Budget-Smoothed Analysis for Submodular Maximization∗

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Abstract

The greedy algorithm for monotone submodular function maximization subject to cardinality constraint is guaranteed to approximate the optimal solution to within a $1 - 1/e$ factor. Although it is well known that this guarantee is essentially tight in the worst case — for greedy and in fact any efficient algorithm, experiments show that greedy performs better in practice. We observe that for many applications in practice, the empirical distribution of the budgets (i.e., cardinality constraints) is supported on a wide range, and moreover, all the existing hardness results in theory break under a large perturbation of the budget.

To understand the effect of the budget from both algorithmic and hardness perspectives, we introduce a new notion of budget-smoothed analysis. We prove that greedy is optimal for every budget distribution, and we give a characterization for the worst-case submodular functions. Based on these results, we show that on the algorithmic side, under realistic budget distributions, greedy and related algorithms enjoy provably better approximation guarantees, that hold even for worst-case functions, and on the hardness side, there exist hard functions that are fairly robust to all the budget distributions.

∗We thank Eric Balkanski and Matt Weinberg for interesting discussions and comments on earlier drafts.
†Supported by NSF CCF-1954927, and a David and Lucile Packard Fellowship.
‡Supported by NSF CCF-1954927.
1 Introduction

Monotone submodular function maximization subject to a cardinality constraint is a fundamental problem in combinatorial optimization with a wide variety of applications including feature selection, sensor placement, influence maximization in social networks, document summarization, etc. (see e.g. [KG14] and references therein). We will use influence maximization in social networks as a running example: an advertiser has a limited budget of \( k \) free product samples that she wishes to distribute to seed consumers, who will then propagate the news about the product to their friends, then their friends’ friends, etc. The standard approach to this problem [KKT15] models the expected final reach of the campaign as a monotone submodular function \( f : \mathcal{P}([n]) \rightarrow \mathbb{R} \) of the set of seed consumers (where \([n]\) is the set of all users in the network). The goal of the optimization problem is to find a set of \( k \) seed consumers that (approximately) maximizes \( f \).

Classic work shows that the simple greedy algorithm achieves a \( 1 - 1/e \)-approximation to the optimal solution in the worst case [NWF78]. Furthermore, this bound is tight for algorithms that make sub-exponential queries to the function [NW78, Von13]; and even succinctly representable functions (e.g. simple models of influence propagation on a social network graph) do not allow better approximation algorithms unless \( P = NP \) [Fei98]. In theory, this tight characterization of the optimal approximation factor is very satisfying.

Given the importance of this problem in practical applications, it is also interesting to ask what is the optimal approximation factor that can be obtained on realistic instances. As one can expect, the performance of the greedy algorithm tends to be significantly better in practice (e.g. [TSP20, BQS21]). When reasoning about real-world instances, there is a natural tradeoff between quality and generality of the guarantees: at one extreme, worst-case analysis only gives a \( (1 - 1/e) \)-approximation but applies to every instance; at the other extreme we could, in principle, empirically evaluate the performance of the greedy algorithm on each instance of interest, but we would have to redo this for every new instance. Ideally, we want to extend the classic worst-case model –while making minimal assumptions– to explain why efficient algorithms like greedy should obtain better-than-\( (1 - 1/e) \)-approximation in practice.

Coming up with useful and realistic assumptions about submodular functions continues to be an interesting and active topic of research. In Section 1.3 we survey several natural restrictions, including recent success stories that allow for improved approximation algorithms [KL14, SVW17, BRS16, IIS16, Yos16, CRV17, TSP20, STY20]. Deferring details for later, we argue that the bottom line of this discussion is that submodular functions are complex objects, and as such modeling their beyond-worst-case behavior is tricky and application-dependent, and moreover, it is often intractable to verify the model assumptions in practice.

In this work, we consider beyond-worst-case analysis (and hardness) of submodular maximization from a novel perspective by focusing on modeling the average-case behavior of a much simpler object: the cardinality constraint. As we now explain, our approach of perturbing the cardinality constraint is motivated by both theory and practice.

In practice, many applications of submodular maximization have multiple users with the same or similar objective but various budgets (i.e., cardinality constraints). In the example about influence maximization, multiple advertisers could advertise and propagate on the same social network and hence maximize essentially the same, possibly worst-case, submodular function. However, their budgets can easily vary by an order of magnitude or more because of different sizes of business or different amounts of funds. A concrete example is the distribution of the campaign budgets of the candidates in the 2020 Democratic Party primary elections [AKS20]. Thus even if the social

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1 In general calculating the approximation factor on a real-world instance requires computing the value of the optimal solution. The recent work [BQS21] provides an instance-specific method to estimate the optimal value.
network/submodular function is worst-case, the “average” advertiser uses an “average” budget which is independent of the social network/submodular function. In a different example about feature selection, the engineers wish to make predictions in the testing phase using a small subset from the high-dimensional feature space that is selected during the training phase, and in the training phase, they apply the same standard machine learning model (e.g., linear regression, logistic regression) to the same standard datasets (e.g., ImageNet [RDS+15]) and hence optimize the same monotone (approximately) submodular objective [DK11, EKDN18]. However, the number of features they want to choose can easily range from one hundred to one million depending on the computational power they have or the model complexity they prefer. Therefore, from a practical point of view, it is interesting to understand whether the average-case behavior of the budget makes the problem of submodular maximization easier to some extent.

In theory, all the known worst-case instances for cardinality-constrained monotone submodular maximization [NWF78, Von13, Fei98] are sensitive to large budget perturbations: even outputting a random solution achieves $\gtrsim 0.95$ approximation, when we perturb (i.e., multiply) the cardinality constraint by a significant multiplicative factor like 0.1 or 10. Hence, from a theoretical point of view, it is interesting to investigate the effect perturbing the cardinality constraint has on the hardness of approximation results.

With the above motivations from theory and practice, we initiate the study of submodular function maximization in a semi-adversarial setting, where the (empirical) distribution of the cardinality constraints is supported on a wide range (e.g. $[x, 10x]$), from both the algorithmic and the hardness perspectives. Namely, we hope to answer the question of whether a large random perturbation of the cardinality constraint allows efficient algorithms to achieve higher optimal approximation ratio or there is a stronger hardness result that is robust to any such perturbations.

To formalize this question, we propose a simple and elegant framework called budget-smoothed analysis. The name is inspired by the celebrated smoothed analysis for linear programming (LP) [ST04]. Admittedly, the analogy is not perfect: we consider much larger perturbations than smoothed analysis for LP. However, much like smoothed analysis for LP, the generative process of random perturbations for budget-smoothed analysis for submodular maximization is grounded in concrete applications (such as viral marketing and feature selection – see discussion above).

**The budget-smoothed analysis model**

We study monotone submodular function maximization subject to a cardinality constraint in the following semi-adversarial setting:

**Definition 1.1 (Budget-smoothed analysis).**

1. The distribution $\tilde{\mathcal{D}}$ of budgets (e.g. uniform over $[x, 10x]$) is given as input to the adversary.

2. The adversary chooses a (monotone submodular) function $f$.

3. The cardinality constraint $k \sim \tilde{\mathcal{D}}$ is drawn at random and given as input to the algorithm.

4. The algorithm (approximately) maximizes $f$ over all sets of size at most $k$.

Clearly, tiny perturbations of the constraint cannot escape the hardness of approximation results, because by submodularity, a $(1 + \varepsilon)$-multiplicative perturbation in budget cannot affect the value of the solution by more than a $(1 + \varepsilon)$-factor.
For any distribution $\tilde{D}$, we’re interested in the expected ratio $R_{ALG}(f, \tilde{D})$ between the value obtained by the algorithm and the optimal solution,

$$R_{ALG}(f, \tilde{D}) := \mathbb{E}_{k \sim \tilde{D}} \left[ \frac{f(ALG)}{f(OPT)} \right].$$

For notational convenience, we make the following change to the above model: Rather than sampling the budget from a distribution, each instance will be characterized by a base budget $k_0$ and a budget perturbation distribution $D$, with the final cardinality constraint being $k := \rho \cdot k_0$ for $\rho \sim D$. This will allow us to talk about a distribution $D$ like “uniform over $[x, 10x]$” while studying the asymptotic complexity as the instance size and cardinality constraint go to infinity.

The budget-smoothed analysis model has the following advantages — First, it is simple and clean in theory, which not only provides a formal setup for studying our aforementioned question but also has the flexibility to be integrated with other models that make beyond-worst-case assumptions about the submodular functions. Second, it is easy-to-apply in practice, since calculating the (empirical) distribution of the budgets is much more tractable than verifying the complex assumptions about the submodular functions.

Our results

We revisit the classic problem of monotone submodular maximization in our model of budget-smoothed analysis, and in particular, we investigate the following fundamental questions:

**Question 1:** What is the optimal efficient algorithm?

**Question 2:** What are the worst-case instances for an arbitrary budget distribution?

**Question 3:** What are the optimal approximation factors for the budget distributions that are supported on a wide range (e.g. $[x, 10x]$) and what is the best budget distribution?

**Remark.** All the hardness results below hold both in the black-box oracle model (for any algorithm that makes a subexponential number of queries), or assuming $P \neq NP$ in the computational model for coverage functions (on a polynomial-size graph).

**Result 1 (main theorem): Optimal approximation algorithms** Our main theorem shows that a large class of algorithms that are (near-)optimal in the classic setting continue to obtain (near-)optimal approximation factors under budget-smoothed analysis for any distribution (Theorem 3.1 and Observation 3.5). This class includes the classic greedy algorithms, as well as (variants of) recent efficient parallel algorithms, and Map-Reduce algorithms (Appendix B). In particular, these algorithms are optimal even in comparison to algorithms that know the budget perturbation distribution $D$. In other words, they intrinsically adjust themselves to the budget distribution optimally. The proof of the main theorem relies on a characterization of the worst-case instances in the model of budget-smoothed analysis, and therefore, we completely answer Questions 1 and 2.

The main theorem and Question 3 set up a win-win scenario for us: either we explain (a fraction of) the success of greedy for some interesting distributions, or we get a stronger hardness result that hold against all the budget distributions. Either way, we would bring new insights to the classic problem of submodular maximization.

Applying the main theorem, we manage to give partial answers (Result 2 and 3) to Question 3 on both positive and negative sides.
Table 1: Empirical Results

| Budget perturbation distribution                          | Worst-case approximation ratio |
|-----------------------------------------------------------|-------------------------------|
| Baseline (no perturbation)                                | 0.6321                        |
| Uniform over $[1, 10]$                                    | 0.6675                        |
| Log-scale-uniform over $[1, 10]$                          | 0.6674                        |
| Log-scale-uniform over $[1, 600]$                         | 0.6808                        |
| Top 10 social/political campaigns on Facebook             | 0.6625                        |
| 2020 Democratic presidential candidates                   | 0.6727                        |

We calculated tight approximation factors for worst-case monotone submodular functions for several exemplary budget distributions. See Section 5.1 for details.

Result 2: Optimal approximation factors  For any budget perturbation distribution $D$, we formulate a simple (but non-convex) mathematical program (Section 5) that computes the optimal possible approximation factor for a given budget distribution. We also include some numerical estimates for natural distributions (Table 1). For the special case of $D$ supported on two budgets, we also give a closed-form solution (Proposition A.4).

These results are interesting on both positive and negative sides — On the positive side, the optimal approximation ratios have modest but non-negligible improvements for many interesting distributions even for worst-case submodular functions, which explain a fraction of the success of greedy algorithms. On the negative side, we pin down the worst-case instances for these distributions which remain significantly hard to approximate (and studying these instances might provide new insights about the structure of beyond-worst-case submodular functions).

Result 3: Bounding the best-case budget distribution  We also prove that for every budget distribution and any efficient algorithm, the optimal budget-smoothed analysis approximation factor is bounded away from 1, and in particular, it is at most 0.9087 (Theorem 4.1).

It is worth mentioning that because the program in Result 2 is non-convex, we are only able to compute the optimal approximation factors after discretizing the budget distributions with a limited number of budgets, and thus, the positive results may still have a lot of room to improve.4 This leaves an interesting open problem: close the gap and give a complete answer to Question 3.

1.1 Broader discussion

Our model of budget-smoothed analysis introduces a new (and more tractable) angle for studying beyond-worst-case analysis and average-case hardness. We believe that there are countless future directions and applications to explore in the broader field of TCS. To exhibit this breadth of possibilities, we mention a couple of preliminary results that we have for other problems that fit into our new model:

- Submodular maximization subject to knapsack constraint. While the optimal $1 - 1/e$ factor can again be recovered in polynomial time, the state-of-the-art algorithms for this problem

  4For comparison, [ST04]’s original polynomial upper bound for smoothed analysis (of a non-trivial variant) of the Simplex algorithm was $O(d^{2.5} n^{8.5} \sigma^{-3} + d^{2} n^{10} \sigma)$ iterations ($d$ is the number of variables, $n$ is the number of constraints, and $\sigma$ is the variance of the perturbation), which is significantly improved now [DHS]. That said, in light of Result 3, we do not expect the positive side of budget-smoothed analysis to fully explain the success of greedy in practice, but explaining a greater fraction of success or showing a more robust hardness result would still be interesting.
are still not completely satisfying [Svi04, EN19, NS20], and the greedy algorithm does not provide any non-trivial approximation guarantee. Our preliminary results show that with budget-smoothed analysis, greedy guarantees a constant factor approximation with knapsack constraint, and in fact to date we haven’t been able to rule out $1 - \frac{1}{e}$ approximation (or better).

- Budget-feasible mechanism design: this is a well-studied problem in algorithmic game theory [Sm10, CGL11, DPS11, BKS12, SM13, CCT14, EG14, GNS14, HIM14, BH16, CC16, NSKK16, ZLM16, ZWG+17, LMSZ17, AGN18, KT18, AKS19, GJLZ19, LZY20]. Under a large market assumption [AGN18] obtain a mechanism with optimal approximation guarantee, incidentally also $1 - \frac{1}{e}$. Our preliminary results show that this mechanism does not improve at all under budget-smoothed analysis. However, the budget-smoothed analysis inspires a new mechanism that is not only optimal for every budget distribution but also instance-optimal among a canonical class of mechanisms, and it also significantly outperforms [AGN18]’s mechanism on realistic distributions empirically.

In hindsight, although the performance improvement guaranteed by budget-smoothed analysis is relatively moderate, we believe that the optimality of an algorithm in the model of budget-smoothed analysis is a theoretical evidence that the algorithm is not just worst-case optimal but also likely to perform favorably on realistic instances. In other words, budget-smoothed analysis offers an analytically approachable beyond-worst-case performance test for the worst-case optimal algorithms of budget-constrained problems, which helps us identify (or design) the “right” algorithm among various worst-case optimal algorithms.

1.2 Roadmap

In Section 1.3 we survey several other approaches to beyond-worst-case submodular maximization. In Section 3 we prove our core technical result, namely that greedy obtains optimal approximation factors for any distribution; in Appendix B we extend this result to other related algorithms. Henceforth, we build on these techniques; in particular we simply analyze the approximation factors of the greedy algorithm. In Section 4 we prove that the optimal approximation factor is bounded away from 1 for any distribution. In Section 5 we characterize the optimal approximation factor by a program, which we then use to simulate several exemplary distributions.

1.3 Beyond-worst-case submodular functions

Due to the popularity of submodular maximization in practice, there is a lot of interest in understanding and designing algorithms for “typical” cases. We discuss a few approaches below. We note that our model of beyond-worst-case cardinality constraint is orthogonal to any assumptions about the submodular function, and in principle could be combined with any of them to obtain even stronger results.

The model most closely in spirit to our smoothed-analysis-like approach is to take a worst-case submodular function and perturb it with random noise. The most straightforward way of doing this is independently perturbing the value of the function for each set. Unfortunately, this breaks the submodularity, which makes the problem significantly harder, even for small perturbations: [HS17] barely recovers the $1 - \frac{1}{e}$ approximation factor in this setting (under further restrictions and with a technically involved algorithm).

Another approach is to consider coverage functions, an important special class of monotone submodular functions. This restriction has been successful for learning submodular functions [BCIWT12]...
but Feige’s NP-hard instance already rules out efficient algorithms with improved approximation ratios for this case. One may combine this restriction with perturbations of the weights of the elements of the ground set; but it is not hard to show that Feige’s instance can be made robust even to very large amounts of noise. Another alternative is to consider special classes of graphs, e.g., power law, small-world, or triangle-dense that are common for social networks [WS98, GRS16]. But again Feige’s instance either already satisfies all of those, or can be adapted to do so.

Another popular restriction of monotone submodular functions is bounded curvature [CC84], which restricts the extent to which ground elements interact; this indeed allows for better algorithms with applications to e.g. maximum entropy sampling [CC84, SVW17, BRS16, HSI6, Yos16]. But bounded curvature seems too restrictive for applications like influence in social networks and consumers’ valuations with diminishing returns.

To cope with the limited applicability of curvature, the original paper of [CC84] also defined a relaxed notion of greedy curvature, which only restricts the interaction between elements selected by the greedy algorithm and elements in the optimal solution. In exciting recent work, [TSP20] define various notions of sharpness which only restricts the interactions of the average element of the optimal solution. Both greedy curvature and sharpness parameters suffer from the disadvantage that they may be intractable to compute (both are assumptions about interaction of elements with the optimal solutions, and if we knew the optimal solution...). Moreover, due to their complicated form it’s hard to heuristically reason about their fit for any particular application. Nevertheless, on the positive side both are more realistic than vanilla curvature assumption, and combining them with our budget-smoothed analysis model is an interesting direction for future research.

[CRV17] study submodular maximization under a stability assumption, i.e. they assume that the optimal subset does not change when the function is perturbed. [TSP20] argue that in the context of submodular maximization, stable instances may fail to capture significant interaction between elements. As in the case of greedy curvature and sharpness, it is also not clear how to compute the stability of a function, or reason about instances that we expect to be stable.

Finally, one setting that is both natural and allows for improved approximation factors is influence maximization in undirected graphs [KL14, STI9, STY20]. Specifically, [KL14] prove that the greedy algorithm obtains a \((1 - 1/e + \varepsilon)\)-approximation (for some small unspecified constant \(\varepsilon > 0\)) for the independent cascade model on undirected graph. [STY20] show that in the linear threshold model the greedy algorithm does not beat the \((1 - 1/e)\)-approximation factor (by any constant, in the worst case).

## 2 Preliminaries

**Definition 2.1.** A function \(f : 2^V \to \mathbb{R}_{\geq 0}\) is submodular if for all \(S \subseteq T \subseteq V\) and \(i \in V \setminus T\), \(f(S \cup \{i\}) - f(S) \geq f(T \cup \{i\}) - f(T)\), where \(V\) is called ground set. Moreover, we denote the marginal gain by \(f(X | S) := f(X \cup S) - f(S)\).

We make the following conventions in this paper—When we say “efficient algorithm”, we mean polynomial time algorithms in the general computation model assuming \(P \neq \text{NP}\), or algorithms using sub-exponential number of function queries in the oracle query model. Moreover, we consider continuous distribution of budget perturbations \(\mathcal{D}\), and we let \(\mathcal{D}(k)\) denote

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4If, for example, we already selected all of a node’s neighbors, the marginal contribution of adding this node is diminished to zero. For consumers’ valuations, the marginal contribution of, e.g. the one-thousandth apple, is again diminished to essentially zero. This means that curvature is unbounded in both settings (see [CC84, SVW17] for formal definitions).
the distribution of budgets in which a budget is sampled by multiplying a random perturbation factor \( \rho \sim D \) with \( k \) (if \( \rho \cdot k \) is fractional, we can round it to an integer). Furthermore, following Definition 1.1, we denote \( \mathcal{R}_{\text{ALG}}(D(k)) := \min_f \mathcal{R}_{\text{ALG}}(f, D(k)) \) s.t. \( f \) is monotone and submodular.

The hard instances in our analysis can be built on top of either Feige’s max-\( k \)-cover instances \cite{Fei98} or Vondrák’s hard instances \cite{Von13}. In the following theorem, we summarize useful properties of these two hardness results.

**Theorem 2.2.** There exists a class of monotone submodular functions \( \mathcal{C} \) such that for every \( \epsilon > 0 \), for any efficient algorithm \( A \) that given a submodular function \( f \) and an integer \( l \) outputs a set \( X_l \) of cardinality \( l \), for every sufficiently large \( k \) that grows with size of instance, there is a submodular function \( f_k \in \mathcal{C} \) such that

(i) for all \( l \leq k \), \( f_k(O_l) = (l/k)f_k(O_k) \), where \( O_l \) is the optimal set that maximizes \( f_k \) among all cardinality-\( l \) sets, and

(ii) for all \( l \), \( f_k(X_l) \leq (1 - e^{-l/k} + \epsilon)f_k(O_k) \).

Next, we state a standard lemma for greedy analysis.

**Lemma 2.3.** Given a monotone submodular \( f \), we let \( X_k \) and \( O_k \) denote the greedy solution and the optimal solution of cardinality \( k \), respectively. Then, for all \( i, k > 0 \), \( f(X_i) - f(X_{i-1}) \geq \frac{1}{k}(f(O_k) - f(X_{i-1})) \).

**Proof.** Let \( x_i \) denote the \( i \)-th element selected by greedy. It holds that

\[
f(X_i) - f(X_{i-1}) = f(x_i \mid X_{i-1}) \geq \frac{1}{k} \cdot \sum_{o \in O_k} f(o \mid X_{i-1}) \geq \frac{1}{k} \cdot f(O_k \mid X_{i-1}) \geq \frac{1}{k} (f(O_k) - f(X_{i-1})),
\]

where the first inequality is by greedy selection, the second is by submodularity, and the third is by monotonicity. \( \square \)

## 3 Greedy is Optimal

In this section, we prove our core technical result: greedy is optimal for submodular maximization with respect to arbitrary distribution of budget perturbations.

**Theorem 3.1.** For any distribution of budget perturbations \( D \), for every \( \epsilon' > 0 \), for any efficient algorithm \( A \), for every sufficiently large \( k \) that grows with the size of instance, it holds that \( \mathcal{R}_A(D(k)) \leq (1 + \epsilon')\mathcal{R}_{\text{greedy}}(D(k)) \).

**Theorem 3.1** follows directly from Theorem 3.2 using a discretization argument.

**Theorem 3.2.** For any perturbation factors \( 0 < \rho_1 < \rho_2 < \cdots < \rho_m \), for every \( \epsilon > 0 \), there exists a sufficiently large \( k \) that grows with the size of instance such that given \( m \) budgets \( k_1 = \rho_1 \cdot k, \ldots, k_m = \rho_m \cdot k \), for any monotone submodular function \( f^{(\text{bad-for-greedy})} \), for any efficient algorithm \( A \), there exists a monotone submodular function \( f \) such that

\(^5\)The implicit dependence of \( k \) on \( \epsilon \) in Theorem 2.2 carries over to the dependence of \( k \) on \( \epsilon' \) in this statement, and therefore, we keep such dependence implicit, and we are mostly interested in the asymptotic result.
(i) **Greedy is (almost) no worse than \( \mathcal{A} \) on \( f \):** for all \( i \in [m] \), the solution \( Y_{k_i} \) computed by \( \mathcal{A} \) for budget \( k_i \) has value \( f(Y_{k_i}) \leq (1 + \epsilon) f(X_{k_i}) \), where \( X_{k_i} \) is the greedy solution for budget \( k_i \).

(ii) **\( f \) is as hard as \( f^{(\text{bad-for-greedy})} \) for greedy:** for all \( i \in [m] \), given budget \( k_i \), the approximation ratio of greedy on \( f \) is at most \( 1 + \epsilon \) times the approximation ratio of greedy on \( f^{(\text{bad-for-greedy})} \).

**Proof of Theorem 3.1** For arbitrarily small \( \tau > 0 \), let \( \rho_{\min} \) and \( \rho_{\max} \) be such that the mass of \( \mathcal{D} \) on \( [\rho_{\min}, \rho_{\max}] \) is at least \( 1 - \tau \). We discretize \( \{\rho_{\min} \cdot k, \rho_{\min} \cdot k + 1, \ldots, \rho_{\max} \cdot k\} \) into \( \{\rho_{\min} \cdot k, (1 + \delta) \rho_{\min} \cdot k, (1 + \delta)^2 \rho_{\min} \cdot k, \ldots, \rho_{\max} \cdot k\} \). Without loss of generality, we assume that there exists \( m \) such that \( (1 + \delta)^{m-1} \rho_{\min} = \rho_{\max} \) and every \( k_i := (1 + \delta)^{i-1} \rho_{\min} \cdot k \) is integral. Let \( f^* \) be the worst-case monotone submodular function for which greedy achieves only \( R_{\text{greedy}}(\mathcal{D}(k)) \) approximation in expectation. By Theorem 3.2, for any efficient algorithm \( \mathcal{A} \), there is a monotone submodular function \( f \) such that for all \( i \in [m] \), the solution \( Y_{k_i} \) outputted by \( \mathcal{A} \) for budget \( k_i \) only achieves \( f(Y_{k_i}) \leq (1 + \epsilon) f(X_{k_i}) \), where \( X_{k_i} \) is the greedy solution for budget \( k_i \), and moreover, for every budget \( k_i \),

\[
\frac{f(X_{k_i})}{f(O_{k_i})} \leq (1 + \epsilon) \frac{f^*(X_{k_i}^*)}{f^*(O_{k_i}^*)}
\]  

(1)

where \( O_{k_i} \) and \( O_{k_i}^* \) denote optimal size-\( k_i \) sets of \( f \) and \( f^* \) respectively, and \( X_{k_i}^* \) denotes the size-\( k_i \) greedy solution for \( f^* \).

Besides, because marginal gain in each iteration of greedy is non-increasing, we have that \( (1 + \delta)f(X_{k_{i-1}}) \geq f(X_{k_i}) \). Furthermore, without loss of generality, we assume that \( f(Y_b) \) is non-decreasing in \( b \), since otherwise, for budget \( b \), we can let the algorithm choose the best solution among \( Y_l \) for all \( l \leq b \) instead. For any \( 2 \leq i \leq m \) and any budget \( b \) such that \( k_{i-1} \leq b \leq k_i \), it follows that

\[
\frac{f(Y_b)}{f(O_b)} \leq \frac{f(Y_{k_i})}{f(O_{k_i})} \leq (1 + \epsilon) \frac{f(X_{k_i})}{f(O_{k_i})} \leq (1 + \epsilon)(1 + \delta) \frac{f(X_{k_{i-1}})}{f(O_{k_{i-1}})} \leq (1 + \epsilon)^2 (1 + \delta) \frac{f^*(X_{k_{i-1}}^*)}{f^*(O_{k_{i-1}}^*)} \leq (1 + \epsilon)^2 (1 + \delta) \frac{f^*(X_b^*)}{f^*(O_b^*)} \leq (1 + \epsilon)^2 (1 + \delta)^2 \frac{f^*(X_b^*)}{f^*(O_b^*)} \leq (1 + \epsilon)^2 (1 + \delta)^2 \frac{f^*(X_b^*)}{f^*(O_b^*)} \leq \frac{b}{k_{i-1}} f^*(O_{k_{i-1}}^*) \text{ by submodularity}
\]

Therefore, for every budget \( b \) in \( \{\rho_{\min} \cdot k, \rho_{\min} \cdot k + 1, \ldots, \rho_{\max} \cdot k\} \), \( \mathcal{A} \) can achieve on \( f \) in expectation at most a factor of \( (1 + \epsilon)^2 (1 + \delta)^2 \) times what greedy achieves on \( f^* \). The proof finishes since \( \delta, \epsilon, \tau \) can be arbitrarily small. \( \square \)
3.1 Proof of Theorem 3.2

In our proof we will not derive the analytic formula of the approximation ratio, but instead, the proof works in a black-box way—First, we introduce an array of parameters such that every instance can be characterized by these parameters, and we can show a parameterized guarantee of the marginal gain for each iteration of greedy. Then, we construct a hard instance characterized by the same parameters such that the best possible marginal gains for this instance always match the parametrized guarantees from greedy. It follows that the performance of greedy is optimal for every budget. Our hard instance has the following nice structure: it is a convex combination of disjoint-support copies of the classic hard instances guaranteed by Theorem 2.2.

Proof of Theorem 3.2 Proof setup: bounding a single step of greedy performance

We first lower bound the single-step performance of greedy solutions. By Lemma 2.3, we have the following performance guarantees for each iteration of greedy,

\[ \forall l \in [m], \quad f(X_i) - f(X_{i-1}) \geq \frac{1}{k_l} \cdot (f(O_{k_l}) - f(X_{i-1})) \quad \text{(l-th guarantee)}, \]

where \( O_{k_l} \) denotes the optimal solution of cardinality \( k_l \), and we call the inequality associated with \( O_{k_l} \) the l-th guarantee. Given any \( 1 \leq l_1 < l_2 \leq m \), if the \( l_2 \)-th guarantee dominates (i.e., is at least as large as the \( l_1 \)-th guarantee) at some iteration \( i \), then the \( l_2 \)-th guarantee will keep dominating the \( l_1 \)-th guarantee for all the iterations \( i' \geq i \), because the two guarantees are linear functions with variable \( f(X_{i-1}) \), and the \( l_2 \)-th guarantee decreases slower than the \( l_1 \)-th guarantee. Therefore, as \( f(X_i) \) increases, the best guarantee can only transit from some \( l \) to some \( l' > l \). Given an instance, we let \( t \leq m - 1 \) be the number of times such transition occurs until \( k_m \)-th iteration and let \( l_1 < l_2 < \cdots < l_t \) be the indices of the corresponding best guarantees.

For \( j \leq t-1 \), let \( F_j \) be the lowest possible value of \( f(X_{i-1}) \), for which the \( j \)-th transition occurs,

\[ \frac{1}{k_{l_j}} \cdot (f(O_{k_{l_j}}) - F_j) = \frac{1}{k_{l_{j+1}}} \cdot (f(O_{k_{l_{j+1}}}) - F_j). \]  

(2)

We will be particularly interested in the quantity \( r_j := F_j/f(O_{k_{l_j}}) \). Plugging into Eq. (2), we have that

\[ r_j = \frac{1 - (k_{l_j}/k_{l_{j+1}}) \cdot (f(O_{k_{l_{j+1}}})/f(O_{k_{l_j}}))}{1 - (k_{l_j}/k_{l_{j+1}})}. \]  

(3)

Lower bounding the total value of the greedy solution recursively

For \( q \geq 0 \), we denote by \( f^{(\text{greedy-lb})}(q) \) the best lower bound induced by the union of “l-th guarantees” on the value of the \( q \)-th iterate of the greedy algorithm, namely

\[ f^{(\text{greedy-lb})}(q) := f^{(\text{greedy-lb})}(q-1) + \max_l \left\{ \frac{1}{k_l} \cdot (f(O_{k_l}) - f^{(\text{greedy-lb})}(q-1)) \right\}. \]

Now we analyze \( f^{(\text{greedy-lb})}(q) \) specifically for the instance with before-mentioned guarantee transitions. We start from the \( l_1 \)-th guarantee and let \( f^{(\text{greedy-lb})}(0) = 0 \). Inductively, suppose that in the current iteration \( q \), the \( l_j \)-th guarantee dominates the others, we apply the \( l_j \)-th guarantee \( f^{(\text{greedy-lb})}(q) - f^{(\text{greedy-lb})}(q-1) = (f(O_{k_{l_j}}) - f^{(\text{greedy-lb})}(q-1))/k_{l_j} \) and continue iteratively until
we reach some $i_j$-th iteration such that $f^{(\text{greedy-lb})}(i_j - 1) \leq r_j \cdot f(O_{k_l}) < f^{(\text{greedy-lb})}(i_j)$. At the $i_j$-th iteration, $l_{j+1}$-th guarantee starts dominating, and thus, we switch to the $l_{j+1}$-th guarantee and continue like above.

**Approximation ratio based on $f^{(\text{greedy-lb})}(q)$'s is determined by $r_j$'s**

We claim that the parameters $r_j$ fully determine the ratio between the greedy lower bound $f^{(\text{greedy-lb})}(k_l)$ and $f(O_{k_l})$ for all $i \in [t]$. To see this, first observe that by [3] we can infer $f(O_{k_{i+1}})/f(O_{k_i})$ from $r_j$. We can assume that $f(O_{k_i})$ is fixed without loss of generality, and then the parameters $r_j$ determine all the remaining $f(O_{k_i})$, i.e., for all $1 < j \leq t$,

$$f(O_{k_i}) = f(O_{k_1}) \cdot \prod_{j' \leq j-1} \left( \frac{k_{l_{j'}+1}}{k_{l_{j'}}} - \left( \frac{k_{l_{j'}+1}}{k_{l_{j'}}} - 1 \right) r_{j'} \right). \quad (4)$$

Moreover, the greedy lower bound $f^{(\text{greedy-lb})}(k_l)$ by definition is a linear combination of the $f(O_{k_i})$'s. Therefore, the ratio between any $f^{(\text{greedy-lb})}(k_l)$ and $f(O_{k_l})$ is fully characterized by $r_j$. In the other words, given an instance, we can get the approximation ratios of the greedy algorithm that depend only on its parameters $r_j$.

By definition of $r_1$, any feasible $r_1$ has to satisfy $r_1 \leq 1$, and by our assumption of the transitions, any feasible $r_j$ should satisfy $r_{j-1} \cdot f(O_{k_{j-1}}) \leq r_j \cdot f(O_{k_j})$ for all $1 < j \leq t$, which is equivalent to

$$r_{j-1} \leq r_j \cdot \left( \frac{f(O_{k_j})}{f(O_{k_{j-1}})} \right) \quad (5)$$

$$= r_j \cdot \left( \frac{k_{l_j}}{k_{l_{j-1}}} - \left( \frac{k_{l_j}}{k_{l_{j-1}}} - 1 \right) r_{j-1} \right) \quad \text{(By Eq. (4))}$$

$$= r_j \cdot \left( 1 + \frac{k_{l_j} - k_{l_{j-1}}}{k_{l_{j-1}}} - \left( \frac{k_{l_j} - k_{l_{j-1}}}{k_{l_{j-1}}} \right) r_{j-1} \right)$$

$$= r_j \cdot \left( 1 + \left( \frac{k_{l_j} - k_{l_{j-1}}}{k_{l_{j-1}}} \right) (1 - r_{j-1}) \right). \quad (6)$$

Next, for any feasible $r_j$'s (and in particular the $r_j$'s that correspond to the arbitrary instance $f^{(\text{bad-for-greedy})}$ in the theorem statement), we construct a hard instance $f$ that is characterized by the same $r_j$'s (i.e., it satisfies Eq. (3) for the given $r_j$'s), such that for the hard instance $f$ and for every budget $k_l$, up to an arbitrarily small multiplicative error, (i) the aforementioned approximation ratio determined by the $r_j$'s is also an upper bound of the approximation ratio of greedy (which implies the second item in the theorem statement, because the approximation ratio determined by the particular $r_j$'s corresponding to $f^{(\text{bad-for-greedy})}$ is a lower bound of the approximation ratio of greedy on $f^{(\text{bad-for-greedy})}$), and (ii) greedy performs at least (almost) as good as the efficient $A$ (which implies the first item in the theorem statement).

**Construction of hard instance**

Let $\Delta_1 = k_{l_1}, \Delta_j = k_{l_j} - k_{l_{j-1}}, \forall 1 < j < t$, and $\Delta_t = k_{l_t} - k_{l_{t-1}}$. We apply Theorem 2.2 to create $t$ hard (with respect to greedy and $A$) functions $f_{\Delta_1}, \ldots, f_{\Delta_t}$ over disjoint ground sets $V_1, \ldots, V_t$. We normalize these functions such that they have the same optimal value 1 (i.e., $f_{\Delta_j}(O^{(j)}) = 1$, where $O^{(j)}$ denotes the optimal size-$\Delta_j$ solution for $f_{\Delta_j}$) and extend them to the ground set $V := \cup_{i=1}^t V_i$.

The final submodular function is

$$f(X) := \sum_{j=1}^t \alpha_j \cdot f_{\Delta_j}(X),$$

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where $\alpha_1 := 1$ and

$$\alpha_j := (\Delta_j / \sum_{s=1}^{j-1} \Delta_s) \cdot \left( \sum_{s=1}^{j-1} \alpha_s \right) \cdot (1 - r_{j-1}), \quad \text{for all } 1 < j \leq t.$$ 

Claim 3.3. For any $r_j$ that satisfy Eq. (3), $r_j \cdot \sum_{s=1}^{j} \alpha_s$ is non-decreasing in $j$.

Proof of Claim 3.3

$$r_j \cdot \sum_{s=1}^{j} \alpha_s = r_j \cdot \left( \sum_{s=1}^{j-1} \alpha_s + \alpha_j \right)$$

$$= r_j \cdot \left( \sum_{s=1}^{j-1} \alpha_s + \frac{\Delta_j}{\sum_{s=1}^{j-1} \Delta_s} \cdot \left( \sum_{s=1}^{j-1} \alpha_s \right) \cdot (1 - r_{j-1}) \right) \quad \text{(Definition of } \alpha_s)$$

$$= r_j \cdot \left( \sum_{s=1}^{j-1} \alpha_s + \frac{k_{j-1} - k_{j-1-1}}{k_{j-1-1}} \cdot \left( \sum_{s=1}^{j-1} \alpha_s \right) \cdot (1 - r_{j-1}) \right) \quad \text{(Telescoping sum)}$$

$$= \left( 1 + \frac{k_{j-1} - k_{j-1-1}}{k_{j-1-1}} \cdot (1 - r_{j-1}) \right) r_j \cdot \sum_{s=1}^{j} \alpha_s$$

$$\geq r_{j-1} \cdot \sum_{s=1}^{j} \alpha_s, \quad \text{ (Ineq. (5))}$$

Because any feasible $r_j$’s that we need to consider satisfy $r_1 \leq 1$ and Eq. (5), it follows by Claim 3.3 and $\sum_{i=1}^{t} \alpha_s - (\sum_{i=1}^{j-1} \Delta_s / \Delta_j) \alpha_j = r_{j-1} \cdot \sum_{s=1}^{j-1} \alpha_s$ that $\alpha_j / \Delta_j$ is decreasing as $j$ increases. Hence, $f(O_{k_i}) = \sum_{i=1}^{j} \alpha_i$ for all $j \in [t]$. Moreover, it is easy to verify that the $r_j$’s indeed characterize the $f$ constructed above in the sense that Eq. (3) holds for the $f$ constructed above.

Upper bounding greedy performance on the hard instance: a single step

First, we analyze the best possible improvement of a single step of greedy on this instance. Suppose that greedy has chosen some size-$i$ set $X_i^{(j)} \subset V_j$, if it chooses another element from $V_j$, then we claim that the marginal gain is almost always $(f_{\Delta_j}(O^{(j)}) - f_{\Delta_j}(X_i^{(j)})) / \Delta_j$ (it is at least this amount by greedy guarantee). Assume otherwise, for some $\gamma, \epsilon_1, \epsilon_2 > 0$, in the first $\gamma \cdot \Delta_j$ iterations when greedy chooses elements from $V_j$, there are more than $\epsilon_1 \cdot \Delta_j$ iterations $i$ in which the marginal gain is larger than $(1 + \epsilon_2) / \Delta_j \cdot (f_{\Delta_j}(O^{(j)}) - f_{\Delta_j}(X_i^{(j)}))$. Suppose that at $(\gamma \cdot \Delta_j)$-th iteration $f_{\Delta_j}(X_i^{(j)}) = c \cdot f_{\Delta_j}(O^{(j)})$, then each of those $\epsilon_1 \cdot \Delta_j$ iterations gets at least an extra $(\epsilon_2 / \Delta_j) \cdot (1 - c) f_{\Delta_j}(O^{(j)})$ in addition to basic greedy guarantee, which implies that $f_{\Delta_j}(X_i^{(j)}) \geq (1 - \epsilon^{-\gamma} + \epsilon_1 \cdot \epsilon_2 (1 - c)) f_{\Delta_j}(O^{(j)})$. Then, for some $\epsilon_3 > 0$, $f_{\Delta_j}(X_i^{(j)}) \geq (1 - \epsilon^{-\gamma} + \epsilon_3) f_{\Delta_j}(O^{(j)})$, which is impossible by Theorem 2.2.

Henceforth, we can assume that the marginal gain for $f_{\Delta_j}$ is always $(f_{\Delta_j}(O^{(j)}) - f_{\Delta_j}(X_i^{(j)})) / \Delta_j$ for the $i$-th iteration when greedy chooses elements from $V_j$, and this will only decrease all the values of interest by an arbitrarily small multiplicative error.
Upper bounding greedy performance on the hard instance: total value

When we start running the greedy algorithm, for a while it only select elements from $V_1$ since those have the highest marginal contribution. Specifically, suppose that at the beginning of the $q$-th step, greedy has selected $X_{q-1} \subset V_1$. Then the best achievable marginal gain of an element from $V_1$ for $f$ is $\alpha_1(f_{\Delta_1}(O^{(1)}) - f_{\Delta_1}(X_{q-1}))/\Delta_1$. In comparison, the best singleton value of an element in $V_2$ is $(\alpha_2/\Delta_2)f_{\Delta_2}(O^{(2)})$, which is dominated by $\alpha_1(f_{\Delta_1}(O^{(1)}) - f_{\Delta_1}(X_{q-1}))/\Delta_1$, when $f_{\Delta_1}(X_{q-1}) \leq r_1 \cdot f_{\Delta_1}(O^{(1)})$, because

$$\frac{\alpha_1}{\Delta_1}(f_{\Delta_1}(O^{(1)}) - f_{\Delta_1}(X_{q-1})) \geq \frac{\alpha_1}{\Delta_1}(f_{\Delta_1}(O^{(1)}) - r_1 \cdot f_{\Delta_1}(O^{(1)})) \quad \text{(By } f_{\Delta_1}(X_{q-1}) \leq r_1 \cdot f_{\Delta_1}(O^{(1)}))$$

$$= \frac{\alpha_1}{\Delta_1}(1 - r_1) \quad \text{(By } f_{\Delta_1}(O^{(1)}) = 1)$$

$$= \frac{\alpha_2}{\Delta_2} \quad \text{(By definition of } \alpha_2)$$

$$= \frac{\alpha_2}{\Delta_2} f_{\Delta_2}(O^{(2)}) \quad \text{(By } f_{\Delta_2}(O^{(2)}) = 1). \quad (7)$$

Thus, when $f(X_{q-1}) = \alpha_1 \cdot f_{\Delta_1}(X_{q-1}) < r_1 \cdot f(O^{(1)})$, greedy should always prefer choosing elements from $V_1$ over $V_2$ (and other $V_i$’s), and the single step improvement is $f(X_q) - f(X_{q-1}) = (f(O^{(1)}) - f(X_{q-1}))/\Delta_1 = (f(O^{(1)}) - f(X_{q-1}))/k_l$ (this matches how $f(\text{greedy-lb})(q)$ changes).

We now analyze what happens when, after running greedy for a while, the marginal contribution from $V_1$-elements decays so that greedy may prefer $V_2$-elements. By Eq. (7), it is when $f_{\Delta_1}(X_{q-1}) = r_1 \cdot f_{\Delta_1}(O^{(1)})$ that the best singleton value of $V_2$-elements $(\alpha_2/\Delta_2)f_{\Delta_2}(O^{(2)})$ becomes equal to the best marginal contribution of a $V_1$-element $\alpha_1(f_{\Delta_1}(O^{(1)}) - f_{\Delta_1}(X_{q-1}))/\Delta_1$. Therefore, once $r_2 \cdot f(O^{(1)} \cup O^{(2)}) > f(X_{q-1}) \geq r_1 \cdot f(O^{(1)})$, greedy should start choosing elements from $V_1$ and $V_2$ alternatively to keep the identity $\alpha_1(f_{\Delta_1}(O^{(1)}) - f_{\Delta_1}(X_{q-1}))/\Delta_1 = \alpha_2(f_{\Delta_2}(O^{(2)}) - f_{\Delta_2}(X_{q-1}))/\Delta_2$ (up to negligible error), it follows that

$$\frac{\alpha_1(f_{\Delta_1}(O^{(1)}) - f_{\Delta_1}(X_{q-1}))}{\Delta_1} = \frac{\alpha_2(f_{\Delta_2}(O^{(2)}) - f_{\Delta_2}(X_{q-1}))}{\Delta_2}$$

$$= \frac{\alpha_1(f_{\Delta_1}(O^{(1)}) - f_{\Delta_1}(X_{q-1}))}{\Delta_1} + \frac{\alpha_2(f_{\Delta_2}(O^{(2)}) - f_{\Delta_2}(X_{q-1}))}{\Delta_2}$$

$$= \frac{f(O^{(1)} \cup O^{(2)}) - f(X_{q-1})}{k_l}.$$
that the greedy performance \( f(X_q) \) changes in exactly the same way as \( f^{(\text{greedy-lb})}(q) \), and hence, the approximation ratio based on \( f^{(\text{greedy-lb})}(q) \)'s is tight for greedy on the hard instance.

How greedy spends the budget

Finally, following the above derivation, we emphasize how greedy spends the budget. As we have shown, for any \( 1 \leq p \leq t \), when \( r_{p-1} \cdot f(\bigcup_{j=1}^{p} O^{(j)}) \leq f(X_{q-1}) \leq r_{p} \cdot f(\bigcup_{j=1}^{p} O^{(j)}) \), greedy splits its budget on \( V_1, \ldots, V_p \) to keep all the \( \alpha_j(f_{\Delta_j}(O^{(j)}) - f_{\Delta_j}(X_{q-1}))/\Delta_j \) approximately equal to each other. Moreover, for any \( j' > p \), the best singleton value \((\alpha_{j'}/\Delta_{j'}) f_{\Delta_{j'}}(O^{(j')}) \) of \( V_{j'} \) is smaller than \( \alpha_j(f_{\Delta_j}(O^{(j)}) - f_{\Delta_j}(X_{q-1}))/\Delta_j \) for any \( j \leq p \). Suppose that greedy has spent budget \( \hat{b}_j \) on \( V_j \) for each \( j \), which implies that \( f_{\Delta_j}(X_{q-1}) = 1 - e^{-\hat{b}_j/\Delta_j} \). Then, we have that \( \alpha_j(f_{\Delta_j}(O^{(j)}) - f_{\Delta_j}(X_{q-1}))/\Delta_j = \frac{d\alpha_j(1-e^{-x/\Delta_j})}{dx} \big|_{x=\hat{b}_j} \), and thus, \( \frac{d\alpha_j(1-e^{-x/\Delta_j})}{dx} \big|_{x=\hat{b}_j} \) for all \( j \leq p \) are equal to each other. Moreover, since \((\alpha_{j'}/\Delta_{j'}) f_{\Delta_{j'}}(O^{(j')}) = \frac{d\alpha_{j'}(1-e^{-x/\Delta_{j'}})}{dx} \big|_{x=0}, \frac{d\alpha_{j'}(1-e^{-x/\Delta_{j'}})}{dx} \big|_{x=\hat{b}_{j'}} \geq \frac{d\alpha_{j'}(1-e^{-x/\Delta_{j'}})}{dx} \big|_{x=0} \) for all \( j \leq p \) and \( j' > p \).

Greedy spends the budget optimally on the hard instance

By Theorem 2.2 and the design of our hard instances, for any budget \( b_j \), the best possible value the efficient algorithm \( A \) can get by spending budget \( b_j \) on \( f_{\Delta_j} \) is \( u_j(b) = (1 - e^{-b_j/\Delta_j})\alpha_j \). Suppose \( A \) spends budget \( b_j \) on each \( f_{\Delta_j} \), where \( \sum_{j=1}^{t} b_j = b \) for some \( b \), then in this case, the best possible value in total is \( \sum_{j=1}^{t} u_j(b_j) \), and hence, in general, the best possible value for budget \( b \) is upper bounded by the maximum of the following program:

\[
\max \sum_{j=1}^{t} u_j(b_j) \quad \text{s.t.} \quad \sum_{j=1}^{t} b_j = b \quad \text{and} \quad \forall j \quad b_j \geq 0.
\]

We observe that for an arbitrary fixed \( b \), the maximizer \( b^*_j \)'s for this program should satisfy that for all positive \( b^*_j \), the derivatives of \( u_j \)'s at \( b^*_j \)'s are equal (notice that the way greedy spends the budget also satisfies this property), and moreover, they are not smaller than the derivative of \( u_j \)'s at 0 for any \( j' \) such that \( b^*_{j'} = 0 \). Otherwise, there must exist \( \frac{du_{j_1}(x)}{dx} \big|_{x=b^*_{j_1}} < \frac{du_{j_2}(x)}{dx} \big|_{x=b^*_{j_2}} \) where \( b^*_{j_1} \) is strictly positive, then increasing \( b^*_{j_2} \) by \( \delta \) and decreasing \( b^*_{j_1} \) by \( \delta \) for sufficiently small \( \delta \) will increase the objective value while preserving the feasibility of \( b^*_j \)'s.

Now we prove that for any fixed \( b \), the \( b^*_j \)'s satisfying the above mentioned property are unique. (Then, it follows that the maximizer matches exactly how greedy spends the budget, and moreover, greedy attains the optimal value of the program.) Suppose that besides \( b^*_j \)'s, \( b^*_j \)'s also satisfy the property. Let \( \text{supp}(b^*) \) be the set of \( j \) such that \( b^*_j > 0 \). We first argue if \( j' \notin \text{supp}(b^*) \), then \( \tilde{b}_{j'} = 0 \). Suppose otherwise, \( \tilde{b}_{j'} > 0 \), then \( \sum_{j \in \text{supp}(b^*)} \tilde{b}_j < t \), and hence, there must exist a \( j \in \text{supp}(b^*) \) such that \( \tilde{b}_j < b^*_j \). By strict concavity of \( u_j \), \( \frac{du_j(x)}{dx} \big|_{x=\tilde{b}_j} > \frac{du_j(x)}{dx} \big|_{x=b^*_j} \). However, since the \( b^*_j \)'s satisfy the above mentioned property and \( b^*_j = 0 \), \( \frac{du_j(x)}{dx} \big|_{x=b^*_j} \geq \frac{du_j(x)}{dx} \big|_{x=0} \), and by strict concavity of \( u_j \), \( \frac{du_j(x)}{dx} \big|_{x=0} > \frac{du_j(x)}{dx} \big|_{x=\tilde{b}_j} \), which gives a contradiction. Furthermore, we can argue that for all \( j \in \text{supp}(b^*) \), \( \tilde{b}_{j'} = b^*_j \), because otherwise, there must exist \( j,j' \in \text{supp}(b^*) \) such that \( b^*_j > \tilde{b}_{j'} \) and \( b^*_j < \tilde{b}_{j'} \), and hence \( \frac{du_j(x)}{dx} \big|_{x=\tilde{b}_{j'}} < \frac{du_j(x)}{dx} \big|_{x=b^*_j} \) and \( \frac{du_j(x)}{dx} \big|_{x=b^*_j} > \frac{du_j(x)}{dx} \big|_{x=\tilde{b}_{j'}} \), which contradicts the property \( \frac{du_j(x)}{dx} \big|_{x=\tilde{b}_j} = \frac{du_j(x)}{dx} \big|_{x=b^*_j} \).
3.2 Two Remarks for Theorem 3.2

One might wonder whether the transitions of the greedy guarantees in the above analysis of Theorem 3.2 always occur in the order 1, 2, ..., m but never in any proper subsequence, namely whether \( r_1 \cdot f(O_{k_1}) \leq r_{i-1} \cdot f(O_{k_{i-1}}) \), which is equivalent to

\[
\frac{f(O_{k_i}) - (k_i/k_{i+1})f(O_{k_{i+1}})}{1 - (k_i/k_{i+1})} \geq \frac{f(O_{k_{i-1}}) - (k_{i-1}/k_i)f(O_{k_i})}{1 - (k_{i-1}/k_i)},
\]

This is equivalent to

\[
f(O_{k_{i+1}}) - f(O_{k_i}) \leq \frac{k_{i+1} - k_i}{k_i - k_{i-1}} \cdot (f(O_{k_i}) - f(O_{k_{i-1}})),
\]

which is actually true for our instances but not in general. See Example 3.4.

Example 3.4. Consider the function \( f \) on the ground set \( \{1, 2, 3, 4\} \) with values \( f(\emptyset) = 0, f(\{1\}) = 1, f(\{2\}) = f(\{3\}) = f(\{4\}) = 1/2, f(\{1, 2\}) = f(\{1, 3\}) = f(\{1, 4\}) = 7/6, f(\{2, 3\}) = f(\{2, 4\}) = f(\{3, 4\}) = 1, f(\{2, 3, 4\}) = 3/2, f(\{1, 2, 3\}) = f(\{1, 2, 4\}) = f(\{1, 3, 4\}) = 4/3 \) and \( f(\{1, 2, 3, 4\}) = 3/2 \). It is straightforward to check that \( f \) is submodular and monotone, and that \( f(O_3) - f(O_2) > f(O_2) - f(O_1) \).

Finally, we end this section with the following observation. In the appendix, we give the proof of this observation and show that many practical algorithms satisfy the condition of this observation.

Observation 3.5. For any perturbation factors \( 0 < \rho_1 < \rho_2 < \cdots < \rho_m \), there exists a sufficiently large \( k \) that grows with the size of instance such that given \( m \) budgets \( k_1 = \rho_1 \cdot k, \ldots, k_m = \rho_m \cdot k \), the optimality described in Theorem 3.2 actually holds for a general class of algorithms such that:

- Given budget \( k_i \), the algorithm \( A \) runs in \( T \) rounds (\( T \) is sufficiently large), of which each round selects about \( k_i/T \) elements.
- For any \( \varepsilon > 0 \), it holds for all \( t \in [T] \), for all \( j \in [m] \), that \( f(X_{tk_i/T}^A) - f(X_{(t-1)k_i/T}^A) \geq ((1 - \varepsilon)\rho_i/(\rho_j T)) \cdot (f(O_{k_j}) - f(X_{tk_i/T}^A)) \), where \( X_s^A \) is the \( s \)-th element chosen by \( A \).

4 A Lower Bound for Every Distribution

The main thesis of this paper is that worst case instances of submodular maximization are really tailored to a specific budget constraint. It is natural to hope that as the distribution of budget perturbation becomes arbitrarily spread (aka arbitrarily far from the worst case single budget), the approximation factor approaches 1. In this section, we give a negative answer to this question.

Theorem 4.1. For any distribution of budget perturbations \( D \), for any efficient algorithm \( A \), for every sufficiently large \( k \) that grows with the size of instance, \( \mathcal{R}_A(D(k)) \leq 0.9087 \).

(We did not seriously try to optimize the constant 0.9087. Computing the optimal constant is an interesting open problem for future work.)

Proof. For arbitrarily small \( \tau > 0 \), let \( \rho_{\min} \) and \( \rho_{\max} \) be such that the mass of \( D \) on \([\rho_{\min}, \rho_{\max}]\) is at least \( 1 - \tau \). Let \( q = 50 \). Let \( K_1 = q^{-(i^*-1)} \cdot k \) where \( i^* \) is the largest \( i \) such that \( q^{-(i-1)} \cdot k \leq \rho_{\min} \cdot k \). Let \( N \) be the smallest \( i \) such that \( q^{-(i-1)} \cdot K_1 \geq \rho_{\max} \cdot k \). We first construct the hard instances, and by Theorem 3.2 it suffices to upper bound the approximation ratio achieved by greedy on these instances.
Construction of hard instances

Let \( K_i = q^{(i-1)} \cdot K_1 \). We use Theorem 2.2 to create hard (with respect to greedy algorithm) functions \( f_{K_i} \) for all \( i \in [N] \) over disjoint ground sets. We normalize these functions such that they have the same optimal value 1 (i.e., \( f_{K_i}(O^{(i)}) = 1 \), where \( O^{(i)} \) denotes the optimal size-\( K_i \) solution for \( f_{K_i} \)) and extend them to the union of all the ground sets. The final submodular function is \( f(X) = \sum_{i=1}^{N} (q/e)^{i-1} \cdot f_{K_i}(X) \).

Upper bounding the approximation ratio on the hard instances

Consider a budget \( K \) between \( \sum_{j=1}^{i} K_j \) and \( \sum_{j=1}^{i+1} K_j \), for any \( i \leq N - 1 \). We first show that the contribution of \( f_{K_j} \) with \( j \leq i - 1 \) is negligible. Notice that the best singleton value of \( f_{K_i} \) is \((q/e)^{j-1}/(q^{j-1} \cdot K_1)\), which is decreasing in \( j \). Hence, we can generously assume that the algorithm spends a budget of size \( \sum_{j=1}^{i} K_j \) getting all the utilities from \( f_{K_j} \) with \( j \leq i - 1 \), which is the best one can hope for. The total value of these \( f_{K_j} \)'s is \( \sum_{j=1}^{i-1} (q/e)^{j-1} = ((q/e)^{i-1} - 1)/(q/e - 1) \), which is less than \( 1/(q/e - 1) < 0.0575 \) fraction of the value of the \( f_{K_i} \). Therefore, the best possible approximation ratio for budget \( K \) is at most the best possible approximation ratio for budget \( K - \sum_{j=1}^{i-1} K_j \) on the \( f_{K_j} \)'s with \( j \geq i \) plus 0.0575.

Furthermore, the best singleton value of \( f_{K_{i+1}} \) is at most \((q/e)^i/(q^i \cdot K_1)\). On the other hand, with budget \( K_i \) on \( f_{K_i} \), greedy can achieve approximation ratio at most \( 1 - 1/e \) by Theorem 2.2 and thus, at the \((K_i+1)\)-th iteration, greedy has marginal gain at least \((1 - (1 - 1/e) \cdot (q/e)^i-1)/(q/e - 1)\), which is equal to the best singleton value of \( f_{K_{i+1}} \). Hence, greedy will not choose anything from \( f_{K_{i+1}} \) until it has selected \( K_i \) elements from \( f_{K_i} \). The remaining budget \( K - \sum_{j=1}^{i} K_j \) is at most \( K_{i+1} \), and it follows for the same reason that greedy will not spend remaining budget on any \( f_{K_j} \)'s with \( j \geq i + 2 \).

It remains to show how greedy performs on \( f_{K_i} \) and \( f_{K_{i+1}} \) with budget \( K' = K - \sum_{j=1}^{i} K_j \). Let \( a = K'/K_i \). Notice that greedy splits its budget in the way that the marginal gain of choosing the next element from \( f_{K_i} \) is approximately equal to that of choosing the next element from \( f_{K_{i+1}} \). This can be expressed as the following equations:

\[
a_1 K_i + a_2 K_i = a K_i,
\]

\[
e^{-a_1} \cdot \frac{e}{q} K_i = e^{-a_2} \frac{K_i}{K_{i+1}} \cdot \frac{1}{K_{i+1}},
\]

where \( a_1 K_i \) and \( a_2 K_i \) are the budgets spent on \( f_{K_i} \) and \( f_{K_{i+1}} \) respectively. The solution is \( a_1 = (a + q)/(q + 1) \) and \( a_2 = (q(a - 1))/(q + 1) \). Hence the approximation ratio of greedy on \( f_{K_i} \) and \( f_{K_{i+1}} \) with budget \( K' \) is at most

\[
\left(1 - e^{-a_1 K_i/K_i}\right) \cdot \frac{e}{q} + \left(1 - e^{-a_2 K_i/K_{i+1}}\right) = \left(1 - e^{-(a+q)/(q+1)}\right) \cdot \frac{e}{q} + \left(1 - e^{-(a-1)/(q+1)}\right) = \frac{e}{q} + \frac{a}{q} - \frac{a-1}{q}.
\]

The maximum is approximately 0.85114 achieved by \( a \approx 9.2199 \). Hence, in total, the approximation ratio for the entire instance is less than 0.8512 + 0.0575 = 0.9087. We have proved that the optimal approximation ratio for the hard instance is less than 0.9087 for any budget in \([\rho_{\text{min}} \cdot k, \rho_{\text{max}} \cdot k]\), and this finishes the proof because \( \tau \) is arbitrarily small.

\[\square\]
5 Numerical Simulation

We formulate a mathematical program that computes the worst possible optimal expected approximation ratio, for any fixed distribution on any fixed choice of \( m \) budgets \( \rho_1 k < \rho_2 k < \cdots < \rho_m k = k \) (we also let \( \rho_0 = 0 \)). We denote the probability of budget \( \rho_i k \) by \( p_i \) for each \( i \).

Reducing the hard instances to a standard form

Recall that the hard instances in the proof of Theorem 3.2 have following form—

\[
\sum_{i=1}^{l} \alpha_i \cdot f(\rho_i - \rho_{i-1}) k, \quad \text{where } l_1, \ldots, l_t \text{ is a subsequence of } 1, \ldots, m, \quad \text{and } f(\rho_i - \rho_{i-1}) k(X) \text{ is the hard submodular function from Theorem 2.2 and it is normalized such that its optimal value for budget } (\rho_i - \rho_{i-1}) k \text{ is 1. Moreover, } \alpha_j / (\rho_j - \rho_{j-1}) \text{ is increasing in } j. \text{ We show that there is a submodular function that is as hard as } f^* \text{ to approximately maximize in the following standard form}^{\text{6}} \quad -f(X) = \sum_{i=1}^{m} \beta_i \cdot f(\rho_i - \rho_{i-1}) k(X), \text{ where } \beta_i \text{'s satisfy}
\]

\[
\frac{\beta_i}{\rho_i - \rho_{i-1}} \geq \frac{\beta_{i+1}}{\rho_{i+1} - \rho_i}, \quad \forall i < m,
\]

where \( f(\rho_i - \rho_{i-1}) k \) is defined analogously to \( f(\rho_i - \rho_{j-1}) k \), and we denote the ground set of \( f(\rho_i - \rho_{i-1}) k \) by \( V_i \).

For \( i \in [m] \) and \( j \) such that \( l_{j-1} < i \leq l_j \), we define

\[
\lambda_i := \frac{\rho_i - \rho_{i-1}}{\rho_j - \rho_{j-1}}.
\]

**Claim 5.1.** Given budget \( x \cdot k \) for any \( x \geq 0 \), the best achievable approximation ratio for \( \sum_{i=l_{j-1}+1}^{l_j} \lambda_i \cdot f(\rho_i - \rho_{i-1}) k(X) \) is equal to that for \( f(\rho_i - \rho_{i-1}) k(X) \).

**Proof.** For any budget \( x \cdot k \), the best achievable approximation ratio for \( \sum_{i=l_{j-1}+1}^{l_j} \lambda_i \cdot f(\rho_i - \rho_{i-1}) k(X) \) is

\[
\max_{x_i \text{ 's}} \sum_{i=l_{j-1}+1}^{l_j} \lambda_i (1 - e^{-\frac{x_i}{\rho_i - \rho_{i-1}}}) \quad \text{s.t. } \sum_{i=l_{j-1}+1}^{l_j} x_i = x \text{ and } x_i \text{'s are non-negative.}
\]

For any feasible \( x_i \text{'s},
\]

\[
\sum_{i=l_{j-1}+1}^{l_j} \lambda_i (1 - e^{-\frac{x_i}{\rho_i - \rho_{i-1}}}) \leq 1 - e^{-\sum_{i=l_{j-1}+1}^{l_j} \frac{\lambda_i x_i}{\rho_i - \rho_{i-1}}} \quad \text{(Jensen’s inequality and } \sum_{i=l_{j-1}+1}^{l_j} \lambda_i = 1) \]

\[
= 1 - e^{-\sum_{i=l_{j-1}+1}^{l_j} \frac{x_i}{\rho_i - \rho_{i-1}}} \quad \text{(By definition of } \lambda_i) \]

\[
= 1 - e^{-\frac{x}{\rho_{l_j} - \rho_{l-1}}} \quad \text{(By } \sum_{i=l_{j-1}+1}^{l_j} x_i = x)\]

Moreover, when \( \frac{x_i}{\rho_i - \rho_{i-1}} \) for all \( i \) are equal to each other, we have that \( \frac{x_i}{\rho_i - \rho_{i-1}} = \frac{\sum_{i=l_{j-1}+1}^{l_j} x_i}{\sum_{i=l_{j-1}+1}^{l_j} \rho_i - \rho_{i-1}} = \frac{x}{\rho_{l_j} - \rho_{l-1}} \), and then \( \sum_{i=l_{j-1}+1}^{l_j} \lambda_i (1 - e^{-\frac{x_i}{\rho_i - \rho_{i-1}}}) = 1 - e^{-\frac{x}{\rho_{l_j} - \rho_{l-1}}} \). Hence, \( 1 - e^{-\frac{x}{\rho_{l_j} - \rho_{l-1}}} \) is exactly the best achievable approximation ratio for \( \sum_{i=l_{j-1}+1}^{l_j} \lambda_i \cdot f(\rho_i - \rho_{i-1}) k(X) \). Notice that it is also the best achievable approximation ratio for \( f(\rho_i - \rho_{i-1}) k(X) \).

\[\square\]
Henceforth, we can replace each $f_{(\rho_i, \rho_{i+1})}$ with $\sum_{j=l_{l+1}}^{l_i} \lambda_i \cdot f_{(\rho_{i-1}, \rho_i)}(X)$ in $f^*$, which reduces $f^*$ to the standard form. Then, Eq. [8] follows by definition of $\lambda_i$’s and the monotonicity of $\alpha_j/\rho_{i-1}$. Finally, we note that Eq. [8] implies that optimal value of the $f^*$ for budget $\rho_k$ is $\sum_{j=1}^i \beta_j$ and that for any $i$, whenever $V_i$ is used by the greedy algorithm, so should the $V_i$’s for any $i' \leq i$.

**Formulating the mathematical program**

For each budget $\rho_i \cdot k$, the best possible approximation ratio is achieved by choosing elements from the first $l$ subsets $V_1, \ldots, V_l$ for certain $l \leq m$ (which we do not know a priori), and the budget should be split in a way such that the marginal contribution from the next element is (approximately) equal among $V_1, \ldots, V_l$. That is,

$$d(\beta_1(1 - e^{-x_j^{(i,l)}/\rho_1})) = d(\beta_j(1 - e^{-x_j^{(i,l)}/(\rho_j - \rho_{j-1})}))/dx_j^{(i,l)}, \forall j \leq l,$$

$$\sum_{j \leq l} x_j^{(i,l)} = \rho_i, \text{ and } x_j^{(i,l)} \geq 0, \forall j \leq l.$$

Solving the system of equations in the above constraint gives us

$$x_1^{(i,l)} = \frac{\rho_i}{\rho_1} - \sum_{j=1}^l \ln \left( \frac{\beta_j \rho_1}{\beta_1 (\rho_j - \rho_{j-1})} \right) \cdot \frac{\rho_j - \rho_{j-1}}{\rho_1}, \quad (9)$$

$$x_j^{(i,l)} = x_1^{(i,l)} + \ln \left( \frac{\beta_j \rho_1}{\beta_1 (\rho_j - \rho_{j-1})} \right), \forall j \leq l.$$

We let $h^{(i,l)}(\beta_1, \ldots, \beta_m)$ denote the approximation ratio achieved by $x_j^{(i,l)}$’s, then it is given by

$$h^{(i,l)}(\beta_1, \ldots, \beta_m) = \frac{\sum_{j=1}^l \beta_j (1 - e^{-x_j^{(i,l)}/\rho_1})}{\sum_{j=1}^l \beta_j} = \frac{\sum_{j=1}^l \beta_j (1 - e^{-x_1^{(i,l)}/\rho_1}) \cdot \frac{\beta_1 (\rho_j - \rho_{j-1})}{\beta_j \rho_1}}{\sum_{j=1}^l \beta_j}.$$  

(Solution of $x_j^{(i,l)}$)

$$= \frac{\sum_{j=1}^l \beta_j - \sum_{j=1}^l \beta_1 (\rho_j - \rho_{j-1}) e^{-x_j^{(i,l)}/\rho_1}}{\sum_{j=1}^l \beta_j} e^{-x_1^{(i,l)}/\rho_1}.$$  

(Telescoping sum)

$$= \frac{\left( \sum_{j=1}^l \beta_j \right) - \beta_1 \cdot \frac{\rho_1}{\rho_i} \cdot e^{-x_1^{(i,l)}/\rho_1}}{\sum_{j=1}^l \beta_j}.$$  

(Solution of $x_1^{(i,l)}$)
Table 2: Campaign Budgets (in millions)

| Candidate | Bennet | Biden | Bloomberg | Buttigieg | Gabbard | Klobuchar | Patrick | Sanders | Steyer | Warren | Yang |
|-----------|--------|-------|-----------|-----------|---------|-----------|---------|---------|--------|--------|------|
| Budget    | 2.6    | 23.3  | 188.4     | 34.1      | 2.9     | 10.1      | 0.9     | 50.1    | 153.7  | 33.7   | 19.2 |

where the nominator is the value achieved by $x_j^{(i,l)}$s, and the denominator is the optimal value. Since we do not know the right choice of $l$ a priori, we will enumerate all possible choices of $l$ and pick the best. Moreover, for every $l \leq m$, we consider $l$ as a candidate choice only if the solutions of $x_j^{(i,l)}$s by Eq. (9) are non-negative, because this holds for the right choice of $l$. (Note that $l = 1$ is always a candidate choice, because it means that all budget are spent on the first sub-instance, and hence $x_1^{(i,1)} = \rho_i \geq 0$.) Therefore, we let $h^{(i)}$ denote the best approximation ratio for budget $\rho_i k$, then it is given by

$$h^{(i)}(\beta_1, \ldots, \beta_m) = \max_{1 \leq l \leq m} h^{(i,l)}(\beta_1, \ldots, \beta_m) \cdot 1[x_l^{(i,l)} \geq 0]$$

$$= \max_{1 \leq l \leq m} h^{(i,l)}(\beta_1, \ldots, \beta_m) - C \cdot 1[x_l^{(i,l)} < 0]$$

(C is a large constant),

where $x_l^{(i,l)}$ can be represented as a function of $\beta_i$’s. Note that we only restrict $x_l^{(i,l)}$ to be non-negative, which actually implies that every $x_j^{(i,l)}$ for $j \leq l$ is non-negative, by Eq.(8) and Eq.(9).

Finally, the expected approximation ratio $h$ is given by $h(\beta_1, \ldots, \beta_m) = \sum_{i=1}^{m} p_i \cdot h^{(i)}(\beta_1, \ldots, \beta_m)$.

Given any fixed $\rho_i$’s, the worst possible optimal average approximation ratio is the result of the following program

$$\min_{\beta_1, \ldots, \beta_m \geq 0} h(\beta_1, \ldots, \beta_m) \text{ s.t. Eq.(8) and Eq.(9)}$$

5.1 Empirical results

We solve this program numerically for various distributions of budget perturbations ($\rho_i$’s); the results are summarized in Table 2.

Canonical distributions  It is natural to ask what is the expected approximation factor when the budget is drawn from uniform over $[x, 10x]$. Since we don’t know how to compute this value exactly, we take discretization of this distribution namely 25 budgets evenly spaced between $x$ and $10x$. Similarly, we experiment with discretizations of log-scale uniform distributions over $[x, 10x]$ and $[x, 600x]$.

Top social/political campaigns on Facebook  With the application of influence maximization on social networks in mind, we use the budgets of the top ten campaigns on Facebook’s database of social/political campaigns.

2020 Democratic Party presidential candidates  We use reported total campaign budgets by candidates in the 2020 Democratic Party primary elections during months October-December 2019 [AKS20] (see Table 2).

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7 We use a somewhat sparse discretization since the program is non-convex.
8 Top ten amount spent during the month before Mar. 26, 2020 in the Facebook Ad report [Fac20].
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arbitrary efficient algorithm) submodular functions $f$. Using Theorem 2.2, we create two hard (with respect to an arbitrary efficient algorithm) submodular functions $f_{k_1}$ and $f_{k_2-k_1}$, with disjoint ground sets $V_1$ and $V_2$. To simplify notation, we let $f_1$ and $f_2$ denote $f_{k_1}$ and $f_{k_2-k_1}$ respectively. Then, we normalize the two functions such that $f_1(O(1)) = f_2(O(2)) = 1$, where $O(1)$ denotes the optimal size-$k_1$ solution for $f_1$ and $O(2)$ is the optimal size-$(k_2 - k_1)$ solution for $f_2$. Furthermore, we extend both the functions to the ground set $V := V_1 \cup V_2$ in a natural way that the sets of $V_2$ have zero value to $f_1$ and vice versa. Finally, for $\alpha > 0$, we define $f : V \rightarrow \mathbb{R}_{\geq 0}$ as $f(X) = \alpha \cdot f_1(X) + f_2(X)$.

### Lemma A.1

For any efficient algorithm $A$, there is a submodular function $f$ constructed as above that has the following properties:

(i) For every $k = \beta \cdot k_2$ with $0 \leq \beta \leq 1$, the optimal value of $f$ with budget $k$ is $\min\{1, \beta/(1 - \rho)\} + \max\{(\beta - 1 + \rho)/\rho, 0\} \cdot \alpha$ if $\alpha < \rho/(1 - \rho)$ and is $\min\{1, \beta/\rho\} \cdot \alpha + \max\{(\beta - \rho)/(1 - \rho), 0\}$ otherwise.

(ii) For every $k = \beta \cdot k_2$ with $0 \leq \beta \leq 1$, the solution value of $A$ on $f$ with budget $k$ is upper bounded by $\max_{0 \leq x \leq \beta} (1 - e^{-x/\rho}) \cdot \alpha + (1 - e^{(x-\beta)/(1-\rho)})$.

**Proof.** (i) By the first property in Theorem 2.2, the optimal values of $f_1$, $f_2$ grow linearly with the budget. Moreover, the marginal gain of an element of $O(1)$ is $\alpha/k_1 = \alpha/(\rho \cdot k_2)$ until we select all $O(1)$ and is zero after that. The marginal gain of an element in $O(2)$ is $1/(k_2 - k_1) = 1/((1 - \rho)k_2)$ until $O(2)$ is exhausted. Hence, if $\alpha < \rho/(1 - \rho)$, the optimal solution to $f$ should prefer the elements of $O(2)$ until it exhausts $O(2)$, and then spend the rest of budget on $O(1)$. Hence, the optimal value is $\min\{1, \beta/(1 - \rho)\} + \max\{(\beta - 1 + \rho)/\rho, 0\} \cdot \alpha$. The other case is similar.
Suppose \( x \cdot k_2 = (x/\rho)k_1 \) elements are chosen from \( V_1 \), then, the remaining \(((\beta-x)/(1-\rho))k_2\) elements are from \( V_2 \). By the second property in Theorem 2, the value we can obtain is at most \((1-e^{-x/\rho}) \cdot \alpha + (1-e^{(x-\beta)/(1-\rho)})\). Therefore, the maximum of this objective is an upper bound of the optimum of \( f \) with budget \( k \).

With this lemma, we can easily prove a parametrized hardness result.

**Proposition A.2.** Given any \( 0 < \rho < 1 \) and \( \alpha(1 - \rho)/\rho \geq 1 \), for any \( \epsilon > 0 \), there is no efficient algorithm can approximate submodular maximization problem for two budgets \( k_2 \) and \( k_1 = \rho \cdot k_2 \), with the average approximation ratio is larger than

\[
\frac{1}{2} \left( 1 - e^{-\rho} \left( \frac{\rho}{\alpha(1-\rho)} \right)^{1-\rho} \right) \alpha + 1 - e^{-\rho} \left( \frac{\alpha(1-\rho)}{\rho} \right)^{\rho} + \epsilon, \tag{10}
\]

if \( \alpha(1 - \rho)/\rho \leq \epsilon \),

\[
\frac{1}{2} \left( 1 - e^{-1} \right) + \frac{\left( 1 - e^{-1} \left( \frac{\rho}{\alpha(1-\rho)} \right)^{1-\rho} \right) \alpha + 1 - e^{-1} \left( \frac{\alpha(1-\rho)}{\rho} \right)^{\rho}}{\alpha + 1} + \epsilon, \tag{11}
\]

if \( \epsilon \leq \alpha(1 - \rho)/\rho \leq e^{1/\rho} \),

\[
\frac{1}{2} \left( 1 - e^{-1} + (1 - e^{-1/\rho}) \cdot \frac{\alpha}{\alpha + 1} \right) + \epsilon, \tag{12}
\]

if \( \alpha(1 - \rho)/\rho \geq e^{1/\rho} \).

**Proof.** Since \( \alpha(1 - \rho)/\rho \geq 1 \), by the first property in Lemma A.1, we know the optimal value for budget \( k_1 \) is \( \alpha \) and that for budget \( k_2 \) is \( \alpha + 1 \). We can maximize the best achievable solution value \((1-e^{-x/\rho}) \cdot \alpha + (1-e^{(x-\beta)/(1-\rho)})\) for \( \beta = \rho \) and 1 by standard calculus. In general, we find that the optimal \( x \) is \((\ln(\alpha(1-\rho)/\rho) + \beta/(1-\rho))/(1/(1-\rho) + 1/\rho)\). Then, we observe that if \( 1 \leq \alpha(1 - \rho)/\rho \leq \epsilon \), the optimal \( x \) for \( \beta = \rho \) is between 0 and \( \rho \), and that for \( \beta = 1 \) is between \( \rho \) and 1. Hence the optimal \( x \) for both \( \beta \) are feasible, and we can calculate the analytic formula of each maximum. Therefore, we have an upper bound of approximation ratio for each budget. Obviously, the average of these two upper bounds, which is given in Eq. (10), is an upper bound for the average approximation ratio. If \( \epsilon \leq \alpha(1 - \rho)/\rho \leq e^{1/\rho} \), then the optimal \( x \) is \( \rho \) when \( \beta = \rho \) and is still \((\ln(\alpha(1-\rho)/\rho) + \beta/(1-\rho))/(1/(1-\rho) + 1/\rho)\) when \( \beta = 1 \). As before, we can calculate the upper bound, which is given in Eq. (11). Finally, if \( \alpha(1 - \rho)/\rho \geq e^{1/\rho} \), then the optimal \( x \) is \( \rho \) when \( \beta = \rho \) and is 1 when \( \beta = 1 \). The corresponding upper bound is given in Eq. (12).
Next, we derive the closed-form parametrized formulas of the approximation ratios of the greedy algorithm for monotone submodular maximization with two budgets, which will match the hardness in Proposition A.2. Before that, we establish a useful lemma for greedy analysis.

**Lemma A.3.** Given the same conditions as in Lemma 2.3, for all \( k_1 = \theta \cdot k_2 \) with \( \theta \geq 0 \) and \( k = \eta \cdot k_1 \) with \( 0 \leq \eta \leq 1 \), the following inequality holds,

\[
f(X_{k_1}) \geq (1 - e^{-\eta \theta}) f(O_{k_2}) + e^{-\eta \theta} \cdot f(X_k),
\]

and in particular, by letting \( \eta = 0 \),

\[
f(X_{k_1}) \geq (1 - e^{-\theta}) f(O_{k_2}).
\]

**Proof.** We start from Lemma 2.3,

\[
f(X_i) - f(X_{i-1}) \geq \frac{1}{k_2} (f(O_{k_2}) - f(X_{i-1})).
\]

We rearrange the terms as follows,

\[
f(O_{k_2}) - f(X_i) \leq \left(1 - \frac{1}{k_2}\right) \cdot (f(O_{k_2}) - f(X_{i-1}))
\]

and we recursively apply this step and get

\[
f(O_{k_2}) - f(X_{k_1}) \leq \left(1 - \frac{1}{k_2}\right)^{k_1-k} \cdot (f(O_{k_2}) - f(X_k))
\]

\[
\leq e^{\theta - \eta \theta} (f(O_{k_2}) - f(X_k)).
\]

The proof finishes by rearranging the terms. \( \square \)

**Proposition A.4.** Given a monotone submodular function \( f \) and two budgets \( k_2 \) and \( k_1 = \rho \cdot k_2 \) with \( 0 < \rho < 1 \), we let \( X_k \) and \( O_k \) denote the greedy solution and the optimal solution of cardinality \( k \). Suppose \( f(O_{k_1}) = c \cdot f(O_{k_2}) \), where \( \rho \leq c \leq 1 \). Then, the greedy algorithm has the following average approximation ratios,

(i)

\[
\frac{1}{2} \left( \left(1 - e^{-\rho} \left( \frac{1 - \rho}{\rho(1/c - 1)} \right)^\rho \cdot \frac{1}{c} + e^{-\rho} \left( \frac{1 - \rho}{\rho(1/c - 1)} \right)^\rho \cdot \frac{1 - \rho/c}{1 - \rho} \right) + \left(1 - e^{-1} \left( \frac{1 - \rho}{\rho(1/c - 1)} \right)^\rho + e^{-1} \left( \frac{1 - \rho}{\rho(1/c - 1)} \right)^\rho \cdot \frac{c - \rho}{1 - \rho} \right) \right),
\]

if \( ((1 - \rho/c)/(1 - \rho)) \leq 1 - e^{-1} \),

(ii)

\[
\frac{1}{2} \left( (1 - e^{-1}) + \left(1 - e^{-1} \left( \frac{1 - \rho}{\rho(1/c - 1)} \right)^\rho \cdot (1 - c) + e^{-1} \left( \frac{1 - \rho}{\rho(1/c - 1)} \right)^{1-\rho} \cdot c \right) \right),
\]

if \( 1 - e^{-1} \leq ((1 - \rho/c)/(1 - \rho)) \leq 1/c \),

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\[
\frac{1}{2} \left( (1 - e^{-1}) + (1 - e^{-1/\rho})c \right),
\]

if \((1 - \rho/c)/(1 - \rho) \geq 1/c\).

Moreover, there is no efficient algorithm can achieve better approximation ratios.

Proof. By Lemma 2.3, we have the following two guarantees,

\[
f(X_i) - f(X_{i-1}) \geq \frac{1}{k_2} (f(O_{k_2}) - f(X_{i-1})),
\]

\[
f(X_i) - f(X_{i-1}) \geq \frac{1}{k_1} (f(O_{k_1}) - f(X_{i-1})).
\]

Observe that \((f(O_{k_1}) - f(X_{i-1}))/k_1 \geq (f(O_{k_2}) - f(X_{i-1}))/k_2\) holds if and only if \(f(X_{i-1}) \leq ((c - \rho)/(1 - \rho))f(O_{k_2})\), where the right hand side is equal to \(((1 - \rho/c)/(1 - \rho))f(O_{k_1})\). We let \(i^* = 1 = \tau \cdot k_1\) be the largest \(i - 1\) such that \(f(X_{i-1}) \leq ((1 - \rho/c)/(1 - \rho))f(O_{k_1})\) holds. We first consider the case where

\[
((1 - \rho/c)/(1 - \rho)) \leq 1 - e^{-1},
\]

which implies that \(\tau \leq 1\), because \(f(X_{k_1}) \geq (1 - e^{-1})f(O_{k_1})\). Then, by Lemma A.3 with \(\theta = \tau, \eta = 0, f(X_{i^*-1}) \geq (1 - e^{-\tau})f(O_{k_1})\). It follows that

\[
((1 - \rho/c)/(1 - \rho)) \geq 1 - e^{-\tau}.
\]

Now we apply Lemma A.3 with \(\theta = \rho, \eta = i^*/k_1 = \tau + o(1)\),

\[
f(X_{k_1}) \geq (1 - e^{-\rho - \rho})f(O_{k_2}) + e^{\rho - \rho} \cdot f(X_{i^*})
\]

\[
\geq (1 - e^{-\rho - \rho})f(O_{k_2}) + e^{\rho - \rho} ((1 - \rho/c)/(1 - \rho))f(O_{k_1})
\]

\[
= ((1 - e^{-\rho - \rho})/c + e^{\rho - \rho} ((1 - \rho/c)/(1 - \rho)))f(O_{k_1})
\]

\[
\geq \left( 1 - e^{-\rho} \left( \frac{1 - \rho}{\rho(1/c - 1)} \right)^\rho \cdot \frac{1}{c} + e^{-\rho} \left( \frac{1 - \rho}{\rho(1/c - 1)} \right)^\rho \cdot \frac{1 - \rho/c}{1 - \rho} \right) f(O_{k_1}),
\]

where the second inequality is by definition of \(i^*\), and the last inequality follows from Eq. (17). Then, we apply Lemma A.3 with \(\theta = 1, \eta = \rho\) and use the previous bound of \(f(X_{k_1})\),

\[
f(X_{k_2}) \geq (1 - e^{\rho - 1})f(O_{k_2}) + e^{\rho - 1} f(X_{k_1})
\]

\[
\geq \left( 1 - e^{\rho - 1} \left( \frac{1 - \rho}{\rho(1/c - 1)} \right)^\rho \cdot \frac{c - \rho}{1 - \rho} \right) f(O_{k_2}).
\]

We relate this case, where we assume Eq. (16), with the first case of Proposition A.2 by noticing that \(c = \alpha/(1 + \alpha)\). It is straightforward to verify the approximation ratios for \(f(X_{k_1})\) and \(f(X_{k_2})\) match the ratios there. Therefore, greedy is optimal for this case. Next, we consider the case where

\[
1 - e^{-1} \leq ((1 - \rho/c)/(1 - \rho)) \leq 1/c,
\]

which implies \(1 \leq \tau \leq 1/\rho\), and Eq. (17) still holds. In this case, we know \(f(X_{k_1}) \geq (1 - e^{-1})f(O_{k_1})\), and similar to before, we apply Lemma A.3 with \(\theta = 1, \eta = i^*/k_2 = \rho \cdot \tau + o(1)\),

\[
f(X_{k_2}) \geq (1 - e^{\rho - \tau - 1})f(O_{k_2}) + e^{\rho - \tau - 1} \cdot f(X_{i^*})
\]
\[ \geq (1 - e^{\rho \tau - 1}) f(O_{k_2}) + e^{\rho \tau - 1} (1 - e^{-\tau}) f(O_{k_1}) \]
\[ = ((1 - e^{\rho \tau - 1}) + e^{\rho \tau - 1} (1 - e^{-\tau}) \cdot c) f(O_{k_2}) \]
\[ \geq \left( 1 - e^{-1} \left( \frac{1 - \rho}{\rho(1/c - 1)} \right)^{\rho} \cdot (1 - c) + e^{-1} \left( \frac{1 - \rho}{\rho(1/c - 1)} \right)^{1 - \rho} \cdot c \right) f(O_{k_2}), \]

where the second inequality is again by Lemma A.3 with \( \theta = 1, \eta = i^*/k_1 = \tau + o(1) \) and the last inequality follows from Eq. (17). It is not hard to verify that this case correspond to the second case of Proposition A.2 and greedy has optimal ratios. Finally, we consider the last case where

\[(1 - \rho/c)/(1 - \rho) \geq 1/c. \tag{19} \]

In this case, we know that \( f(X_{k_1}) \geq (1 - e^{-1}) f(O_{k_1}) \) and \( f(X_{k_2}) \geq (1 - e^{-1/\rho}) f(O_{k_1}) = (1 - e^{-1/\rho}c \cdot f(O_{k_2})) \). We conclude that greedy is optimal by comparing this case with the third case of Proposition A.2.

Figure 1: Optimal average approximation ratios for two budgets.

For every \( 0 < \rho < 1 \), using Proposition A.4, we can compute the worst \( c \) to minimize the approximation ratio, and it follows that the minimal ratio is the best achievable approximation guarantee for submodular maximization with budgets \( k_2 \) and \( k_1 = \rho \cdot k_2 \). It turns out that the first case of Proposition A.4 is always the worst case. We illustrate the best achievable approximation ratios for \( 0.01 \leq \rho \leq 0.99 \) in Figure 1.

B Optimal Algorithms in Practical Settings

In this section, we show that our main result, greedy is optimal for multiple budgets, generalizes to the constant rounds Map-Reduce algorithm in distributed setting [LVIS], and the logarithmic rounds parallel algorithm [BRS19]. We sketch the main ideas behind these algorithms and point out how to adapt them to Observation 3.5. Before that, we provide the proof of Observation 3.5

Observation B.1 (Observation 3.5 restated). For any perturbation factors \( 0 < \rho_1 < \rho_2 < \cdots < \rho_m \), there exists a sufficiently large \( k \) that grows with the size of instance such that given \( m \) budgets \( k_1 = \rho_1 \cdot k, \ldots, k_m = \rho_m \cdot k \), the optimality described in Theorem 3.2 actually holds for a general class of algorithms such that:

- Given budget \( k_i \), the algorithm \( A \) runs in \( T \) rounds (\( T \) is sufficiently large), of which each round selects about \( k_i/T \) elements.
For any $\epsilon > 0$, it holds for all $t \in [T]$, for all $j \in [m]$, that $f(X^A_{tk_i/T} - f(X^A_{(t-1)k_i/T})) \geq ((1 - \epsilon)\rho_j / (\rho_j T)) \cdot f(O_{k_j}) - f(X^A_{tk_i/T}))$, where $X^A_s$ is the $s$-th element chosen by $A$.

**Proof.** We let $\hat{f}(X^A_{tk_i/T})$ denote the lower bound estimate of $f(X^A_{tk_i/T})$ that we get by iteratively applying the best greedy guarantees until the $l$-th iteration ($\hat{f}(X^A_l)$ is defined similarly). Note that the second property in the observation is similar to the performance guarantee of greedy algorithm with respect to each $O_{k_j}$. The only difference is that with respect to any $O_{k_j}$, in average, every element selected in round $t$ of $A$ has the same guarantee ($f(O_{k_j}) - \hat{f}(X^A_{tk_i/T})) / k_j$ (ignore the $1 - \epsilon$ factor), while for each $s \leq k_i / T$, the $((t - 1)k_i / T + s)$-th element selected by standard greedy has guarantee ($f(O_{k_j}) - \hat{f}(X^A_{tk_i/T})) / k_j$. However, we can show that this difference between the two guarantees can be ignored. First, observe that $\hat{f}(X^A_{tk_i/T}) \leq \hat{f}(X^A_{tk_i/T})$ for all $t \leq T$, because greedy has better choices of guarantees than $A$. Moreover, notice that for all $t$, $((t - 1)k_i / T + s)$-th element selected by $A$ has the same guarantee $f(O_{k_j}) - \hat{f}(X^A_{tk_i/T})) / k_j$, and the difference between $f(O_{k_j}) - \hat{f}(X^A_{(t-1)k_i/T})) / k_j$ and $f(O_{k_j}) - \hat{f}(X^A_{tk_i/T})) / k_j$ for any $j$ is upper bounded by $\hat{f}(X^A_{tk_i/T}) - \hat{f}(X^A_{tk_i/T})) / k_1$, which in turn is upper bounded by $\epsilon_t := (\hat{f}(X^A_{tk_i/T}) - \hat{f}(X^A_{tk_i/T})) / k_1$. Furthermore, if we iteratively apply the guarantee for each $t \leq T$ and $s \leq k_i / T$ for $A$ as follows

$$\hat{f}(X^A_{tk_i/T+s}) \geq \hat{f}(X^A_{(t-1)k_i/T+s-1}) - \hat{f}(X^A_{tk_i/T}) / k_j + f(O_{k_j}) / k_j$$

$$= \hat{f}(X^A_{(t-1)k_i/T+s-1}) - \hat{f}(X^A_{(t-1)k_i/T+s-1}) / k_j + f(O_{k_j}) / k_j - \epsilon_t,$$

where for each $t$ and $s$, $j$ is chosen to be same as the best choice of $j$ for greedy in this iteration, then by an inductive argument (base case is $\hat{f}(X^A_{tk_i/T}) = \hat{f}(X^A_{tk_i/T}))$, we have that $\hat{f}(X^A_{(t-1)k_i/T+s}) - \hat{f}(X^A_{(t-1)k_i/T+s}) \leq \sum_{r=1}^{t-1}(k_i / T)\epsilon_r + s \cdot \epsilon_t$ for each $t \leq T$ and $s \leq k_i / T$. By a telescoping sum, $\sum_{t=1}^{T}(k_i / T)\epsilon_t \leq (f(X^A_{k_i}) - f(\emptyset))/\rho_i T)$, which is negligible if $T$ is sufficiently large. Therefore, the final performance guarantee of $A$ is approximately equal to the final greedy guarantee. \hfill \Box

**Map-Reduce algorithm.** Suppose the budget is $k_i$. The setup is that there are $\sqrt{n / k_i}$ machines and a central machine, each with memory $O(\sqrt{n k_i})$. The algorithm has $t = O(1)$ Map-Reduce rounds and maintains a solution set $G$, which is empty initially. At the $l$-th round, the algorithm sets a threshold $\frac{1}{T}(1 - \frac{1}{T})f(O_{k_i})$ and wants to add $\frac{1}{T}$ elements to $G$ (actually, it might differ from this amount, but this is fine as we will explain later in this paragraph), each with marginal gain above the threshold. To achieve this, each machine from its storage selects a candidate set consisting of the elements that have marginal gains above the threshold with respect to $G$ ($G$ is not updated) and sends the candidates to the central machine, and then the central machine enumerates all the candidates and adds the element to $G$ if it has marginal gain above the threshold with respect to the latest $G$. The chosen threshold is actually the greedy guarantee of marginal gain when the cumulative utility reaches $(1 - (1 - \frac{1}{T})f(O_{k_i})$. Hence, during the enumeration procedure on the central machine, either it successfully selects $\frac{k_i}{T}$ elements with marginal contribution above $\frac{1}{T}(1 - \frac{1}{T})f(O_{k_i})$, or there is no such element left, in which case the cumulative utility should already reach $(1 - (1 - \frac{1}{T})f(O_{k_i})$. In either case, we will achieve roughly $(1 - (1 - \frac{1}{T})f(O_{k_i})$ at the end of $l$-th round, and the final $1 - 1/e$ approximation ratio follows by standard greedy analysis. Two issues remain—first, we do not know $f(O_{k_i})$, which can be fixed by standard ”guessing optimal value” trick, second, we need to bound the memory usage. For the ordinary machines, we can simply randomly partition the ground set, and for the central machine, this can be fixed as follows: at the beginning of each round, the central machine samples a random set $S$ of size $4 \sqrt{n k_i}$ and
sequentially adds the elements from $S$ to $G$ if the element has marginal gain above the threshold with respect to the latest $G$; If this procedure ends up selecting at least $k_i$ elements, the algorithm can stop, otherwise it continues as before. Using a martingale argument, it can be shown that if there are many (more than $\sqrt{n k_i}$) candidate elements chosen by the ordinary machines, then with high probability, the central machine should have already chosen at least $k_i$ elements in the above procedure.

In order to apply Observation 3.5, we need the greedy guarantees with respect to all $O_{k_j}$'s. To this end, we can guess $f(O_{k_j})$'s rather than just $f(O_{k_i})$, and moreover, we set the threshold at the $l$-th round as the largest of $\frac{1}{k_j} (f(O_{k_j}) - \hat{f}(X_{k_i,l/t}))$ for all $j$, where $\hat{f}(X_{k_i,l/t})$ is the lower bound estimate of $f(X_{k_i,l/t})$ we get by applying best guarantee for each iteration of greedy. Finally, if we want the algorithm to be oblivious to the budget distribution, we can simply discretize the domain of perturbed budgets and apply above-mentioned trick for the budgets in the discretized domain.

**Parallel algorithm.** Suppose the budget is $k_i$. The parallel algorithm is similar to the MapReduce algorithm. It runs in $t$ rounds and maintains a solution set $G$. In each round, it adds to $G$ a set of $\frac{k_i}{t}$ elements with total marginal gain above the threshold $\frac{1-\epsilon}{t} (f(O_{k_i}) - f(G))$. Specifically, the algorithm first selects a candidate set $X$ by iteratively discarding from $X$ all the elements that have marginal contribution roughly below $\frac{1-\epsilon}{k_i} (f(O_{k_i}) - f(G))$ with respect to the union between $G$ and a random subset of size $\frac{k_i}{t}$ of $X$ until the expected total marginal gain of a random size-$\frac{k_i}{t}$ subset of $X$ achieves the threshold, and then it samples a set of size $\frac{k_i}{t}$ from $X$ and adds it to $G$. Each round terminates quickly because if the expected total marginal gain of a random subset is low in one iteration, then there should be many elements with low marginal contribution, and they will be discarded together in this iteration.

In order to apply our analysis, we can adapt the algorithm similarly to what we did for the MapReduce algorithm, i.e., we set the threshold as the largest of $\frac{1-\epsilon}{k_j} (f(O_{k_j}) - f(G))$ for all $j$. 