Closed timelike geodesics in a gas of cosmic strings

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Abstract. We find a class of solutions of Einstein’s field equations representing spacetime outside a spinning cosmic string surrounded by vacuum energy and a rotating gas of non-spinning strings of limited radial extension. Outside the region with vacuum energy and cosmic strings there is empty space. We show that there exist closed timelike geodesics in this spacetime.

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1. Introduction

The question of the causal structure of spacetime is related to the question of the existence of closed timelike curves (CTCs). One should therefore try to formulate as simple criteria as possible for the existence of such curves. It would be particularly exciting if a free particle could move so that it arrives back at the event it started from. Such a particle must follow a closed timelike geodesic (CTG).

The existence of CTCs has been investigated in some cases. Gödel [1] was the first one to demonstrate the existence of a solution of Einstein’s field equations with CTCs. He showed that such curves exist in a rotating universe model with dust and negative vacuum energy. Already in 1937 van Stockum [2] found the solutions of Einstein’s field equations inside and outside a cylindrically symmetric rotating dust distribution. Tipler [3] showed that the external solution contains CTCs. Gott has shown that CTCs exist in a spacetime with a pair of moving cosmic strings [4]. An eternal time machine in $(2 + 1)$-dimensional anti-de Sitter space has been analysed by DeDeo and Gott [5]. Another example of a spacetime admitting CTCs is Bonnor’s rotating dust metric which has recently been analysed by Collas and Klein [6].

Only a few known solutions contain CTGs. The first one was presented by Soares [7] in 1979. He found a class of cosmological solutions of the Einstein–Maxwell equations with rotating dust and electromagnetic fields. For a subclass of these models, the topology of the spacetime manifold is $S^3 \times R$, and the timelike geodesic curves followed by the dust particles are closed. In this case, the existence of the CTGs depends upon the nontrivial topology of the considered spacetime.

The first one to have demonstrated the existence of CTGs in $R^4$ is Steadman [8]. He has shown that such curves exist in the exterior van Stockum spacetime. It has also been shown by Bonnor and Steadman [9] that a spacetime with two spinning particles, each one with a magnetic moment and a mass equal to their charge, permit special cases in which there exist CTGs. Rosa and Letelier [10] have shown that there exist regions where these curves are stable.

Recently Rosa and Letelier have constructed a new solution of Einstein’s field equations with a spacetime containing a black hole pierced by a spinning cosmic string [11].

In the present paper, we find a class of new solutions of Einstein’s field equations describing a spacetime with CTGs. This spacetime is time independent and cylindrically symmetric. It is filled with a gas of limited radial extension which consists of non-spinning strings parallel to the symmetry axis and vacuum energy, and there is a spinning cosmic string along the axis. Spacetime in the gas is matched continuously to the external metric in empty space outside a spinning cosmic string [12].

2. Physical properties of the reference frame

We shall consider curved spacetime with axial symmetry. The metric is assumed to be stationary. The line element may then be written [13]

$$ds^2 = -f dt^2 + 2k dt d\phi + l d\phi^2 + e^\mu (dr^2 + dz^2),$$

where $f, k, l$ and $\mu$ are functions of $r$ and $z$, and we have used units so that the velocity of light $c = 1$. It may be noted that the Lorentz invariance in the $z$-direction implies that $f = e^\mu$. The coordinates $r, \phi$ and $z$ are comoving in a reference frame $R$. 

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Rindler and Perlick [14] have deduced a formula for the four-acceleration $A_\mu$ and the vorticity $\Omega$ of the four-velocity field of the reference particles in $R$ relative to the local compass of inertia. Applied to the line element (1) one obtains [15]

$$A_\mu = \left(0, \frac{f r}{2 f}, 0, 0\right)$$  \hspace{0.5cm} (2)

and

$$\Omega = \frac{e^{-\mu/2 f}}{2 D} \left(\frac{k}{f}\right)_\nu,$$  \hspace{0.5cm} (3)

where

$$D^2 = f l + k^2 > 0.$$  \hspace{0.5cm} (4)

The condition $fl + k^2 > 0$ comes from the signature of the spacetime metric. Here $\Omega$ represents the local rate of rotation of the reference frame $R$, i.e. the rotation with respect to a ‘compass of inertia’. A more general formula, valid also for time-dependent metrics, has been deduced by Weissenhoff [16]. When $k/f$ is constant, the vorticity is vanishing. Since the metric is time independent, the distances between fixed reference points are also independent of time. In this sense, the reference system is rigid. Thus, when $k/f$ is constant, the swinging plane of the Foucault pendulum, representing a local compass of inertia, will have a fixed direction in $R$.

In order to investigate the kinematics of the frame $R$, we consider a particle moving in a plane $z = \text{constant}$. The Lagrange function of the particle is

$$L = -\frac{1}{2} f t^2 + k t \dot{\phi} + \frac{1}{2} l \dot{\phi}^2 + \frac{1}{2} e^{\mu} r^2.$$  \hspace{0.5cm} (5)

The conserved angular momentum of the particle is

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = l \dot{\phi} + k i,$$  \hspace{0.5cm} (6)

where the dot denotes differentiation with respect to the proper time of the particle.

In the case $l > 0$, the coordinate basis vector $\vec{e}_\phi$ is spacelike, and we can define an angular velocity

$$\omega = \frac{d\phi}{dt} = \frac{\dot{\phi}}{l},$$  \hspace{0.5cm} (7)

giving

$$p_\phi = (l \omega + k)i.$$  \hspace{0.5cm} (8)

A local inertial reference frame is a freely falling, non-rotating frame having $p_\phi = 0$. Hence the inertial frames have an angular velocity

$$\omega = -\frac{k}{l}$$  \hspace{0.5cm} (9)

in the reference frame $R$. This is an expression of the inertial dragging effect. Hence, in the case $l > 0$, the presence of the $d\phi dr$-term in the line element (1) is due to the rotation of the reference
frame $R$ relative to the inertial frames. We will now argue that this interpretation is not valid in the case $l < 0$.

The line element (1) may be written

$$ds^2 = -\frac{D^2}{l}dt^2 + D^2 f^2 + e^\mu (dr^2 + dz^2).$$

(10)

This form of the line element is obtained by replacing the coordinate basis vector field $(\vec{e}_t, \vec{e}_\phi, \vec{e}_r, \vec{e}_z)$ by the orthogonal basis vector field $(\vec{e}_t - (k/l)\vec{e}_\phi, \vec{e}_\phi, \vec{e}_r, \vec{e}_z)$. The vector $\vec{e}_t - (k/l)\vec{e}_\phi$ is timelike and $\vec{e}_\phi, \vec{e}_r$ and $\vec{e}_z$ are spacelike when $l > 0$, which leads to the usual interpretation of $\omega$ as an angular velocity as given in equation (9). However, $\vec{e}_\phi$ is timelike and $\vec{e}_t - (k/l)\vec{e}_\phi, \vec{e}_r$, and $\vec{e}_z$ are spacelike when $l < 0$, making this interpretation impossible.

The line element can also be written in the form

$$ds^2 = - f \left( dt - \frac{k}{f} d\phi \right)^2 + \frac{D^2}{f} d\phi^2 + e^\mu (dr^2 + dz^2)$$

(11)

when $l > -k^2/f$, permitting both positive and negative values of $l$. This is obtained by replacing the coordinate basis vector field $(\vec{e}_t, \vec{e}_\phi, \vec{e}_r, \vec{e}_z)$ with the orthogonal basis vector field $(\vec{e}_t, \vec{e}_\phi + (k/f)\vec{e}_r, \vec{e}_r, \vec{e}_z)$. The three-space spanned by the spacelike basis vectors $(\vec{e}_\phi + (k/f)\vec{e}_r, \vec{e}_r, \vec{e}_z)$ represents the simultaneity space orthogonal to the timelike basis vector $\vec{e}_t$. Hence the form (11) of the line element separates out the three-space orthogonal to the four-velocity of the reference particles.

The expression $dt - (k/f) d\phi$ is an exact differential when $k/f$ is constant. In this case, one can introduce a new time coordinate

$$\hat{t} = t - \frac{k}{f} \phi,$$

(12)

and the line element takes the form

$$ds^2 = - f d\hat{t}^2 + \frac{D^2}{f} d\phi^2 + e^\mu (dr^2 + dz^2).$$

(13)

The clocks showing $\hat{t}$ are locally Einstein synchronized. Hence the coordinate clocks showing $t$ are not Einstein synchronized.

In the case $l > 0$, the reference frame is rotating. Then the Einstein synchronized clocks showing $\hat{t}$ have a gap when one goes around a circle about the axis. In this case, the time $\hat{t}$ is not well-defined globally due to the mentioned gap.

As shown above, when $l < 0$ the reference system is not rotating. In this case, one may Einstein synchronize clocks around a circle. Then the time $\hat{t}$ given by equation (12) is well-defined globally. Hence, although the form (11) of the line element is valid both for $l > 0$ and $l < 0$, it introduces this well-defined coordinate $\hat{t}$ only when $l < 0$.

Locally the situation is then similar to that of a person travelling westwards around the equator in an airplane so that the local time is constant on the Earth at the position of the airplane. This corresponds to keeping the time $t$ constant. However, the proper time of the passenger increases, corresponding to an increase of the time $\hat{t}$ as given in equation (12), when $\phi$ decreases. Going around a circle, there is a gap of the time $t$ on the Earth at the date line.

However, spacetime may be different. In the present paper, we shall consider spacetime outside a spinning cosmic string. Then there is no time gap in $t$ when one goes around the circle.
3. General criteria for CTGs

Let us consider a circular timelike curve in the plane \( z = \text{constant} \). The four-velocity identity applied to the tangent vector field of the curve takes the form

\[
l \dot{\phi}^2 + 2k \dot{t} \dot{\phi} - f \dot{t}^2 = -1,
\]

where we have used units so that the velocity of light \( c = 1 \) and the dot means the derivative with respect to proper time \( \tau \). If \( \dot{t} = 0 \), the curve is closed in spacetime. This condition seems to mean that a particle moving along such a curve is everywhere on the curve at a fixed point of time, i.e., it has an infinitely great velocity. One may wonder if this is in conflict with the special theory of relativity. However, this infinitely great velocity is measured with clocks that are not Einstein synchronized. As measured by the Einstein synchronized clocks, the velocity of the particle is less than the velocity of light. Hence its world line is inside the light cone, which means to say that it is timelike. For such a curve the four-velocity identity reduces to

\[
l \dot{\phi}^2 = -1.
\]

Hence the metric component \( l < 0 \) on the curve.

We shall now consider closed geodesic curves in a plane \( z = \text{constant} \). The radial Lagrange equation is

\[
\frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{r}} \right) = \frac{\partial L}{\partial r}.
\]

In the case of a circular geodesic in the \( z = \text{constant} \) plane, the Lagrangian of a free particle is

\[
L = -\frac{1}{2} f \dot{t}^2 + ki \dot{\phi} + \frac{1}{2} l \dot{\phi}^2,
\]

and the equation takes the form

\[
l \dot{r} \dot{\phi}^2 + 2k \dot{r} \dot{\phi} - f \dot{r} \dot{t}^2 = 0.
\]

The condition that the curve shall be closed is \( \dot{t} = 0 \). From equations (15) and (18) we then have the following two conditions which must be fulfilled on a closed circular geodesic curve with radius \( r = r_2 \):

\[
l(r_2) < 0, \quad l_r(r_2) = 0.
\]

Similarly, from the \( z \)-component of the geodesic equation, we obtain the condition

\[
l_z(r_2) = 0.
\]

The first of the conditions (19) comes from the requirement that the curve shall be closed in spacetime, the second as well as (20), that it is also geodesic.

4. Applications to some known metrics

4.1. The Bonnor metric

This metric represents spacetime in a rotating cloud of dust. It has earlier been investigated by Collas and Klein [6]. Let us apply the conditions (19) and (20) to Bonnor’s metric, having

\[
l = r^2 - 4\hbar^2 r^4 / (r^2 + z^2)^3.
\]
In this case the condition (20) takes the form
\[ l_\perp (r_2) = 24 h^2 r_2^4 z / (r_2^2 + z^2)^4 = 0, \quad (22) \]
which requires that \( z = 0 \). Hence timelike circular closed geodesics in the hyperplane of \( z =\) constant, with centre on the \( z\)-axis, can only exist in the plane \( z = 0 \). The condition \( l_\perp (r_2) = 0 \) takes the form \( r_2^4 = -4 h^2 \), showing that there are no closed, circular geodesics in the plane \( z = 0 \). This shows that the Bonnor metric does not permit the existence of such closed, timelike geodesics. This provides a simple proof of a result that was originally found by Collas and Klein [6].

4.2. The van Stockum spacetime

We shall apply our criteria (19) to a van Stockum metric [2] which has earlier been analysed by Steadman [8], describing an empty space outside a rotating dust cylinder. In this case, the relevant metric function (in our notation) is
\[ l = \frac{R r \sin (3 \epsilon + \log (r / R) \tan \epsilon)}{\sin \epsilon + \sin (3 \epsilon)}, \quad (23) \]
where
\[ \tan \epsilon = (4 a^2 R^2 - 1)^{1/2}, \quad \frac{1}{2} < a R < 1. \quad (24) \]
The constant \( a \) is the spin per mass of the rotating dust.

The condition \( l_\perp (r_2) = 0 \) for a CTG now takes the form
\[ r_2 = R \exp ((k \pi - 4 \epsilon) \cot \epsilon), \quad (25) \]
where \( k \) is an integer. Inserting this into the expression for \( l \) gives
\[ l(r_2) = (-1)^{k+1} \frac{\sin \epsilon}{\sin \epsilon + \sin (3 \epsilon)} r_2 R. \quad (26) \]
Using equation (24) this can be written
\[ l(r_2) = (-1)^{k+1} a^2 R^3 r_2. \quad (27) \]
In addition, the condition \( l(r_2) < 0 \) must be fulfilled. This requires that \( k = 2n \) where \( n \) is an integer. Hence there exists closed timelike circular geodesics with radii
\[ r_2 = R \exp (2(n \pi - 2 \epsilon) \cot \epsilon), \quad (28) \]
where \( n \) is an integer. Since the present metric is a solution of the field equations in empty space outside the dust distribution, the radii must satisfy \( r_2 > R \). Hence \( n \pi > 2 \epsilon \) which demands that \( n \) is a positive integer. We have thereby given a simple demonstration of a result which was first found by Steadman [8].

4.3. The Gödel universe

We shall here apply our criteria to the Gödel universe model [1]. This is a rotating universe filled with dust and vacuum energy. The density of the vacuum energy is negative and has half the value of the density of the dust. Barrow and Tsagas [18] have recently analysed the stability
and dynamics of the Gödel universe and also discussed the existence of CTCs in this universe model. The metric function $g_{\phi\phi}$ is

$$l = 4a^2 \left[ \sinh^2 \left( \frac{r}{2a} \right) - \sinh^4 \left( \frac{r}{2a} \right) \right],$$  \hfill (29)

where $a^2 = 1/\kappa \mu$ and $\mu$ is the density of the dust. Then $l, r = 0$ leads to

$$\sinh \left( \frac{r}{2a} \right) = \frac{1}{\sqrt{2}}.$$  \hfill (30)

Inserting this into equation (29) we get $l = l_0 = a^2 > 0$. Hence closed timelike circular geodesics do not exist in this universe model. However, there is a region with closed timelike non-geodesic curves. In this region $l < 0$, which corresponds to $\sinh(r/2a) > 1$, i.e. $r > 2a \log(1 + \sqrt{2})$.

5. New solutions with CTGs

We shall consider a cylindrically symmetric space outside a spinning cosmic string. The space is filled with vacuum energy and a non-rotating gas of non-spinning strings oriented parallel to the axis. The vacuum energy and the gas of strings have limited extension in the radial direction. We shall find the metric inside the gas by solving Einstein’s field equations, demanding that it matches continuously at the outer boundary of the mass distribution to the metric [12] in empty space outside a spinning cosmic string.

We now assume that $f = e^{\mu} = 1$ and that $k$ is constant. According to equation (3) this means that the vorticity of the four-velocity field of the reference particles vanishes. From equation (2) it also follows that the acceleration of gravity vanishes, so that the gas of strings is in static equilibrium. We shall construct solutions with $l < 0$ in the region with cosmic strings. Hence the reference frame is non-rotating in this region. The line element then takes the form

$$ds^2 = - (dt - k d\phi)^2 + D^2 d\phi^2 + dr^2 + dz^2.$$  \hfill (31)

This form of the metric with $k = \text{constant}$ has recently been discussed by Cooperstock and Tieu [19] and Culetu [20]. In this case, it is possible to introduce a new time coordinate $\hat{t} = t - k\phi$ so that the line element takes the form

$$ds^2 = - d\hat{t}^2 + D^2 d\phi^2 + dr^2 + dz^2.$$  \hfill (32)

The three-space of the gas is given by the last three terms in equation (32).

Going around a curve from $\phi = 0$ to $\phi = 2\pi$, either $t$ or $\hat{t}$ (or both) have to make a gap, and in general there seems to be lacking a criterion to decide which one. It may be a question of defining the topology of the spacetime which is considered.

In the present work, we assume that the spacetime outside the gas which surrounds the string is that of empty space outside a spinning cosmic string [12]. Then the original time coordinate $t$ does not have a gap, whereas $\hat{t}$ has a gap equal to $2\pi k$.

The nonzero contravariant components of the energy–momentum tensor in the gas of strings are

$$T^t_t = \rho_g, \quad T^z_z = p_g.$$  \hfill (33)

We assume that the vacuum energy has a density $\rho_\Lambda > 0$. It is also assumed to have a pressure $p_\Lambda$ only in the axial direction. The nonzero contravariant components of the energy–momentum tensor for the vacuum are

$$T^t_t = \rho_\Lambda, \quad T^z_z = p_\Lambda.$$  \hfill (34)
where \( p_\Lambda = -\rho_\Lambda \) is constant. There is no energy current since the gas is at rest in the reference frame.

Calculating the Einstein tensor for the line element (31), we find that the \( tt \) - and \( zz \) -components of Einstein’s field equations take the form

\[
-\frac{D''}{D} = \kappa (\rho_g - \rho_\Lambda \delta^{tt}) = \kappa (\rho_g + \rho_\Lambda l/D^2)
\]

and

\[
\frac{D''}{D} = \kappa (\rho_g + \rho_\Lambda \delta^{zz}) = \kappa (\rho_g + p_\Lambda),
\]

where \( \kappa \) is Einstein’s gravitational constant. From these equations we obtain

\[
\rho_g = -\left( p_g + k^2/D^2 \rho_\Lambda \right) = k^2/D^2 \rho_\Lambda - p_g,
\]

Note that in the present case the usual coordinate condition \( D(r) = r \) does not permit a non-vanishing density, and therefore cannot be applied in the space inside the gas.

The metric function \( l \) depends upon the density as a function of \( r \) which may be chosen freely. Furthermore,

\[
l(r) = D(r)^2 - k^2.
\]

For a bounded function \( D(r) \) the condition \( l(r) < 0 \) may be fulfilled by choosing \( k \) sufficiently large. Hence there are CTCs in this spacetime. However, these curves will in general not be geodesics. If they shall be geodesics, the additional requirement, \( l'(r) = 2D(r)D'(r) = 0 \) must be met. In order not to have a degenerate metric, we must assume that \( D(r) \neq 0 \). Then the condition \( l'(r) = 0 \) is equivalent to the condition \( D'(r) = 0 \), showing that the two conditions (19) for a closed circular geodesic curve may be fulfilled.

In the situation we consider, there is a spinning cosmic string along the \( z \)-axis. The non-vanishing energy–momentum tensor components of the string are given by

\[
\sqrt{g} T^{tt} = -\sqrt{g} T^{zz} = \lambda_{cs} \delta^{(2)}(r), \quad \sqrt{g} T^{t\phi} = \frac{1}{2} \frac{\partial}{\partial r} \delta^{(2)}(r),
\]

where \( \lambda_{cs} \) and \( J \) are the mass and the angular momentum per unit length of the string [21]. The line element in empty space outside this string is [12]

\[
ds^2 = - \left( dt - J/\lambda_0 d\phi \right)^2 + (1 - \lambda/\lambda_0)^2 r^2 d\phi^2 + dr^2 + dz^2,
\]

where \( \lambda \) is the total mass per unit length of the spinning string on the \( z \)-axis and the gas and vacuum outside it. Furthermore, \( \lambda_0 = \hat{\lambda}/4 \) where \( \hat{\lambda} = c^2/G = 1.3 \times 10^{27} \) kg m\(^{-1} \) is the gravitational line density. This solution has

\[
l(r) = (1 - \lambda/\lambda_0)^2 r^2 - (J/\lambda_0)^2.
\]

Assume that the exterior boundary of the gas is at \( r = r_1 \). Then the components of the metric tensor and their derivatives must be continuous at \( r = r_1 \). Hence

\[
k = J/\lambda_0,
\]

\[
D(r_1) = (1 - \lambda/\lambda_0)r_1,
\]

\[
D'(r_1) = 1 - \lambda/\lambda_0.
\]
We see that $D''$ and $D$ must have equal signs. Otherwise the graph of $D(r)$ curves towards the $r$-axis. In this case, $D(r) = 0$ for a value of $r$ between each CTG where $D'(r) = 0$, and between $r = r_1$ and the geodesic of maximum $r$. These values of $r$ correspond to cylindrical surfaces in spacetime where the metric is degenerate. In order to obtain a connected spacetime, we must therefore assume that $D''/D > 0$. According to equations (35) and (36) this implies that

$$\rho_g < -\frac{l}{D^2}\rho_\Lambda$$

and

$$p_g > -\rho_\Lambda.$$ (45)

Note that when $l < 0$, equation (45) permits a positive mass density of the gas of strings. Also, since $p_\Lambda = -\rho_\Lambda$, it follows from equation (46) that $p_g > 0$. Hence the pressure and the density of the gas of cosmic strings have the same sign. This is unusual. In the absence of the vacuum energy the cosmic gas would, according to equation (37), obey the equation of state $p_g = -\rho_\Lambda$. In the present model, however, equation (37) tells that the equation of state of the gas of cosmic strings is modified by the vacuum energy. Also, there are no cosmic strings in the external region where there is no vacuum energy.

In the present case, the sign of $D(r)$ cannot change, and we can therefore assume that $D(r) > 0$ for every value of $r > 0$. This means that the line density of the string must fulfil $\lambda < \lambda_0$ which secures the possibility of a weak field approximation.

By choosing

$$J = (\lambda_0 - \lambda)r_1,$$ (47)

we obtain $l(r_1) = 0$ from equation (41). With this choice of the angular momentum per unit length of the string, we have that $l(r) > 0$ for $r > r_1$. According to our analysis in section 2, this means that the reference frame $R$ is rotating in this region. From equations (9) and (3) it follows that the angular velocity of the reference frame decreases for increasing $r$ in such a way that the vorticity of the four-velocity field of the reference particles vanishes also in this region. We consider solutions with the property that in the region $0 < r \leq r_1$, the function $D(r)$ has a maximum at $r = r_1$, ensuring that $l < 0$ in the gas of strings. This means that the reference frame is non-rotating in the entire region with vacuum energy and strings.

We shall now discuss two different solutions with these properties.

(i) The pressure $p_g$ of the gas is constant inside a cylindrical surface with radius $r_1$. The total pressure in the combined system of strings and vacuum energy is $p_0 = p_g + p_\Lambda$. Solving the field equations then gives

$$D(r) = \beta \cosh \left(\frac{r - r_1}{r_0} + \alpha\right)$$

for $r \leq r_1$, where $\beta > 0$ and $\alpha$ are integration constants. In the region $r \leq r_1$, the function $D(r)$ has a maximum for $r = r_1$. Thus $l(r) < 0$ for $r < r_1$. Furthermore, $r_0 = 1/\sqrt{k p_0}$, which may be written $r_0 = 0.75(p_1/p_0)^{1/2}10^{13}$ m $= 7.9(p_1/p_0)^{1/2}10^{-4}$ y., where $p_1 = -1$ N m$^{-2}$. Equation (37) shows that the density of the gas is positive when

$$p_g < \frac{k^2}{D^2}\rho_\Lambda.$$ (49)
Applying the conditions (43) and (44) to the solution (48) leads to
\[ \beta \cosh(\alpha) = (1 - \lambda/\lambda_0)r_1, \quad \beta \sinh(\alpha) = (1 - \lambda/\lambda_0)r_0, \]  
showing that \( r_1 > r_0 \). Hence
\[ \beta = (1 - \lambda/\lambda_0)\sqrt{r_1^2 - r_0^2}, \quad \alpha = \arctan\left(\frac{r_0}{r_1}\right). \]  
(51)

It may be noted that if the gas is removed, corresponding to \( r_0 \to \infty \), we get \( \lambda \to \lambda_{cs} \) giving the spacetime outside the string [12].

The condition \( D'(r_2) = 0 \) for a closed circular timelike geodesic with radius \( r_2 \) leads to
\[ r_2 = r_1 - r_0 \alpha. \]  
(52)

In order that \( r_2 > 0 \), the radius \( r_1 \) of the gas must fulfil
\[ r_1 > r_0 \alpha, \]  
(53)

which may be written
\[ \frac{r_1}{r_0} \tanh\left(\frac{r_1}{r_0}\right) > 1. \]  
(54)

This means approximately that \( r_1 > 1.2r_0 \), which implies that \( p_0 = p_g + p_\Lambda > -1.4/\kappa r_1^2 \).

Since \( p_\Lambda < 0 \) this requires \( p_g > 0 \). This shows that there is no CTG without the gas surrounding the spinning string.

(ii) The total pressure \( p = p_g + p_\Lambda \) of the gas and vacuum energy is assumed to be
\[ p(r) = \frac{2}{\kappa [(r - a)^2 + b^2]}, \]  
(55)

where the constant \( b \) represents a length corresponding to the total pressure of the gas and vacuum at the radius \( r = a \),
\[ b^2 = \frac{2}{\kappa p(a)}. \]  
(56)

Note that \( p(r) > 0 \) in accordance with equation (46). We assume that the vacuum energy and the gas extend to a radius
\[ r_1 = \sqrt{a^2 + b^2}. \]  
(57)

Outside this radius the spacetime is described by the metric (40).

The field equation (36) now takes the form
\[ D'' - \frac{2}{(r - a)^2 + b^2} D = 0. \]  
(58)

Introducing
\[ u = \frac{D}{(r - a)^2 + b^2}, \]  
(59)

this equation can be written
\[ \left\{ \left[(r - a)^2 + b^2\right] u'\right\}' = 0, \]  
(60)

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which implies that
\[ \left( (r - a)^2 + b^2 \right)^2 u' = c_1. \]  
(61)
The general solution of this equation is
\[ u = \frac{c_1}{2b^3} \left[ \frac{b(r - a)}{(r - a)^2 + b^2} + \arctan \frac{r - a}{b} \right] + c_2, \]  
(62)
where \( c_1 \) and \( c_2 \) are integration constants. Using equations (59) and (57), the condition (43) leads to
\[ 2(r_1 - a)u(r_1) = 1 - \lambda / \lambda_0. \]  
(63)
Using equations (59) and (61), the condition (44) gives
\[ 2(r_1 - a)u(r_1) + \frac{c_1}{(r_1 - a)^2 + b^2} = 1 - \lambda / \lambda_0. \]  
(64)
Equations (63) and (64) imply that
\[ c_1 = 0, \quad c_2 = \frac{1 - \lambda / \lambda_0}{2(r_1 - a)}, \]  
(65)
so that
\[ D(r) = c_2[(r - a)^2 + b^2]. \]  
(66)
In order that \( D(r) > 0 \), the condition \( \lambda < \lambda_0 \) must be fulfilled since \( r_1 > a \).

In order for the reference frame to be non-rotating in the entire region with vacuum energy and strings, the condition \( l(r) < 0 \) must be satisfied for \( 0 < r \leq r_1 \). Hence, in this region the function \( D(r) \) describing a parabola must have a maximum for \( r = r_1 \), which demands that \( D(0) \leq D(r_1) \) giving \( r_1 \geq 2a \). Inserting the expression (57) for \( r_1 \) this condition may also be expressed as \( b \geq \sqrt{3a} \).

For the present solution the conditions for the existence of CTGs give
\[ r_2 = a, \quad J > D(a)\lambda_0. \]  
(67)
Thus, the constant \( a \) represents the radius of the CTGs. With \( J \) as given in equation (47), and using equations (42) and (38), we have \( J = D(r_1)\lambda_0 \). Hence, the inequality (67) may be written \( D(r_1) > D(a) \), which is clearly satisfied for the parabola described by equation (66).

6. Isometric embedding

In this section, we shall find the embedding surface for the three-space \( \hat{r} = \) constant in the case of (ii). Writing the line element of the spacetime in the gas of cosmic strings in the form (11), we obtain
\[ ds^2 = -d\hat{t}^2 + c_2^2[(r - a)^2 + b^2]^2 d\phi^2 + dr^2 + dz^2, \]  
(68)
where \( \hat{t} = t - (J / \lambda_0)\phi \) and \( c_2 \) is given in equation (65). It may be noted that this metric has a strange property. Consider a circle in the plane \( z = \) constant having its centre on the axis. The quotient between the length of this circle and the diameter tends to infinity in the limit \( r \to 0 \). However, \( r = 0 \) is not part of the spacetime described by the line element (68), since there is
a cosmic string along the $z$-axis. The spatial part of this line element represents the space $S$ defined by simultaneity on Einstein synchronized clocks at rest in the reference frame $R$.

We shall now show that the $r\phi$-surface with $z =$ constant can be isometrically embedded in Euclidean three-space. The line element of this three-space in cylinder coordinates ($\hat{r}, \phi, \hat{z}$) is

$$dl^2 = dr^2 + \hat{r}^2 d\phi^2 + d\hat{z}^2. \quad (69)$$

The line element of the $r\phi$-surface is

$$dl_{r\phi}^2 = dr^2 + c_2^2 [(r - a)^2 + b^2] d\phi^2 = dr^2 + \hat{r}(r)^2 d\phi^2, \quad (70)$$

where we have introduced the function

$$\hat{r}(r) = c_2 [(r - a)^2 + b^2]. \quad (71)$$

The line element (70) can be reorganized as follows:

$$dl_{r\phi}^2 = dr^2 + \hat{r}(r)^2 d\phi^2 + \left[1 - \left(\frac{d\hat{r}}{dr}\right)^2\right] dr^2. \quad (72)$$

This will coincide with the line element in the three-space given by equation (66) if we choose

$$d\hat{z} = \sqrt{1 - \left(\frac{d\hat{r}}{dr}\right)^2} dr. \quad (73)$$

Hence

$$\hat{z} = \int_a^r \sqrt{1 - 4c_2^2(r-a)^2} d\bar{r}, \quad (74)$$

which leads to

$$\hat{z} = \frac{1}{2} (r - a) \sqrt{1 - 4c_2^2(r-a)^2} + \frac{1}{4c_2} \arcsin \{2c_2(r-a)\}, \quad (75)$$

where $0 < r \leq r_1$. The embedding is thus given by (75) together with

$$x = c_2 [(r - a)^2 + b^2] \cos \phi, \quad y = c_2 [(r - a)^2 + b^2] \sin \phi. \quad (76)$$

The embedding of the external space, given by the spatial part of the line element (40), has

$$\hat{z} = \hat{z}(r_1) + \sqrt{\frac{\lambda}{\lambda_0}} \left(2 - \frac{\lambda}{\lambda_0}\right) (r - r_1), \quad (77)$$

for $r \geq r_1$. Hence, it is described by a part of a cone in the embedding diagram. The embedded surface is shown in figure 1.

Expression (77) shows that the total line density $\lambda$ must satisfy $0 < \lambda < 2\lambda_0$ in order that the embedding shall be well-defined.

One may wonder why the embedded surface is bounded below at a value of $\hat{z}$ corresponding to $r = 0$. The reason is that the value $r = 0$ must be excluded from the region with the gas of strings because of the spinning cosmic string at the axis.
Figure 1. The \( r\phi \)-surface of the three-space \( \hat{t} = \text{constant} \) as embedded in Euclidean three-space. The region \( \hat{z} < 0 \) corresponds to \( 0 < r < a \). Note that in the embedding diagram, \( r = 0 \) is represented by the circle at the bottom of the surface.

7. The Sagnac effect

The local vorticity of a velocity field is given by equation (3). Applied to our solutions of Einstein’s field equations in section 5 this formula gives a vanishing vorticity. One may then wonder whether the existence of CTGs is independent of the state of rotation of the space. This would, however, be rather surprising, since CTCs have been shown to exist in some rotating systems, e.g. in the Gödel universe \([1]\), outside a rotating cosmic string \([12]\), and in a rotating distribution of dust \([8, 9]\).

Usually the Sagnac effect is thought of as an effect appearing when the apparatus is fixed in a rotating reference frame. In section 2, we have argued that if \( l > 0 \), the frame \( R \) is rotating, but that \( R \) is not rotating if \( l < 0 \). We shall now investigate whether an experiment with the apparatus fixed in \( R \) will show a Sagnac effect independent of the sign of \( l \). For light moving in a circular path with \( ds = dr = dz = 0 \), the four-velocity identity takes the form (including here the velocity of light \( c \))

\[
-c^2 f i^2 + l\dot{\phi}^2 + 2k i\dot{\phi} = 0. \tag{78}
\]

Introducing the coordinate velocity \( \frac{d\phi}{dt} \) we then get

\[
\left( \frac{d\phi}{dt} \right)^2 + \frac{2kc}{l} \frac{d\phi}{dt} - f \frac{c^2}{l} = 0. \tag{79}
\]

The solutions are

\[
\left( \frac{d\phi}{dt} \right)_{\pm} = -\frac{k \pm D}{l} c. \tag{80}
\]

We shall now consider light emitted in opposite directions along a circle around the axis. Integrating equation (80) around the circle we find the travelling times for the two light signals

\[
t_1 = \frac{2\pi l}{c(D+k)}, \quad t_2 = \frac{2\pi l}{c(D-k)}. \tag{81}
\]
The time difference for light travelling around a circle in opposite directions is
\[
\Delta t = t_2 - t_1 = \frac{4\pi k}{cf},
\] (82)
where we have used that \(D^2 = k^2 + f l\). The Sagnac effect is due to this time difference. In our models we have \(l < 0\). Then it follows from equation (81) that \(t_1 < 0\) and \(t_2 > 0\). Hence the light emitted in the negative \(\phi\)-direction moves backwards in time. This implies the possibility that free material particles can move along closed timelike worldlines.

As an illustration, we may draw the light cones of light moving in the circular paths. For comparison, we first consider the usual case \(l > 0\). Then \(t_1\) and \(t_2\) are both positive. The shape of the light cone is as shown in figure 2(a). In our models with \(l < 0\) the shape of the light cone is as shown in figure 2(b). In equation (78), we see that the first two terms are negative, which implies the condition
\[
i \dot{\phi} > 0.
\] (83)
This shows that \(i\) has opposite signs for light moving in opposite directions. Hence there exist worldlines of light moving in the negative time direction.

Delgado et al [22] have introduced an angular velocity \(\omega_S\) characterizing a global rotation of space, utilizing the Sagnac effect. They compared the Sagnac effect in the Gödel universe with that in a rotating reference frame in flat spacetime in the weak field approximation. In this approximation
\[
\Delta t_{RF} = -4\pi \frac{r^2 \omega_S}{c^2},
\] (84)
in the rotating frame. This gives
\[
\omega_S = \frac{k}{r^2 f}.
\] (85)
By using equation (42) we get for our solutions
\[
\omega_S = \frac{4GJ}{c^2 r^2}.
\] (86)
According to this interpretation space has a global rotation. However, the angular velocity decreases with the distance from the axis in such a way that the local vorticity vanishes. Hence
the existence of a non-vanishing global rotation seems to be decisive for the existence of CTGs in this spacetime.

We have seen in section 2, however, that $R$ can only be interpreted as a rotating frame when $l > 0$. When $l < 0$ the Sagnac effect is not due to a rotation of the reference frame, but is a consequence of the fact that the clocks in $R$ (showing $t$) are not Einstein synchronized. As we noted, they cannot be so, in order that the time $t$ shall not have a gap around a closed circle about the axis. Consequently, the velocity of light is anisotropic as measured with these clocks. This implies a non-vanishing Sagnac effect although the apparatus which is fixed in $R$ does not rotate.

8. Inertial dragging

In section 2, we showed that if $l > 0$, inertial frames will be dragged around the axis in the reference frame $R$ with an angular velocity $\omega = -k/l$. In this case, there is therefore an inertial dragging effect.

We now want to investigate the case $l < 0$. The conserved energy of the particle is

$$p_t = \frac{\partial L}{\partial \dot{t}} = -f \dot{t} + k \dot{\phi}.$$  \hspace{1cm} (87)

Solving equations (6) and (87) with respect to $\dot{t}$ and $\dot{\phi}$, we obtain

$$\dot{t} = \frac{kp_{\phi} - lp_t}{D^2}, \quad \dot{\phi} = \frac{fp_{\phi} + kp_t}{D^2}. \hspace{1cm} (88)$$

The four-velocity identity of the particle is

$$-f \dot{t}^2 + l \dot{\phi}^2 + 2ki \dot{t} \dot{\phi} + c^2 \dot{r}^2 = \epsilon.$$  \hspace{1cm} (89)

where $\epsilon = -1, 0, 1$, respectively, for a timelike, lightlike or spacelike interval in spacetime. Inserting the expressions (88) for $\dot{t}$ and $\dot{\phi}$ leads to

$$fp_{\phi}^2 + 2kp_{\phi}p_t - lp_t^2 = (\epsilon - c^2 \dot{r}^2)D^2. \hspace{1cm} (90)$$

If $p_{\phi} = 0$, then

$$lp_t^2 = (-\epsilon + c^2 \dot{r}^2)D^2. \hspace{1cm} (91)$$

Hence $l < 0$ implies that $\epsilon = 1$. Consequently local inertial frames having $p_{\phi} = 0$, must follow spacelike worldlines in spacetime when $l < 0$. They then have a tachyon-like character. This is not to be interpreted as some sort of super dragging effect. One must rather conclude that when $l < 0$ no inertial frames with velocity $v < c$ exist. This is not due to rotation, but to the way that the clocks in $R$ must be synchronized in order that the time $t$ showed by them shall not have a gap when one passes around a circle about the axis. This property of the spacetime is presumably due to the spinning cosmic string at the axis.

9. Conclusion

In the present paper, we have found a new class of solutions of Einstein’s field equations. They represent cylindrically symmetric spacetimes in a region of limited radial extension with a non-rotating gas of vacuum energy and strings, and with a spinning cosmic string along the symmetry axis. Outside this region there is empty spacetime.

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Timelike circular curves are characterized by the four-velocity identity $\vec{u} \cdot \vec{u} = -1$. Considering a circular timelike curve in the plane $z = \text{constant}$, this equation takes the form (14). Surprisingly, such a timelike curve may fulfill the condition $l = 0$, which means that the curve is closed in spacetime. This requires $l < 0$ in the line element (1). If such a curve shall also be a geodesic, the requirements (19) must also be met.

We have investigated two solutions of the class mentioned above, one with constant pressure, and one in which the pressure is given by equation (55), both containing CTGs. In these models it is possible for a free particle to travel along such a curve and come back to the event it started from.

Rosa and Letelier [23] have shown that the CTGs followed by such free particles are not stable. However, in the present case the force needed to move along a CTC perturbing the geodesic may be made arbitrarily small.

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