A NOTE ON THE SMOOTHNESS OF THE MINKOWSKI FUNCTION

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Abstract. The Minkowski function is a crucial tool used in the study of balanced domains and, more generally, quasi-balanced domains in several complex variables. If a quasi-balanced domain is bounded and pseudoconvex then it is well-known that its Minkowski function is plurisubharmonic. In this short note, we prove that under the additional assumption of smoothness of the boundary, the Minkowski function of a quasi-balanced domain is in fact smooth away from the origin. This allows us to construct a smooth plurisubharmonic defining function for such domains. Our result is new even in the case of balanced domains.

1. Introduction

The study of holomorphic mappings between balanced and quasi-balanced domains pose an interesting challenge. As the automorphism group contains the circle, such domains possess symmetry that often confers strong rigidity on holomorphic mappings between these domains. Indeed, a classical result of Cartan exploits the circle action to show that any automorphism of a bounded balanced domain fixing the origin must be linear. One of the key tools that facilitate the study of balanced and quasi-balanced domains is the Minkowski function. Several generalizations of Cartan’s theorem are now known (Bel82, BP00, Kos14, YZ17), and many of them use the Minkowski function as a central tool in the proofs. The demand of the presence of a circle action is also not too severe and there are several interesting classes of domains that are quasi-balanced. For instance, the symmetrized polydisk and related domains are quasi-balanced domains that have been extensively studied using the Minkowski function (see Nik06, Kos11).

Let \( p_1, p_2, \ldots, p_n \) be relatively prime positive integers. We say that a domain \( D \subset \mathbb{C}^n \) is \((p_1, p_2, \ldots, p_n)\)-balanced (quasi-balanced) if

\[
\lambda \cdot z \in D \quad \forall \lambda \in \mathbb{D} \quad \forall z \in D,
\]

where \( \mathbb{D} \) is the closed unit disk in \( \mathbb{C} \) and for \( z = (z_1, z_2, \ldots, z_n) \in D \), we define

\[
\lambda \cdot z := (\lambda^{p_1} z_1, \lambda^{p_2} z_2, \ldots, \lambda^{p_n} z_n).
\]

If \( p_1 = p_2 = \cdots = p_n = 1 \) above, then we say \( D \) is a balanced domain (also known as a complete circular domain in the literature).

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Given a \((p_1, p_2, \ldots, p_n)\)-balanced domain \(D \subset \mathbb{C}^n\), we define the Minkowski function \(h_D : \mathbb{C}^n \to \mathbb{C}\)

\[
h_D(z) := \inf\{t > 0 : \frac{1}{t} \cdot z \in D\}.
\]

Clearly \(D = \{z \in \mathbb{C}^n : h_D(z) < 1\}\) and \(h_D(\lambda \cdot z) = |\lambda| h_D(z)\). It also turns out that \(h_D\) is plurisubharmonic if \(D\) is additionally pseudoconvex. This fact has been a crucial ingredient in several results on balanced domains; see [Ham00, JP13], for instance.

One natural question that seems to be unanswered (to the best of the authors’ knowledge) in the literature is the following:

Is the Minkowski function of a smoothly bounded pseudoconvex quasi-balanced domain a smooth function near the boundary?

In fact, we found a remark in [GK03, p. 190], with a reference to Hamada’s paper [Ham00], stating that the answer to the above question is no if the domain has only a \(C^1\)-boundary. That the Minkowski function of a balanced and bounded pseudoconvex domain with \(C^1\)-smooth plurisubharmonic defining function is \(C^1\)-smooth on \(\mathbb{C}^n \setminus \{0\}\) has already been established in [Ham00, Proposition 1]. Using the recent work [NZZ17], we are able to prove smoothness of the Minkowski function on \(\mathbb{C}^n \setminus \{0\}\) for any smoothly bounded quasi-balanced domains. The main result of this paper is the following

**Theorem 1.** Let \(D \subset \mathbb{C}^n\) be a smoothly bounded quasi-balanced pseudoconvex domain. Then the Minkowski function \(h_D\) is \(C^\infty\)-smooth on \(\mathbb{C}^n \setminus \{0\}\). Furthermore, the function \(r(z) := h_D(z) - 1\) is a plurisubharmonic defining function for \(D\).

**Remark 2.** By a smoothly bounded domain, we shall mean a bounded domain whose boundary is \(C^\infty\)-smooth.

**Remark 3.** The analogue of the above result for convex domains is well-known. The reader is referred to [KP99, Section 6.3] for details.

## 2. Supporting results

Before we give the proof of Theorem 1, we first give a brief overview of the necessary tools.

We shall now consider the setting in [NZZ17, p. 18, p. 23]. Let \(D \subset \mathbb{C}^n\) be a smoothly bounded domain and let \(G \subset \text{Aut}(D) \cap C^\infty(\overline{D})\) be a compact Lie subgroup of \(\text{Aut}(D)\) in the compact open topology. Consider a continuous representation \(\rho : G \to GL(\mathbb{C}^n)\) of \(G\) and the set

\[
\mathcal{O}(\mathbb{C}^n)^G := \{f \in \mathcal{O}(\mathbb{C}^n) : f \circ \rho(g) = f \text{ for all } g \in G\}
\]
called the set of \(G\)-invariant entire functions.

A domain \(D\) is said to be \(G\)-invariant if \(\rho(g) \cdot D = D\) for all \(g \in G\). We will say that \(G\) acts transversely on \(D\) if for each \(z_0 \in \partial D\) the image of the tangent map \(d\Psi_{z_0} : T_{z_0}G \to T_{z_0}\partial D\) associated to the map \(\Psi_{z_0} : G \to \partial D\) given by \(g \mapsto g(z_0)\), is not contained in \(T_{z_0}\partial D\), the complex tangent space to \(\partial D\) at \(z_0\). We have the following
Let $\psi$ be a defining function for $D$. Consider the representation of the compact lie group $\mathbb{S}^1$ given by

$$\rho(\lambda)(z) = \lambda \cdot z$$

where $\lambda \in \mathbb{S}^1$.

**Proposition 5.** Under the above action, $\mathcal{O}(\mathbb{C}^n)^{\mathbb{S}^1} = \mathbb{C}$.

**Proof.** Consider $f \in \mathcal{O}(\mathbb{C}^n)$ such that $f(\lambda \cdot z) = f(z)$ for every $\lambda \in \mathbb{S}^1$ and for all $z \in D$. Fix $z \in \mathbb{C}^n$ and define a function $g_z : \mathbb{C} \to \mathbb{C}$ given by $g_z(\lambda) = f(\lambda \cdot z)$. Then $g$ is a holomorphic function that is constant on $\mathbb{S}^1$ and hence $g_z \equiv g(0)$. Since our choice of $z$ was arbitrary, we have $f(z) = f(0)$. The constant functions clearly belong to $\mathcal{O}(\mathbb{C}^n)^{\mathbb{S}^1}$. □

That $D$ is $\mathbb{S}^1$-invariant is a direct consequence of the fact that $D$ is $(p_1, p_2, \ldots, p_n)$-balanced. Thus in our case, we can conclude the following

**Corollary 6.** Under the hypotheses on $D$ as in Theorem 1 for each $\xi = (\xi_1, \ldots, \xi_n) \in \partial D$, the vector

$$(ip_1\xi_1, \ldots, ip_n\xi_n) \notin T^C_\xi \partial D.$$  

**Proof.** With $\Psi_\xi : \mathbb{S}^1 \to \partial D$ given by $\Psi_\xi(\lambda) = \lambda \cdot \xi$, the evaluation of the derivative map $d\Psi_\xi(1) = (ip_1\xi_1, \ldots, ip_n\xi_n) \in T_\xi \partial D$. By Result 3, $d\Psi_\xi(1) \notin T^C_\xi \partial D$ as otherwise $d\Psi_\xi(T_{\mathbb{S}^1}) \subset T^C_\xi \partial D$. □

We will use the following version of Hopf’s lemma in the proof of Theorem 1.

**Lemma 7** (Lemma 3, p. 177, [KG89]). Let $D \subset \mathbb{C}^n$ be a smoothly bounded domain and let $r$ be a negative plurisubharmonic function defined on $D$. Then there exists a constant $c > 0$ such that $|r(z)| > c \cdot \text{dist}(z, \partial D)$.

3. PROOF OF THEOREM 1

Let $\psi$ be a defining function for $D$. Consider the map $g \in \mathcal{C}^\infty(\mathbb{C}^n \times \mathbb{R} \setminus \{0\})$ given by

$$g(z, t) := \psi \left( \frac{1}{t} \cdot z \right)$$

Observe that $g(z, \eta_D(z)) = 0$. Let us fix a point $z_0 \in \mathbb{C}^n \setminus \{0\}$. We shall show that $\frac{\partial g}{\partial n}(z_0, \eta_D(z_0)) \neq 0$.

Let us denote the coordinates of $z_0$ by $(z_1, \ldots, z_n)$. Then the point $\xi = (\xi_1, \ldots, \xi_n)$ defined to be $\frac{1}{\eta_D(z_0)} \cdot z_0$ belongs to $\partial D$. A direct calculation gives us that

$$\frac{\partial g}{\partial n}(z_0, \eta_D(z_0)) = \frac{-1}{\eta_D(z_0)} \left( \frac{\partial \psi}{\partial x_1}, \ldots, \frac{\partial \psi}{\partial x_n} \right) \xi \cdot \left( \begin{array}{c} p_1\xi_1 \\ p_2\xi_2 \\ \vdots \\ p_n\xi_n \end{array} \right)$$
If $\frac{\partial g}{\partial t}(z_0, h_D(z_0)) = 0$, then $(p_1\xi_1, \ldots, p_n\xi_n) \in T_\xi \partial D$. Consider the curve
$$\gamma(\theta) = e^{i\theta} \cdot \xi$$
in $\partial D$. Then the corresponding tangent vector $(ip_1\xi_1, \ldots, ip_n\xi_n) \in T_\xi \partial D$ and hence is in the complex tangent space $T_\xi \partial D$ which is a contradiction to Corollary 6. Now by the implicit function theorem, $h_D$ is $C^\infty$-smooth on $\mathbb{C}^n \setminus \{0\}$.

We shall now prove that $r$ is a defining function. We are left with observing that $dr \neq 0$ on $\partial D$. It is easy to see that the normal derivative at every point on the boundary $\partial D$ is bounded below by the constant $c$ by an application of Hopf’s lemma (Lemma 7). Hence $dr \neq 0$ on $\partial D$. □

Our result implies that the main results in [Ham00, HK01] on balanced domains with $C^1$-smooth plurisubharmonic defining function also hold for smoothly bounded balanced pseudoconvex domains.

References

[Bel82] Steven R. Bell, *Proper holomorphic mappings between circular domains*, Comment. Math. Helv. 57 (1982), no. 4, 532–538.

[BP00] François Berteloot and Giorgio Patrizio, *A Cartan theorem for proper holomorphic mappings of complete circular domains*, Adv. Math. 153 (2000), no. 2, 342–352.

[GK03] Ian Graham and Gabriela Kohr, *Geometric function theory in one and higher dimensions*, New York, NY: Marcel Dekker, 2003.

[Ham00] Hidetaka Hamada, *Starlike mappings on bounded balanced domains with $C^1$-plurisubharmonic defining functions*, Pacific J. Math. 194 (2000), no. 2, 359–371.

[HK01] Hidetaka Hamada and Gabriela Kohr, *Some necessary and sufficient conditions for convexity on bounded balanced pseudoconvex domains in $\mathbb{C}^n$*, Complex Variables, Theory Appl. 45 (2000), no. 2, 101–115.

[JP13] Marek Jarnicki and Peter Pflug, *Invariant distances and metrics in complex analysis. 2nd extended ed.*, 2nd extended ed. ed., Berlin: Walter de Gruyter, 2013.

[KG89] G.M. Khenkin and R.V. Gamkrelidze (eds.), *Several complex variables III. Geometric function theory*. Berlin etc.: Springer Verlag, 1989.

[Kos11] Łukasz Kosiński, *Geometry of quasi-circular domains and applications to tetra-block*, Proceedings of the American Mathematical Society 139 (2011), no. 2, 559–569.

[Kos14] Łukasz Kosiński, *Holomorphic mappings preserving Minkowski functionals.*, J. Math. Anal. Appl. 409 (2014), no. 2, 643–648.

[KP99] Steven G. Krantz and Harold R. Parks, *The geometry of domains in space*, Birkhäuser Advanced Texts: Basler Lehrbücher., Birkhäuser Boston, Inc., Boston, MA, 1999.

[Nik06] Nikolai Nikolov, *The symmetrized polydisc cannot be exhausted by domains biholomorphic to convex domains.*, Ann. Pol. Math. 88 (2006), no. 3, 279–283.

[NZZ17] Jiafu Ning, Huiping Zhang, and Xiangyu Zhou, *Proper holomorphic mappings between invariant domains in $\mathbb{C}^n$*, Trans. Amer. Math. Soc. 369 (2017), no. 1, 517–536.

[YZ17] Atsushi Yamamori and Liyou Zhang, *On origin-preserving automorphisms of quasi-circular domains*, The Journal of Geometric Analysis (2017), 1–13.