Functional Schrödinger Representations of Holstein-Primakoff Boson and Slave-Boson Theories for Heisenberg Antiferromagnets

Sul-Ah Ahn, Kwangyl Park and Sung-Ho Suck Salk

Department of Physics, Pohang University of Science and Technology, Pohang 790-784, Korea

(March 24, 2022)

Abstract

We present functional Schrödinger representations of Holstein-Primakoff boson and slave-boson theories for the Heisenberg Hamiltonian and the $t - J$ Hamiltonian respectively. Based on these representations we obtain the dispersion relations of magnons for two dimensional antiferromagnets. By applying the functional Schrödinger representation of the Holstein-Primakoff boson theory to the Heisenberg Hamiltonian, the exchange energy is correctly predicted and the self-energy of quasi-hole is obtained. From the use of the functional Schrödinger representation of the $t - J$ Hamiltonian it is shown that at half-filling the dispersion relation obtained from the slave-boson theory leads to that obtained from the Holstein-Primakoff boson approach.

PACS numbers: 71.10.Fd, 71.27.+a, 75.30.Ds
I. INTRODUCTION

Although widely used in the area of high energy physics, the functional Schrödinger picture theory is relatively new in condensed matter physics. Lately, only a limited number of application to condensed matter physics appeared in the literature. Earlier we treated the $t - J$ Hamiltonian by introducing a slave-boson approach to the functional Schrödinger picture (FSP) theory for the two-dimensional systems of antiferromagnetically correlated systems. The Holstein-Primakoff boson representation is often used to describe the broken symmetry phases of the quantum Heisenberg antiferromagnet. Lately Chang introduced a generalized Holstein-Primakoff representation of the $t - J$ model Hamiltonian in order to describe a higher order (second order) effect on spin waves and antiferromagnetic spin polarons by allowing a systematic perturbative expansion. Here we present the functional Schrödinger representations of both the Holstein-Primakoff boson and slave-boson theories for the Heisenberg Hamiltonian and the $t - J$ Hamiltonian respectively. Based on these representations we derive magnon dispersion relations and present comparison between the two approaches.

II. HOLSTEIN-PRIMAKOFF BOSON THEORY OF HEISENBERG HAMILTONIAN BY FUNCTIONAL SCHRÖDINGER REPRESENTATION; ANTIFERROMAGNETIC MAGNON

To allow symbol definitions for later use, here we choose a brief review on a generalized approach of perturbatively treated Heisenberg Hamiltonian based on a Holstein-Primakoff transformation. The Hilbert space of present interest is spanned by the states, $|a_1\rangle \otimes |a_2\rangle \otimes \cdots \otimes |a_j\rangle \otimes \cdots \otimes |a_N\rangle$, where $|a_j\rangle \in \{ c_{j\uparrow}^\dagger |0\rangle_j, c_{j\downarrow}^\dagger |0\rangle_j, |0\rangle_j \}$ with $c_{j\sigma}|0\rangle_j = 0$. The vacuum state is given by $|0\rangle = \otimes |0\rangle_j$ which satisfies $c_{j\sigma}|0\rangle = 0$. $j$ is the site index and $\sigma$, the spin index. $N$ is the total number of lattice sites. With the local Hubbard operators $X_{j}^{ab} = |a_j\rangle \langle b_j|$, the Heisenberg Hamiltonian for the two-dimensional system of antiferromagnetically correlated...
electrons is written

\[ H_J = \frac{J}{2} \sum_{\langle i,j \rangle} \left( X_i^{\sigma-} X_j^{\sigma-} - X_i^{\sigma+} X_j^{\sigma+} \right), \]  

(1)

where \( J \) is the Heisenberg coupling constant. \( \langle ij \rangle \) stands for summation only over nearest neighbours. The above Hamiltonian includes the contribution of the \( \frac{1}{4} n_i n_j \) term that appears in the usual \( t - J \) Hamiltonian, where \( n_i = \sum_\sigma c_{i\sigma}^\dagger c_{i\sigma} \). Introducing two commuting boson(magnon) operators (\( a_j, b_j \)) and an anticommuting fermion(hole) operator \( f_j \) for a bipartite lattice made of sublattices, \( A \) and \( B \), two sets of Holstein-Primakoff representations for the local Hubbard operators are given in the table below.

| Hubbard operator | \( A \) sublattice | \( B \) sublattice |
|-----------------|---------------------|---------------------|
| \( X_j^{00} \)  | \( f_j^\dagger f_j \) | \( f_j^\dagger f_j \) |
| \( X_j^{01} \)  | \( f_j^\dagger a_j \) | \( \sqrt{2s} f_j^\dagger \sqrt{1 - \frac{1}{2s}} (b_j^\dagger b_j + f_j^\dagger f_j) \) |
| \( X_j^{10} \)  | \( a_j^\dagger f_j \) | \( \sqrt{2s} \sqrt{1 - \frac{1}{2s}} (b_j^\dagger b_j + f_j^\dagger f_j) b_j \) |
| \( X_j^{01} \)  | \( \sqrt{2s} f_j^\dagger \sqrt{1 - \frac{1}{2s}} (a_j^\dagger a_j + f_j^\dagger f_j) \) | \( f_j^\dagger b_j \) |
| \( X_j^{10} \)  | \( \sqrt{2s} \sqrt{1 - \frac{1}{2s}} (a_j^\dagger a_j + f_j^\dagger f_j) f_j \) | \( b_j^\dagger f_j \) |
| \( X_j^{11} \)  | \( a_j^\dagger a_j \) | \( 2s \left[ 1 - \frac{1}{2s} \left( b_j^\dagger b_j + f_j^\dagger f_j \right) \right] \) |
| \( X_j^{11} \)  | \( 2s \left[ 1 - \frac{1}{2s} \left( a_j^\dagger a_j + f_j^\dagger f_j \right) \right] b_j \) | \( b_j^\dagger b_j \) |

For sublattice \( A \), \( a_j(a_j^\dagger) \) is the annihilation(creation) operator of spin-up boson (magnon excitation) and \( f_j(f_j^\dagger) \), the annihilation (creation) operator of spinless fermion (hole excitation) at site \( j \). \( s \) is the spin quantum number. Likewise, for sublattice \( B \), \( b_j(b_j^\dagger) \) is the annihilation(creation) operator of spin-down boson(magnon excitation) and \( f_j(f_j^\dagger) \), the annihilation(creation) operator of spinless fermion(hole excitation) at site \( j \).

Using the local Hubbard operators in the table above, we derive from Eq.(1),

\[ H_J = J \sum_{\langle i,j \rangle} \left\{ sa_i^\dagger \sqrt{1 - \frac{1}{2s} (a_i^\dagger a_i + f_i^\dagger f_i) b_j^\dagger} \sqrt{1 - \frac{1}{2s} (b_j^\dagger b_j + f_j^\dagger f_j)} - \frac{1}{2} a_i^\dagger a_i b_j^\dagger b_j + s \sqrt{1 - \frac{1}{2s} (a_i^\dagger a_i + f_i^\dagger f_i) a_i} \sqrt{1 - \frac{1}{2} (b_j^\dagger b_j + f_j^\dagger f_j)} b_j \right\} \]
The transformed Heisenberg Hamiltonian above is perturbatively treated by using the following expansions:

\[
\sqrt{2s} \left[ 1 - \frac{1}{2s} \left( a_i^\dagger a_i + f_i^\dagger f_i \right) \right] \left[ 1 - \frac{1}{2s} \left( b_j^\dagger b_j + f_j^\dagger f_j \right) \right]
\]

(2)

We consider terms up to order \( \frac{1}{2s} \) in Eq. (3) for the evaluation of Eq. (4). By first taking the Fourier transformation of Eq. (2) and then the Bogoliubov transformation, we obtain the following four terms; 1) the free Hamiltonian \( H = 0 \), the total number of spins. 2) The magnon-magnon interaction, \( H_{M-M} \),

\[
H_{M-M} = -\frac{J_z}{2} \sum_{k_1, k_2} \left[ \sum_{k_1, k_2} \left( \frac{\omega_k}{\omega_{k_1} \omega_{k_2}} \right) \left( c_{k_1}^\dagger c_{k_1} c_{k_2}^\dagger c_{k_2} + d_{k_1}^\dagger d_{k_1} d_{k_2}^\dagger d_{k_2} \right) \right]
\]

(5)

3) the magnon-hole interaction, \( H_{M-H} \),

\[
H_{M-H} = -\frac{J_z}{2} \sum_{k_1, k_2} \frac{\omega_{k_2}}{\omega_{k_1} \omega_{k_2}} c_{k_1}^\dagger c_{k_2} \left( c_{k_1}^\dagger d_{k_2} + d_{k_2}^\dagger c_{k_1} \right),
\]

(6)

and 4) the hole-hole interaction, \( H_{H-H} \),

\[
H_{H-H} = -\frac{J_z}{2} \sum_{k_1, k_2, q} \gamma_q f_{k_1, k_2}^\dagger f_{k_1, k_2}^\dagger f_{k_2, k_2}^\dagger f_{k_2, k_2}^\dagger,
\]

(7)

where \( \omega_k = Jsz \sqrt{1 - \gamma_k^2} \), \( \gamma_k \equiv \frac{1}{2} \sum_{\delta} e^{i k \cdot \delta} = \gamma_{-k} \) with \( z \), the number of nearest neighbors.

For a square lattice we have \( \gamma_k = \frac{1}{2} \left[ \cos(k_x a) + \cos(k_y a) \right] \). The last two terms in Eq. (6) are contributed from the \( \frac{1}{4} n_i n_j \) term that appears in the original \( t-J \) Hamiltonian. This term was not considered in the work of Chang. \( c_k (c_k^\dagger) \) and \( d_k (d_k^\dagger) \) are the annihilation(creation)
operators of the chargeless bosons (quasimagnons) of momentum \( k \), in association with two sublattices \( A \) and \( B \) respectively. \( f_k(f^\dagger_k) \) is the annihilation (creation) operator of a spinless hole (holon) of momentum \( k \).

For the application of the Holstein-Primakoff theory to functional Schrödinger picture (FSP) representation, one has to choose the FSP operator of the charged fermion. On the other hand the charge-neutral fermion FSP operator is used in the slave-boson theory.

In the following, we derive the ground state energy and the dispersion energy of magnon for the two-dimensional systems of antiferromagnetically correlated electrons at and near half filling.

Similarly to the case of relativistic FSP theory we define, for chargeless scalar bosons (magnons),

\[
c_k = \frac{1}{\sqrt{2}} \left( \phi_k^c + \frac{\delta}{\delta \phi_k^c} \right), \quad c^\dagger_k = \frac{1}{\sqrt{2}} \left( \phi_k^c - \frac{\delta}{\delta \phi_k^c} \right),
\]

\[
d_k = \frac{1}{\sqrt{2}} \left( \phi_k^d + \frac{\delta}{\delta \phi_k^d} \right), \quad d^\dagger_k = \frac{1}{\sqrt{2}} \left( \phi_k^d - \frac{\delta}{\delta \phi_k^d} \right),
\]

with \( \phi_k \), the scalar field variable in momentum space, and for charged fermions (holes),

\[
f_k = \frac{1}{\sqrt{2}} \left( u_k + \frac{\delta}{\delta u_k} \right), \quad f^\dagger_k = \frac{1}{\sqrt{2}} \left( u_k^\dagger - \frac{\delta}{\delta u_k} \right),
\]

with \( u_k \), the Grassmann field variable in momentum space. The field variables \( \phi \) and \( u \) defined above satisfy the following commutation and anticommutation relations (see Appendix [A]),

\[
\{ c_k, c^\dagger_k \} = \delta_{k,k'}, \quad \{ d_k, d^\dagger_k \} = \delta_{k,k'}, \quad \{ f_k, f^\dagger_k \} = \delta_{k,k'}.
\]

The functional Schrödinger equation is written,

\[
H[\phi^c, \phi^d, u]\Psi_\Omega[\phi^c, \phi^d, u] = E\Psi_\Omega[\phi^c, \phi^d, u]
\]

where \( \Psi_\Omega \) is the ground state for the square lattice of antiferromagnetic spin order at or near half-filling,

\[
\Psi_\Omega[\phi^c, \phi^d, u] = N_\Omega \exp \left[ -\frac{1}{2} \sum_k \left( \phi^c_{-k} \Omega_k^c \phi^c_k + \phi^d_{-k} \Omega_k^d \phi^d_k \right) + \frac{1}{2} \sum_{k,k'} u^\dagger_{k,k'} \Omega^f_{k,k'} u_{k'} \right]
\]
where $N\Omega$ is a proper normalization constant. By taking the variation of the ground state energy with respect to the Gaussian exponents $\Omega^c$, $\Omega^d$ and $\Omega^f$,

$$E = \frac{\langle \Omega | H | \Omega \rangle}{\langle \Omega | \Omega \rangle} = E[\Omega^c, \Omega^d, \Omega^f]$$

we readily obtain $\Omega^c = \Omega^d = 1$ and $\Omega^f = -1$. For the sake of brevity, the summation symbol $\sum$ is omitted in the above expression (11).

We find from the use of Eq. (8) through (12) that the ground state energy of the total system is (see Appendix A for details)

$$E = Jsz \sum_k \left( \sqrt{1 - \gamma_k^2} + 2D_{k,k} \right) - Js(s + 1)zN$$

$$+ \frac{Jz}{2} \sum_{k_1,k_2,q} \gamma_q (D_{k_2,k_1-q}D_{k_1,k_2+q} - D_{k_1,k_1-q}D_{k_2,k_2+q})$$

(13)

where $D_{k,k'} = \frac{1}{2} \left[ \left( 1 + \Omega^f \right) \left( \Omega^f + \bar{\Omega}^f \right) \left( 1 + \Omega^f \right) \left( \Omega^f + \bar{\Omega}^f \right) \delta_{k,k'} \right] k,k'$. It is of note that at half-filling $\sum_k D_{k,k} = 0$. We readily note from Eq. (13) above that the energy of free magnons at or near half-filling is given by

$$E_0 = \sum_k \omega_k^0 - Js(s + 1)zN,$$

(14.1)

where $\omega_k^0$ is the dispersion energy of magnons,

$$\omega_k^0 = Jsz \sqrt{1 - \gamma_k^2},$$

(14.b)

for a square lattice, the dispersion energy is simply,

$$\omega_k^0 = 2J \sqrt{1 - \gamma_k^2}$$

(14.c)

with $s = \frac{1}{2}$. Eq. (14.1) represents the familiar zero-point energy of the antiferromagnetic system. The last term in Eq. (14.1) is the quantum contribution which is essential for lowering energy.
III. HOLSTEIN-PRIMAKOFF BOSON APPROACH OF $T-J$ HAMILTONIAN
BY FUNCTIONAL SCHRODINGER REPRESENTATION;
ANTIFERROMAGNETIC SPIN POLARON

Based on the $t-J$ Hamiltonian, we write the antiferromagnetic spin polaron Hamiltonian in momentum space:

$$H_{\text{AFP}} = \sum_q \omega^0_q \left[ 1 - \frac{1}{2s} \frac{2}{N} \sum_{q'} \left( \frac{\omega^0_{q'}}{J_{sz}} - 1 \right) \right] \left( c^\dagger_q c_q + d^\dagger_k d_k \right) - \sqrt{2s} t_{sz} \sqrt{\frac{2}{N}}$$

$$\times \left( 1 - \frac{1}{4s} \frac{2}{N} \sum_{k_1} \sinh^2 \theta_{k_1} \right) \left\{ \sum_{k,q} f_k f_{k-q} \left[ g_{cc}(k,q) c^\dagger_q c_q + g_{dd}(k,q) d_q \right] + \text{H.c.} \right\}$$

$$+ J_{sz} \sum_k \left( \frac{\omega^0_k}{J_{sz}} - 1 - 2 f^\dagger_k f_k \right) - J s_{sz}^2 N.$$  (15)

Here $\omega^0_q = J_{sz} \sqrt{1 - \gamma^2_q}$. $t$ is the hopping integral. $\theta_k$ is the mixing angle defined by $\tanh 2\theta_k = \gamma_k$. $g_{cc}$ and $g_{dd}$ are the coupling functions given by

$$g_{cc}(k,q) = \gamma_{k-q} \cosh \theta_q - \gamma_k \sinh \theta_q$$  (16.a),

$$g_{dd}(k,q) = -\gamma_{k-q} \sinh \theta_q + \gamma_k \cosh \theta_q$$  (16.b).

The first and the last two terms in Eq. (15) above are identical to the terms that appear in Eq. (13). It is reminded that the last two terms were not considered in Ref. (4).

In order to properly estimate the influence of fermi-bose (holon-magnon) coupling, we now introduce the shifted wave functional associated with the scalar field,

$$\Psi_\Omega[\phi^c, \phi^d, u] = N_\Omega \exp \left[ -\frac{1}{2} \left( \Omega^c_k (\phi^c_k - \phi^c_{k0})^2 + \Omega^d_k (\phi^d_k - \phi^d_{k0})^2 \right) + \frac{1}{2} u^\dagger_k \Omega f_k u_{k'} \right].$$  (17)

The scalar field $\phi$ is shifted by a constant $\phi_0$. With the use of Eqs. (8) and (17) for Eq. (14), we obtain the ground state energy, $E = \langle H_{\text{AFP}} \rangle / \langle \Omega | \Omega \rangle$,

$$E = \langle H_{\text{AFP}} \rangle / \langle \Omega | \Omega \rangle$$

$$= \sum_q \omega^0_q \left[ 1 - \frac{1}{2s} \frac{2}{N} \sum_{q'} \left( \frac{\omega^0_{q'}}{J_{sz}} - 1 \right) \right] \left\{ \frac{1}{4} \left( 1 - \Omega^c_{q} \right) \Omega^c_{q-1} - \frac{1}{2} \left( 1 - \Omega^c_{q} \right) + \frac{\phi^c_{q0} \phi^c_{q0}}{2} \right\}$$

$$+ \left\{ \frac{1}{4} \left( 1 - \Omega^d_{q} \right) \Omega^d_{q-1} - \frac{1}{2} \left( 1 - \Omega^d_{q} \right) + \frac{\phi^d_{q0} \phi^d_{q0}}{2} \right\} + J s_{sz} \sum_k \left( \frac{\omega^0_k}{J_{sz}} - 1 - 2 D_{k,k} \right) - J s_{sz}^2 N$$

7
We note that

\[ \delta E \]

\[ = \sum_k \omega_k \left[ 1 - \frac{1}{2sN} \sum_{q'} \left( \frac{\omega_{q'}^0}{J_{sz}} - 1 \right) \phi_{q0}^0 - \sqrt{2s} \sqrt{1 - \frac{1}{2sN} \sum_{q'} \sin^2 \theta_{q'}} \right] \]

\[ \times \sum_{k,p} (D_{k,k-p} + \delta_{k,k-p}) g_{cc}(k,p) \frac{1}{\sqrt{2}} \delta_{p,q} \]

\[ = \omega_q \left[ 1 - \frac{1}{2sN} \sum_{q'} \left( \frac{\omega_{q'}^0}{J_{sz}} - 1 \right) \right] \phi_{q0}^0 - \sqrt{2s} \sqrt{1 - \frac{1}{2sN} \sum_{q'} \sin^2 \theta_{q'}} \]

\[ \times \sum_{k,q} (D_{k,k-q} + \delta_{k,k-q}) g_{cc}(k,q) \frac{1}{\sqrt{2}} \]

\[ = 0 \quad (19) \]

Thus from the above expression we readily find the vacuum expectation value, 

\[ \phi_{q0}^c = \frac{\sqrt{s} \sqrt{1 - \frac{1}{2sN} \sum_{q'} \sin^2 \theta_{q'}}}{\omega_q^0 \left[ 1 - \frac{1}{2sN} \sum_{q'} \left( \frac{\omega_{q'}^0}{J_{sz}} - 1 \right) \right]} \sum_{k'} D_{k',k'-q} g_{cc}(k',q). \quad (20) \]

We note that \( D_{k',k'-q} = 0 \) for \( |q| \neq 0 \) and \( \Omega_q^c = \Omega_q^d = 1. \) Realizing the vacuum to vacuum transition, we obtain the ground state energy from the insertion of Eq. (20) into Eq. (18), 

\[ E = J_{sz} \sum_q \sqrt{1 - \gamma_q^2} - J_s(s+1)zN + \sum_k \Sigma^\Omega(k,\omega) \quad (21) \]

where 

\[ \Sigma^\Omega(k,\omega) = -\frac{st^2z^2}{N} \left[ \frac{\left( 1 - \frac{1}{2sN} \sum_{q'} \sin^2 \theta_{q'} \right)^2}{\left[ 1 - \frac{1}{2sN} \sum_{q'} \left( \frac{\omega_{q'}^0}{J_{sz}} - 1 \right) \right]^2} \right. \]

\[ \times \left. \sum_q \left( D_{k,k-q}^2 g_{cc}^2(k,q) + D_{k,k-q}^2 g_{dd}^2(k,q) \right) / \omega_q^0 \right] \quad (22) \]
The first two terms in Eq. (21) are identical to the expressions of Eq. (14.a). It is seen from Eqs. (21) and (22) that the ground state energy is lowered by the self-energy of the holon (spinless hole) which is contributed from coupling to magnons.

IV. SLAVE-BOSON APPROACH OF T − J HAMILTONIAN BY FUNCTIONAL SCHRÖDINGER REPRESENTATION

Here the slave-boson functional Schrödinger representation of the $t − J$ Hamiltonian will be discussed for the hole doped system of antiferromagnetically correlated electrons. In this section emphasis is placed on a rigorous derivation of dispersion relation and its comparison with the Holstein-Primakoff theory of Heisenberg Hamiltonian in the limit of half-filling.

The $t − J$ Hamiltonian in the slave-boson representation is given by

$$H = -t \sum_{\langle ij \rangle \sigma} b_i^\dagger f_j^\dagger f_j f_i^\sigma - \frac{J}{2} \sum_{\langle ij \rangle \sigma} (f_{i\sigma}^\dagger f_{j-\sigma}^\dagger f_{j\sigma} - f_{i\sigma}^\dagger f_{j-\sigma}^\dagger f_{j\sigma} f_{i-\sigma}) - \mu \sum_{i\sigma} f_{i\sigma}^\dagger f_{i\sigma}. \quad (23.a)$$

Allowing a uniform hole doping rate, $\delta = b_i b_j^\dagger$, we rewrite

$$H = -t \delta \sum_{\langle ij \rangle \sigma} f_j^\dagger f_j f_i^\sigma - \frac{J}{2} \sum_{\langle ij \rangle \sigma} (f_{i\sigma}^\dagger f_{j-\sigma}^\dagger f_{j\sigma} - f_{i\sigma}^\dagger f_{j-\sigma}^\dagger f_{j\sigma} f_{i-\sigma}) - \mu \sum_{i\sigma} f_{i\sigma}^\dagger f_{i\sigma}. \quad (23.b)$$

Here $t$ is the hopping strength and $\mu$, the chemical potential. $b_i (b_j^\dagger)$ is the annihilation (creation) operator for a spinless boson and $f_{i\sigma} (f_{i\sigma}^\dagger)$, the annihilation (creation) operator for a chargeless fermion with spin $\sigma$ at site $i$.

Following Floreanini and Jackiw, the chargeless fermion field (spinon) operator $f$ can be written in terms of the Grassmann field variables $u$

$$f_{\alpha} = \frac{1}{\sqrt{2}} \left( u_{\alpha} + \frac{\delta}{\delta u_{\alpha}} \right) \quad (24)$$

where $\alpha$ represents $(i, \sigma)$. We write Gaussian functional,

$$|\Omega\rangle \equiv \langle u|\Psi_\Omega \rangle = N_\Omega \exp\left(\frac{1}{2} u_{\alpha} \Omega_{\alpha\beta} u_{\beta}\right) \equiv N_\Omega \exp\left\{ \frac{1}{2} \bar{u} \Omega u \right\} \quad (25)$$

where $\Omega$ is the antisymmetric 2 $\times$ 2 kernel matrix, and $N_\Omega$, the normalization constant. For brevity the summation sign over the lattice sites is omitted in the above expression. We now introduce Eqs. (24) and (25) into Eq. (24.b) to write the ground state energy,
\[ E = \frac{\langle \Omega | H | \Omega \rangle}{\langle \Omega | \phi \rangle}. \]  

(26)

The resulting ground state energy is, (for derivation, see Appendix B),

\[ E = -\delta \sum_{\langle ij \rangle} D_{\sigma \sigma}(j, i) - \mu \sum_i D_{\sigma \sigma}(i, i) \]

\[ - \frac{J}{2} \sum_{\langle ij \rangle} \{ D_{\sigma \sigma}(i, i) D_{-\sigma \sigma}(j, j) + D_{\sigma \sigma}(j, j) D_{-\sigma \sigma}(i, i) \}, \]  

(27)

where

\[ D_{\sigma \sigma'}(i, j) = \frac{1}{2} \left\{ [I + \Omega(i, j)] \left[ \Omega(i, j) + \overline{\Omega}(i, j) \right]^{-1} [I + \overline{\Omega}(i, j)] \right\}_{\sigma \sigma'} \]  

(28)

with \( \sigma \) and \( \sigma' \), either the spin up or spin down state of electron.

We readily find from the inspection of the chemical potential terms in Eqs. (24) and (27) that the number of electron \( n_i \) at site \( i \) and the local magnetization \( m_i \) at site \( i \) are given by

\[
\begin{align*}
    n_i &= D_{\uparrow \uparrow}(i, i) + D_{\downarrow \downarrow}(i, i) = n_{i \uparrow} + n_{i \downarrow}, \\
    m_i &= D_{\uparrow \uparrow}(i, i) - D_{\downarrow \downarrow}(i, i) = n_{i \uparrow} - n_{i \downarrow}.
\end{align*}
\]  

(29)

From Eq. (29), we obtain

\[
\begin{align*}
    D_{\uparrow \uparrow}(i, i) &= \frac{1}{2} (n_i + m_i), \\
    D_{\downarrow \downarrow}(i, i) &= \frac{1}{2} (n_i - m_i).
\end{align*}
\]  

(30)

Insertion of Eq. (30) into Eq. (27) leads to

\[ E = -t\delta \sum_{\langle ij \rangle} D_{\sigma \sigma}(j, i) - \frac{J}{4} \sum_{\langle ij \rangle} D_{\sigma \sigma}(i, i) (n_j - \sigma m_j) - \frac{J}{2} \sum_{\langle ij \rangle} D_{\sigma \sigma}(j, j) D_{-\sigma \sigma}(i, i) \]

\[ - \mu N(1 - \delta). \]  

(31)

From the Fourier transform of expression (31) above we obtain the following ground state energy (see Appendix B for verification),

\[ E = \pm 2 \sum_{k} \sqrt{\left[ (4t\delta)^2 - (Jm)^2 \right]^2 + J^2 m^2 - \mu(1 - \delta)N - 2J(1 - \delta)^2 N} \]  

(32)

and the dispersion relation,
\[ \omega_k = \pm 2 \sqrt{\left(4t\delta^2 - (Jm)^2 \right) \gamma_k^2 + J^2 m^2} \]  

(33)

with \( \gamma_k = \frac{\left[\cos(k_x a) + \cos(k_y a)\right]}{2} \).

At half-filling, i.e., \( \delta = 0 \), Eq. (33) leads to

\[ \omega_k = \pm 2Jm\sqrt{1 - \gamma_k^2}. \]  

(34)

For the system of paramagnetic state, i.e., \( m = 0 \), we obtain

\[ \omega_k = \pm 8t\delta\gamma_k. \]  

(35)

For the case of vanishing hopping integral or in the limit of large \( J \), that is, \( t/J \ll 1 \), we obtain the dispersion energy of spin waves(magnon) from Eq. (33),

\[ \omega_k = 2Jm\sqrt{1 - \gamma_k^2}. \]  

(36)

Realizing from Eq. (29) that \( m = 1 \) at half-filling, we find that

\[ \omega_k = 2J\sqrt{1 - \gamma_k^2}. \]  

(37)

We find from Eq. (34) above that in the limit of half-filling, that is, \( \delta \to 0 \), the slave-boson theory of FSP leads to the identical dispersion relation (Eq. (14.c)) of magnon obtained from the Holstein-Primakoff theory.

V. SUMMARY

In the present study we showed the functional Schrödinger representations of Holstein-Primakoff boson and slave-boson theories for the Heisenberg Hamiltonian and the \( t - J \) Hamiltonian respectively. From the use of the functional Schrödinger picture(FSP) theory of Holstein-Primakoff boson approach for the Heisenberg Hamiltonian we obtained both the zero-point energy of free magnons and the self-energy of holon(spinless holes). The FSP of slave-boson theory was also introduced into the \( t - J \) Hamiltonian to derive the dispersion relation of magnon. We find that in the limit of half-filling(\( \delta \to 0 \)) the dispersion relation
derived from this approach is identical to the one obtained from the Holstein-Primakoff boson approach. Improvement over the present approach of functional Schrödinger picture theory is desirable to fully account for many-body effects (correlation effects) beyond the mean field level.

ACKNOWLEDGMENTS

One (S.H.S.S.) of us greatly acknowledges the Korean Ministry of Education (BSRI-1998) and the Center for Molecular Science at Korea Advanced Institute of Science and Technology for financial supports. We are grateful to Dr. I. E. Dikstein for helpful discussions.

APPENDIX A: EVALUATIONS OF GAUSSIAN EXPONENTS, $\Omega^C$, $\Omega^D$ AND $\Omega^F$ AND FUNCTIONAL SCHRÖDINGER REPRESENTATIONS OF ONE- AND TWO-BODY TERMS FOR BOTH BOSONS AND FERMIIONS

In real space, we define the ground state Gaussian functional of boson to be

$$\langle \phi | \Omega_b \rangle \equiv \Psi_{\Omega}[\phi] = N_\Omega e^{-\frac{1}{2} \int \int \phi(x) \Omega_b(x-y) \phi(y) \ dx \ dy},$$

(A1)

or in an abbreviated form, we write $\Psi_{\Omega}[\phi] = e^{-\frac{1}{2} \phi^* \Omega_b \phi}$. $N_\Omega$ is the normalization constant.

The ground state functional in momentum space is written

$$\Psi_{\Omega}[\tilde{\phi}] = N_\Omega e^{-\frac{1}{2} \int \phi_k^* \Omega_k \phi_k \ dk},$$

(A2)

or allowing discreteness in momentum $k$,

$$\Psi_{\Omega}[\tilde{\phi}] = N_\Omega \exp \left[ -\frac{1}{2N} \sum_k \tilde{\phi}_k^* \Omega_k \tilde{\phi}_k \right]$$

(A3)

where $N$ is the total number of lattice sites.

Introducing the boson field $\phi$, we define the boson operators in FSP,

$$c(x) = \frac{1}{\sqrt{2}} \left( \phi(x) + \frac{\delta}{\delta \phi(x)} \right)$$

$$c^\dagger (x) = \frac{1}{\sqrt{2}} \left( \phi(x) - \frac{\delta}{\delta \phi(x)} \right)$$

(A4)
which satisfy the commutation relation,

\[
[c(x), c^\dagger(x')] = \delta(x - x').
\] (A5)

The field operators in momentum space are then

\[
c(k) = \frac{1}{\sqrt{2}} \frac{1}{(2\pi)^d} \int d^d x \ e^{i k \cdot x} \left( \phi(x) + \frac{\delta}{\delta \phi(x)} \right),
\]

\[
c^\dagger(k) = \frac{1}{\sqrt{2}} \frac{1}{(2\pi)^d} \int d^d x \ e^{-i k \cdot x} \left( \phi(x) - \frac{\delta}{\delta \phi(x)} \right),
\] (A6)

and thus,

\[
c(k) = \frac{1}{\sqrt{2}} \left( \delta \phi(k) + \frac{\delta}{\delta \phi^*(k)} \right),
\]

\[
c^\dagger(k) = \frac{1}{\sqrt{2}} \left( \delta \phi^*(k) - \frac{\delta}{\delta \phi(k)} \right),
\] (A7)

the use of which satisfies the commutation relation, \([c_k, c^\dagger_{k'}] = \delta_{k,k'}\).

Now we evaluate the vacuum state expectation of one-body term \((\langle c^\dagger_k c_k \rangle)\) in FSP. Using the ground state Gaussian functional

\[
\Psi_\Omega[\tilde{\phi}] = N_\Omega \exp \left[ -\frac{1}{2N} \sum_k \phi_{-p} \Omega_p \phi_p \right],
\]

we obtain

\[
\frac{\delta}{\delta \phi^*_k} \Psi_\Omega[\tilde{\phi}] = \frac{\delta}{\delta \phi_{-k}} \Psi_\Omega[\tilde{\phi}]
\]

\[
= -\frac{1}{2} \sum_p \left( \Omega^*_{-p} \delta_{-k,-p} \phi_p + \Omega^*_{-p} \phi_{-p} \delta_{-k,p} \right) \Psi_\Omega[\tilde{\phi}]
\]

\[
= -\frac{1}{2} \left( \Omega^b_{k} \phi_k + \Omega^b_{-k} \phi_k \right) \Psi_\Omega[\tilde{\phi}].
\]

By using the symmetric condition, \(\Omega^b_{k} = \Omega^b_{-k}\), we obtain

\[
\frac{\delta}{\delta \phi^*_k} \Psi_\Omega[\tilde{\phi}] = -\Omega^b_{k} \phi_k \Psi_\Omega[\tilde{\phi}].
\] (A8)

Thus we find

\[
\langle 0 | c^\dagger_{k_1} c_{k_2} | 0 \rangle \rightarrow \frac{1}{2} \left( 1 - \Omega^b_{k_2} \right) \left( 1 + \Omega^b_{k_1} \right) \int D\phi \Psi^*_{\Omega[\tilde{\phi}]} \phi^*_{k_1} \phi_{k_2} \Phi_\Omega[\tilde{\phi}] - \frac{1}{2} \left( 1 - \Omega^b_{k_2} \right) \delta_{k_1,k_2}
\] (A9)
with

\[
\int D\phi \Psi^*_{\Omega} \hat{\phi}_1 \hat{\phi}_2 \Psi_{\phi} = \int D\phi \ e^{-\phi^* \Omega \phi} \hat{\phi}_1 \hat{\phi}_2 \ e^{-\phi^* \Omega \phi} \\
= \int D\phi \ \phi_1^* \phi_2 \ e^{-2\phi^* \Omega \phi} \\
= \lim_{J \to 0} \frac{\delta^2}{\delta J_1 \delta J_2} \left( \int D\phi \ e^{-2\phi^* \Omega \phi + J^* \phi + J \phi^*} \right) \\
= \lim_{J \to 0} \frac{\delta}{\delta J_1} \left( \frac{1}{2} \Omega_{k_2}^{-1} \delta_{k_1,k_2} + \frac{1}{2} \Omega_{k_2}^{-1} J_{k_2,k_1} e^{\frac{1}{2} J^* \Omega_{k_2}^{-1} J} \right) \\
= \lim_{J \to 0} \left( \frac{1}{2} \Omega_{k_2}^{-1} \delta_{k_1,k_2} - \frac{1}{2} \Omega_{k_2}^{-1} \delta_{k_1,k_2} \right) = \frac{1}{2} \Omega_{k_2}^{-1} \delta_{k_1,k_2}. \tag{A10}
\]

By inserting Eq. (A10) into Eq. (A9), we obtain

\[
\langle 0 | c_{k_1}^\dagger c_{k_2} | 0 \rangle \to \frac{1}{4} \left( 1 - \Omega_{k_2}^b \right) \left( 1 + \Omega_{k_1}^b \right) \Omega_{k_2}^{-1} \delta_{k_1,k_2} - \frac{1}{2} \left( 1 - \Omega_{k_2}^b \right) \delta_{k_1,k_2}. \tag{A11}
\]

For \( k_1 = k_2 \), we get

\[
\langle 0 | c_{k_1}^\dagger c_{k_1} | 0 \rangle \to \frac{1}{4} \left( 1 - \Omega_{k}^b \right) \left( 1 + \Omega_{k}^b \right) \Omega_{k}^{-1} - \frac{1}{2} \left( 1 - \Omega_{k}^b \right) \\
= \frac{1}{4} \left( \Omega_{k}^{-1} - \Omega_{k}^b \right) - \frac{1}{2} \left( 1 - \Omega_{k}^b \right) \Omega_{k} = \frac{\Omega_{k}^{-1} + \Omega_{k} - 2}{4} \tag{A12}
\]

In order to determine the value of \( \Omega^b \) for the energy of free boson we now take the variation of the one-body energy term \( \langle c_{k_1}^\dagger c_{k_1} \rangle = \langle 0 | c_{k_1}^\dagger c_{k_1} | 0 \rangle \) with respect to \( \Omega_k \),

\[
\frac{\partial}{\partial \Omega_k} \langle c_{k_1}^\dagger c_{k_1} \rangle = \frac{1}{4} \left[ -\frac{1}{\left( \Omega_k^b \right)^2} + 1 \right] = 0 \tag{A13}
\]

to obtain

\[
\Omega_k^b = \pm 1.
\]

Here \( \Omega_k^b = -1 \) is discarded as the ground state Gaussian functional cannot be defined. Finally for \( k_1 = k_2 \), Eq. (A11) leads to
\[ \langle 0|c_k \hat{c}_k|0 \rangle = 0 \quad \text{for} \quad \Omega^b_k = 1 \]  \hspace{1cm} (A14)

as expected. In a similar manner we obtain \( \Omega^b_k = 1 \) for the two-body terms.

We note that for fermions

\[ f_a = \frac{1}{\sqrt{2}} \left( u_a + \frac{\delta}{\delta u_a^\dagger} \right) \]

and

\[ \{ f_a, f_b^\dagger \} = \delta_{ab} \]

with \( i = a \) and \( b \) to represent site indices.

We find

\[ \langle \Omega|\Omega \rangle = \int D\mathbf{u} D\mathbf{u}^\dagger \exp[\mathbf{u}^\dagger (\Omega + \overline{\Omega}) \mathbf{u}] = \det^{\frac{1}{2}}(\Omega + \overline{\Omega}). \]  \hspace{1cm} (A15)

Realizing that

\[ f_b|0 \rangle \rightarrow \frac{1}{\sqrt{2}} \left( u_b + \frac{\delta}{\delta u_b^\dagger} \right) \exp(u_c^\dagger \Omega_{cd} u_d) \]

\[ = \frac{1}{\sqrt{2}} (u_b + \delta_{bc} \Omega_{cd} u_d) \exp(u_c^\dagger \Omega_{cd} u_d) \]

\[ = (\delta_{bd} + \Omega_{bd}) u_d \exp(u_c^\dagger \Omega_{cd} u_d) \]

\[ = (I + \Omega)_{bd} u_d \exp(u_c^\dagger \Omega_{cd} u_d) \]

the expectation value of \( \langle f^\dagger f \rangle \) is

\[ \langle f^\dagger_a f_b \rangle \rightarrow D_{ba} \equiv \frac{1}{2} \langle \Omega| \left( u_a^\dagger + \frac{\delta}{\delta u_a} \right) \left( u_b + \frac{\delta}{\delta u_b^\dagger} \right) |\Omega\rangle / \langle \Omega|\Omega \rangle \]

\[ = \frac{1}{2} (I + \overline{\Omega})_{ea} (I + \Omega)_{bd} \int D\mathbf{u} D\mathbf{u}^\dagger u_c^\dagger u_d \exp[\mathbf{u}^\dagger (\overline{\Omega} + \Omega) \mathbf{u}] \]

\[ = \frac{1}{2} (I + \overline{\Omega})_{ea} (I + \Omega)_{bd} \delta_{\eta \overline{\eta}} \delta_{\eta \overline{\eta}} Z_0[\eta, \overline{\eta}]|_{\eta = \overline{\eta} = 0} \]

\[ = \frac{1}{2} (I + \overline{\Omega})_{ea} (I + \Omega)_{bd} \delta_{\eta \overline{\eta}} \delta_{\eta \overline{\eta}} \exp[-\overline{\eta}(\Omega + \overline{\Omega})^{-1}\eta]|_{\eta = \overline{\eta} = 0} \]

\[ = \frac{1}{2} (I + \overline{\Omega})_{ea} (I + \Omega)_{bd} (\Omega + \overline{\Omega})^{-1} \]

\[ = \frac{1}{2} [(I + \Omega)(\Omega + \overline{\Omega})^{-1}(I + \Omega)]_{ba} \]  \hspace{1cm} (A16)
where
\[
Z_0(\eta, \bar{\eta}) = \int Du D\bar{u} \exp\{\int (u^\dagger S u + \bar{\eta} u + u^\dagger \eta)\} = \text{det}(S) \exp\left(-\bar{\eta} S^{-1} \eta\right)
\] (A17)

with \( S = \Omega + \bar{\Omega} \). Similarly we find, for the two-body term,
\[
\langle f_a^\dagger f_b^\dagger f_c f_d \rangle \rightarrow (I + \bar{\Omega})_b(I + \bar{\Omega})_{ja}(I + \Omega)_d f_a(I + \Omega)_c h_c(\Omega + \bar{\Omega})_{fa}(I + \bar{\Omega})^{-1} (\Omega + \bar{\Omega})^{-1}_{hp} (\delta_{pd} \delta_{nj} - \delta_{ni} \delta_{pj})
\]
\[
= [(I + \Omega)_d f_j (\Omega + \bar{\Omega})^{1}_{fa}(I + \bar{\Omega})_{ja}][(I + \Omega)_c h_c(\Omega + \bar{\Omega})_{hp}^{-1} (I + \bar{\Omega})_b]
\]
\[
- [(I + \Omega)_d f_j (\Omega + \bar{\Omega})^{1}_{fa}(I + \bar{\Omega})_{ja}][(I + \Omega)_c h_c(\Omega + \bar{\Omega})_{hp}^{-1} (I + \bar{\Omega})_b]
\]
\[
= D_{da} D_{eb} - D_{db} D_{ca}
\] (A18)

with \( i = a, b, c \) and \( d \), the site index.

With the use of Eq. (8), we obtain in momentum space,
\[
\langle H_0 \rangle = \sum_k \omega_k^0 \left[ 1 - \frac{1}{2sN} \sum_q \left( \frac{\omega_q^0}{J_{sz}} \right)^2 \right] \left( \langle c_k^\dagger c_k \rangle + \langle d_k^\dagger d_k \rangle \right) + J_{sz} \sum_k \left( \frac{\omega_k^0}{J_{sz}} - 1 - 2 \langle f_k^\dagger f_k \rangle \right)
\]
\[
- J s^2 z N
\]
\[
= \sum_k \omega_k^0 \left[ 1 - \frac{1}{2sN} \sum_q \left( \frac{\omega_q^0}{J_{sz}} \right)^2 \right] \left[ \frac{1}{4} (1 - \Omega_k^0) (1 + \Omega_k^0) \Omega_k^{-1} - \frac{1}{2} (1 - \Omega_k^0)
\]
\[
+ \frac{1}{4} (1 - \Omega_k^0) (1 + \Omega_k^0) \Omega_k^{-1} - \frac{1}{2} (1 - \Omega_k^0) \right] + J_{sz} \sum_k \left( \frac{\omega_k^0}{J_{sz}} - 1 - 2 D_{kk} \right)
\]
\[
- J s^2 z N,
\] (A19)

\[
\langle H_{m-m} \rangle = -\frac{J_z}{4} \sum_{k_1, k_2} \left\{ \left( 1 - \frac{\omega_{k_1} \omega_{k_2}}{(J_{sz})^2} \right) \left( \langle c_{k_1}^\dagger c_{k_1} c_{k_2}^\dagger c_{k_2} \rangle + \langle d_{k_1}^\dagger d_{k_1} d_{k_2}^\dagger d_{k_2} \rangle \right)
\]
\[
+ 2 \left( 1 + \frac{\omega_{k_1} \omega_{k_2}}{(J_{sz})^2} \right) \langle c_{k_1}^\dagger c_{k_1} \rangle \langle d_{k_2}^\dagger d_{k_2} \rangle \right\}
\]
\[
= -\frac{J_z}{4} \sum_{k_1, k_2} \left\{ \left( 1 - \frac{\omega_{k_1} \omega_{k_2}}{(J_{sz})^2} \right) \left[ \frac{1}{16} (1 - \Omega_{k_2}^0) (1 + \Omega_{k_2}^0) \right.
\]
\[
\times \left( 1 - \Omega_{k_1}^0 \right) (1 + \Omega_{k_1}^0) \Omega_{k_2}^{-1} \left[ \Omega_{k_1}^{-1} + \Omega_{k_2}^{-1} (\delta_{k_1, k_2})^2 \right]
\]
\[
- \frac{1}{8} (1 - \Omega_{k_2}^0) (1 + \Omega_{k_2}^0) \left( 1 - \Omega_{k_1}^0 \right) \left( \Omega_{k_2}^{-1} + \delta_{k_1, k_2} \Omega_{k_2}^{-1} \delta_{k_1, k_2} \right)
\]
\[
+ \frac{1}{8} (1 - \Omega_{k_2}^0) (1 + \Omega_{k_2}^0) \delta_{k_1, k_2} (1 + \Omega_{k_1}^0) \Omega_{k_2}^{-1} \delta_{k_1, k_2}
\]
\[
- \frac{1}{8} (1 - \Omega_{k_2}^0) (1 + \Omega_{k_2}^0) \left( \delta_{k_1, k_2} \right)^2
\]
\[
- \frac{1}{8} (1 - \Omega_{k_2}^0) (1 - \Omega_{k_1}^0) (1 + \Omega_{k_1}^0) \Omega_{k_2}^{-1} \right
\]
\[ + \frac{1}{4} \left( 1 - \Omega_{k_2}^d \right) \left( 1 - \Omega_{k_1}^d \right) \]
\[ + \frac{1}{16} \left( 1 - \Omega_{k_2}^d \right) \left( 1 + \Omega_{k_2}^d \right) \left( 1 - \Omega_{k_1}^d \right) \left( 1 + \Omega_{k_1}^d \right) \Omega_{k_2}^d \Omega_{k_1}^d^{-1} \left[ \Omega_{k_1}^d \Omega_{k_2}^d \Omega_{k_1}^d \Omega_{k_2}^d \right] \]
\[ + \frac{1}{8} \left( 1 - \Omega_{k_2}^d \right) \left( 1 + \Omega_{k_2}^d \right) \left( 1 - \Omega_{k_1}^d \right) \left( 1 + \Omega_{k_1}^d \right) \Omega_{k_2}^d \Omega_{k_1}^d^{-1} \delta_{k_1,k_2} \]
\[ + \frac{1}{8} \left( 1 - \Omega_{k_2}^d \right) \left( 1 + \Omega_{k_2}^d \right) \delta_{k_1,k_2} \left( 1 + \Omega_{k_1}^d \right) \Omega_{k_2}^d \Omega_{k_1}^d^{-1} \delta_{k_1,k_2} \]
\[ + \frac{1}{4} \left( 1 - \Omega_{k_2}^d \right) \left( 1 + \Omega_{k_2}^d \right) \delta_{k_1,k_2}^2 \]
\[ + \frac{1}{8} \left( 1 - \Omega_{k_2}^d \right) \left( 1 - \Omega_{k_1}^d \right) \left( 1 + \Omega_{k_1}^d \right) \Omega_{k_2}^d \Omega_{k_1}^d^{-1} \]
\[ + \frac{1}{4} \left( 1 - \Omega_{k_2}^d \right) \left( 1 - \Omega_{k_1}^d \right) \left( 1 + \Omega_{k_1}^d \right) \Omega_{k_2}^d \Omega_{k_1}^d^{-1} \]
\[ + 2 \left( 1 + \frac{\omega_{k_1}^0 \omega_{k_2}^d}{\langle Jsz \rangle^2} \right) \left[ \frac{1}{4} \left( 1 - \Omega_{k_2}^c \right) \left( 1 + \Omega_{k_2}^c \right) \Omega_{k_2}^c \Omega_{k_2}^c^{-1} - \frac{1}{2} \left( 1 - \Omega_{k_2}^c \right) \right] \]
\[ \times \left[ \frac{1}{4} \left( 1 - \Omega_{k_2}^d \right) \left( 1 + \Omega_{k_2}^d \right) \Omega_{k_2}^d \Omega_{k_2}^d^{-1} - \frac{1}{2} \left( 1 - \Omega_{k_2}^d \right) \right] \right), \quad (A20) \]

\[ \langle H_{m-h} \rangle = -\frac{J_z}{2} \frac{2}{N} \sum_{k_1,k_2} \frac{\omega_{k_2}^0}{\langle Jsz \rangle} \left\langle f_{k_1}^d f_{k_1} \right\rangle \left( \left\langle c_{k_2}^d c_{k_2} \right\rangle + \left\langle d_{k_2}^d c_{k_2} \right\rangle \right) \]
\[ = -\frac{J_z}{2} \frac{2}{N} \sum_{k_1,k_2} \frac{\omega_{k_2}^0}{\langle Jsz \rangle} \frac{1}{2} \left[ (I + \Omega)^{I}(I + \Omega)^{-1}(I + \Omega)^{-1} \right]_{k_1,k_1} \]
\[ \times \left[ \frac{1}{4} \left( 1 - \Omega_{k_2}^c \right) \left( 1 + \Omega_{k_2}^c \right) \Omega_{k_2}^c \Omega_{k_2}^c^{-1} - \frac{1}{2} \left( 1 - \Omega_{k_2}^c \right) \right] \]
\[ + \frac{1}{4} \left( 1 - \Omega_{k_2}^d \right) \left( 1 + \Omega_{k_2}^d \right) \Omega_{k_2}^d \Omega_{k_2}^d^{-1} - \frac{1}{2} \left( 1 - \Omega_{k_2}^d \right) \right] \right) \]
\[ = -\frac{J_z}{2} \frac{2}{N} \sum_{k_1,k_2} \frac{\omega_{k_2}^0}{\langle Jsz \rangle} \frac{1}{2} \left[ (I + \Omega)^{I}(I + \Omega)^{-1}(I + \Omega)^{-1} \right]_{k_1,k_1} \]
\[ \times \left( \Omega_{k_2}^c \Omega_{k_2}^c \Omega_{k_2}^c \Omega_{k_2}^c \right) \]
\[ + \frac{1}{4} \left( 1 - \Omega_{k_2}^d \right) \left( 1 + \Omega_{k_2}^d \right) \Omega_{k_2}^d \Omega_{k_2}^d^{-1} - \frac{1}{2} \left( 1 - \Omega_{k_2}^d \right) \right) \right), \quad (A21) \]

and

\[ H_{h-h} = -\frac{J_z}{2} \frac{2}{N} \sum_{k_1,k_2} \gamma_q \left\langle f_{k_1}^d f_{k_1}^d f_{k_2}^d f_{k_2}^d \right\rangle \]
\[ = -\frac{J_z}{2} \frac{2}{N} \sum_{k_1,k_2} \gamma_q \left\langle f_{k_1}^d f_{k_1}^d \delta_{k_1,k_2} \Omega_{k_1}^d \Omega_{k_1}^d \Omega_{k_1}^d \Omega_{k_1}^d \right\rangle \]
\[ = -\frac{J_z}{2} \frac{2}{N} \sum_{k_1,k_2} \gamma_q \left\langle f_{k_1}^d f_{k_1}^d \delta_{k_1,k_2} \Omega_{k_1}^d \Omega_{k_1}^d \Omega_{k_1}^d \Omega_{k_1}^d \right\rangle \]
\[ = -\frac{J_z}{2} \frac{2}{N} \sum_{k_1,k_2} \gamma_q \left[ \sum_{k} D_{k,k} - \sum_{k_1,k_2} \left( D_{k_2,k_1} D_{k_2,k_2} + D_{k_1,k_1} D_{k_2,k_2} \right) \right] \] (A22)
where
\[
D_{k,k'} = \frac{1}{2} \left[ (1 + \Omega') (\Omega + \Omega') (1 + \Omega') \right]_{k,k'} = \langle f_{k'}^\dagger f_k \rangle. \tag{A23}
\]

By noting that
\[
\sum_k \gamma_k = \sum_k \gamma_{-k} = 0, \tag{A24}
\]
we obtain
\[
H_{h-h} = \frac{J_z}{2} \sum_{k_1,k_2,q} \gamma_q \left( D_{k_2,k_1-q}D_{k_1,k_2+q} - D_{k_1,k_1-q}D_{k_2,k_2+q} \right). \tag{A25}
\]

By taking the variation of the ground state energy with respect to \( \Omega^d_k \), we find that \( \Omega^d_k = 1 \). The resulting ground state energy is then
\[
E = Js z \sum_k \left( \sqrt{1 - \gamma^2_k} + 2n^f_k \right) - J s (s + 1) z N
\]
\[
+ \frac{J_z}{2} \sum_{k_1,k_2,q} \gamma_q \left( D_{k_2,k_1-q}D_{k_1,k_2+q} - D_{k_1,k_1-q}D_{k_2,k_2+q} \right) \tag{A26}
\]
where \( n^f_k \equiv \langle f_{k}^\dagger f_k \rangle = D_{k,k} \). By taking the variation of \( \langle H_0 \rangle \) in Eq. (A19) with respect to \( \Omega^d_k \) we obtain \( \Omega^d_k = -1 \), which leads to \( D_{k,k'} = 0 \) at half-filling. Thus both the free hole energy term and the hole-hole interaction energy (the second term in Eq. (A26)) term vanish.

**APPENDIX B: THE DISPERSION RELATION OF MAGNONS FROM THE \( T-J \) HAMILTONIAN**

From Eq. (23.b) the expectation value of the \( t-J \) Hamiltonian in slave-boson representation is
\[
E = \langle H \rangle
\]
\[
= -t \delta \sum_{\langle ij \rangle} \langle f_{i\sigma}^\dagger f_{j \sigma} \rangle - \frac{J}{2} \sum_{\langle ij \rangle} \left( \langle f_{i\sigma}^\dagger f_{j \sigma}^\dagger f_{j \bar{\sigma}} f_{i \bar{\sigma}} \rangle - \langle f_{i\sigma}^\dagger f_{j \sigma}^\dagger f_{j \bar{\sigma}} f_{i \bar{\sigma}} \rangle - \langle f_{i\sigma}^\dagger f_{j \sigma}^\dagger f_{j \bar{\sigma}} f_{i \bar{\sigma}} \rangle \right)
\]
\[
- \mu \sum_{i \sigma} \langle f_{i \sigma}^\dagger f_{i \sigma} \rangle. \tag{B1}
\]
In the following we express both \( \langle f^+_a f_b \rangle \) and \( \langle f^+_a f_b f^+_c f_d \rangle \) above in functional Schrödinger picture. Using that

\[
 f_b |0\rangle \rightarrow \frac{1}{\sqrt{2}} \left( u_b + \frac{\delta}{\delta u_b} \right) \exp(\bar{u}_c \Omega_{cd} u_d) \\
 = \frac{1}{\sqrt{2}} (u_b + \delta_{bc} \Omega_{cd} u_d) \exp(\bar{u}_c \Omega_{cd} u_d) \\
 = (\delta_{bd} + \Omega_{bd}) u_d \exp(\bar{u}_c \Omega_{cd} u_d) \\
 = (I + \Omega)_{bd} u_d \exp(\bar{u}_c \Omega_{cd} u_d),
\]

we obtain

\[
 \langle f^+_a f_b \rangle = \frac{\langle \Omega | f^+_a f_b | \Omega \rangle}{\langle \Omega | \Omega \rangle} \\
 = \frac{1}{2} (I + \Omega)_{ba} (I + \Omega)_{bd} \int Du D\bar{u} \bar{u}_e u_d \exp[\bar{u}(\Omega + \Omega)u] \\
 = \frac{1}{2} (I + \Omega)_{ba} (I + \Omega)_{bd} \frac{\delta}{\delta \eta} \frac{\delta}{\delta \bar{\eta}} Z_0[\eta, \bar{\eta}] |_{\eta=\bar{\eta}=0} \\
 = \frac{1}{2} (I + \Omega)_{ba} (I + \Omega)_{bd} (\Omega + \Omega)^{-1} \\
 = \frac{1}{2} (I + \Omega)_{ba} (I + \Omega)_{bd} (\Omega + \Omega)^{-1} (I + \Omega)_{ba} 
\]

(B2)

where

\[
 Z_0[\eta, \bar{\eta}] = \int Du D\bar{u} \exp[\int (\ddot{u} Su + \bar{\eta} u + \ddot{u} \eta)] = \det(S) \exp[-\bar{\eta} S^{-1} \eta].
\]

(B3)

For brevity we define

\[
 D_{ba} = \frac{1}{2} \left[ (I + \Omega) (\Omega + \Omega)^{-1} (I + \Omega) \right]_{ba},
\]

(B4)

with \( S = \Omega + \Omega \), and \( \langle \Omega | \Omega \rangle = \det^{\frac{1}{2}} (\Omega + \Omega) \).

We express

\[
 f_c f_d |0\rangle \rightarrow \frac{1}{2} \left( u_c + \frac{\delta}{\delta \bar{u}_c} \right)(u_d + \frac{\delta}{\delta \bar{u}_d}) \exp(\bar{u}_e \Omega_{ef} u_e) \\
 = \frac{1}{2} (u_c + \frac{\delta}{\delta \bar{u}_c}) (I + \Omega)_{gf} u_f \exp(\bar{u}_g \Omega_{gh} u_h)
\]
\[ \langle \Omega | \Omega \rangle = \det^{2}(\Omega + \Omega) \quad \text{(B6)} \]

Realizing that \( \langle \Omega | \Omega \rangle = \det^{2}(\Omega + \Omega) \), we find from Eq. (B6) and Eq. (B7) that

\[ \langle f_{a}^{\dagger} f_{b}^{\dagger} f_{c} f_{d} \rangle = (I + \Omega)_{ab}(I + \Omega)_{ja}(I + \Omega)_{bf}(I + \Omega)_{hc}(\Omega + \Omega)_{fj}^{-1}(\Omega + \Omega)_{hp}^{-1}(\delta_{pl}\delta_{nj} - \delta_{nl}\delta_{pj}) \]

\[ = [(I + \Omega)_{df}(\Omega + \Omega)_{fj}^{-1}(I + \Omega)_{ja}][(I + \Omega)_{ch}(\Omega + \Omega)_{hj}^{-1}(I + \Omega)_{lb}] \]

\[ - [(I + \Omega)_{df}(\Omega + \Omega)_{fj}^{-1}(I + \Omega)_{ja}][(I + \Omega)_{ch}(\Omega + \Omega)_{hj}^{-1}(I + \Omega)_{lb}] \]

\[ = D_{dc}D_{cb} - D_{db}D_{ca}. \quad \text{(B8)} \]

The substitution of Eqs. (B2) and (B8) into Eq. (B1) leads to

\[ E = -t\delta \sum_{(ij)\sigma} D_{\sigma\sigma}(j, i) - \frac{J}{2} \sum_{(ij)\sigma} \{ [D_{\sigma\sigma}(j, j) - D_{\sigma\sigma}(j, j)] - D_{-\sigma\sigma}(j, i) D_{+\sigma\sigma}(i, j) \}

- [D_{-\sigma\sigma}(i, j) D_{+\sigma\sigma}(j, i) - D_{-\sigma\sigma}(j, i) D_{+\sigma\sigma}(i, j)] - \mu \sum_{i\sigma} D_{\sigma\sigma}(i, i). \quad \text{(B9)} \]
Allowing the global $SU(2)$ symmetry and thus

$$D_{\sigma,-\sigma}(i, j) = \langle f_{j,-\sigma}^\dagger f_{i,\sigma} \rangle = 0,$$  \hspace{1cm} (B10)

Eq. (B9) is rewritten,

$$E = -t\delta \sum_{(ij)\sigma} D_{\sigma\sigma}(j, i) - \frac{J}{2} \sum_{(ij)\sigma} \{D_{\sigma\sigma}(i, i)D_{-\sigma,-\sigma}(j, j) + D_{\sigma\sigma}(j, j)D_{-\sigma,-\sigma}(i, i)\}$$

$$-\mu \sum_{i,\sigma} D_{\sigma\sigma}(i, i).$$  \hspace{1cm} (B11)

From the inspection of the last term, we note that the number of electron $n_i$ at site $i$ and the local/site magnetization $m_i$ at site $i$ are given by

$$n_i = D_{\uparrow\uparrow}(i, i) + D_{\downarrow\downarrow}(i, i) = n_{i\uparrow} + n_{i\downarrow},$$  \hspace{1cm} (B12)

$$m_i = D_{\uparrow\uparrow}(i, i) - D_{\downarrow\downarrow}(i, i) = n_{i\uparrow} - n_{i\downarrow}.$$  \hspace{1cm} (B13)

From Eqs. (B12) and (B13), we obtain

$$D_{\uparrow\uparrow}(i, i) = \frac{1}{2}(n_i + m_i),$$

$$D_{\downarrow\downarrow}(i, i) = \frac{1}{2}(n_i - m_i),$$

$$\sum_{\sigma} D_{\sigma\sigma}(i, i) = D_{\uparrow\uparrow}(i, i) + D_{\downarrow\downarrow}(i, i) = n_i$$

$$\sum_{\sigma} D_{\sigma\sigma}(i, i)D_{-\sigma,-\sigma}(j, j) = D_{\uparrow\uparrow}(i, i)D_{\downarrow\downarrow}(j, j) + D_{\downarrow\downarrow}(i, i)D_{\uparrow\uparrow}(j, j)$$

$$= D_{\uparrow\uparrow}(i, i)\frac{(n_j - m_j)}{2} + D_{\downarrow\downarrow}(i, i)\frac{(n_j + m_j)}{2}$$

$$= \frac{1}{2} \sum_{\sigma} D_{\sigma\sigma}(i, i)(n_j - \sigma m_j)$$  \hspace{1cm} (B14)

with $\sigma = 1(-1)$ for up(down) spin. Using Eq. (B14), we rewrite Eq. (B11),

$$E = -t\delta \sum_{(ij)\sigma} D_{\sigma\sigma}(j, i) - \frac{J}{4} \sum_{(ij)} D_{\sigma\sigma}(i, i)(n_j - \sigma m_j) - \frac{J}{2} \sum_{(ij)\sigma} D_{\sigma\sigma}(j, i)D_{-\sigma,-\sigma}(i, j)$$

$$-\mu \sum_{i} n_i.$$  \hspace{1cm} (B15)

Allowing uniform hole doping and thus $n = n_i = 1 - \delta$, we obtain the total number of electrons.
\[
\sum_i n_i = \sum_i (1 - \delta) = N(1 - \delta) \quad (B16)
\]

and
\[
\sum_{ij} \left[ \sum_{\sigma} D_{\sigma \sigma}(i, i) \right] n_j = \sum_{ij} n_i n_j = \sum_{ij} (1 - \delta)^2 = 4N (1 - \delta)^2 \quad (B17)
\]

With the use of Eqs. (B16) and (B17), Eq. (B13) leads to
\[
E = -t\delta \sum_{ij} D_{\sigma \sigma}(j, i) + \frac{J}{4} \sum_{ij} D_{\sigma \sigma}(i, i) \sigma m_j - \frac{J}{2} \sum_{ij} D_{\sigma \sigma}(j, i) D_{-\sigma -\sigma}(i, j) \\
- JN (1 - \delta)^2 - \mu N(1 - \delta). \quad (B18)
\]

We define
\[
n(i, j) = D_{\uparrow \uparrow}(i, j) + D_{\downarrow \downarrow}(i, j), \\
m(i, j) = D_{\uparrow \downarrow}(i, j) - D_{\downarrow \uparrow}(i, j). \quad (B19)
\]

Then we have
\[
D_{\uparrow \uparrow}(i, j) = \frac{1}{2} [n(i, j) + m(i, j)], \\
D_{\downarrow \downarrow}(i, j) = \frac{1}{2} [n(i, j) - m(i, j)]. \quad (B20)
\]

Using Eq. (B20), we write the third term in Eq. (B18),
\[
\sum_{\langle ij \rangle \sigma} D_{\sigma \sigma}(j, i) D_{-\sigma -\sigma}(i, j) = \sum_{\langle ij \rangle} \left[ D_{\uparrow \uparrow}(j, i) D_{\downarrow \downarrow}(i, j) + D_{\downarrow \uparrow}(j, i) D_{\uparrow \downarrow}(i, j) \right] \\
= \sum_{\langle ij \rangle} \left[ D_{\uparrow \uparrow}(j, i) \cdot \frac{\{n(i, j) - m(i, j)\}}{2} + D_{\downarrow \downarrow}(j, i) \cdot \frac{\{n(i, j) + m(i, j)\}}{2} \right] \\
= \frac{1}{2} \sum_{\langle ij \rangle \sigma} D_{\sigma \sigma}(j, i) [n(i, j) - \sigma m(i, j)], \quad (B21)
\]

The substitution of Eq. (B21) into Eq. (B18) leads to
\[
E = -t\delta \sum_{\langle ij \rangle \sigma} D_{\sigma \sigma}(j, i) + \frac{J}{4} \sum_{\langle ij \rangle \sigma} D_{\sigma \sigma}(i, i) \sigma m_j - \frac{J}{4} \sum_{\langle ij \rangle \sigma} D_{\sigma \sigma}(j, i) [n(i, j) - \sigma m(i, j)] \\
- JN (1 - \delta)^2 - \mu N(1 - \delta). \quad (B22)
\]
We note that the two-dimensional staggered magnetization is represented by \( m_i = m e^{iQ \cdot i} \) with \( Q = (\pi, \pi) \) and \( i = (i_x, i_y) \). Thus Eq. (B22) leads to

\[
E = -t\delta \sum_{\langle ij \rangle \sigma} D_{\sigma\sigma}(j, i) + \frac{Jm}{4} \sum_{\langle ij \rangle \sigma} \sigma D_{\sigma\sigma}(i, i) e^{iQ \cdot j} - \frac{J}{4} \sum_{\langle ij \rangle \sigma} D_{\sigma\sigma}(j, i) [n(i, j) - \sigma m(i, j)] - \mu N(1 - \delta) - JN(1 - \delta)^2
\]

(B23)

where

\[
\sum_{\sigma} D_{\sigma\sigma}(j, i) = D_{\uparrow\uparrow}(j, i) + D_{\downarrow\downarrow}(j, i) = n(j, i),
\]

\[
\sum_{\langle ij \rangle \sigma} D_{\sigma\sigma}(j, i) n(i, j) = \sum_{\langle ij \rangle} n(j, i) n(i, j).
\]

(B24)

Introducing

\[
\sum_{\langle ij \rangle \sigma} D_{\sigma\sigma}(j, i) n(i, j) \simeq \sum_{\langle ij \rangle} n_i n_j = 4N(1 - \delta)^2
\]

(B25)

and

\[
m(i, j) \simeq m_i = me^{iQ \cdot i},
\]

(B26)

we obtain from Eq. (B23),

\[
E = -t\delta \sum_{\langle ij \rangle \sigma} D_{\sigma\sigma}(j, i) + \frac{Jm}{4} \sum_{\langle ij \rangle \sigma} \sigma D_{\sigma\sigma}(i, i) e^{iQ \cdot j} + \frac{Jm}{4} \sum_{\langle ij \rangle \sigma} D_{\sigma\sigma}(j, i) e^{iQ \cdot j} - \mu N(1 - \delta) - 2JN(1 - \delta)^2
\]

\[
= \sum_{\langle ij \rangle \sigma} \left\{ D_{\sigma\sigma}(j, i) \left[ -t\delta + \frac{Jm}{4} \sigma e^{iQ \cdot j} \right] + \frac{Jm}{4} \sigma D_{\sigma\sigma}(i, i) e^{iQ \cdot j} \right\}
\]

\[- \mu N(1 - \delta) - 2JN(1 - \delta)^2.
\]

(B27)

In momentum space, we note that for the case of square lattice,

\[
D_{\sigma\sigma}(j, i) = \frac{1}{(2\pi)^2} \sum_{k, k'} d_{\sigma\sigma}(k, k') e^{i k \cdot j} e^{-i k' \cdot i},
\]

(B28)

and thus

\[
\sum_{\langle ij \rangle} D_{\sigma\sigma}(j, i) = \frac{1}{(2\pi)^2} \sum_{\langle ij \rangle} d_{\sigma\sigma}(k, k') e^{i k \cdot j} e^{-i k' \cdot i}
\]

\[
= \frac{1}{(2\pi)^2} \sum_{k, k'} d_{\sigma\sigma}(k, k') \sum_{\langle ij \rangle} e^{i k \cdot j} e^{-i k' \cdot i}.
\]

(B29)
Using

\[ \sum_{\langle ij \rangle} e^{i\mathbf{k}_i \cdot \mathbf{r}_j} e^{-i\mathbf{k}_i' \cdot \mathbf{r}_j} = \sum_{i,a} e^{i\mathbf{k}_i \cdot (\mathbf{i} + \mathbf{a})} e^{-i\mathbf{k}_i' \cdot \mathbf{i}} \]

\[ = \sum_{i} e^{i(\mathbf{k}_i - \mathbf{k}_i') \cdot \mathbf{i}} \sum_{a} e^{i\mathbf{a}} \]

\[ = (2\pi)^2 \delta(\mathbf{k} - \mathbf{k}') \times 2(\cos k_x a + \cos k_y a), \quad (B30) \]

with \( \mathbf{a} \), the four nearest neighbour sites around site \( i \), we obtain from Eq. (B29),

\[ \sum_{\langle ij \rangle} D_{\sigma\sigma}(j,i) = \sum_{k,k'} d_{\sigma\sigma}(\mathbf{k}, \mathbf{k}') \delta(\mathbf{k} - \mathbf{k}') \times 2(\cos k_x a + \cos k_y a) \]

\[ = \sum_{k} d_{\sigma\sigma}(\mathbf{k}) 2(\cos k_x a + \cos k_y a) \quad (B31) \]

Noting that

\[ \sum_{\langle ij \rangle} D_{\sigma\sigma}(j,i) e^{i\mathbf{Q} \cdot \mathbf{r}_j} = \frac{1}{(2\pi)^2} \sum_{\langle ij \rangle} \sum_{k,k'} d_{\sigma\sigma}(\mathbf{k}, \mathbf{k}') e^{i\mathbf{k}_i \cdot \mathbf{r}_j} e^{-i\mathbf{k}_i' \cdot \mathbf{i}} e^{i\mathbf{Q} \cdot \mathbf{r}_j} \]

\[ = \frac{1}{(2\pi)^2} \sum_{k,k'} \sum_{i,a} d_{\sigma\sigma}(\mathbf{k}, \mathbf{k}') e^{i\mathbf{k}_i \cdot (\mathbf{i} + \mathbf{a})} e^{-i\mathbf{k}_i' \cdot \mathbf{i}} e^{i\mathbf{Q} \cdot \mathbf{r}_j} \]

\[ = \frac{1}{(2\pi)^2} \sum_{k,k'} \sum_{i} d_{\sigma\sigma}(\mathbf{k}, \mathbf{k}') e^{i(\mathbf{k}_i - \mathbf{k}_i') \cdot \mathbf{i}} \sum_{a} e^{i\mathbf{a}} \]

\[ = \sum_{k,k'} d_{\sigma\sigma}(\mathbf{k}, \mathbf{k}') \delta(\mathbf{k} - \mathbf{k}') 2(\cos k_x a + \cos k_y a) \]

\[ = \sum_{k} d_{\sigma\sigma}(\mathbf{k} + \mathbf{Q}) 2(\cos k_x a + \cos k_y a), \quad (B32) \]

and

\[ \sum_{\langle ij \rangle} D_{\sigma\sigma}(i,i) e^{i\mathbf{Q} \cdot \mathbf{r}_j} = \frac{1}{(2\pi)^2} \sum_{\langle ij \rangle} \sum_{k,k'} d_{\sigma\sigma}(\mathbf{k}, \mathbf{k}') e^{i\mathbf{k}_i \cdot \mathbf{r}_j} e^{-i\mathbf{k}_i' \cdot \mathbf{i}} e^{i\mathbf{Q} \cdot (\mathbf{i} + \mathbf{a})} \]

\[ = \frac{1}{(2\pi)^2} \sum_{k,k'} \sum_{i} d_{\sigma\sigma}(\mathbf{k}, \mathbf{k}') e^{i(\mathbf{k}_i + \mathbf{k}_i') \cdot \mathbf{i}} \sum_{a} e^{i\mathbf{Q} \cdot \mathbf{a}} \]

\[ = -4 \sum_{k} d_{\sigma\sigma}(\mathbf{k} + \mathbf{Q}), \quad (B33) \]

we obtain, with the use of Eq. (B31) above,

\[ E = -t\delta \sum_{k,\sigma} 2(\cos k_x a + \cos k_y a) d_{\sigma\sigma}(\mathbf{k}) + \frac{J_m}{2} \sum_{k,\sigma} \sigma (\cos k_x a + \cos k_y a) d_{\sigma\sigma}(\mathbf{k}, \mathbf{k} + \mathbf{Q}) \]

\[ -J_m \sum_{k,\sigma} \sigma d_{\sigma\sigma}(\mathbf{k}, \mathbf{k} + \mathbf{Q}) - \mu N (1 - \delta) - 2J N (1 - \delta)^2. \quad (B34) \]
Introducing $\gamma_k = \frac{(\cos k_x a + \cos k_y a)}{2}$, the above equation is rewritten

$$E = \sum_{k,\sigma} \left[ -4t \delta \gamma_k d_{\sigma\sigma}(k, k) + J m \sigma (\gamma_k - 1) d_{\sigma\sigma}(k, k + Q) \right] - \mu N (1 - \delta) - 2J N (1 - \delta)^2. \quad (B35)$$

We take $k$-sum in the reduced Brillouin zone i.e., half the first Brillouin zone, to write

$$\sum_{k,\sigma} \gamma_k d_{\sigma\sigma}(k, k) = \sum_{k,\sigma} \left[ (\gamma_k - 1) d_{\sigma\sigma}(k, k + Q) + (\gamma_{k+Q} - 1) d_{\sigma\sigma}(k + Q, k + Q) \right] = \gamma_k \sum_{k,\sigma} \left[ d_{\sigma\sigma}(k, k) - d_{\sigma\sigma}(k + Q, k + Q) \right] , \quad (B36)$$

and

$$\sum_{k,\sigma} (\gamma_k - 1) d_{\sigma\sigma}(k, k + Q) = \sum_{k,\sigma} \left[ (\gamma_k - 1) d_{\sigma\sigma}(k, k + Q) + (\gamma_{k+Q} - 1) d_{\sigma\sigma}(k + Q, k + 2Q) \right] = \sum_{k,\sigma} \left[ (\gamma_k - 1) d_{\sigma\sigma}(k, k + Q) - (\gamma_k + 1) d_{\sigma\sigma}(k + Q, k) \right] . \quad (B37)$$

where we used the nesting condition, $\gamma_{k+Q} = -\gamma_k$. The symbol $'$ indicates momentum summation in the reduced Brillouine zone. By applying (B36) and (B37) to (B35), we obtain the ground state energy of the two-dimensional hole-doped systems,

$$E = \sum_{k,\sigma} \left[ -4t \delta \gamma_k d_{\sigma\sigma}(k, k) + J m \sigma (\gamma_k - 1) d_{\sigma\sigma}(k, k + Q) \right] \quad \quad \quad \quad \quad \quad \quad \quad \quad (B38)$$

Realizing the equivalence between $d_{\sigma\sigma}(k,k')$ and $f_k^\dagger f_{k'}$ from the inspection of Eq. (23.b) and Eq. (27), we consider the transformation matrix in

$$\begin{bmatrix} f_k & f_{k+Q}^\dagger \end{bmatrix} \begin{bmatrix} -4t \delta \gamma_k & J m \sigma (\gamma_k - 1) \\ -J m \sigma (\gamma_k + 1) & 4t \delta \gamma_k \end{bmatrix} \begin{bmatrix} f_{k}^\dagger \\ f_{k+Q} \end{bmatrix} ,$$

From the determinant of the above matrix, we obtain

$$\epsilon^2 - (4t \delta \gamma_k)^2 + J^2 m^2 (\gamma_k^2 - 1) = 0. \quad (B39)$$

The dispersion energy is then
\[ \epsilon = \pm \sqrt{(4t\delta\gamma_k)^2 - J^2m^2(\gamma_k^2 - 1)}. \]  

(B40)

Considering summation over the up and down spins, we write the dispersion energy,

\[ \omega_k = \sum_{\sigma} \sqrt{[(4t\delta)^2 - (Jm)^2] \gamma_k^2 + J^2m^2} = 2\sqrt{[(4t\delta)^2 - (Jm)^2] \gamma_k^2 + J^2m^2}. \]  

(B41)

The ground state energy of the hole-doped two-dimensional antiferromagnet is then

\[ E_k = \pm 2 \sum_k \sqrt{[(4t\delta)^2 - (Jm)^2] \gamma_k^2 + J^2m^2 - \mu(1 - \delta) - 2J(1 - \delta)^2}. \]  

(B42)

For the paramagnetic states, \( m = 0 \), we obtain

\[ \omega_k = \pm 4t\delta \gamma_k. \]  

(B43)

For the undoped antiferromagnet, that is, \( \delta = 0 \), the dispersion energy of the antiferromagnetic magnon is

\[ \omega_k = 2Jm \sqrt{1 - \gamma_k^2}. \]  

(B44)
REFERENCES

1. R. Floreanini and R. Jackiw, Phys. Rev. D 37, 2206 (1988); references therein.

2. B. F. Hatfield, in Quantum Field Theory of Point Particles and Strings (Addison-Wesley, New York, 1992); references therein.

3. T. Barnes and G. I. Ghandour, Phys. Rev. D 22, 924 (1980); references therein.

4. K. Zrembo, Mod. Phys. Lett. A 13 (1998) 1709-1718; J. Cruz, J. M. Izquierdo, D. J. Navarro and J. Navarro-Salas, Phys. Rev. D 58 (1998) 044010; D. Vautherin and T. Matsui, Phys. Rev. D 55 (1997) 4492-4495; E. Benedict, R. Jackiw and H.-J. Lee, Phys. Rev. D 54 (1996) 6213-6225; Lee Smolin and Chopin Soo, Nucl. Phys. B 449 (1995) 289.

5. C. Kiefer and A. Wipf, Ann. Phys. 236, 241 (1994).

6. R. Jackiw and A. Kerman, Phys. Lett. A 71, 158 (1979).

7. A. Duncan, H. Meyer-Ortmanns, and R. Roskies, Phys. Rev. D 36, 3788 (1988).

8. O. Eboli, R. Jackiw and S. Y. Pi, Phys. Rev. D 37, 3557 (1988).

9. H. -j. Lee, K. Na and J. H. Yee, Phys. Rev. D. 51, 3125 (1995); S. K. Kim, J. Yang, K. S. Soh and J. H. Yee, Phys. Rev. D. 40, 2647 (1989).

10. H. S. Noh, S. K. You and C. K. Kim, to appear in Int. J. Mod. Phys. B.

11. H. S. Noh, C. K. Kim and K. Nahm, Phys. Lett. A 204, 162 (1995); ibid, A 210, 317 (1996).

12. Sul-Ah Ahn, I. E. Dikstein and Sung Ho Suck Salk, Physica C 282-287, 1707-1708 (1997); in the present paper, a correction in the derivation of dispersion energy is made with full details in Appendix.

13. Zhe Chang, Phys. Rev. B 53, 1171(1996); ibid, B 57, 2979 (1998-I).

14. C. Kittel, in Quantum Theory of Solids (2nd ed., John Wiley & Sons, New York, 1987).
15 F. Onufrieva and J. Rossat-Mignod, Phys. Rev. B 52, 7572 (1995-II); ibid, B 50, 12935 (1994-I).

16 G. Baskaran, Z. Zou and P. W. Anderson, Solid State. Commun. 63, 973 (1987); G. Baskaran and P. W. Anderson, Phys. Rev B 37, 580 (1988).

17 P. A. Bares, G. Blatter, and M. Ogata, Phys. Rev. B. 44, 130 (1991).

18 J. W. Negele and H. Orland, in *Quatum Many Particle Systems* (Addison-Wesley, New York, 1987).