Anomalous diffusion, nonlinear fractional Fokker-Planck equation and solutions

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We obtain new exact classes of solutions for the nonlinear fractional Fokker-Planck-like equation
\[ \frac{\partial}{\partial t} \rho(x,t) = \frac{\partial}{\partial x} \left\{ D(x) \frac{\partial^{\nu-1}}{\partial x^{\nu-1}} [\rho(x,t)]^{\nu} - F(x) \rho(x,t) \right\}, \] (1)

where \( \nu, \mu \in \mathbb{R} \), \( D(x) \propto |x|^{-\theta} \) is a (dimensionless) diffusion coefficient (\( \theta \in \mathbb{R} \)), and \( F(x) \equiv -dV(x)/dx \) is a (dimensionless) external force (drift) associated with the potential \( V(x) \). We use the Riemann-Liouville operator \[ \mathcal{J} \] for the fractional derivative, and we also work with the positive \( x \) axis and, later on, we employ symmetry to extend the results to the entire real axis (in other words, we are working with \( \partial^{\nu-1}/\partial|x|^{\nu-1} \). In addition, the initial condition \( \rho(x,0) = \delta(x) \) and the boundary condition \( \rho(\pm\infty, t) \to 0 \) are used. Notice that the normalization of \( \rho \) is time independent. Indeed, if we write the equation in the form \( \partial_t \rho = \partial_x \mathcal{J} \) and assume the boundary conditions \( \mathcal{J}(\pm\infty, t) \to 0 \), it can be shown that \( \int_{-\infty}^{\infty} dx \rho(x,t) \) is a constant of motion.

As indicated above, Eq.(1) can be used to describe a large class of anomalous diffusion processes since it contains, as a particular case, the porous medium equation, Lévy superdiffusion, as well as a mix of them. For \( \mu = 2 \) and \( \nu = 1 \), Eq.(1) recovers the standard Fokker-Planck equation with a drift term. The particular case \( F(x) = 0 \) (no drift), \( D(x) = \text{constant} \), \( \mu = 2 \), and arbitrary \( \nu \) has been considered by Spohn \textsuperscript{19}. The case \( \mu = 2 \) has also been addressed in \textsuperscript{20} \textsuperscript{19} \textsuperscript{21} \textsuperscript{22} \textsuperscript{23} with simple drifts. The case \( \mu < 2 \) with \( F(x) = 0 \) and \( D = \text{constant} \) has been addressed in \textsuperscript{23}. Our present analysis involves several extensions of these cases, in particular, by employing the external force \( F = -k_1 x + k_\gamma x |x|^{\gamma-1} \). Indeed, a variety of systems, which presents anomalous diffusion, can be described by short or long-range forces when the previous one is employed. This analysis is presented in Sec. II for the case \( \mu = 2 \); the solution is obtained in a closed form. In Sec. III, we also obtain exact results for the case \( \mu \neq 2 \), i.e., for the external drift \( F = -k_1 x \) we ob-
tain solutions for \( \mu \in \mathcal{R} \), and for \( F = -k_1 x + k_\gamma x|^{\gamma -1} \) we find solutions for \( \mu = 0 \) and \( \mu = 1 \). Finally, in Sec. IV we present our conclusions.

II. NONLINEAR FOKKER-PANCKE EQUATION

Before starting our discussion, it is interesting to note that the results presented in [20, 21, 22, 23, 24, 25] may be essentially obtained by using normalized scaled solutions of the type

\[ \rho(x, t) = \frac{1}{\Phi(t)} \Phi \left( \frac{x}{\Phi(t)} \right) \]

in Eq. (1). Similar approach has been employed, for instance, in [3, 24]. To illustrate this procedure let us insert Eq. (2) into Eq. (1) with \( \mu = 2, F(x) = 0 \) and \( D(x) = D = \text{constant} \). Thus,

\[ -\frac{\Phi(t)}{\Phi(t)^2} \cdot \frac{d}{dz}[\rho(z)] = \frac{d}{dz} \left\{ \frac{D}{\Phi(t)^{2 + \nu}} \frac{d}{dz} [\rho(z)]^{\nu} \right\} \]  

In the following, we eliminate the explicitly time dependence on Eq. (4) choosing

\[ [\Phi(t)]^{\nu} \frac{d}{dt} \Phi(t) = k, \]

where \( k \) is a constant. Thus, Eq. (4) replaced into Eq. (3) gives us

\[ -k \frac{d}{dz} \rho(z) = \frac{d}{dz} \left\{ D \frac{d}{dz} [\rho(z)]^{\nu} \right\} \]

From Eq. (5) we obtain

\[ \Phi(t) = [(1 + \nu)kt]^{1/(1+\nu)} \]

where we have adopted the solution which satisfies \( \Phi(0) = 0 \) for \( \nu > -1 \). Finally, the solution of Eq. (6) leads to

\[ \rho(x, t) = \frac{1}{\Phi(t)} \exp_q \left[ -\frac{k}{2 D \nu} \left( \frac{x}{\Phi(t)} \right)^{2} \right], \]

with \( q = 2 - \nu \) and \( \exp_q(x) \equiv [1 + (1-q)x]^{1/(1-q)} \) being the \( q \)-exponential function. This generalized exponential arises within the nonextensive thermostatistical formalism by optimizing, under appropriate constraints, the entropic form \( S_q = (1 - \int dx \rho(x, t)^{q})/(q-1) \) [26].

Firstly, we address our discussion to case \( \mu = 2 \). In this case, Eq. (3) reduces to

\[ -\frac{\Phi(t)}{\Phi(t)^2} \cdot \frac{d}{dz}[\rho(z)] = \frac{d}{dz} \left\{ \frac{D}{\Phi(t)^{2 + \nu}} \frac{d}{dz} [\rho(z)]^{\nu} + \left[ \frac{z_0^k}{\Phi(t)^{2+\nu}} - \frac{z_0^k}{\Phi(t)^{2-\gamma}} \right] \rho(z) \right\} \]

when \( \rho(x, t) \), given by Eq. (7), is employed. We are interested in situations for which the scaled solution of the type indicated in Eq. (8) is still valid, i.e., we would like to uncouple the \( t \) and \( z \) dependence in Eq. (8) as performed in Eq. (3) via Eq. (4). This can be accomplished when \( \gamma + \theta + \nu = 0 \). If this condition is satisfied, we obtain

\[ \frac{\Phi(t)}{\Phi(t)^2} + k_1 = \frac{k'}{\Phi(t)^{2+\nu+\theta}} \]

and

\[ -k' \frac{d}{dz} [\rho(z)] = \frac{D}{\Phi(t)^{2+\nu}} \frac{d}{dz} [\rho(z)]^{\nu} - k_\gamma \rho(z) \]

where \( k' \) is a constant that plays a role analogous to \( k \) in Eq. (4), and it is to be determined through the normalization.

From the solutions of Eq. (9) and Eq. (10), which generalize the results of the previous example, we have

\[ \Phi(t) = k' \left( 1 - e^{-(1+\nu+\theta)k_\gamma t} \right)^{1/(1+\nu+\theta)} \]

and

\[ \rho(x, t) = \frac{1}{\Phi(t)} \exp_q \left[ -\frac{1}{D \nu} \left( \frac{x}{\Phi(t)} \right)^{2+\theta} - k_\gamma \ln_q \left( \frac{|x|}{\Phi(t)} \right) \right], \]

where \( \ln_q x \equiv (x^{1-q} - 1)/(1-q) \) is the \( q \)-logarithm function (that is the inverse function of the \( q \)-exponential).
In the previous study for the case $\mu = 2$, we notice that the analytical solution \[\text{Eq. (13)}\] takes fractal and nonlinear aspects into account, since there is a spatial dependence on the diffusion coefficient ($\theta \neq 0$) and a nonlinear term ($\mu \neq 1$). In fact, it contains results obtained in \[\text{Eq. (14)}\] \[\text{Eq. (20)}\] \[\text{Eq. (22)}\] \[\text{Eq. (25)}\] \[\text{Eq. (27)}\] as particular case. Moreover, it reduces to the Rayleigh process \[\text{Eq. (28)}\] case in the limit $q \to 1$ and $\theta = 0$.

III. NONLINEAR FRACTIONAL FOKKER-PLANCK EQUATION

In order to give an extension for the previous discussion ($\mu = 2$), we address our investigation concerning Eq. \[\text{Eq. (1)}\] by considering $\mu \neq 2$. Without loss of generality we are considering $D = 1$. We follow the procedure employed in \[\text{Ref. 24, 27}\] taking the generic property

$$ \frac{d^\delta}{dz^\delta} G(\alpha x) = \alpha^\delta \frac{d^\delta}{dz^\delta} G(\alpha z) \quad (\delta \in \mathbb{R}) \quad \text{(13)} $$

with $\alpha = ax$ into account. This basic property holds not only for the ordinary derivative but also for all fractional operators that we are considering here. Thus, substituting Eq. \[\text{Eq. (20)}\] into Eq. \[\text{Eq. (1)}\] and imposing

$$ \frac{\dot{\Phi}(t)}{\Phi(t)^2} + \frac{k_1}{\Phi(t)} = -\frac{\kappa'}{\Phi(t)^{\nu+\theta+\mu}} \quad \text{(14)} $$

in order to uncouple $t$ and $z$ dependence, where $\kappa'$ is an arbitrary constant. Therefore, we obtain

$$ \Phi(t) = \left[\kappa_2 - \frac{\kappa'}{k_1} \left(1 - e^{-(\nu+\theta+\mu)k_1 t}\right)\right]^{\nu+\theta+\mu+1} \quad \text{(15)} $$

and

$$ \kappa \frac{d}{dz} [z \tilde{\rho}(z)] = \frac{d}{dz} \left\{z^{-\theta} \frac{d^{\mu-1}}{dz^{\mu-1}} [\tilde{\rho}(z)]^\nu - k_2 z^{-\nu-\theta-\mu+2} \tilde{\rho}(z)\right\} \quad \text{(16)} $$

with $\gamma = -\theta - \nu - \mu + 2$ and $\kappa_2$ being an arbitrary constant. If $k_2 = 0$ we have $\Phi(0) = 0$ for $\nu + \mu + \theta > 1$, recovering Eq. \[\text{Eq. (11)}\] when $\mu \to 2$. By considering $k_2 = 0$ corresponds $\rho(x, 0) \propto \delta(x)$. On the other hand, employing $k_2 \neq 0$ leads to $\rho(x, 0)$ less concentrated than $\delta(x)$. A more tractable equation to $\tilde{\rho}(z)$ is obtained after an integration of Eq. \[\text{Eq. (17)}\], i. e.,

$$ \kappa z \tilde{\rho}(z) = z^{-\theta} \frac{d^{\mu-1}}{dz^{\mu-1}} [\tilde{\rho}(z)]^\nu - k_2 z^{-\nu-\theta-\mu+2} \tilde{\rho}(z) + C \quad \text{(17)} $$

where $C$ is another arbitrary constant. In the following, we address our discussion focusing two situations of Eq.\[\text{Eq. (17)}\]. The first one is characterized by a linear drift, $F = -k_1 x$. In the second one, we consider the drift $F = -k_1 x + k_2 x|x|^{\gamma-1}$ for particular values of $\mu$.

A. The drift $F = -k_1 x$

Let us start our discussion by considering Eq.\[\text{Eq. (17)}\] with $k_1 \neq 0$ and $k_2 = 0$. To solve it, we can use the procedure employed in \[\text{Ref. 23}\] with $C = 0$ and

$$ \nu = \frac{2 - \mu}{1 + \mu + \theta} \quad \text{(18)} $$

In this case, we have

$$ \rho(x, t) = A \frac{\Phi(t)}{\Phi(0)} \left[\frac{z^{(\mu+\theta)(1+\mu+\theta)}}{(1 + \beta z^{(1-\mu)(1+\mu+\theta)})}\right]^{-\frac{1}{\mu+\theta}} \quad \text{(19)} $$

with $\Phi(t)$ given by Eq.\[\text{Eq. (20)}\], $z \equiv x/\Phi(t)$, and

$$ A = \left[\frac{\kappa'}{\Gamma(\alpha + 1)}\right]^{\frac{1 + \mu + \theta}{\mu + \theta}} \quad \text{(20)} $$

Note that Eq.\[\text{Eq. (19)}\] reduces that one obtained in \[\text{Ref. 27}\] in the absence of drift. We notice that the presence of source (absorvent) term, $\alpha \rho (\alpha = \text{const})$, in the right hand of Eq.\[\text{Eq. (11)}\] modifies Eq.\[\text{Eq. (12)}\] as follows:

$$ \Phi(t) = \left[\kappa_2 - \frac{(\mu + \theta + \nu - 1)\kappa'}{(\nu - 1)\alpha + (\mu + \theta + \nu - 1)k_1} \left(\frac{e^{(\nu-1)\alpha t} - e^{-(\mu+\theta+\nu-1)k_1 t}}{\mu + \nu + \theta + 1}\right)^{\mu + \nu + \theta + 1}\right] \quad \text{(21)} $$

B. The drift $F = -k_1 x + k_2 x|x|^{\gamma-1}$

Now, we consider other particular cases, namely $\mu = 0$ and $\mu = 1$. Let us start with $\mu = 0$ and arbitrary $\nu$. The corresponding equation is

$$ \frac{\partial}{\partial t} \rho(x, t) = \frac{\partial}{\partial x} \left\{x^{-\theta} \int_0^x [\rho(y, t)]^\nu \, dy + [k_1 x - \kappa_2 x^{-\nu-\theta+2}] \rho(x, t) \right\} \quad \text{(22)} $$
To solve it let us go back to Eq. (17) with $\mu = 0$ and $C = 0$, i.e.,

$$\kappa z \hat{\rho}(z) = z^{-\theta} \int_0^z d\tilde{z} [\hat{\rho}(\tilde{z})]^{\nu} - \kappa \gamma z^{-\nu-\theta+2} \hat{\rho}(z),$$

(23)

whose solution is given by

$$\hat{\rho}(z) \propto \frac{1}{k z^{1+\theta} + \kappa \gamma z^{-\nu+2}} \left[ 1 + \tilde{C} \int d\tilde{z} \left( k z^{1+\theta} + \kappa \gamma z^{-\nu+2} \right)^{-\nu} \right]^{1/(1-\nu)},$$

(24)

where $\tilde{C}$ is a constant.

Let us now address our analysis to the case $\mu = 1$. It corresponds to investigate the equation

$$\frac{\partial}{\partial t} \rho(x,t) = \frac{\partial}{\partial x} \left\{ x^{-\theta} \rho(x,t) \right\}^{\nu} + \left[ k_{1,x} - \kappa \gamma x^{-\nu-\theta+1} \right] \rho(x,t).$$

(25)

To solve it let us go back to Eq. (17) with $\mu = 0$ and $C = 0$, i.e.,

$$\kappa z \hat{\rho}(z) = z^{-\theta} \int_0^z d\tilde{z} [\hat{\rho}(\tilde{z})]^{\nu} - \kappa \gamma z^{-\nu-\theta+2} \hat{\rho}(z),$$

(23)

whose solution is given by

$$\hat{\rho}(z) \propto \frac{1}{k z^{1+\theta} + \kappa \gamma z^{-\nu+2}} \left[ 1 + \tilde{C} \int d\tilde{z} \left( k z^{1+\theta} + \kappa \gamma z^{-\nu+2} \right)^{-\nu} \right]^{1/(1-\nu)},$$

(24)

where $\tilde{C}$ is a constant.

Let us now address our analysis to the case $\mu = 1$. It corresponds to investigate the equation

$$\frac{\partial}{\partial t} \rho(x,t) = \frac{\partial}{\partial x} \left\{ x^{-\theta} \rho(x,t) \right\}^{\nu} + \left[ k_{1,x} - \kappa \gamma x^{-\nu-\theta+1} \right] \rho(x,t).$$

(25)

To obtain the solution of this equation employing the ansatz (3) we use Eq. (17). It follows that

$$\kappa z \hat{\rho}(z) = z^{-\theta} [\hat{\rho}(z)]^{\nu} - \kappa \gamma z^{-\nu-\theta+1} \hat{\rho}(z) + C.$$

(26)

The solution corresponding to $C = 0$ is

$$\hat{\rho}(z) \propto \left( k z^{1+\theta} + \kappa \gamma z^{-\nu+1} \right)^{1/(1-\nu)}. $$

(27)

Let us finally indicate a connection between the results obtained here and the solutions that arise from the optimization of the nonextensive entropy [4]. These distributions do not coincide for arbitrary value of $x$. However, the comparison between the asymptotic behaviors (large $|x|$) enables us to identify the type of tails. For instance, by taking the behavior exhibited in Eq. (19) for large $|x|$ with the asymptotic behavior $1/|x|^{2/(q-1)}$ into account, which appears in [4] for the entropic problem, we obtain

$$q = \frac{3 + \mu + \theta}{1 + \mu + \theta}.$$ 

(28)

This relation is the same obtained in [27] in the absence of drift. It also recovers that one already established in [23] for $\theta = 0$.

IV. CONCLUSIONS

In summary, we have considered the one-dimensional nonlinear fractional Fokker-Planck equation (Eq. [1]) for some space-dependent (power-law) classes of external drift and diffusion coefficient. We have shown that it admits exact solutions where space scales with a (usually simple) function of time (as indicated in Eq. (2)), and have fully and explicitly worked out some of those. In particular, we first consider the case $\mu = 2$ in the presence of the drift, $F = -k_1 x + \kappa \gamma x |x|^{-\gamma-1}$, with $k_1, \kappa \gamma \neq 0$. After, we analyse the case $\mu \neq 2$ in the presence of the same drift for some situations. We have also discussed the connection with nonextensive statistics, when appropriated, providing the relation between the entropic index $q$ and the exponents appearing in the Fokker-Planck equation. Thus, by extending here results such as those discussed in [23, 24, 25, 27], we hope to make possible further applications to physical systems exhibiting suitable mix of different forms (nonlinear $\nu \neq 1$, fractional derivative $\mu \neq 2$ and fractal $\theta \neq 0$) of anomalous diffusion.

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