Properties of conditionally filtered equations: Conservation, normal modes, and variational formulation

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Conditionally filtered equations have recently been proposed as a basis for modeling the atmospheric boundary layer and convection. Conditional filtering decomposes the fluid into a number of categories or components, such as convective updraughts and the background environment, and derives governing equations for the dynamics of each component. Because of the novelty and unfamiliarity of these equations, it is important to establish some of their physical and mathematical properties and to examine whether their solutions might behave in counterintuitive or even unphysical ways. It is also important to understand the properties of the equations in order to develop suitable numerical solution methods. The conditionally filtered equations are shown to have conservation laws for mass, entropy, momentum or axial angular momentum, energy, and potential vorticity. The normal modes of the conditionally filtered equations include the usual acoustic, inertio-gravity, and Rossby modes of the standard compressible Euler equations. In addition, the equations support modes with different perturbations in the different fluid components that resemble gravity modes and inertial modes but with zero pressure perturbation. These modes make no contribution to the total filter-scale fluid motion, and their amplitude diminishes as the filter scale diminishes. Finally, it is shown that the conditionally filtered equations have a natural variational formulation, which can be used as a basis for systematically deriving consistent approximations.

KEYWORDS
approximate equations, conditional average, convection, Hamilton’s principle

1 | INTRODUCTION

Conditionally filtered equations have recently been proposed as a basis for mathematical and numerical modeling of the atmospheric boundary layer and convection (Thuburn et al., 2018). Conditional filtering itself is an extension of coarse-graining ideas that are commonly used in large-eddy turbulence modeling, which enable one to write down equations of motion valid for a particular scale of motion, with the subgrid-scale terms then appearing on the right-hand side and in need of parameterization – see Leonard (1975), Frisch (1995), and Aluie et al. (2018) for a range of examples. The conditionally filtered equations extend this idea, so that prognostic equations can be constructed for particular fluid types as well as particular scales, for example for different “components” of the fluid, such as convective updraughts, downdraughts, and the background environment. These prognostic equations may then be solved in a numerical model, even when the individual convective updraughts and downdraughts are too small-scale to be resolved.

The conditionally filtered equations provide a natural way of representing qualitatively quite different types of small-scale physical process within the same mathematical framework. For example, local turbulent fluxes might be represented by right-hand side subgrid terms as an eddy diffusion, while fluxes associated with coherent structures such as...
deep boundary-layer thermals or convective updraughts might be represented by one of the fluid components, the dynamics of which is explicitly represented by the left-hand side terms (see Equations 1–5 and Figure 1 below). By making certain approximations to the conditionally filtered equations and certain choices for the parameterized terms, they can be shown to reduce to a typical mass-flux convection scheme, or to a typical eddy diffusion scheme, coupled to resolved-scale dynamics. Thus, the conditionally filtered equations could provide a useful and self-consistent basis for improving the coupling of different parameterization schemes with each other and with the resolved dynamics, or for building unified parameterization schemes that can transition smoothly between different regimes, for example between a dry convective boundary layer and shallow convection. A particular motivation for us is the possibility of extending the dynamical core of a weather or climate model to solve the left-hand sides of the conditionally filtered equations for all fluid components, thus capturing some of the dynamics of convection explicitly. Ultimately, we wish to explore the potential of this approach to improve some of the well-known modeling problems in convection-dynamics coupling, including memory of the dynamical state of convection, the propagation of convective systems to neighboring grid columns, and the horizontal location of compensating subsidence. These motivations are discussed in more detail by Thuburn et al. (2018).

Similar ideas, leading to prognostic equations for multiple fluid components, may be found in the work of Yano et al. (2010) and Yano (2012), and in the prognostic cloud scheme of Randall and Fowler (1999). The conditionally filtered approach, however, is more systematic and leads to consistent prognostic equations for all the dynamical variables, as well as thermodynamic variables and component volume fractions. Similar equation sets are also used for modeling multiphase flows in engineering applications (e.g. Drew, 1983; Abgrall and Karni, 2001). The conditionally filtered compressible Euler equations are given in section 2 below.

The right-hand sides of the conditionally filtered equations represent a range of important, subgrid-scale physical processes, such as local turbulent fluxes and entrainment and detrainment. The eventual applications envisaged for the conditionally filtered equations will depend critically on the choices made to parameterize these terms. The focus of the present article, however, is on the left-hand sides, which represent a modified form of the resolved-scale dynamics of the compressible Euler equations. Complex models, built from multiple components that are themselves complex, can behave in unexpected and unphysical ways if the individual components are not sufficiently well-understood and well-behaved (see e.g. Gross et al., 2017 for some examples). This motivates us to analyse and document some of the physical and mathematical properties of the conditionally filtered equations when their right-hand sides are zero. We consider this an important preliminary, before attempting to increase the complexity of the system by coupling to parameterized right-hand side terms. It is also important to understand the properties of the equations in order to develop suitable numerical solution methods. This article examines their conservation properties and normal modes, and presents a variational formulation.

Conservation properties are fundamental properties of a physical system, and respecting relevant conservation properties is widely regarded as essential in any mathematical model. Budgets of conserved quantities can help us to understand physical mechanisms (e.g. Hoskins et al., 1985; Peixoto and Oort, 1992; Pauluis and Held, 2002), and respecting conservation properties in numerical models can help to ensure their stability and accuracy (e.g. Thuburn, 2008 and references therein). Section 3 discusses conservation of mass, entropy, momentum, energy, and potential vorticity for the conditionally filtered equations.

The conditionally filtered equations have a rather unusual structure, with separate density, entropy, and velocity fields for each fluid component, but a single common pressure field (section 2). This raises the question of what types of motion the equations might support; these might be counterintuitive or even unphysical. One way to address this question is to examine the normal modes of the linearized equations (e.g. Gill, 1982; Vallis, 2017). This is done for the conditionally filtered equations in section 4. Normal modes can also give useful insight in the development of numerical solution methods, including the choice of grid staggering to best capture mode structures (Arakawa and Lamb, 1977; Thuburn et al., 2002), identification of modes that might be most challenging for a numerical method, identification of computational modes, and understanding the structure of the Helmholtz problem that arises for implicit time integration schemes. They are also valuable as test cases for numerical models (e.g. Baldauf and Brdar, 2013; Shamir and Paldor, 2016).

A variational formulation of fluid dynamical equations can be useful in several ways. The conservation properties of the system can be related to certain symmetries of the Lagrangian (e.g. Salmon, 1998). Approximate versions of the governing equations, for example hydrostatic, pseudo-incompressible, or Boussinesq, can be derived in a systematic way by approximating the Lagrangian and the conservation properties will be preserved by the approximation provided the corresponding symmetries are preserved (e.g. Cotter and Holm, 2014; Dubos and Voitus, 2014; Staniforth, 2014; Tort and Dubos, 2014). Such approximate versions of the governing equations might be useful for more idealized modeling or as the basis for simple theoretical models. Section 5 confirms that the conditionally filtered compressible Euler equations can be obtained from a variational formulation.

2 | GOVERNING EQUATIONS

As in the derivation of the coarse-grained equations used in large-eddy simulation (LES), conditional filtering makes use of an Eulerian spatial filter that retains only the flow
variation on scales larger than some filter scale. However, in
to the filter it also employs a set of quasi-Lagrangian
labels $I_i$, $i = 1, \ldots, n$; at any point in the fluid, exactly one
of the $I_i$ is equal to 1 and the rest are equal to 0. In the pro-
posed application, it is envisaged that the labels might be used
to pick out coherent structures in the flow, such as convective
updraughts and downdraughts and their environment. This
quasi-Lagrangian labeling of fluid parcels is intended to cap-
ture, in mathematical form, some of the intuitive ideas behind
the way we think about coherent structures such as cumulus
clouds. For example, we typically think of an air parcel as
retaining its identity as a cloud parcel over some time period
until physical processes such as mixing and evaporation change
its physical properties, at which point it may be relabeled as
an environment parcel.

To proceed, the fluid dynamical equations are multiplied by
each $I_i$ before applying the spatial filter. This then leads to a set
of equations of motion for each fluid component $i$. When the
starting equations are the dry non-hydrostatic compressible
Euler equations, the resulting conditionally filtered equations
are the following (Thuburn et al., 2018):

$$
\sum_{i=1}^{n} \sigma_i = 1, \quad \tag{1}
$$

$$
\frac{\partial}{\partial t} (\rho_i \sigma_i) + \nabla \cdot (\rho_i \sigma_i \mathbf{u}_i) = \sum_{j \neq i} (M_{ij} - M_{ji}) , \quad \tag{2}
$$

$$
\frac{\partial}{\partial t} (\rho_i \eta_i) + \nabla \cdot (\rho_i \sigma_i \mathbf{u}_i) = \sum_{j \neq i} (M_{ij} \eta_{ij} - M_{ji} \eta_{ji}) - \nabla \cdot \mathbf{F}_{SF}^\eta, \quad \tag{3}
$$

$$
\frac{\partial}{\partial t} (\rho_i \sigma_i \mathbf{u}_i) + \nabla \cdot (\rho_i \sigma_i \mathbf{u}_i \mathbf{u}_i) + \sigma_i \nabla p + \rho_i \nabla \Phi
$$

$$
= \sum_{j \neq i} (M_{ij} \mathbf{u}_{ij} - M_{ji} \mathbf{u}_{ji}) - \nabla \cdot \mathbf{F}_{SF}^\mathbf{u} - \mathbf{b}_i - \sum_j \mathbf{d}_{ij}, \quad \tag{4}
$$

$$
\bar{p} - P(\rho_i, \eta_i) = P_{SF}^p. \quad \tag{5}
$$

Here $\sigma_i$, $\rho_i$, $\eta_i$, and $\mathbf{u}_i$ are the volume fraction, density, specific
entropy, and velocity, respectively, of the $i$th fluid com-
ponent on the filter scale, $\bar{p}$ is the filter-scale pressure, and $\Phi$
is the geopotential. See Figure 1 for a schematic illustration of
the meaning of the conditionally filtered fields. Equation 1
expresses the fact that the volume fractions must sum to one,
Equation 2 expresses mass conservation, Equation 3 entropy
conservation, and Equation 4 momentum conservation, while
Equation 5 is a generic form for the equation of state relating
pressure to entropy and density. For simplicity, the Coriolis
terms associated with planetary rotation have been neglected
here. However, it is straightforward to reintroduce them and
we do so for the purpose of section 4 below.

The right-hand sides of the above equations allow for the
possibility that fluid parcels may be relabeled as the flow
evolves; this could represent processes such as entrainment
and detrainment of fluid between convective updraughts and
their environment. Thus, for example, $M_{ij}$ is the rate per unit
volume at which mass is relabeled from type $j$ to type $i$, and $\eta_{ij}$
and $\mathbf{u}_{ij}$ are representative values of specific entropy and veloc-
ity for that relabeled fluid. If the fluid labels $I_i$ were exactly
materially conserved, then the relabeling terms $M_{ij}$ would
vanish. Note also that the time over which a parcel keeps a rec-
ognizable identity is much longer than a model time step—the
lifetimes of small individual clouds are of the order of sev-
eral minutes, but in a model approaching cloud resolution the
time step is measured in seconds. In a climate model, the time
step might be of order tens of minutes, but the cloud popula-
tions at that resolution last for the order of hours. Relabeling,
and its relation to physical processes such as evaporation and
mixing, is discussed further by Thuburn et al. (2018).

As in the equations of LES, subfilter-scale variability
contributes to the filter-scale behavior. Here $\mathbf{F}_{SF}^\eta$ is a
subfilter-scale flux of entropy, $\mathbf{F}_{SF}^\mathbf{u}$ is a subfilter-scale momentum
flux tensor, and $P_{SF}^p$ accounts for variations in pressure
between the fluid components, as well as effects of filtering
a nonlinear equation of state. The right-hand sides cannot be
derived from the equations of motion; rather, they must be
parameterized, as must terms representing similar processes
in, for example, a mass-flux scheme.

Note that the same filter-scale pressure $\bar{p}$ appears in
the pressure-gradient term on the left-hand side of the momentum
equation 4 for every $i$. This is a similar assumption
to that made in conventional parcel arguments, where it is
assumed that the parcel takes on the pressure of the envi-
ronment (e.g. Bohren and Albrecht, 1998). The assumption
may be justified by noting that (in most convective circum-
stances) the acoustic adjustment time—the time required for
an acoustic wave to propagate the width of a cloud and so
remove unbalanced pressure fluctuations—is short compared
with the time-scales of interest. Thus, acoustic oscillations
will very quickly equilibrate the pressure between compo-
ents, and by making the equal-pressure assumption we are
supposing this adjustment to take place instantaneously. A
consequence of the assumption is that the equations do not
support those acoustic modes for which fluid component
$i$ has a different pressure from fluid component $j \neq i$ (see
also section 4). These acoustic modes would, in any case,
have very small amplitude, and resolving them explicitly
would present unnecessary difficulties for numerical solu-
tion methods, with no gain in accuracy. In the Boussinesq
and anelastic approximations, acoustic modes are eliminated
ab initio because the speed of sound is taken to be infinite.
The acoustic adjustment between different fluid components
then occurs instantaneously, and the assumption of the same
filter-scale pressure is a very natural one.

In a convecting fluid, the pressure gradient is not, in fact,
homogeneous on the scale of the convective updraughts, and
rising thermals experience a significant drag due to pressure
variations on the scale of the thermal (e.g. Romps and Charn,
2015). These pressure variations do not represent acoustic
waves; they are present in Boussinesq and anelastic numerical simulations. In the conditionally filtered equations, the fact that the net pressure gradient experienced by fluid $i$ departs from $\nabla p$ is accounted for by the terms $(-b_i - \sum_j d_{ij})$ on the right-hand side. In particular, $d_{ij}$ is minus the pressure drag exerted by fluid $j$ on fluid $i$. These terms have the properties that

$$\sum_i b_i = 0$$

and

$$d_{ij} = -d_{ji}.$$ 

(7)

These terms are not predicted by the conditionally filtered equations and so, in general, must be parameterized, just as the analogous terms are parameterized in typical mass-flux convection schemes (e.g. de Roode et al., 2012 and references therein). In this article, we will be concerned mainly with the left-hand sides of the conditionally filtered equations, so we will often neglect these terms, along with the other right-hand side terms.

It may be useful to note how the conditionally filtered Equations 1–5 are related to the usual filtered single-fluid equations. The conditionally filtered equations reduce to the usual filtered single-fluid equations simply by setting the number of fluid components $n$ to 1; in that case, the fluid relabeling terms $\mathcal{M}_{ij}$ and the terms representing pressure forces between fluid components $b_i$ and $d_{ij}$ all vanish, and $\sigma_1 \equiv 1$. Thuburn et al. (2018) also show that the usual filtered single-fluid equations are obtained by summing the conditionally filtered equations over all fluid components $i$. Note also that, although the left-hand sides of Equations 2–4 are written here in Eulerian flux form, this is not a requirement; it is straightforward to convert them to Lagrangian form, as we do, for example, in Equations 18 and 24 below.

In the absence of the right-hand sides, Equations 1–5 form a closed system (see below). All of the right-hand side terms, on the other hand, must be modelled or parameterized by making some additional assumptions. The present article focuses mainly on the properties of the equations in the absence of the
parameterized terms, whereupon they reduce to
\[ \sum_{i=1}^{n} \sigma_i = 1, \quad (8) \]
\[ \frac{\partial}{\partial t} (\sigma_i \rho_i) + \nabla \cdot (\sigma_i \rho_i \mathbf{u}_i) = 0, \quad (9) \]
\[ \frac{\partial}{\partial t} (\sigma_i \rho_i \eta_i) + \nabla \cdot (\sigma_i \rho_i \mathbf{u}_i \mathbf{n}_i) = 0, \quad (10) \]
\[ \frac{\partial}{\partial t} (\sigma_i \rho_i \mathbf{u}_i) + \nabla \cdot (\sigma_i \rho_i \mathbf{u}_i \mathbf{n}_i) + \rho_i V^2 + \sigma_i \rho_i \mathbf{\Phi} = 0, \quad (11) \]
\[ \overline{\rho} - P(\rho_i, \eta_i) = 0, \quad (12) \]
where Equation 8 is the same as Equation 1 but is included for completeness.

In the case of a single fluid component \( n = 1 \), Equations 8–12 reduce to the usual non-hydrostatic compressible Euler equations. For \( n > 1 \), the equations for different \( i \) are coupled by the common pressure-gradient term \( \nabla \overline{p} \) and the requirement Equation 8 (these two points are related—see section 5). Also, for \( n > 1 \), it is not immediately obvious that Equations 8–12 form a closed system. It can be confirmed, simply by counting, that the number of equations is equal to the number of unknowns. Appendix A outlines how the given equations imply the time evolution of \( \sigma_i, \rho_i \), and \( \overline{p} \) and how they allow \( \overline{p} \) to be diagnosed. For the linearized version of these equations, the fact that a dispersion relation can be derived (section 4) provides further confirmation that they form a closed system.

A potentially useful variant of the conditionally filtered equations, mentioned by Thuburn et al. (2018), is one in which all fluid components are constrained to have identical horizontal velocity: \( \mathbf{v}_i = \overline{\mathbf{v}} \), where \( \mathbf{u}_i = (\mathbf{v}_i, \mathbf{w}_i) \). In this variant, the horizontal components of the interfluid pressure forces \( \mathbf{b}_i + \sum \mathbf{d}_i \) are assumed to be just what is required to maintain the equality of the \( \mathbf{v}_i \). The ansatz that the horizontal velocities are all the same is an additional physical assumption that may be useful in some circumstances, but it is not demanded by the mathematical structure of the equations. Making this assumption does not change the vertical part of 4, namely
\[ \frac{\partial}{\partial t} (\sigma_i \rho_i \mathbf{w}_i) + \nabla \cdot (\sigma_i \rho_i \mathbf{u}_i \mathbf{w}_i) + \sigma_i \overline{p}_z + \sigma_i \rho_i \mathbf{\Phi}_z = \sum_{j=1}^{n} \left[ (\mathbf{M}_j \mathbf{w}_j - \mathbf{M}_j \mathbf{\hat{w}}_j) - \nabla \cdot \mathbf{\mathbf{F}}^s_{\mathbf{SF}} - b_i^{(2)} - \sum_{j} d_{ij}^{(2)} \right] \]
where subscript \( z \) indicates a vertical derivative and superscript \( (z) \) indicates a vertical component. However, the horizontal part is replaced by the sum over all fluid components,
\[ \frac{\partial}{\partial t} (\overline{\rho} \overline{\mathbf{v}}) + \nabla \cdot (\overline{\rho} \overline{\mathbf{v}} \overline{\mathbf{v}}) + \nabla \overline{\mathbf{H}} \overline{\rho} + \overline{\mathbf{H}} \overline{\rho} \nabla \mathbf{\Phi} = -\nabla \cdot \mathbf{\mathbf{F}}^s_{\mathbf{SF}} \]
where \( \mathbf{H} \) is the horizontal part of the gradient,
\[ \overline{\rho} = \sum_i \sigma_i \rho_i \]
\[ \overline{\mathbf{v}} = \sum_i \sigma_i \mathbf{u}_i \]
\[ \overline{\mathbf{w}} = \sum_i \sigma_i \mathbf{w}_i \]
\[ \overline{\mathbf{\mathbf{F}}_{\mathbf{SF}}} = \sum_i \sigma_i \mathbf{\mathbf{F}}^i \]
is the total filter-scale density, and \( \overline{\mathbf{v}}^* \) is the density-weighted filter-scale velocity, given by
\[ \overline{\rho} \overline{\mathbf{v}}^* = \sum_i \sigma_i \rho_i \mathbf{u}_i. \quad (16) \]
The \( \mathbf{b} \) and \( \mathbf{d} \) terms have cancelled in Equation 14 because of Equations 6 and 7, while the relabeling terms also cancel when summed over \( i \) and \( j \). Appendix B summarizes how the main results of the article carry over to this equal-\( \mathbf{v} \), variant.

3 | CONSERVATION PROPERTIES

This section examines the conservation properties of the conditionally filtered equations. We focus on the compressible Euler equations, but similar derivations may be carried out for other, approximate, governing equation sets such as hydrostatic, pseudo-incompressible, or Boussinesq equations.

3.1 | Mass

Equation 9 is manifestly in the form of a conservation law for the mass of the \( i \)th fluid component. If there is no mass flux across domain boundaries, then it implies that the mass of each fluid component is individually conserved, and hence that their sum, the total fluid mass, is also conserved.

If relabeling terms are reintroduced, Equations 9 becomes Equation 2. Then the mass of each fluid component is no longer conserved. However, summing Equation 2 over \( i \) and noting that the relabeling terms then cancel (because they are relabeling terms) gives
\[ \frac{\partial \overline{\rho}}{\partial t} + \nabla \cdot (\overline{\rho} \overline{\mathbf{v}}) = 0, \quad (17) \]
with \( \overline{\rho} \) and \( \overline{\mathbf{v}}^* \) given by Equations 15 and 16. Thus the total fluid mass is conserved even when relabeling terms are included.

3.2 | Entropy

Equation 10 is manifestly in the form of a conservation law for the entropy of the \( i \)th fluid component. If there is no entropy flux across domain boundaries, then it implies that the entropy of each fluid component is individually conserved, and hence that their sum, the total fluid entropy, is also conserved.

Subtracting \( \eta_i \) times Equation 9 from Equation 10 gives
\[ \frac{D \eta_i}{Dt} = \frac{\partial \eta_i}{\partial t} + \mathbf{u}_i \cdot \nabla \eta_i = 0. \quad (18) \]
This shows that \( \eta_i \) is materially conserved following fluid parcels that move with velocity \( \mathbf{u}_i \).

If the subfilter-scale flux term \( \nabla \cdot \mathbf{\mathbf{F}}^s_{\mathbf{SF}} \) is included in Equation 10, then the equation is still in the form of a flux-form conservation law, so the entropy of each fluid component is still conserved in an integral sense, though it is no longer materially conserved. If, in addition, the relabeling terms are included to give Equation 3, then the entropy of each

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1The notation \( \overline{X} \) to indicate a filtered value of \( X \) and \( \overline{\overline{X}} \) to indicate a density-weighted filtered value, so that \( \overline{\overline{X}} = \overline{\rho X} \), is retained for consistency with Thuburn et al. (2018).
3.3 | Momentum

The geopotential gradient $\nabla \Phi$ provides an external force and hence an external source of momentum. Even if this term is ignored for the moment, Equation 11 does not conserve the momentum of the $i$th fluid component, because the $\sigma_i \psi \nabla \bar{P}$ term is not in conservation form. However, the $\sigma_i \psi \nabla \bar{P}$ term does represent a conservative exchange of momentum between different fluid components, as do the $\mathbf{b}_i$, $\mathbf{d}_{ij}$, and relabeling terms. This can be seen by summing Equation 6 over $i$ and using Equations 6 and 7 to obtain

$$\frac{\partial}{\partial t} (\bar{\rho} \mathbf{u}^a) + \mathbf{F}^a = 0,$$

where $\bar{\rho} \mathbf{u}^a$ is given by Equation 16 and

$$\mathbf{F}^a = \bar{\rho} I + \sum_i \left( \sigma_i \rho_i \mathbf{u}_i + \mathbf{F}^a_{\text{SF}} \right)$$

is the total momentum flux tensor, with $I$ the identity matrix. Thus, the total fluid momentum is conserved except for the effect of the external force.

If the Coriolis terms are reintroduced for a rotating planet, then the relevant conserved quantity is the axial angular momentum. The axial angular momentum of the $i$th fluid is not conserved, but it is straightforward to verify that the $\nabla \bar{P}$, $\mathbf{b}_i$, $\mathbf{d}_{ij}$, and relabeling terms all describe conservative transfers between fluid components and the total axial angular momentum is conserved.

3.4 | Energy

In this subsection we ignore the subfilter-scale flux terms and the relabeling terms; in general, they do not conserve the energy of the filter-scale flow. For the moment, the $\mathbf{b}_i$ and $\mathbf{d}_{ij}$ terms are retained.

Subtracting $\mathbf{u}_i$ times Equation 9 from Equation 4 and neglecting $\mathbf{F}^a_{\text{SF}}$ and $\mathcal{M}_{ij}$ gives the advective form of the momentum equation:

$$\sigma_i \rho_i \frac{D}{Dt} \mathbf{u}_i + \sigma_i \psi \nabla \bar{\rho} + \sigma_i \rho_i \nabla \Phi = -\mathbf{b}_i - \sum_j \mathbf{d}_{ij}. \tag{24}$$

Taking the dot product of $\mathbf{u}_i$ with Equation 24 gives

$$\sigma_i \rho_i \frac{D}{Dt} \left( \frac{1}{2} |\mathbf{u}_i|^2 + \Phi \right) + \sigma_i \mathbf{u}_i \cdot \nabla \bar{\rho} = -\mathbf{u}_i \cdot \left( \mathbf{b}_i + \sum_j \mathbf{d}_{ij} \right). \tag{25}$$

Next, defining $e_i(\rho_i, \eta_i)$ to be the specific internal energy of fluid component $i$,

$$\frac{D}{Dt} \frac{\partial e_i}{\partial \rho_i} \bigg|_{\eta_i} + \frac{\partial e_i}{\partial \eta_i} \frac{\partial \eta_i}{\partial \rho_i} \frac{D}{Dt} \rho_i = \left( \frac{\partial \eta_i}{\partial \rho_i} \right) \frac{D}{Dt} \rho_i,$$ \tag{26}

Noting that

$$\frac{\partial e_i}{\partial \rho_i} = \frac{\bar{p}}{\rho_i^2},$$ \tag{27}

and using Equation 9 to obtain the material derivative of $\rho_i$,

$$\sigma_i \rho_i \frac{D}{Dt} \frac{\partial \sigma_i}{\partial t} + \rho_i \frac{\partial \sigma_i}{\partial t} + \rho_i \nabla \cdot (\sigma_i \mathbf{u}_i) = 0,$$ \tag{28}

and Equation 18 for the material derivative of $\eta_i$, Equation 26 becomes

$$\sigma_i \rho_i \frac{D}{Dt} \epsilon_i = -\bar{\rho} \left( \frac{\partial \sigma_i}{\partial t} + \nabla \cdot (\sigma_i \mathbf{u}_i) \right). \tag{29}$$

Adding this result to Equation 25 gives

$$\sigma_i \rho_i \frac{D}{Dt} \left( \epsilon_i + \nabla \cdot (\sigma_i \mathbf{u}_i) + \bar{\rho} \frac{\partial \epsilon_i}{\partial t} \right) = -\mathbf{u}_i \cdot \left( \mathbf{b}_i + \sum_j \mathbf{d}_{ij} \right), \tag{30}$$

where

$$\epsilon_i = \frac{1}{2} |\mathbf{u}_i|^2 + \Phi + e_i$$ \tag{31}

is the total filter-scale energy per unit mass of the $i$th fluid component. Finally, adding $\epsilon_i$ times Equation 9 to Equation 30 gives

$$\frac{\partial}{\partial t} \left( \sigma_i \rho_i \epsilon_i \right) + \nabla \cdot \left( \sigma_i \rho_i \mathbf{u}_i \epsilon_i + \sigma_i \mathbf{u}_i \bar{\rho} \right) + \bar{\rho} \frac{\partial \epsilon_i}{\partial t} \frac{\partial}{\partial t} \rho_i \left( \mathbf{b}_i + \sum_j \mathbf{d}_{ij} \right). \tag{32}$$

The quantity $\sigma_i \rho_i \epsilon_i$ is the contribution from the $i$th fluid component to the total filter-scale energy density. In general, it is not conserved. The term $\bar{\rho} \frac{\partial \epsilon_i}{\partial t}$ represents a conservative exchange of energy between fluid components, since $\sum_i \epsilon_i = 1$ implies $\sum_i \bar{\rho} \frac{\partial \epsilon_i}{\partial t} = 0$. The terms $\mathbf{b}_i + \sum_j \mathbf{d}_{ij}$ will typically tend to reduce differences between the $\mathbf{u}_i$; they thus represent a sink of filter-scale energy and a transfer to subfilter scales. If the $\mathbf{b}_i + \sum_j \mathbf{d}_{ij}$ terms can be ignored, then summing
Equation 32 over \( i \) shows that the total filter-scale energy is conserved:

\[
\frac{\partial}{\partial t} \left( \sum_i \sigma_i \rho_i \varepsilon_i \right) + \nabla \cdot \left( \sum_i \left( \sigma_i \rho_i \mathbf{u}_i \varepsilon_i + \sigma_i \mathbf{u}_i \mathbf{p} \right) \right) = 0. \tag{33}
\]

### 3.5 Potential vorticity

Using standard vector calculus identities, the advective form of the momentum equation 24 may be written in so-called vector-invariant form:

\[
\frac{\partial \mathbf{u}_i}{\partial t} + \nabla \cdot \left( \rho_i \mathbf{u}_i \right) = -\frac{1}{\rho_i} \left( \mathbf{b}_i + \sum_j d_{ij} \right), \tag{34}
\]

For now, suppose the right-hand side can be neglected. Taking the curl and using further vector calculus identities gives the vorticity equation for the \( i \)th fluid component:

\[
\frac{D_i}{{\partial t}} \mathbf{\zeta}_i + \mathbf{\zeta}_i \nabla \cdot \mathbf{u}_i - \nabla \cdot \mathbf{u}_i + \nabla \times \left( \frac{1}{\rho_i} \mathbf{p} \right) = 0. \tag{35}
\]

Rewriting Equation 9 in the form

\[
\frac{D_i}{\partial t} (\sigma_i \rho_i) + \sigma_i \rho_i \nabla \cdot \mathbf{u}_i = 0 \tag{36}
\]

allows the velocity divergence term to be eliminated:

\[
\sigma_i \rho_i \frac{D_i}{\partial t} \left( \frac{\mathbf{\zeta}_i}{\sigma_i \rho_i} \right) - \mathbf{\zeta}_i \cdot \nabla \mathbf{u}_i + \nabla \times \left( \frac{1}{\rho_i} \mathbf{p} \right) = 0. \tag{37}
\]

Now consider a scalar \( \lambda \) that is materially conserved following the velocity field \( \mathbf{u}_i \), i.e. \( D_\lambda \mathbf{\lambda}/\partial t = 0 \). Taking the gradient, expanding, and rearranging gives

\[
\frac{D_i}{\partial t} \nabla \mathbf{\lambda} + (\nabla \mathbf{u}_i) \cdot \nabla \mathbf{\lambda} = 0. \tag{38}
\]

If we construct the quantity

\[
\Pi_i = \frac{\mathbf{\zeta}_i \cdot \nabla \mathbf{\lambda}}{\sigma_i \rho_i} \tag{39}
\]

and use the product rule to evaluate its material derivative, we obtain

\[
\frac{D_i \Pi_i}{\partial t} - \frac{1}{\sigma_i \rho_i^3} \nabla \rho_i \cdot \nabla \mathbf{\mathbf{p}} \cdot \nabla \mathbf{\mathbf{\lambda}} = 0. \tag{40}
\]

If \( \lambda \) is chosen to be the specific entropy \( \eta_i \), or any function of the specific entropy, such as the potential temperature (\( \Pi_i \) is then the potential vorticity of the \( i \)th fluid), then \( \lambda \) can be expressed as a function of \( \rho_i \) and \( \mathbf{p}, \nabla \lambda \) at every point is a linear combination of \( \nabla \rho_i \) and \( \nabla \mathbf{p} \), and so the scalar triple product term in Equation 40 vanishes, leaving

\[
\frac{D_i \Pi_i}{\partial t} = 0. \tag{41}
\]

Thus the potential vorticity of the \( i \)th fluid component is materially conserved following \( \mathbf{u}_i \). This derivation closely parallels the standard textbook derivation of potential vorticity conservation for a single-component fluid (e.g. Vallis, 2017). A notable difference is the appearance of \( \sigma_i \) as well as \( \rho_i \) in the denominator of Equation 39.

If the \( \mathbf{b}_i + \sum_j d_{ij} \) terms cannot be neglected, then they may be carried through the derivation to appear as source terms in Equation 41. The potential vorticity of the \( i \)th fluid component is then no longer materially conserved.

Haynes and McIntyre (1987) showed that potential vorticity satisfies a flux-form conservation law even in the presence of diabatic heating and frictional forces. They also proved the impermeability theorem, that there is no net flux of potential vorticity across an isentropic surface. These results are purely kinematic (Bretherton and Schär, 1993; Vallis, 2017); they do not depend on the governing dynamical equations, only on the definition of potential vorticity and the fact that the vorticity is the curl of a vector and hence divergence-free. It comes as no surprise, then, that the conservation law and impermeability theorem generalize straightforwardly to the conditionally filtered equations, as follows.

The conservation law is obtained from Equation 39, setting \( \lambda = \eta_i \) and using \( \nabla \cdot \mathbf{\zeta}_i = 0 \).

\[
\sigma_i \rho_i \Pi_i = \nabla \cdot (\eta_i \mathbf{\zeta}_i). \tag{42}
\]

Taking the time derivative then gives

\[
\frac{\partial}{\partial t} (\sigma_i \rho_i \Pi_i) + \nabla \cdot \mathbf{F}_i = 0 \tag{43}
\]

where

\[
\mathbf{F}_i = -\frac{\partial}{\partial t} (\eta_i \mathbf{\zeta}_i). \tag{44}
\]

The time derivative in the expression for \( \mathbf{F}_i \) can be removed using the prognostic equations for \( \eta_i \) and \( \mathbf{\zeta}_i \) (including diabatic heating and friction, if present). Note also that the flux is not unique; any divergence-free vector may be added to \( \mathbf{F}_i \), leaving the conservation law intact.

Next consider the integral of potential vorticity within a volume bounded by a surface of constant \( \eta_i \). For example, this surface might envelope the Earth.

\[
\int \sigma_i \rho_i \Pi_i \, dV = \int \nabla \cdot (\eta_i \mathbf{\zeta}_i) \, dV = \int_{\partial V} \eta_i \mathbf{\zeta}_i \cdot d\mathbf{A}. \tag{45}
\]

where the last integral is the area integral of the outward normal component of \( \eta_i \mathbf{\zeta}_i \) over the boundary of the original volume. Since \( \eta_i \) is constant on this boundary, it can be brought outside the integral:

\[
\int \sigma_i \rho_i \Pi_i \, dV = \eta_i \int_{\partial V} \mathbf{\zeta}_i \cdot d\mathbf{A} = \eta_i \int \nabla \cdot \mathbf{\zeta}_i \, dV = 0. \tag{46}
\]

Similarly, the integral of potential vorticity within a volume bounded by a pair of isentropic surfaces must also vanish.

Finally, consider the integral of potential vorticity within a volume that is bounded in part by an isentropic surface \( \mathcal{A} \) on which \( \eta_i = \eta_i^{(\mathcal{A})} = \text{const} \) and in part by a surface \( \mathcal{B} \), such as the ground, on which \( \eta_i = \eta_i^{(\mathcal{B})} \) may vary in space and time. The boundary integral in Equation 45 may be split into two
contributions:
\[
\int \sigma_i \rho_i \Pi_i \, dV = \eta_i^{(A)} \int \xi_i^A \cdot dA + \int \eta_i^{(B)} \xi_i^B \cdot dA = \int \eta_i^{(A)} (\xi_i^B - \xi_i^A) \cdot dA.
\]

Thus the integral of potential vorticity within the volume, and therefore its rate of change, depends only on contributions from surface \(B\); there is no contribution from surface \(A\).

In summary, for the conditionally filtered equations, the potential vorticity of each fluid component \(i\) satisfies a flux-form conservation law and the impermeability theorem.

\section{NORMAL MODES}

In this section, we focus mainly on the case of two-fluid components. The case of more fluid components is discussed briefly at the end. To analyse the normal modes, all of the right-hand side terms in Equations 2–5 are neglected, so the starting point is Equations 8–12. For simplicity, planar geometry is assumed and the equations are written in Cartesian coordinates. However, Coriolis terms are reintroduced, with a linear dependence of the Coriolis parameter on the northward coordinates. However, Coriolis terms are reintroduced, with a linear dependence of the Coriolis parameter on the northward coordinates, i.e. we use a \(\beta\)-plane, because the Coriolis terms and \(\beta\)-effect are crucial to the dynamics of the normal modes.

Small perturbations to a basic state are considered. The case of more fluid components is discussed briefly at the end. To analyse the normal modes, all of the right-hand side terms in Equations 2–5 are neglected, so the starting point is Equations 8–12. For simplicity, planar geometry is assumed and the equations are written in Cartesian coordinates. However, Coriolis terms are reintroduced, with a linear dependence of the Coriolis parameter on the northward coordinate, i.e. we use a \(\beta\)-plane, because the Coriolis terms and \(\beta\)-effect are crucial to the dynamics of the normal modes.

Small perturbations to a basic state are considered. The basic state (indicated by superscript (r)) is at rest and in hydrostatic balance, and the basic state thermodynamic quantities \(\rho^r, \eta^r, \sigma^r\) are identical for the two-fluid components, though their volume fractions \(\sigma_i^r, \sigma_i^r\) might be different. Basic-state quantities are functions of the vertical coordinate \(z\) only.

Equations 8–12 are linearized about the basic state and wavelike solutions proportional to \(\exp[i(kx + ly - \omega t)]\) are sought, where \(k, \omega\) are the horizontal components of the wave vector and \(\omega\) is the frequency. The \(\beta\)-effect is included, while still permitting such wavelike solutions, following the approximation made by Thuburn and Woollings (2005). Including the \(\beta\)-effect is useful for identifying the Rossby modes and distinguishing them from any zero-frequency modes.

The resulting linearized equations are
\[
\sum_{i=1}^{2} \sigma_i = 0, \quad \sigma_i^r \rho_i + \sigma_i^r \rho_i^r = 0, \quad \sigma_i^r \rho_i^r (iku_i + ilv_i) + (\sigma_i^r \rho_i^r w_i)_{z} = 0, \quad \sigma_i^r \rho_i^r = 0, \quad -i \omega \sigma_i^r \rho_i^r + ik \frac{\rho_i^r}{\sigma_i^r} = 0, \quad \sum_{i=1}^{2} \left\{ -i \omega \sigma_i^r \rho_i^r + \sigma_i^r \rho_i^r (iku_i + ilv_i) + \left( \sigma_i^r \rho_i^r w_i \right)_{z} \right\} = 0.
\]
Using Equation 58 to eliminate $\rho$, gives
\[
\sum_{i=1}^{2} \left\{ -i\omega \sigma_i^{(r)} \frac{P}{c^2} + \sigma_i^{(r)} \rho_i^{(r)} (i ku_i + i lv_i) \right. \\
+ \left. \left( \sigma_i^{(r)} \rho_i^{(r)} w_i \right) + \frac{N^2}{g} \sigma_i^{(r)} \rho_i^{(r)} w_i \right\} = 0. \tag{64}
\]

Now $u_i$, $v_i$, and $w_i$ may be eliminated by using Equations 61, 62, and 60, giving an equation in the single unknown $p$:
\[
\sum_{i=1}^{2} \left\{ -i\omega \sigma_i^{(r)} \frac{P}{c^2} + \sigma_i^{(r)} \left( \frac{iK^2 \omega}{\omega^2 - f^2} \right) p \right. \\
- \left( \sigma_i^{(r)} \frac{i \omega}{\omega^2 - N^2} \left( p_i + \frac{g}{c^2} \right) \right) z \right. \\
- \sigma_i^{(r)} \frac{N^2}{g} \frac{i \omega}{\omega^2 - N^2} \left( p_i + \frac{g}{c^2} \right) \right\} = 0. \tag{65}
\]

This equation can be simplified by noting that $\sum \sigma_i^{(r)} = 1$, to obtain
\[
\left( \frac{\omega}{c^2} - \frac{K^2 \omega}{\omega^2 - f^2} \right) \frac{d}{dz} + \frac{N^2}{g} \left( \frac{\omega}{\omega^2 - N^2} \right) \left( \frac{d}{dz} + \frac{g}{c^2} \right) p = 0. \tag{66}
\]

For a general equation of state and for arbitrary basic state profiles, Equation 60 could be solved numerically. Normal modes can be obtained analytically if the perfect gas equation of state is assumed, the basic state is assumed to be isothermal, and $g$ is taken to be constant. In that case, $c^2$ and $N^2$ are constant, and so is the density scale height $H$, which is given by
\[
\frac{1}{H} = \frac{N^2}{g} + \frac{g}{c^2}. \tag{67}
\]

Then Equation 66 is a constant coefficient equation for $p$. The solutions have a simpler structure when expressed in terms of a rescaled variable,
\[
q = p \exp(z/2H). \tag{68}
\]

Equation 66 then reduces to
\[
\left( \frac{\frac{\omega}{c^2} - \frac{K^2 \omega}{\omega^2 - f^2}}{\omega^2 - N^2} \right) q + \left( \frac{\omega}{\omega^2 - N^2} \right) \frac{d^2}{dz^2} - \frac{1}{\Gamma^2} \right) q = 0, \tag{69}
\]

where the inverse length-scale $\Gamma$ is given by
\[
\Gamma = \frac{1}{2} \left( \frac{g}{c^2} - \frac{N^2}{g} \right). \tag{70}
\]

### 4.1 Single-fluid-equivalent modes

Equation 69 has solutions for $q$ proportional to $\exp(imz)$ with real vertical wavenumber $m$. These are internal mode solutions. The allowed values of $m$ are determined by the lower and upper boundary conditions, for example, $w_i = 0$ at $z = 0$ and at some domain height $D$, though we will not dwell on this detail here. Substituting such solutions into Equation 69 and canceling $q$ gives the dispersion relation, relating $\omega$ to $k$, $l$, and $m$:
\[
\frac{\omega}{c^2} - \frac{K^2 \omega}{\omega^2 - f^2} - \omega \frac{m^2 + \Gamma^2}{\omega^2 - N^2} = 0. \tag{71}
\]

It is clear that Equation 71 is a quintic equation for $\omega$, and it is easily confirmed that it is identical to the dispersion relation obtained by Thuburn and Woollings (2005) for the single-fluid-component compressible Euler equations. The five roots for $\omega$ for any given $(k,l,m)$ correspond to five branches of normal modes: eastward- and westward-propagating acoustic modes, eastward- and westward-propagating inertio-gravity modes, and westward-propagating Rossby modes.

To examine the structure of these normal modes, note that Equations 60–62 imply $(u_1, v_1, w_1) = (u_2, v_2, w_2)$. (It has been assumed here that $\omega^2 \neq N^2$ and $\widetilde{\omega}^2 \neq f^2$, but it can be confirmed that such values of $\omega$ are not solutions of Equation 71, except for very special and unrealistic parameter values.) It then follows from Equations 50 and 54 that $\rho_1 = \rho_2$ and $\eta_1 = \eta_2$, while Equation 49 implies that $\sigma_1$ and $\sigma_2$ are determined simply by vertical advection of the background values $\sigma_i^{(r)}$ and $\sigma_i^{(l)}$. Thus, these normal modes have identical perturbations in the two-fluid components. Their structure, as well as their frequency, is exactly that of the normal modes for the single-fluid-component compressible Euler equations.

In other words, the single-fluid normal modes are a subset of the two-fluid normal modes.

The single-fluid compressible Euler equations also support external modes, with zero vertical velocity and entropy perturbation (assuming a rigid-lid upper boundary condition) and exponential profiles of the other perturbation variables. Seeking such modes in the two-fluid case, only the first line is retained on the left-hand sides of Equations 64, 65, 66, and 69, and the dispersion relation becomes
\[
\frac{\omega}{c^2} - \frac{K^2 \omega}{\omega^2 - f^2} = 0. \tag{72}
\]

This is a cubic equation for $\omega$, giving three branches of normal modes: eastward and westward external acoustic modes, and westward external Rossby modes. Again, the frequencies are identical to those in the single-fluid case, and the mode structures are identical in the two-fluid components, so again the single-fluid normal modes are a subset of the two-fluid normal modes.

In order to obtain Equation 71 from Equation 69, it was assumed that $q$ was non-zero in order to cancel $q$. Another way to satisfy Equation 69 is for $q$ to be identically zero. There are then three ways to obtain non-trivial solutions.

### 4.2 Two-fluid gravity modes

To have zero $p$ but non-zero vertical velocity, Equation 60 implies that $\omega^2 = N^2$. Equations 61 and 62 then imply that $u_i = v_i = 0$; the motion is purely vertical. From Equations 58 and 54, the entropy and density perturbations are related to
the vertical velocity perturbation by
\[ \pm i N \eta_i = \mp i N \frac{\rho_i}{\rho^{(r)}} = \frac{N^2}{g} w_i. \]  
(73)

Equation 64 reduces to
\[ \left( \frac{d}{dz} + \frac{N^2}{g} \right) \left( \sum_{i=1}^{\bar{r}} \sigma_i^{(r)} \rho^{(r)} w_i \right) = 0. \]  
(74)

The essential dynamics of these motions involves the coupling between the vertical velocity with buoyancy perturbations, and their structure and frequency are reminiscent of deep internal gravity waves. This justifies our classification of them as two-fluid gravity modes. As for the two-fluid gravity modes, the frequency of these motions is independent of their vertical structure.

The frequency of these motions is independent of their vertical structure. Therefore, there is no unique way to define a set of vertical normal modes. A convenient choice is \( w_1 \propto (\sigma_2^{(r)} / \sigma_1^{(r)})^{1/2} \exp[imz] \), and so forth. This choice ensures that the modes for different \( m \) are indeed mutually orthogonal (i.e. normal) with respect to the energy of the linearized equations:
\[ E_{\text{lin}} = \sum_{i} \left\{ \frac{\sigma_i^{(r)} \rho^{(r)}}{2} \left( |u_i|^2 + \frac{g^2 Q^2}{N^2} |\eta_i|^2 + \frac{|p|^2}{\rho^{(r)} z^2} \right) \right\}, \]  
(77)
and it allows us to discuss the vertical wavenumber \( m \). [The expression Equation 77 reduces to that given by, for example, Phillips (1990) and Thuburn et al. (2002) in the case of a single fluid component.]

For any given \((k, l, m)\), there are two possible frequencies, \( \omega = \pm N \), giving two branches to the dispersion relation. Although the structures and frequencies of these modes resemble those of deep internal gravity modes in some respects, these features hold for all \( m \), including large \( m \), so the mode structures do not, in fact, have to be deep.

Finally, since the frequency of these modes is independent of \( k, l, \) and \( m \), their group velocity is identically zero. They propagate neither horizontally nor vertically.

4.3 Two-fluid inertial modes

To have zero \( p \) with non-zero horizontal velocity, Equations 61 and 62 imply that \( \tilde{\omega}^2 = f^2 \), i.e. \( \omega = \pm f - k \beta / K^2 \). Equation 60 then implies that \( w_1 = 0 \), and hence \( \rho_i = 0 \) and \( \eta_i = 0 \). Either Equation 51 or Equation 52 shows that
\[ v_i = \mp i u_i, \]  
(78)
from which it follows that the horizontal divergence,
\[ \delta_i = i k u_i + i l v_i = i k u_i \pm i l u_i, \]  
(79)
is in quadrature with the vertical component of vorticity,
\[ \zeta_i = i k v_i - i l u_i = \pm k u_i - i l u_i = \mp i \delta_i. \]  
(80)

Equation 49 implies that
\[ \sum_i \sigma_i \delta_i = 0, \]  
(81)
so the net mass flux convergence vanishes everywhere. Combining this with Equation 79 then shows that the net mass flux \( \sum_{i} \sigma_i^{(r)} u_i \) vanishes everywhere. The essential dynamics of these motions involves the coupling between \( u \) and \( v \) via the Coriolis term, and they have structure and frequency resembling inertial modes, but with compensating horizontal mass fluxes in the two-fluid components, rather than in layers at nearby heights. Hence we classify them as two-fluid inertial modes. As for the two-fluid gravity modes, the pressure perturbation vanishes, but now there is the possibility for horizontal coupling through horizontal advection.

As for the two-fluid gravity modes, the frequency of these motions is independent of their vertical structure. Again, there is no unique way to define a set of vertical normal mode structures, but a convenient choice is \( u_1 \propto (\sigma_2^{(r)} / \sigma_1^{(r)})^{1/2} \exp[imz] \), and so forth, so that the modes for different \( m \) are orthogonal with respect to Equation 77.

For any given \((k, l, m)\), there are two branches of these normal modes, with frequencies \( \omega = \pm f - \beta k / K^2 \). The frequency is independent of \( m \), so their vertical group velocity is zero. Their horizontal group velocity is small but non-zero, similar to that of barotropic Rossby waves.

4.4 Relabeling modes

One further branch of modes is possible, in which \( u_i, v_i, w_i, \rho_i, \eta_i \), and \( p \) all vanish. The frequency is zero, but the volume-fraction perturbations are non-zero and satisfy
\[ \sigma_1 = -\sigma_2. \]  
(82)
These represent modes in which some fluid has been relabeled, but the physical state of the system is identical to the basic state. The energy perturbation Equation 77 vanishes for these modes.
4.5 Normal modes for $n > 2$ fluid components

The normal modes for the two-fluid case discussed above generalize in a straightforward way to the case of any number $n$ of fluid components. The derivation of Equation 71 carries through exactly as before, so we have the same branches of single-fluid-equivalent modes: eastward- and westward-propagating acoustic modes, eastward- and westward-propagating inertio-gravity modes, and westward-propagating Rossby modes. As before, $\rho_i$, $\eta_i$, and $u_i$ are all independent of $i$.

The two branches of two-fluid gravity modes become $2(n-1)$ branches of multi-fluid gravity modes. They all have $u_i$ and $v_i$ identically zero, and satisfy $\sum_i \sigma_i^{(r)} \rho_i^{(r)} w_i = 0$. One way to confirm the number of branches is to note that, for $\omega = \pm N$ (hence the factor 2), and for any vertical profile $w_i(z)$, there are $n-1$ linearly independent modes with $\sigma_i^{(r)} w_i^{(r)} + \sigma_j^{(r)} w_j^{(r)} = 0$ and $w_j = 0$ when $j \neq i$, for $i = 2, \ldots, n$. If an orthogonal set of modes is needed, then this can be obtained (non-uniquely) by writing the vertical velocity of the $j$th mode as

$$a_j^{(i)} = a_j^{(i)} f^{(m)}(z),$$

(83)

where

$$f^{(m)}(z) = \left( \sum_i \frac{1}{\sigma_i^{(r)}} \right)^{-1/2} \exp(i m \zeta)$$

(84)

and

$$a_i^{(j)} = \exp \left( \frac{2 \pi i j}{n} \right).$$

(85)

As before, all of these modes have zero group velocity.

In an analogous way, the two branches of two-fluid inertial modes become $2(n-1)$ branches of multi-fluid inertial modes. They all have $\rho_i$, $\eta_i$, and $w_i$ identically zero, and satisfy $\sum_i \sigma_i^{(r)} u_i = 0$. An orthogonal set of modes is obtained by defining the $u_i$ for the $j$th mode as

$$a_j^{(i)} = a_j^{(i)} f^{(m)}(z),$$

(86)

and so forth, with $f^{(m)}(z)$ and $a_i^{(j)}$ as above. As before, all these modes have zero vertical group velocity.

Finally, the branch of two-fluid relabeling modes becomes $n-1$ branches of multi-fluid relabeling modes.

4.6 Normal modes in the Boussinesq equations

To interpret the normal modes, it is helpful to consider the two-fluid Boussinesq equations. These equations eliminate acoustic modes ab initio, and if we restrict attention further to the $f$-plane, we expose the physically new modes of the system more transparently. Allowing density to vary only in terms associated with gravity, the Boussinesq versions of Equations 8–12 are given by the following, now including the Coriolis term:

**Volume fractions must sum to unity:**

$$\sum_{i=1}^{n} \sigma_i = 1,$$

(87)

**Mass or volume conservation:**

$$\frac{\partial \sigma_i}{\partial t} + \nabla \cdot (\sigma_i u_i) = 0,$$

(88)

where $k$ is the unit vector in the vertical, $\phi$ is the deviation of the kinematic filter-scale pressure ($\overline{p_f}$) from a hydrostatic reference state, and $b_i$ is the buoyancy of the $i$th component.

If we take $f$ to be a constant and linearize these equations around a state of rest of given basic-state volume fraction $\sigma_i^{(r)}$ and constant stratification $N^2$, we obtain the following:

**Perturbation volume fractions sum to zero:**

$$\sum_i \sigma_i = 0,$$

(91)

**Mass conservation:**

$$-i \sigma_i + \sigma_i^{(r)} (iku_i + ilv_i + imw_i) = 0,$$

(92)

**$u$ momentum:**

$$-i \sigma_i u_i - f v_i + ik \phi = 0,$$

(93)

**$v$ momentum:**

$$-i \sigma_i v_i + f u_i + il \phi = 0,$$

(94)

**$w$ momentum:**

$$-i \sigma_i w_i + im \phi - b_i = 0,$$

(95)

**Buoyancy:**

$$-i \sigma_i b_i + N^2 w_i = 0.$$
and

\[ \omega^2 = f^2, \quad \sigma_i^{(r)}(ku_1 + hv_1) = -\sigma_i^{(r)}(ku_2 + hv_2). \] (104)

The two-fluid gravity-wave mode, Equation 103, is of particular note. Since \( \sigma_i^{(r)} \) is positive for all \( i \), the mode represents ascending motion by one fluid and descending motion by the other fluid. (The mode is, of course, present in the fully compressible equations, but in the Boussinesq derivation it is seen most plainly.) It is not an unphysical mode, since the conditionally filtered equations represent motion on a large scale. Rather, within that large scale there is an oscillation consisting of ascent of one fluid component and descent of the other. It may be the most important new mode introduced by conditional filtering, since subfilter-scale buoyancy-driven motions such as cumulus convection will project strongly on to this mode.

### 4.7 Behavior in the limit of short filter scale

One of the motivations for the introduction of the conditionally filtered equations was the desire to formulate a mathematical framework that could represent cumulus convection in both the unresolved case, where the scale of convection is much smaller than the filter scale, and the resolved case, where the scale of convection is greater than the filter scale. Rather, within that large scale there is an oscillation consisting of ascent of one fluid component and descent of the other. It may be the most important new mode introduced by conditional filtering, since subfilter-scale buoyancy-driven motions such as cumulus convection will project strongly on to this mode.

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The Lagrangian density is essentially the kinetic energy density minus the potential and internal energy density, but there are different flavours of the idea depending on whether an Eulerian or Lagrangian description of the fluid motion is of interest, and whether constraints such as conservation of mass are imposed through restricting the allowed perturbations $\delta \mathbf{x}$ or through Lagrange multipliers in Equation 108 (e.g. Salmon, 1998).

In this section, we focus on an Eulerian description of the fluid motion, following chapter 7, section 8 of Salmon (1998).

As we would for a single fluid, we impose conservation of mass, and also material conservation of entropy and material conservation of another Lagrangian label, via Lagrange multipliers, for each fluid component $i$. (See Salmon, 1998 for a discussion of how the extra Lagrangian label relates to Lin’s constraint.) Equality of the pressures in the different fluid components and the requirement for the volume fractions to sum to unity are also imposed through Lagrange multipliers. Hence the appropriate expression for $\mathcal{L}$ is

$$\mathcal{L} = \int \int dt \, d\mathbf{x} \, L(\mathbf{x}).$$

(109)

Here, $\phi_i$ is the Lagrange multiplier associated with conservation of mass of the $i$th fluid, $A_i$ is the Lagrange multiplier associated with material conservation of $\eta_i$, $\lambda_i$ is a Lagrangian label for the $i$th fluid and $C_i$ is the Lagrange multiplier associated with material conservation of $\lambda_i$, the $\nu_i$ are a set of Lagrange multipliers associated with the equality of pressure in the different fluid components, and $\mu$ is the Lagrange multiplier associated with the volume fractions summing to unity.

The pressure $p_i$ is related to the internal energy density $e_i$ by

$$p_i = \rho_i^\gamma \left. \frac{\partial e_i}{\partial \rho_i} \right|_{\eta_i}.$$ 

(111)

Hamilton’s principle now states that $\delta \mathcal{L} = 0$ for arbitrary, independent, small variations of $\sigma_i$, $\rho_i$, $\eta_i$, $\mathbf{u}_i$, $\lambda_i$, $\phi_i$, $A_i$, $C_i$, $\nu_i$, and $\mu$. Boundary conditions (in space and time) are assumed to be such that any boundary terms arising through integration by parts vanish.

For variations in $\mu$, $\delta \mathcal{L} = 0$ implies

$$\sum_i \sigma_i - 1 = 0,$$

(112)

in agreement with Equation 8. For variations in $\nu_i$, $\delta \mathcal{L} = 0$ implies

$$p_i = p_1;$$

(113)

thus the pressures in all the fluid components take the same value, which we can call $\overline{p}$ for consistency with the earlier notation. For variations in $A_i$ and $C_i$, $\delta \mathcal{L} = 0$ implies

$$\frac{D_i \eta_i}{Dt} = 0,$$

(114)

consistent with Equation 18, and

$$\frac{D_i \lambda_i}{Dt} = 0.$$

(115)

For variations in $\phi_i$, $\delta \mathcal{L} = 0$ implies, after integration by parts,

$$\frac{\partial}{\partial t} (\sigma_i p_i) + \nabla \cdot (\sigma_i \rho_i \mathbf{u}_i) = 0,$$

(116)

consistent with Equation 9.

For variations in $\sigma_i$ and in $\rho_i$, $\delta \mathcal{L} = 0$ implies

$$\rho_i \left( \frac{1}{2} |\mathbf{u}_i|^2 - \Phi - e_i - \frac{D_i \phi_i}{Dt} \right) - \mu = 0$$

(117)

and

$$\frac{1}{2} |\mathbf{u}_i|^2 - \Phi - e_i - \frac{\overline{p}}{\rho_i} - \frac{D_i \phi_i}{Dt} = 0,$$

(118)

where Equations 114 and 115 have been used to eliminate terms involving $D_i \eta_i / Dt$ and $D_i \lambda_i / Dt$, and Equation 113 has been used to write $p_i = p_1 = \overline{p}$.

Taking Equation 117 minus $\rho_i$ times Equation 118 shows that

$$\mu = \overline{p}.$$ 

(119)
Thus $\bar{p}$ is the Lagrange multiplier corresponding to the requirement for the volume fractions to sum to unity. This is reflected in the fact that the volume fractions summing to unity is crucial for determining $\bar{p}$; see Equation A7. This result is analogous to the well-known interpretation of pressure as the Lagrange multiplier corresponding to the incompressibility condition for an incompressible fluid.

For variations in $\eta_i$ and $\lambda_i$, $\delta \mathcal{L} = 0$ implies, after integration by parts,

$$-\sigma_i \rho_i T_i + \frac{\partial}{\partial t} \left( \sigma_i \rho_i A_i \right) + \nabla \cdot \left( \sigma_i \rho_i A_i \mathbf{u}_i \right) = 0,$$

(120)

and

$$\frac{\partial}{\partial t} \left( \sigma_i \rho_i C_i \right) + \nabla \cdot \left( \sigma_i \rho_i C_i \mathbf{u}_i \right) = 0,$$

(121)

where

$$T_i = \frac{\partial e_i}{\partial \eta_i} \bigg|_{\rho_i}$$

(122)

is the temperature of the $i$th fluid. Finally, for variations in $\mathbf{u}_i$, $\delta \mathcal{L} = 0$ implies

$$\mathbf{u}_i - \nabla \phi_i - A_i \nabla \eta_i - C_i \nabla \lambda_i = 0.$$  

(123)

To obtain the equations of motion, we systematically eliminate the remaining Lagrange multipliers and the materially conserved scalars $\lambda_i$. Taking Equation 120 minus $A_i$ times Equation 116 gives

$$\frac{D_i A_i}{D t} = T_i,$$

(124)

while Equation 121 minus $C_i$ times Equation 116 gives

$$\frac{D_i C_i}{D t} = 0.$$  

(125)

Taking $\partial / \partial t$ of 123 gives

$$\frac{\partial \mathbf{u}_i}{\partial t} - \nabla \frac{\partial \phi_i}{\partial t} - \frac{\partial A_i}{\partial t} \nabla \eta_i - A_i \nabla \frac{\partial \eta_i}{\partial t} - \frac{\partial C_i}{\partial t} \nabla \lambda_i - C_i \nabla \frac{\partial \lambda_i}{\partial t} = 0.$$  

(126)

Taking $\mathbf{u}_i$-Equation 123, subtracting Equation 118, and using Equations 114 and 115 gives

$$\frac{1}{2} \left| \mathbf{u}_i \right|^2 + \Phi + e_i + \frac{\bar{p}}{\rho_i} + \frac{\partial \phi_i}{\partial t} + A_i \frac{\partial \eta_i}{\partial t} + C_i \frac{\partial \lambda_i}{\partial t} = 0.$$  

(127)

Then taking Equation 126 plus the gradient of Equation 127 and using Equations 114, 115, 124, and 125 gives

$$\frac{\partial \mathbf{u}_i}{\partial t} + \nabla \left( \frac{1}{2} \left| \mathbf{u}_i \right|^2 + \Phi + e_i + \frac{\bar{p}}{\rho_i} \right) = (\mathbf{u}_i \cdot \nabla \eta_i) \nabla A_i + (T_i - \mathbf{u}_i \cdot \nabla A_i) \nabla \eta_i + (\mathbf{u}_i \cdot \nabla \lambda_i) \nabla C_i - (\mathbf{u}_i \cdot \nabla C_i) \nabla \lambda_i.$$

(128)

Taking the curl of Equation 123 gives

$$\zeta_i = \nabla \times \mathbf{u}_i = \nabla A_i \times \nabla \eta_i + \nabla C_i \times \nabla \lambda_i,$$

(129)

so that

$$\zeta_i \times \mathbf{u}_i = (\mathbf{u}_i \cdot \nabla A_i) \nabla \eta_i - (\mathbf{u}_i \cdot \nabla \eta_i) \nabla A_i + (\mathbf{u}_i \cdot \nabla C_i) \nabla \lambda_i - (\mathbf{u}_i \cdot \nabla \lambda_i) \nabla C_i.$$  

(130)

Hence, Equation 128 simplifies to

$$\frac{\partial \mathbf{u}_i}{\partial t} + \zeta_i \times \mathbf{u}_i + \nabla \left( \frac{1}{2} \left| \mathbf{u}_i \right|^2 + \Phi + e_i + \frac{\bar{p}}{\rho_i} \right) = T_i \nabla \eta_i.$$  

(131)

Finally, noting that

$$\nabla \left( e_i + \frac{\bar{p}}{\rho_i} \right) = T_i \nabla \eta_i + \frac{1}{\rho_i} \nabla \bar{p},$$

(132)

Equation 131 reduces to

$$\frac{\partial \mathbf{u}_i}{\partial t} + \zeta_i \times \mathbf{u}_i + \frac{1}{\rho_i} \nabla \bar{p} + \nabla \left( \frac{1}{2} \left| \mathbf{u}_i \right|^2 + \Phi \right) = 0,$$

(133)

which agrees with Equation 34 in the absence of its right-hand side.

Equations 112, 116, 114, and 133 derived from the variational method thus agree with the conditionally filtered equations 8, 9, 18, and 34.

6 | SUMMARY AND DISCUSSION

We have documented the conservation properties, normal modes, and a variational formulation of the conditionally filtered equations. The results confirm that these equations have a natural mathematical structure, respecting key physical properties, lending them some credibility for their use in modeling atmospheric flows. In particular, the normal mode results, with real frequency results, with real frequency, imply that the equations are free from spurious unphysical instabilities, at least for small perturbations to a simple basic state. Furthermore, the modes themselves have a sensible physical interpretation. The usual Rossby, inertia-gravity and acoustic modes exist and have the same frequency and structure as in the single-fluid case. In addition, we have identified inertia and gravity modes with zero pressure perturbation in which the fluid components move separately, and in general in opposite vertical and horizontal directions. This is precisely a property one might wish for when modeling subgrid-scale convection, in which some of the subgrid-scale fluid ascends while some of it descends. Furthermore, the amplitude of these modes goes to zero as the filter scale diminishes, which is an attractive property when considering how the fluid system might behave as the model resolution increases.

The availability of a variational formulation implies that a variety of standard approximations, such as hydrostatic or pseudo-incompressible, should be applicable to the conditionally averaged equations, leading to simpler equation sets that might be appropriate for some applications, both theoretical and numerical. We have already begun to experiment with hydrostatic and Boussinesq versions of the conditionally filtered equations. It is even possible to make different approximations in different fluid components: for example, making one component hydrostatic (though some thought must then be given to the relabeling terms if strict energy consistency is required). However, one would of course normally wish for the fluid component that represents convecting fluid to
be treated non-hydrostatically. The results may also be of use in developing and testing numerical methods for the solution of the conditionally filtered equations. For example, numerical methods should respect the conservation properties of the continuous equations, at least to within the numerical truncation error. The normal modes derived here provide known, exact, stable, linear solutions that a numerical method should be able to reproduce.

Finally, the results also give some early indications of the suitability of the conditionally filtered equations for modelling cumulus convection, the application for which they were originally proposed. The multi-fluid gravity modes show that the conditionally filtered equations can capture the essential dynamics of vertical buoyancy-driven motion of one fluid component relative to another, which will be required in order to model convective updraughts and downdraughts. Of course the subfilter-scale terms, interfluid pressure terms, and relabelling terms, that is, the right-hand sides of Equations 2–4, which would need to parameterized, are also of leading-order importance for such flows (Siebesma et al., 2007; de Rooy et al., 2013; Romps and Charn, 2015). On the other hand, the vanishing group velocity of the multi-fluid gravity modes suggests that the conditionally filtered equations would not help us to capture convectively generated gravity waves (e.g. Lane and Moncrieff, 2010) unless those waves project on to the single-fluid-equivalent gravity modes. It is also conceivable that, away from the region of convection, the two-fluid gravity modes and inertial modes might have undesirable behaviour. For example, their dispersion properties might lead to behavior analogous to that of some numerical computational modes. If this turns out to be the case, then some measures to suppress them might be needed. The analysis presented here should, at least, help to identify such problems.

ACKNOWLEDGEMENTS

We are grateful to two anonymous reviewers for their constructive comments on an earlier version of this article. This work was funded by the Natural Environment Research Council under grant NE/N013123/1 as part of the ParaCon programme.

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APPENDIX A: PROGNOSTIC EQUATIONS FOR $\sigma_i$, $\rho_i$, AND $\vec{p}$, AND A DIAGNOSTIC EQUATION FOR $\vec{p}$

It is not immediately obvious how Equations 8–12 imply the time evolution of all variables. As well as the fundamental question of whether the system of equations is closed, this is relevant to the design of numerical methods for the solution of the conditionally averaged equations.

First note that Equation 9 can be expanded as

$$\sigma_i \frac{\partial \rho_i}{\partial t} + \rho_i \frac{\partial \sigma_i}{\partial t} + \nabla \cdot (\sigma_i \rho_i \mathbf{u}_i) = 0. \quad \text{(A1)}$$

The time derivative of Equation 12 can be written

$$\frac{1}{c_i^2 \rho_i} \frac{\partial \vec{p}}{\partial t} = \frac{1}{\rho_i} \frac{\partial \rho_i}{\partial t} + Q_i \frac{\partial \eta_i}{\partial t} \quad \text{(A2)}$$

where $c_i^2 = \partial P/\partial \rho_i |_{\eta_i}$ is the sound speed squared in the $i$th fluid, and $Q_i = -\partial \ln \rho_i / \partial \eta_i |_{\rho_i}$ (compare Equation 54).

Multiplying by $\sigma_i$ and substituting from Equations A1 and 18 gives

$$\frac{\sigma_i}{c_i^2 \rho_i} \frac{\partial \vec{p}}{\partial t} = -\frac{\partial \sigma_i}{\partial t} - \frac{1}{\rho_i} \nabla \cdot (\sigma_i \rho_i \mathbf{u}_i) - \sigma_i Q_i \cdot \nabla \eta_i. \quad \text{(A3)}$$

Summing over $i$ and using Equation 8 gives an equation for the rate of change of $\vec{p}$ in terms of known quantities:

$$\sum_i \frac{\sigma_i}{c_i^2 \rho_i} \frac{\partial \vec{p}}{\partial t} = -\sum_i \frac{1}{\rho_i} \nabla \cdot (\sigma_i \rho_i \mathbf{u}_i) - \sum_i \sigma_i Q_i \cdot \nabla \eta_i. \quad \text{(A4)}$$

Having obtained $\partial \vec{p}/\partial t, \partial \rho_i/\partial t$ follows from Equation A2, and $\partial \sigma_i/\partial t$ follows from Equation A1.

Alternatively, a diagnostic equation for $\vec{p}$ in terms of the predicted quantities $\sigma_i \rho_i$ and $\eta_i$ can be derived as follows. The equation of state can be rearranged to make $\rho_i$ the subject:

$$\rho_i = R(\vec{p}, \eta_i). \quad \text{(A5)}$$

Then

$$\sigma_i = \frac{(\sigma_i \rho_i)}{R(\vec{p}, \eta_i)}. \quad \text{(A6)}$$

and summing over $i$ gives

$$\sum_i \frac{(\sigma_i \rho_i)}{R(\vec{p}, \eta_i)} = 1. \quad \text{(A7)}$$

Thus we have a single equation for the single unknown $\vec{p}$.

In the special case of a perfect gas equation of state, and predicting potential temperature $\theta_i$ instead of entropy $\eta_i$, Equation A7 simplifies to

$$\left( \frac{\vec{p}}{p_0} \right)^{(1-Rd/c_p)} = \frac{Rd}{p_0} \sum_i \sigma_i \rho_i \theta_i. \quad \text{(A8)}$$

where $p_0$ is a constant reference pressure, $Rd$ is the gas constant, and $c_p$ is the specific heat capacity at constant pressure (Thuburn et al., 2018).

APPENDIX B. PROPERTIES OF THE EQUAL-\textit{v}, VARIANT

This Appendix examines briefly how the results discussed in the main body of the article carry over, or are modified, for the variant of the conditionally filtered equations in which all fluid components have the same horizontal velocity.

B1 | Conservation properties

The equal-\textit{v} variant effectively assumes that the $-b_i - \sum_j d_{ij}$ terms on the right-hand side of Equation 4 are exactly what is needed to maintain equality of the \textit{v}$_i$. Since Equations 9, 10, and 18 do not involve $-b_i - \sum_j d_{ij}$, the conservation laws for mass and entropy, and the material conservation of entropy, remain the same as for the full equations. Equation 14 for the evolution of $\nabla \cdot \mathbf{v}$ is obtained by summing Equation 4 over $i$ (neglecting $F_{Si}^{\theta}$ and $M_{ij}$) and using Equations 6 and 7, and is entirely consistent with Equation 22. Therefore, the equal-\textit{v} variant also conserves momentum.
The derivation of the energy equation 32 does not depend on any assumption about the $b_i + \sum_j d_{ij}$ terms, so it holds for the equal-$v_i$ variant too. Because the $b_i + \sum_j d_{ij}$ terms can no longer be assumed zero, we can no longer make the step to Equation 33. However, the contributions to the change in total energy from the horizontal components of $b_i + \sum_j d_{ij}$ sum to zero, leaving

$$\frac{\partial}{\partial t} \left( \sum_i \sigma_i \rho_i \varepsilon_i \right) + \nabla \cdot \left( \sum_i \left( \sigma_i \rho_i \mathbf{u}_i \varepsilon_i + \sigma_i \mathbf{u}_i \bar{p} \right) \right) = - \sum_i w_i \left( b_i^{(z)} + \sum_j d_{ij}^{(z)} \right); \quad (B1)$$

only the vertical components (indicated by superscript $(z)$) contribute to the change in total energy.

Material conservation of potential vorticity (Equation 41) does depend on the vanishing of the $b_i + \sum_j d_{ij}$ terms and therefore no longer holds for the equal-$v_i$ variant. The impermeability theorem, however, involves no assumptions about the forcing terms and continues to hold.

### B2  | Normal modes

The single-fluid-equivalent modes, multi-fluid gravity modes, and relabeling modes found in section 4 all have identical $v_i$ for all $i$. Therefore they continue to exist, with exactly the same frequency and structure, in the equal-$v_i$ variant. The multi-fluid inertial modes, on the other hand, must satisfy $\sum_i \sigma_i^{(r)} v_i = 0$. This could only hold with equal $v_i$ if $v_i = 0$ for all $i$, but then there would be no disturbance at all. Thus, the multi-fluid inertial modes do not exist in the equal-$v_i$ variant. These rather general arguments are confirmed by detailed calculation analogous to that in section 4.

### B3 | Variational formulation

We have not, so far, been able to discover a suitable variational formulation of the equal-$v_i$ variant of the conditionally filtered equations. There appear to be considerable technical subtleties associated with the equal-$v_i$ constraint.