The partial captivity condition for U(1) extensions of expanding maps on the circle

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Abstract
This paper concerns the compact group extension

$$f: \mathbb{T}^2 \to \mathbb{T}^2, \quad f(x, s) = (E(x), s + \tau(x) \mod 1)$$

of an expanding map $E: \mathbb{S}^1 \to \mathbb{S}^1$. The dynamics of $f$ and its stochastic perturbations have previously been studied under the so-called partial captivity condition. Here we prove a supplementary result that shows that partial captivity is a $C^r$ generic condition on $\tau$, once we fix $E$.

Keywords: dynamical system, partially expanding map, partial captivity, transversality
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1. Introduction
Let $E: \mathbb{S}^1 \to \mathbb{S}^1$ be a $C^r$ expanding map on the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ of degree $\ell \geq 2$. For each $\tau: \mathbb{S}^1 \to \mathbb{R}$ in $C^r$, we consider the compact group extension

$$f = f_\tau = f_{E, \tau}: \mathbb{T}^2 \to \mathbb{T}^2, \quad f(x, s) = (E(x), s + \tau(x) \mod 1)$$

where $\mathbb{T}^2$ denotes the torus $\mathbb{S}^1 \times \mathbb{S}^1$. This is one of the typical examples of partially hyperbolic dynamical systems. A naive expectation about the dynamics of $f$ is that we will observe ‘virtually’ random dynamics in the fibers, the randomness being driven by the chaotic dynamics...
of $E$, and consequently that the dynamics of $f$ will be strongly mixing. This is of course not true if the function $\tau$ does not transmit the randomness of the dynamics of $E$ to that in the fibers. Indeed, if $\tau(x) \equiv c$, we will observe just a rigid rotation by $c$ in the fibers. Further, if $\tau$ is cohomologous to a constant, i.e. $\tau(x) = \varphi(E(x)) - \varphi(x) + c$ for some $\varphi \in \mathcal{C}^0(\mathbb{S}^1)$ and $c \in \mathbb{R}$, the dynamics of $f_{E,\tau}$ is conjugated to the case $\tau(x) \equiv c$. To obtain rigorous statements that realize the naive idea described above we must therefore impose some condition on $\tau$. It is of course preferable if such a condition is generic, that is, holds for most systems $f$. It is known that $f$ is exponentially mixing once $\tau$ is not cohomologous to a constant. (See [2, section 3] for instance.) This provides a rather complete description of the mixing property of $f$. However, stronger conditions are needed in order to study finer structures of the dynamics, such as spectral properties of the associated transfer operators.

Below we define and discuss the partial captivity condition for $f$, which implies roughly that the dynamics of $f$ have properties contrary to those in the case $\tau(x) \equiv c$. (However, we wish to emphasize that the partial captivity condition is strictly stronger than not being cohomologous to a constant, see appendix B.) The condition was introduced by Faure [3] to study spectral properties of the associated Perron–Frobenius operators. In the previous paper [4], the first and third author studied fine properties of stochastic perturbations of $f$ again assuming this condition. In this paper, we prove a supplementary result showing that the partial captivity condition is indeed a generic condition.

Before we state the condition we introduce some notation. Note that, by the definition of an expanding map, there are constants $1 < \lambda \leq \Lambda$ such that

$$S \lambda \Lambda \in \mathbb{F} \forall x \in \mathbb{S}^1.$$ 

Let us set

$$\vartheta(\tau, \lambda) = -\lambda^\infty.$$ 

Fix some $R > \|\tau\|_{\infty}$ and put

$$\vartheta(R) := R/(\lambda - 1) > \vartheta(\tau).$$

Then the corresponding cone $\mathcal{K} \subset \mathcal{K} \subset \mathcal{K} \subset \mathcal{K}$ is (forward) invariant under the Jacobian matrix

$$D f(z) = \begin{pmatrix} E'(x) & 0 \\ \tau'(x) & 1 \end{pmatrix} \quad z = (x, s) \in \mathbb{T}^2.$$ 

More precisely we have for all $z \in \mathbb{T}^2$ and $m \geq 1$ that

$$D f^m(z) \mathcal{K} \subset \mathcal{K} \subset \mathcal{K} \subset \mathcal{K} \quad \text{where } R_m = \|\tau\|_{\infty} + \lambda^m \|R - \|\tau\|_{\infty}\| < R. \quad (1)$$

For $z \in \mathbb{T}^2$ and $n \geq 1$, let us consider the images of $\mathcal{K}$ by $D f^n$ in $T_z \mathbb{T}^2$, i.e.

$$D f^n(\zeta) \mathcal{K} \quad \text{for } \zeta \in f^{-n}(z). \quad (2)$$

It is not difficult to see that $\tau$ is cohomologous to a constant if and only if all the cones in (2) have a line in common at every point $z \in \mathbb{T}^2$ and $n \geq 1$. Thus we naturally come to the idea of considering transversality between the cones (2). As a way to quantify this notion, we set

$$n(\tau, R; n) = \sup \sup \#(\zeta \in f^{-n}(z) | v \in D f^n(\zeta) \mathcal{K})$$

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where \( \sup \) denotes the supremum over unit vectors \( v \in \mathbb{R}^2 \). This is sub-multiplicative as a function of \( n \). Indeed, from (1), we have
\[
\tau(\tau; R; n + m) \leq \tau(\tau', R; r; n) \cdot \tau(\tau; R; m) \leq \tau(\tau, R; n) = \tau(\tau, R; m).
\]
By Fekete’s lemma, the limit
\[
\tau := \lim_{n \to \infty} \tau(\tau; R; n)
\]
then exists and is equal to \( \inf R \tau(\tau; R; n) \). Note that we are justified in dropping \( R \) from the notation \( \tau \) since it does not depend on \( R \) in view of (3).

**Definition 1.** We say that \( f = f_{(E, \tau)} \) is partially captive if \( \tau = 1 \). This condition is equivalent to the condition introduced and used in \([3, 4]\). (See also \([5]\). For completeness, the equivalence is demonstrated in appendix A.) However, the partial captivity condition is not proved to be generic in \([3, 4]\) and nowhere else. Here we provide the required:

**Theorem 2.** Let \( r > 2 \) and suppose that the expanding map \( E : S^1 \to S^1 \) is fixed. For every \( q > 1 \), there is an open dense subset \( \mathcal{V} \subset \mathcal{C}^0(S^1) \) such that if \( \tau \in \mathcal{V} \) then
\[
\tau < q.
\]
Consequently, there is a residual subset \( \mathcal{R} \subset \mathcal{C}^0(S^1) \) such that \( \tau = 1 \) for \( \tau \in \mathcal{R} \).

**2. Proof of theorem 2**

We henceforth fix \( R \) and sometimes drop it from the notation.

**2.1. Notation**

Let \( \mathcal{A} = \{0, \ldots, \ell - 1\} \). We suppose that \( 0 \in S^1 \) is one of the fixed points of \( E \). Let \( \mathcal{I}(j) = [\alpha_j, \beta_j] \subset S^1, j \in \mathcal{A}, \) be the semi-open intervals obtained by dividing \( S^1 \) at the \( \ell \) points in \( E^{-1}(0) \), so that \( E \) maps each of them onto \( S^1 \) bijectively. For a word \( \alpha = (\alpha_0, \ldots, \alpha_2, \alpha_1) \in \mathcal{A}^\ell \) with alphabet \( \mathcal{A} \) and of length \( |\alpha| := n \), let \( \mathcal{I}(\alpha) \) be the interval defined by
\[
\mathcal{I}(\alpha) = \bigcap_{j=0}^{n-1} E^{-j}(\mathcal{I}(\alpha_{n-j})).
\]
Clearly \( E^n \) maps each \( \mathcal{I}(\alpha) \) with \( |\alpha| = n \) onto \( S^1 \) bijectively. For each \( 1 \leq p \leq n \), we define \( [\alpha_p]_p \in \mathcal{A}^p \) to be the truncation \( (\alpha_p, \ldots, \alpha_1) \). For each \( x \in S^1 \) and \( \alpha \in \mathcal{A}^p \), we write \( x_{\alpha} \) for the unique point in \( \mathcal{I}(\alpha) \) that is mapped to \( x \) by \( E^n \). The differential \( Df(z) \) actually depends only on the first component \( x \) of \( z = (x, s) \in T^2 \). Thus we will sometimes write \( Df(x) \) by abuse of notation. We also consistently identify the pre-images \( f^{-n}(z) \) and \( E^{-n}(x) \) for \( z = (x, s) \).

**2.2. A difficulty caused by the nonlinearity of \( E \)**

To prove theorem 2, we resolve by perturbations the situation where a unit vector \( v_0 \in \mathbb{R}^2 \) is contained in many of the cones in (2). To this end we note that, because of the nonlinearity of \( E \), there is much variation in the angles of the cones \( \mathcal{D} f^\alpha(x_0) \mathcal{K}_R \) for \( \alpha \in \mathcal{A}^p \). As one can easily imagine, the larger the angle of \( \mathcal{D} f^\alpha(x_0) \mathcal{K}_R \), the more perturbation is needed to resolve the situation \( v_0 \in \mathcal{D} f^\alpha(x_0) \mathcal{K}_R \) for some fixed \( v_0 \). This prevents us from applying the argument in
literally. But this difficulty is compensated by the fact that there are relatively few \( \alpha \in A \) for which \( Df^n(\alpha) \mathcal{H} \) is larger than average. More precisely, we have the following lemma. Note that the angle of \( Df^n(x_n) \mathcal{H} \) is proportional to the reciprocal of \( (E^n)'(x_n) \).

**Lemma 3.** There exists a constant \( C > 0 \) such that, for any \( x \in S^1 \) and \( b > 0 \),

\[
\# \{ y \in E^{-n}(x) \mid (E^n)'(y) \leq e^{bn} \} \leq Ce^{bn}.
\]

**Proof.** By a simple distortion argument along the backward orbits, we have that

\[
C^{-1} \leq \sum_{y \in E^{-n}(x)} (E^n)'(y) \leq C
\]

for a constant \( C \geq 1 \). Then the lemma is a direct consequence.

\[\square\]

### 2.3. Consequences of the condition \( \sigma(\tau) \geq e^\rho \)

Following the argument in Tsujii [5], we begin by analyzing the situation where condition (4) does not hold, that is to say, when \( \sigma(\tau) \geq e^\rho \) for some \( \rho > 0 \). To proceed with the idea described in the previous subsection, we cover the interval \( [\log \lambda, \log \Lambda] \) with open intervals \( I_j = (a_j, b_j), 1 \leq j \leq J \), such that \( |I_j| := |b_j - a_j| < \rho/3 \). We set

\[
N = N(\rho) := \lceil 6\rho^{-1}\log(2\Lambda) \rceil
\]

where \( \lceil t \rceil \) denotes the smallest integer \( \geq t \) for \( t \in \mathbb{R} \). Then we choose an integer \( q \) so large that

\[
(q + 1)N \cdot e^{-q\rho/2} < 1/(4J).
\]

Let \( e^\mu = [e^{a_0}, e^{b_0}] \) for \( I = [a, b] \).

**Proposition 4.** If \( \sigma(\tau) \geq e^\rho \), then we can find an arbitrarily large \( n \) and

- a point \( z_0 = (x, s) \in T^2 \),
- a unit vector \( v_0 \in \mathbb{R}^2 \),
- an integer \( 1 \leq j \leq J \),
- a subset \( B \subset A^f \) with \( \#B = 2(q + 1)N \),
- subsets \( \Sigma(\beta) \subset A^f \) for each \( \beta \in B \) with \( \#\Sigma(\beta) \geq e^\mu \cdot e^{-q\rho/(2J)} \)

such that

- \( [\alpha_0] = \beta \) for \( \alpha \in \Sigma(\beta) \),
- \( v_0 \in Df^n(x_n) \mathcal{H} \) and \( (E^n)'(x_n) \in e^{a_0} \) for \( \alpha \in \Sigma(\beta) \) with \( \beta \in B \).

**Proof.** Since \( \sigma(\tau) \geq e^\rho \), we can find an arbitrarily large \( n \) such that \( \sigma(\tau; n) \geq e^{\rho(2q)} \cdot \sigma(\tau; n - q) \).

By definition, we can find a point \( z_0 = (x, s) \) and a unit vector \( v_0 \in \mathbb{R}^2 \) such that

\[
\# \{ \alpha \in A^f \mid v_0 \in Df^n(x_n) \mathcal{H} \} = \sigma(\tau; n).
\]

Hence, there is a \( 1 \leq j \leq J \) such that the set

\[
B' = \{ \alpha \in A^f \mid v_0 \in Df^n(x_n) \mathcal{H}, (E^n)'(x_n) \in e^{a_0} \}
\]
satisfies \( \#B' \geq \eta(\tau; n)/J \). We divide the set \( B' \) into
\[
\Sigma(\beta) = \{ \alpha \in B' \mid [\alpha]_q = \beta \} \quad \text{for } \beta \in \mathcal{A}.
\]
Note that, for \( \alpha \in \mathcal{A} \) with \( [\alpha]_q = \beta \), we have \( v_0 \in D f^n(x_0) \mathcal{K}_R \) if and only if \( (D f^n(x_0))^{-1}(v_0) \in D f^{n-q}(x_0) \mathcal{K}_R \). This implies
\[
\#\Sigma(\beta) \leq \eta(\tau; n-q) \leq e^{-({\rho/2})q} \cdot \eta(\tau; n).
\]
Note also that we obviously have
\[
\sum_{\beta \in \mathcal{A}} \#\Sigma(\beta) = \#B' \geq \eta(\tau; n)/J.
\]
We now pick the \( 2(q+1)N \) largest sets among \( \Sigma(\beta) \) for \( \beta \in \mathcal{A} \), and let \( B \subset \mathcal{A} \) be the corresponding \( 2(q+1)N \) elements in \( \mathcal{A} \). By the estimates above we have
\[
\#\Sigma(\beta) \geq \eta(\tau; n) \cdot \ell^{-q}/(2J) \quad \text{for } \beta \in B,
\]
because the average of \( \#\Sigma(\beta) \) over the rest \( \mathcal{A} \setminus B \) must be bounded from below by
\[
\ell^{-q}(\eta(\tau; n)/J - e^{-({\rho/2})q} \cdot \eta(\tau; n) \cdot 2(q+1)N) \geq \eta(\tau; n) \cdot \ell^{-q}/(2J)
\]
where the last inequality follows from the choice of \( q \). Since \( \eta(\tau; n) \) is sub-multiplicative, we have \( \eta(\tau; n) \geq e^{\rho n} \) for all \( n \geq 1 \), which completes the proof. \( \square \)

Below we rewrite the conclusion of proposition 4 in a form that is more suitable for the perturbation argument in the next subsection. By choosing \( \varepsilon > 0 \) so small that the intervals \( I_j' = (a_j + \varepsilon, b_j - \varepsilon) \) for \( 1 \leq j \leq J \) still cover \([\log \lambda, \log \Lambda]\), we may assume that the conclusion of proposition 4 holds with \( I_j \) replaced by \( I_j' \subseteq I_j \). Next, we rewrite the condition
\[
v_0 \in D f^n(x_0) \mathcal{K}_R
\]
Let us define
\[
S_0(x; \alpha) = \sum_{k=1}^{n} \frac{\tau'(x_{0k})}{(E^n y(x_{0k}))}.
\]
This is nothing but the slope of the image of the horizontal line by \( D f^n(x_0) \mathcal{K}_R \). In particular, \( v_0 \in D f^n(x_0) \mathcal{K}_R \) is equivalent to \( |S_0(x; \alpha) - S| \leq \partial \mathcal{K} \cdot (E^n y(x_0))^{-1} \), where \( S \) denotes the slope of \( v_0 \). Hence, the conditions \( v_0 \in D f^n(x_0) \mathcal{K}_R \) and \( (E^n y(x_0)) \in e^{\mathcal{K} I_j} \) imply that
\[
|S_0(x; \alpha) - S| \leq \partial \mathcal{K} \cdot (E^n y(x_0))^{-1} \leq \partial \mathcal{K} \cdot e^{-({\alpha/2})\log n}.
\]
Finally, we shift the first component \( x \) of the point \( z_0 = (x, s) \) and the unit vector \( v_0 \) respectively to nearby points in the finite sets
\[
T(n) = \{ x \in S \mid [2\Lambda]^n x \in \mathbb{Z} \}
\]
and
\[
S(n) = \{ (\cos 2\pi \theta, \sin 2\pi \theta) \in S \mid [2\Lambda]^n \theta \in \mathbb{Z} \}.
\]
This will change the values of \( S_0(x; \alpha), (E^n y(x_0)) \) and the slope of \( v_0 \) slightly. However, it is easy to see that the grids \( T(n) \) and \( S(n) \) are fine enough for us to conclude that, after the shift,
\((E^m)'(x_0)\) belongs to the larger interval \(e^{\alpha} P\), and that \(|S_0(x; \alpha) - S| \leq e^{-\alpha} \vartheta\), where \(S\) is the slope of the shifted vector \(v\). Hence, we can replace the parts

\[
\begin{align*}
\text{a point } z_0 \in \mathbb{T}^2, & \quad \text{a unit vector } v_0 \in \mathbb{R}^2, & \quad v_0 = Df^n(x_0) \mathcal{K}_R \\
\end{align*}
\]

in the conclusion of proposition 4 with

\[
\begin{align*}
\text{a point } x \in T(n), & \quad \text{a unit vector } v_0 = (\cos \theta, \sin \theta) \in S(n), \\
|S_0(x; \alpha) - \tan \theta| \leq e^{-\alpha} \vartheta, \\
\end{align*}
\]

respectively.

2.4. Generic perturbations

Let \(\tau \in \mathcal{C}'(\mathbb{S})\). For perturbations of \(\tau\), we will take a set of functions \(\varphi_i \in \mathcal{C}'(\mathbb{S})\), \(1 \leq i \leq m\), and consider the family

\[
\tau_t(x) = \tau(x) + \sum_{i=1}^{m} t_i \varphi_i(x)
\]

parametrized by \(t = (t_i)_{i=1}^{m} \in \mathbb{R}^m\). For a point \(x \in \mathbb{S}\), an integer \(n \geq 1\) and a finite subset \(A \subset \mathcal{A}\) with \(#A = p\), let \(G_{s,A} : \mathbb{R}^n \to \mathbb{R}^p\) be the affine map defined by

\[
G_{s,A}(t) = (S_0(x; \alpha; \tau_i))_{i \in A}
\]

where we used the notation \(S_0(x; \alpha; \tau) = S_0(x; \alpha)\) for the functions given by (6) to indicate the dependence on \(\tau\). For an affine map \(M : E \to F\) between Euclidean spaces, let \(\text{Jac}(M)\) be the modulus of the Jacobian determinant of \(DM|_{\text{ker}(DM)^\perp}\) the restriction of the linear part \(DM\) to the orthogonal complement of its kernel when \(M\) is surjective, and put \(\text{Jac}(M) = 0\) otherwise. Obviously we have \(\text{Jac}(M) \geq \text{Jac}(M|_{L})\) for any subspace \(L \subset E\). Also we have

\[
\text{Leb}_E\{z \in E ||z|| \leq 1, M(z) \in X\} \leq C \text{Jac}(M)^{-1} \text{Leb}_X
\]

where the constant \(C\) depends only on the dimension of \(E\). The next lemma is a slight generalization of [5, proposition 3.4].

**Proposition 5.** For any integers \(p \geq 1\) and \(\nu \geq 1\), there are functions \(\varphi_i \in \mathcal{C}'(\mathbb{S})\), \(1 \leq i \leq m\), such that the following holds true: for any \(x \in \mathbb{S}\) and subset \(B \subset \mathcal{A}\) with \(#B \geq p(\nu + 1)\), there is a subset \(B' \subset B\) with \(#B' = p\) such that we have

\[
\text{Jac}(G_{s,A}) \geq 1
\]

provided that \(A\) is a subset of \(\mathcal{A}\) with \(n \geq \nu\) such that the map \(\alpha \in A \mapsto [\alpha]_B \in B'\) is a one-to-one correspondence.

**Remark 6.** We omit the proof of proposition 5, because it is obtained by translating that of [5, proposition 3.4] almost literally. Note that the map \(E\) is now a general expanding map though it was linear \(x \mapsto \ell x\) in [5, proposition 3.4] and that we have to replace the factor \(\ell^k\) in some places by \((a \mathcal{C}^1\)-approximation\(^4\) of) the derivative of \(E^k\). (Also note that \(x_0\) was denoted \(\alpha(x)\) in [5].)

\(^4\)Since the derivative of \(E^k\) is \(\mathcal{C}^{r-1}\), we actually have to approximate it by a \(\mathcal{C}^r\) function.
2.5. The end of the proof

We take an arbitrary bounded open subset $D$ of $\mathcal{C}^r(S^1)$. Then we take the constant $R > 0$ such that $R > ||r||_{\infty}$ uniformly for $r \in D$. We also take $\rho > 0$ arbitrarily and let $X_\rho$ be the set of $r \in D$ for which $r(\tau) > e^{\rho}$. For the proof of theorem 2, it is enough to show that the complement of $X_\rho$ in $D$ contains a dense subset because the condition $r(\tau) < e^{\rho}$ is an open condition on $\tau$. (Recall that $r(\tau; R; n)$ is sub-multiplicative.) Let $X(n) \subset D$ be the set of $\tau \in D$ for which the conditions in the statement of proposition 4 (modified as described at the end of subsection 2.3) hold for $n$. Suppose $\tau \in D$. To define the family $X_\rho$, we take the functions $C_{\phi} \in \mathcal{C}(S^1)$, $\im_{\phi}/uni2A7D/uni2A7D$, in proposition 5 by choosing $p = N$ and $\nu = q$. Theorem 2 follows from the next proposition.

**Proposition 7.** $\text{Leb}\{t \in [-1, 1]^m | \gamma \in X(n) \text{ for infinitely many } n\} = 0$.

**Proof.** We focus on how quantities below depend on $n$ and use $C$ to denote generic constants which do not depend on $n$. For given $n$, a point $x \in T(n)$, a vector $(\cos \theta, \sin \theta) \in S(n)$, an integer $j \in \mathbb{N}$ and a subset $B \subset A$ with $#B = 2(q + 1)N$, let $X(n; x, \theta, j, B)$ be the set of $\tau \in D$ for which the conditions in the (modified) statement of proposition 4 hold. Let $B' \subset B$ with $#B' = N$ be the subset in proposition 5. (We suppose that $B'$ is chosen uniquely for each $B$.) Let $\Sigma(\beta) \subset A$ be those in proposition 4, but considered now only for $\beta \in B'$. From the choice of the functions $\phi_i \in \mathcal{C}^r(S^1)$, we have

$$\text{Leb}\{t \in [-1, 1]^m | [S_n(x; \alpha_i; \tau_1)] - \tan \theta \leq e^{-qn} \cdot \partial R\} \leq Ce^{-qnN}$$

for any combination $(\alpha_1, \alpha_2, \cdots, \alpha_N)$ of elements in $A^n$ such that

- $(E^n)(x_\alpha) \in e^{\theta n} = [e^{\theta n}, e^{b\theta n}]$,
- $[\alpha_i], 1 \leq i \leq N$, are in one-to-one correspondence with the elements of $B'$.

(This is simply a result of (8) applied to a translate of (7).) On the one hand, the number of possible such combinations $(\alpha_1, \alpha_2, \cdots, \alpha_N)$ is at most $Ce^{b\theta nN}$ by lemma 3. On the other hand, $\tau_1$ belongs to $X(n; x, \theta, j, B)$ only if the condition

$$[S_n(x; \alpha_i; \tau_1)] - \tan \theta \leq e^{-qn} \cdot \partial R$$

holds for at least $(\#\Sigma(\beta))^N$ of these possible combinations. Therefore we have

$$\text{Leb}\{t \in [-1, 1]^m | \gamma \in X(n; x, \theta, j, B)\} \leq \frac{C e^{-qnN} \cdot Ce^{b\theta nN}}{(\#\Sigma(\beta))^N} \leq \frac{Ce^{b\theta nN} \cdot e^{-qnN}}{(e^m \cdot e^{-qnN})^N}$$

Taking the number of possible choices for $x, \theta, j$ and $B$ into account, we obtain

$$\text{Leb}\{t \in [-1, 1]^m | \gamma \in X(n)\} \leq \frac{Ce^{(b - a)nN} \cdot e^{2n \log[2\Lambda]}}{e^{mN}} \leq Ce^{-(\rho/3)nN}$$

where the latter inequality is a consequence of the choice of $N$ and the intervals $I_j$. Therefore the conclusion follows by the Borel–Cantelli lemma.
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Appendix A. The definition of partial captivity

Here we show that our definition of partial captivity indeed coincides with the one introduced by Faure [3, definition 15]. We fix \( \tau \) and mostly suppress it from the notation. For \( \alpha \in A^\infty \), let

\[
S(x; \alpha) = \sum_{k=1}^{\infty} \frac{x'_{(0,k)}}{(E^\tau)'_{(0,k)}}.
\]

Then \( S_\alpha(x; \alpha) \) defined by (6) is the truncation of \( S(x; \alpha) \) of length \( n \). For \( R > 0 \), set

\[
\widetilde{N}_R(n) = \sup_{y, \eta} \{ \alpha \in A^\infty | |\eta - S(y; \alpha)| \leq R \cdot (E^\tau)'(x_\alpha) \}^{-1}.
\]

By [3, proposition 17] it follows that \( f = f_{E_\alpha} \) is partially captive in the sense of Faure if and only if

\[
\lim_{n \to \infty} n^{-1} \log \widetilde{N}_R(n) = 0 \quad \text{(A.1)}
\]

for all \( R > 0 \). (We remark that \( S \) and \( S_\alpha \) appear in [3, 4] with a change of sign.)

We first show that if \( \sigma(\tau) = 1 \) then (A.1) holds. Assume therefore that \( \sigma(\tau) = 1 \) and let \( R > 0 \) be arbitrary. Fix \( z = (x, s) \in T^2 \) and \( \eta \in \mathbb{R} \). Define a unit vector \( v = (\cos \theta, \sin \theta) \) by setting \( \theta = \arctan \frac{\eta}{\sigma} \). Suppose now that \( \alpha \) is a word in \( A^\infty \) such that

\[
|\eta - S(x; \alpha)| \leq R \cdot (E^\tau)'(x_\alpha)^{-1}.
\]

Note that the set of pre-images \( \zeta \in f^{-1}(z) \) is in one-to-one correspondence with \( \alpha \in A^\infty \) via \( \pi_\alpha(\zeta) = x_\alpha \), where \( \pi_\alpha : T^2 \to S^1 \) is the natural projection onto the first coordinate. A simple calculation shows that

\[
|\zeta - S(x; \alpha) - \tan \theta| \leq (R + \sigma)(E^\tau)'(x_\alpha)^{-1}
\]

so \( v \in Df^n(\zeta) X_R \) whenever \( \sigma > 0 \). It follows that \( \widetilde{N}_R(n) \leq \sigma(\tau, R; n) \), so (A.1) holds, as promised.

The reversed implication is proved in an analogue fashion. Given \( v_0 \in Df^n(\zeta) X_R \) with \( v_0 = (\cos \theta_0, \sin \theta_0) \) we set \( \eta_0 = \tan \theta_0 \). With \( \alpha \) given by the correspondence \( \pi_\alpha(\zeta) = x_\alpha \), a calculation similar to the one above shows that

\[
|\eta_0 - S(x; \alpha)| \leq (\sigma + \sigma)(E^\tau)'(x_\alpha)^{-1}
\]

so if \( R > \sigma + \sigma \) then \( \sigma(\tau, R; n) \leq \widetilde{N}_R(n) \). Hence \( \sigma(\tau) = 1 \) by virtue of (A.1).

Appendix B. The transversality condition

In this appendix, we briefly discuss two other quantities defined similarly to \( \sigma(\tau) \). We define

\[
m(\tau, R; n) = \sup_{\zeta \in f^{-n}(z)} \sup_{w \in f^{-n}(\zeta) \cap \partial}$

\[
= \frac{1}{\det Df^n(\zeta)}
\]

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where the sum is taken over \( \zeta \in f^{-n}(z) \) such that \( Df^n(\zeta) \cap Df^n(w) = \emptyset \). We then set

\[
m(\tau) := \lim \sup_{n \to \infty} m(\tau, R; n)\]

which does not depend on the choice of \( R \). Similarly, for \( z \in \mathbb{T}^d \) and \( n \geq 2 \), we set

\[
n(\tau, R; n) = \sup_{\zeta} \sup_{v} \sum_{\zeta \in f^{-n}(z) \cap Df^n(\zeta)} \frac{1}{\det Df^n(\zeta)},
\]

where the supremum \( \sup \) is taken over all unit vectors \( v \) in \( \mathbb{R}^2 \) and the summation is taken over those points \( \zeta \in f^{-n}(z) \) for which \( Df^n(\zeta) \) contains \( v \). This is sub-multiplicative so that we can define

\[
n(\tau) := \lim \sup_{n \to \infty} n(\tau; n)^{\frac{1}{n}}.
\]

Lemma 8. The following are all equivalent:

1. \( \tau \) is cohomologous to a constant,
2. \( m(\tau) = 1 \),
3. \( n(\tau) = 1 \),
4. \( \sigma(\tau) = \ell \).

Proof. For the equivalence of items 1, 2, 3, we refer to [1]. It is clear that item 4 follows from item 1. We prove that \( n(\tau) < 1 \) implies \( \sigma(\tau) < \ell \) to complete the proof. If \( n(\tau) < 1 \), we have \( n(\tau, R; n) < (1 - \varepsilon)^n \) for sufficiently small \( \varepsilon > 0 \) and sufficiently large \( n \). This means that \( \sigma(\tau, R; n) \) cannot be equal to \( \ell^n \) for infinitely many \( n \), because for each such \( n \) we have \( C^{-1} \leq n(\tau, R; n) < (1 - \varepsilon)^n \) by (5), which leads to a contradiction if we choose \( n \) large enough. Hence, \( \sigma(\tau, R; n) < \ell^n \) for sufficiently large \( n \), so \( \sigma(\tau, R; n) \frac{1}{n} \leq \ell - \eta \) for some small \( \eta > 0 \). Since \( \sigma(\tau, R; n) \) is sub-multiplicative, we find that \( \sigma(\tau) \leq \ell - \eta < \ell \).

From the lemma above, we see that the partial captivity condition \( \sigma(\tau) = 1 \) is a much stronger condition than requiring that \( \tau \) not be cohomologous to a constant. Nevertheless, theorem 2 shows that partial captivity is still a generic condition. Finally, setting \( \chi = \lim_{n \to \infty} (\min(E^n)^\frac{1}{n}) \leq \lambda^{-1} < 1 \), we observe that

\[
\chi \leq m(\tau) \leq 1, \quad \chi \leq n(\tau) \leq 1, \quad 1 \leq \sigma(\tau) \leq \ell
\]

in general. The partial captivity condition \( \sigma(\tau) = 1 \) implies that \( m(\tau) \) and \( n(\tau) \) take the smallest possible value, that is, \( m(\tau) = n(\tau) = \chi \).

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