Monotonicity of the Holevo quantity: a necessary condition for equality in terms of a channel and its applications.

M.E. Shirokov
Steklov Mathematical Institute, RAS, Moscow

Abstract
A condition for reversibility (sufficiency) of a channel with respect to a given countable family of states with bounded rank is obtained.
This condition shows that a quantum channel preserving the Holevo quantity of at least one (discrete or continuous) ensemble of states with rank \( \leq r \) has the r-partially entanglement-breaking complementary channel. Several applications of this result are considered. In particular, it is shown that coincidence of the constrained Holevo capacity and the quantum mutual information of a quantum channel at least at one full rank state implies that this channel is entanglement-breaking.

Contents
1 Introduction 2
2 Preliminaries 3
3 A condition for reversibility of a channel with respect to a countable set of states 6

*email:msh@mi.ras.ru
1 Introduction

The Holevo quantity $\chi(\{\pi_i, \rho_i\})$ of an ensemble of quantum states $\{\pi_i, \rho_i\}$ provides an upper bound for accessible classical information which can be obtained by applying a quantum measurement [9]. The fundamental monotonicity property of the relative entropy implies non-increasing of the Holevo quantity under action of an arbitrary quantum channel $\Phi$, that is

$$\chi(\{\pi_i, \Phi(\rho_i)\}) \leq \chi(\{\pi_i, \rho_i\}) \quad (1)$$

for any ensemble of quantum states $\{\pi_i, \rho_i\}$.

Necessary and sufficient conditions for the case of equality in fundamental entropic inequalities of quantum theory have been intensively studied (see [7, 20, 25, 27, 32] and the references therein). In particular, two characterizations of the equality in (1) in finite dimensions are obtained in [7, Examples 4 and 9]. The first one derived from Petz’s theorem (Theorem 3 in Appendix 6.1) states that the equality in (1) holds if and only if

$$\rho_i = A\Phi^*(B\Phi(\rho_i)B)A, \quad A = (\bar{\rho})^{1/2}, \quad B = (\Phi(\bar{\rho}))^{-1/2}, \quad \forall i, \quad (2)$$

where $\Phi^*$ is a dual map to the channel $\Phi$ and $\bar{\rho}$ is the average state of the ensemble $\{\pi_i, \rho_i\}$. The second characterization of the equality in (1) is derived from the characterization of the equality case in the strong subadditivity of the quantum entropy by identifying the channel $\Phi$ with a subchannel of a
partial trace, so it is not clear how to apply this condition to a given quantum channel $\Phi$.

Condition (2) means reversibility (sufficiency) of the channel $\Phi$ with respect to the set $\{\rho_i\}$ of quantum states [19, 21, 24].

In Section 3 we prove a simple necessary condition for reversibility of a channel with respect to a given countable family of states with bounded rank, which implies a necessary condition for the equality in (1) expressed in terms of the channel $\Phi$. The main advantage of this condition consists in possibility to use it in analysis of entropic characteristics of a given quantum channel determined as extremal values of particular functionals depending on the Holevo quantity (such as the Holevo capacity and the related characteristics).

In Section 4 we generalize the above condition to the case of continuous ensembles.

Several applications of the obtained conditions concerning the notions of the Holevo capacity and of the minimal output entropy of a quantum channel as well as properties of the quantum conditional entropy are considered in Section 5. In particular, it is shown that the equality in the general inequality

$$\bar{C}(\Phi, \rho) \leq I(\Phi, \rho),$$

connecting the constrained Holevo capacity $C(\Phi, \rho)$ and the quantum mutual information $I(\Phi, \rho)$ of a quantum channel $\Phi$ at a state $\rho$, implies that the restriction of the channel $\Phi$ to the set of states supported by the subspace $\text{supp}\rho$ is entanglement-breaking.

## 2 Preliminaries

Let $\mathcal{H}, \mathcal{K}$ be either finite dimensional or separable Hilbert spaces, $\mathfrak{B}(\mathcal{H})$ and $\mathfrak{T}(\mathcal{H})$ – the Banach spaces of all bounded operators in $\mathcal{H}$ and of all trace-class operators in $\mathcal{H}$ correspondingly, $\mathfrak{B}_+(\mathcal{H})$ – the positive cone in $\mathfrak{B}(\mathcal{H})$ and $\mathfrak{S}(\mathcal{H})$ – the closed convex subset of $\mathfrak{T}(\mathcal{H})$ consisting of positive operators with unit trace called states [1, 18].

Denote by $I_{\mathcal{H}}$ and $\text{Id}_{\mathcal{H}}$ the unit operator in a Hilbert space $\mathcal{H}$ and the identity transformation of the Banach space $\mathfrak{T}(\mathcal{H})$ correspondingly.

A linear completely positive trace preserving map $\Phi : \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_B)$ is called quantum channel [18]. We will say that the above channel $\Phi$ is isometrically equivalent to the channel $\Phi' : \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}'_B)$ if there is a
partial isometry $W : \mathcal{H}_B \rightarrow \mathcal{H}_{B'}$ such that
\[ \Phi'(A) = W \Phi(A) W^*, \quad \Phi(A) = W^* \Phi'(A) W, \quad A \in \mathfrak{T}(\mathcal{H}_A). \quad (3) \]

For a given channel $\Phi : \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ the Stinespring theorem implies existence of a Hilbert space $\mathcal{H}_E$ and of an isometry $V : \mathcal{H}_A \rightarrow \mathcal{H}_B \otimes \mathcal{H}_E$ such that
\[ \Phi(A) = \operatorname{Tr}_{\mathcal{H}_E} VAV^*, \quad A \in \mathfrak{T}(\mathcal{H}_A). \quad (4) \]

A quantum channel
\[ \mathfrak{T}(\mathcal{H}_A) \ni A \mapsto \hat{\Phi}(A) = \operatorname{Tr}_{\mathcal{H}_B} VAV^* \in \mathfrak{T}(\mathcal{H}_E) \quad (5) \]
is called complementary to the channel $\Phi$ \[12\] The complementary channel is defined uniquely in the following sense: if $\hat{\Phi}' : \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_{E'})$ is a channel defined by \[5\] via the Stinespring isometry $V' : \mathcal{H}_A \rightarrow \mathcal{H}_B \otimes \mathcal{H}_{E'}$, then the channels $\hat{\Phi}$ and $\hat{\Phi}'$ are isometrically equivalent in the sense of \[3\] \[12\].

The Stinespring representation \[4\] is called minimal if the subspace
\[ \mathcal{M} = \{ (X \otimes I_E) V|\varphi \rangle \mid \varphi \in \mathcal{H}_A, X \in \mathfrak{B}(\mathcal{H}_B) \} \]
is dense in $\mathcal{H}_B \otimes \mathcal{H}_E$. The complementary channel $\hat{\Phi}$ defined by \[5\] via the minimal Stinespring representation has the following property:
\[ \hat{\Phi}(\rho) \text{ is a full rank state in } \mathfrak{S}(\mathcal{H}_E) \text{ for any full rank state } \rho \text{ in } \mathfrak{S}(\mathcal{H}_A). \quad (6) \]

The Stinespring representation \[4\] generates the Kraus representation
\[ \Phi(A) = \sum_k V_k AV_k^*, \quad A \in \mathfrak{T}(\mathcal{H}), \quad (7) \]
where $\{V_k\}$ is the set of bounded linear operators from $\mathcal{H}_A$ into $\mathcal{H}_B$ such that $\sum_k V_k^* V_k = I_{\mathcal{H}_A}$ defined by the relation
\[ \langle \varphi | V_k \psi \rangle = \langle \varphi \otimes k | V \psi \rangle, \quad \varphi \in \mathcal{H}_B, \psi \in \mathcal{H}_A, \]
where $\{|k\rangle\}$ is a particular orthonormal basis in the space $\mathcal{H}_E$. The corresponding complementary channel is expressed as follows
\[ \hat{\Phi}(A) = \sum_{k,l} \operatorname{Tr} [V_k AV_l^*] |k\rangle \langle l|, \quad A \in \mathfrak{T}(\mathcal{H}). \quad (8) \]

\[1\] The quantum channel $\hat{\Phi}$ is also called conjugate to the channel $\Phi$ \[16\].
The Schmidt rank of a pure state \( \omega \) in \( \mathcal{S}(\mathcal{H} \otimes \mathcal{K}) \) can be defined as the operator rank of the isomorphic states \( \text{Tr}_{\mathcal{K}} \omega \) and \( \text{Tr}_{\mathcal{H}} \omega \) \[31\].

The Schmidt class \( \mathcal{S}_r \) of order \( r \in \mathbb{N} \) is the minimal convex closed subset of \( \mathcal{S}(\mathcal{H} \otimes \mathcal{K}) \) containing all pure states with the Schmidt rank \( \leq r \), i.e. \( \mathcal{S}_r \) is the convex closure of these pure states \[31, 28\]. In this notation \( \mathcal{S}_1 \) is the set of all separable (non-entangled) states in \( \mathcal{S}(\mathcal{H} \otimes \mathcal{K}) \).

A channel \( \Phi \) is called \textit{entanglement-breaking} if for an arbitrary Hilbert space \( \mathcal{K} \) the state \( \Phi \otimes \text{Id}_\mathcal{K}(\omega) \) is separable for any state \( \omega \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{K}) \) \[15\]. This notion is generalized in \[2\] as follows.

**Definition 1.** A channel \( \Phi : \mathcal{T}(\mathcal{H}_A) \rightarrow \mathcal{T}(\mathcal{H}_B) \) is called \textit{r-partially entanglement-breaking} (briefly \( r \)-PEB) if for an arbitrary Hilbert space \( \mathcal{K} \) the state \( \Phi \otimes \text{Id}_\mathcal{K}(\omega) \) belongs to the Schmidt class \( \mathcal{S}_r \subset \mathcal{S}(\mathcal{H}_B \otimes \mathcal{K}) \) for any state \( \omega \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{K}) \).

In this notation entanglement-breaking channels are \( 1 \)-PEB channels. Properties of \( r \)-PEB channels in finite dimensions are studied in \[2\], where it is proved, in particular, that the class of \( r \)-PEB channels coincides with the class of channels having Kraus representation \[7\] such that \( \text{rank} V_k \leq r \) for all \( k \). But in infinite dimensions the first class is essentially wider than the second one, moreover, for each \( r \) there exist \( r \)-PEB channels such that all operators in any their Kraus representations have infinite rank \[28\].

If a channel \( \Phi \) has Kraus representation \[7\] such that \( \text{rank} V_k = 1 \) for all \( k \) then representation \[8\] shows that the complementary channel \( \widehat{\Phi} \) is pseudo-diagonal in the sense of the following definition \[3, 12, 16\].

**Definition 2.** A channel \( \Phi : \mathcal{S}(\mathcal{H}_A) \rightarrow \mathcal{S}(\mathcal{H}_B) \) is called \textit{pseudo-diagonal} if it has the representation

\[
\Phi(\rho) = \sum_{i,j} c_{ij} \langle \psi_i | \rho | \psi_j \rangle | i \rangle \langle j |, \quad \rho \in \mathcal{S}(\mathcal{H}_A),
\]

where \( \|c_{ij}\| \) is a Gram matrix of some collection of unit vectors, \( \{|\psi_i\rangle\} \) is a collection of vectors in \( \mathcal{H}_A \) satisfying the overcompleteness relation \( \sum_i |\psi_i\rangle \langle \psi_i| = I_{\mathcal{H}_A} \) and \( \{|i\rangle\} \) is an orthonormal basis in \( \mathcal{H}_B \).

\[1\]In finite dimensions the convex closure coincides with the convex hull by the Caratheodory theorem, but in infinite dimensions even the set of all \textit{countable} convex mixtures of pure states with the Schmidt rank \( \leq r \) is a proper subset of \( \mathcal{S}_r \) for each \( r \) \[28\].
Let $H(\rho)$ and $H(\rho\|\sigma)$ be respectively the von Neumann entropy of the state $\rho$ and the quantum relative entropy of the states $\rho$ and $\sigma$ \cite{17, 18, 22}.

A finite or countable collection of states $\{\rho_i\}$ with the corresponding probability distribution $\{\pi_i\}$ is called \textit{ensemble} and denoted $\{\pi_i, \rho_i\}$. The state $\bar{\rho} = \sum_i \pi_i \rho_i$ is called the \textit{average state} of the ensemble $\{\pi_i, \rho_i\}$.

The Holevo quantity of an ensemble $\{\pi_i, \rho_i\}$ is defined as follows

$$\chi(\{\pi_i, \rho_i\}) \doteq \sum_i \pi_i H(\rho_i\|\bar{\rho}) = H(\bar{\rho}) - \sum_i \pi_i H(\rho_i),$$

where the second expression is valid under the condition $H(\bar{\rho}) < +\infty$.

By monotonicity of the relative entropy for an arbitrary quantum channel $\Phi$ we have

$$\chi(\{\pi_i, \Phi(\rho_i)\}) \leq \chi(\{\pi_i, \rho_i\}). \tag{9}$$

\textbf{Remark 1.} If $H(\bar{\rho}) < +\infty$ and $H(\Phi(\bar{\rho})) < +\infty$ then inequality (9) means convexity of the entropy gain $H(\Phi(\rho)) - H(\rho)$ of the channel $\Phi$.

A necessary condition for the equality in (9) expressed in terms of the channel $\Phi$ is obtained in the next section (Corollary \[\[\])

\section{A condition for reversibility of a channel with respect to a countable set of states}

Let $\{\rho_i\}$ be a finite or countable set of states in $\mathcal{S}(\mathcal{H})$ and $\{\pi_i\}$ be a non-generate probability distribution. By Petz’s theorem (Theorem \[\[\] in the Appendix 6.1) if the Holevo quantity of an ensemble $\{\pi_i, \rho_i\}$ is finite then the equality in (9) holds if and only if the channel $\Phi$ is \textit{reversible} with respect to the set $\{\rho_i\}$ in the sense of the following definition.

\textbf{Definition 3.} \cite{21} A channel $\Phi : \mathcal{S}(\mathcal{H}_A) \rightarrow \mathcal{S}(\mathcal{H}_B)$ is reversible with respect to a set $\mathcal{S} \subseteq \mathcal{S}(\mathcal{H}_A)$ if there exists a channel $\Psi : \mathcal{S}(\mathcal{H}_B) \rightarrow \mathcal{S}(\mathcal{H}_A)$ such that $\rho = \Psi \circ \Phi(\rho)$ for all $\rho \in \mathcal{S}$.

The following theorem gives a necessary condition for reversibility of a channel with respect to a countable set of states with bounded rank.

\footnote{This property is also called sufficiency of the channel $\Phi$ with respect to the set $\mathcal{S}$ \cite{19, 24}.}

6
Theorem 1. Let \( \Phi : \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_B) \) be a quantum channel and \( \hat{\Phi} : \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_E) \) be its complementary channel. Let \( \{\rho_i\}_{i=1}^n, n \leq +\infty \), be a set of states in \( \mathcal{S}(\mathcal{H}_A) \) such that \( \sup_i \text{Tr} \rho_i > 0 \) for any nonzero operator \( A \) in \( \mathfrak{B}_A(\mathcal{H}_A) \) and \( \text{rank} \rho_i \leq r \in \mathbb{N} \) for all \( i \).

If the channel \( \Phi \) is reversible with respect to the set \( \{\rho_i\} \) then the channel \( \hat{\Phi} \) has Kraus representation (7) such that \( \text{rank} V_k \leq r \) for all \( k \) and hence \( \hat{\Phi} \) is a \( r \)-partially entanglement-breaking channel (Def[14]).

If the above hypothesis holds with \( r = 1 \), i.e. \( \rho_i = |\varphi_i\rangle\langle \varphi_i| \) for all \( i \), then the channel \( \Phi \) is isometrically equivalent \(^4\) (in the sense of (3)) to the pseudo-diagonal channel

\[
\Phi'(\rho) = \sum_{i,j,k,l} \langle \phi_i|\rho|\phi_k\rangle \langle \psi_{kl}|\psi_{ij}\rangle |i\otimes j\rangle\langle k\otimes l|
\]

from \( \mathcal{S}(\mathcal{H}_A) \) into \( \mathcal{S}(\mathcal{H}_n \otimes \mathcal{H}_B) \), where \( \{|\phi_i\rangle\}_{i=1}^n \) is an overcomplete system of vectors in \( \mathcal{H}_A \) defined by means of an arbitrary non-generate probability distribution \( \{\pi_i\}_{i=1}^n \) as follows

\[
|\phi_i\rangle = \pi_i^{1/2} (\bar{\rho}_\pi)^{-1/2} |\varphi_i\rangle, \quad \bar{\rho}_\pi = \sum_{i=1}^n \pi_i |\varphi_i\rangle\langle \varphi_i|,
\]

\( \{|\psi_{ij}\rangle\} \) is a collection of vectors such that \( \sum_j \|\psi_{ij}\|^2 = 1 \) for all \( i \), \( \{|i\rangle\}_{i=1}^n \) and \( \{|j\rangle\} \) are orthonormal base in \( \mathcal{H}_n \) and in \( \mathcal{H}_B \) correspondingly.

The main assertion of Theorem 1 means that the channel \( \hat{\Phi} \) has the following property: for an arbitrary Hilbert space \( \mathcal{K} \) and any state \( \omega \) in \( \mathcal{S}(\mathcal{H}_A \otimes \mathcal{K}) \) the state \( \hat{\Phi} \otimes \text{Id}_\mathcal{K}(\omega) \) is a countably decomposable state in the Schmidt class \( \mathcal{S}_r \subset \mathcal{S}(\mathcal{H}_E \otimes \mathcal{K}) \), i.e. it can be represented as a countable convex mixture of pure states having the Schmidt rank \( \leq r \) (there exist states in \( \mathcal{S}_r \) which are not countably decomposable \(^{28}\)).

The last assertion of Theorem 1 gives a necessary and sufficient condition for reversibility of the channel \( \Phi \) provided the set \( \{|\varphi_i\rangle\langle \varphi_i|\} \) consists of orthogonal states (in this case \( |\phi_i\rangle = |\varphi_i\rangle \) for all \( i \) and \( \mathcal{H}_n = \mathcal{H}_A \)).

Proof. Let \( \hat{\Phi}(\rho) = \sum_{k=1}^m V_k \rho V_k^* \), \( m \leq +\infty \), be the Kraus representation of the channel \( \hat{\Phi} : \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_E) \) obtained via its minimal Stinespring representation with the isometry \( V : \mathcal{H}_A \to \mathcal{H}_E \otimes \mathcal{H}_C \) (see Section 2). The

\(^4\)By Lemma 1 below the reversibility with respect to a given set of states is a common property for two isometrically equivalent channels.
complementary channel $\Psi = \hat{\Phi}$ to the channel $\hat{\Phi}$ defined via this representation is expressed as follows

$$\mathcal{S}(\mathcal{H}_A) \ni \rho \mapsto \Psi(\rho) = \sum_{k,l=1}^m \text{Tr} V_k \rho V_l^* |k\rangle \langle l| \in \mathcal{S}(\mathcal{H}_C),$$

where $\{|k\rangle\}_{k=1}^m$ is an orthonormal basis in the $m$-dimensional Hilbert space $\mathcal{H}_C$.

Since $\Psi = \hat{\Phi}$, there exists a partial isometry $W : \mathcal{H}_B \to \mathcal{H}_C$ such that

$$\Psi(\rho) = W \Phi(\rho) W^*, \quad \Phi(\rho) = W^* \Psi(\rho) W, \quad \rho \in \mathcal{S}(\mathcal{H}_A).$$

By Lemma 1 below the channel $\Psi$ is reversible with respect to the set $\{\rho_i\}$.

Let $\{\pi_i\}_{i=1}^n$ be an arbitrary non-generate probability distribution and $\bar{\rho}$ be the average state of the ensemble $\{\pi_i, \rho_i\}_{i=1}^n$. By property (6) $\Psi(\bar{\rho})$ is a full rank state in $\mathcal{S}(\mathcal{H}_C)$. By Theorem 3 in [19] the reversibility condition implies $A_i = \Psi^*(\sum_j |\psi_{ij}\rangle\langle \psi_{ij}|)$ for all $i$, where $A_i = \pi_i(\bar{\rho})^{-1/2} \rho_i(\bar{\rho})^{-1/2}$ and $B_i = \pi_i(\Psi(\bar{\rho}))^{-1/2} \Psi(\rho_i)(\Psi(\bar{\rho}))^{-1/2}$ are positive operators in $\mathcal{B}(\mathcal{H}_A)$ and in $\mathcal{B}(\mathcal{H}_C)$ correspondingly.

Note that

$$\Psi^*(A) = \sum_{k,l=1}^m \langle l| A^* V_l V_k, \quad A \in \mathcal{B}(\mathcal{H}_C).$$

Let $B_i = \sum_j |\psi_{ij}\rangle\langle \psi_{ij}|$, where $\{|\psi_{ij}\rangle\}_{j}$ is a set of vectors in $\mathcal{H}_C$, for each $i$. Since $\Psi(\bar{\rho})$ is a full rank state in $\mathcal{S}(\mathcal{H}_C)$, we have

$$\sum_{i,j} |\psi_{ij}\rangle\langle \psi_{ij}| = \sum_i B_i = I_{\mathcal{H}_C}. $$

By Lemma 2 below $\hat{\Phi}(\rho) = \sum_{i,j} W_{ij} \rho W_{ij}^*$, where $W_{ij} = \sum_{k=1}^m |\psi_{ij}\rangle \langle k| V_k$.

Since $A_i = \Psi^*(\sum_j |\psi_{ij}\rangle\langle \psi_{ij}|)$ is an operator of rank $\leq r$ for each $i$ and

$$\Psi^*(|\psi_{ij}\rangle\langle \psi_{ij}|) = \sum_{k,l=1}^m \langle l| \psi_{ij}\rangle\langle \psi_{ij}| k\rangle V_l^* V_k = W_{ij}^* W_{ij},$$

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5 This representation can be obtained by multiplying the both sides of the equality $I_{\mathcal{H}_C} = \sum_j |j\rangle\langle j|$, where $\{|j\rangle\}$ is an arbitrary basis in $\mathcal{H}_C$, by $B_i^{1/2}$. 

8
the family \( \{ W_{ij} \} \) consists of operators of rank \( \leq r \).

If \( \rho_i = |\varphi_i\rangle\langle\varphi_i| \) then \( A_i = |\phi_i\rangle\langle\phi_i| \), where the vector \( |\phi_i\rangle \) is defined by (11). Hence representation (10) can be obtained from the above arguments by using representation (8) for the channel complementary to the channel \( \hat{\Phi}(\rho) = \sum_{i,j} W_{ij} \rho W_{ij}^* \) and by noting that the above partial isometry \( W^* \) is an embedding of \( H_C \) into \( H_B \) (since \( \Psi(\hat{\rho}) \) is a full rank state in \( S(\mathcal{H}_C) \)).

**Lemma 1.** Let \( \Phi : S(\mathcal{H}_A) \rightarrow S(\mathcal{H}_B) \) and \( \Phi' : S(\mathcal{H}_A) \rightarrow S(\mathcal{H}_B') \) be quantum channels isometrically equivalent in the sense of (3). If the channel \( \Phi \) is reversible with respect to a set \( \mathcal{G} \subseteq S(\mathcal{H}_A) \) then the channel \( \Phi' \) is reversible with respect to the set \( \mathcal{G} \) and vice versa.

**Proof.** Let \( \Psi \) be the reverse channel for the channel \( \Phi \), i.e. \( \Psi \circ \Phi(\rho) = \rho \) for all \( \rho \in \mathcal{G} \). Let \( \Theta(\cdot) = W^*(\cdot)W + \sigma \text{Tr}(I_{H_B'} - WW^*)(\cdot) \) be a channel from \( S(\mathcal{H}_B') \) into \( S(\mathcal{H}_B) \), where \( W \) is the partial isometry from (3) and \( \sigma \) is a fixed state in \( S(\mathcal{H}_B) \). Then \( \Psi \circ \Theta \) is a reverse channel for the channel \( \Phi' \). □

**Lemma 2.** Let \( \Phi(\rho) = \sum_{k=1}^{m} V_k \rho V_k^* \) be a quantum channel and \( \{ |k\rangle \}_{k=1}^{m} \) be an orthonormal basis in the \( m \)-dimensional Hilbert space \( H_m \), \( m \leq +\infty \). An arbitrary overcomplete system \( \{ |\psi_i\rangle \} \) of vectors in \( H_m \) generates the Kraus representation \( \Phi(\rho) = \sum_i W_i \rho W_i^* \) of the channel \( \Phi \), where \( W_i = \sum_{k=1}^{m} \langle \psi_i | k \rangle V_k \).

**Proof.** Since \( \sum_i |\psi_i\rangle\langle\psi_i| = I_{H_m} \), we have

\[
\sum_i W_i \rho W_i^* = \sum_{k,l=1}^{m} V_k \rho V_l^* \sum_i \langle \psi_i | k \rangle \langle l | \psi_i \rangle = \sum_{k=1}^{m} V_k \rho V_k^*. \quad \square
\]

By Petz’s theorem (Theorem 3 in Appendix 6.1) Theorem 1 implies the following necessary condition for the equality in (9), which is not sufficient (even in the weak sense) by Remark 3 below.

**Corollary 1.** Let \( \Phi : S(\mathcal{H}_A) \rightarrow S(\mathcal{H}_B) \) be a quantum channel and \( \hat{\Phi} : S(\mathcal{H}_A) \rightarrow S(\mathcal{H}_E) \) be its complementary channel. If there exists an ensemble \( \{ \pi_i, \rho_i \} \) with the full rank average state \( \hat{\rho} \) such that \( \text{rank} \rho_i \leq r \) for all \( i \) and

\[
\chi(\{ \pi_i, \Phi(\rho_i) \}) = \chi(\{ \pi_i, \rho_i \}) < +\infty
\]

then the channel \( \hat{\Phi} \) has Kraus representation \( \{ \] such that \( \text{rank} V_k \leq r \) for all \( k \) and hence \( \hat{\Phi} \) is a \( r \)-partially entanglement-breaking channel (Def.[1]).
Remark 2. By Corollary 1 to prove the strict inequality in (9) for all ensembles \( \{\pi_i, \rho_i\} \) such that \( \text{supp} \bar{\rho} = \mathcal{H}_A \) and \( \text{rank} \rho_i \leq r \) for all \( i \) it suffices to show that the channel \( \hat{\Phi} \) is not \( r \)-partially entanglement-breaking. This can be done by showing existence of a state \( \omega \) in \( \mathcal{S}(\mathcal{H}_A \otimes \mathcal{K}) \) such that

\[
\text{SN}(\hat{\Phi} \otimes \text{Id}_\mathcal{K}(\omega)) > r \quad \text{or} \quad E(\hat{\Phi} \otimes \text{Id}_\mathcal{K}(\omega)) > \log r, \tag{12}
\]

where \( \text{SN} \) is the Schmidt number (defined in [31] and in [28] in finite and in infinite dimensions correspondingly) and \( E \) is any convex entanglement monotone coinciding on the set of pure states with the entropy of a partial state, in particular, \( E = E_{\text{OF}} \) [26].

The condition \( \text{supp} \bar{\rho} = \mathcal{H}_A \) in Corollary 1 can be removed by considering the restrictions of the channels \( \Phi \) and \( \hat{\Phi} \) to the set \( \mathcal{S}(\mathcal{H}_\bar{\rho}) \), where \( \mathcal{H}_\bar{\rho} = \text{supp} \bar{\rho} \). Thus, to prove the strict inequality in (9) for an arbitrary ensemble \( \{\pi_i, \rho_i\} \) such that \( \text{rank} \rho_i \leq r \) for all \( i \) it suffices to show existence of a state \( \omega \) in \( \mathcal{S}(\mathcal{H}_\bar{\rho} \otimes \mathcal{K}) \) such that (12) holds.

The necessity of the condition \( \text{supp} \bar{\rho} = \mathcal{H}_A \) is discussed in Remark 5 in Section 5.1.

We complete this section by the following remark.

Remark 3. There exist quantum channels complementary to entanglement-breaking channels such that the strict inequality holds in (9) for any ensemble of pure states with the full rank average. To show this consider the channel

\[
\Phi(\rho) = \sum_{k=1}^{3} \langle \varphi_k | \rho | \varphi_k \rangle |k\rangle \langle k|,
\]

where \( |\varphi_k\rangle = \sqrt{\frac{2}{3}} \left[ \cos \frac{2}{3} \pi (k - 1), \sin \frac{2}{3} \pi (k - 1) \right]^T \), \( k = 1, 2, 3 \), are vectors in the 2-D space \( \mathcal{H}_A \) and \( \{|k\rangle\}_{k=1}^{3} \) is an orthonormal basis in the 3-D space \( \mathcal{H}_B \).

Suppose there exists an ensemble \( \{\pi_i, \rho_i\} \) of pure states with the full rank average state \( \bar{\rho} \) such that \( \chi(\{\pi_i, \Phi(\rho_i)\}) = \chi(\{\pi_i, \rho_i\}) \). Since \( \Phi(\bar{\rho}) \) is a full rank state and \( \Phi^*(A) = \sum_{k=1}^{3} \langle k|A|k\rangle |\varphi_k\rangle \langle \varphi_k| \), condition (2) implies that \( \text{rank} \Phi(\rho_i) = 1 \) for any \( i \). But this can not be valid, since it is easy to see that \( \text{rank} \Phi(\rho) > 1 \) for any \( \rho \). Hence \( \chi(\{\pi_i, \Phi(\rho_i)\}) < \chi(\{\pi_i, \rho_i\}) \) for any ensemble \( \{\pi_i, \rho_i\} \) of pure states with the full rank average. \( \square \)
4 Continuous ensembles

A continuous (generalized) ensemble of quantum states can be defined as a Borel probability measure \( \mu \) on the set \( \mathcal{S}(\mathcal{H}) \). The Holevo quantity of such ensemble (measure) \( \mu \) is defined as follows (cf. [13])

\[
\chi(\mu) = \int_{\mathcal{S}(\mathcal{H})} H(\rho \| \bar{\rho}(\mu)) \mu(d\rho),
\]

where \( \bar{\rho}(\mu) \) is the barycenter of the measure \( \mu \) defined by the Bochner integral

\[
\bar{\rho}(\mu) = \int_{\mathcal{S}(\mathcal{H})} \rho \mu(d\rho).
\]

If \( H(\bar{\rho}(\mu)) < +\infty \) then \( \chi(\mu) = H(\bar{\rho}(\mu)) - \int_{\mathcal{S}(\mathcal{H})} H(\rho) \mu(d\rho) \) [13].

Denote by \( \mathcal{P}(\mathcal{A}) \) the set of all Borel probability measures on a closed subset \( \mathcal{A} \subset \mathcal{S}(\mathcal{H}) \) endowed with the weak convergence topology [23].

The image of a continuous ensemble \( \mu \in \mathcal{P}(\mathcal{S}(\mathcal{H}_A)) \) under a channel \( \Phi : \mathcal{S}(\mathcal{H}_A) \rightarrow \mathcal{S}(\mathcal{H}_B) \) is a continuous ensemble corresponding to the measure \( \Phi(\mu) := \mu \circ \Phi^{-1} \in \mathcal{P}(\mathcal{S}(\mathcal{H}_B)) \). Its Holevo quantity can be expressed as follows

\[
\chi(\Phi(\mu)) = \int_{\mathcal{S}(\mathcal{H}_A)} H(\Phi(\rho) \| \Phi(\bar{\rho}(\mu))) \mu(d\rho)
\]

\[
= H(\Phi(\bar{\rho}(\mu))) - \int_{\mathcal{S}(\mathcal{H}_A)} H(\Phi(\rho)) \mu(d\rho),
\]

where the second formula is valid under the condition \( H(\Phi(\bar{\rho}(\mu))) < +\infty \).

We will assume in what follows that \( \bar{\rho}(\mu) \) is a full rank state in \( \mathcal{S}(\mathcal{H}_A) \) and that \( \sup_{\rho \in \mathcal{S}(\mathcal{H}_A)} \text{Tr} B \Phi(\rho) > 0 \) for any \( B \in \mathcal{B}_+(\mathcal{H}_B) \setminus \{0\} \) (otherwise we may consider restrictions to smaller subspaces \( \mathcal{H}_A' \subset \mathcal{H}_A \) and \( \mathcal{H}_B' \subset \mathcal{H}_B \)). It follows from these assumptions that \( \Phi(\bar{\rho}(\mu)) \) is a full rank state in \( \mathcal{S}(\mathcal{H}_B) \).

Similarly to the discrete case monotonicity of the relative entropy implies monotonicity of the Holevo quantity for continuous ensembles:

\[
\chi(\Phi(\mu)) \leq \chi(\mu).
\]

By using Petz’s theorem (Theorem 3 in Appendix 6.1) one can obtain the following characterization of the equality in (15):

\[
\chi(\Phi(\mu)) = \chi(\mu).
\]
Proposition 1. Let $\Phi : \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_B)$ be a quantum channel and $\mu$ be a measure in $\mathcal{P}(\mathcal{S}(\mathcal{H}_A))$ such that $\chi(\mu) < +\infty$. Let $\Theta_{\tilde{\rho}(\mu)}$ be the predual channel to the linear completely positive unital map

$$\Theta_{\tilde{\rho}(\mu)}(\cdot) = A\Phi(B(\cdot)B)A, \quad A = [\Phi(\tilde{\rho}(\mu))]^{-1/2}, \ B = [\tilde{\rho}(\mu)]^{1/2}.$$ 

The following statements are equivalent:

(i) $\chi(\Phi(\mu)) = \chi(\mu)$;

(ii) $H(\Phi(\rho)||\Phi(\tilde{\rho}(\mu))) = H(\rho||\tilde{\rho}(\mu))$ for $\mu$-almost all $\rho$ in $\mathcal{S}(\mathcal{H}_A)$;

(iii) $\rho = \Theta_{\tilde{\rho}(\mu)}(\Phi(\rho))$ for $\mu$-almost all $\rho$ in $\mathcal{S}(\mathcal{H}_A)$;

(iv) the channel $\Phi$ is reversible with respect to $\mu$-almost all $\rho$ in $\mathcal{S}(\mathcal{H}_A)$.

In contrast to Theorem 3 in [19], in Proposition 1 it is not assumed that the "dominating" state $\tilde{\rho}(\mu)$ is a countable convex mixture of some states from the support of the measure $\mu$.

Suppose the support $\mathcal{S}_\mu$ of the measure $\mu$ consists of states with rank $\leq r$. By Proposition 1 the equality in (15) implies existence of a subset $\mathcal{S} \subseteq \mathcal{S}_\mu$ such that $\mu(\mathcal{S}) = 1$ and $\rho = \Theta_{\tilde{\rho}(\mu)}(\Phi(\rho))$ for all $\rho \in \mathcal{S}$. By Lemma 2 in [19] there exists an ensemble $\{\pi_i, \rho_i\}$ of states in $\mathcal{S}$ having the average state $\bar{\rho}$ such that $\text{supp}\rho \subseteq \text{supp}\tilde{\rho}$ for all $\rho \in \mathcal{S}$ and hence $\bar{\rho}$ is a full rank state in $\mathcal{S}(\mathcal{H}_A)$ (since $\bar{\rho}(\mu) = \int_\mathcal{S} \rho \mu(d\rho)$ is a full rank state). By applying Theorem 1 to the set $\{\rho_i\}$ we obtain the following continuous version (in fact, a generalization) of Corollary 1.

**Theorem 2.** Let $\Phi : \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_B)$ be a quantum channel and $\tilde{\Phi} : \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_E)$ be its complementary channel. If there exists a measure $\mu \in \mathcal{P}(\mathcal{S}^r)$, where $\mathcal{S}^r = \{\rho \in \mathcal{S}(\mathcal{H}_A) | \text{rank} \rho \leq r\}$, with the full rank barycenter $\bar{\rho}(\mu)$ such that

$$\chi(\Phi(\mu)) = \chi(\mu) < +\infty, \quad \text{(16)}$$

then the channel $\tilde{\Phi}$ has Kraus representation (7) such that $\text{rank} V_k \leq r$ for all $k$ and hence $\tilde{\Phi}$ is a $r$-partially entanglement-breaking channel (Def[17]).

If the above hypothesis holds with $r = 1$ then the channel $\Phi$ is isometrically equivalent to a pseudo-diagonal channel (Def[2]) in the sense of (3).

**Remark 4.** Condition (16) in Theorem 2 can be replaced by the condition of reversibility of the channel $\Phi$ with respect to $\mu$-almost all $\rho$ in $\mathcal{S}(\mathcal{H}_A)$, in which finiteness of $\chi(\mu)$ is not required.
5 Applications

5.1 Finite dimensional channels

In this subsection we consider some implications of Corollary 1 assuming that \( \dim \mathcal{H}_A \) and \( \dim \mathcal{H}_B \) are finite.

The Holevo capacity of the channel \( \Phi : \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_B) \) is defined as follows (cf. [18])

\[
\bar{C}(\Phi) = \sup_{\{\pi_i, \rho_i\}} \chi(\{\pi_i, \Phi(\rho_i)\}).
\] (17)

Monotonicity the Holevo quantity shows that \( \bar{C}(\Phi) \leq \log \dim \mathcal{H}_A \) for any channel \( \Phi \). The equality holds in the above inequality for many quantum channels (for example, for the noiseless channel, for the channel \( \Phi(\rho) = \sum_k \langle k|\rho|k\rangle |k\rangle\langle k|, \) where \( \{|k\rangle\} \) is an orthonormal basis in \( \mathcal{H}_B = \mathcal{H}_A \).

Since the supremum in (17) is always achieved at some ensembles of pure states [30], Corollary 1 with \( r = 1 \) implies the following observation.

**Proposition 2.** Let \( \Phi : \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_B) \) be a quantum channel. If \( \bar{C}(\Phi) = \log \dim \mathcal{H}_A \) then the channel \( \Phi \) is isometrically equivalent to a pseudo-diagonal channel in the sense of [3] and hence it is degradable [3].

Proposition 2 can be used to show positivity of the minimal output entropy

\[
H_{\min}(\Phi) = \min_{\rho \in \mathcal{S}(\mathcal{H}_A)} H(\Phi(\rho))
\]

for a class of quantum channels.

**Corollary 2.** Let \( \Phi : \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_B), \mathcal{H}_B = \mathcal{H}_A, \) be a quantum channel covariant with respect to some irreducible representation \( \{V_g\}_{g \in G} \) of a compact group \( G \) in the sense that \( \Phi(V_g\rho V_g^*) = V_g \Phi(\rho)V_g^* \) for all \( g \in G \). If the channel \( \Phi \) is not isometrically equivalent to a pseudo-diagonal channel (in particular, is not degradable) then \( H_{\min}(\Phi) > 0 \).

**Proof.** It follows from the covariance condition of the corollary that \( \bar{C}(\Phi) = \log \dim \mathcal{H}_A - H_{\min}(\Phi) \) [10]. By Proposition 2 we have \( H_{\min}(\Phi) > 0 \).
\( \square \)

Corollary 2 shows that \( H_{\min}(\Phi) > 0 \) for any unital qubit channel, which is not isometrically equivalent to a pseudo-diagonal channel (in particular, is not degradable).
5.2 Infinite dimensional channels

In this subsection we consider two implications of Theorems 1 and 2 concerning general (finite or infinite dimensional) quantum systems and channels.

5.2.1 Strict decrease of the Holevo quantity under partial trace and strict concavity of the conditional entropy

Since the partial trace \( \mathcal{S}(\mathcal{H} \otimes \mathcal{K}) \ni \rho \mapsto \text{Tr}_{\mathcal{H}} \rho \) is not \( r \)-PEB channel for \( r < \dim \mathcal{K} \), Corollary 1 and Theorem 2 imply the following observations.

**Proposition 3.** Let \( \mathcal{H}_A = \mathcal{H}_B \otimes \mathcal{H}_E \) and \( \Phi(\rho) = \text{Tr}_{\mathcal{H}_E} \rho, \rho \in \mathcal{S}(\mathcal{H}_A) \).

A) \( \chi(\{\pi_i, \Phi(\rho_i)\}) < \chi(\{\pi_i, \rho_i\}) \) for any ensemble \( \{\pi_i, \rho_i\} \) of states in \( \mathcal{S}(\mathcal{H}_A) \) with the full rank average state such that \( \sup_i \text{rank} \rho_i < \dim \mathcal{H}_E \) and \( \chi(\{\pi_i, \rho_i\}) < +\infty \).

B) \( \chi(\Phi(\mu)) < \chi(\mu) \) for any probability measure \( \mu \) on \( \mathcal{S}(\mathcal{H}_A) \) with the full rank barycenter such that \( \sup_{\rho \in \text{supp} \mu} \text{rank} \rho < \dim \mathcal{H}_E \) and \( \chi(\mu) < +\infty \).

**Remark 5.** By the Stinespring representation every quantum channel is isomorphic to a particular subchannel of a partial trace. Since the Holevo quantity does not strict decrease for all channels, Proposition 3 clarifies necessity of the full rank average state condition in Corollary 1 and in Theorem 2.

The conditional entropy of a state \( \rho \) of a composite system \( AB \) is defined as follows

\[
H_{A|B}(\rho) \triangleq H(\rho) - H(\text{Tr}_{\mathcal{H}_A} \rho)
\]

provided

\[
H(\rho) < +\infty \quad \text{and} \quad H(\text{Tr}_{\mathcal{H}_A} \rho) < +\infty. \tag{18}
\]

By Remark 1 concavity of the function \( \rho \mapsto H_{A|B}(\rho) \) on the convex set defined by condition (18) follows from monotonicity of the Holevo quantity. Proposition 3A implies the following strict concavity property of the conditional entropy.

**Corollary 3.** Let \( \rho \) be a full rank state in \( \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B) \) satisfying (18). Then

\[
H_{A|B}(\rho) > \sum_i \pi_i H_{A|B}(\rho_i)
\]

for any ensemble \( \{\pi_i, \rho_i\} \) with the average state \( \rho \) such that \( \text{rank} \rho_i < \dim \mathcal{H}_A \) for all \( i \).
By using Proposition 3B one can obtain a continuous (integral) version of Corollary 3.

It is easy to construct an example showing that the strict concavity property of the conditional entropy stated in Corollary 3 does not hold for arbitrary state \( \rho \) and its convex decomposition.

5.2.2 A necessary condition for the equality \( \bar{C}(\Phi, \rho) = I(\Phi, \rho) \)

The constrained Holevo capacity \( \bar{C}(\Phi, \rho) \) and the quantum mutual information \( I(\Phi, \rho) \) are important entropic characteristics playing the basic roles in expressions for the classical capacity and the classical entanglement-assisted capacity of (constrained or unconstrained) quantum channel \( \Phi \) \cite{4, 11, 18}. In general

\[
\bar{C}(\Phi, \rho) \leq I(\Phi, \rho), \quad \rho \in \mathcal{S}(\mathcal{H}_A) \tag{19}
\]

(this inequality can be proved by using expression (23) below valid under the condition \( H(\rho) < +\infty \) and a simple approximation). But there exist channels \( \Phi \) for which the equality holds in (19) for some states \( \rho \). As the simplest example one can consider the channel \( \Phi(\rho) = \sum_k |k\rangle \langle k| \rho |k\rangle \langle k| \), where \( \{|k\rangle\} \) is an orthonormal basis in \( \mathcal{H}_B = \mathcal{H}_A \). For this channel the equality in (19) holds for any state \( \rho \) diagonalizable in the basis \( \{|k\rangle\} \).

In this subsection we derive from Theorem 2 a necessary condition for the equality in (19) at some state \( \rho \) expressed in terms of the channel \( \Phi \).

The constrained Holevo capacity is defined as follows

\[
\bar{C}(\Phi, \rho) = \sup_{\pi_i, \rho_i, \rho} \chi(\{\pi_i, \Phi(\rho_i)\}) = \sup_{\mu \in \mathcal{P}(\mathcal{S}(\mathcal{H}_A)), \rho(\mu) = \rho} \chi(\Phi(\mu)), \tag{20}
\]

where the second expression can be derived from Corollary 1 in \cite{13} with \( A = \{\rho\} \). If \( H(\Phi(\rho)) < +\infty \) then

\[
\bar{C}(\Phi, \rho) = H(\Phi(\rho)) - \hat{H}_\Phi(\rho),
\]

where \( \hat{H}_\Phi(\rho) = \inf_{\{\pi_i, \rho_i\}, \rho = \mu} \sum_i \pi_i H(\Phi(\rho_i)) \) (the infimum here can be taken over ensembles of pure states by concavity of the function \( \rho \mapsto H(\Phi(\rho)) \)).

\[\text{In} \cite{13} \text{ the constrained Holevo capacity} \bar{C}(\Phi, \rho) \text{ is denoted} \chi_\Phi(\rho) \text{ and called the} \chi \text{-function of the channel} \Phi. \text{ We do not use this notation, since in this paper the symbol} \chi \text{ denotes the Holevo quantity of an ensemble of quantum states.}\]
In finite dimensions the quantum mutual information is defined as follows (cf. [18])

\[
I(\Phi, \rho) = H(\rho) + H(\Phi(\rho)) - H(\hat{\Phi}(\rho)).
\]

(21)

Since in infinite dimensions the terms in the right side of (21) may be infinite, it is reasonable to define the quantum mutual information by the following formula

\[
I(\Phi, \rho) = H(\Phi \otimes \text{Id}_R(\ket{\varphi_\rho}\bra{\varphi_\rho})) \frac{\| \Phi(\rho) \|}{\| \hat{\Phi}(\rho) \|}.
\]

where \( \varphi_\rho \) is a purification vector in \( \mathcal{H}_A \otimes \mathcal{H}_R \) for the state \( \rho \in \mathcal{S}(\mathcal{H}_A) \) and \( \varphi = \text{Tr}_{\mathcal{H}_A} \ket{\varphi_\rho}\bra{\varphi_\rho} \) is a state in \( \mathcal{G}(\mathcal{H}_R) \) isomorphic to \( \rho \). If \( H(\rho) \) and \( H(\Phi(\rho)) \) are finite then the last formula for \( I(\Phi, \rho) \) coincides with (21).

**Proposition 4.** Let \( \Phi : \mathcal{G}(\mathcal{H}_A) \to \mathcal{G}(\mathcal{H}_B) \) be a quantum channel and \( \rho \) be a state in \( \mathcal{S}(\mathcal{H}_A) \) with the support \( \mathcal{H}_\rho \) such that \( H(\rho) < +\infty \) and the following condition holds

\[
\exists \mu \in \mathcal{P}(\mathcal{S}(\mathcal{H}_A)) \text{ such that } \bar{\rho}(\mu) = \rho \text{ and } \bar{C}(\Phi, \rho) = \chi(\Phi(\mu)),
\]

(22)

which means that the supremum in the second expression in (20) is attainable. If \( \bar{C}(\Phi, \rho) = I(\Phi, \rho) < +\infty \) then there exist sets \( \{\varphi_k\} \subset \mathcal{H}_\rho \) and \( \{\psi_k\} \subset \mathcal{H}_B \) such that

\[
\Phi(\sigma) = \sum_k \bra{\varphi_k} \sigma \ket{\varphi_k} \bra{\psi_k} \ket{\psi_k}, \quad \sum_k \| \varphi_k \| = 1 \quad \forall k,
\]

for any state \( \sigma \in \mathcal{S}(\mathcal{H}_\rho) \) and hence \( \Phi|_{\mathcal{S}(\mathcal{H}_\rho)} \) is an entanglement-breaking channel.

Condition (22) is valid if either \( H(\Phi(\rho)) < +\infty \) or one of the functions \( \sigma \mapsto H(\Phi(\sigma)\|\Phi(\rho)) \), \( \sigma \mapsto H(\hat{\Phi}(\sigma)\|\hat{\Phi}(\rho)) \) is continuous and bounded on the set \( \text{extr}\mathcal{G}(\mathcal{H}_A) \).

**Proof.** Without loss of generality we may consider that the measure \( \mu \) in (22) belongs to the set \( \mathcal{P}(\text{extr}\mathcal{G}(\mathcal{H}_A)) \). This follows from convexity of the function \( \sigma \mapsto H(\Phi(\sigma)\|\Phi(\rho)) \), since for an arbitrary measure \( \mu \in \mathcal{P}(\mathcal{G}(\mathcal{H}_A)) \) there exists a measure \( \hat{\mu} \in \mathcal{P}(\text{extr}\mathcal{G}(\mathcal{H}_A)) \) such that \( \bar{\rho}(\hat{\mu}) = \bar{\rho}(\mu) \) and \( \int f(\sigma)\hat{\mu}(d\sigma) \geq \int f(\sigma)\mu(d\sigma) \) for any convex lower semicontinuous nonnegative function \( f \) on \( \mathcal{S}(\mathcal{H}_A) \) (this measure \( \hat{\mu} \) can be constructed by using the arguments from the proof of the Theorem in [13]).

\(^7\)This means that \( \text{Tr}_{\mathcal{H}_R} \ket{\varphi_\rho}\bra{\varphi_\rho} = \rho \).
By Lemma 4 in the Appendix we have

\[ I(\Phi, \rho) = H(\rho) + \bar{C}(\Phi, \rho) - \bar{C}(\hat{\Phi}, \rho) = \bar{C}(\Phi, \rho) + \Delta_\Phi(\rho), \quad (23) \]

where \( \Delta_\Phi(\rho) = H(\rho) - \bar{C}(\hat{\Phi}, \rho) \geq 0 \) (by monotonicity of the Holevo quantity).

Thus \( \bar{C}(\Phi, \rho) = I(\Phi, \rho) \) means \( H(\rho) = \bar{C}(\hat{\Phi}, \rho) \). By the remark after Lemma 4 in the Appendix condition (22) implies that \( \bar{C}(\hat{\Phi}, \rho) = \chi(\mu) \).

Since \( H(\rho) = \chi(\mu) \), equality \( H(\rho) = \bar{C}(\hat{\Phi}, \rho) \) shows that the channel \( \hat{\Phi} \) preserves the Holevo quantity of the measure \( \mu \). By Theorem 2 the restriction of the channel \( \hat{\Phi} = \Phi \) to the set \( \mathcal{S}(\mathcal{H}_\rho) \) has the Kraus representation (7) such that \( \text{rank} V_k = 1 \) for all \( k \).

If \( H(\Phi(\rho)) < +\infty \) then condition (22) holds by Corollary 2 in [13].

If the function \( \sigma \mapsto H(\Phi(\sigma)||\Phi(\rho)) \) is continuous and bounded on the set \( \text{extr} \mathcal{S}(\mathcal{H}_A) \) then the function \( \mathcal{P}(\text{extr} \mathcal{S}(\mathcal{H}_A)) \ni \mu \mapsto \chi(\Phi(\mu)) \) is continuous by the definition of the weak convergence. Since the subset of \( \mathcal{P}(\text{extr} \mathcal{S}(\mathcal{H}_A)) \) consisting of measures with the barycenter \( \rho \) is compact by Proposition 2 in [13], the last function attains its least upper bound on this subset.

If the function \( \sigma \mapsto H(\hat{\Phi}(\sigma)||\hat{\Phi}(\rho)) \) is continuous and bounded on the set \( \text{extr} \mathcal{S}(\mathcal{H}_A) \) then the similar arguments shows attainability of the supremum in the definition of the value \( \bar{C}(\hat{\Phi}, \rho) \), which is equivalent to (22) by the remark after Lemma 4 in the Appendix. □

We complete this subsection by deriving from Proposition 4 a necessary condition for coincidence of the Holevo capacity with the entanglement-assisted classical capacity of the channel \( \Phi \) with the constraint defined by the inequality

\[ \text{Tr} H \rho \leq h, \quad h > 0, \quad (24) \]

where \( H \) is a positive operator – Hamiltonian of the input quantum system.

The operational definitions of the unassisted and the entanglement-assisted classical capacities of a quantum channel with constraint (24) are given in [11], where the corresponding generalizations of the HSW and BSST theorems are proved.

The case of unconstrained finite or infinite dimensional channels can be considered as a partial case of the below observations (by setting \( H = 0 \)).

\[ \text{Speaking about capacities of infinite dimensional quantum channels we have to impose particular constraints on the choice of input code-states to avoid infinite values of the capacities and to be consistent with the physical implementation of the process of information transmission [11].} \]
The Holevo capacity of the channel $\Phi$ with constraint (24) can be defined as follows

$$\bar{C}(\Phi|H,h) = \sup_{TrH\rho \leq h} \bar{C}(\Phi,\rho).$$

By the generalized HSW theorem ([11, Proposition 3]) the classical capacity of the channel $\Phi$ with constraint (24) can be expressed by the following regularization formula

$$C(\Phi|H,h) = \lim_{n \to +\infty} n^{-1}\bar{C}(\Phi^\otimes n|H_n, nh),$$

where $H_n = H \otimes I \otimes ... \otimes I + I \otimes H \otimes I \otimes ... \otimes I + ... + I \otimes ... \otimes I \otimes H$ (each of $n$ summands consists of $n$ multiples).

By the generalized BSST theorem ([11, Proposition 4]) the entanglement-assisted classical capacity of the channel $\Phi$ with constraint (24) is determined as follows

$$C_{ea}(\Phi|H,h) = \sup_{TrH\rho \leq h} I(\Phi,\rho).$$

This expression is proved in [11] under the particular technical conditions on the channel $\Phi$ and the operator $H$, which can be removed by using the approximation method [14]. We will assume that expression (26) is valid.

Proposition 4 implies the following necessary condition for coincidence of $\bar{C}(\Phi|H,h)$ and $C_{ea}(\Phi|H,h)$.

**Corollary 4.** If $\bar{C}(\Phi|H,h) = C_{ea}(\Phi|H,h) < +\infty$ and the supremum in (26) is achieved at a state $\rho_*$ such that $H(\rho_*) < +\infty$ and $H(\Phi(\rho_*)) < +\infty$ then the restriction of the channel $\Phi$ to the set $\mathcal{S}(H_{\rho_*})$, $H_{\rho_*} = \text{supp}\rho_*$, is entanglement-breaking.

Instead of the condition $H(\Phi(\rho_*)) < +\infty$ one can require that one of the functions $\sigma \mapsto H(\Phi(\sigma)\|\Phi(\rho_*))$, $\sigma \mapsto H(\Phi(\sigma)\|\Phi(\rho_*))$ is continuous and bounded on the set $\text{extr}\mathcal{S}(H_\Lambda)$.

**Remark 6.** If $\Phi$ is an unconstrained finite dimensional channel then the condition $\bar{C}(\Phi) = C_{ea}(\Phi)$ means that $\rho_*$ is the average state of an optimal ensemble for the channel $\Phi$, which always exists [30]. Hence Corollary 4 shows that $\bar{C}(\Phi) = C_{ea}(\Phi)$ implies that $\Phi$ is an entanglement-breaking channel if there exists an optimal ensemble for the channel $\Phi$ with the full rank average state. The last condition does not hold in general (see the example of non-entanglement-breaking channel such that $\bar{C}(\Phi) = C_{ea}(\Phi)$ considered in [4]).
If $\Phi$ is an infinite dimensional channel then the additional conditions in Corollary 4 imply existence of an optimal measure $\mu$ for the channel $\Phi$ with constraint (24) such that
\[
\bar{C}(\Phi|_{H,h}) = \int_{S(H_A)} H(\Phi(\sigma)\|\Phi(\rho_*)) \mu(d\sigma), \quad \bar{\rho}(\mu) = \int_{S(H_A)} \sigma \mu(d\sigma) = \rho_*.
\]
These conditions hold if the output entropy of the channel $\Phi$ (the function $\rho \mapsto H(\Phi(\rho))$) is continuous on the subset of $\mathcal{S}(H_A)$ defined by inequality (24).

Example. The additional conditions in Corollary 4 hold for a Gaussian channel $\Phi$ with the power constraint of the form (24), where $H = R^T \epsilon R$ is the many-mode oscillator Hamiltonian (see the remark after Proposition 3 in [13]). In this case the optimal state $\rho_*$ - the barycenter of an optimal measure - always exists. So, if we assume that $\rho_*$ is a Gaussian state, then Corollary 4 shows that $\bar{C}(\Phi|H,h) = C_{\text{ea}}(\Phi|H,h)$ may be valid only if $\Phi$ is an entanglement-breaking channel having the Kraus representation with the operators of rank one.

The above assumption holds provided the conjecture of Gaussian optimizers is valid for the channel $\Phi$ (see [5, 6] and the references therein).

6 Appendix

6.1 Petz’s theorem in infinite dimensions

Monotonicity of the relative entropy means that
\[
H(\Phi(\rho)\|\Phi(\sigma)) \leq H(\rho\|\sigma)
\]
for any channel $\Phi : \mathcal{S}(H_A) \rightarrow \mathcal{S}(H_B)$ and any states $\rho$ and $\sigma$ in $\mathcal{S}(H_A)$.

Since finiteness of $H(\rho\|\sigma)$ implies $\text{supp}\rho \subseteq \text{supp}\sigma$ we will assume in what follows that $\sigma$ and $\Phi(\sigma)$ are full rank states in $\mathcal{S}(H_A)$ and in $\mathcal{S}(H_B)$ correspondingly.

Petz’s theorem characterizing the equality case in (27) can be formulated as follows (where it is assumed that $H(\rho\|\sigma)$ is finite).

**Theorem 3.** The equality holds in (27) if and only if $\Theta_\sigma(\Phi(\rho)) = \rho$, where $\Theta_\sigma$ is a channel from $\mathcal{S}(H_B)$ to $\mathcal{S}(H_A)$ defined by the formula
\[
\Theta_\sigma(\varrho) = [\sigma]^{1/2} \Phi^* \left( [\Phi(\sigma)]^{-1/2}(\varrho) [\Phi(\sigma)]^{-1/2} \right) [\sigma]^{1/2}, \quad \varrho \in \mathcal{S}(H_B).
\]
Note that $\Theta_\sigma(\Phi(\sigma)) = \sigma$, so the above criterion for the equality in (27) can be treated as a reversibility condition (sufficiency of the channel $\Phi$ with respect to the states $\rho$ and $\sigma$ in terms of $[24]$).

The proof of (a generalized version of) Theorem 3 in the finite dimensional case can be found in [8, Theorem in Sec.5.1].

In infinite dimensions finiteness of $H(\rho\|\sigma)$ does not imply that $\lambda \rho \leq \sigma$ for some $\lambda > 0$ and hence the argument of the map $\Phi^*$ in (28) with $\varrho = \Phi(\rho)$ may be an unbounded operator. Nevertheless, we can define the channel $\Theta_\sigma$ as a predual map to the linear completely positive unital map

$$\Theta_\sigma^*(A) = [\Phi(\sigma)]^{-1/2} \Phi\left(\left[\sigma^{1/2} A \sigma^{1/2}\right] \Phi(\sigma)\right)^{-1/2}, \quad A \in \mathcal{B}(\mathcal{H}_A).$$

(29)

This means that we can use formula (28), keeping in mind that $\Phi^*$ is an extension of the dual map to unbounded operators in $\mathcal{H}_B$ (which can be defined by $\Phi^*(\cdot) = \sum_k V_k^*(\cdot)V_k$ via the Kraus representation $\Phi(\cdot) = \sum_k V_k(\cdot)V_k^*$).

With this definition of the channel $\Theta_\sigma$ Theorem 3 is proved in [24] (in the von Neumann algebra settings and with the transition probability instead of the relative entropy) under the condition that $\rho$ is full rank state in $\mathcal{G}(\mathcal{H}_A)$. Since in this paper Theorem 3 is used with the non-full rank state $\rho$, we will show below that it can be derived from Theorem 3 and Proposition 4 in [19].

Consider the ensemble consisting of two states $\rho$ and $\sigma$ with probabilities $t$ and $1 - t$, where $t \in (0, 1)$. Let $\sigma_t = t\rho + (1 - t)\sigma$. By Donald’s identity (Proposition 5.22 in [22]) we have

$$tH(\rho\|\sigma) + (1 - t)H(\sigma\|\sigma) = tH(\rho\|\sigma_t) + (1 - t)H(\sigma\|\sigma_t) + H(\sigma_t\|\sigma)$$

(30)

and

$$tH(\Phi(\rho)\|\Phi(\sigma)) + (1 - t)H(\Phi(\sigma)\|\Phi(\sigma)) = tH(\Phi(\rho)\|\Phi(\sigma_t)) + (1 - t)H(\Phi(\sigma)\|\Phi(\sigma_t)) + H(\Phi(\sigma_t)\|\Phi(\sigma)),$$

(31)

where the left-hand sides are finite and coincide by the condition. Since the first, the second and the third terms in the right-hand side of (30) are not less than the corresponding terms in (31) by monotonicity of the relative entropy, we obtain

$$H(\Phi(\rho)\|\Phi(\sigma_t)) = H(\rho\|\sigma_t) \quad \text{and} \quad H(\Phi(\sigma)\|\Phi(\sigma_t)) = H(\sigma\|\sigma_t).$$

(32)

I would be grateful for any reference on the proof of Theorem 3 in infinite dimensions without the full rank condition on the state $\rho$. 20
Theorem 3 and Proposition 4 in [19] imply \( \rho = \Theta_t(\Phi(\rho)) \) for all \( t \in (0, 1) \), where

\[
\Theta_t(\varrho) = [\sigma_t]^{1/2} \Phi^* ([\Phi(\sigma_t)]^{-1/2}(\varrho) [\Phi(\sigma_t)]^{-1/2}) [\sigma_t]^{1/2}, \quad \varrho \in \mathcal{S}(\mathcal{H}_B).
\]

To complete the proof it suffices to show that

\[
\lim_{t \to +0} \Theta_t = \Theta_\sigma
\]

in the strong convergence topology (in which \( \Phi_n \to \Phi \) means \( \Phi_n(\rho) \to \Phi(\rho) \) for all \( \rho \) [14]), since this implies \( \rho = \lim_{t \to +0} \Theta_t(\Phi(\rho)) = \Theta_\sigma(\Phi(\rho)) \).

Since \( \Theta_t(\Phi(\sigma)) = \sigma \) for all \( t \in (0, 1) \), the set of channels \( \{\Theta_t\}_{t \in (0,1)} \) is relatively compact in the strong convergence topology by Corollary 2 in [14]. Hence there exists a sequence \( \{t_n\} \) converging to zero such that

\[
\lim_{n \to +\infty} \Theta_{t_n} = \Theta_0,
\]

where \( \Theta_0 \) is a particular channel. We will show that \( \Theta_0 = \Theta_\sigma \).

Note that (34) means that the sequence \( \{\Theta^*_n(A)\} \) tends to the operator \( \Theta^*_\sigma(A) \) in the weak operator topology for any positive \( A \in \mathcal{B}(\mathcal{H}_B) \). By Lemma 3 below we have

\[
\lim_{n \to +\infty} [\Phi(\sigma_{t_n})]^{1/2} \Theta^*_n(A) [\Phi(\sigma_{t_n})]^{1/2} = [\Phi(\sigma)]^{1/2} \Theta^*_\sigma(A) [\Phi(\sigma)]^{1/2}
\]

in the Hilbert-Schmidt norm topology. But the explicit form of \( \Theta^*_n \) shows that

\[
[\Phi(\sigma_{t_n})]^{1/2} \Theta^*_n(A) [\Phi(\sigma_{t_n})]^{1/2} = \Phi ([\sigma_{t_n}]^{1/2} A [\sigma_{t_n}]^{1/2})
\]

and since \( \lim_{n \to +\infty} [\sigma_{t_n}]^{1/2} A [\sigma_{t_n}]^{1/2} = [\sigma]^{1/2} A [\sigma]^{1/2} \) in the trace norm topology, the above limit coincides with \( \Phi([\sigma]^{1/2} A [\sigma]^{1/2}) \). So, we have \( \Theta^*_\sigma(A) = \Theta^*_\sigma(A) \) for all \( A \) and hence \( \Theta_0 = \Theta_\sigma \).

The above observation shows that for an arbitrary sequence \( \{t_n\} \) converging to zero any partial limit of the sequence \( \{\Theta_{t_n}\} \) coincides with \( \Theta_\sigma \), which means (33).

**Lemma 3.** Let \( \{\rho_n\} \) be a sequence of states in \( \mathcal{S}(\mathcal{H}) \) converging to a state \( \rho_0 \) and \( \{A_n\} \) be a sequence of operators in the unit ball of \( \mathcal{B}(\mathcal{H}) \) converging to an operator \( A_0 \) in the weak operator topology. Then the sequence

\[\text{[10]}\text{Since this topology coincides with the } \sigma\text{-weak operator topology on the unit ball of } \mathcal{B}(\mathcal{H}_A) \text{.}\]
\{\sqrt{\rho_n}A_n\sqrt{\rho_n}\} converges to the operator \(\sqrt{\rho_0}A_0\sqrt{\rho_0}\) in the Hilbert-Schmidt norm topology.

**Proof.** Since \(\{\rho_n\}_{n \geq 0}\) is a compact set, the compactness criterion for subsets of \(\mathcal{S}(\mathcal{H})\) (see [13, Proposition in the Appendix]) implies that for an arbitrary \(\varepsilon > 0\) there exists a finite rank projector \(P_\varepsilon\) such that \(\text{Tr} \, \overline{P_\varepsilon} \rho_n < \varepsilon\) for all \(n \geq 0\), where \(\overline{P_\varepsilon} = I_H - P_\varepsilon\). We have

\[
\sqrt{\rho_n}A_n \sqrt{\rho_n} = \sqrt{\rho_n}P_\varepsilon A_n P_\varepsilon \sqrt{\rho_n} + \sqrt{\rho_n}P_\varepsilon A_n \overline{P_\varepsilon} \rho_n \sqrt{\rho_n} + \sqrt{\rho_n}P_\varepsilon A_n \overline{P_\varepsilon} \rho_n \sqrt{\rho_n} + \sqrt{\rho_n}P_\varepsilon A_n \overline{P_\varepsilon} \rho_n \sqrt{\rho_n} \quad \text{for all} \quad n \geq 0, \quad (35)
\]

Since \(P_\varepsilon\) has finite rank, \(P_\varepsilon A_n P_\varepsilon\) tends to \(P_\varepsilon A_0 P_\varepsilon\) in the norm topology and hence \(\sqrt{\rho_n}P_\varepsilon A_n P_\varepsilon \sqrt{\rho_n}\) tends to \(\sqrt{\rho_0} P_\varepsilon A_0 P_\varepsilon \sqrt{\rho_0}\) the trace norm topology, while it is easy to show that the Hilbert-Schmidt norm of the other terms in the right-hand side of (35) tends to zero as \(\varepsilon \to 0\) uniformly on \(n\). □

### 6.2 A difference between the Holevo quantities for complementary channels

Let \(\Phi : \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_B)\) be a quantum channel and \(\hat{\Phi} : \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_E)\) be its complementary channel. In finite dimensions the **coherent information** of the channel \(\Phi\) at any state \(\rho\) can be defined as a difference between \(H(\Phi(\rho))\) and \(H(\hat{\Phi}(\rho))\) [18, 29]. Since in infinite dimensions these values may be infinite even for the state \(\rho\) with finite entropy, for any such state the coherent information can be defined via the quantum mutual information as follows

\[
I_c(\Phi, \rho) = I(\Phi, \rho) - H(\rho).
\]

Let \(\rho\) be a state in \(\mathcal{S}(\mathcal{H}_A)\) with finite entropy. By monotonicity of the Holevo quantity the values \(\chi(\Phi(\mu)) \) and \(\chi(\hat{\Phi}(\mu))\) do not exceed \(H(\rho) = \chi(\mu)\) for any measure \(\mu \in \mathcal{P}(\text{extr}\mathcal{S}(\mathcal{H}_A))\) with the barycenter \(\rho\). The following lemma can be considered as a generalized version of the observation in [29].

**Lemma 4.** Let \(\mu\) be a measure in \(\mathcal{P}(\text{extr}\mathcal{S}(\mathcal{H}_A))\) with the barycenter \(\rho\). Then

\[
\chi(\Phi(\mu)) - \chi(\hat{\Phi}(\mu)) = I(\Phi, \rho) - H(\rho) = I_c(\Phi, \rho). \quad (36)
\]

This lemma shows, in particular, that the difference \(\chi(\Phi(\mu)) - \chi(\hat{\Phi}(\mu))\) does not depend on \(\mu\). So, if the supremum in the second expression in (20)
for the value $\bar{C}(\Phi, \rho)$ is achieved at some measure $\mu_*$ then the supremum in
the similar expression for the value $\bar{C}(\tilde{\Phi}, \rho)$ is achieved at this measure $\mu_*$
and vice versa.

**Proof.** If $H(\Phi(\rho)) < +\infty$ then $H(\tilde{\Phi}(\rho)) < +\infty$ by the triangle inequality
and (36) can be derived from (21) by using the second formula in (14) and
by noting that the functions $\rho \mapsto H(\Phi(\rho))$ and $\rho \mapsto H(\tilde{\Phi}(\rho))$
coincide on the
set of pure states. In general case it is necessary to use the approximation
method to prove (36). To realize this method we have to introduce some
additional notions.

Let $\mathfrak{T}_1(\mathcal{H}) = \{ A \in \mathfrak{T}(\mathcal{H}) | A \geq 0, \text{Tr} A \leq 1 \}$. We will use the following
two extensions of the von Neumann entropy to the set $\mathfrak{T}_1(\mathcal{H})$ (cf.[17])

$$S(A) = -\text{Tr} A \log A \quad \text{and} \quad H(A) = S(A) + \text{Tr} A \log \text{Tr} A, \quad \forall A \in \mathfrak{T}_1(\mathcal{H}).$$

Nonnegativity, concavity and lower semicontinuity of the von Neumann entropy imply the same properties of the functions $S$ and $H$ on the set $\mathfrak{T}_1(\mathcal{H})$.

The relative entropy for two operators $A$ and $B$ in $\mathfrak{T}_1(\mathcal{H})$ is defined as follows (cf.[17])

$$H(A \| B) = \sum_i \langle i | (A \log A - A \log B + B - A) | i \rangle,$$

where $\{|i\rangle\}$ is the orthonormal basis of eigenvectors of $A$. By means of
this extension of the relative entropy the Holevo quantity of a measure $\mu$ in
$\mathcal{P}(\mathfrak{T}_1(\mathcal{H}))$ is defined by expression (13).

A completely positive trace-non-increasing linear map $\Phi : \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_B)$ is called quantum operation [18]. For any quantum operation $\Phi$ the
Stinespring representation (4) holds, in which $V$ is a contraction. The comple-
mentary operation $\tilde{\Phi} : \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_E)$ is defined via this representation
by (5).

By the obvious modification of the arguments used in the proof of Proposition 1 in [13] one can show that the function $\mu \mapsto \chi(\mu)$ is lower semicontinuous on the set $\mathcal{P}(\mathfrak{T}_1(\mathcal{H}))$ and that for an arbitrary quantum operation $\Phi$ and a measure $\mu \in \mathcal{P}(\mathfrak{S}(\mathcal{H}_A))$ such that $S(\Phi(\tilde{\rho}(\mu))) < +\infty$ the Holevo quantity of the measure $\Phi(\mu) = \mu \circ \Phi^{-1} \in \mathcal{P}(\mathfrak{T}_1(\mathcal{H}_B))$ can be expressed as follows

$$\chi(\Phi(\mu)) = S(\Phi(\tilde{\rho}(\mu))) - \int_{\mathfrak{S}(\mathcal{H}_A)} S(\Phi(\rho)) \mu(d\rho). \quad (37)$$

23
We are now in a position to prove (36) in general case.

Note that for a given measure \( \mu \in \mathcal{P}(\mathcal{S}(H_A)) \) the function \( \Phi \mapsto \chi(\Phi(\mu)) \) is lower semicontinuous on the set of all quantum operations endowed with the strong convergence topology (in which \( \Phi_n \to \Phi \) means \( \Phi_n(\rho) \to \Phi(\rho) \) for all \( \rho \)). This follows from lower semicontinuity of the functional \( \mu \mapsto \chi(\mu) \) on the set \( \mathcal{P}(\mathcal{T}_1(H_B)) \), since for an arbitrary sequence \( \{\Phi_n\} \) of quantum operations strongly converging to a quantum operation \( \Phi \) the sequence \( \{\Phi_n(\mu)\} \subset \mathcal{P}(\mathcal{T}_1(H_B)) \) weakly converges to the measure \( \Phi(\mu) \) (this can be verified directly by using the definition of the weak convergence and by noting that for sequences of quantum operations the strong convergence is equivalent to the uniform convergence on compact subsets of \( \mathcal{S}(H_A) \)).

Let \( \{P_n\} \) be an increasing sequence of finite rank projectors in \( \mathcal{B}(H_B) \) strongly converging to \( I_B \). Consider the sequence of quantum operations \( \Phi_n = \Pi_n \circ \Phi \), where \( \Pi_n(\cdot) = P_n(\cdot)P_n \). Then

\[
\hat{\Phi}_n(\rho) = \text{Tr}_{H_B} P_n \otimes I_{H_E} V\rho V^* , \quad \rho \in \mathcal{S}(H_A),
\]

where \( V \) is the isometry from Stinespring representation (11) for the channel \( \Phi \).

The sequences \( \{\Phi_n\} \) and \( \{\hat{\Phi}_n\} \) strongly converges to the channels \( \Phi \) and \( \hat{\Phi} \) correspondingly. Let \( \rho = \sum_k \lambda_k |k\rangle \langle k| \) and \( |\varphi_\rho\rangle = \sum_k \sqrt{\lambda_k} |k\rangle \otimes |k\rangle \). Since \( S(\Phi_n(\rho)) < +\infty \), the triangle inequality implies \( S(\hat{\Phi}_n(\rho)) < +\infty \). So, we have

\[
I(\Phi_n, \rho) = H(\Phi_n \otimes \text{Id}_R(|\varphi_\rho\rangle \langle \varphi_\rho|) \| \Phi_n(\rho) \otimes \rho)
\]

\[
= -S(\hat{\Phi}_n(\rho)) + S(\Phi_n(\rho)) + a_n = -\chi(\hat{\Phi}_n(\mu)) + \chi(\Phi_n(\mu)) + a_n,
\]

where \( a_n = -\sum_k \text{Tr}(\Phi_n(|k\rangle \langle k|)) \lambda_k \log \lambda_k \) and the last equality is obtained by using (37) and coincidence of the functions \( \rho \mapsto S(\Phi(\rho)) \) and \( \rho \mapsto S(\hat{\Phi}(\rho)) \) on the set of pure states.

Since the function \( \Phi \mapsto I(\Phi, \rho) \) is lower semicontinuous (by lower semicontinuity of the relative entropy) and \( I(\Phi_n, \rho) \leq I(\Phi, \rho) \) for all \( n \) by monotonicity of the relative entropy under action the quantum operation \( \Pi_n \otimes \text{Id}_K \), we have

\[
\lim_{n \to +\infty} I(\Phi_n, \rho) = I(\Phi, \rho).
\]

We will also show that

\[
\lim_{n \to +\infty} \chi(\Phi_n(\mu)) = \chi(\Phi(\mu)) \quad \text{and} \quad \lim_{n \to +\infty} \chi(\hat{\Phi}_n(\mu)) = \chi(\hat{\Phi}(\mu)).
\]
The first relation in (41) follows from lower semicontinuity of the function $\Phi \mapsto \chi(\Phi(\mu))$ (established before) and the inequality $\chi(\Phi_n(\mu)) \leq \chi(\Phi(\mu))$ valid for all $n$ by monotonicity of the Holevo quantity under action of the quantum operation $\Pi_n$.

To prove the second relation in (41) note that (38) implies $\hat{\Phi}_n(\rho) \leq \hat{\Phi}(\rho)$ for any state $\rho \in \mathcal{S}(H_A)$. Thus Lemma 2 in [14] shows that
\[
\chi(\hat{\Phi}_n(\mu)) \leq \chi(\hat{\Phi}(\mu)) + f(\text{Tr}\hat{\Phi}_n(\rho))
\] (42)
where $f(x) = -2x \log x - (1-x) \log(1-x)$, for any measure $\mu \in \mathcal{P}(\mathcal{S}(H_A))$ with finite support and the barycenter $\rho$. To prove that (42) holds for any measure $\mu \in \mathcal{P}(\mathcal{S}(H_A))$ with the barycenter $\rho$ one can take the sequence $\{\mu_n\}$ of measures with finite support and the barycenter $\rho$ constructed in the proof of Lemma 1 in [13], which weakly converges to the measure $\mu$, and use lower semicontinuity of the function $\mu \mapsto \chi(\Psi(\mu))$, where $\Psi$ is a quantum operation, and the inequality $\chi(\hat{\Phi}(\mu_n)) \leq \chi(\hat{\Phi}(\mu))$ valid for all $n$ by the construction of the sequence $\{\mu_n\}$ and convexity of the relative entropy.

Inequality (42) and lower semicontinuity of the function $\Phi \mapsto \chi(\Phi(\mu))$ imply the second relation in (41).

Since $\{a_n\}$ obviously tends to $H(\rho)$, (39), (40) and (41) imply (36). □

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References

[1] O.Bratteli, D.W.Robinson ”Operators algebras and quantum statistical mechanics”, vol.I, Springer Verlag, New York-Heidelberg-Berlin, 1979.

[2] D.Chruscinski, A.Kossakowski ”On Partially Entanglement Breaking Channels”, Open Sys. Information Dyn., V.13 P.17-26, 2006; arXiv:quant-ph/0511244

[3] T.S.Cubitt, M.B.Ruskai, G.Smith ”The structure of degradable quantum channels”, J. Math. Phys., V.49, 102104, 2008; arXiv:0802.1360
[4] C.H. Bennett, P.W. Shor, J.A. Smolin, A.V. Thapliyal "Entanglement-assisted capacity and the reverse Shannon theorem", IEEE Trans. Inform. Theory; arXiv:quant-ph/0106052.

[5] J. Eisert, M.M. Wolf "Gaussian quantum channels", Quantum Information with Continuous Variables of Atoms and Light, P.23-42 (Imperial College Press, London, 2007); arXiv:quant-ph/0505151.

[6] R. Garcia-Patron, C. Navarrete-Benlloch, S. Lloyd, J.H. Shapiro, N.J. Cerf "Majorization theory approach to the Gaussian channel minimum entropy conjecture", arXiv:1111.1986.

[7] P. Hayden, R. Jozsa, D. Petz, A. Winter "Structure of states which satisfy strong subadditivity of quantum entropy with equality", Commun. Math. Phys., V.246, N 2, P.359-374, 2004; arXiv:quant-ph/0304007.

[8] F. Hiai, M. Mosonyi, D. Petz, C. Beny "Quantum f-divergences and error correction", arXiv:1008.2529.

[9] A.S. Holevo "Some estimates for information quantity transmitted by quantum communication channel", Probl. Inform. Transm., V.9, N3, P.3-11, 1973.

[10] A.S. Holevo "Remarks on the classical capacity of quantum channel", arXiv:quant-ph/0212025.

[11] A.S. Holevo "Classical capacities of quantum channels with constrained inputs", Probability Theory and Applications, 48, N.2, 359-374, 2003, arXiv quant-ph/0211170.

[12] A.S. Holevo "On complementary channels and the additivity problem", Probability Theory and Applications, V.51. N.1. P.134-143, 2006; arXiv:quant-ph/0509101.

[13] A.S. Holevo, M.E. Shirokov "Continuous ensembles and the $\chi$-capacity of infinite dimensional channels", Probability Theory and Applications, 50, N.1, 98-114, 2005; arXiv:quant-ph/0408176.

[14] A.S. Holevo, M.E. Shirokov "On approximation of infinite dimensional quantum channels", Problems of Information Transmission. 2008. V. 44. N. 2. P. 3-22; arXiv:0711.2245.
[15] M.Horodecki, P.W.Shor, M.B.Ruskai "General Entanglement Breaking Channels", Rev. Math. Phys., V.15, P.629-641, 2003; arXiv:quant-ph/0302031

[16] C.King, K.Matsumoto, M.Nathanson, M.B.Ruskai "Properties of Conjugate Channels with Applications to Additivity and Multiplicativity", Markov Process and Related Fields, V.13, P.391-423, 2007; arXiv:quant-ph/0509126

[17] G.Lindblad "Expectations and Entropy Inequalities for Finite Quantum Systems", Commun. Math. Phys., V. 39, P.111-119, 1974.

[18] M.A.Nielsen, I.L.Chuang "Quantum Computation and Quantum Information", Cambridge University Press, 2000.

[19] A.Jencova, D.Petz "Sufficiency in quantum statistical inference", Commun. Math. Phys. 263, P.259-276, 2006; arXiv:math-ph/0412093

[20] A.Jencova, M.B.Ruskai "A Unified Treatment of Convexity of Relative Entropy and Related Trace Functions, with Conditions for Equality", arXiv:0903.2895.

[21] A.Jencova "Reversibility conditions for quantum operations", arXiv:1107.0453.

[22] M.Ohya, D.Petz "Quantum Entropy and Its Use", Texts and Monographs in Physics. Berlin: Springer-Verlag, 1993.

[23] K.Parthasarathy "Probability measures on metric spaces", Academic Press, New York and London, 1967.

[24] D. Petz "Sufficiency of channels over von Neumann algebras", Quart. J. Math. Oxford Ser. (2) V.39, N.153, P.97-108, 1988.

[25] D.Petz "Monotonicity of quantum relative entropy revisited", Rev. Math. Phys., V.15, P.79-91, 2003; arXiv:quant-ph/0209053.

[26] M.B.Plenio, S.Virmani "An introduction to entanglement measures", Quantum Inf. Comput., V.7, P.1-51, 2007; arXiv:quant-ph/0504163
[27] M.B.Ruskai ”Inequalities for Quantum Entropy: A Review with Conditions for Equality”, J. Math. Phys. V.43, P.4358-4375, 2002; arXiv:quant-ph/0205064

[28] M.E.Shirokov ”The Schmidt number and partially entanglement breaking channels in infinite dimensions”, arXiv:1110.4363.

[29] B.Schumacher, M.D.Westmoreland ”Quantum Privacy and Quantum Coherence”, Phys. Rev. Lett., V. 80, P.5695-5697. 1998; arXiv:quant-ph/9709058

[30] B.Schumacher, M.D.Westmoreland ”Optimal signal ensemble”, arXiv:quant-ph/9912122

[31] B.M.Terhal, P.Horodecki ”A Schmidt number for density matrices”, Phys. Rev. A Rapid Communications, 61, 040301, 2000; arXiv:quant-ph/9911117, 2000.

[32] L.Zhang, J.Wu ”Von Neumann Entropy-Preserving Quantum Operations”, Physics Letters A 375(47), P.4163-4165, 2011; arXiv:1104.2992