Boolean Operations, Joins, and the Extended Low Hierarchy

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Abstract

We prove that the join of two sets may actually fall into a lower level of the extended low hierarchy [BBS86] than either of the sets. In particular, there exist sets that are not in the second level of the extended low hierarchy, EL₂, yet their join is in EL₂. That is, in terms of extended lowness, the join operator can lower complexity. Since in a strong intuitive sense the join does not lower complexity, our result suggests that the extended low hierarchy is unnatural as a complexity measure. We also study the closure properties of EL₂ and prove that EL₂ is not closed under certain Boolean operations. To this end, we establish the first known (and optimal) EL₂ lower bounds for certain notions generalizing Selman’s P-selectivity [Sel79], which may be regarded as an interesting result in its own right.

1 Introduction

The low hierarchy [Sch83] provides a yardstick to measure the complexity of sets that are known to be in NP but that are seemingly neither in P nor NP-complete. In order to extend this classification beyond NP, the extended low hierarchy [BBS86] has been introduced (see the surveys [Köb95, Hem93]). An informal way of describing the intuitive nature of these hierarchies might be the following: A set \( A \) that is placed in the \( k \)th level of the low or the extended low hierarchy contains no more information than the empty set relative to the computation of a \( \Sigma^p_k \) machine (see [MS72, Sto77] for the definition of the \( \Sigma \) levels of the polynomial hierarchy), either because \( A \) is so chaotically organized that a \( \Sigma^p_k \) machine is not able to extract useful information from \( A \), or because \( A \) is so simple that it has no useful information to offer a \( \Sigma^p_k \) machine.\(^1\) The low and extended low hierarchies have been very thoroughly investigated in many papers (see, e.g., [Sch83, KS85, BBS86, Sch87, Sch89, Ko91, AH92, ABC, Köb94, SL94, HNOS96]). In light of the informal intuition given above—that classifying the level in the extended low hierarchy of a problem or a class gives insight into the amount of polynomial-hierarchy computational power needed to make access to the problem or the class redundant—one main motivation for the study of the extended low hierarchy is to understand which natural complexity classes and problems easily extend the power of the polynomial hierarchy and which do not. Among the important natural classes and problems that have been carefully classified in these terms are the Graph Isomorphism Problem (which in fact is known to be low), bounded probabilistic polynomial time (BPP), approximate polynomial time (APT), the class of complements of sets having Arthur-Merlin games (coAM), the class of sparse and co-sparse sets, the P-selective sets, and the class of sets having polynomial-size circuits (P/poly).

Another motivation for the study of the low and extended low hierarchies is to relate their properties to other complexity-theoretic concepts. For instance, Schöning showed that the existence of an NP-complete set (under any “reasonable” reducibility) in the low hierarchy implies a collapse of the polynomial hierarchy [Sch83]. Among the most important recent results about extended lowness

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\(^1\)We stress that this is a very loose and informal description. In particular, for the case of the extended low hierarchy, it would be more accurate to say: A set \( A \) that is placed in the \( (k+1) \)st level of the extended low hierarchy, \( k > 1 \), is such that NP\(^k\) contains no more information than SAT ⊕ L relative to the computation of a \( \Sigma^p_k \) machine.
are Sheu and Long’s result that the extended low hierarchy is a strictly infinite hierarchy \cite{SL94} and Köbler’s optimal location of P/poly in the extended low hierarchy \cite{Kob94}. In this note, we seek to further explore the structure of the extended low hierarchy by studying its interactions with such operations as the join. In particular, we prove properties of EL₂ regarding its interaction with the join and with Boolean operations. Our results add to the body of evidence that extended lowness does not provide a natural, intuitive measure of complexity.

In light of the many ways in which extended lowness captures certain concepts of low information content (such as all sparse sets and certain reduction closures of the sparse sets—e.g., the Turing closure of the class of sparse sets, which is known to be equal to P/poly) as well as certain concepts of “almost” feasible computation (such as BPP, APT, and P-selectivity, etc.), it might be tempting to assume that extended lowness would provide a reasonable measure of complexity in the sense that a problem’s property of being extended low indicates that this problem is of “low” complexity. However, in Section 2, we will prove that the join operator can lower difficulty as measured in terms of extended lowness: There exist sets that are not in EL₂, yet their join is in EL₂. Since in a strong intuitive sense the join does not lower complexity, our result suggests that, if one’s intuition about complexity is—as is natural—based on reductions, then the extended low hierarchy is not a natural measure of complexity. Rather, it is a measure that is related to the difficulty of information extraction, and it is in flavor quite orthogonal to more traditional notions of complexity. That is, our result sheds light on the orthogonality of “complexity in terms of reductions” versus “difficulty in terms of non-extended-lowness.” In fact, our result is possible only since the second level of the extended low hierarchy is not closed under polynomial-time many-one reductions (this non-closure is known, see \cite{AH92}, and it also follows as a corollary of our result).

In Section 3, we apply the technique developed in the preceding section to prove that the second level of the extended low hierarchy is not closed under the Boolean operations intersection, union, exclusive-or, and equivalence. Our result will follow from the proof of another result, which establishes the first known (and optimal) EL₂ lower bounds for generalized selectivity-like classes (that generalize Selman’s class of P-selective sets \cite{Sel79}, denoted P-Sel) such as the polynomial-time membership-comparable sets introduced by Ogihara \cite{Ogi95} and the multi-selective sets introduced by Hemaspaandra et al. \cite{HJR96}. These results sharply contrast with the known result that all P-selective sets are in EL₂ and they are thus interesting in their own right.

## 2 Extended Lowness and the Join Operator

The low hierarchy and the extended low hierarchy are defined as follows.

**Definition 1** \cite{Sch83}

1. For each \(k \geq 1\), define \(\text{Low}_k \triangleq \{L \in \text{NP} \mid \Sigma^{p,L}_k = \Sigma^p_k\}\).

2. For each \(k \geq 2\), define \(\text{EL}_k \triangleq \{L \mid \Sigma^{p,L}_k = \Sigma^{p,L\oplus\text{SAT}}_{k-1}\}\), where SAT is the set of all satisfiable Boolean formulas.

For sets \(A\) and \(B\), their join, \(A \oplus B\), is \(\{0x \mid x \in A\} \cup \{1x \mid x \in B\}\). Theorem 2 below establishes that the join operator can lower the difficulty measured in terms of extended lowness.
At first glance, this might seem paradoxical. After all, every set that \( \leq_p \) reduces to a set \( A \) or \( B \) also reduces to \( A \oplus B \), and thus intuition strongly suggests that \( A \oplus B \) must be at least as hard as \( A \) and \( B \), as most complexity lower bounds (e.g., NP-hardness) are defined in terms of reductions. However, extended lowness merely measures the opacity of a set’s internal organization, and thus Theorem 2 is not paradoxical. Rather, Theorem 2 highlights the orthogonality of “complexity in terms of reductions” and “difficulty in terms of non-extended-lowness.” Indeed, note Corollary 3, which was first observed by Allender and Hemaspaandra (then Hemachandra) [AH92]. We interpret Theorem 2 as evidence that extended lowness is not an appropriate, natural complexity measure with regard to even very simple operations such as the join.

**Theorem 2** There exist sets \( A \) and \( B \) such that \( A \not\subseteq \text{EL}_2 \) and \( B \not\subseteq \text{EL}_2 \), and yet \( A \oplus B \in \text{EL}_2 \).

Lemma 3 below will be used in the upcoming proof of Theorem 2. First, we fix some notations. Fix the alphabet \( \Sigma = \{0, 1\} \). Let \( \Sigma^* \) denote the set of all strings over \( \Sigma \). For each set \( L \subseteq \Sigma^* \), \( L^{\leq n} \) is the set of all strings in \( L \) having length \( n \) (less than or equal to \( n \)), and \( |L| \) denotes the cardinality of \( L \). Let \( \Sigma^n \) be a shorthand for \( (\Sigma^*)^n \). Let \( \leq_{\text{lex}} \) denote the standard quasi-lexicographical ordering on \( \Sigma^* \). The census function of a set \( L \) is defined by \( \text{census}_L(0^n) = |L^{\leq n}| \). \( L \) is said to be sparse if there is a polynomial \( p \) such that for every \( n \), \( \text{census}_L(0^n) \leq p(n) \). Let \( \text{SPARSE} \) denote the class of all sparse sets. For each class \( C \) of sets over \( \Sigma \), define \( \text{co} C \equiv \{ L \mid \overline{L} \in C \} \). Let \( \mathbb{N} \) denote the set of non-negative integers. To encode a pair of integers, we use a one-one, onto, polynomial-time computable pairing function, \( (\cdot, \cdot) : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \), that has polynomial-time computable inverses. \( \text{FP} \) denotes the class of polynomial-time computable functions. We shall use the shorthand NPTM to refer to “nondeterministic polynomial-time Turing machine.” For an NPTM \( M \) (an NPTM \( M \) and a set \( A \), respectively), \( L(M) (L(M^A)) \) denotes the set of strings accepted by \( M \) (relative to \( A \)).

**Lemma 3** If \( F \) is a sparse set and \( \text{census}_F \in \text{FP}^{F \oplus \text{SAT}} \), then \( F \in \text{EL}_2 \).

**Proof.** Let \( L \in \text{NP}^{\text{NP}_F} \) via NPTMs \( N_1 \) and \( N_2 \), i.e., \( L = L(N_1^{N(N^F)}) \). Let \( q(n) \) be a polynomial bounding the length of all queries that can be asked in the run of \( N_1^{L(N^F)} \) on inputs of length \( n \). Below we describe an NPTM \( N \) with oracle \( F \oplus \text{SAT} \):

On input \( x \), \( |x| = n \), \( N \) first computes \( \text{census}_F(0^i) \) for each relevant length \( i \leq q(n) \), and then guesses all sparse sets up to length \( q(n) \). Knowing the exact census of \( F \), \( N \) can use the \( F \) part of its oracle to verify whether the guess for \( F^{\leq q(n)} \) is correct, and rejects on all incorrect paths. On the correct path, \( N \) uses itself, the SAT part of its oracle, and the correctly guessed set \( F^{\leq q(n)} \) to simulate the computation of \( N_1^{L(N^F)} \) on input \( x \).

Clearly, \( L(N^{F \oplus \text{SAT}}) = L \). Thus, \( \text{NP}^{\text{NP}_F} \subseteq \text{NP}^{F \oplus \text{SAT}} \), i.e., \( F \in \text{EL}_2 \). \( \square \)

**Proof of Theorem 2.** \( A \equiv \bigcup_{i \geq 0} A_i \) and \( B \equiv \bigcup_{i \geq 0} B_i \) are constructed in stages. In order to show \( A \not\subseteq \text{EL}_2 \) and \( B \not\subseteq \text{EL}_2 \) it suffices to ensure in the construction that \( \text{NP}^A \not\subseteq \text{coNP}^{A \oplus \text{SAT}} \) and \( \text{NP}^B \not\subseteq \text{coNP}^{B \oplus \text{SAT}} \) (and thus, \( \text{NP}^A \not\subseteq \text{NP}^{A \oplus \text{SAT}} \) and \( \text{NP}^B \not\subseteq \text{NP}^{B \oplus \text{SAT}} \)).

\(^2\)For sets \( X \) and \( Y \), \( X \leq_p^m Y \) if and only if there is a polynomial-time computable function \( f \) such that \( X = \{ x \mid f(x) \in Y \} \).
Define function \( t \) inductively by \( t(0) \overset{df}= 2 \) and \( t(i) \overset{df}= 2^{2^{t(i-1)}} \) for \( i \geq 1 \). Let \( \{N_i\}_{i \geq 1} \) be a fixed enumeration of all coNP oracle machines having the property that the runtime of each \( N_i \) is independent of the oracle and each machine appears infinitely often in the enumeration. Define

\[
L_A \overset{df}= \{0^{t(i)} \mid \exists j \geq 1 \left[ i = \langle 0, j \rangle \land \|A \cap \Sigma^t(i)\| \geq 1 \right]\},
\]

\[
L_B \overset{df}= \{0^{t(i)} \mid \exists j \geq 1 \left[ i = \langle 1, j \rangle \land \|B \cap \Sigma^t(i)\| \geq 1 \right]\}.
\]

Clearly, \( L_A \in \text{NP}^A \) and \( L_B \in \text{NP}^B \). In stage \( i \) of the construction, at most one string of length \( t(i) \) will be added to \( A \) and at most one string of length \( t(i) \) will be added to \( B \) in order

1. to ensure \( L(N_{j}^{A_1 \oplus \text{SAT}}) \neq L_A \) if \( i = \langle 0, j \rangle \) (or \( L(N_{j}^{B_1 \oplus \text{SAT}}) \neq L_B \), respectively, if \( i = \langle 1, j \rangle \)), and

2. to encode an easy to find string into \( A \) if \( i = \langle 1, j \rangle \) (or into \( B \) if \( i = \langle 0, j \rangle \)) indicating whether or not some string has been added to \( B \) (or to \( A \)) in (1).

Let \( A_{i-1} \) and \( B_{i-1} \) be the content of \( A \) and \( B \) prior to stage \( i \). Initially, let \( A_0 = B_0 = \emptyset \). Stage \( i \) is as follows:

First assume \( i = \langle 0, j \rangle \) for some \( j \geq 1 \). If it is the case that no path of \( N_{j}^{A_1 \oplus \text{SAT}}(0^{t(i)}) \) can query all strings in \( \Sigma^t(i) - \{0^{t(i)}\} \) and \( N_{j}^{A_1 \oplus \text{SAT}}(0^{t(i)}) \) cannot query any string of length \( t(i+1) \) (otherwise, just skip this stage—we will argue later that the diagonalization still works properly), then simulate \( N_{j}^{A_1 \oplus \text{SAT}} \) on input \( 0^{t(i)} \). If it rejects (in the sense of coNP, i.e., if it has one or more rejecting computation paths), then fix some rejecting path and let \( w_i \) be the smallest string in \( \Sigma^t(i) - \{0^{t(i)}\} \) that is not queried along this path, and set \( A_i := A_{i-1} \cup \{w_i\} \) and \( B_i := B_{i-1} \cup \{0^{t(i)}\} \).

Otherwise (i.e., if \( 0^{t(i)} \in L(N_{j}^{A_1 \oplus \text{SAT}}) \)), set \( A_i := A_{i-1} \) and \( B_i := B_{i-1} \). The case of \( i = \langle 1, j \rangle \) is analogous: just exchange \( A \) and \( B \). This completes the construction of stage \( i \).

Since each machine \( N_i \) appears infinitely often in our enumeration and as the \( t(i) \) are strictly increasing, it is clear that for only a finite number of the \( N_{i_1}, N_{i_2}, \ldots \) that are the same machine as \( N_i \) can it happen that stage \( i_k \) must be skipped (in order to ensure that \( w_{i_k} \), if needed to diagonalize against \( N_{i_k} \), indeed exists, or that the construction stages do not interfere with each other), and thus each machine \( N_i \) is diagonalized against eventually. This proves that \( A \notin \text{EL}_2 \) and \( B \notin \text{EL}_2 \). Now observe that \( A \oplus B \) is sparse and that \( \text{census}_{A \oplus B} \in \text{FP}^{A \oplus B} \). Indeed,

\[
\text{census}_{A \oplus B}(0^n) = 2\left(\|A \cap \{0,00,\ldots,0^n-1\}\| + \|B \cap \{0,00,\ldots,0^n-1\}\|\right).
\]

Thus, by Lemma 3, \( A \oplus B \in \text{EL}_2 \).

\[\square\]

**Corollary 4** [AH92] \( \text{EL}_2 \) is not closed under \( \leq^p_m \)-reductions.

In contrast to the extended low hierarchy, every level of the low hierarchy within NP is clearly closed under \( \leq^p_m \)-reductions. Thus, the low hierarchy analog of Theorem 2 cannot hold.

**Fact 5** \((\forall k \geq 0) \ (\forall A, B) \ [(A \not\in \text{Low}_k \lor B \not\in \text{Low}_k) \implies A \oplus B \not\in \text{Low}_k]\).
**Proof.** Assume $A \oplus B \in \text{Low}_k$. Since for all sets $A$ and $B$, $A \leq_p m A \oplus B$ and $B \leq_p m A \oplus B$, the closure of $\text{Low}_k$ under $\leq_p m$-reductions implies that both $A$ and $B$ are in $\text{Low}_k$. \qed

One of the most interesting open questions related to the results presented in this note is whether the join operator also can raise the difficulty measured in terms of extended lowness. That is, do there exist sets $A$ and $B$ such that $A \in \text{EL}_k$ and $B \in \text{EL}_k$, and yet $A \oplus B \not\in \text{EL}_k$ for, e.g., $k = 2$? Or is the second level of the extended low hierarchy (and more generally, are all levels of this hierarchy) closed under join? Regarding potential generalizations of our result, we conjecture that Theorem 2 can be generalized to higher levels of the extended low hierarchy. Such a result, to be sure, would probably require some new technique such as a clever modification of the lower-bound technique for constant-depth Boolean circuits developed by Yao, H˚ astad, and Ko (see, e.g., [Has89, Ko91]).

### 3 EL₂ is not Closed Under Certain Boolean Connectives

In this section, we will prove that the second level of the extended low hierarchy is not closed under the Boolean connectives union, intersection, exclusive-or, or equivalence. We will do so by combining the technique of the previous section with standard techniques of constructing P-selective sets. To this end, we first seek to improve the known EL₂ lower bounds of P/poly, the well-studied class of sets having polynomial-size circuits [KL80]. To wit, we will show that certain generalizations of the class of P-selective sets, though still contained in P/poly [Ogi95, HJRW96], are not contained in EL₂. As interesting as this result may be in its own right, its proof will even provide us with the means required to show the above-mentioned main result of this section: EL₂ is not closed under certain Boolean connectives (and indeed P-selective sets can be used to witness the non-closure). This extends the main result of Hemaspaandra and Jiang [HJ95], namely that P-Sel is not closed under those Boolean connectives.

Let us first recall the following generalizations of Selman’s P-selectivity. Ogihara introduced the P-membership comparable sets [Ogi95] and the present paper’s authors ([HJRW96], see also [Rot95]) introduced the notion of multi-selectivity as defined in Definition 7.

**Definition 6 [Ogi95]** Fix a positive integer $k$. A function $f$ is called a $k$-membership comparing function for a set $A$ if and only if for every $w_1, \ldots, w_m$ with $m \geq k$,

$$f(w_1, \ldots, w_m) \in \{0, 1\}^m \text{ and } (\chi_A(w_1), \ldots, \chi_A(w_m)) \neq f(w_1, \ldots, w_m),$$

where $\chi_A$ denotes the characteristic function of $A$. If in addition $f \in \text{FP}$, $A$ is said to be polynomial-time $k$-membership comparable. Let P-mc$(k)$ denote the class of all polynomial-time $k$-membership comparable sets.

We can equivalently (i.e., without changing the class) require in the definition that $f(w_1, \ldots, w_m) \neq (\chi_A(w_1), \ldots, \chi_A(w_m))$ must hold only if the inputs $w_1, \ldots, w_m$ happen to be distinct. This is true because if there are $r$ and $t$ with $r \neq t$ and $w_r = w_t$, then $f$ simply outputs a length $m$ string having a “0” at position $r$ and a “1” at position $t$. 

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Definition 7 Fix a positive integer $k$. Given a set $A$, a function $f \in \text{FP}$ is said to be an $S(k)$-selector for $A$ if and only if $f$ satisfies the following property: For each set of distinct input strings $y_1, \ldots, y_n$,

1. $f(y_1, \ldots, y_n) \in \{y_1, \ldots, y_n\}$, and
2. if $\|A \cap \{y_1, \ldots, y_n\}\| \geq k$, then $f(y_1, \ldots, y_n) \in A$.

The class of sets having an $S(k)$-selector is denoted by $S(k)$.

It is easy to see that $P\text{-mc}(1) = P$ and $S(1) = \text{P-Sel}$. Furthermore, though the hierarchies $\bigcup_k P\text{-mc}(k)$ and $\bigcup_k S(k)$ are properly infinite, they both are still contained in $P/\text{poly}$ [Ogi95, HJR96]. Among a number of other results, all the relations between the classes $P\text{-mc}(j)$ and $S(k)$ are completely established in Hemaspaandra et al. [HJR96]. These relations are stated in Lemma 8 below, as they’ll be referred to in the upcoming proof of Theorem 9.

Lemma 8 [HJR96]

1. $P\text{-mc}(2) \not\subseteq \bigcup_{k \geq 1} S(k)$.
2. For each $k \geq 1$, $S(k) \subset P\text{-mc}(k + 1)$ and $S(k) \not\subseteq P\text{-mc}(k)$.

The following result establishes a structural difference between Selman’s $P$-selectivity and the generalized selectivity introduced above: Though clearly $S(1) = \text{P-Sel} \subseteq \text{EL}_2$ [ABG] and $P\text{-mc}(1) = P \subseteq \text{EL}_2$, we show that there are sets (indeed, sparse sets) in $S(2) \cap P\text{-mc}(2)$ that are not in $\text{EL}_2$. Previously, Allender and Hemaspaandra [AH92] have shown that $P/\text{poly}$ (and indeed $\text{SPARSE}$ and $\text{coSPARSE}$) is not contained in $\text{EL}_2$. Theorem 9 and Corollary 10, however, extend this result and give the first known (and optimal) $\text{EL}_2$ lower bounds for generalized selectivity-like classes.

Theorem 9 $\text{SPARSE} \cap S(2) \cap P\text{-mc}(2) \not\subseteq \text{EL}_2$.

Proof. Let $t$ be the function defined in the proof of Theorem 8 that gives triple-exponentially spaced gaps. Let $T_k \df \Sigma^{t(k)}$, for $k \geq 0$, and $T \df \bigcup_{k \geq 0} T_k$. Let $\text{EE}$ be defined as $\bigcup_{c \geq 0} \text{DTIME}[2^{2^{2^c}}]$. We will construct a set $B$ such that

(a) $B \subseteq T$,
(b) $B \in \text{EE}$,
(c) $\|B \cap T_k\| \leq 1$ for each $k \geq 0$, and
(d) $B \not\in \text{EL}_2$.

\footnote{This generalizes to $k$ larger than 1 a result of Ogihara who proves that the $P$-selective sets are strictly contained in $P\text{-mc}(2)$ [Ogi95] as well as the known fact that $P\text{-Sel}$ is strictly larger than $P$ [Sel79].}
Note that it follows from (a), (b), and (c) that $B$ is a sparse set in $S(2)$. Indeed, any input to the $S(2)$-selector for $B$ that is not in $T$ (which can easily be checked) is not in $B$ by (a) and may thus be ignored. If all inputs that are in $T$ are in the same $T_k$ then, by (c), the $S(2)$-promise (that $B$ contains at least two of the inputs) is never satisfied, and the selector may thus output an arbitrary input. On the other hand, if the inputs that are in $T$ fall in more than one $T_k$, then for all inputs of length smaller than the maximum length, it can be decided by brute force whether or not they belong to $B$—this is possible, as $B \in \text{EE}$ and the $T_k$ are triple-exponentially spaced. From these comments, the action of the $S(2)$-selector is clear.

By Lemma 8, $B$ is thus in $\text{P-mc}(k)$ for each $k \geq 3$. But since $S(2)$ and $\text{P-mc}(2)$ are incomparable (again by Lemma 8), we still must argue that $B \in \text{P-mc}(2)$. Again, this follows from (a), (b), and (c), since for any fixed two inputs, $u$ and $v$, if they are of different lengths, then the smaller one can be solved by brute force; and if they have the same length, then it is impossible by (c) that $(x_B(u), x_B(v)) = (1, 1)$. In each case, one out of the four possibilities for the membership of $u$ and $v$ in $B$ can be excluded in polynomial time. Hence, $B \in \text{P-mc}(2)$.

For proving (d), we will construct $B$ such that $\text{NP}^B \not\subseteq \text{coNP}^{B \oplus \text{SAT}}$ (which clearly implies that $\text{NP}^{\text{NP}^B} \not\subseteq \text{NP}^{B \oplus \text{SAT}}$). Define

$$L_B \overset{df}{=} \{0^n \, | \, (\exists x) \, [ |x| = n \land x \in B] \}.$$ 

Clearly, $L_B \in \text{NP}^B$. As in the proof of Theorem 2, let $\{N_i\}_{i \geq 1}$ be a standard enumeration of all coNP oracle machines satisfying the condition that the runtime of each $N_i$ is independent of the oracle and each machine is repeated infinitely often in the enumeration. Let $p_i$ be the polynomial bound on the runtime of $N_i$. The set $B \overset{df}{=} \bigcup_{i \geq 0} B_i$ is constructed in stages. In stage $i$, at most one string of length $n_i$ will be added to $B$, and $B_{i-1}$ will have previously been set to the content of $B$ up to stage $i$. Initially, $B_0 = \emptyset$ and $n_0 = 0$. Stage $i > 0$ is as follows:

Let $n_i$ be the smallest number such that (i) $n_i > n_{i-1}$, (ii) $n_i = t(k)$ for some $k$, and (iii) $2^{n_i} > p_i(n_i)$. Simulate $N_i^{B_{i-1} \oplus \text{SAT}}(0^{n_i})$.

**Case 1:** If $N_i^{B_{i-1} \oplus \text{SAT}}(0^{n_i})$ rejects (in the sense of coNP, i.e., if there are one or more rejecting computation paths), then fix some rejecting path and let $w_i$ be the smallest string of length $n_i$ that is not queried along this path. Note that, by our choice of $n_i$, such a string $w_i$, if needed, must always exist. Set $B_i := B_{i-1} \cup \{w_i\}$.

**Case 2:** If $0^{n_i} \in L(N_i^{B_{i-1} \oplus \text{SAT}})$, then set $B_i := B_{i-1}$.

**Case 3:** If the simulation of $N_i^{B_{i-1} \oplus \text{SAT}}$ on input $0^{n_i}$ fails to be completed in double exponential (say, $2^{100 \cdot 2^{n_i}}$ steps) time (for example, because $N_i$ is huge in size relative to $n_i$), then abort the simulation and set $B_i := B_{i-1}$.

This completes the construction of stage $i$.

Since we have chosen an enumeration such that the same machine as $N_i$ appears infinitely often and as the $n_i$ are strictly increasing, it is clear that for only a finite number of the $N_{i_1}, N_{i_2}, \ldots$ that are the same machine as $N_i$, can Case 3 occur (and thus $N_i$, either directly or via one of its clones,
is diagonalized against eventually). Note that the construction meets requirements (a), (b), and (c) and shows \( L_B \neq L(N_i^{B \oplus \text{SAT}}) \) for each \( i \geq 1 \).

Since \( \text{EL}_2 \) and \( \text{P-mc}(2) \) are both closed under complementation, we have the following corollary.

**Corollary 10** \( \text{coSPARSE} \cap \text{coS}(2) \cap \text{P-mc}(2) \nsubseteq \text{EL}_2 \).

When suitably combined with standard techniques of constructing \( \text{P-selective sets} \), the proof of the previous theorem even proves that the second level of the extended low hierarchy is not closed under a number of Boolean operations, as we have claimed in the beginning of this section. These results extend the main result of Hemaspaandra and Jiang [HJ95], which says that \( \text{P-Sel} \) is not closed under those Boolean connectives.

Let us first adopt and slightly generalize some of the formalism used in [HJ95] so as to suffice for our objective. The intuition is that we want to show that certain widely-spaced and complexity-bounded sets whose definition will be based on the set \( B \) constructed in the previous proof are \( \text{P-selective} \). Fix some complexity-bounding function \( f \) and some wide-spacing function \( \mu \) such that the spacing is at least as wide as given by the following inductive definition: \( \mu(0) = 2 \) and \( \mu(i + 1) = 2^{f(\mu(i))} \) for each \( i \geq 0 \). Now define for each \( k \geq 0 \),

\[
R_k \overset{df}{=} \{ i \mid i \in \mathbb{N} \land \mu(k) \leq i < \mu(k + 1) \},
\]

and the following two classes of languages (where we will implicitly use the standard correspondence between \( \Sigma^* \) and \( \mathbb{N} \)):

\[
\mathcal{C}_1 \overset{df}{=} \{ A \subseteq \mathbb{N} \mid (\forall j \geq 0) [R_{2j} \cap A = \emptyset \land (\forall x, y \in R_{2j+1}) [(x \leq y \land x \in A) \implies y \in A)] \};
\]

\[
\mathcal{C}_2 \overset{df}{=} \{ A \subseteq \mathbb{N} \mid (\forall j \geq 0) [R_{2j} \cap A = \emptyset \land (\forall x, y \in R_{2j+1}) [(x \leq y \land y \in A) \implies x \in A)] \}.
\]

In [HJ95], the following lemma is proven for the particular complexity-bounding function \( f'(n) = 2^{O(n)} \) and for the classes \( \mathcal{C}_1' \) and \( \mathcal{C}_2' \) having implicit in their definition the particular wide-spacing function that is given by \( \mu'(0) = 2 \) and \( \mu'(i + 1) = 2^{2^{f'(i)}} \), \( i \geq 0 \). However, there is nothing special about these functions \( f' \) and \( \mu' \); i.e., for Lemma [1] to hold it suffices that \( f \) and \( \mu \) relate to each other as required above. In light of this, the proof of Lemma [1] is quite analogous to the proof given in [HJ95].

**Lemma 11** \( \mathcal{C}_1 \cap \text{DTIME}[f] \subseteq \text{P-Sel} \) and \( \mathcal{C}_2 \cap \text{DTIME}[f] \subseteq \text{P-Sel} \).

Now we are ready to prove the main result of this section.

**Theorem 12** \( \text{EL}_2 \) is not closed under intersection, union, exclusive-or, or equivalence.
Proof. Using the technique of [HJ95], it is not hard to prove that the set $B$ constructed in the proof of Theorem 9 can be represented as $B = A_1 \cap A_2$ for P-selective sets $A_1$ and $A_2$. More precisely, let

$$A_1 \overset{df}{=} \{ x \mid (\exists w \in B) \left[ |x| = |w| \land x \leq_{\text{lex}} w \right]\},$$

$$A_2 \overset{df}{=} \{ x \mid (\exists w \in B) \left[ |x| = |w| \land w \leq_{\text{lex}} x \right]\}.$$

Since $B \in \text{EE}$ and is triple-exponentially spaced, we have from Lemma 11 that $A_1$ and $A_2$ are in P-Sel and thus in EL$_2$. On the other hand, we have seen in the previous proof that $B = A_1 \cap A_2$ is not in EL$_2$. Similarly, if we define

$$C_1 \overset{df}{=} \{ x \mid (\exists w \in B) \left[ |x| = |w| \land x <_{\text{lex}} w \right]\},$$

$$C_2 \overset{df}{=} \{ x \mid (\exists w \in B) \left[ |x| = |w| \land x \leq_{\text{lex}} w \right]\},$$

we have $B = C_1 \Delta C_2$, where $\Delta$ denotes the exclusive-or operation. As before, $C_1$ and $C_2$ are in P-Sel and thus in EL$_2$. Hence, EL$_2$ is not closed under intersection or exclusive-or. Since EL$_2$ is closed under complementation, it must also fail to be closed under union and equivalence. $\square$

The proof of the above result also gives the following corollary.

**Corollary 13** [HJ95] P-Sel is not closed under intersection, union, exclusive-or, or equivalence.

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