Disorder perturbed Flat Bands II: a search for criticality

Pragya Shukla

Department of Physics, Indian Institute of Technology, Kharagpur, India.

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Abstract

We seek the possibility of a disorder driven transition in a tight-binding lattice with a flat band using complexity parameter approach. Our results indicate the existence of a localized to extended states transition with increasing disorder, insensitive to disorder strength, in weak disorder limit; the spectral statistics at the critical point corresponds to a critical Brownian ensemble, a non-equilibrium universality class of random matrix ensembles, intermediate to Poisson and Gaussian orthogonal ensemble. With increasing disorder, the statistics again approaches Poisson limit indicating a localization → extended → localization transition of the wave-dynamics. Our analysis also reveals a hidden connection of weakly disordered flat bands to a wide-range of other complex systems including standard Anderson Hamiltonian.

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I. INTRODUCTION

A dispersion-less band, also referred as a flat band, appears in crystal lattices under subtle interplay of the system conditions. The onset of disorder may lead to violation of these conditions, lifting the degeneracy of the energy levels and changing the nature of eigenfunction dynamics. The important role played by these bands e.g. in magnetic systems makes it relevant to seek the detailed information about the role played by disorder in changing the physical properties e.g. if varying disorder may lead to a localization to delocalization transition and whether its nature is similar to other disorder driven transitions.

The standard search of a localization to delocalization transition, hereafter referred as LD transition, in a disordered system is based on a range of criteria e.g. the existence of a order parameter, a divergence of correlations length at the critical point, a scaling behavior for finite system sizes and critical exponents of the average physical properties. For complex systems however the fluctuation of physical properties, from one sample to another or even within one sample subjected to a perturbation, are often comparable to their average behavior and their influence on the physical properties can not be ignored. As a consequence, one has to consider criteria based on the distribution of the physical properties \[1\]. In case of systems where the physical properties can in principle be expressed in terms of the eigenvalues and eigenfunctions of a relevant linear operator, it is appropriate to seek criteria based on their joint probability distribution function (JPDF) \[1\].

The definition of criticality in a JPDF of \( N \) variable \( x_1, \ldots, x_N \) is in general based on a single parameter scaling concept \[1\]. The distribution \( P(x_1, \ldots, x_N; t_1, \ldots, t_n) \) that depends on system size \( N \) and a set of \( n \) parameters \( t_1, t_2, \ldots, t_n \) obeys one parameter scaling if for large \( N \) it is approximately a function of only variables \( x_1, \ldots, x_N \) and one scale dependent parameter, say, \( \Lambda \equiv \Lambda(N, t_1, \ldots, t_n) \). For system conditions under which the limit \( \Lambda^* = \lim_{N \to \infty} \Lambda(N) \) exists, the distribution approaches a universal limiting form \( P^*(\{x\} \Lambda^*) = \lim_{N \to \infty} P(\{e\}, \Lambda) f[1] \) and is referred as critical with \( \Lambda^* \) as the critical parameter. In \[2\], we derived a single parametric formulation of the ensemble density of the Hamiltonian for a typical perturbed flat band. As an integration of the ensemble density over all eigenfunction leads to the JPDF of its eigenvalues, this encourages us to search for a single parametric scaling of the latter as well as higher order eigenvalue correlations. The universal limit of these correlation, if it exists, is referred as the critical spectral statistics for the perturbed
The concept of critical spectral statistics was first introduced in [3] in context of metal-insulator transition in disordered Hamiltonians; the study showed that the distribution $P(s)$ of the spacings $s$ between the nearest neighbor eigenvalues of the Hamiltonian turns out to be a universal hybrid of the Wigner-Dyson distribution at small-$s$ and Poisson at large-$s$, with an exponentially decaying tail: $P(s) \sim e^{-\kappa s}$ for $s \gg 1$ with $\kappa$ as a constant. The analytical studies later on indicated the criticality to manifest also through an asymptotically linear behavior of the number variance $\Sigma^2(r)$ (the variance in the number of levels in an spectrum interval of length $rD$) in mean number of levels $r$ with a fractional coefficient.

As indicated by many studies of the transition in disordered systems, the wave-functions at the critical point are multifractal [1]. This led to introduction of the singularity spectrum as the criteria for the criticality. The wave-functions in the delocalized limit are essentially structureless and overlapping almost everywhere which leads to GOE type level repulsion. In localized limit, the wave-functions are typically localized at different basis state with almost negligible overlap which manifests in uncorrelated level-statistics described by Poisson universality class. But the mutifractality leads to an intimate conspiracy between the correlations of energy levels and eigenfunctions. This is because the two fractal wave-functions, irrespective of their sparsity, still overlap strongly which in turn affects the decay of level correlations at long energy ranges. For $|e_n - e_m| \gg \Delta$, the correlation between two wave-functions $\psi_n(r)$ and $\psi_m(r)$ at energy $e_n$ and $e_m$ is given as [4]: $\langle |\psi_n(r)|^2 |\psi_m(r)|^2 \rangle \propto |e_n - e_m|^{1-D_2/d}$. In [4], $\chi$ was suggested to be related to the multifractality of eigenfunctions too: $\chi = \frac{d-D_2}{2d}$ with $D_2$ as the fractal dimension and $d$ as the system-dimension. However numerical studies later on indicated the result to be valid only in the weak-multifractality limit [6].

Our objective in the present work is to analyze the criticality of the spectral statistics and eigenfunctions when a flat band is perturbed by the disorder. In [2], we analyzed the disordered tight binding Hamiltonians with a flat band in the clean limit, using the site basis for their matrix representation. Presence of disorder makes it necessary to consider an ensemble of such Hamiltonians; assuming the Gaussian disorder in on-site energies (and/or interaction strengths, hopping etc) and by representing the non-random matrix elements by a limiting Gaussian, the ensemble density, say $\rho(H)$ with $H$ as the Hamiltonian, was described in [2] by a multi-parametric Gaussian distribution, with uncorrelated or correlated matrix elements. Using the complexity parameter formulation [5] [7–9], the statistics of $\rho(H)$ can
then be mapped to that of a single parametric Brownian ensemble (BE) appearing between Poisson and Wigner-Dyson ensemble \[5,13,14\] (also equivalent to Rosenzweig-Porter model \[16\]). The mapping is achieved by identifying a rescaled complexity parameter \(Y\) of the BE with that of the disordered band. The mapping not only implies connections of the flat band statistics with BE but also with other complex systems under similar global constraints e.g. symmetry conditions and conservation laws. Additionally it also leads to a single parametric formulation of the level density and inverse participation ratio of the perturbed flat band.

In case of the BEs, the existence of a critical statistics and multifractal eigenstates is already known \[5,17\]. Their connection with disorder perturbed lat bands suggests presence of criticality in the latter too. This motivates us to pursue a complexity parameter approach in the present work. Our results indicate existence of a critical statistics for all weak disorders which is in contrast to a single critical point in the disorder driven Anderson transition. Although the disorder independence of the statistics of a weakly disordered flat band was numerically observed in previous studies \[18\]–\[20\], its critical aspects were not explored. Another feature different from the Anderson transition is the following: with increasing disorder, the spectral statistics in a flat band undergoes a Poisson \(\rightarrow\) Brownian ensemble \(\rightarrow\) Poisson transition, implying a localization \(\rightarrow\) extended \(\rightarrow\) localization transition of the eigenstates. As well-known, the standard Anderson transition undergoes a delocalization \(\rightarrow\) localization transition with increasing disorder \[7\]. Notwithstanding these differences, the complexity parameter formulation predicts an Anderson analog of a weakly disordered flat band and also reveals its connection of to a wide range of other ensembles \[11,12,27\] of the same global constraint class; the prediction is verified by a numerical analysis discussed later in the paper. Although the theoretical analysis presented here is based on the Gaussian disorder in flat bands but it can also be extended to other type of disorders \[9\].

The paper is organized as follows. The complexity parameter formulation for the ensemble density of a disordered tight-binding lattice with a flat band is discussed in \[2\]. To avoid the repetition, we directly proceed, in section II, to review the complexity parameter formulation for the statistics of the eigenvalues and eigenfunctions. The information given in section II is used in sections III and IV to seek criticality in the disorder perturbed flat bands and the influence of other neighboring bands on the statistics. A detailed numerical analysis of our theoretical claims is presented in section V and a numerical comparison of disordered flat bands statistics with two other disordered ensembles with same global constraints, namely,
the Anderson ensemble with on-site Gaussian disorder and Rosenzweig-Porter ensemble is discussed in section VI. We conclude in section VII with a brief summary of our main results.

II. CRITICAL SPECTRAL STATISTICS

Consider the Hamiltonian $H$ of a disorder perturbed tight binding lattice with at least one flat band in clean limit: $H = V + U$ with $V$ and $U$ as single particle and two particle parts of $H$. For the cases $U = 0$ or $U \neq 0$, $H$ can be represented as a matrix in the $N$-dimensional basis of the single particle states or Foch states, respectively; here $N$ is a system specific parameter. As discussed in [2] along with many examples, the statistical behavior of the $H$-matrix, with entries $H_{kl}$, can be modeled by a multi-parametric ensemble with probability density $\rho(H,v,b) = C \exp[-\sum_{q=1}^{\beta} \sum_{k \leq l} \frac{1}{2v_{kl;q}}(H_{kl;q} - b_{kl;q})^2]$; here the subscript $q$ refers to the real or imaginary component of the variable. Further the variances $v_{kl;q}$ and mean values $b_{kl;q}$ can take arbitrary values (e.g. $v_{kl;q} \to 0$ for non-random cases). Assuming $e_1, e_2, \ldots, e_N$ and $U_1, \ldots, U_N$ as the eigenvalues and eigenfunctions of $H$, the correlations among their various combinations can then be obtained, in principle, by an integration of $\rho(H)$ over those variables which do not appear in the combination. To study the effect of varying system conditions on the correlations, it is however easier as well as more informative to first derive an evolution equation of $\rho(H)$ which on integration leads to the evolution equations for the correlations. As described in [2], irrespective of the number of changing conditions, the diffusion of $\rho(H)$ undergoes a single parametric evolution

$$\frac{\partial \rho}{\partial Y} = \sum_{k,l,q} \frac{\partial}{\partial H_{kl;q}} \left[ g_{kl} \frac{\partial}{2 \partial H_{kl;q}} + \gamma H_{kl;q} \right] \rho$$

with $Y$ referred as the ensemble complexity parameter of $\rho(H)$:

$$Y = -\frac{1}{\gamma N_{\beta}} \ln \left[ \prod_{k,l} \prod_{q=1}^{\beta} \left| 1 - (2 - \delta_{kl})\gamma v_{kl;q} \right| \left| b_{kl;q} + b_0 \right|^2 \right] + \text{constant.}$$

Here $g_{kl} = 1 + \delta_{kl}$ with $\delta_{kl}$ as a Kronecker delta function and $N_{\beta} = \frac{2N}{\beta}(N + 2 - \beta) + N_b$ and $N_b$ as the total number of $b_{kl;q}$ which are not zero. Further $b_0 = 1$ or 0 if $b_{kl;q} = 0$ or $\neq 0$, respectively and $\gamma$ as an arbitrary constant, marking the end of transition.

The statistical measures of a spectrum basically correspond to the fluctuations of spectral density around its average value and can in principle be obtained from the $n^{th}$ order level-density correlations $R_n(e_1, e_2, \ldots, e_n; Y)$, defined as $R_n = \int \prod_{k=1}^{n} \delta(e_k - \lambda_k) \rho(H; Y) \, DU$. A hierarchical diffusion equation for $R_n$ can be derived by a direct integration of eq.(1) over
$N-n$ eigenvalues and entire eigenvector space, (see eq.(16) of [14]). The specific case of $R_1(e)$ was discussed in detail in [2]; it varies at a scale $Y \sim N\Delta_e^2$. The solution of the equation for $n = 2$ case with Poisson initial conditions is discussed in [14] (see eq.(48) therein). Contrary to $R_1$, $R_n$ with $n > 1$ undergo a rapid evolution at a scale $Y \sim \Delta_e^2$, with $\Delta_e(e)$ as the local mean level spacing in a small energy-range around $e$. For comparison of the local spectral fluctuations around $R_1(e)$, therefore, a rescaling (also referred as unfolding) of the eigenvalues $e_n$ by local mean level spacing $\Delta_e(e)$ is necessary. This leads to a rescaling of both $R_n$ as well as the crossover parameter $Y$, with new correlations $R_n(r_1, \ldots, r_n) = \lim_{N \to \infty} (\Delta_e)^n R_n(e_1, e_2, \ldots e_n)$ and parameter $Λ_e$ given as

$$Λ_e(Y, e) = \frac{|Y - Y_0|}{\Delta_e^2}. \quad (2)$$

Hereafter $Λ_e$ will be referred as the spectral complexity parameter. It must be noted that $Λ_e \to \infty$ leads to a steady state i.e GOE (or GUE in absence of the time-reversal symmetry), $Λ_e \to 0$ corresponds to an initial state.

In principle, all spectral fluctuation measures can be expressed in terms of $R_n$; the spectral statistics as well as its criticality, therefore, depends on the system parameters and energy only through $Λ_e$. For system conditions under which the limit $Λ^* = \lim_{N \to \infty} Λ(N)$ exists, $R_n$ approaches a universal limiting form $R_n^*(r_1, \ldots, r_n; Λ^*) = \lim_{N \to \infty} P(r_1, \ldots, r_n; Λ)$. Clearly the size-dependence of $Λ_e$ plays an important role in locating the critical point which can be explained as follows. The standard definition of a phase transition refers to infinite system sizes (i.e limit $N \to \infty$); the parameter governing the transition is therefore expected to be $N$-independent in this limit. In general, both $Y - Y_0$ as well as $\Delta_e$ and therefore $Λ_e$ can be $N$-dependent. In finite systems, a variation of $N$ therefore leads to a smooth crossover of spectral statistics between an initial state ($Λ_e \to 0$) and the equilibrium ($Λ_e \to \infty$); the intermediate statistics belongs to an infinite family of ensembles, parameterized by $Λ_e$. However, for system-conditions leading to an $N$-independent value of $Λ_e$, say $Λ^*$, the spectral statistics becomes universal for all sizes; the corresponding system conditions can then be referred as the critical conditions with $Λ^*$ as the critical value of $Λ_e$. It should be stressed that the conditions satisfying the critical criteria may not exist in all systems; the critical statistics therefore need not be a generic feature of all systems.

At the critical value $Λ_e = Λ^*_e$, $R_n$ (for $n > 1$) and therefore all spectral fluctuation measures are different from the two end points of the transition i.e $Λ_e = 0$ and $\infty$ and any
one of them can, in principle, be used as a criteria for the critical statistics \[5\]. An important aspect of these measures is their energy-dependence: \( R_n \) retain the dependence through \( \Lambda_e \) even after unfolding and are non-stationary i.e vary along the spectrum \[5\]. Any criteria for the criticality in the spectral statistics can then be defined only locally i.e within the energy range, say \( \delta e_c \), in which \( \Lambda_e \) is almost constant \[5\].

Based on previous studies, two spectral measures namely nearest neighbor spacing distribution \( P(s) \) and the number variance \( \Sigma^2(r) \) are believed to be a good criteria for seeking criticality \[1, 13\]. Here \( P(s) \) measures the probability of a spacing \( s \) between two nearest neighbor energy levels and \( \Sigma^2(r) \) gives the variance of the number of levels in an interval of \( r \) unit mean spacings. As confirmed by several studies in past, the level fluctuations of a system in a fully delocalized wave limit behave similar to that of a Gaussian orthogonal ensemble (GOE)\[1, 13, 26\] with

\[
P(s) = \frac{s}{2} s e^{-\frac{4}{3}s^2} \quad \text{and} \quad \Sigma^2(r) \approx \frac{2}{\pi^2} \left( \ln(2\pi r) + \gamma + 1 - \frac{\pi^2}{8} \right)
\]

with \( \gamma = 0.5772 \). Similarly the fully localized case shows a behavior typical of a set of uncorrelated random levels, that is, exponential decay for \( P(s) \), also referred as Poisson distribution, \( P(s) = e^{-s} \), and \( \Sigma^2(r) = r \) \[1, 13, 26\]. (In case of structured matrices however Poisson spectral statistics may appear along with delocalized eigenfunctions \[25\].

For non-zero, finite \( \Lambda_e \) cases, the exact \( P(s) \) behavior is known only for the matrices of size \( N = 2 \) \[28\]. As \( P(s) \) is dominated by the nearest neighbor pairs of eigenvalues, this result is a good approximation also for \( N \times N \) case, especially in small-\( s \) and small-\( \Lambda_e \)-result \[29\]:

\[
P(s, \Lambda_e) = \frac{s}{4\Lambda_e} \exp \left( -\frac{s^2}{8\Lambda_e} \right) \int_0^\infty dx \, e^{-\frac{8x^2}{s \Lambda_e}} \, I_0 \left( \frac{x s}{4\Lambda_e} \right) \quad \beta = 1 \quad (3)
\]

\[
P(s, \Lambda_e) = \frac{s}{\sqrt{2\pi\Lambda_e}} \exp \left( -\frac{s^2}{8\Lambda_e} \right) \int_0^\infty dx \, \frac{1}{x} e^{-\frac{8x^2}{s \Lambda_e}} \sinh \left( \frac{x s}{4\Lambda_e} \right) \quad \beta = 2 \quad (4)
\]

with \( I_0 \) as the modified Bessel function. Clearly, \( P(s) \), in the critical regime, is a universal hybrid of Wigner-Dyson statistics at small-\( s \) and Poisson at large-\( s \), with an exponentially decaying tail: \( P(s) \sim e^{-\kappa s} \) for \( s \gg 1 \) with \( \kappa \) a constant. Similarly \( \Sigma^2(r) \) for the critical spectral statistics is linear but with fractional coefficient: \( \Sigma^2(r) \sim \chi r \) with \( 0 < \chi < 1 \).

The coefficient \( \chi \), also referred as the level compressibility, is often used as a measure for criticality of the statistics \[1\]. A characteristic of the long-range correlations of levels, \( \chi \) is defined as, in a range \( r \) around energy \( e \), \( \chi(e,r) = 1 - \int_r^s (1 - R_2(e, e + s)) \, ds \). As \( R_2(e, r) \) is related to \( \Sigma_2(e, r) \), \( \chi \) can also be expressed as the \( r \)-rate of change of \( \Sigma_2(e, r) \)\[1, 4\]:

7
\[ \chi = \lim_{r \to \infty} \frac{d\Sigma^2(r)}{dr}. \]

As discussed in [5, 7], \( \chi \) at the critical point \( \Lambda^* \) can be given as

\[
\chi \approx 1 - 4 \pi^2 \Lambda^* \quad \text{small } \Lambda^*
\]

\[
\approx \frac{1}{\beta \pi^2 \Lambda^*} \quad \text{large } \Lambda^*
\]

with \( \chi(e, r, \Lambda = 0) = 1 \) and 0 in Poisson and GOE (or GUE if \( \beta = 2 \)) limits, respectively. \( \chi \) is also believed to be related to the exponential decay rate of \( P(s) \) for large \( s \): \( \chi = \frac{1}{2\kappa}. \)

At the critical point, the fluctuations of eigenvalues measures are in general correlated with those of the eigenfunctions. The spectral features at the criticality are therefore expected to manifest in the eigenfunction measures too. As shown by previous studies [1], this indeed occurs through large fluctuations of their amplitudes at all length scales, and can be characterized by an infinite set of critical exponents related to the scaling of the ensemble averaged, generalized inverse participation ratio (IPR) i.e moments of the wavefunction intensity with system size. At transition, it reveals an anomalous scaling with size \( N \):

\[ \langle I_q \rangle(e) \sim N^{-(q-1)D_q/d} \]

with \( D_q \) as the generalized fractal dimension of the wave-function structure and \( d \) as the system dimension. At critical point, \( D_q \) is a non-trivial function of \( q \), with \( 0 < D_q < d \). The criticality in the statistics also manifests through other eigenfunction fluctuation measures e.g. IPR-distribution or two-point wave-function correlations [6]. A complexity parameter based formulation for these measures is discussed in [10] and [5, 15].

The dimensionality dependence of the critical point in the localization \( \to \) delocalization transitions of the wave-functions is well-established. This can also be seen through \( \Lambda_e \) based formulation where dimensionality enters mainly through \( \Delta_e(e) \). In the localized regime, the latter is a function of the average localization length \( \xi \) and dimensionality \( d \) of the system at energy \( e \). \( \Delta_e(e) = \frac{\Delta(e)N}{\xi} \). Note \( \xi(e) \) is in general a function of dimensionality [1] and can be expressed in terms of the average inverse participation ratio \( \langle I_2 \rangle \) of the eigenfunctions in a small neighborhood of \( e \): \( \xi^d = \left( \langle I_2 \rangle \right)^{-1} \). This along with eq.(2) then results in

\[
\Lambda_e(Y, N, e) = \frac{|Y - Y_0|}{N^2} \left( \frac{R_1}{\langle I_2 \rangle} \right)^2.
\]

As clear from the above, a size-independence of \( \Lambda_e(e) \) i.e existence of \( \Lambda^*(e) \) requires a subtle cancellation of size-dependence among the ensemble complexity parameter, averaged level density and inverse participation ratio.
III. TRANSITION IN AN ISOLATED FLAT BAND

In [2], we obtained the ensemble complexity parameter \( Y \) for a perturbed flat band. For cases, in which disorder \( w \) is the only parameter subjected to variation, \( Y \) turns out to be

\[
Y - Y_0 = \frac{-1}{N} \ln |1 - w^2|, \tag{8}
\]

where \( Y_0 \) corresponds to the unperturbed flat band (disorder \( w = 0 \)) and \( N \) is the number of energy levels in the band.

As discussed in [2], the level density \( R_1 \) for an isolated flat band for arbitrary \( w \) is

\[
R_1(e; w) = \frac{N}{\sqrt{2\pi w^2}} e^{-\frac{e^2}{2w^2}} \tag{9}
\]

Further the inverse participation ratio \( \langle I_2 \rangle \) for arbitrary \( w \) and large \( N \) can be approximated as

\[
\langle I_2 \rangle \approx \frac{6\pi}{N E_c} \frac{u_0}{E_c} e^{-\frac{4u_0}{E_c} + \frac{e^2}{2w^2}} \tag{10}
\]

with \( u_0 \) as the local intensity at \( e = 0 \) and \( \Lambda_f = \frac{4\ln|1-w^2|}{E_c^2} \). Here \( E_c \) is an energy scale associated with the range of level-repulsion around \( e \) and can in general depend on \( e \) as well as \( w \). Eq.(10) is obtained by assuming \( E_c \sim N^{-\mu} \) with \( \mu > 0 \). The assumption is consistent with the definition of \( E_c \); as discussed in [2], \( E_c \sim E_{th} \) with \( E_{th} \) as the Thouless energy: \( E_{th} \sim o(N^{-1}) \) and \( o(N^0) \) for the localized and delocalized dynamics respectively but in partially localized regime \( E_{th} \sim \Delta(e)N^{D_2/d} \), with \( \Delta(e) = (R_1(e))^{-1} \) as the mean level spacing at energy \( e \), \( D_2 \) as the fractal dimension and \( d \) as the physical dimension. Assuming \( \Delta(e) \sim N^{-\eta} \) with \( \eta \) as a system-dependent power, this gives \( E_c \sim N^{-(\eta d-D_2)/d} \) and \( \mu = (\eta d-D_2)/d \). With \( 0 < D_2 < d \), the above assumption is valid at least in flat band regime where \( \eta = 1 \) which gives \( \mu > 0 \).

Substitution of eq.(9), eq.(10) along with eq.(8) in eq.(7) leads to

\[
\Lambda_e(Y, N, e) = \frac{NE_c^2}{72 \pi^3 u_0^2} \ln |1 - w^2| \frac{\ln |1 - w^2|}{w^2} e^{-\frac{16\ln|1-w^2|}{NE_c^2}} e^{\frac{2e^2}{\sqrt{2w^2}}} \tag{11}
\]

As clear from the above, \( \Lambda_e \) depends on the energy \( e \), disorder \( w \) as well as energy scale \( E_c \). To seek the critical point, it is necessary to find specific \( e \) and \( w \) values which results in a \( \Lambda_e \)
size-independent as well as different from the two end-points: \( \lim_{N \to \infty} \Lambda_e \neq 0, \infty \). For further analysis of eq.(11), we consider following energy and disorder regimes:

**Case \( e \sim 0 \):** For large \( N \) and \( E_c \sim N^{-\mu} \) with \( 0 < \mu \leq 1/2 \), one can approximate \( e^{-16 \ln(1-w^2)/NE_c^2} \sim 1 \). This along with eq.(11) then implies disorder-independence of \( \Lambda_e \) for \( e \sqrt{2} < w \): \( \Lambda_e(Y, N, e) = \frac{NE_c^2}{\pi^2 w_0} \). Further for cases with \( \mu = 1/2 \), \( \Lambda_e \) is also size-independent, implying a critical spectral statistics in the bulk of the flat band spectrum (i.e \( e \sim 0 \)). As indicated by our numerical analysis, \( E_c \propto N^{-1/2} \) for the 2 dimensional chequered board lattice in weak disorder limit, and, the fluctuation measures also confirm the criticality of the spectral statistics (see parts (c) and (e) of the figures (2,3)). The details are discussed later in section V.

For large \( w \) and finite \( N \), \( \Lambda_e \) decrease smoothly with increasing \( w \) and therefore the spectral statistics near \( e \sim 0 \) again approaches Poisson limit, implying lack of level-repulsion. This prediction is again consistent with our numerical analysis (see figure 4(c,e) and figure 5(c,e)). Note as displayed in figure 6(a), \( \Delta(e) \propto N^{-1} \) and figure 6(b), \( D_2 \approx 0.5 \) which gives \( E_c \sim N^{-0.75} \), thus implying a size-dependent \( \Lambda_e \) and ruling out the critical statistics; figure 6 indeed confirms the size-dependence of measures. Further in limit \( N \to \infty \), \( \Lambda_e \to 0 \) for any finite \( w > 1 \) which indicates a transition from critical statistics to Poisson at \( w \approx 1 \).

As clear from the above, the statistics undergoes an inverse Anderson transition in the disorder perturbed flat band, with fully localized states at zero disorder becoming partially localized for a weak disorder (\( w < 1 \) in our case). However the usual Anderson transition sets in presence of strong disorder (\( w \approx 1 \)). The statistics now shows two types of disorder driven critical behavior near \( e \sim 0 \): (i) near \( w \sim 0 \), Poisson \( \to \) GOE (or GUE in presence of magnetic field) transition of the level statistics, (ii) for \( w \sim 1 \), the level-statistics transits from GOE/ GUE \( \to \) Poisson.

**Case \( e > 0 \):** The contribution from the term \( e^{-\frac{2e^2}{w^2}} \) in eq.(11) is no longer negligible for small \( w < 2e^2 \) however it becomes insignificant for large \( w^2 > 2e^2 \). At this stage however other \( w \)-dependent terms start affecting the statistics. For finite \( N \) the statistics at \( e > 0 \) therefore changes from Poisson \( \to \) GOE \( \to \) Poisson.

An important point worth emphasizing here is an energy dependence of the spectral statistics for infinite system sizes (\( N \to \infty \)) and for weak disorder: critical near \( e \sim 0 \) if \( E_c(e \sim 0) \propto \frac{1}{\sqrt{N}} \) but Poisson for \( e > 0 \) if \( N \frac{E_c^2}{E_c^2} < 1 \) for \( e > 0 \). This suggests the existence of a mobility edge separating partially localized states from the localized states.
At this stage, it is relevant to indicate the following. As the level density for a flat band
in clean limit can be expressed as a $\delta$-function, irrespective of whether the band is single or
many particle, the formulation derived in [2] remains valid for both type of bands; (although
$Y$ for two cases is different). Similarly the response of the average inverse participation ratio
to weak disorder discussed in [2] is based on a knowledge of initial condition only and not
on the presence or absence of interactions in the band, it is thus applicable for both type of
bands too. This is however not the case for the spectral fluctuations which are governed by
$\Lambda_e$ and therefore dependent on the local mean level spacing $\Delta_e$. For many particle spectrum,
$\Delta_e$ in general depends on many particle localization length which can be varied by tuning
either disorder or interactions. Thus the size-independence of many body $\Delta_e$ can be achieved
in many ways which could as a result lead to more than one critical point.

IV. TRANSITION IN A FLAT BAND WITH OTHER BANDS IN THE NEIGH-
BORHOOD

In presence of other bands, the energy as well as size dependence of $\Lambda_e$, defined in eq.(7)
can vary significantly based on the neighborhood. As calculation of $\Lambda_e$ requires a prior
knowledge of the level densities and IPR, here we consider two examples for which these
measures are discussed in [2]:

(i) two flat bands:

As discussed in [2], $R_1(e)$ can now be expressed as a sum over two Gaussians (originating
from $\delta$-function densities of two fat bands)

$$R_1(e; w) = \frac{N}{2\sqrt{2\pi w^2}} \sum_{k=1}^{2} e^{-\frac{(e-e_k)^2}{2w^2}}$$

with $e_1, e_2$ as the centres of two flat bands. The IPR in large $N$ limit is

$$\left\langle I^2 \right\rangle(e, \Lambda_I) \approx \frac{3}{2} \frac{u_0\sqrt{2\pi}}{R_1 E_c} \sum_{k,l=1}^{2} e^{-\frac{4(e-e_k)}{2w^2}} e^{-\frac{(e_l-e_k)^2}{2w^2}} + \frac{2\Lambda_I}{N} \Theta(e - e_k)$$

with $\Lambda_I = \frac{4\ln|1-w^2|}{E_c^2}$ and $\Theta(x)$ as the step function: $\Theta(x) = 0, 1$ for $x < 0$ and $x > 0$
respectively. Substitution of eq.(12) and eq.(13) in eq.(7) now gives $\Lambda_e$ for this case. A
better insight can however be gained by deriving $\Lambda_e$ in different energy regimes.
Case $e \sim e_k$: For $e \sim e_k$, with $k = 1, 2$, eq.(12) and eq.(13) can be approximated as $R_1(e; w) \approx \frac{N}{2\sqrt{2\pi w}}[1 + e^{-\frac{(e_2-e_1)^2}{2w^2}}]$ and $\langle \mathcal{L}_2 \rangle (e) \approx \frac{6\pi u_0}{N E_c} e^{\frac{8\ln|1-w^2|}{N E_c^2}}$. These on substitution in eq.(7) give

$$
\Lambda_e(Y, N, e) = \frac{N E_c^2}{288 \pi^3 u_0^2} \frac{\ln|1-w^2|}{w^2} \frac{e^{-\frac{16 \ln|1-w^2|}{N E_c^2}}}{1 + e^{-\frac{(e_2-e_1)^2}{2w^2}}} \tag{14}
$$

Clearly, similar to the single band case, here again $\Lambda_e$ is independent of disorder for $w < N^{1-2\mu}$ and in large N limit but it rapidly decreases with larger disorder. The size-independence of $\Lambda_e$ however requires $E_c \propto \frac{1}{\sqrt{N}}$. For $w < 1$, the spectral statistics at the centers of two Gaussian bands (flat ands in clean limit) can therefore be critical as well as disorder independent only if $\mu = 1/2$.

Case $e \sim (e_1+e_2)/2$: For the energies midway between two bands, $R_1$ is very small for $w < 1$ but, contrary to band centre, it increases with increasing $w$ for $w > |e_1 - e_2|$: $R_1 \left( \frac{e_1+e_2}{2} \right) = \frac{N}{\sqrt{2\pi w}} e^{-\frac{(e_2-e_1)^2}{8w^2}}$ and eq.(13) give $\langle \mathcal{L}_2 \rangle (e) \approx \frac{6\pi u_0}{N E_c} e^{\frac{8\ln|1-w^2|}{N E_c^2}} e^{-\frac{2(e_2-e_1)^2}{E_c}} e^{\frac{(e_1-e_2)^2}{8w^2}} \left[ 1 + e^{-\frac{(e_1-e_2)^2}{2w^2}} \right]$. With $Y - Y_0$ given by eq.(8), we now have

$$
\Lambda_e(Y, N, e) = \frac{N E_c^2}{72 \pi^3 u_0^2} \frac{\ln|1-w^2|}{w^2} e^{-\frac{16 \ln|1-w^2|}{N E_c^2}} e^{\frac{4(e_2-e_1)}{E_c}} e^{\frac{-2(e_1-e_2)^2}{2w^2}} \left( 1 + e^{-\frac{(e_1-e_2)^2}{2w^2}} \right)^2 \tag{15}
$$

As clear from the above, here also $\Lambda_e$ become independent of $N$ if $E_c \propto N^{-1/2}$ thus implying critical statistics.

A case of two flat bands was studied in [18] for the 3-dimensional hexagonal diamond lattice. The study indicates $D_2 \approx 2.55$ and 2.61 for $e \sim e_k$ and $e \sim (e_1 + e_2)/2$, respectively. With $\Delta(e) \propto N^{-1}$, $E_c$ for this system is $N^{-0.17}$ and $N^{-0.15}$. Based on our theory, the statistics is predicted to be size as well as disorder dependent near $e \sim (e_1 + e_2)/2$ but disorder-independent near $e \sim e_k$. The display in figures (4,5) of [18] indeed confirms this prediction.

The case of three flat bands was discussed in [22], for a bipartite periodic lattice described by a tight binding, interacting Hamiltonian. The study indicates a localization $\rightarrow$ delocalization transition at the onset of disorder or many body interactions. The possibility of a critical behavior for this case can be explored along the same route as given above.

(iii) a flat band at the edge of a dispersive band: For the combination of a flat band located at $e = 0$ and a dispersive band with the level density $f_d(e)$, the results in section VI of [2] give
\[ R_1(e; w) = \frac{N}{2 \sqrt{2\pi w^2}} e^{-\frac{N^2}{2w^2}} + \frac{N}{2} f_w(e, w, N) \]  

with \( f_w(e, w, N) \) as the dispersive band density at disorder \( w \) and

\[
\langle T_2 \rangle(e, \Lambda_t) \approx \frac{1}{2} R_1 \frac{3\sqrt{2}}{w E_c} \left[ u_0 \sqrt{\pi} + B_1 + B_2 + B_3 \right] e^{-\frac{4e}{E_c} + \frac{2\Lambda_t}{N}}
\]

with \( B_1 = \frac{2u_0 w}{E_c} \sqrt{\frac{\pi N}{\Lambda_t}} \int_{-\infty}^{\infty} dx f_w(x) e^{-\frac{2N^2x^2}{4w^2} + \frac{4\pi}{\Lambda_t}} \), \( B_2 = N \int_{-\infty}^{\infty} dx f_w(x) u_d(x) e^{-\frac{x^2}{w^2} + \frac{4\pi}{\Lambda_t}} \), \( B_3 = \sqrt{2\pi w^2} \int_{-\infty}^{\infty} dx f_w(x) u_d(x) e^{\frac{4\pi}{\Lambda_t}} \) and \( \Lambda_t = \frac{4\ln |1-w^2|}{E_c^2} \). Here \( u_0 \) and \( u_d(e, w) \) are the local eigenfunction intensities in the flat band at disorder \( w = 0 \) and in dispersive band at disorder \( w \). For cases in which \( f_w(e, w, N) \) varying slower than the Gaussians in the related integrals, \( B_1 \) and \( B_2 \) can be approximated as follows: \( B_1 = \pi \sqrt{2} u_0 f_w\left( \frac{\Lambda_t E_c}{4} \right) e^{\frac{2\Lambda_t N}{E_c}} \), \( B_2 = \sqrt{2\pi w^2} u_d\left( \frac{4w^2}{E_c^2} \right) e^{\frac{8\ln |1-w^2|}{E_c^2}} \).

A substitution of eq.\((16)\), eq.\((17)\) along with eq.\((8)\) in eq.\((7)\) give \( \Lambda_e \) for arbitrary energy and disorder but here again it is instructive to analyze the behavior near specific energies:

Case \( e \sim 0 \): Due to almost negligible contribution for weak disorder from the dispersive part near \( e \sim 0 \), one can approximate \( R_1 \approx \frac{N}{2 \sqrt{2\pi w^2}} \) and \( \langle T_2 \rangle \approx \frac{6\pi u_0}{NE_c} \) which in turn gives \( \Lambda_e = \frac{NE_c^2}{288 \pi^2 u_0^2} \). The latter is therefore again size as well as disorder independent indicating criticality near \( e \sim 0 \) for all weak-disorders if \( E_c \propto N^{-1/2} \). As intuitively expected, the behavior of spectral statistics near \( e \sim 0 \) and \( w < 1 \) in this case is analogous to that of the single flat band case.

For large \( w \) (e.g. \( w > 1 \) for the case with \( \mu = 1/2 \)), however the contribution from the dispersive band becomes significant near \( e \sim 0 \). This results in \( R_1(e \sim 0) \approx \frac{N}{2 \sqrt{2\pi w^2}} T_1 \) where \( T_1 = 1 + \sqrt{2\pi w^2} f_w(0, w, N) \) and \( \langle T_2 \rangle \approx \frac{6 \sqrt{\pi}}{NE_c T_0} \left[ u_0 \sqrt{\pi} + B_1 + B_2 + B_3 \right] e^{-\frac{4\pi}{E_c} + \frac{8\ln |1-w^2|}{NE_c^2}} \). These on substitution in eq.\((7)\) give

\[
\Lambda_e(Y, N, e) = \frac{NE_c^2}{288 \pi^2} \frac{T_1^4}{(u_0 \sqrt{\pi} + B_1 + B_2 + B_3)^2} e^{-\frac{8\ln |1-w^2|}{NE_c^2}}
\]

Case \( e > 0 \): Due to weaker contribution from the Gaussian density for \( e > 0 \), the contribution from the dispersive band density need not be negligible and it is appropriate to consider the full form of \( R_1(e) \). The IPR can now be approximated as

\[
\langle T_2 \rangle \approx \frac{6 \sqrt{\pi}}{NE_c T_0} \left[ u_0 \sqrt{\pi} + B_1 + B_2 + B_3 \right] e^{-\frac{4\pi}{E_c} + \frac{8\ln |1-w^2|}{NE_c^2}}
\]
where \( T_0 = e^{-\frac{e^2}{2w^2}} + \sqrt{2\pi w^2} f_w(e) \). The above leads to

\[
\Lambda_e(Y, N, e) = \frac{N E_c^2}{288 \pi^2} \frac{T_0^4}{(u_0 \sqrt{\pi} + B_1 + B_2 + B_3)^2} e^{-\frac{16 \ln|1-w^2|}{N E_c^2}} e^{\frac{w}{E_c}} \tag{20}
\]

As mentioned in [2, 20], the two dimensional chequered board lattice consists of a flat band and a dispersive band in clean limit. Our numerical analysis of the system for \( w < 1 \) indicated \( \Delta(e) \propto N^{-1} \) and \( D_2 \sim 1.2 \) (see figures 2,3 of the present work and figure 4 of [2]), leading to \( E_c \sim N^{-0.6} \) which implies \( \langle \mathcal{I}_2 \rangle \sim N^{-1/2} \), an indicator of partially localized states. Near \( e \sim 0 \) and \( w < 1 \), eq.(18) then gives a size-independent \( \Lambda_e \), implying criticality of the spectral statistics. For \( w^2 = 10 \) and near \( e \sim 0 \), gives \( D_2 \approx 0.5 \) (see figure 5 of present work and also figure 4 of [2]). With \( \Delta(e) \propto N^{-1} \), we now have \( E_c \sim N^{-0.8} \). Clearly \( \Lambda_e \) now rapidly goes to zero with increasing \( N \), implying Poisson spectral statistics. Due to size-dependence of the \( \Lambda_e \) in eq.(20), the statistics is not critical for \( e > 0 \); it is also expected to vary with disorder. The above theoretical predictions are confirmed by our numerical analysis discussed in next section.

V. NUMERICAL ANALYSIS: 2-D CHEQUERED BOARD LATTICE

To verify our theoretical prediction, we pursue a numerical statistical analysis of the eigenvalues and eigenfunctions of the Hamiltonian \( H = \sum_{x,y} N V_{xy} c_x^\dagger c_x \) of a 2-d planer pyrochlore lattice with single orbital per site [2, 20]. With 2-d unit cell labeled as \((m, n)\), one can write a site-index as \( x = (m, n, \alpha) \) with \( \alpha = a, b \) (i.e two atoms per unit cell). The lattice consists of one flat band \( E_f = \epsilon - 2t \) and one dispersive band \( E_d = \epsilon + 2t(\cos k_x + \cos k_y + 1) \) if \( V_{xy} \) satisfies following set of conditions [2, 20]: (i) \( V_{xx} = \epsilon \), (ii) \( V_{xy} = t \) with \( x = (m, n, \alpha) \) if \( y = (m, n, \beta) \) or \((m - 1, n, \beta) \) or \((m, n + 1, \beta) \) with \( \beta = a, b \) and (iii) \( V_{xy} = 0 \) for all other \( x, y \) pairs.

For \( \epsilon = 2, t = 1 \), the Hamiltonian, in absence of disorder, consists of a flat band at \( e = 0 \) and a dispersive band centered at \( e = 4 \). (This can be seen from the band energies \( E_f \) and \( E_d \) given above). The onset of disorder through on-site energies with \( \langle V_{xx} \rangle = \epsilon, \langle V_{xx}^2 \rangle - \langle V_{xx} \rangle^2 = w^2 \) leads to randomization of the Hamiltonian. For the numerical analysis, therefore, we simulate large matrix ensembles of the Hamiltonian, and at many \( w \), for various ensemble-sizes \( M \) (the number of matrices in the ensemble) as well as the matrix-sizes \( N = L^2 \). The energy-sensitivity of the transition (due to energy-dependence of \( \Lambda_e \)) requires the fluctuation
measures analysis at precisely a given value of energy. In order to improve the statistics however a consideration of the averages over an optimized energy range $\Delta E$ is necessary (not too large to avoid mixing of different statistics). For comparison of a measure for different system-sizes $N$ at a given disorder, we have used only 20% levels in our numerical analysis.

In [2], we theoretically analyzed the disorder dependence of level density $R_1$ and average inverse participation ratio $\langle I_2 \rangle$. Our results indicated a disorder insensitivity of these measures in weak disorder limit ($w < 1$). This was also confirmed by their numerical analysis as well as that of $D_q$ displayed in figure 1-4 of [2]. In this section, we numerically analyze the disorder and size dependence of the spectral fluctuations as well as fractal dimensions $D_q$. The figure 1 displays the disorder-dependence of $P(s)$ and $\Sigma^2(r)$ in two energy regimes i.e near $e \sim 0$ and $e \sim 4$ (corresponding to bulk of the flat band and dispersive bands in clean limit). As clear from figures 1(a) and 1(c), for a weak disorder ($w < 1$) and near $e \sim 0$, both measures are insensitive to change in disorder. But as displayed in figures 1(b,d), the statistics in the dispersive case varies with disorder even for weak disorders. A similar result was reported by the numerical study of a 3-dimensional disordered diamond lattice (with two flat bands in the clean limit) [18]. The effect of on-site disorder for the $T_3$ lattice with three flat bands in clean limit, was analyzed in [19]. The results again indicated disorder independence of the fluctuation measures for low disorder $w < 1$ but an increase of localization with $w$ for $w > 1$.

Our next step is to seek criticality in the spectral and eigenfunction statistics. For this purpose, we focus on the size-dependence of $P(S)$, $\chi$ and $D_2$ in two energy regimes $e \sim 0$ and $e \sim 4$; the results for four disorder-strengths, two in weak and two in strong disorder regime, are displayed in figures 2-5. (Here, for clarity of presentation, a comparison with theoretical approximation given by eq.(3) is not displayed). To determine $\Lambda_e$ for these cases, it is numerically easier to use the following expression (instead of the theoretical approximation discussed in the previous section),

$$\Lambda_{e,FE} = \frac{R_1^2}{\langle I_2 \rangle^2} \frac{\ln |1 - w^2|}{N^3}. \quad (21)$$

where $R_1$ and $\langle I_2 \rangle$ are numerically obtained; the corresponding values are given in the captions of figures 2-5. The size-independence as well as location of the curves, intermediate to Poisson and GOE limits in figure 2(b,c,e) is an indicator of the critical spectral statistics and partially localized wave-functions in the weakly disordered flat band bulk; also note
that figures 2(b) and 2(e) give $D_2 \approx 1.2$ and $\chi \approx 0.2$ respectively which agrees well with the prediction based on the weak multifractality relation $D_2 = d(1 - 2\chi)$ for the flat band (note $d = 2$ in our case) [4]. With $\Lambda_e \approx 0.384$ in this case (see caption of figure 2), the numerically obtained $\chi$-value is also consistent with eq.1. In contrast to behavior near $e \sim 0$, the size-dependence of the measures is clearly visible from figures 2(d,f), displaying behavior near $e \sim 4$ which rules out criticality in the dispersive regime.

As shown in figures 3(b,c,e), the critical behavior in the flat band persists even when disorder is varied to $w \sim 10^{-1}$. But in contrast to $w = 10^{-5}$, the statistics in the dispersive regime for $w \sim 10^{-1}$ is now closer to Poisson limit (see figures 3(d, f)); this implies a tendency of the wave-functions in the dispersive band to increasingly localize as $w$ increases. The results given in figures 2 and 3 clearly indicate the reverse trend of the statistics in two bands with increasing disorder in range $0 < w \leq 1$: the flat band bulk undergoes a Poisson to GOE type crossover with increasing $w$ (though never reaching GOE) but the dispersive bulk changes from GOE $\rightarrow$ Poisson $\rightarrow$ GOE limit. (Note in clean limit, the flat band corresponds to Poisson statistics while dispersive band corresponds to that of GOE). For $w > 1$ however bands increasingly overlap with each other and the statistics for both energy ranges approaches Poisson limit with increasing disorder (although at different rate based on energy regime), as expected from a standard Anderson transition. The statistics now seems to be size-independent for all energy ranges. Also note from figures 6(b,e), the relation $D_2 = d(1 - 2\chi)$ is no longer satisfied (here $D_2 \approx 0.8$ and $\chi \approx 0.42$). This is expected because the multifractality in the band is no longer weak.

The Poisson and GOE statistics are believed to be the indicators of localized and delocalized dynamics of the eigenfunctions, with an intermediate statistics indicating partially localized states; (although, as discussed in [25], this relation between in spectral statistics and eigenfunction dynamics is valid only for Hermitian matrices). This implies that, for $w^2 = 10^{-9}$ and 0.01, the states near $e \sim 0$ are extended (although not completely delocalized) but localized near $e \sim 4$ (see parts (c),(d) of figures 2,3). For $w = 1$, however the localization tendency is now reversed, with almost localized near $e \sim 0$ but delocalized near $e \sim 4$. This inverse eigenstate localization tendency at $e \sim 0$ to the at $e \sim 4$ for a given weak disorder hints at the existence of a mobility edge/region. Note beyond $w > 1$, all states are almost localized although the speed of localization is energy-dependent.

Let us now focus on the flat band only. As clear from the above, the behavior near $e \sim 0$
indicates the occurrence of an inverse Anderson transition, with fully/compact localized states at zero disorder becoming partially localized for a non-zero weak disorder ($w < 1$ in our case). However the usual Anderson transition sets in presence of the strong disorder ($w \simeq 1$). The quantum dynamics near $e \sim 0$ now shows two types of critical behavior: (i) at $w = 0$, a localized $\rightarrow$ extended state transition, in weak disorder regime and (ii) an extended state $\rightarrow$ localization transition at $w \approx 1$.

VI. ANALOGY WITH OTHER ENSEMBLES

Based on the complexity parametric formulation, different ensembles subjected to same global constraint (which in this case is the Hermitian nature of $H$-matrix) are expected to undergo similar evolution. This in turn implies an analogy of their statistical measures if the values of their complexity parameters are equal and the initial conditions are statistically analogous. In this section, we verify the analogy by comparing the statistical behavior of weakly disordered flat bands with two other disordered ensembles of real-symmetric matrices, namely, the Anderson ensemble with on-site Gaussian disorder and Rosenzweig-Porter ensemble. Similar to Flat band lattices, both of these ensembles can be expressed as a multi-parametric Gaussian ensemble and the expressions for $Y$ and $\Lambda_{e}$ for them can be easily obtained (see [7], [12] and [5] for details). The two ensembles can briefly be described as follows.

**Anderson Ensemble:** The standard Anderson Hamiltonian $H = \sum_{k=1}^{N} \varepsilon_{k} c_{k}^\dagger c_{k} + \sum_{k,l=n,n}^{N} V_{kl} c_{k}^\dagger c_{l}$ describes the dynamics of an electron moving in a random potential in a $d$-dimensional tight binding lattice with one atom per unit cell. The disorder in the lattice can appear through on-site energies $\varepsilon_{k}$ or hopping $V_{kl}$ between nearest neighbor sites. Here we consider the lattice with $N$ sites, an on-site Gaussian disorder (with $\langle \varepsilon_{k}^{2} \rangle = w^{2}$, $\langle \varepsilon_{k} \rangle = 0$, and a random nearest neighbor hopping ($\langle V_{kl}^{2} \rangle = t f_{0}$, $\langle V_{kl} \rangle = 0$ with $f_{0} = 1$ if the sites $k, l$ are nearest neighbors otherwise it is zero) with $z$ as the number of nearest neighbors. The ensemble density in this case can be written as

$$\rho(H) = \lim_{\sigma \to 0} C_{a} \prod_{k=1}^{N} e^{-\frac{H_{kk}^{2}}{2\sigma^{2}}} \prod_{k,l=n,n}^{N} e^{-\frac{H_{kl}^{2}}{2\sigma^{2}}} \prod_{k,l \neq n,n}^{N} e^{-\frac{H_{kl}^{2}}{2\sigma^{2}}}$$

(22)

with $C_{a}$ as the normalization constant. From eq.(8), the ensemble complexity parameter
in this case is \[7\]

\[Y \approx -\frac{1}{N} \ln \left[|1-w^2| |1-2t|^{z/2}\right] + \text{const}, \quad (23)\]

Here the initial state is chosen as a clean lattice with sufficiently far off atoms resulting in zero hopping (i.e. both \(w = 0\) and \(t = 0\)) which corresponds to a localized eigenfunction dynamics with Poisson spectral statistics. (This choice ensures the analogy of initial statistics with the flat band case). Substitution of eq.\((23)\) in eq.\((2)\) with \(\Delta(e) = \frac{N\langle I_2 \rangle}{R_1}\) and \(\langle I_2 \rangle\) as the typical IPR at \(e\), leads to

\[\Lambda_{e,AE}(Y, N, e) = \frac{R_1^2}{\langle I_2 \rangle^2} \frac{1}{N} \ln(|1-w^2| |1-2t|^{z/2}). \quad (24)\]

Based on the complexity parameter formulation and verified by the numerical analysis, the level density here turns out to be a Gaussian: \(R_1(e) = \frac{N}{\sqrt{2\pi\alpha^2}} e^{-\frac{e^2}{2\alpha^2}} \quad [7]\). As indicated by several studies in past, the localization length \(\xi\) in this case depends on the dimensionality as well as disorder: (i) \(\xi \approx \pi l \approx O(L^0)\) for all \(w\) for \(d = 1\) with \(l\) as the mean free path of the electron in the lattice, (ii) \(\xi \approx \frac{L}{\pi k_F} \approx O(L^0)\) for all \(w\) for \(d = 2\) with \(k_F\) as the Fermi wave-vector and (iii) \(\xi \approx \xi_0(e, w) L^{D_2}\) with \(D_2 = \frac{d}{2}\) for \(w = w^*\) for \(d > 2\). As a consequence, \(\Lambda_e \sim O(1/N)\) for \(d \leq 2\) which implies the statistics approaching an insulator limit a \(N \to \infty\). For \(d > 2\), \(\Lambda_e\) in the spectral bulk is size-independent only for a critical disorder \(w = w^*\) (for a fixed \(t\)), thus indicating only one critical point \([7]\) of transition from delocalized to localized states with increasing disorder.

An important point worth re-emphasizing is here is that notwithstanding the \(N\)-dependence of \(Y - Y_0\) same for AE and the flat bands (discussed in section II.a), the statistics of energy levels and eigenfunctions in the two cases undergoes an inverse transition. This occurs because \(\Lambda_e\), the only parameter governing the spectral statistics, depends on the localization length and mean level density which have different response to disorder in the two cases.

**Rosenzweig Porter Ensemble:** This represents an ensemble of Hermitian matrices with independent, Gaussian distributed matrix elements with zero mean, and different variance for the diagonal and off diagonals. The ensemble density in this case can be given as
\[ \rho(H) \propto \exp \left[ -\frac{1}{2} \sum_{i=1}^{N} H_{ii}^2 - (1 + \mu_0) \sum_{i,j=1;i<j}^{N} H_{ij}^2 \right] \]  \quad (25)

As clear from the above, contrary to multi-parametric dependent Anderson case, the Rosenzweig Porter ensemble (or alternatively the Brownian ensemble (BE) appearing between Poisson and GOE) depends on a single parameter i.e ratio of diagonal to off-diagonal variance (besides matrix size). The choice of initial condition as an ensemble of diagonal matrices \((\mu_0 \to \infty)\) gives \(Y - Y_0 = \frac{1}{4\mu_0}\) and

\[ \Lambda(e, BE) = \frac{Y - Y_0}{\Delta^2} = \frac{R_1^2}{4\mu_0}. \]  \quad (26)

As discussed in [5], the size-dependence of \(R_1(e; \mu_0)\) in this case changes from \(\sqrt{N}\) to \(N\). Further with \(Y - Y_0 \sim N^{-\gamma_0}\), we have \(\Lambda_I \sim \frac{N^{1-\gamma_0}}{\Delta^2}\) and \(\langle I_q \rangle \sim N^{(\gamma-2)(q-1)/2}\) for \(q > 0\).

As discussed in [5], the size-dependence of \(R_1(e; \mu_0)\) in this case changes from \(\sqrt{N}\) to \(N\). Further with \(Y - Y_0 \sim N^{-\gamma_0}\), we have \(\Lambda_I \sim \frac{N^{1-\gamma_0}}{\Delta^2}\) and \(\langle I_q \rangle \sim N^{(\gamma-2)(q-1)/2}\) for \(q > 0\).

Note \(d = 1\) for the BE. A comparison of the above result with \(\langle I_q(\Lambda_I) \rangle \sim N^{-(q-1)D_q/d}\) then gives \(D_q = (2 - \gamma_0)/2\), for \(q > 0\). This in turn indicates the existence of two critical points:

(i) for \(\mu = c_1 N^2\): here \(R_1 = \frac{N}{\sqrt{\pi}} \mathrm{e}^{-\varepsilon^2}\) which gives \(\Lambda(e, BE) = \frac{1}{4\pi c_1} \mathrm{e}^{-\varepsilon^2}\), (ii) for \(\mu = c_2 N^2\) here \(R_1(e) = (b\pi)^{-1} \sqrt{2bN - \varepsilon^2}\) leading to \(\Lambda(e) = \frac{2bN - \varepsilon^2}{\pi^{3/2} N c_2}\) with \(b \sim 2\). The two critical points here corresponds to a transition from localized \(\to\) extended \(\to\) delocalized states with decreasing \(\mu_0\) [5].

**Parametric values for the analogues:** For numerical analysis of Anderson ensemble, we consider a three dimensional cubic lattice with hard wall boundary conditions, on-site Gaussian disorder \(w\) and a random hopping with \(t = 1/12\). For Brownian ensemble, we consider the case with \(\mu_0 = c N^2\). The system parameters for Anderson and Brownian ensemble analogs of a weakly disordered flat band are related through corresponding \(\Lambda\)s as follows:

\[ \Lambda_{e,FE} = \Lambda_{e,AE} = \Lambda_{e,BE}. \]  \quad (27)

with \(\Lambda_{e,FE}, \Lambda_{e,AE}, \Lambda_{e,BE}\) given by eq.(21), eq.(24) and eq.(26) respectively. An important point to note here is that both \(\Lambda_{e,FE}\) and \(\Lambda_{e,BE}\) can be size independent for many \(w\) and \(c\) values respectively. But \(\Lambda_{e,AE}\) is size-independent only for a specific value of disorder. The AE analogue of a FE therefore need not be critical, can be size-dependent and is therefore not unique.
Figure 10 displays a comparison of the nearest neighbor spacing distribution for two cases of disordered chequered board lattice (with Fermi energy in bulk of the flat band) with AE and BE analogues predicted by eq. (27). The numerically obtained values for the analogs near ($e \sim 0$ for each case) are as follows:

(i) **weak disorder analogy:** (a) FE: $N = 1156, w^2 = 0.01, \varepsilon = 2, t = 1, \langle I^{\text{typ}} \rangle = 0.0116, R_1(e) = 0.248 N/w$ which gives $\Lambda_{e,FE} = 0.395$, (b) AE: $N = 512, w^2 = 4.15/6, t = 1/12, z = 6, \langle I^{\text{typ}} \rangle = 0.025, R_1 \approx 0.11/\sqrt{N}$ with $\Lambda_{e,AE} = 0.465$ and (c) BE: $N = 512, c = 0.2$ with $\Lambda_{e,BE} = 0.398$.

(ii) **strong disorder analogy:** (a) FE: $N = 1156, w^2 = 10, \varepsilon = 2, t = 1, \langle I^{\text{typ}} \rangle = 0.1149, R_1(e) = 0.3 N/w$ which gives $\Lambda_{e,FE} = 1.3 \times 10^{-3}$, (b) AE: $N = 512, w^2 = 120.15/6, \langle I^{\text{typ}} \rangle \approx 0.3, R_1 \approx 0.1/N$ with $\Lambda_{e,AE} = 7.58 \times 10^{-4}$ and (c) BE: $N = 512, c = 69.2$ with $\Lambda_{e,BE} = 1.15 \times 10^{-3}$.

The AE and BE analogs for the other flat band cases can similarly be obtained. To confirm that this analogy is not a mere coincidence and exist for other $\Lambda_e$ values too, we compare these ensembles for full crossover from $\Lambda_e = 0 \rightarrow \infty$. One traditionally used measure in this context is the relative behavior of the tail of nearest-neighbor spacing distribution $P(s)$, defined as

$$
\gamma(\delta; \Lambda) = \frac{\int_0^{\delta} (P(s; \Lambda) - P(s; \infty)) ds}{\int_0^{\delta} (P(s; 0) - P(s; \infty)) ds} \quad (28)
$$

with $\delta$ as one of the two crossing points of $P_o(s) = P(s; \infty)$ and $P_p(s) = P(s; 0)$ (here the subscripts $o$ and $p$ refer to the GOE and Poisson cases respectively) \[1\]. As obvious, $\gamma = 0$ and 1 for GOE and Poisson limit respectively and a fractional value of $\gamma$ indicates the probability of small-spacings different from the two limits. Figures 10(c) and 10(d) show a comparison of $\gamma$ for two $\delta$-values for three systems: $\gamma_1 = \gamma(\delta_1)$ and $\gamma_2 = \gamma(\delta_2)$, with $\delta_1 = 0.4699, \delta_2 = 1.9699$; the display confirm our theoretical claim regarding the analogy of the three systems. It must be noted that the $\Lambda_e$ for FE never approaches a value as large as that of AE and BE; following from eq.(11) and eq.(15), it first increases and then decreases beyond a disorder-strength $w \sim 1$. This is contrary to AE and BE for which $\Lambda_e$ decreases with increasing disorder. This behavior is also confirmed by our numerical analysis displayed in figure.
VII. CONCLUSION

In the end we summarize with main insights given by our analysis. Based on complexity parameter formulation, we find that the criticality of the spectral statistics in a weakly disordered flat band (i.e the one perturbed by weak disorder) is sensitive to system conditions which may lead to a localization $\rightarrow$ delocalization $\rightarrow$ localization transition with increasing disorder. The criticality occurs when the spectral complexity parameter becomes size-independent for an arbitrary weak disorder. It must be emphasized however that all weakly disordered flat bands need not be critical. This requires the average uncertainty in the system, measured in the units of local mean level spacing, to become scale-free. Our theoretical predictions are confirmed by the numerical analysis of a chequered board lattice.

The complexity parameter formulation also reveals an interesting feature of the localization to delocalization crossover in finite systems: irrespective of the number of critical points and different equilibrium limits, the statistics of a disordered flat band can be mapped to that of a single parametric Brownian ensemble as well as multi-parametric Anderson ensemble. The analogy of these ensembles to other multi-parametric ensembles intermediate between Poisson and GOE is already known [5, 9, 11, 27]. In fact it seems a wide range of localization $\rightarrow$ delocalization transition can be modeled by a single parameter Brownian ensemble appearing between Poisson and GOE (Rosenzweig-Porter ensemble). This hints at a large scale universality and a hidden web of connection underlying complex systems even in the non-equilibrium regime.

Our study gives rise to many new queries. For example, an important question is whether local weak particle-particle interactions can mimic the role of weak disorder in the flat bands. At least the complexity parameter formulation predicts this to be the case but a thorough investigation of the fluctuations is needed to confirm the prediction. A detailed analysis of the role of the symmetries in flat band physics using complexity parameter approach still remains to be investigated. Our analysis seems to suggest the existence of a mobility edge too however this requires a more thorough investigation. We expect to explore some of these questions in future.

[1] M. Janssen, Phys. Rep. 295, 1, (1998).
[2] P. Shukla, Submitted to Physical Review B, 2018.

[3] B.I. Shklovskii, B. Shapiro, B.R. Sears, P. Lambrianides and H.B. Shore, Phys. Rev. B 47, 11487, (1993).

[4] J.T. Chalker, V.E. Kravtsov and I.V. Lerner, Pis’ma Zh. Eksp. Teor. Fiz. 64, 355 (1996) [JETP Lett. 64, 386, (1996)].

[5] S. Sadhukhan and P. Shukla, Phys. Rev. E, (2017).

[6] A.D. Mirlin and F. Evers, Rev. Mod. Phys.; F. Evers, A. Mildenberger and A.D. Mirlin, Phys. Rev. B 64, 241303, (2001).

[7] P. Shukla, J. Phys.: Condens. Matter 17, 1653, (2005).

[8] P. Shukla, Phys. Rev. E, 71, 026266, (2005).

[9] P. Shukla, J. Phys. A, 41, 304023, (2008).

[10] P. Shukla, Phys. Rev. E, (2007).

[11] P. Shukla, Phys. Rev. E 62, 2098, (2000). R. Dutta and P. Shukla, Phys. Rev. E, 76, 051124, (2007). R. Dutta and P. Shukla, Phys. Rev. E 78, 031115 (2008). M V Berry and P. Shukla, J. Phys. A, 42, 485102, (2009).

[12] P. Shukla, New J. Phys, (2017).

[13] M.L. Mehta, Random Matrices, (2nd ed., Academic Press, N.Y., 1991).

[14] A. Pandey, Chaos, Solitons, Fractals, 5, 1275, (1995).

[15] P. Shukla, J. Phys. A, (2017); Phys. Rev. E, 75, 051113, (2007).

[16] N. Rosenzweig and C.E. Porter, Phys. Rev. 120, 1698 (1960).

[17] V.E. Kravtsov, I.M. Khaymovich, E. Cuevas and M. Amini, New. J. Phys (IOP), (2016).

[18] M. Goda, S. Nishino and H. Matsuda, Phys. Rev. Lett. 96, 126401, (2006).

[19] J. Vidal, P. Butaud, B. Doucot, and R. Mosseri, Phys. Rev. B 64, 155306 (2001); J. Vidal, G. Monatambaux and B. Doucot, Phys. Rev. B 62, R16294, (2000).

[20] J.T. Chalker, T.S. Pickles and P. Shukla, Phys. Rev. B, 82, 104209, (2010).

[21] J. Vidal, R. Mosseri and B. Doucot, Phys. Rev. Lett. 81, 5888, (1998).

[22] J. Vidal, B. Doucot, R. Mosseri and P. Butaud, Phys. Rev. Lett. 85, 3906, (2000).

[23] Z. Gulacsi, Phys. Rev. B 69, 054204, (2004)); Z. Gulacsi, A. Kampf and D. Vollhardt, Phys. Rev. Lett., 99, 026404, (2007).

[24] Z. Gulacsi, A. Kampf and D. Vollhardt, Phys. Rev. Lett., 105, 266403, (2010).

[25] T. Mondal, S. Sadhukhan, P. Shukla, Phys. Rev. E, 2017.
[26] Y.V.Fyodorov and A.D.Mirlin, Int. J. Mod. Phys. B, 8, 3795, (1994).

[27] P. Shukla and S. Sadhukhan, J.Phys.A, 48, 415002, (2015); S. Sadhukhan and P. Shukla, J. Phys. A, 415003, (2015).

[28] V.K.B.Kota and S.Sumedha, Phys. Rev. E, 60, 3405, (1999); E. Caurier, B. Grammaticos and A. Ramani, J. Phys. A 23, 4903, (1990); G.Lenz and F.Haake, Phys. Rev. Lett. 67, 1, (1991)

[29] S.Tomsovic, Ph.D Thesis, University of Rochester (1986); (unpublished). F. Leyvraz and T.H.Seligman, J. Phys. A 23, 1555 (1990).
FIG. 1. **Disorder dependence of spectral measures in two energy ranges:** (a) $P(S)$ in the bulk of flat band ($e \sim 0$), (b) $P(S)$ in the bulk of dispersive band ($e \sim 4$), (c) $\chi(r)$ in the bulk of flat band ($e \sim 0$), (d) $\chi(r)$ in the bulk of dispersive band ($e \sim 4$). Here $W = w^2$ and $P(S)$ refers to the distribution of the nearest-neighbor spacing $S$ and $\chi(r)$ as the spectral compressibility for the unfolded eigenvalues taken from a narrow energy-range around the specific energy for a fixed system size $L = 70$. The total number of eigenvalues used in each case is approximately $10^5$. As clear from parts (a, c), the statistics is near GOE and disorder-insensitive for $w < 1$ but approaches Poisson limit for $w > 1$. Clearly, with $w = 0$ as the Poisson case (due to degeneracy in flat band spectrum), increasing disorder from zero leads to a change of statistics from Poisson to near-GOE to Poisson, which corresponds to a localization-delocalization-localization crossover of the eigenfunctions in the bulk of the flat band. But parts (b,d) indicate a disorder insensitivity as well as inverse crossover in the dispersive band: with $w = 0$ as GOE case, increasing disorder from zero leads to a change of statistics from $\text{GOE} \rightarrow \text{Poisson} \rightarrow \text{GOE}$ which corresponds to a delocalization-localization-crossover of the eigenfunctions in the bulk of the dispersive band.
**FIG. 2.** Critical spectral statistics for weak disorder $w = \sqrt{3} \times 10^{-5}$: (a) Level density in the flat band (inset showing the behavior in the dispersive band) with fit $f(e_w) = \frac{1}{2\sqrt{1.25\pi}} e_w^{-0.8} e_w^2$, (b) $D_q$ in the flat band, (c) $P(S)$ for flat band bulk, (d) $P(S)$ for dispersive band bulk, (e) $\chi(r)$ for flat band bulk, (f) $\chi(r)$ for dispersive band bulk. The parts (c-e) also display the GOE and Poisson limits. Here, with $\langle I_2 \rangle = 0.0116$ and $R_1 \approx 0.248 N_w$, eq. (21) gives $\Lambda_e = 0.395$ near $e \sim 0$. The convergence of the curves for different sizes in parts (c, e) indicates scale-invariance of the statistics in the flat band. The behavior is critical due to $P(S)$ being different from the two endpoints, namely, Poisson and GUE statistics even in large size limit. This is also confirmed by the $\chi$-behavior shown in part (e), approaching a constant value 0.2 for large $r$, and $D_q$ behavior shown in part (b). Note the $\chi$-value is in agreement with eq. (6) and $D_2$ is consistent with relation $D_2 = d(1 - 2\chi)$ with $d = 2$. The survival of scale-invariance and partially localized behavior even for such a weak disorder indicates the critical point of the inverse Anderson transition to occur at zero disorder strength. In contrast, parts (d) and (f) indicate that the bulk statistics in the dispersive band is size-dependent and is not critical. An interesting point to note is that the statistics in flat band is close to GOE but is near Poisson in dispersive band.
FIG. 3. Critical spectral statistics for weak disorder: $w = 0.1$: The details here are same as in figure 2; the fit in part (a) is $f(e_w) = \frac{1}{2\sqrt{1.2} \pi} e^{0.8 e_{2w}}$. Here, with $\langle I_2 \rangle = 0.0116$ and $R_1(e) \approx \frac{0.248 N}{w}$, eq. (21) gives $\Lambda_e \approx 0.395$. As can be seen from parts (b) and (e) $\chi = 0.2, D_2 = 1.2$ near $e \sim 0$ which is again in agreement with eq. (6) as well as relation $D_2 = d(1 - 2\chi)$ with $d = 2$. The analogy of the statistics with the case displayed in figure 2 indicates the disorder insensitivity of the statistics for $w < 1$. 

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FIG. 4. Critical spectral statistics for disorder: $w = 1$: (a) Level density $R_1(e_w)/N$ along with fit $f(e_w) = \frac{1}{2\sqrt{1.15\pi}} e^{-0.76 e_w^2}$ with $e_w = e/w$, (b) $D_q$ near $e \sim 0$, (c) $P(S)$ near $e \sim 0$, (d) $P(S)$ near $e \sim 4$, (e) $\chi(r)$ near $e \sim 0$, (f) $\chi(r)$ near $e \sim 4$. Here, with $\langle I_2 \rangle = 0.0269$ and $R_1(e) \approx \frac{0.261}{w} N$, eq. [21] gives $\Lambda_e = 0.08$ near $e \sim 0$. The parts (b) and (e) now give $D_2 = 0.95, \chi = 0.5$ near $e \sim 0$; this values are no longer consistent with eq. [5] or eq. [6] or relation $D_2 = d(1 - 2\chi)$; the latter is however expected because the $D_2 - \chi$ relation is expected to be valid only for small $\chi$. Further, as can be seen from part (a), the two bands start merging at this disorder strength. In contrast to weak disorder case, the statistics in flat and dispersive bands are now reversed i.e closer to Poisson and GOE respectively. But the survival of scale-invariance and partially localized behavior even for this disorder strength indicates the dominance of flat band spectrum on that of the dispersive band.
FIG. 5. **Critical spectral statistics for strong disorder** $w = \sqrt{10}$: (a) Level density $R_1(e_w)/N$ along with fit $f(e_w) = \frac{1}{2\sqrt{0.82\pi}} e^{-1.2 e_w^2}$ with $e_w = e/w$, (b) $D_q$ near $e \sim 0$, (c) $P(S)$ near $e \sim 0$, (d) $P(S)$ near $e \sim 4$, (e) $\chi(r)$ near $e \sim 0$, (f) $\chi(r)$ near $e \sim 4$. Here with $\langle I_2 \rangle = 0.1149$ and $R_1(e) \approx 0.3 N/w$, eq. (21) gives $\Lambda_e = 1.38 \times 10^{-3}$ near $e \sim 0$. As can be seen from parts (b) and (e) $\chi \approx 0.6$, $D_2 = 1.2$ near $e \sim 0$. Again there is no agreement with eq.(5) or relation $D_2 = d(1-2\chi)$. Clearly eq.(5) seems to be applicable for a much smaller $\Lambda_e$. Further a large $D_2$ value here seems to be the effect of the complete merging between two bands giving rise to a new band. The statistics approaches Poisson regime and an intermediate regime for $e \sim 0$ and $e \sim 4$ respectively, both indicating localized dynamics of wave functions. The system has now reached an insulator limit in the bulk energies but is still partially localized at the edge of the new band.
FIG. 6. Comparison of Flat band ensemble (FE) with AE and BE: Here the parts (a) and (b) display the \( P(s) \) comparison for the AE, BE analogues of a weakly disordered flat band for two disorders \( w^2 = 0.1 \) and 10. The AE and BE analogues have been obtained by the conditions \( \Lambda_{e,FE} = \Lambda_{e,AE} = \Lambda_{e,BE} \) given by eq.(21), eq.(24) and eq.(26) respectively; the system parameter for the three ensembles leading to approximately same \( \Lambda_e \) near \( e \sim 0 \) are as follows: (a) FE: \( N = 1156, w^2 = 10^{-2}, \varepsilon = 2, t = 1, \) AE: \( N = 512, w^2 = \frac{415}{6}, t = \frac{1}{12}, \) BE: \( N = 512, c = 0.2, \) and, (b) FE: \( N = 1156, w^2 = 10, \varepsilon = 2, t = 1, \) AE: \( N = 512, w^2 = \frac{120.15}{6}, t = \frac{1}{12}, \) BE: \( N = 512, c = 69.2. \)

To rule out the accidental coincidence, we also compare \( \gamma_1 = \gamma(0.4699), \gamma_2 = \gamma(1.9699) \) for a range of \( \Lambda_e \) values. The results are displayed in parts (c) and (d), respectively. As clearly visible from the figures, the values for all three ensemble collapse on the same curve for small \( \Lambda_e \) values. But while \( \Lambda_e \) of a flat band decreases for both small and large \( \Lambda_e \), the \( \Lambda_e \) of an AE and BE smoothly increases from 0 to a large value with decreasing disorder. The deviation in their behavior for large \( \Lambda_e \) values is therefore an indicator of the different nature of transition.