UNRAVELING SPONTANEOUSLY BROKEN
ABELIAN CHERN-SIMONS THEORIES∗

Mark de Wild Propitius†
Laboratoire de Physique Théorique et Haute Energies‡
Université Pierre et Marie Curie - PARIS VI
Université Denis Diderot - PARIS VII
4 place Jussieu, Boîte 126, Tour 16, 1er étage
F-75252 Paris CEDEX 05, France

August 1996

In this talk I describe recent work (hep-th/9606029) in which I classified all conceivable
2+1 dimensional Chern-Simons (CS) theories with continuous compact abelian gauge
group or finite abelian gauge group. The CS theories with finite abelian gauge group that
can be obtained from the spontaneous breakdown of a CS theory with gauge group the
direct product of various compact $U(1)$ gauge groups were also identified. Those that can
not be reached in this way are actually the most interesting since they lead to nonabelian
phenomena such as nonabelian braid statistics, Alice fluxes and Cheshire charges and
quite generally lead to dualities with 2+1 dimensional theories with a nonabelian finite
gauge group.

1. Introduction

Ever since the pioneering work of Schonfeld and Deser et al. 1, 2+1 dimensional
Chern-Simons (CS) theories have received quite some attention. The motivations
to study these theories range from applications in knot theory to applications in
condensed matter systems such as fractional quantum Hall liquids. In this talk, I
will discuss the implications of adding a CS term to 2+1 dimensional gauge theories
spontaneously broken down to a finite residual gauge group by means of the Higgs
mechanism. That is, the focus is on models governed by an action of the form

$$S = S_{YMH} + S_{\text{matter}} + S_{CS},$$

where the Yang–Mills–Higgs action $S_{YMH}$ describes the spontaneous breakdown of
some continuous compact gauge group $G$ to a finite subgroup $H$, $S_{\text{matter}}$ a conserved
matter current coupled to the gauge fields and $S_{CS}$ denotes the CS action.

The so-called discrete $H$ gauge theories describing the long distance physics of
the models (1) without CS term have been studied in 2+1 and 3+1 dimensional
space time and are by now completely understood. For a recent review and detailed

∗Based on a talk given at ‘The workshop on low dimensional field theory’, Telluride, Colorado,
USA, August 5 to 17, 1996.
†e-mail: mdwp@lpthe.jussieu.fr
‡Laboratoire associé No. 280 au CNRS
references, see Ref. 2. To sketch the main results, the spectrum features topological defects which in 2+1 dimensional space time appear as vortices carrying magnetic flux labeled by the elements of $H$. If $H$ is nonabelian, the vortices exhibit a non-abelian Aharonov-Bohm (AB) effect: upon braiding two vortices their fluxes affect each other through conjugation. The residual global gauge group $H$ also acts on the fluxes through conjugation, so the different magnetic vortices are labeled by the conjugacy classes of $H$. This is in a nutshell the physics described by the Yang-Mills Higgs part $S_{\text{YM}}$ of the action (1). The matter fields coupled to the gauge fields in $S_{\text{matter}}$ form multiplets which transform irreducibly under $G$. In the broken phase these branch to irreducible representations of the residual gauge group $H$. So the matter fields introduce point charges in the broken phase labeled by the unitary irreducible representations (UIR's) $\Gamma$ of $H$. If such a charge encircles a magnetic flux $h \in H$, it also undergoes an AB effect: it returns transformed by the matrix $\Gamma(h)$. Since all gauge fields are massive, the foregoing AB effects form the only long range interactions among the charges and vortices. The complete spectrum also features dyons obtained by composing the vortices and charges. These are labeled by the conjugacy classes of $H$ paired with a nontrivial centralizer representation $\rho$.

A breakthrough in the understanding of these models was the observation that this spectrum of charges, vortices and dyons together with the spin, braiding and fusion properties of these particles is, in fact, fully described by the representation theory of the quasitriangular Hopf algebra $D(H)$ resulting from Drinfeld’s double construction applied to the algebra $F(H)$ of functions on $H$.

As has been argued in Ref. 5, the presence of a CS term $S_{CS}$ for the broken gauge group $G$ in the action (1) gives rise to additional AB interactions among the vortices which are completely encoded in a 3-cocycle $\omega \in H^3(H, U(1))$ for the residual finite gauge group $H$. The related algebraic structure is the quasi-Hopf algebra $D^\omega(H)$ being a deformation of $D(H)$ by this 3-cocycle $\omega$. In Ref. 5, these general results were just explicitly illustrated by the the abelian CS Higgs model in which the (compact) gauge group $G \simeq U(1)^k$ is broken down to a cyclic subgroup $H \simeq \mathbb{Z}_N^k$. Here, I will summarize the results of my recent paper 6 in which this analysis was extended to spontaneously broken abelian CS theories in full generality. I will be rather burlesque concerning references. For civilized referencing, the reader has to consult Ref. 6. As for conventions, natural units in which $\hbar = c = 1$ are employed throughout. I will exclusively work in 2+1 dimensional Minkowsky space with signature $(+,-,-)$. Spatial coordinates are denoted by $x^1$ and $x^2$ and the time coordinate by $x^0$. Greek indices run from 0 to 2, whereas spatial components are labeled by latin indices $\in 1,2$.

2. The models, their spectrum and the AB interactions

Let us concentrate on the subset of models realizing symmetry breaking schemes $G \simeq U(1)^k \rightarrow H$ with $U(1)^k$ the direct product of $k$ compact $U(1)$ gauge groups and the finite subgroup $H \simeq \mathbb{Z}_{N^{(1)}} \times \cdots \times \mathbb{Z}_{N^{(k)}}$ a direct product of $k$ cyclic groups $\mathbb{Z}_{N^{(i)}}$ of order $N^{(i)}$. So, the Yang–Mills–Higgs part of the action (1) contains $k$
complex scalar Higgs fields $\Phi^{(i)}$ (with $i \in 1, 2, \ldots, k$) carrying charge $N^{(i)}e^{(i)}$ with $e^{(i)}$ the coupling constant for the $i$th compact $U(1)$ gauge field $A^{(i)}_\kappa$, i.e.

$$S_{\text{YMH}} = \int d^3x \left( \sum_{i=1}^k \left( -\frac{1}{4} F^{(i)\kappa\nu} F^{(i)\kappa\nu}_{\kappa\nu} + (D^{\kappa} \Phi^{(i)})^* D^\kappa \Phi^{(i)} - V(|\Phi^{(i)}|) \right) \right),$$  \hspace{1cm} (2)

with $D^\kappa \Phi^{(i)} = (\partial_\kappa + iN^{(i)}e^{(i)}A^{(i)}_\kappa) \Phi^{(i)}$ and $F^{(i)\kappa\nu} = \partial_\kappa A^{(i)}_\nu - \partial_\nu A^{(i)}_\kappa$. All $U(1)$ gauge groups are assumed to be broken down at the same energy scale $M_H = v\sqrt{2\lambda}$. Hence, $V(|\Phi^{(i)}|) = \frac{\lambda}{4}(|\Phi^{(i)}|^2 - v^2)^2$ with $\lambda, v > 0$. In the matter part of (3), we then have $k$ conserved matter currents $j^{(i)}$ coupled to the gauge fields

$$S_{\text{matter}} = \int d^3x \left( -\sum_{i=1}^k j^{(i)\kappa} A^{(i)}_\kappa \right).$$  \hspace{1cm} (3)

The matter charges $q^{(i)}$ introduced by the current $j^{(i)}$ are supposed to be multiples of $e^{(i)}$. Finally, the most general CS action for this theory is of the form

$$S_{\text{CS}} = \int d^3x \left( \sum_{1 \leq i < j \leq k} \mu^{(i)} \frac{\mu^{(ij)}}{2} \epsilon^{\kappa\rho\sigma} A^{(i)}_\kappa \partial_\rho A^{(j)}_\sigma + \mu^{(ij)} \frac{\mu^{(ij)}}{2} \epsilon^{\kappa\rho\sigma} A^{(i)}_\kappa \partial_\rho A^{(j)}_\sigma \right),$$  \hspace{1cm} (4)

with $\mu^{(i)}$ and $\mu^{(ij)}$ the topological masses and $\epsilon^{\kappa\rho\sigma}$ the three dimensional antisymmetric Levi-Civita tensor normalized such that $\epsilon^{012} = 1$. Hence, there are $k$ distinct CS terms $(i)$ describing self couplings of the $U(1)$ gauge fields. In addition, there are $\frac{k(k-1)}{2}$ distinct CS terms $(ij)$ establishing pairwise couplings between different $U(1)$ gauge fields. Note that by a partial integration a CS term $(ij)$ becomes a term $(ji)$ becomes a term $(ji)$, so these terms are equivalent.

Let us also assume that this theory features a family of Dirac monopoles for each compact $U(1)$ gauge group. That is, the spectrum of Dirac monopoles consists of the magnetic charges $g^{(i)} = \frac{2\pi m^{(i)}}{e^{(i)}}$ with $m^{(i)} \in \mathbb{Z}$ for $1 \leq i \leq k$. In this 2+1 dimensional Minkowski setting these monopoles are instantons tunneling between states with flux difference $\Delta \phi^{(i)} = \Delta \int d^2x \epsilon^{kl} \partial_k A^{(i)}_l = \frac{2\pi m^{(i)}}{e^{(i)}}$. It can be shown that a consistent implementation of these monopoles requires that the topological masses in (4) are quantized as

$$\mu^{(i)} = \frac{p^{(i)}e^{(i)}e^{(i)}}{\pi} \quad \text{and} \quad \mu^{(ij)} = \frac{p^{(ij)}e^{(i)}e^{(j)}}{\pi} \quad \text{with} \quad p^{(i)}, p^{(ij)} \in \mathbb{Z}. \hspace{1cm} (5)$$

It is known that in contrast to ordinary 2+1 dimensional QED, the presence of Dirac monopoles in these massive gauge theories does not lead to confinement of the matter charges $g^{(i)}$.

The spectrum of the theory defined by (1) with (2)–(4) contains $k$ different quantized matter charges $q^{(i)} = n^{(i)}e^{(i)}$ with $n^{(i)} \in \mathbb{Z}$, $k$ different vortex species carrying quantized magnetic flux $\phi^{(i)} = \frac{2\pi a^{(i)}}{N^{(i)}e^{(i)}}$ with $a^{(i)} \in \mathbb{Z}$ and dyonic combinations of these charges and vortices. Since all gauge fields are massive, there are no
long range Coulomb interactions between these particles. The remaining long range interactions are AB interactions. As has been explained in \[\Phi\] a counterclockwise monodromy of a vortex $\phi^{(i)}$ and a charge $q^{(i)}$ gives rise to the AB phase $\exp(\mu^{(i)}\phi^{(i)})$ in the wave function. The crucial point was that the Higgs mechanism replaces the fluxes attached to the charges $q^{(i)}$ in the unbroken CS phase \[\Phi\] by screening charges which screen the Coulomb fields around the charges but do not couple to the AB interactions. Hence, contrary to the unbroken CS phase there are no AB interactions among the charges in the CS Higgs phase. Instead, the CS term \(i\) in \[\Phi\] now implies the AB phase $\exp(\mu^{(i)}\phi^{(i)}\phi^{(i)'}\phi^{(j)})$ for a counterclockwise monodromy of two remote vortices $\phi^{(i)}$ and $\phi^{(i)'}$, whereas a CS term \((ij)\) gives rise to the AB phase $\exp(\mu^{(i)}\phi^{(i)}\phi^{(j)})$ for a counterclockwise monodromy of two remote vortices $\phi^{(i)}$ and $\phi^{(j)}$. Let us label the particles in this theory as $(A, n^{(1)} \ldots n^{(k)})$ with $A := (a^{(1)}, \ldots, a^{(k)})$ and $a^{(i)}, n^{(i)} \in \mathbb{Z}$. Upon implementing \[\Phi\], the foregoing AB interactions can then be recapitulated as

$$\mathcal{R}^2 \frac{A^{(1)}}{n^{(1)} \ldots n^{(k)}} \frac{A'}{n^{(1)'} \ldots n^{(k)'}} = \varepsilon_A(A') \Gamma^{n^{(1)} \ldots n^{(k)}}(A') \varepsilon_A(A) \Gamma^{n^{(1)'} \ldots n^{(k)'}}(A).$$

The indices attached to the monodromy operator $\mathcal{R}^2$ express the fact that it acts on the particles $(A, n^{(1)} \ldots n^{(k)})$ and $(A', n^{(1)'} \ldots n^{(k)'})$, whereas

$$\varepsilon_A(A') := \exp \left( \sum_{1 \leq i < j \leq k} \frac{2\pi i \mu^{(i)}}{N^{(i)} N^{(j)}} a^{(i)} a^{(i)'} + \frac{2\pi i \mu^{(j)}}{N^{(i)} N^{(j)}} a^{(i)} a^{(j)'} \right),$$

and $\Gamma^{n^{(1)} \ldots n^{(k)}}(A) := \exp \left( \sum_{i=1}^{k} \frac{2\pi i s^{(i)}}{N^{(i)}} a^{(i)} \right)$. It can also be shown that the particles in this theory satisfy the canonical spin-statistics connection:

$$\exp\left(i \theta_{(A, n^{(1)} \ldots n^{(k)})}\right) = \exp\left(2\pi i s_{(A, n^{(1)} \ldots n^{(k)})}\right) = \varepsilon_A(A) \Gamma^{n^{(1)} \ldots n^{(k)}}(A),$$

with $\exp(i \theta_{(A, n^{(1)} \ldots n^{(k)})})$ the quantum statistics phase resulting from a counterclockwise braid operation $\mathcal{R}$ on two identical particles $(A, n^{(1)} \ldots n^{(k)})$ and $s_{(A, n^{(1)} \ldots n^{(k)})}$ the spin assigned to these particles. Under the remaining (long range) AB interactions \[\Phi\] and \[\Phi\], the charge labels $n^{(i)}$ clearly become $\mathbb{Z}_{N^{(i)}}$ quantum numbers. Also, in the presence of the aforementioned Dirac monopoles the fluxes $a^{(i)}$ are conserved modulo $N^{(i)}$. Specifically, the tunneling events induced by the minimal monopoles read

$$\text{monopole } (i): \left\{ \begin{array}{c}
a^{(i)} \\ n^{(i)}
\end{array} \right\} \rightarrow \left\{ \begin{array}{c}
a^{(i)} - N^{(i)} \\ n^{(i)} + 2p^{(i)}
\end{array} \right\},$$

Hence, the decay of an unstable flux $a^{(i)}$ through a monopole $(i)$ is accompanied by the creation of matter charges of species and strength depending on the CS parameters \[\Phi\]. It is easily verified that these local tunneling events are invisible to the long range AB interactions \[\Phi\] and that the particles connected by the monopoles have the same spin factor \[\Phi\]. So the spectrum of this theory compactifies to
Let \( A, N \in \mathbb{N} \) with \( N \) defined by \( p \) and \( A \). Moreover, it can be shown that the CS parameters \( p^{(i)} \) and \( p^{(ij)} \) become periodic with period \( N^{(i)} \) and greatest common divisor \( \gcd(N^{(i)}, N^{(j)}) \) of \( N^{(i)} \) and \( N^{(j)} \) respectively. That is, up to a relabeling of the dyons, the broken CS theory defined by \( p^{(i)} \) and \( p^{(ij)} \) describes the same spectrum and AB interactions as that defined by \( p^{(i)} + N^{(i)} \) and \( p^{(ij)} + \gcd(N^{(i)}, N^{(j)}) \). Finally, note that the additional AB phases among the vortices and the twisted tunneling properties of the monopoles form the only distinction with the abelian discrete gauge theory describing the long distance physics in the absence of the CS action. As will be explained in the following sections, this distinction is completely encoded in a 3-cocycle for the residual gauge group \( \mathbb{Z}_{N^{(i)}}, \ldots, \mathbb{Z}_{N^{(k)}} \).

3. Group cohomology and symmetry breaking

A deep result due to Dijkgraaf and Witten states that the CS actions \( S_{\text{CS}} \) for a compact gauge group \( G \) are in one–one correspondence with the elements of the cohomology group \( H^3(BG, \mathbb{Z}) \) of the classifying space \( BG \) with integer coefficients \( \mathbb{Z} \). In particular, this classification includes the case of finite groups \( H \). The isomorphism \( H^4(BH, \mathbb{Z}) \cong H^3(H, U(1)) \) which is only valid for finite \( H \) then implies that the different CS theories for a finite gauge group \( H \) correspond to the different elements \( \omega \in H^3(H, U(1)) \), i.e. algebraic 3-cocycles \( \omega \) taking values in \( U(1) \).

One of the new observations of was that the long distance physics of the spontaneously broken model is described by a CS theory with finite gauge group \( H \) and 3-cocycle \( \omega \in H^3(H, U(1)) \) determined by the original CS action \( S_{\text{CS}} \in H^4(BG, \mathbb{Z}) \) for the broken gauge group \( G \) by the natural homomorphism

\[
H^4(BG, \mathbb{Z}) \rightarrow H^3(H, U(1)),
\]

induced by the inclusion \( H \subset G \). The physical picture behind this homomorphism, also known as the restriction, is that the CS term \( S_{\text{CS}} \) gives rise to additional AB interactions among the magnetic vortices which are completely encoded in the 3-cocycle \( \omega \) for the finite residual gauge group \( H \) being the image of \( S_{\text{CS}} \) under the homomorphism.

Let me illustrate these general remarks with the abelian example of the previous section where \( G \cong U(1)^k \) and \( H \cong \mathbb{Z}_{N^{(1)}} \times \cdots \times \mathbb{Z}_{N^{(k)}} \). A simple calculation shows \( H^4(B(U(1)^k), \mathbb{Z}) \cong \mathbb{Z}^{k+1} \). Note that this classification of the CS actions for the compact gauge group \( G \cong U(1)^k \) is indeed in agreement with \( \mathbb{Z}^{k+1} \), i.e. the integral CS parameters \( p^{(i)} \) and \( p^{(ij)} \) provide a complete labeling of the elements of \( H^4(B(U(1)^k), \mathbb{Z}) \). To proceed, it can be shown that for \( H \cong \mathbb{Z}_{N^{(1)}} \times \cdots \times \mathbb{Z}_{N^{(k)}} \)

\[
H^3(H, U(1)) \cong \bigoplus_{1 \leq i < j < l \leq k} \mathbb{Z}_{N^{(i)}} \oplus \mathbb{Z}_{\gcd(N^{(i)}, N^{(j)})} \oplus \mathbb{Z}_{\gcd(N^{(i)}, N^{(j)}, N^{(l)})}.
\]

Let \( A, B \) and \( C \) denote elements of \( \mathbb{Z}_{N^{(1)}} \times \cdots \times \mathbb{Z}_{N^{(k)}} \), so \( A := (a^{(1)}, a^{(2)}, \ldots, a^{(k)}) \) with \( a^{(i)} \in \mathbb{Z}_{N^{(i)}} \) for \( i = 1, \ldots, k \) and similar decompositions for \( B \) and \( C \). I
adopt the additive presentation, i.e. the elements $a^{(i)}$ of $\mathbb{Z}_{N^{(i)}}$ take values in the range $0, \ldots, N^{(i)} - 1$ and group multiplication is defined as: $A \cdot B = [A + B] := ([a^{(i)} + b^{(i)}], \ldots, [a^{(k)} + b^{(k)}])$. The rectangular brackets denote modulo $N^{(i)}$ calculus such that the sum always lies in the range $0, \ldots, N^{(i)} - 1$. The most general 3-cocycle for $\mathbb{Z}_{N^{(1)}} \times \cdots \times \mathbb{Z}_{N^{(k)}}$ can then be presented as some product of

$$
\omega^{(i)}(A, B, C) = \exp \left( \frac{2\pi i p^{(i)}}{N^{(i)}} a^{(i)} \left( b^{(i)} + c^{(i)} - [b^{(i)} + c^{(i)}] \right) \right) \quad (12)
$$

$$
\omega^{(ij)}(A, B, C) = \exp \left( \frac{2\pi i p^{(ij)}}{N^{(i)}N^{(j)}} a^{(i)} \left( b^{(j)} + c^{(j)} - [b^{(j)} + c^{(j)}] \right) \right) \quad (13)
$$

$$
\omega^{(ijkl)}(A, B, C) = \exp \left( \frac{2\pi i p^{(ijkl)}}{\gcd(N^{(i)}, N^{(j)}, N^{(l)})} a^{(i)} b^{(j)} c^{(l)} \right), \quad (14)
$$

with $1 \leq i < j < l \leq k$. The integral parameters $p^{(i)}$, $p^{(ij)}$ and $p^{(ijkl)}$ label the different elements of $\mathbb{Z}$. It can be verified that in agreement with (11) the functions (12), (13) and (14) are periodic in these parameters with period $N^{(i)}$, $\gcd(N^{(i)}, N^{(j)})$ and $\gcd(N^{(i)}, N^{(j)}, N^{(l)})$ respectively. It is also readily checked that these three functions and their products indeed satisfy the 3-cocycle relation

$$
\delta \omega(A, B, C, D) = \frac{\omega(A, B, C) \omega(A, B \cdot C, D) \omega(B, C, D)}{\omega(A, B, C, D)} = 1, \quad (15)
$$

where $\delta$ denotes the coboundary operator.

We are now ready to make the homomorphism (10) accompanying the spontaneous breakdown of the gauge group $U(1)^k$ to $\mathbb{Z}_{N^{(1)}} \times \cdots \times \mathbb{Z}_{N^{(k)}}$ explicit. In terms of the integral CS parameters (5), it takes the form

$$
H^3(BU(1)^k, \mathbb{Z}) \longrightarrow H^3(\mathbb{Z}_{N^{(1)}} \times \cdots \times \mathbb{Z}_{N^{(k)}}, U(1)) \quad (16)
$$

$$
p^{(i)} \longmapsto p^{(i)} \mod N^{(i)} \quad (17)
$$

$$
p^{(ij)} \longmapsto p^{(ij)} \mod \gcd(N^{(i)}, N^{(j)}) \quad (18)
$$

The periodic parameters being the images of this mapping label the 3-cocycles (12) and (13). So the long distance physics of the spontaneously broken $U(1)^k$ CS theory (4)–(5) is described by a $\mathbb{Z}_{N^{(1)}} \times \cdots \times \mathbb{Z}_{N^{(k)}}$ CS theory with 3-cocycle being the product $\omega = \prod_{1 \leq i < j \leq k} \omega^{(i)} \omega^{(ij)}$. That this 3-cocycle indeed leads to the additional AB phases (7) and the twisted tunneling properties (9) will become clear in the next section. Finally, note that the image of (16) does not contain the 3-cocycles (12). In other words, abelian discrete CS theories defined by these 3-cocycles can not be obtained from the spontaneous breakdown of a $U(1)^k$ CS theory.

4. The quasi-quantum double

The quasi-quantum double $D^\omega(H)$ related to a CS theory with finite abelian gauge group $H \simeq \mathbb{Z}_{N^{(1)}} \times \cdots \times \mathbb{Z}_{N^{(k)}}$ and some 3-cocycle $\omega$ is spanned by the elements $\{P_A B\}_{A,B \in H}$ representing a global symmetry transformation $B \in H$ followed by
the operator $P_A$ projecting out the magnetic flux $A \in H$. In this basis, multiplication, and comultiplication are defined as

$$P_A B \cdot P_D C = \delta_{A,D} P_A B \cdot C \ c_A(B, C)$$

$$\Delta(P_A B) = \sum_{C,D=A} P_C B \otimes P_D B \ c_B(C,D),$$

with $c_A(B, C) = \frac{\omega(A,B,C)\omega(B,C,A)}{\omega(B,A,C)}$ and $\delta_{A,B}$ the Kronecker delta. From (13) it follows that $c_A$ satisfies the 2-cocycle relation $\delta c_A(B, C, D) = \frac{c_A(B,C) c_A(C,D)}{c_B(B,C,D)c_A(C,D)} = 1$, which implies that the multiplication (19) is associative and that the comultiplication (20) is quasi-coassociative: $(id \otimes \Delta)\Delta(P_A B) = \varphi \cdot (\Delta \otimes id)\Delta(P_A B) \cdot \varphi^{-1}$ with $\varphi := \sum_{A,B,C} \omega^{-1}(A,B,C) P_A \otimes P_B \otimes P_C$.

The different particles in the associated CS theory are labeled by their magnetic flux $A \in H$ paired with a projective UIR $\alpha$ of $H$ defined as $\alpha(B) \cdot \alpha(C) = c_A(B,C) \alpha(B \cdot C)$. Each particle $(A, \alpha)$ is equipped with an internal Hilbert space $V^A_\alpha$ (spanned by the states $\{|A, \alpha v_j\}_{j=1,...,d_\alpha}$ with $\alpha v_j$ a basis vector and $d_\alpha$ the dimension of $\alpha$) carrying an irreducible representation $\Pi^A_\alpha$ of $D^\omega(H)$ given by

$$\Pi^A_\alpha(P_B C | A, \alpha v_j) = \delta_{A,B} | A, \alpha(C)_{\alpha v_j}.$$  

In the process of rotating a particle $(A, \alpha)$ over an angle of $2\pi$ its charge $\alpha$ is transported around the flux $A$ and as a result picks up a global transformation $\alpha(A)$. With (21) it is easily checked that this AB effect is implemented by the central element $\sum_B P_B B$. Schur’s lemma then implies: $\alpha(A) = e^{2\pi is_{(A,\alpha)}} 1_A$ with $s_{(A,\alpha)}$ the spin assigned to the particle $(A, \alpha)$ and $1_A$ the unit matrix.

The action (21) of $D^\omega(H)$ is extended to two-particle states by means of the comultiplication (20). Specifically, the representation $(\Pi^A_\alpha \otimes \Pi^B_\beta, V^A_\alpha \otimes V^B_\beta)$ of $D^\omega(H)$ related to a system of two particles $(A, \alpha)$ and $(B, \beta)$ is defined by the action $\Pi^A_\alpha \otimes \Pi^B_\beta (\Delta(P_A B))$. The tensor product representation of $D^\omega(H)$ related to a system of three particles $(A, \alpha), (B, \beta)$ and $(C, \gamma)$ may now be constructed through $(\Delta \otimes id)\Delta$ or through $(id \otimes \Delta)\Delta$. The aforementioned quasi-coassociativity relation implies that these two constructions are equivalent.

The braid operator $R$ establishing a counterclockwise interchange of two particles $(A, \alpha)$ and $(B, \beta)$ is defined as

$$R | A, \alpha v_j | B, \beta v_j \rangle = | B, \beta(A)_{\alpha v_j} | A, \alpha v_j \rangle.$$  

The tensor product representation $(\Pi^A_\alpha \otimes \Pi^B_\beta, V^A_\alpha \otimes V^B_\beta)$ in general decomposes into a direct sum of irreducible representations $(\Pi^C_\gamma, V^C_\gamma)$

$$\Pi^A_\alpha \otimes \Pi^B_\beta = \bigoplus_{C,\gamma} N^{AB\gamma}_{\alpha\beta C} \Pi^C_\gamma.$$  

This so-called fusion rule determines which particles $(C, \gamma)$ can be formed in the composition of two particles $(A, \alpha)$ and $(B, \beta)$. The modular matrices $S$ and $T$
associated to the fusion algebra \((23)\) are determined by the braid operator \((22)\) and the spin factors \(e^{2\pi i s(A,\alpha)} := \alpha(A)/d_{\alpha}\)

\[
S_{\alpha\beta} := \frac{1}{|H|} \text{tr} R^{-2}_{\alpha\beta} \quad \text{and} \quad T_{\alpha\beta} := \delta_{\alpha,\beta} \delta^{A,B} \exp\left(2\pi i s(A,\alpha)\right). \tag{24}
\]

\(|H|\) denotes the order of \(H\) and \(\text{tr}\) abbreviates trace. As usual, the multiplicities in \((23)\) can be expressed in terms of the matrix \(S\) by means of Verlinde’s formula:

\[
N_{\alpha\beta\gamma}^{AB} = \sum_{D,\delta} S_{\alpha\delta}^{AD} S_{\beta\delta}^{BD} (S^*)_{\gamma\delta}^{CD} S_{\delta\delta}^{eD}. \tag{25}
\]

Fig. 1. The diagrams in (a) and (b) (with the ribbons representing particle–trajectories) are homotopic. So the result of braiding a particle with two particles separately or with the composite that arises after fusion should be the same.

Fig. 2. The fact that the ribbon diagrams in (a) are homotopic indicates that the result of a counterclockwise monodromy of two particles in a given fusion channel followed by fusion of the pair should be the same as a clockwise rotation of the two particles separately followed by fusion of the pair and a counterclockwise rotation of the composite. The fact that the diagrams in (b) are homotopic implies that the effect of a counterclockwise interchange of two particles in two identical but separate particle/anti-particle pairs should be the same as a counterclockwise rotation of a (anti-)particle in one of these pairs.

It is impossible to do justice to the complete structure of \(D^\omega(H)\) in this limited number of pages. A detailed treatment can be found in [6]. Let me just flash some pictures giving an impression of some relations unmentioned so far. First of all, the comultiplication \((21)\), braid operator \((22)\) and associator \(\varphi\) obey the so-called quasi-triangularity equations expressing the compatibility of fusion and braiding depicted in Fig. 1. As an immediate consequence the braid operators satisfy the quasi–Yang–Baxter equation implying that the multi-particle internal Hilbert spaces carry a (possibly reducible) representation of the braid group. Since the braid operators \((22)\) are of finite order, the more accurate statement is that we are dealing with representations of truncated braid groups being factor groups of the braid group \(\mathbb{Z}_2\times\mathbb{Z}_6\). The quasi-triangularity equations also state that the action of the truncated braid
group commutes with the action of $D^\omega(H)$. Thus the multi-particle internal Hilbert spaces, in fact, carry a (possibly reducible) representation of the direct product of $D^\omega(H)$ and some truncated braid group. Further, to keep track of the writhing of the particle-trajectories and the resulting spin factors these are represented by ribbons. Passing from worldlines to ‘worldribbons’ can only be consistent if the demands in Fig. 2 are met. The consistency demand in Fig. 2(a) is met by the generalized spin-statistics connection $K^{ABC}_\alpha R^2 = e^{2\pi i(s(C,\gamma) - s(A,\omega) - s(\beta,\delta))} K^{AB\gamma}_\alpha$ with $K^{ABC}_\alpha$ the projection on the channel $(C,\gamma)$ in (23) and the demand in Fig. 2(b) by the canonical spin-statistics connection $K^{AAC\gamma}_\alpha R = e^{2\pi i s(A,\omega)} K^{AAC\gamma}_\alpha$ which only holds for the fusion channels $(C,\gamma)$ in which both particles $(A,\alpha)$ are in identical internal quantum states.

Let me finally establish that the CS term (1) in the broken model of section 2 indeed boils down to the 3-cocycle $\omega = \prod_{1 \leq i < j < k} \omega^{(i)} \omega^{(ij)}$ as indicated by (10). To start with, it is readily checked that the 2-cocycle $c_A$ entering (10) and (20) for this $\omega$ is trivial, i.e. $c_A(B, C) = \delta_{A, B} \delta_{A, C} = \frac{\varepsilon_A(B) \varepsilon_A(C)}{\varepsilon_A(H)}$ with $\varepsilon_A$ given by (5). Thus the dyon charges $\alpha$ in (23) for this $\omega$ are trivial projective representations of $H$ of the form $\alpha(C) = \varepsilon_A(C) \Gamma^{(1)} \cdots \Gamma^{(k)}(C)$ with $\Gamma^{(1)} \cdots \Gamma^{(k)}$ the ordinary UIR of $H$ appearing in (6). It is now easily verified that the monodromy operator following from (22) for this case is the same as (8), that the spinfactors $e^{2\pi i s(A,\omega)} = \alpha(A)$ coincide with (8) and that the fusion rules (23) following from (24) and (25) reproduce the tunneling properties (9) of the Dirac monopoles. □

5. Nonabelian electric/magnetic dualities

The 3-cocycles (14) that can not be reached from a spontaneously broken $U(1)^k$ CS theory are actually the most interesting. They render an abelian discrete $H$ gauge theory nonabelian and generally lead to dualities with 2+1 dimensional theories with a nonabelian finite gauge group of the same order as $H$. The point is that the 2-cocycles $c_A$ appearing in (19) and (20) for such a 3-cocycle $\omega^{(ij)}$ are nontrivial, so the dyon charges $\alpha$ in (23) become nontrivial (i.e. higher dimensional) projective UIR’s of $H$. Consequently, the braid operator (22) now generally acts as a matrix leading to the usual host of nonabelian phenomena such as nonabelian braid statistics, nonabelian AB scattering, exchange of nontrivial Cheshire charges and quantum statistics between particle pairs through monodromy processes, etc...

Let me briefly illustrate these general remarks with the simplest example, namely a CS theory with gauge group $H \simeq \mathbb{Z}_2^2 := \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ defined by the corresponding nontrivial 3-cocycle (14). Whereas the ordinary $\mathbb{Z}_2^2$ theory features 64 different singlet particles, it turns out that the spectrum just consists of 22 different particles in the presence of this 3-cocycle $\omega^{(123)}$. Specifically, the dyon charges which formed 1-dimensional UIR’s of $\mathbb{Z}_2^2$ are reorganized into 2-dimensional or doublet projective UIR’s of $\mathbb{Z}_2^2$, i.e. besides the 8 singlets (the vacuum and 7 ordinary nontrivial $\mathbb{Z}_2$ charges) the spectrum now contains 14 doublet dyons carrying a nontrivial (singlet) flux and a nontrivial projective $\mathbb{Z}_2^2$ doublet charge. Further, there are only two nonabelian finite groups of order $|\mathbb{Z}_2^2| = 8$: the dihedral group $D_4$ and...
the double dihedral group $\bar{D}_2$. Like the $\{\mathbb{Z}_2^3, \omega^{(123)}\}$ CS theory, the spectrum of the theories with gauge group $D_4$ and $\bar{D}_2$ both contain 8 singlet particles and 14 doublet particles albeit of different nature. It can be checked that the $\{\mathbb{Z}_2^3, \omega^{(123)}\}$ CS theory is dual to the $D_4$ theory, i.e. the exchange $\{\mathbb{Z}_2^3, \omega^{(123)}\} \leftrightarrow D_4$ corresponds to an invariance of the modular matrices (24) indicating that these two theories indeed describe the same spectrum and the same topological interactions. The duality transformation exchanges the projective dyon charges in the $\{\mathbb{Z}_2^3, \omega^{(123)}\}$ CS theory with the magnetic doublet fluxes in the $D_4$ theory. So we are actually dealing with some kind of nonabelian electric/magnetic duality. Let me also note that adding a 3-cocycle (13) does not spoil this duality, i.e. we also have the dualities $\{\mathbb{Z}_2^3, \omega^{(123)}\omega^{(ij)}\} \leftrightarrow D_4$ with $1 \leq i < j \leq k$. Finally, the $\{\mathbb{Z}_2^3, \omega^{(123)}\omega^{(i)}\}$ CS theories (with $\omega^{(i)}$ given in (13) and $i = 1, 2$ or 3) turn out to be dual to the $\bar{D}_2$ theory.

6. Concluding remarks

Whether the interesting 3-cocycles (13) can be reached from the spontaneous breakdown of a nonabelian CS theory is under investigation. I am currently also working on the generalization of the dualities described in the previous section to abelian finite gauge groups of order higher than $|\mathbb{Z}_2^3| = 8$. Finally, for a (concise) discussion of CS theories in which a nonabelian compact gauge group is spontaneously broken to a nonabelian finite subgroup, the reader is referred to Ref. 9.

Acknowledgements

I would like to thank the organizers for an inspiring workshop. This work was partly supported by an EC grant (contract no. ERBCHBGCT940752).

References

1. J. Schonfeld, *Nucl. Phys. B185* (1981) 157; S. Deser, R. Jackiw and S. Templeton, *Phys. Rev. Lett.* 48 (1982) 975; *Ann. Phys. (NY)* 140 (1982) 372.
2. M. de Wild Propitius and F.A. Bais, *Discrete Gauge Theories*, lectures presented at *The CRM-CAP Summer School on Particles and Fields ’94*, Banff, Canada (16-24 August 1994), preprint PAR–LPTHE 95–46 and ITFA 95–20 (1995) hep-th/9511201.
3. F.A. Bais, P. van Driel and M. de Wild Propitius, *Phys. Lett. B280* (1992) 63.
4. R. Dijkgraaf, V. Pasquier and P. Roche, *Nucl. Phys. B (Proc. Suppl.) 18B* (1990) 60.
5. F.A. Bais, P. van Driel and M. de Wild Propitius, *Nucl. Phys. B393* (1993) 547; F.A. Bais and M. de Wild Propitius, in *The proceedings of the III International Conference on Mathematical Physics, String Theory and Quantum Gravity*, Alushta, 1993, *Theor. Math. Phys.* 98 (1994) 357.
6. M. de Wild Propitius, *Spontaneously broken abelian Chern-Simons theories*, preprint PAR–LPTHE 96–17 and ITFA 96–16 (1996) hep-th/9606025.
7. F.A. Bais, A. Morozov and M. de Wild Propitius, *Phys. Rev. Lett.* 71 (1993) 2383.
8. R. Dijkgraaf and E. Witten, *Comm. Math. Phys.* 129 (1990) 393.
9. M. de Wild Propitius, *Topological Interactions in Broken Gauge Theories*, PhD thesis, Universiteit van Amsterdam, 1995, hep-th/9511195.