NONEXISTENCE OF CUSP CROSS-SECTION OF ONE-CUSPED COMPLETE COMPLEX HYPERBOLIC MANIFOLDS II

YOSHINOBU KAMISHIMA

Abstract. Long and Reid have shown that some compact flat 3-manifold cannot be diffeomorphic to a cusp cross-section of any complete finite volume 1-cusped hyperbolic 4-manifold. Similar to the flat case, we give a negative answer that there exists a 3-dimensional closed Heisenberg infranilmanifold whose diffeomorphism class cannot be arisen as a cusp cross-section of any complete finite volume 1-cusped complex hyperbolic 2-manifold. This is obtained from the formula by the characteristic numbers of bounded domains related to the Burns-Epstein invariant on strictly pseudo-convex CR-manifolds [1],[3]. This paper is a sequel of our paper [11].

Introduction

We shall consider whether every Heisenberg infranilmanifold can be arisen, up to diffeomorphism, as a cusp cross-section of a complete finite volume 1-cusped complex hyperbolic manifold. Long and Reid considered the problem that every compact Riemannian flat manifold is diffeomorphic to a cusp cross-section of a complete finite volume 1-cusped hyperbolic manifold. They have shown it is false for some compact flat 3-manifold [15]. We shall give a negative answer similarly to the flat case.

Theorem. Any 3-dimensional closed Heisenberg infranilmanifold with non-trivial holonomy cannot be diffeomorphic to a cusp cross-section of any complete finite volume 1-cusped complex hyperbolic 2-manifold.

McReynolds informed us that W. Neumann and A. Reid have obtained the similar result.

2. Heisenberg infranilmanifold

Let \( \langle z, w \rangle = \bar{z}_1 \cdot w_1 + \bar{z}_2 \cdot w_2 + \cdots + \bar{z}_n \cdot w_n \) be the Hermitian inner product defined on \( \mathbb{C}^n \). The Heisenberg nilpotent Lie group \( \mathcal{N} \) is the product \( \mathbb{R} \times \mathbb{C}^n \) with group law:

\[
(2.1) \quad (a, z) \cdot (b, w) = (a + b - \text{Im}(z, w), \ z + w).
\]

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It is easy to see that $\mathcal{N}$ is 2-step nilpotent, i.e., $[\mathcal{N}, \mathcal{N}] = (\mathbb{R}, 0) = \mathbb{R}$, which is the central subgroup $C(\mathcal{N})$ of $\mathcal{N}$. This induces a central group extension: $1 \to C(\mathcal{N}) \to \mathcal{N} \xrightarrow{P} \mathbb{C}^n \to 1$. Let $\text{Iso}(\mathbb{H}^{n+1}_C)$ be the full group of the isometries of the complex hyperbolic space $\mathbb{H}^{n+1}_C$. It is isomorphic to $\text{PU}(n + 1, 1) \rtimes \langle \tau \rangle$ where $\tau$ is the (anti-holomorphic) involution induced by the complex conjugation. The Heisenberg rigid motions is defined as a subgroup of the stabilizer $\text{Iso}(\mathbb{H}^{n+1}_C)$ at the point at infinity $\infty$.

**Definition 2.1.** The group of Heisenberg rigid motions $E^r(\mathcal{N})$ is defined to be $\mathcal{N} \times (U(n) \rtimes \langle \tau \rangle)$. A Heisenberg infranilmanifold (respectively orbifold) is a compact manifold (respectively orbifold) $\mathcal{N}/\pi$ such that $\pi$ is a torsionfree (not necessarily torsionfree) discrete cocompact subgroup of $E^r(\mathcal{N})$.

3. **CR-structure on $S^{2n-1} - S^{2n-1}$**

The sphere complement $S^{2n+1} - S^{2n-1}$ is a spherical $CR$ manifold with the transitive group $\text{Aut}_{CR}(S^{2n+1} - S^{2n-1})$ of $CR$ transformations which is isomorphic to the unitary Lorentz group $U(n, 1)$. Note that $S^{2n+1} - S^{2n-1}$ is identified with the $(2n + 1)$-dimensional Lorentz standard space form $V_{-1}^{2n+1}$ of constant sectional curvature $-1$. The center $ZU(n, 1)$ of $U(n, 1)$ is $S^1$. Then $V_{-1}^{2n+1}$ is the total space of the principal $S^1$-bundle over the complex hyperbolic space: $S^1 \to (U(n, 1), V_{-1}^{2n+1}) \xrightarrow{\nu} (\text{PU}(n, 1), \mathbb{H}^n_C)$. If $\omega_H$ is the connection form of the above principal bundle, then it is a contact form on $V_{-1}^{2n+1}$ such that $\text{Null } \omega_H$ is a $CR$ structure. Note that $d\omega_H = \nu^* \Omega_H$ up to constant factor for the Kähler form $\Omega_H$ on $\mathbb{H}^n_C$. Since $U(n, 1) = S^1 \cdot \text{SU}(n, 1)$, the above equivariant principal bundle induces the following commutative fibrations:

$\begin{align*}
\mathbb{Z} & \longrightarrow (\text{SU}(n, 1), \tilde{V}_{-1}^{2n+1}) \xrightarrow{\tilde{\nu}} (\text{PU}(n, 1), \mathbb{H}^n_C) \\
\downarrow & \downarrow \\
\mathbb{Z}/n + 1 & \longrightarrow (\text{SU}(n, 1), V_{-1}^{2n+1}) \xrightarrow{\nu} (\text{PU}(n, 1), \mathbb{H}^n_C).
\end{align*}$

(3.1)

Here $\tilde{\text{SU}}(n, 1)$ is a lift of $\text{SU}(n, 1)$ associated to the covering $\mathbb{Z} \to \tilde{V}_{-1}^{2n+1} \to V_{-1}^{2n+1}$. For a discrete subgroup $G \subset \text{PU}(n + 1, 1)$ such that $\mathbb{H}^{n+1}_C/G$ is a complete finite volume complex hyperbolic orbifold, let $\hat{G} \subset \tilde{\text{SU}}(n, 1)$ be a lift where $1 \to \mathbb{Z} \to \hat{G} \to G \to 1$ is an exact sequence. Then $S^1 \to \tilde{V}_{-1}^{2n+1}/\hat{G} \xrightarrow{\hat{\nu}} \mathbb{H}^{n+1}_C/G$ is an injective Seifert fibration (i.e. the singular fiber bundle with typical fiber is $S^1$. The exceptional fiber is also a circle.)

4. **BURNS AND EPSTEIN’S FORMULA**

In general, the Heisenberg infranilmanifold or its two fold cover at least admits a spherical $CR$-structure, see Definition 2.1. In [2], Burns and Epstein obtained the $CR$-invariant $\mu(M)$ on the 3-dimensional strictly pseudoconvex $CR$-manifolds $M$ provided that the holomorphic line bundle is trivial.
Let $X$ be a compact strictly pseudoconvex complex 2-dimensional manifold with smooth boundary $M$. Then they have shown the following equality in [3]:

$$\int_X c_2 - \frac{1}{3} c_1^2 = \chi(X) - \frac{1}{3} \int_X \bar{c}_1^2 + \mu(M).$$

Here $\bar{c}_1$ is a lift of $c_1$ by the inclusion $j^* : H^2(X, M; \mathbb{R}) \to H^2(X; \mathbb{R})$.

5. Geometric boundary

5.1. One-cusped complex hyperbolic 2-manifold. Let $E^r(\mathcal{N})$ be the group of Heisenberg rigid motions on the 3-dimensional Heisenberg nilpotent Lie group $\mathcal{N}$ and $L : E^r(\mathcal{N}) \to U(1) \rtimes \langle \tau \rangle$ the holonomy homomorphism. Suppose that $M = \mathcal{N}/\Gamma$ is realized as a cusp cross-section of a complete finite volume one-cusped complex hyperbolic 2-manifold $W = \mathbb{H}_\mathbb{C}^2/G$. Put $\bar{W} = \mathbb{H}_\mathbb{C}^2/G - M \times (0, \infty)$ so that $\partial \bar{W} = M$. Then $\bar{W}$ is homotopic to $W$ and $M$ is viewed as a boundary of $\text{Int}\bar{W}$ which supports a complete complex hyperbolic structure. The holonomy group $L(\Gamma)$ of a 3-dimensional compact Heisenberg non-homogeneous infranilmanifold $M = \mathcal{N}/\Gamma$ is a cyclic subgroup of order 2, 3, 4, 6 of $U(1)$ or $L(\Gamma)$ is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2 \subset U(1) \rtimes \langle \tau \rangle$, see [4, 10] for the classification. If $M$ has the holonomy $\mathbb{Z}/2 \times \mathbb{Z}/2$, then $G$ has nontrivial summand in $\langle \tau \rangle$ of $\text{Iso}(\mathbb{H}_\mathbb{C}^2) = PU(2, 1) \rtimes \langle \tau \rangle$.

The two fold cover $W/G \cap PU(2, 1)$ is still a one-cusped complex hyperbolic manifold for which the cusp cross-section is the two fold cover of $M$ whose holonomy group becomes $\mathbb{Z}/2 \subset U(1)$. When the holonomy group belongs to $U(1)$, the spherical $CR$-structure on $M$ is canonically induced from the complex hyperbolic structure on $W$. (Note that $\tau$ does not preserve the $CR$-structure bundle.)

5.2. Integral of $\bar{c}_1^2$. Let $p : \bar{W} \to W$ be the finite covering, say of order $\ell$, whose induced covering $\bar{M}$ of $M$ is now a (homogeneous) nilmanifold (using the separability argument if necessary). Possibly it consists of a finite number of such nilmanifolds. Since $W$ admits a complete Einstein-Kähler metric, we know that $c_2 - \frac{1}{3} c_1^2 = 0$. Moreover, since $\bar{M}$ is a spherical $CR$ manifold with trivial holomorphic line bundle, it follows that $\mu(\bar{M}) = 0$. As in [11], let $j^* : H^2(\bar{W}, M : \mathbb{R}) \to H^2(\bar{W} : \mathbb{R}) = H^2(W : \mathbb{R})$ be the map such that $j^* \bar{c}_1(\bar{W}) = c_1(W)$. Applying (4.1) to $\bar{W}$, we have $\chi(\bar{W}) = \frac{1}{3} \int_{\bar{W}} \bar{c}_1^2$.

As $p^* (\bar{c}_1(\bar{W})) = \bar{c}_1(\bar{W})$ by naturality and $p_* [\bar{W}] = \ell [\bar{W}]$, it follows that

$$\int_{\bar{W}} \bar{c}_1^2 = \langle \bar{c}_1^2(\bar{W}), [\bar{W}] \rangle = \langle \bar{c}_1^2(\bar{W}), [\bar{W}] \rangle.$$

Since $\chi(\bar{W}) = \ell \chi(W)$,

$$3 \chi(W) = \langle \bar{c}_1^2(\bar{W}), [\bar{W}] \rangle.$$

**Proposition 5.1.** If $M = \mathcal{N}/\Gamma$ is realized as a cusp cross-section of a complete finite volume one-cusped complex hyperbolic 2-manifold $W = \mathbb{H}_\mathbb{C}^2/G$, then $\bar{c}_1^2(\bar{W})$ is an integer in $H^4(\bar{W}, M : \mathbb{Z}) = \mathbb{Z}$. 
5.3. Torsion element in $M$. Given a CR-structure on $M$, there is the canonical splitting $TM \otimes \mathbb{C} = B^{1,0} \oplus B^{0,1}$ where $B^{1,0}$ is the holomorphic line bundle. Since $M$ is an infranilmanifold but not homogeneous, $B^{1,0}$ is nontrivial, i.e. $c_1(B^{1,0}) \neq 0$. (In fact, it is a torsion element in $H^2(M : \mathbb{Z})$, because the $\ell$-fold covering $\tilde{M}$ has the trivial holomorphic bundle.) The spherical CR manifold $M$ has a characteristic CR vector field $\xi$. If $e^1$ is the vector field on $M$ pointing outward to $W$, then the distribution $\langle e^1, \xi \rangle$ generates a trivial holomorphic line bundle $T\mathbb{C}^{1,0}$ on $M$ for which $TW \otimes \mathbb{C}|M = B^{1,0} + T\mathbb{C}^{1,0} \oplus B^{0,1} + T\mathbb{C}^{0,1}$. As $i^\ast(c_1(W)) = c_1(B^{1,0} + T\mathbb{C}^{1,0}) = c_1(B^{1,0})$ and $\ell \cdot c_1(B^{1,0}) = 0$, we have $j^\ast \beta = \ell \cdot c_1(W)$ for some integral class $\beta \in H^2(W, M : \mathbb{Z})$.

5.4. $H_1(M : \mathbb{Z})$. Let $1 \to \Delta \to \Gamma \to F \to 1$ be the group extension of the fundamental group $\Gamma = \pi_1(M)$ where $\Delta$ is the maximal normal nilpotent subgroup and $F \cong \mathbb{Z}_\ell$ ($\ell = 2, 3, 4, 6$) or $F \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Recall that $\Delta$ is generated by $\{a, b, c\}$ where $[a, b] = aba^{-1}b^{-1} = e^k$ for some $k > 0$. It follows that $\Delta/\langle \Delta, \Delta \rangle = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_k$. Let $\gamma$ be an element of $\Delta$ which maps to a generator of $\mathbb{Z}_\ell$. A calculation shows that mod $\langle \Delta, \Delta \rangle$,

\begin{align}
\gamma a\gamma^{-1} &= a^{-1}, \quad \gamma b\gamma^{-1} = b^{-1} \quad (\ell = 2), \\
\gamma a\gamma^{-1} &= b, \quad \gamma b\gamma^{-1} = a^{-1}b^{-1} \quad (\ell = 3), \\
\gamma a\gamma^{-1} &= b, \quad \gamma b\gamma^{-1} = a^{-1} \quad (\ell = 4), \\
\gamma a\gamma^{-1} &= b, \quad \gamma b\gamma^{-1} = a^{-1}b \quad (\ell = 6).
\end{align}

When $F = \mathbb{Z}_2 \times \mathbb{Z}_2$, let $\delta$ be an element of $\Gamma$ which goes to another generator of $F$. Then $\gamma\delta\gamma^{-1} = a$, $\gamma b\gamma^{-1} = b^{-1}$ mod $\langle \Delta, \Delta \rangle$. In view of the above relation (5.1), $\gamma$ (also $\delta$) becomes a torsion element of order $m$ in $\Gamma/\langle \Gamma, \Gamma \rangle$ where $m$ is divisible by $\ell$. As $\Gamma$ is generated by $\{a, b, c, \gamma\}$ or $\{a, b, c, \gamma, \delta\}$, it follows that

\begin{equation}
H_1(M : \mathbb{Z}) = \mathbb{Z}_k \oplus \mathbb{Z}_m \oplus \left\{ \begin{array}{l}
\mathbb{Z}_2 \oplus \mathbb{Z}_2 \quad (\ell = 2) \\
\mathbb{Z}_3 \quad (\ell = 3) \\
\mathbb{Z}_2 \quad (\ell = 4) \\
1 \quad (\ell = 6) \\
\mathbb{Z}_2 \oplus \mathbb{Z}_2 \end{array} \right\}.
\end{equation}

In any case, if $\mathcal{N}/\Gamma$ has a nontrivial holonomy group $F$, then $H_1(M : \mathbb{Z})$ is a torsion group.

5.5. Intersection number. Put $\tilde{H}^2(\tilde{W}, M : \mathbb{Z}) = H^2(\tilde{W}, M : \mathbb{Z})/\mathrm{Tor}$ where Tor is the torsion subgroup. We have a nondegenerate inner product $\tilde{H}^2(\tilde{W}, M : \mathbb{Z}) \times \tilde{H}^2(\tilde{W}, M : \mathbb{Z}) \to \mathbb{Z}$ defined by the intersection form

$$(x, y) = \langle x \cup y, [\tilde{W}] \rangle.$$ 

Denote by $\tilde{W} \# \pm \mathbb{C}P^2$ the connected sum of $\tilde{W}$ with $\mathbb{C}P^2 \# - \mathbb{C}P^2$ if necessary. We can assume that $(\ , \ )$ is an indefinite form of odd type, i.e..there are nonzero elements $x, y \in \tilde{H}^2(\tilde{W} \# \pm \mathbb{C}P^2, M : \mathbb{Z}) = \tilde{H}^2(\tilde{W}, M) + \langle 1 \rangle + \langle -1 \rangle$.
such that \((x, x)\) is odd and \((y, y) = 0\). By \((\pm 1)\) we shall mean that it is generated by either \(x_+\) or \(x_-\) of \(\check{H}^2(W \# \mathbb{C}P^2, M : \mathbb{Z})\) such that \((x_-, x_+) = \pm 1\) respectively. Moreover by the classification of nondegenerate indefinite inner product cf. [10], there is an isomorphism preserving the inner product from \(\check{H}^2(W \# \mathbb{C}P^2, M : \mathbb{Z})\) onto

\[
(5.3)\quad m(1) \oplus n(-1) = (1)_1 \oplus \cdots \oplus (1)_m \oplus (-1)_1 \oplus \cdots \oplus (-1)_n
\]

for \((m, n \neq 0)\). Here \((\pm 1)_i\) is the \(i\)-th copy of \((\pm 1)\). Consider the commutative diagram:

\[
\begin{array}{ccc}
H^2(W, M : \mathbb{Z}) & \xrightarrow{\partial} & H^2(W : \mathbb{Z}) \\
\downarrow D & & \downarrow D \\
H_2(W : \mathbb{Z}) & \xrightarrow{j_*} & H_2(W, M : \mathbb{Z}) & \xrightarrow{\partial} & H_1(M : \mathbb{Z}).
\end{array}
\]

It follows from \((5.2)\) that \(j_* : H_2(W : \mathbb{Z}) \to H_2(W, M : \mathbb{Z})\) is injective and is isomorphic if \(\mathbb{Z}\) replaces \(\mathbb{R}\). Similarly note that \(j_* : \check{H}_2(W \# \mathbb{C}P^2 : \mathbb{Z}) \to H_2(W \# \mathbb{C}P^2, M : \mathbb{Z})\) is injective (and an isomorphism for the coefficient \(\mathbb{R}\)). Identified the generators of \(H_2(W \# \mathbb{C}P^2 : \mathbb{Z})\) with the basis \((5.3)\) of \(\check{H}_2(W \# \mathbb{C}P^2, M : \mathbb{Z})\), we may choose the generators \([V_i] \in H_2(W \# \mathbb{C}P^2, M : \mathbb{Z})\) such that

\[
(5.5)\quad j_*((\pm 1)_i) = \ell_i[V_i]
\]

for some \(\ell_i \in \mathbb{Z}\).

5.6. **Canonical bundle.** The circle (line) bundle \(L : S^1 \to \check{V}_1 / \hat{\mathbb{C}}^2 / G = W\) is represented by the Kähler form \(\Omega\) of the Kähler-hyperbolic metric, i.e. \([\Omega] = c_1(L) \in H^2(W : \mathbb{Z})\). Hence \(W = \hat{\mathbb{C}}^2 / G\) is projective-algebraic, i.e. \(W \subset \mathbb{C}P^N\) so \(c_1(W)\) can be represented by \(c_1([V])\) for some divisor \(V\) in \(W\), i.e. \(D(c_1(W)) = [V] \in \check{H}_2(W, M : \mathbb{Z})\), compare [7]. Embed \(V\) into \(W \# \mathbb{C}P^2\) and suppose that

\[
[V] = \sum_i a_i[V_i] \in \check{H}_2(W \# \mathbb{C}P^2, M : \mathbb{Z}).
\]

As \(D \circ i^*c_1(W) = \partial[V]\), it follows \(\ell\partial([V]) = 0\) by the argument of 5.5.3. We observe that \(\partial[V]\) maps into \(Z_m\) in \(H_1(M : \mathbb{Z})\) (cf. (5.2)) and so does each \(\partial[V_i]\). It may occur that \(\partial a_i[V_i] = \partial a_j[V_j]\) for some \(i, j\). So we can write \([V] = k[V_1] + j*\) where \(x \in H_2(W \# \mathbb{C}P^2 : \mathbb{Z})\) and \(V_1\) satisfies that

1. \(\partial V_1 = S^1\) and \(\ell[S^1] = 0\) in \(Z_m \subset H_1(M : \mathbb{Z})\).
2. \(\ell\) is minimal with respect to (1).
3. \((k, l)\) is relatively prime.

5.7. **Realization of \(\check{c}_1\).** As \(\ell\partial[V_1] = 0\) in \(H_1(M : \mathbb{Z})\), there is a surface \(U\) in \(W\) whose cycle \([U]\) \(\in H_2(W \# \mathbb{C}P^2 : \mathbb{Z})\) represents \(j_*[U] = \ell[V_1]\).

Let \([U] = a_1(\pm 1)_1 + a_2(\pm 1)_2 + \cdots + a_s(\pm 1)_s\). Then, \(\ell[V_1] = a_1\ell_1[V_1] + a_2\ell_2[V_2] + \cdots + a_s\ell_s[V_s]\). Since each \([V_i]\) is a generator of \(H_2(W \# \mathbb{C}P^2, M : \mathbb{Z})\), it follows that \(\ell = a_1\ell_1\) and \(a_j = 0\) \((j \neq 1)\). Hence \([U] = a_1(\pm 1)_1\). On
the other hand, note that \( \langle \pm 1 \rangle_1 \) is a cycle of \( \tilde{H}^2(\tilde{W} \# \pm \mathbb{CP}^2, M : \mathbb{R}) \) for which \( j_*(\langle \pm 1 \rangle_1) = \ell_1[V_1] \) by (5.5). Noting that \( \ell \) is minimal by Property (2) of \( \S § 5.6 \) \( \ell_1 \) is divisible by \( \ell \). Therefore \( \ell_1 = \pm \ell \) and \( a_1 = \pm 1 \) so that \([U] = \pm \langle \pm 1 \rangle \). In particular, the intersection number
\[
(U) \cdot [U] = 1.
\]

Put \( y = \frac{k}{\ell} [U] + x \in H_2(\tilde{W} \# \pm \mathbb{CP}^2 : \mathbb{R}) \). Calculate
\[
y \cdot y = \frac{k^2}{\ell^2} [U] \cdot [U] + \frac{2k}{\ell} [U] \cdot x + x \cdot x
\]
\[
= \pm \frac{k^2}{\ell^2} + \frac{2k}{\ell} [U] \cdot x + x \cdot x,
\]

(5.7)
\[
\ell(y \cdot y) = \pm \frac{k^2}{\ell} \mod \mathbb{Z}
\]

Noting that \((k, l) = 1\) by Property (3) of \( \S § 5.6 \) if \( \ell \neq 1 \), \( y \cdot y \) cannot be an integer.

As \( j_* \frac{k}{\ell} [U] = k[V_1] \), note that \( j_* y = k[V_1] + j_* x = [V] \). Consider the following diagram:
\[
\tilde{H}^2(\tilde{W} \# \pm \mathbb{CP}^2 : \mathbb{R}) \xrightarrow{j^*} \tilde{H}^2(\tilde{W} \# \pm \mathbb{CP}^2, M : \mathbb{R})
\]

(5.8)
\[
H_2(W : \mathbb{R}) + \langle 1 \rangle + \langle -1 \rangle \xrightarrow{j_* + \text{id}} H_2(W, M : \mathbb{R}) + \langle 1 \rangle + \langle -1 \rangle.
\]

Let \( y = y_0 + t(1) + s(-1) \) for some \( y_0 \in \tilde{H}_2(\tilde{W} : \mathbb{R}), s, t \in \mathbb{R} \). As \( j_* y = [V] \), it follows that \([V] = j_* y_0 + t(1) + s(-1) \). Noting \([V] \in H_2(W, M : \mathbb{Z})\), we have that \([V] = j_* y_0 \) and \( t = s = 0 \). In particular, this implies that \( y = y_0 \in \tilde{H}_2(\tilde{W} : \mathbb{R}) \). Using the commutative diagram (5.4) and by the fact \( D(c_1(W)) = [V] \), the element \( D^{-1}(y) \in H^2(\tilde{W}, M : \mathbb{R}) \) satisfies that \( j^*(D^{-1}(y)) = c_1(W) \).

On the other hand, recall from the argument of \( [3] \) that the integral \( \langle c_1^2(\tilde{W}), [\tilde{W}] \rangle \) does not depend on the choice of lift \( c_1(\tilde{W}) \) to \( c_1(W) \), so we can choose \( c_1(\tilde{W}) = D^{-1}(y) \in H^2(\tilde{W}, M : \mathbb{R}) \) (cf. \( \S § 5.2 \)). By definition, \( y \cdot y = \langle c_1(W)^2, [W] \rangle \) which is an integer by Proposition (5.1). This contradiction proves \textbf{Theorem}.

\textbf{Remark 5.2.} Neumann and Reid have shown that if an infranil 3-manifold arises as a cusp cross-section of a 1-cusped complex hyperbolic 2-manifold, then the rational Euler number must be 1/3-integral. There are infranilmanifolds which do not satisfy this condition.

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