Yukawa Couplings involving Excited Twisted Sector States for $\mathbb{Z}_M \times \mathbb{Z}_N$ Orbifolds

D. Bailin$^a$, A. Love$^b$ and W.A. Sabra$^b$

$^a$School of Mathematical and Physical Sciences, University of Sussex, Brighton U.K.

$^b$Department of Physics, Royal Holloway and Bedford New College, University of London, Egham, Surrey, U.K.

ABSTRACT

A study is made for $\mathbb{Z}_M \times \mathbb{Z}_N$ orbifolds of the modification of the form of the twisted sector Yukawa couplings when some of the states involved are excited twisted sectors rather than twisted sector ground states.
A knowledge of the Yukawa couplings for orbifold compactified string theory models [1, 2] will be required for a comparison of such models with observation. In particular, the exponential dependence of Yukawa couplings on moduli which can occur when all the states involved are in twisted sectors [3, 4] may have a bearing on hierarchies of quark and lepton masses [5].

Twisted sector Yukawa couplings have already been investigated for both the $\mathbb{Z}_N$ orbifolds [3–11] and for the $\mathbb{Z}_M \times \mathbb{Z}_N$ orbifolds [12–13]. However, the discussion has been mostly limited to couplings involving only twisted sector ground states, though an outline has been given of how twisted sector excited states might be included [3, 4]. Here we shall extend the discussion to Yukawa couplings involving twisted sector excited states of $\mathbb{Z}_M \times \mathbb{Z}_N$ Coxeter orbifolds. It is particularly important to be able to include these excited states in view of the fact that string loop threshold corrections to gauge coupling constants [14,15] consistent with their low energy values have so far always involved modular weights for quarks and leptons requiring the use of excited twisted sector states.

The $\mathbb{Z}_M \times \mathbb{Z}_N$ orbifolds under consideration are those for which the point group is realised in the simplest possible way in terms of the Coxeter elements of Lie algebra root lattices as discussed in the second reference of [12] and summarized in table 1. We are interested in massless states in the twisted sectors of these orbifolds with the $SU(3) \times SU(2) \times U(1)$ quantum numbers of quarks, leptons and electroweak Higgses. Possible (fractional) oscillator numbers $\tilde{N}$ for massless states with these gauge group quantum numbers are bounded by [17]

$$\tilde{N} \leq a_L - \frac{3}{5}, \quad \text{for } Q, u_c \text{ and } e_c$$

and by

$$\tilde{N} \leq a_L - \frac{2}{5}, \quad \text{for } L, d_c \text{ and } H,$$

where $a_L$ is the left mover normal ordering constant for the twisted sector in question. The inequality in (1) and (2) reflects the generic occurrence in orbifold
models, before spontaneous symmetry breaking, of extra $U(1)$ gauge fields to which
the quarks and leptons couple. The bounds (1) and (2) put tight constraints on the
left movers bosonic oscillators that can be deployed in the construction of massless
twisted sector states.

Allowed Yukawa couplings involving excited twisted sector states are restricted
by the observation [4, 19] that the discrete symmetries of the 2-dimension-
al sub-lattices of the 6-dimensional compact manifold are left unbroken by the
construction of the orbifold. If $X \lambda i, i = 1, 2, 3$, are the complex coordinates defining
the 6-dimensional compact manifold and a discrete symmetry acting in the $i$-th
complex plane is of order $P$, then correlation functions involving
$(\partial z X \lambda i) \lambda m (\partial \bar{z} X \lambda i) \lambda n$ are allowed only if

$$m - n = 0 \pmod{P}. \quad (3)$$

Yukawa couplings involving only twisted sector ground states are determined
by bosonic twist field correlation functions [3, 4] of the type

$$Z = \prod_{i=1}^{\lambda 3 \mathcal{Z}_i}, \quad (4)$$

with

$$\mathcal{Z}_i = \langle \sigma_\alpha \lambda i(z_1, \bar{z}_1) \sigma_\beta \lambda i(z_2, \bar{z}_2) \sigma_\gamma \lambda i(z_3, \bar{z}_3) \rangle. \quad (5)$$

where the subscripts $\alpha$, $\beta$ and $\gamma$ on the twist fields are the point group elements for
the three twisted sectors involved, and because of the point group selection rule

$$\gamma = (\alpha \beta) \lambda - 1. \quad (6)$$

If excited twisted sector states are involved then excited twist fields $\tilde{\tau}$ and $\tilde{\tau}'$ are
required which are defined by the operator product expansions [3,4]

\[ \partial \bar{z} X_\sigma(\bar{w}, \bar{w}) \sim (\bar{z} - \bar{w})^{\lambda - \eta_\alpha} \bar{\tau}'_\alpha(\bar{w}, \bar{w}) + .... \]  

(7)

and

\[ \partial \bar{z} \bar{X}_{\sigma}(\bar{w}, \bar{w}) \sim (\bar{z} - \bar{w})^{\lambda - (1 - \eta_\alpha) \bar{\tau}_\alpha(\bar{w}, \bar{w}) + ....} \]  

(8)

where in the \( \alpha \) twisted sector the coordinates \( X_{\lambda i} \) are twisted by \( e^{\lambda - 2\pi i \eta_\alpha \lambda} \) and the index \( i \) labelling the \( i \)-th complex plane of the 6-dimensional compact manifold has been suppressed in (7) and (8). (The excited twist fields \( \bar{\tau}_\alpha \) and \( \bar{\tau}'_\alpha \) are associated with the excited twisted sector states created by a single bosonic oscillator corresponding to moding \( \eta_\alpha \) or \( 1 - \eta_\alpha \). This is the only situation that turns out to be relevant here.) Then, we are interested in correlation functions (4) where at least one of the factors \( Z_i \) involves excited twist fields and is of the form

\[ \left( Z_i^{\lambda 3} \right)_{\text{excited}} = \langle \bar{\tau}_\alpha \lambda i(z_1, \bar{z}_1) \bar{\tau}'_\lambda \sigma \gamma \lambda i(z_3, \bar{z}_3) \rangle, \]  

(9)

up to permutations of \( \alpha, \beta \) and \( \gamma \). If the point group element \( \gamma \) leaves the \( i \)-th complex plane unrotated then \( \sigma \gamma \lambda i \) is trivial and (9) reduces to the excited two-point function

\[ \left( Z_i^{\lambda 2} \right)_{\text{excited}} = \langle \bar{\tau}_\alpha \lambda i(z_1, \bar{z}_1) \bar{\tau}'_\lambda \alpha \lambda - 1(z_2, \bar{z}_2) \rangle, \]  

(10)

which can be normalised to 1 apart from factors of \( (z_1 - z_2) \) and \( (\bar{z}_1 - \bar{z}_2) \) determined by \( SL(2, \mathbb{C}) \) invariance. (The same remark applies to two-point functions where the excited states are created by a product of two bosonic oscillators.) Thus, non-trivial modifications of the Yukawa couplings due to the use of excited twist fields only occur when \( \sigma \gamma \lambda i \) is non-trivial in (9). As an illustration, the sectors of the \( \mathbb{Z}_3 \times \mathbb{Z}_6 \) orbifold for which this happens are displayed in table 2, using the notation \( T_{pq} \) to denote the \( \theta \lambda p \omega \lambda q \) twisted sector.
Normalization of the excited twist fields is necessary in order to create normalized states. The twisted sector mode expansion for $X$ and $\bar{X}$ suggests that the normalization factors for $\tilde{\tau}_i$ and $\tilde{\tau}'_i$ should be essentially $(2\eta_\alpha)\lambda - \frac{1}{2}$ and $(2(1 - \eta_\alpha))\lambda - \frac{1}{2}$, respectively. This can be checked in detail by considering the excited two-point function (10) for $\beta = \alpha\lambda - 1$ which can be derived from the correlation function

$$\bar{g}(\bar{z}, \bar{w}) = -\frac{1}{2} \frac{\langle \partial_z X \partial_{\bar{w}} \bar{X} \sigma_\alpha(z_1, \bar{z}_1)\sigma_{\alpha\lambda-1}(z_2, \bar{z}_2) \rangle}{\langle \sigma_\alpha(z_1, \bar{z}_1)\sigma_{\alpha\lambda-1}(z_2, \bar{z}_2) \rangle},$$  \hspace{1cm} (11)

where we have suppressed the index $i$. With the aid of the operator product expansions (7) and (8), the excited twist field two-point function is related to $\bar{g}(\bar{z}, \bar{w})$ by

$$\langle \tilde{\tau}_\alpha(z_1, \bar{z}_1)\tilde{\tau}'_{\alpha\lambda-1}(z_2, \bar{z}_2) \rangle = -2 \langle \sigma_\alpha(z_1, \bar{z}_1)\sigma_{\alpha\lambda-1}(z_2, \bar{z}_2) \rangle \lim_{\bar{z}_1 \to \bar{z}_2} (\bar{z} - \bar{z}_1)\lambda 1 - \eta_\alpha(\bar{w} - \bar{z}_1)\lambda 1 - \eta_\alpha \bar{g}(\bar{z}, \bar{w}).$$  \hspace{1cm} (12)

The correlation function $\bar{g}(\bar{z}, \bar{w})$ has has the form [3]

$$\bar{g}(\bar{z}, \bar{w}) = (\bar{z} - \bar{z}_1)\lambda - \eta_\alpha(\bar{z} - \bar{z}_2)\lambda - (1 - \eta_\alpha)(\bar{w} - \bar{z}_1)\lambda - (1 - \eta_\alpha)(\bar{w} - \bar{z}_2)\lambda - \eta_\alpha$$

$$\frac{((1 - \eta_\alpha)((\bar{w} - \bar{z}_1)(\bar{z} - \bar{z}_2)) + \eta_\alpha(\bar{z} - \bar{z}_1)(\bar{w} - \bar{z}_2))}{(\bar{z} - \bar{w})\lambda 2}.$$  \hspace{1cm} (13)

Combining (12) and (13) gives

$$\langle \tilde{\tau}_\alpha(z_1)\tilde{\tau}'_{\alpha\lambda-1}(z_2) \rangle = 2\eta_\alpha(-1)\lambda - \eta_\alpha(\bar{z}_2 - \bar{z}_1)\lambda - 2\eta_\alpha(\bar{z}_2 - \bar{z}_1)\lambda - 2\eta_\alpha(z_2 - z_1)\lambda - 4h_\alpha.$$  \hspace{1cm} (14)

where $h_\alpha = \frac{1}{2}\eta_\alpha(1 - \eta_\alpha)$ is the conformal weight of $\sigma_\alpha$. The factors of $(\bar{z}_2 - \bar{z}_1)$ and $(z_2 - z_1)$ are to be expected because of $SL(2, \mathbb{C})$ invariance (see for example, ref [20]). This confirms the remark at the beginning of the paragraph that the normalization factors for $\tilde{\tau}_i$ and $\tilde{\tau}'_i$ are essentially $(2\eta_\alpha)\lambda - \frac{1}{2}$ and $(2(1 - \eta_\alpha))\lambda - \frac{1}{2}$, respectively, but also makes it clear that there is a factor of $(-1)\lambda - \eta_\alpha$ due to the fact that the excited twist fields have a non-zero spin. Such a factor is absorbed in the definition of the in and out states.
The three-point function (9) can be evaluated directly apart from an overall moduli independent normalization factor and this overall normalization can be determined by factorizing a four-point function. Using the operator product expansions (7) and (8),

\[
\left( Z\lambda^3 \right)_{\text{excited}}^{z\rightarrow\bar{z}1, \bar{z}2} = \lim_{z\rightarrow\bar{z}1, \bar{z}2} (\bar{z} - \bar{z}2)\lambda \eta_{\beta}(\bar{w} - \bar{z}1)\lambda 1 - \eta_{\alpha}
\]

\[
\langle \partial_{\bar{z}} X \partial_{\bar{w}} \bar{X} \sigma_{\alpha}(z_1, \bar{z}1)\sigma_{\beta}(z_2, \bar{z}2)\sigma_{(\alpha\beta)\lambda-1}(z_3, \bar{z}3) \rangle.
\]

where the index \( i \) labelling the \( i \)th complex plane has been suppressed in this and subsequent equations. Separating \( X \) into a classical part and a quantum part

\[
X = X_{cl} + X_{qu},
\]

we have

\[
\langle \partial_{\bar{z}} X \partial_{\bar{w}} \bar{X} \sigma_{\alpha}(z_1, \bar{z}1)\sigma_{\beta}(z_2, \bar{z}2)\sigma_{(\alpha\beta)\lambda-1}(z_3, \bar{z}3) \rangle =
\]

\[
\sum_{X_{cl}} e\lambda - S_{cl} \langle \partial_{\bar{z}} X_{qu} \partial_{\bar{w}} \bar{X}_{qu} \rangle_{3-\text{twists}} + \sum_{X_{cl}} e\lambda - S_{cl} \partial_{\bar{z}} X_{cl} \partial_{\bar{w}} \bar{X}_{cl} Z_{qu} \lambda^3,
\]

where

\[
\langle \partial_{\bar{z}} X_{qu} \partial_{\bar{w}} \bar{X}_{qu} \rangle_{3-\text{twists}} = \int \mathcal{D}X_{qu} e\lambda - S_{qu} \partial_{\bar{z}} X_{qu} \partial_{\bar{w}} \bar{X}_{qu}
\]

\[
\sigma_{\alpha}(z_1, \bar{z}1)\sigma_{\beta}(z_2, \bar{z}2)\sigma_{(\alpha\beta)\lambda-1}(z_3, \bar{z}3),
\]

\[
Z_{qu} \lambda^3 = \int \mathcal{D}X_{qu} e\lambda - S_{qu} \sigma_{\alpha}(z_1, \bar{z}1)\sigma_{\beta}(z_2, \bar{z}2)\sigma_{(\alpha\beta)\lambda-1}(z_3, \bar{z}3)
\]

and

\[
S = \frac{1}{\pi} \int d\lambda 2\pi \left( \partial_{\bar{z}} X \partial_{\bar{z}} \bar{X} + \partial_{\bar{z}} X \partial_{\bar{z}} \bar{X} \right).
\]

Continuing to suppress the index \( i \) on \( X\lambda i \), the classical fields consistent with the
operator product expansions (7) and (8) have derivatives of the form
\[ \partial_{\bar{z}} X_{cl} = d(\bar{z} - \bar{z}_1)\lambda - \eta_\alpha (\bar{z} - \bar{z}_2)\lambda - \eta_\beta (\bar{z} - \bar{z}_3)\lambda - (1 - \eta_\alpha - \eta_\beta), \]

and

\[ \partial_{\bar{w}} X_{cl} = a(\bar{w} - \bar{z}_1)\lambda - (1 - \eta_\alpha)(\bar{w} - \bar{z}_2)\lambda - (1 - \eta_\beta)(\bar{w} - \bar{z}_3)\lambda - (\eta_\alpha + \eta_\beta). \]

The constant \( d \) must be chosen to be zero for an acceptable classical solution because the classical action is otherwise divergent. Consequently, the second term in (17) vanishes, and the moduli dependence of \( (Z\lambda^3)_{\text{excited}} \), which is contained in \( \sum_{X_{cl}} e^{\lambda - S_{cl}} \), is exactly the same as for the three-point function with unexcited twist fields.

Determination of the overall normalization of the three-point function, which depends on the twisted sectors involved, can be achieved by considering the four-point function,

\[ (Z\lambda^4)_{\text{excited}} = \langle \bar{\tau}_{\alpha\lambda - 1}(z_1, \bar{z}_1)\sigma_\alpha(z_2, \bar{z}_2)\bar{\tau}'_{\beta\lambda - 1}(z_3, \bar{z}_3)\sigma_\beta(z_4, \bar{z}_4) \rangle. \]

With the aid of the operator product expansions (7) and (8) this can be written as

\[ (Z\lambda^4)_{\text{excited}} = \lim_{\bar{w} \to \bar{z}_1} (\bar{w} - \bar{z}_1)\lambda \eta_\alpha(\bar{w} - \bar{z}_3)\lambda - \eta_\beta \langle \partial_{\bar{z}} X \partial_{\bar{w}} X \sigma_{\alpha\lambda - 1}(z_1, \bar{z}_1)\sigma_\alpha(z_2, \bar{z}_2)\sigma_{\beta\lambda - 1}(z_3, \bar{z}_3)\sigma_\beta(z_4, \bar{z}_4) \rangle. \]

Separating \( X \) into a classical part and quantum part as in (16),

\[ \langle \partial_{\bar{z}} X \partial_{\bar{w}} X \sigma_{\alpha\lambda - 1}(z_1, \bar{z}_1)\sigma_\alpha(z_2, \bar{z}_2)\sigma_{\beta\lambda - 1}(z_3, \bar{z}_3)\sigma_\beta(z_4, \bar{z}_4) \rangle = \]

\[ \sum_{X_{cl}} e^{\lambda - S_{cl}} \langle \partial_{\bar{z}} X_{qu} \partial_{\bar{w}} X_{qu} \rangle_{4-\text{twists}} + \sum_{X_{cl}} e^{\lambda - S_{cl}} \partial_{\bar{z}} X_{cl} \partial_{\bar{w}} X_{cl} Z_{qu} \lambda^4, \]

7
where

\[
\langle \partial_z X_{qu} \partial_{\bar{w}} \bar{X}_{qu} \rangle_{4-\text{twists}} = \\
\int D X_{qu} e^{\lambda S_{qu} \partial_z X_{qu} \partial_{\bar{w}} \bar{X}_{qu} \sigma_{\alpha \lambda - 1}(z_1, \bar{z}_1) \sigma_{\alpha}(z_2, \bar{z}_2) \sigma_{\beta \lambda - 1}(z_3, \bar{z}_3) \sigma_{\beta}(z_4, \bar{z}_4)}
\]

(26)

and

\[
Z_{qu} \lambda^4 = \int D X_{qu} e^{\lambda - S_{qu} \sigma_{\alpha \lambda - 1}(z_1, \bar{z}_1) \sigma_{\alpha}(z_2, \bar{z}_2) \sigma_{\beta \lambda - 1}(z_3, \bar{z}_3) \sigma_{\beta}(z_4, \bar{z}_4)}.
\]

(27)

The first term in (25) can be evaluated from

\[
\bar{h}(\bar{z}, \bar{w}) = -\frac{1}{2} \frac{\langle \partial_z X_{qu} \partial_{\bar{w}} \bar{X}_{qu} \rangle_{4-\text{twists}}}{Z_{qu} \lambda^4}.
\]

(28)

Using the operator product expansion

\[
-\frac{1}{2} \partial_z X \partial_{\bar{w}} \bar{X} \sim \frac{1}{(\bar{z} - \bar{w}) \lambda^2} + \text{finite}
\]

(29)

together with the operator product expansions (7) and (8), we require \( \bar{h}(\bar{z}, \bar{w}) \) to satisfy

\[
\bar{h}(\bar{z}, \bar{w}) \sim \frac{1}{(\bar{z} - \bar{w}) \lambda^2} + \text{finite}, \quad \bar{z} \rightarrow \bar{w}
\]

(30)

\[
\bar{h}(\bar{z}, \bar{w}) \sim (\bar{z} - \bar{z}_1) \lambda - (1 - \eta_{\alpha}), \quad \bar{z} \rightarrow \bar{z}_1
\]

(31)

and

\[
\bar{h}(\bar{z}, \bar{w}) \sim (\bar{w} - \bar{z}_1) \lambda - \eta_{\alpha}, \quad \bar{w} \rightarrow \bar{z}_1
\]

(32)

with similar conditions for \( \bar{z} \rightarrow \bar{z}_2, \bar{z}_3, \bar{z}_4 \) and \( \bar{w} \rightarrow \bar{z}_2, \bar{z}_3, \bar{z}_4 \). Following the methods
of \([3, 10]\), we arrive at
\[
\lim_{\bar{x} \to z_1} \lambda \eta_\alpha (\bar{z} - \bar{z}_3) \lambda 1 - \eta_\beta \bar{h}(\bar{z}, \bar{w}) = (-1) \lambda 3 \eta_\alpha - \eta_\beta (\bar{x}) \lambda \eta_\alpha (1 - \bar{x}) \lambda - \eta_\alpha \left( (\eta_\alpha - \eta_\beta) - (1 - \bar{x}) \partial_{\bar{x}} \log I(x, \bar{x}) \right),
\]
where we used \(SL(2, C)\) invariance to set
\[
z_1 = 0, \ z_2 = x, \ z_3 = 1, \ z_4 = z_\infty,
\]
and
\[
I(x, \bar{x}) = J_2 \bar{G}_1(\bar{x}) H_2(1 - x) + J_1 G_2(x) \bar{H}_1(1 - \bar{x}),
\]
with
\[
J_1 = \frac{\Gamma(\eta_\alpha) \Gamma(1 - \eta_\beta)}{\Gamma(1 + \eta_\alpha - \eta_\beta)}, \quad J_2 = \frac{\Gamma(1 - \eta_\alpha) \Gamma(\eta_\beta)}{\Gamma(1 + \eta_\beta - \eta_\alpha)},
\]
\[
G_1(x) = F(\eta_\alpha, 1 - \eta_\beta; 1; x), \quad G_2(x) = F(1 - \eta_\alpha, \eta_\beta; 1; x)
\]
and
\[
H_1(x) = F(\eta_\alpha, 1 - \eta_\beta; 1 + \eta_\alpha - \eta_\beta; x), \quad H_2(x) = F(1 - \eta_\alpha, \eta_\beta; 1 + \eta_\beta - \eta_\alpha; x).
\]

To factorize \((\mathcal{Z}_4)^{\text{excited}}\) into a product of Yukawa couplings we shall, as in the case with only unexcited twist fields \([3, 10]\) have to take the limit \(z_2 \to z_4\), i.e., \(x \to z_\infty\). In this limit, we find from the asymptotic behaviour of the hypergeometric functions \(F\), that
\[
\lim_{x \to z_\infty} \left( \begin{array}{c}
\bar{w} - \bar{z}_1 \\
\bar{z} - \bar{z}_3
\end{array} \right) \lambda \eta_\alpha (\bar{z} - \bar{z}_3) \lambda 1 - \eta_\beta \bar{h}(\bar{z}, \bar{w}) = \lim_{x \to z_\infty} -2 \mathcal{Z}_{\text{qu}} \lambda 4 (\bar{w} - \bar{z}_1) \lambda \eta_\alpha (\bar{z} - \bar{z}_3) \lambda 1 - \eta_\beta \bar{h}(\bar{z}, \bar{w}) = \lim_{x \to z_\infty} 2 \eta_\beta (-1) \lambda 2 \eta_\alpha - \eta_\beta \mathcal{Z}_{\text{qu}} \lambda 4 \quad \text{for} \ \eta_\alpha < 1 - \eta_\beta,
\]
\[
\lim_{x \to z_\infty} 2 (1 - \eta_\alpha)(-1) \lambda 2 \eta_\alpha - \eta_\beta \mathcal{Z}_{\text{qu}} \lambda 4 \quad \text{for} \ \eta_\alpha > 1 - \eta_\beta.
\]

The second term in (25) can be evaluated by writing \([10]\), consistently with the
operator product expansions (7) and (8),

\[
\partial\bar{z}X_{cl} = c(\bar{z})\lambda\eta_{\alpha\lambda-1}(\bar{z} - \bar{x})\lambda - \eta_{\alpha}(\bar{z} - 1)\lambda\eta_{\beta\lambda-1}(\bar{z} - \bar{z}_\infty)\lambda - \eta_{\beta} \tag{40}
\]

and

\[
\partial\bar{z}\bar{X}_{cl} = \bar{b}(\bar{z})\lambda - \eta_{\alpha}(\bar{z} - \bar{x})\lambda\eta_{\alpha\lambda-1}(\bar{z} - 1)\lambda - \eta_{\beta}(\bar{z} - \bar{z}_\infty)\lambda\eta_{\beta\lambda-1} \tag{41}
\]

Then, \(c\) and \(b\) can be derived in terms of hypergeometric functions from the independent global monodromy conditions, as in the third reference of [11], and taking the limit \(x \to \infty\) we find that the second term in (25) vanishes.

Now, with the aid of (39), we find that for \(x \to z_\infty\),

\[
\begin{align*}
(Z\lambda^4)_{\text{excited}} = \lim_{x \to z_\infty} & 2\eta_{\beta}(-1)\lambda2\eta_{\alpha} - \eta_{\beta}\tilde{Z}_{\text{qu}}\lambda^4 \sum_{X_{cl}} e\lambda - S_{cl}, & \text{for } \eta_{\alpha} < 1 - \eta_{\beta}, \\
& \lim_{x \to z_\infty} 2(1 - \eta_{\alpha})(-1)\lambda2\eta_{\alpha} - \eta_{\beta}\tilde{Z}_{\text{qu}}\lambda^4 \sum_{X_{cl}} e\lambda - S_{cl} & \text{for } \eta_{\alpha} > 1 - \eta_{\beta}.
\end{align*}
\tag{42}
\]

The factorization into a product of three-point functions now proceeds much as in the absence of excited twist fields [10,11], and we conclude that

\[
\frac{\langle \tilde{\tau}_{\alpha\lambda-1}(0)\tilde{\tau}_{\beta\lambda-1}(1)\sigma_{\alpha\beta}(z_\infty) \rangle}{\langle \sigma_{\alpha\lambda-1}(0)\sigma_{\beta\lambda-1}(1)\sigma_{\alpha\beta}(z_\infty) \rangle} = \begin{cases} 
2\eta_{\beta}(-1)\lambda2\eta_{\alpha} - \eta_{\beta} & \text{for } \eta_{\alpha} < 1 - \eta_{\beta}, \\
2(1 - \eta_{\alpha})(-1)\lambda2\eta_{\alpha} - \eta_{\beta} & \text{for } \eta_{\alpha} > 1 - \eta_{\beta}.
\end{cases}
\tag{43}
\]

The relevant Yukawa couplings are those for excited twist fields that create normalized states. Consistently with the remarks following (14) we should define the Yukawa coupling

\[
Y\lambda E_{\alpha\lambda-1,\beta\lambda-1,\alpha\beta} = \frac{1}{2}\eta_{\lambda}(-1/2)(1-\eta_{\alpha})\lambda - 1/2(-1)\lambda\eta_{\beta} - 2\eta_{\alpha}\langle \tilde{\tau}_{\alpha\lambda-1}(0)\tilde{\tau}_{\beta\lambda-1}(1)\sigma_{\alpha\beta}(z_\infty) \rangle
\tag{44}
\]
so that

\[
\frac{Y \lambda E_{\alpha \lambda -1, \beta \lambda -1, \alpha \beta}}{\langle \sigma_{\alpha \lambda -1}(0) \sigma_{\beta \lambda -1}(1) \sigma_{\alpha \beta}(z_\infty) \rangle} = \sqrt{\frac{\eta_\beta}{(1 - \eta_\alpha)}}, \quad \eta_\alpha < 1 - \eta_\beta, \\
\sqrt{\frac{(1 - \eta_\alpha)}{\eta_\beta}}, \quad \eta_\alpha > 1 - \eta_\beta.
\] (45)

This result can be used to obtain the Yukawa couplings for \( \mathbb{Z}_M \times \mathbb{Z}_N \) orbifolds when some of the twisted sector states involved are excited states from the Yukawa couplings between twisted sector ground states evaluated elsewhere [12]. The twist dependent suppression factors that arise may be of significance in obtaining the detailed pattern of quark and lepton masses.

ACKNOWLEDGEMENTS

This work was supported in part by S.E.R.C.
Table Captions

Table 1: Point group elements and lattices for $\mathbb{Z}_M \times \mathbb{Z}_N$ orbifolds. The point group elements $\theta$ and $\omega$ which are of the form $(e^{\lambda 2\pi i\eta_1}, e^{\lambda 2\pi i\eta_2}, e^{\lambda 2\pi i\eta_3})$, are specified in the table by $(\eta_1, \eta_2, \eta_3)$.

Table 2: Possible excited three-point functions $\left(\mathbb{Z}_i \lambda 3\right)_{\text{excited}}$ for the $\mathbb{Z}_3 \times \mathbb{Z}_6$ orbifolds. The $\theta \lambda p \omega \lambda q$ twisted sector is denoted by $T_{pq}$. Factors in $\left(\mathbb{Z}_i \lambda 3\right)_{\text{excited}}$ involving only unexcited twist fields are not displayed, nor are two-point functions involving excited twist fields, which can be normalized to 1. The notation is a slight modification of that of Eq. 9 with the twists $\eta \lambda i_{\alpha}$, $\eta \lambda i_{\beta}$ and $\eta \lambda i_{\gamma}$ displayed as subscripts.
### Table 1

| Point Group   | $\theta$       | $\omega$       | Lattice          |
|---------------|----------------|----------------|-----------------|
| $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $(1, 0, -1)/2$ | $(0, 1, -1)/2$ | $SO(4)\lambda_3$ |
| $\mathbb{Z}_3 \times \mathbb{Z}_3$ | $(1, 0, -1)/3$ | $(0, 1, -1)/3$ | $SU(3)\lambda_3$ |
| $\mathbb{Z}_2 \times \mathbb{Z}_4$ | $(1, 0, -1)/2$ | $(0, 1, -1)/4$ | $SO(4) \times SO(5)\lambda_2$ |
| $\mathbb{Z}_4 \times \mathbb{Z}_4$ | $(1, 0, -1)/4$ | $(0, 1, -1)/4$ | $SO(5)\lambda_3$ |
| $\mathbb{Z}_2 \times \mathbb{Z}_6$ | $(1, 0, -1)/2$ | $(0, 1, -1)/6$ | $SO(4) \times G_2\lambda_2$ |
| $\mathbb{Z}_2 \times \mathbb{Z}_6'$ | $(1, 0, -1)/2$ | $(1, 1, -2)/6$ | $G_2\lambda_3$ |
| $\mathbb{Z}_3 \times \mathbb{Z}_6$ | $(1, 0, -1)/3$ | $(0, 1, -1)/6$ | $SU(3) \times G_2\lambda_2$ |
| $\mathbb{Z}_6 \times \mathbb{Z}_6$ | $(1, 0, -1)/6$ | $(0, 1, -1)/6$ | $G_2\lambda_3$ |

### Table 2

| Yukawa Coupling | Possible excited three-point functions                                                                 |
|-----------------|--------------------------------------------------------------------------------------------------------|
| $T_0T_1T_4T_2$  | $\langle \sigma \lambda_{21/6} \tilde{\tau}' \lambda_{22/3} \tilde{\tau} \lambda_{21/6} \rangle$,     |
|                 | $\langle \tilde{\tau} \lambda_{21/6} \tilde{\tau}' \lambda_{22/3} \sigma \lambda_{21/6} \rangle$     |
| $T_0T_1T_3T_2$  | $\langle \tilde{\tau}' \lambda_{32/3} \tilde{\tau} \lambda_{31/6} \sigma \lambda_{31/6} \rangle$,     |
|                 | $\langle \tilde{\tau}' \lambda_{32/3} \sigma \lambda_{31/6} \tilde{\tau} \lambda_{31/6} \rangle$     |
| $T_0T_1T_4T_2$  | $\langle \tilde{\tau}' \lambda_{22/3} \tilde{\tau} \lambda_{21/6} \sigma \lambda_{21/6} \rangle$,     |
|                 | $\langle \tilde{\tau}' \lambda_{22/3} \sigma \lambda_{21/6} \tilde{\tau} \lambda_{21/6} \rangle$     |
| $T_0T_1T_2T_1$  | $\langle \tilde{\tau} \lambda_{31/6} \tilde{\tau}' \lambda_{32/3} \sigma \lambda_{31/6} \rangle$,     |
|                 | $\langle \sigma \lambda_{31/6} \tilde{\tau}' \lambda_{32/3} \tilde{\tau} \lambda_{31/6} \rangle$     |
| $T_1T_3T_1$     | $\langle \tilde{\tau}' \lambda_{32/3} \tilde{\tau} \lambda_{31/6} \sigma \lambda_{31/6} \rangle$,     |
|                 | $\langle \tilde{\tau}' \lambda_{32/3} \sigma \lambda_{31/6} \tilde{\tau} \lambda_{31/6} \rangle$     |
| $T_1T_2T_1$     | $\langle \tilde{\tau} \lambda_{21/6} \tilde{\tau}' \lambda_{21/6} \lambda_{21/6} \rangle$,           |
|                 | $\langle \sigma \lambda_{21/6} \tilde{\tau}' \lambda_{21/6} \tilde{\tau} \lambda_{21/6} \rangle$     |
REFERENCES

1. L. Dixon, J. A. Harvey, C. Vafa and E. Witten, Nucl. Phys. B261 (1985) 678; B274 (1986) 285.
2. A. Font, L. E. Ibanez, F. Quevedo and A. Sierra, Nucl. Phys. B331 (1991) 421.
3. L. Dixon, D. Friedan, E. Martinec and S. Shenker, Nucl. Phys. B282 (1987) 13.
4. S. Hamidi, and C. Vafa, Nucl. Phys. B279 (1987) 465.
5. L. E. Ibanez, Phys. Lett. B181 (1986) 269.
6. J. A. Casas and C. Munoz, Nucl. Phys. B332 (1990) 189
7. J. A. Casas, F. Gomez and C. Munoz, Phys. Lett. B251 (1990) 99
8. J. A. Casas, F. Gomez and C. Munoz, CERN preprint, TH6194/91.
9. T. Kobayashi and N. Ohtsubu, Kanazawa preprint, DPKU–9103.
10. T. T. Burwick, R. K. Kaiser and H. F. Muller, Nucl. Phys. B355 (1991) 689
11. J. Erler, D. Jungnickel and J. Lauer, Phys. Rev D45 (1992) 3651; S. Stieberger, D. Jungnickel, J. Lauer and M. Spalinski, Mod. Phys. Lett. A7 (1992) 3059; J. Erler, D. Jungnickel, M. Spalinski and S. Stieberger, preprint, MPI–Ph/92–56.
12. D. Bailin, A. Love and W. A. Sabra, Mod. Phys. Lett A6 (1992) 3607; D. Bailin, A. Love and W. A. Sabra, Sussex preprint, SUSX-TH-92/17, to be published in Nucl. Phys. B.
13. S. Stieberger, preprint, TUM–TH–151/92.
14. V. S. Kaplunovsky, Nucl. Phys. B307 (1988) 145; L. J. Dixon, V. S. Kaplunovsky and J. Louis, Nucl. Phys. B355 (1991) 649; J. P. Derendinger, S. Ferrara, C. Kounas and F. Zwirner, Nucl. Phys. B372 (1992) 145, Phys. Lett. B271 (1991) 307.
15. I. Antoniadis, J. Ellis, R. Lacaze and D. V. Nanopoulos, Phys. Lett. B268 (1991) 188; S. Kalara, J. L. Lopez and D. V. Nanopoulos, Phys. Lett. B269 (1991) 84.

16. L. E. Ibanez and D. Lüst and G. G. Ross, Phys. Lett. B272 (1991) 251; D. Bailin and A. Love, Phys. Lett. B278 (1992) 125.

17. L. E. Ibanez and D. Lüst, Nucl. Phys. B382 (1992) 305.

18. D. Bailin and A. Love, Phys. Lett. B292 (1992) 315.

19. A. Font, L. E. Ibanez, H-P. Nilles and F. Quevedo, Nucl. Phys. B307 (1988) 109.

20. A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. B241 (1984) 333.