DERIVATIONS OF QUANTIZATIONS IN CHARACTERISTIC $p$

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Abstract. Let $k$ be an algebraically closed field of odd characteristic. We describe derivations of a large class of quantizations of affine normal Poisson varieties over $k$.

Let $k$ be an algebraically closed field of characteristic $p > 2$. Let $A$ be an associative $k$-algebra and let $Z$ be its center. Then we have the natural restriction map $HH^1(A) \to \text{Der}_k(Z, Z)$ from the first Hochschild cohomology of $A$ over $k$ to $k$-derivations of $Z$. In this note we show that this map is injective for a large class of quantizations of Poisson algebras (Theorem 1) and is an isomorphism for central quotients of the enveloping algebras of semi-simple Lie algebras (Corollary 1). It is well-known that this map is an isomorphism if $A$ is an Azumaya algebra over $Z$. In fact in this case all corresponding Hochschild cohomology groups are isomorphic $HH^*(A) \cong HH^*(Z)$.

Throughout given an element $x \in A$, by $\text{ad}(x)$ we will denote the commutator bracket $[x, -] : A \to A$ as it is customary. Thus we have an injective homomorphism of $Z$-modules $A/Z \xrightarrow{\text{ad}} \text{Der}_k(A, A)$. We have a short exact sequence of $Z$-modules

$$0 \to A/Z \xrightarrow{\text{ad}} \text{Der}_k(A, A) \to HH^1(A) \to 0.$$

We will be interested in determining whether this sequence splits. We will start by recalling how a deformation of $A$ over $W_2(k)$ gives rise to a Poisson bracket on $Z$, where $W_2(k)$ denotes the ring of Witt vectors of length 2 over $k$. Hence $W_2(k)$ is a free $Z/p^2Z$-module and $W_2(k)/pW_2(k) = k$.

Let $A_2$ be a lift of $A$ over $W_2(k)$. Thus $A_2$ is an associative $W_2(k)$-algebra which is free as a $W_2(k)$-module and $A_2/pA_2 = A$. Then we have a derivation $i : Z \to HH^1(A)$ defined as follows. For $z \in Z$, let $\tilde{z} \in A_2$ be a lift of $z$. Then $\text{ad}(\tilde{z})(A_2) \subseteq pA_2$. Hence

$$i(z) = \left(\frac{1}{p}\text{ad}(\tilde{z})\right) \mod p : A \to A$$

is a derivation which is independent of a lift of $z \mod$ inner derivations. The map $i$ restricted on $Z$ gives rise to the Poisson bracket $\{,\} : Z \times Z \to Z$, which we will refer to as the deformation Poisson bracket on $Z$.

Then we have the following

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Lemma 1. Let $A$ be an associative $k$-algebra, and let $A_2$ be its lift over $W_2(k)$. Let $Z$ be the center of $A$. Assume that $Z$ admits a lift as a subalgebra of $A_2$. Assume that Spec$Z$ is a normal variety such that the deformation Poisson bracket on $Z$ is symplectic on the smooth locus of Spec$Z$, and $A$ is a finitely generated Cohen-Macaulay $Z$-module such that Ann$_Z A = 0$. Then the restriction map Der$_k(A, A) \to$ Der$_k(Z, Z)$ admits a $Z$-module splitting.

Proof. Let $\tilde{Z} \subset \tilde{A}$ be an algebra lift of $Z$ over $W_2(k)$. Thus $\tilde{Z}$ is a subalgebra of $\tilde{A}$ free over $W_2(k)$ such that $\tilde{Z}/p\tilde{Z} = Z$. Then we have a map $(\frac{1}{p} \text{ad}) \mod p : \tilde{Z} \to \text{Der}_k(A, A)$. This map clearly factors through a derivation $\tilde{Z}/p\tilde{Z} = Z \to \text{Der}_k(A, A)$ and is a lift of the map $i : Z \to HH^1(A)$ described above. Let $U$ be the smooth locus of Spec$Z$. Thus by the assumption the deformation Poisson bracket of Spec$Z$ is symplectic on $U$. We have the map of coherent sheaves $\tilde{\theta}|_U : \Omega^1_U \to \text{Der}_k(A, A)|_U$, and composing it with the identification by the symplectic form between tangent and cotangent bundles $T^1_U \to \Omega^1_U$, we get a map of coherent sheaves on $U$, $\tau : T^1_U \to \text{Der}(A, A)|_U$. Since codim$(\text{Spec} Z \setminus U) \geq 2$ and Spec$Z$ is a normal variety, then $\Gamma(U, T^1_U) = \text{Der}_k(Z, Z)$. Also $\Gamma(U, A_{1U}) = A$ since $A$ is a Cohen-Macaulay $Z$-module of dimension dim $Z$. Thus we get a map of $Z$-modules $\tau : \text{Der}_k(Z, Z) \to \text{Der}_k(A, A)$ which is a section of the restriction map Der$_k(A, A) \to$ Der$_k(Z, Z)$.

Next we have the following

Lemma 2. Let $A$ be an associative $k$-algebra which is a finite over its center $Z$. Assume that $Z$ is a normal $k$-domain such that $A/Z$ is a Cohen-Macaulay module over $Z$, and Ann$_Z (A/Z) = 0$. Assume moreover that the Azumaya locus of $A$ has a compliment of codimension $\geq 2$ in Spec$Z$. Then the restriction map HH$^1(A) \to$ Der$_k(Z, Z)$ is injective.

Proof. Let $D : A \to A$ be a $k$-derivation such that $D(Z) = 0$. Let $U$ be the Azumaya locus of $A$. Put $Y = \text{Spec} Z \setminus U$. Since $A|_U$ is Azumaya algebra, it follows that there exists $x \in \Gamma(U, A/Z)$ such that $D|_U$ is equal to ad($x$). Since $A/Z$ is Cohen-Macaulay module over $Z$ and Ann$_Z (A/Z) = 0$, it follows that depth$_Y (A/Z) \geq 2$. Thus the standard argument using local cohomology groups [H] implies that $\Gamma(U, A/Z) = A/Z$. It follows that there exists $x \in A$ such that $D - \text{ad}(x)x$ vanishes on $U$. Since $Z$ is normal, it follows that depth$_Y A \geq 2$. Hence $\Gamma(U, A) = A$. Therefore $D - \text{ad}(x) = 0$, hence $D = \text{ad}(x)$ is an inner derivation. We conclude that HH$^1(A) \to$ Der$_k(Z, Z)$ is injective as desired. 

Let an associative $k$-algebra $A$ be equipped with an algebra filtration $1 \in A_0 \subset A_1 \subset \cdots$ such that the associated graded algebra gr$A = \bigoplus_n A_n/A_{n-1}$ is commutative. Then recall that there is a graded Poisson bracket on gr$A$ defined as follows. Given $x \in A_n/A_{n-1}$, $y \in A_m/A_{m-1}$, then their Poisson bracket $\{x, y\}$ is defined to be $[\tilde{x}, \tilde{y}] \in A_{n+m-1}/A_{n+m-2}$, where $\tilde{x} \in A_n$, $\tilde{y} \in A_m$ are lifts of $x$, $y$. In this setting we say that a filtered algebra $A$ as a quantization of a graded
Poisson algebra $\text{gr} A$. This is closely related to deformation quantizations: By taking $\tilde{A}$ to be $(\hbar$-completion of) the Rees algebra of $A : R(A) = \bigoplus_n A_n \otimes \hbar^n$, then $\tilde{A}/\hbar\tilde{A} = \text{gr} A$.

We will need the following computation which relates the deformation Poisson bracket on $Z$ to the Poisson bracket on $\text{gr} A$. This computation is similar and motivated by a result of Kanel-Belov and Kontsevich \[KK\], where the the Poisson bracket on $Z$ was computed when $A$ is the Weyl algebra.

**Lemma 3.** Let $A$ be a filtered $W_2(\mathbb{k})$-algebra, such that $\text{gr} A = B$ is commutative and free over $W_2(\mathbb{k})$. Put $\overline{A} = A/pA, \overline{B} = B/pB$. Let $\overline{Z}$ denote the center of $\overline{A}$. Assume that $\text{gr}(\overline{Z}) = \overline{B}^p$. Then the top degree part of the deformation Poisson bracket on $\overline{Z}$ is equal to $-1$ times the Poisson bracket of $\overline{B}$.

**Proof.** We will verify that given central elements $\bar{x}, \bar{y} \in \overline{Z}$ such that $\text{gr}(\bar{x}) = \bar{a}^p, \text{gr}(\bar{y}) = \bar{b}^p, \bar{a}, \bar{b} \in \overline{B}$, then $\text{gr}([x, y]) = p\{\bar{a}, \bar{b}\}$. Here $x, y \in A$ are lifts of $\bar{x}, \bar{y}$ respectively.

It will be more convenient to work in the deformation quantization setting. Thus we will assume that $A = B[[h]]$ as a a free $W_2[k][[h]]$-module such that

$$A/hA = B, [a, b] = h\{a, b\} \mod h^2, a, b \in B.$$ 

Then by our assumption $\overline{Z} = \{a^p - h^{p-1}a_{[p]}, a \in \overline{B}\}$. Thus $\text{ad}_p(a)^p = \text{ad}_p(a_{[p]})$, here $\text{ad}_p(x)$ denotes the Poisson bracket $\{x, -\}, x \in B$. We will compute the Poisson bracket on $\overline{Z}$ mod $h^{p+1}$. Thus without loss of generality we will put $h^{p+1} = 0$. Let $x = a^p - h^{p-1}a_{[p]}, y = b^p - h^{p-1}b_{[p]}$. We want to compute $[x, y]$. We have

$$[a^p, y] = \text{ad}(a)^p(y) - \sum_{i=1}^{p-1} (-1)^i \binom{p}{i} a^i y a^{p-i}.$$ 

Since $\binom{p}{i} y$ is in the center of $A$, we have that

$$\sum_{i=1}^{p-1} (-1)^i \binom{p}{i} a^i y a^{p-i} = \left( \sum_{i=1}^{p-1} (-1)^i \binom{p}{i} \right) a^p y = 0.$$ 

So $[a^p, y] = \text{ad}(a)^p(y)$. We have

$$\text{ad}(a)^p(y) = \text{ad}(a)^p(b^p) - h^{p-1}\text{ad}(a)^p(b_{[p]}).$$ 

But $\text{ad}(a)^p(a) \subset h^p A$ thus $h^{p-1}\text{ad}(a)(A) = 0$. So $\text{ad}(a)^p(y) = \text{ad}(a)^p(b^p)$. On the other hand

$$[h^{p-1}a_{[p]}, y] = h^{p-1}[a_{[p]}, b^p].$$ 

Now since

$$\text{ad}(a)^p(b^p) = ph^p\text{ad}_p(a)(b^{p-1}\{a, b\}), \quad h^{p-1}[a_{[p]}, b^p] = ph^p\{a_{[p]}, b\}b^{p-1},$$ 

we obtain that

$$[x, y] = ph^p \left( \text{ad}_p(a)^{p-1}(b^{p-1}\{a, b\}) - \{a_{[p]}, b\}b^{p-1} \right) = (p - 1)!\{a, b\}^p.$$
Here we used the following identity. Let \( D : B \to B \) be a derivation, then
\[
D^{p-1}(b^{p-1}D(b)) = (p - 1)!D(b)^p + b^{p-1}D^p(b), \quad b \in B.
\]

Recall that a reduced commutative \( k \)-algebra \( B \) is said to be Frobenius split if the quotient map \( B \to B/B^p \) splits as a \( B^p \)-module homomorphism.

**Theorem 1.** Let \( \text{Spec} B \) be a normal Frobenius split Cohen-Macaulay Poisson variety over \( k \) such that the Poisson bracket on the smooth locus of \( \text{Spec} B \) is symplectic. Let \( A \) be a quantization of \( B \) such that \( \text{gr} Z = B^p \), where \( Z \) is the center of \( A \). Moreover, assume that \( A \) admits a lift to \( W_2(k) \). Let \( U \) denote the smooth locus of \( \text{Spec} Z \).

Then the restriction map \( H^1_{k}(A) \to \text{Der}_k(Z, Z) \) is injective and its cokernel is a quotient of \( \Omega^1(U)/\Omega^1_Z \) as a \( Z \)-module.

**Proof.** It was shown in [T] that the Azumaya locus of \( A \) in \( \text{Spec} Z \) has the compliment of codimension \( \geq 2 \). Normality of \( B \) implies that \( \text{Ann}_{B^p}(B/B^p) = 0 \). Since \( B/B^p \) is a direct summand of \( B \), and \( B \) is a Cohen-Macaulay \( B^p \)-module, it follows that \( B/B^p \) is a Cohen-Macaulay \( B^p \)-module of dimension \( \text{dim}B^p \). Since \( \text{gr}(A/Z) = B/B^p \) and \( \text{gr} Z = B^p \), it follows that \( A/Z \) is a Cohen-Macaulay \( Z \)-module and \( \text{Ann}_Z(A/Z) = 0 \). Now Lemma 3 implies that the Poisson bracket on \( Z \) coming from a lift of \( A \) over \( W_2(k) \) is a deformation of the Poisson bracket on \( B \), hence it is symplectic on an open subset of \( \text{Spec} Z \) whose compliment has codimension \( \geq 2 \). Thus all assumptions of Lemma 2 are satisfied.

Denote by \( P \) the \( Z \)-span of derivations of the form \( a\{b, -\}, a, b \in Z \). Clearly \( P \) is in the image of the restriction \( H^1_{k}(A) \to \text{Der}_k(Z, Z) \). Then we have a \( Z \)-module map \( \Omega^1_Z \to P \subset \text{Der}_k(Z, Z) \) corresponding to the Poisson bracket, and \( \text{Der}_k(Z, Z) \) can be identified with \( \Gamma(U, \Omega) \) via the symplectic pairing. Hence \( \Omega^1(U)/\Omega^1_Z \) maps onto the cokernel of the restriction map \( H^1_{k}(A) \to \text{Der}_k(Z, Z) \).

This result applies to a large class of algebras including symplectic reflection algebra. Our next result shows that the restriction map from Theorem 1 is an isomorphism for the case of central quotients of enveloping algebras of semi-simple Lie algebras. Let us recall their definition and fix the appropriate notations first.

Let \( g \) be a Lie algebra of a connected semi-simple simply connected algebraic group \( G \) over \( k \), assume that \( p \) is large enough relative to \( g \) (for example \( p \) is very good for \( G \).) Let \( Z_0 \subset Z(\mathfrak{u}g) \) denote \( G \)-invariants of the enveloping algebra \( \mathfrak{u}g \) under the adjoint action of \( G \). Let \( \chi : Z_0 \to k \) be a character. Put \( \mathfrak{u}g = \mathfrak{u}g/\text{Ker}(\chi)\mathfrak{u}g \).
Corollary 1. Let $A$ be a quotient enveloping algebra $\mathfrak{U}_k \mathfrak{g}$. Let $Z$ be the center of $A$. Then we have an isomorphism of $Z$-modules $\text{Der}_k(A, A) \cong A/Z \oplus \text{Der}_k(Z, Z)$.

Proof. Let $\tilde{\mathfrak{g}}$ be a Lie algebra lift of $\mathfrak{g}$ over $W_2(\mathbb{k})$. Let $t_1, \ldots, t_n$ be generators of $\ker(\chi)$ and $\tilde{t}_1, \ldots, \tilde{t}_n$ be their lift in $Z(\mathfrak{U}\tilde{\mathfrak{g}})$. Thus $Z(\mathfrak{U}\tilde{\mathfrak{g}})$ is generated by $t_1, \ldots, t_n$ over $Z_p = \{g^p - g[p], g \in \mathfrak{g}\}$. Also, $Z$ is the quotient of $Z_p$. Let $A_2 = \mathfrak{U}\tilde{\mathfrak{g}}/(\tilde{t}_1, \ldots, \tilde{t}_n)$. So $A_2$ is a lift of $A$ over $W_2(\mathbb{k})$. We will show that $Z$ admits an algebra lift in $A_2$. Let $[p] : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ be a lift of the restricted structure map $[p] : \mathfrak{g} \to \mathfrak{g}$. Then if follows from computation in Lemma 3 that

$$[x^p - x[p], y^p - y[p]] = -p([x, y]^p - [x, y][p]) \quad x, y \in \tilde{\mathfrak{g}}.$$

Let $\tilde{\mathfrak{g}}_1$ be a $W_2(\mathbb{k})$-Lie algebra such that $\tilde{\mathfrak{g}}_1 = \mathfrak{g}$ as $W_2(\mathbb{k})$-module and the Lie bracket $[x, y]_{\tilde{\mathfrak{g}}_1}$ is defined as $-p[x, y]_{\mathfrak{g}}$. Thus we have an algebra map $i : \mathfrak{U}\tilde{\mathfrak{g}}_1 \to \mathfrak{U}\tilde{\mathfrak{g}}$ where $i(x) = x^p - x[p], x \in \tilde{\mathfrak{g}}$. Denote the image of $i$ by $S$. Thus $S$ is an algebra lift of $Z_p$ in $\mathfrak{U}\tilde{\mathfrak{g}}$. Let $S_1$ denote the image of $S$ under the quotient map $\mathfrak{U}\tilde{\mathfrak{g}} \to \mathfrak{U}\tilde{\mathfrak{g}}/(\tilde{t}_1, \ldots, \tilde{t}_n) = A_2$. Therefore $S'$ is an algebra lift of $Z$ in $A_2$.

Using the usual PBW filtration of $A$ we have $\text{gr}(A) = k[N]$, where $N$ is the nilpotent cone of $\mathfrak{g}$. Now since $N$ is a Frobenius split normal Cohen-Macaulay variety [BK], and the Poisson bracket on the regular locus of $N$ is symplectic, Theorem 1 and Lemma 1 imply the desired result.

Remark 1. It is known that in characteristic 0 Hochschild cohomology of symplectic reflection algebras $H$ is concentrated in even dimensions [GK], so it has no outer derivations. The same is true for the enveloping algebras $\mathfrak{U}\mathfrak{g}$ and its quotients.

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