We consider $2d$ sigma models with a $D = 2 + N$ - dimensional Minkowski signature target space metric having a covariantly constant null Killing vector. These models are UV finite. The $2 + N$-dimensional target space metric can be explicitly determined for a class of supersymmetric sigma models with $N$-dimensional ‘transverse’ part of the target space being homogeneous Kähler. The corresponding ‘transverse’ sub-theory is an $n = 2$ supersymmetric sigma model with the exact $\beta$-function coinciding with its one-loop expression. For example, the finite $D = 4$ model has $O(3)$ supersymmetric sigma model as its ‘transverse’ part. Moreover, there exists a non-trivial dilaton field such that the Weyl invariance conditions are also satisfied, i.e. the resulting models correspond to string vacua. Generic solutions are represented in terms of the RG flow in ‘transverse’ theory. We suggest a possible application of the constructed Weyl invariant sigma models to quantisation of $2d$ gravity. They may be interpreted as ‘effective actions’ of the quantum $2d$ dilaton gravity coupled to a (non-conformal) $N$-dimensional ‘matter’ theory. The conformal factor of the $2d$ metric and $2d$ ‘dilaton’ are identified with the light cone coordinates of the $2 + N$ - dimensional sigma model.
1. INTRODUCTION

One of the important problems in string theory is to classify possible solutions of the string effective equations, i.e. string vacuum backgrounds which may be represented in terms of Weyl invariant 2d sigma models (for reviews see, e.g., ref.1). Since the string equations (or \(\bar{\beta}\)-functions’) are quite complicated (already at the string tree level) containing all terms in \(\alpha'\) the structure of the space of solutions is poorly understood. Among a few classes of solutions which are explicitly known are: (1) flat space with linear dilaton\(^2/\); (2) group spaces (WZW models)\(^3/\); (3) ‘plane wave’ backgrounds\(^4/\); (4) backgrounds corresponding to gauged WZW theories\(^5/\); (5) various possible direct products (see e.g. second paper in ref.2). In contrast to the first three classes of backgrounds (which can be represented in a form essentially independent of \(\alpha'\)) the backgrounds of the fourth type are non-trivial functions of \(\alpha'\) (see ref.6). There are, of course, many other solutions of the leading order string equations (see, e.g., ref.7) but their generalisations to all orders in \(\alpha'\) (which should exist in perturbation theory) are not explicitly known. One can try to construct new solutions by using various types of duality transformations\(^8,9,10/\). However, since the exact form of the sigma model duality transformations is not explicitly known (except in the first two orders in \(\alpha'\))\(^9/\) all discussed duality rotations of exact string solutions solve string equations only to the leading order \(\alpha'\). Having found an exact string background one is still confronting an additional problem of identifying a conformal theory which it should correspond to. The solution to this problem is known only in the case of (gauged) WZW theories.

In order to understand better gravitational applications of string theory (e.g. string backgrounds related to cosmology, black hole physics or possibly to high energy string scattering\(^11/\)) it is important to find new exact solutions which have physical Minkowski signature. A class of such solutions will be described below. In general, the solutions will be non-trivial functions of \(\alpha'\). We shall present a simple algorithm of their construction in terms of the renormalisation group flow of a non-conformal euclidean 2d theory.
Namely, the following theorem is true\textsuperscript{12,13}: given a non-conformal sigma model with an $N$-dimensional target space with euclidean signature metric there exists a conformal invariant sigma model in $2 + N$ dimensions with Minkowski signature metric. The $2 + N$-dimensional metric depends on only one of the two extra coordinates (it has a covariantly constant null Killing vector) and is expressed in terms of the “running” coupling of the $N$-dimensional theory (the “transverse” part of the metric satisfies a first order renormalisation group - type equation). Thus starting from an arbitrary $N$-dimensional euclidean background one can construct a $2 + N$-dimensional string solution with Minkowski signature.

We shall discuss the $2d$ supersymmetric generalisation of this class of finite sigma models and will show that the $2 + N$-dimensional metric can be explicitly determined in the case when the transverse space is homogeneous Kähler. Then the ‘transverse’ sub-model is $n = 2$ supersymmetric and the expression for the exact $\beta$-function of the transverse theory is known (it coincides with the one-loop result) so that the RG equation is easy to integrate.

Conformal invariant sigma models with a null Killing vector are also of interest in connection with the problem of quantising $2d$ gravity. If one starts with a $2d$ model of gravity coupled to a (non-conformal) $N$-dimensional matter theory it is expected\textsuperscript{14,15,16,17} that the couplings of the matter theory should develop a dependence on the conformal factor such that the resulting ‘quantum action’ is represented by $N + 1$-dimensional Weyl invariant sigma model. This suggestion suffers from the following difficulty: since the Weyl invariance conditions turn out to be second order differential equations in the $N + 1$-th ‘time’ coordinate (conformal factor) there is an ambiguity in choosing a particular solution which satisfies natural initial conditions. This problem is (at least partially) avoided\textsuperscript{12,13} if one considers a model of $2d$ quantum gravity where there is an extra scalar field ($2d$ ‘dilaton’) coupled to $2d$ curvature (see e.g. refs.18,19). The central observation is that the corresponding quantum action can be identified with an action of a conformal
invariant $N+2$-dimensional sigma model with a null Killing vector. The extra scalar field and the conformal factor play the role of the light-cone coordinates $v$ and $u$. The theory is effectively $N+1$-dimensional since the condition of Killing symmetry implies that couplings are $v$-independent. As a result, the conformal invariance equations are first order differential equations in $u$ (in fact, the standard RG equations of the ‘transverse’ $N$-dimensional theory) and their solution satisfying natural initial conditions is unique.

In Sec.2 we shall first show that the sigma models with covariantly constant null Killing vector are UV finite in flat 2d space. In contrast to what happens, for example, in WZW models the divergences will not cancel automatically at each order of perturbation theory but will be absent on shell (i.e. it will be possible to redefine them away)\(^{12}\). The mechanism of finiteness which operates here was already discussed (at the one-loop level) in ref.20. We shall then study the Weyl invariance conditions\(^{21}\) on a sigma model defined on a curved 2-surface and will prove (making use of the general coordinate invariance identities for the Weyl anomaly coefficients\(^{22}\)) that there exists a dilaton field such that the sigma models with a covariantly constant null Killing vector are Weyl invariant\(^ {13}\). That means they represent solutions of string effective equations. In contrast to the previously known string solutions with a null Killing vector\(^4\) which have flat $N$-dimensional space the backgrounds we have found may have an arbitrary transverse space.

A new class of finite supersymmetric sigma models with null Killing vector will be presented in Sec.3. We shall present an explicit expression for the $2+N$-dimensional target space metric (with homogeneous Kähler transverse subspace) which represents an exact solution of superstring theory and consider some of its properties.

A relation to 2d quantum gravity models will be discussed in Sec.4. In particular, we shall consider a generalisation to the case when the sigma model action contains the tachyonic coupling (or a scalar potential).
2. FINITENESS AND WEYL INVARIANCE OF SIGMA MODELS WITH COVARIANTLY CONSTANT NULL KILLING VECTOR

2.1. Proof of Finiteness

The most general $D = N + 2$ dimensional Minkowski signature metric admitting a covariantly constant null Killing vector can be represented in the form

$$ds^2 = \hat{g}_{\mu\nu} dx^\mu dx^\nu = -2dudv + g_{ij}(u,x)dx^i dx^j, \quad (1)$$

$$\mu, \nu = 0, 1, ..., N + 1 \quad i, j = 1, ..., N.$$  

In fact, starting from the null metric

$$ds^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = -2dudv + g_{ij}(u,x)dx^i dx^j + 2A_i(u,x)dx^i du + K(u,x)du^2, \quad (2)$$

one can eliminate $A_i$ and $K$ by a change of coordinates which preserves the “null” structure of (2)\textsuperscript{23}. Thus the most general null metric is parametrized by the functions $g_{ij}(u,x)$. It is important to keep in mind, however, that (1) considered as a generic form of the metric is written using a specific choice of coordinates $v, x^i$. For example, if $g_{ij}(u,x)$ is flat as a function of $x^i$ this does not imply that a generic “null” metric with a flat transverse part is just given by $ds^2 = -2dudv + dx^i dx_i$: transforming the coordinates to make $g_{ij}$ equal to $\delta_{ij}$ we will get back the metric (2) with non-vanishing $A_i$ and $K$.

To establish the UV finiteness of the corresponding sigma model on a flat $2d$ background one should check that there exists a vector $M_\mu$ such that the $\beta$ - function for the target space metric $G_{\mu\nu} = \hat{g}_{\mu\nu} (1)$ vanishes up to the $M_\mu$ - reparametrisation term\textsuperscript{24}/

$$\beta^G_{\mu\nu} + 2D(\mu_M_\nu) = 0 . \quad (3)$$

If (3) is satisfied the divergences can be absorbed into a redefinition of the coordinates $x^\mu$. As we shall see, (3) is indeed satisfied for a particular $g_{ij}(x,u)$ as a function of $u$. Using
that the non-vanishing components of the Christoffel connection and the curvature of \( \hat{g} \) are

\[
\hat{\Gamma}^i_{jk} = \Gamma^i_{jk} \quad \hat{\Gamma}^v_{ij} = \frac{1}{2} \dot{g}_{ij} \quad \hat{\Gamma}^i_{ju} = \frac{1}{2} g^{ik} \dot{g}_{kj} \quad \dot{g}_{ij} = \frac{\partial g_{ij}}{\partial u} ,
\]

(4)

\[
\hat{R}^{ijkl} = R^{ijkl} \quad \hat{R}^{iuvj} = T_{ij} \quad \hat{R}^{uijk} = E_{ijk} ,
\]

(5)

and that \( \beta^G_{iv} = 0, \beta^G_{uv} = 0 \) (this follows from the fact that the \( \beta^G \)-function is constructed in terms of curvature tensors and covariant derivatives) we can rewrite (3) in the ‘component’ form

\[
\beta^g_{ij} + 2D_{(i} M_{j)} - 2\hat{\Gamma}^v_{ij} M_v = 0 ,
\]

(7)

\[
\bar{\beta}^g_{ij} = \dot{g}_{ij} M_v \quad \bar{\beta}^g_{ij} = \beta^g_{ij} + 2D_{(i} M_{j)} ,
\]

(8)

\[
\beta^G_{uu} = -2 \partial_u M_u ,
\]

(9)

\[
\beta^G_{iu} = -\partial_i M_u - \partial_u M_i + g^{jk} \dot{g}_{ij} M_k ,
\]

(10)

Since all the components of \( \beta^G_{\mu\nu} \) do not depend on \( v \), the only \( v \) - dependence that is possible in \( M_\mu \) is a linear \( v \)-term in \( M_u \). Then the general solution of (10) is given by

\[
M_v = mu + p \quad M_u = -mv + Q(u, x) \quad M_i = M_i(u, x) \quad p, m = \text{const.}
\]

(11)

For a given \( g_{ij}(u, x) \) the components \( \beta^G_{uu} \) and \( \beta^G_{iu} \) are some particular \( N + 1 \) functions of \( u \) and \( x \) so that one can always satisfy the equations (8) and (9) by properly choosing \( N + 1 \) functions \( M_u \) and \( M_i \) (once we have solved (8), we can put (9) in the form \( \partial_u M_i + h^i_j(u, x) M_j = E_i(u, x) \) which always has a solution).

Having determined \( M_u \) and \( M_i \) as functionals of \( g_{ij} \) we are left with the final equation (7). It should be interpreted as an equation for \( g_{ij}(u, x) \). Using (11) and introducing

\[
\tau = m^{-1} \ln(mu + p) \quad m \neq 0 \quad \tau = p^{-1} u \quad m = 0
\]

(12)
(to get a Weyl invariant model one should actually set \( m = 0 \), see below) we can represent (7) in the form

\[
\frac{dg_{ij}}{d\tau} = \bar{\beta}_{ij}^g .
\]

Thus we have proved the following statement: if the metric \( g_{ij} \) depends on \( u \) in such a way that it satisfies the standard RG equation of the \( N \) - dimensional sigma model (with some particular reparametrisation vectors \( M_i \)) then the \( 2 + N \) - dimensional sigma model based on (1) is UV finite to all orders of the loop expansion.

Let us now make a number of comments. If \( g_{ij} \) corresponds to a finite \( N \) - dimensional theory, i.e. \( \bar{\beta}_{ij}^g = 0 \) then one should set \( p = 0 \), i.e. a finite \( 2 + N \) - dimensional model is found for arbitrary dependence of \( g_{ij} \) on \( u \). The above argument for finiteness is simplified in the “one-coupling” case when the transverse metric is proportional to a metric \( \gamma_{ij}(x) \) of a symmetric (constant curvature) space

\[
g_{ij}(u, x) = f(u)\gamma_{ij}(x) .
\]

The corresponding model is renormalisable for arbitrary \( f(u) \). To get more explicit formulas let us assume that the transverse space is maximally symmetric, i.e.

\[
R_{ijkl}(\gamma) = \frac{R}{N(N - 1)}(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}) .
\]

Since \( \beta_{iu}^G = 0 \) and scalar functions (e.g. \( \beta_{uu}^G \)) are \( x \) - independent we set \( M_i = 0 \), \( \partial_i M_u = 0 \) and thus solve (7),(8),(9) by\(^{12/}\)

\[
M_v = mu + p , \quad M_u = -mv + Q(u) ,
\]

\[
\dot{Q} = -\frac{1}{2}\beta_{uu}^G(u) = \frac{1}{4}\alpha'N(f^{-1}\ddot{f} - \frac{1}{2}f^{-2}\dot{f}^2) + O(\alpha'^3) ,
\]

\[
M_v\dot{f}\gamma_{ij} = \beta(f)\gamma_{ij} ,
\]

i.e.

\[
\frac{df}{d\tau} = \beta(f) ,
\]
\[ \beta^G_{ij} = \beta^g_{ij} = \beta(f)g_{ij} \quad \beta^g_{ij} = \alpha' R_{ij} + O(\alpha'^2) , \]
\[ \beta(f) = a + (N - 1)^{-1} a^2 f^{-1} \]
\[ + \frac{1}{4}(N - 1)^{-2}(N + 3)a^3 f^{-2} + O(a^4 f^{-3}) , \quad a \equiv \alpha' N^{-1} R . \]

Eq. (16) has the obvious perturbative solution (we choose \( m = 0 \) case in (12))
\[ f(u) = bu + (N - 1)^{-1} \ln u + O(u^{-1}) , \quad b \equiv p^{-1} a . \] (17)

The asymptotic freedom corresponds to \( f \) (i.e. the inverse coupling of the sigma model) growing to infinity at large \( u \). Having found \( f(u) \) from (16) one determines \( Q \) from (15) and thus solves (7)–(10).

We see that the metric of the transverse space (and thus the full metric (1)) is determined by the \( \beta \)-function of the transverse theory. The explicit all order expressions for the latter are not known in bosonic sigma models. On the other hand, there are examples of \( n = 2 \) supersymmetric (\( n \) is the number of 2d supersymmetries) sigma models with homogeneous symmetric Kähler target spaces for which the exact \( \beta \)-function coincides with the one-loop expression25/. As we shall discuss in Sec.3, the metric of the corresponding finite 2 + \( N \)-dimensional \( n = 1 \) supersymmetric sigma models is explicitly given by (14) and the first term in (17).

2.2. Solution of Weyl Invariance Conditions

The UV finiteness of a sigma model in flat 2-space does not in general guarantee that the corresponding model on a curved 2d background is Weyl invariant. The Weyl invariance conditions for the model
\[ I = \frac{1}{4\pi \alpha'} \int d^2 z \sqrt{g} [ \ G_{\mu\nu}(x) \partial_\mu x^{a} \partial^a x^\nu + \alpha' R^{(2)}(x) \phi(x) ] \] (18)
(which are equivalent to the string effective equations) have the following general structure21/
\[ \bar{\beta}^G_{\mu\nu} = \beta^G_{\mu\nu} + 2D_{(\mu}M_{\nu)} = 0 , \] (19)
\[ \beta^G = \beta^\phi + M^\mu \partial_\mu \phi = 0 \quad , \]

\[ \beta^\phi = c - \frac{1}{2} \alpha'^2 D^2 \phi + \frac{1}{16} \alpha'^2 R_{\mu\alpha\beta\gamma} R^{\mu\alpha\beta\gamma} + O(\alpha'^3) \quad , \quad c = \frac{1}{6} (D - 26) \quad , \]

where \( M_\mu \) is not arbitrary but is given by

\[ M_\mu = \alpha' \partial_\mu \phi + \frac{1}{2} W_\mu \quad . \]

Here \( W_\mu \) is a covariant vector constructed of \( G_{\mu\nu} \) only (and determined by the mixing under renormalisation of dimension two composite operators\(^{21/}\)). To prove that sigma model based on \((1)\) is Weyl invariant one needs to show that there exists a dilaton field \( \phi \) such that \( M_\mu \) in \((3)\) can be represented in the form \((21)\).

The Weyl anomaly coefficients \( \bar{\beta}^G_{\mu\nu} \) and \( \bar{\beta}^\phi \) satisfy \( D \) differential identities which can be derived from the condition of non-renormalisation of the trace of the energy-momentum tensor of the sigma model\(^{22/}\). They can be considered to be a consequence of the target space reparametrisation invariance given that \( \bar{\beta}^G_{\mu\nu} \) and \( \bar{\beta}^\phi \) are related to a covariant effective action \( S \)

\[ \frac{\delta S}{\delta \varphi^A} = k_{AB} \bar{\beta}^B \quad , \quad \varphi^A = (G_{\mu\nu}, \phi) \quad , \]

\[ 2D_\mu \frac{\delta S}{\delta G_{\mu\nu}} - \frac{\delta S}{\delta \phi} D^\nu \phi = 0 \quad . \]

In general, the identity has the following structure\(^{22,21/}\)

\[ \partial_\mu \bar{\beta}^\phi - \bar{\beta}^G_{\mu\nu} D^\nu \phi - V^\alpha_\mu \beta^G_{\alpha\beta} = 0 \quad , \]  

where the differential operator \( V^\alpha_\mu \) depends only on \( G_{\mu\nu} \). To the lowest order in \( \alpha' \) one finds\(^{26,21/}\)

\[ \partial_\mu \bar{\beta}^\phi - \bar{\beta}^G_{\mu\nu} D^\nu \phi + \frac{1}{2} D^\nu (\bar{\beta}^G_{\mu\nu} - \frac{1}{2} G_{\mu\nu} G^\lambda_\rho \bar{\beta}^G_{\lambda\rho}) + O(\alpha'^2) = 0 \quad . \]

One of the consequences of \((24)\) is that \( \bar{\beta}^\phi = \text{const} \) once \((19)\) is satisfied. In general, the identity \((24)\) implies that only \( \frac{1}{2} D(D + 1) + 1 - D \) of equations \((19), (20)\) are independent. It may happen, in particular, that if the “transverse” subset of \( \frac{1}{2} (D - 2)(D - 1) \) equations in
(19) and the dilaton equation (20) are solved, the remaining $D$ equations (19) are satisfied automatically.

Let us look for solutions of (19),(20) which have the form

$$F_{\mu\nu} = \hat{g}_{\mu\nu}(u,x) \ , \ \phi = \phi(u,v,x) \ , \ x^\mu = (v,u,x^i) \ , \quad (26)$$

where $\hat{g}_{\mu\nu}$ is given by (1). Since $\beta_{\mu\nu}, W_\mu$ and hence $\beta_{\mu\nu}' = \beta_{\mu\nu}^G + D_{(\mu}W_{\nu)}$ in (19) are covariant functions of the curvature and its derivatives and since the metric has a Killing vector it is easy to see that the $(\mu\nu)$ component of $\beta_{\mu\nu}'$ is identically zero. Then (19) gives the following constraint on the dilaton: $\partial_\mu \partial_\nu \phi = 0$ , i.e.

$$\phi = pv + \phi(u,x) \ , \quad p = \text{const} \ . \quad (27)$$

Here $p$ is an arbitrary integration constant and $\phi(u,x)$ is to be determined. From now on all the functions will depend only on $u$ and $x^i$. Using (4), (27) we can represent the non-trivial components of (19) as follows (we shall put $\alpha' = 1$)

$$\bar{\beta}_i^g - p \dot{g}_i = 0 \ , \quad (28)$$

$$\bar{\beta}_i^g = \beta_{ij}^G + D_i (W_j) + 2D_i D_j \phi \ , \quad (29)$$

$$\beta_{uu}^G + \dot{W}_u + 2 \ddot{\phi} = 0 \ . \quad (30)$$

Equation (20) takes the form

$$\bar{\beta} = c - \gamma \phi + (\partial_\mu \phi)^2 + \frac{1}{2}W_\mu \partial_\mu \phi + \omega$$

$$\quad = \frac{1}{3} + \bar{\beta}' + \frac{1}{2}pM_{ij} \dot{g}_{ij} - \frac{1}{2}pW_u - 2p \dot{\phi} = 0 \ , \quad (31)$$

$$\bar{\beta}' = c' - \gamma' \phi + (\partial_i \phi)^2 + \frac{1}{2}W_i \partial_i \phi + \omega \ , \quad c' = \frac{1}{6}(N - 26) \ , \quad (32)$$

where $\gamma'$ is the ‘anomalous dimension’ differential operator, $\omega$ is a covariant function of $G_{\mu\nu}$ only and the $M_{ij}$-term ($M_{ij} = \frac{1}{2}g_{ij} + ...$) in (31) originates from the linear in $\phi$ term.
\[-\gamma \phi = -\gamma' \phi - M^{ij} D_i D_j \phi + O(D^3 \phi)\] (see ref.13 for details). Being scalar functions of the curvature \(\gamma', \omega\) and hence \(\bar{\beta}^{\phi'}\) do not depend on the derivatives of the metric over \(u\). The functions \(\bar{\beta}^g_{ij}\) and \(\bar{\beta}^{\phi'}\) can be interpreted as the Weyl anomaly coefficients of the “transverse” theory defined by \(g_{ij}(u,x)\) and \(\phi(u,x)\) at fixed \(u\) (\(\frac{1}{3}\) in (31) corresponds to the central charge contribution of the two light-cone coordinates).

Let us first consider the case of non-vanishing \(p\). Then (28) is a first order differential equation for \(g_{ij}(u,x)\) which always has a solution. Eliminating the derivatives of \(g_{ij}\) over \(u\) from (31) using (28) we find a similar first order equation for \(\phi(u,x)\). Eqs. (28),(31) can be interpreted as renormalisation group equations of the “transverse” theory with \(u\) playing the role of the RG “time”\(^{12}/\).

Still there is a question whether the solutions of (28) and (31) satisfy also (29) and (30). It is answered positively\(^{13}/\) using the identity (24). Substituting \(\bar{\beta}^G_{ij} = 0\), \(\bar{\beta}^G_{uu} = 0\) and the expression (27) for the dilaton into (24) one finds\(^{13}/\)

\[
p \bar{\beta}^G_{\nu u} = 0\ , \quad p \bar{\beta}^G_{\nu u} - \bar{\beta}^G_{\nu u} D^\nu \phi - 2V^j u^j \bar{\beta}^G_{\nu u} = 0 \ .
\] (33)

That means that once (28) and (31) are satisfied for non-zero \(p\) the remaining equations (29) and (30) are satisfied as well. The conclusion is that given some initial data \((g_{ij}(x), \phi(x))\) at \(u = 0\) there exists a \(u\)-dependent solution \((g_{ij}(u,x), \phi(u,x))\) of the Weyl invariance conditions (19)–(21).

In the particular case when the transverse space is symmetric (i.e. its metric is given by (14)) the symmetry requires that \(W_i = 0\), \(\bar{\beta}^G_{\nu u} = 0\) and that \(\phi\) is \(x^i\)-independent,

\[
\phi = p v + \phi(u) \ .
\]

The functions which enter the equations for \(f(u)\) and \(\phi(u)\) are

\[
\beta^G_{ij} = \beta(f) \gamma_{ij} \ , \quad \beta^G_{\nu u} = \beta^G_{\nu u}(f) \ , \quad W_u = W_u(f) \ ,
\]

\[
\bar{\beta}^{\phi'} = c' + \omega(f) \ , \quad M^{ij} = \frac{1}{2} f^{-1}[1 + M(f)]\gamma^{ij} \ .
\]
Since eq.(30) is a consequence of (28) and (31) $\beta_{uu}^G$ is not independent and we are left with the following two equations for $f(u)$ and $\phi(u)$ (cf.(15),(16))

$$p\dot{f} = \beta(f) , \quad p\dot{\phi} = \frac{1}{2}c + J(f) ,$$

$$J = \frac{1}{2}\omega(f) + \frac{1}{8}N[1 + M(f)]f^{-1}\beta(f) - \frac{1}{4}pW_u , \quad c = \frac{1}{6} + c' = \frac{1}{6}(N - 24) .$$

As a result, the ‘scale factor’ of the metric $f(u)$ runs according to the standard (“flat space”) RG equation while the dilaton depends on $u$ in such a way as make the total central charge vanish. It is possible to show\(^{12}\) that if (28),(30) are satisfied the central charge of this model $\bar{\beta}^\phi$ is equal to that of the free $2 + N$-dimensional theory plus the contribution of the linear terms in the dilaton. In fact, since $\bar{\beta}^\phi$ is constant on a solution of (28),(30) it can be computed at any value of $u$, e.g. $u = \infty$. Given that all higher loop contributions should vanish in the weak coupling limit of large $u$ (we are assuming that the transverse sigma model is asymptotically free) it is sufficient to compute $\bar{\beta}^\phi$ in the leading order approximation. Representing the dilaton in the form

$$\phi = pv + qu + \Phi(u) ,$$

where $\Phi$ stands for contributions which are due to sigma model interactions (which depend on the coupling $f$, i.e. $\Phi(u) = F(f(u))$) we find that the ‘free theory’ and ‘interaction’ contributions cancel separately, giving

$$\bar{\beta}^\phi = c - 2pq = 0 , \quad p\dot{\Phi} = J(f(u)) .$$

Thus one can satisfy the zero total central charge condition for arbitrary $N$ by a proper choice of the constants $p$ and $q$.

If the “initial” transverse theory is generic, i.e. if $\bar{\beta}^g_{ij}$ in (28) is non-vanishing at $u = 0$ then the solution exists only for a non-zero $p$. If, however, the initial theory is Weyl invariant, i.e.

$$\bar{\beta}^g_{ij}(u = 0) = 0 \quad , \quad \bar{\beta}^\phi(u = 0) = c'' = \text{const} \quad ,$$

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there are two possibilities. For $p \neq 0$ the simplest solution of (19),(20) is the ‘direct product’ one represented by the fixed point of the RG equations (28),(31) $g_{ij}(u, x) = g_{ij}(x)$, $\phi(u, x) = \frac{1}{2p}(\frac{1}{3} + c'' + c')u + \phi(x)$. When the “transverse” theory $(g_{ij}(u, x), \phi(u, x))$ is Weyl invariant at $u = 0$ and $p = 0$ eqs.(28),(31) imply that the initial Weyl invariance conditions (34) are satisfied also for all other values of $u$. Therefore a solution with (34),(27) and $p = 0$ may exist only if the transverse theory is conformal for all $u$. One can also prove the converse: to get a non-trivial solution with a flat $g_{ij}(u, x)$ (more generally, with a conformal transverse theory) one should set $p = 0$. Then (assuming $\phi = \phi(u)$) eqs.(28),(31) are satisfied automatically but since $p = 0$ the identities (33) no longer imply that (29),(30) are also satisfied.

Since (28) holds identically it does not give an equation for $g_{ij}(u, x)$. The same is true for (31): it does not contain terms with $u$ - derivatives and being a constant (as a consequence of (19),(24)) is satisfied for all $u$ if it is satisfied for $u = 0$, i.e. if $\frac{1}{3} + c' = 0$. Instead of $N + 1$ identities in (33) for $p = 0$ we are left with just one. As a result, we get $N$ independent equations (29),(30) ((33) gives a relation between components of (29)) on $\frac{1}{2}N(N + 1) + 1$ functions $g_{ij}(u, x)$, $\phi(u, x)$. Their particular solutions in the case when the transverse metric is flat (and correspondence with the ‘plane-wave’ solutions found previously were studied in detail in ref.13. In that case it is useful to change coordinates, trading the functions $g_{ij}(u, x)$ corresponding to a flat transverse metric for $A_i$ and $K$ in (2), i.e. transforming the metric (1) into the form (2) where $g_{ij}(u, x)$ has its canonical $\delta_{ij}$ form.

The above discussion can be generalised to the case of non-vanishing antisymmetric tensor coupling. Namely, there exist similar solutions of the Weyl invariance conditions with the metric (1), dilaton (27) and the $v$-independent antisymmetric tensor $\hat{B}_{\mu \nu}$: $\hat{B}_{ij} = B_{ij}(u, x)$, $\hat{B}_{iu} = B_i(u, x)$, $\hat{B}_{\mu \nu} = 0$. 

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3. NEW CLASS OF FINITE SUPERSYMMETRIC SIGMA MODELS WITH MINKOWSKI SIGNATURE TARGET SPACE

In Sec. 2 we have shown that it is possible to construct conformal invariant Minkowski signature models in $2 + N$ dimensions from non-conformal Euclidean models in $N$ dimensions. Since the metric and the dilaton of the $2 + N$-dimensional theory are essentially the ‘running’ couplings of the transverse theory their dependence on $u$ is determined by the $\beta$-functions of the transverse theory. The structure of the $\beta$-functions is usually simpler in supersymmetric theories so it is of interest to generalise the above construction to the supersymmetric case. In particular, we would like to make use of the known fact that there are examples of supersymmetric sigma models with homogeneous symmetric Kähler target spaces for which the exact $\beta$-function coincides with the one-loop expression, i.e. is explicitly calculable

The two dimensional ($n = 1$) supersymmetric sigma model can be constructed for an arbitrary metric $G_{\mu\nu}$ of a $D$-dimensional target space. Its superfield action is given by

$$I = \frac{1}{4\pi\alpha'} \int d^2 z d^2 \theta \, G_{\mu\nu}(X) \bar{D}X^\mu \bar{D}X^\nu \ , \quad (37)$$

where

$$X^\mu = x^\mu + \bar{\theta}\psi^\mu + \frac{1}{2}\bar{\theta}\theta F^\mu \ , \quad \bar{D} = \frac{\partial}{\partial \theta} + \bar{\theta}\gamma^a \partial_a \ .$$

The component form of the action is

$$I = \frac{1}{4\pi\alpha'} \int d^2 z \left[ G_{\mu\nu}(x) \partial_a x^\mu \partial^a x^\nu + G_{\mu\nu}(x) \bar{\psi}^\mu \gamma^a D_a \psi^\nu + \frac{1}{6} R_{\mu\nu\lambda\rho} \bar{\psi}^\mu \psi^\lambda \bar{\psi}^\nu \psi^\rho \right] \ . \quad (38)$$

For the metric with the null Killing vector (1) we can represent (37) in terms of the real superfields $U$, $V$ and $X^i$

$$I = \frac{1}{4\pi\alpha'} \int d^2 z d^2 \theta \left[ -2\bar{D}U \bar{D}V + g_{ij}(U,X) \bar{D}X^i \bar{D}X^j \right] \ . \quad (39)$$

The component form of (39) can be found either directly from (39) or by substituting the expressions (1),(4),(5),(6) into (38).
Eqs. (3)–(17) have a straightforward generalisation to the supersymmetric case. In particular, the solution \( g_{ij}(u, x) \) of the condition of finiteness (13) is determined by the \( \beta \)-function of the ‘transverse’ part of (39), i.e. of the supersymmetric model with the metric \( g_{ij}(u, x) \) for constant \( u \). As is well known\(^{28}\), if the transverse space is Kähler the \( N \)-dimensional model is \( n = 2 \) supersymmetric. If it is also a compact symmetric homogeneous space (e.g. \( S^2 = SO(3)/SO(2) \) or \( CP^m \)) then it is very plausible that its \( \beta \)-function is exactly calculable and is given by the one-loop expression\(^{25}\). This was actually proved in ref.25 for the following classes of Kähler manifolds:

\[
M_1 = SO(m + 2)/SO(m) \times SO(2) , \quad N = 2m ;
\]

\[
M_2 = SU(m + k)/SU(m) \times SU(k) \times U(1) , \quad N = 2mk ;
\]

\[
M_3 = Sp(m)/SU(m) \times U(1) , \quad N = m^2 + m ;
\]

\[
M_4 = SO(2m)/SU(m) \times SO(2) , \quad N = m^2 - m .
\]

In that case the transverse part of the metric (14), the \( \beta \)-function (16) and the solution of (13) are given simply by

\[
g_{ij}(u, x) = f(u)\gamma_{ij}(x) , \quad \beta(f) = a , \quad f(u) = bu , \quad b = p^{-1}a .
\] (40)

The constant \( a > 0 \) is determined by the geometry of the transverse space\(^{25}\) (\( a_1(m = 1) = 2 ; a_1(m \geq 2) = m \); \( a_2 = m + k \); \( a_3 = m + 1 \); \( a_4 = m - 1 \)). The constant \( b \) is arbitrary and can be absorbed into a redefinition of the coordinates \( u \) and \( v \). Then the final expression for the Minkowski signature metric of the finite \( 2 + N \)-dimensional supersymmetric sigma model is

\[
ds^2 = -2dudv + u\gamma_{ij}(x)dx^i dx^j
\] (41)

(we have assumed \( u > 0 \)). Note that while the transverse model (with fixed constant \( u \)) is \( n = 2 \) supersymmetric the full \( 2 + N \)-dimensional model apparently has only \( n = 1 \)
supersymmetry. The non-zero components of the curvature of the metric (41) can be found from (5),(6)

$$\hat{R}^i_{jkl} = R^i_{jkl}(\gamma), \quad \hat{R}^i_{uju} = \frac{b}{4u}\gamma_{ij}.$$  \hspace{1cm} (42)

All curvature invariants are singular at $u = 0$. It is still possible that this singularity is harmless in string theory (cf. ref.29).

The simplest non-trivial example of the finite models we have constructed corresponds to the case when the transverse theory is represented by the $O(3)$ supersymmetric sigma model$^{30}$. The resulting metric (41) is that of four ($2 + N = 4$) dimensional space with the transverse part being proportional to the metric on $S^2$,

$$ds^2 = -2dudv + u(d\theta^2 + \sin^2\theta d\varphi^2).$$  \hspace{1cm} (43)

This metric is conformal to the standard metric on the product of the two-dimensional Minkowski space and two-sphere. The corresponding geodesic equations can be easily integrated with the conclusion that the part of space with $u > 0$ is not geodesically complete (replacing the factor $u$ by the modulus $|u|$ apparently introduces additional singularities at $u = 0$).

To find out whether the constructed finite supersymmetric models can be identified with the exact solutions of the superstring effective equations we need to check that these sigma models correspond to Weyl invariant theories on a curved 2d background. It is straightforward to add to (37) the dilaton coupling term $\int d^2z d^2\theta E^{-1}R^{(2)}(X) (E^{-1}$ is the determinant of the $n = 1$ supervielbein) and to generalise the expressions for the Weyl invariance conditions (19)–(21) and the identity (24) to the case of $n = 1$ supersymmetric sigma models$^{31}$. Then the argument in Sec.2.2 can be repeated to prove that for an arbitrary “initial” ($u = 0$) transverse euclidean $n = 1$ supersymmetric model there exist such metric $g_{ij}(u,x)$ and dilaton $\phi(u,x)$ that the corresponding $n = 1$ supersymmetric model with metric (1) is Weyl invariant, i.e. represent a string vacuum.
Let us now specialize to the case when the transverse metric is symmetric Kähler. Then we can apply the discussion of the symmetric transverse space case in Sec.2.1. We conclude that the Weyl invariance conditions are again given by equations (34),(36). The equation on $f$ is the same RG equation as in the finiteness condition so its solution is represented by (40),(41). Since the transverse model is $n = 2$ supersymmetric we can make use of the result$^{31}$ that the dilaton coupling is not renormalised in the $n = 2$ supersymmetric case (in the minimal subtraction scheme). That means that $M$ and $\omega$ in $J$ in (34) should vanish. As a result, the dilaton $\phi$ is given by (cf.(35),(36))

$$
\phi(v, u) = pv + qu + \Phi(u) , \quad \bar{\beta}^\phi = c - 2pq = 0 , \quad c = \frac{1}{4}(N - 8) ,
$$

$$
\dot{\Phi} = I(f(u)) , \quad I = p^{-1}J = \frac{N}{8p}f^{-1}\beta(f) - \frac{1}{4}W_u = \frac{N}{8u} - \frac{1}{4}W_u , \quad f = bu ,
$$

where we have used that in the superstring theory $c = \frac{1}{4}(D - 10)$. Note that differentiating the equation for $\Phi$ in (45) and comparing with (30) gives

$$
\dot{W}_u = -\frac{N}{2p} \frac{d}{du}(f^{-1}\beta) - 2\beta^G_{uu} = \frac{1}{2}Nu^{-2} - 2[\frac{1}{4}Nu^{-2} + O(\alpha' u^{-3})] = O(\alpha'^3 u^{-3})
$$

(46)

(the one-loop term in $\beta^G_{uu}$, i.e. $\alpha'R_{uu}$, is given by (42); see also (15)). Higher loop corrections to $W_u$ and to $\beta^G_{uu}$ are thus directly related. It is easy to see that there is no two-loop term in $\beta^G_{uu}$ in the case of symmetric transverse space; in the bosonic case both $\beta^G_{uu}$ and $W_u$ are non-vanishing in the tree-loop approximation$^{12}$. It is possible that $W_u$ is actually vanishing in the present model. Though the results of ref.32 (a comparison of the perturbative expression for the $\beta^G$-function with superstring effective equations) imply that $W_\mu$ contains a non-zero four loop term in a general $n = 1$ supersymmetric model, $W_\mu$ does vanish in $n = 2$ supersymmetric models$^{30}$. If $W_u = 0$ then the exact expression for the dilaton is (see (44),(45))

$$
\phi(v, u) = \phi_0 + pv + qu + \frac{1}{8}N \ln u .
$$

(47)
The resulting backgrounds (41),(44) or (47) thus represent exact solutions of superstring effective equations with non-trivial dilaton. Note that the string coupling $e^\phi$ is

$$e^\phi = A u^{N/8} e^{(qu+pv)} , \quad A = e^{\phi_0} .$$

(48)

It goes to zero in the strong coupling region $u \to 0$ of the transverse sigma model, i.e. is small near the singularity $u = 0$. If $N < 8$ the constant $q$ in (44),(47) is negative (we are assuming $u > 0$, $v > 0$, $p > 0$) so that the string coupling is also vanishing in the small coupling region $u \to \infty$. In the case of the critical dimension $D = 10$ (or $N = 8$) $q$ must vanish. Then the string coupling is inverse proportional to the sigma model coupling $f^{-1}$,

$$e^\phi = A' f e^{pv} .$$

4. APPLICATION TO 2D QUANTUM GRAVITY

As is well known, the classical gravitational action in $d = 2$ is trivial before one accounts for the (non-local) quantum anomaly term. Introducing an extra scalar field (“2d dilaton”) coupled to the scalar curvature one obtains a non-trivial theory (though still with no propagating degrees of freedom). This theory seems simpler and better defined as a starting point for a (perturbative) quantisation. By redefining the fields one can represent the general action in the form$^{18,19}$

$$S = -\frac{1}{2} \int d^2x \sqrt{g} \left[ \partial^\mu \varphi \partial_\mu \varphi + q \varphi R + V(\varphi) \right] .$$

(49)

For example, the metric – dilaton action which generates the $\sigma$-model Weyl anomaly coefficients in the case of $D = 2$ target space and which has a classical “black hole” solution$^{33}$

$$S = -\frac{1}{2} \int d^2x \sqrt{g} e^{-2\phi} \left[ R + 4(\partial \phi)^2 + c \right] ,$$

(50)

can be represented as (49) with $V = c \exp (\varphi/q)$. By a further redefinition it can be put into the form

$$S = -\frac{1}{2} \int d^2x \sqrt{\hat{g}} (\hat{R} v + c) , \quad \hat{g}_{\mu\nu} = v g_{\mu\nu} , \quad v = e^{-2\phi} .$$

(51)
Let us now switch to ‘world sheet’ notation and consider the metric-scalar $(\hat{\gamma}, v)$ gravitational theory coupled to some extra $N$ “matter” scalar fields which is described by the sigma model

$$I_0 = \frac{1}{4\pi} \int d^2 z \sqrt{\hat{\gamma}} \left[ pv\hat{R}^{(2)} + g_{ij}(x)\partial_a x^i \partial^a x^j + T(x) \right],$$

(52)

In the conformal gauge

$$\hat{\gamma}_{ab} = e^{-2u/p} \gamma_{ab}$$

(52) takes the form

$$I_0 = \frac{1}{4\pi} \int d^2 z \sqrt{\gamma} \left[ -2\partial_a v \partial^a u + g_{ij}(x)\partial_a x^i \partial^a x^j + pvR^{(2)} + T(x) e^{-2u/p} \right].$$

(53)

This model is renormalisable on a flat background with $g_{ij}$ ‘running’ with a cutoff. Once all the fields are quantised one may expect that the ‘effective action’ will be represented by a general sigma model in $2 + N$ dimensions $x^\mu = (u, v, x^i)$. The model should be Weyl invariant with respect to the background metric $\gamma_{ab}$ since the 2d metric itself is an integration variable$^{14,15,16}$. We are implicitly assuming that the theory can be regularised in a way covariant with respect to the original metric $\hat{\gamma}$ so that all the elements of the theory - the action, the measure and the regularisation depend only on the full $\hat{\gamma}$ (and that we are in the phase where 2d metric has zero expectation value). To determine the ‘effective action’ we need to find a solution of the Weyl invariance conditions for the metric, dilaton and tachyon couplings of the $2 + N$-dimensional theory such that at the classical limit they reduce to the couplings in (53). It seems natural to impose an additional assumption that the dependence of the couplings on $v$ in the ‘effective action’ should remain as simple as in (53), i.e. the target space metric and the tachyon should be $v$-independent (the metric will have a Killing vector) while the dilaton will be at most linear in $v$. It is precisely such solutions of the metric and dilaton Weyl invariance conditions (19),(20) that we have studied in Sec.2 (let us first ignore the tachyon coupling term). We have found that the action

$$I = \frac{1}{4\pi} \int d^2 z \sqrt{\gamma} \left[ -2\partial_a v \partial^a u + g_{ij}(u, x)\partial_a x^i \partial^a x^j + (pv + \phi(u, x))R^{(2)} \right]$$

(54)
defines a Weyl invariant quantum theory if the metric \( g_{ij} \) and dilaton \( \phi \) depend on \( u \) according to the first order RG equations (28),(31). The result that \( g_{ij} \) starts running with \( u \) according to the RG equation \( \dot{g}_{ij} \sim R_{ij} + ... \) is very natural given that \( u(z) \) is proportional to the conformal factor of the 2d metric (which should be coupled to a covariant cutoff). At the same time, one would also expect to find the conformal anomaly term \( \sim K(u,x) \partial u \) but it is missing in (54). Note, however, that such term can be generated by a redefinition of the field \( v \). As discussed in ref.13, there is, in fact, an equivalent solution of the conformal invariance conditions (19)–(21) with \( \phi(u,x) = 0 \) but with the metric (1) containing the additional term \( K(u,x)du^2 \) (cf.(2)). The difference between the theory (52) and the standard 2d gravity coupled to a sigma model (where both the anomaly term \( K(u,x)\partial u \partial^a u \) and \( \phi(u,x)R^{(2)} \) should appear in the quantum action\(^{16/} \)) is due to the presence of the extra scalar field \( v \).

Let us now study the solutions of the Weyl invariance condition for the tachyon coupling\(^{34,21/} \) (cf.(19)–(21))

\[
\beta^T = -\gamma T + (\alpha' \partial^\mu \phi + \frac{1}{2} W^\mu) \partial_\mu T - 2T + b(T) \\
= -\frac{1}{2} \alpha' D^2 T + \alpha' \partial^\mu \phi \partial_\mu T - 2T + O(\alpha'^3) + b(T) = 0 \ . \quad (55)
\]

\( \gamma \) is the same differential operator which appeared in (31). \( b(T) \) represents “non-perturbative” corrections which are of higher order in \( T \). If there were no \( v \) coordinate so that the metric of the 1 + \( N \)-dimensional space and the dilaton were given by \( ds^2 = K du^2 + ds_N^2 \) and \( \phi = Ku + ... \) then (55) would reduce to a second order equation in \( u \)\(^{17/} \), \( -\frac{1}{2} K^{-1} \ddot{T} + \dot{T} + ... = 0 \), which would reproduce the standard RG equation only in the “semiclassical” limit of large anomaly coefficient \( K \). On the other hand, if the metric \( G_{\mu\nu} \) is given by (1) and the dilaton is linear in \( v \) (27) then for \( v \)-independent tachyon \( T = T(u,x) \) eq.(55) takes the form similar to (28),(31), i.e. it becomes a first order RG-type equation(cf. ref.17)

\[
p\dot{T} = \beta^T \ . \quad (56)
\]
\(\bar{\beta}^{T'}\) (containing only derivatives over \(x^i\)) denotes the Weyl anomaly coefficient of the ‘transverse’ theory with the coupling \(T(u, x)\) and \(u = \text{const}\) playing the role of the RG “time”. The simplest example of a solution of (55),(56) is found if \(T = T(u)\). Let us first ignore the “non-perturbative” term \(b(T)\). Then (cf.(53)) \[p\dot{T} = -2T, \quad T = T_0e^{-2u/p}.\] (57)

Equivalent solutions in the context of 2\(d\) gravity model were discussed in ref.19. Now it is possible show that \(T\) in (57) solves the full eq.(56) (with all higher order terms included), i.e. that there are no non-perturbative divergences in the model

\[I = \frac{1}{4\pi} \int d^2 z \sqrt{\gamma} \left[-2\partial_a v \partial^a u + pvR^{(2)} + T(u)\right].\] (58)

In fact, \(v\) plays the role of a Lagrange multiplier which makes \(u\) effectively non-propagating so that there are no quantum corrections in the theory (see also ref.35). Then the condition of conformal invariance is equivalent to the classical conformal invariance relation (57). To reconcile this conclusion with the expected presence of \(O(T^2)\) and \(O(\partial T\partial T)\) terms in \(\bar{\beta}^T\), \(\bar{\beta}^\phi\) and \(\bar{\beta}^G\) one is to note that a derivation of such terms (or a proof of correspondence with \(O(T^3)\) terms in the effective action) presumes an analytic continuation in momenta and is not, strictly speaking, valid in the case when \(T\) depends just on one variable (the question of non-perturbative terms in the \(\beta\)-functions should be addressed separately for each 2\(d\) theory corresponding to a particular scalar potential \(T\), see ref.34).

In conclusion, we have suggested a connection between the conformal invariant 2 + \(N\) - dimensional sigma models and the 2\(d\) scalar quantum gravity coupled to non-conformal ‘transverse’ \(N\) - dimensional sigma models. The conformal factor of the 2\(d\) metric is identified not with time but with the light cone coordinate \(u\); this makes the corresponding Weyl invariance conditions first order in \(u\). Given that the target space metric corresponding to 2\(d\) gravity plus scalar matter models has natural Minkowski signature\(^{18}\) it seems important to try to clarify further the connection between the ‘Minkowski’ conformal theories and 2\(d\) quantum gravity.
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