Optimal Intervention in Transportation Networks

Leonardo Cianfanelli \(\text{\textcopyright},\) Member, IEEE, Giacomo Como \(\text{\textcopyright},\) Member, IEEE, Asuman E. Ozdaglar \(\text{\textcopyright},\) and Francesca Parise \(\text{\textcopyright},\) Member, IEEE

Abstract—We study a network design problem (NDP) where the planner aims at selecting the optimal single-link intervention on a transportation network to minimize the travel time under Wardrop equilibrium flows. Our first result is that, if the delay functions are affine and the support of the equilibrium is not modified with interventions, the NDP may be formulated in terms of electrical quantities computed on a related resistor network. In particular, we show that the travel time variation corresponding to an intervention on a given link depends on the effective resistance between the endpoints of the link. We suggest an approach to approximate such an effective resistance by performing only local computation and exploit it to design an efficient algorithm to solve the NDP. We discuss the optimality of this procedure in the limit of infinitely large networks and provide a sufficient condition for its optimality. We then provide numerical simulations, showing that our algorithm achieves good performance even if the equilibrium support varies and the delay functions are nonlinear.

Index Terms—Network analysis and control, network design problems (NDPs), traffic control, transportation networks.

I. INTRODUCTION

Due to the increasing population living in urban areas, many cities are facing the problem of traffic congestion, which leads to increasing levels of pollution and a massive waste of time and money [1]. The problem of mitigating congestion has been tackled in the literature from two main perspectives. One approach is to influence the user behavior by incentive-design mechanisms, for instance, by road tolling [2], [3], [4], [5], [6], [7], information design [8], [9], [10], [11], or lottery rewards [12], to minimize the inefficiencies due to the autonomous uncoordinated decisions of the agents. The second approach is to intervene in the transportation network by building new roads or enlarging the existing ones. The corresponding network design problem (NDP) (i.e., the problem of optimizing the intervention on a transportation network subject to some budget constraints, see, e.g., [13]) is very challenging because of its bilevel nature, i.e., it involves a network intervention optimization problem given the flow distribution for that particular network. We assume that each link of the network is endowed with a delay function and the flow distributes according to a Wardrop equilibrium, taking paths with minimum cost, defined as the sum of the delay functions of the links along the path (see [14] and [15]). A characterization of the Wardrop equilibrium is used to construct the lower level of the bilevel NDP.

In this work, we define an NDP and analyze in detail a special instance of the problem, where the delay functions are affine, and the planner can improve the delay function of a single link. Our objective is to strike a balance between a problem that is simple enough to guarantee tractable analysis, yet rich enough to allow insights for more general classes of NDPs. We then extend the validity of the proposed method by a numerical analysis, showing that good performance is achieved even if the delay functions are nonlinear. For single-link affine NDPs, our first theoretical result provides an analytical characterization of the cost variation (i.e., the total travel time at the equilibrium) corresponding to an intervention on a particular link under a regularity assumption, which states that the set of links carrying positive flow remains unchanged with an intervention. This assumption, which is not new in the traffic equilibrium literature (see, e.g., [16], [17]) leads to a characterization of Wardrop equilibria using a system of linear equations and enables representing single-link interventions as rank-1 perturbations of the system. We show that this assumption is satisfied provided that the total incoming flow to the network is large enough and the network is series–parallel, which may be of independent interest. We exploit the structure of our characterization and the linearity of the delay functions to express the cost variation using the effective resistance of a link (i.e., between the endpoints of the link), defined with respect to a related resistor network, obtained by making the directed transportation network undirected, and assigning a conductance to each link based on the delay function of the link. Computing the effective resistance of a single link requires the solution of a linear system whose dimension scales with the network size (we indistinctly refer to the network size as the cardinality of the node and the link sets, implicitly assuming that transportation

Manuscript received 18 March 2022; revised 17 October 2022; accepted 11 February 2023. Date of publication 22 February 2023; date of current version 5 December 2023. This work was supported in part by the Dipartimento di Scienze Matematiche and the MIUR-funded Progetto di Eccellenza, in part by the MIUR Research Project PRIN 2017 “Advanced Network Control of Future Smart Grids” (http://vectors.dieti.unina.it), and in part by the Compagnia di San Paolo through a Joint Research Project and Project “SMAILE—Simple Methods for Artificial Intelligence Learning and Education.” Recommended by Associate Editor A. A. Malikopoulos. (Corresponding author: Leonardo Cianfanelli.) Leonardo Cianfanelli is with the Dipartimento di Scienze Matematiche, Politecnico University of Turin, 10129 Torino, Italy (e-mail: leonardo.cianfanelli@polito.it). Giacomo Como is with the Dipartimento di Scienze Matematiche, Politecnico University of Turin, 10129 Torino, Italy, and also with the Department of Automatic Control, Lund University, SE-221 00 Lund, Sweden (e-mail: giacomo.como@polito.it). Asuman E. Ozdaglar is with the Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge, MA 02139 USA (e-mail: asuman@mit.edu). Francesca Parise is with the Department of Electrical and Computer Engineering, Cornell University, Ithaca, NY 14850 USA (e-mail: fp264@cornell.edu).

Color versions of one or more figures in this article are available at https://doi.org/10.1109/TAC.2023.3247542.

Digital Object Identifier 10.1109/TAC.2023.3247542

Authorized licensed use limited to the terms of the applicable license agreement with IEEE. Restrictions apply.
networks are sparse in a such a way that the average degree of
the nodes is independent of the number of nodes, inducing
than a proportionality between the number of nodes and links).
Hence, solving the NDP requires the solution of E of these
problems, with E denoting the number of links. Since this can be
computationally intractable for large networks, our second main
result proposes a method based on Rayleigh’s monotonicity
laws to approximate the effective resistance of each link with
a number of iterations independent of the network size, thus
leading to a significant reduction of complexity. The key idea is
that the effective resistance between two adjacent nodes i and
j depends mainly on the local structure of the network around
the two nodes (i.e., the set of nodes \( N_{i,j} \) that are at distance no
greater than a small constant \( d \) from at least one of i and j),
and may, therefore, be approximate by performing only local
computation. Since typically in transportation networks, the
local structure of the network is independent of the network size
(think for instance of a bidimensional square grid), the size of
\( N_{i,j} \) does not scale with the network size; thus, we can guarantee
that the approximation error and computational complexity of
our method also do not scale. Our third main result establishes
sufficient conditions under which the approximation error van-
ishes asymptotically in the limit of infinite networks, proving
that if the related resistor network is recurrent the approximation
error tends to vanish for large distance \( d \). In the conclusive
section, we conduct a numerical analysis on synthetic and real
transportation networks, showing that a good approximation of
the effective resistance of a link can be achieved by looking at
a small portion of the network. Moreover, while several
assumptions are made to establish theoretical results (e.g., affine
delay functions, support of equilibrium flows not varying with
the intervention), we conduct a numerical analysis showing that
good performance is achieved even if some assumptions are
relaxed, i.e., if the delay functions are nonlinear and the support
of the equilibrium is allowed to vary with interventions.

In our work, we consider a special case of NDPs. These
problems have been formalized in the last decades via many
different formulations. Both continuous NDPs \[18\], \[19\], \[20\],
where the budget can be allocated continuously among the
links, and discrete formulations, in which the decision variables
include which new roads to build \[21\], how many lanes to add
to existing roads \[22\], or a mix of those two problems \[23\],
have been considered in the literature, together with dynamical
formulations \[24\], and formulations where the optimum
is achieved by removing, instead of adding, links, because of
Braess’ paradox \[25\], \[26\]. For comprehensive surveys on the
literature on NDP, we refer to the work in \[27\] and \[28\]. We
stress that most of the literature focuses on finding polynomial
algorithms to solve in approximation NDPs in their most general
form. Our main contribution is to provide a tractable approach
to solving a single-link NDP in quasi-linear time, as well as
providing intuition and a completely new formulation. In the
future, we aim at extending our techniques to more general cases,
like the multiple interventions case. In the setting of affine delay
functions, our NDP formulation is also related to the literature
on marginal cost pricing. We assume that interventions modify
the linear coefficient of the delay function of link \( e \) from \( a_e \)
to \( \bar{a}_e \), leading to \( \bar{\tau}_e(f_e) = \tau_e(f_e) - (a_e - \bar{a}_e)f_e \), which is equivalent
to adding a negative marginal cost toll on a link. In the literature,
the problem of optimal toll design has been widely explored,
also dealing with the problem of the support of the Wardrop
equilibrium varying after the intervention, i.e., without imposing
restrictive assumptions. However, most of the toll literature aims
at finding conditions under which a general NP-hard problem
may be solved in polynomial time. The scope of our work is
instead to provide a new formulation to a more tractable problem.
Moreover, to relax the regularity assumption on the support of
the equilibrium, in the toll literature, it is often assumed that the
network has parallel links, which is unrealistic for transportation
networks (see, e.g., \[29\], \[30\]). Our work is also related to the
work in \[16\] and \[17\], where the authors investigate the sign of
total travel time variation when a new path is added to a two-
terminal network, under similar assumptions to ours, providing
sufficient conditions under which Braess’ paradox arises. In our
work, we instead compute the total travel time variation with an
intervention and suggest an efficient algorithm to select the
optimal intervention. As mentioned, the key step of our approach
is to reformulate the NDP in terms of a resistance problem and
also exploits the parallelism between resistor networks and ran-
dom walks. From a methodological perspective, it is worthwhile
mentioning that the relation between Wardrop equilibria and
resistor networks has been first investigated in \[31\] while the
parallelism between random walks and Wardrop equilibria has
been investigated in \[32\], although with different purposes. The
relation between random walks and resistor networks is quite
standard and well-known (see, e.g., \[33\]). To summarize, the
contribution of this article is twofold. From a methodological
perspective, we provide a method to locally approximate the
effective resistance between adjacent nodes, which may be of
independent interest (effective resistance of a link is related to
spanning tree centrality \[34\]). From the NDP perspective, we
provide a new formulation of the NDP in terms of resistor
networks and exploit our methodological result to approximate
efficiently single-link NDPs.

The rest of the article is organized as follows. In Section II, we
define the model and formulate the NDP as a bilevel program. In
Section III, we define single-link NDPs, rephrase the problem
in terms of resistor networks, and discuss the regularity as-
sumption. In Section IV, we provide our method to approximate
the effective resistance of a link and exploit such a method to
construct an algorithm to solve the problem. We then analyze
the asymptotic performance of the proposed method in the limit
of infinite networks in Section V. In Section VI, we provide
numerical simulations. Finally, in the conclusive section, we
summarize the work and discuss future research lines.

A. Notation

We let \( \delta^{(i)} \), \( 1 \), \( 0 \), and \( I \) denote the unitary vector with 1 in
position \( i \) and 0 in all the other positions, the column vector
of all ones, the column vector of all zeros, and the identity
matrix, respectively, where the size of them may be deduced from
the context. \( A^T \) and \( v^T \) denote the transpose of matrix \( A \)
and vector \( v \), respectively. Given a vector \( v \), we let \( I_v \) denote
the matrix whose off-diagonal elements are zero and with diagonal
elements \( (I_v)_{ii} = v_i \).

II. MODEL AND PROBLEM FORMULATION

We model the transportation network as a directed multigraph
\( G = (N, E) \) and denote by \( o, d \in N \) the origin and the destina-
tion of the network. We assume for simplicity of notation that
\( N = \{1, \ldots, N\} \) and \( E = \{1, \ldots, E\} \), and assume that \( o \) and \( d \)
are, respectively, the first and the last node of the network. Every
link $e$ is endowed with a tail $\xi(e)$ and a head $\theta(e)$ in $\mathcal{N}$. We allow multiple links between the same pair of nodes and assume that every link belongs to at least a path from $o$ to $d$, otherwise such a link may be removed without loss of generality. Let $m > 0$ denote the throughput from the origin $o$ to the destination $d$, and $\nu = m(\delta(o) - \delta(d))$ in $\mathbb{R}^N$. Let $\mathcal{P} = \{1, \ldots, P\}$ denote the set of paths from $o$ to $d$. An admissible path flow is a vector $z$ in $\mathbb{R}_+^\mathcal{P}$ satisfying the mass constraint
\[
1^T z = m. \tag{1}
\]
Let $A$ in $\mathbb{R}^{E \times \mathcal{P}}$ denote the link-path incidence matrix, with entries $A_{ep} = 1$ if link $e$ belongs to the path $p$ or 0 otherwise. The path flow induces a unique link flow $f$ in $\mathbb{R}^E$ via
\[
f = Az. \tag{2}
\]
Every link $e$ is endowed with a nonnegative and strictly increasing delay function $\tau_e : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. We assume that the delay functions are in the form $\tau_e(f_e) = \tau_e(0) + a_e(f_e)$, where $\tau_e(0)$ is the travel time of the link when there is no flow on it and $a_e(f_e)$ describes congestion effects, with $a_e(0) = 0$. The cost of path $p$ under flow distribution $f$ is the sum of the delay functions of the links belonging to $p$, i.e.,
\[
c_p(f) = \sum_{e \in \mathcal{E}} A_{ep} \tau_e(f_e). \tag{3}
\]

**Definition 2.1 (Routing game):** A routing game is a triple $(\mathcal{G}, \tau, \nu)$.

A Wardrop equilibrium is a flow distribution such that no one has an incentive in changing path. More precisely, we have the following definition.

**Definition 2.2 (Wardrop equilibrium):** A path flow $z^*$, with associated link flow $f^* = Az^*$, is a Wardrop equilibrium if for every path $p$
\[
z_p^* > 0 \Rightarrow c_p(f^*) = c_q(f^*) \quad \forall q \in \mathcal{P}. \tag{4}
\]

Let $B$ in $\mathbb{R}^{N \times E}$ denote the node-link incidence matrix, with entries $B_{ne} = 1$ if $n = \xi(e), B_{ne} = -1$ if $n = \theta(e)$, or $B_{ne} = 0$ otherwise. It is proved in [14] that a link flow $f^*$ is a Wardrop equilibrium of a routing game if and only if
\[
f^* = \arg \min_{f \in \mathbb{R}^E_+, Bf = \nu} \sum_{e \in \mathcal{E}} f_e \tau_e(s)ds \tag{5}
\]
where $Bf = \nu$ is the projection of (1) on the link set. Since the delay functions are assumed strictly increasing, the objective function in (4) is strictly convex and the Wardrop equilibrium $f^*$ is unique.

**Definition 2.3 (Social cost):** The social cost of a routing game is the total travel time at the equilibrium, i.e.,
\[
C(0) = \sum_{e \in \mathcal{E}} \tau_e(f_e^*). \tag{6}
\]

The third condition, known as complementary slackness, implies that if $\lambda^*_c > 0$, then $f_c^* = 0$, i.e., link $e$ is not used at the equilibrium. We let $\mathcal{E}_+$ denote the set of the links $e$ such that $\lambda^*_c > 0$. The next lemma shows that the social cost may be characterized in terms of the Lagrangian multiplier $\gamma^*$.

**Lemma 1:** Let $(\mathcal{G}, \tau, \nu)$ denote a routing game. Then
\[
C(0) = m(\gamma^*_0 - \gamma^*_d). \tag{7}
\]

**Proof:** See Appendix B.

We consider an NDP where the planner can improve the delay functions of the network with the goal of minimizing a combination of the social cost after the intervention and the cost of the intervention itself. Specifically, let $u$ in $\mathbb{R}^E_+$ denote the intervention vector, with corresponding delay functions
\[
\tau_e^{(u)}(f_e) = \tau_e(0) + \frac{a_u(f_e)}{1 + u_e}. \tag{8}
\]

This type of intervention may correspond for instance to adding lanes to some roads of the network. We let $h_e : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ denote the cost associated with the intervention on link $e$. The goal of the planner is to minimize a combination of the social cost and the intervention cost, where $\alpha \geq 0$ is the tradeoff parameter. More precisely, by letting $f^*(u)$ denote the Wardrop equilibrium corresponding to intervention $u$, the NDP reads as follows.

**Problem 1:** Let $(\mathcal{G}, \tau, \nu)$ be a routing game, and $\alpha \geq 0$ be the tradeoff parameter. The goal is to select $u^*$ such that
\[
u^* \in \arg \min_{u \in \mathbb{R}^E_+, Bf = \nu} \sum_{e \in \mathcal{E}} f_e^{(u)}(\tau_e^{(u)}(f_e^{(u)})) + \alpha h(u) \tag{9}
\]

where $h(u) = \sum_{e \in \mathcal{E}} h_e(u_e)$, and
\[
f^{(u)} = \arg \min_{f \in \mathbb{R}^E_+, Bf = \nu} \sum_{e \in \mathcal{E}} \int_0^{f_e^{(u)}} \tau_e^{(u_e)}(s)ds \tag{10}
\]

**Remark 1:** We stress the fact that Problem 1 is bilevel, in the sense that the planner optimizes the intervention $u$ according to a cost function that depends on the Wardrop equilibrium $f^*(u)$, which, in turn, is the solution of the optimization problem (7), whose objective function depends on the intervention $u$ itself.

**Remark 2:** Problem 1 is not equivalent to the toll design problem. The key difference between the two problems is that tolls modify the Wardrop equilibrium, but the performance of tolls is evaluated with respect to the original delay functions $\tau_e$. On the contrary, in Problem 1, the intervention is evaluated with respect to the new delay functions $\tau_e^{(u_e)}$.

Problem 1 is, in general, nonconvex and hard to solve because of its bilevel nature. For these reasons, in the next section, we shall study a simplified problem where the delay functions are affine and the planner may intervene on one link only. In this setting, we are able to rephrase the problem as a single-level optimization problem and provide an electrical network interpretation of the problem.
III. SINGLE-LINK INTERVENTIONS IN AFFINE NETWORKS

In this section, we provide an electrical network formulation of the NDP under some restrictive assumptions. In particular, we provide a closed formula for the social cost variation in terms of electrical quantities computed on a related resistor network. To this end, we restrict our analysis to the space of feasible interventions $\mathcal{U}$, defined as

$$\mathcal{U} := \{ u : u_e \delta_e^c \text{ for a link } e \in \mathcal{E}, u_e \geq 0 \}.$$  

In other words, $\mathcal{U}$ represents the space of interventions on a single link of the network. We also assume that the delay functions are affine, i.e., $\tau_e(f_e) = a_e f_e + b_e$ for every $e$, and denote by $(G, a, b, \nu)$ routing games with affine delay functions. For an intervention $u$, let $(G, a(u), b, \nu)$ denote the corresponding affine routing game, $C(u)$ denote the corresponding social cost, and $\Delta C(u) = C(0) - C(u)$ denote the social cost gain. Our problem can be expressed as follows.

Problem 2: Let $(G, a, b, \nu)$ be an affine routing game and $\alpha \geq 0$ be the tradeoff parameter. Find

$$u^* \in \arg \max_{u \in \mathcal{U}} (\Delta C(u) - \alpha h(u)).$$

The next example shows that the problem cannot be decoupled by first selecting the optimal link $e^*$ and then the optimal strength of the intervention $u_e^*$.

Example 1: Consider the transportation network in Fig. 1, with linear delay functions $\tau_e(f_e) = a_e f_e$. By some computation, one can prove that

$$\Delta C(u_1, \delta^{(1)}) = m \frac{a_1 a_2^2 a_1}{(a_1 + a_2)(u_1 + 1)a_1 + a_2},$$
$$\Delta C(u_2, \delta^{(2)}) = m \frac{a_2^2 a_2 u_2}{(a_1 + a_2)(u_1 + 1)a_2},$$
$$\Delta C(u_3, \delta^{(3)}) = m \frac{u_3}{a_3 + 1}.$$  

In Fig. 1, the social cost variation corresponding to intervention on every link $e$ is illustrated as functions of $u_e$. Observe that the link that maximizes the social cost gain depends on $u_e$. Thus, the problem cannot be decoupled by first selecting the optimal link $e^*$ and then the optimal $u_e^*$.

Our theoretical results rely on the following technical assumption, stating that the support of the Wardrop equilibrium is not modified with an intervention.

Assumption 1: Let $\mathcal{E}_+(u)$ be the set of links $e$ such that for the routing game $(G, a(F), b, \nu)$ the Lagrangian multiplier $\lambda_e^\ast(u) > 0$. We assume that $\mathcal{E}_+(u) = \mathcal{E}_+$ for every $u$ in $\mathcal{U}$.

Assumption 1 is not new in the literature [16], [17]. We will get back to the assumption in Section III-A. With a slight abuse of notation, from now on let $\mathcal{E}$ denote $\mathcal{E} \setminus \mathcal{E}_+$. We now define a mapping from the transportation network $G$ to an associated resistor network $\mathcal{G}_R$.

Definition 3.1 (Associated resistor network): Given the transportation network $G = (\mathcal{N}, \mathcal{E})$, the associated resistor network $\mathcal{G}_R = (\mathcal{N}, \mathcal{L}, \mathcal{W})$ is constructed as follows.

1) The node set $\mathcal{N}$ is the same.

2) $W \in \mathbb{R}^{N \times N}$ is the conductance matrix, with elements

$$W_{ij} = \begin{cases} \sum_{e \in E} c_{ij}(e) \xi_{ij}(e), & \text{if } i \neq j, \\ 0, & \text{if } i = j. \end{cases}$$

Note that $W$ is symmetric, thus $\mathcal{G}_R$ is undirected. The element $W_{ij}$ has to be interpreted as the conductance between nodes $i$ and $j$.

3) Multiple links connecting the same pair of nodes are not allowed; hence, every link $l$ in $\mathcal{L}$ can be identified by an unordered pair of nodes $(i, j)$, and the set $\mathcal{L}$ is uniquely determined by $W$. Let $\lambda$ denote the cardinality of $\mathcal{L}$. The mapping $M : \mathcal{E} \rightarrow \mathcal{L}$ associates to every link $e$ of the transportation network the corresponding link $l = M(e) = \{\xi(e), \theta(e)\}$ of the resistor network. Note by (8) that $M(e)$ belongs to $\mathcal{L}$ for every $e \in \mathcal{E}$.

Note that the coefficients $a_e$ correspond to resistances in the resistor networks. We let $w = W1$ denote the degree distribution of the resistor network, and $w^* = \max_{i \in \mathcal{N}} w_i$ denote the maximal degree. Before establishing our main result, we define two relevant quantities.

Definition 3.2: Let $v$ in $\mathbb{R}^N$ be the voltage vector on $\mathcal{G}_R$ when a net electrical current $m$ is injected from $o$ to $d$, i.e., $v$ is the unique solution of

$$\sum_{k \in \mathcal{N}} W_{hk}(v_h - v_k) = m(\delta^{(o)} - \delta^{(d)}) \quad \forall h \in \mathcal{N}. \tag{9}$$

For a link $l$, let $y_e$ denote the electrical current flowing from $\xi(e)$ to $\theta(e)$ on link $M(e)$ of $\mathcal{G}_R$, and let $\Delta y_e = v_{\xi(e)} - v_{\theta(e)}$. By Ohm’s law, $\Delta v_e = a_e y_e$.

Definition 3.3: Let $\mathcal{T}$ in $\mathbb{R}^N$ be the voltage vector on $\mathcal{G}_R$ when a unitary current is injected from $i$ to $j$, i.e.,

$$\sum_{k \in \mathcal{N}} W_{hk}(\mathcal{T}_h - \mathcal{T}_k) = \delta^{(i)} - \delta^{(j)} \quad \forall h \in \mathcal{N}. \tag{10}$$

The effective resistance $r_l$ of link $l = \{i, j\}$ in $\mathcal{L}$ is the effective resistance between $i$ and $j$, i.e., $r_l = \mathcal{T}_i - \mathcal{T}_j$. Given a link $e$ in $\mathcal{E}$, we denote by $r_e$ the effective resistance of link $M(e)$ of the associated resistor network.
The next theorem establishes a relation between the social cost gain with a single-link intervention and the associated resistor network.

**Theorem 1:** Let \((G, a, b, \nu)\) be an affine routing game, and let Assumption 1 hold. Then

\[
\Delta C(u, \delta^{(c)}) = a_e f^*_e \frac{y_e}{u_e} + \frac{y_e}{u_e}. \tag{11}
\]

**Proof:** See Appendix B.

The ratio \(r_e/u_e\) belongs to \((0,1]\) and is also known as spanning tree centrality, which measures the fraction of spanning trees including link \(M(e)\) among all spanning trees of the undirected network \(G_R\) [34]. The spanning tree centrality of a link is maximized when removing the link disconnects the network. Theorem 1 states that the social cost variation due to intervention on link \(e\) is as follows.

1) Proportional to \(a_e f^*_e\), which measures the delay at the equilibrium due to congestion on link \(e\).

2) Decreasing in the spanning tree centrality. Intuitively speaking, the benefits of intervention on link \(e\) are larger when the intervention modifies the equilibrium flows so that agents can move from paths not including \(e\) to paths including \(e\), namely when \(f^*_e\) increases after the intervention. This phenomenon does not occur if \(e\) is a bridge, i.e., if \(r_e/u_e = 1\), and occurs largely when many paths from \(\delta(e)\) to \(\theta(e)\) exist, i.e., when \(r_e/u_e\) is small.

3) Proportional to the current \(y_e\). The role of this term is more clear in the special case of linear delay functions. In this case, \(y_e = f^*_e\) for all links \(e\) in \(E \setminus E_e\); hence, \(a_e f^*_e y_e = a_e (f^*_e)^2\), which is the total travel time on link \(e\) before the intervention.

The idea behind the proof is that with affine delay functions the KKT conditions of the Wardrop equilibrium are linear, and under Assumption 1, single-link interventions are equivalent to rank-1 perturbations of the system. Thus, by Lemma 1, we can compute the cost variation by looking at the Lagrangian multiplier \(\gamma^*_e\), and then express such a variation in terms of electrical quantities. In order to solve Problem 2 by the electrical formulation, we need to compute (11) for every link \(e\) in \(E\). The Wardrop equilibrium \(f^*_e\) is assumed to be observable and therefore given. The voltage \(v\) (and thus \(y\)) can be derived by solving the linear system (9) and has to be computed only once. On the contrary, the computation of \(r_e\) must be repeated for every link; hence, it requires to solve L sparse linear systems. To reduce the computational effort, in Section IV, we shall propose a method to approximate the effective resistance of a link that, under a suitable assumption on the sparseness of the network, does not scale with the network size, allowing for a more efficient solution to Problem 2. The next result shows how to compute the derivative of the social cost variation for small interventions.

**Corollary 1:** Let \((G, a, b, \nu)\) be a routing game, and assume that for every \(e\) in \(E\) it holds either \(f^*_e > 0\) or \(\lambda^*_e > 0\). Then

\[
\left. \frac{\partial \Delta C(u)}{\partial u_e} \right|_{u = 0} = \begin{cases} a_e f^*_e y_e & \text{if } \lambda^*_e = 0, \\ 0 & \text{if } \lambda^*_e > 0. \end{cases}
\]

**Proof:** The fact that for every link \(i\) it holds either \(f^*_i > 0\) or \(\lambda^*_i > 0\) implies that for inﬁnitesimal interventions the support of \(f^*\) is not modiﬁed. If \(\lambda^*_k = 0\), then \(f^*_k\) and we can derive the social cost variation in (11) with respect to \(u_k\). The case \(\lambda^*_k > 0\) follows from continuity arguments and from the complementary slackness condition, which implies that \(f^*_e(u) = 0\) in a neighborhood of \(u = 0\).

**Remark 3:** Observe that the derivative of the social cost does not depend on the effective resistance of the link.

### A. On the Validity of Assumption 1

In this section, we discuss Assumption 1. In particular, we show that the assumption is without loss of generality on series–parallel networks if the throughput is sufﬁciently large. We ﬁrst recall the deﬁnition of directed series–parallel networks and then present the result in Proposition 1.

**Deﬁnition 3.4:** A directed network \(G\) is series–parallel if and only if 1) it is composed of two nodes only (o and d), connected by a single link from o to d, or 2) it is the result of connecting two directed series–parallel networks \(G_1\) and \(G_2\) in parallel, by merging \(o_1\) with \(o_2\) and \(d_1\) with \(d_2\), or 3) it is the result of connecting two directed series–parallel networks \(G_1\) and \(G_2\) in series, by merging \(d_1\) with \(o_2\).

**Proposition 1:** Let \((G, a, b, \nu)\) be a routing game. If \(G\) is series–parallel, there exists \(\overline{m}\) such that for every \(m \geq \overline{m}\), \(E_e = \emptyset\). Furthermore, if \(b = 0\), \(E_e = \emptyset\) for every \(m > 0\).

**Proof:** See Appendix B.

**Remark 4:** Proposition 1 implies that Assumption 1 is without loss of generality on series–parallel networks if the throughput is sufﬁciently large.

The next example shows that, if the throughput is not sufﬁciently large, Assumption 1 may be violated.

**Example 2:** Consider the series–parallel network in Fig. 2. Let \(m = 1\), and consider an optimal delay function \(\tau_i(f_i) = f_i + 1, \tau_e(f_e) = f_e + 3/2\). One can verify that \(f^*_1 = 3/4,\ f^*_2 = 1/4,\ \lambda^*_1 = \lambda^*_2 = 0\).

Modifying \(a_i\) from 1 to 1/3 (i.e., with \(u = 2\delta^{(1)}\)), we get \(f^*_1(u) = 1,\ f^*_2(u) = 0,\ \lambda^*_1(u) = 0,\ \lambda^*_2(u) = 1/6\) violating Assumption 1. Proposition 1 proves that this does not occur if \(m\) is sufﬁciently large.

### IV. APPROXIMATE SOLUTION TO PROBLEM 1

As shown in the previous section, Problem 2 may be rephrased in terms of electrical quantities over a related resistor network. Solving the NDP problem in this formulation requires to solve \(L\) linear systems whose dimension scales linearly with \(N\). Since the voltage \(v\) may be computed in quasi-linear time by solving the sparse linear system (9) (see [35] for more details), the computational bottleneck is given by the computation of the effective resistance of every link of the resistor network. The main idea of our method is that, although the effective resistance of a link depends on the entire network, it can be approximate by looking at a local portion of the network only. We then formulate
an algorithm to solve Problem 2 by exploiting our approximation method.

A. Approximating the Effective Resistance

We introduce the following operations on resistor networks.

Definition 4.1 (Cutting at distance \(d\)): A resistor network \(G_R\) is cut at distance \(d\) from link \(l = \{i, j\}\) in \(\mathcal{L}\) if every node at distance greater than \(d\) from link \(l\) (i.e., from both \(i\) and \(j\)) is removed, and every link with at least one endpoint in the set of the removed nodes is removed. Let \(G_{l}^{U_d}\) and \(G_{l}^{L_d}\) denote such a network and the effective resistance of link \(l\) on it, respectively.

Definition 4.2 (Shorting at distance \(d\)): A resistor network \(G_R\) is shorted at distance \(d\) from \(l\) in \(\mathcal{L}\) if all the nodes at distance greater than \(d\) from link \(l\) are shorted together, \(i.e., an infinite conductance is added between each pair of such nodes. Let \(G_{l}^{T_d}\) and \(r_{l}^{T_d}\) denote such a network and the effective resistance of link \(l\) on it, respectively.

We refer to Fig. 3 for an example of these techniques applied to a regular grid. We next prove that \(r_{l}^{U_d}\) and \(r_{l}^{T_d}\) are, respectively, an upper and a lower bound for the effective resistance \(r_{l}\) for every link \(l\). To this end, let us introduce Rayleigh’s monotonicity laws.

Lemma 2 (Rayleigh’s monotonicity laws [36]): If the resistances of one or more links are increased, the effective resistance between two arbitrary nodes cannot decrease. If the resistances of one or more links are decreased, the effective resistance cannot increase.

Proposition 2: Let \(G_R\) be a resistor network. For every link, \(l = \{i, j\}\) in \(\mathcal{L}\)

\[
r_{l}^{U_d} \geq r_{l}^{L_d} \geq r_{l}^{T_d}\quad \forall d \geq d_1 \geq 1.
\]

Moreover

\[
1/w^* \leq r_{l}^{T_d} \leq r_{l}^{U_d} \leq 1/W_{ij}\quad \forall d \geq 1.
\]

Proof: Cutting a network at distance \(d\) is equivalent to setting to infinity the resistance of all the links with at least one endpoint at distance greater than \(d\). Shortening a network at distance \(d\) is equivalent to setting to zero the resistance between any pair of nodes at distance greater than \(d\). Then, by Rayleigh’s monotonicity laws, it follows \(r_{l}^{U_d} \geq r_{l}^{L_d} \geq r_{l}^{T_d}\). Similar arguments may be used to show that if \(d_1 < d_2\), then \(r_{l}^{U_{d_1}} \geq r_{l}^{U_{d_2}}\) and \(r_{l}^{L_{d_1}} \leq r_{l}^{L_{d_2}}\). The right inequality in (12) follows from Rayleigh’s monotonicity laws, by noticing that the effective resistance computed in the network with only nodes \(i\) and \(j\) (which is equal to \(1/W_{ij}\)) is an upper bound for \(r_{l}^{T_d}\). The left inequality follows from noticing that the effective resistance on the network in which every node except \(i\) is shorted with \(j\), which results in a network with only two nodes and a conductance between \(i\) and \(j\) not greater than \(w^*\) (hence, resistance no less than \(1/w^*\)) is a lower bound of \(r_{l}^{T_d}\).

Proposition 2 states that cutting and shorting a network provides upper and lower bounds for the effective resistance of a link. Moreover, the bound gap is a monotone function of the distance \(d\).

B. Our Algorithm

We here propose an algorithm to solve in approximation Problem 2 based on our method for approximating the effective resistance. Our approach is detailed in Algorithm 1.

Notice that the performance of Algorithm 1 depends on the choice of the parameter \(d\). Specifically, the higher \(d\) is, the better is the approximation of the social cost variation.

Theorem 2: Let \(\Delta C_{d}^{(u)}\) be the social cost gain corresponding to intervention \(u = u_e \delta^{(e)}\) as given in Theorem 1, and

\[
\Delta C_{d}^{(u)} = a_{e} f_{e}^* \frac{y_{e}}{u_{e}} + \frac{w_{d}^{e} + w_{d}^{r}}{2\alpha_{e}}.
\]

Algorithm 1: The Proposed Algorithm to Solve Problem 2.

Input: The affine routing game \((G, a, b, \nu)\), the cost functions \(\{h_e\}_{e \in E}\), and the distance \(d \geq 1\) for effective resistance approximation.

Output: The optimal intervention \(u^{sd}\).

1. Construct the associated resistor network \(G_R\).
2. Compute \(v\) and \(y\) by solving (9).
3. for every \(l \in E\) do
   4. Construct \(G_{l}^{U_d}\) and \(G_{l}^{L_d}\).
   5. Compute \(r_{l}^{U_d}\) and \(r_{l}^{L_d}\) on \(G_{l}^{U_d}\) and \(G_{l}^{L_d}\).
   6. end
7. for every \(e \in E\) do
   8. Find \(u_{e}^{sd}\) such that
   9. \[
u_{e}^{sd} = \arg \max_{u_{e} \geq 0} \quad \frac{y_{e}}{u_{e}} + \frac{w_{d}^{e} + w_{d}^{r}}{2\alpha_{e}} - \alpha h_{e}(u_{e}).
\]
   10. end
11. Find \(e^{sd}\) such that
   12. \[
u_{e}^{sd} = \arg \max_{e \in E} \quad \frac{y_{e}}{u_{e}} + \frac{w_{d}^{e} + w_{d}^{r}}{2\alpha_{e}} - \alpha h_{e}(u_{e}^{sd}).
\]
13. The optimal intervention is \(u^{sd} = u_{e}^{sd} \delta^{(e^{sd})}\).
be the social cost gain estimated by Algorithm 1 for a given distance \( d \geq 1 \). Then
\[
\frac{\Delta C^{(u)} - \Delta C^{(d)}_d}{\Delta C^{(u)}} \leq 2 \left( \frac{1}{a_e} + \frac{v'_{e,d} + r'_{e,d}}{2a_e} \right)
\]
where
\[
\epsilon_{ed} := \frac{v'_{e,d} - r'_{e,d}}{a_e}.
\]
Furthermore
\[
\Delta C^{(u)} \geq a_e f_{x} \frac{\gamma_{e}}{a_e} + \frac{v'_{e}}{a_e}.
\]

Proof: See Appendix B.

Assumption 2 is suitable for transportation networks, because of physical constraints not allowing for the degree of the nodes to grow unlimitedly (think for instance of planar grids, where the degree of the nodes is given no matter what the size of the network is, and the local structure of the network around an arbitrary node does not depend on the network size \( N \)). Notice also that, under Assumption 2, \( N \) and \( L \) are proportional.

Proposition 3: Let \( G_R \) be a resistor network corresponding to the transportation network \( G \). Let \( l \in \mathcal{L} \) be an arbitrary link of \( G_R \), and \( N_{e,l} \) denote the set of nodes that are at distance greater than \( d \) from link \( l \). We assume that the network \( G \) is sparse in such a way that the cardinality of \( N_{e,l} \) does not depend on \( N \) for any \( d \).

Assumption 2 is suitable for transportation networks, because of physical constraints not allowing for the degree of the nodes to grow unlimitedly (think for instance of planar grids, where the degree of the nodes is given no matter what the size of the network is, and the local structure of the network around an arbitrary node does not depend on the network size \( N \)). Notice also that, under Assumption 2, \( N \) and \( L \) are proportional.

Proposition 3: Let \( G_R \) be a resistor network corresponding to the transportation network \( G \). Let \( l \in \mathcal{L} \) be an arbitrary link of \( G_R \), and \( N_{e,l} \) denote the set of nodes that are at distance greater than \( d \) from link \( l \). We assume that the network \( G \) is sparse in such a way that the cardinality of \( N_{e,l} \) does not depend on \( N \) for any \( d \).

Proof: See Appendix B.

Remark 5: To the best of our knowledge, the complexity of the most efficient algorithm to compute the spanning tree centrality (or effective resistance) of a link in large networks scales with the number of links [34]. On the contrary, Proposition 3 states that under Assumption 2, the computational time for approximating a single effective resistance does not scale with \( N \). Therefore, approximating all the effective resistances requires a computational time linear in \( N \). Observe that \( v \) (and thus \( y \)) is computed via diagonally dominant, symmetric, and positive-definite linear systems. The design of fast algorithms to solve this class of problem is an active field of research in the last years. To the best of our knowledge, the best algorithm has been provided in [35] and has complexity \( O(M \log^6 N \log 1/\epsilon) \), where \( \epsilon \) is the tolerance error, \( k \) is a constant, and \( M \) is the number of nonzero elements in the matrix of the linear system. Since in our case \( M \) scales with \( L \), and since \( L \) scales with \( N \), under Assumption 2, Algorithm 1 is quasi-linear in \( N \). Step (13) consists in maximizing a function of one variable. Finally, step (14) consists in taking the maximum of \( E \) numbers.

V. Bound Analysis

In this section, we characterize the gap between the bounds on the effective resistance of a link in terms of random walks over the resistor networks \( G_R, G_t^{U_d, L_d} \), and \( G_t^{L_d} \). We then leverage this characterization to provide a sufficient condition on the network under which the bound gap vanishes asymptotically for large distance \( d \). To this end, we introduce the conductance matrix \( W \) of the resistor network as the transition rates of a continuous-time Markov chain whose state space is the node set of the network and introduce the following notation. Let the following hold.

1) \( T_S \) and \( T_{+} \) denote the hitting time (i.e., the first time \( t \geq 0 \) such that the random walk hits the set \( S \subseteq \mathcal{N} \)), and the return time (i.e., the first time \( t > 0 \) such that the random walk visits the set \( S \), respectively.

2) \( N_d \) denote the set of nodes that are at distance \( d \) from link \( l = \{i,j\} \), i.e., at distance \( d \) from \( i \) (or \( j \)) and at distance greater or equal than \( d \) from \( j \) (or \( i \)). Index \( i \) is omitted for simplicity of notation.

3) \( p_k(X), p^{U_d}_k(X), \) and \( p^{L_d}_k(X) \), denote the probability that \( X \) occurs, conditioned on the fact that the random walk starts in \( k \) at time 0 and evolves over the resistor networks \( G_R, G_t^{U_d, L_d} \) and \( G_t^{L_d} \), respectively.

Next result provides a characterization of the bound gap in terms of random walks over \( G_R, G_t^{U_d, L_d} \), and \( G_t^{L_d} \).

Proposition 4: Let \( G_R = (\mathcal{N}, \mathcal{L}, W) \) be a resistor network. For each link, \( l = \{i,j\} \in \mathcal{L} \),
\[
r^{U_d}_l - r^{L_d}_l \leq \max_{g \in \mathcal{N}} \left( p^{U_d}_g(T_i < T_j) - p^{L_d}_g(T_i < T_j) \right)
\]
where the quantities in (16) are computed with respect to the continuous-time Markov chain with transition rates \( W \).

Proof: See Appendix B.

In the next sections, we shall use this result to analyze the asymptotic behavior of the bound gap for an arbitrary link \( l \) in \( \mathcal{L} \) as \( d \to +\infty \), for networks whose node set is infinite and countable. In particular, we show in Section V-A that this error vanishes asymptotically for the class of recurrent networks. The core idea to prove this result is to show that Term 1 vanishes. To generalize our analysis beyond recurrent networks, in Section V-B, we study both Terms 1 and 2 and provide examples showing that all combinations in Table I are possible. In particular, it is possible that the bound gap vanishes asymptotically for nonrecurrent networks (for which Term 1 \( \to 0 \), see [36, Sec. 21.2]) if Term 2 \( \to 0 \).
A. Recurrent Networks

We start by introducing the class of recurrent networks.

Definition 5.1 (Recurrent random walk): A random walk is recurrent if, for every starting point, it visits its starting node infinitely often with probability one [36, Sec. 21.1].

Definition 5.2 (Recurrent network): An infinite resistor network $G_R = (\mathcal{N}, \mathcal{L}, W)$ is recurrent if the random walk on the network is recurrent.

The next theorem states that the bound gap vanishes asymptotically on recurrent networks if the degree of every node is finite. Note that the boundedness of the degree of all the nodes is guaranteed under Assumption 2.

Theorem 3: Let $G_R = (\mathcal{N}, \mathcal{L}, W)$ be an infinite recurrent resistor network, and let $w^* < +\infty$. Then, for every $l$ in $\mathcal{L}$

$$\lim_{d \to +\infty} (r^{U_d}_{l} - r^{L_d}_{l}) = 0.$$ 

Proof: It is proved in [36, Proposition 21.3] that a network is recurrent if and only if

$$\lim_{d \to +\infty} p_l (T_{N_d} < T_j) = 0 \quad \forall l = \{i, j\} \in \mathcal{L}. \quad (17)$$

Observe that, to hit any node in $N_{d+1}$, the random walk starting from $i$ has to hit at least one node in $N_d$. Hence, the sequence $\{p_l (T_{N_d} < T_j)\}_{d=1}^{\infty}$ is nonincreasing in $d$ and the limit in $(17)$ is well defined. Then, from (16) and (17), from the fact that $0 \leq p^{U_d}_g (T_i < T_j) - p^{L_d}_g (T_i < T_j) \leq 1$ for every node $g$, and from the assumptions $w^* < +\infty$ and $W_{ij} > 0$ (recall that $i$ and $j$ are adjacent nodes), it follows

$$\lim_{d \to +\infty} r^{U_d}_{l} - r^{L_d}_{l} \leq \frac{w^*}{(W_{ij})^2} \lim_{d \to +\infty} p_l (T_{N_d} < T_j) = 0.$$

Corollary 2: Let $G$ be a transportation network with recurrent associated resistor network $G_{R}$. Then, for every $u$ in $\mathcal{U}$

$$\lim_{d \to +\infty} \left| \frac{\Delta C^{(u)}(n) - \Delta C^{(u)}(i)}{\Delta C^{(u)}} \right| = 0.$$

Proof: The proof follows from Theorems 2 and 3, which imply $\lim_{d \to +\infty} \epsilon_{e,d} = 0$ for every $e$ in $\mathcal{E}$. Recurrence is a sufficient condition for the approximation error of a link effective resistance to vanish asymptotically, but is not necessary, as discussed in the next section.

B. Beyond Recurrence

We here provide examples of infinite resistor networks for all of the cases in Table I. Observe that, for every link $l = \{i, j\}$ in $\mathcal{L}$, the network cut at distance $d$ from $l$ and the network shorted at distance $d$ from $l$ differ for one node only (denoted by $s$), which is the result of shorting in a unique node all the nodes at distance greater than $d$ from $l$. Intuitively speaking, our conjecture is that Term 2 in (16) is small when the network has many short paths. In this case, adding the node $s$ leads to a small variation of the probability, starting from any node in $N_d$, of hitting $i$ before $j$, thus making Term 2 small. This intuition can be clarified with the next examples.

1) 2-D Grid: Consider an infinite unweighted bidimensional grid as in Fig. 4. This network is very relevant for NDPs since many transportation networks have similar topologies. The network is known to be recurrent [36, Example 21.8]; hence, Theorem 3 guarantees that Term 1 vanishes asymptotically for every link $l = \{i, j\}$. Our conjecture, confirmed by numerical simulations, is that for every node $g$ in $N_d$

$$\lim_{d \to +\infty} p^{U_d}_g (T_i < T_j) = 1/2,$$

$$\lim_{d \to +\infty} p^{L_d}_g (T_i < T_j) = 1/2.$$

Hence, this is a recurrent network for which also Term 2 vanishes asymptotically.

2) 3-D Grid: Consider an infinite unweighted tridimensional grid. This network is not recurrent [36, Example 21.9]; therefore, Term 1 does not vanish asymptotically, and we cannot conclude from Theorem 3 that for every $l = \{i, j\}$ the bound gap vanishes asymptotically. Nonetheless, numerical simulations show that, similarly to the bidimensional grid, for every node $g$ in $N_d$

$$\lim_{d \to +\infty} p^{U_d}_g (T_i < T_j) = 1/2,$$

$$\lim_{d \to +\infty} p^{L_d}_g (T_i < T_j) = 1/2.$$

Hence, this is a nonrecurrent network for which Term 2 (and therefore the bound gap $r^{U}_l - r^{L}_l$) vanishes asymptotically in the limit of infinite distance $d$.

3) Ring: Consider an infinite unweighted ring network, and let us focus on nodes 5 and 6 in Fig. 5. Then

$$p^{U_d}_5 (T_1 < T_2) = 1, \quad p^{U_d}_6 (T_1 < T_2) = 0$$

for each $d$ (even $d \to +\infty$), whereas

$$p^{L_d}_5 (T_1 < T_2) = \frac{d}{2d + 1}, \quad \lim_{d \to +\infty} \frac{1}{2}$$

$$p^{L_d}_6 (T_1 < T_2) = \frac{d + 1}{2d + 1}, \quad \lim_{d \to +\infty} \frac{1}{2}$$
since this case is equivalent to Gambler’s ruin problem (see [36, Proposition 2.1]). Hence, Term 2 does not vanish for the ring. This is due to the fact that all the paths from 5 to 2 in $G_{L^2}$ not including node 1 include the node $s$. Still, Term 1 (and thus the bound gap $r^{U_1}_d - r^{L_1}_d$) vanishes asymptotically by Theorem 3, because the ring network is recurrent.

4) Double Tree Network: The last example illustrates an infinitely large network in which the bound gap does not vanish asymptotically. This network is not relevant for traffic applications since it admits one path only between every pair of nodes but provides an interesting counterexample where the bound gap does not converge asymptotically. The network is composed of two infinite trees starting from node $i$ and $j$, connected by a link $l = \{i, j\}$ (see Fig. 6), and is unweighted. It can be shown that on this network the probability that a random walk, starting from $i$, returns on $i$ is equal to the same quantity computed on a biased random walk over an infinite line (for more details see Appendix C). Since the biased random walk on a line is not recurrent [36, Example 21.2], then the double tree network is nonrecurrent, and Term 1 $\not\rightarrow$ 0. Moreover, we show in Appendix C that

$$\lim_{d \to +\infty} r^{U_1}_d - r^{L_1}_d = \frac{1}{3}$$

thus implying that Term 2 $\not\rightarrow$ 0.

VI. NUMERICAL SIMULATIONS

This section is devoted to numerical simulations. In Section VI-A, we analyze the bound gap for finite distance $d$, both on real and synthetic transportation networks. Then, we discuss in Section VI-B how to adapt our method to more general NDPs with nonlinear delay functions and provide numerical simulations showing that our algorithm may be applied in real scenarios even if the regularity assumption on the Wardrop equilibrium (i.e., Assumption 1) is violated.

A. Effective Resistance Approximation

1) Infinite Grids: Infinite regular grids are relevant networks to test the performance of the bounds on the effective resistance since they are a good proxy for transportation networks. In Table II, the bound gap in a square grid network with unitary conductances is shown. Similar results are obtained in any regular infinite grid. Numerical simulations show that for every link $l$ in $L$

$$\frac{r^{U_1}_d - r_l}{r_l} = \frac{r_l - r^{L_1}_d}{r_l} = O(1/d^2).$$

We emphasize that, despite the network being infinitely large, even at $d = 5$ the bounds are close to the true value effective resistance, which is $1/2$ [37].

2) Oldenburg Transportation Network: In this section, we illustrate the performance of our bounds on the effective resistance of links of the resistor network associated with the transportation network of Oldenburg [38]. The transportation network is composed of 6105 nodes and 7035 links, and the diameter of the associated resistor network (i.e., maximum distance between pair of nodes) is 104. We assume for simplicity $a_e = 1$ for every link $e$ in $E$, but numerical results prove to be robust with respect to some variability in those parameters. The average relative bound gap on the associated resistor network, defined as

$$\Delta R_d := \frac{1}{|L|} \sum_{l \in L} \frac{r^{U_1}_d - r^{L_1}_d}{r_l}$$

is shown in Table III and Fig. 7.

We observe that also in this network the bound gap decreases quickly compared to the diameter of the network.

B. Relaxing Assumptions

The goal of this section is twofold. We first show how to adapt Theorem 1 when the delay functions are nonaffine and validate by numerical analysis the proposed method. We then show that violating Assumption 1 is not a practical issue in real-case scenarios. The numerical example is based on the highway network of Los Angeles (see Fig. 8, [39]). To handle nonlinear delay functions, the main idea is to adapt Theorem 1 by constructing a resistor network and then follow the same steps as in Algorithm 1. To this end, recall that in general $\tau_e(f_e) = a_e(f_e) + \tau_e(0)$, and let us write the KKT conditions

\begin{table}[h]
\centering
\caption{Table of Upper and Lower Bound in Infinite Square Grid}
\begin{tabular}{|c|c|c|c|c|c|}
\hline
$d$ & 1 & 2 & 3 & 4 & 5 \\
\hline
(r^{U_1}_d - r_{1})/r_{1} & 1/5 & 0.0804 & 0.0426 & 0.0262 & 0.0178 \\
(r_{1} - r^{L_1}_d)/r_{1} & 1/5 & 0.0804 & 0.0426 & 0.0262 & 0.0178 \\
\hline
\end{tabular}
\end{table}

\begin{table}[h]
\centering
\caption{Average Relative Error Bound Gap at Distance $d$ on the Oldenburg Network}
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
$d$ & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
$\Delta R_d$ & 0.21 & 0.12 & 0.079 & 0.036 & 0.041 & 0.031 & 0.024 \\
\hline
\end{tabular}
\end{table}
of (4) as follows:

\[
\begin{bmatrix}
\text{diag} \left( \left\{ \frac{\alpha_e(f_e^\ast)}{f_e} \right\}_{e \in E} \right) - (B_\ast)^T & 0 \\
0 & \gamma_\ast
\end{bmatrix}
\begin{bmatrix}
f_e^\ast \\
\gamma_e
\end{bmatrix}
= - \begin{bmatrix}
\tau_e(0) \\
\nu_e
\end{bmatrix}
\]

where \(f^\ast\) and \(\gamma^\ast\) denote the Wardrop equilibrium and the optimal Lagrangian multipliers before the intervention. The KKT conditions suggest that in nonaffine routing games the term \(\alpha_e(f_e^\ast)/f_e\) plays the role of \(\alpha_e\) in affine routing games (see the proof of Theorem 1 in Appendix B for more details). Hence, by following similar steps as in affine routing games, we construct a resistor network with conductance matrix

\[
W_{ij} = \begin{cases}
\sum_{\xi(e) = i, \delta(e) = j} \frac{f_e^\ast}{\alpha_e(f_e^\ast)} & \text{if } i \neq j \\
0 & \text{if } i = j.
\end{cases}
\]

The social cost variation for single-link interventions is then computed by using Theorem 1 with respect to the new resistor network with conductance matrix (18). Observe that, in contrast with the affine case, this method is not exact for nonlinear delay functions since the Wardrop equilibrium (and thus the elements of \(W\)) are modified by interventions, not allowing to leverage the Sherman–Morrison theorem to compute the social cost variation.

To validate our method, we assume that delay functions are in the form \(\tau_e(f_e) = c_e \cdot (f_e)^4 + b_e\) (hence \(\alpha_e(f_e) = c_e \cdot (f_e)^4\)), and consider interventions in the form \(u = 3(\delta^e)\) for every \(e\) in \(E\). Numerical parameters are not reported in the article due to limited space, but the obtained results are robust with respect to a change of numerical values. For every intervention, we compare the social cost variation computed by two methods: 1) by solving the convex optimization (7) and plugging the new equilibrium \(f^\ast(u)\) into the social cost function \((exact)\); 2) via the electrical formulation, i.e., by leveraging Theorem 1 with conductance matrix (18) and ignoring the fact that Assumption 1 may be violated \((approximated)\). Fig. 9 illustrates the social cost variation computed by the two methods corresponding to interventions on the five links of the network that yield the largest cost variation. The numerical simulations show that support of the equilibrium varies with the intervention. Nonetheless, the proposed method approximates quite well the social cost variation and selects the optimal link for the intervention. The implication of combining the results of this section and Section VI-A is that Algorithm 1 should manage to select optimal (or weakly suboptimal) interventions in large transportation networks also when the delay functions are nonlinear, Assumption 1 is violated, and effective resistances are computed at small distance \(d\).

**VII. CONCLUSION**

In this work, we study an NDP where a single link can be improved. Under the assumption that the support of the Wardrop equilibrium is not modified with an intervention, we reformulate the problem in terms of electrical quantities computed on a related resistor network, in particular in terms of the effective resistance of a link. We then provide a method to approximate such an effective resistance by performing only local computation, which may be of separate interest. Based on the electrical formulation and our approximation method for the effective resistance, we propose an efficient algorithm to solve efficiently the NDP. We then show by numerical examples that our method can be adapted to routing games with nonlinear delay functions and achieves good performance even if the support of the equilibrium is modified by the intervention. An interesting direction for the future is a deeper analysis of the tightness of the bounds on effective resistance for finite...
distance \( d \). Future research lines also include extending the analysis to the case of multiple interventions. Indeed, the general problem is not submodular, thus guarantees on the performance of the greedy algorithm are not given. A possible direction is to exploit the closed formula for the social cost derivative to implement gradient descent algorithms. Other directions include extending the theoretical framework to the case of multiple origin-destination pairs and heterogeneous preferences [40], [41].

**Appendix A**

**Preliminaries on Connection Between Green’s Function, Random Walks, and Effective Resistance**

Let \( \mathcal{G}_R = (N, \mathcal{L}, W) \) denote a connected resistor network (this is without loss of generality for resistor network associated with transportation networks), and \( P = \mathbf{I}_w^{-1} W \) the transition probability matrix of the jump chain of the continuous-time Markov chain with rates \( W \). We denote by \( kP \) the matrix obtained by deleting from \( P \) the row and the column referring to the node \( k \). \( kP \) can be thought of as the transition matrix of a killed random walk obtained by creating a cemetery in the node \( k \). We then define Green’s function as

\[
    kG := \sum_{i=0}^{\infty} (kP)^i = (I - kP)^{-1}. \tag{19}
\]

The last inequality in (19) follows from the connectedness of \( \mathcal{G}_R \), which implies that \( kP \) is substochastic and irreducible. Hence, it has a spectral radius smaller than 1 and the inversion is well defined [42]. Since \( (kP)^i_{ij} \) is the probability that the killed random walk starting from \( i \) is in \( j \) after \( t \) steps, \( kG_{ij} \) indicates the expected number of times that the killed random walk visits \( j \) starting from \( i \) before being absorbed in \( k \) [43]. It is known that Green’s function of the random walk on a resistor network can be related to electrical quantities [43]. In particular, with the convention that

\[
kG_{ik} = kG_{ki} = kG_{kk} = 0 \quad \forall i \in N
\]

and it is known that for any node \( k \) and link \( l = \{i, j\} \) in \( \mathcal{L} \)

\[
    r_l = \frac{kG_{ii} - kG_{ij}}{w_i} + \frac{kG_{jj} - kG_{ij}}{w_j} = \frac{1}{w_i p_i (T_j < T_i^+)} = \frac{jG_{ii}}{w_i}
\]

where \( p_i (T_j < T_i^+) \) is defined in Section V, and \( r_l \) is the effective resistance of link \( l \) as defined in Definition 3.3.

### APPENDIX B

**Proof of Lemma 1:** From (5), for all the used links \( e \) in \( \mathcal{E} \)

\[
    \gamma \ast_{\ast} - \gamma \ast_{\theta} = \tau e (f_e\ast).
\]

Consider a path \( p = (e_1, e_2, \ldots, e_s) \), with \( \xi(e_1) = 0, \theta(e_s) = d \), and \( \theta(e_i) = \xi(e_{i+1}) \) for every \( 1 \leq i < s \). Thus, from (3)

\[
c_p(f\ast) = \sum_{i=1}^{s} \tau e_i (f\ast_{e_i}) = \sum_{i=1}^{s} (\gamma \ast_{\xi(e_i)} - \gamma \ast_{\theta(e_i)}) = \gamma \ast_0 - \gamma \ast_d.
\]

Hence, all the used paths at the equilibrium have the same cost \( \gamma \ast_0 - \gamma \ast_d \). Then, the social cost is

\[
    C(0) = \sum_{e \in \mathcal{E}} \tau e (f_e\ast) = \sum_{e \in \mathcal{E}} \tau e (f_e\ast) \sum_{p \in \mathcal{P}} A_{ep} z_p \ast
\]

\[
\sum_{p \in \mathcal{P}} z_p \ast \sum_{e \in \mathcal{E}} A_{ep} \tau e (f_e\ast) = \sum_{p \in \mathcal{P}} z_p \ast \tau e (f_e\ast)
\]

\[
= (\gamma \ast_0 - \gamma \ast_d) \sum_{p \in \mathcal{P}} z_p \ast = m(\gamma \ast_0 - \gamma \ast_d)
\]

where the second equivalence follows from (2), the fourth one from (3), and the last one from (1).

**Proof of Theorem 1:** Consider the KKT conditions (5), and let us remove the links in \( \mathcal{E}_+ \). Thus, the last three conditions of (5) can be ignored without affecting the solution. With a slight abuse of notation, from now on let \( \mathcal{E} \) denote \( \mathcal{E} \setminus \mathcal{E}_+ \). Using the fact that the delay functions are affine, the KKT conditions become

\[
\begin{align*}
    a_e f_e\ast + b_e + \gamma \ast_{\theta(e)} - \gamma \ast_{\xi(e)} &= 0 \quad \forall e \in \mathcal{E} \\
    \sum_{e \in \mathcal{E}, \delta(e) = i} f_e - \sum_{e \in \mathcal{E}, \xi(e) = i} f_e\ast + \nu_i &= 0 \quad \forall i \in N
\end{align*}
\]

where the constraint \( f_e\ast \geq 0 \) can now be removed since the solution to the new KKT conditions gives \( f_e\ast \geq 0 \) for every link \( e \) in \( \mathcal{E}_+ \). Observe that the optimal flow \( f_e\ast \) depends on \( \gamma \ast \) only via the difference \( \gamma \ast_{\xi(e)} - \gamma \ast_{\theta(e)} \), so that \( \gamma \ast \) remains a solution if a constant vector is added to it. This is due to the fact that the matrix \( B \) is not full rank. Observe that removing the last row of \( B \) is equivalent to imposing \( \gamma \ast_d = 0 \). We let \( \gamma \ast_\ast \) and \( \nu \ast \) denote, respectively, \( \gamma \ast \) and \( \nu \) where the last element of both vectors is removed, and let \( B \in \mathbb{R}^{(N-1) \times E} \) denote the node-link incidence matrix where the last row is removed. Finally, we define \( H \in \mathbb{R}^{(N+E-1) \times (N+E-1)} \) as

\[
    H := \begin{bmatrix} I_a & -B_- \end{bmatrix} - B_+^T 0.
\]

With this notation in mind, and assuming \( \gamma \ast_d = 0 \), the KKT conditions may be written in compact form as

\[
    H \begin{bmatrix} f \ast \\ \gamma \ast_d \end{bmatrix} = - \begin{bmatrix} b \\ \nu \ast \end{bmatrix}.
\]

Since we take \( \gamma \ast_d = 0 \), the system has a unique solution, i.e.,

\[
    \begin{bmatrix} f \ast \\ \gamma \ast_d \end{bmatrix} = \begin{bmatrix} KQ^{-1}K^T & I_a \\ Q^{-1}K^T & KQ^{-1} \end{bmatrix} \begin{bmatrix} b \\ \nu \ast \end{bmatrix}
\]

where \( K := I_a^{-1}B_+ \in \mathbb{R}^{E \times (N-1)} \) and \( Q := B_-I_a^{-1}B_+ \in \mathbb{R}^{(N-1) \times (N-1)} \). The invertibility of \( H \) follows from the invertibility of \( I_a \) (the delays are strictly increasing) and from the invertibility of \( Q \) (see [42]), which will be proved in a few lines. From the definitions of \( B_- \) and \( a \), it follows that for every link \( e \)

\[
    K_e := \frac{\left( \delta(\xi(e)) \right)^T - \left( \delta(\theta(e)) \right)^T}{a_e}
\]
with the convention that \( \delta^{(d)} = 0 \cdot 1 \) (since we removed the destination in \( B_+ \)). Moreover

\[
Q_{ij} = \begin{cases} 
-\sum_{\xi(e) = 1} \frac{1}{a_e} & \text{if } i \neq j \\
\frac{1}{a_e} & \text{if } i = j.
\end{cases} \quad \forall i, j \in \mathbb{N} \setminus d
\]

where \( \partial_i \) denotes the in and out neighborhood links of \( i \), i.e.,

\[
\partial_i := \{ e \in \mathcal{E} : B_{ie} \neq 0 \}
\]

Let \( L = I_w - W \) denote the Laplacian of the associated resistor network \( \tilde{G}_R \), and \( dL \) denote its restriction to \( \mathbb{N} \setminus d \). We remark that \( \partial_i \) includes also links pointing to the destination. This allows us to observe that \( Q = dL \), which implies the invertibility of \( Q \).

Let \( I_w^{(u)} \), \( H(u) \), \( Q(u) \), and \( K(u) \) denote the matrix \( I_w \), \( H \), \( Q \), and \( K \) corresponding to the intervention \( u \). Note that an intervention on link \( e \) corresponds to a rank-1 perturbation of \( Q \). In particular

\[
Q^{(u_e, \delta^{(e)})} = Q + \frac{u_e}{a_e} B^e (B^e)^T
\]

where \( B^e \) denotes the \( e \)th column of \( B_- \). Thus, by Sherman–Morrison formula

\[
(Q^{(u_e, \delta^{(e)})})^{-1} = Q^{-1} - \frac{Q^{-1} B^e (B^e)^T Q^{-1}}{\frac{u_e}{a_e} + (B^e)^T Q^{-1} B^e}.
\]

Let for simplicity of notation assume \( \xi(e) = i, \theta(e) = j \). Then

\[
K^{(u_e, \delta^{(e)})} - K = \frac{u_e}{a_e} \delta^{(e)}(\delta^{(i)} - \delta^{(j)})^T = \frac{u_e}{a_e} \delta^{(e)}(B^e)^T
\]

By (22), (25), and (26), we thus get

\[
\gamma_0^* - \gamma_0^{(u_e, \delta^{(e)})} = -\frac{u_e}{a_e} Q^{-1} B^e (\delta^{(e)})^T b + \frac{Q^{-1} B^e (B^e)^T Q^{-1}}{\frac{u_e}{a_e} + (B^e)^T Q^{-1} B^e} \cdot \left( K^T b + \frac{u_e}{a_e} B^e (\delta^{(e)})^T b + \nu_- \right)
\]

where the second equivalence follows from KKT conditions \( Q^{-1}(K^T b + \nu_-) = \gamma_+ \), the last one from \( \gamma_+^* - \gamma_+ = a_e f^*_e + b_e \), and \( v \) is used instead of \( \nu_- \), coherently with the convention \( \delta^{(d)} = 0 \cdot 1 \). The statement then follows from Lemma 1 from \( \gamma_d = 0 \), and from Ohm’s law, i.e., \( v_i - v_j = a_e y_e \).

Proof of Proposition 1: A sufficient condition under which \( \mathcal{E}_+ = \emptyset \) is that the first E components of (23), corresponding to equilibrium link flows, are nonnegative. Indeed, since (4) is strictly convex, if the flow \( f^* \) obtained by (23) is nonnegative, then \( f^* \) is feasible and is the unique Wardrop equilibrium, with \( \lambda^* = 0 \). Links \( e \) with \( \lambda^*_e > 0 \) are those such that \( f^*_e \) computed by (23) is strictly negative. Hence, we aim at finding conditions under which \( f^*_e \geq 0 \) for every \( e \in \mathcal{E} \) according to (23). Let us define \( \bar{v} = v/m \). From (23), (29), and \( \nu_- = m \delta^{(o)} \), it follows that for every link \( e \)

\[
f^*_e = -\frac{b_e}{a_e} + [KQ^{-1}K^T]_e c + [KQ^{-1}]_e (\nu_-)
\]

\[
= -\frac{b_e}{a_e} + [KQ^{-1}K^T]_e c + m \frac{\bar{v}_e - \bar{\delta}_e}{a_e}
\]

where we recall that \( r_e \) denotes the effective resistance of link \( M(e) = \{i, j\} \in \mathcal{L} \), and the last equivalence follows from (21) and from noticing that the definition of \( a \) is coherent with (20). Let \( v_- \) denote the restriction of \( v \) on \( \mathbb{N} \setminus \{d\} \). Definition 3.2 and \( Q = dL \) imply that

\[
v_- = mQ^{-1} \delta^{(o)}.
\]

Plugging this equivalence and (28) in (27), we get

\[
\gamma_0^* - \gamma_0^{(u_e, \delta^{(e)})} = -\frac{u_e}{a_e} (\delta^{(e)})^T Q^{-1} B^e (\delta^{(e)})^T b + \frac{u_e}{a_e} (B^e)^T Q^{-1} B^e \cdot \left( K^T b + \frac{u_e}{a_e} B^e (\delta^{(e)})^T b + \nu_- \right)
\]

\[
= -\frac{u_e}{a_e} \frac{b_e}{m a_e} (v_i - v_j) + \frac{1}{m} \frac{v_i - v_j}{a_e} \left( (B^e)^T \gamma_+ + \frac{b_e}{a_e} r_e \right)
\]

\[
= \frac{1}{m} \frac{v_i - v_j}{a_e} \left( -b_e \gamma^*_i + \gamma^*_j \right)
\]

\[
= \frac{1}{m} \frac{v_i - v_j}{a_e} f^*_e
\]

where the second equivalence follows from KKT conditions. The statement then follows from Lemma 1 from \( \gamma_d = 0 \), and from Ohm’s law, i.e., \( v_i - v_j = a_e y_e \).
Let $\overline{\pi}_e = (b_e - a_e[KQ^{-1}KT]_{ee}b_e)/\Delta \hat{v}_e$. If $\Delta \hat{v}_e > 0$, then for every $m \geq \overline{\pi}_e$ it holds $f^*_e \geq 0$, which in turn implies that if $m \geq \overline{\pi}_e$ then $\overline{\pi}_e = 0$, and otherwise, if the delays are linear, $\Delta \hat{v}_e \geq 0$ implies $f^*_e \geq 0$ and $\overline{\pi}_e = 0$ for every $m \geq 0$, because $b > 0$. We have now to prove that $\Delta \hat{v}_e > 0$. Note by Ohm’s law that $\Delta \hat{v}_e \cdot a_e = \hat{y}_e$, where $\hat{y}_e$ denotes the current flowing on $G_R$ from node $\xi(e)$ to node $\theta(e)$ when unitary current is injected from $o$ to $d$. Then, it suffices to show that $\hat{y}_e > 0$. To this end, observe that if the transportation network is series–parallel, it has a single link $e = (o,d)$, or it can be obtained by connecting in series or in parallel two series–parallel networks. Thus, a series–parallel network can be reduced to a single link from $o$ to $d$ by recursively 1) merging two links $e_1$ and $e_2$ connected in series (i.e., $\xi(e_2) = \theta(e_1)$) into a single link $e_3$, or 2) merging two links $e_1$ and $e_2$ connected in parallel, i.e., with same head and tail, into a single link $e_3$. The transformation 1) results in a resistor network where the links $M(e_1)$ and $M(e_2)$ are replaced by their series composition $M(e_3) = \{\xi(e_1), \theta(e_2)\}$ with current $\hat{y}_e = \hat{y}_e = \hat{y}_e$. Instead, the transformation 2) results in an associated resistor network where the links $M(e_1)$ and $M(e_2)$ are replaced by their parallel composition $M(e_3)$, with $\hat{y}_e > 0$ if and only if $\hat{y}_e, \hat{y}_e > 0$. Thus, in both the cases 1) and 2), $\hat{y}_e > 0$ and only if $\hat{y}_e, \hat{y}_e > 0$. Obviously, when the transportation network is reduced to a single link from $o$ to $d$, the flow on the unique link is positive because $b > 0$. Then, by applying those arguments recursively, for every link $e \in \mathcal{E}$, we get $\hat{y}_e > 0$, which implies by Ohm’s law that $\Delta \hat{v}_e > 0$. Thus, if $m \geq \overline{\pi}_e$ then $f^*_e \geq 0$ and $\overline{\pi}_e = 0$.  

**Proof of Theorem 2:** Consider an intervention $u = u_{e}\delta(e)$. Then

$$\left|\Delta C(u) - \Delta C^d(u)\right| = a_e f_e \left[ \frac{y_e}{1 + a_e} - \frac{y_e}{1 + a_e} + \frac{f^*_e}{a_e} \frac{u_e y_e}{1 + \frac{u_{\Delta e} + u_{\Delta d}}{2 a_e}} \varepsilon_{ed} \right].$$

Notice also that

$$\left| r^U_{e} + r^L_{e} - 2 r_{e} \right| \leq \left| r^U_{e} - r_{e} \right| + \left| r_{e} - r^L_{e} \right| = \frac{r^U_{e} - r_{e}}{a_e} = \varepsilon_{ed}.$$

Putting those two together, and using (11), we get

$$\left| \frac{\Delta C(u) - \Delta C^d(u)}{\Delta C(u)} \right| \leq 2 \left( \frac{\varepsilon_{ed} \left( u_{\Delta e} + u_{\Delta d} \right) {a_e}}{2 a_e} \right).$$

where the last inequality follows from (12). Finally, (15) follows from Theorem 1 and $r^U_{e} \geq r^L_{e}$.

**Proof of Proposition 3:** The cut and shorted networks are obtained by finding the neighborhoods within distance $d$ and $d+1$ from $i,j$, respectively. The neighbors of a node $i$ can be found by checking the nonzero elements of $W(i,:)$, and the borders within distance $d$ can be found by iterating such operation $d$ times. Hence, the time to construct the cut and the shorted network depends on the local structure, which, under Assumption 2, does not depend on the network size. Since the bounds of the effective resistance are computed on these subnetworks, their time complexity and tightness depend on local structure, which, under Assumption 2, is independent of the network size.

**Proof of Proposition 4:** We introduce the following notation:

1. The index $U_d$ and $L_d$ indicate that the random walk takes place over $G^U_i$ and $G^L_i$, respectively. So, for instance, $G^U_{ij}$ denotes the expected number of times that the random walk on the network $G^U_i$, starting from $i$, hits $j$ before hitting $k$.
2. $p_i(T_j = T_S)$, with $j$ in $S$, denotes the probability that the random walk starting from $i$ hits the node $j$ in $S$ before hitting any other node in $S$.

By applying (21) to the effective resistance of link $l = \{i,j\}$ in the shorted and the cut network, it follows

$$r^U_{l} = \frac{G^U_{ij}}{w_i}, \quad r^L_{l} = \frac{G^L_{ij}}{w_j}$$

where we recall that $G^U_{ij}$ and $G^L_{ij}$ are the expected number of visits on $i$, before hitting $j$, starting from $i$, of the random walk defined on $G^U_i$ and $G^L_i$, respectively. The visits on $i$ before hitting $j$ can be divided in two disjoint sets: 1) the visits before hitting $j$ and before visiting any node in $N_d$, and 2) the visits before hitting $j$ but after at least a node in $N_d$ has been visited. Let $G^{U>\text{res}}_{ij}$ denote the expected number of visits to $i$, starting from $i$, before hitting any node in $N_d$ and before hitting the absorbing node $j$ (for simplicity of notation, we omit the index $j$ from now on). Note that $G^U_{ij}$ and $G^L_{ij}$ differ only in the node $s$, which is the node obtained by shorting all the nodes at distance greater than $d$ from $i$ and $j$. Since $s$ cannot be reached before hitting nodes in $N_d$ before, $G^{U>\text{res}}_{ij}$ is equivalent when computed on $G^U_i$ and $G^L_i$. Thus, we can write the following decomposition:

$$G^U_{ij} = G^{U>\text{res}}_{ij} + G^{U>\text{res}}_{ij}$$

$$G^L_{ij} = G^{L>\text{res}}_{ij} + G^{L>\text{res}}_{ij}$$

where $G^{U>\text{res}}_{ij}$ and $G^{L>\text{res}}_{ij}$ indicate, respectively, the expected visits in $i$, starting from $i$, before hitting $j$ and after hitting any node in $N_d$, on $G^U_i$ and $G^L_i$, respectively. This implies by (31)

$$r^U_{l} - r^L_{l} = \frac{G^{U>\text{res}}_{ij}}{w_i} - \frac{G^{L>\text{res}}_{ij}}{w_i}.$$
of a geometric sum. Therefore
\[
G_{ii}^{U > N_d} = \sum_{g \in N_d} p_i(T_g = T_{j:U > N_d}) p_{g:d}^U (T_i < T_j)
\]
(1)
\[
\cdot \sum_{k=1}^{\infty} k \left( p_i^U (T_i^+ < T_j) \right)^{k-1} \left( 1 - p_i^U (T_i^+ < T_j) \right)
\]
(2)
\[
= \sum_{g \in N_d} p_i(T_g = T_{j:U > N_d}) p_{g:d}^U (T_i < T_j)
\]
(3)
\[
= \sum_{g \in N_d} p_i(T_g = T_{j:U > N_d}) p_{g:d}^U (T_i < T_j)
\]
(4)
where
1) probability of hitting \( g \) before hitting \( j \) and any other node in \( N_d \) starting from \( i \);
2) probability of hitting \( i \) before \( j \) starting from \( g \);
3) probability of hitting \( k - 1 \) times \( i \) before hitting \( j \) starting from \( i \);
4) probability of hitting \( j \) before returning in \( i \) starting from \( i \).

Similarly
\[
G_{ii}^{L > N_d} = \sum_{g \in N_d} p_i(T_g = T_{j:L > N_d}) p_{g:d}^L (T_i < T_j)
\]
\[
\cdot \sum_{k=1}^{\infty} k \left( p_i^L (T_i^- < T_j) \right)^{k-1} \left( 1 - p_i^L (T_i^- < T_j) \right)
\]
\[
= \sum_{g \in N_d} p_i(T_g = T_{j:L > N_d}) p_{g:d}^L (T_i < T_j)
\]
\[
= \sum_{g \in N_d} p_i(T_g = T_{j:L > N_d}) p_{g:d}^L (T_i < T_j)
\]
Substituting in (32), we get
\[
r_{l:d}^U - r_{l:d}^L = \frac{1}{w_i} \sum_{g \in N_d} p_i(T_g = T_{j:L > N_d}) \left( p_{g:d}^U (T_i < T_j) - \frac{p_{g:d}^L (T_i < T_j)}{1 - p_i^L (T_i^- < T_j)} \right)
\]
(33)
From (21), it follows
\[
r_{l:d}^U = \frac{1}{u_i p_{l:d}^U (T_j < T_i^+)} = \frac{1}{u_i \left( 1 - p_i^U (T_i^+ < T_j) \right)}
\]
\[
r_{l:d}^L = \frac{1}{u_i p_{l:d}^L (T_j < T_i^+)} = \frac{1}{u_i \left( 1 - p_i^L (T_i^- < T_j) \right)}
\]
Therefore, \( r_{l:d}^U - r_{l:d}^L \) reads
\[
\sum_{g \in N_d} p_i(T_g = T_{j:L > N_d}) \left( p_{g:d}^U (T_i < T_j) - \frac{p_{g:d}^L (T_i < T_j)}{1 - p_i^L (T_i^- < T_j)} \right) \]
\[
= \sum_{g \in N_d} p_i(T_g = T_{j:L > N_d}) \left( p_{g:d}^U (T_i < T_j) - \frac{p_{g:d}^L (T_i < T_j)}{1 - p_i^L (T_i^- < T_j)} \right) r_{l:d}^U
\]
\[
+ \sum_{g \in N_d} p_i(T_g = T_{j:L > N_d}) p_{g:d}^L (T_i < T_j) (r_{l:d}^U - r_{l:d}^L)
\]
\[
\leq \sum_{g \in N_d} p_i(T_g = T_{j:L > N_d}) \left( p_{g:d}^U (T_i < T_j) - \frac{p_{g:d}^L (T_i < T_j)}{1 - p_i^L (T_i^- < T_j)} \right) r_{l:d}^U
\]
Fig. 10. Double tree network is equivalent to a biased random walk like this.
\[
+ \sum_{g \in N_d} p_i(T_g = T_{j:L > N_d}) (r_{l:d}^U - r_{l:d}^L)
\]
\[
= \sum_{g \in N_d} p_i(T_g = T_{j:L > N_d}) \left( p_{g:d}^U (T_i < T_j) - \frac{p_{g:d}^L (T_i < T_j)}{1 - p_i^L (T_i^- < T_j)} \right) r_{l:d}^U
\]
\[
+ \sum_{g \in N_d} p_i(T_g = T_{j:L > N_d}) (r_{l:d}^U - r_{l:d}^L)
\]
where the last inequality follows from \( p_{g:d}^U (T_i < T_j) \leq 1 \) and the last equality from the fact that \( p_i(T_{N_d} < T_j) = \sum_{g \in N_d} p_i(T_g = T_{j:L > N_d}) \).

\[\text{APPENDIX C}
\]
\[\text{MORE DETAILS ON SECTION V-B4}\]
We prove that the double tree network is not recurrent by showing that \( p_i(T_i < T_{N_d}) \) is the same as in a biased random walk. Indeed, from any \( d \), the probability of going from a node at distance \( d \) from \( i \) to a node at distance \( d + 1 \) and \( d - 1 \) are 2/3 and 1/3, respectively. Hence, the double tree is equivalent to a biased random walk on a line as in Fig. 10, which is not recurrent [36, Example 21.2]. Since in the complete network and in the cut network, there are no paths between \( i \) and \( j \) except link \( l = \{ i, j \} \) [see Fig. 11(a) and (b)], \( r_{l:d}^U = 1 \). Computing \( r_{l:d}^U \) is more involved. First, referring to Fig. 11, we note that, because of the symmetry of the network, the effective resistance between \( i \) and \( j \) in the shorted network (c), which is \( r_{l:d}^U \), is equivalent to the effective resistance in (d). Indeed, if we set voltage \( v_i = 1 \) and \( v_j = 0 \), because of symmetry every yellow node has voltage 1/2. Thus, adding infinite conductance between all of them, i.e., shorting them, does not affect the current in the network (this procedure is also known in the literature as gluing, see [36, Sec. 9.4]) and, therefore, the effective resistance. The network (d) is series–parallel, so that the effective resistance can be computed iteratively. Specifically, we refer to Fig. 12 to illustrate the recursion that leads to \( r_{l:d}^U \). From top to bottom, one can see that the first network has effective resistance between the two blue nodes equal to 3. The second network is the parallel composition of two of these, in series with two single links. This procedure is iteratively repeated \( d - 1 \) times (in Fig. 12 only once since \( d = 2 \)), leading to a network that, composed in parallel with a
From above to below: (a) Double tree network. (b) Network in Fig. 11(d). Hence, is the result of the following recursion:

\[
\begin{cases}
  r(0) = 3 \\
  r(n) = 2 + \frac{r(n-1)}{2}, \quad d > n \geq 1 \\
  r^L_d = \left(1 + \frac{2}{d+1}\right)^{-1}
\end{cases}
\]

which has solution

\[
\begin{cases}
  r(n) = \frac{(2d+2)^{d+1} - 1}{2^d}, \quad d > n \geq 1 \\
  r^L_d = \frac{2^{d+1} + 1}{2^{d+2} + 1} \xrightarrow[d \to +\infty]{} \frac{1}{2}
\end{cases}
\]

REFERENCES

[1] European Union, “Urban mobility,” 2023. Accessed: Aug. 31, 2021. [Online]. Available: https://transport.ec.europa.eu/transport-themes/clean-transport-urban-transport/urban-mobility/urban-mobility-package_en

[2] P. N. Brown and J. R. Marden, “Studies on robust social influence mechanisms: Incentives for efficient network routing in uncertain settings,” IEEE Control Syst. Mag., vol. 37, no. 1, pp. 98–115, Feb. 2017.

[3] L. Fleischer, K. Jain, and M. Mahdian, “Tolls for heterogeneous selfish users in multicommodity networks and generalized congestion games,” in Proc. 45th Annu. IEEE Symp. Found. Comput. Sci., 2004, pp. 277–285.

[4] Y. Zhao and K. M. Kockelman, “On-line marginal-cost pricing across networks: Incorporating heterogeneous users and stochastic equilibria,” Transp. Res. Part B: Methodol., vol. 40, no. 5, pp. 424–435, 2006.

[5] R. Cole, Y. Dodis, and T. Roughgarden, “How much can taxes help selfish users?,” J. Comput. Syst. Sci., vol. 72, no. 3, pp. 444–467, 2006.

[6] G. Como, E. Lovisari, and K. Savla, “Convexity and robustness of dynamic network traffic assignment and control of freeway networks,” Transp. Res. Part B, Methodol., vol. 91, pp. 446–465, 2016.

[7] G. Como and R. Maggiestro, “Distributed dynamic pricing of multiscale transportation networks,” IEEE Trans. Autom. Control, vol. 67, no. 4, pp. 1625–1636, Apr. 2022.

[8] S. Das, E. Kamenica, and R. Mirka, “Reducing congestion through information design,” in Proc. 55th Annu. Allerton Conf. Commun., Control, Comput., 2017, pp. 1279–1284.

[9] E. Meigs, P. Parise, A. Ozdaglar, and D. Acemoglu, “Optimal dynamic information provision in traffic routing,” 2020, arXiv:2001.03222.

[10] M. Wu and S. Amin, “Information design for regulating traffic flows under uncertain network state,” in Proc. 57th Annu. Allerton Conf. Commun., Control, Comput., 2019, pp. 671–678.

[11] M. Wu, S. Amin, and A. E. Ozdaglar, “Value of information in Bayesian routing games,” Operations Res., vol. 69, no. 1, pp. 148–163, 2021.

[12] C. Zhu, J. S. Yue, C. V. Mandayam, D. Merugu, H. K. Abadi, and B. Prabhakar, “Reducing road congestion through incentives: A case study,” in Proc. Transp. Res. Board 94th Annu. Meeting, 2015. [Online]. Available: https://trid.trb.org/view/1336629

[13] L. J. LeBlanc, “An algorithm for the discrete network design problem,” Transp. Sci., vol. 9, no. 3, pp. 183–199, 1975.

[14] M. Beckmann, C. McGuire, and C. B. Winston, Studies in the Economics of Transportation, New Haven, CT, USA: Yale Univ. Press, 1956.

[15] J. G. Wardrop, “Road paper. Some theoretical aspects of road traffic research,” Proc. Inst. Civil Eng., vol. 1, no. 3, pp. 325–362, 1952.

[16] R. Steinberg and W. I. Zangwill, “The prevalence of Braess’ paradox,” Transp. Sci., vol. 17, no. 3, pp. 301–318, 1983.

[17] S. Dafermos and A. Nagurney, “On some traffic equilibrium theory paradoxes,” Transp. Res. Part B, Methodol., vol. 18, no. 2, pp. 101–110, 1984.

[18] S. W. Chiou, “Bilevel programming for the continuous transport network design problem,” Transp. Res. Part B, Methodol., vol. 39, no. 4, pp. 361–383, 2005.

[19] C. Li, H. Yang, D. Zha, and Q. Meng, “A global optimization method for continuous network design problems,” Transp. Res. Part B, Methodol., vol. 46, no. 9, pp. 1144–1158, 2012.

[20] G. Wang, Z. Gao, M. Xu, and H. Sun, “Models and a relaxation algorithm for continuous network design problem with a tradable credit scheme and equity constraints,” Comput. Operations Res., vol. 41, pp. 252–261, 2014.

[21] Z. Gao, J. Wu, and H. Sun, “Solution algorithm for the bi-level discrete network design problem,” Transp. Res. Part B, Methodol., vol. 39, no. 6, pp. 479–495, 2005.

[22] S. Wang, Q. Meng, and H. Yang, “Globalization methods for the discrete network design problem,” Transp. Res. Part B, Methodol., vol. 50, pp. 42–60, 2013.

[23] H. Poorzahedy and O. M. Rouhani, “Hybrid meta-heuristic algorithms for solving network design problem,” Eur. J. Oper. Res., vol. 182, no. 2, pp. 578–596, 2007.

[24] P. Fontaine and S. Minner, “A dynamic discrete network design problem for maintenance planning in traffic networks,” Eur. J. Oper. Res., vol. 253, no. 2, pp. 757–772, 2017.

[25] T. Roughgarden, “On the severity of Braess’ paradox: Designing networks for selfish users is hard,” J. Comput. Syst. Sci., vol. 72, no. 5, pp. 922–953, 2006.

[26] D. Fotakis, A. C. Kaporos, and P. G. Spirakis, “Efficient methods for selfish network design,” Theor. Comput. Sci., vol. 448, pp. 9–20, 2012.

[27] H. Yang and M. G. H. Bell, “Models and algorithms for road network design: A review and some new developments,” Transp. Res. Part B, Methodol., vol. 18, no. 3, pp. 257–278, 1994.

[28] R. Z. Farahani, E. Miandoabchi, W. Y. Szeto, and H. Rashidi, “A review of urban transportation network design problems,” Eur. J. Oper. Res., vol. 229, no. 2, pp. 281–302, 2013.
Leonardo Cianfanelli (Member, IEEE) received the B.Sc. (cum laude) degree in physics and astrophysics from Università degli Studi di Firenze, Florence, Italy, in 2014, the M.S. (cum laude) degree in physics of complex systems from Università di Torino, Torino, Italy, in 2017, and the Ph.D. degree in pure and applied mathematics from Politecnico di Torino in 2022. He is currently a Research Assistant with the Department of Mathematical Sciences, Politecnico di Torino. He was a Visiting Student with the Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, Cambridge, MA, USA, in 2008-2011. His research interests include dynamics and control in network systems, with applications to transportation networks and epidemics.

Giacomo Como (Member, IEEE) received the B.Sc., M.S., and Ph.D. degrees in applied mathematics from Politecnico di Torino, Torino, Italy, in 2002, 2004, and 2008, respectively. He is currently a Professor with the Department of Mathematical Sciences, Politecnico di Torino, Torino, and a Senior Lecturer with the Automatic Control Department, Lund University, Lund, Sweden. He was a Visiting Assistant in research with Yale University during 2006-2007 and a Postdoctoral Associate with the Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, Cambridge, MA, USA, from 2016 to 2020. His research interests include dynamics, information, and control in network systems with applications to infrastructure, social, and economic networks.

Prof. Como is currently a Senior Editor for the IEEE TRANSACTIONS ON CONTROL OF NETWORK SYSTEMS, an Associate Editor for Automatica, and the Chair of the IEEE-CSS Technical Committee on Networks and Communications. He was an Associate Editor for the IEEE TRANSACTIONS ON NETWORK SCIENCE AND ENGINEERING from 2015 to 2021 and for the IEEE TRANSACTIONS ON CONTROL OF NETWORK SYSTEMS from 2016 to 2022. He was the IPC Chair of the IFAC Workshop NecSys’15 and a Semiplenary Speaker at the International Symposium MTNS’16. He is the recipient of the 2015 George S. Axelby Outstanding Paper Award.

Asuman E. Ozdaglar received the B.S. degree in electrical engineering from the Middle East Technical University, Ankara, Turkey, in 1996, and the S.M. and Ph.D. degrees in electrical engineering and computer science from the Massachusetts Institute of Technology (MIT), Cambridge, MA, USA, in 1998 and 2003, respectively. She is currently the Distinguished Professor of Engineering with the Electrical Engineering and Computer Science (EECS) Department, MIT. She is also the Department Head of EECS and the Deputy Dean of Academics in the new Schwarzman College of Computing, MIT. She is the coauthor of the book entitled Convex Analysis and Optimization (Athena Scientific, 2003). Her research expertise includes optimization theory and algorithms, with focus on machine learning and large-scale data processing, distributed optimization and control, game theory, with applications in technological, social, and economic networks, and network analysis with special emphasis on contagious processes, systemic risk, and dynamic control.

Dr. Ozdaglar was the recipient of a Microsoft Fellowship, the MIT Graduate Student Council Teaching Award, the NSF Career Award, the 2008 Donald P. Eckman Award of the American Automatic Control Council, the Class of 1943 Career Development Chair, the inaugural Steven and Renee Innovation Fellowship, and the 2014 Spira Teaching Award. She served on the Board of Governors of the Control System Society in 2010 and was an Associate Editor for the IEEE TRANSACTIONS ON AUTOMATIC CONTROL. She was the inaugural area Co-Editor for the area entitled Games, Information and Networks in the journal Operations Research.

Francesca Parise (Member, IEEE) received the B.Sc. and M.Sc. (summa cum laude) degrees in information and automation engineering from the University of Padova, Padua, Italy, in 2010 and 2012, respectively, the graduate degree from the Galilean School of Excellence, University of Padova, in 2013, and the Ph.D. degree in information technology and electrical engineering from the Automatic Control Laboratory, ETH Zurich, Zurich, Switzerland, in 2016. She conducted her master's thesis research with Imperial College London, London, U.K., in 2012.

From 2016 to 2020, she was a Postdoctoral Researcher with the Laboratory for Information and Decision Systems, MIT, and is currently an Assistant Professor of Electrical and Computer Engineering with Cornell University, Ithaca, NY. Her research interests include identification, analysis, and control of multilayer systems, with application to transportation, social, economic networks, and systems biology.