Results on the intersection graphs of subspaces of a vector space

N. Jafari Rad and S. H. Jafari

Department of Mathematics,
Shahrood University of Technology,
Shahrood, Iran
n.jafarirad@shahroodut.ac.ir

Abstract

For a vector space $V$ the intersection graph of subspaces of $V$, denoted by $G(V)$, is the graph whose vertices are in a one-to-one correspondence with proper nontrivial subspaces of $V$ and two distinct vertices are adjacent if and only if the corresponding subspaces of $V$ have a nontrivial (nonzero) intersection. In this paper, we study the clique number, the chromatic number, the domination number and the independence number of the intersection graphs of subspaces of a vector space.

Keywords: Vector space, Subspace, Dimension, Intersection graph, Graph, Domination, Clique, Independence, Matching.

1 Introduction

For graph theory terminology in general we follow [10]. Specifically, let $G = (V, E)$ be a graph with vertex set $V$ of order $n$ and edge set $E$. If $S$ is a subset of $V(G)$, then we denote by $G[S]$ the subgraph of $G$ induced by $S$. A set of vertices $S$ in $G$ is a dominating set, if $N[S] = V(G)$. The domination number, $\gamma(G)$, of $G$ is the minimum cardinality of a dominating set of $G$. A
set of vertices $S$ in $G$ is an independent set, if $G[S]$ has no edge. The independence number, $\alpha(G)$, of $G$ is the maximum cardinality of an independent set of $G$. The clique number of a graph $G$, written $\omega(G)$, is the maximum size of a set of pair-wise adjacent vertices of $G$. A function $f$ defined on $V(G)$ is a proper vertex coloring if $f(v) \neq f(u)$ for any pair of adjacent vertices $u, v$. The (vertex) chromatic number, $\chi(G)$, of $G$ if the minimum $k$ such that there is a proper vertex coloring $f$ on $G$ with $|f(V(G))| = k$.

Let $F = \{S_i : i \in I\}$ be an arbitrary family of sets. The intersection graph $G(F)$ is the one-dimensional skeleton of the nerve of $F$, i.e., $G(F)$ is the graph whose vertices are $S_i$, $i \in I$ and in which the vertices $S_i$ and $S_j$ ($i, j \in I$) are adjacent if and only if $S_i \neq S_j$ and $S_i \cap S_j \neq \emptyset$ [9].

The study of algebraic structures using the properties of graphs has become an exciting research topic in the last few decades years, leading to many fascinating results and questions. It is interesting to study the intersection graphs $G(F)$ when the members of $F$ have an algebraic structure. For references of intersection graphs of algebraic structures see for example [1, 3, 4, 11].

Intersection graphs of subspaces of a vector space are studied by Jafari Rad and Jafari in [6, 7]. For a vector space $V$ the intersection graph of subspaces of $V$, denoted by $G(V)$, is the graph whose vertices are in a one-to-one correspondence with proper nontrivial subspaces of $V$ and two distinct vertices are adjacent if and only if the corresponding subspaces of $V$ have a nontrivial (nonzero) intersection. Clearly the set of vertices is empty if $\dim(V) = 1$. Jafari Rad and Jafari characterized all vector spaces whose intersection graphs are connected, bipartite, complete, Eulerian, or planar.

In this paper, we continue the study of the intersection graph of subspaces of a vector space. We study the chromatic number, the clique number, the domination number, and the independence number in the intersection graph of subspaces of a vector space. Throughout this paper $V$ is a vector space with $\dim(V) = n$ on a finite field $F$ with $|F| = q$. We also denote by $0$ the zero subspace of a vector space.
2 Known results

Let $F$ be a finite field with $|F| = q$, and let $V$ be an $n$-dimensional vector space over $F$. For integer $t \in \{1, 2, ..., n\}$, the number of $t$-dimensional subspaces of $V$ is given in [5] by \[
{n \choose t}_q = \prod_{0 \leq i < t} q^{n-i-1}.\] We suppose that \[
{n \choose 0}_q = 1 \quad \text{and} \quad {n \choose t}_q = 0 \quad \text{if} \quad t \notin \{0, 1, 2, ..., n\}. \] Note that \[
{n \choose t}_q = \left[\frac{n}{n-t}\right]_q \quad \text{for any} \quad t \in \{0, 1, ..., n\}.
\]

Lemma 1 (Jafari Rad and Jafari [7]) If $\dim(W) = m$, then \[
|\{W' : \dim(W') = t, W \cap W' = 0\}| = q^{mt} {n-m \choose t}_q.
\]

Theorem 2 (Jafari Rad and Jafari [7]) If $\dim(W) = m$, then \[
\deg(W) = \sum_{t=0}^{n} {n \choose t}_q - \sum_{t=0}^{n-m} q^{mt} {n-m \choose t}_q - 2.
\]

Theorem 3 (Jafari Rad and Jafari [7]) Let $V$ be a vector space. Then $G(V)$ is connected if and only if $\dim(V) \geq 3$.

We next state some known results of graph theory.

Theorem 4 (Hall’s Marriage Theorem) For $k > 0$, every $k$-regular bipartite graph has a perfect matching.

Lemma 5 ([10]) For every graph $G$, $\chi(G) \geq w(G)$.

Theorem 6 (Brooks, [2]) If $G$ is a connected graph other than a complete graph or an odd cycle, then $\chi(G) \leq \Delta(G)$. 

3
3 New results

Theorem 7 If $n$ is odd, then

$$w(G(V)) = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{i}_q$$

Proof. First notice that if $W_1, W_2$ are two subspaces of $V$ with $\dim(W_i) \geq \frac{n}{2}$ for $i = 1, 2$, then $W_1 \cap W_2 \neq 0$. This means that

$$w(G(V)) \geq |\{W : W \leq V, \dim(W) \geq \frac{n}{2}\}| = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{i}_q.$$

Now we show that $w(G(V)) \leq |\{W : W \leq V, \dim(W) \geq \frac{n}{2}\}|$. Let $\mathcal{A}_t$ be the set of all $t$-dimensional subspaces of $V$ with $1 \leq t \leq n-1$. For $1 \leq t \leq \frac{n}{2}$, let $H_t$ be the induced bipartite subgraph of $G(V)$ with partite sets $\mathcal{A}_t$ and $\mathcal{A}_{n-t}$. Let $\overline{H}_t$ be the complement of $H_t$ in $K_{|\mathcal{A}_t|,|\mathcal{A}_{n-t}|}$. By Lemma 1, $\overline{H}_t$ is a $k$-regular graph with $k = q^{(n-t)}$. By Theorem 4, $\overline{H}_t$ has a perfect matching $M_t$. We deduce that $\chi(G(V)) \leq |\{W : W \leq V, \dim(W) \geq \frac{n}{2}\}|$. By Lemma 5, $w(G(V)) \leq \chi(G(V)) \leq |\{W : W \leq V, \dim(W) \geq \frac{n}{2}\}|$. This completes the proof.

Corollary 8 If $n$ is odd, then $\chi(G(V)) = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{i}_q$.

Theorem 9 If $n$ is even then

$$\sum_{i=1}^{\frac{n-1}{2}} \binom{n}{i}_q + \binom{n-1}{\frac{n-2}{2}}_q \leq w(G(V)) \leq \sum_{i=1}^{\frac{n-1}{2}} \binom{n}{i}_q + \binom{n}{\frac{n}{2}}_q - q^{\frac{n^2}{4}} - 1.$$ 

Proof. First notice that if $W_1, W_2$ are two subspaces of $V$ with $\dim(W_i) > \frac{n}{2}$ for $i = 1, 2$, then $W_1 \cap W_2 \neq 0$. Also if $W_1, W_2$ are two subspaces of $V$ with $\dim(W_1) = \frac{n}{2}$ and $\dim(W_2) > \frac{n}{2}$, then $W_1 \cap W_2 \neq 0$. But for a 1-dimensional subspace $V_1$ of $V$, any subspace of $V_1$ of dimension $\frac{n-2}{2}$ is in the form $\frac{W}{V_1}$.
where \( \text{dim}(W) = \frac{n}{2} \). So there are at least \( \left\lfloor \frac{n-1}{\frac{n-2}{2}} \right\rfloor_q \) subspaces of dimension \( \frac{n}{2} \) which contain \( V_1 \). We deduce that

\[
w(G(V)) \geq |\{W : W \leq V, \text{dim}(W) > \frac{n}{2}\}| + \left\lfloor \frac{n-1}{\frac{n-2}{2}} \right\rfloor_q\]

\[
\geq \sum_{i=1}^{\frac{n}{2}-1} \left\lfloor \frac{n}{i} \right\rfloor_q + \left\lfloor \frac{n-1}{\frac{n-2}{2}} \right\rfloor_q.
\]

Let \( A_t \) be the set of all \( t \)-dimensional subspaces of \( V \) with \( 1 \leq t \leq n-1 \). Let \( G_1 = G[A_{\frac{n}{2}}] \) and \( G_2 = G - G_1 \). As in the proofs of Theorem 7 and Corollary 8, we obtain that \( G_2 \) is \( k \)-regular with \( k = q^{\frac{n}{2}(n-\frac{2}{2})} = q^{\frac{n^2}{4}} \) and \( \chi(G_2) = \sum_{i=1}^{\frac{n}{2}-1} \left\lfloor \frac{n}{i} \right\rfloor_q \). Since \( G_1 \) is not a complete graph or an odd cycle, by Theorems 3 and 6, \( \chi(G_1) \leq \Delta(G_1) = \left\lfloor \frac{n}{\frac{n}{2}} \right\rfloor_q - q^{\frac{n^2}{4}} - 1 \). So \( \chi(G) \leq \sum_{i=1}^{\frac{n}{2}-1} \left\lfloor \frac{n}{i} \right\rfloor_q + \left\lfloor \frac{n}{\frac{n}{2}} \right\rfloor_q - q^{\frac{n^2}{4}} - 1 \). Now the results follows by Lemma 5.

**Corollary 10** If \( n \) is even then

\[
\sum_{i=1}^{\frac{n}{2}-1} \left\lfloor \frac{n}{i} \right\rfloor_q + \left\lfloor \frac{n-1}{\frac{n-2}{2}} \right\rfloor_q \leq \chi(G(V)) \leq \sum_{i=1}^{\frac{n}{2}-1} \left\lfloor \frac{n}{i} \right\rfloor_q + \left\lfloor \frac{n}{\frac{n}{2}} \right\rfloor_q - q^{\frac{n^2}{4}} - 1.
\]

**Theorem 11** \( \gamma(G(V)) = q + 1 \).

**Proof.** Let \( W \) be a subspace of \( V \) with \( \text{dim}(W) = n-2 \). It follows that \( \frac{V}{W} \) has \( q+1 \) subspaces \( \frac{W_1}{W}, \frac{W_2}{W}, ..., \frac{W_{q+1}}{W} \) with \( \text{dim}(\frac{W_i}{W}) = 1 \) for \( i = 1, 2, ..., q+1 \). It is obvious that

\[
\frac{V}{W} = \frac{W_1}{W} \cup \frac{W_2}{W} \cup ... \cup \frac{W_{q+1}}{W}.
\]

Now we can see that

\[
V = W_1 \cup W_2 \cup ... \cup W_{q+1}.
\]

This means that \( \{W_1, W_2, ..., W_{q+1}\} \) is a dominating set for \( G(V) \) and so \( \gamma(G(V)) \leq q + 1 \). On the other hand suppose that \( S = \{V_1, V_2, ..., V_t\} \) is a
minimum dominating set for $G(V)$. If there is a vector $x \not\in (V_1 \cup V_2 \cup \ldots \cup V_t)$, then $\langle x \rangle$ is not dominated by $S$. This contradiction implies that $V_1 \cup V_2 \cup \ldots \cup V_t = V$. Then

$$q^n = |\cup_{i=1}^t V_i| < \sum_{i=1}^t |V_i| \leq \sum_{i=1}^t q^{n-1} - tq^{n-1}.$$  

We deduce that $t \geq q + 1$, and so $\gamma(G(V)) \geq q + 1$. This completes the proof.

**Theorem 12** $\alpha(G(V)) = \frac{q^n - 1}{q - 1}$.

**Problem 13** What is the exact value of $w(G(V))$ for even $n$?

**References**

[1] Bosak, J., *The graphs of semigroups*, in: Theory of Graphs and Application, Academic Press, New York, 1964, 119-125.

[2] Brooks, R. L., *On colouring the nodes of a network*, Proc. Cambridge Phil. Soc. 37 (1941), 194-197.

[3] Chakrabarty, I., Ghosh, S., Mukherjee, T. K., and Sen, M. K., *Intersection graphs of ideals of rings*, Discrete Mathematics 309 (2009), 5381-5392.

[4] Cskny, B., Pollk, G., *The graph of subgroups of a finite group*, Czechoslovak Math. J. 19 (1969), 241-247.

[5] Frankl, P., and Graham, R. L., *Intersection theorems for vector spaces*, European Journal of Combinatorics, 6 (1988), 183-187.

[6] Jafari Rad, N., *Intersection graphs of subspaces of a vector space*, The Second Conference on Algebraic Combinatorics, Ferdowsi University of Mashhad, Iran, May 2010.

[7] Jafari Rad, N. and Jafari, S. H., *Graphs of subspaces of a vector space*, Submitted for publication (2009).
[8] Kenneth, H., and Ray, K., *Linear Algebra* Second edition Prentice-Hall, Inc., Englewood Cliffs, N.J. (1971).

[9] Szpilrajn-Marczewski, E., *Sur deux propriétés des classes d’ensembles*, Fund. Math. 33 (1945), 303-307.

[10] West, D. B., *Introduction To Graph Theory*, Prentice-Hall of India Pvt. Ltd, 2003.

[11] Zelinka, B., Intersection graphs of finite abelian groups, Czechoslovak Math. J. 25 (2) (1975), 171-174.