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On the Partial Sums and Marcinkiewicz and Fejér Means on the One- and Two-dimensional One-parameter Martingale Hardy Spaces

Georgian PhD Thesis

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**Abstract**

Unlike the classical theory of Fourier series which deals with decomposition of a function into continuous waves, the Walsh functions are rectangular waves. Such waves have already been used frequently in the theory of signal transmission, codic theory, cryptography, filtering, image enhancement and digital signal processing.

The problems we have studied in this PhD thesis are central to Mathematical Analysis. They involve techniques which have been developed a great deal during the last three decades.

In this PhD thesis we are dealing with convergence and summability of partial sums, Fejér and Marcinkiewicz means with respect to one- and two-dimensional Walsh-Fourier series on the martingale Hardy spaces.

This thesis is focus to achieve the following main results:

To find estimation of convergence and divergence of the subsequences of partial sums of the one-dimensional Walsh-Fourier series on the martingale Hardy spaces $H_p(G)$, when $0 < p \leq 1$.

To find necessary and sufficient conditions in terms of modulus of continuity of martingale Hardy spaces, for which subsequences of partial sums of the one-dimensional Walsh-Fourier series convergence in $H_p(G)$ norm, when $0 < p \leq 1$.

To find estimation of convergence and divergence of the subsequences of Fejér means of the one-dimensional Walsh-Fourier series on the martingale Hardy spaces $H_p(G)$, when $0 < p \leq 1/2$.

To find necessary and sufficient conditions in terms of modulus of continuity of martingale Hardy spaces, for which subsequences of Fejér means of the one-dimensional Walsh-Fourier series converge in $H_p(G)$ norm, when $0 < p \leq 1/2$.

To prove strong convergence of one-dimensional Fejér means with respect to Walsh system on the martingale Hardy spaces $H_p(G)$, when $0 < p \leq 1/2$.

To prove strong convergence of diagonal partial sums with respect to the two-dimensional Walsh-Fourier series on the martingale Hardy spaces $H_p(G^2)$, when $0 < p < 1$.

To prove strong convergence of Marcinkiewicz means with respect to the two-dimensional Walsh-Fourier series in $H_{2/3}(G^2)$ norm.

To find necessary and sufficient conditions in terms of modulus of continuity of Hardy spaces, for which Marcinkiewicz means of the two-dimensional Walsh-Fourier series converge in $H_{2/3}(G^2)$ norm.
Key words: Walsh group, Walsh system, $L_p$ space, weak-$L_p$ space, modulus of continuity, Walsh-Fourier coefficients, Walsh-Fourier series, partial sums, Lebesgue constants, Fejér means, Marcinkiewicz means, dyadic martingale, the one-dimensional Hardy space, the two-dimensional Hardy space, maximal operator, strong convergence.
Preface

The classical theory of Fourier series deals with decomposition of a function into sinusoidal waves. Unlike these continuous waves the Vilenkin (Walsh) functions are rectangular waves. Such waves have already been used frequently in the theory of signal transmission, multiplexing, filtering, image enhancement, coding theory, digital signal processing and pattern recognition. The development of the theory of Vilenkin-Fourier series has been strongly influenced by the classical theory of trigonometric series. Because of this it is inevitable to compare results of Vilenkin series to those on trigonometric series. There are many similarities between these theories, but there exist differences also. Much of these can be explained by modern abstract harmonic analysis, which studies orthonormal systems from the point of view of the structure of a topological group.

The problems studied in this PhD thesis are very important in Mathematical Analysis and its applications. In particular, we consider convergence and summability of partial sums, Fejér and Marcinkiewicz means with respect to the one- and two-dimensional Walsh-Fourier series in the martingale Hardy spaces. According to the problems considered in this PhD thesis, widely are used methods of real analysis combined with methods of abstract and non-linear harmonic analysis together with theory of approximation. Other research methods include theory of function spaces. They involve techniques which have been developed a great deal during the last three decades.

This PhD thesis consists the following chapters:

Preliminaries

Partial sums with respect to the one-dimensional Walsh-Fourier series on the martingale Hardy spaces

Fejér means means with respect to the one-dimensional Walsh-Fourier series on the martingale Hardy spaces

Convergence and summability of partial sums with respect to the two-dimensional Walsh-Fourier series on the martingale Hardy spaces

In Chapter 1 we first present some classical results and well-known facts, which are very important in the theory of Fourier analysis and is also very crucial for the further investigation of problems considered in this thesis. Moreover, there are presented results which are proved in the next chapters of this thesis and is emphasized the actuality of them, there is also shown some connections with known results.

In Chapter 2 we first define Walsh group and system, which are important to develop theory of harmonic analysis on locally compact Abelian groups. We consider some expressions and estimations of Lebesgue constants and Dirichlet kernels, give basic definitions and notation of the theory of martingale Hardy spaces and fundamental theorems which are very
important to prove main results of this thesis. We also construct martingales which we use to prove sharpness of our positive results. Next, we find rate of convergence and divergence of the subsequences of partial sums with respect to the one-dimensional Walsh-Fourier series on the martingale Hardy spaces for $0 < p \leq 1$. Finally, we apply these results to find necessary and sufficient conditions for the modulus of continuity which provide norm convergence of subsequences of the partial sums in the martingale Hardy spaces, for $0 < p \leq 1$.

In Chapter 3 we investigate Fejér means with respect to the one-dimensional Walsh-Fourier series. First, we consider some expressions and estimations of Fejér kernels and find rate of convergence and divergence of the subsequences of Fejér means in the martingale Hardy spaces for $0 < p \leq 1/2$. After that, we apply these results to find necessary and sufficient conditions for the modulus of continuity, which provide convergence of subsequences of Fejér means in the martingale Hardy spaces. Finally, we prove some new strong convergence theorems of Fejér means, for $0 < p \leq 1/2$. We also prove sharpness of all our main results in this Chapter.

In Chapter 4 we investigate basic definitions and notation of partial sums and Marcinkiewicz means with respect to the two-dimensional Walsh-Fourier series. First, we consider some expressions and estimations of Marcinkiewicz kernels, give basic definitions and notation of the theory of the two-dimensional martingale Hardy spaces and fundamental theorems which are very important to prove main results of this thesis. Next, we present and prove strong convergence results of diagonal partial sums with respect to the two-dimensional Walsh-Fourier series in the martingale Hardy spaces for $0 < p < 1$. Moreover, we consider strong convergence results of Marcinkiewicz means with respect to the two-dimensional Walsh-Fourier series for $p = 2/3$ and find necessary and sufficient conditions for the modulus of continuity, which provide convergence in $H_{2/3}(G^2)$ norm of Marcinkiewicz means.

This PhD thesis is written as a monograph based on the following publications:

[1] G. Tephnadze, Strong convergence of two-dimensional Walsh-Fourier series, Ukr. Math. J., 65, (6), (2013), 822-834.

[2] G. Tephnadze, Strong convergence theorems of Walsh-Fejr means, Acta Math. Hung., 142 (1) (2014), 244259.

[3] K. Nagy and G. Tephnadze, Approximation by Walsh-Marcinkiewicz means on the Hardy space $H_{2/3}$, Kyoto J. Math., 54 (3), (2014), 641-652.

[4] G. Tephnadze, On the partial sums of Walsh-Fourier series, Colloquium Mathematicum, 141, 2 (2015), 227-242.

[5] G. Tephnadze, On the convergence of Fejr means of Walsh-Fourier series in the space $H_p$, J. Contemp. Math. Anal., 51, 2 (2016), 51-63.

[6] K. Nagy and G. Tephnadze, Strong convergence theorem for Walsh-Marcinkiewicz means, Math. Inequal. Appl., 19, 1 (2016), 185195.
1 Preliminaries

It is well-known that (for details see e.g. [30] and [47]) for every \( p > 1 \) there exists an absolute constant \( c_p \), depending only on \( p \), such that

\[
\| S_n f \|_p \leq c_p \| f \|_p , \quad \text{when } p > 1 \text{ and } f \in H_1(G).
\]

Moreover, Watari [76] (see also Gosselin [31] and Young [85]) proved that there exists an absolute constant \( c \) such that, for \( n = 1, 2, \ldots, \)

\[
\lambda \mu (|S_n f| > \lambda \lambda) \leq c \| f \|_1 , \quad f \in L_1(G_m), \quad \lambda > 0.
\]

On the other hand, it is also well-known that (for details see e.g. [1] and [47]) Walsh system is not Schauder basis in \( L_1(G) \) space. Moreover, there exists function \( f \in H_1(G) \), such that partial sums with respect to Walsh system are not uniformly bounded in \( L_1(G) \).

By applying Lebesgue constants

\[
L(n) := \| D_n \|_1
\]

we easily obtain that (for details see e.g. [2] and [47]) subsequences of partial sums \( S_{n_k} f \) with respect to Walsh system converge to \( f \) in \( L_1 \) norm if and only if

\[
\sup_{k \in \mathbb{N}} L(n_k) \leq c < \infty . \tag{1.1}
\]

Since \( n \)-th Lebesgue constant with respect to Walsh system, where

\[
n = \sum_{j=0}^{\infty} n_j 2^j , (n_j \in \mathbb{Z}_2)
\]

can be estimated by variation of natural number

\[
V(n) = n_0 + \sum_{k=1}^{\infty} |n_k - n_{k-1}|
\]

and it is also well known that (for details see e.g. [12] and [47]) the following two-sided estimate is true

\[
\frac{1}{8} V(n) \leq L(n) \leq V(n)
\]

to obtain convergence of subsequences of partial sums \( S_{n_k} f \) with respect to Walsh system of \( f \in L_1 \) in \( f \in L_1 \)-norm. Condition (1.1) can be replaced by

\[
\sup_{k \in \mathbb{N}} V(n_k) \leq c < \infty
\]
Partial Sums and Marcinkiewicz and Fejér Means

It follows that (for details see e.g. [47] and [78]) subsequence of partial sums $S_{2^n}$ are bounded from $H_p(G)$ to $H_p(G)$ for every $p > 0$, from which we obtain that

$$\|S_{2^n} f - f\|_{H_p(G)} \to 0, \text{ as } n \to \infty,$$  \hfill (1.2)

On the other hand, (see e.g. [61]) there exist a martingale $f \in H_p(G)$ ($0 < p < 1$), such that

$$\sup_{n \in \mathbb{N}} \|S_{2^n+1} f\|_{weak-L_p(G)} = \infty.$$

The main reason of divergence of subsequence $S_{2^n+1} f$ of partial sums is that (for details see [62]) Fourier coefficients of $f \in H_p(G)$ are not uniformly bounded when $0 < p < 1$.

When $0 < p < 1$ in [74] was investigated boundedness of subsequences of partial sums with respect to Walsh system from $H_p(G)$ to $H_p(G)$. In particular, the following result is true:

**Theorem T1.** Let $0 < p < 1$ and $f \in H_p(G)$. Then there exists an absolute constant $c_p$, depending only on $p$, such that

$$\|S_{m_k} f\|_{H_p(G)} \leq c_p \|f\|_{H_p(G)}$$

if and only if the following condition holds

$$\sup_{k \in \mathbb{N}} d(m_k) < c < \infty,$$  \hfill (1.3)

where

$$d(m_k) := |m_k| - \langle m_k \rangle.$$

In particular, Theorem T1 immediately follows:

**Theorem T2.** Let $p > 0$ and $f \in H_p(G)$. Then there exists a absolute constant $c_p$, depending only on $p$, such that

$$\|S_{2^n} f\|_{H_p(G)} \leq c_p \|f\|_{H_p(G)}$$

and

$$\|S_{2^n+2^n-1} f\|_{H_p(G)} \leq c_p \|f\|_{H_p(G)}.$$

On the other hand, we have the following result:

**Theorem T3.** Let $p > 0$. Then there exists a martingale $f \in H_p(G)$, such that

$$\sup_{n \in \mathbb{N}} \|S_{2^n+1} f\|_{H_p(G)} = \infty.$$

Taking into account these results it is interesting to find behaviour of rate of divergence of subsequences of partials sums with respect to Walsh system of martingale $f \in H_p(G)$ in the martingale Hardy spaces $H_p(G)$.

*Hardy spaces*
In the second chapter of this thesis (see also [63]) we investigate above mentioned problem. For $0 < p < 1$ we have the following result:

**Theorem 2.10.** Let $f \in H_p(G)$. Then there exists a absolute constant $c_p$, depending only on $p$, such that the following inequality is true

$$\|S_n f\|_{H_p(G)} \leq c_p 2^{d(n)(1/p-1)} \|f\|_{H_p(G)}.$$ \hfill (1.4)

On the other hand, if $0 < p < 1$, $\{m_k : k \geq 0\}$ be increasing subsequence of natural numbers, such that

$$\sup_{k \in \mathbb{N}} d(m_k) = \infty$$ \hfill (1.5)

and $\Phi : \mathbb{N}_+ \to [1, \infty)$ be non-decreasing function satisfying the condition

$$\lim_{k \to \infty} \frac{d(m_k)(1/p-1)}{\Phi (m_k)} = \infty,$$

then there exists a martingale $f \in H_p(G)$, such that

$$\sup_{k \in \mathbb{N}} \left\| \frac{S_{m_k} f}{\Phi (m_k)} \right\|_{\text{weak-}L^p(G)} = \infty.$$

Theorem 2.11 easily follows the following corollary:

**Corollary 2.11.** Let $0 < p < 1$ and $f \in H_p(G)$. Then there exists a absolute constant $c_p$, depending only on $p$, such that

$$\|S_n f\|_{H_p(G)} \leq c_p \left\{ n \mu \{\text{supp} \ (D_n)\} \right\}^{1/p-1} \|f\|_{H_p(G)}.$$

On the other hand, if $0 < p < 1$ and $\{m_k : k \geq 0\}$ be increasing sequence of natural numbers, such that

$$\sup_{k \in \mathbb{N}} m_k \mu \{\text{supp} \ (D_{m_k})\} = \infty$$

and $\Phi : \mathbb{N}_+ \to [1, \infty)$ be non-decreasing function satisfying the condition

$$\lim_{k \to \infty} \frac{m_k \mu \{\text{supp} \ (D_{m_k})\}^{1/p-1}}{\Phi (m_k)} = \infty,$$

then there exists a martingale $f \in H_p(G)$, such that

$$\sup_{k \in \mathbb{N}} \left\| \frac{S_{m_k} f}{\Phi (m_k)} \right\|_{\text{weak-}L^p(G)} = \infty.$$

In particular, we also get the proofs of Theorem T1 and Theorem T2.

In the second chapter of this thesis we also investigate case $p = 1$. In this case the following result is true:

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Theorem 2.12. Let $n \in \mathbb{N}_+$ and $f \in H_1(G)$. Then there exists a absolute constant $c$, such that
\[
\|S_n f\|_{H_1(G)} \leq c V(n) \|f\|_{H_1(G)}.
\] (1.6)

Moreover, if $\{m_k : k \geq 0\}$ be increasing sequence of natural numbers $\mathbb{N}_+$, such that
\[
\sup_{k \in \mathbb{N}} V(m_k) = \infty
\]
and $\Phi : \mathbb{N}_+ \to [1, \infty)$ be non-decreasing function satisfying the condition
\[
\lim_{k \to \infty} \frac{V(m_k)}{\Phi(m_k)} = \infty.
\]

Then there exists a martingale $f \in H_1(G)$, such that
\[
\|S_{m_k} f\|_{\Phi(m_k)} = \infty.
\]

When $0 < p < 1$ in [74] was proved boundedness of maximal operators of subsequences of partial sums from $H_p(G)$ to $L_p(G)$. In particular, the following is true:

**Theorem T4.** Let $0 < p < 1$ and $f \in H_p(G)$. Then the maximal operator
\[
\sup_{k \in \mathbb{N}} |S_{m_k} f|
\]
is bounded from $H_p(G)$ to $L_p(G)$, if and only if condition (1.3) is fulfilled.

In the special cases we obtain that the following is true:

**Theorem T5.** Let $p > 0$ and $f \in H_p(G)$. Then there exists an absolute constant $c_p$, depending only on $p$, such that
\[
\left\| \sup_{n \in \mathbb{N}} |S_{2^n} f| \right\|_p \leq c_p \|f\|_{H_p(G)}
\] (1.7)
and
\[
\left\| \sup_{n \in \mathbb{N}} |S_{2^n + 2^n - 1} f| \right\|_p \leq c_p \|f\|_{H_p(G)}.
\]

On the other hand we have the following result:

**Theorem T6.** Let $p > 0$. Then there exists a martingale $f \in H_p(G)$, such that
\[
\left\| \sup_{n \in \mathbb{N}} |S_{2^n + 1} f| \right\|_p = \infty.
\]

Above mentioned condition (1.3) is sufficient condition for the case $p = 1$ also, but there exist subsequences which do not satisfy this condition, but maximal operators of these
subsequences of partial sums with respect to Walsh system are not bounded from $H_1(G)$ to $L_1(G)$.

Such necessary and sufficient conditions which provides boundedness of maximal operators of subsequences of partial sums with respect to Walsh system from $H_1(G)$ to $L_1(G)$ is open problem.

In [62] and [74] was investigated boundedness of weighted maximal operators from $H_p(G)$ to $L_p(G)$, when $0 < p \leq 1$:

**Theorem T7.** Let $0 < p \leq 1$. Then weighted maximal operator

$$\tilde{S}_{p,f}^* := \sup_{n \in \mathbb{N}_+} \frac{|S_n f|}{(n + 1)^{1/p - 1} \log[p](n + 1)}$$

is bounded from $H_p(G)$ to $L_p(G)$, where $[p]$ denotes integer part of $p$.

Moreover, for any non-decreasing function $\varphi : \mathbb{N}_+ \rightarrow [1, \infty)$ satisfying the condition

$$\lim_{n \to \infty} \frac{(n + 1)^{1/p - 1} \log[p](n + 1)}{\varphi(n + 1)} = +\infty,$$

there exists a martingale $f \in H_p(G)$ $(0 < p \leq 1)$, such that

$$\sup_{n \in \mathbb{N}} \left\| \frac{S_n f}{\varphi(n)} \right\|_p = \infty.$$

According to negative result for weighted maximal operator of partial sums of Walsh-Fourier series we immediately get the following result:

**Theorem S1.** There exists a martingale $f \in H_p(G)$, $(0 < p \leq 1)$, such that

$$\sup_{n \in \mathbb{N}} \| S_n f \|_p = \infty.$$

On the other hand, boundedness of weighted maximal operators immediately follows the following estimation:

**Theorem S2.** Let $0 < p \leq 1$. Then there exists a absolute constant $c_p$, depending only on $p$, such that

$$\| S_n f \|_p \leq c_p (n + 1)^{1/p - 1} \log[p](n + 1) \| f \|_{H_p(G)}, \text{ for } 0 < p \leq 1,$$

where $[p]$ denotes integer part of $p$.

By applying this inequality (see [60]) we find necessary and sufficient conditions for martingale $f \in H_p(G)$ for which partial sums with respect to Walsh system of martingale $f \in H_p(G)$ converge in $H_p(G)$ norm.

**Theorem T8.** Let $0 < p \leq 1$, $[p]$ denotes integer part of $p$, $f \in H_p(G)$ and

$$\omega_{H_p(G)} \left( \frac{1}{2N}, f \right) = o \left( \frac{1}{2^N(1/p - 1) N^{[p]}} \right), \text{ as } N \to \infty.$$
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Then
\[ \| S_n f - f \|_p \to 0, \text{ as } n \to \infty. \]

Moreover, there exists a martingale \( f \in H_p(G) \), where \( 0 < p < 1 \), such that
\[ \omega_{H_p(G)} \left( \frac{1}{2^N}, f \right) = O \left( \frac{1}{2^{N(1/p-1)} N^{[p]}} \right), \text{ as } N \to \infty \]

and
\[ \| S_n f - f \|_{\text{weak-}L_p(G)} \not\to 0, \text{ as } n \to \infty. \]

By taking these results into account, it is interesting to find necessary and sufficient conditions for modulus of continuity, such that subsequences of partial sums with respect to Walsh system of martingale \( f \in H_p(G) \) converge in \( H_p(G) \) norm.

In the second chapter of this thesis (see also [63]) we investigate this problem. By combining inequalities (1.4) and (1.6) we get the following theorem:

**Theorem 2.16.** Let \( 2^k < n \leq 2^{k+1} \). Then there exists an absolute constant \( c_p \), depending only on \( p \), such that
\[ \| S_n f - f \|_{H_p(G)} \leq c_p 2^{d(n)(1/p-1)} \omega_{H_p(G)} \left( \frac{1}{2^k}, f \right), \quad (0 < p < 1) \]

and
\[ \| S_n f - f \|_{H_1(G)} \leq c_1 V(n) \omega_{H_1(G)} \left( \frac{1}{2^k}, f \right) \]

(1.8)

By applying inequality (1.8) the following result is proved in the second chapter:

**Theorem 2.17.** Let \( 0 < p < 1 \), \( f \in H_p(G) \) and \( \{ m_k : k \geq 0 \} \) be increasing sequence of natural number satisfying the condition
\[ \omega_{H_p(G)} \left( \frac{1}{2^{|m_k|}}, f \right) = o \left( \frac{1}{2^{d(m_k)(1/p-1)}} \right) \text{ as } k \to \infty. \]

Then
\[ \| S_{m_k} f - f \|_{H_p(G)} \to 0 \text{ as } k \to \infty. \]

(1.10)

On the other hand, if \( \{ m_k : k \geq 0 \} \) be increasing sequence of natural numbers satisfying the condition (1.5), then there exists a martingale \( f \in H_p(G) \) and subsequence \( \{ \alpha_k : k \geq 0 \} \subset \{ m_k : k \geq 0 \} \), for which
\[ \omega_{H_p(G)} \left( \frac{1}{2^{|\alpha_k|}}, f \right) = O \left( \frac{1}{2^{d(\alpha_k)(1/p-1)}} \right) \text{ as } k \to \infty \]

and
\[ \limsup_{k \to \infty} \| S_{\alpha_k} f - f \|_{\text{weak-}L_p(G)} > c_p > 0 \text{ as } k \to \infty. \]

(1.11)
where $c_p$ is an absolute constant depending only on $p$.

According to this theorem we immediately get that the following result is true:

**Corollary 2.18.** Let $0 < p < 1$, $f \in H_p(G)$ and $\{m_k : k \geq 0\}$ be increasing sequence of natural number, satisfying the condition

$$\omega_{H_p(G)} \left( \frac{1}{2^{|m_k|}}, f \right) = o \left( \frac{1}{(m_k \mu (\text{supp} D_{m_k}))^{1/p-1}} \right), \text{ as } k \to \infty.$$

Then (1.10) holds.

On the other hand, if $\{m_k : k \geq 0\}$ be increasing sequence of natural number, satisfying the condition

$$\lim_{k \to \infty} \left( \frac{m_k \mu (\text{supp} D_{m_k}))^{1/p-1}}{\Phi(m_k)} \right) = \infty,$$

then there exists a martingale $f \in H_p(G)$ and subsequence $\{\alpha_k : k \geq 0\} \subset \{m_k : k \geq 0\}$ such that

$$\omega_{H_p(G)} \left( \frac{1}{2^{|\alpha_k|}}, f \right) = O \left( \frac{1}{(\alpha_k \mu (\text{supp} D_{\alpha_k}))^{1/p-1}} \right), \text{ as } k \to \infty$$

and (1.11) holds.

By applying (1.9) we prove that the following is true:

**Theorem 2.19.** Let $f \in H_1(G)$ and $\{m_k : k \geq 0\}$ be increasing sequence of natural number, satisfying the condition

$$\omega_{H_1(G)} \left( \frac{1}{2^{|m_k|}}, f \right) = o \left( \frac{1}{V(m_k)} \right) \text{ as } k \to \infty.$$

Then

$$\|S_{m_k} f - f\|_{H_1(G)} \to 0 \text{ as } k \to \infty.$$

Moreover, if $\{m_k : k \geq 0\}$ be increasing sequence of natural number, satisfying the condition (1.5), then there exists a martingale $f \in H_1(G)$ and subsequence $\{\alpha_k : k \geq 0\} \subset \{m_k : k \geq 0\}$ for which

$$\omega_{H_1(G)} \left( \frac{1}{2^{|\alpha_k|}}, f \right) = O \left( \frac{1}{V(\alpha_k)} \right) \text{ as } k \to \infty$$

and

$$\limsup_{k \to \infty} \|S_{\alpha_k} f - f\|_1 > c > 0 \text{ as } k \to \infty,$$

where $c$ is an absolute constant.

By applying Theorem 2.17 and Theorem 2.19 we immediately get proof of Theorem T8.
Weisz [79] consider convergence in norm of Fejér means of the one-dimensional Walsh-Fourier and proved the following:

**Theorem We1.** Let \( p > 1/2 \) and \( f \in H_p(G) \). Then there exists a absolute constant \( c_p \), depending only on \( p \), such that
\[
\|\sigma_k f\|_{H_p(G)} \leq c_p f_{H_p(G)}.
\]

Weisz (for details see e.g. [78]) also consider boundedness of subsequences of Fejér means \( \sigma_{2^n} \) of the one-dimensional Walsh-Fourier series from \( H_p(G) \) to \( H_p(G) \) when \( p > 0 \):

**Theorem We2.** Let \( p > 0 \) and \( f \in H_p(G) \). Then
\[
\|\sigma_{2^k} f - f\|_{H_p(G)} \to 0, \quad \text{as} \quad k \to \infty.
\] (1.12)

On the other hand, in [56] was proved the following result:

**Theorem T9.** There exists a martingale \( f \in H_p(G) \) \((0 < p \leq 1/2)\) such that
\[
\sup_{n \in \mathbb{N}} \|\sigma_{2^n+1} f\|_{H_p(G)} = \infty.
\]

Goginava [27] (see also [44]) proved that the following result is true:

**Theorem Gog1.** Let \( 0 < p \leq 1 \). Then the sequence of operators \( |\sigma_{2^n} f| \) are not bounded from \( H_p(G) \) to \( H_p(G) \).

When \( 0 < p < 1/2 \) then in [45] was proved bondedness of subsequences of Fejér means of the one-dimensional Walsh-Fourier from \( H_p(G) \) to \( H_p(G) \). In particular, the following is true:

**Theorem T10.** Let \( 0 < p < 1/2 \) and \( f \in H_p(G) \). Then there exists a absolute constant \( c_p \), depending only on \( p \), such that
\[
\|\sigma_{mk} f\|_{H_p(G)} \leq c_p f_{H_p(G)}
\]
estimation holds if and only if the condition (1.3) is fulfilled.

Theorem T10 immediately follows theorem of Weisz (see Theorem We2) and and also interesting results:

**Theorem T11.** Let \( p > 0 \) and \( f \in H_p(G) \). Then there exists an absolute constant \( c_p \), depending only on \( p \), such that
\[
\|\sigma_{2^n} f\|_{H_p(G)} \leq c_p f_{H_p(G)}
\]

and
\[
\|\sigma_{2^n+2^n-1} f\|_{H_p(G)} \leq c_p f_{H_p(G)}.
\]

On the other hand, we have the following result:
Theorem T12. Let $p > 0$. Then there exists a martingale $f \in H_p(G)$, such that
\[
\sup_{n \in \mathbb{N}} \left\| \sigma_{2^n+1} f \right\|_{H_p(G)} = \infty.
\]

According to above mentioned results it is interesting to find rate of divergence of subsequences $\sigma_{n_k} f$ of Fejér means of the one-dimensional Walsh-Fourier series in the Hardy spaces $H_p(G)$.

In the third chapter of this thesis (see also [64]) we find rate of divergence of subsequences of Fejér means of the one-dimensional Walsh-Fourier series on the martingale Hardy spaces $H_p(G)$, when $0 < p \leq 1/2$.

First, we consider case $p = 1/2$:

**Theorem 3.28.** Let $n \in \mathbb{N}_+$ and $f \in H_{1/2}(G)$. Then there exists an absolute constant $c$, such that
\[
\left\| \sigma_n f \right\|_{H_{1/2}(G)} \leq c V^2(n) \left\| f \right\|_{H_{1/2}(G)}.
\]

Moreover, if \( \{m_k : k \geq 0\} \) be increasing sequence of natural numbers, such that
\[
\sup_{k \in \mathbb{N}} V(m_k) = \infty
\]
and $\Phi : \mathbb{N}_+ \to [1, \infty]$ be non-decreasing function satisfying the conditions
\[
\lim_{k \to \infty} \frac{V^2(m_k)}{\Phi(m_k)} = \infty,
\]
then there exists a martingale $f \in H_{1/2}(G)$, such that
\[
\sup_{k \in \mathbb{N}} \left\| \sigma_{m_k} f \right\|_{1/2} = \infty.
\]

There was also considered case $0 < p < 1/2$ and was proved that the following is true:

**Theorem 3.29.** Let $0 < p < 1/2$ and $f \in H_p(G)$. Then there exists an absolute constant $c_p$, depending only on $p$ such that
\[
\left\| \sigma_n f \right\|_{H_p(G)} \leq c_p 2^{d(n)(1/p-2)} \left\| f \right\|_{H_p(G)}.
\]

On the other hand, if $0 < p < 1/2$, \( \{m_k : k \geq 0\} \) be increasing sequence of natural numbers satisfying the condition (1.5) and $\Phi : \mathbb{N}_+ \to [1, \infty)$ be non-decreasing function such that
\[
\lim_{k \to \infty} \frac{2^{d(m_k)(1/p-2)}}{\Phi(m_k)} = \infty,
\]
then there exists a martingale $f \in H_p(G)$, such that
\[
\sup_{k \in \mathbb{N}} \left\| \sigma_{m_k} f \right\|_{\text{weak-}L_p(G)} = \infty.
\]
From these results also follows proof of Theorem We2.

In 1975 Schipp [46] (see also [86]) proved that the maximal operator of Fejér means $\sigma^*$ is of type weak-(1,1):

$$\mu(\sigma^* f > \lambda) \leq \frac{c}{\lambda}\|f\|_1, \quad (\lambda > 0).$$

By using Marcinkiewicz interpolation theorem it follows that $\sigma^*$ is of strong type-$(p,p)$, when $p > 1$:

$$\|\sigma^* f\|_p \leq c\|f\|_p, \quad (p > 1).$$

The boundedness does not hold for $p = 1$, but Fujji [15] (see also [84]) proved that maximal operator of Fejér means is bounded from $H_1(G)$ to $L_1(G)$. Weisz in [80] generalized result of Fujji and proved that maximal operator of Fejér means is bounded from $H_p(G)$ to $L_p(G)$, when $p > 1/2$. Simon [48] construct the counterexample, which shows that boundedness does not hold when $0 < p < 1/2$. Goginava [21] (see also [9] and [10]) generalized this result for $0 < p \leq 1/2$ and proved that the following is true:

**Theorem Gog2.** There exists a martingale $f \in H_p(G)$ $(0 < p \leq 1/2)$ such that

$$\sup_{n \in \mathbb{N}} \|\sigma_n f\|_p = \infty.$$

Weisz [81] (see also Goginava [23]) proved that the following is true:

**Theorem We3.** Let $f \in H_{1/2}(G)$. Then there exists an absolute constant $c$, such that

$$\|\sigma^* f\|_{L_{1/2}(G)} \leq c \|f\|_{H_{1/2}(G)}.$$

In [45] was considered boundedness of maximal operators of subsequences of Fejér means of the one-dimensional Walsh-Fourier series from $H_p(G)$ to $L_p(G)$ for $0 < p < 1/2$. In particular, the following is true:

**Theorem T13.** Let $0 < p < 1/2$ and $f \in H_p(G)$. Then the maximal operator

$$\tilde{\sigma}^* f := \sup_{k \in \mathbb{N}} |\sigma_{m_k} f|$$

is bounded from $H_p(G)$ to $L_p(G)$ if and only if when condition (1.3) is fulfilled.

As consequences the following results are true:

**Theorem T14.** Let $p > 0$ and $f \in H_p(G)$. Then there exists an absolute constant $c_p$ depending only on $p$, such that

$$\left\| \sup_{n \in \mathbb{N}} |\sigma_{2^n} f| \right\|_p \leq c_p \|f\|_{H_p(G)} \quad (1.15)$$

and

$$\left\| \sup_{n \in \mathbb{N}} |\sigma_{2^n + 2^n - 1} f| \right\|_p \leq c_p \|f\|_{H_p(G)}.$$
On the other hand, we have the following negative result:

**Theorem T15.** Let $0 < p < 1/2$. Then there exists a martingale $f \in H_p(G)$, such that

$$
\left\| \sup_{n \in \mathbb{N}} |\sigma_{2^{n+1}} f| \right\|_p = \infty.
$$

above mentioned condition is sufficient for the case $p = 1/2$ also, but there exists subsequences, which do not satisfy condition (1.3), but maximal operator of subsequences of Fejér means of the one-dimensional Walsh-Fourier series are bounded from $H_{1/2}(G)$ to $L_{1/2}(G)$.

However, it is open problem to find necessary and sufficient conditions on the indexes, which provide boundedness of maximal operator of subsequences of Fejér means of the one-dimensional Walsh-Fourier series from $H_{1/2}(G)$ to $L_{1/2}(G)$.

In [22] and [56] (see also [43], [58], [29] and [55]) is proved that the following is true:

**Theorem GT1.** Let $0 < p \leq 1/2$ and $f \in H_p(G)$. Then the maximal operator

$$
\tilde{\sigma}_p f := \sup_{n \in \mathbb{N}} \frac{|\sigma_n f|}{(n+1)^{1/p-2} \log^{2[1/2+p]} (n+1)}
$$

is bounded from $H_p(G)$ to $L_p(G)$.

Moreover, for any nondecreasing function $\varphi : \mathbb{N} \rightarrow [1, \infty)$ satisfying the condition

$$
\lim_{n \to \infty} \frac{(n+1)^{1/p-2} \log^{2[1/2+p]} (n+1)}{\varphi(n)} = +\infty,
$$

there exists a martingale $f \in H_p(G)$, $(0 < p < 1/2)$ such that

$$
\sup_{n \in \mathbb{N}} \left\| \frac{\sigma_n f}{\varphi(n)} \right\|_p = \infty.
$$

From the divergence of weighted maximal operators we immediately get that there exists a martingale $f \in H_p(G)$ $(0 < p \leq 1/2)$, such that

$$
\sup_{n \in \mathbb{N}} \left\| \sigma_n f \right\|_p = \infty.
$$

and from the boundedness results of weighted maximal operators we immediately get that for any $f \in H_p(G)$ there exists an absolute constant $c_p$, such that the following inequality holds true:

$$
\left\| \sigma_n f \right\|_p \leq c_p n^{1/p-2} \log^{2[1/2+p]} (n+1) \left\| f \right\|_{H_p(G)}, \quad 0 < p \leq 1/2.
$$

(1.16)

By applying inequality (1.16) in [60] was found necessary and sufficient conditions for modulus of continuity of martingale $f \in H_p(G)$, for which Fejér means of the one-dimensional Walsh-Fourier series converge in $H_p(G)$ norm.
Theorem T16. Let $0 < p \leq 1/2$, $f \in H_p(G)$ and 

$$
\omega_{H_p(G)} \left( \frac{1}{2N}, f \right) = o \left( \frac{1}{2^{N(1/p-2)} N^{1/2} [1/p]} \right), \quad \text{as } N \to \infty.
$$

Then

$$
\| \sigma_n f - f \|_p \to 0, \quad \text{as } n \to \infty.
$$

Moreover, there exists a martingale $f \in H_p(G)$, for which

$$
\omega_{H_{1/2}(G)} \left( \frac{1}{2N}, f \right) = O \left( \frac{1}{2^{N(1/p-2)} N^{1/2} [1/p]} \right), \quad \text{as } N \to \infty
$$

and

$$
\| \sigma_n f - f \|_p \not\to 0, \quad \text{as } n \to \infty.
$$

According above mentioned results, it is interesting to find necessary and sufficient conditions for the modulus of continuity, for which subsequences $\sigma_{n_k} f$ of Fejé r means of the one-dimensional Walsh-Fourier series converge in $H_p(G)$ norm.

In the third chapter of this thesis we find necessary and sufficient conditions for the modulus of continuity, for which subsequences $\sigma_{n_k} f$ of Fejé r means of the one-dimensional Walsh-Fourier series converge in $H_p(G)$ norm (see also [64]).

By applying inequality (1.13) for the case $p = 1/2$ the following necessary and sufficient conditions are found:

Theorem 3.33. Let $f \in H_{1/2}(G)$ and $\{m_k : k \geq 0\}$ be increasing sequence of natural numbers, such that

$$
\omega_{H_{1/2}(G)} \left( \frac{1}{2^{m_k}}, f \right) = o \left( \frac{1}{V^2(m_k)} \right), \quad \text{as } k \to \infty.
$$

Then

$$
\| \sigma_{m_k} f - f \|_{H_{1/2}(G)} \to 0 \quad \text{as } k \to \infty.
$$

Moreover, if $\{m_k : k \geq 0\}$ be increasing sequence of natural numbers, such that (1.5) holds true, then there exists a martingale $f \in H_{1/2}(G)$ and subsequence $\{\alpha_k : k \geq 0\} \subset \{m_k : k \geq 0\}$ such that

$$
\omega_{H_{1/2}(G)} \left( \frac{1}{2^{\alpha_k}}, f \right) = O \left( \frac{1}{V^2(\alpha_k)} \right), \quad \text{as } k \to \infty
$$

and

$$
\limsup_{k \to \infty} \| \sigma_{\alpha_k} f - f \|_{1/2} \geq c > 0 \quad \text{as } k \to \infty,
$$

where $c$ is an absolute constant.

By applying inequality (1.14) we also investigate case $0 < p < 1/2$. In the third chapter of this thesis we prove that the following is true:

Hardy spaces...
**Theorem 3.34.** Let $0 < p < 1/2$, $f \in H_p(G)$ and $\{m_k : k \geq 0\}$ be increasing sequence of natural numbers, such that

$$\omega_{H_p(G)}\left(\frac{1}{2|m_k|}, f\right) = o\left(\frac{1}{2d(m_k)(1/p-2)}\right), \text{ as } k \to \infty.$$  

Then

$$\|\sigma_{m_k} f - f\|_{H_p(G)} \to 0, \text{ as } k \to \infty.$$  

On the other hand, if $\{m_k : k \geq 0\}$ be increasing sequence of natural numbers satisfying the condition (1.5), then there exists a martingale $f \in H_p(G)$ and subsequence $\{\alpha_k : k \geq 0\} \subset \{m_k : k \geq 0\}$, for which

$$\omega_{H_p(G)}\left(\frac{1}{2|\alpha_k|}, f\right) = O\left(\frac{1}{2d(\alpha_k)(1/p-2)}\right), \text{ as } k \to \infty$$  

and

$$\limsup_{k \to \infty} \|\sigma_{\alpha_k} f - f\|_{\text{weak-}L_p(G)} > c_p > 0, \text{ as } k \to \infty,$$

where $c_p$ is constant depending only on $p$.

However, Simon in [49] and [51] (see also [13, 52]) consider strong convergence theorems of the one-dimensional Walsh-Fouriere series and proved the following:

**Theorem Si1.** Let $0 < p \leq 1$ and $f \in H_1(G)$. Then there exists an absolute constant $c_p$ depending only on $p$, such that the following inequality is true:

$$\frac{1}{\log |p|} \sum_{k=1}^{n} \|S_k f\|_{H_p(G)} \leq c_p \|f\|_{H_p(G)}.$$  

Analogical result for trigonometric system was proved in [53], for unbounded Walsh systems in [17].

In [57] was proved that the following is true:

**Theorem T17.** for any $0 < p < 1$ and non-decreasing function $\varphi : \mathbb{N}_+ \to [1, \infty)$ satisfying the condition

$$\lim_{n \to \infty} \frac{n^{2-p}}{\varphi(n)} = +\infty,$$

there exists a martingale $f \in H_p(G)$, such that

$$\sum_{k=1}^{\infty} \frac{\|S_k f\|^p_{\text{weak-}L_p(G)}}{\varphi(k)} = \infty, \quad (0 < p < 1).$$

Theorem Si1 follows that if $f \in H_1(G)$ then the following equalities are true:

$$\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\|S_k f - f\|_1}{k} = 0.$$
and
\[ \lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\|S_k f\|_1}{k} = \|f\|_{H_p(G)}. \]

When \(0 < p < 1\) and \(f \in H_p(G)\), then Theorem S1 follows that there exists an absolute constant \(c_p\) depending only on \(p\), such that
\[ \frac{1}{n^{1/2-p/2}} \sum_{k=1}^{n} \frac{\|S_k f\|_p^p}{k^{3/2-p/2}} \leq c_p \|f\|_p^p. \]

Moreover,
\[ \frac{1}{n^{1/2-p/2}} \sum_{k=1}^{n} \frac{\|S_k f - f\|_p^p}{k^{3/2-p}} = 0. \]

It follows the following equality
\[ \frac{1}{n^{1/2-p/2}} \sum_{k=1}^{n} \frac{\|S_k f\|_p^p}{k^{3/2-p/2}} = \|f\|_p^p. \]

In the third chapter of this thesis we consider strong convergence results of Fejér means of the one-dimensional Walsh-Fourier series. According to Theorem We1 and Theorem Gog2 we only have to consider case \(0 < p \leq 1/2\) (for details see [59], see also [4], [5], [7], [8], [12]):

**Theorem 3.37.** Let \(0 < p \leq 1/2\) and \(f \in H_p(G)\). Then there exists an absolute constant \(c_p\) depending only on \(p\), such that
\[ \frac{1}{\log^{1/2+p} n} \sum_{m=1}^{n} \frac{\|\sigma_m f\|_p^p}{m^{2-2p}} \leq c_p \|f\|_p^p. \]

Moreover, let \(0 < p < 1/2\) and \(\Phi : \mathbb{N}_+ \to [1, \infty)\) be non-decreasing, non-negative function, such that \(\Phi(n) \uparrow \infty\) and
\[ \lim_{k \to \infty} \frac{k^{2-2p}}{\Phi(k)} = \infty. \]

Then there exists a martingale \(f \in H_p(G)\), such that
\[ \sum_{m=1}^{\infty} \frac{\|\sigma_m f\|_{\text{weak-}L_p(G)}^p}{\Phi(m)} = \infty. \]

When \(p = 1/2\) is was also proved that the following is true:

**Theorem 3.38.** Let \(f \in H_{1/2}(G)\). Then
\[ \sup_{n \in \mathbb{N}_+} \|f\|_{H_{1/2}(G)} \leq \sup_{n \in \mathbb{N}_+} \frac{1}{n} \sum_{m=1}^{n} \|\sigma_m f\|_{1/2}^{1/2} = \infty. \]
Theorem 3.37 follows that if \( f \in H_{1/2}(G) \) then the following equalities are true:

\[
\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\|\sigma_k f - f\|_{H_{1/2}(G)}^{1/2}}{k} = 0
\]

and

\[
\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\|\sigma_k f\|_{H_{1/2}(G)}^{1/2}}{k} = \|f\|_{H_{1/2}(G)}^{1/2}.
\]

When \( 0 < p < 1/2 \) and \( f \in H_p(G) \), then Theorem 3.37 follows that there exists an absolute constant \( c_p \) depending only on \( p \), such that

\[
\frac{1}{n^{1/2-p}} \sum_{k=1}^{n} \frac{\|\sigma_k f\|_{H_p(G)}^p}{k^{3/2-p}} \leq c_p \|f\|_{H_p(G)}^p.
\]

Moreover,

\[
\frac{1}{n^{1/2-p}} \sum_{k=1}^{n} \frac{\|\sigma_k f - f\|_{H_p(G)}^p}{k^{3/2-p}} = 0.
\]

It follows that

\[
\frac{1}{n^{1/2-p}} \sum_{k=1}^{n} \frac{\|\sigma_k f\|_{H_p(G)}^p}{k^{3/2-p}} = \|f\|_{H_p(G)}^p.
\]

For the two-dimensional case (for details see e.g. [47] and [78]) the following is true:

**Theorem S3.** Let \( p > 0 \) and \( f \in H_p(G^2) \). Then

\[
\|S_{2^n, 2^n} f - f\|_p \to 0, \quad \text{as } n \to \infty.
\]

Moreover,

**Theorem S4.** Let \( p > 0 \) and \( f \in H_p(G^2) \). Then there exists an absolute constant \( c_p \) depending only on \( p \), such that

\[
\left\| \sup_{n \in \mathbb{N}} |S_{2^n, 2^n} f| \right\|_p \leq c_p \|f\|_{H_p(G^2)}, \quad (1.18)
\]

By applying Theorem S4 we can conclude that the following holds true (for details see e.g. [47] and [78]):

**Theorem S5.** Let \( p > 0 \) and \( f \in H_p(G^2) \). Then there exists an absolute constant \( c_p \) depending only on \( p \), such that

\[
\|S_{2^n, 2^n} f\|_{H_p(G^2)} \leq c_p \|f\|_{H_p(G^2)}, \quad (1.19)
\]

On the other hand, (see [61]) the following is true:
Theorem T18. Let $0 < p \leq 1$. Then there exists a martingale $f \in H_p(G^2)$ such that
\[ \sup_{n \in \mathbb{N}} \| S_{n,n} f \|_p = \infty. \]

However, for the two-dimensional case Weisz [77] proved the following:

Theorem We4. Let $\alpha \geq 0$ and $f \in H_p(G^2)$. Then there exists an absolute constant $c_p$ depending only on $p$, such that
\[ \sup_{n,m \geq 2} \left( \frac{1}{\log n \log m} \right)^{\lfloor p \rfloor} \sum_{2^{-\alpha} \leq k/l \leq 2^\alpha, (k,l) \leq (n,m)} \| S_{k,l} f \|^p_{H_p(G^2)} \leq c_p \| f \|^p_{H_p(G^2)}, \]

where $0 < p \leq 1$ and $\lfloor p \rfloor$ denotes integer part of real number $p$.

Moreover, sharpness of of the rates of weights are proved [72] was proved that rate of the $(k,l)^{2-p}$ is sharp.

Goginava and Gogoladze in [28] generalized this result in the case when $\alpha = 0$:

Theorem GG1. Let $f \in H_1(G^2)$. Then there exists an absolute constant $c$, such that
\[ \sum_{n=1}^{\infty} \frac{\| S_{n,n} f \|_1}{n \log^2 n} \leq c \| f \|_{H_1(G^2)}. \]

In [65] was proved that rate of the weights $\{ n \log^2 n \}_n^\infty$ is sharp. The following is true:

Theorem T19. Let $\Phi : \mathbb{N} \to [1, \infty)$ be non-decreasing function satisfying the condition $\lim_{n \to \infty} \Phi(n) = +\infty$. Then
\[ \sup_{\| f \|_{H_1(G^2)} \leq 1} \sum_{n=1}^{\infty} \frac{\| S_{n,n} f \|_1}{n \log^2 (n + 1)} \Phi(n) = \infty. \]

Theorem GG1 follows that if $f \in H_1(G^2)$, then
\[ \frac{1}{\log^{1/2} n} \sum_{k=1}^{n} \frac{\| S_{k,k} f \|_{H_1(G^2)}}{k \log^{3/2} k} \leq c \| f \|_{H_1(G^2)}, \]

Moreover,
\[ \lim_{n \to \infty} \frac{1}{\log^{1/2} n} \sum_{k=1}^{n} \frac{\| S_{k,k} f - f \|_{H_1(G^2)}^{2/3}}{k \log^{3/2} k} = 0. \]

It follows the following equality
\[ \lim_{n \to \infty} \frac{1}{\log^{1/2} n} \sum_{k=1}^{n} \frac{\| S_{k,k} f \|^2_{H_2/3(G^2)}}{k \log^{3/2} k} = \| f \|_{H_2/3(G^2)}. \]
In the fourth chapter (see also [66]) of this thesis we consider strong convergence of Marcinkiewicz means with respect to the two-dimensional partial sums of Walsh-Fourier series when $0 < p < 1$:

**Theorem 4.49** Let $0 < p < 1$ and $f \in H_p(G^2)$. Then there exists an absolute constant $c_p$ depending only on $p$, such that

$$
\sum_{n=1}^{\infty} \frac{\|S_{n,n}f\|_p^p}{n^{3-2p}} \leq c_p \|f\|_{H_p(G^2)}^p.
$$

Moreover, if $0 < p < 1$ and $\Phi : \mathbb{N} \to [1, \infty)$ be a non-decreasing function satisfying condition $\lim_{n \to \infty} \Phi(n) = +\infty$, then there exists a martingale $f \in H_p(G^2)$ such that

$$
\sum_{n=1}^{\infty} \frac{\|S_{n,n}f\|_{\text{weak}-L_p(G^2)}^p \Phi(n)}{n^{3-2p}} = \infty.
$$

Theorem 4.49 follows that, if $0 < p < 1$ and $f \in H_p(G^2)$, then there exists an absolute constant $c_p$ depending only on $p$, such that

$$
\frac{1}{n^{1/2-p}} \sum_{k=1}^{n} \frac{\|S_{k,k}f\|_{H_p(G^2)}^p}{k^{3/2-p}} \leq c_p \|f\|_{H_p(G^2)}^p.
$$

Moreover,

$$
\frac{1}{n^{1/2-p}} \sum_{k=1}^{n} \frac{\|S_{k,k}f - f\|_{H_p(G^2)}^p}{k^{3/2-p}} = 0.
$$

It follows the following equality

$$
\frac{1}{n^{1/2-p}} \sum_{k=1}^{n} \frac{\|S_{k,k}f\|_{H_p(G)}^p}{k^{3/2-p}} = \|f\|_{H_p(G)}^p.
$$

Weisz (for details see e.g. [78]) consider Marcinkiewicz means with respect to the two-dimensional partial sums of Walsh-Fourier series and proved the following:

**Theorem We5.** Let $p > 2/3$ and $f \in H_p(G^2)$. Then there exists an absolute constant $c_p$ depending only on $p$, such that

$$
\|M_n f - f\|_{H_p(G^2)} \to 0, \quad n \to \infty.
$$

Goginava [26] proved that the following is true:

**Theorem Gog2.** Let $0 < p \leq 2/3$. Then there exists a martingale $f \in H_p(G^2)$, such that

$$
\sup_{n \in \mathbb{N}} \|M_n f\|_{H_p(G^2)} = \infty.
$$
Goginava in [24] consider subsequence \( M_{2n} \) of Marcinkiewicz means with respect to the two-dimensional partial sums of Walsh-Fourier series and proved that the following is true:

**Theorem Gog3.** Let \( p > 1/2 \) and \( f \in H_p(G^2) \). Then

\[
\| M_{2n} f - f \|_{H_p(G^2)} \to 0, \quad \text{as} \quad k \to \infty. \tag{1.21}
\]

Moreover, there exists a martingale \( f \in H_p(G^2) \), \((0 < p \leq 1/2)\) such that

\[
\sup_{n \in \mathbb{N}} \| M_{2n} f \|_{H_p(G^2)} = \infty.
\]

In [37] was investigated strong convergence theorems of Marcinkiewicz means with respect to the two-dimensional partial sums of Walsh-Fourier series when \( 0 < p < 2/3 \):

**Theorem NT1.** Let \( 0 < p < 2/3 \) and \( f \in H_p(G^2) \). Then there exists an absolute constant \( c_p \) depending only on \( p \), such that

\[
\sum_{m=1}^{\infty} \frac{\| M_m f \|^p_{H_p(G^2)}}{m^{3-3p}} \leq c_p \| f \|^p_{H_p(G^2)},
\]

Moreover, if \( 0 < p < 2/3 \) and \( \Phi : \mathbb{N}_+ \to [1, \infty) \) be non-decreasing function satisfying the condition \( \Phi(n) \uparrow \infty \) and

\[
\lim_{k \to \infty} \frac{k^{3-3p}}{\Phi(k)} = \infty,
\]

then there exists a martingale \( f \in H_p(G^2) \), such that

\[
\sum_{m=1}^{\infty} \frac{\| M_m f \|^p_{\text{weak-}L_p(G^2)}}{\Phi(m)} = \infty.
\]

Theorem NT1 follows that if \( 0 < p < 2/3 \) and \( f \in H_p(G^2) \), then there exists an absolute constant \( c_p \) depending only on \( p \), such that

\[
\frac{1}{n^{1-3p/2}} \sum_{k=1}^{n} \frac{\| M_k f \|^p_{H_p(G^2)}}{k^{2-3p/2}} \leq c_p \| f \|^p_{H_p(G^2)},
\]

Moreover,

\[
\frac{1}{n^{1-3p/2}} \sum_{k=1}^{n} \frac{\| M_k f - f \|^p_{H_p(G^2)}}{k^{2-3p/2}} = 0,
\]

It follows the following equality:

\[
\frac{1}{n^{1-3p/2}} \sum_{k=1}^{n} \frac{\| M_k f \|^p_{H_p(G)}}{k^{2-3p/2}} = \| f \|^p_{H_p(G)}.
\]
In the fourth chapter (see [35]) we consider strong convergence results of Marcinkiewicz means with respect to the two-dimensional partial sums of Walsh-Fourier series, when \( p = 2/3 \):

**Theorem 4.50** Let \( f \in H_{2/3}(G^2) \). Then there exists an absolute constant \( c \), such that

\[
\frac{1}{\log n} \sum_{m=1}^{n} \| M_m f \|_{H_{2/3}(G^2)}^{2/3} \leq c \| f \|_{H_{2/3}(G^2)}^{2/3}.
\]

From these results we obtain that, if \( f \in H_{2/3}(G^2) \) then

\[
\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \| M_k f - f \|_{H_{2/3}(G^2)}^{2/3} = 0
\]

and

\[
\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \| M_k f \|_{H_{2/3}(G^2)}^{2/3} = \| f \|_{H_{2/3}(G^2)}^{2/3}.
\]

For the two-dimensional case Weisz [83] proved that the following is true:

**Theorem We6.** Let \( p > 2/3 \) and \( f \in H_p(G^2) \). Then the maximal operator of Marcinkiewicz means with respect to the two-dimensional partial sums of Walsh-Fourier series \( M^* f \) is bounded from \( H_p(G^2) \) to \( L_p(G^2) \):

\[
\| M^* f \|_p \leq c_p \| f \|_{H_p(G^2)},
\]

where \( c_p \) is an absolute constant, depending only on \( p \).

Goginava [25] also proved that the following is true:

**Theorem Gog4.** Let \( f \in H_{2/3}(G^2) \). Then there exists an absolute constant \( c \), such that

\[
\| M^* f \|_{weak-L_{2/3}(G^2)} \leq c \| f \|_{H_{2/3}}.
\]

Moreover, there exists a martingale \( f \in H_p(G^2) \), \( (0 < p \leq 2/3) \) such that

\[
\sup_{n \in \mathbb{N}} \| M_n f \|_p = \infty.
\]

Goginava [26] also consider restricted maximal operator of Marcinkiewicz means with respect to the two-dimensional partial sums of Walsh-Fourier series \( \sup_{n \in \mathbb{N}} | M_{2^n} f | \) and show that the following is true:

**Theorem Gog5.** Let \( p > 1/2 \) and \( f \in H_p(G^2) \). Then there exists an absolute constant \( c_p \) depending only on \( p \), such that

\[
\left\| \sup_{n \in \mathbb{N}} | M_{2^n} f | \right\|_p \leq c_p \| f \|_{H_p(G^2)},
\]

(1.22)
Moreover, there exists a martingale \( f \in H_p(G^2), \ (0 < p \leq 1/2) \) such that
\[
\sup_{n \in \mathbb{N}} \| M_n f \|_p = \infty.
\]

In [34] and [37] we investigate boundedness of weighted maximal operators when \( 0 < p \leq 2/3 \):

**Theorem NT2.** Let \( 0 < p \leq 2/3 \). Then the maximal operator
\[
\tilde{M}^* := \sup_{n \in \mathbb{N}} \frac{|M_n|}{(n + 1)^{2/p-3} \log^{3[1/3+p]/2} (n + 1)}
\]
is bounded from \( H_p(G^2) \) to \( L_p(G^2) \).

Moreover, if \( \varphi : \mathbb{N} \to [1, \infty) \) be non-decreasing function satisfying the condition
\[
\lim_{n \to \infty} \frac{n^{2/p-3} \log^{3[1/3+p]/2} (n)}{\varphi(n)} = +\infty,
\]
then
\[
\sup_{n \in \mathbb{N}} \left\| \frac{M_n f}{\varphi(n)} \right\|_p = \infty.
\]

From Theorem NT2 we get that for \( 0 < p \leq 1/2 \) and \( f \in H_p(G^2) \), there exists an absolute constant \( c_p \), depending only on \( p \), such that:
\[
\| M_n f \|_p \leq c_p (n + 1)^{2/p-3} \log^{3[1/3+p]/2} (n + 1) \| f \|_{H_p(G^2)} . 
\]

By applying inequality (1.23) in [37] (see also [38]) we obtain necessary and sufficient conditions for modulus of continuity of martingale \( f \in H_p(G^2) \), for which Marcinkiewicz means with respect to the two-dimensional partial sums of Walsh-Fourier series of \( f \in H_p(G^2) \) converge in \( H_p(G^2) \) norm.

**Theorem NT3.** Let \( 1/2 < p < 2/3 \), \( f \in H_p(G^2) \) and
\[
\omega_{H_p(G^2)} \left( \frac{1}{2k}, f \right) = o \left( \frac{1}{2k(2/p-3)} \right), \text{ as } k \to \infty.
\]

Then
\[
\| M_n f - f \|_{H_p(G^2)} \to 0, \text{ as } n \to \infty.
\]

Moreover, if \( 0 < p < 2/3 \), then there exists a martingale \( f \in H_p(G^2) \), such that
\[
\omega_{H_p(G^2)} \left( \frac{1}{2k}, f \right) = O \left( \frac{1}{2k(2/p-3)} \right), \text{ as } k \to \infty
\]
and
\[
\| M_n f - f \|_{\text{weak-}L_p(G^2)} \to 0, \text{ as } n \to \infty.
\]
Partial Sums and Marcinkiewicz and Fejér Means

If we apply again (1.23) and improve method which was investigated in [37] in the fourth chapter (see also [36]) we obtain that the following is true:

**Theorem 4.52.** Let \( f \in H_{2/3}(G^2) \) and

\[
\omega_{H_{2/3}(G^2)} \left( \frac{1}{2^k}, f \right) = o \left( \frac{1}{k^{3/2}} \right), \text{ as } k \to \infty.
\]

Then

\[
\| M_n f - f \|_{H_{2/3}(G^2)} \to 0, \text{ as } n \to \infty.
\]

On the other hand, there exists a martingale \( f \in H_{2/3}(G^2) \), such that

\[
\omega_{H_{2/3}(G^2)} \left( \frac{1}{2^k}, f \right) = O \left( \frac{1}{k^{3/2}} \right), \text{ as } k \to \infty
\]

and

\[
\| M_n f - f \|_{2/3} \nrightarrow 0, \text{ as } n \to \infty.
\]
2 Partial sums with respect to the one-dimensional Walsh-Fourier series on the martingale Hardy spaces

2.1 Basic notations

Denote by $\mathbb{N}_+$ the set of the positive integers and by $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$ the set of non-negative integers. Denote by $\mathbb{Z}_2$ the additive group of integers modulo-2, which contains only two elements $\mathbb{Z}_2 := \{0, 1\}$, group operation is modulo-2 sum and all sets are open.

Define the group $G$ as the complete direct product of the groups $\mathbb{Z}_2$ with the product of the discrete topologies $\mathbb{Z}_2$. The direct product $\mu$ of the measures $\mu_n \left( \{j\} \right) := 1/2$, $(j \in \mathbb{Z}_2)$ is the Haar measure on $G$ with $\mu \left( G \right) = 1$.

The elements of $G$ are represented by sequences $x := (x_0, x_1, ..., x_j, ...) \ (x_k = 0, 1)$.

It is easy to give a base for the neighbourhood of $G$

$$ I_0 (x) := G, $$

$$ I_n (x) := \{y \in G \mid y_0 = x_0, ..., y_{n-1} = x_{n-1}\} \ (x \in G, \ n \in \mathbb{N}). $$

Set $I_n := I_n (0)$ for any $n \in \mathbb{N}$ and $\overline{I}_n := G \setminus I_n$.

It is evident that

$$ \overline{I}_M = \left( \bigcup_{k=0}^{M-2} \bigcup_{l=k+1}^{M-1} I_{l+1} (e_k + e_l) \right) \bigcup \left( \bigcup_{k=0}^{M-1} I_M (e_k) \right) = \bigcup_{k=0}^{M-1} I_k \setminus I_{k+1}. \quad (2.1) $$

If $n \in \mathbb{N}$, then it can be uniquely expressed as

$$ n = \sum_{k=0}^{\infty} n_j 2^j $$

where $n_j \in \mathbb{Z}_2 \ (j \in \mathbb{N})$ and only a finite number of $n_j$s differ from zero.

Set

$$ \langle n \rangle := \min \{j \in \mathbb{N}, n_j \neq 0\} \ \text{and} \ |n| := \max \{j \in \mathbb{N}, n_j \neq 0\}, $$

It is evident that $2^{|n|} \leq n \leq 2^{|n|+1}$.

Let

$$ d \left( n \right) := |n| - \langle n \rangle , \ \text{for any} \ n \in \mathbb{N}. $$

Denote by $V (n)$ variation of natural number $n \in \mathbb{N}$

$$ V \left( n \right) = n_0 + \sum_{k=1}^{\infty} |n_k - n_{k-1}|. $$
Define $k$-th Rademacher functions by
\[ r_k(x) := (-1)^{x_k} \quad (x \in G, \ k \in \mathbb{N}). \]

By using Rademacher functions we define Walsh system $w := (w_n : n \in \mathbb{N}) G$ as:
\[ w_n(x) := \prod_{k=0}^{\infty} r_k^n(x) = r_{|n|}(-1) \sum_{k=0}^{n_k x_k} (n \in \mathbb{N}). \]

The norm (quasi-norm) of space $L_p(G)$, $(0 < p < \infty)$ is defined as
\[ \|f\|_p^p := \left( \int_G |f(x)|^p d\mu(x) \right). \]
and norm (quasi-norm) of space weak $- L_p(G)$ is defined by
\[ \|f\|_{\text{weak} - L_p(G)}^p := \sup_{\lambda > 0} \lambda^p \mu(x \in G : |f| > \lambda) < +\infty. \]

Walsh system is orthonormal and complete in $L_2(G)$ (see [47]).
For any $f \in L_1(G)$ the numbers
\[ \hat{f}(n) := \int_G f(x) w_n(x) d\mu(x) \]
are called $n$-th Walsh-Fourier coefficient of $f$.
$n$-th partial sum is denoted by
\[ S_n(f ; x) := \sum_{i=0}^{n-1} \hat{f}(i) w_i(x). \]
Dirichlet kernels are defined by
\[ D_n(x) := \sum_{i=0}^{n-1} w_i(x). \]
We also define the following maximal operators
\[ S^*f = \sup_{n \in \mathbb{N}} |S_n f| \]
\[ S^* f = \sup_{n \in \mathbb{N}} \sum_{i=0}^{2^n-1} |S_{2^n} f|. \]
The $\sigma$-algebra generated by the intervals $I_n(x)$ with measure $2^{-n}$ is denoted by $F_n$ ($n \in \mathbb{N}$). Conditional exponential operator with respect to $F_n$ ($n \in \mathbb{N}$) is denoted by $E_n$ and it is given by

$$
E_n f(x) = S_{2^n} f(x) = \sum_{k=0}^{2^n-1} \hat{f}(k) w_k(x) = \frac{1}{|I_n(x)|} \int_{I_n(x)} f(x) d\mu(x),
$$

where $|I_n(x)| = 2^{-n}$ denotes length of set $I_n(x)$.

Sequence $f = (f_n, n \in \mathbb{N})$ of functions $f_n \in L_1(G)$ is called dyadic martingale (for details see [39], [47]) if

(i) $f_n$ is measurable with respect to $\sigma$-algebras $F_n$ for any $n \in \mathbb{N}$,

(ii) $E_n f_m = f_n$ for any $n \leq m$.

The maximal function of a martingale $f$ is defined by

$$
\hat{f}^* = \sup_{n \in \mathbb{N}} |f_n|.
$$

In the case $f \in L_1(G)$, the maximal functions are also be given by:

$$
\hat{f}^*(x) = \sup_{n \in \mathbb{N}} \frac{1}{\mu(I_n(x))} \left| \int_{I_n(x)} f(u) d\mu(u) \right|.
$$

For $0 < p < \infty$ the Hardy martingale space $H_p(G)$ consists of all martingales, for which

$$
\|f\|_{H_p(G)} := \|\hat{f}^*\|_p < \infty.
$$

A bounded measurable function $a$ is said to be a $p$-atom if there exists an dyadic interval $I$, such that

$$
\int_I a d\mu = 0, \quad \|a\|_{\infty} \leq \mu(I)^{-1/p}, \quad \text{supp}(a) \subset I.
$$

It is easy to show that for martingale $f = (f_n, n \in \mathbb{N})$ and for any $k \in \mathbb{N}$ there exists a limit

$$
\hat{f}(k) := \lim_{n \to \infty} \int_G f_n(x) w_k(x) d\mu(x)
$$

and it is called $k$-th Walsh-Fourier coefficients of $f$.

If $f_0 \in L_1(G)$ and $f := (E_n f_0 : n \in \mathbb{N})$ is regular martingale then

$$
\hat{f}(k) = \int_G f(x) w_k(x) d\mu(x) = \hat{f}_0(k), \quad k \in \mathbb{N}.
$$
The modulus of continuity in $H_p(G)$ space is defined by

$$\omega_{H_p(G)} \left( \frac{1}{2^n}, f \right) := \| f - S_{2^n} f \|_{H_p(G)}.$$

It is important to describe how can be understood difference $f - S_{2^n} f$, where $f$ be martingale $S_{2^n} f$ is a function:

**Remark 2.1** Let $0 < p \leq 1$. Since

$$S_{2^n} f = f^{(n)} \in L_1(G), \text{ where } f = (f^{(n)} : n \in \mathbb{N}) \in H_p(G)$$

and

$$\left( S_{2^k} f^{(n)} : k \in \mathbb{N} \right) = (S_{2^k} S_{2^n}, k \in \mathbb{N})$$

$$= (S_{2^n} f, \ldots, S_{2^n-1} f, S_{2^n} f, S_{2^n} f, \ldots)$$

$$= (f^{(0)}, \ldots, f^{(n-1)}, f^{(n)}, f^{(n)}, \ldots).$$

*Under the difference $f - S_{2^n} f$ we mean the following martingale:*

$$f := ((f - S_{2^n} f)^{(k)}, k \in \mathbb{N})$$

*where*

$$(f - S_{2^n} f)^{(k)} = \begin{cases} 0, & k = 0, \ldots, n, \\ f^{(k)} - f^{(n)}, & k \geq n + 1, \end{cases}$$

Consequently, the norm $\| f - S_{2^n} f \|_{H_p(G)}$ is understood as $H_p$-norm of

$$f - S_{2^n} f = ((f - S_{2^n} f)^{(k)}, k \in \mathbb{N})$$

Watari [75] showed that there are strong connections between

$$\omega_p \left( \frac{1}{2^n}, f \right), \ E_{2^n} (L_p, f) \quad \text{and} \quad \| f - S_{2^n} f \|_p, \ p \geq 1, \ n \in \mathbb{N}.$$

In particular,

$$\frac{1}{2} \omega_p \left( \frac{1}{2^n}, f \right) \leq \| f - S_{2^n} f \|_p \leq \omega_p \left( \frac{1}{2^n}, f \right)$$

and

$$\frac{1}{2} \| f - S_{2^n} f \|_p \leq E_{2^n} (L_p, f) \leq \| f - S_{2^n} f \|_p.$$
2.2 AUXILIARY LEMMAS

First we present and prove equalities and estimations of Dirichlet kernel and Lebesgue constants with respect to the one-dimensional Walsh-Fourier systems (see Lemmas 2.2-3.26).

First equality of the following Lemma is proved in [47] and second identity is proved in [18]:

**Lemma 2.2** Let \( j, n \in \mathbb{N} \). Then

\[
D_{j+2^n} = D_{2^n} + w_{2^n} D_j, \text{ when } j \leq 2^n,
\]

and

\[
D_{2^n - j} = D_{2^n} - \psi_{2^n-1} D_j, \text{ when } j < 2^n.
\]

The following estimation of Dirichlet kernel with respect to the one-dimensional Walsh-Fourier systems is proved in [47]:

**Lemma 2.3** Let \( n \in \mathbb{N} \). Then

\[
D_{2^n} (x) = \begin{cases} 
2^n, & \text{if } x \in I_n, \\
0, & \text{if } x \notin I_n,
\end{cases}
\]

and

\[
D_n = w_n \sum_{k=0}^{\infty} n_k r_k D_{2^k} = w_n \sum_{k=0}^{\infty} n_k (D_{2^{k+1}} - D_{2^k}), \text{ for } n = \sum_{i=0}^{\infty} n_i 2^i.
\]

The following two-sided estimations of Lebesgue constants with respect to the one-dimensional Walsh-Fourier systems is proved in [47] and second equality is proved in [14]:

**Lemma 2.4** Let \( n \in \mathbb{N} \). Then

\[
\frac{1}{8} V(n) \leq \| D_n \|_1 \leq V(n)
\]

and

\[
\frac{1}{n \log n} \sum_{k=1}^{n} V(k) = \frac{1}{4 \log 2} + o(1).
\]

Hardy martingale space \( H^p (G) \) for any \( 0 < p \leq 1 \) can be characterized by simple functions which are called \( p \)-atoms. The following is true (for details see [50], [78] and [82]):

**Lemma 2.5** A martingale \( f = (f_n, n \in \mathbb{N}) \) belongs to \( H^p (G) \) \((0 < p \leq 1)\) if and only if there exists a sequence of \( p \)-atoms of \( (a_k, k \in \mathbb{N}) \) and sequence of real numbers \( (\mu_k, k \in \mathbb{N}) \) such that for all \( n \in \mathbb{N} \),
\[ \sum_{k=0}^{\infty} \mu_k S_{2^n} a_k = f_n \quad (2.2) \]

and

\[ \sum_{k=0}^{\infty} |\mu_k|^p < \infty. \]

Moreover,

\[ \|f\|_{H_p(G)} \sim \inf \left( \sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p}, \]

where the infimum is taken over all decomposition of \( f \) of the form (2.2).

The next five Examples of martingales will be used many times to prove sharpness of our main results. Such counterexamples first appear in the papers of Goginava [24] (see also [23]). Such constructions of martingales are also used in the papers [3], [4], [11], [32], [33], [36], [40], [41], [54], [59], [63], [64], [66], [67], [68], [72], [73], [71], [73]. For the one-dimensional case we use martingales which were used in [74]. So, we leave out the details of proof.

**Example 2.6** Let \( 0 < p \leq 1 \), \( \{\lambda_k : k \in \mathbb{N}\} \) be sequence of real numbers

\[ \sum_{k=0}^{\infty} |\lambda_k|^p \leq c_p < \infty \quad (2.3) \]

and \( \{a_k : k \in \mathbb{N}\} \) be sequence of \( p \)-atoms, given by

\[ a_k(x) := 2^{(1/p-1)|\alpha_k|} \left( D_{2^{\lfloor |\alpha_k|\rfloor + 1}}(x) - D_{2^{\lfloor |\alpha_k|\rfloor}}(x) \right), \]

where \( |\alpha_k| := \max \{ j \in \mathbb{N} : (\alpha_k)_j \neq 0 \} \) and \( (\alpha_k)_j \) denotes \( j \)-th binary coefficients of real number of \( \alpha_k \in \mathbb{N}_+ \). Then \( f = (f_n : n \in \mathbb{N}) \), where

\[ f_n(x) := \sum_{\{k : |\alpha_k| < n\}} \lambda_k a_k(x). \]

is martingale, which belongs to \( H_p(G) \) for any \( 0 < p \leq 1 \).

It is easy to show that

\[ \hat{f}(j) \]

\[ = \begin{cases} 
\lambda_k 2^{(1/p-1)|\alpha_k|}, & j \in \left\{ \sum_{k=1}^{\infty} 2^{|\alpha_k|}, \ldots, 2^{|\alpha_k|+1} - 1 \right\}, \ k \in \mathbb{N}_+, \\
0, & j \notin \bigcup_{k=1}^{\infty} \left\{ \sum_{k=1}^{\infty} 2^{|\alpha_k|}, \ldots, 2^{|\alpha_k|+1} - 1 \right\}. 
\end{cases} \]
Let $2^{|\alpha_{l-1}|+1} \leq j \leq 2^{|\alpha_l|}$, $l \in \mathbb{N}_+$. Then
\[
S_j f = S_{2^{|\alpha_{l-1}|+1}} = \sum_{\eta=0}^{l-1} \lambda_{\eta} 2^{(|\alpha_{\eta}|+1)(1/p-1)} (D_{2^{|\alpha_{\eta}|+1}} - D_{2^{|\alpha_{\eta}|}}).
\] (2.5)

Let $2^{|\alpha_l|} \leq j < 2^{|\alpha_l|+1}$, $l \in \mathbb{N}_+$. Then
\[
S_j f = S_{2^{|\alpha_l|}} + \lambda_l 2^{(1/p-1)|\alpha_l|} w_{2^{|\alpha_l|}} D_{j-2^{|\alpha_l|}}
\]
\[
= \sum_{\eta=0}^{l-1} \lambda_{\eta} 2^{(|\alpha_{\eta}|+1)(1/p-1)} (D_{2^{|\alpha_{\eta}|+1}} - D_{2^{|\alpha_{\eta}|}}) + \lambda_l 2^{(1/p-1)|\alpha_l|} w_{2^{|\alpha_l|}} D_{j-2^{|\alpha_l|}}.
\] (2.6)

Moreover, for the modulus of continuity for $0 < p \leq 1$ we have the following estimation:
\[
\omega_{H_p} \left( \frac{1}{2^n} f \right) = O \left( \sum_{\{k: |\alpha_k| \geq n\}} |\lambda_k|^p \right)^{1/p}, \text{ as } n \to \infty.
\] (2.7)

By applying Lemma 2.5 we easily obtain that the following is true (see [82]):

**Lemma 2.7** Let $0 < p \leq 1$ and $T$ be $\sigma$-sub-linear operator, such that, for any $p$-atom $a$,
\[
\int_G |T a(x)|^p \, d\mu(x) \leq c_p < \infty.
\]

Then
\[
\|T f\|_p \leq c_p \|f\|_{H_p(G)}. \tag{2.8}
\]

In addition, if $T$ is bounded from $L_\infty(G)$ to $L_\infty(G)$ then to prove (2.8) it is suffices to show that
\[
\int_I |T a(x)|^p \, d\mu(x) \leq c_p < \infty,
\]
for every $p$-atom $a$, where $I$ denotes support of the atom $a$.

In the concrete cases the norm of Hardy martingale spaces can be calculated by simpler formulas (for details see [50], [78] and [79]):
Lemma 2.8 If $g \in L^1(G)$ and $f := (E_n g : n \in \mathbb{N})$ be regular martingale, then $H_p(G)$ for $0 < p \leq 1$ norm can be calculated by

$$\|f\|_{H_p(G)} = \left\| \sup_{n \in \mathbb{N}} |S_{2^n} g| \right\|_p.$$ 

The following lemmas are proved in [59], [63], [64].

Lemma 2.9 Let $0 < p \leq 1$, $2^k \leq n < 2^{k+1}$ and $S_n f$ be $n$-th partial sum with respect to the one-dimensional Walsh-Fourier series, where $f \in H_p(G)$. Then for any fixed $n \in \mathbb{N}$,

$$\|S_n f\|_{H_p(G)}^p \leq \left\| \sup_{0 \leq l \leq k} |S_{2^l} f| \right\|_p^p + \|S_n f\|_p^p$$

Proof: Let consider the following martingales

$$f_\# := (S_{2^k} S_n f, k \in \mathbb{N}_+)$$

$= (S_{2^0}, S_{2^k} f, S_n f, \ldots, S_n f, \ldots)$. 

Hence, Lemma 2.8 immediately follows that

$$\|S_n f\|_{H_p(G)}^p \leq \left\| \sup_{0 \leq l \leq k} |S_{2^l} f| \right\|_p^p + \|S_n f\|_p^p$$

$$\leq \left\| S_\# f \right\|_p^p + \|S_n f\|_p^p.$$ 

Lemma is proved.

2.3 BOUNDEDNESS OF SUBSEQUENCES OF PARTIAL SUMS WITH RESPECT TO THE ONE-DIMENSIONAL WALSH-FOURIER SERIES ON THE MARTINGALE HARDY SPACES

In this section we consider boundedness of subsequences of partial sums with respect to the one-dimensional Walsh-Fourier series in the martingale Hardy spaces (for details see [63]).

Theorem 2.10 a) Let $0 < p < 1$ and $f \in H_p(G)$. Then there exists an absolute constant $c_p$ depending only on $p$, such that

$$\|S_n f\|_{H_p(G)} \leq c_p 2^{d(n)(1/p - 1)} \|f\|_{H_p(G)}.$$ 

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b) Let \(0 < p < 1\), \(\{m_k : k \in \mathbb{N}_+\}\) be non-negative, increasing sequence of natural numbers such that
\[
\sup_{k \in \mathbb{N}} d(m_k) = \infty
\]
(2.9)
and \(\Phi : \mathbb{N}_+ \to [1, \infty)\) be non-decreasing function satisfying the condition
\[
\lim_{k \to \infty} 2^{d(m_k)(1/p - 1)} \Phi(m_k) = \infty.
\]
(2.10)

Then there exists a martingale \(f \in H_p(G)\) such that
\[
\sup_{k \in \mathbb{N}} \left\| \frac{S_{m_k}f}{\Phi(m_k)} \right\|_{\text{weak}-L_p(G)} = \infty.
\]

**Proof:** Suppose that
\[
\left\| 2^{(1-1/p)d(n)} S_n f \right\|_p \leq c_p \left\| f \right\|_{H_p(G)}.
\]
(2.11)

By combining Lemma 2.9 and inequalities (1.7) and (2.11), since \(2^{(1-1/p)d(n)} \leq c_p\) we obtain that
\[
\left\| 2^{(1-1/p)d(n)} S_n f \right\|_{H_p(G)} \leq \left\| 2^{(1-1/p)d(n)} S_n f \right\|_p + \left\| 2^{(1-1/p)d(n)} \tilde{S}_n^* f \right\|_p
\]
\[
\leq c_p \left\| f \right\|_{H_p(G)} + c_p \left\| \tilde{S}_n^* f \right\|_p
\]
\[
\leq c_p \left\| f \right\|_{H_p(G)}.
\]
(2.12)

By combining Lemma 2.7 and (2.12) it is suficies to show that
\[
\int_G \left| 2^{(1-1/p)d(n)} S_n a \right|^p d\mu \leq c_p < \infty,
\]
(2.13)
for every \(p\)-atom \(a\), with support \(I\), such that \(\mu(I) = 2^{-M}\).

Without loss the generality we may assume that \(p\)-atom \(a\) has support \(I = I_M\). Then it is easy to see that \(S_n a = 0\), where \(2^M \geq n\). So, we may assume that \(2^M < n\). Since \(\|a\|_\infty \leq 2^{M/p}\) we can conclude that
\[
\left| 2^{(1-1/p)d(n)} S_n a (x) \right|
\]
\[
\leq 2^{(1-1/p)d(n)} \|a\|_\infty \int_{I_M} |D_n (x + t)| d\mu (t)
\]
\[
\leq 2^{M/p} 2^{(1-1/p)d(n)} \int_{I_M} |D_n (x + t)| d\mu (t).
\]
(2.14)
Let \( x \in I_M \). Since \( V(n) \leq 2d(n) \), by using first estimations of Lemma 2.4 we can conclude that

\[
|2^{(1-1/p)d(n)}S_n a| \\
\leq 2^{M/p} 2^{(1-1/p)d(n)} V(n) \\
\leq 2^{M/p} d(n) 2^{(1-1/p)d(n)}
\]

and

\[
\int_{I_M} |2^{(1-1/p)d(n)}S_n a|^p \, d\mu \\
\leq d(n) 2^{(1-1/p)d(n)} < c_p < \infty.
\]  

(2.15)

Let \( t \in I_M \) and \( x \in I_s \setminus I_{s+1} \), where \( 0 \leq s \leq M - 1 < \langle n \rangle \) or \( 0 \leq s < \langle n \rangle \leq M - 1 \). Then \( x + t \in I_s \setminus I_{s+1} \) and if we use both equality of Lemma 2.3 we get that \( D_n (x + t) = 0 \) and it follows that

\[
|2^{(1-1/p)d(n)}S_n a (x)| = 0.
\]

(2.16)

Let \( x \in I_s \setminus I_{s+1}, \langle n \rangle \leq s \leq M - 1 \). Then \( x + t \in I_s \setminus I_{s+1} \), where \( t \in I_M \). Then by using again both equality of Lemma 2.3 we have that

\[
|D_n (x + t)| \leq \sum_{j=0}^{s} n_j 2^j \leq c 2^s.
\]

If we apply again (2.14) we can conclude that

\[
|2^{(1-1/p)d(n)}S_n a (x)| \\
\leq 2^{(1-1/p)d(n)} 2^{M/p} 2^s \\
\leq 2^{\langle n \rangle (1/p-1)} 2^{M(1/p-1)} 2^s \\
\leq 2^{\langle n \rangle (1/p-1)} 2^s.
\]

(2.17)

By identity (2.1) and inequalities (2.16) and (2.17) we find that

\[
\int_{I_M} |2^{(1-1/p)d(n)}S_n a (x)|^p \, d\mu (x) \\
= \sum_{s=\langle n \rangle}^{M-1} \int_{I_s \setminus I_{s+1}} |2^{\langle n \rangle (1/p-1)} 2^s|^p \, d\mu (x) \\
\leq c \sum_{s=\langle n \rangle}^{M-1} 2^{\langle n \rangle (1-p)} \leq c_p < \infty.
\]
Partial Sums and Marcinkiewicz and Fejér Means

Now, we prove part b) of Theorem 2.10. By using condition (2.10) there exists sequence of natural numbers \( \{\alpha_k : k \in \mathbb{N}_+\} \subset \{m_k : k \in \mathbb{N}_+\} \), such that

\[
\sum_{\eta=0}^{\infty} \frac{\Phi^{p/2}(\alpha_\eta)}{2^{d(\alpha_\eta)(1-p)/2}} < \infty,
\]

(2.18)

Let \( f = (f_n, n \in \mathbb{N}_+) \in H_p(G) \) be a martingale from the Example 2.6, where

\[
\lambda_k = \frac{\Phi^{1/2}(\alpha_k)}{2^{d(\alpha_k)(1/p-1)/2}}.
\]

(2.19)

Then, if we use (2.18) we obtain that condition (2.3) is fulfilled and it follows that \( f = (f_n, n \in \mathbb{N}_+) \in H_p(G) \).

If we apply (2.4) when \( \lambda_k \) are given by the formula (2.19) then we get that

\[
\widehat{f}(j) = \begin{cases} 
\Phi^{1/2}(\alpha_k)2^{(|\alpha_k|+(\alpha_k))(1/p-1)/2}, & \text{if } j \in \{2^{\lfloor \alpha_k \rfloor}, ..., 2^{\lfloor \alpha_k \rfloor+1} - 1\}, \ k \in \mathbb{N}_+ \\
0, & \text{if } j \not\in \bigcup_{k=0}^{\infty} \{2^{\lfloor \alpha_k \rfloor}, ..., 2^{\lfloor \alpha_k \rfloor+1} - 1\}.
\end{cases}
\]

(2.20)

In the view of (2.6) when \( \lambda_k \) are given by (2.19) we get that

\[
\frac{S_{\alpha_k}f}{\Phi(\alpha_k)} = \frac{1}{\Phi(\alpha_k)} \sum_{\eta=0}^{k-1} \Phi^{1/2}(\alpha_\eta)2^{(|\alpha_\eta|+(\alpha_\eta))(1/p-1)/2} (D_{2^{\lfloor \alpha_\eta \rfloor+1}} - D_{2^{\lfloor \alpha_\eta \rfloor}}) + \frac{2^{(|\alpha_k|+(\alpha_k))(1/p-1)/2} w_{2^{\lfloor \alpha_k \rfloor}} D_{\alpha_k-2^{\lfloor \alpha_k \rfloor}}}{\Phi^{1/2}(\alpha_k)} =: I + II.
\]

(2.21)

by using (2.18) for I we have that

\[
\|I\|_{L^p(G)}^p \leq \frac{1}{\Phi(\alpha_k)} \sum_{\eta=0}^{k-1} \Phi^{p/2}(\alpha_\eta)2^{d(\alpha_\eta)(1/p-1)/2} \|2^{\lfloor \alpha_\eta \rfloor}(D_{2^{\lfloor \alpha_\eta \rfloor+1}} - D_{2^{\lfloor \alpha_\eta \rfloor}})\|_{L^p(G)}^p \leq \frac{1}{\Phi(\alpha_k)} \sum_{\eta=0}^{\infty} \Phi^{p/2}(\alpha_\eta)2^{d(\alpha_\eta)(1/p-1)/2} \leq c < \infty.
\]

(2.22)

Let \( x \in I_{\langle \alpha_k \rangle} \setminus I_{\langle \alpha_k \rangle+1} \). Since \( |\alpha_k| \neq \langle \alpha_k \rangle \) and \( \langle \alpha_k - 2^{\lfloor \alpha_k \rfloor} \rangle = \langle \alpha_k \rangle \).

Hardy spaces
By using both inequalities of Lemma 2.3 we get that

\[
|D_{α_k-2|α_k|}(x)| = \left| (D_{2(α_k)+1}(x) - D_{2|α_k|}(x)) + \sum_{j=|α_k|+1}^{2|α_k|} (α_k)_j (D_{2j+1}(x) - D_{2j}(x)) \right|
\]

and

\[
|II| = \frac{2^{((α_k)+(α_k))(1/p-1)/2} \mu \left\{ x \in G : |II| \geq \frac{2^{((α_k)+(α_k))(1/p-1)/2} 2^{(α_k)(1/p+1)/2}}{\Phi^{1/2}(α_k)} \right\}^{1/p}}{\Phi^{1/2}(α_k)} \to \infty, \text{ as } k \to \infty.
\]

The proof of Theorem 2.10 is complete.

\[\textbf{Corollary 2.11} \ a) \ Let \ n \in \mathbb{N}^+, \ 0 < p < 1 \text{ and } f \in H_p(G). \ Then \ there \ exists \ an \ absolute \ constant \ c_p, \ depending \ only \ on \ p \ such \ that \]

\[
\|S_n f\|_{H_p(G)} \leq c_p \left( n \mu \{\text{supp } (D_n)\} \right)^{1/p-1} \|f\|_{H_p(G)}.
\]

\[b) \ Let \ 0 < p < 1, \ \{m_k : k \in \mathbb{N}^+\} \ \text{be increasing sequence of natural numbers, such that} \]

\[
\sup_{k \in \mathbb{N}^+} m_k \mu \{\text{supp } (D_{m_k})\} = \infty \quad (2.25)
\]

\[\text{and } \Phi : \mathbb{N}^+ \to [1, \infty) \ \text{be non-decreasing function satisfying the condition} \]

\[
\lim_{k \to \infty} \left( \frac{m_k \mu \{\text{supp } (D_{m_k})\}^{1/p-1}}{\Phi(m_k)} \right)^{1/p-1} = \infty. \quad (2.26)
\]
Then there exists a martingale \( f \in H_p(G) \) such that

\[
\sup_{k \in \mathbb{N}} \left\| \frac{S_{m_k} f}{\Phi(m_k)} \right\|_{\text{weak-}L_p(G)} = \infty.
\]

**Proof:** By applying both inequalities of Lemma 2.3 we get that

\[
I_{(n)} \setminus I_{(n)+1} \subset \text{supp} \{D_n\} \subset I_{(n)} \quad \text{and} \quad 2^{-(n)-1} \leq \mu \{\text{supp} (D_n)\} \leq 2^{-(n)}.
\]

Hence,

\[
\frac{2^{d(n)(1/p-1)}}{4} \leq \left( n\mu \{\text{supp} (D_n)\} \right)^{1/p-1} \leq 2^{d(n)(1/p-1)}.
\]

Corollary 2.11 is proved. \( \square \)

**Theorem 2.12** a) Let \( n \in \mathbb{N}_+ \) and \( f \in H_1(G) \). Then there exists an absolute constant \( c \), such that

\[
\|S_n f\|_{H_1(G)} \leq cV(n) \|f\|_{H_1(G)}.
\]

b) Let \( \{m_k : k \in \mathbb{N}_+\} \) be non-negative increasing sequence of natural numbers such that

\[
\sup_{k \in \mathbb{N}} V(m_k) = \infty \quad (2.27)
\]

and \( \Phi : \mathbb{N}_+ \to [1, \infty) \) be non-decreasing function satisfying the condition

\[
\lim_{k \to \infty} \frac{V(m_k)}{\Phi(m_k)} = \infty. \quad (2.28)
\]

Then there exists a martingale \( f \in H_1(G) \), such that

\[
\sup_{k \in \mathbb{N}} \left\| \frac{S_{m_k} f}{\Phi(m_k)} \right\|_1 = \infty.
\]

**Proof:** Since

\[
\left\| \frac{S_n f}{V(n)} \right\|_1 \leq \|f\|_1 \leq \|f\|_{H_1(G)} \quad (2.29)
\]

by combining Lemmas 2.9 and (2.29) we can conclude that

\[
\left\| \frac{S_n f}{V(n)} \right\|_{H_1(G)} \leq \left\| \frac{S_n f}{V(n)} \right\|_1 + \frac{1}{V(n)} \left\| \tilde{S}_\# f \right\|_1 \leq c \|f\|_{H_1(G)} + c \left\| \tilde{S}_\# f \right\|_1 \leq c \|f\|_{H_1(G)}. \quad (2.30)
\]
Now prove second part of Theorem 2.12. Let \( \{m_k : k \in \mathbb{N}_+\} \) be increasing sequence of natural numbers and function \( \Phi : \mathbb{N}_+ \to [1, \infty) \) satisfies conditions (2.27) and (2.28). Then there exists non-negative, increasing sequence \( \{\alpha_k : k \in \mathbb{N}_+\} \subset \{m_k : k \in \mathbb{N}_+\} \) such that

\[
\sum_{k=1}^{\infty} \frac{\Phi^{1/2}(\alpha_k)}{V^{1/2}(\alpha_k)} \leq \beta < \infty. \tag{2.31}
\]

Let \( f = (f_n, n \in \mathbb{N}_+) \) be martingale from Example 2.6, where

\[
\lambda_k = \frac{\Phi^{1/2}(\alpha_k)}{V^{1/2}(\alpha_k)}. \tag{2.32}
\]

By applying condition (2.31) we can conclude that condition (2.3) is fulfilled and it follows that \( f = (f_n, n \in \mathbb{N}_+) \in H_1(G) \).

In the view of (2.4) when \( \lambda_k \) are given by (2.32) we get that

\[
\hat{f}(j) = \begin{cases} 
\frac{\Phi^{1/2}(\alpha_k)}{V^{1/2}(\alpha_k)} & \text{if } j \in \{2^{\lceil \alpha_k \rceil}, \ldots, 2^{\lfloor \alpha_k \rfloor + 1} - 1\}, \, k = 0, 1, \ldots \\
0 & \text{if } j \notin \bigcup_{k=0}^{\infty} \{2^{\lceil \alpha_k \rceil}, \ldots, 2^{\lfloor \alpha_k \rfloor + 1} - 1\}. 
\end{cases} \tag{2.33}
\]

Analogously to (2.21) if we apply (2.6) when \( \lambda_k \) are given by (2.32) we get that

\[
S_{\alpha_k}f = \sum_{\eta=0}^{k-1} \frac{\Phi^{1/2}(\alpha_\eta)}{V^{1/2}(\alpha_\eta)} \left( D_{2^{\lceil \alpha_\eta \rceil} + 1} - D_{2^{\lfloor \alpha_\eta \rfloor}} \right) + \frac{\Phi^{1/2}(\alpha_k)}{V^{1/2}(\alpha_k)} w_{2^{\lceil \alpha_k \rceil}} D_{\alpha_k - 2^{\lfloor \alpha_k \rfloor}}.
\]

By applying first estimation of Lemma 2.4 and (2.31) we can conclude that

\[
\frac{\|S_{\alpha_k}f\|_1}{\Phi^{1/2}(\alpha_k)} \geq \frac{\Phi^{1/2}(\alpha_k)}{V^{1/2}(\alpha_k)} \left| D_{\alpha_k - 2^{\lfloor \alpha_k \rfloor}} \right|_1 \\
- \frac{1}{\Phi(\alpha_k)} \sum_{\eta=0}^{k-1} \frac{\Phi^{1/2}(\alpha_\eta)}{V^{1/2}(\alpha_\eta)} \left| D_{2^{\lceil \alpha_\eta \rceil} + 1} - D_{2^{\lfloor \alpha_\eta \rfloor}} \right|_1 \\
+ \frac{V(\alpha_k - 2^{\lfloor \alpha_k \rfloor})}{8\Phi(\alpha_k)} \Phi^{1/2}(\alpha_k) \\
- \frac{1}{\Phi(\alpha_k)} \sum_{\eta=0}^{\infty} \frac{\Phi^{1/2}(\alpha_\eta)}{V^{1/2}(\alpha_\eta)} \\
\geq \frac{cV^{1/2}(\alpha_k)}{\Phi^{1/2}(\alpha_k)} \to \infty, \text{ as } k \to \infty.
\]

Theorem 2.12 is proved.
Corollary 2.13 Let $n \in \mathbb{N}$, $0 < p \leq 1$ and $f \in H_p(G)$. Then there exists an absolute constant $c_p$, depending only on $p$, such that

$$\|S_{2^n} f\|_{H_p(G)} \leq c_p \|f\|_{H_p(G)}.$$  \hfill (2.34)

**Proof:** To prove Theorem 2.13 we only have to show that

$$|2^n| = n, \quad \langle 2^n \rangle = n - 1 \quad d(2^n) = 0.$$  

By applying first part of Theorem 2.10 we immediately get that (2.34) for any $0 < p \leq 1$ and proof of Corollary 2.13 is proved. \hfill ■

Corollary 2.14 Let $n \in \mathbb{N}$, $0 < p \leq 1$ and $f \in H_p(G)$. Then there exists an absolute constant $c_p$, depending only on $p$ such that

$$\|S_{2^n + 2^{n-1}} f\|_{H_p(G)} \leq c_p \|f\|_{H_p(G)}.$$  \hfill (2.35)

**Proof:** Since

$$|2^n + 2^{n-1}| = n, \quad \langle 2^n + 2^{n-1} \rangle = n - 1 \quad d(2^n + 2^{n-1}) = 1$$

by first part of Theorem 2.10 we get that (2.35) holds, for any $0 < p \leq 1$ and proof of Corollary 2.14 is complete. \hfill ■

Corollary 2.15 Let $n \in \mathbb{N}$ and $0 < p < 1$. Then there exists a martingale $f \in H_p(G)$, such that

$$\sup_{n \in \mathbb{N}} \|S_{2^n + 1} f\|_{\text{weak-}L_p(G)} = \infty.$$  \hfill (2.36)

On the other hand, there exists an absolute constant $c$, such that

$$\|S_{2^n + 1} f\|_{H_1(G)} \leq c \|f\|_{H_1(G)}.$$  \hfill (2.37)

**Proof:** Since

$$|2^n + 1| = n, \quad \langle 2^n + 1 \rangle = 0 \quad d(2^n + 1) = n.$$  \hfill (2.38)

by applying second part of Theorem 2.10 we get that there exists a martingale $f = (f_n, n \in \mathbb{N}) \in H_p(G)$, for $0 < p < 1$, such that (2.36) holds.

On the other hand, proof of (2.37) follows simple observation that

$$V(2^n + 1) = 4 < \infty.$$  

Corollary 2.15 is proved. \hfill ■
2.4 Modulus of Continuity and Convergence in Norm of Subsequences of Partial Sums with Respect to the One-Dimensional Walsh-Fourier Series on the Martingale Hardy Spaces

In this section we apply Theorem 2.10 and Theorem 2.12 to find necessary and sufficient conditions for modulus of continuity, for which subsequences of partial sums with respect to the one-dimensional Walsh-Fourier series are bounded in the martingale Hardy spaces.

First, we prove the following estimation:

**Theorem 2.16** Let \( n \in \mathbb{N}_+ \) and \( 2^k < n \leq 2^{k+1} \). Then there exists an absolute constant \( c_p \), depending only on \( p \) such that

\[
\| S_n f - f \|_{H^p(G)} \leq c_p 2^{d(n)(1/p-1)} \omega_{H^p(G)} \left( \frac{1}{2^k}, f \right), \quad (f \in H^p(G)) \quad (0 < p < 1) \quad (2.39)
\]

and

\[
\| S_n f - f \|_{H^1(G)} \leq c_1 V(n) \omega_{H^1(G)} \left( \frac{1}{2^k}, f \right), \quad (f \in H^1(G)). \quad (2.40)
\]

**Proof:** Let \( 0 < p < 1 \) and \( 2^k < n \leq 2^{k+1} \). By applying first part of Theorem 2.10 we get that

\[
\| S_n f - f \|_{H^p(G)} \leq c_p 2^{d(n)(1/p-1)} \omega_{H^p(G)} \left( \frac{1}{2^k}, f \right) \quad (f \in H^p(G)) \quad (0 < p < 1) \quad (2.41)
\]

\[
\| S_n f - f \|_{H^1(G)} \leq c_1 V(n) \omega_{H^1(G)} \left( \frac{1}{2^k}, f \right) \quad (f \in H^1(G)). \quad (2.40)
\]

The proof of (2.40) is analogical to (2.39). Analogously to (2.39) we can also prove estimation (2.40). So, we leave out the details.

Theorem 2.16 is proved.

**Theorem 2.17** a) Let \( k \in \mathbb{N}_+, \) \( 0 < p < 1 \), \( f \in H^p(G) \) and \( \{m_k : k \in \mathbb{N}_+\} \) be increasing sequence of natural numbers, such that

\[
\omega_{H^p(G)} \left( \frac{1}{2|m_k|}, f \right) = o \left( \frac{1}{2^{d(m_k)(1/p-1)}} \right) \quad \text{as} \quad k \to \infty. \quad (2.42)
\]

Then

\[
\| S_{m_k} f - f \|_{H^p(G)} \to 0 \quad \text{as} \quad k \to \infty. \quad (2.43)
\]
b) Let \( \{m_k : k \in \mathbb{N}_+\} \) be increasing sequence of natural numbers, such that condition (2.9) is fulfilled. Then there exists a martingale \( f \in H_p(G) \) and increasing sequence of natural numbers \( \{\alpha_k : k \in \mathbb{N}_+\} \subset \{m_k : k \in \mathbb{N}_+\} \), such that

\[
\omega_{H_p(G)} \left( \frac{1}{2|\alpha_k|}, f \right) = O \left( \frac{1}{2^{d(\alpha_k)/(1/p-1)}} \right) \quad \text{as} \quad k \to \infty
\]

and

\[
\limsup_{k \to \infty} \|S_{\alpha_k} f - f\|_{\text{weak-}L_p(G)} > c_p > 0, \quad \text{as} \quad k \to \infty,
\]

(2.44)

where \( c_p \) is an absolute constant depending only on \( p \).

**Proof:** Let \( 0 < p < 1, f \in H_p(G) \) and \( \{m_k : k \in \mathbb{N}_+\} \) be increasing sequence of natural numbers, such that condition (2.42) is fulfilled. By combining Theorem 2.16 and estimation (2.39) we get that (2.43) holds true.

Now, prove second part of Theorem 2.17. In the view of (2.9) we simply get that there exists sequence \( \{\alpha_k : k \in \mathbb{N}_+\} \subset \{m_k : k \in \mathbb{N}_+\} \), such that

\[
2^{d(\alpha_k)} \uparrow \infty, \quad \text{as} \quad k \to \infty, \quad 2^{2(1/p-1)d(\alpha_k)} \leq 2^{(1/p-1)d(\alpha_{k+1})}.
\]

(2.45)

Let \( f = (f_n, n \in \mathbb{N}) \) be a martingale from Example 2.6, such that

\[
\lambda_i = 2^{-((1/p-1)d(\alpha_i))}.
\]

(2.46)

By applying (2.45) we obtain that condition (2.3) is fulfilled and it follows that \( f \in H_p(G) \).

By applying (2.4), where \( \lambda_k \) are given by (2.46), then

\[
\hat{f}(j) = \begin{cases} 2^{(1/p-1)d(\alpha_k)}, & \text{if } j \in \{2^{\alpha_k}, \ldots, 2^{\alpha_k+1} - 1\}, k \in \mathbb{N}_+, \\ 0, & \text{if } j \notin \bigcup_{k=0}^{\infty} \{2^{\alpha_k}, \ldots, 2^{\alpha_k+1} - 1\}. \end{cases}
\]

(2.47)

By combining (2.45) and (2.7) we have that

\[
\omega_{H_p(G)} \left( \frac{1}{2|\alpha_k|}, f \right) \leq \sum_{i=k}^{\infty} \frac{1}{2^{(1/p-1)d(\alpha_i)}} = O \left( \frac{1}{2^{(1/p-1)d(\alpha_k)}} \right), \quad \text{as} \quad k \to \infty.
\]

(2.48)

By using (2.23) we get that

\[
\left| D_{\alpha_k - 2^{(\alpha_k)}} \right| \geq 2^{(\alpha_k)}, \quad \text{where} \quad I_{(\alpha_k)} \setminus I_{(\alpha_k)+1}.
\]
In the view of (2.6) we can conclude that

\[ S_{\alpha_k} f = S_{2^{\alpha_k}} f + 2^{(1/p - 1)\alpha_k} w_{2^{\alpha_k}} D_{\alpha_k - 2^{\alpha_k}}, \]

Since

\[ \| D_{\alpha_k} \|_{\text{weak}-L_p(G)} \geq 2^{(\alpha_k)\mu \{ x \in I_{\alpha_k} \setminus I_{\alpha_k+1} : |D_{\alpha_k}| \geq 2^{(\alpha_k)} \}^{1/p} \]

\[ \geq 2^{(\alpha_k)(\mu \{ I_{\alpha_k} \setminus I_{\alpha_k+1} \})^{1/p}} \geq 2^{(\alpha_k)(1-1/p)}, \]

if we apply (1.2) (see also Theorem T2) we obtain that

\[ \| f - S_{\alpha_k} f \|^p_{\text{weak}-L_p(G)} \]

\[ \geq 2^{(1-p)(\alpha_k)} \| w_{2^{\alpha_k}} D_{\alpha_k - 2^{\alpha_k}} \|^p_{\text{weak}-L_p(G)} \]

\[ \geq c - o(1) > c > 0, \quad \text{as} \quad k \to \infty. \]

Proof of Theorem 2.17 is complete. ■

**Corollary 2.18** a) Let \( 0 < p < 1, f \in H_p(G) \) and \( \{m_k : k \in \mathbb{N}_+\} \) be increasing sequence of natural numbers, such that

\[ \omega_{H_p(G)} \left( \frac{1}{2^{m_k}}, f \right) = o \left( \frac{1}{(m_k \mu (\text{supp} D_{m_k})^{1/p - 1}} \right) \quad \text{as} \quad k \to \infty. \]

Then (2.43) holds.

b) Let \( \{m_k : k \in \mathbb{N}_+\} \) be increasing sequence of natural numbers, such that

\[ \sup_{k \in \mathbb{N}_+} m_k \mu (\text{supp} D_{m_k}) = \infty. \]

Then there exist a martingale \( f \in H_p(G) \) and sequence \( \{\alpha_k : k \in \mathbb{N}_+\} \subset \{m_k : k \in \mathbb{N}_+\} \), such that

\[ \omega_{H_p(G)} \left( \frac{1}{2^{\alpha_k}}, f \right) = O \left( \frac{1}{(\alpha_k \mu (\text{supp} D_{\alpha_k})^{1/p - 1}} \right) \quad \text{as} \quad k \to \infty \]

and (2.44) holds.

**Theorem 2.19** a) Let \( f \in H_1(G) \) and \( \{m_k : k \in \mathbb{N}_+\} \) be increasing sequence of natural numbers, such that

\[ \omega_{H_1(G)} \left( \frac{1}{2^{m_k}}, f \right) = o \left( \frac{1}{V(m_k)} \right) \quad \text{as} \quad k \to \infty. \]
Then
\[ \|S_{m_k} f - f\|_{H_1(G)} \to 0 \quad \text{as} \quad k \to \infty. \] (2.52)

b) Let \( \{m_k : k \in \mathbb{N}_+\} \) be increasing sequence of natural numbers, such that condition (2.27) is fulfilled. Then there exists a martingale \( f \in H_1(G) \) and increasing sequence of natural numbers \( \{\alpha_k : k \in \mathbb{N}_+\} \subset \{m_k : k \in \mathbb{N}_+\} \) such that
\[ \omega_{H_1(G)} \left( \frac{1}{2|\alpha_k|} f \right) = O \left( \frac{1}{V(\alpha_k)} \right) \quad \text{as} \quad k \to \infty \]
and
\[ \limsup_{k \to \infty} \|S_{\alpha_k} f - f\|_1 > c > 0 \quad \text{as} \quad k \to \infty, \] (2.53)
where \( c \) is an absolute constant.

**Proof:** Let \( f \in H_1(G) \) and \( \{m_k : k \in \mathbb{N}_+\} \) be increasing sequence of natural numbers, such that (2.51). By applying Theorem 2.16 we get that condition (2.52) is fulfilled.

Now, we prove second part of Theorem 2.19. By applying (2.27) we conclude that there exists sequence \( \{\alpha_k : k \in \mathbb{N}_+\} \subset \{m_k : k \in \mathbb{N}_+\} \), such that
\[ V(\alpha_k) \uparrow \infty, \quad \text{as} \quad k \to \infty \quad \text{and} \quad V^2(\alpha_k) \leq V(\alpha_{k+1}) \quad k \in \mathbb{N}_+. \] (2.54)

Let \( f = (f_n, n \in \mathbb{N}_+) \) be a martingale from the Example 2.6, where
\[ \lambda_k = \frac{1}{V(\alpha_k)}. \]

By applying (2.54) we conclude that (2.3) is fulfilled and we conclude that \( f = (f_n, n \in \mathbb{N}_+) \in H_1(G) \).

In the view of (2.4) we have that
\[ \hat{f}(j) = \begin{cases} \frac{1}{V(\alpha_k)}, & j \in \{2^{|\alpha_k|}, \ldots, 2^{|\alpha_k|+1} - 1\}, \quad k = 0, 1, \ldots \\ 0, & j \notin \bigcup_{k=0}^{\infty} \{2^{|\alpha_k|}, \ldots, 2^{|\alpha_k|+1} - 1\}. \end{cases} \] (2.55)

According to (2.7) we get that
\[ w_{H_1(G)}(1/2^n, f) \]
\[ = \|f - S_{2^n} f\|_{H_1(G)} \leq \sum_{i=n+1}^{\infty} \frac{1}{V(\alpha_i)} \]
\[ = O \left( \frac{1}{V(\alpha_n)} \right), \quad \text{as} \quad n \to \infty. \] (2.56)
By applying (2.6) we can conclude that
\[ S_{\alpha_k}f = S_{2^{\alpha_k}|f|} + \frac{w_{2^{\alpha_k}|D_{\alpha_k-2^{\alpha_k}}}}{V(\alpha_k)}, \]

If we use (1.2) and Theorem T2 we get that
\[
\|f - S_{\alpha_k}f\|_1 \\
\geq \|\frac{w_{2^{\alpha_k}|D_{\alpha_k-2^{\alpha_k}}}}{V(\alpha_k)}\|_1 - \|f - S_{2^{\alpha_k}|f|}\|_1 \\
\geq \frac{V(\alpha_k - 2^{\alpha_k})}{8V(\alpha_k)} - o(1) > c > 0, \text{ as } k \to \infty.
\]

The proof of Theorem 2.19 is proved.

Theorem 3.34 follows the following corollaries which are [61]:

**Corollary 2.20**

a) Let \(0 < p < 1, f \in H_p(G)\) and
\[ \omega_{H_p(G)} \left( \frac{1}{2^k}, f \right) = o \left( \frac{1}{2^{k(1/p-1)}} \right), \text{ as } k \to \infty. \]

Then
\[ \|S_kf - f\|_{H_p(G)} \to 0, \text{ as } k \to \infty. \]

b) There exists a martingale \(f \in H_p(G)\) \((0 < p < 1)\), such that
\[ \omega_{H_p(G)} \left( \frac{1}{2^k}, f \right) = O \left( \frac{1}{2^{k(1/p-1)}} \right), \text{ as } k \to \infty \]

and
\[ \|S_kf - f\|_{weak-L_p(G)} \to 0, \text{ as } k \to \infty. \]

**Corollary 2.21**

a) Let \(f \in H_1(G)\) and
\[ \omega_{H_1(G)} \left( \frac{1}{2^k}, f \right) = o \left( \frac{1}{k} \right), \text{ as } k \to \infty. \]

Then
\[ \|S_kf - f\|_{H_1(G)} \to 0, \text{ as } k \to \infty. \]

b) There exists a martingale \(f \in H_1(G)\), such that
\[ \omega_{H_1(G)} \left( \frac{1}{2^k}, f \right) = O \left( \frac{1}{k} \right), \text{ as } k \to \infty \]

and
\[ \|S_kf - f\|_1 \to 0, \text{ as } k \to \infty. \]
3 Fejér means with respect to the one-dimensional Walsh-Fourier series on the martingale Hardy spaces

3.1 Basic notations

For the one-dimensional case Fejér means with respect to the one-dimensional Walsh-Fourier series $\sigma_n$ is defined by:

$$\sigma_n f(x) = \frac{1}{n} \sum_{k=1}^{n} S_k f(x) \quad (n \in \mathbb{N}_+).$$

The following equality is true (for details see [2] and [47]):

$$\sigma_n f(x) = \frac{1}{n} \sum_{k=0}^{n-1} (D_k * f)(x)$$

$$= (f * K_n)(x) = \int_{G_m} f(t) K_n(x-t) d\mu(t).$$

where

$$K_n(x) = \frac{1}{n} \sum_{k=1}^{n} D_k(x) \quad (n \in \mathbb{N}_+).$$

In the literature $K_n$ is called $n$-th Fejér kernel.

We also define the following maximal operators

$$\sigma^* f = \sup_{n \in \mathbb{N}} |\sigma_n f|$$

$$\tilde{\sigma}^*_# f = \sup_{n \in \mathbb{N}} |\sigma_{2n} f|.$$

For any natural number $n \in \mathbb{N}$ we also need the following expression

$$n = \sum_{i=1}^{s} 2^{n_i}, \quad n_1 < n_2 < ... < n_s.$$  

Set

$$n^{(i)} := 2^{n_1} + ... + 2^{n_{i-1}}, \quad i = 2, ..., s$$

and

$$A_{0,2} := \left\{ n \in \mathbb{N} : n = 2^0 + 2^2 + \sum_{i=3}^{s} 2^{n_i} \right\}.$$
Then, for any natural number \( n \in \mathbb{N} \) there exists numbers
\[
0 \leq l_1 \leq m_1 \leq l_2 - 2 < l_2 \leq m_2 \leq ... \leq l_s - 2 < l_s \leq m_s
\]
such that it can be written as
\[
n = \sum_{i=1}^{s} \sum_{k=l_i}^{m_i} 2^k,
\]
where \( s \) is depending on \( n \).

It is evident that
\[
s \leq V (n) \leq 2s + 1.
\]

### 3.2 Auxiliary Lemmas

The following equality and estimation of Fejér kernels with respect to the one-dimensional Walsh-Fourier series is proved in [47]:

**Lemma 3.22** Let \( n \in \mathbb{N} \) and \( n = \sum_{i=1}^{s} 2^{n_i}, \) \( n_1 < n_2 < ... < n_s. \) Then
\[
nK_n = \sum_{r=1}^{s} \left( \prod_{j=r+1}^{s} w_{2^n j} \right) 2^{n_r} K_{2^{n_r}} + \sum_{t=2}^{s} \left( \prod_{j=t+1}^{s} w_{2^n j} \right) n^{(t)} D_{2^{n_t}},
\]

and
\[
\sup_{n \in \mathbb{N}} \int_{G} |K_n (x)| \, d\mu (x) \leq c < \infty,
\]
where \( c \) is an absolute constant.

The following equality is proved in [47] (see also [16]):

**Lemma 3.23** Let \( n > t \) and \( t, n \in \mathbb{N} \). Then we have the following expression for \( 2^n \)-th Fejér kernels with respect to the one-dimensional Walsh-Fourier series:
\[
K_{2^n} (x) = \begin{cases} 
2^{t-1}, & \text{if } x \in I_n (e_t), \\
\frac{2^{t+k}}{2}, & \text{if } x \in I_n, \\
0, & \text{otherwise}.
\end{cases}
\]

The following estimation is proved by Goginava [22]:

**Lemma 3.24** Let \( x \in I_{l+1} (e_k + e_l), \) \( k = 0, ..., M - 2, \) \( l = 0, ..., M - 1. \) Then
\[
\int_{I_M} |K_n (x + t)| \, d\mu (t) \leq \frac{c2^{l+k}}{n2^M}, \text{ where } n > 2^M.
\]

Let \( x \in I_M (e_k), m = 0, ..., M - 1. \) Then
\[
\int_{I_M} |K_n (x + t)| \, d\mu (t) \leq \frac{c2^k}{2^M}, \text{ for } n > 2^M,
\]
where \( c \) is an absolute constant.
The following estimations of Fejér kernels with respect to the one-dimensional Walsh-Fourier series is proved in [64]:

**Lemma 3.25** Let

\[ n = \sum_{i=1}^{r} \sum_{k=l_i}^{m_i} 2^k, \]

where

\[ m_1 \geq l_1 > l_1 - 2 \geq m_2 \geq l_2 > l_2 - 2 > ... > m_s \geq l_s \geq 0. \]

Then

\[ |nK_n| \leq c \sum_{A=1}^{r} \left( 2^{l_A} |K_{2^l_A}| + 2^{m_A} |K_{2^{m_A}}| + 2^{l_A} \sum_{k=l_A}^{m_A} D_{2^k} \right) + cV(n), \]

where \( c \) is an absolute constant.

**Proof:** Let

\[ n = \sum_{i=1}^{r} 2^{n_i}, n_1 > n_2 > ... > n_r \geq 0. \]

By using Lemma 3.22 for \( n \)-th Fejér kernels we can conclude that

\[ nK_n = \sum_{A=1}^{r} \left( \prod_{j=1}^{A-1} w_{2^{n_j}} \right) \left( (2^{n_A} K_{2^{n_A}} + (2^{n_A} - 1) D_{2^{n_A}}) \right) \]

\[ - \sum_{A=1}^{r} \left( \prod_{j=1}^{A-1} w_{2^{n_j}} \right) \left( 2^{n_A} - 1 - n^{(A)} \right) \right) D_{2^{n_A}} = I_1 - I_2. \]

For \( I_1 \) we have the following equality

\[ I_1 = \sum_{v=1}^{r} \left( \prod_{j=1}^{v-1} m_j \right) \left( \prod_{j=1}^{l_v} w_{2^{j}} \right) \left( \prod_{k=l_v}^{m_v} w_{2^{j}} \right) \left( 2^{k} K_{2^k} - (2^k - 1) D_{2^k} \right) \]

\[ = \sum_{v=1}^{r} \left( \prod_{j=1}^{v-1} m_j \right) \left( \prod_{k=0}^{l_v-1} \left( \prod_{j=k+1}^{m_v} w_{2^{j}} \right) \left( 2^{k} K_{2^k} - (2^k - 1) D_{2^k} \right) \right) \]

\[ = \sum_{v=1}^{r} \left( \prod_{j=1}^{v-1} m_j \right) \left( \prod_{j=1}^{l_v-1} \left( \prod_{j=k+1}^{m_v} w_{2^{j}} \right) \left( 2^{k} K_{2^k} - (2^k - 1) D_{2^k} \right) \right) \]

\[ - \sum_{v=1}^{r} \left( \prod_{j=1}^{v-1} m_j \right) \left( \prod_{j=1}^{l_v-1} \left( \prod_{j=k+1}^{m_v} w_{2^{j}} \right) \left( 2^{k} K_{2^k} - (2^k - 1) D_{2^k} \right) \right). \]
Since
\[ 2^n - 1 = \sum_{k=0}^{n-1} 2^k \]
and
\[ (2^n - 1) K_{2^n-1} = \sum_{k=0}^{n-1} \left( \prod_{j=k+1}^{n-1} w_{2j} \right) (2^k K_{2^k} - (2^k - 1) D_{2^k}) , \]
we obtain that
\[
I_1 = \sum_{v=1}^{r} \left( \prod_{j=1}^{v} \prod_{l_{j}} m_{j} \right) (2^{m_{v}} - 1) K_{2^{m_{v}+1}-1} - \sum_{v=1}^{r} \left( \prod_{j=1}^{v} \prod_{l_{j}} m_{j} \right) (2^{l_{v}} - 1) K_{2^{l_{v}-1}} .
\]

If we apply estimations
\[ |K_{2^n}| \leq c |K_{2^{n-1}}| \]
and
\[ |K_{2^{n-1}}| \leq c |K_{2^n}| + c \]
we get that
\[
|I_1| \leq c \sum_{v=1}^{r} \left( 2^{l_{v}} |K_{2^{l_{v}}}| + 2^{m_{v}} |K_{2^{m_{v}}}| + cr \right) . \tag{3.1}
\]

Let \( l_{j} < n_{A} \leq m_{j} \), where \( j = 1, \ldots, s \). Then
\[
n^{(A)} \geq \sum_{v=l_{j}}^{n_{A}-1} 2^v \geq 2^{n_{A}} - 2^{l_{j}}
\]
and
\[
2^{n_{A}} - 1 - n^{(A)} \leq 2^{l_{j}} .
\]

If \( l_{j} = n_{A} \) where \( j = 1, \ldots, s \), then
\[
n^{(A)} \leq 2^{m_{j-1}+1} < 2^{l_{j}} .
\]

By using these estimations we can conclude that
\[
|I_2| \leq c \sum_{v=1}^{r} 2^{l_{v}} \sum_{k=l_{v}}^{m_{v}} D_{2^k} . \tag{3.2}
\]

By combining (3.1)-(3.2) we get the proof of Lemma 3.25.

The following estimations of Fejér kernels with respect to the one-dimensional Walsh-Fourier series is proved in [64]:
Lemma 3.26 Let

\[ n = \sum_{i=1}^{s} \sum_{k=l_i}^{m_i} 2^k, \]

where

\[ 0 \leq l_1 \leq m_1 \leq l_2 - 2 < l_2 \leq m_2 \leq \ldots \leq l_s - 2 < l_s \leq m_s. \]

Then

\[ n |K_n (x)| \geq \frac{2^{2l_i}}{16}, \quad \text{for} \quad x \in I_{l_i+1} (e_{l_i-1} + e_{l_i}). \]

Proof: If we apply Lemma 3.22 for \( n = \sum_{i=1}^{s} \sum_{k=l_i}^{m_i} 2^k \) we can write that

\[
K_n = \sum_{r=1}^{s} \sum_{k=l_r}^{m_r} \left( \prod_{j=r+1}^{s} \prod_{q=l_j}^{m_j} w_{2q} \prod_{j=k+1}^{m_r} w_{2j} \right) 2^k K_{2^k}
\]

\[
+ \sum_{r=1}^{s} \sum_{k=l_r}^{m_r} \left( \prod_{j=r+1}^{s} \prod_{q=l_j}^{m_j} w_{2q} \prod_{j=k+1}^{m_r} w_{2j} \right) \left( \sum_{t=1}^{r-1} \sum_{q=l_t}^{m_t} 2^d + \sum_{q=l_r}^{r-1} 2^d \right) D_{2^k}.
\]

Let \( x \in I_{l_i+1} (e_{l_i-1} + e_{l_i}). \) Then

\[ n |K_n| \geq |2^{l_i} K_{2^{l_i}}| - \sum_{r=1}^{i-1} \sum_{k=l_r}^{m_r} |2^k K_{2^k}| - \sum_{r=1}^{i-1} \sum_{k=l_r}^{m_r} |2^k D_{2^k}| = I - II - III. \]

Lemma 3.23 follows that

\[ I = |2^{l_i} K_{2^{l_i}} (x)| = \frac{2^{2l_i}}{4}. \]  \( (3.3) \)

Since \( m_{i-1} \leq l_i - 2 \), we easily obtain that the following estimation is true:

\[ II \leq \sum_{n=0}^{l_i-2} |2^n K_{2^n} (x)| \]

\[ \leq \sum_{n=0}^{l_i-2} 2^n \frac{(2^n + 1)}{2} \]

\[ \leq \frac{2^{2l_i}}{24} + \frac{2^{l_i}}{4} - \frac{2}{3}. \]  \( (3.4) \)

For \( III \) we get that

\[ III \leq \sum_{k=0}^{l_i-2} |2^k D_{2^k} (x)| \leq \sum_{k=0}^{l_i-2} 4^k = \frac{2^{2l_i}}{12} - \frac{1}{3}. \]  \( (3.5) \)
By combining (3.3-3.5) we can conclude that
\[ n |K_n(x)| \geq I - II - III \geq \frac{2^{2l_i}}{8} - \frac{2^{l_i}}{4} + 1. \] (3.6)

Suppose that \( l_i \geq 2 \). Then
\[ n |K_n(x)| \geq \frac{2^{2l_i}}{8} - \frac{2^{2l_i}}{16} \geq \frac{2^{2l_i}}{16}. \]

If \( l_i = 0 \) or \( l_i = 1 \), then by applying (3.6) we get that
\[ n |K_n(x)| \geq \frac{7}{8} \geq \frac{2^{2l_i}}{16}, \]
Lemma is proved.

The following estimations of Fejér kernels with respect to the one-dimensional Walsh-Fourier series is proved in [64] (see also [74]):

**Lemma 3.27** Let \( 0 < p \leq 1 \), \( 2^k \leq n < 2^{k+1} \) and \( \sigma_n f \) be Fejér means with respect to the one-dimensional Walsh-Fourier series, where \( f \in H_p(G) \). Then, for any fixed \( n \in \mathbb{N} \),
\[
\| \sigma_n f \|_{H_p(G)} \leq \| \sup_{0 \leq l \leq k} |\sigma_{2l} f| \p \| + \| \sup_{0 \leq l \leq k} |S_{2l} f| \p \| + \| \sigma_n f \|_p \\
\leq \| \tilde{\sigma}^* f \p \| + \| \tilde{S}^* f \p \| + \| \sigma_n f \|_p .
\]

**Proof:** Let consider the following martingale
\[
f# = \left( S_{2^k} \sigma_n f, \ k \in \mathbb{N} \right) \left( \frac{2^0 \sigma_{2^0} f}{n}, \ldots, \frac{2^k \sigma_{2^k} f}{n}, \ldots, \frac{(n-2^0)S_{2^0} f}{n}, \ldots, \frac{(n-2^k)S_{2^k} f}{n}, \sigma_n f, \ldots, \sigma_n f, \ldots \right)
\]
By using Lemma 2.8 we immediately get
\[
\| \sigma_n f \|_{H_p(G^2)}^p \leq \| \sup_{0 \leq l \leq k} |\sigma_{2l} f| \p \| + \| \sup_{0 \leq l \leq k} |S_{2l} f| \p \| + \| S_n f \p \|_p \\
\leq \| \tilde{\sigma}^* f \p \|_p + \| \tilde{S}^* f \p \|_p + \| \sigma_n f \p \|_p .
\]
Lemma is proved.
3.3 **Boundedness of subsequences of Fejér means with respect to the one-dimensional Walsh-Fourier series on the martingale Hardy spaces**

In this section we study boundedness of subsequences of Fejér means with respect to the one-dimensional Walsh-Fourier series in the martingale Hardy spaces (For details see [64]).

First, we consider case $p = 1/2$. The following estimation is true:

**Theorem 3.28**

a) Let $f \in H_{1/2}(G)$. Then there exists an absolute constant $c$, such that

$$
\| \sigma_n f \|_{H_{1/2}(G)} \leq c V^2(n) \| f \|_{H_{1/2}(G)}.
$$

b) Let $\{n_k : k \in \mathbb{N}_+\}$ be increasing sequence of natural numbers, such that $\sup_{k \in \mathbb{N}_+} V(n_k) = \infty$ and $\Phi : \mathbb{N}_+ \to [1, \infty)$ be non-decreasing function satisfying the conditions $\Phi(n) \uparrow \infty$ and

$$
\lim_{k \to \infty} \frac{V^2(n_k)}{\Phi(n_k)} = \infty.
$$

(3.7)

Then there exists a martingale $f \in H_{1/2}(G)$, such that

$$
\sup_{k \in \mathbb{N}} \left\| \frac{\sigma_{n_k} f}{\Phi(n_k)} \right\|_{1/2} = \infty.
$$

**Proof:** Suppose that

$$
\left\| \frac{\sigma_n f}{V^2(n)} \right\|_{1/2} \leq c \| f \|_{H_{1/2}(G)}. \tag{3.8}
$$

By combining estimations (1.7), (1.15) and Lemma 3.27 we can conclude that

$$
\left\| \frac{\sigma_n f}{V^2(n)} \right\|_{H_{1/2}(G)} \leq \left\| \frac{\sigma_n f}{V^2(n)} \right\|_{1/2}^{1/2} + \frac{1}{V^2(n)} \left\| \sigma_n^* f \right\|_{1/2}^{1/2} + \frac{1}{V^2(n)} \left\| S_n^* f \right\|_{1/2}^{1/2} \tag{3.9}
$$

\[ \leq \left\| \frac{\sigma_n f}{V^2(n)} \right\|_{1/2}^{1/2} + \left\| \sigma_n^* f \right\|_{1/2}^{1/2} + \left\| S_n^* f \right\|_{1/2}^{1/2} \leq c \| f \|_{H_{1/2}(G)}^{1/2}. \]

By combining Lemma 2.7 and (3.9), Theorem 3.28 will be proved if we show that

$$
\int_{\mathcal{M}} \left( \frac{|\sigma_n a|}{V^2(n)} \right)^{1/2} d\mu \leq c < \infty,
$$

for any $1/2$-atom $a$. 

*Hardy spaces*
Without loss the generality we may assume that $a$ is $1/2$-atom, with support $I$, for which $\mu (I) = 2^{-M}$, $I = I_M$. It is easy to check that $\sigma_n (a) = 0$, when $n \leq 2^M$. Therefore, we may assume that $n > 2^M$. Set
\[ II_{\alpha A}^1 (x) := 2^M \int_{I_M} 2^{\alpha A} |K_{2^\alpha A} (x + t)| d\mu (t), \]
\[ II_{\alpha A}^2 (x) = 2^M \int_{I_M} 2^{\alpha A} \sum_{k=1}^{m_A} D_{2k} (x + t) d\mu (t). \]

Let $x \in I_M$. Since $\sigma_n$ is bounded from $L_\infty (G)$ to $L_\infty (G)$, for $n > 2^M$ and $\|a\|_\infty \leq 2^{2M}$, by using Lemma 3.24 we can conclude that
\[
\frac{|\sigma_n a (x)|}{V^2 (n)} \leq \frac{c}{V^2 (n)} \int_{I_M} |a (x)| |K_n (x + t)| d\mu (t)
\]
\[
\leq \frac{c \|a\|_\infty}{V^2 (n)} \int_{I_M} |K_n (x + t)| d\mu (t)
\]
\[
\leq \frac{c 2^M}{V^2 (n)} \int_{I_M} |K_n (x + t)| d\mu (t)
\]
\[
\leq \frac{c 2^M}{V^2 (n)} \left\{ \sum_{A=1}^{s} \int_{I_M} 2^{\alpha A} |K_{2^\alpha A} (x + t)| d\mu (t) + \int_{I_M} 2^{\alpha A} |K_{2^{\alpha A}} (x + t)| d\mu (t) \right\}
\]
\[
+ \frac{c 2^M}{V^2 (n)} \sum_{A=1}^{s} \int_{I_M} 2^{\alpha A} \sum_{k=1}^{m_A} D_{2k} (x + t) d\mu (t) + \frac{c 2^M}{V^2 (n)} \int_{I_M} V (n) d\mu (t)
\]
\[
= \frac{c}{V^2 (n)} \sum_{A=1}^{s} (II_{\alpha A}^1 (x) + II_{\alpha A}^2 (x) + II_{\alpha A}^2 (x)) + c.
\]

Hence,
\[
\frac{1}{|I_M|} \int_{I_M} \left| \frac{\sigma_n a (x)}{V^2 (n)} \right|^{1/2} d\mu (x)
\]
\[
\leq \frac{c}{V (n)} \left( \sum_{A=1}^{s} \int_{I_M} |II_{\alpha A}^1 (x)|^{1/2} d\mu (x) + \int_{I_M} |II_{\alpha A}^1 (x)|^{1/2} d\mu (x) + \int_{I_M} |II_{\alpha A}^2 (x)|^{1/2} d\mu (x) \right) + c.
\]

Since $s \leq 4V (n)$ we obtain that Theorem 3.28 will be proved if we show that
\[
\int_{I_M} |II_{\alpha A}^1 (x)|^{1/2} d\mu (x) \leq c < \infty, \quad \int_{I_M} |II_{\alpha A}^2 (x)|^{1/2} d\mu (x) \leq c < \infty, \quad (3.10)
\]
where \( \alpha_A = l_A \) or \( \alpha_A = m_A \), \( A = 1, \ldots, s \).

Let \( t \in I_M \) and \( x \in I_{l+1}(e_k+e_l), 0 \leq k < l < \alpha_A \leq M \) or \( 0 \leq k < l \leq M \leq \alpha_A \). Since \( x+t \in I_{l+1}(e_k+e_l) \), by applying Lemma 3.23 we can conclude that

\[
K_{2^\alpha_A}(x+t) = 0 \quad \text{and} \quad II_{\alpha_A}^1(x) = 0. \tag{3.11}
\]

Let \( x \in I_{l+1}(e_k+e_l), 0 \leq k < \alpha_A \leq l \leq M \). Then \( x+t \in I_{l+1}(e_k+e_l) \), where \( t \in I_M \) and if we apply again Lemma 3.23 we get that

\[
2^{\alpha_A} |K_{2^\alpha_A}(x+t)| \leq 2^{\alpha_A+k} \quad \text{and} \quad II_{\alpha_A}^1(x) \leq 2^{\alpha_A+k}. \tag{3.12}
\]

Analogously to (3.12) for \( 0 \leq \alpha_A \leq k < l \leq M \) we can prove that

\[
2^{\alpha_A} |K_{2^\alpha_A}(x+t)| \leq 2^{2\alpha_A}, \quad II_{\alpha_A}^1(x) \leq 2^{2\alpha_A}, \quad t \in I_M, \ x \in I_{l+1}(e_k+e_l). \tag{3.13}
\]

Let \( 0 \leq \alpha_A \leq M-1 \), where \( A = 1, \ldots, s \). According to (2.1) and (3.11-3.13) we find that

\[
\int_{I_M} |II_{\alpha_A}^1(x)|^{1/2} d\mu(x)
= \sum_{k=0}^{M-2} \sum_{l=k+1}^{M-1} \int_{I_{l+1}(e_k+e_l)} |II_{\alpha_A}^1(x)|^{1/2} d\mu(x)
+ \sum_{k=0}^{M-1} \int_{I_M(e_k)} |II_{\alpha_A}^1(x)|^{1/2} d\mu(x)
\leq c \sum_{k=0}^{\alpha_A-1} \sum_{l=\alpha_A+1}^{M-1} \int_{I_{l+1}(e_k+e_l)} 2^{(\alpha_A+k)/2} d\mu(x)
+ c \sum_{k=0}^{M-2} \sum_{l=\alpha_A+1}^{M-1} \int_{I_{l+1}(e_k+e_l)} 2^{\alpha_A} d\mu(x)
+ c \sum_{k=0}^{\alpha_A-1} \sum_{l=\alpha_A+1}^{M-1} \int_{I_M(e_k)} 2^{(\alpha_A+k)/2} d\mu(x)
+ c \sum_{k=0}^{\alpha_A-1} \sum_{l=\alpha_A+1}^{M-1} \int_{I_M(e_k)} 2^{\alpha_A} d\mu(x)
\leq c \sum_{k=0}^{\alpha_A-1} \sum_{l=\alpha_A+1}^{M-1} \frac{2^{(\alpha_A+k)/2}}{2^l} + c \sum_{k=\alpha_A}^{M-2} \sum_{l=\alpha_A+1}^{M-1} \frac{2^{\alpha_A}}{2^l}
+ c \sum_{k=0}^{\alpha_A-1} \frac{2^{(\alpha_A+k)/2}}{2M} + c \sum_{k=\alpha_A}^{M-2} \frac{2^{\alpha_A}}{2M} \leq c < \infty.
\]

Let \( \alpha_A \geq M \). Analogously to \( II_{\alpha_A}^1(x) \) we can prove (3.10), for \( A = 1, \ldots, s \).
Now, prove boundedness of $II^2_{l_A}$. Let $t \in I_M$ and $x \in I_i \setminus I_{i+1}$, $i \leq l_A - 1$. Since $x + t \in I_i \setminus I_{i+1}$, if we apply first equality of Lemma 2.3 we get that

$$II^2_{l_A}(x) = 0. \quad (3.14)$$

Let $x \in I_i \setminus I_{i+1}$, $l_A \leq i \leq m_A$. Since $n \geq 2^M$ and $t \in I_M$, if we apply first equality of Lemma 2.3 we get that

$$II^2_{l_A}(x) \leq 2^M \int_{I_M} 2^{l_A} \sum_{k=l_A}^i D_{2^k}(x+t) \, d\mu(t) \leq c2^{l_A+i}. \quad (3.15)$$

Let $x \in I_i \setminus I_{i+1}$, $m_A < i \leq M - 1$. Then $x + t \in I_i \setminus I_{i+1}$, for any $t \in I_M$ and by first equality of Lemma 2.3 we have that

$$II^2_{l_A}(x) \leq c2^M \int_{I_M} 2^{l_A+m_A} \leq c2^{l_A+m_A}. \quad (3.16)$$

Let $0 \leq l_A \leq m_A \leq M$. Then, in the view of (2.1) and (3.14-3.16) we can conclude that

$$\int_{I_M} |II^2_{l_A}(x)|^{1/2} \, d\mu(x) = \left( \sum_{i=0}^{l_A-1} + \sum_{i=l_A}^{m_A} \sum_{i=m_A+1}^{M-1} \right) \int_{I_i \setminus I_{i+1}} |II^2_{l_A}(x)|^{1/2} \, d\mu(x)
\leq c \sum_{i=l_A}^{m_A} \int_{I_i \setminus I_{i+1}} 2^{(l_A+i)/2} \, d\mu(x)
+ c \sum_{i=m_A+1}^{M-1} \int_{I_i \setminus I_{i+1}} 2^{(l_A+m_A)/2} \, d\mu(x)
\leq c \sum_{i=l_A}^{m_A} 2^{(l_A+i)/2} \frac{1}{2^i}
+ c \sum_{i=m_A+1}^{M-1} 2^{(l_A+m_A)/2} \frac{1}{2^i} \leq c < \infty.$$

Analogously, we can prove same estimations in the cases $0 \leq l_A \leq M < m_A$ and $M \leq l_A \leq m_A$.

Now, we prove part b) of Theorem 3.28. According to (3.7), there exists increasing sequence $\{\alpha_k : k \in \mathbb{N}_+\} \subset \{n_k : k \in \mathbb{N}_+\}$ of natural numbers such that

$$\sum_{k=1}^{\infty} \Phi^{1/4}(\alpha_k) V^{1/2}(\alpha_k) \leq c < \infty. \quad (3.17)$$
Let \( f = (f_n, n \in \mathbb{N}_+) \) be martingale form Example 2.6, where

\[
\lambda_k := \Phi_1^{1/2}(\alpha_k) / V(\alpha_k).
\]

According to (3.17) we get that condition (2.3) is fulfilled and it follows that \( f = (f_n, n \in \mathbb{N}_+) \).

By applying (2.4) we get that

\[
\hat{f}(j) = \begin{cases} 
2^{2|\alpha_k|} \Phi_1^{1/2}(\alpha_k) / V(\alpha_k), & j \in \{2^{|\alpha_k|}, \ldots, 2^{|\alpha_k|+1} - 1\}, \ k \in \mathbb{N}_+ \\
0, & j \notin \bigcup_{k=0}^{\infty} \{2^{|\alpha_k|}, \ldots, 2^{|\alpha_k|+1} - 1\}.
\end{cases}
\]

Let \( 2^{|\alpha_k|} < j < \alpha_k \). If we apply (2.6) we get that

\[
S_j f = S_{2^{|\alpha_k|}} f + \frac{w_{2^{|\alpha_k|}} D_{j-2^{|\alpha_k|}} \Phi_1^{1/2}(\alpha_k)}{V(\alpha_k)} \tag{3.19}
\]

Hence,

\[
\sigma_{\alpha_k} f \Phi(\alpha_k) = \frac{\sigma_{2^{|\alpha_k|}} f}{\Phi(\alpha_k)} \tag{3.20}
\]

\[
= \frac{1}{\Phi(\alpha_k)} \sum_{j=1}^{2^{|\alpha_k|}} S_j f + \frac{1}{\Phi(\alpha_k)} \sum_{j=2^{|\alpha_k|}+1}^{\alpha_k} S_j f
\]

\[
= \frac{\sigma_{2^{|\alpha_k|}} f}{\Phi(\alpha_k)} + \frac{(\alpha_k - 2^{|\alpha_k|}) S_{2^{|\alpha_k|}} f}{\Phi(\alpha_k)}
\]

\[
+ \frac{w_{2^{|\alpha_k|}} \phi_{2^{|\alpha_k|}} \Phi_1^{1/2}(\alpha_k)}{\Phi(\alpha_k) V(\alpha_k)} \sum_{j=2^{|\alpha_k|}+1}^{\alpha_k} D_{j-2^{|\alpha_k|}}
\]

\[
= \text{III}_1 + \text{III}_2 + \text{III}_3.
\]
For $III_3$ we can conclude that

$$|III_3| = \frac{2^{|\alpha_k|}\Phi^{1/2}(\alpha_k)}{\Phi(\alpha_k)V(\alpha_k)/\alpha_k} \sum_{j=1}^{\alpha_k-2^{|\alpha_k|}} D_j$$

$$= \frac{2^{|\alpha_k|}\Phi^{1/2}(\alpha_k)}{\Phi(\alpha_k)V(\alpha_k)/\alpha_k} (\alpha_k - 2^{|\alpha_k|}) \left| V_{\alpha_k-2^{|\alpha_k|}} \right|$$

$$\geq c (\alpha_k - 2^{|\alpha_k|}) \left| V_{\alpha_k-2^{|\alpha_k|}} \right| \Phi^{1/2}(\alpha_k)V(\alpha_k).$$

Let

$$\alpha_k = \sum_{i=1}^{r_k} \sum_{k=l_i^k}^{m_i^k} 2^k,$$

where

$$m_i^k \geq l_i^k > l_i^k - 2 \geq m_2^k \geq l_2^k - 2 > \ldots \geq m_s^k \geq l_s^k \geq 0.$$ 

Since (see theorems 2.10 and 3.28)

$$||III_1||_{1/2} \leq c, \ ||III_2||_{1/2} \leq c,$$

and

$$\mu \left\{ E_{l_i^k} \right\} \geq 1/2^{l_i^k - 1},$$

By combining (3.20), (3.21) and Lemma 3.26 we get that

$$\int_G |\sigma_{\alpha_k} f(x)/\Phi(\alpha_k)|^{1/2} d\mu(x) \geq ||III_3||^{1/2}_{1/2} - ||III_2||^{1/2}_{1/2} - ||III_1||^{1/2}_{1/2}$$

$$\geq c \sum_{i=2}^{r_k-2} \int_{E_{l_i^k}} \left| 2^{2l_i^k} / \left( \Phi^{1/2}(\alpha_k)V(\alpha_k) \right) \right|^{1/2} d\mu(x) - 2c$$

$$\geq c \sum_{i=2}^{r_k-2} 1/ \left( V^{1/2}(\alpha_k) \Phi^{1/4}(\alpha_k) \right) - 2c$$

$$\geq cr_k/ \left( V^{1/2}(\alpha_k) \Phi^{1/4}(\alpha_k) \right)$$

$$\geq cV^{1/2}(\alpha_k)/\Phi^{1/4}(\alpha_k) \rightarrow \infty, \text{ as } k \to \infty.$$ 

Theorem 3.28 is proved.
Theorem 3.29  a) Let $0 < p < 1/2$, $f \in H_p(G)$. Then there exists an absolute constant $c_p$ depending only on $p$ such that

$$
\|\sigma_n f\|_{H_p(G)} \leq c_p 2^{d(n)(1/p - 2)} \|f\|_{H_p(G)} .
$$

b) Let $0 < p < 1/2$ and $\Phi(n) : \mathbb{N}_+ \to [1, \infty)$ be non-decreasing function such that

$$
\sup_{k \in \mathbb{N}_+} d(n_k) = \infty, \quad \lim_{k \to \infty} 2^{d(n_k)(1/p - 2)} \Phi(n_k) = \infty.
$$

Then there exist a martingale $f \in H_p(G)$, such that

$$
\sup_{k \in \mathbb{N}_+} \left\| \frac{\sigma_{n_k} f}{\Phi(n_k)} \right\|_{weak-L_p(G)} = \infty.
$$

Proof: Let $n \in \mathbb{N}$. Analogously to (3.9) it is sufficient to prove that

$$
\int_{I_M} \left( 2^{d(n)(2-1/p)} |\sigma_n(a)| \right)^p d\mu \leq c_p < \infty,
$$

for every $p$-atom $a$, where $I$ denotes support of the atom.

Analogously to Theorem 3.28 we may assume that $a$ is $p$-atom with support $I = I_M$, $\mu(I_M) = 2^{-M}$ and $n > 2^M$. Since $\|a\|_\infty \leq 2^{M/p}$ we can conclude that

$$
2^{d(n)(2-1/p)} |\sigma_n a| \\
\leq 2^{d(n)(2-1/p)} \|a\|_\infty \int_{I_M} |K_n(x + t)| d\mu(t) \\
\leq 2^{d(n)(2-1/p)} 2^{M/p} \int_{I_M} |K_n(x + t)| d\mu(t) .
$$

Let $x \in I_{l+1}(e_k + e_l)$, $0 \leq k, l \leq [n] \leq M$. Then, by applying Lemma 3.23 we get that $K_n(x + t) = 0$, where $t \in I_M$ and hence,

$$
2^{d(n)(2-1/p)} |\sigma_n a| = 0. \quad (3.23)
$$

Let $x \in I_{l+1}(e_k + e_l)$, $[n] \leq k, l \leq M$ or $k \leq [n] \leq l \leq M$. Then Lemma 3.25 follows that

$$
2^{d(n)(2-1/p)} |\sigma_n a| \\
\leq 2^{d(n)(2-1/p)} 2^{M(1/p - 2) + k + l} \\
\leq c_p 2^{[n](1/p - 2) + k + l} . \quad (3.24)
$$
By combining (2.1), (3.23) and (3.24) we can conclude that

\[
\begin{align*}
\int_{I_M} \left| 2^{d(n)(2-1/p)} \sigma_n a (x) \right|^p \, d\mu (x) \\
\leq \left( \sum_{k=0}^{M-1} \sum_{l=k+1}^{[n]-1} + \sum_{k=0}^{[n]-1M-1} M-2 \sum_{l=k+1}^{M-1} \right) \int_{I_{l+1}(e_k+\varepsilon_1)} \left| 2^{d(n)(2-1/p)} \sigma_n a (x) \right|^p \, d\mu (x) \\
+ \sum_{k=0}^{M-1} \int_{I_M(e_k)} \left| 2^{d(n)(2-1/p)} \sigma_n a (x) \right|^p \, d\mu (x) \\
\leq c_p \sum_{k=0}^{M-1} \sum_{l=k+1}^{[n]} \frac{1}{2^{l}} 2^{[n](2p-1)} 2^{p(k+l)} \\
+ c_p \sum_{k=0}^{M-1} \sum_{l=[n]+1}^{M-1} \frac{1}{2^{l}} 2^{[n](2p-1)} 2^{p(k+l)} \\
+ c_p 2^{[n](2p-1)} \sum_{k=0}^{[n]} 2^{p(k+M)} < c_p < \infty.
\end{align*}
\]

Now, we prove part b) of Theorem 3.29. According to (3.22) there exists an increasing sequence of natural numbers \( \{ \alpha_k : k \in \mathbb{N}_+ \} \subset \{ n_k : k \in \mathbb{N}_+ \} \), such that \( \alpha_0 \geq 3 \) and

\[
\sum_{\eta=0}^{\infty} u^{-p} (\alpha_\eta) < c_p < \infty, \quad u (\alpha_k) = 2^{d(\alpha_k)(1/p-2)/2} / \Phi^{1/2} (\alpha_k).
\]  

(3.25)

Let \( f \) be martingale from Example 2.6, where

\[
\lambda_k = u^{-1} (\alpha_k),
\]

If we apply (3.25) we get that (2.3) is fulfilled and it follows that \( f \in H_p(G) \). According to (2.4) we have that

\[
\hat{f}(j) = \begin{cases} 2^{\alpha_k(1/p-1)/u (\alpha_k)}, & j \in \{ 2^{\alpha_k}, \ldots, 2^{\alpha_k+1} - 1 \}, \ k \in \mathbb{N}_+, \\ 0, & j \notin \bigcup_{k=0}^{\infty} \{ 2^{\alpha_k}, \ldots, 2^{\alpha_k+1} - 1 \}.
\end{cases}
\]

(3.26)

Let \( 2^{\alpha_k} < j < \alpha_k \). Then, analogously to (3.19) and (3.20), if we apply (3.26) we get...
that
\[
\frac{\sigma_{\alpha_k} f}{\Phi(\alpha_k)} = \frac{\sigma_{2^{[\alpha_k]}} f}{\Phi(\alpha_k) \alpha_k} + \frac{(\alpha_k - 2^{[\alpha_k]}) S_{2^{[\alpha_k]}} f}{\Phi(\alpha_k) \alpha_k} + \frac{2^{[\alpha_k] (1/p - 1)}}{\Phi(\alpha_k) \alpha_k} \sum_{j=2^{[\alpha_k]}}^{\alpha_k - 1} (D_j - D_{2^{[\alpha_k]}})
\]
\[= IV_1 + IV_2 + IV_3.\]

Let \( \alpha_k \in \mathbb{N} \) and \( E_{[\alpha_k]} := I_{[\alpha_k]+1} (e_{[\alpha_k]-1} + e_{[\alpha_k]}) \). Since \( [\alpha_k - 2^{[\alpha_k]}] = [\alpha_k] \), analogously to (3.21), if we apply Lemma 3.26 for \( IV_3 \) we have the following estimation
\[
|IV_3| = \frac{2^{[\alpha_k] (1/p - 1)}}{\Phi(\alpha_k) \alpha_k} \left| K_{\alpha_k - 2^{[\alpha_k]}} \right|
\]
\[= \frac{2^{[\alpha_k] (1/p - 1)}}{\Phi(\alpha_k) \alpha_k} \left| K_{[\alpha_k]} \right|
\]
\[\geq \frac{2^{[\alpha_k] (1/p - 2)/2^{[\alpha_k] - 4}}}{\Phi(\alpha_k) \alpha_k} \left| g_{[\alpha_k]} \right| \Phi^{1/2}(\alpha_k).
\]

Hence,
\[
\|IV_3\|^p_{weak-L_p(G)} \geq \left( \frac{2^{[\alpha_k] (1/p - 2)/2^{[\alpha_k] - 4)}}{\Phi^{1/2}(\alpha_k)} \right)^p \mu \left\{ x \in G : |IV_3| \geq \frac{2^{[\alpha_k] (1/p - 2)/2^{[\alpha_k] - 4}}}{\Phi^{1/2}(\alpha_k)} \right\}
\]
\[\geq c_p \left( \frac{2^{[\alpha_k] (1/p - 2)/\Phi^{1/2}(\alpha_k)} \right)^p \mu(E_{[\alpha_k]}).
\]
\[\geq c_p \left( \frac{2^{(\alpha_k - [\alpha_k]) (1/p - 2)/\Phi(\alpha_k)}}{\Phi(\alpha_k)} \right)^{p/2}
\]
\[= c_p \left( \frac{2^{d(\alpha_k) (1/p - 2)/\Phi(\alpha_k)}}{\Phi(\alpha_k)} \right)^{p/2} \rightarrow \infty, \text{ as } k \rightarrow \infty.
\]

By combining Corollary 2.13 and first part of Theorem 3.29 we find that
\[
\|IV_1\|_{weak-L_p(G)} \leq c_p < \infty, \quad \|IV_2\|_{weak-L_p(G)} \leq c_p < \infty.
\]

On the other hand, for sufficiently large \( n \) we can conclude that
\[
\|\sigma_{\alpha_k} f\|^p_{weak-L_p(G)} \geq \|IV_3\|^p_{weak-L_p(G)} - \|IV_2\|^p_{weak-L_p(G)} - \|IV_1\|^p_{weak-L_p(G)}
\]
\[\geq \frac{1}{2} \|IV_3\|^p_{weak-L_p(G)} \rightarrow \infty, \text{ as } k \rightarrow \infty.
\]
Theorem 3.29 is proved.

The proofs of Corollaries 3.30-3.32 are similar to the proofs of Corollaries 2.13-2.15. So, we leave out the details of proofs and just present these results:

**Corollary 3.30** Let $p > 0$ and $f \in H_p(G)$. Then

$$\|\sigma_{2k} f - f\|_{H_p(G)} \to 0, \text{ as } k \to \infty.$$  

**Corollary 3.31** Let $p > 0$ and $f \in H_p(G)$. Then

$$\|\sigma_{2k+2^k-1} f - f\|_{H_p(G)} \to 0, \text{ as } k \to \infty.$$  

**Corollary 3.32** Let $0 < p < 1/2$. Then there exists a martingale $f \in H_p(G)$, such that

$$\|\sigma_{2^{k+1}} f - f\|_{\text{weak--}L_p(G)} \nrightarrow 0, \text{ as } k \to \infty.$$  

On the other hand, for any $f \in H_{1/2}(G)$ the following is true:

$$\|\sigma_{2^{k+1}} f - f\|_{H_{1/2}(G)} \to 0, \text{ as } k \to \infty.$$  

3.4 Modulus of Continuity and Convergence in Norm of Subsequences of Fejér Means with Respect to the One-Dimensional Walsh-Fourier Series on the Martingale Hardy Spaces

In this section we apply Theorem 3.28 and Theorem 3.29 to find necessary and sufficient conditions for modulus of continuity of martingale $f \in H_p(G)$, for which subsequences of Fejér means converge in $H_p$-norm.

First, we prove the following result:

**Theorem 3.33**  

a) Let $f \in H_{1/2}(G)$, $\sup_{k \in \mathbb{N}_+} V(n_k) = \infty$ and

$$\omega_{H_p(G)} \left(1/2^{\lfloor n_k \rfloor}, f\right) = o \left(1/V^2(n_k)\right), \text{ as } k \to \infty.$$  

Then

$$\|\sigma_{n_k} f - f\|_{H_{1/2}(G)} \to 0, \text{ as } k \to \infty.$$  

b) Let $\sup_{k \in \mathbb{N}_+} V(n_k) = \infty$. Then there exists a martingale $f \in H_{1/2}(G)$, such that

$$\omega_{H_{1/2}(G)} \left(1/2^{\lfloor n_k \rfloor}, f\right) = O \left(1/V^2(n_k)\right), \text{ as } k \to \infty$$  

and

$$\|\sigma_{n_k} f - f\|_{H_{1/2}(G)} \nrightarrow 0, \text{ as } k \to \infty.$$  

---

*Hardy spaces*  

G. Tephnadze
**Proof:** Let \( f \in H_{1/2}(G) \) and \( 2^k < n \leq 2^{k+1} \). Then

\[
\| \sigma_n f - f \|_{H_{1/2}(G)}^{1/2} \\
\leq \| \sigma_n f - \sigma_n S_{2^k} f \|_{H_{1/2}(G)}^{1/2} \\
+ \| \sigma_n S_{2^k} f - S_{2^k} f \|_{H_{1/2}(G)}^{1/2} \\
+ \| S_{2^k} f - f \|_{H_{1/2}(G)}^{1/2} \\
= \| \sigma_n (S_{2^k} f - f) \|_{H_{1/2}(G)}^{1/2} \\
+ \| S_{2^k} f - f \|_{H_{1/2}(G)}^{1/2} \\
+ \| \sigma_n S_{2^k} f - S_{2^k} f \|_{H_{1/2}(G)}^{1/2} \\
\leq c (V(n) + 1) \omega_{H_{1/2}(G)}^{1/2} (1/2^k, f) \\
+ \| \sigma_n S_{2^k} f - S_{2^k} f \|_{H_{1/2}(G)}^{1/2}.
\]

It is evident that

\[
\sigma_n S_{2^k} f - S_{2^k} f \\
= \frac{2^k}{n} (S_{2^k} \sigma_{2^k} f - S_{2^k} f) \\
= \frac{2^k}{n} S_{2^k} (\sigma_{2^k} f - f).
\]

Let \( p > 0 \). By combining Corollaries 2.13 and 3.30 we can conclude that

\[
\| \sigma_n S_{2^k} f - S_{2^k} f \|_{H_{1/2}(G)}^{1/2} \\
\leq \frac{2^k}{n^{1/2}} \| S_{2^k} (\sigma_{2^k} f - f) \|_{H_{1/2}(G)}^{1/2} \\
\leq \| \sigma_{2^k} f - f \|_{H_{1/2}(G)}^{1/2} \to 0, \text{ as } k \to \infty.
\]

Now, we prove part b) of Theorem 3.33. Since \( \sup_{k \in \mathbb{N}_+} V(\alpha_k) = \infty \), then there exists a martingale \( \{ \alpha_k : k \in \mathbb{N}_+ \} \subset \{ n_k : k \in \mathbb{N}_+ \} \) such that \( V(\alpha_k) \uparrow \infty \), as \( k \to \infty \) and

\[
V^2(\alpha_k) \leq V(\alpha_{k+1}). \tag{3.30}
\]

Let \( f \) be martingale from Example 2.6, where

\[
\lambda_k = V^{-2}(\alpha_k),
\]

\[Hardy spaces\]
If we apply (3.30) we get that condition (2.3) is fulfilled and it follows that \( f \in H_p(G) \).

By using (2.4) we find that
\[
\hat{f}(j) = \begin{cases} 
2^{|\alpha_k|}/V^2(\alpha_k), & j \in \{2^{|\alpha_k|}, \ldots, 2^{|\alpha_k|+1} - 1\}, \ k \in \mathbb{N}_+ \\
0, & j \notin \bigcup_{k=0}^{\infty} \{2^{|\alpha_k|}, \ldots, 2^{|\alpha_k|+1} - 1\}.
\end{cases}
\] (3.31)

By combining (2.7) and (3.30) we can conclude that
\[
w_{H_{1/2}(G)}(1/2^n, f) = \| f - S_{2^n}f \|_{H_{1/2}(G)} \leq \sum_{i=n+1}^{\infty} 1/V^2(\alpha_i) = O \left( 1/V^2(\alpha_n) \right), \ \text{as} \ n \to \infty.
\] (3.32)

Let \( 2^{\alpha_k} < j \leq \alpha_k \). By using (2.6) we get that
\[
S_j f = S_{2^{\alpha_k}} f + \frac{2^{\alpha_k}|w_{2^{\alpha_k}} D_{j-2^{\alpha_k}}|}{V^2(\alpha_k)}.
\]

Hence,
\[
\sigma_{\alpha_k} f - f = \frac{2^{\alpha_k}}{\alpha_k} \left( \sigma_{2^{\alpha_k}} f - f \right) + \frac{\alpha_k - 2^{\alpha_k}}{\alpha_k} \left( S_{2^{\alpha_k}} f - f \right) + \frac{2^{\alpha_k}|w_{2^{\alpha_k}} (\alpha_{k} - 2^{\alpha_{k}}) K_{\alpha_{k} - 2^{\alpha_{k}}}}{\alpha_k V^2(\alpha_k)}.
\] (3.33)

According to (1.2), (1.12) and (3.33) we have that
\[
\| \sigma_{\alpha_k} f - f \|^{1/2} \geq \frac{c}{V(\alpha_k)} \| (\alpha_k - 2^{\alpha_k}) K_{\alpha_{k} - 2^{\alpha_{k}}} \|^{1/2} \left( \frac{2^{\alpha_k}}{\alpha_k} \right)^{1/2} \| \sigma_{2^{\alpha_k}} f - f \|^{1/2} \left( \frac{\alpha_k - 2^{\alpha_k}}{\alpha_k} \right)^{1/2} \| S_{2^{\alpha_k}} f - f \|^{1/2}.
\] (3.34)

Let
\[
\alpha_k = \sum_{i=1}^{r_k} \sum_{k=l_i}^{m_k} 2^k.
\]
where
\[ m_1^k \geq t_1^k > t_1^k - 2 \geq m_2^k \geq t_2^k - 2 > \ldots > m_s^k \geq t_s^k \geq 0 \]
and
\[ E_{i}^{k} := I_{t_{i-1}^k + 1} \left( e_{i-1}^k + e_{i}^k \right) . \]

By using Lemma 3.26 we get that
\[
\int_{G} \left| (\alpha_{k} - 2^{\lceil|\alpha_{k}|\rceil}) K_{\alpha_{k} - 2^{\lceil|\alpha_{k}|\rceil}}(x) \right|^{1/2} d\mu \geq \frac{1}{16} \sum_{i=2}^{r_{k} - 2} \int_{E_{i}^{k}} \left| (\alpha_{k} - 2^{\lceil|\alpha_{k}|\rceil}) K_{\alpha_{k} - 2^{\lceil|\alpha_{k}|\rceil}}(x) \right|^{1/2} d\mu(x) \geq \frac{1}{16} \sum_{i=2}^{r_{k} - 2} 2^{l_{i}^k} \geq c_{r_{k}} \geq c V (\alpha_{k}).
\] (3.35)

By combining estimations (3.34-3.35), Corollaries 2.13 and 3.30 we get that (3.29) holds true and Theorem 3.33 is proved.

**Theorem 3.34**  

a) Let \( 0 < p < 1/2 \), \( f \in H_{p}(G) \), \( \sup_{k \in \mathbb{N}^+} d(n_{k}) = \infty \) and
\[
\omega_{H_{p}(G)} \left( 1/2^{\lceil|n_{k}|\rceil}, f \right) = o \left( 1/2^{d(n_{k})(1/p - 2)} \right), \text{ as } k \to \infty.
\] (3.36)

Then
\[
\|\sigma_{n_{k}} f - f\|_{H_{p}(G)} \to 0, \text{ as } k \to \infty.
\] (3.37)

b) Let \( \sup_{k \in \mathbb{N}^+} d(n_{k}) = \infty. \) Then there exists a martingale \( f \in H_{p}(G) \) \( (0 < p < 1/2) \), such that
\[
\omega_{H_{p}(G)} \left( 1/2^{\lceil|n_{k}|\rceil}, f \right) = O \left( 1/2^{d(n_{k})(1/p - 2)} \right), \text{ as } k \to \infty
\] (3.38)
and
\[
\|\sigma_{n_{k}} f - f\|_{w_{p_{L_{p}(G)}}} \to 0, \text{ as } k \to \infty.
\] (3.39)

**Proof:**  
Let \( 0 < p < 1/2 \). Then under condition (3.36) if we repeat steps of the proof of Theorem 3.33, we immediately get that (3.37) holds.

Let prove part b) of Theorem 3.34. Since \( \sup_{k} d(n_{k}) = \infty \), there exists \( \{\alpha_{k} : k \in \mathbb{N}^+\} \subset \{n_{k} : k \in \mathbb{N}^+\} \) such that \( \sup_{k \in \mathbb{N}^+} d(\alpha_{k}) = \infty \) and
\[
2^{2^{d(n_{k})(1/p - 2)}} \leq 2^{d(n_{k+1})(1/p - 2)}. \] (3.40)

Let \( f \) be a martingale from Lemma 2.6, where
\[
\lambda_{k} = 2^{-(1/p - 2)d(n_{k})}.
\]
If we use (3.40) we conclude that condition (2.3) is fulfilled and it follows that \( f \in H_p(G) \).

According to (2.4) we get that
\[
\hat{f}(j) = \begin{cases} 
2^{(1/p - 2)|\alpha_k|}, & j \in \left\{ 2^{\alpha_k}, \ldots, 2^{\alpha_k + 1} - 1 \right\}, \ k \in \mathbb{N}_+ \\
0, & j \notin \bigcup_{n=0}^{\infty} \left\{ 2^{\alpha_n}, \ldots, 2^{\alpha_n + 1} - 1 \right\}.
\end{cases} \tag{3.41}
\]

By combining (2.7) and (3.40) we have that
\[
\omega_{H_p(G)}(1/2^{\alpha_k}, f) \leq \sum_{i=k}^{\infty} 1/2^{d(\alpha_i)(1/p-2)} = O \left( 1/2^{d(\alpha_k)(1/p-2)} \right) \text{ as } k \to \infty. \tag{3.42}
\]

Analogously to the proof of previous theorem, if we use also Corollaries 2.13 and 3.30, for the sufficiently large \( k \) we can conclude that
\[
\| \sigma_{\alpha_k} f - f \|_{\text{weak}-L_p(G)} \geq 2^{(1-2p)|\alpha_k|} \left\| (\alpha_k - 2^{\alpha_k}) K_{\alpha_k-2^{\alpha_k}} \right\|_{\text{weak}-L_p(G)}
\]

Let \( x \in E[\alpha_k] \). Lemma 3.26 follows that
\[
\mu \left( x \in G : (\alpha_k - 2^{\alpha_k}) \left| K_{\alpha_k-2^{\alpha_k}} \right| \geq 2^{2^{\alpha_k} - 4} \right) \geq 1/2^{\alpha_k} - 4
\]
and
\[
2^{2\alpha_k - 4} \mu \left( x \in G : (\alpha_k - 2^{\alpha_k}) \left| K_{\alpha_k-2^{\alpha_k}} \right| \geq 2^{2^{\alpha_k} - 4} \right) \geq 2^{(2p-1)\alpha_k - 4}. \tag{3.44}
\]

Hence, by combining (1.2), (1.12), (3.43) and (3.44) we get that
\[
\| \sigma_{\alpha_k} f - f \|_{\text{weak}-L_p(G)} \not\to 0, \text{ as } k \to \infty.
\]
The proof of Theorem 3.34 is complete. \( \blacksquare \)

By using Theorem 3.34 we easily get an important result which was proved in [60]:

\[
\text{Hardy spaces G. Tephnadze ,}
\]
Corollary 3.35  a) Let \( f \in H_{1/2}^1(G) \) and
\[
\omega_{H_{1/2}^1(G)} \left( 1/2^k, f \right) = o \left( \frac{1}{k^2} \right), \quad \text{as } k \to \infty.
\]
Then
\[
\| \sigma_k f - f \|_{H_{1/2}^1(G)} \to 0, \quad \text{as } k \to \infty.
\]

b) There exists a martingale \( f \in H_{1/2}^1(G) \), for which
\[
\omega_{H_{1/2}^1(G)} \left( 1/2^k, f \right) = O \left( \frac{1}{k^2} \right), \quad \text{as } k \to \infty
\]
and
\[
\| \sigma_k f - f \|_{1/2} \to 0, \quad \text{as } k \to \infty.
\]

Corollary 3.36  a) Let \( 0 < p < 1/2 \), \( f \in H_p^p(G) \) and
\[
\omega_{H_p^p(G)} \left( 1/2^k, f \right) = o \left( \frac{1}{2^{k(1/p-2)}} \right), \quad \text{as } k \to \infty.
\]
Then
\[
\| \sigma_k f - f \|_{H_p^p(G)} \to 0, \quad \text{as } k \to \infty.
\]

b) Then there exists a martingale \( f \in H_p^p(G) \) (\( 0 < p < 1/2 \)), for which
\[
\omega_{H_p^p(G)} \left( 1/2^k, f \right) = O \left( \frac{1}{2^{k(1/p-2)}} \right), \quad \text{as } k \to \infty
\]
and
\[
\| \sigma_k f - f \|_{\text{weak-}L_p^p(G)} \not\to 0, \quad \text{as } k \to \infty.
\]

3.5 Strong convergence of Fejér means with respect to the one-dimensional Walsh-Fourier series on the martingale Hardy spaces

In this section we consider strong convergence results of Fejér means with respect to the one-dimensional Walsh-Fourier series in the martingale Hardy spaces, when \( 0 < p \leq 1/2 \) (for details see [59]).

The following is true:

Theorem 3.37  a) Let \( 0 < p \leq 1/2 \) and \( f \in H_p^p(G) \). Then there exists a constant \( c_p \), depending only on \( p \), such that
\[
\frac{1}{\log^{1/2+p} n} \sum_{m=1}^{n} \left\| \sigma_m f \right\|_{H_p^p(G)}^p \leq c_p \left\| f \right\|_{H_p^p(G)}^p.
\]
b) Let $0 < p < 1/2$, $\Phi : \mathbb{N}_+ \to [1, \infty)$ be non-decreasing function, such that $\Phi (n) \to \infty$ and
\[
\lim_{k \to \infty} \frac{k^{2-2p}}{\Phi (k)} = \infty.
\]

Then there exists a martingale $f \in H_p (G)$, such that
\[
\sum_{m=1}^{\infty} \frac{\| \sigma_m f \|^p_{\text{weak-}L_p (G)}}{\Phi (m)} = \infty.
\]

**Proof:** Suppose that
\[
\frac{1}{\log^{1/2+p} n} \sum_{m=1}^{n} \frac{\| \sigma_m f \|^p_{H_p (G)}}{m^{2-2p}} \leq c_p \| f \|^p_{H_p (G)}.
\]

By combining (1.7), (1.15) and Lemma 3.27 we can conclude that
\[
\frac{1}{\log^{1/2+p} n} \sum_{m=1}^{n} \frac{\| \sigma_m f \|^p_{H_p (G)}}{m^{2-2p}} \leq \frac{1}{\log^{1/2+p} n} \sum_{m=1}^{n} \frac{\| \sigma_m f \|^p_{H_p (G)}}{m^{2-2p}} + \| \widetilde{\sigma}_n^* f \|_{H_p (G)} + \| \overline{\sigma}_n^* f \|_{H_p (G)}
\]
\[
\leq c_p \| f \|^p_{H_p (G)}.
\]

According to Lemma 2.7 and (3.45) Theorem 3.37 will be proved if we show that
\[
\frac{1}{\log^{1/2+p} n} \sum_{m=1}^{n} \frac{\| \sigma_m a \|^p_{H_p (G)}}{m^{2-2p}} \leq c < \infty, \quad m = 2, 3, ...
\]
for any $p$-atom $a$. We may assume that $a$ is $p$-atom, with support $I$, $\mu (I) = 2^{-M}$ and $I = I_M$. It is evident that $\sigma_n (a) = 0$, when $n \leq 2^M$. Therefore, we may assume that $n > 2^M$.

Let $x \in I_M$. Since $\sigma_n$ is bounded from $L_\infty (G)$ to $L_\infty (G)$ (The boundedness follows fact that Fejér kernels are uniformly bounded in the space $L_1 (G)$, which is proved in Lemma 3.22) and $\| a \|_\infty \leq 2^{M/p}$ we can conclude that
\[
\int_{I_M} |\sigma_m a (x)|^p d\mu (x) \leq \| \sigma_m a \|_{L_\infty}^p / 2^M
\]
\[
\leq \| a \|_{L_\infty}^p / 2^M \leq c < \infty, \quad 0 < p \leq 1/2.
\]

Let $0 < p \leq 1/2$. Then
\[
\frac{1}{\log^{1/2+p} n} \sum_{m=1}^{n} \frac{\int_{I_M} |\sigma_m a (x)|^p d\mu (x)}{m^{2-2p}}
\]
Partial Sums and Marcinkiewicz and Fejér Means

\[ \leq \frac{c}{\log^{1/2+p} n} \sum_{m=1}^{n} \frac{1}{m^{2-2p}} \leq c < \infty. \]

It is evident that

\[ |\sigma_m a(x)| \leq \int_{I_M} |a(t)||K_m(x+t)| d\mu(t) \]

\[ \leq 2^{M/p} \int_{I_M} |K_m(x+t)| d\mu(t). \]

Lemma 3.23 follows that

\[ |\sigma_m a(x)| \leq \frac{c 2^{k+l} 2^{M(1/p - 1)}}{m}, \quad x \in I_{l+1}(e_k + e_l), \ 0 \leq k < l < M \]  

and

\[ |\sigma_m a(x)| \leq c 2^{M(1/p - 1)} 2^k, \quad x \in I_M(e_k), \ 0 \leq k < M. \]

If we use identity (2.1) and (3.46-3.47) we get that

\[ \int_{I_M} |\sigma_m a(x)|^p d\mu(x) \]

\[ = \sum_{k=0}^{M-2} \sum_{l=k+1}^{M-1} \int_{I_{l+1}(e_k + e_l)} |\sigma_m a(x)|^p d\mu(x) \]

\[ + \sum_{k=0}^{M-1} \int_{I_M(e_k)} |\sigma_m a(x)|^p d\mu(x) \]

\[ \leq c \sum_{k=0}^{M-2} \sum_{l=k+1}^{M-1} \frac{1}{2^l} \frac{2^{p(k+l)} 2^{M(1-p)}}{m^p} + c \sum_{k=0}^{M-1} \frac{1}{2^M 2^{M(1-p)} 2^p} \]

\[ \leq \frac{c 2^{M(1-p)} M^{-2} 2^{M(1-p)}}{m^p} \sum_{k=0}^{M-2} \sum_{l=k+1}^{M-1} \frac{2^{p(k+l)}}{2^l} + c \sum_{k=0}^{M-1} \frac{2^p k}{2^p M} \]

\[ \leq \frac{c 2^{M(1-p)} M^{1/2+p}}{m^p} + c. \]

Hence,

\[ \frac{1}{\log^{1/2+p} n} \sum_{m=2M+1}^{n} \frac{\int_{I_M} |\sigma_m a(x)|^p d\mu(x)}{m^{2-2p}} \]

\[ \leq \frac{1}{\log^{1/2+p} n} \left( \sum_{m=2M+1}^{n} \frac{c 2^{M(1-p)} M^{1/2+p}}{m^{2-p}} + \sum_{m=2M+1}^{n} \frac{c}{m^{2-2p}} \right) < c < \infty. \]
The proof of part a) of theorem 3.37 is complete.

Now, we prove part b) of Theorem 3.37. Let \( \Phi(n) \) non-decreasing function satisfying the condition
\[
\lim_{k \to \infty} \frac{2((|n_k|+1)(2-2p))}{\Phi(2|n_k|+1)} = \infty. \tag{3.49}
\]

According to (3.49), there exists an increasing sequence \( \{\alpha_k : k \in \mathbb{N}_+\} \subset \{n_k : k \in \mathbb{N}_+\} \) such that
\[
|\alpha_k| \geq 2, \quad \text{where } k \in \mathbb{N}_+ \tag{3.50}
\]
and
\[
\sum_{\eta=0}^{\infty} \Phi^{1/2}(2|\alpha_\eta|+1) \frac{2(|\alpha_\eta|+1)(1-p)}{2(|\alpha_\eta|+1)(1-p)} = 2^{1-p} \sum_{\eta=0}^{\infty} \Phi^{1/2}(2|\alpha_\eta|+1) < c < \infty. \tag{3.51}
\]

Let \( f = (f_n, n \in \mathbb{N}_+) \in H_p(G) \) be a martingale from the Example 2.6, where
\[
\lambda_k = \frac{\Phi^{1/2p}(2|\alpha_k|+1)}{2(|\alpha_k|)(1/p-1)}
\]
By combining (2.3) and (3.51) we get that \( f \in H_p(G) \). According to (2.4) we have that
\[
\hat{f}(j) = \begin{cases} 
\Phi^{1/2}(2|\alpha_k|+1) & \text{if } j \in \{2|\alpha_k|, ..., 2|\alpha_k|+1-1\}, k \in \mathbb{N}_+, \\
0 & \text{if } j \notin \bigcup_{k=0}^{\infty} \{2|\alpha_k|, ..., 2|\alpha_k|+1-1\}.
\end{cases} \tag{3.52}
\]
Let \( 2^{|\alpha_k|} < n < 2^{|\alpha_k|+1} \). Then
\[
\sigma_n f = \frac{1}{n} \sum_{j=1}^{2^{|\alpha_k|}} S_j f + \frac{1}{n} \sum_{j=2^{|\alpha_k|}+1}^{n} S_j f = III + IV. \tag{3.53}
\]
It is evident that
\[
S_j f = 0, \quad \text{if } 0 \leq j \leq 2^{|\alpha_1|} \tag{3.54}
\]
Let \( 2^{|\alpha_s|} < j \leq 2^{|\alpha_s|+1} \), where \( s = 1, 2, ..., k \). If we apply (2.6) we get that
\[
S_j f = \sum_{\eta=0}^{s-1} \Phi^{1/2p}(2|\alpha_\eta|+1) \left(D_{2|\alpha_\eta|+1} - D_{2|\alpha_\eta|}ight) + \Phi^{1/2p}(2|\alpha_s|+1) w_{2|\alpha_s|} D_{j-2|\alpha_s|}. \tag{3.55}
\]
Let $2^{|\alpha_s|+1} \leq j \leq 2^{|\alpha_s+1|}$, $s = 0, 1, \ldots, k - 1$. Then if we use (2.5) we can conclude that

$$S_j f = \sum_{\eta=0}^{s} \Phi^{1/2p} \left(2^{|\alpha_\eta|+1}\right) \left(D_{2^{|\alpha_\eta|+1}} - D_{2^{|\alpha_\eta|}}\right).$$

(3.56)

Let $x \in I_2 (e_0 + e_1)$. Since (see Lemmas 2.3 and 3.23)

$$D_{2^n} (x) = K_{2^n} (x) = 0, \text{ where } n \geq 2$$

by combining (3.50) and (3.54-3.57) we get that

$$III = \frac{1}{n} \sum_{\eta=0}^{k-1} \Phi^{1/2p} \left(2^{|\alpha_\eta|+1}\right) \sum_{v=2^{|\alpha_\eta|+1}}^{2^{|\alpha_\eta|+1}} D_v (x)$$

(3.58)

$$= \frac{1}{n} \sum_{\eta=0}^{k-1} \Phi^{1/2p} \left(2^{|\alpha_\eta|+1}\right) \left(2^{|\alpha_\eta|+1} K_{2^{|\alpha_\eta|+1}} (x) - 2^{|\alpha_\eta|} K_{2^{|\alpha_\eta|}} (x)\right) = 0.$$

If we use (3.55) when $s = k$ for $IV$ we can write that

$$IV = \frac{n - 2^{|\alpha_k|}}{n} \sum_{\eta=0}^{k-1} \Phi^{1/2p} \left(2^{|\alpha_\eta|+1}\right) \left(D_{2^{|\alpha_\eta|+1}} - D_{2^{|\alpha_\eta|}}\right)$$

(3.59)

$$+ \frac{\Phi^{1/2p} \left(2^{|\alpha_k|+1}\right)}{n} \sum_{j=2^{|\alpha_k|+1}}^{n} w_{2^{|\alpha_k|}} D_{j-2^{|\alpha_k|}}$$

$$= IV_1 + IV_2.$$

By combining (3.50) and (3.57) we can conclude that

$$IV_1 = 0, \text{ where } x \in I_2 (e_0 + e_1).$$

(3.60)

Let $\alpha_k \in \mathbb{A}_{0,2}$, $2^{|\alpha_k|} < n < 2^{|\alpha_k|+1}$ and $x \in I_2 (e_0 + e_1)$. Since $n - 2^{|\alpha_k|} \in \mathbb{A}_{0,2}$, Lemmas 2.2 and 3.22 and (3.57) follows that

$$|IV_2| = \frac{\Phi^{1/2p} \left(2^{|\alpha_k|+1}\right)}{n} \sum_{j=1}^{n-2^{|\alpha_k|}} D_j (x)$$

$$= \frac{\Phi^{1/2p} \left(2^{|\alpha_k|+1}\right)}{n} \left(n - 2^{|\alpha_k|}\right) K_{n-2^{|\alpha_k|}} (x)$$

$$\geq \frac{\Phi^{1/2p} \left(2^{|\alpha_k|+1}\right)}{2^{|\alpha_k|+1}}.$$
Let $0 < p < 1/2$ and $n \in \mathbb{A}_{0,2}$. By combining (3.53-3.61) we get that

$$
\|\sigma_n f\|_{weak-L_p(G)}^p \geq \frac{c_p \Phi^{1/2}(2^{\alpha_k+1})}{2^{\rho(\alpha_k+1)}} \mu \left\{ x \in I_2 (e_0 + e_1) : |\sigma_n f| \geq \frac{c_p \Phi^{1/2p}(2^{\alpha_k+1})}{2^{\alpha_k+1}} \right\}
$$

$$
\geq \frac{c_p \Phi^{1/2}(2^{\alpha_k+1})}{2^{\rho(\alpha_k+1)}} \mu \left\{ I_2 (e_0 + e_1) \right\}
$$

$$
\geq \frac{c_p \Phi^{1/2}(2^{\alpha_k+1})}{2^{\rho(\alpha_k+1)}}.
$$

Hence,

$$
\sum_{n=1}^{\infty} \frac{\|\sigma_n f\|_{weak-L_p(G)}^p}{\Phi (n)} \geq \sum_{\{ n \in \mathbb{A}_{0,2} : 2^{\alpha_k} \leq n < 2^{\alpha_k+1} \}} \frac{\|\sigma_n f\|_{weak-L_p(G)}^p}{\Phi (n)} \geq \frac{1}{\Phi^{1/2}(2^{\alpha_k+1})} \sum_{\{ n \in \mathbb{A}_{0,2} : 2^{\alpha_k} \leq n < 2^{\alpha_k+1} \}} \frac{1}{2^{\rho(\alpha_k+1)}} \geq \frac{c_p 2^{1-p(\alpha_k+1)}}{\Phi^{1/2}(2^{\alpha_k+1})} \rightarrow \infty, \quad as \quad k \rightarrow \infty.
$$

The proof of Theorem 3.37 is complete. \(\blacksquare\)

**Theorem 3.38** Let $f \in H_{1/2}(G)$. Then

$$
\sup_{n \in \mathbb{N}^+} \sup_{\|f\|_{H_p} \leq 1} \frac{1}{n} \sum_{m=1}^{n} \|\sigma_m f\|_{1/2}^{1/2} = \infty.
$$

**Proof:** Let $0 < p \leq 1$ and

$$
f_k(x) := 2^k (D_{2k+1}(x) - D_{2k}(x))
$$

Since

$$
\text{supp}(f_k) = I_k, \quad \int_{I_k} a_k \, d\mu = 0
$$

and

$$
\|f_k\|_{\infty} \leq 2^{2k} = (\text{supp} f_k)^{-2},
$$

we conclude that $f_k$ is $1/2$-atom, for every $k \in \mathbb{N}$.
Moreover, if we use orthogonality of Walsh functions we get that

\[
S_{2^n} (f_k, x) = \begin{cases} 
0, & n = 0, \ldots, k, \\
(D_{2^{k+1}}(x) - D_{2^k}(x)), & n \geq k + 1,
\end{cases}
\]

and

\[
\sup_{n \in \mathbb{N}} |S_{2^n} (f_k, x)| = |(D_{2^{k+1}}(x) - D_{2^k}(x))|,
\]

where \(x \in G\).

By combining first equality of Lemma 2.2 and Lemma 2.8 we obtain that

\[
\|a_k\|_{H^p(G)} = 2^k \left\| \sup_{n \in \mathbb{N}} |S_{2^n} (D_{2^{k+1}}(x) - D_{2^k}(x))| \right\|_{1/2}
\]

\[
= 2^k \| (D_{2^{k+1}}(x) - D_{2^k}(x)) \|_{1/2}
\]

\[
= 2^k \| D_{2^k}(x) \|_{1/2}
\]

\[
\leq 2^k \cdot 2^{-k} \leq 1.
\]

It is easy to show that

\[
\hat{f}_m(i) = \begin{cases} 
2^m, & \text{if } i = 2^m, \ldots, 2^{m+1} - 1, \\
0, & \text{otherwise}
\end{cases}
\]

(3.62)

and

\[
S_{f_m} = \begin{cases} 
2^m (D_{i} - D_{2^m}), & \text{if } i = 2^m + 1, \ldots, 2^{m+1} - 1, \\
f_m, & \text{if } i \geq 2^{m+1}, \\
0, & \text{otherwise}.
\end{cases}
\]

(3.63)

Let \(0 < n < 2^m\). By using first equality of Lemma 2.2 we have that

\[
|\sigma_{n+2^m} f_m| \leq \frac{1}{n + 2^m} \left| \sum_{j=2^m+1}^{n+2^m} S_j f_m \right|
\]

(3.64)

\[
= \frac{1}{n + 2^m} \left| 2^m \sum_{j=2^m+1}^{n+2^m} (D_j - D_{2^m}) \right|
\]

\[
= \frac{1}{n + 2^m} \left| 2^m \sum_{j=1}^{n} (D_{j+2^m} - D_{2^m}) \right|
\]

\[
= \frac{1}{n + 2^m} \left| 2^m \sum_{j=1}^{n} D_j \right| = \frac{2^m}{n + 2^m} n |K_n|.
\]
Let
\[ n = \sum_{i=1}^{s} \sum_{k=l_i}^{m_i} 2^k, \]
where
\[ 0 \leq l_1 \leq m_1 \leq l_2 - 2 < m_2 \leq \ldots \leq l_s - 2 < m_s. \]
By applying Lemma 3.26 and (3.64) we find that
\[ |\sigma_{n+2^m} f_m(x)| \geq c2^{2l_i}, \quad \text{where} \quad x \in I_{l_i+1}(e_{l_i-1} + e_i). \]
Hence,
\[
\int_G |\sigma_{n+2^m} f_m(x)|^{1/2} d\mu(x) \\
\geq \sum_{i=0}^{s} \int_{I_{l_i+1}(e_{l_i-1} + e_i)} |\sigma_{n+2^m} f_m(x)|^{1/2} d\mu(x) \\
\geq c \sum_{i=0}^{s} \frac{1}{2^{l_i}} 2^{2l_i} \geq cs \geq cV(n).
\]
According to the second estimation of Lemma 2.4 we can conclude that
\[
\sup_{n \in \mathbb{N}_+} \sup_{\|f\|_{H^p} \leq 1} \frac{1}{n} \sum_{k=1}^{n} \|\sigma_k f\|_{1/2}^{1/2} \\
\geq \frac{1}{2^{m+1}} \sum_{k=2^m+1}^{2^{m+1}-1} \|\sigma_k f_m\|_{1/2}^{1/2} \\
\geq \frac{c}{2^{m+1}} \sum_{k=2^m+1}^{2^{m+1}-1} V(k - 2^m) \\
\geq \frac{c}{2^{m+1}} \sum_{k=1}^{2^m-1} V(k) \geq c \log m \to \infty, \quad \text{as} \quad m \to \infty.
\]
The proof is complete.
4 CONVERGENCE AND SUMMABILITY OF PARTIAL SUMS WITH RESPECT TO THE TWO-DIMENSIONAL WALSH-FOURIER SERIES ON THE MARTINGALE HARDY SPACES

4.1 BASIC NOTATIONS

Let denote by \( \overline{x} \) the two-dimensional vector \( \overline{x} := (x^1, x^2) \) and by \( G^2 \) the direct product of two Walsh groups. Let \( I_n := I_n (0) \times I_n (0) \) for any \( n \in \mathbb{N} \) and \( J_n := G^2 \setminus I_n \).

The norms (or quasi-norms) of the spaces of \( L^p(G^2) \) space is defined by

\[
\|f\|_p := \left( \int_{G^2} |f(\overline{x})|^p d\mu(\overline{x}) \right)^{1/p} \quad (0 < p < \infty).
\]

The space weak - \( L^p(G^2) \) consists of all functions \( f \) for which

\[
\|f\|_{\text{weak-}L^p(G^2)} := \sup_{\lambda > 0} \lambda \mu(\|f\| > \lambda)^{1/p} < +\infty.
\]

Two-dimensional Walsh system is defined by

\[
w_{n_1,n_2}(x^1, x^2) := w_{n_1}(x^1)w_{n_2}(x^2).
\]

The two-dimensional Walsh system is orthonormal and complete \( L^2(G^2) \) (see [47]).

For \( f \in L^1(G^2) \) the following number

\[
\hat{f}(n_1, n_2) := \int_{G^2} f(\overline{x})w_{n_1,n_2}(\overline{x}) d\mu(\overline{x})
\]

is called \((n_1, n_2)\)-th Fourier coefficients of function \( f \).

\((n_1, n_2)\)-th rectangular partial sum \( S_{n_1,n_2} \) of function \( f \) is defined by:

\[
S_{n_1,n_2}(f; \overline{x}) := \sum_{i_1=0}^{n_1-1} \sum_{i_2=0}^{n_2-1} \hat{f}(i_1, i_2) w_{i_1,i_2}(\overline{x}).
\]

\( n \)-th Marcinkiewicz (Marcinkiewicz-Fejér) means of the two-dimensional Walsh-Fourier series of function \( f \) is defined by

\[
\mathcal{M}_n(f; \overline{x}) := \frac{1}{n} \sum_{k=0}^{n} S_{k,...,k}(f; \overline{x}).
\]

Dirichlet and Marcinkiewicz kernels of the two-dimensional Walsh-Fourier series are defined by

\[
D_{n_1,n_2}(\overline{x}) = D_{n_1}(x^1) D_{n_2}(x^2).
\]
and

\[ K_n(\overrightarrow{x}) := \frac{1}{n} \sum_{k=0}^{n} D_{k,k}(\overrightarrow{x}). \]

For the partial sums of the two-dimensional Walsh-Fourier let as define

\[ S_{M}^{(1)} f(x^1, x^2) := \int_{G} f(s, x^2) D_{M}(x^1 + s) \, d\mu(s) \]

and

\[ S_{N}^{(2)} f(x^1, x^2) := \int_{G} f(x^1, t) D_{N}(x^2 + t) \, d\mu(t). \]

For the partial sums of the two-dimensional Walsh-Fourier let define the following maximal operators \( S_{\#}^* \) and \( \tilde{S}_{\#}^* \) by

\[ S_{\#}^*(f; x^1, x^2) := \sup_{n \in \mathbb{N}_+} |S_{n,n}(f; x^1, x^2)| \]

and

\[ \tilde{S}_{\#}^*(f; x^1, x^2) := \sup_{n \in \mathbb{N}} |S_{2^n,2^n}(f; x_1, x_2)|. \]

We define the maximal operator \( \mathcal{M}^* \) and restricted maximal operator \( \mathcal{\tilde{M}}_{\#}^* \) of Marcinkiewicz means by

\[ \mathcal{M}^*(f; x^1, x^2) := \sup_{n \in \mathbb{N}_+} |\mathcal{M}_{n}(f; x_1, x_2)| \]

and

\[ \mathcal{\tilde{M}}_{\#}^*(f; x^1, x^2) := \sup_{n \in \mathbb{N}} |\mathcal{M}_{2^n}(f; x_1, x_2)|. \]

For the partial sums of the two-dimensional Walsh-Fourier let define the following weighted maximal operators

\[ \mathcal{\tilde{M}}^*(f; x_1, x_2) := \sup_{n \in \mathbb{N}_+} \frac{|\mathcal{M}_{n}(f; x_1, x_2)|}{\log^{3/2}(n + 1)} \]

and

\[ \mathcal{\tilde{M}}_{\#}^{*,p}(f; x_1, x_2) = \sup_{n \in \mathbb{N}_+} \frac{|\mathcal{M}_{n}(f; x_1, x_2)|}{(n + 1)^{2/p-3}}. \]

The \( \sigma \)-algebra generated by the two-dimensional cubes

\[ I_n(\overrightarrow{x}) := I_n(x^1) \times I_n(x^2), \]

is defined by \( F_n(n \in \mathbb{N}) \).
The conditional expectation operator with respect to \( F_n (n \in \mathbb{N}) \) is denoted by \( E_n \) and in our concrete case we have the following explicit expression for it:

\[
E_n f (\vec{x}) = S_{2n,2n} f (\vec{x}) = \sum_{k_1=0}^{2^n-1} \sum_{k_2=0}^{2^n-1} \hat{f} (k_1, k_2) w_{k_1,k_2} (\vec{x}) = \frac{1}{|I_n^2 (\vec{x})|} \int_{I_n^2 (\vec{x})} f (\vec{x}) d\mu (\vec{x}),
\]

where \( |I_n^2 (\vec{x})| = 2^{-2n} \) denotes measure of cube \( I_n^2 (\vec{x}) \).

Sequence \( f = (f_n, n \in \mathbb{N}) \) of functions \( f_n \in L^1 (G^2) \) is called dyadic martingales if (for details see [47])

(i) \( f_n \) is measurable with respect to \( \sigma \)-algebra \( F_n \), for any \( n \in \mathbb{N} \),

(ii) \( E_n f_m = f_n \) for every \( n \leq m \).

The maximal function of martingale \( f \) is defined by

\[
f^* = \sup_{n \in \mathbb{N}} |f_n|.
\]

If \( f \in L^1 (G^2) \), then it is well-known that the maximal operator is defined by

\[
f^* (\vec{x}) = \sup_{n \in \mathbb{N}} \frac{1}{\mu (I_n^2 (\vec{x}))} \left| \int_{I_n^2 (\vec{x})} f (\vec{u}) d\mu (\vec{u}) \right|.
\]

For \( 0 < p < \infty \) the one-parameter martingale Hardy space \( H_p (G^2) \) consist of all martingales for which

\[
\| f \|_{H_p (G^2)} := \| f^* \|_p < \infty.
\]

Next, we define \( p \)-atoms, which are very important to characterize martingale Hardy spaces.

A function \( a \) is called a \( p \)-atom, if there exists an interval \( I^2 \), such that

\[
\int_{I^2} a d\mu = 0, \quad \|a\|_\infty \leq \mu (I^2)^{-1/p}, \quad \text{supp} \ (a) \subset I^2.
\]

It is easy to check that for every martingale \( f = (f_n, n \in \mathbb{N}) \) and for every \( (k_1, k_2) \in \mathbb{N}^2 \) the limit

\[
\hat{f} (k_1, k_2) := \lim_{n_1,n_2 \to \infty} \int_{G^2} f_{n_1,n_2} (\vec{x}) w_{k_1,k_2} (\vec{x}) d\mu (\vec{x})
\]

exists and it is called \((k_1, k_2)\)-th Walsh-Fourier coefficients of \( f \).
If \( f \in L_1(G^2) \) and \((E_nf : n \in \mathbb{N})\) is regular martingale, then
\[
\hat{f}(k_1, k_2) = \int_{G^2} f(\overrightarrow{x}) w_{k_1, k_2}(\overrightarrow{x}) d\mu(\overrightarrow{x}) = \hat{f}(k_1, k_2), \quad k_1, k_2 \in \mathbb{N}.
\]

For the two-dimensional case modulus of continuity in \( H_p(G^2) \) spaces can be defined as
\[
\omega_{H_p(G^2)} \left( \frac{1}{2^n}, f \right) := \| f - S_{2^n,2^n}f \|_{H_p(G^2)}.
\]

It is necessary to describe how can be understood difference \( f - S_{2^n,2^n}f \) where \( f \) is martingale and \( S_{2^n,2^n}f \) is function. The following is true:

**Remark 4.39** Let \( 0 < p \leq 1 \). Since
\[
S_{2^n,2^n}f = f^{(n)} \in L_1(G^2), \quad \text{where} \quad f = (f^{(n)} : n \in \mathbb{N}) \in H_p(G^2)
\]
and
\[
(S_{2^n,2^n}f^{(n)} : k \in \mathbb{N}) = (S_{2^n,2^n}f^{(0)}, \ldots, S_{2^n,2^n}f^{(n-1)}, S_{2^n,2^n}f^{(n)}, S_{2^n,2^n}f, \ldots)
\]
we obtain that \( f - S_{2^n,2^n}f \) is martingale for which
\[
(f - S_{2^n,2^n}f)^{(k)} = \begin{cases} 
0, & k = 0, \ldots, n, \\
\hat{f}^{(k)} - f^{(n)}, & k \geq n + 1,
\end{cases}
\]
and norm
\[
\| f - S_{2^n,2^n}f \|_{H_p(G^2)}
\]
can be understood as \( H_p(G^2) \) norm of martingale
\[
f - S_{2^n,2^n}f = ((f - S_{2^n,2^n}f)^{(k)} : k \in \mathbb{N})
\]

### 4.2 Auxiliary Lemmas

In the following lemmas we investigate estimations of Marcinkiewicz means of the two-dimensional Walsh-Fourier series (see Lemma 4.40-Lemma 4.43).

Glukhov [19] proved that the following is true:

**Lemma 4.40** There exists an absolute constant \( c \), such that
\[
\sup_{n \in \mathbb{N}} \int_{G^2} |K_n(x^1, x^2)| d\mu(x^1, x^2) \leq c.
\]
The following lemma is proved in [25]:

**Lemma 4.41** Let \( n \geq 2^N \), \((x^1, x^2) \in (I_1 \setminus I_{l+1}) \times (I_m \setminus I_{m+1})\) and \( 0 \leq l \leq m < N \). Then

\[
\int_{I_N \times I_N} |K_n(x^1 + t^1, x^2 + t^2)|d\mu(t^1, t^2) \\
\leq \frac{c}{n2^N} \left\{ 2^{l_1-m_1} \sum_{r_1=l_1+1}^{m_1+1} 2^{r_1} D_{2^{m_1+1}}(x^1 + e_{l_1} + e_{r_1}) \sum_{s=m_2+1}^{N} D_{2^s}(x^2 + e_{m_2} + x_{m_2+1,s-1}) \\
+ 2^{l_1+m_2} \sum_{s=l_1}^{m_2} \sum_{r_1=l_1+1}^{m_2+1} D_{2^s}(x^1 + e_{l_1} + e_{r_1}) \right\},
\]

where

\[
x_{i,j} := \sum_{s=i}^{j} x_s e_s, \quad (x_{i,i-1} = 0).
\]

For our further investigation we need the following lemma (for details see [25]):

**Lemma 4.42** Let \((x^1, x^2) \in (I_1 \setminus I_{m_2}) \times (I_m \setminus I_{m_2+1})\) and \( 0 \leq m_2 < N \) Then

\[
\int_{I_N \times I_N} |K_n(x^1 + t^1, x^2 + t^2)|d\mu(t^1, t^2) \\
\leq \frac{c}{n2^N} \sum_{s=m}^{2N-1} D_{2^s}(x^2 + e_{m_2}), \quad \text{when } n > 2^N.
\]

We also need the following lemma proved by Goginava [20]:

**Lemma 4.43** Let

\[
x^1 \in I_{4A} \left(0, ..., 0, x^1_{4m} = 1, 0, ..., 0, x^1_{4l} = 1, x^1_{4l+1}, ..., x^1_{4A-1}\right)
\]

and

\[
x^2 \in I_{4A} \left(0, ..., 0, x^2_{4l} = 1, x^1_{4l+1}, ..., x^1_{4q-1}, 1 - x^1_{4q}, x^2_{4q+1}, ..., x^2_{4A-1}\right).
\]

Then

\[
n_{A-1} |K_{n_{A-1}} (x^1, x^2)| \geq 2^{4q+4l+4m-3},
\]

where

\[
n_A = 2^{4A} + 2^{4A-4} + ... + 2^4 + 2^0.
\]

Hardy martingale spaces \( H_p (G^2) \) have atomic decomposition for \( 0 < p \leq 1 \). The following is true (for details see [50], [78] and [79]):
Lemma 4.44 A martingale \( f = (f_n, \ n \in \mathbb{N}) \) belongs to \( H_p(G^2) \) \((0 < p \leq 1)\) if and only if there exist a sequence of \( p \)-atoms \( (a_k, \ k \in \mathbb{N}) \) and sequence of real numbers \( (\mu_k, \ k \in \mathbb{N}) \) such that for every \( n \in \mathbb{N} \)

\[
\sum_{k=0}^{\infty} \mu_k S_{2^n, 2^n} a_k = f_n \tag{4.1}
\]

and

\[
\sum_{k=0}^{\infty} |\mu_k|^p < \infty.
\]

Moreover,

\[
\|f\|_{H_p(G^2)} \sim \inf \left( \sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p},
\]

where infimum is taken over all decomposition of \( f \) of the form (4.1).

Lemma 4.45 Let \( 0 < p \leq 1 \) and \( T \) be \( \sigma \)-sublinear operator, such that

\[
\int_{G^2} |Ta(\vec{x})|^p \, d\mu(\vec{x}) \leq c_p < \infty,
\]

for any \( p \)-atom \( a \). Then there exists an absolute constant \( c_p \), such that

\[
\|Tf\|_p \leq c_p \|f\|_{H_p(G)}.
\]

In addition, if \( T \) is bounded from \( L_\infty(G^2) \) to \( L_\infty(G^2) \), then we only have to prove that

\[
\int_{I^2} |Ta(\vec{x})|^p \, d\mu(\vec{x}) \leq c_p < \infty,
\]

for every \( p \)-atom \( a \), where \( I^2 \) denotes support of \( a \) and \( \bar{T}^2 := G^2 \setminus I^2 \).

In special cases there exists simpler ways how to calculate \( H_p(G^2) \)-norm of martingale \( f \in H_p(G^2) \) (For details see e.g. [50], [78] and [79]):

Lemma 4.46 Let \( g \in L_1(G^2) \) and \( f := (E_n g : n \in \mathbb{N}) \) be regular martingale. Then \( H_p(G^2) \) \((0 < p \leq 1)\) norm is calculated by

\[
\|f\|_{H_p(G^2)} = \left\| \sup_{n \in \mathbb{N}} |S_{2^n, 2^n} g| \right\|_p.
\]

Proofs of Lemma 4.47 and Lemma 4.48 are proved in [35], [36], [66].
Lemma 4.47 Let \(0 < p \leq 1\), \(2^k \leq n < 2^{k+1}\) and \(S_{n,n}f\) be \((n,n)\)-th partial sum, where \(f \in H_p(G^2)\). Then for any fixed \(n \in \mathbb{N}\), we have the following estimation:

\[
\|S_{n,n}f\|_{H_p(G^2)} \leq \left\| \sup_{0 \leq t \leq k} |S_{2^t,2^t}f| \right\|_p + \|S_{n,n}f\|_p \\
\leq \|\tilde{S}^*_f\|_p + \|S_{n,n}f\|_p.
\]

**Proof:** Let consider the following martingale

\[
f_\# := \left(S_{2^k,2^k}S_{n,n}f, \ k \in \mathbb{N}_+ \right) = \left(S_{2^0,2^0}, S_{2^k,2^k}f, ..., S_{n,f}, ..., S_{n,n}f, ...ight).
\]

By using Lemma 4.46 we immediately get that

\[
\|S_{n,n}f\|_{H_p(G^2)}^p \leq \left\| \sup_{0 \leq t \leq k} |S_{2^t,2^t}f| \right\|_p^p + \|S_{n,n}f\|_p^p \\
\leq \|\tilde{S}^*_f\|_p^p + \|S_{n,n}f\|_p^p.
\]

The proof is complete. \(\blacksquare\)

Lemma 4.48 Let \(0 < p \leq 1\), \(2^k \leq n < 2^{k+1}\) and \(\mathcal{M}_n f\) be \(n\)-th Marcinkiewicz means, where \(f \in H_p(G^2)\). Then for any fixed \(n \in \mathbb{N}\), we get that

\[
\|\mathcal{M}_n f\|_{H_p(G^2)}^p \leq \left\| \sup_{0 \leq t \leq k} |\mathcal{M}_{2^t}f| \right\|_p^p + \left\| \sup_{0 \leq t \leq k} |S_{2^t,2^t}f| \right\|_p^p + \|\mathcal{M}_n f\|_p^p \\
\leq \|\tilde{\mathcal{M}}^*_f\|_p^p + \|\tilde{S}^*_f\|_p^p + \|\mathcal{M}_n f\|_p^p.
\]

**Proof:** Let consider the following martingale

\[
f_\# = \left(\frac{2^0, \mathcal{M}_{2^0}}{n} + \frac{(n-2^0)S_{2^0}f}{n}, ..., \frac{2^k, \mathcal{M}_{2^k}f}{n} + \frac{(n-2^k)S_{2^k}f}{n}, \mathcal{M}_n f, ..., \mathcal{M}_n f, ... \right).
\]

According to Lemma 4.46 we immediately get

\[
\|\mathcal{M}_n f\|_{H_p(G^2)}^p \leq \left\| \sup_{0 \leq t \leq k} |\mathcal{M}_{2^t}f| \right\|_p^p + \left\| \sup_{0 \leq t \leq k} |S_{2^t,2^t}f| \right\|_p^p + \|S_{n,n}f\|_p^p \\
\leq \|\tilde{\mathcal{M}}^*_f\|_p^p + \|\tilde{S}^*_f\|_p^p + \|\mathcal{M}_n f\|_p^p.
\]

The proof is complete. \(\blacksquare\)
4.3 **Strong Convergence of Partial Sums with Respect to the Two-Dimensional Walsh-Fourier Series on the Martingale Hardy Spaces**

In this section we investigate strong convergence of partial sums with respect to the two-dimensional Walsh-Fourier series on the martingale Hardy spaces when $0 < p \leq 1$ (see [66]). The following theorem is true:

**Theorem 4.49** a) Let $0 < p < 1$ and $f \in H_p(G^2)$. Then there exists an absolute constant $c_p$, depending only on $p$, such that

$$\sum_{n=1}^{\infty} \frac{\|S_{n,n}f\|_{H_p(G^2)}^p}{n^{3-2p}} \leq c_p \|f\|_{H_p(G^2)}^p.$$  

b) Let $0 < p < 1$ and $\Phi : \mathbb{N} \rightarrow [1, \infty)$ be non-decreasing function, satisfying the condition

$$\lim_{n \to \infty} \Phi(n) = +\infty. \quad (4.3)$$

Then there exists a martingale $f \in H_p(G^2)$, such that

$$\sum_{n=1}^{\infty} \frac{\|S_{n,n}f\|_{L_p(G^2)}^p \Phi(n)}{n^{3-2p}} = \infty.$$  

**Proof:** Suppose that

$$\sum_{n=1}^{\infty} \frac{\|S_{n,n}f\|_p^p}{n^{3-2p}} \leq c_p \|f\|_{H_p(G^2)}^p.$$  

By combining Lemma 2.9 and inequality (1.18) we can conclude that

$$\sum_{n=1}^{\infty} \frac{\|S_{n,n}f\|_{H_p(G^2)}^p}{n^{3-2p}} \leq \sum_{n=1}^{\infty} \frac{\|S_{n,n}f\|_p^p}{n^{3-2p}} + \|\tilde{S}_n f\|_p^p \leq \|f\|_{H_p(G^2)}^p.$$  

According to Lemma 4.45 we only have to prove that

$$\sum_{n=1}^{\infty} \frac{\|S_{n,n}a\|_p^p}{n^{3-2p}} \leq c_p < \infty, \quad (4.4)$$

for every $p$-atom $a$.

Let $a$ be $p$-atom with support $I_N (z^1) \times I_N (z^2)$, where $\mu (I_N) = \mu (I_N) = 2^{-N}$. Without loss the generality we may assume that $z^1 = z^2 = 0$.

Let $(x^1, x^2) \in \mathcal{T}_N \times \mathcal{T}_N$. Then

$$D_{2i} \left( x^1 + t^1 \right) 1_{I_N} (t^1) = 0, \quad \text{when } i \geq N.$$  

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and
\[ D_{2i} \left( x^2 + t^2 \right) 1_{I_N} (t^2) = 0, \quad \text{when } i \geq N. \]

If we apply \( w_{2i} (x^i + t^i) = w_{2i} (x^i) \), where \( t^i \in I_N \), \( i = 1 \lor 2 \) and \( j < N \) according to both equality of Lemma 2.3 we get that
\[
S_{n,n} a(x^1, x^2)
= \int_{G \times G} a(t^1, t^2) D_n (x^1 + t^1) D_n (x^2 + t^2) \, d\mu(t^1, t^2)
\]
\[
= \int_{I_N \times I_N} a(t^1, t^2) D_n (x^1 + t^1) D_n (x^2 + t^2) \, d\mu(t^1, t^2)
\]
\[
= \int_{I_N \times I_N} a(t^1, t^2) w_n (x^1 + t^1 + x^2 + t^2) \sum_{i=0}^{N-1} n_i w_{2i} (x^1 + t^1) D_{2i} (x^1 + t^1)
\]
\[
\quad \times \sum_{j=0}^{N-1} n_j w_{2j} (x^2 + t^2) D_{2j} (x^2 + t^2) \, d\mu(t^1, t^2)
\]
\[
= w_n (x^1) \sum_{i=0}^{N-1} n_i w_{2i} (x^2) D_{2i} (x^1) \sum_{j=0}^{N-1} n_j w_{2j} (x^2) D_{2j} (x^2)
\]
\[
\times \int_{I_N \times I_N} a(t^1, t^2) w_n (t^1 + t^2) \, d\mu(t^1, t^2)
\]
\[
= w_n (x^1 + x^2) \sum_{i=0}^{N-1} n_i w_{2i} (x^1) D_{2i} (x^1) \sum_{j=0}^{N-1} n_j w_{2j} (x^2) D_{2j} (x^2)
\]
\[
\times \left( \int_{I_N} \left( \int_{I_N} a(t^2 + \tau, t^2) \, d\mu(t^2) \right) w_n (\tau) \, d\mu(\tau) \right)
\]
\[
= w_n (x^1 + x^2) \sum_{i=0}^{N-1} n_i w_{2i} (x^1) D_{2i} (x^1) \sum_{j=0}^{N-1} n_j w_{2j} (x^2) D_{2j} (x^2)
\int_{I_N} \Phi(\tau) w_n (\tau) \, d\mu(\tau)
\]
\[
= w_n (x^1 + x^2) \sum_{i=0}^{N-1} n_i w_{2i} (x^1) D_{2i} (x^1) \sum_{j=0}^{N-1} n_j w_{2j} (x^2) D_{2j} (x^2) \Phi(n),
\]

where
\[
\Phi(\tau) = \int_{I_N} a(t^i + \tau, t^i) \, d\mu(t^i) \quad \text{and } i = 1 \lor 2.
\]

Let \( x \in I_s \setminus I_{s+1} \), where \( i = 1 \lor 2 \). By using again Lemma 2.3 can conclude that
\[
\sum_{i=0}^{N-1} D_{2i} (x) \leq c2^s.
\]
By using (2.1) we have that

\[
\int_{I_N} \left( \sum_{i=0}^{N-1} D_{2i}(x) \right)^p d\mu(x) \tag{4.5}
\]

\[
\leq c_p \sum_{s=0}^{N-1} \int_{I_s \setminus I_{s+1}} 2^{ps} d\mu(x)
\]

\[
\leq c_p \sum_{s=0}^{\infty} 2^{(p-1)s}
\]

\[
< c_p < \infty, \quad 0 < p < 1.
\]

From (4.5) we get that

\[
\sum_{n=1}^{\infty} \frac{1}{n^{3-2p}} \int_{I_N \times I_N} |S_{n,n} a(x^1, x^2)|^p d\mu(x^1, x^2)
\]

\[
\leq \sum_{n=1}^{\infty} \frac{\left| \tilde{\Phi}(n) \right|^p}{n^{3-2p}} \int_{T_N} \left( \sum_{i=0}^{N-1} D_{2i}(x^1) \right)^p d\mu(x^1) \int_{T_N} \left( \sum_{i=0}^{N-1} D_{2i}(x^2) \right)^p d\mu(x^2)
\]

\[
\leq c_p \sum_{n=1}^{\infty} \frac{\left| \tilde{\Phi}(n) \right|^p}{n^{3-2p}}.
\]

Let \( n < 2^N \). Since \( w_n(\tau) = 1 \), where \( \tau \in I_N \) we obtain that

\[
\tilde{\Phi}(n) = \int_{I_N} \Phi(\tau) w_n(\tau) d\mu(\tau)
\]

\[
= \int_{I_N} \left( \int_{I_N} a(t^2 + \tau, t^2) d\mu(t^2) \right) w_n(\tau) d\mu(\tau)
\]

\[
= \int_{I_N \times I_N} a(t^1, t^2) d\mu(t^1, t^2) = 0.
\]
So, we may assume that \( n \geq 2^N \). If we apply Holder’s inequality we get that

\[
\sum_{n=1}^{\infty} \left| \frac{\hat{\Phi}(n)}{n^{3-2p}} \right|^p \leq \left( \sum_{n=2^N}^{\infty} \left| \frac{\hat{\Phi}(n)}{n^{3-2p}} \right|^2 \right)^{p/2} \left( \sum_{n=2^N}^{\infty} \frac{1}{n^{1(3-2p)/(2/(2-p))}} \right)^{(2-p)/2} \leq \left( \frac{1}{2^N(2(3-2p)/(2-p)-1)} \right)^{(2-p)/2} \left( \int_G \left| \Phi(\tau) \right|^2 d\mu(\tau) \right)^{p/2} \leq \frac{c_p}{2^N(4-3p)/2} \left( \int_{I_N} \left\| a(t^1, t^2) \right\|_\infty \right)^{p/2} \left( \frac{1}{2^Np/2} \right)^{1/2} \left( \frac{1}{2^Np} \right)^{1/2} \leq \frac{c_p}{2^N(4-3p)/2} 2^{12N} \frac{1}{2^{3pN/2}} c_p < c_p < \infty.
\]

Let \((x^1, x^2) \in \overline{T}_N \times I_N\). Then

\[
S_{n,n}a(x^1, x^2) = w_n(x^1) \sum_{j=0}^{N-1} n_j w_{2j} (x^1) D_{2j} (x^1) \times \int_{G \times G} a(t^1, t^2) w_n(t^1) D_n (x^2 + t^2) d\mu(t^1, t^2)
\]

\[
= w_n(x^1) \sum_{j=0}^{N-1} n_j w_{2j} (x^1) D_{2j} (x^1) \int_{G} S_{n}^{(2)} a(t^1, x^2) w_n(t^1) d\mu(t^1)
\]

\[
= w_n(x^1) \sum_{j=0}^{N-1} n_j w_{2j} (x^1) D_{2j} (x^1) \widehat{S}_{n}^{(2)} a(n, x^2).
\]
By applying (4.5) we get that

\[
\sum_{n=1}^{\infty} \frac{1}{n^{3-2p}} \int_{I_N \times I_N} |S_{n,n,a}(x^1, x^2)|^p \, d\mu(x^1, x^2)
\leq \sum_{n=1}^{\infty} \frac{1}{n^{3-2p}} \int_{I_N \times I_N} \left( \sum_{j=0}^{N-1} D_{2j}^2(x^1) \left| \widehat{S}_{n}^{(2)}(n, x^2) \right| \right)^p \, d\mu(x^1, x^2)
\leq \sum_{n=1}^{\infty} \frac{1}{n^{3-2p}} \int_{I_N} \left( \sum_{i=0}^{N-1} D_{2i}^2(x^1) \right)^p \, d\mu(x^1) \cdot \int_{I_N} \left| \widehat{S}_{n}^{(2)}(n, x^2) \right|^p \, d\mu(x^2)
\leq \sum_{n=1}^{\infty} \frac{1}{n^{3-2p}} \int_{I_N} \left| \widehat{S}_{n}^{(2)}(n, x^2) \right|^p \, d\mu(x^2).
\]

Let \( n < 2^N \). By applying definition of \( p \)-atom we get that

\[
\widehat{S}_{n}^{(2)}(n, x^2) = \int_{G} \left( \int_{G} a(t^1, t^2) D_{n}(x^2 + t^2) \, d\mu(t^2) \right) w_n(t^1) \, d\mu(t^1)
\leq D_{n}(x^2) \int_{I_N \times I_N} a(t^1, t^2) \, d\mu(t^1, t^2) = 0.
\]

Therefore, we can suppose that \( n \geq 2^N \). If follows that

\[
\sum_{n=1}^{\infty} \frac{1}{n^{3-2p}} \int_{I_N \times I_N} |S_{n,n,a}(x^1, x^2)|^p \, d\mu(x^1, x^2)
\leq \sum_{n=2^N}^{\infty} \frac{1}{n^{3-2p}} \int_{I_N} \left| \widehat{S}_{n}^{(2)}(n, x^2) \right|^p \, d\mu(x^2)
\]

Since

\[
\left\| \widehat{S}_{n}^{(2)}(n, x^2) \right\|_2 \leq c \, \|a\|_2
\]
if we use Holder’s inequality we can conclude that

\[
\int_{I_N} \left| \hat{S}_n^{(2)} a \left( n, x^2 \right) \right|^p d\mu \left( x^2 \right)
\]

\[
\leq \frac{c_p}{2^{N(1-p)}} \left( \int_{I_N} \left| \hat{S}_n^{(2)} a \left( n, x^2 \right) \right| d\mu \left( x^2 \right) \right)^p
\]

\[
= \frac{c_p}{2^{N(1-p)}} \left( \int_{I_N} \int_{I_N} S_n^{(2)} a \left( t^1, x^2 \right) w_n \left( t^1 \right) d\mu \left( t^1 \right) d\mu \left( x^2 \right) \right)^p
\]

\[
= \frac{c_p}{2^{N(1-p)}} \left( \int_{I_N} \int_{I_N} \left( \int_{I_N} a \left( t^1, t^2 \right) D_n \left( x^2 + t^2 \right) d\mu \left( t^2 \right) \right) w_n \left( t^1 \right) d\mu \left( t^1 \right) d\mu \left( x^2 \right) \right)^p
\]

\[
\leq \frac{c_p}{2^{N(1-p)}} \left( \int_{I_N} \int_{I_N} \int_{I_N} a \left( t^1, t^2 \right) D_n \left( x^2 + t^2 \right) d\mu \left( t^2 \right) d\mu \left( x^2 \right) d\mu \left( t^1 \right) \right)^p
\]

\[
\leq \frac{c_p}{2^{N(1-p)}} \left( \frac{1}{2^{N/2}} \int_{I_N} \int_{I_N} \int_{I_N} a \left( t^1, t^2 \right) D_n \left( x^2 + t^2 \right) d\mu \left( t^2 \right)^2 d\mu \left( x^2 \right) d\mu \left( t^1 \right) \right)^{1/2}^p
\]

\[
\leq \frac{c_p}{2^{N(1-p)}} \left( \frac{1}{2^{N/2}} \int_{I_N} \int_{I_N} \int_{I_N} a \left( t^1, t^2 \right)^2 d\mu \left( t^2 \right) d\mu \left( t^1 \right) \right)^{1/2}^p
\]

\[
\leq \frac{c_p}{2^{N(1-p)}} \left( \frac{1}{2^{N/2}} \frac{1}{N^{1/(2p)}} \frac{1}{2^{N/2}} \right)^p \leq \frac{c_p}{2^{N(1-p)}} \left( \frac{2^{N/p}}{2^N} \right)^p \leq c_p 2^{N(1-p)}.
\]

Hence,

\[
\sum_{n=1}^{\infty} \frac{1}{n^{1/(2p-2)}} \int_{I_N \times I_N} \left| S_{n,n}^1 a \left( x^1, x^2 \right) \right|^p d\mu \left( x^1, x^2 \right) \leq c_p 2^{N(1-p)}.
\]

(4.7)

\[
\sum_{n=1}^{\infty} \frac{1}{n^{3/(2p)}} \int_{I_N \times I_N} \left| S_{n,n} a \left( x^1, x^2 \right) \right|^p d\mu \left( x^1, x^2 \right) \leq c_p 2^{N(1-p)}.
\]

Analogously, we can prove that

\[
\sum_{n=1}^{\infty} \frac{1}{n^{1/(2p-2)}} \int_{I_N \times I_N} \left| S_{n,n} a \left( x^1, x^2 \right) \right|^p d\mu \left( x^1, x^2 \right) \leq c_p < \infty.
\]

(4.8)
Let \((x^1, x^2) \in I_N \times I_N\). Then by the definition of \(p\)-atom we get that

\[
\int_{I_N \times I_N} |S_{n,n,a} \left( x^1, x^2 \right) |^p \, d\mu \left( x^1, x^2 \right)
\]

\[
\leq \frac{1}{2N(2-p)} \left( \int_{I_N \times I_N} |S_{n,n,a} \left( x^1, x^2 \right) |^2 \, d\mu \left( x^1, x^2 \right) \right)^{p/2}
\]

\[
\leq \frac{1}{2N(2-p)} \left( \int_{I_N \times I_N} |a \left( x^1, x^2 \right) |^2 \, d\mu \left( x^1, x^2 \right) \right)^{p/2}
\]

\[
\leq \frac{\|a\|_\infty^p}{2N(2-p)} \frac{1}{2np} \leq c_p < \infty.
\]

It follows that

\[
\sum_{n=1}^{\infty} \frac{1}{n^{3-2p}} \int_{I_N \times I_N} |S_{n,n,a} \left( x^1, x^2 \right) | \, d\mu \left( x^1, x^2 \right) \leq c_p \sum_{n=1}^{\infty} \frac{1}{n^{3-2p}} \leq c_p < \infty.
\] (4.9)

By combining (4.4-4.9) we get that Theorem 4.49 is proved.

Let \(0 < p < 1\) and \(\Phi \left( n \right)\) satisfies condition 4.3. Then there exists increasing sequence of natural numbers \(\{\alpha_k : k \in \mathbb{N}_+\}\) such that

\[
\alpha_0 \geq 2
\]

and

\[
\sum_{k=0}^{\infty} \Phi^{-p/4} \left( 2^{2\alpha_k} \right) < \infty.
\] (4.10)

Let \(f = (f_n, n \in \mathbb{N}_+)\) be a martingale

\[
f_n \left( x^1, x^2 \right) = \sum_{k=1}^{n} \lambda_k a_k \left( x^1, x^2 \right)
\]

where

\[
a_k \left( x^1, x^2 \right) = 2^{2\alpha_k \left( 1/p - 1 \right)} \left( D_{2^{2\alpha_k+1}} \left( x^1 \right) - D_{2\alpha_k} \left( x^1 \right) \right) \left( D_{2^{2\alpha_k+1}} \left( x^2 \right) - D_{2\alpha_k} \left( x^2 \right) \right)
\]

and

\[
\lambda_k = \Phi^{-1/4} \left( 2^{2\alpha_k} \right).
\]
By applying (4.10) and Lemma 4.44 we obtain that \( f \in H_p(G^2) \).

It is evident that

\[
\hat{f}(i, j) = \begin{cases} 
\frac{2^{2\alpha_k(2/p-2)}}{\Phi^{1/4}(2^{2\alpha_k})} \Phi^{1/4}(2^{2\alpha_k})(i, j) \in \{2^{2\alpha_k}, \ldots, 2^{2\alpha_k+1} - 1\}, \, k \in \mathbb{N}_+, \\
0, \\
(i, j) \notin \bigcup_{k=1}^{\infty} \{2^{2\alpha_k}, \ldots, 2^{2\alpha_k+1} - 1\}^2.
\end{cases}
\] (4.11)

Let \( 2^{2\alpha_k} < n < 2^{2\alpha_k+1} \). By combining (4.11) and first equality of Lemma 2.2 we get that

\[
S_{n,n}f(x^1, x^2) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \hat{f}(i, j)w_i(x^1)w_j(x^2) \\
+ \sum_{i=2^{2\alpha_k}}^{n-1} \sum_{j=2^{2\alpha_k}}^{n-1} \hat{f}(i, j)w_i(x^1)w_j(x^2) \\
= \sum_{\eta=0}^{2^{2\alpha_k}+1-1} \sum_{i=2^{2\alpha_k}}^{\eta} \sum_{j=2^{2\alpha_k}}^{\eta} \frac{2^{2\alpha_k(2/p-2)}}{\Phi^{1/4}(2^{2\alpha_k})}w_i(x^1)w_j(x^2) \\
+ \sum_{i=2^{2\alpha_k}}^{n-1} \sum_{j=2^{2\alpha_k}}^{n-1} \frac{2^{2\alpha_k(2/p-2)}}{\Phi^{1/4}(2^{2\alpha_k})}w_i(x^1)w_j(x^2) \\
= \sum_{\eta=0}^{2^{2\alpha_k}+1-1} \sum_{i=2^{2\alpha_k}}^{\eta} \sum_{j=2^{2\alpha_k}}^{\eta} \frac{2^{2\alpha_k(2/p-2)}}{\Phi^{1/4}(2^{2\alpha_k})}(D_{2^{2\alpha_k}+1}(x^1) - D_{2^{2\alpha_k}}(x^1))(D_{2^{2\alpha_k}+1}(x^2) - D_{2^{2\alpha_k}}(x^2)) \\
+ \frac{2^{2\alpha_k(2/p-2)}}{\Phi^{1/4}(2^{2\alpha_k})}(D_n(x^1) - D_{2^{2\alpha_k}}(x^1))(D_n(x^2) - D_{2^{2\alpha_k}}(x^2)) \\
= I + II.
\] (4.12)

Let \((x^1, x^2) \in (G \setminus \Gamma)^2\) and \( n \) is odd number. Since \( n - 2^{2\alpha_k} \) is also odd, according to both equality of Lemma 2.3 we get that
\[ |II| = \frac{2^{2\alpha_k (2/p - 2)}}{\Phi^{1/4} (2^{2\alpha_k})} | w_{2^{2\alpha_k}} (x^1) D_{n - 2^{2\alpha_k}} (x^1) w_{2^{2\alpha_k}} (x^2) D_{n - 2^{2\alpha_k}} (x^2) | \]
\[ = \frac{2^{2\alpha_k (2/p - 2)}}{\Phi^{1/4} (2^{2\alpha_k})} | w_{2^{2\alpha_k}} (x^1) w_{n - 2^{2\alpha_k}} (x^1) D_1 (x^1) w_{2^{2\alpha_k}} (x^2) w_{n - 2^{2\alpha_k}} (x^2) D_1 (x^2) | \]
\[ = \frac{2^{2\alpha_k (2/p - 2)}}{\Phi^{1/4} (2^{2\alpha_k})}. \]

If we apply again second equality of Lemma 2.3 according to condition \( \alpha_n \geq 2 \) \( (n \in \mathbb{N}) \) for \( I \) we can conclude that

\[ I = \sum_{n=0}^{k-1} \frac{2^{2\alpha_k (2/p - 2)}}{\Phi^{1/4} (2^{2\alpha_k})} (D_{2^{2\alpha_k} n + 1} (x^1) - D_{2^{2\alpha_k} n} (x^1)) (D_{2^{2\alpha_k} n + 1} (x^2) - D_{2^{2\alpha_k} n} (x^2)) = 0. \]

Hence,
\[ \left\| S_{n,n} f (x^1, x^2) \right\|_{weak-L_p(G^2)} \geq \frac{c_p 2^{2\alpha_k (2/p - 2)}}{\Phi^{1/4} (2^{2\alpha_k})} \left\| (G \backslash I_1)^2 \right\| \]

By using (4.15) we get that
\[ \sum_{n=1}^{2^{2\alpha_k + 1} - 1} \frac{\left\| S_{n,n} f \right\|_{weak-L_p(G^2)}^p \Phi(n)}{n^{3-2p}} \]
\[ \geq \sum_{n=2^{2\alpha_k} + 1}^{2^{2\alpha_k + 1} - 1} \frac{\left\| S_{n,n} f \right\|_{weak-L_p(G^2)}^p \Phi(n)}{n^{3-2p}} \]
\[ \geq c_p \Phi (2^{2\alpha_k}) \sum_{n=2^{2\alpha_k - 1} + 1}^{2^{2\alpha_k} - 1} \frac{\left\| S_{2n+1,2n+1} f \right\|_{weak-L_p(G^2)}^p}{(2n + 1)^{3-2p}} \]
\[ \geq c_p \Phi (2^{2\alpha_k}) \Phi^{1/4} (2^{2\alpha_k}) \sum_{n=2^{2\alpha_k - 1} + 1}^{2^{2\alpha_k} - 1} \frac{1}{(2n + 1)^{3-2p}} \]
\[ \geq c_p \Phi^{3/4} (2^{2\alpha_k}) \to \infty, \quad \text{as} \quad k \to \infty. \]
4.4 Strong convergence of Marcinkiewicz means with respect to the two-
dimensional Walsh-Fourier series on the martingale Hardy spaces

In this section we consider strong convergence of Marcinkiewicz means with respect
to the two-dimensional Walsh-Fourier series in the martingale Hardy spaces for
\( p = \frac{2}{3} \) (for details see Nagy and Tephnadze [35]).

**Theorem 4.50** Let \( f \in H_{2/3}(G^2) \). Then there exists an absolute constant \( c \), such that

\[
\frac{1}{\log n} \sum_{m=1}^{n} \frac{\| \mathcal{M}_m f \|_{H_{2/3}(G^2)}^{2/3}}{m} \leq c \| f \|_{H_{2/3}(G^2)}^{2/3}.
\]

**Proof:** Suppose that

\[
\frac{1}{\log n} \sum_{m=1}^{n} \frac{\| \mathcal{M}_m f \|_{H_{2/3}(G^2)}^{2/3}}{m} \leq c \| f \|_{H_{2/3}(G^2)}^{2/3}.
\] (4.17)

By combining Lemma 4.48 and inequalities (1.18), (1.22), (4.17) we get that

\[
\frac{1}{\log n} \sum_{m=1}^{n} \frac{\| \mathcal{M}_m f \|_{H_{2/3}(G^2)}^{2/3}}{m} \leq \frac{1}{\log n} \sum_{m=1}^{n} \frac{\| \mathcal{M}_m f \|_{H_{2/3}(G^2)}^{2/3}}{m} + \left\| \mathcal{S}_{\#} f \right\|_{H_{2/3}(G^2)}^{2/3} \leq \| f \|_{H_{2/3}(G^2)}^{2/3}.
\] (4.18)

Since \( \mathcal{M}_n \) is (see Lemma 4.40) bounded from \( L_\infty(G^2) \) to \( L_\infty(G^2) \), if we use Lemma 4.45 we only have to prove that

\[
\frac{1}{\log n} \sum_{m=1}^{n} \frac{\| \mathcal{M}_m a \|_{H_{2/3}(G^2)}^{2/3}}{m} < c < \infty,
\]

for every \( 2/3 \)-atom \( a \).

Let \( a \) be \( 2/3 \)-atom with support \( I^2 \), where \( \mu(I^2) = 2^{-2N} \). Without loss the generality we may assume that \( I^2 := I_N^2 \). It is easy to show that \( \mathcal{M}_n a = 0 \) for \( n \leq 2^N \). So, we may assume that \( n > 2^N \).

Hardy spaces
We can write that

\[
\frac{1}{\log n} \sum_{m=1}^{n} \left\| \mathcal{M}_m a \right\|_{2/3}^{2/3} \leq \frac{1}{\log n} \sum_{m=2N}^{n} \left\| \mathcal{M}_m a \right\|_{2/3}^{2/3} \leq \frac{1}{\log n} \sum_{m=2N}^{n} \int_{I_N \times I_N} \left| \mathcal{M}_m a \right|^{2/3} \frac{1}{m} \, d\mu
\]

\[
+ \frac{1}{\log n} \sum_{m=2N}^{n} \int_{I_N \times I_N} \left| \mathcal{M}_m a \right|^{2/3} \frac{1}{m} \, d\mu
\]

\[
+ \frac{1}{\log n} \sum_{m=2N}^{n} \int_{I_N \times I_N} \left| \mathcal{M}_m a \right|^{2/3} \frac{1}{m} \, d\mu
\]

\[
+ \frac{1}{\log n} \sum_{m=2N}^{n} \int_{I_N \times I_N} \left| \mathcal{M}_m a \right|^{2/3} \frac{1}{m} \, d\mu
\]

\[=: I_1 + I_2 + I_3 + I_4.\]

By applying Lemma 4.40 we have that

\[I_1 \leq \frac{1}{\log n} \sum_{m=2N}^{\infty} \int_{I_N \times I_N} \left| \mathcal{M}_m a \right|^{2/3} \frac{1}{m} \, d\mu\]

\[\leq \frac{1}{\log n} \sum_{m=2N}^{\infty} \frac{1}{m} \left\| a \right\|_{2/3}^{2/3} / 2^{2N}\]

\[\leq \frac{1}{\log n} \sum_{m=2N}^{n} \frac{1}{m} < c < \infty.\]

Now, we estimate \(I_2\). Set

\[J_t := I_t \setminus I_{t+1}, \quad (t \in \mathbb{N}).\]

We introduce \(I_N\) and \(J_{m^2}\) as the following disjoint union:

\[I_N = \bigcup_{m^2=0}^{N-1} J_m, \quad J_m = \bigcup_{q^2=m^2+1}^{N} I_{m^2,q^2}^{N},\]

(4.19)

where

\[I_{m^2,q^2}^{N} := \begin{cases} I_{q^2+1}(0, \ldots, 0, x_{m^2} = 1, 0, \ldots, 0, x_{q^2} = 1), & \text{where } m^2 < q^2 < N, \\ I_{N}(0, \ldots, 0, x_{m^2} = 1, 0, \ldots, 0), & \text{where } q^2 = N. \end{cases}\]

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Let \((x^1, x^2) \in I_N \times I_N^{m_2, q_2}\). According to Lemma 4.42 we can conclude that
\[
|\mathcal{M}_n a(x^1, x^2)|
\leq \|a\|_\infty \int_{I_N \times I_N} |K_n(x^1 + t^1, x^2 + t^2)| d\mu(t^1, t^2)
\leq c 2^{3N} \frac{2^{m_2}}{n 2^N} \sum_{m_2=0}^{q_2} D_{2^q}(x^2 + \epsilon_{m_2})
\leq c 2^{2N+m_2} \frac{1}{n} \sum_{m=2^N}^{q_2} 2^q
\leq c 2^{2N+m_2+q_2} \frac{1}{n}.
\]

Hence,
\[
I_2 \leq \frac{c 4^{2N/3}}{\log n} \sum_{m=2^N}^{n} \sum_{m_2=0}^{q_2} \sum_{m_2=0}^{q_2} \int_{I_N \times I_N^{m_2, q_2}} \frac{|\mathcal{M}_m a|^{2/3}}{m} d\mu
\leq \frac{c 4^{2N/3}}{\log n} \sum_{m=2^N}^{n} \sum_{m_2=0}^{q_2} \sum_{m_2=0}^{q_2} \int_{I_N \times I_N^{m_2, q_2}} \frac{2^{2(m^2+q^2)/3}}{m^{5/3}} d\mu
\leq \frac{c 4^{2N/3}}{\log n} \sum_{m=2^N}^{n} \sum_{m_2=0}^{q_2} \sum_{m_2=0}^{q_2} 2^{2m_2^2/3} \sum_{q_2=m_2+1}^{N} 2^{-q^2/3}
\leq \frac{c 4^{2N/3}}{\log n} \sum_{m=2^N}^{n} \frac{2^{N/3}}{m^{5/3}}
\leq c 2^{2N/3} \frac{1}{\log n} \sum_{m=2^N}^{n} \frac{1}{m^{5m/3}}
\leq \frac{c}{N}.
\]

Analogously, we can prove that \(I_3 \leq c < \infty\).

Next we prove boundedness of \(I_4\). If we apply (4.19) we get that
\[
I_4 \leq \frac{1}{\log n} \sum_{m=2^N}^{n} \sum_{t_1=0}^{N-1} \sum_{t_2=0}^{l_1-1} \int_{J_{t_1} \times J_{t_2}} \frac{|\mathcal{M}_m a|^{2/3}}{m} d\mu
+ \frac{1}{\log n} \sum_{m=2^N}^{n} \sum_{t_1=0}^{N-1} \sum_{t_2=0}^{l_2-1} \int_{J_{t_1} \times J_{t_2}} \frac{|\mathcal{M}_m a|^{2/3}}{m} d\mu
=: I_{4,1} + I_{4,2}.
\]
Let consider $I_{4.2}$ (Analogously we can estimate $I_{4.1}$). For $(x^1, x^2) \in J_{l_1} \times J_{m_2}$ if we apply Lemma 4.41 we obtain that

$$|\mathcal{M}_n a(x^1, x^2)| \leq \|a\|_\infty \int_{I_{l_1} \times I_{m_2}} |K_n(x^1 + t^1, x^2 + t^2)| d\mu(t^1, t^2) \leq \frac{2^{N+l_1-m_2}}{n} \sum_{r^1 = l_1+1}^{m_2+1} 2^{r^1} D_{2^{r^1}+1}(x^1 + e_{r^1} + e_{r^1}) \sum_{s=m_2+1}^{N} D_{2^s}(x^2 + e_{m_2} + x_{m_2+1,s-1})$$

$$+ \frac{2^{N+l_1+m_2}}{n} \sum_{s=l_1}^{m_2} \sum_{r^1 = l_1+1}^{s} D_{2^s}(x^1 + e_{r^1} + e_{r^1}).$$

It is evident that

$$\int_{J_{l_1} \times J_{m_2}} D_{2^{r^1}+1}^{2/3}(x^1 + e_{r^1} + e_{r^1}) D_{2^{r^1}+1}^{2/3}(x^2 + e_{m_2} + x_{m_2+1,s-1}) d\mu(x^1, x^2) \leq c 2^{s/3-m_2-3-t^1} \leq c2^{-(m_2+s)/3}$$

and

$$\int_{J_{l_1} \times J_{m_2}} D_{2^{m_2}+1}^{2/3}(x^1 + e_{r^1} + e_{r^1}) d\mu(x^1, x^2) \leq c 2^{s/3-m_2-3-t^1} \leq c 2^{-m_2-s/3}.$$

Hence,

$$\int_{J_{l_1} \times J_{m_2}} |\mathcal{M}_m a|^{2/3} d\mu$$

$$\leq c 2^{(N+l_1-m_2)/3} \frac{m_2+1}{m_2^{2/3}} \sum_{r^1 = l_1+1}^{m_2+1} \sum_{s=m_2+1}^{N} 2^{r^1/3} \times$$

$$\times \int_{J_{l_1} \times J_{m_2}} D_{2^{r^1}+1}^{2/3}(x^1 + e_{r^1} + e_{r^1}) D_{2^{r^1}+1}^{2/3}(x^2 + e_{m_2} + x_{m_2+1,s-1}) d\mu(x^1, x^2)$$

$$+ \frac{c 2^{N+l_1+m_2}}{m_2^{2/3}} \sum_{s=l_1}^{m_2} \sum_{r^1 = l_1+1}^{s} \int_{J_{l_1} \times J_{m_2}} D_{2^s}^{2/3}(x^1 + e_{r^1} + e_{r^1}) d\mu(x^1, x^2)$$

$$\leq c 2^{(N+l_1-m_2)/3} \frac{m_2+1}{m_2^{2/3}} \sum_{r^1 = l_1+1}^{m_2+1} 2^{r^1/3} \sum_{s=m_2+1}^{N} 2^{-(m_2+s)/3} + \frac{c 2^{N+l_1+m_2}}{m_2^{2/3}} \sum_{s=l_1}^{m_2} \sum_{r^1 = l_1+1}^{s} 2^{-m_2-s/3}$$

$$\leq c 2^{(N+l_1-m_2)/3} \frac{m_2+1}{m_2^{2/3}} 2^{-2m_2/3} \sum_{r^1 = l_1+1}^{m_2+1} 2^{r^1/3} + \frac{c 2^{N+l_1+m_2}}{m_2^{2/3}} \sum_{s=l_1}^{m_2} (s - l_1 - 1) 2^{-m_2-s/3}$$

$$\leq c 2^{(N+l_1-m_2)/3} \frac{m_2+1}{m_2^{2/3}} + c 2^{(2N+l_1-m_2)/3}.$$
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and

\[
I_{4,2} \leq \frac{c}{\log n} \sum_{m=2^N}^{n} \frac{1}{m} \sum_{l=0}^{N-1} \sum_{m^2+l}^{N-1} \frac{2^{2(N+m^2+l)/3} + 2^{(2N-m^2+l)/3}}{m^{2/3}} \leq c. \]

\[\text{span}\]

**Corollary 4.51** Let \( f \in H_{2/3}(G^2) \). Then

\[
\lim_{n \to \infty} \frac{1}{\log n} \sum_{m=1}^{n} \frac{\|M_m f - f\|_{H_{2/3}(G^2)}^{2/3}}{m} = 0
\]

and

\[
\lim_{n \to \infty} \frac{1}{\log n} \sum_{m=1}^{n} \frac{\|M_m f\|_{H_{2/3}(G^2)}^{2/3}}{m} = \|f\|_{H_{2/3}(G^2)}^{2/3}.
\]

4.5 Modulus of Continuity and Convergence in Norm of Marcinkiewicz Means with Respect to the Two-Dimensional Walsh-Fourier Series on the Martingale Hardy Spaces

In this section we investigate necessary and sufficient conditions for modulus of continuity, which provide convergence in norm of Marcinkiewicz means with respect to the two-dimensional Walsh-Fourier series in \( H_{2/3} \)-norm (For details see [37]).

**Theorem 4.52** a) Let \( f \in H_{2/3}(G^2) \) and

\[
\omega_{H_{2/3}(G^2)} \left( \frac{1}{2^k}, f \right) = o \left( \frac{1}{k^{3/2}} \right), \quad \text{as} \quad k \to \infty. \quad (4.20)
\]

Then

\[
\|M_n f - f\|_{H_{2/3}(G^2)} \to 0, \quad \text{as} \quad n \to \infty.
\]

b) There exists a martingale \( f \in H_{2/3}(G^2) \), such that

\[
\omega_{H_{2/3}(G^2)} \left( \frac{1}{2^{2^k}}, f \right) = O \left( \frac{1}{2^{3k/2}} \right), \quad \text{as} \quad k \to \infty
\]

and

\[
\|M_n f - f\|_{2/3} \to 0 \quad \text{as} \quad n \to \infty.
\]
**Proof:** In [34] it was proved that (see inequality (1.23)) the following inequality is true:

\[
\|M_n f\|_{2/3} \leq c \log^{2/3} (n + 1) \|f\|_{H_{2/3}(G^2)}.
\] (4.21)

If we apply inequality (1.22) and Lemma 4.48 according to (4.21) we get the following estimation

\[
\|M_n f\|_{H_{2/3}(G^2)}^{2/3} \leq \|M_n f\|_{2/3}^{2/3} + \|\widetilde{M}_n f\|_{2/3}^{2/3} + \|\widetilde{S}_n f\|_{2/3}^{2/3} \\
\leq c \log (n + 1) \|f\|_{H_{2/3}(G^2)}^{2/3} + c \|f\|_{H_{2/3}(G^2)}^{2/3} \\
\leq c \log (n + 1) \|f\|_{H_{2/3}(G^2)}^{2/3}.
\] (4.22)

Let \(2^N < n \leq 2^{N+1}\). If we use (4.22) by simple calculations we have that

\[
\|M_n f - f\|_{H_{2/3}(G^2)}^{2/3} \\
\leq \|M_n f - M_n S_{2^N, 2^N} f\|_{H_{2/3}(G^2)}^{2/3} \\
+ \|M_n S_{2^N, 2^N} f - S_{2^N, 2^N} f\|_{H_{2/3}(G^2)}^{2/3} \\
+ \|S_{2^N, 2^N} f - f\|_{H_{2/3}(G^2)}^{2/3} \\
= \|M_n (S_{2^N, 2^N} f - f)\|_{H_{2/3}}^{2/3} \\
+ \|M_n S_{2^N, 2^N} f - S_{2^N, 2^N} f\|_{H_{2/3}(G^2)}^{2/3} \\
+ \|S_{2^N, 2^N} f - f\|_{H_{2/3}(G^2)}^{2/3} \\
\leq c \left( \log (n + 1) + 1 \right) \omega_{H_{2/3}(G^2)}^{2/3} \left( \frac{1}{2^N}, f \right) \\
+ \|M_n S_{2^N, 2^N} f - S_{2^N, 2^N} f\|_{H_{2/3}(G^2)}^{2/3}.
\]
Let $2^N < n \leq 2^{N+1}$. Then it is evident that

\[
\mathcal{M}_n S_{2^N} f - S_{2^N} f = \frac{1}{n} \sum_{k=0}^{n} S_{k,k} S_{2^N} f \\
+ \frac{1}{n} \sum_{k=2^N+1}^{n} S_{k,k} S_{2^N} f - S_{2^N} f \\
= \frac{1}{n} \sum_{k=0}^{n} S_{k,k} f \\
+ \frac{(n - 2^N)}{n} S_{2^N} f - S_{2^N} f \\
= \frac{2^N}{n} (\mathcal{M}_2 f - S_{2^N} f) \\
= \frac{2^N}{n} (S_{2^N} \mathcal{M}_2 f - S_{2^N} f) \\
= \frac{2^N}{n} S_{2^N} (\mathcal{M}_2 f - f).
\]

By combining (1.19) and (1.21) we get that

\[
\|\mathcal{M}_n S_{2^N} f - S_{2^N} f\|_{H_{2/3}(G^2)}^{2/3} \\
\leq \left(\frac{2^N}{n}\right)^{2/3} \|S_{2^N} (\mathcal{M}_2 f - f)\|_{H_{2/3}(G^2)}^{2/3} \\
\leq \|S_{2^N} (\mathcal{M}_2 f - f)\|_{H_{2/3}(G^2)}^{2/3} \\
\leq \|\mathcal{M}_2 f - f\|_{H_{2/3}(G^2)}^{2/3} \to 0, \text{ where } k \to \infty,
\]

Hence, we immediately get that if

\[
\omega_{H_{2/3}(G^2)} \left(\frac{1}{2n^3} f\right) = o \left(\frac{1}{n^{3/2}}\right), \text{ as } n \to \infty,
\]

then

\[
\|\mathcal{M}_n f - f\|_{H_{2/3}(G^2)} \to 0, \text{ as } n \to \infty.
\]

Now, prove part b) of Theorem 4.52. Let

\[
a_i(x^1, x^2) = 2^{2^i} \left(D_{2^{2i+1}}(x^1) - D_{2^{2i}}(x^1)\right) \left(D_{2^{2i+1}}(x^2) - D_{2^{2i}}(x^2)\right)
\]

and

\[
f_n(x^1, x^2) = \sum_{i=1}^{n} a_i(x^1, x^2) \frac{1}{2^{3i/2}}.
\]
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Since

\[ \sum_{i=1}^{\infty} \left( \frac{1}{2^{3i/2}} \right)^{2/3} < c < \infty \]

\[ S_{2^n, 2^n} a_k (x^1, x^2) = \begin{cases} a_k (x^1, x^2), & \text{if } 2^k \leq n, \\ 0, & \text{if } 2^k > n \end{cases} \]

and

\[ \text{supp } a_k = I_{2^k}, \]

\[ \int_{I_{2^k}} a_k d\mu = 0, \]

\[ \|a_k\|_{\infty} \leq \mu(\text{supp } a_k)^{-3/2}, \]

by using Lemma 4.44, we conclude that \( f \in H_{2/3} \).

On the other hand, if we apply Remark 4.39 we immediately get that

\[
\begin{align*}
 f - S_{2^n, 2^n} f &= (f^{(1)} - S_{2^n, 2^n} f^{(1)}), 
   \ldots, f^{(n)} - S_{2^n, 2^n} f^{(n)}, \ldots, f^{(n+k)} - S_{2^n, 2^n} f^{(n+k)}, \ldots \\
 &= (0, \ldots, 0, f^{(n+1)} - f^{(n)}, \ldots, f^{(n+k)} - f^{(n)}, \ldots) \\
 &= \left(0, \ldots, 0, \sum_{i=\log n+1}^{\log n+k} a_i (x), \ldots\right), \quad k \in \mathbb{N}_+,
\end{align*}
\]

Hence,

\[
\omega_{H_{2/3}\left(G^2\right)} \left( \frac{1}{2^n}, f \right) \leq \sum_{i=\lceil \log n \rceil}^{\infty} \frac{1}{2^{3i/2}} = O \left( \frac{1}{n^{3/2}} \right) \quad \text{as } k \to \infty.
\]

where \( \lceil \log n \rceil \) denotes integer part of \( \log n \).

Set

\[
n_{2^{A-2}} = 2^{4 \cdot 2^{A-2}} + 2^{4 \cdot 2^{A-2} - 4} + \ldots + 2^4 + 2^0
\]

\[
= 2^4 + 2^{A-4} + \ldots + 2^4 + 2^0.
\]

If we use Lemma 4.43 we get that

\[
\mathcal{M}_{n_{2^{k-2}}} f - f = f - S_{2^{k}} f
\]

\[
= \frac{2^k}{n_{2^{k-2}}} \mathcal{M}_{n_{2^k}} f + \frac{1}{n_{2^{k-2}}} \sum_{j=2^k+1}^{n_{2^{k-2}}} S_{j,j} f
\]

\[
- \frac{2^k}{n_{2^{k-2}}} f - \frac{n_{2^{k-2}-1}}{n_{2^{k-2}}} f.
\]
It is evident that
\[
\hat{f}(i, j) = \begin{cases} 
\frac{2^{nk}}{2^{nk/2}}, & (i, j) \in \left\{2^{2k}, ..., 2^{2k+1} - 1 \right\}, \quad k \in \mathbb{N} \\
0, & (i, j) \notin \bigcup_{k=0}^{\infty} \left\{2^{2k}, ..., 2^{2k+1} - 1 \right\}.
\end{cases} \tag{4.25}
\]

Let \(2^{2k} < j \leq n_{2k-1} \). Since \(w_{v+2^{2k}} = w_{2^{2k}} w_v\), when \(v < 2^{2k}\), if we apply (4.25) and first equality of Lemma 2.2 we obtain that

\[
S_{j,j} f(x_1, x_2) = S_{2^{2k}, 2^{2k}} f(x_1, x_2) + \sum_{v=2^{2k}}^{j-1} \sum_{s=2^{2k}}^{j-1} \hat{f}(v, s) w_{v,s}(x_1, x_2)
\]

\[
= S_{2^{2k}, 2^{2k}} f(x_1, x_2) + \frac{2^{3k/2}}{n_{2k-2}} \sum_{v=0}^{j-2^{2k} - 1} \sum_{s=0}^{j-2^{2k} - 1} w_{v+2^{2k}}(x_1) w_{s+2^{2k}}(x_2)
\]

\[
= S_{2^{2k}, 2^{2k}} f(x_1, x_2) + \frac{2^{2k} w_{2^{2k}}(x_1) w_{2^{2k}}(x_2)}{n_{2k-2} 2^{3k/2}} \sum_{j=1}^{n_{2k-2} - 1} D_{j,j}(x_1, x_2)
\]

Hence,

\[
\frac{1}{n_{2k-2}} \sum_{j=2^{2k+1}}^{n_{2k-2}} S_{j,j} f(x_1, x_2) = \frac{n_{2k-2} - 1}{n_{2k-2}} S_{2^{2k}, 2^{2k}} f(x_1, x_2) + \frac{2^{2k} w_{2^{2k}}(x_1) w_{2^{2k}}(x_2)}{n_{2k-2} 2^{3k/2}} \sum_{j=1}^{n_{2k-2} - 1} D_{j,j}(x_1, x_2)
\]

\[
= \frac{n_{2k-2} - 1}{n_{2k-2}} S_{2^{2k}, 2^{2k}} f(x_1, x_2) + \frac{2^{2k} w_{2^{2k}}(x_1) w_{2^{2k}}(x_2)}{n_{2k-1} 2^{3k/2}} \frac{n_{2k-2} - 1}{n_{2k-2}} K_{n_{2k-2} - 1}(x_1, x_2).
\]
By applying (4.24) we have that

\[
\|\mathcal{M}_{n_{2^k-2}} f - f \|^2/3 \geq \frac{c}{2^k} \|n_{2^k-2}^{-1} K_{n_{2^k-2}} \|^{2/3} \\
- \left( \frac{2^{2k}}{n_{2^k-2}} \right)^{2/3} \|M_{2^k} f - f \|^2/3 \\
- \left( \frac{n_{2^k-2}^{-1}}{n_{2^k-2}} \right)^{2/3} \|S_{2^k, 2^k} f - f \|^2/3 \\
\geq \frac{c}{2^k} \|n_{2^k-2}^{-1} K_{n_{2^k-2}} \|^{2/3} \\
- \|M_{2^k} f - f \|^2/3 \\
- \|S_{2^k, 2^k} f - f \|^2/3.
\]

Set

\[
x^1 \in I^{m,l}_{2^{k-2}} \\
= : I_{2^{k-2}} \left( 0, \ldots, 0, x^1_{4m} = 1, 0, \ldots, 0, x^1_{4d+1}, \ldots, x^1_{2^{k-2}-1} \right)
\]

and

\[
x^2 \in J^{l,q}_{2^{k-2}} \\
= : I_{2^{k-2}} \left( 0, \ldots, 0, x^2_{4d} = 1, x^1_{4d+1}, \ldots, x^1_{4q-1}, 1 - x^1_{4q}, x^2_{4q+1}, \ldots, x^2_{2^{k-2}-1} \right),
\]

According to Lemma 4.43 we get that

\[
n_{2^{k-2}}^{-1} \left| K_{2^{k-2}} \left( x^1, x^2 \right) \right| \geq 2^{4q+4d+4m-3}.
\]

Hence,

\[
\int_G \left( n_{2^{k-2}-1} \left| K_{n_{2^{k-2}-2}} \left( x^1, x^2 \right) \right| \right)^{2/3} d\mu(x^1, x^2) \\
\geq c \sum_{m=1}^{2^{k-2}-3} \sum_{l=1}^{2^{k-2}-2} \sum_{q=1}^{2^{k-2}-2} \sum_{x^1_{4d+1}=0}^{1} \sum_{x^1_{2^{k-2}-1}=0}^{1} \sum_{x^2_{2^{k-2}-1}=0}^{1} \\
\int_{I^{m,l}_{2^{k-2}}} \left( n_{2^{k-2}-1} \left| K_{n_{2^{k-2}-2}} \left( x^1, x^2 \right) \right| \right)^{2/3} d\mu(x^1, x^2) \\
\geq c \sum_{m=1}^{2^{k-2}-3} \sum_{l=1}^{2^{k-2}-2} \sum_{q=1}^{2^{k-2}-2} \sum_{x^1_{4d+1}=0}^{1} \sum_{x^1_{2^{k-2}-1}=0}^{1} \sum_{x^2_{2^{k-2}-1}=0}^{1} \mu \left( I^{m,l}_{2^{k-2}} \right) \left( 8q+8l+8m \right)/3
\]
\[ \geq c \sum_{m=1}^{2^{k-2}-3} \sum_{l=m+1}^{2^{k-2}-2} \sum_{q=l+1}^{2^{k-2}-4l} \left( 2 \frac{(8q+8l+8m)}{3} 2^{2k-2-4l/3} 2^{2k-2-4q/3} \right)^2 \]

\[ \geq c \sum_{m=1}^{2^{k-2}-3} 2^{8m/3} \sum_{l=m+1}^{2^{k-2}-2} 2^{-4l/3} \sum_{q=l+1}^{2^{k-2}-4q/3} \]

\[ \geq c \sum_{m=1}^{2^{k-2}-3} 1 \geq c2^k. \]

By combining (1.17), (1.21) and (4.26) we can conclude that

\[ \limsup_{k \to \infty} \| M_{n_{2^k-2}} f - f \|_{2/3} \geq c > 0. \]

Theorem 4.52 is proved.
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