VANISHING THEOREMS ON COMPACT CKL HERMITIAN
MANIFOLDS

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Abstract. We show that, under the definiteness of holomorphic sectional curvature, the
spaces of some holomorphic tensor fields on compact Chern-Kähler-like Hermitian manifolds
are trivial. These can be viewed as counterparts to Bochner’s classical vanishing theorems
on compact Kähler manifolds under the definiteness of Ricci curvature or the existence of
Kähler-Einstein metrics. Our proof is inspired by and based on some ideas due to X. Yang
and L. Ni-F. Zheng.

1. Introduction and main results

Throughout this article denote by \((M, \omega)\) a compact connected complex manifold of complex
dimension \(n \geq 2\) endowed with a Hermitian metric whose associated positive \((1,1)\)-form is \(\omega\).
By abuse of notation, \(\omega\) itself is also called the Hermitian metric. Let \(TM\) and \(T^*M\) be the
holomorphic tangent and cotangent bundles of \(M\) respectively, and
\[
\Gamma^p_q(M) := H^0(M, (TM)^\otimes p \otimes (T^*M)^\otimes q), \quad (p, q \in \mathbb{Z}_{\geq 0})
\]
the space of \((p,q)\)-type holomorphic tensor fields on \(M\).

When \((M,\omega)\) is Kähler, Bochner noticed that (see [YB53, §8]) definiteness properties of the
Ricci curvature or existence of Kähler-Einstein metrics impose heavy restrictions on \(\Gamma^p_q(M)\).
The main idea of the proof is, for \(T \in \Gamma^p_q(M)\), the eigenvalues of the Ricci curvature are
involved in the difference \(\Delta |T|^2 - |\nabla T|^2\), where \(\Delta(\cdot)\) is the Laplacian operator and \(\nabla(\cdot)\) the
Levi-Civita connection of \(\omega\). This trick and its later various variants are called the Bochner
techniques and have far-reaching impacts in and beyond differential geometry (see [Wu88]).
Thanks to the later celebrated Calabi-Yau theorem and Aubin-Yau theorem ([Yau77]), (Some
of) Bochner’s results can now be reformulated in terms of definiteness of the first Chern class
\(c_1(M)\) as follows ([Ko80-1], [Ko80-2], [KH83, p. 57]).

**Theorem 1.1** (Bochner, Calabi-Yau, Aubin-Yau). Let \((M, \omega)\) be a compact Kähler manifold.

1. If \(c_1(M)\) is quasi-positive, then \(\Gamma^p_q(M) = 0\) when \(q >> p\), and, \(\Gamma^0_q(M) = 0\) when \(q \geq 1\).
   In particular, the Hodge numbers \(h^{q,0}(M) = 0\) when \(q \geq 1\).
2. If \(c_1(M)\) is quasi-negative, then \(\Gamma^p_q(M) = 0\) when \(p >> q\), and, \(\Gamma^p_0(M) = 0\) when
   \(p \geq 1\).
3. If \(c_1(M) < 0\), then \(\Gamma^p_q(M) = 0\) when \(p > q\).

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11722109).
Remark 1.2. (1) Quasi-positivity (resp. quasi-negativity) of $c_1(M)$ means that there exists a closed $(1, 1)$-form representing $c_1(M)$ which is nonnegative (resp. non-positive) everywhere and positive (resp. negative) somewhere.

(2) The qualitative conditions “$q >> p$” and “$p >> q$” can be made precise in terms of the maximum and minimum of the eigenvalues of the Ricci curvature. Details can be found in [YB53, §8] and [KH83].

For Kähler manifolds the relationship between Ricci curvature (“Ric” for short) and holomorphic sectional curvature (“HSC” for short) is quite subtle and even mysterious. On one hand, both $\text{Ric}(\omega)$ and $\text{HSC}(\omega)$ dominate and are dominated by scalar curvature and holomorphic bisectional curvature respectively. On the other hand, in general $\text{Ric}(\omega)$ and $\text{HSC}(\omega)$ do not dominate each other. Nevertheless, a recent breakthrough due to Wu-Yau, Tosatti-Yang and Diverio-Trapani ([WY16-1], [TY17], [DT19], [WY16-2]) tells us that the quasi-negativity of $\text{HSC}(\omega)$ implies $c_1 < 0$. As a consequence, the conclusions of Parts (2) and (3) in Theorem 1.1 remain true under the assumption of $\text{HSC}(\omega)$ being quasi-negative. In view of this, one may wonder whether similar statement holds true for compact Kähler manifolds with positive HSC. However, it turns out that ([Hi75], [Ya16, p. 949]) the Hirzebruch surfaces $M = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-k) \oplus \mathcal{O}_{\mathbb{P}^1})$ ($k \geq 2$) admit Kähler metrics with positive HSC but $c_1(M)$ are not positive.

Even if in general the positivity of HSC cannot imply that of the first Chern class, we may still ask if the conclusions of Part (1) in Theorem 1.1 hold true under the assumption of HSC being positive. In his recent work [Ya18], X.K. Yang proved a conjecture of S.-T. Yau ([Yau82, Problem 47]), which states that a compact Kähler manifold $(M, \omega)$ with $\text{HSC}(\omega) > 0$ must be projective and thus gives a metric criterion of the projectivity. Indeed Yang showed that the condition of $\text{HSC}(\omega) > 0$ leads to the Hodge numbers $h^{0,0}(M) = 0$ for $1 \leq q \leq n$ via the notion of RC-positivity introduced by himself ([Ya18, Thm 1.7]). In particular $h^{2,0}(M) = 0$ implies the projectivity of $M$ due to a well-known result of Kodaira ([MK71, p. 143]). Note that $\Gamma_q^0(M) = 0$ yields $h^{q,0}(M) = 0$ and hence the vanishing results Yang obtained provide evidence towards the expected conclusions of Part (1) in Theorem 1.1.

In another recent work [NZ18], L. Ni and F.Y. Zheng proved that the projectivity remains true under a weaker positivity: the 2nd scalar curvature $S_2(\omega)$. they indeed introduced in [NZ18] a family of $S_k(\omega)$ ($1 \leq k \leq n$), called $k$-th scalar curvatures, by averaging $\text{HSC}(\omega)$ over $k$-dimensional subspaces of the tangent spaces, and the positivity of $\text{HSC}(\omega)$ implies that of $S_2(\omega)$, and showed that ([NZ18, Thm 1.1]) the Hodge numbers $h^{q,0}(M) = 0$ for $2 \leq q \leq n$ whenever $S_2(\omega) > 0$.

The purpose of this note is, by building on some ideas in [Ya18] and [NZ18], to show that the conclusions of Part (1) in Theorem 1.1 still hold true under the assumption of HSC being positive. In fact, we will show them in a more general setting. For this purpose, let us recall the following notion, which was introduced and investigated by B. Yang and F. Zheng in [YZ18].

**Definition 1.3.** Let $(M, \omega)$ be a Hermitian manifold and $R$ the curvature tensor of the Chern connection, which is the unique canonical connection compatible with both the metric and the complex structure. The Hermitian metric $\omega$ is called Chern-Kähler-like (CKL for short) if

\[
R(X, \nabla X, Z, \nabla Z) = R(Z, \nabla Z, X, \nabla X)
\]
for any \((1,0)\)-type tangent vectors \(X, Y, Z\) and \(W\).

**Remark 1.4.** When \(\omega\) is Kähler, Chern connection coincides with Levi-Civita connection and Hence \(R\) is the usual curvature tensor, which clearly satisfies (1.1). Note that, by taking complex conjugations, (1.1) implies that

\[
R(X, \overline{Y}, Z, W) = \overline{R(Y, X, W, Z)} = \overline{R(W, X, Y, Z)} = R(X, W, Z, \overline{Y}).
\]

Therefore the condition (1.1) ensures that \(R\) obeys all the symmetries satisfied by the curvature tensor of a Kähler metric and thus the term CKL is justified.

As pointed out in [YZ18, p. 1197], there are plenty of non-Kähler Hermitian metrics which are CKL. On the other hand, it turns out that ([YZ18, Thm 3]) a CKL Hermitian metric \(\omega\) must be balanced, i.e., \(d\omega^{n-1} = 0\). Hence CKL Hermitian metrics interpolate between Kähler and balanced metrics.

For a Hermitian manifold \((M, \omega)\), \(x \in M\) and \(v \in T_xM - \{0\}\), the holomorphic sectional curvature of \(\omega\), denoted by \(HSC(\omega)\), at the point \(x\) and the direction \(v\) is defined by

\[
H_x(v) := \frac{R(v, \overline{v}, v, \overline{v})}{|v|^4}.
\]

HSC(\(\omega)\) is called positive, denoted by \(HSC(\omega) > 0\), if \(H_x(v) > 0\) for any \(x \in M\) and \(v \in T_xM - \{0\}\). Note that this \(H_x(\cdot)\) is indeed defined on \(\mathbb{P}(T_xM)\), the projectivation of \(T_xM\). Thus the maximal and minimal values of \(H_x\) can be attained, a fact which will be used later.

With these notions in mind, our main result in this note is the following theorem, which can be viewed as counterparts to Theorem 1.1.

**Theorem 1.5.** Let \((M, \omega)\) be a compact CKL Hermitian manifold.

1. If \(HSC(\omega) > 0\), then \(\Gamma_q^p(M) = 0\) when \(q >> p\), and, \(\Gamma_0^0(M) = 0\) when \(q \geq 1\). In particular, the Hodge numbers \(h^{q,0}(M) = 0\) when \(1 \leq q \leq n\).

2. If \(HSC(\omega) < 0\), then \(\Gamma_q^p(M) = 0\) when \(p >> q\), and, \(\Gamma_0^0(M) = 0\) when \(p \geq 1\).

**Remark 1.6.** (1) As remarked above, when the metric \(\omega\) is Kähler, Part (2) in Theorem 1.5 follows from Theorem 1.1 and the recent result due to Wu-Yau et. al. However, if the CKL metric \(\omega\) is non-Kähler, the statements in Part (2) are also new.

(2) The qualitative conditions “\(q >> p\)” and “\(p >> q\)” in Theorem 1.5 can be made more quantitative. Details can be found in Theorem 3.1 and its proof.

2. Preliminaries

We briefly collect in this section some basic facts on Hermitian holomorphic vector bundles and Hermitian manifolds in the form we shall use to prove Theorem 1.5. A thorough treatment can be found in [Ko87].

Let \((E^*, h) \to M\) be a Hermitian holomorphic vector bundle of rank \(r\) on an \(n\)-dimensional compact complex manifold \(M\) endowed with the canonical Chern connection \(\nabla\) and the curvature tensor

\[
R := \nabla^2 \in \Gamma(\Lambda^{1,1}M \otimes E^* \otimes E).
\]

Here and throughout this section we use \(\Gamma(\cdot)\) to denote the space of smooth sections for vector bundles and the notation \(\Gamma_q^p(M)\) is reserved to denote that of the holomorphic \((p,q)\)-tensor fields on \(M\) as before.
Under a local frame field \( \{s_1, \ldots, s_r\} \) of \( E \), whose dual coframe is denoted by \( \{s_1^*, \ldots, s_r^*\} \), and local coordinates \( \{z^1, \ldots, z^n\} \) on \( M \), the curvature tensor \( R \) and the Hermitian metric \( h \) can be written as

\[
\begin{align*}
R &=: \Omega^\beta_a s^*_a \otimes s_\beta =: R^\beta^\gamma_i dz^i \wedge d\bar{z}^j \otimes s^*_a \otimes s_\beta, \\
h &= (h_{\alpha \bar{\beta}}) := (h(s_\alpha, s_\beta)), \\
R^\gamma_i_{\alpha \bar{\beta}} &=: R^\gamma_i_{\bar{\alpha} \beta}.
\end{align*}
\] (2.1)

Here and in what follows we always adopt the Einstein summation convention. For simplicity we sometimes use \( <\cdot, \cdot> \) to denote the Hermitian metric \( h(\cdot, \cdot) \) and the induced metrics on various vector bundles arising naturally from \( E \).

Recall that for \( \eta = \eta^\alpha s_\alpha \in \Gamma(E) \),

\[
R(\eta) = (\Omega^\beta_a s^*_a \otimes s_\beta)(\eta^\gamma s_\gamma) = \Omega^\beta_a \eta^\alpha s_\beta,
\]

and \( u \in u^i \frac{\partial}{\partial z^i}, v \in v^j \frac{\partial}{\partial \bar{z}^j} \),

\[
R_{\bar{u} \bar{v}}(\eta) = \Omega^\beta_a (u, v) \eta^\alpha s_\beta = \Omega^\beta_a \eta^\alpha s_\beta,
\]

and thus

\[
< R_{\bar{u} \bar{v}}(\eta), \xi > = < R^\beta_{\bar{\alpha} \bar{\beta}}, u^i \bar{v}^j \eta^\alpha s_\beta, \xi^\gamma s_\gamma >
\] (2.2)

\[
= R^\gamma_{\alpha \bar{\beta}} u^i \bar{v}^j \eta^\alpha \xi^\gamma h_{\beta \bar{\gamma}}
\]

\[
= R^\gamma_{\alpha \bar{\beta}} u^i \bar{v}^j \eta^\alpha \xi^\beta.
\]

A direct consequence of (2.2) is that \( R_{\bar{u} \bar{v}}(\cdot) \) is a Hermitian transformation:

\[
< R_{\bar{u} \bar{v}}(\eta), \xi > = < \eta, R_{\bar{u} \bar{v}}(\xi) >, \quad \forall \eta, \xi \in \Gamma(E).
\] (2.3)

The following two lemmas are crucial to our proof.

**Lemma 2.1.** Let \( \eta \in \Gamma(E) \) be holomorphic and the maximum of \( |\eta| := < \eta, \eta >^{\frac{1}{2}} \) is attained at \( x \in M \). Then

\[
< R_{\bar{u} \bar{v}}(\eta), \eta > \bigg|_x \geq 0, \quad \forall u \in T_x M.
\] (2.4)

**Remark 2.2.** In terms of local coordinates, (2.4) is equivalent to the fact that

\[
(R^\gamma_{\alpha \bar{\beta}} \eta^\alpha \eta^\beta) \bigg|_x \geq 0
\]

as a Hermitian matrix.

**Proof.** A well-known Bochner-type formula reads ([Ko87, p.50])

\[
\partial \bar{\partial} |\eta|^2 = < \nabla \eta, \nabla \eta > - < R(\eta), \eta >,
\] (2.5)

where \( < R(\eta), \eta > \) is understood to pair the elements in \( \Gamma(E) \) and maintain those in \( \Gamma(\Lambda^{1,1} M) \). The holomorphicity of \( \eta \) implies that \( \nabla \eta \in \Gamma(\Lambda^{1,0} M \otimes E) \) and hence

\[
< \nabla \eta, \nabla \eta > \in \Gamma(\Lambda^{1,1} M)
\]

is similarly understood. The reader is referred to [Ko87, p.50] for (2.5) in terms of local coordinates.
For any vector $u \in T_x M$ we apply $u \wedge \bar{u}$ to evaluate both sides of (2.5) to yield
\[
< R_{u\bar{u}}(\eta), \eta > |_x = \left( - \sqrt{-1} \partial \bar{\partial} |\eta|^2 \right) |_x \left( \frac{1}{\sqrt{-1}} u \wedge \bar{u} \right) + |\nabla_u \eta|^2.
\]
Since the maximum of $|\eta|$ is attained at $x$, the maximum principle implies that
\[
\left( - \sqrt{-1} \partial \bar{\partial} |\eta|^2 \right) |_x \geq 0
\]
as a $(1,1)$-form, and hence we arrive at the desired result. \qed

**Remark 2.3.** The above proof of using $\partial \bar{\partial}$-Bochner formula and applying the maximum principle to part of directions are indeed inspired by some arguments in [Ya18, Prop. 4.2] and [NZ18, Lemma 2.1]. Similar techniques and ideas can also be found in [An12], [AC11], [Liu16] and [Ni13].

The following lemma is parallel to [Ya18, Lemma 6.1], where the conclusion was stated for Kähler manifolds.

**Lemma 2.4.** Let $(M, \omega)$ be a compact CKL Hermitian manifold, $x \in M$, and the unit vector $u \in T_x M$ minimizes (resp. maximizes) the holomorphic sectional curvature at $x$. Then
\[
R(u, \bar{u}, v, \bar{v}) \geq (\leq) \frac{1 + |\langle u, v \rangle|^2}{2} H_x(u), \quad \forall \text{ unit vector } v \in T_x M.
\]

**Remark 2.5.** As pointed out in [Ya18, p.198], the tricks and various variants used in the proof of Lemma 2.4 can be found in [Go98, p.312], [Br10, p.136], [BYT13, Lemma 1.4] and [Ya17, Lemma 4.1]. We remark that similar tricks can also be found in [BG63] and [Gr70, §2], to the author’s best knowledge. Although [Ya18, Lemma 6.1] is stated for Kähler manifolds, we will see in the course of the proof below that what it really needs is various Kähler-type symmetries of the curvature tensor $R$, which are satisfied by CKL Hermitian metrics as explained in Remark 1.4. For the sake of the reader’s convenience as well as completeness, we still include a proof below.

**Proof.** Assume that a unit vector $w \in T_x M$ is such that $\langle u, w \rangle = 0$. Note that under this assumption we have
\[
\left| (\cos \theta) u + (\sin \theta) w \right| = \left| (\cos \theta) u + (\sqrt{-1} \sin \theta) w \right| = 1, \quad \forall \theta \in \mathbb{R}.
\]
We consider
\[
f(\theta) := H_x((\cos \theta) u + (\sin \theta) w) = R((\cos \theta) u + (\sin \theta) w, \ldots, (\cos \theta) u + (\sin \theta) w),
\]
and
\[
g(\theta) := H_x((\cos \theta) u + (\sqrt{-1} \sin \theta) w) = R((\cos \theta) u + (\sqrt{-1} \sin \theta) w, \ldots, (\cos \theta) u + (\sqrt{-1} \sin \theta) w).
\]
Since $u$ minimizes (resp. maximizes) $H_x(\cdot)$ and $f(0) = g(0) = H_x(u)$, $\theta = 0$ attains the minimal (resp. maximal) value of $f(\theta)$ and $g(\theta)$. This means that
\[
f'(0) = g'(0) = 0
\]
and
\[
f''(0) \geq (\leq) g''(0) \geq (\leq) 0.
\]
Denote by $R_{1111} := R(u, \bar{u}, u, \bar{u})$, $R_{1121} := R(u, \bar{u}, w, \bar{u})$ and so on. Direct calculations, together with various Kähler-like symmetries of $R_{ijkl}$ ensured by the CKL condition, show that

\begin{equation}
(2.9) \quad f'(0) = R_{1112} + R_{1121}, \quad g'(0) = -R_{1112} + R_{1121}
\end{equation}

and

\begin{equation}
(2.10) \quad \left\{ \begin{array}{l}
f''(0) = 4R_{1122} + R_{1212} + R_{2121} - 2R_{1111} \\
g''(0) = 4R_{1122} - R_{1212} - R_{2121} - 2R_{1111}.
\end{array} \right.
\end{equation}

Combining (2.7) with (2.9) implies that

\begin{equation}
(2.11) \quad R_{1112} = R_{1121} = 0,
\end{equation}

and (2.8) with (2.10) leads to

\begin{equation}
(2.12) \quad 2R_{1122} \geq \left( \leq \right) R_{1111}.
\end{equation}

Now for any unit vector $v \in T_x M$, we can choose a unit vector $w \in T_x M$ such that $<u, w> = 0$ and $v = au + bw$ with $|a|^2 + |b|^2 = 1$. Then

\[
R(u, \bar{u}, v, \bar{v}) = R(u, \bar{u}, au + bw, \bar{au + bw}) = |a|^2 R_{1111} + |b|^2 R_{1122} \quad \text{ (by (2.11))}
\]

\[
\geq (|a|^2 + \frac{|b|^2}{2}) R_{1111} \quad \text{ (by (2.12))}
\]

\[
= 1 + \frac{|<u, v>|^2}{2} H_x(u),
\]

which yields the desired inequality (2.6).

\[\square\]

3. Proof of Theorem 1.5

We can now proceed to prove the main result in this note, Theorem 1.5.

Assume temporarily that $(M^n, \omega)$ is a general compact Hermitian manifold and $T \in \Gamma^p_q(M)$ a $(p, q)$-type holomorphic tensor field on it. Let $x \in M$ and unit vector $u \in T_x M$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_x M$ and $\{\theta^1, \ldots, \theta^n\}$ an orthonormal basis of $T^*_x M$ dual to $\{e_i\}$. Denote by

\[
T|_x = T^\alpha_1 \ldots \alpha_p \epsilon^\alpha_1 \otimes \cdots \otimes \epsilon^\alpha_p \otimes \theta^{\beta_1} \otimes \cdots \otimes \theta^{\beta_q}.
\]

Recall from (2.3) that $R_{u\bar{u}}(\cdot)$ is a Hermitian transformation and hence its eigenvalues are all real, say $\lambda_i = \lambda_i(x, u)$ ($1 \leq i \leq n$). Without loss of generality we may choose the above orthonormal basis $\{e_1, \ldots, e_n\}$ such that

\begin{equation}
(3.1) \quad R_{u\bar{u}}(e_i) = \lambda_i e_i, \quad \lambda_i \in \mathbb{R}, \quad 1 \leq i \leq n.
\end{equation}

Note that the induced actions of $R_{u\bar{u}}$ on $\theta^1, \ldots, \theta^n$ are given by

\[
R_{u\bar{u}}(\theta^i) = -\lambda_i \theta^i, \quad 1 \leq i \leq n,
\]
and hence

\[
R_{\bar{u}u}(T\big|_x) = \sum_{\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q} \left( \sum_{i=1}^{p} \lambda_{\alpha_i} - \sum_{j=1}^{q} \lambda_{\beta_j} \right) T^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q} e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_p} \otimes \theta_{\beta_1} \otimes \cdots \otimes \theta_{\beta_q}.
\]

This yields that

\[
< R_{\bar{u}u}(T\big|_x), T\big|_x > = \sum_{\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q} \left( \sum_{i=1}^{p} \lambda_{\alpha_i} - \sum_{j=1}^{q} \lambda_{\beta_j} \right) | T^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q} |^2.
\]

Let

\[
\lambda_{\max}(x, u) := \max \{ \lambda_i(x, u) \}, \quad \lambda_{\min}(x, u) := \min \{ \lambda_i(x, u) \}
\]

and

\[
\lambda_{\max} := \max_{x \in M, u \in T_x M; |u| = 1} \lambda_{\max}(x, u), \quad \lambda_{\min} := \min_{x \in M, u \in T_x M; |u| = 1} \lambda_{\min}(x, u).
\]

We remark that \(\lambda_{\max}(x, u)\) and \(\lambda_{\min}(x, u)\) are in general continuous functions and may not be smooth. Nevertheless, continuity is enough to guarantee that \(\lambda_{\max}\) and \(\lambda_{\min}\) can both be attained as the maximum and minimum in (3.4) are over the unit sphere bundle of \(TM\), which is compact.

Similarly define

\[
H_{\max} := \max_{x \in M, u \in T_x M; |u| = 1} H_x(u), \quad H_{\min} := \min_{x \in M, u \in T_x M; |u| = 1} H_x(u).
\]

For the same reason as above \(H_{\max}\) and \(H_{\min}\) can also be attained as they are smooth. Thus the positivity (resp. negativity) of \(\text{HSC}(\omega)\) implies that of \(H_{\min}\) (resp. \(H_{\max}\)).

Now we are ready to prove Theorem 1.5 in a more quantitative version, that is

**Theorem 3.1.** Let \((M, \omega)\) be a compact CKL Hermitian manifold.

1. If \(\text{HSC}(\omega) > 0\), then \(\Gamma^{p}_q(M) = 0\) whenever \(2p \lambda_{\max} < q H_{\min}\). In particular, \(\Gamma^{p}_q(M) = 0\)
   whenever \(q \geq 1\). Consequently the Hodge numbers \(h^{q,0}(M) = 0\) for \(1 \leq q \leq n\).
2. If \(\text{HSC}(\omega) < 0\), then \(\Gamma^{p}_q(M) = 0\) whenever \(p H_{\max} < 2q \lambda_{\min}\). In particular, \(\Gamma^{p}_q(M) = 0\)
   whenever \(p \geq 1\).

**Proof.** Assume that \(\text{HSC}(\omega) > 0\). Let \(T \in \Gamma^{p}_q(M)\), the maximal value of \(|T|\) be attained at \(x \in M\), and the unit vector \(u \in T_x M\) minimize \(H_x\).

First by the definitions (3.3) and (3.4) the eigenvalues \(\lambda_i(x, u)\) of the Hermitian transformation \(R_{\bar{u}u}\) at \(T_x M\) are bounded above by \(\lambda_{\max}: \lambda_i(x, u) \leq \lambda_{\max}\).
Secondly, these eigenvalues $\lambda_i(x, u)$ are bounded below by $\frac{1}{2}H_{\min}$, which is positive under the assumption of $HSC(\omega) > 0$. Indeed,

$$\lambda_i(x, u) = \langle Ru_i, e_i \rangle \quad \text{(by (3.1))}$$

$$= R(u, \bar{u}, e_i, \bar{e}_i) \geq \frac{1 + |\langle u, e_i \rangle|^2}{2}H_x(u) \quad \text{(by (2.6))}$$

$$\geq \frac{1}{2}H_{\min} \quad \text{(by (3.5))}$$

$$\geq 0.$$

In summary, under the condition of $HSC(\omega) > 0$, we have

$$\lambda_{\max} \geq \lambda_i(x, u) \geq \frac{1}{2}H_{\min} > 0, \quad \forall 1 \leq i \leq n. \quad (3.6)$$

Note that the key point here is that the two positive bounds $\lambda_{\max}$ and $\frac{1}{2}H_{\min}$ are independent of the choice of the pair $(x, u)$ and hence depends only on the metric $\omega$. Therefore,

$$\langle Ru_i(T), T \rangle \big|_x = \sum_{\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q} (\sum_{i=1}^p \lambda_{\alpha_i} - \sum_{j=1}^q \lambda_{\beta_j}) |T_{\alpha_1 \ldots \alpha_p}^{\beta_1 \ldots \beta_q}|^2 \quad \text{(by (3.2))}$$

$$\leq \sum_{\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q} (p \cdot \lambda_{\max} - q \cdot \frac{1}{2}H_{\min}) |T_{\alpha_1 \ldots \alpha_p}^{\beta_1 \ldots \beta_q}|^2 \quad \text{(by (3.6))}$$

$$\leq 0. \quad \text{(when } 2p\lambda_{\max} < qH_{\min}) \quad (3.7)$$

However, (2.4) in Lemma 2.1 implies that

$$\langle Ru_i(T), T \rangle \big|_x \geq 0,$$

which, together with (3.7), in turn yields that

$$\langle Ru_i(T), T \rangle \big|_x = 0.$$

Under the condition of the strict inequality $2p\lambda_{\max} < qH_{\min}$, the equality case of (3.7) occurs only if all $|T_{\alpha_1 \ldots \alpha_p}^{\beta_1 \ldots \beta_q}| = 0$ at $x$, i.e., $T(x) = 0$. The maximum of $|T|$ at $x$ then lead to $T \equiv 0$. This completes the first part in Theorem 3.1.

The proof of the second part is completely analogous. Under the condition of HSC being negative, we assume that the maximal value of $|T|$ be attained at $x$ and the unit vector $u \in T_x M$ maximizes $H_x$.

Due to the same reasoning as in the first part we have

$$\lambda_{\min} \leq \lambda_i(x, u) \leq \frac{1}{2}H_{\max} < 0, \quad (1 \leq i \leq n)\quad (1)$$
and thus, when $pH_{\text{max}} < 2q\lambda_{\text{min}}$,

$$< R_{u\bar{u}}(T), T >|_x = \sum_{\alpha_1, \ldots, \alpha_p}^{\alpha_1, \ldots, \alpha_p} \left( \sum_{i=1}^{p} \lambda_{\alpha_i} - \sum_{j=1}^{q} \lambda_{\beta_j} \right) \left| T_{\alpha_1 \ldots \alpha_p}^{\beta_1 \ldots \beta_q} \right|^2$$

$$\leq \sum_{\alpha_1, \ldots, \alpha_p}^{\alpha_1, \ldots, \alpha_p} \left( p \cdot \frac{1}{2} H_{\text{max}} - q \cdot \lambda_{\text{min}} \right) \left| T_{\alpha_1 \ldots \alpha_p}^{\beta_1 \ldots \beta_q} \right|^2$$

$$\leq 0.$$

The same arguments as above again lead to $T \equiv 0$, which completes the proof of the second part. □

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