Hydrodynamic Theory of Granular Solids: 
Permanent, Transient and Granular Elasticity

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Abstract

Although fully elastic when static, granular media become transiently elastic when being slowly sheared – during which both the elastic energy and stress relax. Starting from this observation, we cogently derive the framework for granular hydrodynamics, a set of differential equations consistent with general principles of physics, especially reversible and irreversible thermodynamics. In addition, an expression for the granular elastic energy is reviewed and further discussed.

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I. INTRODUCTION

In granular media, although the grains roll and slide, in addition to being compressed and sheared, only the latter, the deformation of the grains, leads to reversible energy storage that sustains a static, elastic stress, while rolling and sliding heats up the system. The granular strain field $\varepsilon_{ij}$, therefore, has two contributions, an elastic one $u_{ij}$ accounting for deformation of the grains, and a plastic one $p_{ij}$ for the rest, where $\varepsilon_{ij} = u_{ij} + p_{ij}$. The elastic energy $w(u_{ij})$ is a function of $u_{ij}$, not of $\varepsilon_{ij}$, and the elastic contribution to the stress $\sigma_{ij}$ is given as $\pi_{ij}(u_{ij}) \equiv - \partial w / \partial u_{ij}$. With $\sigma_{ij} = \pi_{ij}$ in statics, stress balance $\nabla_j \sigma_{ij} = 0$ may be closed with $\pi_{ij} = \pi_{ij}(u_{ij})$ and uniquely determined employing appropriate boundary conditions [1, 2]. Because the plastic part of the strain needed for arriving at a given stress state is quite irrelevant for its determination, one may with certain justification consider static granular media, say a sand pile at rest, as fully elastic.

If this sand pile is perturbed by periodic tapping, circumstances change qualitatively. Its conic form will then degrade until the surface becomes flat. This is because part of the grains in the pile loose contact with each other temporarily, during which their deformation decreases. This implies a relaxing elastic strain $u_{ij}$, and correspondingly, smaller elastic energy $w(u_{ij})$ and static stress $\pi_{ij}(u_{ij})$. Since the sand pile is no longer able to sustain static stresses, it is now a transiently elastic system, same as polymers – though the respective microscopic mechanisms are of course very different: temporary unjamming and rearrangement of the grains versus slow disentanglement of polymer strands. Note that flattening a sand pile implies sizable granular rearrangement, requiring a considerable portion of plastic strain $p_{ij}$.

Quantifying the random motion of the grains as granular temperature $T_g$, we may take the relaxation time $\tau$ of the elastic strain $u_{ij}$ as a function of $T_g$, with $\tau(T_g) \to \infty$ for $T_g \to 0$. For vanishing $T_g$, there is no strain relaxation, the deformation of the grains are maintained, the sand pile keeps its conic shape, and the system is elastic. For finite $T_g$, the elasticity turns transient, with $u_{ij}$, $\pi_{ij}(u_{ij})$ and $w(u_{ij})$ relaxing.

When granular media are being slowly sheared, circumstances are similar. In addition to moving with the large scale velocity $v_i$, the grains also move and slip in deviation of it. This allows temporary, partial unjamming, and implies a finite $T_g$, both again lead to transient elasticity. Since $T_g$ is not always an externally imposed parameter, as with tapping, but
frequently internally produced, especially by shear flows, it is an independent variable of the granular hydrodynamic theory, to be accounted for by its own equation of motion. More specifically, the production of $T_g$ by shear flows should have great similarities to viscous heat production in normal fluids.

Granular media has different phases that, in dependence of the grain’s ratio of deformation to kinetic energy, may loosely be referred to as gaseous, liquid and solid. Moving fast and being free most of the time, the grains in the gaseous phase have much kinetic, but next to none elastic, energy $[3]$. In the denser liquid phase, say in chute flows, there is less kinetic energy, more deformation, and a rich rheology that has been scrutinized recently $[4]$. In granular statics, with the grains deformed but stationary, the energy is all elastic. This state is legitimately referred to as solid because static shear stresses are sustained. If granular solid is slowly sheared, the predominant part of the energy remains elastic. As discussed, the system is transiently elastic, or quasi-solid. In this paper, we focus on the last two cases, and for simplicity refer to both as the solid granular phase.

The transition between permanent and transient elasticity is a crucial key to understanding granular solids. And remarkably, it is as input quite sufficient for a formal and cogent derivation of the framework for granular solid hydrodynamics – if one takes careful notice of all general principles of physics, especially symmetry and thermodynamic considerations. This is the first part of the present paper. The second part deals with an concrete expression for the granular elastic energy, how this expression is supported by extensive experimental data from granular statics. This is important because general principles only confines the structure of the hydrodynamic theory – they yield a framework into which many different theories fit. The three sets of differential equations given below need the input of specific expressions for the thermodynamic energy and the transport coefficients. Only when their functional dependence on the thermodynamic variables is given, do the theories attain predictive power.

In the following, we first recall the hydrodynamic theory of permanent and transient elasticity, in § IA and § IB then merge both to form granular hydrodynamics, in § IC All equations in these three subsections are valid irrespective what form the energy $w$ has. A specific energy density suitable for granular media is then reviewed and further discussed in § III

In an accompanying paper $[5]$, we compare hypoplasticity $[6]$, a state of the art engineering
model on the behavior of granular solids, with granular solid hydrodynamics as derived here, and specified using the elastic energy of §III.

II. ELASTICITY THEORY

A. Permanent Elasticity

The conserved, thermodynamic energy density \( w \) of solids is a function of the symmetric strain field \( u_{ij} = u_{ji} \), and of the densities of entropy \( s \), mass \( \rho \), momentum \( g \). So we write (neglecting gravity)

\[
dw = Tds + \mu dg_i + v_i dg_i - \pi_{ij} du_{ij},
\]

(1)
denoting \( T(s, \rho, u_{ij}) \equiv \partial w/\partial s \), \( \mu(s, \rho, g_i, u_{ij}) \equiv \partial w/\partial \rho \), \( v_i \equiv \partial w/\partial g_i = g_i/\rho \), \( \pi_{ij}(s, \rho, u_{ij}) \equiv -\partial w/\partial u_{ij} \). The equations of motion for the energy and its variables are

\[
\frac{\partial}{\partial t} w + \nabla Q_i = 0,
\]

(2)
\[
\frac{\partial}{\partial t} s + \nabla f_i = \frac{R}{T},
\]

(3)
\[
\frac{\partial}{\partial t} \rho + \nabla j_i = 0,
\]

(4)
\[
\frac{\partial}{\partial t} g_i + \nabla \sigma_{ij} = 0,
\]

(5)
\[
\frac{\partial}{\partial t} u_{ij} - v_{ij} + \left[ \frac{1}{2} \nabla_j v_j + u_{ik} \nabla_j v_k + i \leftrightarrow j \right] = 0,
\]

(6)
where \( \frac{\partial}{\partial t} = \frac{\partial}{\partial t} + v_k \nabla_k \) and \( v_{ij} \equiv \frac{1}{2} (\nabla_i v_j + \nabla_j v_i) \). Expressing conservation laws and entropy production, the first four equations are quite general and shared by all hydrodynamic theories. Alone, they describe normal fluids and represent the simplest hydrodynamic theory. The fifth equation is characteristic of elastic systems, especially ones that break the translational symmetry spontaneously. (More on why Eq (4) must have the above form is given in [7].) Inserting Eqs (2-4) into the temporal derivative of Eq (1),

\[
\frac{\partial}{\partial t} w = T \frac{\partial}{\partial t} s + \mu \frac{\partial}{\partial t} \rho + v_i \frac{\partial}{\partial t} g_i - \pi_{ij} \frac{\partial}{\partial t} u_{ij},
\]

(7)
and introducing the notations: \( f_i^D, \sigma_{ij}^D \), taking them to be given as

\[
f_i \equiv sv_i - f_i^D,
\]

(8)
\[
\sigma_{ij} \equiv \pi_{ij} - \pi_{ik} u_{jk} - \pi_{jk} u_{ik}
\]
\[
+ (Ts + v_i g_i + \mu \rho + g_i v_j - w) - \sigma_{ij}^D,
\]

(9)
we obtain

\[
\nabla Q_i = \nabla_i (Tf_i + \mu j_i + v_j \sigma_{ij} - y_j \pi_{ij})
\]

(10)
\[
+ f_i^D \nabla_i T + \sigma_{ij}^D u_{ij} + y_i \nabla_j \pi_{ij} - R.
\]
Clearly, one can write the left hand side of Eq (5) as the divergence of something, plus something else that vanishes in equilibrium (because the so-called thermodynamic forces, $\nabla_i T, v_{ij}$ and $\nabla_j \pi_{ij}$ do). Therefore, an inviting possibility is to identify the first with the energy flux $Q_i$, and the second with the entropy production $R$, a quantity that also vanishes in equilibrium,

$$Q_i = Tf_i + \mu j_i + v_j \sigma_{ij} - y_j \pi_{ij}, \quad (9)$$

$$R = f_i^D \nabla_i T + \sigma_{ij}^D v_{ij} + y_i \nabla_j \pi_{ij}. \quad (10)$$

This identification is in fact unique. It is easy to verify that, as long as the energy $w$ remains general, unspecified, there is no other way to write the left hand side of Eq (5) as the sum of a divergence and an expression that vanishes in equilibrium.

Taking in Eq (10) ($\nabla_i T, v_{ij}, \nabla_j \pi_{ij}$) as the thermodynamic forces, ($f_i^D, \sigma_{ij}^D, y_i$) as the fluxes, and forming each into a 12-component vector, $\vec{Y}$ and $\vec{Z}$, the Onsager force-flux relation gives their linear connection as,

$$\vec{Z} = \hat{c} \cdot \vec{Y}, \quad (11)$$

where $\hat{c}$ is the transport matrix, with diagonal elements that are positive, and off-diagonal ones that satisfy the Onsager reciprocity relation. The simplest example for $\hat{c}$ has only diagonal elements, all positive scalars,

$$f_i^D = \kappa \nabla_i T, \quad (12)$$

$$\sigma_{ij}^D = \zeta v_{\ell \ell} \delta_{ij} + \eta v^0_{ij}, \quad (13)$$

$$y_i = \beta^P \nabla_j \pi_{ij}. \quad (14)$$

Accounting for heat conduction and viscous stress, the first two equations are shared by all hydrodynamic theory. (The superscript $^0$, here and below, denotes the traceless part of a tensor, eg. $v^0_{ij} \equiv v_{ij} - \frac{1}{3} \delta_{ij} v_{\ell \ell}$. ) The third accounts for permeation and defect motion, and is specific to elastic media [8], see section II A 1.

All elements of the matrix $\hat{c}$, usually referred to as transport coefficients, are functions of the thermodynamic variables, $s, \rho, u_{ij}$, or alternatively, of the conjugate variables, $T, \mu, \pi_{ij}$. In the generally accepted and above employed linear version of the Onsager relation, they do not depend on thermodynamic forces, $\nabla_i T, v_{ij}, \nabla_j \pi_{ij}$. So we may take the coefficients $\kappa, \eta, \zeta$ and $\beta^P$ to depend on the temperature, the pressure, and scalar combinations of the stress, such as $\pi_{\ell \ell}$ and $\pi^2_s \equiv \pi^0_{ij} \pi^0_{ij}$. 

5
1. **Solid Creep Motion**

Enforcing a steady velocity at the surface of granular solid, the velocity field is observed to penetrate rather deep into the bulk of the granular medium, with a magnitude that decays exponentially with depth \[9\]. The usual collective modes of velocities in hydrodynamic theories of elastic media are of course such that they reduce to a constant velocity when stationary (sound), or one that varies linearly in space (shear diffusion). But there is also a less-known one that decays exponentially, a consequence of Eq \((14)\) and the less studied permeation coefficient \(\beta^P\). We shall refer to this mode as “solid creep motion.”

Linearized with respect to velocity, Eq \((14)\) reduces, for the stationary case \(\partial u_{ij}/\partial t = 0\), to

\[
v_{ij} = \frac{1}{2} \beta^P \nabla_k (\nabla_i \pi_{jk} + \nabla_j \pi_{ik}),
\]

implying that mass and shear flows are possible without any changes in the elastic strain field, or equivalently, in the elastic stress and elastic energy. Similarly, momentum conservation, or Eq \((3)\), linearized and in steady flow, \(\partial(\rho v_i)/\partial t = 0\), reduces to

\[
\nabla_j (D\delta_{ij} + \pi_{ij} - \eta v_{ij}^0) = 0
\]

(16)

(where \(D\delta_{ij}\) stands for the diagonal terms that do not concern us here). Now, consider a half space \(y > 0\) filled with solid, which has its surface at \(y = 0\), moving with a given velocity along \(x\). Permitting only a \(y\)-dependence in this one-dimensional geometry, we have

\[
v_{xy} = \frac{1}{2} \beta^P \nabla^2 y \pi_{xy}, \quad \nabla_y (\pi_{xy} - \eta v_{xy}^0) = 0.
\]

(17)

These two equations clearly imply exponentially decaying velocity \(v_x\) and change of the elastic stress \(\delta \pi_{xy}\),

\[
v_x, \delta \pi_{xy} \sim \exp \frac{-y}{\sqrt{\frac{1}{2} \eta \beta^P}}.
\]

(18)

In granular medium, this behavior will be modified, because the elasticity there is transient rather than permanent. But should solid creep motion retains its qualitative behavior under certain circumstances, Eq \((18)\) would constitute a natural explanation of granular creep flow.
B. Transient Elasticity

Although the equations of the last section are fairly general and account for all kinds of elasticity, linear as well as nonlinear, they do exclude transient elasticity, such as realized in polymers. In these, elasticity arises from entanglement of polymer strands, which are stretched and sheared, if not given enough time to disentangle. But if given enough time, the deformation, with it also the associated energy and stress, relax. So the system is to be accounted for by a set of equations which reduce to those of the last section for small time spans, but allow the deformation $u_{ij}$ to relax for longer time spans.

The independent variables remain the same, so do the conservation laws. So Eqs (1,2,3) are unchanged, but Eq (4) is modified to allow for a relaxation term $X_{ij}$

$$\frac{d}{dt} u_{ij} - v_{ij} + \left[ \frac{1}{2} \nabla_i y_{ij} + u_{ik} \nabla_j v_k + i \leftrightarrow j \right] = X_{ij}. \quad (19)$$

The same calculation of Eq (5), with the same notation of Eq (6,7), then leads to the same energy flux $Q_i$, but a modified entropy production,

$$R = f^D_i \nabla_i T + \sigma_{ij}^D v_{ij} + y_i \nabla_j \pi_{ij} + X_{ij} \pi_{ij}. \quad (20)$$

This implies $\pi_{ij}$ is now not only a conjugate variable, but also a thermodynamic force, increasing the dimension of the 12-component vector $\vec{Y}$ in Eq (11) by another 6 components. Similarly, the vector $\vec{Z}$ is also increased by the 6 components of $X_{ij}$, and $\hat{c}$ is now a 18 x 18-matrix. Other from that, Eq (11) still holds. The simplest, diagonal and scalar example is again given by Eqs (12, 13, 14), in addition to

$$X_{ij} = \beta \pi_{ij}^0 + \beta_1 \pi_{\ell \ell} \delta_{ij}, \quad (21)$$

a term that permits $u_{ij}$ to relax, as long as $\pi_{ij}$ is nonzero.

As discussed in the last paragraph of § II A, the transport coefficients $\beta, \beta_1$ are functions of the thermodynamic variable $u_{ij}$, or equivalently, of $\pi_{ij} = \pi_{ij}(u_{ij})$. This remains true even though $\pi_{ij}$ is now also part of $R$, Eq (20), and hence an additional thermodynamic force.

A point worth clarifying concerns the plastic strain $p_{ij}$: The total strain $\varepsilon_{ij} = u_{ij} + p_{ij}$, a purely kinematic quantity, obeys the equation

$$\frac{d}{dt} \varepsilon_{ij} + \left[ \varepsilon_{ik} \nabla_j v_k + i \leftrightarrow j \right] = v_{ij}. \quad (22)$$
So, as a result of Eq (19), the plastic strain is determined by

\[
\frac{d}{dt} p_{ij} + \left[ -\frac{1}{2} \nabla_i y_j + p_{ik} \nabla_j v_k + i \leftrightarrow j \right] = -X_{ij}.
\]  

(23)

If a transiently elastic medium is quickly and uniformly deformed, such that there is no time for relaxation, \( X_{ij} \approx 0 \), we have \( \varepsilon_{ij} = u_{ij} \) after the deformation. Holding it for a while, \( v_{ij} = 0 \), the elastic deformation \( u_{ij} \) relaxes to zero, while the plastic one \( p_{ij} \) grows accordingly, until one replaces the other completely, and we have \( p_{ij} = \varepsilon_{ij} \). The system now stays where it is, and the initial displacement is referred to as “plastic,” rather than elastic, because it does not have the tendency to return to the original position.

Essentially this set of equations, as specified in this section, was recently shown well able of accounting for the full range of polymers’ non-Newtonian behavior, including shear-thinning, elongational strain-hardening, rod climbing (the Weissenberg effect), and various empirical rules such as Cox-Merz and First Gleissle Mirror Rule [7, 10].

C. Granular Elasticity

As discussed, sand and other granular media display both elastic and transiently elastic behavior—depending on whether the granular temperature \( T_g \) vanishes or not. Including the density of granular entropy \( s_g \) as an additional, independent thermodynamic variables, the Gibbs relation of Eq (1) now reads

\[
dw = T ds + T_g ds_g + \mu d\rho + v_i dg_i - \pi_{ij} du_{ij}.
\]  

(24)

Granular temperature is not a new concept. Haff, also Jenkin and Savage [3], were probably the first to introduce it in the context of granular gas, using it to denote the average kinetic energy of the grains. Hence \( T_g \sim \epsilon_k \), where \( \epsilon_k \) is the kinetic energy density. Nowadays, this \( T_g \) is routinely used in considering granular gas and liquid [11]. Note that given this interpretation of \( T_g \), we have \( T_g = \partial \epsilon_k / \partial s_g \sim \partial T_g / \partial s_g \), and the granular entropy is uniquely determined, \( s_g \sim \ln T_g \). More recently, there is much discussion of a configurational entropy \( S_c \) in the literature. The original concept by Edwards was to approximate grains as infinitely rigid and all configurations as having identical energy [12], so \( S_c \) is a function only of the system’s volume. When relaxing the rigidity approximation, and allowing the elastic energy to vary, \( S_c \) is again a function of energy and volume \( S_c = S_c(E, V) \), and a configurational temperature is naturally given as \( T_c^{-1} = \partial S_c / \partial E \) (see [13] for a review).
In thermodynamics, the energy change $dw$ from all microscopic, implicit variables is subsumed as $Tds$, with $s$ the entropy and $T \equiv \partial w/\partial s$ its conjugate variable. From this, we divide out the kinetic energy of granular random motion, executed by the grains in deviation from the ordered, large-scale motion, denoting it as $T_gds_g$, and calling $s_g$ and $T_g \equiv \partial w/\partial s_g$ granular entropy and temperature, respectively. In other words, we consider two heat reservoirs, the first containing the energy of granular random motion, the second the rest of all microscopic degrees of freedom, especially phonons. In equilibrium, $T_g = T$, and $s_g$ is part of $s$. (In fact, we may simply forget $s_g$, since it has far less degrees of freedom.) But when the granular system is being tapped or sheared, and $T_g$ is many orders of magnitude larger than $T$, then this leaky, intermediary heat reservoir can no longer be ignored. As $s_g$ then serves as a nonhydrodynamic, macroscopically slow degree of freedom, with $T_g$ its conjugate variable.

Taking $s_g$ as the part of the entropy accounting for the granular kinetic energy, our definition is fairly close to the entropy of granular gas discussed above, as given by Haff, though its functional dependence will probably be modified, because it must be evaluated taking into consideration the effect of excluded volumes – an overwhelming one in the dense solid phase [see Eq (9) of the third of [11]]. The concept of configurational entropy, on the other hand, is closer to our second heat reservoir, the true entropy $s$, see section 6 of the first, and section 10 of the third, reference [2], for a discussion of their relationship.

The functional dependence of $s_g(T_g)$, more precisely, the equation of state $T_g = T_g(s, s_g, \rho, u_{ij})$, is given once the energy $w$ is known. Although all equations of this and the last two sections remain valid irrespective of what special expression one chooses for $w$, concrete predictions certainly depend on it. Since it appears difficult, at least at present, to evaluate $w$ microscopically, one may alternatively employ experimental data in conjunction with general considerations to narrow down its possibility. We shall examine $w$’s dependence on $u_{ij}$ and $\rho$ in the next section, but defer that on $s_g$ to a future publication.

Taking the balance equation for $s_g$, in the uniform case, as $\frac{\partial}{\partial t} s_g = R_g/T_g$, we first of all need $R_g$ to contain the term $-\gamma(T_g - T)^2$. This is because being a slow, nonhydrodynamic variable, the equation of motion for $s_g$ should have the usual relaxation form, $\frac{\partial}{\partial t} s_g = -\gamma(T_g - T)$, pushing $T_g$ towards the ambient temperature $T$. (Since any random motion of the grains implies such improbably high $T_g$, neglecting $T$ in this expression is always an excellent approximation. We shall therefore from here on always write $\frac{\partial}{\partial t} s_g = -\gamma T_g$.)
Second, with the heat bath divided into two parts, viscous heat production should fill both baths simultaneously. Therefore, we keep the term $\sigma_{ij}^{D}v_{ij}$ in $R$, with $\sigma_{ij}^{D} = \eta v_{ij}^0 + \zeta v_{\ell\ell}$, and write the analogous one, $\Sigma_{ij}^{D}v_{ij}$ into $R_g$, with $\Sigma_{ij}^{D} = \eta_g v_{ij}^0 + \zeta v_{\ell\ell}$ denoting the viscous stress contribution from exciting granular random motion. The magnitude of the four viscosities depend on microscopic details and cannot be decided on general principles. For instance, while $\eta$ is probably a small quantity compared to $\eta_g$ in dry sand, because macroscopic shear flows excite granular random motion first, $\eta$ should be quite a bit larger in wet sand: A macroscopic shear flow implies much stronger microscopic shear flows in the fluid layers between grains, and the energy dissipated in these layers should predominantly go to $s$, rather than to $s_g$ first.

Third, granular entropy production $R_g$ should have the term $\kappa_g \nabla_i T_g^2$, from an inhomogeneous granular temperature, in exact analogy to the term $\kappa \nabla_i T^2$ in $R$. So the final expression should be $R_g = \Sigma_{ij}^{D}v_{ij} + \kappa_g \nabla_i T_g^2 - \gamma T_g^2$. A direct and desirable consequence of this expression is that for stationarity, $\frac{\partial}{\partial t} s_g = R_g/T_g = 0$, and a constant $T_g$, any shear flows excite the granular temperature of $\gamma T_g^2 = \eta_g v_{ij}^0 v_{ij}^0 + \zeta v_{\ell\ell}^2$, which is (as discussed) what renders granular elasticity transient.

We do not have good reasons for ruling out a term in $R_g$ analogous to $y_i \nabla_j \pi_{ij}$, or one $\sim \nabla_i T_g^2$ in $R$. But neither is there any experimental evidence demanding their existence. So although both are allowed for the general case, they are left out here for the simplicity of display. On the other hand, a term in $R_g$ analogous to $X_{ij}\pi_{ij}$ cannot exist, because we would then have $\gamma T_g^2 = X_{ij}\pi_{ij}$ for granular statics, implying a finite $T_g$ and decaying sand piles.

Given the above consideration specifying $R_g$, we may embark on the derivation of the equations of motion for granular elasticity, in the same way as above. We start from the following equations,

\begin{equation}
\frac{\partial}{\partial t} w + \nabla_i Q_i = 0, \quad \frac{\partial}{\partial t} \rho + \nabla_i j_i = 0, \tag{25}
\end{equation}

\begin{equation}
\frac{\partial}{\partial t} s + \nabla_i f_i = R/T, \quad \frac{\partial}{\partial t} s_g + \nabla_i F_i = R_g/T_g, \tag{26}
\end{equation}

\begin{equation}
\frac{\partial}{\partial t} g_i + \nabla_j \sigma_{ij} = 0, \tag{27}
\end{equation}

\begin{equation}
\frac{\partial}{\partial t} u_{ij} - v_{ij} + \left[ \frac{1}{2} \nabla_i y_j + u_{ik} \nabla_j v_k + i \leftrightarrow j \right] = X_{ij}. \tag{28}
\end{equation}

Inserting these into Eq (24),

\begin{equation}
\frac{\partial}{\partial t} w = T \frac{\partial}{\partial t} s + T_g \frac{\partial}{\partial t} s_g + \mu \frac{\partial}{\partial t} \rho + v_i \frac{\partial}{\partial t} g_i - \pi_{ij} \frac{\partial}{\partial t} u_{ij}, \tag{29}
\end{equation}

10
using the notations

\[ f_i \equiv s v_i - f^D_i, \quad F_i \equiv s_g v_i - F^D_i, \quad (30) \]

\[ \sigma_{ij} \equiv (-w + T s + v_i g_i + \mu \rho + T_g s_g) \delta_{ij} + \pi_{ij} - \pi_{ik} u_{jk} - \pi_{jk} u_{ik} + g_i v_j - \sigma^D_{ij} = \Sigma^D_{ij}, \quad (31) \]

we obtain

\[ \nabla_i Q_i = \nabla_i (T f_i + T_g F_i + \mu j_i + v_j \sigma_{ij} - y_j \pi_{ij}) \quad (32) \]

\[ -R + f^D_i \nabla_i T + y_i \nabla_j \pi_{ij} + \sigma^D_{ij} v_{ij} + X_{ij} \pi_{ij} + \gamma T_g \]

\[ -R_g + \Sigma^D_{ij} v_{ij} + F^D_i \nabla_i T_g - \gamma T_g^2 \]

and deduce

\[ Q_i = T f_i + T_g F_i + \mu j_i + v_j \sigma_{ij} - y_j \pi_{ij}, \quad (33) \]

\[ R = f^D_i \nabla_i T + y_i \nabla_j \pi_{ij} + \sigma^D_{ij} v_{ij} + X_{ij} \pi_{ij} + \gamma T_g^2, \quad (34) \]

\[ R_g = \Sigma^D_{ij} v_{ij} + F^D_i \nabla_i T_g - \gamma T_g^2. \quad (35) \]

Given the expressions for \( R \), we may take flux vector as \( \vec{Z} = (f^D_i, y_i, \sigma^D_{ij}, X_{ij}) \), the force vectors as \( \vec{Y} = (\nabla_i T, \nabla_j \pi_{ij}, v_{ij}, \pi_{ij}) \), and again formulate the Onsager force-flux relation as \( \vec{Z} = \hat{c} \cdot \vec{Y} \). Analogously, given \( R_g \), we have \( \vec{Z}_g = \hat{c}_g \cdot \vec{Y}_g \), where \( \vec{Z}_g = (F^D_i, \Sigma^D_{ij}) \) and \( \vec{Y}_g = (\nabla_i T_g, v_{ij}) \). In addition, we require

\[ X_{ij} \to 0 \quad \text{for} \quad T_g \to 0, \quad (36) \]

to ensure permanent elasticity in granular statics.

This completes the derivation and presentation of the structure of a hydrodynamics of permanent elasticity at \( T_g = 0 \), and transient elasticity at finite \( T_g \). To find Granular solid hydrodynamics, we still need to specify the energy \( w \), and the functional dependence of the transport matrices, \( \hat{c}, \hat{c}_g \). Instead of a microscopic derivation of these quantities starting from some specific interaction, we employ general considerations (such as requiring \( w \) to have a positive curvature where the system is stable, see § 3.3A) and experimental data to narrow down the possibilities. Hereby, \( w \) may be determined by static data alone, but \( \hat{c}, \hat{c}_g \).
must be considered using data from granular dynamics. The simplest example is again given by \( \hat{c}, \hat{c}_g \) being both diagonal,

\[
\begin{align*}
 f_i^D &= \kappa \nabla_i T, \\
 F_i^D &= \kappa_g \nabla_i T_g, \\
 y_i &= \beta^P \nabla_j \pi_{ij}, \\
 \Sigma_{ij}^D &= \zeta_g v_{ij} \delta_{ij} + \eta_g v_{ij}^0, \\
 \sigma_{ij}^D &= \zeta v_{ij} \delta_{ij} + \eta v_{ij}^0, \\
 X_{ij} &= \beta \pi_{ij}^0 + \beta_1 \delta_{ij} \pi_{ij}.
\end{align*}
\]

(37)

In the next section, § III, an energy expression appropriate for granular media is presented, and shown to account for important features of granular statics. For the homogeneous case, with \( \nabla_i T, \nabla_i T_g, \nabla_j \pi_{ij} = 0 \), we propose to combine this \( w \) with the following transport structure, diagonal except for the two terms preceded by \( \alpha \),

\[
\begin{align*}
 \sigma_{ij}^D + \Sigma_{ij}^D &= (\zeta + \zeta_g) v_{ij} \delta_{ij} + (\eta + \eta_g) v_{ij}^0 + \alpha \pi_{ij}, \\
 X_{ij} &= -\alpha v_{ij} - u_{ij}^0 \frac{\tau}{\tau_1} - u_{ij} \frac{\delta_{ij}}{\tau_1}.
\end{align*}
\]

(40)

The first equation is simply a sum of the two dissipative stress contributions. The second equation uses the specific form of \( w \), a result of which is

\[
\pi_{ij} \equiv -\frac{\partial w}{\partial u_{ij}} = \sqrt{\Delta} (B \Delta \delta_{ij} - 2A u_{ij}^0) + A \frac{u_{ij}^2}{2\sqrt{\Delta}} \delta_{ij},
\]

(42)

see Eq (52) below. So the relaxation times are given as

\[
\frac{1}{\tau} \equiv 2\beta A \sqrt{\Delta}, \quad \frac{1}{\tau_1} \equiv 3\beta_1 \sqrt{\Delta} \left( B + \frac{A u_{ij}^2}{2\Delta^2} \right).
\]

(43)

Obviously, a simplification is given by taking either \( \beta \) and \( \beta_1 \), or \( \tau \) and \( \tau_1 \), as independent from \( u_{ij} \). Choosing the second possibility, and taking \( \tau, \tau_1 \) as proportional to \( T_g \), all other coefficients (ie. \( \zeta, \zeta_g, \eta, \eta_g, \alpha \)) as constant gives us a complete and well specified theory. As will be shown in an accompanying paper [5], this choice leads to a surprisingly good agreement with hypoplasticity [6], a modern engineering theory widely employed to model solid granular behavior, especially triaxial experiments.

1. Granular Gas

Since we are considering a hydrodynamic theory, we should expect the equations as given above to easily connect to that of granular gas, such as given in [3] by Haff. Taking the
elastic strain to relax infinitely fast, $\tau, \tau_1 \to 0$, essentially eliminates $u_{ij}$ as an independent variable. As a result, we have $w = w(T, T_g, \rho)$ in the rest frame, and only Eqs (25,26,27) remain as equations of motion, with the dissipative currents given by the second of Eqs (37), and the first of Eqs (38). Following Haff, we may take $w \sim T_g$, $s_g \sim \ln T_g$, and the term $(T_g s_g + \mu \rho - w) \delta_{ij}$ as the main contribution to the pressure [see Eq (31)]; also

$$\zeta_g, \eta_g, \kappa_g T_g, \gamma \sim \rho \sqrt{T_g}. \tag{44}$$

[Because $s$ is not included as an independent variable, the first of Eqs (26), is ignored in [3], as are $\kappa, \eta, \zeta$. Moreover, $\Sigma_{ij}^D$ are included only in $R_g$, not in the stress flux $\sigma_{ij}$, which is perhaps not quite consistent. The general gist, however, is certainly the same.]

III. A GRANULAR ENERGY EXPRESSION

Linear elasticity is a simple, consistent and complete theory. It starts with an energy $w$ that depends on the strain, $u_{ij} = \frac{1}{2}(\nabla_i U_j + \nabla_j U_i)$, with $U_i$ the displacement vector,

$$w = \frac{1}{2} K \Delta^2 + \mu u_s^2 \quad (\Delta \equiv -u_{\ell \ell}, \; u_s \equiv \sqrt{u_{ij}^0 u_{ij}^0}), \tag{45}$$

see [14]. $K, \mu > 0$ are two material-dependent constants, referred to as the bulk and shear modulus. ($u_{\ell \ell}$ is the trace of $u_{ij}$, and $u_{ij}^0 \equiv u_{ij} - \frac{1}{3} u_{\ell \ell} \delta_{ij}$ its traceless part.) The stress-strain relation is obtained as a derivative,

$$\sigma_{ij} = \pi_{ij} \equiv -\frac{\partial w}{\partial u_{ij}} = K \Delta \delta_{ij} - 2 \mu u_{ij}^0, \tag{46}$$

which contains the pressure $P$ and the scalar shear stress $\sigma_s$,

$$P \equiv \frac{1}{3} \sigma_{\ell \ell} = K \Delta, \quad \sigma_s \equiv \sqrt{\sigma_{ij}^0 \sigma_{ij}^0} = 2 \mu u_s, \tag{47}$$

both employed frequently below. Note that as there is no difference between $\sigma_{ij}$ and $\pi_{ij}$ in statics, we shall use them interchangeably here, in § III

Some ramifications of linear elasticity are: (1) Since the stress $\sigma_{ij}$ is given as a function of three variables, $U_i$, the three components of the force balance $\nabla_j \sigma_{ij} = \rho G_i$ (with $\rho$ the density and $G_i$ the gravitational constant) suffice to uniquely determine $U_i$, from which the stress $\sigma_{ij}$ may be calculated for arbitrary geometry. (2) The inverse compliance tensor, $M_{ijk\ell}$,
linking the increments of stress and strain, \( d\sigma_{ij} \) and \( du_{k\ell} \), is both isotropic and constant,

\[
d\sigma_{ij} = \frac{\partial\sigma_{ij}}{\partial u_{k\ell}} du_{k\ell} \equiv M_{ijk\ell} du_{k\ell},
\]

(48)

\[
M_{ijk\ell} = K\delta_{ij}\delta_{k\ell} - \mu(\delta_{ik}\delta_{j\ell} + \delta_{jk}\delta_{i\ell}).
\]

(49)

(3) As the pressure \( P = K\Delta \) does not depend on the shear \( u_s \), there is no volume dilatancy, \( (\partial P/\partial u_s)|_\Delta = 0 \). (4) Yield is not predicted. [Note that while the points (2), (3), (4) depend on the form of the energy \( w \), the statement under (1) is quite general.]

These equations account well for ordinary solids, but not for granular systems. Sand displays volume dilatancy, possesses a compliance tensor with significant stress-induced anisotropy, and most importantly, never strays far from yield, displaying significant irreversible, fluid-like, plastic movements in its vicinity.

The first attempt to modify linear elasticity, so as to better account for granular behavior, was due to Boussinesq [15]. He assumed, around 1874, stress-dependent elastic moduli, \( K, \mu \sim \Delta^{1/2} \sim P^{1/3} \), in Eq (46),

\[
\sigma_{ij} \sim \sqrt{\Delta} \left( \Delta \delta_{ij} - \frac{3 - 6\nu}{1 + \nu} u_{ij}^0 \right), \quad \frac{3 - 6\nu}{1 + \nu} = \frac{2\mu}{K},
\]

(50)

with \( \nu \) the constant Poisson ratio. This nonlinear stress-strain relation, sometimes referred to as the “quasi-elastic model,” is employed to understand granular compression [16] and sound velocity [17]. Unfortunately, the above failure list of linear elasticity remains partly intact: • As \( P \) remains a function of \( \Delta \) alone, dilatancy vanishes, \( \partial P/\partial u_s|_\Delta = 0 \). • Yield must still be postulated. In addition, Eq (50) contains a basic deficiency: No energy \( w \) exists such that \( \sigma_{ij} = -\partial w/\partial u_{ij} \) holds, because the associated Maxwell relation is violated, \( \partial\sigma_{ij}/\partial u_{tk} \neq \partial\sigma_{tk}/\partial u_{ij} \).

We choose the granular elastic energy to be [1]

\[
w = \sqrt{\Delta} \left( \frac{3}{2} B \Delta^2 + A u_s^2 \right),
\]

(51)

with \( A, B > 0 \) denoting two material constants. The associated stress is

\[
\sigma_{ij} = \sqrt{\Delta}(B\Delta \delta_{ij} - 2A u_{ij}^0) + A \frac{u_s^2}{2\sqrt{\Delta}} \delta_{ij}.
\]

(52)

As compared to Eq (50), the only difference is the last term \( \sim u_s^2/\sqrt{\Delta} \). This is, however, amazingly useful in accounting for granular behavior. As we shall see, it yields volume dilatancy, shear-induced anisotropy, and above all, predicts yield at the Coulomb condition,

\[
\sigma_s/P = \sqrt{2A/B}.
\]

(53)
In granular materials, there is a regime in which dissipation is insignificant and elastic responses dominant: small-amplitude perturbations from given points in the stress space. This is shown by Kuwano and Jardine [18] experimentally, who observed that stress increments become reversible if the strain fluctuations are around $10^{-4}$. It is also corroborated by Alonso-Marroquin and Herrmann [19] in molecular-dynamic simulations: Reducing elastic strains to $10^{-6}$, the irreversible plastic contributions are found around $10^{-14}$, implying a line as the stress-strain response, rather than the usual ellipse at higher amplitudes.

This fact is important because it makes a direct verification of Eq (52) possible: Measure $d\sigma_{ij} = (\partial\sigma_{ij}/\partial u_{k\ell}) du_{k\ell}$ and $du_{k\ell}$ independently, and compare the result to $M_{ijkl} \equiv \partial\sigma_{ij}/\partial u_{k\ell}$ as calculated from Eq (52). The data in [18] are extensive, comprising of 36 independent components of $M_{ijkl}$, all as functions of pressure, shear and the void ratio $e$. Comparing these data to the calculate $M_{ijkl}$ is the main result of this section, and represents an ambitious test of the energy $w$, Eq (51): Energy and stress of Eqs (51,52) depend only on two material parameters, $A$ and $B$, with their ratio fixed by the yield condition, Eq (53). Since the Ham river sand used in the experiment has a Coulomb yield angle of around $28^\circ$, implying $\xi \equiv B/A = 5/3$, only $A$, a scale factor and a measure of the total hardness, is left as an adjustable parameter. Taking $A = 5100$ Mpa, we find satisfactory agreement with their data at all values of pressure and shear, for the void ratio $e = 0.66$ — except close to yield which, due to increased plastic contributions, represents an especially difficult experimental regime. Because Kuwano and Jardine noticed that $e$ only alters the total hardness, by the factor $f \equiv (2.17 - e)^2/(1 + e)$, taking $A, B \sim f$ achieves agreement with respect to any other values of $e$ as well. Similar agreement to their data on ballotini (glass beads) was achieved by taking $A = 4200$ Mpa. Therefore, we take

$$A = A_0 \times \frac{(2.17 - e)^2}{1.3736(1 + e)}, \quad \xi \equiv \frac{B}{A} = \frac{5}{3}$$

(54)

with $A_0 = 5100$ and 4200 Mpa being the value of $A$ for $e = 0.66$, for Ham river sand and ballotini, respectively.

Given this experimental support on the functional dependence of $\sigma_{ij}$ on $U_k$, we have employed Eq (52) to evaluate static stress distributions in silos, sand piles and under point loads, not surprisingly with rather satisfactory results, see [2]. Note that Eq (52) does not contain any fit parameters: $\xi = 5/3$ is fixed by the yield angle, while $A_0$, as a scale factor, does not enter the stress distribution at all. (Given a solution, one may change the strain
by the factor $\alpha$, and $A_0$ by $\alpha^{-1.5}$, with the stress unchanged and still a solution, provided the boundary conditions are the usual ones, either given in terms of stresses or require that the displacement vanishes.)

A. Yield and Energetic Instability

A thermodynamic energy must be a convex function of state variables to ensure stability – this is why compressibility and specific heat are always positive, cf. [20]. Being a quadratic function of $\Delta$ and $u_s$, the energy of linear elasticity, Eq (45), is always convex. Conversely, the granular energy, Eq (51), is convex if and only if

\[
\begin{align*}
\left(\frac{\partial^2 w}{\partial \Delta^2}\right)_{u_s} &\geq 0, \quad \left(\frac{\partial^2 w}{\partial u_s^2}\right)_\Delta \geq 0, \\
\left(\frac{\partial^2 w}{\partial \Delta \partial u_s}\right)^2 &\leq \left(\frac{\partial^2 w}{\partial \Delta^2}\right)_{u_s} \left(\frac{\partial^2 w}{\partial u_s^2}\right)_\Delta
\end{align*}
\]

(55) (56)

hold. (See appendix on some subtleties in this context.) More explicitly, this implies

\[
u_s^2/\Delta^2 \leq 2B/A,
\]

(57)
drawing the boundary for the region of stable strains. Deriving $4P/\sigma_s = (\Delta/u_s) \times (2B/A + u_s^2/\Delta^2)$ from Eq (52), and inserting $u_s^2/\Delta^2 = 2B/A$ into it, Eq (53), the Drucker-Prager version of the Coulomb yield condition (cf. Schofield & Wroth, 1968; Huang, 1983) is obtained. The actual Coulomb yield condition, $\sigma_s/P = (\sqrt{18 + 6L^2 \sin \varphi_c})/(L \sin \varphi_c + 3)$, where $L \equiv \sqrt{3} \tan \left[\frac{1}{3} \arcsin (\sqrt{6} \sigma_0^{ij} \sigma_0^{jk} \sigma_0^{ki}/\sigma_3^2)\right]$ denotes the Lode parameter, would only result if terms $\sim u_{ij}^0 u_{jk}^0 u_{ki}^0$ are included in Eq (51).

In a classic paper, Goddard [17] started from Hertz contacts between grains, and considered the structure of the energy and stress. He concluded that, if the topology of the grain contacts do not change with stress, the energy is a homogeneous function of degree 5/2 in the strain $u_{ij}$, of the form $w = \Delta^{2.5} \times g(u_s^2/\Delta^2, u_{ij}^0 u_{jk}^0 u_{ki}^0/\Delta^3)$, where $g$ is an arbitrary function. As Eq (7) is clearly a special case of this general energy, we take this as a further, microscopically founded support for our starting point.

There is an instructive analogy between the granular stress-strain relation, Eq (52), and the van der Waals equation of state for real gases. The Boyle’s law is stable everywhere while the van der Waals equation has a non-physical zone, the liquid-gas instability, in which the compressibility is negative. Similarly, the Hooke’s law is stable everywhere, but the granular
stress-strain relation has a forbidden region, that of yield. Note

$$ \frac{\partial P}{\partial \Delta} \bigg|_{\sigma_s} \geq 0 \quad (58) $$

is implied by Eqs (55, 56), see appendix, so this forbidden region is also characterized by a negative compressibility. The actual innovation of the van der Waals theory is the fact that the condition for the onset of the liquid-gas transition, instead of being an extra input, is implied by the free energy. Similarly, yield is now a result of elasticity.

**B. Granular Stress-Strain Relation**

The granular stress-strain relation, Eq (52), and the definitions of Eq (47) imply

$$ P = \Delta^{3/2} (B + \frac{1}{2} A u_s^2 / \Delta^2), \quad (59) $$

$$ \sigma_s = 2 A \Delta^{1/2} u_s. \quad (60) $$

Eliminating $\Delta$, we obtain

$$ B \sigma_s^4 - 8 A^3 P u_s^3 \sigma_s + 8 A^5 u_s^6 = 0. \quad (61) $$

Fig. plots $\sigma_s$ versus $u_s$ for the fixed pressure of $P = 0.1$ Mpa. Note how remarkably linear.
FIG. 2: Thick line: Pressure versus compression at fixed shear. Dashed lines represent unstable states. Thin straight line: The same curve for linear elasticity. Insert: The analogous instability in the isothermal curve of the van der Waals equation of state.

The plot is – almost until yield, where the curve turns back abruptly. (Dashed lines are used throughout for unstable states.) This behavior is approximated by the elastoplastic model, frequently used in soil mechanics: Linear elasticity followed by yield and flat plastic motion, see the lower inserts in Fig. 1. Nonlinearity is relevant only when yield is close.

If instead $u_s$ is eliminated from Eqs (59, 60), the expression

$$\sigma_s^2 + 8AB\Delta^3 - 8AP\Delta^{3/2} = 0$$

allows a plot of pressure $P$ versus compression $\Delta$, at given $\sigma_s = 0.1$ Mpa, see Fig. 2. The pressure increases with the compression, implying a positive compressibility, only in the region of large $\Delta$. The compressibility is negative where $\Delta$ is small, and the stability condition, Eq (53) or (58), is violated. The van der Waals equation of state, $(P - a/v^2)(v - b) = RT$, is quite similar, where $1/v$ corresponds to $\Delta$, $R$ is the gas constant and $v$ the molar volume, see eg. [20]. The system can be either in the dense liquid state or the rarefied gaseous phase, with the zone in between forbidden, see insert of Fig. 2.

Alternatively, we may plot $\Delta$ versus $u_s$ at fixed $P$, or $P$ versus $\sigma_s$ at fixed $\Delta$, see Figs. 3 and 4, both showing clear evidence of “volume dilatancy,” the fact (first noticed by Reynold) that granular systems expand with shear, or $\partial\Delta/\partial u_s|_P \neq 0$, or $\partial P/\partial \sigma_s|_\Delta \neq 0$. For linear
elasticity, these plots are simply horizontal, and the derivatives vanish. If the Boussinesq model, Eq (50), were employed, all four plots would be indistinguishable from those of linear elasticity. So the last term of Eq (52) is indeed essential. (Plastic motion, not considered here, contribute to additional dilatancy, and may dominate.)

C. Shear-Dependence of the Elastic Moduli

The Hooke’s law, Eq (46), \( \sigma_{ij} = K\delta_{ij} - 2\mu u_{ij}^0 \), may be written as

\[
    u_{ij} = \frac{\nu}{E} \sigma_{nn} \delta_{ij} - \frac{\sigma_{ij}}{2\mu},
\]

with the Poisson ratio \( \nu \) and the Young modulus \( E \) given as

\[
    E = \frac{9\mu K}{3K + \mu}, \quad \nu = \frac{3K - 2\mu}{6K + 2\mu}. \tag{64}
\]

Requiring the granular stress-strain relation Eq (52) to assume these familiar forms, either Eq (46) or (63), leads to strain-dependency of \( K, \mu \),

\[
    K = \Delta^{1/2} \left( B + \frac{1}{2} \mathcal{A} u_s^2 / \Delta^2 \right), \tag{65}
\]

\[
    \mu = \mathcal{A} \Delta^{1/2}, \tag{66}
\]

FIG. 3: Compression \( \Delta \) versus shear strain \( u_s \), at fixed pressure. The dashed line is again unstable.

In linear elasticity, the same curve is a horizontal straight line.
FIG. 4: Pressure $P$ versus shear stress $\sigma_s$, at fixed compression. The dashed line is unstable. In linear elasticity, the same curve is a horizontal straight line.

and via Eq (64) also of $E, \nu$. As this is an intuitive way to characterize nonlinear elastic behavior, we shall consider their shear and pressure dependency more closely here. Using Eqs (59,60), we write these moduli as

$$\mu = \tilde{\mu} \xi^{1/3}, \quad K = \tilde{K} \xi^{-2/3},$$

$$E = \tilde{E} \frac{3B + A}{3B + A \xi^{1/3}}, \quad \nu = \frac{3B - 2A \xi}{6B + 2A \xi};$$

(67)

where $\xi$ quantifies shear,

$$\xi = \frac{1}{2} \left[ 1 \pm \sqrt{1 - \left( \frac{B}{2A} \right) \left( \sigma_s/P \right)^2} \right],$$

(68)

and $\tilde{\mu}, \tilde{K}, \tilde{E}, \tilde{\nu}$ denote the respective value without shear, at $\xi = 1$,

$$\tilde{\mu} = A \left( \frac{P}{B} \right)^{\frac{1}{3}}, \quad \tilde{K} = B \left( \frac{P}{B} \right)^{\frac{2}{3}}, \quad \tilde{E} = \frac{9AB}{3B + A} \left( \frac{P}{B} \right)^{\frac{1}{3}},$$

(69)

see Fig. 5. (The positive sign in Eq (68) is the stable branch, which meets the unstable branch with the negative sign at yield, where the square root vanishes.)

As mentioned in the introduction, the $P^{1/3}$-dependence of the twiddled letters is well-known. For typical granular behavior, however, the more relevant dependence is that on shear, which derives – same as yield and dilatancy – from the last term of Eq (52).
FIG. 5: Variations of $K, \mu, E, \nu$ with $\sigma_s/P$. The moduli are rescaled by their values at $\sigma_s = 0$, denoted respectively with a twiddle. Their variation $\sim P^{1/3}$ is shown in the insert.

D. The Compliance Tensor

1. Theoretical Expressions

Starting from Eq (52), the tensor $M_{ijkl}$ of Eq (48) is calculated as

$$M_{ijkl} = A\sqrt{\Delta} [(u_s^2/4\Delta^2 + 4/3 - 3B/2A)\delta_{ij}\delta_{kl}$$

$$-\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + (u_{ij}\delta_{kl} + \delta_{ij}u_{kl})/\Delta].$$

The compliance tensor $\lambda_{ijk\ell}$, defined via

$$d\sigma_{ij} = \lambda_{ijk\ell}d\sigma_{k\ell},$$

(71)
is obtained by inverting $M_{ijkl}$.

$$
\lambda_{ijkl} = \frac{[Au_s^2 + 2 (A - B) \Delta^2] \delta_{kl} \delta_{ij} - \delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk}}{6A\Delta^{1/2} (Au_s^2 - 2B\Delta^2)}
+ \frac{u_i \Delta \delta_{kl} + u_k \Delta \delta_{ij} + u_j u_k}{3\Delta^{1/2} (Au_s^2 - 2B\Delta^2)},
$$

$$
\lambda_{ijkl} = \frac{9A^5 \sigma_s^2 + 8 (4A - 9B) \mu^6 \delta_{kl} \delta_{ij} - \delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk}}{54\mu (A^5 \sigma_s^2 - 8\mu^6 B)}
- \frac{4A^5 \mu^3 (\sigma_{ij} \delta_{kl} + \sigma_{kl} \delta_{ij}) - 3A^5 \sigma_{ij}^2 \sigma_{kl}^0}{9\mu (A^5 \sigma_s^2 - 8\mu^6 B)}. 
$$

In the first expression $\lambda_{ijkl}$ is strain-, in the second stress-dependent — where the conversion is calculated using $\Delta = \mu^2 / A^2$, $u_i^0 = -\frac{1}{2} \sigma_{ij}^0 / \mu$, $u_s = \frac{1}{2} \sigma_s / \mu$, with $\mu = A(\xi \mu / B)^{1/3}$, cf. Eqs (67, 69). The second expression – a surprisingly complicated one if the starting expression for the energy serves as a benchmark – is what may be compared to experiments directly.

Before we do this, it is useful to pause and notice that the last term of both Eq (70) and (73) deviates structurally from the isotropic form of Eq (49). More generally, for an isotropic medium and in the presence of pure compression ($\sigma_{ij}^0 = 0$, $P \neq 0$), we may (quite independent of the specific form of the elastic energy) take $\lambda_{ijkl}$ to be

$$
\lambda_{ijkl}^0 = \lambda_1 \delta_{ij} \delta_{kl} + \lambda_2 (\delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk}),
$$

with $\lambda_1$, $\lambda_2$ arbitrary scalar functions of $\Delta$, $u_s$, and the Lode parameter $L$. This is because • both $\sigma_{ij}$ and $u_{kl}$ are symmetric, hence $\lambda_{ijkl} = \lambda_{jikl} = \lambda_{ijkl}$; • the Maxwell relation holds, $\partial^2 w / \partial u_{ij} \partial u_{lk} = \partial^2 w / \partial u_{lk} \partial u_{ij}$, hence $\lambda_{ijkl} = \lambda_{lkij}$. In the presence of shear, $\sigma_{ij}^0 \neq 0$, $\lambda_{ijkl}$ can take on many more terms. To linear order in $\sigma_{ij}^0$, these are

$$
\lambda_3 (\sigma_{ij}^0 \delta_{kl} + \delta_{ij} \sigma_{kl}^0) + \lambda_4 (\sigma_{ik}^0 \delta_{j\ell} + \sigma_{i\ell}^0 \delta_{jk} + \sigma_{j\ell}^0 \delta_{ik} + \sigma_{jk}^0 \delta_{i\ell}).
$$

To second order, we may substitute all above $\sigma_{ij}^0$ with $\sigma_{ik}^0 \sigma_{kj}^0$, and also add the terms: $\sigma_{ij}^0 \sigma_{kl}^0$ and $\sigma_{ik}^0 \sigma_{j\ell}^0 + \sigma_{j\ell}^0 \sigma_{ik}^0$. We shall refer to $\lambda_{ijkl}^0$ as being isotropic, and the $\sigma_{ij}^0$-dependent ones as displaying “shear-induced anisotropy.” If the medium were inherently anisotropic, say because the grains are pressed into some quasi-periodic array, leading to a preferred direction $\mathbf{n}$, the above expression is more complicated, because $\delta_{ij}$ in Eq (74) may now be substituted by three different tensors: $\delta_{ij} - n_i n_j$, $n_i n_j$, and $\epsilon_{ijk} n_k$. For triclinic symmetry and without the Maxwell relation, all 36 elements of $\lambda_{ijkl}$ are independent — even in the absence of shear. As mentioned, this “fabric anisotropy” is not included in the present consideration, because the starting expression for the energy, Eq (51), is isotropic.
2. Comparison with Experiments

Because $\sigma_{ij}$ and $u_{ij}$ are symmetric, each characterized by six independent components, Eq \((71)\) may be written as a vector equation, $d\vec{u} = \hat{\lambda} d\vec{\sigma}$, with $\hat{\lambda}$ a 6x6 matrix, and $du, d\sigma$ given as in Eq \((75)\). In the so-called “principle system” of coordinates, in which $\sigma_{ij}$ is diagonal (but not $\delta\sigma_{ij}$), Kuwano and Jardine take this vector equation to be given as \([18]\)

$$
\begin{pmatrix}
    du_{11} \\
    du_{22} \\
    du_{33} \\
    2du_{23} \\
    2du_{13} \\
    2du_{12}
\end{pmatrix}
= 
\begin{pmatrix}
    \hat{C} & 0 & 0 & 0 & 0 & 0 \\
    0 & \hat{C} & 0 & 0 & 0 & 0 \\
    0 & 0 & \hat{C} & 0 & 0 & 0 \\
    0 & 0 & 0 & G_{23}^{-1} & 0 & 0 \\
    0 & 0 & 0 & 0 & G_{13}^{-1} & 0 \\
    0 & 0 & 0 & 0 & 0 & 2G_{12}^{-1}
\end{pmatrix}
\begin{pmatrix}
    d\sigma_{11} \\
    d\sigma_{22} \\
    d\sigma_{33} \\
    -d\sigma_{23} \\
    -d\sigma_{13} \\
    -d\sigma_{12}
\end{pmatrix}
$$

(75)

with

$$
\hat{C} = 
\begin{pmatrix}
    -1/E_1 & \nu_{12}/E_2 & \nu_{13}/E_3 \\
    \nu_{21}/E_1 & -1/E_2 & \nu_{23}/E_3 \\
    \nu_{31}/E_1 & \nu_{32}/E_2 & -1/E_3
\end{pmatrix}
$$

(76)

$G_{ij}$ is referred to as the shear modulus in the $i - j$ plane, $E_i$ the Young modulus along $i$, and $\nu_{ij}$ the Poisson ratio for “the effect of the $i$-strain on $j$-strain.” Identifying these moduli with components of the $\lambda_{ijkl}$ tensor,

$$
G_{ij} = -1/4\lambda_{ijij}, \quad E_i = -1/\lambda_{iiii}, \quad \nu_{ij} = -\lambda_{ijij}/\lambda_{jjjj}
$$

(77)

(for $i \neq j$ and without summation over $i$ or $j$), we may employ Eq \((73)\) to obtain

$$
G_{13} = G_{23} = G_{12} = \mu, \quad E_i = \frac{27\mu (A^5\sigma_i^2 - 8\mu^6B)}{9A^5\sigma_i^2 - 72\mu^6B - A_s^2 i}; \quad \nu_{ij} = \frac{19A^5\sigma_i^2 - 72\mu^6B + 2A_s s_{i} s_{j}}{2(9A^5\sigma_i^2 - 72\mu^6B - A_s^2 s_{i} s_{j})}
$$

(78)

(79)

(80)

with $\mu = A(\xi P/B)^{1/3}, s_i \equiv 3A^2\sigma_i^0 - 4\mu^3, \sigma_i^0 \equiv \sigma_i - P$, and $\sigma_i$ denoting the three diagonal components of $\sigma_{ij}$ in the principle system. Before embarking on a comparison, we shall first
establish a few qualitative features from theory: • Without shear, $\sigma_i^0 \rightarrow 0$, all $E_i$ are equal,

$$E_i \rightarrow E_{sec} = \frac{27AB}{2A + 9B} \left( \frac{P}{B} \right)^{\frac{1}{3}},$$

(81)

where $E_{sec}$ is called the secant Young modulus. Same holds for the Poisson ratios,

$$\nu_{ij} \rightarrow \tilde{\nu}^* = \frac{19B - 4A}{29B + 2A}.$$  

(82)

(Note $\tilde{\nu}^*$ differs from $\tilde{\nu}$, and $E_{sec}$ from $\tilde{E}$, by a constant factor.) • Because of Eq (74) and irrespective of the energy specified, we have $E_1 = E_2 = E_3$, $\nu_{12} = \nu_{13} = \nu_{23}$, and $G_{12} = G_{13} = G_{23}$ in the absence of shear, $\sigma_i^0 = 0$. Any discrepancy with experiment therefore implies fabric anisotropy. • Finite shear will split $E_i$ and $\nu_{ij}$, but not $G_{ij}$, cf. Eq (78) — though this is an energy-related feature. • Because of the Maxwell relation, the matrix $\hat{\lambda}$ of Eq (75) is symmetric, implying especially (no summation)

$$\nu_{ij}E_i = \nu_{ji}E_j.$$  

(83)

This symmetry was noted by Love (1927) and adopted by Kuwano and Jardine in interpreting their data [18]. • The moduli $E, \mu, \nu$ are related as $E = 2\mu (\nu + 1)$, see Eqs.(64). A similar relation holds for $\mu, E_i, \nu_{ik}$ [no summation, see Eqs (79,80)],

$$E_i (6\mu \nu_{ij} - E_j)^2 = 4E_j (3\mu - E_i) (3\mu - E_j)$$  

(84)

It is important to realize that all formulas of this section hold not only for Cartesian coordinates, $i \rightarrow x, y, z$, but also for cylindrical ones, $i \rightarrow z, \rho, \varphi$. Taking $\Delta = u_{\varphi\varphi} + u_{\rho\rho} + u_{zz}$, and similarly for $u_s$, we may again start from the same energy, Eq (51), and derive all the results here. [Spatial differentiation is what mars the similarity. Yet once the strain components $u_{\rho\rho}, u_{\varphi\varphi}$... are given, no spatial differentiation is needed.] The one difference is, for any constant $\sigma_{ij}$ in Cartesian coordinates, there is always a principle system. In cylindrical coordinates, this holds only if the stress is also cylindrically symmetric. In other words, only if the stress is uniaxially diagonal, $\sigma_{ij} = \text{diag}(\sigma_1, \sigma_2, \sigma_3)$ with $\sigma_2 = \sigma_1$ in Cartesian coordinates, will it be diagonal cylindrically.

Because Kuwano and Jardine [18] used an axialsymmetric device for their measurements, the stress they apply is indeed cylindrically symmetric, with: $G_{pz} = G_{pz}$, $E_\rho = E_\varphi$, $\nu_{pz} = \nu_{\varphi z}$, $\nu_{z\rho} = \nu_{z\varphi}$, cf. Eqs.(78,80) noting $s_\rho = s_\varphi$. In addition, Eq (83) leads to $\nu_{\rho\varphi} = \nu_{\varphi\rho}$. Following them, we refer to the response coefficients being measured as: $G_{hh} \equiv G_{\rho\varphi}$, $G_{vh} \equiv$
FIG. 6: Variation with pressure $P$ of the shear moduli $G_{vh}, G_{hh}$, Young moduli $E_v, E_h$ and Poisson ratios $\nu_{vh}, \nu_{hh}$ (insert), at $\sigma_h/\sigma_v = 0.45$. Symbols are the same data on Ham River sand, at a void ratio of 0.66 by Kuwano & Jardine, (2002).

$G_{\rho z} = G_{\varphi z}$, $E_h \equiv E_\rho = E_{\varphi}$, $E_v \equiv E_z$, $\nu_{hh} \equiv \nu_{\rho \varphi} = \nu_{\varphi \rho}$, $\nu_{hv} \equiv \nu_{\rho z} = \nu_{\varphi z}$, $\nu_{vh} \equiv \nu_{z \rho} = \nu_{z \varphi}$, where $h$ is the horizontal directions, either $\rho $ or $\varphi $, and $v$ the vertical direction $z$, see the cylinder of Fig. 6. The main plots of Fig. 6 compare the theoretical curve [calculated by taking $\sigma_\rho = \sigma_\varphi = \sigma_h$ and $\sigma_z = \sigma_v$ in Eqs. (78-80)] and the experimental data [measured with Ham River sand] of $E_h, E_v, G_{vh}, G_{hh}$, as functions of $P$, for $\sigma_h = 0.45 \sigma_v$. The insert shows the same comparison for $\nu_{vh}, \nu_{hh}$. We especially note that theory and experiment agree on the ordering of the induced anisotropy, ie $\nu_{vh} > \nu_{hh}$, $E_v > E_h$ and $G_{hh} \approx G_{vh}$, which are pairwise equal in linear elasticity and the Boussinesq model. (The slight difference between $G_{hh}, G_{vh}$ is, as mentioned, the result of fabric anisotropy present in the sample.) For a theory without any useful fit parameter, the agreement must be considered a convincing verification of the elastic approach which, instead of postulating the stress-dependence of 21 (or even 36).
FIG. 7: Variation of Young and shear moduli, $E_{sec}$ and $\mu$, with pressure $P$, for the case of vanishing shear, $\sigma_v = \sigma_h$. The dotted lines are the empirical formula of Kuwano & Jardine (2002), for the Ham River sand at the void ratio $e = 0.66$. The split is proof of fabric anisotropy.

independent components of $\lambda_{ijkl}$ directly, looks for one appropriate scalar expression for the energy $w$. Even if it is heavy-handedly simplified, a large number of geometric correlation is preserved by the mere fact that $\lambda_{ijkl}$ is obtained via a double differentiation. This must be the main reason why the calculated $\lambda_{ijkl}$ stands up so surprisingly well when compared to the extensive data of [18].

Kuwano and Jardine [18] employ the following empirical formulas (in Mpa) for the Ham
River sand,

\[ E_v = 204f \left( \frac{\sigma_v}{P_a} \right)^{0.52} \]  
\[ E_h = 174f \left( \frac{\sigma_h}{P_a} \right)^{0.53} \]  
\[ G_{vh} = 72f \left( \frac{\sigma_v}{P_a} \right)^{0.32} \left( \frac{\sigma_h}{P_a} \right)^{0.2} \]  
\[ G_{hh} = 81f \left( \frac{\sigma_v}{P_a} \right)^{-0.04} \left( \frac{\sigma_h}{P_a} \right)^{0.53} \]

where \( P_a = 0.1013 \text{ Mpa} \) is the atmospheric pressure and \( f = (2.17 - e)^2 / (1 + e) \). \( (f = 1.3736 \) for the void ratio \( e = 0.66.\) \) Fig. 7 shows the theoretical and experimental values for \( E_h, E_v, G_{vh} \) and \( G_{hh} \), as functions of \( P \) for the isotropic case \( \sigma_h = \sigma_v \). The fact that \( E_h, E_v \) and \( G_{vh} \), \( G_{hh} \) are pairwise different, indicates (as discussed above) fabric anisotropy. Moreover, the theoretical curves are \( \sim P^{1/3} \), yet experimental ones seem to back a larger power: \( \sim P^{1/2} \).

As discussed, • this is a known contradiction between Hertz contact and sound data, with possible explanations provided by Goddard\[17\] and de Gennes [21], • and a question of simplicity versus accuracy in the present approach.

Fig 8 displays the effect of shear on different moduli, with \( \sigma_h \neq \sigma_v \). The upper, middle and lower figures respectively plot the Young moduli \( E_i \), the shear modulus \( \mu \) (both scaled by their isotropic values, \( E_{sec}, \tilde{\mu} \)), and the Poisson ratios \( \nu_{ij} \). In agreement with the empirical formulas Eqs (85-88), \( E_v \) increases with \( \sigma_s / P \), while \( E_h \) decreases, in the region away from yield. As yield is approached, both drop quickly to zero. This critical, pre-yield behavior is clearly absent for the empirical formulas and is of interests for future experiments. In theory, \( G_{vh}, G_{hh} \) are equal, decreasing with \( \sigma_s / P \) moderately, by less than 20%. In experiments, the shear moduli are split, with one increasing, the other decreasing. The discrepancy between the theory and experiment is in the range from \( \sigma_s / P = 0 \) to 0.6 within 20%. This need not be a result from fabric anisotropy, as a more complicated energy expression will also do.

Variation of the Poisson ratios \( \nu_{vh}, \nu_{hv}, \nu_{hh} \) is given by Eq (80). As depicted, \( \nu_{vh} \) and \( \nu_{hv} \) increase, while \( \nu_{hh} \) decreases, with \( \sigma_s / P \), all being divergent at yield. No empirical formulae for the ratios are given in \[18\], and the two circles in the plot simply depict the values from the insert of Fig 1. However, \( \nu_{hh} = E_h / (2G_{hh}) - 1 \) was assumed to hold by the authors, and interestingly, it may be derived by taking \( i = h, j = h \) in Eq (84), yielding \( \nu_{hh} = E_h / (2\mu) - 1 \).

Assuming that both coefficients \( A, B \) of Eq (51) are proportional to \( f \) of Eqs (85-88), agreement between experiment and theory is extended to all values of the void ratio. Comparison was also made to Kuwano and Jardine’s data gained using glass ballotini \[18\]. Taking
FIG. 8: Upper, middle, and lower figures show the Young moduli, shear moduli and Poisson ratios as functions of $\sigma_s/P$. The dotted lines present the empirical formulas of Kuwano & Jardine (2002) for the Ram River sand, at the void ratio $e = 0.66$.

$A = 4200, B = \frac{5}{3}A = 7000$, we find similar agreement.
E. The Elastic Part of Flow Rules

The increment relation, Eq (48), may also be written in the matrix form \( \overrightarrow{\sigma} = \hat{M} \overrightarrow{u} \), with \( \hat{M} \) a symmetric 6 \( \times \) 6 matrix, and \( \overrightarrow{\sigma}, \overrightarrow{u} \) still given as in Eq (75). The determinant, \( \text{det} \hat{M} = 9A^5 (2B\Delta^2 - Au_s^2) \Delta \), calculated from Eq (70), vanishes at the yield surface, \( Au_s^2 = 2B\Delta^2 \), because an Eigenvalue, call it \( m_1 \), also does. (This is not a coincidence as \( \hat{M} \) is the Jacobian matrix of the energy function, which is positive only in the stable region. It may be of interest to note that the determinant of the Boussinesq model, \( \text{det} \hat{M} = 9A^5 (3B + 4A) \Delta^3 \), never vanishes.) The associated Eigenvector \( \overrightarrow{m}_1 \) points along the direction at which a finite deformation \( \overrightarrow{u} \neq 0 \) may take place under constant stress \( d\sigma_{ij} = 0 \). We refer to \( \overrightarrow{m}_1 \) as the elastic flow direction, since \( \overrightarrow{m}_1 \parallel d\overrightarrow{u} \) is only the elastic contribution of the strain. Setting \( d\sigma_{ij} = 0 \) in Eq (48) and using \( Au_s^2 = 2B\Delta^2 \), we obtain

\[
du_{ij} = -\frac{1}{2} \left( \delta_{ij} + \frac{u_{ij}}{\Delta} \right) d\Delta = \left( \frac{\sqrt{B}}{2A} \frac{\sigma_{ij}^0}{\sigma_s} - \frac{\delta_{ij}}{3} \right) d\Delta.
\]

The calculated \( du_{ij} \rightarrow d\overrightarrow{u} \) is the Eigenvector \( \overrightarrow{m}_1 \). Remarkably, one can rewrite this equation as \( du_{ij}/d\Delta = \partial g/\partial \sigma_{ij} \), or

\[
\overrightarrow{m}_1 \parallel \partial g/\partial \overrightarrow{\sigma}, \quad \text{with} \quad g = \sqrt{B/2A}\sigma_s - P, \quad (89)
\]

implying that the elastic flow direction is perpendicular to the yield surface, as defined by the equation \( g = 0 \). If the plastic contribution to the strain field may be neglected for some reasons, this property is referred to as the associated flow rule see [22, 23].

APPENDIX A: ENERGETIC STABILITY

In the main text, we considered the convexity of the energy with respect to the variables \( u_s \) and \( \Delta \). Relevant is the convexity with respect to \( u_{ij} \). As the transformation between these two sets of variables is nonlinear, we bear the burden of proof that both are equivalent.

Thermodynamic stability requires the elastic energy to be a convex function of its six strain variables, or linear combinations of them. This means that all eigenvalues of the Jacobian matrix \( \partial^2 w/\partial X_\alpha \partial X_\beta \) are positive. We take: \( X_1 = u_{xy}, X_2 = u_{xz}, X_3 = u_{yz}, \ X_4 = (u_{xx} - u_{zz})/2, \ X_5 = (u_{xy} - 2u_{yy} + u_{zz})/(2\sqrt{3}), \ X_6 = -u_{xx} - u_{yy} - u_{zz}, \) with \( Q = u_s^2 = 2\sum_{\alpha=1}^5 X_\alpha^2 \). For an energy of the form \( w = w(\Delta, Q) = w(X_6, Q) \) and denoting
\( f \equiv 4\partial w/\partial Q, a \equiv \partial^2 w/\partial \Delta^2, b \equiv 4\partial^2 w/\partial Q\partial \Delta, c = 16\partial^2 w/\partial Q^2 \), the Jacobian matrix is
\[
\begin{pmatrix}
  f + cX_1^2 & cX_1X_2 & cX_1X_3 & cX_1X_4 & cX_1X_5 & bX_1 \\
  cX_1X_2 & f + cX_2^2 & cX_2X_3 & cX_2X_4 & cX_2X_5 & bX_2 \\
  cX_1X_3 & cX_2X_3 & f + cX_3^2 & cX_3X_4 & cX_3X_5 & bX_3 \\
  cX_1X_4 & cX_2X_4 & cX_3X_4 & f + cX_4^2 & cX_4X_5 & bX_4 \\
  cX_1X_5 & cX_2X_5 & cX_3X_5 & cX_4X_5 & f + cX_5^2 & bX_5 \\
  bX_1 & bX_2 & bX_3 & bX_4 & bX_5 & a
\end{pmatrix}
\]
with its six eigenvalues given as \( f_{1-4} = f \) and
\[
 f_\pm = \frac{f + a + cQ}{2} \pm \frac{1}{2} \sqrt{\left( f - a + \frac{cQ}{2} \right)^2 + 2b^2Q}. \tag{A1}
\]
They are all positive if, and only if, \( f > 0, 2af + acQ - b^2Q > 0, f + a + cQ/2 > 0 \), or equivalently,
\[
\begin{align*}
  \frac{\partial w}{\partial Q} &> 0, \quad 4\frac{\partial w}{\partial Q} + \frac{\partial^2 w}{\partial \Delta^2} + 8Q\frac{\partial^2 w}{\partial Q^2} > 0, \tag{A2} \\
  \frac{\partial^2 w}{\partial \Delta^2} \frac{\partial w}{\partial Q} + 2Q\frac{\partial^2 w}{\partial Q^2} \frac{\partial^2 w}{\partial \Delta^2} - 2Q \left( \frac{\partial^2 w}{\partial Q \partial \Delta} \right)^2 &> 0, \tag{A3}
\end{align*}
\]
Because \( u_s^2 = Q \), or \( 2u_s(\partial w/\partial Q) = \partial w/\partial u_s, 2u_s(\partial^2 w/\partial \Delta \partial Q) = \partial^2 w/\partial \Delta \partial u_s, 4u_sQ \times (\partial^2 w/\partial Q^2) = u_s(\partial^2 w/\partial u_s^2) - \partial w/\partial u_s \), these conditions are equivalent to Eqs (55, 56), or
\[
\begin{align*}
  \frac{\partial w}{\partial \Delta} &> 0, \quad \frac{\partial^2 w}{\partial \Delta^2} > 0, \quad \frac{\partial^2 w}{\partial u_s^2} > 0, \quad \frac{\partial^2 w}{\partial \Delta^2} \frac{\partial^2 w}{\partial u_s^2} > \left( \frac{\partial^2 w}{\partial u_s \partial \Delta} \right)^2. \tag{A4, A5}
\end{align*}
\]
For the energy of Eq (51), the inequalities (A4) imply \( A > 0, B > 0 \), while Eq (A5) gives the yield condition (57). Using \( P = \partial w/\partial \Delta \) and \( \sigma_s = \partial w/\partial u_s \) we can also write Eqs (A4, A5) as
\[
\begin{align*}
  \left( \frac{\partial P}{\partial \Delta} \right)_{u_s} &> 0, \quad \left( \frac{\partial \sigma_s}{\partial u_s} \right)_{\Delta} > 0, \tag{A6} \\
  \left( \frac{\partial P}{\partial \Delta} \right)_{u_s} \left( \frac{\partial \sigma_s}{\partial u_s} \right)_{\Delta} &> \left( \frac{\partial P}{\partial u_s} \right)_{\Delta}^2 \tag{A7}
\end{align*}
\]
The Maxwell relation \( \partial P/\partial u_s|_{\Delta} = \partial \sigma_s/\partial \Delta|_{u_s} \) and the thermodynamic identities
\( \partial P/\partial \Delta|_{u_s} = \partial P/\partial \Delta|_{\sigma_s} + \partial P/\partial \sigma_s|_{\Delta} \cdot \partial \sigma_s/\partial \Delta|_{u_s}, \partial P/\partial u_s|_{\Delta} = \partial P/\partial \sigma_s|_{\Delta} \cdot \partial \sigma_s/\partial u_s|_{\Delta} \), imply an alternative stability condition,
\[
\left( \frac{\partial P}{\partial \Delta} \right)_{\sigma_s} > 0. \tag{A8}
\]
[1] Y.M. Jiang, M. Liu, Phys. Rev. Lett., 91, 144301 (2003), 93, 148001(2004); Eur. Phys. J. E., 1292-8941(2007).

[2] D.O. Krimer, M. Pfitzner, K. Bräuer, Y. Jiang, M. Liu, Phys. Rev. E74, 061310 (2006); K. Bräuer, M. Pfitzner, D.O. Krimer, M. Mayer, Y. Jiang, M. Liu, Phys. Rev. E74, 061311 (2006); Y.M. Jiang, M. Liu, Eur. Phys. J. E., 1292-8941(2007).

[3] P. K. Haff, J. Fluid Mech., 134, 401(1983); J. T. Jenkins and S. B. Savage, J. Fluid Mech., 130, 187(1983).

[4] L.E. Silbert, D. Ertas, G.S. Grest, T.C. Halsey, D. Levine, S.J. Plimpton, Phys. Rev. E 64, 051302 (2001); GDR MiDi group, Eur. Phys. J. E 14, 341 (2004); P.Jop, Y. Forterre, O. Pouliquen, Nature 441, 727, 2006.

[5] Y.M. Jiang, M. Liu, Cond-Mat, arXiv:0706.1354

[6] D. Kolymbas, Introduction to Hypoplasticity, (Balkema, Rotterdam, 2000); W. Wu and D. Kolymbas, in Constitutive Modelling of Granular Materials ed Kolymbas, (Springer, Berlin, 2000), and references therein.

[7] H. Temmen, H. Pleiner, M. Liu and H.R. Brand, Phys. Rev. Lett. 84, 3228 (2000) and 86, 745 (2001); H. Pleiner, M. Liu and H.R. Brand, Acta Rheol. 39, 560 (2000)

[8] G.H. Helfrich, Phys.Rev.Lett 23, 372 (1969); P.G. de Gennes, J. Prost, The Physics of Liquid Crystals (Clarendon Press, Oxford, 1993); H. Pleiner, H.R. Brand, in Pattern Formation in Liquid Crystals eds. A. Buka, L. Kramer, (Springer, New York, 1996); P.C. Martin, O. Parodi, and P.S. Pershan, Phys. Rev. A 6, 2401 (1972).

[9] T. Komatsu, S. Inagaki, N. Nakagawa, and S. Nasuno, Phys. Rev. Lett. 86, 1757 (2001).

[10] O. Müller, PhD-thesis, (2006), University Tübingen; O. Müller, M. Liu, H. Pleiner and H.R. Brand, preprint.

[11] L. Bocquet, J. Errami, and T. C. Lubensky, Phys. Rev. Lett., 89, 184301 (2002); W. Losert, L. Bocquet, T. C. Lubensky, and J. P. Gollub, Phys. Rev. Lett., 85, 1428 (2000); L. Bocquet, W. Losert, D. Schalk, T. C. Lubensky, and J. P. Gollub, Phys. Rev., E 65, 11307 (2001); I. Goldhirsh, Annu. Reev. Fluid Mech., 35, 267(2003).

[12] S.F. Edwards, R.B.S. Oakeshott, Physica A 157, 1080 (1989); A. Metha, S.F. Edwards, Physica A 157, 1091.
[13] P. Richard, M. Nicodemi, R. Delannay, P. Ribiere and D. Bideau, Nature Materials, 4, 121 (2005).

[14] L. D. Landau, and E. M. Lifshitz, Theory of Elasticity (New York, Pergamon Press, 3rd edn. 1986)

[15] G. Gudehus, in In Constitutive relations for soils, eds G. Gudehus, F. Darve, and I. Vardoulakis, (Rotterdam, Balkema, 1984).

[16] P. Evesque and P. G. de Gennes, C. R. Acad. Sci. Paris, 326, 761 (1998); P. G. de Gennes, Rev. Mod. Phys., 71, 347 (1999).

[17] J. D. Goddard, Proc. R. Soc. London, A430, 105 (1990).

[18] R. Kuwano, and R. J. Jardine, Géotechnique, 52, 727 (2002).

[19] F. Alonso-Marroquin, H.J. Herrmann, Phys. Rev. Lett. 92, 054301 (2004).

[20] H. B. Callen, Thermodynamics (New York, John Wiley & Sons, 1960).

[21] de Gennes, P. G. Europhys. Lett., 35, 145 (1996).

[22] A. Schofield and P. Wroth, Critical state soil state mechanics, (London, Mcgrau-Hill 1968).

[23] W. X. Huang, Engineering properties of soil, in Chinese (Beijing, Hydroelectricity publishing, 1983).