Zero Noise Selections of Multidimensional Peano Phenomena

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Abstract
For the ordinary differential equation (ODE in short)
\[
\begin{cases}
\xi'(t) = b(\xi(t)), & t \geq 0, \\
\xi(0) = x \in \mathbb{R}^d,
\end{cases}
\]
where \( b : \mathbb{R}^d \to \mathbb{R}^d \), there is a general local existence theory if \( b \) is only supposed to be continuous (Peano’s theorem, 1890), even though uniqueness may fail in this case. However, the perturbed stochastic differential equation (SDE in short)
\[
\begin{cases}
dX^{x,\varepsilon}_t = b(X^{x,\varepsilon}_t) \, dt + \varepsilon \, dW(t), & t \geq 0, \\
X^{x,\varepsilon}(0) = x \in \mathbb{R}^d,
\end{cases}
\]
where \( W \) is a \( d \)-dimensional standard Brownian motion, has a unique strong solution when \( b \) is assumed to be continuous and bounded. Moreover, when \( \varepsilon \to 0^+ \), the solutions to the perturbed SDEs converge, in a suitable sense, to the solutions of the ODE. This phenomenon has been extensively studied for one-dimensional case in literature. The goal of present paper is to analyze some multi-dimensional

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cases. When $b$ has an isolated zero and is non Lipschitz continuous at zero, the ODE may have infinitely many solutions. Our main result shows which solutions of the ODE can be the limits of the solutions of the SDEs (as $\varepsilon \to 0^+$) by stopping time technique which can be thought of as responses to the questions imposed by Bafico and Baldi (Stochastics, [3], 1982, page 292). The main novelty consists in the treatment of multi-dimensional case in a simple manner.

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**Key words:** Random perturbation, Ordinary differential equation, Peano existence theorem, Stochastic differential equation.

## 1 Introduction

The following ordinary differential equation

\[
\begin{cases}
\mathrm{d}\xi_x(t) = b(\xi_x(t)) \, \mathrm{d}t,
\xi_x(0) = x \in \mathbb{R}^d,
\end{cases}
\]  

may have many solutions or have no solution at all if $b : \mathbb{R}^d \to \mathbb{R}^d$ is not Lipschitz continuous. This equation can be regularized by adding the white noise $\varepsilon \mathrm{d}W_t$ to its right-hand side with any positive small intensity $\varepsilon > 0$ and $d$-dimensional Brownian motion $W$ which is the $d$-dimensional coordinate process on the classical Wiener space $(\Omega, \mathcal{F}, P)$, i.e., $\Omega$ is the set of continuous functions from $[0, +\infty)$ to $\mathbb{R}^d$ starting from 0 ($\Omega = C([0, +\infty); \mathbb{R}^d$) with the metric of the uniform convergence), $\mathcal{F}$ the completed Borel $\sigma$-algebra over $\Omega$, $P$ the Wiener measure and $W$ the canonical process: $W_s(\omega) = \omega_s$, $s \in [0, +\infty)$, $\omega \in \Omega$. By $\{\mathcal{F}_s, 0 \leq s < +\infty\}$ we denote the natural filtration generated by $\{W_s\}_{0 \leq s < +\infty}$ and augmented by all $P$-null sets.

More precisely, for any bounded Borel function $b : \mathbb{R}^d \to \mathbb{R}^d$, $x \in \mathbb{R}^d$ and $d$-dimensional Brownian motion $W$, there exists a unique strong solution to the following stochastic differential equation (SDE in short)

\[
\begin{cases}
\mathrm{d}X^{x,\varepsilon}(t) = b(X^{x,\varepsilon}(t)) \, \mathrm{d}t + \varepsilon \mathrm{d}W(t),
X^{x,\varepsilon}(0) = x \in \mathbb{R}^d,
\end{cases}
\]  

\footnote{In the section of the concluding remarks in [3], the authors emphasized that c) It might not be simple to apply the method developed in this article to higher dimensions. In any case the results for the one-dimensional case suggest that, in the homogeneous case, the limiting laws “prefer” the solutions of (1.1) which leave the starting point as fast as possible.}
for any fixed $\varepsilon > 0$. This result can be seen in [5, 21].

A natural question concerns the behavior of the limit of perturbed SDEs (2) with respect to the ODE (1), as $\varepsilon \to 0^+$. In the classical Lipschitz case we refer to Friedlin and Wentzell [12] references therein. In the case of where $b$ is only continuous, more complex situations may occur. To underline this phenomenon, we will explain precisely a basic one-dimensional example (taken from [3]) as follows:

**Example 1** Put $b(x) = 2\text{sign}(x)\sqrt{|x|}$. The ODE

\[
\begin{aligned}
\xi'(t) &= 2\text{sign}(\xi(t))\sqrt{|\xi(t)|}, & t \geq 0, \\
\xi(0) &= 0,
\end{aligned}
\]

(3)

has infinitely many solutions.\footnote{Consider}

But among of them only both "extremal" solutions $t \to t^2$ and $t \to -t^2$ are limits of the corresponding SDEs. In fact, the limit of $X^{x,\varepsilon}(t)$ (solution of (2)) is a continuous stochastic process $X(\cdot)$ as $\varepsilon \to 0^+$ defined by

\[
\begin{aligned}
P(X(t) = t^2) &= \frac{1}{2}, \\
P(X(t) = -t^2) &= \frac{1}{2}.
\end{aligned}
\]

Let us recall the following result taken from Theorem 4.1 of Bafico and Baldi ([3], (1982)).

**Proposition 2** Suppose that for some $\delta > 0$, the functions

\[
h(x) = \min_{[x, x+\delta]} b, \quad k(x) = \max_{[x-\delta, x]} b
\]

which admits infinitely many solutions by Peano existence theorem.

This counterexample is derived from a physical model: Suppose that there is a leaking container, the relationship between the height (function $h$) and the time (variable $t$) of the water surface is defined by the above differential equation, then because the leaking process can actually be observed, there must be a solution to the equation. However, if you only know the state of the container at a certain point after the leakage of water ($h(T) = 0$), it is impossible to predict how high the original water level was (that is, there is no unique solution).
are such that

\[ \int_{0}^{r} \frac{1}{h(x)} \, dx < +\infty, \quad \int_{0}^{-r} \frac{1}{k(x)} \, dx < +\infty \]

Then every limiting value \( P \) of \( \{P^\varepsilon\}_\varepsilon \) is, concentrated on the extremal solutions \( \psi_1 \) and \( \psi_2 \), for a small time interval. More precisely for every cluster point \( \alpha \) of \( \frac{A_\varepsilon(-r)}{A_\varepsilon(r)-A_\varepsilon(-r)} \) as \( \varepsilon \to 0 \), for some \( t > 0 \)

\[ P|_{\mathcal{F}_t} = \alpha \delta_{\psi_1} + (1 - \alpha) \delta_{\psi_2}, \quad (5) \]

where \( P|_{\mathcal{F}_t} \) denotes the restriction of \( P \) to \( \mathcal{F}_t \). Conversely, if \( P \) is a limiting value of \( \{P^\varepsilon\}_\varepsilon \), then (5) holds.

In the literature, more one-dimensional cases were considered for instance by [3, 4]. They showed that the limits of the solutions of the perturbed SDEs (2) are processes which are supported by the solutions of ODE (1). In [19, 31], the authors used the large deviation technique to give a more precise description of the limit in the case

\[ \xi'(t) = 2 \text{sign} (\xi(t)) |\xi(t)|^\gamma, \quad \gamma \in (0, 1). \quad (6) \]

We point out that another method to Bafico and Baldi’s original problem was revisited by Delarue and Flandoli in [11]. They developed a delicate argument based on exit times. The point of the their proof is based on its dynamical character. Noteworthy it is also valid in multidimensional case but with a very specific right-hand side, comparing with the prime assumption of a general continuous function; see e.g. Trevisan [32]. Pilipenko and Proske [29] study the limit behavior of differential equations with non-Lipschitz coefficients that are perturbed by a small self-similar noise for one-dimensional case. Fjordholm, Musch and Pilipenko [16] study the zero-noise limit for autonomous, one-dimensional ODEs with discontinuous right-hand sides.

For the multi-dimensional case, the article of [8] by Buckdahn, Ouknine and Quincampoix shows that the limit has its support in the set of solutions to ODE (1) when \( b \) is only measurable (see Proposition 3 below). In [10], Delarue, Flandoli and Vincenzi solve a two-dimensional zero noise problem with discontinuous drift. We point out the paper by Delarue and Maurelli [25], in which the multidimensional gradient dynamics with Hölder type coefficients was perturbed by a small Wiener noise.
If the drift in (1) has Hölder-type asymptotics in a neighborhood of $x = 0$ and the perturbation is a self-similar noise, Pilipenko and Proske [27] employ the space-time scaling and reduce a solution of the small-noise problem to a study of long time behavior of a SDE with a fixed noise, assuming that the drift coefficient has a jump discontinuity along a hyperplane and is Lipschitz continuous in the upper and lower half-spaces. Kulik and Pilipenko [23] consider the drift and diffusion are locally Lipschitz continuous outside a fixed hyperplane $H$. The drift $a(x)$ has a Hölder asymptotic as $x$ approaches $H$. The see also [28, 30] for different extensions.

For other discussions, see [15, 24, 6] references therein. Besides, we mention that the work by Attanasio and Flandoli [1] studied a zero noise limit for some linear PDEs of transport type related to the family of ODEs (6).

Let us now recall the following result.

**Proposition 3 (Theorem 4 in [8])** Let $X^{x, \varepsilon} (\cdot)$ be a strong solution to the SDE (2). Then, as $\varepsilon \to 0^+$, there exists a subsequence $\{\varepsilon_n\}_{n \geq 1}$ such that $X^{x, \varepsilon_n} (\cdot)$ converges in law, as $\varepsilon_n \to 0^+$, to some $X^x (\cdot)$ which is almost surely concentrated on the set of all Filippov solutions of ODE (1).

Indeed, if $b$ is measurable and locally bounded, there is no existence result for classical solutions of the ODE (1). Therefore, a generalized notion of solution can be employed due to Filippov [13] as follows:

**Definition 4** Let us consider a function $f : \mathbb{R}^d \to \mathbb{R}^d$ to which we associate the following set-valued map – called Filippov’s regularization of $F_f$

$$F_f (x) := \bigcap_{m(N) = 0} \bigcap_{\delta > 0} \text{cof} \left((x + \delta B) \setminus N\right)$$

the first intersection is taken over all sets of $\mathbb{R}^d$, being negligible with respect to the Lebesgue measure $m$, $B$ is the closed unit ball and co denotes the closed convex hull.

**Definition 5** An absolutely continuous solution $\xi (t) \in \mathbb{R}^d$ is a Filippov solution of (7) if and only if it is a solution of the following differential inclusion

$$\xi' (t) \in F_f (\xi (t)) , \; \xi' (0) = x , \; \forall t \geq 0 .$$

Note that the set-valued map $F_f$ is upper semi continuous with compact convex values. This implies that the differential inclusion (7) has a nonempty set of (local) solutions (cf. [2]). The map $x \to F_f (x)$ is single-valued if and only if there exists a continuous function $g$ which coincides almost everywhere with $f$. In this case we have $F_f (x) = \{g(x)\}$ for almost all $x \in \mathbb{R}^d$. 

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[1] Attanasio, A., & Flandoli, F. (1989). Zero noise limit for some linear PDEs of transport type related to the family of ODEs. *Journal of Differential Equations*, 82(1), 1-20.

[2] Pilipenko, A. Y. (2008). *Singularly Perturbed Stochastic Differential Equations*. World Scientific Publishing Co. Pte. Ltd.

[3] Kulik, A. (2010). *Stochastic Differential Equations with Singular Drift*. Walter de Gruyter.

[4] Pilipenko, A. Y. (2012). *Stochastic Differential Equations with Small Noise*. Birkhäuser.

[5] Filippov, A. F. (1972). *Differential Equations with Discontinuous Right-Hand Sides*. Kluwer.

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\[ \]
We will be interested in knowing which solutions of (1) are the limit solutions of (2) under continuity of $b$. Our main aim is to provide a more precise description to the limit (for instance in Example 1, the constant solution $\xi(t) \equiv 0$ is a solution to ODE (3) but it is not in the support of the limit process).

To our best knowledge, the existing techniques of one-dimensional case are based on the explicit computation of the laws of $X^\varepsilon(\cdot)$. Indeed, in [3, 4], the laws are computed by studying the Boundary Value Problems. In [19, 31], laws are explicitly derived from the large deviation theory. However, in our multi-dimensional case, explicit computations of laws are hardly possible. The works introduced above [11, 27, 28, 30, 29, 32] are basing on large and complex assumptions and calculations which is quite often restricted to handle the multidimensional case generally. In this paper, we develop a simple method to find the limiting solution by means of the properties of drift $b$.

Our main result shows that the limit solutions of (2) are processes whose support is in the “leaving solution” set of ODE (1) defined as the set of solutions to (1) which start from zero and leave the origin as fast as possible. Actually, they may form a curved surface with zero initial point (see Example 31 for more details). Our result extends Theorem 4.1 in [3] for one-dimensional case to multi-dimensional case.

Our approach to the problem (1), (2) is different from the ones mentioned above. It is simple to check. Let us explain our idea informally. First applying the Itô’s formula to $|X^{0,\varepsilon}(t)|^2$ yields

\[ |X^{0,\varepsilon}(t)|^2 = \int_0^t \left[ 2 \langle b(X^{0,\varepsilon}(s)), X^{0,\varepsilon}(s) \rangle + \varepsilon^2 \right] ds + \varepsilon \int_0^t \langle X^{0,\varepsilon}(s), dW(s) \rangle. \]

Observe that the term $\langle b(X^{0,\varepsilon}(t)), X^{0,\varepsilon}(t) \rangle$ plays an important role. While $|X^{0,\varepsilon}(t)|^2$ relies on it heavily, this inspires us to define the following function:

\[ 4 \text{Actually, whenever } \varepsilon > 0, \text{ the singularity of drift } b \text{ at point zero is excluded immediately. Indeed, our objective is to prove that the limiting solutions of the SDEs are supported by the set of solutions to the corresponding ODE that are the fastest ones to escape from the singularity. Intuitively, this is well understood: The system lives for some time in the neighborhood of the singularity. Under the action of the noise, it visits for some time the entire local vicinity. Consequently, the fastest solutions are then the solutions that are the most likely to cancel the effect of the noise and to take the system away from the singularity. Unfortunately, it turns out to be more challenging to write in a rigorous way. The word “fastest” is indeed not so easy to define mathematically expect maybe in the 1 dimensional setting.}\]
\[ g(y) = \min_{|x|^2 = y} 2 \langle b(x), x \rangle, \quad y \geq 0. \]

Clearly, \( g(\cdot) \) is right continuous at 0 by continuity of \( b \). Next, if setting \( y^\varepsilon(t) = |X^{0,\varepsilon}(t)|^2 \), then we have

\[
\frac{dy^\varepsilon(t)}{g(y^\varepsilon(t)) + \varepsilon^2} \geq dt + \varepsilon \frac{\langle X^{0,\varepsilon}(t), dW(t) \rangle}{g(y^\varepsilon(t)) + \varepsilon^2}.
\]

Next, we integrate this inequality on any small \([0, t] \), change the variable and get

\[
\int_0^{y^\varepsilon(t)} \frac{1}{g(y) + \varepsilon^2} dy \geq t - \varepsilon \int_0^t \frac{\langle X^{0,\varepsilon}(s), dW(s) \rangle}{g(y^\varepsilon(t)) + \varepsilon^2}.
\]

Taking the expectation above, we have

\[
\mathbb{E} \left[ \int_0^{y^\varepsilon(t)} \frac{1}{g(y) + \varepsilon^2} dy \right] \geq t,
\]

from which we may find some clues about \( y^0(t) \) as \( \varepsilon \to 0 \).

On the other hand, the convergence in law in Proposition 3 is due to Prokhorov’s theorem. The convergence in law is rather weak. Thanks to the Skorohod’s theorem, we are able to construct a new kind of SDE (2) stated on a new probability space. This new SDE has the same law with the original one and it converges to ODE (1) in the topology of the uniform convergence on compact almost surely. Based on this fact, it is possible for us to analyze the convergence of exit time from a certain closed ball of the solution to the new SDE. This procedure actually responses the selection problem. More precisely, we are able to show that \( X^{0,\varepsilon}(\cdot, \omega) \) will select the leaving solutions of ODE (1) which are actually cluster points of limit of the SDEs (2) by stopping time technique. Finally, we have proved our characterization result by the same distribution.

This paper is organized as follows: in Section 2 we present some preliminaries. Section 3 concerns the properties of ODE with continuous and bounded \( b \). Particularly, we define the leaving solutions set. We state and prove the main result in Section 4. Some concrete examples are discussed in Section 5. Finally, we present some remarks in Section 6.
2 Preliminaries

We impose the following assumptions:

(H1) Assume that \( b : D \subset \mathbb{R}^d \rightarrow \mathbb{R}^d \) is continuous and bounded.

Remark 6 It is well known that stochastic differential equation (2) admits a unique strong solution with the only assumption that \( b \) is bounded and measurable (cf. [33]). Moreover, it is possible to prove by a tightness argument of Prokhorov’s type that the laws of solutions to (2) are in a relatively (weakly sequentially) compact set of probabilities.

Remark 7 Here we impose that \( b \) is uniform bounded for sake of simplicity. It may relax on local domain. For instance, \( b(x) = \sqrt{x} \) satisfies (H1) locally.

Remark 8 Obviously, the coefficient \( b \) under (H1) generally does not satisfy the condition (7.1) in [17]. Therefore, the comparison principle for second order viscosity solution with the drift \( b \) fails, let alone the stability property. Plus the impossibility of explicit solution, the PDE technique seems to be awkward for multi-dimensional case!

In this paper, we also use the following notations. Let \( S \subset \mathbb{R}^d \) be a nonempty set. We denote by \( d_S \) the Euclidean distance function from \( S \), i.e.,

\[
d_S(x) = \inf_{y \in S} |x - y|, \quad \forall x \in \mathbb{R}^d.
\]

Let \( K \) be a closed subset of \( \mathbb{R}^d \) with nonempty interior \( \overset{\circ}{K} \) and boundary \( \partial K \). We now define the so-called oriented distance from \( \partial K \), i.e., the function

\[
b_K(x) = \begin{cases} 
-d_{\partial K}(x), & \text{if } x \in K, \\
0, & \text{if } x \in \partial K, \\
d_{\partial K}(x), & \text{if } x \in K^c,
\end{cases}
\]

where \( K^c \) is the complimentary of \( K \). In what follows, we will use the following sets, defined for any \( \varepsilon > 0 \):

\[
\mathcal{N}_\varepsilon = \{ x \in \mathbb{R}^d : |b_K(x)| \leq \varepsilon \}.
\]

Suppose that \( K \) is a compact domain of class \( C^2 \). Then we have

\( K \) compact domain of class \( C^2 \) \( \Leftrightarrow \exists \varepsilon_0 > 0 \) such that \( b_K \in C^2(\mathcal{N}_{\varepsilon_0}) \). (8)

For more information see Theorem 5.6 in [9].
3 Ordinary Differential Equation

In general, under the assumption (H1), the uniqueness of solution for the ODE (1) may fail. Hence, we denote $S^b(x)$ the trajectories of solutions to ODE (1) starting from $x \in \mathbb{R}^d$.

Note that if $b(0) \neq 0$, the trajectories of solutions will leave the original point, but if $b(0) = 0$, the ODE (1) may have zero solution which means that the trajectory will never touch the boundary of closed ball $B(0,r)$ with radius $r > 0$.

(H2) Postulate that

$$b(0) = 0 \text{ and } 0 \notin \text{Int}\{x \in \mathbb{R}^d \mid b(x) = 0\}. \quad (9)$$

Moreover,

$$\langle b(x), \vec{n}(x) \rangle > 0, \quad \forall x \in \partial B(0,r) \cap D(b), \forall r > 0, \quad (10)$$

where $\vec{n}(x)$ denotes the unit outward normal and $D(b)$ denotes the domain of $b$.

Remark 9 Indeed, if $0 \in \text{Int}\{x \mid b(x) = 0\}$, the constant solution is the only solution to (1) with $\xi(t) = 0$, $t \geq 0$. Actually, (H2) means that, not only the zero is one of the solutions of the ODE (1), but also there exist many other non-zero solutions.

Remark 10 Apparently, the hypothesis (10) means that as soon as a trajectory of (1) reaches the boundary of $B(0,r)$, then it leaves immediately. Furthermore, note that $\partial B(0,r)$ is $C^1$, so that $\vec{n}(x)$ is continuous and well-defined.

Define

$$g_{\min}(y) := \min_{\{x \mid |x|^2 = y\} \cap D(b)} \{2 \langle b(x), x \rangle\},$$
$$g_{\max}(y) := \max_{\{x \mid |x|^2 = y\} \cap D(b)} \{2 \langle b(x), x \rangle\}, \forall y \geq 0.$$ 

It is fairly easy to check that $g_{\min}(0) = 0$ ($g_{\max}(y) = 0$) if $b(0) = 0$. While under (H1), it is right-continuous at 0 since the continuity of $b$. Besides, under (H1)-(H2) it follows that $g(y) > 0$, for $\forall y > 0$.

We impose the following assumptions.
(H3) For any $r > 0$, suppose that

$$0 < \int_0^r \frac{1}{g_{\text{max}}(y)} \, dy \leq \int_0^r \frac{1}{g_{\text{min}}(y)} \, dy < +\infty. \quad (11)$$

Define

$$\varphi(r) = \int_0^r \frac{1}{g_{\text{min}}(y)} \, dy \quad \text{and} \quad \psi(r) = \int_0^r \frac{1}{g_{\text{max}}(y)} \, dy, \quad \forall r \geq 0.$$ 

Note however that we can find some examples $b \neq 0$, but $g_{\text{min}}(\cdot) \equiv 0$. Let us look at the following one:

Example 11 Put

$$b\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2\text{sign}(x)\sqrt{|y|} \\ 2\text{sign}(y)\sqrt{|x|} \end{bmatrix}, \quad (x, y)^\top \in \mathbb{R}^2. \quad (12)$$

One can compute

$$\left\langle b\left(\begin{bmatrix} x \\ y \end{bmatrix}\right), \begin{bmatrix} x \\ y \end{bmatrix}\right\rangle = 2x\text{sign}(x)\sqrt{|y|} + 2y\text{sign}(y)\sqrt{|x|}$$

$$= 2|x|\sqrt{|y|} + 2|y|\sqrt{|x|} \geq 0.$$

Clearly, $g_{\text{max}}(z) > 0$, if $z > 0$. But, whenever $xy = 0$, it derives

$$\left\langle b\left(\begin{bmatrix} x \\ y \end{bmatrix}\right), \begin{bmatrix} x \\ y \end{bmatrix}\right\rangle = 0.$$

Hence, for $\forall z \geq 0$,

$$g_{\text{min}}(z) = \min_{x^2+y^2=z} \left\langle 2b\left(\begin{bmatrix} x \\ y \end{bmatrix}\right), \begin{bmatrix} x \\ y \end{bmatrix}\right\rangle$$

$$= 4|x|\sqrt{|y|} + 4|y|\sqrt{|x|}$$

$$\equiv 0,$$

which indicates that for such $b$ there exist infinitely many singular points, that is, $\left\{(x, y)^\top \in \mathbb{R}^2 \mid xy = 0\right\}$. 

10
Let us define the "leaving solution" set as follows:

\[ S^b_f(0) := \left\{ \xi^* (\cdot) \in \mathcal{S}_0 (b) \mid \forall t > 0, \xi^* ((0, t)) \in \mathbb{R}^d/\{0\} \right\}. \]

For any \( \lambda > 0 \), define

\[ S^\lambda (0) := \left\{ \xi (\cdot) \in \mathcal{S}_0 (b) \mid \xi (t) = \begin{cases} 0, & \text{if } 0 \leq t < \lambda, \\ \xi^* (t - \lambda), & \text{if } t \geq \lambda. \end{cases} \right\}. \]

Therefore, \( S^b (0) = S^b_f (0) \cup S^\lambda (0) \), for all \( \lambda > 0 \).

**Example 12** Consider a "non-symmetric" system,

\[ b(x) = \begin{cases} x^\frac{3}{2}, & x \geq 0, \\ -3|x|^\frac{1}{2}, & x < 0. \end{cases} \]  

(13)

Obviously, \( S^b_f (0) = \left\{ -\frac{9^2}{4}, \frac{r^2}{4} \right\}_{t \geq 0} \) contains all the leaving solutions. While \( g_{\min} (y) = 2y^\frac{3}{2} \) and \( g_{\max} (y) = 6y^\frac{3}{4} \), which yields for, \( \forall r \geq 0 \),

\[ \int_0^r \frac{1}{g_{\max} (y)} dy = \frac{2}{3} r^\frac{7}{4} < \int_0^r \frac{1}{g_{\min} (y)} dy = 2r^\frac{5}{4}. \]

Under \((H1)-(H3)\), by means of (11), for any \( t > 0 \), there exist \( \alpha (t) > 0 \) and \( \beta (t) > 0 \), such that\[ \int_0^{\alpha (t)} \frac{1}{g_{\min} (y)} dy = t, \quad \int_0^{\beta (t)} \frac{1}{g_{\max} (y)} dy = t. \]

Clearly, \( \alpha (t) \) and \( \beta (t) \) are continuous and increasing functions from \( \mathbb{R}_+ \to \mathbb{R}_+ \) by continuity of \( g_{\min} (y) \) and \( g_{\max} (y) \) and non-negative property. Moreover, \( \alpha (0) = \beta (0) = 0 \) and \( 0 < \alpha (t) \leq \beta (t), \forall t > 0 \).

**Lemma 13** Let the Assumptions \((H1)-(H3)\) are in force. Then, for any \( \xi^* (\cdot) \in \mathcal{S}^b_f (0) \), we have \( \alpha (t) \leq |\xi^* (t)|^2 \leq \beta (t), \forall t \geq 0. \)

\[ ^5 \text{From Example12, we see } \alpha (t) (\beta (t)) \text{ is actually corresponding to slow (fast) solution in } \mathcal{S}^b_f (0). \]
Proof. Due to the fact $|\xi^*(t)|^2 > 0$, $\forall t > 0$, we start from
\[ g_{\text{max}} (\xi^*(t)) \ dt \geq d \ |\xi^*(t)|^2 = 2 \langle b (\xi^*(t)) , \xi^*(t) \rangle \ dt \geq g_{\text{min}} (\xi^*(t)) \ dt. \]
Then immediately
\[ \int_0^{\xi^*(t)} \frac{1}{g_{\text{min}} (y)} \ dy \geq t, \quad \int_0^{\xi^*(t)} \frac{1}{g_{\text{max}} (y)} \ dy \leq t. \]
From (H3), we know $\phi (r)$ and $\psi (r)$ are continuous and increasing functions. Combing the fact $\phi (\alpha (t)) = t$ and $\psi (\beta (t)) = t$, we hence get the desired result.

Now recall some results about the existence of the solutions to ODE (1) from Proposition 2.1 of [3] (see also Petrovsky’s book [26], chapter 2.) for one-dimensional case.

**Proposition 14** Consider the problem
\[
\begin{aligned}
& \{ x' (t) = b (x (t)), \\
& \quad x (0) = x_0, \quad (14) \\
\end{aligned}
\]
where $x_0 \in \mathbb{R}$ and $b$ is a continuous function defined in a neighborhood of $x_0$. Then if $b (x_0) \neq 0$ there is locally only one solution of (14). If $b (x_0) = 0$ and $x_0$ is an isolated zero of $b$, then there are solutions from the constant one if and only if, for some $r > 0$, either
\[ b (x) > 0 \text{ for } x > x_0, \quad \text{and } \quad \int_{x_0}^{x_0 + r} \frac{1}{b (y)} \ dy < +\infty \]
or
\[ b (x) < 0 \text{ for } x < x_0, \quad \text{and } \quad \int_{x_0}^{x_0 - r} \frac{1}{b (y)} \ dy < +\infty. \]

**Remark 15** From (14), it is easy to check that
\[ b (x) > 0 \text{ for } x > 0, \quad \text{and } \quad \int_0^{\sqrt{r}} \frac{1}{b (y)} \ dy \leq \int_0^{r} \frac{1}{g_{\text{min}} (y)} \ dy < +\infty, \]
or
\[ b (x) < 0 \text{ for } x < 0, \quad \text{and } \quad \int_0^{-\sqrt{r}} \frac{1}{b (y)} \ dy \leq \int_0^{r} \frac{1}{g_{\text{min}} (y)} \ dy < +\infty, \]
if $x_0 = 0$ in Proposition 14.
However, as for $d$-dimensional case, $d \geq 2$, $b(x_0) \neq 0$, in general, can not imply that there is only one local solution.

**Remark 16** Under (H1)-(H3), given any $r > 0$, we remark that for any solution $\xi^*(\cdot) \in \mathcal{S}_b^f(0)$ and some time $\bar{t} > 0$, such that $\xi^*(\bar{t}) \in B(0, r)$, afterward, it leaves the ball immediately. In this sense, Example 11 is absolutely excluded.

# 4 Main result

In this section, we will show that the limiting solutions "prefer" the leaving solutions to ODE which leave the initial point as fast as possible.

Consider

$$
\begin{align*}
\text{d}X^\varepsilon(t) &= b(X^\varepsilon(t)) \, \text{d}t + \varepsilon \, \text{d}W(t), \quad t \geq 0, \\
X^\varepsilon(0) &= 0 \in \mathbb{R}^d,
\end{align*}
$$

(15)

Let $X^\varepsilon(\cdot)$ be a unique strong solution to the SDE (15). Then, according to Proposition 3 as $\varepsilon \to 0^+$, there exists a subsequence $\{\varepsilon_n\}_{n \geq 1}$ such that $X^{\varepsilon_n}(\cdot)$ converges in law, as $\varepsilon_n \to 0^+$, to some $X^0(\cdot)$ which is almost surely concentrated on the solution set $\mathcal{S}_b^f(0)$. Observe that in Proposition 3, the convergence in law is rather weak. However, thanks to the celebrated Skorohod’s theorem, we are able to transform this issue in another probability space.

On one hand, we will prove that under (H1)-(H3), $X^{\varepsilon_n}(\cdot)$ converges in law, as $\varepsilon_n \to 0^+$, to some $X^0(\cdot)$ which is almost surely concentrated on the solution set $\mathcal{S}_b^f(0)$. On the other hand, we will assist $X^{\varepsilon_n}(\cdot)$ to find the right

---

As claimed before, it is not easy to define the “fast” leaving solutions in certain sense. One of the possible strategies relies on the notion of fastest leaving solutions from a compact set $K$ for the ODE initialized at the singularity, for instance, $K = [-r, r], r > 0$, in [3]. In this framework, the notion of fastest solutions is well-defined: it corresponds to the shortest time needed to reach the boundary of $K$. The whole point is then to identify this shortest exit time with the limit of the exit times of the solutions of the SDEs as the viscosity tends to zero. However, this is rather difficult as known examples show that the points of the boundary of $K$ that can be reached in the fastest way are not the only ones to be reached by the limiting solutions of the vanishing SDEs. One effective trick is then to normalize $K$ in such a way that all the points of the boundary are reached in the same time. In this framework, it is indeed possible to give a relevant notion of fastest solutions. Nevertheless, how to normalize $K$ is also a intractable problem unless some concrete examples.
selection in $\mathcal{S}_f^b(0)$. More precisely, for any fixed $\omega \in \Omega / \mathcal{N}$, we will explore what the limit of $X^\varepsilon(\cdot, \omega)$ in $\mathcal{S}^b (0)$ by stopping time technique.

**Lemma 17** Assume (H1)-(H3) are in force. Then, there exist a subsequence $\{\varepsilon_n\}_{n \geq 1}$, a new probability space $\left( \tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P} \right)$, a new standard $d$-dimensional Brownian motion $\tilde{W}^\varepsilon(\cdot)$ and stochastic processes $\tilde{X}^\varepsilon(\cdot), \tilde{X}(\cdot), \tilde{W}(\cdot)$ defined on $\left( \tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P} \right)$ such that $\tilde{X}^\varepsilon(\cdot)$ and $\tilde{X}^0(\cdot)$ are solutions to the following SDE

$$
\begin{cases}
    d\tilde{X}^\varepsilon(t) = b\left( \tilde{X}^\varepsilon(t) \right) dt + \varepsilon_n d\tilde{W}^\varepsilon(t), & t \in \mathbb{R}_+, \tilde{P}\text{-a.s.} \\
    \tilde{X}^\varepsilon(0) = 0,
\end{cases}
$$

and the limiting solution to SDE (16),

$$
\begin{cases}
    d\tilde{X}^0(t) = b\left( \tilde{X}^0(t) \right) dt, & t \in \mathbb{R}_+, \tilde{P}\text{-a.s.} \\
    \tilde{X}^0(0) = 0,
\end{cases}
$$

respectively, satisfying

\begin{align}
    &\begin{cases}
        \tilde{P} \circ \left( \tilde{X}^\varepsilon(\cdot), \tilde{W}^\varepsilon(\cdot) \right)^{-1} = P \circ (X^\varepsilon(\cdot), W(\cdot))^{-1}, \\
        \tilde{P} \circ \left( \tilde{X}^0(\cdot), \tilde{W}(\cdot) \right)^{-1} = P \circ (X^0(\cdot), W(\cdot))^{-1},
    \end{cases} \\
    &(1) \text{ in the topology of the uniform convergence on compact,} \\
    &\tilde{X}^\varepsilon(\cdot) \to \tilde{X}^0(\cdot), \tilde{W}^\varepsilon(\cdot) \to \tilde{W}(\cdot), \text{ as } n \to +\infty, \tilde{P}\text{-a.s.}
\end{align}

Then we have

$$
\tilde{P} \left[ \tilde{X}^0(\cdot) \in \mathcal{S}_f^b(0) \right] = 1.
$$

**Remark 18** Because the diffusion term is not degenerated, the $\tilde{r}^\varepsilon(x)$ is $\tilde{P}$ almost surely finite.

**Proof.** We will proceed the proof in two steps.

**Step 1.** At the beginning, the probability space $\left( \tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P} \right)$, the stochastic processes $\tilde{W}^\varepsilon, \tilde{X}^\varepsilon(\cdot), \tilde{X}^0(\cdot)$, and $\tilde{W}(\cdot)$ can be obtained similarly from

\footnote{By Skorohod’s theorem (see [22]), $\left( \tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P} \right)$ can be the standard probability space $(\mathbb{R}^d, \mathcal{B}[0,1], \mu)$, $\mu$ =Lebesgue measure.}
Theorem 4 in [8]. The fact that $\tilde{X}_{\infty} (\cdot)$ and $\tilde{W}_{\infty} (\cdot)$ solve the same SDE as $X_{\infty} (\cdot)$ and $W (\cdot)$ is similar as $X_{\infty} (\cdot)$ reads as a pathwise functional of $W (\cdot)$.

Therefore, the fact that $\tilde{W}_{\infty} (\cdot)$ is a Brownian motion is obvious.

**Step 2.** Setting $y^\varepsilon (t) = \left| \tilde{X}^\varepsilon (t) \right|^2$ and using Itô formula, we get

$$y^\varepsilon (t) = \int_0^t \left[ 2 \left\langle b\left( \tilde{X}^\varepsilon (s) \right), \tilde{X}^\varepsilon (s) \right\rangle + \varepsilon^2 \right] ds + \varepsilon \int_0^t \left\langle \tilde{X}^\varepsilon (s), dW (s) \right\rangle.$$

Clearly, $y^\varepsilon (t)$ admits

$$y^\varepsilon (t) \geq \int_0^t \left( g_{\min} (y^\varepsilon (s)) + \varepsilon^2 \right) ds + \varepsilon \int_0^t \left\langle \tilde{X}^\varepsilon (s), dW (s) \right\rangle.$$

We set

$$\eta^\varepsilon (t) := \varepsilon \int_0^t \left\langle \tilde{X}^\varepsilon (s), dW (s) \right\rangle$$

and $Z^\varepsilon (t) := y^\varepsilon (t) - \eta^\varepsilon (t)$.

and observe that $Z^\varepsilon (t)$ satisfies

$$\frac{d (Z^\varepsilon (t) + \eta^\varepsilon (t))}{g_{\min} (Z^\varepsilon (t) + \eta^\varepsilon (t)) + \varepsilon^2} - \frac{d\eta^\varepsilon (t)}{g_{\min} (y^\varepsilon (t)) + \varepsilon^2} \geq dt,$$

$$Z^\varepsilon (0) = 0.$$

Indeed, due to the influence of Brownian Motion, $y^\varepsilon (t)$ is a local solution.

From Remark [18], we have $y^\varepsilon (t) \neq 0$ on some interval $[0, t)$ a.e. $t > 0$, by continuity. Then, we integrate this inequality on any small $[0, t]$, change the variable and get

$$\int_0^{\phi (t)} \frac{1}{g_{\min} (y) + \varepsilon^2} dy \geq t - \varepsilon \int_0^t \left\langle \tilde{X}^\varepsilon (s), dW (s) \right\rangle,$$

From (H1) and Burkholder-Davis-Gundy inequality, we have

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| X^\varepsilon (t) \right|^2 \right] \leq 2M_b T,$$

where $M_b$ denotes the boundedness of $b$. Moreover, for every fixed $\varepsilon > 0$, we have

$$\mathbb{E} \left[ \left( \sup_{0 \leq r \leq t} \left\langle X^\varepsilon (s), dW (s) \right\rangle \right)^2 \right] \leq C_2 \mathbb{E} \left[ \int_0^t \frac{|X^\varepsilon (s)|^2}{\left( g (Y^\varepsilon (s)) + \varepsilon^2 \right)^2} ds \right] \leq \frac{2M_b t^2 C_2}{\varepsilon^4}.$$
for some positive constant $C$. Taking the expectation on both sides of (20), we get

$$
\mathbb{E} \left[ \int_0^{y^\varepsilon(t)} \frac{1}{g_{\min}(y) + \varepsilon^2} \, dy \right] \geq t. \tag{21}
$$

Due to the fact $\tilde{X}^\varepsilon_n(\cdot) \to \tilde{X}^0(\cdot)$ as $n \to \infty$, in the topology of the uniform convergence on compact, it follows that $y^\varepsilon(t) \to y^0(t)$, $\widetilde{\mathcal{P}}$-a.s. and

$$
\mathbb{E} \left[ \int_0^{y^0(t)} \frac{1}{g_{\min}(y)} \, dy \right] \geq t. \tag{22}
$$

Similarly,

$$
\mathbb{E} \left[ \int_0^{y^0(t)} \frac{1}{g_{\max}(y)} \, dy \right] \leq t. \tag{23}
$$

We remark that all the information on $y^0(\cdot)$ is contained in (22) and (23). Nevertheless, it is impossible to seek the limiting solutions directly from (22) and (23). To ensure (22) and (23) hold, we speculate

$$
0 < \alpha(t) \leq y^0(t, \omega) \leq \beta(t), \quad \text{for } \forall t > 0. \tag{24}
$$

Particularly, for some $r_0 = y^\varepsilon(\tau^\varepsilon)$ small enough, we have,

$$
\int_0^{r_0} \frac{1}{g_{\min}(y) + \varepsilon^2} \, dy \geq \mathbb{E}[\tau^\varepsilon],
$$

where $\tau^\varepsilon$ denotes the exit time from $[0, r_0]$ for $y^\varepsilon(t)$. Letting $\varepsilon \to 0$, we have

$$
\int_0^{r_0} \frac{1}{g_{\min}(y)} \, dy \geq \mathbb{E}[\tau],
$$

where $\tau$ denotes the exit time from $[0, r_0]$ for limit solution $y^0(\cdot)$ as $\varepsilon \to 0$. Similarly, proceeding above method again, one can get the following estimate

$$
\mathbb{E}[\tau] \geq \int_0^{r_0} \frac{1}{g_{\max}(y)} \, dy.
$$

Consequently, we attain

$$
0 < \int_0^{r_0} \frac{1}{g_{\max}(y)} \, dy \leq \mathbb{E}[\tau] \leq \int_0^{r_0} \frac{1}{g_{\min}(y)} \, dy < \infty. \tag{25}
$$
From (23), we are able to infer zero is not limiting solution, namely,
\[ \text{if } y^0 (\cdot) \equiv 0 \text{ then } \mathbb{E}[\tau] = +\infty, \]
which leads to a contradiction. This indicates the limiting solutions are non-zero. The following lemma describes a crucial character of non-zero limiting solution.

**Lemma 19** Suppose that, at time \( t > 0 \), \( y^0 (t, \omega) > 0 \). Then
\[
0 < \alpha (t) \leq y^0 (t, \omega) \leq \beta (t).
\]

The proof of Lemma 19 can be scheduled in Appendix.

We now prove (19) holds. All the trajectories in \( S_b (0) \) are analyzed and verified seriously in the following cases:

i) For any \( \xi^* (\cdot) \in S^b_f (0) \), where \( \xi^* (\cdot) \) denotes the solution corresponding to \( \alpha (\cdot) \), consider a solution like
\[
\xi^\lambda (t) = \begin{cases} 
0, & \text{if } 0 \leq t \leq \lambda, \ \forall \lambda > 0, \\
\xi^* (t - \lambda), & \text{if } t > \lambda.
\end{cases}
\]
First, \( |\xi^\lambda (t)|^2 \leq \beta (t), \forall t \geq 0 \). Next, there exists time \( t^\lambda > 0 \) such that \( |\xi^\lambda (t^\lambda)|^2 = \alpha (t^\lambda) \). Taking any time \( \bar{t} \in (\lambda, t^\lambda) \), we have \( |\xi^\lambda (\bar{t})|^2 > 0 \), but, \( |\xi^\lambda (\bar{t})|^2 < \alpha (\bar{t}) \) which leads to a contradiction to Lemma 19. So this case can’t be the limiting solutions and excluded completely;

ii) Fixing \( \alpha (\cdot) \), we define a moving solution like above
\[
\alpha^\lambda (t) = \begin{cases} 
0, & \text{if } 0 \leq t \leq \lambda, \ \forall \lambda > 0, \\
\alpha (t - \lambda), & \text{if } t > \lambda.
\end{cases}
\]
Apparently, for any \( t > \lambda \), we have \( \alpha^\lambda (t) > 0 \), but \( \alpha^\lambda (t) < \alpha (t) \), which also makes a contradiction to Lemma 19. Likewise this case is also refused totally.

Basing on i)-ii), the solution set \( S^b_\lambda (0) \) is undoubtedly excluded from limiting solutions. Let us next focus on the last solutions set \( S^b_f (0) \), potential limiting solutions.

iii) For every \( \xi^* (\cdot) \in S^b_f (0) \), by virtue of Lemma 19 we have (24) holds. Furthermore, (22) and (23) holds simultaneously.
Basing on i)-iii), as a result, we claim that \((19)\) holds. The proof is thus complete. ■

**Remark 20** The relation \((21)\), \((11)\) and Lemma 19 play significant roles in the process of proof. However, it is necessary to point out that certain discontinuous drift can ensure the continuity of \(g_{\min} (\cdot)\) and \(g_{\max} (\cdot)\), even \((11)\) holds (see Example 22 below).

Let us present several examples for reader’s convenience.

**Example 21** Let \(b = 0\). Then \(g_{\min} (y) = g_{\max} (y) = 0\). From \((21)\), we have
\[
E \left[ y^\varepsilon (t) \right] = \varepsilon^2 t,
\]
from which we derive that zero is the unique limiting solution.

**Example 22** Consider \(b (x) = \text{sign} (x)\). The Filippov set value map is defined by
\[
F_{\text{sign} (\cdot)} (x) = \begin{cases} 
1 & \text{if } x > 0, \\
[-1, 1] & \text{if } x = 0, \\
-1 & \text{if } x < 0.
\end{cases}
\]
One can verify that \(\xi (t) \equiv 0\), \(\xi (t) = t\) and \(\xi (t) \equiv -t\) are Filippov solutions to
\[
\begin{cases} 
\xi ^t (t) \in F_{\text{sign} (\cdot)} (\xi (t)), \\
\xi (t) = 0.
\end{cases}
\]
Due to the symmetry, \(g_{\min} (y) = g_{\max} (y) = 2\sqrt{y}\). From \((27)\), we have
\[
E \left[ \int_0^{y^0 (t)} \frac{1}{2\sqrt{y} + \varepsilon^2} dy \right] = t.
\]
Letting \(\varepsilon \to 0\), we derive
\[
E \left[ y^0 (t) \right] = t^2 > 0,
\]
which means that \(\xi (t) = -t\) and \(\xi (t) = t\) are limiting solutions.

**Example 23** Put
\[
b (x) = \begin{cases} 
2x^2, & x \geq 0, \\
|x|^4, & x < 0.
\end{cases}
\]
Theorem 5.2 in Bafico & Baldi ([3], 1982) provided the answer: Every limiting value of \( \{ P^\varepsilon \}_\varepsilon \) is concentrated on the extremal solution. Note that the solutions set to drift (26) is the same as \( b(x) = 2x^{1/2}, x \geq 0 \). So we set the domain for (26) \( D_b = \mathbb{R}_+ \), in this way, \( g_{\min}(y) = g_{\max}(y) = 2y^{1/4} \), we get the same conclusion.

Next we will response the selection problem: For any \( \omega \), which solution \( \xi^* (\cdot) \in S^b_f(0) \) will the sample point \( \omega \) select? Due to the technical limitations, we can only deal with continuous drift of \( b \) currently.

**Lemma 24** Assume (H1)-(H3) are in force. We have

\[
\lim_{n \to +\infty} \tilde{\tau}^\varepsilon_n = \tilde{\tau}^0 < \infty, \text{ } \tilde{P}\text{-a.s.},
\]

where the exit time \( \tilde{\tau}^\varepsilon_n \) and \( \tilde{\tau}^0 \) are defined as follows: for some \( r > 0 \)

\[
\tilde{\tau}^\varepsilon_n := \inf \left\{ t \geq 0 : \tilde{X}^\varepsilon_n (t) \notin B(0, r) \right\},
\]

\[
\tilde{\tau}^0 := \inf \left\{ t \geq 0 : \tilde{X}^0(t) \notin B(0, r) \right\}.
\]

**Remark 25** The mathematical implication of (27) is embodied in pursuing of the right selection: For any \( \omega \in \tilde{\Omega}/\mathcal{N} \) fixed, from (19), we know \( \tilde{X}^\varepsilon_n (\cdot, \omega) \to \tilde{X}^0 (\cdot, \omega) \in S^b_f(0), \text{ } \tilde{P}\text{-a.s.} \). A natural question arises: Which solution in \( S^b_f(0) \) can be the right candidate for \( \tilde{X}^\varepsilon_n (\cdot, \omega) \)? The conclusion (27) exactly provides the answer: The trajectory \( \tilde{X}^0 (\cdot, \omega) \) the corresponding to the stopping time \( \tilde{\tau}^0 \) is the right one.

**Proof.** Let us consider our issue on the new probability space \( (\tilde{\Omega}, \tilde{F}, \tilde{P}) \) again defined in Lemma 17. By virtue of the assumptions (H1)-(H3), we have (19) holds since for any \( \xi^* (\cdot) \in S^b_f(0), \exists t^* > 0, \text{ such that } \xi^* (t^*) \in \partial B(0, r) \) for some \( r > 0 \).

Therefore, for arbitrarily given \( T > 0 \),

\[ \tilde{X}^\varepsilon_n (\cdot) \to \tilde{X}^0 (\cdot) \text{ in } C \left( [0, T] ; \mathbb{R}^d \right), \text{ } \tilde{P}\text{-a.s.} \]

We prove (27) by contradiction. Now set

\[ A_1 := \left\{ \omega \in \tilde{\Omega} \left| \limsup_{n \to +\infty} \tilde{\tau}^\varepsilon_n (\omega) > \tilde{\tau}(\omega) \right. \right\} . \]
Suppose that $\tilde{P}(\mathcal{A}) > 0$. For any $\omega \in \mathcal{A}$ set $\mathcal{X}(t) = \tilde{X}(\tau(\omega) + t)$, $\forall t \geq 0$. Immediately,

$$\mathcal{X}(t) - \mathcal{X}(0) = \int_{0}^{t} b(\mathcal{X}(s)) \, ds, \quad \mathcal{X}(0) \in \partial \mathcal{K}, \, \forall t \geq 0.$$ 

Define a closed set $\mathcal{N}_{\mu} = \{ x \in \mathbb{R}^{d} : \left| b_{B(0,r)}(x) \right| \leq \mu \}$, where $\mu > 0$ is small enough such that $b_{B(0,r)} \in C^{2}(\mathcal{N}_{\mu})$ since $B(0,r)$ is a compact domain of class $C^{2}$ (see (8) in Section 2). By assumption (H3), we get

$$2\alpha := \inf_{x \in \partial B(0,r)} \left\langle b(x), \nabla b_{B(0,r)}(x) \right\rangle > 0,$$

for some positive $\alpha > 0$.

Note that Lipschitz constant of $b_{B(0,r)}(x)$ is 1. Taking $\eta \in (0, \mu)$ small enough, such that $\forall y \in \mathcal{N}_{\eta}$, we have $\left\langle b(y), \nabla b_{B(0,r)}(y) \right\rangle > \alpha$ by continuity of $b$. □

In particular, picking $\tilde{t}_{\eta} > 0$ such that $\forall t \in [0, \tilde{t}_{\eta}]$, we have

$$-\eta \leq b_{B(0,r)}(\mathcal{X}^{x}(t)) \leq \eta. \quad (31)$$

Indeed, (31) yields that $|\mathcal{X}(t)| \leq r + \eta$, which implies that

$$|\mathcal{X}(0)| + \left| \int_{0}^{t} b(\mathcal{X}(s)) \, ds \right| \leq r + tM_{b} \leq r + \eta.$$ 

Consequently, we have $\tilde{t}_{\eta} = \frac{\eta}{M_{b}}$, which is independent of $x$. Also note that

$$\frac{1}{2} \frac{d}{dt} b_{B(0,r)}(\mathcal{X}(t)) = \left\langle \nabla b_{B(0,r)}(\mathcal{X}(t)), \mathcal{X}'(t) \right\rangle = \left\langle \nabla b_{B(0,r)}(\mathcal{X}(t)), b(\mathcal{X}(t)) \right\rangle > \alpha,$$

and

$$b_{B(0,r)}(\mathcal{X}(0)) = 0, \quad \text{since } \mathcal{X}(0) \in \partial \mathcal{K}.$$ 

Thus

$$b_{B(0,r)}(\mathcal{X}(t)) \geq 2\alpha t, \quad t \in [0, \tilde{t}_{\eta}].$$
We have, for \( \forall \omega \in \mathcal{A} \),
\[
\left| \tilde{X}^{\varepsilon_n} (\tilde{\tau} (\omega) + \tilde{t}_\eta) - \mathcal{X} (\tilde{t}_\eta) \right| < \alpha \tilde{t}_\eta,
\]
for
\[
 n > N_1 (\omega, \tilde{t}_\eta)
\]
large enough depending only on \( \omega \in \tilde{\Omega}/\mathcal{N} \) and \( \tilde{t}_\eta \) since the metric of the uniform convergence.

Also observe that
\[
\mathcal{X} (\tilde{t}_\eta) = \tilde{X} (\tilde{\tau} (\omega) + \tilde{t}_\eta).
\]

Hence, for any \( \omega \in \mathcal{A} \), we have
\[
\mathcal{B}_{B(0,r)} \left( \tilde{X}^{\varepsilon_n} (\tilde{\tau} (\omega) + \tilde{t}_\eta) \right) \geq \mathcal{B}_{B(0,r)} (\mathcal{X} (\tilde{t}_\eta)) - \left| \mathcal{X} (\tilde{t}_\eta) - \tilde{X}^{\varepsilon_n} (\tilde{\tau} (\omega) + \tilde{t}_\eta) \right|
\]
\[
\geq 2\alpha \tilde{t}_\eta - \alpha \tilde{t}_\eta > 0,
\]
which implies that
\[
\tilde{\tau}^{\varepsilon_n} (\omega) \leq \tilde{\tau} (\omega) + \tilde{t}_\eta.
\]

Passing to \( \limsup \) as \( n \to +\infty \) followed by \( \eta \to 0 \), of course, \( \tilde{t}_\eta \to 0 \), we obtain \( \limsup_{n \to +\infty} \tilde{\tau}^{\varepsilon_n} (\omega) \leq \tilde{\tau} (\omega) \), which is a contradiction to the definition of \( \omega \in \mathcal{A} \).

Now we define
\[
\mathcal{A}_2 := \left\{ \omega \in \tilde{\Omega} \left| \liminf_{n \to +\infty} \tilde{\tau}^{\varepsilon_n} (\omega) < \tilde{\tau} (\omega) \right. \right\}.
\]

Now fixing \( \omega \in \mathcal{A}_2 \), for any \( \delta > 0 \) small enough such that \( \delta \in (0, \tilde{\tau} (\omega)) \), it is easy to prove by absurdum that there exists
\[
n > N_2 (\omega, \delta)
\]
since the topology of the uniform convergence on compact, such that
\[
\left| \tilde{X}^{\varepsilon_n} (\tilde{\tau} (\omega) - \delta) - \tilde{X} (\tilde{\tau} (\omega) - \delta) \right| + \mathcal{B}_{B(0,r)} \left( \tilde{X} (\tilde{\tau} (\omega) - \delta) \right) < 0,
\]
from which we conclude that
\[
\tilde{\tau}^{\varepsilon_n} (\omega) \geq \tilde{\tau} (\omega) - \delta.
\]
Since arbitrary of $\delta$, we have $\lim \inf_{n \to +\infty} \tilde{\tau}^{\varepsilon_n} (\omega) \geq \tilde{\tau} (\omega)$, which is a contradiction to the definition of $\omega \in \mathcal{A}_2$.

Now fixing $\omega \in \tilde{\Omega} / \mathcal{N}$, from (32) and (33) by the method used above, we set

$$\beta := \min \{ \bar{t}_n, \delta \} \quad \text{and} \quad N_3 (\omega, \beta) := \max \{ N_1 (\omega, \bar{t}_n), N_2 (\omega, \delta) \}. \quad (34)$$

Immediately, for $n > N_3 (\omega, \beta)$ large enough, we have

$$|\tilde{\tau}^{\varepsilon_n} (\omega) - \tilde{\tau} (\omega)| < \beta. \quad (35)$$

Now we end the proof by observing that

$$\tilde{P} \left[ \lim_{n \to +\infty} \tilde{\tau}^{\varepsilon_n} = \tilde{\tau} \right] = 1. \quad (36)$$

The proof is complete. ■

**Remark 26** Lemma 17 just answers the small random perturbation problem of Theorem 1.1 in [4]. In fact, the second inequality in (25) can guarantee zero can’t be the limiting solution $\tilde{P}$-a.s.. Therefore, whenever $g_{\min} (\cdot) \equiv 0$, it must have

$$y^\varepsilon (t) \geq t \varepsilon^2 + \varepsilon \int_0^t \left\langle \tilde{X}^\varepsilon (s), dW (s) \right\rangle,$$

which carries out nothing information on limiting solution as $\varepsilon \to 0$.

Our main result in this paper is the following one which actually extends Theorem 4.1 in [3]:

**Theorem 27** Let the Assumptions (H1)-(H3) are in force. Let $X^\varepsilon (\cdot)$ be a strong solution to the SDE (2) with initial condition $x = 0$. Then, as $\varepsilon \to 0^+$, there exists a subsequence $\varepsilon_n \to 0^+$ as $n \to +\infty$, such that $X^{\varepsilon_n} (\cdot)$ converges in law, to some $X^0 (\cdot)$ which belongs to the leaving solution set $S^b_f (0)$ of ODE (1), i.e.,

$$P \left[ X^0 (\cdot) \in S^b_f (0) \right] = 1. \quad (37)$$

Furthermore, $X^{\varepsilon_n} (\cdot)$ will select such a limit process $X^0 (\cdot)$ as $n \to \infty$, in the following sense:

$$P \left[ \lim_{n \to +\infty} \tau^{\varepsilon_n} = \tau^0 \right] = 1, \quad (38)$$
where the exit time $\tau^\varepsilon_n$ and $\tau^0$ are defined as follows: for some $r > 0$

$$\tau^\varepsilon_n := \inf \left\{ t \geq 0 : X^\varepsilon_n (t) \notin B(0, r) \right\}, \quad (39)$$

$$\tau^0 := \inf \left\{ t \geq 0 : X^0 (t) \notin B(0, r) \right\}. \quad (40)$$

**Remark 28** In other words, the stopping time $\tau^\varepsilon_n$ corresponding to $X^\varepsilon_n (\cdot)$ defined in (39) converges to certain $\tau$ defined in (40), whose associated trajectory $\xi^*(\cdot)$ is our target!

**Proof.** Because $\tilde{X}^0 (\cdot)$ and $X^0 (\cdot)$ have the same law (see the second assertion of (18)), by Lemma 17, we can conclude that

$$P \left[ X^0 (\cdot) \in S^b_f (0) \right] = \tilde{P} \left[ \tilde{X}^0 (\cdot) \in S^b_f (0) \right] = 1,$$

and

$$P \left[ \lim_{n \to +\infty} \tau^\varepsilon_n = \tau^0 \right] = \tilde{P} \left[ \lim_{n \to +\infty} \tilde{\tau}^\varepsilon_n = \tilde{\tau}^0 \right] = 1. \quad (41)$$

We thus complete the proof. ■

We have found the limiting solutions, i.e., $S^b_f (0)$. However, the following example shows that the probability concentrated on the extremal solution could be zero.

**Example 29** Let

$$b(x) = \begin{cases} -x^\alpha \log x, & x \geq 0, \\ -|x|^\beta, & x < 0, \end{cases}$$

with $\alpha < \beta < 1$. By Lemma 4.2 in [3], Bafico & Baldi (1982) have proven that in a small time interval there is exactly one limiting value which gives mass only on the upper extremal solution.

## 5 Examples

In this section, we illustrate the result of Section 5 by looking at some examples and provide a concrete example whose trajectories of leaving solution are non-symmetric in one-dimensional perturbed SDE. Simultaneously, we give the explicit leaving solutions and validate our major theoretical result (Theorem 27).
Example 30 Consider again the dynamic systems (13) of Example 12. As \( \varepsilon \to 0 \), there are two limiting solutions

\[
\begin{align*}
  x_1 (t) &= \frac{t^2}{4}, & x_2 (t) &= -\frac{9t^2}{4}, & t \geq 0,
\end{align*}
\]

with probability \( \frac{1}{1+3\varepsilon^2} \) \( \frac{3\varepsilon^2}{1+3\varepsilon^2} \), respectively (for more details see [3]).

We end this section with a two-dimensional coupled case which shows infinitely many leaving solutions.

Example 31 Consider a two-dimensional SDE. Let \( W^1 (\cdot) \) and \( W^2 (\cdot) \) be two independent Brownian motions,

\[
\begin{align*}
  dX^\varepsilon_1 (t) &= \frac{2X^\varepsilon_1 (t)}{\left((X^\varepsilon_1 (t))^2 + (X^\varepsilon_2 (t))^2\right)^{\frac{3}{4}}} dt + \varepsilon dW^1 (t), \\
  dX^\varepsilon_2 (t) &= \frac{2X^\varepsilon_2 (t)}{\left((X^\varepsilon_1 (t))^2 + (X^\varepsilon_2 (t))^2\right)^{\frac{3}{4}}} dt + \varepsilon dW^2 (t), \\
  X^\varepsilon_1 (0) &= 0, & X^\varepsilon_2 (0) &= 0.
\end{align*}
\]

The corresponding ODE is

\[
\begin{align*}
  dx_1 (t) &= \frac{2x_1 (t)}{\left(x_1 (t))^2 + (x_2 (t))^2\right)^{\frac{3}{4}}} dt, \\
  dx_2 (t) &= \frac{2x_2 (t)}{\left((x_1 (t))^2 + (x_2 (t))^2\right)^{\frac{3}{4}}} dt, \\
  x_1 (0) &= 0, & x_2 (0) &= 0.
\end{align*}
\]

The solutions are zero or

\[
X^{\theta, \lambda} (t) = \begin{cases} 
  x_1 (t) = \begin{cases} 
    0, & \text{if } 0 \leq t \leq \lambda, \\
    (t - \lambda)^2 \sin \theta, & \text{if } t > \lambda,
  \end{cases} \\
  x_2 (t) = \begin{cases} 
    0, & \text{if } 0 \leq t \leq \lambda, \\
    (t - \lambda)^2 \cos \theta, & \text{if } t > \lambda.
  \end{cases}
\end{cases}
\]

On one hand,

\[
\left\langle b \left( \begin{bmatrix} 2x_1 \\ (x_1^2 + x_2^2)^{\frac{3}{4}} \\ (x_1^2 + x_2^2)^{\frac{3}{4}} \end{bmatrix} \right) \right| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle = 2 \left( x_1^2 + x_2^2 \right)^{\frac{3}{2}} > 0, \text{ if } x_1^2 + x_2^2 > 0.
\]

On the other hand, it is easy to show \( g_{\min} (y) = 4y^{\frac{3}{4}} \) and for any \( r > 0 \), \( \int_0^r \frac{1}{g_{\min}(y)} dy = r^{\frac{4}{3}} \). So the hypotheses (H1)-(H3) hold. Therefore, the limit processes \( X^0 (\cdot) \) are supported by \( \{ X^{\theta, 0} (t) \}_{t \geq 0, \theta \in [0, 2\pi]} \) with the uniform law on \([0, 2\pi]\).
6 Concluding remarks

In this paper, we develop a method to seek the limiting solution of noise perturbed SDE with continuous and locally bounded drift \( b \). The main tools are based on the Prokhorov’s theorem, Skorohod’s theorem and stopping time technique. Still and all, there are large interesting issues imposed as follows:

(i) Theorem 27 states the limiting solutions (leaving solutions) can be successfully obtained via the convergence of stopping time of \( X^\varepsilon (\cdot) \). Naturally, one must concern on the probability happening on this process. See e.g. Example 30 for only two leaving solutions. As for multi-dimensional case, how to compute it? From Example 31 we see under some symmetric structure, the probability on limiting solutions exhibits the uniform law on \([0, 2\pi]\). However, the general case at present is unexplored.

(ii) In Example 11 the drift \( b \) has infinitely many singular points, namely, \( g_{\min}(\cdot) \equiv 0 \). At least, due to its symmetrical structure, once again we conjecture the limiting solutions are \((t^2, t^2), (-t^2, t^2), (t^2, -t^2), (-t^2, -t^2)\), respectively. We will attack this issue in our future work.

(iii) It might be not easy to employ our idea to study the discontinuous framework, even only measurable and bounded. Indeed, Filippov solution is a very good choice. Nevertheless, the related tools should be developed simultaneously.

A Appendix

A.1 Proof of Lemma 19

Proof. For reader’s convenience, recall

\[
\int_0^{y^\varepsilon(t)} \frac{1}{g_{\min}(y) + \varepsilon^2} \, dy \geq t - \varepsilon \int_0^t \frac{\langle \tilde{X}^\varepsilon(s), dW(s) \rangle}{g_{\min}(y^\varepsilon(s)) + \varepsilon^2}. \tag{43}
\]

If \( y^0(t, \omega) > 0 \), then \( g_{\min}(y^0(t)) > 0 \). Immediately, letting \( \varepsilon \to 0 \) in (43), we have

\[
\frac{\langle \tilde{X}^\varepsilon(s), dW(s) \rangle}{g_{\min}(y^\varepsilon(t)) + \varepsilon^2} \to \frac{\langle \tilde{X}^0(s), dW(s) \rangle}{g_{\min}(y^0(t))}.
\]
Consequently
\[ \int_0^{y^0(t,\omega)} \frac{1}{g_{\min}(y)} dy \geq t. \] (44)

Similarly,
\[ \int_0^{y^0(t,\omega)} \frac{1}{g_{\max}(y)} dy \leq t. \] (45)

Therefore,
\[ \alpha(t) \leq y^0(t,\omega) \leq \beta(t). \]

We end the proof. ■

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References

[1] S. Attanasio, F. Flandoli, (2009), Zero-noise solutions of linear transport equations without uniqueness: an example, *C. R. Math. Acad. Sci. Paris* 347, no. 13-14, 753–756.

[2] J.-P. Aubin, A. Cellina, Differential Inclusions, Grundlehren Math. Wiss., vol. 264, Springer-Verlag, Berlin, 1984.

[3] R. Bafico, and P. Baldi, (1982), Small random perturbation of Peano Phenomena, *Stochastics*, 6, 279-292.

[4] R. Bafico, (1980), On the convergence of the weak solutions of stochastic differential equations when the noise intensity goes to zero, *Boll. Unione Mat. Ital. Sez. B* 17, 308-324.

[5] K. Bahlali, (1999), Flows of homeomorphisms of stochastic differential equations with measurable drift. *Stochastics and Stochastics Reports*, Vol. 67, 53-82.

[6] V. S. Borkar, K. Suresh Kumar, (2010), A new Markov selection procedure for degenerate diffusions, *J. Theoret. Probab.* 23, no. 3, 729–747.
[7] P. Baldi and L. Caramellino, (2011), General Freidlin-Wentzell large deviations and positive diffusions, *Statist. Probab. Lett.* 81 no. 8, 1218–1229.

[8] R. Buckdahn, Y. Ouknine, and M. Quincampoix, (2009), On limiting values of stochastic differential equations with small noise intensity tending to zero, *Bull. Sci. Math* 133: 229-237.

[9] M. C. Delfour, J. -P. Zolesio, (1994), Shape analysis via oriented distance functions, *J. Funct. Anal.* 123, 129-201.

[10] F. Delarue, F. Flandoli, and D. Vincenzi, (2014), Noise prevents collapse of Vlasov-Poisson point charges, *Comm. Pure Appl. Math.* Volume 67, Issue10, October, Pages 1700-1736

[11] F. Delarue, F. Flandoli, (2014), The transition point in the zero-noise limit for a 1D Peano example, *Discrete and Continuous Dynamical Systems*, 34(10): 4071-4083.

[12] M. I. Friedlin, A. D. Wentzell, Random perturbations of dynamical systems, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 260, Springer-Verlag, New York, 1984.

[13] A.F. Filippov, Differential Equations with Discontinuous Right-Hand Sides, Math. Appl. Soviet Ser., vol. 18, Kluwer

[14] Academic Publishers, Dordrecht, 1988.

[15] F. Flandoli, Random Perturbation of PDEs and Fluid Dynamic Models, Springer, 2011.

[16] U. S. Fjordholm, M. Musch, A. Pilipenko, (2023), The zero-noise limit of SDEs with $L^\infty$ drift, [arXiv:2205.15082v1 [math.PR]].

[17] W. H. Fleming, H.M. Soner, Controlled Markov Processes and Viscosity Solutions, Springer, 2006.

[18] D. Gilbarg, N. Trudinger, *Elliptic Partial Differential Equations of Second Order*. Springer.
[19] M. Gradinaru, S. Herrmann, B. Roynette, (2001), A singular large deviations phenomenon, *Ann. Inst. H. Poincaré Probab. Statist.* 37, (5), 555-580.

[20] G. Gabor, M. Quincampoix, (2002), On existence of solutions to differential equations or inclusions remaining in a prescribed closed subset of a finite-dimensional space, *Journal of Differential Equations* 185, 483-512.

[21] I. Gyöngy, T. Martinez, (2001), On stochastic differential equations with locally unbounded drift, *Czechoslovak Math. J.* (4), 51, (126), 763-783.

[22] N. Ikeda, S. Watanabe, Stochastic differential equations and diffusion processes, Second Edition.

[23] A. Kulik, A. Pilipenko, (2021), On Regularization by a Small Noise of Multidimensional Odes with Non-Lipschitz Coefficients. *Ukr Math J* 72, 1445–1481.

[24] O. Menoukeu-Pamen, T. Meyer-Brandis, F. Proske, (2010), A Gel’fand triple approach to the small noise problem for discontinuous ODE’s. Dept. of Math./Cma Univ. of Oslo pure mathematics No 25, ISSN, 0806-2439.

[25] M. Maurelli F. Delarue. Zero noise limit for multidimensional sdes driven by a pointy gradient. arXiv preprint, [arXiv:1909.08702](https://arxiv.org/abs/1909.08702), 201

[26] I. G. Petrovsky, Ordinary Differential Equations, Prentice-Hall, 1966.

[27] A. Pilipenko and F. N. Proske, (2018), On a selection problem for small noise perturbation in the multidimensional case. *Stochastics and Dynamics*, Vol. 18, No. 06, 1850045.

[28] I. Pavlyukevich, A. Pilipenko, (2020). Generalized Peano problem with Levy noise. *Electronic Communications in Probability*, 25.

[29] A. Yu. Pilipenko, F.N. Proske, (2018) On perturbations of an ODE with non-Lipschitz coefficients by a small self-similar noise, *Statistics & Probability Letters*, Volume 132, January 2018, Pages 62-73.

[30] Pilipenko, A., Proske, F. N. (2021). Small Noise Perturbations in Multidimensional Case. arXiv preprint [arXiv:2106.09935](https://arxiv.org/abs/2106.09935).
[31] S. Herrmann, (2001), Phénomène de Peano et grandes déviations, *C. R. Acad. Sci. Paris Sér. I Math.* 332 no. 11, 1019–1024.

[32] D. Trevisan, (2013), Zero noise limits using local times, *Electron. Commun. Probab.* 18 no. 31, 1–7.

[33] A.K. Zvonkin, (1974), A transformation of the phase space of a diffusion process that removes the drift, *Mat. Sb.* (1), 93, (135).