Loop products on level homology and closed geodesics

Arun Maiti

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Abstract

In this paper we discuss how loop products on level (co)homology can be used to answer questions about existence of closed geodesics on Riemannian manifolds. We furnish proofs of some of the results stated in the paper 'Loop products and closed geodesics' by M. Goresky and N. Hingston.

In the beginning of the 21st century D. Sullivan introduced various operations on homology of free loop space which are also known as the string topology operations. In [GH09], M. Goresky and N. Hingston reinterpreted and studied properties of the operations. In this paper we are interested in two operations among them, one is a product on homology known as the Chas-Sullivan product, and the other is a product on cohomology known as the Goresky-Hingston product. In [GH09], the authors also defined the two operations on level and sublevel homologies, and showed some applications. In this paper we furnish proofs of two such applications (theorems) whose statements appear in the same article without proofs. The theorems guarantee existence of infinitely many closed geodesics when the Chas-Sullivan product or the Goresky-Hingston product admits a non-nilpotent sublevel homology or cohomology class respectively given by a closed geodesic. The author claims no originality of the results stated in this article.

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1 Free loop space

Let $M$ be an $n$-dimensional connected oriented closed Riemannian manifold. Let $\Lambda(M)$ be the free loop space which is a completion of all piecewise smooth curves with same initial and end point,

$$\Lambda(M) = \{ \alpha \in H^1([0,1], M) | \alpha(0) = \alpha(1) \}.$$ 

We denote the free loop space of $M$ simply by $\Lambda$ when there is no confusion about the underlying manifold. It admits a structure of a Hilbert manifold (see [K]). We
also denote by \( \Lambda_0 \) the space of constant loops in \( \Lambda \). We have the evaluation mappings defined by \( ev_s : \Lambda \to M, \ ev_s(\alpha) = \alpha(s) \). Let us define

\[
\Theta = \{ \alpha, \beta \in \Lambda \times \Lambda | \alpha(0) = \beta(0) \}
\]

and

\[
\Theta^{>0,>0} = \{ \alpha, \beta \in \Theta \subset \Lambda \times \Lambda | \alpha, \beta \in \Lambda - \Lambda_0 \}.
\]

Let \( \phi_2 : \Theta \to \Lambda \) be the usual concatenation of loops and we define the cutting map \( c_2 : \Theta_t \to \Lambda \times \Lambda, c_2(\alpha)(s) = (\alpha(st), \alpha(t + s(1-t))) \).

### 1.1 The Chas-Sullivan product

There is a pullback square

\[
\begin{array}{ccc}
\Theta^{>0,>0} & \to & \Lambda \times \Lambda \\
\downarrow & & \downarrow \\
M^{>0,>0} & \to & M \times M
\end{array}
\]

where \( D \) is the diagonal embedding. We know that \( ev_0 \times ev_0 : \Lambda \times \Lambda \to M \times M \) is a fibre bundle, so \( \Theta \to \Lambda \times \Lambda \) can be viewed as a codimension \( n \) embedding.

The Chas-Sullivan product, which we denote by \( \star \), is defined by the following composition:

\[
H^i(\Lambda) \times H^j(\Lambda) \xrightarrow{\Sigma} H_{i+j}(\Lambda) \to H_{i+j}(\Lambda, \Lambda - \Theta) \xrightarrow{\mu} H_{i+j-n}(\Theta) \xrightarrow{(\phi_2)} H_{i+j-n}(\Lambda)
\]

where \( \mu \) is the Thom class for the embedding \( \Theta \hookrightarrow \Lambda \times \Lambda \).

### 1.2 The Goresky-Hingston product

Let

\[
\Gamma : \Lambda \times [0,1] \to \Lambda, \quad \Gamma(\alpha, s) = \alpha \circ \theta_{1/2 \to s},
\]

where \( \theta = \theta_{1/2 \to s} : [0,1] \to [0,1] \), be the reparametrization function that is linear on \([0,1/2]\), linear on \([1/2,1]\), has \( \theta(0) = 0, \theta(1) = 1 \), and \( \theta(1/2) = s \). The Goresky-Hingston product

\[
H^i(\Lambda, \Lambda_0) \times H^j(\Lambda, \Lambda_0) \xrightarrow{\partial} H^{i+j+n-1}(\Lambda, \Lambda_0)
\]

is defined by the following composition

\[
H^i(\Lambda, \Lambda_0) \times H^j(\Lambda, \Lambda_0) \xrightarrow{\Sigma} H^{i+j}(\Lambda \times \Lambda) \xrightarrow{\tau} H^{i+j+n}(\Theta, \Theta \to \Theta^{>0,>0}) \xrightarrow{\Gamma_*} H^{i+j+n}(\Lambda \times I, \Lambda_0 \times I \cup I \times I) \cong H^{i+j+n-1}(\Lambda, \Lambda_0),
\]

where \( \tau \) is the Thom isomorphism. The product and its properties were studied in detail in [GH09].
1.3 Closed geodesics and index growth

The energy and length of an element $\alpha \in \Lambda$ are defined respectively by

$$E(\alpha) = \int_0^1 g(\alpha'(t), \alpha'(t)) dt \quad \text{and} \quad L(\alpha) = \int_0^1 \sqrt{g(\alpha'(t), \alpha'(t))} dt.$$ (1.3.1)

The energy of a loop depends on its parametrisation, but the length does not. There is an obvious circle action on the free loop space defined by translation

$$S^1 \times \Lambda \to \Lambda, (e^{2\pi it}, \alpha) \to \beta, \beta(s) = \alpha(s + t).$$

Both the energy and length function are invariant under the $S^1$ action.

The Chas-Sullivan product and the Goresky-Hingston product behaves better with level homology or cohomology when the levels are defined with respect to the square root of the energy function instead of the energy function. Let $F = \sqrt{E}$ be the square root of the energy function. The critical points of $F$ are closed geodesics. We recall that the index of a closed geodesic $\gamma$ is the dimension of a maximal subspace of the tangent space $T_\gamma(\Lambda)$ at $\gamma$ of $\Lambda$ on which the Hessian $d^2F(\gamma)$ is negative definite, and the nullity of $\gamma$ is $\dim T_0^0(\Lambda) - 1$, where $T_0^0$ is the null space of the Hessian $d^2F(\gamma)$. The $-1$ is incorporated to account for the fact that every nonconstant closed geodesic $\gamma$ occurs in an $S^1$ orbit of closed geodesic. A critical point of $F$ is called nondegenerate if the nullity is zero, in other words the $S^1$ orbit of the critical point is a nondegenerate Morse-Bott critical submanifold.

We know that iterates of closed geodesics are also closed geodesics. The following theorem gives us estimates of indices of iterates.

**Theorem 1.1.** ([GH09]) Let $\gamma$ be a closed geodesic with index $\lambda$ and nullity $\nu$. Let $\lambda_m$ resp. $\nu_m$ are respectively index resp. nullity of the $m$-fold iterates $\gamma^m$. Then $\nu_m \leq 2(n - 1)$ for all $m$ and

$$|\lambda_m - m\lambda| \leq (m - 1)(n - 1),$$

$$|\lambda_m + \nu_m - m(\lambda + \nu)| \leq (m - 1)(n - 1),$$ (1.3.2)

The average index

$$\bar{\lambda} = \lim_{m \to \infty} \frac{\lambda_m}{m}$$

exists and

$$|\bar{\lambda} - \lambda| \leq n - 1 \quad \text{and} \quad |\bar{\lambda} - \lambda + \nu| \leq n - 1$$

Furthermore, if $\gamma$ and $\gamma^2$ are nondegenerate critical points then the inequalities in (1.3.2) are strict, and for any $k \geq 0$, there exists $M$ such that for all $m \geq M$,

$$|\lambda_m - m\lambda| \leq (m - 1)(n - 1) - k,$$

$$|\lambda_m + \nu_m - m(\lambda + \nu)| \leq (m - 1)(n - 1) - k.$$ (1.3.3)

So the greatest and smallest possible values of $\lambda_m$ satisfying (1.3.2) is

$$\lambda_m^{\min} = m\lambda - (m - 1)(n - 1),$$

$$\lambda_m^{\max} = m\lambda + (m - 1)(n - 1).$$ (1.3.4)
We say the iterate $\gamma^r$ has maximal (resp. minimal) index growth up to level $k$ if for all $m \leq rk$,

$$\lambda_m^{\text{min}} = m\lambda \pm (m-1)(n-1).$$

(1.3.5)

1.4 String products on the level homology

For any $a, 0 \leq a \leq \infty$, the level sets of $F = \sqrt{E}$ are defined by

$$\Lambda^{\leq a} = \{ \alpha \in \Lambda | F(\alpha) \leq a \} \quad \text{and} \quad \Lambda^{<a} = \{ \alpha \in \Lambda | F(\alpha) < a \}. $$

Let $\gamma$ be a nondegenerate critical point of $F$ of index $\lambda$, $\Sigma \subset \Lambda$ denotes the $S^1$ orbit of $\gamma$. Let $\Gamma^+, \Gamma^0$ and $\Gamma^-$ be the positive, null, and negative eigenvectors of the Hessian of $F$. The tangent bundle of $T\Lambda|_\Sigma$ decomposes into an orthogonal sum of vector bundles $T\Lambda|_\Sigma = \Gamma^+ \oplus \Gamma^0 \oplus \Gamma^-$. The inclusion $\Sigma \to \Lambda$ induces an isomorphism $T\Sigma \cong \Gamma^0$. So the normal bundle of $\Sigma$ in $\Lambda$ can be identified with $\Gamma^+ \oplus \Gamma^-$. The homology groups and cohomology groups of the pair $(\Lambda^{\leq a}, \Lambda^{<a})$ can be computed using Morse theory.

**Theorem 1.2.** ([RP63]) Assume that the Riemannian metric on $M$ is chosen so that all the geodesics are nondegenerate.

1) If there are no critical values of $F$ in $[a, b]$ then $\Lambda^{\leq b}$ retracts onto $\Lambda^{\leq a}$.
2) If $c, a < c < b$ is the only critical value of $F$ in the interval $[a, b]$. Denote by $N_1, N_1, \cdots, N_r$ the components of $\text{Crit}(F)$ at level $c$, with respective indices $\lambda_1, \lambda_2, \cdots, \lambda_r$, and $\Gamma^-_i \to N_i, i = 1, \cdots, r$ are the negative bundles. Then $\Lambda^{\leq b}$ retracts onto a space homeomorphic to $\Lambda^{\leq a}$ with a disc bundles $D\Gamma^-_i$ of dimension $\lambda_i$ disjointly attached along their boundaries.

**Corollary 1.3.** Suppose $\bar{\gamma}$ be the nondegenerate critical orbit of a critical point $\gamma$ of index $\lambda$. Let $G = \mathbb{Z}$ when negative bundle $\Gamma^-$ oriented and $G = \mathbb{Z}_2$ otherwise.

Then the local level homology groups are given by

$$H_i(\Lambda^{\leq c} \cup \bar{\gamma}, \Lambda^{<c}) \cong H_i(\Lambda^{\leq c}, \Lambda^{<c} - \bar{\gamma}) \cong \begin{cases} G & \text{for } i = \lambda, \lambda + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $U_\gamma \subset \Lambda$ be neighborhood of $\bar{\gamma}$ including no other critical point of $F$. Then the inclusion $\Lambda^{\leq c} \cap U_\gamma \hookrightarrow \Lambda^{\leq c}$ induces an isomorphism

$$H_i(\Lambda^{\leq c}, \Lambda^{<c} - \bar{\gamma}) \cong H_i(\Lambda^{\leq c} \cap U_\gamma, (\Lambda^{\leq c} - \bar{\gamma}) \cap U_\gamma).$$

(1.4.1)

Let $\Sigma^{=a} \subset \Lambda^{=a}, a \in \mathbb{R}$ consists of all critical points of $F$ at level $a \in \mathbb{R}$. Suppose $U \subset \Lambda$ be neighborhood of $\Sigma^{=a}$ including no other critical point of $F$. Then the inclusion $(\Lambda^{\leq c} - \Sigma^{=a}) \cap U \hookrightarrow \Lambda^{\leq c}$ induces an isomorphism

$$H_i(\Lambda^{\leq a}, \Lambda^{\leq a} - \Sigma^{=a}) \cong H_i(\Lambda^{\leq a} \cap U, (\Lambda^{\leq a} - \Sigma^{=a}) \cap U).$$

(1.4.2)
For a level homology class $\alpha$ the critical level of $\alpha$ is defined by
$$cr(\alpha) = \inf\{a \in \mathbb{R} : \alpha \in \text{Image}(H_i(\Lambda^{\leq a}) \to H_i(\Lambda))\}.$$ 

The Chas-Sullivan product extends to a product on level homologies:
$$H_i(\Lambda^{\leq a}) \times H_j(\Lambda^{\leq b}) \to H_{i+j}(\Lambda^{\leq a+b}) \quad (1.4.3)$$

$$H_i(\Lambda^{\leq a}, \Lambda^{< a}) \times H_j(\Lambda^{\leq b}, \Lambda^{< b}) \to H_{i+j}(\Lambda^{\leq a+b}, \Lambda^{< a+b}) \quad (1.4.4)$$

A homology class $\beta \in H_i(\Lambda^{\leq a}, \Lambda^{< a})$ is called level-nilpotent if there exists $m$ such that the Chas-Sullivan product in level homology vanishes
$$0 = \beta^m \in H_{mi+(m-1)n}(\Lambda^{\leq ma}, \Lambda^{< ma}).$$

A homology class $\alpha \in H_i(\Lambda)$ is called level nilpotent if there exists $m$ such that $cr(\alpha^m) < mcr(\alpha)$.

The following theorem shows how level homology classes behaves with respect to the Chas-Sullivan product.

**Theorem 1.4** ([GH09], Theorem 7.3). *Let $F : \Lambda \to \mathbb{R}$ as above. Suppose that all the critical orbits of $F$ are isolated and nondegenerate. Then every homology class $\alpha \in H_i(\Lambda)$ is level-nilpotent, and for every $a \in \mathbb{R}$, every level homology class $\beta \in H_i(\Lambda^{\leq a}, \Lambda^{< a})$ is level-nilpotent.*

Similar to the Chas-Sullivan product the Goresky-Hingston product also extends to level cohomology.
$$H^i(\Lambda^{\leq a}) \times H^j(\Lambda^{\leq b}) \to H^{i+j+n-1}(\Lambda^{\leq a+b}) \quad (1.4.5)$$

$$H^i(\Lambda^{\leq a}, \Lambda^{< a}) \times H^j(\Lambda^{\leq b}, \Lambda^{< b}) \to H^{i+j}(\Lambda^{\leq a+b}, \Lambda^{< a+b}) \quad (1.4.6)$$

For a cohomology class $\alpha \in H^j(\Lambda, \Lambda_0)$ critical level of $\alpha$ is defined by
$$cr(\alpha) = \sup\{\ker(H^j(\Lambda, \Lambda_0) \to H^j(\Lambda^{< b}))\}$$

A class $\alpha \in H^i(\Lambda, \Lambda_0)$ is level nilpotent if there exist $m$ so that $cr(\alpha^m) > mcr(\alpha)$. A cohomology class $\beta \in H^i(\Lambda^{\leq a}, \Lambda^{< a})$ is called level nilpotent if there exists $m$ such that $\beta^m = 0$ in $H^{mi+(m-1)n-1}(\Lambda^{\leq ma}, \Lambda^{< ma})$.

Let $\alpha \in H^i(\Lambda, \Lambda_0)$ and $\beta \in H^i(\Lambda^{\leq a}, \Lambda^{< a})$ are associated if there exist an associating class $\omega \in H^i(\Lambda, \Lambda^{< a})$ with

$\omega \longrightarrow \alpha \quad H^i(\Lambda, \Lambda^{< a}) \longrightarrow H^i(\Lambda, \Lambda_0)$

$\downarrow \quad \text{in} \quad \downarrow$

$\beta \quad H^i(\Lambda^{\leq a}, \Lambda^{< a})$

**Lemma 1.5.** Suppose $\alpha \in H^i(\Lambda, \Lambda_0)$ and $\beta \in H^i(\Lambda^{\leq a}, \Lambda^{< a})$ are associated, where $\alpha = cr(\alpha)$. If $\beta$ is level nilpotent, then $\alpha$ is also level nilpotent.
Let $\mathcal{A} \in \Lambda$ be the space of all loops parametrized proportional to arc length. Then $\mathcal{A}$ is homotopy equivalent to $\Lambda$.

For $A, B \subset \Lambda$ we write $A \times_M B = (A \times B) \cap \Theta$. Let $\phi_{min} : (\Theta - \Lambda_0) \to \Lambda$ defined by $\phi_{min}(\alpha, \beta) = \phi_{s}(\alpha, \beta)$ with $s = \frac{F(\alpha)}{F(\alpha) + F(\beta)}$, which extends continuously to $\Lambda_0$ giving a map $\phi_{min} : \Theta \to \Lambda$. Define $A \ast B = \Phi_{min}(A \times_M B)$.

**Proposition 1.6.** Suppose that $\alpha \in H^{i}(\Lambda, \Lambda - A)$ and $\beta \in H^{i}(\Lambda, \Lambda - B)$ with $A$ and $B$ closed subset of $\Lambda - \Lambda_0$. Then $\alpha \circ \beta \in H^{i}(\Lambda, \Lambda - A \ast B)$.

The statement of the following theorem stated in [GH09], theorem 11.3, however proof was omitted, here we give a proof.

**Theorem 1.7.** Suppose that all critical points of the function $F : \Lambda \to \mathbb{R}$ are nondegenerate. Then every class $\alpha \in H^{i}(\Lambda, \Lambda_0)$ is level-nilpotent and for any $a \in \mathbb{R}$, every class $\beta \in H^{i}(\Lambda^{\leq a}, \Lambda^{< a})$ is level-nilpotent.

**Proof.** By the above lemma, it suffices to prove that every class $\beta \in H^{i}(\Lambda^{\leq a}, \Lambda^{< a})$ is level nilpotent, where $a \in \mathbb{R}$ is a nondegenerate critical value. Since all the critical orbits are nondegenerate, therefore the critical set $\Sigma^{= a}$ of the $S^1$ orbit of any closed geodesics, say, $\gamma_1, \gamma_2, \cdots, \gamma_s$. Let $S^1$ orbit of $\gamma_j$ is denoted by $\bar{\gamma}_j$. We may choose the ordering so that there exists $r \leq s$ such that $H^{i}(\Lambda^{\leq a}, \Lambda^{a} - \bar{\gamma}_j) \neq 0$ if and only if $1 \leq j \leq s$. Let $\Sigma_0^{= a} \subset \Sigma^{= a}$ consists $S^1$ orbits of $\gamma_1, \gamma_2, \cdots, \gamma_r$. For $1 \leq j \leq r$, we can choose open neighborhood $U_j \in \Lambda$ be a neighborhood of $S^1$ orbit of $\gamma_j$, denoted by $\bar{\gamma}_j$, such that $U_j \cap U_k = \phi$ for $j \neq k$. The level cohomology groups can be expressed as direct sum

$$H^{i}(\Lambda^{\leq a}, \Lambda^{< a}) \cong \bigoplus_{j=1}^{r} H^{i}(\Lambda^{\leq a} \cap U_j, \Lambda^{< a} \cap U_j) \cong \bigoplus_{j=1}^{r} H^{i}(\Lambda^{\leq a} \cap U_j, \Lambda^{< a} - \Sigma^{= a}).$$

Using the above proposition and cohomology version of [1.4.1] and [1.4.2] we have

$$b^{\circ m} \in H^{b}(\Lambda^{\leq m a}, \Lambda^{\leq ma} - (\Sigma^{= a})^{*m}) \cong \bigoplus_{j=1}^{r} H^{b}(U_{j,m}, U_{j,m} - \bar{\gamma}_j^m) \cong \bigoplus_{j=1}^{r} H^{i}(\Lambda^{\leq a} \cap U_j, \Lambda^{< a} - \bar{\gamma}_j^m),$$

where $U_{j,m}$ are disjoint neighborhoods of the $\bar{\gamma}_j^m$ containing no other critical points and $b = mi + (m - 1)(n - 1)$.

Let $\lambda_m$ denote the index of $\gamma_j^m$. Suppose $j$th summand in equation [1.4.7] is not zero, then $\lambda_m$ equals to either $b - 1$ or $b$, so $\lambda_m = mi + (m - 1)(n - 1)$, or $\lambda_m = mi + (m - 1)(n - 1) - 1$. On the other hand, since $\beta \in H^{i}(\Lambda^{\leq a}, \Lambda^{< a}) \neq 0$ therefore $\lambda_1 \in \{i - 1, i\}$ and we know from equation [1.3.2] that for any $k \geq 0$ there exist $m$ such that $\lambda_m \leq m\lambda_1 + (m - 1)(n - 1) - k$. Now if $i = \lambda_1$ or $i = \lambda_1 + 1$, then for a large $k$ we arrive at a contradiction. This concludes the proof.

$$\square$$

2 **Existence of closed geodesics**

A closed geodesic $\gamma \in \Lambda$ is called prime if it is not iterate of another closed geodesic. In Riemannian geometry one often ask, given a closed Riemannian manifold, does
there exist a closed geodesic, or how many prime closed geodesic are there. It is expected that any compact simply connected Riemannian manifold admits infinitely many prime closed geodesics. Below we gather some of the known results.

If the sequence of Betti numbers $b_k(\Lambda; F)$ with coefficients in some field $F$ is unbounded, the answer is affirmative (Gromoll-Meyer theorem [GM]). This is in particular true for manifolds whose rational cohomology ring requires at least two generators (PMS), and also for globally symmetric spaces of rank $> 1$ (Zi).

Among the spaces which do not satisfy the assumptions of the Gromoll-Meyer theorem, the problem is solved in the affirmative for the 2-sphere ([H93]). It is solved in the affirmative for a generic metric on 2-sphere ([HR]). To the author’s knowledge the problem in the case of the $n$-spheres $S^n$, $n \geq 3$ remains still open.

We state here two theorems of N. Hingston, which can be applied to show that any Riemannian 2-sphere admits infinitely many closed geodesics.

**Theorem 2.1.** ([H93]) Let $\gamma$ be an isolated closed geodesic of a $n$-dimensional compact manifold $M$ of index $\lambda$, nullity $\eta$, and $F(\gamma) = a$. Assume that
1) $\gamma$ has non-trivial local level cohomology class i.e.
$$H^\lambda(\Lambda^{<a} \cup \gamma, \Lambda^{<a}) \neq 0$$
2) $(\text{index} + \text{nullity})(\gamma^m) \geq m(\lambda + \eta) + (n - 1)(m - 1)$ for all $m \geq 1$.
Then for any $\epsilon > 0$, if $m \in \mathbb{Z}$ is sufficiently large, there exists closed geodesic with length in the open interval $(ma - \epsilon, ma)$. It follows that $M$ has infinitely many closed geodesics.

A corollary of the theorem is that suppose $M$ is a two-sphere and suppose that the ’shortest" Lusternik-Schnirelmann closed geodesic (obtained by minimax) is isolated and "nonrotating" then the above conclusion holds with $\lambda = 1$.

**Theorem 2.2.** ([H97]) Let $\gamma$ be an isolated closed geodesic of a $n$-dimensional compact manifold $M$ of length $a$, index $\lambda$, nullity $\eta$, length($\gamma$) = $L$, and $F(\gamma) = a$. Assume that
1) $\gamma$ has non-trivial local level homology class i.e.
$$H_{\lambda+\eta}(\Lambda^{<a} \cup \gamma, \Lambda^{<a}) \neq 0$$
2) $(\text{index} + \text{nullity})(\gamma^m) \leq m(\lambda + \eta) - (n - 1)(m - 1)$ for all $m \geq 1$.
Then $M$ admits infinitely many closed geodesics. Furthermore for any $\epsilon > 0$, if $m \in \mathbb{Z}$ is sufficiently large, then there exists a closed geodesic with length in the open interval $(mL, mL + \epsilon)$.

A nondegenerate closed geodesic is isolated and satisfies the condition (1) of both the theorem. Notice that by [13.2] for nondegenerate closed geodesics, condition (2) of the first theorem can be satisfied only when the index growth is maximal, and condition (2) of the second theorem can be satisfied only when the sum of index and nullity growth is minimal.

Below we prove our main theorems which were stated in [GH09] without any proof.
Theorem 2.3. ([GH09], Theorem 12.7) Let $\gamma$ be an isolated closed geodesic with nonnilpotent level homology with respect to the Chas-Sullivan product. Let $L = \text{length}(\gamma)$. Then for any $\epsilon > 0$, if $m \in \mathbb{Z}$ is sufficiently large, there is a closed geodesic with length in the open interval $(mL, mL + \epsilon)$. It follows that $M$ has infinitely many closed geodesics.

Proof. Let $\gamma$ be an isolated closed geodesic with nonnilpotent level homology class $x \in H_j(\Lambda^a \cup \gamma, \Lambda^a)$, where $a = F(\gamma)$. Denote $\lambda_m = \text{index}(\gamma^m)$, $\nu_m = \text{nullity}(\gamma^m)$. Since $x^m \in H_{mj - n(m - 1)}(\Lambda^{\leq ma}, \Lambda^{\leq ma})$ is non-trivial homology class for all $m > 1$, it follows that

$$\lambda_m \leq mj - n(m - 1) \leq \lambda_m + \nu_m + 1$$

for all $m$. This implies that the average index of $\gamma$

$$\bar{\lambda} = \lim_{m \to \infty} \frac{\lambda_m}{m} = j - n$$

We have the following fact about the average index

$$\lambda_m + \nu_m \leq m\bar{\lambda} + (n - 1).$$

Now combining the above two equations and the equation 1.3.2 we have

$$mj - n(m - 1) - 1 \leq \lambda_m + \nu_m \leq mj - n + (n - 1) \Rightarrow \lambda_m + \nu_m = mj - n + (n - 1).$$

For $m = 1$, this gives $\lambda_1 + \nu_1 = j - 1$, so

$$\lambda_m + \nu_m = mj - n + (n - 1) = m(\lambda_1 + \nu_1) - (m - 1)(n - 1).$$

Therefore the geodesic $\gamma$ satisfies the hypothesis of the theorem 2.2 and this concludes the proof the theorem.

Theorem 2.4. ([GH09], Theorem 12.8) Let $\gamma$ be a nondegenerate closed geodesic with nonnilpotent level cohomology with respect to the Goresky-Hingston product. Let $L = \text{length}(\gamma)$. Then for any $\epsilon > 0$, if $m \in \mathbb{Z}$ is sufficiently large, there exists a closed geodesic with length in the open interval $(mL - \epsilon, mL)$. It follows that $M$ has infinitely many closed geodesics.

Proof. Let $\gamma$ be an isolated closed geodesic with nonnilpotent level cohomology class $x \in H^j(\Lambda^{\leq a}, \Lambda^a)$, where $a = F(\gamma)$. We use the same notations as in the last theorem. Since $x^m \in H^{mj + (m - 1)(n - 1)}(\Lambda^{\leq ma}, \Lambda^{< ma})$ is a non-trivial class, it follows that

$$\lambda_m \leq mj + (m - 1)(n - 1) \leq \lambda_m + \nu_m + 1$$

for all $m$. We know that $\nu_m \leq 2n - 1$ for all $m$ [see [MR]], so the average index

$$\bar{\lambda} = j + n - 1.$$
We also have the following fact about the average index from 1.3.2
\[ m\bar{\lambda} - (n - 1) \leq \lambda_m. \]

Now combining the above two equations and the equation 1.3.2 we obtain
\[ m(j + n - 1) - (n - 1) \leq \lambda_m \leq mj + (m - 1)(n - 1) \Rightarrow \lambda_m = mj + (m - 1)(n - 1). \]

Putting \( m = 1 \) gives us \( \lambda = \lambda_1 = j \), so
\[ \lambda_m = m(\text{index}(\gamma)) + (m - 1)(n - 1) \text{ for all } m. \]

Therefore, the geodesic \( \gamma \) satisfies the hypothesis of the theorem 2.1 and this concludes the proof of the theorem.

The above two theorem shows that the Chas-Sullivan and the Goresky-Hingston product are in some sense dual to each other.

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