MODELING STUDENT ENGAGEMENT USING OPTIMAL CONTROL THEORY

DEBRA LEWIS

Mathematics Department, University of California Santa Cruz
Santa Cruz, CA 95064, USA

Dedicated to Professor Anthony Bloch on the occasion of his 65th birthday.

Abstract. Student engagement in learning a prescribed body of knowledge can be modeled using optimal control theory, with a scalar state variable representing mastery, or self-perceived mastery, of the material and control representing the instantaneous cognitive effort devoted to the learning task. The relevant costs include emotional and external penalties for incomplete mastery, reduced availability of cognitive resources for other activities, and psychological stresses related to engagement with the learning task. Application of Pontryagin’s maximum principle to some simple models of engagement yields solutions of the synthesis problem mimicking familiar behaviors including avoidance, procrastination, and increasing commitment in response to increasing mastery.

1. Introduction. In the academic community, there is a widespread tendency to regard students’ procrastination and avoidance as illogical behavior to be lamented, shrugged off, or perhaps abated by means of scaffolding and inspiring pedagogy. Assertions such as “as a form of self-regulation failure, procrastination has a great deal to do with short-term mood repair and emotion regulation” [26] suggest that students with properly functioning self-regulation and generally good moods don’t procrastinate, but low engagement is sufficiently common that regarding such behavior as a bug, rather than a feature, is problematic.

Advances in neuroscience, including fMRI studies of brain activity immediately prior to and during stressful cognitive tasks, in combination with the increasing prevalence of machine learning and artificial intelligence in diverse contexts, have led to optimization-based models of cognitive performance. Models based on value of computation—the difference between the expected gain resulting from a computation and the expected cost of carrying out that computation—have been successfully ported from their origins in machine learning to predictive models of human cognition in various tasks. (See, e.g., [11].) The authors of an in-depth survey [25] of recent research on cognitive effort as a reward-based choice assert that we are constitutively reluctant to mobilize all available cognitive resources. That is, mental effort is inherently aversive or costly... incentives are found regularly to improve cognitive performance, suggesting that individuals can increase their control allocation when higher

2020 Mathematics Subject Classification. Primary: 49N90, 91E40, 93B50.
Key words and phrases. Optimal control theory, learning models, cognitive resources, Pontryagin’s maximum principle, Hamiltonian dynamics.

* Corresponding author: Debra Lewis.
incentives are on offer (i.e., they are not constrained by ability) but hold back from doing so.

Psychological and social factors such as math anxiety and stereotype threat make some cognitive tasks torturous for many students. Experiments using fMRI have validated the colloquial characterization of various psychological or social stressors (e.g., rejection) as painful, in that regions of the brain associated with physical pain and injury perception show significant increases in activity when these psychological stresses are induced in human experimental subjects. (See, e.g., [17].) In particular, among individuals who identify as math anxious, anticipation of a math problem has been shown to trigger increased activity in these pain-processing regions [7]; this finding offers a plausible partial explanation for the relationship between math anxiety and math avoidance [6].

Anxiety-inducing learning activity can carry other physical, mental, and social penalties. Stress typically raises cortisol levels, which in turn negatively influences working memory and innovation in problem solving, leading to reduced performance in many cognitive tasks [19]. Hence cognitive effort invested in an anxiety-inducing learning task not only depletes a valuable and limited resource, but can significantly degrade the efficacy of cognition in other tasks. For students who identify with groups burdened with negative stereotypes regarding their abilities pertinent to the task at hand, stereotype threat is another potential cost of engagement [18]. Stereotype vigilance can impair working memory and increase susceptibility to other stressors such as math anxiety; repeated activation of stereotype vigilance can lead to alienation and avoidance [24].

We develop some simple optimal control models intended to capture key aspects of the interactions described above. The instantaneous cost functions used here combine three terms:

- a solely state-dependent term that models the costs (both internal and external) of the current state,
- a solely control-dependent term that models ‘life balance’ costs—the time and cognitive effort invested in studying isn’t available for other purposes (e.g. job, child care, recreation)
- a term modeling the psychological costs of studying; e.g., anxiety and stress resulting from low self-efficacy [4].

Interventions intended to benefit students by reducing one of these cost components can indirectly influence the other components. For example, a scholarship could reduce a student’s life balance costs, but alter the psychological stakes—if the student interprets the scholarship as indicative of their recognized abilities and potential, it could increase their self-efficacy and make studying less stressful; on the other hand, if the student has low self-efficacy, fears of failure and consequent loss of the scholarship—guilt over “letting down” the awarding agency if the funding continues despite a period of poor performance—could make both low mastery and studying more stressful, and hence more costly. Even highly simplified models can provide insight into the range of likely consequences of interventions, and guide targeted support of at risk students.

2. Pontryagin’s maximum principle and the synthesis problem. Pontryagin’s maximum principle [23] relates optimal control to Hamiltonian dynamics. The appeal of this approach lies primarily in its constructive nature—well-known results and techniques for boundary value problems and Hamiltonian dynamics can be used
to construct the pool of possibly optimal trajectories. We will consider general solutions of the Hamiltonian systems associated to optimal control problems, not just the globally optimal ones. In particular, in some parts of our analysis we will drop the usual end condition specifying the terminal values of the state variables, reflecting the reality that many students never reach the externally imposed content mastery goal.

The set $S$ of possible states of a control system is typically a manifold (possibly with boundary), and the admissible control values for a given state value $m$ are elements of the state-dependent admissible control region $A_m$ for $m$. (See, e.g., [5, 28]). The set $A$ of admissible state and control pairs is given by

$$A := \{(m, u) : m \in S \text{ and } u \in A_m\}$$

and the evolution of the state variable $m$ with respect to the independent variable $t$ (e.g. time) is determined by a controlled vector field $X$, i.e. a map $X : A \to TS$ satisfying $X(m, u) \in T_mS$ for all $(m, u) \in A$.

Given a controlled vector field $X$, state boundary data $m_0$ and $m_f$, and specified duration $t_f \in \mathbb{R}^+$, we denote by $A_X(m_0, m_f, t_f)$ the set of curves $(m, u) : [0, t_f] \to A$ satisfying

1. $m$ is differentiable,
2. $u$ is piecewise continuous,
3. $m'(t) = X(m, u)$, $m(0) = m_0$, and $m(t_f) = m_f$.

Given an instantaneous cost function $C \in C^0(A, \mathbb{R})$, a solution of the fixed time optimal control problem for $X$, boundary data $m_0$ and $m_f$, and duration $t_f$ is an element $(m_*, u_*)$ of $A_X(m_0, m_f, t_f)$ satisfying

$$\int_0^{t_f} C(m_*(t), u_*(t))dt \leq \int_0^{t_f} C(m(t), u(t))dt$$

for all $(m, u) \in A_X(m_0, m_f, t_f)$. Solutions of the corresponding free end-time optimal control problem minimize the total cost over

$$A_X(m_0, m_f) := \bigcup_{t_f \in \mathbb{R}^+} A_X(m_0, m_f, t_f),$$

with the total cost of each element of $A_X(m_0, m_f)$ being obtained by integration over that element’s domain.

The cost function $C$ and controlled vector field $X$ can be used to construct a time-dependent Hamiltonian system on the cotangent bundle $T^*S$ of the state space, where the time dependence arises through a choice of control curve. We briefly describe the construction of the associated system of differential equations in the special case that $S$ is an open subset of a finite dimensional vector space $V$, identifying the tangent bundle $TS$ with its natural trivialization $S \times V$. If the base point $m$ is clear in context, we will denote elements of $TS$ simply as elements of $V$. In particular, we will identify the controlled vector field $X$ with a map from $A$ to $V$. We assume that there exists $\ell \in \mathbb{N}$ such that for every $m \in S$, the associated admissible control region $A_m$ is a compact subset of $\mathbb{R}^\ell$. See, e.g., [5] for the analogous constructions and results in more general settings.

If we regard the controlled vector field $X$ as taking values in $V$ and define

$$B := \{(m, \lambda, u) : (m, u) \in A \text{ and } \lambda \in V^*\},$$

we can define $\widetilde{H} : B \to \mathbb{R}$ by

$$\widetilde{H}(m, \lambda, u) := \lambda(X(m, u)) - C(m, u).$$

(2)
We assume that
1. \( X \in C^1(\mathcal{A}, V) \),
2. \( C \) is continuously differentiable with respect to the state variable \( m \), and
3. \( C \) is continuously differentiable with respect to the control \( u \) on \( \mathcal{A} \setminus (\bigcup_{m \in \mathcal{S}} \partial \mathcal{A}_m) \), with well-defined limiting behavior of \( \frac{\partial C}{\partial u}(m, u) \) as \( u \) approaches \( \partial \mathcal{A}_m \); if \( \frac{\partial C}{\partial u}(m, u_*) \) does not exist at \( u_* \in \partial \mathcal{A}_m \), \( \lim_{u \to u_*} \| \frac{\partial C}{\partial u}(m, u) \| = \infty \).

Allowing blow up of the partial derivative of the cost function with respect to the control permits costs that discourage attainment of control values on the boundary of the admissible control regions without incentivizing consistently miserly control investment. We will illustrate this approach in subsequent sections.

Given \( (m, u) \in \mathcal{A}_X(m_0, m_f, t_f) \) satisfying (1), Pontryagin’s maximum principle implies that there is an associated curve \( \lambda : [0, t_f] \to V^* \) such that \( \tilde{z} := (m, \lambda, u) : [0, t_f] \to B \) satisfies
\[
\begin{align*}
\dot{m}(t) &= \frac{\partial \tilde{H}}{\partial \lambda}(\tilde{z}(t)) \\
\dot{\lambda}(t) &= -\frac{\partial \tilde{H}}{\partial m}(\tilde{z}(t))
\end{align*}
\]
(3)
where \( \frac{\partial \tilde{H}}{\partial \lambda} \) takes values in \( V \), and
\[
\tilde{H}(\tilde{z}(t)) = \max_{u \in \mathcal{A}(m(t))} \tilde{H}(m(t), \lambda(t), u)
\]
(4)
for \( 0 \leq t \leq t_f \). The optimal control value \( u(t) \) at time \( t \) either satisfies the criticality condition
\[
\frac{\partial \tilde{H}}{\partial m}(m(t), \lambda(t), u(t)) = 0
\]
(5)
or lies on the boundary of the admissible control region \( \mathcal{A}(m(t)) \).

Curves \( \tilde{z} \) satisfying (3) and (4) are solutions of the synthesis problem [23] determined by \( X \) and \( C \) if \( \tilde{H}(\tilde{z}(t_f)) = 0 \). Solutions yielding nonzero values of \( \tilde{H}(\tilde{z}(t_f)) \) are solutions of the corresponding fixed time synthesis problem. Pontryagin’s conditions are necessary, but not sufficient, for optimality; solutions of the synthesis problem need not be globally optimal solutions of the control problem. We will focus on the synthesis problem in developing and analyzing our models, regarding it as determining a dynamical system and studying the associated initial value problems.

**Remark 1.** An elementary but influential application of optimal control theory to economics determines decisions optimizing profit over time; the scalar state variable is the current capital, the control encodes decisions influencing changes in capital, the cost is the negation of profit, and the auxiliary variable \( \lambda \) is the marginal value of capital. In Dorfman’s seminal work on optimal control in economics, the criticality condition (5) for this application has the following interpretation: “The firm should choose [the control] at every moment so that the marginal immediate gain just equals the marginal long-run cost” [8]. The mastery-based models considered in §4.1–§6 can be loosely interpreted as academic morphs of this economic control system, with subject mastery serving as capital.

### 3. Biased moderation incentives.
We now develop a family of psychological study cost functions \( \psi \) intended to qualitatively capture aversion to cognitive effort investment in the learning task at hand, using cost terms rewarding sub-maximal control effort in time-critical control systems. Such cost modifications, called moderation incentives, were developed to capture the trade-offs for a rational biological agent between avoidance of potentially fatal situational hazards and long-term harm.
resulting from overtaxed muscles and joints [15]. For example, ‘bang-bang’ solutions for a biomechanical system, regarded as a mathematical idealization of an extremely rapid transition, may carry a significantly higher risk of minor musculoskeletal damage than solutions assuming more slowly varying, smaller magnitude control values. On the other hand, a close approach to the boundary of the admissible control region may be appropriate when making the best of a bad situation—a sprain or microfracture sustained in escaping a life-threatening emergency usually isn’t a fate worse than death.

3.1. Moderation incentives. We briefly review the cost terms used in [15] to model reluctance to push a physical system—particularly a biomechanical system—to its limits in modified time minimization problems [23, 13]. Moderated cost functions are constructed by subtracting from a cost function modeling a ‘do or die’ approach a term equaling zero on the boundary of the admissible control region and taking nonnegative values on the interior of those regions. Selection of an appropriate moderating term is essential; a penalty function taking extremely large values near the boundaries of the admissible control regions may yield unrealistically leisurely solutions, while very low near-boundary costs may result in crisis-level responses in almost all situations.

Given an admissible space $\mathcal{A}$, $\bar{C} \in C^0(\mathcal{A}, [0, \infty))$ is a moderation incentive for $\mathcal{A}$ if for all $m \in S$

1. $u \in \partial \mathcal{A}_m$ implies $\bar{C}(m, u) = 0$ and
2. $\bar{C}$ satisfies the cost differentiability conditions given in §2.

If the unmoderated system has cost function $\bar{C}$, the cost function for the moderated system is $C = \bar{C} - \bar{C}$. Thus the moderation incentive favors control values on the interior of each admissible control region $\mathcal{A}_m$. Note that a moderation incentive is not required to have a finite derivative on $\partial \mathcal{A}_m$; if $\bar{C} \in C^1(\mathcal{A}, \mathbb{R})$, unbounded growth of $\bar{C}^{\frac{\partial}{\partial u}}$ as the control $u$ approaches $\partial \mathcal{A}_m$ can ensure that control values satisfying the control criticality condition (5) lie in the interior of $\mathcal{A}_m$.

Remark 2. In [15], the unmoderated systems have purely state-dependent cost functions, while the learning systems considered here include a control-dependent life balance term in the unmoderated cost function. Moderation incentives can be added to systems with unmoderated costs depending nontrivially on the control, but conditions under which the optimal control is uniquely determined then depend crucially on the form of the control dependence in the original cost function. In the situations considered in §4.1–6, we will restrict our attention to systems for which we can implement a change of variables yielding purely state-dependent unmoderated costs.

A pertinent criterion in constructing or selecting a moderation incentive for a particular control system is the tractability of the resulting optimal control and evolution equations. One of the advantages of the quadratic control cost terms frequently used in geometric optimal control for systems with controlled velocities and unbounded admissible control regions is the simplicity of the determination of the optimal control value: if $X(m, u) = u$ and the control cost corresponds to the kinetic energy of a simple mechanical system, the optimal control value is simply the inverse Legendre transform of the auxiliary variable, which facilitates application of powerful results from geometric Hamiltonian mechanics. (See, e.g., [5, 21, 3].) Families of moderation incentives generating differentiable optimal controls that are
rescalings of the inverse Legendre transform of the auxiliary variable are constructed in [15].

**Example 1.** We consider an optimal control system for which the admissible control regions are the unit balls in $\mathbb{R}^f$ with respect to a family of state-dependent inner products. Specifically, there is a map $S \in C^1(S, S^+)$, where $S^+$ denotes the space of real $\ell \times \ell$ symmetric positive definite matrices, such that for each $m \in S$, the admissible control region $A_m$ is the unit ball with respect to the norm $\| \|_m$ on $\mathbb{R}^f$ determined by $S_m$.

Given maps $\xi \in C_0([0,1], [0,1])$ and $\mu \in C_1(S, \mathbb{R}^+)$, if $\xi$ is continuously differentiable on $[0,1)$, $\xi(1) = 0$, and $\lim_{s \to 1} |\xi'(s)| = \infty$, then

$$\tilde{C}(m, u) := \mu(m) \xi(\|u\|_m)$$  \hspace{1cm} (6)

is a moderation incentive. The scaling factor $\mu$ is called the *moderation parameter*. These moderation incentives discourage peak control values; see [15] for details.

Given a moderation parameter $\mu$, the associated parametrized family of *elliptic moderation incentives* $\tilde{C}_{\alpha,p}$ is defined using (6) and the functions

$$\xi_{\alpha,p}(s) := \frac{1}{p\alpha} (1 - s^p)^\alpha$$  \hspace{1cm} (7)

for $0 < \alpha < 1 < \ell \leq p$, excluding $\alpha = 1 = p$. The moderation incentives $\tilde{C}_{\alpha,p}$ yield tractable expressions for the optimal control as a function of the state and auxiliary variables, while offering a wide range of responses as the parameters $\alpha$ and $p$ are varied. For example, under appropriate conditions the controls determined by $\xi_{1,2}$ determine a homotopy between optimal controls determined by a ‘kinetic energy’ control cost for $X(m, u) = u$ and a logarithmic penalty function [15].

### 3.2. Biased moderation incentives in psychological cost of study functions.

The admissible control regions for the mastery-based control systems we will consider in sections 4 and 5 are not unit balls—we do not have a meaningful interpretation of negative study effort. We are interested in models in which the psychological cost of one endpoint of the admissible control interval is lower than that of the other, so shifting the controls so as to obtain admissible control regions equal to the unit ball in $\mathbb{R}$ is not an appealing approach. However, if we generalize the notion of a moderation incentive by relaxing the assumption that the incentive equals zero on the boundaries of the admissible control regions, we can easily adapt the existing families of moderation incentives to the new setting.

In keeping with the experimental studies suggesting that humans are generally averse to substantial conscious cognitive effort [25], we will primarily be interested in positive biased moderation incentives in our psychological cost terms, since a positive moderation incentive rewards lower cognitive effort. However, some students experience internalized social pressures that can make *not* studying stressful; familial or cultural expectations can create a sense of guilt or anxiety. Hence we will not require that biased moderation incentives be nonnegative, but will focus on the nonnegative case in §5 and §6.

Given an admissible space $\mathcal{A}$ and subsets $D_m \subseteq \partial A_m$ for each $m \in M$, $\tilde{C} \in C^0(\mathcal{A}, \mathbb{R})$ is a *biased moderation incentive* for $\mathcal{A}$ and $\{D_m\}_{m \in S}$ if for all $m \in S$

1. $u \in D_m$ implies $\tilde{C}(m, u) = 0$ and
2. \( \tilde{C} \) is continuously differentiable with respect to the control \( u \) on \( A \setminus (\cup_{m \in S} D_m) \), with well-defined limiting behavior of \( \frac{\partial \tilde{C}}{\partial u}(m, u) \) as \( u \) approaches \( D_m \); if \( \frac{\partial \tilde{C}}{\partial u} \)

   does not exist at \( u \in D_m \), \( \lim_{u \to u_*} \left\| \frac{\partial \phi}{\partial u}(m, u) \right\| = \infty \).

**Example 2.** A pertinent generalization of situations in which the admissible control regions are closed balls includes restrictions on some components of the control vectors with respect to specified bases. Given a map \( S \in C^1(S, S_{++}^\ell) \) as in the previous example, and maps \( v_i \in C^1(S, \mathbb{R}^\ell) \), \( i = 1, \ldots, j \) for some \( j \in \mathbb{N} \), such that \( \{v_1(m), \ldots, v_j(m)\} \) is linearly independent for all \( m \in S \), if

\[
A_m = \left\{ u \in \mathbb{R}^\ell : \|u\|_m \leq 1 \quad \text{and} \quad 0 \leq \min_{1 \leq i \leq j} v_i(m) \cdot u \right\},
\]

and

\[
D_m = \{ u \in A_m : \|u\|_m = 1 \}
\]

for all \( m \in S \), then the restrictions of the moderation incentives described in the previous example to \( A \) determined by (8) are biased moderation incentives.

4. **Control-affine progress and linear life balance in subject mastery models.** We now introduce some simplifying assumptions about the cost function terms and the admissible control regions:

- The admissible control regions \( A_m \) are given by (8) for some \( S \in C^1(S, S_{++}^\ell) \) and \( v_i \in C^1(A, \mathbb{R}^\ell) \), \( i = 1, \ldots, j \).
- The system is control-affine: such that

\[
X(m, u) = (L(m))(u) + Y(m),
\]

for \( L \in C^1(S, C(\mathbb{R}^\ell, V)) \) and drift field \( Y \in C^1(A, V) \).
- The cost function is the sum of three contributing costs:

  1. a purely state dependent cost \( \gamma \in C^1(S, \mathbb{R}) \),
  2. a linear life balance cost \( \beta \), specifically \( \beta(u) = \beta \cdot u \) for some \( \beta \in \mathbb{R}^\ell \), and
  3. a psychological cost \( \psi \) of the form

\[
\psi(m, u) = \psi_{pe}(m) - \mu(m)\xi(\|u\|_m),
\]

where \( \psi_{pe} \in C^1(S, \mathbb{R}) \) denotes the psychological cost of maximal study effort \( \|u\|_m = 1 \), \( \mu \in C^1(S, \mathbb{R}) \), and \( \xi \in C^0([0, 1], [0, 1]) \)

- (a) is continuously differentiable on \([0, 1] \),
- (b) satisfies \( \xi(0) = 1 \), \( \xi'(0) = 0 \), and \( \xi(1) = 0 \), and
- (c) either \( \xi'(1) \) exists or \( \lim_{s \to 1} |\xi'(s)| = \infty \).

Thus \( C \in C^0(A, \mathbb{R}) \) satisfies

\[
C(m, u) = \gamma(m) + \beta(u) + \psi(m, u) = \gamma(m) + \beta \cdot u + \psi_{pe}(m) - \mu(m)\xi(\|u\|_m). \quad (11)
\]

When seeking the value of the control \( u \in A_m \) maximizing \( \tilde{H}(m, \lambda, u) \) for given \( m \in S \) and \( \lambda \in V^* \), it suffices to consider \( \nu : B \to \mathbb{R} \) given by

\[
\nu(m, \lambda, u) := (L(m)\lambda - \beta) \cdot u + \mu(m)\xi(\|u\|_m), \quad (12)
\]

where we identify \( L(m)\lambda \) with an element of \( \mathbb{R}^\ell \) using the Euclidean inner product on \( \mathbb{R}^\ell \), since

\[
\tilde{H}(m, \lambda, u) - \nu(m, \lambda, u) = \lambda(Y(m)) - \gamma(m) - \psi_{pe}(m) \quad (13)
\]
is independent of the control \( u \).
We define \( \hat{u} : S \times V^* \to \mathbb{R}^\ell \) by
\[
\hat{u}(m, \lambda) := S^{-1}_m (L(m)^* \lambda - \beta).
\] (14)
If \( \hat{u}(m, \lambda) \neq 0 \), then \((m, \lambda, u)\) satisfies (5) for \( \bar{H} \) iff \( u = \sigma \hat{u}(m, \lambda) \) for \( \sigma \in \mathbb{R} \) satisfying
\[
\mu(m) \sigma \frac{\xi'(s)}{s} \bigg|_{s = \|\hat{u}(m, \lambda)\|_m} = -1.
\] (15)
If we define \( w \in C^0([0, 1), \mathbb{R}) \) by
\[
w(s) := \xi(s) - s \xi'(s),
\] (16)
then \( \nu(m, \lambda, u) = \mu(m) w(\|u\|_m) \) if \( u = 0 \) or \( u = \sigma \hat{u}(m, \lambda) \) for \( \sigma \) satisfying (15).
If \( \hat{u}(m, \lambda) = 0 \) and \( \mu(m) \neq 0 \), then
\begin{enumerate}
\item Pontryagin’s control criticality equation (5) for \( \bar{H} \) is satisfied by \((m, \lambda, u)\) iff \( \|u\|_m \) is a critical point of \( \xi \). Note that since \( \xi'(0) = 0 \), the trivial control \( u = 0 \) satisfies the criticality condition.
\item \( u \) maximizes \( \nu(m, \lambda, u) \) iff \( \|u\|_m \) maximizes \( \mu(m) \xi \).
\end{enumerate}
If \( \hat{u}(m, \lambda) = 0 \) and \( \mu(m) = 0 \), \( \nu(m, \lambda, \cdot) \) is identically zero.

**Remark 3.** \( w(s) \) can be regarded as the evaluation at the origin of the first order Taylor series expansion of \( \xi \) at \( s \). Thus for \( \mu(m) > 0 \), \( u_* \) satisfying (15) yields a greater value of \( w \), and hence of \( \nu \) and \( \bar{H} \), than \( u = 0 \) iff the affine approximation of \( \xi \) at \( \|u_*\|_m \) overestimates the value of \( \xi \) at 0. Loosely interpreted, a critical study investment is preferable to inaction when the estimate based on the marginal relief experienced by a student working at that intensity overvalues the psychological relief they would obtain by not studying at all.

If \( \xi \) is twice differentiable on \((0, s)\), there exists \( 0 < \bar{s} < s \) such that
\[
1 = \xi(0) = \xi(u) + \xi'(s)(0 - s) + \frac{1}{2} \xi''(\bar{s})(0 - s)^2
\]
and hence
\[
w(s) = \xi(s) - s \xi'(s) = 1 - \frac{1}{2} \xi''(\bar{s}) s^2.
\]
It follows that if \( \xi'' \) is defined and positive (respectively negative) on \((0, 1)\), then \( w \) is monotonic on \((0, 1)\).

**Example 3.** The elliptic moderation incentive (7) with \( \alpha = \frac{1}{2} \) and \( p = 2 \), and its biased analogs, yields particularly simple expressions for the constructions described in this section. For notational convenience, we set
\[
\xi_\circ(s) := \xi_{\frac{1}{2}}, 2(s) = \sqrt{1 - s^2},
\] (17)
with
\[
\xi_\circ'(s) = - \frac{s}{\sqrt{1 - s^2}} \quad \text{and} \quad w_\circ(s) = \xi_\circ(s) - s \xi_\circ'(s) = \frac{1}{\sqrt{1 - s^2}}.
\] (18)
In particular, \( \xi_\circ(0) = 0 \), \( \lim_{s \to 1} \xi_\circ'(s) = -\infty \), and \( \xi_\circ \) is invertible, with
\[
(\xi_\circ')^{-1}(x) = - \frac{x}{\sqrt{1 + x^2}} \quad \text{and} \quad \xi_\circ \left( (\xi_\circ')^{-1}(x) \right) = \frac{1}{\sqrt{1 + x^2}}.
\]
It follows that if \( \mu(m) \neq 0 \), then (15) has the unique solution
\[
\sigma(m, \lambda) := \frac{\text{sgn}(\mu(m))}{\sqrt{\|\hat{u}(m, \lambda)\|_m^2 + \mu(m)^2}},
\]
where \( \text{sgn} \) is the sign function.
which satisfies

$$\|\sigma(m, \lambda) \hat{u}(m, \lambda)\|_m < 1 \quad \text{and} \quad \mu(m) w_\nu(\|\sigma(m, \lambda) \hat{u}(m, \lambda)\|_m) = \frac{1}{\sigma(m, \lambda)}.$$ 

The rescaling of $\hat{u}(m, \lambda)$ by $\sigma(m, \lambda)$ is an admissible control if it satisfies the constraints determined by $\nu_i(m)$, $i = 1, \ldots, j$.

4.1. Scalar mastery-based learning models. We now further specialize and simplify the models under consideration by restricting our attention to systems with a scalar state variable, subject mastery; mastery may be objective, as measured by an external authority (e.g. a course instructor) or subjective, as perceived by the student. We let $S$ be an open interval in $V = \mathbb{R}$ containing the interval $[0, M]$ for some $M > 0$ representing complete mastery of the pertinent skills and/or knowledge. We will not require that our systems have equilibria at 0 and $M$, given that “knows nothing about the subject” and “knows everything there is to know about it” are typically arbitrarily determined extremes and those boundaries are rarely recognizable to students; we will simply stop the clock when a solution reaches 0 or $M$. For comparison, in population models, extinction of a population is typically a clear-cut state, and model behavior near that state should qualitatively reflect the associated biological realities.

We consider a scalar control taking values in the unit interval $[0, 1]$ for all states $m$; $u(t)$ represents the fraction of the total cognitive effort available to the student at time $t$ that is invested in studying the pertinent material. Thus $A = S \times [0, 1]$.

Since $S$ and $A_m$ are intervals, $L(m)$ and $S_m$ can be identified with scalars; we assume that $L(m) = p(m)$ for some function $p \in C^1(S, \mathbb{R}^+)$, i.e. studying always yields some increase in mastery, and that $S_m = 1$, and hence $\|u\|_m = u$, for all $(m, u) \in A$. With an eye towards a rescaling of time and the exploitation of convenient trigonometric identities in some of the examples, we express the drift field $Y$ in terms of a rescaling by $p$ of a trigonometric function applied to a ‘drift angle’. Specifically, given $\phi \in C^1(S, [0, \frac{\pi}{2}])$, we define

$$u_\phi := \sin \circ \phi$$

and set

$$Y(m) = -p(m) u_\phi(m);$$

it follows that (10) takes the form

$$X(m, u) = p(m)(u - u_\phi(m)). \quad (19)$$

We refer to $\phi$ as the drift angle.

We now implement a change of variables suggested by (12) and (14), and an associated rescaling of the independent variable $t$. The new variables facilitate analysis of the dynamics of (3) with cost function (11).

**Proposition 1.** Given a piecewise differentiable curve $\tilde{z} = (m, \lambda, u) : [0, t_f] \to B$, if we define $b : [0, t_f] \to \mathbb{R}$ and $\tau : [0, t_f] \to [0, \infty)$ by

$$b(t) := p(m(t)) \lambda(t) - \beta \quad \text{and} \quad \tau(t) := \int_0^t p(m(s)) ds,$$

and set $\tau_f := \tau(t_f)$, then $\tilde{z}$ is a solution of the fixed time synthesis problem determined by (2), (11), and (19) iff $z = (m, b, u) \circ \tau^{-1} : [0, \tau_f] \to B$ is a solution of the fixed time synthesis problem determined by the controlled vector field
If $0 < u < 1$ implies that either

$$b + \mu(m)\xi'(u) \neq 0 \quad \text{or} \quad \mu(m)w(u) < \max\{\mu(m), b\},$$

then the optimal control value for the pair $(m, b)$ is 0 if $\mu(m) > b$ and 1 if $b > \mu(m)$. On the other hand, if

1. $\{u \in (0, 1) : b + \mu(m)\xi'(u) = 0\}$ is nonempty,
2. $\mu(m)w$ is maximized at a unique point $u_c$ in that set, and
3. $\mu(m)w(u_c) > \max\{\mu(m), b\},$

then the optimal control value for $(m, b)$ is $u_c$.

If the maximum of $\mu(m)w$ over the set of critical control values is achieved at multiple elements of the set described in (i), or if any of $\mu(m)$, $b$, and $\mu(m)w(u_c)$ are equal, then the optimal control value for $(m, b)$ is not unique.

**Example 4.** The optimal control value can readily be determined for $\xi_\phi$ using the criteria developed above. If we define $u_c : (\mathbb{R}^+) \times (0, 1)$ by

$$u_c(\mu, b) := (\xi_\phi')^{-1}\left(-\frac{b}{\mu}\right) = \left(1 + \left(\frac{b}{\mu}\right)^2\right)^{-1/2},$$

then the optimal control value for $\xi_\phi$ is $u_c$. For example, when $\xi_\phi$ is indicative of a quadratic cost, the optimal control is

$$u_c(\mu, b) := \left(1 + \left(\frac{b}{\mu}\right)^2\right)^{-1/2},$$

which is a standard result from optimal control theory.
then

\[
\{ u \in (0,1) : b + \mu \xi'(u) = 0 \} = \begin{cases} 
\{ u_{c}(\mu,b) \} & b \mu > 0 \\
\emptyset & b \mu \leq 0 \text{ and } (b,\mu) \neq (0,0) \\
(0,1) & (b,\mu) = (0,0) 
\end{cases}.
\]

For \((\mu,b) \in (\mathbb{R}^+)^2 \cup (-\mathbb{R}^+)^2\), substitution of \(u_{c}\) into (18) yields

\[
\mu w(u_{c}(\mu,b)) = \frac{\mu}{\sqrt{1 - u_{c}(\mu,b)^2}} = \text{sgn}(\mu)\sqrt{\mu^2 + b^2}.
\]

Hence if we define \(u_{*} \in C^{0}((\mathcal{S} \times \mathbb{R}) \setminus \{(m,\mu(m)) : \mu(m) \leq 0\},[0,1])\) by

\[
u_{*}(m,b) := u_{\text{opt}}(\mu(m),b)
\]

for

\[
u_{\text{opt}}(\mu,b) := \begin{cases} 
0 & b \leq 0 \text{ and } b < \mu \\
u_{c}(\mu,b) & 0 < \min\{\mu,b\} \\
1 & \mu \leq 0 \text{ and } \mu < b
\end{cases},
\]

then \(u_{*}\) gives the optimal control values on its domain. Multiple control values maximize (22) when \(\mu(m) = b \leq 0\); both 0 and 1 maximize (22) for \(\mu(m) = b < 0\), and (22) equals 0 for any value of \(u\) if \((b,\mu) = (0,0)\).

We briefly comment on some implications of the qualitative behavior of \(u_{\text{opt}}\), as shown in Figure 1. If \(b\) is interpreted as the ‘external’ marginal academic value of study effort, shifted downward by the life balance proportionality factor \(\beta\), we can regard positive values of \(b\) as indicative that additional study would be beneficial on purely pragmatic grounds, while negative values of \(b\) indicate that additional study isn’t worthwhile from a practical perspective. The moderation parameter \(\mu\) determines the student’s psychological sensitivity to changes in study effort. The greater the relief obtained from submaximal study investment, the greater the marginal practical value required to make studying worthwhile.
5. “I prefer not to”—strictly positive moderation parameters. When $\mu(m) \neq 0$, the optimality condition (22) depends on $b$ and $m$ only through the ratio $\frac{b}{\mu(m)}$ and the signs of $\mu(m)$ and $b$. We now consider the case of strictly positive $\mu$—studying less than the maximum possible provides some psychological relief—and perform a second change of variables, replacing $b$ with $r$, which further simplifies some aspects of the analysis and suggests possible “dimensionless” parameters for use in appropriate situations.

It follows immediately from (21) that

$$H(m, \mu(m)r, u) = \mu(m) (r (u - u_0(m)) + \xi(u)) - \tilde{C}(m),$$

and hence if $\mu(m) > 0$, then

$$H(m, \mu(m)r, u_*) = \max_{0 \leq u \leq 1} H(m, \mu(m)r, u)$$

$$\iff r u_* + \xi(u_*) = \max_{0 \leq u \leq 1} (r u + \xi(u))$$

and

$$\frac{\partial H}{\partial m}(m, \mu(m)r, u) = 0 \iff r + \xi'(u) = 0.$$ (27)

We construct a family of Hamiltonian functions with $r$-dependent terms modeled on the moderation potentials for systems with unbiased moderation incentives [15], with adaptations for biased moderation incentives. Given $v : \mathbb{R} \to [0, 1]$ and $\theta \in (0, \frac{\pi}{2})$, define $K_0$ and $K_\theta : \mathbb{R} \to \mathbb{R}$ by

$$K_0(r) := rv(r) + \xi(v(r)) \quad \text{and} \quad K_\theta(r) := K_0(r) - r \sin \theta.$$ (28)

To motivate this construction, note that if $v$ and $\xi$ are differentiable at $r$, and $v$ satisfies either (27) or $v'(r) = 0$ at $r$, then

$$K_\theta'(r) = v(r) + (r + \xi'(v(r)) v'(r)) v'(r) = v(r).$$

As we will see in Proposition 2, if $v(r)$ satisfies the maximality condition (26) for all $r$ and $K_0$ is an antiderivative of $v$, neither $\xi$ nor $v$ need be differentiable to determine the control values for a solution of the synthesis problem.

Next we define the parametrized families of functions $\tilde{C}_x : \mathcal{S} \to \mathbb{R}$ and $\tilde{H}_x : \mathcal{S} \times \mathbb{R} \to \mathbb{R}$, for $x \in \mathbb{R}$, by

$$\tilde{C}_x(m) := \frac{x - \tilde{C}(m)}{\mu(m)} \quad \text{and} \quad \tilde{H}_x(m, r) := K_{\phi(m)}(r) - \tilde{C}_x(m).$$ (29)

Differentiation of the equality

$$H(m, \mu(m)r, v(r)) - \mu(m)\tilde{H}_x(m, r) = x$$

with respect to $m$ yields

$$\frac{\partial H}{\partial m}(m, \mu(m)r, v(r)) - \mu(m)\frac{\partial \tilde{H}_x}{\partial m}(m, r)$$

$$= \mu'(m) \left( \xi(v(r)) - \tilde{C}_x(m) \right)$$

$$= \mu'(m) \left( \tilde{H}_x(m, r) - r(v(r) - u_0(m)) \right).$$

---

1 Bartleby’s (in)famous demurral [20] epitomizes rational, but ultimately fatal, inaction. Bartleby’s polite but unshakeable disengagement from his stultifying work progresses to passive suicide by starvation.
If $K_0$ is differentiable at $r$, then it follows immediately from differentiation of (30) with respect to $r$ that
\[
\frac{\partial H}{\partial r}(m, \mu(m)r, v(r)) - \frac{\partial H}{\partial r}(m, r) = v(r) - K'_0(r). \tag{32}
\]

We will now show that if we can first identify an appropriate control function $v$, then given initial data $m_0$ and $b_0$, we can construct a solution of the synthesis problem determined by $H$ from a solution of the canonical Hamiltonian system determined by $\tilde{H}_x$ for a value of $x$ depending on the initial data. The key advantage of this two stage approach is that the conditions on $v$ depend only on the function $\xi$; the state-dependent aspects of the Hamiltonian do not come into play.

**Proposition 2.** If $v \in C^0([0, 1])$ satisfies
\[
r v(r) + \xi(v(r)) = \max_{0 \leq u \leq 1} (ru + \xi(u)) \tag{33}
\]
for all $r \in \mathbb{R}$, and $K_0$ given by (28) is an anti-derivative of $v$, then $\tilde{H}_x \in C^1(S \times \mathbb{R}, \mathbb{R})$ for any $x \in \mathbb{R}$. Given $(m_0, r_0) \in S \times \mathbb{R}$, if $(m, r) : [0, \tau] \rightarrow S \times \mathbb{R}$ is the solution satisfying $m(0) = m_0$ and $r(0) = r_0$ of the canonical Hamiltonian system determined by $\tilde{H}_{h_0}$ for $h_0 := \mu(m_0)\tilde{H}_0(m_0, r_0)$ (34) then $(m, (\mu \circ m)r, v \circ r)$ is a solution of the synthesis problem determined by (21).

**Proof.** The mastery evolution equation
\[
m' = \frac{\partial H_{h_0}}{\partial r}(m, r) = \frac{\partial H}{\partial r}(m, \mu(m)r, v(r)) \tag{35}
\]
follows directly from (32).

By construction, $\tilde{H}_{h_0}(m(0), r(0)) = 0$; hence conservation of $\tilde{H}_{h_0}$ along solutions of the canonical Hamiltonian system determined by $\tilde{H}_{h_0}$ implies that $\tilde{H}_{h_0} \circ (m, r)$ equals zero. Thus it follows from (31) and Hamilton’s equations for $(m, r)$ that $b = (\mu \circ m)r$ satisfies
\[
b' = \mu'(m)m'r + \mu(m)r' = \mu'(m)\frac{\partial H_{h_0}}{\partial r}(m, r) + \mu(m)\frac{\partial H_{h_0}}{\partial m}(m, r)
\]
\[= -\frac{\partial H}{\partial m}(m, b, v(r)).
\]

The maximality condition (33) on $v$ implies that
\[
b v(r) + \mu(m)\xi(v(r)) = \mu(m)(ru(r) + \xi(v(r))) = \mu(m) \max_{0 \leq u \leq 1} (ru + \xi(u)) = \max_{0 \leq u \leq 1} (bu + \mu(m)\xi(u)).
\]

Hence $(m, b, v \circ r)$ is a solution of the synthesis problem determined by (21).}

\section*{5.1. Piecewise reduction to quadrature for $\xi_\varphi$.} Once again, we use the biased elliptic moderation incentive $\xi_\varphi(u) = \sqrt{1 - u^2}$ to illustrate the new constructions. Equation (24) implies that for the biased moderation incentive $\xi_\varphi$,
\[
v_\varphi(r) := \begin{cases} 
0 & r \leq 0 \\
\frac{r}{\sqrt{1 + r^2}} & r > 0
\end{cases} \tag{36}
\]
Figure 2. Left: $K_\theta(r)$ for $r$ near $\tan \frac{\pi}{10}$, right: $\nu_\circ(r)$ associated to $\xi_\circ(u) = \sqrt{1-u^2}$.

is the unique control function satisfying (33); (28) yields

$$K_0(r) = \begin{cases} 1 & r \leq 0 \\ \frac{1}{\sqrt{1+r^2}} & r > 0 \end{cases}. \]

Since $K_0 \in C^1(\mathbb{R}, \mathbb{R})$, with $K_0' = \nu_\circ$, $\nu_\circ$ satisfies the conditions of Proposition 2.

If $\phi$ is strictly positive, $\dot{H}_x$ shares key qualitative features with a one degree of freedom simple mechanical system: for fixed $m$ with $\phi(m) > 0$, $K_\phi(m)$ is convex, strictly convex on $[0, \infty)$, and achieves its minimum at the unique value of $r$ for which the state component of the Hamiltonian vector field equals zero. (See Figure 2 for a representative graph of $K_\theta(r)$.)

The generic level sets of $\dot{H}_x$ are the unions of the graphs of two functions of $m$—one associated to trajectory segments with nondecreasing $m$ and the other to segments with nonincreasing $m$. Given initial data $(m_0, r_0)$, we can construct two parametrized pairs of functions from $S$ to $\mathbb{R}$ that determine the evolution of the state variable $m$ as a function of $\tau$; analogous to the classical ‘reduction to quadrature’ of a one degree of freedom simple mechanical system via partial decoupling of the differential equations. In the next two paragraphs we outline the algebraic details of this process.

For $\theta \in (0, \frac{\pi}{2})$, set $r_\theta := \tan \theta$ and define $r^\pm_\theta \in C^0([\cos \theta, \infty), \mathbb{R})$ by

$$r^+_\theta(c) := \frac{c}{\cos \theta} r_\theta + \frac{1}{\cos \theta} \sqrt{\left(\frac{c}{\cos \theta}\right)^2 - 1},$$

$$r^-_\theta(c) := \begin{cases} \frac{c}{\cos \theta} r_\theta - \frac{1}{\cos \theta} \sqrt{\left(\frac{c}{\cos \theta}\right)^2 - 1} & c < 1 \\ \frac{1-c}{\sin \theta} & c \geq 1. \end{cases}$$

For positive $\theta$, the functions $K_\theta$ and $r^+_\theta$ have the following properties

1. $K_\theta$ is strictly convex, with global minimum $\cos \theta = K_\theta(r_\theta)$,
2. $c > \cos \theta$ implies $r^-_\theta(c) < r_\theta < r^+_\theta(c)$ and $K^{-1}_\theta(c) = \{r^-_\theta(c), r^+_\theta(c)\}$,
3. $r^+_\theta(\cos \theta) = r_\theta$, and
4. $\nu(r_\theta) = \sin \theta$.

It follows from (i) that if $(m, r) : [0, \tau] \to S \times \mathbb{R}$ is a solution of the Hamiltonian system determined by $\dot{H}_{h_0}$, with initial data $(m_0, r_0)$, then for $h_0$ given by (34), we have

$$\dot{C}_{h_0} \circ m \geq \cos \phi \circ m \quad (37)$$
Figure 3. Graphs of $X_{h_0}^\pm$ for $\mathcal{S} = [0,1]$, $\gamma(m) = (1 - m)^2$, $\mu(m) = \frac{1}{2} \left( 1 - \frac{m^2}{2} \right)^2$, constant functions $\psi_{pe} = \frac{1}{2}$ and $\phi = \frac{\pi}{12}$, and representative values of $h_0$ for which $\hat{C}_{h_0}$ has a strict global minimum $c_{h_0}$. Solid curves: $X_{h_0}^+$; dashed curves: $X_{h_0}^-$. and there is an function $\zeta : [0, \tau_f] \to \{-, +\}$ such that the functions $\rho_{h_0}^\pm, X_{h_0}^\pm \in C^0(\mathcal{S}, \mathbb{R})$ given by

$$
\rho_{h_0}^\pm(m) := r_{\phi(m)}^\pm(\hat{c}_{h_0}(m)) \quad \text{and} \quad X_{h_0}^\pm := \nu \circ \rho_{\phi, h_0}^\pm - u_{\phi} \quad (38)
$$

satisfy

$$
m'(\tau) = X_{\hat{C}_{h_0}}(m(\tau)) \quad \text{and} \quad r(\tau) = \rho_{h_0}^\pm(m(\tau))
$$
for $0 \leq \tau \leq \tau_f$. $\zeta$ can change value only at points $\tau_*$ for which $m_* = m(\tau_*)$ satisfies

$$
\hat{C}_{h_0}(m_*) = \cos \phi(m_*), \quad \text{and hence} \quad X_{h_0}^\pm(m_*) = 0 \quad \text{and} \quad \rho_{h_0}^\pm(m_*) = r_{\phi(m_*)}.
$$

If $\phi$ is not strictly positive, the reduction to a partially decoupled system of differential equations can break down, since

$$
K_0^{-1}(c) = \begin{cases} 
\emptyset & c < 1 \\
(-\infty, 0] & c = 1 \\
\sqrt{c^2 - 1} & c > 1
\end{cases}
$$

If $(m, r)$ is a solution of the Hamiltonian system determined by $\tilde{H}_{h_0}$, with initial data $(m_0, r_0)$, conservation of $\tilde{H}_{h_0}$ implies that $\hat{C}_{h_0}(m(\tau)) \geq 1$ when $\phi(m(\tau)) = 0$. If $\hat{C}_{h_0}(m(\tau)) = 1$ and $\phi(m(\tau)) = 0$ for some $\tau$, then $m'(\tau) = 0; r(\tau)$ cannot be determined solely from conservation of $\tilde{H}_{h_0}$ and continuity of $r$—we only know that $r(\tau) \notin \mathbb{R}^+$.  

6. Example: Qualitative analysis for strictly positive $\mu$, $\xi$, constant positive drift angle, and $\hat{C}_{h_0}$ with a strict global minimum. For illustrative purposes, we now describe some of the qualitative behavior of trajectories for costs, moderation parameters, and initial data such that $\hat{C}_{h_0}$ achieves its strict global minimum $c_{h_0}$ at a single point $0 < m_c < M$. Note that $\hat{C}_{h_0}$ can have a strict unique minimum for monotonically decreasing $\hat{C}$ and $\mu$, so models in which mastery-dependent costs and aversion to studying decrease as mastery increases can fall into this category, provided that appropriate conditions on the relative rates of decrease are satisfied. The trajectories described here determine solutions of the synthesis problem for the associated Hamiltonian, but need not be associated to globally optimal solutions. If $\tau_f$ is specified in advance, the clock may run out before the behavior described below is completed; if either 0 (total ignorance of the material) or $M$ (complete mastery of the material) is reached, we simply stop the clock.  

In this simple proof of concept example, we assume the drift angle $\phi$ is constant and strictly positive. This choice of drift field yields a separable Hamiltonian system with trajectories that can be readily qualitatively characterized and classified using
elementary methods. A bifurcation analysis considering more general vector fields and cost functions could be carried out to determine the extent to which these models are representative of nearby systems with non-constant drift angles, but is beyond the scope of this paper.

**Remark 4.** There is a mature body of literature on mathematical models of forgetting in specific memorization tasks that can be consulted in the development of more realistic drift fields. Ebbinghaus’ classic forgetting curve \( \frac{1}{1+\kappa (\ln t)^q} \), for appropriate constants \( \kappa \) and \( q \), estimates the difference in the time required to memorize a sequence of nonsense syllables and the time required to relearn that sequence after time \( t \) away from the task [9, 10, 22]. More recent models of forgetting provide better estimates for more realistic learning tasks (see, e.g., [2]); however, there does not appear to be consensus on the influence, if any, of current mastery on memory [16, 27]. When modeling students’ subjective perception of their mastery, decay may combine actual loss of knowledge and/or skills with devaluation of retained abilities—low self-efficacy may cause the student’s estimate of their knowledge and abilities may decrease over time, even if they haven’t actually forgotten material.

We can use elementary phase portrait sketching techniques for separable Hamiltonian systems on \( S \times \mathbb{R} \) to determine the qualitative behavior of the solutions of the synthesis problem for the constant drift field angle \( \phi \). Given initial data \( m_0 \) and \( r_0 \), we can compute \( h_0 \) using (34) and determine the key features of the graph of \( \mathcal{C}_{h_0} \), particularly the location of any local or global maxima or minima. Since the solution \( (m, r) : [0, \tau_f] \rightarrow S \times \mathbb{R} \) of the canonical Hamiltonian system determined by \( \mathcal{H}_{h_0} \) with initial data \( m_0 \) and \( r_0 \) lies within the zero level set of \( \mathcal{H}_{h_0} \),

\[
K_\phi \circ r = \mathcal{C}_{h_0} \circ m. \quad (39)
\]

Hence we can determine the behavior of a solution \((m, r)\) by sketching the graphs of \( K_\phi \) and \( \mathcal{C}_{h_0} \) side-by-side and visualizing a moving horizontal line passing through the points \((r, K_\phi(r))\) and \((m, \mathcal{C}_{h_0}(m))\), as illustrated in Figure 4. As shown in §5.1, \( K_\phi \) associated to \( \xi_5 \) has a global minimum of \( \cos \phi \) at \( r_\phi = \tan \phi \); for small positive \( \phi \), \( r_\phi \) is the unique critical point of \( K_\phi \). The direction of movement along the graph of \( \mathcal{C}_{h_0} \) is readily read off from the graph of \( K_\phi \): \( m \) decreases if \( r < r_\phi \) and increases if \( r > r_\phi \).

If \( m < \bar{m}_c \), then \( r' = \mathcal{C}_{h_0}'(m) < 0 \), and hence \( r \) decreases; if \( m > \bar{m}_c \), then \( r \) increases; \( r' = 0 \) when \( m = \bar{m}_c \). It follows that if \((m_0 - \bar{m}_c)(r_0 - r_\phi) > 0\), the long term outcomes do not depend on \( c_{h_0} \).

- If \( m_0 > \bar{m}_c \) and \( r_0 > r_\phi \), both \( m \) and \( r \) will increase until \( m \) reaches \( M \). This situation corresponds to a student with relatively high initial mastery who always studies enough to make progress; as their mastery increases, so does their study effort.
- If \( m_0 < \bar{m}_c \) and \( r_0 < r_\phi \), both \( m \) and \( r \) will decrease until \( m \) reaches zero. If \( 0 < r_0 < r_\phi \), then the control \( v(r) \) will initially be nonzero, but insufficient to maintain mastery; once \( r \) is non-positive, the optimal control value is zero. This situation corresponds to a student with relatively low initial mastery and little or no study effort; the student loses mastery at all times, leading eventually to a complete loss of mastery. The student may make a slight effort early on, but not enough to retain or gain mastery.

The outcomes for \((m_0 - \bar{m}_c)(r_0 - r_\phi) < 0\) depend on the values of \( c_{h_0} = \mathcal{C}_{h_0}(m_c) \) and \( K_\phi(r_\phi) = \cos \phi \). Qualitatively different behavior arises for \( c_{h_0} \) greater than,
equal to, or less than \( \cos \phi \). See Figure 3 for representative graphs of \( X_{\text{ho}}^{\pm} \), and Figure 4 for graphs of \( K_\phi \) and \( \tilde{\mathcal{C}}_{\text{ho}} \).

If \( m_0 < m_c \) and \( r_0 > r_\phi \), then initially \( m \) is increasing and \( r \) is decreasing.

- If \( c_{\text{ho}} > \cos \phi \), then \( r \) is still greater than \( r_\phi \) when \( \tilde{\mathcal{C}}_{\text{ho}} \) attains its minimum at \( m = m_c \); hence \( m \) continues to increase, now with increasing \( r \) as well.

At first the student’s progress slows until it momentarily ceases; the student then begins to gain mastery; the rate of progress increases until the student achieves complete mastery. Colloquially, after a period of increasing disengagement, the student eventually wakes up, smells the coffee, and gets back to business.

- If \( c_{\text{ho}} = \cos \phi \), then \((m_c, r_\phi)\) is an equilibrium; hence \( r \) approaches \( r_\phi \) as \( m \) approaches \( m_c \), but neither value is attained.

The student’s efforts decrease as their mastery increases, approaching the minimum study effort required to avoid losing ground as their mastery approaches the critical value \( m_c \). This is a “good enough, I just need to pass the class” approach.

- If \( c_{\text{ho}} < \cos \phi \), there exists a point \( m_0 < m_\phi < m_c \) satisfying \( \tilde{\mathcal{C}}_{\text{ho}} (m_\phi) = \cos \phi \). (If \( \tilde{\mathcal{C}}_{\text{ho}} (m) < \cos \phi \) for all \( m \in [0, m_c) \), then \( m_0 < m_c \) would not be possible.) As \( m \) approaches \( m_\phi \), \( r \) approaches \( r_\phi \), and hence \( m' \) approaches zero; since \( r' = \tilde{\mathcal{C}}_{\text{ho}}' \circ m \) is still negative, \( r \) continues to decrease and \( m \) begins to decrease; eventually all mastery is lost.

The student initially makes progress, but their study effort decreases over time; at some point, they stop making progress and begin to lose mastery; they eventually stop studying.
If $m_0 > m_c$ and $r_0 < r_\phi$, then initially $m$ is decreasing and $r$ is increasing.

- If $c_{ho} > \cos \phi$, then $r$ is still less than $r_\phi$ when $\tilde{C}_{ho}$ attains its minimum at $m = m_c$; hence $m$ continues to decrease, now with decreasing $r$ as well.

  If $c_{ho} \geq 1$, the student never studies. If $1 > c_{ho} > \cos \phi$, the student initially studies a small amount, but not enough to prevent loss of mastery; they eventually stop studying and ghost the task.

- If $c_{ho} = \cos \phi$, then $m$ decreases towards $m_c$ as $r$ increases towards $r_\phi$, but neither value is attained.

  If $r_0$ is negative, the student initially invests no effort; eventually the student studies a small amount, but not enough to prevent loss of mastery; over time their mastery approaches the “good enough” value $m_c$. This is consistent with an “I just need some easy units” approach.

- If $c_{ho} < \cos \phi$, then there exists a point $m_c < m_\phi^+ < m_0$ satisfying $\tilde{C}_{ho}(m_\phi^+) = \cos \phi$. (If $\tilde{C}_{ho}(m) > \cos \phi$ for all $m \in (m_c, M)$, then $m_c < m_0$ is impossible.)

  As $m$ approaches $m_\phi^+$, $r$ approaches $r_\phi$, and hence $m'$ approaches zero; since $r' = \tilde{C}_{ho}'(m)$ is still positive, $r$ continues to increase and $m$ begins to increase.

  Initially the student doesn’t study at all ($r_0 \leq 0$) or doesn’t study enough to avoid losing mastery ($0 < r_0 < r_\phi$); when mastery drops to $m_\phi^+$, the student begins to study enough to make progress, and continues to increase their efforts as their mastery increases. This behavior resembles that of the overconfident hare of “The Tortoise and the Hare”.

7. **Conclusions.** Optimal control models can shed light on behaviors that may seem unreasonable, showing that under appropriate circumstances these behaviors may appear to follow the best path forward. Even highly idealized models can suggest lines of further inquiry and experimentation. In mechanical systems, certain combinations (e.g. the Reynolds number of a fluid) of several parameters describing the original physical system may suffice to determine the behavior of the system, allowing researchers to infer the behavior of all systems within a class determined by the relevant combination of values from experiments conducted with a convenient member of that class. Experiments involving interventions intended to improve learning outcomes could potentially be more cost effective and more broadly relevant if analogous equivalence relations could be identified for models of engagement and learning.

  Identification of globally optimal solutions appears to require numerical calculations even for the simple models used here. We intend to carry out numerical experiments for representative fixed and free end-time problems in future work. Time dependent cost terms would significantly increase the realism of the models used here, as would end costs (e.g. final exams), and could be readily implemented using numerical approximations.

  Extension of the constructions developed here to a multidimensional state space is a natural and conceptually straightforward progression. Overall and task-specific self-efficacy—“the belief in one’s capabilities to organize and execute the courses of action required to manage prospective situations” [4]—plays an important role in student engagement. Students with low self-efficacy typically predict worse outcomes from their learning activities than are predicted by students with high self-efficacy, and self-efficacy theory asserts that “the most effective way of developing a strong sense of efficacy is through mastery experiences” [4]. Hence inclusion of


self-efficacy as an additional state variable would be a promising next step in the development of more realistic models of engagement. Multitasking, with different levels of current mastery for each task, is an essential aspect of most students’ learning experiences. Identification of conditions leading to a balanced mastery portfolio, or to a ‘play to your strengths’ focus on the learning tasks with the greatest initial mastery or subject-specific self-efficacy to the detriment of the tasks with relatively low initial mastery or self-efficacy could suggest interventions for at-risk students.

Given our intended use of models of learning behavior in increasing equity in STEM instruction, we seek mathematical models of forgetting that include pertinent psychological, social, and life balance factors. Stress significantly influences key mechanisms of memory retention, organization, and retrieval [14, 1, 12]. In particular, stress can inhibit valuable ‘decluttering’ processes that facilitate efficient access to useful knowledge by suppressing unused knowledge [14]; thus models combining multitasking, with differing levels of payoff for success in different tasks, with stress and self-efficacy state variables could have practical value.

REFERENCES

[1] M. C. Anderson, R. A. Bjork and E. L. Bjork, Remembering can cause forgetting: Retrieval dynamics in long-term memory, *Journal of Experimental Psychology. Learning, Memory, and Cognition*, 20 (1994), 1063–1087.

[2] L. Averell and A. Heathcote, The form of the forgetting curve and the fate of memories, *Journal of Mathematical Psychology*, 55 (2011), 25–35.

[3] J. Baillieul and J. Willems, *Mathematical Control Theory*, Springer, 1999.

[4] A. Bandura, *Self-Efficacy in Changing Societies*, Cambridge University Press, 1995.

[5] A. M. Bloch, *Nonholonomic Mechanics and Control*, Springer, 2003.

[6] H. Chang and S. L. Beilock, The math anxiety-math performance link and its relation to individual and environmental factors: a review of current behavioral and psychophysiological research, *Current Opinion in Behavioral Sciences*, 10 (2016), 33–38.

[7] K. Choe, J. Jenifer, C. Rozek, M. Berman and S. Beilock, Calculated avoidance: Math anxiety predicts math avoidance in effort-based decision-making, *Science Advances*, 5, 2019.

[8] R. Dorfman, An economic interpretation of optimal control theory, *The American Economic Review*, 59 (1969), 817–831.

[9] H. Ebbinghaus, *Über das Gedächtnis*, Dunker, 1885.

[10] E. O. Finkenbinder, The curve of forgetting, *The American Journal of Psychology*, 24 (1913), 8–32.

[11] T. L. Griffiths, F. Lieder and N. D. Goodman, Rational use of cognitive resources: Levels of analysis between the computational and the algorithmic, *Topics in Cognitive Science*, 11 (2015), 217–229.

[12] K. Hall, E. Fawcett, K. Hourihan and J. Fawcett, Emotional memories are (usually) harder to forget: A meta-analysis of the item-method directed forgetting literature, *Psychonomic Bulletin & Review*, 28 (2021), 1313–1326.

[13] D. Kirk, *Optimal Control Theory: An Introduction*, Dover Publications, Inc., 2004.

[14] S. Koessler, H. Engler, C. Riether and J. Kissler, No retrieval-induced forgetting under stress, *Psychological Science*, 20 (2009), 1356–1363.

[15] D. Lewis, A soothing invisible hand: Moderation potentials in optimal control, in *Geometry, Mechanics, and Dynamics: The Legacy of Jerry Marsden*, Springer, 2015, 257–284.

[16] G. R. Loftus, Evaluating forgetting curves, *Journal of Experimental Psychology*, 11 (1985), 397–406.

[17] I. Lyons and S. Beilock, When math hurts: Math anxiety predicts pain network activation in anticipation of doing math, *PLOS ONE*, 7.

[18] E. Maloney, M. Schaeffer and S. Beilock, Mathematics anxiety and stereotype threat: Shared mechanisms, negative consequences, and promising interventions, *Research in Mathematics Education*, 15 (2013), 115–128.
[19] A. Mattarella-Micke, J. Mateo, M. Kozak, K. Foster and S. Beilock, Choke or thrive? The relation between salivary cortisol and math performance depends on individual differences in working memory and math anxiety, *Emotion*, 11 (2011), 1000–1005.

[20] H. Melville, Bartleby, the scrivener: A story of Wall Street, *Putnam’s Magazine*, November and December (1853), 446–457 and 609–615.

[21] R. Montgomery, Optimal control of deformable bodies and its relation to gauge theory, *Math. Sci. Res. Inst. Publ.*, 22 (1991), 403–438.

[22] J. Murre and J. Dros, Replication and analysis of Ebbinghaus’ forgetting curve, *PLOS ONE*, 10 (2015).

[23] L. Pontryagin, V. Boltyanskii, R. Gamkrelidze and E. Mishchenko, *The Mathematical Theory of Optimal Processes*, Interscience Publishers, 1962.

[24] R. Rydell, A. McConnell and S. Beilock, Multiple social identities and stereotype threat: Imbalance, accessibility, and working memory, *Journal of Personality and Social Psychology*, 96 (2009), 949–966.

[25] A. Shenhav, S. Musslick, F. Lieder, W. Kool, T. Griffiths, J. Cohen and M. Botvinick, Toward a rational and mechanistic account of mental effort, *Annual Review of Neuroscience*, 40 (2017), 99–124.

[26] F. Sirois and T. Pychyl, Procrastination and the priority of short-term mood regulation: Consequences for future self, *Social and Personality Psychology Compass*, 7 (2013), 115–127.

[27] N. Slamecka, On comparing rates of forgetting: Comment on Loftus (1985), *Journal of Experimental Psychology. Learning, Memory, and Cognition*, 11 (1985), 812–816.

[28] E. Sontag, Integrability of certain distributions associated with actions on manifolds and applications to control problems, in *Nonlinear Controlability and Optimal Control*, Marcel Dekker, Inc., 133 (1990), 81–131.

Received June 2021; revised November 2021; early access January 2022.

E-mail address: lewis@ucsc.edu