HAUSDORFF DIMENSION FOR ERGODIC MEASURES OF INTERVAL EXCHANGE TRANSFORMATIONS

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ABSTRACT. I show that there exist minimal interval exchange transformations with an ergodic measure whose Hausdorff dimension is arbitrarily small, even 0. I will also show that in particular cases one can bound the Hausdorff dimension between $\frac{1}{2r+4}$ and $\frac{1}{r}$ for any $r$ greater than 1.

1. INTRODUCTION

This paper is concerned with determining the possible Hausdorff dimensions for ergodic measures of interval exchange transformations (IETs). In general, almost all IETs (with irreducible permutation) are uniquely ergodic with respect to Lebesgue measure [V2], [M] (see also [B1] for an elementary proof) and therefore their only ergodic measure has Hausdorff dimension 1. In fact, since a minimal $d$ interval IET has at most $\left\lfloor \frac{d}{2} \right\rfloor$ (probability) ergodic measures [V1], [Ka], the smallest number of intervals for which one can non-trivially consider the Hausdorff dimension of ergodic measures is 4.

In a celebrated 1977 paper Michael Keane provided a method for constructing minimal 4 IETs that are not uniquely ergodic [K]. This followed an earlier construction of Keynes and Newton, who showed a minimal 5 IET could be not uniquely ergodic [KN]. Using Keane's construction I will present several results on the Hausdorff dimensions of ergodic measures of non-uniquely ergodic minimal IETs. By the estimate in the previous paragraph, minimal 4-IETs have at most $2^d$ (probability) ergodic measures [V1], [Ka], the smallest number of intervals for which one can non-trivially consider the Hausdorff dimension of ergodic measures is 4.

Some of the results contained in the paper are:

1) That one can build minimal IETs that have an ergodic measure with arbitrarily small (even 0) Hausdorff dimension (see Theorem 2 and Corollary 1).

2) That one can place the Hausdorff dimension of particular IETs between $\frac{1}{r}$ and $\frac{1}{2r+4}$ for $r \geq 1$.

3) That a large class of Keane's non-uniquely ergodic IETs have both ergodic measures having Hausdorff dimension 1 (see Remark 1).

Prior to this work it was known that if the lengths of the intervals are algebraic numbers then the Hausdorff Dimension is greater than 0 [B2]. However, it was not known if the Hausdorff dimension could ever be less than 1.
The plan of the paper is as follows: The second section reviews IETs and Keane’s paper. The third section introduces terminology and notation that will be used throughout the remainder of the paper. The fourth section states the theorems of the paper. The fifth section presents preliminary lemmas. The sixth section presents proofs of theorems and a couple of results that they imply.

2. Review of IETs and Michael Keane’s Results

**Definition 1.** Given \( L = (l_1, l_2, \ldots, l_n) \) where \( l_i > 0 \), \( l_1 + \ldots + l_n = 1 \) we can obtain \( n \) subintervals of the unit interval \( I_1 = [0, l_1), I_2 = [l_1, l_1 + l_2), \ldots, I_n = [l_1 + \ldots l_{n-1}, 1) \). If we are also given a permutation on \( n \) letters \( \pi \) we obtain an \( n \) Interval Exchange Transformation \( T: [0, 1) \to [0, 1) \) which exchanges the intervals \( I_i \) according to \( \pi \). That is, if \( x \in I_j \) then

\[
T(x) = x - \sum_{k<j} l_k + \sum_{\pi(k')<\pi(j)} l_{k'}.
\]

Keane’s paper and this one are concerned with 4-IETs with permutation (4213). Keane relied on the induced map on the fourth interval for his result. He showed that by choosing the lengths appropriately one could ensure that this induced map had the permutation (2431). Name these in reverse order and we once again get a IET whose landing pattern is given by the columns of following matrix:

\[
A_{m,n} = \begin{pmatrix}
0 & 0 & 1 & 1 \\
m - 1 & m & 0 & 0 \\
n & n & n - 1 & n \\
1 & 1 & 1 & 1
\end{pmatrix} ; \quad m, n \in \mathbb{N} = \{1, 2, \ldots \}.
\]

For instance the second (after renaming) subinterval of the induced map visits the intervals of the original IET according to the pattern \([0 m n 1]\) before returning to the 4th interval. That is, it does not land in the first interval, it lands a total of \( m \) times in the second interval, \( n \) times in the third interval and returns to the fourth interval. One can now repeat this procedure on the 4th interval (once again after renaming) of our induced map with a new matrix \( A_{m,n} \) and on the 4th interval of this induced map with another matrix \( A_{m,n} \) and so on.

The IETs that have this property for the matrices \( A_{m_1, n_1}, \ldots, A_{m_k, n_k} \) are those contained in the image of the 3-simplex under the map \( A_{m_1, n_1} A_{m_2, n_2} \ldots A_{m_k, n_k} \) (see Definition 7). Michael Keane showed that if we choose our \( m_i, n_i \) appropriately we get an IET with two ergodic measures. In particular, if one chooses our lengths according to the vector \( \lim_{k \to \infty} A_{m_1, n_1} A_{m_2, n_2} \ldots A_{m_k, n_k} e_3 \) (where \( e_3 = [0 0 1 1]^T \)) one gets that one of our measures is Lebesgue measure and the other is singular with respect to Lebesgue measure. The singular measure assigns weights according to the vector \( \lim_{k \to \infty} A_{m_1, n_1} A_{m_2, n_2} \ldots A_{m_k, n_k} e_2 \) (where \( e_2 = [0 1 0 0]^T \)). Following Keane we will denote the Lebesgue measure as \( \lambda_3 \) and the singular measure \( \lambda_2 \). The conditions Keane gives are as follows (for notation see Definition 7):

**Theorem 1.** If one chooses \( 3(n_k + 1) \leq m_k \leq \frac{1}{2} (n_k + 1) \) and \( n_1 \geq 10 \) then the IET with lengths determined by the vector \( \lim_{k \to \infty} A_{m_1, n_1} A_{m_2, n_2} \ldots A_{m_k, n_k} e_3 \) is not uniquely ergodic.
This is Theorem 5 of [K] and these conditions are assumed to be satisfied for the remainder of the paper. The following two results of Keane are key in proving this result and will be used in this paper (for notation see Definition 2):

**Lemma 1.** \[ \frac{\lambda_3(I_k^{(k)})}{\lambda_3(I^{(k)})} \leq \frac{2m_k}{(n_k+1)(n_k+1)}. \]

This result is in the proof of Lemma 3 of [K].

**Lemma 2.** \[ \frac{\lambda_2(I_k^{(k)})}{\lambda_2(I^{(k)})} \geq \frac{1}{3}. \]

This is Lemma 4 of [K].

### 3. Definitions and Notation

**Definition 2.** \( I^{(k)} \) denotes the \( k \)th induced interval and \( I^{(k)}_j \) denotes the \( j \)th subinterval of \( I^{(k)} \).

Note: \( I^{(k)}_4 = I^{(k+1)} \). Also note that in [K] this notation is reversed, so in his paper \( I^{(j)}_k \) is the \( j \)th subinterval of the \( k \)th induced interval.

**Definition 3.** \( |v|_1 \) is the sum of the absolute values of the entries of the vector \( v \).

**Definition 4.** \( B_k = A_{m_1,n_1}A_{m_2,n_2}\ldots A_{m_k,n_k} \).

This matrix describes the travel of the subintervals of \( I^{(k)} \) until they land in \( I^{(k)} \) again. That is, the number of times each subinterval of \( I^{(k)} \) lands in our initial subintervals before returning to \( I^{(k)} \).

**Definition 5.** \( b_{t,i} \) denotes \( |B_t e_i|_1 \).

As above, \( e_i \) denotes the column vector where the \( i \)th entry is 1 and all other entries are 0.

**Definition 6.** \( O(I_j^{(k)}) \) denotes the union of images of \( I_j^{(k)} \) that \( B_k e_j \) counts.

That is, \( O(I_j^{(k)}) = \bigcup_{l=0}^{b_{k,j}-1} T^l(I_j^{(k)}) \).

**Definition 7.** Let \( \tilde{A}_{m_k,n_k}(v) = \frac{A_{m_k,n_k}(v)}{|A_{m_k,n_k}(v)|} \).

\( \tilde{A}_{m_k,n_k} \) maps vectors in the unit 3-simplex to vectors in the unit 3-simplex. This ensures that the measures obtained by the conditions of the theorems are probability measures.

Note: \( \tilde{A}_{m_k,n_k}(u + v) \neq \tilde{A}_{m_k,n_k}(u) + \tilde{A}_{m_k,n_k}(v) \) in general.

**Definition 8.** \( S = \bigcap_{k=1}^{\infty} \bigcup_{r=k}^{\infty} O(I_2^{(r)}) \).

This is the set of all points which lie in \( O(I_2^{(k)}) \) for infinitely many \( k \).

**Definition 9.** If \( M \subset [0,1) \) is set, \( H_{dim}(M) \) denotes the Hausdorff dimension of \( M \).

For a definition of Hausdorff dimension and an introduction to it see [F].
Definition 10. The Hausdorff dimension of a probability measure $\mu$ is
$$H_{\dim}(\mu) = \inf\{H_{\dim}(M) : M \text{ is Borel and } \mu(M) = 1\}.$$

4. New Results

Theorem 2. If an IET has lengths determined by the vector
$$\lim_{k \to \infty} \bar{A}_{m_1, n_1} \bar{A}_{m_2, n_2} \ldots \bar{A}_{m_k, n_k} e_3$$
and there exists $N$ such that $n_{k+1} \geq (b_{k,2})^r 2^r m_k$ for all $k \geq N$, then the Hausdorff dimension of $\lambda_2$, the other ergodic measure, is less than or equal to $\frac{1}{2}$.

The condition for Theorem 2 along with Lemma 1 implies that $\lambda_3(I_{2}^{(k)}) \leq \frac{1}{(b_{k,2})^r 2^r}$ for $k \geq N$. This fact is crucial for the proof.

Theorem 3. If an IET has lengths determined by the vector
$$\lim_{k \to \infty} \bar{A}_{m_1, n_1} \bar{A}_{m_2, n_2} \ldots \bar{A}_{m_k, n_k} e_3$$
and there exists $N$ such that $b_{k+1,2} \leq (b_{k,2})^r$, $m_k \geq k^2 n_k$ for all $k \geq N$, then the Hausdorff dimension of $\lambda_2$, the other ergodic measure, is greater than or equal to $\frac{1}{2r}$.

As the next theorem suggests, there is a gap between the $r$ in Theorem 2 and in Theorem 3. In general one can have $r_2 + 2 \geq r_3$ (where $r_2$ is the $r$ in Theorem 2 and $r_3$ is the $r$ in Theorem 3). This is done by setting $n_k = k^2 (b_{k-1,2})^r 2^r m_{k-1}$ and $m_k = k^2 n_k$. Theorem 2 provides the upper bound and Theorem 3 provides the lower bound. In particular,

Theorem 4. If an IET has lengths determined by the vector
$$\lim_{k \to \infty} \bar{A}_{m_1, n_1} \bar{A}_{m_2, n_2} \ldots \bar{A}_{m_k, n_k} e_3,$$
with $n_k = 9^{k-1}$ and $m_k = 9^{k-1} + k$, then $\frac{1}{8} \leq H_{\dim}(\lambda_2) \leq \frac{1}{4}$.

5. Preliminary Lemmas

First, a strengthening of Lemma 2.

Lemma 3. $\lambda_2(O(I_{2}^{(k)})) = b_{k,2} \lambda_2(I_{2}^{(k)})$ is greater than $\frac{1}{3}$ for all $k \geq 0$.

Proof: I begin by showing that $b_{k,2} \geq b_{k,i}$ by comparing the entries of $B_{k,2} e_2$ and $B_{k,i} e_i$. $b_{k,2} > b_{k,1}$ because the second entry of $A_{m_k, n_k} e_2 = m_k > m_k - 1$ and $m_k - 1$ is the second entry of $A_{m_k, n_k} e_1$. $A_{m_k, n_k} e_2$ agrees with $A_{m_k, n_k} e_1$ in all other entries. $b_{k,2} \geq b_{k,j}$ for $j = 3, 4$ because $A_{m_k, n_k} e_2 \geq A_{m_k, n_k} e_j$ in all entries but the first and $m_k A_{m_k-1, n_k-1} e_2 > A_{m_k-1, n_k-1} e_1$ in all entries (the second entry of $A_{m_k, n_k} e_j$ is 0 and the second entry of $A_{m_k, n_k} e_2$ is $m_k e_2$ and also the first entry of $A_{m_k, n_k} e_j = 1$). This argument shows that $A_{m_k-1, n_k-1} A_{m_k, n_k} e_2$ has each entry greater than or equal to the corresponding entries of $A_{m_k-1, n_k-1} A_{m_k, n_k} e_j$ for $j = 3, 4$.

We also have that $\lambda_2(I_{2}^{(k)}) > \frac{1}{3} \lambda_2(I^{(k)})$ by Lemma 2. Therefore, because our IET is minimal, we have,

$1 = \lambda_2([0, 1]) = \lambda_2(O(I_{1}^{(k)})) + \lambda_2(O(I_{2}^{(k)})) + \lambda_2(O(I_{3}^{(k)})) + \lambda_2(O(I_{4}^{(k)}))$

$= b_{k,1} \lambda_2(I_{1}^{(k)}) + b_{k,2} \lambda_2(I_{2}^{(k)}) + b_{k,3} \lambda_2(I_{3}^{(k)}) + b_{k,4} \lambda_2(I_{4}^{(k)})$, 


Lemma 6. Under the conditions of Theorem 3, for

$$-T$$

for all

$$k$$

Lemma 7. $$\lambda_i T$$

This is because the image of

$$2$$

belongs to

$$2$$

Lemma 8. Proof: Observe that

$$1$$

Lemma 4. $$\lambda_2(S) = 1.$$ 

Lemma 5. Under the conditions of Theorem 3

$$\frac{\lambda_2(I_2^{(k)})}{\lambda_2(I^{(k)})} > \frac{k^2}{k^2 + 4}.$$ 

Lemma 6. Under the conditions of Theorem 3, for $$\lambda_2$$ almost all points $$x$$,

$$T^{-r}(x) \in I_2^{(k)}$$

and $$T^s(x) \in I_2^{(k)}$$ for some $$0 \leq r, s \leq b_{k,2}$$ for all but finitely many $$k$$.

This lemma says that for $$\lambda_2$$ almost every $$x$$ there exists $$N_x$$ such that

$$T^i(x) \in O(I_2^{(k)})$$

for $$k \geq N_x$$ and $$0 \leq i \leq b_{k,2}$$.

Lemma 7. $$\lambda_3(I_4^{(k)}) \geq \lambda_3(I^{(k+1)}),$$

Proof: Observe that

$$\lambda_3(I_1^{(k)}) = \lambda_3(I^{(k)}) \frac{\lambda_3(I^{(k+1)}))}{\lambda_3(I^{(k)})}.$$ 

Also, $$b_{k,2} \geq b_{k,1}$$ (see Lemma 3) and $$T$$ is minimal implies $$\lambda_3(I^{(k)}) \geq \frac{1}{b_{k,2}}$$.

This lemma is similar to Lemma 1 in [K].

Lemma 8. The images of $$I_2^{(k)}$$ in $$O(I_2^{(k)})$$ are never immediately adjacent.

Proof: By Keane’s construction $$I_2^{(k)}$$ will always be bordered on both sides by $$I_1^{(k)}$$ or on one side by $$I_1^{(k)}$$ and the other by $$I_4^{(k)}$$ or on one side by $$I_4^{(k)}$$ and the other by $$I_3^{(k)}$$. This is because the image of $$I_2^{(k)}$$ that are inside a subinterval of $$I^{(k-1)}$$ have this property. Also the only subinterval of $$I^{(k-1)}$$ which has an image of $$I_2^{(k)}$$ on its boundary is $$I_2^{(k-1)}$$ (its left and right boundary are both images of $$I_2^{(k)}$$). The result follows by induction on $$k$$. Just for reference, $$I_1^{(k-1)}$$’s boundary blocks are $$I_1^{(k-1)}$$ and $$I_4^{(k)}$$, $$I_3^{(k-1)}$$’s are $$I_1^{(k)}$$ and $$I_4^{(k)}$$ and $$I_4^{(k-1)}$$’s boundary blocks are $$I_4^{(k)}$$ and $$I_1^{(k)}$$.

6. Proofs of the Theorems

Proposition 1. Under the conditions of Theorem 3, $$\lambda_2$$ a.e. $$x$$ satisfies
\[
\liminf_{s \to \infty} s^{2r+\varepsilon}|T^s(x) - x| = \infty.
\]

Proof: Given \(x\), pick \(k'\) so that we have \(x\) satisfying Lemma 6 for all \(k > k'\). By Lemma 8 this means that for \(s \leq b_{k,2}\) we have \(|T^s(x) - x| \geq \min \{\lambda_3(I^{(k)}_i)\}\). This is because all of the images of \(x\) lie in separate images of \(I^{(k)}_2\), which are separated by the image of some \(I^{(k)}_i\) for \(i \neq 2\). \(I^{(k)}_1\) has the smallest \(\lambda_3\) measure of these subintervals (subintervals that are not \(I^{(k)}_2\)). \(\lambda_3(I^{(k)}_1)\) gives a lower bound. By Lemma 7, the fact that \(b_{k,2} \geq b_{k,i}\) and the fact that \(\lim_{k \to \infty} \lambda_3(I^{(k)}_i) = 1\), it follows that \(\lambda_3(I^{(k)}_1) \geq \frac{1}{(b_{k+1,2})^{1+\varepsilon}}\), eventually. (Indeed, \(b_{k,2} \geq b_{k,i}\) so \(\lambda_3(I^{(k)}_1) \geq \frac{1}{b_{k,2}}\).) Thus, \(|T^s(x) - x| \geq \frac{1}{(b_{k+1,2})^{1+\varepsilon}}\). So if the conditions of Theorem 3 are satisfied and \(b_{k,2} \leq s \leq b_{k+1,2}\), then we have

\[
\begin{align*}
(b_{k,2})^{2r+4r\varepsilon} &\geq (b_{k,2})^{2r+4r\varepsilon} \frac{1}{(b_{k+1,2})^{1+\varepsilon}} \geq (b_{k,2})^{2r+4r\varepsilon} \frac{1}{((b_{k,2})^{2r})^{1+\varepsilon}} = (b_{k,2})^{2r+4r\varepsilon + 2r}, \\
(b_{k,2})^{2r+4r\varepsilon} &\geq (b_{k,2})^{2r+4r\varepsilon + 2r} = (b_{k,2})^{2r},
\end{align*}
\]

which goes to infinity with \(k\).

Theorem 3 can now be proved with the assistance of Theorem 1.3 in [B2]. Put in the language of this paper it states:

**Theorem 5.** If the Hausdorff dimension of an invariant measure \(\mu\) for a dynamical system \(T : [0,1) \to [0,1)\) is less than \(\alpha\) then \(\lim \{n^{\frac{1}{r}}|T^n(x) - x|\} = 0\) for \(\mu\) almost every \(x\).

This proves Theorem 3 because it shows that \(H_{\dim}(\lambda_2) \geq \frac{1}{2r+\varepsilon}\) for any \(\varepsilon\).

**Remark 1.** Theorem 3 also shows that if \(m_k \geq k^\alpha n_k\) then, unless one stipulates much faster growth than Keane does, \(H_{\dim}(\lambda_2) = 1\). This is because if \(n_k\) grows exponentially so does \(m_k\). This implies that \(b_{k,2}\) grows like \(c k^2\), which means that for any \(\varepsilon > 0\) eventually \((b_{k,2})^{1+\varepsilon} > b_{k+1,2}\).

Proof of Theorem 2: By Lemma 4 it suffices to show \(H_{\dim}(S) \leq \frac{1}{r}\). Observe that covering \(O(I^{(k)}_2)\) with images of \(I^{(k)}_2\), performing the Hausdorff \(\frac{1}{r}\)-dimensional estimate gives a number less than \(b_{k,2}(\frac{1}{(b_{k,2})^{2r+\varepsilon}}) = 2^{-k}\). By summing \(\lambda_2(O(I^{(k)}_2))\) over \(k \geq L\) the Hausdorff \(\frac{1}{r}\)-dimensional measure of \(S\) is less than \(2^{-L+1}\) for any \(L\). So \(H_{\dim}(S) \leq \frac{1}{r}\).

**Corollary 1.** If the conditions of Theorem 2 are satisfied and additionally, \(n_{k+1} = (b_{k,2})^k\), then \(H_{\dim}(\lambda_2) = 0\).

Proof: The conditions of Theorem 2 are satisfied for any \(r\) by picking \(N\) big enough.

In order to prove Theorem 4, we obtain coarse estimates on \(b_{k,2}\).

**Lemma 9.** \(b_{k,2} \geq m_1 m_2 \ldots m_k\).

Indeed, the second entry of \(B_k e_2\) is bigger than this.
Lemma 10. \(b_{k,2} \leq (m_1 + n_1 + 1)(m_2 + n_2 + 1)...(m_k + n_k + 1)\).

Proof: Observe that \(A_{m,n}c_2 \geq A_{m,n}e_j\), which means \(|A_{m,n}v|_1 \leq |v|_1|A_{m,n}e_1|\).

The lemma follows from this fact and induction.

Proof of Theorem 4: Observe \((m_i + n_i + 1) \leq 2m_i\). So by Lemmas 9 and 10 under the conditions of Theorem 4 we have

\[
g^{(4^k-1)^{\frac{1}{2} + \frac{k(k+1)}{2}}} \leq b_{k,2} \leq 2^k g^{(4^k-1)^{\frac{1}{2} + \frac{k(k+1)}{2}}}.
\]

In view of the fact that

\[
(2^k g^{(4^k-1)^{\frac{1}{2} + \frac{k(k+1)}{2}}})^2 2^{2k} g^{4^{k-1}+k} < g^k
\]

for large \(k\), the conditions of Theorem 2 are satisfied with \(r_1 = 2\). Also notice that

\[
(2^k g^{(4^k-1)^{\frac{1}{2} + \frac{k(k+1)}{2}}} + (k+1)(k+2)^{\frac{1}{2}})^{4+\epsilon} \geq 2^{k+1} g^{(4^{k+1}-1)^{\frac{1}{2} + \frac{k(k+1)}{2}}} + (k+1)(k+2)^{\frac{1}{2}}
\]

for large \(k\), satisfying the conditions for Theorem 3 with \(r_2 = 4 + \epsilon\). This gives Theorem 4.

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