SCHRÖDINGER EQUATIONS WITH VANISHING POTENTIALS INVOLVING BREZIS-KAMIN TYPE PROBLEMS

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Abstract. We prove the existence of a bounded positive solution for the following stationary Schrödinger equation

\[-\Delta u + V(x)u = f(x, u), \quad x \in \mathbb{R}^n, \quad n \geq 3,
\]

where $V$ is a vanishing potential and $f$ has a sublinear growth at the origin (for example if $f(x,u)$ is a concave function near the origen). For this purpose we use a Brezis-Kamin argument included in [6]. In addition, if $f$ has a superlinear growth at infinity, besides the first solution, we obtain a second solution. For this we introduce an auxiliary equation which is variational, however new difficulties appear when handling the compactness. For instance, our approach can be applied for nonlinearities of the type $\rho(x)f(u)$ where $f$ is a concave-convex function and $\rho$ satisfies the (H) property introduced in [6]. We also note that we do not impose any integrability assumptions on the function $\rho$, which is imposed in most works.

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1. Introduction.

1.1. Motivation of our results. We study existence of positive solutions for the semilinear Schrödinger equations

\[-\Delta u + V(x)u = f(x, u) \text{ in } \mathbb{R}^n, \quad n \geq 3,\]  

(P)

where V is a continuous and nonnegative vanishing potential, that is, \(\lim_{|x| \to \infty} V(x) = 0\), and \(f(x, u)\) is a Carathéodory function. The main models of \(f(x, u)\) studied here are

I. \(\rho(x)u^q\)  
II. \(\lambda \rho(x)(u + 1)^p\)  
III. \(\lambda \rho(x)(u^q + u^p)\),

where \(0 < q < 1 < p < \frac{n+2}{n-2}\) and \(\rho\) satisfies the property (H) introduced by Brezis and Kamin [6]: a function \(\rho \in L^\infty_{\text{loc}}(\mathbb{R}^n), \rho \geq 0\), has the property (H) if the linear problem

\[-\Delta u = \rho \text{ in } \mathbb{R}^n\]  

(1)

has a bounded solution.

In the case of model I, in the celebrated paper [6], Brezis and Kamin proved that the sublinear problem

\[-\Delta u = \rho(x)u^q \text{ in } \mathbb{R}^n, \quad n \geq 3,\]  

(2)

has a bounded positive solution if and only if \(\rho\) has the property (H). One interesting detail is that the authors proved that the problem (2) has a bounded solution if and only if

\[U(x) := \frac{1}{|x|^{n-2}} \ast \rho \in L^\infty(\mathbb{R}^n),\]  

(3)

and thus \(U\) is a bounded solution of (1) (see [6, Lemma A.1]). Moreover, notice that when \(\rho\) is a radially symmetric function satisfying (3), then for all \(x \in \mathbb{R}^n\) with \(|x| = r\), we have \(U(x) = U(r)\) where

\[U(r) := \int_r^{+\infty} \left[ s^{1-n} \int_0^s t^{n-1} \rho(t) \, dt \right] ds,\]

which allows us to show, as observed in [6], that a bounded solution for problem (1) exists for potentials like

\[\rho(x) = \frac{1}{1 + |x|^\beta}, \quad \text{for any } \beta > 2,\]  

(4)

while no bounded solutions exist for \(\beta \leq 2\).

Considering the property (H) and inspired by [6] we give necessary and sufficient conditions for problem (P) in order to obtain a bounded and positive solution for a general class of nonlinearities \(f(x, s)\), which includes I, II and III. To obtain a second solution for models II and III, our interest is to use the variational method. Since the associated functional is not well defined, we will introduce an auxiliary equation, which according to our studies done in this work is well defined and variational. However, new difficulties appear in the study of the compactness of the functional, as it will be seen later. It is worth observing that one of the main novelties here is the fact that we do not impose any integrability assumption on the function \(\rho\), which according to our knowledge, is imposed in all works of Schrödinger equations involving nonlinearities of type II or III (see [3, 4, 14, 15, 17]).

Bahrouni et al [3] considered the problem (P) for certain types of concave nonlinearities, which are similar to type I:

\[f(x, u) = a(x)|u|^{q-1}u + h(x).\]
Among other hypotheses, it is assumed that $V$ is a non-symmetric potential and bounded from below by a positive constant outside a ball centered at zero, and that $a$ is an indefinite weight, which is bounded from above by a negative constant outside a ball or $a \in L^{\frac{2}{2-q}}(\mathbb{R}^n)$, $0 < q < 1$, while $h \neq 0$ is a nonnegative perturbation in $L^2(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Using variational methods, they proved the existence of two solutions with negative and positive energies, one of these solutions being nonnegative.

On the other hand, after a literature review, we have noticed that in the study of the existence of positive solutions for stationary Schrödinger equations involving concave–convex nonlinearities, in all works it is assumed that the potential $V$ is bounded from below by a positive constant near to infinity, and also include certain integrability restrictions on the concave term (see [8, 14, 15, 17]). For instance, in [8], Chabrowski and do Ó studied the problem (P) with
\[ f(x, u) = \lambda u^q + \mu u^p \]
where $0 < q < 1 < p$, $\lambda, \mu > 0$ are parameters and $V(x)$ is positive, locally Hölder continuous and bounded from below by a positive constant outside a ball. Using the method of successive approximations and the monotone method of sub and supersolutions, they proved the existence results depending on the range of parameters $\lambda$ and $\mu$. On the other hand, T-F Wu [17], by using variational methods, the author studied multiplicity of positive solutions for the problem (P) when
\[ f(x, u) = f_{\lambda}(x)u^{q-1} + g_{\mu}(x)u^{p-1} \]
where $V \equiv 1$, $1 < q < 2 < p < 2^*$ and the parameters $\lambda, \mu \geq 0$. Among other hypotheses, it is assumed that $f_{\lambda}(x) \in L^{\frac{p}{p-q}}(\mathbb{R}^n)$ is a sign-changing function depending on the parameter $\lambda$ and $g_{\mu}(x) = a(x) + \mu b(x)$, where $a, b \in C(\mathbb{R}^n)$, $b(x) \to 0$ and $a(x) \to 1$ as $|x| \to \infty$. In [15], Liu and Wang studied problem (P) for odd nonlinearities $f(x, u)$ involving a combination of concave and convex terms. A typical model they considered was
\[ f(x, u) = \lambda |u|^{q-2}u + \mu u + \nu |u|^{p-2}u, \]
where $1 < q < 2 < p < 2^*$ and $\lambda, \mu$ and $\nu$ are parameters. To deal with the concave term the authors considered potentials $V \in C(\mathbb{R}^n, \mathbb{R})$, $V(x) \geq 1$ and $\int_{\mathbb{R}^n} V(x)^{-1}dx < \infty$. Using variational methods they shown the existence and multiplicity of positive and nodal solutions for the problem (P), depending on the range of the parameters $\lambda, \mu$ and $\nu$.

Before stating our results, let us first observe that, as far as we know, this work is the first study of concave–convex problems in the literature involving vanishing potential and without any restrictions on the integrability of the function $\rho$.

1.2. The linear Schrödinger equation. As in [6], in our approach it is fundamental to study the existence of bounded solutions for the following linear Schrödinger equation
\[ -\Delta u + V(x)u = \rho(x) \text{ in } \mathbb{R}^n. \] (LS)
A simple computation shows that the function $u \in C^2(\mathbb{R}^n)$ given by
\[ u(x) = (|x|^\beta + 1)^{-\gamma}, \]
where $\beta > 2$ and $\gamma > 0$, is a classical solution for the linear Schrödinger equation (LS) for $V$ and $\rho$ given by:

$$V(x) = \frac{\beta^2 \gamma (\gamma + 1)|x|^{2(\beta - 1)}}{(|x|^\beta + 1)^2} \quad \text{and} \quad \rho(x) = \frac{\gamma \beta (\beta + n - 2)|x|^{\beta - 2}}{(|x|^\beta + 1)^{\gamma + 1}},$$

while the linear Schrödinger equation (LS) for

$$V(x) = \frac{1}{1 + |x|^\alpha} \quad \text{and} \quad \rho(x) = \frac{1}{1 + |x|^\beta},$$

does not have any bounded solution, for any $\alpha > \beta > 0$ and $\beta \leq 2$ (see Example 1). This means that both the existence and nonexistence of bounded positive solutions for (LS) is related to the growth of $V$ and $\rho$.

In this context we introduce a “compatibility” condition between $\rho$ and $V$ which allows us to obtain bounded positive solutions for the linear Schrödinger equation (LS).

**Definition 1.1.** Suppose that $\rho$ has the property (H) and let $U$ be the bounded solution of $-\Delta U = \rho(x)$ in $\mathbb{R}^n$. We say that $V$ and $\rho$ are compatible if

$$\frac{1}{|x|^{n-2}} * (VU) \in L^\infty(\mathbb{R}^n).$$

(5)

Notice that the statement (5) says that the product $VU$ also has the property (H).

**Remark 1.** Considering $\rho$ as in (4) it is possible to give several examples of $V$ that are compatible with $\rho$. In fact, let $U$ be a bounded solution of (1). From [6, Remark 3] we know that $U(x) \sim |x|^{-(\beta - 2)}$ at infinity. Thus, assuming that $V \in L^\infty_{\text{loc}}(\mathbb{R}^n)$ and $V(x) \sim |x|^{-\alpha}$ at infinity, we have that $\rho$ and $V$ are compatible if $\alpha + \beta > 4$ (see Remark 3).

Our first result establish that the compatibility between $V$ and $\rho$ is sufficient to guarantee the existence of a bounded and positive solution for the linear Schrödinger equation (LS).

**Theorem 1.2.** According to the Definition 1.1, if $\rho$ and $V$ are compatible, then the linear Schrödinger equation (LS) has a bounded solution.

It is not so clear for us that the converse of Theorem (1.2) holds, i.e., if compatibility is equivalent to the existence of bounded and positive solution for the linear Schrödinger equation (LS). In this direction, our next result shows a situation where the non compatibility between $V$ and $\rho$ implies nonexistence of bounded positive solutions of the problem (LS), but with the additional hypothesis that $V$ goes to zero more rapidly than $\rho$ at infinity.

**Theorem 1.3.** Let $\rho \in L^\infty(\mathbb{R}^n)$ be a positive potential such that does not satisfy property (H) and $V$ satisfying

$$\lim_{|x| \to \infty} \frac{V(x)}{\rho(x)} = 0.$$ 

(6)

Then, the linear Schrödinger equation (LS) does not have positive and bounded solutions.
1.3. Existence of a bounded solution for Equation (P). The second part of our study is about the existence of a bounded and positive solution for the nonlinear Schrödinger equation (P). We suppose that the potential $V$ is a continuous and nonnegative vanishing potential and the nonlinearity $f$ is a continuous function satisfying local hypotheses of sublinearity near the origin. More precisely, we suppose that there exists $\rho \in L^\infty_{\text{loc}}(\mathbb{R}^n)$, $\rho \neq 0$ and $\rho \geq 0$, satisfying (H) and a constant $a > 0$ such that:

(f1) There exists a constant $c > 0$ such that
$$f(x, u) \leq c\rho(x)(u + 1) \quad \forall x \in \mathbb{R}^n, u \in (0, a).$$

(f2) For a.e. $x \in \mathbb{R}^n$, the functions
(i) $u \mapsto f(x, u)$ is increasing on $(0, a)$ and;
(ii) $u \mapsto f(x, u)/u$ is decreasing on $(0, a)$.

(f3) There exists a function $f_0 : \mathbb{R} \to [0, +\infty)$ such that:
(i) $u \mapsto f_0(u)/u$ is non-increasing on $(0, a);
(ii) \lim_{u \to 0^+} f_0(u)/u = +\infty;
(iii) f(x, u) \geq \rho(x)f_0(u)$ for all $u \in [0, a)$;
(f4) There exists $\xi \in (0, a]$ such that
$$\frac{f(x, \xi)}{\xi} < \frac{\rho(x)}{\|U\|_{\infty}} \quad \text{a.e. } x \in \mathbb{R}^n,$$

where $U$ is the convolution given in (3).

Hypotheses (f1) and (f2) imply that $f$ is semi-linear near the origin and guarantee existence and uniqueness of positive solutions of problem (P) on balls with zero Dirichlet boundary condition. Hypotheses (f3) and (f4) allow us to apply the sub-supersolution method.

The classical sublinear nonlinearity $f(x, u) = \rho(x)u^q$ studied in [6], which correspond to our first model, verifies the hypotheses (f1) − (f4). Our other two main model nonlinearities
$$f(x, u) = \lambda \rho(x)(1 + u)^p \quad \text{and} \quad f(x, u) = \lambda \rho(x)(u^q + u^p),$$
also check the hypotheses (f1) − (f4) for $0 < q < 1 < p < 2^* - 1$ and $\lambda > 0$ sufficiently small.

In this context we have the following result.

Theorem 1.4. Suppose that the hypotheses (f1) − (f4) holds with $\rho$ and $V$ compatible. Then, problem (P) has a bounded solution. Moreover, the solution obtained is minimal.

As in [6], Theorem 1.4 is obtained via monotonicity method. In fact, we employ the sub-supersolutions technique to obtain a non-decreasing sequence of bounded solutions in balls $B_R$ that converges to a bounded solution of (P), when $R$ goes to infinity.

Remark 2. The solution in Theorem 1.4 is indeed a weak solution. Furthermore, since $V, \rho \in L^\infty_{\text{loc}}(\mathbb{R}^n)$, this solution has at least $C^{1,\alpha}$ regularity ([13, Theorem 3.13]). If we impose more restrictions on $V$ and $\rho$ (Hölder continuity, for example) it is possible to conclude that the solution is indeed a classical solution.

We also prove a converse of Theorem 1.4, when $f_0$ is a positive smooth function verifying the following additional assumptions:
The function $H(t) := t/f_0(t)$ is concave near the origin. In other words, the function $h(t) := H'(t)$ is non-increasing on $(0, a)$;

There exists $0 < \theta \leq 1$ such that

$$\theta H(t) \leq h(t)t \quad \text{for all} \quad t \in (0, a).$$

Note that the functions $f_0(t) = t^q$ and $f_0(t) = t^q + t^p$ satisfy the hypotheses $(f_5)$ and $(f_6)$, where $0 < q < 1 < p$.

The following converse of Theorem 1.4 holds.

**Theorem 1.5.** Assume that $(f_1) - (f_6)$ holds. If $(P)$ has a bounded solution, then $(LS)$ has a positive bounded solution.

Finally, assuming $\rho \in L^\infty(\mathbb{R}^n)$, $\rho \geq 0$, $\rho \neq 0$ such that

$$0 < \rho(x) \leq \frac{k}{1 + |x|^\alpha} \quad \text{in} \quad \mathbb{R}^n,$$

for constants $k > 0$ and $\beta > 2$ we will establish the existence of at least two solutions for two families of superlinear Schrödinger equations. We observe that $\rho$ is integrable only for $\beta > n$, but here we also consider $2 < \beta \leq n$.

**1.4. Two solutions involving nonlinearities of type II.** The first nonlinear Schrödinger equation such that we obtain two positive solutions is the following:

$$
\begin{cases}
-\Delta u + V(x)u = \lambda \rho(x)(u + 1)^p \quad \text{in} \quad \mathbb{R}^n, \\
u(0) > 0 \quad \text{in} \quad \mathbb{R}^n, \\
u(x) \to 0 \quad \text{as} \quad |x| \to \infty,
\end{cases}
$$

where $1 < p < 2^* - 1$ and $2^* := 2n/(n - 2)$, $n \geq 3$, is the critical Sobolev exponent. These problems on bounded domain and zero Dirichlet boundary condition was studied in [7, 11] when $\rho(x) \equiv 1$ and in [10] for variable coefficient $\rho(x)$. Our main result concerning Problem $(P_{\lambda,p})$ is the following.

**Theorem 1.6.** Assume that $\rho$ satisfies $(H_\rho)$ and $V$ is a nonnegative and continuous potential such that

$$\frac{a}{1 + |x|^\alpha} \leq V(x) \leq \frac{A}{1 + |x|^\alpha}, \quad \text{for all} \quad x \in \mathbb{R}^n,$$

for some constants $a, A > 0$, $\alpha \in (0, 2]$, with $\alpha + \beta > 4$. Then, there exists $\Lambda > 0$ such that problem $(P_{\lambda,p})$ has at least two positive solutions $u_{1,\lambda} < u_{2,\lambda}$ in $\mathbb{R}^n$, for any $\lambda \in (0, \Lambda)$. Furthermore

$$u_{1,\lambda}(x) \leq c_\lambda U(x) \quad \text{for all} \quad x \in \mathbb{R}^n,$$

where $c_\lambda \to 0$ as $\lambda \to 0$.

**1.5. Two solutions involving nonlinearities of type III.** The second family involves concave-convex nonlinearities:

$$
\begin{cases}
-\Delta u + V(x)u = \lambda \rho(x)(u^q + u^p) \quad \text{in} \quad \mathbb{R}^n, \\
u > 0 \quad \text{in} \quad \mathbb{R}^n, \\
u(x) \to 0 \quad \text{as} \quad |x| \to \infty,
\end{cases}
$$

where $1 < q < 2^* - 1$ and $2^* := 2n/(n - 2)$, $n \geq 3$, is the critical Sobolev exponent. These problems on bounded domain and zero Dirichlet boundary condition was studied in [7, 11] when $\rho(x) \equiv 1$ and in [10] for variable coefficient $\rho(x)$. Our main result concerning Problem $(P_{\lambda,p,q})$ is the following.
where \(0 < q < 1 < p < 2^*-1\). This type of problem on bounded domain was studied in [1], where the authors established existence and multiplicity results for the semilinear elliptic problem

\[
\begin{aligned}
-\Delta u &= \lambda u^q + u^p & \text{in } \Omega, \\
\lambda > 0 & & \text{in } \Omega, \\
\lambda = 0 & & \text{on } \partial \Omega.
\end{aligned}
\]  

They shown, in particular, that if \(\lambda \leq \frac{(n+2)}{(n-2)}\), there exists a constant \(\Lambda > 0\) such that (7) has at least two solutions for \(\lambda < \Lambda\), at least one for \(\lambda = \Lambda\) and no solution for \(\lambda > \Lambda\). This result of global multiplicity on bounded domain was extended in [10] to the case of nonlinearity with variable coefficient \(f_\lambda(x,u) = \lambda a(x)u^q + b(x)u^p\), where \(a(x)\) is restricted to be nonnegative and \(b(x)\) is allowed to change sign. See also [9].

The main result involving \((P_{\lambda,p,q})\) is the following:

**Theorem 1.7.** Assume that \(\rho\) satisfies \((H_\rho)\) and

\[
\frac{c_0}{1 + |x|^\beta} \leq \rho(x) \quad \text{for all } x \in \mathbb{R}^n,
\]

for some \(c_0 > 0\), and also assume \(V\) is a continuous potential verifying

\[
\frac{c_\mu}{1 + |x|^2} \leq V(x) \leq \frac{\mu}{1 + |x|^2},
\]

for all \(x \in \mathbb{R}^n\), \((H_V)\)

and some constant \(0 < c < 1\). Then, there exist \(\mu^*, \lambda^* > 0\) and \(q^* \in (0,1)\) such that \((P_{\lambda,p,q})\) has at least two positive solutions \(u_{1,\lambda} < u_{2,\lambda}\) in \(\mathbb{R}^n\), for any \(\mu \in (0,\mu^*)\), \(q \in (0,q^*)\) and \(\lambda \in (0,\lambda^*)\). Furthermore, for each \(\lambda \in (0,\lambda^*)\) there exists \(c_\lambda > 0\), independent of \(p\) and \(q\), such that

\[
u_{1,\lambda}(x) \leq c_\lambda U(x) \quad \text{for all } x \in \mathbb{R}^n,
\]

where \(c_\lambda \to 0\) as \(\lambda \to 0\).

Theorem 1.7 can be applied to many situations. Since our potential \(V\) depends on the parameter \(\mu\), in what follows we write \(V_\mu\) in place of \(V\). For example, as we will prove later, Theorem 1.7 applies in the case

\[
V_\mu(x) = \frac{\mu}{1 + |x|^2} \quad \text{and} \quad \rho(x) = \frac{a}{1 + |x|^\beta},
\]

for constants \(\mu, a > 0\) and \(\beta > 2\).

1.6. **The approach to get the second solution and some comments.** The existence of the first solution of Theorems 1.6 and 1.7 is a consequence of Theorem 1.4, due to the fact that both nonlinearities satisfy the conditions \((f_1) - (f_4)\), for \(\lambda > 0\) sufficiently small. The second solution is obtained by using variational methods and some properties of the first solution. It is well known that the main difficulty to use variational methods for problems on unbounded domains is the lack of compactness. In these two families of problems we consider here, besides the lack of compacity, we have an extra difficulty because it is not so clear that the energy functionals associated with \((P_{\lambda,p})\) and \((P_{\lambda,p,q})\) are well defined, which is related to the fact that \(\rho(x)\) is not necessarily an integrable function. To overcome this difficulty, we
consider the following auxiliary equation
\[
\begin{cases}
-\Delta w + V(x)w = g_{\lambda}(x, w) & \text{in } \mathbb{R}^n, \\
w > 0 & \text{in } \mathbb{R}^n, \\
w(x) \to 0 & \text{as } |x| \to \infty,
\end{cases}
\]
where \( g_{\lambda}(x, s) := \lambda \rho(x)(f(u_{1,\lambda}(x) + s) - f(u_{1,\lambda}(x))) \), \( f(s) = (s + 1)^p \) or \( f(s) = (s^q + s^p) \), \( s \geq 0 \), and \( u_{1,\lambda} \) is the first solution. This will allow us to obtain the second solution for \((P_{\lambda,p})\) and \((P_{\lambda,p,q})\) of the form \( u_{1,\lambda} + w \), where \( w \) is the solution for the auxiliary equation \((P_{g_{\lambda}})\).

We observed that for problem \((P_{\lambda,p,q})\), under the hypotheses on \( V \) and \( \rho \) given in Theorem 1.4, it is possible to prove that the energy functional associated to \((P_{g_{\lambda}})\) is well defined and verifies the mountain pass geometry, but the nonlinearity \( g_{\lambda} \) does not satisfy the classical Ambrosetti-Rabinowitz condition usually imposed in unbounded domain:
\[
\theta G_{\lambda}(x, s) \leq g_{\lambda}(x, s)s, \quad \text{for all } x \in \mathbb{R}^n \text{ and } s \geq 0, \quad (AR)
\]
which is usually required to prove the boundedness of Palais-Smale sequences. To overcome this difficult, we take advantage of the properties of the first solution \( u_{1,\lambda} \) (for details, see Section 4).

1.7. The outline of this paper. In Section 2 we study the existence and nonexistence of bounded solutions for the linear Schrödinger equation \((LS)\). We prove Theorems 1.2 and 1.3, and we present more examples related to the results. In Section 3 we prove some uniqueness result of bounded solutions for a nonlinear Schrödinger equation on bounded domains and we use it to prove Theorems 1.4 and 1.5. In Section 4 we formulate a variational framework for the auxiliary problem \((P_{g_{\lambda}})\) and we prove Theorem 1.6 and Theorem 1.7.

2. Linear Schrödinger equation. In this section, we establish the existence and nonexistence of bounded solutions for the Linear Schrödinger Equation \((LS)\).

2.1. Existence of bounded solution. Let us first prove a result that guarantees the compatibility of the pair \( \rho \) and \( V \) when they satisfies the hypotheses contained in both Theorem 1.6 and Theorem 1.7.

Lemma 2.1. Suppose that \( \rho \) satisfies \((H_{\rho})\) and the potential \( V \) satisfies
\[
\frac{c_1}{1 + |x|^{\alpha}} \leq V(x) \leq \frac{c_2}{1 + |x|^{\alpha}},
\]
for some constants \( c_1, c_2 > 0 \) and \( \alpha > 0 \) with \( \alpha + \beta > 4 \). Then \( \rho \) and \( V \) are compatible.

Proof. Let \( U \in L^\infty(\mathbb{R}^n) \) be the convolution given in \((3)\). Notice first that from \([6]\) there exists \( c > 0 \) such that
\[
U(x) \leq \frac{c}{1 + |x|^\beta - 2} \quad \text{for all } x \in \mathbb{R}^n.
\]
Thus, from the hypothesis on \( V \) we also have
\[
V(x)U(x) \leq \frac{c}{1 + |x|^\alpha + \beta - 2} \quad \text{for all } x \in \mathbb{R}^n,
\]
which implies
\[
\frac{c}{|x|^{n-2}} * [V(x)U(x)] \in L^\infty(\mathbb{R}^n),
\]
whenever \( \alpha + \beta > 4 \).

We now prove the existence of a bounded and positive solution for the linear Schrödinger equation (LS).

**Proof of Theorem 1.2.** Let \( u_R \) be a nonnegative solution of the problem

\[
(R) \quad \begin{cases} -\Delta u_R + V(x)u_R = \rho(x) & \text{in } B_R, \\ u_R = 0 & \text{on } \partial B_R. \end{cases}
\]

Since \( V \in L^\infty(B_R) \) and \( \rho(x) \geq 0, \ \rho \neq 0 \), it follows that \( u_R \geq 0 \) in \( B_R, \ u_R \neq 0 \). Note that

\[ -\Delta u_R \leq -\Delta u_R + V(x)u_R = \rho(x) = -\Delta U \text{ in } B_R, \]

where \( U \) is given in (3). Furthermore, \( u_R = 0 < U \) on \( \partial B_R \). Thus, by the classical maximum principle \( u_R \leq U \). Clearly if \( R' > R \) thus \( u_R \geq u_R \) in \( B_R \). In fact, let us denote by \( \bar{u}_R \) the null extension of \( u_R \) to \( B_{R'} \), then we have

\[ -\Delta (u_{R'} - \bar{u}_R)(x) + V(x)(u_{R'}(x) - \bar{u}_R(x)) \geq 0, \]

with \( u_{R'} - \bar{u}_R = u_{R'} - 0 \geq 0 \) on \( \partial B_R \). The maximum principle implies that \( u_{R'}(x) \geq u_R(x) \) in \( B_R \) (see [12, 16]). Taking \( v_R := U - u_R \), since \( u_R \) is a solution of \((R)\) and \( -\Delta U = \rho(x) \), we can write the equation

\[
(P_{V,R}) \quad \begin{cases} -\Delta v_R = V(x)u_R & \text{in } B_R, \\ v_R = U & \text{on } \partial B_R. \end{cases}
\]

Furthermore, \( v_R \leq U \) for all \( R > 0 \) and \( v_R \geq v_{R'} \) for \( R \leq R' \). We have the following representation formula to \( v_R \) (see [12, p. 56]):

\[ v_R(x) = c_1 \int_{\partial B_R} \frac{R^2 - |x|^2}{R|x - y|^n} U(y)dy + c_2 \int_{B_R} G_R(x, y)V(y)u_R(y)dy, \]

where \( G_R \) is the Green function in \( B_R \). Now, let \( U_V := \lim_{R \to \infty} u_R \). Using monotone convergence, we obtain

\[ \int_{\partial B_R} G_R(x, y)V(y)u_R(y)dy \to \int_{\mathbb{R}^n} \frac{V(y)U_V(y)}{|x - y|^{n-2}} dy \text{ as } R \to \infty. \]

Since \( |x - y| \geq |y| - |x| = R - |x| \) for any \( |y| = R \), it follows that

\[ \frac{1}{|x - y|^n} \leq \frac{1}{(R - |x|)^n}, \]

for large values of \( R \), which implies

\[ \int_{\partial B_R} \frac{R^2 - |x|^2}{R|x - y|^n} U(y)dy \leq \frac{1}{R^{n-1}} \int_{\partial B_R} \frac{(R^2 - |x|^2)R^{n-2}}{(R - |x|)^n} U(y)dy. \]

From [6, Lemma A5] and the last inequality we have

\[ \int_{\partial B_R} \frac{R^2 - |x|^2}{R|x - y|^n} U(y)dy \leq \frac{(R^2 - |x|^2)R^{n-2}}{(R - |x|)^n} \int_{\partial B_R} U(y)dy \to 0, \ \ R \to \infty. \]

Therefore, \( v := \lim_{R \to \infty} v_R \) is given by

\[ v(x) = c \int_{\mathbb{R}^n} \frac{V(y)U_V(y)}{|x - y|^{n-2}} dy =: \frac{c}{|x|^{n-2}} * (V(x)U_V(x)). \]
Using that \( U_V \leq U \) and the compatibility between \( V \) and \( \rho \), we get \( v \in L^\infty(\mathbb{R}^n) \).

From [6, Lemma A.1], the function \( v \) satisfies

\[- \Delta v = V(x)U_V(x) \text{ in } \mathbb{R}^n, \tag{8}\]

and

\[ v = \lim_{R \to \infty} v_R = \lim_{R \to \infty} (U - u_R) = U - U_V. \tag{9}\]

Thus, equation (8) we obtain

\[-\Delta(U - U_V) = -\Delta U + \Delta U_V = \rho(x) + \Delta U_V = V(x)U_V \text{ in } \mathbb{R}^n,\]

or, equivalently,

\[-\Delta U_V + V(x)U_V = \rho(x) \text{ in } \mathbb{R}^n.\]

Finally, the inequalities \( 0 \leq v \leq U \) implies

\[ \lim_{|x| \to \infty} v(x) = 0. \]

Therefore,

\[ \lim_{|x| \to \infty} U_V(x) = \lim_{|x| \to \infty} [U(x) - v(x)] = 0, \]

finishing the proof of our theorem. \( \square \)

As an application of our study we can construct many examples of linear Schrödinger equations (LS) which have at least one positive bounded solution.

**Remark 3.** Let \( \alpha > \gamma \geq 0 \) and \( \beta > 2 \) be constants and consider the problem

\[- \Delta u + \frac{|x|\gamma}{1 + |x|^\alpha} u = \frac{1}{1 + |x|^2} x \in \mathbb{R}^n. \tag{10}\]

By the argument above, the solution \( U_V \) can be written in the following integral form

\[ U_V(x) = U(x) - c \int_{\mathbb{R}^n} \frac{U_V(y)|y|\gamma}{(1 + |y|\alpha)|x - y|^{n-2}} dy, \]

where \( U \) is a bounded positive solution for (1). The function \( U_V \) is a solution if this integral above defines an \( L^\infty(\mathbb{R}^n) \) function. Thus, the natural question is: for which values of \( \alpha, \beta \) and \( \gamma \) does it occur? Note that \( V(x) \sim |x|^{-\alpha} \) at infinity and, as we know \( U(x) \sim |x|^{-\alpha-\gamma} \) at infinity - see [6, Remark 3]. It implies that \( \rho \) and \( V \) are compatible if \( \alpha + \beta > 4 + \gamma \). in this case, Problem (10) has a positive bounded solution.

We finish this section with a study about (LS) when \( \rho \) satisfies the hypothesis (\( H_\rho \)) and \( V \) satisfies (\( H^\alpha_V \)). More precisely, we prove that \( V \) and \( \rho \) are compatible according to the Definition 1.1 and the solution \( u \) from (LS) behaves like \( U \) at infinity, where \( U \) is the bounded solution of the Poisson equation (1), obtained by Brezis and Kamin in [6].

**Lemma 2.2.** Assume hypotheses (\( H^\alpha_V \)), (\( H_\rho \)) and (\( H'_\rho \)), and let \( U \) be the solution of (1). Then the solution \( U_\mu \) of (LS) satisfies

\[(1 - \epsilon \mu)U(x) \leq U_\mu(x) \leq U(x), \]

for all \( x \in \mathbb{R}^n \) and some constants \( \epsilon > 0 \). In addition, \( U_\mu \to U \) as \( \mu \to 0 \) uniformly in \( x \).
Proof. Note that \((H_2)\) and \((H_\rho)\) imply that \(V\) and \(\rho\) are compatibles for any \(\mu > 0\). The solution \(U_\mu\) of \((LS)\) satisfies
\[
U_\mu(x) = U(x) - c \int_{\mathbb{R}^n} \frac{U_\mu(y)V(y)}{|x-y|^{n-2}} dy
\]
or equivalently
\[
\frac{U_\mu(x)}{U(x)} = 1 - c \frac{\int_{\mathbb{R}^n} U_\mu(y)V(y)}{|x-y|^{n-2}} dy.
\]
The hypothesis \((H_\rho)\) implies that there exists \(c_1 > 0\) such that
\[
\frac{U(x)}{1+|x|^2} \leq \frac{c_1}{1+|x|^2}, \quad \text{for all } x \in \mathbb{R}^n.
\]
Then, we have
\[
\int_{\mathbb{R}^n} \frac{U_\mu(y)V(y)}{|x-y|^{n-2}} dy \leq \int_{\mathbb{R}^n} \frac{U(y)}{|x-y|^{n-2}} \frac{\mu}{(1+|y|^2)} dy \leq \mu \int_{\mathbb{R}^n} \frac{1}{(1+|y|^2)|x-y|^{n-2}} dy = c\mu U_0(x),
\]
where \(U_0\) be the unique bounded positive solution of
\[
-\Delta u = \frac{1}{1+|x|^2}, \quad \text{in } \mathbb{R}^n,
\]
which satisfies \(U_0(x) \to 0\) as \(|x| \to \infty\). Since \((H_\rho')\) holds it is easy to see that \(c_0 U_0 \leq U\), and then
\[
\frac{c}{U(x)} \int_{\mathbb{R}^n} \frac{U_\mu(y)V(y)}{|x-y|^{n-2}} dy \leq c' \mu,
\]
for some constant \(c' > 0\). Therefore
\[
\frac{U_\mu(x)}{U(x)} \geq (1 - c\mu),
\]
for some constant \(c > 0\). \(\square\)

2.2. Nonexistence of bounded solutions. In this section we prove Theorem 1.3. We begin by proving some generalizations of results from [6, Appendix I] related to the equation
\[
-\Delta u = f \quad \text{in } \mathbb{R}^n,
\]
by supposing that \(f \in L^\infty_{\text{loc}}(\mathbb{R}^n)\) is nonnegative only outside of some ball centered at the origin.

Lemma 2.3. Suppose that there exists a constant \(M > 0\) such that
\[
f(x) \geq 0 \quad \text{a.e. in } |x| \geq M,
\]
and that \(f\) is not identically zero, then equation (12) has a bounded solution iff
\[
\frac{1}{|x|^{n-2}} * f \in L^\infty(\mathbb{R}^n).
\]
Proof. Let \( u_R \) be the solution of
\[
\begin{aligned}
-\Delta u_R &= f(x) \text{ in } B_R, \\
u_R &= 0 \text{ on } \partial B_R.
\end{aligned}
\] (14)
For any \( R' > R > M, R \) sufficiently large we have \( u_R \neq 0 \). Furthermore
\[
\begin{aligned}
-\Delta (u_{R'} - u_R) &= \chi_{B_{R'} \setminus B_R}(x)f(x) \geq 0 \text{ in } B_{R'}, \\
u_{R'} - u_R &= 0 \text{ on } \partial B_{R'}.
\end{aligned}
\]
By the classical maximum principle \( u_R \) is a nondecreasing sequence of functions (bounded from below by \( u_M \) which can change sign!). Moreover \( u_R \) is given by
\[
\begin{aligned}
u_R(x) &= \int_{B_R} G_R(x,y)f(y)dy
\end{aligned}
\] (15)
where \( G_R \) is the Green function to \( B_R \) and zero boundary condition. Let \( u_\infty(x) = \lim_{R \to \infty} u_R(x) \) (possibly \( +\infty \)).

Since
\[
|G_R(x,y)f(y)| \leq \left( \frac{2n-2c}{M^{n-2}} + \Gamma(|x-y|) \right) \|f\|_{L^\infty(B_M)} \in L^1(B_M)
\]
and
\[
G_R(x,y)f(y) \to \frac{c}{|x-y|^{n-2}}f(y) \quad \text{a.e. } y \in B_M,
\]
by dominated convergence theorem, we have
\[
\int_{B_M} G_R(x,y)f(y)dy \to \int_{B_M} \frac{c}{|x-y|^{n-2}}f(y)dy, \quad \text{as } R \to \infty.
\] (16)
On the other hand, by monotone convergence theorem
\[
\int_{B_M \setminus B_M} G_R(x,y)f(y)dy \to \int_{\mathbb{R}^n \setminus B_M} \frac{c}{|x-y|^{n-2}}f(y)dy
\] (17)
as \( R \to \infty \) (possibly \( +\infty \)). From (16) and (17), we conclude that
\[
u_\infty(x) = c \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}}f(y)dy = \frac{c}{|x|^{n-2}} * f \quad \text{possibly } +\infty.
\] (18)
As in [6], there are only two possibilities \( u_\infty(x) = +\infty \) for all \( x \) or \( u_\infty(x) < +\infty \) for all \( x \). The rest of the proof is now similar to the proof of Lemma A.1 in [6].

Now we use Lemma 2.3 to prove the Theorem 1.3.

Proof of Theorem 1.3. Suppose that \( \rho \in L^\infty(\mathbb{R}^n) \) is a positive potential which does not satisfy property (H) and assume (6). Suppose by contradiction that there exists a bounded and positive solution \( u \) of (LS). Thus
\[
-\Delta u \geq f(x)
\]
where \( f(x) := \rho(x) - V(x)u \| u \|_{L^\infty}, \ x \in \mathbb{R}^n \). By assumption (6), it follows that \( f(x) \geq 0 \) for all \( |x| > M \) and some constant \( M > 0 \). Thus, the convolution \( u_\infty \) in (18) satisfies \( u_\infty \leq u \in L^\infty(\mathbb{R}^n) \), so it is a bounded solution for problem (12). On the other hand, since
\[
f(y) \geq \frac{\rho(y)}{2} \quad \text{for all } |y| \geq M,
\]
we have
\[ \int_{|y| \geq M} \frac{1}{|x-y|^{n-2}} f(y) dy \geq \frac{1}{2} \int_{|y| \geq M} \frac{1}{|x-y|^{n-2}} \rho(y) dy = +\infty. \]
Thus \( u_{\infty}(x) = +\infty \) for all \( x \), which is a contradiction.

**Example 1.** As an application of Theorem 1.3, if \( \beta < \alpha \) and \( \beta \leq 2 \), then the linear Schrödinger equation
\[ -\Delta u + \frac{1}{1+|x|^\alpha} u = \frac{1}{1+|x|^\beta} \text{ in } \mathbb{R}^n, \tag{19} \]
has no bounded positive solution.

3. Bounded solutions for nonlinear Schrödinger equation. In this section, inspired by [6], we establish the existence of a positive and bounded solution for the nonlinear Schrödinger equation
\[ -\Delta u + V(x)u = f(x,u) \text{ in } \mathbb{R}^n, \tag{P} \]
where \( V \) is a nonnegative potential and the nonlinearity \( f \) satisfies all the hypotheses \((f_1) - (f_4)\). More precisely, we will prove Theorem 1.4 which is crucial in the next section. For this purpose we need some uniqueness result for Problem \((P)\) on a bounded domain with a Dirichlet boundary condition.

3.1. Krasnosel’skii type uniqueness result. Consider the Dirichlet problem
\[
\begin{cases}
-\Delta u + V(x)u = f(x,u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{P_\Omega}
\]
where \( \Omega \) is a smooth bounded domain. Our aim is to establish an uniqueness result of solutions with small \( L^\infty \)-norm for the problem \((P_\Omega)\).

**Lemma 3.1.** Assume that function the \( f \) satisfies \((f_1) - (f_2)\). Then problem \((P_\Omega)\) has a unique positive and bounded solution with \( L^\infty \)-norm smaller than \( a \).

**Proof.** As in [5], suppose that \( u_1 \) and \( u_2 \) are two positive solutions of \((P_\Omega)\) with \( L^\infty \)-norm smaller than \( a \). Define
\[ \Lambda := \{ t \in [0,1] : tu_1 \leq u_2 \leq a \text{ on } \Omega \}. \]
We observe that \( 0 \in \Lambda \). Furthermore, if \( 0 < \epsilon_0 \in \Lambda \), then for every \( \epsilon \in (0,\epsilon_0) \) we have \( \epsilon \in \Lambda \). Hence, \( t_0 u_1 \leq u_2 \) on \( \Omega \) where \( t_0 := \max \Lambda \). We claim that \( 1 \in \Lambda \). In fact, assume by contradiction that \( t_0 < 1 \). The idea is to show the existence of some \( \epsilon > 0 \) such that \( (t_0 + \epsilon) u_1 \leq u_2 \), which contradicts the choice of \( t_0 \). For this purpose we will use the maximum principle. Since \( f \) is increasing
\[ -\Delta (u_2 - t_0 u_1) + V(x)(u_2 - t_0 u_1) = f(x,u_2) - t_0 f(x,u_1) \geq f(x,t_0 u_1) - t_0 f(x,u_1) \geq 0, \]
where the last inequality occurs because \( t_0 < 1 \) and \( u \mapsto f(x,u)/u \) is decreasing on \((a, +\infty)\). This implies
\[ (i) \ u_2 - t_0 u_1 \equiv 0, \text{ or} \]
\[ (ii) \ u_2 - t_0 u_1 > 0 \text{ in } \Omega \text{ and } \frac{\partial(u_2 - t_0 u_1)}{\partial \nu} < 0 \text{ on } \partial\Omega. \]
If (i) occurs, we must have $f(x, t_0 u_1) = t_0 f(x, u_1)$ which implies

$$\frac{f(x, t_0 u_1)}{t_0 u_1} = \frac{f(x, u_1)}{u_1}$$

with $t_0 < 1$. From our hypothesis, this is impossible. If (ii) occurs, there exists $\epsilon > 0$ such that $\epsilon u_1 \leq u_2 - t_0 u_1$ on $\Omega$, or equivalently $(t_0 + \epsilon) u_1 \leq u_2$ on $\Omega$, which is a contradiction.

Now we are ready to prove Theorem 1.4.

3.2. Proof of Theorem 1.4. Let

$$BR = \{x \in \mathbb{R}^n : |x| < R\}$$

and $u_R$ be the unique solution of the following Dirichlet problem on a ball which is given by Lemma 3.1:

$$(P_R) \begin{cases} -\Delta u_R + V(x) u_R = f(x, u_R) & \text{in } BR, \\
 u_R = 0 & \text{on } \partial BR. \end{cases}$$

We claim that the sequence $u_R$ is increasing with $R$. In fact, if $R' > R$ then $u_{R'}$ is a supersolution for Problem $(P_R)$. Thus, we will construct a subsolution $\underline{u}$ for $Problem (P_R)$ with $\underline{u} \leq u_{R'}$. Indeed, in view of $(f_3)$, for $\underline{u}$ we may take $\epsilon \varphi_1$ where $\varphi_1$ satisfies

$$\begin{cases} -\Delta \varphi_1 + V(x) \varphi_1 = \lambda \rho \varphi_1 & \text{in } BR, \\
 \varphi_1 = 0 & \text{on } \partial BR, \end{cases}$$

and $\epsilon$ sufficiently small such that $\lambda_1 < f_0(\epsilon \|\varphi_1\|_{\infty})/\epsilon \|\varphi_1\|_{\infty}$. This implies that there is a solution $u$ for the Problem $(P_R)$ between $\underline{u}$ and $u_{R'}$. Since the unique solution is $u_R$ it follows that $u_R < u_{R'}$ in $BR$. We now prove that the sequence $u_R$ remains bounded as $R \to +\infty$. Consider $UV$ be a bounded solution of $(LS)$. Notice that

$$-\Delta CU + V(x)CU \geq f(x, CU)$$

if, and only if,

$$C \rho(x) \geq f(x, CU).$$

Since $f$ is increasing, the last inequality occurs if

$$C \rho(x) \geq f(x, C\|U\|_{\infty})$$

or

$$\frac{\rho(x)}{\|U\|_{\infty}} \geq \frac{f(x, C\|U\|_{\infty})}{C\|U\|_{\infty}}.$$
or equivalently
\[ u_R(x) = \int_{\mathbb{R}^n} G_R(x, y) f(y, u_R(y)) \chi_{B_R}(y) dy - \int_{\mathbb{R}^n} G_R(x, y) V(y) u_R(y) \chi_{B_R}(y) dy, \]
where \( \chi_{B_R} \) denotes the characteristic function of \( B_R \). Now, given \( x \in \mathbb{R}^n \), let us study the convergence of these two integrals above as \( R \to \infty \). Once that \( u_R(y) \to u(y) \) for a.e. \( y \in B_R \) we have
\[ G_R(x, y)V(y)u_R(y)\chi_{B_R}(y) \to \frac{cV(y)u(y)}{|x - y|^{n-2}} \text{ for a.e. } y \in \mathbb{R}^n, \]
as \( R \to +\infty \) and since \( u_R \) is a nondecreasing sequence on \( R \), we have by monotone convergence
\[ \int_{\mathbb{R}^n} G_R(x, y)V(y)u_R(y)\chi_{B_R}(y) dy \to c \int_{\mathbb{R}^n} \frac{V(y)u(y)}{|x - y|^{n-2}} dy \]
as \( R \to +\infty \). Note that \( u \leq CU \) with the fact that \( V \) and \( \rho \) are compatible, imply
\[ \int_{\mathbb{R}^n} \frac{V(y)u(y)}{|x - y|^{n-2}} dy \leq C \int_{\mathbb{R}^n} \frac{V(y)U(y)}{|x - y|^{n-2}} dy \in L^\infty(\mathbb{R}^n). \]
On the other hand, for each \( x \in \mathbb{R}^n \) we have
\[ G_R(x, y)f(y, u_R(y))\chi_{B_R}(y) \to \frac{cf(y, u(y))}{|x - y|^{n-2}} \text{ for a.e. } y \in \mathbb{R}^n, \]
as \( R \to \infty \). Since \( f(y, \cdot) \) is nondecreasing, for \( R < R' \) we have
\[ f(y, u_R(y)) \leq f(y, u_{R'}(y)), \text{ for all } y \in B_R. \]
Thus, by monotone convergence we also have
\[ \int_{\mathbb{R}^n} G_R(x, y)f(y, u_R(y))\chi_{B_R}(y) dy \to c \int_{\mathbb{R}^n} \frac{f(y, u(y))}{|x - y|^{n-2}} dy \]
as \( R \to +\infty \). Therefore, from (23) and (24) we have
\[ u(x) = c \int_{\mathbb{R}^n} \frac{f(y, u(y)) - V(y)u(y)}{|x - y|^{n-2}} dy, \]
which implies
\[ -\Delta u + V(x)u = f(x, u) \text{ in } \mathbb{R}^n, \]
with \( u \leq CU \) so that \( u \in L^\infty(\mathbb{R}^n) \). Clearly \( u \) is the minimal solution of (P). \( \square \)

3.3. Proof of Theorem 1.5. Suppose that \( u \) is a bounded positive solution of (P) and denote
\[ v = \frac{1}{\theta}H(u). \]
Then by (f3) and (f5) it follows that
\[ -\Delta v + V(x)v = -H''(u)\theta^{-1}|\nabla u|^2 + V(x)v - h(u)\theta^{-1}\Delta u \]
\[ \geq V(x)v - h(u)\theta^{-1} \Delta u \]
\[ \geq \theta^{-1} \left( \frac{H(u)}{u} - h(u) \right) V(x)u + h(u)\theta^{-1} f(x, u). \]
Using hypothesis (f3) (part (iii)), we obtain
\[ h(u)\theta^{-1} f(x, u) \geq \rho(x)f_0(u)h(u)\theta^{-1}. \]
This inequality and \((f_6)\) implies
\[-\Delta v + V(x)v \geq \rho(x)f_0(u)h(u)\theta^{-1} = \frac{h(u)u}{\partial H(u)}\rho(x) \geq \rho(x).\]

Finally, note that the solution \(w\) of the problem
\[
\begin{cases}
-\Delta w_R + V(x)w_R = \rho & \text{in } B_R, \\
w_R = 0 & \text{on } \partial B_R,
\end{cases}
\]
satisfies \(w_R \leq v\) and \(w_R\) increases as \(R \to \infty\) to a bounded solution of \((LS)\).

\(\square\)

4. Existence of two solutions. This section is devoted to the proof of Theorems 1.6 and 1.7. Note that both nonlinearities \(f(x,s) = \lambda \rho(x)(s^q + s^p)\) and \(f(x,s) = \lambda \rho(x)(1+s)^p\), where \(0 < q < 1 < p < 2^* - 1\), there exists \(a > 0\) such that \((f_1) - (f_3)\) are satisfied for \(s \in (0,a)\), with \(a\) independent on \(q\) and \(p\). Furthermore, there exists a constant \(\lambda^* > 0\) such that the condition \((f_3)\) is verified for \(\lambda \in (0,\lambda^*)\) on the interval \((0,a)\). It is easily seen that Lemma 2.1 guarantees the compatibility between \(\rho\) and \(V\) under the hypotheses of Theorems 1.6 and 1.7. Thus, applying Theorem 1.4 there exists a bounded and positive solution for both problems \((P_{\lambda,p})\) and \((P_{\lambda,p,q})\), which we will denote by \(u_{1,\lambda}\).

The following elementary inequalities will be useful for our purposes:
\[
(a + s)^r - a^r \leq \begin{cases} r(a + s)^{r-1}s, & \text{for all } s \geq 0, a > 0 \text{ if } r \geq 1; \\ ra^{r-1}s, & \text{for all } s \geq 0, a > 0 \text{ if } 0 < r \leq 1. \end{cases} \tag{EI}
\]

To obtain a second solution, we define the function \(g_\lambda : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}\) by
\[
g_\lambda(x,s) = f(x,u_{1,\lambda} + s) - f(x,u_{1,\lambda}),
\]
and consider the auxiliary problem
\[
\begin{cases}
-\Delta w + V(x)w = g_\lambda(x,w) & \text{in } \mathbb{R}^n, \\
w > 0 & \text{in } \mathbb{R}^n, \\
w(x) \to 0 & \text{as } |x| \to \infty. 
\end{cases}
\tag{Pg_\lambda}
\]

It is easy to see that if \(w\) is solution of \((Pg_\lambda)\), then \(u_{1,\lambda} + w\) is a solution of Problem \((P_{\lambda,p})\) (or \((P_{\lambda,p,q})\)).

As usual, let us denote by \(H^1(\mathbb{R}^n)\) the standard Sobolev space, which is the completion of \(C_0^\infty(\mathbb{R}^n)\) with respect to the norm \(\|u\|_{L^1} = \|\nabla u\|_2 + \|u\|_2\) and by \(H\) the following subspace of \(H^1(\mathbb{R}^n)\)
\[
H = \{u \in H^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} V(x)u^2dx < \infty\},
\]
dowered with the inner product
\[
\langle u, v \rangle = \int_{\mathbb{R}^n} (\nabla u \nabla v + V(x)uv)dx
\]
and its correspondent norm \(\|u\| = \langle u, u \rangle^{1/2}\). Using the fact that \(V\) is bounded, we obtain
\[
\|u\| \leq C\|u\|_{L^1}, \quad \text{for all } u \in H^1(\mathbb{R}^n).
\]
On the other hand, since \(V\) is vanishing, the norms \(\|\cdot\|\) and \(\|\cdot\|_{H^1}\) are not equivalent.
The energy functional associated with problem \( \text{(P}_{g_\lambda} \text{)} \) is
\[
J_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^n} (|\nabla u|^2 + V(x)u^2)\,dx - \int_{\mathbb{R}^n} G_\lambda(x,u)\,dx,
\]
where \( G_\lambda(x,u) = \int_0^u g_\lambda(x,t)\,dt \) and \( u \in H^1(\mathbb{R}^n) \). In the next section we prove that the auxiliary functional \( J_\lambda \) is well defined. In fact, \( J_\lambda \) is of class \( C^1(H^1(\mathbb{R}^n), \mathbb{R}) \) and has the mountain pass geometry.

In order to prove Theorem 1.6 and 1.7 we will need some embedding result. Denote by \( L^p_\rho \) the weighted space of measurable functions \( u : \mathbb{R}^n \to \mathbb{R} \) such that
\[
|u|_{r,\rho} = \left[ \int_{\mathbb{R}^n} \rho(x)|u|^r\,dx \right]^{1/r} < \infty.
\]
The following embedding result was proved in [2].

**Theorem 4.1.** Under the hypotheses \( (H_\rho) \) and \( (H_\varphi) \), the embedding \( \mathcal{H} \subset L^{p+1}_\rho \) is continuous provided that \( 1 \leq p \leq 2^* - 1 \), i.e., there exists \( C > 0 \) such that
\[
|u|_{p+1,\rho} \leq C\|u\|, \quad \text{for all } u \in \mathcal{H}.
\]
Furthermore, the embedding of \( \mathcal{H} \) into \( L^{p+1}_\rho \) is compact when \( 1 < p < 2^* - 1 \).

4.1. **Second solution of \( \text{(P}_{\lambda,\rho} \text{)} \).** Since \( p > 1 \), from (EI) it follows that
\[
(u_{1,\lambda}(x) + s + 1)^p - (u_{1,\lambda}(x) + 1)^p \leq ps(u_{1,\lambda}(x) + 1 + s)^{p-1}.
\]
Using that \( u_{1,\lambda} \in L^\infty(\mathbb{R}^n) \) we obtain the following growth estimate
\[
|(u_{1,\lambda}(x) + s + 1)^p - (u_{1,\lambda}(x) + 1)^p| \leq C(s + s^p), \quad (26)
\]
for some constant \( C > 0 \). On the other hand, since \( 0 < \alpha \leq 2 < \beta \) we have
\[
\delta = \inf_{x \in \mathbb{R}^n} \frac{1 + |x|^{\beta}}{1 + |x|^{\alpha}} > 0. \quad (27)
\]
Thus, there exists a constant \( c > 0 \) such that
\[
\rho(x) \leq cV(x), \quad \text{for all } x \in \mathbb{R}^n. \quad (28)
\]
From (26) and (28) we obtain
\[
\int_{\mathbb{R}^n} G_\lambda(x,u)\,dx \leq c\lambda \int_{\mathbb{R}^n} (V(x)u^2 + \frac{\rho(x)}{p+1}|u|^{p+1})\,dx. \quad (29)
\]
Therefore, the energy functional \( J_\lambda \) is well defined on \( \mathcal{H} \) and critical points of \( J_\lambda \) are weak solutions for \( \text{(P}_{g_\lambda} \text{)} \).

We establish the existence of a second solution as follows. From inequality (29), we have
\[
J_\lambda(u) \geq \frac{1}{2} \int_{\mathbb{R}^n} (|\nabla u|^2 + (1 - C\lambda)V(x)u^2)\,dx - \frac{C\lambda}{p+1} \int_{\mathbb{R}^n} \rho(x)|u|^{p+1}\,dx.
\]
This fact together with the Theorem 4.1, we obtain the following estimate
\[
J_\lambda(u) \geq \frac{1}{4}\|u\|^2 - \frac{C\lambda}{p+1}\|u\|^{p+1}.
\]
On the other hand, we can easily prove that there exists \( t_0 > 0 \) such that
\[
G_\lambda(x,t) \geq \lambda\rho(x)t^{p+1},
\]
for a.e. \( x \in \mathbb{R}^n \) and all \( t > t_0 \). Then, taking a smooth and positive function \( \varphi \) with compact support in \( \mathbb{R}^n \) we can prove that \( J_\lambda(t\varphi) \to -\infty \) as \( t \to +\infty \).
Now let us prove that \( J_\lambda \) satisfies the (PS) condition. Since our nonlinearity \( g_\lambda \) does not satisfy the classical Ambrosetti-Rabinowitz condition, we will introduce an appropriate argument to prove the boundedness of (PS) sequences. In fact, let us define
\[
g(x, s) = f(u_{1, \lambda} + s) - f(u_{1, \lambda}),
\]
where \( f(s) = (s + 1)^p \), and notice that for \( 2 < \theta < p + 1 \) holds
\[
g(x, s) - \theta G(x, s) = [(u_{1, \lambda} + 1 + s)^p - (u_{1, \lambda} + 1)^p] s + \theta (u_{1, \lambda} + 1)^p s - \frac{\theta}{p + 1} [(u_{1, \lambda} + 1 + s)^{p+1} - (u_{1, \lambda} + 1)^{p+1}].
\]
Using (EI), we have
\[
g(x, s) - \theta G(x, s) \geq -\theta s [(u_{1, \lambda} + 1 + s)^p + (u_{1, \lambda} + 1)^p].
\]
and
\[
g(x, s) - \theta G(x, s) \geq -\theta p(u_{1, \lambda} + 1 + s)^{p-1} s^2,
\]
for all \( s \geq 0 \) and \( x \in \mathbb{R}^n \). Note that it is not difficult to prove that for some \( s_0 > 0 \) independent on \( x \), we have
\[
g(x, s) - \theta G(x, s) \geq 0 \quad \text{for all} \quad s \geq s_0, \; x \in \mathbb{R}^n.
\]
On the other hand, from (30) there exists a constant \( c > 0 \) such that
\[
g(x, s) - \theta G(x, s) \geq -cs^2 \quad \text{for all} \quad 0 \leq s \leq s_0.
\]
Therefore,
\[
g(x, s) - \theta G(x, s) \geq -cs^2 \quad \text{for all} \quad s \geq 0.
\]
Now, let \( (u_n) \) be a (PS) sequence. Then, using (31)
\[
c + o_n(1) = J_\lambda(u_n) - \frac{1}{\theta} J'_\lambda(u_n) u_n
\]
\[
\geq \left( \frac{1}{2} - \frac{1}{\theta} \right) \int_{\mathbb{R}^n} (|\nabla u_n|^2 + V(x) u_n^2) dx
\]
\[
+ \lambda \int_{\mathbb{R}^n} \rho(x) \left[ \frac{1}{\theta} g_\lambda(x, u_n) u_n - G_\lambda(x, u_n) \right] dx
\]
\[
\geq \left( \frac{1}{2} - \frac{1}{\theta} \right) \int_{\mathbb{R}^n} (|\nabla u_n|^2 + V(x) u_n^2) dx - c\lambda \int_{\mathbb{R}^n} \rho(x) u_n^2 dx.
\]
Using inequality (28) we get
\[
c + o_n(1) \geq \left[ \left( \frac{1}{2} - \frac{1}{\theta} \right) - c\lambda \right] \int_{\mathbb{R}^n} (|\nabla u_n|^2 + V(x) u_n^2).
\]
Since \( 2 < \theta < p + 1 \) we can take \( \lambda > 0 \) small enough such that
\[
\left[ \left( \frac{1}{2} - \frac{1}{\theta} \right) - c\lambda \right] > 0,
\]
and then \( (u_n) \) is bounded in \( \mathcal{H} \). Since \( \mathcal{H} \) is compactly embedding in \( L^{p+1}_p \), the Mountain Pass Theorem can thus be applied to obtain a solution \( w \) of Problem \( (P_{g_\lambda}) \).
4.2. Second solution for \((P_{\lambda,p,q})\). The equation \((P_{\lambda,p,q})\) is the Euler-Lagrange equation formally associated with a functional given by

\[ I_{\lambda}(u) = \frac{1}{2} \|u\|^2 - \lambda \int_{\mathbb{R}^n} \rho(x) \left( \frac{u^{q+1}}{q+1} + \frac{u^{p+1}}{p+1} \right) dx. \]

However, since \(q + 1 < 2\), it is not so clear that the integral

\[ \int_{\mathbb{R}^n} \rho(x) u^{q+1} dx, \]

which appears in the definition of \(I_{\lambda}\), is finite for \(u \in \mathcal{H}\). This fact motivates us to work with the auxiliary functional \(J_{\lambda}\). To prove that \(J_{\lambda}\) is well defined on the space \(\mathcal{H}\) we note that the following inequality holds

\[ |(u_{1,\lambda} + s)^q + (u_{1,\lambda} + s)^p - u_{1,\lambda}^q - u_{1,\lambda}^p| \leq 2qu_{1,\lambda}^{q-1} s + cs^p, \quad (32) \]

for some constant \(c > 0\). In fact, since \(0 < q < 1 < p\) using (EI) twice, we obtain

\[ (u_{1,\lambda} + s)^q + (u_{1,\lambda} + s)^p - u_{1,\lambda}^q - u_{1,\lambda}^p \leq qu_{1,\lambda}^{q-1} s + p(u_{1,\lambda} + s)^{p-1} s. \]

Let \(\epsilon > 0\), since \(q - 1 < 0\), for \(s \geq \epsilon\) we have

\[ qu_{1,\lambda}^{q-1} s + p(u_{1,\lambda} + s)^{p-1} s \leq qu_{1,\lambda}^{q-1} s + p \left( \frac{\|u_{1,\lambda}\|_{\infty}}{\epsilon} + 1 \right)^{p-1} s^p. \]

On the other hand, for \(0 \leq s \leq \epsilon\),

\[ qu_{1,\lambda}^{q-1} s + p(u_{1,\lambda} + s)^{p-1} s \leq [qu_{1,\lambda}^{q-1} + p(u_{1,\lambda} + \epsilon)^{p-1}] s. \]

Considering \(\|u_{1,\lambda}\|_{\infty} + \epsilon < [q/p]^{1/(p-q)}\) we have

\[ p(u_{1,\lambda} + \epsilon)^{p-1} < q(u_{1,\lambda} + \epsilon)^{q-1}, \]

and since \(q(u_{1,\lambda} + \epsilon)^{q-1} < qu_{1,\lambda}^{q-1}\), we obtain \(p(u_{1,\lambda} + \epsilon)^{p-1} < qu_{1,\lambda}^{q-1}\). Thus, \(0 \leq s \leq \epsilon\) implies

\[ qu_{1,\lambda}^{q-1} s + p(u_{1,\lambda} + s)^{p-1} s \leq 2qu_{1,\lambda}^{q-1} s, \]

from where we conclude (32).

From the definition of \(G_{\lambda}(x,u)\) and (32) we obtain

\[ G_{\lambda}(x,u) \leq \lambda \rho(x)(qu_{1,\lambda}^{q-1} u^2 + C|u|^{p+1}). \quad (33) \]

Since \(0 < q < 1\), in order to conclude that the energy functional \(J_{\lambda}\) is well defined and has the mountain pass geometry we need to analyse a lower estimate for \(u_{1,\lambda}\).

**Lemma 4.2.** Let \(u_{1,\lambda}\) be the first solution for problem \((P_{\lambda,p,q})\), then

\[ [\lambda(1-q)]^{1/(1-q)} U_{V}^{1-\lambda} \leq u_{1,\lambda} \text{ in } \mathbb{R}^n, \]

where \(U_{V}\) is the bounded and positive solution for \((LS)\).

**Proof of Lemma 4.2.** Let \(u_{1,\lambda}\) be a bounded and positive solution of \((P_{\lambda,p,q})\), with \(f(s) = (s^q + s^p), s \geq 0\), and define

\[ v_{\lambda} := \frac{1}{\lambda(1-q)} u_{1,\lambda}^{1-q}, \]

where \(U_{V}\) is the bounded and positive solution for \((LS)\).
which verifies
\[-\Delta v_\lambda = \frac{qu_0^{-q-1}|\nabla u_0|^2 - \Delta u_0}{\lambda u_0}\]
\[\geq \rho(x) - \frac{V(x)u_0^{1-q}}{\lambda} = \rho(x) - (1-q)V(x)v_\lambda \text{ in } B_R,
\]
and then
\[-\Delta v_\lambda + (1-q)V(x)v_\lambda \geq \rho(x) \text{ in } B_R.\]
Since $0 < q < 1$ and $V$ be a non negative potential, we also have
\[-\Delta v_\lambda + V(x)v_\lambda \geq \rho(x) \text{ in } B_R.\]
Thus, by the maximum principle (see [12, 16]) the solution $U_{V,R}$ of the problem
\[
\begin{cases}
-\Delta U_{V,R} + V(x)U_{V,R} = \rho(x) & \text{in } B_R, \\
U_{V,R} = 0 & \text{on } \partial B_R,
\end{cases}
\]
satisfies
\[U_{V,R} \leq v_\lambda \text{ in } B_R.\]
Passing to the limit as $R \to \infty$ we obtain
\[U_V \leq v_\lambda \text{ in } \mathbb{R}^n.\]
Therefore
\[\lambda (1-q)^{1/\gamma} V_\mu(x) u_\lambda(x) \leq u_{1,\lambda} \text{ in } \mathbb{R}^n.\]

Now, we prove that the functional $J_\lambda$ is well defined in $\mathcal{H}$. In fact, from Lemma 4.2 we have
\[\lambda u_\lambda^{1/\gamma} \leq \frac{1}{1-q}.\]
Thus, from (33) and (35), it follows that
\[\int_{\mathbb{R}^n} G_\lambda(x,u)dx \leq \lambda q \int_{\mathbb{R}^n} \rho(x)u_\lambda^{1+1/q}u^2 + \lambda c \int_{\mathbb{R}^n} \rho(x)|u|^{p+1}dx\]
\[\leq \frac{q}{(1-q)^{1/\gamma}} \int_{\mathbb{R}^n} \rho(x) V(x) u_\lambda(x) u^2 + \lambda c \int_{\mathbb{R}^n} \rho(x)|u|^{p+1}dx.\]

We observe that the hypotheses $(H^2_\lambda)$ and $(H_\rho)$, imply that there exists $c_\mu > 0$ such that
\[\rho(x) V(x) U_\mu(x) \leq c_\mu, \text{ for all } x \in \mathbb{R}^n,
\]
and each $\mu \in (0,\mu^*)$. In fact, by Lemma 2.2
\[\limsup_{|x| \to \infty} \frac{\rho(x)}{V(x) U_\mu(x)} \leq \limsup_{|x| \to \infty} \frac{\rho(x)(1+|x|^2)}{c_1 \mu (1-c_\mu U(x))} \leq \tilde{c}_\mu \limsup_{|x| \to \infty} \frac{(1+|x|^2)|x|^{2-2}}{1+|x|^2} \leq \infty.
\]
Thus, from estimate (37) we have
\[\int_{\mathbb{R}^n} \frac{\rho(x)}{V(x) U_\mu(x)} u^2 dx = \int_{\mathbb{R}^n} \frac{\rho(x)}{V(x) V_\mu(x)} V(x) u^2 dx\]
\[
\leq c \int_{\mathbb{R}^n} V(x)u^2 dx.
\]

Hence
\[
\int_{\mathbb{R}^n} \frac{\rho(x)}{U_V(x)} u^2 dx \leq c\|u\|^2, \quad \text{for any} \quad u \in \mathcal{H}. \quad (39)
\]

Therefore, the functional \( J_\lambda \) is well defined in \( \mathcal{H} \). Using standard arguments, we can see that \( J_\lambda \in C^1(\mathcal{H}, \mathbb{R}) \) and for all \( u, v \in \mathcal{H} \), we have
\[
J_\lambda'(u)(v) = \langle u, v \rangle - \int_{\mathbb{R}^n} g_\lambda(x, u)v dx.
\]

Consequently, critical points of \( J_\lambda \) are weak solutions of \((P_{g_\lambda})\). Now, let us show that \( J_\lambda \) has the mountain pass geometry for sufficiently small values of \( q \in (0,1) \).

Using (39) in (36) and the definition of \( J_\lambda \) we obtain
\[
J_\lambda(u) \geq \frac{1}{2} \left( 1 - \frac{cq}{1-q} \right) \|u\|^2 - \lambda C\|u\|^p.
\]

Defining \( q^* := 1/(1 + c) < 1 \) we have
\[
\left( 1 - \frac{cq}{1-q} \right) > 0
\]
for any \( q \in (0, q^*) \). In this case, for \( \|u\| = r \) sufficiently small, we have
\[
J_\lambda(u) \geq \delta_0 > 0.
\]

for some constant \( \delta_0 \), and thus \( J_\lambda \) satisfies the first mountain pass geometry. To prove the second part of the geometry, note first that the definition of \( f_\lambda \) implies the following inequality for \( G_\lambda \):
\[
G_\lambda(x, s) \geq c_1 \lambda \rho(x)s^{p+1} - c_2 \rho(x),
\]
for constants \( c_1, c_2 > 0 \). Thus, considering a nonnegative function \( \varphi \in C_0^\infty(\mathbb{R}^n) \backslash \{0\} \), we obtain
\[
J_\lambda(u) = \frac{t^2}{2} \|\varphi\|^2 - \int_{\text{supp } \varphi} G_\lambda(x, t\varphi) dx
\leq \frac{t^2}{2} \|\varphi\|^2 - c_1 \lambda t^{p+1} \int_{\text{supp } \varphi} \rho(x)\varphi^{p+1} dx + c_2 \lambda \|\|\varphi\|\|_{\infty} |\text{supp } \varphi| dx
< 0,
\]
for \( t > 0 \) sufficiently large.

Now let us prove that \( J_\lambda \) satisfies the \((PS)\) condition. By Theorem 4.1, it is sufficient to prove that \((PS)\) sequences are bounded in \( \mathcal{H} \). Since our nonlinearity \( g_\lambda \) does not verify the classical Ambrosetti-Rabinowitz condition, we develop a technique to prove the boundedness of the \((PS)\) sequence for our associated functional.

Now, let us define \( g(x, s) = f(u_{1,\lambda} + s) - f(u_{1,\lambda}) \), where \( f(s) = (s^q + s^p) \), and notice that
\[
g(x, s) - \theta G(x, s) = \left[ (u_{1,\lambda} + s)^q - u_{1,\lambda}^q \right] s + \left[ (u_{1,\lambda} + s)^p - u_{1,\lambda}^p \right] s
+ \theta u_{1,\lambda}^q s - \frac{\theta}{q+1} \left[ (u_{1,\lambda} + s)^{q+1} - u_{1,\lambda}^{q+1} \right]
+ \theta u_{1,\lambda}^p s - \frac{\theta}{p+1} \left[ (u_{1,\lambda} + s)^{p+1} - u_{1,\lambda}^{p+1} \right]
\]
\[
\begin{align*}
\geq \theta u_{1,\lambda}^q s - \frac{\theta}{q+1} \left[ (u_{1,\lambda} + s)^q + 1 - u_{1,\lambda}^{q+1} \right] \\
+ \frac{\theta u_{1,\lambda}^p s}{p+1} - \frac{\theta}{p+1} \left[ (u_{1,\lambda} + s)^p + 1 - u_{1,\lambda}^{p+1} \right].
\end{align*}
\]

Using \( \text{(EI)} \), we have
\[
g(x, s) - \theta G(x, s) \geq -\theta (u_{1,\lambda} + s)^q s + \theta u_{1,\lambda}^q s - \theta (u_{1,\lambda} + s)^p s + \theta u_{1,\lambda}^p s
\]
or, equivalently,
\[
g(x, s) - \theta G(x, s) \geq -\theta \left[ (u_{1,\lambda} + s)^q - u_{1,\lambda}^q \right] s - \theta \left[ (u_{1,\lambda} + s)^p - u_{1,\lambda}^p \right] s.
\]

Using again \( \text{(EI)} \), we obtain
\[
g(x, s) - \theta G(x, s) \geq -\theta q u_{1,\lambda}^{q-1} s^2 - \theta p (u_{1,\lambda} + s)^{p-1} s^2
\]
for all \( s \geq 0 \) and \( x \in \mathbb{R}^n \). Note that it is not difficult to prove that for \( 2 < \theta < p + 1 \) and some \( s_0 > 0 \) independent of \( x \), holds
\[
g(x, s) - \theta G(x, s) \geq 0 \quad \text{for all} \quad s \geq s_0, \quad x \in \mathbb{R}^n.
\]

On the other hand, from \( \text{(40)} \), we have
\[
g(x, s) - \theta G(x, s) \geq -\theta u_{1,\lambda}^{q-1} s^2 \quad \text{for all} \quad 0 \leq s \leq s_0.
\]
Therefore
\[
g(x, s) - \theta G(x, s) \geq -\theta u_{1,\lambda}^{q-1} s^2 \quad \text{for all} \quad s \geq 0.
\]

Thus, let \( (u_n) \) be a \( (PS) \) sequence. Then, using \( \text{(41)} \)
\[
c + o_n(1) = J_\lambda(u_n) - \frac{1}{\theta} \mathcal{J}_\lambda'(u_n) u_n
\]
\[
\geq \left( \frac{1}{\theta} - 1 \right) \int_{\mathbb{R}^n} (|\nabla u_n|^2 + V(x) u_n^2) \, dx
\]
\[
+ \lambda \int_{\mathbb{R}^n} \rho(x) \left[ \frac{1}{\theta} g_\lambda(x, u_n) u_n - G_\lambda(x, u_n) \right] \, dx
\]
\[
\geq \left( \frac{1}{\theta} - 1 \right) \int_{\mathbb{R}^n} (|\nabla u_n|^2 + V(x) u_n^2) - c\lambda \int_{\mathbb{R}^n} \rho(x) u_{1,\lambda}^{q-1} u_n^2 \, dx.
\]

Using similar arguments to obtain \( \text{(39)} \), we have
\[
c + o_n(1) \geq \left( \frac{1}{\theta} - 1 \right) \int_{\mathbb{R}^n} (|\nabla u_n|^2 + V(x) u_n^2) - c\lambda \int_{\mathbb{R}^n} V(x) u_n^2 \, dx
\]
\[
\geq \left[ \left( \frac{1}{\theta} - 1 \right) - \frac{cq}{1-q} \right] \int_{\mathbb{R}^n} (|\nabla u_n|^2 + V(x) u_n^2).
\]

Since \( 2 < \theta \) we can take \( q > 0 \) small enough such that
\[
\left[ \left( \frac{1}{\theta} - 1 \right) - \frac{cq}{1-q} \right] > 0,
\]
then we conclude that \( (u_n) \) is bounded in \( \mathcal{H} \). Since \( \mathcal{H} \) is compactly embedding in \( L_{p+1}^p \), the classical Mountain Pass Theorem can thus be applied to obtain a solution \( w \) for problem \( (Pg_\lambda) \).

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