Numerical study of the vertical vibration of a vehicle model with variable speed

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Abstract. The paper deals with the numerical analysis of the general model of vehicle oscillations considering the non-stationarity of random excitation. The model parameters of the applied railway vehicle are deterministic functions. The non-stationary random function will be modelled by the variable speed of the vehicle and the vertical unevenness of the track. The so-called evolutionary Gaussian random process will be considering. The proposed comparative study of the dynamics of the vertical motion of the analysed railway vehicle will be realized using Monte Carlo simulation and a numerical procedure based on the theory of Markov processes. The originality of the article can be found in the implementation and algorithmization of the principles of solving non-stationary oscillation problems of machines. A universal methodology applicable in the dynamics of machines of various purposes is presented.

Keywords: random vibration, Monte Carlo simulation, Markov process, non-stationary random process, stochastic analysis, mean response, covariance response

1. Introduction

Studies and research in the field of stochastic dynamics often assume Gaussian stationary excitations. But only a few random processes in engineering practice are really Gaussian and stationary. Stochastic loadings will be interpreted not only as external forces, but also as external effects. For example Bolotin defines random excitation as follows: loading due to atmospheric turbulence, acoustic loading, loading due to pulsation in a turbulent boundary layer, loading due to pressure of sea waves, loading of transport machines due to unevenness of track, and seismic loading. On the other hand, the exact solutions in linear or non-linear random vibration analyses are very limited and they are usually based on the assumptions that the excitation is Gaussian white noise and the structures can be modelled as single degree-of-freedom systems. It is that to solve the multi-degree-of-freedom dynamics systems is possible only by numerical approaches. In the case of stochastic systems (especially non-linear) we encounter the approaches, such as tangent linearization method (TLM) [2,3], statistical linearization method (SLM) in various modifications [4-11], statistical quadratization method (SQM) [12], the Markov process approach (MPT) [13-15], functional method of Volterra and Wiener (FMVW) [3], asymptotic method of Krylov-Bogoljubov-Mitropolsky (ASM) [16], perturbation method (PM) and its modifications [9,15,17].
Thanks to computer techniques, Monte Carlo simulation method (MCS) is very popular and frequently applied [16,18-21]. Although this method is straightforward and does not have such limitations, it generally time-consuming and costly. In view of these difficulties, approximate methods, including PM, TLM, SLM, SQM can be advantageous. Some authors look for the new approaches of the solution by combining the Monte Carlo method with other methods [19,22].

2. Mathematical model and solution methods
To construct a mathematical model of a system for dynamic analysis, it is necessary to idealize the inertia, damping and stiffness properties by discrete or continuous elements. Usually the first step is to construct a physical model that may be an assemblage of discrete elements such as mass, springs and dashpots, continuous elements such as bars, beams, shells and volumes, or a combination of both discrete and continuous elements. The application of the fundamental laws of mechanics yields a set of generally non-linear differential equations

\[ \dot{x}(t) + A(x,t) \cdot x(t) = f(t), \]  

where \( x(t) \) is the response vector corresponding to the random excitation vector \( f(t) \), \( A(x,t) \) is the real or complex structural matrix of order \( n \times n \). \( A(x,t) \) may be linear or non-linear, depending on the nature of the problem. Many mechanical models are linear thanks to their analytical simplicity and the fact that they yield realistic results for large class problems.

There are, however, a number of problems for which linear models do not yield acceptable results, so that it becomes necessary to construct non-linear models. It means, if \( A(x,t) \) is non-linear, we can apply well-known approximate methods (PM, TLM, SLM or SQM). Using linearization techniques we get the statistically equivalent structural matrix \( A \).

This study presents two approaches in determining the response of a system modelled by equation (1) at first by the Markov processes theory and secondly by Monte Carlo simulation.

2.1. Solution by Markov process theory
The Markov process formulation requires the idealization that the excitation is independent at two instants of time regardless of how close they are (delta correlation) [15]. This assumption [15,17], which is clearly physically unrealisable, leads to such models as white noise and processes obtained by linearly filtering white noise.

Let us consider the system of first-order differential equations (1) with initial conditions \( x(0) = 0 \) and force excitation \( f(t) = y(t) \cdot p(t) \). The force \( f(t) \) is a modulated evolutionary process vector with a deterministic vector function \( y(t) \) and stationary random process \( p(t) \) with zero mean.

The mean response of \( x(t) \) [15] is

\[ E[x] + A \cdot E[x] = E[f], \]  

where \( E[...] \) is the mean value operator. The covariance response of \( x(t) \) is

\[ \dot{K}(t) + A \cdot K(t) + (A \cdot K(t))^{T} = b \cdot y^{T} + y \cdot b^{T}, \]  

where \( K(t) \) is the covariance matrix,

\[ K(t) = E[ x \cdot x^{T} ] \quad \text{and} \quad K(0) = 0, \]  

and

\[ b(t) = h(t) \cdot \int_{0}^{t} h^{-1}(u) \cdot y(u) \cdot E[p(u) \cdot p(t)] \cdot du. \]
Equations (2) and (3) imply that mean vector and covariance matrix are the time functions. Matrix $h(t)$ is so-called the fundamental solution matrix or impulse response matrix. If it is assumed that $p(t)$ is white noise with $E[p(t_1)p(t_2)] = 2\pi \Phi_0 \delta(t_2-t_1)$, then the equation (3) can be expressed as

$$K(t) + A \cdot K(t) + (A \cdot K(t))^T = 2\pi \cdot \Phi_0 \cdot y(t) \cdot y^T(t),$$

(4)

where $\Phi_0$ is the power spectral density of $p(t)$. The acceptable solution of the equation (4) is possible to make by special numerical approach.

Let us consider Crank-Nicolson integration method. The discrete time derivation is given by

$$\dot{K}(t) = \frac{2}{\Delta} \cdot [K(t) - K(t - \Delta)] - \dot{K}(t - \Delta),$$

(5)

where $\Delta$ is the time step of the integration method. Using the equations (5) and (4) we can write

$$\left( \frac{2}{\Delta} \cdot I + A \right) \cdot K(t) + [A \cdot K(t)]^T = 2\pi \cdot \Phi_0 \cdot y(t) \cdot y^T(t) + \frac{2}{\Delta} \cdot K(t - \Delta) + K(t - \Delta),$$

(6)

where $I$ is the identity matrix. The equation (6) is so-called Lyapunov equation in generally form. In each time step is necessary to use the special numerical algorithm created in MATLAB.

2.2. Solution by Monte Carlo simulation

The alternative way of the previous approach is to use the Monte Carlo method. With the advent of recent computational facilities, this method becomes ever more attractive [19,20,23]. The results are determined from the series of numerical analyses of equation (1) (approximately 100-1000 realisations of random excitation). It is recommended to generate about 5000 random values of excitation function (defined by power spectral density $S_{ff}(\omega)$) for each realisation. Simulation of input non-stationary evolutionary Gaussian process $f(t)$ with zero mean can be formulated by

$$f(t) = c(t) \cdot y(t) = \sqrt{2} \cdot \sum_{k=1}^{N} S_{ff}(\omega_k) \cdot \Delta \omega \cdot \cos(\omega_k \cdot t - \phi_k) \cdot y(t),$$

(7)

where $\phi_k$ is uniformly distributed random number ($0 \leq \phi_k \leq 2\pi$), $y(t)$ is a deterministically modulating time function, $c(t)$ is zero mean stationary process with power spectral density $S_{cc}(\omega)$. If $S_{cc}(\omega)$ is constant then $c(t)$ is white noise. The evolutionary power spectral density of input loading $f(t)$ is $S_{ff}(\omega, t) = y^2(t) \cdot S_{cc}(\omega)$.

3. Modelled engineering problem - Statistical analysis of railway vehicle vertical vibration

A vehicle moving on railway track causes vibrations. Since the profile of a track is a random function of the spatial coordinates, these vibrations are also random. We shall assume that the motion of the vehicle in the horizontal direction is non-uniform (changeable speed, although more important non-stationarity can be the changeable track quality).

Using previous theory we shall solve the response of the simple vehicle model (Fig. 1) under non-stationary random excitation. Let us determine the first and second statistical moments (i.e. the mean vector and the covariance matrix) of the response of the mechanical model on Fig. 1.

The structural parameters are:

- mass of bogie, $m_1 = 3000$ kg,
- mass of body of coach, $m_2 = 13000$ kg,
- damping coefficient in vertical direction, $b_1 = 120000$ Nsm$^{-1}$,
- damping coefficient in vertical direction, $b_2 = 100000$ Nsm$^{-1}$,
- vertical stiffness, $k_1 = 3000000$ Nm$^{-1}$,
- vertical stiffness, $k_2 = 1500000$ Nm$^{-1}$.
Let us consider the approximation of the power spectral density of vertical unevenness $u(t)$ of track in due order ORE B 176 in the form

$$S_{uu}(\lambda) = \frac{A \cdot b^2}{(\lambda^2 + a^2)(\lambda^2 + b^2)},$$

where $a = 0.0206$, $b = 0.8246$, $A = 4.032 \cdot 10^{-7}$ for a good track and $A = 1.08 \cdot 10^{-8}$ for a bad track. $\lambda$ is the length frequency.

If the vehicle speed is time function $v = v(t)$ and

$$\lambda = \frac{\omega}{v},$$

then

$$S_{uu}(\omega, t) = \frac{1}{v(t)} \cdot \frac{A \cdot b^2}{\frac{\omega^2}{v^2(t)} + a^2 \cdot \left(\frac{\omega^2}{v^2(t)} + b^2\right)},$$

where $\omega$ is the circular frequency.

Applying the Markov process theory we shall need to use the assumption of an evolutionary random excitation with a deterministic modulated function and white noise process. Therefore, it is necessary to define the filter parameters of the excitation function. A commonly used filter in modelling of the earthquake ground motion is the Kanai-Tajimi filter governed by the following differential equation

$$m_e \dot{u} + b_e \dot{u} + k_e u = w(t),$$

where $w(t)$ is well-known Gaussian white noise process with constant power spectral density $S_0$. The frequency response function of the filter can be expressed as

$$H(\omega) = \frac{1}{k_e - \omega^2 \cdot m_e + i \cdot \omega \cdot b_e}.$$  

Comparing the power spectral density of $u(t)$ from (11) and (10) we get

$$S_0 = \frac{1}{v} \cdot \frac{A \cdot b^2}{\frac{\omega^2}{v^2} + a^2 \cdot \left(\frac{\omega^2}{v^2} + b^2\right)}.$$
From (13) it is clear that

\[ S_0 = A \cdot b^2, \quad m_e = \frac{1}{\sqrt{v^3}}, \quad b_e = \frac{(a+b)^2}{v}, \quad k_e = a \cdot b \cdot \sqrt{v}. \]  

(14)

Let us construct the equations of motion

\[ \begin{bmatrix} m_1 & 0 & x_1 \\ 0 & m_2 & x_2 \\ 0 & 0 & \frac{1}{\sqrt{v^3(t)}} \end{bmatrix} + \begin{bmatrix} b_1 + b_2 - b_2 \\ b_2 \\ 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix} + \begin{bmatrix} k_1 + k_2 - k_2 \\ -k_2 \\ 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix} = \begin{bmatrix} h_1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} u \\ 0 \end{bmatrix} + \begin{bmatrix} k_1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} u \\ 0 \end{bmatrix}. \]

(15)

Combining (11) and (15) the equations of motion can be expressed as

\[ \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & \frac{1}{\sqrt{v^3(t)}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix} + \begin{bmatrix} b_1 + b_2 - b_2 \\ -b_2 \\ 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix} + \begin{bmatrix} k_1 + k_2 - k_2 \\ -k_2 \\ 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ k_1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} b_2 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} a \cdot b \cdot \sqrt{v(t)} \end{bmatrix} \begin{bmatrix} u \end{bmatrix}. \]

(16)

Let us constitute the 2-dimensional Markov vector by substitute:

\[ y_1 = x_1, \quad y_2 = x_2, \quad y_3 = u, \quad y_4 = x_1, \quad y_5 = x_2, \quad y_6 = u, \]

then the final equations are given by

\[ \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{k_1 + k_2}{m_1} & \frac{k_2}{m_1} & -\frac{k_1}{m_1} & -\frac{b_1 + b_2}{m_1} & \frac{b_2}{m_1} & \frac{b_1}{m_1} \\ -\frac{k_2}{m_2} & \frac{k_1}{m_2} & 0 & -\frac{b_2}{m_2} & \frac{b_1}{m_2} & 0 \\ -\frac{a \cdot b \cdot \sqrt{v(t)}}{m_2} & \frac{a \cdot b \cdot \sqrt{v(t)}}{m_2} & 0 & 0 & (a+b)^2 \cdot \sqrt{v(t)} & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \sqrt{v^3(t)} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sqrt{v^3(t)} \end{bmatrix}, \]

(17)

or

\[ \dot{y}(t) = A(t) \cdot y(t) + b(t) \cdot w(t). \]

(18)

Considering \( E(y) = 0 \) we obtain the covariance response by using (4) as follows

\[ \dot{E}[y \cdot y^T] + A \cdot E[y \cdot y^T] + (A \cdot E[y \cdot y^T])^T = 2\pi \cdot S_0 \cdot b(t) \cdot b^T(t). \]

The numerical solution can be realising by (6). If

\[ v(t) = \frac{100}{3.6} \cdot \left[ 1 + 0.4 \cdot \sin \left( \frac{\pi \cdot t}{T} \right) \right], \]

then the time modulation function is

\[ b^T(t) = \sqrt{\frac{100}{3.6} \cdot \left[ 1 + 0.4 \cdot \sin \left( \frac{\pi \cdot t}{T} \right) \right]^3}, \]
where $T (= 120 \text{ s})$ is duration of the simulation. The mark of ride quality $W_z$ can be expressed as

$$W_z = 3.17 \cdot \left[ E\left( x_2^2 \right) \right]^{0.15}.$$  \hspace{1cm} (20)

The results of the solution of equations (18) and (19) are shown in graphic form on Figs. 2 to 5. We compare the standard deviation of vertical displacements and accelerations of mass bodies 1 and 2 for the track quality parameter $A = 4.032 \cdot 10^{-7}$ (good track) and $A = 1.08 \cdot 10^{-8}$ (bad track). Figure 6 shows the behaviour of the mark of ride quality $W_z$ of mass body $m_2$.

**Figure 2.** Time behaviour of the standard deviation of displacement $x_1$
- Monte Carlo simulation
- Markov process theory

**Figure 3.** Time behaviour of the standard deviation of displacement $x_2$
- Monte Carlo simulation
- Markov process theory
Figure 4. Time behaviour of the standard deviation of acceleration $d^2x/dt^2$

- Monte Carlo simulation
- Markov process theory

Figure 5. Time behaviour of the standard deviation of acceleration $d^2x/dt^2$

- Monte Carlo simulation
- Markov process theory

Figure 6. Time behaviour of the mark of ride quality of body of coach

- Monte Carlo simulation
- Markov process theory
4. Conclusion

The paper presents partial results from numerical analyses for simpler dynamic models of machines with a precondition for successful implementation into complex nonlinear models with a large number of degrees of freedom (for example more detailed vehicle models) or implementation in continuum mechanics (FEM, etc.).

In our numerical study a non-stationary vibration description is extended to the dynamic analyses of vehicles by using the Markov process theory and “classic” Monte Carlo approach, which eliminate the traditional restriction of constant speed (or the track quality) during the period oscillation.

Particularly, after a series of numerical analyses (Monte Carlo method), the presented Markov vector approach is very effective and rapid with respect to the computational time (approximately fifty times more rapid). The Monte Carlo simulation is presented to check the accuracy of the results, which show a fairly good comparison.

Finally, it should be emphasized that these statistically responses are very useful for estimating the reliability of the vehicles structures.

5. References

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