Serap Bulut

Fekete–Szegö problem for subclasses of analytic functions defined by Komatu integral operator

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Abstract Using the Komatu integral operator, new subclasses of analytic functions are introduced. For these classes, several Fekete–Szegö type coefficient inequalities are derived.

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1 Introduction and definitions

Let $A$ denote the class of functions of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$$

which are analytic in the unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$ 

Also let $S$ denote the subclass of $A$ consisting of univalent functions in $\mathbb{U}$.

Fekete and Szegö proved a noticeable result that the estimate

$$|a_3 - \lambda a_2^2| \leq 1 + 2 \exp\left(\frac{-2\lambda}{1 - \lambda}\right)$$

holds for $f \in S$ and for $0 \leq \lambda \leq 1$. This inequality is sharp for each $\lambda$ (see [8]). The coefficient functional

$$\phi_{\lambda}(f) = a_3 - \lambda a_2^2 = \frac{1}{6} \left( f'''(0) - \frac{3\lambda}{2} (f''(0))^2 \right)$$

on $f \in A$ represents various geometric quantities as well as in the sense that this behaves well with respect to the rotation, namely

$$\phi_{\lambda}(e^{-i\theta} f(e^{i\theta} z)) = e^{2i\theta} \phi_{\lambda}(f) \quad (\theta \in \mathbb{R}).$$

S. Bulut (✉)
Civil Aviation College, Kocaeli University, Arslanbey Campus, 41285 İzmit-Kocaeli, Turkey
E-mail: bulutserap@yahoo.com; serap.bulut@kocaeli.edu.tr

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In fact, other than the simplest case when
\[ \phi_0(f) = a_3, \]
we have several important ones. For example,
\[ \phi_1(f) = a_3 - a_2^2 \]
represents \( S_f(0)/6 \), where \( S_f \) denotes the Schwarzian derivative
\[
S_f(z) = \left( \frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2.
\]
Moreover, the first two non-trivial coefficients of the \( n \)-th root transform
\[
(f(z^n))^\frac{1}{n} = z + c_{n+1}z^{n+1} + c_{2n+1}z^{2n+1} + \ldots
\]
of \( f \) with the power series (1.1), are written by
\[ c_{n+1} = \frac{a_2}{n} \]
and
\[ c_{2n+1} = \frac{a_3}{n} + \frac{(n-1)a_2^2}{2n}, \]
so that
\[ a_3 - \lambda a_2^2 = n(c_{2n+1} - \mu c_{n+1}^2), \]
where
\[ \mu = \lambda n + \frac{n-1}{2}. \]

Thus, it is quite natural to ask about inequalities for \( \phi_\lambda \) corresponding to subclasses of \( S \). This is called Fekete–Szegö problem. Actually, many authors have considered this problem for typical classes of univalent functions (see, for instance \([1–6, 8, 11–13, 15, 16]\)).

Recently, Komatu \([14]\) introduced a certain integral operator \( L^\delta_a \) defined by
\[
L^\delta_a f(z) = \frac{a^\delta}{\Gamma(\delta)} \int_0^1 t^{a-2} \left( \log \frac{1}{t} \right)^{\delta-1} f(zt) \, dt,
\]
where
\[ a > 0; \ \delta \geq 0; \ \ f(z) \in A; \ z \in U. \]

Thus, if \( f \in A \) is of the form (1.1), then it is easily seen from (1.2) that (see \([14]\))
\[
L^\delta_a f(z) = z + \sum_{n=2}^{\infty} \left( \frac{a}{a+n-1} \right)^\delta a_n z^n. \tag{1.3}
\]

Using the relation (1.3), it is easy to verify that
\[
z(L^\delta_a f(z))' = aL^\delta_a f(z) - (a-1)L^{\delta+1}_a f(z) \tag{1.4}
\]
and
\[
L^\delta_a (zf'(z)) = z(L^\delta_a f(z))'. \tag{1.5}
\]

We note that:
For $a = 1$ and $\delta = k$ ($k$ is any integer), the multiplier transformation $L_1^k f(z) = I^k f(z)$ was studied by Flett [9] and Salagean [18];

(ii) For $a = 1$ and $\delta = -k$ ($k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$), the differential operator $L_1^{-k} f(z) = D^k f(z)$ was studied by Salagean [18];

(iii) For $a = 2$ and $\delta = k$ ($k$ is any integer), the operator $L_2^k f(z) = L_2^k f(z)$ was studied by Uralegaddi and Somanatha [19];

(iv) For $a = 2$, the multiplier transformation $L_2^k f(z) = I^k f(z)$ was studied by Jung et al. [10].

Using the operator $L_\delta^a$, we now introduce the following classes:

**Definition 1.1** We say that a function $f \in A$ is in the class $S_{a,\delta}(b)$ if

$$\Re \left\{ 1 + \frac{1}{b} \left( \frac{z(L_\delta^a f(z))'}{L_\delta^a f(z)} - 1 \right) \right\} > 0$$

$(a > 0; \; \delta \geq 0; \; b \in \mathbb{C}\setminus\{0\}; \; z \in \mathbb{U})$.

**Definition 1.2** We say that a function $f \in A$ is in the class $C_{a,\delta}(b)$ if

$$\Re \left\{ 1 + \frac{1}{b} \left( \frac{z(L_\delta^a f(z))''}{(L_\delta^a f(z))'} \right) \right\} > 0$$

$(a > 0; \; \delta \geq 0; \; b \in \mathbb{C}\setminus\{0\}; \; z \in \mathbb{U})$.

Note that $f \in C_{a,\delta}(b) \iff zf' \in S_{a,\delta}(b)$. (1.6)

In particular, we have starlike and convex function classes,

$$S_{a,0}(1) = S^* \quad \text{and} \quad C_{a,0}(1) = C,$$

respectively.

We denote by $P$ a class of the analytic functions in $\mathbb{U}$ with

$p(0) = 1 \quad \text{and} \quad \Re \{p(z)\} > 0$.

We shall require the following lemmas.

**Lemma 1.3** [7] Let $p \in P$ with $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$. Then

$$|c_n| \leq 2 \quad (n \geq 1).$$

**Lemma 1.4** [17] Let $p \in P$ with $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$. Then for any complex number $v$

$$|c_2 - vc_1^2| \leq 2 \max\{1, |2v - 1|\},$$

and the result is sharp for the functions given by

$$p(z) = \frac{1 + z^2}{1 - z^2}, \quad p(z) = \frac{1 + z}{1 - z}.$$
2 Main results

**Theorem 2.1** Let $a > 0; \delta \geq 0; b \in \mathbb{C}\setminus\{0\}$. If $f \in S_{a,\delta}(b)$, then

$$|a_2| \leq 2|b|\left(\frac{a + 1}{a}\right)^\delta,$$

(2.1)

$$|a_3| \leq |b|\left(\frac{a + 2}{a}\right)^\delta \max\{1, |1 + 2b|\},$$

(2.2)

and

$$\left|a_3 - \frac{1}{2} \left(\frac{a(a + 2)}{(a + 1)^2}\right)^\delta a_2^2\right| \leq |b|\left(\frac{a + 2}{a}\right)^\delta.$$

Proof Denote

$$L_a^\delta f(z) = z + A_2z^2 + A_3z^3 + \cdots.$$

Then by (1.3), we can write

$$A_2 = \left(\frac{a}{a + 1}\right)^\delta a_2, \quad A_3 = \left(\frac{a}{a + 2}\right)^\delta a_3.$$

(2.3)

By the definition of the class $S_{a,\delta}(b)$, there exists $p \in \mathcal{P}$ such that

$$\frac{z(L_a^\delta f(z))'}{L_a^\delta f(z)} = 1 - b + b p(z),$$

so that

$$\frac{z(1 + 2A_2z + 3A_3z^2 + \cdots)}{z + A_2z^2 + A_3z^3 + \cdots} = 1 - b + b(1 + c_1 z + c_2 z^2 + \cdots),$$

which implies the equality

$$z + 2A_2z^2 + 3A_3z^3 + \cdots = z + (A_2 + bc_1)z^2 + (A_3 + bc_1A_2 + bc_2)z^3 + \cdots.$$

Equating the coefficients of both sides, we have

$$A_2 = bc_1, \quad A_3 = \frac{b}{2}(c_2 + bc_2^2),$$

(2.4)

so that, on account of (2.3)

$$a_2 = b \left(\frac{a + 1}{a}\right)^\delta c_1, \quad a_3 = \frac{b}{2} \left(\frac{a + 2}{a}\right)^\delta (c_2 + bc_2^2).$$

(2.5)

Taking into account (2.5) and Lemma 1.3, we obtain

$$|a_2| \leq 2|b|\left(\frac{a + 1}{a}\right)^\delta,$$

(2.6)

and Lemma 1.4

$$|a_3| = \left|\frac{b}{2} \left(\frac{a + 2}{a}\right)^\delta (c_2 + bc_2^2)\right| \leq |b|\left(\frac{a + 2}{a}\right)^\delta \max\{1, |1 + 2b|\}.$$
Moreover, by Lemma 1.3
\[
\left| a_3 - \frac{1}{2} \left( \frac{a(a+2)}{(a+1)^2} \right)^{\delta} a_2^2 \right| = \left| \frac{b}{2} \left( \frac{a+2}{a} \right)^{\delta} (c_2 + bc_1^2) - \frac{b^2 c_1^2}{2} \left( \frac{a(a+2)}{(a+1)^2} \right)^{\delta} \left( \frac{a+1}{a} \right)^{\delta} \right| \\
= \left| \frac{bc_2}{2} \left( \frac{a+2}{a} \right)^{\delta} \right| \\
\leq |b| \left( \frac{a+2}{a} \right)^{\delta}
\]
as asserted. \(\square\)

Now, we consider functional \(|a_3 - \mu a_2^2|\) for complex \(\mu\).

**Theorem 2.2** Let \(a > 0; \delta \geq 0; b \in \mathbb{C} \setminus \{0\}\). If \(f \in \mathcal{S}_{a,\delta}(b)\), then for \(\mu \in \mathbb{C}\) we have
\[
|a_3 - \mu a_2^2| \leq |b| \left( \frac{a+2}{a} \right)^{\delta} \max \left\{ 1, \left| 1 + 2b - 4\mu b \left( \frac{(a+1)^2}{a(a+2)} \right)^{\delta} \right| \right\}.
\]
Moreover for each \(\mu\), there is a function in \(\mathcal{S}_{a,\delta}(b)\) such that equality holds.

**Proof** Taking into account (2.5) we have
\[
a_3 - \mu a_2^2 = \frac{b}{2} \left( \frac{a+2}{a} \right)^{\delta} (c_2 + bc_1^2) - \mu b^2 c_1^2 \left( \frac{a+1}{a} \right)^{\delta} \]
(2.7)
where
\[
\tau = -b + 2\mu b \left( \frac{(a+1)^2}{a(a+2)} \right)^{\delta}.
\]
Then, with the aid of Lemma 1.4, we obtain
\[
|a_3 - \mu a_2^2| \leq |b| \left( \frac{a+2}{a} \right)^{\delta} \max \left\{ 1, \left| 1 + 2b - 4\mu b \left( \frac{(a+1)^2}{a(a+2)} \right)^{\delta} \right| \right\}, \quad (2.8)
\]
as asserted. An examination of the proof shows that equality is attained for the first case when \(c_1 = 0\) and \(c_2 = 2\) and the corresponding \(f \in \mathcal{S}_{a,\delta}(b)\) is given by
\[
\frac{z(L_a f(z))'}{L_a f(z)} = \frac{1 + (2b - 1)z^2}{1 - z^2}, \quad (2.9)
\]
and likewise for the second case when \(c_1 = c_2 = 2\) the corresponding \(f \in \mathcal{S}_{a,\delta}(b)\) is given by
\[
\frac{z(L_a f(z))'}{L_a f(z)} = \frac{1 + (2b - 1)z}{1 - z}, \quad (2.10)
\]
respectively. \(\square\)

Taking \(\delta = 0\) and \(b = 1\) in Theorem 2.2, we have

**Corollary 2.3** [12] If \(f \in \mathcal{S}^*\), then for \(\mu \in \mathbb{C}\) we have
\[
|a_3 - \mu a_2^2| \leq \max\{1, |4\mu - 3|\}.
\]
Moreover for each \(\mu\), there is a function in \(\mathcal{S}^*\) such that equality holds.
We next consider the case when $\mu$ and $b$ are real. Then we have:

**Theorem 2.4** Let $a > 0; \delta \geq 0; b > 0$. If $f \in S_{a, \delta}(b)$, then for $\mu \in \mathbb{R}$, we have

$$|a_3 - \mu a_2^2| \leq \begin{cases} 
 b \left( \frac{a+2}{a} \right)^\delta \left[ 1 + 2b - 4\mu b \left( \frac{(a+1)^2}{a(a+2)} \right)^\delta \right] & \text{if } \mu \leq \frac{1}{4} \left( \frac{a(a+2)}{(a+1)^2} \right)^\delta \\
 b \left( \frac{a+2}{a} \right)^\delta \left[ -1 - 2b + 4\mu b \left( \frac{(a+1)^2}{a(a+2)} \right)^\delta \right] & \text{if } \frac{1}{4} \left( \frac{a(a+2)}{(a+1)^2} \right)^\delta \leq \mu \leq \frac{1+b}{2b} \left( \frac{a(a+2)}{(a+1)^2} \right)^\delta \\
 b \left( \frac{a+2}{a} \right)^\delta \left[ 1 - 2b + 4\mu b \left( \frac{(a+1)^2}{a(a+2)} \right)^\delta \right] & \text{if } \mu \geq \frac{1+b}{2b} \left( \frac{a(a+2)}{(a+1)^2} \right)^\delta 
\end{cases} \quad (2.11)$$

Moreover for each $\mu$, there is a function in $S_{a, \delta}(b)$ such that equality holds.

**Proof** By (2.7), we obtain

$$a_3 - \mu a_2^2 = \frac{b}{2} \left( \frac{a+2}{a} \right)^\delta \left[ c_2 - \frac{c_1^2}{2} + \frac{c_1^2}{2} \left( 1 + 2b - 4\mu b \left( \frac{(a+1)^2}{a(a+2)} \right)^\delta \right) \right]. \quad (2.12)$$

First, let $\mu \leq \frac{1}{4} \left( \frac{a(a+2)}{(a+1)^2} \right)^\delta$. In this case, by (2.12), Lemma 1.3 and Lemma 1.5 give

$$|a_3 - \mu a_2^2| \leq \frac{b}{2} \left( \frac{a+2}{a} \right)^\delta \left[ 2 - \left| c_1 \right|^2 + \frac{\left| c_1 \right|^2}{2} \left( 1 + 2b - 4\mu b \left( \frac{(a+1)^2}{a(a+2)} \right)^\delta \right) \right]$$

$$\leq b \left( \frac{a+2}{a} \right)^\delta \left[ 1 + 2b - 4\mu b \left( \frac{(a+1)^2}{a(a+2)} \right)^\delta \right].$$

Now let $\frac{1}{4} \left( \frac{a(a+2)}{(a+1)^2} \right)^\delta \leq \mu \leq \frac{1+b}{2b} \left( \frac{a(a+2)}{(a+1)^2} \right)^\delta$. Then, using the above calculations, we get

$$|a_3 - \mu a_2^2| \leq b \left( \frac{a+2}{a} \right)^\delta.$$

Finally, if $\mu \geq \frac{1+b}{2b} \left( \frac{a(a+2)}{(a+1)^2} \right)^\delta$, then we obtain

$$|a_3 - \mu a_2^2| \leq \frac{b}{2} \left( \frac{a+2}{a} \right)^\delta \left[ 2 + \left| c_1 \right|^2 \left( -1 - 2b + 4\mu b \left( \frac{(a+1)^2}{a(a+2)} \right)^\delta \right) \right]$$

$$\leq b \left( \frac{a+2}{a} \right)^\delta \left[ -1 - 2b + 4\mu b \left( \frac{(a+1)^2}{a(a+2)} \right)^\delta \right].$$

Equality is attained for the second case on choosing $c_1 = 0, c_2 = 2$ in (2.9) and in (2.10) $c_1 = c_2 = 2; c_1 = 2i, c_2 = -2$ for the first and third case, respectively. Thus the proof is complete.

Using the relation (1.6), we easily obtain bounds of coefficients and a solution of the Fekete–Szegö problem in $C_{a, \delta}(b)$.

**Theorem 2.5** Let $a > 0; \delta \geq 0; b \in \mathbb{C}\{0\}$. If $f \in C_{a, \delta}(b)$, then

$$|a_2| \leq \left| b \right| \left( \frac{a+1}{a} \right)^\delta,$$

$$|a_3| \leq \left| b \right| \left( \frac{a+2}{a} \right)^\delta \max\{1, \left| 1+2b \right|\},$$

and

$$\left| a_3 - \frac{2}{3} \left( \frac{a(a+2)}{(a+1)^2} \right)^\delta a_2^2 \right| \leq \left| b \right| \left( \frac{a+2}{a} \right)^\delta.$$
Theorem 2.6 Let $a > 0; \delta \geq 0; b \in \mathbb{C}\{0\}$. If $f \in C_{a,\delta}(b)$, then for $\mu \in \mathbb{C}$ we have

$$|a_3 - \mu a_2^2| \leq \frac{|b|}{3} \left(\frac{a + 2}{a}\right)\delta \max \left\{1, \left|1 + 2b - 3\mu b \left(\frac{a + 1}{a(a + 2)}\right)^{\delta}\right|\right\}.$$ 

Moreover for each $\mu$, there is a function in $C_{a,\delta}(b)$ such that equality holds.

Taking $\delta = 0$ and $b = 1$ in Theorem 2.6, we have

Corollary 2.7 [12] If $f \in \mathbb{C}$, then for $\mu \in \mathbb{C}$ we have

$$|a_3 - \mu a_2^2| \leq \max \left\{\frac{1}{3}, |\mu - 1|\right\}.$$ 

Moreover for each $\mu$, there is a function in $\mathbb{C}$ such that equality holds.

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