GLOBAL EXISTENCE AND DECAY ESTIMATE OF CLASSICAL SOLUTIONS TO THE COMPRESSIBLE VISCOELASTIC FLOWS WITH SELF-GRAVITATING

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Abstract. In this paper, we consider the initial value problem for the compressible viscoelastic flows with self-gravitating in $\mathbb{R}^n$ ($n \geq 3$). Global existence and decay rates of classical solutions are established. The corresponding linear equations becomes two similar equations by using Hodge decomposition and then the solutions operator is derived. The proof is mainly based on the decay properties of the solutions operator and energy method. The decay properties of the solutions operator may be derived from the pointwise estimate of the solution operator to two linear wave equations.

1. Introduction. The compressible viscoelastic flows with self-gravitating in multi-dimensional space is governed by

$$
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho u) &= 0, \\
\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla P(\rho) &= \mu_1 \Delta u + \mu_2 \nabla (\nabla \cdot u) + \nabla \cdot (\rho FF^T) + \rho \nabla \Phi, \\
\partial_t F + u \cdot \nabla F &= \nabla u F, \\
\Delta \Phi &= \rho - \bar{\rho}, \quad \lim_{|x| \to \infty} \Phi = 0
\end{align*}
$$

(1)

The variables are the density $\rho$, the velocity $u$, the deformation tensor $F$ and the electrostatic potential $\Phi$. Furthermore, $P = P(\rho)$ is the pressure function satisfying $P'(\rho) > 0$ for $\rho > 0$. The viscosity coefficients satisfy $\mu_1 > 0$, $2\mu_1 + n\mu_2 > 0$.

The compressible viscoelastic flows with self-gravitating have strong physical background, we may refer to [20]. For instance in semiconductor devices, it can be used to simulate the transport of charged particles under the electric field of electrostatic potential force. When $\mu_1 = \mu_2 = 0$, (1) reduce to self-gravitating Hookean elastodynamics. The three dimension Hookean elastodynamics has been studied and global classical solutions have been established by Hu [2]. The compressible viscoelastic flows with self-gravitating may be viewed as the compressible viscoelastic equations coupled with the self-consistent Poisson equation. Since the 80's of the last century, the local existence, global existence and asymptotic behavior

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of solutions to the initial value problem for the compressible viscoelastic equations have been investigated extensively, we refer to [3, 4, 5, 6, 7, 8, 19, 26]. The global existence with initial data close to an equilibrium state has been established in Besov spaces by Hu and Wang [4]. Several important estimates are also achieved, including a smoothing effect on the velocity, and the $L^1$-decay of the density and deformation gradient. The existence and uniqueness of the local strong solutions near the equilibrium have been established in [5]. We also refer to [3] for local large solutions. Local well-posedness of solutions in critical spaces has been obtained by Qian and Zhang [19], provided that the initial density is bounded away from zero. Moreover, global solutions for small data were also established. Recently, Hu and Guo [7] established global existence and optimal decay rates of solutions.

When the density is constant, the compressible viscoelastic equations is reduced to the incompressible viscoelastic equations. Many mathematican’s investigated the incompressible viscoelastic equations and lots of interesting results have been established, we may refer to [1, 9, 10, 11, 12, 13, 14, 15, 17, 16, 18, 27]. Local existence with large initial data and global existence with small initial data were established in [1, 12, 18]. To obtain global solutions, some important relations are also used to prove that some linear terms are in fact high order terms. Lei, Liu and Zhou [13] find $E = F - I$ satisfies the relation

$$\nabla_k E^{ij} - \nabla_j E^{ik} = E^{lj} \nabla_l E^{ik} - E^{lk} \nabla_l E^{ij},$$

which implies $\nabla \times E$ is a high order term.

Let $m = \rho u$ be the momentum, the compressible elastodynamics equations (1) may be written as

$$\begin{align*}
\partial_t \rho + \nabla \cdot m &= 0, \\
\partial_t m + \nabla \cdot (\frac{m \otimes m}{\rho}) + \nabla P(\rho) &= \mu_1 \Delta (\frac{m}{\rho}) + \mu_2 \nabla \cdot \left( \frac{m}{\rho} \right) + \nabla \cdot (\rho F F^T) + \rho \nabla \Phi, \\
\partial_t F + \frac{m}{\rho} \cdot \nabla F &= \nabla \left( \frac{m}{\rho} \right) F, \\
\Delta \Phi &= \rho - \bar{\rho}, \quad \lim_{|x| \to \infty} \Phi = 0
\end{align*}$$

(2)

In this paper, we investigate global existence and optimal decay of classical solutions to (2) with the following initial value

$$t=0: \rho = \partial x_1 \rho_0(x), \quad m = m_0(x), \quad F = F_0(x), \quad x \in \mathbb{R}^n$$

(3)

Our main goals of this paper are to prove global existence and the optimal decay estimate of solutions to the initial value problem (2), (3). By introducing the Hodge decomposition, (2) may be written as the system (13) whose linear system are decoupled. By the decay estimate of solution operator to (13) and energy method, global existence and the optimal decay estimate of solutions are established. For the details, please refer to Theorem 4.2, 5.1 and Remark 1, 2.

There are also two-folds in our present paper: firstly, it is very difficult to obtain the solutions operator to (2) since (2) has $n^2 + n + 1$ equations and the unknown functions are coupled. To overcome this difficulty, we introduce the Hodge decomposition and use some special relations (8), (9), then (2) may be written as the system (13) whose linear system are decoupled; Secondly, we clarify the decay property of the solutions operator by investigating the solutions operator to two linear wave equation in (15) and (20). The advantage of this method in the paper is avoiding the complex calculation by using Taylor formula.
The paper is organized as follows. We make rearrangement for the problem by using the Hodge decomposition and some relations in Section 2. In Section 3, we discuss the decay properties of solution operator to (13). Section 4 is devoted to establishing global existence and asymptotic behavior of solutions in odd space dimensions, while Section 5 is devoted to global existence and asymptotic behavior of solutions in even space dimensions.

2. Rearrangement of (2). Without loss of generality, we assume $\bar{\rho} = 1$ and $P'(1) = 1$. Let $\sigma = \rho - \bar{\rho} = \rho - 1$ and $E = F - I$, then (2) may be transformed into

$$\begin{cases}
\partial_t \sigma + \nabla \cdot m = 0, \\
\partial_t m + \nabla \sigma - \nabla \Phi = \mu_1 \Delta m + \mu_2 \nabla(\nabla \cdot m) + \nabla \cdot E + f, \\
\partial_t E - \nabla m = g, \\
\Delta \Phi = \sigma, \quad \lim_{|x| \to \infty} \Phi = 0.
\end{cases}$$

Here

$$f = -\mu_1 \Delta\left(\frac{\sigma m}{\sigma + 1}\right) - \mu_2 \nabla\left(\frac{\sigma m}{\sigma + 1}\right) - \nabla \cdot \left(\frac{m \otimes m}{\sigma + 1}\right) - \nabla (P(\sigma + 1) - 1 - \sigma) + \sigma \nabla \Phi + \nabla \cdot (\sigma EE^T + EE^T + \sigma E)$$

and

$$g = \nabla\left(\frac{m}{\sigma + 1}\right)E - \nabla\left(\frac{\sigma m}{\sigma + 1}\right) - \frac{1}{\sigma + 1} m \cdot \nabla E.$$

Instituting the last equation in (4) into the second equation in (4), we arrive at

$$\begin{cases}
\partial_t \sigma + \nabla \cdot m = 0, \\
\partial_t m + \nabla \sigma - \nabla (-\Delta)^{-1} \sigma = \mu_1 \Delta m + \mu_2 \nabla(\nabla \cdot m) + \nabla \cdot E + f, \\
\partial_t E - \nabla m = g.
\end{cases}$$

The following results have been established in [4] and [19].

**Lemma 2.1.** Assume that $(\rho, u, F, \Phi)$ is a solution to (1). Then the following identity

$$\nabla \cdot (\rho F) = 0$$

and

$$F^{ik} \nabla_i F^{ij} = F^{ij} \nabla_i F^{ik}$$

hold for all time $t > 0$ if it initially satisfies (7).

(7) implies for all $t > 0$

$$\nabla_k E^{ij} + E^{ik} \nabla_i E^{ij} = \nabla_j E^{ik} + E^{lj} \nabla_l E^{ik}.$$

Thus we have

$$\begin{align*}
\nabla_j \nabla_k E^{ik} - \nabla_i \nabla_k E^{ik} &= \nabla_j \nabla_k E^{ik} - \nabla_k \nabla_i E^{jk} \\
&= \nabla_k \nabla_k E^{ij} - \nabla_k \nabla_k E^{kj} + \nabla_k (E^{ik} \nabla_i E^{ij} - E^{ij} \nabla_i E^{ik}) \\
&- \nabla_k (E^{ik} \nabla_i E^{ij} - E^{ij} \nabla_i E^{jk}) \\
&= \Delta (E^{ij} - E^{ji}) + \nabla_k (E^{ik} \nabla_i E^{ij} - E^{ij} \nabla_i E^{ik}) \\
&- \nabla_k (E^{ik} \nabla_i E^{ij} - E^{ij} \nabla_i E^{jk}).
\end{align*}$$
By (6), we have
\[ \nabla \cdot (\nabla \cdot E) = \nabla \cdot (\nabla \cdot E^T) = \nabla \cdot \left[ (\sigma + 1)(E + I)^T \right] - \nabla \cdot (\sigma I + \sigma E^T) = -\Delta \sigma - \nabla \cdot (\sigma E). \] (10)

Owing to (9) and (10), we get
\[ \Delta \nabla \cdot E = -\Delta \nabla \sigma - \nabla \nabla \cdot (\sigma E) + \Delta \nabla \cdot (E - E^T) + \nabla \cdot \mathcal{L}, \] (11)
where \( \mathcal{L} \) is antisymmetric matrix, which is defined by
\[ \mathcal{L}^{ij} = \nabla_k (E^{ik} \nabla_i E^{jk} - E^{ik} \nabla_i E^{jk}) - \nabla_k (E^{ik} \nabla_i E^{jk} - E^{ik} \nabla_i E^{jk}). \] (12)

Let
\[ \Lambda^s v = \mathcal{F}^{-1}(|\xi|^s \hat{v}), \quad \Omega = \Lambda^{-1}(\nabla \cdot m), \quad \Gamma = \Lambda^{-1}(\nabla \times m). \]

Then using (9) and (10), (5) may be transformed into
\[ \begin{cases} 
\partial_t \sigma + \Lambda \Omega = 0, \\
\partial_t \Omega - (\mu_1 + \mu_2) \Delta \Omega - 2\Lambda \sigma - \Lambda^{-1} \sigma = \Lambda^{-1} \nabla \cdot \left( f - \nabla \cdot (\sigma E) \right), \\
\partial_t \Gamma - \mu_1 \Delta \Gamma + \Lambda (E - E^T) = \Lambda^{-1} (\mathcal{L} + \nabla \times f), \\
\partial_t (E - E^T) + \Lambda \Gamma = g_1 - g_1^T - \frac{1}{\sigma + 1} m \cdot (E - E^T),
\end{cases} \] (13)
where \( \mathcal{L} \) is antisymmetric matrix, which is given by (12) and
\[ g_1 = \nabla \left( \frac{m}{\sigma + 1} E \right) - \nabla \left( \frac{\sigma m}{\sigma + 1} \right). \]

The following Lemma comes from [7].

**Lemma 2.2.** Assume that \( \| (\sigma, m, E, \nabla \Phi) \|_{H^s} \) is suitably small, then we have
\[ \| \partial_x^l E(t) \|_{L^2} \leq C \| \partial_x^l (\sigma, E - E^T)(t) \|_{L^2} \]
for \( 1 \leq l \leq s \).

3. ** Decay properties of solution operator to (13).** This section is devoted to study the decay properties of solution operator to (13). To do so, we first consider the linear equations of (13)
\[ \begin{cases} 
\partial_t \sigma + \Lambda \Omega = 0, \\
\partial_t \Omega - (\mu_1 + \mu_2) \Delta \Omega - 2\Lambda \sigma - \Lambda^{-1} \sigma = 0, \\
\partial_t \Gamma - \mu_1 \Delta \Gamma + \Lambda (E - E^T) = 0, \\
\partial_t (E - E^T) + \Lambda \Gamma = 0.
\end{cases} \] (14)

Noting that the first equation and second equation in (14) is coupled, \( \sigma \) satisfies the following problem
\[ \begin{cases} 
\sigma_{tt} - (\mu_1 + \mu_2) \Delta \sigma_t + 2\Lambda^2 \sigma + \sigma = 0, \\
t = 0: \quad \sigma = \sigma_0, \quad \sigma_t = -\Delta \Omega_0.
\end{cases} \] (15)

By Fourier transform, we have
\[ \hat{\sigma} = -\mathcal{G}(\xi, t) \hat{\xi} \hat{\Omega}_0 + \hat{\mathbb{H}}(\xi, t) \hat{\sigma}_0, \] (16)
where
\[ \mathcal{G}(\xi, t) = \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}, \quad \mathbb{H}(\xi, t) = \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} \] (17)
and
\[ \lambda_{\pm} = \frac{-(\mu_1 + \mu_2)|\xi|^2 \pm \sqrt{(\mu_1 + \mu_2)^2|\xi|^4 - 4(2|\xi|^2 + 1)}}{2}. \]

\(\Omega\) satisfies the following problem
\[ \begin{cases} \Omega_{tt} - (\mu_1 + \mu_2)\Delta \Omega_t + 2\lambda^2 \Omega + \Omega = 0, \\ t = 0 : \Omega = \Omega_0, \quad \Omega_t = (\mu_1 + \mu_2)\Delta \Omega_0 + 2\lambda \sigma_0 + \Delta^{-1} \sigma_0. \end{cases} \]  
(18)

Similarly, we arrive at
\[ \hat{\Omega} = \hat{G}(\xi, t) \left(2|\xi| + \frac{1}{|\xi|}\right)\hat{\sigma}_0 + \left(\hat{H}(\xi, t) - (\mu_1 + \mu_2)|\xi|^2 \hat{G}(\xi, t)\right)\hat{\Omega}_0. \]  
(19)

\(\Gamma\) satisfies the following problem
\[ \begin{cases} \Gamma_{tt} - \mu_1 \Delta \Gamma_t + \Lambda^2 \Gamma = 0, \\ t = 0 : \Gamma = \Gamma_0, \quad \Gamma_t = \mu_1 \Delta \Gamma_0 + \Lambda (E_t - E_0^T). \end{cases} \]  
(20)

By Fourier transform, we have
\[ \hat{\Gamma} = \hat{G}(\xi, t)|\xi|(\hat{E}_0 - \hat{E}_0^T) + \left(\hat{H}(\xi, t) - \mu_1 |\xi|^2 \hat{G}(\xi, t)\right)\hat{\Gamma}_0, \]  
(21)

where
\[ \hat{G}(\xi, t) = \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}, \quad \hat{H}(\xi, t) = \frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-}. \]  
(22)

and
\[ \lambda_{\pm} = \frac{-\mu_1 |\xi|^2 \pm \sqrt{\mu^2 |\xi|^4 - 4|\xi|^2}}{2}. \]

\(E - E^T\) satisfies the following problem
\[ \begin{cases} (E - E^T)_{tt} - \mu_1 \Delta (E - E^T)_t + \Lambda^2 (E - E^T) = 0, \\ t = 0 : E - E^T = E_0 - E_0^T, \quad (E - E^T)_t = -\Delta \Gamma_0. \end{cases} \]  
(23)

By Fourier transform, we have
\[ \hat{E} - \hat{E}^T = -\hat{G}(\xi, t)|\xi|\hat{\Gamma}_0 + \hat{H}(\xi, t)(\hat{E}_0 - \hat{E}_0^T). \]  
(24)

Owing to (16), (19), (21) and (24), we have
\[ \begin{pmatrix} \dot{\hat{\Omega}} \\ \hat{\Omega}_0 \\ \hat{\Gamma} \\ \hat{E} - \hat{E}^T \end{pmatrix} = \hat{G}(\xi, t) \begin{pmatrix} \hat{\sigma}_0 \\ \Omega_0 \\ \hat{\Gamma}_0 \\ \hat{E}_0 - \hat{E}_0^T \end{pmatrix} = \begin{pmatrix} \hat{G}_{11} & \hat{G}_{12} & 0 & 0 \\ \hat{G}_{21} & \hat{G}_{22} & 0 & 0 \\ 0 & 0 & \hat{G}_{33} & \hat{G}_{34} \\ 0 & 0 & \hat{G}_{43} & \hat{G}_{44} \end{pmatrix} \begin{pmatrix} \hat{\sigma}_0 \\ \Omega_0 \\ \Gamma_0 \\ \hat{E}_0 - \hat{E}_0^T \end{pmatrix}, \]  
(25)

where
\[ \hat{G}_{11} = \hat{H}, \quad \hat{G}_{12} = -|\xi| \hat{G}, \quad \hat{G}_{21} = (2|\xi| + \frac{1}{|\xi|}) \hat{G}, \quad \hat{G}_{22} = \hat{H} - (\mu_1 + \mu_2)|\xi|^2 \hat{G}, \]  
\[ \hat{G}_{33} = \hat{H}, \quad \hat{G}_{34} = -|\xi| \hat{G}, \quad \hat{G}_{43} = |\xi| \hat{G}, \quad \hat{G}_{44} = \hat{H} - \mu_1 |\xi|^2 \hat{G}. \]  
(26)

Taking \(\mathcal{F}^{-1}\) to (25), we arrive at
\[ \begin{pmatrix} \sigma \\ \Omega \\ \Gamma \\ E - E^T \end{pmatrix} = \mathcal{F}(t) \begin{pmatrix} \sigma_0 \\ \Omega_0 \\ \Gamma_0 \\ E_0 - E_0^T \end{pmatrix}, \]  
(27)

where
\[ \mathcal{F}(t) = \mathcal{F}^{-1}(\hat{G}(\xi, t)). \]
In order to study the decay properties of solution operator \( G \), it is suffice to study the decay properties of \( G, H, \hat{G} \) and \( H \) since the solution operator \( \mathcal{G} \) is given in term of \( G, H, \hat{G} \) and \( H \). Therefore, we firstly study the decay properties of \( G \) and \( H \). Taking Fourier transform of (15), we have

\[
\begin{cases}
\hat{\sigma}_{tt} + (\mu_1 + \mu_2)|\xi|^2\hat{\sigma} + 2|\xi|^2\hat{\sigma} + \hat{\sigma} = 0. & t = 0; \hat{\sigma} = \hat{\sigma}_0, \hat{\sigma}_t = -|\xi|^2(t) \hat{\sigma}_0.
\end{cases}
\]  

(28)

**Lemma 3.1.** Let \( \sigma \) be the solution to the problem (15). Then its Fourier image \( \hat{\sigma} \) verifies the pointwise estimate

\[
|\hat{\sigma}(\xi, t)|^2 + (1 + |\xi|^2)2|\hat{\sigma}(\xi, t)|^2 \leq C e^{-c \omega(\xi)t}(|\xi|^2\hat{\Omega}_0|^2 + (1 + |\xi|^2)|\hat{\sigma}_0|^2),
\]

for \( \xi \in \mathbb{R}^n \) and \( t \geq 0 \), where \( \omega(\xi) = \frac{|\xi|^2}{1 + |\xi|^2} \).

*Proof.* The proof may be found in [23]. Here we omit the details. \( \Box \)

By the method in [21] and [22], the pointwise estimate (29) together with the solution formula ((16)) to the problem (28) give the corresponding pointwise estimates for \( \hat{G} \) and \( \hat{H} \). The result is stated as follows.

**Lemma 3.2.** Let \( \hat{G} \) and \( \hat{H} \) be the fundamental solutions of (15) in the Fourier space, which are given explicitly in (17). Then we have the pointwise estimates

\[
|\hat{G}(\xi, t)| \leq C (1 + |\xi|^2)^{-1} e^{-c \omega(\xi)t},
\]

\[
|\hat{H}(\xi, t)| \leq C e^{-c \omega(\xi)t},
\]

(30)

for \( \xi \in \mathbb{R}^n \) and \( t \geq 0 \), where \( \omega(\xi) = \frac{|\xi|^2}{1 + |\xi|^2} \).

**Lemma 3.3.** Let \( e = E - E^T \) be the solution to the problem (23). Then its Fourier image \( \hat{e} \) verifies the pointwise estimate

\[
|\hat{e}(\xi, t)|^2 + |\xi|^2(1 + |\xi|^2)|\hat{e}(\xi, t)|^2 \leq C e^{-c \omega(\xi)t}(|\xi|^2|\hat{\Gamma}_0|^2 + |\xi|^2(1 + |\xi|^2)|\hat{E}_0 - \hat{E}_0^T|^2),
\]

(31)

for \( \xi \in \mathbb{R}^n \) and \( t \geq 0 \), where \( \omega(\xi) = \frac{|\xi|^2}{1 + |\xi|^2} \).

*Proof.* The proof may be found in [24]. Here we omit the details. \( \Box \)

Similar to the proof of Lemma 3.2, by Lemma 3.3, it is not difficult to get

**Lemma 3.4.** Let \( \hat{G} \) and \( \hat{H} \) be the fundamental solutions to (23) in the Fourier space, which are given explicitly in (22). Then we have the pointwise estimates of \( \hat{G} \) and \( \hat{H} \)

\[
|\hat{G}(\xi, t)| \leq C |\xi|^{-1}(1 + |\xi|^2)^{-\frac{1}{2}} e^{-c \omega(\xi)t},
\]

\[
|\hat{H}(\xi, t)| \leq C e^{-c \omega(\xi)t},
\]

(32)

for \( \xi \in \mathbb{R}^n \) and \( t \geq 0 \), where \( \omega(\xi) = \frac{|\xi|^2}{1 + |\xi|^2} \).

**Lemma 3.5.** Let \( 1 \leq p \leq 2 \), and let \( k, j \) and \( l \) be nonnegative integers. Assume that \( \varphi \in W^{j,p} \cap H^{k+l+2} \). Then we have

\[
\|\partial_x^k G(t) \ast \varphi\|_{L^2} \leq C (1 + t)^{-\frac{k}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{k+2}{2}} \|\partial_x^j \varphi\|_{L^p} + C e^{-ct} \|\partial_x^{k+l+2} \varphi\|_{L^2},
\]

\[
\|\partial_x^k H(t) \ast \varphi\|_{L^2} \leq C (1 + t)^{-\frac{k}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{k+2}{2}} \|\partial_x^j \varphi\|_{L^p} + C e^{-ct} \|\partial_x^{k+l+2} \varphi\|_{L^2}
\]

(33)
\[ \|\partial^k_z G(t) * \phi\|_{L^2} \leq C(1 + t)^{-\frac{q}{2}(\frac{1}{2} - \frac{1}{q})} \|\partial^2_z \phi\|_{L^p} + Ce^{-ct}\|\partial^{k+l-2}_x \varphi\|_{L^2}, \quad (35) \]

\[ \|\partial^k_z \mathcal{H}(t) * \phi\|_{L^2} \leq C(1 + t)^{-\frac{q}{2}(\frac{1}{2} - \frac{1}{q})} \|\partial^2_z \phi\|_{L^p} + Ce^{-ct}\|\partial^{k+l}_x \varphi\|_{L^2} \quad (36) \]

for \(0 \leq j \leq k\) in (33), (34) and (36), \(0 \leq j \leq k - 1\) in (35), where \(k + l - 2 \geq 0\) in (33) and (35).

Next we state the decay property of solution operator \( \mathcal{G} \) to (13). By Lemma 3.5 and (26), it is not difficult to derive the following decay estimate for solution operator \( \mathcal{G} \).

**Lemma 3.6.** Let \( 1 \leq p \leq 2 \), and let \( k, j \) and \( l \) be nonnegative integers. Assume that all norms appearing on the right-hand side of the following inequalities are bounded. Then we have

\[ \|\partial^k_z \mathcal{G}_{11}(t) * \partial_x \phi\|_{L^2} \leq C(1 + t)^{-\frac{q}{2}(\frac{1}{2} - \frac{1}{q})} \|\partial^2_z \phi\|_{L^p} + Ce^{-ct}\|\partial^{k+l}_x \phi\|_{L^2} \quad (37) \]

\[ \|\partial^k_z \mathcal{G}_{12}(t) * \psi\|_{L^2} \leq C(1 + t)^{-\frac{q}{2}(\frac{1}{2} - \frac{1}{q})} \|\partial^2_z \psi\|_{L^p} + Ce^{-ct}\|\partial^{k+l-1}_x \psi\|_{L^2} \quad (38) \]

\[ \|\partial^k_z \mathcal{G}_{21}(t) * \partial_x \phi\|_{L^2} \leq C(1 + t)^{-\frac{q}{2}(\frac{1}{2} - \frac{1}{q})} \|\partial^2_z \phi\|_{L^p} + Ce^{-ct}\|\partial^{k+l-1}_x \phi\|_{L^2} \quad (39) \]

\[ \|\partial^k_z \mathcal{G}_{22}(t) * \psi\|_{L^2} \leq C(1 + t)^{-\frac{q}{2}(\frac{1}{2} - \frac{1}{q})} \|\partial^2_z \psi\|_{L^p} + Ce^{-ct}\|\partial^{k+l-1}_x \psi\|_{L^2} \quad (40) \]

\[ \|\partial^k_z \mathcal{G}_{33}(t) * \tilde{\phi}\|_{L^2} \leq C(1 + t)^{-\frac{q}{2}(\frac{1}{2} - \frac{1}{q})} \|\partial^2_z \tilde{\phi}\|_{L^p} + Ce^{-ct}\|\partial^{k+l}_x \tilde{\phi}\|_{L^2} \quad (41) \]

\[ \|\partial^k_z \mathcal{G}_{34}(t) * \psi\|_{L^2} \leq C(1 + t)^{-\frac{q}{2}(\frac{1}{2} - \frac{1}{q})} \|\partial^2_z \psi\|_{L^p} + Ce^{-ct}\|\partial^{k+l-1}_x \psi\|_{L^2} \quad (42) \]

\[ \|\partial^k_z \mathcal{G}_{43}(t) * \tilde{\phi}\|_{L^2} \leq C(1 + t)^{-\frac{q}{2}(\frac{1}{2} - \frac{1}{q})} \|\partial^2_z \tilde{\phi}\|_{L^p} + Ce^{-ct}\|\partial^{k+l}_x \tilde{\phi}\|_{L^2} \quad (43) \]

\[ \|\partial^k_z \mathcal{G}_{44}(t) * \tilde{\psi}\|_{L^2} \leq C(1 + t)^{-\frac{q}{2}(\frac{1}{2} - \frac{1}{q})} \|\partial^2_z \tilde{\psi}\|_{L^p} + Ce^{-ct}\|\partial^{k+l}_x \tilde{\psi}\|_{L^2} \quad (44) \]

for \(0 \leq j \leq k + 1\) in (37), (38), \(0 \leq j \leq k\) in (39)-(44).

4. **Asymptotic behavior of solutions in odd space dimensions.** The purpose of this section is to prove global existence and asymptotic decay of solutions to the initial value problem (2), (3) in odd space dimensions. We need the following lemma for composite functions, which can be found in Lemma 5.2.6 on pp188 of [25].

**Lemma 4.1.** Suppose that \( f = f(v) \) is smooth, where \( v = (v_1, \ldots, v_N) \) is a vector function. Assume that \( f(v) = O(|v|^{1+\sigma}) \) for \( |v| \to 0 \), where \( \sigma \geq 1 \) is an integer. Let \( v \in L^\infty \) and \( \|v\|_{L^\infty} \leq M_0 \) for a positive constant \( M_0 \). Let \( 1 \leq p, q, r \leq +\infty \) and \( \frac{1}{p} + \frac{1}{q} = \frac{1}{r} \), and let \( k \geq 0 \) be an integer. Then we have

\[ \|\partial^k_z f(v)\|_{L^p} \leq C\|v\|_{L^q}^{1-1}\|v\|_{L^r}\|\partial^k_z v\|_{L^r} \]

where \( C = C(M_0) \) is a constant depending on \( M_0 \).

Therefore, the solutions to the problem (2), (3) have the form

\[
\begin{pmatrix}
\sigma \\
\Omega \\
\Gamma \\
E - ET^T
\end{pmatrix}
= \mathcal{G}(t) \ast
\begin{pmatrix}
\sigma_0 \\
\Omega_0 \\
\Gamma_0 \\
E_0 - E_0^T
\end{pmatrix}
+ \int_0^t \mathcal{G}(t - \tau) \ast
\begin{pmatrix}
\Lambda^{-1} \nabla \cdot (f - \nabla \cdot (\sigma E)) \\
\Lambda^{-1}(\mathcal{L} + \nabla \times f) \\
g_1 - g_1 - \frac{1}{\sigma + 1} m \cdot \nabla (E - ET^T)
\end{pmatrix}(\tau) d\tau, \quad (45)
\]
where $\sigma_0(x) = \rho_0(x, 0) - 1 = \partial_{x_1}(\rho_0(x) - x_1)$.

(45) is rewritten as

$$
\begin{align*}
\sigma(t) &= G_{11}(t) \ast \sigma + G_{12}(t) \ast \Omega_0 + \int_0^t G_{12}(t - \tau) \ast \Lambda^{-1} \nabla \cdot \left( f - \nabla \cdot (\sigma E) \right)(\tau) d\tau, \\
\Omega(t) &= G_{21}(t) \ast \sigma + G_{22}(t) \ast \Omega_0 + \int_0^t G_{22}(t - \tau) \ast \Lambda^{-1} \nabla \cdot \left( f - \nabla \cdot (\sigma E) \right)(\tau) d\tau, \\
\Gamma(t) &= G_{33}(t) \ast \Gamma_0 + G_{34}(t) \ast (E_0 - E_0^T) + \int_0^t G_{33}(t - \tau) \ast \Lambda^{-1} (\mathcal{L} + \nabla \times f)(\tau) d\tau \\
&+ \int_0^t G_{34}(t - \tau) \ast \left( g_1 - g_1^T - \frac{1}{\sigma + 1} m \cdot \nabla (E - E^T) \right)(\tau) d\tau, \\
(E - E^T)(t) &= G_{43}(t) \ast \Gamma_0 + G_{44}(t) \ast (E_0 - E_0^T) \\
&+ \int_0^t G_{43}(t - \tau) \ast \Lambda^{-1} (\mathcal{L} + \nabla \times f)(\tau) d\tau \\
&+ \int_0^t G_{44}(t - \tau) \ast \left( g_1 - g_1^T - \frac{1}{\sigma + 1} m \cdot \nabla (E - E^T) \right)(\tau) d\tau.
\end{align*}
$$

(46)

where

$$
\begin{align*}
f &= -\nabla \cdot (\nabla (-\Delta)^{-1}\sigma \otimes \nabla (-\Delta)^{-1}\sigma) + \frac{1}{2} \nabla (|\nabla (-\Delta)^{-1}\sigma|^2 I_{n \times n}) \\
&+ \nabla (P(\sigma + 1) - P(1) - P'(1)\sigma) - \text{div}(m \otimes \frac{m}{\sigma + 1}) \\
&- \mu_1 \Delta(\frac{\sigma m}{\sigma + 1}) - \mu_2 \nabla \cdot (\frac{\sigma m}{\sigma + 1}).
\end{align*}
$$

(47)

When $|U| = |(\sigma, m, F - I, \nabla \Phi)|$ is small enough, we may write

$$
f = \sum_i \partial_{x_i} f_1 + \sum_{i, j} \partial_{x_i x_j} f_2
$$

with

$$
f_1 = O(|U|^2), \quad f_2 = O(|U|^2).
$$

(48)

**Theorem 4.2.** Let $n \geq 3$ be an odd integer and $s \geq \frac{1}{2}(n + 5)$. Let $p \in [1, \frac{2n}{n + 2})$. Suppose that $(\rho_0 - x_1, F_0 - I, m_0) \in L^p$, $(\partial_{x_1}(\rho_0 - x_1), F_0 - I, m_0) \in H^s$ and put $E_0 = ||(\rho_0 - x_1, F_0 - I, m_0)||_{L^p} + ||(\partial_{x_1}(\rho_0 - x_1), F_0 - I, m_0)||_{H^s}$. Then there is a positive constant $\delta_0$ such that if $E_0 \leq \delta_0$, then the problem (2), (3) has a unique global solution $(\rho - 1, m, F - I, \nabla \Phi)$ with $(\rho - 1, m, F - I, \nabla \Phi) \in C^0([0, +\infty); H^s \times H^s \times H^{s+1})$. For $i \leq \frac{n + 1}{2}$, the solution verifies the decay estimates

$$
\|\partial_x^i (\rho - 1)(t)\|_{L^2} \leq CE_0(1 + t)^{-\frac{n - 1}{2} + \frac{i - 1}{4}}, \quad \|\partial_x^i (m, F - I, \nabla \Phi)(t)\|_{L^2} \leq CE_0(1 + t)^{-\frac{n - 1}{2} + \frac{i - 1}{4}}.
$$

(49)

Moreover, for $\frac{n + 1}{2} \leq k \leq s - 2$, we have

$$
\|\partial_x^k (\rho - 1, m, F - I, \nabla \Phi)(t)\|_{L^2} \leq CE_0(1 + t)^{-\frac{n - 1}{2} + \frac{k - 1}{4}}.
$$

(50)

**Remark 1.** Under the same assumptions of Theorem 4.2, for $2 \leq q \leq 2n$, by Gagliardo-Nirenberg inequality, $L^q$ decay estimate

$$
\|\rho - 1\|_{L^q} \leq CE_0(1 + t)^{-\frac{n - 1}{2} + \frac{q - 1}{2}}, \quad \|m, F - I, \nabla \Phi\|_{L^q} \leq CE_0(1 + t)^{-\frac{n - 1}{2} + \frac{q - 1}{2}}
$$

(51)

hold.

**Proof.** The local existence of classical solutions to the problem (2), (3) can be established by applying the contracting map argument. To prove global classical solution to the problem (2), (3) and the decay estimate (49), (50), we need to
establish the uniform a priori estimates and obtain the decay estimate of the solution to (46). To do so, we introduce the following quantity

\[ X(t) = \sum_{i \leq \frac{n+1}{2}} \sup_{0 \leq \tau \leq t} \left\{ (1 + \tau)^{\frac{n}{p} - \frac{3}{2}} + \frac{3}{2} \right\} \left\langle \partial^k_x \sigma(\tau) \right\rangle_{L^2} \]

\[ + (1 + \tau)^{\frac{n}{p} - \frac{3}{2}} \left\langle \partial^k_x (m, E, \nabla \Phi)(\tau) \right\rangle_{L^2} \]

\[ + \sum_{\frac{n+1}{2} \leq k \leq s - 2} \sup_{0 \leq \tau \leq t} \left\langle \partial^k_x \sigma(\tau) \right\rangle_{L^2} \]

\[ + \sum_{n = s - 1} \sup_{0 \leq \tau \leq t} \left\langle \partial^k_x (m, E, \nabla \Phi)(\tau) \right\rangle_{L^2}. \]  

(52)

Differentiating the first equality in (46) \( k(\frac{n+1}{2} \leq k \leq s - 2) \) times with respect to \( x \) and taking the \( L^2 \) norm, we have

\[ \left\langle \partial^k_x \sigma(t) \right\rangle_{L^2} \leq \left\langle \partial^k_x \Phi_1(t) * \sigma_0 \right\rangle_{L^2} + \left\langle \partial^k_x \Phi_2(t) * \Omega_0 \right\rangle_{L^2} \]

\[ + \int_0^t \left\langle \partial^k_x \Phi_2(t - \tau) * \Lambda^{-1} \nabla \cdot (f - \nabla \cdot (\sigma E)) \right\rangle_{L^2} d\tau \]

\[ =: I_1 + I_2 + I_3. \]  

(54)

We apply (37) with \( j = 0, l = 0 \) to \( I_1 \), it yields

\[ I_1 \leq C(1 + t)^{-\frac{3}{2} (\frac{1}{p} - \frac{1}{2}) - \frac{n+1}{p}} \left\| \rho_0 - x_1 \right\|_{L^p} + C e^{-ct} \left\| \partial^k_x \partial_x (\rho_0 - x_1) \right\|_{L^2} \]

\[ \leq C(1 + t)^{-\frac{3}{2} (\frac{1}{p} - \frac{1}{2}) - \frac{n+1}{p}} E_0. \]  

(55)

By using (38) with \( j = 0, l = 1 \) to \( I_2 \), it yields

\[ I_2 \leq C(1 + t)^{-\frac{3}{2} (\frac{1}{p} - \frac{1}{2}) - \frac{n+1}{p}} \left\| \Omega_0 \right\|_{L^p} + C e^{-ct} \left\| \partial^k_x \Omega_0 \right\|_{L^2} \]

\[ \leq C(1 + t)^{-\frac{3}{2} (\frac{1}{p} - \frac{1}{2}) - \frac{n+1}{p}} E_0. \]  

(56)

Gagliardo-Nirenberg inequality and (52) give

\[ \left\| \sigma(\tau) \right\|_{L^\infty} \leq C \left\| \partial^\frac{n+1}{2} \sigma(\tau) \right\|_{L^2}^{\frac{1}{2}} \left\| \partial^\frac{n+1}{2} \sigma(\tau) \right\|_{L^2}^{\frac{1}{2}} \leq C X(t)(1 + \tau)^{-\frac{3}{2} (\frac{1}{p} - \frac{1}{2}) - \frac{n+1}{p}}, \]

\[ \left\| m(\tau) \right\|_{L^\infty} \leq C \left\| \partial^\frac{n+1}{2} m(\tau) \right\|_{L^2}^{\frac{1}{2}} \left\| \partial^\frac{n+1}{2} m(\tau) \right\|_{L^2}^{\frac{1}{2}} \leq C X(t)(1 + \tau)^{-\frac{3}{2} (\frac{1}{p} - \frac{1}{2}) - \frac{n+1}{p}}, \]

\[ \left\| E(\tau) \right\|_{L^\infty} \leq C \left\| \partial^\frac{n+1}{2} E(\tau) \right\|_{L^2}^{\frac{1}{2}} \left\| \partial^\frac{n+1}{2} E(\tau) \right\|_{L^2}^{\frac{1}{2}} \leq C X(t)(1 + \tau)^{-\frac{3}{2} (\frac{1}{p} - \frac{1}{2}) - \frac{n+1}{p}}, \]

\[ \left\| \nabla \Phi(\tau) \right\|_{L^\infty} \leq C \left\| \partial^\frac{n+1}{2} \nabla \Phi(\tau) \right\|_{L^2}^{\frac{1}{2}} \left\| \partial^\frac{n+1}{2} \nabla \Phi(\tau) \right\|_{L^2}^{\frac{1}{2}} \leq C X(t)(1 + \tau)^{-\frac{3}{2} (\frac{1}{p} - \frac{1}{2}) - \frac{n+1}{p}}. \]

(57)

We estimate the nonlinear term \( I_3 \). We divide \( I_3 \) into two parts and write \( I_3 = I_{31} + I_{32} \), where \( I_{31} \) and \( I_{32} \) are corresponding to the time intervals \([0, t/2]\) and \([t/2, t]\), respectively. For the term \( I_{31} \), we apply (38) with \( j = 0 \) and \( l = 0 \). This yields

\[ I_{31} \leq C \int_0^{t/2} (1 + t - \tau)^{-\frac{3}{2} (\frac{1}{p} - \frac{1}{2}) - \frac{n+1}{p}} \left\| U^2(\tau) \right\|_{L^p} d\tau \]

\[ + C \int_0^{t/2} e^{-c(t-\tau)} \left( \left\| \partial^k_x |U|^2(\tau) \right\|_{L^2} + \left\| \partial^{k+1} |U|^2(\tau) \right\|_{L^2} \right) d\tau. \]

(58)
It follows from Lemma 4.1 and (57), (52) that

\[ \| U \|^2 (\tau) \|_{L^p} \leq C \| U (\tau) \|^2 \|_{L^\infty} \leq C \| U (\tau) \|^2 \|_{L^2} \]
\[ \leq CX^2 (t) (1 + \tau)^{-n (\frac{1}{p} - \frac{1}{2}) - \frac{(n - 1)(p - 1)}{2p}}, \]
\[ \| \partial_x^k |U|^2 (\tau) \|_{L^2} \leq C \| U (\tau) \|_{L^\infty} \| \partial_x^k U (\tau) \|_{L^2} \leq CX^2 (t) (1 + \tau)^{-n (\frac{1}{p} - \frac{1}{2}) - \frac{n - 1}{4}}, \]
\[ \| \partial_x^{k+1} |U|^2 (\tau) \|_{L^2} \leq C \| U (\tau) \|_{L^\infty} \| \partial_x^{k+1} U (\tau) \|_{L^2} \leq CX^2 (t) (1 + \tau)^{- \frac{3}{4} (\frac{1}{p} - \frac{1}{2}) - \frac{n - 1}{4}}, \]
\[ \| \partial_x^{k+2} |U|^2 (\tau) \|_{L^2} \leq C \| U (\tau) \|_{L^\infty} \| \partial_x^{k+2} U \|_{L^2} \leq CX^2 (t) (1 + \tau)^{- \frac{3}{4} (\frac{1}{p} - \frac{1}{2}) - \frac{n - 1}{4}}. \]

Inserting the estimate (59) into (58) and noting \( p \in [1, \frac{2n}{n+2}) \), we obtain

\[ I_{31} \leq CX^2 (t) \int_0^\frac{1}{t} \left( 1 + t - \tau \right)^{- \frac{n}{2} (\frac{1}{p} - \frac{1}{2}) - \frac{3k}{2} (1 + \tau)^{- n (\frac{1}{p} - \frac{1}{2}) - \frac{(n - 1)(p - 1)}{2p}} d\tau \]
\[ + CX^2 (t) \int_0^\frac{1}{t} e^{-c(t-\tau)} (1 + \tau)^{-n (\frac{1}{p} - \frac{1}{2}) - \frac{n - 1}{4}} d\tau \]
\[ + CX^2 (t) \int_0^\frac{1}{t} e^{-c(t-\tau)} (1 + \tau)^{- \frac{3}{4} (\frac{1}{p} - \frac{1}{2}) - \frac{n - 1}{4}} d\tau \]
\[ \leq CX^2 (t) (1 + t)^{- \frac{3}{4} (\frac{1}{p} - \frac{1}{2}) - \frac{n - 1}{4}}. \]

For the term \( I_{32} \), we apply (38) with \( p = 2, j = k \) and \( l = 0 \). This yields

\[ I_{32} \leq C \int_0^t (1 + t - \tau)^{-1} \| \partial_x^k |U|^2 (\tau) \|_{L^2} d\tau \]
\[ + C \int_0^t e^{-c(t-\tau)} \left( \| \partial_x^k |U|^2 (\tau) \|_{L^2} + \| \partial_x^{k+1} |U|^2 (\tau) \|_{L^2} \right) d\tau \]
\[ \leq CX^2 (t) \int_0^t (1 + t - \tau)^{-1} (1 + \tau)^{- n (\frac{1}{p} - \frac{1}{2}) - \frac{n - 1}{4}} d\tau \]
\[ + CX^2 (t) \int_0^t e^{-c(t-\tau)} (1 + \tau)^{- \frac{3}{4} (\frac{1}{p} - \frac{1}{2}) - \frac{n - 1}{4}} d\tau \]
\[ + CX^2 (t) \int_0^t e^{-c(t-\tau)} (1 + \tau)^{- \frac{3}{4} (\frac{1}{p} - \frac{1}{2}) - \frac{n - 1}{4}} d\tau \]
\[ \leq CX^2 (t) (1 + t)^{- \frac{3}{4} (\frac{1}{p} - \frac{1}{2}) - \frac{n - 1}{4}}. \]

where we have used (59).

We apply \( \partial_x^k (\frac{n+1}{2}) \leq k \leq s - 2 \) to the second equality in (46) and take the \( L^2 \) norm, it gives

\[ \| \partial_x^k \Omega (t) \|_{L^2} \leq \| \partial_x^k \mathcal{G}_{21} (t) * \sigma_0 \|_{L^2} + \| \partial_x^k \mathcal{G}_{22} (t) * \Omega_0 \|_{L^2} \]
\[ + \int_0^t \| \partial_x^k \mathcal{G}_{22} (t - \tau) \Lambda^{-1} \nabla \cdot (f - \nabla \cdot (\sigma E))(\tau) \|_{L^2} d\tau \]
\[ =: J_1 + J_2 + J_3. \]
Making use of (39) with $j = 0$, $l = 1$ to $J_1$, it yields

$$J_1 \leq C(1 + t)^{-\frac{5}{2}(\frac{1}{2} - \frac{1}{2})} \|\partial_x \phi\|_{L^p} + C e^{-ct} \|\partial_x^k \phi\|_{L^2}$$

$$\leq C(1 + t)^{-\frac{5}{2}(\frac{1}{2} - \frac{1}{2}) - \frac{n+1}{4}} E_0.$$  (63)

We obtain from (40) with $j = 0$, $l = 0$

$$J_2 \leq C(1 + t)^{-\frac{5}{2}(\frac{1}{2} - \frac{1}{2})} \|\Omega_0\|_{L^p} + C e^{-ct} \|\partial_x^k \Omega_0\|_{L^2}$$

$$\leq C(1 + t)^{-\frac{5}{2}(\frac{1}{2} - \frac{1}{2}) - \frac{n+1}{4}} E_0.$$  (64)

Similar to $I_3$, we write $J_3 = J_{31} + J_{32}$. We estimate the term $J_{31}$ by using (40) with $j = 0$, $l = 0$, (59) and $p \in [1, \frac{2n}{n+2})$

$$J_{31} \leq C \int_0^T (1 + t - \tau)^{-\frac{5}{2}(\frac{1}{2} - \frac{1}{2})} \|U_\tau^2(\tau)\|_{L^p} d\tau$$

$$+ C \int_0^T e^{-c(t-\tau)} \|\partial_x^{k+1} U^2(\tau)\|_{L^2} + \|\partial_x^{k+2} U^2(\tau)\|_{L^2} d\tau$$

$$\leq C X^2(t) \int_0^T (1 + t - \tau)^{-\frac{5}{2}(\frac{1}{2} - \frac{1}{2}) - \frac{k+1}{4}} (1 + \tau)^{-n(\frac{1}{2} - \frac{1}{2}) - \frac{(n+1)(p-1)}{2p}} d\tau$$

$$\leq C X^2(t)(1 + t)^{-\frac{5}{2}(\frac{1}{2} - \frac{1}{2}) - \frac{n+1}{4}}.$$  (65)

By applying (40) with $p = 2$, $j = k$, $l = 0$ and (59) to the term $J_{32}$, we arrive at

$$J_{32} \leq C \int_0^T (1 + t - \tau)^{-\frac{5}{2}} \|\partial_x^k U^2(\tau)\|_{L^2} d\tau$$

$$+ C \int_0^T e^{-c(t-\tau)} \|\partial_x^{k+1} U^2(\tau)\|_{L^2} + \|\partial_x^{k+2} U^2(\tau)\|_{L^2} d\tau$$

$$\leq C X^2(t) \int_0^T (1 + t - \tau)^{-\frac{5}{2}} (1 + \tau)^{-n(\frac{1}{2} - \frac{1}{2}) - \frac{n+1}{4}} d\tau$$

$$\leq C X^2(t)(1 + t)^{-\frac{5}{2}(\frac{1}{2} - \frac{1}{2}) - \frac{n+1}{4}}.$$  (66)

We apply $\partial_x^k (\frac{n+1}{2}) \leq k \leq s - 2$ to the third equality in (66) and take the $L^2$ norm, it gives

$$\|\partial_x^k \Gamma(t)\|_{L^2} \leq \|\partial_x^k \mathcal{G}_{33}(t) * \Gamma_0\|_{L^2} + \|\partial_x^k \mathcal{G}_{34}(t) * (E_0 - E_0^T)\|_{L^2}$$

$$+ \int_0^t \|\partial_x^k \mathcal{G}_{33}(t - \tau) * \Lambda^{-1} (\mathcal{L} + V \times f)(\tau)\|_{L^2} d\tau$$

$$+ \int_0^t \|\partial_x^k \mathcal{G}_{34}(t - \tau) * (g - g^T)(\tau)\|_{L^2} d\tau$$

$$= H_1 + H_2 + H_3 + H_4.$$  (67)
Making use of (41) with \( j = 0, \ l = 0 \) to \( H_1 \), it yields

\[
H_1 \leq C(1 + t)^{-\frac{\alpha}{2} \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{1}{2}} \| \Gamma_0 \|_{L^p} + C e^{-ct} \| \partial_x^k \Gamma_0 \|_{L^2} \\
\leq C(1 + t)^{-\frac{\alpha}{2} \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{n - 1}{2p}} E_0.
\]

(68)

We obtain from (42) with \( j = 0, \ l = 1 \)

\[
H_2 \leq C(1 + t)^{-\frac{\alpha}{2} \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{1}{2}} \| E_0 - E_0^T \|_{L^p} + C e^{-ct} \| \partial_x^k (E_0 - E_0^T) \|_{L^2} \\
\leq C(1 + t)^{-\frac{\alpha}{2} \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{n - 1}{2p}} E_0.
\]

(69)

Similarly, we write \( H_3 = H_{31} + H_{32} \). We estimate the term \( H_{31} \) by using (41) with \( j = 0, \ l = 0, (59) \) and \( p \in [1, \frac{2n}{n+2}] \)

\[
H_{31} \leq C \int_0^\tau (1 + t - \tau)^{-\frac{\alpha}{2} \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{k+1}{2}} || U \|_{L^p}^2 \| d\tau \\
+ C \int_0^\tau (1 + t - \tau)^{-\frac{\alpha}{2} \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{1}{2}} || \nabla U \|_{L^p}^2 \| d\tau \\
+ C \int_0^\tau e^{-c(1 - \tau)} (|| \partial_x^{k+1} U \|_{L^p}^2 + || \partial_x^{k+2} U \|_{L^p}^2) d\tau \\
+ C \int_0^\tau e^{-c(1 - \tau)} || \partial_x^{k} (\nabla U) \|_{L^2} \| d\tau \\
\leq CX^2(t) \int_0^\tau (1 + t - \tau)^{-\frac{\alpha}{2} \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{k+1}{2}} (1 + \tau)^{-n \left( \frac{1}{2} - \frac{1}{q} \right) - \frac{(n-1)(p-1)}{2p}} d\tau
\]

(70)

\[
+ CX^2(t) \int_0^\tau (1 + t - \tau)^{-\frac{\alpha}{2} \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{k+1}{2}} (1 + \tau)^{-n \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{(n-1)(p-1)}{2p}} d\tau
\]

\[
+ CX^2(t) \int_0^\tau e^{-c(1 - \tau)} (1 + t - \tau)^{-\frac{\alpha}{2} \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{k+1}{2}} d\tau
\]

\[
+ CX^2(t) \int_0^\tau e^{-c(1 - \tau)} (1 + t - \tau)^{-\frac{\alpha}{2} \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{k+1}{2}} d\tau
\]

\[
\leq CX^2(t)(1 + t)^{-\frac{\alpha}{2} \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{k+1}{2}}.
\]

Here we have used

\[
|| \nabla U \|_{L^p} \leq || U \|_{L^\frac{2p}{2p-2}} || \nabla U \|_{L^2} \leq C || U \|_{L^\infty}^{\frac{2p-2}{2p}} || U \|_{L^2} \| \nabla U \|_{L^2} \leq CX^2(t)(1 + t)^{-n \left( \frac{1}{2} - \frac{1}{q} \right) - \frac{(n-1)(p-1)}{2p}}
\]

(71)

and

\[
|| \partial_x^{k} (\nabla U) \|_{L^2} \leq C (|| U \|_{L^\infty} \| \partial_x^{k+1} U \|_{L^2} + || \nabla U \|_{L^\infty} \| \partial_x^{k} U \|_{L^2})
\]

\[
\leq CX^2(t) \left( (1 + t)^{-\frac{\alpha}{2} \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{n - 1}{2p}} + (1 + t)^{-\frac{\alpha}{2} \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{n - 1}{2p}} \right).
\]

(72)
By applying (41) with \( p = 2, l = 0 \) and (59) to the term \( H_{32} \), we arrive at

\[
H_{32} \leq C \int_0^t (1 + t - \tau)^{-\frac{1}{2}} \| \partial_x^k |U|^2(\tau) \|_{L^2} d\tau \\
+ C \int_0^t (1 + t - \tau)^{-\frac{2}{2} + \frac{1}{2} - \frac{k}{2}} \| U \nabla U(\tau) \|_{L^p} d\tau \\
+ C \int_0^t e^{-c(t-\tau)} (\| \partial_x^{k+1} |U|^2(\tau) \|_{L^2} + \| \partial_x^{k+2} |U|^2(\tau) \|_{L^2}) d\tau \\
+ C \int_0^t e^{-c(t-\tau)} \| \partial_x^k (U \nabla U)(\tau) \|_{L^2} d\tau \\
\leq C X^2(t) \int_0^t (1 + t - \tau)^{-\frac{1}{2}} (1 + \tau)^{-n(\frac{1}{2} - \frac{1}{2}) - \frac{n-1}{4p}} d\tau \\
+ C X^2(t) \int_0^t (1 + t - \tau)^{-\frac{2}{2} + \frac{1}{2} - \frac{k}{2}} (1 + \tau)^{-n(\frac{1}{2} - \frac{1}{2}) - \frac{n-1}{4p}} d\tau \\
+ C X^2(t) \int_0^t e^{-c(t-\tau)} (1 + \tau)^{-\frac{2}{2} + \frac{1}{2} - \frac{k}{2}} d\tau \\
+ C X^2(t) \int_0^t e^{-c(t-\tau)} (1 + \tau)^{-n(\frac{1}{2} - \frac{1}{2}) - \frac{n-1}{4p}} d\tau \\
\leq C X^2(t)(1 + t)^{-\frac{2}{2} + \frac{1}{2} - \frac{k}{2}}.
\] (73)

We also write \( H_4 = H_{41} + H_{42} \). We estimate the term \( H_{41} \) by using (42) with \( j = 0, l = 0, (59) \) and \( p \in [1, \frac{2n}{n+2}] \)

\[
H_{41} \leq C \int_0^t (1 + t - \tau)^{-\frac{2}{2} + \frac{1}{2} - \frac{k}{2}} \| U \nabla U(\tau) \|_{L^p} d\tau \\
+ C \int_0^t (1 + t - \tau)^{-\frac{2}{2} + \frac{1}{2} - \frac{k}{2}} \| \partial_x^k (U \nabla U)(\tau) \|_{L^2} d\tau \\
+ C \int_0^t e^{-c(t-\tau)} (\| \partial_x^{k+1} |U|^2(\tau) \|_{L^2} + \| \partial_x^{k+2} |U|^2(\tau) \|_{L^2}) d\tau \\
\leq C X^2(t) \int_0^t (1 + t - \tau)^{-\frac{2}{2} + \frac{1}{2} - \frac{k}{2}} (1 + \tau)^{-n(\frac{1}{2} - \frac{1}{2}) - \frac{n-1}{4p}} d\tau \\
+ C X^2(t) \int_0^t e^{-c(t-\tau)} (1 + \tau)^{-\frac{2}{2} + \frac{1}{2} - \frac{k}{2}} d\tau \\
+ C X^2(t) \int_0^t e^{-c(t-\tau)} (1 + \tau)^{-n(\frac{1}{2} - \frac{1}{2}) - \frac{n-1}{4p}} d\tau \\
\leq C X^2(t)(1 + t)^{-\frac{2}{2} + \frac{1}{2} - \frac{k}{2}}.
\] (74)
By applying (42) and (59) to the term $H_{42}$, we obtain

\[
H_{42} \leq C \int_{\frac{t}{2}}^{t} (1 + t - \tau)^{-\frac{1}{2}} \| \partial_x^k |U|^2(\tau) \|_{L^2} d\tau \\
+ C \int_{\frac{t}{2}}^{t} (1 + t - \tau)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}} \| \nabla U(\tau) \|_{L^p} d\tau \\
+ C \int_{\frac{t}{2}}^{t} e^{-c(t-\tau)} \| \partial_x^k |U|^2(\tau) \|_{L^2} d\tau + C \int_{\frac{t}{2}}^{t} e^{-c(t-\tau)} \| \partial_x^k (U \nabla U)(\tau) \|_{L^2} d\tau \\
\leq CX^2(t) \int_{\frac{t}{2}}^{t} (1 + t - \tau)^{-\frac{1}{2}} (1 + \tau)^{-\frac{n}{2} - \frac{1}{2}} \frac{n-1}{2} - \frac{1}{2} d\tau
\]

\[
(75)
\]

We apply $\partial_x^k (\frac{n}{2} + 1 \leq k \leq s - 2)$ to the last equality in (46) and take the $L^2$ norm, it gives

\[
\| \partial_x^k (E - E^T)(t) \|_{L^2} \leq \| \partial_x^k \mathcal{G}_{43}(t) \ast \Gamma_0 \|_{L^2} + \| \partial_x^k \mathcal{G}_{44}(t) \ast (E_0 - E_0^T) \|_{L^2}
\\
+ \int_{0}^{1} \| \partial_x^k \mathcal{G}_{43}(t - \tau) \ast \Lambda^{-1}(\mathcal{L} + \nabla \times f)(\tau) \|_{L^2} d\tau
\\
+ \int_{0}^{1} \| \partial_x^k \mathcal{G}_{44}(t - \tau) \ast (g - g^T)(\tau) \|_{L^2} d\tau
\\
=: + K_1 + K_2 + K_3 + K_4.
\]

Making use of (43) with $j = 0, l = 1$ to $K_1$, it yields

\[
K_1 \leq C (1 + t)^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{k}{2}} \| \Gamma_0 \|_{L^p} + C e^{-ct} \| \partial_x^k \Gamma_0 \|_{L^2}
\\
\leq C (1 + t)^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{k}{2}} E_0.
\]

\[
(77)
\]

We obtain from (44) with $j = 0, l = 0$

\[
K_2 \leq C (1 + t)^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{k}{2}} \| E_0 - E_0^T \|_{L^p} + C e^{-ct} \| \partial_x^k (E_0 - E_0^T) \|_{L^2}
\\
\leq C (1 + t)^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{k}{2}} E_0.
\]

\[
(78)
\]
We also write $K_3$ into two parts and writing $K_3 = K_{31} + K_{32}$. We estimate the term $K_{31}$ by using (43) with $j = 0, l = 0, (59), (71), (72)$ and $p \in [1, \frac{2n}{n+2})$

\[
K_{31} \leq C \int_0^t (1 + t - \tau)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{k+1}{2}} \|U|^2(\tau)\|_{L^p} d\tau \\
+ C \int_0^t (1 + t - \tau)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{k+1}{2}} \|U\nabla U(\tau)\|_{L^p} d\tau \\
+ C \int_0^t e^{-c(t-\tau)} \|\partial_x^k [U]^2(\tau)\|_{L^2} d\tau + C \int_0^t e^{-c(t-\tau)} \|\partial_x^k (U\nabla U)(\tau))\|_{L^2} d\tau \\
\leq CX^2(t) \int_0^t (1 + t - \tau)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{k+1}{2}} (1 + \tau)^{-(n(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2} - \frac{(n-1)(p-1)}{2p})} d\tau \\
+ CX^2(t) \int_0^t e^{-c(t-\tau)} (1 + \tau)^{-(n(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2} - \frac{(n-1)(p-1)}{2p})} d\tau \\
+ CX^2(t) \int_0^t e^{-c(t-\tau)} (1 + \tau)^{-(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}} d\tau \\
\leq CX^2(t)(1 + t)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{n+1}{2}} \quad (79)
\]

By applying (43) and (59), (71), (72) to the term $K_{32}$, we arrive at

\[
K_{32} \leq C \int_0^t (1 + t - \tau)^{-\frac{3}{2}} \|\partial_x^k [U]^2(\tau)\|_{L^2} d\tau \\
+ C \int_0^t (1 + t - \tau)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{k+1}{2}} \|U\nabla U(\tau)\|_{L^p} d\tau \\
+ C \int_0^t e^{-c(t-\tau)} \|\partial_x^k [U]^2(\tau)\|_{L^2} d\tau + C \int_0^t e^{-c(t-\tau)} \|\partial_x^k (U\nabla U)(\tau))\|_{L^2} d\tau \\
\leq CX^2(t) \int_0^t (1 + t - \tau)^{-\frac{3}{2}} (1 + \tau)^{-(n(\frac{1}{p} - \frac{1}{2}) - \frac{n+1}{2})} d\tau \\
+ CX^2(t) \int_0^t (1 + t - \tau)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{k+1}{2}} (1 + \tau)^{-(n(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2} - \frac{(n-1)(p-1)}{2p})} d\tau \\
+ CX^2(t) \int_0^t e^{-c(t-\tau)} (1 + \tau)^{-(n(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2} - \frac{(n-1)(p-1)}{2p})} d\tau \\
+ CX^2(t) \int_0^t e^{-c(t-\tau)} (1 + \tau)^{-(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}} d\tau \\
\leq CX^2(t)(1 + t)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{n+1}{2}} \quad (80)
\]

In the same way, we write $K_4$ into two parts and writing $K_4 = K_{41} + K_{42}$. We estimate the term $K_{41}$ by using (44) with $j = 0, l = 0, (59), (71), (72)$ and
\[ p \in [1, \frac{2n}{n+2}) \]

\[ K_{41} \leq C \int_{\frac{t}{2}}^{t} (1 + t - \tau)^{-\frac{2}{n}(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2} + \frac{1}{p} + \frac{1}{2}} \|U\|^2(\tau)\|_{L^p} d\tau + C \int_{\frac{t}{2}}^{t} (1 + t - \tau)^{-\frac{2}{n}(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2} + \frac{1}{p} + \frac{1}{2}} \|U\|^2(\tau)\|_{L^p} d\tau 
\]

By applying (44) with \( p = 2, j = 0, l = 0 \) and (59), (71), (72) to the term \( K_{42} \), we arrive at

\[ K_{42} \leq C \int_{\frac{t}{2}}^{t} (1 + t - \tau)^{-\frac{1}{2}(1 + \tau) - \frac{n}{2} + \frac{1}{2}} \|\partial_x^k U\|^2(\tau)\|_{L^2} d\tau 
\]

\[ + C \int_{\frac{t}{2}}^{t} (1 + t - \tau)^{-\frac{1}{2}(1 + \tau) - \frac{n}{2} + \frac{1}{2}} \|U\|_{L^p} d\tau + C \int_{\frac{t}{2}}^{t} e^{-c(t-\tau)} \|\partial_x^k (U \nabla U)\|_{L^2} d\tau + C \int_{\frac{t}{2}}^{t} e^{-c(t-\tau)} \|\partial_x^k (U \nabla U)\|_{L^2} d\tau 
\]

\[ \leq C X^2(t) \int_{\frac{t}{2}}^{t} (1 + t - \tau)^{-\frac{1}{2}(1 + \tau) - \frac{n}{2} + \frac{1}{2}} d\tau 
\]

\[ + C X^2(t) \int_{\frac{t}{2}}^{t} e^{-c(t-\tau)} (1 + \tau)^{-\frac{n}{2} - \frac{1}{2}} d\tau 
\]

\[ + C X^2(t) \int_{\frac{t}{2}}^{t} e^{-c(t-\tau)} (1 + \tau)^{-\frac{n}{2} - \frac{1}{2}} d\tau 
\]

\[ \leq C X^2(t) (1 + t)^{-\frac{1}{2}(1 + \tau) - \frac{n}{2} + \frac{1}{2}}. \]
In (54), we replace $k$ by $l(t < \frac{n-1}{2})$, it gives

\[
\|\partial_x^l \sigma(t)\|_{L^2} \leq \|\partial_x^l \mathcal{G}_1(t) \ast \sigma_0\|_{L^2} + \|\partial_x^l \mathcal{G}_2(t) \ast \Omega_0\|_{L^2} \\
+ \int_0^t \|\partial_x^l \mathcal{G}_3(t-\tau) \ast \Lambda^{-1} \nabla \cdot (f - \nabla \cdot (\sigma E))(\tau)\|_{L^2} d\tau
\]  

(83)

\[
=:L_1 + L_2 + L_3.
\]

For the term $L_1$, thanks to (37) with $j = 0$, $l = 0$, we have

\[
L_1 \leq C(1 + t)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{n-2}{2}} \|\mathcal{G}_1\|_{L^p} + C e^{-ct} \|\partial_x \mathcal{G}_1(\rho_0 - x_1)\|_{L^2} \\
\leq C(1 + t)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{n-2}{2}} E_0.
\]

(84)

By exploiting (38) with $j = 0$, $l = 1$, it yields

\[
L_2 \leq C(1 + t)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{n-2}{2}} \|\mathcal{G}_2\|_{L^p} + C e^{-ct} \|\partial_x \mathcal{G}_2(\Omega_0)\|_{L^2} \leq C(1 + t)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{n-2}{2}} E_0.
\]

(85)

As before, dividing $L$ into two parts and writing $L_3 = L_{31} + L_{32}$. By (38) with $j = 0$, $l = 0$ and noting $p \in [1, \frac{2n}{n+2}]$, it deduces that

\[
L_{31} \leq C \int_0^t (1 + t - \tau)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{n-2}{2}} \|U(\tau)\|_{L^p} d\tau \\
+ C \int_0^t e^{-c(t-\tau)} (\|\partial_x^l U(\tau)\|_{L^2} + \|\partial_x^{l+1} U(\tau)\|_{L^2}) d\tau \\
\leq C X^2(t) \int_0^t (1 + t - \tau)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{n-2}{2}} (1 + \tau)^{-n(\frac{1}{p} - \frac{1}{2}) - \frac{n-1}{4}} d\tau \\
+ C X^2(t) \int_0^t e^{-c(t-\tau)} (1 + \tau)^{-n(\frac{1}{p} - \frac{1}{2}) - \frac{n-1}{4}} d\tau \\
+ C X^2(t) \int_0^t e^{-c(t-\tau)} (1 + \tau)^{-n(\frac{1}{p} - \frac{1}{2}) - \frac{n-1}{4}} d\tau \\
\leq C X^2(t)(1 + t)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{n-2}{2}},
\]

where we have used

\[
\|U^2(\tau)\|_{L^p} \leq C \|U(\tau)\|_{L^p}^{\frac{2p-1}{p-1}} \|U(\tau)\|_{L^2}^{\frac{p-1}{p-1}} \\
\leq C X^2(t)(1 + \tau)^{-n(\frac{1}{p} - \frac{1}{2}) - \frac{n-1}{4}}(p-1),
\]

\[
\|\partial_x U^2(\tau)\|_{L^2} \leq C \|U(\tau)\|_{L^p} \|\partial_x U(\tau)\|_{L^2} \leq C X^2(t)(1 + \tau)^{-n(\frac{1}{p} - \frac{1}{2}) - \frac{n-1}{4}},
\]

\[
\|\partial_x^{l+1} U^2(\tau)\|_{L^2} \leq C \|U(\tau)\|_{L^p} \|\partial_x^{l+1} U(\tau)\|_{L^2} \leq C X^2(t)(1 + \tau)^{-n(\frac{1}{p} - \frac{1}{2}) - \frac{n-1}{4}}.
\]

(87)

Noting that $\nu + 1 \leq s - 2$, theses estimates may be derived from Lemma 4.1 and (57), (52).
In what follows, we estimate $L_{32}$. (38) with $p = 2$, $j = l$, $l = 0$ and (87) give
\[
L_{32} \leq C \int_{\frac{1}{2}}^{t} (1 + t - \tau)^{-1} \| \partial_x^2 \| \mathcal{U}^2(\tau) \|_{L^2} d\tau \\
+ C \int_{\frac{1}{2}}^{t} e^{-c(t - \tau)} \left( \| \partial_x^1 \| \mathcal{U}^2(\tau) \|_{L^2} + \| \partial_x^2 \| \mathcal{U}^2(\tau) \|_{L^2} \right) d\tau \\
\leq CX^2(t) \int_{\frac{1}{2}}^{t} (1 + t - \tau)^{-1} (1 + \tau)^{-n(\frac{1}{2} - \frac{1}{2} - \frac{n}{2} - \frac{1}{2} - \frac{n}{2})} d\tau \\
+ CX^2(t) \int_{\frac{1}{2}}^{t} e^{-c(t - \tau)} (1 + \tau)^{-n(\frac{1}{2} - \frac{1}{2} - \frac{n}{2} - \frac{1}{2} - \frac{n}{2})} d\tau \\
+ CX^2(t) \int_{\frac{1}{2}}^{t} e^{-c(t - \tau)} (1 + \tau)^{-n(\frac{1}{2} - \frac{1}{2} - \frac{n}{2} - \frac{1}{2} - \frac{n}{2})} d\tau \\
\leq CX^2(t) (1 + t)^{-\frac{n}{2} (\frac{1}{2} - \frac{1}{2} - \frac{n}{2} - \frac{1}{2} - \frac{n}{2})}.
\]
Replacing $k$ by $i \leq \frac{n-1}{2}$ in (62), we have
\[
\| \partial_x^2 \mathcal{O}(t) \|_{L^2} \leq \| \partial_x^2 \mathcal{G}_1(t) * \mathcal{G}_0 \|_{L^2} + \| \partial_x^2 \mathcal{G}_2(t) * m_0 \|_{L^2} \\
+ \int_{0}^{t} \| \partial_x^2 \mathcal{G}_2(t - \tau) * \Lambda^{-1} \nabla \cdot (f - \nabla \cdot (\sigma E))(\tau) \|_{L^2} d\tau \\
= M_1 + M_2 + M_3.
\]
We apply (39) with $j = 0$, $l = 1$ to $M_1$, it yields
\[
M_1 \leq C(1 + t)^{-\frac{n}{2} (\frac{1}{2} - \frac{1}{2} - \frac{n}{2} - \frac{1}{2} - \frac{n}{2})} \| \rho_0 - x_1 \|_{L^p} + C e^{-c t} \| \partial_x^2 \mathcal{O}_1(t) \|_{L^2} \\
\leq C(1 + t)^{-\frac{n}{2} (\frac{1}{2} - \frac{1}{2} - \frac{n}{2} - \frac{1}{2} - \frac{n}{2})} E_0.
\]
By using (40) with $j = 0$, $l = 0$ to $M_2$, we deduce that
\[
M_2 \leq C(1 + t)^{-\frac{n}{2} (\frac{1}{2} - \frac{1}{2} - \frac{n}{2} - \frac{1}{2} - \frac{n}{2})} \| \mathcal{O}_0 \|_{L^p} + C e^{-c t} \| \partial_x^2 \mathcal{O}_0 \|_{L^2} \leq C(1 + t)^{-\frac{n}{2} (\frac{1}{2} - \frac{1}{2} - \frac{n}{2} - \frac{1}{2} - \frac{n}{2})} E_0.
\]
We write $M_3$ as $M_3 = M_{31} + M_{32}$. From (40) with $j = 0$, $l = 0$ and $p \in [1, \frac{2n}{n+2})$, we have
\[
M_{31} \leq C \int_{0}^{\frac{1}{2}} (1 + t - \tau)^{-\frac{n}{2} (\frac{1}{2} - \frac{1}{2} - \frac{n}{2} - \frac{1}{2} - \frac{n}{2})} \| \mathcal{U}^2(\tau) \|_{L^2} d\tau \\
+ C \int_{0}^{\frac{1}{2}} e^{-c(t - \tau)} \left( \| \partial_x^1 \mathcal{U}^2(\tau) \|_{L^2} + \| \partial_x^2 \mathcal{U}^2(\tau) \|_{L^2} \right) d\tau \\
\leq CX^2(t) \int_{0}^{\frac{1}{2}} (1 + t - \tau)^{-\frac{n}{2} (\frac{1}{2} - \frac{1}{2} - \frac{n}{2} - \frac{1}{2} - \frac{n}{2})} (1 + \tau)^{-n(\frac{1}{2} - \frac{1}{2} - \frac{n}{2} - \frac{1}{2} - \frac{n}{2})} d\tau \\
+ CX^2(t) \int_{0}^{\frac{1}{2}} e^{-c(t - \tau)} (1 + \tau)^{-n(\frac{1}{2} - \frac{1}{2} - \frac{n}{2} - \frac{1}{2} - \frac{n}{2})} d\tau \\
+ CX^2(t) \int_{0}^{\frac{1}{2}} e^{-c(t - \tau)} (1 + \tau)^{-\frac{n}{2} (\frac{1}{2} - \frac{1}{2} - \frac{n}{2} - \frac{1}{2} - \frac{n}{2})} d\tau \\
\leq CX^2(t) (1 + t)^{-\frac{n}{2} (\frac{1}{2} - \frac{1}{2} - \frac{n}{2} - \frac{1}{2} - \frac{n}{2})}.
\]
where we have used the first and third inequalities in 87 and
\[
\| \partial_x^2 \mathcal{U}^2(\tau) \|_{L^2} \leq C \| \mathcal{U}(\tau) \|_{L^\infty} \| \partial_x^2 \mathcal{U}(\tau) \|_{L^2} \leq CX^2(t) (1 + \tau)^{-\frac{n}{2} (\frac{1}{2} - \frac{1}{2} - \frac{n}{2} - \frac{1}{2} - \frac{n}{2})}.
\]
For the term $M_{32}$, we apply (40) with $p = 2, j = \iota, l = 0$ and (87), (93). This yields

$$M_{32} \leq C \int_t^{t_2} (1 + t - \tau)^{-\frac{1}{2}} \| \partial_x^l |U|^2(\tau) \|_{L^2} d\tau$$

$$+ C \int_t^{t_2} e^{-c(t-\tau)}(\| \partial_x^{l+1} |U|^2(\tau) \|_{L^2} + \| \partial_x^{l+2} |U|^2(\tau) \|_{L^2}) d\tau$$

$$\leq C X^2(t) \int_t^{t_2} (1 + t - \tau)^{-\frac{1}{2}} (1 + \tau)^{-n(\frac{1}{p} - \frac{1}{2}) - \frac{n+1}{4} - \frac{\iota}{2}} d\tau$$

$$+ C X^2(t) \int_t^{t_2} e^{-c(t-\tau)} (1 + \tau)^{-n(\frac{1}{p} - \frac{1}{2}) - \frac{n+1}{4} - \frac{\iota}{2}} d\tau$$

$$+ C X^2(t) \int_t^{t_2} e^{-c(t-\tau)} (1 + \tau)^{-\frac{1}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{n+1}{4} - \frac{\iota}{2}} d\tau$$

$$\leq C X^2(t) (1 + t)^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{\iota}{2}}. \quad (94)$$

We apply $\partial_x^l (t \leq \frac{n-1}{2})$ to the third equality in (46) and take the $L^2$ norm, it gives

$$\| \partial_x^l \Gamma(t) \|_{L^2} \leq \| \partial_x^l \mathcal{G}_{33}(t) * \Gamma_0 \|_{L^2} + \| \partial_x^l \mathcal{G}_{34}(t) * (E_0 - E_0^T) \|_{L^2}$$

$$+ \int_0^t \| \partial_x^l \mathcal{G}_{33}(t - \tau) * \mathcal{L} + \nabla \times f \|_{L^2} d\tau$$

$$+ \int_0^t \| \partial_x^l \mathcal{G}_{34}(t - \tau) * (g - g^T) \|_{L^2} d\tau +$$

$$= : N_1 + N_2 + N_3 + N_4. \quad (95)$$

Making use of (41) with $j = 0, l = 0$ to $K_1$, it yields

$$N_1 \leq C (1 + t)^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{\iota}{2}} E_0. \quad (96)$$

We obtain from (42) with $j = 0, l = 1$

$$N_2 \leq C (1 + t)^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{\iota}{2}} E_0. \quad (97)$$
We also write \( N_3 = N_{31} + N_{32} \). We estimate the term \( N_{31} \) by using (41) with \( j = 0, l = 0, (71), (72), (87), (93) \) and \( p \in \left[ 1, \frac{2n}{n+2} \right) \):

\[
N_{31} \leq C \int_0^\frac{t}{2} (1 + t - \tau)^{-\frac{n}{2} \left( \frac{1}{p} - \frac{1}{4} \right)} \frac{1}{\| U^2(\tau) \|_{L^p}} d\tau
\]

\[
+ C \int_0^\frac{t}{2} (1 + t - \tau)^{-\frac{n}{2} \left( \frac{1}{p} - \frac{1}{4} \right)} \| U \nabla U(\tau) \|_{L^p} d\tau
\]

\[
+ C \int_0^\frac{t}{2} e^{-c(t-\tau)} \left( \| \partial_x^{p+1} U^2(\tau) \|_{L^2} + \| \partial_x^{p+2} U^2(\tau) \|_{L^2} \right) d\tau
\]

\[
+ C \int_0^\frac{t}{2} e^{-c(t-\tau)} \| \partial_x(U \nabla U)(\tau) \|_{L^2} d\tau
\]

\[
\leq C X^2(t) \int_0^\frac{t}{2} (1 + t - \tau)^{-\frac{n}{2} \left( \frac{1}{p} - \frac{1}{4} \right)} \frac{1}{\| U \nabla U(\tau) \|_{L^p}} d\tau
\]

\[
+ C X^2(t) \int_0^\frac{t}{2} e^{-c(t-\tau)} \left( \| \partial_x^{p+1} U^2(\tau) \|_{L^2} + \| \partial_x^{p+2} U^2(\tau) \|_{L^2} \right) d\tau
\]

\[
+ C X^2(t) \int_0^\frac{t}{2} e^{-c(t-\tau)} \| \partial_x(U \nabla U)(\tau) \|_{L^2} d\tau
\]

\[
\leq C X^2(t) \int_0^\frac{t}{2} (1 + t - \tau)^{-\frac{n}{2} \left( \frac{1}{p} - \frac{1}{4} \right) - \frac{1}{2}} d\tau
\]  \( \text{(98)} \)

By applying (41) and (71), (72), (87), (93) to the term \( J_{32} \), we arrive at

\[
N_{32} \leq C \int_\frac{t}{2}^t (1 + t - \tau)^{-\frac{n}{2} \left( \frac{1}{p} - \frac{1}{4} \right) - \frac{1}{2}} \| \partial_x U^2(\tau) \|_{L^p} d\tau
\]

\[
+ C \int_\frac{t}{2}^t (1 + t - \tau)^{-\frac{n}{2} \left( \frac{1}{p} - \frac{1}{4} \right) - \frac{1}{2}} \| U \nabla U(\tau) \|_{L^p} d\tau
\]

\[
+ C \int_\frac{t}{2}^t e^{-c(t-\tau)} \left( \| \partial_x^{p+1} U^2(\tau) \|_{L^2} + \| \partial_x^{p+2} U^2(\tau) \|_{L^2} \right) d\tau
\]

\[
+ C \int_\frac{t}{2}^t e^{-c(t-\tau)} \| \partial_x(U \nabla U)(\tau) \|_{L^2} d\tau
\]

\[
\leq C X^2(t) \int_\frac{t}{2}^t (1 + t - \tau)^{-\frac{n}{2} \left( \frac{1}{p} - \frac{1}{4} \right) - \frac{1}{2}} d\tau
\]  \( \text{(99)} \)
Similarly, we divide $N_4$ into two parts and writing $N_4 = N_{41} + N_{42}$. We estimate the term $N_{41}$ by using (42) with $j = 0$, $l = 0$, (71), (72), (87) and $p \in [1, \frac{2n}{n+2})$

\[
N_{41} \leq C \int_0^{\frac{t}{2}} (1 + t - \tau)^{-\frac{n}{2} \left( \frac{1}{p} - \frac{1}{2} \right)} \left\| |U|^2(\tau) \right\|_{L^p} d\tau
+ C \int_0^{\frac{t}{2}} (1 + t - \tau)^{-\frac{n}{2} \left( \frac{1}{p} - \frac{1}{2} \right)} \left\| U \nabla U(\tau) \right\|_{L^p} d\tau
+ C \int_0^{\frac{t}{2}} e^{-c(t-\tau)} \left\| \partial_x^2 |U|^2(\tau) \right\|_{L^2} d\tau + C \int_0^{\frac{t}{2}} e^{-c(t-\tau)} \left\| \partial_x^2 (U \nabla U)(\tau) \right\|_{L^2} d\tau
\leq CX^2(t) \int_0^{\frac{t}{2}} (1 + t - \tau)^{-\frac{n}{2} \left( \frac{1}{p} - \frac{1}{2} \right)} \left( 1 + \tau \right)^{-n(\frac{1}{p} - \frac{1}{2}) - \frac{2}{p} - \frac{n-1}{2}} d\tau
+ CX^2(t) \int_0^{\frac{t}{2}} e^{-c(t-\tau)} (1 + \tau)^{-n(\frac{1}{p} - \frac{1}{2}) - \frac{2}{p} - \frac{n-1}{2} d\tau
+ CX^2(t) \int_0^{\frac{t}{2}} e^{-c(t-\tau)} (1 + \tau)^{-\frac{n}{2} \left( \frac{1}{p} - \frac{1}{2} \right)} d\tau
\leq CX^2(t)(1 + t)^{-\frac{n}{2} \left( \frac{1}{p} - \frac{1}{2} \right)}
\]

By applying (42), (71), (72), (87) to the term $K_{32}$, we arrive at

\[
N_{42} \leq C \int_0^{\frac{t}{2}} (1 + t - \tau)^{-\frac{n}{2} \left( \frac{1}{p} - \frac{1}{2} \right)} \left\| \partial_x^2 |U|^2(\tau) \right\|_{L^2} d\tau
+ C \int_0^{\frac{t}{2}} (1 + t - \tau)^{-\frac{n}{2} \left( \frac{1}{p} - \frac{1}{2} \right)} \left\| U \nabla U(\tau) \right\|_{L^p} d\tau
+ C \int_0^{\frac{t}{2}} e^{-c(t-\tau)} \left\| \partial_x^2 |U|^2(\tau) \right\|_{L^2} d\tau + C \int_0^{\frac{t}{2}} e^{-c(t-\tau)} \left\| \partial_x^2 (U \nabla U)(\tau) \right\|_{L^2} d\tau
\leq CX^2(t) \int_0^{\frac{t}{2}} (1 + t - \tau)^{-\frac{n}{2} \left( \frac{1}{p} - \frac{1}{2} \right)} \left( 1 + \tau \right)^{-n(\frac{1}{p} - \frac{1}{2}) - \frac{2}{p} - \frac{n-1}{2}} d\tau
+ CX^2(t) \int_0^{\frac{t}{2}} e^{-c(t-\tau)} (1 + \tau)^{-n(\frac{1}{p} - \frac{1}{2}) - \frac{2}{p} - \frac{n-1}{2} d\tau
+ CX^2(t) \int_0^{\frac{t}{2}} e^{-c(t-\tau)} (1 + \tau)^{-\frac{n}{2} \left( \frac{1}{p} - \frac{1}{2} \right)} d\tau
\leq CX^2(t)(1 + t)^{-\frac{n}{2} \left( \frac{1}{p} - \frac{1}{2} \right)}
\]
We apply $\partial_x^k (t \leq \frac{n-1}{2})$ to the last equality in (46) and take the $L^2$ norm, it gives

$$\|\partial_x^k (E - E^T) (t)\|_{L^2} \leq \|\partial_x^k \mathcal{G}_{43} (t) * \Gamma_0 \|_{L^2} + \|\partial_x^k \mathcal{G}_{44} (t) * (E_0 - E_0^T)\|_{L^2}$$

$$+ \int_0^t \|\partial_x^k \mathcal{G}_{43} (t - \tau) * \Lambda^{-1} (\mathcal{L} + \nabla \times f) (\tau)\|_{L^2} d\tau$$

$$+ \int_0^t \|\partial_x^k \mathcal{G}_{44} (t - \tau) * (g - g^T) (\tau)\|_{L^2} d\tau +$$

$$=: O_1 + O_2 + O_3 + O_4.$$  \(\tag{102}\)

Making use of (43) with $j = 0, l = 1$ to $O_1$, it yields

$$O_1 \leq C (1 + t)^{-\frac{\alpha}{2} - \frac{1}{2}} E_0. \quad \text{(103)}$$

We obtain from (44) with $j = 0, l = 0$

$$O_2 \leq C (1 + t)^{-\frac{\alpha}{2} - \frac{1}{2}} E_0. \quad \text{(104)}$$

As before, $O_3$ may be written as $O_3 = O_{31} + O_{32}$. We estimate the term $O_{31}$ by using (43) with $j = 0, l = 0, (71), (72), (87)$ and $p \in [1, \frac{2n}{n+2}]$

$$O_{31} \leq C \int_0^t \frac{1}{2} (1 + t - \tau)^{-\frac{\alpha}{2} - \frac{1}{2} - \frac{n+1}{p}} \|U \|_{L^p} \| \mathcal{U} \|_{L^p} d\tau$$

$$+ C \int_0^t \frac{1}{2} (1 + t - \tau)^{-\frac{\alpha}{2} - \frac{1}{2} - \frac{n+1}{p}} \| U \nabla U \|_{L^p} d\tau$$

$$+ C \int_0^t e^{-c (t - \tau)} (\| \partial_x^k U \|_{L^2} + \| \partial_x^{k+1} U \|_{L^2}) d\tau$$

$$+ C \int_0^t e^{-c (t - \tau)} \| \partial_x^k (U \nabla U) \|_{L^2} d\tau$$

$$\leq C X^2 (t) \int_0^t \frac{1}{2} (1 + t - \tau)^{-\frac{\alpha}{2} - \frac{1}{2} - \frac{n+1}{p}} (1 + \tau)^{-n \left(\frac{1}{2} - \frac{1}{p}\right)} d\tau$$

$$+ C X^2 (t) \int_0^t \frac{1}{2} (1 + t - \tau)^{-\frac{\alpha}{2} - \frac{1}{2} - \frac{n+1}{p}} (1 + \tau)^{-n \left(\frac{1}{2} - \frac{1}{p}\right) - \frac{n+1}{2p}} d\tau$$

$$+ C X^2 (t) \int_0^t e^{-c (t - \tau)} \left\{ (1 + \tau)^{-n \left(\frac{1}{2} - \frac{1}{p}\right) - \frac{n+1}{2p}} + (1 + \tau)^{-n \left(\frac{1}{2} - \frac{1}{p}\right) - \frac{n+1}{2p} - \frac{1}{p}} \right\} d\tau$$

$$+ C X^2 (t) \int_0^t e^{-c (t - \tau)} (1 + \tau)^{-\frac{\alpha}{2} - \frac{1}{2} - \frac{n+1}{p}} d\tau$$

$$\leq C X^2 (t) (1 + t)^{-\frac{\alpha}{2} - \frac{1}{2} - \frac{n+1}{p}} \quad \text{(105)}$$
By applying (43), (71), (72), (87) to the term $O_{32}$, we arrive at

$$O_{32} \leq C \int_{\frac{t}{2}}^{t} (1 + t - \tau)^{-\frac{3}{4}} \| \partial_x^4 |U|^2(\tau) \|_{L^2} d\tau$$

$$+ C \int_{\frac{t}{2}}^{t} (1 + t - \tau)^{-\frac{3}{4} (\frac{1}{p} - \frac{1}{2})} \| U \nabla U(\tau) \|_{L^p} d\tau$$

$$+ C \int_{\frac{t}{2}}^{t} e^{-c(t-\tau)} \left( \| \partial_x^4 |U|^2(\tau) \|_{L^2} + \| \partial_x^{l+1} |U|^2(\tau) \|_{L^2} \right) d\tau$$

$$+ C \int_{\frac{t}{2}}^{t} e^{-c(t-\tau)} \| \partial_x^3 (U \nabla U)(\tau) \|_{L^2} d\tau$$

$$\leq C X^2(t) \int_{\frac{t}{2}}^{t} (1 + t - \tau)^{-\frac{3}{4} (1 + \tau)^{-n(\frac{1}{p} - \frac{1}{2}) - \frac{n-1}{2p}} - \frac{n-1}{2} d\tau$$

$$+ C X^2(t) \int_{\frac{t}{2}}^{t} e^{-c(t-\tau)} \left( \| \partial_x^{l+1} |U|^2(\tau) \|_{L^2} + \| \partial_x^{l+1} (U \nabla U)(\tau) \|_{L^2} \right) d\tau$$

$$\leq C X^2(t)(1 + t)^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{3}{4}}.$$

(106)

Likewise, we also write $O_{41} = O_{411} + O_{412}$. We estimate the term $O_{41}$ by using (44) with $j = 0$, $l = 0$, (71), (72), (87) and $p \in [1, \frac{2n}{n+2})$

$$O_{41} \leq C \int_{0}^{\frac{t}{2}} (1 + t - \tau)^{-\frac{3}{4} (\frac{1}{p} - \frac{1}{2})} \| |U|^2(\tau) \|_{L^p} d\tau$$

$$+ C \int_{0}^{\frac{t}{2}} (1 + t - \tau)^{-\frac{3}{4} (\frac{1}{p} - \frac{1}{2})} \| U \nabla U(\tau) \|_{L^p} d\tau$$

$$+ C \int_{0}^{\frac{t}{2}} e^{-c(t-\tau)} \| \partial_x^{l+1} |U|^2(\tau) \|_{L^2} d\tau + C \int_{0}^{\frac{t}{2}} e^{-c(t-\tau)} \| \partial_x^{l+1} (U \nabla U)(\tau) \|_{L^2} d\tau$$

$$\leq C X^2(t) \int_{0}^{\frac{t}{2}} (1 + t - \tau)^{-\frac{3}{4} (\frac{1}{p} - \frac{1}{2})} \| |U|^2(\tau) \|_{L^p} d\tau$$

$$+ C X^2(t) \int_{0}^{\frac{t}{2}} e^{-c(t-\tau)} \| \partial_x^{l+1} |U|^2(\tau) \|_{L^2} d\tau$$

$$+ C X^2(t) \int_{0}^{\frac{t}{2}} e^{-c(t-\tau)} \| \partial_x^{l+1} (U \nabla U)(\tau) \|_{L^2} d\tau$$

$$\leq C X^2(t)(1 + t)^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{3}{4}}.$$

(107)
By applying (44) with $p = 2, j = 0, l = 0$ and (71), (72), (87) to the term $O_{42}$, we arrive at

$$O_{42} \leq C \int_{\tau}^{t}(1 + t - \tau)^{-\frac{1}{2}} \|\partial_x^k U\|^2(\tau)\|_{L^2}d\tau$$

$$+ C \int_{\tau}^{t}(1 + t - \tau)^{-\frac{1}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}}\|U\nabla U(\tau)\|_{L^p}d\tau$$

$$+ C \int_{\tau}^{t}e^{c(t-\tau)}\|\partial_x^k U|U|^2(\tau)\|_{L^2}d\tau + C \int_{\tau}^{t}e^{c(t-\tau)}\|\partial_x^k (U\nabla U)(\tau)\|_{L^2}d\tau$$

$$\leq CX^2(t) \int_{\tau}^{t}(1 + t - \tau)^{-\frac{1}{2}}(1 + \tau)^{-n(\frac{1}{p} - \frac{1}{2}) - \frac{n-1}{2}}\|\partial_x^k U\|^2(\tau)\|_{L^2}d\tau$$

$$+ CX^2(t) \int_{\tau}^{t}(1 + t - \tau)^{-\frac{1}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}}(1 + \tau)^{-n(\frac{1}{p} - \frac{1}{2}) - \frac{n-1}{2}}\|\partial_x^k (U\nabla U)(\tau)\|_{L^2}d\tau$$

$$+ CX^2(t) \int_{\tau}^{t}e^{c(t-\tau)(1 + \tau)^{-n(\frac{1}{p} - \frac{1}{2}) - \frac{n-1}{2}}}d\tau$$

$$+ CX^2(t) \int_{\tau}^{t}e^{c(t-\tau)}(1 + \tau)^{-\frac{1}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}}d\tau$$

$$\leq CX^2(t)(1 + t)^{-\frac{1}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}}. \quad (108)$$

Finally, we note

$$\|\partial_x^k \nabla \Phi(t)\|_{L^2} = \|\partial_x^{k-1} \Phi(t)\|_{L^2}. \quad (109)$$

On one hand, noting that

$$m = - \Lambda^{-1} \nabla \Omega + \Lambda^{-1} \nabla \Gamma,$$

then

$$\|\partial_x^k m(t)\|_{L^2} \leq \|\partial_x^k \Omega(t)\|_{L^2} + \|\partial_x^k \Gamma(t)\|_{L^2}. \quad (110)$$

We arrive at from (8)

$$\|\Lambda^{-1} \nabla \times E(t)\|_{L^2} \leq C \|\Lambda^{-1} (E \nabla E)(t)\|_{L^2} \leq C \|E \nabla E(t)\|_{L^2}$$

$$\leq C \|E(t)\|_{L^\infty} \|\nabla E(t)\|_{L^\frac{2p}{p+2}}$$

$$\leq CX^2(t)(1 + t)^{-\frac{1}{2}(\frac{1}{p} - \frac{1}{2})}. \quad (111)$$

It follows from (11) and (103)-(108) that

$$\|\Lambda^{-1} \nabla \cdot E(t)\|_{L^2} \leq C \|\Lambda^{-1} \nabla \cdot E(t)\|_{L^2} + C \|\sigma E(t)\|_{L^2} + \|\Lambda^{-2} \mathcal{L}\|_{L^2}$$

$$\leq CE_0 + CX^2(t)(1 + t)^{-\frac{1}{2}(\frac{1}{p} - \frac{1}{2})}.$$
integrating with respect to \( x \), using integration by parts, we have

\[
\frac{1}{2} \frac{d}{dt} (\| \sigma(t) \|^2_{L^2} + \| m(t) \|^2_{L^2} + \| E(t) \|^2_{L^2} + \| \nabla \Phi(t) \|^2_{L^2} + \mu_1 \| \nabla m \|^2_{L^2} + \mu_2 \| \nabla \cdot m \|^2_{L^2})
\]

\[
= \int_{\mathbb{R}^n} (f \cdot m + g \cdot E) dx.
\]

(112)

Thanks to integration by parts and (57), we arrive at

\[
\int_{\mathbb{R}^n} (f \cdot m + g \cdot E) dx \leq C(1 + t)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{n-1}{4}} X(t) \| \sigma \|, \ m, \ E, \ \nabla \Phi, \nabla m \|_{L^2}^2
\]

(113)

Substituting (113) into (112) yields

\[
\frac{1}{2} \frac{d}{dt} (\| \sigma(t) \|^2_{L^2} + \| m(t) \|^2_{L^2} + \| \nabla \Phi(t) \|^2_{L^2} + \mu_1 \| \nabla m \|^2_{L^2} + \mu_2 \| \nabla \cdot m \|^2_{L^2})
\]

\[
\leq C(1 + t)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{n-1}{4}} X(t) \| \sigma \|, \ m, \ E, \ \nabla \Phi, \nabla m \|_{L^2}^2.
\]

(114)

Similarly, we obtain

\[
\frac{1}{2} \frac{d}{dt} (\| \nabla^k \sigma(t) \|^2_{L^2} + \| \nabla^k m(t) \|^2_{L^2} + \| \nabla^{k+1} \Phi(t) \|^2_{L^2} + \mu_1 \| \nabla^{k+1} m \|^2_{L^2} + \mu_2 \| \nabla \cdot \nabla^{k} m \|^2_{L^2})
\]

\[
\leq C(1 + t)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{n-1}{4}} X(t) \| \nabla^k \sigma \|, \ \nabla^k m, \ \nabla^k E, \ \nabla^{k+1} \Phi, \nabla^{k+1} m \|^2_{L^2}
\]

(115)

for \( k = 0, \ldots, s \).

We claim that for any \( t \in [0, T] \), it holds

\[
X(t) \leq CE_0,
\]

(116)

provided \( E_0 \) is small enough.

Let

\[
Y(t) = \| \sigma(t) \|^2_{H^k} + \| m(t) \|^2_{H^k} + \| E(t) \|^2_{H^k} + \| \nabla \Phi(t) \|^2_{H^k}, \ k = 0, \cdots, s.
\]

(117)

From (115) - (117), we infer that

\[
\frac{d}{dt} Y(t) + \| \nabla m \|^2_{H^k} + \| \nabla \cdot m \|^2_{H^k} \leq C(1 + t)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{n-1}{4}} X(t) Y(t).
\]

Gronwall inequality gives

\[
Y(t) \leq (\| \sigma_0 \|^2_{H^k} + \| m_0 \|^2_{H^k} + \| E_0 \|^2_{H^k} + \| \nabla \Phi(0) \|^2_{H^k}) C^2 \int_0^t (1 + \tau)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{n-1}{4}} X(\tau) d\tau.
\]

(118)

Noting that \( \frac{3}{2}(\frac{1}{p} - \frac{1}{2}) + \frac{n-1}{4} > 1 \), then from (118) and the a priori assumption (116) we have

\[
Y(t) \leq CE_0.
\]

(119)

Combining (54)-(56), (60)-(70), (73)-(86) and (88)-(92), (94)-(111), (119), we arrive at

\[
X(t) \leq CE_0 + C X^2(t),
\]

(120)

from which we can deduce \( X(t) \leq CE_0 \), provided that \( E_0 \) is suitably small. Thus, we complete the proof of the claim (116). So, by the local existence and the closure of the a priori estimate, the global existence of smooth solution to the problem (2), (3) follows from the standard continuity argument. Meanwhile, we also prove the decay estimate (49) and (50). Therefore, Theorem 4.2 is proved.
5. Asymptotic behavior of solutions in odd space dimensions. The purpose of this section is to prove global existence and asymptotic decay of solutions to the initial value problem (2), (3) in even space dimensions. We state our result as follows:

**Theorem 5.1.** Let \( n \geq 4 \) be an even integer and \( s \geq \frac{n}{2} + 3 \). Let \( p \in [1, \frac{2n}{n+2}) \) Suppose that \( (\rho_0 - x_1, F_0 - I, m_0) \in L^p, (\partial x_1(\rho_0 - x_1), F_0 - I, m_0) \in H^s \) and put \( E_1 = \| (\rho_0 - x_1, F_0 - I, m_0) \|_{L^p} + \| (\partial x_1(\rho_0 - x_1), F_0 - I, m_0) \|_{H^s} \). Then there is a positive constant \( \delta_1 \) such that if \( E_1 \leq \delta_1 \), then the problem (2), (3) has a unique global solution \((\rho - 1, m, F - I, \nabla \Phi)\) with \((\rho - 1, m, F - I, \nabla \Phi) \in C^0([0, +\infty); H^s \times H^s \times H^s \times H^{s+1})\). For \( t \leq \frac{n+2}{2} \), the solution verifies the decay estimates

\[
\| \partial_x^j (\rho - 1)(t) \|_{L^2} \leq CE_1(1 + t)^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{n+2}{2}}, \tag{121}
\]

\[
\| \partial_x^j (m, F - I, \nabla \Phi)(t) \|_{L^2} \leq CE_1(1 + t)^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{n+2}{2}}. \tag{122}
\]

Moreover, for \( \frac{n}{2} \leq k \leq s - 2 \), we have

\[
\| \partial_x^k (\rho - 1, m, F - I, \nabla \Phi)(t) \|_{L^2} \leq CE_1(1 + t)^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{n+2}{2}}. \tag{123}
\]

**Remark 2.** Under the same assumptions of Theorem 5.1, for \( 2 \leq q \leq n \), by Gagliardo-Nirenberg inequality, \( L^q \) decay estimate

\[
\| (\rho - 1)(t) \|_{L^q} \leq CE_1(1 + t)^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{n+2}{2}}, \tag{124}
\]

\[
\| m, F - I, \nabla \Phi(t) \|_{L^q} \leq CE_1(1 + t)^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{n+2}{2}}. \tag{125}
\]

**Proof.** We may extend local solutions to global solutions and prove decay estimate of solutions to the problem (2), (3) by establishing the uniform a priori estimates. To do so, we define the following norm

\[
X(t) = \sum \sup \{ (1 + \tau)^{\frac{n+2}{2}\left(\frac{1}{p} - \frac{1}{2}\right) + \frac{n+2}{2}\| \partial_x^2 \sigma(\tau) \|_{L^2} + (1 + \tau)^{\frac{n+2}{2}\left(\frac{1}{p} - \frac{1}{2}\right) + \frac{n+2}{2}\| \partial_x^2 (m, E, \nabla \Phi)(\tau) \|_{L^2} + \sum \sup (1 + \tau)^{\frac{n+2}{2}\left(\frac{1}{p} - \frac{1}{2}\right) + \frac{n+2}{2}\| \partial_x^2(\sigma, m, E, \nabla \Phi)(\tau) \|_{L^2} + \sum \sup (1 + \tau)^{\frac{n+2}{2}\left(\frac{1}{p} - \frac{1}{2}\right) + \frac{n+2}{2}\| \partial_x^2(\sigma, m, E, \nabla \Phi)(\tau) \|_{L^2}.
\]

The proof of Theorem 5.1 is similar to the proof of Theorem 4.2, it only need replace (57) by the following Gagliardo-Nirenberg inequalities

\[
\| \sigma(\tau) \|_{L^\infty} \leq C \| \partial_x^{\frac{n+2}{2}} \sigma(\tau) \|_{L^2} + \| \partial_x^{\frac{n+2}{2}} \sigma(\tau) \|_{L^2} \leq CX(t)(1 + \tau)^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{n+2}{2}}, \tag{126}
\]

\[
\| m(\tau) \|_{L^\infty} \leq C \| \partial_x^{\frac{n+2}{2}} m(\tau) \|_{L^2} + \| \partial_x^{\frac{n+2}{2}} m(\tau) \|_{L^2} \leq CX(t)(1 + \tau)^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{n+2}{2}}, \tag{127}
\]

\[
\| E(\tau) \|_{L^\infty} \leq C \| \partial_x^{\frac{n+2}{2}} m(\tau) \|_{L^2} + \| \partial_x^{\frac{n+2}{2}} m(\tau) \|_{L^2} \leq CX(t)(1 + \tau)^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{n+2}{2}}, \tag{128}
\]

\[
\| \nabla \Phi(\tau) \|_{L^\infty} \leq C \| \partial_x^{\frac{n+2}{2}} \nabla \Phi(\tau) \|_{L^2} + \| \partial_x^{\frac{n+2}{2}} \nabla \Phi(\tau) \|_{L^2} \leq CX(t)(1 + \tau)^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{n+2}{2}}. \tag{129}
\]

We omit the details. The proof is completed. \(\square\)
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