Bessel and Struve Related Integrals

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Abstract

Analytic expressions for integrals which arise in a theory of atomic structure due to Schwinger and Englert are evaluated in terms of Bessel and Struve functions.

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Integrals which arise within a theory of atomic structure due to Schwinger and Englert [1] are investigated in the work below. In keeping with the spirit of their approach, analytic expressions for those integral are presented below. We denote the integrals in question by

\[ S(z, \zeta) = \int_{0}^{\pi/2} \cos \theta \sin^2 \theta \sin(z \cos \theta) \sin(\zeta \cos^2 \theta) \, d\theta, \]

\[ C(z, \zeta) = \int_{0}^{\pi/2} \cos \theta \sin^2 \theta \cos(z \cos \theta) \cos(\zeta \cos^2 \theta) \, d\theta. \]

These integrals will be shown to involve higher derivatives of the Bessel functions of the first kind i.e. \( J_\nu(z) \) and of the Struve functions \( H_\nu(z) \) [2] respectively. The derivatives in turn can be reduced to expressions containing the corresponding Bessel and Struve functions of orders zero and one.

1 The \( S(z, \zeta) \) Integral

In the case of the “sine” integral \( S(z, \zeta) \) we begin by expanding the term \( \sin(\zeta \cos^2 \theta) \) in a power series in \( \zeta \) to get

\[ S(z, \zeta) = \sum_{\kappa=0}^{\infty} \frac{(-1)^\kappa 2^\kappa + 1}{(2\kappa + 1)!} \int_{0}^{\pi/2} \cos^{4\kappa + 3} \theta \sin^2 \theta \sin(z \cos \theta) \, d\theta. \]

(1)

From the theory of Bessel functions [3] we have the related integral

\[ J_1(z)/z = \frac{2}{\pi} \int_{0}^{\pi/2} \cos(z \cos \theta) \sin^2 \theta \, d\theta. \]

Differentiation of this relation with respect to \( z \), \( 4\kappa + 3 \) times produces the integrals occurring in (1). We have

\[ S(z, \zeta) = \frac{\pi}{2} \sum_{\kappa=0}^{\infty} \frac{(-1)^\kappa 2^\kappa + 1}{(2\kappa + 1)!} \frac{d^{4\kappa + 3} [J_1(z)/z]}{dz^{4\kappa + 3}}, \]

an expression which requires analytic expressions for the higher derivatives of \( J_1(z)/z \).
1.1 The Higher Derivatives of $J_1(z)/z$

Once again, using an identity from the theory of Bessel functions of the first kind [4] i.e.

$$\left(\frac{1}{z} \frac{d}{dz}\right)^k \frac{J_\nu(z)}{z}\nu = (-1)^k \frac{J_{\nu+k}(z)}{z^{\nu+k}},$$

we will make use of that relation which in this application is more usefully rewritten as

$$\frac{d^k}{d(z^2)^k} \left( \frac{J_1(z)}{z} \right) = \left( -\frac{1}{2} \right)^k \frac{J_{k+1}(z)}{z^{k+1}}.$$  \hspace{1cm} (2)

In this form equation (2) makes it possible to obtain the required higher derivatives. Using Faà di Bruno’s [5] formula for the differentiation of a function of a function (i.e. a composite function) we have

$$\frac{d^k}{dz^k} \left( \frac{J_1(z)}{z} \right) = k! \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(2i)^k}{i! (5-2i)!} \frac{d^{i-1}}{d(z^2)^{k-i}} \left( \frac{J_1(z)}{z} \right).$$

Using this together with equation (2), the sought after expression for the higher derivatives of $J_1(z)/z$ is

$$\frac{d^k}{dz^k} \left( \frac{J_1(z)}{z} \right) = 2(-1)^k k! \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^i}{i! (5-2i)!} \frac{J_{k+1-i}(z)}{(2z)^{k+1}},$$  \hspace{1cm} (3)

where $\lfloor z \rfloor$ i.e. the floor of $z$, is the largest integer $\leq z$. The Bessel functions of order $k + 1 - i$ on the right-hand side of equation (3) are reducible to expressions which contains the products of each of the functions i.e. $J_0(z)$, and $J_1(z)$ with corresponding polynomials containing powers of $1/z$.

Using recurrence relations (for decreasing distant neighbors) for the Bessel functions [6] i.e.

$$J_\nu(z) = C_n(\nu, z) J_{\nu-n}(z) - C_{n-1}(\nu, z) J_{\nu-n-1}(z),$$

with

$$C_0(\nu, z) = 1, C_1(\nu, z) = 2(\nu - 1)/z, \hspace{1cm} C_n(\nu, z) = \frac{2(\nu-n)}{z} C_{n-1}(\nu, z) - C_{n-2}(\nu, z),$$

we have for this application the required reduced recurrence relation

$$J_\nu(z) = C_{\nu-1}(\nu, z) J_1(z) - C_{\nu-2}(\nu, z) J_0(z).$$  \hspace{1cm} (4)

The difference equation for $C_n(\nu, z)$ has solutions

$$C_n(\nu, z) = (-2/z)^n (1 - \nu)_n \cdot {}_2F_3\left(\begin{array}{c} \nu - n, \frac{z}{2} \\ \nu, -\nu, -z^2 \end{array} \right),$$

where $(a)_n$ is the Pochhammer symbol and ${}_2F_3$ is a generalized hypergeometric function.

In the case of the $J_0(z)$ prefactor i.e. $C_{\nu-2}(\nu, z)$, the hypergeometric function is reducible in the case of integer $\nu$ (after some algebra), to a polynomial in $1/z$ i.e.

$$C_{\nu-2}(\nu, z) = \sum_{j=0}^{\lceil \frac{\nu - 1}{2} \rceil} \frac{\Gamma(\nu - j) \Gamma(\nu - 1 - j)}{\Gamma(\nu - 2 - j)} \frac{(-1)^j}{j! (j + 1)!} \left( \frac{2}{z} \right)^{\nu - 2j} = R^{(0)}(\nu, z),$$

where $\lceil z \rceil$ is the ceiling of $z$, i.e. the smallest integer $\geq$ to $z$. The polynomials $C_{\nu-2}(\nu, z)$ are in fact special cases of the well known Lommel polynomials i.e. $R_{\nu-2,2}(z)$ [7]. For simplicity in notation we will denote these special cases of the Lommel polynomials by $R^{(0)}(\nu, z)$.

In a similar way, the prefactor of $J_1(z)$ i.e. $C_{\nu-1}(\nu, z)$ reduces to

$$C_{\nu-1}(\nu, z) = \sum_{j=0}^{\lceil \frac{\nu - 1}{2} \rceil} \frac{\Gamma(\nu - j)^2}{\Gamma(\nu - 2j)} (-1)^j \left( \frac{2}{z} \right)^{\nu - 2j} = R^{(1)}(\nu, z),$$
which are the Lommel polynomials \( R_{\nu-1,1}(z) \) and will be referred to as \( R^{(1)}(\nu, z) \). Using (3,4) the \( k^{th} \) derivative of \( J_1(z)/z \) is then given by

\[
\frac{d^k}{dz^k} \left( \frac{J_1(z)}{z} \right) = (-1)^k \left[ \mathcal{P}_1(k, z) J_1(z) - \mathcal{P}_0(k, z) J_0(z) \right],
\]

where \( \mathcal{P}_1(k, z) \) and \( \mathcal{P}_0(k, z) \) are polynomials in \( 1/z \) i.e.

\[
\mathcal{P}_1(k, z) = 2k! \sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{(-1)^i \mathcal{R}^{(1)}(k + 1 - i, z)}{(2z)^{i+1}},
\]

\[
\mathcal{P}_0(k, z) = 2k! \sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{(-1)^i \mathcal{R}^{(0)}(k + 1 - i, z)}{(2z)^{i+1}}.
\]

The \( J_1(z)/z \) derivatives are seen to be oscillatory functions of \( z \) with small and rapidly decreasing amplitudes. Direct calculation of the \( J_1(z)/z \) derivatives shows that they either vanish at small \( z \) in the case of odd values of \( k \) or for even \( k = 2m \) have values \((-1)^m \Gamma(m + 1/2)/2\sqrt{\pi}(m + 1)! \).

The integral \( S(z, \zeta) \) is then given by

\[
S(z, \zeta) = \frac{\pi}{2} \sum_{\kappa=0}^{\infty} \frac{(-1)^\kappa 2^\kappa + 1}{(2\kappa + 1)!} \left[ \mathcal{P}_0(4\kappa + 3, z) J_0(z) - \mathcal{P}_1(4\kappa + 3, z) J_1(z) \right],
\]

where

\[
\mathcal{P}_0(4\kappa + 3, z) = 2(4\kappa + 3)! \sum_{i=0}^{2\kappa + 1} \frac{(-1)^i \mathcal{R}^{(0)}(4\kappa + 4 - i, z)}{(2z)^{i+1}},
\]

\[
\mathcal{P}_1(4\kappa + 3, z) = 2(4\kappa + 3)! \sum_{i=0}^{2\kappa + 1} \frac{(-1)^i \mathcal{R}^{(1)}(4\kappa + 4 - i, z)}{(2z)^{i+1}}.
\]

\section{The \( C(z, \zeta) \) Integral}

The “cosine” integral \( C(z, \zeta) \) is treated in a manner similar to the “sine” integral \( S(z, \zeta) \) discussed above i.e.

\[
C(z, \zeta) = \sum_{\kappa=0}^{\infty} \frac{(-1)^\kappa 2^\kappa}{(2\kappa)!} \int_0^{\pi/2} \cos^{4\kappa + 1} \theta \sin^2 \theta \cos(z \cos \theta) \, d\theta.
\]

Here however, the analysis is considerably more complicated due to the occurrence of the Struve functions \[^8\]. The integral representation for the Struve function \( H_1(z)/z \)

\[
H_1(z)/z = \frac{2}{\pi} \int_0^{\pi/2} \sin(z \cos \theta) \sin^2 \theta \, d\theta,
\]

upon differentiation \( 4\kappa + 1 \) times with respect to \( z \) gives the integrals occurring in (5) with the result that

\[
C(z, \zeta) = \frac{\pi}{2} \sum_{\kappa=0}^{\infty} \frac{(-1)^\kappa 2^\kappa}{(2\kappa)!} \frac{d^{4\kappa + 1} [H_1(z)/z]}{dz^{4\kappa + 1}}.
\]

\subsection{The Higher Derivatives of \( H_1(z)/z \)}

In the case of the Struve functions, relations analogous to those in equation (2) do not exist. However, using the recurrence relation \[^8\]

\[
H_1(z) = \frac{2}{\pi} - H_{-1}(z),
\]
we get
\[
\frac{d^k}{dz^k} \left( \frac{H_1(z)}{z} \right) = \frac{2k!}{\pi} \frac{(-1)^k}{z^{k+1}} - \frac{d^k}{dz^k} \left( \frac{H_{-1}(z)}{z} \right).
\]  
(6)

The identity [9]
\[
\left( \frac{1}{z} \frac{d}{dz} \right)^k \frac{H_{-1}(z)}{z} = \frac{H_{-k-1}(z)}{z^{k+1}},
\]
analogous to the one involving the Bessel function used above will be useful in this case. Rewritten as
\[
\frac{d^k}{d(z^2)^k} \left( \frac{H_{-1}(z)}{z} \right) = \left( \frac{1}{2} \right)^k \frac{H_{-k-1}(z)}{z^{k+1}},
\]
it will prove helpful here. Using the Faà di Bruno formula as employed above, together with the latter relation we get
\[
\frac{d^k}{dz^k} \left( \frac{H_{-1}(z)}{z} \right) = 2k! \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \frac{1}{i!(k-2i)!} \frac{H_{-k-1+i}(z)}{(2z)^{i+1}}.
\]

Using (6), we have the required derivatives i.e.
\[
\frac{d^k}{dz^k} \left( \frac{H_1(z)}{z} \right) = \frac{2k!}{\pi} \frac{(-1)^k}{z^{k+1}} - 2k! \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \frac{1}{i!(k-2i)!} \frac{H_{-k-1+i}(z)}{(2z)^{i+1}}.
\]  
(7)

The Struve functions occurring in (7) can also be reduced to expressions containing only \(H_0(z)\) and \(H_1(z)\). To do that, we note as a first step that the Struve functions with negative integer orders are related to those with the corresponding positive orders by the relations [10]
\[
H_{-\nu}(z) = (-1)^\nu H_\nu(z) + \sum_{j=0}^{\nu-1} \frac{(-1)^j}{\Gamma(j+3/2)\Gamma(j+3/2-\nu)} \left( \frac{z}{2} \right)^{2j+1-\nu}.
\]

As a result (7) becomes
\[
\frac{d^k}{dz^k} \left( \frac{H_1(z)}{z} \right) = 2k! (-1)^k \left\{ \frac{1}{\pi} \frac{1}{z^{k+1}} + \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{(-1)^i}{i!(k-2i)!} \frac{H_{k+1-i}(z)}{(2z)^{i+1}} \right\}
\]
\[
- k! \left( \frac{1}{2} \right)^{k+1} \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{(1/2)^{2i+1}}{i!(k-2i)!} \sum_{j=0}^{k-i} \frac{(-1)^j (z/2)^{2j}}{\Gamma(j+3/2)\Gamma(j+1/2+i-k)}.
\]  
(8)

The last (double) sum in equation (8) can be rewritten in ascending powers of \(z/2\) as
\[
-k! \left( \frac{1}{2} \right)^{k+1} \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{(-1)^j (z/2)^{2j}}{\Gamma(j+3/2)} \sum_{i=0}^{k-j} \frac{(1/2)^{2i+1}}{i!(k-2i)!\Gamma(j+1/2+i-k)}.
\]

The first of these two sum makes a contribution only when \(j = 0\), all other terms vanishing. The surviving term from that sum cancels the leading term in (8) and we get
\[
\frac{d^k}{dz^k} \left( \frac{H_1(z)}{z} \right) =
\]
\[
k! (-1)^k \left[ 2 \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{(-1)^i}{i!(k-2i)!} \frac{H_{k+1-i}(z)}{(2z)^{i+1}} - \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{(-1)^j (z/2)^{k-1-2j}}{\Gamma(k+3/2-j)} \sum_{i=0}^{j} \frac{(1/2)^{2i+1}}{i!(k-2i)!\Gamma(i+1/2-j)} \right].
\]  
(9)
As in the case of the “sine” integral \( S(z, \zeta) \) it is necessary to use recurrence relations (for decreasing distant neighbors) for the Struve functions \( H_{k+1-j}(z) \) in order to rewrite those functions in terms of \( H_0(z) \) and \( H_1(z) \). The Wolfram Function Site [11] has given the expressions (due to Yu A. Brychkov) needed for the reduction of the \( H_{k+1-j}(z) \) functions i.e.

\[
H_\nu(z) = C_{\nu-1}(\nu, z) H_1(z) - C_{\nu-2}(\nu, z) H_0(z) + S(\nu, z),
\]

where

\[
S(\nu, z) = \frac{1}{\sqrt{\pi}} \sum_{j=0}^{\nu-2} \frac{(z/2)^{\nu-1-j}}{\Gamma(\nu+1/2-j)} C_j(\nu, z),
\]

\[
C_j(\nu, z) = \left(-2/z\right)^j (1 - \nu) j \cdot 2F3\left(\frac{j+1}{1-\nu, -j, \nu-j}\right).
\]

This is a similar but more complicated set of relations than those encountered in the case of the \( J_\nu(z) \) functions. In spite of that, we see that the prefactors of \( H_0(z) \) and \( H_1(z) \) in (10) are identical to the corresponding ones for \( J_0(z) \) and \( J_1(z) \) in the reduced recurrence relations for \( J_\nu(z) \). In this case however, an additional term i.e. \( S(\nu, z) \), occurs and requires special attention.

In the case of the \( H_0(z) \) prefactor i.e. \( C_{\nu-2}(\nu, z) \), the corresponding hypergeometric function (as was seen above in the case of the Bessel functions) reduces to a polynomial in \( 1/z \) i.e.

\[
C_{\nu-2}(\nu, z) = \sum_{k=0}^{[\nu-1]} \frac{\Gamma(\nu-k) \Gamma(\nu-1-k)}{\Gamma(\nu-1-2k)} \frac{(-1)^{k+1}}{k!(\nu+1)!} \left(\frac{2}{z}\right)^{\nu-2-2k} = R_{\nu-2,2}(z).
\]

In a similar way the prefactor of \( H_1(z) \) i.e. \( C_{\nu-1}(\nu, z) \) reduces to

\[
C_{\nu-1}(\nu, z) = \sum_{k=0}^{[\nu-1]} \frac{\Gamma(\nu-k) \Gamma(\nu-2k)}{\Gamma(\nu-2k)} \frac{(-1)^k}{k!^2} \left(\frac{2}{z}\right)^{\nu-1-2k} = R_{\nu-1,1}(z).
\]

A table containing the first eight of the Lamell polynomials \( R_{\nu-1,1}(z) \), \( R_{\nu-2,2}(z) \) have been given below in the appendix.

### 2.2 The sum \( S(\nu, z) \)

For either even or odd values of \( \nu \) it is possible to simplify the term \( S(\nu, z) \) in (10). Substitution of the expression for \( C_j(\nu, z) \) in (11) into \( S(\nu, z) \) and simplifying the result gives

\[
S(\nu, z) = \frac{1}{\sqrt{\pi}} \sum_{j=0}^{\nu-2} \frac{(z/2)^{\nu-1-j}}{\Gamma(\nu+1/2-j)} 2^{2\nu-1-2j} j^{\nu-1-j} 2F3\left(\frac{j+1}{1-\nu, -j, \nu-j}\right).
\]

### 2.3 The subcases \( S(\nu, z) \) with \( \nu = 2m, \nu = 2m+1 \) for integer \( m \)

In the case where \( \nu = 2m \) we have

\[
S(2m, z) = \frac{1}{\pi} \sum_{j=0}^{2m-2} \frac{\Gamma(2m)}{\Gamma(4m-2j)} 2^{4m-1-2j} j^{2m-1-j} 2F3\left(\frac{j+1}{1-2m, -j, 2m-j}\right).
\]

Separating the sum over \( j \) into its even and odd terms we have

\[
S(2m, z) = \frac{1}{\pi} \sum_{i=0}^{m-1} \frac{\Gamma(2m)}{\Gamma(4m-4i)} 2^{4m-1-4i} i^{2m-1-4i} 2F3\left(\frac{i+1}{1-2m, -i, -i, z^2}\right) + \frac{1}{\pi} \sum_{i=0}^{m-2} \frac{\Gamma(2m)}{\Gamma(4m-4i-2)} 2^{4m-3-4i} i^{3m-3-4i} 2F3\left(\frac{-i-i-i}{1-2m, -i-i, 2m-2i}\right).
\]

The hypergeometric functions in the two sum in equation (12) reduce to

\[
2F3\left(\frac{-i+i+i}{1-2m, -i, -i, z^2}\right) = \frac{\Gamma(2m-2i)}{\Gamma(2m)} \sum_{\kappa=0}^{i} \frac{(-1)^{\kappa} \Gamma(2m-2i) \Gamma(2i+1-\kappa) (z/2)^{2\kappa}}{\Gamma(2i+1-2\kappa) \Gamma(2m-2i+2\kappa) \kappa!}.
\]
and

\[ _2F_3\left(\frac{-i, -\frac{i}{2}, -z^2}{1-2i, \frac{1}{2}, -2i}\right) = \frac{\Gamma(2m - 1 - 2i)}{\Gamma(2m)} \sum_{\kappa=0}^{i} \frac{(-1)^\kappa \Gamma(2m - \kappa) \Gamma(2i + 2 - \kappa)}{\Gamma(2i + 2 - 2\kappa) \Gamma(\kappa + 2m - 1 - 2i) \kappa!} \frac{(z/2)^{2\kappa}}{\kappa!}, \]

respectively. Recombining these terms we get for \( S(2m, z) \) the expression

\[ S(2m, z) = \frac{2^{4m-1}}{\pi} \sum_{j=0}^{2m-2} \frac{\Gamma(2m - 2j) 2^{-2j}}{\Gamma(4m - 2j)} \sum_{\kappa=0}^{j} \frac{(-1)^\kappa \Gamma(2m - \kappa) \Gamma(j + 1 - \kappa) (z/2)^{2\kappa-2j+2m-1}}{\Gamma(j + 1 - 2\kappa) \Gamma(\kappa + 2m - j) \kappa!}. \]

A similar analysis of the sum \( S(2m + 1, z) \) allows one to conclude that in each case, a single relation valid for arbitrary \( j \) and \( \nu \) is obtainable i.e.

\[ _2F_3\left(\frac{i, \frac{i}{2}, -z^2}{1, \frac{1}{2}, -2i}\right) = \frac{\Gamma(\nu - j)}{\Gamma(\nu)} \sum_{\kappa=0}^{j} \frac{(-1)^\kappa \Gamma(\nu - \kappa) \Gamma(j + 1 - \kappa) (z/2)^{2\kappa}}{\Gamma(j + 1 - 2\kappa) \Gamma(\kappa + \nu - j) \kappa!}. \]

As a result,

\[ S(\nu, z) = \frac{2^{2\nu-1}}{\pi} \sum_{j=0}^{\nu-2} \frac{\Gamma(\nu - j)}{\Gamma(2\nu - 2j)} \sum_{\kappa=0}^{j+1} \frac{(-1)^\kappa \Gamma(\nu - \kappa) \Gamma(j + 1 - \kappa) (z/2)^{2\kappa-2j+\nu-1}}{\Gamma(j + 1 - 2\kappa) \Gamma(\kappa + \nu - j) \kappa!}. \]

If the order of summation in the double sum in equation (13) is interchanged, that expression (for \( \nu \geq 2 \)) can be written in ascending powers of \( 2/z \) as

\[ S(\nu, z) = \frac{1}{\pi} \left(\frac{z}{2}\right)^{\nu-2} \sum_{\mu=0}^{\nu-2} \frac{\Gamma(\mu + 1/2)}{\Gamma(\nu + 1/2 - \mu)} \left(\frac{2}{z}\right)^{2\mu} + \frac{1}{\sqrt{\pi}} \left(\frac{z}{2}\right)^{\nu-1} \sum_{\mu=\nu-1}^{\nu-2} \frac{\mu!}{\Gamma(\nu - \mu)} \left(\sum_{i=0}^{\mu} \frac{(-1)^i \Gamma(\nu - i)}{\Gamma(\nu + 1/2 - \mu - i) \Gamma(\mu + 1 - i)!} \right) \left(\frac{2}{z}\right)^{2\mu}. \]

This can be rewritten as

\[ S(\nu, z) = \frac{1}{\pi} \left(\frac{z}{2}\right)^{\nu-1} \sum_{\mu=0}^{\nu-1} \frac{\Gamma(\mu + 1/2)}{\Gamma(\nu + 1/2 - \mu)} \left(\frac{2}{z}\right)^{2\mu} + \frac{1}{\sqrt{\pi}} \left(\frac{z}{2}\right)^{3\nu} \sum_{\mu=0}^{\nu-1} \left( \sum_{i=0}^{\mu} \frac{(-1)^i \Gamma(\nu - i)}{\Gamma(\nu + 5/2 - i) \Gamma(\nu + 1 - \mu - i)!} \right) \left(\frac{\nu - 2 - \mu}{\Gamma(\mu + 2)} \right) \left(\frac{2}{z}\right)^{2\mu}, \]

where we have used the relation

\[ \left[ N/2 \right] + \left[ N/2 \right] = N, \]

valid for integers (which follows from the properties \[12\] of the ceiling and floor functions) in the first summation’s limits and we have reindexed the second i.e. the double sum over \( \mu \). Gathering the terms found above, the final expression for \( H_\nu(z) \) is

\[ H_\nu(z) = H_1(z) \sum_{\kappa=0}^{\nu-1} \frac{\Gamma(\nu - \kappa)^2 (-1)^\kappa \Gamma(2\kappa)}{\kappa!^2} \left(\frac{2}{z}\right)^{\nu-2\kappa} - H_0(z) \sum_{\kappa=0}^{\nu-1} \frac{\Gamma(\nu - \kappa) \Gamma(\nu - 1 - \kappa) (-1)^{\kappa+1} \Gamma(\nu - \kappa)}{\kappa! (\kappa + 1)!} \left(\frac{2}{z}\right)^{\nu-2-2\kappa} + S(\nu, z), \]

with \( S(\nu, z) \) given in (14).

Finally, using (9, 15) the expression for the higher derivative of \( H_1(z)/z \) is given by
\[
\frac{(-1)^k}{k!} \frac{d^k}{dz^k} \left( \frac{H_1(z)}{z} \right) = \left\{ \left[ \sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{(-1/4)^i}{i!} \left( \frac{1}{2} \right)^i \sum_{\nu=0}^{i} \frac{\Gamma(k+1-i-\nu)!}{\Gamma(k+1-i-2\nu)!} \frac{(-1)^\nu}{\nu!} \left( \frac{z}{2} \right)^{2\nu-k-1} \right] \right\} H_1(z) - \left\{ \left[ \sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{(-1/4)^i}{i!} \left( \frac{1}{2} \right)^i \sum_{\nu=0}^{i} \frac{\Gamma(k+1-i-\nu)!}{\Gamma(k+1-i-2\nu)!} \frac{(-1)^\nu}{\nu!} \left( \frac{z}{2} \right)^{2\nu-k} \right] \right\} H_0(z) \\
+ \sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{(-1)^i}{i!} \left( \frac{1}{2} \right)^i \frac{\Gamma(k+1-i-\nu)!}{\Gamma(k+1-i-2\nu)!} \left( \frac{z}{2} \right)^{2\nu-1} \right\} H_0(z) \\
+ \frac{2}{z} \sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{(-1)^i}{i!} \left( \frac{1}{2} \right)^i \frac{\Gamma(k+1-i-\nu)!}{\Gamma(k+1-i-2\nu)!} \left( \frac{z}{2} \right)^{2\nu-1} \right\} H_0(z) \\
+ \frac{2}{z} \sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{(-1)^i}{i!} \left( \frac{1}{2} \right)^i \frac{\Gamma(k+1-i-\nu)!}{\Gamma(k+1-i-2\nu)!} \left( \frac{z}{2} \right)^{2\nu-1} \right\} H_0(z)
\]

The prefactors of \(H_1(z)\) and \(H_0(z)\) in equation (16) are now seen to be double sums, and can be simplified when the orders of the summations are interchanged. Furthermore, when \(S(k+1-i, z)\) as represented in (14) is substituted into (16) and the orders of summation are interchanged, internal cancellation of many of the resulting terms occurs and we obtain

\[
(-1)^k \frac{d^k}{dz^k} \left( \frac{H_1(z)}{z} \right) = H_0(z) \sigma_0(k, z) \cdot (2/z)^k + H_1(z) \sigma_1(k, z) \cdot (2/z)^{k+1} + \sigma_2(k, z) \cdot (2/z)^{k-1},
\]

where \(\sigma_0(k, z), \sigma_1(k, z)\), and \(\sigma_2(k, z)\) are polynomials in \(z\) i.e.

\[
\sigma_0(k, z) = k! \left\{ \left[ \sum_{\nu=0}^{\left\lfloor k/2 \right\rfloor} \left( \frac{1}{2} \right)^k \sum_{i=0}^{\left\lfloor k/2 \right\rfloor} \frac{(-1/4)^i}{i!} \frac{(k-\nu-i)!}{(k-1-2\nu-i)!} \right] \frac{(-1)^\nu}{\nu!} \left( \frac{z}{2} \right)^{2\nu} \right\}
\]

\[
\sigma_1(k, z) = k! \left\{ \left[ \sum_{\nu=0}^{\left\lfloor k/2 \right\rfloor} \left( \frac{1}{2} \right)^k \sum_{i=0}^{\left\lfloor k/2 \right\rfloor} \frac{(-1/4)^i}{i!} \frac{(k-\nu-i)!}{(k-1-2\nu-i)!} \right] \frac{(-1)^\nu}{\nu!} \left( \frac{z}{2} \right)^{2\nu} \right\}
\]

\[
\sigma_2(k, z) = k! \left\{ \left[ \sum_{\nu=0}^{\left\lfloor k/2 \right\rfloor} \frac{(-1)^\nu}{\nu!} \left( \frac{z}{2} \right)^{2\nu-2} \right] \right\}
\]

where \(\delta[i, j]\) is the Kronecker delta (introduced to prevent over counting in a sum) and when it occurs \(1/0!\) is counted as zero. The coefficients \(\mathcal{C}(k, a, \nu)\) are defined as

\[
\mathcal{C}(k, a, \nu) = \sum_{j=0}^{a} \frac{(-1/4)^j}{j!} \frac{(k-\nu-j)!}{(k-2j)!} \sum_{i=0}^{j} \frac{(-1)^i}{i!} \frac{\sqrt{\pi}}{(k-\nu-j-i)! \Gamma(\nu + 3/2 - i)}
\]

Direct calculation of the derivatives \(d^k [H_1(z)/z]/d z^k\) shows that these quantities like those involving \(J_1(z)/z\) also behave as oscillatory functions of \(z\) with small and rapidly decreasing amplitudes. At \(z = 0\) the amplitude is zero for even \(k\) and for \(k = 2m+1\) the amplitude is \((-1)^m m! / 2\sqrt{\pi} \Gamma(m+5/2)\). In the case of both higher derivatives, the finite values of their amplitudes at the origin are of order \(1/m^{3/2}\) for large \(m\). Further, as anticipated in the work above, the prefactors of \(H_0(z)\) and \(H_1(z)\) and those of \(J_0(z)\) and \(J_1(z)\) are seen to be identical. As a result,
alternative representations of the quantities \( \sigma_i(k, z)/z^{k+1} \) and \( P_i(k, z) \) i.e. as explicit polynomials in \( 1/z \) or as sums of Lommel polynomials have been given.

Finally the expression for \( C(z, \zeta) \) is (with some additional cancellation occurring because of the odd values of \( k \) encountered in this application)

\[
C(z, \zeta) = \pi \sum_{\kappa=0}^{\infty} \frac{(-1)^{\kappa+1} 2^\kappa}{(2\kappa)!} \left[ H_0(z) \sigma_0(4\kappa + 1, z) \cdot (2/z)^{4\kappa+1} + H_1(z) \sigma_1(4\kappa + 1, z) \cdot (2/z)^{4\kappa+2} \right],
\]

with

\[
\sigma_0(4\kappa + 1, z) = (4\kappa + 1)! \sum_{\nu=0}^{\kappa} \frac{1}{2} \sum_{i=0}^{2\kappa+1} \frac{(-1/4)^i}{i! (4\kappa + 1 - 2i)!} \left( \frac{(4\kappa + 1 - \nu - i)! (4\kappa - \nu - i)!}{(4\kappa - 2\nu - i)!} \right) \frac{(-1)^{\nu+1}}{\nu! (\nu + 1)!} \left( \frac{z}{2} \right)^{2\nu} + \frac{2\kappa}{4\kappa + 1 - 2\nu} \sum_{i=0}^{\nu} \frac{(-1/4)^i}{i! (4\kappa + 1 - 2i)!} \left( \frac{(4\kappa + 1 - \nu - i)! (4\kappa - \nu - i)!}{(4\kappa - 2\nu - i)!} \right) \frac{(-1)^{\nu+1}}{\nu! (\nu + 1)!} \left( \frac{z}{2} \right)^{2\nu},
\]

\[
\sigma_1(4\kappa + 1, z) = (4\kappa + 1)! \sum_{\nu=0}^{\kappa} \frac{1}{2} \sum_{i=0}^{2\kappa+1} \frac{(-1/4)^i}{i! (4\kappa + 1 - 2i)!} \left( \frac{(4\kappa + 1 - \nu - i)! (4\kappa - \nu - i)!}{(4\kappa - 2\nu - i)!} \right) \frac{(-1)^{\nu+1}}{\nu! (\nu + 1)!} \left( \frac{z}{2} \right)^{2\nu} + \frac{2\kappa}{4\kappa + 2 - 2\nu} \sum_{i=0}^{\nu} \frac{(-1/4)^i}{i! (4\kappa + 1 - 2i)!} \left( \frac{(4\kappa + 1 - \nu - i)! (4\kappa - \nu - i)!}{(4\kappa - 2\nu - i)!} \right) \frac{(-1)^{\nu+1}}{\nu! (\nu + 1)!} \left( \frac{z}{2} \right)^{2\nu},
\]

\[
\sigma_2(4\kappa + 1, z) = \frac{(4\kappa + 1)!}{2\pi} \sum_{\nu=1}^{\kappa} \frac{\zeta(4\kappa + 1, 2\kappa + 1, \nu)}{\nu!} \left( \frac{z}{2} \right)^{2\nu-2} + \frac{2\kappa}{4\kappa + 3 - 2\nu} \sum_{i=4\kappa + 3 - 2\nu}^{\nu} \frac{(-1/4)^i}{i! (4\kappa + 1 - 2i)!} \frac{\Gamma(4\kappa + 3/2 - \nu - i)}{\Gamma(\nu + 3/2)} \left( \frac{z}{2} \right)^{2\nu-2}.
\]

Appendix

In the table below we list a few of the Lommel polynomials referred to in the text.

| \( \nu \) | \( R_{\nu-1,1}(z) \) | \( R_{\nu-2,2}(z) \) |
|---|---|---|
| 1 | 1 | \( z \) |
| 2 | 2/z | 1 |
| 3 | 8/z^2 - 1 | 4/z |
| 4 | 48/z^3 - 8/z | 24/z^2 - 1 |
| 5 | 384/z^4 - 72/z^2 + 1 | 192/z^3 - 12/z |
| 6 | 3840/z^5 - 768/z^3 + 18/z | 1920/z^4 - 144/z^2 + 1 |
| 7 | 46080/z^6 - 9600/z^4 + 288/z^2 - 1 | 23040/z^5 - 1920/z^3 + 24/z |
| 8 | 645120/z^7 - 138240/z^5 + 4800/z^3 - 32/z | 322560/z^6 - 28800/z^4 + 480/z^2 - 1 |

References

[1] B.-G. Englert and J. Schwinger, Phys. Rev. A 32, 31,(1985).

[2] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, National Bureau of Standards (1972), pp. 358, 496.

[3] R. P. Soni, F. Oberhettinger, W. Magnus, Formulas and Theorems for the Special Functions of Mathematical Physics, Springer-Verlag, (1966), p. 79, 3.6.2.

[4] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, National Bureau of Standards (1972), p. 361.
This relation follows from a comparison of the infinite series representations for $H_\nu$ and $H_{-\nu}$. See reference [8], p. 496, 12.1.3. Writing out the first $\nu$ terms in the series representation for $H_{-\nu}$ and comparing the remaining terms with the series for $H_\nu$ produces the desired relation.