Spontaneous Radiation and Amplification of Kelvin Waves on Quantized Vortices in Bose–Einstein Condensates

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We propose a new type of Landau instability in trapped Bose-Einstein condensates by a helically moving environment. In the presence of quantized vortices, the instability can cause spontaneous radiation and amplification of Kelvin waves. This study gives the microscopic understanding of the Donnelly–Glaberson instability which was known as a hydrodynamic instability in superfluid helium. The Donnelly–Glaberson instability can be a powerful tool for observing the dispersion relation of Kelvin waves, vortex reconnections, and quantum turbulence in atomic Bose–Einstein condensates.

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Even if an external environment, such as a container wall, is moving in the laboratory frame, superfluid component can remain at rest in thermal equilibrium, which is frictionless flow. The thermodynamic stability of the frictionless flow is discussed with the thermodynamic energy

\[ E(V, \Omega) = E_0 - V \cdot P - \Omega \cdot L, \]  

where \( E_0, P \) and \( L \) are the energy, the momentum, and the angular momentum of the superfluid in the laboratory frame, respectively. Here, \( V \) and \( \Omega \) refer to the linear velocity and the rotational frequency of the motion of the environment, respectively. The second term on the right-hand side of Eq. (1) is essential to the discussion for superfluid although it is usually neglected for ordinary liquid [1]. The frictionless flows are realized at local minina of the thermodynamic energy of Eq. (1). This kind of thermodynamic consideration has been applied only to the translational \( (V \neq 0 \text{ and } \Omega = 0) \) and the rotational \( (V = 0 \text{ and } \Omega \neq 0) \) cases, which has been thoroughly studied in superfluid helium and atomic Bose-Einstein condensates (BECs) [2]. The former was applied to the famous discussion of Landau critical velocity for bulk superfluid. The latter was discussed for understanding the anomalous moment of inertia and the vortex lattice formation. The coupled situation \( (V \neq 0 \text{ and } \Omega \neq 0) \), which has not been known well to date, reveals a new aspect of superfluidity.

In this work, we theoretically study the superfluidity under a helically moving environment with \( V \parallel \Omega \) in the presence of quantized vortices in atomic BECs. It is revealed that helical vortex modes (Kelvin waves) \( \xi \) can be spontaneously radiated and amplified due to the Landau instability. This study gives a microscopic understanding of the Donnelly–Glaberson (DG) instability, which was phenomenologically understood as simple hydrodynamic instability causing the amplification of Kelvin waves on vortex lines in superfluid helium [4,5].

This paper is organized as follows. First, we introduce the DG instability in superfluid helium with a phenomenological model for comparing with the instability in atomic BECs. Next, the DG instability is microscopically discussed in atomic BECs with the Gross–Pitaevskii (GP) and the Bogoliubov–de Gennes (BdG) models. Finally, it is shown that the DG instability can be applied to the observation of the dispersion relation of Kelvin waves, vortex reconnections [6], and quantum turbulence [7] in atomic BECs.

We now introduce an intuitive description of the DG instability in the simplest case of axial normal flow along an isolated vortex line under the vortex filament model in superfluid helium [8]. When the line is deformed into a helix with the amplitude \( \epsilon \), the wave vector \( k \), and the angular velocity \( \omega \), the position \( s \) of the line may be parameterized with \( \xi \) as \( s(\xi, t) = \epsilon \cos(kz(\xi) - \omega t)x + \epsilon \sin(kz(\xi) - \omega t)y + z(\xi)z \). In the localized induction approximation which neglects interactions between vortices [8], the equation of motion of vortex lines is written as

\[ \frac{ds}{dt} = v_i + \alpha s^1 \times (v_N - v_i), \]

where we use \( s^n = d^n s/dk^n \), the mutual friction coefficient \( \alpha(T) \geq 0 \), the normal fluid velocity \( v_N \), and the local self-induced velocity \( v_i \). When \( \epsilon k \ll 1 \), the self-induced velocity is linearized to \( v_i = \beta s^0 \times s^0 \approx \beta k^2 \epsilon \sin(kz(\xi) - \omega t)x - \beta k^2 \epsilon \cos(kz(\xi) - \omega t)y \) with \( \beta = \frac{4\pi}{ka} \ln(ka) \), the circulation quantum \( \kappa \), and vortex core radius \( a \). If the normal component is negligible at \( T = 0 \), \( \alpha(T = 0) = 0 \), the Kelvin wave propagates keeping its initial configuration and rotating with frequency \( \omega_0 = \beta k^2 \). At finite temperatures under the helical normal-fluid flow with \( v_N = \Omega\hat{z} \times \hat{r} + V\hat{z} \), where \( \Omega \) and \( V \) are positive constants, the initial stage of the dynamics is governed by

\[ \frac{d\epsilon}{dt} = -\alpha(\omega_0 + \Omega - kV)\epsilon. \]  

The mode with \( k \) is amplified or damped for \( \omega + \Omega - kV < 0 \) or \( > 0 \), respectively. Therefore, the vortex line becomes unstable when the velocity \( V \) exceeds the DG critical velocity

\[ V_{DG} = \min_k \left( \frac{\omega_0 + \Omega}{k} \right). \]  

Since the instability is determined by the local configuration of the vortex lines, a similar mechanism can be ap-
plied to each vortex line in more complicated cases such as vortex lattices and vortex tangles by considering the total velocity field originating from all vortices beyond the localized induction approximation [9,10]. Thus, the DG instability plays an important role in the vortex dynamics in superfluid $^4$He and $^3$He-B at finite temperatures.

Let us analyze the DG instability from a thermodynamic point of view, which is not clear in the last paragraph. When the normal component, which is regarded as an external environment, moves helically with the velocity $\Omega \hat{z} \times \mathbf{r} + V \hat{z}$, we have the thermodynamic energy $E(V, \Omega) = E_0 - VP_z - \Omega L_z$ for the superfluid component. Here, $P_z$ and $L_z$ are the momentum and the angular momentum of the superfluid component along the $z$-axis, respectively. Then, we suppose that the disappearance of the local minimum of the thermodynamic energy leads to the onset of the amplification of Kelvin waves. This kind of thermodynamic instability can be described by the Landau instability. However, it is difficult to analyze the instability from this viewpoint in superfluid helium systems where the vortex dynamics was analyzed with the phenomenological model. This presents a contrast with atomic BECs, where calculations from first principles give exact solutions of vortex states and makes us possible to discuss Kelvin waves microscopically [11,12,13,14]. By considering this phenomenon in atomic BECs, we reveal the microscopic physics of the DG instability below.

Let us consider quantized vortices in trapped BECs. For simplicity we assume periodic systems along the rotation axis and use an axisymmetric harmonic potential $V(\mathbf{r}) = M \omega_h^2 \rho^2/2$ with cylindrical coordinates $(\rho, \theta, z)$, where $M$ and $\omega_h$ are the atomic mass and trapping frequency, respectively. The thermodynamic energy for the superfluid component is well described by the macroscopic wave function $\Psi$. Under the constraint that the total number of particles $N$ in the system is constant, in the helically moving frame we have the thermodynamic energy

$$K(V, \Omega) = K_0 - VP_z - \Omega L_z$$

(4)

with $K_0 = E_0 - \mu N$, where $\mu$ is the chemical potential. From the potential of Eq. (4), we obtain the time-dependent GP equation

$$i \frac{\partial}{\partial t} \Psi = \left( -\frac{1}{2} \nabla^2 + \frac{1}{2} \rho^2 + g|\Psi|^2 - \mu - V \hat{p}_z - \Omega L_z \right) \Psi$$

(5)

with $\hat{p}_z = -i \nabla_z$ and $L_z = -i \mathbf{r} \times \nabla$, where the units of energy, length, and time are given by the corresponding scales of the harmonic potential as $\hbar \omega_h$, $\hbar/\sqrt{M \omega_h}$, and $\hbar \omega_h^{-1}$, respectively. The wave function is normalized as $\int d\rho d\theta \int_L^{2\pi} dz |\Psi(\mathbf{r})|^2 = 1$, where $L$ is the periodicity along the $z$-axis. The atomic interaction is characterized by $g$, which is proportional to the s-wave scattering length $a$ as $g = 4\pi a N / b_\perp > 0$.

First, we discuss the simplest case of a straight vortex line located along the $z$-axis. The wave function $\Psi_0$ in the stationary state can be written in an axisymmetric form as $\Psi_0 = \psi_0(\rho) e^{i\theta}$. Due to the symmetry of the wave function, the chemical potential $\mu$ in the co-moving frame may be defined as

$$\mu \equiv \mu(V, \Omega) = \mu(0, 0) - \Omega;$$

(6)

$\psi_0$ is then independent of $V$ and $\Omega$.

We now represent the collective modes with the perturbed wave function $\Psi = \Psi_0 + \delta \Psi$. Then, a collective excitation with frequency $\omega$ is written as

$$\delta \Psi = e^{i\theta} \left[ u_{k,l}(\rho) e^{i(kz + l\theta - \omega t)} - v_{k,l}(\rho) e^{-i(kz + l\theta - \omega t)} \right] \Psi_0,$$

(7)

where $k$ and $l$ refer to the wave number and the angular quantum number of the excitation along the $z$-axis, respectively. Here, we consider only the lowest modes along the radial direction. The normalization is $\int d\rho [u_{k,l}^2(\rho) + v_{k,l}^2(\rho)] = \eta \delta_{k,k'} \delta_{l,l'}$, where $\eta > 0$ and $\delta_{k,l'}$ is the Kronecker delta.

The lowest modes can be classified into three groups by the angular quantum number $l$. One group is Kelvin waves with $l = -1$, which deforms the vortex line into a helix [11]. The second is density waves with $l = 0$, which propagate keeping the condensate density axisymmetric. We call this mode the ‘varicose wave’ analog to classical fluid, for which the core diameter varies as the wave propagates [8]. The last group consists of surface waves with $l \neq -1, 0$, which disturb the condensate density only near the surface of the condensate.

Linearizing the GP Eq. (5) with Eq. (7), we obtain the BdG equations

$$(\omega + kV + i\Omega) \begin{pmatrix} u_{k,l} \\ v_{k,l} \end{pmatrix} = \begin{pmatrix} \hat{h}_+ - g\psi_0^2 & -\hat{h}_- \\ g\psi_0^* & -\hat{h}_- \end{pmatrix} \begin{pmatrix} u_{k,l} \\ v_{k,l} \end{pmatrix},$$

(8)

where $\hat{h}_\pm = \pm \frac{1}{2} \left[ (d^2/d\rho^2) + (d/d\rho - ((\pm 1)/2)^2 - k^2)^2 + 2g|\psi_0|^2 - \mu(0, 0) \right]$. Analogous to Eq. (6), we can define the frequency $\omega$ in the co-moving frame as

$$\omega \equiv \omega_{k,l}(V, \Omega) = \omega_{k,l}^0 - kV - i\Omega,$$

(9)

where $\omega_{k,l}^0 \equiv \omega_{k,l}(0, 0)$ is the dispersion relation in the laboratory frame. Then, $u_{k,l}$ and $v_{k,l}$ are independent of $V$ and $\Omega$. Figure 1 shows the dispersion relation $\omega_{k,l}^0$ for various $l$ by numerically solving Eq. (8) and Eqs. (3) for $g2\Omega \equiv g/L = 500$.

If there is at least one mode with $\omega < 0$, the stationary state $\Psi_0$ is a saddle point of the thermodynamic energy of Eq. (4). Then, the state becomes thermodynamically unstable due to the Landau instability and the mode should be spontaneously radiated and amplified to decrease the thermodynamic energy of Eq. (4). The stability of the single-vortex states for $V = 0$ was investigated in Ref. [12]. For $V = 0$, the frequency of Kelvin waves becomes negative for small wave number when $\Omega < \Omega_V \equiv -\omega_{0,-1}^0$, and the frequency of some surface wave with $l > 0$ becomes negative when $\Omega$ exceeds $\Omega_V \equiv \min \left( \frac{\omega_{0,l}^0}{l} \right) (l > 0)$. Thus the single-vortex states
can be stabilized when $\Omega_L < \Omega < \Omega_U$. However, even if $\Omega_L < \Omega < \Omega_U$, the single-vortex states become unstable in the presence of $V$. The frequency $\omega_{k,l}(V, \Omega)$ of the mode with $l$ becomes negative when $V$ exceeds the Landau critical velocity

$$V_i(\Omega) = \min_k \left( \frac{\omega^0_{k,l} - l\Omega}{k} \right) = \frac{\omega^0_{k,l} - l\Omega}{k_l}, \quad (10)$$

where $k_l$ is the critical wave number. Then, the mode with the angular quantum number $l$ and the wave number $k_l$ can be spontaneously radiated due to the Landau instability. In particular, Kelvin waves with wave number $k_{-1}$ are spontaneously radiated when $V$ exceeds $V_{-1}$, which represents the onset of the DG instability due to the Landau instability. In fact, we see that the critical velocity $V_{-1}$ for Kelvin waves has the same form as the DG criterion of Eq. (3) in the vortex filament model. In this way, we derived a microscopic understanding of the DG instability from the thermodynamic consideration.

To observe the DG instability, the condition $V_{-1} < V_l (l \neq -1)$ must be satisfied, because the single-vortex states become unstable due to the Landau instability exciting the other modes if $V_{l\neq-1} < V < V_{-1}$. Figure 2 shows the critical velocity $V_l$ for various $l$ and the critical wave number $k_{-1}$ obtained by Eq. (10) with the result of Fig. 1. The critical velocities for surface waves with $l < -1$ are always higher than $V_{-1}$, not shown in Fig. 2 (a). The critical velocity $V_0$ for varicose waves is typically higher than $V_{-1}$. This is because the dispersion relation of varicose waves is phonon-like while that of Kelvin waves is quadratic for small wave numbers (see Fig. 1). While the critical velocity $V_{-1}$ for Kelvin waves monotonically increases with $\Omega$, $V_{l>0}$ for surface waves decreases. The increase in $\Omega$ reduces $V_{l>0}$ for some $l$ to be equal to $V_{-1}$ at $\Omega = \Omega_M$. Thus the DG instability can be observed in the region $\Omega_L < \Omega < \Omega_M$. This region generally appears because the condition $\Omega_L < \Omega_U$ is satisfied $^{[15]}$ and $\Omega_M$ is always between $\Omega_L$ and $\Omega_U$ for positive $g$.

Next, we discuss the amplification of Kelvin waves after spontaneous radiation due to the Landau instability. In order to understand the amplification process qualitatively, we include the dissipation term phenomenologically in the left side of Eq. (3).

$$(i - \gamma) \frac{\partial \Psi}{\partial t} = \left( -\frac{\nabla^2}{2} + \frac{g^2}{2} + g|\Psi|^2 - \mu - V \hat{p}_z - \Omega \hat{L}_z \right) \Psi(11)$$

where $\gamma (> 0)$ is the dissipation term. The value of $\gamma$ should be determined from the interaction between an external environment and the condensate. The role of the environment, which dissipates the thermodynamic energy, and the condensate density is broken and the excited states are no longer eigenstates of the angular momentum, which makes the above analysis of the BdG equations very complicated. It is convenient in this case to use the imaginary time propagation (ITP) of the GP Eq. (5). The ITP makes it possible to obtain the smallest critical velocity $V_c = \min(V_l)$. We numerically investigate how $V_c$ depends on $\Omega$ for $g_{2D} = 500$ and which modes are excited via the instability for states with multiple vortices (Fig. 3). The critical velocity for each state has a maximum at a certain value of $\Omega$. The Kelvin and surface waves are amplified to the left and right side
of each maximum, respectively. We plot the ITP results for single-vortex states as cross marks in Fig. 2 (a), compared to BdG analysis results. Both results are consistent, which shows that the ITP analysis is available.

**FIG. 3:** Smallest critical velocity $V_c(\Omega) = \min(V_i)$ for vortex states with $n_v(=1, 2, 3, 4)$ vortices obtained by the ITP analysis.

When the Kelvin waves are amplified to the order of $1$ to $2$, vortex reconnection takes place (c,d,e), leading to complex dynamics (f).

**FIG. 4:** Nonlinear vortex dynamics caused by the DG instability in a trapped BEC with two vortices. After Kelvin waves are amplified (a,b), vortex reconnection takes place (c,d,e), leading to complex dynamics (f).

As discussed for superfluid helium [9], the DG instability in vortex lattices can lead to a dense vortex tangle, namely quantum turbulence. A similar scenario is expected for atomic BECs. In fact, we numerically obtained a transition from a vortex lattice (Fig. 5 left) to a dense vortex tangle (Fig. 5 right) after the amplification of Kelvin waves on each vortex (Fig. 5 center). The DG instability gives the possibility of realizing quantum turbulence in atomic BECs [20].

In conclusion, we theoretically discussed a new type of Landau instability in atomic BECs driven by a helically moving environment. In particular, we studied the Landau instability of Kelvin waves, which gives a microscopic understanding of the DG instability. The DG instability is possible for both single- and multi-vortex states. These phenomena lead to the direct observation of the dispersion relation of Kelvin waves, vortex reconnections, and quantum turbulence.

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[1] L. D. Landau and E. M. Lifshitz, *Statistical Physics*, Course of Theoretical Physics, Vol. 5, third ed., part 1 (Pergamon, New York, 1980).
[2] L. P. Pitaevskii and S. Stringari, *Bose-Einstein Condensation* (Oxford University Press, Oxford, 2003).
[3] W. Thomson, *Phil Mag.* 10, 155 (1880).
[4] W. I. Glaberson, W.W. Johnson, and R.M. Ostermeier, Phys. Rev. Lett. 33, 1197 (1974).
[5] R. J. Donnelly, *Quantized Vortices in Helium II* (Cambridge University Press, 1991).
[6] S. Ogawa, M. Tsubota, and Y. Hattori, J. Phys. Soc. Jpn. 71, 813 (2002).
[7] Prog. Low Temp. Phys., ed. W. P. Halperin and M. Tsubota (Elsevier, Amsterdam, 2008) Vol. 16.
[8] M. Tsubota, J. Phys. Soc. Jpn. 77, 111006 (2008).
[9] M. Tsubota, T. Araki, and C. F. Barenghi, Phys. Rev. Lett. 90, 205301 (2003).
[10] A. P. Finne et al., Phys. Rev. Lett. 96, 085301 (2006).
[11] L. P. Pitaevskii, Zh. Éksp. Teor. Fiz. 40, 646 (1961) [Sov. Phys. JETP 13, 451 (1961)].
[12] T. Mizushima, M. Ichioka, and K. Machida, Phys. Rev. Lett. 90, 180401 (2003).
[13] A. L. Fetter, Phys. Rev. A 69, 043617 (2004).
[14] T. P. Simula, T. Mizushima, and K. Machida, Phys. Rev. Lett. 101, 020402 (2008).
[15] T. Isoshima and K. Machida, J. Phys. Soc. Jpn. 68, 487 (1999).
[16] K. Kasamatsu, M. Tsubota, and M. Ueda, Phys. Rev. A 67, 033610 (2003).
[17] C. Raman, J. R. Abo-Shaeer, J. M. Vogels, K. Xu, and
W. Ketterle, Phys. Rev. Lett. 87, 210402 (2001).

[18] D. E. Miller et al., Phys. Rev. Lett. 99, 070402 (2007).

[19] V. Bretin, P. Rosenbusch, F. Chevy, G. V. Shlyapnikov, and J. Dalibard, Phys. Rev. Lett. 90, 100403 (2003).

[20] M. Kobayashi and M. Tsubota, Phys. Rev. A 76, 045603 (2007).