INJECTIVITY RADIUS BOUND OF RICCI FLOW WITH POSITIVE RICCI CURVATURE AND APPLICATIONS

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Abstract. In this paper, we study the injectivity radius bound for 3-d Ricci flow. We also give an example showing that the positivity of Ricci curvature cannot be weakened into non-negativity. As applications we show the long time existence of the Ricci flow with positive Ricci curvature. We also partially settle a question in page 302 (line -13) of the book of Chow-Lu-Ni (2006).

Mathematics Subject Classification 2000: 53Cxx,35Jxx

Keywords: injectivity radius bound, Ricci flow, positive Ricci curvature, compactness

1. Introduction

In this paper we study the injectivity radius bound for 3-d Ricci flow with positive Ricci curvature. We prove the following result.

Theorem 1. Assume that $(M, g(t))$, $t \in [0, T)$, is a 3-d simply connected complete non-compact n-dimensional Ricci flow with bounded curvature, i.e. $|Rm(g)| \leq B$ for some constant $B > 0$ on $M$ and with positive Ricci curvature $Rc > 0$. Then for each $t > 0$, $(M, g(t))$ is non-collapse everywhere on $M$. Namely, there is a positive constant $c = c(t) > 0$ such that the injectivity radius $injrad(x,g(t)) \geq c(t)$.

This result is inspired from the works [12] [13] [14]. As applications, we continue our study of the global existence of the Ricci flow on the 3-dimensional complete non-compact Riemannian manifold $(M, g_0)$ with positive Ricci curvature. We assume that the Riemannian curvature of $(M, g_0)$ decays at infinity, namely,

$|Rm(g_0)(x)| \to 0$

as the distance $d_{g_0}(x, x_0) \to \infty$.

We shall prove the following result.

Theorem 2. Assume that $(M, g_0)$ is a 3-dimensional complete non-compact Riemannian manifold $(M, g_0)$ with positive Ricci curvature and with the condition (1). Then there is a global Ricci flow with the same properties of $(M, g_0)$. 

The research is partially supported by the National Natural Science Foundation of China 10631020 and SRFDP 20090002110019.
It is known by results of R.Hamilton ([6] [8]) and W.Shi that the Ricci flow exists and preserve the above two conditions. R.Hamilton asks if the Ricci flow exists globally. The above result settles his question. Without the assumption of behavior about the curvature at infinity, there is a finite blow-up Ricci flow, which is the standard solution constructed by Perelman [18].

We have proven the global existence of the Ricci flow under the positive curvature assumption on \((M, g_0)\). The positive curvature condition is for the injectivity radius bound used for the compactness theorem [8] [11].

We now recall the definition of type III Ricci flow. A global Ricci flow \((M, g(t))\) is called Type III if there exists a constant \(A > 0\) such that

\[
\sup_{M^n \times [0, \infty)} t|Rm(g(t))| = A < \infty.
\]

We also partially settle a question in page 302 (line -13) of the book [5].

**Theorem 3.** Given a 3-d type III Ricci flow with positive Ricci curvature and with a sequence \((x_i, t_i) \in M \times [0, \infty)\) and \(t_i K_i \geq c\) where \(K_i = |Rm(x_i, t_i)|\) for some positive constant \(c > 0\). Then the family of Ricci flows

\[
g_i(t) = K_i g(t_i + K_i^{-1}t)
\]

has a convergent subsequence in the sense of Cheeger-Gromov.

The interesting part of the result Theorem 1 above lies in that we don’t assume any non-collapse for the initial metric. The key part of our proof is to use the Gauss-Bonnet formula to the minimal surface at infinity by using the simply connectedness at infinity proved by Schoen-Yau [20]. We also point out that there is a new interesting result of X.C. Rong [19] about non-collapsing for Riemannian manifolds with almost-negative curvature and with connectivity at infinity.

Recall that for the metric \(g\) with positive Ricci curvature \(Rc(g) > 0\), the volume quotient

\[
\frac{Vol(B_g(x, R))}{\omega_n R^n}
\]

is well defined monotone non-increasing function in \(R \in (0, \infty)\) via the Bishop theorem. One may use the lower bound of this quotient to get the lower injectivity bound. However, near the infinity, this quantity may be very small.

The key result Theorem 1 is proved in next section. We also give an example showing that the positivity of Ricci curvature can not be weakened into non-negativity. The proofs of other results are discussed in the last section.
2. Proof of Theorem 1 and its remarks

Recall that for any \( b > 0 \), the metric \( g_b = b^2 g \), we have the following relations of the metric balls (see page 253 in [5]) that

\[
B_g(x, r) = B_{g_b}(x, br), \quad Vol_{g_b}(B_{g_b}(x, br)) = b^n Vol_g(B_g(x, r)),
\]

\[
|R_{mg}(x)| = b^{-2}|R_{mg}(x)|,
\]

and

\[
\text{injrad}(x, g_b) = b \cdot \text{injrad}(x, g).
\]

We now prove Theorem 1.

Proof. We argue by contradiction. Assume the conclusion of Proposition 1 is not true for some \( t_0 \in (0, T) \). Then there exists \( x_j \to \infty \) such that

\[
\text{injrad}(x_j, g(t_0)) \to 0
\]

and a minimizing geodesic \( \sigma_j \) based at \( x_j \) such that \( L(\sigma_j) = 2\text{injrad}(x_j, g(t_0)) \). Set \( \lambda_j = \text{injrad}(x_j, g(t_0)) \). We now normalize the Ricci flow \( g(t) \) at \( (x_j, t_0) \) by

\[
g_j(t) = \lambda_j^{-2} g(\lambda_j^2 t + t_0).
\]

Then the Ricci flow \( g_j(t) \) has

\[
\text{injrad}(x_j, g_j(0)) = 1
\]

and

\[
|R_m(g_j(t))| \leq B\lambda_j^2 \to 0.
\]

Using the Cheeger-Gromov-Hamilton convergence theorem we may assume that

\[(M, g_j(t), x_j) \to (M_\infty, g_\infty(t), x_\infty)\]

in the sense of \( C^2 \) Cheeger-Gromov sense ([8], [4], [5]) and \( Rm(g_\infty) = 0 \). Hence \( B_{g_\infty}(0)(x_\infty, 1) \) is the unit euclidean ball and

\[
Vol(B_{g_\infty}(0)(x_\infty, 1)) = \omega_n.
\]

We now recall a key fact from Schoen-Yau’s work [20] about the geometry of the complete non-compact Riemannian manifold \((M^3, g)\) of positive Ricci curvature that \( M \) is diffeomorphic to \( R^3 \). Recall here that \( M \) is simply connected at infinity if there are no compact set \( K \subset M \) and a sequence of Jordan curves \( \{\sigma_j\} \) tending uniformly to infinity such that any sequence of disks \( \{D_j\} \) with \( \partial D_j = \sigma_j \) has \( D_j \cap K \neq \emptyset \) for each \( j \).

Using \( \text{injrad}(x_j, g_j) = 1 \) we get a closed minimizing geodesic \( \gamma_j = (\sigma_j) \) in \((M, g_j)\) based at \( x_j \) (and the limit \( \gamma_\infty \) of \( \gamma_j \) is a true closed geodesic in flat space \((M_\infty, g_\infty)\)). Smoothing \( \gamma_j \) (at \( x_j \), which may be a corner point) and using the simply-connectedness at infinity of \((M, g(t_0))\) (by the well-known theorem of Schoen-Yau [20]) we can bound it by a minimizing (immersed) minimal disk \( \Sigma_j \) (and its existence is guaranteed by the works of S.Hildebrandt and C.B.Morrey) with its area bounded above by some uniform constant, saying \( 8\pi \) for large \( j \). We remark that the area bound comes from a reversible construction from the fact (Lemma 9.5.1(b) in [17]) that
\[
\lim_{j \to \infty} A(\Sigma_j) = A(\Sigma_\infty), \quad \text{where } \Sigma_\infty \text{ is the minimizing minimal disk bounded by the loop } \gamma_\infty \text{ in } (M_\infty, g_\infty). \quad \text{One can also get the area bound by using the pull-back of the surface } \Sigma_\infty \text{ to the space } (M, g_j(t)) \text{ as the comparison surface. The geometric picture of this construction is that the limit space } M_\infty \text{ can not be } (S^1 \times \mathbb{R}) \times \mathbb{R}. \quad \text{For otherwise, we would have that } \gamma_j \text{ has its limit } S^1 \quad \text{and the } \Sigma_j \text{ converges to } S^1 \times \mathbb{R}. \quad \text{Hence, the area of } \Sigma_j \text{ would be large.}
\]

Since the limit of Busemann functions \( B_{\gamma_j} \) corresponding to \( \gamma_j \) is the coordinate function in the real line \( \mathbb{R} \), we can use the cut of regular level sets of the Busemann function \( B_{\gamma_j} \) to strictly reduce \( \Sigma_j \) in its area, which is a contradiction to the minimizing property of \( \Sigma_j \) (being an area minimizing disk). See also the remark below and the references [7] [18] for related idea. On the surface \( \Sigma_j \) with the unit normal vector \( N \) and the second fundamental form \( A_j \) of \( \Sigma_j \subset M \), we know that
\[
Rcg_j(N, N) + \frac{1}{2}|A_j|^2 = \frac{R(g_j)}{2} - K_j.
\]
Here \( K_j \) is the Gauss curvature of the minimal surface \( \Sigma_j \). Then using \( Rc > 0 \) we have
\[
\frac{1}{2}|A|^2 + K_j \leq \frac{R(g_j)}{2} \to 0
\]
and by using the uniform area bound of \( \Sigma_j \),
\[
\int_{\Sigma_j} K_j \leq \frac{1}{2} \int_{\Sigma_j} R(g_j) \leq \sup_{\Sigma_j} R(g_j) A(\Sigma_j) \to 0.
\]

Recall that the Gauss-bonnet formula (and here we may assume that \( \Sigma_j \)'s are embedded)
\[
\int_{\Sigma_j} K_j + \int_{\gamma_j} k_{g_j} = 2\pi,
\]
which gives us a contradiction since both terms in left side tend to zero or less than zero as \( j \to \infty \).

We then complete the proof of Theorem 1. \( \square \)

We give two remarks about the argument above. One is that the important step in getting the minimizing minimal disk bounding \( \gamma_i \) with uniform bounded area in the above argument is using the simply-connectedness at infinity of \( M \) and non-splitting of \( M \). Recall here that for a given ray \( \tilde{\gamma}(t) \) with \( \tilde{\gamma}(0) = x_0 \), the Busemann function is defined by
\[
B_{\tilde{\gamma}}(x) = \lim_{t \to \infty} \left[ t - d(x, \tilde{\gamma}(t)) \right],
\]
where \( d(x, x_0) \) is the distance function in \( (M, g) \). The sequence \( (x_i) \) determines a ray \( \tilde{\gamma}(t) \) and we write by \( B_{\tilde{\gamma}} \) \( (B_i) \) the corresponding Busemann function in \( (M, g_j) \) \( (M, g_i, x_i) \). By the simply-connectedness at infinity (based on the fact that the level sets of the Busemann function \( B_i \) are of positive mean curvature, see line 7 in page 221 in [20], and correspondingly the level sets of the distance function \( d(x) := d(x, x_0) \) are of almost negative mean curvature when \( d(x) \) large) and Bernstein type theorem (Theorem 2 in
we can always find minimizing minimal disks $D_i$ bounding $\gamma_i$ in $(M, x_i)$ (according to the works of Hildebrandt [9] and C.B.Morrey [16]) such that there are only finite $D_i$ can interest any fixed compact subset $K \subset M$ (and otherwise there is a non-trivial stable minimal surface in $(M, g)$, which is impossible [20]). We can show that for $\text{dist}(\gamma_i, \gamma_j)$ large, there is a contractible domain $\Omega(i, j)$ such that both $D_i$ and $D_j$ are in the boundary of $\Omega(i, j)$. This implies that we find at least one comparison disk $G_i$ spanning $\gamma_i$ with uniform area bound such that it lies between two level sets $a_i + L_i$ for some $a_i \gg 1$ and $0 < L_i \leq 2$ (because the half length of $\gamma_i$ is one). Another way to find such comparison surface is below. Note that the limit surface of $D_i$ is a flat disk in the flat space $(M_\infty, g_\infty, x_\infty)$. Using this limit surface we can also construct a comparison surface to each $D_i$ such that the area of $D_i$ is uniformly bounded. This step can not be carried through to 3-d manifold $M$ with non-negative Ricci curvature since we may not find a minimizing disk bounding $\gamma_i$ in $M_i$ with uniform bounded area. Here is the example that $M = N \times R$ with $N = R^2$ equipped with cigar metric $h$. With this metric, $M$ has non-negative Ricci curvature and it does not have the strong contractible property. In this example, we let $x_i$ in $N$ going to infinity. Let $\gamma_i$ be a closed geodesic realizing $\text{injrad}(x_i) \approx 1$. The area of the disk $D_i$ bounding $\gamma_i$ goes to infinity with $i \to \infty$. Moreover $(N, h, x_i)$ converges to a flat cylinder $S^1 \times R$ and $\gamma_i$ converges to the $S^1$ factor of the cylinder, which does not bound a disk. Now if $M = R^3 = N \times R$ and $g = h + ds^2$ on $M$, consider $(0, x_i)$ in $M$ going to infinity and the limit here is the flat $S^1 \times R^2$, and $\gamma_i$ converges to the $S^1$ factor, which does not bound a disk. The key point for this example is that with the metric $g = h + ds^2$, $R^3$ has a splitting structure, which makes the minimizing disk have no area bound.

The other is that once we have the uniform injectivity radius bound of $g(t_0)$, for some $t_0 > 0$, we have the the uniform injectivity radius bound $c(t_0, B)$ of $g(t)$ for $t > t_0$. The constant $c(t_0, B)$ depends only on $t_0, B$ by using the estimates of Carron Proposition 4 in [2]).

3. Remarks about proofs of other results

The key step in the argument about the generalized version of the result in [15] is about the injectivity radius bound, which now can be obtained via Theorem [1]. Then the proof of Theorem [2] is almost the same as in [15]. So we omit the detail.

The proof of Theorem [3] follows easily from Hamilton’s compactness theorem of Ricci flow [8, 5].

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