Planar graphs are $9/2$-colorable

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Joint with Landon Rabern
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Connections in Discrete Math
Simon Fraser
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The 5 Color Theorem

Fact 1: Every $n$-vertex triangulation has $3n - 6$ edges.

Cor: $K_5$ is non-planar. (Since $3(5) - 6 = 9 < 10 = \binom{5}{2}$.)

Thm: Every planar graph $G$ is 5-colorable.

Pf: Add edges to get a triangulation. Now $\sum_{v \in V} d(v) = 2|E| = 2(3n - 6) < 6n$. So some vertex $v$ is a 5-vertex. When $v$ is a 4-vertex, we 5-color $G - v$ by induction, then color $v$. Now, since $K_5$ is non-planar, $v$ has non-adjacent neighbors $w_1$ and $w_2$. Contract $vw_1$ and $vw_2$; 5-color by induction. This gives 5-coloring of $G - v$. Now extend to $v$, since $w_1$ and $w_2$ have same color. ■
The 5 Color Theorem

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So some vertex $v$ is a $5^-$-vertex.
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$$\sum_{v \in V} d(v) = 2|E| = 2(3n - 6) < 6n.$$

So some vertex $v$ is a $5^{-}$-vertex. When $v$ is a $4^{-}$-vertex, we 5-color $G - v$ by induction, then color $v$. 

\[ 
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{k5.png}
\end{array}
\]
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\begin{center}
\begin{tikzpicture}
\node[draw, circle] (v) at (0,0) {};
\node[draw, circle] (w1) at (1,1) {};
\node[draw, circle] (w2) at (1,-1) {};
\draw (v) -- (w1);
\draw (v) -- (w2);
\end{tikzpicture}
\end{center}
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![Diagram](image)
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![Diagram showing the process of 5-coloring a triangulated graph](image)
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So some vertex \( v \) is a \( 5^- \)-vertex. When \( v \) is a \( 4^- \)-vertex, we 5-color \( G - v \) by induction, then color \( v \). Now, since \( K_5 \) is non-planar, \( v \) has non-adjacent neighbors \( w_1 \) and \( w_2 \).

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Between 4 Color Theorem and 5 Color Theorem

4CT is hard and 5CT is easy. What's in between?

- Two-fold coloring: color vertex “half red and half blue”
- 5CT implies that 10 colors suffice
- 4CT implies that 8 colors suffice
- $9_{\leq 2}$CT will show that 9 colors suffice.

Def: The Kneser graph $K_t^k$ has as vertices the $k$-element subsets of $\{1, \ldots, t\}$. Vertices are adjacent whenever their sets are disjoint.

We'll show that planar graphs have a map to $K_9^{2\leq 2}$. $G$ is $t$-colorable iff $G$ has homomorphism to $K_t^t$. 
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\]

\[\text{Want } f: V(G) \to V(K_t): f(u) \neq f(v) \iff uv \in E(G).\]

We’ll show that planar graphs have a map to \(K_{9:2}\).

\(G\) is \(t\)-colorable iff \(G\) has homomorphism to \(K_t\).
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Want $f : V(G) \rightarrow V(K_{t:k})$ where $f(u)f(v) \in E(K_{t:k})$ if $uv \in E(G)$.
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Want \(f : V(G) \rightarrow V(K_{t:k})\) where \(f(u)f(v) \in E(K_{t:k})\) if \(uv \in E(G)\).

We’ll show that planar graphs have a map to \(K_{9:2}\). \(G\) is \(t\)-colorable iff \(G\) has homomorphism to \(K_t\).
9/2-coloring planar graphs

Thm:

Every planar graph has a homomorphism to $K_9:2$.

Pf:

Induction on $n$, like 5CT. If we can't do induction, then $G$:

1. has minimum degree 5
2. has no separating triangle
3. can't have "too many 6-vertices near each other"
   ▶ has no 5-vertex with a 5-nbr and a non-adjacent 6-nbr
   ▶ has no 6-vertex with two non-adjacent 6-nbrs
   ▶ has no 7-vertex with a 5-nbr and two non-adjacent 6-nbrs
   if so, then contract some non-adjacent pairs of nbrs;
   color smaller graph by induction, then extend to $G$

Use discharging method to contradict (1), (2), or (3).

▶ each $v$ gets $\text{ch}(v) = d(v) - 6$, so

$\sum_{v \in V} \text{ch}(v) = -12$

▶ redistribute charge, so every vertex finishes nonnegative

▶ Now $-12 = \sum_{v \in V} \text{ch}(v) = \sum_{v \in V} \text{ch}^*(v) \geq 0$.

Contradiction!
9/2-coloring planar graphs

**Thm:** Every planar graph has a homomorphism to $K_{9:2}$.
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   - has no 6-vertex with two non-adjacent $6^-$-nhrs

Use discharging method to contradict (1), (2), or (3).

$\triangleright$ each $v$ gets $ch(v) = d(v) - 6$, so $\sum v \in V ch(v) = -12$

$\triangleright$ redistribute charge, so every vertex finishes nonnegative

$\triangleright$ Now $-12 = \sum v \in V ch(v) = \sum v \in V ch^*(v) \geq 0$

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   - has no 7-vertex with a 5-nbr and two non-adjacent $6^-$-nbrs
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   - has no 6-vertex with two non-adjacent 6−-nbrs
   - has no 7-vertex with a 5-nbr and two non-adjacent 6−-nbrs

if so, then contract some non-adjacent pairs of nbrs; color smaller graph by induction, then extend to $G$
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   - has no 7-vertex with a 5-nbr and two non-adjacent $6^-$-nbs

if so, then contract some non-adjacent pairs of nbs; color smaller graph by induction, then extend to $G$

Use discharging method to contradict (1), (2), or (3).

- each $v$ gets $ch(v) = d(v) - 6$, so $\sum_{v \in V} ch(v) = -12$
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Thm: Every planar graph has a homomorphism to $K_{9:2}$.

Pf: Induction on $n$, like 5CT. If we can’t do induction, then $G$:

1. has minimum degree 5
2. has no separating triangle
3. can’t have “too many 6⁻-vertices near each other”
   ▶ has no 5-vertex with a 5-nbr and a non-adjacent 6⁻-nbr
   ▶ has no 6-vertex with two non-adjacent 6⁻-nhrs
   ▶ has no 7-vertex with a 5-nbr and two non-adjacent 6⁻-nhrs

if so, then contract some non-adjacent pairs of nhrs; color smaller graph by induction, then extend to $G$

Use discharging method to contradict (1), (2), or (3).

▶ each $v$ gets $ch(v) = d(v) - 6$, so $\sum_{v \in V} ch(v) = -12$
▶ redistribute charge, so every vertex finishes nonnegative
9/2-coloring planar graphs

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**Pf:** Induction on $n$, like 5CT. If we can’t do induction, then $G$:

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3. can’t have “too many 6−-vertices near each other”
   - has no 5-vertex with a 5-nbr and a non-adjacent 6−-nbr
   - has no 6-vertex with two non-adjacent 6−-nbrs
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Use discharging method to contradict (1), (2), or (3).

- each $v$ gets $ch(v) = d(v) - 6$, so $\sum_{v \in V} ch(v) = -12$
- redistribute charge, so every vertex finishes nonnegative
- Now $-12 = \sum_{v \in V} ch(v) = \sum_{v \in V} ch^*(v) \geq 0$, 
\textbf{9/2-coloring planar graphs}

\textbf{Thm:} Every planar graph has a homomorphism to $K_{9:2}$.

\textbf{Pf:} Induction on $n$, like 5CT. If we can’t do induction, then $G$:

1. has minimum degree 5
2. has no separating triangle
3. can’t have “too many $6^-$-vertices near each other”
   - has no 5-vertex with a 5-nbr and a non-adjacent $6^-$-nbr
   - has no 6-vertex with two non-adjacent $6^-$-nbrs
   - has no 7-vertex with a 5-nbr and two non-adjacent $6^-$-nbrs

if so, then contract some non-adjacent pairs of nbrs; color smaller graph by induction, then extend to $G$

Use \textcolor{red}{discharging method} to contradict (1), (2), or (3).

- each $v$ gets $ch(v) = d(v) - 6$, so $\sum_{v \in V} ch(v) = -12$
- redistribute charge, so every vertex finishes nonnegative
- Now $-12 = \sum_{v \in V} ch(v) = \sum_{v \in V} ch^*(v) \geq 0$, \textcolor{red}{Contradiction!}
Too many 6^-vertices near each other
Too many $6^-$-vertices near each other

**Key Fact:** Denote the center vertex of $\xymatrix{&*{2}{(a)}\ar@{-}[r] &*{2}{(b)} &*{2}{(c)}}$ by $v$ and the other vertices by $u_1, u_2, u_3$. 

Proof:

Give $v$ a color available for at most one $u_i$, say $u_1$. Since $5 > 3$, now give $v$ another color not available for $u_1$. Now color each $u_i$. 

\[ \begin{array}{c}
\xymatrix{ &*{2}{(a)} \
&*{2}{(b)} \\
*{2}{(c)} &*{2}{(a)} \\
*{2}{(b)} &*{2}{(c)} } 
\end{array} \]
Too many $6^-$-vertices near each other

**Key Fact:** Denote the center vertex of $\leq$ by $v$ and the other vertices by $u_1, u_2, u_3$. If $v$ has 5 allowable colors and each $u_i$ has 3 allowable colors, then we can color each vertex with 2 colors, such that no color appears on both ends of an edge.

**Proof:**

Give $v$ a color available for at most one $u_i$, say $u_1$.

Now give $v$ another color not available for $u_1$.

Now color each $u_i$. 
Too many $6^-$-vertices near each other

**Key Fact:** Denote the center vertex of \( \leq \) by \( \nu \) and the other vertices by \( u_1, u_2, u_3 \). If \( \nu \) has 5 allowable colors and each \( u_i \) has 3 allowable colors, then we can color each vertex with 2 colors, such that no color appears on both ends of an edge.

**Pf:** Give \( \nu \) a color available for at most one \( u_i \), say \( u_1 \).
**Key Fact:** Denote the center vertex of $\leq$ by $v$ and the other vertices by $u_1, u_2, u_3$. If $v$ has 5 allowable colors and each $u_i$ has 3 allowable colors, then we can color each vertex with 2 colors, such that no color appears on both ends of an edge.

**Pf:** Give $v$ a color available for at most one $u_i$, say $u_1$. $2(5) > 3(3)$
Too many 6−-vertices near each other

**Key Fact:** Denote the center vertex of $\begin{array}{c} \text{A} \\ \text{B} \end{array}$ by $v$ and the other vertices by $u_1, u_2, u_3$. If $v$ has 5 allowable colors and each $u_i$ has 3 allowable colors, then we can color each vertex with 2 colors, such that no color appears on both ends of an edge.

**Pf:** Give $v$ a color available for at most one $u_i$, say $u_1$. $2(5) > 3(3)$ Now give $v$ another color not available for $u_1$. 

\[ \begin{array}{c} \text{A} \\ \text{B} \\ \text{A} \\ \text{B} \\ \text{A} \\ \text{B} \end{array} \]
Key Fact: Denote the center vertex of $6^-$ by $v$ and the other vertices by $u_1, u_2, u_3$. If $v$ has 5 allowable colors and each $u_i$ has 3 allowable colors, then we can color each vertex with 2 colors, such that no color appears on both ends of an edge.

Pf: Give $v$ a color available for at most one $u_i$, say $u_1$. $2(5) > 3(3)$
Now give $v$ another color not available for $u_1$. Now color each $u_i$. 
Too many $6^-$-vertices near each other

**Key Fact:** Denote the center vertex of $\blacklozenge$ by $v$ and the other vertices by $u_1, u_2, u_3$. If $v$ has 5 allowable colors and each $u_i$ has 3 allowable colors, then we can color each vertex with 2 colors, such that no color appears on both ends of an edge.

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Now give $v$ another color not available for $u_1$. Now color each $u_i$. 

![Graph diagram]
**Too many 6−-vertices near each other**

**Key Fact:** Denote the center vertex of \( \triangle \) by \( v \) and the other vertices by \( u_1, u_2, u_3 \). If \( v \) has 5 allowable colors and each \( u_i \) has 3 allowable colors, then we can color each vertex with 2 colors, such that no color appears on both ends of an edge.

**Pf:** Give \( v \) a color available for at most one \( u_i \), say \( u_1 \). \( 2(5) > 3(3) \) Now give \( v \) another color not available for \( u_1 \). Now color each \( u_i \).
Discharging

Each vertex gets charge \( \text{ch}(v) = d(v) - 6 \).

Now 5-vertices need 1 from nbrs.

Def: \( H_v \) is subgraph induced by 6-nbrs of \( v \).

If \( d_{H_v}(w) = 0 \), then \( w \) is isolated nbr of \( v \);
otherwise \( w \) is non-isolated nbr of \( v \).

A non-isolated 5-nbr of vertex \( v \) is crowded (w.r.t. \( v \)) if it has two 6-nbrs in \( H_v \).

(R1) Each 8-vertex gives charge \( \frac{1}{2} \) to each isolated 5-nbr and charge \( \frac{1}{4} \) to each non-isolated 5-nbr.

(R2) Each 7-vertex gives charge \( \frac{1}{2} \) to each isolated 5-nbr, charge 0 to each crowded 5-nbr and charge \( \frac{1}{4} \) to each remaining 5-nbr.

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Now show that \( \text{ch}^*(v) \geq 0 \) for all \( v \).
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**Def:** $H_v$ is subgraph induced by $6^-$-nbrs of $v$. If $d_{H_v}(w) = 0$, then $w$ is isolated nbr of $v$; otherwise $w$ is non-isolated nbr of $v$. 

![Diagram showing vertex $v$ connected to vertices 5, 6, and 7, with arrows indicating directions.](image)
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