$U$-duality extension of Drinfel’d double

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Abstract

A family of algebras $\mathcal{E}_n$ that extends the Lie algebra of the Drinfel’d double is proposed. This allows us to systematically construct the generalized frame fields $E_A^I$ which realize the proposed algebra by means of the generalized Lie derivative, i.e., $\hat{\mathcal{L}}_{E_A} E_B^I = -\mathcal{F}_{AB}^\ C E_C^I$. By construction, the generalized frame fields include a twist by a Nambu–Poisson tensor. A possible application to the non-Abelian extension of $U$-duality and a generalization of the Yang–Baxter deformation are also discussed.
1 Introduction

The familiar $T$-duality is a symmetry of string theory when the target space has commuting (or Abelian) Killing vectors. An extension of the $T$-duality, where the Killing vectors do not commute with each other, has been proposed in [1–6], and it is known as non-Abelian $T$-duality (see [7] for a list of references). Subsequently, a further extension, called the Poisson–Lie (PL) $T$-duality has been found in [8], and it can be applied to a more general class of target spaces. As it has been discovered there, there is a group structure of the Drinfel’d double behind the PL $T$-duality, and the symmetry of the PL $T$-duality can be understood as a freedom in the choice of the physical subgroup $G$ out of the doubled Lie group.

In the case of Abelian $T$-duality, the double extension of the target space has been proposed in various contexts (see for example [10–14]). The geometry of the doubled space has been studied in [15–17], and more recently, the idea has been developed in the context of double field theory (DFT) [18,19]. In the original formulation of DFT, the symmetry of Abelian $T$-duality is manifest, but the non-Abelian $T$-duality or the PL $T$-duality has not been clearly discussed. A new formulation of DFT on group manifolds (called DFT$_{WZW}$) has been developed in [20–22], and in the recent works [23,24], the PL $T$-duality has been studied in the framework of DFT$_{WZW}$. In more recent papers [7,25], the non-Abelian $T$-duality and the PL $T$-duality have been discussed by using another approach, called the gauged DFT [26–31] (see also [32,33] for recent discussion on the Drinfel’d double and related aspects in DFT). Thus, DFT is now not restricted to Abelian $T$-duality but can be applied also to the non-Abelian extensions.

When the target space has $D$ Abelian Killing vectors, the $T$-duality group of type II superstring theory is $O(D,D)$. As it is well-known, this $T$-duality group is only a subgroup of a larger duality group, called the $U$-duality group. The $U$-duality group is $E_n (n \equiv D + 1)$, which is summarized in Table 1.1. In order to manifest the $U$-duality symmetry, the doubled space is not enough. As it has been discussed in [34–39], we need to extend the $n$-dimensional space (with Abelian Killing vectors) into an extended space with dimension $D_n$, which is the dimension of the vector representation $R_1$ of the $U$-duality group (see Table 1.1). For higher $n$, it is much larger than the doubled space with dimension $2D$, and the extended space is called the exceptional space for the obvious reason. The $U$-duality-manifest formulation of

| $n$ | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|-----|-----|-----|-----|-----|-----|-----|-----|
| $E_n$ | $SL(2) \times \mathbb{R}^+$ | $SL(3) \times SL(2)$ | $SL(5)$ | $SO(5,5)$ | $E_6$ | $E_7$ | $E_8$ |
| $D_n$ | 3   | 6   | 10  | 16  | 27  | 56  | 248 |
| $d_n$ | 2   | 3   | 5   | 10  | 27  | 133 | 3875+1 |

Table 1.1: The $U$-duality group $E_n$ for each $n$ and dimensions of two representations, known as the $R_1$-representation and the $R_2$-representation.
supergravities have been studied in [36,39–49], and more recently, a formulation using a similar language to DFT has been developed in [50–57], which is called the exceptional field theory (EFT). Similar to the case of DFT, for the consistency of the theory, we need to choose a physical subspace from the extended space, and all of the supergravity fields are defined on the physical subspace. In the case of DFT, the maximal dimension allowed for the consistency is always \( D \)-dimensional, but in the case of EFT, there are two maximal choices, \( n \)-dimensions and \( D(=n-1) \)-dimensions [58]. If we adopt the former choice, the target space of M-theory is reproduced while the latter reproduces that of type IIB theory [58]. In this sense, EFT unifies the geometry of M-theory and type IIB string theory, and the structure of the exceptional space is much richer than that of the double space.

Unlike the case of \( T \)-duality, there is no concrete proposal for the extension of \( U \)-duality when the Killing vectors are non-Abelian. Originally, the non-Abelian \( T \)-duality has been discovered by introducing certain gauge fields (associated with the Killing vectors) into the string sigma model. This allows us to reformulate the string theory as a gauged sigma model, which reduces to the standard string sigma model if we first eliminate certain auxiliary fields. On the other hand, if we eliminate the gauge fields first, the string sigma model on the dual geometry is recovered [3–6], and in this sense, the gauged sigma model connects the original geometry and the dual geometry. If we try to apply the same procedure to the membrane sigma model, we face a difficulty (see [7]). In order to formulate the non-Abelian extension of \( U \)-duality, the approach of the PL \( T \)-duality will be more useful. For this purpose, we need to extend the exceptional space to some extended group manifold, similar to the Drinfel’d double. Such an extension has been studied in [59, 60], but the relation to the Drinfel’d double is not so clear and no proposal has been made for the extension of the PL \( T \)-duality.

In this paper, by using the idea of EFT, we propose a family of Leibniz algebra \( \mathcal{E}_n \) which contains the Lie algebra of the Drinfel’d double as a subalgebra in a particular case. For simplicity, we restrict our analysis to the case \( n \leq 4 \). In that case, the generators of the algebra \( T_A \) \((A = 1, \ldots, D_n)\) can be parameterized as \( \{T_A\} = \{T_a, T^{a_1a_2}\} \), where \( a = 1, \ldots, n \) and \( T^{a_1a_2} = -T^{a_2a_1} \). The algebra can be expressed as

\[
T_a \circ T_b = f_{ab}^c T_c, \\
T_a \circ b_{b_1b_2} = f_a^{b_1b_2c} T_c + 2 f_{ac}[b_1 T^{b_2}c], \\
T^{a_1a_2} \circ T_b = -f_{b}^{a_1a_2c} T_c + 3 f_{c_1c_2}[a_1 \delta^{a_2}_{b}] T^{c_1c_2}, \\
T^{a_1a_2} \circ b_{b_1b_2} = -2 f_{d}^{a_1a_2}[b_1 T^{b_2}d],
\]

where \( f_{ab}^c = f_{[ab]}^c \) and \( f_{a}^{b_1b_2b_3} = f_a[b_1b_2b_3] \), and the following bilinear forms are defined:

\[
\langle T_a, b_{b_1b_2} \rangle_c = 2! \delta^c_{a} \delta^{b_1}_{b_2} , \quad \langle T^{a_1a_2}, b_{b_1b_2} \rangle_{c_1\cdots c_4} = 4! \delta^{c_1}_{a_1} \delta^{c_2}_{a_2} \delta^{c_3}_{b_1} \delta^{c_4}_{b_2} ,
\]

which naturally extend the standard bilinear form \( \langle \cdot, \cdot \rangle \) of the Drinfel’d double.
By using the proposed algebra, we can systematically construct the generalized frame fields $E_A^M(x)$ which satisfy
\[
\hat{\mathcal{L}}_{E_A} E_B^I = -\mathcal{F}_{ABC} E_C^I,
\] (1.3)
where $\mathcal{F}_{ABC}$ is the structure constant of the proposed Leibniz algebra $T_A \circ T_B = \mathcal{F}_{ABC} T_C$. Here, $\hat{\mathcal{L}}_V$ is the generalized Lie derivative in EFT, which generates the gauge transformations in EFT. The systematic construction of $E_A^I$ is indispensable to perform the PL $T$-duality or its extension PL $T$-plurality \([60]\), and we expect that the $U$-duality extension presented in this paper will be useful in studying the non-Abelian extension of $U$-duality.

The structure of this paper is as follows. In section 2, we briefly summarize the idea of the PL $T$-duality. In particular, we explain how the relation (1.3) is important in the PL $T$-duality. In section 3, we find the algebra $\mathcal{E}_n$ and study its detailed properties. The construction of the generalized frame fields is explained in section 4. In section 5, we show several examples of the $\mathcal{E}_n$ algebra. Section 6 is devoted to the summary and discussion.

2 Poisson–Lie $T$-duality

In this section, we review the PL $T$-duality by using the language of DFT.

**Basics definitions in DFT:** Let us set up basic definitions of DFT. We consider a doubled space which has the generalized coordinates $(x^M) = (x^m, \tilde{x}_m)$ ($m = 1, \ldots, D$). The metric and the B-field are packaged into the generalized metric,
\[
(\mathcal{H}_{MN}) = \begin{pmatrix}
\delta_{mn} - B_{mp} g^{pq} B_{qn} & -B_{mp} g^{mn} \\
g^{mp} B_{pn} & g^{mn}
\end{pmatrix},
\] (2.1)
and the dilaton $\Phi$ is redefined into the $T$-duality-invariant combination,
\[
e^{-2d} \equiv e^{-2\Phi} \sqrt{|g|},
\] (2.2)
where $d(x)$ is called the DFT dilaton. We denote the $O(D, D)$-invariant metric as
\[
(\eta_{MN}) \equiv \begin{pmatrix}
0 & \delta_m^m \\
\delta^n_m & 0
\end{pmatrix}, \quad (\eta^{MN}) \equiv \begin{pmatrix}
0 & \delta^m_n \\
\delta^m_n & 0
\end{pmatrix},
\] (2.3)
and use these to raise or lower the indices $M, N$. The fields $\mathcal{H}_{MN}(x)$ and $d(x)$ are formally defined on the doubled space, but for the consistency, we impose the section condition,
\[
\eta^{MN} \partial_M A(x) \partial_N B(x) = 0, \quad \eta^{MN} \partial_M \partial_N A(x) = 0,
\] (2.4)
for arbitrary fields $A(x)$ and $B(x)$. According to the section condition, all of the fields can depend only on a set of $D$ coordinates, and in this paper, we choose $x^m$ as such $D$ coordinates. Any other choices can be mapped to this choice by performing a $T$-duality transformation.
A generalization of the Lie derivative, called the generalized Lie derivative is defined as

\[ \hat{\mathcal{L}}_V W^M \equiv V^N \partial_N W^M - (\partial_N V^M - \partial^M V_N) W^N, \tag{2.5} \]

which generates the gauge transformations in DFT. Under the section where all fields depend only on the physical coordinates \( x^m \), the generalized Lie derivative reduces to

\[ \hat{\mathcal{L}}_V W^M = \begin{pmatrix} \mathcal{L}_v w^m \\ (\mathcal{L}_v \tilde{w}_1 - \ell_w d\tilde{v}_1)_m \end{pmatrix}, \tag{2.6} \]

where we have parameterized the generalized vectors as \((V^M) = (v^m, \tilde{v}_m)\) and \((W^M) = (w^m, \tilde{w}_m)\) and denoted the 1-forms as \(\tilde{v}_1 \equiv \tilde{v}_m dx^m\) and \(\tilde{w}_1 \equiv \tilde{w}_m dx^m\).

**Abelian T-duality:** Now, let us consider the T-duality by using the above notation. In order to perform the standard Abelian T-duality, the generalized metric and the DFT dilaton are required to be constant,

\[ \mathcal{H}_{MN}(x) = \mathcal{\hat{H}}_{MN}, \quad d(x) = \hat{d}, \tag{2.7} \]

in a certain adapted coordinate system. In this constant background, equations of motion of DFT are trivially satisfied, and a constant O\((D, D)\) transformation

\[ \mathcal{H}_{MN}' = C^P_M C^Q_N \mathcal{\hat{H}}_{PQ}, \quad \hat{d}' = \hat{d}, \tag{2.8} \]

maps the solution to another constant solution.

**Poisson–Lie T-dualizable backgrounds:** When we consider the PL T-duality, the target space is allowed to be non-constant. The generalized metric \(\mathcal{H}_{MN}\) can be twisted by a non-constant matrix \(E^A_M(x)\) and the DFT dilaton also can have a non-constant factor,

\[ \mathcal{H}_{MN}(x) = E^A_M(x) E^B_N(x) \hat{\mathcal{H}}_{AB}, \quad e^{-2d(x)} = |\ell(x)| e^{-2\hat{d}}, \tag{2.9} \]

where \(|\ell(x)| \equiv |\det(\ell^a_m)|\) and the matrices, \(E^A_M(x)\) and \(\ell^a_m(x)\) are defined as follows. First, we introduce the Lie algebra of the Drinfel’d double,

\[ [T_a, T_b] = f^{abc} T_c, \quad [T_a, \bar{T}^b] = \bar{f}^{abc} T_c - f_{ac}^b \bar{T}^c, \quad [\bar{T}^a, \bar{T}^b] = \bar{f}^{ab}_c \bar{T}^c, \tag{2.10} \]

which is equipped with the \(ad\)-invariant bilinear form,

\[ \langle T_A, T_B \rangle = \eta_{AB}, \quad (\eta_{AB}) = \begin{pmatrix} 0 & \delta^b_a \\ \delta^a_b & 0 \end{pmatrix}, \quad (T_A) \equiv (T_a, \bar{T}^a). \tag{2.11} \]

The indices \(A, B\) are raised or lowered by using the metric \(\eta_{AB}\). The structure constants \(f^{abc}\) and \(\bar{f}^{ab}_c\) can be chosen arbitrarily as long as Jacobi identities are satisfied. Secondly, we
decompose the algebra into two subalgebras \( \mathfrak{g} \oplus \tilde{\mathfrak{g}} \), where \( \mathfrak{g} \) is the “physical algebra” spanned by \( T_a \) and the dual algebra \( \tilde{\mathfrak{g}} \) is spanned by \( \tilde{T}_a \). Each of these is maximally isotropic for the bilinear form \( \langle \cdot , \cdot \rangle \). We then define a group element \( g \) by using the generators of the physical subalgebra, for example, \( g = e^{x^a T_a} \), and define the left- and right-invariant forms as

\[
\ell \equiv e^a_m T_a dx^m \equiv g^{-1} dg, \quad r \equiv r^a_m T_a dx^m \equiv dg g^{-1}.
\]

The left- and right-invariant vectors are denoted by \( v^a_m \) and \( e^a_m \) (\( \ell^a_m v^b_m = \delta^a_b \) and \( r^a_m e^b_m = \delta^a_b \)). We also parameterize the adjoint action of \( g^{-1} \) on the generators of the Drinfel’d double as

\[
\lambda^{-1}_A g \equiv M_{AB} T_B, \quad M \equiv \left( \begin{array}{cc} \delta^c_a & 0 \\ -\Pi^{ac} & \delta^a_c \\ 
0 & (a^{-1})_b^c
\end{array} \right).
\]

Finally, by using the above quantities, we define the twist matrix as

\[
(E_M^A) \equiv \left( \begin{array}{cc} \gamma^a_m & 0 \\ -e^a_m \Pi^{ca} & e^m_a \end{array} \right).
\]

Once the structure constants \( f^{abc} \) and \( \tilde{f}^{abc} \) and the parameterization of \( g \) are given, the matrices \( E_M^A(x) \) and \( \lambda^a_m(x) \) are uniquely obtained. Then, by using these matrices, the PL \( T \)-dualizable background is expressed as (2.9). For later convenience, it is useful to note that at the identity \( g = 1 \) (which corresponds to \( x^a = 0 \) when \( g = e^{x^a T_a} \)), we have

\[
\Pi^{ab}(x) \big|_{g=1} = 0, \quad a^b_a(x) \big|_{g=1} = \delta^b_a,
\]

by their definitions. We also note that the Abelian \( T \)-dualizable background, i.e., the constant background (2.7), is reproduced as a particular case, \( f^{abc} = 0 \) and \( \tilde{f}^{abc} = 0 \) with \( g = e^{x^a T_a} \).

**Poisson–Lie \( T \)-duality:** We denote the Lie algebra of the Drinfel’d double as

\[
[T_A, T_B] = \mathcal{F}_{AB}^C T_C,
\]

where

\[
\mathcal{F}_{ab}^c = f^{abc}, \quad \mathcal{F}_{abc} = 0, \quad \mathcal{F}_a^{bc} = f^{bac}, \quad \mathcal{F}_a^b = -f^{bc}_a, \quad \mathcal{F}^{ab}_c = \tilde{f}^{abc}, \quad \mathcal{F}^{abc} = 0.
\]

For simplicity, we here suppose that the structure constant of the dual algebra is unimodular \( \tilde{f}^{ba}_c = 0 \) \footnote{See \[7,24\] for the PL \( T \)-duality for non-unimodular cases.}. Then, the equations of motion of DFT for a general PL \( T \)-dualizable background (2.9) reduce to the following algebraic equations:

\[
\frac{1}{12} \mathcal{F}_{ABC} \mathcal{F}_{DEF} \left( 3 \hat{\mathcal{H}}^{AD} \eta^{BE} \eta^{CF} - \hat{\mathcal{H}}^{AD} \hat{\mathcal{H}}^{BE} \hat{\mathcal{H}}^{CF} \right) = 0, \quad \frac{1}{2} \left( \eta^{CE} \eta^{DF} - \hat{\mathcal{H}}^{CE} \hat{\mathcal{H}}^{DF} \right) \hat{\mathcal{H}}^{[A} \mathcal{F}_{CD}^{B]} \mathcal{F}_{EFG} = 0.
\]
They are manifestly covariant under constant \(O(D,D)\) transformations:\footnote{The matrix \(C_A^B\) should be chosen such that the structure constant \(\mathcal{F}_{ABC}^C\) has the form of the Drinfel’d double \footnote{To be more precise, the PL \(T\)-duality is a particular transformation \((C_A^B) = \left(\begin{array}{cc} 0 & I_d \\ I_d & 0 \end{array} \right)\). In particular, when \(\tilde{f}^{\hat{a}}_{\hat{b}c} = 0\), it is called the non-Abelian \(T\)-duality. A general \(O(D,D)\) transformation is called the PL \(T\)-plurality transformation \footnote{In this paper, we denote an arbitrary \(O(D,D)\) transformation as the PL \(T\)-duality.} , but in this paper, we denote an arbitrary \(O(D,D)\) transformation as the PL \(T\)-duality.}}

\[
\hat{\mathcal{H}}_{AB} = C_A^C C_B^D \hat{\mathcal{H}}_{CD}, \quad \hat{d} = \hat{d}, \quad \hat{\mathcal{F}}_{ABC} = C_A^E C_B^F C_C^G \hat{\mathcal{F}}_{EFG}.
\] (2.19)

This is the PL \(T\)-duality, which extends the Abelian \(T\)-duality \footnote{In this paper, we denote an arbitrary \(O(D,D)\) transformation as the PL \(T\)-duality.}.

In DFT, the tensor \(\mathcal{F}_{ABC}^C\) is called the generalized flux, and it is generally defined as

\[
\hat{\mathcal{L}}_{EA} E_B^M = -\mathcal{F}_{ABC}^C E_C^M, \quad (2.20)
\]

Here, the generalized frame fields \(E_A = (E_A^M)\) correspond to the inverse of the twist matrix \(E_M^A\) given in \footnote{In this paper, we denote an arbitrary \(O(D,D)\) transformation as the PL \(T\)-duality.}. Even in general spacetimes where the generalized flux \(\mathcal{F}_{ABC}^C\) is not constant, the equations of motion of DFT can be expressed by using \(\mathcal{F}_{ABC}^C\). However, they contain the derivative of the generalized flux \(\partial_M \mathcal{F}_{ABC}^C\) and are much more complicated than \footnote{In this paper, we denote an arbitrary \(O(D,D)\) transformation as the PL \(T\)-duality.}. In such general cases, the \(O(D,D)\) transformation \footnote{In this paper, we denote an arbitrary \(O(D,D)\) transformation as the PL \(T\)-duality.} is not a symmetry of DFT. Therefore, the constancy of the generalized flux \(\mathcal{F}_{ABC}^C\) is crucial for the PL \(T\)-duality.

**Dual geometry:** Under a PL \(T\)-duality, the structure constant \(\mathcal{F}_{ABC}^C\) is mapped to another one \(\mathcal{F}_{ABC'}^{C'}\). It is associated with a new set of the generalized frame fields \(E_A'\) satisfying

\[
\hat{\mathcal{L}}_{E_A'} E_B'^M = -\mathcal{F}_{ABC}^C E_C'^M. \quad (2.21)
\]

Then, by using the new twist matrix \(E_A'^M\) and the relation \footnote{In this paper, we denote an arbitrary \(O(D,D)\) transformation as the PL \(T\)-duality.}, the metric, \(B\)-field, and the dilaton in the dual geometry are obtained as

\[
\hat{\mathcal{H}}_{MN}' = E_M^A E_N^B \hat{\mathcal{H}}_{AB}, \quad e^{-2\hat{d}'} = e^{-2\hat{d}} |\det(\epsilon_m^a)|. \quad (2.22)
\]

A possible problem in the PL \(T\)-duality is the explicit construction of \(E_A'^M\) satisfying \footnote{In this paper, we denote an arbitrary \(O(D,D)\) transformation as the PL \(T\)-duality.}. However, it is not a problem if we use the algebra of the Drinfel’d double. Under the PL \(T\)-duality \footnote{In this paper, we denote an arbitrary \(O(D,D)\) transformation as the PL \(T\)-duality.}, the generators are also redefined as \(T'_A = C_A^B T_B\), and they satisfy

\[
[T'_A, T'_B] = \mathcal{F}_{ABC}^C T_C, \quad (T'_A, T'_B) = \eta_{AB}. \quad (2.23)
\]

Then, by decomposing the new generators as \((T'_A) = (T'_a, \tilde{T}^{\hat{a}}_a)\), we obtain a new physical subalgebra \(g'\) spanned by \(T'_a\). By parameterizing a group element as before, such as \(g' = e^{c_{\hat{a}} T_{\hat{a}}}\), we can again obtain the matrices \(\Pi^{\hat{a}\hat{b}}, \epsilon_{\hat{a}}^m,\) and \(r_{\hat{a}m}\). Then, the new generalized frame fields,

\[
(E_A'^M) \equiv \left(\begin{array}{cc} \epsilon^m_a & 0 \\ 0 & \Pi^{ac} \epsilon^m_c r_{\hat{a}m} \end{array} \right), \quad (2.24)
\]

satisfy the expected relation \footnote{In this paper, we denote an arbitrary \(O(D,D)\) transformation as the PL \(T\)-duality.}. In this manner, we can systematically construct the dual geometry, and the construction method of \(E_A'^M\) is important for the PL \(T\)-duality.
A short summary: As shortly reviewed in this section, the PL $T$-duality is a constant $O(D,D)$ rotation of the indices $A, B$. We can perform this duality when the generalized frame fields satisfy the relation (2.20) by using the structure constant $F_{ABC}$ of a Drinfel’d double. The algebra of the Drinfel’d double provide a systematic way to construct the generalized frame fields satisfying (2.21), and the dual geometry can be explicitly constructed.

In the next section, we introduce an extension of the Drinfel’d double and explain a systematic way to construct a generalized frame fields satisfying

$$\hat{\mathcal{L}}_E A_B^I = - F_{AB}^C E_C^I \quad (F_{AB}^C : \text{constant}),$$

by means of the generalized Lie derivative in EFT.

3 Leibniz algebra based on $U$-duality

Here, we propose a Leibniz algebra $\mathcal{E}_n$ by using the generalized Lie derivative in the $E_n$ EFT. For this purpose, let us begin with a quick introduction to EFT.

Basic definitions in EFT: As we have explained in the introduction, in EFT, we introduce an exceptional space with dimension $D_n$. When we adopt the M-theory picture, we decompose the generalized coordinates $x^I$ ($I = 1, \ldots, D_n$) as

$$(x^I) = (x^i, \frac{y_{i_1 i_2}}{\sqrt{2!}}, \frac{y_{i_1 i_2 i_3}}{\sqrt{3!}}, \cdots) \quad (i = 1, \ldots, n),$$

where the multiple indices are totally antisymmetric and the numerical factors are introduced for convenience. The ellipses are not necessary as far as we consider the cases $n \leq 6$. The supergravity fields such as the metric and gauge potentials are contained in the generalized metric $\mathcal{M}_{IJ}$, which extends the one $\mathcal{H}_{MN}$ in DFT. The fields are formally defined on the exceptional space, but the extension of the section condition (2.4) again restricts the coordinate dependence. In order to reproduce M-theory, we choose $x^i$ as the physical coordinates and any more coordinate dependence is not allowed by the section condition. Thus, in the following discussion, we eliminate the coordinate dependence on the dual coordinates,

$$\frac{\partial}{\partial y_{i_1 i_2}} = 0, \quad \frac{\partial}{\partial y_{i_1 \cdots i_5}} = 0, \quad \cdots.$$ 

(3.2)

Similar to DFT, the generalized Lie derivative in EFT is defined as

$$\hat{\mathcal{L}}_V W^I = V^J \partial_J W^I - W^J \partial_J V^I + Y_{KL}^{IJ} \partial_J V^K W^L,$$

where $Y_{KL}^{IJ}$ is an invariant tensor satisfying $\hat{\mathcal{L}}_V Y_{KL}^{IJ} = 0$. For our purpose, it is enough to know the expression under the situation where all fields depend only on the physical coordinates $x^i$.  

In that case, the generalized Lie derivative is expressed as (see [61] for our convention)

\[
\hat{\mathcal{L}}_V W^I = \left( \frac{\mathcal{L}_V W^I}{(\mathcal{L}_V w_2 - e_2 dw_2)_{i_1 i_2} \sqrt{2!}} \right) \left( \frac{\sqrt{2!}}{\mathcal{L}_V w_5 + dw_2 \wedge w_5 - e_2 dw_5)_{i_1 \cdots i_5} \sqrt{2!}} \right),
\]

where the two arbitrary generalized vectors \( V^I \) and \( W^I \) are parameterized as

\[
V^I = \begin{pmatrix} v^i \\ \sqrt{2!} \end{pmatrix}, \quad W^I = \begin{pmatrix} u^i \\ \sqrt{2!} \end{pmatrix}, \quad \hat{\mathcal{L}}_V W^I = \left( \frac{\mathcal{L}_V w^i (\mathcal{L}_V w_2 - e_2 dw_2)_{i_1 i_2}}{\sqrt{2!}} \right) \left( \frac{\sqrt{2!}}{\mathcal{L}_V w_5 + dw_2 \wedge w_5 - e_2 dw_5)_{i_1 \cdots i_5} \sqrt{2!}} \right).
\]

and we have defined \( v_p \equiv \frac{1}{\sqrt{2!}} v_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \) and similar for \( w_p \). We note that the expression [3.4] coincides with the Dorfman derivative in generalized geometry [47, 49].

In order to simplify our discussion, we restrict our attention to \( n \leq 4 \). Then, terms with five (or more) antisymmetrized indices identically vanish (e.g. \( v_{1 \cdots 5} = 0 \)) and the above generalized vectors reduce to

\[
V^I = \begin{pmatrix} v^i \\ \sqrt{2!} \end{pmatrix}, \quad W^I = \begin{pmatrix} u^i \\ \sqrt{2!} \end{pmatrix}, \quad \hat{\mathcal{L}}_V W^I = \left( \frac{\mathcal{L}_V w^i (\mathcal{L}_V w_2 - e_2 dw_2)_{i_1 i_2}}{\sqrt{2!}} \right) \left( \frac{\sqrt{2!}}{\mathcal{L}_V w_5 + dw_2 \wedge w_5 - e_2 dw_5)_{i_1 \cdots i_5} \sqrt{2!}} \right).
\]

**Generalized frame fields in EFT:** In order to consider the relation (2.24) in EFT, let us introduce certain generalized frame fields \( E_A^I \) in EFT. By considering the analogy with the DFT case (2.24), we consider the following parameterization:

\[
E_A^I \equiv \begin{pmatrix} E_a^I \\ \sqrt{2!} \end{pmatrix} \equiv \begin{pmatrix} \delta^b_a & 0 \\ -\Pi^{a_1 a_2 b} \delta_{b_1 b_2} & \delta_{b_1 b_2} \end{pmatrix} \begin{pmatrix} e_b^i \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ r_{[a_1} r_{b_2]} & r_{[a_1} r_{b_2]} \end{pmatrix} = \begin{pmatrix} e_a^i \\ \sqrt{2!} \Pi^{a_1 a_2 b} e^i_b \begin{pmatrix} 0 \\ r_{[a_1} r_{b_2]} \end{pmatrix} \end{pmatrix},
\]

where \( \delta^{a_1 \cdots a_n}_{b_1 \cdots b_n} \equiv \delta^{[a_1} \cdots \delta^{a_n]}_{b_1 \cdots b_n} \) and \( \Pi^{a_1 a_2 a_3} = \Pi^{[a_1 a_2 a_3]} \), and \( e_a^m \) (or \( r_a^m \)) is a certain right-invariant vector (or 1-form) satisfying

\[
\mathcal{L}_{e_a} e_b^m = -f_{ab}^c e_c^m, \quad \text{dr}^a = \frac{1}{2} f_{bc}^a r^b \wedge r^c, \quad \iota_{e_a} r^b = \delta_a^b.
\]

Then, using (3.3), we can compute the generalized Lie derivative as follows:

\[
\hat{\mathcal{L}}_{E_A} E_B^I = -X_{AB}^C \cdot E_C^I,
\]

8
Accordingly, in the following, we investigate a Leibniz algebra satisfying
\[ X_{a b c}^e = f_{a b}^c, \quad X_{abc_1c_2} = 0, \]
\[ X_a b_i b_j c = D_a \Pi^{b_i b_j} c - 3 f_{ad}^c \Pi^{b_i b_j} d, \]
\[ X_a b_i b_j c_{1 c_2} = 4 f_{ad}^e \delta_{b_i}^{b_j} \delta_{c_1 c_2}^f, \]
\[ X^{a_1 a_2 b_i} c = -(D_a \Pi^{a_1 a_2} c - 3 f_{bd}^c \Pi^{a_1 a_2} d - f_{d_1 d_2}^{a_1} \delta_b^{a_2} \Pi^{d_1 d_2} c), \]
\[ X^{a_1 a_2 b_i c_{1 c_2}} = 6 f_{b c_1}^{a_1} \delta_{c_2}^{a_2}, \]

and \( D_a \equiv e_i^a \partial_i \). In general, the generalized flux \( X_{ABC} = X_{ABC}^C \) is not constant.\(^4\)

Unlike the DFT case, the first two indices are not antisymmetric \( X_{ABC} = X_{ABC}^C \neq X_{[ABC]}^C \) and even if we find a certain situation where \( X_{ABC} \) is constant, the algebra is not a Lie algebra. Accordingly, in the following, we investigate a Leibniz algebra satisfying
\[ T_A \circ T_B = \mathcal{F}_{ABC} T_C. \]

Here, \( T_A \circ T_B \neq -T_B \circ T_A \) but the Leibniz identity,
\[ X \circ (Y \circ Z) - Y \circ (X \circ Z) = (X \circ Y) \circ Z, \]
is satisfied similar to the case of the generalized Lie derivative,
\[ \hat{L}_{V_1} \hat{L}_{V_2} W = \hat{L}_{V_2} \hat{L}_{V_1} W = \hat{L}_{V_1 V_2} W. \]

**Construction of the algebra \( \mathcal{E}_n \):** Here we take a heuristic approach to find the Leibniz algebra \( \mathcal{E}_n \). First, we suppose that the generalized flux \( X_{ABC} \) is constant, and assume that there exists an algebra \((3.18)\) with \( \mathcal{F}_{ABC} = X_{ABC}^C \). Secondly, we assume that \( \Pi^{a_1 a_2 a_3} = 0 \) at a certain point \( x^a = 0 \), which corresponds to \((2.15)\). We further assume that the so-called R-flux \( X^{a_1 a_2 b_i b_j} c \) vanishes. At least when \( e_i^a = \delta_i^a \), this requirement is precisely a condition for \( \Pi^{a_1 a_2 a_3} \) to be a Nambu–Poisson tensor \((3.14)\), and the condition \( X^{a_1 a_2 b_i b_j} c = 0 \) will be understood as a natural generalization of the definition of the Nambu–Poisson tensor. In the case of the Drinfel’d double, the bi-vector \( \Pi^{ab} \) has been a Poisson tensor, and in our setup, the Poisson tensor is naturally extended to the Nambu–Poisson tensor.

\(^4\)See [62][63] for computation of the generalized flux in the SL(5) EFT in a more general setup.
Under these assumptions, the generalized flux $X_{AB}^C$ at the point $x^a = 0$ reduces to

\begin{align}
X_{ab}^c &= f_{ab}^c, \\
X_{abc_{1}c_2} &= 0, \\
X_a b_1 b_2 c &= D_a \Pi_{b_1 b_2 c} = X_a [b_1 b_2 c], \\
X_a b_1 b_2 c_{1}c_2 &= 4 f_a c d \delta_{b_2}^{d} \delta_{c_{1}c_2}, \\
X^a_{12} a_{2} b_{1} c &= -D_b \Pi^{a b} c = -X_b a_{1} a_{2} c, \\
X^{a_{1} a_{2}} b_{c_{1}c_2} &= 6 f_{b_{c_{1}}} \delta_{a_{2}}^{c_{2}}, \\
X^{a_{1} a_{2}} b_{1} b_{2} c &= 0, \\
X^{a_{1} a_{2}} b_{1} b_{2} c_{1}c_2 &= -4 D_d \Pi^{a_{1} a_{2}} b_{1} \delta_{b_{2}}^{d} = -4 X_d a_{1} a_{2} b_{1} \delta_{b_{2}}^{d}. \\
\end{align}

This suggests us to define a new Leibniz algebra $\mathcal{E}_n$ as

\begin{align}
T_a \circ T_b &= f_{ab}^c T_c, \\
T_a \circ T^{b_1 b_2} &= f_{a} b_{1} b_{2} c T_c + 2 f_{a c} b_{1} T^{b_2} c, \\
T^{a_{1} a_{2}} b_{1} b_{2} c &= -f^{a_{1} a_{2} c} b_{1} T_{c_{1}c_2}, \\
T^{a_{1} a_{2}} b_{1} b_{2} c &= -2 f^{a_{1} a_{2} b_{1} T^{b_2}} d, \\
\end{align}

where $f_{ab}^c = f_{[ab]}^c$ and $f_{a} b_{1} b_{2} b_{3} = f_{a} [b_{1} b_{2} b_{3}]$. The Leibniz identity

\begin{align}
X \circ (Y \circ Z) &= (X \circ Y) \circ Z + Y \circ (X \circ Z),
\end{align}

for the generators $T_a$ and $T^{a_{1} a_{2}}$ requires the following relations:

\begin{align}
0 &= f_{[ab]} f_{c} d, \\
0 &= f_{bc} \delta_{e} f_{e} a_{1} a_{2} d + 6 f_{e[} [d] f_{c] a_{1} a_{2} e, \\
0 &= f_{d_1 d_2} \delta_{b} f_{c} d_{1} d_{2} e, \\
0 &= 3 f_{[d_1 d_2} \delta_{b} f_{e} c e b_{1} b_{2} + 4 f_{e f} a_{1} f_{c a_2} e [b_{1} \delta_{b_2}^{d} f], \\
0 &= f_{c e a_{1} a_{2}} f_{e} d_{1} b_{2} - 3 f_{e e b_{1} b_{2}} f_{e} d_{1} a_{1} a_{2}. \\
\end{align}

A bilinear form: We also introduce the bilinear form, which extends the bilinear form $\langle \cdot, \cdot \rangle$ of the Drinfel’d double. A natural extension of the bilinear form has been known in EFT\footnote{This bilinear form has been studied also in a mathematical literature \cite{55}.}

\begin{align}
\langle V, W \rangle_K &= \eta_{IJ,K} V^I W^J, \\
\end{align}

where $\eta_{IJ,K}$ connects a product of two $R_1$-representation and another representation, called the $R_2$-representation (see for example \cite{51}) whose dimension $d_n$ is given in Table\textsuperscript{1.1}. Namely,
the additional index $\mathcal{K}$ appended to the bilinear form transforms in the $R_2$-representation. This index can be decomposed as (see [61] for the explicit form of $\eta_{IJ;K}$)

$$ (\eta_{IJ;K}) = (\eta_{IJ;k_k}; \frac{\eta_{IJ;k_k} k_k}{\sqrt{4!}}, \cdots), $$

and in our case $n \leq 4$, it is enough to consider the first two components,

$$ \langle \cdot, \cdot \rangle_\mathcal{K} = \left( \langle \cdot, \cdot \rangle_k, \frac{\langle \cdot, \cdot \rangle_k k_k}{\sqrt{4!}}, \cdots \right). $$

The bilinear form takes the form

$$ \langle V, W \rangle_k = \left(\eta_{IJ;k_k} v^i_a + \eta_{IJ;k_k} w^i_a\right), $$

for arbitrary two vectors $V^I$ and $W^I$ parameterized as (3.6). Under an arbitrary $E_n U$-duality transformation $\Lambda$, the tensor $\eta_{IJ;K}$ behaves as

$$ \Lambda^I J \Lambda^J L \Lambda^K L \eta_{L_1 L_2;L} = \eta_{IJ;K}, $$

where $\Lambda^I J$ and $\Lambda^J L$ denote the same $E_n$ transformation in the $R_1$- and $R_2$-representation, respectively.

Now, we introduce a matrix,

$$ E_A^I = \left( \begin{array}{cc} \delta_a^b & -4a_{[b} \Pi^{b_1 b_2 b_3 b_4]} \delta_{a_1 a_2 a_3 a_4} \vspace{1mm} \end{array} \right) \left( \begin{array}{c} e_i^b \
0 \
\delta_{a_1 \cdots a_4}^{b_1 \cdots b_4} \end{array} \right), $$

which satisfies

$$ E_A^I E_B^J E_C^K \eta_{IJ;K} = \eta_{AB;C}, $$

and redefine the bilinear form as

$$ \langle \cdot, \cdot \rangle_A \equiv \left( \langle \cdot, \cdot \rangle_a, \frac{\langle \cdot, \cdot \rangle_a a_a}{\sqrt{4!}}, \cdots \right) = E_A^I \langle \cdot, \cdot \rangle_I. $$

Then, the bilinear form for the generalized frame fields (3.7) becomes

$$ \langle E_a, E_b^{b_1 b_2} \rangle_c = 2! \delta_a^{b_1 b_2}, \quad \langle E^{a_1 a_2}, E^{b_1 b_2}_{c_1 \cdots c_4} \rangle = 4! \delta_{a_1 a_2}^{a_1 a_2 b_1 b_2}. $$

Identifying the generalized frame fields $E_A$ with the $E_n$ generator $T_A$, we define the following bilinear form for the generators:

$$ \langle T_a, T_b^{b_1 b_2} \rangle_c = 2! \delta_a^{b_1 b_2}, \quad \langle T^{a_1 a_2}, T^{b_1 b_2}_{c_1 \cdots c_4} \rangle = 4! \delta_{a_1 a_2}^{a_1 a_2 b_1 b_2}. $$

We note that the subalgebra spanned by $\{T_a\}$ is maximally isotropic for the bilinear form.

In fact, the isotropy shows that the subalgebra is a Lie algebra $T_a \circ T_b = [T_a, T_b]$, where

$$ [T_A, T_B] = \frac{1}{2} (T_A \circ T_B - T_B \circ T_A). $$
This can be understood from the explicit form of the generalized Lie derivative (3.6), namely,

\[
\hat{L}_V W^I = \left( [v, w]^I_{[w_2 - w_2 + d(t_1 w_2)]_1]_2} \right). \tag{3.47}
\]

When \( V^I \) and \( W^I \) satisfy \( \langle V, W \rangle_A = 0 \), we have

\[
t_w v_2 = \frac{1}{2} (t_w v_2 - t_v w_2), \tag{3.48}
\]

and the generalized Lie derivative satisfies \( \hat{L}_V W^I = -\hat{L}_W V^I \). Accordingly, for a set of the generalized frame fields \( \{ E_α \} \) forming an isotropic subalgebra, we have

\[
\hat{L}_{E_α} E_b^I = \frac{1}{2} (\hat{L}_{E_α} E_b^I - \hat{L}_{E_b} E_α^I) = -X_{[αb]}^c E_c^I, \tag{3.49}
\]

and the subalgebra is a Lie algebra. This property plays an important role when we explicitly construct the generalized frame fields.

**\( E_n \) generators:** For the sake of clarity, let us explain our convention for the \( E_n \) generators. We decompose the \( E_n \) generators \( \{ t_α \} \) (\( α = 1, \ldots, \dim E_n \)) for \( n \leq 4 \) as \([10]\)

\[
(t_α) \equiv (K^c_d, \frac{R^{c1e2c3}}{\sqrt{3!}}, \frac{R_{c1e2c3}}{\sqrt{3!}}). \tag{3.50}
\]

Their matrix representations \( (t_α)_A^B \) in the \( R_1 \)-representation are given as follows\(^6\)

\[
(K_c^d)_A^B \equiv (\hat{K}_c^d)_A^B + \frac{\delta_d^l}{9-n} \delta_A^B, \quad (\hat{K}_c^d)_A^B \equiv \begin{pmatrix} \delta_c^e \delta_b^f - 2 \delta_{de} \delta_{bf} & 0 \\ 0 & -2 \delta_{de} \delta_{bf} \end{pmatrix},
\]

\[
(R^{c1e2c3})_A^B \equiv \begin{pmatrix} 3! \delta_{ce1} \delta_{de2} \delta_{bf3} & 0 \\ 0 & 0 \end{pmatrix}, \quad (R_{c1e2c3})_A^B \equiv \begin{pmatrix} 0 & 0 \\ 3! \delta_{ce1} \delta_{de2} \delta_{bf3} & 0 \end{pmatrix}. \tag{3.51}
\]

The matrix representations \( (t_α)_A^B \) in the \( R_2 \)-representation are

\[
(K_c^d)_A^B \equiv (\hat{K}_c^d)_A^B - \frac{2 \delta_d^l}{9-n} \delta_A^B, \quad (\hat{K}_c^d)_A^B \equiv \begin{pmatrix} \delta_c^e \delta_b^f \delta_{de} - 4 \delta_{de} \delta_{bf} \delta_{de} & 0 \\ 0 & 0 \end{pmatrix},
\]

\[
(R^{c1e2c3})_A^B \equiv \begin{pmatrix} 0 & 0 \\ 4! \delta_{ce1} \delta_{de2} \delta_{bf3} \delta_{de} & 0 \end{pmatrix}, \quad (R_{c1e2c3})_A^B \equiv \begin{pmatrix} 0 & 0 \\ 4! \delta_{ce1} \delta_{de2} \delta_{bf3} \delta_{de} & 0 \end{pmatrix}. \tag{3.52}
\]

Now, let us rewrite the \( E_n \) algebra. If we express the algebra as

\[
T_C \circ T_A = (T_C)_A^B T_B, \tag{3.53}
\]

\(^6\)The second term in the \( \text{GL}(n) \) generator \( (K^a)_A^B \), which is proportional to the identity matrix \( \delta_A^B \), is necessary for the commutator \([R^{c1e2c3}, R_{d_1d_2d_3}]\) to be expanded by the generator \( K^a \).
They can be expressed as

\[(T_c)_A^B = \begin{pmatrix}
    f_{ca}^b & -2 f_c[b_1 | a_1 \delta_{a_2}^{b_2}]
    \\
    \frac{f_c[a_2 | b_2]}{\sqrt{2}} & 0
\end{pmatrix},
\]

(3.54)

\[(T^{c_1c_2})_A^B = \begin{pmatrix}
    -f_a c_1 c_2 b & 6 f[b_1 b_2 | c_1 \delta_{a_2}^{b_2}]
    \\
    0 & -2 f_d c_1 c_2 | a_1 \delta_{a_2}^{b_2}]
\end{pmatrix}.
\]

(3.55)

In general, they are not exactly \(E_n\) U-duality transformations, because \(K^a b\) is not an \(E_n\) generator. Thus, suggested by [47][52], we introduce an additional generator \((t_0)_A^B \equiv -\delta_A^B\) for \(\mathbb{R}_+ [52]\), and express the algebra as

\[(T_c)_A^B = f_{cd}^e (K^d e)_A^B + \frac{1}{3!} f_c d_1 d_2 d_3 (R_{d_1 d_2 d_3})_A^B + \frac{f_{cd}^d}{9 - n} (t_0)_A^B,
\]

(3.56)

\[(T^{c_1c_2})_A^B = -f_d c_1 c_2 e (K^d e)_A^B + f_{d_1 d_2} [c_1 \delta_{d_3}^{c_2}] (R_{d_1 d_2 d_3})_A^B - \frac{f_d c_1 c_2 d}{9 - n} (t_0)_A^B.
\]

(3.57)

The last term in each line appears due to the fact that the generalized frame fields \(E_A^I\) has a density weight \(\frac{1}{9 - n}\). Then, the \(E_n\) algebra can be also expressed as

\[T_A \circ T_B = [\Theta_A^B (t_0)_B^C + \Theta_A (t_0)_B^C] T_C,
\]

(3.60)

where \(\Theta_A^B\) and \(\Theta_A\) are constants. If we decompose the index \(\hat{a}\) as

\[(\Theta_A^B) = (\Theta_A^B)^b_a, \quad (\Theta_A^B)^a_b = \frac{[\Theta_A^B]_a | b_1 a_2 a_3}{\sqrt{3!}}, \quad [\Theta_A^B]_a | b_1 a_2 a_3 \]

(3.61)

their components are

\[\theta_a = \frac{f_{ed}^d}{9 - n}, \quad \theta^{a_1 a_2} = \frac{f_d c_1 c_2 d}{9 - n}.
\]

Then, we can easily obtain the matrices \((T_c)_A^B\) in the \(R_2\)-representation as follows:

\[(T_c)_A^B = f_{cd}^e (K^d e)_A^B + \frac{1}{3!} f_c d_1 d_2 d_3 (R_{d_1 d_2 d_3})_A^B + \frac{f_{cd}^d}{9 - n} (t_0)_A^B,
\]

(3.62)

\[(T^{c_1c_2})_A^B = -f_d c_1 c_2 e (K^d e)_A^B + f_{d_1 d_2} [c_1 \delta_{d_3}^{c_2}] (R_{d_1 d_2 d_3})_A^B - \frac{f_d c_1 c_2 d}{9 - n} (t_0)_A^B
\]

(3.63)
where \((t_0)_{ij}^R = 2 \delta_{ij}^R\). The invariance of the bilinear form under \(E_n \times \mathbb{R}^+\) transformations leads to the following identity:

\[
\langle T_A \circ T_B, T_B \rangle_D + \langle T_A, T_C \circ T_B \rangle_D + (T_C)_{D \lambda} \langle T_A, T_B \rangle_E = 0. \tag{3.65}
\]

**Lie algebra of the Drinfel’d double:** If we decompose the generators as \(\{T_a\} = \{\dot{T}_a, T_a\}\) and \(\{T_{ab}\} = \{\dot{T}_{a\dot{b}}, T_{a\dot{b}}\}\) \((\dot{a} = 1, \ldots, n - 1)\) and require

\[
f_{ab}^c = 0, \quad f_{a\dot{b}}^{\dot{c}} = 0, \quad f_{\dot{a}b\dot{c}}^z = 0, \quad f_{\dot{a}b\dot{c}\dot{b}} = 0, \tag{3.66}
\]

the subalgebra spanned by

\[
(T_{\dot{A}}) \equiv (T_{\dot{a}}, T_{\dot{a}}) \quad (T_{\dot{a}} \equiv T_{\dot{a}z}), \tag{3.67}
\]

becomes

\[
T_{\dot{a}} \circ T_{\dot{b}} = f_{\dot{a}\dot{b}}^{\dot{c}} T_{\dot{c}}, \quad T_{\dot{a}} \circ T_{\dot{b}} = \tilde{f}_{\dot{a}\dot{b}}^{\dot{c}} T_{\dot{c}} - f_{\dot{a}\dot{b}}^{\dot{c}} T_{\dot{c}} = -T_{\dot{b}} \circ T_{\dot{a}}, \quad T_{\dot{a}} \circ T_{\dot{b}} = \tilde{f}_{\dot{a}\dot{b}}^{\dot{c}} T_{\dot{d}}, \tag{3.68}
\]

where \(\tilde{f}_{\dot{a}\dot{b}}^{\dot{c}} = -f_{\dot{a}\dot{b}}^{\dot{c}}\). This is precisely the Lie algebra of the Drinfel’d double. Moreover, we can easily see that the bilinear form reduces to that of the Drinfel’d double,

\[
\langle T_{\dot{a}}, T_{\dot{b}} \rangle \equiv \langle T_{\dot{a}}, T_{\dot{b}} \rangle_z = \delta_{\dot{a}}^{\dot{b}}. \tag{3.69}
\]

The invariance \([3.65]\) reduces to the standard \(ad\)-invariance,

\[
\langle T_C \circ T_A, T_B \rangle + \langle T_A, T_C \circ T_B \rangle = 0. \tag{3.70}
\]

In this sense, the Leibniz algebra \(E_n\) is an extension of the Lie algebra of the Drinfel’d double.

It is noted that there exist certain Drinfel’d doubles, which are not straightforwardly embedded into the \(E_n\) algebra. When the assumption \([3.66]\) is satisfied, the Leibniz identity \([3.33]\) for the restricted generators \(T_{\dot{A}}\) is automatically satisfied. However, if we require the Leibniz identity \([3.33]\) for the full \(E_n\) generators, \([3.33]\) is equivalent to \(f_{\dot{d}_1 \dot{d}_2}^{\dot{a} \dot{a}} f_{\dot{d}_1 \dot{d}_2}^{\dot{b} \dot{b}} = 0\). Then, even under the assumption \([3.66]\), we obtain a constraint

\[
f_{\dot{c}_1 \dot{c}_2}^{\dot{a}} \tilde{f}_{\dot{c}_1 \dot{c}_2}^{\dot{b}} = 0, \tag{3.71}
\]

for the structure constants of the Drinfel’d double (the Leibniz identity \([3.34]\) also may give an additional constraint). As we discuss in section \([\text{6}]\) in the context of the Yang–Baxter (YB) deformation, the condition \([3.71]\) is equivalent to the requirement that the classical \(r\)-matrix is unimodular. This means that, when the classical \(r\)-matrix is non-unimodular, the Lie algebra of the corresponding Drinfel’d double cannot be embedded into the \(E_n\) algebra. This may be related to the fact \([\text{66}]\) that the YB deformation for a non-unimodular \(r\)-matrix generally produces a solution of the generalized supergravity \([\text{67,68}]\), and the fact that the embedding of the generalized supergravity into EFT is non-trivial \([\text{69}]\) (see section \([\text{6}]\) for further discussion).
**U-duality transformation:** Let us consider a redefinition of the $E_n$ generators,  
\[ T'_A = C_A^B T_B , \]  
where $C_A^B$ is an element of the $E_n$ group. We also redefine the bilinear-form as  
\[ \langle \cdot, \cdot \rangle'_A = C_A^B \langle \cdot, \cdot \rangle'_B , \]  
by acting the same $E_n$ transformation in the $R_2$-representation. Then, the physical subalgebra is maximally isotropic even after the redefinition,  
\[ \langle T'_a, T'_b \rangle' = 0 . \]  
On the other hand, the $E_n$ algebra is transformed as  
\[ T'_A \circ T'_B = [\Theta^\alpha_A (t_\alpha)_{BC} + \theta'_{A} (t_{0})_{B} C] T'_C , \]  
where we have defined  
\[ \Theta^\alpha_A = C_A^B \Theta^\hat{\beta}_B C \hat{\beta}_B , \]  
\[ C_A^C (t_\alpha)_{C} D (C^{-1})_D B \equiv C_{\alpha}^\hat{\beta} (t_{\hat{\beta}}) A B , \]  
\[ \theta'_A = C_A^B \theta_B . \]  

If fact, the particular forms of $\Theta^\alpha_A$ and $\theta_A$ given in (3.62) are not preserved under a general $U$-duality transformation. For example, we are assuming $[\Theta_a]_{c_1c_2c_3} = 0$, but it can appear under a general redefinition (see section 3 for such an example). The situation is the same as the Drinfel’d double. In the case of the Drinfel’d double, an extension of the algebra including the non-vanishing $H$-flux (which corresponds to $[\Theta_a]_{c_1c_2c_3}$) has been discussed in [24], but here we do not consider such extension. Rather, we restrict the $U$-duality transformation such that $\Theta^\alpha_A$ and $\theta'_A$ have the same form as (3.62) by using new structure constants $f'_a^{bc}$. Even under such restriction, the allowed $U$-duality symmetry is much larger than the case of the PL $T$-duality.

### 4 Generalized frame fields

In this section, we present a systematic construction method of the generalized frame fields $E_A^I$, which is analogous to the one known in the PL $T$-duality. Then, by following the approach of [70], we show that the $E_A^I$ indeed satisfy the desired relation,  
\[ \mathcal{L}_{E_A} E_B^I = -X_{AB}^C E_C^I , \]  
where $X_{AB}^C$ is the structure constant $F_{AB}^C$ of the Leibniz algebra $E_n$.

Let us prepare a set of generators $T_a$ associated with a maximal isotropic subalgebra. As already explained, the subalgebra is a Lie algebra, and we can parameterize an element of the Lie group $G$ as usual, e.g., $g = e^{x^a T_a}$. We define the left-/right-invariant 1-forms/vectors as  
\[ g^{-1} dg \equiv \ell^i_a T_a dx^i , \quad dg g^{-1} \equiv r^a_i T_a dx^i , \quad \ell^a_i v_b^i = r^a_i e_b^i = \delta^a_b , \]
which satisfy

\[ [v_a, v_b]^i = f_{abc}^i e_c^i, \quad [e_a, e_b]^i = -f_{abc}^i e_c^i. \]  

(4.3)

Then, we define the action of \( g^{-1}(x) \equiv e^{h(x)} \) on \( T_A \) as

\[
g^{-1}(x) \circ T_A = 1 + h \circ T_A + \frac{1}{2!} h \circ (h \circ T_A) + \frac{1}{3!} h \circ (h \circ (h \circ T_A)) + \cdots
\]

\[
\equiv M_A^B(x) T_B.
\]

(4.4)

Since the infinitesimal transformation is an \( E_n \times \mathbb{R}^+ \) transformation of the lower-triangular form \([3.64]\), the matrix \( M_A^B \) can be generally parameterized as

\[
M_A^B = \begin{pmatrix} a_{ab} & 0 \\ -\frac{\Pi^{[a_1a_2c]}e_{bc}^i}{\sqrt{2!}} & (a^{-1})_{[a_1} \frac{c}{a_2]} \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{\Pi^{[a_1a_2c]}e_{bc}^i}{\sqrt{2!}} \end{pmatrix}.
\]

(4.5)

Then, we define the generalized frame fields as

\[
E_A^I \equiv M_A^B L_B^I = \begin{pmatrix} e_a^i \\ \frac{\Pi^{[a_1a_2c]}e_{bc}^i}{\sqrt{2!}} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{\Pi^{[a_1a_2c]}e_{bc}^i}{\sqrt{2!}} \end{pmatrix},
\]

(4.6)

where the matrix \( L_A^I \) is defined by

\[
L_A^I = \begin{pmatrix} 0 \\ \frac{\Pi^{[a_1a_2c]}e_{bc}^i}{\sqrt{2!}} \end{pmatrix}.
\]

(4.7)

and we have used \( \ell_i^a = a_b^a r_i^b \) in the second equality of (4.6). This matrix \( E_A^I \) plays the role of the desired generalized frame fields as we show below. For this purpose, let us find several identities by following [70].

Differential identities: Differentiating the definition (4.4) of the matrix \( M_A^B \), we obtain

\[
\partial_i g^{-1}(x) \circ T_A = \partial_i M_A^B(x) T_B.
\]

(4.8)

The left-hand side can be evaluated as

\[
\partial_i g^{-1} \circ T_A = -g^{-1} \circ \partial_i g \circ g^{-1} \circ T_A = -\left( \ell_i^d T_d \right) \circ (M_A^B T_B) = -\ell_i^d M_A^B (T_d) B^C T_C
\]

\[
= \ell_i^d \left( \frac{\Pi^{[a_1a_2c]} a_{bc} f_{ab}^{[a_1} (a^{-1})_{b2]} b_{c]} d_{c1} (a_1)_{b2} a_2 f_{d}^{b_{1} b_{2} c} 2 (a^{-1})_{[a_1} a_1 b_{2]} a_2 f_{d}^{b_{1} b_{2} c} d_{c1} (a_1)_{b2} a_2 f_{d}^{b_{1} b_{2} c} a_3 b_{c} d_{d}^{b_{1} b_{2} b_{3}} \right) T_C,
\]

(4.9)

and (4.8) gives the following identities:

\[
D_e a_a^b = a_d^d a_c^e f_{de}^b, \quad D_e \Pi^{[a_1a_2a_3} = (a^{-1})_{b_1} a_1 (a^{-1})_{b_2} a_2 a_3 a_c^d f_{d}^{b_{1} b_{2} b_{3}}.
\]

(4.10)
**Algebraic identities:** In order find further relations, we consider the identity,

$$(g \circ T_A) \circ (g \circ T_B) = g \circ (T_A \circ T_B) ,$$  \hspace{1cm} (4.11)

which follows from the Leibniz identity. For convenience, we decompose this identity as

$$\langle (g \circ T_A) \circ (g \circ T_B), T_C \rangle_D = \langle g \circ (T_A \circ T_B), T_C \rangle_D .$$ \hspace{1cm} (4.12)

The component \( \{A, B, C, D\} = \{a_1a_2, b, c, d\} \) or \( \{A, B, C, D\} = \{a, b_1b_2, c, d\} \) leads to

$$(a^{-1})_{a}^{e} (a^{-1})_{b}^{f} a_{g} f_{e} f_{g} = f_{ab} c .$$ \hspace{1cm} (4.13)

On the other hand, the component \( \{A, B, C, D\} = \{a, b_1b_2, c_1c_2, d\} \) additionally requires

$$a_{a}^{c} (a^{-1})_{f}^{b_1} (a^{-1})_{g}^{b_2} (a^{-1})_{h}^{b_3} f_{i} f_{j} f_{k} = f_{a}^{b_1b_2b_3} + 3 f_{ac}^{b_1} \Pi^{b_3c} ,$$ \hspace{1cm} (4.14)

and the component \( \{A, B, C, D\} = \{a_1a_2, b, c_1c_2, d\} \) also requires

$$f_{e_{1\ell 2}}^{[a_1 \delta_{b}^{a_2}] \Pi^{c_2]c_2} = 0 \iff f_{ab}^{c} \Pi^{c} = 0 .$$ \hspace{1cm} (4.15)

The component \( \{A, B, C, D\} = \{a_1a_2, b_1b_2, c, d\} \) further gives

$$3 \left( f_{e_{[c}^{a_1} \delta_{d]}^{a_2} \Pi^{b_1b_2]e} - f_{e_{[c}^{a_2} \delta_{d]}^{a_1} \Pi^{b_1b_2]e} \right) + f_{e_{d}^{[a_1 \Pi^{a_2]}b_1b_2] = 0 .}$$ \hspace{1cm} (4.16)

Finally, the component \( \{A, B, C, D\} = \{a_1a_2, b_1b_2, c_1c_2, d\} \) gives

$$f_{d}^{b_1b_2c} \Pi^{a_1a_2d} - 3 f_{d}^{a_1a_2[b_1 \Pi^{b_2c]}d = 3 f_{d}^{c} \Pi^{b_1b_2]d \Pi^{a_1a_2e} - 3 f_{d}^{a_1 \Pi^{a_2[}d[b_1 \Pi^{b_2c]c}} ,}$$ \hspace{1cm} (4.17)

and they are all identities coming from (4.11).

**Computation of \( X_{AB}^{C} \):** By using the differential and algebraic identities, we can easily show \( X_{a}^{b_1b_2c} = f_{a}^{b_1b_2c} \) and \( X^{a_1a_2b}_b = -f_{b}^{a_1a_2c} \). The derivation of

$$X^{a_1a_2b_1b_2c} = 0 , \quad X^{a_1a_2b_1b_2c_1c_2} = -4 f_{d}^{a_1a_2[b_1 \delta_{c_1c_2]d} ,}$$ \hspace{1cm} (4.18)

requires a slightly longer computation. The former requires the identity (4.17) while the latter requires (4.16). In this way, we have shown the desired relation \( X_{AB}^{C} = F_{AB}^{c} \).

In summary, by using the \( E_{n} \) algebra, we have explained a systematic construction of the generalized frame fields \( E_{A}^{I} \), which satisfy the algebra of \( E_{n} \) by means of the generalized Lie derivative. The construction is a straightforward extension of the procedure known in the PL T-duality, and we expect that this extension plays an important role in formulating the U-duality extension of the PL T-duality.
5 Examples of $\mathcal{E}_n$ algebra

5.1 3D algebra $\mathcal{E}_2$

When $n = 2$, we obtain a three-dimensional algebra with generators $\{T_A\} = \{T_1, T_2, T^{12}\}$. By denoting $f_{12}^1 = a$ and $f_{12}^2 = b$, we obtain

\begin{align*}
T_1 \circ T_2 &= a T_1 + b T_2 = [T_1, T_2], \\
T_1 \circ T^{12} &= -b T^{12}, \quad T_2 \circ T^{12} = a T^{12}, \quad T^{12} \circ T_A = 0. \tag{5.1}
\end{align*}

The non-vanishing components of the bilinear form are

$$
\langle T_1, T^{12} \rangle_2 = -1, \quad \langle T_2, T^{12} \rangle_1 = 1. \tag{5.3}
$$

This is not an interesting example, but it is a good example to clearly see the existence of another maximal isotropic subalgebra. As we can clearly see from (5.3), the generator $T^{12}$ has non-vanishing inner products with other generators. This shows that the Abelian algebra generated by $\{T_a\} = \{T^{12}\}$ is another maximal isotropic subalgebra. Similarly, $\mathcal{E}_n$ always has two types of maximal isotropic subalgebras with dimension $n$ and $n - 1$.

5.2 6D algebra $\mathcal{E}_3$

The algebra $\mathcal{E}_3$ is a six-dimensional algebra with generators $\{T_A\} = \{T_1, T_2, T_3, T^{12}, T^{13}, T^{23}\}$. The structure constants $f_{abc}$ have 9 components and $f_{abcd}$ have 3 components. According to the Bianchi classification, the 3D Lie algebra $f_{abc}$ has been classified. It is interesting to classify the additional structure constants $f_{abcd}$ for each physical 3D Lie algebra.

5.3 10D algebra $\mathcal{E}_4$

**M-theory frame I:** In $n = 4$, the algebra $\mathcal{E}_4$ is ten-dimensional and the structure is much richer. As a particular example, we here consider the case,

$$
f_{ab}^c = 0, \quad f_1^{234} = a, \quad f_1^{134} = b, \quad f_2^{234} = c, \quad f_2^{134} = d, \tag{5.4}
$$

which satisfies the Leibniz identity. If we introduce an additional non-vanishing component for $f_a^{b_1 b_2 b_3}$, the Leibniz identity is broken, and in that sense it contains a maximal set of components under $f_{ab}^c = 0$. Using the generators $T_a$, we parameterize an element of the physical subgroup as $g = e^{x^a T_a}$. As it is Abelian, the left-/right invariant forms are trivial,

$$
\ell = r = T_a \, dx^a. \tag{5.5}
$$

On the other hand, the tensor $\Pi^{i_1 i_2 i_3} \equiv e_{i_1}^{a_1} e_{i_2}^{a_2} e_{i_3}^{a_3} \Pi^{a_1 a_2 a_3}$ has the form

$$
\Pi = [(b \, x^1 + d \, x^2) \, \partial_1 + (a \, x^1 + c \, x^2) \, \partial_2] \wedge \partial_3 \wedge \partial_4. \tag{5.6}
$$
By construction, the $R$-flux $X^{a_1a_2b_1b_2c}$ should vanish, and it satisfies
\[ \Pi^{ijjk} \partial_k \Pi^{ijjl} - 3 \Pi^{ijjl} \partial_k \Pi^{ijjk} = 0. \] (5.7)

In order for this to be a Nambu–Poisson tensor, the algebraic or quadratic identity,
\[ \Pi^{k[ij} \Pi^{l]}_{jk} + \Pi^{l[ij} \Pi^{k]}_{jk} = 0, \] (5.8)

should be satisfied [64] (see also [71]). In this example, it is indeed satisfied and the above
$\Pi$ is a Nambu–Poisson tensor. In general, we have not checked the quadratic identity, but it
may follow from a certain requirement such as the Leibniz identity.

By using the trivial right-invariant vector and the Nambu–Poisson structure, the general-
ized frame fields become
\[ E_A^M = \left( \begin{array}{c} e_m^a \\ -\frac{\Pi^{a_1a_2m}}{\sqrt{2!}} \\ -\frac{\Pi^{a_1a_2m} r_{a_1 a_2} r_{m_1 m_2}}{\sqrt{2!}} \end{array} \right) \] (5.9)

As we have generally proven, this satisfies the relation $\hat{E}_A E_B^J = -F_{AB}^C E_C^J$ for the structure constants given in (5.4).

**M-theory frame II:** Let us consider a redefinition,
\[
\begin{pmatrix}
T_1 \\
T_2 \\
T_3 \\
T_4 \\
T_{t12} \\
T_{t13} \\
T_{t14} \\
T_{t23} \\
T_{t34}
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
T_1 \\
T_2 \\
T_3 \\
T_4 \\
T_{t12} \\
T_{t13} \\
T_{t14} \\
T_{t23} \\
T_{t34}
\end{pmatrix}.
\] (5.10)

This is a map
\[
M\text{-theory} \rightarrow \text{Type IIA} \rightarrow \text{Type IIA} \rightarrow \text{M-theory},
\] (5.11)

corresponding to a double Abelian $T$-duality, and it is a particular $U$-duality transformation.

After the redefinition, we find the new physical generators satisfy
\[ [T'_a, T'_b] = f'_a{}^c T'_c + \frac{1}{2T} f'_a{}^{b c} T_{t12}^{b c}, \] (5.12)

where $f'_{a_1\ldots a_4} \equiv f'_{[a_1\ldots a_4]}$ and
\[ f'_{13} = b, \quad f'_{23} = -c, \quad f'_{34} = c, \quad f'_{1234} = -d, \quad f'_{1234} = -a. \] (5.13)
The component $f'_{abc}c_2$ is not allowed in the $E_4$ algebra, and we can consider this $U$-duality transformation only when $a = 0$. Moreover, the algebra of other generators further requires $c = 0$. Under $a = c = 0$, the $U$-duality converts the structure constants of the $E_4$ algebra as

$$f'_{13}^1 = b, \quad f'_{123}^{12} = -d.$$  \hspace{1cm} (5.14)

Again, we can easily construct the generalized frame fields realizing this algebra.

**Type IIB frame:** Let us consider another redefinition of the $E_4$ generators,

$$\{T_A\} \equiv \{T_1, T_2, T^{34}, T^{14}, T^{24}, T_3, -T^{13}, -T^{23}, T_4, T^{12}\}.  \hspace{1cm} (5.15)$$

This map has been considered in [72], which connects the M-theory picture and the type IIB picture (see [41, 58] for earlier discussion). This is not a $U$-duality transformation but rather corresponds to a change in the picture, from M-theory to type IIB theory.

Under $a = c = 0$, the $U$-duality converts the structure constants of the $E_4$ algebra as

$$f'_{13}^1 = b, \quad f'_{123}^{12} = -d.$$  \hspace{1cm} (5.14)

Again, we can easily construct the generalized frame fields realizing this algebra.

The flux $f_{1234}^1 = a$ corresponds to the $Q$-flux $Q_1^{23} = -a$ in type IIA theory and the double $T$-duality transforms it to the $H$-flux $H_{123} = -a$. The 11D uplift corresponds to $f'_{1234} = -a$. The reason may be understood as follows. Originally, the II$^*_{12}z^{23}$ has the $x^2$-dependence, but under the double $T$-dualities, $x^2$ is mapped to $y_{24}$. The dependence on the dual coordinate breaks our assumption (3.2). Accordingly, the resulting algebra has a different form from $E_4$.

One can see that the bilinear form is not invariant under the transformation (see [74] for the transformation rule of the index $A$ under this redefinition).
the M-theory picture, we can construct a solution of EFT that is twisted by the matrix \((5.9)\) with \(a = 0\), \(b = -1\), \(c = 1\), and \(d = 0\). Under the change of the generators, the solution is mapped to the type IIB solution of \([60]\). However, it is not so interesting because it is nothing more than the straightforward 11D uplift of the PL T-duality. In the type IIA picture, \((5.4)\) reduces to the Lie algebra of the Drinfel’d double [recall \((3.68)\) and choose \(z = 4\)]

\[
f_{abc} = 0, \quad \tilde{f}^{13} = 1, \quad \tilde{f}^{23} = -1,
\]

where the physical algebra is Abelian and the dual algebra is Bianchi type \(6_0\). Then, the redefinition \((5.15)\) corresponds to a non-Abelian T-duality. In order to find genuinely U-duality examples, it is important to study the detailed classification of the \(E_n\) algebra.

\section{Summary and Discussion}

\textbf{Summary:} When we perform the PL T-duality, a systematic construction of the generalized frame fields satisfying \(\hat{L}_{E_A} E_B^M = -F_{ABC} E_C^M\) with a constant \(F_{ABC}\) is useful. In this paper, by considering the U-duality extension of the PL T-duality, we have proposed a Leibniz algebra \(E_n\), which extends the Lie algebra of the Drinfel’d double. Then, we have shown that this provides a systematic way to construct the generalized frame fields in EFT, which satisfy the \(E_n\) algebra \(\hat{L}_{E_A} E_B^I = -F_{ABC} E_C^I\) by means of the generalized Lie derivative in EFT.

\textbf{Straightforward extensions:} In this paper, we have concentrated on the case \(n \leq 4\), but the extension to higher \(n\) will be straightforward. Since the generators \(T_A\) are transforming in the \(R_1\)-representation, for higher \(n\), we introduce the following generators:

\[
\{T_A\} = \{T_a, T^{a_1a_2}, T^{a_1\cdots a_5}, T^{a_1\cdots a_7}, \ldots \}.
\]

According to the success of the \(E_{11}\) conjecture \([35, 40]\), it will be possible to extend \(n\) up to \(n = 11\). Here we have almost restricted to the M-theory picture, but if we consider the type IIB picture, the generators are parameterized as

\[
\{T_A\} = \{T_a, T^a, T^{a_1a_2a_3}, T^{a_1\cdots a_5}, T^{a_1\cdots a_6}, \ldots \}.
\]

The invariant bilinear forms in both the M-theory/type IIB pictures are also well studied in EFT (see \([61]\) for \(n \leq 7\)). The algebra should always have the form

\[
T_A \circ T_B = [\Theta_A^C (t_a) B^C + \theta_A^C (t_0) B^C] T_C,
\]

and what we need to do for higher \(n\) will be to consistently find the constants \(\Theta_A^C\) and \(\theta_A^C\). The construction method of \(E_A^I\) also will be straightforwardly extended to higher \(n\).
Towards non-Abelian $U$-duality: The most interesting application of our result is the $U$-duality extensions of the PL $T$-duality (which may be called the Nambu–Leibniz $U$-duality). In order to study non-trivial examples of such $U$-duality, the decompositions of the $\mathcal{E}_n$ algebra into the physical and the dual subalgebras need to be classified. In the case of the Drinfel’d double, such decomposition is known as the Manin triple, and the classification for six-dimensional case has been worked out in [73]. The extension of such classification for each $\mathcal{E}_n$ algebra is important. A major difference from the case of the Drinfel’d double is in the existence of the two types of subalgebras with dimension $n$ and $n - 1$. Another difference is that the dual algebra of $\mathcal{E}_n$ (generated by $T^{a_1a_2}$) is not maximally isotropic and accordingly is a Leibniz algebra. Namely, unlike the case of the Drinfel’d double, the $\mathcal{E}_n$ algebra is decomposed into an $n$-dimensional physical Lie algebra and a $(D_n - n)$-dimensional dual Leibniz algebra. It is also noted that the $\mathcal{E}_n$ algebra in the M-theory picture and the type IIB picture may not be exactly the same in general. In the M-theory picture, we introduced the structure constants $f_{ca}^b$ and $f_{c[a_1a_2a_3}^a$ corresponding to the $E_n$ generators $K^a_b$ and $R_{a_1a_2a_3}$ but do not introduce $f_{c[a_1a_2a_3}$ that corresponds to $R^{a_1a_2a_3}$. On the other hand, in the type IIB picture, we may introduce $f_{c[a}^b$, $f_c^{(a|\beta)}$, and $f_{c[a_1a_2]}^a$ corresponding to the $E_n$ generators $K^a_{b}$, $R_{(a|\beta)}$, and $R_{a_1a_2}$, but will not introduce $f_{c[a_1a_2]}^a$ that is associated with $R^{a_1a_2}$. Then, the number of the structure constants does not match between the two pictures. It may coincide after imposing the Leibniz identity but it is not obvious and is important to study the correspondence in detail.

It is also important to study the flux-formulation of EFT. In the case of gauged DFT [26–31], the action and equations of motion are expressed purely by using the generalized flux $F_{ABC}$ (and additional flux $F_A$). Moreover, when the flux is constant, the equations of motion reduce to the algebraic equations (2.18). A similar analysis has been done in [52], and the action of EFT is expressed by the generalized fluxes $X_{ABC}$. If the equations of motion are also expressed by using the fluxes, and if they reduce to simple algebraic equations when $X_{ABC}$ is constant, we can clearly see the symmetry of the non-Abelian $U$-duality.

Duality in the membrane sigma model: It is important to study the duality symmetry also in the context of the membrane sigma model. Originally, the PL $T$-dualizability condition has been found in the form,

$$\mathcal{L}_{v_a} E_{mn} = -\delta^{bc}_a E_{mp} v^p_b v^q_c E_{qn},$$

where $E_{mn} \equiv g_{mn} - B_{mn}$. By solving the differential equation with the help of the Drinfel’d double, the twist matrix (2.39) has been obtained. The condition (6.4) shows that the equations of motion of the string sigma model are expressed as a Maurer–Cartan equation,

$$dJ_a - \frac{1}{2} \delta^{bc}_a J_b \wedge J_c = 0, \quad J_a \equiv v^m_a (g_{mn} * dx^n - B_{mn} dx^n),$$

22
and this plays an important role in realizing the PL $T$-duality as a symmetry in the equation of motion of string theory. When the matrix $E_{mn}$ is invertible, (6.4) is equivalent to

$$\mathcal{L}_{v_a} E^{mn} = \tilde{f}^{bc} v_b^m v_c^n, \quad (6.6)$$

and if we define a dual metric $\tilde{g}_{mn}$ and a bi-vector $\beta^{mn}$ through the relation

$$\tilde{g}^{mn} + \beta^{mn} = (g + B)^{-1} \quad (6.7)$$

the dualizability condition is expressed as

$$\mathcal{L}_{v_a} \tilde{g}_{mn} = 0, \quad \mathcal{L}_{v_a} \beta^{mn} = -\tilde{f}^{bc} v_b^m v_c^n. \quad (6.8)$$

In fact, we can easily find a similar relation in our setup. If we define the generalized metric as

$$M_{IJ} = |v|^2 E_I^A E_J^B \mathcal{M}_{AB} \quad (6.9)$$

and if we define a dual metric $\tilde{g}_{ij}$ and a bi-vector $\beta^{ij}$ through the relation

$$\tilde{g}_{ij} = e_i^a e_j^b \kappa_{ab}, \quad \Omega^{ij \ell i_3} = \Pi^{ij \ell} i_3 \equiv c^{i_1} e^{i_2} e^{i_3} \Pi a^{i_2} i_3. \quad (6.10)$$

Then, we can show the following relation, which is the M-theory uplift of (6.8):

$$\mathcal{L}_{v_a} \tilde{g}_{ij} = 0, \quad \mathcal{L}_{v_a} \Omega^{ij \ell i_3} = \mathcal{L}_{v_a} \Omega^{ij} i_3 = f_a^{b_1 b_2} v_{b_1}^{i_1} v_{b_2}^{i_2} v_{i_3}^{i_3}. \quad (6.11)$$

It is interesting to study the implication of these relations in the context of the membrane sigma model. Perhaps, the equations of motion of the membrane sigma model can be expressed in a similar form as (6.5) and it may help to discuss the non-Abelian $U$-duality in the context of the membrane sigma model.

**Generalized Yang–Baxter deformation:** Another related direction is a generalization of the YB deformation [74–78]. As it has been observed in [79], the YB deformation is a coordinate-dependent $\beta$-deformation $\beta^{mn} \rightarrow \beta^{mn} = \beta^{mn} + r^{mn}$, associated with a bi-Killing vector $r^{mn} \equiv r_{ab} v_a^m v_b^n$, where $r_{ab} = -r_{ba}$ is a constant matrix. Here, the set of vector fields $v_a^m$ satisfies the algebra $[v_a, v_b]^m = f_{ab}^c v_c^m$ and the Killing equation $\mathcal{L}_{v_a} (g + B)_{mn} = 0$ in the undeformed background. In this case, the YB-deformed background satisfies the PL $T$-dualizability condition (6.8) with the dual structure constant given by

$$f^{b_1 b_2} = 2 r^{c\ell} f_{a\ell} b_2. \quad (6.11)$$

Interestingly, when the matrix $r^{ab}$ satisfies the homogeneous classical YB equations (CYBE),

$$f_{d_1 d_2}^{a} r^{b_1 d_1 r c_{d_2}} + f_{d_1 d_2}^{b} r^{c_{d_1} r a_{d_2}} + f_{d_1 d_2}^{c} r^{a_{d_1} r b_{d_2}} = 0, \quad (6.12)$$

23
the YB deformation always maps a DFT solution to another DFT solution. The reason can be clearly understood by noticing that the YB deformation is a specific PL $T$-duality.

Before the YB deformation, the background satisfies $\mathcal{L}_{v_a}(g + B)_{mn} = 0$ and this shows that $\tilde{f}^{ab} = 0$. Namely, in the original background, which is described by the generalized metric $\mathcal{H}_{MN} = E_M^A E_N^B \mathcal{H}_{AB}$, the fields $E_A^M$, $g_{mn}$, and $\beta^{mn}$ have the following form:

$$(E_A^M) = \left( \begin{array}{cc} e^a_m & 0 \\ 0 & r^a_m \end{array} \right), \quad g_{mn} = \kappa_{m}^{ab} \kappa_{n}^{ab} = \ell_{m}^{ab} \kappa_{n}^{ab}, \quad \beta^{mn} = e^m_a e^n_b \beta^{ab},$$

(6.13)

where $\kappa_{ab}$ and $\beta^{ab}$ are constant, and $\kappa_{ab}$ is supposed to be an invariant metric of the isometry algebra. The YB deformation corresponds to the PL $T$-duality (2.19) with

$$(C_A^B) = \begin{pmatrix} \delta^b_a & 0 \\ -r_{ab} & \delta^a_b \end{pmatrix}.$$  

(6.14)

Under this transformation, the generators become $T'_a = T_a$ and $T'^{ab} = T^{ab} - r^{ab} T_b$ and the constant fields are transformed as $\kappa_{ab} \rightarrow \kappa'_{ab} = \kappa_{ab}$ and $\beta^{ab} \rightarrow \beta'^{ab} = \beta^{ab} + r^{ab}$. By requiring that the new generators $T'_A$ satisfy the Lie algebra of the Drinfel’d double with $\tilde{f}^{b_1 b_2} \rightarrow$ given by (6.11), the matrix $r^{ab}$ must be a classical $r$-matrix satisfying (6.12). Then, using the systematic construction of $E'_A^M$, we can in principle compute the generalized frame fields

$$E'_A^M = \begin{pmatrix} \delta^b_a & 0 \\ \Pi^{ab}(x) & \delta^a_m \end{pmatrix} \begin{pmatrix} e^m_b & 0 \\ 0 & r^b_m \end{pmatrix},$$

(6.15)

and the $\beta$-field in the deformed background can be computed as

$$\beta'^{mn} = \beta^{mn} + \pi^{mn}, \quad \pi^{mn} = e^m_a e^n_b \left( \Pi^{ab} + r^{ab} \right).$$

(6.16)

The great benefit of the YB deformation is that we do not need to compute $\pi^{mn}$. It is simply given by $\pi^{mn} = r^{mn}$ because $r^{mn}$ solves the differential equation $\mathcal{L}_{v_a} r^{mn} = -\tilde{f}^{bc} v_b^a v_c^m$, and $\pi^{mn} = r^{mn}$ is trivially satisfied at the identity $g = 1$ [recall (2.15)]. Thus, once we find a classical $r$-matrix, we can easily generate a new solution. In this sense, the YB deformation is a systematic way to perform the PL $T$-duality (6.14) and the homogeneous CYBE ensure that the structure of the Drinfel’d double is preserved under the deformation.

Recently, an 11D extension of this YB deformation has been studied in [80]. There, the YB deformation is generalized to the $\Omega$-deformation $\Omega^{ij} \rightarrow \Omega^{ij} + \rho^{ij}$ associated with a tri-Killing vector

$$\rho^{ij} = \rho^{a_1 a_2 a_3} v^{a_1}_{a_2} v^{a_2}_{a_3} v^{a_3},$$

(6.17)

where $\rho^{a_1 a_2 a_3}$ is a certain constant. By assuming the Killing equations ($\mathcal{L}_{v_a} g_{ij} = 0$ and $\mathcal{L}_{v_a} C_{ij} = 0$) in the undeformed background, the $\Omega$-deformed background satisfies the
relation (6.11) with the dual structure constant given by

\[ f_a^{b_1 b_2 b_3} = 3 \rho^{c|b_1 b_2} f_{a c}^{b_3} \]  

(6.18)

Similar to the case of the YB deformation, this also can be understood as a specific non-Abelian U-duality transformation [3.72] with

\[ (C_A^B) = \begin{pmatrix} \delta_a^b & 0 \\ \rho^{a_1 a_2 b} \sqrt{2!} & \delta_{b_1 b_2} \end{pmatrix}. \]  

(6.19)

By requiring that the redefined generators \( T'_a = T_a \) and \( T'^{a_1 a_2} = T^{a_1 a_2} + \rho^{a_1 a_2 b} T_b \) to satisfy the \( \mathcal{E}_n \) algebra, we obtain

\[ f_{b_1 b_2}^{a} \rho^{b_1 b_2 c} = 0, \quad 3 \left( f_{d|c_1}^{[a_1} \delta^{[a_2}_{c_2]} \rho^{b_1 b_2]d} - f_{d|c_2}^{[a_1} \delta^{[a_2}_{c_2]} \rho^{b_1 b_2]d} \right) + f_{c_1 c_2}^{[a_1} \rho^{a_2]b_1 b_2} = 0, \]
\[ 4 f_{d_1 d_2}^{[a_1} \rho^{a_2]d_1 b_1} \rho^{b_2]c d_2} + 3 f_{d_1 d_2}^{[b_1} \rho^{b_2]c d_1} \rho^{a_1 a_2 d_2} = 0. \]  

(6.20)

The last requirement is a natural generalization of the homogeneous CYBE and the former two equations are intrinsic to the \( \mathcal{E}_n \) algebra [which correspond to (1.15) and (1.16)]. Again, the \( \Omega \)-field after the deformation is given by

\[ \Omega^{i_1 i_2 i_3} = \Omega^{i_1 i_2 i_3} + e^{i_1}_{a_1} e^{i_2}_{a_2} e^{i_3}_{a_3} (\Pi^{a_1 a_2 a_3} + \rho^{a_1 a_2 a_3}) = \Omega^{i_1 i_2 i_3} + \rho^{a_1 a_2 a_3} v^{i_1}_{a_1} v^{i_2}_{a_2} v^{i_3}_{a_3}. \]  

(6.21)

Namely, once we found a solution of the generalized CYBE [6.20], we can easily obtain the deformed background without computing the matrix \( M_A^B \). If we could show that the non-Abelian U-duality transformation [3.72] is a solution generating transformation in EFT, this tri-Killing deformation is also a solution transformation in EFT. In order to consider concrete applications, it is important to classify the solutions of the generalized CYBE [6.20].

The tri-Killing deformation can be understood as the M-theory uplift of the YB deformation in type IIA theory, where the parameter \( \rho^{a_1 a_2 a_3} \) is related to the \( r \)-matrix as \( \rho^{d b z} = r^{d b} \).

Then, the first equation in (6.20) is reduced to the unimodularity condition \( f_{b_1 b_2}^{a} \rho^{b_1 b_2} = 0 \)[10]. This unimodularity condition is precisely the condition for the YB-deformed background to satisfy the supergravity equations of motion [66]. However, the Lie algebra of the Drinfel’d double itself is consistently defined even when the unimodularity is violated. Moreover, as it is shown in [66], even in the non-unimodular case, the YB-deformed background does satisfy the equations of motion of the generalized supergravity [67][68], and it is also a solution of DFT [31]. Then, a natural question is why the non-unimodular cases are excluded from the tri-vector deformation based on the \( \mathcal{E}_n \) algebra. This can be understood as follows.

In the case of non-unimodular YB deformations, the deformed geometries are solutions of the generalized supergravity, which means that the dilaton in type IIA theory acquires a dependence on the dual coordinates \( \tilde{x}_m = y_m \) [82]. In the case of DFT, the dilaton is not

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[10] The Leibniz identity [3.71] for the dual structure constant [6.11] also reproduces the same condition.
contained in the generalized frame fields $E_A^M$ and this does not cause any problem in realizing the algebra of the Drinfel’d double as $\hat{\mathcal{E}}_{AB} = -\mathcal{F}^C_{AB} E^C_M$. However, in EFT, the dilaton is contained in the generalized frame fields $E_A^I$ and the dual-coordinate dependence conflicts with our assumption. This will be the reason why the Drinfel’d double associated with non-unimodular YB deformation cannot be embedded into the $\mathcal{E}_n$ algebra. In order to study the non-unimodular YB deformation in the context of EFT, it may be necessary to deform the $\mathcal{E}_n$ algebra by changing the choice of the section. Such deformation of the $\mathcal{E}_n$ algebra may be realized also by considering a deformation of the generalized Lie derivative as it has been considered because the introduction of the dual-coordinate dependence is equivalent to the introduction of the deformation parameters.

**Connection with mathematics:** It is interesting to investigate connections with various known facts in the mathematical literature. As we have mentioned, the $\mathcal{E}_n$ algebra is related to the Nambu–Poisson tensor, and various results on the Nambu–Poisson group (see for example [55, 71, 85]) will be useful to clarify the structure of the $\mathcal{E}_n$ algebra. In addition, the $\mathcal{E}_n$ algebra is a Leibniz algebra (rather than a Lie algebra), and it seems to have an intricate global structure. The detailed study of such global structure is also an interesting future direction.

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