Scaling properties of fractal-like structures

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Abstract. This work is aimed at solving one of the fundamental problems of modern fractal optics. The goal is to establish general laws that relate structural features of fractal-like systems and their associated spectral characteristics. Establishing such laws allows finding new diagnostic approaches for exploration of fractal-like systems and their formation processes. Nanocluster systems and multilayer structures, also with inclusion of metamaterial inserts, were chosen as the objects of research. A wide variety of applications of varying physical nature is enabled as fractal optics successfully penetrate into interdisciplinary technologies and theoretical concepts. In this work, particular interest is paid to scaling properties of structures of the studied systems and their optical characteristics. Based on theoretically obtained amplitude and phase distributions of light waves that diffract or interfere in the analyzed systems, the features of scaling in light fields are determined. When analyzing the obtained modeling results, a comparison is made between the structural features of the studied systems revealed by the scaling parameters in diffraction patterns and transmission spectra.

1. Introduction

Fractal-like elements with complex spatial structure have found a wide application in different optical devices, created particularly on the basis of modern nanotechnologies [1]. Their improvement is closely concerned with the development and improvement of diagnostic methods providing to define quality of considered systems. Among them the fractal-like multilayer systems with the metamaterial insertions and lattice structures, representing the self-organized nanocluster dendrite-type systems, are widely distributed [2]-[3].

In the process of multilayer systems characteristics study has been found, that the deviation from periodicity leads to the appearance of the whole set of unique features, being of practical and theoretical interest. Particularly, aperiodic multilayer systems (AMS) are used for creating wide-range reflectors, optical switches, x-ray optics elements, antennas with new optical features, highly sensitive sensors for biological and chemical agents detection and other devices [1]. Often aperiodicity is connected with the presence of fractal features appearing in the objects structure and their

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characteristics. As fractality affects significantly on the structures optical properties there exists the necessity in the determining of the range of the influence of different factors on the self-similarity of their characteristics.

In spite of certain interest to the study of fractal multilayer structures with metamaterials and nanocluster systems of dendrite type [4]-[5] the whole number of questions concerning the investigation of their scaling features is still studied insufficiently.

There is a number of publications dedicated to the study of fractal features of the dendrite cluster structure growth, for example [2], [6]-[7]. But there exist not enough works concerning the comparison of fractal characteristics of dendrite structures and their diffraction patterns [8]. For diagnostic applications it is useful to determine the sustainable connection between scaling properties of nanodendrite structures and their spectral characteristics (diffraction patterns). The fractal properties obtained can be used for the statement and solution of the inverse problem of correspondence of the registered light fields and the certain forms of dendrite cluster structures, and also for the refinement of electrophysical interaction mechanism of nanoparticles description.

The main goal of this work was the consideration of the set of weakly studied questions concerning the analysis and determining the scaling properties of dendrites and transmission spectra of multilayer systems with metamaterials. In particular the scaling parameters of optical characteristics of multilayer Fibonacci system with metamaterials at small losses were obtained using numerical modeling. The scaling parameters of optical characteristics were compared with the scaling parameters of analyzed structures.

### 2. The pattern analysis of scaling features of the structures

This work is devoted to a fundamental problem of modern fractal optics related to the finding of general physical features determining the sustainable connection between aperiodicity of the objects structure of different physical nature and fractality of their characteristics. The existence of such connection in many cases was declared before in the course of theoretical and experimental studies [9]-[10].

Using the elements of pattern analysis, it is possible to obtain the quantitative parameters of the connection mentioned above and significantly refine the diagnostic methods in terms of fast information extraction even in the presence of disturbance factors The pattern analysis became an attribute of many research works carried out in different fields of science (first of all in biology and medicine). But it is not widely used for the study of spectral characteristics of fractal-like objects and processes of their formation.

In the same time the data of researching diffraction and interference on aperiodic fractal-like structures with self-similarity symmetry, including one dimensional quasi-crystals and photon crystals, show the presence of sustainable formations in their optical characteristics, whose form is strictly individual with respect to considering object [10].

In the Figure 1 the fragments of the structures of one dimensional fractal objects created using the features of the Cantor set are presented. In this case the initial levels of the analyzed system according to block building method are the following: $S_0 = A$; $S_1 = ABA$; $S_2 = ABABBBABA$ etc. The transition from lower level to higher level is made using the elements substitution: $A \rightarrow ABA$; $B \rightarrow BBB$.

The calculation of optical characteristics of study objects with aperiodic structure was based on finding their Fourier images and scattering spectra of multilayer systems using the matrix method [11]. For the reflection coefficient the logarithmic presentation was used: $r(\Omega) = -\ln(1 - R)$, where $R$ is the power reflection coefficient, $\Omega = \omega/\omega_0$ is the normalized frequency, $\omega_0$ corresponds to equal optical thickness of the layers. In the Figure 1 the optical characteristics of some analyzed objects, possessing the expressed symmetry of self-similarity and fractal features of the structure, are presented.

The results of the study showed that the most sustainable invariant for the self-similarity assessment of both aperiodic objects structures and their different optical characteristics is the scaling coefficient $\zeta$. 

that is determined by the ratio of self-similar domains. For example, \( \zeta = a_1 f_1 / b_1 c_1 / c_1 d_1 = 3 \) (Figure 1, b). The scaling coefficient for Cantor systems is also equal \( \zeta = 3 \), that follows from the general principle of building such objects (Figure 1, a).

![Figure 1](image)

**Figure 1.** Optical characteristics of fractal objects on the basis of properties of Cantor numerical set. Fragments of the structures of the 1D Cantor lattice (circles) and the multilayer Cantor system (a). Fourier transform of the 1D Cantor lattice (b). The reflection spectrum of a multilayer Cantor system with the same optical thicknesses \( J = 243 \) layers, \( N_A = 2, N_B = 3, \Omega = \omega/\omega_0, \omega \) – incident wave frequency, \( \omega_0 \) – normalizing frequency) (c).

From the Figure 1 one can see the structural similarity of optical characteristics of different types (Fourier images and transmission spectra) of the fully dielectric Cantor structure. That’s why the scaling parameters of its different optical characteristics turn out to be close to the value of \( \zeta = 3 \). The similar scaling properties also have other fractal-like systems and their optical features. For example, the considered before [9] details of the self-similarity symmetry manifestation in AMS let to claim that the structures of Thue-Morse and the systems of double-period can be characterized by the scaling coefficient \( \zeta = 2 \), and the Fibonacci structure – by the coefficient \( \zeta = \Phi = 1.62 \).

3. **Optical characteristics of multilayer systems with metamaterials**

Using the geometrical changes in the studying multilayer structures appears to be possible to make an intentional effect on their spectral characteristics. Changes in the analyzing multilayer structure were inserted by transforming its geometrical building principle.

The multilayer structures are often presented as a system of block elements \( A \) and \( B \) that alternate following the certain law and correspond to different levels of generation [1]. For example, the initial levels of aperiodic Fibonacci system consist of elements \( S_0 = B, S_1 = A, S_2 = AB \). At a transition to higher structural level the following rules of substitution are used: \( A \rightarrow AB, B \rightarrow A \). The blocks of the
first structural levels of Thue-Morse AMS have the form \( S_0 = A, S_1 = AB, S_2 = ABBA \). The transition to each higher structural level may be performed using the rules of substitution: \( A \rightarrow AB, B \rightarrow BA \). The values of \( A, B \) and their sequence determine the alternation of layers with different refraction indexes \( N_A \) and \( N_B \) in the AMS. In this work the refraction index of the media surrounding the AMS was taken to be \( N_C = 1 \).

In the framework of the study the model of AMS where the layers \( A \) are made of metamaterial that is characterized by a negative refraction index in a certain spectral interval is considered. The layers \( B \) are made of dielectric with constant positive refraction index. The possible geometric configuration of metamaterial is presented in [12].

The permittivity \( \varepsilon \) and permeability \( \mu \) of the metamaterial is determined by the following relations [13]:

\[
\varepsilon_A(f) = 1 + \frac{5^2}{0.9^2 - f^2 - if\gamma} + \frac{10^2}{11.5^2 - f^2 - if\gamma},
\]

\[
\mu_A(f) = 1 + \frac{3^2}{0.902^2 - f^2 - if\gamma},
\]

where \( f \) and \( \gamma \) are the frequency and the attenuation parameter in GHz respectively, \( i = \sqrt{-1} \). The discretization of \( f \) value is defined with the relation \( f_k = 1.5(1 + 0.0033k) \), where \( k = 0 \ldots \tilde{N}_{\text{max}} \) are the discretization coefficients, \( \tilde{N}_{\text{max}} \) is a given integer value limiting the frequency interval.

For calculating reflection and transmission spectra of AMS the matrix method [11] using Fresnel’s law for metamaterials [14] was applied. It was also suggested that the wave of unit amplitude was forming at the output of the studied systems. The amplitudes of electric field of reflected wave \( E_r^{(t)} \) and transmitted wave \( E_t^{(t)} \) taking into account boundary conditions on the media interface with the number \( m \) have the form:

\[
\begin{bmatrix}
E_0^{(t)} \\
E_0^{(r')}
\end{bmatrix}
= \Delta_1 \Delta_2 \Delta_3 \ldots \Delta_{m-1}
\begin{bmatrix}
E_0^{(t)} \\
E_0^{(r')}
\end{bmatrix}
= \Delta
\begin{bmatrix}
\frac{1}{2} \left( 1 + \frac{N_m}{N_{m-1}} \right) \\
\frac{1}{2} \left( 1 - \frac{N_m}{N_{m-1}} \right)
\end{bmatrix},
\]

where \( \Delta = \prod_{j=1}^{m-1} \Delta_j \), indexes \( t \) and \( r' \) correspond to the waves propagating in forward and reverse directions respectively, the sign \( \leftrightarrow \) in the subscripts denote that the fields are defined at the left-hand side and right-hand side of the layers interface. The value of the power transmission coefficient \( T_k \) can be found using the value of \( E_0^{(t)} \) obtained from the equation (3):

\[
T_k = \frac{1}{|E_0^{(t)}|^2},
\]

For more distinct manifestation of self-similar details of spectral dependences the logarithmic presentation \( r = -\ln(T_k) \) was used.

The results of numerical calculations showed that insertion of metamaterials significantly changes and complicates the structure of spectral characteristics of multilayer structures. These changes may be
analyzed on the basis of pattern method [10] that is based on fixing and determining features of separate self-similar fragments (patterns) of the structures and the characteristics of the considered distributions. By registering the presence and the form of a pattern in the characteristics of the AMS one can perform identification of their structural features.

During the study of optical characteristics of aperiodic multilayer structures with different geometry of layers distribution it was found that patterns formation at negative refraction indexes was peculiar only to structures with not equal number of layers of different type. This is due to the strong effect of phase compensation in the systems with equal number of layers of different type that is typical for the transmission spectra of periodic structures with metamaterials [15].

For example, the spectral characteristic \( r(k) \) of AMS of Thue-Morse, whose properties are considered in [5], [9] is presented in Figure 2. As one can see from Figure 2, for negative refraction indexes the patterns \( ab, cd \) will not be formed. The simplified dispersive model (Figure 2) lets selectively describe the influence of the fact of metalayers presence in the multilayer system. Patterns in the given graph occur only in the area, where \( \varepsilon_A > 0, \mu_A > 0 \). Their form completely corresponds to the patterns that are observed in the dielectric AMS of Thue-Morse.

![Figure 2](image_url)

Figure 2. The spectral characteristic \( r(k) \) of Thue-Morse system in the one-dimensional approach: \( [\varepsilon_A = 9, [\mu_A = 1, \varepsilon_B = 2.25, \mu_B = 1 \text{. The patterns in the spectrum are characterized by the maxima } abcd \text{. Thickness of forming layers are equal } d_A = 2.4 \text{ sm, } d_B = 4.8 \text{ sm. The number of layers: } J = 32 \).]

Quantitative estimations of the structural correspondence of detected pattern formations in the spectral characteristics of studied systems were carried out using correlation analysis and determining of scaling coefficients. The dependence of reflection coefficient \( r(k) \) for Fibonacci AMS taking into account dispersion effects and losses (1)-(2) is presented in Figure 3. In Figure 3, b the change of the form of the most sustainable to losses fragment of spectral dependence \( r(k) \) is shown. The coefficients of cross-correlation \( K \) between the curves 1 – 4 and the curve in the region \( cd \) for \( \gamma = 0 \) (Figure 3, a), were found in the interval \( K = 0.99 - 0.97 \) for small losses \( \gamma \).

The local scaling coefficients in the transition domain \( cd \), determined from the relationship of distances between peaks, have the values \( \zeta = cd / cf = 1.62 \text{ and } \zeta' = cf / fd = 1.6 \) (Figure 3). In the same time in the area with positive refraction index the maxima also subject to the Golden section rule [3]. In the multilayer Fibonacci system considered the number of layers \( A \) exceeds the number of layers \( B \) approximately in \( \zeta \) times. Here \( \zeta = 1.62 \) is the scaling coefficient of Fibonacci structure that is equal to the Golden section coefficient \( \Phi \).

In the case of presence of metalayers in AMS it is necessary to correctly take into account the effects of phase compensation that can completely suppress the patterns formation (in the system of Thue-Morse).
Figure 3. Transformation of the patterns shape in the spectral characteristics of a multilayer Fibonacci system with dispersion effects (a). Number of layers $J = 64$, $n_b = 1.5$, $cd\,,\,cefd$ are the patterns. The dependencies of the reflection coefficient $r$ for the pattern $cd$:

1. $r = 0.01$ GHz, $k = 1.5$, $n_b = 1.5$ GHz, $3 - \gamma = 0.1$ GHz, $4 - \gamma = 0.2$ GHz. $k$ and $k_1$ are sampling coefficients for $f$ (b).

4. The scaling properties of dendrite structures

In the framework of the study the possible variant of scaling estimation for dendrite structures was proposed. The scaling parameters of dendrites formed within the modified algorithms of ballistic aggregation (BA) and diffusion limited by aggregation (DLA) were determined using obtained values for the boundaries of areas of instability for the dependencies of the form $D(N)$ [8]: 1500 particles for the BA model (Figure 4) and 2100 particles for the modification of DLA model, where $D$ is cluster fractal dimension, $N$ is the number of particles in the dendrite.

The results of numerical simulation showed that mass fractal dimensions $D$ of the dendrite structure and their diffraction patterns in the near zone had close values. Cluster fractal dimension of dendrite tends to the value 1.7 with increasing $N$ (in the frameworks of suggested modification of DLA model) and to the value 1.9 (in the case of non-mesh BA model), what does not contradict the literary data [2], [6].

The structure of dendrite corresponding to modification of BA model [8] is presented in Figure 4 specifying the areas denoted by dotted lines for calculation of scaling. Division of the dendrite structure into scaling regions is performed by detecting the number of particles exceeding the instability zone. The area $OA$ contains 1500 particles, the area $OB$ contains 3000 particles, the area $OC$ contains 4500 particles etc. For calculating the DLA model corresponding areas contained 2100, 4200 and 6300 particles, respectively.

The scaling parameters were determined using the formula (Figure 4):
According to (5) the following scaling parameters of dendrite structures (Figure 4) were calculated: 
\( \zeta_1 = 1.3 \pm 0.1, \quad \zeta_2 = 1.6 \pm 0.07, \quad \zeta_3 = 1.85 \pm 0.1, \quad \zeta_4 = 2.1 \pm 0.1 \) for BA model; and 
\( \zeta_1 = 1.4 \pm 0.1, \quad \zeta_2 = 1.8 \pm 0.1, \quad \zeta_3 = 2.1 \pm 0.1, \quad \zeta_4 = 2.4 \pm 0.1 \) for DLA model. The averaging was performed using 30 realizations of dendrite.

The presented scaling coefficients characterizing the scale invariance combined with the determining the cluster fractal dimension and scaling area may be used for identification of fractal features of dendrite systems.

5. Conclusions
The obtained results showed that changes in geometrical principle of building multilayer systems, in particular, creating of some layers using metamaterials, effects significantly on the scaling in their characteristics.

The study of fractal features sustainability of AMS optical characteristics taking into account the influence of phase effects including the effect of phase compensation showed that the presence of metamaterial layers has a noticeable effect on scaling manifestation in optical characteristics of considered systems, and in some cases even fully suppresses it due to phase compensation. This fact must be taken into account in the process of patterns fixation in reflection spectra of AMS for determining the features of their structure.

The performed analysis of fractal properties of dendrites with different geometry points out the possibility of establishing a one-to-one relationship between their structural features and diffraction patterns using the definition of mass fractal dimensions, domains and scaling parameters \( \zeta \).

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