Cosmological Neutrinos

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Plan

• Lecture 1: Cosmological effects of neutrinos in linear perturbation theory
• Lecture 2: Non-linear regime
• Lecture 3: Neutrinos in Intergalactic space
• Lecture 4: New ways of probing neutrino masses
Lesgourgues and Pastor 2006 review on the arXiv

Wong https://arxiv.org/pdf/1111.1436.pdf

Lesgourgues, Mangano, Miele, Pastor “Neutrino Cosmology” 2013 Cambridge University Press

Ma & Bertschinger https://arxiv.org/pdf/astro-ph/9506072.pdf

Drop me an email if you wish further references: viel AT sissa.it
• Lecture 1: Cosmological effects of neutrinos in linear perturbation theory

• Lecture 2: Non-linear regime

• Lecture 3: Neutrinos in Intergalactic space

• Lecture 4: New ways of probing neutrino masses
Boundary conditions from particle physics

\[ 0.056 \, (0.095) \, \text{eV} \lesssim \sum_i m_i \lesssim 6 \, \text{eV} \]

We will see that cosmology will be sensitive to total neutrino mass
The neutrino background: neutrino decoupling

Firm established prediction of the standard Big Bang model

\[ f_{eq}(p) = \left[ \exp \left( \frac{p - \mu_\nu}{T} \right) + 1 \right]^{-1} \]

With chemical potential \( \sim 0 \) BBN sets tight limits

\[ n_\nu = \frac{g}{(2\pi)^3} \int d^3p \ f_\nu(E, T_\nu), \quad \rho_\nu = \frac{g}{(2\pi)^3} \int d^3p \ E \ f_\nu(E, T_\nu), \]

Weak interaction rate \quad Hubble parameter

\[ \Gamma_\nu = \langle \sigma_\nu n_\nu \rangle \quad H = \sqrt{\frac{8\pi \rho}{3M_P^2}} \]

\[ T_{\text{dec}} \sim \text{MeV} \quad \text{[at 1 sec]} \]

After decoupling: \( f_{eq} \) is preserved because \( T \) and \( p \) scale as \( 1/a \)

No dependence on the mass

This means that momentum distribution is exact even in the epochs of structure formation!

Alpher, Follin, Herman 1953
CvB
The neutrino background: energy densities

After neutrino decoupling photon temperature drops below electron mass, e+e- annihilation heat the plasma [we are at T~0.5 MeV or so]

\[
n_\nu = \frac{3}{11} n_\gamma = \frac{6\zeta(3)}{11\pi^2} T_\gamma^3
\]

\[
n_\nu / n_\gamma = 0.68
\]

At any time after electron Positron annihilation

\[
\rho_\nu(m_\nu \ll T_\nu) = \frac{7\pi^2}{120} \left(\frac{4}{11}\right)^{4/3} T_\gamma^4
\]

\[
\rho_\nu(m_\nu \gg T_\nu) = m_\nu n_\nu .
\]

Two well-defined limits for matter and radiation

Note: there are small non-thermal distortions in the neutrino FD spectrum and a slight increase in the photon neutrino temperature due to relic interactions of e+ e- with neutrinos [e.g. Dolgov 02]
The neutrino background: today

Extrapolating to today:

\[ n_\nu = 339.5 \text{ cm}^3 \text{ average} \]

\[ T_\gamma = 2.73 \text{ K} \rightarrow T_\nu = 1.95 \text{ K} \]

\[ \langle p \rangle = 1.7 \times 10^{-4} \text{ eV} = 3.15 \, T_\nu \]

Note: clustering in the local Universe can slightly change this number

\[ z_{nr} = \left( \frac{m_\nu}{5.28 \times 10^{-4} \text{ eV}} \right) \left( \frac{T_\nu^a}{T_\nu} \right) - 1 \]

\[ \Omega_\nu h^2 = \frac{m_\nu}{93.14 \text{ eV}} \]

\[ \Omega_\nu h^2 \geq 6 \times 10^{-4} \text{ (NO), or } \geq 10^{-3} \text{ (IO)} \]

\[ \Omega_\nu > 0.5\% \text{ of matter components – sub-dominant matter component} \]

- Hot DM
  - 5\textsuperscript{th} most abundant Universe component by energy
  - 2\textsuperscript{nd} most abundant by number density

Very small numbers \( \rightarrow \) direct detection difficult
The neutrino background – radiation era

\[ \rho_R = \left[ 1 + \frac{7}{8} \left( \frac{4}{11} \right)^{4/3} N_{\text{eff}} \right] \rho_\gamma \]

Extra radiation contribution

\( N_{\text{eff}} = \) energy density of neutrinos + other light particles / energy density of 1 neutrino family neglecting e+e->v

Neff is accurately theoretically estimated to be 3.044 and also measured from cosmological observations

From CMB Planck 2018

\( N_{\text{eff}} = 3.00^{+0.57}_{-0.53} \) (95 %, Planck TT+lowE),

\( N_{\text{eff}} = 2.92^{+0.36}_{-0.37} \) (95 %, Planck TT,TE,EE+lowE),

Also BBN provides constraints and the error bar can be reduced by a factor 2 see Yeh+22

Convincing proof of existence of cosmic neutrino background
Remarkable success of cosmological data in probing particle dark matter (including Non-standard heat injection in the primordial Universe)

Neff now detected at >10-20σ

But there is more:
Detected anisotropies/perturbations in the fluid and their imprint on the CMB (especially polarization)

Everything consistent with a Relativistic fluid with no viscosity!
$N_{\text{eff}}$ to test particle (new) physics

Constraints on additional relativistic particles

$$\Delta N_{\text{eff}} = g \left[ \frac{43}{4 g_s} \right]^{4/3} \times \begin{cases} 
4/7 & \text{boson,} \\
1/2 & \text{fermion,}
\end{cases}$$

Fully thermalized relics

Evolution of effective degrees of freedom for SM particles vs photon temperature

Expected $\Delta N_{\text{eff}}$ today
For species decoupling
From thermal equilibrium
At $T_\gamma$
Evolution of energy densities in a neutrino Universe

Normal hierarchy with $m_1=0$ eV, $m_2=0.009$ eV, $m_3=0.05$ eV
The perturbed Universe

1. The metric

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu = a^2(\tau)[(1 + 2\phi)d\tau^2 - (1 - 2\psi)\delta_{ij}dx^i dx^j] \]

2. The tensor

\[ \delta T^0_0 = \delta \rho \ , \quad \text{Energy density perturbation} \]
\[ \delta T^0_i = (\bar{\rho} + \bar{p}) v_i^{||} , \quad v_i^{||} \text{ Longitudinal component of velocity field} \]
\[ \delta T^i_j = -\delta p \delta^i_j + \Sigma^i_j , \quad \delta p \text{ pressure perturbation, traceless and longitudinal component of the 3x3 tensor} \]

\[ \Sigma^i_j = (\partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2)\tilde{\sigma} \]

3. New Variables

Velocity divergence

\[ \theta \equiv \sum_i \partial_i v_i = \nabla^2 \tilde{u} , \]

Shear (anisotropic) stress

\[ (\bar{\rho} + \bar{p}) \nabla^2 \sigma \equiv -\sum_{i,j} (\partial_i \partial_j - \frac{1}{3} \nabla^2 \delta_{ij}) \Sigma^i_j = -\frac{2}{3} \nabla^4 \tilde{\sigma} \]
4. Perturbed Einstein equation for the scalar sector (which imply conservation of total energy momentum-tensor )

\[
\delta G^0_0 = 2a^{-2} \left\{ -3 \left( \frac{\dot{a}}{a} \right)^2 \phi - 3\frac{\dot{a}}{a} \psi + \nabla^2 \psi \right\} = 8\pi G \delta \rho ,
\]

\[
\delta G^0_i = 2a^{-2} \partial_i \left\{ \frac{\dot{a}}{a} \phi + \psi \right\} = 8\pi G \left( \bar{\rho} + \bar{p} \right) v_i ,
\]

\[
\delta G^i_j = -2a^{-2} \left\{ \left[ \left( \frac{2\ddot{a}}{a} - \left( \frac{\dot{a}}{a} \right)^2 \right) \phi + \frac{\dot{a}}{a} \left( \dot{\phi} + 2\psi \right) + \ddot{\psi} + \frac{1}{3} \nabla^2 (\phi - \psi) \right] \delta^i_j 
\]

\[
- \frac{1}{2} \left( \partial_i \partial_j - \frac{1}{3} \nabla^2 \delta^i_j \right) (\phi - \psi) \right\} = 8\pi G \left( -\delta \rho \delta^i_j + \Sigma^i_j \right)
\]

5. Perturbed Einstein equations + change of variable + let's move to Fourier space

\[
-3 \left( \frac{\dot{a}}{a} \right)^2 \phi - 3\frac{\dot{a}}{a} \dot{\psi} - k^2 \psi = 4\pi G a^2 \bar{\rho} \delta ,
\]

\[
-k^2 \left( \frac{\dot{a}}{a} \phi + \dot{\psi} \right) = 4\pi G a^2 \left( \bar{\rho} + \bar{p} \right) \theta ,
\]

\[
\left( 2\frac{\ddot{a}}{a} - \left( \frac{\dot{a}}{a} \right)^2 \right) \phi + \frac{\dot{a}}{a} \left( \dot{\phi} + 2\dot{\psi} \right) + \ddot{\psi} - \frac{k^2}{3} (\phi - \psi) = 4\pi G a^2 \delta p ,
\]

\[
k^2 (\phi - \psi) = 12\pi G a^2 \left( \bar{\rho} + \bar{p} \right) \sigma
\]
6. ….but we are dealing with fluids: continuity for each fluid component

\[ \delta = (1 + w)(\theta + 3\psi) \]

7. … and Euler for each fluid component

\[ \dot{\theta} = \frac{\dot{a}}{a}(3w - 1)\theta - \frac{\dot{w}}{1 + w}\theta - k^2\phi - k^2\sigma - \frac{w}{1 + w}k^2\delta \]

8. … and a perfect fluid \(\rightarrow\) energy-momentum tensor is diagonal and isotropic

\[ T^{\mu\nu} = -p \ g^{\mu\nu} + (\rho + p)U^\mu U^\nu \]

\[ U^\mu = dx^\mu/[a(1 + \phi)d\tau] \] and obtain:

\[ U^\mu = \left(a^{-1}[1 - \phi], a^{-1}v^i\right), \quad T^0_0 = \rho, \quad T^i_0 = v^i, \quad T^i_i = -p \]
9. Now solve Einstein equations in a **neutrinoless** Universe with the perturbed energy-momentum tensor

\[
\delta T^0_0 = \delta \rho_r + \delta \rho_m , \\
\partial^i (\delta T^0_i) = (\bar{\rho}_r + \bar{\rho}_r) \theta_r + \bar{\rho}_m \theta_m = \frac{4}{3} \bar{\rho}_r \theta_r + \bar{\rho}_m \theta_m , \\
\delta T^i_i = -\delta \rho_r = -\frac{1}{3} \delta \rho_r
\]

At this point it is very important to define a **Jeans length**

**Causal Horizon/Particle Horizon**  
(maximum physical scale at which a signal can propagate)

\[
d(t_i, t) = a(t) \int_{t_i}^{t} dx = a(t) \int_{t_i}^{t} \frac{v dt'}{a(t')}
\]

**Hubble radius/Particle Horizon**  
(for \( a \sim t^n \) and \( n < 1 \))

\[
R_H(t) = \frac{t}{n} , \quad d_H(t \gg t_i) \simeq \frac{t}{1 - n} .
\]

**Acoustic perturbations** \( c_s \)  
**Sound Horizon** \( c_s/H(t) \)

\[
k_J(t) = \left( \frac{4\pi G \bar{\rho}(t) a^2(t)}{c_s^2(t)} \right)^{1/2} , \quad \lambda_J(t) = 2\pi \frac{a(t)}{k_J(t)} = 2\pi \sqrt{\frac{2}{3} \frac{c_s(t)}{H(t)}}
\]
10. Evolution of perturbation in a fluid with perturbations propagating with sound speed $c_s$

Two regimes

Large scales $k < k_J$ pressure unimportant (Jeans unstable)

Small scales Modes with $k > k_J$ will oscillate
With frequency $k x c_s$ (competition
between pressure and gravity) $\rightarrow$ they are Jeans stable

$$\ddot{\delta} + \frac{\dot{a}}{a} \dot{\delta} + (k^2 - k_J^2) c_s^2 \delta = 0$$

11. Jeans instability is a key ingredient for structure formation
before recombination $c_s \sim c$ the photon-baryon fluid oscillates on scales smaller than $\lambda_J$
after recombination $c_s \rightarrow 0$ $\lambda_J \rightarrow 0$ and structure can grow

More in Enzo Branchini’s lectures
The perturbed Universe - VI

In a neutrinoless Universe

More in Douglas Scott’s lectures
The perturbed Universe: adding neutrinos

At least 2 neutrinos should be matter – an extra matter component, this implies that they turned non-relativistic during matter domination z<3000

\[ q_i = \alpha p_i, \quad P_i = a(1 - \psi)p_i \quad f(x^i, P_j, \tau) = f_0(P, \tau) \]

Canonical conjugate of the comoving coordinate \( x^i \)

In absence of perturbation pi will decade as \( 1/a \) while \( P_i \) will stay constant

\[ \epsilon = a(p^2 + m^2)^{1/2} = (q^2 + a^2 m^2)^{1/2} \]

\[ T^0_0 = \bar{\rho}_\nu = \frac{4\pi}{a^4} \int q^2 dq \epsilon f_0(q), \quad f_0(q) = \frac{1}{eq/aT_\nu + 1} \]

\[ T^i_i = -\bar{p}_\nu = -\frac{4\pi}{3a^4} \int q^2 dq \frac{q^2}{\epsilon} f_0(q) \]

Spatial perturbations in the metric will induce variations in neutrino phase-space density

Depending on time, space and momentum \( \rightarrow \) this will impact on the energy momentum tensor

BUT

Scalar sector of the tensor will contain now the anisotropic stress (different w.r.t. perfect fluid)
The perturbed Universe: the energy momentum tensor

\[ f(x^i, P_j, \tau) = f_0(q) [1 + \Psi(x^i, q_j, \tau)] \quad \text{with} \quad P_j = (1 - \Psi)q_j. \]

\[
T_{\mu\nu} = \int dP_1 dP_2 dP_3 (-g)^{-1/2} \frac{P_\mu P_\nu}{P^0} f(x^i, P_j, \tau)
\]

\[
T^0_0(x^i) = a^{-4} \int q^2 dq d\Omega \epsilon f_0(q) [1 + \Psi(x^i, q\hat{n}_j, \tau)] ,
\]

\[
T^0_i(x^i) = a^{-4} \int q^2 dq d\Omega q \hat{n}_i f_0(q) \Psi(x^i, q\hat{n}_j, \tau) ,
\]

\[
T^i_j(x^i) = -a^{-4} \int q^2 dq d\Omega \frac{q^2}{\epsilon} \hat{n}_i \hat{n}_j f_0(q) [1 + \Psi(x^i, q\hat{n}_j, \tau)] ,
\]

Collisionless fluid with no Microscopic interactions and No bulk motions

d\Omega is the differential of the momentum direction \(\hat{n}_j = q_j/q\)

\[ (-g)^{-1/2} = a^{-4}(1 - \phi + 3\psi) \]

These are now the perturbed components

\[
\delta \rho_\nu = a^{-4} \int q^2 dq d\Omega \epsilon f_0 \Psi , \quad \delta P_\nu = \frac{1}{3} a^{-4} \int q^2 dq d\Omega \frac{q^2}{\epsilon} f_0 \Psi , \quad (92)
\]

\[
\delta T^0_0 = a^{-4} \int q^2 dq d\Omega q \hat{n}_i f_0 \Psi , \quad \Sigma^i_j = -a^{-4} \int q^2 dq d\Omega \frac{q^2}{\epsilon} (\hat{n}_i \hat{n}_j - \frac{1}{3} \delta_{ij}) f_0 \Psi .
\]
As we defined the Jeans length we can now replace $c_s$ with $v_{\text{thermal}}$

$$k_{FS}(t) = \left( \frac{4 \pi G \bar{\rho}(t) a^2(t)}{v_{\text{th}}^2(t)} \right)^{1/2}, \quad \lambda_{FS}(t) = 2 \pi \frac{a(t)}{k_{FS}(t)} = 2 \pi \sqrt{\frac{2 v_{\text{th}}(t)}{3 H(t)}}$$

$$v_{\text{th}} = \frac{\langle p \rangle}{m} \simeq \frac{3 T_{\nu}}{m} = \frac{3 T_{\nu}^0}{m} \left( \frac{a_0}{a} \right) \simeq 150 (1 + z) \left( \frac{1 \text{eV}}{m} \right) \text{km s}^{-1}$$

$$\lambda_{FS}(t) = 7.7 \frac{1 + z}{\sqrt{\Omega_\Lambda + \Omega_m (1 + z)^3}} \left( \frac{1 \text{eV}}{m} \right) h^{-1} \text{Mpc},$$

$$k_{FS}(t) = 0.82 \frac{\sqrt{\Omega_\Lambda + \Omega_m (1 + z)^3}}{(1 + z)^2} \left( \frac{m}{1 \text{eV}} \right) h \text{Mpc}^{-1},$$
The perturbed Universe: neutrino free streaming - II

Matteo Viel

These scales are physical (not comoving) and set by particle physics and have to be compared to cosmic expansion.
Before non-relativistic transition $\lambda_{FS} \sim t$ and $a \sim t^{1/2}$
After non-relativistic transition free-streaming scale increases $\lambda_{FS} \sim 1/(aH) \sim t^{1/3}$
But comoving $\lambda_{FS}/a \sim t^{-1/3}$, because $a \sim t^{2/3}$, comoving free streaming scale decreases!

Thus, if neutrinos become relativistic during MD the comoving free-streaming scale passes through a minimum at $k=k_{nr}$ when $m=\langle p \rangle=3T\nu=2000$ (m/1eV)

$$k_{nr} \simeq 0.018 \Omega_m^{1/2} \left( \frac{m}{1 \text{ eV}} \right)^{1/2} h \text{ Mpc}^{-1}$$

Damping of perturbations below this scale
1- Neutrinos cannot be confined into these smaller scales
2- Metric perturbations will be damped at these scales by gravitational back-reaction
3- Modes at $k<k_{nr}$ will evolve like in a neutrinoless universe
The perturbed Universe: neutrino free streaming - III

Taken from Palanque-Delabrouille PONT17 talk
The perturbed Universe: neutrino free streaming - IV

Matteo Viel

After decoupling neutrino free-stream

Free streaming scale is a dynamical quantity which quantifies which scales free-streaming can be neglected in the evolution equations at any given time

\[ \lambda_{FS} \equiv 2\pi \sqrt{\frac{2}{3} \frac{c_\nu(\eta)}{H(\eta)}} \]

Free streaming horizon is the average distance travelled by neutrinos between the early universe and a given time, displaying the scales that can be affected at all times

\[ d_{FS} = ar_{FS} \equiv a \int_{\eta_{in}}^{\eta} c_\nu(\eta) d\eta \]

Comoving horizon scale

Free streaming horizon is the key physical quantity however this role is also taken in the literature by \( 2\pi/k_{nr} \)

NOTE: Neutrinos in the keV range will become non-relativistic in the RD era where \( c_\nu \sim 1/a \sim 1/\eta \) and \( H \sim \eta^{-2} \) Thus free streaming scale increases like \( \eta \). While after equality it will decrease. Maximum is in this case reached between equality and non-relativistic transition In this case \( d_{FS} \) can be much larger than \( 1/k_{nr} \) (and grows logarithmically)
Vlasov (neutrino) equation

\[
\frac{Df}{d\tau} = \frac{\partial f}{\partial \tau} + \frac{dx^i}{d\tau} \frac{\partial f}{\partial x^i} + \frac{dq}{d\tau} \frac{\partial f}{\partial q} + \frac{dn_i}{d\tau} \frac{\partial f}{\partial n_i} = 0
\]

1. Perturb f and keep linear order only
2. Fill with the metric
3. Go to Fourier

General case with non-zero Non-gravitational interactions C[f]

\[
\frac{\partial f}{\partial \tau} + \frac{dx}{d\tau} \cdot \frac{\partial f}{\partial x} + \frac{dq}{d\tau} \cdot \frac{\partial f}{\partial q} = C[f]
\]

\[
\dot{\Psi} - i \frac{q}{\epsilon} (\vec{k} \cdot \hat{n}) \Psi = - \left( \psi + i \frac{\epsilon}{q} (\vec{k} \cdot \hat{n}) \phi \right) \frac{\partial \ln f_0}{\partial \ln q}.
\]

Neutrino phase space

Metric
Vlasov (neutrino) equation in MD regime

\[
\frac{dq}{d\tau} = q\dot{\psi} + (q^2 + a^2 m^2)^{1/2} \hat{n}_i \partial_i \phi \quad \text{Geodesic equation}
\]

\[
\Psi(k, q, \hat{n}, \tau) = \sum_{l=0}^{\infty} (-i)^l (2l + 1) \Psi_l(k, q, \tau) P_l(k \cdot \hat{n}) . \quad \text{Legendre polynomials } P_l
\]

Then, the Vlasov equation becomes

\[
\dot{\Psi}_0 = \frac{q k}{\epsilon} \Psi_1 - \psi \frac{\partial \ln f_0}{\partial \ln q} ,
\]

\[
\dot{\Psi}_1 = \frac{q k}{3 \epsilon} (\Psi_2 - 2\Psi_0) + \frac{\epsilon k}{3 q} \phi \frac{\partial \ln f_0}{\partial \ln q} ,
\]

\[
\dot{\Psi}_l = \frac{q k}{(2l + 1) \epsilon} [(l + 1) \Psi_{l+1} - l \Psi_{l-1}] , \quad l \geq 2 .
\]

Infinite series of differential equations
With multipoles related to physical quantities

\[
\delta \rho_\nu = 4\pi a^{-4} \int q^2 dq \epsilon f_0(q) \Psi_0 ,
\]

\[
\delta p_\nu = \frac{4\pi}{3} a^{-4} \int q^2 dq \frac{q^2}{\epsilon} f_0(q) \Psi_0 ,
\]

\[
(\bar{\rho}_\nu + \bar{p}_\nu) \theta_\nu = 4\pi k a^{-4} \int q^2 dq q f_0(q) \Psi_1 ,
\]

\[
(\bar{\rho}_\nu + \bar{p}_\nu) \sigma_\nu = -\frac{8\pi}{3} a^{-4} \int q^2 dq \frac{q^2}{\epsilon} f_0(q) \Psi_2 .
\]
This is for the mode $k=0.1 \, h/\text{Mpc}$ and $m=0.1 \, \text{eV}$ using adiabatic initial conditions set by inflation as initial conditions. Thin lines are the massless case.

Isotropic stress perturbation. And shear stress start to be subdominant. After NR transition, damping is clearly visible. After non-relativistic transition $a/a_0=5 \times 10^{-3}$.

CMBFAST, CLASS, CAMB. Public available codes.
Vlasov (neutrino) equation in MD regime - III

Well in the matter dominated regime, things get simpler

\[ \dot{\delta}_\nu = \theta_\nu + 3\psi , \]
\[ \dot{\theta}_\nu = -\frac{\dot{a}}{a} \theta_\nu - k^2 \phi . \]

\[ \psi \text{ and } \phi \sim \text{const and } a \sim \tau^2 \]

\[ \ddot{\delta}_\nu + \frac{\dot{a}}{a} \delta_\nu = -k^2 \phi + 3(\psi + \frac{\dot{a}}{a} \psi) \]

Neutrinos grow like matter!

\[ \delta_\nu \longrightarrow -\frac{2}{3} \left( \frac{k}{aH} \right)^2 \phi \propto a \]

This above is solution of Poisson Equation in a MD Universe

For \( k > k_{\text{nr}} \)

Note that \( \delta_\nu \) can grow faster than \( \sim a \)

For a short time due to the \( \ln a \) term
From the above equations and neglecting neutrinos’ backreaction on cdm

\[
P(k) = \left\langle \left( \frac{\delta \rho_{\text{cdm}} + \delta \rho_b + \delta \rho_{\nu}}{\rho_{\text{cdm}} + \rho_b + \rho_{\nu}} \right)^2 \right\rangle
\]

\[
= \left\langle \left( \frac{\Omega_{\text{cdm}} \delta_{\text{cdm}} + \Omega_b \delta_b + \Omega_{\nu} \delta_{\nu}}{\Omega_{\text{cdm}} + \Omega_b + \Omega_{\nu}} \right)^2 \right\rangle
\]

\[
= \begin{cases} 
\left\langle \delta_{\text{cdm}}^2 \right\rangle & \text{for } k < k_{\text{nr}}, \\
[1 - \Omega_{\nu}/\Omega_{\text{m}}]^2 \left\langle \delta_{\text{cdm}}^2 \right\rangle & \text{for } k \gg k_{\text{nr}},
\end{cases}
\]

Factor in front of Pcdm is \( \sim 2f_{\nu} \) however…. We are lucky the actual effect will be 8 times larger… why?
1. Effects from homogenous pressure and density $\rightarrow$ Hubble expansion
2. Gravitational back-reaction on metric perturbations through modification of energy-momentum tensor

To check for 1. we can omit the $\delta v$ in Poission equation

\[
\ddot{\delta}_{\text{cdm}} + \frac{\dot{a}}{a} \dot{\delta}_{\text{cdm}} = -k^2 \phi + 3(\psi + \frac{\dot{a}}{a} \psi)
\]

\[
\ddot{\delta}_{\text{cdm}} + \frac{2}{\tau} \dot{\delta}_{\text{cdm}} - \frac{6}{\tau^2} \delta_{\text{cdm}} = 0 ,
\]

\[
f_\nu \equiv \frac{\rho_\nu}{(\rho_{\text{cdm}} + \rho_b + \rho_\nu)} = \frac{\Omega_\nu}{\Omega_m}
\]

\[
\delta_{\text{cdm}} \propto \tau^{2p}
\]

\[
\delta_{\text{cdm}} \propto a^{p-1} \simeq a^{\frac{1-3}{5} f_\nu - \frac{3}{5} f_\nu}
\]

Poisson eq.

\[
\delta_{\text{cdm}} \propto [a g(a)]^{p-1} \simeq [a g(a)]^{1-\frac{3}{5} f_\nu} .
\]
Matter power spectrum from massive vs massless vs

1. Large scales $k < k_{nr}$

2. $k \gg k_{nr}$ and $k \gg k_{eq}$ for $a < a_{nr}$

$$\delta_{\text{cdm}}^{f_{\nu}}[a] = \delta_{\text{cdm}}^{f_{\nu}=0}[(1 - f_{\nu})a] \quad a_{eq}^{f_{\nu}}/a_{eq}^{f_{\nu}=0} = (1 - f_{\nu})^{-1}$$

$$\delta_{\text{cdm}}^{f_{\nu}}[a_{0}] = \left(\frac{a_{0} g(a_{0})}{a_{nr}}\right)^{1 - \frac{6}{5} f_{\nu}} \delta_{\text{cdm}}^{f_{\nu}}[a_{nr}]$$

$$\frac{\delta_{\text{cdm}}^{f_{\nu}}[a_{0}]}{\delta_{\text{cdm}}^{f_{\nu}=0}[a_{0}]} = (1 - f_{\nu})^{1/2} \left(\frac{a_{0} g(a_{0})}{a_{nr}}\right)^{-\frac{3}{5} f_{\nu}}$$

$$\frac{P(k)^{f_{\nu}}}{P(k)^{f_{\nu}=0}} = (1 - f_{\nu})^{3} \left(\frac{a_{0} g(a_{0})}{a_{nr}}\right)^{-\frac{6}{5} f_{\nu}} = (1 - f_{\nu})^{3} \left[1.9 \times 10^{5} g(a_{0}) \omega_{m} f_{\nu}/N_{\nu}\right]^{-\frac{6}{5} f_{\nu}}$$

Small few % effect? Yes!
But "integrated" through structure formation era i.e. $P(k)$ is $P(k,z)$

But… is it really small? ---> ask also Enzo Branchini!
Matter power spectrum from massive vs massless vs

In practice: numerical solutions → popular Boltzmann solvers like CMBFAST and CLASS

Massless neutrino universe

Massive neutrino universe: $f_\nu = 0.1$

Equality takes place a bit later

Massive neutrino universe: $k = 1 \text{ h/Mpc}$

Massive neutrino universe: $k = 10^{-3} \text{ h/Mpc}$
Matter power spectrum from massive vs massless vs

Suppression factor computed today at $k=10 \, h/\text{Mpc}$

Little sensitivity to the mass splitting
Matter power spectrum from massive vs massless vs Neutrino free streaming effect

From 0.15 eV to 1.5eV total neutrino mass
Matter power spectrum from massive vs massless vs Neutirino free streaming effect
Matter power spectrum from massive vs massless vs
Matter power spectrum from massive vs massless vs

MASS SPLITTING

\[
\frac{\rho}{\rho_{0/m_0/m_{0q}}} \quad \text{vs} \quad k \ (h/\text{Mpc})
\]

TOTAL MASS = 0.12 eV
Matter power spectrum from massive vs massless vs

For CMB: Douglas Scott
Paolo Natoli
Matter power spectrum from massive vs massless vs

For CMB: Douglas Scott
Paolo Natoli
Matter power spectrum from massive vs massless vs

\[ P(k) \ (\text{Mpc}/h)^3 \]

- \( f_\nu = 0 \)
- \( f_\nu = 0.1 \), fixed \((\omega_c, \Omega_\Lambda)\)
- \( f_\nu = 0.1 \), fixed \((\omega_c, h)\)
Matter power spectrum from massive vs massless vs
Matter power spectrum from massive vs massless vs
Recap - key moments

1. Relativistic neutrino contribution to cosmic expansion

2. Neutrino free streaming slows down CMB photon clustering

3. Metric fluctuations During non-relativistic neutrino transition (early ISW)

4. Neutrino free streaming slows down ordinary matter clustering

5. Non-relativistic neutrino contribution to late expansion rate

BBN CMB LSS
Recap - key take home messages

1. Neutrino number density is large, like photons, in terms of number density is the second most abundant species
   \[ n_\gamma \sim n_\nu \sim 10^{10} n_{\text{atoms, e-}} \]

2. Unlike other particles they become non-relativistic after decoupling and do not annihilate

3. Looking at the whole Universe from large to small scales they can be probed