Integral models of certain PEL Shimura varieties with $\Gamma_1(p)$-type level structure

Richard Shadrach

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Abstract

We study $p$-adic integral models of certain PEL Shimura varieties with level subgroup at $p$ given by the pro-unipotent radical of an Iwahori. We will consider two cases: the case of Shimura varieties associated with unitary groups that split over an unramified extension of $\mathbb{Q}_p$ and the case of Siegel modular varieties. We construct local models, i.e. simpler schemes which are étale locally isomorphic to the integral models. Our integral models are defined by a moduli scheme using the notion of an Oort-Tate generator of a group scheme. We use these local models to find a resolution of the integral model in the case of the Siegel modular variety of genus 2. The resolution is regular with special fiber a nonreduced divisor with normal crossings.

1 Introduction

In the arithmetic study of Shimura varieties, one seeks to have a model of the Shimura variety over the ring of integers $\mathcal{O}_E$, where $E$ is the completion of the reflex field $\mathbb{E}$ at some finite place $p$. Denote by $\text{Sh}_K(G,X)$ the Shimura variety given by the Shimura datum $(G,X)$ and choice of an open compact subgroup $K = \prod \ell K_{\ell} \subset G(\mathcal{A}_f)$, where $\mathcal{A}_f$ is the ring of finite rational adèles. For Shimura varieties of PEL-type, which are moduli spaces of abelian varieties with certain (polarization, endomorphism, and level) structures, one can define such an integral model by proposing a moduli problem over $\mathcal{O}_E$. The study of such models began with modular curves by Shimura and Deligne-Rapoport. More generally, Langlands, Kottwitz, Rapoport-Zink, Chai, and others studied these models for various types of PEL Shimura varieties. The reduction modulo $p$ of these integral models is nonsingular if the factor $K_p \subset G(\mathbb{Q}_p)$ is chosen to be “hyperspecial” for the rational prime $p$ lying under $p$. However if the level subgroup $K_p$ is not hyperspecial, usually (although not always) singularities occur. It is important to determine what kinds of singularities can occur, and this is expected to be influenced by the level subgroup $K_p$.

In order to study the singularities of these integral models, significant progress has been made by finding “local models”. These are schemes defined in simpler terms which control the singularities of the integral model. They first appeared in [DP] for Hilbert modular varieties and in [dJ2] for Siegel modular varieties with Iwahori level subgroup. More generally, Rapoport and Zink have constructed local models for PEL Shimura varieties with parahoric level subgroup [RZ].
Görtz has shown that in the case of a Shimura variety of PEL-type associated with a unitary group which splits over an unramified extension of $\mathbb{Q}_p$, the Rapoport-Zink local models are flat with reduced special fiber [Gor1]. In [Gor2], the same is shown for the local models of Siegel modular varieties. On the other hand, Pappas has shown that these local models can fail to be flat in the case of a ramified extension [Pap2]. In [PR1], [PR2], and [PR3], Pappas and Rapoport give alternative definitions of the local models which are flat. More recently in [PZ], Pappas and Zhu have given a general group-theoretic definition of the local models which, for PEL cases, agrees with the Rapoport-Zink local models in the unramified case and the alternative definitions in the ramified case.

Throughout this article, $K_p$ is assumed to be either an Iwahori subgroup of $G(\mathbb{Q}_p)$ or the pro-unipotent radical of an Iwahori subgroup. There is some ambiguity in calling these $\Gamma_0(p)$-level structure and $\Gamma_1(p)$-level structure respectively; indeed one may consider more generally a parahoric subgroup. As such, we will call the former $Iw_0(p)$-level structure and the latter $Iw_1(p)$-level structure. In all the situations we consider, $G = G_{Q_p}$ extends to a reductive group over $\mathbb{Z}_p$ and one can take an Iwahori subgroup as being the inverse image of a Borel subgroup of $G(F_p)$ under the reduction $G(\mathbb{Z}_p) \to G(F_p)$.

We will also take $K^p = \prod_{\ell \neq p} K_\ell$ to be a sufficiently small open compact subgroup of $G(\mathbb{A}_f)$ so that the moduli problems we consider below are represented by schemes.

Haines and Rapoport [HR], interested in determining the local factor of the zeta function associated with the Shimura variety, constructed affine schemes which are étale locally isomorphic to integral models of certain Shimura varieties with $Iw_1(p)$-level structure. This follows the older works of Pappas [Pap1] and Harris-Taylor [HT].

Haines and Rapoport consider the case of a Shimura variety associated with a unitary group which splits locally at $p$ given by a division algebra $B$ defined over an imaginary quadratic extension of $\mathbb{Q}$. The cocharacter associated with the Shimura datum is assumed to be “of Drinfeld type”.

In this article, we will consider $Iw_1(p)$-level structure for two particular types of PEL Shimura varieties. First the unitary case, where the division algebra $B$ has center $F$, an imaginary quadratic extension of a totally real finite extension $F^+$ of $\mathbb{Q}$ with $(p)$ is unramified in $F^+$. We will make assumptions on $p$ so that the unitary group $G$ in the Shimura datum splits over an unramified extension of $\mathbb{Q}_p$ as $GL_n \times G_{m}$. The second case is that of the Siegel modular varieties where the group in the Shimura datum is $G = GSp_{2n}$. We will refer to this as the symplectic case.

The moduli problem defining the integral model with $Iw_0(p)$-level structure is given in terms of chains of isogenies of abelian schemes with certain additional structures. We write $\mathcal{A}^{GL}_{Iw}$ and $\mathcal{A}^{GSp}_{Iw}$ for the scheme representing this moduli problem in the unitary and symplectic cases respectively. Then in these two cases, the moduli problem defining the integral model with $Iw_1(p)$-level structure is given by also including choices of “Oort-Tate generators” for certain group schemes associated with the kernels of the isogenies (see Section 3.2 for the notion of an Oort-Tate generator). Let $\mathcal{A}^{GL}_{Iw}$ and $\mathcal{A}^{GSp}_{Iw}$ denote the schemes representing these moduli problems in each case respectively. To study the singularities of $\mathcal{A}^{GL}_{Iw}$ and $\mathcal{A}^{GSp}_{Iw}$, we will construct étale local models.

**Definition 1.1.** Let $X$ and $M$ be schemes. We say that $M$ is an étale local model of $X$ if there exists an étale cover $V \to X$ and an étale morphism $V \to M$.

In order to describe our results in the unitary case, we begin by recalling the local model...
By loc. cit. we can take is an étale local model of the single closed point, with $U$ an affine open neighborhood of the “worst point”, i.e. the unique cell which consists of a $SL$ of rank $r$ of rank $r$, of the local model can be embedded into the affine flag variety for $\mathcal{A}^\text{GL}_0$ of $\mathcal{A}^\text{GL}_0$ is determined by giving a diagram

$$
\begin{array}{cccccccc}
\mathcal{O}^n_S & \xrightarrow{\varphi_0} & \mathcal{O}^n_S & \xrightarrow{\varphi_1} & \cdots & \xrightarrow{\varphi_{n-2}} & \mathcal{O}^n_S & \xrightarrow{\varphi_{n-1}} & \mathcal{O}^n_S \\
\downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow \\
\mathcal{F}_0 & \to & \mathcal{F}_1 & \to & \cdots & \to & \mathcal{F}_{n-1} & \to & \mathcal{F}_n \\
\end{array}
$$

where $\varphi_i$ is given by the matrix $\text{diag}(p^{i+1}, 1^{n-i-1})$ with respect to the standard basis, $\mathcal{F}_i$ is an $\mathcal{O}_S$-submodule of $\mathcal{O}^n_S$, and Zariski locally on $S$, $\mathcal{F}_i$ is a direct summand of $\mathcal{O}^n_S$ of rank $r$. With $S = \mathcal{M}^\text{loc}_\text{GL}$, the determinants

$$
\bigwedge^{\text{top}} \mathcal{F}_i \to \bigwedge^{\text{top}} \mathcal{F}_{i+1}
$$

and

$$
\bigwedge^{\text{top}} \mathcal{O}^n_S/\mathcal{F}_i \to \bigwedge^{\text{top}} \mathcal{O}^n_S/\mathcal{F}_{i+1}
$$

determine global sections $q_i$ and $q_i^*$ of the universal line bundles

$$
q_i = \left(\bigwedge^{\text{top}} \mathcal{F}_i\right)^{-1} \otimes \bigwedge^{\text{top}} \mathcal{F}_{i+1}
$$

and

$$
q_i^* = \left(\bigwedge^{\text{top}} \mathcal{O}^n_S/\mathcal{F}_i\right)^{-1} \otimes \bigwedge^{\text{top}} \mathcal{O}^n_S/\mathcal{F}_{i+1}
$$

respectively.

The special fiber of the local model can be embedded into the affine flag variety for $SL_n$ and identified with a disjoint union of Schubert cells [Gör1]. Let $U \subset \mathcal{M}^\text{loc}_\text{GL}$ be an affine open neighborhood of the “worst point”, i.e. the unique cell which consists of a single closed point, with $U$ sufficiently small so that each $Q^*_i$ is trivial. Choosing such a trivialization, we can then identify the sections $q_i^*$ with regular functions on $U$.

**Theorem 1.2.** The scheme

$$
U_1 = \text{Spec}_U \left(\mathcal{O}[u_0, \ldots, u_{n-1}]/(u_0^{p-1} - q_0^*, \ldots, u_{n-1}^{p-1} - q_{n-1}^*)\right)
$$

is an étale local model of $\mathcal{A}^\text{GL}_1$.

By loc. cit. we can take $U = \text{Spec}(B_{\text{GL}})$ where

$$
B_{\text{GL}} = \mathbb{Z}[[a_{jk}^i, i = 0, \ldots, n - 1, j = 1, \ldots, n - r, k = 1, \ldots, r]]/I
$$

and $I$ is the ideal generated by the entries of certain matrices. To make the above theorem completely explicit, we will compute the $q_i^*$ with respect to this presentation. They are given by the strikingly simple expression $q_i^* = a_{n-r}^{i+1}$ for $0 \leq i \leq n - 1$, where the upper index is taken modulo $n$. As a result, the integral models with Iw$_1(p)$-level structure are reasonably well-behaved and can be explicitly analyzed.
For the symplectic case, the moduli problem defining the integral model $A^{\text{GSp}}_0$ is again given in terms of chains of isogenies of abelian schemes with certain additional structures. Our construction of the local models for $A^{\text{GSp}}_1$ is similar to that of the unitary case. In particular, they are explicitly defined as well.

It is also of interest to have certain resolutions of the integral model of the Shimura variety with “nice” singularities, for example one which is semi-stable or locally toroidal. This problem was considered in the case of $Iw_0(p)$-level structure by Genestier [Gen], Faltings [Fal], de Jong [dJ1], and Götz [Gör3] among others. Using the explicitly defined local model, and in particular the rather simple expression for $q_1^*$, we will construct a resolution of $A^{\text{GSp}}_1$ in the case $n = 2$.

**Theorem 1.3.** Let $A_1$ denote the moduli scheme for the Siegel modular variety of genus 2 with $Iw_1(p)$-level structure. There is a regular scheme $\tilde{A}_1$ with special fiber a nonreduced divisor with normal crossings\(^1\) that supports a proper birational morphism $\tilde{A}_1 \to A_1$.

Moreover, we will find the number of geometric irreducible components of $\tilde{A}_1 \otimes \mathbb{F}_p$, and describe how they intersect using a “dual complex”, see Theorem 5.24 for details.

We now outline the construction of $\tilde{A}_1$. It begins with a known semi-stable resolution $\tilde{A}_0 \to A_0$ [dJ1]. This gives a modification (i.e. proper birational morphism) $A_1 \times_{A_0} \tilde{A}_0 \to \tilde{A}_1$. The scheme $A_1 \times_{A_0} \tilde{A}_0$ is not normal. Let $Z$ be the reduced closed subscheme of $\tilde{A}_0$ whose support is the locus of closed points where all of the corresponding kernels of the isogenies are infinitesimal. Also let $W$ be the unique irreducible component of the special fiber of $\tilde{A}_0$ where each kernel is generically isomorphic to $\mu_p$. Take the strict transform of $Z$ and $W$ with respect to the morphism $\tilde{A}_0 \to A_0$. Denote by $Z'$ and $W'$ the reduced inverse image of these strict transforms with respect to the projection $A_1 \times_{A_0} \tilde{A}_0 \to \tilde{A}_0$. Consider the modification given by the blowup of $A_1 \times_{A_0} \tilde{A}_0$ along $Z'$:

$$\text{Bl}_{Z'}(A_1 \times_{A_0} \tilde{A}_0) \to A_1 \times_{A_0} \tilde{A}_0.$$ 

We will see that $\text{Bl}_{Z'}(A_1 \times_{A_0} \tilde{A}_0)$ is normal. Let $W''$ denote the strict transform of $W'$ with respect to the modification $\text{Bl}_{Z'}(A_1 \times_{A_0} \tilde{A}_0) \to A_1 \times_{A_0} \tilde{A}_0$. We arrive at $\tilde{A}_1$ by first blowing up $\text{Bl}_{Z'}(A_1 \times_{A_0} \tilde{A}_0)$ along $W''$, and then blowing up each resulting modification along the strict transform of $W''$ stopping after a total of $p - 2$ blowups. Carrying out the corresponding process on the local model, we will show by explicit computation that the resulting resolution of the local model is regular with special fiber a nonreduced divisor with normal crossings. It will then follow that $\tilde{A}_1$ has these properties as well. By keeping track of how certain subschemes transform in each step of the above process, with much of this information coming from the explicit computation of the modifications of the local model, we will be able to find the number of geometric irreducible components of $\tilde{A}_1 \otimes \mathbb{F}_p$ as well as describe how they intersect.

In closing we mention that as this article was prepared, T. Haines and B. Stroh announced a similar construction of local models in order to prove the analogue of the Kottwitz nearby cycles conjecture. They relate their local models to “enhanced” affine flag varieties.

\(^1\)In this article, by a “nonreduced divisor with normal crossings” we mean a divisor $D$ such that in the completion of the local ring at any closed point, $D$ is given by $\mathcal{Z}(f_1^{e_1} \cdots f_t^{e_t})$ where $\{f_1, \ldots, f_t\}$ are part of a regular system of parameters and the integers $e_i$ are greater than zero.
Finally, I would like to thank G. Pappas for introducing me to this area of mathematics and for his invaluable support. I would also like to thank M. Rapoport for a useful conversation, T. Haines and B. Stroh for communicating their results, and U. Görtz for providing the source for Figure 1 to which some modifications were made.

2 Review of the unitary case with $Iw_0(p)$-level structure

2.1 The integral model $A_0^{GL}$

Let $F$ be an imaginary quadratic extension of some totally real finite extension $F^+/\mathbb{Q}$, $D$ a finite dimensional division algebra with center $F$, $*$ an involution on $D$ which induces the nontrivial element of $\text{Gal}(F/F^+)$ on $F$, and $h_0 : C \to D_\mathbb{R}$ an $\mathbb{R}$-algebra homomorphism such that $h_0(z)^* = h_0(z)$ and the involution $x \to h_0(i)^{-1}x^*h_0(i)$ is positive. We make the following assumptions:

- $p$ is an odd rational prime, unramified in $F^+$, such that each factor of $(p)$ in $F^+$ splits in $F$; and
- the division algebra $D$ splits over $\mathbb{Q}_p$.

We first consider the case $F^+ = \mathbb{Q}$. In the general case, after an unramified base extension the local model (described below) becomes isomorphic to a product of local models in the case $F^+ = \mathbb{Q}$, indexed by the factors of $(p)$ in $F^+$. We will return to the general case in Section 3.3.

The datum $(D,*,h_0)$ induces a PEL Shimura datum. We will briefly recall the crucial points, see [Hai2] Section 5 for details. Set $B = D^{opp}$ and $V = D$ viewed as a left $B$-module using right multiplications. Let $G$ be the reductive group given by $G(R) = \{ x \in D_R : x^*x \in R^+ \}$ for a $\mathbb{Q}$-algebra $R$ and set $G = G_{\mathbb{Q}_p}$. Fixing once and for all the embeddings $\mathbb{C} \hookrightarrow \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, let $\mu : \mathbb{Q}^{m,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$ be the minuscule cocharacter induced by $h_0$. We have that $h_0$ induces the decomposition $V_\mathbb{C} = V^+ \oplus V^-$ where $h_0(z \otimes 1)$ acts by $z$ on $V^+$ and by $\overline{z}$ on $V^-$. Choose an isomorphism $D_\mathbb{C} \cong M_n(\mathbb{C}) \times M_n(\mathbb{C})$ such that $h_0(z \otimes 1) = \text{diag}(\overline{z}^{n-r},z^r) \times \text{diag}(z^{n-r},\overline{z}^r)$ for some $1 \leq r \leq n-1$. Then we have $\mu(z) = \text{diag}(1^{n-r},(z^{-1})^r) \times \text{diag}((z^{-1})^{n-r},1^r)$ which we denote by $\mu = (0^{n-r},(-1)^r)$.

With $(p) = pp^*$ in $F$, we have $F_{\mathbb{Q}_p} = F_p \times F_{p^*}$, making $D_{\mathbb{Q}_p} = D_p \times D_{p^*}$. Note that $F_{p^*} = F_{p^*} = \mathbb{Q}_p$ and so $D_p$ and $D_{p^*}$ are $\mathbb{Q}_p$-algebras with $*$ inducing $D_p \cong D_{p^*}^{opp}$. The second assumption above means $D_p \cong M_n(\mathbb{Q}_p)$ and thus $G \cong \text{GL}_n,\mathbb{Q}_p \times \mathbb{G}_m,\mathbb{Q}_p$. With $G$ split over $\mathbb{Q}_p$ we have that $E$, the reflex field at $p$, is $\mathbb{Q}_p$. Using the fixed embeddings $\mathbb{C} \hookrightarrow \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, let $E' \subset \overline{\mathbb{Q}}_p$ be a finite extension of $E$ over which we have the decomposition $V_{E'} = V^+ \oplus V^-$. To define the integral model, we must specify a lattice chain $L$ and a maximal order $O_B \subset B$. Let $\xi \in D^{\times}$ be such that $\xi^* = -\xi$ and $\iota(x) = \xi x^* \xi^{-1}$ is a positive involution on $D$. Then $(\cdot,\cdot) : D \times D \to \mathbb{Q}$ defined by $(x,y) = \text{Tr}_{D/\mathbb{Q}}(x\xi y^*)$ is a nondegenerate alternating pairing. Fix an isomorphism $D_{\mathbb{Q}_p} = D_p \times D_{p^*} \cong M_n(\mathbb{Q}_p) \times M_n(\mathbb{Q}_p)$ so that $\iota$ goes to the standard involution $(X,Y) \to (Y^t,X^t)$. Set $(\chi^t,-\chi)$ to be the image of $\xi$ under this isomorphism with $\chi \in M_n(\mathbb{Q}_p)$. Define $\Lambda_t$ to be the $Z_p$-lattice chain in
$M_n(\mathbb{Q}_p)$ given by $\Lambda_i = \text{diag}(p^{-1}, 1^{n-i})M_n(\mathbb{Z}_p)$ for $0 \leq i \leq n-1$ and extend it periodically to all $i \in \mathbb{Z}$ by $\Lambda_{n+i} = p^{-1}\Lambda_i$. Likewise, $\Lambda^*_i = \chi^{-1}\text{diag}(1, (p^{-1})^{n-i})M_n(\mathbb{Z}_p)$ for $0 \leq i \leq n-1$ and again extended periodically. One may check that $(\Lambda_i \oplus \Lambda^*_i)^\perp = \Lambda_{-i} \oplus \Lambda^*_{-i}$ where

$$(\Lambda_i \oplus \Lambda^*_i)^\perp = \{x \in D_{\mathbb{Q}_p} : (x, y) \in \mathbb{Z}_p \text{ for all } y \in \Lambda_i \oplus \Lambda^*_i\},$$

and thus the lattice chain $L = (\Lambda_i \oplus \Lambda^*_i)_i$ is self-dual. Finally the maximal order $O_B \subset B$ is chosen so that under the identification $D_{\mathbb{Q}_p} \cong M_n(\mathbb{Q}_p) \times M_n(\mathbb{Q}_p)$, we have $O_B \otimes \mathbb{Z}_p = M^n_{opp}(\mathbb{Z}_p) \times M^n_{opp}(\mathbb{Z}_p)$.

$K$ is taken to be a compact open subgroup of $G(A_f)$ of the form $K = K^p K^p$, where $K^p \subset \mathbb{G}(A^p)$ is a sufficiently small compact open subgroup and $K_p = \text{Aut}_{O^p}(L)$, the automorphism group of the polarized multichain $L$.

With the data above, the integral model $A^\text{GL}_0$ of the Shimura variety is given in [RZ, Definition 6.9]. It is the moduli scheme over $\text{Spec}(\mathbb{Z})$ representing the following functor. For any $\mathbb{Z}_p$-scheme $S$, $A^\text{GL}_0(S) = A^\text{GL}_0(S) \circ \gamma(S)$ is the set of tuples $(A_\bullet, \lambda, \bar{\eta})$, up to isomorphism, where

(i) $A_\bullet$ is a chain $A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} \ldots \xrightarrow{\alpha_{n-1}} A_n$ of $n$-dimensional abelian schemes over $S$, determined up to prime-to-$p$ isogeny, with each morphism $\alpha_i$ an isogeny of degree $p^{2n}$;

(ii) each $A_i$ is equipped with an $O_B \otimes \mathbb{Z}_p$-action that commutes with the isogenies $\alpha_i$;

(iii) there are “periodicity isomorphisms” $\theta_p : A_{i+n} \xrightarrow{\sim} A_i$ such that for each $i$ the composition

$$A_i \twoheadrightarrow A_{i+1} \twoheadrightarrow \ldots \twoheadrightarrow A_{i+n} \xrightarrow{\theta_p} A_i$$

is multiplication by $p$;

(iv) the action of $O_B \otimes \mathbb{Z}_p$ satisfies the Kottwitz condition: for each $i$

$$\det_{O_B}(b; \text{Lie}(A_i)) = \det_{E'}(b; V^+) \quad \text{for all } b \in O_B;
$$

(v) $\bar{\lambda}$ is a $\mathbb{Q}$-homogeneous class of principal polarizations [RZ, Definition 6.7]; and

(vi) $\bar{\eta}$ is a $K^p$-level structure [Kot, Section 5].

2.2 Local model diagram

As constructed in [RZ, Chapter 3] and [Pap2, Theorem 2.2], there is a local model diagram

$$A^\text{GL}_0 \xrightarrow{\Phi} A^\text{GL}_0 \xrightarrow{\Phi} M^\text{loc}_{\text{GL}}$$

described as follows. Let $(A_\bullet, \bar{\lambda}, \bar{\eta}) \in A^\text{GL}_0(S)$. For an abelian scheme $A/S$ of relative dimension $n$, denote by $H^1_{\text{dR}}(A/S)^\vee$ the $O_S$-dual of the de Rham cohomology sheaf. It is a locally free $O_S$-module of rank $2n^2$ [BBM, Section 2.5], and the collection $(H^1_{\text{dR}}(A_i/S)^\vee)$, gives a polarized multichain of $O_B \otimes \mathbb{Z}_p$, $O_S$-modules of type $(\mathcal{L})$ [RZ, 3.23b]. Now $A^\text{GL}_0$ is the scheme representing the functor which associates with an
\( \mathcal{O}_E \)-scheme \( S \) the set of tuples \((A_\bullet, \bar{\lambda}, \bar{\eta}, \{\gamma_i\})\), up to isomorphism, where \((A_\bullet, \bar{\lambda}, \bar{\eta}) \in A_{0GL}(S) \) and

\[
\gamma_i : H^1_{dR}(A_i/S)^\vee \to \Lambda_i \otimes \mathbb{Z}_p \mathcal{O}_S
\]

is an isomorphism of polarized multichains of \( \mathcal{O}_B \otimes \mathbb{Z}_p \mathcal{O}_S \)-modules. Define the morphism

\[
\Phi : \overline{\mathcal{A}}^{GL}_0 \to \mathcal{A}_{0GL} \quad \text{by} \quad (A_\bullet, \bar{\lambda}, \bar{\eta}, \{\gamma_i\}) \to (A_\bullet, \bar{\lambda}, \bar{\eta}).
\]

Let \( \mathcal{G} \) be the smooth affine group scheme where \( \mathcal{G}(S) = \text{Aut}((\Lambda_i \otimes \mathcal{O}_S)) \) for an \( \mathcal{O}_E \)-scheme \( S \) \cite[Theorem 3.11]{RZ}. Then \( \Phi \) is a smooth surjective \( \mathcal{G} \)-torsor \cite[Theorem 2.2]{Pap2}.

The local model \( M_{GL}^{loc} \) is the \( \mathcal{O}_E \)-scheme representing the moduli problem given in \cite[Definition 3.27]{RZ}, which is equivalent to the following. See \cite[Section 6.3.3]{Hai2} for details.

**Definition 2.1.** With \( S \) an \( \mathcal{O}_E \)-scheme, an \( S \)-valued point of \( M_{GL}^{loc} \) is given by the following data.

- A functor from the category \( \mathcal{L} \) to the category of \( \mathcal{O}_B \otimes \mathbb{Z}_p \mathcal{O}_S \)-modules on \( S \)
  \[
  \Lambda \to \omega_\Lambda, \quad \Lambda \in \mathcal{L}.
  \]
- A morphism of functors \( \psi_\Lambda : \omega_\Lambda \to \Lambda \otimes \mathbb{Z}_p \mathcal{O}_S \).

We require that the following conditions are satisfied.

(i) The morphisms \( \psi_\Lambda \) are injective.

(ii) The quotients \( t_\Lambda := (\Lambda \otimes \mathbb{Z}_p \mathcal{O}_S)/\psi(\omega_\Lambda) \) are locally free \( \mathcal{O}_S \)-modules of finite rank.

For the action of \( \mathcal{O}_B \) on \( t_\Lambda \), we have the Kottwitz condition:

\[
\det_{\mathcal{O}_S}(b|t_\Lambda) = \det_{E^0}(b|V^+) \quad \text{for all} \quad b \in \mathcal{O}_B.
\]

(iii) For each \( \Lambda \in \mathcal{L} \), \( \omega^{perp}_\Lambda = \omega_{\Lambda^\perp} \) where for an open subscheme \( U \subset S \),

\[
\omega^{perp}_\Lambda(U) = \left\{ y \in (\Lambda^\perp \otimes \mathbb{Z}_p \mathcal{O}_S)(U) : (x, y) = 0 \quad \text{for all} \quad x \in \omega_\Lambda(U) \right\}.
\]

Note that \( \mathcal{G} \) acts on the local model by acting on \( \psi_\Lambda \) through its natural action on \( \Lambda \otimes \mathcal{O}_S \). To define the morphism \( \Psi \), set \( \omega_{A_i} = \text{Lie}(A_i)^\vee \) which is a locally free sheaf of rank \( n^2 \) on \( S \). By \cite[Prop. 5.1.10]{BBM}, we have the Hodge filtration

\[
0 \to \omega_{\hat{A}_i} \to H^1_{dR}(A_i/S)^\vee \to \text{Lie}(A_i) \to 0
\]

where \( \hat{A}_i \) denotes the dual abelian scheme of \( A_i \). We can now associate with each point \((A_\bullet, \bar{\lambda}, \bar{\eta}, \{\gamma_i\}) \in \overline{\mathcal{A}}^{GL}_0(S) \) the collection of injective morphisms \( \omega_{\hat{A}_i} \to H^1_{dR}(A_i/S)^\vee \cong \Lambda_i \otimes \mathbb{Z}_p \mathcal{O}_S \) and this defines the morphism \( \Psi : \overline{\mathcal{A}}^{GL}_0 \to M_{GL}^{loc} \).

**Theorem 2.2.** \cite[Chapter 3]{RZ} (see also \cite[Section 2]{Pap2} and \cite[Section 6]{Hai2}) The diagram above is a “local model diagram”. Specifically \( \Psi \) is a surjective \( \mathcal{G} \)-torsor, \( \Psi \) is a smooth \( \mathcal{G} \)-equivariant morphism, and there exists an étale cover \( \varphi : V \to \mathcal{A}_{0GL} \) with a section \( \sigma : V \to \mathcal{A}_{0GL} \) of \( \Phi \) such that \( \Psi \circ \sigma \) is étale.
Remark 2.3. Given a closed point $x$ of $\mathcal{A}_0^{GL}$, we will say that a closed point $y$ of $M_{GL}^{\text{loc}}$ corresponds to $x$ if there exists $\varphi$ and $\sigma$ as in the above theorem, along with a closed point $v$ of $V$ such that $\varphi(v) = x$ and $\Psi \circ \sigma(v) = y$.

Recall the identification $\mathcal{O}_B \otimes \mathbb{Z}_p \cong M_{n}^{\text{opp}}(\mathbb{Z}_p) \times M_{n}^{\text{opp}}(\mathbb{Z}_p)$. With the $\mathcal{O}_B \otimes \mathbb{Z}_p$-action on the sheaves in the moduli problem defining $M_{GL}^{\text{loc}}$, by Morita equivalence and the Kottwitz condition we get the following. Let $e_{11} \in M_{n}^{\text{opp}}(\mathbb{Z}_p) \times M_{n}^{\text{opp}}(\mathbb{Z}_p)$ denote the idempotent of the first factor and write $\Lambda_{i,S} = \Lambda_i \otimes_{\mathbb{Z}_p} \mathcal{O}_S$.

Proposition 2.4. An $S$-valued point of $M_{GL}^{\text{loc}}$ is determined by a diagram

\[
\begin{array}{cccc}
e_{11} \Lambda_{0,S} & \varphi_0 & & e_{11} \Lambda_{1,S} & \varphi_1 & & \cdots & \varphi_{n-1} & e_{11} \Lambda_{n,S} \\
\mathcal{F}_0 & \rightarrow & \mathcal{F}_1 & \rightarrow & \cdots & \rightarrow & \mathcal{F}_n
\end{array}
\]

where

- $\varphi_i$ is the morphism induced from the inclusion $\Lambda_i \subset \Lambda_{i+1}$; and
- $\mathcal{F}_i$ is a locally free $\mathcal{O}_S$-module which is Zariski locally a direct summand of $\Lambda_{i,S}$ of rank $r$.

Proof. An $S$-valued point of $M_{GL}^{\text{loc}}$ as in Definition 2.4 is given by the data $\{\omega_i \mapsto \Lambda_{i,S}\}$. Setting $\mathcal{F}_i = e_{11} \omega_i$ gives the diagram above. See [Hai2, Section 6.3.3] for details. $\square$

3 The integral and local model of $\mathcal{A}_1^{GL}$

Throughout this section, $S$ is an $\mathcal{O}_E$-scheme and $k$ is an algebraically closed field.

3.1 The group schemes $G_i$

In order to define $\mathcal{A}_1^{GL}$, we will first associate with an $S$-valued point of $\mathcal{A}_0^{GL}$ a collection of finite flat group schemes $\{G_i\}_{i=0}^{n-1}$ each of order $p$. Then, given a geometric point $x : \text{Spec}(k) \rightarrow \mathcal{A}_0^{GL}$ we will determine the isomorphism type of the group schemes $G_i$ from the data given by a corresponding geometric point $y : \text{Spec}(k) \rightarrow M_{GL}^{\text{loc}}$ in the sense of Remark 2.3.

Let $(A_1, \lambda, \eta) \in \mathcal{A}_0^{GL}(S)$. For each $i \in \mathbb{Z}$, consider the $p$-divisible group $A_i(p^\infty) = \lim_{\longrightarrow} A_i[p^n]$. Note that $H_i := \ker(A_i \rightarrow A_{i+1})$ is contained in $A_i(p^\infty)$. With $e_{11} \in M_{n}^{\text{opp}}(\mathbb{Z}_p) \times M_{n}^{\text{opp}}(\mathbb{Z}_p)$ the idempotent of the first factor, the action of $\mathcal{O}_B \otimes \mathbb{Z}_p$ gives a chain

\[
e_{11} A_0(p^\infty) \rightarrow e_{11} A_1(p^\infty) \rightarrow \cdots \rightarrow e_{11} A_{n-1}(p^\infty) \rightarrow e_{11} A_n(p^\infty)
\]

of isogenies of degree $p$ whose composition with the map $e_{11} A_n(p^\infty) \xrightarrow{\sim} e_{11} A_0(p^\infty)$ (induced by the periodicity isomorphism) is multiplication by $p$. For $0 \leq \ell \leq n-1$, we define $G_{\ell}$ to be the finite flat group scheme of order $p$ given by

\[
G_{\ell} = \ker(e_{11} A_{\ell}(p^\infty) \rightarrow e_{11} A_{\ell+1}(p^\infty)).
\]
Definition 3.1. For a group scheme $G/S$, define $\omega_G = \omega_{G/S}$ to be the sheaf on $S$ given by $\varepsilon^*(\Omega^1_{G/S})$ where $\varepsilon : S \to G$ is the identity section.

When $S$ is affine and $G/S$ is a finite flat group scheme, we denote by $G^*$ the Cartier dual of $G$.

Proposition 3.2. With $S = \text{Spec}(k)$, let $x : S \to A^G_0$ be a geometric point and $y : S \to M^\text{loc}_{\text{GL}}$ a corresponding geometric point in the sense of Remark 2.3. Let $\{\omega_i \to \Lambda_{i,S}\}$ be the data induced by $y$ as in Definition 2.1 and set $F_i = e_{11}\omega_i$. Then we have the following:

(i) $\dim_k \omega_{H_i}^* = \dim_k (\omega_{i+1}/\varphi_i(\omega_i))$;
(ii) $\dim_k \omega_{H_i} = \dim_k (\Lambda_{i+1,S}/(\varphi_i(\Lambda_{i,S}) + \omega_{i+1}))$;
(iii) $\dim_k \omega_G^* = \dim_k (\varphi_i(F_{i+1}/\varphi_i(F_i)))$; and
(iv) $\dim_k \omega_G = \dim_k e_{11}(\Lambda_{i+1,S}/(\varphi_i(e_{11}\Lambda_i,S) + F_{i+1}))$.

Proof. Start with the standard exact sequence of Kähler differentials induced by the morphisms $\hat{A}_{i+1} \to \hat{A}_i \to S$ and $A_i \to A_{i+1} \to S$ respectively. Pull these sequences of sheaves back to $S$ by the appropriate identity section. Then (i) and (ii) quickly follow from the isomorphisms

$$\omega_{H_i}^* \cong \varepsilon^*_{\hat{A}_{i+1}}(\Omega^1_{\Lambda_{i+1}/\hat{A}_i}) \quad \text{and} \quad \omega_{H_i} \cong \varepsilon^*_{\Lambda_i}(\Omega^1_{\Lambda_i/A_{i+1}}).$$

Now (iii) and (iv) follow from the functorality of the decompositions.

For a general $S$-valued point $y : S \to M^\text{loc}_{\text{GL}}$, the maps $\varphi_i : F_i \to F_{i+1}$ and $\varphi_i^* : e_{11}\Lambda_i,S/F_i \to e_{11}\Lambda_{i+1,S}/F_{i+1}$ induce global sections $q_i$ and $q_i^*$ of the line bundles

$$Q_i = \left(\bigwedge^\text{top} F_i\right)^{-1} \otimes F_{i+1} \quad \text{and} \quad Q_i^* = \left(\bigwedge^\text{top} e_{11}\Lambda_i,S/F_i\right)^{-1} \otimes \bigwedge^\text{top} e_{11}\Lambda_{i+1,S}/F_{i+1}$$

respectively. In the case $S = \text{Spec}(k)$, $q_i$ and $q_i^*$ are the determinants of the corresponding linear maps. Note that $\Lambda_{i,S} \to \Lambda_{i+1,S}$ carries $F_i$ into $F_{i+1}$, and hence the determinant of $e_{11}\Lambda_{i,S} \to e_{11}\Lambda_{i+1,S}$ is equal to the product of the determinants of $F_i \to F_{i+1}$ and $e_{11}\Lambda_{i,S}/F_i \to e_{11}\Lambda_{i+1,S}/F_{i+1}$. Thus $q_i \otimes q_i^* = p$.

Proposition 3.3. With the same hypotheses as the previous proposition, denote by $q_i$ and $q_i^*$ the global sections induced by $y$ as described above.

(i) $q_i = 0$ if and only if $\dim_k \omega_{G_i^*} = 1$.
(ii) $q_i^* = 0$ if and only if $\dim_k \omega_{G_i} = 1$.

Proof. This follows immediately from Proposition 3.2 as $q_i \neq 0$ if and only if $F_i$ is carried isomorphically onto $F_{i+1}$, and similarly with $q_i^*$.

If $\text{char}(k) \neq p$, then any finite flat group scheme of order $p$ over $\text{Spec}(k)$ is isomorphic to the constant group scheme $\mathbb{Z}/p\mathbb{Z}$. If $\text{char}(k) = p$, then up to isomorphism there are three finite flat group schemes $G$ of order $p$ over $\text{Spec}(k)$: $\mathbb{Z}/p\mathbb{Z}$, $\mu_p$, and $\alpha_p$ [OT, Lemma 1]. The Cartier dual of $\mathbb{Z}/p\mathbb{Z}$ is $\mu_p$, and the Cartier dual of $\alpha_p$ is $\alpha_p$ itself. It is easy to compute the following table.
\[
\begin{array}{|c|c|c|c|}
\hline
G & \mu_p & \mathbb{Z}/p\mathbb{Z} & \alpha_p \\
(\dim_k \omega_G, \dim_k \omega_G^*) & (1, 0) & (0, 1) & (1, 1) \\
\hline
\end{array}
\]

Thus knowing \( q_i \) and \( q_i^* \), one can determine the isomorphism type of \( G_i \).

**Corollary 3.4.** Consider the divisor on \( S \) defined by the vanishing of \( q_i^* \). Then the support of this divisor is precisely the locus given by the collection of closed points where the corresponding group scheme \( G_i \) is infinitesimal.

### 3.2 The integral model \( \mathcal{A}_1^{GL} \)

To define \( \mathcal{A}_1^{GL} \), we will use the theory of Oort-Tate. We recall the concise summary given in [HR, Theorem 3.3.1].

**Theorem 3.5.** [OT] Let \( OT \) be the \( \mathbb{Z}_p \)-stack representing finite flat group schemes of order \( p \).

(i) \( OT \) is an Artin stack isomorphic to

\[
[(\text{Spec} \mathbb{Z}_p[X,Y]/(XY - w_p))/\mathbb{G}_m]
\]

where \( \mathbb{G}_m \) acts via \( \lambda \cdot (X,Y) = (\lambda^{p-1} X, \lambda^1 Y) \). Here \( w_p \) denotes an explicit element of \( p\mathbb{Z}_p^\times \) given in loc. cit.

(ii) The universal group scheme \( G_{OT} \) over \( OT \) is

\[
G_{OT} = [(\text{Spec}_{OT} \mathcal{O}[Z]/(Z^p - XZ))/\mathbb{G}_m],
\]

(where \( \mathbb{G}_m \) acts via \( Z \to \lambda Z \)), with identity section \( Z = 0 \).

(iii) Cartier duality acts on \( OT \) by interchanging \( X \) and \( Y \).

As in [HR, Section 3.3], we denote by \( G_{OT}^\times \) the closed subscheme of \( G_{OT} \) defined by the ideal \( (Z^{p-1} - X) \). The morphism \( G_{OT}^\times \to OT \) is relatively representable, flat, and finite of degree \( p - 1 \).

**Definition 3.6.** Let \( \varphi : S \to OT \) be a morphism, \( G = S \times_{OT} G_{OT} \), and \( G^\times = S \times_{OT} G_{OT}^\times \). We say that a section \( c \in G(S) \) is an Oort-Tate generator if \( c \in G^\times(S) \).

The following definition and proposition solely serve to elucidate the notion of an Oort-Tate generator.

**Definition 3.7.** [KM, Section 1.8] Let \( Z \) be a finite locally free scheme over \( S \) of rank \( N \geq 1 \). Then we say that a set of sections \( P_1, \ldots, P_N \) in \( Z(S) \) is a full set of sections of \( Z/S \) if for every affine \( S \)-scheme \( T = \text{Spec}(R) \), and for every function \( f \in B = \Gamma(Z \times_S T, \mathcal{O}_{Z \times_S T}) \), we have

\[
\text{Norm}_{B/R}(f) = \prod_{i=1}^{N} f(P_i).
\]

**Proposition 3.8.** [HR, Remark 3.3.2] (cf. [Pap1, Section 5.1]) Let \( S \) be a \( \mathbb{Z}_p \)-scheme and \( \pi : G \to S \) a finite flat group scheme of order \( p \) with finite presentation over \( S \). Then \( c \in G(S) \) is an Oort-Tate generator if and only if the collection \( \{0, c, [2]c, \ldots, [p-1]c\} \) forms a full set of sections.
Example 3.9. Let $k$ be an algebraically closed field of characteristic $p$ and $G$ a finite flat group scheme of order $p$ over $\text{Spec}(k)$. Then the identity section $\varepsilon : \text{Spec}(k) \to G$ is an Oort-Tate generator if $G \cong \mu_p$ or $G \cong \alpha_p$, but it is not an Oort-Tate generator if $G \cong \mathbb{Z}/p\mathbb{Z}$.

Given a point of $A_0^{\text{GL}}$, we have associated with it the group schemes $\{G_i\}_{i=0}^{n-1}$ in the previous section. This induces the morphism $\varphi$ in the following definition.

**Definition 3.10.** $A_1^{\text{GL}}$ is the fibered product

$$
A_1^{\text{GL}} \to G_{\text{OT}}^x \times_{\mathbb{Z}/p} \cdots \times_{\mathbb{Z}/p} G_{\text{OT}}^x,
$$

In the next section, we will present the $i$th factor of the two fiber products in the diagram above as follows.

$$
OT = [\text{Spec} \mathbb{Z}_p[X_{i-1}, Y_{i-1}] / (X_{i-1}Y_{i-1} - w_p)] / G_m
$$

$$
G_{\text{OT}}^x = [\text{Spec}_{\text{OT}} \mathcal{O}[Z_{i-1}] / (Z_{i-1}^{p-1} - X_{i-1})] / G_m
$$

### 3.3 Local model of $A_1^{\text{GL}}$

Let $x : \text{Spec}(k) \to A_0^{\text{GL}}$ be a geometric point. By Theorem 2.2, there exists an étale neighborhood $V \to A_0^{\text{GL}}$ of $x$ and a section $\sigma : V \to A_0^{\text{GL}}$ of $A_0^{\text{GL}} \to A_0$ such that the composition $V \xrightarrow{\sigma} A_0^{\text{GL}} \xrightarrow{\psi} M_{\text{GL}}^{\text{loc}}$ is étale. Set $\psi = \Psi \circ \sigma$ and assume that $\psi$ factors through an open subscheme $U \subset M_{\text{GL}}^{\text{loc}}$ where each universal line bundle $\mathcal{Q}_i^*$ on $M_{\text{GL}}^{\text{loc}}$ is trivial. Choosing a trivialization, we will identify the global section $q_i^*$ of $\mathcal{Q}_i^*$ with a regular function on $U$. Note that if $\mathcal{Q}_i^*$ is trivial then so is $\mathcal{Q}_i$, and thus we also have $q_i \in \Gamma(U, \mathcal{O}_U)$. Consider the following diagram

$$
U \xleftarrow{\varphi_i} V \to A_0^{\text{GL}} \xrightarrow{\varphi} OT
$$

where $\varphi_i : A_0^{\text{GL}} \to OT$ is $\varphi$ followed by the $i$th projection.

**Proposition 3.11.** Let $\rho_i : V \to OT$ be the morphism in the diagram above. It is given by

$$
\rho_i^*(X_i) = \varepsilon_i \psi^*(q_i^*) \quad \text{and} \quad \rho_i^*(Y_i) = w_p \varepsilon_i^{-1} \psi^*(q_i)
$$

where $\varepsilon_i$ is a unit in $V$ and $w_p$ is as in Theorem 3.3.

**Proof.** The special fiber of $M_{\text{GL}}^{\text{loc}}$, and therefore of $V$, is reduced [Gor1, Theorem 4.25]. From the equalities $q_i q_i^* = p$ and $X_i Y_i = \omega_i p$, we have that the divisors defined by the vanishing of global sections $Z(\psi^*(q_i^*))$ and $Z(\rho_i^*(X_i))$ are reduced. By Corollary 3.4 and Example 3.9, the locus where $\psi^*(q_i^*)$ vanishes agrees with the locus where $\rho_i^*(X_i)$ vanishes. Therefore $Z(\psi^*(q_i^*)) = Z(\rho_i^*(X_i))$.

$M_{\text{GL}}^{\text{loc}}$ is of finite type over $\text{Spec}(\mathbb{Z}_p)$, its generic fiber is smooth and hence normal, and it is flat over $\text{Spec}(\mathbb{Z}_p)$ with reduced special fiber [Gor1, Theorem 4.25]. It follows that
$M_{GL}^{lc}$ is normal \cite[Proposition 8.2]{PZ}. Thus $V$ is normal as well, and so the equality of divisors above implies $\psi^*(q_i^*)$ and $\rho_i^*(X_i)$ are equal up to a unit, say $\varepsilon_i$. Similar statements apply to $\psi^*(q_i)$ and $\rho_i^*(Y_i)$, giving that $\psi^*(q_i)$ and $\rho_i^*(Y_i)$ are equal up to a unit. This unit must be $w_p\varepsilon_i^{-1}$ because $X_i Y_i = w_p p$ and $q_i q_i^* = p$.

**Proposition 3.12.** Let $\tilde{\eta} : \text{Spec}(k) \to A_1^{GL}$ be a geometric point and set $x = \pi(\tilde{\eta})$. Let $V \to A_0^{GL}$ be an étale neighborhood of $x$ which carries an étale morphism $\psi : V \to M_{GL}^{lc}$ as described above. Suppose $V \to M_{GL}^{lc}$ factors through an open affine subscheme $U \subset M_{GL}^{lc}$ on which $Q_i^*$ is trivial for each $i$. Set

$$U_1 = \text{Spec}_U \left( \mathcal{O}[u_0, \ldots, u_{n-1}] / \left( u_0^{p-1} - q_0^*, \ldots, u_{n-1}^{p-1} - q_{n-1}^* \right) \right).$$

Then there exists an étale neighborhood $\tilde{V}$ of $\tilde{\eta}$ and an étale morphism $\tilde{\psi} : \tilde{V} \to U_1$.

**Proof.** Set $\tilde{V} = V \times_{A_0^{GL}} A_1^{GL}$. Consider the diagram

$$U_1 \xleftarrow{\cdots} \tilde{V} \to A_1^{GL} \to G^\chi_{OT} \times \cdots \times G^\chi_{OT} \to OT \times \cdots \times OT$$

where the two right squares are cartesian. Denote by $\eta$ the morphism $\tilde{V} \to G^\chi_{OT} \times \cdots \times G^\chi_{OT}$. The morphism $G^\chi_{OT} \to OT$ is relatively representable and thus $\tilde{V}$ is isomorphic to

$$\text{Spec}_V \left( \mathcal{O}[u_0, \ldots, u_{n-1}] / \left( u_0^{p-1} - \rho_0^*(X_0), \ldots, u_{n-1}^{p-1} - \rho_{n-1}^*(X_{n-1}) \right) \right).$$

With the notation as in the previous proposition, by shrinking $V$ we can choose for each $i$ a $(p - 1)$th root of $\varepsilon_i$, denote it by $\tau_i$. Let $\tilde{V} \to U_1$ be the morphism induced by the ring homomorphism $\Gamma(U_1, \mathcal{O}_{U_1}) \to \Gamma(\tilde{V}, \mathcal{O}_{\tilde{V}})$ which sends $u_i$ to $\tau_i u_i$ for each $i$. This is well-defined by the previous proposition and it follows that $\tilde{V} \cong U_1 \times_{U} V$. Therefore by the above diagram, the morphism $\tilde{V} \to U_1$ is étale.

**Remark 3.13.** Given a covering of $M_{GL}^{lc}$ by affine open subschemes $\{U_j\}$ such that $Q_i^*$ is trivial on each $U_j$ for every $i$, it is tempting to hope that one may glue together the schemes defined in the proposition above to get a connected scheme $M_1^{lc}$ which is an étale local model of $A_1^{GL}$. However this is not possible. Indeed, suppose that $M_1^{lc}$ is such a connected scheme. Then for each $j$, $U_j \times_{M_{GL}^{lc}} M_1^{lc} \cong \text{Spec}_{U_j} \left( \mathcal{O}[u_0, \ldots, u_{n-1}] / \left( u_0^{p-1} - q_0^*, \ldots, u_{n-1}^{p-1} - q_{n-1}^* \right) \right).$

The sections $q_0, \ldots, q_{n-1}$ vanish only on the special fiber. Therefore the restriction of $M_1^{lc} \to M_{GL}^{lc}$ to the generic fiber is finite étale. However the generic fiber of $M_{GL}^{lc}$ is the Grassmannian $\text{Gr}(n, r)$ and is therefore simply connected. This easily leads to a contradiction.

Let $\mathcal{F}_{SL}$ denote the affine flag variety associated with $SL_n$, noting that the smooth affine group scheme $\mathcal{G}$ from Section 2.2 acts on $\mathcal{F}_{SL}$ \cite[Section 4]{Gor1}. The following theorem is extracted from \cite{Gor1}.
Theorem 3.14. There is a \( G \)-equivariant embedding \( M^{\text{loc}}_{GL} \otimes \mathbb{F}_p \hookrightarrow \mathcal{F}_{SL} \). The stratification of \( \mathcal{F}_{SL} \) by Schubert cells induces a stratification of \( M^{\text{loc}}_{GL} \otimes \mathbb{F}_p \). There is a unique stratum of \( M^{\text{loc}}_{GL} \otimes \mathbb{F}_p \) which consists of a single closed point, called the “worst point”. Any open subscheme of \( M^{\text{loc}}_{GL} \) containing the worst point is an étale local model of \( A_0 \).

In loc. cit., such an open subscheme \( U^\tau_{GL} \) is defined and a presentation is computed as follows. An alcove of \( \text{SL}_n \) is given by a collection \( \{x_0, \ldots, x_{n-1}\} \) where \( x_i \in \mathbb{Z}^n \), written as \( (x_i(1), \ldots, x_i(n)) \), satisfying the following two conditions. Setting \( x_n = x_0 + (1, \ldots, 1) \), for \( 0 \leq i \leq n - 1 \) we require \( x_i(j) \leq x_{i+1}(j) \) for all \( 1 \leq j \leq n \) and

\[
\sum_{j=1}^n x_{i+1}(j) = 1 + \sum_{j=1}^n x_i(j).
\]

Consider the alcoves

\[
\omega = (\omega_0, \ldots, \omega_{n-1}) \quad \text{with} \quad \omega_i = (1^i, 0^{n-i})
\]

\[
\tau = ((1^r, 0^{n-r}), (1^{r+1}, 0^{n-r-1}), \ldots, (2^{r-1}, 1^{n-r+1})).
\]

Recall that the size of an alcove \( x \) is \( \sum_j x_0(j) - \omega_0(j) \) and \( x \) is \( \mu \)-admissible means \( x \leq \sigma(\omega) \) for some \( \sigma \in S_n \), where \( \leq \) is the Bruhat order and \( S_n \) is the group of permutations on \( n \) letters.

The affine Weyl group of \( \text{SL}_n \) is isomorphic to \( S_n \) and acts simply transitively on the collection of alcoves of size \( r \). Thus fixing the base alcove \( \tau \) we may identify the two sets. Let \( \{e_1, \ldots, e_n\} \) be the canonical basis of \( E^n \) and fix the basis of the \( \mathcal{O}_E \)-module \( e_{11} \Lambda_i \) where

\[
e_i^j = p^{-1} e_j \quad \text{for} \quad 1 \leq j \leq i \quad \text{and} \quad e_i^j = e_j \quad \text{for} \quad i < j \leq n.
\]

Definition 3.15. \([\text{Gör1]}\) Definition 4.4/ With respect to each \( \mu \)-admissible alcove \( x = (x_0, \ldots, x_{n-1}) \), we define an open subscheme of \( M^{\text{loc}}_{GL} \), denoted \( U^\tau_x \), which consists of the points \( (\mathcal{F}_i)_x \) such that for all \( i \) the quotient \( e_{11} \Lambda_i / \mathcal{F}_i \) is generated by those \( e_i^j \) with \( \omega_i(j) = x_i(j) \).

The open subscheme \( U^\tau_{GL} \) contains the “worst point” and thus serves as an étale local model of \( A^{GL}_0 \). Henceforth we shall denote \( U^\tau_{GL} \) by \( U^GL \). It is immediate that the line bundles \( \mathcal{Q}_i \) and \( \mathcal{Q}_i^* \) are trivial over \( U^GL_0 \).

Theorem 3.16. Choose a trivialization of each \( \mathcal{Q}_i^* \mid U^GL_0 \) and identify \( q_i \) and \( q_i^* \) with regular functions on \( U^GL_0 \). The scheme

\[
U^GL_1 = \text{Spec}_{U^GL_0} \left( \mathcal{O}[u_0, \ldots, u_{n-1}] / \left( u_0^{p-1} - q_0^*, \ldots, u_{n-1}^{p-1} - q_{n-1}^* \right) \right)
\]

is an étale local model of \( A^{GL}_1 \).

To make the above theorem completely explicit, we now describe the presentation of \( U^GL_0 \) computed in \([\text{Gör1}]\). As \( M^{\text{loc}}_{GL} \) is a closed subscheme of the \( n \)-fold product of \( \text{Gr}(n, r) \), we represent a point of \( M^{\text{loc}}_{GL} \) by giving \( \{F_i\}_{i=0}^{n-1} \) where each subspace \( F_i \) is \( r \)-dimensional, and we represent \( F_i \) as the column space of the \( n \times r \) matrix \( (a_{jk}^i) \). One can check that \( (F_i)_x \in U^GL_0 \) implies the \( r \times r \) submatrix given by rows \( i+1 \) to \( r+i \) (taken cyclically, so row \( n+1 \) is row 1) of \( F_i \) is invertible for \( 0 \leq i \leq n-1 \). As
such, we require this submatrix to be the identity matrix. For example, \( F_0 \) and \( F_1 \) are represented by

\[
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
a_{11}^0 & a_{12}^0 & \cdots & a_{1r}^0 \\
a_{n-r,1}^0 & a_{n-r,2}^0 & \cdots & a_{n-r,r}^0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
a_{n-r,1}^1 & a_{n-r,2}^1 & \cdots & a_{n-r,r}^1 \\
1 & 1 & \cdots & 1 \\
a_{11}^1 & a_{12}^1 & \cdots & a_{1r}^1 \\
\vdots & \vdots & \ddots & \vdots \\
a_{n-r-1,1}^1 & a_{n-r-1,2}^1 & \cdots & a_{n-r-1,r}^1
\end{pmatrix}
\]

Note that by requiring the matrix to have a specific \( r \times r \) submatrix be the identity, all the entries of the matrix are uniquely determined by its column space. By abuse of notation, we will use \( F_i \) to denote both the subspace and the matrix representing it. To express the condition \( F_i \) is mapped into \( F_{i+1} \), we must have \( \varphi_i(F_i) = F_{i+1}A_i \) for some \( r \times r \) matrix \( A_i \). As \( F_{i+1} \) has an \( r \times r \) submatrix being the identity, \( A_i \) is determined:

\[
A_i = \begin{pmatrix}
0 & 1 & \cdots & 1 \\
& \ddots & \ddots & \vdots \\
& \cdots & 0 & 1 \\
a_{11}^i & a_{12}^i & \cdots & a_{1r}^i
\end{pmatrix}
\]

From [Gör1] Proposition 4.13, \( U_0^{\text{GL}} \cong \text{Spec}(B_{\text{GL}}) \) with

\[
B_{\text{GL}} = \mathbb{Z}_p[a_{1k}^i; i = 0, \ldots, n-1, k = 1, \ldots, r]/I
\]

where \( I \) is the ideal generated by the entries of the matrices

\[
A_{n-1}A_{n-2} \cdots A_0 - p \cdot \text{Id}, \quad A_{n-2} \cdots A_0 A_{n-1} - p \cdot \text{Id}, \quad \ldots, \quad A_0 A_{n-1} \cdots A_1 - p \cdot \text{Id}.
\]

**Lemma 3.17.** With the presentation of \( U_0^{\text{GL}} \) as described above, let \( \{ F_i \subset e_{11} \Lambda_i, S \} \) correspond to a geometric point \( \text{Spec}(k) \to U_0^{\text{GL}} \).

(i) The map \( F_i \to F_{i+1} \) is an isomorphism if and only if \( a_{11}^i \neq 0 \).

(ii) The map \( e_{11} \Lambda_i / F_i \to e_{11} \Lambda_{i+1} / F_{i+1} \) is an isomorphism if and only if \( a_{n-r,r}^{i+1} \neq 0 \), where the upper index is taken modulo \( n \) with the standard representatives \( \{0, \ldots, n-1\} \).

**Proof.**

(i) The map \( F_i \to F_{i+1} \) will be an isomorphism if and only if \( \det(A_i) \neq 0 \), which is if and only if \( a_{11}^i \neq 0 \).

(ii) We may take \( \{ \bar{e}_j^{i+1}, \ldots, \bar{e}_j^{i+n} \} \) to be a basis of \( e_{11} \Lambda_i / F_i \), where \( \bar{e}_j^i \) denotes the reduction of \( e_j^i \) modulo \( F_i \). Here the upper index is taken modulo \( n \) with the standard set of representatives \( \{0, \ldots, n-1\} \) while the lower index is taken modulo \( n \) with the representatives \( \{1, \ldots, n\} \). Then the matrix representing the
map \( e_{i+1} \Lambda_i / F_i \to e_{i+1} \Lambda_{i+1} / F_{i+1} \) with respect to these bases is
\[
\begin{pmatrix}
-a_{1r}^{i+1} & 1 \\
-a_{2r}^{i+1} & 1 \\
\vdots & \ddots \\
-a_{n-r,r}^{i+1} & 1
\end{pmatrix}.
\]

From this, we see that the map \( e_{i+1} \Lambda_i / F_i \to e_{i+1} \Lambda_{i+1} / F_{i+1} \) is an isomorphism if and only if \( a_{n-r,r}^{i+1} \neq 0 \).

\[\square\]

It follows from the lemma that, up to a unit, \( q_i^* = a_{n-r,r}^{i+1} \) for \( 0 \leq i \leq n-1 \). Combining this with Theorem 3.16 we get the following.

**Theorem 3.18.** The scheme
\[
U_{1}^{GL} = \text{Spec } \left( B_{GL}[u_0, \ldots, u_{n-1}]/(u_0^{p-1} - a_{n-r,r}^{1}, \ldots, u_{n-1}^{p-1} - a_{n-r,r}^{n}) \right)
\]
is an étale local model of \( A_1^{GL} \), where the upper index of \( a_{jk}^{i} \) is taken modulo \( n \) with the standard representatives \( \{0, \ldots, n-1\} \).

### 3.4 The general case

We have thus far assumed that \( F^+ = \mathbb{Q} \). In this section, we extend the results to the more general case where \( F^+ \) is a totally real finite extension of \( \mathbb{Q} \).

Let \( F \) be an imaginary quadratic extension of \( F^+ \), with \( p \) an odd rational prime unramified in \( F^+ \) such that each factor of \( (p) \) in \( F^+ \) splits in \( F \). So write \( (p) = \prod_j p_j \) in \( F^+ \) and \( p_j = p_j^* \) in \( F \). Then
\[
F_{Q,p} = \prod_j F_{p_j} \times F_{p_j^*} \quad \text{making} \quad D_{Q,p} = \prod_j D_{p_j} \times D_{p_j^*}.
\]

Here \( D_{p_j} \) and \( D_{p_j^*} \) are each respectively a central simple \( F_{p_j} \) and \( F_{p_j^*} \) algebra with the involution \(*\) giving \( D_{p_j} \cong D_{p_j^*}^{opp} \). We furthermore assume that \( D_{Q,p} \) splits, i.e. \( D_{p_j} \cong M_n(F_{p_j}) \) for every \( j \). The splitting of \( D_{Q,p} \) gives
\[
G_{Q,p} = \prod_j G_{Q,p,j}
\]
where each factor may be written as
\[
G_{Q,p,j}(R) = \left\{ (x_1, x_2) \in (D_{p_j} \times R)^\times \times (D_{p_j^*} \times R)^\times : x_1 = c(x_2^*)^{-1} \text{ for some } c \in R^\times \right\}
\]
for a \( \mathbb{Q}_p \)-algebra \( R \). Then
\[
G_{Q,p,j} \cong D_{p_j}^{\times} \times G_m \cong GL_n \times G_m.
\]

Let \( \mu_j \) be the cocharacter \( \mu : \mathbb{G}_m \times \mathbb{G}_m \to G_{Q,p,j} \) composed with the projection onto the \( j \)th factor in the decomposition above. Then \( \mu = \prod_j \mu_j \). The periodic lattice chain \( \mathcal{L} \) is taken to be the product over \( j \) of those describe in Section 2.1.
With this data the integral model of the Shimura variety $A^\GL_0$ is the scheme representing the moduli problem given in [RZ] Definition 6.9, similar to the moduli problem in Section 2.1. We then define $A^\GL_1$ as in Definition 3.10. After an unramified base extension, the local model $M^\loc_j$ as given in Definition 2.1 is a product of the local models in the case $F^+ = \Q$, indexed by the factors of $(p)$ in $F^+$. Let $M^\loc_j$ denote the $j$th factor. For $0 \leq i \leq n-1$, we have the universal line bundle $Q^*_{j,i}$ on $M^\loc_j$ with the global section $q^*_{j,i}$ as defined in Section 3.1. Let $U = \prod_j U_j$ be an affine open subscheme of $M^\GL_0$ such that $U_j \subset M^\loc_j$ and each $Q^*_{j,i}$ is trivial on $U_j$. Choosing a trivialization, we identify the $q^*_{j,i}$ with regular functions on $U_j$.

**Theorem 3.19.** Let $U_1 = \prod_j U_{1,j}$, where

$$U_{1,j} = \Spec_{U_j} \left( \mathcal{O}[u_0, \ldots, u_{n-1}]/(u_0^{p-1} - q^*_{j,0}, \ldots, u_{n-1}^{p-1} - q^*_{j,n-1}) \right).$$

Then $U_1$ is an étale model of $A^\GL_1$.

### 4 Integral and local model of $A^\GSp_1$

Let $\{e_1, \ldots, e_{2n}\}$ be the standard basis of $V = \Q_p^{2n}$ and equip $V$ with the standard symplectic pairing $(\cdot, \cdot)$ where $(e_i, e_{2n+1-j}) = \delta_{ij}$. Define the $\Z_p$-lattice chain $\Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_{2n-1}$ where

$$\Lambda_i = \langle p^{-1}e_1, \ldots, p^{-1}e_i, e_{i+1}, \ldots, e_{2n} \rangle \subset \Q_p^{2n}$$

and extend it periodically to all $i \in \Z$ by $\Lambda_{i+2n} = p^{-1}\Lambda_i$. Then $L = (\Lambda_i)_i$ is a full periodic self-dual lattice chain with $\Lambda^\perp = \Lambda^{-1}$. Take $G = \GSp_{2n}$ with minuscule cocharacter $\mu = (0^n, (-1)^n)$. The reflex field at $p$ is $\Q_p$, $K^p$ is a sufficiently small compact open subgroup of $G(A^p_f)$, and $K_p = \Aut(L)$, the automorphism group of the polarized multichain $L$.

With this data, the moduli problem [RZ] Definition 6.9 for the integral model $A^\GSp_0$ is equivalent to the following. For any $\Z_p$-scheme $S$, $A^\GSp_0(S) = A^\GSp_{0,K^p}(S)$ is the set of tuples $(A_0, \lambda_0, \lambda_n, \tilde{\eta})$, up to isomorphism, where

(i) $A_0$ is a chain $A_0 \overset{\alpha_0}{\to} A_1 \overset{\alpha_1}{\to} \cdots \overset{\alpha_{n-1}}{\to} A_n$ of $n$-dimensional abelian schemes over $S$ with each morphism $\alpha_i$ an isogeny of degree $p$;

(ii) denoting the dual abelian scheme of $A_i$ by $\hat{A}_i$, the morphisms $\lambda_0 : A_0 \to A_0$ and $\lambda_n : A_n \to A_n$ are principal polarizations making the loop starting at any $A_i$ or $\hat{A}_i$ in the diagram

$$
\begin{array}{ccccccccc}
A_0 & \overset{\alpha_0}{\to} & A_1 & \overset{\alpha_1}{\to} & \cdots & \overset{\alpha_{n-1}}{\to} & A_n \\
\lambda_0^{-1} \downarrow & & & & & & & \lambda_n \\
\hat{A}_0 & \overset{\alpha_0^\vee}{\leftarrow} & \hat{A}_1 & \overset{\alpha_1^\vee}{\leftarrow} & \cdots & \overset{\alpha_{n-1}^\vee}{\leftarrow} & \hat{A}_n
\end{array}
$$

multiplication by $p$; and

(iii) $\tilde{\eta}$ is a $K^p$-level structure on $A_0$ [Kol Section 5].
There is an alternative description of this moduli problem in terms of chains of finite flat subgroup schemes instead of chains of isogenies \cite[Section 1]{J}. The local model $M_{\text{GSp}}^\rm{loc}$ is the $\mathbb{Z}_p$-scheme representing the following functor. An $S$-valued point of $M_{\text{GSp}}^\rm{loc}$ is given by a commutative diagram

$$
\begin{array}{cccccccc}
\Lambda_0,S & \longrightarrow & \Lambda_1,S & \longrightarrow & \cdots & \longrightarrow & \Lambda_{2n-1},S & \longrightarrow & p^{-1}\Lambda_0,S \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathcal{F}_0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \cdots & \longrightarrow & \mathcal{F}_{2n-1} & \longrightarrow & p^{-1}\mathcal{F}_0
\end{array}
$$

where $\Lambda_i,S = \Lambda_i \otimes_{\mathbb{Z}_p} \mathcal{O}_S$, the morphisms $\Lambda_i,S \rightarrow \Lambda_{i+1},S$ are induced by the inclusions $\Lambda_i \subset \Lambda_{i+1}$, $\mathcal{F}_i$ are locally free $\mathcal{O}_S$-submodules of rank $n$ which are Zariski-locally direct summands of $\Lambda_{i,S}$, and the $\mathcal{F}_i$ satisfy the following duality condition: the map

$$
\mathcal{F}_i \rightarrow \Lambda_i,S \xrightarrow{\sim} \mathcal{F}_{2n-i} \rightarrow \hat{\mathcal{F}}_{2n-i}
$$

is zero for all $i$. Here $\mathcal{H}om_{\mathcal{O}_S}(\cdot, \mathcal{O}_S)$ and $\Lambda_i,S \xrightarrow{\sim} \hat{\Lambda}_{2n-i,S}$ is induced from $\Lambda_i^\perp = p^{-1}\Lambda_i$. It follows that for $0 \leq i \leq n-1$, $\mathcal{F}_{2n-i}$ is determined by $\mathcal{F}_i$.

$M_{\text{GSp}}^\rm{loc}$ is readily seen to be representable as a closed subscheme of a product of Grassmannians and it is also a closed subscheme of $M_{\text{GL}}^\rm{loc}$. Using the description of the open subscheme $U_{\text{GL}}^0 = \text{Spec}(B_{\text{GL}}) \subset M_{\text{GL}}^\rm{loc}$ from Section 3.3, the duality condition imposes the following additional equations for $1 \leq i \leq n-1$ \cite[5.1]{Gor2}.

$$
a_{j,k}^{2n-i} = \varepsilon_{j,k}a_{n-k+1,n-j+1}^i \quad \text{with} \quad \varepsilon_{j,k} = \begin{cases} 
1 & \text{if } j, k \leq i \text{ or } j, k \geq i+1 \\
-1 & \text{otherwise}
\end{cases}
$$

We denote the resulting ring by $B_{\text{GSp}}$ and set $U_{\text{GSp}}^0 = \text{Spec}(B_{\text{GSp}})$. Then the results from Section 3.1 on the dimension of the invariant differentials of the group schemes $\{G_i\}_{i=0}^{n-1}$ now defined as $G_i = \ker(A_i \rightarrow A_{i+1})$ carry over. This collection of group schemes induces the morphism $\varphi$ in the following definition.

**Definition 4.1.** $A_{1,\text{GSp}}$ is the fibered product

$$
\begin{array}{ccc}
A_{1,\text{GSp}} & \longrightarrow & G_{\text{OT}}^\times \times \cdots \times G_{\text{OT}}^\times \\
\pi & & \downarrow \\
A_{0,\text{GSp}} & \varphi & \longrightarrow \text{OT} \times \cdots \times \text{OT}.
\end{array}
$$

We also define $Q_i$, $Q_i^\star$ and their global sections $q_i, q_i^\star$ as in Section 3.1. As $M_{\text{GSp}}^\rm{loc}$ is flat over $\text{Spec}(\mathbb{Z}_p)$ and the special fiber is reduced \cite{Gor2}, the results from Section 3.3 carry over in the appropriate manner.

**Theorem 4.2.** The scheme

$$
U_{1,\text{GSp}} = \text{Spec} \left( B_{\text{GSp}}[u_0, \ldots, u_{n-1}]/(u_0^{p-1} - a_0^{n,n}, \ldots, u_{n-1}^{p-1} - a_n^{n,n}) \right)
$$

is an étalé local model of $A_{1,\text{GSp}}$, where the upper index of $a_{j,k}^i$ is taken modulo $n$ with the standard representatives $\{0, \ldots, n-1\}$. 

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5 A Resolution of $\mathcal{A}_1^{\text{GSp}}$ for $n = 2$

In this section we will only be utilizing the integral and local models associated with $G = \text{GSp}_4$ and $K^p = K(N)$, the principal congruence subgroup where the integer $N$ is such that $p \nmid N$ and $N \geq 3$. As such, all unnecessary subscripts and superscripts will be removed from the notation. Let $\mathbb{Q}_p$ denote the completion of the maximal unramified extension of the reflex field $E = \mathbb{Q}_p$, and let $\hat{\mathbb{Z}}_p \subset \mathbb{Q}_p$ be its ring integers. All schemes in this section are of finite type over $\text{Spec}(\hat{\mathbb{Z}}_p)$.

The connected components of $\mathcal{A}_0 \otimes_{\mathbb{Z}_p} \hat{\mathbb{Z}}_p$ are indexed by the primitive $N$th roots of unity [Hai1, Section 2.1]. The number of connected components of the KR-strata (see below) in each connected component of $\mathcal{A}_0 \otimes_{\mathbb{Z}_p} \hat{\mathbb{Z}}_p$ is the same. As will be seen, it thus suffices to consider a single connected component of $\mathcal{A}_0 \otimes_{\mathbb{Z}_p} \hat{\mathbb{Z}}_p$. We abuse the notation by denoting such a connected component of $\mathcal{A}_0 \otimes_{\mathbb{Z}_p} \hat{\mathbb{Z}}_p$ by $\mathcal{A}_0$. Likewise we write $\mathcal{A}_1$ for the inverse image of this connected component with respect to the morphism $\mathcal{A}_1 \otimes_{\mathbb{Z}_p} \hat{\mathbb{Z}}_p \to \mathcal{A}_0 \otimes_{\mathbb{Z}_p} \hat{\mathbb{Z}}_p$ and $M^{\text{loc}}$ for $M^{\text{loc}} \otimes_{\mathbb{Z}_p} \hat{\mathbb{Z}}_p$. Note that $\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is an algebraic closure of $\mathbb{F}_p$, written as $\overline{\mathbb{F}}_p$, and that $\mathcal{A}_0 \otimes \overline{\mathbb{F}}_p$ is (geometrically) connected [Hai2, Lemma 13.2].

At each stage of the construction, we will first work on a local model and then carry this over to the corresponding integral model by producing a “linear modification” in the sense of [Pap2, Section 1]. Thus when performing a blowup of a local model, we require that the subscheme being blown up corresponds to a subscheme of the integral model. In order to understand more of the global structure of the resolution, such as the number of irreducible components and how they intersect, it is also necessary to track how certain subschemes transform with each modification.

For either a locally closed subset or a subscheme $Z \subset X$, $Z^{\text{red}}$ denotes the corresponding reduced scheme. For a morphism of schemes $f : Y \to X$, $f^{-1}(Z)$ always means the scheme-theoretic pullback, i.e. $Y \times_X Z$, unless otherwise noted.

**Definition 5.1.** [EH, Definition IV-15] Let $X$ be a scheme, $Z \subset X$ a subscheme. We say that $Z$ is Cartier at a closed point $x$ of $X$ if in an affine open neighborhood of $x$, $Z$ is the zero locus of a single regular function which is not a zero divisor. We say that $Z$ is a Cartier subscheme of $X$ if $Z$ is Cartier at all closed points of $X$.

**Definition 5.2.** Let $\rho : X' \to X$ be a modification, i.e. a proper birational morphism.

- If $\rho$ is given by the blowup of a closed subscheme $Z$ of $X$, then $Z$ is called a center of $\rho$.
- The true center of $\rho$, denoted by $C_\rho$, is defined to be the closed reduced subscheme of $X$ given set-theoretically by the complement of the maximal open subscheme where $\rho$ is an isomorphism.
- The fundamental center of $\rho$, denoted by $C_\rho^{\text{fund}}$, is defined to be the reduced subscheme of $C_\rho$ whose support is given by the closed points with fiber of dimension at least one.
- The residual locus of $\rho$, denoted by $C_\rho^{\text{res}}$, is defined to be the complement $C_\rho \setminus C_\rho^{\text{fund}}$.
- The exceptional locus of $\rho$ is $\rho^{-1}(C_\rho)^{\text{red}}$. 

- The strict transform of a subscheme $W \subset X$, denoted by $\text{ST}_\rho(W)$, is defined as either $\rho^{-1}(W)_{\text{red}}$ if $W \subset C_\rho$ or the Zariski closure of $\rho^{-1}(W \setminus C_\rho)$ inside of $X'$ with reduced scheme structure if $W \not\subset C_\rho$.

Remark 5.3. For a modification $\rho$ with a center $Z$, we will often say “the center” when there is a canonical choice of $Z$. The subscript $\rho$ will be dropped from $C_\rho, C^\text{fund}_\rho, C^\text{res}_\rho$, and $\text{ST}_\rho(W)$ when the morphism $\rho$ is understood. By upper semi-continuity of the dimension of the fiber, the fundamental center $C^\text{fund}$ is a closed subscheme of $C$. Since all schemes are assumed to be of finite type over $\text{Spec}(\mathbb{Z})$, the fiber over a closed point of $C^\text{res}$ is a finite collection of closed points.

Definition 5.4. Let $M$ be an étale local model of $X$. Fix a triple $(V, \varphi, \psi)$ where $\varphi : V \to X$ is an étale cover and $\psi : V \to M$ is an étale morphism. We say that a subscheme $Z \subset M$ étale locally corresponds to a subscheme $Z \subset X$ with respect to $(V, \varphi, \psi)$ if $\varphi^{-1}(Z) = \psi^{-1}(Z)$ as subschemes of $V$.

We now briefly mention some facts that will be used implicitly throughout the construction. With $(V, \varphi, \psi)$ fixed as in the definition above, if $Z_1, Z_2 \subset X$ étale locally correspond to $Z_1, Z_2 \subset M$ respectively, then so do their union, intersection, complement, and Zariski-closure. Also $\text{Bl}_M(Z_1)$ is an étale local model of $\text{Bl}_X(Z_1)$ by taking

$$V' := \text{Bl}_X(Z_1) \times_X V = \text{Bl}_M(Z_1) \times_M V$$

with $\varphi' : V' \to \text{Bl}_X(Z_1)$ and $\psi' : V' \to \text{Bl}_M(Z_1)$ the pullbacks of $\varphi$ and $\psi$ respectively. With respect to an étale morphism, the pullback of the locus where a subscheme is Cartier is precisely the locus where the pullback of the subscheme is Cartier. Thus with respect to $(V, \varphi, \psi)$ and the induced $(V', \varphi', \psi')$, we get that the true centers, fundamental centers, residual loci, and exceptional loci of the morphisms $\rho_X : \text{Bl}_X(Z) \to X$ and $\rho_M : \text{Bl}_M(Z) \to M$ étale locally correspond. Furthermore, it is easy to show that $\text{ST}_{\rho_X}(Z_2)$ étale locally corresponds to $\text{ST}_{\rho_M}(Z_2)$, again with respect to $(V', \varphi', \psi')$.

Lemma 5.5. Let $M$ and $M'$ be étale local models of $X$ and $X'$ respectively. Suppose there is an étale cover $\varphi : V \to X$ and an étale morphism $\psi : V \to M$ along with the morphisms $\rho_X$, $\rho_V$, and $\rho_M$ giving the diagram

$$\begin{array}{ccc}
X' & \xleftarrow{\varphi'} & V' \\
\rho_X \downarrow & & \rho_V \downarrow \rho_M \\
X & \xleftarrow{\varphi} & V \\
& & \psi
\end{array}$$

where the left and right squares are cartesian. Let $v$ be a closed point of $V$. Then with $x = \varphi(v)$ and $y = \psi(v)$, we have $\rho_X^{-1}(x) \cong \rho_M^{-1}(y)$.

Proof. Since $\varphi$ and $\psi$ are étale, $k(x) = k(v) = k(y)$ where $k(\cdot)$ denotes the residue field.
Thus
\[
\rho_X^{-1}(x) = X' \times_X \text{Spec}(k(x)) = (X' \times_X V) \times_V \text{Spec}(k(x)) = (M' \times_M V) \times_V \text{Spec}(k(x)) = M' \times_M \text{Spec}(k(x)) = (M' \times_M \text{Spec}(k(y))) \times_{\text{Spec}(k(y))} \text{Spec}(k(x)) = \rho_M^{-1}(y) \times_{\text{Spec}(k(y))} \text{Spec}(k(x)) = \rho_M^{-1}(y).
\]

Throughout the construction, we will work with a fixed étale cover of the integral model and a fixed étale morphism to the local model. We now describe the covers and morphisms to be used in each step. Choose an étale cover \( \varphi : V \to A_0 \) and a section \( \sigma : V \to A_0 \) such that \( \psi := \Psi \circ \sigma \) factors through \( U_0 \subset M^{\text{loc}} \). Since \( \Psi : \tilde{A}_0 \to M^{\text{loc}} \) is surjective [Gen Proposition 1.3.2], we also choose \( V \) and the morphisms above such that \( \psi \) surjects onto \( U_0 \).

In Step I we construct \( A'_0 = \text{Bl}_{A_0}(Z_1) \) and \( U'_0 = \text{Bl}_{U_0}(Z_1) \), taking the induced étale cover \( \varphi' : V' \to A'_0 \) and étale morphism \( \psi' : V' \to U'_0 \) associated with the blowup as described above. In Step II, the schemes constructed are \( A''_0 = A_1 \times_{A_0} A'_0 \) and \( U''_0 = U_1 \times_{U_0} U'_0 \). The étale cover and morphism in Step II are given by the top horizontal arrows of the diagram
\[
\begin{array}{ccc}
A''_0 & \xrightarrow{\varphi''} & V'' \\
\rho''_A & \downarrow & \Downarrow \rho''_U \\
A'_0 & \xrightarrow{\varphi'} & V' \\
\rho_A & \downarrow & \Downarrow \rho_U \\
A_0 & \xrightarrow{\varphi} & V \\
\end{array}
\]
where both squares are cartesian. With each step after Step II being given by a blowup, we again take the induced étale cover and étale morphism. Then Lemma 5.5 immediately gives the following.

**Proposition 5.6.** Let \( X \) and \( X' \) be two successive integral models mentioned above, so in particular we have a morphism \( \rho_X : X' \to X \). Denote by \( \rho_M : M' \to M \) the morphism between the corresponding local models, where they are étale local models with respect to \( (V, \varphi, \psi) \) and \( (V', \varphi', \psi') \). For \( x \) a closed point of \( X \), choose a closed point \( v \) of \( V \) with \( x = \varphi(v) \) and set \( y = \psi(v) \). Then \( \rho_X^{-1}(x) \cong \rho_M^{-1}(y) \).

**Notation 5.7.** Roman letters such as \( Z, C, \) and \( E \) will be used to denote subschemes of the local models. Calligraphic letters such as \( \mathcal{Z}, \mathcal{C}, \) and \( \mathcal{E} \) denote subschemes of the integral models that étale locally correspond to their Roman counterparts. In general, \( C \) and \( C \) will denote true centers, \( E \) and \( E \) will denote exceptional loci, and \( Z_{ij} \) and \( \mathcal{Z}_{ij} \) will denote irreducible components of the special fiber.

The schemes constructed in each additional step will be decorated with an additional tick mark \( ' \), and the superscript \( [i] \) denotes \( i \) tick marks (e.g. \( A_0^{[0]} = A_0, A_0^{[1]} = A'_0 \)).

The integral models that will be constructed are
\[
A_1^{[p+1]} \xrightarrow{\rho_A^{[p+1]}} A_1^{[p]} \xrightarrow{\rho_A^{[p]}} \cdots \xrightarrow{\rho_A^{[1]}} A_1^{[1]} \xrightarrow{\rho_A^{[1]}} A_1^{[0]} \xrightarrow{\rho_A^{[0]}} A_0^{[0]} = A_0 \]
Figure 1: $\mu$-admissible set for $\text{Sp}_4$

with their corresponding étale local models

$U_1^{[p+1]} \xrightarrow{\rho_1^{[p+1]}} U_1^{[p]} \xrightarrow{\rho_1^{[p]}} \ldots \xrightarrow{\rho_1^{[4]}} U_1^{[4]} \xrightarrow{\rho_1^{[4]}} U_1'' \xrightarrow{\rho_1''} U_0 \xrightarrow{\rho_1} U_0$.

Moreover, any subscheme will also be decorated by tick marks, so $Z''$ signifies that $Z'' \subset A''$.

As mentioned before, it will be necessary to observe how certain subschemes of $A_0$ transform (either their strict transform or scheme-theoretic inverse image) in each step. To keep track of this, we will use a subscript to denote which step the subscheme will be used in. So for example, $C_4$ is a subscheme of $A_0$. The $C$ indicates it will transform to be the true center of some blowup, and the 4 indicates that it will become the true center of the blowup $A_1^{[4]} \to A_1^{[3]}$. So in this example, we start with the strict transform $C_4' = \text{ST}_{\rho_A}(C_4)$ and then $C_4'' = (\rho_A')^{-1}(C_4')^{\text{red}}$. Finally $C_4''' = \text{ST}_{\rho_A''}(C_4'')$ and, as we will show, this is the true center of $A_1^{[4]} \to A_1'''$.

The subschemes that will be blown up arise from subschemes of $A_0$ that are the union of certain KR-strata. These strata are indexed by the $\mu$-admissible alcoves of $\text{Sp}_{2n}$. An alcove of $\text{Sp}_{2n}$ is an alcove $x$ of $\text{SL}_{2n}$ (see Section 3.3) that satisfies the following duality condition: there exists a $c \in \mathbb{Z}$ such that $x_i(j) - x_{2n-i}(2n - j + 1) = c$ for $0 \leq i \leq 2n$ and $1 \leq j \leq 2n$. For $x \in \text{Adm}(\mu)$, $A_x$ will denote the KR-stratum corresponding to $x$ and $n_x$ denotes the number of connected components of $A_x$. The affine Weyl group $W^{\text{aff}}$ of $\text{Sp}_{2n}$ acts simply transitively on the alcoves of size $n$, and hence fixing the base alcove $\tau$ from Section 3.3 with $r = n$ we can identify these two sets. $W^{\text{aff}}$ is isomorphic to the subgroup of $\mathbb{Z}^{2n} \rtimes S_{2n}$ generated by the simple affine reflections $s_0, \ldots, s_n$ where

$s_i = (i, i + 1)(2n + 1 - i, 2n - i) \quad \text{for} \quad 1 \leq i \leq n - 1$

$s_0 = (-1, 0, \ldots, 0, 1)(1, 2n) \quad s_n = (n, n + 1)$.

The alcoves that make up the $\mu$-admissible set are shown in Figure 1 in various shades of gray. The dimension of the stratum corresponding to $w \in W^{\text{aff}}$ is equal to $\ell(w)$,
where \( \ell(\cdot) \) is the length with respect to the Bruhat order. In particular the stratum \( S_\tau \) corresponding to the base alcove \( \tau \) is the unique KR-stratum of dimension zero. The irreducible components are the extreme alcoves which are shaded medium gray. The \( p \)-rank zero locus is pictured in dark gray, given by \( A_{s_0s_2\tau} \cup A_{s_1\tau} \).

5.1 Step 0: \( U_0, A_0, \) and \( A_1 \)

5.1.1 Description of the local model \( U_0 \)

From Section 4, a presentation of \( U_0 \) is given as a closed subscheme of \( \text{Spec}(\tilde{\mathbb{Z}}_p[a_{i,k}; i = 0, \ldots, 3, j, k = 1, 2]) \). By setting
\[
x = a_{22}^1, \quad y = a_{11}^0, \quad a = a_{12}^0, \quad b = a_{12}^2, \quad c = -a_{12}^1,
\]
we arrive at the étale local model derived in [dJ2]:
\[
U_0 = \text{Spec}(B) \quad \text{where} \quad B = \tilde{\mathbb{Z}}_p[x,y,a,b,c]/(xy - p, ax + by + abc).
\]

With this presentation we have that, up to a unit,
\[
q_0 = y, \quad q_0^* = x, \quad q_1 = y + ac, \quad q_1^* = x + bc.
\]

There are four irreducible components of \( U_0 \otimes \mathbb{F}_p \):
\[
Z_{00} = Z(y,a), \quad Z_{01} = Z(y, x + bc), \quad Z_{10} = Z(x, y + ac), \quad Z_{11} = Z(x, b).
\]

5.1.2 Description of the integral model \( A_0 \)

As in [dJ2 Section 5], using the local model diagram we define \( Z_{ij} = \Phi(\Psi^{-1}(Z_{ij}))^{\text{red}} \). Since there are precisely four irreducible components of \( A_0 \otimes \mathbb{F}_p \) [Yu Theorem 1.1], they are given by \( Z_{ij} \).

Proposition 5.8. \( Z_{ij} \) étale locally corresponds to \( Z_{ij} \). Moreover this holds for arbitrary unions, intersections, and complements of the \( Z_{ij} \), e.g. \( Z_{11} \cup (Z_{01} \cap Z_{10}) \) étale locally corresponds to \( Z_{11} \cup (Z_{01} \cap Z_{10}) \), where each is given the reduced scheme structure.

We now define the subschemes that will be used in the steps throughout the construction of the resolution.

| Local model \( U_0 \) | Integral model \( A_0 \) |
|-----------------------|-----------------------|
| \( Z_1 = Z(x,b) \)   | \( Z_1 = Z_{11} \)    |
| \( C_1 = Z(x,y,a,b) \)| \( C_1 = Z_{00} \cap Z_{01} \cap Z_{10} \cap Z_{11} \) |
| \( Z_3 = Z(x,bc) \)  | \( Z_3 = Z_{11} \cup (Z_{01} \cap Z_{10}) \) |
| \( C_3 = Z(x,bc) \)  | \( C_3 = Z_{11} \cup (Z_{01} \cap Z_{10}) \) |
| \( Z_4 = Z(x,b) \)   | \( Z_4 = Z_{11} \)    |
| \( C_4 = Z(x,y,b,c) \)| \( C_4 = (Z_{01} \cap Z_{10} \cap Z_{11}) \setminus Z_{00} \) |

Using Proposition 5.8, it is easy to check that the subschemes on the left étale locally correspond to those on the right.
Proposition 5.9. Writing each subscheme below in terms of KR-strata, we have the following.

\[
Z_{11} = \frac{A_{s_2 s_1 s_2 \tau}}{A_{s_0 s_2 \tau} \cup A_{s_1 \tau}}
\]

\[
Z_{01} \cap Z_{10} = \frac{A_{s_0} \cup A_{s_1 \tau}}{A_{s_2 \tau}}
\]

\[
(Z_{01} \cap Z_{10} \cap Z_{11}) \setminus Z_{00} = \frac{A_{s_1 \tau}}{A_{s_2 \tau}}
\]

Proof. The locus of \( U_0 \) corresponding to the \( p \)-rank zero locus of \( A_0 \) is \( Z(q_0, q_1, q_0^*, q_1^*) = Z_{01} \cap Z_{10} \). Hence by Proposition 5.8 \( Z_{01} \cap Z_{10} \) is the \( p \)-rank zero locus of \( A_0 \) and this is given by \( \frac{A_{s_0 s_2 \tau} \cup A_{s_1 \tau}}{A_{s_2 \tau}} \) [GY Proposition 2.7].

To show \( Z_{11} = \frac{A_{s_2 s_1 s_2 \tau}}{A_{s_0 s_2 \tau} \cup A_{s_1 \tau}} \), we choose the closed point \( x = a = b = c = 0 \) and \( y = 1 \) of \( U_0 \) which lies solely on the irreducible component \( Z_{11} \) of the special fiber. This point corresponds to the flag

\[
F_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad F_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad F_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

and in the notation of [Gor1 Section 4] (cf. [Gor3 Section 4.3]) this gives the associated lattice chain

\[
\begin{pmatrix} 1 & \pi & 1 \\ 1 & \pi & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & \pi^{-1} \\ 1 & \pi^{-1} & 1 \end{pmatrix}
\]

where we omit any entry which is zero. Our chosen point corresponds to a closed point lying on \( A_{s_2 s_1 s_2 \tau} \) if and only if there is an element \( b \) in the Iwahori subgroup such that \( b \cdot s_2 s_1 s_2 \tau \) corresponds to the same lattice chain as above. With \( s_2 s_1 s_2 \tau \) corresponding to the lattice chain

\[
\begin{pmatrix} \pi & \pi & 1 \\ \pi & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \pi & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & \pi^{-1} \\ 1 & \pi^{-1} & 1 \end{pmatrix},
\]

it is easy to check that

\[
b = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}
\]

suffices. Therefore \( Z_{11} = \frac{A_{s_2 s_1 s_2 \tau}}{A_{s_0 s_2 \tau} \cup A_{s_1 \tau}} \). Since \( Z_{00} \cap Z_{11} \) is one dimensional, it must be that \( Z_{00} \cap Z_{11} \) is one dimensional as well. From inspection of the \( \mu \)-admissible set we see that \( Z_{00} = \frac{A_{s_0 s_1 s_0 \tau}}{A_{s_2 s_1 s_2 \tau}} \). The equality \( Z_{00} \cap Z_{01} \cap Z_{10} \cap Z_{11} = A_{\tau} \cup A_{s_1 \tau} \) can be read off the \( \mu \)-admissible set as well. Finally \( Z_{01} \cap Z_{10} \cap Z_{11} = A_{\tau} \cup A_{s_1 \tau} \cup A_{s_2 \tau} \) and thus we have \( (Z_{01} \cap Z_{10} \cap Z_{11}) \setminus Z_{00} = A_{s_2 \tau} \).

\[\square\]

Proposition 5.10. The number of connected and irreducible components of the subschemes of \( A_0 \) are as follows.
\[
\begin{array}{c|c|c}
\text{Subscheme of } A_0 & \text{# of connected components} & \text{# irreducible components} \\
\hline
\mathcal{Z}_1 & 1 & 1 \\
\mathcal{C}_1 & n_{s_1 \tau} & n_{s_1 \tau} \\
\mathcal{Z}_3 & 1 & 1 + n_{s_0 s_2 \tau} \\
\mathcal{C}_3 & 1 & 1 + n_{s_0 s_2 \tau} \\
\mathcal{Z}_4 & 1 & 1 \\
\mathcal{C}_4 & n_{s_2 \tau} & n_{s_2 \tau} \\
\end{array}
\]

Proof. $\mathcal{Z}_1 = \mathcal{Z}_4 = \mathcal{Z}_{11}$: This is an irreducible component.

$\mathcal{Z}_3 = \mathcal{C}_3 = \mathcal{Z}_{11} \cup (\mathcal{Z}_{01} \cap \mathcal{Z}_{10})$: To see this subscheme is connected, it suffices to show that each connected component of $\mathcal{Z}_{01} \cap \mathcal{Z}_{10}$ meets $\mathcal{Z}_{11}$. Let $\mathcal{W}$ be such a connected component. Since $\mathcal{Z}_{01} \cap \mathcal{Z}_{10}$ is a union of KR-strata, by possibly shrinking $\mathcal{W}$ we may assume $\mathcal{W}$ is a connected component of some KR-stratum. By [GY Theorem 6.4], $\mathcal{W} \cap A_{r} \neq \emptyset$ where $\mathcal{W}$ is the Zariski closure of $\mathcal{W}$ inside of $A_0$. As $A_{r} \subset \mathcal{Z}_{11}$, the claim follows.

To find the number of irreducible components, note that $\mathcal{Z}_{11} \cup (\mathcal{Z}_{01} \cap \mathcal{Z}_{10})$ is a union of three and two dimensional irreducible components: the unique three dimensional component is $\mathcal{Z}_{11}$ and the two dimensional components are given by the irreducible components of $\mathcal{Z}_{01} \cap \mathcal{Z}_{10} \setminus \mathcal{Z}_{11} = \overline{A_{s_0 s_2 \tau}}$. Inspection of $U_0$ shows that $(\mathcal{Z}_{01} \cap \mathcal{Z}_{10}) \setminus \mathcal{Z}_{11}$ and hence $\mathcal{Z}_{01} \cap \mathcal{Z}_{10} \setminus \mathcal{Z}_{11}$ is smooth. Thus each of the $n_{s_0 s_2 \tau}$ connected components of $\overline{A_{s_0 s_2 \tau}}$ is irreducible. The result now follows.

$\mathcal{C}_1 = \mathcal{Z}_{00} \cap \mathcal{Z}_{01} \cap \mathcal{Z}_{10} \cap \mathcal{Z}_{11}$: From Proposition 5.9 this subscheme is given by $\overline{A_{s_1 \tau}}$. Since $\mathcal{C}_1$ is smooth, each connected component of $\mathcal{C}_1 = \overline{A_{s_1 \tau}}$ is irreducible. Of course, $\overline{A_{s_1 \tau}}$ has the same number of connected components as $\overline{A_{s_1 \tau}}$.

$\mathcal{C}_4$: This follows by a similar argument to that given for the statement about $\mathcal{C}_1$. \[\square\]

5.1.3 Description of the integral model $A_1$

As remarked in the previous section, $A_0 \otimes \mathbb{F}_p$ has four irreducible components: $\mathcal{Z}_{00}$, $\mathcal{Z}_{01}$, $\mathcal{Z}_{10}$, and $\mathcal{Z}_{11}$. To determine the irreducible components of $A_1 \otimes \mathbb{F}_p$, we will use the following lemma. Recall that

\[U_1 = \text{Spec} \left( \mathbb{Z}_p[x, y, a, b, c, u, v]/(xy - p, ax + by + abc, u^{p-1} - x, v^{p-1} - x - bc) \right).\]

Lemma 5.11. The irreducible components of $U_1 \otimes \mathbb{F}_p$ are normal.

Proof. The four irreducible components of $U_1 \otimes \mathbb{F}_p$ correspond to the ideals $(u, v, b), (u, y + ac), (y, v),$ and $(y, a)$. The first irreducible component is smooth, while the other three are respectively the spectra of the rings

\[\mathbb{F}_p[a, b, c, v]/(v^{p-1} - bc), \mathbb{F}_p[a, b, c, u]/(u^{p-1} - bc), \mathbb{F}_p[b, c, u, v]/(v^{p-1} - u^{p-1} - bc).\]

Each is a complete intersection and hence Cohen-Macaulay. Using the Jacobian Criterion, we see that the singular locus has codimension greater than one. By Serre’s Criterion [Mat] Theorem 23.8], each is normal. \[\square\]

Proposition 5.12. $A_1 \otimes \mathbb{F}_p$ is connected, equidimensional of dimension three, and the irreducible components are normal. Furthermore, $A_1 \otimes \mathbb{F}_p$ and has precisely four irreducible components given by $\pi^{-1}(\mathcal{Z}_{ij})^{\text{red}}$ where $\pi : A_1 \to \mathcal{A}_0$. 

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Proof. That $\mathcal{A}_1 \otimes \overline{\mathbb{F}}_p$ is equidimensional of dimension three is immediate from inspection of the local model $U_1 \otimes \overline{\mathbb{F}}_p$. As $\mathcal{A}_r$ is in the $p$-rank zero locus of $\mathcal{A}_0$, the fiber above a closed point of $\mathcal{A}_r$ with respect to $\pi$ consists of a single closed point. Since $\pi$ is finite and surjective, each irreducible component of $\mathcal{A}_1 \otimes \overline{\mathbb{F}}_p$ maps surjectively onto an irreducible component of $\mathcal{A}_0 \otimes \overline{\mathbb{F}}_p$. As every irreducible component of $\mathcal{A}_0 \otimes \overline{\mathbb{F}}_p$ contains $\mathcal{A}_r$ [GY, Theorem 6.4], we conclude that $\mathcal{A}_1 \otimes \overline{\mathbb{F}}_p$ is connected.

For $0 \leq i, j \leq 1$, fix an irreducible component $Z'_{ij}$ of $\mathcal{A}_1 \otimes \overline{\mathbb{F}}_p$ that maps onto $Z_{ij}$ and suppose there is a fifth irreducible component $Z'$ of $\mathcal{A}_1 \otimes \overline{\mathbb{F}}_p$. As mentioned above, each irreducible component of $\mathcal{A}_0 \otimes \overline{\mathbb{F}}_p$ contains $\mathcal{A}_r$, and from this it follows that $Z'_{00}, Z'_{01}, Z'_{10}, Z'_{11}$, and $Z'$ all simultaneously intersect at some closed point $x$ of $\mathcal{A}_1 \otimes \overline{\mathbb{F}}_p$. We now show that this is not possible.

Let $\mathcal{A}_1 \xrightarrow{\varphi} V \xrightarrow{\psi} U_1$ be an étale cover of $\mathcal{A}_1$ with étale morphism to $U_1$. Choose a closed point $v$ of $V$ such that $\varphi(v) = x$ and set $y = \psi(v)$. Since the irreducible components of $U_1 \otimes \overline{\mathbb{F}}_p$ are integral, normal, and excellent, the completion at $y$ of any irreducible component that $y$ lies on is also integral and normal. Hence there are at most four irreducible components of $V$ passing through $v$. Therefore the number of irreducible components of $\mathcal{A}_1 \otimes \overline{\mathbb{F}}_p$ passing through $x$ is at most four, giving the contradiction that we are seeking. \hfill $\square$

5.2 Step I: Semi-stable resolution of $\mathcal{A}_0$ [dJ1]

Set $\mathcal{A}_0' = \text{Bl}_{\mathcal{A}_0}(Z_1)$ and $U_0' = \text{Bl}_{U_0}(Z_1)$.

5.2.1 Description of $U_0'$

An easy calculation shows

$$U_0' = \text{Proj}_{\mathcal{O}_{U_0}} \left( B[\bar{x}, \bar{b}] / (a\bar{x} + b\bar{y} + abc, x\bar{b} - \bar{x}b) \right)$$

where $\bar{x}$ and $\bar{b}$ are of grade one. Inspection reveals that the true center of $\rho_U' : U_0' \to U_0$ is given by $C_1 = Z(x, y, a, b)$ and is of dimension one, the true center of $\rho_U'$ is equal to its fundamental center, the exceptional locus of $\rho_U'$ is two dimensional, and $U_0' \otimes \overline{\mathbb{F}}_p$ is equidimensional of dimension three. The strict transforms of the closed subschemes given in Step 0 are

$$Z_3' = C_3' = Z(x, b) \cup Z(x, y, c, \bar{x}), \quad Z_4' = Z(x, b), \quad C_4' = Z(x, y, b, c).$$

5.2.2 Description of $\mathcal{A}_0'$

With $U_0'$ an étale local model of $\mathcal{A}_0'$, Proposition 5.8 gives that the true center of $\mathcal{A}_0' \to \mathcal{A}_0$ is $C_1$. By the remarks in the previous section, $\mathcal{A}_0' \otimes \overline{\mathbb{F}}_p$ is equidimensional of dimension three and the exceptional locus of $\mathcal{A}_0' \to \mathcal{A}_0$ is two dimensional. Thus no irreducible component of $\mathcal{A}_0' \otimes \overline{\mathbb{F}}_p$ is contained in the exceptional locus. Therefore $\mathcal{A}_0' \otimes \overline{\mathbb{F}}_p$ has four irreducible components, each being given by the strict transform of an irreducible component of $\mathcal{A}_0 \otimes \overline{\mathbb{F}}_p$. We denote the strict transform of $Z_{ij}$ by $Z'_{ij}$.

**Proposition 5.13.** The number of connected and irreducible components of the subschemes of $\mathcal{A}_0'$ are as follows.
Proof. \( Z'_3 = C'_3 \): We start by showing \( Z'_3 \) is connected. From the proof of Proposition 5.10 \( Z_3 = Z_{11} \cup (Z_{10} \cap Z_{11}) \) is a union of three and two dimensional components, and they intersect in a one dimensional closed subscheme. We claim that this one dimensional subscheme intersects with the true center \( C_1 = Z_{00} \cap Z_{01} \cap Z_{10} \cap Z_{11} \) in a zero dimensional subscheme. This can easily be seen by writing each as a union of KR-strata. Indeed, set \( W = Z_{01} \cap Z_{10} \setminus Z_{11} \) which is equidimensional of dimension two. Then \( Z_3 = Z_{11} \cup W \) and from Proposition 5.9 each is given as a union of KR-strata as follows.

\[
\begin{align*}
Z_{11} &= A_r \cup A_{s_1 r} \cup A_{s_2 r} \cup A_{s_2 s_1 r} \cup A_{s_2 s_1 s_2 r} \\
W &= A_r \cup A_{s_0 r} \cup A_{s_2 r} \cup A_{s_0 s_2 r} \\
C_1 &= A_r \cup A_{s_1 r} 
\end{align*}
\]

Therefore the one dimensional subscheme \( Z_{11} \cap W \) intersects with \( C_1 \) in \( A_r \), a zero dimensional subscheme as claimed. With \( Z_{11} \) and \( W \) smooth, it follows immediately that \( Z_3 \setminus C_1 \) is connected. Therefore \( Z'_3 = C'_3 \) is connected.

To see that \( Z'_4 = C'_4 \) has \( n_{s_0 s_2} + 1 \) irreducible components, from the proof of Proposition 5.10 we have \( Z_4 \) is a union of \( n_{s_0 s_2} + 1 \) irreducible components and each is of dimension two or three. Since the true center of \( A'_0 \rightarrow A_0 \) is of dimension one, each such irreducible component is not contained in the true center. Hence the strict transform of each irreducible component is irreducible as well and the claim follows.

\( Z'_4 \): Note that \( Z_4 = Z_{11} \) is irreducible and three dimensional giving that \( Z_4 \setminus C_1 \) is irreducible as well. It follows immediately that \( Z'_4 \) is irreducible.

\( C'_4 \): As \( C_4 \) is smooth, each irreducible component is a connected component. Also, \( C_4 \) intersects the true center of \( A'_0 \rightarrow A_0 \) in a zero-dimensional subscheme and therefore strict transform of each irreducible component of \( C_4 \) is irreducible.

\[\square\]

5.3 Step II: Fiber with \( A_1 \)

Set \( A''_1 = A_1 \times_{A_0} A'_0 \) and \( U''_1 = U_1 \times_{U_0} U'_0 \).

5.3.1 Description of \( U''_1 \)

As in Theorem 4.2 \( U_1 \) is given in the chosen presentation by adjoining to \( U_0 \) the variables \( u \) and \( v \), along with the relations \( u^{p-1} - x \) and \( v^{p-1} - (x + bc) \). We thus have

\[
U''_1 = \text{Proj} \left( B[u, v][\overline{x}, \overline{b}] / (a\overline{x} + \overline{b}y + abc, x\overline{b} - \overline{x}b, u^{p-1} - x, v^{p-1} - (x + bc) \right)
\]

where \( B[u, v] \) is of grade 0 while \( \overline{x} \) and \( \overline{b} \) are of grade 1. Calculating the reduced inverse images under \( U''_1 \rightarrow U'_0 \) we have

\[
Z''_3 = C''_3 = Z(u, v, b) \cup Z(u, v, y, c, \overline{x}), \quad Z''_4 = Z(u, v, b), \quad C''_4 = Z(u, v, y, b, c, \overline{x}).
\]
Note that $Z(u, v, b) \cup Z(u, v, c, \bar{x}) = Z(u, v)$ as the relation $v^{p-1} - w^{p-1} - bc$ implies that if $u$ and $v$ are zero, then $bc = 0$ giving the two components.

### 5.3.2 Description of $A''_i$

With $A''_i = A_1 \times_{A_0} A'_i$, the projection $A''_i \to A_1$ is proper and birational and so it is a modification. Also note that the projection $\rho'_A : A''_i \to A'_0$ is finite and flat.

**Remark 5.14.** As claimed in the introduction, $A''_i$ is not normal. Indeed, it suffices to show that $U''_i$ is not normal. Consider the irreducible component $Z''_{ij}$ of $U''_i \otimes \mathbb{F}_p$. In the local ring of the generic point of this component, the maximal ideal is given by $(u, v)$. This ideal is not principal since $u \notin (v)$ and $v \notin (u)$, and hence the local ring is not regular. Therefore $U''_i$ is not normal by Serre’s Criterion [Mat, Theorem 23.8].

**Proposition 5.15.** Set $Z''_{ij} = (\rho''_A)^{-1}(Z'_{ij})$. Each $Z''_{ij}$ is an irreducible component of $A''_i \otimes \mathbb{F}_p$, and these give all the irreducible components of $A''_i \otimes \mathbb{F}_p$.

**Proof.** From Proposition 5.12 we have that $W_{ij} = \pi^{-1}(Z_{ij})$ is irreducible, where $\pi : A_1 \to A_0$. Note that the morphism $A''_i \to A_1$ is a modification with true center of dimension at most one. As such, $W_{ij}$ is not contained in the true center and therefore its strict transform $W''_{ij}$ with respect to $A''_i \to A_1$ is irreducible.

Set $\mathcal{U} = A_0 \setminus C_1$, $\mathcal{U}' = (\rho''_A)^{-1}(\mathcal{U})$, and $\mathcal{U}'' = (\rho''_A)^{-1}(\mathcal{U}')$. Then $Z''_{ij} \cap \mathcal{U}'' = W''_{ij} \cap \mathcal{U}''$ because both can be described as the reduced inverse image of $Z_{ij} \cap \mathcal{U}$ under the two paths in the following cartesian diagram.

\[ \begin{array}{ccc}
A''_i & \longrightarrow & A'_0 \\
\downarrow & & \downarrow \\
A_1 & \longrightarrow & A_0
\end{array} \]

As sets we have

\[
\begin{align*}
Z''_{ij} &= (\rho''_A)^{-1}(Z'_{ij}) \\
&= (\rho''_A)^{-1}\left(Z''_{ij} \cap \mathcal{U}'\right) \\
&= (\rho''_A)^{-1}\left(Z''_{ij} \cap \mathcal{U}''\right) \\
&= (\rho''_A)^{-1}(Z'_{ij} \cap (\rho''_A)^{-1}(\mathcal{U}')) \\
&= Z''_{ij} \cap \mathcal{U}''.
\end{align*}
\]

It thus suffices to show that $Z''_{ij} \cap \mathcal{U}''$ is irreducible. But this is immediate since $Z''_{ij} \cap \mathcal{U}'' = W''_{ij} \cap \mathcal{U}''$ with $W''_{ij}$ irreducible. That the collection $\{Z''_{ij}\}$ gives all the irreducible components is immediate.

**Proposition 5.16.** The number of connected and irreducible components of the subschemes of $A''_i$ are as follows.

| Closed subscheme of $A''_i$ | # connected components | # irreducible components |
|-----------------------------|------------------------|--------------------------|
| $Z''_{ij}$ = $(\rho''_A)^{-1}\left(Z''_{ij}\right)$ | 1 | $n_{s2\tau} + 1$ |
| $C''_{ij}$ = $(\rho''_A)^{-1}\left(C''_{ij}\right)$ | 1 | $n_{s2\tau} + 1$ |
| $Z''_{ij}$ = $(\rho''_A)^{-1}\left(Z''_{ij}\right)$ | 1 | 1 |
| $C''_{ij}$ = $(\rho''_A)^{-1}\left(C''_{ij}\right)$ | $n_{s2\tau}$ | $n_{s2\tau}$ |
Proof. \( Z''_3 = C''_4 \): Let \( W' \subset Z''_3 \) be an irreducible component. Then we claim that \((\rho''_4)^{-1}(W')^{\text{red}} \) is irreducible. As shown in the proof of Proposition 5.13, \( W' \) arises as the strict transform of an irreducible component of \( Z_3 \). First consider the case where \( W' \) is the strict transform of \( Z_{11} \). Then \( W' = Z'_{11} \) and it was already shown that \((\rho''_4)^{-1}(Z'_{11})\) is irreducible.

So assume now that \( W' \) is the strict transform of some two dimensional irreducible component of \( Z_3 \). From the proof of Proposition 5.10 it must be that this two dimensional component of \( Z_3 \) is contained in \( Z_{01} \cap Z_{10} \), and hence \( \rho'_3(W') \subset Z_{01} \cap Z_{10} \).

Recall the following diagram.

\[
\begin{array}{ccc}
A'' & \xrightarrow{\rho''_A} & A' \\
\downarrow & & \downarrow \\
A_1 & \xrightarrow{\pi} & A_0 \\
\end{array}
\]

Let \( x' \) be a closed point of \( W' \) and \( x = \rho'_A(x') \). Then \( x \) lies in the \( p \)-rank zero locus and it follows that the fiber \( \pi^{-1}(x) \) consists of a single closed point. Therefore the fiber \((\rho''_4)^{-1}(x')\) also consists of a single closed point. With \( \rho''_4 \) finite, it must be that \((\rho''_4)^{-1}(W)\) is irreducible.

Thus we conclude that each irreducible component of \( Z''_4 = C''_3 \) arises as the reduced inverse image of an irreducible component of \( Z''_3 \), and therefore the number of irreducible components of \( Z''_4 \) is \( n_{s_a} + 2τ + 1 \). That \( Z''_4 = C''_3 \) is connected follows from the fact that the fiber above any closed point of a two dimension component of \( Z''_3 \) with respect to the morphism \( \rho''_4 \) consists of a single closed point.

\( Z''_4 : \) Since \( Z'_4 = Z''_{11} \), that \( Z''_4 \) is irreducible was shown in the previous proposition.

\( C'_4 : \) Let \( W' \) be a connected component of \( C'_4 \). From the proof of Proposition 5.10, \( W' \) is irreducible and arises as the strict transform of some irreducible component of \( C_4 \subset A_0 \). Let \( x' \) be a closed point of \( W' \). Then \( x = \rho'_A(x') \) lies in \( C_4 \) and hence is contained in the \( p \)-rank zero locus. As such, \( \pi^{-1}(x) \) consists of a single closed point and it follows that the fiber \((\rho''_4)^{-1}(W)\) also consists of a single closed point. Hence the reduced inverse image of \( W' \) under \( A''_4 \rightarrow A_0 \) is irreducible. Therefore, \( C''_4 \) has the same number of connected and irreducible components as \( C'_4 \), namely \( n_{s_2 τ} \).

\[ \square \]

### 5.4 Step III: Blowup of \( Z'_3 \)

Set \( A''_1 = \text{Bl}_{A''_4}(Z''_4) \) and \( U''_1 = \text{Bl}_{U''_4}(Z''_3) \).

#### 5.4.1 Description of \( U''_1 \)

To simplify the notation, let \( X = U''_1 \). A presentation of \( X \) is given by the closed subscheme of \( \text{Proj}_{U''_4}(O[\bar{u}, \bar{v}]) \), where \( \bar{u} \) and \( \bar{v} \) are of grade 1, cut out by the following equations.

\[
u\bar{v} - \bar{u}v, \quad (\bar{x} + bc)\bar{u}^{p-1} - \bar{x}\bar{v}^{p-1}, \quad y\bar{u}^{p-1} - (y + ac)\bar{v}^{p-1}
\]

\( X \) is covered by the four standard affine charts.
We claim that if \( \rho \) is an irreducible component of \( Z \) we have
\[
\text{that the strict transforms of } \rho \text{ is equidimensional of dimension three.}
\]
As the exceptional locus maps surjectively onto the true center \( C \) by the previous section, the exceptional locus is equidimensional of dimension three. Note that the residual locus is equidimensional of dimension three, and the exceptional locus is equidimensional of three dimensional. Indeed, since \( C'' \) is smooth of dimension two and \( C'' \) intersects with the fundamental center of \( U'' \) in a smooth one dimensional subscheme. Taking the strict transform with respect to \( \rho '' \) we have
\[
Z'' = Z(u,v,b), \quad C'' = Z(u,v,b,c,\bar{x}).
\]

**5.4.2 Description of \( A'' \)**

\( A'' \) has \( U'' \) as an étale local model and it is immediate that \( A'' \) is normal. As the true center of \( U'' \) is \( C'' \), the fundamental center of \( Z'' \) is \( Z'' \cap Z'' \), and the residual locus is \( Z'' \setminus (Z'' \cap Z'' \binfty) \). The fundamental center is two dimensional and smooth, the residual locus is three dimensional, and the exceptional locus is equidimensional of dimension three. Indeed, since \( C'' \) is smooth of dimension two and \( C'' \) intersects with the fundamental center of \( U'' \) in a smooth one dimensional subscheme.

**Proposition 5.17.** \( A'' \otimes \bar{F}_p \) has precisely \( 4 + n_{s0s2 \tau} \) irreducible components. Three are given by the strict transforms of \( Z''_0, Z''_1, \) and \( Z''_0 \). The other \( 4 + n_{s0s2 \tau} \) are contained in the exceptional locus: one lying above \( Z''_0 \) and one lying above each two dimensional irreducible component of \( Z''_0 \).

**Proof.** That the strict transforms of \( Z''_0, Z''_1, \) and \( Z''_0 \) are irreducible follows immediately from the fact that they are not contained in the true center \( C'' \).

By the previous section, the exceptional locus \( E'' \) of \( A'' \) is equidimensional of dimension three. As the exceptional locus maps surjectively onto the true center \( C'' \) which has \( 1 + n_{s0s2 \tau} \) irreducible components by Proposition 5.10, we conclude that \( E'' \) must consist of at least \( 1 + n_{s0s2 \tau} \) irreducible components. Denote these irreducible components by \( \{W''_i\} \). Without loss of generality assume that \( \rho''(W''_1) \subset Z''_1 \) and \( \rho''(W''_2), \ldots, \rho''(W''_{1 + n_{s0s2 \tau}}) \) are each contained in a unique two dimensional irreducible component of \( Z''_0 \).

We claim that if \( \rho''(W''_1) \subset Z''_1 \), then \( \rho''(W''_1) = Z''_1 \). Indeed, since \( \rho'' \) is proper it suffices to show that \( \rho''(W''_1) \) is three dimensional. As the fiber above any closed point of \( U''_1 \) with respect to \( \rho'' \) is at most one dimensional, we conclude that the same is true for the fibers of \( \rho'' \) and thus \( \rho''(W''_1) \) is at least two dimensional. From Step II we have that \( Z''_1 = Z(u,v,b) \), and from the previous section \( C'' \) is contained in \( Z'' \).

They intersect in a one dimensional scheme, and it follows that the intersection of \( Z''_1 \) and \( Z''_2 \) is at most one dimensional. We can cover \( X_0 \) with two open subschemes each defined respectively by the condition \( \bar{b} \neq 0 \) and \( \bar{c} \neq 0 \). These open subschemes are
\[
\begin{align*}
X'_0 & = \begin{cases} \bar{b} \neq 0 & \overline{p}[a,u,\bar{b}+1,v]/(u^{p-1}(a)\bar{b}^{p-1} - p) \\
X''_0 & = \begin{cases} \bar{c} \neq 0 & \overline{p}[a,u,\bar{b},v]/(u^{p-1}(a)\bar{b}^{p-1} - p). 
\end{cases}
\end{cases}
\end{align*}
\]

Since \( X'_0 \subset X_0 \) as an open subscheme, \( X_0 \) is covered by \( X'_0, X_0, X_0, \) and \( X_1 \). By inspection, we see that \( X = U'' \) is normal, the true center of \( U'' \) is \( C'' \), and the fundamental center is \( Z''(u,v,\bar{x},y,c) = Z''_0 \cap Z''_0 \), and the residual locus is \( Z''(Z''_0 \cap Z''_1) \). The fundamental center is two dimensional and smooth, the residual locus is three dimensional, and the exceptional locus is equidimensional of dimension three. Indeed, since \( C'' \) is smooth of dimension two and \( C'' \) intersects with the fundamental center of \( U'' \) in a smooth one dimensional subscheme. Taking the strict transform with respect to \( \rho'' \) we have
\[
Z'' = Z(u,v,b), \quad C'' = Z(u,v,b,c,\bar{x}).
\]
with $C''_{\text{red},3}$ is one dimensional. Thus it must be that $\rho''_{\text{red}}(W''')$ is three dimensional and the claim follows.

Now consider a closed point $x''$ of $C'''_3 \setminus Z'''_{11}$. From Proposition 5.6 and inspection of $U'''_1 \to U''_1$, we see that the fiber above $x''$ is connected and smooth. It follows that for each irreducible component of $C'''_3 \setminus Z'''_{11}$, there is a single irreducible component of the exceptional locus of $\rho'''_{\text{red}}$ mapping surjectively onto it. Recall that we have labeled these components $W'''_2, \ldots, W'''_{n+\tau}$.

We claim that $(\rho'''_{\text{red}})^{-1}(Z'''_{11})$ is irreducible, i.e. that $(\rho'''_{\text{red}})^{-1}(Z'''_{11}) = W'''_1$. Indeed, since $(\rho'''_{\text{red}})^{-1}(Z'''_{11})$ is smooth and equidimensional of dimension three, so is $(\rho'''_{\text{red}})^{-1}(Z'''_{11})$. This implies that $(\rho'''_{\text{red}})^{-1}(Z'''_{11})$ is a disjoint union of irreducible components of $A'''_1 \otimes \mathbb{F}_p$. As each of these irreducible components maps into $Z'''_{11}$, they must map surjectively onto $Z'''_{11}$. It follows that there is a closed point $x''$ of $Z'''_1 \cap C'''_3 \setminus Z'''_{11}$ contained in the image of every irreducible component. From Proposition 5.6 and inspection of the local model, we have the fiber above $x''$ is connected, and hence $(\rho'''_{\text{red}})^{-1}(Z'''_{11})$ is connected. The claim follows immediately.

Now suppose there exists another irreducible component $W'''_{2+n+\tau}$ of $A'''_1 \otimes \mathbb{F}_p$. By the above, it must be that $W'''_{2+n+\tau}$ is mapped via $\rho'''_{\text{red}}$ into $Z'''_{11}$. But then it follows that

$$W'''_{2+n+\tau} \subset (\rho'''_{\text{red}})^{-1}(Z'''_{11}) = W'''_1$$

and therefore $W'''_{2+n+\tau} = W'''_1$. □

**Proposition 5.18.** The number of connected and irreducible components of the subschemes of $A'''_1$ are as follows.

| Closed subscheme of $A'''_1$ | # connected components | # irreducible components |
|-----------------------------|------------------------|-------------------------|
| $Z'''_1 = ST(Z'''_1)$       | 1                      | 1                       |
| $C'''_4 = ST(C'''_4)$       | $n + \tau$            | $n + \tau$             |

**Proof.** That $Z'''_1$ is irreducible was shown in the proof of the previous proposition.

As $C'''_4$ and $C'''_4$ are both smooth, $C'''_4$ and $C'''_4$ are smooth as well giving that each connected component is irreducible. Let $W'' \subset C'''_4$ be some connected component. Recalling that $C'''_4$ has $n + \tau$ such connected components and that $C'''_4$ is contained in the true center of $A'''_1 \to A'''_1$, the proposition will follow by showing that the inverse image of $W''$ with respect to $A'''_1 \to A'''_1$ is connected. But this is immediate since the fiber above every closed point of $C'''_4$ with respect to the morphism $U'''_1 \to U'''_1$ is connected. □

### 5.5 Step IV: $p - 2$ blowups of $Z'''_{11}$.

In this last step we define $A_1'^{[i]}$ for $4 \leq i \leq p + 1$ by first blowing up $Z_4'''$ in $A'''_1$, and then blowing up the strict transform of $Z_4'''$ in each successive modification. Likewise we define $U_1'^{[i]}$ for $4 \leq i \leq p + 1$ by blowing up $Z_4'''$ in $U_1'''$, and then blowing up the strict transform of $Z_4'''$ in each successive modification.
5.5.1 Description of $U_1^{[i]}$

Recall that $Z_{11}'''$ is given by $Z(b, u, v)$. $Z_{11}'''$ may be described on each affine chart in the cover of $X = U_1'''$ by giving its corresponding ideal.

| Chart | $X_{00}'$ | $X_{01}$ | $X_{10}$ | $X_{11}$ |
|-------|----------|---------|---------|---------|
| Ideal | $(u)$    | $(v)$   | $(b, u)$| $(b, v)$|

As $Z_{11}'''$ is Cartier on $X_{00}'$, and $X_{01}$, the blowups of $Z_{11}'''$ and its strict transforms are isomorphisms over these open subschemes. Focusing now on $X_{10}$ and $X_{11}$, each of these two charts are given by a scheme with the presentation

$$Y = \text{Spec}(A), \quad A = \tilde{\mathbb{Z}}_p[x_1, x_2, x_3, x_4, u]/(x_1x_2^2x_3^3x_4 - p, u^{p-1} - x_1x_2)$$

and $Z_{3}'''$ is given by the subscheme $W$ corresponding to the ideal $(x_1, u)$ in this presentation. To describe the blowups, we write $[x_1^{[1]} : u^{[1]}], [x_1^{[2]} : u^{[2]}]$, etc. for projective coordinates.

**Proposition 5.19.** Set $Y^{[0]} = Y$, $W_0 = W$, and for $1 \leq i \leq p - 2$, define $Y^{[i]}$ inside $Y \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ by

$$u^{[i]}u^{p-i-1} - x_1^{[i]}x_2, \quad u^{[1]}x_1 - x_1^{[1]}u^{[1]},$$

$$u^{[j-1]}x_1^{[j]} - x_1^{[j-1]}x_1^{[j]} \quad \text{for} \quad 2 \leq j \leq i$$

Let $W_i$ be the strict transform of $W_{i-1}$ in $Y^{[i]}$ for each $i \geq 1$. Then for $1 \leq i \leq p - 2$ we have the following.

(i) $Y^{[i]} \cong \text{Bl}_{W_{i-1}}(Y^{[i-1]})$.

(ii) The true center of $Y^{[i]} \to Y^{[i-1]}$ is one dimensional and smooth.

(iii) The fundamental center of $Y^{[i]} \to Y^{[i-1]}$ is equal to its true center.

(iv) The exceptional locus of $Y^{[i]} \to Y^{[i-1]}$ is smooth and two dimensional.

Furthermore, $Y^{[p-2]}$ is regular with special fiber a nonreduced divisor with normal crossings.

**Proof.** (i) We proceed by induction. So assume $W_{i-1}$ corresponds to the ideal $(x_1^{[i-1]}, u)$ which is certainly true for $i = 1$. By explicit computation, the claimed equations are part of those defining $\text{Bl}_{W_{i-1}}(Y^{[i-1]})$. The standard affine charts of $Y^{[i]}$, indexed by $1 \leq k \leq i + 1$, are described by the conditions

$$u^{[j]} \neq 0 \quad \text{for} \quad 1 \leq j < k \quad \text{and} \quad x_1^{[j]} \neq 0 \quad \text{for} \quad k \leq j \leq i.$$

In order to explicitly write them, we must consider three cases.

$k = 1$: The equations of $Y^{[i]}$ become

$$x_2 = x_1^{p-2}\left(\frac{u^{[1]}}{x_1^{[1]}}\right)^{p-1} \quad u = x_1\frac{u^{[1]}}{x_1} \quad \frac{u^{[j]}}{x_1^{[j]}} = x_1^{j-1}\left(\frac{u^{[1]}}{x_1^{[1]}}\right)^j \quad 2 \leq j \leq i$$

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and the coordinate ring is
\[ \tilde{\mathbb{Z}}_p \left[ x_1, \frac{u^{[1]}}{x_1^{[1]}}, x_3, x_4 \right] / \left( x_1^{2p-3} \left( \frac{u^{[1]}}{x_1^{[1]}} \right)^{2p-2} x_3^{p-1} x_4 - p \right). \]

1 < k ≤ i: The equations of \( Y^{[i]} \) become
\[ x_1 = \left( \frac{x_1^{[k-1]}}{u^{[k-1]}} \right)^k \left( \frac{u^{[k]}}{x_1^{[k]}} \right)^{k-1} \] \[ x_2 = \left( \frac{x_1^{[k-1]}}{u^{[k-1]}} \right)^{p-k} \left( \frac{u^{[k]}}{x_1^{[k]}} \right)^{p-k} \] \[ u = \frac{x_1^{[k-1]}}{u^{[k-1]}} \frac{u^{[k]}}{x_1^{[k]}} \]

and the coordinate ring is
\[ \tilde{\mathbb{Z}}_p \left[ \frac{x_1^{[k-1]}}{u^{[k-1]}}, \frac{u^{[k]}}{x_1^{[k]}}, x_3, x_4 \right] / \left( x_1^{2p-2} \left( \frac{u^{[k]}}{x_1^{[k]}} \right)^{2p-1} x_3^{p-1} x_4 - p \right). \]

k = i + 1: The equations of \( Y^{[i]} \) become
\[ x_1 = u \frac{x_1^{[i]}}{u^{[i]}} - x_3^{[i]} x_2 = 0 \]
\[ u^{[j]} = u^{i-j+1} \frac{x_1^{[i+1]}}{u^{[i+1]}} \] for 2 ≤ j ≤ i

and the coordinate ring is
\[ \tilde{\mathbb{Z}}_p \left[ \frac{x_1^{[i]}}{u^{[i]}}, x_2, x_3, x_4, u \right] / \left( u \frac{x_1^{[i]}}{u^{[i]}} x_2 x_3^{p-1} x_4 - p, u^{p-i-1} x_1^{[i]} - \frac{x_1^{[i]}}{u^{[i]}} x_2 \right). \]

Note that each chart is integral. Since the equations defining \( Y^{[i]} \) are part of those defining \( \text{Bl}_{W_{i-1}}(Y^{[i-1]}) \), there is a closed immersion \( \iota : \text{Bl}_{W_{i-1}}(Y^{[i-1]}) \to Y^{[i]} \) which is an isomorphism on the generic fiber. With \( Y^{[i]} \)-integral and of the same dimension as \( \text{Bl}_{W_{i-1}}(Y^{[i-1]}) \), this implies \( \iota \) is an isomorphism.

To complete the induction, we must show that the strict transform of the subscheme of \( Y^{[i-1]} \) given by \( Z(x_1^{[i-1]}, u) \) corresponds to the subscheme of \( Y^{[i]} \) given by \( Z(x_1^{[i]}, u) \). From the charts above, the true center of \( Y^{[i]} \to Y^{[i-1]} \) is given by \( Z(x_1^{[i-1]}, x_2, u) \). Thus outside of the true center we have that \( x_2 \) is invertible, and so from the relation \( u^{[i]} u^{p-i-1} - x_1^{[i]} x_2 \) of \( Y^{[i]} \) we get that \( x_1^{[i]} \) is in the ideal defining the strict transform. As subschemes of \( Y^{[i]} \), \( Z(x_1^{[i-1]}, x_1^{[i]}, u) = Z(x_1^{[i]}, u) \). This subscheme is irreducible of dimension three, and therefore we conclude it must be the strict transform of \( W_{i-1} \).

The remainder of the proposition now follows from inspection of the above charts. □
Using the explicit equations above, we record certain aspects of the irreducible components of the special fiber.

**Lemma 5.20.** The irreducible components of $U_1^{[p-2]} \otimes \overline{\mathbb{F}}_p$ are described as follows.

- There are $p + 3$ components. Three components are given by $Z(\tilde{u})$, $Z(\tilde{v})$ and $Z(a)$. We index the other components by $1 \leq i \leq p$. For $1 \leq i \leq p - 1$, the $i$th irreducible component is given by
  \[ Z_i = Z(u, b, \tilde{x}, b^{[1]}, b^{[2]}, \ldots, b^{[i-2]}, u^{[i]}, u^{[i+1]}, \ldots, u^{[p-2]}) \]
  and the $p$th irreducible component is given by
  \[ Z_p = Z(u, b, b^{[1]}, b^{[2]}, \ldots, b^{[p-2]}). \]

- The components $Z(\tilde{u})$, $Z(\tilde{v})$, $Z(a)$, and $Z_i$ have multiplicity $p - 1$, $p - 1$, 1, and $2p - i - 1$ respectively. In particular, $Z_{p-1}$ is the only component with multiplicity divisible by $p$.

- The components $Z_1$ and $Z_p$ are isomorphic to $\mathbb{A}_F^3$. The components $Z_i$ with $2 \leq i \leq p - 1$ are isomorphic to $\mathbb{P}_F^1 \times \mathbb{A}_F^2$.

- The components intersect as indicated in the following “dual complex”, drawn for $p = 5$. Each vertex represents an irreducible component and the label indicates its multiplicity. Each edge indicates that the two irreducible components intersect.

Moreover, a full subgraph that is a $k$-simplex indicates a $(k + 1)$-fold intersection of the corresponding irreducible components, and conversely.

- An intersection of components as indicated by a $k$-simplex has dimension $3 - k$ over $\text{Spec}(\mathbb{F}_p)$.

### 5.5.2 Description of $A^{[i]}_1$

**Proposition 5.21.** For $4 \leq i \leq p + 1$, the scheme $A^{[i]}_1 \otimes \overline{\mathbb{F}}_p$ has $4 + n_{s_0s_2\tau} + (i - 3) \cdot n_{s_2\tau}$ irreducible components.

**Proof.** We recall the following facts:
(i) \( C_4^{[3]} \) has \( n_{s_2^7} \) connected components and each is smooth of dimension two;

(ii) for \( 4 \leq i \leq p + 1 \), the fiber over a closed point of the true center of \( U_1^{[i]} \to U_1^{[i-1]} \)

is one dimensional, smooth, and connected; and

(iii) for \( 4 \leq i \leq p + 1 \), \( U_1^{[i]} \otimes \overline{\mathbb{F}}_p \) is equidimensional of dimension three.

We proceed by induction, starting with the modification \( A_1^{[4]} \to A_1^{[3]} \). Facts (i) and

(ii) imply that the exceptional locus of \( A_1^{[4]} \to A_1^{[3]} \) has the same number of connected components as the true center \( C_4^{[3]} \), and furthermore inspection of \( U_1^{[4]} \) shows that each such connected component is three dimensional and smooth. By (iii), each of

these connected components is an irreducible component of \( A_1^{[4]} \otimes \overline{\mathbb{F}}_p \), with all of the

other irreducible components of \( A_1^{[4]} \otimes \overline{\mathbb{F}}_p \), being the strict transforms of the irreducible components of \( A_1^{[3]} \otimes \overline{\mathbb{F}}_p \). Therefore there are \( 4 + n_{s_0 s_2^7} + n_{s_2^7} \) irreducible components of \( A_1^{[4]} \otimes \overline{\mathbb{F}}_p \).

Now assume that the result is true for \( i - 1 \) with \( 4 < i \leq p + 1 \). We must show that

\( C_i^{[p-1]} \) has \( n_{s_2^7} \) connected components. Indeed then the induction will follow using the same argument as in the above paragraph, noting that \( C_i^{[p-1]} \) is two dimensional and smooth since this is true for \( C_i^{[p-1]} \). Let \( Z_i^{[i-1]} \) denote the strict transform of

\( Z_i^{[i]} \) with respect to the modification \( A_i^{[i-1]} \to A_i^{[p-1]} \), and likewise with \( Z_i^{[i]} \) and the modification \( U_i^{[i-1]} \to U_i^{[p-1]} \). With \( Z_i^{[i-1]} \), \( E_i^{[i-1]} \), and \( C_i^{[i-1]} \) étale locally corresponding to \( Z_i^{[i-1]} \), \( E_i^{[i-1]} \), and \( C_i^{[i-1]} \) respectively, from \( C_i^{[i-1]} = Z_i^{[i-1]} \cap \mathcal{E}_i^{[i-1]} \) we get that

\( C_i^{[i-1]} = Z_i^{[i-1]} \cap \mathcal{E}_i^{[i-1]} \).

Consider a connected component of \( C_i^{[i-2]} \). Note that such a component is irreducible because \( C_i^{[i-2]} \) is smooth. The fiber above this component with respect to the morphism \( A_i^{[i-1]} \to A_i^{[i-2]} \) is connected by Zariski’s Main Theorem since \( A_i^{[i-2]} \) is normal. Thus \( E_i^{[i-1]} \) has the same number of connected components as \( C_i^{[i-2]} \). Now \( Z_i^{[i-1]} = \text{ST}(Z_i^{[i-2]}) \) maps surjectively onto \( Z_i^{[i-2]} \) via \( \rho_i^{[i-1]} \), and hence the image meets each connected component of \( C_i^{[i-2]} \). As such, \( Z_i^{[i-1]} \) meets each connected component of \( C_i^{[i-1]} \). Therefore \( C_i^{[i-1]} = Z_i^{[i-1]} \cap \mathcal{E}_i^{[i-1]} \) has the same number of connected components as \( C_i^{[i-2]} \); namely \( n_{s_2^7} \). \( \square \)

The information on the irreducible components of \( U_1^{[p+1]} \otimes \overline{\mathbb{F}}_p \) can now be carried over to the integral model. Namely the multiplicities of the components can be computed étale locally, and two irreducible components of \( A_1^{[p+1]} \otimes \overline{\mathbb{F}}_p \) intersect if and only if they correspond to irreducible components of \( U_1^{[p+1]} \otimes \overline{\mathbb{F}}_p \) that intersect. We first construct the graph that will give the dual complex.

**Definition 5.22.** Let \( p \) be an odd rational prime and \( K^p = K(N) \subset G(\mathbb{A}_F^p) \), so that \( K^p \) determines the numbers \( n_{s_2^7} \) and \( n_{s_0 s_2^7} \). We then define the vertex-labeled graph \( \Gamma_{p, K^p} \) as follows.

(i) Begin with \( n_{s_2^7} \) batons, each having \( p - 2 \) vertices. Label the vertices \( 2p - 3, 2p - 4, \ldots, p \) from head to tail.
Example: $p = 5, n_{s_2\tau} = 2$.

(ii) Add one vertex labeled $2p - 2$ (top left) and attach edges between this vertex and the heads of the batons. Add two more vertices labeled $p - 1$ (bottom left and top right) and connect these two vertices to every vertex in the batons, as well as the unique vertex labeled $2p - 2$. Add $n_{s_0s_2\tau}$ vertices labeled $p - 1$ (bottom right) and attach edges between these and the tails of the batons, as well as the two vertices labeled $p - 1$ from before.

Example: $p = 5, n_{s_2\tau} = 2, n_{s_0s_2\tau} = 3$.

(iii) Add one vertex labeled $1$ and attach edges from this to every vertex constructed in the above two steps.

Example: $p = 5, n_{s_2\tau} = 2, n_{s_0s_2\tau} = 3$.

**Definition 5.23.** We define the following subsets of the vertices of $\Gamma_{p,K_p}$.

- The batons consist of the vertices given in step one above. They may be identified as the vertices with label in $[p, 2p - 3]$. 

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• The front consists of the vertices labeled \( p - 1 \) on the bottom right of the diagram directly above. They may be identified as the vertices of label \( p - 1 \) and edge degree \( 3 + n_{s_2} \) that share edges with precisely two vertices labeled 4.

• The sides consist of the vertices labeled \( p - 1 \) which are not in the front.

Recall that we are writing \( \mathcal{A}_0 \) for what is really a single connected component of \( \mathcal{A}_0 \otimes \mathbb{Z}_p \hat{\mathbb{Z}}_p \), and similarly with \( \mathcal{A}_1 \).

**Theorem 5.24.** \( \mathcal{A}_1^{[p+1]} \to \mathcal{A}_1 \) is a resolution of singularities and the special fiber of \( \mathcal{A}_1^{[p+1]} \) is a nonreduced divisor with normal crossings. \( \mathcal{A}_1^{[p+1]} \otimes \overline{\mathbb{F}}_p \) has \( 4 + n_{s_0 s_2} + (p - 2) \cdot n_{s_2} \) irreducible components whose intersections are described by the vertex-labeled graph \( \Gamma_{p,K_p} \) as follows.

(i) Each vertex represents an irreducible component. The label of the vertex is the multiplicity of the component.

(ii) A full subgraph of \( \Gamma_{p,K_p} \) which is a \( k \)-simplex indicates a \( (k+1) \)-fold intersection of the corresponding irreducible components, and conversely. Such an intersection has dimension \( 3 - k \) over \( \text{Spec}(\overline{\mathbb{F}}_p) \).

(iii) Let \( x^{[p-2]} \) be a closed point of \( \mathcal{A}_1^{[p-2]} \) and \( \{e_1, \ldots, e_t\} \) be the multiset of the multiplicities of the irreducible components which \( x^{[p-2]} \) lies on. Then there is an étale neighborhood of \( x^{[p-2]} \) of the form

\[
\text{Spec}(\mathbb{Z}_p[x_1, x_2, x_3, x_4]/(x_1^{e_1} \cdots x_t^{e_t} - p)).
\]

(iv) The following table gives the image of each irreducible component under the map \( \mathcal{A}_1^{[p+1]} \to \mathcal{A}_0 \).

| Description         | Image                                                                 |
|---------------------|----------------------------------------------------------------------|
| Front               | Each irreducible component surjects onto a connected component of \( \mathcal{A}_{s_0 s_2} \) |
| Sides               | These two irreducible components surject onto the irreducible components \( \mathcal{Z}_{01} \) and \( \mathcal{Z}_{10} \) respectively. |
| Vertex labeled 2p − 2 | Surjects onto \( \mathcal{Z}_{11} \).                              |
| Vertex labeled 1    | Surjects onto \( \mathcal{Z}_{00} \).                              |
| Batons              | Fix a baton \( B \). The irreducible components corresponding to the vertices in \( B \) all surject onto the same connected component of \( \mathcal{A}_{s_2} \). This induces a bijection between the set of batons and the set of connected components of \( \mathcal{A}_{s_2} \). |

**References**

[BBM] P. Berthelot, L. Breen, and W. Messing. *Théorie de Dieudonné cristalline. II*, volume 930 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1982.

[dJ1] A. J. de Jong. Talk given in Wuppertal. 1991.
A. J. de Jong. The moduli spaces of principally polarized abelian varieties with \(\Gamma_0(p)\)-level structure. *J. Algebraic Geom.*, 2(4):667–688, 1993.

P. Deligne and G. Pappas. Singularités des espaces de modules de Hilbert, en les caractéristiques divisant le discriminant. *Compositio Math.*, 90(1):59–79, 1994.

D. Eisenbud and J. Harris. *The geometry of schemes*, volume 197 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

G. Faltings. Toroidal resolutions for some matrix singularities. In *Moduli of abelian varieties (Texel Island, 1999)*, volume 195 of *Progr. Math.* pages 157–184. Birkhäuser, Basel, 2001.

A. Genestier. Un modèle semi-stable de la variété de Siegel de genre 3 avec structures de niveau de type \(\Gamma_0(p)\). *Compositio Math.*, 123(3):303–328, 2000.

U. Görtz. On the flatness of models of certain Shimura varieties of PEL-type. *Math. Ann.*, 321(3):689–727, 2001.

U. Görtz. On the flatness of local models for the symplectic group. *Adv. Math.*, 176(1):89–115, 2003.

U. Görtz. Computing the alternating trace of Frobenius on the sheaves of nearby cycles on local models for GL\(_4\) and GL\(_5\). *J. Algebra*, 278(1):148–172, 2004.

U. Görtz and C. Yu. The supersingular locus in Siegel modular varieties with Iwahori level structure. *Math. Ann.*, 353(2):465–498, 2012.

T. Haines. On connected components of Shimura varieties. *Canad. J. Math.*, 54(2):352–395, 2002.

T. Haines. Introduction to Shimura varieties with bad reduction of parahoric type. In *Harmonic analysis, the trace formula, and Shimura varieties*, volume 4 of *Clay Math. Proc.*, pages 583–642. Amer. Math. Soc., Providence, RI, 2005.

T. Haines and M. Rapoport. Shimura varieties with \(\Gamma_1(p)\)-level via Hecke algebra isomorphisms: The Drinfeld case. *Ann. Scient. Ecole Norm. Sup.*, 45(4):719–785, 2012.

M. Harris and R. Taylor. Regular models of certain Shimura varieties. *Asian J. Math.*, 6(1):61–94, 2002.

N. Katz and B. Mazur. *Arithmetic moduli of elliptic curves*, volume 108 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1985.

R. Kottwitz. Points on some Shimura varieties over finite fields. *J. Amer. Math. Soc.*, 5(2):373–444, 1992.

H. Matsumura. *Commutative algebra*, volume 56 of *Mathematics Lecture Note Series*. Benjamin/Cummings Publishing Co., Inc., Reading, Mass., second edition, 1980.
[OT] F. Oort and J. Tate. Group schemes of prime order. *Ann. Sci. École Norm. Sup. (4)*, 3:1–21, 1970.

[Pap1] G. Pappas. Arithmetic models for Hilbert modular varieties. *Compositio Math.*, 98(1):43–76, 1995.

[Pap2] G. Pappas. On the arithmetic moduli schemes of PEL Shimura varieties. *J. Algebraic Geom.*, 9(3):577–605, 2000.

[PR1] G. Pappas and M. Rapoport. Local models in the ramified case. I. The EL-case. *J. Algebraic Geom.*, 12(1):107–145, 2003.

[PR2] G. Pappas and M. Rapoport. Local models in the ramified case. II. Splitting models. *Duke Math. J.*, 127(2):193–250, 2005.

[PR3] G. Pappas and M. Rapoport. Local models in the ramified case. III. Unitary groups. *J. Inst. Math. Jussieu*, 8(3):507–564, 2009.

[PZ] G. Pappas and X. Zhu. Local models of Shimura varieties and a conjecture of Kottwitz. *Invent. Math.*, 194(1):147–254, 2013.

[RZ] M. Rapoport and T. Zink. *Period spaces for p-divisible groups*, volume 141 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1996.

[Yu] C. Yu. Irreducibility and p-adic monodromies on the Siegel moduli spaces. *Adv. Math.*, 218(4):1253–1285, 2008.