Compactness of certain class of singular minimal hypersurfaces

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Abstract

Given a closed Riemannian manifold \((N^{n+1}, g)\), \(n + 1 \geq 3\) we prove the compactness of the space of singular, minimal hypersurfaces in \(N\) whose volumes are uniformly bounded from above and the \(p\)-th Jacobi eigenvalue \(\lambda_p\)'s are uniformly bounded from below. This generalizes the results of Sharp [Sha17] and Ambrozio-Carlotto-Sharp [ACS16] in higher dimensions.

1 Introduction

A hypersurface of a Riemannian manifold \((N^{n+1}, g)\) is called minimal if it is a critical point of the \(n\)-dimensional area functional. By the combined works of Almgren [Alm65], Pitts [Pit81] and Schoen-Simon [SS81] one gets the following Theorem.

**Theorem 1.1** ([Alm65], [Pit81], [SS81]). Let \((N^{n+1}, g)\) be an arbitrary closed Riemannian manifold with \(n + 1 \geq 3\). Then \(N\) contains a singular, minimal hypersurface which is smooth and embedded outside a singular set of Hausdorff dimension at most \(n-7\). In particular, when \(3 \leq n+1 \leq 7\) there exists a smooth, closed, embedded, minimal hypersurface in \(N\).

Recently, Almgren-Pitts min-max theory has been further developed to show that minimal hypersurfaces exist in abundance when the ambient dimension \(3 \leq n + 1 \leq 7\). By the results of Marques-Neves [MN17] and Song [Son18] every closed Riemannian manifold \(N\) of dimension \(3 \leq n + 1 \leq 7\) contains infinitely many minimal hypersurfaces. Moreover, Irie, Marques and Neves have shown that [MNS17] for a generic metric the union of all closed, minimal hypersurfaces is dense in \(N\); this theorem was later improved by Marques, Neves and Song in [MNS17] where they proved that for a generic metric there exists an equidistributed sequence of closed, minimal hypersurfaces in \(N\). The Weyl law for the volume spectrum proved by Liokumovich, Marques and Neves [LMN18] played a major role in the arguments of [LMN18] and [MNS17]. There is yet another proof of the existence of infinitely many closed, minimal hypersurfaces for a generic metric on \(N\) which follows from the papers by Marques-Neves [MN17] and Zhou [Zho19]. The reason of the upper bound of the dimension \(n + 1 \leq 7\)

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in [Son18, IMN18, MNS17, Zho19] is that the space of singular, minimal hypersurfaces is not well understood unlike the smooth case ([Whi91, Whi17]).

Using the Allen-Cahn equation Chodosh and Mantoulidis [CM18] have proved the existence of infinitely many minimal surfaces for generic metrics in dimension 3; Gaspar and Guaraco [GG18] have given alternative proofs of the above mentioned density and equidistribution theorems.

In higher dimensions, Li [Li19] has proved that a closed manifold $M_{n+1}, n + 1 \geq 8$ equipped with a generic Riemannian metric contains infinitely many singular, minimal hypersurfaces with optimal regularity (i.e. the singular set has Hausdorff dimension at most $n - 7$).

One of the key ingredients of the papers [Son18, IMN18, MNS17, Zho19] is Sharp’s compactness theorem [Sha17] which asserts certain compactness properties of the set of smooth, closed, minimal hypersurfaces in a Riemannian manifold $(M^{n+1}, g), 3 \leq n + 1 \leq 7$ with bounded volume and index. This result was generalized by Ambrozio, Carlotto and Sharp [ACS16] where instead of bounded volume and index, an upper bound of the volume and a lower bound of the $p$-th Jacobi eigenvalue $\lambda_p$ (for some $p \in \mathbb{N}$) was assumed. (We note that for a smooth, closed minimal hypersurface $\Sigma$, $\text{Ind}(\Sigma) \leq I$ is equivalent to $\lambda_{I+1}(\Sigma) \geq 0$.)

In the present article we will suitably generalize the results of [Sha17] and [ACS16] in higher dimensions; for that, we need to consider the minimal hypersurfaces which may have singularities. We will state the notion of the index and the $p$-th Jacobi eigenvalue for a stationary $n$-varifold and prove the following Theorem.

**Theorem 1.2.** Let $\{M_k\}_{k=1}^{\infty}$ be a sequence of closed, connected, singular, minimal hypersurfaces in a closed Riemannian manifold $(N^{n+1}, g), n + 1 \geq 3$. Let $V_k = |M_k|$, the varifold associated to $M_k$. Suppose, there exist $\Lambda > 0, \alpha \geq 0, p \in \mathbb{N}$ such that for all $k$

- $\mathcal{H}^{n-2}(\text{sing}(M_k)) = 0$
- $\mathcal{H}^{n}(M_k) = \|V_k\|(N) \leq \Lambda$
- $\lambda_p(V_k) \geq -\alpha$

Then there is a stationary, integral varifold $V$ such that possibly after passing to a subsequence, $V_k \rightharpoonup V$ in the $\mathbf{F}$ metric. Moreover, denoting $M = \text{spt}(V)$ we have

- $\|V\|(N) \leq \Lambda$
- $\lambda_p(V) \geq -\alpha$
- $\mathcal{H}^{s}(\text{sing}(M)) = 0 \quad \forall s > n - 7$
- **The convergence is smooth and graphical over the compact subsets of $\text{reg}(M) \setminus \mathcal{Y}$ where $\mathcal{Y}$ is a finite subset of $\text{reg}(M)$ with $|\mathcal{Y}| \leq p - 1$.**

From the definitions of the index and the Jacobi eigenvalue, it will be clear that
$\text{Ind}(V) \leq I$ if and only if $\lambda_{l+1}(V) \geq 0$. Therefore, Theorem 1.2 has the following Theorem as a corollary which generalizes Sharp’s compactness theorem [Sha17] in higher dimensions.

**Theorem 1.3.** Let $\{M_k\}_{k=1}^{\infty}$ be a sequence of closed, connected, singular, minimal hypersurfaces in a closed Riemannian manifold $(N^{n+1}, g)$, $n + 1 \geq 3$. Suppose for all $k$, $H^{n-2}(\text{sing}(M_k)) = 0$, $H^n(M_k) \leq \Lambda$ and $\text{Ind}(|M_k|) \leq I$. Then there is a stationary, integral varifold $V$ such that possibly after passing to a subsequence, $|M_k| \rightarrow V$ in the $F$ metric, $\|V\|(N) \leq \Lambda$ and $\text{Ind}(V) \leq I$. Further, if $M = \text{spt}(V)$ then $H^s(\text{sing}(M)) = 0$ $\forall s > n-7$ and the convergence is smooth and graphical over the compact subsets of $\text{reg}(M) \setminus \mathcal{Y}$, where $\mathcal{Y}$ is a finite subset of $\text{reg}(M)$ with $|\mathcal{Y}| \leq I$.

The space of minimal hypersurfaces with bounded volume and index is particularly interesting; due to the work of Marques and Neves [MN16], the minimal hypersurfaces constructed by the min-max procedure have bounded volume and index. More precisely, they have proved the following Theorem.

**Theorem 1.4 ([MN16]).** Suppose $(N^{n+1}, g)$ is a closed Riemannian manifold, $n+1 \geq 3$. Let $X$ be an $m$ dimensional simplicial complex and $\Pi$ be a $F$-homotopy class of continuous maps from $X$ to $\mathbb{Z}_n(N; F; \mathbb{Z}_2)$. We define

$$L(\Pi) = \inf_{\Phi \in \Pi} \sup_{x \in X} \{M(\Phi(x))\}$$

Then there is a stationary, integral varifold $V$ with $\text{spt}(V) = \Sigma$ such that

- $\|V\|(N) = L(\Pi)$
- $\text{Ind}(V) \leq m$
- $H^s(\text{sing}(\Sigma)) = 0$ $\forall s > n-7$.

The index upper bound of the minimal hypersurfaces in the Allen-Cahn settings has been proved by Gaspar [Gas17] and Hiesmayr [Hie17].

If we take $M_k$ to be $M$ for all $k$ in Theorem 1.2, we get the following regularity result.

**Proposition 1.5.** Let $M^n$ be a singular, minimal hypersurface in $(N^{n+1}, g)$, $n + 1 \geq 3$. Suppose, $H^{n-2}(\text{sing}(M)) = 0$ and $\lambda_p(|M|) > -\infty$ for some $p$. Then $H^s(\text{sing}(M)) = 0$ $\forall s > n-7$.

The proof of Theorem 1.2 is very similar to that of [Sha17] and [ACS16]. However, for the sake of completeness we will give a self-contained proof of it.

**Acknowledgements.** I am very grateful to my advisor Prof. Fernando Codá Marques for many helpful discussions and for his support and guidance. I also thank Yangyang Li and Antoine Song for answering some of my questions. The author is partially supported by NSF grant DMS-1811840.
2 Notations and Preliminaries

2.1 Notations

Here we summarize the notations which will be frequently used later.

\[ V_n(U) \] the set of \( n \)-varifolds in \( U \)
\[ IV_n(U) \] the set of integral \( n \)-varifolds in \( U \)
\( \mathcal{H}^s \) the Hausdorff measure of dimension \( s \)
\( \| V \| \) the Radon measure associated to the varifold \( V \)
\( | \Sigma | \) the varifold associated to a singular hypersurface \( \Sigma \)
\( \delta^2 V \) 2-nd variaration of the stationary varifold \( V \)
\( \text{Ind}(\cdot) \) index (of a stationary hypersurface or varifold)
\( \lambda_k(\cdot) \) \( k \)-th Jacobi eigenvalue (of a stationary hypersurface or varifold)
\( \text{sing}(\Sigma) \) the singular part of \( \Sigma \)
\( \text{reg}(\Sigma) \) the regular part of \( \Sigma \)
\( B(p, r) \) open ball of radius \( r \) centered at \( p \)

2.2 Preliminaries from geometric measure theory

Here we will briefly discuss the notion of varifold and various related concepts; further details can be found in Simon’s book [Sim].

Given a Riemannian manifold \( (U^{n+1}, g) \) let \( G_k(U) \) denote the Grassmanian bundle of \( k \)-dimensional hyperplanes over \( U \). A \( k \)-varifold in \( U \) is a positive Radon measure on \( G_k(U) \). The topology on the space of \( k \)-varifolds \( V_k(U) \) is given by the weak* topology i.e. a net \( \{ V_i \}_{i \in I} \subset V_k(U) \) converges to \( V \) iff

\[ \int_{G_k(U)} f(x, \omega) dV_i(x, \omega) \longrightarrow \int_{G_k(U)} f(x, \omega) dV(x, \omega) \]

for all \( f \in C_c(G_k(U)) \). This topology is metrizable and the metric is denoted by \( F \). If \( V \in V_k(U) \) and \( \pi : G_k(U) \longrightarrow U \) denotes the canonical projection then \( \| V \| = \pi_! V \) is a Radon measure on \( U \); \( \| V \|(A) = V(\pi^{-1}(A)) \).

If \( \varphi : U \longrightarrow U' \) is a diffeomorphism and \( V \in V_k(U) \), we define \( \varphi_* V \in V_k(U') \) by the following formula

\[ (\varphi_* V)(g) = \int_{G_k(U)} g(\varphi(x), dx_\varphi) J_\varphi(x, \omega) dV(x, \omega) \]

where

\[ J_\varphi(x, \omega) = \left( \det \left( \left( d_{x\varphi} \varphi \right)^t \circ \left( d_x \varphi \right) \right) \right)^{1/2} \]

is the Jacobian factor and \( g \in C_c(G_k(U')) \). Given a compactly supported, smooth vector-field \( X \) on \( U \) let \( \varphi_t \) denote the flow of \( X \); the first variation and second variation of \( V \) are given by
\[ \delta V(X) = \frac{d}{dt} \bigg|_0 \| (\varphi_t)_* V \| (U) \quad ; \quad \delta^2 V(X, X) = \frac{d^2}{dt^2} \bigg|_0 \| (\varphi_t)_* V \| (U) \]

We say that \( V \) is stationary if \( \delta V(X) = 0 \) for all \( X \) and a stationary varifold \( V \) is called stable if \( \delta^2 V(X, X) \geq 0 \) for all \( X \).

Given a \( k \)-rectifiable set \( S \subset U \) and a non-negative function \( \theta \in L^1_{\text{loc}}(S, \mathcal{H}^k S) \) we define the \( k \)-varifold \( v(S, \theta) \) by

\[ v(S, \theta)(f) = \int_S f(x, T_x S) \theta(x) d\mathcal{H}^k(x) \]

where \( T_x S \) denotes the tangent space of \( S \) at \( x \) which exists \( \mathcal{H}^k S \)-a.e. \( V \) is called an integral \( k \)-varifold if \( V = v(S, \theta) \) for some \( S \) and \( \theta \) with \( \theta \) taking non-negative integer values \( \mathcal{H}^k S \)-a.e.

In the present article we will only deal with \( n \)-varifolds and from now on we will simply write ‘varifold’ instead of ‘\( n \)-varifold’. Given \( A \subset U \) we define the regular and singular part of \( A \)

\[ \text{reg}(A) = \{ x \in A : \exists \text{ open } P \subset U \text{ containing } x \text{ such that } P \cap A \text{ is a smooth, embedded hypersurface} \} \]

and

\[ \text{sing}(A) = A \setminus \text{reg}(A). \]

Further, by a singular, minimal hypersurface \( \Sigma \) in \( U \) we will mean that \( \Sigma \subset U \) is a closed, \( n \)-rectifiable set with \( \mathcal{H}^{n-1}(\text{sing}(\Sigma)) = 0 \) and \( |\Sigma| = v(\Sigma, 1_\Sigma) \) (where \( 1_\Sigma \) is the constant function 1) is stationary. By [Ilm96] (Equation (4)), \( |\Sigma| \) is stationary in \( U \) if and only if \( \text{reg}(\Sigma) \) is a smooth, minimal hypersurface and \( \mathcal{H}^n(\Sigma \cap B(x, r)) \leq C(U') r^n \) for all \( B(x, r) \subset U' \subset U \).

3 Index and Jacobi eigenvalues of a stationary varifold

We will now state the notion of the index and the Jacobi eigenvalues of a stationary varifold following the paper by Marques and Neves [MN16]. The definition is motivated by the following min-max characterization of \( \lambda_k(\Sigma) \) when \( \Sigma^n \subset (U^{n+1}, g) \) is a smooth, minimal hypersurface.

\[ \lambda_k(\Sigma) = \inf_{\dim(V) = k} \sup_{X \in V \setminus \{0\}} \left( \frac{\delta^2 \Sigma(X, X)}{\int_{\Sigma} |X|^2 d\mathcal{H}^n} \right) \]

The infimum is over all the \( k \)-dimensional linear subspaces \( V \subset \Gamma_c(N\Sigma) = \) compactly supported smooth sections of \( N\Sigma \). Therefore, \( \lambda_k(\Sigma) < a \) if there is a \( k \)-dimensional subspace \( V \subset \Gamma_c(N\Sigma) \) such that for all \( X \in V \setminus \{0\} \),

\[ \delta^2 \Sigma(X, X) < a \int_{\Sigma} |X|^2 d\mathcal{H}^n \]
Given a Riemannian manifold \((U^{n+1}, g)\) and \(k \in \mathbb{N} \ (0 \not\in \mathbb{N})\), a \(k\)-parameter family of diffeomorphisms is a smooth map \(F : \overline{B}^k(0,1) \subset \mathbb{R}^k \rightarrow \text{Diff}(U)\) such that

- \(F_v(= F(v)) = (F_{-v})^{-1} \ \forall v \in \overline{B}^k(0,1)\) and \(F_0 = \text{Id}\)
- there exists open \(U' \subset U\) such that \(F_v|_{U \setminus U'} = \text{Id} \ \forall v \in \overline{B}^k(0,1)\)

If \(F\) is a \(k\)-parameter family of diffeomorphisms, we define the vector-fields \(Y_i, i = 1, ..., k\) by

\[
Y_i|_p = \frac{d}{dt} \bigg|_0 F_{\varepsilon_i}(p)
\]

Suppose further, we have a stationary varifold \(V\) in \(U\). Then we define a smooth function \(A^V\) and a quadratic form \(K^V\) as follows.

\[
A^V : \overline{B}^k(0,1) \rightarrow [0, \infty), \quad A^V(v) = \| (F_v)_\# V \|(U)
\]

\[
K^V : \mathbb{R}^k \times \mathbb{R}^k \rightarrow \mathbb{R}, \quad K^V(u, u) = \sum_i u_i Y_i^2_{L^2(U, \|V\|)}
\]

**Remark 3.1.** If \(V_i \rightarrow V\) in the \(F\) metric then \(A^V_i \rightarrow A^V\) ([Pit81], Section 2.3) in the smooth topology and also \(K^{V_i} \rightarrow K^V\) smoothly on compact subsets.

**Definition 3.2.** Given a stationary varifold \(V\) in \(U\), \(k \in \mathbb{N}\) and \(\alpha \geq 0\) we say that \(\lambda_k(V) < -\alpha\) if there exists a \(k\)-parameter family of diffeomorphisms \(F\) such that

\[
D^2A^V|_0 (u, u) < -\alpha K^V(u, u)
\]

for all \(u \in \mathbb{R}^k \setminus \{0\}\) or equivalently for all \(u \in \mathbb{R}^k\) with \(\|u\| = 1\). Further, by \(\lambda_k(V) \geq -\alpha\) we will mean that \(\lambda_k(V) < -\alpha\) does not hold. If \(\lambda_k(V) < -\alpha\) then restricting \(F\) to \(\overline{B}^{k-1}(0,1) \subset \overline{B}^k(0,1)\) we get that \(\lambda_{k-1}(V) < -\alpha\) as well. Therefore, it will be natural to define

\[
\text{Ind}(V) = \begin{cases} 
0 & \text{if } \{I \in \mathbb{N} : \lambda_I(V) < 0\} = \emptyset, \\
\sup\{I \in \mathbb{N} : \lambda_I(V) < 0\} & \text{otherwise}.
\end{cases}
\]

Hence, \(\text{Ind}(V) \leq I\) is equivalent to \(\lambda_{I+1}(V) \geq 0\). Further, \(\text{Ind}(V) = 0\) iff \(\lambda_1(V) \geq 0\) iff \(V\) is stable.

**Remark 3.3.** By Remark 3.1 and from the above definition it is clear that whenever \(\lambda_k(V) < -\alpha\) and \(F(V, V')\) is sufficiently small, we have \(\lambda_k(V') < -\alpha\) as well.

**Proposition 3.4.** Given \(\Lambda > 0\), \(k \in \mathbb{N}\) and \(\alpha \geq 0\), the following sets are compact with respect to the \(F\) metric topology.

\[
\mathcal{M}_U(\Lambda, k, \alpha) = \{V \in \mathcal{V}_n(U) : V \text{ is stationary, } \|V\| \leq \Lambda \text{ and } \lambda_k(V) \geq -\alpha\} \subset \mathcal{V}_n(U)
\]

\[
\mathcal{M}'_U(\Lambda, k, \alpha) = \{V \in IV_n(U) : V \text{ is stationary, } \|V\| \leq \Lambda \text{ and } \lambda_k(V) \geq -\alpha\} \subset IV_n(U)
\]
Proof. By the standard compactness theorems, if \( \{V_i\}_{i=1}^{\infty} \) is a sequence of stationary varifolds with \( \|V_i\| \leq \Lambda \) then up to a subsequence \( V_i \) converges to a stationary varifold \( V \) in the \( F \) metric with \( \|V\| \leq \Lambda \). Further, by Allard’s theorem \( [A172] \) if \( V_i \)’s are integral varifolds, \( V \) is also an integral varifold. Moreover, by Remark 3.3 if \( \lambda_k(V) < -\alpha \) then for \( i \) large \( \lambda_k(V_i) < -\alpha \) as well. Hence, \( \lambda_k(V) \geq -\alpha \). 

\[ \square \]

**Theorem 3.5.** Let \( \Sigma \) be a singular, minimal hypersurface in \( U, V = |\Sigma| \) and \( H^{n-2}(\text{sing}(\Sigma)) = 0 \). Then, \( \lambda_k(V) < -\alpha \leq 0 \iff \lambda_k(\text{reg}(\Sigma)) < -\alpha \). Hence, \( \text{Ind}(V) = \text{Ind}(\text{reg}(\Sigma)) \); therefore, \( V \) is stable iff \( \text{reg}(\Sigma) \) is stable.

**Proof.** We note that

\[
D^2A^V\bigg|_0 (u, u) = \frac{d^2}{dt^2} \bigg|_0 A^V(tu, tu) = \frac{d^2}{dt^2} \| (F tu) \# V \|(U) = \delta^2 V\left( \sum_i u_i X_i, \sum_i u_i X_i \right) 
\]

(3.1)

Let \( \lambda_k(\text{reg}(\Sigma)) < -\alpha \). Then there are \( k \) linearly independent, compactly supported normal vector-fields on \( \text{reg}(\Sigma) \) say \( X_1, X_2, ..., X_k \) such that

\[ \delta^2 V(X, X) < -\alpha \| X \|^2_{L^2(\Sigma)} \]

for any non-zero vector-field \( X \) in the span of \( \{X_i\}_{i=1}^k \). We extend each \( X_i \) to a compactly supported, globally defined vector-field on \( U \) and continue to call it by \( X_i \); we define \( F_v = \Phi \Sigma_i u_i X_i \), where \( \Phi X \) denotes the time 1 flow of the vector-field \( X \). Let us check that this choice of \( F \) indeed works. Clearly, \( F_{-v} = F_v^{-1} \).

By (3.1)

\[
D^2A^V\bigg|_0 (u, u) = \delta^2 V\left( \sum_i u_i X_i, \sum_i u_i X_i \right) < -\alpha \| \sum_i u_i X_i \|^2_{L^2(\Sigma)} 
\]

For the converse, we consider the following.

\[ \delta^2 V(X, X) = \int_{\Sigma} \left( (\text{div}_\Sigma X)^2 + \sum_{i=1}^n |(\nabla \tau_i X)^{-1}|^2 - \sum_{i,j=1}^n \langle \tau_i, \nabla \tau_j X \rangle \langle \tau_j, \nabla \tau_i X \rangle \right) dH^n \]

where \( \{\tau_i\}_{i=1}^n \) is an orthonormal basis of \( T_2 \Sigma \). Since \( H^{n-2}(\text{sing}(\Sigma)) = 0 \), given any \( \delta > 0 \) and \( 0 < \kappa < 1 \) we can choose balls \( \{B(x_i, r_i)\}_{i=1}^K \) such that each \( r_i < \kappa \), \( \text{sing}(\Sigma) \subset \cup_i B(x_i, r_i) \) and \( \sum_i r_i^{n-2} < \delta \). Therefore, \( \sum_i r_i^{n-2} < \delta \) and \( \sum_i r_i^\kappa < \delta \) as well. We choose smooth cut-off functions \( 0 \leq \zeta_i \leq 1 \) on \( U \) such that

\[ \zeta_i = \begin{cases} 
0 & \text{on } B(x_i, r_i) \\
1 & \text{outside } B(x_i, 2r_i) 
\end{cases} \]
and $|\nabla \zeta| \leq 2/r_i$ (this can be ensured by choosing $\kappa$ sufficiently small). Let $\zeta_\delta = \min_i \zeta_i$. From the second variation formula, we see that

$$\left| \delta^2 V(X, X) - \delta^2 V(\zeta_\delta X, \zeta_\delta X) \right| \leq \int_{\Sigma} \left( (1 - \zeta_\delta^2) + |\nabla \zeta_\delta| + |\nabla \zeta_\delta|^2 \right) f(X, \nabla X) \, dH^n$$

(3.2)

where $f$ is an expression involving $X$ and $\nabla X$. By the monotonicity formula, the R.H.S. of this equation is bounded by $C\delta$ for some constant $C$ depending only on $(U, g)$, $n$, $\|V\|(U)$ and $\|X\|_{C^1}$.

Therefore, for $u \in \mathbb{R}^k$ with $\|u\| = 1$

$$\delta^2 V \left( \zeta_\delta \sum_i u_i Y_i^+, \zeta_\delta \sum_i u_i Y_i^+ \right) = \delta^2 V \left( \zeta_\delta \sum_i u_i Y_i, \zeta_\delta \sum_i u_i Y_i \right)$$

(3.3)

$$\leq \delta^2 V \left( \sum_i u_i Y_i, \sum_i u_i Y_i \right) + C\delta$$

(3.4)

Here $C$ depends only on $(U, g)$, $n$, $\|V\|(U)$ and $\|Y_i\|_{C^1}$ and not on $u$.

We assume $\lambda_k(V) < -\alpha$. By Definition 3.2 and equation 3.1 for all $u \in \mathbb{R}^k$ with $\|u\| = 1$,

$$\delta^2 V \left( \sum_i u_i Y_i, \sum_i u_i Y_i \right) < -\alpha \|\sum_i u_i Y_i\|_{L^2(\Sigma)}^2 \implies \sum_i u_i Y_i \neq 0$$

(3.5)

Therefore,

$$\sup_{\|u\| = 1} \frac{\delta^2 V \left( \sum_i u_i Y_i, \sum_i u_i Y_i \right)}{\|\sum_i u_i Y_i\|_{L^2}^2} \leq -\alpha - 2\varepsilon$$

(3.6)

for some $\varepsilon > 0$. Hence, using 3.4 for all $\|u\| = 1$ and $\delta$ sufficiently small

$$\delta^2 V \left( \zeta_\delta \sum_i u_i Y_i^+, \zeta_\delta \sum_i u_i Y_i^+ \right)$$

$$\leq \delta^2 V \left( \sum_i u_i Y_i^+, \sum_i u_i Y_i \right) + C\delta$$

$$\leq (-\alpha - 2\varepsilon) \int_{\Sigma} \left| \sum_i u_i Y_i \right|^2 \, dH^n + C\delta$$

$$\leq (-\alpha - 2\varepsilon) \int_{\Sigma} \left| \sum_i u_i Y_i \zeta_\delta \right|^2 \, dH^n + C\delta$$

$$< (-\alpha - \varepsilon) \int_{\Sigma} \left| \sum_i u_i Y_i \zeta_\delta \right|^2 \, dH^n$$

(3.7)

$$\leq (-\alpha - \varepsilon) \left\| \sum_i u_i Y_i^+ \zeta_\delta \right\|_{L^2(\Sigma)}^2$$

(3.8)

\footnote{Though $\zeta_\delta$ is only a Lipschitz continuous function, its use in the subsequent calculations can be justified by an approximation argument.}
To justify 3.7, we observe:

\[
\int_\Sigma \left| \sum_i u_i Y_i \right|^2 dH^n = \int_\Sigma \left| \sum_i u_i Y_i \right|^2 dH^n + \int_\Sigma (\sum_i Y_i^2 - 1) \sum_i u_i Y_i dH^n \\
\geq \inf_{\|u\|=1} \int_\Sigma \left| \sum_i u_i Y_i \right|^2 dH^n - C \sum_i r_n^2 \quad \text{(using monotonicity formula)} \\
\geq \theta - C\delta \quad \text{(for some } \theta > 0 \text{ by 3.5 and } C \text{ is independent of } u) 
\]

Clearly 3.8 implies that \( \{\xi_i^i\}_{i=1}^k \) are linearly independent normal vector fields on \( reg(\Sigma) \) and \( \lambda_k(reg(\Sigma)) < -\alpha \).

In view of the above Theorem 3.5, we will use the terms \( \lambda_k(V) \) and \( \lambda_k(reg(\Sigma)) \) interchangeably.

### 4 Modifications of the results of Schoen-Simon [SS81]

Suppose the unit ball \( B^{n+1}(0,1) \subset \mathbb{R}^{n+1} \) is equipped with a Riemannian metric \( g; \mu_1 \) is a constant such that if \( g = g_{ij} dx^i dx^j \)

\[
\sup_{B^{n+1}(0,1)} \left| \frac{\partial g_{ij}}{\partial x_k} \right| \leq \mu_1 \quad \sup_{B^{n+1}(0,1)} \left| \frac{\partial^2 g_{ij}}{\partial x_k \partial x_l} \right| \leq \mu_1^2 
\]

(4.1)

**Theorem 4.1** (Modification of Schoen-Simon [SS81], Theorem 1 ; page 747).

Suppose \( \Sigma \) is a singular, minimal hypersurface in \( (B^{n+1}(0,1), g) \) satisfying \( H^{n-2}(sing(\Sigma)) = 0, H^n(\Sigma) \leq \mu \) and \( \lambda_1(\Sigma) \geq -\alpha \) for some \( \alpha \geq 0 \). Then there exist \( \delta_0 \in (0,1), r_0 \in (0,1/4) \) and \( c > 0 \) depending only on \( n, \mu, \mu_1, \alpha \) such that the following holds. If \( x = (x', x_{n+1}) \in \Sigma \cap B^{n+1}(0,1/4), p \leq r_0, \Sigma' \) is the connected component of \( \Sigma \cap C(x, \rho) \) (\( C(x, \rho) \) is the cylinder on \( B^n(x, \rho) \)) containing \( x \) and

\[
\sup_{y = (y', y_{n+1}) \in \Sigma'} |y_{n+1} - x_{n+1}| \leq \delta_0 \rho \quad (*)
\]

then, \( \Sigma' \cap C(x, \rho/2) \) consists of disjoint union of graphs of functions \( u_1 < u_2 < \ldots < u_k \) defined on \( B^n(x, \rho/2) \) satisfying the following estimate.

\[
\sup_{B^n(x, \rho/2)} (|Du_i| + \rho |DDu_i|) \leq c\delta_0 
\]

for \( i = 1, 2, \ldots, k \). In particular, \( \Sigma \) is smooth near \( x \) and the second fundamental form \( |A_2(x)| \leq c/\rho \) (for possibly a different constant \( c \)).
 Remark 4.2. The difference between the above Theorem 4.1 and Theorem 1 of Schoen-Simon [SS81] is that instead of assuming $|\Sigma|$ is stable we have assumed that $\lambda_1(|\Sigma|) \geq -\alpha$. Indeed under this weaker assumption, the stability inequality (1.17) of Schoen-Simon [SS81] (page 746) continues to hold with the constant $c_5$ replaced by $c_5 + \frac{\alpha}{\rho l}$; therefore all the successive calculations in the paper [SS81] go through.

Theorem 4.3 (Modification of Schoen-Simon [SS81], Theorem 2; page 784). Let $\{\Sigma_q\}$ be a sequence of singular, minimal hypersurfaces in $(\mathbb{R}^{n+1}, 0, y)$ such that $\mathcal{H}^{n-2}(\text{sing}(\Sigma_q)) = 0$, $\lambda_1(\Sigma_q) \geq -\alpha$ and $|\Sigma_q|$ converges to a varifold $W$; $0 \in \text{spt}(W) = \Sigma$. Then $\mathcal{H}^*(\text{sing}(\Sigma) \cap B^{n+1}(0, \frac{1}{2})) = 0$ for all $s > n - 7$.

Proof. As before, The difference between the above Theorem 4.3 and Theorem 2 of Schoen-Simon [SS81] is that instead of assuming $|\Sigma|$ is stable we have assumed that $\lambda_1(|\Sigma|) \geq -\alpha$.

The proof of Theorem 2 of Schoen-Simon [SS81] goes as follows. By the successive blow-up argument, one arrives at a varifold $W_1$ which is a stationary, integral, codimension 1 cone in $\mathbb{R}^{n-l+1}$ such that $\text{sing}(W_1) = \{0\}$ and

$$\mathbb{R}^l \times W_1 = \lim_{m \to \infty} J_{\#} \circ \tau_{y_m, \#} \circ \mu_{r_m, \#} |\Sigma_{q_m}|$$  \hspace{1cm} (4.2)

for some sequence of points $y_m \in \mathbb{R}^{n+1}$ and positive real numbers $r_m$ and some subsequence $\{\Sigma_{q_m}\} \subset \{\Sigma_q\}$; $|y_m|, r_m \to \infty$. Here $J$ is some orthogonal transformation of $\mathbb{R}^{n+1}$, $\tau_y$ denotes the translation of $\mathbb{R}^{n+1}$ which brings $y$ to the origin, $\mu_r$ is the multiplication (scaling) by $r$. It is shown that (equation (5.22) of Schoen-Simon) $l \geq s$ for every $s$ such that $\mathcal{H}^*(\text{sing}(\Sigma) \cap B^{n+1}(0, \frac{1}{2})) > 0$ (hence, one only needs to show that $l \leq n - 7$) and $l \leq n - 3$. Upto this point, the only facts about $\Sigma_q$ which are used: $\Sigma_q$ satisfies the stability inequality (1.17) and Theorem 1 of Schoen-Simon [SS81]. After this, stability of $|\Sigma_q|$ is used to conclude that $\mathbb{R}^l \times W_1$ is stable. Hence, $W_1$ is smooth, stable codimension 1 cone in $\mathbb{R}^{n-l+1}$ with a singularity at origin. Therefore, $n-l \geq 7$ i.e. $l \leq n - 7$.

In our context of Theorem 4.3 the above mentioned proof can be modified as follows. As noted above, stability inequality (1.17) and Theorem 1 of Schoen-Simon [SS81] continue to hold under the weaker assumption $\lambda_1(|\Sigma_q|) \geq -\alpha$. Moreover

$$\lambda_1(|\Sigma_q|) \geq -\alpha \implies \lambda_1(J_{\#} \circ \tau_{y, \#} \circ \mu_{r, \#} |\Sigma_{q_m}|) \geq \frac{-\alpha}{r_m}$$ \hspace{1cm} (4.3)

can be justified, for example, using Theorem 3.5. If $\lambda_1(\mathbb{R}^l \times W_1) < -\varepsilon$ for some $\varepsilon > 0$. (This can be seen from the proof of Theorem 3.5). Since $\mathbb{R}^l \times W_1$ is the varifold limit of $J_{\#} \circ \tau_{y, \#} \circ \mu_{r, \#} |\Sigma_{q_m}|$, in view of Remark 3.3 for all large $m$,

$$\lambda_1(J_{\#} \circ \tau_{y, \#} \circ \mu_{r, \#} |\Sigma_{q_m}|) < -\varepsilon$$

This contradicts (4.3) as $\lim_{m \to \infty} r_m = \infty$. Hence, $\lambda_1(\mathbb{R}^l \times W_1) \geq 0$ i.e. $\mathbb{R}^l \times W_1$ is stable. \(\square\}
To prove the graphical convergence part of Theorem 1.2 we will need the following Lemma which is a consequence of Theorem 4.1.

**Lemma 4.4.** Let $(N^{n+1}, g)$ be a closed Riemannian manifold. Let $\{\Sigma_q\}$ be a sequence of singular, minimal hypersurfaces in $N$ with $H^{n-2}(\text{sing}(\Sigma_q)) = 0$ for all $q$. We also assume that $W_q = |\Sigma_q|$ varifold converges to a stationary, integral varifold $W$, $\Sigma = \text{spt}(W)$ and $\Sigma_q$ converges to $\Sigma$ in the Hausdorff topology. Then, for every $N$ sequence of singular, minimal hypersurfaces in $\Sigma_q$ we can choose $R > q$ all $\mu_0$ such that

$\lim_{t \to 0} \frac{W((B^N(x_0,t)))}{t^n} \leq \Theta + 1/2 \quad \forall t \leq \rho_0$  \hspace{1cm} (4.4)

Let $\delta_0$ and $r_0$ be the constants which are provided by Theorem 4.1 when we set $\mu = \Theta + 1$ and $\mu_1 = 1$. Let $s_0 = 1/\delta_0$. For $0 < a < 1/\delta_0$ identifying $\mathbb{R}^{n+1}$ with $T_{x_0}N$ and $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ is identified with $T_{x_0}M$ we define

$\Phi_a : B^{n+1}(0,2) \to B^N(x_0,2a), \quad \Phi_a(v) = \exp_{x_0}(av), \quad g_a = \Phi_a^* g$

We can choose $R > 0$ so that

- $R < \min \left\{ \frac{1}{\delta_0} \text{inj}_N, \frac{1}{2} d^N(x_0, \text{sing}(\Sigma)), 1, \rho_0 \right\}$

and whenever $r \leq R$

- $(B^{n+1}(0,1), g_0)$ satisfies 4.1 with $\mu_1 = 1$
- $\Sigma = \Phi_a^{-1}(\Sigma \cap B^N(x_0, 2r)) \subset B^{n+1}(0,2) \cap \{x : |x_{n+1}| < s_0\}$

We will show that the above choice of $R = R(x_0)$ works. Let us fix an $r \leq R$. We define

$C = B^n(0,1) \times [-1,1] \subset B^{n+1}(0,2), \quad C' = C \setminus \{x : |x_{n+1}| < s_0\}$.

Then

$C' \cap \left( \tilde{\Sigma} \cup \partial B^{n+1}(0,2) \right) = \emptyset; \quad d := \text{dist}_{g_0} \left( C', \left( \tilde{\Sigma} \cup \partial B^{n+1}(0,2) \right) \right)$

We choose $q_0$ (depending on $r$) so that for all $q \geq q_0$

- $0 < \frac{\|W_q\|(B^N(x_0,r))}{r^{n+1}} \leq \Theta + 1$ \hspace{1cm} \[\text{This is possible since } \Sigma \text{ is smooth near } x_0\]
- $\lambda_1(W_q, B^N(x_0,r)) \geq -\alpha$ \hspace{1cm} \[\text{This is possible because of 4.3 and varifold convergence of } W_q \text{ to } W\]
- The Hausdorff distance $d_H(\Sigma_q, \Sigma) < dr$
Hence, denoting $\bar{\Sigma}_q = \Phi^{-1}(\Sigma_q \cap B^N(x_0, 2r))$, for all $q \geq q_0$ we have $\bar{\Sigma}_q \cap B^{n+1}(0, 1) \neq \emptyset$,

$$\mathcal{H}^n(\bar{\Sigma}_q \cap B^{n+1}(0, 1)) = \mathcal{H}^n(\Sigma_q \cap B^N(x_0, r)) \leq \Theta + 1,$$

$$\lambda_1(|\bar{\Sigma}_q \cap B^{n+1}(0, 1)|) \geq -\alpha r \geq -\alpha$$

and

$$\bar{\Sigma}_q \cap C' = \emptyset \quad \text{i.e.} \quad (\bar{\Sigma}_q \cap C) \subset C \cap \{x : |x_{n+1}| < s_0\} \quad (4.5)$$

We can now apply Theorem 4.1 to the singular, minimal hypersurface $(\bar{\Sigma}_q \cap B^{n+1}(0, 1)) \subset (B^{n+1}(0, 1), g_r)$ for $q \geq q_0$. Since, $B^{n+1}(0, 1) \subset C$, equation 4.5 implies that for all $q \geq q_0$, $x \in \bar{\Sigma}_q \cap B^{n+1}(0, 1/4)$ and for $\rho = r_0$ the oscillation bound (*) of Theorem 4.1 is satisfied in the cylinder $C(x, r_0)$. The counterpart of the Arzela-Ascoli theorem for smooth, minimal hypersurface gives that in a smaller ball $B^{n+1}(0, 1/5)$, $\bar{\Sigma}_q$ converges to $\bar{\Sigma}$ smoothly and graphically. (Here we do not have to pass to a further subsequence since we already know that $\bar{\Sigma}_q$ Hausdorff converges to $\bar{\Sigma}$). When we rescale back to go back to $N$, we get that in the ball $B^N(x_0, r/5)$, $\Sigma_q$ converges to $\Sigma$ smoothly and graphically.

\[\square\]

5 Proof of the main theorem

In this section we will give a proof of Theorem 1.2.

By Allard’s compactness theorem [All72], possibly after passing to a subsequence, $V_k \rightarrow V$ in the $F$ metric with $V$ is a stationary, integral varifold and $\|V\|(N) \leq \Lambda$. If $\lambda_p(V) < -\alpha$ then $\lambda_p(V_k) < -\alpha$ for $k$ sufficiently large by Remark 3.3. Hence, $\lambda_p(V) \geq -\alpha$. We also know that $\text{spt}(V_k) = M_k$ converges to $\text{spt}(V) = M$ in the Hausdorff topology.

**Lemma 5.1.** Let $W$ be a stationary varifold in $U^{n+1}$ with $\lambda_p(W) \geq -\alpha$, $\alpha \geq 0$. Let $U_1, \ldots, U_p$ be mutually disjoint open subsets of $U$ such that $\|W\|(U_j) \neq 0$ for each $j = 1, \ldots, p$. Then there exists $i \in \{1, \ldots, p\}$ such that $\lambda_1(W \nabla U_i) \geq -\alpha$.

**Proof.** Let us assume that for all $i = 1, \ldots, p$ we have $\lambda_1(W \nabla U_i) < -\alpha$. Hence, from Definition 3.2 and equation 3.1 we have maps

$$F^i : T^i \rightarrow \text{Diff}(U_i)$$

This follows from $d_H(\Sigma_q, \Sigma) < dr$
and vector-fields $Y^i$ compactly supported in $U_i$

$$Y^i|_x = \frac{d}{dt}|_0 F^i_t(x)$$

such that

$$\delta^2 W(Y^i, Y^i) < -\alpha \|Y^i\|^2_{L^2}$$

We define

$$F : \mathcal{B}^p \rightarrow \text{Diff}(U), \quad F_v = \Phi^{\sum_i v_i Y^i}$$

where $\Phi^X$ denotes the time 1 flow of the vector field $X$. For this choice of $F$ we have

$$D^2A^W|_0(u, u) = \delta^2 W \left( \sum_i u_i Y^i, \sum_i u_i Y^i \right)$$

$$= \sum_i u_i^2 \delta^2 W(Y^i, Y^i) \quad \text{(since spt}(Y^i)\text{'s are mutually disjoint)}$$

$$< -\alpha \sum_i u_i^2 \|Y^i\|^2_{L^2}$$

$$= -\alpha \sum_i u_i Y^i \|Y^i\|^2_{L^2} \quad \text{(since spt}(Y^i)\text{'s are mutually disjoint)}$$

$$= -\alpha K^W(u, u)$$

This contradicts $\lambda_p(W) \geq -\alpha$. \hfill \Box

Returning back to the proof of the main Theorem 1.2, let $G \subset M$ be the set of points $x \in M$ for which there exists $r = r(x) > 0$ and some subsequence $\{V_{k'}\} \subset \{V_k\}$ such that for each $k'$, $\lambda_1(V_{k'}, B^N(x, 2r)) \geq -\alpha$. Therefore, using Theorem 4.3, $H^s(\text{sing}(M) \cap B^N(x, r(x))) = 0$ for all $x \in G$ and $s > n - 7$.

**Lemma 5.2.** The set $M \setminus G$ has atmost $p - 1$ points.

**Proof.** Suppose there exists $p$ points $\{x_i\}_{i=1}^p \subset M \setminus G$. Let $t$ be small enough so that the normal geodesic balls $\{B^N(x_i, t)\}_{i=1}^p$ are mutually disjoint. By the definition of $G$, there exists $k_0$ such that for all $k \geq k_0$, $\lambda_1(B^N(x_i, t)) < -\alpha$ for each $i$. By Lemma 5.1, this implies $\lambda_p(V_k) < -\alpha$ for all $k \geq k_0$, a contradiction. \hfill \Box

We note that

$$\text{sing}(M) \cap G = \bigcup_{x \in \text{sing}(M) \cap G} (\text{sing}(M) \cap B^N(x, r(x))) \quad (5.1)$$

We can extract a countable subcover of the R.H.S. of (5.1) and write

$$\text{sing}(M) \cap G = \bigcup_{i=1}^\infty (\text{sing}(M) \cap B^N(x_i, r(x_i))) \quad (5.2)$$

\[\text{This is possible because } \mathcal{N}, \text{ being a manifold, is second-countable and a subspace of a second-countable space is second-countable.}\]
where each $x_i$ is in $\text{sing}(M) \cap G$. By the definition of $G$, for $s > n - 7$, $\mathcal{H}^s(\text{sing}(M) \cap B^N(x_l, r(x_l))) = 0$ for each $i$. Therefore, Lemma 5.2 and Lemma 5.2 imply that $\mathcal{H}^s(\text{sing}(M)) = 0$. (Here we are implicitly assuming that $n \geq 7$ so that for $s > n - 7 \geq 0$, $\mathcal{H}^s(M \setminus G) = 0$; when $n < 7$ further arguments are required to show that $M$ is smooth at the points of $M \setminus G$ as explained in [ACS16].)

We can now complete the proof of graphical convergence. We will produce a set $\mathcal{Y} \subset \text{reg}(M)$ and a subsequence $\{M_l\} \subset \{M_k\}$ such that $M_l$ converges smoothly and graphically on compact subsets of $\text{reg}(M) \setminus \mathcal{Y}$. Let $X$ be a countable, dense subset of $\text{reg}(M)$ and

$$\mathcal{B} = \{B^N(x, r) : x \in X, r \in \mathbb{Q}^+, r < d^N(x, \text{sing}(M)), r < \text{inj}_N\}$$

Then $\mathcal{B}$ is a countable collection of balls, say, $\mathcal{B} = \{B_i\}_{i=1}^\infty$. We will mark each $B_1$ as good or bad and to each $B_i$ we will assign an infinite index set $I_i \subset \mathbb{N}$ as follows. At the first step we examine whether there exists an infinite set $J \subset \mathbb{N}$ such that $\{M_j\}_{j \in J}$ converges to $M$ smoothly in $B_1$. If it exists we mark $B_1$ as good and define $I_1$ to be that $J$. Otherwise we mark $B_1$ as bad and define $I_1$ to be $\mathbb{N}$. Suppose we have marked $B_{i-1}$ as good or bad and defined $I_{i-1}$. Then we examine whether there exists an infinite set $J \subset I_{i-1}$ such that $\{M_j\}_{j \in J}$ converges to $M$ smoothly in $B_{i-1}$. If it exists we mark $B_i$ as good and define $I_i$ to be that $J$. Otherwise we mark $B_i$ as bad and define $I_i$ to be $I_{i-1}$.

Let $\mathcal{G}$ be the union of good balls. Denoting $\mathcal{Y} = \text{reg}(M) \setminus \mathcal{G}$, we claim that $|\mathcal{Y}| \leq p - 1$. Otherwise, there exists $p$ distinct points $x_1, ..., x_p$ in $\mathcal{Y}$. To arrive at a contradiction we will apply Lemma 4.4 to the sequence of singular, minimal hypersurfaces $\{M_k\}$: $\{M_k\}$ converges in the varifold sense to $V$ which is supported on $M$; Lemma 4.4 provides a function $R : \text{reg}(M) \rightarrow \mathbb{R}^+$. Let $\tau$ be a positive number such that $5\tau \leq R(x_l)$ for each $l$ and the balls $\{B^N(x_l, 5\tau)\}_{l=1}^\infty$ are mutually disjoint. There exists $B_{i_1} \in \mathcal{B}$ such that $x_l \in B_{i_1} \subset B^N(x_l, \tau)$. As $x_l \in \mathcal{Y}$, $B_{i_1}$ is a bad ball. Hence $\{M_j\}_{j \in I_{i_1}}$ does not have a subsequence which smoothly converges to $M$ in $B^N(x_l, \tau)$. Therefore, by Lemma 4.4, $\lambda_1(V_j \mathcal{L} B^N(x_l, 5\tau)) < -\alpha$ for all large $j \in I_{i_1}$. Without loss of generality, we can assume that $i_1 < ... < i_p$ so that $I_{i_1} \supset ... \supset I_{i_p}$. Hence $\lambda_1(V_j \mathcal{L} B^N(x_l, 5\tau)) < -\alpha$ for each $l = 1, ..., p$ and for all large $j \in I_{i_p}$. By Lemma 5.1 this gives $\lambda_p(V_j) < -\alpha$ for all large $j \in I_{i_p}$, a contradiction. Hence, $|\mathcal{Y}| \leq p - 1$.

By a diagonal argument, we can choose an infinite set $I \subset \mathbb{N}$ such that $|I \setminus I_i|$ is finite for all $i$. Then, by the definition of good ball, $\{M_i\}_{i \in I}$ is a sequence which converges smoothly and graphically on the compact subsets of $\text{reg}(M) \cap \mathcal{G} = \text{reg}(M) \setminus \mathcal{Y}$.

6 Some further remarks

Besides the main compactness Theorems of [Sha17] and [ACS16], some additional results proved in these two papers can be suitably generalized in higher
dimensions. In this last section, we will state them as a sequence of remarks. Below we will assume that $M_k$’s and $M$ are as in Theorem 1.2 (and Theorem 1.3).

**Remark 6.1.** We have assumed that each $M_k$ is connected. Since $\{M_k\}$ converges to $M$ in the Hausdorff distance, this implies that $M$ is connected as well. From [Ilm96] (Theorem A (ii)) it follows that $reg(M)$ and hence $reg(M) \setminus \mathcal{Y}$ is also connected. Therefore, the number of sheets in the graphical convergence is constant over $reg(M) \setminus \mathcal{Y}$. In particular, $V = m|M|$ for some $m \in \mathbb{N}$.

**Remark 6.2.** If the number of sheets in the graphical convergence is 1, then $\mathcal{Y} = \emptyset$. This is Claim 4 in [Sha17] and the proof presented there works in our case as well.

**Remark 6.3.** Suppose $reg(M)$ is two sided. If the number of sheets in the graphical convergence is at least 2 or if the number of sheets is 1 and $M_k \cap M = \emptyset$ for large $k$, we can construct a positive Jacobi field on $reg(M)$. In this case, $reg(M)$ and hence $M$ is stable. The proof is same as presented in [Sha17].

**Remark 6.4.** Continuing with Remark 6.3, suppose $\text{Ric}(N, g) > 0$. Then the convergence of $M_k$ to $M$ is always single sheeted. This can be thought of as a higher dimensional analogue of Choi-Schoen [CS85] which asserts that in a three manifold with positive Ricci curvature, the space of closed, embedded minimal surfaces with bounded genus is compact in the smooth topology. Indeed, in our case, if $H_n(N, \mathbb{Z}_2) = 0$ then $reg(M)$ is two sided: hence, if the number of sheets is $\geq 2$, $M$ is stable by the above Remark 6.3. However, as proved in [Zho17] (Lemma 2.8) positive Ricci curvature of $(N, g)$ implies that $M$ can not be stable. The general case can be obtained by lifting $M_k$’s and $M$ to the universal cover $\tilde{N}$ of $N$ (by [Fra66] and Lemma 2.10 of [Zho17] the lifts $\tilde{M}_k, \tilde{M}$ are connected).

**Remark 6.5.** Theorems 1.2 and 1.3 hold in the varying metric set-up. More precisely, instead of assuming $|M_k|$ is stationary with respect to the fixed metric $g$ if we assume that $|M_k|$ is stationary with respect to the metric $g_k$ and $g_k$ converges to $g$ in $C^3$, Theorems 1.2 and 1.3 continue to hold.

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