DORROH EXTENSIONS OF ALGEBRAS AND COALGEBRAS, II

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Abstract. In this paper, we study Dorroh extensions of bialgebras and Hopf algebras. Let \((H, I)\) be both a Dorroh pair of algebras and a Dorroh pair of coalgebras. We give necessary and sufficient conditions for \(H \bowtie_d I\) to be a bialgebra and a Hopf algebra, respectively. We also describe all ideals of Dorroh extensions of algebras and subcoalgebras of Dorroh extensions of coalgebras and compute these ideals and subcoalgebras for some concrete examples.

Introduction

Dorroh [6] gave a general way to embed a ring \(I\) without identity into a ring with an identity \(\mathbb{Z} \oplus I\), which is now called a Dorroh extension of \(I\). In classical ring theory, Dorroh extension has become an important method of constructing new rings and of analyzing properties of rings. Many rings, such as trivial extension of a ring with a bimodule, triangular matrix ring, \(\mathbb{N}\)-graded ring, can be regarded as Dorroh extensions of rings. There are many papers to study Dorroh extensions of algebras, see [2, 3, 4, 5, 7, 8, 11].

In [13], we investigated Dorroh extensions of algebras, Dorroh extensions of coalgebras and the duality between them. In this paper, we continue the study. The paper is organized as follows. In Section 1, we study Dorroh extensions of bialgebras and Hopf algebras. Let \((H, I)\) be meanwhile a Dorroh pair of algebras and a Dorroh pair of coalgebras. Then \(H \bowtie_d I\) is both an algebra and a coalgebra. We give necessary and sufficient conditions for \(H \bowtie_d I\) to be a bialgebra and a Hopf algebra, respectively. In Section 2, we investigate the ideals of Dorroh extensions of algebras. For a given algebra Dorroh extension \(A \bowtie_d I\), we describe the structures of ideals of \(A \bowtie_d I\). In Section 3, we investigate the subcoalgebras of Dorroh extensions of coalgebras. Given a coalgebra Dorroh extension \(C \bowtie_d P\), we describe the structures of subcoalgebra of \(C \bowtie_d P\). In Section 4, as an application, we compute the ideals of unitization \(k \bowtie_d I\) of an algebra \(I\) without identity and the ideals of the trivial extension \(A \bowtie M\) of an algebra \(A\). We also describe all subcoalgebras of counitization \(k \bowtie_d P\) of a coalgebra \(P\) without counit and the subcoalgebras of the trivial extension \(C \bowtie M\) of a coalgebra \(C\).

Throughout this paper, we work over a field \(k\). All algebras are not necessarily unital, and coalgebras are not necessarily counital as in the previous paper [13], unless otherwise stated. However, Hopf algebras have always identities and counits. For basic facts about coalgebras and Hopf algebras, the reader can refer to the books [1][9][12]. For simplicity, we use \(1\) denote the identity maps on vector spaces. When an algebra \(A\) is unital, a left (resp., right) \(A\)-module \(M\) is always assumed to be unital, i.e., \(1_A m = m\) (resp., \(m 1_A = m\)) for any \(m \in M\). For a coalgebra \(C\) and \(c \in C\), we write \(\Delta(c) = \sum c_1 \otimes c_2\). If \((M, \rho)\) is a right (resp., left) \(C\)-comodule, we write \(\rho(m) = \sum m_{(0)} \otimes m_{(1)}\) (resp., \(\rho(m) = \sum m_{(-1)} \otimes m_{(0)}\)), \(m \in M\). When \(C\) is counital, a right (resp., left) \(C\)-comodule \(M\) is always assumed to be counital, i.e., \(\sum m_{(0)} \varepsilon(m_{(1)}) = m\) (resp., \(\sum \varepsilon(m_{-1}) m_{(0)} = m\)), \(m \in M\).

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1. DORROH EXTENSIONS OF HOPF ALGEBRAS

In this section, a bialgebra does not necessarily have an identity nor a counit.

Recall from [13] that \((H, I)\) is called a Dorroh pair of algebras if \(H\) and \(I\) are algebras and \(I\) is an \(H\)-bimodule such that the module actions are compatible with the multiplication of \(I\), that is,

\[
a(xy) = (ax)y, \quad (xa)y = x(ay), \quad (xy)a = x(ya),
\]

for all \(a \in H\) and \(x, y \in I\). In this case, one can construct an associative algebra \(H \bowtie_d I\) as follows: \(H \bowtie_d I = H \oplus I\) as a vector space, the multiplication of \(H \bowtie_d I\) is given by

\[
(a, x)(b, y) = (ab, ay + xb + xy).
\]

\(H \bowtie_d I\) is an algebra Dorroh extension of \(H\) by \(I\), where \(H\) and \(I\) are regarded as subspaces of \(H \bowtie_d I\) in a canonical way. Moreover, any algebra Dorroh extension of \(H\) is isomorphic to some \(H \bowtie_d I\) as an algebra.

Dually, \((H, I)\) is called a Dorroh pair of coalgebras if \(H\) and \(I\) are coalgebras and \(I\) is an \(H\)-bicomodule such that the comodule coactions are compatible with the comultiplication of \(I\), that is,

\[
\begin{align*}
\sum x_1 \otimes x_2(0) \otimes x_2(1) &= \sum x_{(0)1} \otimes x_{(0)2} \otimes x_{(1)}, \\
\sum x_{(1-)} \otimes x_{(0)} \otimes x_2 &= \sum x_{(1-)} \otimes x_{(0)1} \otimes x_{(0)2}, \\
\sum x_{(0)} \otimes x_{(1)} \otimes x_2 &= \sum x_1 \otimes x_{2(1-)} \otimes x_{2(0)},
\end{align*}
\]

for all \(x \in I\). In this case, one can a coassociative coalgebra \(H \bowtie_d I\) as folows: \(H \bowtie_d I = H \oplus I\) as vector spaces, the comultiplication is given by

\[
\Delta(h, x) = \sum (h_1, 0) \otimes (h_2, 0) + \sum (x_{(1-)}, 0) \otimes (0, x_{(0)}) + \sum (0, x_{(0)}) \otimes (x_{(1)}, 0) + \sum (0, x_1) \otimes (0, x_2).
\]

\(H \bowtie_d I\) is a coalgebra Dorroh extension of \(H\) by \(I\). Moreover, any coalgebra Dorroh extension of \(H\) is isomorphic to some \(H \bowtie_d I\) as a coalgebra.

For a Dorroh pair \((H, I)\) of algebras or coalgebras, let \(\tau_H : H \rightarrow H \bowtie_d I, h \mapsto (h, 0)\) and \(\tau_I : I \rightarrow H \bowtie_d I, x \mapsto (0, x)\) be the canonical injections, respectively, and let \(\pi_H : H \bowtie_d I \rightarrow H, (h, x) \mapsto h\) and \(\pi_I : H \bowtie_d I \rightarrow I, (h, x) \mapsto x\) be the canonical projections, respectively. For any subspaces \(U \subseteq H\) and \(V \subseteq I\), denote \(\tau_H(U)\) by \((U, 0)\), \(\tau_I(V)\) by \((0, V)\) and \(\pi_H(U) + \pi_I(V)\) by \((U, V)\). In what follows, we will always use such symbols.

**Definition 1.1.** Let \(H\) be a bialgebra. A bialgebra \(B\) is called a bialgebra Dorroh extension of \(H\) if \(H\) is a subbialgebra of \(B\) and there exists a biideal \(I\) of \(B\) such that \(B = H \oplus I\) as vector spaces. In this case, \(B\) is also called a bialgebra Dorroh extension of \(H\) by \(I\).

Let \((H, I)\) be both a Dorroh pair of algebras and a Dorroh pair of coalgebras. Then \(H \bowtie_d I\) is both an algebra and a coalgebra as stated above. We will give some necessary and sufficient conditions for \(H \bowtie_d I\) to be a bialgebra or a Hopf algebra.

**Proposition 1.2.** Let \((H, I)\) be both a Dorroh pair of algebras and a Dorroh pair of coalgebras. Then \(H \bowtie_d I\) is a bialgebra if and only if the following are satisfied:

(a) \(H\) is a bialgebra, i.e., \(\forall a, b \in H,\)

\[
\sum (ab)_1 \otimes (ab)_2 = \sum a_1 b_1 \otimes a_2 b_2.
\]
(b) The comultiplication of $I$, the left and right $H$-comodule structure maps of $I$ are all $H$-bimodule homomorphism, i.e., $\forall a, b \in H$ and $x, y \in I$.

\[
\sum (ay)_1 \otimes (ay)_2 = \sum a_1 y_1 \otimes a_2 y_2, \\
\sum (xb)_1 \otimes (xb)_2 = \sum x_1 b_1 \otimes x_2 b_2, \\
\sum (ay)(-1) \otimes (ay)(0) = \sum a_1 y(-1) \otimes a_2 y(0), \\
\sum (xb)(-1) \otimes (xb)(0) = \sum x(-1)b_1 \otimes x(0)b_2, \\
\sum (ay)(0) \otimes (ay)(1) = \sum a_1 y(0) \otimes a_2 y(1), \\
\sum (xb)(0) \otimes (xb)(1) = \sum x(0)b_1 \otimes x(1)b_2.
\]

(c) The left and right $H$-comodule structure maps of $I$ are both algebra homomorphisms, i.e., $\forall x, y \in I$.

\[
\sum (xy)(-1) \otimes (xy)(0) = \sum x(-1)y(-1) \otimes x(0)y(0), \\
\sum (xy)(0) \otimes (xy)(1) = \sum x(0)y(0) \otimes x(1)y(1).
\]

(d) The all structure maps of $I$ (multiplication, comultiplication, $H$-bimodule and $H$-bicocomodule) satisfy the following compatible condition: $\forall x, y \in I$.

\[
\sum (xy)_1 \otimes (xy)_2 = \sum x(-1)_1 y_0 \otimes x(0)_1 y(1) + \sum x(-1)_2 y_1 \otimes x(0)_2 y_2 + \sum x(0)_0 y(-1) \otimes x(1)_0 y(0)
\]

\[
+ \sum x(0)_1 y_1 \otimes x(1)_2 y_2 + \sum x(1)_1 y(-1) \otimes x(2)_0 y_0 + \sum x(1)_2 y_0 \otimes x(2)_2 y(1) + \sum x(1)_1 y_1 \otimes x(2)_2 y_2.
\]

In this case, if we regard $H$ and $I$ as subspaces of $H \ltimes_d I$ via $\tau_H$ and $\tau_I$ respectively, then $H \ltimes_d I$ is a bialgebra Dorroh extension of $H$ by $I$.

**Proof.** Note that $H \ltimes_d I$ is a bialgebra if and only if $\Delta((a, x)(b, y)) = \Delta(a, x)\Delta(b, y)$ for any $(a, x), (b, y) \in H \ltimes_d I$. Clearly, $\Delta((a, x)(b, y)) = \Delta(a, x)\Delta(b, y)$ if and only if $\Delta((a, 0)(b, 0)) = \Delta(a, 0)\Delta(b, 0), \Delta((a, 0)(0, y)) = \Delta(a, 0)\Delta(0, y), \Delta((0, x)(b, 0)) = \Delta(0, x)\Delta(b, 0)$ and $\Delta((0, x)(0, y)) = \Delta(0, x)\Delta(0, y)$. Then a straightforward computation shows that $\Delta((a, 0)(b, 0)) = \Delta(a, 0)\Delta(b, 0)$ is equivalent to Eq. (1). Similarly, $\Delta((a, 0)(0, y)) = \Delta(a, 0)\Delta(0, y)$ is equivalent to the three equations Eq. (2), Eq. (4) and Eq. (6), $\Delta((0, x)(b, 0)) = \Delta(0, x)\Delta(b, 0)$ is equivalent to the three equations Eq. (5), Eq. (7) and Eq. (9), and $\Delta((0, x)(0, y)) = \Delta(0, x)\Delta(0, y)$ is equivalent to the three equations Eq. (3), Eq. 9 and Eq. (10).

Now assume that $H \ltimes_d I$ is a bialgebra. Then $H$ is also a bialgebra by (a). By regarding $H = \tau_H(H)$ and $I = \tau_I(I)$, one can see that $H$ is a subbialgebra of $H \ltimes_d I$ and $I$ is a biideal of $H \ltimes_d I$. Therefore, $H \ltimes_d I$ is a bialgebra Dorroh extension of $H$ by $I$.

**Definition 1.3.** A pair $(H, I)$ as stated in Proposition 1.2 is called a Dorroh pair of bialgebras.

**Remark 1.4.** Let $(H, I)$ be a Dorroh pair of bialgebras. Then

(a) Though $I$ is both an algebra and a coalgebra, $I$ is not a bialgebra unless $\sum x(-1)_1 y_0 \otimes x(0)_1 y(1) + \sum x(-1)_2 y_1 \otimes x(0)_2 y_2 + \sum x(-1)_3 y(1) \otimes x(0)_3 y(0) + \sum x(-1)_4 y(0) \otimes x(0)_4 y(0) + \sum x(0)_0 y(0) \otimes x(0)_0 y(0) + \sum x(0)_1 y(1) \otimes x(0)_1 y(1) = 0$ for all $x, y \in I$.

(b) $I$ is an $H$-Hopf bimodule by Eqs. (4)–(7). $I$ is a left $H$-module coalgebra and a right $H$-module coalgebra by Eq. (2) and Eq. (3). Moreover, $I$ is a left $H$-comodule algebra and a right $H$-comodule algebra by Eq. (5) and Eq. (2).

(c) The comultiplication $\Delta_I$ of $I$ is different from $\Delta_{H \ltimes_d I}$, the restriction of the comultiplication $\Delta_{H \ltimes_d I}$ of $H \ltimes_d I$ on $I$. 

**Proposition 1.5.** Let $A$ be a bialgebra Dorroh extension of $H$ by $I$. Then $(H, I)$ is a Dorroh pair of bialgebras and $A = H \bowtie_d I$ as bialgebras, where the comultiplication $\Delta_l$ and the comodule structure maps $\rho_l$ and $\rho_r$ of $I$ are given respectively by the following compositions

$$
\Delta_l : I \hookrightarrow A \overset{\Delta}{\to} A \otimes A \overset{\pi_l \otimes \pi_r}{\to} I \otimes I,
\rho_l : I \hookrightarrow A \overset{\Delta}{\to} A \otimes A \overset{\pi_l \otimes \pi_r}{\to} I \otimes I,
\rho_r : I \hookrightarrow A \overset{\Delta}{\to} A \otimes A \overset{\pi_l \otimes \pi_r}{\to} I \otimes H,
$$

$\pi_H : A \to H$ and $\pi_I : A \to I$ are the projections corresponding to the direct sum decomposition $A = H \oplus I$ of vector spaces.

**Proof.** By [13], one knows that $(H, I)$ is both a Dorroh pair of algebras and a Dorroh pair of coalgebras. Moreover, the canonical linear isomorphism $H \bowtie_d I \to A$, $(h, x) \mapsto h + x$ is both an algebra isomorphism and a coalgebra isomorphism. This implies that $H \bowtie_d I$ is a bialgebra since $A$ is a bialgebra. Hence $(H, I)$ is a Dorroh pair of bialgebras by Proposition [12] \qed

In the following proposition, we describe the relation between the braided bialgebra $B$ and the bialgebra $A = H \bowtie_d I$.

**Proposition 1.6.** Assume that $H$ is a Hopf algebra with antipode $S$. Then the following hold.

(a) Let $(H, I)$ be a Dorroh pair of bialgebras and $A = H \bowtie_d I$. Then with the above notations, $B = (k1, I^{colH})$ and $A = B^h_d H$ as bialgebras, where $I^{colH} = \{ x \in I \mid \rho_l(x) = x \otimes 1 \}$, the $H$-coinvariants of $I$.

(b) Let $B$ be a braided bialgebra in the Yetter-Drinfeld category $^H_H \mathcal{YD}$ and $A = B^h_d H$. Then $(H, I)$ is a Dorroh pair of bialgebras and $A = H \bowtie_d I$ as bialgebras, where $I = \ker(\varepsilon_B)B^h$ and $\varepsilon_B$ is the counit of $B$.

**Proof.** (a) Let $(h, x) \in A$. Then $(h, x) \in B$ if and only if $(1 \otimes \pi_H)\Delta(h, x) = (h, x) \otimes 1$. Now we have $(1 \otimes \pi_H)\Delta(h, x) = \sum(h_1, 0) \otimes h_2 + \sum(0, x_{(0)}) \otimes x_{(1)}$. Suppose $\sum(h_1, 0) \otimes h_2 + \sum(0, x_{(0)}) \otimes x_{(1)} = (h, x) \otimes 1$. Then by applying $\pi_H \otimes 1$ and $\pi_l \otimes 1$ to the equation respectively, one gets $\sum h_1 \otimes h_2 = h \otimes 1$ and $\sum x_{(0)} \otimes x_{(1)} = x \otimes 1$. Hence $h = \varepsilon_H(h)1 \in k1$ and $x \in I^{colH}$, i.e., $(h, x) \in (k1, I^{colH})$. Conversely, if $(h, x) \in (k1, I^{colH})$, then $(h, x) \in B$ by the discussion above. Therefore, $B = (k1, I^{colH})$ and $A = B^h_d H$ as bialgebras.

(b) Let $\iota : H \to B^h_d H = A$ and $\pi : A = B^h_d H \to H$ be defined by $\iota(h) = 1\# h$ and $\pi(b \# h) = \varepsilon_B(b)h$ respectively, where $h \in H$ and $b \in B$. Then by [10], $\iota$ and $\pi$ are bialgebra maps and $\pi \circ \iota = 1$. Regarding $\iota$ as an embedding and $H = \iota(H)$, then $A$ is a bialgebra Dorroh extension of $H$ by $I$ with...
Let \( I = \text{Ker}(\pi) = \text{Ker}(\varepsilon_H) \# H \). By Proposition 1.5, \((H, I)\) is a Dorroh pair of bialgebras and \( A \cong H \bowtie_d I \) as bialgebras.

**Definition 1.7.** Let \( H \) be a Hopf algebra. A Hopf algebra \( A \) is called a Hopf algebra Dorroh extension of \( H \) if \( H \) is a Hopf subalgebra of \( A \) and there exists a Hopf ideal \( I \) of \( A \) such that \( A = H \oplus I \) as vector spaces. In this case, \( A \) is also called a Hopf algebra Dorroh extension of \( H \) by \( I \).

Let \((H, I)\) be a Dorroh pair of bialgebras. Then by Proposition 1.2, \( H \bowtie_d I \) is a bialgebra Dorroh extension of \( H \) by \( I \). That is, under the identifications \( H = \tau_H(H) \) and \( I = \tau_I(I) \), \( H \) is a subbialgebra of \( H \bowtie_d I \) and \( I \) is a bideal of \( H \bowtie_d I \).

**Proposition 1.8.** Let \((H, I)\) be a Dorroh pair of bialgebras. Assume that \( H \) is a Hopf algebra with antipode \( S_H \). Then \( H \bowtie_d I \) is a Hopf algebra Dorroh extension of \( H \) by \( I \) if and only if there exists a linear endomorphism \( S_I \) of \( I \) such that for any \( x \in I \),

\[
\sum S_H(x_{(-1)})x_{(0)} + \sum S_I(x_{(0)})x_{(1)} + \sum S_I(x_{(1)})x_{(2)} = 0,
\]

\[
\sum x_{(-1)}S_I(x_{(0)}) + \sum x_{(0)}S_H(x_{(1)}) + \sum x_{(1)}S_I(x_{(2)}) = 0.
\]

**Proof.** By Proposition 1.2, \( H \bowtie_d I \) is a bialgebra Dorroh extension of \( H \) by \( I \), where \( H \) and \( I \) are regarded as subspaces of \( H \bowtie_d I \) under the identifications \( H = \tau_H(H) \) and \( I = \tau_I(I) \). Since \( H \) is a Hopf algebra, \( H \bowtie_d I \) has an identity \((1_H, 0)\) and a counit \((\varepsilon_H, 0)\).

Assume that there exists a linear endomorphism \( S_I \) of \( I \) such that

\[
\sum S_H(x_{(-1)})x_{(0)} + \sum S_I(x_{(0)})x_{(1)} + \sum S_I(x_{(1)})x_{(2)} = 0,
\]

\[
\sum x_{(-1)}S_I(x_{(0)}) + \sum x_{(0)}S_H(x_{(1)}) + \sum x_{(1)}S_I(x_{(2)}) = 0,
\]

for any \( x \in I \). Define a linear map \( S : H \bowtie_d I \to H \bowtie_d I \) by \( S(h, x) = (S_H(h), S_I(x)) \) for any \( (h, x) \in H \bowtie_d I \). Then for any \((h, x) \in H \bowtie_d I \),

\[
\sum S((h, x)(1))(h, x) = \sum S(h_{(1)})(0, x_{(2)}) + \sum S(h_{(2)})(0, x_{(1)}),
\]

\[
= \sum S(h_{(1)})(0, x_{(2)}) + \sum S(h_{(2)})(0, x_{(1)}),
\]

\[
= \sum \tau_H(h_{(1)})(0, x_{(2)}) + \sum \tau_I(h_{(2)})(0, x_{(1)}),
\]

and similarly \( \sum \tau_H(h_{(1)})(0, x_{(2)}) = \varepsilon(h, x)(1_H, 0) \). Thus, \( H \bowtie_d I \) is a Hopf algebra. Clearly, \( H \) is a Hopf subalgebra of \( H \bowtie_d I \) and \( I \) is a Hopf ideal of \( H \bowtie_d I \). Hence \( H \bowtie_d I \) is a Hopf algebra Dorroh extension of \( H \) by \( I \).

Conversely, assume that \( H \bowtie_d I \) is a Hopf algebra Dorroh extension of \( H \) by \( I \). Then \( H \) is a Hopf subalgebra of \( H \bowtie_d I \) and \( I \) is a Hopf ideal of \( H \bowtie_d I \). Hence \( S(h, x) = (S_H(h), S_I(x)) \) for any \( h \in H \), and there is a linear endomorphism \( S_I \) of \( I \) such that \( S(0, x) = (0, S_I(x)) \) for any \( x \in I \), where \( S \) is the antipode of \( H \bowtie_d I \). Now let \( x \in I \). Then \( \sum S((0, x)(1))(0, x) = \varepsilon(0, x)(1_H, 0) = 0 \). On the other hand, we have

\[
\sum S((0, x)(1))(0, x) = \sum S_{H}(x_{(-1)}), 0)(0, x_{(0)}) + \sum S(0, x_{(0)})(x_{(1)}, 0) + \sum S(0, x_{(1)})(0, x_{(2)}),
\]

\[
= \sum S_{H}(x_{(-1)}), 0)(0, x_{(0)}) + \sum S_{I}(x_{(0)})(x_{(1)}, 0) + \sum S_{I}(x_{(1)})(0, x_{(2)}),
\]

\[
= \sum S_{H}(x_{(-1)}), x_{(0)} + \sum S_{I}(x_{(0)})(x_{(1)}, 0) + \sum S_{I}(x_{(1)})(0, x_{(2)}),
\]

\[
= (0, S_{H}(x_{(-1)}), x_{(0)}) + \sum S_{I}(x_{(0)})(x_{(1)}, 0) + \sum S_{I}(x_{(1)})(0, x_{(2)}),
\]

It follows that \( \sum S_{H}(x_{(-1)}), x_{(0)} + \sum S_{I}(x_{(0)})(x_{(1)}, 0) + \sum S_{I}(x_{(1)})(0, x_{(2)}) = 0 \). Similarly, from \( \sum (0, x_{(1)})(0, x_{(2)}) = \varepsilon(0, x)(1_H, 0) = 0 \), one gets \( \sum x_{(-1)}S_I(x_{(0)}) + \sum x_{(0)}S_H(x_{(1)}) + \sum x_{(1)}S_I(x_{(2)}) = 0 \).
Definition 1.9. A pair \((H, I)\) as stated in Proposition 1.8 is called a Dorroh pair of Hopf algebras.

Corollary 1.10. Let \((H, I)\) be a Dorroh pair of Hopf algebras. With the notations of Proposition 1.8 for any \(h, x, y \in I\), we have

(a) \(S_I(hx) = S_I(x)S_H(h), S_I(xh) = S_H(h)S_I(x)\) and \(S_I(xy) = S_I(y)S_I(x)\),
(b) \(\sum S_I(x(0)) \otimes S_I(x(1)) = \sum S_I(x(0)) \otimes S_H(x(-1)), \sum S_I(x(-1)) \otimes S_I(x(0)) = \sum S_H(x(1)) \otimes S_I(x(0))\)

Proof. By Proposition 1.8 \(H \bowtie_d I\) is a Hopf algebra with the antipode \(S\) given by \(S(h, x) = (S_H(h), S_I(x))\), \((h, x) \in H \bowtie_d I\). Let \(h \in H\) and \(x, y \in I\). Since \(S\) is an algebra antipomorphism, we have \(S((h, 0)(0, x)) = S(0, x)S(h, 0) = S(h, 0)S(0, x)\). They are equivalent to \(S_I(hx) = S_I(x)S_H(h), S_I(xh) = S_H(h)S_I(x)\) and \(S_I(xy) = S_I(y)S_I(x)\), respectively. This shows (a). Since \(S\) is also a coalgebra antipomorphism, we have \(\sum (S(0, x)I) \otimes (S(0, x)I) = \sum (S((0, x)I)^{0}) \otimes (S((0, x)I)^{1})\), that is,

\[
\sum (S_I(x(-1), 0)) \otimes (S_I(x(0))) + \sum (S_I(x(0)) \otimes (S_I(x(1), 0)) + \sum (S_I(x(1)) \otimes (S_I(x(0)))) = \sum (0, S_I(x(0))) \otimes (S_I(x(-1), 0)) + \sum (S_H(x(1)), 0) \otimes (S_I(x(0))) + \sum (S_I(x(1)) \otimes (S_I(x(2)))) = \sum (S_I(x(2)) \otimes S_I(x(1))).
\]

Applying \(\pi_1 \otimes \pi_2, \pi_2 \otimes \pi_1\) to the above equation respectively, one gets \(\sum (S_I(x(0)) \otimes S_I(x(1)) = \sum S_H(x(1)) \otimes S_I(x(0))\) and \(\sum S_I(x(1)) \otimes S_I(x(2)) = \sum S_I(x(2)) \otimes S_I(x(1))\). This shows (b). \(\square\)

Proposition 1.11. Let \(A\) be a Hopf algebra Dorroh extension of \(H\) by \(I\). Then \((H, I)\) is a Dorroh pair of Hopf algebras and \(A \cong H \bowtie_d I\) as Hopf algebras, where the comultiplication \(\Delta_A\) of \(I\) and the comodule structure maps of \(I\) are given in Proposition 1.5.

Proof. By Proposition 1.5 and its proof, \((H, I)\) is a Dorroh pair of bialgebras and there is a bialgebra isomorphism \(\phi: H \bowtie_d I \rightarrow A\). Since \(A\) is a Hopf algebra, \(H \bowtie_d I\) is also a Hopf algebra with the antipode \(S\) given by \(S = \phi^{-1} \circ S_A \circ \phi\) and \(\phi\) is a Hopf algebra isomorphism, where \(S_A\) is the antipode of \(A\). Since \(A\) is a Hopf algebra Dorroh extension of \(H\) by \(I\), \(H \bowtie_d I\) is a Hopf algebra Dorroh extension of \(\phi^{-1}(H)\) by \(\phi^{-1}(I)\). Clearly, \(\phi^{-1}(H) = \tau_H(H)\) and \(\phi^{-1}(I) = \tau_I(I)\). Thus, under the identifications \(H = \tau_H(H)\) and \(I = \tau_I(I)\), \(H \bowtie_d I\) is a Hopf algebra Dorroh extension of \(H\) by \(I\), and so \((H, I)\) is a Dorroh pair of Hopf algebras by Proposition 1.8. \(\square\)

In the rest of this section, we give some examples of Dorroh pairs (or Dorroh extensions) of bialgebras and Hopf algebras.

Example 1.12. (a) Let \(I\) be both an algebra and a coalgebra. Assume that the multiplication and the comultiplication of \(I\) satisfy

\[
\Delta(x) = x \otimes y + y \otimes x + \sum y_1 \otimes y_2 + \sum x_1 \otimes y_2 + \sum x_1 y \otimes x_2 + \sum x_1 y_1 \otimes x_2 y_2
\]

for all \(x, y \in I\). Here \(I\) is a bialgebra. Note that the ground field \(k\) is a Hopf algebra as usual. Hence \(I\) is a \(k\)-Hopf bimodule with the trivial comodule structures. One can check that \((k, I)\) is a Dorroh pair of bialgebras, and so \(k \bowtie_d I\) is a usual bialgebra with the identity \((1, 0)\) and the counit \(\varepsilon\) given by \(\varepsilon(\alpha, x) = \alpha, (\alpha, x) \in k \bowtie_d I\). Moreover, the multiplication and comultiplication of \(k \bowtie_d I\) are given by

\[
(\alpha, x)(\beta, y) = (\alpha \beta, \alpha y + \beta x + xy),
\]

\[
\Delta(\alpha, x) = (\alpha, 0) \otimes (1, 0) + (1, 0) \otimes (0, x) + (0, x) \otimes (1, 0) + \sum (0, x_1) \otimes (0, x_2).
\]

Clearly \((\alpha, x)\) is a group-like element if and only if \(\alpha = 1\) and \(\Delta(x) = x \otimes x\).

If there exists a linear map \(S_I: I \rightarrow I\) such that \(\sum S_I(x_1) = \sum x_1 S_I(x_2) = -S_I(x) - x\), then it follows from Proposition 1.8 that \(k \bowtie_d I\) is a Hopf algebra with the antipode \(S\) given by \(S(\alpha, x) = (\alpha, S_I(x))\).
(b) Let $H$ be a bialgebra and $M$ an $H$-Hopf bimodule. Define a multiplication on $M$ by $xy = 0$, $x, y \in M$, and a comultiplication on $M$ by $\Delta_M = 0$, respectively. Then $M$ is both an algebra and a coalgebra. From Proposition 2.2 and Proposition 1.3, one obtains the results in [14, Theorem 11 and Corollary 12] as follows.

(i) The trivial extension $H \langle M$ is a bialgebra if and only if $M$ is an $H$-Hopf bimodule and satisfies
\[ \sum m_{(m)}x(y) \otimes m_{(y)}x(1) + \sum m_{(m)}x(1) \otimes m_{(y)}x(0) = 0 \text{ for all } m, x \in M. \]

(ii) Suppose the trivial extension $H \langle M$ is a bialgebra. If $H$ is a Hopf algebra, then so is $H \angle M$.

(c) Let $H = \oplus_{i=1}^{\infty} H_i$ be an $\mathbb{N}$-graded Hopf algebra. Set $I := \oplus_{i=1}^{\infty} H_i$. Then $(H_0, I)$ is a Dorroh pair of Hopf algebras and $H \equiv H_0 \ltimes_d I$ as Hopf algebras.

2. The ideals of algebra Dorroh extensions

Throughout this section, let $(A, I)$ be a Dorroh pair of algebras. Then $A \ltimes_d I$ is an algebra Dorroh extension of $A$ by $I$ as stated in the last section. We will study the ideals of $A \ltimes_d I$.

Let $K \subseteq A \ltimes_d I$ be a subspace. Put
\[
B := \{ a \in A | (a, x) \in K \text{ for some } x \in I \}, \quad J := \{ x \in I | (a, x) \in K \text{ for some } a \in A \}, \quad Z := \{ a \in A | (a, 0) \in K \}, \quad L := \{ x \in I | (0, x) \in K \}.
\]

Then $B$ and $Z$ are subspaces of $A$ and $Z \subseteq B$, $J$ and $L$ are subspaces of $I$ and $L \subseteq J$. Define a linear map $\varphi : J \to B/Z \subseteq A/Z$ by $\varphi(x) = a + Z$ if $(a, -x) \in K$. Obviously, $\varphi$ is a well-defined linear map.

Proposition 2.1. With the above notations, $K$ is an ideal of $A \ltimes_d I$ if and only if the following (a), (b) and (c) hold if and only if the following (a), (b') and (c') hold.

(a) $Z$ and $B$ are both ideals of $A$, so $B/Z$ is a well-defined $A$-Algebra.

(b) $J$ is an $A$-subAlgebra of $I$.

(b') $J$ is an $A$-subbimodule of $I$.

(c) $\varphi : J \to B/Z$ is an $A$-Algebra homomorphism, and if $\varphi(x) = a + Z$ then $ay \in \ker(\varphi)$ for any $y \in I$.

(c') $\varphi : J \to B/Z$ is an $A$-bimodule homomorphism, and if $\varphi(x) = a + Z$ then $ay \in \ker(\varphi)$ for any $y \in I$.

Proof. It is similar to the proof of [8, Proposition 5]. Here we only prove that (a), (b') and (c') imply that $K$ is an ideal of $A \ltimes_d I$.

Let $(a, -x) \in K$. Then $a \in B$, $x \in J$ and $\varphi(x) = a + Z$. Let $b \in A$ and $y \in I$. Then $ab \in B$ by (a), and $xb \in J$ by (b'). By (c'), $\varphi(ab) = \varphi(a)b = (a + Z)b = ab + Z$, and hence $(ab, -xb) \in K$. Again by (c'), $ay \in \ker(\varphi)$. Hence $ay \in J$ and $\varphi(ay) = 0 + Z$. This implies $(0, -ay + xy) \in K$. Therefore, $(a, -x)(b, y) = (ab, ay - xb - xy) = (ab, -xb) - (0, -ay + xy) \in K$. Similarly, one can show that $(b, y)(a, -x) \in K$. Thus, $K$ is an ideal of $A \ltimes_d I$. \qed
By Proposition 2.1 for any subspaces $Z \subseteq B \subseteq A$ and $J \subseteq I$, if there exists a linear surjection $\varphi : J \to B/Z$ such that (a), (b') and (c') (or (a), (b) and (c)) in Proposition 2.1 are satisfied, then the following set is an ideal of $A \ltimes_d I$:
$$[(a, -x) \in A \ltimes_d I | a \in B, x \in J, \varphi(x) = a + Z].$$
Conversely, any ideal of $A \ltimes_d I$ has this form.

We can also use the induced isomorphism $\overline{\varphi} : J/L \to B/Z$ to replace the homomorphism $\varphi$.

**Corollary 2.2.** With the above notations, $K$ is an ideal of $A \ltimes_d I$ if and only if the following (a), (b) and (c) hold if and only if the following (a), (b') and (c') hold.

(a) $Z$ and $B$ are both ideals of $A$, so $B/Z$ is a well-defined $A$-Algebra.
(b) $J$ is an $A$-subAlgebra of $I$ and $L$ is an $A$-Ideal of $I$, so $J/L$ is a well-defined $A$-Algebra.
(b') $J$ and $L$ are both $A$-subbimodules of $I$, so $J/L$ is a well-defined $A$-bimodule.
(c) $\overline{\varphi} : J/L \to B/Z$ is an $A$-Algebra isomorphism, and if $\overline{\varphi}(x + L) = a + Z$ then $ay - xy, xa - yx \in L$ for any $y \in I$.
(c') $\overline{\varphi} : J/L \to B/Z$ is an $A$-bimodule isomorphism, and if $\overline{\varphi}(x + L) = a + Z$ then $ay - xy, xa - yx \in L$ for any $y \in I$.

**Proof.** It follows from Proposition 2.1 and $\text{Ker}(\varphi) = L$. \hfill $\square$

Let $\tau_A : A \to A \ltimes_d I$ and $\tau_I : I \to A \ltimes_d I$ be the canonical inclusions, and let $\pi_A : A \ltimes_d I \to A$ and $\pi_I : A \ltimes_d I \to I$ be the canonical projections as in the last section.

**Lemma 2.3.** With the notations above, $\tau_A$, $\tau_I$ and $\pi_A$ are algebra homomorphisms, and the sequences
$$0 \to L \xrightarrow{\tau_I} K \xrightarrow{\pi_I} B \to 0, \quad 0 \to Z \xrightarrow{\tau_I} K \xrightarrow{\pi_I} J \to 0$$
are exact, where $K$ is a subspace of $A \ltimes_d I$, and $Z \subseteq B \subseteq A$ and $L \subseteq J \subseteq I$ are subspaces constructed from $K$ as before.

**Proof.** By the definition of $A \ltimes_d I$, one can see that $\tau_A$, $\tau_I$ and $\pi_A$ are algebra homomorphisms. By the definitions of $B$, $J$, $Z$ and $L$, we have $\pi_A(K) = B$, $(0, I) \cap K = (0, L)$, $\pi_I(K) = J$ and $(A, 0) \cap K = (Z, 0)$. Hence the exactness of the sequences in the lemma follows form that of the sequences
$$0 \to I \xrightarrow{\tau_I} A \ltimes_d I \xrightarrow{\pi_I} A \to 0, \quad 0 \to A \xrightarrow{\tau_I} A \ltimes_d I \xrightarrow{\pi_I} I \to 0.$$ 

\hfill $\square$

By Lemma 2.3, we have an algebra isomorphism $K/(0, L) \cong B$ and a vector space isomorphism $K/(Z, 0) \cong J$. Note that $\pi_I$ is generally not an algebra homomorphism unless $AI = IA = 0$, i.e., the module actions are zero.

**Proposition 2.4.** Assume that $K$ is an ideal of $A \ltimes_d I$. Let $Z \subseteq B \subseteq A$ and $L \subseteq J \subseteq I$ be subspaces constructed from $K$ as before. Then $(Z, L)$ is an ideal of $K$ and $K/(Z, L) \cong B/Z \cong J/L$ as algebras.

**Proof.** Let $z \in Z$, $x \in L$ and $(a, -y) \in K$. Then by Corollary 2.2 (a) and (c), we have $az, za \in Z$ and $ax - yx, xa - xy \in L$. Since $(z, 0) \in K$, $(0, y)(z, 0) = (0, yz) \in K$ and $(z, 0)(0, y) = (0, zy) \in K$, and so $zy, yz \in L$. Hence $(a, -y)(z, x) = (az, ax - yz - yx) \in (Z, L)$ and $(z, x)(a, -y) = (za, xa - zy - xy) \in (Z, L)$. This shows that $(Z, L)$ is an ideal of $K$.

By Lemma 2.3, $\pi_A : K \to B$ is an algebra epimorphism with $\text{Ker}(\pi_A) : K \to B) = (0, L)$. Since $(Z, L) \supseteq (0, L)$ and $\pi_A(Z, L) = Z$, $\pi_A$ induces an algebra isomorphism
$$\overline{\pi_A} : K/(Z, L) \to B/Z, (a, -y) + (Z, L) \mapsto a + Z.$$

By Corollary 2.2 (c), $B/Z \cong J/L$ as algebras. \hfill $\square$
3. The subcoalgebras of coalgebra Dorroh extensions

Throughout this section, let $(C, P)$ be a Dorroh pair of coalgebras. Then $C \bowtie_d P$ is a coalgebra Dorroh extension of $C$ by $P$ as stated in Section 1. We shall study the subcoalgebras of $C \bowtie_d P$.

Let $T$ be a subspace of $C \bowtie_d P$. Put
\[
D := \{ c \in C | (c, p) \in T \text{ for some } p \in P \},
\]
\[
Q := \{ p \in P | (c, p) \in T \text{ for some } c \in C \},
\]
\[
E := \{ c \in C | (c, 0) \in T \},
\]
\[
R := \{ p \in P | (0, p) \in T \}.
\]

Then $D$ and $E$ are subspaces of $C$ and $E \subseteq D$, and $Q$ and $R$ are subspaces of $P$ and $R \subseteq Q$. Define a linear map $\eta : D \to P/R$ by $\eta(c) = p + R$ if $(c, p) \in T$. Clearly, $\eta$ is a well-defined linear map. $\text{Im}(\eta) = Q/R$ and $\text{Ker}(\eta) = E$. Thus, $\eta$ induces a linear isomorphism $\overline{\eta} : D/E \to Q/R$ given by $\overline{\eta}(c + E) = p + R$ when $(c, p) \in T$. In this case, the subspace $T$ can be described as follows:
\[
T = \{ (c, p) \in C \bowtie_d P | c \in D, p \in Q, \eta(c) = p + R \}
= \{ (c, p) \in C \bowtie_d P | c \in D, p \in Q, \overline{\eta}(c + E) = p + R \}.
\]

Let $\tau_C : C \to C \bowtie_d P$ and $\tau_P : P \to C \bowtie_d P$ be the canonical inclusions, and let $\pi_C : C \bowtie_d P \to C$ and $\pi_P : C \bowtie_d P \to P$ be the canonical projections as before.

In the rest of this section, unless otherwise stated, we fix the above notations.

**Lemma 3.1.** $\tau_C, \pi_C$ and $\pi_P$ are coalgebra homomorphisms, and the sequences
\[
0 \to R \xrightarrow{\rho_l} T \xrightarrow{\tau_C} D \to 0, \quad 0 \to E \xrightarrow{\rho_r} T \xrightarrow{\tau_P} Q \to 0
\]
are exact.

**Proof.** It is straightforward to check that $\tau_C, \pi_C$ and $\pi_P$ are coalgebra homomorphisms. The exactness of the two sequences follows from an argument similar to the proof of Lemma 2.3. \qed

Hence $T/(0, R) \cong D, T/(E, 0) \cong Q$ as vector spaces. Note that $\tau_P$ is generally not a coalgebra homomorphism.

Since $\pi_C$ is a coalgebra homomorphism, $C \bowtie_d P$ becomes a C-bicomodule with the comodule structure maps given by $\rho_l = (\pi_C \otimes 1)\Delta$ and $\rho_r = (1 \otimes \pi_C)\Delta$, respectively. That is, $\rho_l(c, p) = (\pi_C \otimes 1)\Delta(c, p) = \sum c_1 \otimes (c_2, 0) + \sum p_{(-1)} \otimes (0, p_{(0)})$ and $\rho_r(c, p) = (1 \otimes \pi_C)\Delta(c, p) = \sum (c_1, 0) \otimes c_2 + \sum (0, p_{(0)}) \otimes p_{(1)}$, where $(c, p) \in C \bowtie_d P$. In this case, $\pi_P : C \bowtie_d P \to P$ is a C-bicomodule homomorphism. Obviously, $C \bowtie_d P$ is a C-Coalgebra.

Summarizing the discussion above, one gets the following lemma.

**Lemma 3.2.** $C \bowtie_d P$ is a C-Coalgebra with the comodule structure maps $\rho_l = (\pi_C \otimes 1)\Delta$ and $\rho_r = (1 \otimes \pi_C)\Delta$. Moreover, $\pi_P : C \bowtie_d P \to P$ is a C-Coalgebra homomorphism.

Suppose that $R$ is a $C$-Coideal of $P$. Then $P/R$ becomes a $C$-Coalgebra and $(C, P/R)$ is also a Dorroh pair of coalgebras. Moreover, the canonical epimorphism $\pi : P \to P/R$ is a $C$-Coalgebra homomorphism, which induces a coalgebra epimorphism $(1, \pi) : C \bowtie_d P \to C \bowtie_d (P/R), (c, p) \mapsto (c, \pi(p))$. Clearly, $\text{Ker}(1, \pi) = (0, R)$, a coideal of $C \bowtie_d P$. Hence $(1, \pi)$ induces a coalgebra isomorphism $(C \bowtie_d P)/(0, R) \to C \bowtie_d (P/R), (c, p) + (0, R) \mapsto (c, \pi(p)) = (c, p + R)$. We identify $(C \bowtie_d P)/(0, R)$ with $C \bowtie_d (P/R)$ via the isomorphism. In this case, the map $(1, \pi) : C \bowtie_d P \to C \bowtie_d (P/R)$ is exactly the canonical epimorphism $C \bowtie_d P \to (C \bowtie_d P)/(0, R)$, denoted simply by $\pi$. Moreover, for any $(c, p) \in T$, $\pi(c, p) = (c, p + R) = (c, \eta(c))$. Let $\pi_C : C \bowtie_d (P/R) \to C$ and $\pi_{P/R} : C \bowtie_d (P/R) \to P/R$ denote the corresponding projections, respectively.
Proposition 3.3. \( T \) is a subcoalgebra of \( C \kappa_d P \) if and only if the following hold:

(a) \( D \) is a subcoalgebra of \( C \).

(b) \( Q \) is a \( C \)-subcoalgebra of \( P \) and \( R \) a \( C \)-Coideal of \( Q \), and \( \rho_i^\ell(Q) \subseteq D \otimes Q \) and \( \rho_i^r(Q) \subseteq Q \otimes D \), where \( \rho_i^\ell \) and \( \rho_i^r \) are the \( C \)-module structure maps of \( P \). Hence \( Q \) and \( Q/R \) are both \( D \)-coalgebras.

(c) \( \eta : D \to Q/R \) is a \( C \)-bicomodule (or \( D \)-bicocomodule) homomorphism, and

\[
\sum \eta(p_{(-1)}) \otimes p_{(0)} = \sum \pi(p_1) \otimes p_2, \quad \sum p_{(0)} \otimes \eta(p_{(1)}) = \sum p_1 \otimes \pi(p_2), \quad \forall p \in Q.
\]

If this is the case, then \( \eta \) is a coalgebra homomorphism, and hence it is a \( C \)-Coalgebra (or \( D \)-Coalgebra) homomorphism.

Proof. Assume that \( T \) is a subcoalgebra of \( C \kappa_d P \). By Lemma 3.1, \( D = \pi_C(T) \), \( Q = \pi_P(T) \), and \( \pi_C \) and \( \pi_P \) are \( C \)-coalgebras. It follows that \( D \) is a subcoalgebra of \( C \) and \( Q \) is a subcoalgebra of \( P \). By Lemma 3.2, \( C \kappa_d \) is a \( C \)-Coalgebra and \( \pi_P : C \kappa_d P \to P \) is a \( C \)-Coalgebra homomorphism. Since \( T \) is a subcoalgebra of \( C \kappa_d P \), \( T \) becomes a \( C \)-subcoalgebra of \( C \kappa_d P \). Hence \( Q = \pi_P(T) \) is a \( C \)-subcoalgebra of \( P \), the restriction map \( \pi_P : T \to Q \) is a \( C \)-Coalgebra homomorphism.

Since \( T \) is a subcoalgebra and \((0, P)\) is a coideal of \( C \kappa_d P \), \((0, R) \cap T \) is a coideal of \( T \), and so \( R = \pi_R(0, R) \) is a coideal of \( Q = \pi_P(T) \).

Now we have \( \rho_i(T) = (\pi_C \otimes 1)\Delta(T) \subseteq (\pi_C \otimes 1)(T \otimes T) = D \otimes T \), and similarly \( \rho_i(T) \subseteq T \otimes D \). Since \( \pi_P \) is a \( C \)-bicomodule homomorphism, \( \rho_i^\ell(Q) = \rho_i^r(\pi_P(T)) = (1 \otimes \pi_P)(\rho_i(T)) \subseteq (1 \otimes \pi_P)(D \otimes T) = D \otimes Q \), and similarly \( \rho_i^r(Q) \subseteq Q \otimes D \).

Since \((0, R)\) is a coideal of \( T \), \( \Delta(0, R) \subseteq T \otimes (0, R) + (0, R) \otimes T \). Hence \( \rho_i(0, R) = (\pi_C \otimes 1)\Delta(0, R) \subseteq (\pi_C \otimes 1)(T \otimes (0, R) + (0, R) \otimes T) = D \otimes (0, R) \), and similarly \( \rho_i(0, R) \subseteq (0, R) \otimes D \). So \((0, R)\) is a \( C \)-subcoalgebra of \( T \). Since \( \pi_P \) is a \( C \)-bicomodule homomorphism, \( R = \pi_P(0, R) \) is \( C \)-subbicomodule of \( P \). This proves (a) and (b).

Let \( c \in D \) (resp., \( p \in Q \)). Then there exists some \( p \in Q \) (resp., \( c \in D \)) such that \((c, p) \in T \), and hence \( \eta(c) = p + R = \pi(p) \). Since \( T \) is a subcoalgebra of \( C \kappa_d P \), there exist elements \((c_i^\ell, p_i^\ell), (c_i^r, p_i^r) \) in \( T \), \( 1 \leq i \leq n \), such that \( \Delta(c, p) = \sum_{i=1}^n (c_i^\ell, p_i^\ell) \otimes (c_i^r, p_i^r) \). Moreover, \( \eta(c_i^\ell(p_i^\ell)) = \pi(p_i^\ell) \) and \( \eta(c_i^r(p_i^r)) = \pi(p_i^r) \), \( 1 \leq i \leq n \). On the other hand, we have \( \Delta(c, p) = \sum (c_1, 0) \otimes (c_2, 0) + \sum (p_{(-1)}, 0) \otimes (0, p_0) + \sum (0, p_{(0)}) \otimes (p_{(1)}, 0) + (0, p_1) \otimes (0, p_2) \). Hence

\[
\sum_{i=1}^n (c_i^\ell(p_i^\ell)) = \sum (c_1, 0) \otimes (c_2, 0) + \sum (p_{(-1)}, 0) \otimes (0, p_0) + \sum (0, p_{(0)}) \otimes (p_{(1)}, 0) + (0, p_1) \otimes (0, p_2).
\]

Applying \( \pi_C \otimes \pi_C, \pi_C \otimes \pi_P, \pi_P \otimes \pi_C \) and \( \pi_P \otimes \pi_P \) to the above equation respectively, one gets the following equations:

\[
\sum_{i=1}^n c_i^\ell \otimes c_i^r = \sum c_1 \otimes c_2, \quad \sum_{i=1}^n c_i^\ell \otimes p_i^r = \sum p_{(-1)} \otimes p_{(0)}, \quad \sum_{i=1}^n p_i^\ell \otimes c_i^r = \sum p_0 \otimes p_{(1)}, \quad \sum_{i=1}^n p_i^\ell \otimes p_i^r = \sum p_1 \otimes p_2.
\]

Hence \( \sum \eta(c_{(-1)} \otimes \eta(c_{(0)}) = \sum (p_{(-1)} \otimes \pi(p_{(0)}) = \sum p_{(-1)} \otimes \pi(p_{(0)}) = \sum_{i=1}^n c_i^\ell \otimes \pi(p_i^r) = \sum_{i=1}^n c_i^\ell \otimes \eta(c_i^r) = \sum c_1 \otimes \eta(c_2) \). Similarly, one can check that \( \sum \eta(c_{(0)}) \otimes \eta(c_{(1)}) = \sum \eta(c_1) \otimes c_2 \). Therefore, \( \eta \) is a \( C \)-bicomodule homomorphism. We also have \( \sum \eta(p_{(-1)}) \otimes p_{(0)} = \sum_{i=1}^n \eta(c_i^r) \otimes p_i^r = \sum_{i=1}^n \pi(p_i^r) \otimes p_i^r = \sum \eta(p_1) \otimes p_2 \), and similarly \( \sum p_0 \otimes \eta(p_{(1)}) = \sum p_1 \otimes \pi(p_2) \). This shows (c).

Conversely, assume that (a), (b) and (c) are satisfied. Let \((c, p) \in T \). Then \( c = D \), \( p \in Q \) and \( \eta(c) = p + R = \pi(p) \in Q/R \). By (a) and (b), \( \sum c_1 \otimes c_2 \in D \otimes D, \sum p_1 \otimes p_2 \in Q \otimes Q, \sum p_{(-1)} \otimes p_{(0)} \in D \otimes Q \) and \( \sum p_{(0)} \otimes p_{(1)} \in Q \otimes D \). By (c), we have \( \sum c_1 \otimes c_2 = \sum p_{(-1)} \otimes \pi(p_{(0)}), \sum \eta(c_1) \otimes c_2 = \sum \pi(p_{(0)}) \otimes p_{(1)}, \sum \eta(p_{(-1)}) \otimes \pi(p_{(0)}) = \sum \eta(p_1) \otimes p_2 \). This shows (c).
\[ \sum \eta(p_{(-1)}) \otimes p_{(0)} = \sum \pi(p_1) \otimes p_2 \text{ and } \sum p_{(0)} \otimes \eta(p_{(1)}) = \sum p_1 \otimes \pi(p_2). \] Thus, we have
\[
(\pi \otimes 1)\Delta(c, p) = (\pi \otimes 1)(\sum (c_1, 0) \otimes (c_2, 0) + \sum (p_{(-1)}, 0) \otimes (0, p_{(0)}))
+ \sum (0, p_{(0)}) \otimes (p_{(1)}, 0) + \sum (0, p_1) \otimes (0, p_2))
= \sum (c_1, \pi(0)) \otimes (c_2, 0) + \sum (p_{(-1)}, \pi(0)) \otimes (0, p_{(0)})
+ \sum (0, \pi(p_{(0)})) \otimes (p_{(1)}, 0) + \sum (0, \pi(p_1)) \otimes (0, p_2)
= \sum (c_1, \eta(0)) \otimes (c_2, 0) + \sum (p_{(-1)}, \eta(0)) \otimes (0, p_{(0)})
+ \sum (0, \eta(c_1)) \otimes (c_2, 0) + \sum (0, \eta(p_{(-1)})) \otimes (0, p_{(0)})
\]
This implies \((\pi \otimes 1)\Delta(c, p) \in \pi T \otimes (C \otimes P).\)

The canonical projection \(\pi T \otimes \pi Q \rightarrow \pi T \otimes \pi Q \otimes \pi T \otimes \pi Q = (\pi T \otimes \pi Q) \otimes (\pi T \otimes \pi Q) \subset (\pi T \otimes \pi Q) \otimes (\pi T \otimes \pi Q) \subset T \otimes (C \otimes T), \)
\[ \Delta(c, p) \in T \otimes (C \otimes T) \cap ((C \otimes T) \otimes T) = T \otimes T, \]
and so \(T \) is a subcoalgebra of \(C \otimes T.\)

Furthermore, we have \(\sum \eta(c_1) \otimes \eta(c_2) = (\eta \otimes 1)(\sum c_1 \otimes \eta(c_2)) = (\eta \otimes 1)(\sum c_{(-1)} \otimes \pi(p_{(0)})) = (1 \otimes \pi)(\sum \eta(p_{(-1)}) \otimes p_{(0)}) = (1 \otimes \pi)(\sum \pi(p_1) \otimes p_2) = \sum \pi(p_1) \otimes \pi(p_2) = \sum \eta(c_1) \otimes \eta(c_2).\)

Hence \(\eta\) is a coalgebra homomorphism. □

By Proposition \[3.3\] for any subspaces \(D \subseteq C\) and \(R \subseteq Q \subseteq P,\) if there exists a linear surjection \(\eta : D \rightarrow Q/R,\) such that (a), (b) and (c) in Proposition \[3.3\] are satisfied, then the following set is a subcoalgebra of \(C \otimes P:\)
\[ \{(c, p) \in C \otimes P | c \in D, p \in Q, \eta(c) = p + R\}. \]
Conversely, any subcoalgebra of \(C \otimes P\) has this form.

If \(T\) is a subcoalgebra of \(C \otimes P,\) then \(\text{Ker}(\eta) = E\) is a subcoalgebra, and hence a coideal of \(C\) (or \(D\)). Thus, one gets the following corollary.

**Corollary 3.4.** \(T\) is a subcoalgebra of \(C \otimes P\) if and only if the following hold:

(a) \(D\) and \(E\) are subcoalgebras of \(C.\)
(b) \(Q\) is a \(C\)-subcoalgebra of \(P\) and \(R\) a \(C\)-coideal of \(Q,\) and \(p^C_1(Q) \subseteq D \otimes Q\) and \(p^C_2(Q) \subseteq Q \otimes D,\)
where \(p^C_1\) and \(p^C_2\) are the \(C\)-comodule structure maps of \(P.\) Hence \(Q\) and \(Q/R\) are both \(D\)-coalgebras.
(c) \(\overline{\eta} : D/E \rightarrow Q/R\) is a \(C\)-bicomodule (or \(D\)-bicomodule) isomorphism, and
\[
\sum \overline{\eta}(p_{(-1)}) \otimes p_{(0)} = \sum \pi(p_{(1)}) \otimes p_2, \sum p_{(0)} \otimes \overline{\eta}(p_{(1)}) = \sum p_1 \otimes \pi(p_2), \forall p \in Q,
\]
where \(\overline{\eta}\) denotes the image of \(x \in D\) under the canonical projection \(D \rightarrow D/E.\)

If this is the case, then \(\overline{\eta}\) is a coalgebra isomorphism, and hence it is a \(C\)-Coalgebra (or \(D\)-Coalgebra) isomorphism.

**Proof.** It follows from Proposition \[3.3\] and \(\text{Ker}(\eta) = E.\) □

**Corollary 3.5.** If \(T\) is a subcoalgebra of \(C \otimes P,\) then \((E, R)\) is a coideal of \(T\) and
\[ T/(E, R) \cong D/E \cong Q/R \]
as coalgebras.

**Proof.** Suppose that \(T\) is a subcoalgebra of \(C \otimes P.\) By Corollary \[3.4\] \(E\) and \(D\) are subcoalgebras of \(C,\)
\(Q\) is a subcoalgebra of \(P\) and \(R\) is a coideal of \(Q.\) Moreover, \(D/E \cong Q/R\) as coalgebras. Hence
the canonical projection \(D \rightarrow D/E, d \mapsto d + E\) is a coalgebra epimorphism. Then by Lemma \[3.1\] the
composition \( \theta : T \xrightarrow{\zeta} D \to D/E, (c, p) \mapsto c + E \) is a coalgebra epimorphism. Let \((c, p) \in T\). Then \(c \in D, p \in Q, \eta(c) = p + R \subset Q/R\) and \(\theta(c, p) = c + E \in D/E\). Since \(\text{Ker}(\eta) = E, (c, p) \in \text{Ker}(\theta)\) if and only if \(c \in E\). Hence \(\text{Ker}(\theta) = \{(c, p) | c \in E, p \in R\} = (E, R)\). It follows that \((E, R)\) is a coideal of \(T\) and \(T/(E, R) \cong D/E\) as coalgebras.

4. Some applications

4.1. The ideals of \(k \triangleleft_d I\). Let \(I\) be an algebra without identity. Then we can construct an algebra Dorroh extension \(k \triangleleft_d I\), which is a \(k\)-algebra with the identity \((1, 0)\).

Let \(K\) be an ideal of \(k \triangleleft_d I\). Then \(K = \{(x, -x) | x \in I, \varphi(x) = 0\}\) by Proposition 2.1, where \(Z \subseteq B \subseteq k\) and \(L \subseteq J \subseteq I\) are subspaces, and \(\varphi : J \to B/Z\) is a linear surjection with \(\text{Ker}(\varphi) = L\) such that \((a), (b)\) and \((c)\) in Proposition 2.1 are satisfied. Clearly, \(B = Z = 0\), or \(B = Z = k\), or \(B = k\) and \(Z = 0\).

If \(B = Z = 0\), then \(J = L\) is an ideal of \(I\) and \(K = (0, L) = (0, J)\). If \(B = Z = k\), then \(\varphi(0) = 1 + Z\). Hence \(L = J\) by Proposition 2.1(c), and so \(J = I\). Thus, \(K = k \triangleleft_d I\). If \(B = k\) and \(Z = 0\), then \(J\) is an ideal of \(I\), \(\varphi : J \to k\) is an algebra epimorphism such that \(\varphi(x) = \alpha\) implies \(ay - xy, ay - yx \in \text{Ker}(\varphi)\) for any \(y \in I\). Moreover, \(K = \{(x, -x) | x \in J, \varphi(x) = \alpha\}\).

Summarizing the above discussion, we have the following proposition.

Proposition 4.1. Let \(K\) be a subspace of \(k \triangleleft_d I\). Then \(K\) is an ideal of \(k \triangleleft_d I\) if and only if one of the following hold.

(a) \(K = (0, L),\) where \(L\) is an ideal of \(I\).
(b) \(K = k \triangleleft_d I\).
(c) \(K = \{(x, -x) | x \in J, \varphi(x) = \alpha\},\) where \(J\) is an subalgebra of \(I\), \(\varphi : J \to k\) is an algebra epimorphism, and if \(\varphi(x) = \alpha\) then \(\varphi(ay - xy) = \varphi(ay - yx) = 0\) for any \(y \in I\).

By Proposition 4.1 a nontrivial ideal of \(k \triangleleft_d I\) may not be contained in \((0, I)\). In the following, we give such an example.

Example 4.2. Let \(U\) be an algebra without identity. Then \(I = k \times U\) is also an algebra without identity. Consider the Dorroh extension \(k \triangleleft_d I = k \triangleleft_d (k \times U)\). Let \(J = I\). Define \(\varphi : J \to k\) by \(\varphi(\alpha, u) = \alpha\). Then \(\varphi\) is an algebra homomorphism. If \(\varphi(x) = \alpha\), then \(x = (\alpha, u) \in J\) for some \(u \in U\). In this case, for any \(y = (\beta, v) \in I = k \times U\), \(ay - x = (\alpha \beta, v) - (\beta \alpha, uv) = (0, av - uv)\) and \(ay - yx = (\alpha \beta, av) - (\beta \alpha, vu) = (0, av - vu)\), and hence \(\varphi(ay - xy) = \varphi(ay - yx) = 0\). Thus, by Proposition 4.1(c), the following set is an ideal of \(k \triangleleft_d I = k \triangleleft_d (k \times U)\):

\[
K = \{(\alpha, -x) | x = (\alpha, u) \in J\} = \{(\alpha, -\alpha, u) | \alpha \in k, u \in U\}.
\]

4.2. The ideals of trivial algebra extensions. Let \(A\) be a unital algebra and \(M\) a unital \(A\)-bimodule. Then by [13] Example 1.3(c)], the trivial algebra extension \(A \rtimes M\) is also an algebra Dorroh extension of \(A\). Let \(K\) be a subspace of \(A \rtimes M\). Then by the discussion in Section 2 there are subspaces \(Z \subseteq B \subseteq A\) and \(J \subseteq M\), and a linear surjection \(\varphi : J \to B/Z\) such that \(K = \{(a, -x) | a \in B, x \in J, \varphi(x) = a + Z\}\). With these notations, we have the following corollary.

Corollary 4.3. \(K\) is an ideal of \(A \rtimes M\) if and only if the following hold.

(a) \(B\) and \(Z\) are ideals of \(A\), and \(B^2 \subseteq Z\).
(b) \(J\) is an \(A\)-subbimodules of \(M\).
(c) \(\varphi\) is an \(A\)-bimodule epimorphism, and \(BM \subseteq \text{Ker}(\varphi)\) and \(MB \subseteq \text{Ker}(\varphi)\).

Proof. By [13] Example 1.3(c)], \(M\) is an algebra with the multiplication given by \(xy = 0\) for any \(x, y \in M\). Hence \(M^2 = 0, ay - xy = ay\) and \(ya - yx = ya\) for any \(a \in A\) and \(x, y \in M\).
If $K$ is an ideal of $A \ltimes M$, then (a), (b), (c) in Proposition 4.4 are satisfied. Hence $B$ and $Z$ are ideals of $A$, $J$ is an $A$-subalgebra of $M$ and $\varphi: J \to B/Z$ is an $A$-Algebra homomorphism. Moreover, if $\varphi(x) = y + Z$ then $ay, a_1 y \in \text{Ker}(\varphi)$ for any $y \in M$ since $ay - ay = ay$ and $y_1 y = y$. Hence $BM \subseteq \text{Ker}(\varphi)$ and $MB \subseteq \text{Ker}(\varphi)$ since $\varphi$ is surjective. By $M^2 = 0$, $J^2 = 0$. Since $\varphi$ is an algebra epimorphism, $(B/Z)^2 = 0$, i.e., $B^2 \subseteq Z$. Thus, (a), (b) and (c) hold. Conversely, if (a), (b) and (c) hold, then $K$ is an ideal of $A \ltimes M$ by Proposition 4.4.

4.3. The subcoalgebras of $k \ltimes_d P$. Let $P$ be a coalgebra without counit. By [13] Example 2.11(a), there is a coalgebra Dorroh extension $k \ltimes_d P$, which is a coalgebra with the counit $\varepsilon$ given by $\varepsilon(\alpha, p) = \alpha$, $\forall \alpha \in k, p \in P$.

Let $T$ be a subcoalgebra of $k \ltimes_d P$. Then 

$$T = \{(\alpha, p) \in k \ltimes_d P| \alpha \in D, p \in Q, \eta(\alpha) = p + R\},$$

where $E \subseteq D \subseteq k$ and $R \subseteq Q \subseteq P$ are subspaces, and $\eta: D \to Q/R$ is a linear surjection with $\text{Ker}(\eta) = E$ such that (a), (b) and (c) in Proposition 4.3 are satisfied. Clearly, $D = E = 0$, or $D = E = k$, or $D = k$ and $E = 0$.

If $D = E = 0$, then $Q = R$ since $\eta$ is surjective. By Proposition 4.3(b), $\rho^1_1(Q) \subseteq D \otimes Q = 0$. However, $\rho^1_1(p) = 1 \otimes p \in \otimes Q$ for any $p \in Q$. Hence $Q = 0$, and so $T = 0$.

If $D = E = k$, then $Q = R$ since $\eta$ is surjective and $\text{Ker}(\eta) = E$. By Proposition 4.3(b), $Q$ is a subcoalgebra of $P$. Hence $T = \{(\alpha, p)|\alpha \in k, p \in Q\}$ for some subcoalgebra $Q$ of $P$.

If $D = k$ and $E = 0$, then $\dim(Q/R) = 1$ since $\eta$ is surjective and $\text{Ker}(\eta) = E$. By Proposition 4.3(b) and (c), $Q$ is a subcoalgebra of $P$. $R$ is a coideal of $Q$ (or $P$), and $\eta$ is a linear isomorphism. Moreover, $\eta(1) \otimes p = \sum p_1 \otimes p_2$ and $p \otimes \eta(1) = \sum p_1 \otimes p_2$ for any $p \in Q$, where $\pi: Q \to Q/R$ is the canonical projection. Choose an element $x \in Q$ such that $\eta(1) = \pi(x) = x + R$. Then $\eta(1) \otimes p = \sum p_1 \otimes p_2$ and $p \otimes \eta(1) = \sum p_1 \otimes \pi(p_2)$ are equivalent to $\Delta(p) - x \otimes p \in R \otimes Q$ and $\Delta(p) - p \otimes x \in Q \otimes R$, respectively, where $p \in Q$. When $p = x$, the two equations are both equivalent to $x \otimes x - \sum x_1 \otimes x_2 \in R \otimes R$.

**Proposition 4.4.** Let $T$ be a subspace of $k \ltimes_d P$. Then $T$ is a subcoalgebra of $k \ltimes_d P$ if and only if one of the following hold.

(a) $T = 0$.
(b) $T = (k, Q)$, where $Q$ is a subcoalgebra of $P$.
(c) $T = \{(\alpha, \alpha x + p)|\alpha \in k, p \in R\} = k(1, x) + (0, R)$, where $R$ is a coideal of $P$ and $x \in P \setminus R$ such that $\Delta(x) - x \otimes x \in R \otimes R$, $\Delta(p) - x \otimes p \in R \otimes (kx + R)$ and $\Delta(p) - p \otimes x \in (kx + R) \otimes R$ for any $p \in R$.

**Proof.** Suppose that $R$ is a coideal of $P$ and $x \in P \setminus R$ such that $\Delta(x) - x \otimes x \in R \otimes R$, $\Delta(p) - x \otimes p \in R \otimes (kx + R)$ and $\Delta(p) - p \otimes x \in (kx + R) \otimes R$ for any $p \in R$. Let $Q = kx + R$. Then $\dim(Q/R) = 1$ and $\Delta(x), \Delta(p) \in Q \otimes Q$ for any $p \in R$. Hence $\Delta(Q) \subseteq Q \otimes Q$, i.e., $Q$ is a subcoalgebra of $P$. Clearly, $R$ is a coideal of $Q$. Moreover, $\Delta(p) - x \otimes p \in R \otimes Q$ and $\Delta(p) - p \otimes x \in Q \otimes R$ for any $p \in Q$. Thus, the proposition follows from the above discussion.

4.4. The subcoalgebras of trivial coalgebra extensions. Let $C$ be a counital coalgebra and $M$ a counital $C$-bicomodule. Then by [13] Example 2.11(c)], the trivial coalgebra extension $C \ltimes M$ is a coalgebra Dorroh extension, where $M$ is a coalgebra with $\Delta_M = 0$. Let $T$ be a subspace of $C \ltimes M$. Then by the discussion in Section 4.4 there are subspaces $E \subseteq D \subseteq C$ and $R \subseteq Q \subseteq M$, and a linear surjection $\eta: D \to Q/R$ with $\text{Ker}(\eta) = E$ such that $T = \{(c, m) \in C \ltimes M|c \in D, m \in Q, \eta(c) = m + R\}$. With these notations, we have the following corollary.

**Corollary 4.5.** $T$ is a subcoalgebra of $C \ltimes M$ if and only if the following hold.
(a) $E$ and $D$ are subcoalgebras of $C$ and $\Delta(D) \subseteq E \otimes D + D \otimes E$.
(b) $Q$ and $R$ are $C$-subbicodules of $M$, and $\rho_l(Q) \subseteq E \otimes Q$ and $\rho_r(Q) \subseteq Q \otimes E$, where $\rho_l$ and $\rho_r$ are the $C$-comodule structure maps of $M$.
(c) $\eta$ is a $C$-bicodule homomorphism.

Proof. Note that $\Delta_M(m) = \sum m_1 \otimes m_2 = 0$ for any $m \in M$ as stated above.

If $T$ is a subcoalgebra of $C \bowtie M$, then (a), (b) and (c) in Proposition 3.3 are satisfied. Hence $E$ and $D$ are subcoalgebras of $C$, $Q$ and $R$ are $C$-subbicodules of $M$, and $\eta : D \to Q/R$ is a $C$-bicodule homomorphism. Moreover, $\sum \eta(m_{-1}) \otimes m_0 = 0$ and $\sum m_0 \otimes \eta(m_{1}) = 0$ for any $m \in Q$. Therefore, $\rho_l(m) \in \text{Ker}(\eta \otimes 1) = E \otimes Q$ and $\rho_r(m) \in \text{Ker}(1 \otimes \eta) = Q \otimes E$ for any $m \in Q$. That is, $\rho_l(Q) \subseteq E \otimes Q$ and $\rho_r(Q) \subseteq Q \otimes E$. Note that $Q/R$ is a coalgebra with the comultiplication $\Delta_{Q/R} = 0$. By Proposition 3.3, $\eta$ is a homomorphism of coalgebra (without counit). Hence $(\eta \otimes \eta)(\Delta(D)) = \Delta_{Q/R}(\eta(D)) = 0$, and so $\Delta(D) \subseteq \text{Ker}(\eta \otimes \eta) = E \otimes D + D \otimes E$. Thus, (a), (b) and (c) hold. Conversely, if (a), (b) and (c) hold, then $T$ is a subcoalgebra of $C \bowtie M$ by Proposition 3.3 and the proof above. □

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