Noncommutative $SO(2,3)$ gauge theory and noncommutative gravity

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Abstract

In this paper noncommutative gravity is constructed as a gauge theory of the noncommutative $SO(2,3)$ group, while the noncommutativity is canonical (constant). The Seiberg-Witten map is used to express noncommutative fields in terms of the corresponding commutative fields. The commutative limit of the model is the Einstein-Hilbert action with the cosmological constant term and the topological Gauss-Bonnet term. We calculate the second order correction to this model and obtain terms that are of zeroth to fourth power in the curvature tensor and torsion. Trying to relate our results with $f(R)$ and $f(T)$ models, we analyze different limits of our model. In the limit of big cosmological constant and vanishing torsion we obtain a $x$-dependent correction to the cosmological constant, i.e. noncommutativity leads to a $x$-dependent cosmological constant. We also discuss the limit of small cosmological constant and vanishing torsion and the teleparallel limit.

Keywords: gauge theory of gravity, Seiberg-Witten map, expansion in powers of curvature

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1 Introduction

General Relativity (GR) is widely accepted as a classical (low energy/large scale) description of the geometric properties of space-time and is experimentally very well tested. However, the rapid development of observational cosmology during the last 20 years has led to data that cannot be explained by GR only. The most important of these are the two phases of acceleration: inflation in the very early Universe and the accelerated expansion of the Universe today. There are various attempts to explain these two phases: cosmological constant, scalar field $\phi$, $f(R)$ and $f(T)$ theories and some other modifications of GR.

In addition to these problems, no consistent (renormalizable) quantum field theory of gravity has been constructed yet. Some candidates for a quantum gravity are string theory, quantum loop gravity, dynamical triangularization, . . . . Combining problems of divergences in quantum field theory (QFT) and singularities in GR leads to discretized geometry [1]. Motivated by quantum mechanics and Heisenberg uncertainty relations, noncommutative (NC) spaces can be defined [2]. Then the nonzero commutation relations between coordinates lead to discretization of space-time. Unfortunately, it is not yet clear how to formulate a gravity theory on NC spaces (NC gravity) and there are various proposals in the literature. One can follow the twist approach in which the commutative diffeomorphisms are replaced by the twisted diffeomorphisms [3]. However, a full understanding of the twisted symmetries is still missing. Having in mind that the NC gauge theories can be consistently defined, many authors consider NC gravity as a gauge theory of the Lorentz/Poincaré group. These approaches are based on hermitian metrics [4] or vielbeins [5, 6, 7]. One can also construct emergent gravity from noncommutative gauge theory and matrix models, see [8]. Finally, there is the approach of NC differential geometry and frame formalism [9].

Recently, a lot of attention has been given to the anti de Sitter (AdS) gauge theory and to its application to GR [10], quantization of gravity [11], AdS/CFT correspondence and its applications [12]. In our previous paper [13], we begun the study of noncommutative (NC) gravity based on the AdS gauge group. We started with the MacDowell-Mansouri action on the commutative space-time and generalized it to the NC MacDowell-Mansouri action on the canonically deformed space. One of the drawbacks of our approach was that we had to assume from the beginning that in the commutative limit torsion vanishes. The other disadvantage was that we introduced noncommutativity "in the middle": the symmetry breaking from $SO(2, 3)$ to $SO(1, 3)$ was performed in the commutative model. The obtained $SO(1, 3)$ invariant action was a basis for a noncommutative gravity action. Using the enveloping algebra approach and the Seiberg-Witten (SW) map [14, 15] we constructed the NC gravity action invariant under the NC $SO(1, 3)$, gauge symmetry. The deformation of theory has not been introduced from the very beginning, mostly for technical reasons: complicated calculations and gauge non-invariant expressions.

Nevertheless, it is of importance to have more general and more systematic results. Therefore, in this paper we analyze the full NC $SO(2, 3)$, gauge theory and perform symmetry breaking after introducing the noncommutative deformation. The NC space-
time we work with is the canonically deformed, with the Moyal-Weyl $\star$-product given by
\[ f(x) \star g(x) = e^{\frac{i}{2} \theta_{\alpha\beta} \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial y^\beta} f(y) |_{y \to x}}. \tag{1.1} \]
Here $\theta^\mu\nu$ is a constant antisymmetric matrix which is considered to be a small deformation parameter. Indices $\mu, \nu$ take values 0, 1, 2, 3 and the four dimensional Minkowski metric is $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$.

In the next section we shortly describe the commutative $SO(2,3)$ gravity theory as given in the literature and adapted to our notation. In Section 3 the NC $SO(2,3)_\star$ gauge theory is constructed using the enveloping algebra approach and the SW map. We then expand the NC action to the second order in the deformation parameter $\theta^{\alpha\beta}$ and calculate correction terms to the commutative action. We obtain that the first order corrections vanish, thus we confirm the results already present in the literature. The second order correction is calculated using the method of composite fields developed in [16]. The correction terms we obtain are of zeroth to fourth power in the curvature tensor and torsion. They are written in a manifestly covariant way. However, the full result is very cumbersome and it is difficult to discuss its physical implications. Fortunately, having three different scales in the model enables us to discuss different limits of our model. To be able to compare our results with $f(R)$ models present in the literature, in Section 4 we analyze the limit of big cosmological constant and vanishing torsion and the limit of small cosmological constant and vanishing torsion. In the limit of big cosmological constant we obtain a $x$-dependent correction for the cosmological constant and we analyze possible modifications of the zeroth order solution of vacuum Einstein equations. Finally, we discuss the teleparallel limit, the limit in which curvature vanishes and torsion is different from zero. Again we try to compare our results with the existing results for $f(T)$ theories.

2 Commutative gravity as AdS gauge theory

In order to establish the notation, in this section we briefly review the AdS gauge theory on four-dimensional Minkowski space-time. More details about this construction can be found in [13].

We assume that space-time has the structure of four-dimensional Minkowski space-time $M_4$ and follow the usual steps for constructing a gauge theory on $M_4$ taking the $SO(2,3)$ group as the gauge group. The gauge field is $SO(2,3)$-valued
\[ \omega_\mu = \frac{1}{2} \omega^{AB}_\mu M_{AB} , \tag{2.2} \]
with the generators of the $SO(2,3)$ group denoted by $M_{AB}$. The algebra is given by
\[ [M_{AB}, M_{CD}] = i(\eta_{AD}M_{BC} + \eta_{BC}M_{AD} - \eta_{AC}M_{BD} - \eta_{BD}M_{AC}) . \tag{2.3} \]
The 5D metric is $\eta_{AB} = \text{diag}(+,-,-,-,+)$. Indices $A,B,\ldots$ take values 0, 1, 2, 3, 5, while indices $a,b,\ldots$ take values 0, 1, 2, 3. A representation of this algebra is given by
\[ M_{ab} = \frac{i}{4} [\gamma_a, \gamma_b] = \frac{1}{2} \sigma_{ab} , \]
\[ M_{5a} = \frac{1}{2} \gamma_a , \] (2.4)

where \( \gamma_a \) are four dimensional Dirac gamma matrices. Then the gauge potential \( \omega^{AB}_\mu \) decomposes into \( \omega^{ab}_\mu \) and \( \omega^{a5}_\mu = \frac{1}{4} e^a_\mu \)

\[ \omega_\mu = \frac{1}{2} \omega^{AB}_\mu M_{AB} = \frac{1}{4} \omega^{ab}_\mu \sigma_{ab} - \frac{1}{2 l} e^a_\mu \gamma_a . \] (2.5)

The parameter \( l \) has dimension of length, while fields \( e^a_\mu \) are dimensionless. The meaning of the parameter \( l \) will be clear at the end of this section. Under the infinitesimal gauge transformations the gauge potential transforms as

\[ \delta \omega_\mu = \partial_\mu \epsilon - i [\omega_\mu, \epsilon] , \] (2.6)

with the gauge parameter denoted by \( \epsilon = \frac{1}{4} e^{AB} M_{AB} \). The field strength tensor is defined in the standard way as

\[ F_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu - i [\omega_\mu, \omega_\nu] = \frac{1}{2} F^{AB}_{\mu\nu} M_{AB} . \] (2.7)

Just like the gauge potential, the components of the field strength tensor \( F^{AB}_{\mu\nu} \) decompose into \( F^{ab}_{\mu\nu} \) and \( F^{a5}_{\mu\nu} \). It is easy to show that

\[ F_{\mu\nu} = \left( R^{ab}_{\mu\nu} - \frac{1}{l^2} (e^a_\mu \epsilon_\nu - e^b_\mu \epsilon_\nu) \right) \frac{\sigma_{ab}}{2} - F^{a5}_{\mu\nu} \gamma_a , \] (2.8)

where

\[ R^{ab}_{\mu\nu} = \partial_\mu \omega^{ab}_\nu - \partial_\nu \omega^{ab}_\mu + \omega^{ac}_\mu \omega^{cb}_\nu - \omega^{bc}_\mu \omega^{ca}_\nu , \] (2.9)

\[ l F^{a5}_{\mu\nu} = D_\mu e^a_\nu - D_\nu e^a_\mu = T^a_{\mu\nu} . \] (2.10)

Under the infinitesimal gauge transformation the field strength transforms covariantly

\[ \delta \epsilon F_{\mu\nu} = i [\epsilon, F_{\mu\nu}] . \] (2.11)

Equations (2.5), (2.8), (2.9) and (2.10) suggest that one can identify \( \omega^{ab}_\mu \) with the spin connection of the Poincaré gauge theory, \( e^a_\mu \) with the vierbeins, \( R^{ab}_{\mu\nu} \) with the curvature tensor and \( l F^{a5}_{\mu\nu} \) with the torsion.

Indeed, it was shown in the seventies that one can do such identification and relate AdS gauge theory with GR. Different ways were discussed in the literature, see [17, 18, 19]. One way is to start from the action which contains a scalar field

\[ S = \frac{il}{64 \pi G_N} \text{Tr} \int d^4x \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \phi , \] (2.12)

where \( G_N \) is the Newton gravitational constant. The auxiliary field \( \phi = \phi^A \Gamma_A \), \( \Gamma_A = (i \gamma_a \gamma_5, \gamma_5) \) transforms in the adjoint representation of \( SO(2,3) \)

\[ \delta \phi = i [\epsilon, \phi] . \] (2.13)
The action (2.12) is invariant under the $SO(2,3)$ gauge symmetry. However, if we break the symmetry and restrict the field $\phi$ to be $\phi^a = 0$, $\phi^5 = l$ then the symmetry of the action is reduced to the $SO(1,3)$ gauge symmetry. The constraint on the field $\phi$ can be introduced in various ways via a Lagrange multiplier or dynamically [17]. The action obtained after symmetry breaking is then given by

$$S = \frac{-1}{16 \pi G_N} \int d^4x \left[ \frac{l^2}{16} \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} R^a_{\mu\nu} R^b_{\rho\sigma} + eR + 2e\Lambda \right], \quad (2.14)$$

where $\Lambda = -3/l^2$ and $e = \det(e^a_{\mu})$. In the first line we inserted expansions (2.8) and (2.9) and after some standard manipulation with indices and traces we obtained the second line. The action (2.14) appeared for the first time in the paper by MacDowell and Mansouri [18].

This action is written in the first order formalism: the spin connection $\omega_{ab}^\mu$ and the vierbeins $e^a_{\mu}$ are independent fields. The corresponding equations of motion give vanishing torsion and enable to express the spin connection as a function of vierbeins. Inserting the solution for the spin connection in the action (2.14) gives the action in the second order formalism: the only dynamical (propagating) field is the metric $g_{\mu\nu} = \eta_{ab} e^a_{\mu} e^b_{\nu}$. In that case the first term in (2.14) is the Gauss-Bonnet term; it is a topological term and does not contribute to the equations of motion. The second term is the Einstein-Hilbert action, while the last term is the cosmological constant term. Therefore, after symmetry breaking the action (2.12) describes GR with the negative cosmological constant and the topological Gauss-Bonnet term. We see that the parameter $l$ is related with the cosmological constant and the radius of AdS space-time. AdS space is a solution of Einstein vacuum equations obtained from this action.

3 The NC $SO(2,3)_*$ gauge theory

In this section we generalize the model (2.12) to the noncommutative case. We work with the simplest form of noncommutativity, canonical or $\theta$-constant noncommutativity. Following the approach of deformation quantization we represent noncommutative functions as functions of commuting coordinates and algebra multiplication with the Moyal-Weyl $\star$-product (1.1). The noncommutativity (deformation) is encoded in the $\star$-product, while all variables (fields) are functions of commuting coordinates. Integration is well defined since the usual integral is cyclic:

$$\int d^4x (f \star g \star h) = \int d^4x (h \star f \star g) + \text{boundary terms.} \quad (3.15)$$

In general, depending on the behavior of fields at the boundary, these boundary terms can be different from zero. The boundary terms do not influence equations of motion, but might be needed to have a well defined variational principle, that is a well defined
functional derivative of the action\(^1\). For example, in the case of Einstein-Hilbert action the necessary boundary term is the Gibbons-Hawking-York term [20, 21]. In this paper we calculate the NC gravity action up to second order in the deformation parameter and the equations of motion which follow from it. Therefore, in the following we will omit boundary terms. Their analysis in the noncommutative theories is nontrivial. It is a subject for itself and we postpone it for future work.

In particular, from (3.15) we have \(\int d^4x (f \star g) = \int d^4x (g \star f) = \int d^4xfg\). Note that the volume element \(d^4x\) is not \(\star\)-multiplied with the functions under the integral.

### 3.1 The Seiberg-Witten map

In order to construct the NC \(SO(2,3)_\star\) gauge theory we use the enveloping algebra approach and the Seiberg-Witten map developed in [14, 15]. Under the infinitesimal NC \(SO(2,3)_\star\) gauge transformations the NC gauge field \(\hat{\omega}_\mu\) transforms as

\[\delta^\star \hat{\omega}_\mu = \partial_\mu \hat{\Lambda}_\epsilon + i[\hat{\Lambda}_\epsilon \star \hat{\omega}_\mu],\]

with the NC gauge parameter \(\hat{\Lambda}_\epsilon\). In the commutative limit \(\hat{\Lambda}_\epsilon\) reduces to the commutative gauge parameter \(\epsilon = \frac{1}{2} \epsilon^{AB} M_{AB}\). We demand consistency of the NC gauge transformations

\[[\delta^\star_1 \star \delta^\star_2] = \delta^\star_{[\epsilon_1, \epsilon_2]} .\]

This will be the case provided that the gauge parameter \(\hat{\Lambda}_\epsilon\) is in the enveloping algebra of the \(so(2,3)\) algebra\(^2\). However, an enveloping algebra is infinitely dimensional and the resulting theory seems to have infinitely many degrees of freedom. This problem is solved by the Seiberg-Witten map. The idea of the Seiberg-Witten map is that the NC gauge transformations are induced by the corresponding commutative gauge transformations

\[\hat{\omega}_\mu(\omega) + \delta^\star_\epsilon \hat{\omega}_\mu(\omega) = \hat{\omega}_\mu(\omega + \delta_\epsilon \omega),\]

with the commutative gauge field \(\omega_\mu\) and the commutative gauge parameter \(\epsilon\). As a result of this, all noncommutative variables (gauge parameter, fields) can be expressed

\[^1\text{The functional variation of a general action for a field } \phi \text{ can be given by } \delta S = \int d^4x \left(E \delta \phi + \partial_\mu (B_1 \delta \phi) + \partial_\mu (B_2 \delta \partial_\mu \phi)\right)\].

The first term gives the equations of motion \(E = 0\). The term with \(B_1\) is a surface term and it vanishes since the variation of the field \(\phi\) is zero at the boundary. The term with \(B_2\) is again a surface term. However, it does not vanishes, since the variation of \(\partial_\mu \phi\) does not have to be zero at the boundary. To cancel this term, one adds a boundary term to the starting action. Then the new action has a well defined functional derivative. This situation is typical in gravity theories on commutative space-time.

\[^2\text{Note that in (3.16) } \star\text{-commutators appear. These commutators do not close in the Lie algebra. Namely, if } A = A^a T^a \text{ and } B = B^a T^a \text{ then } [A \star B] = \frac{1}{2} (A^a \star B^b + B^b \star A^a)[T^a, T^b] + \frac{1}{2} (A^a \star B^b - B^b \star A^a)(T^a, T^b).\]

Only in the case of \(U(N)\) in the fundamental representation the anticommutator of generators is still in the corresponding Lie algebra.
in terms of the corresponding commutative variables and their derivatives as power series in the noncommutativity parameter $\theta^{\alpha\beta}$.

In the case of NC gauge parameter the expansion is

$$\hat{\Lambda}_\epsilon = \Lambda^{(0)} + \Lambda^{(1)} + \Lambda^{(2)} \ldots .$$  \hspace{1cm} (3.19)

Inserting this expansion into (3.17) and expanding all $\star$-products gives a variational equation for the NC gauge parameter. This equation can be solved to all orders of the deformation parameter. The zeroth order solution is the commutative gauge parameter $\epsilon$, as mentioned earlier. The recursive relation between the $n$th and the $(n+1)$st order solution is given by [22], [23]

$$\hat{\Lambda}^{(n+1)} = \frac{1}{4(n+1)}\theta^{\kappa\lambda}\{\hat{\omega}_\kappa \star \partial_\lambda \hat{\epsilon}\}^{(n)} ,$$  \hspace{1cm} (3.20)

where $(A \star B)^{(n)} = A^{(n)}B^{(0)} + A^{(n-1)}B^{(1)} + \ldots + A^{(0)}B^{(n-1)} + A^{(1)}B^{(n-2)} + \ldots$ includes all possible terms of order $n$. Expanding this recursive relation we obtain

$$\hat{\Lambda} = \epsilon - \frac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha, \partial_\beta \epsilon\} + \mathcal{O}(\theta^2)$$  \hspace{1cm} (3.21)

$$= \frac{1}{2}\Lambda^{AB}M_{AB} + \Lambda^A \Gamma_A + \Lambda I$$

$$= \frac{1}{4}\Lambda^{ab}\sigma_{ab} + \Lambda^a \gamma_a + \tilde{\Lambda}^a \gamma_a\gamma_5 + \tilde{\Lambda} \gamma_5 + \Lambda I .$$  \hspace{1cm} (3.22)

For example, $\Lambda^{AB}(0) = \epsilon^{AB}$, while $\Lambda^{A}(0) = 0$. From the first line it is obvious that $\hat{\Lambda}$ is enveloping algebra valued$^3$.

Solving the equation

$$\hat{\omega}_\mu(\omega) + \delta_\mu \hat{\omega}_\mu(\omega) = \hat{\omega}_\mu(\omega + \delta_\omega)$$  \hspace{1cm} (3.23)

order by order in the NC parameter the noncommutative gauge field $\hat{\omega}_\mu$ is expressed in terms of the commutative gauge field $\omega_\mu$. The recursive relation in this case is given by

$$\hat{\omega}_\mu^{(n+1)} = \frac{1}{4(n+1)}\theta^{\kappa\lambda}\{\hat{\omega}_\kappa \star \partial_\lambda \hat{\omega}_\mu + \hat{\hat{F}}_{\lambda\mu}\}^{(n)} .$$  \hspace{1cm} (3.24)

The gauge field $\hat{\omega}_\mu$ is of the form

$$\hat{\omega}_\mu = \omega_\mu - \frac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha, \partial_\beta \omega_\mu + F_{\lambda\mu}\} + \mathcal{O}(\theta^2)$$  \hspace{1cm} (3.25)

$$= \frac{1}{4}\omega^{ab}\sigma_{ab} + \omega^a \gamma_a + \tilde{\omega}_\mu \gamma_a \gamma_5 + \tilde{\omega}_\mu \gamma_5 + \omega_\mu I .$$  \hspace{1cm} (3.26)

The NC field strength tensor is defined as

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{\omega}_\nu - \partial_\nu \hat{\omega}_\mu - i[\hat{\omega}_\mu \star \hat{\omega}_\nu]$$  \hspace{1cm} (3.27)

$^3$The advantage of working with the $\gamma$-matrix representation is that the enveloping algebra is finite.
and
\[ \delta^* \hat{F}_{\mu\nu} = i[\hat{\Lambda}_\kappa \hat{F}_{\mu\nu}] . \] (3.28)

The SW map solution for \( \hat{F}_{\mu\nu} \) follows from the definition (3.27), using the result (3.24).

The recursive formula is
\[ \hat{F}^{(n+1)}_{\mu\nu} = -\frac{1}{4(n+1)} \theta^{\kappa\lambda} \left( \{ \hat{\omega}_\kappa \hat{F}_{\mu\nu} + \partial_\lambda \hat{F}_{\mu\nu} \} \right)^{(n)} \]
\[ + \frac{1}{2(n+1)} \theta^{\kappa\lambda} \left( \{ \hat{F}_{\mu\kappa} \hat{F}_{\nu\lambda} \} \right)^{(n)} . \] (3.29)

Note that we do not put a "hat" on the covariant derivative \( D_\mu \), the meaning of \( D_\mu \) is defined by the expression it acts on:
\[ D_\lambda \hat{F}_{\mu\nu} = \partial_\lambda \hat{F}_{\mu\nu} - i[\hat{\omega}_\lambda \hat{F}_{\mu\nu}] \] and
\[ D_\lambda \phi = \partial_\lambda \phi - i[\omega_\lambda \phi] \].

Using the previous results we find the recursive relation
\[ \hat{\phi}^{(n+1)} = -\frac{1}{4(n+1)} \theta^{\kappa\lambda} \left( \{ \hat{\omega}_\kappa \partial_\lambda \hat{\phi} + D_\lambda \hat{\phi} \} \right)^{(n)} , \] (3.33)

with \( D_\lambda \hat{\phi} = \partial_\lambda \hat{\phi} - i[\omega_\lambda \hat{\phi}] \) and \( D_\lambda \phi = \partial_\lambda \phi - i[\omega_\lambda \phi] \). The solution for \( \hat{\phi} \) has the following structure
\[ \hat{\phi} = \phi - \frac{1}{4} \theta^{\kappa\lambda} \{ \omega_\kappa, \partial_\lambda \phi + D_\lambda \phi \} + \mathcal{O}(\theta^2) \] (3.34)
\[ = \phi^a \gamma_a \gamma_5 + \phi \gamma_5 + \frac{1}{4} \phi^a \sigma_{ab} + \phi^a \gamma_a + \mathcal{O}(\theta^2) . \] (3.35)

Note that a term proportional to the unit matrix is absent from the first order solution. This is a consequence of the algebra of \( M_{AB} \) and \( \Gamma_A \) matrices, see the list of identities in Appendix. The term proportional to the unit matrix will appear in the second and higher orders.

3.2 The NC AdS action

The NC action is now given by
\[ S_{NC} = \frac{i l}{64\pi G_N} \text{Tr} \int d^4 x e^{\mu\nu\rho\sigma} \hat{F}_{\mu\nu} \hat{F}_{\rho\sigma} \hat{\phi} . \] (3.36)
The $\star$-product is the Moyal-Weyl $\star$-product (1.1), fields with a "hat" are NC fields and we will use the SW map solutions (3.30), (3.34). Using the transformation laws (3.28), (3.32) and the cyclicity of the integral (3.15) one can show that this action is invariant under the NC SO(2,3)$_\star$ gauge transformations$^4$. In the limit $\theta^{\alpha\beta} \to 0$ the action (3.36) reduces to the commutative action (2.12).

Let us now calculate the first order correction to (2.12). To this end, we calculate

$$
(F_{\mu\nu} \star F_{\rho\sigma})(1) = F_{\mu\nu}^{(1)} F_{\rho\sigma} + F_{\mu\nu} F_{\rho\sigma}^{(1)} + \frac{i}{2} \theta^{\alpha\beta} \partial_\alpha F_{\mu\nu} \partial_\beta F_{\rho\sigma}
$$

and

$$
(F_{\mu\nu} \star F_{\rho\sigma} \star \hat{\phi})(1) = (F_{\mu\nu} \star F_{\rho\sigma})(1) \phi + (F_{\mu\nu} F_{\rho\sigma}) \phi^{(1)} + \frac{i}{2} \theta^{\alpha\beta} \partial_\alpha (F_{\mu\nu} F_{\rho\sigma} \partial_\beta \phi
$$

Then, the first order of the NC action (3.36) is given by

$$
S_{NC}^{(1)} = \frac{i}{64 \pi G_N} \text{Tr} \int d^4 x \epsilon^{\mu\nu\rho\sigma} (\hat{F}_{\mu\nu} \star \hat{F}_{\rho\sigma} \star \hat{\phi})^{(1)}
$$

$$
= \frac{i}{64 \pi G_N} \theta^{\alpha\beta} \text{Tr} \int d^4 x \epsilon^{\mu\nu\rho\sigma} \left( - \frac{1}{4} F_{\mu\nu} F_{\rho\sigma} \{ F_{\alpha\beta}, \phi \} + \frac{i}{2} D_\alpha F_{\mu\nu} D_\beta F_{\rho\sigma} \phi + \frac{1}{2} \{ F_{\alpha\mu}, F_{\beta\nu} \} F_{\rho\sigma} \phi + \frac{1}{2} F_{\mu\nu} \{ F_{\alpha\rho}, F_{\beta\sigma} \} \phi \right)
$$

$^4$The NC gauge variation of the action (3.36) is given by

$$
\delta^{\star} S_{NC} \sim \int d^4 x \epsilon^{\mu\nu\rho\sigma} \text{Tr} [\Lambda_\varepsilon \hat{F}_{\mu\nu} \star \hat{F}_{\rho\sigma} \star \hat{\phi}] = \int d\Sigma_\mu K^\mu
$$

where $K^\mu$ is a function of $\varepsilon$, $\omega_\mu$, $F_{\mu\nu}$, their derivatives and $\theta^{\alpha\beta}$. This is not unusual in the field theory. For example, the variation of the commutative Einstein-Hilbert action under infinitesimal diffeomorphisms generated by the vector field $\xi = \xi^\mu \partial_\mu$ is different form zero and is given by a surface term. This surface term vanishes if one demands that $\xi^\mu = 0$ at the boundary. This is a standard textbook procedure, see [24] and references therein. Therefore, if we demand that the gauge parameter $\varepsilon$ has appropriate behavior at the boundary, the surface term vanishes and the variation of the action (3.36) is zero. A detailed analysis of boundary terms is required if one discussed conserved quantities, especially in gravity, see Chapter 7 in [25].
We performed one partial integration to obtain (3.39) from (3.38). After explicitly calculating the traces of the products of gamma matrices, we obtain $S^{(1)}_{NC} = 0$. In this way, once again we confirm the statement already present in the literature: if the reality of action is imposed, then there is no first order correction for the NC gravity action. This results seems to be model independent [5, 7, 26].

Similarly to the SW map solutions, there is a recursive relation between the $(n+1)$st and the $n$th order of the expansion of (3.36). In particular, the second order correction is given by

$$S^{(2)}_{NC} = \frac{i l}{128 \pi G_N} \theta^{\alpha \beta} \int d^4 x e^{\mu \rho \sigma} \left( - \frac{1}{4} \hat{F}_{\mu \nu} \star \hat{F}_{\rho \sigma} \star \{ \hat{F}_{\alpha \beta} \star \hat{\phi} \right)$$

+ \frac{i}{2} D_{\alpha} \hat{F}_{\mu \nu} \star D_{\beta} \hat{F}_{\rho \sigma} \star \phi$$

+ \frac{1}{2} \{ \hat{F}_{\alpha \mu} \star \hat{F}_{\beta \nu} \} \star \hat{F}_{\rho \sigma} \star \hat{\phi} + \frac{1}{2} \hat{F}_{\mu \nu} \star \{ \hat{F}_{\alpha \rho} \star \hat{F}_{\beta \sigma} \} \star \hat{\phi} \right)^{(1)} .

In order to calculate this expression, we have to expand the $\star$-products in (3.40) and use the SW map solutions (3.30) and (3.34). However, inserting these solutions straightforwardly into (3.40) gives lots of noncovariant terms: terms with partial derivatives and the "naked" gauge field $\omega_{\mu}$. To avoid this, we use the method of composite fields [16] which enables to write the result in a manifestly gauge covariant way. The first, third and fourth term can be calculated similarly to what has been done in (3.37) and (3.38) and we will not go into details here. The useful formula is

$$(\hat{F}_{\mu \nu} \star \hat{F}_{\rho \sigma} \star \hat{F}_{\alpha \beta} \star \hat{\phi})^{(1)} = - \frac{1}{4} \theta^{\kappa \lambda} \{ \omega_{\kappa}, \partial_{\lambda} (D_{\alpha} F_{\mu \nu} F_{\beta \mu} F_{\alpha \beta} \phi) \}$$

$$+ \frac{i}{2} \theta^{\kappa \lambda} D_{\kappa} F_{\mu \nu} D_{\lambda} (F_{\rho \sigma} F_{\alpha \beta} \phi) + \frac{i}{2} \theta^{\kappa \lambda} F_{\mu \nu} \left( D_{\kappa} (F_{\rho \sigma} F_{\alpha \beta}) D_{\lambda} \phi + D_{\kappa} F_{\rho \sigma} D_{\lambda} F_{\alpha \beta} \phi \right)$$

$$+ \frac{1}{2} \theta^{\kappa \lambda} \left( \{ F_{\kappa \mu}, F_{\lambda \nu} \} F_{\rho \sigma} F_{\alpha \beta} + F_{\mu \nu} \{ F_{\kappa \rho}, F_{\lambda \sigma} \} F_{\alpha \beta} + F_{\mu \nu} F_{\rho \sigma} \{ F_{\kappa \alpha}, F_{\lambda \beta} \} \right) \phi .$$

The second term containing the covariant derivative $D_{\mu}$ we calculate in details. Starting from

$$(D_{\alpha} \hat{F}_{\mu \nu})^{(1)} = - \frac{1}{4} \theta^{\kappa \lambda} \{ \omega_{\kappa}, D_{\lambda} (D_{\alpha} F_{\mu \nu} + D_{\lambda} (D_{\alpha} F_{\mu \nu})) \} + \frac{1}{2} \theta^{\kappa \lambda} \{ F_{\kappa \alpha}, D_{\lambda} F_{\mu \nu} \}$$

$$+ \frac{1}{2} \theta^{\kappa \lambda} \{ D_{\alpha} F_{\kappa \mu}, F_{\lambda \nu} \} + \frac{1}{2} \theta^{\kappa \lambda} \{ F_{\mu \nu}, D_{\alpha} F_{\lambda \nu} \} ,

we obtain

$$(D_{\alpha} F_{\mu \nu} \star D_{\beta} \hat{F}_{\rho \sigma})^{(1)} = - \frac{1}{4} \theta^{\kappa \lambda} \{ \omega_{\kappa}, D_{\lambda} (D_{\alpha} F_{\mu \nu} D_{\beta} F_{\rho \sigma} + D_{\lambda} (D_{\alpha} F_{\mu \nu} D_{\beta} F_{\rho \sigma})) \}$$

$$+ \frac{i}{2} \theta^{\kappa \lambda} (D_{\kappa} D_{\alpha} F_{\mu \nu}) (D_{\lambda} D_{\beta} F_{\rho \sigma})$$

$$+ \frac{1}{2} \theta^{\kappa \lambda} \left( \{ F_{\kappa \alpha}, D_{\lambda} F_{\mu \nu} \} + \{ D_{\alpha} F_{\kappa \mu}, F_{\lambda \nu} \} + \{ F_{\kappa \mu}, D_{\alpha} F_{\lambda \nu} \} \right) (D_{\beta} F_{\rho \sigma})$$

$$+ \frac{1}{2} \theta^{\kappa \lambda} (D_{\alpha} F_{\mu \nu}) \left( \{ F_{\kappa \beta}, D_{\lambda} F_{\rho \sigma} \} + \{ D_{\beta} F_{\kappa \rho}, F_{\lambda \sigma} \} + \{ F_{\kappa \rho}, D_{\beta} F_{\lambda \sigma} \} \right) .$$
Collecting the results for all three terms in (3.40) we obtain traces is given by

This expanded action is obviously invariant under the commutative \(SO(2,3)\) gauge transformations, as guaranteed by the SW map. In order to break this symmetry down to \(SO(1,3)\) we have to constrain the commutative field \(\phi\) to be of the form \(\phi = (0, 0, 0, 0)\). In this way, in the limit \(\theta^{\alpha\beta} \to 0\) we obtain the commutative action (2.14). The second order correction after the symmetry breaking and after calculating traces is given by

\[
S_{NC}^{(2)} = \frac{i l}{64\pi G_N} \theta^{\alpha\beta} \theta^{\kappa\lambda} \varepsilon_{\mu\nu\rho\sigma} \varepsilon_{abcd} \int d^4 x \frac{1}{1256} \left( F_{\mu\nu}^{ab} F_{\rho\sigma}^{cd} F_{\alpha\beta}^{mn} F_{\kappa\lambda mn} - 8 F_{\mu\nu}^{ab} F_{\rho\sigma}^{cd} F_{\alpha\beta}^{mn} F_{\kappa\lambda}^{5} + F_{\mu\nu}^{ab} F_{\rho\sigma}^{cd} (F_{\mu\nu}^{mn} F_{\rho\sigma mn} + 2 F_{\mu\nu}^{5} F_{\rho\sigma mn} \right)
\]

\[
- \frac{1}{32} \left( F_{\mu\nu}^{ab} F_{\rho\sigma}^{cd} F_{\alpha\beta}^{mn} F_{\kappa\lambda mn} + 2 F_{\alpha\beta}^{ab} F_{\rho\sigma}^{cd} F_{\kappa\lambda}^{m5} F_{\lambda mn} \right)
\]
\[ + F_{\kappa\alpha} \frac{ab}{F_{\lambda\alpha}^{cd}} F_{\alpha\beta}^{mn} F_{\rho\sigma mn} \]

\[ - \frac{1}{128} \left( F_{\kappa\alpha} \frac{ab}{F_{\lambda\beta}^{cd}} (F_{\mu\nu}^{mn} F_{\rho\sigma mn} + 2F_{\kappa\alpha}^{m5} F_{\lambda\beta}^{5m}) \right) \]

\[ + 2F_{\mu\nu}^{m5} F_{\rho\sigma mn} \]

\[ + \frac{1}{16} F_{\alpha\beta} \left( (D_{\kappa} F_{\mu\nu})^{cm} (D_{\lambda} F_{\rho\sigma})^{d} + (D_{\kappa} F_{\mu\nu})^{c5} (D_{\lambda} F_{\rho\sigma})^{d5} \right) \]

\[ - \frac{1}{16} \left( (D_{\kappa} F_{\mu\nu})^{ab} (D_{\lambda} F_{\alpha\beta})^{d5} F_{\rho\sigma}^{c5} + (D_{\kappa} F_{\mu\nu})^{a5} (D_{\lambda} F_{\alpha\beta})^{b5} F_{\rho\sigma}^{cd} \right) \]

\[ + \frac{1}{16} F_{\alpha\beta} F_{\beta\gamma} F_{\kappa\rho}^{mn} F_{\lambda\sigma mn} + 2F_{\kappa\rho}^{m5} F_{\lambda\sigma m5} \]

\[ + \frac{1}{16} \left( (D_{\kappa} F_{\mu\nu})^{ab} (D_{\lambda} F_{\alpha\beta})^{d5} F_{\rho\sigma}^{c5} + (D_{\kappa} F_{\mu\nu})^{a5} (D_{\lambda} F_{\alpha\beta})^{b5} F_{\rho\sigma}^{cd} \right) \]

\[ - \frac{1}{8} \left( (D_{\kappa} F_{\alpha\mu})^{cm} (D_{\lambda} F_{\beta\nu})^{d} + (D_{\kappa} F_{\alpha\mu})^{c5} (D_{\lambda} F_{\beta\nu})^{d5} \right) \]

\[ + \frac{1}{2} \left( (D_{\kappa} F_{\mu\nu})^{ab} (D_{\lambda} F_{\rho\sigma})^{d5} F_{\rho\sigma}^{c5} + (D_{\kappa} F_{\mu\nu})^{a5} (D_{\lambda} F_{\rho\sigma})^{b5} F_{\rho\sigma}^{cd} \right) \]

\[ - \frac{1}{32} (D_{\kappa} D_{\lambda} F_{\mu\nu})^{ab} (D_{\lambda} D_{\beta} F_{\rho\sigma})^{cd} \} \tag{3.46} \]

Here \( D_{\alpha} F_{\mu\nu} \) is the \( SO(2,3) \) covariant derivative and its components are

\[
(D_{\alpha} F_{\mu\nu})^{ab} = \nabla_{\alpha} F_{\mu\nu}^{ab} - \frac{1}{i^2} (\epsilon_{a}^{\alpha} T_{\mu\nu}^{b} - \epsilon_{b}^{\alpha} T_{\mu\nu}^{a})
\]

\[
(D_{\alpha} F_{\mu\nu})^{a5} = \frac{1}{i} (\nabla_{\alpha} T_{\mu\nu}^{a} + \epsilon_{m}^{a} F_{\mu\nu m}^{a}) \),
\]

\[
(D_{\kappa} D_{\alpha} F_{\mu\nu})^{ab} = \nabla_{\kappa} \nabla_{\alpha} F_{\mu\nu}^{ab} - \frac{1}{i^2} (\nabla_{\kappa} \epsilon_{a}^{\alpha} T_{\mu\nu}^{b} - (\nabla_{\kappa} \epsilon_{b}^{\alpha} T_{\mu\nu}^{a} + \epsilon_{a}^{\alpha} (\nabla_{\kappa} T_{\mu\nu}^{b}) - \epsilon_{b}^{\alpha} (\nabla_{\kappa} T_{\mu\nu}^{a} + \epsilon_{a}^{\alpha} F_{\mu\nu m}^{b} - \epsilon_{b}^{\alpha} F_{\mu\nu m}^{a}) \right) ,
\]

with the \( SO(1,3) \) covariant derivative \( \nabla_{\alpha} F_{\mu\nu}^{ab} = \partial_{\alpha} F_{\mu\nu}^{ab} + \omega_{\alpha}^{ac} F_{\mu\nu c}^{b} - \omega_{\alpha}^{bc} F_{\mu\nu c}^{a} \) and \( \nabla_{\alpha} T_{\mu\nu}^{a} = \partial_{\alpha} T_{\mu\nu}^{a} + \omega_{\alpha}^{ac} T_{\mu\nu c} \).

### 4 Discussion

Starting from the NC \( SO(2,3) \) gauge theory, in this paper we constructed a model of NC gravity. Our construction is based on the enveloping algebra approach and the SW map. Assuming that the noncommutativity is very small, we expanded the NC gravity action (3.36) in orders of the noncommutativity parameter \( \theta^{05} \). The first order correction vanishes; the second order correction is non-zero and we calculated it explicitly (3.45). It is obvious that this correction is invariant under the commutative
SO(2, 3) gauge transformations. After the symmetry breaking the action (3.45) becomes (3.46) and the symmetry is reduced to the commutative \(SO(1, 3)\). The result (3.46) is written in the first order formalism, the spin connection \(\omega^{ab}_{\mu}\) and the vierbeins \(e^{\mu}_a\) are independent fields. Unlike in our previous paper, the torsion appears explicitly in (3.46).

However, the result (3.46) we obtained for the NC gravity action is very cumbersome; it is hard to immediately see and discuss any physical implications from it. Therefore we analyze different limits of our theory. There are three different scales in the model and they are related with the following three parameters: the cosmological constant \(\Lambda = -\frac{3}{l^2}\), the NC parameter \(\theta^{\alpha\beta}\) and the powers of the curvature tensor (powers of derivatives). Depending on the values these parameters take different limits of the model are obtained. Looking closely at (3.45) we see that separate terms can be grouped depending on the powers of dimensionless quantity \(Rl^2\). There are five different types:

\[
\frac{\theta^2}{l^6} \left(1, Rl^2, R^2l^4, R^3l^6, R^4l^8\right).
\] (4.47)

At higher energies, higher powers of curvature (which correspond to the higher powers of derivatives) are dominant and for some fixed \(l\) the leading term is \(R^4l^8\). On the other hand, at low energies, lower powers of curvature are dominant and the leading term is \(Rl^2\). The cosmological constant in zeroth order is given by \(\Lambda = -\frac{3}{l^2}\). The limit of the big/small cosmological constant is obtained taking \(l\) to be small/big. The NC parameter \(\theta^{\alpha\beta}\) is taken to be very small, but the dimensionless quantity \(\theta^2\sim \theta^2\Lambda^2\) can be fine-tuned to be smaller, greater or equal 1, depending on the value of \(l\). In the following we discuss three different expansions.

**Expansion 1**

Let us first assume that we are interested in the limit of small curvature, vanishing torsion and big cosmological constant. This limit cannot be used to describe Universe today\(^5\) but it could have been relevant in some phases of its evolution. In that case, from (3.45) we include only terms which are of zeroth, first and second order in the curvature. The result is given by

\[
S = S^{(0)} + S^{(2)},
\]

\[
S^{(0)} = -\frac{1}{16\pi G_N} \int d^4x \left[ \frac{l^2}{16} \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} R^{ab}_{\mu\nu} R^{cd}_{\rho\sigma} + \eta^{R} + 2\eta^{\Lambda} \right],
\]

\[
S^{(2)} = 3\theta^{\alpha\beta}\theta^{\kappa\lambda} \frac{64\pi G_N l^6}{l^4} \int d^4x \sqrt{-g} g_{\alpha\kappa} g_{\beta\lambda}
- \theta^{\alpha\beta}\theta^{\kappa\lambda} \frac{64\pi G_N l^6}{16} \int d^4x \sqrt{-g} \left(3g_{\alpha\kappa} R_{\beta\lambda} + 3R_{\alpha\beta\kappa\lambda} - 2R_{\alpha\kappa\beta\lambda}\right)
+ \theta^{\alpha\beta}\theta^{\kappa\lambda} \frac{256\pi G_N l^6}{256} \int d^4x \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} \hat{e}_\alpha^d \hat{e}_\mu \hat{e}_\rho \nabla_\gamma R_{\beta\lambda} - 2\epsilon^{a\nu}_{\mu} \epsilon^{b\rho}_{\nu} g_{\beta\rho}
\]

\(^5\)We work with the negative cosmological constant, but all the results can be easily generalized to the case of positive cosmological constant.
Therefore the affine connection (Christoffel symbols) \( \Gamma^\sigma_{\mu \nu} \) (3.36) is invariant under the twisted diffeomorphisms. How the symmetry breaking and make the diffeomorphism non-invariance explicit. On the other hand, the action

\[
S^{(2)} = \frac{3 \theta^{\alpha \beta} \theta^{\kappa \lambda}}{64 \pi G_N l^6} \int d^4 x \sqrt{-g} g_{\alpha \kappa} g_{\beta \lambda}.
\]

This action is invariant under the \( SO(1,3) \) gauge symmetry. However, due to the noncommutativity it is no longer invariant under the diffeomorphism symmetry. The non-invariant terms manifest in two ways. Firstly, there are tensors contracted with the NC parameter \( \theta^{\alpha \beta} \) such as \( \theta^{\alpha \beta} \theta^{\kappa \lambda} R_{\alpha \kappa \beta \lambda} \). Since \( \theta^{\alpha \beta} \) is not a tensor under the diffeomorphism symmetry (it is a non-transforming constant matrix), those terms are also not tensors. Then there are terms in which \( SO(1,3) \) covariant derivatives of vierbeins appear. Using the metricity condition

\[
\nabla^\mu a^\rho = \partial_\mu a^\rho + \omega^b_{\mu \rho} a^b - \Gamma^\sigma_{\mu \rho} a^\sigma = 0
\]

the \( SO(1,3) \) covariant derivative can be written as

\[
\nabla_\mu a^\rho = \partial_\mu a^\rho + \omega^b_{\mu \rho} a^b = \Gamma^\sigma_{\mu \rho} a^\sigma.
\]

Therefore the affine connection (Christoffel symbols) \( \Gamma^\sigma_{\mu \rho} \) appears explicitly in (4.48). Some of the terms can be grouped to to the curvature tensor, but some will remain and make the diffeomorphism non-invariance explicit. On the other hand, the action (3.36) is invariant under the twisted diffeomorphisms. How the symmetry breaking affects this invariance is not clear yet and remains to be studied further.

The assumption of the big cosmological constant leads to \( 1 \gg l^2 R \gg \frac{\theta^2}{l^4} \) and therefore selects the leading order terms to be proportional to \( 1/l^6 \). Explicitly

\[
S^{(2)} = \int d^4 x \sqrt{-g} g_{\alpha \kappa} g_{\beta \lambda}.
\]
The equation of motion following from (4.51) is obtained varying with respect to \(g_{\mu\nu}\) and it is given by
\[
R_{\rho\sigma} - \frac{1}{2} g_{\rho\sigma} (R + 2\Lambda) + \frac{3}{4l^6} \theta^{\alpha\beta} \theta^{\kappa\lambda} \left( \frac{1}{2} g_{\rho\sigma} g_{\alpha\kappa} g_{\beta\lambda} + 2 g_{\beta\lambda} g_{\alpha\rho} g_{\kappa\sigma} \right) = 0. 
\] (4.52)

The cosmological constant is modified in this model, it becomes \(x\)-dependent
\[
\Lambda(x) = \Lambda - \frac{3}{8} \frac{\theta^{\alpha\beta} \theta^{\kappa\lambda} l^6}{g_{\alpha\kappa} g_{\beta\lambda}}. 
\] (4.53)

To see if this modification can "flatten" the starting commutative space (AdS is the zeroth order solution of equations (4.52)) we check whether the Minkowski space-time is a solution of (4.52). That is, we look for \(\theta^{\alpha\beta}\) such that \(g_{\mu\nu} = \eta_{\mu\nu}\) is a solution of (4.52). Unfortunately, this is not the case. The easiest way to see this is to look at the equation obtained by contracting (4.52)
\[
R = 4\Lambda + \frac{3}{l^6} \theta^{\alpha\beta} \theta^{\kappa\lambda} g_{\alpha\kappa} g_{\beta\lambda}. 
\] (4.54)

Demanding that \(g_{\mu\nu} = \eta_{\mu\nu}\) is a solution of this equation, gives \(\frac{\theta^2}{l^4} = 2\), which is in contradiction with the assumption that \(1 \gg l^2 R \gg \frac{\theta^2}{l^4}\). Therefore we conclude that the Minkowski space-time cannot be a solution of (4.52).

We know that the zeroth order solution of (4.52) is AdS space-time, \(g^{(0)}_{\mu\nu} = g^{AdS}_{\mu\nu}\). Making an expansion \(g_{\mu\nu} = g^{AdS}_{\mu\nu} + \varepsilon h_{\mu\nu}\) with a small parameter \(\varepsilon\) we can linearize the equations around this solution. This has to be done very carefully and we postpone the calculation for the next paper.

Note that in our previous paper [13] we did not obtain a \(x\)-dependent correction to the cosmological constant. This shows once more that the deformation and symmetry breaking do not commute, instead they lead to different models.

**Expansion 2**

Our next choice is the limit of the small cosmological constant and vanishing torsion. It can be relevant when describing Universe today or in some of its earlier phases. In that case the second order correction for the NC gravity action is given by
\[
S^{(2)} = -\frac{l^2 \theta^{\alpha\beta} \theta^{\kappa\lambda}}{64\pi G_N} \int d^4 x e \left[ \left( \frac{1}{64} R_{\alpha\beta\gamma\delta} R_{\kappa\lambda}^{\gamma\delta} + \frac{1}{32} R_{\kappa\alpha\gamma\delta} R_{\lambda\beta}^{\gamma\delta} \right) \left( R^2 + 4 R_{\rho\mu} R^{\rho\mu} + R_{\rho\mu\rho\sigma} R^{\rho\mu\rho\sigma} \right) 
\right. \\
+ R_{\rho\mu\gamma\delta} R_{\rho\sigma}^{\rho\sigma} \left( \frac{1}{32} \left( 2 R_{\mu\rho} R_{\nu\sigma} - R_{\alpha\beta}^{\mu\rho} R_{\kappa\lambda}^{\nu\sigma} \right) 
\right. \\
+ \frac{1}{16} R_{\kappa\alpha\rho\sigma} R_{\lambda\beta}^{\mu\nu} - \frac{1}{8} R_{\kappa\alpha\sigma\rho} R_{\lambda\beta}^{\mu\nu} \left( R_{\alpha\mu\rho\sigma} R_{\beta}^{\mu\nu} + R_{\alpha\mu\nu} R_{\beta}^{\mu\nu} - R_{\alpha\rho} R_{\beta\sigma} \right) \\
\left. + \frac{1}{4} R_{\kappa\alpha\gamma\delta} R_{\rho\mu\gamma\delta} \left( R_{\alpha\mu\rho\sigma} R_{\beta}^{\mu\nu} + R_{\alpha\mu\nu} R_{\beta}^{\mu\nu} - R_{\alpha\rho} R_{\beta\sigma} \right) \right].
\]
The correction terms are of the third and fourth power of curvature. Since the cosmological constant is small, we can assume that the zeroth order solution is the Minkowski space-time, \( g_{\mu\nu}^{(0)} \approx \eta_{\mu\nu} \). Then we can expand around this solution assuming \( g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu} \) and look for the equation for \( h_{\mu\nu} \) and its solutions. This we postpone for the next paper.

**Expansion 3**

Our final example is the "NC teleparallel" solution: we assume \( R_{\mu\nu}^{ab} = 0 \) and \( T_{\mu\nu}^a \neq 0 \). The second order correction is given by

\[
S_T = \frac{-\theta^{\alpha\beta}\theta^{\mu\lambda}}{64\pi G N t^6} \int d^4x \sqrt{-g} \left[ \frac{1}{8} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} + \frac{1}{2} \theta_{\alpha\beta\mu\nu} \theta^{\alpha\beta\mu\nu} + \frac{1}{4} \theta_{\alpha\beta\mu\nu} \theta^{\alpha\beta\mu\nu} \right]
\]

\[
+ \frac{1}{8} R_{\alpha\gamma\beta\delta} R^{\alpha\gamma\beta\delta} (R_{\kappa\lambda\mu\nu} + R_{\kappa\lambda\rho\sigma} R^{\rho\sigma\mu\nu} + 4R_{\kappa\lambda\rho\sigma} R^{\rho\sigma\mu\nu})
\]

\[
+ \frac{1}{2} R_{\alpha\gamma\beta\delta} R^{\alpha\gamma\beta\delta} (R_{\kappa\mu\rho\sigma} R^{\rho\sigma\mu\nu} + R_{\kappa\mu\sigma\rho} R^{\rho\sigma\mu\nu} + R_{\kappa\sigma\rho\mu} R^{\rho\sigma\mu\nu})
\]

\[
- \frac{1}{4} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} (R_{\mu\nu\rho\sigma} R^{\rho\sigma\mu\nu} + 2R_{\mu\nu\rho\sigma} R^{\rho\sigma\mu\nu} + 2R_{\mu\nu\rho\sigma} R^{\rho\sigma\mu\nu} + R_{\mu\nu\sigma\rho} R^{\rho\sigma\mu\nu})
\]

\[
- \frac{1}{4} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \left[ R_{\mu\nu\rho\sigma} R^{\rho\sigma\mu\nu} + 2R_{\mu\nu\rho\sigma} R^{\rho\sigma\mu\nu} + 2R_{\mu\nu\rho\sigma} R^{\rho\sigma\mu\nu} + R_{\mu\nu\sigma\rho} R^{\rho\sigma\mu\nu} \right]
\]

\[
\frac{-12 \theta^{\alpha\beta\kappa\lambda}}{64\pi G N} \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} \int d^4x \left[ \frac{1}{16} R_{\alpha\beta}(\nabla_{\kappa} R_{\mu\nu})^{cm}(\nabla_{\lambda} R_{\rho\sigma})^{md} \right.
\]

\[
- \frac{1}{8} R_{\alpha\beta}(\nabla_{\kappa} R_{\mu\nu})^{cm}(\nabla_{\lambda} R_{\rho\sigma})^{md} - \frac{1}{32} (\nabla_{\kappa} \nabla_{\alpha} R_{\mu\nu})^{ab}(\nabla_{\lambda} \nabla_{\beta} R_{\rho\sigma})^{cd} \right]
\]

\[(4.55)\]

The correction terms are of the third and fourth power of curvature. Since the cosmological constant is small, we can assume that the zeroth order solution is the Minkowski space-time, \( g_{\mu\nu}^{(0)} \approx \eta_{\mu\nu} \). Then we can expand around this solution assuming \( g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu} \) and look for the equation for \( h_{\mu\nu} \) and its solutions. This we postpone for the next paper.
The zeroth order action in this limit reduces to the cosmological constant term. That means that there is no kinetic term for the vierbeins in the zeroth order. We would have to look for it in (4.56). Another way to obtain it would be to start from a different commutative action, the one that includes the kinetic term for vierbeins.

The analysis of diffeomorphism invariance of (4.55) and (4.56) remains the same as in the first example (4.48): the commutative diffeomorphism symmetry is broken, while the invariance under the twisted diffeomorphism symmetry remains to be understood better.

Finally, let us comment that our results (4.48), (4.55) and (4.56) cannot be related with $f(R)$ and $f(T)$ theories. Some of the indices on the curvature tensor and torsion will always be contracted with the NC parameter $\theta^{\alpha \beta}$, this is a consequence of the SW map. Therefore, it seems impossible to construct invariants of curvature tensor or torsion alone.

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A  AdS algebra and the $\gamma$-matrices

Algebra relations$^6$:

\[
\{M_{AB}, \Gamma_C\} = i \epsilon_{ABCDE} M^{DE}
\]

\[
\{M_{AB}, M_{CD}\} = \frac{i}{2} \epsilon_{ABCDE} \Gamma^E + \frac{1}{2} (\eta_{AC} \eta_{BD} - \eta_{AD} \eta_{BC})
\]

\[
[M_{AB}, \Gamma_C] = i(\eta_{BC} \Gamma_A - \eta_{AC} \Gamma_B)
\]

\[
\Gamma^A_\alpha = -\gamma_0 \Gamma_A \gamma_0
\]

\[
M_{AB}^\dagger = \gamma_0 M_{AB} \gamma_0
\]

\[
\{\sigma_{ab}, \sigma_{cd}\} = 2(\eta_{ac} \eta_{bd} - \eta_{ad} \eta_{bc} + i \epsilon_{abcd} \gamma_5)
\]

\[
[\sigma_{ab}, \gamma_c] = 2i(\eta_{bc} \gamma_a - \eta_{ac} \gamma_b)
\]

\[
\{\sigma_{ab}, \gamma_c\} = 2 \epsilon_{abcd} \gamma^5 \gamma^d
\]

Identities with traces:

\[
\text{Tr}(\Gamma_A \Gamma_B) = 4 \eta_{AB}
\]

\[
\text{Tr}(\Gamma_A) = \text{Tr}(\Gamma_A \Gamma_B \Gamma_C) = 0
\]

$^6 \epsilon^{01235} = +1, \epsilon^{0123} = 1$
\[
\text{Tr}(\Gamma_A \Gamma_B \Gamma_C \Gamma_D) = 4(\eta_{AB} \eta_{CD} - \eta_{AC} \eta_{BD} + \eta_{AD} \eta_{CB}) \\
\text{Tr}(\Gamma_A \Gamma_B \Gamma_C \Gamma_D \Gamma_E) = -4i \epsilon_{ABCDE} \\
\text{Tr}(M_{AB} M_{CD} \Gamma_E) = i \epsilon_{ABCDE} \\
\text{Tr}(M_{AB} M_{CD}) = -\eta_{AD} \eta_{CB} + \eta_{AC} \eta_{BD} 
\] (1.58)

References

[1] S. Doplicher, K. Fredenhagen and J. E. Roberts, The Quantum structure of spacetime at the Planck scale and quantum fields, Commun. Math. Phys. 172, 187 (1995), [hep-th/0303037].

[2] A. Connes, Non-commutative Geometry, Academic Press (1994).

J. Madore, An Introduction to Noncommutative Differential Geometry and its Physical Applications, 2nd Edition, Cambridge Univ. Press (1999).

P. Aschieri, M. Dimitrijević, P. Kulish, F. Lizzi and J. Wess Noncommutative spacetimes: Symmetries in noncommutative geometry and field theory, Lecture notes in physics 774, Springer (2009).

[3] P. Aschieri, C. Blohmann, M. Dimitrijević, F. Meyer, P. Schupp and J. Wess, A Gravity Theory on Noncommutative Spaces, Class. Quant. Grav. 22, 3511 (2005), [hep-th/0504183].

P. Aschieri, M. Dimitrijević, F. Meyer and J. Wess, Noncommutative Geometry and Gravity, Class. Quant. Grav. 23, 1883 (2006), [hep-th/0510059].

[4] A. H. Chamseddine, Complexified gravity in noncommutative spaces, Commun. Math. Phys. 218, 283 (2001) [hep-th/0005222].

[5] A. H. Chamseddine, Deforming Einstein’s gravity, Phys. Lett. B 504 33 (2001), [hep-th/0009153].

[6] M. A. Cardella and D. Zanon, Noncommutative deformation of four-dimensional gravity, Class. Quant. Grav. 20, L95 (2003), [hep-th/0212071].

A. H. Chamseddine, SL(2,c) gravity with complex vierbein and its noncommutative extension, Phys. Rev. D69, 024015 (2004), [hep-th/0309166].

[7] P. Aschieri and L. Castellani, Noncommutative D = 4 gravity coupled to fermions JHEP, 0906, 086 (2009), [arXiv:0902.3823].

[8] H. S. Yang, Emergent gravity from noncommutative spacetime, Int. J. Mod. Phys. A24, 4473 (2009), [hep-th/0611174].

H. Steinacker, Emergent Geometry and Gravity from Matrix Models: an Introduction, Class. Quant. Grav. 27, 133001 (2010), [arXiv:1003.4134].
[9] M. Burić and J. Madore, *Spherically Symmetric Noncommutative Space: d = 4*, Eur. Phys. J. C58, 347 (2008), [arXiv: 0807.0960].
M. Burić and J. Madore, *On noncommutative spherically symmetric spaces*, arXiv:1401.3652.

[10] A. H. Chamseddine and V. Mukhanov, *Who Ordered the Anti-de Sitter Tangent Group?*, JHEP 1311 095 (2013), [arXiv:1308.3199].
A. H. Chamseddine and V. Mukhanov, *Gravity with de Sitter and Unitary Tangent Groups*, JHEP 1003, 033 (2010), [arXiv:1002.0541].

[11] J. W. Barrett and S. Kerr, *Gauge gravity and discrete quantum models*, arXiv:1309.1660.

[12] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, *Large N Field Theories, String Theory and Gravity*, Phys. Rept. 323, 183 (2000), [hep-th/9905111].

[13] M. Dimitrijević, V. Radovanović and H. Štefančić, *AdS-inspired noncommutative gravity on the Moyal plane*, Phys. Rev. D 86, 105041 (2012), [arXiv:1207.4675].

[14] B. Jurčo, S. Schraml, P. Schupp and J. Wess, *Enveloping algebra valued gauge transformations for non-abelian gauge groups on non-commutative spaces*, Eur. Phys. J. C17, 521 (2000), [hep-th/0006246].
B. Jurčo, L. Möller, S. Schraml, P. Schupp and J. Wess, *Construction of non-Abelian gauge theories on noncommutative spaces*, Eur. Phys. J. C21, 383 (2001), [hep-th/0104153].

[15] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, JHEP 09, 032 (1999), [hep-th/9908142].

[16] P. Aschieri, L. Castellani and M. Dimitrijević, *Noncommutative gravity at second order via Seiberg-Witten map*, Phys. Rev. D 87, 024017 (2013), [arXiv:1207.4346]

[17] K. S. Stelle and P. C. West, *Spontaneously broken de Sitter symmetry and the gravitational holonomy group*, Phys. Rev D 21, 1466 (1980).

[18] S. W. MacDowell and F. Mansouri, *Unified geometrical theory of gravity and supergravity*, Phys. Rev. Lett. 38, 739 (1977).

[19] P. K. Townsend, *Small-scale structure of spacetime as the origin of the gravitation constant*, Phys. Rev. D 15, 2795 (1977).

[20] J. W. York, *Role of conformal three-geometry in the dynamics of gravitation*, Phys. Rev. Lett. 28, (1972).
G. W. Gibbons, S. W. Hawking, *Action integrals and partition functions in quantum gravity*, Phys. Rev. D 15, 2752 (1977).
[21] R. Wald, *General Relativity*, University of Chicago Press, Chicago and London (1984).

[22] K. Ulker and B. Yapiskan, *Seiberg-Witten maps to all orders*, Phys. Rev. D 77, 065006 (2008), [arXiv: 0712.0506].

[23] P. Aschieri and L. Castellani, *Noncommutative gravity coupled to fermions: second order expansion via Seiberg-Witten map*, JHEP 1207 184 (2012), [arXiv:1111.4822].

[24] T. P. Sotiriou, S. Liberati, *Field equations from a surface term*, Phys.Rev. D74, 044016 (2006).

[25] M. Blagojević, *Gravitation and Gauge Symmetries*, Institute of Physics Publication, Bristol (2002).

[26] P. Mukherjee and A. Saha, *A Note on the noncommutative correction to gravity*, Phys. Rev D 74, 027702 (2006), [hep-th/0605287].