ON THE CONSTRUCTION OF INTEGRABLE DILUTE A-D-E MODELS.

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Abstract

We give an integrable extension of the lattice models recently considered by I. Kostov in his study of strings in discrete space. These models are IRF models with spins variables living in any connected graph, the vertex model underlying these models being the Izergin-Korepin model. When the graph is taken to be the simply laced Dynkin Diagrams, it is conjectured that these models possess critical regimes which are the dilute phase of SOS models of ADE type.

1 Introduction

During these last few years there has been an incredible amount of work studying the connections between integrable lattice models and conformal field theory. One particularly fruitful but difficult problem consists of finding in each universality class of critical phenomena an integrable spin model. In the case where the central charge satisfies \( c < 1 \) this problem is almost completely understood. The aim of this letter is to provide the still missing unitary integrable lattice model having \( c < 1 \).

There exists a classification of unitary conformal field theory of central charge \( c < 1 \) [CIZ] based on a complete list of modular invariant partition functions. The central charge of these CFT are equal to \( c = 1 - 6\frac{(p-p')^2}{pp'} \) with \( |p - p'| = 1 \). The modular invariants are classified by a pair \((A_{p-1}, G_{p'}-1)\) with \( G \) being a simple Lie algebra of \( A, D, E \) type, \((p, p')\) indexing the largest exponents of the corresponding Lie algebras.

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V. Pasquier has shown [Pa1] that it is possible to generalize the construction of RSOS models of Andrews-Baxter-Forrester [ABF] to any simply laced Dynkin Diagram. The definition of these models is as follows. Let \((C_{ij})\) denote a symmetric matrix with coefficients equal to 0 or 1, this matrix define a graph: points of the graph are the entries of the matrix and two points \(i, j\) are connected by a line of the graph if the matrix element \(C_{ij}\) is different of zero. We will suppose that the corresponding graph \(G\) is connected. In that case the Perron Frobenius theorem asserts that there exists a greatest positive eigenvalue \(\beta\) of \(C\) which is non degenerate and that the corresponding eigenstate can be chosen such that all is components \((S_i)\) are positive. The spin variables of this model are points of the graph \(G\), the Boltzmann weights associated to a configuration of spins surrounding a plaquette is defined by:

\[
W(a, b, c, d) = \delta_{b,d} + \frac{X (S_b S_d)^{1/2}}{S_a} \delta_{a,c} 
\]

(1)

with the restriction that \((a, b)\) is a couple of connected points on the graph, as well as \((b, c), (c, d)\) and \((d, a)\). We can represent the Boltzmann weight by the following picture

\[
W(a, b, c, d) = \begin{array}{c}
\begin{array}{c}
\text{a} \\
\begin{array}{c}
\text{b}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{d}
\end{array}
\begin{array}{c}
\text{c}
\end{array}
\end{array}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{a} \\
\begin{array}{c}
\text{b}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{d}
\end{array}
\begin{array}{c}
\text{c}
\end{array}
\end{array}
\end{array}
\end{array}
\]

(2)

where the lines inside the square separate connected spins on the graph \(G\).

These weights satisfy Yang-Baxter equation, i.e the equation:

\[
\sum_g W''(a_1, a_2, a_3, g)W'(g, a_3, a_4, a_5)W(a_1, g, a_5, a_6) = \]

\[
\sum_g W(a_2, a_3, a_4, g)W'(a_1, a_2, g, a_6)W''(a_6, g, a_4, a_5)
\]

(3)

provided that the relation \(X'(1 - \frac{1}{\beta} XX'') = X + XX''\) is satisfied.

When \(X = \beta\) this model is isotropic and is in a critical regime when \(\beta \leq 2\). This last constraint implies that the graph is a Dynkin Diagram of ADE type or an extended Dynkin diagram \(\hat{A}\hat{D}\hat{E}\). It can be shown [Pa2] that the partition function on a torus of the continuum limit of these models are the modular invariants of \((A_p - 1, G_{p' - 1})\) type with \(p < p'\). There are strong evidence that the statistical models associated to the other branch \(p > p'\) can be described by a dilute version of these ADE SOS models [Ni]. An important step has been pushed forward by I.Kostov in [Ko1]. In this work he defines a class of statistical models where spins are points of a graph \(G\) and the Boltzmann weights are taken to be:
$W(a, b, c, d) = \begin{array}{c}
\begin{array}{cc}
a & d \\
b & c \\
\end{array}
\end{array} +
\begin{array}{c}
\begin{array}{cc}
a & d \\
b & c \\
\end{array}
\end{array} +
\begin{array}{c}
\begin{array}{cc}
a & d \\
b & c \\
\end{array}
\end{array}$ (4)

with

$\begin{array}{c}
\begin{array}{cc}
a & d \\
b & c \\
\end{array}
\end{array} = \frac{1}{T} \delta_{b,d}C_{ab}C_{bc}$, 
$\begin{array}{c}
\begin{array}{cc}
a & d \\
b & c \\
\end{array}
\end{array} = \frac{1}{T} \delta_{a,c}C_{ab}C_{ad}(\frac{S_bS_d}{S_aS_c})^{1/2}$ (5)

$\begin{array}{c}
\begin{array}{cc}
a & d \\
b & c \\
\end{array}
\end{array} = \frac{1}{T} \delta_{b,c}C_{ab}$, 
$\begin{array}{c}
\begin{array}{cc}
a & d \\
b & c \\
\end{array}
\end{array} = \frac{1}{T} \delta_{a,b}C_{bc}$ (6)

$\begin{array}{c}
\begin{array}{cc}
a & d \\
b & c \\
\end{array}
\end{array} = \frac{1}{T} \delta_{a,d}C_{ad}(\frac{S_b}{S_a})^{1/2}$, 
$\begin{array}{c}
\begin{array}{cc}
a & d \\
b & c \\
\end{array}
\end{array} = \frac{1}{T} \delta_{a,b}C_{cd}(\frac{S_d}{S_a})^{1/2}$ (7)

$\begin{array}{c}
\begin{array}{cc}
a & d \\
b & c \\
\end{array}
\end{array} = \delta_{a,b} \delta_{b,c} \delta_{c,d}$ (8)
where $T$ is the temperature of the statistical system. When $T$ goes to zero the Boltzmann weights are those of the type described by (1), when $T$ goes to infinity the configuration of spins freeze to a constant and there are no long range correlations. He expects that there exists a critical temperature $T_c$ at which there is a phase transition between these two phases. At this point there is a new critical regime, which is called the dilute phase [Ni].

He argues that the partition function of these models at $T_c$ is indeed the modular invariants $(A_{p-1}, G_{p'-1})$ with $p > p'$. Unfortunately the Boltzmann weights are not written in a way that could satisfy Yang-Baxter equation, one problem being that there is no spectral parameter. In this letter we describe a family of integrable models such that the Boltzmann weights at the isotropic point is of the type (1).

2 Description of the solution

We have searched a solution of Yang Baxter equation of the type:

$$W(a, b, c, d) = A\delta_{ab}\delta_{bc}\delta_{cd} + C_1\delta_{b,d}C_{ab}C_{bc} + C_2\delta_{a,c}C_{ab}C_{ad}\left(\frac{S_bS_d}{S_aS_c}\right)^{1/2} +$$

$$B_1\delta_{b,c}\delta_{c,d}C_{ab} + B_3\delta_{a,b}\delta_{a,d}C_{bc} + B_2\delta_{a,d}\delta_{d,c}C_{ab}\left(\frac{S_b}{S_a}\right)^{1/2} + B_4\delta_{a,b}\delta_{b,c}C_{cd}\left(\frac{S_d}{S_a}\right)^{1/2} +$$

$$D_1\delta_{a,b}\delta_{cd}C_{bc} + D_2\delta_{a,d}\delta_{b,c}C_{ab}$$

(9)

The coefficients $A$, $B_1$, $B_2$, $B_3$, $B_4$, $C_1$, $C_2$, $D_1$, $D_2$ being unknown variables. We have taken a slighter generalisation of the weights (5-8) by allowing new types of Boltzmann weights ($D_1$ and $D_2$) otherwise there is no interesting solution. Notice that these Boltzmann weights do not depend on the spin variables. After having written the whole family of Yang-Baxter equations, one discovers that all factors depending on $S_i$ disappear and combine nicely leaving sometimes a coefficient $\beta$. The system of algebraic equations one obtains consits in 49 equations. The easiest to handle are the four

$$B_1B_3'B_1'' = B_3B_1'B_3''$$

(10)

$$B_2B_4'B_1'' = B_4B_1'B_4''$$

(11)

$$B_3B_4'B_2'' = B_1B_2'B_4''$$

(12)

$$B_4B_3'B_2'' = B_2B_1'B_4''$$

(13)

from which one deduces that $B_1 = B_3$ and $B_2 = \Gamma B_4$ with $\Gamma$ constant.

The whole set of algebraic equations reduces then to :

$$-B_1C_1'B_1'' + C_1B_1'C_1'' - D_2B_1'D_1'' = 0$$

(14)
\[-B_2 D'_1 B'' + D_2 B'_1 C''_1 - C_2 B'_2 D''_1 = 0 \]  
\[-D_1 C'_1 B''_1 + C_1 D'_1 B''_1 - B_1 D'_1 D''_1 = 0 \]  
\[-D_2 B'_1 B'' + D_1 D'_1 C''_1 - B_1 C'_1 D''_1 = 0 \]  
\[-C_2 B'_2 B'' + B_2 C'_2 C''_1 - B_2 D'_2 D''_2 = 0 \]  
\[-B_1 C'_2 B''_1 + C_2 D'_1 B''_1 + B_4 B'_2 D''_2 = 0 \]  
\[-D_1 B'_2 B'' + B_1 D'_2 C''_2 + B_1 C'_2 D''_2 = 0 \]  
\[-B_1 D'_1 B'' + D_2 B'_2 C''_2 - C_1 B'_2 D''_1 = 0 \]  
\[-C_1 C'_2 B'' + D_1 D'_1 B'' + B_1 B'_2 C''_2 = 0 \]  
\[D_2 A'B''_1 - AD'_1 B'' + B_4 B'_2 D''_1 + B_1 B'_1 D''_2 = 0 \]  
\[-D_1 B'_2 A'' + B_2 D'_2 B''_1 - B_1 D'_1 B''_2 + A B'_2 D''_2 = 0 \]  
\[B_1 D'_1 A'' - D_1 B'_1 B''_1 + D_2 B'_1 D''_1 - B_1 A'D''_1 = 0 \]  
\[-B_2 B'_1 B''_1 + C_1 C'_2 C''_1 - B C_2 C'_1 C''_2 - C_2 C'_2 C''_2 - C_2 C'_1 C''_1 - C_1 C'_1 C''_2 = 0 \]  
\[-B_2 B'_1 A'' - C_1 C'_1 B''_2 - C_2 C'_2 B''_2 + B_1 B'_2 C''_1 - B C_2 C'_1 B''_2 = 0 \]  
\[C_1 B'_1 B'' - A B'_1 B'' - B_2 C'_1 C''_2 - B_2 C'_2 C''_2 - B_2 C'_1 C''_1 - C_1 C'_1 C''_2 = 0 \]  
\[-A B'_1 A'' + B_1 A'B''_1 - B_2 B'_2 B''_1 - B_2 C'_2 C''_2 + B_2 D'_1 D''_1 = 0 \]  
\[-B_1 C'_2 B''_1 + B_2 A'B''_2 + B C_2 B'_1 C''_2 + B_1 C'_1 C''_1 + B_1 B'_1 C''_2 = 0 \]  
\[-B_1 B'_2 A'' + A A'B''_2 - D_2 D'_2 B''_2 + B_2 B'_1 C''_2 + B_2 B'_1 C''_1 = 0 \]  
\[-B_2 A'A'' - C_1 B'_1 B''_2 - B C_2 B'_1 B''_2 + A B'_2 B''_1 + B_2 D'_1 D''_1 = 0 \]  

One solves this system through the following steps.

We first find the invariants, i.e. rational functions of the unknowns which are equal for the unprime variables and for the prime variables. These functions will define an algebraic manifold whose points are the spectral parameter.

By eliminating the variables with double primes in the system (14, 15, 16) and (17, 18, 19), we obtain that:

\[\theta_2 = \frac{C_1 C_2 - D_2^2}{B_1 B_2} \]  
\[\Omega_2 = \frac{B_1 D_2}{C_1 B_2} + \theta_2 \frac{D_1}{C_1} \text{ and } \tilde{\Omega}_2 = \frac{B_2 D_2}{C_2 B_1} + \Gamma \theta_2 \frac{D_1}{C_2} \]  

are invariants.

By eliminating the unprime variables in the system (16, 20, 22) and (14, 17, 21), we obtain that:

\[\theta_1 = \frac{C_1 C_2 - D_2^2}{B_1 B_2} \]  
\[\Omega_1 = \frac{B_1 D_1}{C_1 B_2} + \theta_1 \frac{D_2}{C_1} \text{ and } \tilde{\Omega}_1 = \frac{B_2 D_1}{C_2 B_1} + \theta_1 \Gamma \frac{D_2}{C_2} \]
are invariants.

By eliminating the prime variables in the system (15, 18, 20) and (19, 21, 22), we obtain that:

\[
\Omega^2 \Gamma = B^2_2 D^1_2 + \theta^2 \Omega^1 \Gamma = B^2_2 D^1_2 + \Gamma \theta^1 \Omega^1 \Gamma
\]

are invariants.

The only interesting solution corresponds to \( D_1 = D_2 = D \) from which we deduce that:

\[
C_1 C_2 - D^2 = \theta^2 B_1 B_2
\]

\[
\frac{D}{C_1} (\theta + B_1) = \frac{D}{C_2} (\theta + \Gamma^{-1} B_2) = \Omega^2
\]

We can now eliminate the double prime variables in (20, 21, 22) and one must have that

\[
\frac{C^2_2 + D^2 - \Gamma^{-1} B^2_2}{2C_2 D} = \frac{C^2_1 + D^2 - B^2_1}{2DC_1} = \Delta.
\]

As usual one can put \( \Delta = \frac{1}{2} (q^2 + q^{-2}) \) and one is led to parametrize

\[
D = \rho_1 (x_1 - x_1^{-1}) = \rho_2 (x_2 - x_2^{-1})
\]

\[
C_1 = \rho_1 (q^2 x_1 - q^{-2} x_1^{-1})
\]

\[
C_2 = \rho_2 (q^2 x_2 - q^{-2} x_2^{-1})
\]

\[
B_1 = \rho_1 (q^2 - q^{-2}), B_2 = \Gamma^{1/2} \rho_2 (q^2 - q^{-2})
\]

The equations (14...22) are then equivalent to the fact that the product \( x_1 x_2 \) is fixed and equal to a constant \( \lambda \), function of the invariant \( \Delta, \Gamma, \Omega_1 \). Defining \( x = x_1 \) the system (14..22) is now equivalent to the simple relation \( x' = x x'' \). Finally equation (26) implies that \( \lambda = q^{-3} \), and \( \beta = -(q^4 + q^{-4}) \).

A is finally obtained by solving (27).

The final result is the parametrisation of weights up to a constant:

\[
A(x) = q^3 x^2 + q^{-3} x^{-2} + (q + q^{-1})(1 - q^4 - q^{-4})
\]

\[
B_1(x) = B_3(x) = (q^2 - q^{-2})(q^{-3} x^{-1} - q^3 x)
\]

\[
\Gamma^{1/2} B_2(x) = \Gamma^{-1/2} B_4(x) = (x - x^{-1})(q^2 - q^{-2})
\]

\[
C_1(x) = (q^2 x - q^{-2} x^{-1})(q^{-3} x^{-1} - q^3 x)
\]

\[
C_2(x) = (x - x^{-1})(q^{-1} x^{-1} - q x)
\]

\[
D_1(x) = D_2(x) = (x - x^{-1})(q^{-3} x^{-1} - q^3 x).
\]

It is then quite easy to show that Yang-Baxter equations are satisfied using this parametrisation.
Remark: we have discarded the solution corresponding to \( D_1 = D_2 = 0 \) otherwise we would have obtained the ordinary SOS model (1) with an adjacency matrix equal to \( G + 1 \) which clearly has always \( \beta > 2 \) and which do not possess a critical regime.

We have then obtained an integrable IRF model, which we will call diluted, associated to any graph \( G \). This is very reminiscent of the construction of the RSOS models (1) sketched in the introduction. It is well known that the ordinary \( A_n \) models can be obtained by a vertex -IRF transformation from the 6 vertex model. One can ask the same question in the case of these diluted model when associated to \( A_n \) Dynkin diagrams. The vertex model one obtains is the Izergin-Korepin model [IK]. As shown by M.Jimbo [Ji] the R matrix of the Izergin-Korepin model is the R matrix in the fundamental representation (of dimension 3) of the deformation of the Kac-Moody algebra \( U_k(A_2^{(2)}) \). If we set \( k = -q^2 \) in the parametrisation of M.Jimbo we obtain up to a normalization constant the vertex model one can straightforwardly obtain from the Boltzmann weights in the \( A_n \) case we have computed (one just has to get rid of the factor \( S_a \)).

3 Conclusion

Although our study has just been an algebraic one, what remains to be done is the study of the thermodynamical properties of these models. One has to study the critical regimes of these models and give a proof that when the graph \( G \) is taken to be a Dynkin diagram of \( ADE \) type then there exists a critical regime where it is described by the minimal conformal field theories in the branch \( p > p' \). We would then have a one to one correspondance between minimal unitary CFT and a family of lattice integrable models.

The study of the perturbed minimal models by \( \phi_{1,2} \) has shown that the \( S \)-matrix of these fields theories are the RSOS version of the R-matrix of Izergin-Korepin [Sm]. It would be interesting to understand the connection, if any, between these field theories and the lattice models we have built.

From a pure mathematical point of view the ordinary ADE models provide a natural representation of the Temperley-Lieb algebra on the space of paths of any graph \( G \). As shown by V.Jones in [Jo], this algebra is a central tool in the study of the inclusion of hyperfinite factors. Because our construction of integrable models works as well for any graph \( G \) it is rather natural to wonder if these models can have a natural interpretation in terms of inclusion of factors.

Note added:

After the completion of this work we received a preprint of S.O.Warnaar, B.Nienhuis and K.A.Seaton [WNS] where they give the same formula (9) in a different parametrisation. In order to get their weights from (45,50) one has to first take the opposite of \( D \) and then put \( q = i e^{i\lambda} \) and \( x = e^{iu} \).
They have obtained an off critical extension of $A_n$ models, and they give results on the study of the critical regimes of the statistical models defined by (9).

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4 **References**

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