Algebraic Cobordism in mixed characteristic

Markus Spitzweck

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Abstract

We compute the geometric part of algebraic cobordism over Dedekind domains of mixed characteristic after inverting the positive residue characteristics and prove cases of a Conjecture of Voevodsky relating this geometric part to the Lazard ring for regular local bases. The method is by analyzing the slice tower of algebraic cobordism, relying on the Hopkins-Morel isomorphism from the quotient of the algebraic cobordism spectrum by the generators of the Lazard ring to the motivic Eilenberg-MacLane spectrum, again after inverting the positive residue characteristics.

1 Introduction

Algebraic cobordism is a theory for smooth schemes over a base scheme $S$ defined by a motivic ring spectrum $\text{MGL}_S$ in the stable motivic homotopy category $\text{SH}(S)$. It is the motivic counterpart of complex cobordism $\text{MU}$. A famous Theorem of Quillen states that the natural map from the Lazard ring $L_\ast$ classifying formal group laws to the coefficients of $\text{MU}$ is an isomorphism, moreover $L_\ast \cong \mathbb{Z}[x_1,x_2,x_3,\ldots]$ with $\text{deg}(x_i) = i$ (here we divide the usual topological grading by 2).

For an oriented motivic ring spectrum $E$ the geometric part $E_{(2,1)} \ast$ of the coefficients also carries a formal group law constructed in the exact same way as in topology by evaluating the theory on $P^\infty$ and using that $P^\infty$ is naturally endowed with a multiplication.

Thus there is a classifying map $L_\ast \to E_{(2,1)} \ast$. It is known that for $E = \text{MGL}_k$ for a field $k$ of characteristic 0 this map is an isomorphism using the Hopkins-Morel isomorphism, see [2, Proposition 8.2]. More generally in [3] it is shown that over such fields the Levine-Morel algebraic cobordism $\Omega^\ast(-)$ is isomorphic to $\text{MGL}_{(2,1)}(\mathbb{Z})(-)$ on smooth schemes over $k$. If the base field $k$ has positive characteristic the map $L_\ast \to \text{MGL}_{(2,1)}(\mathbb{Z})$ becomes at least an isomorphism after inverting the characteristic, see again [2, Proposition 8.2].

The main ingredient in the proof is that the Hopkins-Morel isomorphism yields a computation of the slices of $\text{MGL}_S$ with respect to Voevodsky’s slice filtration, that $\text{MGL}_S$ is complete with respect to this filtration and that the slices have a simple form, namely they are shifted twists of the motivic Eilenberg-MacLane spectrum.

The facts about the slices of $\text{MGL}_S$ hold more generally true over spectra $S$ of Dedekind domains of mixed characteristic (after inverting the positive residue characteristics), using the motivic Eilenberg-MacLane spectrum introduced in [4]. The main new input of this note is that in this case $\text{MGL}_S$ is also complete.
with respect to the slice filtration (Corollary 5.9), a consequence of the fact that MGL is connective with respect to the homotopy sheaves, see Proposition 5.8.

This yields a computation of the geometric part of the homotopy groups of MGL (Theorem 6.5), again after inverting the residue characteristics. In our formulation we always assume a Hopkins-Morel isomorphism for the given coefficients, hoping that the Hopkins-Morel isomorphism will be settled completely in the future.

We prove cases of a Conjecture of Voevodsky ([7, Conjecture 1]), see Theorem 6.7, comparing the Lazard ring to (MGL)_2, for S the spectrum of a regular local ring.

We also give applications to some homotopy groups or sheaves of MGL outside the geometric diagonal, see section 7, and discuss generalizations of our results to motivic Landweber spectra.

We note that the observation that the Hopkins-Morel isomorphism yields the computation of the zero-slice of the sphere spectrum (after inverting suitable primes), see Theorem 3.1, was independently made by Oliver Röndigs.

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2 Preliminaries

By a base scheme we always mean a separated Noetherian scheme of finite Krull dimension. For a base scheme S we let SH(S) be the stable motivic homotopy category.

We let MZS ∈ SH(S) be the motivic Eilenberg-MacLane spectrum over S constructed in [4]. Also we let M(r) ∈ D(Sh(SmS Z, Z, Z)) (for notation see [4]) be the motivic complexes of weight r ∈ Z, so as a Gm,S-spectrum MZS has M(r)[r] in level r. If S is the spectrum of a Dedekind domain of mixed characteristic we note that M(0) = S0Z, thus for X ∈ SmS we have H^0,0(X, Z) = Zπ0(X). Also M(1) ≃ O∗[-1], so H^1,1(X, Z) ≃ O∗(X) and H^2,1(X, Z) ≃ Pic(X). We have M(r) ≃ 0 for r < 0.

For general S we denote by MGLS ∈ SH(S) the algebraic cobordism spectrum. There is a natural map L* → (MGLS)∗, where L* denotes the Lazard ring. Fixing generators x_i ∈ L_i there is a map

ΦS: MGLS/(x_1, x_2, ...) MGLS → MZS,

see [4] §11.1, which is an isomorphism after inverting all positive residue characteristics of S, see [4] Theorem 11.3].

For any ring or abelian group R we let MR ∈ SH(S) be the Moore spectrum on R and MR the version of MZ with R-coefficients.
3 Slices

For $i \in \mathbb{Z}$ denote by $f_i$ resp. $l_i$ the $i$-th colocalization resp. localization functor for Voevodsky’s motivic slice filtration on $\text{SH}(S)$. For any $E \in \text{SH}(S)$ and $k \geq n$ we set $E\langle n, k \rangle := l_{k+1}(f_n(E))$. Thus we have exact triangles

$$f_{k+1}(E) \rightarrow f_n(E) \rightarrow E\langle n, k \rangle \rightarrow f_{k+1}(E)[1]$$

and $s_n(E) = E\langle n, n \rangle$.

We note that all these functors commute with homotopy colimits.

Theorem 3.1. Let $X$ be an essentially smooth scheme over a Dedekind domain of mixed characteristic and $R$ a localization of $\mathbb{Z}$ such that $\Phi_X \wedge MR$ is an isomorphism (e.g. if every positive residue characteristic of $X$ is invertible in $R$). Then

$$s_0MR \cong s_0(MGL_X \wedge MR) \cong MR_X.$$

More generally

$$s_n(MGL_X \wedge MR) \cong \Sigma^{2n,n}MR_X \otimes L_n.$$

Proof. The first isomorphism of the first line follows from [5, Corollary 3.3]. From the assumption that $\Phi_X \wedge MR$ is an isomorphism it follows that the map $MGL_X \wedge MR \rightarrow MR_X$ induces an isomorphism on zero-slices and that $MR_X$ is effective. Moreover $l_1MZ_X \cong MZ_X$, since negative weight motivic cohomology vanishes in our situation. Thus the second isomorphism of the first line follows.

The second line is a version of [5, Theorem 4.7] with $R$-coefficients.

Remark 3.2. It is then also possible to determine the slices of motivic Landweber spectra with $R$-coefficients, see [6], for example of $KGL_X \wedge MR$.

4 Subcategories of the stable motivic homotopy category

Fix a base scheme $S$. We let $\text{SH}(S)_{\geq n}$ be the $\geq n$ part (in the homological sense) of $\text{SH}(S)$ with respect to the homotopy $t$-structure, see e.g. [2, §2.1]. Thus $\text{SH}(S)_{\geq n}$ is generated by homotopy colimits and extensions by the objects $\Sigma^p,q \Sigma^\infty X$ for $X \in \text{Sm}_S$ and $p - q \geq n$.

For each $E \in \text{SH}(S)$ we let $\pi^\text{pre}_{p,q}(E)$ be the presheaf

$$X \mapsto \text{Hom}_{\text{SH}(S)}(\Sigma^p,q \Sigma^\infty X, E)$$

on $\text{Sm}_S$. Let $\pi_{p,q}(E)$ be the sheafification of $\pi^\text{pre}_{p,q}(E)$ with respect to the Nisnevich topology. We also set $\pi_{p,q}(E) := \pi^\text{pre}_{p,q}(E)(S) = E_{p,q}$.

We let $\text{SH}(S)_{h\geq n}$ be the full subcategory of $\text{SH}(S)$ of objects $E$ such that $\pi_{p,q}(E) = 0$ for $p - q < n$.

Lemma 4.1. The categories $\text{SH}(S)_{h\geq n}$ are closed under homotopy colimits and extensions in $\text{SH}(S)$.

Proof. Th functors $\pi_{p,q}$ respect sums. Moreover the long exact sequences of homotopy sheaves associated to an exact triangle in $\text{SH}(S)$ show that $\text{SH}(S)_{h\geq n}$ is closed under cofibers and extensions. This shows the claim.
Proposition 4.2. Let \( i: Z \to S \) be a closed inclusion of base schemes. Then \( i_*(\mathcal{H}(Z)_{h_{2n}}) \subseteq \mathcal{H}(S)_{h_{2n}} \).

Proof. Let \( E \in \mathcal{H}(Z)_{h_{2n}} \). Let \( Y \) be the spectrum of the henselization of a local ring of a scheme from \( \text{Sm}_S \). Then \( Y_Z := Y \times_S Z \) is also the spectrum of a henselian local ring, and \( \pi_{p,q}^{\text{pre}}(i_*(E))(Y_Z) \equiv \pi_{p,q}^{\text{pre}}(E)(Y_Z) = 0 \) for \( p - q < n \) (the first isomorphism holds since \( i_* \) commutes with homotopy colimits).

We let \( \mathcal{H}_S^{St}(S) \) be the homotopy category of presheaves of \( S^1 \)-spectra on \( \text{Sm}_S \) localized with respect to the Nisnevich topology, and \( \mathcal{H}_S^{St}(S) \) the further \( K^1 \)-localization of that category.

We let \( \mathcal{H}_S^{St}(S)_{2n} \) be the \( n \) part (in the homological sense) of \( \mathcal{H}_S^{St}(S) \) with respect to the standard \( t \)-structure, and for \( E \in \mathcal{H}_S^{St}(S) \) we let \( E_{2n} \) and \( E_{\leq n} \) be the corresponding truncations. We let \( E_{=n} := (E_{2n})_{S_n} \).

As above for \( E \in \mathcal{H}_S^{St}(S) \) we have the presheaves \( \pi_k^{\text{pre}}(E) \) and the sheaves \( E_k \). For \( E \in \mathcal{H}_S^{St}(S) \) we have \( E \in \mathcal{H}_S^{St}(S)_{2n} \) if and only if \( \pi_k(E) = 0 \) for \( k < n \).

Note that \( \mathcal{H}_S^{St}(S)_{2n} \) is generated by homotopy colimits and extensions by the objects \( \Sigma^m \Sigma^\infty_X \), \( X \in \text{Sm}_S \), thus the canonical functor \( \sigma: \mathcal{H}_S^{St}(S) \to \mathcal{H}(S) \) sends \( \mathcal{H}_S^{St}(S)_{2n} \) to \( \mathcal{H}(S)_{2n} \).

Lemma 4.3. We have \( \mathcal{H}(S)_{h_{2n}} \subseteq \mathcal{H}(S)_{2n} \). If \( S \) is the spectrum of a field then the inclusion is an equality.

Proof. Let \( E \in \mathcal{H}(S)_{h_{2n}} \). For any \( i \in \mathbb{N} \) let \( E_i \) be the image of \( \Sigma^i E \) in \( \mathcal{H}_S^{St}(S) \).

By assumption we have \( E_i \in \mathcal{H}_S^{St}(S)_{2n} \). Thus \( \Sigma^{i-1} \sigma(E_i) \in \mathcal{H}(S)_{2n} \). The proof of the first statement concludes by noting that \( E \equiv \text{hocolim}_{i \to \infty} \Sigma^{i-1} \sigma(E_i) \).

The second statement is [2, Theorem 2.3].

Lemma 4.4. Let \( E \in \mathcal{H}_S^{St}(S) \). Then \( E \to \text{holim}_{m \to \infty} E_{\leq m} \) is an isomorphism.

Proof. We show that for all \( n \in \mathbb{Z} \) we have \( \pi_n(E) \equiv \pi_n(\text{holim}_{m \to \infty} E_{\leq m}) \). Fix \( n \in \mathbb{Z} \) and let \( X \in \text{Sm}_S \) be of dimension \( d \). We are ready if we show that \( \pi_n(E)|_{X_{Nis}} \equiv \pi_n(\text{holim}_{k \to \infty} E_{\leq k})|_{X_{Nis}} \).

For \( m > n + d \) we have \( \pi_k(\text{holim}_{m \to \infty} E_{\leq m})|_{X_{Nis}} \equiv \pi_k(\text{holim}_{k \to \infty} E_{\leq k})|_{X_{Nis}} \).

For \( m > n + d \) we have \( \pi_k(\text{holim}_{m \to \infty} E_{\leq m})|_{X_{Nis}} \equiv \pi_k(\text{holim}_{k \to \infty} E_{\leq k})|_{X_{Nis}} \). Using the Milnor exact sequence this shows that \( \pi_k(\text{holim}_{k \to \infty} E_{\leq k})|_{X_{Nis}} \equiv \pi_k(\text{holim}_{k \to \infty} E_{\leq k})|_{X_{Nis}} \).

for \( m > n + d \). Sheafifying proves \((*)\).

Corollary 4.5. Let

\[
\cdots \to E_{i+1} \to E_i \to E_{i-1} \to \cdots \to E_1 \to E_0
\]

be an inverse system of objects in \( \mathcal{H}(S) \). Suppose for each \( n \in \mathbb{N} \) there is an \( N \in \mathbb{N} \) such that \( E_i \in \mathcal{H}(S)_{h_{2n}} \) for \( i \geq N \). Then \( \text{holim}_{i \to \infty} E_i \equiv 0 \).
Proof. Fix $q \in \mathbb{Z}$ and let $F_i$ be the image of $\Sigma^{p,q} F_i$ in $\text{SH}^S(S)$. We are ready if we show $\text{holim}_{i \to \infty} F_i \cong 0$. By assumption for every $n \in \mathbb{N}$ there is a $N \in \mathbb{N}$ such that $F_i \in \text{SH}^S(S)_{2n}$ for each $i \geq N$. By Lemma 4.3 we have $F_i \cong \text{holim}_{i \to \infty}(F_i)_{\leq k}$. Thus

$$\text{holim}_{i \to \infty} F_i \cong \text{holim}_{i \to \infty} \text{holim}_{k \to \infty} (F_i)_{\leq k} \cong \text{holim}_{k \to \infty} (\text{holim}_i (F_i))_{\leq k} \cong \text{holim}_k 0 \cong 0.$$ 

\[ \square \]

We also have the

**Corollary 4.6.** Let $E \in \text{SH}(S)_{\geq n}$ and $X \in \text{Sm}_S$ of dimension $d$. Then

$$\pi_{p,q}^E(E)(X) = 0$$

for $p - q < n - d$.

**Proposition 4.7.** Let

$$\cdots \to E_{i+1} \to E_i \to E_{i-1} \to \cdots \to E_1 \to E_0$$

be an inverse system of objects in $\text{SH}(S)_{\geq n}$. Suppose for each $p,q \in \mathbb{Z}$ and $d \in \mathbb{N}$ there is an $N \in \mathbb{N}$ such that for $X \in \text{Sm}_S$ of dimension $d$ the map

$$\pi_{p,q}^E(E_{i+1})(X) \to \pi_{p,q}^E(E_i)(X)$$

is an isomorphism for all $i \geq N$. Then $\text{holim}_{i \to \infty} E_i \in \text{SH}(S)_{\geq n}$. (Here the latter homotopy limit is computed in $\text{SH}(S)$.)

**Proof.** Let $p,q \in \mathbb{Z}$, $d \in \mathbb{N}$ and $X \in \text{Sm}_S$ of dimension $d$. Choose $N \in \mathbb{N}$ such that for any $Y \in \text{Sm}_S$ of dimension $\leq d$ the map

$$\pi_{p,q}^E(E_{i+1})(Y) \to \pi_{p,q}^E(E_i)(Y)$$

is an isomorphism for all $i \geq N$. We claim that

$$\pi_{p,q}^E(\text{holim}_k E_k)|_{X_{\text{Nis}}} \cong \pi_{p,q}^E(E_i)|_{X_{\text{Nis}}}$$

for all $i \geq N$. For every $Y \in X_{\text{Nis}}$ we have the Milnor short exact sequence

$$0 \to \lim_i \pi_{p,q}^E(E_i)(Y) \to \pi_{p,q}^E(\text{holim}_i E_i)(Y) \to \lim_i \pi_{p,q}^E(E_i)(Y) \to 0.$$ 

The $\lim^1$-term vanishes because the inverse system of abelian groups stabilizes by assumption. Sheafifying we see that $\pi_{p,q}^E(\text{holim}_k E_k)|_{X_{\text{Nis}}} \cong \pi_{p,q}^E(E_i)|_{X_{\text{Nis}}}$ for $i \geq N$, in particular $\pi_{p,q}^E(\text{holim}_k E_k)|_{X_{\text{Nis}}} = 0$ in case $p-q < n$. Since this is true for all $X \in \text{Sm}_S$ we conclude $\pi_{p,q}^E(\text{holim}_k E_k) = 0$ for $p-q < n$. \[ \square \]

5 Connectivity of algebraic cobordism

**Lemma 5.1.** Let $X$ be a smooth scheme over a Dedekind domain of mixed characteristic or over a field. Then for any abelian group $A$ we have $\text{MA}_X \in \text{SH}(X)_{\geq 0}$. 

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Proof. This follows from the fact that the motivic complexes $\mathcal{M}(r)$ have vanishing $i$-th cohomology sheaf for $i > r$, see \cite[Corollary 4.4]{1}.

**Proposition 5.2.** Let $S$ be the spectrum of a discrete valuation ring of mixed characteristic, $j: \eta \to S$ the inclusion of the generic point. Then for any abelian group $A$ we have $j_* \mathcal{M}_S A \in \mathcal{SH}(S)_{h \geq 0}$.

Proof. Let $i: s \to S$ be the inclusion of the closed point. We have an exact triangle

$$i_* i^! \mathcal{M}_S \to \mathcal{M}_S \to j_* \mathcal{M}_\eta \to i_* i^! \mathcal{M}_S[1]$$

and an isomorphism $i^! \mathcal{M}_S \cong \mathcal{M}_s (-1)[-2] \in \mathcal{SH}(s)_{h \geq -1}$, see \cite[Theorem 7.4]{3}. We conclude with Proposition 4.2 and Lemma 5.1.

**Lemma 5.3.** Let the situation be as in Proposition 5.2. Then

$$j_* \mathcal{M}_{\eta} (0,n) \wedge M_A \in \mathcal{SH}(S)_{h \geq 0}$$

for all $n \geq 0$.

Proof. We can assume $A = \mathbb{Z}$. Since $\eta$ is of characteristic 0 we have $s_n \mathcal{M}_{\eta} \cong \Sigma^{2n} \mathbb{Z} \otimes L_n$. Moreover we have exact triangles

$$s_n \mathcal{M}_{\eta} \to \mathcal{M}_{\eta} (0,n) \to \mathcal{M}_{\eta} (0,n-1) \to s_n \mathcal{M}_{\eta}[1].$$

Applying $j_*$ to these triangles and using Proposition 5.2, one concludes by induction on $n$.

**Lemma 5.4.** Let the situation be as in Proposition 5.2. Let $p, q \in \mathbb{Z}$ and $X \in \mathcal{Sm}_S$ of dimension $d$. Then

$$\pi_{p,q}^\text{pre} (j_* \mathcal{M}_{\eta} (0,n+1))(X) \to \pi_{p,q}^\text{pre} (j_* \mathcal{M}_{\eta} (0,n))(X)$$

is an isomorphism for $n \geq p - q + d$.

Proof. Consider the exact triangle

$$j_* s_{n+1} \mathcal{M}_{\eta} \to j_* \mathcal{M}_{\eta} (0,n+1) \to j_* \mathcal{M}_{\eta} (0,n) \to s_{n+1} \mathcal{M}_{\eta}[1].$$

We have

$$\pi_{p,q}^\text{pre} (j_* s_{n+1} \mathcal{M}_{\eta})(X) \cong H_{\text{mot}}^{2(n+1)-p,n+1-q}(X, L_{n+1}).$$

The latter group vanishes for $2(n+1) - p > n + 1 - q + d$, showing the claim.

**Lemma 5.5.** Let the situation be as in Proposition 5.2. Then $j_* \mathcal{M}_{\eta} \in \mathcal{SH}(S)_{h \geq 0}$.

Proof. Consider the inverse system

$$\cdots \to j_* \mathcal{M}_{\eta} (0,n+1) \to j_* \mathcal{M}_{\eta} (0,n) \to \cdots \to j_* s_{0} \mathcal{M}_{\eta}$$

in $\mathcal{SH}(S)$. Since $j_*$ preserves homotopy limits the homotopy limit over this system is $j_* \mathcal{M}_{\eta}$, using \cite[Corollary 2.4 and Lemma 8.10 or Theorem 8.12]{2}. By Lemma 5.3 every object of this system is in $\mathcal{SH}(S)_{h \geq 0}$. Moreover by Lemma 5.4 the assumptions of Proposition 4.7 are satisfied. Thus this Proposition implies the claim.
Let the situation be as in Proposition 5.3 and let $i : s \to S$ be the inclusion of the closed point. Then $i^! \SH_\ast \in \SH_\ast_{S_{2}^0}$.

**Proof.** Note first that $i^!$ sends $\SH(S)_{\geq 0}$ to $\SH(S)_{\geq 20}$. We have $\SH_\ast \in \SH_\ast(S)_{\geq 0}$ and by Lemma 5.5 also $j_* \SH_\ast \in \SH_\ast(S)_{\geq 20} \subset \SH_\ast(S)_{\geq 0}$. Applying $i^*$ to the exact triangle

$$i^* i^! \SH_\ast \to \SH_\ast \to j_* \SH_\ast \to i^* i^! \SH_\ast[1]$$

shows the claim.

**Lemma 5.7.** Let $S$ be the spectrum of a discrete valuation ring of mixed characteristic. Then $\SH_\ast \in \SH_\ast(S)_{S_{2}^0}$.

**Proof.** Let the notation be as above. The claim follows from the exact triangle

$$i^* i^! \SH_\ast \to \SH_\ast \to j_* \SH_\ast \to i^* i^! \SH_\ast[1],$$

Lemma 5.5, Proposition 5.6 and Proposition 1.6.

**Proposition 5.8.** Let $S$ be the spectrum of a Dedekind domain of mixed characteristic. Then $\SH_\ast \in \SH_\ast(S)_{S_{2}^0}$

**Proof.** The henselization of a local ring of a scheme in $\text{Sm}_S$ lies over a local ring of $S$, thus the claim follows from Lemma 5.4.

Compare the following result to [3, Conjecture 15].

**Corollary 5.9.** Let $S$ be the spectrum of a Dedekind domain of mixed characteristic. Let $R$ be a localization of $\mathbb{Z}$ that $\Phi \times \SH_\ast \cong \SH_\ast \times \Phi$ is an isomorphism. Then for any $R$-module $A$ we have

$$f_* \SH_\ast \wedge A \cong \lim_{k \to \infty} \SH_\ast(n_\wedge k) \wedge A.$$  

**Proof.** Under the assumption we have $f_* \SH_\ast \wedge A \in \SH_\ast(S)_{\geq 2k-1}$, since this is a homotopy colimit of objects of the form $\Sigma^{2k} \wedge \SH_\ast \wedge A \in \SH_\ast(S)_{\geq 2k}$, see the proof of [5, Theorem 4.7], using Proposition 5.8. Thus by Corollary 5.3 we have $\lim_{k \to \infty} f_* \SH_\ast \wedge A = 0$ implying the claim.

**Remark 5.10.** A similar result holds for motivic Landweber spectra using the same argument as in the proof of [2, Lemma 8.11]. For example $KGL \wedge A$ is complete with respect to the slice filtration.

### 6 The geometric part of algebraic cobordism

**Lemma 6.1.** Let $S$ be the spectrum of a Dedekind domain of mixed characteristic. Let $R$ be a localization of $\mathbb{Z}$ such that $\Phi \times \SH_\ast \cong \SH_\ast \times \Phi$ is an isomorphism. Let $p, q \in \mathbb{Z}$ and $X \in \text{Sm}_S$. Then for any $R$-module $A$ the inverse system of abelian groups $(\pi_{n 0}^{\wedge p q}(\SH_\ast(0, k) \wedge A)(X))_k$ eventually becomes constant for $k \to \infty$.

**Proof.** This follows from the exact triangle

$$s_k \SH_\ast \wedge A \to \SH_\ast(0, k) \wedge A \to \SH_\ast(0, k-1) \wedge A \to s_k \SH_\ast \wedge A[1]$$

and $s_k \SH_\ast \wedge A \cong \Sigma^{2k-1} \wedge A \wedge L_k$ since $\pi_{n 0}^{\wedge p q}(\Sigma^{2k+1-j} \wedge A)(X) = 0$, $j \geq 0$, for $k$ big enough.
Corollary 6.2. Let $S$ be the spectrum of a Dedekind domain of mixed characteristic. Let $R$ be a localization of $\mathbb{Z}$ such that $\Phi_S \wedge M_R$ is an isomorphism. Let $p, q \in \mathbb{Z}$ and $X \in \text{Sm}_S$. Then for any $R$-module $A$ the canonical map
\[
\pi_{p,q}^\text{pre}(\text{MGL}_S \wedge M_A)(X) \to \lim_k \pi_{p,q}^\text{pre}(\text{MGL}_S(0,k) \wedge M_A)(X)
\]
is an isomorphism.

Proof. This follows from Corollary 5.9, the Milnor short exact sequence and Lemma 6.1.

Lemma 6.3. Let $S$ be the spectrum of a Dedekind domain of mixed characteristic. Let $R$ be a localization of $\mathbb{Z}$ such that $\Phi_S \wedge M_R$ is an isomorphism. Let $n \in \mathbb{Z}$. Then for $k \geq n + 1$ and any $R$-module $A$ the natural map
\[
\pi_{2n,n}^\text{MGL}_S(n,k + 1) \wedge M_A \to \pi_{2n,n}^\text{MGL}_S(n,k) \wedge M_A
\]
is an isomorphism.

Proof. This follows from the exact sequence
\[
\pi_{2n,n}^\text{MGL}_S(n,k + 1) \wedge M_A \to \pi_{2n,n}^\text{MGL}_S(n,k) \wedge M_A \to \pi_{2n-1,n}^\text{MGL}_S \wedge M_A
\]
and the fact that the two outer terms are 0 for $k \geq n + 1$.

Corollary 6.4. Let $S$ be the spectrum of a Dedekind domain of mixed characteristic. Let $R$ be a localization of $\mathbb{Z}$ such that $\Phi_S \wedge M_R$ is an isomorphism. Then for any $R$-module $A$ the canonical map
\[
\pi_{2n,n}^\text{MGL}_S \wedge M_A \to \pi_{2n,n}^\text{MGL}_S(n,n) \wedge M_A
\]
is an isomorphism.

Proof. This follows from Corollary 6.2 and Lemma 6.3.

Theorem 6.5. Let $S$ be the spectrum of a Dedekind domain of mixed characteristic. Let $R$ be a localization of $\mathbb{Z}$ such that $\Phi_S \wedge M_R$ is an isomorphism. Then for every $n \in \mathbb{Z}$ and $R$-module $A$ there is a canonical isomorphism
\[
\pi_{2n,n}^\text{MGL}_S \wedge M_A \cong L_n \otimes A \oplus L_{n+1} \otimes \text{Pic}(S) \otimes A.
\]

Proof. We have the exact sequence
\[
\pi_{2n+1,n}^\text{MGL}_S \wedge M_A \to \pi_{2n,n}^\text{MGL}_S \wedge M_A \to \pi_{2n,n}^\text{MGL}_S(n,n+1) \wedge M_A
\]
\[
\to \pi_{2n,n}^\text{MGL}_S \wedge M_A \to \pi_{2n-1,n}^\text{MGL}_S \wedge M_A.
\]
The two outer terms are 0. Also $\pi_{0,0}^\text{MGL}_S \wedge M_A \cong \text{Pic}(S) \otimes A$. Moreover there is a canonical map $L_n \otimes A \to \pi_{2n,n}^\text{MGL}_S \wedge M_A$ splitting the resulting short exact sequence, whence the claim follows from Corollary 6.3.

Corollary 6.6. Let $S$ be the spectrum of a Dedekind domain of mixed characteristic and $R$ the localization of $\mathbb{Z}$ obtained by inverting all positive residue characteristics of $S$. Then
\[
(\pi_{2n,n}^\text{MGL}_S) \otimes R \cong (L_n \otimes L_{n+1} \otimes \text{Pic}(S)) \otimes R.
\]
We have the following case of a Conjecture of Voevodsky (see [7, Conjecture 1]):

**Theorem 6.7.** Let \( S = \text{Spec}(R) \), where \( R \) is a (regular) Noetherian local ring which is regular over some discrete valuation ring of mixed characteristic. Then the natural map

\[
L_* \rightarrow (\text{MGL}_S)_{2*,*}
\]

becomes an isomorphism after inverting the residue characteristic of the closed point of \( S \).

**Proof.** By Popescu’s Theorem \( R \) is a filtered colimit of smooth algebras over a discrete valuation ring \( V \) of mixed characteristic. Thus we are reduced to the case where \( R \) is the local ring of a scheme \( X \in \text{Sm}_{\text{Spec}(V)} \) by a colimit argument. Let \( p \) be the residue characteristic of the closed point of \( \text{Spec}(V) \). By the same type of argument as above and the vanishing of \((p,q)\)-motivic cohomology of such local rings for \( p > q \) we have

\[
(\text{MGL}_S)_{2n,n}[1/p] \cong (s_n\text{MGL}_S)_{2n,n}[1/p] \cong L_n[1/p],
\]

using that for a fixed dimension only a fixed finite number of slices of \( \text{MGL}_S[1/p] \) contribute to the value of \( \pi^{\text{pre}}_{2n,n}(\text{MGL}_S[1/p]) \) on schemes of that dimension.

More generally we have

**Proposition 6.8.** Let \( S \) be as in the previous Theorem and \( E \in \text{SH}(S) \) a motivic Landweber spectrum modelled on \( E_{2*,*}^{\text{top}} \). Then the natural map

\[
E_{2*,*}^{\text{top}} \rightarrow E_{2*,*}
\]

is an isomorphism after inverting the residue characteristic of the closed point of \( S \).

**Proof.** This follows from the definition of motivic Landweber spectrum.

### 7 Some other parts of algebraic cobordism

We have the following vanishing result:

**Proposition 7.1.** Let \( S \) be the spectrum of a Dedekind domain of mixed characteristic. Let \( R \) be a localization of \( \mathbb{Z} \) such that \( \Phi_S \wedge M_R \) is an isomorphism. Then for any \( p, q \in \mathbb{Z} \) and \( R \)-module \( A \) we have \( \pi^{\text{pre}}_{p,q}(\text{MGL}_S \wedge M_A) \equiv 0 \) for \( p < 2q \) or \( p < q \). In particular we have \( \text{MGL}_S \wedge M_R \in \text{SH}(S)_{h \geq 0} \).

**Proof.** Let \( p, q \in \mathbb{Z} \) satisfying the condition of the statement. Let \( d \in \mathbb{N} \). Then there is a \( N \geq q \) such that for any scheme of dimension \( \leq d \) and \( k \geq N \) the map

\[
\pi^{\text{pre}}_{p,q}(\text{MGL}_S \wedge M_A)(X) \rightarrow \pi^{\text{pre}}_{p,q}(\text{MGL}_S(0,k) \wedge M_A)(X)
\]

is an isomorphism. The assertion then follows by an induction argument on \( i \) showing that \( \pi_{p,q}(\text{MGL}_S(q,q+i) \wedge M_A) = 0 \).

Generalizing the argument given in the last proof we get
Lemma 7.2. Let $S$ be the spectrum of a Dedekind domain of mixed characteristic. Let $R$ be a localization of $\mathbb{Z}$ such that $\Phi_S \otimes M_R$ is an isomorphism. Then for any $p, q \in \mathbb{Z}$ and $R$-module $A$ we have

$$\pi_{p,q}(\text{MGL}_S \otimes M_A) \cong \text{lim}_k \pi_{p,q}(\text{MGL}_S(0,k) \otimes M_A) \cong \pi_{p,q}(\text{MGL}_S(\max(0,q),n) \otimes M_A)$$

for $n \geq p - q$ or $n \geq p - 2q$.

Corollary 7.3. Let $S$ be the spectrum of a Dedekind domain of mixed characteristic. Let $R$ be a localization of $\mathbb{Z}$ such that $\Phi_S \otimes M_R$ is an isomorphism. Then for any $R$-module $A$ and $n \in \mathbb{Z}$ we have $\pi_{-n,n}(\text{MGL}_S \otimes M_A) \cong K^{-n}_M \otimes A$, where $K^{-n}_M$ is the $(-n)$-th Milnor-$K$-theory sheaf defined via the degree $(-n,-n)$-motivic cohomology.

Corollary 7.4. Let $S$ be the spectrum of a Dedekind domain of mixed characteristic. Let $R$ be a localization of $\mathbb{Z}$ such that $\Phi_S \otimes M_R$ is an isomorphism. Then for any $R$-module $A$ and $n \in \mathbb{Z}$ we have $\pi_{n,n}(\text{MGL}_S \otimes M_A) \cong \mathcal{O}^* \otimes L_{n+1} \otimes A$.

Proof. By Lemma 7.2 we have

$$\pi_{n+1,n}(\text{MGL}_S \otimes M_A) \cong \pi_{n+1,n}(\text{MGL}_S(n,n+1) \otimes M_A).$$

The long exact sequence of sheaves associated to the exact triangle

$$s_n \text{MGL}_S \otimes M_A \to \text{MGL}_S(n,n+1) \otimes M_A \to s_n \text{MGL}_S \otimes M_A \to s_{n+1} \text{MGL}_S \otimes M_A[1]$$

together with

$$\pi_{n+1,n}(s_n \text{MGL}_S \otimes M_A[-1]) \cong \pi_{n+1,n}(s_n \text{MGL}_S \otimes M_A) = 0$$

and

$$\pi_{n,0}(\Sigma^{1,1}M_S) \cong \mathcal{O}^* \otimes A$$

gives the result. \qed

Similarly we get

Corollary 7.5. Let $S$ be the spectrum of a Dedekind domain of mixed characteristic. Let $R$ be a localization of $\mathbb{Z}$ such that $\Phi_S \otimes M_R$ is an isomorphism. Then for any $R$-module $A$ and $n \in \mathbb{Z}$ there is an exact sequence

$$K^{-n}_M \otimes A \to \pi_{n+1,n}(\text{MGL}_S \otimes M_A) \to H^{n-1,n}_{\text{mot}}(-, A) \to 0,$$

where the latter group denotes the motivic cohomology sheaf in degrees $(-n-1,-n)$ and $A$-coefficients.

We also have
Proposition 7.7. Let $S$ be the spectrum of a Dedekind domain of mixed characteristic. Let $R$ be a localization of $\mathbb{Z}$ such that $\Phi_S \wedge M_R$ is an isomorphism. Then for any $R$-module $A$ and $n \in \mathbb{Z}$ there is an exact sequence

$$H^{3,2}(S) \otimes A \otimes L_{n+2} \to \tau_{2n+1,n} MGL_S \to H^{1,1}(S,A) \otimes L_{n+1} \to 0.$$ 

If $A$ is torsionfree the first map is also injective.

Proposition 7.8. Let $S$ be the spectrum of a Dedekind domain of mixed characteristic. Let $R$ be a localization of $\mathbb{Z}$ such that $\Phi_S \wedge M_R$ is an isomorphism. Let $X$ be an essentially smooth scheme over $S$. Then for any $R$-module $A$, $n \in \mathbb{Z}$ and $i \geq 2$ we have $MGL^2_{S}(X,A) = 0$.

Proof. This follows from the above considerations and the fact that for $X \in \text{Sm}_S$ we have $H^{p,q}(X) = 0$ for $p \geq 2q + 2$, since motivic cohomology is computed as hypercohomology over $S$ of the Bloch-Levine cycle complexes.

We leave assertions about the groups $\pi_{2n+1,n} MGL_S \wedge M_A$, $\pi_{n,n} MGL_S \wedge M_A$ and $\pi_{n-1,n} MGL_S \wedge M_A$ to the interested reader.

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Fakultät für Mathematik, Universität Osnabrück, Germany.

e-mail: markus.spitzweck@uni-osnabrueck.de