Subalgebras and finitistic dimensions of Artin algebras*

Aiping Zhang¹, Shunhua Zhang ²

1 School of Mathematics and Statistics, Shandong University at Weihai, 264209, China
2 School of Mathematics, Shandong University, Jinan 250100, China

Abstract. Let $A$ be an Artin algebra. We investigate subalgebras of $A$ with certain conditions and obtain some classes of algebras whose finitistic dimensions are finite.

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1 Introduction

Let $A$ be an Artin algebra, $A$-mod the category of finitely generated left $A$-modules, and $A$-ind a full subcategory of $A$-mod containing exactly one representative of each isomorphism class of indecomposable $A$-modules. We denote the projective dimension of an $A$-module $X$ by $\text{pd}_A X$.

Email addresses: pingping326@163.com(A.Zhang ), shzhang@sdu.edu.cn(S.Zhang)

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Let $A$ be an Artin algebra. Recall from [1] that the finitistic dimension of $A$, denoted by $\text{fin.dim } A$, is defined as

$$\text{fin.dim } A = \sup \{ \text{pd}_A M \mid M \in A - \text{mod}, \text{pd}_A M < \infty \}.$$ 

The finitistic dimension conjecture claims that every Artin algebra has a finite finitistic dimension.

So far, only a few classes of algebras were known to have finite finitistic dimensions. For example, monomial algebras [3], algebras where the cube of the radical is zero [4], and the algebras given in [5,6,7,8,9,10,11,12]. However, the finitistic dimension conjecture is still open and it is far from to be proven.

Let $A$ be an Artin algebra, and $0 \to A A \to I_0 \to I_1 \to \cdots$ be the minimal injective resolution of $A$. Nakayama conjectured in [13] that $A$ is a self-injective algebra whenever all $I_j$ is projective. Up to now, Nakayama conjecture is still open. It is well known that finitistic dimension conjecture implies Nakayama conjecture, and this motivated further research on finitistic dimension conjecture. We refer to [8,9,10,11,12,14] for the background and some new progress about this conjecture.

According to Xi in [9], the finitistic dimension conjecture is equivalent to the following statement: if $B$ is a subalgebra of $A$ such that $\text{rad } B$ is a left ideal in $A$, then $B$ has finite finitistic dimension whenever $A$ has finite finitistic dimension.

In this paper, we investigate the finitistic dimensions of Artin algebras by using Igusa-Todorov function defined in [5] and obtain some classes of algebras with finite finitistic dimensions. The paper is arranged as follows. In Section 2 we collect some definitions and results needed for our research, and give a different proof for a well known fact (Theorem 2.5). In Section 3, we obtain some classes of algebras with finite finitistic dimensions, which gives a partly positive answer to the question 2 mentioned in [10].
2 Preliminaries

Throughout this paper, we always assume that $A$ is an Artin algebra. We denote the global dimension of $A$ by $\text{gl.dim } A$ and the Jacobson radical of $A$ by $\text{rad } A$. For an $A$-module $M$, we denote by $\text{add } M$ the full subcategory having as objects the direct sums of indecomposable summands of $M$, by $\Omega^i M$ the $i$th syzygy of $M$. Then, $\mathcal{P} = \text{add } A A$ is the full subcategory consisting of all finitely generated projective $A$-modules, and $\mathcal{I} = \text{add } A D A$ is the full subcategory consisting of all finitely generated injective $A$-modules, where $D : A - \text{mod } \to A^{\text{op}} - \text{mod}$ is the standard duality, and $A^{\text{op}}$ is the opposite algebra of $A$. Given two homomorphisms $f : L \to M$ and $g : M \to N$, the composition of $f$ and $g$ is denoted by $gf$. We follow the standard terminology and notation used in the representation theory of algebras, see [15] and [16].

An $A$-module $V$ is called a generator – cogenerator if every indecomposable projective module and every indecomposable injective module is isomorphic to a summand of $V$. Recall from [18] that the number

$$\text{rep.dim } A = \inf \{ \text{gl.dim } \text{End}_A(V) \mid V \text{ is a generator – cogenerator } \}$$

is called the representation dimension of an Artin algebra $A$.

**Lemma 2.1.** Let $M$ be an $A$-module and there is an exact sequence $0 \to X_s \to \cdots \to X_1 \to X_0 \to M \to 0$. If $\text{pd}_A X_i \leq k$, $i = 0, \cdots, s$, then $\text{pd}_A M \leq s + k$. $\square$

The following two Lemmas proved in [18] and [9] will be used later.

**Lemma 2.2.** Let $V$ be a generator-cogenerator of $A$-mod and $n \geq 3$ an integer. The following two statements are equivalent:

(1) For any $X \in A\text{-ind}$, there is an exact sequence

$$0 \to V_{n-2} \to \cdots \to V_1 \to V_0 \to X \to 0$$


with \( V_i \in \text{add} (A^i) \) for \( j = 0, \cdots, n-2 \), such that

\[
0 \to \text{Hom}_A(V, V_{n-2}) \to \cdots \to \text{Hom}_A(V, V_1) \to \text{Hom}_A(V, V_0) \to \text{Hom}_A(V, X) \to 0
\]

is exact.

(2) \( \text{gl.dim } \text{End } _AV \leq n. \) \( \square \)

**Lemma 2.3.** Suppose \( B \) is a subalgebra of \( A \) such that \( \text{rad } B \) is a left ideal in \( A \). For any \( B \)-module \( X \) and integer \( i \geq 2 \), there is a projective \( A \)-module \( Q \) and an \( A \)-module \( Z \) such that \( \Omega^i_B(X) \cong \Omega^2_A(Z) \oplus Q \) as \( A \)-modules. If \( \text{rad } B \) is an ideal in \( A \), then there is an exact sequence of \( A \)-modules

\[
0 \to \Omega^i_B(X) \to \Omega^2_A(Y) \oplus P \to S \to 0,
\]

where \( P \) is projective, and \( S \) is an \( A \)-module such that \( B^2S \) is semi-simple. In particular, if \( \text{rad } B = \text{rad } A \), the module \( S \) is even a semi-simple \( A \)-module. \( \square \)

Let \( K(A) \) be the free abelian group with the basis of non-isomorphism classes of non-projective indecomposable \( A \)-modules in \( A \)-mod. Igusa and Todorov in [5] define a function \( \psi_A \) on \( K(A) \), which depends on the algebra \( A \) and take values of non-negative integers. Now \( \psi_A \) is called Igusa-Todorov function, and it is a powerful tool to show the finiteness of the finitistic dimensions. The following lemma collects some important properties of this Igusa-Todorov function.

**Lemma 2.4.** Let \( A \) be an Artin algebra, and \( \psi_A \) be the corresponding Igusa-Todorov function. Let \( M, X, Y, Z \) be \( A \)-modules in \( A \)-mod.

(1) \( \psi_A(M) = \text{pd } M \) provided \( \text{pd } M < \infty. \)

(2) \( \psi_A(X) \leq \psi_A(X \oplus Y). \)

(3) If \( 0 \to X \to Y \to Z \to 0 \) is an exact sequence in \( A \)-mod and \( \text{pd } Z < \infty \), then \( \text{pd } Z \leq \psi_A(X \oplus Y) + 1. \)
Let $\mathcal{X}$ be a full subcategory of $A$-mod. When we say that $\mathcal{X}$ is a full subcategory, we always mean that $\mathcal{X}$ is closed under direct summands. We denote by gen $\mathcal{X}$ (cogen $\mathcal{X}$) the full subcategory of $A$-mod generated (cogenerated) by $\mathcal{X}$, see [17] and [16]. If $\mathcal{X} = \{M\}$, we set $\mathcal{X} = M$ and denote gen $\mathcal{X}$ (cogen $\mathcal{X}$) by gen $M$ (cogen $M$). If $\mathcal{X}$ contains only finite non-isomorphic indecomposable $A$-modules, we call $\mathcal{X}$ is of finite type.

It has been shown in [20] that rep.dim $A$ is at most 3 whenever gen $DA$ is finite, then in this case, according to [5], fin.dim $A$ is finite. Now, we give a different proof for this result by using Igusa-Todorov function.

**Theorem 2.5.** Let $A$ be an Artin algebra. If gen $DA$ is of finite type, then fin.dim $A$ is finite.

**Proof.** Let $X$ be an $A$-module with finite projective dimension. Let $i : X \longrightarrow E(X)$ be the injective envelope of $X$, then we have an exact sequence $0 \longrightarrow X \overset{i}{\longrightarrow} E(X) \longrightarrow \text{coker } i \longrightarrow 0$.

We may assume that $M_1, \cdots, M_t$ are a complete list of pairwise non-isomorphic indecomposable $A$-modules in gen $DA$. Since the modules $E, \text{coker } i$ lie in gen $DA$, we may write $E(X) = \bigoplus_{i=1}^{t} M_{ti}^{ti}$, coker $i = \bigoplus_{j=1}^{s} M_{sj}^{sj}$. We denote by $a = \max\{ t_i + s_i \}$. By Lemma 2.4, we know that

\[
\text{pd } X \leq \psi_A(\Omega(E(X) \oplus \text{coker } i)) + 1
\]
\[
= \psi_A(\Omega(E(X)) \oplus \Omega(\text{coker } i)) + 1
\]
\[
= \psi_A(\Omega(\bigoplus_{i=1}^{t} M_{ti}^{ti}) \oplus \Omega(\bigoplus_{j=1}^{t} M_{sj}^{sj})) + 1
\]
\[
= \psi_A(\bigoplus_{i=1}^{t} \Omega(M_i)^{t_i} \oplus \bigoplus_{j=1}^{t} \Omega(M_j)^{s_j}) + 1
\]
\[
= \psi_A(\bigoplus_{i=1}^{t} \Omega(M_i)^{t_i+s_i}) + 1
\]
\[
\leq \psi_A(\bigoplus_{i=1}^{t} \Omega(M_i)^a) + 1
\]
\[
= \psi_A(\bigoplus_{i=1}^{t} \Omega(M_i)) + 1.
\]
Thus \( \text{fin.dim } A \leq \psi_A \left( \bigoplus_{i=1}^{t} \Omega(M_i) \right) + 1 \), it follows that the finitistic dimension of \( A \) is finite. \( \square \)

3 Main results

In this section, we investigate the finitistic dimensions of subalgebras of an Artin algebra with certain conditions and give some examples to show how our results are applied.

We replace the condition \( \text{gl.dim } A \leq 1 \) of Theorem 3.3 in [10] by \( \text{rep.dim } A \leq 3 \) and obtain the following result.

**Theorem 3.1.** Let \( A_0 = B \subseteq A_1 \subseteq \cdots \subseteq A_{s-1} \subseteq A_s = A \) be a chain of subalgebras of \( A \), \( \text{rad } (A_{i-1}) \) is a left ideal in \( A_i \) for all \( i \) and \( \text{pd}_{A_{i-1}} A_i < \infty \) for all \( 1 \leq i \leq s - 1 \). If \( \text{rep.dim } A \leq 3 \), then \( \text{fin.dim } B < \infty \).

**Proof.** According to the proof of Theorem 3.1 in [10]. We know that \( \text{pd}_B A_j < \infty \) for all \( 1 \leq j \leq s - 1 \). Suppose \( M \) is a \( B \)-module, \( \text{pd}_B M < \infty \). We denote by \( \Omega_i \) the first syzygy operator of \( A_i \)-modules. By Lemma 2.3, \( \Omega^2_0(M) \) is an \( A_1 \)-module. Similarly \( \Omega^2_j \cdots \Omega^2_1\Omega^2_0(M) \) is an \( A_{j+1} \)-module, we have the following exact sequences:

\[
0 \rightarrow \Omega^2_0(M) \rightarrow P_0(1) \rightarrow P_0(0) \rightarrow M \rightarrow 0,
\]

\[
0 \rightarrow \Omega^2_1\Omega^2_0(M) \rightarrow P_1(1) \rightarrow P_1(0) \rightarrow \Omega^2_0(M) \rightarrow 0,
\]

\[
0 \rightarrow \Omega^2_2\Omega^2_1\Omega^2_0(M) \rightarrow P_2(1) \rightarrow P_2(0) \rightarrow \Omega^2_1\Omega^2_0(M) \rightarrow 0,
\]

\[
\vdots
\]

\[
0 \rightarrow \Omega^2_{s-2}\cdots\Omega^2_0(M) \rightarrow P_{s-1}(1) \rightarrow P_{s-1}(0) \rightarrow \Omega^2_{s-2}\cdots\Omega^2_0(M) \rightarrow 0, \quad (**)
\]

where \( P_0(1), P_0(0) \) are projective \( A_0 \)-modules, \( P_1(1), P_1(0) \) are projective \( A_1 \)-modules, \( P_2(1), P_2(0) \) are projective \( A_2 \)-modules, \( \cdots, P_{s-1}(1), P_{s-1}(0) \) are projective \( A_{s-1} \)-modules.
Thus we have the following long exact sequence
\[
(1) \quad 0 \to \Omega^2_{s-1}\Omega^2_{s-2}\cdots\Omega^2_0(M) \to P_{s-1}(1) \to P_{s-1}(0) \to \cdots \to P_1(1) \to P_1(0) \to P_0(1) \to P_0(0) \to M \to 0.
\]

By (**) we know that $B$-module $\Omega^2_{s-1}\Omega^2_{s-2}\cdots\Omega^2_0(M)$ has finite projective dimension.

It follows from Lemma 2.2 and the inequality $\text{rep. dim } A \leq 3$ that there exists a generator-cogenerator $V$ for $A$-$\text{mod}$, such that for any $A$-module $X$, there is an exact sequence $0 \to V_1 \to V_0 \to X \to 0$, with $V_1, V_0 \in \text{add } V$, such that
\[
0 \to \text{Hom}_A(V, V_1) \to \text{Hom}_A(V, V_0) \to \text{Hom}_A(V, X) \to 0
\]
is exact.

Obviously $\Omega^2_{s-1}\Omega^2_{s-2}\cdots\Omega^2_0(M)$ is an $A$-module and there is a short exact sequence
\[
0 \to V_1 \to V_0 \to \Omega^2_{s-1}\Omega^2_{s-2}\cdots\Omega^2_0(M) \to 0
\]
of $A$-module with $V_1, V_0 \in \text{add } V$. By Lemma 2.4, we know that
\[
\text{pd}_B \Omega^2_{s-1}\Omega^2_{s-2}\cdots\Omega^2_0(M) \leq \psi_B(V_1 \oplus V_0) + 1 \\
\leq \psi_B(V) + 1.
\]

By the long exact sequence (1), we have
\[
\text{pd}_B M \leq 2s + \max\{ \text{pd}_B \Omega^2_{s-1}\Omega^2_{s-2}\cdots\Omega^2_0(M), \text{pd}_B (P_j(i)) \}_{j=0, \ldots, s-1} \}
\leq 2s + \max\{ \psi_B(V) + 1, \text{pd}_B (A_j) \}_{j=1, \ldots, s-1} \}
\]
Thus fin.dim $B$ is finite.

When $s = 2$, we obtain the following consequence.

**Corollary 3.2.** Let $C \subseteq B \subseteq A$ be a chain of subalgebras of an Artin algebra $A$ such that rad $C$ is a left ideal in $B$, rad $B$ is a left ideal in $A$. If $\text{pd}_C B < \infty$ and $\text{rep. dim } A \leq 3$, then fin.dim $C$ is finite.
Example 1. Let $A$ be an algebra (over a field) given by the following quiver with relations: $cd = ef$, $a^4 = ba = 0$.

Now we use the method "gluing of idempotents" to construct subalgebras of $A$, which have the same radical $\text{rad } A$, see [10] for details.

Let $B$ be the subalgebra of $A$ given by the following quiver with relations: $cd = ef$, $a^4 = ba = ca = bd = ad = 0$:

Now we consider the subalgebra $C$ of $B$, which is given by quiver and relations $cd = ef$, $a^4 = ba = ca = bd = ad = 0$, $gf = hf$.

By Corollary 2.4 in [19], rep. dim $A \leq 3$, and it is easy to see that $\text{pd}_C B = 2$. Then we have that fin. dim $C < \infty$ by Corollary 3.2.
Theorem 3.3. Let $B$ be a subalgebra of an Artin algebra $A$ such that $\text{rad} B$ is an ideal in $A$. If $\text{add}\{\Omega^2_A(M) \mid M \in A - \text{mod}\}$ is of finite type, then $\text{fin.dim} B$ is finite.

Proof. Suppose $X$ is a $B$-module with finite projective dimension, by Lemma 2.3, there is an exact sequence of $A$-modules:

\[ (*) \quad 0 \to \Omega_B^2(X) \to \Omega_A^2(Y) \oplus P \to S \to 0, \]

where $AP$ is projective, and $S$ is an $A$-module such that $BS$ is semisimple.

Since $\text{add}\{\Omega_A^2(M) \mid M \in A - \text{mod}\}$ is of finite type, we may assume that $M_1, \ldots, M_t$ are a complete list of pairwise non-isomorphic indecomposable $A$-modules. Obviously $\Omega_A^2(Y)$ lie in $\text{add}\{\Omega_A^2(M) \mid M \in A - \text{mod}\}$, so we may write $\Omega_A^2(Y) = \bigoplus_{i=1}^t M_i^t$.

By Lemma 2.4 and $(*)$, we know that

\[
\text{pd}_B X \leq \text{pd} \Omega_B^2(X) + 2 \\
\leq \psi_B(\Omega_B(\Omega_A^2(Y) \oplus P \oplus S)) + 3 \\
= \psi_B((\bigoplus_{i=1}^t M_i^t) \oplus P \oplus \Omega_B(S)) + 3 \\
\leq \psi_B(\Omega_B(M_1) \oplus \cdots \oplus \Omega_B(M_t) \oplus \Omega_B(A) \oplus \Omega_B(B/\text{rad}B)) + 3.
\]

Thus $\text{fin.dim} B$ is finite. \qed

Corollary 3.4. Let $B$ be a subalgebra of an Artin algebra $A$ such that $\text{rad} B$ is an ideal in $A$. If $\text{gl.dim} A \leq 2$, then $\text{fin.dim} B$ is finite. \qed

Theorem 3.5. Let $C \subseteq B \subseteq A$ be a chain of subalgebras of an Artin algebra $A$ such that $\text{rad} C$ is a left ideal in $B$, $\text{rad} B$ is a left ideal in $A$. If $\text{cogen} A$ is of finite type, then $\text{fin.dim} C$ is finite.

Proof. Suppose $X$ is a $C$-module with finite projective dimension. By Lemma 2.3, $\Omega_C^2(X)$ is a $B$-module, we have the following exact sequence

\[ 0 \to \Omega_B \Omega_C^2(X) \to P \to \Omega_C^2(X) \to 0, \]
of $B$-modules where $P$ is a projective $B$-module. By Lemma 2.3, there is a $B$-module $Y$ and a projective $B$-module $Q'$ such that $\Omega^2_B(X) = \Omega_B(Y) \oplus Q'$. Thus the above exact sequence can be rewritten as

$$0 \rightarrow \Omega^2_B(Y) \rightarrow P \rightarrow \Omega^2_C(X) \rightarrow 0,$$

by Lemma 2.3 again, there is an $A$-module $Z$ and a projective $A$-module $Q$ such that $\Omega^2_B(Y) = \Omega_A(Z) \oplus Q$, so we have the following exact sequence

$$0 \rightarrow \Omega_A(Z) \oplus Q \rightarrow P \rightarrow \Omega^2_C(X) \rightarrow 0.$$

Since cogen $A$ is of finite type, we may assume that $M_1, \cdots, M_t$ are a complete list of pairwise non-isomorphic indecomposable $A$-modules, $\Omega_A(Z) \in \text{cogen } A$, $Q \in \text{cogen } A$, we may write $\Omega_A(Z) = \bigoplus_{i=1}^t M_i^{x_i}$, $Q = \bigoplus_{j=1}^s M_j^{y_j}$, By Lemma 2.4, we know that

$$\text{pd} \ C.X \leq \text{pd} \ \Omega^2_C(X) + 2 \leq \psi_C(\Omega_A(Z) \oplus Q \oplus P) + 3 \leq \psi_C(\bigoplus_{i=1}^t M_i^{x_i} \oplus \bigoplus_{j=1}^s M_j^{y_j} \oplus B) + 3 \leq \psi_C(M_1 \oplus \cdots \oplus M_t \oplus B) + 3.$$

Thus fin.dim $C$ is finite. 

**Remark.** we should point that Corollary 3.2 and Theorem 3.5 give partial answer to the question 2 in [10].

**Corollary 3.6.** Let $C \subseteq B \subseteq A$ be a chain of subalgebras of an Artin algebra $A$ such that rad $C$ is a left ideal in $B$, rad $B$ is a left ideal in $A$. If $A$ is a hereditary Artin algebra, then fin.dim $C$ is finite.
Example 2. Let $A$ be an algebra (over a field) given by the following quivers with relations: $a^4 = 0$, $cd = ef$.

We use the method "gluing of idempotents" (see [10]) to construct subalgebras of $A$ as following.

Let $B$ be the subalgebra of $A$ given by the following quivers with relations: $a^4 = ba = 0$, $ca = ef$, $g_1 f = g_2 f$, $g_3 f = g_4 f$.

Let $C$ be the subalgebra of $B$ given by the following quiver with relations $a^4 = ba = bd = ca = ad = 0$, $cd = ef$, $g_1 f = g_2 f = g_3 f = g_4 f$.

Then we get a chain of subalgebras of $A$, $C \subseteq B \subseteq A$, such that $\text{rad } C$ is a left ideal of $B$, $\text{rad } B$ is a left ideal of $A$. It is easy to see that $\text{cogen } A$ is of finite type. By Theorem 3.5, we know that $\text{fin.dim } C < \infty$. 
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