LOCALISED HILBERT MODULES AND WEAK NONCOMMUTATIVE CARTAN PAIRS

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ABSTRACT. We define the localisation of a Hilbert module in analogy to the local multiplier algebra. We use properties of this localisation to enrich non-closed actions on C*-algebras to closed actions on local multiplier algebras, and descend known results on such closed actions down to their unclosed counterparts. We define weak Cartan inclusions and characterise them as crossed products by inverse semigroup actions. We show that in the commutative case we show that weak Cartan subalgebras are maximal abelian, thereby generalising the case studied by Renault.

1. INTRODUCTION

One of the most common ways to construct interesting C*-algebras is to build crossed products from dynamical systems. Given a pair of C*-algebras A ⊆ B one can ask whether B ‘acts’ in a meaningful way on A, and if so can B be explicitly described in terms of these dynamics. With no further conditions on the C*-algebras A ⊆ B, the answer is probably not. For this reason, many authors have explored the question by adding several helper-conditions to the inclusion A ⊆ B and in various capacities have answered the question positively.

Two of these helper conditions have been so far inescapable. The first is that A should contain an approximate identity for B (non-degeneracy), and the second is that B should be generated as a C*-algebra by normalisers of A. A normaliser of the inclusion A ⊆ B is an element m ∈ B such that m*Am and mAm* are contained in A. This ensures that one can consider closed A-submodules of normalisers, called slices, without losing information about B not encoded by these. There are typically two more ingredients in the definition of these pairs, although these are the conditions which are modified most often in papers on the topic. One usually requires that A is maximal in some way in B, and that there is a conditional expectation of some kind P : B → A. The flavour of maximality changes from author to author, many papers building stronger results from the same conditions or showing the results hold under weaker ones. The expectation is sometimes removed in favour of some other condition at surface level, but often one can build kinds of conditional expectations using the replaced condition.

One of the first bids to describe these pairs of C*-algebras was by Kumjian [12] and Renault [18] to study commutative Cartan subalgebras. The types of pairs A ⊆ B that Renault considered were such that A was a maximal commutative subalgebra (masa) in B, and there was a faithful conditional expectation P : B → A. In combination with results from [12], Renault was able to describe these pairs in terms of twisted groupoid C*-algebras for étale, locally compact, Hausdorff, second countable, effective groupoids. This was later expanded upon by Exel [7] to allow for noncommutative Cartan subalgebras by replacing (masa) with a virtual maximality condition and to describe them as reduced section algebras of Fell bundles over inverse semigroups. This was later expanded upon by Kwaśniewski and Meyer in [13], who showed that the Cartan pairs of Exel could be described as crossed products by closed and purely outer actions, as well as showing that Exel’s virtual maximality condition is equivalent to a number of other conditions on the inclusion.
Another condition one may put on the pair $A \subseteq B$ include aperiodicity, which is explored in more detail in [14], [15], and [16]. This condition guarantees that there is at most one pseudo-expectation $E : B \to I(A)$, that is, a conditional expectation taking values in the injective hull of $A$. One particular subset of these expectations are expectations that take values in the local multiplier algebra $M_{\text{loc}}(A)$ of $A$, which embeds in the injective hull of $A$ (cf. [10]).

In this article we aim to mimic results of the previously mentioned authors and papers while adjusting some of the conditions. We shall consider inclusions $A \subseteq B$ that are aperiodic and have a faithful conditional expectation taking values in the local multiplier algebra of $A$. One of the challenges that arises is that the conditional expectation $E : B \to M_{\text{loc}}(A)$ maps part of $B$ isomorphically onto a subalgebra of $M_{\text{loc}}(A)$. This is undesirable since the inclusion $A \subseteq M_{\text{loc}}(A)$ is quite badly behaved, as $M_{\text{loc}}(A)$ does not have interesting dynamics on $A$. Thus we shall insist on a technical condition limiting the multiplicative domain of $E$ as described by Choi [3]. This ensures that the ‘intersection’ between $B$ and $M_{\text{loc}}(A)$ is as small as possible, namely is exactly $A$ itself.

One of the main tools we use throughout this article is an analogous construction of the local multiplier algebra applied to Hilbert bimodules. Given a Hilbert $A$-$B$-bimodule $X$, one can construct a Hilbert $M_{\text{loc}}(A)$-$M_{\text{loc}}(B)$-bimodule; the localisation of $X$ analogously to the construction of the local multiplier algebra. We prove some useful properties of the localisation of a Hilbert bimodule, following Ara and Matthew [1]. We also show how some bimodule-specific properties interact with that localisation construction.

Our main application of the localisation technique is to take an aperiodic action of an inverse semigroup by Hilbert bimodules on a $C^*$-algebra $A$, and gain a corresponding Fell bundle over an inverse semigroup with unit fibre $M_{\text{loc}}(A)$. To then gain an inverse semigroup action on $M_{\text{loc}}(A)$ is not immediate, but does follow from an application of the main theorem of [3]. We can then show that the localised action on $M_{\text{loc}}(A)$ gives rise to an Exel-Cartan inclusion, and so we then have the results of [7], [13] which we can apply to the localised inclusion. We then show that with another technical condition (that the canonical conditional expectation have minimal multiplicative domain), much of the structure of the localised Cartan inclusion descends to the original action in which we take interest.

The condition that a given generalised conditional expectation has minimal multiplicative domain is somewhat technical and not too well studied. For this reason, we sought to find if it is implied by another condition which is either already used in this context or more reasonably expected. To this end, we introduce the criterion of effectivity for slices in a regular non-degenerate inclusion. Briefly, a slice $X$ for the pair $A \subseteq B$ is effective if the only subslices of $X$ that act trivially on the spectrum of $A$ are ideals in $A$. This mimics the condition of effectivity for groupoids seen in [15] and [14], where the only bisections contained in the isotropy bundle of a groupoid are then contained in the units. We then also show that for an étale effective groupoid with locally compact Hausdorff unit space, the inclusion of the algebra of continuous functions on the unit into the essential groupoid $C^*$-algebra (as defined in [14]) is effective.

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2. Localised bimodules and actions

Analogous to the construction of the local multiplier algebra of a $C^*$-algebra (cf. [1] Definition 2.3.1), we define a localisation for Hilbert $C^*$-modules. We recall some key properties of the local multiplier algebra both from [1] and others that follow from short arguments. We also briefly study the relationship between a $C^*$-algebra and its local
multiplier algebra, and show how some of these properties translate into the Hilbert module setting. Additionally, we discuss some properties specific to Hilbert (bi)modules and how these affect properties of localised modules.

2.1. Local multiplier algebras. Throughout $A$ will denote a fixed $C^*$-algebra, and $\mathcal{I}(A)$ will be the lattice of essential ideals of $A$ ordered by containment. Note that $\mathcal{I}(A)$ is a directed set, as the intersection of two essential ideals is itself an essential ideal.

**Definition 2.1** ([1] Definition 2.3.1]). The restriction maps $M(I) \to M(J)$ for $J \subseteq I, I, J \in \mathcal{I}(A)$ are injective and give rise to the inductive limit

$$M_{\text{loc}}(A) := \lim_{I \in \mathcal{I}(A)} M(I),$$

which we call the local multiplier algebra of $A$.

By definition $M_{\text{loc}}(A)$ contains $M(I)$ for all essential ideals $I \lhd A$. Moreover even if $J \lhd A$ is not essential, the ideal $J \oplus J^\perp$ is essential and we have $M(J) \subseteq M(J \oplus M(J^\perp)) = M(J \oplus J^\perp)$, which then includes into $M_{\text{loc}}(A)$. In this fashion, one may think of $M_{\text{loc}}(A)$ as the $C^*$-algebra generated by multipliers on ideals of $A$. Considering that ideals of $A$ are in bijective correspondence with open subsets of the spectrum $\hat{A}$ of $A$, we see that $M_{\text{loc}}(A)$ is densely spanned by multipliers defined on these open subsets, hence the name local.

**Lemma 2.2** ([1] Lemma 2.3.2]). If $\bar{I} \lhd M_{\text{loc}}(A)$ is an ideal then $\bar{I} \cap A = \{0\}$ if and only if $\bar{I} = 0$. Moreover, if $\bar{I}$ is essential then so is $\bar{I} \cap A$.

**Lemma 2.3** ([1] Lemma 2.3.6]). For each $I \in \mathcal{I}(A)$ we have $M_{\text{loc}}(I) = M_{\text{loc}}(A)$. Let $(A_i)$ be a family of $C^*$-algebras. Then

$$M_{\text{loc}} \left( \bigoplus_i A_i \right) = \prod_i M_{\text{loc}}(A_i).$$

In the setting where $A = C_0(U)$ is a commutative $C^*$-algebra, essential ideals are of the form $C_0(V)$ for dense open subsets $V \subseteq U$, whereby the local multiplier algebra of $A$ loses the information of $\partial V$ for every dense open $V \subseteq U$. In the noncommutative setting, one may think of the quotient $A/I$ by an essential ideal $I \in \mathcal{I}(A)$ as the ‘boundary’ of $I$. Lemma 2.3 implies that even if $A/I$ is non-trivial, the quotient $M_{\text{loc}}(A)/M_{\text{loc}}(I)$ is always zero.

**Lemma 2.4.** Let $I \lhd A$ be an ideal. Then $A \cap M_{\text{loc}}(I) = A \cap M(I)$, where the intersection is taken in $M_{\text{loc}}(A)$.

**Proof.** Since $I \oplus I^\perp$ is an essential ideal in $A$ we have the inclusion $A \subseteq M(I \oplus I^\perp) = M(I) \oplus M(I^\perp)$ in $M_{\text{loc}}(A)$. The left and right summands each embed respectively into the orthogonal summands $M_{\text{loc}}(I)$ and $M_{\text{loc}}(I^\perp)$, and so $A \cap M_{\text{loc}}(I) \subseteq (M(I) \oplus M(I^\perp)) \cap M_{\text{loc}}(I) = M(I)$. Thus $A \cap M_{\text{loc}}(I) = A \cap M(I) \subseteq A \cap M(I)$. The reverse inclusion holds as $M(I) \subseteq M_{\text{loc}}(I)$. \hfill \Box

**Lemma 2.5.** Let $\bar{I} \lhd M_{\text{loc}}(A)$ be an ideal. Then $\bar{I} \subseteq M_{\text{loc}}(A \cap \bar{I})$.

**Proof.** Lemma 2.3 implies $M_{\text{loc}}(A) = M_{\text{loc}}(\bar{I} \cap A) \oplus M_{\text{loc}}((\bar{I} \cap A)^\perp)$. Set $K := \bar{I} \cap M_{\text{loc}}((\bar{I} \cap A)^\perp)$. This is an ideal in $M_{\text{loc}}(A)$ and so Lemma 2.2 gives $K \cap A = \{0\}$ if and only if $K = \{0\}$. Lemma 2.4 then implies $K \cap A = \bar{I} \cap M_{\text{loc}}((\bar{I} \cap A)^\perp) \cap A = (\bar{I} \cap A) \cap M((\bar{I} \cap A)^\perp) = \{0\}$, so $\bar{I}$ must be contained in $M_{\text{loc}}((\bar{I} \cap A))$. \hfill \Box
2.2. Module localisation definition and properties. Let $A$ and $B$ be $C^*$-algebras.

**Notation 2.6.** Let $X$ and $Y$ are right Hilbert $B$-modules. Recall the rank-one operators $X \to Y$ of the form $z \to y \cdot \langle x, z \rangle$ for $x, z \in X$ and $y \in Y$. The space $\mathcal{K}(X, Y)$ of compact operators $X \to Y$ is the completion of the span of rank-one operators $X \to Y$, in the operator norm $\|T\| = \sup_{\|x\|=1} \|Tx\|$. We denote the space of adjointable operators $X \to Y$ by $L(X, Y)$. If $X$ and $Y$ are Hilbert $A$-$B$-bimodules, and we wish to distinguish between operators that are left-adjointable and right-adjointable, we shall write

$$\mathcal{K}_R(X, Y) = \{ f : X \to Y | f \text{ is compact in the right Hilbert module structure} \}$$
$$\mathcal{L}_R(X, Y) = \{ f : X \to Y | f \text{ is adjointable in the right Hilbert module structure} \}$$
$$\mathcal{K}_L(X, Y) = \{ f : X \to Y | f \text{ is compact in the left Hilbert module structure} \}$$
$$\mathcal{L}_L(X, Y) = \{ f : X \to Y | f \text{ is adjointable in the left Hilbert module structure} \}.$$

In the case where $X$ and $Y$ are bimodules, and no subscript of $L$ or $R$ is specified as in $\mathcal{K}(X, Y)$ and $\mathcal{L}(X, Y)$, we assume these refer to the spaces of maps compatible with the right Hilbert module structure of $X$ and $Y$.

Let $X$ be a right Hilbert $B$-module. For each $J \in \mathcal{I}_c(B)$ we identify $X \cdot J$ with the module $\mathcal{K}(J, X \cdot J)$ of compact operators $J \to X \cdot J$ via the isomorphism $\xi \cdot a \mapsto [\xi] \cdot a$, that is, treating elements of $X \cdot J$ as left creation operators. From this we gain the right Hilbert $M(J)$-module $\mathcal{L}(J, X \cdot J)$ of adjointable operators $J \to X \cdot J$, containing $X \cdot J$ via embedding as compact operators. This serves as the module analogue of the multiplier algebra of an essential ideal for our construction.

If $K \subseteq J \ll B$ are both essential ideals, then any adjointable operator $T \in \mathcal{L}(J, X \cdot J)$ restricts to a map $T|_K : K \to X \cdot J$. Moreover, $T|_K$ has range contained in $X \cdot K$ since each $a \in K$ can be written as $a = a_1a_2$ for some $a_1, a_2 \in K$ and we have $T|_K a = (Ta_1)a_2 \in X \cdot J \cdot K = X \cdot K$. The restriction $T|_K$ is adjointable from $K \to X \cdot K$, with adjoint $T^*|_{X \cdot K}$, which takes values in $K$ since $T^*(X \cdot K) = T^*(X \cdot J) \cdot K \subseteq K$. Thus, restriction of operators gives rise to a module homomorphism $\mathcal{L}(J, X \cdot J) \to \mathcal{L}(K, X \cdot K)$. If $T \in \mathcal{L}(J, X \cdot J)$ restricts to the zero operator on $K$, then $\{0\} = T^*(X \cdot K) = T^*(X \cdot J) \cdot K$, whereby $T^*(X \cdot J)$ is an ideal that annihilates $K$. But $K$ has zero annihilator in $J$ as $K$ is essential in $J$, whereby $T^* = 0$. In particular, $T = 0$ if and only if $T|_K = 0$, and the map $\mathcal{L}(J, X \cdot J) \to \mathcal{L}(K, X \cdot K)$ is injective. These restriction maps give rise to an inductive system over essential ideals in $B$, ordered by reverse inclusion: $I \subseteq J$ if and only if $J \ll I$.

**Definition 2.7.** Let $X$ be a right Hilbert $B$-module. We define the *(right)-localisation* of $X$ as the inductive limit

$$X_{\text{loc}} := \lim_{J \ll \mathcal{I}_c(B)} \mathcal{L}_R(J, X \cdot J)$$

taken over essential ideals $J \ll B$.

Analogously, if $X$ is a left Hilbert $A$-module we define the *(left)-localisation* of $X$ as

$$\text{loc} X := \lim_{I \ll \mathcal{I}_c(A)} \mathcal{L}_L(I, I \cdot X).$$

Taking the case $X = B$, with the canonical $B$-bimodule structure inherited from multiplication, we see that $B_{\text{loc}} = \text{loc} B = M_{\text{loc}}(B)$, so this construction generalises the local multiplier algebra.

**Lemma 2.8.** Let $X$ be a right Hilbert $B$-module. For each ideal $J \ll B$ there is an isomorphism $\mathcal{L}(J, X \cdot J)^* \cong \mathcal{L}(J, J \cdot X^*)$ of left Hilbert $B$-modules. In particular $(X_{\text{loc}})^* \cong \text{loc}(X^*)$.

**Proof.** For $T \in \mathcal{L}(J, X \cdot J)$, define $S_T : J \to J \cdot X^* = (X \cdot J)^*$ by $S_T a := (Ta)^*$. Then the map $T \mapsto S_T$ gives an anti-isomorphism between $\mathcal{L}(J, X \cdot J)$ and $\mathcal{L}(J, J \cdot X^*)$. □
Many of the following results may be generalised to left Hilbert modules using either Lemma 2.8 or symmetry arguments.

**Lemma 2.9.** Let $X$ be a right Hilbert $B$-module. For any $J \in \mathcal{I}(B)$ we have $(X \cdot J)_{\text{loc}} = X_{\text{loc}}$.

**Proof.** This follows because the collection of essential ideals of $J$ is cofinal in $\mathcal{I}(A)$ over which the inductive limit is taken. \hfill \Box

In the case where $X$ is a Hilbert bimodule, all submodules are of the form $X \cdot J$ for some ideal $J$. In particular, a submodule $Y \subseteq X$ has zero orthogonal complement in $X$ if and only if the source ideal of $Y$ is an essential ideal of the source of $X$. In the bimodule case, this then gives that the localisation of any submodule with zero orthogonal complement is equal to the localisation of the whole module.

**Corollary 2.10.** For any ideal $J \ll B$ we have $X_{\text{loc}} = (X \cdot (J \oplus J^\perp))_{\text{loc}}$.

**Lemma 2.11.** Let $X$ and $Y$ be right Hilbert $B$-modules. Then $(X \oplus Y)_{\text{loc}} = X_{\text{loc}} \oplus Y_{\text{loc}}$.

**Proof.** This follows because the map $\mathcal{L}(J, X \cdot J) \oplus \mathcal{L}(J, Y \cdot J) \to \mathcal{L}(J, (X \oplus Y) \cdot J)$; $(T, S) \mapsto T + S$ is an isomorphism entwining the restriction maps for all ideals $J \ll B$. \hfill \Box

**Lemma 2.12.** The module $X_{\text{loc}}$ is a right Hilbert $M_{\text{loc}}(B)$-module.

**Proof.** We show that $\mathcal{L}(J, X \cdot J)$ is a right Hilbert $M(J)$-module for each essential ideal $J \ll A$. The right multiplication is given by $T(a)b = T(ab)$ for $T \in \mathcal{L}(J, X \cdot J)$ and $a \in J$, and $b \in M(J)$. For $T, S \in \mathcal{L}(J, X \cdot J)$, the inner product is given by the $(T, S) := T^*S : J \to J$, which is an adjointable operator $J \to J$ (considered as a right Hilbert $J$-module). These are exactly multipliers and $\mathcal{L}(J) = M(J)$. For $T \in \mathcal{L}(J, X \cdot J)$ the norm of $T$ satisfies (considering $a \in J$)

$$||T||^2 = \sup_{||a||=1} ||T(a)||^2 = \sup_{||a||=1} ||(T(a), T(a))|| = \sup_{||a||=1} ||a^*T^*Ta|| = ||\sqrt{T^*T}a||^2 = ||(T, T)||,$$

where one notes that (by representing on Hilbert spaces) $T^*T$ is a positive element of $M(J)$.

Thus the norms on $\mathcal{L}(J, X \cdot J)$ and $M(J)$ are compatible. The inductive limit structure is preserved since all the inductive limit maps are restrictions of operators, which clearly preserve the right actions and inner products. \hfill \Box

**Lemma 2.13.** Let $X$ be a Hilbert $A$-$B$-bimodule and let $Y \subseteq X$ be a closed Hilbert submodule. Then $(Y^\perp)_{\text{loc}} = (Y_{\text{loc}})^\perp$ and $X_{\text{loc}} = Y_{\text{loc}} \oplus Y_{\text{loc}}^\perp$.

**Proof.** We have $Y = X \cdot s(Y)$ and so by Lemma 2.11 we gain

$$X_{\text{loc}} = X \cdot (s(Y) \oplus s(Y)^\perp) = Y_{\text{loc}} \oplus (X \cdot s(Y)^\perp)_{\text{loc}}.$$

The right summand must then be equal to both $(Y^\perp)_{\text{loc}}$ and $(Y_{\text{loc}})^\perp$. \hfill \Box

**Lemma 2.14.** Let $X$ be a Hilbert $A$-$B$-bimodule and let $J \ll B$ be an ideal. Then $(X \cdot J)_{\text{loc}} = X_{\text{loc}} \cdot M_{\text{loc}}(J)$.

**Proof.** First we show that $(X \cdot J)_{\text{loc}} \subseteq X_{\text{loc}} \cdot M_{\text{loc}}(J)$. Fix an essential ideal $J' \ll B$, and define $J'_{\text{ess}} := J' \oplus (J' J)^\perp$, and note $J'_{\text{ess}}$ is an essential ideal of $A$. Fix $\xi \in \mathcal{L}(J'_{\text{ess}}, (X \cdot J) \cdot J'_{\text{ess}})$. Then $\xi = \xi_{J' J} \oplus \xi_{J' J}^\perp$ and since $J'_{\text{ess}} = J' J$ we have $\xi_{J' J}^\perp = 0$, giving $\xi = \xi \cdot 1_J$.

Thus $\xi \in \mathcal{L}(J'_{\text{ess}}, (X \cdot J) J'_{\text{ess}}) \cdot M_{\text{loc}}(J) \subseteq (X \cdot J)_{\text{loc}} \cdot M_{\text{loc}}(J)$. This gives $(X \cdot J)_{\text{loc}} \subseteq X_{\text{loc}} \cdot M_{\text{loc}}(J)$.

Lemmas 2.11 and 2.13 imply $X_{\text{loc}} \cdot M_{\text{loc}}(J) \oplus X_{\text{loc}} \cdot M_{\text{loc}}(J^\perp) = X_{\text{loc}} = (X \cdot J)_{\text{loc}} \oplus (X \cdot J^\perp)_{\text{loc}}$. We also have $X_{\text{loc}} \cdot M_{\text{loc}}(J) \cap (X \cdot J^\perp)_{\text{loc}} \subseteq X_{\text{loc}} \cdot M_{\text{loc}}(J) \cap X_{\text{loc}} \cdot M_{\text{loc}}(J^\perp) = \{0\}$ by the above argument (applied to $J^\perp$). Thus $X_{\text{loc}} \cdot M_{\text{loc}}(J) \subseteq (X \cdot J)_{\text{loc}}$ must hold. \hfill \Box
Lemma 2.15. Let $X$ be a Hilbert $A$-$B$-bimodule. Then $X_{\text{loc}}$ and $\text{loc}X$ are Hilbert $M_{\text{loc}}(A)$-$M_{\text{loc}}(B)$-bimodules.

Proof. We show that $X_{\text{loc}}$ is a left Hilbert $M_{\text{loc}}(A)$-module. The case for $\text{loc}X$ then follows by a symmetric argument, or from Lemma 2.13. For each essential ideal $J \lhd B$, let $I_J := r(X \cdot J) \oplus r(X \cdot J)^{\perp}$. Then $I_J$ is an essential ideal in $A$ and satisfies $I_J \cdot X = X \cdot J$. For $\tau \in M(I_J)$ and $T \in \mathcal{L}(J, X \cdot J)$ we define $(\tau T)a := \tau(Ta)$. This bilinear map commutes with the maps in the inductive system, giving the left action of $M_{\text{loc}}(A)$ on $X_{\text{loc}}$. For $S, T \in \mathcal{L}(J, X \cdot T)$, we define the left inner product $\langle \langle T, S \rangle \rangle$ as the element of $M(r(X \cdot J))$ corresponding to $TS^*$ under the isomorphism $\mathcal{L}(X \cdot J) \cong M(r(X \cdot J))$ arising from $X \cdot J \cong K_{\tau}(J, X \cdot J)$.

We now show that if $X$ is a Hilbert bimodule, then the left and right localisations agree.

Proposition 2.16. Let $X$ be a Hilbert $A$-$B$-bimodule. Then for each ideal $J \lhd B$ and the corresponding ideal $I_J := r(X \cdot J)$, there is an isomorphism $\mathcal{L}_{\text{loc}}(J, X \cdot J) \cong \mathcal{L}_L(I_J, I_J \cdot X)$ of Hilbert $M(I_J)$-$M(I_J)$-bimodules that preserves the inductive limit structures of $X_{\text{loc}}$ and $\text{loc}X$. In particular, $\text{loc}X \cong X_{\text{loc}}$.

Proof. Note that $X \cdot J = I_J \cdot X$ is isomorphic to both $K_{\tau}(J, X \cdot J)$ and $K_L(I_J, I_J \cdot X)$ via the assignment of $x \in X \cdot J$ to its left (respectively right) creation operator. For $x \in X \cdot J$ let $\langle x \rangle$ and $\langle \langle x \rangle \rangle$ denote the left and right $x$-creation operators in $K_{\tau}(J, X \cdot J)$ and $K_L(I_J, I_J \cdot X)$. Then the map $x \mapsto \langle x \rangle$ is an isomorphism $K_{\tau}(J, X \cdot J) \cong K_L(I_J, I_J \cdot X)$. We shall extend this to an isomorphism of the respective modules of adjointable operators.

For each $T \in \mathcal{L}_L(J, X \cdot J)$ and $a \in I_J$ we have $aT = r(X \cdot J) \cdot \mathcal{L}_L(J, X \cdot J) = K_{\tau}(J, X \cdot J)$. In particular, there exists a unique element $x \in X \cdot J$ such that $a \cdot T = \langle x \rangle$. In light of this, for each $a \in I_J$ and $T \in \mathcal{L}_L(J, X \cdot J)$ we identify $aT$ with its corresponding vector.

Define $\hat{T} : I_J \to X \cdot J$ by $\hat{T}(a) = aT$. This map has adjoint $\hat{T}^* : X \cdot J \to I_J$ given by $\hat{T}^*x = xT^*$, where $xT^*$ is the unique element of $I_J$ corresponding to the operator $\langle x \rangle \circ T^* : I_J \to I_J$ (note that this operator is compact since $\langle x \rangle$ is).

We claim the assignment $\alpha : T \mapsto \hat{T}$ is the desired isomorphism. If $\tau \in M(I_J)$ and $\sigma \in M(J)$, then for $a \in I_J$ and $b \in J$ we have $[\tau \hat{T} \sigma(a)b] = \tau a Tb \sigma = (\tau [\hat{T}(a)] \sigma)b$ since the left and right actions of $M(I_J)$ and $M(J)$ are given by post-multiplication, thus $\alpha$ is a Hilbert bimodule homomorphism.

To see that $\alpha$ is isometric, fix $T, S \in \mathcal{L}_L(J, X \cdot J)$. For $a \in I_J$ and $x \in X \cdot J$ we compute $\langle \langle T, S \rangle \rangle(a) = aT^*S = a(T, S)$ and $\langle \langle T, S \rangle \rangle(x) = \langle x \rangle \circ T^* \cdot S^* \cdot \langle x \rangle \circ \langle T, S \rangle$.

To see that $\alpha$ is a bijection, for $S \in \mathcal{L}_L(I_J, I_J \cdot X)$ define $\hat{S} : J \to K_{\tau}(I_J, I_J \cdot X)$ by $[\hat{S}(a)]x = S(a)b$ for all $a \in I_J$ and $b \in B$. The map $\hat{S}(b)$ belongs to $K_{\tau}(I_J, I_J \cdot X)$ by a symmetric argument to above, and for all $a \in I_J$ and $b \in I_J$ we see that $[\hat{T}(b)]a = [\hat{T}(a)]b = aTb$. In particular, picking an approximate unit $(e_\lambda) \subseteq I_J$ we see that $[\hat{T}(b)]e_\lambda = e_\lambda Tb \to Tb$, whereby $\hat{T} = T$ so $\alpha$ is a bijection.

Lastly, fix an essential ideal $J' \lhd J$ and note $I' = r(X \cdot J')$ is essential in $I$. If $T \in \mathcal{L}_L(J, X \cdot J)$ then $T|_{I'}$ gives the operator $T|_{I'} \in \mathcal{L}_L(I', X \cdot J)$. For any $a \in I'$ and $b \in J'$ we have $[\hat{T}|_{I'}(a)]b = aT|_{I'}b = aTb = [\hat{T}(a)]b$, so $T|_{I'} = \hat{T}|_{I'}$. Thus $\alpha$ preserves the inductive limit structures and induces an isomorphism of $\text{loc}X$ and $X_{\text{loc}}$.

Proposition 2.16 allows us to suppress the left- and right- prefixes of the localisation of a bimodule. In principal these differ as sets, but we are mostly only concerned with the isomorphism class of the module. This is a problem that does not arise in the case of local multiplier algebras, as these (when considered as Hilbert bimodules) are entirely symmetric.

Lemma 2.17. Let $X$ be a Hilbert $A$-$B$-bimodule and let $Y \subseteq X_{\text{loc}}$ be a Hilbert $M_{\text{loc}}(A)$-$M_{\text{loc}}(B)$-subbimodule of $X_{\text{loc}}$. Then $Y = \{0\}$ if and only if $Y \cap X = \{0\}$.
Suppose by Lemma 2.14.

**Proof.** We have

\[ X \cap Y = X \cap (X \circ \cdot s(Y)) \]

\[ \supseteq (X \cdot (s(Y) \cap B)) \cap (X \circ \cdot s(Y)) \]

\[ = X \cdot (s(Y) \cap B). \]

Suppose \( X \cdot (s(Y) \cap B) \) is zero. Then \( s(Y) \cap B \subseteq s(X)^\perp \), and so by Lemma 2.5 we have

\[ s(Y) \subseteq M_{\text{loc}}(s(X)^\perp), \]

and so \( Y = X \circ \cdot s(Y) \subseteq X \circ \cdot M_{\text{loc}}(s(X)^\perp) = (X \cdot s(X)^\perp)_{\text{loc}} = \{0\} \) by Lemma 2.14.

**Lemma 2.17** is the bimodule analogue of detection of ideals for an inclusion \( A \subseteq M_{\text{loc}}(A) \).

It is worth noting that this argument requires \( X \) to be a Hilbert bimodule, since without this condition it is no longer true in general that \( Y = X \cdot s(Y) \) (for example, all closed subspaces of a Hilbert space have the same source ideal \( \mathbb{C} \)). This shall not be a problem for us, as in later sections we shall exclusively examine Hilbert bimodules.

Although localisation commutes with right multiplication by an ideal (as in Lemma 2.14), this is not true for balanced tensor products of Hilbert bimodules. One particular consequence of this is that if \( X \) is a Morita equivalence \( A-B \)-bimodule, it is not always true that \( X_{\text{loc}} \) induces a Morita equivalence between \( M_{\text{loc}}(A) \) and \( M_{\text{loc}}(B) \). This failure occurs because \( X_{\text{loc}} \) need not be full, even when \( X \) is.

**Example 2.18.** Let \( X = \mathcal{H} \) be an infinite-dimensional Hilbert space. Then \( \mathcal{H} \) induces a Morita equivalence between \( \mathbb{C} \) and \( r(\mathcal{H}) = \mathcal{K}(\mathcal{H}) \). Clearly, \( \mathcal{H}_{\text{loc}} = L(\mathbb{C}, \mathcal{H}) \cong \mathbb{C} \) as \( \mathbb{C} \) is simple and unital, so \( \mathcal{H}_{\text{loc}} \otimes \mathbb{C} \mathcal{H}_{\text{loc}}^* \cong \mathcal{K}(\mathcal{H}) \). However, \( (\mathcal{H} \otimes \mathcal{H}^*)_{\text{loc}} \cong \mathcal{K}(\mathcal{H})_{\text{loc}} = B(\mathcal{H}) \).

**Proposition 2.19.** Let \( X \) be a Hilbert \( A-B \)-bimodule and let \( Y \) be a Hilbert \( B-C \)-bimodule. There is an isometric bimodule map \( \Phi : X_{\text{loc}} \otimes M_{\text{loc}}(B) Y_{\text{loc}} \to (X \otimes B Y)_{\text{loc}} \) that restricts to the identity on \( X \otimes B Y \subseteq X_{\text{loc}} \otimes M_{\text{loc}}(B) Y_{\text{loc}} \).

**Proof.** Fix \( J \in \mathcal{I}_e(B) \) and \( K \in \mathcal{I}_e(C) \). Then, there is \( J' \in \mathcal{I}_e(C) \) such that \( J \cdot Y = Y \cdot J' \), and consequently \( X \cdot J' \otimes B Y = X \otimes B Y \cdot J' K \). Fix \( \xi \in \mathcal{L}(J, X \cdot J) \) and \( \eta \in \mathcal{L}(K, Y \cdot K) \). Denote by \( \text{mult}_{J' K, Y} : J' K \otimes B Y \to J' K \cdot Y \) the unitary map implementing the left multiplication of \( B \) on \( Y \). Identify \( \eta \) with \( \eta_{J' K} \in \mathcal{L}(J' K, Y \cdot J' K) \) and define \( \varphi_{\xi, \eta} := (\xi \otimes 1) \circ \text{mult}_{J' K, Y}^{-1} \cdot \eta : J' K \to (X \otimes B Y) J' K \). This is an adjointable map since each \( \xi \otimes 1, \text{mult}_{J' K, Y}, \eta \) are. We claim there is a bimodule homomorphism \( \Phi : X_{\text{loc}} \otimes M_{\text{loc}}(B) Y_{\text{loc}} \to (X \otimes B Y)_{\text{loc}} \) that sends elementary tensors \( \xi \otimes \eta \mapsto \varphi_{\xi, \eta} \) for \( \xi \in \mathcal{L}(J, X \cdot J) \) and \( \eta \in \mathcal{L}(Y, Y \cdot K) \). First, note from the definition that for \( \tau \in M(r(X \cdot J)), \omega \in M(J \cap r(Y \cdot J' K)) \) and \( \sigma \in M(J' K) \), we have \( \varphi_{\tau \xi, \omega \eta, \sigma} = \tau \varphi_{\xi, \omega} \cdot \sigma \), so the assignment \( (\xi, \eta) \mapsto \varphi_{\xi, \eta} \) respects the balanced tensor product structure and ensures that \( \Phi \) will be a bimodule homomorphism. To see the assignment is isometric, consider \( J' \) as above and fix \( \xi_1, \xi_2 \in \mathcal{L}(J, X \cdot J) \) and \( \eta_1, \eta_2 \in \mathcal{L}(J' K, Y \cdot J' K) \). For \( b \in J' K \) we compute

\[ \langle \varphi_{\xi_1, \eta_1}, \varphi_{\xi_2, \eta_2} \rangle(b) = \varphi_{\xi_1, \eta_1}^* \varphi_{\xi_2, \eta_2}(b) \]

\[ = (\eta_1^* \circ \text{mult}_{J' K, Y} \circ \xi_1^* \otimes 1)(\xi_2(a) \otimes y), \]

where \( a \cdot y = \eta_2(b) \),

\[ = \eta_1^* (\xi_1^* \otimes \xi_2(a)) y \]

\[ = \eta_1^* (\xi_1^* \otimes \eta_2)(b) \]

\[ = \langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle(b). \]

Lastly, for \( x \in X, y \in Y \), and \( c \in C \) we have \( \varphi_{x, y}(c) = x \otimes y \cdot c \), so \( \Phi \) restricts to the identity on \( X \otimes B Y \). □

The map given in Proposition 2.19 will in many cases not be adjointable. To see this, recall Example 2.18 and note that \( \mathcal{K}(\mathcal{H}) \) embeds identically into \( B(\mathcal{H}) \), but since \( \mathcal{K}(\mathcal{H}) \) is an essential ideal this map cannot have an adjoint.
The image of $X_{\text{loc}} \otimes_{M_{\text{loc}}(B)} Y_{\text{loc}}$ in the localisation of the tensor product does however ‘detect’ all Hilbert $M_{\text{loc}}(A) \cdot M_{\text{loc}}(C)$-subbimodules of $(X \otimes B Y)_{\text{loc}}$. This follows as $X_{\text{loc}} \otimes_{M_{\text{loc}}(B)} Y_{\text{loc}}$ contains $X \otimes B Y$, which detects such subbimodules by Lemma 2.17. This then implies that $(X_{\text{loc}} \otimes_{M_{\text{loc}}(B)} Y_{\text{loc}})_{\perp} = \{0\}$, giving that $s(X_{\text{loc}} \otimes_{M_{\text{loc}}(B)} Y_{\text{loc}})$ is an essential ideal of $s((X \otimes B Y)_{\text{loc}})$.

2.3. Non-triviality conditions for bimodules. One may also ask how this localisation process alters non-triviality conditions for bimodules. In particular, we are interested in two particular conditions for Hilbert bimodules: pure outerness and aperiodicity. Pure outerness is of interest to us for the classification of noncommutative Cartan pairs (cf. [7], [13]). Aperiodicity is explored in [14] and grants us two properties of interest: unique-ness of conditional expectations taking value in the injective envelope of a $C^*$-algebra, in particular, the local multiplier algebra ([10, Theorem 1], [16]), and pure outerness of the localisation of a bimodule (see Proposition 2.23 ahead). In later sections we shall use this to construct Cartan pairs by localising a special class of actions that do not alone give rise to Cartan inclusions.

Definition 2.20. Let $X$ be a Banach $A$-bimodule. We say that $X$ is purely outer if the only ideal of $J \triangleleft A$ such that $A \cdot J \cong J$ is $J = \{0\}$.

Definition 2.21 ([15]). Let $X$ be a normed $A$-bimodule. We say that $x \in X$ satisfies Kishimoto’s condition if for all $\varepsilon > 0$ and non-zero hereditary subalgebras $D \subseteq A$, there exists $a \in D$ with $a \geq 0$ and $||a|| = 1$, such that $||axa|| < \varepsilon$. We say that $X$ is aperiodic if all $x \in X$ satisfy Kishimoto’s condition.

Lemma 2.22 ([14, Lemma 5.12]). Subbimodules, quotient bimodules, extensions, finite direct sums, and inductive limits of aperiodic normed $A$-bimodules remain aperiodic. If $f : X \rightarrow Y$ is a bounded $A$-bimodule homomorphism with dense range and $X$ is aperiodic, then so is $Y$. If $D \subseteq A$ is hereditary, then an aperiodic $A$-bimodule is also aperiodic as a $D$-bimodule. If $J \in \mathcal{I}_e(A)$ and $X$ an $A$-bimodule, then $JXJ$ is aperiodic as a $J$-bimodule if and only if $X$ is aperiodic as an $A$-bimodule.

By [14 Lemma 5.10] no positive non-zero element of $A$ satisfies Kishimoto’s condition when considering $A$ as an $A$-bimodule. In particular, no ideals of $A$ are aperiodic other than the zero ideal. Thus, if $X$ is an aperiodic bimodule and $J \triangleleft A$ is an ideal such that $X \cdot J \cong J$, then $J$ is also aperiodic, giving $J = 0$. Thus, aperiodicity implies pure outerness. The converse is in general false, however if $A$ contains a simple essential ideal then they are equivalent ([15 Theorem 8.1]).

Proposition 2.23. Let $X$ be an aperiodic Hilbert $A$-bimodule. Then $X_{\text{loc}}$ is a purely outer $M_{\text{loc}}(A)$-bimodule.

Proof. Let $\tilde{I} \triangleleft M_{\text{loc}}(A)$ be an ideal such that $X_{\text{loc}} \cdot \tilde{I} \cong \tilde{I}$. Let $\Phi : X_{\text{loc}} \cdot \tilde{I} \rightarrow \tilde{I}$ be such an isomorphism. Then $\Phi$ restricts to an injective $A$-bimodule map $\phi : X \cap (X_{\text{loc}} \cdot \tilde{I}) \rightarrow M_{\text{loc}}(A)$. By [16 Proposition 3.16] the image of $\phi$ is zero, and so $X \cap (X_{\text{loc}} \cdot \tilde{I}) = \{0\}$. Lemma 2.17 then gives $\tilde{I} \approx X_{\text{loc}} \cdot \tilde{I} = \{0\}$. □

3. Localising inverse semigroup actions

We shall recall the definition of an inverse semigroup action on a $C^*$-algebra $A$ by Hilbert bimodules, and then apply our localisation construction. This will not immediately give an action on $M_{\text{loc}}(A)$, but after some technical arguments we do gain this.
3.1. Inverse semigroup actions and crossed products. Throughout this article, \( A \) shall denote a \( C^* \)-algebra and \( S \) shall denote a unital inverse semigroup. There is a canonical partial order on \( S \) given by \( t \leq u \) if and only if \( t = ut^*t \) for \( u, t \in S \).

**Definition 3.1** (cf. [5] Definition 4.7]). An action \( \mathcal{E} = (\mathcal{E}_t, \mu_{t,u})_{t,u \in S} \) of \( S \) on \( A \) by Hilbert bimodules consists of

- Hilbert \( A \)-bimodules \( \mathcal{E}_t \) for each \( t \in S \); and
- bimodule isomorphisms \( \mu_{t,u} : \mathcal{E}_t \otimes_A \mathcal{E}_u \to \mathcal{E}_{tu} \);

satisfying

(i) \( \mathcal{E}_1 = A \), with the canonical \( A \)-bimodule structure;
(ii) the maps \( \mu_{t,t} : A \otimes_A \mathcal{E}_t \to \mathcal{E}_t \) and \( \mu_{t,1} : \mathcal{E}_t \otimes_A A \to \mathcal{E}_t \) are the canonical isomorphisms coming from the respective left and right actions of \( A \) on \( \mathcal{E}_t \); and
(iii) associativity: for all \( t, u, v \in S \), the following diagram commutes

\[
\begin{array}{ccc}
(\mathcal{E}_t \otimes_A \mathcal{E}_u) \otimes_A \mathcal{E}_v & \xrightarrow{\mu_{t,u} \otimes_A \text{id}_{\mathcal{E}_v}} & \mathcal{E}_{tu} \otimes_A \mathcal{E}_v \\
\text{id}_{\mathcal{E}_t} \otimes_A \mathcal{E}_u & \xrightarrow{\mathcal{E}_t \otimes_A \mu_{u,v}} & \mathcal{E}_t \otimes_A \mathcal{E}_{uv} \\
\end{array}
\]

If \( S \) has a zero element, we then require \( \mathcal{E}_0 = \{0\} \).

We shall now give the necessary concepts to build the full and essential crossed products for inverse semigroup actions. These are the same definitions as in [13 Section 2.2] and [14 Section 4], and we refer the reader there for a more in depth explanation.

If \( t \leq u \) for \( t, u \in S \) and \( \mathcal{E} \) is an action of \( S \) on \( A \), then there is an inclusion map \( \mathcal{E}_t \hookrightarrow \mathcal{E}_u \) gained from the multiplication maps. This restricts to an isomorphism \( j_{u,t} : \mathcal{E}_t \to \mathcal{E}_u, s(\mathcal{E}_t) = r(\mathcal{E}_t) \cdot \mathcal{E}_u \). For each \( v \leq t, u \), there is an isomorphism \( \vartheta^v_{t,u} : \mathcal{E}_u \cdot s(\mathcal{E}_v) \to \mathcal{E}_t \cdot s(\mathcal{E}_v) \) given by \( \vartheta^v_{t,u} := j_{t,v} \circ j^{-1}_{u,v} \). Define

\[
I_{t,u} = \sum_{v \leq r, u} s(\mathcal{E}_v),
\]

the closed ideal generated by \( s(\mathcal{E}_v) \) for all \( v \leq t, u \). We call the ideal \( I_{t,u} \) the intersection ideal for \( t, u \). This is contained in \( s(\mathcal{E}_u) \cap s(\mathcal{E}_t) \) and the inclusion may be strict. There is a unique Hilbert bimodule isomorphism \( \vartheta^v_{t,u} : \mathcal{E}_u \cdot I_{t,u} \to \mathcal{E}_t \cdot I_{t,u} \) which for each \( v \leq t, u \) restricts to \( \vartheta^v_{t,u} \) on \( \mathcal{E}_t \cdot s(\mathcal{E}_v) \) by [5 Lemma 2.4]. The algebraic crossed product \( A \rtimes_{\text{alg}} S \) is defined as the quotient of \( \bigoplus_{t \in S} \mathcal{E}_t \) by the linear span of \( \vartheta_{u,t}(\xi)\delta_u - \xi\delta_t \) for \( u, t \in S \) and \( \xi \in \mathcal{E}_t \cdot I_{t,u} \). \( A \rtimes_{\text{alg}} S \) is a *-algebra with multiplication and involution induced by the maps \( \mu_{t,u} \) and the involutions \( \mathcal{E}_t^* \to \mathcal{E}_t^* \). There is a maximal \( C^* \)-norm on \( A \rtimes_{\text{alg}} S \).

**Definition 3.2** ([13 Definition 2.7]). The full crossed product \( A \rtimes S \) of the action \( \mathcal{E} \) is defined as the maximal \( C^* \)-completion of the *-algebra \( A \rtimes_{\text{alg}} S \).

We also define some non-triviality conditions for inverse semigroup actions. Kwaśniewski and Meyer ([13, 14]) show that actions satisfying these conditions give rise to inclusions of \( C^* \)-algebras satisfying certain maximality conditions that are useful in the analysis of Cartan-like pairs.

**Definition 3.3** ([14 Definitions 6.1, 6.9]). Let \( \mathcal{E} \) be an action of an inverse semigroup \( S \) on \( A \). We say that the action is purely outer if \( \mathcal{E}_t \cdot I_{t,t}^- \) is a purely outer \( A \)-bimodule for each \( t \in S \). We say that the action is aperiodic if \( \mathcal{E}_t \cdot I_{t,t}^- \) is an aperiodic \( A \)-bimodule for each \( t \in S \).

A definition of the reduced crossed product for such an action can be found in [13 Section 2.2]. This involves the construction of a weak conditional expectation \( E : A \rtimes S \to A^\text{u} \) taking values in the enveloping von Neumann algebra of \( A \). One may also consider
essentially defined conditional expectations or local expectations that instead take values in the local multiplier algebra $M_{\text{loc}}(A)$ of $A$.

**Definition 3.4.** Let $A \subseteq B$ be an inclusion of $C^*$-algebras. An essentially defined conditional expectation or local expectation is a positive contractive linear map $E : B \to M_{\text{loc}}(A)$ such that $E$ restricts to the identity on $A$. We say that $E$ is faithful if $E(b^*b) = 0$ implies $b = 0$ for all $b \in B$, and $E$ is almost faithful if $E((ba)^*ba) = 0$ for all $a \in B$ implies $b = 0$ for all $b \in B$.

The dense subalgebra $A \times_{\text{alg}} S$ of $A \rtimes S$ is exactly the span of the bimodules $E_t$ under the maps $E_t : A \times_{\text{alg}} S$, $\xi \mapsto [\xi \delta_t]$. Each of these maps is injective, so we identify each $E_t$ with its image in $A \rtimes_{\text{alg}} S$. We define a local expectation $EL : A \rtimes S \to M_{\text{loc}}(A)$ as follows: let $t \in S$. Recalling the isomorphism $\vartheta_{1,t} : E_t \cdot I_{1,t} \to I_{1,t}$, each $\xi \in E_t$ defines a multiplier of $\mathcal{M}(I_{1,t})$ via $EL(\xi) : a \mapsto \vartheta_{1,t}(\xi \cdot a)$. By [14, Proposition 4.4] this extends to a local expectation $EL : A \rtimes S \to M_{\text{loc}}(A)$, which we call the canonical local expectation. We denote by $N_{EL}$ the largest ideal contained in $\ker(EL)$.

**Definition 3.5.** The essential crossed product $A \times_{\text{ess}} S$ is defined as the quotient $(A \rtimes S)/N_{EL}$. The local expectation $EL$ descends to a local expectation $A \times_{\text{ess}} S \to M_{\text{loc}}(A)$, which we also denote by $EL$.

**Theorem 3.6 ([14, Theorem 4.12]).** The canonical local expectation $EL : A \times_{\text{ess}} S \to M_{\text{loc}}(A)$ is faithful.

3.2. The dual groupoid to an inverse semigroup action. If $\mathcal{E}$ is an action of $S$ on $A$, there is an induced action $\hat{\mathcal{E}} = (\hat{E}_t)_{t \in S}$ of $S$ on $\hat{A}$ such that $\hat{E}_t : s(\hat{E}_t) \to r(\hat{E}_t)$ for each $t \in S$. The construction of the transformation groupoid $\hat{A} \rtimes S$ associated to $\mathcal{E}$ can be found in [13, Section 2.3].

**Definition 3.7 ([14]).** We call $\hat{\mathcal{E}}$ the dual action to the action $\mathcal{E}$ of $S$ on $A$. The transformation groupoid $\hat{A} \rtimes S$ is called the dual groupoid.

The unit space of the $\hat{A} \rtimes S$ is homeomorphic to $\hat{A}$ via the map $\hat{A} \to (\hat{A} \rtimes S)^{(0)}$, $[\pi] \mapsto [1, [\pi]]$. We often identify $\hat{A}$ and $(\hat{A} \rtimes S)^{(0)}$ under this map.

We also recall the following proposition from [13] relating the dual groupoid of an action to the action.

**Proposition 3.8 ([13, Proposition 2.17]).** Let $\mathcal{E}$ be an action of $S$ on $A$. The following are equivalent:

1. the canonical weak conditional expectation $E : A \rtimes S \to A^w$ takes values in $A$ (cf. [14, Lemma 4.5]), that is, $E$ is a genuine expectation;
2. the unit space $\hat{A}$ in the dual groupoid $\hat{A} \rtimes S$ is closed;
3. the intersection ideals $I_{1,t}$ for $t \in S$ are each complemented in $s(E_t)$.

In light of this, we make the following definition.

**Definition 3.9 ([13, Definition 2.16]).** Let $\mathcal{E}$ be an action of $S$ on $A$. We say that the action $\mathcal{E}$ is closed if any of the equivalent conditions in Proposition 3.8 hold.

Unfortunately, one cannot simply take an action $\mathcal{E}$ of $S$ on $A$, localise the module, and then gain an action on $M_{\text{loc}}(A)$. The main problem that occurs is that the map in Proposition 2.11 is not always an isomorphism, so we do not always have $(E_t)_{\text{loc}} \otimes M_{\text{loc}}(A) \cong (E_t \otimes A \mathcal{E}_u)_{\text{loc}}$. Our solution is to instead create a non-saturated Fell bundle with these localised modules, and then gain a saturated Fell bundle using [3, Theorem 7.2], which is then equivalent to an inverse semigroup action on $M_{\text{loc}}(A)$. 
Definition 3.10 ([2, Definition 2.10]). Let $S$ be an inverse semigroup. A Fell bundle over $S$ is a collection $\mathcal{A} = (A_t)_{t \in S}$ of Banach spaces $A_t$ together with a multiplication $\cdot : A_t \times A_u \rightarrow A_{tu}$ for each $t \in S$, linear maps $j_{t,u} : A_u \rightarrow A_t$ for $t,u \in S$, and an involution $^* : A_t \rightarrow A^*_t$ for each $t \in S$ satisfying the following:

(i) The multiplication is bilinear from $A_t \times A_u$ to $A_{tu}$ for all $t,u \in S$;

(ii) The multiplication is associative;

(iii) $||a \cdot b|| \leq ||a|| \cdot ||b||$ for all $a,b \in \bigcup_{t \in S} A_t$;

(iv) $^*$ is conjugate linear on each $A_t$;

(v) $(a^*)^* = a$, $||a^*|| = ||a||$, and $(a \cdot b)^* = b^* \cdot a^*$ for all $a,b \in \bigcup_{t \in S} A_t$;

(vi) $||a^*a|| = ||a||^2$ and $a^*a$ is a positive element in the $C^*$-algebra $A_{A\tau}$ for all $t \in S$ and $a \in A_t$.

(vii) $j_{t,u}$ is an isometric linear map for all $t,u \in S$ with $u \leq t$;

(viii) If $v \leq u \leq t$ then $j_{t,v} = j_{t,u} \circ j_{u,v}$;

(ix) If $s \leq t$ and $u \leq v$ in $S$, then $j_{t,s}(a) \cdot j_{v,u}(b) = j_{tv,us}(a \cdot b)$ for all $a \in A_s$ and $b \in A_v$;

(x) If $s \leq t$ then $j_{t,s}(a)^* = j_{t,s}(a^*)$ for all $a \in A_s$.

If $A_t$, $A_u$ spans a dense subspace of $A_{tu}$ for all $s,t \in S$, we say that the Fell bundle $\mathcal{A}$ is saturated. If $S$ is unital we call $A_1$ the unit fibre of the Fell bundle $\mathcal{A}$.

One can build a $C^*$-algebra out of the sections of a Fell bundle.

Definition 3.11 ([2, Definition 3.4], [13, Definition 2.7]). Let $\mathcal{A}$ be a Fell bundle over a unital inverse semigroup $S$ with unit fibre $\mathcal{A}_1 = A_1$. Let $\mathcal{L}(\mathcal{A}) := \bigoplus_{t \in S} A_t$ and $N := \spn\{a_1 \delta_s - j_{s,t}(a_1) \delta_t : s,t \in S, s \leq t, a_s \in \mathcal{A}_s\}$. The full cross sectional $C^*$-algebra $C^*(\mathcal{A})$ is defined as the maximal $C^*$-completion of $\mathcal{L}(\mathcal{A})/N$ with multiplication and involution inherited from the Fell bundle.

Each fibre $A_t$ of the Fell bundle $\mathcal{A}$ embeds canonically in $\mathcal{L}(\mathcal{A})/N$ via $A_t \ni x \mapsto [x \delta_1] \in \mathcal{L}(\mathcal{A})/N$, and so embeds in the full $C^*$-algebra $C^*(\mathcal{A})$. Identifying each fibre with its image in $C^*(\mathcal{A})$ we see that $\mathcal{E}_t \cdot I_{t,u} = \mathcal{E}_u \cdot I_{t,u} = \mathcal{E}_t \cap \mathcal{E}_u$ for each $t,u \in S$. In particular, we have $I_{1,t} = \mathcal{E}_t \cap A_1$.

Proposition 3.12. Let $\mathcal{A}$ be a $C^*$-algebra, $S$ an inverse semigroup, and $\mathcal{E} = (\mathcal{E}_t, \mu_{t,u})_{t,u \in S}$ an action of $S$ on $\mathcal{A}$ by Hilbert bimodules. There exists a Fell bundle over $\mathcal{A}$ over $S$ with unit fibre $M_{loc}(\mathcal{A})$ such that $\mathcal{A}_t = (\mathcal{E}_t)_loc$ for each $t \in S$.

Proof. Define $\mathcal{A}_t := (\mathcal{E}_t)_loc$. For $t,u \in S$ we define the multiplication by $\xi_t \cdot \eta_u := \mu_{t,u}(\xi_t \otimes_{M_{loc}(\mathcal{A})} \eta_u)$, $\xi_t \in \mathcal{A}_t$, $\eta_u \in \mathcal{A}_u$, where $\mu_{t,u}$ is the map induced by $\mu_{t,u}$ using Proposition 2.19.

One readily checks that this satisfies the axioms of a Fell bundle. □

Definition 3.13. We call the Fell bundle $\mathcal{A}$ defined in Proposition 3.12 the localised Fell bundle of the action $\mathcal{E}$.

Lemma 3.14. For each $t \in S$ let $I_{1,t} := \sum_{s \leq 1,t} s(\mathcal{A}_s)$ be the intersection ideal for $1,t \in S$ for the localised Fell bundle $\mathcal{A}$. Then $I_{1,t} = M_{loc}(I_{1,t})$.

Proof. For any $v \leq 1,t$ we have $\mathcal{E}_v \cong s(\mathcal{E}_v)$ and so for $a \in s(\mathcal{A}_s)$ and $b \in I_{1,t}^\perp$ we have $ab = ba = 0$ as $I_{1,t}^\perp$ annihilates $\mathcal{E}_v \cong s(\mathcal{E}_v)$, which detects subbimodules (or ideals) in $\mathcal{A}_v \cong M_{loc}(s(\mathcal{E}_v))$. Thus $a \in (I_{1,t}^\perp)^\perp = M_{loc}(I_{1,t})^\perp = M_{loc}(I_{1,t})$ by Lemma 2.13. This gives $s(\mathcal{A}_s) \subseteq M_{loc}(I_{1,t})$ for each $v \leq 1,t$, giving $I_{1,t} \subseteq M_{loc}(I_{1,t})$.

For the reverse inclusion, we see that $I_{1,t} \cong \mathcal{A}_t \cdot I_{1,t}$, and this is in fact equality in $C^*(\mathcal{A})$. For $J \subseteq I_{1,t}$ and $\tau \in M(J \cdot I_{1,t})$ we see there is an adjoinable map $\tau' : I_{1,t}J \rightarrow \mathcal{E}_t \cdot I_{1,t}J$ defined by $\tau'(a) := \vartheta_{1,t}(\tau a)$. Thus, $\tau' = \vartheta_{1,t} \circ \tau \in \mathcal{L}(I_{1,t}J, \mathcal{E}_t \cdot I_{1,t}J) \subseteq \mathcal{E}_t$. The mapping $\tau \mapsto \tau'$ is compatible with the inductive limit structures of $M_{loc}(I_{1,t})$ and $\mathcal{A}_t$, and is an isomorphism as $\vartheta_{1,t}$ is a Hilbert bimodule isomorphism.
There are isomorphisms $A_t \cdot I_{1,t} \to A_t \cap M_{\text{loc}}(A) \subseteq C^*(A)$ and $M_{\text{loc}}(I_{1,t}) \to M_{\text{loc}}(E_t \cap A) \subseteq C^*(A)$. In $C^*(A)$ we have $\tau = \vartheta_{I_{1,t}}$ giving $M(I_{1,t},J) \subseteq A_t \cap M_{\text{loc}}(A) = A_t \cdot I_{1,t}$ in $C^*(E)$ for each essential ideal $J \lhd A$. Thus, $M_{\text{loc}}(I_{1,t})$ is isomorphic to a submodule of $A_t \cap M_{\text{loc}}(A) = A_t \cdot I_{1,t}$ in $C^*(A)$. Taking sources of these Hilbert bimodules gives $M_{\text{loc}}(I_{1,t}) \subseteq I_{1,t}$, as required. \hfill \Box

**Corollary 3.15.** The canonical weak conditional expectation $\tilde{E} : C^*(A) \to M_{\text{loc}}(A)^\sigma$ and local expectation $\tilde{E}L : C^*(A) \to M_{\text{loc}}(M_{\text{loc}}(A))$ are genuine (that is, take values in $M_{\text{loc}}(A)$, and agree).

**Proof.** Using the characterisation of of $\tilde{E}L$ in [3, Lemma 4.5] and the definition of $\tilde{E}L$, we see that for $t \in S$ and $\xi \in A_t$ we have

$$\tilde{E}(\xi) = \tilde{E}L(\xi) = \xi \cdot 1_{I_{1,t}},$$

where $1_{I_{1,t}}$ is the unit in $M_{\text{loc}}(I_{1,t})$, which is equal to $I_{1,t}$ by Lemma [3.15]. \hfill \Box

The canonical weak expectation mentioned in Corollary 3.15 is that defined in [3, Lemma 4.5]. The definition of this expectation is not given in here, since under the conditions we impose this expectation is equal to the canonical local expectation.

We shall now use [3, Theorem 7.2] to gain a saturated Fell bundle $\tilde{E}$ over an inverse semigroup $\tilde{S}$, such that any constructed $C^*$-algebras from $A$ and $\tilde{E}$ are isomorphic. In the case where the original action on $A$ is aperiodic, this isomorphism will also preserve the conditional expectation, and the action constructed will be closed and purely outer.

**Proposition 3.16.** Let $E$ be an action of $S$ on $A$. There exists an inverse semigroup $\tilde{S}$ and a closed inverse semigroup action $\tilde{E}$ of $\tilde{S}$ on $M_{\text{loc}}(A)$ such that $M_{\text{loc}}(A) \times \tilde{S} \cong C^*(A)$ via an isomorphism that entwines the canonical local conditional expectations and maps $M_{\text{loc}}(A)$ identically to itself. Moreover, if $E$ is an aperiodic action, then the action $\tilde{E}$ is purely outer.

**Proof.** The saturated Fell bundle exists by [3, Theorem 7.2] and the bimodules associated to the action take the form $\tilde{E}_x = r(A_{t_1}) \ldots r(A_{t_n})A_t$ for some $t_1, \ldots, t_n, t \in S$, and the isomorphism described in [3, Theorem 7.2] maps each $A_t$ identically to itself. We see that the isomorphism $C^*(E) \rightarrow M_{\text{loc}}(A) \times \tilde{S}$ must preserve the conditional expectations, since on each fibre $A_x$ we have that the conditional expectation is the restriction of $\tilde{E}$ on $A_t$ to a submodule. The conditional expectation is genuine on $C^*(A)$ by Corollary 3.15, so the action $\tilde{E}$ is closed. If $E$ is an aperiodic action, then each of these bimodules acts purely outerly since each each $A_t \cdot I_{1,t} = (E_t \cdot I_{1,t})_{\text{loc}}$ is purely outer by Lemma [2.23]. \hfill \Box

**Remark 3.17.** The inverse semigroup $\tilde{S}$ in Proposition 3.16 is called the prefix expansion of $S$ and is taken from [3] and is the inverse semigroup generated by $S$ under alternative relations. These relations give rise to more idempotents in $\tilde{S}$, as well as a canonical injective partial homomorphism $\pi : S \to \tilde{S}$, which is an injective map satisfying $\pi(t)^* = \pi(t^*)$, $\pi(tu) = \pi(t)\pi(u)$ for all $t, u \in S$, and if $t \leq u$ then $\pi(t) \leq \pi(u)$. In the context of a localised action, we have shown that $\tilde{E}_x = r(A_{t_1}) \ldots r(A_{t_n})A_t = \tilde{E}_x$ for all $t, u \in S$, and $\tilde{E}_x \subseteq \tilde{E}_u$ in $C^*(E)$ whenever $t \leq u$, which is exactly describes a partial homomorphism.

**Definition 3.18.** Let $E$ be an action of $S$ on $A$. We call the action $\tilde{E}$ of $\tilde{S}$ on $M_{\text{loc}}(A)$ the localised action or localisation of $E$.

The localisation of an aperiodic action $E$ gives rise to a Cartan inclusion $M_{\text{loc}}(A) \subseteq M_{\text{loc}}(A) \times \tilde{S}$ in the sense of Exel [7] by [13, Theorem 4.3]. The inclusion we are interested in however is $A \subseteq A \times_{\text{ess}} S$, so we require a way to descend the structure of the localised inclusion to the original inclusion. To do this, we need that the local expectation $A \times \tilde{S} \to M_{\text{loc}}(A)$ is unique. We shall later use that this follows if the inclusion $A \subseteq A \times_{\text{ess}} S$ is...
between to a generalised expectation (that is, an expectation taking values in some algebra $\tilde{\mathcal{A}}$).

**Definition 4.1.** Let $A \subseteq M_{\text{loc}}(A) \rtimes S$ be encoded in the one algebra $\tilde{\mathcal{A}}$.

Theorem 3.19. There is an injective homomorphism $A \rtimes_{\text{ess}} S \rightarrow M_{\text{loc}}(A) \rtimes S$ that commutes with the inclusions $\mathcal{E}_t \rightarrow \tilde{\mathcal{E}}_t \subseteq M_{\text{loc}}(A) \rtimes S$ for each $t \in S$. That is, for each $t \in S$ the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{E}_t & \longrightarrow & A \rtimes_{\text{ess}} S \\
\downarrow & & \downarrow \\
\tilde{\mathcal{E}}_t & \longrightarrow & M_{\text{loc}}(A) \rtimes S
\end{array}
$$

**Proof.** Recall that $A \rtimes S$ is defined as the maximal $C^*$-completion of $A \rtimes_{\text{alg}} S$, which is in turn the quotient of $\bigoplus_{t \in S} \mathcal{E}_t$ by $\mathcal{N} := \text{span} \{ \xi_t \delta_t - \tilde{\varphi}_{s,t}(\xi_t) \delta_s : t, s \in S, \xi_t \in \mathcal{E}_t \cdot I_{t,s} \}$. Similarly, $M_{\text{loc}}(A) \rtimes S$ is the quotient of $\bigoplus_{t \in S} \tilde{\mathcal{E}}_t$ by the ideal $\tilde{\mathcal{N}} = \text{span} \{ \xi_t \delta_t - \tilde{\varphi}_{s,t}(\xi_t) \delta_s : t, s \in S, \xi_t \in \tilde{\mathcal{E}}_t \cdot M_{\text{loc}}(I_{t,s}) \}$. The inclusion $\bigoplus_{t \in S} \mathcal{E}_t \rightarrow \bigoplus_{t \in S} \tilde{\mathcal{E}}_t$ descends to both quotients since under this inclusion as we have $\mathcal{N} \subseteq \mathcal{N}$ whereby we gain a map $A \rtimes_{\text{alg}} S \rightarrow M_{\text{loc}}(A) \rtimes_{\text{alg}} S$. This map then extends to a homomorphism $i : A \rtimes S \rightarrow M_{\text{loc}}(A) \rtimes S$.

To see this map descends further to the essential and reduced crossed products, we first show that the expectations agree. Since $\mathcal{E}$ is an aperiodic action, $A \subseteq A \rtimes S$ is an aperiodic inclusion by [14, Proposition 6.3], and hence $\text{EL} : A \rtimes S \rightarrow M_{\text{loc}}(A)$ is the unique pseudo-expectation for this inclusion by [15, Theorem 3.6]. Thus if $\tilde{\mathcal{E}} : M_{\text{loc}}(A) \rtimes S \rightarrow M_{\text{loc}}(A)$ is the canonical expectation for the localised action, we must have $\text{EL} = \tilde{\mathcal{E}} \circ i$. Since $A \rtimes_{\text{ess}} S$ is the quotient of $A \rtimes S$ by the ideal $N_{\text{EL}} = \{ a \in A \rtimes S : \text{EL}(a^*a) = 0 \} = \{ a \in A \rtimes S : \tilde{\mathcal{E}} \circ i(a^*a) = 0 \}$ we have $i(N_{\text{EL}}) \subseteq N_{\tilde{\mathcal{E}}}$, so $i$ descends to a map on $A \rtimes_{\text{ess}} S$.

Lastly we show that $i$ is injective on $A \rtimes_{\text{ess}} S$. To see this, we note that an element $a \in A \rtimes_{\text{ess}} S$ is mapped to zero under $i$ if and only if it satisfies $\tilde{\mathcal{E}}(i(a^*a)) = \text{EL}(a^*a) = 0$, giving $a = 0$. $\square$

Theorem 3.19 allows the dynamic and algebraic structures of the actions $\mathcal{E}$ and $\tilde{\mathcal{E}}$ to be encoded in the one algebra $M_{\text{loc}}(A) \rtimes S$, and provides a setting useful for computations. Throughout the rest of this article we identify the modules $\mathcal{E}_t, \tilde{\mathcal{E}}_t$, and the algebras $A, M_{\text{loc}}(A), A \rtimes_{\text{ess}} S$ with their images in $M_{\text{loc}}(A) \rtimes S$ via Theorem 3.19.

4. Inclusions with minimal multiplicative domain

With Theorem 3.19 we see in particular that the inclusion $A \subseteq A \rtimes_{\text{ess}} S$ embeds into $M_{\text{loc}}(A) \rtimes S$. As briefly mentioned, the inclusion $A \subseteq M_{\text{loc}}(A)$ is problematic, and so we wish to ensure that the intersection of $A \rtimes_{\text{ess}} S$ and $M_{\text{loc}}(A)$ in $M_{\text{loc}}(A) \rtimes S$ is as small as possible. The local conditional expectation $\text{EL} : A \rtimes_{\text{ess}} S \rightarrow M_{\text{loc}}(A)$ restricts to a $^*$-homomorphism on at least $A$, since $\text{EL}|_A = \text{id}_A$, and so to minimise the intersection between $M_{\text{loc}}(A)$ and $A \rtimes_{\text{ess}} S$ we investigate the circumstances under which $\text{EL}$ restricts to a $^*$-homomorphism only on $A$.

4.1. Minimal multiplicative domain.

**Definition 4.1.** Let $A \subseteq B$ be an inclusion of $C^*$-algebras and let $E : B \rightarrow \hat{A}$ be a generalised expectation (that is, an expectation taking values in some algebra $\hat{A}$ containing $A$). We say that $E$ has **minimal multiplicative domain** or is (MMD) if the the following subset is equal to $A$.

$$
\mu(E) := \{ b \in B : E(b^*b) = E(b^*)E(b), E(bb^*) = E(b)E(b^*) \}.
$$

The set $\mu(E)$ for an expectation $E$ is the (two-sided) multiplicative domain of $E$; the $C^*$-subalgebra of $B$ to which $E$ restricts to a $^*$-homomorphism. This characterisation is adapted from Choi [6, Theorem 3.1] in which the multiplicative domain for 2-positive maps...
between $C^*$-algebras is defined. This formulation applies here as generalised expectations are completely positive by definition, and the two-sided condition ensures that $\mu(E)$ is $\ast$-closed. The ‘minimal’ descriptor of the definition of (MMD) stems from the fact that we always have $A \subseteq \mu(E)$, as $E$ restricts to the identity on $A$.

If $E$ is faithful, then $E$ restricts to an injective $\ast$-homomorphism $\mu(E) \to \hat{A}$. Thus, in the case of a faithful genuine conditional expectation $E : B \to A$, the multiplicative domain of $E$ is always minimal.

**Lemma 4.2.** Let $\mathcal{E}$ be an aperiodic action of $S$ on $A$ and let $\hat{\mathcal{E}}$ be its localisation. Consider $A \times_{\text{ess}} S$ as a subalgebra of $M_{\text{loc}}(A) \times_r \hat{S}$ as in Theorem 3.19. Then the multiplicative domain of $E \mathcal{L} A \times_{\text{ess}} S \to M_{\text{loc}}(A)$ is equal to the intersection $M_{\text{loc}}(A) \cap A \times_{\text{ess}} S$, where the intersection is taken in $M_{\text{loc}}(A) \times_r \hat{S}$.

**Proof.** Let $\hat{\mathcal{E}} : M_{\text{loc}}(A) \times_r \hat{S} \to M_{\text{loc}}(A)$ be the canonical conditional expectation for the action $\hat{\mathcal{E}}$. The inclusion $A \subseteq A \times_{\text{ess}} S$ is aperiodic by [14] Proposition 6.3, and so the local expectation $EL$ is unique by [15] Theorem 3.6. Since $\hat{\mathcal{E}}$ also restricts to a local expectation $A \times_{\text{ess}} S \to M_{\text{loc}}(A)$, we have that $EL = \hat{E}|_{A \times_{\text{ess}} S}$. Since $\hat{\mathcal{E}}$ is a faithful genuine expectation for the inclusion $M_{\text{loc}}(A) \subseteq M_{\text{loc}}(A) \times_r \hat{S}$ we have $\mu(\hat{\mathcal{E}}) = M_{\text{loc}}(A)$. Thus

$$\mu(EL) = \mu(\hat{E}|_{A \times_{\text{ess}} S}) = \mu(\hat{\mathcal{E}}) \cap A \times_{\text{ess}} S = M_{\text{loc}}(A) \cap A \times_{\text{ess}} S.$$

**Corollary 4.3.** The local expectation $EL$ for the inclusion $A \subseteq A \times_{\text{ess}} S$ has minimal multiplicative domain if and only if $M_{\text{loc}}(A) \cap A \times_{\text{ess}} S = A$.

Corollary 4.3 does not work generally for the modules $\mathcal{E}_t \subseteq A \times_{\text{ess}} S$: it fails even for non-unital ideals $I \triangleleft A$ since Lemma 2.2 gives $M_{\text{loc}}(I) \cap A \times_{\text{ess}} S = M(I) \cap A \triangleleft I$. However, we always will have containment $\mathcal{E}_t \subseteq \hat{\mathcal{E}}_t \cap A \times_{\text{ess}} S$.

**Lemma 4.4.** Let $\mathcal{E}$ be an aperiodic action such that $E \mathcal{L} A \times_{\text{ess}} S \to M_{\text{loc}}(A)$ has minimal multiplicative domain. For $t \in S$ we have $\mathcal{E}_t = (\hat{\mathcal{E}}_t \cap A \times_{\text{ess}} S) \cdot s(\mathcal{E}_t)$.

**Proof.** The inclusion $\mathcal{E}_t \subseteq (\hat{\mathcal{E}}_t \cap A \times_{\text{ess}} S) \cdot s(\mathcal{E}_t)$ follows since $\mathcal{E}_t$ is contained in both $\hat{\mathcal{E}}_t$ and $A \times_{\text{ess}} S$, and the fact that $\mathcal{E}_t = \mathcal{E}_t \cdot s(\mathcal{E}_t)$. We shall show that $\hat{\mathcal{E}}_t \cap A \times_{\text{ess}} S$ is a Hilbert $A$-bimodule, and then the reverse inclusion follows as the source ideals will be equal.

For $\xi, \eta \in \hat{\mathcal{E}}_t \cap A \times_{\text{ess}} S$, both $\langle \xi, \eta \rangle = \xi^* \eta + \xi \eta^*$ and $\langle \xi, \eta \rangle = \xi^* \eta^*$ belong to $M_{\text{loc}}(A)$ as $\hat{\mathcal{E}}_t$ is a Hilbert $M_{\text{loc}}(A)$-bimodule, and belong to $A \times_{\text{ess}} S$ as $\xi, \eta \in A \times_{\text{ess}} S$ which is closed under its own multiplication. Thus the inner products of $\hat{\mathcal{E}}_t \cap A \times_{\text{ess}} S$ take values in $M_{\text{loc}}(A) \cap A \times_{\text{ess}} S$, which is $A$ by Corollary 4.3. Then $\hat{\mathcal{E}}_t \cap A \times_{\text{ess}} S$ is closed under the left and right $A$-multiplications as both $\hat{\mathcal{E}}_t$ and $A \times_{\text{ess}} S$ are, and is norm-closed as an intersection of closed subsets of $M_{\text{loc}}(A) \times_r \hat{S}$. \[\square\]

### 4.2. Slice reconstruction

Renault [18], and earlier Kumjian [12] considered inclusions of $C^*$-algebras $A \subseteq B$ such that the larger algebra $B$ is densely spanned by elements that normalise $A$ in the following sense: $n \in B$ is a normaliser if $n^* An, nAn^* \subseteq A$. This has since been taken as a standard assumption for Cartan pair-like objects in [7], [13], and many others. For an action $\mathcal{E}$ of a unital inverse semigroup $S$ on a $C^*$-algebra $A$ (or more generally a Fell bundle over $S$ with unit fibre $A$), the inclusion $A \subseteq A \times S$ satisfies this property. This is because each $\mathcal{E}_t$ carries an $A$-bimodule structure, and together they span a dense subspace of $A \times S$.

**Definition 4.5.** Let $A \subseteq B$ be an inclusion of $C^*$-algebras. We call the inclusion regular if the set of normalisers $N(A, B) = \{ n \in B : n^* An, nAn^* \subseteq A \}$ spans a dense subspace of $B$. Closed subspaces $M \subseteq N(A, B)$ such that $AM, MA \subseteq M$ are called slices, and the collection of slices for the inclusion $A \subseteq B$ is denoted $\mathcal{S}(A, B)$. A subslice $N$ of $M$ is a slice $N$ contained in another slice $M$. 

---
If $A \subseteq B$ is a non-degenerate inclusion then $M^*M, MM^* \subseteq A$ for any slice $M \in \mathcal{S}(A, B)$, giving each slice a Hilbert $A$-bimodule structure with inner products induced from the multiplication in $B$. The set $\mathcal{S}(A, B)$ then becomes an inverse semigroup with operation $M \cdot N := \text{span} MN$ and *-operation given by the adjoint in $B$. For brevity of notation we write $MN$ to denote $M \cdot N$, the closed span of products of elements in $M$ and $N$. This gives rise to an action of $\mathcal{S}(A, B)$ on $A$, where the bimodules for the action are the slices, and the multiplication isomorphisms are induced by the multiplication in $B$. In the case where one has a closed and purely outer action $E$ of $S$ on $A$, Kwaśniewski and Meyer showed one can reconstruct slices for the inclusion $A \subseteq A \rtimes S$ from the bimodules $E_t$.

In general if $A \subseteq B$ is a regular non-degenerate inclusion then the slice inverse semigroup $\mathcal{S}(A, B)$ acts tautologically on $A$ via the action $E_X = X$ for slices $X \in \mathcal{S}(A, B)$, and multiplication maps given by the multiplication in $B$. In this case we gain a canonical map $\mathcal{S}(A, B) \to \text{Bis}(A \rtimes S(A, B))$ sending a slice to its corresponding bisection of the dual groupoid of the action on $A$. This map is not in general injective, but does preserve the inverse semigroup structure.

**Lemma 4.6.** Let $A \subseteq B$ be a regular non-degenerate inclusion of $C^*$-algebras and let $\hat{A} \rtimes \mathcal{S}(A, B)$ be the dual groupoid for the tautological action of $\mathcal{S}(A, B)$ on $A$. The map 

$$^\circ: \mathcal{S}(A, B) \to \text{Bis}(A \rtimes \mathcal{S}(A, B)), X \mapsto \hat{X} := \{(X, [\pi]) : [\pi] \in s(X)\},$$

is a semigroup homomorphism. Moreover, if $X, Y \in \mathcal{S}(A, B)$ such that $Y \subseteq X$, then $\hat{Y} \subseteq \hat{X}$, and $s(Y)$ is an essential ideal of $s(X)$ if and only if $\hat{Y}$ is a dense open subset of $\hat{X}$.

**Proof.** First note that the unit $A \in \mathcal{S}(A, B)$ is mapped to $\hat{A}$, the unit in $\text{Bis}(\hat{A} \rtimes \mathcal{S}(A, B))$, and the zero $\{0\}$ in $\mathcal{S}(A, B)$ is mapped to the empty set, which is the zero in $\text{Bis}(\hat{A} \rtimes \mathcal{S}(A, B))$.

Fix $X, Y \in \mathcal{S}(A, B)$. We see that $[Z, [\pi]] \in X\hat{Y}$ if and only if $[Z, [\pi]] = [X, [Y \otimes \rho]] \cdot [Y, [\rho]] = [XY, [\rho]]$ for some $[\rho] \in s(\hat{Y})$ such that $[Y \otimes \rho] \in s(X)$, giving $X\hat{Y} \subseteq XY$. Conversely, if $[XY, [\pi]] \in \hat{X}\hat{Y}$, then in particular $XY \neq \{0\}$, so $s(XY) = s(s(X) \cdot Y)$ contains $[\pi]$. This gives $[Y \otimes [\pi]] \in \tau(s(X) \cdot Y) \subseteq s(X)$, so $[X, [Y \otimes [\pi]]] \in \hat{X}$. Thus we have 

$$[XY, [\pi]] = [X, [Y \otimes [\pi]]] \cdot [Y, [\pi]] \in XY,$$

so $X\hat{Y} = XY$.

If $Y \subseteq X$ are slices then $X = Y \cdot s(Y)$, and we have $\hat{Y} = Xs(\hat{Y}) \subseteq \hat{X}$ as $s(\hat{Y})$ is an idempotent in $\text{Bis}(\hat{A} \rtimes \mathcal{S}(A, B))$. Now suppose that $s(Y) \triangleleft s(X)$ is essential. Any open subset $U \subseteq s(\hat{X})$ gives rise to a corresponding ideal $I \triangleleft s(X)$ such that $\hat{I} = U$. Then $s(Y) \cap I = \{0\}$ if and only if $I = \{0\}$ as $s(Y)$ is essential, so we have $s(\hat{Y}) \cap U = s(\hat{Y}) \cap I = \emptyset$ if and only if $I = \{0\}$. Thus $s(\hat{Y})$ is dense in $s(\hat{X})$, so $\hat{Y}$ is dense in $\hat{X}$ since $\hat{A} \rtimes \mathcal{S}(A, B)$ is étale and the source map restricts to a homeomorphism $\hat{X} \to s(\hat{X})$. Now suppose $\hat{Y} \subseteq \hat{X}$ is open and dense. Since the source map in $\hat{A} \rtimes \mathcal{S}(A, B)$ restricts to a homeomorphism on $\hat{X}$, we see that $s(\hat{Y}) \subseteq s(\hat{X})$ is open and dense. Any non-zero ideal $I \triangleleft s(X)$ then gives non-empty open $I \subseteq s(X)$, which then has non-empty intersection with $s(\hat{Y})$ by density. Thus the ideal $I \cap s(Y)$ has at least one non-zero representation, implying $I \cap s(Y) \neq \{0\}$, so $s(Y)$ is essential.

One of the statements in [13] Theorem 5.6] is that the bisection inverse semigroup $\text{Bis}(\hat{A} \rtimes S)$ for a closed and purely outer action is isomorphic to the slice inverse semigroup $\mathcal{S}(A, A \rtimes S)$. Slices for the inclusion $A \subseteq A \rtimes S$ can then be recovered from their intersections with the bimodules $E_t$.

**Lemma 4.7.** Let $E : S \rightarrow A$ be a closed and purely outer action. Let $X \subseteq A \rtimes S$ be a slice. Then $X = \sum_{t \in S} X \cap E_t$. 


Proof. The inclusion \( \sum_{t \in S} X \cap \mathcal{E}_t \subseteq X \) is clear. The proof of [13, Theorem 5.6] shows that \( X \) is the closure of the span of slices \( \mathcal{E}_t \cdot J_t \) for some \( t \in T \subseteq S \) and ideals \( J_t \subset A \). Each \( \mathcal{E}_t \cdot J_t \) is contained in \( \mathcal{E}_t \) and \( X \), so we gain the desired result.

\[ \]

**Corollary 4.8.** Let \( \mathcal{E} \) be an aperiodic action such that \( EL : A \rtimes_{\text{ess}} S \to M_{\text{loc}}(A) \) has minimal multiplicative domain. Let \( \tilde{Y} \subseteq M_{\text{loc}}(A) \rtimes_r \hat{S} \) be a slice for the localised inclusion \( M_{\text{loc}}(A) \subseteq M_{\text{loc}}(A) \rtimes_r \hat{S} \). Then \( \tilde{Y} = \{ 0 \} \) if and only if \( \tilde{Y} \cap A \rtimes_{\text{ess}} S = \{ 0 \} \), where the intersection is taken in \( M_{\text{loc}}(A) \rtimes_r \hat{S} \) using Theorem 3.19.

**Proof.** If \( \tilde{Y} = \{ 0 \} \) then so is \( \tilde{Y} \cap A \rtimes_{\text{ess}} S \). Conversely if \( \tilde{Y} \neq \{ 0 \} \) then for some \( t \in S \) we have \( \tilde{Y} \cap \mathcal{E}_t \neq \{ 0 \} \), which is a subbimodule of \( \mathcal{E}_t \). Lemma 2.11 then gives \( \{ 0 \} \neq \tilde{Y} \cap \mathcal{E}_t \cap \mathcal{E}_t \subseteq \tilde{Y} \cap A \rtimes_{\text{ess}} S \).

**Corollary 4.9.** Let \( \mathcal{E} \) be an aperiodic action such that \( EL : A \rtimes_{\text{ess}} S \to M_{\text{loc}}(A) \) has minimal multiplicative domain. Let \( X \subseteq \mathcal{X} \rtimes_{\text{ess}} S \) be a slice and let \( \tilde{X} = \text{span}(\mathcal{X} \rtimes_{\text{ess}} S) \cdot X \cdot M_{\text{loc}}(A) \) be the slice in \( M_{\text{loc}}(A) \rtimes_r \hat{S} \) generated by \( X \). Then for any Hilbert-A-subbimodule \( Y \subseteq \tilde{X} \cap A \rtimes_{\text{ess}} S \) we have \( Y = \{ 0 \} \) if and only if \( Y \cap X = \{ 0 \} \).

**Proof.** The source ideal of \( \tilde{X} \) annihilates \( M_{\text{loc}}(s(X)\uparrow) \) in \( M_{\text{loc}}(A) \) since

\[ \tilde{X} \cdot s(X)\uparrow / \text{span}(\mathcal{X} \rtimes_{\text{ess}} S) = \{ 0 \}. \]

Thus \( s(\tilde{X}) \subseteq M_{\text{loc}}(s(X)\uparrow) / \text{span}(\mathcal{X} \rtimes_{\text{ess}} S) \). We then have \( s(\tilde{X} \cap A \rtimes_{\text{ess}} S) \subseteq M_{\text{loc}}(s(X)) \cap A \rtimes_{\text{ess}} S \), which is equal to \( M(s(X)) \cap A \) by Lemma 2.4 and Corollary 4.3. If \( Y \subseteq \tilde{X} \cap A \rtimes_{\text{ess}} S \) is a non-zero Hilbert-A-subbimodule, then \( Y \cdot s(X) \neq \{ 0 \} \) as \( s(X) \) is essential in \( s(\tilde{X} \cap A \rtimes_{\text{ess}} S) \). Thus \( Y \cdot s(X) \subseteq (\tilde{X} \cap A \rtimes_{\text{ess}} S) \cdot s(X) = X \), so \( \{ 0 \} \neq Y \cdot s(X) \subseteq X \cap Y \). Contrapositively, \( Y \cap X = \{ 0 \} \) gives \( Y = \{ 0 \} \).

Now let \( \mathcal{E} \) be an aperiodic action with local expectation \( EL : A \rtimes_{\text{ess}} S \to M_{\text{loc}}(A) \). If \( X \subseteq \mathcal{X} \rtimes_{\text{ess}} S \) is a slice, one would want to analyse a corresponding slice \( \tilde{X} \subseteq M_{\text{loc}}(A) \rtimes_r \hat{S} \) encoding \( X \).

**Lemma 4.10.** Let \( \mathcal{E} \) be an aperiodic action such that the local expectation \( EL : A \rtimes_{\text{ess}} S \to M_{\text{loc}}(A) \) is \((\text{MMD})\). Let \( X \subseteq \mathcal{X} \rtimes_{\text{ess}} S \) be a slice for the inclusion \( A \subseteq \mathcal{X} \rtimes_{\text{ess}} S \). Considering \( A \rtimes_{\text{ess}} S \subseteq M_{\text{loc}}(A) \rtimes_r \hat{S} \) as in Theorem 3.19, \( \tilde{X} := \text{span}(M_{\text{loc}}(A) \cdot X) \cdot \mathcal{X} \cdot M_{\text{loc}}(A) \) is a slice for \( M_{\text{loc}}(A) \subseteq M_{\text{loc}}(A) \rtimes_r \hat{S} \) and satisfies \( X = (\tilde{X} \cap A \rtimes_{\text{ess}} S) \cdot s(X) \).

**Proof.** We first show that \( \tilde{X} \) is a slice for \( M_{\text{loc}}(A) \subseteq M_{\text{loc}}(A) \rtimes_r \hat{S} \). Fix \( x, y \in X \) and \( a \in M(J) \) for some \( J \in I_0(A) \). Let \( I := s(J \cdot X) \oplus s(J \cdot X)^\perp \). This ideal is essential in \( A \) by construction, and by the Cohen-Hewitt Factorisation Theorem for all \( b \in I \) there exists \( z \in X \) and \( c \in J \) such that \( yb = cz \). We then have \( (x^a y)b = x^a (ac)z = X^a M(J) JX = X^a JX = s(J \cdot X) \subseteq I \). Thus \( x^a y \) is a multiplier on \( I \), so belongs to the local multiplier algebra. Since \( ||x^a y|| \leq ||x|| \cdot ||y|| \cdot ||a|| \), it follows that \( x^a M_{\text{loc}}(A)y \subseteq M_{\text{loc}}(A) \) for all \( x, y \in X \). Thus \( \tilde{X}^a M_{\text{loc}}(A) \tilde{X} = \text{span}(M_{\text{loc}}(A) \cdot X^a \cdot M_{\text{loc}}(A) \cdot X \cdot M_{\text{loc}}(A) \subseteq M_{\text{loc}}(A)^3 = M_{\text{loc}}(A) \), so \( \tilde{X} \) is a slice.

Similarly to Lemma 4.4, the inclusion \( X \subseteq (\tilde{X} \cap A \rtimes_{\text{ess}} S) \cdot s(X) \) is clear. For the reverse inclusion, we see that \( X \cap A \rtimes_{\text{ess}} S \) has inner products taking value in \( M_{\text{loc}}(A) \cap A \rtimes_{\text{ess}} S \), which is equal to \( A \) by Corollary 4.3. By cutting down with the source ideal \( s(X) \), we then gain the desired equality.

**Lemma 4.11.** Let \( \mathcal{E} \) be an aperiodic action such that the local expectation \( EL : A \rtimes_{\text{ess}} S \to M_{\text{loc}}(A) \) is \((\text{MMD})\). Let \( X \subseteq A \rtimes_{\text{ess}} S \) be a slice for the inclusion \( A \subseteq A \rtimes_{\text{ess}} S \). Then \( X = \{ 0 \} \) if and only if \( X \cap \mathcal{E}_t = \{ 0 \} \) for all \( t \in S \).

**Proof.** The ‘only if’ direction is clear. Suppose now \( X \cap \mathcal{E}_t = \{ 0 \} \) for all \( t \in S \). Let \( \tilde{X} := \text{span}(M_{\text{loc}}(A) \cdot X \cdot M_{\text{loc}}(A) \) be the slice for the larger inclusion \( M_{\text{loc}}(A) \subseteq M_{\text{loc}}(A) \rtimes_r \hat{S} \).
generated by $X$. The equality $X \cap \mathcal{E}_t = \{0\}$ implies that $X \cap (\mathcal{E}_t \cap \tilde{X} \cap A \times_{\text{ess}} S) = \{0\}$ for each $t \in S$, and so by Corollary 4.9 we have $\{0\} = \mathcal{E}_t \cap \tilde{X} \cap A \times_{\text{ess}} S = \mathcal{E}_t \cap \tilde{X} \supset \mathcal{E}_t \cap \tilde{X} \cap \mathcal{E}_t$. This implies that $\mathcal{E}_t \cap \tilde{X}$ is zero, as this is a submodule of $\tilde{X}$ and $\mathcal{E}_t$ detects submodules of its localization by Lemma 2.17. This holds for each $t \in S$ so Lemma 4.11 gives $\tilde{X} = \{0\}$, which in turn gives $X = \{0\}$.

**Corollary 4.12.** Let $\mathcal{E}$ be an aperiodic action such that the local expectation $EL : A \times_{\text{ess}} S \to M_{\text{loc}}(A)$ is (MMD). For $X \in S(A, A \times_{\text{ess}} S)$, the subslice $X_{\text{ess}} := \sum_{t \in S} X \cap \mathcal{E}_t$ has zero orthogonal complement in $X$, and so has non-zero intersection with all non-zero subslices of $X$.

**Proof.** We see $X_{\text{ess}} \cap X$ satisfies $X_{\text{ess}} \cap X \cap \mathcal{E}_t = \bigcap_{u \in S} (X \cap \mathcal{E}_u)^{\perp} \cap (X \cap \mathcal{E}_t) = \{0\}$, so $X_{\text{ess}} \cap X = \{0\}$ by Lemma 4.11. If $Y \subseteq X$ is a non-zero subslice, then $Y \cap X_{\text{ess}} = Y \cap \sum_{t \in S} X \cap \mathcal{E}_t = \sum_{t \in S} Y \cap \mathcal{E}_t$, which is zero if and only if $Y = \{0\}$ by Lemma 4.11.

The topology of $\tilde{A} \rtimes S(A, B)$ has a basis given by slices of the inclusion $A \subseteq B$. Lemma 4.11 shows that any non-zero slice $X \in S(A, B)$ intersects at least one $\mathcal{E}_t$ for some $t \in S$. In the topology of the groupoid, we see then that any open subset of $\tilde{A} \rtimes S(A, B)$ must intersect the open bisection defined by $\mathcal{E}_t$, thus the bisections defined by $\mathcal{E}_t, t \in S$ from the action $\mathcal{E}$ cover a dense subset of the dual groupoid for the full slice action.

**Corollary 4.13.** Let $\mathcal{E}$ be an aperiodic action such that the local expectation $EL : A \times_{\text{ess}} S \to M_{\text{loc}}(A)$ is (MMD). Let $B = A \times_{\text{ess}} S$. The canonical groupoid homomorphism $\phi : \tilde{A} \rtimes S(A, B) \to \tilde{A} \rtimes S(A, B)$, $\phi[t, [\pi]] = [\mathcal{E}_t, [\pi]]$ has dense open range.

**Proof.** First note that $A \rtimes S(A, B)$ has a basis given by open bisections of the form $\tilde{X} = \{[X, [\pi]] : X \in S(A, B), [\pi] \in s(\tilde{X})\}$ by construction, so it suffices to show that the image of $\phi$ intersects each $\tilde{X}$. For each $X \in S(A, B)$ we see that $\tilde{X} = \emptyset$ if and only if $X = \{0\}$, which by Lemma 4.11 occurs if and only $X \cap \mathcal{E}_t = \{0\}$ for all $t \in S$. This then gives $X \cap \mathcal{E}_t = \tilde{X} \cap \mathcal{E}_t = \emptyset$ if and only if $X = \emptyset$, and since each $\mathcal{E}_t$ lies in the range of $\phi$, we see that $\phi$ has dense image. Each $\phi[t, [\pi]]$ belongs to $\mathcal{E}_t$ for each $[t, [\pi]] \in A \rtimes S$ and $\bigcup_{t \in S} \mathcal{E}_t$ is open in $A \rtimes S(A, B)$ as each $\mathcal{E}_t$ is an open bisection, and so the image of $\phi$ is open.

5. Effective inclusions

Analogous to the definition for groupoids (cf. [13, Definition 6.5]), we define what it means for a Hilbert $A$-bimodule to be effective. We also define corresponding effective actions and inclusions. Our interest in such inclusions is that they combine aperiodicity and minimal multiplicative domain conditions into a condition on the slices of an inclusion of $C^*$-algebras which is more practical to check. The main theorem of this section shows a class of inclusions $A \subseteq B$ of $C^*$-algebras take the form of essential crossed products of inverse semigroup actions, and that the same crossed product results from any choice of spanning inverse subsemigroup $S(A, B)$. We also show that under certain technical conditions, if an inclusion is spanned by effective slices then all slices are effective.

Throughout this section we fix a $C^*$-algebra $A$ and an action $\mathcal{E}$ of a unital inverse semigroup $S$ on $A$ by Hilbert bimodules.

**Definition 5.1.** Let $X$ be a Hilbert $A$-bimodule. We say that $X$ is effective if for all closed submodules $Y \subseteq X$ the induced partial homeomorphism $h_Y : s(Y) \to r(Y)$, $h_Y[\pi] = [Y \otimes \pi]$, is not equal to the identity map $\text{id}_{s(Y)}$.

We say that an the inverse semigroup action $\mathcal{E}$ of $S$ on $A$ is effective if for all $t \in S$, any submodule $Y \subseteq \mathcal{E}_t$ with $h_Y = \text{id}_{s(Y)}$ satisfies $Y \subseteq \mathcal{E}_t \cdot I_{1,t}$. 


We say that an inclusion $A \subseteq B$ of $C^*$-algebras is effective if it is regular, non-degenerate, and if any slice $M \in \mathcal{S}(A, B)$ satisfying $h_M = \id_{s(M)}$ is contained in $A$.

**Remark 5.2.** Considering the dual groupoid $\hat{A} \rtimes \mathcal{S}(A, B)$ for a regular non-degenerate inclusion $A \subseteq B$ we see that a slice $M \in \mathcal{S}(A, B)$ satisfies $s(M) = r(M) \subseteq \hat{A}$ precisely when $h_M = \id_{s(M)}$. If the inclusion is effective, then this ensures that $M \subseteq \hat{A} \cong (\hat{A} \rtimes \mathcal{S}(A, B))(0)$. If the bisection inverse semigroup $\text{Bis}(A \rtimes \mathcal{S}(A, B))$ is isomorphic to $\mathcal{S}(A, B)$ (for example in the case of [13, Theorem 5.6]) we see that the groupoid $A \rtimes \mathcal{S}(A, B)$ is effective exactly when the inclusions $A \subseteq B$ is effective.

**Lemma 5.3.** Let $X$ be an effective Hilbert $A$-bimodule. Then $X$ is aperiodic.

**Proof.** For each open subset $U \subseteq s(X)$ let $I_U \triangleleft A$ be the ideal such that $\hat{I}_U = U$. Then $X_{s(U)} = X \cdot I_U = \id_{s(U)}$ since $X$ is effective, so there exists $[\pi] \in U$ such that $X[\pi] \neq X[\pi]$. This shows that $X$ is topologically non-trivial (cf. [15, Definition 2.13]) and so $X$ is aperiodic by [10, Theorem 4.7].

**Corollary 5.4.** Let $E$ be an effective action of $S$ on $A$. Then $E$ is aperiodic.

**Proof.** For each $t \in S$ we have by definition that $E_t \cdot I_{1,t}$ is an essential part of $E_t$, so in particular $E_t \cdot I_{1,t}$ is effective as an $A$-bimodule. By Lemma 5.3 we see that $E_t \cdot I_{1,t}$ is aperiodic for each $t \in S$, so the action is aperiodic.

**Lemma 5.5.** Let $A \subseteq B$ be an effective inclusion. Then the dual groupoid $A \rtimes \mathcal{S}(A, B)$ is effective.

**Proof.** Since open bisections of the form $M = ([M, [\pi]] : [\pi] \in s(M))$ for slices $M \in \mathcal{S}(A, B)$ form a basis for the topology on $A \rtimes S$, it suffices to show any $M$ contained in isotropy is contained in $A$. If $M$ is contained in isotropy then $M \subseteq A$ since the inclusion $A \subseteq B$ is effective, whereby $M \subseteq A$.

We also see that just as effective slices are aperiodic, effective inclusions are also aperiodic.

**Lemma 5.6.** Let $A \subseteq B$ be an effective inclusion. Then it is aperiodic.

**Proof.** For any slice $X \in \mathcal{S}(A, B)$ we have that $X : (X \cap A) \hookrightarrow X/A = X/(X \cap A)$ as $(X \cdot (X \cap A))^{-1}/A$. Let $J_X = (X \cap A) \oplus (X \cap A)^{-1}$. Then we see that $J \cdot X : (X \cdot (X \cap A))^{-1}/A \subseteq X : (X \cdot (X \cap A)^{-1}$ is an aperiodic $A$-bimodule, and so is an aperiodic $J$-bimodule. Lemma 2.22 then shows that $X/A$ is an aperiodic $A$-bimodule, and since such slices $X \in \mathcal{S}(A, B)$ span $B$ we see that $B/A$ is aperiodic by [15, Lemma 4.2].

We now show the relationship between such actions and inclusions justifying naming them both effective, at least in the case where the local expectation $EL : A \rtimes_{\text{ess}} S \to M_{\text{loc}}(A)$ has minimal multiplicative domain. To do this, first we need a technical lemma.

**Lemma 5.7.** Let $A \subseteq B$ be a regular non-degenerate inclusion of $C^*$-algebras with faithful local expectation $E : B \to M_{\text{loc}}(A)$. If $M \in \mathcal{S}(A, B)$ has a subslice $N$ such that $N \subseteq \mu(E)$ and $N^{-1} \cap M = \{0\}$, then $M \subseteq \mu(E)$.

**Proof.** Fix such $M \in \mathcal{S}(A, B)$. It follows that $s(N)$ is an essential ideal of $s(M)$ as $s(N^{-1} \cap M) = s(N^{-1}) \cap s(M) = \{0\}$. Thus for any $m \in M$ and $n \in N$ we have $(E(m^*m) - E(m^*)E(m))n = E(m^*m)E(mn) - E(m^*)E(mn) = 0$, as $N \subseteq \mu(E)$. Hence $(E(m^*m) - E(m^*)E(m))$ annihilates $E(N)$, whereby it annihilates $M_{\text{loc}}(s(M))$ as $E(N)^{-1}$ contains $M_{\text{loc}}(s(N))$ and $M_{\text{loc}}(s(N)) = M_{\text{loc}}(s(M))$ since $s(N) \triangleleft s(M)$ is essential. This holds for all $m \in M$, and so for any $b \in s(M)$, we see that

$$E((mb)^*mb) - E((mb)^*)E(mb) = b^* (E(m^*m) - E(m^*)E(m))b = 0.$$
By the Cohen-Hewitt factorisation theorem all elements of $M$ are of the form $mb$ for some such $m \in M$ and $b \in s(M)$. Thus $M$ is contained in the multiplicative domain of $E$. 

**Lemma 5.8.** Let $E$ be an effective action of $S$ on $A$ such that the canonical local expectation $EL : A \rightharpoonup_{ess} S \to M_{loc}(A)$ has minimal multiplicative domain. Then $A \subseteq A \rightharpoonup_{ess} S$ is effective.

**Proof.** Fix a slice $X \in S(A, A \rightharpoonup_{ess} S)$ with $M = \text{id}_{\overline{S(M)}}$. Then for each $t \in S$ the subslice $M \cap E_t$ acts trivially on $A$, and so $M \cap E_t$ is contained in $A$ since the action is effective. By Corollary 4.12 the slice $M_S := \sum_{t \in S} M \cap E_t$ detects subslices of $M$, and is contained in $\mu(E)$, so by Lemma 5.7 we have $M \subseteq \mu(E) = A$. 

Given a regular non-degenerate inclusion $A \subseteq B$, being able to examine every slice to see if the inclusion is effective is not so practical. Fortunately, it suffices to know that enough effective slices exist.

The following theorem is an important stepping stone to the main results of this article.

**Theorem 5.9.** Let $A \subseteq B$ be a regular non-degenerate inclusion of $C^*$-algebras, and let $E : B \to M_{loc}(A)$ be a faithful local expectation. Suppose there is a densely spanning inverse subsemigroup $S \subseteq S(A, B)$ that acts aperiodically on $A$. Then there is an isomorphism $\varphi : A \rightharpoonup_{ess} S \to B$ that restricts to the identity on $A$ and entwines $E$ with the canonical local expectation $EL$. In particular, the inclusion $A \subseteq B$ is aperiodic.

**Proof.** The inclusion $A \subseteq A \rightharpoonup S$ is aperiodic by [14, Proposition 6.3], so the local expectation $EL : A \rightharpoonup S \to M_{loc}(A)$ is unique by [16, Theorem 3.6]. Let $\Phi : A \rightharpoonup S \to B$ be the canonical $^*$-homomorphism. This is surjective because it spans each of the slices in $S$. Then $E \circ \Phi : A \rightharpoonup S \to M_{loc}(A)$ is a local expectation, and so is equal to $EL$ by uniqueness.

The map $\Phi$ then descends to an isomorphism $\varphi : A \rightharpoonup S/\ker(\Phi) \to B$, so it suffices to show that the kernel of $\Phi$ is the largest ideal $N_{EL}$ contained in the kernel of $EL$, as quotienting by this ideal gives the essential crossed product. Fix $x \in \ker(\Phi)$. Then $EL(x^*x) = E(\Phi(x^*x)) = 0$ so $x \in N_{EL}$. Conversely, $x \in N_{EL}$ if and only if $0 = EL((xy)^*xy)$ by [14, Proposition 3.5], so $0 = EL((xy)^*xy) = E(\Phi(xy)^*\Phi(xy))$. Since $E$ is faithful, it is almost faithful by [14, Corollary 3.7], and since $\Phi$ is surjective we see that $\Phi(x) = 0$. Thus $\Phi$ descends to an isomorphism $\varphi : A \rightharpoonup_{ess} S \to B$. The inclusion $A \subseteq A \rightharpoonup_{ess} S$ then aperiodic since the quotient map $q : A \rightharpoonup S \to A \rightharpoonup_{ess} S$ descends to a bounded surjective bimodule map $(A \rightharpoonup S)/A \to (A \rightharpoonup_{ess} S)/A$, and the image of an aperiodic bimodule is aperiodic by Lemma 5.22.

At no point does this theorem invoke effectivity, the only required property for the inclusion is enough aperiodic slices, which is a weaker condition by Lemma 5.3 and Corollary 5.4. Aperiodicity and Kishimoto’s condition are however quite technical conditions to place on the slices, and effectiveness can in practice often be known a priori. An example of when one knows that an inclusion is effective a priori comes from examining groupoid $C^*$-algebras of effective groupoids, which we shall do in a later section.

**Corollary 5.10.** If the inverse subsemigroup in Theorem 5.9 acts effectively and the expectation $E : B \to M_{loc}(A)$ has minimal multiplicative domain, then the inclusion $A \subseteq B$ is effective.

**Proof.** Applying Theorem 5.9 and Lemma 5.8 gives effectivity of $A \subseteq A \rightharpoonup_{ess} S$.

**Corollary 5.11.** Let $A \subseteq B$ be a regular non-degenerate inclusion and suppose $E : B \to M_{loc}(A)$ is a faithful local expectation with minimal multiplicative domain. Suppose $S \subseteq S(A, B)$ is a densely spanning inverse subsemigroup that acts effectively on $A$. Then all densely spanning subsemigroups of $S(A, B)$ act effectively on $A$ and give rise to the same essential crossed product.
Proof. We have that \( A \subseteq B \) is effective and \( B \cong A \rtimes_{\text{ess}} S \) by Theorem 5.9 and Corollary 5.10. Then \( S(A, B) \) acts effectively on \( A \), hence aperiodically by Corollary 5.4. Applying Theorem 5.9 again gives \( A \rtimes_{\text{ess}} S(A, B) \cong B \cong A \rtimes_{\text{ess}} S \). All spanning subsemigroups of \( S(A, B) \) then also act effectively on \( A \), and all give rise to essential crossed products isomorphic to \( B \).

This motivates the following definition.

**Definition 5.12.** Let \( A \subseteq B \) be a regular, non-degenerate inclusion of \( C^* \)-algebras. We say the pair \((A, B)\) is a weak Cartan inclusion if

1. (WC1) the inclusion \( A \subseteq B \) is effective;
2. (WC2) there is a faithful local conditional expectation \( E : B \to M_{\text{loc}}(A) \) with minimal multiplicative domain.

If it is known that the conditional expectation has minimal multiplicative domain, then Theorem 5.9 and Corollaries 5.10 and 5.11 show that weak Cartan pairs are regular non-degenerate inclusions with enough effective slices, guaranteeing that all slices for the inclusion are effective. Using this, condition (WC1) in Definition 5.12 can be replaced with the condition that there exists a spanning inverse subsemigroup of slices that act effectively on the subalgebra.

### 5.1. The dual groupoid of an effective action

If \( \mathcal{E} \) is a closed and purely outer action of \( S \) on \( A \), there is an isomorphism \( \hat{\mathcal{A}} \rtimes S \cong \hat{\mathcal{A}} \rtimes S(A, B) \) by [14] Theorem 5.6. We are unfortunately not able to acquire the full power of this theorem. The crux of why this is the case is illustrated in Lemma 4.11 and Corollary 4.12, and how these do not quite give the same consequences as Lemma 4.7. When the action is not closed, it may not be possible for slices of the form \( \mathcal{E}_t \) to encapsulate all the data of \( S(A, A \rtimes_{\text{ess}} S) \), but only the interior parts. This occurs since the localisation process we use does not distinguish between a given Hilbert bimodule \( X \) and its subbimodules of the form \( X \cdot I \) for \( I \in \mathcal{T}_e(A) \), or in the groupoid picture, boundaries of open bisections are not distinguished.

**Lemma 5.13.** Let \( \mathcal{E} \) be an effective action of \( S \) on \( A \). The canonical groupoid homomorphism \( \phi : \hat{\mathcal{A}} \rtimes S \to \hat{\mathcal{A}} \rtimes S(A, B) \), \( \phi[t, [\pi]] = [\mathcal{E}_t, [\pi]] \), is injective.

*Proof.* We first note that since both \( \hat{\mathcal{A}} \rtimes S \) and \( \hat{\mathcal{A}} \rtimes S(A, B) \) have homeomorphic unit space and \( \phi \) restricts to a homeomorphism of the unit spaces, it suffices to show that \( \phi[t, [\pi]] \in (\hat{\mathcal{A}} \rtimes S(A, B))^{(0)} \) implies \( [t, [\pi]] \in (\hat{\mathcal{A}} \rtimes S)^{(0)} \). Fix \( [t, [\pi]] \in \hat{\mathcal{A}} \rtimes S \) such that \( [\mathcal{E}_t, [\pi]] \in (\hat{\mathcal{A}} \rtimes S(A, B))^{(0)} \). This occurs if and only if \( \mathcal{E}_t \cdot I = \mathcal{E}_t [\pi] = \text{id}_I \) for some ideal \( I \triangleleft A \) with \( [\pi] \in \hat{I} \). Since the action \( \mathcal{E} \) is effective, it follows that \( \mathcal{E}_t \cdot I \subseteq \hat{I} \cap A = \mathcal{E}_t \cdot I_{1,t} \). In particular, \( [\pi] \in \hat{I}_{1,t} \) so there exists \( v \in S \), \( v \leq 1, t \) such that \( [\pi] \in s(\mathcal{E}_v) \) and we have \( [t, [\pi]] = [tv^*v, [\pi]] = [v, [\pi]] \), which belongs to the unit space of \( A \rtimes S \) as \( v \leq 1 \) is an idempotent. □

We see with Lemma 5.13 that one may consider \( \hat{\mathcal{A}} \rtimes S \subseteq \hat{\mathcal{A}} \rtimes S(A, B) \) as a dense open subgroupoid.

### 5.2. Maximal abelian subalgebras

Renault [18] included the condition that a Cartan subalgebra should be a maximal abelian subalgebra, or *masa.* If in the prospective inclusion \( A \subseteq B \) the subalgebra \( A \) is commutative, then so is the local multiplier algebra of \( A \) as an inductive limit of commutative algebras. Thus we quickly see that if \( A \) is maximal abelian in \( B \), then any faithful local expectation \( E : B \to M_{\text{loc}}(A) \) will have minimal multiplicative domain, since the multiplicative domain \( \mu(E) \) is isomorphic to a subalgebra of \( M_{\text{loc}}(A) \) and contains \( A \). What is perhaps less obvious is that maximal abelian is equivalent to the inclusion being effective. Before we show this, we first state a technical result on maximal abelian subalgebras.
Lemma 5.14. Let $A \subseteq B$ be a regular non-degenerate inclusion of $C^*$-algebras. Then $A \subseteq B$ is maximal abelian if and only if $M(A) \subseteq M(B)$ is.

Proof. First note that $M(A) \subseteq M(B)$ makes sense since $A \subseteq B$ is non-degenerate, so multipliers of $A$ extend to multipliers of $AB = B$.

Suppose that $A \subseteq B$ is maximal abelian. Fix unitary $u \in M(B)$ that commutes with $M(A)$. Then $u$ commutes with $A$, so for each $a \in A$ we see that $u a u^{-1}$ commutes with $A$, whereby $u a$ belongs to $A$ by hypothesis. Thus $u A \subseteq A$, so $u$ defines a multiplier $u' \in M(A)$. We claim $u$ and $u'$ are equal. Let $(e_\lambda) \subseteq A$ be an approximate unit for $A$. Then for any $b \in B$ we have

$$(u - u')b = \lim_{\lambda} (u - u')e_\lambda b = 0,$$

which implies $u = u'$. Since the commutant of $M(A)$ in $M(B)$ is a unital $C^*$-algebra, it is spanned by unitaries, so we see that $M(A)$ is maximal abelian.

Now suppose $M(A) \subseteq M(B)$ is maximal abelian. Fix $b \in B$ that commutes with $A$. Then $b$ commutes with $M(A)$, and so $b$ is contained in $M(A)$ as $M(A)$ is maximal abelian in $M(B)$. Since $A \subseteq B$ is a non-degenerate inclusion, letting $(e_\lambda)$ be an approximate unit for $A$ we see that $b = \lim_{\lambda} be_\lambda \in A$. Thus $A \subseteq B$ is maximal abelian. □

Theorem 5.15. Let $A \subseteq B$ be a regular non-degenerate inclusion of $C^*$-algebras. Suppose that $A = C_0(X)$ is commutative. Then $A \subseteq B$ is effective if and only if $A$ is maximal abelian in $B$.

Proof. Suppose first that $A \subseteq B$ is an effective inclusion. We shall show that the inclusion $M(A) \subseteq M(B)$ is maximal abelian and employ Lemma 5.14. Fix a unitary $u \in M(B)$ that commutes with $M(A)$. Then $u$ commutes with $A$, and $u A$ is a slice for the inclusion $A \subseteq B$. For any $x \in X = \hat{A}$ and any $f \in C_0(X)$ we see that $(u*fu)(x) = u*uf(x) = f(x)$, whereby the partial homeomorphism $h_{uA}$ induced by $u A$ is trivial. Thus $u A \subseteq A$ since the inclusion $A \subseteq B$ is effective, and so we have that $u$ is given by a multiplier on $A$. Since the commutant of $M(A)$ in $M(B)$ is a unital $C^*$-algebra, it is spanned by the unitaries in contains, whereby $M(A)$ is maximal abelian in $M(B)$. Lemma 5.14 then implies that $A \subseteq B$ is maximal abelian.

Now suppose that $A \subseteq B$ is maximal abelian. Let $M \subseteq B$ be a slice such that $h_M = 1_{\mathcal{A}(M)}$. Fix $n \in M$. Let $\alpha_n : \text{supp}(n^*n) \to \text{supp}(nn^*)$ be the partial homeomorphism of $X$ from [22, 1.6] characterised by

$$(n^*fn)(x) = f(\alpha_n(x))n^*n(x), \quad \text{for all } f \in C_0(X), x \in \text{supp}(n^*n).$$

The normaliser $n$ generates a subslice of $M$, we see that the partial homeomorphism $\alpha_n$ is the restriction of $h_M$ to $\text{supp}(n^*n)$. This implies that $\alpha_n$ is the identity map on $\text{supp}(n^*n)$. Thus, for any positive $f \in C_0(X)$ and $x \in \text{supp}(n^*n)$ we have

$$(n^*fn)(x) = f(x)n^*n(x) = n^*nf(x).$$

If $x \notin \text{supp}(n^*n)$ then we see $|(n^*fn)(x)| \leq |(n^*||f||1_Xn)(x)| = ||f|| \cdot |n^*n(x)| = 0$. This holds for all positive $f \in C_0(X)$, and the map $f \to n^*fn$ is linear thus giving $n^*fn = fn^*n = n^*nf$ for all $f \in C_0(X)$. Thus for any $f \in C_0(X)$ we compute

$$(fn - nf)^*(fn - nf) = n^*f^*fn - f^*n^*fn - n^*f^*fn + f^*n^*nf = 0.$$ 

This then implies that $fn = nf$, whereby $n$ commutes with $C_0(X)$ and so belongs to $C_0(X)$ since the inclusion is maximal abelian. This holds for all $n \in M$, whereby $M$ is contained in $A$ and so $A \subseteq B$ is an effective inclusion. □

Corollary 5.16. Let $A \subseteq B$ be a regular non-degenerate inclusion of $C^*$-algebras with faithful local expectation $E : B \to M_{\text{loc}}(A)$. Then $A \subseteq B$ is a weak Cartan inclusion if and only if $A$ is maximal abelian in $B$. 
Corollary 5.10 shows that in the case where the subalgebra $A$ is commutative, the definition of a weak Cartan inclusion differs from Renault’s definition [18, Definition 5.1] in only one way: the conditional expectation may take values in the local multiplier algebra. In particular, if one has a true Cartan pair in the sense of Renault, then this pair will also be a weak Cartan pair.

6. Étale Groupoid $C^*$-algebras and Effective Inclusions

$C^*$-algebras of étale groupoids provide a wealth of examples of Cartan-like inclusions of $C^*$-algebras. Every Cartan pair in the sense of Renault is given by a twisted groupoid and the inclusion of functions on its unit space to the reduced twisted groupoid $C^*$-algebra [18, Theorem 5.6], and so it behooves us to explore more examples of this form by relaxing some of the conditions imposed by Renault.

Throughout this section we take $G$ to be an étale groupoid with locally compact and Hausdorff unit space $G(0) = U$. Note that the whole of $G$ need not be Hausdorff. By [14, Lemma 7.3, Proposition 7.6] we can consider any Fell bundle over $G$ as a saturated Fell bundle over an inverse semigroup $S$, which in turn can be considered an inverse semigroup action by [5, Theorem 4.8], relating it to earlier sections of this article. In this section we shall consider such Fell bundles over groupoids.

Definition 6.1 ([2, Definition 2.8], [14, Definition 7.1]). A Fell bundle over $G$ is an upper semi-continuous bundle $A = (A_s)_{s \in G}$ of Banach spaces with continuous involution $*: A \to A$ and continuous multiplication

$$\{(a, b) \in A \times A : a \in A_\gamma, b \in A_\eta, \gamma, \eta \in G, s(\gamma) = r(\eta)\} \to A,$$

which is associative whenever defined. Further, the fibres $A_x$ over units $x \in U$ must be $C^*$-algebras, and each $A_x$ must be a Hilbert $A_{s(\gamma)} - A_{r(\gamma)}$-bimodule under the left and right respective inner products given by $\langle \langle \xi, \eta \rangle \rangle = \xi^* \eta$ and $\langle \xi, \eta \rangle = \xi \eta^*$.

The bundle is saturated if $A_\gamma A_\eta$ densely spans $A_{\gamma \eta}$ for all composable $\gamma, \eta \in G$.

In [14, Section 7.2] the authors define a convolution algebra out of quasi-continuous sections of $A$. A section is elementary quasi-continuous if it has compact support contained in a bisection of $G$, on which it is continuous. A quasi-continuous section is a sum of elementary quasi-continuous sections, and we write $\mathcal{S}(G, A)$ for the space of quasi-continuous sections. This carries the convolution product and involution

$$(f * g)(\gamma) := \sum_{r(\eta) = r(\gamma)} f(\eta)g(\eta^{-1}\gamma), \quad (f^*)(\gamma) = f(\gamma^{-1}).$$

The full section $C^*$-algebra of the Fell bundle $A$ over $G$ is defined as the maximal $C^*$-completion of this and is denoted $C^*(G, A)$.

Following [14, Section 7.3], for each $x \in U$ there is a Hilbert $A_x$-module $\ell^2(G_x, A)$ and a canonical representation $\lambda_x: C^*(G, A) \to \mathcal{L}(\ell^2(G_x, A))$. The reduced $C^*$-norm on $\mathcal{S}(G, A)$ is defined as

$$\|f\|_r := \sup\{|f_x(f)|\},$$

and the reduced section $C^*$-algebra is defined as the completion $C^*_r(G, A)$ of $\mathcal{S}(G, A)$ under this norm.

Proposition 6.2. Let $A$ be a Fell bundle over $G$ and let $A$ be the $C_0(U)$-algebra corresponding to the bundle of $C^*$-algebras $(A_x)_{x \in U}$. There is a unital inverse semigroup $S$ and an action of $S$ on $A$ such that $C^*(G, A) \cong A \rtimes S$ and $C^*_r(G, A) \cong A \rtimes_r S$.

Proof. Combine [14, Lemma 7.3], [14, Proposition 7.6], and [14, Proposition 7.9].
Kwaśniewski and Meyer then use this connection between section algebras over groupoids and crossed product algebras by inverse semigroups to form the essential groupoid C*-algebra.

**Definition 6.3** ([14, Definition 7.12]). The essential section C*-algebra $C^*_\text{ess}(G, \mathcal{A})$ is defined as the quotient of $C^*(G, \mathcal{A})$ corresponding to $A \rtimes_{\text{ess}} S$ under the isomorphism of Proposition 6.2.

Under this definition, $C^*_\text{ess}(G, \mathcal{A})$ is the quotient of $C^*(G, \mathcal{A})$ determined by the canonical local expectation $EL : C^*(G, \mathcal{A}) \cong A \rtimes S \to M_{\text{loc}}(A)$, and we can moreover consider $C^*_\text{ess}(G, \mathcal{A})$ as a quotient of $C^*_r(G, \mathcal{A})$ using Proposition 6.2. Kwaśniewski and Meyer call elements of the kernel of the map $C^*_r(G, \mathcal{A}) \to C^*_\text{ess}(G, \mathcal{A})$ singular, following Exel and Pitts [9] and denote the kernel of this map by $J_{\text{sing}}$. There is also by [14] Proposition 7.9 an injective norm-decreasing homomorphism $j : C^*_r(G, \mathcal{A}) \to \mathfrak{B}(G, \mathcal{A})$, whereby we can consider elements of $C^*_r(G, \mathcal{A})$ as (Borel) sections $G \to \mathcal{A}$.

If we consider the trivial Fell line bundle over $G$, we recover the conventional groupoid C*-algebra, so we have the embedding $j : C^*_r(G) \to \mathfrak{B}(G)$. For a locally compact Hausdorff space $U$, Gonschor identifies the local multiplier algebra of $C_0(U)$ with the injective hull $I(C_0(U))$, and shows these are both isomorphic to the Borel functions modulo the ideal of functions with meagre support. Thus we have

$$M_{\text{loc}}(C_0(U)) = \mathfrak{B}(U)/\mathfrak{M}(U) = I(C_0(U)),$$

(cf. [11] Theorem 1]). In the case of a trivial Fell line bundle over $G$ we then gain the local expectation

$$C^*_r(G) \to \mathfrak{B}(G) \xrightarrow{j} \mathfrak{B}(U) \xrightarrow{q} \mathfrak{B}(U)/\mathfrak{M}(U) = M_{\text{loc}}(C_0(U)),$$

where $q : \mathfrak{B}(U) \to \mathfrak{B}(U)/\mathfrak{M}(U)$ is the quotient map. In particular, if the inclusion $C_0(U) \subseteq C^*_r(G)$ is aperiodic, then this expectation will be the same as the canonical local expectation induced by $EL : A \rtimes S \to M_{\text{loc}}(A)$ under the isomorphism in Proposition 6.2 and so we denote it too by $EL : C^*_r(G) \to M_{\text{loc}}(C_0(U))$. This leads to the following characterisation of the multiplicative domain of $EL$.

**Lemma 6.4.** Consider $C^*_r(G)$ as the respective reduced section algebra for the trivial Fell line bundle over $G$. Then the multiplicative domain of $EL : C^*_r(G) \to M_{\text{loc}}(C_0(U))$ is given by

$$\mu(EL) = \{ f \in C^*_r(G) : \text{supp}(j(f)) \cap U \text{ is meagre} \}.$$

**Proof.** Following the argument in the proof of [14] Proposition 7.10], we first note that if $f \in \mathfrak{S}(G)$ then we have

$$j(f^* f)(x) = \sum_{r(\gamma) = x} \overline{f(\gamma^{-1})} f(\gamma) = \sum_{s(\gamma) = x} |f(\gamma)|^2. \quad (1)$$

The norm of the left hand side is bounded by $||f||^2$, and the norm on the right hand side is the square of the norm of $\lambda_x(f)$ in $l^2(G_x)$, which is bounded above by the reduced norm. By continuity, we see for all $f \in C^*_r(G)$ that

$$j(f^* f)(x) = \sum_{s(\gamma) = x} |f(\gamma)|^2.$$

Suppose now $f$ has meagre support outside of $U$. Since $f$ is approximated as a countable sum of quasi-continuous functions on $G$, we see that the support of $f$ is contained in countably many bisections $(V_n)_{n \in \mathbb{N}}$ of $G$. For each of these bisections $V_n$, the source of
Lemma 6.4. Thus for all \(g\) then equal to zero in \(B\) generated by \(f\).

Proof. If \(V\) such that \(\supp(f)\) has meagre support then \(f\) belongs to \(\mu(EL)\) by [6] Theorem 3.1.

Now suppose \(f \in \mu(EL)\). We see by the above argument that
\[
\supp((j(f^* f) - j(f^*)j(f))(x)) = \left( \sum_{s(\gamma) = x} |f(\gamma)|^2 \right) - |f(x)|^2,
\]
which is non-zero if and only if there is some \(\gamma \in \supp(j(f))\setminus U\) with \(s(\gamma) = x\). This shows that the support of \((j(f^* f) - j(f^*)j(f))(x))\) is exactly \(s(\supp(j(f))\setminus U)\), which we have shown to be meagre. Thus \(f\) belongs to \(\mu(EL)\) by [6] Theorem 3.1.

This gives rise to the following characterisation of the essential groupoid \(C^*\)-algebra.

Lemma 6.5. An element \(f \in C^*_\text{r}(G)\) belongs to the largest ideal \(N_{EL}\) contained in \(\ker(EL)\) if and only if \(j(f)\) has meagre support. In particular, two elements \(f, g \in C^*_\text{r}(G)\) are equal in \(C^*_\text{ess}(G)\) if and only if \(j(f - g)\) has meagre support.

Proof. If \(j(f)\) has meagre support then \(f\) belongs to the multiplicative domain of \(EL\) by Lemma 6.4. Thus for all \(g \in C^*_\text{r}(G)\) we have \(EL(fg) = EL(f)EL(g) = EL(gf)\), which is then equal to zero in \(\mathfrak{B}(U)/\mathfrak{M}(U)\) as \(j(f)\) has meagre support. This shows that the ideal generated by \(f\) is contained in \(\ker(EL)\), whereby \(f \in N_{EL}\).

Conversely suppose \(j(f)\) does not have meagre support. Since \(j(f)\) has support contained in countably many bisections of \(G\), we see that there exists some bisection \(V \subseteq G\) such that \(V \cap \supp(j(f))\) is not meagre, and so the support of \(f^* f\) has non-meagre intersection with \(V^{-1}V\) which is open in \(U\). Thus \(EL(f^* f) \not= 0\), in particular \(f\) cannot belong to an ideal contained in \(\ker(EL)\). \(\square\)

In general if \(G\) is an étale groupoid such that the unit space is dense in \(G\), then the compliment of the unit space is meagre so by Lemma 6.5 the essential groupoid \(C^*\)-algebra only detects functions on the unit space. We then gain the trivial inclusion \(C_0(U) = C^*_\text{ess}(G)\), which always has minimal multiplicative domain. The conditional expectation having minimal multiplicative domain does not imply that the groupoid is effective, as Hausdorff étale groupoids always have closed and open unit space but there are myriad families of such groupoids which are not effective.

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