Localization in periodically modulated speckle potentials

O.S. Vershinina 1, E.A. Kozinov 1,2, T.V. Laptyeva1,2, S.V. Denisov2,3, and M.V. Ivanchenko 21

1 Institute of Supercomputing Technologies, Lobachevsky State University of Nizhny Novgorod, Russia
2 Institute of Information Technologies, Mathematics and Mechanics, Lobachevsky State University of Nizhny Novgorod, Russia
3 Department of Theoretical Physics, University of Augsburg, Germany

Disorder in a 1D quantum lattice induces Anderson localization of the eigenstates and drastically alters transport properties of the lattice. In the original Anderson model, the addition of a periodic driving increases, in a certain range of the driving’s frequency and amplitude, localization length of the appearing Floquet eigenstates. We go beyond the uncorrelated disorder case and address the experimentally relevant situation when spatial correlations are present in the lattice potential. Their presence induces the creation of an effective mobility edge in the energy spectrum of the system. We find that a slow driving leads to resonant hybridization of the Floquet states, by increasing both the participation numbers and effective widths of the states in the strongly localized band and decreasing values of these characteristics for the states in the quasi-extended band. Strong driving homogenizes the bands, so that the Floquet states loose compactness and tend to be spatially smeared. In the basis of the stationary Hamiltonian, these states retain localization in terms of participation number but become de-localized and spectrum-wide in term of their effective widths. Signatures of thermalization are also observed.

Anderson localization in disordered systems is a fundamental phenomenon that is still posing new puzzles and bringing new surprises [1,2,3]. The original problem of non-interacting quantum particles [1] was studied thoroughly and has been placed in a broad context, resulting in experimental observations of the localization with matter [2][3], electromagnetic [9], and acoustic waves [10].

The effect of periodic modulations on the localization also received considerable attention. It was found that the localization length increases under the low frequency driving (though non-monotonously with the driving amplitude) and decreases in the opposite limit of the fast driving [11]. The increase of the localization length was attributed to the induced interaction between the particle path channels, with those characterized by weakest localization properties making a dominant contribution. In contrast, the high-frequency driving diminishes time-averaged hopping amplitudes [11][13] and enhance the localization, an effect reminiscent of the dynamic localization [14][15]. Recently, it has been shown that the multi-frequency driving can substantially increase the localization length [16], and the complete de-localization can be achieved with driven quasi-periodic potentials [17].

The existing results, however, address the original Anderson set-up, with on-site energies being random and uncorrelated variables. At the same time, the presence of correlations is inherent to the optical speckle potentials, used in the experiments with atomic Bose-Einstein condensates [18]. Importantly, a finite correlation length leads to emergence of an effective mobility edge separating the bands with localization lengths differing by orders of magnitude [19][21].

Application of periodic modulations to a system with correlated disorder evokes a set of intriguing questions. For example, will the driving modify quasi-extended states in a way different from it does in the case of uncorrelated disorder? Can the driving induce a coupling across the mobility edge? Can the co-existence of the strongly localized and quasi-extended states change the localization picture known for the modulated system with the uncorrelated disorder?

In this Letter we study the fate of localization in periodically and slowly driven 1D speckle potential. We find that the driving creates a more intricate picture than in the case of uncorrelated disorder. Increase of the driving amplitude leads to the appearance of complex structure in the set of the induced Floquet states. Participation numbers grow for strongly localized states and decrease for quasi-extended. At the same time both kinds of states loose compactness and become sparse and weakly localized. Similar transformation occurs in the basis of the stationary Hamiltonian, where the Floquet states become de-localized in terms of the effective width. These changes are underpinned by resonant hybridization of Floquet states, as our numerical results and perturbative arguments indicate. Strong driving leads to a mixing of states, by inducing a homogeneity of their localization properties and brings traits of thermalization, as supported by their center of mass and average energy statistics.

We consider the dynamics of a quantum particle in a speckle potential described by a periodically-modulated tight-binding Hamiltonian,

$$H(t) = \sum_j \epsilon_j \left( 1 + A \cos \omega t \right) b_j^\dagger b_j + \frac{1}{2} \left( b_j^\dagger b_{j+1} + b_{j+1}^\dagger b_j \right),$$  

(1)

where $b_j$ and $b_j^\dagger$ are the annihilation and creation operators of a particle at a site $j$, $j = 1,...,N$, and $\epsilon_j > 0$ are correlated random on-site energies of speckle potential [18]. $A$ and $\omega$ are the amplitude and frequency of the
The localization length of the latter is one or two orders of magnitude larger than that of the former so that in finite systems the quasi-extended states can be treated as effectively de-localized [21]. At higher energies, the strong localization is restored as a “semi-classical” localization.

Dynamics of the driven system can be fully evaluated in terms of the so-called “Floquet eigenstates” [22,24]. They are the eigenstates of the unitary Floquet propagator $U_T = T \exp \left[ -\frac{i}{\hbar} \int_0^T H(\tau) d\tau \right]$, where $T$ is the time-ordering operator. The particular structure of the Floquet propagator, and thus the localization properties of Floquet states, depend on the parameters of the driving. This dependence— in the context of localization— is the main objective of our study.

The Floquet states are found by numerically integrating Eq. (2) over one period of driving, by using a high-precision Magnus-Chebyshev scheme [25] with 200 integration steps per period. We choose $N/D = 4$ so that the correlation length $\xi \approx 3.5$ and make the core simulations for $N = 1600 \approx 450\xi$. The disorder strength is set to $I_0 = 0.2$ so that $\lambda^M \approx 0.5$, ensuring a substantial part of the states of both kinds; see Fig.1(a). Finally, zero boundary conditions are assumed.

Localization properties are quantified by the standard measures used in the field. Participation number is normalized by the system size, $P = 1/(N \sum |\psi_j|^4)$, so that it manifests the effective ratio of non-zero components in a Floquet vector. Their second moment, $m_2 = \sum (j - j')^2 |\psi_j|^2$, defines an effective width, $L = \sqrt{2m_2}/N$ (also normalized matching an effective uniform brick). Finally, we calculate compactness index, $\zeta = (PN)^2/m_2$, which ranges from $\zeta \sim 0$ for sparsely populated states (a few effectively non-zero well-separated elements) to $\zeta \sim 3$ for random sets to $\zeta = 12$ for an ideally compact state (uniformly populated neighbor elements). These quantities are averaged over a period of driving [26]. Note, that for a localized state, which length does not scale with the system size, one has $P, L \sim 1/N$, while $\zeta$ will not change.

Numerical results reveal two opposite trends in the localization of Floquet states (depending on their positions with respect to the mobility edge) when the modulation amplitude increases, $A \gg 0.01$; see Fig.2. Participation number increases for the strongly localized states and decreases for the quasi-extended ones, almost equating them when the driving is strong, $A = 1$, Fig.2(a). Similar scenario is observed for the effective widths of the states although strong driving settles it at the values much greater than participation numbers, see Fig.2(b).

The emerging complex structure of Floquet states is manifested in the compactness index, which rapidly drops below $\zeta = 1$ for both kinds of Floquet states, Fig.2(c). This reflects the hybridization of states and indicates development of the highly fragmented spatial structure. Local outbursts in the site population numbers, scattered over the lattice, give an ultimate image of Floquet states in the direct basis under strong driving. Numerics for a larger system size, $N = 6400, D = 1600$, reveals qualitatively the same results and a fair quantitative match, when $P$ and $L$ are scaled with the system size as described in the above, see Fig.2(a,b,c).

Further insight can be obtained by using the basis of
the stationary Hamiltonian, \{ζ^{(l)}\}, such that \( H \zeta^{(l)} = \lambda_l \zeta^{(l)} \), \( A = 0 \). By using expansion \( \psi = \sum_l \varphi^{(l)} \), Eq. (2) can be rewritten as

\[ i \dot{\varphi}_k = \lambda_k \varphi_k + A \cos \omega t \sum_{k'} J_{k,k'} \varphi_{k'} , \tag{3} \]

where \( J_{k,k'} = \sum_j \epsilon_j \zeta_j^{(k)} \zeta_j^{(k')} \) are the driving-induced overlap coefficients, and the stationary states are numbered according to the eigenvalues, \( \lambda_k \), in increasing order.

Localization in this basis can also be characterized by \( P, \) \( L \), and \( \zeta \). For zero driving the Floquet states coincide with the stationary basis, and are, therefore, compact: \( P = 1 \), while \( m_2, \) \( L \) and \( \zeta \) are ill-defined. As the driving amplitude increases, we observe that participation number grows for all states, along with the effective widths. Notably, the former remains much less than one, indicating that the Floquet states expand over a small subpart of the whole basis only, Fig. 2(d). At the same time, the latter becomes of the order of the system size, manifesting hybridization between the states from different parts of the spectrum, Fig. 2(e). Remarkably, unlike in the direct basis, the normalized effective width, \( L \), does not decrease as the system size grows, while the normalized participation number does, \( P \sim 1/N \), see results for \( N = 6400, D = 1600 \) presented with Fig. 2(d,e). Accordingly, most of the Floquet states are 'smeared' over the whole energy spectrum; in that sense they are de-localized, although they are remaining sparse in terms of the compactness index \( \zeta \sim 1/N^2 \), see Fig. 2(f).

To explain these findings, we analyze the Floquet states \( F^{(l)}_k(t) \),

\[ F^{(l)}_k(t) = \Phi^{(l)}_k(t) \exp(-ist), \tag{4} \]

where \( \Phi^{(l)}_k(t) = T = 2\pi/\omega \)-periodic functions, eigenstates of the Floquet propagator, and \( s_t \) are quasi-energies. Obviously, in the absence of driving, \( A = 0 \), the Floquet states correspond to the stationary states, \( \Phi^{(l)}_k = \delta_{k,l} \), and the quasi-energies are given by the respective eigenvalues (energies), \( s_l = \lambda_l \).

Next we expand the Floquet states in the basis of the eigenstates of the stationary Hamiltonian, Eq. (3).

In this picture, the Floquet states are results of driving-induced cross-breeding between the eigenstates. By using the standard perturbation theory, we expand \( \Phi_k(t) \) into Fourier series and express Floquet states as

\[ F_k(t) = \sum_n \phi^{(n)}_k \exp(-i\omega t) \exp(-ist). \tag{5} \]

Substituting (5) into (3), we get

\[ s\phi_k^{(n)} = (\lambda_l - n\omega)\phi_k^{(n)} + \frac{1}{2} A \sum_{k'} J_{k,k'} \left( \phi_{k'}^{(n-1)} + \phi_{k'}^{(n+1)} \right). \tag{6} \]
FIG. 3. The probability density function $\rho(\lambda, E)$ obtained by expanding Floquet states in the basis of the stationary Hamiltonian, where $E$ is the energy of Floquet states averaged over the period of the modulations and $\lambda$ is the eigen-energy of the eigen-states of the stationary Hamiltonian. The distribution are calculated for different amplitude of driving, (a) $A = 0.01$, (b) $A = 0.1$ and (c) $A = 1.0$. Here $I_0 = 0.2, \omega = 0.5, N = 1600, D = 400$, and $N_r = 100$. (d) Probability density distributions for the Floquet state specific energy (lines correspond to $N = 1600, D = 400, N_r = 100$, symbols to $N = 6400, D = 1600, N_r = 1$). Emergence and sharpening of the maximum indicates homogenization of the states and thermalization of the system.

Taking the $l$-th Floquet state for $A = 0$ as a zero-order approximation, $\phi_k^{(0)} = \delta_{k,l}$ and $s = \lambda_l$, for the first order we obtain non-zero amplitudes

$$\phi_k^{(\pm 1)} = \frac{AJ_{k,l}}{2(\lambda_l - \lambda_k \pm \omega)}.$$  \hspace{1cm} (7)

The net norm of the first order corrections is, therefore,

$$\|\Delta \phi_k^{(l)}\| = \frac{1}{4} A^2 \sum_k \frac{J_{k,l}^2}{(\lambda_l - \lambda_k \pm \omega)^2},$$  \hspace{1cm} (8)

and provides with the necessary validity condition $\|\Delta \phi_k^{(l)}\| \ll 1$.

Note that the higher order resonances will also be induced in chain, until the multiple of the driving frequency stays within the spectrum bound. Clearly, the degree of hybridization between two Floquet states depends on both the overlap coefficient between the states, $J_{k,l}$, and the distance to the resonance, $|\lambda_k - \lambda_l| \approx n\omega$, $n \in \mathbb{N}$. To secure the latter possibility, the driving has to be slow so that its frequency does not exceed the characteristic spectrum width. Furthermore, it can be expected that most of the strongly localized states would not interact significantly, as both the near-resonance and overlap conditions would rarely be satisfied at the same time.

A localized state occupies a certain localization volume $V_L \ll N$, and normalization requires that the non-zero elements scale as $|z| \sim V_L^{-1/2}$. We assume that the signatures of the eigenstates elements $z_j^{(k)}$ and $z_j^{(l)}$ are not correlated, and obtain $|J_{k,l}| \sim V_L^{-1/2} \Theta(k,l)$, where $\Theta(k,l) = 1$ if the eigenstates overlap, and 0 otherwise \[27\]. As the probability of the latter scales as $\sim V_L/N$, we finally obtain $|J_{k,l}| \sim V_L^{-1/2}/N$. In turn, the contribution of the frequency denominator can be estimated as follows. Since the density of states scales linearly with $N$, the eigenstate spacing behaves as $\sim 1/N$, and the squared resonance mismatch in the denominator yields
\( \sim 1/N^2 \). Thus we arrive at \( \| \Delta \phi^{(i)} \| \sim A^2 V_L \). It indicates that the first corrections due to driving-induced interactions between strongly localized states remain finite (even in the limit \( N \to \infty \)), which is consistent with the moderate influence of driving, found in the case of the Anderson (uncorrelated) disorder [11-13].

The presence of effective mobility edge and quasi-extended band due to correlations in disorder, enriches the picture. Indeed, many quasi-extended modes would fall within the same localization volume \( V_E \gg V_L \) and interact efficiently. Moreover, they may couple across the mobility edge to the strongly localized ones, and affect the latter. Repeating the above arguments, for the overlap integrals between quasi-extended states or between such a state and a strongly localized one, we obtain \( \| \Delta \phi^{(i)} \| \sim A^2 V_E \). Given \( V_E/V_L \sim 10^4 \ldots 10^5 \) (cf. Fig.1), it follows that reshaping of Floquet states is dominated by the coupling to and between quasi-extended modes.

Now one can explain the observed changes in the localization properties in both bands. It is straightforward to see that in a hybrid state, composed of a localized and a quasi-extended ones, most contribution to participation number stems from the former due to large local amplitudes. On the contrary, most contribution to the effective width is produced by the latter. Accordingly, the majorities of the Floquet states eventually acquires relatively low participation numbers and relatively high effective widths, thus becoming sparse, see Fig.2(a-c).

Still, in the thermodynamic limit \( N \to \infty \), such hybridization cannot lead to a complete de-localization in the direct basis as even quasi-extended states still have finite, albeit huge, localization length [19-21].

Moreover, since the stationary spectrum is bounded, the density of states scales linearly with the system size. It follows that in the basis of the stationary Hamiltonian the distance between the resonantly hybridizing states will also scale as \( \sim N \), and the normalized effective width of the resulting Floquet states remains finite as \( N \to \infty \), in contrast to the direct basis. In result, the numerically observed de-localization in the stationary basis, \( L \sim 1 \), persists with the up-scaling of \( N \), while the normalized participation numbers and compactness decay, \( P \sim 1/N, \zeta \sim 1/N^2 \); see Fig.2(e).

Now we evaluate the structure of Floquet states in the eigenbasis of the stationary system. We calculate the period-averaged norm of each element of the Floquet state, \( \| F_k^{(i)} \| = \frac{1}{T} \int_0^T | F_k^{(i)}(t) |^2 dt \), and its average energy, \( E_i = \sum_k \lambda_k \| F_k^{(i)} \| \). We define the probability density function

\[
\rho(\lambda, E) = \lim_{\Delta \lambda, \Delta E \to 0} \frac{\langle \| F_k^{(i)} \| \rangle_{\lambda_k \in [\lambda-\Delta \lambda, \lambda+\Delta \lambda], E_i \in [E, E+\Delta E]}}{\Delta \lambda \Delta E},
\]

which, additionally averaged over disorder realizations, essentially gives the contribution (weight) of stationary eigenstates with the energy \( \lambda \) to (in) the Floquet states with the average energy \( E \).

Figures 2(a)-(c) illustrate the modification in the Floquet states by slow modulations of the frequency \( \omega = 0.5 \) (which is smaller than the characteristic spectrum width). For weak driving, \( A = 0.01 \), the Floquet states are only slightly perturbed stationary eigenstates, with tangible distortions appearing, however, when the near-resonance condition holds, see Fig.3(a). The main effect is observed for the quasi-extended states. Stronger modulations, \( A = 0.1 \), induce a progressive resonant hybridization of all states, including the strongly localized ones, see Fig.3(b). As the driving amplitude grows to \( A = 1.0 \), the Floquet states become smeared over the stationary basis and completely loose their initial forms, see Fig.3(c).

As de-localization of Floquet states in the basis of the stationary Hamiltonian is associated with thermalization of the system [28,30], it is tempting to search for it in our model. Usually, one checks if expectation values of certain observables are almost independent of a particular type of the Floquet state, so we choose to follow the period-averaged energy, \( E_i \). The results presented with Fig.3(d) show the development of a pronounced peak in distribution about \( E = 1.25 \). Therefore, a progressively greater part of Floquet states assumes the average particle energy about the band center, which is a trait of the thermalization.

In conclusion, we studied the fate of the single-particle localization in a periodically-driven 1D lattice with a correlated disorder potential (which is the most conventional model of speckled optical lattices [18]). We demonstrate that the existence of the quasi-extended states, appearing due to the presence of spatial correlations in the potential, drastically changes localization properties of the Floquet states. In the direct space we observe a trend to the homogeneity; namely, the decrease of localization for the strongly localized states and the enhancement of localization for the quasi-extended states, which relies on the resonant hybridization between the states from different bands. Ultimately, in the limit of strong driving, most of the Floquet states completely loose their original properties and become weakly localized in the direct space. Moreover, they also de-localize in the basis of the stationary Hamiltonian and exhibit signatures of the thermalization. Our results can be straightforwardly generalized to the case of non-harmonic periodic driving [16,17]. Finally, although the mobility edge in the used model is only an effective one (so that the quasi-extended states are not completely de-localized), our finding constitutes a step towards a much debated problem of the phenomenon of driven many-body localization in the presence of mobility edge [30,32].

The authors acknowledge support of Russian Science Foundation grant No. 15-12-20029 (analytical part) and Ministry of Education and Science of the Russian Federation Research Assignment No. 1.115.2014/K (compu-
Numerical simulations were performed on the Lobachevsky University and Moscow State University Supercomputers and in the Joint Supercomputing Center of RAS.

[1] B. Kramer and A. MacKinnon, Rep. Prog. Phys. 56, 1469 (1993).
[2] F. Evers and A. Mirlin, Rev. Mod. Phys. 80, 1355 (2008).
[3] 50 Years of Anderson Localization, ed. by E. Abrahams (World Scientific, 2010).
[4] P. W. Anderson, Phys. Rev. 109, 1492 (1958).
[5] J. Billy, V. Josse, Z. Zuo, A. Bernard, B. Hambrecht, P. Lugan, D. Clément, L. Sanchez-Palencia, P. Bouyer, and A. Aspect, Nature 453, 891 (2008).
[6] G. Roati, C. D’Errico, L. Fallani, M. Fattori, C. Fort, M. Zaccanti, G. Modugno, M. Modugno, and M. Inguscio, Nature 453, 895 (2008).
[7] S. S. Kondov, W. R. McGehee, J. J. Zirbel, and B. DeMarco, Science 334, 66 (2011).
[8] F. Jendrzejewski, A. Bernard, K. Muller, P. Cheinet, V. Josse, M. Piraud, L. Pezze, L. Sanchez-Palencia, A. Aspect, and P. Bouyer, Nat. Physics 8, 398 (2012).
[9] M. Segev, Y. Silberberg, and D.N. Christodoulides, Nat. Photonics 7, 197 (2013).
[10] H. Hu, A. Strybulevych, J. H. Page, S. E. Skipetrov, and B. A. Van Tiggelen, Nat. Physics 4, 945 (2008).
[11] D.F. Martinez, R.A. Molina, Phys. Rev. B 73 073104 (2006); D.F. Martinez, R.A. Molina, Eur. Phys. J. B 52, 281 (2006).
[12] M. Holthaus, G.H. Ristow, D.W. Hone, Phys. Rev. Lett. 75 (1995) 3914.
[13] M. Holthaus, D.W. Hone, Phil. Mag. Part B 74 (1996) 105.
[14] D.H. Dunlap, V.M. Kenkre, Phys. Rev. B 34 3625 (1986).
[15] F. Grossmann, T. Dittrich, P. Jung, P. Hänggi, Phys. Rev. Lett. 67 516 (1991).
[16] H. Hatami, C. Danieli, J.D. Bodyfelt, and S. Flach Phys. Rev. E 93, 062205 (2016).
[17] L. Morales-Molina, E. Doernner, C. Danieli, and S. Flach, Phys. Rev. A 90, 043630 (2014).
[18] M. Modugno, Phys. Rev. A 73, 013606 (2006).
[19] P. Lugan, A. Aspect, L. Sanchez-Palencia, D. Delande, B. Gremaud, C. A. Müller, and C. Miniatura, Phys. Rev. A 80, 023605 (2009).
[20] G.M. Falco, A.A. Fedorenko, J. Giacomelli, and M. Modugno, Phys. Rev. A 82, 053405 (2010).
[21] J. Giacomelli, Physica A 404, 158 (2014).
[22] J. H. Shirley, Phys. Rev. 138, B979 (1965).
[23] H. Sambe, Phys. Rev. A 7, 2203 (1973).
[24] M. Grifoni, P. Hänggi, Phys. Rep. 304, 229 (1998).
[25] T.V. Laptyeva, E.A. Kozinov, I.B. Meyerov, M.V. Ivanchenko, S. Denisov, P. Hänggi, Comp. Phys. Comm. 201, 85 (2016).
[26] We also verified the qualitative and quantitative similarity of all studied characteristics, when calculated at a fixed stroboscopic moment of time or being averaged over the period of driving. We use the averaged characteristics throughout the paper.
[27] This approximation is sufficient for our purposes, though more care should probably be taken in some more subtle situations, especially for quasi-extended states, which reportedly display traits of multi-fractality [21].
[28] A. Lazarides, A. Das, R. Moessner, Phys. Rev. E 90, 012110 (2014).
[29] L. D’Alessio and M. Rigol, Phys. Rev. X 4, 041048 (2014).
[30] A. Lazarides, A. Das, R. Moessner, Phys. Rev. Lett. 115, 030402 (2015).
[31] J. Santos, R.A. Molina, J. Ortigoso, and M. Rodriguez Phys. Rev. A 80, 063602 (2009).
[32] P. Ponte, Z. Papić, F. Huveneers, D. Abanin, Phys. Rev. Lett. 114, 140401 (2015).