ON THE AFFINE REPRESENTATIONS OF THE TREFOIL KNOT GROUP

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Abstract. The complete classification of representations of the Trefoil knot group $G$ in $S^3$ and $SL(2, \mathbb{R})$, their affine deformations, and some geometric interpretations of the results, are given. Among other results, we also obtain the classification up to conjugacy of the non cyclic groups of affine Euclidean isometries generated by two isometries $\mu$ and $\nu$ such that $\mu^2 = \nu^3 = 1$, in particular those which are crystallographic. We also prove that there are no affine crystallographic groups in the three dimensional Minkowski space which are quotients of $G$.

1. Introduction

The representation of a knot group in the group of isometries of a geometric manifold is important in order to obtain invariants of the knot and also in order to relate geometric structures with the knot.

In [8] the varieties $V(I_cG)$ and $V(I_{ca}G)$ of c-representations and affine c-representations (resp.) of a two-generator group in a quaternion algebra are defined. A c-representation is a representation where the image of the generators are conjugate elements. We gave there the c-representation associated to each point in the varieties and also the complete classification of c-representations of $G$ in $S^3$ and $SL(2, \mathbb{R})$.

In this article we apply the results of [8] to the group $G$ of the Trefoil knot, giving the complete classification of representations of $G$ in $S^3$ and $SL(2, \mathbb{R})$. We classify their affine deformations and we give some geometric interpretations of the results. We also obtain as a consequence the classification up to conjugacy of the non cyclic groups of affine Euclidean isometries which are quotients of $G$, indeed those which are generated by two isometries $\mu$ and $\nu$ such that $\mu^2 = \nu^3 = 1$ (Theorem 10), in particular those which are crystallographic (Theorem 13). We also prove that there are no affine crystallographic groups in the three dimensional Minkowski space which are quotients of $G$, by using Mess’s theorem ([11], [12]) and Margulis invariant ([9], [10]).

This paper is organized as follows. In Sec. 2 we recall some concepts contained in [8], about quaternion algebras, the varieties $V(I_cG)$ and $V(I_{ca}G)$, of c-representations and affine c-representations of a group $G$ in a quaternion algebra. We include

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as Theorem [1] the result in [3] giving explicitly the complete classification of representations of $G$ in $S^3$ and $SL(2, \mathbb{R})$. In Sec. 3 we apply Theorem [1] to the Trefoil knot group $G(3_1)$ and we describe the five occurring cases of representations associated to the real points of the algebraic variety $V(G(3_1))$. We give also a geometric interpretation of the image of the representation in each of these five cases as the holonomy of a 2-dimensional geometric cone-manifold. The geometry of these cone-manifolds is spherical, Euclidean or hyperbolic. In Sec. 4 we obtain all the representations of the Trefoil knot group $G(3_1)$ in the affine isometry group $A(H)$ of a quaternion algebra $H$. In particular we obtain the representations in the 3-dimensional affine Euclidean (and Lorentz) isometry group. Finally we study the Euclidean and Lorentz crystallographic groups which are images of representations of $G(3_1)$ and we deduce some interesting consequences.

2. Preliminaires

2.1. Quaternion algebras. Recall that the quaternion algebra $H = \left( \frac{k[x,y]}{x^2 + y^2} \right)$ is the $k$-algebra on two generators $i, j$ with the defining relations:

$$i^2 = \mu, \quad j^2 = \nu \quad \text{and} \quad ij = -ji.$$

Then $H$ also is a four dimensional vector space over $k$, with basis $\{1, i, j, ij\}$. Given a quaternion $A = \alpha + \beta i + \gamma j + \delta ij$, $A \in H = \left( \frac{k[x,y]}{x^2 + y^2} \right) = \langle 1, \ i, \ j, \ ij \rangle$, we use the notation $A^+ = \alpha$, and $A^- = \beta i + \gamma j + \delta ij$. Then, $A = A^+ - A^-$. The conjugate of $A$, is by definition, the quaternion $\overline{A} := A^+ - A^- - \alpha i - \gamma j - \delta ij$. The trace of $A$ is $T(A) = A + \overline{A}$. Therefore $T(A) = 2A^+ = 2\alpha \in k$. The norm of $A$ is $N(A) := A\overline{A} = A A$. Observe that the norm can be considered as a quadratic form on $H$:

$$N(A) = (\alpha + \beta i + \gamma j + \delta ij)(\alpha - \beta i - \gamma j - \delta ij) = \alpha^2 - \beta^2 \mu - \gamma^2 \nu + \delta^2 \mu \nu \in k.$$

We denote by $(H, N)$ the quadratic structure in $H$ defined by the norm.

In the quaternion algebra $M(2, k) = \left( \frac{k[x,y]}{x^2 + y^2} \right)$ the trace is the usual trace of the matrix, and the norm is the determinant of the matrix.

There are two important subsets in the quaternion algebra $H = \left( \frac{k[x,y]}{x^2 + y^2} \right)$: the pure quaternions $H_0 = \{ A \in H : A^+ = 0 \}$ (a 3-dimensional vector space over $k$ generated by $\{i, j, ij\}$), and the unit quaternions $U_1$ (the multiplicative group of quaternions with norm 1). The norm induces a quadratic structure on the pure quaternions $(H_0, N)$.

There exists a homomorphism $c : U_1 \rightarrow SO(H_0, N)$ such that $c(A)$ acts on $H_0$ by conjugation: $c(A)(B^+) = AB^-\overline{A}$. This homomorphism permits us to associate to each representation $\rho : G \rightarrow U_1$ a linear isometry $c \circ \rho$ of the metric space $(H_0, N)$.

The equiform group or group of similarities $\mathcal{E}q(H)$ of a quaternion algebra $H$, is the semidirect product $H_0 \rtimes U$, where $U$ is the multiplicative group of invertible elements in $H$. This is the group whose underlying space is $H_0 \times U$ and the product is

$$\mathcal{E}q(H) \times \mathcal{E}q(H) \rightarrow \mathcal{E}q(H) \quad ((v, A), (w, B)) \rightarrow (v + c(A)(w), AB)$$

The group of affine isometries $\mathcal{A}(H)$ of a quaternion algebra is the subgroup of $\mathcal{E}q(H)$ which is the semidirect product $H_0 \rtimes U_1$. 
The group $\mathcal{A}(H)$ defines a left action on the 3-dimensional vector space $H_0$:

$$\Psi : \mathcal{A}(H_0) \times H_0 \rightarrow H_0$$

$$(v, A), u \rightarrow (v, A)u := v + c(A)(u)$$

For an element $(v, A) \in \mathcal{E}_q(H)$, $A$ is the linear part of $(v, A)$, $N(A)$ is the homothetic factor, and $v$ is the translational part. For an element $(v, A) \in \mathcal{A}(H)$, the vector $v \in H_0$ can be decomposed in a unique way as the orthogonal sum of two vectors, one of them in the $A^-$ direction:

$$v = sA^- + v^\perp, \quad \langle v^\perp, A^- \rangle = 0$$

Then

$$(v, A) = (v^\perp, 1)(sA^-, A)$$

The element $(v^\perp, 1)$ is a translation in $H_0$. The restriction of the action of $(sA^-, A)$ on the line generated by $A^-$ is a translation with vector $sA^-$. We define $sA^-$ as the vector shift of the element $(v, A)$. The length $\sigma$ of the vector shift will be called the shift of the element $(v, A)$.

The action of $(v, A)$ leaves (globally) invariant an affine line parallel to $A^-$ and its action on this line is a translation with vector the shift $sA^-$. This invariant affine line will be call the axis of $(v, A)$. Then the action of $(v, A)$ on the axis of $(v, A)$ is a translation by $\sigma$.

The following Table contains three important examples, which are the only quaternion algebras over $\mathbb{R}$ and $\mathbb{C}$ up to isomorphism.

| $H$ | $U_1$ | $H_0$ | $(H_0, N)$ |
|-----|-------|-------|------------|
| $M(2, \mathbb{C}) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ | $SL(2, \mathbb{C})$ | $\mathbb{C}^3$ | Complex Euclidean space |
| $M(2, \mathbb{R}) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ | $SL(2, \mathbb{R})$ | $\mathbb{R}^3$ | $E^{2,1}$ Minkowski space |
| $\mathbb{H} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ | $S^3 = SU(2, \mathbb{R})$ | $\mathbb{R}^3$ | $E^3$ (Real) Euclidean space |

In fact, $M(2, \mathbb{R})$ and the Hamilton quaternions $\mathbb{H} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ are each isomorphic to an $\mathbb{R}$-subalgebra of $M(2, \mathbb{C})$.

### 2.2. The varieties $V(\mathcal{T}_G^c)$ and $V(\mathcal{T}_{aG})$

Let $G$ be a group given by the presentation

$$G = \langle a, b : w(a, b) \rangle$$

where $w$ is a word in $a$ and $b$. A homomorphism

$$\rho : G \rightarrow U_1$$

such that $\rho(a) = A$ and $\rho(b) = B$ are conjugate elements in $U_1$ is called here a $c$-representation. Then, since $A$ and $B$ are conjugate quaternions, $A^+ = B^+$.

Set

$$x = A^+ = B^+ \quad \text{and} \quad y = -(A^- B^-)^+.$$ 

We say that $\rho : G \rightarrow U_1$ realizes $(x, y)$. It is proven in [3] that if $U_1$ is the group of unit quaternions in $M(2, \mathbb{C}) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, then there is an algorithm to construct an ideal $\mathcal{T}_G^c$ generated by four polynomials

$$\{p_1(x, y), p_2(x, y), p_3(x, y), p_4(x, y)\}$$

with integer coefficients defining the algebraic variety of $c$-representations of $G$: $V(\mathcal{T}_G^c)$. It is characterized as follows:

- The set of points $(x, y) \in V(\mathcal{T}_G^c) : y^2 \neq (1 - x^2)^2$ coincides with the pairs $(x, y)$ for which there exists an irreducible $c$-representation $\rho : G \rightarrow U_1$, unique up to conjugation in $U_1$, realizing $(x, y)$.  

Theorem 1 (8)

- The set of points \( \{(x, y) \in V(\mathcal{I}_G) : y^2 = (1-x^2)^2, \quad x^2 \neq 1\} \) coincides with the pairs \((x, y)\) for which there exists an almost irreducible c-representation \( \rho : G \to U_1 \), unique up to conjugation in \( U_1 \), realizing \((x, y)\).

- The set of points \( \{(x, y) \in V(\mathcal{I}_G) : y = 0, \quad x^2 = 1\} \) coincides with the pairs \((x, y)\) for which neither irreducible nor almost-irreducible c-representations \( \rho : G \to U_1 \) realizing \((x, y)\) exist.

The real points in \( V(\mathcal{I}_G) \), excepting the cases \( \{(x, \pm(1-x^2))|x^2 \leq 1\} \), correspond to irreducible c-representations in \( S^3 \) and irreducible (or almost-irreducible) c-representations in \( SL(2, \mathbb{R}) \) according to [8, Th.4]:

**Theorem 1 (8).** Let \( G \) be a group given by the presentation

\[ G = |a, b : w(a, b)| \]

where \( w \) is a word in \( a \) and \( b \). If \((x_0, y_0)\) is a real point of the algebraic variety \( V(\mathcal{I}_G) \) we distinguish two cases:

1. If

\[ 1 - x_0^2 > 0, \quad (1 - x_0^2)^2 > y_0^2 \]

there exists an irreducible c-representation \( \rho : G \to U_1, \quad U_1 = S^3 \subset \mathbb{H} \), unique up to conjugation in \( U_1 = S^3 \), realizing \((x_0, y_0)\). Namely:

\[
A = x_0 + \frac{1}{\sqrt{1-x_0^2}}(y_0 + \sqrt{(1-x_0^2)^2 - y_0^2}) \\
B = x_0 + \sqrt{1-x_0^2}, \quad \sqrt{1-x_0^2} > 0
\]

\[
(A^{-1}B^{-1})^- = -\sqrt{(1-x_0^2)^2 - y_0^2}ij
\]

2. The remaining cases. Then excepting the case

\[ 1 - x_0^2 > 0, \quad (1 - x_0^2)^2 = y_0^2 \]

and the case

\[ x_0^2 = 1, \quad y_0 = 0 \]

there exists an irreducible (or almost-irreducible) c-representation \( \rho : G \to U_1, \quad U_1 = SL(2, \mathbb{R}) \subset (\frac{1}{3}) \) realizing \((x_0, y_0)\). Moreover two such homomorphisms are equal up to conjugation in the group of quaternions with norm \( \pm 1, \ U_{\pm 1} \). Specifically:

(2.1) If

\[ 1 - x_0^2 > 0, \quad (1 - x_0^2)^2 < y_0^2 \]

set

\[
A = x_0 + \sqrt{1-x_0^2}I, \quad \sqrt{1-x_0^2} > 0 \\
B = x_0 + \frac{1}{\sqrt{1-x_0^2}}(y_0 + \sqrt{y_0^2 - (1-x_0^2)^2}J)
\]

Then \( \rho : G \to U_1 \) is irreducible, and

\[
(A^{-1}B^{-1})^- = +\sqrt{y_0^2 - (1-x_0^2)^2}IJ
\]

(2.2) If

\[ 1 - x_0^2 < 0, \quad (1 - x_0^2)^2 < y_0^2 \]

there are two subcases:
(2.2.1) \( y_0 < 0 \). Set

\[
\begin{align*}
A &= x_0 + \sqrt{x_0^2 - 1}, \quad \sqrt{x_0^2 - 1} > 0 \\
B &= x_0 - \frac{1}{\sqrt{x_0^2 - 1}} \left( \sqrt{y_0^2 - (x_0^2 - 1)^2} I + y_0 J \right)
\end{align*}
\]

\((A^{-1} B)^- = -\sqrt{y_0^2 - (x_0^2 - 1)^2} IJ\)

(2.2.2) \( y_0 > 0 \). Set

\[
\begin{align*}
A &= x_0 + \sqrt{x_0^2 - 1}, \quad \sqrt{x_0^2 - 1} > 0 \\
B &= x_0 + \frac{1}{\sqrt{x_0^2 - 1}} \left( \sqrt{y_0^2 - (x_0^2 - 1)^2} I - y_0 J \right)
\end{align*}
\]

\((A^{-1} B)^- = +\sqrt{y_0^2 - (x_0^2 - 1)^2} IJ\)

In both subcases \( \rho : G \rightarrow U_1 \) is irreducible.

(2.3) If \( 1 - x_0^2 < 0, \ (1 - x_0^2)^2 > y_0^2 \)

set

\[
\begin{align*}
A &= x_0 + \sqrt{x_0^2 - 1}, \quad \sqrt{x_0^2 - 1} > 0 \\
B &= x_0 + \frac{1}{\sqrt{x_0^2 - 1}} \left( -y_0 J + \sqrt{(x_0^2 - 1)^2 - y_0^2} IJ \right)
\end{align*}
\]

Then \( \rho : G \rightarrow U_1 \) is irreducible and

\((A^{-1} B)^- = -\sqrt{(x_0^2 - 1)^2 - y_0^2} IJ\)

(2.4) If \( 1 - x_0^2 < 0, \ (1 - x_0^2)^2 = y_0^2 \)

set

\[
\begin{align*}
A &= x_0 + \sqrt{x_0^2 - 1}, \quad \sqrt{x_0^2 - 1} > 0 \\
B &= x_0 + (IJ + I) - \frac{y_0}{\sqrt{x_0^2 - 1}} J
\end{align*}
\]

Then \( \rho : G \rightarrow U_1 \) is almost-irreducible, and

\((A^{-1} B)^- = -\sqrt{(x_0^2 - 1)(I + J)}\).

(2.5) If \( 1 - x_0^2 = 0, \ y_0 \neq 0 \)

set

\[
\begin{align*}
A &= x_0 + I + J \\
B &= x_0 + \frac{y_0}{2} (I - J)
\end{align*}
\]

Then \( \rho : G \rightarrow U_1 \) is irreducible, and

\((A^{-1} B)^- = -y_0 IJ\)

\( \Box \)
We also want to study the \( c \)-representations of \( G \) in the affine group \( A(H) \) of a quaternion algebra \( H \), that is representations of \( G \) in the affine group \( A(H) \) of a quaternion algebra \( H \) such that the generators \( a \) and \( b \) go to conjugate elements, up to conjugation in \( \mathcal{E}q(H) \). Given a such \( c \)-representation \( \rho : G \to A(H) \), if \( \rho(a) \), \( \rho(b) \) has translational part different from 0, we can suppose that

\[
\rho : G \to A(H) \\
a \to (sA^-, A) \\
b \to (sB^- + (A^-B^-)^-, B)
\]

where \( (A, B) \) is an irreducible pair of conjugate unit quaternions. (A pair \( (A, B) \) is irreducible if \( \{A^-, B^-, (A^-B^-)\} \) is a basis of \( H_0 \).)

Because \( \rho \) is a homomorphism of the semidirect product \( H_0 \rtimes U_1 = A(H) \), we have

\[
\rho(w(a, b)) = (\frac{\partial w}{\partial a} \big|_\phi \circ v + \frac{\partial w}{\partial b} \big|_\phi \circ u, w(A, B)) = (0, I) \tag{1}
\]

where \( \frac{\partial w}{\partial a} \big|_\phi \) is the Fox derivative of the word \( w(a, b) \) with respect to \( a \), and evaluated by \( \phi \) such that \( \phi(a) = A, \phi(b) = B \). (See [3].) The equation (1) yields two relations between the parameters

\[
x = A^+ = B^+ \\
y = -(A^-B^-)^+ \\
s = \text{vector shift parameter}
\]

the relations are

\[
w(A, B) = I \tag{2}
\]

\[
\frac{\partial w}{\partial a} \big|_\phi \circ v + \frac{\partial w}{\partial b} \big|_\phi \circ u = 0 \tag{3}
\]

The relation (2) yields the ideal \( I_G^c = \{p_i(x, y) \mid i \in \{1, 2, 3, 4\} \} \), and it defines \( V(I_G^c) \) the algebraic variety of \( c \)-representations of \( G \) in \( SL(2, \mathbb{C}) \).

The relation (3) produces four polynomials in \( x, y, s \): \( \{q_j(x, y, s) \mid j \in \{1, 2, 3, 4\} \} \).

The ideal

\[
I_{aG}^c = \{p_i(x, y), q_j(x, y, s) \mid i, j \in \{1, 2, 3, 4\} \}
\]

defines an algebraic variety, that we call \( V_a(I_{aG}^c) \) the variety of affine \( c \)-representations of \( G \) in \( A(H) \) up to conjugation in \( \mathcal{E}q(H) \).

2.3. The group of the trefoil knot. Consider the standard presentation of the group of the trefoil knot \( 3_1 \) as the 2-bridge knot \( 3/1 \). Figure [4] (See [6].):

\[
G(3_1) = \langle a, b; aba = bab \rangle
\]

where \( a \) and \( b \) are meridians. This knot is also the toroidal \( \{3, 2\} \). As such, it has the following presentation:

\[
G(3_1) = \langle F, D; F^3 = D^2 \rangle \tag{4}
\]

where \( F = ab \) and \( D = aba \). It is easy to show that the element \( C := F^3 = D^2 \) belongs to the center of \( G(3_1) \).
3. The representations of $G(3_1)$ in $U_1$

Let

$$\rho: \ G(3_1) = \langle a, b; aba = bab \rangle \longrightarrow \ U_1$$

$$a \quad \rightarrow \quad \rho(a) = A$$

$$b \quad \rightarrow \quad \rho(b) = B$$

denote a representation of $G = G(3_1)$ in the unit group of a quaternion algebra $H$. This always is a $c$-representation because $a$ and $b$ are conjugate elements in $G$. Moreover if $\rho$ is irreducible (that is, if $\{A^-, B^-, (A^-B^-)^-\}$ is a basis of $H_0$, the $3$-dimensional vector space of pure quaternions), then $\{A, B\}$ generates $H$ as an algebra. Since $\rho(C)$ commutes with $A$ and $B$, it belongs to the center of $H$. Hence $\rho(C) = \pm 1$. Applying the algorithm of [8, Th.3] to the presentation

$$G(3_1) = \langle a, b; aba = bab \rangle$$

we obtain $4$ polynomials, two of which are zero and the other two coincide up to sign with

$$2x^2 - 2y - 1.$$  

Therefore $I_G = (2x^2 - 2y - 1)$. The real part of the algebraic variety $V(I_G)$ is the parabola $y = \frac{2x^2-1}{2}$ depicted in Figure 2.
Theorem 1 establishes the different cases of representations associated to real points of the algebraic variety $V(I_G)$. Figure 3 shows a pattern on the real plane with coordinates $x$ and $\frac{y}{1-x^2}$. The plane is divided in several labeled regions by labeled segments. The label corresponds to the case described in Theorem 1. Points in the segments between region (1) and (2.3), which have no label, correspond to almost-irreducible representations in $SL(2,\mathbb{C})$. Therefore, to apply Theorem 1 to the algebraic variety $V(I_G)$ it suffices to study the graphic of $\frac{y}{1-x^2}$ as a function of $x$ over the pattern.

Figure 3. The pattern for different cases.

Figure 4 shows $\frac{y}{1-x^2}$ as a function of $x$ for the algebraic variety $V(I_G)$ of the Trefoil knot.

Then, according with Theorem 1 there are several cases:

3.1. **Case 1.** Region (1).

$$x \in \left(\frac{-\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right) \iff \left\{1-x^2 > 0, (1-x^2)^2 > y^2\right\},$$
where
\[ y = \frac{2x^2 - 1}{2}, \quad u = 1 - x^2 \]

There exists an irreducible c-representation \( \rho_x : G(3) \rightarrow S^3 \) realizing \((x, y)\), unique up to conjugation in \( S^3 \), such that
\[
\rho_x(a) = A = x + \frac{x^2 - 1}{\sqrt{1 - x^2}} i + \frac{i}{\sqrt{1 - x^2}} j + \frac{j}{\sqrt{1 - x^2}} k
\]
\[
\rho_x(b) = B = x + \sqrt{1 - x^2} i, \quad \sqrt{1 - x^2} > 0
\] (6)

The composition of \( \rho_x \) with \( c : S^3 \rightarrow SO(3) \), where \( c(X) \), \( X \in S^3 \), acts on \( P \in H_0 \cong E^3 \) by conjugation, defines the representation \( \rho'_x = c \circ \rho_x : G(3) \rightarrow SO(3) \).

In linear notation, where \{X, Y, Z\} is the coordinate system in \( E^3 \) associated to the basis \{-ij, j, i\} we have
\[
\rho'_x(a) = m_x(a) \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix}
\]
\[
\rho'_x(b) = m_x(b) \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix}
\]

where
\[
m_x(a) = \begin{pmatrix}
\frac{2x^2 - 1}{\sqrt{1 - x^2}} & \frac{x^2 - 2x}{2x^2 - 1} & \frac{x^2 - 2x}{2x^2 - 1} \\
\frac{x^2 - 2x}{2x^2 - 1} & \frac{1 - x^2}{2} & \frac{1 - x^2}{2} \\
-x & \frac{1 - x^2}{2} & \frac{1 - x^2}{2}
\end{pmatrix}
\]

and
\[
m_x(b) = \begin{pmatrix}
\frac{2x^2 - 1}{2x^2 - 1} & \frac{-2x\sqrt{1 - x^2}}{1 - x^2} & 0 \\
\frac{-2x\sqrt{1 - x^2}}{1 - x^2} & \frac{2x^2 - 1}{1 - x^2} & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

The maps \( \rho'_x(a) \) and \( \rho'_x(b) \) are right rotations of angle \( \alpha \) around the axes \( A^+ \) and \( B^- \) where \( x = A^+ = B^+ = \cos \frac{\alpha}{2} \). Moreover, \( \rho'_x(C) \) is the identity. Hence \( \rho'_x : G(3) \rightarrow SO(3) \) factors through the group \( C_2 \times C_3 = |F, D; F^3 = D^2 = 1| \). Thus \( \rho'_x(F) = \rho'_x(ab) \) is a 3-fold rotation and \( \rho'_x(D) = \rho'_x((aba)) \) is a 2-fold rotation.

The angle \( \omega \) between the axes of \( \rho'_x(a) \) and \( \rho'_x(b) \) is given by
\[
\cos \omega = \frac{y}{u} = \frac{x^2 - 1}{1 - x^2}
\]

The geometrical meaning of the representation \( \rho'_x : G(3) \rightarrow SO(3) \) for \( -\frac{\sqrt{3}}{2} < x < \frac{\sqrt{3}}{2} \) is the following.

**Theorem 2.** The image of \( \rho'_x : G(3) \rightarrow SO(3) \) , \( -\frac{\sqrt{3}}{2} < x < \frac{\sqrt{3}}{2} \), is isomorphic to the holonomy of the 2-dimensional spherical cone-manifold \( S^2_{\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}}, \alpha \), where \( x = \cos \frac{\alpha}{2} \), \( \frac{180}{\alpha} > \alpha > \frac{360}{\alpha} \).

**Proof.** The 2-dimensional spherical cone-manifold \( S^2_{\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}}, \alpha \) is the result of identifying the edges of the spherical triangle \( T(\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}) \) as in the Figure [6]. The
holonomy of $S_{\pi, 2\pi/3, \alpha}^2$ is generated by rotations of angle $\alpha$ in the vertices $P$ and $Q$. The distance $r$ between $P$ and $Q$ is calculated by spherical trigonometry:

$$\cos r = \frac{\cos^2 \frac{\pi}{3} + \cos 2\frac{\alpha}{2}}{\sin^2 \frac{\alpha}{2}} = \frac{x^2 - \frac{1}{4}}{1 - x^2} = \cos \omega$$

Then $r = \omega$. Therefore the points $P$ and $Q$ are the intersection with $S^2$ of the axes $A^-$ and $B^-$ of the two generators $\rho'_x(a)$ and $\rho'_x(b)$ of the subgroup $\rho'_x(G(3/1))$. □

![Figure 5. The spherical conemanifold $S_{\pi, 2\pi/3, \alpha}$.](image)

Remark 1. The point $R$ (resp. $M$ the mid-point between $P$ and $Q$) is the intersection with $S^2$ of the axis of the 3-fold (resp. 2-fold) rotation $\rho'_x(F) = \rho'_x(ab)$ (resp. $\rho'_x(D) = \rho'_x(aba)$).

3.2. Case 2. Points in the no-label segments between region (1) and region (2.1).

$$(x, y) = (\pm \sqrt{\frac{3}{2}}, \frac{1}{4}) \iff \{1 - x^2 > 0, (1 - x^2)^2 = y^2\}.$$

There exists an almost-irreducible c-representation $\rho_c : G(3_1) \rightarrow U_1 \subset \mathbb{C}^1$ realizing $(x, y)$, unique up to conjugation in $U_1$, such that:

$$\rho_{\pm \sqrt{3}/2}(a) = A = \frac{\pm \sqrt{3}}{2} + \frac{\sqrt{-1}}{2} IJ$$

$$\rho_{\pm \sqrt{3}/2}(b) = B = \frac{\pm \sqrt{3}}{2} - \frac{1}{2} I + \frac{1}{2} J + \frac{\sqrt{-1}}{2} IJ$$

This representation cannot be conjugated to any real representation. Under the isomorphism $U_1 \approx SL(2, \mathbb{C})$ we have

$$\rho_{\pm \sqrt{3}/2} : G(3_1) \rightarrow SL(2, \mathbb{C})$$

$$a \rightarrow A = \begin{pmatrix} \frac{\pm \sqrt{3}}{2} + \frac{\sqrt{-1}}{2} & 0 \\ 0 & \frac{\pm \sqrt{3}}{2} - \frac{\sqrt{-1}}{2} \end{pmatrix}$$

$$b \rightarrow B = \begin{pmatrix} \frac{\pm \sqrt{3}}{2} + \frac{\sqrt{-1}}{2} & 0 \\ 1 & \frac{\pm \sqrt{3}}{2} - \frac{\sqrt{-1}}{2} \end{pmatrix}$$

The composition of $\rho_{\sqrt{3}/2}$ with $c : U_1 \rightarrow PSL(2, \mathbb{C})$, defines the representations $\rho'_{\sqrt{3}/2} = c \circ \rho_{\sqrt{3}/2} : G(3_1) \rightarrow PSL(2, \mathbb{C})$ where (up to conjugation with $w = \frac{1}{z}$)

$\rho'_{\sqrt{3}/2}(a)$ is the $\frac{2\pi}{6}$-rotation of $\mathbb{C}P^1$ around the point 0:

$$w = e^{i\frac{2\pi}{6}}$$
and \( \rho'_{\sqrt{3}/2}(b) \) is the \( \frac{2\pi}{6} \)-rotation of \( \mathbb{CP}^1 \) around the point \( i \):

\[
w = e^{\frac{2\pi}{3}} z + e^\pi
\]

We see, in fact, that

**Theorem 3.** The image of \( \rho'_{\sqrt{3}/2} : G(3_1) \rightarrow PSL(2, \mathbb{C}) \) acts on the Euclidean plane \( \mathbb{C} \) as the holonomy of the Euclidean crystallographic orbifold \( S_{2,3,6}^2 \).

**Remark 2.** The barycenter of the triangle \( \{0, i, e^{i\pi/6}\} \) is the center of the 3-fold rotation \( \rho'(F) = \rho'_{\sqrt{3}/2}(ab) \) and the point \( \frac{1}{2} \) is the center of the 2-fold rotation \( \rho'(D) = \rho'_{\sqrt{3}/2}(aba) \). See Figure 6(a).

**Remark 3.** The image of \( \rho'_{\sqrt{3}/2} : G(3_1) \rightarrow PSL(2, \mathbb{C}) \) can be interpreted as the holonomy of the Euclidean crystallographic conemanifold \( S_{2,\pi/2,2\pi/3,10\pi/6}^2 \). See Figure 6(b).

### 3.3. Case 3. Region (2.1).

\[
x \in (-1, -\sqrt{3}/2) \cup (\sqrt{3}/2, 1) \iff \{1 - x^2 > 0, (1 - x^2)^2 < y^2\}.
\]

There exists an irreducible c-representation \( \rho_x : G(3_1) \rightarrow SL(2, \mathbb{R}) = U_1 \subset \left( -\frac{1}{\sqrt{2}} \right) \) realizing \( (x, y) \), unique up to conjugation in \( SL(2, \mathbb{R}) \), such that

\[
\rho_x(a) = A = x + \sqrt{1-x^2}i, \quad \sqrt{1-x^2} > 0
\]

\[
\rho_x(b) = B = x + \frac{3x^2-1}{2\sqrt{1-x^2}}i + \frac{1}{2}\sqrt{\frac{3+3x^2}{1-x^2}}j.
\]

The composition of \( \rho_x \) with \( c : SL(2, \mathbb{R}) \rightarrow SO^0(1,2) \simeq Iso^+ (\mathbb{H}^2) \), where \( c(X) \), \( X \in SL(2, \mathbb{R}) \), acts on \( P \in H_0 \simeq E^{1,2} \) by conjugation, defines the representation \( \rho'_x = c \circ \rho_x : G(3_1) \rightarrow SO^0(1,2) \) in affine linear notation, where \( \{X, Y, Z\} \) is the coordinate system associated to the basis \( \{-ij, j, i\} \)

\[
\rho'_x(a) = m_x(a) \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix}
\]

\[
\rho'_x(b) = m_x(b) \begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} X' \\ Z' \end{pmatrix}
\]
such that the matrices of $\rho'_x(a)$ and $\rho'_x(b)$ are respectively:

$$m_x(a) = \begin{pmatrix} 2x^2 - 1 & -2x\sqrt{1 - x^2} & 0 \\ 2x\sqrt{1 - x^2} & 2x^2 - 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$m_x(b) = \begin{pmatrix} 2x^2 - 1 & \frac{x - 2x^3}{\sqrt{1 - x^2}} & \frac{x\sqrt{3 - 4x^2}}{2(1 - x^2)} \\ \frac{x(2x^2 - 1)}{\sqrt{1 - x^2}} & \frac{2 - 8x^2}{2 - 2x^2} & \frac{8x^2 - 5}{2(1 - x^2)} \\ \frac{x\sqrt{3 - 4x^2}}{2(1 - x^2)} & \frac{8x^2 - 5}{2(1 - x^2)} & \frac{2 - 8x^2}{2 - 2x^2} \end{pmatrix}.$$ 

The maps $\rho'_x(a)$ and $\rho'_x(b)$ are right (spherical) rotations on $H_0 \cong E_1^{1,2}$ of angle $\alpha$ around the time-like axes $A^{-}$ and $B^{-}$ where $x = A^+ = B^+ = \cos \frac{\alpha}{2}$. See Figure 7.

Moreover, $\rho'_x(C)$ is the identity. Hence $\rho'_x : G(3_1) \longrightarrow SO^0(1, 2)$ factors through the group $C_2 \ast C_3 = |F, D; F^3 = D^2 = 1|$. Thus $\rho'_x(F) = \rho'_x(ab)$ is a 3-fold rotation and $\rho'_x(D) = \rho'_x(aba)$ is a 2-fold rotation.

The distance $d$ (measured in the hyperbolic plane) between the axes of $\rho'_x(a)$ and $\rho'_x(b)$ is given by

$$\cosh d = \frac{y}{u} = \frac{x^2 - \frac{1}{2}}{-x^2 + 1}$$

The geometrical meaning of the representation $\rho'_x : G(3_1) \longrightarrow SO^0(1, 2) \cong Iso^+(\mathbb{H}^2)$ for $x \in (\frac{\sqrt{3}}{2}, 1)$ is the following.

Figure 7. Case 3.

Figure 8. The hyperbolic cone-manifold $S^{2}_{\frac{2\alpha}{\pi}, \frac{2\alpha}{\pi}, \alpha}$. 

**Theorem 4.** The image of $\rho'_x, \sqrt{2} < x < 1$, is a subgroup of $SO^0(1,2)$ isomorphic to the holonomy of the 2-dimensional hyperbolic cone-manifold $S^2_{\frac{\pi}{2}, \frac{\pi}{2}, \alpha}$, where $x = \cos \alpha \left( \frac{\pi}{2} > \alpha > 0 \right)$.

**Proof.** The 2-dimensional hyperbolic cone-manifold $S^2_{\frac{\pi}{2}, \frac{\pi}{2}, \alpha}$ is the result of identifying the edges of the hyperbolic triangle $T \left( \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2} \right)$ as in Figure 3. The holonomy of $S^2_{\frac{\pi}{2}, \frac{\pi}{2}, \alpha}$ is generated by rotations of angle $\alpha$ in the vertices $P$ and $Q$. The distance $d'$ between $P$ and $Q$ is calculated by hyperbolic trigonometry:

$$\cosh d' = \frac{\cos \frac{\alpha}{2} + \cos \frac{\alpha}{2}}{\sin \frac{\alpha}{2}} = \frac{x^2 - \frac{1}{2}}{-x^2 + 1} = \cosh d$$

Thus $d' = d$. Therefore the points $P$ and $Q$ are the intersection with the upper sheet of the two sheeted hyperboloid (model of the 2-dimensional hyperbolic plane) of the axis $A^-$ and $B^-$ of the two generators of the subgroup $\rho'_x(G(3,1))$. \hfill $\square$

**Remark 4.** The point $R$ (resp. $M$ the mid-point between $P$ and $Q$) is the intersection with $\mathbb{H}^2$ of the axis of the 3-fold (resp. 2-fold) rotation $\rho'_x(F) = \rho'_x(ab)$ (resp. $\rho'_x(D) = \rho'_x(aba)$).

3.4. **Case 4.** Segment (2.5).

$$(x, y) = (\pm 1, \frac{1}{2}) \iff \{1 - x^2 = 0, (1 - x^2)^2 < y^2\}.$$  

There exists an irreducible c-representation $\rho_x : G(3,1) \to SL(2,\mathbb{R}) = U_1 \subset \left( \begin{smallmatrix} 1 & i \cr -i & 1 \end{smallmatrix} \right)$ realizing $(x, y)$, unique up to conjugation in $SL(2,\mathbb{R})$, such that

$$\rho_x(a) = A = \pm 1 + i + j$$
$$\rho_x(b) = B = \pm 1 + \frac{1}{2}(i - j)$$

The composition of $\rho_x$ with $c : SL(2,\mathbb{R}) \to SO^0(1,2) \cong Iso^+(\mathbb{H}^2)$, defines the representation $\rho'_x = c \circ \rho_x : G(3,1) \to SO^0(1,2)$ such that the matrices of $\rho'_x(a)$ and $\rho'_x(b)$ are respectively:

$$m_x(a) = m(-1,1; \pm 1,1,1,0) = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{pmatrix}$$

and

$$m_x(b) = m(-1,1; \pm 1,\frac{1}{4},\frac{1}{4},0) = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

where $\{X, Y, Z\}$ is the coordinate system associated to the basis $\{-ij, j, i\}$. The maps $\rho'_x(a)$ and $\rho'_x(b)$ are parabolic rotations on $H_0 \cong E^{1,2}$ around the light-like axes $A^-$ and $B^-$. See Figure 9.

Therefore the image of $\rho'_x : G(3,1) \to Iso^+(\mathbb{H}^2)$ is conjugate to the action of the modular group $PSL(2,\mathbb{Z})$ in $\mathbb{H}^2$. Thus

**Theorem 5.** The image of $\rho'_3 : G(3,1) \to SO^0(1,2)$ acts on the hyperbolic plane $\mathbb{H}^2$ as the holonomy of the hyperbolic orbifold $S^2_{2,3,\infty}$. 
3.5. **Case 5.** Region (2.2).

\( x \in (-\infty,-1) \cup (1, \infty) \iff \{ 1 - x^2 < 0, (1 - x^2)^2 < y^2 \} \quad (y > 0). \)

There exists an irreducible c-representation \( \rho_x : G(3/1) \to SL(2, \mathbb{R}) = U_1 \subset (\frac{-1}{\mathbb{R}}) \) realizing \((x, y)\), unique up to conjugation in \( SL(2, \mathbb{R}) \), such that

\[
\rho_x(a) = A = x + \sqrt{x^2 - 1}i, \quad \sqrt{x^2 - 1} > 0 \\
\rho_x(b) = B = x - \sqrt{\frac{2x^2 - 1}{2x^2 - 2}}i - \frac{2x^2 - 1}{2\sqrt{2x^2 - 1}}j
\]

The composition of \( \rho_x \) with \( c : SL(2, \mathbb{R}) \to SO^0(1, 2) \cong Iso^+(\mathbb{H}^2) \), defines the representation \( \rho'_x = c \circ \rho_x : G(3/1) \to SO^0(1, 2) \) such that the matrices of \( \rho'_x(a) \) and \( \rho'_x(b) \) are respectively:

\[
m_x(a) = \begin{pmatrix}
2x^2 - 1 & 0 & 2x\sqrt{x^2 - 1} & 0 \\
0 & 1 & 0 & (3-4x^2)\sqrt{x^2 - 1} \\
2x\sqrt{x^2 - 1} & 0 & 2x^2 - 1 & 0 \\
2x^2 - 1 & 0 & 2x\sqrt{x^2 - 1} & 0
\end{pmatrix}
\]

\[
m_x(b) = \begin{pmatrix}
2x^2 - 1 & x\sqrt{\frac{3+4x^2}{x^2-1}} & -x(2x^2-1)\sqrt{\frac{3+4x^2}{x^2-1}} \\
x\sqrt{\frac{3+4x^2}{x^2-1}} & 1 - 2x^2 & \sqrt{\frac{3-4x^2}{2(x^2-1)}} & \sqrt{\frac{3-4x^2}{2(x^2-1)}} \\
-\sqrt{\frac{3+4x^2}{x^2-1}} & 2x^2 - 2 & 2x^2 - 1 & \frac{1+2x^2 - 4x^4}{2-2x^2} \\
\frac{1+2x^2 - 4x^4}{2-2x^2} & \frac{1+2x^2 - 4x^4}{2-2x^2} & \frac{1+2x^2 - 4x^4}{2-2x^2} & \frac{1+2x^2 - 4x^4}{2-2x^2}
\end{pmatrix}
\]
The maps $\rho'_x(a)$ and $\rho'_x(b)$ are hyperbolic rotations on $H_0 \cong E^{1,2}$ moving $\delta$ along the polars of the space-like vectors $A^-$ and $B^-$ where $x = A^+ = B^+ = \cosh \frac{\delta}{2}$. See Figure 11.

Moreover, $\rho'_x(C)$ is the identity. Hence $\rho'_x : G(3_1) \to SO^0(1,2)$ factors through the group $C_2 \ast C_3 = \langle F, D; F^3 = D^2 = 1 \rangle$. Thus $\rho'_x(F) = \rho'_x(ab)$ is a 3-fold rotation and $\rho'_x(D) = \rho'_x(aba)$ is a 2-fold rotation.

The distance $d$ (measured in the hyperbolic plane) between the polars of the axes of $\rho'_x(a)$ and $\rho'_x(b)$ is given by

$$\cosh d = \frac{y}{x^2 - 1} = \frac{x^2 - \frac{1}{2}}{x^2 - 1}$$

The geometrical meaning of the representation $\rho'_x : G(3_1) \to SO^0(1,2) \cong Iso^+(\mathbb{H}^2)$ for $x \in (1, \infty)$ is the following.

**Theorem 6.** The image of $\rho'_x$, $1 < x < \infty$, is a subgroup of $SO^0(1,2)$ isomorphic to the holonomy of the 2-dimensional hyperbolic orbifold $O^2_{2,3,\delta}$ where $O^2$ is an open disk and $\delta$ is the length of the closed geodesic at the end of $O^2$, where $x = \cosh \frac{\delta}{2}$.

**Proof.** The 2-dimensional hyperbolic cone-manifold $O^2_{2,3,\delta}$ is the result of identifying the edges of the hyperbolic triangle $T(\frac{2\pi}{3}, \frac{\delta}{2}, \frac{\delta}{2})$ as in Figure 12 where $P$ and $Q$ are ultrainfinite points such that the length of the segments $T(\frac{2\pi}{3}, \frac{\delta}{2}, \frac{\delta}{2}) \cap P^\perp$ and
\( T(\frac{\pi}{4}, \frac{\delta}{2}, \frac{\delta}{2}) \cap Q^\perp \) are both \( \frac{\delta}{2} \) (\( P^\perp \) denotes the polar of \( P \)). The holonomy of \( O_{2,3,3}^2 \) is generated by hyperbolic rotations of length \( \delta \) around the vertices \( P \) and \( Q \). The distance \( r \) between \( P^\perp \) and \( Q^\perp \) is calculated by hyperbolic trigonometry:

\[
\cosh r = \frac{\cosh^2 \frac{\delta}{2} + \cos \frac{2\pi}{3}}{2} = \frac{x^2 - \frac{1}{2}}{x^2 - 1} = \cosh d
\]

Hence \( r = d \). \( \square \)

**Remark 5.** The point \( R \) (resp. the mid-point between \( P^\perp \) and \( Q^\perp \)) is the intersection with \( \mathbb{H}^2 \) of the axis of the 3-fold (resp. 2-fold) rotation.

### 3.6. Some particular cases.

**Case 1:** We have seen that the image of \( \rho'_x : G(3_1) \longrightarrow SO(3) \), \( 0 \leq x < \frac{\sqrt{3}}{2} \), is isomorphic to the holonomy of the 2-dimensional spherical cone-manifold \( S^2_{\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}} \). The orbifolds among these cone-manifolds are \( S_{232}, S_{233}, S_{234}, S_{235} \). The fundamental groups of these orbifolds are the group of direct symmetries of the equilateral triangle (in \( E^3 \)); the tetrahedron; the octahedron; and the icosahedron, respectively. These groups are isomorphic to, respectively, \( \Sigma_3, A_4, \Sigma_4 \) and \( A_5 \).

**Case 2:** The image of the almost-irreducible c-representation \( \rho'_x : G(3_1) \longrightarrow PSL(2, \mathbb{C}) \) acts on the Euclidean plane \( \mathbb{C} \) as the holonomy of the Euclidean crystallographic orbifold \( S_{236} \).

**Case 3:** The image of \( \rho'_x : \frac{\sqrt{3}}{2} \leq x < 1 \), is a subgroup of \( SO^0(1,2) \) isomorphic to the holonomy of the 2-dimensional hyperbolic cone-manifold \( S^2_{\frac{\pi}{2}, \frac{\pi}{2}, \alpha} \), where \( x = \cos \frac{\pi}{2} \). The orbifolds among these cone-manifolds are \( S_{23p} \), for all \( p \geq 7 \). The orbifold \( S_{237} \) is the (orientable) hyperbolic orbifold of minimal area.

**Case 4:** The image of \( \rho'_1 : G(3_1) \longrightarrow SO^0(1,2) \) acts on the hyperbolic plane \( \mathbb{H}^2 \) as the holonomy of the hyperbolic open orbifold (with finite volume) \( S_{238} \).

**Case 5:** The image of \( \rho'_x : 1 < x < \infty \), is a subgroup of \( SO^0(1,2) \) isomorphic to the holonomy of the 2-dimensional open hyperbolic orbifold (with infinite volume) \( O_{234} \) where \( O \) is an open disk and \( \delta \) is the length of the closed geodesic at the end of \( O \), where \( x = \cosh \frac{\delta}{2} \).

**Theorem 7.** Every image of \( \rho'_x : G(3_1) \longrightarrow SO^0(1,2) \), where \( x \geq 1 \) contains a two generator free subgroup of finite index.

**Proof.** Theorem 6 shows that the image of \( \rho'_x : G(3_1) \longrightarrow SO^0(1,2) \), \( 1 < x < \infty \), is a subgroup of \( SO^0(1,2) \) isomorphic to the holonomy of the 2-dimensional open hyperbolic orbifold (with infinite volume) \( O_{235} \) where \( O \) is an open disk and \( \delta \) is the length of the closed geodesic at the end of \( O \), where \( x = \cosh \frac{\delta}{2} \). Figure 12. The image of \( \rho'_1 : G(3_1) \longrightarrow SO^0(1,2) \), is a subgroup of \( SO^0(1,2) \) isomorphic to the holonomy of the 2-dimensional open hyperbolic orbifold (with finite volume) \( S^2_{\frac{\pi}{2}, \frac{\pi}{2}, \infty} \), with underlying space a punctured 2-sphere. Figure 10. Let us denote \( S^2_{\frac{\pi}{2}, \frac{\pi}{2}, \infty} \) by \( O_{230} \) to unify notation.
Consider the homomorphism
\[
\Omega : \pi_1^0(O_{23\delta}) = |F, D; F^3 = D^2 = 1| \rightarrow \Sigma_6
\]
\[
\begin{align*}
F & \rightarrow (123)(456) \\
D & \rightarrow (15)(24)(36)
\end{align*}
\]
where \( \delta \geq 0 \). It defines a covering of orbifolds
\[
p_\Omega : O_{2\delta,2\delta,2\delta} \rightarrow O_{23\delta}
\]
where \( O_{2\delta,2\delta,2\delta} \) is a 2-sphere with 3 holes and the length of the closed geodesic at every hole is \( 2\delta \) if \( \delta > 0 \) and \( O_{0,0,0} \) is a 3-punctured 2-sphere. Figure 13.

The hyperbolic manifold \( O_{2\delta,2\delta,2\delta} \).

The fundamental group of \( O_{2\delta,2\delta,2\delta} \), \( \pi_1^0(O_{2\delta,2\delta,2\delta}) \), is a free subgroup of \( \rho'_x(G(3_1)) \) generated by \( g_1 = F^{-2}DFD^{-1} = FDFD \) and \( g_2 = F^{-1}DF^2D^{-1} = F^{-1}DF^{-1}D \). We can write the two generators of \( \pi_1^0(O_{2\delta,2\delta,2\delta}) \) in terms of \( \rho'_x(a) = A \) and \( \rho'_x(b) = B \), because \( F = AB \) and \( D = ABA = BAB \). Then
\[
\begin{align*}
g_1 &= FDFD = ABABAABABA = B^{-1}B^{-1} \\
g_2 &= F^{-1}DF^{-1}D = B^{-1}A^{-1}ABA^{-1}A^{-1}ABA = AA
\end{align*}
\]
We conclude that the subgroup of \( \rho'_x(G(3_1)) \) generated by \( \{ \rho'_x(b^{-2}), \rho'_x(a^2) \} \) is an index 6 free subgroup of rank 2 generated by \( \{ B^{-2}, A^2 \} \). □

4. THE REPRESENTATIONS OF \( G(3_1) \) IN \( A(H) \)

In this section we obtain all the representations of the trefoil knot group in the affine isometry group \( A(H) \) of a quaternion algebra \( H \). They include all the representations in the 3-dimensional affine Euclidean isometries \( \mathcal{E}(E^3) \) and all the representations in the 3-dimensional affine Lorentz isometries \( \mathcal{L}(E^{1,2}) \). The representations in the affine isometry group of a quaternion algebra are affine deformations of representations in the unit quaternions group of \( H \), computed in the above section.

Let
\[
\rho : G \rightarrow A(H) \quad \begin{align*}
a & \rightarrow \rho(a) = (sA^- , A) \\
b & \rightarrow \rho(b) = (sB^- + (A^- B^-)^-, B)
\end{align*}
\]
be a representation of \( G \) in the affine group of a quaternion algebra \( H \). The composition of \( \rho \) with the projection \( \pi_2 \) on the second factor of \( A(H) = H_0 \rtimes U_1 \) gives
the linear part of ρ and it is a representation $\hat{\rho}$ on the group of unitary quaternions.

$$\hat{\rho} = \pi_2 \circ \rho : \quad G \longrightarrow U_1$$

$$a \rightarrow A$$

$$b \rightarrow B$$

The composition of ρ with the projection $\pi_1$ on the first factor of $A(H) = H_0 \times U_1$ gives the translational part of ρ:

$$v_\rho = \pi_1 \circ \rho : \quad G \longrightarrow H_0$$

$$a \rightarrow sA^-$$

$$b \rightarrow sB^- + (A^- B^-)^-$$

Therefore $\hat{\rho}$ corresponds to a point in the character variety $V(I_G)$ of representations of G and it is determined by the characters $x$ and $y$. We are interested in the classes of affine deformations up to conjugation in the equiform group $Eq(H) = H_0 \times U$, where $U$ is the group of invertible elements. Each of these classes is determined by the parameter $s$.

The relation $sA$ in the case of the Trefoil knot group shows that the parameter $s$ satisfies the equation

$$4x^2 + 4sx - 3 = 0 \quad (8)$$

**Theorem 8.** Every representation $\rho_x : G(3/1) \longrightarrow A(H) = H_0 \times U_1$ defined by

$$\rho_x(a) = (sA^-, A)$$

$$\rho_x(b) = ((sB^- + (A^- B^-)^-), B)$$

where $A$ and $B$ are independent unit quaternions, factors through the group $C_2 * C_3$.

**Proof.** The group $G(3_1)$ has also the presentation $\langle F, D; F^3 = D^2 \rangle$ (Recall (3).)

The group $C_2 * C_3$ is a quotient of $G(3_1) = \langle F, D; F^3 = D^2 \rangle$.

$$q : G(3/1) = \langle F, D; F^3 = D^2 \rangle \longrightarrow C_2 * C_3 = \langle F, D; F^3 = D^2 = 1 \rangle$$

Then to prove that there exists a homomorphism

$$\lambda_x : C_2 * C_3 = \langle F, D; F^3 = D^2 = 1 \rangle \longrightarrow A(H)$$

such that $\lambda_x \circ q = \rho_x$ it is enough to show that $\rho_x(D^2) = (0, 1)$. Recall that the element $C = D^2 = F^3$ belongs to the center of the group $G(3_1)$, therefore $\rho_x(C)$ commutes with every element of $\rho_x(G(3_1))$, in particular with $\rho_x(a)$ and $\rho_x(b)$. Let us consider first the linear part $\hat{\rho}_x = \pi_2 \circ \rho_x : G(3_1) \longrightarrow U_1$. Because $A$ and $B$ are independent, $A^- \neq \pm B^-$, the only element of $U_1$ commuting with $A$ and $B$ is the identity 1. Therefore $\hat{\rho}_x(C) = 1$, and $\rho_x(C)$ is a translation. If $\rho_x(C) = (v, 1)$ and $\rho_x(a) = (sA^-, A)$ commute then

$$(v, 1)(sA^-, A) = (sA^-, A)(v, 1) \Rightarrow (v, 1)(sA^-, A) = (sA^- + A \circ v, A)$$

$$\Rightarrow v = A \circ v = AvA^{-1}$$

we deduce that $v = 0$ or $v$ and $A^-$ are dependent. An analogous computation with $\rho_x(b)$ yields $v = 0$ or $v$ and $B^-$ are dependent. As $A$ and $B$ are independent, $v = 0$. We have proved that $\rho_x(C) = (0, 1)$.

Moreover, every homomorphism $\lambda : C_2 * C_3 = \langle F, D; F^3 = D^2 = 1 \rangle \longrightarrow A(H)$ induces a representation $\rho : G(3/1) \longrightarrow A(H)$ such that $\rho = \lambda \circ q$. Therefore we can apply the results on representations of $G(3_1)$ in $A(H)$ to representations of $C_2 * C_3$ in $A(H)$. 


Corollary 1. Every non cyclic subgroup $S$ in $A(H)$ generated by two isometries $\mu \neq 1, \nu \neq 1$ such that $\mu^2 = \nu^3 = 1$ is necessarily generated by two conjugate elements in $A(H)$.

Proof. The subgroup $S$ is the image of a representation

$$
\lambda : C_2 \ast C_3 = \langle F, D ; F^3 = D^2 = 1 \rangle \longrightarrow A(H)
$$

$$
\begin{array}{c|c}
D & \mu \\
F & \nu
\end{array}
$$

Then it defines a representation $\rho = \lambda \circ \phi : G(3/1) \longrightarrow A(H)$ with the same image $S$. As $G(3_1)$ is generated by two conjugate elements $a$ and $b$, the group $S$ is generated by the two conjugate elements $\rho(a)$ and $\rho(b)$.

4.1. The Hamilton quaternion algebra $\mathbb{H} = (\mathbb{R} \{-1, -1\})$. Let us study first the affine deformations of representations corresponding to points in the character variety belonging to Case 1: $x \in (\frac{-x^2}{2}, \frac{1}{2}) \iff \{1 - x^2 > 0, (1 - x^2)^2 > y^2\}$, where the quaternion algebra is the Hamilton quaternion algebra $\mathbb{H} = (\mathbb{R} \{-1, -1\})$. Therefore $U_1 = S^3, H_0 = E^3$.

Recall that in this case, the geometrical meaning of parameters $x, y, u, s, \sigma$ is the following.

Let $\alpha$ be the angle of the right rotation around the axis $A^-$ corresponding to the action of $A = A^+ + A^-$ in $H_0 \cong E^3$. Let $\omega$ be the angle defined by $A^-$ and $B^-$. See Figure 14. Then

$$
x = A^+ = B^+ = \cos \frac{\alpha}{2}
$$

$$
u = N(A^-) = N(B^-) = 1 - x^2 = \sin^2 \frac{\alpha}{2}
$$

$$
y = -(A^- B^-)^+ = \langle A^-, B^- \rangle = u \cos \omega
$$

The shift $\sigma$ of the element $(sA^-, A)$ or any element of $A(H)$ conjugate to $(sA^-, A)$ is given by

$$
\sigma = s \sqrt{u} = s \sqrt{1 - x^2}
$$
The distance $\delta$ between the axes of $(sA^-, A)$ and $(sB^- + (A^- B^-)^-, B)$ is half of the length of the vector $(A^- B^-)^-$. See Figure 14.

$$N((A^- B^-)^-) = -y^2 + u^2 \implies \delta = \frac{\sqrt{u^2 - y^2}}{2}$$

(11)

The following Theorem gives the affine deformations of the subgroups of $SO(3, \mathbb{R})$ which are the image of representations of $G(3_1)$, or equivalently the affine deformations of the holonomies of the spherical conemanifold $S^2_{\frac{2\pi}{2\pi}, \frac{\alpha}{2\pi}}, \alpha < \frac{19\pi}{6}$.

**Theorem 9.** For each $x \in (-\sqrt{3}/2, \sqrt{3}/2)$, there exists a representation $\rho_x : G(3/1) \rightarrow \mathcal{E}(3)$ unique up to conjugation in $\mathcal{E}(3)$ such that

$$\rho_x(a) = (sA^-, A) \quad \rho_x(b) = ((sB^- + (A^- B^-)^-, B)$$

where

$$A = x + 2x^2 \frac{1}{\sqrt{1-x^2} + x^2} \quad B = x + \sqrt{1-x^2}i, \quad \sqrt{1-x^2} > 0$$

In affine linear notation, where $\{X, Y, Z\}$ is the coordinate system associated to the basis $\{-i, j, i\}$

$$\rho_x(a) = M_x(a) = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix}$$

$$\rho_x(b) = M_x(b) = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix}$$

where

$$M_x(a) = \begin{pmatrix} 2x^2 - 1 & \frac{x-2x^3}{\sqrt{1-x^2}} & \frac{x^2-3x^3}{2(1-x^2)} & 0 \\ \frac{2x^2-1}{\sqrt{1-x^2}} & \frac{-x^2+2x^2+1}{2-x^2} & \frac{(1-x^2)}{2(x^2-1)} & \frac{\sqrt{3-4x^2}}{8\sqrt{1-x^2}} \\ -x^2 & \frac{1-x^2}{2(x^2-1)} & \frac{1-2x^2}{2x^2-2} & \frac{8x^2+10x^2-3}{8\sqrt{1-x^2}} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(12)

$$M_x(b) = \begin{pmatrix} 2x^2 - 1 & -2x\sqrt{1-x^2} & 0 & -\frac{1}{2}\sqrt{3-4x^2} \\ 2x\sqrt{1-x^2} & 2x^2 - 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{(3-4x^2)\sqrt{1-x^2}}{4x} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(13)

The distance $\delta$ and the angle $\omega$ between the axis of $\rho_x(a)$ and $\rho_x(b)$ are given by

$$\delta = \frac{\sqrt{3-4x^2}}{4}$$

$$\cos \omega = \frac{2x^2 - 1}{2 - 2x^2}$$

The shift $\sigma$ is

$$\sigma = \left(\frac{3}{4x} - x\right)\sqrt{1-x^2}$$
Proof. The values of $A$ and $B$ are given by (6). From the polynomial (8) we obtain the value of $s$

$$s = \frac{3 - 4x^2}{4x}$$

The associated representation $\rho_x : G(3/1) \rightarrow E(E^3)$ such that

$$\rho_x(a) = (sA^-, A)$$

$$\rho_x(b) = ((sB^- + (A^-B^-)^-, B)$$

is unique up to conjugation in $E(E^3)$ (\cite{6}, Th.5) and $(A^-B^-)^- = -\sqrt{1 - x^2} - y^2i j$.

The matrices $M_x(a)$ and $M_x(b)$ as affine transformations are given by

$$M_x(a) = \begin{pmatrix}
  m_x(-1, -1; x, -\frac{y}{\sqrt{1-x^2}}; \sqrt{(1-x^2)^2-y^2}, 0) & 0 \\
  0 & \frac{s}{\sqrt{1-x^2}} \sqrt{(1-x^2)^2-y^2} \\
  0 & 0 & 0 & 1
\end{pmatrix}$$

$$M_x(b) = \begin{pmatrix}
  m_x(-1, -1; x, \sqrt{1-x^2}; 0, 0) & -\sqrt{(1-x^2)^2-y^2} \\
  0 & \frac{s}{\sqrt{1-x^2}} \sqrt{(1-x^2)^2-y^2} \\
  0 & 0 & 0 & 1
\end{pmatrix}$$

The values of $\delta$, $\cos \omega$ and $\sigma$ are given by (11), (9) and (10).

Remark 6. The representation

$$\tilde{\rho}_0 : G(3/1) \rightarrow S^3$$

$$a \rightarrow \frac{1}{2} i + \frac{\sqrt{7}}{2} j$$

$$b \rightarrow \frac{1}{i}$$

does not have any affine deformation, because the polynomial (8) is never zero for $x = 0$.

The following theorem states the classification up to conjugation of the non cyclic subgroups of $E(E^3)$ generated by two isometries $\mu \neq 1$, $v \neq 1$ such that $\mu^2 = v^3 = 1$.

Theorem 10. Every non cyclic subgroup $S$ in the group of direct Euclidean isometries $E(E^3)$ generated by two isometries $\mu \neq 1$, $v \neq 1$ such that $\mu^2 = v^3 = 1$ is one of the following

1. If $S$ has a fixed point, then $S$ is conjugate in $E(E^3)$ to the holonomy of the 2-dimensional spherical conemanifold $S^2_{\frac{2\pi}{3}}, \alpha$, $\frac{2\pi}{3} < \alpha < \frac{10\pi}{6}$.

2. If $S$ has no fixed points, then three cases are possible

2.a) $S$ is conjugate in $E(E^3)$ to the subgroup generated by $M_x(a)$ and $M_x(b)$, $x \in (0, \sqrt{3}/2)$, Sec. (12) and (13).

2.b) $S$ is conjugate in $E(E^3)$ to the natural extension to $E(E^3)$ of the holonomy of the 2-dimensional Euclidean orbifold $S^2_{\frac{2\pi}{36}}$.

2.c) $S$ is conjugate in $E(E^3)$ to the Euclidean crystallographic group $P6_1$.

Proof. If $S$ has a fixed point, the translational part of $S$ is 0. Then $S \subset SO(3)$, and therefore it is conjugate to the image of

$$\tilde{\rho}'_x = c \circ \tilde{\rho}_x : G(3/1) \rightarrow SO(3)$$
for some \( x \in (-\sqrt{3}/2, \sqrt{3}/2) \). Theorem 2 shows that \( \tilde{\rho}_x^* \) \((G(3/1)) \) is isomorphic to the holonomy of the 2-dimensional spherical conemanifold \( S^2_x \mathbb{Z}_{\alpha} \), where \( x = \cos \frac{\alpha}{2} \).

If \( S \) has no fixed points, we analyze the relative position of the axes of the generators \( \rho(a) \) and \( \rho(b) \). If these axes are not parallel, then \( S \) is conjugate to the image of \( \rho_x : G(3/1) \rightarrow E(3) \) for some \( x \in [0, \sqrt{3}/2) \). Theorem 3 gives the generators \( M_x(a) \) and \( M_x(b) \).

If the axes are parallel, we show in the following that \( \pi_2(S) \) is generated by two conjugate rotations \( (\pi_2 \circ \rho)(a) \) and \( (\pi_2 \circ \rho)(b) \) with the same axis. Therefore \( \hat{\rho} \) is a reducible representation.

\[
C_2 \ast C_3 \xrightarrow{\lambda} E(3) \xrightarrow{\pi_2} SO(3)
\]

Then \( \pi_2 \circ \lambda \) factors through the abelianized group \( C_6 \) of \( C_2 \ast C_3 \). Therefore \( \pi_2(S) \) is a cyclic group of order dividing 6. The elements \( (\pi_2 \circ \rho)(a) \) and \( (\pi_2 \circ \rho)(b) \) are both elements of order 1, 2, 3 or 6. Up to similarity we assume that the axis of \( \rho(a) \) is the \( Z \) axis and the axis of \( \rho(b) \) is the line parallel to the \( Z \) axis through the point \((1, 0, 0)\). In affine notation

\[
\rho(a) = \begin{pmatrix}
\cos \alpha & -\sin \alpha & 0 & 0 \\
\sin \alpha & \cos \alpha & 0 & 0 \\
0 & 0 & 1 & \sigma \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
\rho(b) = \begin{pmatrix}
\cos \alpha & -\sin \alpha & 0 & 1 - \cos \alpha \\
\sin \alpha & \cos \alpha & 0 & -\sin \alpha \\
0 & 0 & 1 & \sigma \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

where \( \alpha \) is the angle of rotation around the axis and \( \sigma \) is the shift.

The relation \( aba = bab \) implies

\[
\rho(aba) - \rho(bab) = \begin{pmatrix}
0 & 0 & 0 & (1 - 2 \cos \alpha)^2 \sin \alpha \\
0 & 0 & 0 & (1 - 2 \cos \alpha)^2 \sin \alpha \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\Rightarrow \ \cos \alpha = \frac{1}{2} \ \Rightarrow \ \alpha = \frac{2\pi}{6} \ \Rightarrow \ x = \frac{\sqrt{3}}{2}
\]

If the parameter \( \sigma \) is 0, then \( S \) is conjugate to the holonomy of the Euclidean 2-dimensional orbifold \( S_{236} \).

If the parameter \( \sigma \) is different from 0, we assume up to similarity that \( \sigma = 1/6 \). Then \( S \) is conjugate to the Euclidean crystallographic group \( P6_1 \). To prove this, we compute some elements and their axes.

\[
A = \rho(a) = \begin{pmatrix}
\frac{1}{\sqrt{3}} & -\frac{\sqrt{2}}{2} & 0 & 0 \\
\frac{\sqrt{2}}{2} & \frac{1}{\sqrt{3}} & 0 & 0 \\
0 & 0 & 1 & \frac{1}{6} \\
0 & 0 & 0 & 1
\end{pmatrix}, \ B = \rho(b) = \begin{pmatrix}
\frac{1}{\sqrt{3}} & -\frac{\sqrt{2}}{2} & 0 & \frac{3}{2} \\
\frac{\sqrt{2}}{2} & \frac{1}{\sqrt{3}} & 0 & -\frac{\sqrt{2}}{2} \\
0 & 0 & 1 & \frac{1}{6} \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
The element $A^6 = B^6$ is a translation by vector $(0, 0, 1)$.

$$ABA = \begin{pmatrix}
-1 & 0 & 0 & \frac{3\sqrt{2}}{2} \\
0 & -1 & 0 & \frac{\sqrt{2}}{2} \\
0 & 0 & -1 & \frac{\sqrt{2}}{2} \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad (ABA)^2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

Then $ABA$ is a rotation by $\pi$ with shift $(0, 0, \frac{1}{2})$. The axis of $ABA$ is obtained by solving the equation

$$ABA \begin{pmatrix} x_1 \\
x_2 \\
x_3 \\
1 \end{pmatrix} - \begin{pmatrix} x_1 \\
x_2 \\
x_3 \\
1 \end{pmatrix} = \begin{pmatrix} 0 \\
0 \\
\frac{1}{2} \\
0 \end{pmatrix}$$

Then $x_1 = \frac{3}{4}$, $x_2 = \frac{\sqrt{3}}{4}$. The axis of $ABA$ is the line $\left(\frac{3}{4}, \frac{\sqrt{3}}{4}, t\right)$.

$$AB = \begin{pmatrix}
-\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & \frac{3\sqrt{2}}{2} \\
\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & \frac{\sqrt{2}}{2} \\
0 & 0 & 1 & \frac{3\sqrt{2}}{2} \\
0 & 0 & 0 & 1
\end{pmatrix}$$

The isometry $AB$ is a rotation by $\frac{2\pi}{3}$ with shift $(0, 0, \frac{1}{4})$. The axis of $AB$ is the line $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, t\right)$. These data correspond to the crystallographic group $P6_1$ number 169 in the Tables [1].

The group $P6_1$ is also $\pi_1(\hat{S}_{2,3,6_1})$, the fundamental group of the 3-dimensional Euclidean orbifold $E^3/P6_1$ denoted by $\hat{S}_{2,3,6_1}$, see Figure 15.

![Figure 15. The crystallographic group $P6_1$.](image)

**Theorem 11.** The homomorphism

$$\begin{pmatrix}
\rho : & G(3/1) & \longrightarrow & P6_1 & \subset & E(E^3) \\
ap & \rightarrow & A \\
b & \rightarrow & B
\end{pmatrix}$$

where $A$ and $B$ are given by [14], factors through $\pi_1(M_0)$, where $M_0$ is the 3-manifold obtained from $S^3$ by 0-surgery in the trefoil knot.
Proof. The manifold $M_0$ is the result of pasting a solid torus $T$ to the exterior of the trefoil knot $K$ such that the meridian of the torus $T$ is mapped to the canonical longitude $l_c$ of $K$. The element of $G(3/1)$ represented by $l_c$ is $a^{-4}baab$. Then

$$\pi_1(M_0) = \left| a, b; aba = bab, a^{-4}baab \right|$$

The image $\rho(l_c) = A^{-4}BAAB = I_{4 \times 4}$, therefore the homomorphism $\rho$ factors through $\pi_1(M_0)$:

$$\rho = \rho' \circ \eta$$

Corollary 2. The exterior of the trefoil knot has a Euclidean structure whose completion gives a complete Euclidean structure in $M_0$.

The following Theorem gives more information on this manifold $M_0$.

**Theorem 12.** The manifold $M_0$ is the spherical tangent bundle of the 2-dimensional Euclidean orbifold $S_{2,3,6}$.

Proof. The sphere $S^3$ has the Seifert manifold structure $(Ooo|0; (3, -1), (2, 1))$, with the trefoil knot $K$ as general fibre. The result of 0-surgery in a general fibre produces a Seifert manifold $(Ooo|0; (3, -1), (2, 1), (\alpha, \beta))$, where the pair $(\alpha, \beta)$ can be easily computed as follows. Let $Q_1, Q_2$ be simple closed curves in $S^2$, the base of the Seifert fibration $(Ooo|0; (3, -1), (2, 1))$, which are meridians of a general fibre $H$, the exceptional fibre $(3, -1)$, and the exceptional fibre $(2, 1)$, respectively. Then, the first homology group of $(S^3 \setminus K)$ has the following presentation

$$H_1(S^3 \setminus K) = |Q, Q_1, Q_2, H; 3Q_1 - H = 0, 2Q_2 + H = 0, Q + Q_1 + Q_2 = 0|$$

Let $l$ be the canonical longitude of $K = H$, then in $M_0$ we have that

$$l = \alpha Q + \beta H = \alpha(-Q_1 - Q_2) + \beta 3Q_1 = Q_1(3\beta - \alpha) - \alpha Q_2 = 0$$

This implies that

$$(3\beta - \alpha)Q_1 = \alpha Q_2 \Rightarrow -\alpha Q_1 = \alpha Q_2 + 2\beta Q_2$$

But also $2Q_2 = -3Q_1$. Therefore

$$\frac{3\beta - \alpha}{\alpha} = -\frac{3}{2} \Rightarrow 6\beta - 2\alpha = -3\alpha \Rightarrow 6\beta = -\alpha \Rightarrow \alpha = 6, \beta = -1.$$ 

Using Seifert signature equivalence, we have that

$$M_0 = (Ooo|0; (3, -1), (2, 1), (6, -1)) = (Ooo|1; (3, -1), (2, -1), (6, -1))$$

which is orientation reversing equivalent to

$$(Ooo|1; (3, 1), (2, 1), (6, 1)) = ST(S_{2,3,6})$$

the spherical tangent of the Euclidean orbifold $S_{6,3,2}$ since

$$\chi = -1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 0$$

Remark 7. The manifold $M_0$ is a torus bundle over $S^1$ with periodic monodromy of order 6 ([12]).
Theorem 10 can be used to identify all the 3-dimensional orientation preserving Euclidean crystallographic groups generated by one element of order two and one other of order three.

**Theorem 13.** The only 3-dimensional orientation preserving Euclidean crystallographic groups generated by one element of order two and one other of order three, up to similarity, are $I2_13$, $P4_132(P4_132)$ and $P6_1$.

**Proof.** Assume $S$ is conjugate to the subgroup $\rho_x(G(3/1))$ generated by $M_x(a)$ and $M_x(b)$, $x = \cos \frac{\alpha}{2} \in (0, \sqrt{3}/2)$, where $M_x(a)$ and $M_x(b)$ are given by (12) and (13). A necessary condition for $S$ be Euclidean crystallographic group is $\alpha \in \{ \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{2\pi}{2} \}$.

Case $x = 0$. This case is impossible. (Remark 6)

Case $x = \frac{1}{2}$. Then

$$\alpha = \frac{2\pi}{3}, \quad \cos \omega = -\frac{1}{3}, \quad \sigma = \frac{\sqrt{3}}{2}, \quad \delta = \frac{\sqrt{2}}{4}.$$

We will prove that $\rho_{\frac{1}{2}}(G(3/1))$ is the Euclidean crystallographic group $I2_13$ (number 199 in Tables [1]).

(a) $\rho_{\frac{1}{2}}(G(3/1)) \leq I2_13$. To compare both groups, we conjugate $\rho_{\frac{1}{2}}(G(3/1))$ by a similarity so that

$$\alpha = \frac{2\pi}{3}, \quad \cos \omega = -\frac{1}{3}, \quad \sigma = \frac{\sqrt{3}}{6}, \quad \delta = \frac{\sqrt{2}}{12}.$$

The axes $\rho_{\frac{1}{2}}(a)$ and $\rho_{\frac{1}{2}}(b)$ are depicted in Figure 16. It can be checked that the two axes in Figure 16 have distance $\frac{\sqrt{2}}{12} = \delta$ and the angle $\omega$ is

$$\cos \omega = 2 \cos^2 \frac{\omega}{2} - 1 = 2 \left( \frac{1}{\sqrt{3}} \right)^2 - 1 = -\frac{1}{3}.$$

The shift $\sigma = \frac{\sqrt{3}}{6}$ is the length of the diagonal of the cube with edge length $\frac{1}{6}$. Therefore $\rho_{\frac{1}{2}}(a^{-1})$ and $\rho_{\frac{1}{2}}(b)$ are the screws axes in $I2_13$ (Figure 17) denoted by $A^{-1}$ and $B$.

![Figure 16. The axes of $\rho_{\frac{1}{2}}(a)$ and $\rho_{\frac{1}{2}}(b)$.]
(b) $\rho_\frac{\sqrt{2}}{2}(G(3/1)) = I_23$. The group $I_23$ satisfies the following short exact sequence

$$0 \longrightarrow T^3 \longrightarrow I_23 \overset{p}{\longrightarrow} S_{233} \longrightarrow 1$$

where $T^3$ is the translational subgroup of a cube with edge length 1 as fundamental domain and $S_{233}$, the linear quotient, is the holonomy of the 2-dimensional spherical orbifold denoted also by $S_{233}$. It is clear that $p(\rho_\frac{\sqrt{2}}{2}(a^{-1})) = \rho_\frac{\sqrt{2}}{2'}(a^{-1})$ and $p(\rho_\frac{\sqrt{2}}{2}(b)) = \rho_\frac{\sqrt{2}}{2'}(b)$ generate $S_{233}$ by Theorem 2. Therefore it suffices to prove that $T^3 \leq \rho_\frac{\sqrt{2}}{2}(G(3/1))$ and to find enough elements: The element $A^3$ is a translation with vector $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Figure 17 shows the elements $AB$, $BA$ (3-fold rotations), $AAB$ (2-fold rotation), $ABA$, $BA^{-1}$ (2-screw rotations).

Case $x = \frac{\sqrt{2}}{2}$. Then

$$\alpha = \frac{2\pi}{4}, \quad \cos \omega = 0, \quad \sigma = \frac{1}{4}, \quad \delta = \frac{1}{4}$$

The group $P4_132$ is depicted in Figure 18. Up to similarity we assume that the axes $\rho_\frac{\sqrt{2}}{2}(a)$ and $\rho_\frac{\sqrt{2}}{2}(b)$ are the ones depicted in Figure 18, so they are the $4_1$ axes denoted by $A$ and $B$ in Figure 18.

In affine notation

$$A = \rho_\frac{\sqrt{2}}{2}(a) = \begin{pmatrix} 0 & -1 & 0 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = \rho_\frac{\sqrt{2}}{2}(b) = \begin{pmatrix} 0 & 0 & 1 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & \frac{1}{\sqrt{2}} \\ -1 & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(15)

This proves that $\rho_\frac{\sqrt{2}}{2}(G(3/1)) \leq P4_132$, because $\rho_\frac{\sqrt{2}}{2}(G(3/1))$ is generated by $A$ and $B$, both elements of $P4_132$. To prove the equality, we can compute...
Figure 18. The axes of $\rho_{\frac{\pi}{2}}(a)$ and $\rho_{\frac{\pi}{2}}(b)$.

Figure 19. The crystallographic group $I4_{1}32$.

the others elements of $P4_{1}32$. For instance, the element

$$\begin{pmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

is a rotation by $2\pi/3$ with axis $(t, t, t)$. The element

$$\begin{pmatrix}
-1 & 0 & 0 & \frac{1}{2} \\
0 & 0 & 1 & -\frac{1}{2} \\
0 & 1 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 1
\end{pmatrix}$$
is a rotation by $\pi$ with axis $(1/8, t, t + 1/4)$. Observe that $A^4$ and $B^4$ are translations. The subgroup generated by $A^4$ and $B^4$ has a cube as fundamental domain, with edge length one.

Case $x = \sqrt{3}/2$. Theorem [10] 2.c) shows that $S$ is conjugate to $P6_1$. □

**Remark 8.** The space $E^3/I2_3$ is the Euclidean orbifold $Q_1$ with underlying space $S^3$ and singular set the rational link 10/3 whose two components have isotropies of orders 2 and 3. See Figure [20].

**Remark 9.** The space $E^3/P4_132$ is the Euclidean orbifold $Q_2$ with underlying space $S^3$ and singular set the graph depicted in Figure [20]. This orbifold is 2-fold covered by the Euclidean orbifold $E^3/P2_1$ with underlying space $S^3$ and singular set the Figure Eight knot with isotropy of order 3.

![Figure 20. Euclidean orbifolds.](image)

4.2. **Case 2:** $|x| = \sqrt{3}/2$. The almost-irreducible representation $\rho_{\frac{\sqrt{3}}{2}} : G(3/1) \rightarrow SL(2, \mathbb{C})$ given by (7) do not have a proper affine deformation, because the polynomial $s$ in this case gives $s = 0$. But, by Theorem [10] there exists an affine deformation of a reducible representation $\rho_{\frac{\sqrt{3}}{2}} : G(3/1) \rightarrow SO(3)$, whose image is the crystallographic group $P6_1$.

4.3. **The quaternion algebra** $(-\frac{1,1}{\mathbb{R}})$. Next we analyze the affine deformations of representations corresponding to points in the character variety belonging to Cases 3, 4 and 5: $|x| > \sqrt{3}$, where the quaternion algebra is $M(2, \mathbb{R}) = (-\frac{1,1}{\mathbb{R}})$. Therefore $U_1 = SL(2, \mathbb{R})$, $H_0 = E^{1,2}$, the Minkowski space, and $A(H) = L(E^{1,2})$.

The meaning of the parameters depends on the sign of $x - 1$.

Case 3 : $\sqrt{3}/2 < x < 1 \implies x - 1 < 0$, then $A^-$ is a vector inside the nullcone (a time-like vector) and $A$ acts as a right spherical rotation around the axis $A^-$ with angle $\alpha$, where $x = \cos\frac{\alpha}{2}$. Let $d$ be the hyperbolic distance between the projection of $A^-$ and $B^-$ on the hyperbolic plane (pure unit
where the values of \( A, B \)

Theorem 14. For each Case 4 : 1, then \( A^- \) belongs to the nullcone (a light-like vector) and \( A \) acts as a parabolic transformation around the axis \( A^- \).

Case 5: \( 1 < x \Leftrightarrow x - 1 > 0 \), then \( A^- \) is a vector outside the nullcone (a space-like vector) and \( A \) acts as a right hyperbolic rotation around the axis \( A^- \) with distance \( \partial \), where \( x = \cosh \frac{\partial}{2} \). The meaning of the parameter \( y \) depends on the sign of \( y^2 - (x^2 - 1)^2 \).

(a) If \( y^2 - (x^2 - 1)^2 > 0 \), \( y^2 = (x^2 - 1)^2 \cos^2 d \), where \( d \) is the distance between the polars of \( A^- \) and \( B^- \).

(b) If \( y^2 - (x^2 - 1)^2 = 0 \) then \( y^2 = (x^2 - 1)^2 \).

(c) If \( y^2 - (x^2 - 1)^2 < 0 \), \( y^2 = (x^2 - 1)^2 \cos \theta \), where \( \theta \) is the angle between the polars of \( A^- \) and \( B^- \).

Therefore

\[
\begin{align*}
x &= A^+ = B^+ = \cosh \frac{\partial}{2} \\
y^2 &> (x^2 - 1)^2 \quad \Rightarrow \quad y^2 = (x^2 - 1)^2 \cos^2 d \\
y^2 &< (x^2 - 1)^2 \quad \Rightarrow \quad y^2 = (x^2 - 1)^2 \cos \theta
\end{align*}
\]

(17)

The shift \( \sigma \) of the element \((sA^-, A)\) and the distance \( \delta \) between the axes of \((sA^-, A)\) and \((sB^- + (A^-B^-), B)\) are as in the case 1, that is

\[
\sigma = s\sqrt{u} = s\sqrt{1 - x^2}.
\]

\[
N((A^-B^-)) = -y^2 + u^2 \quad \Rightarrow \quad \delta = \frac{\sqrt{u^2 - y^2}}{2}
\]

(19)

Theorem 14. For each \( x \in (\sqrt{3}/2, \infty) \), there exists a representation \( \rho_x : G(3/1) \longrightarrow \mathcal{L}(E^{1,2}) \) unique up to conjugation in \( \mathcal{L}(E^{1,2}) \) such that

\[
\begin{align*}
\rho_x(a) &= (sA^-, A) \\
\rho_x(b) &= (sB^- + (A^-B^-), B)
\end{align*}
\]

where the values of \( A, B \in SL(2, \mathbb{R}) \) are the following

(1) For \( \frac{\sqrt{3}}{2} < x < 1 \).

\[
\begin{align*}
A &= x + \sqrt{1 - x^2}i, \quad \sqrt{1 - x^2} > 0 \\
B &= x + \frac{2x^2 - 1}{2\sqrt{1 - x^2}}i + \frac{1}{2} \sqrt{\frac{4x^2 - 3}{1 - x^2}}j
\end{align*}
\]

(2) For \( x = 1 \Rightarrow y = \frac{1}{2} \).

\[
\begin{align*}
A &= 1 + i + j \\
B &= 1 + \frac{1}{4}(i - j)
\end{align*}
\]

(3) For \( x > 1, \ y = \frac{2x^2 - 1}{2} > 0 \).

\[
\begin{align*}
\tilde{\rho}_x(a) &= A = x + \sqrt{x^2 - 1}i, \quad \sqrt{x^2 - 1} > 0 \\
\tilde{\rho}_x(b) &= B = x - \frac{1}{2} \sqrt{\frac{4x^2 - 3}{1 - x^2}}i - \frac{2x^2 - 1}{2\sqrt{1 - x^2}}j
\end{align*}
\]
In affine linear notation, where \( \{X, Y, Z\} \) is the coordinate system associated to the basis \( \{-ij, j, i\} \)

\[
\rho_x(a) = M_x(a) \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{pmatrix} X' \\ Y' \\ Z' \\ 1 \end{pmatrix}
\]

where,

1. For \( \frac{\sqrt{3}}{2} < x < 1 \)

\[
M_x(a) = \begin{pmatrix} 2x^2 - 1 & -2x\sqrt{1-x^2} & 0 & 0 \\ 2x\sqrt{1-x^2} & 2x^2 - 1 & 0 & 0 \\ 0 & 0 & 1 & (3-4x^2)\sqrt{1-x^2}^{\frac{3}{2}} \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

2. For \( x = 1 \implies y = \frac{1}{2} \)

\[
M_x(a) = \begin{pmatrix} 1 & -2 & 2 & 0 \\ 2 & -1 & 2 & -\frac{1}{4} \\ 2 & -2 & 3 & -\frac{1}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

3. For \( x > 1, y = \frac{2x^2-1}{2} > 0 \)

\[
M_x(a) = \begin{pmatrix} 2x^2 - 1 & 0 & 2x\sqrt{x^2-1} & 0 \\ 0 & 1 & 0 & \frac{(3-4x^2)\sqrt{x^2-1}}{4x} \\ 2x\sqrt{x^2-1} & 0 & 2x^2 - 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]
The distance $\delta$ and the angle $\omega$ between the axis of $\rho_x(a)$ and $\rho_x(b)$ are given by

$$
\delta = \frac{\sqrt{3 - 4x^2}}{4}
$$

$$
\cos \omega = \frac{2x^2 - 1}{2 - 2x^2}
$$

The shift $\sigma$ is

$$
\sigma = \left(\frac{3}{4x} - x\right)\sqrt{1 - x^2}
$$

Proof. Theorem 1 gives the values of $A$ and $B$ as a function of $x$ and $y$ and also the values of $(A^{-}B^{-})^-$ in each case:

\[
\begin{align*}
\frac{\sqrt{3}}{2} &< x < 1 \implies (A^{-}B^{-})^- = \sqrt{y^2 - (1 - x^2)^2}i \bar{j} \\
x &= 1 \implies (A^{-}B^{-})^- = -yi \bar{j} \\
x > 1 &\implies (A^{-}B^{-})^- = \sqrt{y^2 - (x^2 - 1)^2}i \bar{j}
\end{align*}
\]

From the polynomial defining the character variety \(\mathfrak{c}\) we obtain the value of $y$

$$
y = \frac{2x^2 - 1}{2}
$$

From the polynomial \(\mathfrak{c}\) we obtain the value of $s$

$$
s = \frac{3 - 4x^2}{4x}
$$

Theorem 5] defines the representation $\rho_x : G(3/1) \rightarrow \mathcal{L}(E^{1,2})$ unique up to conjugation in $\mathcal{L}(E^{1,2})$ such that

$$
\rho_x(a) = (sA^{-}, A) \\
\rho_x(b) = ((sB^{-} + (A^{-}B^{-})^-), B)
$$

□

As in the Euclidean case we are interested in representations in $\mathcal{L}(E^{1,2})$ whose image is an affine crystallographic group. This concept is defined by Fried and Goldman as a subgroup of the affine group $A(H)$ acting properly discontinuously and with compact orbit space. An affine crystallographic group is the fundamental group of a flat affine manifold. The following theorem analyzes some examples in case 1 ($x < 1$) of Theorem 14, where generators go to elements with linear part $A$ and $B$ such that $A^{-}$ and $B^{-}$ are timelike vectors.

**Theorem 15.** The image of the representation $\rho_x : G(3/1) \rightarrow \mathcal{L}(E^{1,2})$, where $x = \cos \frac{2\pi}{2n}$, $n \geq 7$, is not a properly discontinuous subgroup of $\mathcal{L}(E^{1,2})$.

Proof. The linear quotient of $\rho_x(G(3/1))$ is the group $S_{23n} \subset SO(1, 2)$. The group $S_{23n}$, been the holonomy group of the 2-dimensional hyperbolic orbifold denoted by the same name $S_{23n}$, is a cocompact group, therefore Mess’s Theorem (\[\mathfrak{c}\], \[\mathfrak{c}\]) says that $Im(\rho_x)$ is not a properly discontinuous subgroup of $\mathcal{L}(E^{1,2})$.

□
To analyze the discrete condition of representations in case 3 \((x > 1)\) of Theorem 14 we can use the Margulis invariant \([10, 12, 11]\). Recall that an element of \(O(1,2)\) is hyperbolic if it has three distinct real eigenvalues. A subgroup \(G \subset O(1,2)\) is purely hyperbolic if every element is hyperbolic.

**Theorem 16.** Every image of \(\rho_x : G(3/1) \to \mathcal{L}(E^{1,2})\), where \(x > 1\) contains an affine deformation of a purely hyperbolic subgroup of finite index.

**Proof.** Theorem 13 shows that the image of the linear quotient of \(\rho_x\), \(\rho_x : G(3/1) \to SO^0(1,2), 1 < x < \infty\), is an index 6 free subgroup generated by \(\{\phi_x(b^2), \phi_x(a^2)\}\). All the elements of this subgroup are hyperbolic transformations.

We conclude that the subgroup of \(\rho_x(G(3/1))\) generated by \(\{\phi_x(b^2), \phi_x(a^2)\}\) is an affine deformation of the purely hyperbolic free subgroup of rank 2 generated by \(\{B^{-2}, A^2\}\).

Margulis defines an invariant \(\alpha_\phi : G \to \mathbb{R}\) of an affine deformation \(\phi\) of a purely hyperbolic subgroup \(G \subset SO^0(1,2)\) as follows. Every element \(g \in G\) has three distinct positive real eigenvalues \(\lambda(g) < 1 < \lambda(g)^{-1}\). Choose an eigenvector \(x^-(g)\) for \(\lambda(g)\) and an eigenvector \(x^+(g)\) for \(\lambda(g)^{-1}\), both in the same component \(\mathcal{N}_g\) of the complement of 0 in the nullcone. Consider the unique eigenvector \(x^0(g)\) for \(g\) with eigenvalue 1 such that \(|x^0(g)| = -1\) and \(\{x^-(g), x^+(g), x^0(g)\}\) is a positively oriented basis of \(E^{1,2}\).

If \(\phi\) is a hyperbolic deformation of \(G\), then \(\alpha_\phi\) is defined as

\[
\alpha_\phi : G \to \mathbb{R}, \quad g \to Q(x^0(g), \phi(g)(x) - x)
\]

for any \(x \in E^{1,2}\), where \(Q(-, -)\) is the bilinear quadratic form defining the Minkowski metric. It has been proven \([10]\) that \(\alpha\) is a complete invariant of conjugacy class of the affine deformation. The following theorem of Margulis can be used to check the proper condition of an affine deformation.

**Theorem 17** (Margulis). Let \(G\) be a purely hyperbolic subgroup of \(SO^0(1,2)\), and \(\phi : G \to \mathcal{L}(E^{1,2})\) an affine deformation. If there exist \(g_1, g_2 \in G\) such that \(\alpha_\phi(g_1) > 0 > \alpha_\phi(g_2)\), then \(\phi\) is not proper.

**Theorem 18.** The image of the representation \(\rho_x : G(3/1) \to \mathcal{L}(E^{1,2})\), where \(x > 1\), is not a properly discontinuous subgroup of \(\mathcal{L}(E^{1,2})\).

**Proof.** We compute the Margulis invariants of the elements \(g_2 = \hat{\rho}_x(a^2)\) and \(g_3 = \hat{\rho}_x(a^2b^2)\) in the purely hyperbolic group \(\pi_1^x(O_{25,25,25}) \subset \hat{\rho}_x(G(3/1))\). Observe that the element

\[
g_2 = \hat{\rho}_x(a^2) = \begin{pmatrix}
1 - 8x^2 + 8x^4 & 0 & 4x\sqrt{x^2 - 1}(2x^2 - 1) \\
0 & 1 & 0 \\
4x\sqrt{x^2 - 1}(2x^2 - 1) & 0 & 1 - 8x^2 + 8x^4
\end{pmatrix}
\]

has the following eigenvalues

\[
\lambda(g_2) = 1 - 8x^2 + 8x^4 - \sqrt{(1 - 8x^2 + 8x^4)^2 - 1} < 1
\]

\[
1 < (\lambda(g_2))^{-1} = 1 - 8x^2 + 8x^4 + \sqrt{(1 - 8x^2 + 8x^4)^2 - 1}.
\]

We choose the eigenvector \(x^-(g_2) = \{-1, 0, 1\}\), \(x^+(g_2) = \{1, 0, 1\}\). Therefore the vector \(x^0(g_2) = \{0, 1, 0\}\) is the unique eigenvector such that \(|x^0(g_2)| = -1\) and
For the following theorem, the image of \( \rho_x(a^2)(x) - x \) for any \( x \in E^{1,2} \) is

\[
\alpha_{\phi_x}(g_2) = (0, 1, 0) \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = -s_2
\]

where \( \{s_1, s_2, s_3\} \) are the \( \{X, Y, Z\} \) coordinates of \( \rho_x(a^2)(x) - x \) for \( x = \{t_1, t_2, t_3\} \).

The computation gives

\[
s_2 = \frac{(3 - 4x^2)\sqrt{x^2 - 1}}{2x}.
\]

Therefore

\[
\alpha_{\phi}(g_2) = \frac{(3 - 4x^2)\sqrt{x^2 - 1}}{2x}.
\]

This value is only zero for \( x = \pm \frac{\sqrt{2}}{2}, \pm 1 \), then it has the same sign for all \( x > 1 \).

For \( x = 2 \), it is \( \frac{13\sqrt{2}}{4} > 0 \). Then

\[
\alpha_{\phi_x}(g_2) > 0, \quad x > 1.
\]

We have used the computer program Mathematica to do an analogous, but much more complicate computation for \( g_3 = \rho_x(a^2b^7) \). We found

\[
\alpha_{\phi_x}(g_3) = \frac{3 + x^2(-1 - 22x^2 + 36x^4 - 16x^6 - 8x\sqrt{x^2 - 1}(9 + 4x^2(-21 + 63x^2 - 76x^4 + 32x^6)))}{8x\sqrt{(x^2 - 1)^3}}
\]

which is never zero for \( x > 1 \), and for \( x = 2 \) it is equal to \(-\frac{143(15 + 7616\sqrt{2})}{144\sqrt{3}} < 0 \).

Therefore \( \alpha_{\rho_x}(g_3) < 0 \), for all \( x > 1 \). We conclude that \( \alpha_{\rho_x}(g_2) > 0 > \alpha_{\rho_x}(g_3) \) for all \( x > 1 \). Then we apply the above Margulis’s Theorem to deduce that \( \rho_x(G(3/1)) \) contains a subgroup with no proper action.

For case 2 (\( x = 1 \)) in Theorem 14, where generators go to elements with linear part \( A \) and \( B \) such that \( A^- \) and \( B^- \) are null vectors, we know by Theorem 7 that the image of \( \rho_1 : G(3) \rightarrow \mathcal{L}(E^{1,2}) \), contains an affine deformation of a free subgroup of index 6 generated by the two parabolic elements \( \{B^-2, A^2\} \). To analyze the discrete condition of the representation \( \rho_1 : G(3) \rightarrow \mathcal{L}(E^{1,2}) \) one could use the generalization of the Margulis invariant method obtained in [2] for subgroups generated by two parabolic elements. But in this case it does not work because the generalized Margulis invariant of the parabolics elements \( A^2 \) and \( B^-2 \) is 0, as is the Margulis invariant of hyperbolic elements \( A^{2n}B^2 \). The reason is that each of the these elements has a line of fixed points. Moreover this property for some element implies a non properly discontinuously action. Therefore we can establish the following theorem.

**Theorem 19.** The image of the representation \( \rho_1 : G(3) \rightarrow \mathcal{L}(E^{1,2}) \), is not a properly discontinuous subgroup of \( \mathcal{L}(E^{1,2}) \).

**Proof.** The elements

\[
\rho_1(a^n) = \begin{pmatrix} 1 & -2 & 2 & 0 \\ 2 & -1 & 2 & -1 \\ 2 & -2 & 3 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}^n
\]
fix each point in the line \((1/8, t, t)\), and the action in a neighborhood of this line is not discontinuous.

**Corollary 3.** There is no affine crystallographic group in \(\mathcal{L}(E^{1,2})\) which is a quotient of \(G(3_1)\).

**Corollary 4.** There is no affine crystallographic group in \(\mathcal{L}(E^{1,2})\) generated by two isometries \(\mu\) and \(\nu\) such that \(\mu^2 = \nu^3 = 1\).

**Proof.** This is a consequence of Corollary 3 and Corollary 1.

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