Reduction of plane-radial freezing - thawing problems to plane-parallel ones

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Abstract. Construction in northern conditions is complicated by the processes of seasonal freezing and thawing of soils. This factor is of particular importance in the area of permafrost distribution. The paper deals with the problems that arise when laying communication networks. Any engineering solution must be justified by a heat engineering calculation. In one case, it is necessary to exclude the thawing of water or sewer networks, in the other case – to prevent the destructive effect of the heating main on the permafrost soil. The most common mathematical model of the heat transfer process in the presence of phase transitions is the Stefan problem. When calculating the processes of freezing-thawing near the pipes of communications, it is rational to set the problem in cylindrical coordinates, using the property of plane-radial symmetry. But the Stefan problem has an exact solution only in Cartesian coordinates, for the case of plane-parallel symmetry. For this case, many approximate dependencies are also obtained. The paper presents a method that allows using the results of solving the plane-parallel Stefan problem to solve the plane-radial problem with the same values of the input parameters. Approximate values of the coordinate of the phase transition front as a function of time for the plane-parallel and plane-radial areas are obtained using the method of sequential change of stationary states. A one-to-one relationship is established between the obtained values. If there is a solution obtained by any method for a plane-parallel area, and then using this dependence, it can be applied to a plane-radial area. This technique is suitable not only for the Stefan problem, but also for many nonlinear heat conduction problems. The results given in the paper are of practical importance: using the above method, almost any solution obtained for one type of symmetry directly extends to other types of symmetry.

1. Introduction

Laying of communications in northern conditions has a number of features. On the one hand, it is necessary to exclude thawing of water and sewer networks, and on the other hand, to prevent the destructive effect of the heating main on the frozen ground. The choice of the laying scheme in each specific case should be justified by economic and technical calculations. If we are talking about the cryolite zone, then the underground laying requires, first of all, a forecast of the thermal regime of the soil. The main purpose of calculations is usually to establish the boundaries of areas with different
aggregate state of the moisture contained in the soil. Often it is also necessary to determine the coordinates of the front of the phase transition of water in pipes.

The most commonly accepted mathematical model of the heat transfer process in the presence of phase transitions is the Stefan problem. In many cases, a perfectly valid approximation is the one-dimensional one-phase Stefan problem in a plane-parallel or plane-radial formulation. The most studied problem is the Stefan problem in Cartesian coordinates [1, 2].

The one-dimensional plane-parallel Stefan problem under a constant boundary condition has an exact solution, and for other cases a large number of approximate methods have been developed that give good results [3, 4, 5]. For the plane-radial case, an exact solution is not obtained, and many approximate methods [6] are either not suitable at all, or significantly lose their effectiveness [7, 8].

It should be noted that during the construction and operation of particularly critical structures, individual modeling of the temperature field is carried out with the use of modern computing tools [9].

In this paper, we present a method that allows using the results of solving the plane-parallel Stefan problem to solve the plane-radial problem for the same values of the input parameters.

2. Materials and methods

The one-dimensional one-phase Stefan problem can be formulated as follows:

\[ \frac{\partial t_i}{\partial \tau} = \frac{\lambda}{c \, \chi^i} \frac{\partial}{\partial x} \left( x \frac{\partial t_i}{\partial x} \right), \quad x \in [a, b], \quad \tau > 0. \]  

(1)

Condition at the mobile boundary \( \eta_i \) of the phase transition:

\[ -\lambda \chi^i \frac{\partial t_i}{\partial x} = \kappa \frac{d \eta_i}{d \tau} \]  

(2)

Initial condition:

\[ t_i(x, 0) = 0 \]  

(3)

Conditions on stationary borders:

at \( x = b \)

\[ t_i(b, \tau) = 0, \]  

(4)

at \( x = a \)

\[ -a \frac{\partial t_i}{\partial x} + ht_i = hf(\tau), \quad i = 0; 1, \]  

(5)

where \( x \) – spatial coordinate; \( \tau \) – time; \( \eta_i \) – coordinate of the phase transition boundary; \( c \) – heat capacity; \( \lambda \) – heat capacity coefficient; \( \kappa \) – latent heat of the phase transition.

The case \( i = 0 \) corresponds to a plane-parallel problem, \( i = 1 \) – a plane-radial one. The conditions of the 1st kind in (5) are obtained at \( h \to \infty \).

Our task is to establish an approximate dependence \( \varphi(\eta_0, \eta_1) = 0 \), from which the known \( \eta_0 \) can \( \eta_1 \) be calculated. By the method of successive changes of stationary states we can find approximate values \( s \approx \eta_0, R \approx \eta_1 \). For the values \( s, R \) we can obtain an exact ratio

\[ \varphi(s, R) \]  

(6)
The established form of dependence (6) is then used for an approximate calculation \( \eta_1 \), if \( \eta_0 \) is found, for example, from the exact solution of the Stefan problem.

The method is described using the example of the following two tasks.

2.1. The problem of thawing the soil around the pipeline laid in the frozen ground.

This process can be described in the first approximation by the single-phase planar-radial Stefan problem (1) – (5) (i=1) at

\[
\begin{align*}
f(\tau) &= t_c = \text{const}, \quad a = r, \quad b \to \infty, \\
\end{align*}
\]

(7)

where \( t_c \) – temperature of the liquid (gas) in the pipe; \( r \) – the outer radius of the pipe.

The main purpose of the calculations is to determine the dependence \( \eta_1(\tau) \). Following the method of sequential change of stationary states, we assume that for any value of the front coordinate \( R \), the specific heat flow from the pipe to the ground (neglecting the thermal conductivity in the pipe wall) is equal to [10]:

\[
q = \frac{2\pi t_\infty}{1 + \frac{1}{\alpha r} \ln \frac{R}{r}},
\]

(8)

where \( \alpha \) – heat transfer coefficient.

With the assumptions made about the quasi-stationary nature of the process, all this heat is spent on thawing the soil, so we can make the following differential equation

\[
\frac{2\pi t_\infty}{1 + \frac{1}{\alpha r} \ln \frac{R}{r}} = 2\pi R \kappa \frac{dR}{d\tau}, \quad \tau \geq 0
\]

(9)

at the initial condition \( R(0) = r \).

We integrate equation (9) and get

\[
\tau \frac{4\lambda t_\infty}{r^2 \kappa} = 2\rho^2 \ln \rho - k \rho^2 + k,
\]

(10)

where

\[
\rho = \frac{R}{r}, \quad k = 1 - \frac{2\lambda}{\alpha r} = 1 - \frac{2}{\kappa h},
\]

(11-12)

The solution of the plane-parallel problem in the quasi-stationary approximation (1) – (5) at \( i = 0, a = r, b \to \infty \) the same time it gives an expression for the coordinate \( s \):

\[
\tau \frac{2\lambda t_\infty}{\kappa} = r(1 - k)(s - r) + (s - r)^2.
\]

(13)

We find the value \( \tau \) from (13), substitute it in (10) and get the desired ratio

\[
2 \frac{r(1 - k)(s - r) + (s - r)^2}{r^2} = 2\rho^2 \ln \rho - k \rho^2 + k.
\]

(14)

If the exact solution of the plane-parallel problem is now obtained, then substituting in (14) instead \( \eta_0 \) of \( s \), we get an approximate value \( \eta_1 \approx R \).
2.2. The problem of freezing water in the pipe.
It is solved similarly to the first problem.

The expression linking the values $R$ and $s$ has the form

$$2r(1-k_1)(s-r)+(s-r)^2 = 2\rho^2 \ln \rho - k_1 \rho^2 + k_1,$$

where

$$k_1 = 1+2/h.$$  \hspace{1cm} (16)

Formula (15) should be used in the same way as formula (14).

3. Results and discussions

3.1. The problem of thawing the soil around the pipeline laid in the frozen ground.
To illustrate the method, the values $\eta_0$, $\eta_1$ at $r = 1$, $b \to \infty$ (all values are dimensionless) are determined with the same accuracy. According to the formula (14), the value $R$ at $s = \eta_0$, is calculated. In Table 1, the values $\eta_1$ and $R$ at $h=1$ and $h \to \infty$, are given and it is seen that the error does not exceed 4% (Figure 1,2). Taking into account that the solution of the problem (1) – (5) in any case is approximate, we can recommend using the formula (14) in technical calculations.

Table 1. Stefan’s external problem.

| $\tau$ | $\eta_1$ | $R$ | $\eta_1$ | $R$ |
|--------|----------|-----|----------|-----|
| 1.85   | 1.80     | 4.78| 2.75     | 2.68|
| 2.50   | 2.46     | 10.79| 3.47     | 3.41|
| 3.80   | 3.68     | 19.20| 4.20     | 4.10|
| 5.00   | 4.83     | 30.07| 4.90     | 4.80|
| 5.58   | 5.40     | 43.30| 5.60     | 5.45|
| 6.71   | 6.50     | 59.00| 6.25     | 6.09|
| 7.27   | 7.00     | 77.00| 6.87     | 6.70|
| 7.81   | 7.54     | 97.00| 7.50     | 7.34|
| 8.90   | 8.60     | 120.00| 8.15    | 7.95|
| 9.95   | 9.60     | 146.00| 8.80    | 8.55|

Figure 1. Stefan’s external problem (at $h=1$).

Figure 2. Stefan’s external problem (at $h \to \infty$).
3.2. The problem of thawing the soil around the pipeline laid in the frozen ground.

To illustrate the method, the values \( \eta_0, \eta_1 \) at \( r = 1 \) (all values are dimensionless) are determined with the same accuracy. The formula (15) is used to calculate the value of \( R \) at \( s = \eta_0 \). Table 2 shows the values of \( \eta_1 \) and \( R \) at \( h=1 \) and \( h \to \infty \). It is showed that the error does not exceed 6% (Figure 3, 4). Taking into account that the solution of the problem (1) – (5) in any case is approximate, we can recommend using the formula (15) in technical calculations.

| \( h=1 \) | \( h \to \infty \) |
|---|---|
| \( \tau \) | \( \eta_1 \) | \( R \) | \( \tau \) | \( \eta_1 \) | \( R \) |
| 0.10 | 0.95 | 0.95 | 0.003 | 0.95 | 0.95 |
| 0.22 | 0.89 | 0.89 | 0.013 | 0.90 | 0.90 |
| 0.35 | 0.84 | 0.84 | 0.030 | 0.85 | 0.85 |
| 0.48 | 0.78 | 0.78 | 0.052 | 0.79 | 0.79 |
| 0.62 | 0.72 | 0.71 | 0.081 | 0.74 | 0.74 |
| 0.76 | 0.65 | 0.64 | 0.115 | 0.68 | 0.68 |
| 0.91 | 0.58 | 0.57 | 0.203 | 0.56 | 0.56 |
| 1.08 | 0.51 | 0.50 | 0.256 | 0.51 | 0.50 |
| 1.24 | 0.43 | 0.41 | 0.315 | 0.44 | 0.43 |
| 1.41 | 0.31 | 0.33 | 0.379 | 0.38 | 0.36 |

4. Summary

Formulas in the form of (14) and (15) reflect the actual physical analogy between the processes of heat transfer with a phase transition at different types of symmetry. In our opinion, the solution of the internal Stefan problem by the proposed method is of particular interest, as the arsenal of solutions for internal problems is much poorer than for external ones.

Obviously, expressions like (14), (15) can be obtained under more complex boundary conditions. When calculating two-phase Stefan problems in a bounded domain, they can use the formula (15), as in general, in such problems, the second phase does not have a significant effect. In the semi-infinite case, with significant deviations of the temperature of the second phase from the temperature of the phase transition, it cannot be neglected. Therefore, when deriving the ratio between \( R \) and \( s \), it is recommended to use the second Leybenzon method [11].

The application value of the obtained results cannot be doubted. Using this method, almost any solution obtained for one of the types of symmetry directly extends to other types of symmetry. They can, for example, link the solution of an external problem to the solution of an internal one. If
necessary, formulas of the form (14) and (15) can also be obtained for problems with central symmetry (problem (1)-(5) at i=2).

5. References
[1] Tikhanov A N and Samarsky A A 2004 Equations of mathematical physics (Moscow: Moscow State University, Science)
[2] Meyrmanov A M 1986 Stefan's problem (Novosibirsk: Science)
[3] Karslow G S and Jaeger D 1964 Thermal conductivity of solids (Moscow: Science)
[4] Sigunov Yu A 2009 Methods for solving the classical Stefan problem (Surgut: Surgut State Pedagogical University)
[5] Johansson B T, Lesnic D and Reeve T 2011 A method of fundamental solutions for the one-dimensional inverse Stefan problem Applied Mathematical Modelling 35(9) 4367-4378
[6] Aksenov B G, Karyakin Y E and Karyakina S V 2018 A Multifront Problem of Freezing-Thawing Moist Soil IOP Conf. Series: Materials Science and Engineering 463(2) 022024 doi:10.1088/1757-899X/463/2/022024
[7] Aksenov B G, Fomina V V and Bogunova A A 2020 Freezing and thawing of wet soil from the surface and around the underground pipeline IOP Conf. Series: Materials Science and Engineering 753 042002 doi:10.1088/1757-899X/753/4/042002
[8] Fomina V V, Aksenov B G, Stepanov O A, Mironov V V and Abrosimova S A 2020 Solving problems of soil freezing-thawing for heat and gas supply systems The Eurasian Scientific Journal 5(12) https://esj.today/PDF/14SAVN520.pdf
[9] Anikin G V, Grigoriev B V, Spasennikova K A and Yanbikova Y F 2017 The calculation of temperature field in soils under the base of oilreservoir at the Varandey oil field MATEC Web of Conferences 106 02005 doi: 10.1051/matecconf/20171060
[10] Isachenko V P, Osipova V A and Sukomel A S 2014 Heat transfer: textbook for universities (Moscow: TPH “Aris”)
[11] Dostavalov B N and Kudryavtsev V A 1967 General permafrost study: a textbook for students of geological specialties (Moscow: Moscow State University)