Calculation of a beam with external reinforcement. Contact layer model

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Abstract. The article is devoted to the stress-strain state study of the beam, with external reinforcement in the tensile zone. The classical Euler-Bernoulli hypotheses are used for the beam. The interaction of the beam with the reinforcement layer is carried out using the contact layer. In order to simplify the previously obtained solution, the flexural stiffness of the reinforcement layer is not taken into account. As a result, the solution of the resolving equations system for this model has been obtained. It is shown that the obtained solution allows satisfying all boundary conditions, including the equality to zero of the shear stresses at the corner points. The influence of various mechanical and geometric parameters on the stress-strain state of the beam, the reinforcement layer and the contact layer is analyzed. The influence of the introduced simplifications on the solution results is analyzed.

Introduction
Increasing and restoring the bearing capacity of the existing beams is one of the most common construction tasks. One of the solutions to this problem is an external amplification device. In this case, as a rule, the stretched zone of the beams is strengthened. The figure below shows a general view of the model of a beam with external reinforcement in the tension zone.

Figure 1. General view of the beam model with external reinforcement

The classical Euler-Bernoulli hypotheses are used for the beam taking into account axial displacements $u_0(x,0)=u_{0,0}(x)$. 
The contact layer is considered as an orthotropic medium, with such properties that it can be represented as a set of elastic short rods not connected to each other. In such a medium, only shear stresses can occur in the shear plane and normal stresses perpendicular to the shear plane. The main characteristics of the contact layer to be determined from the macro experiment are its rigidity and thickness [5, 6, 13].

Reinforcement layer height $h_1$ is generally much less than the height of the beam section $h_0$, i.e., $h_0 \gg h_1$. This allows to neglect bending stiffness $D_1$ of the reinforcement layer, as well as the bending moments and shear forces arising in it. As a result, a uniaxial stress-strain state appears in the reinforcement layer.

The influence of this simplification on the stress-strain state of the model as a whole (in comparison with the model taking into account the bending stiffness of the reinforcement layer) will be analyzed below.

1. Resolving Equations

The system of resolving equations (initial) for a two-layer beam was obtained earlier in [1]. Let us rewrite it without changes.

$$
\begin{aligned}
D_0 \frac{d^4 v_0}{dx^4} + \frac{\eta G^*}{h^*} (v_0 - v_1) + B_0 \left( \frac{h_0}{2} + h^* \right) \frac{d^3 u_{0,0}}{dx^3} = q, \\
D_1 \frac{d^4 v_1}{dx^4} + \frac{\eta G^*}{h^*} (v_0 - v_1) + B_1 \left( \frac{h_0}{2} + h^* \right) \frac{d^3 u_{0,0}}{dx^3} = 0, \\
\frac{d^4 u_{0,0}}{dx^4} B_0 \left( h^* \right)^2 - \frac{d^2 u_{0,0}}{dx^2} B_0 + \frac{B_0 + B_1}{B_1 h^*} \frac{dv}{dx} h^* \left( \frac{h_0}{2} + h^* \right) \frac{d^3 u_{0,0}}{dx^3} + \frac{B_0 + B_1}{B_0 h^*} \frac{dv}{dx} h^* \left( \frac{h_0}{2} + h^* \right) = 0,
\end{aligned}
$$

wherein $D_0 = E_0 h_0^3 / 12$; $D_1 = E_1 h_1^3 / 12$; $B_0 = E_0 h_0$; $B_1 = E_1 h_1$; $E^* = \eta G^*$. The beam width $b$ is taken equal to one. Shear stresses in the contact layer are related to the longitudinal force $N_0$ by the expression:

$$
\tau^* = -\frac{dN_0}{dx}.
$$

Longitudinal forces are related to axial displacements by the equality:

$$
N_0 = B_0 \frac{du_{0,0}}{dx}.
$$

Based on the static equilibrium conditions of the system as a whole $N_1 = -N_0$.

To solve the problem formulated in this work, the following assumptions should be made in the original system of equations (1).

1) To neglect the bending stiffness of the layer 1 (reinforcement layer), compared to the beam bending stiffness, i.e. to accept $D_1 = 0$.

2) In view of assumption 1, the vertical displacements of layer 1 are equal to the layer 0 displacements $v_1 = v_0 = v$.

Taking these assumptions and equality (3) into consideration, the system of equations (1) will take the form:

$$
\begin{aligned}
D_0 \frac{d^4 v}{dx^4} + \left( \frac{h_0}{2} + h^* \right) \frac{d^2 N_0}{dx^2} = q, \\
\left( \frac{h^*}{2} \right)^2 \frac{d^4 N_0}{dx^4} - \frac{d^2 N_0}{dx^2} + \frac{1}{G^*} \left( \frac{1}{B_0} + \frac{1}{B_1} \right) N_0 - \frac{1}{h^*} \left( \frac{h_0}{2} + h^* \right) \frac{d^2 v}{dx^2} = 0.
\end{aligned}
$$
2. The problem solution

The system of equations (4) can be reduced to one resolving equation for the function \( \nu \).

\[
\frac{d^8\nu}{dx^8} - 2\lambda_1 \frac{d^6\nu}{dx^6} + 2\lambda_1\lambda_2\lambda_3\lambda_5 \frac{d^4\nu}{dx^4} = 2\lambda_1\lambda_2\lambda_3\psi.
\]

This equation uses the notation below.

\[
\lambda_1 = \left(\frac{h}{r}\right)^2; \quad \lambda_2 = \frac{G^*D_0}{h^*}; \quad \lambda_3 = \frac{B_0 + B_1}{B_0B_1D_0}; \quad \lambda_4 = \frac{h_0 + h_1 + 2h^*}{D_0}; \quad \lambda_5 = \frac{\lambda_4^2}{4\lambda_3} + 1; \quad \psi = \frac{q}{D_0}.
\]

The solution to the equation (5) is the following expression:

\[
\nu = \frac{1}{s_1^2} \left( C_1 e^{s_1 x} + C_3 e^{-s_1 x} \right) + \frac{1}{s_2^4} \left( C_2 e^{s_2 x} + C_4 e^{-s_2 x} \right) + \frac{C_5 x^3}{6} + \frac{C_6 x^2}{2} + \frac{C_7 x + C_8}{24\lambda_5^2} + \frac{\psi x^4}{24\lambda_5^3},
\]

in which

\[
s_1 = \sqrt{\lambda_1 - \sqrt{\lambda_1 (\lambda_1 - 2\lambda_2\lambda_3\lambda_5)}}, \quad s_2 = \sqrt{\lambda_1 + \sqrt{\lambda_1 (\lambda_1 - 2\lambda_2\lambda_3\lambda_5)}}.
\]

Longitudinal force \( N_0 \) expressed by the displacement function \( \nu \) from the system of equations (4).

\[
N_0 = \eta_1 \frac{d^2\nu}{dx^2} + \eta_2 \frac{d^4\nu}{dx^4} + \eta_3 \frac{d^2\nu}{dx^2} + \eta_2\psi,
\]

in which

\[
\eta_1 = \frac{1}{\lambda_2h_2\lambda_3\lambda_5}; \quad \eta_1 = \frac{2}{\lambda_2h_2\lambda_3\lambda_5}; \quad \eta_3 = \frac{\lambda_4}{2\lambda_3}.
\]

After substitution (6) in (7), we get the final expression for the longitudinal force \( N_0 \).

\[
N_0 = \frac{\chi_1}{s_1^3} \left( C_1 e^{s_1 x} + C_3 e^{-s_1 x} \right) + \frac{\chi_2}{s_2^4} \left( C_2 e^{s_2 x} + C_4 e^{-s_2 x} \right) + \eta_3 C_5 x + \eta_3 C_6 + \frac{\eta_3 x^2 - 2\eta_3 \psi (1 - \psi)}{2\lambda_5^3}.
\]

in which

\[
\chi_1 = \eta_1 s_1^4 - \eta_2 s_1^2 + \eta_3; \quad \chi_2 = \eta_1 s_1^2 - \eta_2 s_2^2 + \eta_3.
\]

Shear stresses in the contact layer are determined from the expression (2) taking into account (8).

\[
\tau^* = \frac{\chi_1}{s_1^3} \left( C_1 e^{s_1 x} - C_3 e^{-s_1 x} \right) - \frac{\chi_2}{s_2^2} \left( C_2 e^{s_2 x} - C_4 e^{-s_2 x} \right) - \eta_3 C_5 - \frac{\eta_3 x^2 - \eta_3 \psi}{\lambda_5^3}.
\]

Stresses and displacements in the beam and reinforcement layer are determined from the expressions

\[
u_0 = u_{0,0} - y_0 \frac{dv}{dx}; \quad \nu_0 = \frac{B_0}{B_1} u_{0,0}; \quad \sigma_{x,0} = E_0 \left( \frac{du_{0,0}}{dx} - y_0 \frac{d^2\nu_0}{dx^2} \right); \quad \sigma_{x,1} = E_1 \frac{du_{0,1}}{dx} + \frac{\tau_{xy,0}}{2}; \quad \tau_{xy,0} = E_0 \left( \frac{\eta_0 - \frac{h_0^2}{4}}{3} \right) \frac{d^4\nu_0}{dx^4} + E_0 \left( \frac{\eta_0 - \frac{h_0^2}{4}}{4} \right) \frac{d^3\nu_0}{dx^3} - \frac{y_0 d\tau^*}{2} - \frac{q}{2}; \quad \sigma_{x,1} = \eta_0 \frac{h_0}{2} \frac{h_0}{2}.
\]

The equivalently static internal forces in the beam and the reinforcement layer are determined from the stresses (10), by integrating over the area of the layer. Let us write them down.

\[
N_k = \int_{-h/2}^{h/2} \sigma_{x,k} dy; \quad M_0 = \int_{-h/2}^{h/2} \tau_{xy,0} dy; \quad Q_k = \int_{-h/2}^{h/2} \tau_{xy,0} dy.
\]

Considering (10), we find:

\[
N_0 = B_0 \frac{du_{0,0}}{dx}; \quad M_0 = -D_0 \frac{d^2\nu_0}{dx^2}; \quad Q_k = -D_0 \frac{d^3\nu_0}{dx^3} + \frac{\tau^* h_0}{2}.
\]

The total forces in the beam are recorded below:
\[ N_{\text{beam}} = 0; \quad M_{\text{beam}} = -D_0 \frac{d^2 v}{dx^2} - N_1 \frac{h_0}{2} + 2h_k + h_l; \quad Q_{\text{beam}} = -D_0 \frac{d^3 v}{dx^3} \frac{dN_1}{dx} \frac{h_0}{2} + 2h_k + h_l. \]  \hspace{1cm} (12)

The total forces in the beam calculated using the expressions (12), should correspond to the forces occurring in a single-layer beam (due to static equivalence). The fulfillment of this condition is one of the most important checks of the obtained solution.

3. Border conditions

Unknown constants of integration included in (6), (8) and (9) are determined from 8 boundary conditions. As an example, let us consider a hinged-supported beam with edges free from the longitudinal forces.

\begin{align*}
\nu_0 \left(\pm \frac{l}{2}\right) &= 0; \quad M_0 \left(\pm \frac{l}{2}\right) = 0; \\
N_0 \left(\pm \frac{l}{2}\right) &= 0; \quad \tau^* \left(\pm \frac{l}{2}\right) = 0.
\end{align*} \hspace{1cm} (13)

These boundary conditions correspond to the symmetric stress-strain state of the beam.

Analyzing the boundary conditions (13) and the expressions (6), (8) and (9) we come to the conclusion that to satisfy the boundary conditions it is necessary to take
\[ C_1 = C_3; \quad C_2 = C_4; \quad C_5 = 0; \quad C_7 = 0. \] \hspace{1cm} (14)

Taking into account (14) the expressions (6), (8) and (9) take the form:
\begin{align*}
v &= \frac{2}{s_1^2} \frac{C_1 \cosh(s_1 x)}{2} + \frac{2}{s_2^2} \frac{C_2 \cosh(s_2 x)}{2} + C_8 + \frac{\psi \lambda x^2}{24 \lambda_5}; \\
\frac{-M_0}{D_0} &= \frac{d^2 v}{dx^2} = \frac{2}{s_1^2} \frac{C_1 \cosh(s_1 x)}{2} + \frac{2}{s_2^2} \frac{C_2 \cosh(s_2 x)}{2} + C_6 + \frac{\psi \lambda x^2}{2 \lambda_5}; \\
N_0 &= \frac{2x_1}{s_1^2} \frac{C_1 \cosh(s_1 x)}{2} + \frac{2x_2}{s_2^2} \frac{C_2 \cosh(s_2 x)}{2} + \eta_3 C_6 + \psi \frac{\eta_3 x^2}{2 \lambda_5} - 2 \eta_2 (1 - \lambda_5); \\
\tau^* &= \frac{2x_1}{s_1^2} \frac{C_1 \cosh(s_1 x)}{2} - \frac{2x_2}{s_2^2} \frac{C_2 \cosh(s_2 x)}{2} - \psi \frac{\eta_3 x}{\lambda_5}.
\end{align*} \hspace{1cm} (15)

Constants \( C_1, C_2, C_6, C_8 \), included in the formulas (15), are determined from the boundary conditions (13) on the right (or left) edge of the beam. In matrix form, the system of linear algebraic equations to determine the integration constants is as follows:
\[ A \cdot c = b \Rightarrow c = A^{-1} b, \] \hspace{1cm} (16)

in which
Solving a system of linear algebraic equations (16) for given values of geometrical and physical-mechanical parameters of the model is not difficult. However, the general form of this solution is quite cumbersome, so it is not presented here. The numerical solution examples of a similar problem can be found in [4].

4. Calculation example

Let us consider a model of a two-layer beam with the following geometrical and physical and mechanical parameters:

\[ E_0 = 1 \cdot 10^4 \text{MPa}; \ E_1 = 2 \cdot 10^5 \text{MPa}; \ G^* = 200 \text{MPa}; \ \eta = 2; \]

\[ h_0 = 20 \text{mm}; \ h_1 = 2 \text{mm}; \ h^* = 1 \text{mm}; \ l = 200 \text{mm}. \]

These parameters will be called basic. We take the value of the applied load equal to one.

Figure 3 shows the graphs of the displacements and internal forces distribution in the beam, the reinforcement layer and the contact layer, calculated by the formulas (10).

To assess the fulfillment of the conditions for static equilibrium of the beam as a whole, Figure 4 shows the graphs of the internal forces distribution, calculated by the formulas (12).

The calculation results for three-layer plates and shells are given in the works [17, 18, 19].
The greatest difference in the results obtained from solving the system of equations (1) and (4), observed for tangential stresses and transverse forces in a narrow zone of the edge effect, presented in the figures below.

**Figure 5.** Transverse forces in the beam.
- 1 – from system solution (1).
- 2 – from system solution (4)

It is obvious that the decisive influence on the differences in the stress-strain state determined from the solution of the system of equations (1) and (4), the thickness (and, accordingly, bending stiffness) of the reinforcement layer. Analysis of the influence of the reinforcement layer thickness on the error in calculating the vertical displacements of the middle of the beam and the maximum value of shear stresses in the contact layer are shown in Figures 7 and 8.

**Figure 7.** Calculation error of vertical displacements

\[ \frac{\Delta v}{v} \]

- 1– \( \frac{E_1}{E_0} = 1 \)
- 2– \( \frac{E_1}{E_0} = 10 \)
- 3– \( \frac{E_1}{E_0} = 20 \)

**Figure 8.** The error in calculating the maximum shear stresses.

\[ \Delta \tau^* \]

- 1– \( \frac{E_1}{E_0} = 1 \)
- 2– \( \frac{E_1}{E_0} = 10 \)
- 3– \( \frac{E_1}{E_0} = 20 \)

The graphs of the calculation error dependence for bending moments, longitudinal forces and displacements have a character that qualitatively and quantitatively coincides with the graphs shown in Figure 7.

**Summary**

A solution that makes it possible to predict the stress-strain state of beams with external reinforcement in the stretched zone based on the contact layer model has been obtained in this work.

Other models of interaction of layers (including elastic-plastic) can be found in [7, 14, 15, 16].

It is shown that the simplifications introduced into the original system of equations (1) (taking into account the bending stiffness of the reinforcement layer) do not significantly affect the results of
determining the forces and stresses in the structure with the reinforcement layer thickness not exceeding 10% of the beam height. In this case, all boundary conditions are satisfied and the conditions for static equilibrium of the beam as a whole are satisfied.

References

[1] Turusov R, Andreev V and Tsybin N 2020 IOP Conference Series Materials Science and Engineering 913 032053.
[2] Andreev V, Tsybin N and Turusov R 2019 E3S Web of Conferences 97 04071.
[3] Tsybin N, Turusov R and Andreev V 2018 IOP Conference Series Materials Science and Engineering 456 012063.
[4] Andreev V, Turusov R and Tsybin N 2016 Procedia Engineering 153 59.
[5] Turusov R, Bogachev E and Andreev V 2019 J. Phys.: Conf. Ser. 1425 012188.
[6] Turusov R, Bogachev E and Elakov A 2016 Mechanics of composite materials and structures 22 3 430-51.
[7] Glagolev V and Markin A 2018 J. Phys.: Conf. Ser. 973 012003.
[8] Korol E 2018 IOP Conference Series Materials Science and Engineering, 456 012075.
[9] Korol E, Tho V 2020 IOP Conference Series Materials Science and Engineering 775 012115.
[10] Chepurnenko A, Andreev V, Beskopylny A and Jazyev B 2016 MATEC Web of Conferences 67 06059.
[11] Krivoshapko S, Shambina S and Hyeng C 2017 Materials Science Forum 895 45.
[12] Andreev V, Turusov R and Tsybin N 2018 MATEC Web of Conferences 251 04066.
[13] Turusov R 2019 Materials Science Forum 974 638.
[14] Sebastianiuk P, Perkowski D and Kulchytsky-Zhyhailo R 2020 International Journal of Mechanical Sciences 182 105734.
[15] Anasiewicz K, Kuczmaszewski J 2019. Materials. 12(23) 3911.
[16] Glagolev V, Markin A and Fursaev A 2016 PNRPU Mechanics Bulletin 2 34.
[17] Chepurnenko A, Mailyan L and Jazyev B 2016 Procedia Engineering 165 990.
[18] Yazyev B, Chepurnenko A and Savchenko A 2018 Materials Science Forum 935 144.
[19] Chepurnenko A 2017 Magazine of Civil Engineering 76 156.