Information-theoretical Limits of Recursive Estimation and Closed-loop Control in High-contrast Imaging

Leonid Pogorelyuk1, Laurent Pueyo2, Jared R. Males3, Kerri Cahoy1, and N. Jeremy Kasdin4

1 Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139, USA
2 Space Telescope Science Institute, 3800 San Martin Drive, Baltimore, MD 21218, USA
3 Steward Observatory, University of Arizona, 933 N. Cherry Avenue, Tucson, AZ 85721, USA
4 University of San Francisco, College of Arts and Sciences, 2130 Fulton Street, San Francisco, CA 94117, USA

Received 2020 October 24; revised 2021 June 25; accepted 2021 July 6; published 2021 October 5

Abstract

A lower bound on unbiased estimates of wave front errors (WFEs) is presented for the linear regime of small perturbation and active control of a high-contrast region (dark hole). Analytical approximations and algorithms for computing the closed-loop covariance of the WFE modes are provided for discrete- and continuous-time linear WFE dynamics. Our analysis applies to both image-plane and non-common-path wave front sensing (WFS) with Poisson-distributed measurements and noise sources (i.e., photon-counting mode). Under this assumption, we show that recursive estimation benefits from infinitesimally short exposure times, is more accurate than batch estimation and, for high-order WFE drift dynamical processes, scales better than batch estimation with amplitude and star brightness. These newly derived contrast scaling laws are a generalization of previously known theoretical and numerical results for turbulence-driven adaptive optics. For space-based coronagraphs, we propose a scheme for combining models of WFE drift, low-order non-common-path WFS (LOWFS) and high-order image-plane WFS (HOWFS) into closed-loop contrast estimates. We also analyze the impact of residual low-order WFE, sensor noise, and other sources incoherent with the star, on closed-loop dark hole maintenance and the resulting contrast. As an application example, our model suggests that the Roman Space Telescope might operate in a regime that is dominated by incoherent sources rather than WFE drift, where the WFE drift can be actively rejected throughout the observations with residuals significantly dimmer than the incoherent sources. The models proposed in this paper make possible the assessment of the closed-loop contrast of coronagraphs with combined LOWFS and HOWFS capabilities, and thus help estimate WFE stability requirements of future instruments.

Unified Astronomy Thesaurus concepts: Coronagraphic imaging (313); Space telescopes (1547); Ground telescopes (687)

1. Introduction

Wave front instability is a major limiting factor on the contrast during coronagraphic observations. In ground-based telescopes, atmospheric turbulence gives rise to fast wave front aberrations that are mostly counteracted by adaptive optics (AO; Roddier 1999). AO uses wave front sensors and deformable mirrors (DMs) to continuously estimate, predict, and correct the wave front errors (WFEs) using a natural or a laser guide star. In the Extreme AO regime (smallest WFE possible over a small field of view), the correction precision is fundamentally limited by the photon flux available for wave front sensing (WFS), along with the spatiotemporal properties of atmospheric turbulence (Guyon 2005; Cavarroc et al. 2006), resulting in contrasts of about $10^{-6}$ with current 8 m class telescopes (Macintosh et al. 2015; Currie et al. 2018).

In the absence of atmosphere, space telescopes are expected to achieve contrasts that are better by at least two orders of magnitude (Demers et al. 2015; Mennesson et al. 2016; Bolcar et al. 2017), eventually enabling the detection of exo-Earths (Pueyo et al. 2019; Stark et al. 2019). However, even in pristine environments such as Lagrange L2, space-based observatories are subject to small thermal and mechanical disturbances that can result in significant variations of the telescope wave front and instrument’s starlight suppression (Shaklan et al. 2011; Patterson et al. 2015; Perrin et al. 2018). As a result, when high-contrast imaging of exoplanets is considered, wave front stability is one of the main drivers of observatories’ overall structural designs (Coyle et al. 2019). It also drives the way the data will be collected, e.g., observation scenarios (Bailey et al. 2018; Laginja et al. 2021), which in turn can reduce exoplanet yields due to the overheads necessary to maintain as stable as possible of a wave front and calibrate remaining variation using post-processing (Stark et al. 2019).

Increasing the precision of real-time wave front correction is a topic of much research (Jovanovic et al. 2018; Snik et al. 2018). Proposed hardware improvements include faster computers and DMs (Macintosh et al. 2018), improved architectures of sensors (N’Diaye et al. 2013; Correia et al. 2020) and cameras (Baudoz et al. 2005; Bottom et al. 2016), and use of guide stars from a space-based laser (Douglas et al. 2019). Algorithmic approaches aim at exploiting all of the available information for a given system. They may, for example, utilize the available post-coronagraphic images (Paul et al. 2013; Martinache et al. 2014; Miller et al. 2017), or incorporate WFE dynamics via recursive estimation and predictive control (Kulcsár et al. 2012; Males & Guyon 2018; Pogorelyuk & Kasdin 2019).

Yet, predicting performances, e.g., computing contrast curves as a function of optical model parameters, WFE spatiotemporal profiles, control algorithms, and their parameters, is a complex task. It typically requires running full-model simulations with various parameter combinations and, possibly, different timescales. In this work, the authors propose an information-theoretical approach to approximating bounds on the residual WFE given linear models of their dynamics and...
of detector sensitivities (at a wave front sensor and/or image plane). The more general treatment of wave front dynamics presented here allows for the examination of both batch and recursive estimation and both common and non-common path control loops.

In Section 2 we outline a technique for computing a bound on the variance of the WFE estimates based on the Cramér-Rao inequality (Rao 1945; Cramér 1946). The discrete-time and continuous-time versions of the variance bounds can then be used to estimate the residual starlight intensity in the image plane (or contrast). In Section 3, closed-form expressions for the contrast are derived for some special cases. In particular, we provide scaling laws for the WFE variance as a function of drift magnitude, power spectral density (PSD), star brightness, and detector noise. The newly derived scaling laws are then compared to those of existing AO systems, and find broad overall agreement. Section 4 contains application examples in the context of space-based coronagraphs. It discusses the connection between low- and high-order wave front sensing (LOWFS and HOWFS), and presents HOWFS closed-loop bounds for the Nancy Grace Roman Space Telescope (RST). Section 5 summarizes the work.

2. Approximate Bounds of Unbiased WFE Mode Estimates

Throughout the paper we assume that the telescope operates in a steady-state linear regime after achieving its best contrast. In the case of the RST, for example, our analysis does not apply to dark hole creation (Krist et al. 2016) via pair probing and EFC (Giv'e'on et al. 2011). Instead, the focus of Section 2.1 is on the slow “drift” of wave front aberrations during the long scientific observation (tens of hours) of a relatively dim target.

We work under the assumption that a nominal dark hole has been generated using the methods above. When seeking to maintain this dark hole in the presence of thermal or mechanical drifts, such as in the optical tube assembly (OTA), the information “about” WFE modes (and hence the ability to correct them), diminishes as the time since they were last estimated increases. However, the information contained in each WFS measurement that can be used to correct the WFE estimates increases as a function of exposure time. Formulating this information balance allows, under certain assumptions, for the estimation of a bound on the residual (closed-loop) WFE and, hence, the contrast.

Throughout the discussion, the $r$ WFE modes coefficients will be denoted as $\epsilon$ (before correction, or open-loop) and $\epsilon_{\text{CL}}$ (closed-loop). The closed-loop contrast depends on how far on average $\epsilon_{\text{CL}}$ is from zero (its temporal covariance) and the sensitivity of the image-plane speckles to $\epsilon_{\text{CL}}$, denoted by $G_{\text{IP}}$. These and other symbols are also defined in Table 1. The challenging part, however, is determining the covariance of $\epsilon_{\text{CL}}$ that ties directly to contrast, without full end-to-end simulations of the closed-loop wave front sensor and DM operations. Indeed, even under the assumption of a perfect controller, closed-loop wave front covariance still depends on open-loop wave front properties, wave front sensor architecture, reconstruction algorithm, incident flux and detector properties. In this paper we present a theoretical framework that captures all of these parameters while circumventing the need for full closed-loop simulations.

Note that, as depicted in Figure 1, the WFS can be performed either in the image plane, denoted by the superscript $\cdot_{\text{IP}}$, or at a dedicated non-common-path wave front sensor, denoted by $\cdot_{\text{WS}}$. When the analysis is applicable to both cases, we drop the superscript (i.e., we use a general electric field WFE sensitivity matrix $G$ instead of $G_{\text{IP}}$ or $G_{\text{WS}}$). Additional sensor quantities that affect estimation include the static electric field, $E_0$ (the constant-in-time zeroth order term in the expansion of the electric field in terms of $e$), and photon sources that are not

| Symbol | Units | Domain | Definition |
|--------|-------|--------|------------|
| $t_x$ | s | R | Sampling time |
| $\epsilon$ | nm | R$^r$ | Controllable wave front modes |
| $\tau$ | s$^{-1}$ | N | Number of wave front modes |
| $N_0$ | N/A | N | Photon flux from star at the primary mirror |
| $G_i$ | nm$^{-1}$ | R$^{2 \times r}$ | Sensitivity of electric field to wave front modes |
| $c$ | N/A | N | Number of wavelengths incoherently summed |
| $E_{\text{est}}$ | i | R | Static (and uncontrollable) component of the E-field |
| $\lambda_i = N_0 \| G_i \epsilon + E_{\text{est}} \|^2$ | s$^{-1}$ | R | Photon flux at pixel $i$ |
| $D_i$ | s$^{-1}$ | R | Flux from sources incoherent with speckles |
| $y_i \sim \text{poisson}(\lambda_i + D_i, t_i)$ | 1 | N | Measured number of photons at pixel $i$ |
| $\Lambda^2 = \sum_i (G_i \epsilon)^2$ | nm$^2$ | R | Sensitivity to mode $j$ |
| $P, P^2$ | nm$^2$ | R$^{r \times r}, R^l$ | Wave front estimate error covariance |
| $Q$ | nm$^2$ | R$^{r \times r}$ | Wave front drift covariance |
| $Z$ | nm$^2$ | R$^{l \times l}$ | Fisher information that all $y_i$ carry about $\epsilon$ |
| $\Xi_{\text{WS}}$, $\xi_{\text{WS}}$ | nm$^2$ s$^{-1}$ | R$^{r \times r}, R^l$ | Wave front drift diffusion matrix/coefficient |
| $\gamma, \phi$ | N/A | N/A | Superscript for WFS quantities |
| $\gamma'$ | N/A | N/A | Superscript for image-plane quantities |
| $\gamma_{\text{WS}}$ | R | Average contrast |
| $\gamma_{\text{WS}}$ | R | Average raw contrast |
| $\gamma_{\text{WS}}$ | R | Frequency, knee frequency |
| $f$, $\nu$ | mm$^2$ s$^{-1}$ | R | Wave front PSD in the limit $f = 0$ |
| $u$ | nm s$^{-1}$ | R | Order of WFE dynamics |
| $v, w$ | s$^{-1}$ | R | Continuous-time white noise |
affected by control, $D$ (e.g., dark current and post-LOWFS residuals in the image plane, see Section 4.2).

In Section 2.1, we focus on the “simplest” drift scenario, under a discrete-time approximation, for which we derive an implicit equation that relates open- and closed-loop WFE modes’ covariance per iteration of the WFS system. The impact of each WFE mode on the contrast is assumed to be proportional to its closed-loop covariance. This formulation also assumes an open-loop temporal PSD inversely proportional to frequency, scaling as $1/f$. In Section 2.2, we extend our theoretical bound to any generic open-loop temporal PSD, using a continuous-time formulation that is more suitable for our theoretical bound to any generic open-loop temporal PSD, scaling as $1/f$.

### 2.1. Derivation of Bounds for Brownian Motion WFE Drift (Discrete-time Formulation)

Below, we first describe our assumptions for the open- and closed-loop WFE coefficients, $\epsilon_k$ and $\epsilon_{k}^{CL}$, where $k$ denotes the number of the exposure. We then relate $\epsilon_{k}^{CL}$ to the number of photons detected during the $t_i$ long exposures, and the Fisher information $\mathcal{I}_k$ contained within those WFS measurements (either using the science camera or a dedicated sensor) about $\epsilon_{k}^{CL}$. This relationship allows us to use the Cramér–Rao inequality to incorporate the uncertainties due to both WFE drift and shot noise into a single implicit equation from which the covariance of $\epsilon_{k}^{CL}$ (denoted by $P_{\epsilon_k}$) can be estimated. Finally, the estimate of the “average” $P_{\epsilon_k}$ in steady state ($k \to \infty$) is used to get the bounds on the closed-loop contrast.

#### 2.1.1. WFE Modes Drift

We begin with a simple Brownian motion (Durrett 2019) model for the evolution of high-order WFE modes in the context of space-based coronagraphs. This assumption leads to linear growth of uncertainty in the intensity—an approximation that is commonly used when evaluating exoplanet detection performance (Nemati et al. 2020). Formally, the $r$ WFE mode coefficients, $\epsilon_r = \epsilon_r (k \cdot t_i) \in \mathbb{R}^r$, are such that their increments are normally (and independently) distributed with some drift covariance $Q \in \mathbb{R}^{r \times r}$,

$$\epsilon_{k+1} - \epsilon_k \sim \mathcal{N}(0, Q), Q(t_i) > 0.$$  

We make the additional simplifying assumption that there exists an unbiased estimate of WFE modes, $\hat{\epsilon}_k$, whose error is also normally distributed with covariance $P_{\hat{\epsilon}_k}$,

$$\hat{\epsilon}_k - \epsilon_k \sim \mathcal{N}(0, P_{\hat{\epsilon}_k}), P_{\hat{\epsilon}_k} > 0,$$

(independently of the WFE increments). Furthermore, the DMs are assumed to be able to perfectly reproduce the WFE modes. Although, due to the imperfect knowledge of these modes and the inability of the estimator to predict their increments, the corrections are slightly off. We call them the “closed-loop” WFE modes,

$$\epsilon_{k+1}^{CL} = \epsilon_{k+1} - \hat{\epsilon}_k,$$

and they are, too, normally distributed with,

$$\epsilon_{k+1}^{CL} \sim \mathcal{N}(0, P_{\hat{\epsilon}_k} + Q).$$  

Note that $Q$ may now also “contain” actuator drift, i.e., the wave front changes faster if each DM actuator also exhibits Brownian motion on top of the prescribed commands (more complex DM dynamics can be treated in a manner suggested in Section 2.2).

#### 2.1.2. Measurements Model and Fisher Information

Here, we relate the closed-loop WFE modes $\epsilon_{k}^{CL}$ to the probabilistic photon measurements. Our goal is to find an
expression for the information $I$ that the measurements carry about the modes, to be later used in the Cramér–Rao inequality. The discussion is constrained to the linear regime where the sensitivity of the field to the WFE modes at detector pixel $i$ is $G_i \in \mathbb{R}^{2c \times r}$ and $c$ is the number of wavelengths in the spectral discretization in the model. Together with the static (and presumably known) component, $E_{0,i} \in \mathbb{R}^{2c}$, the electric field at pixel $i$ is given by $G_i e^{\text{CL}} + E_{0,i} \in \mathbb{R}^{2c}$.

The fields are scaled such that the photon arrival rate (intensity) at pixel $i$ is given by

$$I_i = N_S \| G_i e^{\text{CL}} + E_{0,i} \|^2,$$

where $N_S$ is the photon flux from the star integrated over the primary mirror of the telescope and propagated thorough the various optics (reflective surfaces, masks) between the primary and the WFS detector. Photon flux from external sources such as zodiacal dust, $D_{\text{ext}}$, is presumably fixed and known (or, at least, its average contribution can be canceled out by image subtraction). The flux of internal sources of photoelectrons such as clock-induced charge and dark current (Harding et al. 2016) is denoted as $D_{\text{int}}$ and is also assumed to be known.

The probability distribution of the measured number of photons, $y_i$, in photon-counting mode can be a complex function of $I_i$, $D_{\text{ext}}$, $D_{\text{int}}$ and sampling time $t_s$ (Hirsch et al. 2013; Hu et al. 2021). Here, we assume it is the Poisson distribution,

$$\text{pmf}(y_i) = \frac{1}{y_i!} (I_i + D_i) y_i e^{-(I_i + D_i)t_s},$$

with $D_i = D_{\text{ext}} + D_{\text{int}}$, although the analysis below can be repeated with any probability mass function, pmf($y_i$). This assumption holds well if continuously distributed noise sources (such as clock-induced charge) are small enough (Wilkins et al. 2014) as to not cause confusion with the number of detected photons. Besides simplifying the discussion, it is justifiable in the context of finding lower bounds on contrast, and leads to a conclusion that shorter exposure times are always preferable (see Section 3.1).

The Fisher information that the measured number of photons, $\{y_i\}$, carry about the WFE modes, $e^{\text{CL}}$, is given by

$$I = \sum_i E_{y_i}(\frac{\partial \log \text{pmf}(y_i)}{\partial e^{\text{CL}}} )^T \left( \frac{\partial \log \text{pmf}(y_i)}{\partial e^{\text{CL}}} \right) \in \mathbb{R}^{c \times c},$$

where $E_{y_i}(\cdot)$ denotes the expectation w.r.t. $y_i$. In particular, with pmf($y_i$) given by Equation (2),

$$I = \sum_i \frac{4N_S t_s}{\| G_i e^{\text{CL}} + E_{0,i} \|^2 + N_S^{-1} D_i}$$

$$\times G_i^T (G_i e^{\text{CL}} + E_{0,i}) (G_i e^{\text{CL}} + E_{0,i})^T G_i.$$

This information can be used to compute an estimate of the WFE modes based on a single sensing iteration (such methods will be referred to as “batch estimation”), or combined with the information contained in previous estimates (i.e., “recursive estimation”).

2.1.3. An Implicit Equation to Estimate the Covariance of the Closed-loop WFE Modes

We are now ready to combine the probabilistic assumptions about the evolution of the WFE modes leading to Equation (1) with the measurement model that gives Equation (3), to get an equation from which a bound on the WFE modes covariance $P$ can be estimated. From Equation (1), the Fisher information that the estimate, $\hat{e}_k$, carries about the modes, $e^{\text{CL}}_{k+1}$, is $(P_k + Q)^{-1}$. Together with the information contained in the new measurements, $\hat{I}_{k+1}$, the information about the new estimate $(\hat{e}_{k+1})$ is therefore $\hat{I}_{k+1} + (P_k + Q)^{-1}$.

First, we apply the Cramér–Rao inequality (Rao 1945; Cramér 1946), which states that the variance $P_{k+1}$ of the unbiased (recursive) estimate, $\hat{e}_{k+1}$, is greater than the reciprocal of the Fisher information,

$$P_{k+1} \geq (\hat{I}_{k+1} + (P_k + Q)^{-1})^{-1}.$$

This inequality captures the fundamental trade-off associated with closed-loop WFS or AO operations: the information about the open-loop drift obtained during a sensing exposure, $\hat{I}_{k+1}$, competes with the accrued open-loop variance during that exposure, $Q$. The closed-loop variance at iteration $k + 1$ cannot be smaller than the combination of these two phenomena. Since we are interested in an estimate of the covariance, $P$, of the residual WFE modes in steady-state operation, we assume that it does not change much ($P_{k+1} \approx P_k \approx P$). It is therefore reasonable to approximate the Fisher information with a constant that is equal to the average of Equation (3) across “all” exposures (expectation),

$$\hat{I}_{k+1} \approx \hat{I}_k \approx E_{e^{\text{CL}}}(\hat{I} | P + Q),$$

where $E_{e^{\text{CL}}}(\cdot | P + Q)$ denotes expectation w.r.t. $e^{\text{CL}} \sim \mathcal{N}(0, P + Q)$. (Here we implicitly assumed that the WFEs remain constant throughout the exposure. A more complete analysis is given in Appendix A.1 and leads to qualitatively identical conclusions.)

Finally, in order to estimate a lower bound on $P$, we replace the Cramér–Rao inequality with an equality and solve it in a slightly modified form,

$$P^{-1} - (P + Q)^{-1} = E_{e^{\text{CL}}}(\hat{I} | P + Q).$$

Note that the averaging in the above equation makes it independent of the time varying $e^{\text{CL}}$ and therefore self-contained, although it also makes finding a solution more challenging, as discussed in Section 2.3. For batch estimation (when all information contained in previous estimates is discarded), the bound can be found by solving

$$P_{\text{batch}}^{-1} = E_{e^{\text{CL}}}(\hat{I} | P_{\text{batch}} + Q),$$

instead.

2.1.4. An Expression for the Closed-loop Contrast

Equipped with an estimate of the steady-state residual WFE covariance $P + Q$ (calculated in Section 2.3 based on Equation (4)), we wish to find the average contrast across the image plane, $C$. 

First, the intensity at the image plane (denoted by $I^\text{IP}$) at pixel $i$ can be averaged w.r.t. the WFE modes $\epsilon^\text{CL}$.

$$E_c \{ I^\text{IP} | P + Q \} = N^\text{IP}_s \cdot \text{trace}(G^\text{IP}(P + Q)G^\text{IP}) + \| E_{0,i}^\text{IP} \|^2,$$

which follows directly from Equation (1) and the definitions of $I$, and $E_c$ (the cross term $E_c \{ E_{0,i}^\text{IP}(G^\text{IP} \epsilon^\text{CL})^T | P + Q \}$ is zero because $\epsilon^\text{CL}$ is zero mean and $E_{0,i}$ is constant). Note that $N^\text{IP}_s$ now refers to the star’s photon flux at the primary mirror, but only in the bandwidth of the sensors at the image-plane detector.

While the time-averaged pixel-wise intensity can be used to compute contrast curves, it is also useful to have a single scalar that describes the closed-loop performance of the coronagraph. To this end, we define the average contrast as the sum of all intensities (except exoplanets) across all image-plane pixels, normalized by the photon flux from the star at the primary mirror (in the bandwidth of the image-plane detectors),

$$C = \frac{\sum_i \{E_c \{ I^\text{IP} | P + Q \} + D^\text{IP}_i \}}{N^\text{IP}_s}.$$  

In terms of the WFE covariance, the (average) contrast is given by

$$C = C_0 + \sum_i \left[ \frac{D^\text{IP}_i}{N^\text{IP}_s} + \text{trace}(G^\text{IP}(P + Q)G^\text{IP}) \right].$$  

where $C_0 = \sum_i \| E_{0,i}^\text{IP} \|^2$ is the (average) raw contrast in the absence of WFEs and incoherent sources. Note that the error in the speckle’s contribution to the contrast is directly proportional to the error in the covariance estimate $P + Q$.

### 2.2. Continuous-time Formulation

The discussion in Section 2.1 can be repeated with more general dynamical models for WFE drift suitable for finding bounds on contrasts of ground-based telescopes. Below, this is illustrated with a continuous-time system that may approximate a larger family of temporal PSDs that are typical of AO (see Section 3.2). We extend the analysis to include the time derivatives of the WFE modes, $\frac{d}{dt} \epsilon, \frac{d^2}{dt^2} \epsilon, \ldots$, and their estimates, $\frac{d}{dt} \hat{\epsilon}, \frac{d^2}{dt^2} \hat{\epsilon}, \ldots$. Our goal is to get their closed-loop covariances, i.e., a higher-order continuous-time equivalent of Equation (4) from which they can be found.

We assume that the dynamics of the open-loop WFE modes, $\epsilon$, are linear and given by a $\gamma$th-order transfer function between white noise, $v$, and $\epsilon$. This can be stated as

$$\begin{bmatrix}
\frac{d}{dt} \epsilon \\
\vdots \\
\frac{d^\gamma}{dt^\gamma} \epsilon
\end{bmatrix} = A \begin{bmatrix}
\epsilon \\
\vdots \\
\frac{d^{\gamma-1}}{dt^{\gamma-1}} \epsilon
\end{bmatrix} + Bv(t),$$  

where $v(t) \in \mathbb{R}^4$, and $A \in \mathbb{R}^{\gamma \times \gamma}$ and $B \in \mathbb{R}^{\gamma \times 4}$ are some matrices describing the temporal evolution of the WFE and how it is forced by the white noise. Instead of the discrete-time error covariance $P + Q$ of the WFE modes estimate, a continuous-time covariance $P_{k+1}$ will be used. However, one has to keep in mind the uncertainties in the estimates of the derivatives of the WFE modes as well. We assume that the full-state estimate (including time derivatives) is normally distributed with covariance $\Sigma$, i.e.,

$$\begin{bmatrix}
\frac{d}{dt} \hat{\epsilon} \\
\vdots \\
\frac{d^\gamma}{dt^\gamma} \hat{\epsilon}
\end{bmatrix} \sim \mathcal{N}(0, \Sigma).$$  

The $P_{k+1} \in \mathbb{R}^{\gamma \times \gamma}$ matrix is then a sub-matrix of $P \in \mathbb{R}^{\gamma \times \gamma}$ appearing first on its main diagonal. The steady-state information rate is given by dividing Equation (3) by $t$, and taking the expectation w.r.t. $\epsilon^\text{CL} \sim \mathcal{N}(0, \Pi)$,

$$\dot{\Pi}(\Pi_{k+1}) = E_{\epsilon^\text{CL}} \left[ \sum_i \left( \frac{G^\text{IP}\Pi_{k+1}}{N^\text{IP}_s} \right)^2 + N^\text{IP}_s^2 D_i - \Pi \right] \times G_i^T(G_i\epsilon^\text{CL} + E_{0,i})(G_i\epsilon^\text{CL} + E_{0,i})^T G_i \Pi_{k+1}. $$

We then derive the continuous version of Equation (4), where we use $\Pi$ instead of $P$. Here we do not provide details and instead direct the reader to the derivation of the Kalman-Bucy filter (see, for example, Stengel 1994). In steady state, $\Pi$ is the solution of

$$0 = \Pi A + \Pi A^T + BB^T - \Pi \dot{\Pi}(\Pi_{k+1}) \Pi,$$

which needs to be solved instead of Equation (4) in the continuous-time case. The contrast is then given by

$$C = C_0 + \sum_i \left[ \frac{D^\text{IP}_i}{N^\text{IP}_s} + \text{trace}(G^\text{IP}(P_{k+1},(G^\text{IP})^T) \right].$$

### 2.3. Implementation

Knowing the parameters of the linearized system (WFE sensitivity $G$, static field $E_\alpha$, fluxes $N_i$, and drift parameters $D_i$), it is sufficient to find the closed-loop WFE covariance, $P_{k+1}$, via Equation (4) (or, Equation (8) in the continuous-time case where $A$ and $B$ describe the dynamics instead of $Q$). However, the equation is challenging to solve as is, since it involves the expectation $E_{\epsilon^\text{CL}}$— an integral that depends nonlinearly on the unknown matrix, $P$. Instead, we propose a random-sampling and an analytical-approximation approach for the discrete-time and continuous-time cases, respectively.

#### 2.3.1. A Random-sampling Approach to Finding the Discrete-time WFE Covariance

We introduce an iterative algorithm for computing the steady-state WFE covariance estimates, $P_{k+1}$, based on Equation (4). Instead of explicitly computing $E_{\epsilon^\text{CL}}$, the algorithm samples the WFE coefficients, $E^\text{CL}_{k+1}$, given the covariance $P_{k}$, and uses them to compute the Fisher information, $I_{k+1}$, which is then used to compute $P_{k}$.
Algorithm 1. Discrete Time (Brownian Motion)

1. Initialize $P_0 = 0$
2. Sample $\epsilon_{k+1}^{\text{CL}} \sim \mathcal{N}(0, P_0 + Q)$
3. Compute $\bar{P}_{k+1}$ via Equation (3)
4. Advance via $P_{k+1} = (\bar{P}_{k+1} + Q)^{-1}(\bar{P}_{k+1} + T_{\text{int}})$ (for batch estimation, via $P_{k+1} = \bar{P}_{k+1}$)
5. Repeat steps 2 to 4 until the average of the covariance estimate

\[ P = \frac{1}{k} \sum_{i=1}^{k+1} P_i \] has converged ($\Delta P$ remains arbitrarily small)

Note that the covariances $P_k$ depend on the randomly sampled $\epsilon_{k}^{\text{CL}}$ and are therefore also random, although their average tends to converge to $P_0$—the final covariance estimate. After computing $P$, the contrast can be found via Equation (6), in which $G_{\text{DP}}$ stands for the image-plane sensitivity.

2.3.2. An Analytical Approximation of the Fisher Information

Instead of random sampling, based on Equation (3), one may approximate the expected information, $E_c \{ \mathcal{I} | P_{k+1} + Q \}$, to get a smooth convergence at the expense of some precision. This is achieved by replacing $\| G_{\epsilon} \epsilon_{k}^{\text{CL}} + E_{0,i} \|^2$ and $(G_{i} \epsilon_{k}^{\text{CL}} + E_{0,i})(G_{i} \epsilon_{k}^{\text{CL}} + E_{0,i})^T$ by their expectation,

\[ E_c \{ \mathcal{I} | P + Q \} \approx \sum_i \frac{4N}{N} \frac{\text{trace} \{ G_{i}(P + Q) G_{i}^T + E_{0,i} E_{0,i}^T \} + N^{-1} D_{i}}{G_{i}^T (G_{i}(P + Q) G_{i}^T + E_{0,i} E_{0,i}^T) G_{i}} \] (9)

Figure 2 illustrates the difference between computing $P$ based on random sampling of WFE modes (step 3 of Algorithm 1) and the above analytical approximation. The latter is almost as precise, and will be used in the analysis in the next section.

2.3.3. An Analytical-approximation Approach to Finding the Continuous-time WFE Covariance

In order to solve Equation (8), one has to propagate the full-state covariance matrix, $\Pi$, in continuous time. The algorithm below does that by first approximating the information rate via Equation (9), and then updating the estimate of $\Pi$ via the forward Euler method. The time step $\Delta t$ needs to be small enough so that $\Pi$ does not diverge, and we suspect that more sophisticated numerical schemes can result in faster convergence.

Algorithm 2. Continuous Time (Arbitrary Linear Dynamics)

1. Initialize $\Pi(t = 0)$ and pick $\Delta t$
2. Compute $\mathcal{I}(t)$ via

\[ \mathcal{I}(t) = \sum_i \frac{4N}{N} \frac{\text{trace} \{ G_{i}(P + Q) G_{i}^T + E_{0,i} E_{0,i}^T \} + N^{-1} D_{i}}{G_{i}^T (G_{i}(P + Q) G_{i}^T + E_{0,i} E_{0,i}^T) G_{i}} \] (9)

3. Advance via

\[ \Pi(t + \Delta t) = \Pi(t) + \left[ A \Pi + \Pi A^T + B B^T - \Pi \left[ \begin{array}{ccc} \mathcal{I}(t) & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{array} \right] \right] \Delta t \] (10)

4. Repeat steps 2 and 3 until $\Pi$ has converged

Note that since we used an analytical approximation of the Fisher information $\mathcal{I}$, the covariance $\Pi$ itself is converging. Instead, one could sample $\mathcal{I}$ and then time average $\Pi$ as in Algorithm 1; this would however result in an algorithm with two nested iterative loops that we do not describe here.

3. Special Cases

Given a photon flux, raw contrast level, WFE drift statistics, and corresponding sensitivity matrices for both wave front sensor and coronagraph, one can bound the contrast achievable by WFS and control. This is done by numerically solving for residual WFE covariances ($P + Q$ or $\Pi$) as proposed in Section 2 and illustrated in Section 4. In this section, however, we first discuss some special cases in which the covariances in Equations (4) and (8) have analytical solutions. Besides providing some theoretical insights, we re-derive results from the AO literature and show that our approach is consistent with and a generalization of previous work.

3.1. Brownian Motion of Orthogonal Modes

In the context of space-based coronagraphs, we explore the asymptotic behavior of the bound derived in Section 2.1 for
recursive estimation. In particular, it will be shown that the best contrast is “achieved” in the limit of very short exposure time. Additionally, we will draw a distinction between regimes in which the image-plane intensity is dominated by the initial speckle field (static speckles), wave front instabilities (dynamic speckles), or Poisson-distributed incoherent sources (sensor noise, etc.). We first make some simplifying assumptions, which allow us to decouple the WFE modes. Then, treating each mode separately, we get analytical expressions for the closed-loop WFE covariances and contrasts in cases when the incoherent sources are negligible or in the limit of zero exposure time.

The Brownian motion model is arguably the simplest nonstationary process that can describe non-smooth WFE drift that arises from structural deformations (for example, see Figure 8(c) in Section 4.3). In that case, the open-loop covariance of the WFE increments between adjacent frames, $Q$, is proportional to the sampling time, $t_s$. This can be expressed as $Q = t_s \Xi$, where $\Xi$ is a diffusion matrix that is a property of just the wave front instabilities.

Note that we have the freedom to choose both $Q$ and the basis of the WFE modes (the matrices $G_i$), as long as we keep the covariances of the increments for the electric fields, $G_i Q G_i^T$, constant. As a result, in the monochromatic case ($c = 1$), we may, without loss of generality, choose orthogonal WFE modes whose drift is uncorrelated,

$$Q = \begin{bmatrix} q_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & q_r^2 \end{bmatrix}, \quad \Xi = \begin{bmatrix} \xi_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \xi_r^2 \end{bmatrix},$$

$$\sum_i G_i^T G_i = \begin{bmatrix} \Lambda_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Lambda_r^2 \end{bmatrix} \quad (10)$$

(since the symmetric matrix $GQG^T$, with $G = [G_1^T G_2^T \cdots G_r^T]^T$, always has an orthogonal decomposition). Here $\Lambda_j$ can stand for either the sensitivity at the wave front sensor, $\Lambda_j^{\text{N_t}}$, or at the image plane, $\Lambda_j^{\text{IP}}$.

The major assumption in this subsection is that the WFE modes are “easily distinguishable” by the sensor. Formally, the assumption is that the Fisher information matrix $\mathcal{I}$ has no cross (off-diagonal) terms. Hence, the steady-state closed-loop WFE modes are also not correlated (as a consequence of Equation (4) with diagonal $Q$ and $\mathcal{I}$),

$$\mathcal{I} = \begin{bmatrix} \mathcal{I}_I & \vdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathcal{I}_I \end{bmatrix}, \quad P = \begin{bmatrix} p_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & p_r^2 \end{bmatrix}.$$  

To find bounds on the error variances $p_j$, we start from the approximate Equation (9), and make a further simplifying assumption by replacing the summation of fractions by a fraction of summations. This gives yet another approximation of the Fisher information,

$$\mathcal{I}_j \approx 4N_t^3 \frac{\sum_{l=1}^{r} (p_j^2 + q_j^2) \Lambda_l^2 + \frac{1}{2} \|E_0\|^2}{2\sum_{l=1}^{r} (p_j^2 + q_j^2) \Lambda_l^2 + \|E_0\|^2 + N_t \|D\|^2}, \quad (11)$$

where $\|E_0\|^2 = \sum_l \|E_{0l}\|^2$ and $D = \sum_l D_l$. The WFE variances, $p_j^2$, are the solutions of Equation (4), which now take a diagonal form,

$$p_j^{-2} - (p_j^2 + q_j^2)^{-1} = \mathcal{I}_j, \quad (12)$$

and the contrast in Equation (6) is then given by

$$C \approx C_0 + 2\sum_j (p_j^2 + q_j^2)(\Lambda_{j, \text{IP}}^2)^2 + \frac{D_{j, \text{IP}}^2}{N_t^2}. \quad (13)$$

Equations (11) and (12) for all $1 \leq j \leq r$ are coupled (all $p_j$ depend on one another), although they become decoupled in the cases discussed below.

3.1.1. Negligible Incoherent Sources: Recursive Estimation

One case in which we can express the contrast in terms of the system parameters is when the flux of photons from incoherent sources $D$ is negligible compared to the flux from the coherent speckles. This can be stated as $N_t^{-1}D \ll \sum_{l=1}^{r} q_j^2 \Lambda_l^2 + \|E_0\|^2$. In this case, Equation (11) greatly simplifies, and the information about each mode becomes independent of the other modes, $\mathcal{I}_j \approx 2N_t^3 \Lambda_j^2$. The estimation error variances are then given by

$$p_j^2 \approx \frac{1}{2} \left( \frac{\|E_0\|^2}{2N_t} - 1 \right) q_j^2,$$

and contrast is given by

$$C \approx C_0 + \sum_j \frac{1}{N_t} \left( \frac{1}{2N_t} + 1 \right) (\Lambda_{j, \text{IP}}^2)^2, \quad (14)$$

and does not contain the negligible incoherent sources.

Since $q_j^2 = \xi_j^2 t_s$ where $\xi_j$ are some diffusion coefficients per Equation (10), one can show that the contrast’s infimum (greatest lower bound) is at the limit $t_s = 0$,

$$p_j \xrightarrow{t_s \to 0} \frac{\xi_j}{\sqrt{2N_t \Lambda_j}}, \quad (15)$$

$$C \xrightarrow{t_s \to 0} C_0 + \frac{2}{N_t} \sum_j \frac{\xi_j^2 (\Lambda_{j, \text{IP}}^2)^2}{\Lambda_j}. \quad (16)$$

This is as expected for this limiting case, as we assumed that the variance of the measurement noise is proportional to exposure time and ignored the photon-counting confusion associated with fixed readout noise. Intuitively, if photons/electrons from all sources are Poisson distributed, one loses information about their arrival times by increasing $t_s$, thus decreasing the information rate and consequently worsening the closed-loop contrast. Since a recursive estimator remembers the measurement history, infinitely small exposures do not result in lower-quality information for correction updates (in total).

3.1.2. Negligible Incoherent Sources: Batch Estimation

Similarly to the previous case, but with the bound in Equation (5) instead, the variance of the batch estimate is

$$(p_j^{-2})_{\text{batch}} = \mathcal{I}_j \approx 2N_t^3 \Lambda_j^2. \quad (17)$$
The corresponding contrast,

\[ C_{\text{batch}} \approx C_0 + \sum_j \left( \frac{1}{N_j t_j \Lambda_j^p} + 2\xi_j^2 \delta \right) \left( \Lambda_j^p \right)^2, \]

is unbounded (becomes worse) as \( t_s \to 0 \).

It is customary to optimize the sampling time for batch estimation (see, for example, Guyon 2005) by solving \( dC_{\text{batch}}/dt_s = 0 \):

\[ (t_s)_{\text{min}} = \sqrt{2N_s \sum_j \xi_j^2 \left( \Lambda_j^p \right)^2}, \]

\[ (C_{\text{batch}})_{\text{min}} \approx C_0 + 2 \frac{\sqrt{\sum_j \xi_j^2 \left( \Lambda_j^p \right)^2}}{N_s} \cdot \sqrt{\sum_j \xi_j^2 \left( \Lambda_j^p \right)^2}. \]

In the hypothetical case of a single mode (e.g., \( r = 1 \)), we find an expression for optimal exposure time that is similar to the one derived by Guyon (2005), \( (t_s)_{\text{min}} = \frac{1}{\sqrt{2 \xi_1 N_1}}, \)
which captures the optimal balance between noise in sensing exposures (which decreases with \( t_s \)) and uncorrected wave front drift during exposures (increases with \( t_s \)). Our more general analysis is thus capable of capturing the limiting cases already described in the literature. Note that the contrast contribution of a single mode is larger when using batch estimation when compared to recursive schemes,

\[ (C_{\text{batch}})_{\text{min}} - C_0 = 2 \left( (C_{\text{recursive}})_{\text{min}} - C_0 \right). \]

This factor of two solely corresponds to the contrast improvement associated with recursive estimation for a fixed wave front drift per WFS iteration. In practice, AO systems are limited by control lag (neglected in this paper; Petit et al. 2014 ), which can be alleviated using predictive control. Moreover, for requirement setting exercises, such as discussed in Coyle et al. (2019), recursive estimators enable faster sensing exposures, which turn into relaxed absolute drifts (in wave fronts per unit of time).

3.1.3. The Limit \( t_s \to 0 \) (Recursive Estimation)

Analytical solutions for the limiting contrast formalism can also be found in the presence of non-negligible incoherent sources \((D > 0)\), assuming that they are zero mean (e.g., their systematic component has been subtracted via preliminary detector calibrations) and their stochastic component follows a Poisson distribution. We prove in Appendix A.2 that in this case, the best contrast is still achieved as \( t_s \to 0 \) when using recursive estimators. Intuitively, this means that when every photon is counted individually, longer exposure times increase the probability of confusion between arrival times of distinct photons, which leads to loss of information and less accurate wave front estimation. From the hardware perspective, this \( t_s \to 0 \) regime requires sensors whose readout noise decreases with exposure time. Ideally, each photon’s arrival time would be tagged (see Meeker et al. 2018). Whether a particular detector+estimation algorithm can operate close to the \( t_s = 0 \) limit depends on its implementation and the expected number of photons per measurement. The full discussion is beyond the scope of this paper, but we suspect that propagating the full conditional probability distribution of WFE modes is stable (albeit computationally infeasible) for arbitrarily small \( t_s \).

We will now assume \( D > 0 \) and take the limit of Equations (11) and (12) as \( t_s \to 0 \). We seek to solve for all of the WFE variances \( p_j \). To do this, we write the wave front drift as \( q_j^2 = \xi_j^2 t_s \), combine Equations (11) and (12), and consider the limiting case \( t_s \to 0 \), giving

\[ \xi_j^2 p_j^4 = 4N_s \frac{\sum_j p_j^2 \Lambda_j^2 + \frac{\|E_0\|^2}{2}}{\Lambda_j^2 + \|E_0\|^2 + N_s D}. \]

(18)

The presence of \( D \) in the denominator of Equation (18) precludes the simplifications carried out in Section 3.1.1. However, we can still decouple this equation using the following change of variables:

\[ \begin{align*}
\bar{p}_j^2 &= \sqrt{\frac{N_s}{\lambda_j}} \bar{p}_j^2, \\
\sigma_0 &= \sqrt{\frac{N_s}{\lambda_j^2}} \|E_0\|^2, \\
\delta &= \frac{D}{\sqrt{N_s \sum_j \xi_j^2 \lambda_j}},
\end{align*} \]

(19)

In this modified space, all quantities are normalized by the photon rate. The closed-loop variance of each individual mode is also normalized by its drift and wave front sensor sensitivity, and both the static and incoherent intensities are normalized by the cumulative effect of all modal drifts at the wave front sensor. Because all of these quantities are scaled by the speckle drift, the limiting case \( \sigma_0, \delta \ll 1 \) corresponds to drift-dominated observations (negligible incoherent noise and static contrast). After some algebra, it can be shown by direct substitution that the \( r \) coupled equations given by Equation (18) \( \forall j \), are equivalent to \( r \) un-coupled equations, which we write as a single cubic equation in \( \bar{p}_j \) = \( p_j \), \( \forall j \)

\[ \bar{p}_j^6 + \sigma_0 \bar{p}_j^4 - \bar{p}_j^2 - \sigma_0 - \delta = 0. \]

(20)

The effect of the incoherent sources (including measurement noise) now becomes apparent. When it is absent, \( \delta \ll 1 \), \( \bar{p}_j^2 = 1 \) is a direct solution of Equation (20). Consequently, the estimation error, and thus closed-loop variance, converges to the value in Equation (15) regardless of the magnitude of the static intensity, \( \sigma_0 \). When incoherent sources are dominant, \( \delta \gg \max\{1, \sigma_0\} \), the variance increases proportionally to its cubic root, \( \bar{p}_j^2 \sim \sqrt{\delta} \). These two asymptotic regimes can be identified in Figure 3 , which illustrates the fundamental limits in normalized wave front closed-loop variance and associated contrast. Note that for Figure 3(b) we have assumed that the loop is closed in the image plane \((\lambda_j = \lambda_j^p, E_0 = E_0^p)\), and defined the normalized contrast as

\[ \sigma = \sigma_j^p + (\bar{p}_j^2)^2 + \delta^p = \frac{\sqrt{N_s}}{\sqrt{2 \sum_j \xi_j^2 \lambda_j^p}} \mathcal{C}. \]

(21)

Figure 3(b) shows that contrast can be limited by either static intensity/speckles \((\sigma_0^p \gg \max\{1, \delta^p\}, \) left-hand side, top two curves), incoherent sources \((\delta^p \gg \max\{1, \sigma_0^p\}, \) right-hand side), or wave front instabilities \((\max\{\sigma_0^p, \delta^p\} \ll 1, \) left-hand side, bottom curve). The post-processing contrast, however, will be affected differently by the time varying speckles
than by the static speckles or by the constant incoherent flux. We leave the analysis of the post-processing contrast for future work.

3.2. Higher-order Drift of a Single Mode

3.2.1. Assumptions for Continuous Time

We now consider the case of continuous time. The Brownian motion description of drifts, in which the open-loop variance increases linearly as sensing exposure time, implicitly assumes a $1/f^2$ underlying PSD of wavefront noise. It is a narrow assumption that, for instance, is not readily applicable to ground-based AO systems that seek to correct for atmospheric turbulence. Here, we apply the tools described in Section 2 to derive semi-analytical contrast limits for AO systems, or any WFS system correcting continuous-time disturbances, and compare those to realistic end-to-end closed-loop simulations. We derive approximate scaling laws for the dependency of the closed-loop contrast on WFE drift PSD slope and star brightness. In this section we describe the general procedure underlying these derivations, but leave out the most technical details. We summarize our results in Table 2 and, similarly to Males & Guyon (2018), we arrive at the conclusion that estimators/controllers that take into account higher-order WFE dynamics are more accurate than low-order controllers (batch estimators) and exhibit more favorable scaling laws.

For the remainder of this section, we ignore realistic effects such as incoherent sources, AO-loop time delays and spatiotemporal coupling between WFE modes. To keep this exercise tractable, we consider a single real mode ($r = 1$) with some open-loop PSD that decays as $f^{-2}$ and is equal to $\theta^2$ when $f \to 0$. We wish to derive the relationship between closed-loop contrast and open-loop PSD (described by $V$, $\theta$, and $f_0$), the WFE sensitivities $\Lambda^{WS}$ and $\Lambda^{IP}$, and fluxes. Note that here we distinguish between the flux at the wavefront sensor $N^{WS}$ and at the image plane $N^{IP}$, since they are typically not in the same band in the context of ground-based AO. The temporal PSD for this mode is written as:

$$\text{PSD}^{\text{OL}}(f) = \left( \frac{\theta}{1 + \frac{f}{f_0}} \right)^2, \quad \gamma \in \mathbb{N}, \quad (22)$$

which corresponds to a $\gamma$ th-order low-pass filter applied to white noise. Again, we start with the information rate in the absence of incoherent sources, and write the continuous-time equivalent (i.e., $\dot{I} \sim \frac{dI}{dt}$) of Equation (11),

$$\dot{I} = 2N^{WS}(\Lambda^{WS})^2.$$

Note that this information rate does not depend on the static contrast, due to the peculiar property of Poisson distribution whose information does not depend on the magnitude of the underlying electric field (the trace of $\mathcal{I}$ in Equation (3),

| Assumption | Photon Flux ($N^\gamma$) | Drift PSD ($\theta^\gamma$) |
|------------|-------------------------|-----------------------------|
| Batch Estimation | $\gamma \leq 1$ | $\Delta C \propto \theta^\gamma$ |
| Simple Integrator | $\gamma \geq 2$ | $\Delta C \propto \theta^\gamma$ |
| Theoretical Bound | $\gamma \geq 2$ | $\Delta C \propto \theta^\gamma$ |

Note. $\Delta C$ is proportional to $f_0$ in all cases. The scaling laws have been previously derived in Guyon (2005) for batch estimators with $\gamma \geq 2$ and are shown in Figure 5 of Douglas et al. (2019) for the simple integrator with $\gamma = \frac{1}{\alpha} = 1$. We only prove the theoretical bound for integer $\gamma$. 

![Graph](image-url)
assuming \( D_i = 0 \). In principle, deriving contrast limits in the continuous case can be achieved by injecting this expression for the Fischer information into Equation (8) and solving for \( \Pi_{1,1} \). When \( r = 1 \), this exercise is tractable analytically; however, it becomes increasingly technical as the steepness of the PSD power law \( \gamma \) increases.

3.2.2. Continuous-time Brownian Motion

For the sake of clarity, we first tackle the \( \gamma = 1 \) continuous case, which can be treated using a simple extension of our previous results. We follow the derivation in Section 3.1, this time using a continuous-time formulation for the drift: \( \xi^2 \sim \frac{d\xi}{dt} \). It can be shown by substituting \( \xi_j \) in Equation (16) with its expression as a function of \( \theta \) and \( f_0 \) (given \( \theta = f_0^{-1} \xi_j \)), that the continuous formulation of our fundamental limit case is

\[
\Delta C_{\text{recursive}} = C - C_0 = \left( \frac{A^p}{N^{\text{WS}}} \right)^2 \cdot \frac{f_0}{N_S^{\text{WS}}} \cdot (2N_S^{\text{WS}} \theta^2 (A^{\text{WS}})^2)^\frac{1}{2},
\]

and

\[
(\Delta C_{\text{batch}})_{\text{min}} = 2\Delta C_{\text{recursive}}.
\]

To simplify notation, we denote

\[
\bar{\theta}^2 \equiv 2N_S^{\text{WS}} \theta^2 (A^{\text{WS}})^2,
\]

\[
\Delta \bar{C} \equiv \left( \frac{A^{\text{WS}}}{N^{\text{IP}}} \right)^2 \frac{f_0}{N_S^{\text{WS}}} \Delta C,
\]

where \( \bar{\theta}^2 \) is the drift intensity normalized by WFE sensitivity \( (A^{\text{WS}}) \) and flux \( (N_S^{\text{WS}}) \); note that \( N_S^{\text{IP}} \) cancels out, and \( \Delta \bar{C} \) is the contrast “contribution” normalized by the ratio of WFE sensitivities (wave front sensor and image plane) and by the ratio of flux to WFE PSD knee frequency \( (f_0) \). Just as we did in Section 3.1, we now use these normalized quantities for the remainder of this section.

3.2.3. Higher-order Power Laws: Batch Estimation

We now consider the more general case of \( \gamma \geq 2 \) and first address theoretical bounds in the case of a batch estimator. Calculations in this case are analogous to our derivations using a discrete-time formulation. That is, the contrast limit can be calculated by balancing the information content in the sensing exposure with the stochastic drift occurring during that duration,

\[
\Delta C_{\text{batch}} = 2\left( p_{\text{batch}}^2 + u(\theta, f_0, \gamma)^2 \right) (A^p)^2.
\]

Now that open-loop variance is not an affine function of time, we consider the average stochastic drift \( u(\theta, f_0, \gamma) = \langle |d\epsilon|/dt \rangle \), which is the only relevant quantity when using a batch estimator that averages out higher-order wave front dynamics. Using dimensional analysis, one can show that this average drift scales as

\[
u(\theta, f_0, \gamma)^2 = a_1^2 \bar{\theta}^2 f_0^3,
\]

where \( a_1 \) is some dimensionless constant. The contrast contribution of the single mode with batch estimation is thus

\[
\Delta C_{\text{batch}} = 2\left( \frac{1}{N_S^{\text{WS}}} \frac{1}{t_f (A^{\text{WS}})^2} + a_1^2 \bar{\theta}^2 f_0^3 \right) (A^p)^2,
\]

where we used Equation (17) to substitute the closed-loop variance, \( p_{\text{batch}}^2 \), with the Fischer information. By differentiating w.r.t. \( f_0 \), the minimum contribution is, up to some constant (see also Guyon 2005),

\[
(\Delta C_{\text{batch}})_{\text{min}} \sim \bar{\theta}^2 f_0^3,
\]

and is obtained when

\[
(t_f)_{\text{min}} \sim \theta^{-2} f_0^{-1}.
\]

3.2.4. Higher-order Power Laws: Recursive Estimation

To analyze recursive estimation, we cannot simply consider average drifts; we have to actually solve for \( \Pi_{1,1} \) in Equation (8). To do so, we note that the PSD in Equation (22) corresponds to an integrator of order \( \gamma \) of white noise, \( v(t) \) s.t. \( \int t^{\gamma+1} v dt \sim N(0, \Delta t) \). Stated in terms of Equation (7), the PSD corresponds to,

\[
\begin{bmatrix}
\frac{d}{dt}^\epsilon \\
\frac{d^2}{dt^2} \\
\vdots \\
\frac{d^\gamma}{dt^\gamma}
\end{bmatrix}
= \begin{bmatrix}
-f_0 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & f_0 & 1 \\
0 & \cdots & 0 & f_0^{-1}
\end{bmatrix}
A(f_0)
\begin{bmatrix}
\frac{d}{dt}^\epsilon \\
\frac{d^2}{dt^2} \\
\vdots \\
\frac{d^\gamma}{dt^\gamma}
\end{bmatrix}
+ \begin{bmatrix}
\theta_0^\epsilon \\
\vdots \\
0 \\
\theta_0^\gamma
\end{bmatrix}
v(t).
\]

(24)

We consider the regime for which \( f \gg f_0 \). This corresponds to the case of short WFS timescales (much shorter than \( f_0^{-1} \)), which will benefit most from using a recursive estimator; for longer timescales, using a batch estimator is sufficient. For short timescales, the solutions of Equation (8) are insensitive to the precise values of \( f_0 \) in the \( A \) matrix, and WFE covariance (first block of the full-state covariance \( \Pi \)) is given, up to some constant, by

\[
\Pi_{1,1} \sim (\theta^2 f_0^2 \gamma^2 f_0^{1-2\gamma} \bar{\theta}^2)^{\frac{1}{2}}.
\]

For brevity, we leave out of this paper the somewhat technical proof of this relation for integer \( \gamma \) (the proof does not hold for non-integer \( \gamma \), but we use this relation for comparison purposes in the remainder of this section nevertheless). Plugging this expression into the continuous-time expression for the contrast limit without incoherent noise, \( \Delta C_{\text{recursive}} = 2\Pi_{1,1}(A^p)^2 \), we find the normalized scaling law

\[
\Delta C_{\text{recursive}} \sim \bar{\theta}^\gamma f_0^2,
\]

(25)

and therefore the (not normalized) contrast satisfies the following proportionality:

\[
\Delta C_{\text{recursive}} \propto N_S^{-\frac{1}{2}} f_0^{2}, \quad \Delta C_{\text{recursive}} \propto f_0.
\]
3.2.5. Summary of Analytical Results in the Continuous Case

The scaling laws derived in the continuous case are summarized in Table 2 and illustrated in Figure 4. A few broad conclusions can be drawn from this work. In all cases, recursive estimation with (close to) zero exposure time is more accurate than batch estimation with its corresponding optimal exposure time. Naturally, the normalized closed-loop contrast, $\Delta C$, increases with the normalized drift magnitude $\hat{\theta}^2$ for various open-loop WFE PSDs of the form of Equation (22). While batch estimation becomes more accurate if WFE drift is once differentiable ($\gamma = 2$), it exhibits the same scaling for smoother dynamics ($\gamma \geq 2$). However, the bound on recursive estimation always scales more favorably with higher orders of the drift dynamics such as the $\gamma = \frac{17}{6}$ associated with von Kármán turbulence (Hardy 1998).

3.2.6. Comparison with End-to-end Adaptive Optics Simulations

We now compare the scaling laws resulting from our derivations (summarized in Table 2) to more realistic AO simulations. So far we have worked under the assumption that the sole source of closed-loop variance is noise in the wavefront estimate. Comparisons with end-to-end simulations require also including errors stemming from the control law. In this context, we first discuss the performance of the simple integrator (SI)—the simplest control law that incorporates all measurements. We then compare SI to a linear predictor (LP; Males & Guyon 2018) and the bound in Expression (25).

With some abuse of notation, we define the SI AO control/estimation law as

$$\hat{\epsilon}(t) \propto \int_0^t \Delta \hat{y}(\tau)d\tau$$

where $\hat{\epsilon}$ is the estimate of the WFE mode and $\Delta \hat{y}$ is the measured deviation of the intensity from its nominal flat-wave front value at the WFS (even though this would classically be considered a control rather than estimation law, there is effectively no distinction between the two since we assumed direct influence of the DM on the WFE with no time varying irregularities). In terms of the normalized quantities in Equation (23), we show in Appendix A.3 that the closed-loop contrast contribution with the SI is, up to some constant,

$$\Delta C^{\text{CL,SI}} \sim \begin{cases} \hat{\theta} & \gamma = 1 \\ \hat{\theta}^2 & \gamma \geq 2 \end{cases} .$$

Note that although the SI contrast power law is the same as for batch estimation, the SI benefits from reducing the exposure time, while for batch estimators, the optimal exposure time is finite. We also report these results in Table 2.

We can then compare our results to performances of the ground-based AO-fed coronagraph analyzed using the semi-analytic framework from Males & Guyon (2018). We filtered the temporal power spectra of Fourier modes in von Kármán turbulence (Hardy 1998) by optimized control laws (see Figure 5(a)), and determined the post-coronagraph contrast from the residual variance in each mode assuming an ideal coronagraph. The control laws (SI and LP) were optimized to minimize variance per Fourier mode. We varied WFS exposure times, guide star brightness, and the Fried parameter, which changes the open-loop variance. To simplify analysis, zero loop delay was assumed, except for the sample-and-hold from finite integration.

Figure 5(b) shows the dependency of the residual WFE covariance of both controllers on the sampling time. Since the simple integrator is a recursive estimator, its accuracy becomes better with decreasing sampling time. When $\gamma = 1$, the SI is the optimal recursive estimator. This is not the case when $\gamma > 1$; for instance its first-order dynamics make it suboptimal in the $\gamma = \frac{17}{6}$ case, and it is therefore less accurate than the
higher-order linear predictor. On the other hand, the nonzero optimal sampling time of the linear predictor suggests that it is not "purely" recursive. Figures 5(c) and (d) show that the scaling of the closed-loop contrast of the SI matches the analytically derived Expression (26) (slopes represented by solid gray lines with arbitrary offsets). The LP (green circles) scales more favorably than the SI but not as well as the theoretical bound for recursive estimation (slopes represented by dashed-dotted gray lines).

**4. Space-based Coronagraph Applications**

We now focus on space-based applications of our novel formulation and relate the theory in Section 2 to commonly used single-pixel based estimation in the focal plane (Section 4.1), combining estimated bounds from LOWFS and HOWFS (Section 4.2), and estimating the closed-loop contrast of the RST (Section 4.3). Consistent with the results in Section 3, our numerical simulations show that batch estimation is less "efficient" than recursive estimation, that the closed-loop contrast of the RST is dominated by the incoherent sources, and that its "dynamic" portion scales proportionally to the cubic root of the sources’ combined intensity.

**4.1. Brownian Motion of the Electric Field of a Single Image-plane Pixel**

In image-plane WFS and control, it is common, for estimation purposes, to treat each detector pixel separately.
The random unit vector \( \mathbf{E} \), which stands for the Fisher information about the electric field at a single pixel, \( \mathbf{E} = \mathbf{G} \), in coronagraph images, instead of the WFE mode coefficients, \( \mathbf{e} \). Instead of \( \mathcal{I} \), we can use the matrix \( \mathcal{E} \), which are assumed to be normally distributed with equal probability (the \( \mathcal{N}_p \) superscript will be dropped throughout this example). Here, \( \mathcal{N}_p \) denotes the magnitude of the probes (such that if \( \mathcal{E}^{CL} = 0 \), the contrast is \( \mathcal{C} = \mathcal{C}_p \)).

Neglecting incoherent sources (\( D = 0 \)), the information about \( \mathcal{E}^{CL} \) contained in each photon count is (similar to Equation (3), but dimensionless)

\[
\mathcal{E} = 4\mathcal{N}_t \mathbf{t} \left( \frac{\mathcal{E}^{CL} + \mathbf{E}_p}{\| \mathcal{E}^{CL} + \mathbf{E}_p \|} \right) \left( \frac{\mathcal{E}^{CL} + \mathbf{E}_p}{\| \mathcal{E}^{CL} + \mathbf{E}_p \|} \right)^T.
\]

The random unit vector \( \left( \mathcal{E}^{CL} + \mathbf{E}_p \right) / \| \mathcal{E}^{CL} + \mathbf{E}_p \| \in \mathbb{R}^2 \) has a probability density function that is symmetric w.r.t. rotations by \( \pi/2 \), hence

\[
\mathbb{E}_EE_\mathcal{E} \left( \mathcal{E} \right) = 4\mathcal{N}_t \mathbf{t} \text{cov} \left( \frac{\mathcal{E}^{CL} + \mathbf{E}_p}{\| \mathcal{E}^{CL} + \mathbf{E}_p \|} \right)
= 2\mathcal{N}_t \left[ \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right].
\]

In accordance with Section 2.1, we further split the closed-loop electric field into drift and estimation errors,

\[
\mathcal{E}^{CL} = \mathbf{E}(t + \mathbf{t}_j) - \mathbf{E}(t) = \mathbf{E}(t + \mathbf{t}_j) - \mathbf{E}(t) + \mathbf{E}(t) - \hat{\mathbf{E}}(t),
\]

which are assumed to be normally distributed with

\[
\text{cov} \left( \mathbf{E}(t + \mathbf{t}_j) - \mathbf{E}(t) \right) = \mathcal{N}_\mathbf{t}_j \mathbf{Q} = \mathcal{N}_\mathbf{t}_j \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{Q},
\]

\[
\text{cov} \left( \mathbf{E}(t) - \hat{\mathbf{E}}(t) \right) = \mathbf{P} = \mathcal{N}_\mathbf{t}_j \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{P}.
\]

The bounds on batch and recursive estimation error variances are given in Equations (4) and (5),

\[
\mathbf{P}_\text{batch}^2 = \frac{1}{2\mathcal{N}_\mathbf{t}_j},
\]

\[
\mathbf{P}_\text{recursive}^2 = \frac{1}{2} \left( 1 + \frac{\mathcal{N}_\mathbf{t}_j}{\mathcal{N}_\mathbf{t}_j^2} - 1 \right) \mathcal{N}_\mathbf{t}_j \mathbf{t}_j,
\]

and the corresponding contrasts (Equation (6)) are

\[
\mathcal{C}_\text{batch} = \frac{1}{\mathcal{N}_\mathbf{t}_j} + \frac{2 \mathcal{N}_\mathbf{t}_j^2 \mathbf{t}_j^2}{\mathcal{N}_\mathbf{t}_j^2},
\]

\[
\mathcal{C}_\text{recursive} = \frac{1}{\mathcal{N}_\mathbf{t}_j^2} + \frac{2 \mathcal{N}_\mathbf{t}_j^2 \mathbf{t}_j^2}{\mathcal{N}_\mathbf{t}_j^2}.
\]

The lower bounds that are obtained when infimizing with respect to \( t_j \) are

\[
\mathcal{C}_\text{batch} - \mathcal{C}_p = 2 \mathcal{N}_\mathbf{t}_j \mathbf{t}_j^2 / \mathcal{N}_\mathbf{t}_j^2
\]

\[
\mathcal{C}_\text{recursive} - \mathcal{C}_p = \mathcal{N}_\mathbf{t}_j \mathbf{t}_j^2 / \mathcal{N}_\mathbf{t}_j^2
\]

Yet, the electric field at a single pixel cannot be estimated with \( \mathcal{C}_p = 0 \) (no probing) due to phase ambiguity. Therefore, in order to compare our analytical limits to practical implementations, we conducted a series of simulations of pair probing (Give’on et al. 2011) and an extended Kalman filter (EKF; Pogorelyuk & Kasdin 2019). In these simulations, we fixed \( \Lambda^2 \mathcal{C}_p^2 = 1 \), varied \( \mathcal{N}_\mathbf{t}_j \) and \( t_j \) between \( 2^{-5} \) s and \( 25 \) s, and varied \( \mathcal{N}_\mathbf{t}_j \) between \( 2^{-5} \) and \( 25 \). The temporal discretization of the simulation was \( 2^{-8} \) s, and the photon counts during a single exposure were stacked.

Figure 6 shows the contrast \( \mathcal{C} \), normalized by the recursive estimation bound, as a function of sampling time \( \tau_j \), normalized by the optimal sampling time for batch estimation,

\[
\mathcal{C}_p = \sqrt{\mathcal{N}_\mathbf{t}_j \mathbf{t}_j^2},
\]

When the EKF was stable at short exposure times, it outperformed batch estimation by a factor of about two. The general behavior of normalized contrast as a function of normalized exposure time does follow the expected theoretical bound for both pair probing and EKF, up to a multiplicative constant. Indeed, both achieved contrast worse than their corresponding analytical bounds. This discrepancy highlights the improvement that might be achieved when using more optimal dark hole maintenance algorithms. Moving forward, we encourage future innovations in this field to be benchmarked against our theoretical bound.

4.2. Combining Bounds for Low- and High-order Wave Front Sensing (Space Coronagraphs)

So far we have considered the case of a single WFS loop correcting corrugation due to atmospheric turbulence or internal drifts. In practice, future space-based observatories might operate using several nested closed loops operating at timescales spanning a few orders of magnitude. For instance, the LOWNFS loop of RST will operate at \( > 10 \) Hz to counteract the fast line-of-sight disturbance by the reaction wheels (Shi et al. 2017), while higher-order wave front disturbances due to thermal deformation of the OTA are slower by at least two
orders of magnitude (Krist et al. 2018). As a result, one cannot assume that the post-LOWFS residual WFE modes (jitter) are quasi-static during the minutes-long exposures. Fortunately, the large separation between timescales permits treating the jitter residual as an additional source of incoherent light, while higher-order modes remain decoupled and evolve slowly (see Pogorelyuk et al. 2020, and note that the incoherent intensity associated with jitter changes over time as reaction wheels build up momentum). We can thus also apply our methodology to derive theoretical bounds under this more realistic scenario.

Below we outline the key steps to be undertaken to do so, but we leave comparisons between bounds and realistic simulations to a future publication.

We illustrate how the contrast bounds can be found sequentially, first for LOWFS and then for HOWFS, taking into account the influence of the former on the latter (this can be extended to segmented telescopes that might require three nested WFS loops, some out of band). To simplify notations, we assume that the WFE can be split (Figure 7) into fast and slow modes handled by LOWFS and HOWFS, respectively,

\[ \epsilon = \begin{bmatrix} \epsilon_{\text{fast}} \\ \epsilon_{\text{slow}} \end{bmatrix}. \]

This distinction is suitable, for example, for RST where the residual fast modes, \( \epsilon_{\text{CL,fast}} \), have a zero mean over the sampling time of the slow loop \( (t_{\text{slow}}) \),

\[ \langle \epsilon_{\text{CL,fast}} \rangle = \frac{1}{t_{\text{slow}}} \int_{t_{\text{slow}}}^{t_{\text{slow}} + t_{\text{slow}}} \epsilon_{\text{CL,fast}} \, dt \approx 0, \]

where \( k \) denotes the exposure number (otherwise, the analysis remains valid but the notations become cumbersome). We also assume that LOWFS and HOWFS operate in the same spectral band, hence \( \bar{N}_{\text{IP}} = N_{\text{WS}} = N_{\text{S}} \). We proceed by splitting the
sensitivities of the wave front sensor at the image plane based on LOWFS (fast) and HWFS (slow) modes

\[
G_{t,\text{fast}}^{\text{WS}} \equiv \begin{bmatrix} G_{t,\text{sl}}^{\text{WS}} \\ G_{t,\text{slow}}^{\text{WS}} \end{bmatrix}, \quad G_{t,\text{fast}}^{\text{IP}} \equiv \begin{bmatrix} G_{t,\text{sl}}^{\text{IP}} \\ G_{t,\text{slow}}^{\text{IP}} \end{bmatrix},
\]

and “closing the loops” separately,

\[
e^{\text{CL}}(t) = \begin{bmatrix} e^{\text{CL,fast}} \\ e^{\text{CL,slow}} \end{bmatrix} = e - \begin{bmatrix} \hat{e}_{t,\text{fast}}^{\text{WS}}(t) \\ \hat{e}_{t,\text{slow}}^{\text{WS}}(t) \end{bmatrix},
\]

\[t \in [t_k, t_{k+1}).\]

Here, to illustrate that different temporal analyses can be combined, \( \hat{e}_{t,\text{fast}}^{\text{WS}}(t) \) is treated in continuous time (similarly to AO in Section 3.2), and \( \hat{e}_{t,\text{slow}}^{\text{WS}}(t) \) in discrete time.

The fast loop can be analyzed with the formalism presented in Sections 2.1 or 2.2, with the intensity and photon counts at the wave front sensor given by

\[
I_{t,\text{fast}}^{\text{WS}} = N_z \| G_{t,\text{fast}}^{\text{WS}} e_{t,\text{fast}}^{\text{CL}} + E_{t,0,i}^{\text{WS}} \|^2,
\]

\[
\gamma_{t,\text{fast}}^{\text{WS}} \sim \text{poisson}(I_{t,\text{fast}}^{\text{WS}} + D_{t,\text{ext},i}^{\text{WS}} + D_{t,\text{int},i}^{\text{WS}}),
\]

where \( D_{t,\text{ext},i}^{\text{ext}} \) includes zodi, etc., and \( D_{t,\text{int},i}^{\text{int}} \) induces dark current, etc. The covariance of the closed-loop fast WFE residuals, \( \Pi_{t,\text{fast}} \), t. e. \( \epsilon_{t,\text{fast}}^{\text{CL}} \sim N(0, \Pi_{t,\text{fast}}) \), can be found as prescribed in Section 2.3.

In the image plane, the average fast WFE modes are effectively zero, \( \epsilon_{t,\text{fast}}^{\text{CL}} = 0 \); hence, they do not contribute to the intensity of the coherent speckles. However, their average intensity contribution is positive,

\[
D_{t,\text{int},i}^{\text{IP}} \equiv \text{trace}(G_{t,\text{fast}}^{\text{IP}} \Pi_{t,\text{fast}} (G_{t,\text{fast}}^{\text{IP}})^T) > 0,
\]

and is “seen” by the slow loop as an additional incoherent source. This leads to the following expression for image-plane intensity and photon counts:

\[
I_{t,\text{slow}}^{\text{IP}} = N_z \| G_{t,\text{slow}}^{\text{CL}} e_{t,\text{slow}}^{\text{WS}} + E_{t,0,i}^{\text{WS}} \|^2,
\]

\[
\gamma_{t,\text{slow}}^{\text{IP}} \sim \text{poisson}(I_{t,\text{slow}}^{\text{IP}} + D_{t,\text{ext},i}^{\text{WS}} + D_{t,\text{int},i}^{\text{WS}} + D_{t,\text{int},i}^{\text{IP}}),
\]

which can then be used to find the contrast bounds per Section 2.3.

4.3. Closed-loop Speckle Floor for the RST

For our final application, we compute bounds on the steady-state speckle intensity that can be maintained on RST with image-plane HOWFS, i.e., without periodically pointing at a reference star to recreate the dark hole (Bailey et al. 2018). Our analysis is based on the publicly available OS 9 simulation (Krist 2020), and it suggests that the speckles can be maintained continuously below the dominant detector noise for the parameters of this particular scenario.

We first need to extract the information about the statistical properties of the open-loop drifts from the OS9 data post-coronagraph electric fields (instead of the underlying wavefronts, although starting with wave front would yield similar

![Figure 8. Singular value decomposition of the image-plane electric field increments in RST observation scenario 9 (Krist 2020). (a) The singular values. (b) The largest WFE mode (one of the channels). (c) The evolution of the largest mode (proportional to its contribution to the electric field). Only times at which the telescope was pointing at the target star are shown, and the sequences are split according to its roll angle.](image-url)
The images in Krist (2020) correspond to $t_e = 5$ min long exposures on the target star, 47 UMa. In this scenario, the photon flux from the star was $N_0 = 8.2 \times 10^7$ s$^{-1}$, and we estimated the detector noise (i.e., “incoherent” flux, $D_i$) to be 1.3 electrons per exposure at each pixel, based on the images in OS 9. The electric fields were scaled to give the correct image intensities when squared and multiplied by $N_0$.

This time, we do not provide any algebraically derived limits, and compute the closed-loop bounds via Algorithm 1. For Figure 9 we varied the relative contribution of detector noise, $D_i$, to examine its effects on the closed-loop speckles and the total intensity at the image plane. At the above mentioned level of $D_i = 1.3$, the incoherent sources constituted over 85% of the electrons in the dark hole. In a hypothetical scenario where a 10 times brighter target is observed instead, the majority of the electrons would come from static speckles (the contrast floor achieved when creating the dark hole). In that case, the dynamic speckles driven by wave front instabilities would be accurately estimated and well constrained. However, in accordance with Figure 3(a) and surrounding discussion, the variance of the closed-loop WFE increases proportionally to the cubic root of the incoherent intensity. As a result, even if detector noise is known and uniform in time, it may have an adverse affect on the systematic error in post-processing.

Figure 9 also shows the closed-loop intensities obtained by an EKF of the WFE modes (see Appendix A.4). Similarly to the example in Section 4.1, the qualitative behavior of the EKF is generally consistent with the analytical bound, although there is a factor of three discrepancy between the two (in the limit of low detector noise). We suspect that a better result could be achieved if the dither, which is necessary for phase diversity, is optimized by some sophisticated choice of DM actuations. Nevertheless, the intensity remains dominated by incoherent sources, or static speckles in the limit of negligible detector noise.

We conclude that it is possible, at least in theory, to maintain a steady contrast throughout the nominal RST observation sequence by closing the loop in the image plane. Changing the

---

**Figure 9.** Estimated image-plane intensity as a function of detector noise, based on RST OS 9 open-loop simulations (horizontal dotted lines), closed-loop analytical bounds (solid red line and dashed green line), and EKF simulations (triangles and circles). In the given data (vertical dotted line), sources internal to the telescope (clock-induced charge, etc.) are dominant in both closed- and open-loop observation. When observing brighter stars (left side), static speckles become dominant. In any case, closing the loop (circles and dashed green line) does not significantly impact the intensity and would therefore be preferable to a lower-duty-cycle open-loop approach.

---

results). To do so, we picked 12 uninterrupted sequences of image-plane electric fields (at $0.1 \lambda/D$ resolution) and images (at $0.42 \lambda/D$ resolution) during which the telescope had a fixed alignment, pointing at a target star. Each such sequence contains between $K = 17$ and $K = 21$ electric fields taken 5 minutes apart and corresponding to $c_{pol} = 4$ polarizations and $c_{wvl} = 9$ wavelengths. The electric fields were resampled to the resolution of the images, and only $N_{pix} = 1604$ pixels between 3 and 10$\lambda/D$ were selected, giving the following vector sequences:

$$\{E_{k,1}\}_{k=1}^{K} = \{E_{k,2}\}_{k=1}^{K} \subset \mathbb{R}^{2c_{pol}c_{wvl}N_{pix}}.$$  

In order to compute the WFE drift modes from the simulated data, we arranged the electric field increments into a $115468 \times 228$ matrix,

$$\Delta Y = [\cdots E_{k+1,1} - E_{k,1} \cdots \cdots E_{k+1,12} - E_{k,12} \cdots]$$

where $r = K_1 - 1 + \cdots + K_{12} - 1 = 228$ is the number of empirical WFE modes. Assuming that the modes exhibit Brownian motion, the singular value decomposition of the increments matrix, $\Delta Y = U S \Sigma V^T$, gives estimates of the WFE sensitivity matrix and drift covariance,

$$G^{\text{IP}} = U \in \mathbb{R}^{2c_{pol}c_{wvl}N_{pix} \times r},$$

$$Q(5 \text{ min}) = \frac{1}{r - 1} \Sigma^2 \in \mathbb{R}^{r \times r}.$$  

Figure 8(c) shows the evolution of the largest mode (Figure 8(b), which appears to be neither differentiable, nor discontinuous, thus, at least partially, justifying the Brownian motion in Section 2.1. The static electric field estimate is found by projecting the dynamics modes out, i.e.,

$$E_0 = E_{1,1} = U U^T E_{1,1}$$

(this estimate depends on the frame number, but the variations between frames are insignificant in OS 9).
We proposed a method for computing a lower bound on the variance of post-LOWFS and post-HOWFS wave front modes. The method yields contrast estimates that reproduce previous theoretical work (Guyon 2005) in some bounding cases, generalize it to recursive estimation and non-atmospheric WFE, are consistent with end-to-end AO simulations, and are consistent with dark hole maintenance simulations of the RST based on OS 9. Our analytical approach avoids joint end-to-end simulations of the coronagraph with its wave front control loops. As a result, the optics need to be propagated just once when computing WFE sensitivity matrices, even when assessing a large number of observation scenarios.

Using this approach, we showed that recursive estimation that takes into account WFE dynamics gives the best contrast, and it gives derived power laws of their dependencies on photon flux, detector noise, and temporal PSD of the WFE. Based on RST OS 9, we predict that it should be possible to continuously reject high-order wave front perturbations due to thermal drift of the OTA with negligible contrast loss. The analysis of post-processing the signal-to-noise ratio as a function of the residual wave front variance is left for future work.

The basic implicit equation for a bound on closed-loop WFE variance is derived in Section 2.1 for when the open-loop WFE modes exhibit Brownian motion and all noise sources are Poisson-distributed. This bound relies on the average Fisher information contained in sensor photon counts and the Cramér–Rao inequality. In Section 2.2, it is extended to linear dynamics of an arbitrary order and continuous in time. Two algorithms to approximately compute these bounds are given in Section 2.3.

If the WFE drift modes are “decoupled” in the sense described in Section 3.1, it becomes possible to derive closed-form expressions for their residual variance in some special cases. In particular, it is shown that the best contrast is achieved in the limit of zero exposure time, and that batch estimation is less “efficient” than recursive estimation. When incoherent sources are dominant, the WFE variance increases proportionally to the cubic root of the incoherent intensity (a detail that might play a role in post-processing where the two sources have qualitatively different behaviors).

In Section 3.2, we derive the scaling of closed-loop contrasts with respect to WFE drift magnitude and star brightness under some special assumptions on the open-loop PSD. Our results generalize previous derivations and numerical studies of AO systems, and suggest that currently existing methods do not yet reach theoretical performance limits. Specifically, for WFEs with PSDs that decay rapidly with frequency, the recursive estimation bounds have more favorable scaling laws than both batch estimation and more modern controllers.

Section 4.1 compares the analytical bounds to recursive (EKF) and batch (pair probing) estimation algorithms for a theoretical single-pixel system. Although the bounds are not tight, their qualitative behavior matches simulation results. In Section 4.2, we consider a joint analysis of fast LOWFS and much slower HOWFS loops. The combined bounds can be found by first computing the LOWFS residuals, which then appear as an incoherent source when computing the final contrast estimates.

In Section 4.3, based on OS 9, we estimate a bound on the image-plane intensity that could be maintained by RST while continuously observing the target star, 47 Um. In this scenario, the dominant source of electrons is internal to the telescope (i.e., Poisson-distributed dark current). The contributions of dynamic speckles and DM probes necessary for WFS are less significant. As a result, we conclude that it should, at least in theory, be possible to observe a dim target star continuously without periodically switching to a reference star for the purpose of dark hole maintenance. In our HOWFS numerical simulations, the error covariances of the EKF were larger than the analytical bound by a factor of up to three. Since it is necessary, for estimation purposes, to introduce phase diversity via DM probing or dithering, we speculate that the proposed lower bound is unattainable.

### Appendix

#### Derivations

**A.1. Recursive WFE Covariance for Finite Exposure Time**

Equation (4) is the key equation that we use throughout the paper that relates the closed-loop WFE modes covariance, \( P + Q \), to the average information obtained from measurements, \( E_{ki} \{ T \} P + Q \).

\[
P^{-1} - (P + Q)^{-1} = E_{ki} \{ T \} P + Q.
\]  

(4)

It implicitly approximates the WFE modes as fixed throughout the exposure, and equal to their value at the end of the exposure. In practice, the WFE covariance increases linearly from \( P \) at the beginning of the exposure, to \( P + Q \) at the end, giving a time-averaged covariance of \( P + \frac{1}{2} Q \). Moreover, fluxes also vary in time throughout the exposure, making the co-added photon counts less indicative of the flux at the end.

Here, for completeness, we provide a more subtle analysis that takes WFE drift during the exposure into account. It results in the following relation for \( P \):

\[
P = P + Q - \left( P + \frac{1}{2} Q \right) \times \left( P + E_{ki} \{ T^{-1} \} P + \frac{1}{2} Q \right)^{-1} \left( P + \frac{1}{2} Q \right).
\]  

(A1)

and can be used in Algorithm 1 instead of the less precise Equation (4). Additionally, instead of the average contrast at the end of the exposure, Equation (6), one can measure performance based on the average contrast throughout.

\[
C = C_0 + \sum_i \left[ \frac{D_{iP}}{N_{iP}} \right. \\
+ \text{trace} \left\{ G_{iP} \left( P + \frac{1}{2} Q \right) G_{iP}^\dagger \right\}. \]
\]  

(A2)

While these expressions are more accurate, they are also cumbersome and less intuitive. They give the same estimates as Equations (4) and (6) in the limit of short exposure time \( t_e \to 0 \) where \( T^{-1} \gg P \gg Q \), and differ only slightly in the limit
Therefore, the a posteriori maximum-likelihood estimate of $\hat{\epsilon}_{k+1} = \hat{\epsilon}_{OL, k+1} \hat{\epsilon}_{OL, \text{batch}}$ is

$$
\hat{\epsilon}_{k+1} = \hat{\epsilon}_{OL, k+1} + \sum_{m=1}^{M} \hat{\epsilon}_{OL, \text{batch}}
$$

In the limit $M \to \infty$, we have $\sum_{m=1}^{M} \frac{M - m + 1}{M^2} \to \frac{1}{2}$ and $\sum_{m=1}^{M} \frac{M - m + 1}{M^3} \to \frac{1}{3}$, hence

$$
\hat{\epsilon}_{k+1} \approx \left( P_k + \frac{1}{2} Q \right)
$$

where we replaced $\frac{1}{M^2} \sum_{m=1}^{M} (t_i \hat{I}(\epsilon_{CL, \text{batch}}))^{-1}$ with its approximate value in the middle of the exposure, $E_c \hat{I}(\epsilon_{CL, \text{batch}}) = \frac{1}{3} Q$. To compute the closed-loop covariance at the beginning of the $k+1$ exposure, $P_{k+1} = \text{cov} \hat{\epsilon}_{k+1}$, note that

$$
P_{k+1} = P_k + Q - \left( P_k + \frac{1}{2} Q \right)
$$

Using the expression for $\hat{\epsilon}_{OL, \text{batch}}$ in Equation (A3) and $\epsilon_{k+1} = \epsilon_{OL} + \sum_{m=1}^{M} \epsilon_{k+1}$, one can derive the expression for $P_{k+1}$.

Equation (A1) then follows as the steady-state case limit, $P_{k+1} = P_k$.

We now compare Equations (A1) and (A2) to Equations (4) and (6) under the assumptions of Section 3.1. The information, contrast, and covariance expressions in Equations (11)–(13) become

$$\mathcal{I}_j \approx 4N_S t_s \frac{\sum_{i=1}^{N_j} (p_i^2 + \frac{1}{2} q_i^2) \Lambda_j}{2 \sum_{i=1}^{N_j} (p_i^2 + \frac{1}{2} q_i^2) \Lambda_j + \| E_0 \|^2} \Lambda_j \Lambda_j^{-1} \Lambda_j^{-1} D, \quad p_j^2 \approx p_j^2 + q_j^2 - \frac{(p_j^2 + \frac{1}{2} q_j^2)^2}{p_j^2 + \mathcal{I}_j^{-1} + \frac{1}{2} q_j^2},$$

$$C = C_0 + 2\sum_j (p_j^2 + \frac{1}{2} q_j^2) (\Lambda_j^0)^2 + \frac{D^{IP}}{N_S^{IP}}.$$

In the case of short exposure time, we have

$$q_j^2 = \xi_j^2 t_s \ll p_j^2 \ll t_s^{-1} \xi_j^{-1} = \mathcal{I}_j^{-1},$$

as $t_s$ becomes small. Then, $\mathcal{I}_j$ and $C$ do not explicitly depend on $q_j^2$, and their expressions above become identical to
Equations (11) and (13). Equation (12) also converges to its finite-exposure equivalent since,
\[ p_j^{-2} - (p_j^{-2} + \xi_j^2 t_j)^{-1} \approx p_j^{-2} - p_j^{-2}(1 - p_j^{-2} \xi_j^2 t_j) \]
\[ = p_j^{-4} \xi_j^2 t_j \]
and
\[ \frac{(p_j^{-2} + \frac{1}{2} \xi_j^2 t_j)^2}{p_j^{-2} + \bar{X}_j^{-1} + \frac{1}{2} \xi_j^2 t_j} \approx I_j p_j^4. \]

We conclude that the two approaches give the same bounds at the short-exposure limit, which is where the recursive estimator is optimal.

In the case of long exposure time, \( \bar{N}_s^{-1} D \ll \xi_j^2 t_s \) as \( t_s \) becomes large. We have \( I_j \approx 2\bar{N}_s t_s \bar{\Lambda}_j^2 \) and can solve for \( p_j \) and \( C \).
\[ p_j^2 = q_j^2 \frac{1}{12} + \frac{1}{2\bar{N}_s t_s \bar{\Lambda}_j^2 q_j^2} \approx \sqrt{\frac{12}{12}} q_j^2, \]
\[ C - C_0 = \sum_j \left( \frac{1}{3} + \frac{2}{\bar{N}_s t_s \bar{\Lambda}_j^2 q_j^2} + 1 \right) (\bar{\Lambda}_j^p q_j)^2 \]
\[ \approx \left( 1 + \frac{1}{3} \right) \sum_j (\bar{\Lambda}_j^p q_j)^2. \]

Note that the contrast loss is smaller by a factor of about 1.3 than the one obtained from Equation (14),
\[ C - C_0 = \sum_j \left( \frac{1}{3} + \frac{2}{\bar{N}_s t_s \bar{\Lambda}_j^2 q_j^2} + 1 \right) (\bar{\Lambda}_j^p q_j)^2 \]
\[ \approx 2 \sum_j (\bar{\Lambda}_j^p q_j)^2. \]

### A.2. Optimality of Zero Exposure Time in the Presence of Poisson-distributed Noise Sources

We begin with the assumptions in Section 3.1 and wish to prove that the contrast \( C \) given by Equation (13) achieves its infimum w.r.t exposure time \( t_s \) at the limit \( t_s \rightarrow 0 \). In particular, we assume that all intensity sources are Poisson distributed and that the wave front modes drift independently via Brownian motion \( q_j = \xi_j^2 t_s \). We will only show that the one-sided derivative of the contrast at \( t_s = 0 \) is positive,
\[ \frac{\partial C}{\partial t_s} = 2 \sum_j (\bar{\Lambda}_j^p q_j)^2 \left( \frac{\partial p_j^2}{\partial t_s} \right) \bigg|_{t_s = 0} + \xi_j^2 > 0. \]

Combining Equations (11) and (12), we get
\[ \frac{p_j^{-2} - (p_j^{-2} + \xi_j^2 t_j)^{-1}}{2} = 4\bar{N}_s t_s \frac{\sum_{l=1}^{j-1} (p_l^{-2} + \xi_l^2 t_l) \bar{\Lambda}_j^2 + \frac{1}{2} \|E_0\|^2}{2\sum_{l=1}^{j-1} (p_l^{-2} + \xi_l^2 t_l) \bar{\Lambda}_j^2 + \|E_0\|^2 + \bar{N}_s^{-1} D}. \]
Equation (20),
\[(\bar{p}^2 + \sigma_0)(\bar{p}^4 - 1) = \delta > 0.\]
Then, the assumption \(\frac{\partial \bar{c}}{\partial \bar{e}} \leq 0\) leads to
\[1 - \bar{p}^4 \sum I_i^2 \left( \frac{\partial \bar{c}}{\partial \bar{e}} + \bar{\epsilon}_i \right) \geq 0,
\]
and therefore the right-hand side of Equation (A5) is also nonnegative. It follows that
\[2 \frac{\partial \bar{c}}{\partial \bar{e}} + \bar{\epsilon}_i \geq 0 \text{ or } \frac{\partial \bar{c}}{\partial \bar{e}} \geq - \frac{1}{2} \bar{\epsilon}_i \]
for all \(j\) and thus
\[0 \geq \frac{1}{2} \frac{\partial C}{\partial \bar{e}} = \sum I_i^2 \left( \frac{\partial \bar{c}}{\partial \bar{e}} + \bar{\epsilon}_i \right) \geq \sum I_i^2 \left( - \frac{1}{2} \bar{\epsilon}_i + \bar{\epsilon}_i \right) > 0,
\]
which is the desired contradiction.

A.3. Closed-loop Single WFE Mode Variance with a Simple Integrator

To derive the performance of the simple integrator, we denote the transfer function corresponding to Equation (24) as
\[\epsilon_v(s) = [1 \quad 0 \quad \cdots \quad 0]sI - A^{-1}B = -\theta(-f_0)^{\gamma} (s + f_0)^{\gamma},\]
where \(\epsilon\) is a single \((r = 1)\) open-loop WFE mode, and \(v\) is white noise. In AO, the deviation of WFS measurement from its nominal value, \(\Delta \tilde{y}\), is small and approximately linear in the closed-loop WFE, \(\epsilon_{CL} \in \mathbb{R}\). In other words, we assume that \(\|G_i \epsilon_{CL}\| < \|E_{0,i}\||\) and that the transfer function between the WFE and the measurement is a constant,
\[\frac{\Delta \tilde{y}}{\epsilon_{CL}}(s) = \frac{\partial \Sigma_i L_i}{\partial \epsilon_{CL}} = 2N_s \sum_i (E_{0,i})^T G_{i,WS},\]
Additionally, the shot noise of constant magnitude depends on the intensity at the WFS (assuming a perfect intensity source), which can be stated as,
\[\frac{\Delta \tilde{y}}{w}(s) = \sqrt{N_s \sum_i (E_{0,i})^T E_{0,i}},\]
where \(w\) is also white noise with \(\int_0^{\Delta t} w dt \sim N(0, \Delta t)\). The above transfer functions are a special case of the AO loop presented in Males & Guyon (2018) without WFS and DM delays.

The control law
\[\epsilon_{CL} = \epsilon - \tilde{\epsilon},\]
is specified (up to an initial condition) via the transfer function of the estimator, \(\frac{\tilde{\epsilon}}{\Delta \tilde{y}}(s)\), and yields the following closed-loop WFE dependency on open-loop WFE and shot noise,
\[\epsilon_{CL} = \frac{\epsilon_{v}(s)}{1 + \frac{\tilde{\epsilon}}{\Delta \tilde{y}}(s) \frac{\Delta \tilde{y}}{\epsilon_{CL}}(s)} v - \frac{\hat{\epsilon}_{v}(s) \frac{\Delta \tilde{y}}{w}(s)}{1 + \frac{\tilde{\epsilon}}{\Delta \tilde{y}}(s) \frac{\Delta \tilde{y}}{\epsilon_{CL}}(s)} w.\]
Since \(v, w\) are independent white noise, the closed-loop PSD is given by,
\[\text{PSD}_{CL}(f) = \left| \frac{\epsilon_{v}(f)}{1 + \frac{\tilde{\epsilon}}{\Delta \tilde{y}}(f) \frac{\Delta \tilde{y}}{\epsilon_{CL}}(f)} + \frac{\hat{\epsilon}_{v}(f) \frac{\Delta \tilde{y}}{w}(f)}{1 + \frac{\tilde{\epsilon}}{\Delta \tilde{y}}(f) \frac{\Delta \tilde{y}}{\epsilon_{CL}}(f)} \right|^2\]
and the variance and contrast contribution by
\[\text{var}\{\epsilon_{CL}\} = \int_{-\infty}^{\infty} \text{PSD}_{CL}(f) df,
\]
\[\Delta C = 2 \text{var}\{\epsilon_{CL}\}(\text{AP})^2\]
In the case of a simple integrator control/estimation law parameterized by \(f_{SI}\),
\[\frac{\hat{\epsilon}}{\Delta \tilde{y}}(s) = \frac{f_{SI}}{2N_s \sum_i (E_{0,i})^T G_{i,WS}} s^{-1},\]
the variance is
\[\text{var}\{\epsilon_{CL,SI}\} \approx \theta^2 \int_{-\infty}^{\infty} \frac{f_{SI}^2}{(f + f_{SI})^2(s + f_0)^{2\gamma}} df + \hat{\theta}^{-1} \int_{-\infty}^{\infty} \frac{f_{SI}^2}{(f + f_{SI})^2} df, \quad (A6)\]
where we made the approximation
\[\hat{\theta} \approx \frac{4N_s \sum_i (E_{0,i})^T G_{i,WS}^2}{\sum_i (E_{0,i})^T (E_{0,i})},\]
by switching the order of summation and division, as we did in Equation (11). Again we constrain the discussion to the “pure-integrator” regime, \(\theta^2 \hat{\theta} \gg 1\), for which the following limit is applicable:
\[\lim \theta f_0^{\frac{1}{2\gamma}} \int_{-\infty}^{\infty} \frac{f^2}{(f + f_{SI})^2(s + f_0)^{2\gamma}} df \]
\[\sim \frac{\theta f_0^2}{f_{SI}} \gamma = 1\]
\[\frac{\theta f_0^3}{f_{SI}^2} \gamma \geq 2\]

The variance in Equation (A6) can be optimized w.r.t. \(f_{SI}\) resulting in
\[\text{minvar}\{\epsilon_{CL,SI}\} \sim \left(\frac{\theta f_0^{\frac{1}{2\gamma}}}{f_{SI}}\right)^{\gamma = 1} \left(\frac{\theta f_0^3}{f_{SI}^2}\right)^{\gamma \geq 2}\]
up to some constant. In terms of contrast and the normalized quantities defined in Equation (23), this gives Equation (26).

A.4. EKF of OS 9 WFE Modes

We detail the EKF corresponding to the WFE dynamics used in Section 2.1 to compute closed-loop intensity estimates in Figure 9. Similarly to Pogorelyuk & Kasdin (2019), we approximate the measurement equation (i.e., Equation (2)) with a normal distribution,
\[y_i \sim N((I_i + D_i)T_s, (I_i + D_i)T_s).\]
In vector notation,
\[
G = \begin{bmatrix} G_1 & \cdots & G_{2n_Npix} \end{bmatrix} \in \mathbb{R}^{2n_Npix \times r},
\]
\[
M = \begin{bmatrix} I_{2r} & \cdots & I_{2r} \end{bmatrix} \in \mathbb{R}^{n_Npix \times 2n_Npix},
\]
\[
D = \begin{bmatrix} D_1 & \cdots & D_1 \end{bmatrix} \in \mathbb{R}^{n_Npix},
\]
\[
I_k = \hat{N}_g M \cdot (G(\hat{e}_{k+1} + \mathbf{u}_k) + E_0)\sigma_k^2 \in \mathbb{R}^{n_Npix},
\]
\[
y_k \sim \mathcal{N}(\mathbf{0} + D) t_1, \quad \text{diag}((I_k + D) t_1),
\]
where \( I_{2r} \in \mathbb{R}^{1 \times 2r} \) is a row vector of ones (hence \( M \) is a matrix that sums the squared real and imaginary parts of electric fields of all wavelengths), \( \mathbf{u} \) is DM control in WFE basis, \( \sigma_k^2 \) stands for element-wise squaring, and diag \( \cdot \) yields a diagonal matrix with the elements of its argument on the diagonal.

To avoid confusion with previous definitions, we denote EKF’s covariance (approximation) as \( P \in \mathbb{R}^{r \times r} \) and note that it refers to open-loop modes. It is advanced together with the WFE estimate via (see Stengel 1994)
\[
\hat{p}_{k+1|k} = \hat{p}_{k|k} + Q,
\]
\[
\hat{p}_{k+1|k} = \hat{p}_{k+1|k} - \hat{K}_{k+1} \hat{H}_{k+1} \hat{p}_{k+1|k},
\]
\[
\hat{e}_{k+1|k} = \hat{e}_{k|k},
\]
\[
\hat{e}_{k+1|k} = \hat{e}_{k+1|k} + \hat{K}_{k+1} (\hat{y}_{k+1} - \hat{y}_{k|1}),
\]
with \( \hat{K}_{k+1} \in \mathbb{R}^{r \times n_{pix}} \), \( \hat{H}_{k+1} \in \mathbb{R}^{n_{pix} \times r} \) and \( \hat{y}_{k+1} \in \mathbb{R}^{n_{pix}} \) defined next.

The predicted photon count is given by
\[
\hat{y}_{k+1} = \hat{N}_g t_1 M \cdot (G(\hat{e}_{k+1|k} + \mathbf{u}_{k+1}) + E_0)\sigma_k^2 + D,
\]
its sensitivity to WFE is
\[
\hat{H}_{k+1} = \frac{\partial \hat{y}_{k+1}}{\partial \hat{e}_{k+1|k}}\left(\hat{H}_{k+1} + \hat{p}_{k+1|k} \hat{H}_{k+1}^T + \text{diag}(\hat{y}_{k+1})\right)^{-1},
\]
and the Kalman gain is
\[
\hat{K}_{k+1} = \hat{p}_{k+1|k} \hat{H}_{k+1}^T \times (\hat{H}_{k+1} + \hat{p}_{k+1|k} \hat{H}_{k+1}^T + \text{diag}(\hat{y}_{k+1}))^{-1}.
\]

Finally, a control law \( \mathbf{u}_{k+1}(\hat{e}_{k+1|k}) \) must be provided. For Section 4.3 we sampled
\[
\mathbf{u}_{k+1} - \hat{e}_{k+1|k} \sim \mathcal{N}(\mathbf{0}, \sigma_u Q),
\]
where the dithering magnitude, \( \sigma_u > 1 \), was chosen empirically to give the best contrast.

**References**

Bailey, V. P., Bottom, M., Cady, E., et al. 2018, ProC SPIE, 10698, 106986P
Balcer, C. R., Boccacetti, A., Baudrand, J., & Rouan, D. 2005, in IAU Colloq., 200, Direct Imaging of Exoplanets: Science & Techniques, ed. C. Aime & F. Vakili (Cambridge: Cambridge Univ. Press), 553
Bolcar, M. R., Aloezos, S., Bly, V. T., et al. 2017, ProC SPIE, 10398, 1039809
Bottom, M., Wallace, J. K., Bartos, R. D., Shelton, J. C., & Serabyn, E. 2016, MNRAS, 464, 2937
Cavarroc, C., Boccacetti, A., Baudroz, P., Fusco, T., & Rouan, D. 2006, A&A, 447, 397
Correia, C. M., Fauvarque, O., Bond, C. Z., et al. 2020, MNRAS, 495, 4380
Coyle, L. E., Knight, J. S., Pueyo, L., et al. 2019, ProC SPIE, 11115, 111150R
Cramér, H. 1946, Scand. Actuar. J., 1946, 85
Currie, T., Brandt, T. D., Uyama, T., et al. 2018, AJ, 156, 291
Demers, R. T., Dekens, F., Calvet, R., et al. 2015, ProC SPIE, 9605, 960502
Douglas, E. S., Males, J. R., Clark, J., et al. 2019, AJ, 157, 36
Durret, R. 2019, Probability: Theory and Examples, Vol. 49 (Cambridge: Cambridge Univ. Press)

Give’on, A., Kern, B. D., & Shalak, S. B. 2011, ProC SPIE, 8151, 815110
Guyon, O. 2005, ApJ, 629, 592
Harding, L. K., Demers, R., Hoenk, M. E., et al. 2016, JATIS, 2, 011007
Hardy, J. W. 1998, Adaptive Optics for Astronomical Telescopes, Vol. 16 (Oxford: Oxford Univ. Press)
Hirsch, M., Wareham, R. J., Martin-Fernandez, M. L., Hobson, M. P., & Rolfe, D. J. 2013, PLoS ONE, 8, e53671
Hu, M., Sun, H., Hansen, A., & Kasdin, N. J. 2021, JATIS, 7, 028006
Jovanovic, N., Absil, O., Baudroz, P., et al. 2018, ProC SPIE, 10703, 107031J
Krist, J., Effinger, R., Kern, B., et al. 2018, ProC SPIE, 10698, 106982K
Krist, J. E. 2020, Observing Scenario (OS) 9 time series simulations for the Hybrid Lyot Coronagraph Band 1, https://wfristrp.ipac.caltech.edu/sims/Coronagraph_public_images.html#CGI_OS
Krisciunas, K., Efthymiopoulos, C., & Gagné, R. 2015, ApJ, 806, 154

**ORCID iDs**

Leonid Pogorelyuk https://orcid.org/0000-0001-6387-9444
Jared R. Males https://orcid.org/0000-0002-2346-3441
Kerri Cahoy https://orcid.org/0000-0002-7791-5124