Birationally Rigid Finite Covers of the Projective Space

A. V. Pukhlikov\textsuperscript{a}

Received January 7, 2019; revised May 1, 2019; accepted August 26, 2019

Abstract—In this paper we prove birational superrigidity of finite covers of degree \(d\) of the \(M\)-dimensional projective space of index 1, where \(d \geq 5\) and \(M \geq 10\), that have at most quadratic singularities of rank \(\geq 7\) and satisfy certain regularity conditions. Up to now, only cyclic covers have been studied in this respect. The set of varieties that have worse singularities or do not satisfy the regularity conditions is of codimension \(\geq (M - 4)(M - 5)/2 + 1\) in the natural parameter space of the family.

DOI: 10.1134/S0081543819060142

1. INTRODUCTION

1.1. Statement of the main result.  Let us fix integers \(d \geq 5\) and \(l \geq 2\) with \((d, l) \neq (5, 2)\). Set \(M = (d - 1)l\), so that \(M \geq 10\). In the present paper we study \(d\)-sheeted covers of the complex projective space \(\mathbb{P} = \mathbb{P}^M\) with at most quadratic singularities of rank \(\geq 7\) that are Fano varieties of index 1. Such covers have a convenient representation: let

\[ \mathbb{P} = \mathbb{P}(1, \ldots, 1, l) = \mathbb{P}(1^{M+1}, l) \]

be the weighted projective space with homogeneous coordinates \(x_0, \ldots, x_M, \xi\), where \(x_i\) are of weight 1 and \(\xi\) is of weight \(l\). Furthermore, let

\[ F(x_*, \xi) = \xi^d + A_1(x_*)\xi^{d-1} + \ldots + A_d(x_*) \]

be a (quasi)homogeneous polynomial of degree \(dl\), that is, \(A_i(x_0, \ldots, x_M)\) is a homogeneous polynomial of degree \(il\) for \(i = 1, \ldots, d\). The space

\[ \mathcal{F} = \prod_{i=1}^d H^0(\mathbb{P}, \mathcal{O}_\mathbb{P}(il)) \]

parametrizes all such polynomials. If the hypersurface

\[ V = \{ F = 0 \} \subset \mathbb{P} \]

has at most quadratic singularities of rank \(\geq 7\), then the set \(\text{Sing}\, V\) of singular points is of codimension \(\geq 6\) in \(V\), so that by Grothendieck’s theorem [1] the variety \(V\) is factorial. (By a quadratic singularity we mean a hypersurface singularity whose local equation starts with a quadratic component; the rank condition pertains to that quadratic form; see Subsection 2.3 for details.) Since the property to have at most quadratic singularities of rank \(\geq r\) is stable with respect to blow-ups (see [13, Sect. 3.1]), the singularities of the variety \(V\) are terminal. Now

\[ \text{Pic}\, V = \mathbb{Z}H, \quad K_V = -H, \]

where \(H\) is the class of a hyperplane section, so that \(V\) is a primitive Fano variety of index 1.

\textsuperscript{a} Department of Mathematical Sciences, The University of Liverpool, Liverpool, L69 7ZL, UK.

E-mail address: pukh@liverpool.ac.uk

232
Let \( o^* = (0 : \ldots : 0 : 1) = (0^{M+1} : 1) \in \mathbb{P} \) be the unique singular point of the weighted projective space \( \mathbb{P} \). Obviously, \( o^* \notin V \). Consider the projection

\[
\pi_p : \mathbb{P} \setminus \{o^*\} \to \mathbb{P}, \quad \pi_p((x_0 : \ldots : x_M : \xi)) = (x_0 : \ldots : x_M),
\]

"from the point \( o^* \)." Obviously,

\[
\pi = \pi_p|_V : V \to \mathbb{P}
\]
is a \( d \)-sheeted ramified cover of the projective space. (In particular, \( H \) is the \( \pi \)-pull-back of the class of a hyperplane in \( \mathbb{P} \) onto \( V \).

Now let us state the main result of the paper. We identify a polynomial \( F \in \mathcal{F} \) and the corresponding closed set \( \{F = 0\} \), which enables us to write \( V \in \mathcal{F} \).

**Theorem 1.1.** There is a Zariski open subset \( \mathcal{F}_{\text{reg}} \subset \mathcal{F} \) such that

1. every \( V \in \mathcal{F}_{\text{reg}} \) is a factorial Fano variety of index 1 with terminal singularities;
2. the inequality

\[
\text{codim}(\mathcal{F} \setminus \mathcal{F}_{\text{reg}}) \subset \mathcal{F}) \geq \frac{1}{2}(M-4)(M-5) + 1
\]
holds;
3. every variety \( V \in \mathcal{F}_{\text{reg}} \) is birationally superrigid.

**Corollary 1.1.** Let \( V \in \mathcal{F}_{\text{reg}} \). The following assertions hold.

1. Every birational map \( \chi : V \dashrightarrow V' \) onto a Fano variety \( V' \) with \( \mathbb{Q} \)-factorial terminal singularities and Picard number 1 is a (biregular) isomorphism.
2. There is no rational dominant map \( V \dashrightarrow S \) onto a positive-dimensional variety \( S \) whose fibre has negative Kodaira dimension. Therefore, on \( V \) there are no structures of a rationally connected fibre space and of a Mori fibre space over a positive-dimensional base. In particular, \( V \) has no structure of a conic bundle and \( V \) is nonrational.
3. The groups of birational and biregular automorphisms of the variety \( V \) coincide:

\[
\text{Bir} V = \text{Aut} V.
\]

**Proof.** All these assertions are standard implications of birational superrigidity (see [12, Ch. 2, Sect. 1]). \( \square \)

Note that every automorphism of the variety \( V \in \mathcal{F}_{\text{reg}} \) is induced by an automorphism of the ambient weighted projective space \( \mathbb{P} \).

**1.2. Structure of the paper.** In Subsection 1.3 we list known results on birational superrigidity of finite covers of index 1. All previous results pertain to cyclic covers (for cyclic covers the standard procedure of constructing hypertangent divisors works well [12, Ch. 3], whereas in the case of an arbitrary cover this is not the case).

In Section 2 we give a precise definition of the set \( \mathcal{F}_{\text{reg}} \). This definition includes several conditions, one of which is the condition of having at most quadratic singular points of rank \( \geq 7 \). In addition, we need at every point \( o \in V \) a certain regularity condition, which is similar but not identical to the usual regularity conditions on which the technique of hypertangent divisors is based.

In Section 3 we prove assertion (ii) of Theorem 1.1. As usual (see [12, Ch. 3]), we estimate the codimension of the set of hypersurfaces \( V \) containing a fixed point \( o \) that are not regular at this point. After that it is not difficult to globalize the estimate for the codimension.
In Section 4 we prove the birational superrigidity of regular hypersurfaces $V$. Assuming that assertion (iii) of Theorem 1.1 is not true, we obtain the existence of a mobile linear system $\Sigma \subset |nH|$ with a maximal singularity. In order to prove the birational superrigidity, we have to exclude all possible types of maximal singularities. The main technical ingredients are the $8n^2$-inequality for a nonsingular point $o$ of the hypersurface $V$ and the recently discovered generalized $4n^2$-inequality for a complete intersection singularity, and, of course, the hypertangent divisors. It is impossible to apply the well-known technique of hypertangent divisors directly to noncyclic covers; the essence of this paper is precisely to modify that technique so as to apply it not to the variety $V$ itself but to its intersection with another hypersurface in the weighted projective space.

1.3. Historical remarks. The double covers of the projective space of index 1 are ideal objects for the theory of birational rigidity, due to their low degree. Soon after the classical paper [7], the birational superrigidity of nonsingular double covers of the projective space $\mathbb{P}^3$ branched over a sextic (“sextic double solids”) was shown in [6]. In an arbitrary dimension the birational superrigidity of nonsingular double spaces of index 1 was proved in [9], and of those with certain types of singularities, in [10, 2, 4, 5, 8]. The cyclic covers of an arbitrary degree were studied in [11] in a much more general context, and triple cyclic covers with double points were considered in [3]. However, until the present paper, noncyclic covers have never been studied. The reason is that the technique of hypertangent divisors cannot be directly applied to noncyclic covers, because in the weighted projective representation there is a coordinate of weight $\geq 2$. The aim of the present paper is to overcome this obstacle.

2. REGULAR VARIETIES

In this section we carry out some preliminary work that we need to study finite covers of the projective space. In Subsection 2.1 we consider systems of affine and homogeneous coordinates on $\mathbb{P}$ and hypersurfaces in $\mathbb{P}$ that “substitute” for hyperplanes of the ordinary projective space. In Subsection 2.2 we consider in more detail the local equation of the hypersurface $V \in \mathcal{F}$ with respect to the affine coordinates on a suitable open subset in $\mathbb{P}$. On that basis, in Subsection 2.3 we state the regularity conditions defining the subset $\mathcal{F}_{\text{reg}} \subset \mathcal{F}$. Assertion (i) of Theorem 1.1 follows immediately from the statement of these conditions.

2.1. Preliminary remarks. Let

$$f(z_*) = q_\mu(z_*) + \ldots + q_N(z_*),$$

be a polynomial in the variables $z_1, \ldots, z_M$, which is decomposed into homogeneous components $q_i$ of degree $i \geq 1$ (so that $f(0, \ldots, 0) = 0$). Set $o = (0, \ldots, 0) \in \mathbb{A}^M_z$.

Definition 2.1. The affine hypersurface $\{f = 0\}$ is $k$-regular at the point $o$, where $\mu \leq k \leq N$, if the homogeneous polynomials

$$q_\mu, \ldots, q_k$$

form a regular sequence in $\mathcal{O}_o,\mathbb{A}^M$.

Obviously, the condition of $k$-regularity at the point $o$ means that the system of equations

$$q_\mu = \ldots = q_k = 0$$

defines a closed subset (cone) of codimension $k - \mu + 1$ in $\mathbb{A}^M$. This condition is meaningful only for $k - \mu + 1 \leq M$.

Now let us consider hypersurfaces in the weighted projective space $\overline{\mathbb{P}}$. 
**Proposition 2.1.** For every homogeneous polynomial \( \gamma(x_0, \ldots, x_M) \) of degree 1, the equation \( \xi = \gamma(x_*) \) defines a hypersurface \( R_\gamma \subset \mathbb{P} \) that does not contain the point \( o^* = (0^{M+1}:1) \). The projection \( \pi_\gamma|_{R_\gamma} \) is an isomorphism of \( R_\gamma \) and \( \mathbb{P} = \mathbb{P}^M \).

**Proof.** This is obvious. □

In the affine chart \( \{x_0 \neq 0\} \subset \mathbb{P} \) with the natural affine coordinates \( z_i = x_i/x_0 \) and \( y = \xi/x_0^l \), the projection \( \pi_\gamma \) takes the form of the usual projection

\[
\mathbb{A}^M_{z_1, \ldots, z_M, y} \to \mathbb{A}^M_{z_1, \ldots, z_M}, \quad (z_1, \ldots, z_M, y) \mapsto (z_1, \ldots, z_M),
\]

where \( \mathbb{A}^M \) is the affine chart \( \{x_0 \neq 0\} \) in \( \mathbb{P} \). Obviously, the affine hypersurface \( R_\gamma \cap \{x_0 \neq 0\} \) is given by the equation \( y = g(z_1, \ldots, z_M) \), where \( g(z_*) = \gamma(1, z_1, \ldots, z_M) \).

Now let us consider the singularities of the hypersurface \( V = \{F = 0\} \) and its sections. Taking into account the (quasi-homogeneous) Euler identity, we find that the closed set \( \text{Sing} V \) of singular points of \( V \) is given by the system of equations

\[
\frac{\partial F}{\partial x_0} = \ldots = \frac{\partial F}{\partial x_M} = \frac{\partial F}{\partial \xi} = 0.
\]

**Proposition 2.2.** Let \( R \cap \mathbb{P} \) be either the \( \pi_\gamma \)-preimage of a hyperplane in \( \mathbb{P} \) or a hypersurface \( R_\gamma \) defined above. Then

\[
\dim \text{Sing}(V \cap R) \leq \dim \text{Sing} V + 1.
\]

**Proof.** If \( R \) is the \( \pi_\gamma \)-preimage of a hyperplane, we repeat word for word the well-known argument for the usual projective space. Let \( R = R_\gamma \). Then the intersection \( V \cap R \) is isomorphic to the hypersurface

\[
F(x_0, \ldots, x_M, \gamma(x_0, \ldots, x_M)) = 0
\]

in the projective space \( \mathbb{P} \), whose singularities are given by the equations

\[
\frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial \xi} \frac{\partial \gamma}{\partial x_i} = 0, \quad i = 0, \ldots, M.
\]

Therefore, the intersection of \( \text{Sing}(V \cap R) \subset \mathbb{P} \) with the hypersurface \( \{\partial F/\partial \xi = 0\} \) is contained in \( \text{Sing} V \). □

**Definition 2.2.** A nonsingular point \( o \in V \) is a point of the first type if \( \partial F/\partial \xi \neq 0 \) at \( o \), and of the second type if \( \partial F/\partial \xi = 0 \) at \( o \).

At a point of the second type, the hypersurface given by the equation

\[
\sum_{i=0}^{M} x_i \frac{\partial F}{\partial x_i}(o) = 0
\]

is the natural “tangent hyperplane” \( T_o V \subset \mathbb{P} \), so that the projection \( \pi: V \to \mathbb{P} \) is ramified at that point (therefore, the nonsingular points of the second type form the ramification divisor of the projection \( \pi \) restricted onto the nonsingular part \( V \setminus \text{Sing} V \) of \( V \)). At a point of the first (main) type, there is a whole family of candidates for the role of the tangent hyperplane, and they are given by sections of the sheaf \( \mathcal{O}_\mathbb{P}(l) \), because they include the coordinate \( \xi \): they are of the form \( R_\gamma \) for suitable polynomials \( \gamma \). In order to present the local equation of the hypersurface \( V \) more precisely, one needs to consider affine coordinates.
2.2. Affine coordinates. Let \( o \in V \) be a point. Let us choose projective coordinates on \( \mathbb{P} \) in such a way that \( o = (1:0: \ldots : \beta) = (1:0^M: \beta) \) for some \( \beta \in \mathbb{C} \). Replacing the coordinate \( \xi \) by \( \xi' = \xi - \beta x_0^d \), we may assume that \( \beta = 0 \). Now in the affine coordinates \( z_1, \ldots, z_M, y \) we have \( o = (0, \ldots, 0) \) and \( V \cap \{ x_0 \neq 0 \} \) is given by the equation \( f = 0 \) with
\[
f = y^d + a_1(z_*)y^{d-1} + \ldots + a_{d-1}(z_*)y + a_d(z_*),
\]
where the polynomial \( a_i(z_*) \) has degree \( i \). Write down
\[
a_i(z_*) = a_{i,0} + a_{i,1}(z_*) + \ldots + a_{i,d}(z_*),
\]
where \( a_{i,j} \) is a homogeneous polynomial of degree \( j \). In particular, \( a_{d,0} = 0 \). The point \( o \in V \) is nonsingular if and only if the linear form
\[
a_{d-1,0}y + a_{d,1}(z_*)
\]
is not identically zero, and in that case the point \( o \) is a point of the first type if \( a_{d-1,0} \neq 0 \) and of the second type if \( a_{d-1,0} = 0 \). If \( o \in V \) is a nonsingular point of the first type, then \( z_1, \ldots, z_M \) is a coordinate system on the (affine) tangent space \( T_o V \). In particular, every linear subspace \( \Lambda \subset T_o V \) is given by a system of linear equations that depend only on \( z_* \), and for any nonzero linear form \( h(z_*) \) the intersection
\[
V \cap \{ h = 0 \}
\]
is nonsingular at the point \( o \). (The last intersection can also be understood as the intersection with the hyperplane \( \{ h(x_1, \ldots, x_M) = 0 \} \) in \( \mathbb{P} \).)

Now if \( o \in V \) is a nonsingular point of the second type, then \( z_1, \ldots, z_M \), restricted to \( T_o V \), are linearly dependent. A typical linear subspace \( \Lambda \subset T_o V \) is given by a system of linear equations one of which is of the form \( y - h(z_*) = 0 \) (where the form \( h \) can be identically zero) and the rest depend only on \( z_* \).

With some abuse of notation, we will use the same symbol \( V \) both for the original hypersurface and for its affine part \( V \cap \{ x_0 \neq 0 \} \). For a linear form \( h(z_*) \) (possibly, identically zero) by \( V_h \) we denote the intersection
\[
V \cap R_\gamma,
\]
where \( \gamma = h(x_1, \ldots, x_M)x_0^{-l-1} \) in the coordinate system \( (x_0 : \ldots : x_M : \xi') \) described above. For the affine part of this variety we will use the same notation \( V_h \). Without special comments we consider \( V_h \) to be embedded in \( \mathbb{P} \) or \( \mathbb{A}^M \), depending on the situation.

The intersection with a hyperplane \( \{ h = 0 \} \) (for a nonzero form \( h \)) will be denoted using the restriction symbol as \( \mid \{ h = 0 \} \); again, we use this notation both in the affine and projective context.

2.3. Regularity conditions. Let us formulate the regularity conditions for a point \( o \in V \). These conditions vary slightly depending on whether the point \( o \) is a nonsingular point of the first or second type or a singularity.

Assume that the point \( o \in V \) is nonsingular.

(R1) The hypersurface \( V_h \) is \( [(3d)/8] \)-regular (at \( o \)) for every linear form \( h(z_1, \ldots, z_M) \).

(We use the notation and conventions of Subsection 2.2.)

If the point \( o \in V \) is nonsingular of the second type, that is, \( a_{d-1,0} = 0 \), then, along with condition (R1), one more condition is needed. Recall that in this case the tangent hyperplane \( T_o V \) is given in the affine coordinates by the equation \( a_{d,1}(z_*) = 0 \).

(R1.2) For every linear form \( h(z_1, \ldots, z_M) \) that is linearly independent with \( a_{d,1}(z_*) \), the hypersurface \( V_0 |_{\{ h = 0 \}} \) is \( [(3d)/8] \)-regular.

(The last hypersurface is contained in \( \{ h = 0 \} = \mathbb{A}^{M-1} \).)
Now assume that the point \( o \in V \) is singular. The affine equation of \( V \) at \( o \) starts with the quadratic form
\[
a_{d-2,0}y^2 + a_{d-1,1}(z_*)y + a_{d,2}(z_*).
\]

(R2) The rank of the form (2.1) is at least 7, and the variety \( V_0 \) is \([dl/2]\)-regular at \( o \).

**Definition 2.3.** A hypersurface \( V \in \mathcal{F} \) is *regular* if condition (R1) holds at every nonsingular point, condition (R1.2) additionally holds at every nonsingular point of the second type, and condition (R2) holds at every singular point.

The set of regular hypersurfaces is denoted by \( \mathcal{F}_{\text{reg}} \). Since every hypersurface \( V \in \mathcal{F}_{\text{reg}} \) is either nonsingular or has at most quadratic singularities of rank \( \geq 7 \), assertion (i) of Theorem 1.1 holds. Assertion (ii) of Theorem 1.1 is proved in Section 3.

### 3. CODIMENSION OF THE NONREGULAR SET

The aim of this section is to prove assertion (ii) of Theorem 1.1. In Subsection 3.1 we localize the task: we reduce it to a similar problem for a fixed point \( o \in \overline{P} \). In Subsection 3.2 we recall the methods of estimating the violation of the regularity condition. In Subsection 3.3 we establish local estimates, which completes the proof.

**3.1. Local problem.** Fix a point \( o \in \overline{P}, o \neq o^* \). Let \( F(o) \subset \mathcal{F} \) be the subset (hyperplane) of polynomials that vanish at \( o \) and \( \mathcal{F}_{\text{reg}}(o) \subset F(o) \) be the subset of polynomials satisfying the corresponding regularity condition at that point. Set
\[
F_{\text{non-reg}}(o) = F(o) \setminus \mathcal{F}_{\text{reg}}(o).
\]

Obviously,
\[
F \setminus \mathcal{F}_{\text{reg}} \subset \bigcup_{o \in \overline{P} \setminus \{o^*\}} F_{\text{non-reg}}(o),
\]
so that, in view of the equality \( \text{codim}(F(o) \subset \mathcal{F}) = 1 \), assertion (ii) of Theorem 1.1 is implied by the following local fact.

**Proposition 3.1.** The following inequality holds:
\[
\text{codim}(F_{\text{non-reg}}(o) \subset F(o)) \geq \frac{(M - 4)(M - 5)}{2} + M + 1.
\]

It is this inequality that we prove below. For each of the regularity conditions stated in Subsection 2.3, we have to check that the violation of this condition imposes at least \( (M - 4)(M - 5)/2 + M + 1 \) independent conditions on the coefficients of the polynomial \( F \in F(o) \). As we will see below, the minimum number of independent conditions corresponds to the violation of condition (R2).

**3.2. Methods for estimating the codimension.** We will use two well-known methods. The first method has been repeatedly used (see [12, Ch. 3, Sect. 1]). Let, as in Subsection 2.1,
\[
f(z_*) = q_{\mu}(z_*) + \ldots + q_{N}(z_*)
\]
be a polynomial in the affine coordinates \( z_1, \ldots, z_M \) and \( \mu \leq k \leq M + 1 - \mu \). By \( P_{i,M} \) we denote the space of homogeneous polynomials of degree \( i \in \mathbb{Z}_+ \) in \( z_1, \ldots, z_M \), and for \( i < j \) by \( P_{[i,j],M} \) we denote the product
\[
\prod_{a=i}^{j} P_{a,M}.
\]
so that \( f \in \mathcal{P}_{[\mu,N],M} \). Repeating the arguments of [12, Ch. 3, Sect. 1.3] word for word, we find that the codimension of the set of polynomials \( f \) that do not satisfy the condition of \( k \)-regularity with respect to the space \( \mathcal{P}_{[\mu,N],M} \) is at least

\[
\min_{\mu \leq i \leq k} \left( M - 1 + \mu \right) \frac{i}{i}.
\]

Taking into account the well-known behaviour of the binomial coefficients, we conclude that this minimum is attained either for \( i = \mu \) or for \( i = k \). It is easy to determine at which of these two values the minimum is attained. Note that if the point \((0, \ldots, 0)\) is known to be nonsingular on the hypersurface \( \{ f = 0 \} \), that is, \( \mu = 1 \) and \( q_1 \neq 0 \), then we can fix \( q_1 \) and restrict \( q_i \), \( i \geq 2 \), onto the hyperplane \( \{ q_1 = 0 \} \). In that case the codimension is at least \( \left( \frac{M}{2} \right) \) for \( k \leq M - 2 \). We will need only this estimate.

The second method is the well-known fact that the codimension of the set of quadratic forms of rank \( \leq r \) in the space of all quadratic forms in \( N \) variables is

\[
\frac{(N - r)(N - r + 1)}{2}.
\]

### 3.3. Proof of Proposition 3.1.

In the notation of Subsections 2.2 and 2.3, let us consider one by one the violation of each regularity condition at the point \( o \). Consider first the hypersurfaces that are nonsingular at this point.

The hypersurface \( V = \{ f = 0 \} \subset \mathbb{A}^{M+1} \) is uniquely determined by the set of polynomials

\[
a_1(z_1), \ldots, a_d(z_1),
\]

where \( \text{deg} \ a_i \leq il \). The hypersurface \( V_h \subset \mathbb{A}^M \) for some linear form \( h(z_1) \) is given by the equation \( f_h = 0 \), where

\[
f_h = h(z_1)^d + a_1(z_1)h(z_1)d^{-1} + \ldots + a_d(z_1).
\]

Fix a form \( h \) such that \( V_h \) is nonsingular at the point \( o \). Since the polynomial \( a_d(z_1, \ldots, z_M) \) of degree \( \leq dl \) is arbitrary, in the representation

\[
f_h = q_1(z_1) + q_2(z_1) + \ldots + q_d(z_1),
\]

where \( \text{deg} \ q_i = i \), the homogeneous components \( q_i \) are arbitrary and do not depend on each other. Therefore, we can apply the first method to estimate the codimension, as described in Subsection 3.2. By assumption, \( q_1 \neq 0 \), so that we fix the hyperplane \( \{ q_1 = 0 \} \) and restrict the polynomial \( f_h \) onto this hyperplane:

\[
\left. f_h \right|_{\{ q_1 = 0 \}} = \bar{q}_2 + \ldots + \bar{q}_d.
\]

Since \( \left[ (3dl)/8 \right] \leq M - 2 \), the codimension of the set of polynomials \( \left. f_h \right|_{\{ q_1 = 0 \}} \) that do not satisfy the condition of \( \left[ (3dl)/8 \right] \)-regularity is \( \left( \frac{M}{2} \right) \). The codimension of the set of nonregular polynomials \( f_h \) is the same. Since \( h \) varies in an \( M \)-dimensional family, the codimension of the set of polynomials \( f \) that do not satisfy condition (R1) is

\[
\left( \frac{M}{2} \right) - M.
\]

In a similar way we estimate the codimension of the set of polynomials that do not satisfy condition (R1.2): the nonregularity of the hypersurface \( V_0 \|_{\{ h = 0 \}} \) gives

\[
\left( \frac{M - 1}{2} \right) - M + 1
\]
independent conditions for \( f \). (The additional codimension +1 comes from the equality \( a_{d-1,0} = 0 \) for a nonsingular point of the second type.)

Now let us consider the hypersurfaces \( V \) that are singular at the point \( o \). Since in that case \( a_{d-1,0} = 0 \) and \( a_{d,1}(z_x) \equiv 0 \), we have \( M + 1 \) additional independent conditions for \( f \). If the rank of the quadratic form (2.1) does not exceed 6, we thus obtain

\[
\binom{M-4}{2} + M + 1
\]

independent conditions for \( f \).

Obviously, \( \lceil dl/2 \rceil \leq M - 2 \). Thus, if \( V_0 \) is a nonregular hypersurface, we obtain

\[
\binom{M+1}{2} + M + 1
\]

independent conditions for \( f \).

Comparing the results obtained above and choosing the smallest one for \( M \geq 10 \) (which corresponds to the violation of the rank condition for the quadratic singularity), we complete the proof of Proposition 3.1 (and assertion (ii) of Theorem 1.1). □

Remark 3.1. We excluded the option \((d, l) = (5, 2)\) from consideration for the only reason: it violates the uniformity of the statement of assertion (ii) of Theorem 1.1. For \( M = 8 \) the minimum of the codimension corresponds to the violation of condition (R1.2) and is equal to 14. With this modification the assertions of Theorem 1.1 are true for the values \((d, l) = (5, 2)\) as well.

4. BIRATIONAL SUPERRIGIDITY

In this section we prove the birational superrigidity of a regular hypersurface \( V \). In Subsection 4.1 we recall the key concept of maximal singularity and exclude certain types of maximal singularities. The remaining types of singularities are classified, and then we start to exclude maximal singularities of “general position” (Proposition 4.2). In order to complete this work, we need the technique of hypertangent divisors, which is recalled in Subsection 4.2. Finally, in Subsection 4.3 we exclude maximal singularities of all remaining types, which completes the proof of Theorem 1.1.

4.1. Maximal singularities. Let \( V \in \mathcal{F}_{\text{reg}} \) be a fixed regular variety. Assume that \( V \) is not birationally superrigid. It is well known that in this case on \( V \) there is a mobile linear system \( \Sigma \subseteq |nH| \) with a maximal singularity: for some birational morphism \( \varphi: \tilde{V} \to V \), where \( \tilde{V} \) is a nonsingular projective variety, and for some \( \varphi \)-exceptional prime divisor \( Q \subset \tilde{V} \), the Noether–Fano inequality holds:

\[
\text{ord}_Q \varphi^* \Sigma > n \cdot a(Q, V),
\]

where \( a(Q, V) \) is the discrepancy of \( Q \) with respect to \( V \) (see, for instance, [12, Ch. 2]). Let \( B = \varphi(Q) \subset V \) be the centre of the maximal singularity \( Q \). If \( B \not\subset \text{Sing} V \), then we have the inequality

\[
\text{mult}_B \Sigma > n. \quad (4.1)
\]

**Proposition 4.1.** The codimension of the subvariety \( B \subset V \) is at least 5.

**Proof.** Assume the contrary:

\[
\text{codim}(B \subset V) \leq 4.
\]

Then \( B \not\subset \text{Sing} V \), so that inequality (4.1) holds. For a general polynomial \( g(x_0, \ldots, x_M) \) of degree \( l \), by Bertini’s theorem

\[
\text{codim}(\text{Sing} V_g \subset V_g) = \text{codim}(\text{Sing} V \subset V) \geq 6.
\]

PROCEEDINGS OF THE STEKLOV INSTITUTE OF MATHEMATICS Vol. 307 2019
Set $B_g = B \cap V_g$ and $\Sigma_g = \Sigma|_{V_g}$, where $\Sigma_g < |nH_g|$ is a mobile linear system on $V_g$ and $H_g$ is the class of a hyperplane section of the hypersurface $V_g \subset \mathbb{P}$. Let $P \subset \mathbb{P}$ be a general 6-plane. The variety

$$V_P = V_g \cap P$$

is a nonsingular hypersurface in $P \cong \mathbb{P}^6$. On $V_P$ there is a mobile linear system $\Sigma_P = \Sigma|_{V_P} \subset |nH_P|$, where $\text{Pic} V_P = \mathbb{Z}H_P$ and, moreover, $\text{mult}_{B \cap P} \Sigma_P > n$. However, $B \cap P$ is positive-dimensional, so that the last inequality cannot be true (this is a well-known fact for any nonsingular hypersurface in the projective space, and in fact it is sufficient for the linear system $\Sigma_P$ to be nonempty; see, for instance, [12, Ch. 2, Sect. 2]). We have obtained a contradiction, which completes the proof of the proposition. □

Starting from this moment, we assume that $\text{codim}(B \subset V) \geq 5$. Fix a point $o \in B$ of general position. There are three options:

(I.1) the point $o \notin \text{Sing} V$ is nonsingular of the first type;

I(I.2) the point $o \notin \text{Sing} V$ is nonsingular of the second type;

(II) the point $o \in \text{Sing} V$ is a quadratic singularity.

We have to exclude each of them.

Let us consider first the nonsingular cases. Set $Z = (D_1 \circ D_2)$ to be the self-intersection of the linear system $\Sigma$, where $D_1, D_2 \in \Sigma$ are general divisors. The effective cycle $Z \sim n^2H^2$ of codimension 2 satisfies the (classical) $4n^2$-inequality

$$\text{mult}_o Z > 4n^2$$

and the $8n^2$-inequality

$$\text{mult}_o Z + \text{mult}_\Lambda Z^+ > 8n^2; \quad (4.2)$$

here $\Lambda \subset E$ is some linear subspace of codimension 2, $E = \varepsilon^{-1}(o) \subset V^+$ is the exceptional divisor of the blow-up $\varepsilon: V^+ \to V$ of the point $o$, and $E \cong \mathbb{P}^{M-1}$ (see, for instance, [12, Ch. 2, Sects. 2, 4]).

**Proposition 4.2.** Case (I.1) is impossible.

**Proof.** Assume the contrary: case (I.1) takes place. In the affine coordinates (see Subsection 2.2) the tangent hyperplane $T_o V$ is given by the equation $a_{d-1,0}g + a_{d,1}(z_*) = 0$ with $a_{d-1,0} \neq 0$, so that $z_1, \ldots, z_M$ is a coordinate system on $T_o V$ and $(z_1: \ldots: z_M)$ is a homogeneous coordinate system on $E = \mathbb{P}(T_o V)$. Let

$$\Lambda = \{h_1(z_*) = h_2(z_*) = 0\},$$

where $h_1$ and $h_2$ are linearly independent forms. Let $h = \lambda_1h_1 + \lambda_2h_2$ be a general form in the pencil.

Since $\deg Z = \deg_H Z = dn^2$, inequality (4.2) can be rewritten in the form

$$\text{mult}_o Z + \text{mult}_\Lambda Z^+ > \frac{8}{d} \deg Z. \quad (4.3)$$

This inequality is linear in $Z$. Therefore, there is an irreducible component of $Z$ satisfying this inequality. In order not to make the notation too complicated, let us simply assume that the cycle $Z$ itself is an irreducible subvariety.

**Lemma 4.1.** The subvariety

$$V \cap \{h_1(z_*) = h_2(z_*) = 0\}$$

is irreducible, nonsingular at the point $o$, and different from $Z$. 

---

**PROCEEDINGS OF THE STEKLOV INSTITUTE OF MATHEMATICS** Vol. 307 2019

---

**A. V. PUKHLIKOV**
Remark 4.1. In the statement of the lemma, the coordinates $z_*$ are viewed as affine coordinates on $K_{z_*}^M$. We also used $z_*$ above as homogeneous coordinates on $E$. The subvariety $V \cap \{h_1 = h_2 = 0\}$ is understood as a projective subvariety in $\mathbb{P}$, that is, the closure of the corresponding affine set. These changes from the affine notation to the projective one are obvious and do not require special explanations.

Proof of Lemma 4.1. Non-singularity at the point $o$ is obvious, and irreducibility follows from Proposition 2.2 and the assumption on the rank of the quadratic points (the codimension of the singular set $\operatorname{Sing} V$ is at least 6). Finally, the subvariety $V \cap \{h_1 = h_2 = 0\}$ has degree $d$ and multiplicity 1 at $o$, and its strict transform on $V^+$ has multiplicity precisely 1 along $\Lambda$, so that this subvariety does not satisfy inequality (4.3). Therefore, it is not equal to $Z$. □

Set

$$g(z_*) = -\frac{1}{a_{d-1,0}}a_{d,1}(z_*) + h(z_*).$$

By the lemma, $Z \not\subset V_g$, so that the effective cycle of scheme-theoretic intersection $(Z \circ V_g)$ is well defined. It satisfies the inequality

$$\operatorname{mult}_o(Z \circ V_g) > \frac{8}{dl} \deg(Z \circ V_g)$$

(since $V_g \sim lH$ and $V_g^+$ contains $\Lambda$ by the choice of the form $h$). Since the inequality is linear, there is an irreducible subvariety $Y \subset V_g$ of codimension 2 (a component of the effective cycle $(Z \circ V_g)$) that satisfies the inequality

$$\frac{\operatorname{mult}_o Y}{\deg Y} > \frac{8}{dl} \quad (4.4)$$

(as usual, by $\operatorname{mult}_o/\deg$ we mean the ratio of the multiplicity at the point $o$ to the $H$-degree). The subvariety $Y$ is contained in $V_g$, which is an irreducible hypersurface of degree $dl$ in the projective space $\mathbb{P}$. By condition (R1) this hypersurface is $k$-regular at $o$, where $k = [(3dl)/8]$.

4.2. Technique of hypertangent divisors. Consider $V_g$ as a hypersurface in the projective space $\mathbb{P}$, let us decompose its equation into homogeneous components in $z_*:

$$f_g = q_1 + q_2 + \ldots + q_{dl}.$$

Let us construct the hypertangent linear systems on $V_g$ at the point $o$:

$$\Lambda_i = \left\{ \sum_{j=1}^{i} s_{i-j}(q_1 + \ldots + q_j) \bigg|_{V_g} = 0 \right\},$$

where $s_a$ independently of each other run through the space $\mathcal{P}_{a,M}$ (see [12, Ch. 3] for details and examples). By condition (R1),

$$\operatorname{codim}_o(Bs \Lambda_i \subset V_g) = i \quad \text{for} \quad i = 1, \ldots, k - 1.$$

Now, applying the technique of hypertangent divisors in the usual way, let us construct a sequence of irreducible subvarieties $Y_2, Y_3, \ldots, Y_{k-1}$ of codimension $\operatorname{codim}(Y_i \subset V_g) = i$, where $Y_2 = Y$ and the last variety in this sequence satisfies the inequality

$$\frac{\operatorname{mult}_o Y_{k-1}}{\deg Y_{k-1}} > \frac{8}{dl} \cdot \frac{4}{3} \cdot \frac{5}{4} \cdot \ldots \cdot \frac{k}{k-1} \geq 1,$$

since $8k \geq 3dl$ by assumption. This gives the required contradiction and completes the proof of Proposition 4.2. □
4.3. Exclusion of the remaining options. In order to exclude cases (I.2) and (II), we repeat the arguments used in the proof of Proposition 4.2 with some modifications. We only consider in detail those modifications.

Proposition 4.3. Case (I.2) is impossible.

Proof. Assume the contrary. The tangent hyperplane $T_o V$ is given by the equation $a_{d,1}(z_*) = 0$. For the subspace $\Lambda \subset E$ there are two options:

(I.2.1) the subspace $\Lambda$ is given by the equations

$$y - h_1(z_*) = 0, \quad h_2(z_*) = 0,$$

where the forms $h_2$ and $a_{d,1}$ are linearly independent. If $h_1$ and $h_2$ are linearly dependent, then we may assume that $h_1 \equiv 0$;

(I.2.2) the subspace $\Lambda$ is given by the equations

$$h_1(z_*) = 0, \quad h_2(z_*) = 0,$$

where the forms $h_1$, $h_2$, and $a_{d,1}$ are linearly independent.

Assume first that case (I.2.1) takes place.

Lemma 4.2. The subvariety

$$V \cap \{y - h_1(z_*) = h_2(z_*) = 0\}$$

is irreducible, nonsingular at the point $o$, and different from $Z$.

The proof is completely similar to that of Lemma 4.1 and we omit it.

Now set $g(z_*) = h_1(z_*) + \lambda h_2(z_*)$ for a sufficiently general value $\lambda \in \mathbb{C}$. A contradiction is obtained by exactly the same arguments as in case (I.1). We have shown that case (I.2.1) is impossible.

Assume now that case (I.2.2) takes place. Let

$$h = \lambda_1 h_1 + \lambda_2 h_2 \in \langle h_1, h_2 \rangle$$

be a general form. Again it is easy to check that

$$Z \nsubseteq V|_{\{h=0\}},$$

so that the effective cycle $Z|_{\{h=0\}} = (Z \circ V|_{\{h=0\}})$ has codimension 2 on the irreducible hypersurface $V|_{\{h=0\}} \subset V$ and satisfies the inequality

$$\text{mult}_o Z|_{\{h=0\}} > \frac{8}{d} \deg Z|_{\{h=0\}}.$$

Since this inequality is linear, we may assume that the cycle $Z|_{\{h=0\}}$ is an irreducible subvariety. Now if

$$Z|_{\{h=0\}} \nsubseteq V_0,$$

then set $Y$ to be the irreducible component of the effective cycle $(Z|_{\{h=0\}} \circ V_0)$ with the maximal value of the ratio $\text{mult}_o / \deg$. If, on the contrary, $Z|_{\{h=0\}} \subset V_0$, then set $Y = Z|_{\{h=0\}}$. In any case, $Y \subset V_0|_{\{h=0\}}$ is a subvariety of codimension 1 or 2 that satisfies inequality (4.4). Now, using condition (R1.2), we argue in exactly the same way as in the proof of Proposition 4.2. This completes the proof of Proposition 4.3. □
**Proposition 4.4.** Case (II) is impossible.

**Proof.** Assume the contrary: $B \subset \text{Sing} V$, so that the point $o$ is a quadratic singularity of the variety $V$. By condition (R2) the point $o \in V$ is a quadratic singularity of rank $\geq 7$, so that we can apply the generalized $4n^2$-inequality [14] and conclude that 

$$\text{mult}_o Z > 4n^2 \cdot \text{mult}_o V = 8n^2;$$

hence

$$\text{mult}_o Z > \frac{8}{d} \deg Z.$$ 

Now we argue as in the proof of Proposition 4.3: since the inequality is linear in $Z$, we may assume that $Z$ is an irreducible subvariety of codimension 2. If $Z \not\subset V_0$, then we set $Y$ to be a component of the effective cycle $(Z \circ V_0)$ with the maximal value of the ratio $\text{mult}_o/\deg$. If $Z \subset V_0$, then we set $Y = Z$. Now we complete the proof in exactly the same way as the proof of Proposition 4.3.

The only difference is that now the variety $V_0$ is singular at $o$: it has a quadratic singularity (of rank $\geq 5$), so that if the hypersurface $V_0 \subset \mathbb{P}$ is given (in the affine coordinates) by the equation

$$q_2 + q_3 + \ldots + q_{dl} = 0,$$

then the hypertangent systems are of the form

$$\Lambda_i = \left\{ \sum_{j=2}^{i} s_{i-j}(q_2 + \ldots + q_j)|_{V_0} = 0 \right\},$$

where $i \geq 2$ and the condition for the hypersurface $V_0$ to be $k$-regular, where $k = \lceil dl/2 \rceil$, leads in the notation of the proof of Proposition 4.2 to the inequality

$$\frac{\text{mult}_o}{\deg Y_{k-1}} > \frac{8}{dl} \frac{5}{4} \frac{6}{5} \ldots \frac{k}{k-1} \geq 1,$$

since $2k \geq dl$. □

This completes the proof of Theorem 1.1.

**Remark 4.2.** From the very beginning we assumed that $d \geq 5$. The double covers are cyclic covers and their superrigidity is well known (see Subsection 1.3). If $d \in \{3, 4\}$, then birational superrigidity follows just from the condition that $V$ has at most quadratic singularities of rank $\geq 7$. Indeed, if $B \subset V$ is the centre of an infinitely near maximal singularity and $\text{codim}(B \subset V) \geq 3$, then either $B \not\subset \text{Sing} V$ and the ordinary $4n^2$-inequality holds,

$$\text{mult}_B Z > 4n^2,$$

or $B \subset \text{Sing} V$ and the generalized $4n^2$-inequality holds, which in this case takes the form of the estimate

$$\text{mult}_B Z > 8n^2.$$ 

In any case, $\text{mult}_B Z > \deg_H Z$, which is impossible (the linear system $|H|$ is free and defines the finite morphism $\pi: V \to \mathbb{P}$). Thus, for $d = 3$ or 4 the superrigidity holds under much weaker assumptions on the variety $V$.

**ACKNOWLEDGMENTS**

The author is grateful to the colleagues in the Departments of Algebraic Geometry and Algebra of the Steklov Mathematical Institute for their interest in his work, as well as to the colleagues in algebraic geometry at the University of Liverpool for general support.
The work was supported by the Leverhulme Trust (Research Project Grant RPG-2016-279).

REFERENCES

1. F. Call and G. Lyubeznik, “A simple proof of Grothendieck’s theorem on the parafactoriality of local rings,” in Commutative Algebra: Syzygies, Multiplicities, and Birational Algebra: AMS-IMS-SIAM Summer Res. Conf., 1992 (Am. Math. Soc., Providence, RI, 1994), Contemp. Math. 159, pp. 15–18.

2. I. A. Cheltsov, “Double space with double line,” Sb. Math. 195 (10), 1503–1544 (2004) [transl. from Mat. Sb. 195 (10), 109–156 (2004)].

3. I. A. Cheltsov, “Birationally superrigid cyclic triple spaces,” Izv. Math. 68 (6), 1229–1275 (2004) [transl. from Izv. Ross. Akad. Nauk, Ser. Mat. 68 (6), 169–220 (2004)].

4. I. Cheltsov, “On nodal sextic fivefold,” Math. Nachr. 280 (12), 1344–1353 (2007).

5. I. Cheltsov and J. Park, “Sextic double solids,” in Cohomological and Geometric Approaches to Rationality Problems: New Perspectives (Birkhäuser, Boston, 2010), Prog. Math. 282, pp. 75–132.

6. V. A. Iskovskikh, “Birational automorphisms of three-dimensional algebraic varieties,” J. Sov. Math. 13, 815–868 (1980) [transl. from Itogi Nauki Tekh., Ser.: Sovrem. Probl. Mat. 12, 159–236 (1979)].

7. V. A. Iskovskikh and Ju. I. Manin, “Three-dimensional quartics and counterexamples to the Lüroth problem,” Math. USSR, Sb. 86 (1), 140–166 (1971) [transl. from Mat. Sb. 15 (1), 141–166 (1971)].

8. R. Mullany, “Fano double spaces with a big singular locus,” Math. Notes 87 (3), 444–448 (2010).

9. A. V. Pukhlikov, “Birational automorphisms of a double space and double quadric,” Math. USSR, Izv. 32 (1), 233–243 (1989) [transl. from Izv. Akad. Nauk SSSR, Ser. Mat. 52 (1), 229–239 (1988)].

10. A. V. Pukhlikov, “Birational automorphisms of double spaces with singularities,” in Algebraic Geometry–2 (VINITI, Moscow, 2001), Itogi Nauki Tekh., Ser.: Sovrem. Mat. Prilozh., Temat. Obz. 24, pp. 177–196. Engl. transl. in J. Math. Sci. 85 (4), 2128–2141 (1997).

11. A. V. Pukhlikov, “Birational geometry of algebraic varieties with a pencil of Fano cyclic covers,” Pure Appl. Math. Q. 5 (2), 641–700 (2009).

12. A. Pukhlikov, Birationally Rigid Varieties (Am. Math. Soc., Providence, RI, 2013), Math. Surv. Monogr. 190.

13. A. V. Pukhlikov, “Birationally rigid Fano fibre spaces. II,” Izv. Math. 79 (4), 809–837 (2015) [transl. from Izv. Ross. Akad. Nauk, Ser. Mat. 79 (4), 175–204 (2015)].

14. A. V. Pukhlikov, “The $4n^2$-inequality for complete intersection singularities,” Arnold Math. J. 3 (2), 187–196 (2017).

This article was submitted by the author simultaneously in Russian and English