which is not yet available\footnote{Note added in version 3: This list has now been worked out by several authors; the first were V.G. Kac and A.N. Rudakov: Complexes of modules over exceptional Lie superalgebras $E(3|8)$ and $E(5|10)$, math-ph/0112022 (2001). As explained in [28], the hypercharge assignments did not quite work out the way I hoped.} However, the analogous list for $\text{vle}(3|6)$ has recently been worked out by Kac and Rudakov [18, 19], so a $\text{vle}(3|6)$ gauge theory can be written down at this time [28].

But even before its construction, we know enough about the exceptional Lie superalgebras to make some non-trivial, and falsifiable, predictions.

Let me sketch the construction of a $\mathfrak{g}$ gauge theory, where $\mathfrak{g} = \text{vle}(3|6)$ or $\mathfrak{g} = \text{mb}(3|8)$. Let $\mathfrak{g}$ denote a $\mathfrak{g}_0 = sl(3) \oplus sl(2) \oplus gl(1)$ module. The $\mathfrak{g}$ tensor modules $T(\mathfrak{g})$ are $\mathfrak{g}$-valued functions $\psi(\theta,u,\vartheta)$, where $(\theta,u,\vartheta) \in \mathbb{C}^{3|8}$ are the coordinates in the internal directions ($(\theta,u) \in \mathbb{C}^{3|6}$ for $\text{vle}(3|6)$). The corresponding $\text{map}(4,\mathfrak{g})$ tensor module, also denoted by $T(\mathfrak{g})$, is either irreducible, or contains an irreducible submodule consisting of tensors satisfying $\nabla \psi(x,\theta,u,\vartheta) = 0$. Here $x = (x^\mu) \in \mathbb{R}^4$ is a point in spacetime and $\nabla$ is a morphism inherited from $\mathfrak{g}$, i.e. a differential operator acting on the internal directions only. Kac and Rudakov call the modules $I(\mathfrak{g}) = \ker \nabla \subset T(\mathfrak{g})$ (cohomology is almost always absent) “degenerate irreducible modules”, but I will use the shorter name form modules\footnote{More precisely, one should call $T(\mathfrak{g})$ a form module if it is reducible, and its irreducible quotient $I(\mathfrak{g}) = \ker \nabla \subset T(\mathfrak{g})$ a closed form module. The distinction should be clear from the context.}, because we can think of $\psi$ as a differential form and of $\nabla$ as the exterior derivative.
Theorem \[18, 19\]: The \( \mathfrak{sl}(3|6) \) form modules are

\[
\begin{align*}
\Omega_A(p, r) &= I(p, 0; r; \frac{2}{3}p - r), \\
\Omega_B(p, r) &= I(p, 0; r; \frac{2}{3}p + r + 2), \\
\Omega_C(q, r) &= I(0, q; r; -\frac{2}{3}q - r - 2), \\
\Omega_D(q, r) &= I(0, q; r; -\frac{2}{3}q + r),
\end{align*}
\]

where \((p, q; r; y)\) is an \( sl(3) \oplus sl(2) \oplus gl(1) \) lowest weight \((p, q) \in \mathbb{N}^2\) for \( sl(3) \), \( r \in \mathbb{N} \) for \( sl(2) \), and \( y \in \mathbb{C} \) for \( gl(1) \).

The \( gl(1) \) generator is the operator which computes the Weisfeiler grading; up to normalization, it can be identified with weak hypercharge: \( Y = Z/3 \) \[16\]. By the Gell-Mann-Nishiyima formula, the electric charge in \((p, q; r; y)\) ranges from \( y/2 - r/2 \) to \( y/2 + r/2 \) in integer steps.

We identify the fermions (quarks and leptons) in the first generation with form modules, and additionally assume that they transform as spinors under the Lorentz group in the usual way. This leads to the following assignment of fermions:

| Multiplet | Charges | Form |
|-----------|---------|------|
| \((0, 1; 1/3)\) | \(\frac{2}{3}, -\frac{1}{3}\) | \((u_L, c_L, t_L, b_L)\) \(\Omega_D(1, 1)\) |
| \((1, 0; 1; -\frac{1}{3})\) | \(-\frac{2}{3}, \frac{1}{3}\) | \((\bar{u}_R, \bar{c}_R, \bar{t}_R, \bar{b}_R)\) \(\Omega_A(1, 1)\) |
| \((1, 0; 0; -\frac{4}{3})\) | \(-\frac{2}{3}\) | \(\bar{u}_L, \bar{c}_L, \bar{t}_L\) |
| \((0, 1; 0; \frac{4}{3})\) | \(\frac{2}{3}\) | \(u_R, c_R, t_R\) |
| \((0, 1; 0; -\frac{4}{3})\) | \(-\frac{1}{3}\) | \(d_R, s_R, b_R\) \(\Omega_D(1, 0)\) |
| \((1, 0; 0; \frac{2}{3})\) | \(\frac{1}{3}\) | \(\bar{d}_L, \bar{s}_L, \bar{b}_L\) \(\Omega_A(1, 0)\) |
| \((0, 0; 0; -1)\) | \(0, -1\) | \((\nu_e L, \nu_\mu L, \nu_\tau L)\) \(\Omega_A(0, 1)\) |
| \((0, 0; 1; 1)\) | \(0, 1\) | \((\bar{\nu}_e R, \bar{\nu}_\mu R, \bar{\nu}_\tau R)\) \(\Omega_D(0, 1)\) |
| \((0, 0; 0; 2)\) | \(1\) | \((\bar{e}_L, \bar{\mu}_L, \bar{\tau}_L)\) \(\Omega_C(0, 0)\) |
| \((0, 0; 0; -2)\) | \(-1\) | \((e_R, \mu_R, \tau_R)\) \(\Omega_B(0, 0)\) |

In the usual \( g_0 = sl(3) \oplus sl(2) \oplus gl(1) \) gauge theory, the gauge bosons are map(4, \( g_0 \)) connections, which can be regarded as functions \( A^a_\mu(x) \), with \( a \) a
index. From this one can construct the covariant derivative \( \partial_\mu + A_\mu^a(x)T_a \), where \( T_a \) are the \( g_0 \) generators. In a \( g \) gauge theory, the gauge bosons must analogously be taken as map \((4, g)\) connections, i.e. \( g_0 \)-valued functions \( A_\mu^a(x, \theta, u, \vartheta) \) with a twisted action of \( g \). The map \((4, g)\) covariant derivative \( D_\mu = \partial_\mu + A_\mu^a(x, \theta, u, \vartheta)T_a \) acts on tensor and form modules, making \( D_\mu \psi \) transform as a vector-spinor under the Lorentz group, and as \( \psi \) itself under \( g \). The map \((4, g)\) action on \( A_\mu^a \) follows from the definition \( A_\mu^aT_a = D_\mu - \partial_\mu \), and the curvature is \( F_{\mu\nu} = [D_\mu, D_\nu] \), as usual. The equations of motion are thus the same as in the standard model, except for the replacements \( \psi(x) \to \psi(x, \theta, u, \vartheta) \), \( A_\mu^a(x) \to A_\mu^a(x, \theta, u, \vartheta) \), and the additional conditions \( \nabla \psi(x, \theta, u, \vartheta) = 0 \).

Since fermions in several different form modules are introduced, the natural question is which multiplets to consider. I suggest that one must choose a fundamental set of form modules (actually two sets, one for each helicity), from which all forms can be built by taking appropriate “wedge products”, i.e. bilinear maps

\[ \wedge : I(\rho_1) \times I(\rho_2) \longrightarrow I(\rho_3) \subset T(\rho_1) \otimes T(\rho_2) = T(\rho_1 \otimes \rho_2), \]

where \( \rho_3 \subset \rho_1 \otimes \rho_2 \) is such that \( T(\rho_3) \) is reducible. Unfortunately, these wedge products are not known, even for \( \wlc(3|6) \), so this idea remains a hypothesis.

The \( \wlc(3|6) \) gauge theory leads to several predictions, which are in rough agreement with experiments:

1. Since \( g_0 = sl(3) \oplus sl(2) \oplus gl(1) \), many good properties are inherited from the standard model.

2. Under the restriction to \( g_0 \), a \( g \) module \( I(\rho) \) decomposes into many \( g_0 \) modules, several of which may be isomorphic to \( \rho \). These may be identified with different generations of the same fermion. Thus I suggest unification of e.g. \( (d_R, s_R, b_R) \) into the same \( \wlc(3|6) \) multiplet \( \Omega_D(1, 0) \). However, since the isomorphic \( g_0 \) modules sit in different ways in the parent \( g \) module, the properties of the different generations should be different.

3. The fermions in the first generation belong to different \( g \) multiplets. In particular, the proton is stable because quarks and leptons are not unified into the same multiplet.

4. The \( g \) connections are “fatter” than the \( g_0 \) connections, being functions of the internal coordinates as well, but they have no new components.
Thus there are no new unobserved gauge bosons. This rules out e.g. technicolor scenarios.

5. A particle/anti-particle asymmetry has been built into the theory: \( g \) consists of subspaces \( g_k \), where the \( Z \) eigenvalue \( k \) ranges from \(-3\) (\(-2\) for \( \mathfrak{sl}(3|6) \)) to \(+\infty\), so this algebra is not symmetric under the reflection \( Z \to -Z \). A related observation is that in the definition \( \mathfrak{mb}(3|8) = (\mathbf{2}, \mathbf{3}, \mathbf{3}^* \otimes \mathbf{2}, \mathfrak{sl}(3) \oplus \mathfrak{sl}(2) \oplus \mathfrak{gl}(1))_* \), the fundamental \( \mathfrak{sl}(3) \) modules \( \mathbf{3} \) and \( \mathbf{3}^* \) enter asymmetrically. This asymmetry should be reflected in nature, maybe in the relative abundance of matter and anti-matter.

6. CP amounts to the permutations \( \Omega_A(p,r) \leftrightarrow \Omega_D(p,r) \), \( \Omega_B(p,r) \leftrightarrow \Omega_C(p,r) \), but the directions of the morphisms \( \nabla \) are unchanged. Equivalently, if a particle is described as \( \ker \nabla = \text{im} \nabla' \), its CP conjugate is \( \text{im} \nabla = \ker \nabla' \), i.e. the kernel and image are interchanged. Thus the theory predicts manifest CP violation, without the need for \( \theta \) vacua. However, CPT is conserved because \( T \) reverses the direction of all morphisms.

7. \( \mathfrak{vle}(3|6) \) and \( \mathfrak{mb}(3|8) \) shed no light on the origin of masses. A Higgs particle (a boson of type \( I(0,0;0;1;1) = \Omega_D(0,1) \), i.e. an \( \mathfrak{sl}(2) \) doublet with charges 0 and +1) can be added by hand in the same way as in the standard model, but this is no more (and no less) satisfactory than in the standard model.

8. One could add a fermion in \( \Omega_A(0,0) = \Omega_D(0,0) = I(0,0;0;0) \). It would appear as a sterile neutrino.

Despite these successes, the \( \mathfrak{vle}(3|6) \) theory is fundamentally flawed. In the table above, we see that the right-handed \( u \) quark \( u_R = (0,1;0;\frac{4}{3}) \) and its anti-particle \( \bar{u}_L = (1,0;0;\frac{4}{3}) \) do not correspond to form modules. The absence of a right-handed \( u \) quark is of course fatal for the \( \mathfrak{vle}(3|6) \) theory. However, this result has no bearing on \( \mathfrak{mb}(3|8) \), because its list of form modules has not yet been worked out, and one may hope that all quarks and leptons in the table above will correspond to forms. Conversely, we can turn this physical requirement into a conjecture about the list of form modules for \( \mathfrak{mb}(3|8) \). Fortunately, this problem is of considerable mathematical interest in its own right, so algebraists are likely to attack it soon.

Finally some history. In 1977, Kac classified the finite-dimensional Lie superalgebras \( \mathfrak{L} \), and conjectured that the infinite-dimensional case would
be the obvious super analogue to Cartan’s list \((W_n, S_n, H_n, K_n)\) \([3]\). However, it was immediately pointed out \([1, 29]\) that there are also odd versions of \(H_n\) and \(K_n\); the odd Hamiltonian algebra was introduced in physics soon thereafter by Batalin and Vilkovisky. The deformations were also found \([30]\). It was a complete surprise when Shchepochkina found three exceptions \([38]\), followed by two more \([33, 40, 41]\); the exception \(\mathfrak{as}(1|6)\) was independently found by Cheng and Kac \([7]\). Important techniques were developed by Leites and Shchepochkina \([31]\), and the classification was known to them in 1996 (announced in \([40]\)), but I have only seen their paper in preprint form \([32]\). Meanwhile, Kac also worked out the classification \([15]\), together with Cheng \([8]\).

### 4 Discussion

In the introduction, I made the claim that \(\mathfrak{mb}(3|8)\) and the multi-dimensional Virasoro algebra are essentially the deepest and most general symmetries possible. This statement assumes that one interprets the word “symmetry” in a conservative sense, to mean semisimple Lie algebras or superalgebras, finite-dimensional algebras or infinite-dimensional algebras of vector fields, and abelian extensions thereof. One can view this list as the definition of the present paradigm. This list covers many algebraic structures, including all infinitesimal quantum deformations, which are always of the form

\[
\begin{align*}
{[J^a, J^b]} &= f^{abc} J^c + k^{ab} i h E^i + O(h^2), \\
{[J^a, hE^j]} &= g^{aij} h E^j + O(h^2), \\
{[hE^i, hE^j]} &= 0 + O(h^2).
\end{align*}
\]

We see that every deformation reduces to an abelian (but not necessarily central) extension in the \(h \to 0\) limit.

However, my definition of symmetry excludes algebras of exponential growth, e.g. non-affine Kac-Moody algebras. The reason is that I think of algebras as infinitesimal transformation groups. In particular, \(\mathfrak{vect}(n|m)\) is the Lie algebra of the diffeomorphism group in \(n|m\) dimensions, and this algebra is of polynomial growth. It is difficult to visualize any interesting symmetry that is essentially bigger than the full diffeomorphism group, maybe acting on the total space of some bundle over the base manifold. That the lack of an explicit realization is a problem for non-affine Kac-Moody algebras has been drastically formulated by Kac: “It is a well kept secret that the theory of Kac-Moody algebras has been a disaster.” \([14]\).
If one stays within the present paradigm, there are really no unknown possibilities; classifications exist. However, if there is a paradigm shift, all bets are off. Paradigm shifts have occurred in the past. Examples are given by the transitions from Lie algebras to superalgebras, from semisimple algebras to central extensions, and presumably also from central extensions to abelian but non-central Virasoro-like extensions.

What makes me believe that there will be no further paradigm shift is the convergence between the deepest algebraic structures and the deepest experimental physics: both mathematics and physics seem to require four dimensions and the $sl(3) \oplus sl(2) \oplus gl(1)$ symmetry at their deepest levels. But even if these structures turn out to be unrelated to physics, it is still worthwhile to pursue them. Hardly anything in mathematics is more natural than the representation theory of interesting algebras, so the efforts will not be wasted.

References

[1] Alekseevsky D., Leites D. and Shchepochkina I.: Examples of simple Lie superalgebras of vector fields. C. R. Acad. Bulg. Sci., 34 1187–1190, (1980) (in Russian)

[2] Belavin A.A., Polyakov A.M. and Zamolodchikov A.M.: Infinite conformal symmetry in two-dimensional quantum field theory. Nucl. Phys. B 241, 333–380 (1984)

[3] Berman, S. and Billig, Y.: Irreducible representations for toroidal Lie algebras. J. Algebra 221, 188–231 (1999)

[4] Berman, S., Billig, Y. and Szmi, J.; Vertex operator algebras and the representation theory of toroidal algebras. math.QA/0101094 (2001)

[5] Billig, Y.: Principal vertex operator representations for toroidal Lie algebras. J. Math. Phys. 7, 3844–3864 (1998)

[6] Cartan, E.: Les groupes des transformatoins continus, infinis, simples. Ann. Sci. Ecole Norm. Sup. 26, 93–161 (1909)

[7] Cheng, S.-J. and Kac, V.G.: A new $N = 6$ superconformal algebra. Comm. Math. Phys. 186, 219–231 (1997)

14 Abelian extensions of superalgebras are not yet classified.
[8] Cheng, S.-J. and Kac, V.G.: Structure of some $\mathbb{Z}$-graded Lie superalgebras of vector fields. Transformation groups 4, 219–272 (1999)

[9] Dzhumadildaev A.: Central extensions and invariant forms of Cartan type Lie algebras of positive characteristic. Funct. Anal. Appl. 24, 331–332 (1985)

[10] Dzhumadildaev A.: Virasoro type Lie algebras and deformations. Z. Phys. C 72, 509–517 (1996)

[11] Di Francesco, P., Mathieu, P., and Sénéchal, D.: Conformal field theory. New York: Springer-Verlag (1996)

[12] Henneaux, M. and Teitelboim, C.: Quantization of gauge systems. Princeton Univ. Press (1992)

[13] Kac, V.G.: Lie superalgebras. Adv. Math. 26, 8–96 (1977)

[14] Kac, V.G.: The idea of locality. q-alg/9709008 (1997)

[15] Kac, V.G.: Classification of infinite-dimensional simple linearly compact Lie superalgebras. Adv. Math. 139, 1–55 (1998)

[16] Kac, V.G.: Classification of infinite-dimensional simple groups of supersymmetry and quantum field theory. math.QA/9912235 (1999)

[17] Kassel, C.: Kahler differentials and coverings of complex simple Lie algebras extended over a commutative algebra. J. Pure and Appl. Algebra 34, 256–275 (1985)

[18] Kac, V.G. and Rudakov, A.N.: Representations of the Exceptional Lie Superalgebra $E(3,6)$: I. Degeneracy Conditions. math-ph/0012043 (2000)

[19] Kac, V.G. and Rudakov, A.N.: Representations of the Exceptional Lie Superalgebra $E(3,6)$: II. Four series of degenerate modules. math-ph/0012050 (2000)

[20] Larsson, T.A.: Central and non-central extensions of multi-graded Lie algebras. J. Phys. A. 25, 1177–1184 (1992)

[21] Larsson, T.A.: Lowest-energy representations of non-centrally extended diffeomorphism algebras. Commun. Math. Phys. 201, 461–470 (1999) physics/9705040
[22] Larsson, T.A.: Fock representations of non-centrally extended super-diffeomorphism algebras. [physics/9710023](1997)

[23] Larsson, T.A.: Extended diffeomorphism algebras and trajectories in jet space. Commun. Math. Phys. 214, 469–491 (2000) [math-ph/9810003]

[24] Larsson, T.A.: Quantum physics as the projective representation theory of Noether symmetries. [math-ph/9908028](1999)

[25] Larsson, T.A.: Extensions of diffeomorphism and current algebras. [math-ph/0002018](2000)

[26] Larsson, T.A.: Multi-dimensional diffeomorphism and current algebras from Virasoro and Kac-Moody Currents. [math-ph/0101007](2001)

[27] Larsson, T.A.: Structures preserved by consistently graded Lie superalgebras. [math-ph/0106004](2001)

[28] Larsson, T.A.: A \text{sl}(3|6) gauge theory. In preparation.

[29] Leites, D.: New Lie superalgebras and mechanics. Soviet Math. Dokl. 18, 1277-1280 (1977)

[30] Leites, D.: Lie superalgebras. J. Soviet Math. 30, 2481–2512 (1985)

[31] Leites, D. and Shchepochkina, I.: Towards classification of simple vectorial Lie superalgebras. In Leites D. (ed.): Seminar on Supermanifolds, Reports of Stockholm University, 30/1988-13 and 31/1988-14 (1988)

[32] Leites, D. and Shchepochkina, I.: Classification of the simple Lie superalgebras of vector fields. preprint (2000)

[33] Moody, R.V., Rao, S.E., and Yokonoma, T.: Toroidal Lie algebras and vertex representations. Geom. Ded. 35, 283–307 (1990)

[34] Rao, S.E., Moody, R.V., and Yokonuma, T.: Lie algebras and Weyl groups arising from vertex operator representations. Nova J. of Algebra and Geometry 1, 15–57 (1992)

[35] Rao, S.E. and Moody, R.V.: Vertex representations for N-toroidal Lie algebras and a generalization of the Virasoro algebra. Commun. Math. Phys. 159, 239–264 (1994)

\footnote{Note added in version 3: This paper is subsumed by Exceptional Lie superalgebras, invariant morphisms, and a second-gauged standard model. [math-ph/0202023]}
[36] Rudakov, A.N.: *Irreducible representations of infinite-dimensional Lie algebras of the Cartan type*. Math. USSR-Izvestia 8, 835–866 (1974)

[37] Schroer, B.: *Facts and Fictions about Anti deSitter Spacetimes with Local Quantum Matter*. hep-th/9911100

[38] Shchepochkina, I.: *Exceptional simple infinite dimensional Lie superalgebras*. C. R. Acad. Bulg. Sci. 36, 313–314 (1983) (in Russian)

[39] Shchepochkina, I.: *Maximal subalgebras of simple Lie superalgebras*. In Leites D. (ed.): *Seminar on Supermanifolds*, 1–34 (1987–1990), 32/1988-15, Reports of Stockholm University, 1–43. hep-th/9702120

[40] Shchepochkina, I.: *The five exceptional simple Lie superalgebras of vector fields*. hep-th/9702121 (a preliminary version of [41])

[41] Shchepochkina, I.: *Five exceptional simple Lie superalgebras of vector fields and their fourteen regradings*. Represent. Theory, 3, 373–415 (1999)

[42] Woit, P.: *String theory: an evaluation*. physics/0102051