GLOBAL EXISTENCE OF IDEAL INVICID COMPRESSIBLE AND HEAT CONDUCTIVE FLUIDS WITH RADIAL SYMMETRY.

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Abstract. In this paper, we study the global existence of classical solutions to the three dimensional ideal invicid compressible and heat conductive fluids with radial symmetrical data in $H^s(\mathbb{R}^3)$. Our proof is based on the symmetric hyperbolic structure of the system.

1. Introduction

The motion for a compressible viscous, heat-conductive, isotropic Newtonian fluid is described by the system of equations

$$
\begin{cases}
\rho_t + \nabla \cdot (\rho u) = 0 \\
\rho u_t + \rho u \cdot \nabla u + \nabla p - \left( \mu' - \frac{2}{3} \mu \right) \nabla (\nabla \cdot u) - \nabla \cdot (\mu (\nabla u + (\nabla u)^T)) = 0 \\
\rho \left( \frac{|u|^2}{2} + e \right)_t + \nabla \cdot \left( \rho u \left( \frac{|u|^2}{2} + e \right) + pu \right) - \nabla \cdot (\mu (\nabla u + (\nabla u)^T) u) \\
+ \left( \mu' - \frac{2}{3} \mu \right) u (\nabla \cdot u) = \nabla \cdot (\kappa \nabla T),
\end{cases}
$$

(1.1)

where $t \geq 0$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\rho > 0$ denotes the density, $u = (u_1, u_2, u_3)$ the fluid velocity, $T > 0$ the absolute temperature, $e$ denotes the internal energy, and $p$ denotes the pressure. And the positive constants $\mu, \mu'$ satisfying

$$
\mu > 0, \quad \mu' + \frac{2}{3} \mu \geq 0
$$

describe the viscosity.

There are very rich results about compressible Navier-Stokes system, such as small classical solutions with finite energy by Matsumura-Nishida [15], see also Huang-Li [8] about the case of vacuum, weak, finite-energy solutions by Lions [14], variational solutions by Feireisl [3] and Feireisl-Novotný-Petzeltová [4], solutions in Besov spaces with the interpolation index one by Chikami-Danchin [1], Danchin [2], self-similar solutions by Guo-Jiang [6], Li-Chen-Xie [12] (density-dependent viscosity) and Germain-Iwabuchi [5].

There are also some literature related to the vacuum. Xin [16] proved the non-existence of smooth solutions for the initial density with the compact support. Hoff and Smoller [7] considered 1-D barotropic Navier-Stokes equations and showed that the persistency of the almost everywhere positivity of the density can prevent the formulation of vacuum state. Jang and Masmoudi [9] obtained local solutions of the 3D compressible Euler equations under the barotropic condition with a physical vacuum, see also [10] about problems of vacuum state. Recently, Lai, Liu and Tarfulea [11] studied the derivation of some non-isothermal hydrodynamic models (including non-isothermal ideal gas) and established the corresponding maximum principle.

In the classical paper of Matsumura-Nishida [15], they proved the global existence of classical solutions with small data of $O(\varepsilon)$ in $H^s$, where $\varepsilon$ depends on $\mu$, $\mu'$ and $\kappa$. The main purpose of this paper is to improve the result of [15] in radial symmetry case.

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In this case we can take the small constant $\varepsilon$ independent of $\mu$ and $\mu'$, and only depends on $\kappa$. More precisely, we can set $\mu = \mu' = 0$ and (1.1) will thus reduce to the following system

$$
\begin{align*}
\rho_t + \nabla \cdot (\rho u) &= 0 \\
(\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla p &= 0 \\
\left( \rho \left( \frac{1}{2} |u|^2 + e \right) \right)_t + \nabla \cdot \left( \rho u \left( \frac{1}{2} |u|^2 + e \right) + pu \right) &= \nabla \cdot (\kappa \nabla T),
\end{align*}
$$

(1.2)

We assume the following conditions on (1.2):

1. The gas is ideal: $p = RT \rho$, where $R$ is a positive constant;
2. The gas is polytropic: $e = c_V T$, where $c_V$ is a positive constant which denotes the specific heat at constant volume.

Assume that the positive constants $R$, $c_V$, $\kappa = 1$, then the system (1.2) can be written in the following form

$$
\begin{align*}
\rho_t + \nabla \cdot (\rho u) &= 0 \\
u_t + u \cdot \nabla u + \frac{1}{\rho} \nabla (\rho T) &= 0 \\
T_t + u \cdot \nabla T + T (\nabla \cdot u) &= \frac{\Delta T}{\rho},
\end{align*}
$$

(1.3)

Suppose that the initial data

$$
\rho(0, x) = 1 + a_0(r), \quad T(0, x) = 1 + \theta_0(r), \quad u(0, x) = u_0(r) = u_0(r) \omega
$$

(1.4)

satisfy

$$
\|a_0\|_{H^s}^2 + \|u_0\|_{H^s}^2 + \|\theta_0\|_{H^s}^2 \leq \varepsilon^2,
$$

where $s > 5$ is an integer, $r = |x|$ and $\omega = \frac{x}{|x|}$, and $\varepsilon > 0$ is a small constant.

By the uniqueness of classical solutions, the solutions must have the following form

$$
\rho = 1 + a(t, r), \quad T = 1 + \theta(t, r), \quad u = u(t, r) \omega,
$$

as a result, we obtain

$$
\nabla \times u \equiv 0.
$$

So we may consider the follow system.

$$
\begin{align*}
a_t + u \cdot \nabla a + (1 + a)(\nabla \cdot u) &= 0 \\
u_t + u \cdot \nabla u + \nabla \theta + \frac{1 + \theta}{1 + a} \nabla a &= 0 \\
\theta_t + u \cdot \nabla \theta + (1 + \theta)(\nabla \cdot u) &= \frac{\Delta \theta}{1 + a}
\end{align*}
$$

(1.5)

with the condition

$$
\nabla \times u \equiv 0.
$$

(1.6)

Our main result can be stated as follows.
**Theorem 1.1.** Consider the Cauchy problem of the three dimensional system (1.3)-(1.6) (or (1.5)-(1.6)) with data (1.4). Then there exists a constant $\varepsilon_0 > 0$ such that for $\forall \varepsilon < \varepsilon_0$, the system (1.3)-(1.6) (or (1.5)-(1.6)) admits a global solution 

\[
(a,u) \in L^\infty(\mathbb{R}_+; H^s(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+; H^s(\mathbb{R}^3)),
\]
and

\[
\theta \in L^\infty(\mathbb{R}_+; H^s(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+; H^{s+1}(\mathbb{R}^3)).
\]

As Thm 1.1 shows, heat conduction effect alone can prevent the formation of shock despite the lack of viscosity.

**Remark 1.1.** It’s clear that the solution of (1.2) have the following conservation laws

\[
\frac{d}{dt} \int a \, dx \equiv 0,
\]
and

\[
\frac{d}{dt} \int (a+1) u \, dx \equiv 0,
\]
and

\[
\frac{d}{dt} \int \left( \frac{|u|^2}{2} + \frac{a|u|^2}{2} + a\theta + \theta \right) \, dx \equiv 0.
\]

We set

\[
E_{k,1}(t) \triangleq \sum_{|\alpha| \leq k} \sup_{\tau \in [0,t]} \left( \|\partial^\alpha a(\tau)\|_{L^2}^2 + \|\partial^\alpha u(\tau)\|_{L^2}^2 + \|\partial^\alpha \theta(\tau)\|_{L^2}^2 \right)
\]

\[
+ \sum_{|\alpha| \leq k} \int_0^t \|\nabla \partial^\alpha \theta(\tau)\|_{L^2}^2 \, d\tau
\]

for $0 \leq k \leq s$, and

\[
E_{k,2}(t) \triangleq \sum_{|\alpha| \leq k-1} \int_0^t \left( \|\nabla \partial^\alpha a(\tau)\|_{L^2}^2 + \|\nabla \partial^\alpha u(\tau)\|_{L^2}^2 \right) \, d\tau
\]

for $1 \leq k \leq s$, where

\[
\partial = (\partial_t, \partial_{x_1}, \partial_{x_2}, \partial_{x_3}).
\]

According to (1.4), it’s clear that $\exists \ M > 0$ such that

\[
E_{s,1}(0) + E_{s,2}(0) \leq M^2 \varepsilon^2.
\]

Due to the local existence result, there exists a positive time $t_* \leq +\infty$ such that

\[
t_* = \max \left\{ t \geq 0 \mid E_{s,1}(\tau) + E_{s,2}(\tau) \leq \varepsilon, \ \forall \ \tau \in [0,t_*) \right\}.
\]  

(1.7)

We have the following lemma.

**Lemma 1.1.** Let

\[
S = \ln \left( \frac{T}{\rho} \right) = \ln \left( \frac{1+\theta}{1+a} \right)
\]

(1.8)
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denotes the entropy of unit mass, then the entropy of the system increases.

Proof. The entropy of unit volume is
\[\rho S = \rho \ln \left(\frac{T}{\rho}\right) = \rho \ln T - \rho \ln \rho,\]
and we can establish the evolution equation of \(\rho S\):
\[\partial_t (\rho S) = \rho_t S + \rho S_t = -S \nabla \cdot (\rho u) + \nabla \cdot (\rho u) + \frac{\rho T_t}{T},\]
then the third equation of (1.3) gives the result
\[\frac{d}{dt} \int \rho S dx = \int \rho u \cdot \nabla S + \frac{\rho T_t}{T} dx = \int \frac{\Delta T}{\rho} dx = \int \frac{|\nabla T|^2}{T^2} dx \geq 0.\]
This completes the proof of Lemma 1.1. \(\square\)

2. Basic Energy Estimate

By Lemma 1.1, we have
\[
\frac{d}{dt} \int (1+a) \ln \left(\frac{1+a}{1+\theta}\right) + \int \frac{\nabla \theta^2}{(1+\theta)^2} dx = 0. \tag{2.1}
\]
Making linear combination of (2.1) and the conservation quantities, we obtain
\[
\frac{d}{dt} \int (1+a) \ln \left(\frac{1+a}{1+\theta}\right) + \left(\frac{|u|^2}{2} + a \frac{|u|^2}{2} - a + a \theta + \theta\right) + \int \frac{\nabla \theta^2}{(1+\theta)^2} dx = 0. \tag{2.2}
\]
Making a Taylor expansion of (1.8) with respect to \(a\) and \(\theta\), we have
\[
\ln \left(\frac{1+a}{1+\theta}\right) = a - \theta - \frac{a^2}{2} + \frac{\theta^2}{2} + r(a, \theta),
\]
where the remainder \(r(a, \theta)\) satisfies
\[r(a, \theta) = O(a^3 + \theta^3), \quad |a| + |\theta| \to 0.\]
Go back to (2.2), we get
\[
\|a(t)\|_{L^2} + \|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla \theta(\tau)\|_{L^2}^2 d\tau
= E_{0,1}(0) + \int a_0(\theta_0^2 + |u_0|^2 - a_0^2) + (1 + a_0)r(a_0, \theta_0) dx
- \int a(\theta^2 + |u|^2 - a^2) + (1 + a)r(a, \theta) dx + 2 \int_0^t \int \theta |\nabla \theta|^2 \frac{(2+\theta)}{(1+\theta)^2} dxd\tau.
\]
By (1.7), we have
\[
\|a\|_{L^\infty} + \|\theta\|_{L^\infty} \leq C \sqrt{E_{2,1}(t)} \leq C \varepsilon^{\frac{1}{2}},
\]
this gives the result
\[
E_{0,1}(t) \leq E_{0,1}(0) + E_{2,1}^{3/2}(0) + E_{2,1}^{3/2}(t), \tag{2.3}
\]
here and hereafter \(A \lesssim B\) means \(A \leq CB\) with a positive constant \(C\).
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3. The Estimate of $E_k$

Firstly, we write the equations of $a$ and $u$ in (1.5) in the following form of symmetric hyperbolic systems.

$$A_0(U, \theta) U_t + \sum_{j=1}^{3} A_j(U, \theta) \partial_j U + F = 0,$$

(3.1)

where

$$U = \begin{pmatrix} a \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad A_0 = \begin{pmatrix} \frac{1+a}{1+a} & 0 & 0 & 0 \\ 0 & 1+a & 0 & 0 \\ 0 & 0 & 1+a & 0 \\ 0 & 0 & 0 & 1+a \end{pmatrix},$$

and

$$A_j = \begin{pmatrix} \frac{1+a}{1+a} u_j & (1+\theta)\delta_{1j} & (1+\theta)\delta_{2j} & (1+\theta)\delta_{3j} \\ (1+\theta)\delta_{1j} & (1+a)u_j & 0 & 0 \\ (1+\theta)\delta_{2j} & 0 & (1+a)u_j & 0 \\ (1+\theta)\delta_{3j} & 0 & 0 & (1+a)u_j \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ (1+a)\theta x_1 \\ (1+a)\theta x_2 \\ (1+a)\theta x_3 \end{pmatrix}.$$

By applying $\partial^\alpha$ to (3.1), where the multi-index $\alpha$ satisfying $0 < |\alpha| \leq k$, and the positive integer $k \leq s$, we obtain

$$A_0 \partial^\alpha U_t + \sum_{j=1}^{3} A_j \partial_j \partial^\alpha U = (A_0 \partial^\alpha U_t - \partial^\alpha (A_0 U_t)) + \sum_{j=1}^{3} (A_j \partial_j \partial^\alpha U - \partial^\alpha (A_j \partial_j U)) - \partial^\alpha F.$$

Then we take the $L^2$ inner product of the above equation with $\partial^\alpha U$ and integrate with respect to $t$. By the symmetry of $A_j$ and $A_0$, we have the following energy estimate

$$\int \partial^\alpha U^T A_0 \partial^\alpha U dx$$

$$= 2 \int_0^t \int \partial^\alpha U^T \left( (A_0 \partial^\alpha U_t - \partial^\alpha (A_0 U_t)) + \sum_{j=1}^{3} (A_j \partial_j \partial^\alpha U - \partial^\alpha (A_j \partial_j U)) \right) dx d\tau$$

$$+ \int \partial^\alpha U^T A_0 (U_0, \theta_0) \partial^\alpha U_0 dx + \int_0^t \int \partial^\alpha U^T \left( \partial_t A_0 + \sum_{j=1}^{3} \partial_j A_j \right) \partial^\alpha U dx d\tau$$

$$- 2 \int_0^t \int \partial^\alpha F \cdot \partial^\alpha U dx d\tau,$$

where

$$-2 \int_0^t \int \partial^\alpha F \cdot \partial^\alpha U dx d\tau = -2 \int_0^t \int \partial^\alpha u \cdot \nabla \partial^\alpha \theta + \partial^\alpha u \cdot \partial^\alpha (a \nabla \theta) dx d\tau.$$

On the other hand, we make energy estimate of $\theta$ to obtain

$$\int |\partial^\alpha \theta|^2 dx + 2 \int_0^t \int |\nabla \partial^\alpha \theta|^2 dx d\tau$$

$$= \int |\partial^\alpha \theta_0|^2 dx + 2 \int_0^t \int \partial^\alpha u \cdot \nabla \partial^\alpha \theta dx + 2 \int_0^t \int \partial^\alpha (\partial \theta u) \cdot \nabla \partial^\alpha \theta dx d\tau$$

$$+ 2 \int_0^t \int \frac{\partial^\alpha \theta \nabla \partial^\alpha \theta \cdot \nabla a}{(1+a)^2} + a |\nabla \partial^\alpha \theta|^2 + \partial^\alpha \theta \left( \frac{\Delta \theta}{1+a} - \frac{\Delta \partial^\alpha \theta}{1+a} \right) dx d\tau.$$
Adding (3.2) to (3.3), we get
\[
\int \partial^\alpha U^T A_0 \partial^\alpha U \, dx + \int |\partial^\alpha \theta|^2 \, dx + 2 \int_0^t \int |\nabla \partial^\alpha \theta|^2 \, dx \, d\tau
\]
\[
= \int \partial^\alpha U^T_0 A_0(U_0, \theta_0) \partial^\alpha U_0 \, dx + \int |\partial^\alpha \theta_0|^2 \, dx
\]
\[
+ 2 \int_0^t \int \partial^\alpha (\theta u) \cdot \nabla \partial^\alpha \theta - \partial^\alpha u \cdot \partial^\alpha (a \nabla \theta) \, dx \, d\tau
\]
\[
+ 2 \int_0^t \int \frac{\partial^\alpha \theta \nabla \partial^\alpha \theta \cdot \nabla a}{(1 + a)^2} + \frac{a|\nabla \partial^\alpha \theta|^2}{1 + a} + \partial^\alpha \theta \left( \partial^\alpha \left( \frac{\Delta \theta}{1 + a} - \frac{\Delta \partial^\alpha \theta}{1 + a} \right) \right) \, dx \, d\tau
\]
\[
+ 2 \int_0^t \int \partial^\alpha U^T \left( A_0 \partial^\alpha U_t - \partial^\alpha (A_0 U_t) + \sum_{j=1}^3 (A_j \partial_j \partial^\alpha U - \partial^\alpha (A_j \partial_j U)) \right) \, dx \, d\tau
\]
\[
+ \int_0^t \int \partial^\alpha U^T \left( \partial_t A_0 + \sum_{j=1}^3 \partial_j A_j \right) \partial^\alpha U \, dx \, d\tau.
\]

We have the following lemma from [13] to deal with the nonlinear terms.

**Lemma 3.1.** For $\forall N \in \mathbb{N}_+$, we have
\[
\|fg\|_{H^N} \lesssim \left( \sum_{|\alpha_1| \leq \lfloor \frac{N-1}{2} \rfloor} \|\partial^{\alpha_1} f\|_{L^\infty} \right) \left( \sum_{|\alpha_2| \leq N} \|\partial^{\alpha_2} g\|_{L^\infty} \right)
\]
\[
+ \left( \sum_{|\alpha_3| \leq \lfloor \frac{N-1}{2} \rfloor} \|\partial^{\alpha_3} g\|_{L^\infty} \right) \left( \sum_{|\alpha_4| \leq N} \|\partial^{\alpha_4} f\|_{L^\infty} \right).
\]

For any multi-index $\beta$ satisfying $|\beta| = N > 0$, we have
\[
\|\partial^{\beta} (fg) - f \partial^{\beta} g\|_{L^2} \lesssim \left( \sum_{|\beta_1| \leq \lfloor \frac{N-1}{2} \rfloor} \|\partial^{\beta_1} f\|_{L^\infty} \right) \left( \sum_{|\beta_3| \leq N-1} \|\partial^{\beta_3} g\|_{L^2} \right)
\]
\[
+ \left( \sum_{|\beta_2| \leq \lfloor \frac{N-1}{2} \rfloor} \|\partial^{\beta_2} g\|_{L^\infty} \right) \left( \sum_{|\beta_4| \leq N} \|\partial^{\beta_4} f\|_{L^2} \right).
\]

Recall that $A_0$ is a positive definite matrix. By Lemma 3.1 and the Sobolev imbedding theorems, we obtain
\[
(1 - C\varepsilon) \int |\partial^\alpha U|^2 \, dx + \int |\partial^\alpha \theta|^2 \, dx + 2 \int_0^t \int |\nabla \partial^\alpha \theta|^2 \, dx \, d\tau
\]
\[
\lesssim E_{k,1}(0) + E_{\{k/2+5/2\},1}^{1/2}(E_{k,1}(t) + E_{k,2}(t))
\]

Thus we get
\[
E_{k,1}(t) \lesssim E_{k,1}(0) + E_{\{k/2+5/2\},1}^{1/2}(E_{k,1}(t) + E_{k,2}(t)).
\]

(3.4)

To estimate $E_{k,2}(t)$, we set
\[
B_i(U, \theta) = A_i(U, \theta) - A_i(0,0), \quad 0 \leq i \leq 3,
\]
then we can rewrite (3.1), and apply $\partial^\beta (|\beta| \leq k - 1)$ to get
\[
\partial^\beta U_t + \sum_{j=1}^{3} A_j(0,0) \partial^\beta \partial_j U
\]
\[
= \left( \frac{\partial^\beta a_t + \partial^\beta (\nabla \cdot u)}{\partial^\beta u_t + \nabla \partial^\beta a} \right) = -\partial^\beta \left( B_0 U_t + \sum_{j=1}^{3} B_j \partial_j U + F \right). \tag{3.5}
\]
Taking inner product of (3.5) with the following vector
\[
\partial^\beta V \triangleq (-\partial^\beta (\nabla \cdot u), \nabla \partial^\beta a),
\]
and integrate with respect to $t$, we get
\[
\int \partial^\beta u \cdot \nabla \partial^\beta a \, dx + \int_{0}^{t} \int |\nabla \partial^\beta a|^2 - |\partial^\beta (\nabla \cdot u)|^2 + \nabla \partial^\beta a \cdot \nabla \partial^\beta \theta \, dx \, d\tau
\]
\[
= \int \partial^\beta u_0 \cdot \nabla \partial^\beta a_0 \, dx - \int_{0}^{t} \int \nabla \partial^\beta a \cdot \partial^\beta (a \nabla \theta) \, dx \, d\tau
\]
\[
- \int_{0}^{t} \int \partial^\beta V \cdot \partial^\beta \left( B_0 U_t + \sum_{j=1}^{3} B_j \partial_j U \right) \, dx \, d\tau. \tag{3.6}
\]
Taking inner product of (3.5) with the following vector
\[
\partial^\beta W \triangleq (0, -\nabla \partial^\beta \theta),
\]
we have
\[
- \int \partial^\beta u_t \cdot \nabla \partial^\beta \theta \, dx - \int \partial^\beta a \cdot \nabla \partial^\beta \theta + |\nabla \partial^\beta \theta|^2 \, dx
\]
\[
= \int \nabla \partial^\beta \theta \cdot \partial^\beta \left( u \cdot \nabla u + \frac{\theta - a}{1 + a} \nabla a \right) \, dx. \tag{3.7}
\]
Then we take inner product of the equation of $\partial^\beta \theta$, which is
\[
\partial^\beta \theta_t + \partial^\beta (\nabla \cdot u) - \Delta \partial^\beta \theta = -\partial^\beta \left( \frac{a \Delta \theta}{1 + a} + \nabla \cdot (\theta u) \right),
\]
with $\partial^\beta (\nabla \cdot u)$ to obtain
\[
- \int \partial^\beta u \cdot \nabla \partial^\beta \theta \, dx + \int_{0}^{t} \int |\partial^\beta (\nabla \cdot u)|^2 \, dx \, d\tau - \int \partial^\beta (\nabla \cdot u) \Delta \partial^\beta \theta \, dx \, d\tau
\]
\[
= - \int \partial^\beta (\nabla \cdot u) \partial^\beta \left( \frac{a \Delta \theta}{1 + a} + (\nabla \cdot (\theta u)) \right) \, dx. \tag{3.8}
\]
Adding (3.7) to (3.8) and integrating with respect to $t$, we get
\[
- \int_{0}^{t} \int \partial^\beta u \cdot \nabla \partial^\beta \theta \, dx + \int_{0}^{t} \int |\partial^\beta (\nabla \cdot u)|^2 \, dx \, d\tau - |\nabla \partial^\beta \theta|^2 \, dx \, d\tau
\]
\[
- \int_{0}^{t} \int \nabla \partial^\beta a \cdot \nabla \partial^\beta \theta \, dx \, d\tau - \int_{0}^{t} \int \partial^\beta (\nabla \cdot u) \Delta \partial^\beta \theta \, dx \, d\tau
\]
\[
= - \int \partial^\beta u_0 \cdot \nabla \partial^\beta \theta_0 \, dx + \int_{0}^{t} \int \nabla \partial^\beta \theta \cdot \partial^\beta \left( u \cdot \nabla u + \frac{\theta - a}{1 + a} \nabla a \right) \, dx \, d\tau
\]
\[
- \int_{0}^{t} \int \partial^\beta (\nabla \cdot u) \partial^\beta \left( \frac{a \Delta \theta}{1 + a} + \nabla \cdot (\theta u) \right) \, dx \, d\tau. \tag{3.9}
\]
Adding (3.6) to (3.9), we have
\[
\int_0^t \partial^\beta u \cdot \nabla \partial^\beta (a - \theta) dx + \int_0^t \int \left| \nabla \partial^\beta a \right|^2 - \left| \nabla \partial^\beta \theta \right|^2 - \partial^\beta (\nabla \cdot u) \Delta \partial^\beta \theta dxd\tau
= \int \partial^\beta u_0 \cdot \nabla \partial^\beta (a_0 - \theta_0) dx - \int_0^t \int \nabla \partial^\beta a \cdot \partial^\beta (a \nabla \theta) dxd\tau
- \int_0^t \int \partial^\beta \nabla \cdot \partial^\beta \left( B_0 U_t + \sum_{j=1}^3 B_j \partial_j U \right) dxd\tau
+ \int_0^t \int \nabla \partial^\beta \theta \cdot \partial^\beta \left( u \cdot \nabla u + \frac{\theta - a}{1 + a} \nabla a \right) dxd\tau
- \int_0^t \int \partial^\beta (\nabla \cdot u) \partial^\beta \left( \frac{a \Delta \theta}{1 + a} + \nabla \cdot (\theta u) \right) dxd\tau.
\]

Thus we have
\[
\int_0^t \int \left| \nabla \partial^\beta a \right|^2 dxd\tau \leq \int_0^t \int \left| \nabla \partial^\beta \theta \right|^2 + \frac{1}{2} \left| \partial^\beta (\nabla \cdot u) \right|^2 + \frac{1}{2} \left| \Delta \partial^\beta \theta \right|^2 dxd\tau
+ \int \partial^\beta u_0 \cdot \nabla \partial^\beta (a_0 - \theta_0) dx - \int \partial^\beta u \cdot \nabla \partial^\beta (a - \theta) dx
- \int_0^t \int \partial^\beta \nabla \cdot \partial^\beta \left( B_0 U_t + \sum_{j=1}^3 B_j \partial_j U \right) dxd\tau
- \int_0^t \int \partial^\beta (\nabla \cdot u) \partial^\beta \left( \frac{a \Delta \theta}{1 + a} + \nabla \cdot (\theta u) \right) dxd\tau
+ \int_0^t \int \nabla \partial^\beta \theta \cdot \partial^\beta \left( u \cdot \nabla u + \frac{\theta - a}{1 + a} \nabla a \right) dxd\tau
- \int_0^t \int \nabla \partial^\beta a \cdot \partial^\beta (a \nabla \theta) dxd\tau.
\]

Now go back to (3.6), we have
\[
\frac{1}{2} \int_0^t \int \left| \partial^\beta (\nabla \cdot u) \right|^2 dxd\tau \leq \int_0^t \int \frac{3}{4} \left| \nabla \partial^\beta a \right|^2 + \frac{1}{4} \left| \nabla \partial^\beta \theta \right|^2 dxd\tau
+ \frac{1}{2} \int_0^t \int \partial^\beta \nabla \cdot \partial^\beta \left( B_0 U_t + \sum_{j=1}^3 B_j \partial_j U \right) dxd\tau
+ \frac{1}{2} \int_0^t \int \partial^\beta u \cdot \nabla \partial^\beta adx - \int \partial^\beta u_0 \cdot \nabla \partial^\beta a_0 dx
+ \frac{1}{2} \int_0^t \int \nabla \partial^\beta a \cdot \partial^\beta (a \nabla \theta) dxd\tau.
\]

Substituting (3.12) into (3.11), by (3.4) we have
\[
\sum_{|\beta| \leq k-1} \int_0^t \int \left| \nabla \partial^\beta a \right|^2 dxd\tau
\leq E_{k,1}(t) + E_{k,1}(0) + E_{[k/2+5/2],1}^{1/2}(t) \left( E_{k,1}(t) + E_{k,2}(t) \right)
\leq E_{k,1}(0) + E_{[k/2+5/2],1}^{1/2}(t) \left( E_{k,1}(t) + E_{k,2}(t) \right).
\]
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Substituting (3.13) into (3.12), similarly by (3.4) we have

$$\sum_{|\beta|\leq k-1} \int_0^t \int |\partial^\beta (\nabla \cdot u)|^2 dxd\tau \leq E_{k,1}(t) + E_{k,2}(t) + E_{[k/2+5/2],1}^1(t) \left( E_{k,1}(t) + E_{k,2}(t) \right).$$

(3.14)

Note that by (1.6) and Hodge decomposition, we have

$$\sum_{|\beta|\leq k-1} \int_0^t \int |\partial^\beta (\nabla \cdot u)|^2 dxd\tau = \sum_{|\beta|\leq k-1} \int_0^t \int |\partial^\beta (\nabla u)|^2 dxd\tau.$$

Adding (3.4), (3.13) and (3.14), one has

$$E_{k,1}(t) + E_{k,2}(t) \leq E_{k,1}(0) + E_{[k/2+5/2],1}^1(t) \left( E_{k,1}(t) + E_{k,2}(t) \right), \quad 1 \leq k \leq s.$$  

(3.15)

Note that $s > 5$, by (2.3) and (3.15) we arrive at

$$E_s(t) \triangleq E_{s,1}(t) + E_{s,2}(t) \leq E_s(0) + E_s^{3/2}(0) + E_s^{3/2}(t) \leq C \left( M^2 \varepsilon^2 + M^3 \varepsilon^3 + \varepsilon^{\frac{3}{2}} \right).$$

(3.16)

Now we give the proof of Thm 1.1.

Proof. Assume that $t_* < \infty$ in (1.7). We take $\varepsilon > 0$ small enough, then (3.16) gives

$$E_s(t_*) \leq 2C \varepsilon^{\frac{3}{2}} < \varepsilon,$$

this contradicts our assumption (1.7). Thus we have

$$E_s(t) \leq \varepsilon, \quad \forall \ t \geq 0,$$

which completes the proof of Thm 1.1. $\Box$

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