SERIES WITH SUMMANDS INVOLVING HARMONIC NUMBERS

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Abstract. For each positive integer \( m \), the \( m \)th order harmonic numbers are given by

\[
H^{(m)}_n = \sum_{0 < k \leq n} \frac{1}{k^m} \quad (n = 0, 1, 2, \ldots).
\]

We discover exact values of some series involving harmonic numbers of order not exceeding three. For example, we conjecture that

\[
\sum_{k=0}^{\infty} (6k + 1) \left( \frac{\binom{2k}{k}}{256^k} \right) \left( H^{(3)}_{2k} - \frac{7}{64} H^{(3)}_k \right) = \frac{25\zeta(3)}{8\pi} - G,
\]

where \( G \) denotes the Catalan constant \( \sum_{k=0}^{\infty} (-1)^k/(2k+1)^2 \). This paper contains 66 conjectures posed by the author since October 2022.

1. Introduction

The usual harmonic numbers are those rational numbers

\[
H_n = \sum_{0 < k \leq n} \frac{1}{k} \quad (n = 0, 1, 2, \ldots).
\]

For each \( m \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\} \), the harmonic numbers of order \( m \) are defined by

\[
H^{(m)}_n = \sum_{0 < k \leq n} \frac{1}{k^m} \quad (n \in \mathbb{N} = \{0, 1, 2, \ldots\}).
\]

For any \( m, n \in \mathbb{Z}^+ \), we clearly have

\[
\sum_{k=1}^{n} \frac{1}{(2k - 1)^m} = \sum_{k=1}^{2n} \frac{1}{k^m} - \sum_{j=1}^{n} \frac{1}{(2j)^m} = H^{(m)}_{2n} - \frac{1}{2m} H^{(m)}_{n}.
\]

J. Wolstenholme [46] established two fundamental congruences for harmonic numbers:

\[
H_{p-1} \equiv 0 \pmod{p^2} \quad \text{and} \quad H^{(2)}_{p-1} \equiv 0 \pmod{p}
\]

for any prime \( p > 3 \). For series and congruences involving harmonic numbers, one may consult [28, 35, 40], [39, Section 10.5], and the recent preprint [4] solving various conjectures of the author.

Key words and phrases. Harmonic numbers, series for \( \pi \), Dirichlet \( L \)-functions, combinatorial identities, congruences.

2020 Mathematics Subject Classification. Primary 11B65, 05A19; Secondary 11A07, 11B68.

Supported by the Natural Science Foundation of China (grant no. 11971222).
In 2012, K. N. Boyadzhiev [7] proved that
\[ \sum_{k=0}^{\infty} \binom{2k}{k} H_k x^k = \frac{2}{\sqrt{1 - 4x}} \log \frac{1 + \sqrt{1 - 4x}}{2\sqrt{1 - 4x}} \quad \text{for } x \in \left(-\frac{1}{4}, \frac{1}{4}\right). \tag{1.1} \]

In 2016, H. Chen [8] deduced that
\[ \sum_{k=0}^{\infty} \binom{2k}{k} H_{2k} x^k = \frac{1}{\sqrt{1 - 4x}} \log \frac{1 + \sqrt{1 - 4x}}{2(1 - 4x)} \quad \text{for } x \in \left(-\frac{1}{4}, \frac{1}{4}\right). \tag{1.2} \]

It is well known that
\[ 2 \arcsin \frac{x}{2} = \sum_{k=0}^{\infty} \frac{\binom{2k}{k} x^{2k+1}}{(2k + 1)16^k} \quad \text{for } |x| \leq 2; \]

in particular,
\[ \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2k + 1)16^k} = \frac{\pi}{3} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2k + 1)8^k} = \frac{\pi}{2\sqrt{2}}. \]

The author [26, Theorem 1.1(ii)] determined
\[ \sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k + 1)16^k} \quad \text{and} \quad \sum_{k=(p+1)/2}^{p-1} \frac{\binom{2k}{k}}{(2k + 1)16^k} \]
modulo \(p^2\) for any prime \(p > 3\). By [6], we have
\[ \left( \arcsin \frac{x}{2} \right)^3 = 3 \sum_{k=0}^{\infty} \frac{\binom{2k}{k} x^{2k+1}}{(2k + 1)16^k} \sum_{0 \leq j < k} \frac{1}{(2j + 1)^2} \tag{1.3} \]
and
\[ \left( \arcsin \frac{x}{2} \right)^4 = \frac{3}{2} \sum_{k=0}^{\infty} \frac{H_{k-1}^{(2)} x^{2k}}{k^2 \binom{2k}{k}} \tag{1.4} \]
for \(|x| \leq 2\). In particular,
\[ \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2k + 1)16^k} \left( H_{2k}^{(2)} - \frac{1}{4} H_k^{(2)} \right) = \frac{\pi^3}{648}, \]
\[ \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2k + 1)8^k} \left( H_{2k}^{(2)} - \frac{1}{4} H_k^{(2)} \right) = \frac{\sqrt{2} \pi^3}{384}. \]

and
\[ \sum_{k=1}^{\infty} \frac{H_{k-1}^{(2)}}{k^2 \binom{2k}{k}} = \frac{\pi^4}{1944}. \]

In view of (1.3), we have the following result.

**Theorem 1.1.** If \(|x| < 2\), then
\[ \frac{(\arcsin(x/2))^2}{\sqrt{4 - x^2}} = \sum_{k=1}^{\infty} \frac{\binom{2k}{k} x^{2k}}{16^k} \left( H_{2k}^{(2)} - \frac{1}{4} H_k^{(2)} \right). \tag{1.5} \]
Proof. By taking derivatives of both sides of (1.3), we get
\[ 3 \left( \text{arcsin} \left( \frac{x}{2} \right) \right)^2 \times \frac{1/2}{\sqrt{1 - (x/2)^2}} = 3 \sum_{k=0}^{\infty} \frac{(2k)x^{2k}}{16^k} \sum_{0 \leq j < k} \frac{1}{(2j + 1)^2} \]
and hence
\[ \frac{\left( \text{arcsin}(x/2) \right)^2}{\sqrt{4 - x^2}} = \sum_{k=1}^{\infty} \frac{(2k)x^k}{16^k} \sum_{j=1}^{k} \frac{1}{(2j - 1)^2}, \]
which is equivalent to (1.5). \qed

Motivated by the Ramanujan series
\[ \sum_{k=0}^{\infty} (6k + 1) \frac{(2k)^3}{(-512)^k} = \frac{2\sqrt{2}}{\pi} \] (1.6)
(cf. [24]), L. Long [22] conjectured the congruence
\[ \sum_{k=0}^{(p-1)/2} (6k + 1) \frac{(2k)^3}{(-512)^k} \sum_{j=1}^{k} \left( \frac{1}{(2j - 1)^2} - \frac{1}{16j^2} \right) \equiv 0 \pmod{p} \] (1.7)
for any odd prime \( p \), which was confirmed by H. Swisher [41] in 2015. Note that (1.7) can be rewritten as
\[ \sum_{k=0}^{(p-1)/2} (6k + 1) \frac{(2k)^3}{(-512)^k} \left( H_{2k}^{(2)} - \frac{5}{16} H_{k}^{(2)} \right) \equiv 0 \pmod{p}. \]

In 2022 C. Wei [44] deduced the two identities
\[ \sum_{k=0}^{\infty} (6k + 1) \frac{(2k)^3}{(-512)^k} \left( H_{2k}^{(2)} - \frac{5}{16} H_{k}^{(2)} \right) = -\frac{\sqrt{2}}{48} \pi \]
and
\[ \sum_{k=0}^{\infty} (6k + 1) \frac{(2k)^3}{256^k} \left( H_{2k}^{(2)} - \frac{5}{16} H_{k}^{(2)} \right) = \frac{\pi}{12} \]
conjectured by V.J.W. Guo and X. Lian [18], as well as their \( q \)-analogues.

Motivated by Bauer’s series
\[ \sum_{k=0}^{\infty} (4k + 1) \frac{(2k)^3}{(-64)^k} = \frac{2}{\pi} \] (1.8)
and Ramanujan’s series
\[ \sum_{k=0}^{\infty} (8k + 1) \frac{(2k)^2 (4k)}{48^{2k}} = \frac{2\sqrt{3}}{\pi}, \] (1.9)
Wei and G. Ruan [45] proved the two new identities
\[ \sum_{k=1}^{\infty} (4k + 1) \frac{(2k)^3}{(-64)^k} \sum_{j=1}^{2k} \frac{(-1)^j}{j^2} = \frac{\pi}{12} \]
and
\[ \sum_{k=1}^{\infty} (8k+1) \frac{(2k)^2 (4k)}{48^{2k}} \sum_{j=1}^{k} \left( \frac{1}{(2j-1)^2} - \frac{1}{36j^2} \right) = \frac{\sqrt{3} \pi}{54}, \]
i.e.,
\[ \sum_{k=0}^{\infty} (4k+1) \frac{(2k)^3}{(-64)^k} \left( H_{2k}^{(2)} - \frac{1}{2} H_k^{(2)} \right) = -\frac{\pi}{12} \quad (1.10) \]
and
\[ \sum_{k=0}^{\infty} (8k+1) \frac{(2k)^2 (4k)}{48^{2k}} \left( H_{2k}^{(2)} - \frac{5}{18} H_k^{(2)} \right) = \frac{\sqrt{3} \pi}{54}. \quad (1.11) \]

In 1997 van Hamme [42] thought that series for powers of \( \pi = \Gamma(1/2)^2 \) should have their \( p \)-adic analogues involving the \( p \)-adic Gamma function \( \Gamma_p(x) \), where \( p \) is an odd prime. Note that for any odd prime \( p \) we have
\[ \Gamma_p \left( \frac{1}{2} \right)^2 = (-1)^{(p+1)/2} = -\left( \frac{-1}{p} \right), \]
where \( \left( \frac{\cdot}{p} \right) \) denotes the Legendre symbol. However, van Hamme’s philosophy fails for some Ramanujan-type series for \( 1/\pi \). For example, T. Huber, D. Schultz and D. Ye [21] used modular forms to obtain that
\[ \sum_{k=0}^{\infty} (6k+1) \frac{a_k}{16^k} = \frac{16}{\pi}, \]
where \( a_0 = 1, \ a_1 = 4, \ a_2 = 20 \) and
\[ (n+1)^3 a_{n+1} = 4(2n+1)(2n^2+2n+1)a_n - 16n(4n^2+1)a_{n-1} + 8(2n-1)^3 a_{n-2} \]
for all \( n = 2, 3, \ldots; \) but for a general odd prime \( p \) we even cannot find any pattern for \( \sum_{k=0}^{p-1} (6k+1) a_k/16^k \) modulo \( p \).

The Bernoulli numbers \( B_0, B_1, B_2, \ldots \) are defined by
\[ \frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \quad (0 < |x| < 2\pi). \]
Equivalently,
\[ B_0 = 1, \ \text{and} \ \sum_{k=0}^{n} \binom{n + 1}{k} B_k = 0 \quad \text{for} \ n = 1, 2, 3, \ldots. \]

In 1900 J.W.L. Glaiser [12] proved that
\[ H_{p-1} \equiv -\frac{p^2}{3} B_{p-3} \ (\text{mod} \ p^3) \quad \text{and} \quad H_{p-1}^{(2)} \equiv \frac{2}{3} p B_{p-3} \ (\text{mod} \ p^2) \]
for any prime \( p > 3 \). The Bernoulli polynomials are given by
\[ B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k} \ (n \in \mathbb{N}). \]
The Euler numbers $E_0, E_1, E_2, \ldots$ are defined by
\[
\frac{2}{e^x + e^{-x}} = \sum_{n=0}^{\infty} \frac{E_n}{n!} x^n \quad (|x| < \frac{\pi}{2}).
\]
Clearly $E_{2n+1} = 0$ for all $n \in \mathbb{N}$. It is also known that
\[
\sum_{k=0}^{n} \binom{2n}{2k} E_{2k} = 0 \quad \text{for each } n \in \mathbb{N}.
\]

The Euler polynomials are given by
\[
E_n(x) = \sum_{k=0}^{n} \binom{n}{k} E_k \left( x - \frac{1}{2} \right)^{n-k} \quad (n \in \mathbb{N}).
\]

The author [27, 31] first observed that Ramanujan-type series have corresponding congruences involving Bernoulli or Euler polynomials.

Now we introduce some notations throughout this paper. The Riemann zeta function is defined by
\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{with } \Re(s) > 1.
\]

The Dirichlet beta function is given by
\[
\beta(m) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^m} \quad (m = 1, 2, 3, \ldots).
\]

Note that $G = \beta(2)$ is the Catalan constant. We also adopt the notation
\[
K = L \left( 2, \left( \frac{-3}{.} \right) \right) = \sum_{n=1}^{\infty} \frac{\left( \frac{4}{k} \right)}{k^2}
\]
with $\left( \frac{a}{b} \right)$ the Kronecker symbol. For a prime $p$ and an integer $a \not\equiv 0 \pmod{p}$, we use $q_p(a)$ to denote the Fermat quotient $(a^{p-1} - 1)/p$. Many congruences in later sections involve Fermat quotients.

In Sections 2–4, we will propose 66 new conjectures on series and related congruences with summands involving not only harmonic numbers of order at most three, but also products of several binomial coefficients. All the conjectures have been checked via Mathematica.

### 2. Series with Summands Containing One or Two Binomial Coefficients

**Conjecture 2.1** (2022-10-12). (i) We have
\[
\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3} \binom{2k}{k} \left( H_{2k-1}^{(2)} - \frac{123}{16} H_{k-1}^{(2)} \right) = \frac{451}{40} \zeta(5) - \frac{14}{15} \pi^2 \zeta(3). \quad (2.1)
\]
(ii) For any prime \( p > 5 \), we have
\[
\sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k^3 \binom{2k}{k}} \left( 16 H_{2k-1}^{(2)} - 123 H_{k-1}^{(2)} \right) \equiv -542 B_{p-5} \pmod{p} \quad (2.2)
\]
and
\[
p \sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} \binom{2k}{k} \left( 16 H_{2k}^{(2)} - 123 H_k^{(2)} \right) \equiv 192 \frac{H_{p-1}}{p^2} \pmod{p^2}. \quad (2.3)
\]

**Remark 2.1.** In 1979 R. Apéry [2] proved the irrationality of \( \zeta(3) = \sum_{n=1}^{\infty} 1/n^3 \) via the identity
\[
\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}} = \frac{2}{5} \zeta(3).
\]
In 2014 the author [32] proved the congruence
\[
\sum_{k=1}^{(p-1)/2} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}} \equiv 2 B_{p-3} \pmod{p}
\]
for any prime \( p > 5 \). The author’s conjectural identity (cf. [33])
\[
\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}} (H_{2k} + 4 H_k) = \frac{2\pi^4}{75}
\]
was proved by W. Chu [9] as well as K. C. Au [3, Prop. 7.14]. After seeing an earlier arXiv version of this paper, Au [4, Corollary 2.9] confirmed the author’s conjectural identity (2.1).

**Conjecture 2.2 (2022-11-14).** We have the identity
\[
\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{8^k} \left( H_{2k}^{(3)} - \frac{1}{8} H_k^{(3)} \right) = \frac{35\sqrt{2}}{64} \zeta(3) - \frac{\sqrt{2}}{8} \pi G. \quad (2.4)
\]

**Remark 2.2.** Applying (1.1) and (1.2) with \( x = 1/8 \), we see that
\[
\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{8^k} H_k = -\sqrt{2} \log(12 - 8\sqrt{2}) \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{8^k} H_{2k} = \frac{\log(3/2 + \sqrt{2})}{\sqrt{2}}.
\]
In contrast with (2.4), we have
\[
\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{8^k} \left( H_{2k}^{(2)} - \frac{1}{4} H_k^{(2)} \right) = \frac{\pi^2}{16\sqrt{2}}
\]
by applying (1.5) with \( x = \sqrt{2} \).

**Conjecture 2.3 (2022-11-14).** We have the identity
\[
\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{16^k} \left( H_{2k}^{(3)} - \frac{1}{8} H_k^{(3)} \right) = \frac{2\zeta(3)}{3\sqrt{3}} - \frac{\pi K}{8}. \quad (2.5)
\]
Remark 2.3. Applying (1.1) and (1.2) with \(x = 1/8\), we see that
\[
\sum_{k=0}^{\infty} \binom{2k}{k} \frac{k}{16^k} H_k = -\frac{2}{\sqrt{3}} \log(84 - 48\sqrt{3}) \quad \text{and} \quad \sum_{k=0}^{\infty} \binom{2k}{k} \frac{k}{16^k} H_{2k} = \frac{\log((7 + 4\sqrt{3})/9)}{\sqrt{3}}.
\]
In contrast with (2.5), we have
\[
\sum_{k=0}^{\infty} \binom{2k}{k} \frac{k}{16^k} \left( H_{2k}^{(2)} - \frac{1}{4} H_k^{(2)} \right) = \frac{\pi^2}{36\sqrt{3}}
\]
by applying (1.5) with \(x = 1\).

Conjecture 2.4 (2023-05-28). (i) We have
\[
\sum_{k=1}^{\infty} \frac{H_{3k} - H_k}{k2^k \binom{3k}{k}} = \frac{2}{5}(G + \log^2 2) - \frac{\pi^2}{24} \quad (2.6)
\]
and
\[
\sum_{k=1}^{\infty} \frac{H_{3k} - H_k}{k2^k \binom{3k}{k}} = \frac{11}{4} \zeta(3) - \frac{\pi^2}{24} \log 2 - \pi G. \quad (2.7)
\]
(ii) For any prime \(p > 5\) with \(p \equiv 1 \pmod{4}\), we have
\[
p^{-1} \sum_{k=1}^{p-1} \frac{H_{3k} - H_k}{k2^k \binom{3k}{k}} \equiv \frac{7}{10} q_p(2) \pmod{p}. \quad (2.8)
\]
Also, for each odd prime \(p\) we have
\[
p^{-1} \sum_{k=1}^{p-1} \frac{H_{3k} - H_k}{k2^k \binom{3k}{k}} \equiv -\frac{q_p(2)}{4} \pmod{p}. \quad (2.9)
\]

Remark 2.4. The author’s conjectural identities
\[
\sum_{k=1}^{\infty} \frac{H_{2k} - H_k}{k2^k \binom{3k}{k}} = \frac{3}{10} \log^2 2 + \frac{\pi}{20} \log 2 - \frac{\pi^2}{60}
\]
and
\[
\sum_{k=1}^{\infty} \frac{H_{2k} - H_k}{k2^k \binom{3k}{k}} = \frac{33}{32} \zeta(3) + \frac{\pi^2}{24} \log 2 - \frac{\pi G}{2}
\]
(cf. [39, Conjecture 10.61]) was confirmed by Au [3] in 2022.

Conjecture 2.5 (2023-05-28). (i) We have
\[
\sum_{k=1}^{\infty} \frac{25k - 3}{2^k \binom{3k}{k}} (H_{3k} - 8H_{2k} + 7H_k) = 2G - 2(\pi + 9) \log 2. \quad (2.10)
\]
(ii) For any odd prime \(p\), we have the congruence
\[
p^2 \sum_{k=1}^{\infty} \frac{25k - 3}{2^k \binom{3k}{k}} (H_{3k} - 8H_{2k} + 7H_k) \equiv -\left(\frac{-1}{p}\right) \frac{9}{4} \pmod{p}. \quad (2.11)
\]
Remark 2.5. In 1974 R. W. Gosper announced the identity
\[ \sum_{k=0}^{\infty} 25k - 3 \frac{(3k)}{2k} \frac{2k}{27k} = \pi \frac{2}{2}. \]
an elegant proof of which can be found in [1].

Conjecture 2.6 (2023-05-28). (i) We have
\[ \sum_{k=0}^{\infty} 2^k \frac{3k}{27} H_{2k} = 3 \frac{1}{2} \left( 1 + \sqrt{3} \right) \log \left( 1 + \sqrt{3} \right) - \sqrt{3} \log 2 \] (2.12)
and
\[ \sum_{k=0}^{\infty} 2^k \frac{3k}{27} H_{3k} = 1 + \frac{\sqrt{3}}{2} \left( 2 \log \left( 1 + \sqrt{3} \right) - \frac{\log 3}{2} \right) - \sqrt{3} \log 2. \] (2.13)

(ii) For any prime \( p > 3 \), we have
\[ \sum_{k=(p+1)/2}^{p-1} \frac{3k}{27} \left( 2 \right)^k \left( \frac{2}{27} \right)^k \equiv \frac{1 - \left( \frac{2}{p} \right)}{3} \, (\text{mod} \, p). \] (2.14)

Remark 2.6. For any positive integer \( n \), we clearly have
\[ H_n = \sum_{k=0}^{n-1} \frac{1}{k+1} = \sum_{k=0}^{n-1} \int_0^1 t^k \, dt = \int_0^1 \sum_{k=0}^{n-1} t^k \, dt = \int_0^1 \frac{1 - t^n}{1 - t} \, dt. \]

Using this trick we can deduce that
\[ \sum_{k=0}^{\infty} 2^k \frac{(3k)}{27k} H_k = \frac{3}{4} \left( 1 - \sqrt{3} \right) \log 4 - \left( 1 + \sqrt{3} \right) \log 3 \right) + 2\sqrt{3} \log (1 + \sqrt{3}). \]

Conjecture 2.7 (2023-05-28). We have
\[ \sum_{k=0}^{\infty} \frac{(3k)}{k} \left( \frac{3 + \sqrt{5}}{54} \right)^k (H_{3k} - H_{2k}) = \phi (\log 3 - 2 \log \phi), \] (2.15)
where \( \phi \) denotes the golden ratio \( (1 + \sqrt{5})/2 \approx 1.618 \ldots \)

Remark 2.7. Mathematica yields that
\[ \sum_{k=0}^{\infty} \frac{(3k)}{k} \left( \frac{4x}{27} \right)^k \frac{\cos \arcsin \sqrt{x}}{3} \]
for any \( x \in (-1, 1) \). Applying this with \( x = ((1 + \sqrt{5})/4)^2 \) we obtain that
\[ \sum_{k=0}^{\infty} \frac{(3k)}{k} \left( \frac{3 + \sqrt{5}}{54} \right)^k = \frac{\cos(\pi/10)}{\sqrt{(5 + \sqrt{5})/8}} = \frac{\sqrt{(5 + \sqrt{5})/8}}{\sqrt{(5 - \sqrt{5})/8}} = \phi. \]
**Conjecture 2.8** (2023-05-30). If \((1 - \sqrt{2})/2 < x < 1/2\), then

\[
\sum_{k=0}^{\infty} \binom{4k}{2k} \left( \frac{x(1-x)}{4} \right)^k (2H_{4k} - 3H_{2k} + H_k) = \frac{\sqrt{1-x}}{2x-1} \log(1-x) \tag{2.16}
\]

**Remark 2.8.** For any \(x \in (-1, 1)\), we have

\[
\sum_{k=0}^{\infty} \binom{4k}{2k} \left( \frac{x}{16} \right)^k = \sqrt{\frac{1 + \sqrt{1-x}}{2(1-x)}}
\]

which can be proved directly or via Mathematica. In particular,

\[
\sum_{k=0}^{\infty} \binom{4k}{2k} \left( \frac{3}{64} \right)^k = \sqrt{3}.
\]

For \(x \in (-1, 1)\), we obviously have

\[
\sum_{k=0}^{\infty} \binom{2k}{k} H_k \left( \frac{x}{4} \right)^k + \left( -\frac{x}{4} \right)^k = 2 \sum_{k=0}^{\infty} \binom{4k}{2k} H_{2k} \left( \frac{x}{4} \right)^{2k}
\]

and

\[
\sum_{k=0}^{\infty} \binom{2k}{k} H_{2k} \left( \frac{x}{4} \right)^k + \left( -\frac{x}{4} \right)^k = 2 \sum_{k=0}^{\infty} \binom{4k}{2k} H_{4k} \left( \frac{x}{4} \right)^{2k},
\]

and hence we may find closed formulas for the two series

\[
\sum_{k=0}^{\infty} \binom{4k}{2k} H_{2k} \left( \frac{x}{4} \right)^{2k} \quad \text{and} \quad \sum_{k=0}^{\infty} \binom{2k}{k} H_{4k} \left( \frac{x}{4} \right)^{2k}
\]

by using (1.1) and (1.2). In particular,

\[
\sum_{k=0}^{\infty} \binom{4k}{2k} \left( \frac{3}{64} \right)^k H_{2k} = 2(1 + \sqrt{3}) \log(1 + \sqrt{3}) - (1 + 3\sqrt{3}) \log 2 + \frac{\sqrt{3} - 1}{2} \log 3
\]

and

\[
\sum_{k=0}^{\infty} \binom{4k}{2k} \left( \frac{3}{64} \right)^k H_{4k} = (2 + \sqrt{3}) \log(1 + \sqrt{3}) - (1 + \sqrt{3}) \log 2 + \frac{\sqrt{3} - 1}{4} \log 3.
\]
With the aid of Mathematica, we obtain that
\[
\sum_{k=0}^{\infty} \left(\frac{4k}{2k}\right)^k \left(\frac{3}{64}\right)^k H_k = \sum_{k=1}^{\infty} \left(\frac{4k}{2k}\right)^k \left(\frac{3}{64}\right)^k \int_0^1 \frac{1 - t^k}{1 - t} dt
\]
\[
= \int_0^1 \frac{1}{1 - t} \sum_{k=1}^{\infty} \left(\frac{4k}{2k}\right)^k \left(\frac{3}{64}\right)^k (1 - t^k) dt
\]
\[
= \int_0^1 \frac{1}{1 - t} \left(\sqrt{3} - \frac{\sqrt{2 + \sqrt{4 - 3t}}}{\sqrt{4 - 3t}}\right) dt
\]
\[
= \log \frac{2 + \sqrt{3}}{3} + \sqrt{3} \log \frac{7 + 4\sqrt{3}}{8}
\]
\[
= (2 + 4\sqrt{3}) \log(1 + \sqrt{3}) - \log 3 - (1 + 5\sqrt{3}) \log 2.
\]

**Conjecture 2.9 (2022-12-30).** We have
\[
\sum_{k=0}^{\infty} \left(\frac{2k}{k}\right)^2 (-16)^k (2H_{2k} - H_k) = -\frac{(\log 2) \Gamma(1/4)^2}{4\pi \sqrt{2\pi}}
\]  
(2.17)

and
\[
\sum_{k=0}^{\infty} \left(\frac{2k}{k}\right)^2 \frac{32^k}{32^k} (2H_{2k} - H_k) = \frac{(\log 2) \Gamma(1/4)^2}{4\pi \sqrt{\pi}},
\]  
(2.18)

where $\Gamma(x)$ is the well-known Gamma function.

**Remark 2.9.** Mathematica yields the identities
\[
\sum_{k=0}^{\infty} \left(\frac{2k}{k}\right)^2 (-16)^k = \frac{\Gamma(1/4)^2}{2\pi \sqrt{\pi}} \text{ and } \sum_{k=0}^{\infty} \left(\frac{2k}{k}\right)^2 \frac{32^k}{32^k} = \frac{\Gamma(1/4)^2}{2\pi \sqrt{\pi}}.
\]

For any prime $p \equiv 3 \pmod{4}$, the author [30] proved that
\[
\sum_{k=0}^{p-1} \left(\frac{2k}{k}\right)^2 (-16)^k \equiv \frac{(-1)^{(p-3)/4}}{(p+1)/2} (mod \ p^2),
\]
and Z.-H. Sun [25] confirmed the author’s conjecture (cf. [26, Conjecture 5.5])
\[
\sum_{k=0}^{p-1} \left(\frac{2k}{k}\right)^2 \frac{32^k}{32^k} \equiv 0 (mod \ p^2).
\]

For any prime $p \equiv 1 \pmod{4}$ with $p = x^2 + y^2$ ($x, y \in \mathbb{Z}$ and $x \equiv 1 \pmod{4}$), Z.-H. Sun [25] confirmed the author’s conjecture (cf. [26, Conjecture 5.5])
\[
(-1)^{(p-1)/4} \sum_{k=0}^{p-1} \left(\frac{2k}{k}\right)^2 (-16)^k \equiv \sum_{k=0}^{p-1} \left(\frac{2k}{k}\right)^2 \frac{32^k}{32^k} \equiv 2x - \frac{p}{2x} (mod \ p^2).
\]
Conjecture 2.10 (2022-11-14). We have the identity
\[ \sum_{k=0}^{\infty} \frac{(2k)^2}{32^k} \left( H_{2k}^{(2)} - \frac{1}{4} H_k^{(2)} \right) = \Gamma \left( \frac{1}{4} \right)^2 \frac{\pi^2}{32\sqrt{\pi}} - 8G. \] (2.19)

Remark 2.10. In contrast with (2.19), Mathematica yields that
\[ \sum_{k=0}^{\infty} \frac{(2k)^2}{32^k} = \frac{\Gamma(1/4)}{\sqrt{2\pi}\Gamma(3/4)} = \frac{\Gamma(1/4)^2}{2\pi\sqrt{\pi}}. \]

Conjecture 2.11 (2022-12-30). We have
\[ \sum_{k=0}^{\infty} \frac{(2k)(3k)}{54^k} (3H_{3k} - H_k) = \frac{(3\log 2)\Gamma(1/3)^3}{4\pi^2\sqrt{2}}. \] (2.20)

Remark 2.11. Mathematica yields the identity
\[ \sum_{k=0}^{\infty} \frac{(2k)(3k)}{54^k} = \frac{3\Gamma(1/3)^3}{4\pi^2\sqrt{2}}. \]

By [30, Corollary 1.3], for any prime \( p > 3 \) with \( p \equiv 2 \pmod{3} \), we have
\[ \sum_{k=0}^{p-1} \frac{(2k)(3k)}{(-216)^k} \equiv 0 \pmod{p^2}. \]

Conjecture 2.12 (2022-12-30). We have
\[ \sum_{k=0}^{\infty} \frac{(2k)(3k)}{(-216)^k} (3H_{3k} - H_k) = \left( \log \frac{8}{9} \right) \sum_{k=0}^{\infty} \frac{(2k)(3k)}{(-216)^k}. \] (2.21)

Remark 2.12. For any prime \( p > 3 \), we have
\[ \sum_{k=0}^{p-1} \frac{(2k)(3k)}{(-216)^k} \equiv \frac{p}{3} \sum_{k=0}^{p-1} \frac{(2k)(3k)}{24^k} \pmod{p^2} \]
by [30, Corollary 1.4], and
\[ \sum_{k=0}^{p-1} \frac{(2k)(3k)}{24^k} \equiv \begin{cases} \frac{(2p-2)/3}{(p-1)/3} \pmod{p^2} & \text{if } p \equiv 1 \pmod{3}, \\ \frac{p}{(2p+2)/3} \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}, \end{cases} \]

as conjectured by the author [27, Conjecture 5.13] and proved by C. Wang and Sun [43, Theorem 1.2].

Conjecture 2.13 (2022-12-30). We have
\[ \sum_{k=0}^{\infty} \frac{(4k)(2k)}{128^k} (2H_{4k} - H_{2k}) = \frac{(\log 2)\sqrt{\pi}}{2\Gamma(5/8)\Gamma(7/8)}. \] (2.22)
Remark 2.13. Mathematica yields the identity
\[ \sum_{k=0}^{\infty} \frac{(4k)(2k)}{128^k} = \sqrt{\pi} \frac{\Gamma(5/8)\Gamma(7/8)}{\Gamma(128)} . \]

By [30, Corollary 1.3], for any prime \( p \equiv 5, 7 \mod 8 \) we have
\[ \sum_{k=0}^{p-1} \frac{(4k)(2k)}{128^k} \equiv 0 \mod p^2. \]

Conjecture 2.14 (2022-12-30). We have
\[ \sum_{k=0}^{\infty} \frac{(4k)(2k)}{72^k} (2H_{4k} - H_{2k}) = (\log 3) \sum_{k=0}^{\infty} \frac{(4k)(2k)}{72^k} \]
and
\[ \sum_{k=0}^{\infty} \frac{(4k)(2k)}{576^k} (2H_{4k} - H_{2k}) = \frac{1}{2} \left( \log \frac{9}{8} \right) \sum_{k=0}^{\infty} \frac{(4k)(2k)}{576^k}. \]

Remark 2.14. Let \( p > 3 \) be a prime. By [30, Corollary 1.4],
\[ \sum_{k=0}^{p-1} \frac{(4k)(2k)}{576^k} \equiv \left( \frac{-2}{p} \right) \sum_{k=0}^{p-1} \frac{(4k)(2k)}{72^k} \mod p^2. \]

We also have
\[ \left( \frac{6}{p} \right) \sum_{k=0}^{p-1} \frac{(4k)(2k)}{72^k} \]
\[ \equiv \begin{cases} 2x - \frac{p}{2x} \mod p^2 & \text{if } p = x^2 + y^2, x, y \in \mathbb{Z} \text{ and } 4 \mid x - 1, \\ \frac{2p}{3^{(p+1)/2}} \mod p^2 & \text{if } p \equiv 3 \mod 4, \end{cases} \]
as conjectured by the author [27, Conjecture 5.14(iii)] and proved by Wang and Sun [43, Theorem 5.2 and Remark 5.2].

Conjecture 2.15 (2022-12-30). We have
\[ \sum_{k=0}^{\infty} \frac{(4k)(2k)}{(-192)^k} (2H_{4k} - H_{2k}) = \frac{1}{2} \left( \log \frac{3}{4} \right) \sum_{k=0}^{\infty} \frac{(4k)(2k)}{(-192)^k}. \]

Remark 2.15. Let \( p > 3 \) be a prime. By [30, Corollary 1.4], we have
\[ \sum_{k=0}^{p-1} \frac{(4k)(2k)}{(-192)^k} \equiv \left( \frac{-2}{p} \right) \sum_{k=0}^{p-1} \frac{(4k)(2k)}{48^k} \mod p^2. \]
If \( p = x^2 + 3y^2 \) with \( x, y \in \mathbb{Z} \) and \( x \equiv 1 \mod 3 \), then
\[ \sum_{k=0}^{p-1} \frac{(4k)(2k)}{48^k} \equiv 2x - \frac{p}{2x} \mod p^2, \]
as conjectured by the author [27, Conjecture 5.14] and confirmed by G.-S. Mao and H. Pan [23]. If \( p \equiv 2 \pmod{3} \), then
\[
\sum_{k=0}^{p-1} \frac{\binom{4k}{2k}\binom{2k}{k}}{48^k} \equiv \frac{3p}{2^{(p+1)/2}} \pmod{p^2}
\]
as conjectured by the author [27, Conjecture 5.14] and confirmed by Wang and Sun [43].

**Conjecture 2.16** (2022-12-30). We have
\[
\sum_{k=0}^{\infty} \frac{\binom{4k}{2k}\binom{2k}{k}}{(-4032)^k}(2H_{4k} - H_{2k}) = \frac{1}{2} \left( \log \frac{63}{64} \right) \sum_{k=0}^{\infty} \frac{\binom{4k}{2k}\binom{2k}{k}}{(-4032)^k}.
\]  

(2.26)

**Remark 2.16.** Let \( p > 3 \) be a prime with \( p \neq 7 \). By [30, Corollary 1.4],
\[
\sum_{k=0}^{p-1} \frac{\binom{4k}{2k}\binom{2k}{k}}{(-4032)^k} \equiv \left( \frac{-2}{p} \right) \sum_{k=0}^{p-1} \frac{\binom{4k}{2k}\binom{2k}{k}}{63^k} \pmod{p^2}.
\]
The author [27, Conjecture 5.14(ii)] conjectured that
\[
\sum_{k=0}^{p-1} \frac{\binom{4k}{2k}\binom{2k}{k}}{63^k} \equiv \begin{cases} 
\binom{5}{2} (2x - \frac{k}{x^2}) \pmod{p^2} & \text{if } p = x^2 + 7y^2 \text{ with } x, y \in \mathbb{Z} \text{ and } \binom{5}{2} = 1, \\
0 \pmod{p} & \text{if } \binom{5}{2} = -1.
\end{cases}
\]

**Conjecture 2.17** (2022-12-30). We have
\[
\sum_{k=0}^{\infty} \frac{\binom{6k}{3k}\binom{3k}{k}}{864^k}(6H_{6k} - 3H_{3k} - 2H_{2k} + H_k) = \frac{(\log 2)\sqrt{\pi}}{\Gamma(7/12)\Gamma(11/12)}.
\]

(2.27)

**Remark 2.17.** Mathematica yields the identity
\[
\sum_{k=0}^{\infty} \frac{\binom{6k}{3k}\binom{3k}{k}}{864^k} = \frac{\sqrt{\pi}}{\Gamma(7/12)\Gamma(11/12)}.
\]

By [30, Corollary 1.3], for any prime \( p > 3 \) with \( p \equiv 3 \pmod{4} \) we have
\[
\sum_{k=0}^{p-1} \frac{\binom{6k}{3k}\binom{3k}{k}}{864^k} \equiv 0 \pmod{p^2}.
\]

3. Series and congruences with summands containing 3 or 4 binomial coefficients

In 1993 D. Zeilberger [47] used the WZ method to establish the identity
\[
\sum_{k=1}^{\infty} \frac{21k - 8}{k^3 \binom{2k}{3}^3} = \frac{\pi^2}{6}.
\]
The author [34] proved that
\[
\sum_{k=1}^{(p-1)/2} \frac{21k - 8}{k^3(2k)^3} = -\left(\frac{-1}{p}\right) 4E_{p-3} \pmod{p}
\]
for any prime \(p > 3\).

**Conjecture 3.1** (2022-10-11). (i) We have the identity
\[
\sum_{k=1}^{\infty} \frac{21k - 8}{k^3(2k)^3} \left(H_{2k-1}^{(2)} - \frac{25}{8} H_{k-1}^{(2)}\right) = \frac{47\pi^4}{2880}.
\]

(ii) For any prime \(p > 3\), we have
\[
\sum_{k=1}^{(p-1)/2} (21k + 8) \binom{2k}{k}^3 \left(H_{2k}^{(2)} - \frac{25}{8} H_k^{(2)}\right) \equiv 32p \left(\frac{-1}{p}\right) E_{p-3} \pmod{p^2}
\]
and
\[
\sum_{k=0}^{p-1} (21k + 8) \binom{2k}{k}^3 \left(H_{2k}^{(2)} + \frac{25}{8} H_k^{(2)}\right) \equiv -48H_{p-1} + \frac{246}{5} p^4 B_{p-5} \pmod{p^5}.
\]

(iii) For each prime \(p > 5\), we have
\[
\sum_{k=0}^{p-1} \binom{2k}{k}^3 (21k + 8)(H_{2k} - H_k + 7) \equiv -112pH_{p-1} \pmod{p^5}.
\]

**Remark 3.1.** After seeing an earlier arXiv version of this paper, K. C. Au [4, Corollary 2.3] confirmed the author’s conjectural identity (3.1), and proved an identity (after the proof of [4, Theorem 2.2]) which has the equivalent form
\[
\sum_{k=1}^{\infty} \frac{(21k - 8)(H_{2k-1} - H_{k-1}) - 7/2}{k^3(2k)^3} = \zeta(3).
\]

**Conjecture 3.2** (2022-10-11). (i) We have
\[
\sum_{k=1}^{\infty} \frac{21k - 8}{k^3(2k)^3} \left(H_{2k-1}^{(3)} + \frac{43}{8} H_{k-1}^{(3)}\right) = \frac{711}{28} \zeta(5) - \frac{29}{14} \pi^2 \zeta(3).
\]

(ii) For any prime \(p > 7\), we have
\[
\sum_{k=0}^{(p-1)/2} (21k + 8) \binom{2k}{k}^3 \left(H_{2k}^{(3)} + \frac{43}{8} H_k^{(3)}\right) \equiv 32 \left(\frac{-1}{p}\right) E_{p-3} \pmod{p}.
\]

and
\[
\sum_{k=0}^{p-1} (21k + 8) \binom{2k}{k}^3 \left(H_{2k}^{(3)} + \frac{43}{8} H_k^{(3)}\right) \equiv -\frac{120}{7} pB_{p-3} \pmod{p^2}.
\]

**Remark 3.2.** The identity (3.5) looks quite challenging.
Conjecture 3.3 (2022-10-13). (i) We have
\[
\sum_{k=1}^{\infty} \frac{(3k - 1)(-8)^k}{k^3 \binom{2k}{k}^3} \left( H_{2k-1}^{(2)} - \frac{5}{4} H_{k-1}^{(2)} \right) = -2\beta(4). \tag{3.8}
\]

(ii) For any prime \( p > 3 \), we have
\[
\sum_{k=0}^{(p-1)/2} (3k+1) \frac{\binom{2k}{k}^3}{(-8)^k} \left( H_{2k}^{(2)} - \frac{5}{4} H_k^{(2)} \right) \equiv \left( \frac{2}{p} \right) \frac{p}{4} E_{p-3} \left( \frac{1}{4} \right) \pmod{p^2} \tag{3.9}
\]
and
\[
\sum_{k=0}^{p-1} (3k+1) \frac{\binom{2k}{k}^3}{(-8)^k} \left( H_{2k}^{(2)} - \frac{5}{4} H_k^{(2)} \right) \equiv p E_{p-3} \pmod{p^2}. \tag{3.10}
\]

(iii) Let \( p \) be an odd prime. Then
\[
\sum_{k=0}^{(p-1)/2} (3k+1) \frac{\binom{2k}{k}^3}{(-8)^k} (2(3k+1)(H_{2k} - H_k) + 1) \equiv \left( \frac{-1}{p} \right) 2^{p-1} \pmod{p^2} \tag{3.11}
\]
and
\[
\sum_{k=0}^{p-1} (3k+1) \frac{\binom{2k}{k}^3}{(-8)^k} \left( H_{2k}^{(3)} + \frac{7}{8} H_k^{(3)} \right) \equiv 0 \pmod{p}. \tag{3.12}
\]

Remark 3.3. In 2008, J. Guillera [14] used the WZ method to find the identity
\[
\sum_{k=1}^{\infty} \frac{(3k - 1)(-8)^k}{k^3 \binom{2k}{k}^3} = -2G.
\]
The identity (3.8) provides a fast converging series for computing the constant \( \beta(4) \). We are unable to find the exact values of the series
\[
\sum_{k=1}^{\infty} \frac{(-8)^k}{k^3 \binom{2k}{k}^3} (2(3k - 1)(H_{2k-1} - H_{k-1}) - 1)
\]
and
\[
\sum_{k=1}^{\infty} \frac{(3k - 1)(-8)^k}{k^3 \binom{2k}{k}^3} \left( H_{2k-1}^{(3)} + \frac{7}{8} H_{k-1}^{(3)} \right).
\]

Conjecture 3.4 (2022-10-11). (i) We have the identity
\[
\sum_{k=1}^{\infty} \frac{(3k - 1)16^k}{k^3 \binom{2k}{k}^3} \left( H_{2k-1}^{(2)} - \frac{5}{4} H_{k-1}^{(2)} \right) = \frac{\pi^4}{24}. \tag{3.13}
\]

(ii) Let \( p > 3 \) be a prime. Then
\[
\sum_{k=0}^{(p-1)/2} (3k+1) \frac{\binom{2k}{k}^3}{16^k} \left( H_{2k}^{(2)} - \frac{5}{4} H_k^{(2)} \right) \equiv 2p \left( \frac{-1}{p} \right) E_{p-3} \pmod{p^2} \tag{3.14}
\]
and
\[ \sum_{k=0}^{p-1} (3k+1) \frac{(2k)^3}{16^k} \left( H_{2k}^{(2)} - \frac{5}{4} H_k^{(2)} \right) \equiv \frac{7}{6} p^2 B_{p-3} \pmod{p^3}. \tag{3.15} \]

(iii) Let \( p \) be an odd prime. Then
\[ \sum_{k=1}^{p-1} \frac{(2k)^3}{16^k} (2(3k+1)(H_{2k} - H_k) + 1) \equiv \frac{4}{3} p q_p(2) - \frac{2}{3} p^2 q_p(2)^2 \pmod{p^3}. \tag{3.16} \]

If \( p \equiv 2 \pmod{3} \), then
\[ \sum_{k=0}^{(p-1)/2} \frac{(2k)^3}{16^k} \left( H_{2k}^{(2)} - \frac{5}{4} H_k^{(2)} \right) \equiv 0 \pmod{p}. \tag{3.17} \]

**Remark 3.4.** In 2008 J. Guillera [14, Identity 1] used the WZ method to establish the identity
\[ \sum_{k=1}^\infty \frac{(3k-1)16^k}{k^3(2k)_3} = \frac{\pi^2}{2}. \]

Two \( q \)-analogues of this identity were given by Q.-H. Hou, C. Krattenthaler and Z.-W. Sun [20]. Guo and Lian [18] proved that the two sides of (3.15) are congruent modulo \( p^2 \) for any prime \( p > 3 \). After seeing an earlier arXiv version of this paper, Au [4] confirmed the author’s conjectural identity (3.13), and proved an identity (after the proof of [4, Corollary 2.3]) which has the equation form
\[ \sum_{k=1}^\infty \frac{16^k}{k^3(2k)_3} \left( (3k-1)(H_{2k-1} - H_{k-1}) - \frac{1}{2} \right) = \frac{\pi^2}{3} \log 2 + \frac{7}{6} \zeta(3). \]

**Conjecture 3.5** (2022-10-11). (i) We have
\[ \sum_{k=1}^\infty \frac{(3k-1)16^k}{k^3(2k)_3} \left( H_{2k-1}^{(3)} + \frac{7}{8} H_{k-1}^{(3)} \right) = \frac{\pi^2}{2} \zeta(3). \tag{3.18} \]

(ii) For any odd prime \( p \), we have
\[ \sum_{k=1}^{(p-1)/2} (3k+1) \frac{(2k)^3}{16^k} \left( H_{2k}^{(3)} + \frac{7}{8} H_k^{(3)} \right) \equiv 2 \left( \frac{1}{p} \right) E_{p-3} \pmod{p} \tag{3.19} \]

and
\[ \sum_{k=0}^{p-1} (3k+1) \frac{(2k)^3}{16^k} \left( H_{2k}^{(3)} + \frac{7}{8} H_k^{(3)} \right) \equiv \frac{3}{2} p B_{p-3} \pmod{p^2}. \tag{3.20} \]

**Remark 3.5.** Conjecture 3.5 looks more challenging than Conjecture 3.4.
Conjecture 3.6 (2022-10-16). (i) We have
\[
\sum_{k=1}^{\infty} \frac{(4k-1)(-64)^k}{k^3 \binom{2k}{k}^3} \left( H_{2k-1}^{(2)} - \frac{1}{2} H_{k-1}^{(2)} \right) = -16\beta(4). \tag{3.21}
\]

(ii) For any prime \( p > 3 \), we have
\[
\sum_{k=0}^{(p-1)/2} (4k + 1) \frac{\binom{2k}{k}^3}{(-64)^k} \left( H_{2k}^{(2)} - \frac{1}{2} H_k^{(2)} \right) \equiv pE_{p-3} \pmod{p^2}, \tag{3.22}
\]
\[
\sum_{k=0}^{p-1} (4k + 1) \frac{\binom{2k}{k}^3}{(-64)^k} \left( H_{2k}^{(2)} - \frac{1}{2} H_k^{(2)} \right) \equiv pE_{p-3} \pmod{p^2}. \tag{3.23}
\]

Remark 3.6. J. Guillera \[14\] proved that
\[
\sum_{k=1}^{\infty} \frac{(4k-1)(-64)^k}{k^3 \binom{2k}{k}^3} = -16G.
\]
The congruence (3.22) is motivated by (1.10). After seeing an earlier arXiv version of this paper, Au \[4\, Corollary 2.11\] confirmed the author’s conjectural identity (3.21). It seems that for any \( m, n \in \mathbb{Z}^+ \) and odd prime \( p \), we have
\[
\sum_{k=0}^{(p-1)/2} (4k + 1) \frac{\binom{2k}{k}^n}{(-4)^k} \left( H_{2k}^{(2m)} - H_k^{(2m)} \right) \equiv 0 \pmod{p}. \tag{3.24}
\]

Conjecture 3.7 (2022-12-05). (i) We have
\[
\sum_{k=1}^{\infty} \frac{8^k ((10k - 3)(H_{2k-1} - H_{k-1}) - 1)}{k^3 \binom{2k}{k} \binom{3k}{k}} = \frac{7}{2} \zeta(3) \tag{3.25}
\]
and
\[
\sum_{k=1}^{\infty} \frac{8^k ((10k - 3)(H_{2k-1} - H_{k-1}) - 8/3)}{k^3 \binom{2k}{k} \binom{3k}{k}} = \frac{2\pi^2 \log 2 + 7\zeta(3)}{4}. \tag{3.26}
\]

(ii) For any odd prime \( p \), we have
\[
\sum_{k=1}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{8^k} ((10k + 3)(H_{2k} - H_k) + 1) \equiv \frac{63}{8} p^3 B_{p-3} \pmod{p^4} \tag{3.27}
\]
and
\[
\sum_{k=1}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{8^k} (3(10k + 3)(H_{3k} - H_k) + 8) \equiv 9p q_p(2) - \frac{9}{2} p^2 q_p(2) \pmod{p^3}. \tag{3.28}
\]
Remark 3.7. As conjectured by the author [27] and confirmed by J. Guillera and M. Rogers [17], we have
\[
\sum_{k=1}^{\infty} \frac{(10k - 3)8^k}{k^3 {2k \choose k}^2 {3k \choose k}^2} = \frac{\pi^2}{2}.
\]

Conjecture 3.8 (2022-12-05). (i) We have
\[
\sum_{k=1}^{\infty} \frac{(-27)^k((15k - 4)(3H_{3k-1} - H_{k-1}) - 9)}{k^3 {2k \choose k}^2 {3k \choose k}^2} = -\frac{4\pi^3}{\sqrt{3}}. \tag{3.29}
\]

(ii) For any prime \( p > 3 \), we have
\[
\sum_{k=0}^{p-1} \frac{(2k)^2 (3k)^2}{(-27)^k} \left( (15k + 4)(3H_{3k} - H_k) + 9 \right) \equiv 9 \left( \frac{P}{3} \right) + 6p^2 B_{p-2} \left( \frac{1}{3} \right) \pmod{p^3}. \tag{3.30}
\]

Remark 3.8. As conjectured by the author [27] and confirmed by Kh. Hessami Pilehrood and T. Hessami Pilehrood [19], we have
\[
\sum_{k=1}^{\infty} \frac{(15k - 4)(-27)^{k-1}}{k^3 {2k \choose k}^2 {3k \choose k}^2} = K.
\]

Conjecture 3.9 (2022-12-05). (i) We have
\[
\sum_{k=1}^{\infty} \frac{64^{k-1}((11k - 3)(2H_{2k-1} + H_{k-1}) - 4)}{k^3 {2k \choose k}^2 {3k \choose k}^2} = \frac{7}{2} \zeta(3) \tag{3.31}
\]
and
\[
\sum_{k=1}^{\infty} \frac{64^{k-1}((11k - 3)(3H_{3k-1} - 6H_{k-1}) - 7)}{k^3 {2k \choose k}^2 {3k \choose k}^2} = \frac{6\pi^2 \log 2 - 21\zeta(3)}{8}. \tag{3.32}
\]

(ii) For any odd prime \( p \), we have
\[
\sum_{k=1}^{p-1} \frac{(2k)^2 (3k)^2}{64^k} \left( (11k + 3)(2H_{2k} + H_k) + 4 \right) \equiv 21p^3 B_p (2) \pmod{p^4} \tag{3.33}
\]
and
\[
\sum_{k=1}^{p-1} \frac{(2k)^2 (3k)^2}{64^k} \left( (11k + 3)(3H_{3k} - 6H_k) + 7 \right) \equiv 18p q_p(2) + 9p^2 q_p(2)^2 \pmod{p^3}. \tag{3.34}
\]

Remark 3.9. As conjectured by the author [27] and confirmed by J. Guillera [16], we have
\[
\sum_{k=1}^{\infty} \frac{(11k - 3)64^k}{k^3 {2k \choose k}^2 {3k \choose k}^2} = 8\pi^2.
\]
Conjecture 3.10 (2022-12-09). (i) We have
\[
\sum_{k=1}^{\infty} \frac{81^k((35k - 8)(H_{4k-1} - H_{k-1}) - 35/4)}{k^3\binom{2k}{k}^2\binom{4k}{2k}} = 12\pi^2 \log 3 + 39\zeta(3). \tag{3.35}
\]
(ii) For any prime \( p > 3 \), we have
\[
\sum_{k=1}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{81^k} (4(35k + 8)(H_{4k} - H_k) + 35)
\equiv 32(3^{p-1} - 1) - 16(3^{p-1} - 1)^2 \pmod{p^3}. \tag{3.36}
\]

Remark 3.10. As conjectured by the author [27] and confirmed in [17], we have
\[
\sum_{k=1}^{\infty} \frac{(35k - 8)81^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}} = 12\pi^2.
\]

Conjecture 3.11 (2023-01-14). For any prime \( p > 3 \), we have
\[
\sum_{k=1}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-144)^k} (4(5k + 1)(H_{4k} - H_k) + 5)
\equiv \left(\frac{p}{3}\right)(5 + 2p(2q_p(2) + q_p(3))) \pmod{p^2}. \tag{3.37}
\]

Remark 3.11. As conjectured by the author [27] and confirmed in [17], we have
\[
\sum_{k=1}^{\infty} \frac{(5k - 1)(-144)^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}} = -\frac{45}{2}K.
\]

We are unable to find the exact value of the series
\[
\sum_{k=1}^{\infty} \frac{(-144)^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}} (4(5k - 1)(H_{4k-1} - H_{k-1}) - 5).
\]

The classical rational Ramanujan-type series for \( 1/\pi \) have the following four forms:
\[
\sum_{k=0}^{\infty} \frac{(ak + b)(2k)^3}{m^k} = \frac{c\sqrt{d}}{\pi}, \tag{3.38}
\]
\[
\sum_{k=0}^{\infty} \frac{(ak + b)(2k)^2(3k)}{m^k} = \frac{c\sqrt{d}}{\pi}, \tag{3.39}
\]
\[
\sum_{k=0}^{\infty} \frac{(ak + b)(2k)^2(4k)}{m^k} = \frac{c\sqrt{d}}{\pi}, \tag{3.40}
\]
\[
\sum_{k=0}^{\infty} \frac{(ak + b)(2k)(3k)(6k)}{m^k} = \frac{c\sqrt{d}}{\pi}. \tag{3.41}
\]
where \( a, b, m \in \mathbb{Z}, \ am \neq 0, \ c \in \mathbb{Q} \setminus \{0\} \), and \( d \) is a positive squarefree integer. It is known that there are totally 36 such series, see, e.g., S. Cooper [11, Chapter 14].

For a positive integer \( m \), can we find similar series for \((\log m)/\pi\)? Motivated by Ramanujan-type series of the forms (3.38)-(3.41), the author formulated the following general conjecture.

**Conjecture 3.12** (General Conjecture, 2022-12-08). (i) If we have an identity (3.38) with \( a, b, m \in \mathbb{Z}, \ am \neq 0, \ c \in \mathbb{Q} \setminus \{0\} \), and \( d \in \mathbb{Z}^+ \) squarefree, then

\[
\sum_{k=0}^{\infty} \frac{(2k)^3}{m^k} (6(ak+b)(H_{2k} - H_k) + a) = \frac{c\sqrt{d}}{\pi} \log |m|, \tag{3.42}
\]

and

\[
\sum_{k=0}^{p-1} \frac{(2k)^3}{m^k} (6(ak+b)(H_{2k} - H_k) + a) \equiv \left(\frac{-d}{p}\right) (a + b(m^{p-1} - 1)) \pmod{p^2} \tag{3.43}
\]

for any prime \( p \nmid dm \).

(ii) If we have an identity (3.39) with \( a, b, m \in \mathbb{Z}, \ am \neq 0, \ c \in \mathbb{Q} \setminus \{0\} \), and \( d \in \mathbb{Z}^+ \) squarefree, then

\[
\sum_{k=0}^{\infty} \frac{(2k)^2(3k)}{m^k} ((ak+b)(3H_{3k} + 2H_{2k} - 5H_k) + a) = \frac{c\sqrt{d}}{\pi} \log |m|, \tag{3.44}
\]

and

\[
\sum_{k=0}^{p-1} \frac{(2k)^2(3k)}{m^k} ((ak+b)(3H_{3k} + 2H_{2k} - 5H_k) + a) \equiv \left(\frac{-d}{p}\right) (a + b(m^{p-1} - 1)) \pmod{p^2} \tag{3.45}
\]

for any odd prime \( p \nmid dm \).

(iii) If we have an identity (3.40) with \( a, b, m \in \mathbb{Z}, \ am \neq 0, \ c \in \mathbb{Q} \setminus \{0\} \), and \( d \in \mathbb{Z}^+ \) squarefree, then

\[
\sum_{k=0}^{\infty} \frac{(2k)^2(4k)}{m^k} (4(ak+b)(H_{4k} - H_k) + a) = \frac{c\sqrt{d}}{\pi} \log |m|, \tag{3.46}
\]

and

\[
\sum_{k=0}^{p-1} \frac{(2k)^2(4k)}{m^k} (4(ak+b)(H_{4k} - H_k) + a) \equiv \left(\frac{-d}{p}\right) (a + b(m^{p-1} - 1)) \pmod{p^2} \tag{3.47}
\]

for any odd prime \( p \nmid dm \).
(iv) If we have an identity (3.41) with \(a, b, m \in \mathbb{Z}, am \neq 0, c \in \mathbb{Q} \setminus \{0\}\), and \(d \in \mathbb{Z}^+\) squarefree, then
\[
\sum_{k=0}^{\infty} \frac{{2k \choose k} {3k \choose k} {6k \choose 3k}}{m^k} (3(ak + b)(2H_{6k} - H_{3k} - H_k) + a) = \frac{c\sqrt{d}}{\pi} \log |m|, \tag{3.48}
\]
and
\[
\sum_{k=0}^{p-1} \frac{{2k \choose k} {3k \choose k} {6k \choose 3k}}{m^k} (3(ak + b)(2H_{6k} - H_{3k} - H_k) + a) \equiv \left(\frac{-d}{p}\right) (a + b(m^{p-1} - 1)) \pmod{p^2} \tag{3.49}
\]
for any odd prime \(p \nmid dm\).

Remark 3.12. Ramanujan [24] found the irrational series
\[
\sum_{k=0}^{\infty} \left(k + \frac{31}{270 + 48\sqrt{5}}\right) {2k \choose k}^3 = \frac{16}{(15 + 21\sqrt{5})\pi}.
\]
In the spirit of part (i) of our general conjecture, we guess that
\[
\sum_{k=0}^{\infty} \frac{{2k \choose k}^3}{(2^{20}/(\sqrt{5} - 1)^8)^k} \left(6 \left(k + \frac{31}{270 + 48\sqrt{5}}\right) (H_{2k} - H_k) + 1\right) = \frac{16}{(15 + 21\sqrt{5})\pi} \times \log \frac{2^{20}}{(\sqrt{5} - 1)^8},
\]
which can be easily checked via Mathematica.

Example 3.1. In view of the Ramanujan series (cf. [11, Chapter 14] and [24])
\[
\sum_{k=0}^{\infty} (5k + 1) \frac{{2k \choose k} {3k \choose k}}{(-192)^k} = \frac{4\sqrt{3}}{\pi}
\]
and the known identity
\[
\sum_{k=0}^{\infty} \frac{{2k \choose k}^2 {3k \choose k}}{(-192)^k} = \frac{4\sqrt{\pi}}{3\sqrt{3} \Gamma(5/6)^3}
\]
(cf. (14.29) of [11, p. 624]), by part (ii) of Conjecture 3.12 we should have
\[
\sum_{k=0}^{\infty} \frac{{2k \choose k}^2 {3k \choose k}}{(-192)^k} (5k + 1)(3H_{3k} + 2H_{2k} - 5H_k) = \frac{4\log 192}{\sqrt{3} \pi} - \frac{20\sqrt{\pi}}{3\sqrt{3} \Gamma(5/6)^3}.
\]

Conjecture 3.13 (2022-10-16). We have
\[
\sum_{k=0}^{\infty} (4k + 1) \frac{{2k \choose k}^3}{(-64)^k} H_{2k}^{(3)} = \frac{15\zeta(3)}{4\pi} - 2G. \tag{3.50}
\]
**Remark 3.13.** For any \( m, n \in \mathbb{Z}^+ \) and odd prime \( p \) not dividing \( 2^{2m-1} - 1 \), we have \( H^{(2m-1)}_{p-1} \equiv 0 \pmod{p} \) since

\[
\sum_{j=1}^{p-1} \frac{1}{(2j)^{2m-1}} \equiv \sum_{k=1}^{p-1} \frac{1}{k^{2m-1}} \pmod{p},
\]

thus

\[
\sum_{k=0}^{(p-1)/2} (4k + 1) \binom{(p - 1)/2}{k}^n H_{2k}^{(2m-1)}
\]

\[
= \sum_{k=0}^{(p-1)/2} \left( 4 \left( \frac{p - 1}{2} - k \right) + 1 \right) \binom{(p - 1)/2}{k}^n H_{p-1-2k}^{(2m-1)}
\]

\[
\equiv - \sum_{k=0}^{(p-1)/2} (4k + 1) \binom{(p - 1)/2}{k}^n H_{2k}^{(2m-1)} \pmod{p}
\]

and hence

\[
\sum_{k=0}^{(p-1)/2} (4k + 1) \binom{(2k)^n}{k} H_{2k}^{(2m-1)} \equiv 0 \pmod{p}. \tag{3.51}
\]

(Note that \( (-1)^{k} = \binom{2k}{k}/(-4)^{k} \) for any \( k \in \mathbb{N} \).)

**Conjecture 3.14 (2022-12-04).** (i) We have

\[
\sum_{k=0}^{\infty} (6k + 1) \frac{(2k)^3}{256^k} H_k = \frac{4}{3} \cdot \frac{\sqrt{2} \sqrt[3]{\pi}}{\Gamma(5/6)^3} - \frac{8 \log 2}{\pi}
\]

and

\[
\sum_{k=0}^{\infty} (6k + 1) \frac{(2k)^3}{256^k} H_{2k} = \frac{2}{3} \cdot \frac{\sqrt{2} \sqrt[3]{\pi}}{\Gamma(5/6)^3} - \frac{8 \log 2}{3 \pi}.
\]

(ii) Let \( p \) be an odd prime. Then

\[
\sum_{k=0}^{(p-1)/2} \frac{(2k)^3}{256^k} ((6k + 1)(3H_{2k} - H_k) - 1) \equiv (-1)^{(p+1)/2} \pmod{p^4}. \tag{3.54}
\]

If \( p > 3 \), then

\[
\sum_{k=0}^{(p-1)/2} \frac{(2k)^3}{256^k} ((6k + 1)(H_{2k} - H_k) + 1)
\]

\[
\equiv \left( \frac{-1}{p} \right) \left( 1 + \frac{4}{3} p q_p(2) - \frac{2}{3} p^2 q_p(2)^2 \right) \pmod{p^3}. \tag{3.55}
\]

**Remark 3.14.** It is known (cf. (14.27) of [11, p. 623]) that

\[
\sum_{k=0}^{\infty} \frac{(2k)^3}{256^k} = \frac{2}{3} \cdot \frac{\sqrt{2} \sqrt[3]{\pi}}{\Gamma(5/6)^3}.
\]
In view of this, (3.52) and (3.53) together implies the identities
\[
\sum_{k=0}^{\infty} \frac{(2k)^3}{256^k} ((6k + 1)(3H_{2k} - H_k) - 1) = 0
\]
and
\[
\sum_{k=0}^{\infty} \frac{(2k)^3}{256^k} ((6k + 1)(H_{2k} - H_k) + 1) = \frac{16 \log 2}{3\pi}.
\]
The last identity is also implied by Conjecture 3.12(i).

**Conjecture 3.15** (2022-10-12). Let \( p \) be any odd prime. 
(i) We have
\[
\sum_{k=0}^{(p-1)/2} (6k + 1) \frac{(2k)^3}{256^k} \left( H_{2k}^{(2)} - \frac{5}{16} H_k^{(2)} \right) \equiv \frac{7}{24} \left( \frac{-1}{p} \right) p^2 B_{p-3} \pmod{p^3}.
\]
(3.56)

If \( p > 3 \), then
\[
\sum_{k=0}^{p-1} (6k + 1) \frac{(2k)^3}{256^k} \left( H_{2k}^{(2)} - \frac{5}{16} H_k^{(2)} \right) \equiv -pE_{p-3} \pmod{p^2}.
\]
(3.57)

(ii) If \( p \equiv 2 \pmod{3} \), then
\[
\sum_{k=0}^{(p-1)/2} \frac{(2k)^3}{256^k} \left( H_{2k}^{(2)} - \frac{5}{16} H_k^{(2)} \right) \equiv 0 \pmod{p}.
\]
(3.58)

**Remark 3.15.** For any prime \( p > 3 \), Guo and Lian [18] proved that the two sides of (3.56) are congruent modulo \( p^2 \).

**Conjecture 3.16** (2022-10-11). (i) We have the identity
\[
\sum_{k=0}^{\infty} (6k + 1) \frac{(2k)^3}{256^k} \left( H_{2k}^{(3)} - \frac{7}{64} H_k^{(3)} \right) = \frac{25\zeta(3)}{8\pi} - G.
\]
(3.59)

(ii) Let \( p \) be an odd prime. Then
\[
\sum_{k=0}^{(p-1)/2} (6k + 1) \frac{(2k)^3}{256^k} \left( H_{2k}^{(3)} - \frac{7}{64} H_k^{(3)} \right) \equiv -\frac{1}{2} E_{p-3} \pmod{p}
\]
(3.60)

and
\[
\sum_{k=0}^{p-1} (6k + 1) \frac{(2k)^3}{256^k} \left( H_{2k}^{(3)} - \frac{7}{64} H_k^{(3)} \right) \equiv -\frac{3}{2} E_{p-3} \pmod{p}.
\]
(3.61)

**Remark 3.16.** For any \( k \in \mathbb{Z}^+ \), it is easy to see that
\[
H_{2k}^{(3)} - \frac{7}{64} H_k^{(3)} = \sum_{j=1}^{k} \left( \frac{1}{(2j-1)^3} + \frac{1}{(4j)^3} \right).
\]
Conjecture 3.17 (2022-10-12). Let \( p > 3 \) be a prime. Then
\[
\sum_{k=0}^{(p-1)/2} (6k + 1) \left( -\frac{512}{5} \right)^k \left( H_{2k}^{(2)} - \frac{5}{16} H_k^{(2)} \right) \equiv \frac{2}{p} \frac{p}{4} E_{p-3} \pmod{p^2}
\]
(3.62)
and
\[
\sum_{k=0}^{p-1} (6k + 1) \left( -\frac{512}{5} \right)^k \left( H_{2k}^{(2)} - \frac{5}{16} H_k^{(2)} \right) \equiv \frac{p}{16} E_{p-3} \left( \frac{1}{4} \right) \pmod{p^2}.
\]
(3.63)

Remark 3.17. Note that (3.62) is stronger than (1.7).

Conjecture 3.18 (2022-10-16). (i) We have the identity
\[
\sum_{k=0}^{\infty} (6k + 1) \left( -\frac{512}{5} \right)^k \left( H_{2k}^{(3)} - \frac{7}{64} H_k^{(3)} \right) = \frac{57}{16} \cdot \frac{\zeta(3)}{\sqrt{2 \pi}} - L,
\]
where
\[
L = L \left( 2, \left( \frac{-8}{-1} \right) \right) = \sum_{n=1}^{\infty} \frac{(-8)^n}{n^2} = \sum_{k=0}^{\infty} \frac{(-1)^{k(k-1)/2}}{(2k+1)^2}
\]
(3.64)
with \((\frac{-8}{-1})\) the Kronecker symbol.

(ii) Let \( p \) be an odd prime. Then
\[
\sum_{k=0}^{p-1} (6k + 1) \left( -\frac{512}{5} \right)^k \left( H_{2k}^{(3)} - \frac{7}{64} H_k^{(3)} \right) \equiv 0 \pmod{p}.
\]
(3.65)

Remark 3.18. For any odd prime \( p \), we are unable to find a closed form for the left-hand side of the congruence (3.65) modulo \( p^2 \).

Conjecture 3.19 (2022-12-09). For any prime \( p > 3 \), we have
\[
\sum_{k=0}^{(p-1)/2} \left( \frac{2k}{k} \right)^3 \frac{4096k}{(42k + 5)(H_{2k} - H_k) + 7} \equiv \left( \frac{-1}{p} \right) (7 + 10p q_p(2) - 5p^2 q_p(2)^2) \pmod{p^3}.
\]
(3.66)

Remark 3.19. In view of the Ramanujan series (cf. [24])
\[
\sum_{k=0}^{\infty} (42k + 5) \left( \frac{2k}{k} \right)^3 \frac{1}{4096k} = \frac{16}{\pi},
\]
by part (i) of Conjecture 3.12 we should have
\[
\sum_{k=0}^{\infty} \left( \frac{2k}{k} \right)^3 \frac{4096k}{(42k + 5)(H_{2k} - H_k) + 7} = \frac{32 \log 2}{\pi},
\]
and
\[ \sum_{k=0}^{p-1} \frac{(2k)^3}{4096^k} (6(42k + 5)(H_{2k} - H_k) + 42) \]
\[ \equiv \left( \frac{-1}{p} \right) (42 + 5p q_p(2^{12})) \equiv \left( \frac{-1}{p} \right) (42 + 60p q_p(2)) \pmod{p^2} \]
for any odd prime \( p \).

**Conjecture 3.20** (2022-10-11). (i) We have the identity
\[ \sum_{k=0}^{\infty} (42k + 5) \frac{(2k)^3}{4096^k} \left( H_{2k}^{(2)} - \frac{25}{92} H_k^{(2)} \right) = \frac{2\pi}{69}. \] (3.67)

(ii) Let \( p > 3 \) be a prime. If \( p \neq 23 \), then
\[ \sum_{k=0}^{(p-1)/2} (42k + 5) \frac{(2k)^3}{4096^k} \left( H_{2k}^{(2)} - \frac{25}{92} H_k^{(2)} \right) \]
\[ \equiv \left( \frac{-1}{p} \right) \frac{3}{20} (p^4 B_{p-5} - 5H_{p-1}) \pmod{p^5}. \] (3.68)

Also,
\[ \sum_{k=0}^{p-1} (42k + 5) \frac{(2k)^3}{4096^k} \left( H_{2k}^{(2)} - \frac{25}{92} H_k^{(2)} \right) \equiv -pE_{p-3} \pmod{p^2}. \] (3.69)

**Remark 3.20.** It is interesting to compare this conjecture with Remark 3.19.

**Conjecture 3.21** (2022-10-12). (i) We have
\[ \sum_{k=0}^{\infty} (42k + 5) \frac{(2k)^3}{4096^k} \left( H_{2k}^{(3)} - \frac{43}{352} H_k^{(3)} \right) = \frac{555}{77} \cdot \frac{\zeta(3)}{\pi} - \frac{32}{11} G. \] (3.70)

(ii) For any prime \( p > 7 \), we have
\[ \sum_{k=0}^{(p-1)/2} (42k + 5) \frac{(2k)^3}{4096^k} \left( 11H_{2k}^{(3)} - \frac{43}{32} H_k^{(3)} \right) \equiv -16E_{p-3} \pmod{p} \] (3.71)
and
\[ \sum_{k=0}^{p-1} (42k + 5) \frac{(2k)^3}{4096^k} \left( 11H_{2k}^{(3)} - \frac{43}{32} H_k^{(3)} \right) \equiv -27E_{p-3} \pmod{p}. \] (3.72)

**Remark 3.21.** Conjecture 3.21 looks quite challenging.

**Conjecture 3.22** (2022-12-05). (i) We have
\[ \sum_{k=0}^{\infty} \frac{(2k)^2 (3k)}{216^k} ((6k + 1)(H_{2k} - 2H_k) + 3) = \frac{9\sqrt{3} \log 3}{2\pi} \] (3.73)
and
\[
\sum_{k=0}^{\infty} \frac{(2k)^2 (3k)}{216^k} (6k + 1)(3H_{3k} - H_k) = \frac{9\sqrt{3}\log 2}{\pi}. 
\] (3.74)

(ii) For any prime \( p > 3 \), we have
\[
\sum_{k=0}^{p-1} \frac{(2k)^2 (3k)}{216^k} (6k + 1)(H_{2k} - 2H_k + 3) \equiv \left( \frac{p}{3} \right) \frac{3p + 3}{2} \pmod{p^2} 
\] (3.75)
and
\[
\sum_{k=0}^{p-1} \frac{(2k)^2 (3k)}{216^k} (6k + 1)(3H_{3k} - H_k) \equiv 3 \left( \frac{p}{3} \right) p q_p(2) \pmod{p^2}. 
\] (3.76)

Remark 3.22. This is motivated by the Ramanujan series (cf. [11, Chapter 14] and [24])
\[
\sum_{k=0}^{\infty} (6k + 1) \frac{(2k)^2 (3k)}{216^k} = \frac{3\sqrt{3}}{\pi}.
\]

Conjecture 3.23 (2022-12-04). (i) We have
\[
\sum_{k=0}^{\infty} \frac{(2k)^2 (4k)}{(-1024)^k} (20k + 3)(H_{2k} - 3H_k) + 12) \equiv \frac{56\log 2}{\pi}. 
\] (3.77)

(ii) For any odd prime \( p \), we have
\[
\sum_{k=0}^{p-1} \frac{(2k)^2 (4k)}{(-1024)^k} (20k + 3)(H_{2k} - 3H_k) + 12) \equiv \left( \frac{-1}{p} \right) (12 + 8p q_p(2)) \pmod{p^2}. 
\] (3.78)

Remark 3.23. This is motivated by the Ramanujan series (cf. [11, Chapter 14] and [24])
\[
\sum_{k=0}^{\infty} (20k + 3) \frac{(2k)^2 (4k)}{(-1024)^k} = \frac{8}{\pi}.
\]

Conjecture 3.24 (2022-12-04). (i) We have
\[
\sum_{k=0}^{\infty} \frac{(2k)^2 (4k)}{48^{2k}} ((8k + 1)(3H_{2k} - 4H_k) + 6) = \frac{16\sqrt{3}\log 2}{\pi}. 
\] (3.79)

(ii) For any prime \( p > 3 \), we have
\[
\sum_{k=0}^{p-1} \frac{(2k)^2 (4k)}{48^{2k}} ((8k + 1)(3H_{2k} - 4H_k) + 6) \equiv \left( \frac{p}{3} \right) (6 + 8p q_p(2)) \pmod{p^2}. 
\] (3.80)

Remark 3.24. This is motivated by the Ramanujan series (1.9).
Conjecture 3.25 (2022-10-15). For any prime $p > 3$, we have
\[
\sum_{k=0}^{(p-1)/2}(8k+1)\binom{2k}{k}^2\binom{4k}{2k}\frac{H_{2k}^{(2)} - \frac{5}{18}H_k^{(2)}}{48^{2k}} \equiv \frac{p}{36}B_{p-2}\left(\frac{1}{3}\right) \pmod{p^2},
\]
\[
\sum_{k=0}^{p-1}(8k+1)\binom{2k}{k}^2\binom{4k}{2k}\frac{H_{2k}^{(2)} - \frac{5}{18}H_k^{(2)}}{48^{2k}} \equiv \frac{5}{24}pB_{p-2}\left(\frac{1}{3}\right) \pmod{p^2}.
\]

Remark 3.25. The congruence (3.81) is motivated by (1.11).

Conjecture 3.26 (2022-10-19). (i) For any prime $p > 3$, we have
\[
\sum_{k=0}^{(p-1)/2}(4k+1)\binom{2k}{k}^4\frac{H_{2k}^{(2)} - \frac{1}{2}H_k^{(2)}}{256^k} \equiv \sum_{k=0}^{p-1}(4k+1)\binom{2k}{k}^4\frac{H_{2k}^{(2)} - \frac{1}{2}H_k^{(2)}}{256^k} \equiv \frac{7}{6}p^2B_{p-3} \pmod{p^3}.
\]

(ii) For any odd prime $p$, we have
\[
\sum_{k=0}^{(p-1)/2}(4k+1)\binom{2k}{k}^4H_{2k}^{(3)} \equiv \sum_{k=0}^{p-1}(4k+1)\binom{2k}{k}^4H_{2k}^{(3)} \equiv \frac{3}{2}pB_{p-3} \pmod{p^2}.
\]

Remark 3.26. Guo and Lian [18, (1.7)] proved that for any prime $p > 3$ we have
\[
\sum_{k=0}^{(p-1)/2}(4k+1)\binom{2k}{k}^4\left(H_{2k}^{(2)} - \frac{1}{2}H_k^{(2)}\right) \equiv 0 \pmod{p^2}.
\]

4. Series and congruences with summands containing at least five binomial coefficients

The following two conjectures are motivated by the identity
\[
\sum_{k=1}^{\infty} (-1)^k\frac{(205k^2 - 160k + 32)}{k^5\binom{2k}{k}^5} = -2\zeta(3)
\]
established by T. Amdeberhan and D. Zeilberger [5] in 1997 via the WZ method.

Conjecture 4.1 (2022-12-09). (i) We have
\[
\sum_{k=1}^{\infty} (-1)^{k-1}\frac{1}{k^5\binom{2k}{k}^5}((205k^2 - 160k + 32)(H_{2k-1} - H_{k-1}) - 41k + 16) = \frac{\pi^4}{60}.
\]
For any prime $p > 5$, we have
\[
\sum_{k=0}^{p-1} (-1)^k \binom{2k}{k} \left(205k^2 + 160k + 32\right) (H_{2k} - H_k) + 41k + 16 \equiv 16p + 64p^2 H_{p-1} \pmod{p^6}.
\] (4.2)

Conjecture 4.2 (2022-12-09). (i) We have
\[
\sum_{k=1}^{\infty} (-1)^k \binom{205k^2 - 160k + 32}{2k} \left(4H_{2k-1}^{(2)} - 12H_{k-1}^{(2)} - 43\right) \equiv -8\zeta(5) \pmod{p^6}.
\] (4.3)

(ii) Let $p > 3$ be a prime. Then
\[
\sum_{k=1}^{(p-1)/2} (-1)^k \binom{205k^2 - 160k + 32}{2k} \left(4H_{2k-1}^{(2)} - 12H_{k-1}^{(2)} - 43\right) \equiv -200B_{p-5} \pmod{p},
\] (4.4)
and
\[
\sum_{k=1}^{p-1} (-1)^k \binom{2k}{k} \left(205k^2 + 160k + 32\right) (4H_{2k}^{(2)} - 12H_k^{(2)}) + 43 \equiv 256pH_{p-1} \pmod{p^5}.
\] (4.5)

The following two conjectures are motivated by the identity
\[
\sum_{k=1}^{\infty} \frac{(10k^2 - 6k + 1)(-256)^k}{k^5 \binom{2k}{k}^5} = -28\zeta(3)
\]
(cf. [14, Identity 8]).

Conjecture 4.3 (2022-12-09). (i) We have
\[
\sum_{k=1}^{\infty} \frac{(-256)^k}{k^5 \binom{2k}{k}^5} \left(10k^2 - 6k + 1\right) (2H_{2k-1} - H_{k-1} - 3k + 1) = -\frac{\pi^4}{2}.
\] (4.6)

(ii) For any prime $p > 3$, we have
\[
\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^5}{(-256)^k} \left((10k^2 + 6k + 1)(2H_{2k} - H_k) + 3k + 1\right) \equiv p + \frac{14}{3}p^4B_{p-3} \pmod{p^5}.
\] (4.7)

Remark 4.1. For any prime $p > 3$, the author [36, Conjecture 31(ii)] conjectured the congruence
\[
\sum_{k=(p+1)/2}^{p-1} \frac{\binom{2k}{k}^5}{(-256)^k}(10k^2 + 6k + 1) \equiv -\frac{7}{2}p^5B_{p-3} \pmod{p^6}
\]
which implies that
\[
\sum_{k=(p+1)/2}^{p-1} \frac{(2k)^5}{(-256)^k} \left( (10k^2 + 6k + 1)(2H_{2k} - H_k) + 3k + 1 \right)
= \sum_{k=(p+1)/2}^{p-1} \frac{(2k)^5}{(-256)^k} \cdot \frac{2}{p} \equiv -7p^4B_{p-3} \pmod{p^5}.
\]

**Conjecture 4.4 (2022-12-09).** (i) We have
\[
\sum_{k=1}^{\infty} \frac{(-256)^k((10k^2 - 6k + 1)(4H_{2k-1}^{(2)} - 3H_{k-1}^{(2)}) - 2)}{k^5\binom{2k}{k}^5} = -124\zeta(5). \tag{4.8}
\]

(ii) For any prime \( p > 3 \), we have
\[
\sum_{k=1}^{(p-1)/2} \frac{(-256)^k}{k^5\binom{2k}{k}^5} \left( (10k^2 - 6k + 1)(4H_{2k-1}^{(2)} - 3H_{k-1}^{(2)}) - 2 \right) \equiv -124B_{p-5} \pmod{p} \tag{4.9}
\]
and
\[
\sum_{k=1}^{(p-1)/2} \frac{(2k)^5}{(-256)^k} \left( (10k^2 + 6k + 1)(4H_{2k}^{(2)} - 3H_k^{(2)}) + 2 \right) \equiv \frac{28}{3}p^3B_{p-3} \pmod{p^4}. \tag{4.10}
\]

The following two conjectures are motivated by the identity
\[
\sum_{k=1}^{\infty} \frac{(28k^2 - 18k + 3)(-64)^k}{k^5\binom{2k}{k}^4\binom{3k}{k}} = -14\zeta(3)
\]
conjectured by the author [29] and confirmed recently by Au [4].

**Conjecture 4.5 (2022-12-09).** (i) We have
\[
\sum_{k=1}^{\infty} \frac{(-64)^k}{k^5\binom{2k}{k}^4\binom{3k}{k}} \left( (28k^2 - 18k + 3)(4H_{2k-1} - 3H_{k-1}) - 20k + 6 \right) = \frac{\pi^4}{2}. \tag{4.11}
\]

(ii) For any odd prime \( p \), we have
\[
\sum_{k=0}^{p-1} \frac{(2k)^4\binom{3k}{k}}{(-64)^k} \left( (28k^2 + 18k + 3)(4H_{2k} - 3H_k) + 20k + 6 \right) \equiv 6p - 14p^4B_{p-3} \pmod{p^5}. \tag{4.12}
\]

**Conjecture 4.6 (2022-12-09).** (i) We have
\[
\sum_{k=1}^{\infty} \frac{(-64)^k((28k^2 - 18k + 3)(2H_{2k-1}^{(2)} - 3H_{k-1}^{(2)}) - 2)}{k^5\binom{2k}{k}^4\binom{3k}{k}} = -31\zeta(5). \tag{4.13}
\]
(ii) For any odd prime $p$, we have
\[
\sum_{k=1}^{p-1} \frac{(2k)^4}{(-64)^k} \left( (28k^2 + 18k + 3)(2H_{2k}^{(2)} - 3H_k^{(2)}) + 2 \right) \equiv -7p^3 B_{p-3} \pmod{p^4}.
\]

The following two conjectures are motivated by the identity
\[
\sum_{k=0}^{\infty} (20k^2 + 8k + 1) \frac{(2k)^5}{(-4096)^k} = \frac{8}{\pi^2}
\]
(cf. [14, Identity 8]).

**Conjecture 4.7** (2022-12-09). (i) We have
\[
\sum_{k=0}^{\infty} \frac{(2k)^5}{(-4096)^k} ((20k^2 + 8k + 1)H_k - 6k - 1) = -\frac{16 \log 2}{\pi^2}
\]
and
\[
\sum_{k=0}^{\infty} \frac{(2k)^5}{(-4096)^k} (5(20k^2 + 8k + 1)H_{2k} - 10k - 1) = -\frac{32 \log 2}{\pi^2}.
\]

(ii) For any prime $p > 3$, we have
\[
\sum_{k=0}^{(p-1)/2} \frac{(2k)^5}{(-4096)^k} ((20k^2 + 8k + 1)H_k - 6k - 1) \equiv -p - 2p^2 q_p(2) + p^3 q_p(2)^2 - \frac{2}{3} p^4 q_p(2)^3 \pmod{p^5},
\]
and
\[
\sum_{k=0}^{(p-1)/2} \frac{(2k)^5}{(-4096)^k} (5(20k^2 + 8k + 1)H_{2k} - 10k - 1) \equiv -p - 4p^2 q_p(2) + 2p^3 q_p(2)^2 \pmod{p^4}.
\]

**Conjecture 4.8** (2022-12-09). (i) We have
\[
\sum_{k=1}^{\infty} \frac{(2k)^5}{(-4096)^k} ((20k^2 + 8k + 1)(8H_{2k}^{(2)} - 3H_k^{(2)}) + 4) = -\frac{4}{3}.
\]

(ii) For any prime $p > 3$, we have
\[
\sum_{k=1}^{(p-1)/2} \frac{(2k)^5}{(-4096)^k} ((20k^2 + 8k + 1)(8H_{2k}^{(2)} - 3H_k^{(2)}) + 4) \equiv \frac{14}{3} p^3 B_{p-3} \pmod{p^4}.
\]
and
\[
\sum_{k=1}^{p-1} \frac{(2k\binom{k}{5})^5}{(-4096)^k} ((20k^2 + 8k + 1)(8H_k^{(2)} - 3H_k^{(2)}) + 4) \equiv -\frac{28}{3} p^3 B_{p-3} \pmod{p^4}.
\]

(4.21)

The following two conjectures are motivated by the identity
\[
\sum_{k=0}^{\infty} \frac{(2k\binom{k}{5})^5}{(-2^{20})^k} \frac{(820k^2 + 180k + 13)}{(2k\binom{k}{5})(2k\binom{k}{5})} = \frac{128}{\pi^2}
\]
(cf. [10, Identity 9]).

**Conjecture 4.9** (2022-12-09). (i) We have
\[
\sum_{k=0}^{\infty} \frac{(2k\binom{k}{5})^5}{(-2^{20})^k} \frac{(820k^2 + 180k + 13)(H_{2k} - H_k) + 164k + 18)}{(2k\binom{k}{5})(2k\binom{k}{5})} = \frac{256 \log 2}{\pi^2}.
\]

(4.22)

(ii) For any odd prime \(p\), we have
\[
\sum_{k=0}^{(p-1)/2} \frac{(2k\binom{k}{5})^5}{(-2^{20})^k} (820k^2 + 180k + 13)(H_{2k} - H_k) + 164k + 18) \equiv 18p + 26p^2 q_p(2) - 13p^3 q_p(2)^2 \pmod{p^4}.
\]

(4.23)

**Conjecture 4.10** (2022-12-09). (i) We have
\[
\sum_{k=1}^{\infty} \frac{(2k\binom{k}{5})^5}{(-2^{20})^k} ((820k^2 + 180k + 13)(11H_{2k}^{(2)} - 3H_k^{(2)}) + 43) = -\frac{1}{3}.
\]

(4.24)

(ii) Let \(p > 3\) be a prime. Then
\[
\sum_{k=1}^{p-1} \frac{(2k\binom{k}{5})^5}{(-2^{20})^k} ((820k^2 + 180k + 13)(11H_{2k}^{(2)} - 3H_k^{(2)}) + 43) \equiv -\frac{77}{6} p^3 B_{p-3} \pmod{p^4},
\]

(4.25)

and
\[
\sum_{k=1}^{(p-1)/2} \frac{(2k\binom{k}{5})^5}{(-2^{20})^k} ((820k^2 + 180k + 13)(11H_{2k}^{(2)} - 3H_k^{(2)}) + 43) \equiv -\frac{11}{4} p H_{p-1} \pmod{p^5}
\]

(4.26)

if \(p > 5\).

The following two conjectures are motivated by the known identity
\[
\sum_{k=0}^{\infty} \frac{(74k^2 + 27k + 3)(2k\binom{k}{5})^4(3k\binom{k}{5})}{4096^k} = \frac{48}{\pi^2}
\]
(cf. [15]).
Conjecture 4.11 (2022-12-09). (i) We have
\[ \sum_{k=0}^{\infty} \frac{(\binom{2k}{k}^4 \binom{3k}{k}^4)}{4096^k} \left( (74k^2 + 27k + 3)H_{2k} - 17k - 3 \right) = 0 \] (4.27)
and
\[ \sum_{k=0}^{\infty} \frac{(\binom{2k}{k}^4 \binom{3k}{k}^4)}{4096^k} \left( (74k^2 + 27k + 3)(51H_{3k} + 250H_{2k} - 153H_k) + 15 \right) \\
= \frac{9792 \log 2}{\pi^2}. \] (4.28)

(ii) For any odd prime \( p \), we have
\[ \sum_{k=0}^{p-1} \frac{(\binom{2k}{k}^4 \binom{3k}{k}^4)}{4096^k} \left( (74k^2 + 27k + 3)H_{2k} - 17k - 3 \right) \equiv -3p + 7p^4B_{p-3} \pmod{p^5}, \] (4.29)

and
\[ \sum_{k=0}^{p-1} \frac{(\binom{2k}{k}^4 \binom{3k}{k}^4)}{4096^k} \left( (74k^2 + 27k + 3)(51H_{3k} + 250H_{2k} - 153H_k) + 15 \right) \\
\equiv 15p + 612p^2g_p(2) - 306p^3q_p(2)^2 \pmod{p^4}. \] (4.30)

Conjecture 4.12 (2022-12-09). (i) We have
\[ \sum_{k=0}^{\infty} \frac{(\binom{2k}{k}^4 \binom{3k}{k}^4)}{4096^k} \left( (74k^2 + 27k + 3)(92H_{2k}^{(2)} - 33H_k^{(2)}) + 112 \right) = 160. \] (4.31)

(ii) For any odd prime \( p \), we have
\[ \sum_{k=1}^{p-1} \frac{(\binom{2k}{k}^4 \binom{3k}{k}^4)}{4096^k} \left( (74k^2 + 27k + 3)(92H_{2k}^{(2)} - 33H_k^{(2)}) + 112 \right) \\
\equiv 64p^3B_{p-3} \pmod{p^4}. \] (4.32)

The following two conjectures are motivated by the identity
\[ \sum_{k=0}^{\infty} (120k^2 + 34k + 3) \frac{(\binom{2k}{k}^4 \binom{4k}{2k}^4)}{2^{10k}} = \frac{32}{\pi^2} \] (cf. [14, Identity 10]).

Conjecture 4.13 (2022-12-09). (i) We have
\[ \sum_{k=0}^{\infty} \frac{(\binom{2k}{k}^4 \binom{4k}{2k}^4)}{2^{10k}}(2(120k^2 + 34k + 3)H_{4k} - 16k - 1) = 0 \] (4.33)
and
\[ \sum_{k=0}^{\infty} \frac{(\binom{2k}{k}^4 \binom{4k}{2k}^4)}{2^{10k}}((120k^2 + 34k + 3)(H_{2k} - 2H_k) + 68k + 9) = \frac{128 \log 2}{\pi^2}. \] (4.34)
(ii) Let $p$ be an odd prime. Then
\[
\sum_{k=0}^{p-1} \frac{{2k \choose k}^4 {4k \choose 2k}}{2^{16k}} ((120k^2 + 34k + 3)(H_{2k} - 2H_k) + 68k + 9) \equiv 9p + 12p^2 q_p(2) - 6p^3 q_p(2)^2 \pmod{p^4},
\]
(4.35)
and
\[
\sum_{k=0}^{p-1} \frac{{2k \choose k}^4 {4k \choose 2k}}{2^{16k}} (2(120k^2 + 34k + 3)H_{4k} - 16k - 1) \equiv -p + \frac{77}{6} p^4 B_{p-3} \pmod{p^5}
\]
(4.36)
if $p > 3$.

**Conjecture 4.14** (2022-12-09). (i) We have
\[
\sum_{k=1}^{\infty} \frac{{2k \choose k}^4 {4k \choose 2k}}{2^{16k}} ((120k^2 + 34k + 3)(23H_{2k}^{(2)} - 7H_k^{(2)}) + 24) = \frac{16}{3}.
\]
(4.37)
(ii) Let $p$ be an odd prime. Then
\[
\sum_{k=1}^{p-1} \frac{{2k \choose k}^4 {4k \choose 2k}}{2^{16k}} ((120k^2 + 34k + 3)(23H_{2k}^{(2)} - 7H_k^{(2)}) + 24) \equiv -p + \frac{77}{6} p^4 B_{p-3} \pmod{p^5},
\]
(4.38)
and
\[
\sum_{k=1}^{(p-1)/2} \frac{{2k \choose k}^4 {4k \choose 2k}}{2^{16k}} ((120k^2 + 34k + 3)(23H_{2k}^{(2)} - 7H_k^{(2)}) + 24) \equiv -23pH_{p-1} \pmod{p^5}
\]
(4.39)
if $p \neq 5$.

**Conjecture 4.15** (2022-12-09). (i) We have
\[
\sum_{k=0}^{\infty} \frac{{2k \choose k}^3 {3k \choose k}^3 {4k \choose 2k}}{(-24^4)^k} ((252k^2 + 63k + 5)(4H_{4k} + 3H_{3k} - 7H_k) + 504k + 63) = \frac{192 \log 24}{\pi^2}.
\]
(4.40)
(ii) For any prime $p > 3$, we have
\[
\sum_{k=0}^{p-1} \frac{{2k \choose k}^3 {3k \choose k}^3 {4k \choose 2k}}{(-24^4)^k} ((252k^2 + 63k + 5)(4H_{4k} + 3H_{3k} - 7H_k) + 504k + 63) \equiv 63p + 5p^2 q_p(24^4) - \frac{5}{2} p^3 q_p(24^4)^2 \pmod{p^4}.
\]
(4.41)
Remark 4.2. Conjecture 4.15 is motivated by the identity
\[ \sum_{k=0}^{\infty} (252k^2 + 63k + 5) \frac{(2k)^3 (3k)^2 (6k)}{(-24^k)^k} = \frac{48}{\pi^2} \]
(cf. [10]).

Conjecture 4.16 (2023-01-16). (i) We have
\[ \sum_{k=0}^{\infty} \frac{(2k)^2 (3k)^2 (6k)}{10^{6k}} \left( 3(532k^2 + 126k + 9)(H_{6k} - H_k) + 532k + 63 \right) = 1125 \log 10 \frac{4}{\pi^2} \]
\[ (4.42) \]

(ii) For any odd prime \( p \neq 5 \), we have
\[ \sum_{k=0}^{p-1} \frac{(2k)^2 (3k)^2 (6k)}{10^{6k}} \left( 3(532k^2 + 126k + 9)(H_{6k} - H_k) + 532k + 63 \right) \equiv 63p + \frac{9}{2} p^2 q_p(2^{18}) - \frac{9}{4} p^3 q_p(2^{18})^2 \pmod{p^4}. \]
\[ (4.43) \]

Remark 4.3. Conjecture 4.16 is motivated by the identity
\[ \sum_{k=0}^{\infty} (532k^2 + 126k + 9) \frac{(2k)^2 (3k)^2 (6k)}{10^{6k}} = 375 \frac{4}{\pi^2} \]
(cf. [10]).

Conjecture 4.17 (2023-01-17). (i) We have
\[ \sum_{k=0}^{\infty} \frac{(2k)^2 (3k)^2 (6k)}{(-2^{18})^k} \left( 6(1930k^2 + 549k + 45)(H_{6k} - H_k) + 3860k + 549 \right) = \frac{6912 \log 2}{\pi^2} \]
\[ (4.44) \]

(ii) For any odd prime \( p \), we have
\[ \sum_{k=0}^{p-1} \frac{(2k)^2 (3k)^2 (6k)}{(-2^{18})^k} \left( 6(1930k^2 + 549k + 45)(H_{6k} - H_k) + 3860k + 549 \right) \equiv 549p + 45p^2 q_p(2^{18}) - \frac{45}{2} p^3 q_p(2^{18})^2 \pmod{p^4}. \]
\[ (4.45) \]

Remark 4.4. Conjecture 4.17 is motivated by the identity
\[ \sum_{k=0}^{\infty} (1930k^2 + 549k + 45) \frac{(2k)^2 (3k)^2 (6k)}{(-2^{18})^k} = 384 \frac{4}{\pi^2} \]
(cf. [10]).
Conjecture 4.18 (2023-01-17). (i) We have
\[
\sum_{k=0}^{\infty} \frac{(2k)^2 (3k)^2 (6k)^2}{(-2183653)^k} \left(2(5418k^2 + 693k + 29)(H_{6k} - H_k) + 3612k + 231\right)
\]
\[
= \frac{128\sqrt{5}}{\pi^2} \log(26325).
\]
(4.46)

(ii) For any prime \(p > 5\), we have
\[
\sum_{k=0}^{p-1} \frac{(2k)^2 (3k)^2 (6k)^2}{(-2183653)^k} \left(2(5418k^2 + 693k + 29)(H_{6k} - H_k) + 3612k + 231\right)
\]
\[
\equiv \left(\frac{5}{p}\right) \left(231p + \frac{29}{3} p^2 q_p(2^{18}3^65^3) - \frac{29}{6} p^3 q_p(2^{18}3^65^3)^2\right) \pmod{p^4}.
\]
(4.47)

Remark 4.5. Conjecture 4.18 is motivated by the identity
\[
\sum_{k=0}^{\infty} (5418k^2 + 693k + 29) \frac{(2k)^2 (3k)^2 (6k)^2}{(-2183653)^k} = \frac{128\sqrt{5}}{\pi^2}
\]  
(cf. [10]).

Conjecture 4.19 (2023-01-17). For \(k \in \mathbb{N}\), set
\[
H(k) := 6H_{6k} + 4H_{4k} - 3H_{3k} - 2H_{2k} - 5H_k.
\]

(i) We have
\[
\sum_{k=0}^{\infty} \frac{(2k)^2 (3k)^2 (4k)^2 (6k)^2}{(-22233)^k} ((1640k^2 + 278k + 15)H(k) + 3280k + 278)
\]
\[
= \frac{256}{\sqrt{3\pi^2}} \log(223^3).
\]
(4.48)

(ii) For any prime \(p > 3\), we have
\[
\sum_{k=0}^{p-1} \frac{(2k)^2 (3k)^2 (4k)^2 (6k)^2}{(-22233)^k} ((1640k^2 + 278k + 15)H(k) + 3280k + 278)
\]
\[
\equiv \left(\frac{3}{p}\right) \left(278p + 15p^2 q_p(2223^3) - \frac{15}{2} p^3 q_p(2223^3)^2\right) \pmod{p^4}.
\]
(4.49)

Remark 4.6. Conjecture 4.19 is motivated by the identity
\[
\sum_{k=0}^{\infty} (1640k^2 + 278k + 15) \frac{(2k)^2 (3k)^2 (4k)^2 (6k)^2}{(-22233)^k} = \frac{256}{\sqrt{3\pi^2}}
\]
(cf. [10]).
Conjecture 4.20 (2023-01-17). For $k \in \mathbb{N}$, set
\[ H(k) := 4H_{8k} - 2H_{4k} + H_{2k} - 3H_k. \]

(i) We have
\[
\sum_{k=0}^{\infty} \left( \frac{2k}{k} \right)^3 \left( \frac{4k}{2k} \right)^8 \left( \frac{8k}{4k} \right)^k \left( (1920k^2 + 304k + 15)H(k) + 1920k + 152 \right)
= \frac{56\sqrt{7}}{\pi^2} (9 \log 2 + 2 \log 7).
\]

(ii) For any odd prime $p \neq 7$, we have
\[
\sum_{k=0}^{p-1} \left( \frac{2k}{k} \right)^3 \left( \frac{4k}{2k} \right)^8 \left( \frac{8k}{4k} \right)^k \left( (1920k^2 + 304k + 15)H(k) + 1920k + 152 \right)
\equiv \left( \frac{7}{p} \right) \left( 152p + \frac{15}{2} p^2 q_p (2^{18}7^4) - \frac{15}{4} p^3 q_p (2^{18}7^4)^2 \right) \pmod{p^4}.
\]

Remark 4.7. Conjecture 4.20 is motivated by the identity
\[
\sum_{k=0}^{\infty} (1920k^2 + 304k + 15) \left( \frac{2k}{k} \right)^3 \left( \frac{4k}{2k} \right)^8 \left( \frac{8k}{4k} \right)^k = \frac{56\sqrt{7}}{\pi^2}
\]
(cf. [10]).

Conjecture 4.21 (2022-12-09). (i) We have
\[
\sum_{k=1}^{\infty} \frac{256^k}{k^7 \binom{2k}{k}} \left( (21k^3 - 22k^2 + 8k - 1)(4H_{2k-1}^{(2)} - 5H_{k-1}^{(2)}) - 6k + 2 \right) = \frac{\pi^6}{24}.
\]

(ii) For any odd prime $p$, we have
\[
\sum_{k=0}^{(p-1)/2} \frac{2k}{k^7 \binom{2k}{k}} \left( (21k^3 + 22k^2 + 8k + 1)(4H_{2k}^{(2)} - 5H_k^{(2)}) + 6k + 2 \right)
\equiv 2p \pmod{p^5}.
\]

Remark 4.8. This is motivated by the identity
\[
\sum_{k=1}^{\infty} \frac{(21k^3 - 22k^2 + 8k - 1)256^k}{k^7 \binom{2k}{k}} = \frac{\pi^4}{8}
\]
conjectured by Guillera [13].

The following two conjectures are motivated by the identity
\[
\sum_{k=0}^{\infty} (168k^3 + 76k^2 + 14k + 1) \frac{2k}{2^{20k}} \binom{2k}{k}^7 = \frac{32}{\pi^3}
\]
conjectured by B. Gourevich (cf. [10]).
**Conjecture 4.22** (2022-12-09). (i) We have
\[
\sum_{k=0}^{\infty} \binom{2k}{k}^7 \frac{1}{2^{20k}} \left(7(168k^3 + 76k^2 + 14k + 1)(H_{2k} - H_k) + 252k^2 + 76k + 7\right) = \frac{320 \log 2}{\pi^3}.
\] (4.55)

(ii) For any prime \( p > 5 \), we have
\[
\sum_{k=0}^{(p-1)/2} \binom{2k}{k}^7 \frac{1}{2^{20k}} \left(7(168k^3 + 76k^2 + 14k + 1)(H_{2k} - H_k) + 252k^2 + 76k + 7\right)
\equiv \left(-\frac{1}{p}\right) \left(7p^2 + 10p^3 q_p(2) - 5p^4 q_p(2)^2 + \frac{10}{3} p^5 q_p(2)^3 - \frac{5}{2} p^6 q_p(2)^4\right) \pmod{p^7}.
\] (4.56)

**Conjecture 4.23** (2022-12-09). (i) We have
\[
\sum_{k=0}^{\infty} \binom{2k}{k}^7 \frac{1}{2^{20k}} \left((168k^3 + 76k^2 + 14k + 1)(16H_{2k}^{(2)} - 5H_k^{(2)}) + 8(6k + 1)\right) = \frac{80}{3\pi}.
\] (4.57)

(ii) For any prime \( p > 5 \), we have
\[
\sum_{k=0}^{(p-1)/2} \binom{2k}{k}^7 \frac{1}{2^{20k}} \left((168k^3 + 76k^2 + 14k + 1)(16H_{2k}^{(2)} - 5H_k^{(2)}) + 8(6k + 1)\right)
\equiv \left(-\frac{1}{p}\right) 8p \pmod{p^6}.
\] (4.58)

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