Dilatonic gravity near two dimensions as a string theory

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Abstract

Using the renormalization-group formalism, a sigma model of a special type—in which the metric and the dilaton depend explicitly on one of the string coordinates only—is investigated near two dimensions. It is seen that dilatonic gravity coupled to $N$ scalar fields can be expressed in this form, using a string parametrization, and that it possesses the usual UV fixed point. However, in this stringy parametrization of the theory the fixed point for the scalar-dilaton coupling turns out to be trivial, while the fixed point for the gravitational coupling remains the same as in previous studies being, in particular, non-trivial.
1 Introduction

Having its origin in work by Weinberg [1], where he proposed a program that should lead to the realization of an asymptotically safe quantum gravity (QG), a strong belief exists nowadays that a consistent theory of QG might be actually constructed by means of a process of analytical continuation from two to four dimensions. Further work in this direction has shown however (see [1] and references therein), that $(2 + \epsilon)$-dimensional Einsteinian gravity ought to be probably ruled out from this program, as a consequence of its non-renormalizability at the two-loop level (see the paper by Jack and Jones in Ref. [1]). Recently, there has been the suggestion [2]-[4] to study dilatonic gravity (with matter) in $2 + \epsilon$ dimensions. This theory has a smooth behavior in the limit $\epsilon \to 0$ and is renormalizable near two dimensions. The existence of a non-trivial ultraviolet fixed point (a saddle point actually) has been observed in Refs. [3]-[5] for different versions of dilatonic gravity with matter near two dimensions. However, in dilatonic gravity the beta-functions depend explicitly on the background dilaton field, as in string theory [5, 6]. Both are defined through the use of the standard off-shell effective action and, as a result, they are dependent on the gauge and also on the field parametrization. Hence, the position of the fixed point in dilatonic gravity is gauge dependent too [3]. It is interesting to study this issue further, and to try to ascertain if dilatonic gravity is really an asymptotically safe theory, e.g. of the kind mentioned above.

In this letter we shall study dilatonic gravity with scalar matter near two dimensions. We shall use a parametrization of a special kind (a stringy parametrization), in which dilatonic gravity is represented under the form of the standard sigma model [7, 8]. Then, the renormalization of this sigma model near two dimensions will be discussed and the corresponding beta-functions will be obtained. By studying dilatonic gravity in $2 + \epsilon$ dimensions in this stringy parametrization we will find that an UV non-trivial fixed point for the gravitational coupling constant appears which is the same as the one obtained in Refs. [2]-[4]. However, the fixed point for the dilaton-scalar coupling will turn out to be a trivial one, contrary to the situation that has been observed in Refs. [2]-[4]. To finish, we will briefly comment on this original new result.
2 A sigma-model of a special type in $2 + \epsilon$ dimensions

Our starting point will be the standard $\sigma$-model action corresponding to string theory

$$S = \int d^d x \sqrt{-g} \left[ \frac{1}{2} G_{ij}(X) g^{\mu \nu} \partial_\mu X^i \partial_\nu X^j + \frac{1}{16\pi G} R(\phi) \right],$$

where $i, j = 1, \ldots, D$, $g^{\mu \nu}(x)$ ($\mu, \nu = 1, \ldots, d, d = 2 + \epsilon$) is the $(2 + \epsilon)$-dimensional metric, $R$ the corresponding curvature, and $\phi(X)$ is the dilaton. In the standard sigma-model approach to string theory it is $d = 2$, and in the conformal gauge the one-loop effective action is given by

$$\Gamma_{\text{div}} = \frac{1}{2\pi\epsilon} \int d^2 x \sqrt{-g} \left[ \frac{1}{2} \beta_{ij}(X) g^{\mu \nu} \partial_\mu X^i \partial_\nu X^j + \beta_{\phi}(X) R \right],$$

where

$$\beta_{ij}(X) = R_{ij}, \quad \beta_{\phi}(X) = \frac{26 - D}{12} - 8\pi \frac{1}{G} \Box^{(D)}(X) \phi,$$

Notice that the beta-functions of the above sigma model (which are not the RG $\beta$-functions in our discussion) are generically different from the Weyl anomaly coefficients, and also, that the loop-expansion parameter in our analysis is $G$.

We now use the fact that the theory for $d = 2 + \epsilon$ has a smooth limit for $\epsilon \to 0$. As has been shown, there are no problems when $\epsilon \to 0$ in this theory, unlike what happens in the case of Einstein’s gravity near two dimensions. In other words, one can use the results of the calculation of the effective action at exactly-two dimensions, in order to obtain the one-loop effective action near two dimensions, as it was the case with dilatonic gravity. Then, the renormalization of the theory can be carried out in close analogy with the case of dilatonic gravity.

For simplicity we will consider the situation where the sigma-model under study is such that $G_{ij}$ and $\Gamma_{\text{div}}$ (i.e., $\beta_{\phi}$ and $\beta_{ij}$) depend explicitly on a single field $\varphi \in \{X^i\}$ only. The dilaton in $\Gamma_{\text{div}}$ is also conveniently parametrized as $\phi(X) \to e^{-2\phi(X)}$ and is to be considered as depending only on the field $\varphi$, so that in the end it is always possible to write $\phi(X) = e^{-2\varphi}$.

The renormalization of the fields can be done as in dilatonic gravity, that is

$$g_{0\mu\nu} = g_{\mu\nu} e^{-2\Lambda(\varphi)} \quad (\Lambda(0) = 0),$$

$$\varphi_0 = \varphi + f(\varphi) \quad (f(0) = 0),$$

$$G_{0ij} = G_d(\varphi) Z_{ij}(\varphi),$$

where $Z_{ij} = \delta_{ij} + \tilde{Z}_{ij}$. In this case the bare action $S_0$ is given by

$$S_0 = \int d^d x \sqrt{-g} \left\{ \frac{1}{16\pi G_0} R e^{-2\varphi - \epsilon\Lambda - 2f} \right\}.$$
\[ + \frac{\epsilon + 1}{16\pi G_0} \left[ 4\Lambda' + \epsilon(\Lambda')^2 + 4f'\Lambda' \right] e^{-2\phi - \epsilon\Lambda - 2f} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \\
+ \frac{1}{2} G_{ij} Z_{ij}(\varphi) e^{-\epsilon\Lambda} g^{\mu\nu} \partial_\mu X^i \partial_\nu X^j \right], \]

(5)

where \( \Lambda' = \partial \Lambda / \partial \varphi \). One has to compare (5) with \( S_0 = S + S_{\text{counter}} \), where \( S_{\text{counter}} = -\mu \epsilon \Gamma_{\text{div}} \).

By just matching corresponding terms, we get

\[
\frac{1}{16\pi G_0} e^{-2\phi - \epsilon\Lambda - 2f} = \mu \epsilon \left( \frac{1}{16\pi G} e^{-2\phi} - \frac{1}{2\pi \epsilon} \beta_{\phi}(\varphi) \right),
\]

\[
\frac{\epsilon + 1}{16\pi G_0} \left[ 4\Lambda' + \epsilon(\Lambda')^2 + 4f'\Lambda' \right] e^{-2\phi - \epsilon\Lambda - 2f} + \frac{1}{2} G_{ij} Z_{ij}(\varphi) e^{-\epsilon\Lambda} = \frac{1}{2} G_{\varphi \varphi} - \frac{\mu \epsilon}{4\pi \epsilon} \beta_{\varphi \varphi},
\]

\[
G_{ij} Z_{ij}(\varphi) e^{-\epsilon\Lambda} = G_{ij} - \frac{\mu \epsilon}{2\pi \epsilon} \beta_{ij},
\]

(6)

where in the last expression the case \( i = j = \varphi \) is excluded and the index \( i \) corresponding to \( \varphi \in \{X^i\} \) is also denoted as \( \varphi \).

We will now take into account the fact that \( G \) is the loop expansion parameter, and that \( f, \Lambda \sim O(G) \). Dropping terms of higher order in \( G \) and following the procedure of Ref. [2], Eqs. (6) can be written as

\[
\frac{1}{16\pi G_0} e^{-2\phi - \epsilon\Lambda(\varphi) - 2f(\varphi)} = \mu \epsilon \left( \frac{1}{16\pi G} e^{-2\phi} - \frac{1}{2\pi \epsilon} \beta_{\phi}(\varphi) \right),
\]

\[
\frac{\epsilon + 1}{4\pi G_0} \Lambda'(\varphi) e^{-2\phi} + \frac{1}{2} G_{\varphi \varphi} Z_{\varphi \varphi}(\varphi) e^{-\epsilon\Lambda(\varphi)} = \frac{1}{2} G_{\varphi \varphi} - \frac{\mu \epsilon}{4\pi \epsilon} \beta_{\varphi \varphi},
\]

\[
G_{ij} Z_{ij}(\varphi) e^{-\epsilon\Lambda(\varphi)} = G_{ij} - \frac{\mu \epsilon}{2\pi \epsilon} \beta_{ij}(\varphi),
\]

(7)

where expansion of the exponential functions up to linear terms in \( G \) is to be understood.

From Eqs. (7) one can get explicit relations between the non-renormalized and the renormalized parameters. In particular,

\[
\frac{1}{G_0} = \mu \epsilon \left( \frac{1}{G} - \frac{8}{\epsilon \beta_{\phi}(0)} \right).
\]

(8)

For the rest of the parameters of the theory the renormalization can be performed in different ways (that turn out to be ambiguous). Hence, from now on we will restrict our considerations to an even smaller class of sigma models, where \( G_{\varphi \varphi} - G_{\varphi \varphi} Z_{\varphi \varphi} = 0 \) and \( \beta_{ij} = 0 \) (notice that this case is consistent with the choice of \( G \) as the loop expansion parameter). Then \( \Lambda(\varphi) \) can be easily found, from the second of Eqs. (4), to be:

\[
\Lambda(\varphi) = - \frac{G}{\epsilon + 1} \int_0^\varphi e^{2\phi'} \frac{\beta_{\varphi \varphi}(\varphi')}{\epsilon} d\phi'.
\]

(9)
Substituting (8) and (9) into the first and into the last of Eqs. (7) one finds, to leading order in $G$,

$$f(\varphi) = \frac{4G\beta(0)}{\epsilon} \left( e^{2\varphi} - 1 \right) + \frac{G}{2(\epsilon + 1)} \int_0^\varphi e^{2\varphi'} \beta_{\varphi\varphi}(\varphi') \, d\varphi' + \frac{4G}{\epsilon} e^{2\varphi} \left[ \beta(\varphi) - \beta(0) \right],$$

$$\tilde{Z}_j^k(\varphi) = -\delta_j^k \frac{G}{\epsilon + 1} \int_0^\varphi e^{2\varphi'} \beta_{\varphi\varphi}(\varphi') \, d\varphi'. \quad (10)$$

We can now turn to the evaluation of the beta functions of the sigma model under discussion. We get for $G$

$$\beta_G = \mu \frac{\partial G}{\partial \mu} = \epsilon G - 8G^2 \beta(0). \quad (11)$$

This beta function comes from the renormalization of the dilaton in the original string theory. In the calculation of the beta-function for the metric we will be interested in its dilatonic (i.e. $\varphi$-) dependence only. In this case it is enough to consider the renormalization of $G_{ij}$ which depends on $\varphi_0$ (to leading order in $G$):

$$G_{0ij}(\varphi_0) = G_{ij}(\varphi) - \frac{G_{ij}(\varphi)G}{\epsilon + 1} \int_0^\varphi e^{2\varphi'} \beta_{\varphi\varphi}(\varphi') \, d\varphi'$$

$$= G_{ij}(\varphi_0) - f(\varphi_0)G_{ij}'(\varphi_0) - \frac{G_{ij}(\varphi)G}{\epsilon + 1} \int_0^\varphi e^{2\varphi'} \beta_{\varphi\varphi}(\varphi') \, d\varphi'. \quad (12)$$

Then,

$$\beta_{G_{ij}} = \mu \frac{\partial G_{ij}(\varphi_0)}{\partial \mu} = \epsilon f(\varphi_0)G_{ij}'(\varphi_0) + \frac{\epsilon G_{ij}(\varphi)G}{\epsilon + 1} \int_0^\varphi e^{2\varphi'} \beta_{\varphi\varphi}(\varphi') \, d\varphi'. \quad (13)$$

With this beta function we terminate here the construction of the string RG near two dimensions. We will turn now to consider a physically interesting example.

### 3 Dilatonic gravity near two dimensions as a string theory

Dilatonic gravity provides an interesting example of a string theory of the above type near two dimensions. The corresponding Lagrangian can be written as

$$L = C(\varphi)R - \frac{1}{2} e^{-2\varphi(\varphi)} \partial_\mu \chi_a \partial^\nu \chi^a, \quad (14)$$

where $\varphi$ is the dilaton and $\chi_a$ are $N$ scalar fields. This is a popular toy model for the study of quantum gravity and black hole physics. Now, using the method of Ref. [10] in the conformal gauge

$$g_{\mu\nu} = e^{2\sigma} \bar{g}_{\mu\nu}, \quad (15)$$
one can represent the action of dilatonic gravity as a sigma model, in the form

\[ S = \int d^2x \sqrt{-g} \left[ \frac{1}{2} G_{ij}(X) \tilde{g}^{\mu\nu} \partial_\mu X^i \partial_\nu X^j + R\phi(X) \right], \tag{16} \]

where

\[
G_{ij} = \begin{pmatrix}
0 & 2C'(\varphi) & 0 \\
2C'(\varphi) & 0 & 0 \\
-\cdots & -\cdots & -\cdots \\
0 & 0 & -e^{-2\phi(\varphi)}
\end{pmatrix}, \quad X^i = \{\varphi, \sigma, \chi_a\}, \quad \phi(X) = C(\varphi). \tag{17}
\]

Using Weyl rescaling of the two-dimensional metric one can parametrize the initial Lagrangian (14) so that

\[ C(\varphi) = e^{-2\varphi}/(16\pi G). \]

One can easily realize that the sigma model (16) is of the same type as the one described in the previous section. Turning to the construction of the geometry of such a sigma model, we obtain the stringy \( \beta \)-functions:

\[
\beta_\varphi(\varphi) \equiv \beta_\varphi(0) = \frac{24 - N}{12}, \\
\beta_{\varphi\varphi} = R_{\varphi\varphi} = N \left[ (\varphi')^2(\varphi) - \varphi''(\varphi) - 2\varphi'(\varphi) \right], \tag{18}
\]

and \( \beta_{ij} = 0 \) for the rest of the indexes \( i, j \). Notice that \( \Gamma_{\text{div}} \) in this model has been calculated also in Refs. [11], [3] and [2], in different gauges. The results in the three references are all different from one another, and also different from Eq. (18) (note, however, the coincidence of (18) with the result in Ref. [3] in the gauge \( \alpha = 0 \)). This should not be considered as strange, taking into account the fact that \( \Gamma_{\text{div}} \) is gauge dependent off-shell and that the effective action is also parametrization dependent. Once we use the classical equations of motion, all the calculations ([11, 3, 2] and (18)) lead to the same effective action on shell, what can be checked easily (see the Appendix).

Substituting (18) into (11) and (13), we get

\[ \beta_G = \epsilon G - \frac{2(24 - N)}{3} G^2. \tag{19} \]

Taking into account that the non-diagonal terms \( G_{\varphi\sigma} \) should not be considered in (13), and choosing the particular Ansatz \( \phi(\varphi) = \lambda \varphi \), one finds (by substituting (18) and (14) into (13))

\[ \beta_{\phi\varphi} = G(1 - e^{2\varphi}) \left[ \frac{24 - n}{3} 2\lambda + \frac{N\epsilon}{2(\epsilon + 1)} \lambda(\lambda - 1)(\lambda - 2) \right]. \tag{20} \]

As the one-loop effective action (2) is different from the result one obtains in covariant gauges, Eq. (20) differs slightly from the corresponding results in Refs. [2, 3].
Solving Eq. (19) we find that $G = 0$ is an IR stable fixed point and that

$$G^* = \frac{3\epsilon}{2(24 - n)}, \quad \epsilon > 0, \quad N < 24,$$

is a non-trivial ultraviolet fixed point. Both $\beta_G$ and the corresponding fixed point $G^*$ are gauge parametrization independent, as they should be.

From (20) we find that, due to the stringy form of $\Gamma_{\text{div}}$, the non-trivial perturbative fixed point of order $\lambda^* \sim \epsilon$ found in Ref. [2] (it had been pointed out already [3] that the position of this fixed point depends very much on the gauge) disappears completely, and only the solution $\lambda^* = 0$ remains. However, the other imaginary ultraviolet fixed points, of order $\lambda^* \sim \epsilon^{-1/2}$, mentioned in Ref. [2] show up here as well:

$$\lambda_{1,2} = \pm 2i \sqrt{\frac{24 - N}{3N\epsilon}} + \mathcal{O}(\epsilon^0).$$

We have thus shown that in the string-like formalism the non-trivial ultraviolet fixed point $\lambda^* \sim \epsilon$ of dilatonic gravity (which was observed in the other formalism) does not appear in the present model, which yields simply the ‘trivial’ solution $\lambda^* = 0$.

4 Discussion

Probably the main result that can be extracted from the preceding study of a special type of sigma model near two dimensions is the one that comes about from the consideration of the specific example of dilatonic gravity in $d = 2 + \epsilon$ using the stringy parametrization. Namely, the fact that the position of the UV fixed point for the scalar dilaton coupling has collapsed to zero. In this way we see that the non-trivial fixed point corresponding to this coupling constant —which had been found in previous works by different authors (in other parametrizations)— turns out to be, in the end, a trivial fixed point. At the same time, the position of the fixed point corresponding to the gravitational coupling constant does not change. In any case, the nature of the $\lambda$ fixed point does not change either —it continues to be a UV saddle point as in Refs. [2]-[4].

It would be of interest to develop a RG formalism for the study of more general sigma models than the class of sigma models considered in the present work —near two dimensions and in a unified stringy way. Then one could hope to understand which type of fixed point for the scalar-dilaton coupling (the trivial or the non-trivial one) turns out to be the most acceptable physically. In particular, adopting the point of view that string theory may prove
to be the fundamental ‘theory of everything’, we believe that the trivial fixed point for the scalar-dilaton coupling is in fact the physical UV fixed point.

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A Appendix

In this Appendix we will show that the one-loop off-shell effective action —which depends both on the gauge and on the parametrization— in dilatonic gravity [14] leads to the same on-shell result. As it follows from [18], in the stringy parametrization of dilatonic gravity the one-loop effective action is given by

$$\Gamma_{\text{div}} = \frac{1}{2\pi \epsilon} \int d^2 x \sqrt{-g} \left\{ \frac{24 - N}{12} R + \frac{N}{2} \left[ (\phi')^2 (\phi) - \phi'' (\phi) - 2 \phi' (\phi) \right] g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi \right\}.$$  \hspace{1cm} (A.1)

At the same time, in the covariant gauge of Refs. [11, 3], the result of the calculation of $\Gamma_{\text{div}}$ in the same theory is

$$\Gamma_{\text{div}} = \frac{1}{2\pi \epsilon} \int d^2 x \sqrt{-g} \left\{ \frac{24 - N}{12} R - \frac{1}{2} \left[ 8 - N (\phi')^2 (\phi) \right] g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi \right\}.$$  \hspace{1cm} (A.2)

As we clearly see, these two off-shell effective actions differ because of their gauge and parametrization dependences.

Working, for simplicity, in the conformal gauge ($\bar{g}_{\mu \nu} = \eta_{\mu \nu}$), we use with (A.1) the classical equations of motion:

$$\delta L \over\delta \phi = \frac{1}{16\pi G} e^{-2\phi} (-2\Delta \sigma) - \frac{1}{2} \phi' e^{-2\phi} (\partial \chi_a)^2 = 0,$$

$$\delta L \over\delta \sigma = \Delta e^{-2\phi} = 0.$$  \hspace{1cm} (A.3)

From the second of Eqs. (A.3) we find that the term $(\phi'' + 2\phi') g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi$ in (A.1) is a boundary term and can therefore be dropped out. Moreover, the term $-4 (\partial \phi)^2$ in (A.2) is also a boundary term. As a result, all (A.1), (A.2) and the corresponding $\Gamma_{\text{div}}$ of Ref. [4] lead to a unique result on-shell (after dropping boundary terms), which is

$$\Gamma_{\text{on-shell}} \over\text{div} = \frac{1}{2\pi \epsilon} \int d^2 x \sqrt{-g} \frac{N}{2} (\phi')^2 (\phi) (\partial \phi)^2.$$  \hspace{1cm} (A.4)

Hence, the corresponding $S$-matrix is in fact both gauge and parametrization independent, as it should be from general considerations.
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