D-ORTHOGONAL POLYNOMIALS, TODA LATTICE AND VIRASORO SYMMETRIES

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Abstract. The subject of this paper is a connection between d-orthogonal polynomials and the Toda lattice hierarchy. In more details we consider some polynomial systems similar to Hermite polynomials, but satisfying $d+2$-term recurrence relation, $d > 1$. Any such polynomial system defines a solution of the Toda lattice hierarchy. However we impose also the condition that the polynomials are also eigenfunctions of a differential operator, i.e. a bispectral problem. This leads to a solution of the Toda lattice hierarchy, enjoying a number of special properties. In particular the corresponding tau-functions $\tau_m$ satisfy the Virasoro constraints. The most spectacular feature of these tau-functions is that all of them are partition functions of matrix models. Some of them are well known matrix models - e.g. Kontsevich model, Kontsevich-Penner models, $r$-spin models, etc. A remarkable phenomenon is that the solution corresponding to $d = 2$ contains two famous tau functions describing the intersection numbers on moduli spaces of compact Riemann surfaces and of open Riemann surfaces.

1. Introduction

In 1975 J. Moser published the paper [34], in which he made explicit the connection between orthogonal polynomials and the Toda lattice. Namely, starting with any system of orthogonal polynomials he pointed out that the Toda flow of the coefficients of the 3-term recurrence relation can be expressed in terms of the Stieltjes function (the generating function of the moments of the corresponding measure) of the polynomial system. Later this research was continued both for orthogonal polynomials and for their generalizations (like multiple orthogonal polynomials), see e.g. [3, 7, 35].

The present paper also deals with integrable systems of Toda type and their connections with polynomial systems which are generalizations of orthogonal polynomials. Before explaining the main results we give a brief account of the needed notions.

In 1986 Duistermaat and Grünbaum [21] introduced the notion of bispectral operators. The operators $L(x)$ in the variable $x$ and $\Lambda(z)$ in $z$ are called bispectral if there exists a joint eigenfunction $\Psi(x, z)$ such that

$$L(x)\Psi(x, z) = f(z)\Psi(x, z)$$
$$\Lambda(z)\Psi(x, z) = \theta(x)\Psi(x, z).$$

Here $f(z)$ and $\theta(x)$ are some nonconstant functions. I don’t specify the type of the operators, neither the variables as they can be of different types. Some examples will be more helpful to clarify the situation.

2010 Mathematics Subject Classification. 34L20 (Primary); 30C15, 33E05 (Secondary).

Key words and phrases. d-orthogonal polynomials, bispectral problem, Toda lattice, Virasoro constraints, matrix models.
1) $\psi(x,z) = e^{xz}$, $L = \partial_x$, $\Lambda = \partial_z$, $x,z \in \mathbb{C}$.

$$Le^{xz} = ze^{xz}, \quad \Lambda e^{xz} = xe^{xz}.$$  

2) Airy operator

$$(\partial_x^2 - x)Ai(x + z) = zAi(x + z)$$

$$(\partial_z^2 - z)Ai(x + z) = xAi(x + z).$$

3) Let $T$ be the shift operator in $n \in \mathbb{Z}$; $T \pm 1 f(n) = f(n \pm 1)$ and $L = x\partial_x$, $x \in \mathbb{C}$.

Then

$$Lx^n = nx^n, \quad Tx^n = x \cdot x^n.$$  

4) Let $H_n(x), n = 0,1,\ldots$ be the Hermite polynomials. Then

$$(-\partial_x^2 + x\partial_x)H_n(x) = nH_n(x), \quad (T - nT^{-1})H_n(x) = xH_n(x).$$

The bispectral operators have a number of connections with other fields of research such as integrable systems (KP hierarchy, Sato’s Grassmannian and the Calogero-Moser systems, orthogonal polynomials, representation theory of $W_{1+\infty}$ and Virasoro algebras, ideal structure and automorphisms of the first Weyl algebra, etc. See [8, 9, 10, 13, 11]

In fact the examples 2) Airy functions and 4) Hermite polynomials are obtained via automorphisms of the Weyl algebra from 1) and respectively from 3). This will be demonstrated in the next section in more general context.

The main objects in the paper are the polynomial systems $P_n(x)$, called $d$-orthogonal polynomials with some extra structure - they enjoy the generalized Bochner property, see below. $d$-orthogonal polynomials are orthogonal with respect to $d$ measures (see for details Definition 2.1) rather than one. Then using these polynomials we are going to find special solutions of the Toda lattice hierarchy which lead to important matrix models as explained in Section 8.

As in the case of orthogonal polynomials this property has purely algebraic expression. Due to P. Maroni [32], we can use as a definition of $d$-orthogonality the following one: the polynomials satisfy a $d + 2$-term recursion relation of the form

$$x \cdot P_n(x) = P_{n+1}(x) + \sum_{j=0}^{d} \gamma_j(n) P_{n-j}(x).$$

with constants $\gamma_j(n)$, independent of $x$, $\gamma_d(n) \neq 0$ for $n \geq d$.

**Generalized Bochner problem (GBP).** Find systems of polynomials $P_n(x)$, $n = 0,1,\ldots$ that are eigenfunctions of a differential operator $L$ of order $m$ with eigenvalues $\lambda(n)$ depending on the discrete variable $n$ (the index):

$$LP_n(x) = \lambda(n)P_n(x)$$

and which at the same time are eigenfunctions of a difference operator, i.e. that satisfy a finite-term (of fixed length $d + 2$), recursion relation of the form

$$x \cdot P_n(x) = P_{n+1}(x) + \sum_{j=0}^{d} \gamma_j(n) P_{n-j}(x).$$

In the case of classical orthogonal polynomials Bochner’s theorem shows that all their properties follow from the fact that they are eigenfunctions of a differential operator.
The above generalization produces $d$-orthogonal polynomials analogs of the classical orthogonal polynomials, which have very similar properties.

We found a large class of $d$-orthogonal polynomials with this property (conjecturally all). See Section 2 or [26] for details.

The construction goes as follows. We start with Weyl algebra $W_1$, i.e. the algebra of differential operators in one variable $x$ with polynomial coefficients. It has a natural representation in the space of all polynomials $\mathbb{C}[x]$.

Consider the trivial polynomial system $\{x^n\}$, $n = 0, 1, 2, \ldots$. The polynomials $x^n$ are eigenfunctions of the differential operator $H = x\partial_x$ with eigenvalues $n$. They also satisfy the trivial recurrence relation

$$x \cdot x^n = x^{n+1} = T x^n,$$

where $T$ is the shift operator $T f(n) = f(n+1)$. Also notice the identity $\partial_x x^n = n x^{n-1}$.

We see that the operators $H$ and $T$ are bispectral. Now we will explain how to construct non-trivial solutions of the bispectral problem.

It is known (and easy to see), cf. [20], that given a polynomial $q(\partial_x)$ without constant term, one can construct an automorphism of $W_1$ by the formula

$$\sigma A := e^{\text{ad}_{q(\partial_x)}} A = \sum_{j=0}^{\infty} \frac{\text{ad}^j_{q(\partial_x)} A}{j!}, \quad A \in W_1,$$

where $\text{ad}_B A = [B, A]$ and $\text{ad}^j_{B} A = [B, \text{ad}^{j-1}_{B} A]$. Then following [8] we can use $\sigma$ to obtain a new pair of bispectral operators with non-trivial polynomial system $P_n(x)$, being the common eigenfuction. Namely we put

$$P_n(x) = \sum_{j=0}^{\infty} \frac{q(\partial_x)^j x^n}{j!}.$$

It is obvious that the above sum is finite and $P_n(x)$ is a polynomial of degree $n$. Also introduce the operators

$$L = \sigma H, \quad D = \sigma^{-1} x.$$

For example if $q(\partial_x) = -\partial^2_x / 2$ we obtain that

$$L = -\partial^2_x + x \partial_x \quad \sigma \partial_x = \partial_x \quad \text{and} \quad \sigma^{-1} x = x + \partial_x.$$

Then

$$LP_n(x) = nP_n(x)$$

$$xP_n(x) = [T + n(n-1)T^{-1}]P_n(x)$$

For the last identity, see [26] or the next section for details. Thus we obtain the classical Hermite polynomials. The case of general polynomial $q$ is treated in the same manner. In this paper we consider the special case of $q(\partial_x) = \partial^d_x / (d+1)$, $d \in \mathbb{N}$, $d \geq 1$. Notice that we obtain simultaneously all ingredients - the polynomial system, the differential operator and the recurrence relation.

Our next step is to construct the measures or which is equivalent - their weights. This is done in Section 3. It turns out that they play the same role as the polynomials in the construction of solutions of Toda hierarchy. This can be seen from their integral representations, which are of the same class as the polynomials:
\[ \nu(n, x) = \int_C z^{-n-1} \exp \left( -\frac{z^{d+1}}{d+1} + xz \right) dz \]

with an appropriate contour \( C \). The polynomials correspond to closed contour around \( z = 0 \) and \( n = 0, 1, \ldots \). For the weights we have to choose a contour, going to infinity in both directions, on which the integrand tends to zero and \( n = -1, -2, \ldots \). However the above integrals make sense for all values of \( n \in \mathbb{Z} \) when we chose the contour as for the weights. Moreover the set of functions \( \nu(n, x), n \in \mathbb{Z} \) satisfy both the recurrence relations and the differential equation, i.e. they give another solution of the bispectral problem, this time the range of the variable \( n \) is \( \mathbb{Z} \). Let us formulate the corresponding result, proven in Section 4.

**Lemma 1.1.** (i) The functions \( \nu(n, x) \) satisfy the equation

\[ -\partial_x^{d+1} \nu(n, x) + x \partial_x \nu(n, x) = n \nu(n, x). \]  

(ii) They satisfy the recurrence relation

\[ x \nu(n, x) = (n + 1) \nu(n + 1, x) + \nu(n - d, x). \]

(iii) The lowering operator \( \partial_x \) acts on the solutions \( \nu(n, x) \) as

\[ \partial_x \nu(n, x) = \nu(n - 1, x). \]

In the theory of classical orthogonal polynomials functions with such properties are called functions of the second kind [37]. Each contour \( C \) for which the integral is convergent defines such type of functions.

This gives a bi-infinite system of functions, from which we can construct a solution of the the Toda hierarchy.

We know (after Sato), see [36] [17] that both Toda and KP hierarchy of equations can be written as evolution equations in a certain infinite-dimensional Grassmannian - Sato Grassmannian. Each point of the Sato’s Grassmannian corresponds to a plane \( W \), which consists of functions in an infinite-dimensional linear space. We will be interested in planes possessing certain symmetries.

Let us recall the idea of these symmetries on a simple example. In the studies of generalized Kontsevich models it turned out that a very important ingredient is the notion of Kac-Schwarz [30] operators \( A_j \). Roughly speaking they act on a solution of Airy equations, spanning the entire plane \( W \) and leaving it invariant: \( A_j W \subset W \).

The above constructed functions \( \nu(n, x) \) span a flag \( F = \ldots W_n \subset W_{n+1} \subset \ldots \) of planes \( W_n = (\nu(n, x), \nu(n - 1, x), \ldots) \). Each of these planes gives rise to a tau function \( \tau(n, t) \), where \( t = (t_1, t_2, \ldots) \) via the boson-fermion correspondence.

In the simplest case of \( d = 2 \) the differential operator is

\[ L = -\partial_x^3 + x \partial_x. \]

It is easy to check that the corresponding function \( \nu(-1, x) \) is the Airy function \( \text{Ai}(x) \). It is the first weight function, defining the measure for the \( d \)-orthogonal polynomials \( P_n(x), d = 2 \), which are eigenfunctions of the differential operator. The rest of the weights are given by the derivatives of \( \text{Ai}(x) \).
The next result in the paper shows that all the tau-functions satisfy the Virasoro constraints. Virasoro algebra is an algebra spanned by operators $L_j, j \in \mathbb{Z}$ and the central element $c$ and satisfying the commutator relations

$$[L_k, L_m] = (k - m)L_{k+m} - \frac{k^3 - k}{12}\delta_{k,-m}c.$$  

In the bosonic representation the operators $L_j$ are differential operators of order one or two in infinite number of variables $t_1, t_2, \ldots$. The Virasoro constraints are equations of the type

$$L_j \tau(n, t) = 0, \text{ for } j = -1, 1, 2, \ldots$$

while the function $\tau(n, t)$ is an eigenvector of the operator $L_0$. For the case of $d = 3$ and $n = -1$ the result belongs to Witten and Kontsevich [44, 31] (although in different terminology and notations) For $d = 3$ and $n = 0$ the above result belongs to Alexandrov [5].

The corresponding tau-function $\tau(-1, t)$ has a deep and beautiful algebro-geometric meaning as conjectured by Witten [44] and proved by Kontsevich [31]. Namely it gives the intersection theory of the moduli spaces of compact Riemann surfaces.

The function $\tau(0, t)$ also has algebro-geometric meaning as proved by Alexandrov, Buryak, Tessler [5, 15]. It describes the intersection theory of moduli spaces of open Riemann surfaces. See the cited papers and the references therein.

The above tau-functions have appeared in the cited references as solutions of other integrable hierarchies - KP or extended KP. Here they appear as one solution of the Toda lattice $\tau(n, t)$.

For the other values of $d$ at least one of the tau-functions also has algebro-geometric meaning. For $d > 2$ the function $\tau(-1, t)$ is connected with the so called $r$-spin structures, introduced by Witten [44], see also [22].

Let us point that the connection of the bispectral problem with the solutions of integrable systems exhibiting Kac-Scwartz symmetry has been discussed in many different places. The first documented one that I know is in [2]. Here we present other instances of such connections.

Acknowledgements. This research has been partially supported by the Grant No DN 02-5 of the Bulgarian Fund "Scientific research".

2. Preliminaries

2.1. $d$-orthogonal polynomials.

Definition 2.1. Let $\{P_n(x), n = 0, 1, \ldots\}$ be a family of monic polynomials such that $\deg P_n = n$. The polynomials are $d$-orthogonal iff there exist $d$ functionals $\mathcal{L}_j, j = 0, \ldots, d - 1$ on the space of all polynomials $\mathbb{C}[x]$ such that

$$\begin{cases} \mathcal{L}_j(P_nP_m) = 0, m > nd + j, n \geq 0, \\ \mathcal{L}_j(P_nP_{nd+j}) \neq 0, n \geq 0, \end{cases}$$

for each $j \in N_{d+1} := \{0, \ldots, d-1\}$. When $d = 1$ this is the ordinary notion of orthogonal polynomials.
Notice that $L_j(P_j) \neq 0$, $j = 0, \ldots, d - 1$. The notion of $d$-orthogonal polynomials was introduced by J. Van Iseghem. \[41\].

Let $v(x)$ be the weight (function) on a subset $U \subset \mathbb{R}$, which defines the functional

$$L(P) = \int_U v(x)P(x)dx.$$ 

Instead of $L(P)$ we are going to use the notation

$$\langle v(x), P(x) \rangle,$$

which is very convenient for algebraic manipulations.

One can show easily that for each $n$ there exist polynomials $A_0^{(n)}(x), \ldots, A_{d-1}^{(n)}$ such that the weights $v_n = \sum_{i=0}^{d-1} v_i A_i^{(n)}$ have the property

$$\langle v_j(x), P_m(x) \rangle = \delta_{j,m} c_j \neq 0.$$

The orthogonality connected with $d$ functionals rather than with only one, gives the name of $d$-orthogonal polynomials.

Norming suitably the weights $v_j$ we see that they define the bilinear forms with the properties

$$\langle v_j(x), P_m(x) \rangle = \delta_{j,m}.$$

A very important theorem of P. Maroni \[32\] is the following one.

**Theorem 2.2.** A polynomial system $\{P_n(x)\}$, $n \geq 0$, is $d$-orthogonal if and only if the polynomials satisfy a $d + 2$-term recurrence relation of the form

$$xP_n(x) = P_{n+1} + \sum_{j=0}^{d} \gamma_j(n)P_{n-j}(x)$$

with constants $\gamma_j(n)$, independent of $x$, $\gamma_d(n) \neq 0$, $n \geq d$.

2.2. **Bochner’s property.** In this subsection we briefly recall some notions and results from \[26, 27\] about $d$-orthogonal polynomials, which are eigenfunctions of a differential operator. We have called this property ”Bochner’s property” in analogy with the classical theorem by Bochner \[13\].

The differential operators with polynomial coefficients form the Weyl algebra $W_1$. It is spanned by the two generators $x, \partial_x$.

For any polynomial $q(\partial_x) = \sum_{j=1}^{d+1} a_j \partial_x^j \in \mathbb{C}[\partial_x]$ (notice that the coefficients $a_j$ are constants) we define the automorphism of $W_1$

$$\sigma_q(A) = e^{ad_q(\partial_x)}(A) = e^q A e^{-q}, \ A \in W_1.$$ 

Here $ad_A(B) = [A, B] = AB - BA$, $A, B \in W_1$.

For any polynomial $q(\partial_x) = \sum_{j=1}^{d+1} a_j \partial_x^j$ without a constant term we defined the automorphism $\sigma_q = e^{q(\partial_x)}$ of $W_1$.

Then we found the images of the generators $x, \partial_x$ of $W_1$ under the automorphism $\sigma_q$:

\[1\] The constant term contributes only to multiplication of the polynomials by a constant. On the other hand without it the formulas and the arguments are simpler.
Lemma 2.3.  
\[
\begin{aligned}
\sigma_q(\partial_x) &= \partial_x \\
\sigma_q(x) &= x + q'(\partial_x).
\end{aligned}
\] □

Let us introduce an auxiliary algebra \( R_2 \) defined over \( \mathbb{F} \) with generators \( T, T^{-1}, \hat{n} \) subject to the relations
\[
T \cdot T^{-1} = T^{-1} \cdot T = 1, \quad [T, \hat{n}] = T, \quad [T^{-1}, \hat{n}] = -T^{-1}.
\]

One easily sees that \([T, \hat{n}T^{-1}] = 1\). Hence the operators \( T, \hat{n}T^{-1} \) define a realization of the Weyl algebra. We can introduce another algebra \( B_2 \) as follows.

First we define an anti-homomorphism \( b \), i.e. a map \( b: W_1 \rightarrow R_2 \) satisfying
\[
b(m_1 \cdot m_2) = b(m_2) \cdot b(m_1),
\]
for each \( m_1, m_2 \in W_1 \) by:
\[
\begin{aligned}
b(x) &= T \\
b(\partial_x) &= \hat{n}T^{-1}
\end{aligned}
\]

The algebra \( B_2 \) will be the image \( b(W_1) \) of \( W_1 \). Then \( b: W_1 \rightarrow B_2 \) is an anti-isomorphism and in particular \( b^{-1}: B_2 \rightarrow W_1 \) is well defined.

We start with the polynomial system \( \psi(x,n) = x^n \). Then the corresponding operators \( \partial_x, \hat{x} \) act on \( \mathbb{C}[x] \). In the same way we represent the algebra \( B_2 \) in \( \mathbb{C}[x] \) by realizing \( T \) and \( T^{-1} \) as the shift operators acting on functions \( f(n) \) by \( T^\pm f(n) = f(n \pm 1) \). Finally \( \hat{n} \) denotes the operator of multiplication by \( n \). With this notation we have

Lemma 2.4.  
\[
\begin{aligned}
\partial_x \psi(x,n) &= \hat{n}T^{-1}\psi(x,n) \\
x\psi(x,n) &= T\psi(x,n) \\
x\partial_x \psi(x,n) &= n\psi(x,n).
\end{aligned}
\] □

Using \( q(\partial_x) \) we define another polynomial system \( P_n(x) \) by
\[
P_n(x) = e^{q(\partial_x)}\psi(x,n) = \sum_{j=0}^{\infty} \frac{q(\partial_x)^j \psi(x,n)}{j!}.
\]

Notice that the above series is in fact finite as the operator \( q(\partial_x) \) reduces the degree of any polynomial. Let us denote the operator \( \sigma_q(x\partial_x) \) by \( L \). We proved in [26] among other statements the following

Theorem 2.5. The polynomials \( P_n(x) \) have the following properties:

(i) They are eigenfunctions of the differential operator
\[
L := q'(\partial_x)\partial + x\partial
\]
with eigenvalues \( \lambda(n) = n \).

(ii) They have the lowering operator \( \partial_x \), i.e.
\[
\partial_x P_n(x) = nP_{n-1}.
\]
(iii) They satisfy the recurrence relation

\begin{equation}
(2.8) \quad xP_n(x) = P_{n+1} + \sum_{j=0}^{d} ja_n(n-1) \ldots (n-j)P_{n-j}.
\end{equation}

Proof. We repeat the proof from [26] as here it is simpler.

(i) \( LP_n(x) = (e^{q(\partial)} \cdot x\partial \cdot e^{-q(\partial)}) \cdot (e^{q(\partial)} x^n) = e^{q(\partial)} x\partial x^n = nP_n(x) \)

(ii) is obvious.

(iii) This requires more computations but it is similar to (i). One has to compute \( \sigma^{-1}x = e^{-q(\partial)} \cdot x \cdot e^{q(\partial)} \) and then use the obvious \( x \cdot x^n = x^{n+1} \). Then the expression \( \sigma^{-1}x \cdot x^n \) has to be written only in terms of \( T \) and \( nT^{-1} \), using that \( x \cdot x^n = x^{n+1} \) and \( \partial x^n = nx^{n-1} \). In details the computation is:

\[
\begin{align*}
x \cdot P_n(x) &= x \cdot e^{q(\partial)} x^n = e^{q(\partial)} e^{-q(\partial)} \cdot x \cdot e^{q(\partial)} x^n \\
&= e^{q(\partial)} (\sigma^{-1}x) \cdot x^n = e^{q(\partial)} [x + \sum_{j=0}^{d} \gamma_j \partial^j] \cdot x^n \\
&= e^{q(\partial)} \cdot [x^{n+1} + \sum_{j=0}^{d} \gamma_j n(n-1) \ldots (n-j)x^{n-j}].
\end{align*}
\]

\[\Box\]

Example 2.6. \( q(\partial_x) = -\partial_x^{d+1}/(d+1) \) corresponds to Gould-Hopper polynomials [24]. When \( d = 1 \) these are the Hermite polynomials. The operator \( L \) reads

\[ L = x\partial_x - \partial_x^{d+1}. \]

For the recurrence relation we need to compute \( \sigma^{-1}x \). Similarly to the above we obtain that

\[ \sigma^{-1}x = x + n(n-1) \ldots (n-d)\partial^{n-d}, \]

which gives

\[ xP_n = P_{n+1} + n(n-1) \ldots (n-d)P_{n-d}. \]

3. Weights

3.1. Pearson equations. In what follows with an abuse of language we will not distinguish between a functional and its weight \( v(x) \). Let \( \{P_j(x)\} \), \( j = 0, 1, \ldots \) be a system of polynomials \( \deg P_j(x) = j \). We introduce the dual system of functionals \( v_j(x) \) on \( \mathbb{C}[x] \), such that

\[ \langle v_j, P_n(x) \rangle = \delta_{jn}, \quad j, n = 0, 1, \ldots. \]

For each functional \( v \) we can define new functionals \( xv \) and \( \partial_x v \) by

\[ \langle xv, P_n(x) \rangle = \langle v, xP_n(x) \rangle, \quad \langle \partial_x v, P_n(x) \rangle = \langle v, -\partial_x P_n(x) \rangle. \]

Below we will determine explicitly the dual system \( v_j(x) \) for the polynomials \( \{P_j(x)\} \), defined by \( q(\partial_x) \).
Lemma 3.1. (i) The functional $v_0(x)$ is a solution of the differential equation

\[(q'(-\partial_x) + x)v_0 = 0.\]

(ii) The functionals $v_j(x)$ are given by

\[v_j(x) = \frac{(-1)^j}{j!} \partial^j v_0(x).\]

Proof. From the properties (1.3) of the polynomials $P_n(x)$ we see that

\[\frac{1}{n+1} \partial_x P_{n+1}(x) = P_n(x).\]

(i) We start with the functional $v_0$. It satisfies

\[\langle v_0, P_n(x) \rangle = \delta_{0n}.\]

Let us use the differential operator $L$. We have $LP_n(x) = nP_n(x)$. We will compute the action of $v_0$ on the system $P_n(x)$.

Denote by $L^*$ the formal adjoint to $L$, i.e. $L^*$ satisfies the identity

\[\langle f, LQ(x) \rangle = \langle L^* f, Q(x) \rangle\]

for any polynomials $Q(x)$ and $f(x)$. Using that $LP_n(x) = nP_n(x)$ we obtain

\[\langle v_0, LP_n(x) \rangle = \langle v_0, nP_n(x) \rangle = 0, \text{ for all } n \geq 0.\]

Hence

\[0 = \langle L^* v_0, P_n(x) \rangle = 0, \text{ for all } n \geq 0,\]

where the operator $L^*$ explicitly reads

\[L^* = -\partial_x q'(-\partial_x) - \partial_x x.\]

We see that the functional $L^* v_0 \equiv 0$. Integrating once the function $\partial_x[q'(-\partial_x) + x]v_0(x) = 0$ we obtain that

\[(q'(-\partial_x) + x)v_0 = c, \quad c \in \mathbb{C}.\]

We will compute the constant by applying the functional $(q'(-\partial_x) + x)v_0$ to the system $P_n(x)$, $n = 0, 1, \ldots$. We have

\[\langle c, P_n(x) \rangle = \langle (q'(-\partial_x) + x)v_0, P_n(x) \rangle = \langle (q'(-\partial_x) + x)v_0, \frac{1}{n+1} \partial_x P_{n+1}(x) \rangle = \langle v_0, \frac{1}{n+1} LP_{n+1}(x) \rangle = \langle v_0, \frac{n+1}{n+1} P_{n+1}(x) \rangle = \langle v_0, P_{n+1}(x) \rangle = 0.\]

Hence $c = 0$ and we see that $v_0(x)$ satisfies the equation (3.9).
Next consider
\[ 1 = \delta_{00} = \langle v_0, P_0(x) \rangle = \langle v_0, \partial_x P_1(x) \rangle = \langle -\partial_x v_0, P_1(x) \rangle, \]
i.e.
\[ \langle -\partial_x v_0, P_1(x) \rangle = 1. \]

Using that
\[ \delta_{0n} = \langle v_0, P_n(x) \rangle = \langle -\partial_x v_0, P_{n+1}/(n + 1)(x) \rangle = 0 \]
for \( n > 0 \) we obtain that \( v_1 = -\partial_x v_0 \). We continue by induction to see
\[ v_j(x) = \frac{(-1)^j}{j!} \partial^j v_0(x). \]

The functions \( v_j(x) \) are similar to Airy function and also have explicit representations in terms of definite integrals. We shall derive them in the next section.

Using the bi-orthogonality property and the recurrence relation for the polynomials it is easy to show that the functions \( v_j \) also satisfy recurrence relation of the form:
\[ xv_m(x) = v_{m-1}(x) + \sum_{j=0}^{d-1} b_j(m)v_{m+j}(x), \]
see e.g. [18, 38] for details and more general results.

4. Integral representations

4.1. Integral representations for the weights. We will find integral representations of the weights of the polynomials corresponding to \( q(\partial_x) = -\partial_x^{d+1}/(d+1) \). As we know, it is enough to find one for \( v_0(x) \).

Lemma 4.1. The weight \( v_0(x) \) is given by the formula
\[ v_0(x) = \int_C \exp \left( -z^{d+1}/d + xz \right) dz. \]
with an appropriate contour \( C \), see Fig. 1.

Proof. Denote by \( \hat{v}_0(z) \) the Laplace transform of \( v_0(x) \), i.e.
\[ \hat{v}_0(z) = \int_C e^{-zx} v_0(x) dx. \]
Then performing the Laplace transform on the equation (3.9), where \( q(z) = -z^{d+1}/(d+1) \), we obtain
\[ [-z^d - \partial_z]\hat{v}_0(z) = 0. \]
Solving the differential equation we obtain (up to a multiplicative constant)
\[ \hat{v}_0(z) = \exp \left( -z^{d+1}/d + 1 \right). \]
Finally the inverse Laplace transform applied to \( \hat{v}_0(z) \) gives
\[ v_0(x) = \int_C \exp \left( -\frac{z^{d+1}}{d+1} + xz \right) dz, \]

where the contour \( C \) is chosen so that the integral to be convergent. All such contours give solutions. For example on fig. 1 we have taken the union of two rays \( R_1 \cup R_{-1} \), where

\[ R_k = R \exp(\pm 2\pi i/(d+1)), \quad k = \pm 1 \quad \text{and} \quad R \in [0, \infty). \]

We see that on the rays \( R_k \) the function \( \text{Re} \left[ -\frac{z^{d+1}}{d+1} + xz \right] = -R^{d+1}(1/(d+1) + O(R^{-1})) \), as a function of \( z \). Hence the integral is convergent.

The rest of the weights are given by

\[ v_j(x) = \frac{(-1)^j}{j!} \int_C z^j \exp \left( -\frac{z^{d+1}}{d+1} + xz \right) dz, \]

which follows from Lemma 3.1 (ii).

4.2. Integral representations for the polynomials. Consider the following scalar integrals with suitable contours \( C \)

\[ w_n(x) = \int_C z^{-n-1} \exp \left( -\frac{z^{d+1}}{d+1} + xz \right) dz, \quad n \in \mathbb{Z}_+. \]

Lemma 4.2. The function \( w_n(x) \) satisfies the equation

\[ (-\partial_x^{d+1} + x\partial_x) w_n(x) = nw_n(x). \]

Proof. It is an easy computation using integration by parts. On one hand we have that

\[ -\partial_x^{d+1} w_n(x) = \int_C z^{-n-1} (-z^{d+1}) \exp \left( -\frac{z^{d+1}}{d+1} + xz \right) dz. \]

On the other hand integrate by parts the expression

\[ x\partial_x w_n(x) = \int_C z^{-n} \exp \left( -\frac{z^{d+1}}{d+1} + xz \right) d(xz). \]
This gives

\[
\int_C z^{-n} \exp\left( -\frac{z^{d+1}}{d+1} \right) d\exp(zx)
= - \int_C [(n)z^{-n-1} - zd^{-n}] \exp\left( -\frac{z^{d+1}}{d+1} + zx \right) dz.
\]

Hence

\[
(-\partial_x^{d+1} + x\partial_x)w_n(x) = nw_n(x).
\]
\[(4.15) \quad x \nu(s, x) = (s + 1) \nu(s + 1, x) + \nu(s - d, x).\]

(iii) The lowering operator \(\partial_x\) acts on the solutions \(\nu(s, x)\) as

\[(4.16) \quad \partial_x \nu(s, x) = \nu(s - 1, x).\]

**Proof.** (i) The proof repeats the polynomial case.

(ii) Integrating by parts the expression \(x \nu(s, x)\) we obtain

\[
\int_C z^{-s-1} \exp\left(-\frac{z^{d+1}}{d+1}\right) d\exp(xz) =
\]

\[
= -\int_C \left[(-s - 1)z^{-s-2} - z^{-s-1+d}\right] \exp\left(-\frac{z^{d+1}}{d+1} + xz\right) dz
\]

\[
= \int_C (s + 1)z^{-s-2} \exp\left(-\frac{z^{d+1}}{d+1} + xz\right) dz
\]

\[
+ \int_C z^{-s+d-1} \exp\left(-\frac{z^{d+1}}{d+1} + xz\right) dz =
\]

\[
= (s + 1)\nu(s, x) + \nu(s - d, x).
\]

(iii) Differentiating the integral gives

\[
\partial_x \nu(s, x) = \int_C z^{-s} \exp\left(-\frac{z^{d+1}}{d+1} + zx\right) dz = \nu(s - 1, x).
\]

**Remark 4.5.** The functions \(\nu(n, x)\), where \(n\) is a negative integer correspond to the weights, while \(n \geq 0\) correspond to the polynomials (when we take a closed contour around 0) or to the functions of the second kind. In this way formula (4.13) gives integral representations of the same kind both for the weights and the functions of the second kind (the polynomials).

We can rescale \(\nu(s, x)\) introducing \(u(s, x)\) by \(\nu(s, x) = a(s)u(s, x)\) so that the recurrence (4.15) takes the original form

\[
xu(s, x) = u(s + 1, x) + b(s - d)u(s - d, x).
\]

In this way we will obtain the recurrence for the monic polynomials \(P_n(x)\). However we will work primarily with (1.2).

5. **Preliminaries on infinite dimensional Lie algebras**

5.1. **The Lie algebra \(\mathfrak{gl}_\infty\) and its representations.** By \(\mathfrak{gl}_\infty\) we denote the algebra of doubly infinite matrices with only finite number of nonzero elements. We will need also the algebra \(\bar{\mathfrak{a}}_\infty\) defined by the condition that the nonzero diagonals are finitely many:

\[
\bar{\mathfrak{a}}_\infty = \{(a_{ij})|a_{ij} = 0 \text{ for } |i - j| >> 0\}.
\]

The fermionic Fock space is defined as follows. Introduce a vector space \(V\) with a fixed basis \(v_j, j \in \mathbb{Z}:\)
The algebra $\mathfrak{gl}_\infty$ acts on $V$ in the standard way.

Consider the vectors of the form

$$v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \ldots$$

such that $i_0 > i_{-1} > i_{-2} \ldots$ and $i_{-k} = -k$ for $k$ big enough. We consider the space $F^{(0)}$ spanned by the above vectors with coefficients in $\mathbb{C}$. The element

$$|0\rangle = v_0 \wedge v_{-1} \wedge v_{-2} \wedge \ldots$$

is called the 0-vacuum.

Exactly in the same way we define the $m$-th vacuum by

$$|m\rangle = v_m \wedge v_{m-1} \wedge v_{m-2} \wedge \ldots$$

and the corresponding space $F^{(m)}$. The direct sum

$$F = \bigoplus_{m \in \mathbb{Z}} F^{(m)}.$$ 

will be called fermionic Fock space.

Next we define an action of $\mathfrak{gl}_\infty$ on $F$ as follows. Let $a \in \mathfrak{gl}_\infty$. Take a vector $v_{i_0} \wedge v_{i_1} \wedge \ldots$ and define the map $r(a) : F^{(m)} \rightarrow F^{(m+1)}$ as

$$r(a)(v_{i_0} \wedge v_{i_1} \wedge \ldots) = av_{i_0} \wedge v_{i_1} \wedge \ldots + v_{i_0} \wedge av_{i_1} \wedge \ldots + \ldots$$

and by linearity.

We also will need the group-like operator $R(\exp(a)) : F^m \rightarrow F^m$, where $a \in \mathfrak{gl}_\infty$. It is defined by the action

$$R(\exp(a)(v_{i_0} \wedge v_{i_1} \wedge \ldots)) = \exp(a)v_{i_0} \wedge \exp(a)v_{i_1} \wedge \ldots$$

We will need slightly more general group $G_\infty$ of matrices $g$ with $\det g \neq 0$, which have only finite number of diagonals below the main one but any number above it. It is obvious that the above formula holds for them. Next we define the bosonic Fock spaces $B^{(m)}$ for $m \in \mathbb{Z}$. Introduce the formal variable $Q$ and put

$$B^{(m)} := \mathbb{C}[Q^m, t_1, t_2, \ldots].$$

The element $Q^m \cdot 1 \in B^{(m)}$ is called the $m$-th vacuum.

Introduce the oscillator algebra $A$ spanned by operators $a_j$, $j \in \mathbb{Z}$ and an operator $c$, called central charge subject to the identities

$$[a_j, a_k] = i\delta_{j,-k} \cdot c, \quad [c, a_j] = 0.$$ 

This algebra can be represented in both the bosonic and the fermionic Fock spaces.

If we want to represent the oscillator algebra in the fermionic Fock space in terms of the operators $r(a)$ we see that the operator $r(I)$ (here $I$ is the identity matrix) is not well defined. So we need a regularization. We will recall its definition (cf. [29]) for the algebra $\bar{a}_\infty$.

Let $E_{ij} \in \mathfrak{gl}_\infty$ be the matrix with 1 on the intersection of the $i$-th row and the $j$-th column. Put $A_j = \sum_{i+j}^+ E_{i,i+j}$. 

$$V = \bigoplus_{j \in \mathbb{Z}} \mathbb{C}v_j.$$
Define the operator $\hat{r}(E_{ij})$ by

$$\hat{r}_m(E_{ij}) = \begin{cases} r_m(E_{ij}), & \text{if } i \neq j \text{ or } i = j < 0 \\ r_m(E_{ii}) - id, & \text{if } i \geq 0. \end{cases}$$

and for the entire algebra $\tilde{a}_\infty$ continue by linearity.

Notice that the elements $\hat{r}(\Lambda_j) \in \tilde{a}_\infty$, $j \in \mathbb{Z}$ together with the central element $c \in \tilde{a}_\infty$ satisfy the commutation relation

$$[\hat{r}(\Lambda_i), \hat{r}(\Lambda_j)] = i\delta_{i,j}c.$$ 

We obtain a highest weight representation of $\mathcal{A}$ in the fermionic Fock space noticing that the operators $\hat{r}(\Lambda_j)$ act on the vacuum as

$$\hat{r}_m(\Lambda_j) |m\rangle = 0, \ j = 1, 2, \ldots$$

On the other hand the elements $\hat{r}_m(\Lambda_{-i}) \ldots \hat{r}_m(\Lambda_{-j_k}) |m\rangle$ span $F^{(m)}$ so we have a highest weight representation of the oscillator algebra.

Similarly we can define a representation of $\mathcal{A}$ in the bosonic spaces (5.17), using the operators $J_k = \partial t_k$ and $J_{-k} = k t_k$ for $k \geq 0$.

Notice that the operators $J_k$, $k > 0$ act on the vacuum $Q^m \cdot 1$ by $J_k Q^m \cdot 1 = 0$ and the operators $J_{-k}$, $k > 0$ acting on it span the entire bosonic space $B^{(m)}$.

It is known that the two representations of $\mathcal{A}$ in the bosonic and in the fermionic Fock spaces are equivalent, see [29]. The map between $F^{(m)}$ and $B^{(m)}$ is called boson-fermion correspondence. Let us denote it by $\sigma_m$.

Our next goal is to introduce the Virasoro algebra $Vir$. It is the algebra spanned by a set of operators $L_k$, $k \in \mathbb{Z}$ and $c$ (the central element) with the following commutation relation

$$[L_k, L_m] = (k - m)L_{k+m} + \frac{k^3 - k}{12}\delta_{k,-m}c.$$ 

We aim at defining two representations of this algebra. First we introduce the bosonic field

$$J(z) = \sum_{m \in \mathbb{Z}} J_m z^{-1-m}$$

and define the operators $L_k$ by

$$\frac{1}{2} : J(z)^2 : = \sum_{m \in \mathbb{Z}} L_m z^{-m-2},$$

where

$$:J_k J_l: = \begin{cases} J_k J_l, & \text{if } k < 0 \\ J_l J_k, & \text{if } k > 0 . \end{cases}$$

Explicitly the operators $L_m$ are given by

$$L_m = \sum_{l+k=-m} lkt_{l}t_{k} + \sum_{k=1}^\infty kt_{k}\partial_{k+m} + \frac{1}{2} \sum_{k+l=m} \partial_{l}^2.$$
One can show that they satisfy the commutation relation of the Virasoro algebra.

We can define a fermionic representation of $\text{Vir}$. Let us take the space $V$ to be spanned by functions $v_k(z) = z^k + O(z^{k-1}), \ k \in \mathbb{Z}$. We see that the operators $d_k = -z^{k+1} \partial_z$ act on it and satisfy the commutation relation

$$[d_k, d_m] = (k - m)d_{k+m},$$

which defines centerless algebra. Then we can define action on the corresponding fermionic Fock space $F^{(m)}$. It is easy to show that the operators $\hat{r}(d_k), \ k \in \mathbb{Z}$ and $c = c_m$, (where $c_m$ is not important here) define a representation of the Virasoro algebra.

One can show that the above described representations of the Virasoro algebra in $F^{(m)}$ and $B^{(m)}$ are also equivalent, cf. [29].

Below we will present the formula for this equivalence in an appropriate basis. Let us take instead of $d_k$ the elements $b_k = -\partial_z z^{k+1}$. Then

$$Y_{b_k} = \text{Res}_z \left( z^{k+1} : \frac{(J(z) + \partial_z)J(z)}{2} : \right).$$

**Remark 5.1.** We point out that there are subalgebras of Vir, which are isomorphic to Vir, cf. [29]. Namely let $p \in \mathbb{N}$ be fixed. Consider the elements $L_{pj}, \ j \in \mathbb{Z}$. These elements satisfy the commutation relations

$$[L_{pk}, L_{pn}] = p(k-n)L_{pk+pn} + \frac{(pk)^3 - pk}{12}c_\delta_{k,-n}.$$ 

If we put

$$\tilde{c} = pc, \ \tilde{L}_j = \frac{1}{p}L_{pj} \ \text{for} \ j \neq 0, \ \text{and} \ \tilde{L}_0 = \frac{1}{p}(L_0 + \frac{p^2c - c}{24})$$

we see that the new operators $\tilde{L}_j$ and $\tilde{c}$ satisfy the Virasoro commutation relations [5.18]. In what follows we are going to use this subalgebra.

### 6. Toda flows

In this section we briefly recall the formalism of the 2D-Toda hierarchy following [40, 39]. It is formulated in terms of difference operators.

A pseudo-difference operator is a finite linear combination of the form

$$A = \sum_{n=-\infty}^{N} a_n(s)\Lambda^n \ \text{(operator of } (-\infty, N] \text{ type)}$$

and

$$A = \sum_{n=M}^{\infty} a_n(s)\Lambda^n \ \text{(operator of } [M, +\infty) \text{ type).}$$

These operators are analogs of the pseudo-differential operators for the theory of KP-hierarchy, [41, 42, 19, 19]. We briefly describe the construction.

Let the pseudo-difference operator $A$ have the form

$$A = \Lambda + \sum_{n=-\infty}^{0} a_n(s)\Lambda^n.$$
The operator $A$ can be conjugated to $\Lambda$ by some operator $W$ of the form

$$W = 1 + \sum_{j=0}^{\infty} \gamma_j \Lambda^{-j},$$

called wave operator.

For an operator

$$A = \sum_{j=0}^{N} \gamma_j \Lambda^j$$
as usually we define $A_+$ to be the part of $A$ containing the nonnegative powers of $\Lambda$:

$$A_+ = \sum_{j=0}^{N} \gamma_j \Lambda^j$$

and $A_- = A - A_+$. Then the Toda flows are given by

$$A_{t_k} = [(A^k)_+, A].$$

These flows act on the wave operator $W$ as

$$W_{t_k} = -(A^k)_- W.$$

Consider a recurrence relation of the form

$$xw_n(x) = w_{n+1}(x) + \sum_{j=0}^{d} \gamma_j(n)w_{n-j}(n).$$

It defines a difference operator $D$

$$D = \Lambda + \sum_{j=0}^{d} \gamma_j \Lambda^{-j},$$
such that $Dw = xw$ with $w = (\ldots, w_{n-1}, w_n, w_{n+1}, \ldots)^T$. $D$ can be conjugated to $\Lambda$ using an operator of the form $\overline{W} = 1 + \sum_{j=0}^{\infty} \delta_j \Lambda^{-j}$ or to $\Lambda^{-d}$ by using $W = \sum_{j=0}^{\infty} \delta_j \Lambda^j$, $\delta_0 \neq 0$.

$$D = \overline{W} \Lambda \overline{W}^{-1} = W \Lambda^{-d} W^{-1}.$$ 

One can easily show that the definition of $D^+_a$ is correct.

The 2D-Toda hierarchy is the system of equations

$$\partial_{t_k} D = [(D^k)_+, D]$$

$$\partial_{t_k} D = [((D^k)^{+,\delta})_+, D].$$

Notice that in our case we can assume that $\bar{t}_k = t_{kd}$. For this reason from now on we will work only with the set of variables $t_k$.

We will apply this in the situation from Lemma 4.4. It shows that we can construct the vector

$$\nu = (\ldots, \nu(s + n - 1, x), \nu(s, n, x), \nu(s + n + 1, x), \ldots)^T, \ n \in \mathbb{Z}.$$ 

According to (1.2) we have
\[ x\nu(s, x) = (s + 1)\nu(s + 1, x) + \nu(s - d, x), \]

which can be written as

\[ x\nu = D\nu \]

where the difference operator is given by

\[ D = AA + \Lambda^{-d}, A = \text{diag}(\ldots, s - 1, s, s + 1, \ldots). \]

Next we put \( x = y^d \), which gives the formula

\[ y^d\nu(s) = (s + 1)\nu(s + 1) + \nu(s - d). \] (6.20)

There exists a tau-function \( \tau(s, t, x) \) such that the corresponding solution of the hierarchy (i.e. the coefficients of \( D \)) is given in terms of this function. While we don’t need the explicit form of \( D \) we do need the connection of \( \tau(s, t, x) \) with the vector \( w \). Namely the boson-fermion correspondence maps a plane of Sato’s Grassmannian (or wedge product) to the tau-function. We explain the construction more precisely.

Each vacuum \(|m\rangle \) corresponds to the vector \((w_m, w_{m-1}, \ldots)\). Then the boson-fermion correspondence maps \(|m\rangle \) to \( \tau(m, t) \).

7. Virasoro constraints

We denote by \( L_m \) the operators of the Virasoro algebra. Our approach will be a standard one - to obtain them from differential operators \( A_j \) acting on the functions \( w(s, y) \). It is enough to do this for \( j = -1, 0, 1, 2 \) and use the identity \([L_k, L_m] = (k - m)L_{k+m}\) to make induction. Then we transport their action on the corresponding fermionic Fock space.

Let us first define the fermionic Fock space. Put \( v_j = w(j, y) \) and

\[ \mathbb{V} = \bigoplus_{j \in \mathbb{Z}} \mathbb{C}v_j. \]

Let \( m \) be an integer. The fermionic Fock space \( F^{(m)} \) is spanned by the wedge products

\[ v_{i_m} \wedge v_{i_{m-1}} \wedge \ldots, \]

where \( i_m > i_{m-1} > \ldots \) and \( i_k = k \) for \( k << m \).

Define the operators \( A_{-1}, A_0, A_1 \) that act on \( w(s, y) \) as follows

\[ A_{-1}w(s, y) = -\frac{1}{d}y^{-d+1}\partial_y w(s, y) \]
\[ A_0w(s, y) = -\frac{1}{d}y\partial_y w(s, y) \]
\[ A_1w(s, y) = -\frac{1}{d}y^d[y\partial_y - ds]w(s, y). \]

Simple computations show that these operators satisfy the following commutation relations.
\[ [A_0, A_{-1}] = A_{-1} \]
\[ [A_1, A_0] = A_1 \]
\[ [A_1, A_{-1}] = 2A_0. \]

(7.21)

Our goal will be to present the differential operators \( A_j, j = -1, 0, 1 \) as linear operators (eliminating the differentiation) in the basis \( v_k \). Then we will transport them to the fermionic Fock space \( F^{(m)} \) to obtain the Virasoro operators \( L_j \). However we will see that with \( L_2 \) the situation is more subtle - it does not come from any differential operator.

First we write \( A_j, j = -1, 0, 1 \) in the space spanned by \( w(n, y), n \in \mathbb{Z} \).

**Lemma 7.1.** The following identities hold

(i) \( A_{-1}w(s, y) = -w(s - 1, y) \)

(ii) \( A_0w(s, y) = -[sw(s, y) + w(s - d - 1, y)] \)

(iii) \( A_1w(s, y) = -[(s - d)w(s - d) + w(s - 2d - 1)]. \)

(7.22)

**Proof.** Using that

\[ dx \partial_x = y \partial_y. \]

we obtain

\[ A_{-1}w(s, y) = -d^{-1}y^{1-d} \partial_y w(s, y) = -w(s - 1, y). \]

The second identity follows from the first one by applying the recurrence relation once:

\[ A_0w(s, y) = y^d A_{-1}w(s, y) = -w(s - 1, y) \]

\[ = -[sw(s, y) + w(s - d - 1)]. \]

Repeat the above procedure to obtain the third relation:

\[ A_1w(s, y) = -y^d w(s - d - 1) \]

\[ = -[(s - d)w(s - d) + w(s - 2d - 1)]. \]

\[ \square \]

**Corollary 7.2.** In the basis \( v_j \) the operators \( A_k \) are given by

(i) \( A_{-1}v_j = -v_{j-1} \)

(ii) \( A_0v_j = -[jv_j + v_{j-d-1}] \)

(iii) \( A_1v_j = -(j - d)v_{j-d} - v_{j-2d-1}. \)

(7.23)

We want to transport the action of the operators \( A_k \) on the vacuum

\[ |m \rangle = v_m \wedge v_{m-1} \wedge \ldots \]
Lemma 7.3. We have

\[
\hat{r}_m(A_{-1}) |m\rangle = 0
\]
\[
\hat{r}_m(A_0) |m\rangle = \frac{(m + 1)m}{2} |m\rangle
\]
\[
\hat{r}_m(A_1) |m\rangle = 0.
\]  
(7.24)

Proof. The first and the third identities are obvious. For the second one we need to apply the regularization procedure.

The operator corresponding to Virasoro \( L_2 \) will be obtained following an idea from [25] (see also [5]). It will be constructed as a sum of two operators, acting on the fermionic spaces \( F^{(m)} \). The first one will be the image of some differential operator \( \tilde{A}_2 \). The second one will be \( \beta [\hat{r}(y^d)]^2 \), where \( \beta \) is some constant, which will be determined later. Notice that \( \beta [\hat{r}(y^d)]^2 \) is not an operator that comes from differential one.

We define the operator \( \tilde{A}_2 \) as

\[
\tilde{A}_2 = y^{2d}[A_0 + \alpha],
\]
where \( \alpha \in \mathbb{C} \) will be determined later. Now put

\[
L^F_2 = \hat{r}_m(\tilde{A}_2) - [\hat{r}_m(y^d)]^2.
\]

Lemma 7.4. If \( \beta = -1, \alpha = 2 \) then

\[
L^F_2 |m\rangle = 0.
\]

Proof. First let us compute action of \( \tilde{A}_2 \) on the vectors \( v_j \). We use the recurrence relation.

\[
\tilde{A}_2 v_j = y^{2d}[A_0 + \alpha] v_j =
\]
\[
= -y^{2d}[(j - \alpha) v_j + v_{j-d-1}].
\]

Then using the recurrence relation twice we obtain

\[
\tilde{A}_2 v_j = -y^{d}[(j - \alpha)(j + 1)v_{j+1} + \mathcal{L}(j - d)]
\]
\[
= -(j - \alpha)(j + 1)v_{j+2} + \mathcal{L}(j - d + 1),
\]
where the symbol \( \mathcal{L}(n) \) denotes terms \( v_n \) with indexes \( n \) and lower.

Then for \( \hat{r}_m(\tilde{A}_2) |m\rangle \) we obtain

\[
\hat{r}_m(\tilde{A}_2) |m\rangle = (m - \alpha)(m + 1)v_{m+2} \wedge v_{m-1} \wedge \ldots
\]
\[
- (m - 1 - \alpha)mv_{m+1} \wedge v_m \wedge v_{m-2} \wedge \ldots
\]

For the computation of \( [\hat{r}_m(y^d)]^2 \) we will use the following formula (see, e.g. [4])

\[
[\hat{r}(b)]^2(v_m \wedge v_{m-1} \wedge \ldots) =
\]
\[
= \sum_{-\infty}^{m}(v_m \wedge v_{m-1} \wedge \ldots \wedge b^2 v_l \wedge \ldots) + 2 \sum_{k \geq l} (v_m \wedge v_{m-1} \wedge \ldots \wedge bv_k \ldots \wedge bv_l \wedge \ldots).
\]
Using this formula we obtain

\[(\tilde{r}_n(y^d))^2[ v_{m-1} \land v_{m-1} \land \ldots] \]

\[= (m + 2)(m + 1)v_{m+2} \land v_{m-1} \land v_{m-2} \ldots + m(m + 1)v_{m} \land v_{m+1} \land v_{m-2} \ldots \]

\[+ 2(m + 1)m v_{m+1} \land v_{m} \land v_{m-2} \ldots \]

\[= (m + 2)(m + 1)v_{m+2} \land v_{m-1} \land v_{m-2} \ldots + m(m + 1)v_{m+1} \land v_{m} \land v_{m-2} \ldots \]

Finally for \(\tilde{r}(\tilde{A}_2) + \beta(\tilde{r}_m(y^d))^2\) we obtain

\[\langle \tilde{r}(\tilde{A}_2) + \beta(\tilde{r}_m(y^d))^2 | m \rangle = \langle (m - \alpha)(m + 1) + \beta(m + 2)(m + 1) \rangle v_{m+2} \land v_{m-1} \land \ldots \]

\[+ \langle (m - 1 - \alpha)m + \beta m(m + 1) \rangle v_{m+1} \land v_{m} \land \ldots \]

If we want that the above terms to cancel we have to put

\[\beta = -1, \alpha = -2.\]

\[\square\]

**Theorem 7.5.** The bosonic representation of the operators \(A_j, j = 0, \pm 1\) are as follows

\[Y_{A_{-1}} = \frac{1}{d} \left[ \sum_{j=d}^{\infty} j t_j \partial_{t_{j-d}} + \frac{1}{2} \sum_{j=1}^{d-1} j t_j (d - j) t_{d-j} \right] \]

\[Y_{A_0} = \frac{1}{d} \sum_{j=1}^{\infty} j t_j + \partial_t \]

\[Y_{A_1} = \frac{1}{d} \left[ \sum_{j=1}^{\infty} j t_j \partial_{t_{j+d}} - \frac{1}{2} \sum_{j=1}^{d-1} \partial_{t_j} \partial_{t_{d-j}} \right] \]

The Virasoro operators \(L_j\) for \(j = 0, \pm 1\) are given by \(L_j = Y_{A_j}\).

**Proof.** The boson-fermion correspondence (5.19) gives

\[Y_{A_{-1}} = \frac{1}{d} \text{Res}_y \left( y^{1-d} : (J(y) + \partial_y)J(y) : \right) \]

\[= \frac{1}{d} \left[ \sum_{j=d}^{\infty} j t_j \partial_{t_{j-d}} + \frac{1}{2} \sum_{j=1}^{d-1} j t_j (d - j) t_{d-j} \right]. \]

Next consider

\[Y_{A_0} = -\frac{1}{d} \text{Res}_y \left( y^d : (J(y) + \partial_y)J(y) : \right) = -\sum_{j=1}^{\infty} j t_{j+d} \partial_{t_j}. \]

For the computation of \(Y_{A_1}\), we need the formula

\[ (7.25) \quad Y_{y^d} = -\text{Res}_y \left( y^d J(y) \right) = \partial_{t^d}. \]

Then we get
\[
Y_{A_1} = -\frac{1}{d} \text{Res}_y \left( y^{d+1} : \frac{(J(y) + \partial_y)J(y)}{2} \right) - m \text{Res}_y \left( y^d J(y) \right)
\]
\[
= - \sum_{j=1}^{\infty} j t_j \partial_t + \frac{1}{2} \sum_{j=1}^{d-1} \partial_t \partial_{t_{d-j}} + m \partial_t^2,
\]

From the above formulas we obtain the explicit expressions for \(Y_{A_j}, j = 0, \pm 1\).

The last statement follows from the commutation relations for the operators \(A_j\).

Now we turn to \(L^F_2\).

**Lemma 7.6.** The bosonic representation of \(L^F_2\) is given by

\[
L_2 = \frac{1}{d} \left[ \sum_{j=1}^{\infty} j t_j \partial_t \partial_{t_{j+2d}} + \frac{1}{2} \sum_{j=1}^{2d-1} \partial_t \partial_{t_{2d-j}} + \partial_t^2 \right].
\]

**Proof.** The same computation as above gives

\[
Y_{A_2} = \frac{1}{d} \text{Res}_y \left( y^{2d+1} : \frac{(J(y) + \partial_y)J(y)}{2} + 2y^d J(y) \right)
\]
\[
= \frac{1}{d} \left[ \sum_{j=1}^{\infty} j t_j \partial_t \partial_{t_{j+2d}} + \frac{1}{2} \sum_{j=1}^{2d-1} \partial_t \partial_{t_{2d-j}} + 2 \partial_t^2 \right].
\]

From this and from (7.25) we find

\[
L_2 = \frac{1}{d} \left[ \sum_{j=1}^{\infty} j t_j \partial_t \partial_{t_{j+2d}} + \frac{1}{2} \sum_{j=1}^{2d-1} \partial_t \partial_{t_{2d-j}} + \partial_t^2 \right].
\]

We sum up the obtained results in the next theorem.

**Theorem 7.7.** The tau functions \(\tau_n\) satisfy the Virasoro constraints

\[
L_{-1} \tau_m = 0
\]
\[
L_0 \tau_m = \frac{m(m+1)}{2} \tau_m
\]
\[
L_j \tau_m = 0, j = 1, 2, \ldots
\]

8. Discussion

The goal of this section is to review well known facts about some of the above tau-functions and to put all of them in more general context. Following \[31, 2, 33, 5\] etc. one can show that all of them can be represented as matrix integrals of the form

\[
(8.26) \quad Z(M) = \det(M) C \int \exp \left( - \text{Tr} \left( \frac{\Phi^{d+1}}{d+1} + M \Phi + (m+1) \log(\Phi) \right) \right) [d\Phi].
\]

Here \(M\) is a diagonal matrix \(M = \text{diag}(\mu_1, \mu_2, \ldots, \mu_N)\), \(\det M \neq 0, C \neq 0\) is some constant, which is irrelevant for the tau-function. The integration is taken on the space of Hermitian \(N \times N\) matrices. The connection with the tau function \(\tau_m(t_1, t_2, \ldots)\) is the
following. Let us define the parameters \( t_1, t_2, \ldots \) by the well known Miwa parametrization:

\[
t_j = \frac{1}{k} \text{Tr} M^{-j}.
\]

Then it is known that the function \( \tau(t) = Z(M) \) up to irrelevant multiplicative constant. The parametrization for any fixed number of variables \( t_1, t_2, \ldots \) does not depend on the size \( N \) of the matrices provided it is large enough. For more details see [33].

To be precise, it is proved for few cases. However they all can be done more or less following the same pattern.

Some of these matrix integrals describe beautiful algebraic geometry - intersection numbers on different moduli spaces. Namely introduce the function (it is rather asymptotic expansion)

\[
F = \log(Z(t)) = \sum_{\alpha} B_{\alpha} t^\alpha.
\]

Then the coefficients \( B_{\alpha} \) give intersection numbers in different situations.

The case of \( d = 2 \) is the most important. When \( m = -1 \) it corresponds to the Kontsevich integral [31][44], which describes the intersection theory on all moduli spaces \( \mathcal{M}_{g,n} \) of compact Riemann surfaces of genus \( g \) and \( n \) marked points. The case of \( m = 0 \) describes the intersection theory of open Riemann surfaces. These are Riemann surfaces from which a finite number of discs have been deleted. For more details of this very recent theory see [5][14] and the references therein.

Other cases of interest include arbitrary \( d \) and \( m = -1 \). These are the partition function for \( r \)-spin structures of type \( A \) (here \( r = d + 1 \)). These were introduced by Witten [44] who also formulated the conjecture that the corresponding intersection theory is governed by the model (8.26) with \( d > 2 \) and \( m = -1 \). The conjecture was proved in [22].

Recently there was a suggestion [12] that the case of \( m = 0 \) will produce the open analog for the \( r \)-spin structures.

It is natural to ask if the rest of the tau-functions have some algebro-geometric meaning. This is not clear even for the case \( d = 2 \).

Finally let us recall that instead of integer \( m \) we can take any complex number (see (4.13)). Then we will obtain the extended Toda hierarchy, see [10]. It contains other flows. They deserve further studies.

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