Characters of tangent spaces at torus fixed points and 3d-mirror symmetry

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Abstract

Let $X$ be a Nakajima quiver variety and $X'$ its 3d-mirror. We consider the action of the Picard torus $K = \text{Pic}(X) \otimes \mathbb{C}^\times$ on $X'$. Assuming that $(X')^K$ is finite, we propose a formula for the $K$-character of the tangent spaces at the fixed points in terms of certain enumerative invariants of $X$ known as vertex functions.

1 Overview

1.1

In this paper, we study symplectic varieties which appear as Higgs and Coulomb branches of certain three-dimensional gauge theories with $\mathcal{N} = 4$ supersymmetry. These theories were considered, for example, in [BDG15, GW09, HW97, IS96]. We assume that the physical theories under consideration are of quiver type, in which case the Higgs branches are known as Nakajima quiver varieties, see [Nak98, Gin12] and Section 2 in [MO12] for an introduction. In this subsection we recall the most basic facts about these varieties important in the constructions below.

The Nakajima varieties are smooth quasi-projective symplectic varieties equipped with a natural action of an algebraic torus $T$. The torus $T$ acts on a Nakajima variety $X$ by scaling the symplectic form $\omega \in H^2(X, \mathbb{C})$. We denote by $\mathfrak{h} \in \text{char}(T)$ the character of the one-dimensional $T$-module $\mathbb{C}\omega$. We denote by $A = \ker(\mathfrak{h}) \subset T$ the subtorus preserving the symplectic form and by $C^\times_{\mathfrak{h}} := T/A$ the corresponding one-dimensional factor.

\footnote{We believe that our main conjecture holds in full generality. We restrict the exposition to the quiver varieties for the sake of simplicity of the exposition.}
The Nakajima quiver varieties are examples of symplectic resolutions and thus their cohomology are even [Kal09].

The Nakajima varieties are defined as quotients by a group

\[ G = \prod_{i \in I} GL(v_i) \]

where \( I \) denotes the (finite) set of vertices of the corresponding quiver. This means that \( X \) is naturally equipped with a set of rank \( v_i \) tautological vector bundles \( V_i \).

**Theorem 1** ([MN18]). If \( X \) is a Nakajima variety then \( K^\text{alg}_T(X) = K^\text{top}_T(X) \) is generated by Schur functors of tautological bundles \( V_i, i \in I \).

We will use \( K_T(X) \) to denote the \( T \)-equivariant \( K \)-theory ring of \( X \). Due to the last theorem we do not distinguish between the algebraic and the topological versions. This theorem implies that:

\[
K_T(X) = \mathbb{Z}[x_{i,j}^\pm, a^\pm, \hbar^\pm] / R
\]

where \( x_{i,j}, i \in I, j = 1, \ldots, v_i \) denote the Grothendieck roots of the tautological bundles, \( a, \hbar \) stand for the equivariant parameters of \( T \) and \( R \) denotes certain ideal. If \( X^T \) is finite, then the ideal \( R \) can be described as the ideal of Laurent polynomials whose restrictions (\( \prod \)) to all fixed points in \( X^T \) vanish.

As a corollary of Theorem 1

\[
\text{Pic}(X) \cong \mathbb{Z}^{|I|}
\]

is a finite dimensional lattice generated by the classes of the tautological line bundles

\[
\mathcal{L}_i = \det(V_i) = x_{i,1} \cdots x_{i,v_i}, \quad i \in I.
\]

**1.2**

We denote \( K = \text{Pic}(X) \otimes \mathbb{C}^\times \). The coordinates on torus \( K \) are called \( \text{K"ahler parameters of} \ X \) (or Fayet–Iliopoulos parameters in physical literature). We use symbols \( z = (z_i), i \in I \) to denote these characters.

The coordinates on \( T = \mathbb{C}^\times_\hbar \times A \) are called \( \text{equivariant parameters} \) (or mass parameters in physical literature) and are denoted by symbols \( \hbar \) and \( a = (a_i), i = 1, \ldots, \dim(A) \).
1.3

Assume $X^T$ is finite. The real Lie algebra $\text{Lie}_\mathbb{R}(A)$ is naturally equipped with a set of hyperplanes $\{\omega^\perp\}$, where $\omega$ runs over the set of $A$-characters appearing as $A$-weights of the tangent spaces $T_pX$ for $p \in X^T$. The complement of these hyperplanes is a union of connected components

$$\text{Lie}_\mathbb{R}(A) \setminus \{\omega^\perp\} = \bigcup \mathcal{C}$$

which are called chambers. For a chamber $\mathcal{C}$ we have the corresponding decomposition

$$T_pX = N_p^+ \oplus N_p^-$$

where $N_p^+$, $N_p^-$ are the subspaces whose $A$-characters take positive or negative values on $\mathcal{C}$, respectively. The subspaces $N_p^\pm$ are sometimes referred to as attracting or repelling directions of the tangent space corresponding to $\mathcal{C}$.

1.4

Recall that the vertex function $V(a, z)$ of $X$ is a $K$-theoretic analog of Givental’s $J$-function in Gromov-Witten theory. It is defined by the equivariant count of quasimaps from a rational curve $\mathbb{P}^1$ to $X$. For the definitions of quasimap moduli spaces and vertex functions see Section 7.2 in [Oko15], Section 3 in [AFO18] or Section 6 in [AO16].

By definition, the vertex function is a power series in the Kähler parameters:

$$V(a, z) \in K_{T\times C_q^x}(X)_{\text{loc}}[[z]]$$

where $K_{T\times C_q^x}(X)_{\text{loc}}$ denotes the localized equivariant $K$-theory ring of $X$ and $C_q^x$ is a one-dimensional torus acting on the domain of the quasimaps $\mathbb{P}^1$ by scaling the homogeneous coordinates such that the character of $T_0\mathbb{P}^1$ is equal to $q$.

The classes of the fixed points $p \in X^T$ provide a natural basis in localized $K$-theory. The components of the vertex function in this basis have the following form:

$$V_p(a, z) = \sum_{d \in C_{\text{eff}}(X) \cap H^2(X,\mathbb{Z})} c_d(a) z^d$$
where the degrees $d$ run over a certain cone $C_{\text{eff}}(X) \subset H^2(X, \mathbb{R}) = \text{Lie}_\mathbb{R}(K)$ spanned by effective curves$^2$, and $c_d(a) \in \mathbb{Z}(a, q, \hbar)$ are some rational functions.

Alternatively, the vertex function can be defined as the index of the 3d gauge theory on $\mathbb{C} \times S^1$. The choice of the fixed point $p$ in (3) corresponds then to the choice of the vacuum of the theory at infinity of $\mathbb{C} \times S^1$, see Section 8.5 in [AFO18] for a physical discussion in this direction.

Finally, we also mention that the vertex functions play an important role in the theory of integrable systems known as $XXZ$-spin chains, which were recently understood as the quantum $K$-theory of Nakajima varieties, see [NS09b, NS09a, AO17, PSZ16, KPSZ17, GK13].

1.5

We will be interested in specializations of the vertex functions corresponding to vanishing equivariant parameters. More precisely, let $\sigma : w \in \mathbb{C}^\times \to A$ be a cocharacter from a chamber $\mathcal{C}$. We define

$$V_p(0_c, z) := \lim_{w \to 0} V_p(\sigma(w), z) \in \mathbb{Z}(q, \hbar)[[z]].$$

It follows from the construction of the virtual structure sheaf on the moduli space of quasimaps that these limits exist and are well defined for all chambers.

1.6

3d-mirror symmetry is a conjecture which claims that for every symplectic variety from the class discussed above there exists a dual variety $X'$ called the 3d-mirror of $X$. The geometries of $X$ and $X'$ intimately related.

A 3d mirror $X'$ can be characterized in the language of elliptic stable envelopes [RSVZ19b, RSVZ19a]. The definition in Section 1.1 in [RSVZ19a] is especially convenient for us. This duality, among other things, provides the following data:

$^2$The choice of the cone $C_{\text{eff}}(X)$ corresponds to the choice of the stability parameter for the Nakajima variety $X$.

$^3$See also the talk by A. Okounkov “Enumerative symplectic duality” at the MSRI workshop Structures in Enumerative Geometry in April 2018 for the first discussion of these ideas (available online).
An isomorphism of tori
\[ \kappa : A' \times K' \times \mathbb{C}_q^\times \times \mathbb{C}_q^\times \rightarrow A \times K \times \mathbb{C}_q^\times \times \mathbb{C}_q^\times \] (4)
where \( K' \) and \( A' \times \mathbb{C}_q^\times = T' \) are the Kähler and the equivariant tori of \( X' \). In coordinates, the map \( \kappa \) is an invertible monomial transformation of the form
\[ z_i \rightarrow h^{n_i} \prod_j a_j^{m_{i,j}}, \quad a_i \rightarrow (h'/q)^{r_i} \prod_j z_i^{s_{i,j}}, \quad h \rightarrow q/h' \] (5)
for some \( n_i, r_i, m_{i,j}, s_{i,j} \in \mathbb{Z} \).

A bijection of sets of fixed points
\[ b : X^T \rightarrow (X')^{T'} \] (6)
The restriction of (5) to \( \ker(q) \cap \ker(h) \) provides an isomorphism of tori:
\[ \bar{\kappa} : A \rightarrow K', \quad K \rightarrow A' \]
One often refers to this by saying that “3d mirror symmetry identifies equivariant parameters with Kähler parameters of the dual theory”. In addition, it provides an identification of chambers with effective cones of the 3d mirror variety:
\[ d\bar{\kappa}(\mathcal{C}) = C_{\text{eff}}(X'), \quad d\bar{\kappa}(C_{\text{eff}}(X)) = \mathcal{C}', \]
where \( d\bar{\kappa} \) is the induced map of the Lie algebras. It is therefore more natural to think that the 3d mirror symmetry provides a pair \((X', \mathcal{C}')\) for each pair \((X, \mathcal{C})\).

1.7
The isomorphism (4) provides the dual variety \( X' \) with an action of the torus \( K \). Given a fixed point \( p \in X^T \) it is therefore natural to ask

**What is the \( K \)-character of the tangent space \( T_{b(p)}X' \) in terms of \( X \)?**

The 3d-mirror of a symplectic variety is not known in most cases. Therefore, the above question is non-trivial. The answer to this question provides new interesting information about the geometry of \( X' \).

In this paper, we propose a combinatorial formula for the character of \( T_{b(p)}X' \). The feature of our formula is that it describes the corresponding character purely in terms of the vertex functions of \( X \). In particular, to compute the characters of \( T_{b(p)}X' \), one does not need to know what \( X' \) is.
1.8

For $N \in K_{T'}(pt)$ given by a polynomial $N = w_1 + \cdots + w_m$ we abbreviate

$$\Xi(b, N) = \xi(b, w_1) \cdots \xi(b, w_m),$$

where

$$\xi(b, w) = \frac{\varphi(bw)}{\varphi(w)} \quad \text{and} \quad \varphi(w) = \prod_{n=0}^{\infty} (1 - wq^n)$$

is the $q$-analog of the Gamma function (we assume $0 < |q| < 1$ so that the last infinite product is convergent). By the $q$-binomial theorem

$$\xi(b, w) = \sum_{n=0}^{\infty} \frac{(b)_n}{(q)_n} w^n$$

where

$$(a)_n = \frac{\varphi(a)}{\varphi(aq^n)} = \begin{cases} (1-a) \cdots (1-aq^{n-1}), & n \geq 0 \\ \frac{1}{(1-aq^n) \cdots (1-aq^{-1})}, & n < 0 \end{cases}$$

and thus we may think of $\Xi(b, N)$ as a power series in $w_i$. Similarly, we denote

$$\Phi(N) = \varphi(w_1) \cdots \varphi(w_m),$$

and extend it by multiplicativity to polynomials with negative coefficients. For example:

$$\Phi(a + 2b - 3c) = \frac{\varphi(a)\varphi(b)^2}{\varphi(c)^3}.$$

1.9

Let $(X, \mathcal{C})$ and $(X', \mathcal{C}')$ be pairs of symplectic varieties and chambers related by 3d-mirror symmetry. For $p \in X^T$ let $b(p)$ be the corresponding fixed point on the dual variety $X'$. The chamber $\mathcal{C}' = d\bar{\kappa}(C_{\text{eff}}(X))$ provides a decomposition

$$T_{b(p)}X' = N^+_{b(p)} \oplus N^-_{b(p)}.$$

We identify these spaces with their $K$-theory classes $N^\pm_{b(p)} \in K_{T'}(pt)$ (i.e., simply the $T'$-character of this vector space)
Conjecture 1. The vertex functions of $X$ with vanishing equivariant parameters are given by the Taylor series expansions of the following functions

$$
\kappa^* V_p(0 \xi, z) = \Xi((q/h') (N_{b(p)}^-)^*)
$$

where $\kappa^*$ stands for substitution (5) and $(N_{b(p)}^-)^*$ is the $\mathbb{T}'$-module dual to $N_{b(p)}^-$ (i.e. the weights of $(N_{b(p)}^-)^*$ are inverses of the weights of $N_{b(p)}^-$).

Below we check this conjecture in several cases by explicit computation.

Note 1. Note that the right side of (9) is a product of $\dim(X'/2)$ $q$-binomials (7) with $b = \hbar = q/h'$. Thus, the conjecture provides a non-trivial summation formula for the power series defining the vertex functions.

Note 2. Note that the left side of the proposed identity is defined purely in terms of $X$. Computing the vertex function of $X$ we can therefore obtain the character of $N_{b(p)}^-$ and thus the character of the whole tangent space on the dual side:

$$T_{b(p)} X' = N_{b(p)}^- + (N_{b(p)}^-)^* / h'.$$

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2 Vertex functions for Nakajima varieties

In this section we recall how to compute vertex function (3) for a Nakajima variety $X$. For more details we refer to Section 7.2 in [Oko15], Section 3 in [AFO18] or Section 6 in [AO16].

2.1

Let $X$ be a Nakajima variety. We denote by $Q$ be the corresponding quiver and by $I$ the set of vertices of $Q$. Let $V_i$ and $W_i$, $i \in I$ be the vector spaces
and framing vector spaces corresponding to $X$. We denote $v_i = \dim V_i$, $w_i = \dim W_i$.

Let us fix an orientation of $Q$ and define the polarization by the following virtual vector space

$$P = \left( \bigoplus_{i \to j} V_i^* \otimes V_j \right) \bigoplus \left( \bigoplus_{i \in I} W_i^* \otimes V_i \right) - \bigoplus_{i \in I} V_i^* \otimes V_i$$

where $i \to j$ denotes the sum running over arrows of $Q$ (with fixed orientation). We denote by $\mathcal{P} \in K_T(X)$ the virtual bundle associated to $P$. Explicitly, this K-theory class is given by the following Laurent polynomial:

$$\mathcal{P} = \sum_{i \to j} \left( \sum_{m=1}^{v_i} x_{i,m}^{-1} \right) \left( \sum_{m=1}^{v_j} x_{j,m} \right) + \sum_{i \in I} \left( \sum_{m=1}^{w_i} a_{i,m}^{-1} \right) \left( \sum_{m=1}^{v_i} x_{i,m} \right) - \sum_{i \in I} \left( \sum_{m=1}^{v_i} x_{i,m}^{-1} \right) \left( \sum_{m=1}^{v_i} x_{i,m} \right)$$

where $x_{i,j}$ and $a_{i,j}$ denote the Grothendieck roots of the tautological bundles $V_i$ and the equivariant parameters associated to the framings $W_i$.

2.2

For the tautological line bundles $\mathcal{L}$ we define the following formal expression in Grothendieck roots

$$e(x, z) := \exp \left( \frac{1}{\ln q} \sum_{i \in I} \ln(z_i) \ln(\mathcal{L}_i) \right) = \exp \left( \frac{1}{\ln q} \sum_{i \in I} \sum_{j=1}^{v_i} \ln(z_i) \ln(x_{i,j}) \right)$$

where $x$ abbreviates the set of all Grothendieck roots. This function has the following transformation properties:

$$e(x_{1,1}, \ldots, x_{i,j} q, \ldots, z) = z_i e(x_{1,1}, \ldots, x_{i,j}, \ldots, z)$$

and similarly

$$e(x, \ldots, z_1, \ldots, z_i q, \ldots, z_{|I|}) = \mathcal{L}_i e(x, \ldots, z_1, \ldots, z_i, \ldots, z_{|I|}).$$
2.3

Let \( p \in X^T \) be a fixed point on the quiver variety. The restriction of a tautological bundle \( \mathcal{V}_i \) to a fixed point provides a polynomial:

\[
\mathcal{V}_i|_p = t_{i,1} + \cdots + t_{i,\dim \mathcal{V}_i} \in K_T(pt).
\]

This polynomial can be defined as “evaluation” of the Grothendieck roots of \( \mathcal{V}_i \) at the corresponding fixed point:

\[
x_{i,j} \rightarrow x_{i,j}(p) := t_{i,j} \in K_T(pt).
\]  

(11)

Note that all \( K \)-theory classes are symmetric polynomials in \( x_{i,j} \) and thus the last operation is well defined. We denote by \( \mathbf{x}(p) \) the set of \( x_{i,j}(p) \).

2.4

Up to normalizing prefactor, the vertex function for Nakajima varieties equals

\[
\tilde{V}_p(a, z) = \prod_{i,j} x_{i,j}(p) \int_0^a d_q x_{i,j} \Phi \left( (q - \hbar) \mathcal{P} \right) e(x, z)
\]

(12)

where the integral symbols denote the Jackson \( q \)-integral over all Grothendieck roots:

\[
\int_0^a d_q x f(x) := \sum_{n=0}^{\infty} f(aq^n),
\]

\( \mathcal{P} \) is the polarization (10) expressed in Grothendieck roots and \( \Phi \) is defined by (8). The normalized vertex function equals:

\[
V_p(a, z) = \frac{1}{\Phi \left( (q - \hbar) \mathcal{P}(p) \right) e(x(p), z)} \tilde{V}_p(a, z),
\]

(13)

where \( \mathcal{P}(p) \) is the result substituting (11) into the polarization. Thanks to the identity \( \varphi(a)/\varphi(aq^d) = (a)_d \) the function \( V_p(a, z) \) is a power series in \( z \) with coefficients in \( \mathbb{Z}(a, \hbar, q) \). The normalization (13) has the effect of starting the power series with 1:

\[
V_p(a, z) = 1 + O(z) \in \mathbb{Z}(a, \hbar, q)[[z]].
\]
Theorem 2 (Theorem 4 in [AO16]). For all \( p \in X^T \) the functions (12) are solutions of certain \( q \)-difference equations in the equivariant and Kähler parameters with regular singularities.

For Nakajima quiver varieties the corresponding \( q \)-difference equations were described in [OS16]. As a corollary, the power series (13) converges to an analytical function in some neighborhood \( |z_i| < \epsilon_i \) with \( 0 < \epsilon_i \).

In the following sections we compute the vertex functions explicitly for several Nakajima varieties and check the identity (9).

3  Cotangent Bundle of \( Gr(k, n) \)

3.1
Let \( X = T^*Gr(k, n) \), the cotangent bundle of the Grassmannian of \( k \)-planes in \( n \)-dimensional space. \( X \) is an example of a Nakajima quiver variety, see Section 3 in [RSVZ19b]. The corresponding quiver \( Q \) consists of a single vertex. The associated representation of the quiver consists of a vector space \( V \) with \( \dim V = k \) and a framing space \( W \) with \( \dim W = n \). Let \( A = (\mathbb{C}^\times)^n \) be the torus acting on \( \mathbb{C}^n \) by

\[
(z_1, \ldots, z_n) \rightarrow (z_1a_1^{-1}, \ldots, z_na_n^{-1}).
\]

\( A \) induces a natural action on \( X \). Let \( \mathbb{C}^\times_h \) be the torus which acts on \( X \) by

\[
\eta_p \rightarrow h^{-1}\eta_p
\]

where \( \eta_p \) is a covector at a point \( p \in X \). Thus, the torus \( T = A \times \mathbb{C}^\times_h \) acts on \( X \). One can check that \( A \) preserves the symplectic form, while \( \mathbb{C}^\times_h \) scales it by \( h \).

3.2
The set of fixed points \( X^T \) corresponds to the \( k \)-dimensional coordinate subspaces in \( X \), and thus fixed points \( p \in X^T \) are naturally labeled by size \( k \) subsets of \( \{1, \ldots, n\} \). As a Nakajima quiver variety, \( X \) comes equipped with a rank \( k \)-tautological vector bundle \( V \). Let \( p = \{p_1, \ldots, p_k\} \subset \{1, \ldots, n\} \).
The $T$-character of the $p_i$-th coordinate direction is $a_{p_i}^{-1}$. Thus the restriction of the tautological bundle to the fixed point $p$ is given by

$$V_p = \sum_{i=1}^{k} a_{p_i}^{-1}.$$ 

Thus, the restriction of the Grothendieck roots in (11) has the form:

$$x_i(p) = a_{p_i}^{-1}. \quad (14)$$

3.3

In this case, the polarization $\langle \mathcal{P} \rangle$ takes the form

$$P = W^* \otimes V - V^* \otimes V$$

and we have

$$\Phi((q - h)\mathcal{P}) = \prod_{i=1}^{n} \prod_{j=1}^{k} \frac{\phi(qa_i x_j)}{\phi(ha_i x_j)} \prod_{i,j=1}^{k} \frac{\phi(h x_i / x_j)}{\phi(q x_i / x_j)}$$

Thus, the vertex functions associated to $T^* Gr(k, n)$ are given by the power series:

$$V_p(a_1, \ldots, a_n, z) = \sum_{d_1, \ldots, d_k=0}^{\infty} \left( \prod_{i=1}^{n} \prod_{j=1}^{k} \frac{(ha_i / a_p)_d}{(qa_i / a_p)_d} \prod_{i,j=1}^{k} \frac{(qa_{p_i} / a_p)_d}{(ha_{p_i} / a_p)_d} \right) z^{d_1 + \ldots + d_k}.$$ 

Note 3. In the case of $k = 1$, the fixed point $p$ is just an integer $1 \leq p \leq n$, and the vertex function is

$$V_p(a_1, \ldots, a_n, z) = \sum_{d=0}^{\infty} \left( \prod_{i=1}^{n} \frac{(ha_i / a_p)_d}{(qa_i / a_p)_d} \right) z^d,$$

which is an example of the well-known $q$-hypergeometric series (see [GR90]).
3.4

The torus $A$ is $n$-dimensional, and we choose the chamber $C$ associated to the cocharacter $w \mapsto (w^{-1}, w^{-2}, \ldots, w^{-n})$. Then

$$V_p(0, z) = \lim_{w \to 0} V_p(w^{-1}, \ldots, w^{-n}, z)$$

In terms of equivariant parameters, this limit corresponds to $\frac{a_i}{a_j} \to 0$ if $i < j$. Substituting (14) and taking the limit, we compute

$$V_p(0, z) = \sum_{d_1, \ldots, d_k = 0}^\infty \left( \prod_{i=1}^k \frac{(\hbar)_d}{(q)_d} \prod_{j=i+1}^n \left( \frac{\hbar}{q} \right)^{d_j} \prod_{i=1}^k \prod_{j=i+1}^k \left( \frac{q}{\hbar} \right)^{d_i-d_j} \right) z^{d_1+\cdots+d_k}$$

which simplifies to

$$V_p(0, z) = \sum_{d_1, \ldots, d_k = 0}^\infty \left( \prod_{i=1}^k \frac{(\hbar)_d}{(q)_d} \left( \frac{\hbar}{q} \right)^{n-p_i-k-i+1} \right) z^{d_i}.$$

Interchanging the order of the summation and the product and using the $q$-binomial theorem, we obtain

$$V_p(0, z) = \prod_{i=1}^k \xi \left( \hbar, \left( \frac{\hbar}{q} \right)^{n-p_i+k+2i-1} z \right). \quad (15)$$

3.5

For $n \geq 2k$ the 3d mirror dual $X'$ of $T^*Gr(k, n)$ is the Nakajima quiver variety associated to the $A_{n-1}$ quiver with dimension vector $v = (1, 2, \ldots, k-1, k, \ldots, k, k-1, \ldots, 2, 1)$

with one dimensional framings at positions $k$ and $n-k$ and trivial framings elsewhere (see Section 4 of [RSVZ19b] for details). The fixed points of $X'$ are labeled by Young diagrams $\lambda$ which fit into the rectangle with dimensions $(n-k) \times k$. The boundary of such a diagram is a piecewise linear curve with $n-k$ horizontal segments and $k$ vertical segments. The positions of the $k$ vertical segments naturally determine a $k$-tuple $p_\lambda \subset \{1, 2, \ldots, n\}$. The bijection on fixed points (6) has the form (see also Section 6.1 in [RSVZ19b]):

$$b : \lambda \to p_\lambda$$
The isomorphism (5) between the equivariant parameters of $X'$ and the Kähler parameters of $X$ is the following:

$$
\kappa^* : z \to a' \hbar'^{k-1}, \ h \to q/\hbar'
$$

where $a' = a_1'/a_2'$ is the equivariant parameters of $X'$ (i.e. coordinates on the framing torus $A' = (\mathbb{C}^\times)^2$). Using this transformation we can rewrite (15) as follows:

$$
\kappa^* \left( V_p(0, z) \right) = \Xi \left( q/\hbar', \sum_{i=1}^{k} a' \hbar'^{2k-n+p_i-2i} \right).
$$

We choose the chamber $\mathcal{C}'$ given by the cocharacter $w \mapsto (a_1', a_2') = (1, w^{-1})$. The characters of the tangent spaces at the fixed points and of repelling and attracting parts of it corresponding to $\mathcal{C}'$ were computed in Section 4.4 of [RSVZ19b]:

$$
N^{\nu^+}_\lambda = \sum_{i=1}^{k} \frac{a_1'}{a_2'} \hbar'^{2k-n+p_i-2i-1}, \quad N^{\nu^-}_\lambda = \sum_{i=1}^{k} \frac{a_2'}{a_1'} \hbar'^{-2k-n-p_i+2i},
$$

where $p = b(\lambda)$. Thus, the character of the dual representation equals:

$$
(N^{\nu^-}_\lambda)^* = \sum_{i=1}^{k} \hbar'^{2k-n+p_i-2i}
$$

We, therefore have:

$$
\kappa^* \left( V_p(0, z) \right) = \Xi(q/\hbar', (N^{\nu^-}_\lambda)^*).
$$

which proves Conjecture 1 in this case.

**Note 4.** As we mentioned above, the dual variety $X'$ is a Nakajima variety if $n \geq 2k$. In the case $n < 2k$ the variety $X'$ is not known to us. Conjecture 1, however, still holds in this case.

For example, our formula (15) implies that in the case of the zero-dimensional Nakajima variety $X = T^*Gr(n, n)$, the dual variety $X' \cong \mathbb{C}^{2n}$ is a 2n-dimensional symplectic vector space equipped with an action of two-dimensional torus $T' = \mathbb{C}^\times_a \times \mathbb{C}_n^\times$, so that

$$
\text{char}_{T'}(\mathbb{C}^{2n}) = \sum_{i=1}^{n} (a' \hbar'^{m-i-1} + \hbar'^{-n+i}/a') \in K_{T'}(pt).
$$

We believe that the 3d mirrors can be constructed as bow varieties [Che11, NT16].
4 Hilbert scheme of points on $\mathbb{C}^2$

4.1

In this section, $X$ denotes the Hilbert scheme of $n$ points on the complex plane. This is a smooth, symplectic, quasiprojective variety which parametrizes the polynomial ideals of codimension $n$ [Nak99, Nak16]:

$$X = \{ \mathcal{J} \subset \mathbb{C}[x, y] : \dim_{\mathbb{C}}(\mathbb{C}[x, y]/\mathcal{J}) = n \}.$$ 

Let $T \cong (\mathbb{C}^\times)^2$ be a two-dimensional torus acting on $\mathbb{C}[x, y]$ by:

$$(x, y) \rightarrow (xt_1^{-1}, yt_2^{-1}).$$ (16)

This action induces an action of $T$ on $X$. The one-dimensional space spanned by a symplectic form $\omega \subset H^2(X, \mathbb{C})$ is a natural $T$-module. We denote by $\mathbb{h}$ the $T$-character of $\mathbb{C}\omega$. From our normalization (16) we find:

$$\mathbb{h}^{-1} = t_1t_2 \in K_T(pt) = \mathbb{Z}[t_1^\pm 1, t_2^\pm 1].$$

We denote by

$$A = \ker(\mathbb{h}) \subset T$$

the one-dimensional subtorus preserving the symplectic form on $X$. We denote the coordinate on $A$ by $a$, such that

$$t_1 = \frac{a}{\sqrt{\mathbb{h}}}, \quad t_2 = \frac{1}{a\sqrt{\mathbb{h}}}.$$ (17)

4.2

The set of fixed points $X^T = X^A$ is a finite set labeled by Young diagrams with $n$ boxes. A Young diagram $\lambda$ corresponds to the $T$-invariant ideal generated by monomials:

$$\mathcal{J}_\lambda = \{ x^{\lambda_1}, x^{\lambda_1}y, x^{\lambda_2}y^2 \ldots \}.$$ 

The Hilbert scheme is equipped with a rank $n$-tautological vector bundle $\mathcal{V}$ with fibers at a point $\mathcal{J} \in X$ given by

$$\mathcal{V}|_\mathcal{J} = \mathbb{C}[x, y]/\mathcal{J}.$$ 

\footnote{We may assume $\mathbb{h}^{1/2}$ exists by passing to the double cover of $T$ if needed.}
Let $\square = (m, n)$ be a box with coordinates $m$ and $n$ in a Young diagram $\lambda$. The $T$-character of the corresponding monomial equals $t_1^{-(n-1)} t_2^{-(m-1)}$. We conclude that restriction of the tautological bundle to the fixed point $\lambda$ is given by the sum of the characters of monomials over the boxes in $\lambda$:

$$V|_{\mathcal{J}_\lambda} = \sum_{(m,n) \in \lambda} t_1^{-(n-1)} t_2^{-(m-1)}.$$ 

Thus, the restriction of Grothendieck roots in (11) has the form:

$$x_i(\lambda) = t_1^{-(n_i-1)} t_2^{-(m_i-1)},$$

where $(m_i, n_i)$ are the coordinates of the $i$-th box in $\lambda$. Again, the order of boxes is not important, because all tautological classes are symmetric in $x_i$'s.

### 4.3

The characters of the tangent space at a fixed point $\lambda$ can be computed using, for example, a free resolution of the ideal $\mathcal{J}_\lambda$, see Section 3.4.19 [Oko15]. This gives the following explicit formula:

$$T_\lambda X = \sum_{i \in \lambda} t_1^{-l_\lambda(i)} t_2^{-a_\lambda(i)+1} + t_2^{-a_\lambda(i)} t_1^{-l_\lambda(i)+1} \in K_T(pt),$$

where the sum runs over the boxes of the Young diagram $\lambda$ and

$$l_\lambda(\square) = \lambda_i - j, \quad a_\lambda(\square) = \lambda'_j - i$$

stand for the standard leg and arm length of a box $i$ in $\lambda$ ($\lambda'$ denotes the transposition of $\lambda$). The change of variables (17) provides a decomposition of this character into a sum of characters with positive and negative powers of $a$:

$$T_\lambda X = N^+_\lambda + N^-_\lambda$$

where

$$N^+_\lambda = \sum_{i \in \lambda} t_2^{-a_\lambda(i)} t_1^{-l_\lambda(i)+1}, \quad N^-_\lambda = \sum_{i \in \lambda} t_1^{-a_\lambda(i)} t_2^{-l_\lambda(i)+1}. \quad (18)$$
4.4

The Hilbert scheme is an example of a Nakajima quiver variety, see Section 3.3 in [Smi18]. The corresponding quiver $Q$ consists of one vertex and one loop connecting the vertex with itself. The associated representation of the quiver consists of a vector space $V$ with $\dim V = n$ and a framing space $W$ with $\dim W = 1$.

The polarization (10) in this case has the form (see Section 3.4 in [Smi18]):

$$P = W^* \otimes V + V^* \otimes Vt_1 - V^* \otimes V$$

We compute:

$$\Phi((q - \hbar)\mathcal{P}) = \prod_{i=1}^{n} \varphi(qx_i) \prod_{i,j=1}^{n} \varphi(qt_1x_i/x_j) \varphi(hx_i/x_j).$$

Thus, the vertex functions associated to the Hilbert scheme $X$ are given by the following power series:

$$V_\lambda(a, z) = \sum_{d_1, \ldots, d_n=0}^{\infty} \left( \prod_{i=1}^{n} \frac{(hx_i(\lambda))_{d_i}}{qx_i(\lambda)} \prod_{i,j=1}^{n} \frac{(ht_1x_i(\lambda)/x_j(\lambda))_{d_i-d_j}}{(qt_1x_i(\lambda)/x_j(\lambda))_{d_i-d_j}} \frac{(hx_i(\lambda)/x_j(\lambda))_{d_i-d_j}}{(qtx_i(\lambda)/x_j(\lambda))_{d_i-d_j}} \right) z^{d_1+\cdots+d_n}$$

which is a power series in $z$ with coefficients in $\mathbb{Z}(h, q, a)$.

4.5

In the case of the Hilbert scheme $X$, the torus $A$ preserving the symplectic form is one-dimensional and the chamber decomposition (2) takes the form:

$$\text{Lie}_\mathbb{R}(A) \setminus \{\omega^\perp\} = \mathbb{R} \setminus \{0\} = \mathfrak{c}_+ \cup \mathfrak{c}_-.$$

Let us analyze the case $\mathfrak{c} = \mathfrak{c}_+$ which corresponds to the cocharacters $\{a \to 0\}$ so that

$$V_\lambda(0, z) = \lim_{a \to 0} V_\lambda(a, z).$$

It is expected that the Hilbert scheme $X$ is self-dual with respect to $3d$ mirror symmetry\footnote{To the best of authors knowledge this is not proved.}:

$$X' \cong X.$$
We denote by \( t'_1, t'_2, \hbar', a' \) the equivariant parameters of \( X' \). As in (17) these parameters are related by
\[
t'_1 = \frac{a'}{\sqrt{\hbar'}}, \quad t'_1 = \frac{1}{a'\sqrt{\hbar'}}.
\]
We consider the trivial bijection on the fixed points (6)
\[
b = id : \lambda \mapsto \lambda,
\]
and the change of variables (5):
\[
\kappa^* : z \rightarrow a'\sqrt{\hbar'}, \quad h \rightarrow \frac{q}{h'},
\]
(20)
As above, let \( \kappa^* \mathbf{V}_\lambda(0_\epsilon, z) \) denote the result of the substitution (20) into the power series given by the limit (19). Thus, \( \kappa^* \mathbf{V}_\lambda(0_\epsilon, z) \) is a power series in \( a' \):
\[
\kappa^* \mathbf{V}(0_\epsilon, z) \in \mathbb{Z}(\sqrt{\hbar'}, q)[[a']].
\]
Then, the equality (9) in this case is equivalent to the following identity
\[
\kappa^* \mathbf{V}(0_\epsilon, z) = \Xi(q/\hbar', (N^-_\lambda)^*) = \prod_{i \in \lambda} \xi(q/\hbar', t'_1 t'_2^{-a_\lambda(i)-1})
\]
(21)
where \( N^-_\lambda \) is from (18). This is a quite nontrivial summation formula for the power series \( \mathbf{V}_\lambda(0_\epsilon, z) \). This identity can be proved by induction on the number of boxes. We plan to give a general proof of such identities arising for \( A_n \) and affine \( \hat{A}_n \) quiver varieties in the sequel paper.

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