Structured Time-Delay Models for Dynamical Systems with Connections to Frenet-Serret Frame

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Abstract

Time-delay embeddings and dimensionality reduction are powerful techniques for discovering effective coordinate systems to represent the dynamics of physical systems. Recently, it has been shown that models identified by dynamic mode decomposition (DMD) on time-delay coordinates provide linear representations of strongly nonlinear systems, in the so-called Hankel alternative view of Koopman (HAVOK) approach. Curiously, the resulting linear model has a matrix representation that is approximately antisymmetric and tridiagonal with a zero diagonal; for chaotic systems, there is an additional forcing term in the last component. In this paper, we establish a new theoretical connection between HAVOK and the Frenet-Serret frame from differential geometry, and also develop an improved algorithm to identify more stable and accurate models from less data. In particular, we show that the sub- and super-diagonal entries of the linear model correspond to the intrinsic curvatures in Frenet-Serret frame. Based on this connection, we modify the algorithm to promote this antisymmetric structure, even in the noisy, low-data limit. We demonstrate this improved modeling procedure on data from several nonlinear synthetic and real-world examples.

Keywords: Dynamic mode decomposition, Time-delay coordinates, Frenet-Serret, Koopman operator, Hankel matrix.

1 Introduction

Discovering meaningful models of complex, nonlinear systems from measurement data has the potential to improve characterization, prediction, and control. Focus has increasingly turned from first-principles modeling towards data-driven techniques to discover governing equations that are as simple as possible while accurately describing the data [1−4]. However, available measurements may not be in the right coordinates for which the system admits a simple representation. Thus, considerable effort has gone into learning effective coordinate transformations of the measurement data [5−7], especially those that allow nonlinear dynamics to be approximated by a linear system. These coordinates are related to eigenfunctions of the Koopman operator [8, 13], with dynamic mode decomposition (DMD) [14] being the leading computational algorithm for high-dimensional spatiotemporal data [11, 13, 15]. For low-dimensional data, time-delay embedding [16] has been

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shown to provide accurate linear models of nonlinear systems \cite{5,17,18}. Linear time-delay models have a rich history \cite{19,20}, and recently, DMD on delay coordinates \cite{15,21} has been rigorously connected to these linearizing coordinate systems in the Hankel alternative view of Koopman (HAVOK) approach \cite{5,7,17}. In this work, we establish a new connection between HAVOK and the Frenet-Serret frame from differential geometry, which inspires an extension to the algorithm that improves the stability of these models.

Time-delay embedding is a widely used technique to characterize dynamical systems from limited measurements. In delay embedding, incomplete measurements are used to reconstruct a representation of the latent high-dimensional system by augmenting the present measurement with a time-history of previous measurements. Takens showed that under certain conditions, time-delay embedding produces an attractor that is diffeomorphic to the attractor of the latent system \cite{16}. Time-delay embeddings have also been extensively used for signal processing and modeling \cite{19,20,22,27}, for example, in singular spectrum analysis (SSA) \cite{19,22} and the eigensystem realization algorithm (ERA) \cite{20}. In both cases, a time history of augmented delay vectors are arranged as columns of a Hankel matrix, and the singular value decomposition (SVD) is used to extract eigen-time-delay coordinates in a dimensionality reduction stage. More recently, these historical approaches have been connected to the modern DMD algorithm \cite{15}, and it has become commonplace to compute DMD models on time delay coordinates \cite{15,21}. The HAVOK approach established a rigorous connection between DMD on delay coordinates and eigenfunctions of the Koopman operator \cite{5}; HAVOK \cite{5} is also referred to as Hankel DMD \cite{17} or delay DMD \cite{15}.

HAVOK produces linear models where the matrix representation of the dynamics has a peculiar and particular structure. These matrices tend to be skew-symmetric and dominantly tridiagonal, with zero diagonal (see Fig. 2 for an example). In the original HAVOK paper, this structure was observed in some systems, but not others, with the structure being more pronounced in noise-free examples with an abundance of data. It has been unclear how to interpret this structure and whether or not it is a universal feature of HAVOK models. Moreover, the eigen-time-delay modes closely resemble Legendre polynomials; these polynomials were explored further in Kamb et al. \cite{28}. The present work directly resolves this mysterious structure by establishing a connection to the Frenet-Serret frame from differential geometry.

The structure of HAVOK models may be understood by introducing intrinsic coordinates from differential geometry \cite{29}. One popular set of intrinsic coordinates is the Frenet-Serret frame, which is formed by applying the Gram-Schmidt procedure to the derivatives of the trajectory $\dot{x}(t), \ddot{x}(t), \ldots$ \cite{30,31,32}. Alvarez-Vizoso et al. \cite{33} showed that the SVD of trajectory data converges locally to the Frenet-Serret frame in the limit of an infinitesimal time step. The Frenet-Serret frame results in an orthogonal basis of polynomials, which we will connect to the observed Legendre basis of HAVOK \cite{5,28}. Moreover, we show that the dynamics, when represented in these coordinates, have the same tridiagonal structure as the HAVOK models. Importantly, the terms along the sub- and super-diagonals have a specific physical interpretation as intrinsic curvatures. By enforcing this structure, HAVOK models are more robust to noisy and limited data.

In this work, we present a new theoretical connection between time-delay embedding models and the Frenet-Serret frame from differential geometry. Our unifying perspective sheds light on the antisymmetric, tridiagonal structure of the HAVOK model. We use this understanding to develop structured HAVOK models that are more accurate for noisy and limited data. Section 2 provides a review of dimensionality reduction methods, time delay embeddings, and the Frenet-Serret frame. This section also discusses current connections between these fields. In Section 3, we establish...
the main result of this work, connecting linear time-delay models with the Frenet-Serret frame, explaining the tridiagonal, antisymmetric structure seen in Figure 2. We then illustrate this theory on a synthetic example. In Section 4 we explore the limitations and requirements of the theory, giving recommendations for achieving this structure in practice. In Section 5 based on this theory, we develop a modified HAVOK method, called structured HAVOK (sHAVOK), which promotes tridiagonal, antisymmetric models. We demonstrate this approach on three nonlinear synthetic examples and two real-world datasets, namely measurements of a double pendulum experiment and measles outbreak data, and show that sHAVOK yields more stable and accurate models from significantly less data.

2 Related Work

Our work relates and extends results from three fields: dimensionality reduction, time-delay embedding, and the Frenet-Serret coordinate frame from differential geometry. There is an extensive literature on each of these fields, and here we give a brief introduction of the related work to establish a common notation on which we build a unifying framework in Section 3.

2.1 Dimensionality Reduction

Recent advancements in sensor and measurement technologies have led to a significant increase in the collection of time-series data from complex, spatio-temporal systems. Although such data is typically high dimensional, in many cases it can be well approximated with a low dimensional representation. One central goal is to learn the underlying structure of this data. Although there are many data-driven dimensionality reduction methods, here we focus on linear techniques because of their effectiveness and analytic tractability. In particular, given a data matrix $\mathbf{X} \in \mathbb{R}^{m \times n}$, the

Figure 1: In this work, we unify key results from dimensionality reduction, time-delay embedding and the Frenet-Serret frame to show that a dynamical system may be decomposed into a sparse linear model plus a forcing term. Further, this linear model has a particular structure: it is an antisymmetric tridiagonal matrix with nonzero elements only along the super- and sub-diagonals. These nonzero elements are interpretable as they are intrinsic curvatures of the system in the Frenet-Serret frame.
goal of these techniques is to decompose $X$ into the matrix product

$$X = UV^\top,$$  

(1)

where $U \in \mathbb{R}^{m \times k}$ and $V \in \mathbb{R}^{n \times k}$ are low rank ($k < \min(m, n)$). The task of solving for $U$ and $V$ is highly underdetermined, and different solutions may be obtained when different assumptions are made.

Here we review two popular linear dimensionality reduction techniques: singular value decomposition (SVD) [34, 35] and dynamic mode decomposition (DMD) [13, 15, 36]. Both of these methods are key components of the HAVOK algorithm and play a key role in determining the underlying tridiagonal antisymmetric structure in Figure 2.

### 2.1.1 Singular Value Decomposition (SVD)

The SVD is one of the most popular dimensionality reduction methods, and it has been applied in a wide range of applications, including genomics [37], physics [38], and image processing [39]. SVD is the underlying algorithm for principal component analysis (PCA).

Given the data matrix $X \in \mathbb{R}^{m \times n}$, the SVD decomposes $X$ into the product of three matrices,

$$X = U\Sigma V^\top,$$

where $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ are unitary matrices, and $\Sigma \in \mathbb{R}^{m \times n}$ is a diagonal matrix with nonnegative entries [34, 35]. We denote the $i$th columns of $U$ and $V$ as $u_i$ and $v_i$, respectively. The diagonal elements of $\Sigma$, $\sigma_i$, are known as the singular values of $X$, and they are written in descending order.

The rank of the data is defined to be $R$, which equals the number of nonzero singular values. Consider the low rank matrix approximation

$$X_r = \sum_{j=1}^{r} u_j \sigma_j v_j^T,$$

with $r \leq R$. An important property of $X_r$ is that it is the best rank $r$ approximation to $X$ in the least squares sense. In other words,

$$X_r = \arg\min_Y \|X - Y\| \text{ such that rank}(Y) = r,$$

with respect to both the $l_2$ and Frobenius norms. Further, the relative error in this rank-$r$ approximation using the $l_2$ norm is

$$\frac{\|X - X_r\|_{l_2}}{\|X\|_{l_2}} = \frac{\sigma_{r+1}}{\sigma_1}. $$  

(2)

From (2), we immediately see that if the singular values decay rapidly, ($\sigma_{j+1} \ll \sigma_j$), then $X_r$ is a good low-rank approximation to $X$. This property makes the SVD a popular tool for compressing data.
2.1.2 Dynamic Mode Decomposition (DMD)

DMD \[13–15\] is another linear dimensionality reduction technique that incorporates an assumption that the measurements are time series data generated by a linear dynamical system in time. DMD has become a popular tool for modeling dynamical systems in such diverse fields, including fluid mechanics \[11, 14\], neuroscience \[21\], disease modeling \[40\], robotics \[41\], plasma modeling \[42\], resolvent analysis \[43\], and computer vision \[44, 45\].

Like the SVD, for DMD we begin with a data matrix \(X \in \mathbb{R}^{m \times n}\). Here we assume that our data is generated by an unknown dynamical system so that the columns of \(X\), \(x(t_k)\), are time snapshots related by the map \(x(t_{k+1}) = F(x(t_k))\). While \(F\) may be nonlinear, the goal of DMD is to determine the best-fit linear operator \(A : \mathbb{R}^m \to \mathbb{R}^m\) such that

\[
x(t_{k+1}) \approx Ax(t_k).
\]

If we define the two time-shifted data matrices,

\[
X_1^{n-1} = \begin{bmatrix} x(t_1) & x(t_2) & \cdots & x(t_{n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ x(t_1) & x(t_2) & \cdots & x(t_{n-1}) \end{bmatrix}, \quad \text{and} \quad X_2^n = \begin{bmatrix} x(t_2) & x(t_3) & \cdots & x(t_n) \\ \vdots & \vdots & \ddots & \vdots \\ x(t_2) & x(t_3) & \cdots & x(t_n) \end{bmatrix},
\]

then we can equivalently define \(A \in \mathbb{R}^{m \times m}\) to be the operator such that

\[
X_2^n \approx AX_1^{n-1}.
\]

It follows that \(A\) is the solution to the minimization problem

\[
A = \min_{A'} \|X_2^n - A'X_1^{n-1}\|_F,
\]

where \(\|\cdot\|_F\) denotes the Frobenius norm.

A unique solution to this problem can be obtained using the exact DMD method and the Moore-Penrose pseudo-inverse \(\hat{A} = X_2^n (X_1^{n-1})^\dagger\) \[13, 15\]. Alternative algorithms have been shown to perform better for noisy measurement data, including optimized DMD \[46\], forward-backward DMD \[47\], and total-least squares DMD \[48\].

One key benefit of DMD is that it builds an explicit temporal model and supports short-term future state prediction. Defining \(\{\lambda_j\}\) and \(\{v_j\}\) to be the eigenvalues and eigenvectors of \(A\), respectively, then we can write

\[
x(t_k) = \sum_{j=1}^r v_j e^{\omega_j t_k},
\]

where \(\omega_j = \ln(\lambda_j)/\Delta t\) are eigenvalues normalized by the sampling interval \(\Delta t\), and the eigenvectors are normalized such that \(\sum_{j=1}^r v_j = x(t_1)\). Thus, to compute the state at an arbitrary time \(t\), we can simply evaluate (3) at that time. Further, letting \(v_j\) be the columns of \(U\) and \(\{\exp(\omega_j t_k)\}^{(k = 1, \ldots, r)}\) be the columns of \(V\), then we can express data in the form of (1).

2.2 Time Delay Embedding

Suppose we are interested in a dynamical system

\[
\frac{d\xi}{dt} = F(\xi),
\]
where $\xi(t) \in \mathbb{R}^l$ are states whose dynamics are governed by some unknown nonlinear differential equation. Typically, we measure some possibly nonlinear projection of $\xi$, $x(\xi) \in \mathbb{R}^d$ at discrete time points $t = 0, \Delta t, \ldots, q\Delta t$. In general, the dimensionality of the underlying dynamics is unknown, and the choice of measurements are limited by practical constraints. Consequently, it is difficult to know whether the measurements $x$ are sufficient for modeling the system. For example, $d$ may be smaller than $m$. In this work we are primarily interested in the case of $d = 1$; in other words, we have only a single one-dimensional time series measurement for the system.

We can construct an embedding of our system using successive time delays of the measurement $x$, at $x(t-\tau)$. Given a single measurement of our dynamical system $x(t) \in \mathbb{R}$, for $t = 0, \Delta t, \ldots, q\Delta t$, we can form the Hankel matrix $H \in \mathbb{R}^{m \times n}$ by stacking time shifted snapshots of $x$,

$$
H = \begin{bmatrix}
x_1 & x_2 & x_3 & x_4 & \cdots & x_n \\
x_2 & x_3 & x_4 & x_5 & \cdots & x_{n+1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
x_m & x_{m+1} & x_{m+2} & x_{m+3} & \cdots & x_{q+1}
\end{bmatrix}.
$$

(4)

Each column may be thought of as an augmented state space that includes a short, $m$-dimensional trajectory in time. Our data matrix $H$ is then this $m$-dimensional trajectory measured over $n$ snapshots in time.

There are several key benefits of using time delay embeddings. Most notably, given a chaotic attractor, Taken’s embedding theorem states that a sufficiently high dimensional time delay embedding of the system is diffeomorphic to the original attractor [16], as illustrated in Figure 1. In addition, recent results have shown that time delay matrices are guaranteed to have strongly decaying singular value spectra. In particular, Beckerman et al. [50] prove the following theorem:

**Theorem 1.** Let $H_n \in \mathbb{R}^{n \times n}$ be a positive definite Hankel matrix, with singular values $\sigma_1, \ldots, \sigma_n$. Then $\sigma_j \leq C\rho^{-j/\log n}\sigma_1$ for constants $C$ and $\rho$ and for $j = 1, \ldots, n$.

Equivalently, $H_n$ can be approximated up to an accuracy of $\epsilon \|H_n\|_2$ by a rank $O(\log n \log 1/\epsilon)$ matrix. From this, we see that $H_n$ can be well-approximated by a low-rank matrix.

Many methods have been developed to take advantage of this structure of the Hankel matrix, including the eigensystem realization algorithm (ERA) [20], singular spectrum analysis (SSA) [19], and nonlinear Laplacian spectrum analysis [22]. DMD may also be computed on delay coordinates from the Hankel matrix [15, 21, 51], and it has been shown that this approach may provide a Koopman invariant subspace [5, 52]. In addition, this structure has also been incorporated into neural network architectures [53].

### 2.3 HAVOK: Dimensionality Reduction and Time Delay Embeddings

Leveraging dimensionality reduction and time delay embeddings, the Hankel alternative view of Koopman (HAVOK) algorithm constructs low dimensional models of dynamical systems [5]. Specifically, HAVOK learns effective measurement coordinates of the system and estimate its intrinsic dimensionality. Remarkably, HAVOK models are simple, consisting of a linear model and a forcing term that can be used for short term forecasting.

We illustrate this method in Figure 2 for the Lorenz system (see section 5.2 for details about this system). To do so, we begin with a one dimensional time series $x(t)$ for $t = 0, \Delta t, \ldots, q\Delta t$. We
Forcing Regression on Dynamics

\[ \frac{d}{dt} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{r-1} \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{r-1} \end{bmatrix} + v_r \]

Linear Dynamics

\[ H = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & \cdots & x_m \\ x_2 & x_3 & x_4 & x_5 & \cdots & x_{m+1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_m & x_{m+1} & x_{m+2} & x_{m+3} & \cdots & x_q \end{bmatrix} \]

SVD

\[ H = U \Sigma V^\top. \]

If \( H \) is sufficiently low rank (with rank \( r \)), then we need only consider the reduced SVD,

\[ H_r = U_r \Sigma_r V_r^\top, \]

where \( U_r \in \mathbb{R}^{m \times r} \) and \( V_r \in \mathbb{R}^{n \times r} \) are orthogonal matrices and \( \Sigma_r \in \mathbb{R}^{r \times r} \) is diagonal. Rearranging the terms, \( V_r^\top = \Sigma_r^{-1} U^\top H_r \) and we can think of

\[ V_r^\top = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \tag{5} \]

as a lower dimensional representation of our high dimensional trajectory. For quasi-periodic systems, the SVD decomposition of the Hankel matrix results in principal component trajectories (PCT) \[51\], which reconstruct dynamical trajectories in terms of periodic orbits.

To discover the linear dynamics, we apply DMD. In particular, we construct the time shifted matrices,

\[ V_1 = \begin{bmatrix} v_1 & v_2 & \cdots & v_{n-1} \end{bmatrix} \text{ and } V_2 = \begin{bmatrix} v_2 & v_3 & \cdots & v_n \end{bmatrix}. \tag{6} \]

We then compute the linear approximation \( \hat{A} \) such that \( V_2 = \hat{A} V_1 \), where \( \hat{A} = V_2 V_1^\top \). This yields a model \( \dot{v}_{i+1} = \hat{A} v_i \).

In the continuous case,

\[ \dot{v}(t) = A v(t) \tag{7} \]

which is related to first order in \( \Delta t \) to the discrete case by

\[ A \approx (\hat{A} - I) / \Delta t. \]
For a general nonlinear dynamical system, this linear model yields a poor reconstruction. Instead, [5] proposed a linear model plus a nonlinear forcing term in the last component of $\mathbf{v}$ (Figure 2):

$$\dot{\mathbf{v}}(t) = \mathbf{A}\mathbf{v}(t) + \mathbf{B}\gamma_r(t),$$

where $\mathbf{v}(t) \in \mathbb{R}^{r-1}$, $\mathbf{A} \in \mathbb{R}^{r-1 \times r-1}$, and $\mathbf{B} \in \mathbb{R}^{r-1}$. In this case, $\mathbf{V}_2$ is defined as columns 2 to $n$ of the SVD singular vectors with an $r - 1$ rank truncation $\mathbf{V}_{1 - 1}^\top$. $\hat{\mathbf{A}}, \hat{\mathbf{B}} \in \mathbb{R}^{r-1 \times r-1}$ and $\hat{\mathbf{B}} \in \mathbb{R}^{r-1 \times 1}$ are computed as $\mathbf{V}_2 \mathbf{V}_1^\top$. The continuous analog of $\hat{\mathbf{B}}, \hat{\mathbf{B}}$, is computed by $\mathbf{B} \approx (\hat{\mathbf{B}} - I)/\Delta t$.

HAVOK was shown to be a successful model for a variety of systems, including a double pendulum and switchings of Earth’s magnetic field. In addition, the linear portion of the HAVOK model has been observed to adopt a very particular structure: the dynamics matrix was antisymmetric, with nonzero elements only on the superdiagonal and subdiagonal (Figure 2).

Much work has been done to study the properties of HAVOK. Arbabi et al. [17] showed that, in the limit of an infinite number of time delays ($m \to \infty$), $\mathbf{A}$ converges to the Koopman operator for ergodic systems. Bozzo et al. [55] showed that in a similar limit, for periodic data, HAVOK converges to the temporal discrete Fourier transform. Kamb et al. [28] connects HAVOK to the use of convolutional coordinates. The primary goal of this current work is to connect HAVOK to the concept of curvature in differential geometry, and with these new insights, improve the HAVOK algorithm to take advantage of this structure in the dynamics matrix. In contrast with much of the previous work, we focus on the limit where only small amounts of noisy data are available.

### 2.4 The Frenet-Serret Coordinate Frame

Suppose we have a smooth curve $\gamma(t) \in \mathbb{R}^m$ measured over some time interval $t \in [a, b]$. As before, we would like to determine an effective set of coordinates in which to represent our data. When using SVD or DMD, the basis discovered corresponds to the spatial modes of the data and is constant in time. However, for many systems, it is sometimes natural to express both the coordinates and basis as functions of time [50, 56]. One popular method for developing this noninertial frame is the Frenet-Serret coordinate system, which has been applied in a wide range of fields, including robotics [58, 59], aerodynamics [60], and general relativity [61, 62].

Let us assume that $\gamma(t)$ has $r$ nonzero continuous derivatives, $\gamma'(t), \gamma''(t), \ldots, \gamma^{(r)}(t)$. We further assume that these derivatives are linearly independent and $\|\gamma'(t)\| \neq 0$ for all $t$. Using the Gram-Schmidt process, we can form the orthonormal basis, $e_1, e_2, \ldots, e_r$,

$$e_1(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|},$$

$$e_2(t) = \frac{\gamma''(t) - \langle \gamma''(t), e_1(t) \rangle e_1(t)}{\|\gamma''(t) - \langle \gamma''(t), e_1(t) \rangle e_1(t)\|},$$

$$\vdots$$

$$e_r(t) = \frac{\gamma^{(r)}(t) - \sum_{k=1}^{r-1} \langle \gamma^{(r)}(t), e_k(t) \rangle e_k(t)}{\|\gamma^{(r)}(t) - \sum_{k=1}^{r-1} \langle \gamma^{(r)}(t), e_k(t) \rangle e_k(t)\|}.$$  

Here $\langle \cdot, \cdot \rangle$ denotes an inner product, and we choose $r \leq m$ so that these vectors are linearly independent and hence form an orthonormal basis basis. This set of basis vectors define the Frenet-Serret frame.
To derive the evolution of this basis, let us define the matrix formed by stacking these vectors
\[ Q(t) = [e_1(t), e_2(t), \ldots, e_r(t)]^T \in \mathbb{R}^{r \times m}, \]
so that \( Q(t) \) satisfies the following time-varying linear dynamics,
\[ \frac{dQ}{dt} = \|\gamma'(t)\| K(t)Q, \tag{10} \]
where \( K(t) \in \mathbb{R}^{r \times r} \).

By factoring out the term \( \|\gamma'(t)\| \) from \( K(t) \), it is guaranteed that \( K(t) \) does not depend on the parametrization of the curve (i.e. the speed of the trajectory), but only on its geometry. The matrix \( K(t) \) is highly structured and sparse; the nonzero elements of \( \kappa_i(t) \) are defined to be the curvatures of the trajectory. The curvatures \( \kappa_i(t) \) combined with the basis vectors \( e_i(t) \) define the Frenet-Serret apparatus, which fully characterizes the trajectory up to translation [33].

To understand the structure of \( K(t) \) we derive two key properties:

1. \( K_{i,j}(t) = -K_{j,i}(t) \) (antisymmetry):

   Proof. Since \( r \leq m \), then by construction \( Q(t) \) is a unitary matrix with \( QQ^\top = I \). Taking the derivative with respect to \( t \), \( \frac{dQ}{dt}Q^\top + Q^\top \frac{dQ}{dt} = 0 \), or equivalently

   \[ \frac{dQ}{dt}Q^\top = -\left( \frac{dQ}{dt}Q^\top \right)^\top. \]

   Since \( Q \) is unitary, then \( Q^{-1} = Q^\top \), and hence

   \[ K(t) = \frac{1}{\|\gamma'(t)\|} \frac{dQ}{dt}Q^\top, \]

   from which we immediately see that \( K(t) = -K(t)^\top. \)

2. \( K_{i,j}(t) = 0 \) for \( j \geq i + 2 \):

   We first note that since \( e_i(t) \in \text{span}\{\gamma'(t), \ldots, \gamma^i(t)\} \), its derivative must satisfy \( e'_i(t) \in \text{span}\{\gamma'(t), \ldots, \gamma^{i+1}(t)\} \). Now by construction, using the Gram-Schmidt method, \( e_j \) is orthogonal to \( \text{span}\{\gamma'(t), \ldots, \gamma^{i+1}(t)\} \) for \( j \geq i + 2 \). Since \( e'_i(t) \) is in the span of this set, then \( e_j \) must be orthogonal to \( e'_i \) for \( j \geq i + 2 \). Thus, \( K_{i,j}(t) = \langle e'_i(t), e_j \rangle = 0 \) for \( j \geq i + 2 \).

With these two constraints, \( K(t) \) takes the form,

\[ K(t) = \begin{bmatrix} 0 & \kappa_1(t) & 0 & \cdots & 0 \\ -\kappa_1(t) & \ddots & \ddots & \ddots & 0 \\ \iddots & \ddots & 0 & \kappa_{r-1}(t) \\ 0 & \cdots & -\kappa_{r-1}(t) & 0 \end{bmatrix}. \tag{11} \]

Thus \( K(t) \) is antisymmetric with nonzero elements only along the superdiagonal and subdiagonal, and the values \( \kappa_1(t), \ldots, \kappa_{r-1}(t) \) are defined to be the curvatures of the trajectory.

From a geometric perspective, \( e_1(t), \ldots, e_r(t) \) form an instantaneous (local) coordinate frame, which moves with the trajectory. The curvatures define how quickly this frame changes with time. If the trajectory is a straight line the curvatures are all zero. If \( \kappa_1 \) is constant and nonzero, while all other curvatures are zero, then the trajectory lies on a circle. If \( \kappa_1 \) and \( \kappa_2 \) are constant and nonzero with all other curvatures zero, then the trajectory lies on a helix. Comparing the structure of \( (11) \) to Figure 2, we immediately see a similarity. Over the following sections we will shed light on this connection.
2.5 SVD and Curvature

Given time series data, the SVD constructs an orthonormal basis that is fixed in time, whereas the Frenet-Serret frame constructs an orthonormal basis that moves with the trajectory. In recent work, Alvarez-Vizoso et al. [33] showed how these frames are related. In particular, the Frenet-Serret frame converges to the SVD frame in the limit as the time interval of the trajectory goes to zero.

To understand this further, consider a trajectory $\gamma(t) \in \mathbb{R}^m$ as described in Section 2.4. If we assume that our measurements are from a small neighborhood $t \in (-\epsilon, \epsilon)$ (where $\epsilon \ll 1$), then $\gamma(t)$ is well-approximated by its Taylor expansion,

$$
\gamma(t) - \gamma(0) = \gamma'(0)t + \frac{\gamma''(0)}{2}t^2 + \frac{\gamma'''(0)}{6}t^3 + \ldots
$$

Writing this in matrix form, we have that

$$
\gamma(t) - \gamma(0) = \begin{bmatrix}
\gamma'(0) & \gamma''(0) & \gamma'''(0) & \cdots
\end{bmatrix} \begin{bmatrix}
1 & \frac{1}{2} & \frac{1}{6} & \\
\Gamma & \Sigma & T^r & \\
0 & 0 & 0 & \\
\end{bmatrix}.
$$

(12)

Recall one key property of the SVD is that the $r$th rank truncation in the expansion is the best rank-$r$ approximation to the data in the least squares sense. Since $\epsilon \ll 1$, then each subsequent term in this expansion is much smaller than the previous term,

$$
\|\gamma'(0)t\|_2 \ll \|\frac{\gamma''(0)}{2}t^2\|_2 \ll \|\frac{\gamma'''(0)}{6}t^3\|_2 \ll \ldots.
$$

(13)

From this, we see that the expansion in (12) is strongly related to the SVD. However, in the SVD we have the constraint that the $U$ and $V$ matrices are orthogonal, while for the Taylor expansion $\Gamma$ and $T$ have no such constraint. Alvarez et al. [33] show that in the limit as $\epsilon \to 0$, then $U$ is the result of applying the Gram-Schmidt process to the columns of $\Gamma$, and $V$ is the result of applying the Gram-Schmidt process to the columns of $T$. Comparing this to above, we see that

$$
U = \begin{bmatrix}
e_1(0) & e_2(0) & e_3(0) & \cdots
\end{bmatrix}
$$

and

$$
V = \begin{bmatrix}
p_1(t) & p_2(t) & p_3(t) & \cdots
\end{bmatrix},
$$

where $e_1(t), e_2(t), \ldots, e_r(t)$ is the basis for the Frenet-Serret frame defined in (9) and

$$
p_i(t) = \frac{t^i - \sum_{j=1}^{i-1} \langle t^i, p_j(t) \rangle p_j(t)}{\|t^i - \sum_{j=1}^{i-1} \langle t^i, p_j(t) \rangle p_j(t)\|} \quad \text{for } i = 1, 2, 3, \ldots
$$

(14)

We note that the $p_i(t)$’s form a set of orthogonal polynomials independent of the dataset. In this limit, the curvatures depend solely on the singular values,

$$
\kappa_i(t) = \sqrt{a_i} \frac{\sigma_{i+1}}{\sigma_1(t)\sigma_i(t)}, \text{ where } a_{i-1} = \left(\frac{i}{i+(-1)^i}\right)^2 \frac{4i^2-1}{3}.
$$
3 Unifying Singular Value Decomposition, Time Delay Embeddings, and the Frenet-Serret Frame

In this section, we show that time series data from a dynamical system may be decomposed into a sparse linear dynamical model with nonlinear forcing, and the nonzero elements along the sub- and super-diagonals of the linear part of this model have a clear geometric meaning: they are curvatures of the system. In Section 3.1 we combine key results about the Frenet-Serret frame, time delays, and SVD to explain this structure. Following this theory, Section 3.2 illustrates this approach with a simple synthetic example. The decomposition yields a set of orthogonal polynomials that form a coordinate basis for the time-delay embedding. In Section 3.3 we explicitly describe these polynomials and compare their properties to the Legendre polynomials.
3.1 Connecting SVD, Time Delay Embeddings, and Frenet-Serret Frame

Here we connect the properties of the SVD, time delay embeddings, and the Frenet-Serret to decompose a dynamical model into a linear dynamical model with nonlinear forcing, where the linear model is both antisymmetric and tridiagonal. To do this, we follow the steps of the HAVOK method with slight modifications and show how they give rise to these structured dynamics. This process is illustrated in Figure 3.

Following the notation introduced in Section 2.3, let’s begin with the time series $x(t)$ for $t = 0, \Delta t, \ldots, q\Delta t$. We construct a time delay embedding $H \in \mathbb{R}^{m \times n}$, where we assume $m \ll n$.

Next we compute the SVD of $H$ and show that the singular vectors correspond to the Frenet-Serret frame at a fixed point in time. In particular, to compute the SVD of this matrix, we consider the transpose $H^\top \in \mathbb{R}^{n \times m}$, which is also be a Hankel matrix. Thus, the columns of $h(t)$ can be thought of as a trajectory $h(t) \in \mathbb{R}^n$ for $t = 0, \Delta t, (m-1)\Delta t$. For simplicity, we shift the origin of time so that $h(t)$ spans $t = -(m-1)\Delta t/2, \ldots, 0, \ldots (m-1)\Delta t/2$, and we denote $h(i\Delta t)$ as $h_i$. In this form,

$$H^\top = \begin{bmatrix} h_{(-m+1)/2} & \cdots & h_0 & \cdots & h_{(m-1)/2} \end{bmatrix}.$$

Subtracting the central column $h_0$ from $H^\top$ (or equivalently, the central row of $H$) yields the centered matrix

$$\bar{H}^\top = H^\top - h_0 1^\top.$$

We can then express $h_i$ as a Taylor expansion about $h_0$,

$$h_i - h_0 = h_0' i\Delta t + \frac{1}{2} h_0''(i\Delta t)^2 + \frac{1}{3!} h_0'''(i\Delta t)^3 + \cdots.$$

With this in mind, applying the results of [33] described in Section 2.5 yields the SVD

$$\bar{H}^\top = \begin{bmatrix} e_0^1 & e_0^2 & e_0^3 & \cdots \end{bmatrix} \Sigma \begin{bmatrix} \sigma_1 & \sigma_2 & \cdots & \sigma_3 \\ \sigma_3 & \sigma_2 & \cdots & \sigma_1 \\ \vdots & \ddots & \ddots & \vdots \\ \sigma_1 & \cdots & \sigma_1 \end{bmatrix} \begin{bmatrix} -p_1 & - & -p_1 \\ -p_2 & - & -p_2 \\ \vdots & \ddots & \vdots \\ -p_1 & - & -p_1 \end{bmatrix} \begin{bmatrix} \end{bmatrix} U^\top.$$

The singular vectors in $V$ correspond to the Frenet-Serret frame (the Gram-Schmidt method applied to the vectors, $h_0', h_0'', h_0'''$),

$$e_0 = \frac{h_0'}{||h_0'||},$$

$$e_i^j = \frac{h_0^{(i)} - \sum_{j=1}^{i-1} \langle h_0^{(i)}, e_0^j \rangle e_0^j}{||h_0^{(i)} - \sum_{j=1}^{i-1} \langle h_0^{(i)}, e_0^j \rangle e_0^j||}.$$

---

1 We define the left singular matrix as $V$ and the right singular matrix as $U$. This definition can be thought of as taking the SVD of the transpose of the matrix $H - 1h_0'$. This keeps the definitions of the matrices more inline with the notation used in HAVOK.
The matrix $U$ is similarly defined by the discrete orthogonal polynomials

$$p_1 = \frac{1}{c_1} p$$

$$p_i = \frac{1}{c_i} \left( p_i - \sum_{j=1}^{i-1} (p_i, p_j) p_j \right),$$

where $p$ is the vector

$$p = \begin{bmatrix} (-m + 1)/2 & (-m + 2)/2 & \cdots & 0 & \cdots & (m - 2)/2 & (m - 1)/2 \end{bmatrix},$$

and where $c_i$ is a normalization constant so that $(p_i, p_i) = 1$. Note that $p^i$ here means raise $p$ to the power $i$ element-wise. These polynomials are similar to the discrete orthogonal polynomials defined in [63], except $p$ is the normalized ones vector $\frac{1}{c_i} [1 \cdots 1]$. These polynomials will be discussed further in Section 3.3.

Next, we build a regression model of the dynamics. We first consider the case where the system is closed (i.e. $H$ has rank $r$). Thinking of $V$ as the Frenet-Serret frame at a fixed point in time, then following the Frenet-Serret equations (10),

$$\frac{dV}{dt} = AV^\top,$$  \hspace{1cm} (18)

where $A = \|h'_0\| K$. Here $K$ is a constant tridiagonal and antisymmetric matrix, which corresponds to the curvatures at $t = 0$.

From the dual perspective, we can think about this set of vectors $e^0$ as an $r$-dimensional time series over $n$ snapshots in time,

$$V^\top = \begin{bmatrix} -v_1(t) & - & \cdots & - & -e_0^1 & - \\ -v_2(t) & - & \cdots & - & -e_0^2 & - \\ \vdots & \cdots & \ddots & \cdots & \vdots & \vdots \\ -v_r(t) & - & \cdots & - & -e_0^r & - \end{bmatrix} \in \mathbb{R}^{r \times n}. \hspace{1cm} (19)$$

Here $v(t) = [v_1(t), v_2(t), \cdots v_r(t)]^\top \in \mathbb{R}^r$ denotes the $r$-dimensional trajectory, which corresponds to the $r$-dimensional coordinates considered in [5] for HAVOK. From (18), these dynamics must therefore satisfy

$$\dot{v}(t) = Av(t),$$

where $A$ is a skew-symmetric tridiagonal matrix. If the system is not closed, the dynamics take the form

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \vdots \\ \dot{v}_r \\ \dot{v}_{r+1} \\ \vdots \end{bmatrix} = \|h'_0\| \begin{bmatrix} 0 & \kappa_1 & & & & \\ -\kappa_1 & 0 & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & 0 & \ddots & \ddots & \kappa_r \\ & & -\kappa_{r-1} & 0 & \ddots & \kappa_r \\ & & & -\kappa_r & 0 & \ddots \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_r \\ v_{r+1} \\ \vdots \end{bmatrix}. \hspace{1cm} 13$$
We note that, due to the tridiagonal structure of $K$, the governing dynamics of the first $r - 1$ coordinates $v_1(t), \ldots, v_{r-1}(t)$ are the same as in the unforced case. The dynamics of the last coordinate includes an additional term $\dot{v}_r = -\kappa_{r-1}v_{r-1} + \kappa_{r+1}v_{r+1}$. The dynamics therefore take the form,

$$\frac{dv}{dt} = Av(t) + Bv_{r+1}(t),$$

where $B$ is a vector that is nonzero only its last coordinate. Thus, we recover a model as in (8), but with the desired tridiagonal skew-symmetric structure. The matrix of curvatures is simply given by $K = A/\|h_0\|$. To compute the derivative of the state we now treat $V^\top$ as a 4 dimensional trajectory with 9961 snapshots. Applying DMD to $V^\top$ yields the $A$ matrix,

$$A = \begin{bmatrix}
-1.245 \times 10^{-3} & 1.205 \times 10^{-2} & 4.033 \times 10^{-6} & 1.444 \times 10^{-7} \\
-1.224 \times 10^{-2} & 3.529 \times 10^{-4} & 4.458 \times 10^{-3} & 2.283 \times 10^{-6} \\
-9.390 \times 10^{-4} & -3.467 \times 10^{-3} & 5.758 \times 10^{-4} & 6.617 \times 10^{-3} \\
3.970 \times 10^{-4} & -6.568 \times 10^{-4} & -7.451 \times 10^{-3} & 2.835 \times 10^{-4}
\end{bmatrix}. \tag{22}$$

In summary, we have shown here that the trajectories of singular vectors $v(t)$ from a time-delay embedding are governed by approximately tridiagonal antisymmetric dynamics, with a forcing term nonzero only in the last component. Comparing these steps to those described in Section 2.3, we see that the estimation of $K$ is nearly identical to the steps in HAVOK. In particular, $\|h_0\|K$ is the linear dynamics matrix $A$ in HAVOK. The only difference is the centering step in (15), which is further discussed in Section 3.3.

### 3.2 HAVOK Computes Approximate Curvatures in a Synthetic Example

To illustrate the correspondence between nonzero elements of the HAVOK dynamics matrix and curvatures, we start by considering an analytically tractable synthetic example. We start by applying the steps of HAVOK as described in [5] with an additional centering step. The resultant modes and terms on the sub- and superdiagonals of the dynamics matrix are then compared to curvatures computed with an analytic expression, and we show that they are approximately the same, scaled by a factor of $\|h_0\|$.

We consider data from the one dimensional system governed by

$$x(t) = \sin(t) + \sin(2t),$$

for $t \in [0, 10]$ and sampled at $\Delta t = 0.001$. Following HAVOK, we form the time delay matrix $H \in \mathbb{R}^{41 \times 9961}$ then center the data, subtracting the middle row $h_0$ from all other rows, which forms $\hat{H}$. We next apply the SVD to $\hat{H}^\top = V \Sigma U^\top$.

Figure 4 shows the columns of $U \in \mathbb{R}^{41 \times 4}$ and the columns of $V \in \mathbb{R}^{9961 \times 4}$. The columns of $U$ correspond to the orthogonal polynomials described in Section 3.3 and the columns of $V$ are the instantaneous basis vectors $e_i$ for the 9961 dimensional Frenet-Serret frame.

To compute the derivative of the state we now treat $V$ as a 4 dimensional trajectory with 9961 snapshots. Applying DMD to $V$ yields the $A$ matrix,
This matrix is approximately antisymmetric and tridiagonal as we expect.

Next, we compute the Frenet-Serret frame for the time delay embedding using analytic expressions and show that HAVOK indeed extracts the curvatures of the system multiplied by $\|h_0\|$. Forming the time delay matrix, we can easily compute

$$h_0 = [\sin(t) + \sin(2t) \text{ for } t \in [0.02, 0.021, \ldots, 9.98]]$$

and the corresponding derivatives,

$$\dot{h}_0 = [\cos(t) + 2 \cos(2t) \text{ for } t \in [0.02, 0.021, \ldots, 9.98]]$$
$$\ddot{h}_0 = [-\sin(t) - 4 \sin(2t) \text{ for } t \in [0.02, 0.021, \ldots, 9.98]]$$
$$\dddot{h}_0 = [-\cos(t) - 8 \cos(2t) \text{ for } t \in [0.02, 0.021, \ldots, 9.98]]$$
$$\dddot{h}_0^{(4)} = [\sin(t) + 16 \sin(2t) \text{ for } t \in [0.02, 0.021, \ldots, 9.98]].$$

The 5th derivative $h^{(5)}$ is given by $\cos(t) + 32 \cos(2t)$ and can be expressed as a linear combination of the previous derivatives, namely, $h^{(5)}_0 = -5\dddot{h}_0 - 4\dot{h}_0$. This can also be shown using the fact that $x(t)$ satisfies the 4th order ordinary differential equation $x^{(4)} + 5x + 4x = 0$.

Since only the first four derivatives are linearly independent, only the first three curvatures are nonzero. Further, exact values of the first three curvatures can be computed analytically using the following formulas from [64],

$$\kappa_1 = \sqrt{\frac{\det([h_0 \quad \dot{h}_0 \quad \ddot{h}_0 \quad \dddot{h}_0])}{\|h_0\|^{3/2}}}$$
$$\kappa_2 = \sqrt{\frac{\det([h_0 \quad \dot{h}_0 \quad \ddot{h}_0 \quad \dddot{h}_0 \quad \dddot{h}_0^{(4)}])}{\det([h_0 \quad \dot{h}_0 \quad \ddot{h}_0 \quad \dddot{h}_0])\|h_0\|}}$$
$$\kappa_3 = \frac{\sqrt{\det([h_0 \quad \dot{h}_0 \quad \ddot{h}_0 \quad \dddot{h}_0 \quad \dddot{h}_0^{(4)}])\det([h_0 \quad \dot{h}_0 \quad \ddot{h}_0 \quad \dddot{h}_0 \quad \dddot{h}_0^{(4)}])}}{\det([h_0 \quad \dot{h}_0 \quad \ddot{h}_0 \quad \dddot{h}_0 \quad \dddot{h}_0^{(4)}])\|h_0\|}.$$ 

These formulas yield the values $\kappa_1 = 1.205 \times 10^{-2}$, $\kappa_2 = 4.46 \times 10^{-3}$, and $\kappa_3 = 6.62 \times 10^{-3}$.

As expected, these curvature values are very close to those computed with HAVOK, highlighted in [22]. In particular, the superdiagonal entries of the matrix appear to be a very good approximations to the curvatures. The reasons why the superdiagonal, but not the subdiagonal, is so close in value to the true curvatures is not yet well understood. Further, in Section 5, we use the theoretical insights from Section 3.1 to propose a modification to the HAVOK algorithm that yields an even better approximation to curvatures in the Frenet-Serret frame.

### 3.3 Orthogonal Polynomials and Centering

In the decomposition in [16], we define a set of orthonormal polynomials. Here we discuss the properties of these polynomials, comparing them to the Legendre polynomials and providing explicit expressions for the first several terms in this series.

In Section 3.1, we apply the SVD to the centered matrix $H$, as in [16]. The columns of $U$ in this decomposition yield a set of orthonormal polynomials, which are defined by [14]. In the continuous case, the inner product in [14] is $\langle a(t), b(t) \rangle = \int_p a(t)b(t)dt$, while in the discrete case
Figure 4: Frenet-Serret frame (left) and corresponding orthogonal polynomials (right) for HAVOK applied to time-series generated by \( x(t) = \sin(t) + \sin(2t) \). The orthogonal polynomials and the Frenet-Serret frame are the right singular vectors \( U \) and left singular vectors \( V \) of \( \bar{H} \), respectively.

\[ \langle a, b \rangle = \sum_{j=-p}^{p} a_j b_j \]

The first five polynomials in the discrete case may be found in Appendix A. The first five of these polynomials \( p_i(x) \) in the continuous case are:

\[
egin{align*}
p_1(x) &= \frac{x}{c_1(p)}, \quad \text{where } c_1(p) = \frac{\sqrt{6}}{3} \\
p_2(x) &= \frac{x^2}{c_2(p)}, \quad \text{where } c_2(p) = \frac{\sqrt{10}}{5} \\
p_3(x) &= \frac{1}{c_3(p)} \left( x^3 - \frac{3}{5} p^2 x \right), \quad \text{where } c_3(p) = \frac{2 \sqrt{14}}{35} \\
p_4(x) &= \frac{1}{c_4(p)} \left( x^4 - \frac{5}{7} p^2 x^2 \right), \quad \text{where } c_4(p) = \frac{2 \sqrt{2}}{21} \\
p_5(x) &= \frac{1}{c_5(p)} \left( x^5 + \frac{5}{21} p^4 x - \frac{10}{9} p^2 x^3 \right), \quad \text{where } c_5(p) = \frac{8 \sqrt{22}}{693}.
\end{align*}
\]

By construction, \( p_i(t) \) form a set of orthonormal polynomials, where \( p_i(t) \) has degree \( i \).

Interestingly, these orthogonal polynomials are similar to the Legendre polynomials \( l_i \) \cite{65, 66}, which are defined by the recursive relation

\[
\begin{align*}
l_1 &= \frac{1}{c_1} \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \\
l_i &= \frac{1}{p_i} \left( p_i - \sum_{k=1}^{i-1} \langle p_i, l_k \rangle \right),
\end{align*}
\]

where \( p \) is as defined in \cite{17}. For the corresponding Legendre polynomials normalized over \([-p, p]\), we refer the reader to \cite{63}.

The key difference between these two sets of polynomials is that the first polynomial \( p_1 \) is linear, while the first Legendre polynomial is constant (i.e., corresponding in the discrete case to the normalized ones vector). In particular, if \( H \) is not centered before decomposition by SVD, the resulting columns of \( U \) will be the Legendre polynomials. However, without centering, the resulting
Figure 5: Increasing sampling frequency and number of columns yields more structured HAVOK models for the Lorenz system. Given the Hankel matrix $H$, the linear dynamical model is plotted for values of sampling period $\Delta t$ equal to 0.01, 0.005, 0.001, 0.0005 for a fixed number of rows and fixed time span of measurement (top). Similarly, the model is plotted for values of number of columns $n$ equal to 1001, 2001, 5001, and 10001 for fixed sampling frequency and time span of measurement $q\Delta t$ (bottom). As we increase the sampling frequency and the number of columns of the data, $A$ becomes more antisymmetric with nonzero elements only on the super- and sub-diagonals. These trends illustrate the results in Section 4.

$V$ will no longer be the Frenet-Serret frame. Instead, the resulting frame corresponds to applying the Gram-Schmidt method to the set $\{\gamma(t), \gamma'(t), \gamma''(t), \ldots\}$ instead of $\{\gamma'(t), \gamma''(t), \gamma'''(t), \ldots\}$. Recently it has been shown that using centering as a preprocessing step is beneficial for the dynamic mode decomposition [67]. That being said, since the derivation of the tridiagonal and antisymmetric structure seen in the Frenet-Serret frame is based on the properties of the derivatives and orthogonality, this same structure can be computed without the centering step.

### 4 Limits and Requirements

Section 3.1 has shown how HAVOK yields a good approximation to the Frenet-Serret frame in the limit that the time interval spanned by each row $H$ goes to zero. To be more precise, HAVOK yields the Frenet-Serret frame if (13) is satisfied. However, this property can be difficult to check in practice. Here we establish several rules for choosing and structuring the data so that the HAVOK dynamics matrix adopts the structure we expect from theory.

**Choose $\Delta t$ to be small.** The specific constraint we have from (13) is

$$\|h_0(t_i)\| \gg \left\| \frac{h''_0}{2} t_i \right\| \gg \left\| \frac{h''''_0}{6} t_i^3 \right\| \gg \cdots \gg \left\| \frac{h^{(k)}_0}{k!} t_i^k \right\|,$$

for $-m\Delta t/2 \leq t_i \leq m\Delta t/2$ or more simply $|t_i| \leq m\Delta t$, where $\Delta t$ is the sampling period of the
data and $m$ is the number of delays in the Hankel matrix $H$. If we assume that $m\Delta t < 1$, then rearranging,

$$m\Delta t \ll \frac{2 \|h_0\|}{\|h_0''\|} \cdot \frac{3 \|h_0'\|}{\|h_0''\|} \cdots \frac{k \|h_0^{(k-1)}\|}{\|h_0^{(k)}\|}.$$

(23)

In practice, since the series of ratios of derivatives defined in (23) grows, it is only necessary to check the first inequality. By choosing the sampling period of the data to be small, we can constrain the data to satisfy this inequality. To illustrate the effect of decreasing $\Delta t$, Figure 5 (top) shows the dynamics matrices $A$ computed by the HAVOK algorithm for the Lorenz system for a fixed number of rows of data and fixed time span of the simulation. As $\Delta t$ becomes smaller, $A$ becomes more structured in that it is antisymmetric and tridiagonal.

Choose the number of columns $n$ to be large. The number of columns comes into the Taylor expansion through the derivatives $\|h_0^{(k)}\|$, since $h_0^{(k)} \in \mathbb{R}^n$.

For the synthetic example $x(t) = \sin(t) + 2\sin(t)$, we can show that the ratio $2 \|h_0'\| / \|h_0''\|$ saturates to a fixed value in the limit as $n$ goes to infinity (see Appendix B). However, for short time series (small values of $n$), this ratio can be arbitrarily small, and hence (23) will be difficult to satisfy.

We illustrate this in Figure 5 using data from the Lorenz system. We compute and plot the HAVOK linear dynamics matrix for a varying number of columns $n$, while fixing the sampling frequency and time span of measurements $q\Delta t$. We see that as we increase the number of columns, the dynamics becomes more skew symmetric and tridiagonal. In general, due to practical constraints and restrictions, it may be difficult to guarantee that given data satisfies these two requirements. In Sections 4.1 and 5, we propose methods to tackle this challenge.

### 4.1 Interpolation

From the first requirement, we see that the sampling frequency $\Delta t$ needs to be sufficiently small to recover the antisymmetric structure in $A$. However, in practice, it is not always possible to satisfy this sampling criterion.

One solution to remedy this is to use data interpolation. To be precise, we can increase the sampling rate by spline interpolation, then construct $H$ from the interpolated data that satisfies (23). The ratio of the derivatives $\|h_0'\| / \|h_0''\|$, $\|h_0''\| / \|h_0'''\|$, ..., may also contain some dependence on $\Delta t$, but we observe that this dependence is not significantly affected in practice.

As an example, we consider a set of time series measurements generated from the Lorenz system (see Section 5 for more details about this system). We start with a sampling period of $\Delta t = 0.1$ (Figure 6, top row). Note that here we have simulated the Lorenz system at high temporal resolution then subsampled to produce this timeseries data. Applying HAVOK with centering and $m = 201$, we see that $A$ is not antisymmetric and the columns of $U$ are not the orthogonal polynomials like in the synthetic example shown in Figure 4.

Next, we apply cubic spline interpolation to this data, evaluating at a sampling rate of $\Delta t = 0.001$ (Figure 6, bottom row). We note that, especially for real-world data with measurement noise, this interpolation procedure also serves to smooth the data, making the computation of its derivatives more tractable [68]. Applying HAVOK to this interpolated data yields a new antisymmetric $A$ matrix and the $U$ corresponds to the orthogonal polynomials described in Section 3.3.
Figure 6: In the case where a dynamical system is sparsely sampled, interpolation can be used to recover a more tridiagonal and antisymmetric matrix for the linear model in HAVOK. First, we simulate the Lorenz system, measuring \( x(t) \) with a sampling period of \( \Delta t = 0.1 \). The resulting dynamics model \( A \) and corresponding singular vectors of \( U \) are plotted. Due to the low sampling frequency these values do not satisfy the requirements in (23). Consequently the dynamics matrix is not antisymmetric and the singular vectors do not correspond to the orthogonal polynomials in Section 3.3. Next, the data is interpolated using cubic splines and subsequently sampled using a sampling period of \( \Delta t = 0.001 \). In this case the data satisfies the assumptions in (23), which yields the tridiagonal antisymmetric structure for \( A \) and orthogonal polynomials for \( U \) as predicted.

## 5 Promoting structure in the HAVOK decomposition

HAVOK yields a linear model of a dynamical system explained by the Frenet-Serret frame, and by leveraging these theoretical connections, here we propose a modification of the HAVOK algorithm to promote this antisymmetric structure. We refer to this algorithm as structured HAVOK (sHAVOK) and describe it in Section 5.1. Compared to HAVOK, sHAVOK yields structured dynamics matrices that better approximate the Frenet-Serret frame and more closely estimate the curvatures. Importantly, sHAVOK also produces better models of the system using significantly less data. We demonstrate its application to three nonlinear synthetic example systems in Section 5.2 and two real-world datasets in Section 5.3.

### 5.1 The Structured HAVOK (sHAVOK) Algorithm

We propose a modification to the HAVOK algorithm that more closely induces the antisymmetric structure in the dynamics matrix, especially for shorter data with a smaller number of delays \( n \).
The key innovation in sHAVOK is the application of two SVD’s applied separately to time-shifted Hankel matrices (compare Figure 2 and Figure 7). This simple modification enforces that the singular vector bases on which the dynamics matrix is computed are orthogonal, and thus more closely approximate the Frenet-Serret frame.

Building on the HAVOK algorithm as summarized in Section 2.3, we focus on the step where the singular vectors $V$ are split into $V_1$ and $V_2$. In the Frenet-Serret framework, we are interested in the evolution of the orthonormal frame $e_1(t), e_2(t), \ldots, e_r(t)$. In HAVOK, $V_1$ and $V_2$ correspond to instances of this orthonormal frame.

Although $V$ is a unitary matrix, $V_1$ and $V_2$—which each consist of removing a column from $V$—are not. To enforce this orthogonality, we propose to split $H$ into two time-shifted matrices $\tilde{H}_1$ and $\tilde{H}_2$ (Figure 7) and then compute two SVDs with rank truncation $r$,

$$\tilde{H}_1 = U_1 \Sigma_1 V_1^\top \quad \text{and} \quad \tilde{H}_2 = U_2 \Sigma_2 V_2^\top.$$ 

By construction, $V_1$ and $V_2$ are now orthogonal matrices.

Like in HAVOK, our goal is to estimate the dynamics matrix $A$ such that

$$\dot{v}(t) = Av(t).$$
To do so, we use the matrices $V_1$ and $V_2$ to construct the state and its derivative,

$$V = V_1 \quad dV = \frac{V_2 - V_1}{\Delta t}.$$ 

$A$ then satisfies

$$A = \frac{dV}{dt} V^\top \approx (V_2^\top - V_1^\top) / \Delta t V_1^\top$$

(24)

If this system is not closed (nonzero forcing term), then $V_2$ is defined as columns 2 to $n - 1$ of the SVD singular vectors with an $r - 1$ rank truncation $V_{r-1}^\top$, and $A \in \mathbb{R}^{r-1 \times r-1}$ and $B \in \mathbb{R}^{r-1 \times 1}$ are computed as $[\hat{A}, \hat{B}] = V_2^\top V_1$. The corresponding pseudocode is elaborated in Appendix C.

As a simple analytic example, we apply sHAVOK to the same system described in Section 3.2 generated by $x(t) = \sin(t) + \sin(2t)$. The resulting dynamics matrix is

$$A = \begin{bmatrix}
-1.116 \times 10^{-5} & 1.204 \times 10^{-2} & -1.227 \times 10^{-5} & 8.728 \times 10^{-8} \\
-1.204 \times 10^{-2} & -1.269 \times 10^{-5} & 4.458 \times 10^{-3} & 4.650 \times 10^{-6} \\
2.053 \times 10^{-5} & -4.458 \times 10^{-3} & -4.897 \times 10^{-6} & 6.617 \times 10^{-3} \\
-9.956 \times 10^{-8} & -1.118 \times 10^{-7} & -6.617 \times 10^{-3} & -3.368 \times 10^{-6}
\end{bmatrix}.$$ 

We see immediately that, with this small modification, $A$ has become much more structured compared to (22). Specifically, the estimates of the curvatures both below and above the diagonal are now equal, and the rest of the elements in the matrix, which should be zero, are almost all smaller by an order of magnitude. In addition, the curvatures are equal to the true analytic values up to three decimal places.

5.2 Comparison of HAVOK and sHAVOK for Three Synthetic Examples

The results of HAVOK and sHAVOK converge in the limit of infinite data, and the models they produce are most different in cases of shorter time series data, where we may not have measurements over long periods of time. Using synthetic data from three nonlinear example systems, we compute models using both methods and compare the corresponding dynamics matrices $A$ (Figure 8). In every case, the $A$ matrix computed using the sHAVOK algorithm is more antisymmetric and has a stronger tridiagonal structure than the corresponding matrix computed using HAVOK.

In addition to the dynamics matrices, we also show in Figure 8 the eigenvalues of $A$, $\omega_k \in \mathbb{C}$ for $k = 1, \ldots r$ for HAVOK (teal) and sHAVOK (maroon). We additionally plot the eigenvalues (black crosses) corresponding to those computed from the data measured in the large data limit, but at the same sampling frequency. In this large data limit, both sHAVOK and HAVOK yield the same antisymmetric tridiagonal dynamics matrix and corresponding eigenvalues. Comparing the eigenvalues, we immediately see that eigenvalues from sHAVOK more closely match those computed in the large data limit. Thus, even with a short trajectory, we can still recover models and key features of the underlying dynamics. Below, we describe each of the systems and their configurations.

Lorenz Attractor: We first illustrate these two methods on the Lorenz system. Originally developed in the fluids community, the Lorenz (1963) system is governed by three first order
Figure 8: Structured HAVOK (sHAVOK) yields more structured models from short trajectories than HAVOK on three examples. For each system, we simulated a trajectory extracting a single coordinate in time (gray). We then apply HAVOK and sHAVOK to data \( x(t) \) from a short subset of this trajectory, shown in black. The middle columns show the resulting dynamics matrices \( A \) from the models. Compared to HAVOK, the resulting model for sHAVOK consistently shows stronger structure in that they are antisymmetric with nonzero elements only along the sub- and super-diagonals. The corresponding eigenvalue spectra of \( A \) for HAVOK and sHAVOK are plotted in teal and maroon, respectively, in addition to eigenvalues from HAVOK for the full (gray) trajectory. In all cases, the sHAVOK eigenvalues are much closer in value to those from the long trajectory limit than HAVOK.

differential equations \[ [69]: \]

\[
\begin{align*}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= x(\rho - z) - y \\
\dot{z} &= xy - \beta z.
\end{align*}
\]

The Lorenz system has since been used to model systems in a wide variety of fields, including chemistry [70], optics [71], and circuits [72].
We simulate 3,000 samples with initial condition \([-8,8,27]\) and a stepsize of \(\Delta t = 0.001\), measuring the variable \(x(t)\). We use the common parameters \(\sigma = 10, \rho = 28, \) and \(\beta = 8/3\). This trajectory is shown in Figure 8 and corresponds to a few oscillations about a fixed point. We compare the spectra to that of a longer trajectory containing 300,000 samples, which we take to be an approximation of the true spectrum of the system.

**Rössler Attractor:** The Rössler attractor is given by the following nonlinear differential equations [73, 74]:

\[
\begin{align*}
\dot{x} &= -y - z \\
\dot{y} &= x + ay \\
\dot{z} &= b + z(x - c).
\end{align*}
\]

We choose to measure the variable \(x(t)\). This attractor is a canonical example of chaos, like the Lorenz attractor. Here we perform a simulation with 70,000 samples and a stepsize of \(\Delta t = 0.001\). We choose the following common values of \(a = 0.1, b = 0.1 \) and \(c = 14\) and the initial condition \(x_0 = y_0 = z_0 = 1\). We similarly plot the trajectory and dynamics matrices. We compare the spectra in this case to a longer trajectory using a simulation for 300,000 samples.

**Double Pendulum:** The double pendulum is a similar nonlinear differential equation, which models the motion of a pendulum which is connected at the end to another pendulum [75]. This system is typically represented by its Lagrangian,

\[
L = \frac{1}{6} ml^2 \left( \dot{\theta}_2^2 + 4 \dot{\theta}_1^2 + 3 \dot{\theta}_1 \dot{\theta}_2 \cos (\theta_1 - \theta_2) \right) + \frac{1}{2} mgl (3 \cos \theta_1 + \cos \theta_2), \tag{25}
\]

where \(\theta_1\) and \(\theta_2\) are the angles between the top and bottom pendula and the vertical axis, respectively. \(m\) is the mass at the end of each pendulum, \(l\) is the length of each pendulum and \(g\) is the acceleration constant due to gravity. Using the Euler-Lagrange equations,

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = 0 \text{ for } i = 1, 2,
\]

we can construct two second order differential equations of motion.

The trajectory is computed using a variational integrator to approximate

\[
\delta \int_a^b L(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) dt = 0.
\]

We simulate this system with a stepsize of \(\Delta t = 0.001\) and for 1200 samples. We choose \(m_1 = m_2 = l_1 = l_2 = 1\) and \(g = 10\), and use initial conditions \(\theta_1 = \theta_2 = \pi/2, \dot{\theta}_1 = -0.01\) and \(\dot{\theta}_2 = -0.005\). As our measurement for HAVOK and sHAVOK we use \(x(t) = \sin(\theta_1(t))\) and compare our data to a long trajectory containing 100,000 samples.

### 5.3 sHAVOK Applied to Real-world Datasets

Here we apply sHAVOK to two real world time series datasets, the trajectory of a double pendulum and measles outbreak data. Similar to the synthetic examples, we find that the the dynamics matrix from sHAVOK is much more antisymmetric and tridiagonal compared to the dynamics matrix for HAVOK. In both cases, some of the HAVOK eigenvalues contain positive real components; in
Figure 9: Comparison of HAVOK and structured HAVOK (sHAVOK) for two real world systems: a double pendulum and measles outbreak data. For each system, we measure a trajectory extracting a single coordinate (gray). We then apply HAVOK and sHAVOK to a subset of this trajectory, shown in black. The $A$ matrices for the resulting linear dynamical models are shown.

sHAVOK yields models with an antisymmetric structure, with nonzero elements only along the subdiagonal and superdiagonal. The corresponding eigenvalue spectra for HAVOK and sHAVOK are additionally plotted in teal and maroon, respectively, along with eigenvalues from HAVOK for a long trajectory. In both cases, the eigenvalues of sHAVOK are much closer in value to those in the long trajectory limit than HAVOK. Some of the eigenvalues of HAVOK are unstable and have positive real components. The corresponding reconstructions of the first singular vector of the corresponding Hankel matrices are shown along with the real data. Note that the HAVOK models are unstable, growing exponentially due to the unstable eigenvalues, while the sHAVOK models do not. Credit for images on left: (double pendulum) and (measles) CDC/ Cynthia S. Goldsmith; William Bellini, Ph.D.

other words, these models have unstable dynamics. However, the sHAVOK spectra do not contain positive real components, resulting in much more accurate and stable models (Figure 9).

**Double Pendulum:** We first look at measurements of a double pendulum. A picture of the setup can be found in Figure 9. The Lagrangian in this case is very similar to that in...
One key difference in the synthetic case is that all of the mass is contained at the joints, while in this experiment, the mass is spread over each arm. To accommodate this, the Lagrangian can be slightly modified,

\[ L = \frac{1}{2} \left( m_1 (\dot{x}_1^2 + \dot{y}_1^2) + m_2 (\dot{x}_2^2 + \dot{y}_2^2) \right) + \frac{1}{2} \left( I_1 \dot{\theta}_1^2 + I_2 \dot{\theta}_2^2 \right) - (m_1 y_1 + m_2 y_2) g, \]

where \( x_1 = a_1 \sin(\theta_1) \), \( x_2 = l_1 \sin(\theta_1) + a_2 \sin(\theta_2) \), \( y_1 = a_1 \cos(\theta_1) \), and \( y_2 = l_1 \cos(\theta_1) + a_2 \cos(\theta_2) \). \( m_1 \) and \( m_2 \) are the masses of the pendula, \( l_1 \) and \( l_2 \) are the lengths of the pendula, \( a_1 \) and \( a_2 \) are the distances from the joints to the center of masses of each arm, and \( I_1 \) and \( I_2 \) are the moments of inertia for each arm. When \( m_1 = m_2 = m \), \( a_1 = a_2 = l_1 = l_2 \), and \( I_1 = I_2 = ml^2 \) we recover (25). We sample the data at \( \Delta t = 0.001s \) and plot \( \sin(\theta_2(t)) \) over a 15s time interval. The data over this interval appears approximately periodic.

Measles Outbreaks: As a second example we apply measles outbreak data from New York City between 1928 to 1964 [77]. The case history of measles over time has been shown to exhibit chaotic behavior [78, 79], and [5] applied HAVOK to measles data and successfully showed that the method could extract transient behavior.

For both systems, we apply sHAVOK to a subset of the data corresponding to the black trajectories \( x(t) \) shown in Figure 9. We then compare that to HAVOK applied over the same interval. We use \( m = 101 \) delays with a \( r = 5 \) rank truncation for the double pendulum, and \( m = 51 \) delays and a \( r = 6 \) rank truncation for the measles data. For the measles data, prior to applying sHAVOK and HAVOK the data is first interpolated and sampled at a rate of \( \Delta t = 0.0018 \) years. Like in previous examples, the resulting sHAVOK dynamics is tridiagonal and antisymmetric while the HAVOK dynamics matrix is not. Next, we plot the corresponding spectra for these two methods, in addition to the eigenvalues applied to HAVOK over the entire time series. Most noticeably, the eigenvalues from sHAVOK are closer to the long data limit values. In addition, two of the HAVOK eigenvalues lie to the right of the real axis, and thus have positive real components. All of the sHAVOK eigenvalues, on the other hand, have negative real components. This difference is most prominent in the reconstructions of the first singular vector. In particular, since two of the eigenvalues from HAVOK are positive, the reconstructed time series grows exponentially. In contrast, for sHAVOK the corresponding time-series remains bounded providing a much better model of the true data.

6 Discussion

In this paper, we describe a new theoretical connection between models constructed from time-delay embeddings, specifically using the HAVOK approach, and the Frenet-Serret frame from differential geometry. This unifying perspective explains the peculiar antisymmetric, tridiagonal structure of HAVOK models: namely, the sub- and super-diagonal entries of the linear model correspond to the intrinsic curvatures in the Frenet-Serret frame. Inspired by this theoretical insight, we develop an extension we call structured HAVOK that effectively yields models with this structure. Importantly, we demonstrate that this modified algorithm improves the stability and accuracy of time-delay embedding models, especially when data is noisy and limited in length. All code is available at [https://github.com/sethhirsh/shAVOK](https://github.com/sethhirsh/shAVOK).

Establishing theoretical connections between time-delay embedding, dimensionality reduction, and differential geometry opens the door for a wide variety of applications and future work. By
understanding this new perspective, we now better understand the requirements and limitations of HAVOK and have proposed simple modifications to the method which improve its performance on data. However, the full implications of this theory remain unknown. Differential geometry, dimensionality reduction and time delay embeddings are all well-established fields, and by understanding these connections we can develop more robust and interpretable methods for modeling time series.

For instance, by connecting HAVOK to the Frenet-Serret frame, we recognize the importance of enforcing orthogonality for \( V_1 \) and \( V_2 \) and inspired development of sHAVOK. With this theory, we can incorporate further improvements on the method. For example, sHAVOK can be thought of as a first order forward difference method, approximating the derivative and state by \( (V_2 - V_1) / \Delta t \) and \( V_1 \), respectively. By employing a central difference scheme, such as approximating the state by \( V \), we have observed this to further enforce the antisymmetry in the dynamics matrix and move the corresponding eigenvalues towards the imaginary axis.

Throughout this analysis, we have focused purely on linear methods. In recent years, nonlinear methods for dimensionality reduction, such as autoencoders and diffusion maps, have gained popularity \[7, 80, 81\]. Nonlinear models similarly benefit from promoting sparsity and interpretability. By understanding the structures of linear models, we hope to generalize these methods to create more accurate and robust methods that can accurately model a greater class of functions.

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A Discrete Orthogonal Polynomials

In Section 3.3 we introduced a set of orthogonal polynomials that appear in HAVOK, and listed these polynomials in the continuous case. The first five polynomials in the discrete case are listed below.

\[
\begin{align*}
    p_1(n) &= \frac{n}{c_1} \\
    p_2(n) &= \frac{n^2}{c_2} \\
    p_3(n) &= \frac{1}{c_3} \left( n^3 - n \left( \frac{3p^2 + 3p - 1}{5} \right) \right) \\
    p_4(n) &= \frac{1}{c_4} \left( n^4 - \frac{5n^2 \left( 3p^4 + 6p^3 - 3p + 1 \right)}{7 \left( 3p^2 + 3p - 1 \right)} \right) \\
    p_5(n) &= \frac{1}{c_5} \left( 5 \left( \frac{n(3p^2 + 3p - 1)}{5} - n^3 \right) \left( 2p^2 + 2p - 3 \right) - \frac{n \left( 3p^4 + 6p^3 - 3p + 1 \right)}{7} + n^5 \right)
\end{align*}
\]

\[
\begin{align*}
    c_1 &= \sqrt{\frac{p \left( 2p + 1 \right) \left( p + 1 \right)}{3}} \\
    c_2 &= \sqrt{\frac{p \left( 2p + 1 \right) \left( p + 1 \right) \left( 3p^2 + 3p - 1 \right)}{15}} \\
    c_3 &= \sqrt{\frac{p \left( 2p - 1 \right) \left( 2p + 1 \right) \left( 2p + 3 \right) \left( p - 1 \right) \left( p + 1 \right) \left( p + 2 \right)}{175}} \\
    c_4 &= \sqrt{\frac{p \left( 2p - 1 \right) \left( 2p + 1 \right) \left( 2p + 3 \right) \left( p - 1 \right) \left( p + 1 \right) \left( p + 2 \right) \left( 15p^4 + 30p^3 - 35p^2 - 50p + 12 \right)}{2205 \left( 3p^2 + 3p - 1 \right)}} \\
    c_5 &= \sqrt{\frac{4p \left( 2p - 1 \right) \left( 2p + 1 \right) \left( 2p - 3 \right) \left( 2p + 3 \right) \left( 2p + 5 \right) \left( p - 1 \right) \left( p + 1 \right) \left( p - 2 \right) \left( p + 2 \right) \left( p + 3 \right)}{43659}}
\end{align*}
\]
B  Column Rule for Synthetic Example

In Section 4, we state that when applying HAVOK to the synthetic example in 3.2 in the limit as the number of columns $n$ in the Hankel matrix $H$ goes to infinity, the derivatives in (23) converge to fixed values. Here we prove that the first ratio in the series $\frac{\|h_0''\|}{\|h_0'\|}$ approaches a constant as $n \to \infty$. Further terms in the sequence, can be shown to have the same behavior using a similar proof.

We start with the system $x(t) = \sin(t) + \sin(2t)$. The central row of the matrix $h_0$ will be of the form $x(t)$ for some $a, b \in \mathbb{Z}$ such that

$$t = [a\Delta t \ (a+1)\Delta t \ (a+2)\Delta t \ \ldots \ b\Delta t].$$

In particular $b = n + a$. Thus, showing that the limit as $n \to \infty$ is equivalent to the limit as $b \to \infty$.

\[
\frac{\|h_0''\|}{\|h'_0\|} = \frac{|| - \sin(t) - 4 \sin(2t)||}{|| \cos(t) + 2 \cos(2t)||}
\]

\[
= \frac{|| [-\sin(a\Delta t) - 4 \sin(2a\Delta t) \ \ldots \ -\sin(b\Delta t) - 4 \sin(2b\Delta t)] ||}{|| \cos(a\Delta t) + 2 \cos(2a\Delta t) \ \ldots \ \cos(b\Delta t) + 2 \cos(2b\Delta t) ||}
\]

\[
= \sqrt{\frac{\sum_{k=a}^{b} (\sin(k\Delta t) + 4 \sin(2k\Delta t))^2}{\sum_{k=a}^{b} (\cos(k\Delta t) + 2 \cos(2k\Delta t))^2}}
\]

\[
= \sqrt{\frac{\sum_{k=a}^{b} (\sin^2(k\Delta t) + 8 \sin(k\Delta t) \sin(2k\Delta t) + 16 \sin^2(2k\Delta t))}{\sum_{k=a}^{b} (\cos^2(k\Delta t) + 4 \cos(k\Delta t) \cos(2k\Delta t) + 4 \cos^2(2k\Delta t))}}
\]

\[
= \sqrt{\frac{\sum_{k=a}^{b} \left(\frac{17}{2} + 4 \cos(k\Delta t) - \frac{1}{2} \cos(2k\Delta t) - 4 \cos(3k\Delta t) - 8 \cos(4k\Delta t)\right)}}{\sum_{k=a}^{b} \left(\frac{5}{2} + 2 \cos(k\Delta t) + \frac{1}{2} \cos(2k\Delta t) + 2 \cos(3k\Delta t) + 2 \cos(4k\Delta t)\right)}}
\]

In the last step we have used the trigonometric identities $\sin^2(a) = \frac{1}{2}(1 - \cos(2a))$, and $\cos^2(a) = \frac{1}{2}(1 + \cos(2a))$.

Using \[82\], we have the identity

\[
\sum_{k=0}^{q} \cos(Bk) = \frac{\sin \left[ \left( \frac{q+1}{2} \right) B \right] \cos \left[ \left( \frac{q}{2} \right) B \right]}{\sin \left( \frac{B}{2} \right)}, \quad B, q \in \mathbb{R}.
\]

\[
\sum_{k=a}^{b} \cos(Bk) = \frac{\sin \left[ \left( \frac{b+1}{2} \right) B \right] \cos \left[ \left( \frac{b}{2} \right) B \right] - \sin \left[ \left( \frac{a}{2} \right) B \right] \cos \left[ \left( \frac{a-1}{2} \right) B \right]}{\sin \left( \frac{B}{2} \right)}, \quad B, a, b \in \mathbb{R}.
\]
Defining $g(b)$ and $h(b)$ as the numerator and denominator under the radical,

$$
g(b) = \sum_{k=a}^{b} (4 \cos(k\Delta t) - \frac{1}{2} \cos(2k\Delta t) - 4 \cos(3k\Delta t) - 8 \cos(4k\Delta t))$$

$$= \frac{4(\sin((b+1)(\frac{\Delta t}{2})) \cos(b(\frac{\Delta t}{2})) - \sin[a(\frac{\Delta t}{2})] \cos((a-1)(\frac{\Delta t}{2}))]}{\sin[\frac{\Delta t}{2}]}$$

$$- \frac{\sin((b+1)\Delta t) \cos[b\Delta t] - \sin[a\Delta t] \cos[(a-1)\Delta t]}{\sin[\Delta t]}$$

$$- \frac{4(\sin((b+1)(\frac{3\Delta t}{2})) \cos(b(\frac{3\Delta t}{2})) - \sin[a(\frac{3\Delta t}{2})] \cos((a-1)(\frac{3\Delta t}{2}))]}{\sin[\frac{3\Delta t}{2}]}$$

$$- \frac{8(\sin((b+1)(2\Delta t)) \cos[b(2\Delta t)] - \sin[a(2\Delta t)] \cos((a-1)(2\Delta t)]}{\sin[2\Delta t]}$$

$$h(b) = \sum_{k=a}^{b} (2 \cos(k\Delta t) + \frac{1}{2} \cos(2k\Delta t) + 2 \cos(3k\Delta t) + 2 \cos(4k\Delta t))$$

$$= \frac{2(\sin((b+1)(\frac{\Delta t}{2})) \cos(b(\frac{\Delta t}{2})) - \sin[a(\frac{\Delta t}{2})] \cos((a-1)(\frac{\Delta t}{2}))]}{\sin[\frac{\Delta t}{2}]}$$

$$+ \frac{\sin((b+1)\Delta t) \cos[b\Delta t] - \sin[a\Delta t] \cos[(a-1)\Delta t]}{2\sin[\Delta t]}$$

$$+ \frac{2(\sin((b+1)(\frac{3\Delta t}{2})) \cos(b(\frac{3\Delta t}{2})) - \sin[a(\frac{3\Delta t}{2})] \cos((a-1)(\frac{3\Delta t}{2}))]}{\sin[\frac{3\Delta t}{2}]}$$

$$+ \frac{2(\sin((b+1)(2\Delta t)) \cos[b(2\Delta t)] - \sin[a(2\Delta t)] \cos((a-1)(2\Delta t)]}{\sin[2\Delta t]}.$$

Note that we have the following:

$$\lim_{b \to \infty} \frac{g(b)}{b} = 0 \quad \text{and} \quad \lim_{b \to \infty} \frac{h(b)}{b} = 0.$$

Using this fact, then

$$\lim_{b \to \infty} 2\|h''_0\| = 2 \lim_{b \to \infty} \sqrt{\sum_{k=a}^{b} \frac{17}{2} - \frac{5}{2}} (b-a+1) + g(b)$$

$$= 2 \lim_{b \to \infty} \sqrt{\frac{17}{2} (b-a+1) + g(b)}$$

$$= 2 \lim_{b \to \infty} \sqrt{\frac{17}{2} - \frac{17a}{2b} + \frac{17}{2b} + \frac{g(b)}{b}}$$

$$= 2 \sqrt{\frac{17}{5}}.$$
C Structured HAVOK (sHAVOK) algorithm

Here we present pseudocode for the sHAVOK algorithms with and without forcing terms.

Algorithm 1 Structured HAVOK (sHAVOK) without forcing

Input: Measured signal $x(t)$, number of delays $m$, and rank of Hankel Matrix $r$.
Output: Dynamics matrix $\hat{A} \in \mathbb{R}^{r \times r}$.

$H := \text{Hankel}(x(t), m)$
$H_1 := H[:, 1 : n - 1]$  
$H_2 := H[:, 2 : n]$  
$U_1 \Sigma_1 V_1^T := \text{SVD}(H_1, r)$
$U_2 \Sigma_2 V_2^T := \text{SVD}(H_2, r)$
$\hat{A} := V_2^T V_1$

Algorithm 2 Structured HAVOK (sHAVOK) with forcing

Input: Measured signal $x(t)$, number of delays $m$, and rank of Hankel Matrix $r$.
Output: Dynamics matrix $\hat{A} \in \mathbb{R}^{r - 1 \times r - 1}$ and forcing term $\hat{B} \in \mathbb{R}^{r - 1}$.

$H := \text{Hankel}(x(t), m)$
$H_1 := H[:, 1 : n - 1]$  
$H_2 := H[:, 2 : n]$  
$U_1 \Sigma_1 V_1^T := \text{SVD}(H_1, r)$
$U_2 \Sigma_2 V_2^T := \text{SVD}(H_2, r - 1)$
$[\hat{A}, \hat{B}] := V_2^T V_1$