Thermodynamics in $f(R)$ gravity in the Palatini formalism

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Abstract. We investigate thermodynamics of the apparent horizon in $f(R)$ gravity in the Palatini formalism with non-equilibrium and equilibrium descriptions. We demonstrate that it is more transparent to understand the horizon entropy in the equilibrium framework than that in the non-equilibrium one. Furthermore, we show that the second law of thermodynamics can be explicitly verified in both phantom and non-phantom phases for the same temperature of the universe outside and inside the apparent horizon.

Keywords: modified gravity, quantum black holes, dark energy theory
1 Introduction

There are two representative approaches to account for the current accelerated expansion [1, 2] of the universe [3–15]. One is the introduction of “dark energy” in the framework of general relativity. The other is the investigation of a modified gravitational theory, such as \( f(R) \) gravity [16–19]. In this study, we concentrate on thermodynamics in the modified gravitational theory. The fundamental connection between gravitation and thermodynamics has been suggested by the discovery of black hole thermodynamics [20] with black hole entropy [21] and Hawking temperature [22]. Its application to the cosmological event horizon of de Sitter space has been explored [23] (for recent reviews, see [24]). It was shown that the Einstein equation can be derived from the proportionality of the entropy to the horizon area together with the Clausius relation in thermodynamics [25]. This consequence has been applied to various cosmological settings [26–30]. In ref. [29], it was shown that if the entropy of the apparent horizon in the Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime is proportional to the apparent horizon area, Friedmann equations follow from the first law of thermodynamics.

Thermodynamics in dynamical black holes [31–33] and spherically symmetric spacetimes [34] has also been developed. The first law of the ordinary equilibrium thermodynamics has been discussed in \( f(R) \) gravity, scalar-tensor theories, the Gauss-Bonnet gravity and more general Lovelock gravity [35–39], while the second law has been studied in the accelerating universe, \( f(R) \) gravity, the Gauss-Bonnet gravity and the Lovelock gravity in refs. [40–42], respectively. Thermodynamics in braneworld scenario [43], loop quantum cosmology [44–46] and Hořava-Lifshitz gravity [47] as well as its properties of dark energy [48–50] and those in cosmological scenarios with interactions between dark energy and dark matter [51, 52] have been also examined. Incidentally, the horizon entropy in the four-dimensional modified gravity [53] and the quantum logarithmic correction to the expression of the horizon entropy in cosmological contexts [44, 54–56] have been investigated. In the framework of general relativity, the considerations on the sign of entropy [57, 58] and its evolution [59] in the
phantom phase have been performed. Moreover, the conditions \cite{60} that the black hole entropy can be positive in the $f(R)$ gravity models \cite{61–63} with passing the solar-system tests and cosmological bounds and the evolution of the black hole entropy in the ghost condensate scenario \cite{64} have been studied.

It was pointed out \cite{65} that in $f(R)$ gravity, a non-equilibrium thermodynamic treatment should be required in order to derive the corresponding gravitational field equation by using the procedure in ref. \cite{25}. This point was reanalyzed in $f(R)$ gravity \cite{66} as well as scalar-tensor theories \cite{67} and elaborated by considering the role of gravitational dissipation \cite{68}. By taking into account the non-equilibrium thermodynamic treatment, the studies on the first \cite{69} and second \cite{70} laws of thermodynamics on the apparent horizon in generalized theories of gravitation and the investigations on thermodynamics \cite{71} in a $f(R)$ gravity model \cite{72} with realizing a crossing of the phantom divide from the non-phantom (quintessence) phase to the phantom one have been performed. The non-equilibrium correction \cite{65} has been reinterpreted through the introduction of a mass-like function \cite{73} and by other approaches \cite{74–76}. The formulation in ref. \cite{25} was also extended to the more general extended gravity theory \cite{77,78} (for related discussions, see \cite{79}).

Recently, it has been shown that it is possible to obtain a picture of equilibrium thermodynamics in the FRW background for modified gravity theories with the Lagrangian density $f(R, \phi, X)$ (including $f(R)$ gravity and scalar-tensor theories), where $X = -(1/2) g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi$ is the kinetic term of a scalar field $\phi$ ($\nabla_\mu$ is the covariant derivative operator associated with the metric tensor $g_{\mu\nu}$), due to a suitable redefinition of an energy momentum tensor of the “dark” component that respects a local energy conservation \cite{80}.

The previous studies on thermodynamics in $f(R)$ gravity have been executed in the standard metric formalism, in which the connection is defined in terms of the metric. There exist other formalisms to derive the gravitational field equation from the action, such as the Palatini formalism, in which the connection and the metric are assumed to be independent variables (for more detailed explanations on the Palatini formalism, see \cite{10}). As a result, the gravitational field equation of $f(R)$ gravity in the Palatini formalism is different from that in the metric one. Furthermore, $f(R)$ gravity \cite{81–85} and more general gravitational theories \cite{86,87} in the Palatini formalism have been examined. Bouncing cosmologies in the Palatini $f(R)$ gravity \cite{88} and the dynamical aspects of generalized Palatini theories of gravity \cite{89} have been also investigated.

It should be mentioned that the Palatini $f(R)$ gravity (with the exception of those with very heavily suppressed correction which would only become significant in the very far UV) is in conflict with Solar system tests \cite{83} and particle physics \cite{90,91}. Moreover, it is plagued by surface singularities in stars and spherical matter configurations \cite{92} and it does not necessarily have a well posed Cauchy problem in the presence of matter \cite{10}. However, it should be cautioned that some of the arguments may not be generic. For example, most of criticism were based on \( f(R) = R - \beta/R^n \), where $\beta$ and $n$ are constants, while the against of the Palatini $f(R)$ gravity based on particle physics may be premature as the Higgs sector is not well understood yet.

On the other hand, although there have been many studies on thermodynamics in $f(R)$ gravity in the literature, an equilibrium thermodynamics description of $f(R)$ gravity in the Palatini formalism has not been done yet. In particular, it is interesting to ask whether the non-equilibrium and equilibrium descriptions of thermodynamics can be given in the Palatini formalism as the metric one \cite{80}. For the interest of the formalism, in this paper we explore both non-equilibrium and equilibrium descriptions of thermodynamics in the
Palatini formalism of $f(R)$ gravity. We also show that the second law of thermodynamics in $f(R)$ gravity can be explicitly verified in the phantom phase as well as the non-phantom (quintessence) one if the temperature of the universe inside the horizon is equal to that of the apparent horizon. We use units of $k_B = c = \hbar = 1$ and denote the gravitational constant $8\pi G$ by $\kappa^2 \equiv 8\pi / M_{\text{Pl}}^2$ with the Planck mass of $M_{\text{Pl}} = G^{-1/2} = 1.2 \times 10^{19}\text{GeV}$.

The paper is organized as follows. In section II, we explain $f(R)$ gravity in the Palatini formalism and study the first and second laws of thermodynamics of the apparent horizon in $f(R)$ gravity under a non-equilibrium description of thermodynamics. In section III, we illustrate an equilibrium description of thermodynamics by redefining the energy density and pressure of dark components in $f(R)$ gravity in the Palatini formalism. Finally, conclusions are given in section IV.

2 Non-equilibrium description of thermodynamics in $f(R)$ gravity in the Palatini formalism

2.1 $f(R)$ gravity in the Palatini formalism

The action of $f(R)$ gravity in the Palatini formalism with matter is written as

$$I = \int d^4x \sqrt{-g} \left[ \frac{f(R)}{2\kappa^2} + L_{\text{matter}} \right], \quad (2.1)$$

where $g$ is the determinant of the metric tensor $g_{\mu\nu}$ and $L_{\text{matter}}$ is the matter Lagrangian. In the Palatini formalism, the connection and the metric tensor $g_{\mu\nu}$ are treated as independent variables. We denote the Ricci tensor constructed with this independent connection as $\mathcal{R}_{\mu\nu}$ and the corresponding Ricci scalar as $\mathcal{R} = g^{\mu\nu} \mathcal{R}_{\mu\nu}$. In general, $\mathcal{R}_{\mu\nu}$ is different from the Ricci tensor $R_{\mu\nu}$ constructed with the Levi-Civita connection of the metric. Here, the Ricci scalar $R$ is given by $R = g^{\mu\nu} R_{\mu\nu}$ and $f(\mathcal{R})$ is an arbitrary function of $\mathcal{R}$.

Taking the variation of the action in eq. (2.1) with respect to $g_{\mu\nu}$, one obtains [10]

$$F(\mathcal{R}) \mathcal{R}_{\mu\nu} - \frac{1}{2} f(\mathcal{R}) g_{\mu\nu} = \kappa^2 T_{\mu\nu}^{(\text{matter})}, \quad (2.2)$$

where $F(\mathcal{R}) \equiv df(\mathcal{R})/d\mathcal{R}$. Here, $T_{\mu\nu}^{(\text{matter})}$ is the contribution to the energy-momentum tensor from all perfect fluids of ordinary matter (radiation and non-relativistic matter) with $\rho_f$ and $P_f$ being the energy density and pressure of all ordinary matters, respectively.

Varying the action in eq. (2.1) with respect to the connection and using eq. (2.2), one finds [10]

$$F G_{\mu\nu} = \kappa^2 T_{\mu\nu}^{(\text{matter})} - \frac{1}{2} g_{\mu\nu} (FR - f) + \nabla_\mu \nabla_\nu F - g_{\mu\nu} \Box F - \frac{3}{2F} \left[ \nabla_\mu F \nabla_\nu F - \frac{1}{2} g_{\mu\nu} (\nabla F)^2 \right], \quad (2.3)$$

where $G_{\mu\nu} = R_{\mu\nu} - (1/2) g_{\mu\nu} R$ is the Einstein tensor, $\nabla_\mu$ is the covariant derivative operator associated with the Levi-Civita connection of the metric tensor $g_{\mu\nu}$, $\Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$ is the covariant d’Alembertian for a scalar field, and $(\nabla F)^2 = g^{\mu\nu} \nabla_\mu F \nabla_\nu F$.

We assume the four-dimensional Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime with the metric,

$$ds^2 = h_{\alpha\beta} dx^\alpha dx^\beta + r^2 d\Omega^2, \quad (2.4)$$
where $\tilde{r} = a(t)r$, $x^0 = t$ and $x^1 = r$ with the two dimensional metric $h_{\alpha\beta} = \text{diag}(-1, a^2(t)/(1 - Kr^2))$. Here, $a(t)$ is the scale factor, $K$ is the cosmic curvature, and $d\Omega^2$ is the metric of two-dimensional sphere with unit radius. In the FLR W background (2.4), from eq. (2.3) we obtain the following gravitational field equations:

$$3F \left( H^2 + \frac{K}{a^2} - \frac{\kappa^2}{3F} \rho_t + \frac{1}{2} (FR - f) - 3H\dot{F} - \frac{3}{4} \dot{F}^2 \right),$$

(2.5)

$$-2F \left( \dot{H} - \frac{K}{a^2} - \frac{\kappa^2}{2F} (\rho_t + P_t) + \dot{F} - H\dot{F} - \frac{3}{2F} \dot{F}^2 \right),$$

(2.6)

where $\dot{H} = \dot{a}/a$ is the Hubble parameter and the dot denotes the time derivative of $\partial/\partial t$. We note that the perfect fluid satisfies the continuity equation

$$\dot{\rho}_t + 3H (\rho_t + P_t) = 0.$$  

(2.7)

Equations (2.5) and (2.6) can be rewritten as

$$H^2 + \frac{K}{a^2} = \frac{\kappa^2}{3F} (\dot{\rho}_d + \rho_t),$$

(2.8)

$$\dot{H} - \frac{K}{a^2} = -\frac{\kappa^2}{2F} (\dot{\rho}_d + \dot{P}_d + \rho_t + P_t),$$

(2.9)

where $\dot{\rho}_d$ and $\dot{P}_d$ are the energy density and pressure of “dark” components, given by

$$\dot{\rho}_d \equiv \frac{1}{\kappa^2} \left[ \frac{1}{2} (FR - f) - 3H\dot{F} - \frac{3}{4} \dot{F}^2 \right],$$

(2.10)

$$\dot{P}_d \equiv \frac{1}{\kappa^2} \left[ -\frac{1}{2} (FR - f) + \dot{F} + 2H\dot{F} - \frac{3}{4} \dot{F}^2 \right].$$

(2.11)

Here, a hat denotes quantities in the non-equilibrium description of thermodynamics. Note that $\dot{\rho}_d$ and $\dot{P}_d$ are originated from the energy-momentum tensor $\hat{T}^{(d)}_{\mu\nu}$, defined by

$$\hat{T}^{(d)}_{\mu\nu} \equiv \frac{1}{\kappa^2} \left\{ \frac{1}{2} g_{\mu\nu} (FR - f) + \nabla_{\mu} \nabla_{\nu} F - g_{\mu\nu} \Box F - \frac{3}{2F} \left[ \nabla_{\mu} F \nabla_{\nu} F - \frac{1}{2} g_{\mu\nu} (\nabla F)^2 \right] \right\},$$

(2.12)

where the Einstein equation is given by

$$G_{\mu\nu} = \frac{\kappa^2}{F} \left( T^{(\text{matter})}_{\mu\nu} + \hat{T}^{(d)}_{\mu\nu} \right).$$

(2.13)

Relation between $\mathcal{R}_{\mu\nu}$ and $R_{\mu\nu}$ as well as $\mathcal{R}$ and $R$ are given by [10]

$$\mathcal{R}_{\mu\nu} = R_{\mu\nu} + \frac{3}{2F^2} \nabla_{\mu} F \nabla_{\nu} F - \frac{1}{F} \left( \nabla_{\mu} \nabla_{\nu} F + \frac{1}{2} g_{\mu\nu} \Box F \right),$$

(2.14)

$$\mathcal{R} = R + \frac{3}{2F^2} (\nabla F)^2 - \frac{3}{F} \Box F.$$

(2.15)

In the FLRW background (2.4), the scalar curvature in the metric formalism is written as

$$R = 6 \left( 2H^2 + H + K/a^2 \right).$$

Using this expression and eq. (2.15), we get

$$\mathcal{R} = 6 \left( 2H^2 + \dot{H} + \frac{K}{a^2} \right) + \frac{3}{F} \left( -\frac{1}{2F} + \dot{F} + 3H\dot{F} \right).$$

(2.16)
From eqs. (2.10), (2.11) and (2.16), we find that \( \dot{\rho}_d \) and \( \dot{P}_d \) satisfy the following equation:

\[
\dot{\rho}_d + 3H \left( \dot{\rho}_d + \dot{P}_d \right) = \frac{3}{\kappa^2} \left( H^2 + \frac{K}{a^2} \right) \dot{F}.
\]

(2.17)

In \( f(R) \) gravity, since \( \dot{F} \neq 0 \), the right-hand side (r.h.s.) of eq. (2.17) does not vanish, so that the standard continuity equation does not hold.

### 2.2 First law of thermodynamics

We study the thermodynamic property of \( f(R) \) gravity. The dynamical apparent horizon is determined by the relation \( h^{\alpha\beta} \partial_\alpha \tilde{r} \partial_\beta \tilde{r} = 0 \). According to the recent type Ia Supernovae data, it is suggested that in the accelerating universe the enveloping surface should be the apparent horizon rather than the event one from the thermodynamic point of view [40]. In the FLRW spacetime, the radius \( \tilde{r}_A \) of the apparent horizon is given by

\[
\tilde{r}_A = \left( H^2 + \frac{K}{a^2} \right)^{-1/2}.
\]

(2.18)

Taking the time derivative of this relation, we obtain

\[
- \frac{d\tilde{r}_A}{\tilde{r}_A^2} = \left( \dot{H} - \frac{K}{a^2} \right) H dt.
\]

(2.19)

Combining eqs. (2.9) and (2.10), we find

\[
\frac{F}{4\pi G} d\tilde{r}_A = \tilde{r}_A^3 H \left( \dot{\rho}_d + \dot{P}_d + \rho_t + P_t \right) dt.
\]

(2.20)

In general relativity, the Bekenstein-Hawking horizon entropy is expressed as \( S = A/(4G) \), where \( A = 4\pi \tilde{r}_A^2 \) is the area of the apparent horizon [20–22]. The Bekenstein-Hawking entropy \( S = A/(4G) \) is a global geometric quantity which is proportional to the horizon area \( A \) with a constant coefficient \( 1/(4G) \). This quantity is not directly affected by the difference of gravitational theories, i.e., the difference of the derivative of the Lagrangian density \( f \) with respect to \( R, F \). On the other hand, in the context of modified gravity theories including \( f(R) \) gravity a horizon entropy \( \dot{S} \) associated with a Noether charge has been proposed by Wald [93]. The Wald entropy \( \dot{S} \) is a local quantity defined in terms of quantities on the bifurcate Killing horizon. More specifically, it depends on the variation of the Lagrangian density of gravitational theories with respect to the Riemann tensor. This is equivalent to \( \dot{S} = A/(4G_{\text{eff}}) \), where \( G_{\text{eff}} = G/F \) is the effective gravitational coupling in \( f(R) \) gravity [94].

By using eq. (2.20) and the Wald entropy in \( f(R) \) gravity in the Palatini formalism [95]

\[
\dot{S} = \frac{FA}{4G},
\]

(2.21)

we get

\[
\frac{1}{2\pi \tilde{r}_A} d\tilde{S} = 4\pi \tilde{r}_A^3 H \left( \dot{\rho}_d + \dot{P}_d + \rho_t + P_t \right) dt + \frac{\tilde{r}_A}{2G} dF.
\]

(2.22)

We note that the Wald entropy in \( f(R) \) gravity in the metric formalism has the same form as in eq. (2.21) [93, 96, 97].
The associated temperature of the apparent horizon has the following Hawking temperature $T$ determined through the surface gravity $\kappa_{sg}$:

$$T = \frac{|\kappa_{sg}|}{2\pi},$$  \hspace{1cm} (2.23)

$$\kappa_{sg} = \frac{1}{2\sqrt{-h}}\partial_{\alpha} \left( \sqrt{-h} h^{\alpha\beta} \partial_{\beta} \hat{r} \right),$$  \hspace{1cm} (2.24)

where $h$ is the determinant of the metric $h_{\alpha\beta}$ and $\kappa_{sg}$ is given by [29]

$$\kappa_{sg} = -\frac{1}{r_A} \left( 1 - \frac{\dot{r}_A}{2H\hat{r}_A} \right) = -\frac{\dot{r}_A}{2} \left( 2H^2 + \dot{H} + \frac{K}{a^2} \right)$$

$$= -\frac{2\pi G}{3F} \hat{r}_A \left( \dot{\rho}_T - 3\dot{P}_T \right)$$  \hspace{1cm} (2.25)

with $\dot{\rho}_t \equiv \dot{\rho}_d + \rho_I$ and $\dot{P}_t \equiv \dot{P}_d + P_I$, denoted as the total energy density and pressure of the universe, respectively. It follows from eq. (2.25) that if the total equation of state (EoS) $w_t = \dot{P}_t/\dot{\rho}_t$ satisfies $w_t \leq 1/3$, one has $\kappa_{sg} \leq 0$, which is the case for the standard cosmology. Using eqs. (2.23) and (2.25), the horizon temperature is given by

$$T = \frac{1}{2\pi \hat{r}_A} \left( 1 - \frac{\dot{r}_A}{2H\hat{r}_A} \right).$$  \hspace{1cm} (2.26)

Multiplying the term $1 - \dot{r}_A/(2H\hat{r}_A)$ for eq. (2.22), we obtain

$$Td\hat{S} = 4\pi \tilde{r}_A^2 H \left( \dot{\rho}_d + \dot{\hat{P}}_d + \rho_I + P_I \right) dt - 2\pi \tilde{r}_A^2 \left( \dot{\rho}_d + \dot{\hat{P}}_d + \rho_I + P_I \right) d\hat{r}_A + \frac{T}{G} \pi \tilde{r}_A^2 dF. \hspace{1cm} (2.27)$$

In general relativity, the Misner-Sharp energy [98, 99] is defined as $E = \tilde{r}_A/(2G)$. In $f(R)$ gravity, this may be extended to the form [69, 70, 73] (for related works, see also refs. [100, 101])

$$\hat{E} = \frac{\tilde{r}_A F}{2G}. \hspace{1cm} (2.28)$$

From eqs. (2.18) and (2.28), we find

$$\hat{E} = \frac{\tilde{r}_A F}{2G} = V \frac{3F \left( H^2 + \frac{K}{a^2} \right)}{8\pi G} = V \left( \dot{\rho}_d + \rho_I \right), \hspace{1cm} (2.29)$$

where $V = 4\pi \tilde{r}_A^3/3$ is the volume inside the apparent horizon. The last equality in eq. (2.29) means that $\hat{E}$ corresponds to the total intrinsic energy. It is clear from eq. (2.29) that $\hat{E} > 0$ and therefore $F > 0$, which is consistent with the fact that the effective gravitational coupling in $f(R)$ gravity $G_{\text{eff}} = G/F$ should be positive [10]. In other words, the graviton is not a ghost in the sense of quantum theory [62]. Using eqs. (2.7) and (2.17), we get

$$d\hat{E} = -4\pi \tilde{r}_A^2 H \left( \dot{\rho}_d + \dot{\hat{P}}_d + \rho_I + P_I \right) dt + 4\pi \tilde{r}_A^2 \left( \dot{\rho}_d + \rho_I \right) d\hat{r}_A + \frac{\tilde{r}_A}{2G} dF. \hspace{1cm} (2.30)$$

It follows from eqs. (2.27) and (2.30) that

$$Td\hat{S} = d\hat{E} + 2\pi \tilde{r}_A^2 \left( \dot{\rho}_d + \rho_I - \dot{\hat{P}}_d - P_I \right) d\hat{r}_A + \frac{\tilde{r}_A}{2G} \left( 1 + 2\pi \tilde{r}_A T \right) dF. \hspace{1cm} (2.31)$$
By introducing the work density \[32, 33, 67\]
\[
\hat{W} \equiv -\frac{1}{2} \left( \hat{T}^{(\text{matter})\alpha\beta} h_{\alpha\beta} + \hat{T}^{(d)\alpha\beta} h_{\alpha\beta} \right),
\]
\[
= \frac{1}{2} \left( \hat{\rho}_d + \rho_t - \hat{P}_d - P_t \right),
\]
eq (2.32)

eq. (2.31) is reduced to
\[
Td\hat{S} = -d\hat{E} + \hat{W}dV + \frac{\hat{r}_A}{2G} (1 + 2\pi \hat{r}_A T) dF,
\]
which can be rewritten in the form
\[
Td\hat{S} + Td_i\hat{S} = -d\hat{E} + \hat{W}dV,
\]
where
\[
d_i\hat{S} = -\frac{1}{T} \frac{\hat{r}_A}{2G} (1 + 2\pi \hat{r}_A T) dF = -\left( \frac{\hat{E} + \hat{S}}{T} \right) \frac{dF}{F},
\]
\[
= -\frac{\pi}{G} \frac{4H^2 + \dot{H} + 3K/a^2}{(H^2 + K/a^2) \left( 2H^2 + \dot{H} + K/a^2 \right)} dF.
\]
\[
= (2.35)
\]
Equation (2.36) agrees with the result of ref. \[70\] for \(K = 0\) obtained in \(f(R)\) gravity.

The new term \(d_i\hat{S}\) can be interpreted as a term of entropy produced in the non-equilibrium thermodynamics. General relativity with \(F = \text{constant}\) leads to \(d_i\hat{S} = 0\), which implies that the first-law of equilibrium thermodynamics holds. On the other hand, in \(f(R)\) gravity the additional term in eq. (2.36) appears because \(dF \neq 0\).

### 2.3 Second law of thermodynamics

To investigate the second law of thermodynamics in \(f(R)\) gravity, we start with the Gibbs equation in terms of all matter and energy fluid, given by
\[
T d\hat{S}_t = d(\hat{\rho}_t V) + \hat{P}_t dV = V d\hat{\rho}_t + \left( \hat{\rho}_t + \hat{P}_t \right) dV,
\]
where \(T\) and \(\hat{S}_t\) denote the temperature and entropy of total energy inside the horizon, respectively. Here, we have assumed the same temperature between the outside and inside of the apparent horizon. To obey the second law of thermodynamics in \(f(R)\) gravity, we require that \[70\]
\[
\frac{d\hat{S}_t}{dt} + \frac{d}{dt} \frac{d_i\hat{S}_t}{dt} + \frac{d\hat{S}_t}{dt} \geq 0,
\]
which leads to the condition \[71\]
\[
J \equiv \left( \frac{\dot{H} - K}{a^2} \right)^2 F \geq 0.
\]
\[
= (2.38)
\]

In the FLRW background (2.4), the effective EoS \(w_{\text{eff}}\) is given by \[8\]
\[
w_{\text{eff}} = -1 - 2\dot{H} / (3H^2).
\]
For \(H < 0\), \(w_{\text{eff}} > -1\), which corresponds to the non-phantom (quintessence) phase, while for \(H > 0\), \(w_{\text{eff}} < -1\), which corresponds to the phantom phase. It follows from eq. (2.39) that \(J \geq 0\) even in the phantom phase. As a consequence, the second law of thermodynamics in \(f(R)\) gravity can be satisfied in both phantom and non-phantom phases. This consequence is compatible with a phantom model with ordinary thermodynamics proposed in ref. \[102\].

\[
= (2.39)
\]
3 Equilibrium description of thermodynamics in $f(R)$ gravity in the Palatini formalism

The reason why there exists a non-equilibrium entropy production term $d_i\delta S$ is that $\dot{\rho}_d$ and $\dot{P}_d$ defined in eqs. (2.10) and (2.11) obey eq. (2.17), whose r.h.s. does not vanish in $f(R)$ gravity because $\dot{F} \neq 0$. In other words, the standard continuity equation does not hold. In this section, we redefine the energy density and pressure of dark components to satisfy the continuity equation so that there is no extra entropy production term, which is referred as the equilibrium description. We will discuss such equilibrium description of thermodynamics in $f(R)$ gravity in the Palatini formalism.

3.1 First law of thermodynamics in equilibrium description

We rewrite eqs. (2.5) and (2.6) in the following forms:

\[
3F_0 \left( H^2 + \frac{K}{a^2} \right) = \kappa^2 (\rho_d + \rho_t), \tag{3.1}
\]

\[
-2F_0 \left( \dot{H} - \frac{K}{a^2} \right) = \kappa^2 (\rho_d + P_d + \rho_t + P_t), \tag{3.2}
\]

where $F_0$ is some constant, and $\rho_d$ and $P_d$ are the energy density and pressure of dark components redefined as

\[
\rho_d = \frac{1}{\kappa^2} \left[ \frac{1}{2} (FR - f) - 3H \dot{F} + 3(F_0 - F) \left( H^2 + \frac{K}{a^2} \right) - \frac{3}{4} \dot{F}^2 \right], \tag{3.3}
\]

\[
P_d = \frac{1}{\kappa^2} \left[ -\frac{1}{2} (FR - f) + \ddot{F} + 2H \dot{F} - (F_0 - F) \left( 2\dot{H} + 3H^2 + \frac{K}{a^2} \right) - \frac{3}{4} \dot{F}^2 \right]. \tag{3.4}
\]

It follows from eqs. (3.3) and (3.4) that the standard continuity equation can be satisfied

\[
\dot{\rho}_d + 3H (\rho_d + P_d) = 0. \tag{3.5}
\]

In this representation, eq. (2.20) becomes

\[
\frac{F_0}{4\pi G} d\tilde{r}_A = \tilde{r}_A^3 H (\rho_d + P_d + \rho_t + P_t) dt. \tag{3.6}
\]

By introducing the horizon entropy $S$ in the form

\[
S = \frac{F_0 A}{4G}, \tag{3.7}
\]

and using eq. (3.6), we obtain

\[
\frac{1}{2\pi \tilde{r}_A} dS = 4\pi \tilde{r}_A^3 H (\rho_d + P_d + \rho_t + P_t) dt. \tag{3.8}
\]

From the horizon temperature in eq. (2.26) and eq. (3.8), we find

\[
TdS = 4\pi \tilde{r}_A^3 H (\rho_d + P_d + \rho_t + P_t) dt - 2\pi \tilde{r}_A^2 (\rho_d + P_d + \rho_t + P_t) d\tilde{r}_A. \tag{3.9}
\]
By defining the Misner-Sharp energy as
\[ E = \frac{F_0 \tilde{r}_A}{2G} = V (\rho_d + \rho_t), \] (3.10)
we get
\[ dE = -4\pi \tilde{r}_A^2 H (\rho_d + P_d + \rho_t + P_t) dt + 4\pi \tilde{r}_A^2 (\rho_d + \rho_t) d\tilde{r}_A, \] (3.11)
where there does not exist any additional term proportional to \( dF \) on the r.h.s. due to the continuity equation (3.5). By combining eqs. (3.9) and (3.11), we obtain the following equation corresponding to the first law of equilibrium thermodynamics:
\[ T dS = -dE + W dV, \] (3.12)
where the work density \( W \) is defined by
\[ W = \frac{1}{2} (\rho_d + \rho_t - P_d - P_l). \] (3.13)
As a result, an equilibrium description of thermodynamics can be derived by redefining the energy density \( \rho_d \) and the pressure \( P_d \) so that the continuity equation (3.5) can be met.

Note that the constant \( F_0 \) can be chosen arbitrarily as long as \( F_0 > 0 \) in order to ensure that the sign of the Friedmann equation (3.1) does not change. It is considered that the natural choice is \( F_0 = 1 \) because in this case the entropy \( S \) and the Misner-Sharp energy \( E \) reduce to the standard forms in the Einstein gravity: \( S = A/(4G) \) and \( E = \tilde{r}_A/(2G) \), respectively.

From eqs. (3.1), (3.2) and (3.8), we obtain
\[ \dot{S} = 8\pi^2 H \tilde{r}_A^4 (\rho_d + \rho_t + P_d + P_l) = -\frac{2\pi F_0}{G} \frac{H (H - K/a^2)}{(H^2 + K/a^2)^2}. \] (3.14)
Thus, the horizon entropy increases as long as the null energy condition \( \rho_t + P_l \equiv \rho_d + \rho_t + P_d + P_l \geq 0 \) is satisfied.

The reasons why the above equilibrium picture of thermodynamics can be realized are as follows: The first is that there exists an energy momentum tensor \( T_{\mu\nu}^{(d)} \) satisfying the local conservation law \( \nabla^\mu T_{\mu\nu}^{(d)} = 0 \). The second is that the entropy \( S \) is given by eq. (3.7) as in general relativity, corresponding to the Einstein equation in the form
\[ G_{\mu\nu} = \frac{8\pi G}{F_0} \left( T_{\mu\nu}^{(\text{matter})} + T_{\mu\nu}^{(d)} \right), \] (3.15)
where
\[ T_{\mu\nu}^{(d)} = \frac{1}{\kappa^2} \left\{ -\frac{1}{2} g_{\mu\nu} [F (R - R) + F_0 R - f] + \nabla_\mu \nabla_\nu F - g_{\mu\nu} \Box F + (F_0 - F) R_{\mu\nu} - \frac{3}{2F} \left[ \nabla_\mu F \nabla_\nu F - \frac{1}{2} g_{\mu\nu} (\nabla F)^2 \right]\right\}. \] (3.16)
It is clear that the local conservation of \( T_{\mu\nu}^{(d)} \) follows from eq. (3.15) due to the relations \( \nabla^\mu G_{\mu\nu} = 0 \) and \( \nabla^\mu T_{\mu\nu}^{(\text{matter})} = 0 \). By using the conserved energy momentum tensor \( T_{\mu\nu}^{(d)} \) and
the horizon entropy \( S \) defined in eq. (3.7) and following the method in ref. [25], the Einstein equation (3.15) can also be derived.

It can be shown that the horizon entropy \( S \) in the equilibrium description has the following relation with \( \hat{S} \) in the non-equilibrium description [80]:

\[
dS = d\hat{S} + d_i \hat{S} + \frac{\tilde{r}_A}{2GT} dF - \frac{2\pi (F_0 - F)}{G} \frac{H (\dot{H} - K/a^2)}{(H^2 + K/a^2)^2} dt. \tag{3.17}
\]

By using the relations (2.36) and (3.8), eq. (3.17) is rewritten to the following form:

\[
dS = \frac{F_0}{F} d\hat{S} + \frac{F_0}{F} \frac{2H^2 + H + K/a^2}{4H^2 + H + 3K/a^2} d_i \hat{S}, \tag{3.18}
\]

where

\[
d_i \hat{S} = -\frac{6\pi}{G} \frac{4H^2 + \dot{H} + 3K/a^2}{H^2 + K/a^2} dF. \tag{3.19}
\]

The difference between \( S \) and \( \hat{S} \) appears in \( f(R) \) gravity due to \( dF \neq 0 \), although \( S \) is identical to \( \hat{S} \) in general relativity \((F = F_0 = 1)\). From eq. (3.18), we see that the change of the horizon entropy \( S \) in the equilibrium framework involves the information of both \( d\hat{S} \) and \( d_i \hat{S} \) in the non-equilibrium framework.

In the flat FLRW spacetime \((K = 0)\), the Bekenstein-Hawking entropy is simply proportional to the inverse squared of the expansion rate of the universe \((S \propto H^{-2})\) independent of gravitational theories. Hence \( S \) grows as long as \( H \) decreases, whereas the increase of \( H \) leads to the decrease of \( S \) (as in the case of superinflation). In other words, this property mimics the standard general relativistic picture with energy density \( \rho_\text{d} \) and pressure \( P_\text{d} \) of dark components defined in eqs. (3.3) and (3.4). It can be clearly understood that the Bekenstein-Hawking entropy grows for \( w_\text{d} = P_\text{d}/\rho_\text{d} > -1 \), where \( \rho_\text{d} \equiv \rho_\text{d} + \rho_\text{t} \) and \( P_\text{d} \equiv P_\text{d} + P_\text{t} \), and that it decreases for \( w_\text{d} < -1 \).

The Wald entropy carries the information of gravitational theories through the dependence \( \hat{S} \propto F H^{-2} \). For example, in a model of \( f(R) \) gravity \( f(R) = R + \alpha R^n \), where \( \alpha \) and \( n \) are constants, it follows that \( \hat{S} \propto H^{2(n-2)} \) and hence \( \hat{S} \) grows apart from \( n = 2 \) because \( H \) increases (decreases) for \( n > 2 \) (\( n < 2 \)). This evolution is different from that of the Bekenstein-Hawking entropy. The introduction of the entropy production term \( d_i \hat{S} \) in the non-equilibrium framework allows us to have a connection with the equilibrium picture based on the Bekenstein-Hawking entropy as given in eq. (3.17). Thus, it is considered that our equilibrium description of thermodynamics is useful not only to provide the general relativistic analogue of the horizon entropy irrespective of gravitational theories but also to understand the non-equilibrium thermodynamics deeper in connection with the standard equilibrium framework.

Finally, we remark that the EoS of dark components in the non-equilibrium description is different from that in the equilibrium description. It follows from eqs. (2.10) and (2.11) that the EoS of dark components in the non-equilibrium description \( \tilde{w}_\text{d} = \tilde{P}_\text{d}/\tilde{\rho}_\text{d} \) is given by

\[
\tilde{w}_\text{d} = -\frac{2 (F R - f) + 4 \dot{F} + 8H \dot{F} - 3 \dot{F}^2 / F}{2 (F R - f) - 12H \dot{F} - 3 \dot{F}^2 / F}. \tag{3.20}
\]
On the other hand, by using eqs. (3.3) and (3.4), the EoS of dark components in the equilibrium description \( w_d = P_d/\rho_d \) is expressed as

\[
w_d = \frac{-2 (FR - f) + 4F + 8H\dot{F} - 4(F_0 - F) \left(2\dot{H} + 3H^2 + K/a^2\right) - 3\dot{F}^2/F}{2 (FR - f) - 12H\dot{F} + 12 (F_0 - F) (H^2 + K/a^2) - 3F^2/F}.
\]

(3.21)

It is clear from eqs. (3.20) and (3.21) that in general \( \dot{w}_d \neq w_d \) except general relativity with \( F = F_0 = \text{constant} \). Thus, in order to compare the EoS of dark components predicted by a theory of \( f(R) \) gravity with the observations, it is necessary to derive the expression of the EoS of dark components in both non-equilibrium description and equilibrium one. This is another important physical motivation to consider both non-equilibrium and equilibrium descriptions.

### 3.2 Second law of thermodynamics in equilibrium description

To examine the second law of thermodynamics in the equilibrium description, we write the Gibbs equation in terms of all matter and energy fluid as

\[
TdS_t = d(\rho_t V) + P_t dV = V d\rho_t + (\rho_t + P_t) dV.
\]

(3.22)

The second law of thermodynamics can be described by

\[
\frac{dS}{dt} + \frac{dS_t}{dt} \geq 0,
\]

(3.23)

which gives

\[
\frac{12\pi F_0}{G} \frac{H \left(\dot{H} - K/a^2\right)^2}{(H^2 + K/a^2)^2} \frac{1}{R} \geq 0
\]

(3.24)

by using \( V = 4\pi \tilde{r}_A^3/3 \), \( R = 6 \left(\dot{H} + 2H^2 + K/a^2\right) \), and eqs. (2.26), (3.2) and (3.14). It is clear that in the flat FLRW spacetime \( (K = 0) \), the second law of thermodynamics can be met in both non-phantom and phantom phases, which is the same as the non-equilibrium description. This also agrees with the argument in refs. [48, 51]. We remark that the above consequence can be shown explicitly only for the same temperature of the universe outside and inside the apparent horizon. As a result, a unified understanding between non-equilibrium and equilibrium pictures of thermodynamics has been obtained.

### 4 Conclusion

In the present paper, we have studied the first and second laws of thermodynamics of the apparent horizon in \( f(R) \) gravity in the Palatini formalism. We have explored both non-equilibrium and equilibrium descriptions of thermodynamics in \( f(R) \) gravity. The equilibrium framework is more transparent than the non-equilibrium one because in the equilibrium description the horizon entropy is described by the single expression \( S \), whereas in the non-equilibrium description it consists of two contributions, the change of the horizon entropy \( d\hat{S} \) and the term produced in the non-equilibrium thermodynamics \( d_i\hat{S} \). We have also shown that the second law of thermodynamics can be satisfied in not only the non-phantom phase but also the phantom one, provided that the temperature of the universe inside the horizon is equal to that of the apparent horizon.
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