Effective Characteristics of a Layered Hollow Cylindrical Body of Elastic Creeping Materials

Tatyana Bobyleva 1, Alexei Shamaev 2, 3

1 Moscow State University of Civil Engineering, Yaroslavskoye Shosse, 26, Moscow, 129337, Russia
2 Ishlinsky Institute for Problems in Mechanics of the Russian Academy of Sciences, Pr. Vernadskogo, 101-1, Moscow, 119526, Russia
3 Lomonosov Moscow State University, GSP-1, Leninskie Gory, Moscow, 119991, Russia

tatyana2211@outlook.com

Abstract. The article deals with the problem of a layered composite pipe. The layers alternate in pairs and consist of elastic-creeping materials. The work is devoted to the study of the mechanical properties of the composite material with the help of its microstructure. Processes are studied with a typical pipe radius which is much more than the typical winding width. Heterogeneous layered material behaves as a certain “effective” material without layers in such processes. The method of calculation of effective moduli based on mathematical homogenization theory is described. Creep kernels are given by the sum of a finite number of decreasing exponential functions.

1. Introduction
Structures made of composite materials are widely used in modern construction. They are usually produced on the basis of polymers, which are characterized by phenomena of creep and relaxation. To improve the strength characteristics of pipes in their manufacture, several layers of different materials are used, which together provide high tightness and strength. The layers often differ significantly in rigidity. Examples include fibrous composites, formed by longitudinal-transverse winding. Often the calculation of such materials is impractical without building a homogenized model. Finite element method (FEM) for direct solving of such problem may demand extremely expensive computational efforts. The principles of averaging composite materials were set forth, for example, in the works [1,2]. They lead to simpler equations with averaged (effective) coefficients. Solutions of averaged equations are close to solutions of the original ones. In many cases, that difference has the same order as the periodic cell size. Often the homogenized elastic environment is anisotropic.

In the present work we consider the problem of elastic creep pipe which is described by a system of integro-differential equations with nonlocal members of the convolution type. It is also possible to construct an averaged system for such problems. Elastic media modelling for long time intervals to study the strength of these environments is of interest [3, 4].

In this paper we consider a layered pipe in which rapid change in material properties occurs along the radial direction. Such pipe constructions are often used in practice.

It should be noted that the use of exponential-type kernels as the kernels of creep and relaxation gives the opportunity to explicitly construct averaged system for a homogeneous environment. In contrast to
the previous problems, the equilibrium equations contain a function defining the distance from the pipe cross-section center. The pipe layers are located along the level surfaces of this function. Volterra correspondence principle can be applied to the solution of such elastic-creeping problems. In the paper at first the Laplace transform was applied to all governing equations. As a result, all coefficients depend on the transform parameter. Because the array has layered structure, the theory of averaging is applied to the obtained equations. As a result the equilibrium problem for a homogeneous elastic array is obtained. Then the inverse Laplace transform is applied to it to get final solution.

2. Problem specification and decision
Consider the system of equilibrium equations of the theory of elasticity [1] in Cartesian coordinates \( x_1, x_2, x_3 \):

\[
\frac{\partial}{\partial x_j} \left( A^{ij} \frac{\partial \vec{u}}{\partial x_i} \right) = \vec{f},
\]

here vector \( \vec{u} \) is the displacement vector, \( \vec{f} \) is the vector of mass forces, \( i, j = 1, 2, 3 \). We consider the elastic-creeping material therefore components of the stress tensor \( \sigma_{ij} (i, j = 1 \div 3) \) are determined both at a given time and by the entire history of the body deformation. Therefore, the equations of state connecting the components of the strain and stress tensors for each layer are as follows [3,4]:

\[
\sigma_{ij}^{(s)} = a_{ijkh}^{(s)} \cdot \epsilon_{kh}^{(s)},
\]

where \( a_{ijkh} = c_{ijkh} \delta(t) + d_{ijkh}, \epsilon_{kh}^{(s)} = \frac{1}{2} \left( \frac{\partial u_k^{(s)}}{\partial x_h} + \frac{\partial u_h^{(s)}}{\partial x_k} \right), (k, h = 1 \div 3), (s = 1, 2 \text{ is the layer number}), \) \( u_k \) are components of the displacement vector, \( c_{ijkh}^{(s)} \) are components of the elastic modulus tensor, \( \delta(t) \) is Dirac-delta, \( d_{ijkh}^{(s)}(t, \tau) \) are Volterra integral operators, namely

\[
d_{ijkh}^{(s)} \cdot \epsilon_{kh}^{(s)} = \int_0^t d_{ijkh}^{(s)}(t-\tau) \epsilon_{kh}^{(s)}(\tau) d\tau
\]

variable \( t \) specifies time. (Einstein convention for repeated indices is used.) Relaxation kernels \( d_{ijkh}^{(s)}(t-\tau) \) depend on the difference \( t - \tau \). This follows from the condition of invariance of the quantity \( \sigma_{ij} \) with respect to the origin of time \( t \). In this article, the relaxation kernels are taken in exponential form, since such kernels can be recommended for analysis of long-term deformation processes.

The ideal contact conditions are assumed on the layers surfaces, namely: components of the displacement and the components of stress in radial direction are continuous.

We consider isotropic materials, therefore, the components of the elastic tensors \( c_{ijkh} \) and relaxation kernel tensors \( d_{ijkh} \) in (2) have the form [3]:

\[
c_{ijkh} = \lambda \delta_{ij} \delta_{kh} + \mu(\delta_{ik} \delta_{jh} + \delta_{ih} \delta_{jk}),
\]

\[
d_{ijkh} = -(D_v(t) - \frac{1}{3} D_{sh}(t)) \delta_{ij} \delta_{kh} - \frac{1}{2} D_{sh}(t)(\delta_{ik} \delta_{jh} + \delta_{ih} \delta_{jk}).
\]

We denote here by \( D_{sh} \) and \( D_v \) the regular part of the shear and the bulk relaxation respectively, by \( \delta_{ij} \) Kronecker symbol. Suppose that the amplitude of a bulk relaxation kernel is proportional to the
amplitude of the shear relaxation kernel with a proportionality coefficient $k_s$ for each layer: 
$(D_s)_s = k_s(D_{sh})_s$, $k_s$ is a constant, $k_s > 0$ $(s = 1, 2)$. Further, $D_{sh}$ is denoted by $D$. Let us choose creep kernel of exponential type for each layer: $D_i = g_i e^{-\alpha_i t}$, where $\alpha_i, g_i$ are constants, $\alpha_i > 0$, $g_i > 0$, $(i = 1, 2)$. We apply the Laplace transform in the time domain to the equations (1)

$$
\tilde{f}(p) = \int_0^\infty f(t) e^{-pt} dt.
$$

The result is the system of elasticity theory with a complex parameter $p$, in which the matrices $A^{i,j}$ have the following form

$$
\begin{align*}
\tilde{A}_1^{11} &= \begin{pmatrix}
L - \frac{G}{p+\alpha} & 0 & 0 \\
0 & \mu - \frac{z}{p+\alpha} & 0 \\
0 & 0 & \mu - \frac{z}{p+\alpha}
\end{pmatrix}, & \quad \tilde{A}_1^{12} &= \begin{pmatrix}
\mu - \frac{z}{p+\alpha} & 0 & 0 \\
0 & L - \frac{G}{p+\alpha} & 0 \\
0 & 0 & \mu - \frac{z}{p+\alpha}
\end{pmatrix}, \\
\tilde{A}_1^{13} &= \begin{pmatrix}
\mu - \frac{z}{p+\alpha} & 0 & 0 \\
0 & \mu - \frac{z}{p+\alpha} & 0 \\
0 & 0 & L - \frac{G}{p+\alpha}
\end{pmatrix},
\end{align*}
\quad \tilde{A}_1^{21} = (\tilde{A}_1^{12})^T, \quad \tilde{A}_1^{12} = (\tilde{A}_1^{21})^T, \quad \tilde{A}_1^{13} = (\tilde{A}_1^{13})^T,
$$

$$
\begin{align*}
\tilde{A}_1^{22} &= \begin{pmatrix}
\mu - \frac{z}{p+\alpha} & 0 & 0 \\
0 & L - \frac{G}{p+\alpha} & 0 \\
0 & 0 & \mu - \frac{z}{p+\alpha}
\end{pmatrix}, & \quad \tilde{A}_1^{12} &= \begin{pmatrix}
0 & \lambda - \frac{j}{p+\alpha} & 0 \\
0 & \mu - \frac{z}{p+\alpha} & 0 \\
0 & 0 & \mu - \frac{z}{p+\alpha}
\end{pmatrix},
\end{align*}
\quad \tilde{A}_1^{23} = (\tilde{A}_1^{12})^T, \quad \tilde{A}_1^{12} = (\tilde{A}_1^{23})^T, \quad \tilde{A}_1^{23} = (\tilde{A}_1^{23})^T,
$$

here T means matrix transposition and $L = \lambda + 2\mu$, $G = g \left( k + \frac{2}{3} \right)$, $j = g \left( k - \frac{1}{3} \right)$, $z = g \frac{g}{2}$.

Let us consider a layered pipe made from elastic-creeping materials. Let the typical layer width is much less than the length scale of the body in radial direction, then the behavior of the material can be approximately described by averaged equations which correspond to a certain homogeneous material without layers. The elastic moduli of this material are referred to as effective.
For a three-dimensional hollow cylinder (pipe) \( P \equiv \{ r_0 < x_1^2 + x_2^2 < R^2, \ x_3 \in (0; z) \} \), we consider the system of equilibrium equations of the theory of elasticity [1] in Cartesian coordinates \( x_1, x_2, x_3 \) as follows

\[
\frac{\partial}{\partial x_j} \left( A^{ij} \left( \frac{\varphi(x)}{\varepsilon} \right) \frac{\partial \tilde{u}}{\partial x_i} \right) = \tilde{f} ,
\]

\((i, j = 1, 2, 3)\). This system is similar to (1). The dependence of the matrices \( A^{ij}(\varphi) \) on the scalar variable \( \varphi \) is assumed to be a periodic, \( \varphi = \varphi(x_1, x_2) \) is the smooth function, also vector \( \tilde{u} \) is the displacement vector, \( \tilde{f} \) is the vector of mass forces, \( \varepsilon \) is the small parameter, \( \varepsilon \in (0;1) \). The case of a pipe with a periodic layer structure corresponds to the next function \( \varphi = r = \sqrt{x_1^2 + x_2^2} \). Thin layers are located along the level surfaces of this function.

The set of matrices \( \hat{A}^{is} \) describing the “averaged” or “effective” properties of an elastic cylindrical body \( P \) is given by the formulas [1]

\[
\hat{A}^{is} = \langle B' \rangle^s \langle B^0 \rangle^{-1} \langle B' \rangle - \langle B'' \rangle ,
\]

where \( \langle B' \rangle^s \) is the matrix conjugate to \( \langle B' \rangle \), and \( \langle f \rangle \) means the period average of the function periodic in the variable \( v \): \( \langle f \rangle = \frac{1}{\omega} \int_0^\omega f(v)dv \), \( \omega \) is the period value. Further, we will denote as \( \varphi_k \) the derivative of the function \( \varphi \) with respect to the variable \( x_k \) \((k = 1, 2, 3)\), and also \( y = (x_1, x_2, x_3) \). The formula [10] contains the following matrices

\[
B^0_k(\varphi, y) = B^0 \left( \frac{\varphi}{\varepsilon}, y \right) = \left[ \varphi_i( y ) \varphi_k( y ) A^{ij} \left( \frac{\varphi}{\varepsilon} \right) \right]^{-1} ,
\]

\[
B'_k(\varphi, y) = B' \left( \frac{\varphi}{\varepsilon}, y \right) = \left[ \varphi_i( y ) \varphi_k( y ) A^{ij} \left( \frac{\varphi}{\varepsilon} \right) \right]^{-1} \varphi_p( y ) A^{ip} \left( \frac{\varphi}{\varepsilon} \right) ,
\]

\[
B''_k(\varphi, y) = B'' \left( \frac{\varphi}{\varepsilon}, y \right) = \varphi_i( y ) A^{ij} \left( \frac{\varphi}{\varepsilon} \right) \varphi_p( y ) A^{ip} \left( \frac{\varphi}{\varepsilon} \right) - A'' \left( \frac{\varphi}{\varepsilon} \right) .
\]

We have according to the formula (10):

\[
B^0_k(\varphi, y) = \frac{1}{\Delta} \begin{bmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix} .
\]

The elements of the matrix \( b_{ij} \) and the determinant \( \Delta \) are as follows:
\begin{equation}
\begin{align*}
\lambda_1 &= \frac{\epsilon}{\epsilon}, \\
\lambda_2 &= \frac{1 - \nu}{1 - \nu}, \\
\mu_1 &= \frac{\epsilon}{\epsilon}, \\
\mu_2 &= \frac{1 - \nu}{1 - \nu}, \\
D_1(t) &= [D_1(t), \xi \in [0; h]], \\
D_2(t) &= [D_2(t), \xi \in [1 - h; 1]].
\end{align*}
\end{equation}

\begin{equation}
\psi = \langle \psi \rangle = \int_0^1 \psi(\xi) d\xi
\end{equation}

and performing the operations on the formula (9), we obtain the matrix \( \hat{A} \) of effective coefficients. In packages of symbolic calculations, we can produce the necessary actions, including averaging. Then it is necessary to perform the inverse Laplace transform. To do this, each element of the resulting matrix must be decomposed into the simplest fractions. Further, the implementation of the inverse Laplace transform becomes obvious, since in our problem the inverse Laplace transform of the sum of the simplest fractions is the sum of the time-dependent exponential functions. That is, the task will be solved.

3. Conclusions

In this paper, a new method is proposed for determining the “effective” coefficients of the stress-strain state of a layered tube composed of elastic-creeping materials. The case of creep (relaxation) kernels in the form of decreasing exponential functions is considered. These are frequently used functions for describing long-term processes of changing displacements, stresses and deformations. Exponential functions as nuclei of creep kernels (relaxation) allow us to obtain analytical expressions for slowly varying characteristics (elastic and creeping) of a cylindrical hollow body made of composite material. Thus, it is possible to reduce the problem of stress-strain state to a numerical solution using large finite elements, the linear dimensions of which can be several times greater than the thickness of one layer. In this article, for brevity, we have given explicit expressions for only one of the matrices, namely \( B_0 \), which is used to obtain averaged characteristics. The calculation of other characteristics leads to very cumbersome expressions, the algorithm for obtaining which is obvious. Symbolic mathematics of known computer packages can be applied for the calculations.
References

[1] O. A. Oleynik, G. A. Iosif’y an and A. S. Shamaev, Mathematical problems in elasticity and homogenization, North-Holland: Elsevier, 1992.

[2] B. E. Pobedrya, Mechanics of composite materials, Moscow: MSU, 1984.

[3] A. A. Ilyushin and B. E. Pobedrya, Foundations of the mathematical theory of thermovisco-elasticity, Moscow: Nauka, 1970.

[4] Yu. N. Rabotnov, Elements of hereditary solid mechanics, Moscow: Mir, 1980.

[5] T. N. Bobyleva and A. S. Shamaev, Effective characteristics of a layered tube consisting of elastic-creeping materials, MATEC Web of Conf., vol. 251, p. 04039, 2018.

[6] T. N. Bobyleva and A. S. Shamaev, An efficient algorithm for calculating rheological parameters of layered soil media composed from elastic-creeping materials, Soil Mechanics and Foundation Engineering, vol. 54 (4), p.p. 224-230, 2017.