Finite-variable logics do not have weak Beth definability property

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Abstract

We prove that \( n \)-variable logics do not have the weak Beth definability property, for all \( n \geq 3 \). This was known for \( n = 3 \) (Ildikó Sain and András Simon [19]), and for \( n \geq 5 \) (Ian Hodkinson, [12]). Neither of the previous proofs works for \( n = 4 \). In this paper we settle the case of \( n = 4 \), and we give a uniform, simpler proof for all \( n \geq 3 \). The case for \( n = 2 \) is still open.

1 Introduction

Definability theory is one of the most exciting and important parts of logic. It concerns concept formation and structuring our knowledge by investigating the category of theories. Implicit definitions are important in understanding concept formation and explicit definitions are vital ingredients of interpretations between theories. This has applications in the methodology of sciences [4, 6, 15].

Beth definability theorem for first-order logic (FOL) states that each implicit definition is equivalent to an explicit one, modulo theories. Investigating whether this theorem holds for fragments of first-order logic gives information about complexity of the explicit definition equivalent to the implicit one. Beth definability property is equivalent to surjectivity of epimorphisms in the associated class of algebras (a theorem of Németi [17], see also [13, 7, 18]).

Failure of Beth definability property for the finite variable fragments was first proved in 1983 [3] (for all \( n \geq 2 \)) by showing that epimorphisms are...
not surjective in finite-dimensional cylindric algebras, see [2]. That proof, translated to logic, relies inherently on the fact that the implicit definition it uses is not satisfiable in each model of the theory. The question came up whether the so-called weak Beth definability property holds for finite-variable fragments. Weak Beth definability property differs from the original Beth definability property in that we require not only the uniqueness, but also the existence of the implicitly defined relation. In some sense, the weak Beth definability property is more intuitive, and is considered to be more important than the (strong) Beth definability property, see e.g., [5].

In this paper we prove that $n$-variable logics do not have the weak Beth definability property either, for all $n \geq 3$. This means that there are a first-order logic theory, and an implicit definition that has exactly one solution in each model of the theory, such that both the theory and the implicit definition are written up with using $n$ variables only, yet any explicit definition equivalent to this implicit one has to use more than $n$ variables. For more on finite variable logics and the Beth definability properties see [12] and the remarks at the end of this paper.

2 The Main Theorem

The $n$-variable fragment $L_n$ of a FOL language $L$, where $n$ is any finite number, is the set of all formulas in $L$ which use $n$ variables only (free or bound). To make this more concrete, we may assume that $L$ uses the variables $v_0, v_1, ...$, while $L_n$ uses only the variables $v_0, v_1, ..., v_{n-1}$. In finite variable fragments we do not allow function or constant symbols, but we allow equality. Here is a definition of the formulas of $L_n$:

$R(v_{i_1}, ..., v_{i_k})$ is a formula of $L_n$ if $R$ is a $k$-place relation symbol and $i_1, ..., i_k < n$.

$v_i = v_j$ is a formula of $L_n$ if $i, j < n$.

$\neg \varphi, \varphi \land \psi, \exists v_i \varphi$ are formulas of $L_n$ whenever $\varphi, \psi$ are formulas of $L_n$ and $i < n$.

The above are all the formulas of $L_n$. We use other logical connectives, e.g., $\forall v_i, \lor, \rightarrow$ as derived ones. Models, satisfiability of formulas under evaluations
of the variables, validity in \( \mathcal{L}_n \) are the same as in FOL. The following theorem says that \( \mathcal{L}_n \) does not have even the weak Beth Definability Property whenever \( n \geq 3 \): 

**Theorem 2.1 (No weak Beth Property for \( \mathcal{L}_n \))** Let \( n \geq 3 \). There are a theory \( \text{Th} \) in the language of an \( n \)-place relation symbol \( R \) and a binary relation symbol \( S \), and a theory \( \Sigma(D) \) in the language of \( \text{Th} \) enriched with a unary relation symbol \( D \) such that

in each model of \( \text{Th} \) there is a unique relation \( D \) for which \( \Sigma(D) \) holds (we call such \( \Sigma(D) \) a strong implicit definition of \( D \) in \( \text{Th} \))

there is no explicit definition for \( D \) in \( \text{Th} \), i.e., for each \( n \)-variable formula \( \varphi \) in the language of \( \text{Th} \) we have

\[
\text{Th} \cup \Sigma(D) \not\models \forall v_0[D(v_0) \leftrightarrow \varphi].
\]

**Proof.** We write out the proof in detail for \( n = 3 \). Generalizing this proof to all \( n \geq 3 \) will be easy. We will often write \( x, y, z \) for \( v_0, v_1, v_2 \) and we will write simply \( R \) for \( R(x, y, z) \). We will use \( U_0(x), U_1(y), U_2(z) \) to be abbreviations of the formulas on the right-hand sides of the respective \( \leftrightarrow \)'s below:

\[
U_0(x) : \leftrightarrow \exists yzR, \quad U_1(y) : \leftrightarrow \exists xzR, \quad U_2(z) : \leftrightarrow \exists xyR.
\]

These formulas express the domain of \( R \), i.e., the first projection of \( R \), and the second and third projections of \( R \). We will include formulas into \( \text{Th} \) that express that \( U_0, U_1, U_2 \) are sets of cardinalities 3, 2, 2 respectively, and they form a partition of the universe. We will formulate these properties with 3 variables after describing the main part of the construction. Let us introduce the abbreviations \( T \) and \( \text{big}(R) \) as

\[
T : \leftrightarrow U_0(x) \land U_1(y) \land U_2(z), \quad \text{and}
\]

\[
\text{big}(R) : \leftrightarrow \bigwedge \{ \exists v_i R \leftrightarrow \exists v_i (T \land \neg R) : i = 0, 1, 2 \}.
\]

In the above, \( T \) is the “rectangular hull” of \( R \), and \( \text{big}(R) \) expresses that \( R \) cuts this hull into two parts each of which is sensitive in the sense that as soon as we quantify over them, the information on how \( R \) cuts \( T \) into two parts disappears. (Note that \( \text{big}(R) \) implies that \( \exists v_i R \leftrightarrow \exists v_i T \leftrightarrow \exists v_i (T \land \neg R) \).)
Assume that $|U_0| = 3, |U_1| = 2, |U_2| = 2$ and $\text{partition}(U_0, U_1, U_2)$ are formulas in $\mathcal{L}_3$ that express the associated meanings. Then we define

$$\text{Th} := \{|U_0| = 3, |U_1| = 2, |U_2| = 2, \text{partition}(U_0, U_1, U_2), \text{big}(R)\}.$$ 

We will show that Th has exactly one model, up to isomorphism. But before doing that, let us turn to expressing the properties we promised about the $U_i$’s with using three variables.

We will use Tarski’s way of substituting one variable for the other. I.e., we introduce the abbreviations

$$U_1(x) :\leftrightarrow \exists y(x = y \land U_1(y)), \quad U_2(x) :\leftrightarrow \exists z(x = z \land U_2(z)).$$

We now can express that $U_0, U_1, U_2$ form a partition of the universe:

$$\forall x(U_0(x) \lor U_1(x) \lor U_2(x)), \quad \forall x(U_i(x) \rightarrow \neg U_j(x)) \quad \text{for} \quad i \neq j, \quad i, j < 3.$$ 

For expressing the sizes of the sets $U_i$ we will use the abbreviations

$$U_1(z) :\leftrightarrow \exists z(z = y \land U_1(y)), \quad U_2(y) :\leftrightarrow \exists z(y = z \land U_2(z)).$$

Now, for $i = 1, 2$ we define the formulas

$$|U_i| \leq 2 :\leftrightarrow \neg \exists xyz(x \neq y \land x \neq y \neq z \land U_i(x) \land U_i(y) \land U_i(z)),$$

$$|U_i| \geq 2 :\leftrightarrow \exists xyz(x \neq y \land U_i(x) \land U_i(y)),
|U_i| = 2 :\leftrightarrow |U_i| \geq 2 \land |U_i| \leq 2.$$ 

It remains to express that $U_0$ has exactly three elements. In $\mathcal{L}_n$ with $n \geq 4$ we can express $|U_0| = 3$ similarly to the above, but in $\mathcal{L}_3$ we have to use another tool. For expressing in $\mathcal{L}_3$ that $U_0$ has exactly 3 elements, we will use the binary relation $S$. (This is the sole use of $S$ in Th, for $n \geq 4$ we can omit $S$ from the language.) We are going to express that $S$ is a cycle of order 3 on $U_0$. The following formulas express that $S$ is a function on $U_0$ without a fixed point:

$$\forall x \exists y S(x, y), \quad S(x, y) \land S(x, z) \rightarrow y = z, \quad S(x, y) \rightarrow (U_0(x) \land U_0(y) \land x \neq y).$$

The following formula expresses that $U_0$ consists of exactly one 3-cycle of $S$:

$$S(x, y) \leftrightarrow \exists z(S(y, z) \land S(z, x)), \quad S(x, y) \lor S(y, x) \lor x = y.$$
In the above, we used Tarski-style substitution of variables without mentioning (e.g., \( U_0(y) \)) and we omitted universal quantifiers in front of formulas (e.g., we wrote \( S(x, y) \land S(x, z) \rightarrow y = z \) in place of \( \forall xy(S(x, y) \land S(x, z) \rightarrow y = z) \)). This expresses that \( U_0 \) has exactly 3 elements.

We turn to showing that \( \text{Th} \) has exactly one model up to isomorphism. Let \( \mathfrak{M} = \langle M, R, S \rangle \models \text{Th} \). Let \( U_i, T \) be defined as above. Then \( M \) is the disjoint union of the \( U_i \)'s, and the sizes of the \( U_i \)'s for \( i = 0, 1, 2 \) are 3,2,2 respectively. (So \( M \) has 7 elements.) Let \( U_1 = \{b_0, b_1\} \), let \( c, d \) be the two elements of \( U_2 \) and let

\[
X := \{u \in U_0 : \langle u, b_0, c \rangle \in R\}.
\]

By \( \mathfrak{M} \models \text{big}(R) \) and \( |U_2| = 2 \) we have that \( \langle u, b_0, d \rangle \notin R \) if \( u \in X \) and \( \langle u, b_0, d \rangle \in R \) if \( u \in U_0 - X \). Hence

\[
U_0 - X = \{u \in U_0 : \langle u, b_0, d \rangle \in R\}.
\]

Also, by \( \mathfrak{M} \models \text{big}(R) \), \( X \) has one, or \( X \) has two elements (it cannot be that \( X \) has 0 or 3 elements). If \( |X| = 1 \) then let’s use the notation \( c_0 = c, c_1 = d \), and if \( |X| = 2 \) then let \( c_0 = d, c_1 = c \). Let us name the elements of \( U_0 \) as \( a_0, a_1, a_2 \) such that \( X = \{a_0\} \) if \( |X| = 1 \), \( X = \{a_1, a_2\} \) if \( |X| = 2 \) and \( S = \{\langle a_i, a_j \rangle : j = i + 1 \mod 3 \text{ and } i, j \leq 3\} \). This can be done by \( \mathfrak{M} \models \text{Th} \). The setting so far determines \( R \) by \( \mathfrak{M} \models \text{big}(R) \), as follows. For all \( i \leq 2, j, k \leq 1 \) we have \( \langle a_i, b_j, c_k \rangle \in R \) if and only if \( \langle a_i, b_{j+1(\mod 2)}, c_k \rangle \in T - R \) if and only if \( \langle a_i, b_j, c_{k+1(\mod 2)} \rangle \in T - R \). This is so by \( \mathfrak{M} \models \text{big}(R) \) and by \( |U_i| = 2 \) for \( i = 1, 2 \). From this we have that

\[
R = \{\langle u, b_i, c_j \rangle : u = a_0 \text{ and } i + j = 0(\mod 2)\} \cup \{\langle u, b_i, c_j \rangle : u = a_1 \lor u = a_2 \text{ and } i + j = 1(\mod 2)\}.
\]

We have seen that all models of \( \text{Th} \) are isomorphic to each other. The above also show that there is no automorphism of \( \mathfrak{M} \) that would move \( \{a_0\} \).

We are ready to formulate our implicit definition \( \Sigma(D) \). We design \( \Sigma(D) \) so that, by using the above notation, it specifies \( \{a_0\} \). We will write \( D \) in place of \( D(x) \).

\[
\Sigma(D) := \{ T \land \neg D \land R \rightarrow \forall x(T \land \neg D \rightarrow R), \quad T \land \neg D \land \neg R \rightarrow \forall x(T \land \neg D \rightarrow \neg R), \quad D \rightarrow U_0(x), \quad |D| = 1 \}.
\]
Then in each model of $\text{Th}$ there is exactly one unary relation $D$ for which $\Sigma(D)$ holds, namely $D$ has to be the unary relation $\{a_0\} \subseteq U_0$. Thus $\Sigma(D)$ is a strong implicit definition of $D$ in $\text{Th}$.

It remains to show that $\Sigma$ cannot be made explicit in $L_3$, i.e., there is no 3-variable formula $\varphi$ in the language of $\text{Th}$ for which $\text{Th} \cup \Sigma(D) \models D \leftrightarrow \varphi$. Our plan is to list all the $L_3$-definable relations in the above model and observe that $\{a_0\}$, the relation $\Sigma$ defines, is not among them. For any $\varphi \in L_3$ define

$$\text{mn}(\varphi) := \{(a, b, c) : \mathfrak{M} \models \varphi[a, b, c]\}.$$ 

In the above, $\mathfrak{M} \models \varphi[a, b, c]$ denotes that the formula $\varphi$ is true in $\mathfrak{M}$ when the variables $v_0, v_1, v_2$ are evaluated to $a, b, c$ respectively, and $\text{mn}$ abbreviates “meaning”. Let $A := \{\text{mn}(\varphi) : \varphi \in L_3\}$.

Clearly, $A$ is closed under the set Boolean operations because

$$\text{mn}(\varphi \land \psi) = \text{mn}(\varphi) \cap \text{mn}(\psi),$$
$$\text{mn}(\neg \varphi) = M^3 - \text{mn}(\varphi),$$

and so $A$ is closed under intersection and complementation w.r.t. $M^3$, the set of all $M$-termed 3-sequences. Since $M$ is finite, this implies that $A$ is atomic and the elements of $A$ are exactly the unions of some atoms.

We will list all the atoms of $A$. It is easy to see that the elements $U_i \times U_j \times U_k$ for $i, j, k \leq 2$ are all in $A$ and they form a partition of $M^3$. To list the atoms of $A$, we will list the atoms below each $U_i \times U_j \times U_k$ by specifying a partition of each. For $i, j, k \leq 2$ let’s abbreviate the sequence $\langle i, j, k \rangle$ by $ijk$.

$U_0 \times U_1 \times U_2$ is $T$, and the partition of $T$ will be $\{R, T - R\}$. For $ijk$ a permutation of 012, the partition of $U_i \times U_j \times U_k$, the permuted version of $T$, will be the correspondingly permuted versions of $R$ and $T - R$. Formally: Assume $i, j, k$ are all distinct, i.e., they form a permutation of 0, 1, 2. We define

$$X(ijk, r) := \{\langle u_i, u_j, u_k \rangle : \langle u_0, u_1, u_2 \rangle \in R\},$$
$$X(ijk, -r) := \{\langle u_i, u_j, u_k \rangle \in U_i \times U_j \times U_k : \langle u_0, u_1, u_2 \rangle \notin R\}.$$ 

We note that

$$X(012, r) = R, \quad \text{and} \quad X(012, -r) = T - R.$$
Note that
\[ mn(R(v_i, v_j, v_k)) = X(ijk, r), \]
and the same for \(-r\) in place of \(r\), so \(X(ijk, r), X(ijk, -r)\) are elements of \(A\).

Assume now that \(ijk\) is not repetition-free, i.e., \(|\{i, j, k\}| < 3\). In these cases the blocks of the partition of \(U_i \times U_j \times U_k\) will be put together from partitions of \(U_m \times U_n\) \((m, n < 3)\). Recall that \(S = \{(a_0, a_1), (a_1, a_2), (a_2, a_0)\}\). We define
\[
\overline{S} := \{(a, b) : (b, a) \in S\}, \quad id_i := \{(a, a) : a \in U_i\}, \quad di_i := \{(a, b) : a \neq b, \ a, b \in U_i\}.
\]

Above, \(id_i, di_i\) abbreviate “identity on \(U_i\),” and “diversity on \(U_i\),” respectively, and \(\overline{S}\) is the inverse of \(S\). Since \(S\) is a cycle on the three-element set \(U_0\), its inverse \(\overline{S}\) is its complement in the diversity element of \(U_0\), so \(\{S, \overline{S}, id_0\}\) is a partition of \(U_0 \times U_0\). Also, \(\{di_i, id_i\}\) is a partition of \(U_i \times U_i\) for \(i = 1, 2\). We are ready to define the “binary partitions” as follows
\[
Rel_{00} := \{S, \overline{S}, id_0\}, \quad Rel_{11} := \{di_1, id_1\}, \quad Rel_{22} := \{di_2, id_2\}, \\
Rel_{ij} := \{U_i \times U_j\} \text{ for } i \neq j.
\]

Note that for all \(e \in Rel_{ij}, e' \in Rel_{jk}\) we have \(e \circ e' \in Rel_{ik}\), where \(\circ\) denotes the operation of composing binary relations. In general, when \(|\{i, j, k\}| < 3\) and \(e = \langle e_0, e_1 \rangle \in Rel_{ij} \times Rel_{jk}\) we define
\[
X(ijk, e) := \{(a, b, c) \in U_i \times U_j \times U_k : (a, b) \in e_0, \ (b, c) \in e_1\}.
\]

Notice that we already defined \(X(ijk, e)\) for the case when \(i, j, k\) are distinct and \(e \in \{r, -r\}\). Let \(choice(e, ijk)\) denote \(e \in \{r, -r\}\) when \(ijk\) is repetition-free, and \(e = \langle e_0, e_1 \rangle, \ e_0 \in Rel_{ij}, \ e_1 \in Rel_{jk}\) otherwise. Define
\[
B := \{X(ijk, e) : i, j, k \leq 2, \ choice(e, ijk)\}, \\
C := \{\bigcup Y : Y \subseteq B\}.
\]

The following notation will be convenient when \(choice(e, ijk)\) and \(ijk\) is not repetition-free.
\[
e_{01} := e_0, \quad e_{12} := e_1, \quad e_{02} := e_0 \circ e_1, \\
e_{ij} := \overline{S} \text{ when } i > j \text{ and } e_{ji} = S, \\
e_{ij} := e_{ji} \text{ when } i > j \text{ and } e_{ji} \neq S.
\]
The intuitive meaning of $e_{ij}$ is that $\langle a_i, a_j \rangle \in e_{ij}$ whenever $\langle a_0, a_1 \rangle \in e_0$ and $\langle a_1, a_2 \rangle \in e_1$.

We want to prove that $A = C$. We show $A \subseteq C$ by showing $\text{mn}(\varphi) \in C$ for all $\varphi \in \mathcal{L}_3$, by induction on $\varphi$. Atomic formulas:

\[
\begin{align*}
\text{mn}(R(v_i, v_j, v_k)) &= X(ijk, r) \quad \text{when } |\{i, j, k\}| = 3, \\
\text{mn}(R(v_i, v_j, v_k)) &= \emptyset \quad \text{otherwise,} \\
\text{mn}(S(v_i, v_j)) &= \bigcup \{X(n_1 n_2 n_3, e) : n_i = n_j = 0, e_{n_i n_j} = S\}, \\
\text{mn}(v_i = v_j) &= \bigcup \{X(n_1 n_2 n_3, e) : n_i = n_j, e_{n_i n_j} \in \{\text{id}_0, \text{id}_1, \text{id}_2\}\}.
\end{align*}
\]

Clearly, $M^3 \in C$, and $C$ is closed under complementation with respect to $M^3$ and intersection, because $B$ is finite and its elements form a partition of $M^3$. Thus,

\[
\begin{align*}
\text{mn}(\neg \varphi) &\in C, \\
\text{mn}(\varphi \land \psi) &\in C \quad \text{whenever } \text{mn}(\varphi), \text{mn}(\psi) \in C.
\end{align*}
\]

To deal with the existential quantifiers, let us define for arbitrary $H \subseteq M^3$

\[
\begin{align*}
C_0 H &= \{\langle a, b, c \rangle \in M^3 : \langle a', b, c \rangle \in H \text{ for some } a'\}, \\
C_1 H &= \{\langle a, b, c \rangle \in M^3 : \langle a, b', c \rangle \in H \text{ for some } b'\}, \\
C_2 H &= \{\langle a, b, c \rangle \in M^3 : \langle a, b, c' \rangle \in H \text{ for some } c'\},
\end{align*}
\]

Then we have, by the definition of the meaning of the existential quantifiers, that for all $i \leq 2$

\[
\text{mn}(\exists v_i \varphi) = C_i \text{mn}(\varphi).
\]

Thus, to show that

\[
\text{mn}(\exists v_i \varphi) \in C \quad \text{whenever } \text{mn}(\varphi) \in C
\]

it is enough to show that $C$ is closed under $C_i$, i.e., $C_i X \subseteq C$ whenever $X \in C$ (and $i \leq 2$). Since $C_i$ is additive, i.e., $C_i (X \cup Y) = C_i X \cup C_i Y$, it is enough to show that

\[
\begin{align*}
C_m X(ijk, e) &\subseteq C \quad \text{for all } i, j, k, m \leq 2, \text{ and good choice } e \text{ for } ijk.
\end{align*}
\]

Assume $i, j, k$ are distinct and $e \in \{r, -r\}$. Then by $\mathfrak{M} \models \text{big}(R)$

\[
\begin{align*}
C_0 X(ijk, e) &= M \times U_j \times U_k, \\
C_1 X(ijk, e) &= U_i \times M \times U_k, \\
C_2 X(ijk, e) &= U_i \times U_j \times M.
\end{align*}
\]
It is easy to check that \( U_i \times U_j \times U_k \in C \) for all \( i, j, k \), and hence \( V_0 \times V_1 \times V_2 \in C \) whenever the \( V_i \) are unions of \( U_0, U_1, U_2 \). When \( i, j, k \) are not all distinct
\[
C_0 X(ijk, e) = M \times e_{12} = \{ \langle a, b, c \rangle : \langle b, c \rangle \in e_{12} \} = \bigcup \{ X(mjk, e') : m \leq 2, e'_{12} = e_{12} \},
\]
\[
C_1 X(ijk, e) = \{ \langle a, b, c \rangle : \langle a, c \rangle \in e_{02} \} = \bigcup \{ X(imk, e') : m \leq 2, e'_{02} = e_{02} \},
\]
\[
C_2 X(ijk, e) = \bigcup \{ X(ijm, e') : m \leq 2, e'_{01} = e_{01} \}.
\]

We have seen that \( A \subseteq C \).

To show that \( C \subseteq A \) we have to check that each \( X(ijk, e) \) is the meaning of a formula \( \varphi \in L_3 \) in \( \mathcal{M} \). We already did this for \( X(ijk, r) \), \( i, j, k \) distinct. For \( ijk = 000 \) and \( e = \langle S, S \rangle \)
\[
X(000, \langle S, S \rangle) = mn(U_0(x) \land U_0(y) \land U_0(z) \land S(x, y) \land S(y, z)),
\]
where \( U_0(x) = \exists yzR, \ U_0(y) = \exists x(x = y \land U_0(x)), \ U_0(z) = \exists x(x = z \land U_0(x)) \)
are the abbreviations introduced before. The other cases are similar, we leave checking them to the reader.

Finally, to show that \( mn(D(x)) = \{ \langle a_0, b, c \rangle : b, c \in M \} \notin A \), observe that the domain of each element in \( B \) either contains \( U_0 \) or else is disjoint from it, and therefore the same holds for their unions. Clearly, this is not true for \( mn(D(x)) \). This shows that \( mn(D) \notin A \), i.e., \( D \) cannot be explicitly defined in \( \mathcal{M} \). Since \( \mathcal{M} \) is a model of \( \mathcal{Th} \), this means that \( \Sigma(D) \) is not equivalent to any explicit definition that contains only 3 variables.

To generalize the construction and the proof from \( n = 3 \) to \( n \geq 4 \) is straightforward. In the general case \( M \) has \( 2n + 1 \) elements, it is the disjoint union of sets \( U_0, U_1, \ldots, U_{n-1} \) of sizes \( 3, 2, \ldots, 2 \) respectively and \( R = \{ s \in U_0 \times \cdots \times U_{n-1} : (s_0 = a_0 \land \Sigma \{ a_i : 1 \leq i < n \} \text{ is even}) \lor (s_0 \in \{ a_1, a_2 \} \land \Sigma \{ a_i : 1 \leq i < n \} \text{ is odd}) \} \).

There is a FOL-formula \( \varphi(v_0) \) for \( \mathcal{Th} \) and \( \Sigma(D) \) as in Thm 2.1 which explicitly defines \( D(v_0) \), since the Beth definability theorem holds for FOL. The above theorem then implies that this explicit definition has to use more than \( n \) variables. Thus, both the theory and the implicit definition use only \( n \) variables, but any equivalent explicit definition has to use more than \( n \) variables.
variables. In our example, $D(v_0)$ can be defined by using $n + 1$ variables. Ian Hodkinson [12], by using a construction from [10], proved that for any number $k$ there are also a theory and a (weak) implicit definition using only $n$ variables such that any explicit definition this implicit definition is equivalent to has to use more than $n + k$ variables.

Theorem 2.1 implies (the known fact) that Craig’s Interpolation Theorem does not hold for $n$-variable logic, either, for $n \geq 3$. This is so because in the standard proof of the Beth’s Definability Theorem in, e.g., [8, Thm.2.2.22], the explicit definition is constructed from an interpoland. Complexity investigations for Craig’s theorem were done earlier, see, e.g., Daniel Mundici [16].

The proof given here proves more than what Theorem 2.1 states. In the proof, Th and $\Sigma(D)$ are written in the so-called restricted $n$-variable logic, and $\Sigma(D)$ is not equivalent to any $n$-variable formula using even infinitary conjunctions and disjunctions in a finite model of Th. A formula is called restricted if substitution of variables is not allowed in it, i.e., it uses relational atomic formulas of form $R(v_0, \ldots, v_k)$ only (and it does not contain subformulas of form $R(v_{i0}, \ldots, v_{ik})$ where $\langle i0, \ldots, ik \rangle \neq \langle 0, \ldots, k \rangle$), see [11, Part II, sec.4.3]. Thus the weak Beth definability property fails for a wide variety of logics, from the restricted $n$-variable fragment with finite models only, to $L_{\infty, \omega}^n$.

The variant of $L_n$ in which we allow only models of size $\leq n + 1$ has the strong Beth definability property, for all $n$, this is proved in [2]. Another variant of $L_n$ that has the strong Beth definability property is when we allow models of all sizes but in a model truth is defined by using only a set of selected (so-called admissible) evaluations of the variables (a generalized model then is a pair consisting of a model in the usual sense and this set of admissible evaluations). The so-called Guarded fragments of $n$-variable logics also have the strong Beth definability property. For more on this see [1] [9] [14].

We note that $L_2$ does not have the strong Beth definability property (this is proved in [2]), and we do not know whether it has the weak one. There are indications that it might have. If so, $L_2$ would be a natural example of a logic distinguishing the two Beth definability properties. At present, we only have artificial examples for this, see Chapter XVIII by Makowsky, J. in [5, p.689, item 4.2.2(v)].
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