Linear Contextual Bandits with Knapsacks

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Abstract

We consider the linear contextual bandit problem with resource consumption, in addition to reward generation. In each round, the outcome of pulling an arm is a reward as well as a vector of resource consumptions. The expected values of these outcomes depend linearly on the context of that arm. The budget/capacity constraints require that the total consumption doesn’t exceed the budget for each resource. The objective is once again to maximize the total reward. This problem turns out to be a common generalization of classic linear contextual bandits (linContextual) [7, 16, 1], bandits with knapsacks (BwK) [3, 10], and the online stochastic packing problem (OSPP) [4, 19]. We present algorithms with near-optimal regret bounds for this problem. Our bounds compare favorably to results on the unstructured version of the problem [5, 11] where the relation between the contexts and the outcomes could be arbitrary, but the algorithm only competes against a fixed set of policies accessible through an optimization oracle. We combine techniques from the work on linContextual, BwK and OSPP in a nontrivial manner while also tackling new difficulties that are not present in any of these special cases.

1 Introduction

In the contextual bandit problem [7, 13, 21, 2], the decision maker observes a sequence of contexts (or features). In every round she needs to pull one out of $K$ arms, after observing the context for that round. The outcome of pulling an arm may be used along with the contexts to decide future arms. Contextual bandit problems have found many useful applications such as online recommendation systems, online advertising, and clinical trials, where the decision in every round needs to be customized to the features of the user being served. The linear contextual bandit problem [11, 7, 16] is a special case of the contextual bandit problem, where the outcome is linear in the feature vector encoding the context. As pointed by [2], contextual bandit problems represent a natural half-way point between supervised learning and reinforcement learning: the use of features to encode contexts and the models for the relation between these feature vectors and the outcome are often inherited from supervised learning, while managing the exploration-exploitation tradeoff is necessary to ensure good performance in reinforcement learning. The linear contextual bandit problem can thus be thought of as a midway between the linear regression model of supervised learning, and reinforcement learning.

Recently, there has been a significant interest in introducing multiple “global constraints” in the standard bandit setting [10, 3, 11, 5]. Such constraints are crucial for many important real-world applications. For example, in clinical trials, the treatment plans may be constrained by the total availability of medical facilities, drugs and other resources. In online advertising, there are budget constraints that restrict the number of times an ad is shown. Other applications include dynamic pricing, dynamic procurement, crowdsourcing, etc.; see [10, 4] for many such examples.

In this paper, we consider linear contextual bandit with knapsacks (henceforth, linCBwK) problem. In this problem, the context vectors are generated i.i.d. in every round from some unknown distribution, and on picking an arm, a reward and a consumption vector is observed, which depend linearly on the context vector. The aim of the decision maker is to maximize a total reward while ensuring the the total consumption of every resource remains within a given budget. Below, we give a more precise definition of this problem.

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We use the following notational convention throughout: vectors are denoted by bold face lower case letters, while matrices are denoted by regular face upper case letters. Other quantities such as sets, scalars, etc. may be of either case, but never bold faced. All vectors are column vectors, i.e., a vector in \( n \times 1 \) dimensions is treated as an \( n \times 1 \) matrix. The transpose of matrix \( A \) is \( A^\top \).

**Definition 1** (linCBwK). There are \( K \) “arms”, which we identify with the set \([K]\). The algorithm is initially given as input a budget \( B \in \mathbb{R}_+ \). In every round \( t \), the algorithm first observes context \( x_t(a) \in [0,1]^m \) for every arm \( a \), and then chooses an arm \( a_t \in [K] \), and finally observes a reward \( r_t(a_t) \in [0,1] \) and a \( d \)-dimensional consumption vector \( v_t(a_t) \in [0,1]^d \). The algorithm has a “no-op” option, which is to pick none of the arms and get 0 reward and 0 consumption. The goal of the algorithm is to pick arms such that the total reward \( \sum_{t=1}^T r_t(a_t) \) is maximized, while ensuring that the total consumption does not exceed budget, i.e., \( \sum_t v_t(a_t) \leq B1 \).

We make the following stochastic assumption for context, reward, consumption vectors. In every round \( t \), the tuple \( \{x_t(a), r_t(a), v_t(a)\}_{a=1}^K \) is generated from an unknown distribution \( D \), independent of everything in previous rounds. Also, there exists an unknown vector \( \mu_* \in [0,1]^m \) and matrix \( W_* \in [0,1]^{m \times d} \) such that for every arm \( a \), given contexts \( x_t(a) \), and history \( H_{t-1} \) before time \( t \),

\[
    \mathbb{E}[r_t(a)|x_t(a), H_{t-1}] = \mu_*^\top x_t(a), \quad \mathbb{E}[v_t(a)|x_t(a), H_{t-1}] = W_*^\top x_t(a).
\]

(1)

For succinctness, we will denote the tuple of contexts for \( K \) arms at time \( t \) as matrix \( X_t \in [0,1]^{m \times K} \), with \( x_t(a) \) being the \( a^{th} \) column of this matrix. Similarly, rewards are represented as vector \( r_t \in [0,1]^K \), and consumption vectors are represented as matrix \( V_t \in [0,1]^{d \times K} \).

As we discuss later in the text, the assumption in equation (1) forms the primary distinction between our linear contextual bandit setting and the general contextual bandit setting considered in [5]. Exploiting this linearity assumption will allow us to generate regret bounds which do not depend on number of arms \( K \), rendering it to be especially useful when number of arms is large. Some examples include recommendation systems with large number of products (e.g., retail products, travel packages, ad creatives, sponsored facebook posts). Another advantage over using general contextual bandit setting of [5] is that we don’t need an oracle access to a certain optimization problem, which is required to solve an NP-Hard problem in this case. (See Section [B] for a more detailed discussion.)

We compare the performance of an algorithm to that of the optimal adaptive policy that knows the distribution \( D \) and the parameters \( (\mu_*, W_*) \), and can take into account the history upto that point as well as the current context to decide (possibly with randomization) which arm to pull at time \( t \). However, it is easier to work with an upper bound on this, which is the optimal expected reward of a static policy that is required to satisfy the constraints only in expectation. This technique has been used in several related problems and is standard by now [13][10].

**Definition 2** (Optimal Static Policy). Consider any policy that is context dependent but non-adaptive: for a policy \( \pi \), let \( \pi(X) \in \Delta^{K+1} \) (the unit simplex) denote the probability distribution over arms played (plus no-op) when the context is \( X \in \mathcal{X} \). Define \( r(\pi) \) and \( v(\pi) \) to be the expected reward and consumption vector of policy \( \pi \), respectively, i.e.

\[
    r(\pi) := \mathbb{E}_{(X,r,V)\sim D}[r\pi(X)] = \mathbb{E}_{X\sim D}[\mu_*^\top X \pi(X)].
\]

(2)

\[
    v(\pi) := \mathbb{E}_{(X,r,V)\sim D}[V\pi(X)] = \mathbb{E}_{X\sim D}[W_*^\top X \pi(X)].
\]

(3)

Let \( \pi^* := \arg \max_{\pi} \quad T r(\pi) \quad \text{such that} \quad T v(\pi) \leq B1 \)

(4)

be the optimal static policy. Note that since no-op is allowed, a feasible policy always exists. We denote the value of this optimal static policy by \( \text{OPT} := T r(\pi^*) \).

Following lemma proves that \( \text{OPT} \) upper bounds the value of optimal adaptive policy. The proof is in Appendix [B]
Lemma 1. Let $\overline{OPT}$ denote the value of optimal adaptive policy that knows the distribution $\mathcal{D}$ and parameters $\mu_*, W_*$. We show that there exists a static policy $\pi^*$ such that $\text{Tr}(\pi^*) \geq \overline{OPT}$, and $\text{Tr}(\pi) \leq B$.

Definition 3 (Regret). Let $a_t$ be the arm played at time $t$ by the algorithm. Then, regret is defined as

$$
\text{regret}(T) := \overline{OPT} - \sum_{t=1}^{T} r_t(a_t)
$$

1.1 Main results

Our main result is an algorithm with near-optimal regret bound for linCBwK.

Theorem 1. There is an algorithm for linCBwK such that if $B > mT^{3/4}$, then with probability at least $1 - \delta$,

$$
\text{regret}(T) = O \left( \left( \frac{OPT}{B} + 1 \right) m \sqrt{\ln(dT/\delta) \ln(T)} \right).
$$

Relation to general contextual bandits. There have been recent papers \cite{5, 11} that solve problems similar to linCBwK but for general contextual bandits. Here the relation between contexts and outcome vectors is arbitrary and the algorithms compete with an arbitrary fixed set of context dependent policies $\mathcal{P}$ accessible via an optimization oracle, with regret bounds being $O \left( \left( \frac{OPT}{B} + 1 \right) \sqrt{KT \log(|\mathcal{P}|/\delta)} \right)$. These approaches could potentially be applied to the linear setting using a set $\mathcal{P}$ of linear context dependent policies.

Comparing their bounds with ours, in our results, essentially a $\sqrt{K}$ factor is replaced by a factor of $m$. Most importantly, we have no dependence on $K$ which enables us to consider problems with large action spaces. In any case, both $K$ and $\log(|\mathcal{P}|)$ are at least $m$, so their bounds are no smaller.

Further, suppose that we want to use their result with the set of linear policies, i.e., policies of the form

$$
\arg \max_{a \in \{1, \ldots, K\}} \{ x_t(a)^T \theta \},
$$

for some fixed $\theta \in \mathbb{R}^m$. Then, their algorithms would require access to an “Arg-Max Oracle” that can find the best such policy (maximizing total reward) for a given set of contexts and rewards (no resource consumption). We show that in fact the optimization problem underlying such an “Arg-Max Oracle” problem is NP-Hard, making such an approach computationally expensive. (Proof is in Appendix C.)

The only downside to our results is that we need the budget $B$ to be $\Omega(mT^{3/4})$. Getting similar bounds for budgets as small as $B = \Theta(m\sqrt{T})$ is an interesting open problem. (This also indicates that this is indeed a harder problem than all the special cases.)

Near-optimality of regret bounds. In \cite{17}, it was shown that for the linear contextual bandits problem, no online algorithm can achieve a regret bound better than $\Omega(m\sqrt{T})$. In fact, they prove this lower bound for linear contextual bandits with static contexts. Since that problem is a special case of the linCBwK problem with $d = 1$, this shows that the dependence on $m$ and $T$ in the above regret bound is optimal up to log factors.

For general contextual bandits with resource constraints, the bounds of \cite{5, 11} are near optimal.

Relation to BwK \cite{3} and OSPP \cite{4}. It is easy to see that the linCBwK problem is a generalization of the linear contextual bandits problem \cite{1, 7, 16}. There, the outcome is scalar and the goal is to simply maximize the sum of these. Remarkably, the linCBwK problem also turns out to be a common generalization of bandits with knapsacks (BwK) problem considered in \cite{10, 3}, and the online stochastic packing problem (OSPP) studied by \cite{18, 6, 22, 19, 4}. In both BwK and OSPP, the outcome of every round $t$ is a reward $r_t$ and a vector $v_t$, and the goal of the algorithm is to maximize $\sum_{t=1}^{T} r_t$ while ensuring that $\sum_{t=1}^{T} v_t \leq B \mathbf{1}$. The problems differ in how these rewards and vectors are picked. In the OSPP problem, in every round $t$, the algorithm may pick any reward-vector pair from a given set $A_t$ of $d + 1$-dimensional vectors. The set

\footnote{Similar to the regret bounds for linear contextual bandits \cite{1, 7, 16}.}
Confidence Ellipsoid around the ℓµ. The following result from [1] shows that the algorithm picks an arm and a reward, vector pair is drawn from the corresponding distribution for that arm. This corresponds to the special case of linCBwK, where m = K and the context X = I, the identity matrix, for all t.

We use techniques from all three special cases: our algorithms follow the primal-dual paradigm using an online learning algorithm to search the dual space, that was established in [3]. In order to deal with linear contexts, we use techniques from [14] to estimate the weight matrix W∗, and define “optimistic estimates” of W∗. We also use the technique of combining the objective and the constraints using a certain tradeoff parameter and that was introduced in [4]. Further new difficulties arise, such as in estimating the optimum value from the first few rounds, a task that follows from standard techniques in each of the special cases but is very challenging here. We develop a new way of exploration that uses the linear structure, so that one can evaluate all possible choices that could have led to an optimum solution on the historic sample. This technique might be of independent interest in estimating optimum values. One can see that the problem is indeed more than the sum of its parts, from the fact that we get the optimal bound for linCBwK only when B ≥ Ω(mT3/4), unlike either special case for which the optimal bound holds for all B (but is meaningful only for B = Ω(mT)).

The approach in [3] (for BwK) extends to the case of “static” contexts1 where each arm has a context that doesn’t change over time. The OSPP of [4] is not a special case of linCBwK with static contexts; this is one indication of the additional difficulty of dynamic over static contexts.

Other related work. Budget constraints in a bandit setting has received considerable attention, but most of the early work focussed on special cases such as a single budget constraint in the regular (non-contextual) setting 20 23 26 29 35 36. Recently, [38] showed an O(√T) regret in the linear contextual setting with a single budget constraint, when costs depend only on contexts and not arms. Budget constraints that arise in particular applications such as online advertising [14] [31], dynamic pricing [8] [12] and crowdsourcing [9] [33] [34] have also been considered. There has also been a long line of work studying special cases of the OSCP problem 18 19 22 6 28 24 37 30 27 15.

Due to space constraints, we have eliminated many proofs from the main text. All the missing proofs are in the appendix.

2 Preliminaries

2.1 Confidence Ellipsoid

Consider a stochastic process which in each round t, generates a pair of observations (rt, yt), such that rt is an unknown linear function of yt plus some 0-mean bounded noise, i.e., rt = µ⊤yt + ηt, where yt, µ∗ ∈ Rm, |ηt| ≤ 2R, and E[ηt|y1, r1, . . . , yt−1, rt−1, yt] = 0.

At any time t, a high confidence estimate of the unknown vector µ∗ can be obtained by building a “Confidence Ellipsoid” around the ℓ2-regularized least-square estimate µt constructed from the observations made so far. This technique is common in prior work on linear contextual bandits (e.g., in [2] [16] [1]). For any regularization parameter λ > 0, let

Mi := λI + ∑t−1 i=1 yi−1yi⊤, and µt := Mi−1 ∑t−1 i=1 yi−1rt.

The following result from [1] shows that µ∗ lies with high probability in an ellipsoid with center µt. For any positive semi-definite (PSD) matrix M, define the M-norm as ||µ||M := √µ⊤Mµ. The confidence ellipsoid at time t is defined as

Ct := \{µ ∈ Rm : ||µ − µt||M ≤ R √m ln((1+tm/λ)/δ) + √λm\}.

1It was incorrectly claimed in [3] that the approach can be extended to dynamic contexts without much modifications.

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Lemma 2 (Theorem 2 of [1]). If \( \forall t, \| \mu_t \|_2 \leq \sqrt{m} \) and \( \| y_t \|_2 \leq \sqrt{m} \), then with prob. \( 1 - \delta \), \( \mu \in C_t \).

Another useful observation about this construction is stated below. It first appeared as Lemma 11 of [1], and was also proved as Lemma 3 in [16].

Lemma 3 (Lemma 11 of [1]). \( \sum_{t=1}^T \| y_t \|_{M_t^{-1}} \leq \sqrt{mT \ln(T)} \).

As a corollary of the above two lemmas, we obtain a bound on the total error in the estimate provided by “any point” from the confidence ellipsoid. (Proof is in Appendix D.)

Corollary 1. For \( t = 1, \ldots, T \), let \( \hat{\mu}_t \in C_t \) be a point in the confidence ellipsoid, with \( \lambda = 1, 2R = 1 \). Then, with probability \( 1 - \delta \),

\[
\sum_{t=1}^T |\hat{\mu}_t^\top y_t - \mu_\star^\top y_t| \leq 2m \sqrt{T \ln \left( \frac{(1+Tm)/\delta}{\ln(T)} \right)}.
\]

2.2 Online Learning

The online convex optimization (OCO ) problem considers a \( T \) round game played between a learner and an adversary, where in round \( t \), the learner chooses a \( \theta_t \in \Omega \), and then the adversary picks a concave function \( g_t(\theta_t) : \Omega \rightarrow \mathbb{R} \). The learner’s choice \( \theta_t \) may only depend on learner’s and adversary’s choices in previous rounds. The goal of the learner is to minimize regret defined as the difference between the learner’s objective value and the value of the best single choice on hindsight:

\[ R(T) := \max_{\theta \in \Omega} \sum_{t=1}^T g_t(\theta) - \sum_{t=1}^T g_t(\theta_t). \]

In particular, we will use linear reward functions with values in \([-1,1]\], and domain \( \Omega \) is the unit simplex in \( d + 1 \) dimensions. The algorithm online mirror descent (OMD ) has very fast per step update rules, and provides the following regret guarantees for this setting.

Lemma 4. [32] The online mirror-descent algorithm for the OCO problem achieves regret

\[ R(T) = O(\sqrt{\log(d)T}). \]

We actually need the domain to be

\[ \Omega = \{ \theta : \| \theta \|_1 \leq 1, \theta \geq 0 \}. \]

This is a special case of a unit simplex in \( d + 1 \) dimensions, by letting the rewards on one of the dimensions always be zero. For the rest of the paper, we assume that the OMD algorithm is using this domain.

3 Algorithm

3.1 Optimistic estimates of unknown parameters

Let \( a_t \) denote the arm played by the algorithm at time \( t \). In the beginning of every round, we use the outcomes and contexts from previous rounds to construct a confidence ellipsoid for \( \mu_\star \) and every column of \( W_\star \). The construction of confidence ellipsoid for \( \mu_\star \) follows directly from the techniques in Section 2.1 with \( y_t = x_t(a_t) \) and \( r_t \) being reward at time \( t \). To construct a confidence ellipsoid for a column \( j \) of \( W_\star \), we use the techniques in Section 2.1 while substituting \( y_t = x_t(a_t) \) and \( r_t = v_t(a_t) \), for every \( j \).

As in Section 2.1, let \( M_t := I + \sum_{i=1}^{t-1} x_i(a_i)x_i(a_i)\top \), and construct the regularized least squares estimate for \( \mu_\star, W_\star \), respectively, as

\[
\hat{\mu}_t := M_t^{-1} \sum_{i=1}^{t-1} x_i(a_i)r_i(a_i)\top, \quad \hat{W}_t := M_t^{-1} \sum_{i=1}^{t-1} x_i(a_i)v_i(a_i)\top.
\]
Define confidence ellipsoid for parameter $\mu_\star$ as

$$C_{t,0} := \left\{ \mu \in \mathbb{R}^m : \|\mu - \hat{\mu}\|_M \leq \sqrt{m \ln \left( \frac{(d+tm^2)\delta}{\delta} \right)} + \sqrt{m} \right\},$$

and optimistic estimate of $\mu_\star$ for every arm $a$ as:

$$\tilde{\mu}_t(a) := \arg \max_{\mu \in C_{t,0}} x_t(a)^\top \mu. \quad (7)$$

Let $w_j$ denote the $j^{th}$ column of a matrix $W$. We define a confidence ellipsoid for each column $j$, as

$$C_{t,j} := \left\{ w \in \mathbb{R}^m : \|w - \hat{w}_{t,j}\|_M \leq \sqrt{m \ln \left( \frac{(d+tm^2)\delta}{\delta} \right)} + \sqrt{m} \right\},$$

and denote by $G_t$, the Cartesian product of all these ellipsoids: $G_t := \{ W \in \mathbb{R}^{m \times d} : w_j \in C_{t,j}\}$. Note that Lemma 2 implies $W_\star \in G_t$ with probability $1 - \delta$. Now, given a vector $\theta_t \in \mathbb{R}^d$, we define the optimistic estimate of weight matrix at time $t$ w.r.t. $\theta_t$, for every arm $a \in [K]$, as:

$$\hat{W}_t(a) := \arg \min_{W \in G_t} x_t(a)^\top W \theta_t. \quad (8)$$

Intuitively, for reward we want an upper confidence bound and for consumption we want a lower confidence bound as an optimistic estimate. This intuition aligns with the above definitions, where the maximizer was used in case of reward and a minimizer was used for consumption. The utility and precise meaning of $\theta_t$ will become clearer when we describe the algorithm and present regret analysis.

Using the definition of $\tilde{\mu}_t, \hat{W}_t$, along with the results in Lemma 2 and Corollary 1 about confidence ellipsoids, the following can be derived.

**Corollary 2.** With probability $1 - \delta$, for any sequence of $\theta_1, \theta_2, \ldots, \theta_T$,

1. $x_t(a)^\top \tilde{\mu}_t(a) \geq x_t(a)^\top \mu_\star$, for all arms $a \in [K]$, for all time $t$.
2. $x_t(a)^\top \hat{W}_t(a) \theta_t \leq x_t(a)^\top W_\star \theta_t$, for all arms $a \in [K]$, for all time $t$.
3. $|\sum_{t=1}^T (\tilde{\mu}_t(a_t) - \mu_\star)^\top x_t(a_t)| \leq \left( 2m \sqrt{T \ln \left( \frac{1+tm}{\delta} \right)} \ln(T) \right).$
4. $\|\sum_{t=1}^T (\hat{W}_t(a_t) - W_\star)^\top x_t(a_t)\| \leq \|1_d\| \left( 2m \sqrt{T \ln \left( \frac{(d+tm^2)\delta}{\delta} \right)} \ln(T) \right).$

Essentially, the first two claims ensure that we have optimistic estimates, and the last two claims ensure that the estimates quickly converge to the true parameters.

### 3.2 The core algorithm

In this section, we present an algorithm, and analysis, under the assumption that a certain parameter $Z$ is given. Later, we show how to use the first $T_0$ rounds to estimate $Z$, and also bound the additional regret due to these $T_0$ rounds. We define $Z$ now.

**Assumption 1.** Assume we are given $Z$ such that $\frac{OPT}{B} \leq Z \leq O\left( \frac{OPT}{B} + 1 \right)$.

The algorithm constructs estimates $\hat{\mu}_t$ and $\hat{W}_t$ as in Section 3.1. It also runs the OMD algorithm for an instance of the online learning problem, over the unit simplex. The vector played by the online learning algorithm in time step $t$ is $\theta_t$. After observing the context, the optimistic estimates for each arm are then constructed using $\theta_t$, as defined in (7) and (8). Intuitively, $\theta_t$ is used here as a multiplier to combine different columns of the weight matrix, to get an optimistic weight vector for every arm. An adjusted estimated reward for arm $a$ is then defined by using $B$ to combine optimistic estimate of reward with optimistic estimate of consumption, as $(x_t(a)^\top \hat{\mu}_t(a)) - Z (x_t(a)^\top \hat{W}_t(a) \theta_t)$. The algorithm chooses the arm which appears to be the best according to adjusted estimated reward. After observing the resulting reward and consumption vectors, the estimates are updated. The online learning algorithm is advanced by one step, by defining the profit vector to be $v_t(a_t) - \frac{B}{T} 1$. The algorithm ends either after $T$ time steps or as soon as the total consumption exceeds the budget along some dimension.
We provide a sketch of the proof here, with the full proof in Appendix E. Let Algorithm for linCBwK, with given $\mu_t$ is similar to the online stochastic packing problem; if the actual reward and consumption vectors were $\tilde{r}_t(a_t)$ and $v_t(a_t)$, then it would be exactly like that problem. We adapt techniques from [4]: use the OCO algorithm and the Z parameter to combine constraints into the objective. If a dimension is being consumed too fast, then the multiplier for that dimension should increase, making the algorithm to pick arms that are not likely to consume too much along this dimension.

**Algorithm 1** Algorithm for linCBwK, with given $Z$

Initialize $\theta_1$ as per the OCO algorithm.
Initialize $Z$ such that $\frac{\text{OPT}}{B} \leq Z \leq O\left(\frac{\text{OPT}}{B} + 1\right)$.

for all $t = 1, \ldots, T$ do

Observe $x_t$.
For every $a \in [K]$, compute $\tilde{\mu}_t(a)$ and $\tilde{W}_t(a)$ as per (7) and (8) respectively.
Play the arm $a_t := \arg \max_{a \in [K]} x_t(a)^T (\tilde{\mu}_t(a) - ZW_t(a)\theta_t)$.
Observe $r_t(a_t)$ and $v_t(a_t)$.
If for some $j = 1, d$, $\sum_{t' \leq t} v_t(a_{t'}) \cdot e_j \geq B$ then EXIT.
Use $x_t(a_t), r_t(a_t)$ and $v_t(a_t)$ to obtain $\tilde{\mu}_{t+1}, \tilde{W}_{t+1}$ and $G_{t+1}$.
Update $\theta_{t+1}$ as per the OCO algorithm with $g_t(\theta_t) := \theta_t \cdot (v_t(a_t) - \frac{B}{T} 1)$.
end for

**Theorem 2.** Given a $Z$ as per Assumption 1 Algorithm 1 achieves the following bounds, given that $\mathcal{R}(T)$ is the regret of the OCO algorithm, with probability $1 - \delta$:

$$\text{regret}(T) \leq O \left( (\frac{\text{OPT}}{B} + 1)m \sqrt{T \ln(dT/\delta) \ln(T)} \right).$$

(Proof Sketch) We provide a sketch of the proof here, with the full proof in Appendix E. Let $\tau$ be the stopping time of the algorithm. The proof is in 3 steps:

**Step 1:** Since $\mathbb{E}[v_t(a_t) | x_t, a_t, H_{t-1}] = W_t^* x_t(a_t)$, we apply Azuma-Hoeffding to get that with high probability $\|\sum_{t=1}^\tau v_t(a_t) - W_t^* x_t(a_t)\|_\infty$ is small. Similarly, a lower bound on the sum of $\mu_t^T x_t(a_t)$ is sufficient.

**Step 2:** From Corollary E with high probability, we can bound $\|\sum_{t=1}^T (W_t - \tilde{W}_t(a_t))^T x_t(a_t)\|_\infty$. It is therefore sufficient to work with the sum of the vectors $\tilde{W}_t(a_t)^T x_t(a_t)$, and similarly $\tilde{\mu}_t(a_t)^T x_t(a_t)$.

**Step 3:** The proof is completed by showing the desired bound on $\text{OPT} - \sum_{t=1}^\tau \tilde{\mu}_t(a_t)^T x_t(a_t)$. This part is similar to the online stochastic packing problem; if the actual reward and consumption vectors were $\tilde{\mu}_t(a_t)^T x_t(a_t)$ and $\tilde{W}_t(a_t)^T x_t(a_t)$, then it would be exactly like that problem. We adapt techniques from [4]: use the OCO algorithm and the Z parameter to combine constraints into the objective. If a dimension is being consumed too fast, then the multiplier for that dimension should increase, making the algorithm to pick arms that are not likely to consume too much along this dimension.

### 3.3 Algorithm with Z computation

In this section, we present a modification of Algorithm 1 which computes the required parameter $Z$ and therefore does not need to be provided with a $Z$ as input, as assumed previously in Assumption 1. The algorithm computes $Z$ using the observations from first $T_0$ rounds. Once $Z$ is computed, the algorithm from the previous section can be run for the remaining time steps. However, it needs to be modified slightly to take into account the budget consumed during the first $T_0$ rounds. We handle this by using a smaller budget $B' = B - T_0$ in the computations for remaining rounds. The modified algorithm is given below.

**Algorithm 2** Algorithm for linCBwK, with $Z$ computation

Inputs: $B, T_0, B' = B - T_0$

Using observations from first $T_0$ rounds, compute $Z$ such that $\frac{\text{OPT}}{B} \leq Z \leq O\left(\frac{\text{OPT}}{B} + 1\right)$.
Run Algorithm 1 for $T - T_0$ rounds and budget $B'$. 

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Next, we provide details of the first $T_0$ rounds, and choice of $T_0$.

We provide a method that takes advantage of the linear structure of the problem, and explores in the $m$-dimensional space of contexts and weight vectors to obtain bounds independent of $K$. We use the following procedure. In every round $t = 1, \ldots, T_0$, after observing $X_t$, let $p_t \in \Delta^K$ be

$$p_t := \arg \max_{p \in \Delta^K} \|X_t p\|_{M_t^{-1}},$$

where $M_t := \sum_{i=1}^{t-1} (X_i p_i) (X_i p_i)^\top$. \hfill (9)

Select arm $a_t = a$ with probability $p_t(a)$. In fact, since $M_t$ is a PSD matrix, due to convexity of the function $\|X_t p\|_{M_t^{-1}}^2$, it is the same as playing $a_t = \arg \max_{a \in \Delta} \|x_t(a)\|_{M_t^{-1}}$. Construct estimates $\hat{\mu}, \hat{\mu}_t$ of $\mu, \mu_*$ at time $t$ as

$$\hat{\mu}_t := M_t^{-1} \sum_{i=1}^{t-1} (X_i p_i) r_i(a_i), \quad \hat{W}_t := M_t^{-1} \sum_{i=1}^{t-1} (X_i p_i) \nu_i(a_i)^\top.$$ 

And, for some value of $\gamma$ defined later, obtain an estimate $\hat{\text{OPT}}^\gamma$ of $\text{OPT}$ as:

$$\hat{\text{OPT}}^\gamma := \max_{\pi} \sum_{i=1}^{T_0} \frac{1}{T_0} \sum_{t=1}^{T_0} \hat{\mu}_t^\top X_t \pi(X_t) \leq B + \gamma.$$ \hfill (11)

For an intuition about the choice of arm in (9), observe from the discussion in Section 2.1 that every column $w_x^a$ of $W_*$ is guaranteed to lie inside the confidence ellipsoid centered at column $\hat{w}^a_t$ of $\hat{W}_t$, namely the ellipsoid, $\|w - \hat{w}_t^a\|_{M_t} \leq 4m \ln(Tm/\delta)$. Note that this ellipsoid has principle axes as eigenvectors of $M_t$, and the length of semi-principle axes is given by inverse eigenvalues of $M_t$. Therefore, by maximizing $\|X_t p\|_{M_t^{-1}}$ we are choosing the context closest to the direction of the longest principal axes of the confidence ellipsoid, i.e. in the direction of maximum uncertainty. Intuitively, this corresponds to pure exploration: by making an observation in the direction where uncertainty is large we can reduce the uncertainty in our estimate most effectively.

A more algebraic explanation is as follows. For a good estimation of $\text{OPT}$ by $\hat{\text{OPT}}^\gamma$, we want the estimates $\hat{W}_t$ and $W_*$ (and, $\hat{\mu}$ and $\mu_*$) to be close enough so that $\|\sum_{t=1}^{T_0} (\hat{W}_t - W_*)^\top X_t \pi(X_t)\|_{\infty}$ (and, $|\sum_{t=1}^{T_0} (\hat{\mu}_t - \mu_*)^\top X_t \pi(X_t)|$) is small for all policies $\pi$, and in particular for sample optimal policies. Now, using Cauchy-Schwartz these are bounded by

$$\sum_{t=1}^{T_0} \|\hat{\mu}_t - \mu_*\|_{M_t} \|X_t \pi(X_t)\|_{M_t^{-1}},$$

and

$$\sum_{t=1}^{T_0} \|\hat{W}_t - W_*\|_{M_t} \|X_t \pi(X_t)\|_{M_t^{-1}},$$

where we define $\|W\|_M$, the $M$-norm of matrix $W$, to be the max of column-wise $M$-norms. Using Lemma 2, the term $\|\hat{\mu}_t - \mu_*\|_{M_t}$ is bounded by $2\sqrt{m \ln(T_0m/\delta)}$, and $\|\hat{W}_t - W_*\|_{M_t}$ is bounded by $2\sqrt{m \ln(T_0md/\delta)}$, with probability $1 - \delta$. Lemma 3 bounds the second term $\|\sum_{t=1}^{T_0} X_t \pi(X_t)\|_{M_t^{-1}}$ but only when $\pi$ is the played policy. This is where we use that the played policy $p_t$ was chosen to maximize $\|X_t p_t\|_{M_t^{-1}}$, so that $\sum_{t=1}^{T_0} \|X_t \pi(X_t)\|_{M_t^{-1}} \leq \sum_{t=1}^{T_0} \|X_t p_t\|_{M_t^{-1}}$ and the bound $\sum_{t=1}^{T_0} \|X_t p_t\|_{M_t^{-1}} \leq \sqrt{m T_0 \ln(T_0)}$ given by Lemma 3 actually bounds $\sum_{t=1}^{T_0} \|X_t \pi(X_t)\|_{M_t^{-1}}$ for all $\pi$. Combining, we get a bound of $2m T_0 \ln(T_0) \ln(T_0d/\delta)$ on deviations $\|\sum_{t=1}^{T_0} (\hat{W}_t - \hat{W}_*)^\top X_t \pi(X_t)\|_{\infty}$ and $|\sum_{t=1}^{T_0} (\hat{\mu}_t - \mu_*)^\top X_t \pi(X_t)|$ for all $\pi$.

We prove the following lemma.

**Lemma 5.** For $\gamma = \left(\frac{1}{T_0}\right) 2m \sqrt{T_0 \ln(T_0) \ln(T_0d/\delta)}$, with probability $1 - O(\delta)$,

$$\text{OPT} - 2\gamma \leq \hat{\text{OPT}}^\gamma \leq \text{OPT} + 9\gamma \left(\frac{\hat{\text{OPT}}}{B} + 1\right).$$

**Corollary 3.** Set $Z = \frac{(\text{OPT}^\gamma + 2\gamma)}{B} + 1$, with above value of $\gamma$. Then, with probability $1 - O(\delta)$,

$$\hat{\text{OPT}} + 1 \leq Z \leq (1 + \frac{11\gamma}{B})(\frac{\text{OPT}}{B} + 1).$$
Corollary 3 implies that as long as $B \geq \gamma$, i.e., $B \geq \Omega(\frac{mT}{\sqrt{T_0}})$, $Z$ is a constant factor approximation of $\frac{OPT_B}{B} + 1 \geq Z^*$, therefore Theorem 2 should provide an $\tilde{O}\left(\left(\frac{OPT_B}{B} + 1\right)m\sqrt{T}\right)$ regret bound. However, this bound does not account for the budget consumed in the first $T_0$ rounds. Considering that (at most) $T_0$ amount can be consumed from the budget in the first $T_0$ rounds, we have an additional regret of $\frac{OPT_B}{B}T_0$. Further, since we have $B' = B - T_0$ budget for remaining $T - T_0$ rounds, we need a $Z$ that satisfies the required assumption for $B'$ instead of $B$ (i.e., we need $\frac{OPT_B}{B'} \leq Z \leq O(1)\left(\frac{OPT_B}{B'} + 1\right)$). If $B \geq 2T_0$, then, $B' \geq B/2$, and using 2 times the $Z$ computed in Corollary 3 would satisfy the required assumption.

Together, these observations give Theorem 3.

**Theorem 3.** Using Algorithm 2 with $T_0$ such that $B > \max\{2T_0, mT/\sqrt{T_0}\}$, and twice the $Z$ given by Corollary 3, we get a high probability regret bound of

$$\tilde{O}\left((\frac{OPT_B}{B} + 1)\left(T_0 + m\sqrt{T}\right)\right).$$

In particular, using $T_0 = \sqrt{T}$, and assuming $B > mT^{3/4}$ gives a regret bound of

$$\tilde{O}\left((\frac{OPT_B}{B} + 1)\left(m\sqrt{T}\right)\right).$$

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Appendix

A Concentration Inequalities

Lemma 6 (Azuma-Hoeffding inequality). If a super-martingale \((Y_t; t \geq 0)\), corresponding to filtration \(\mathcal{F}_t\), satisfies \(|Y_t - Y_{t-1}| \leq c_t\) for some constant \(c_t\), for all \(t = 1, \ldots, T\), then for any \(a \geq 0\),
\[
\Pr(Y_T - Y_0 \geq a) \leq e^{-\frac{a^2}{2 \sum_{t=1}^T c_t^2}}.
\]

B Benchmark

Proof of Lemma 1. For an instantiation \(\omega = (X_t, V_t)_{t=1}^T\) of the sequence of inputs, let vector \(p^*_t(\omega) \in \Delta^{K+1}\) denote the distribution over actions (plus no-op) taken by the optimal adaptive policy at time \(t\). Then,
\[
\text{OPT} = E_{\omega \sim D}[\sum_{t=1}^T r_t^\top p^*_t(\omega)] \quad (12)
\]
Also, since this is a feasible policy,
\[
E_{\omega \sim D}[\sum_{t=1}^T V_t^\top p^*_t(\omega)] \leq B \quad (13)
\]
Construct a static context dependent policy \(\pi^*\) as follows: for any \(X \in [0,1]^{m \times K}\), define
\[
\pi^*(X) := \frac{1}{T} \sum_{t=1}^T E_{\omega}[p^*_t(\omega)|X_t = X].
\]
Intuitively, \(\pi^*(X)\) denotes (in hindsight) the probability that the optimal adaptive policy takes an action \(a\) when presented with a context \(X\), averaged over all time steps. Now, by definition of \(r(\pi), v(\pi)\), from above definition of \(\pi^*\), and (12), (13),
\[
Tr(\pi^*) = T E_{X \sim D}[\mu^\top X \pi^*(X)] = E_{\omega}[\sum_{t=1}^T V_t^\top p^*_t(\omega)] = \text{OPT};
\]
\[
Tv(\pi^*) = T E_{X \sim D}[W_t^\top X \pi^*(X)] = E_{\omega}[\sum_{t=1}^T V_t^\top p^*_t(\omega)] \leq B1.
\]

C Hardness of linear AMO

In this section we show that finding the best linear policy is NP-Hard. The input to the problem is, for each \(t \in [T]\), and each arm \(a \in [K]\), a context \(x_t(a) \in [0,1]^m\), and a reward \(r_t(a) \in [-1,1]\). The output is a vector \(\theta \in \mathbb{R}^m\) that maximizes \(\sum_t r_t(a_t)\) where
\[
a_t = \max_{a \in [K]} \{x_t(a)^\top \theta\}.
\]

We give a reduction from the problem of learning halfspaces with noise \([25]\). The input to this problem is for some integer \(n\), for each \(i \in [n]\), a vector \(z_i \in [0,1]^m\), and \(y_i \in \{-1,+1\}\). The output is a vector \(\theta \in \mathbb{R}^m\) that maximizes
\[
\sum_{i=1}^n \text{sign}(z_i^\top \theta) y_i.
\]
Given an instance of the problem of learning halfspaces with noise, construct an instance of the linear AMO as follows. The time horizon \(T = n\), and the number of arms \(K = 2\). For each \(t \in [T]\), the context of
the first arm, \( x_t(1) = z_t \), and its reward \( r_t(1) = y_t \). The context of the second arm, \( x_t(2) = 0 \), the all zeroes vector, and the reward \( r_t(2) \) is also 0.

The total reward of a linear policy w.r.t a vector \( \theta \) for this instance is

\[
|\{i : \text{sign}(z_i^T \theta) = 1, y_i = 1\}| - |\{i : \text{sign}(z_i^T \theta) = 1, y_i = -1\}|.
\]

It is easy to see that this is an affine transformation of the objective for the problem of learning halfspaces with noise.

### D Confidence ellipsoids

**Proof of Corollary 1.** The following holds with probability \( 1 - \delta \).

\[
\sum_{i=1}^{T} |\tilde{\mu}_i^T x_t - \mu_*^T x_t| \leq \sum_{i=1}^{T} \|\tilde{\mu}_i - \mu_*\|_{M_t} \|x_t\|_{M_t^{-1}} \leq \left(\sqrt{m \ln \left(\frac{1 + tm}{\delta}\right)} + \sqrt{m}\right) \sqrt{mT \ln(T)}.
\]

The inequality in the first line is a matrix-norm version of Cauchy-Schwarz (Lemma 7). The inequality in the second line is due to Lemmas 2 and 3. The lemma follows from multiplying out the two factors in the second line.

\[\square\]

**Lemma 7.** For any positive definite matrix \( M \in \mathbb{R}^{n \times n} \) and any two vectors \( a, b \in \mathbb{R}^n \), \( |a^T b| \leq \|a\|_M \|b\|_{M^{-1}} \).

**Proof.** Since \( M \) is positive definite, there exists a matrix \( M_{1/2} \) such that \( M = M_{1/2} M_{1/2}^T \). Further, \( M^{-1} = M_{-1/2} M_{-1/2}^T \).

\[
\|a^T M_{1/2}\|^2 = a^T M_{1/2} M_{1/2}^T a = a^T M a = \|a\|^2_M.
\]

Similarly, \( \|M_{-1/2} b\|^2 = \|b\|^2_{-1/2} \). Now applying Cauchy-Schwarz, we get that

\[
|a^T b| = |a^T M_{1/2} M_{-1/2} b| \leq \|a^T M_{1/2}\| \|M_{-1/2} b\| = \|a\|_M \|b\|_{-1/2}.
\]

\[\square\]

**Proof of Corollary 2.** Here, the first claim follows simply from definition of \( \tilde{W}_t(a) \) and the observation that with probability \( 1 - \delta \), \( W^* \in G_t \). To obtain the second claim, apply Corollary 1 with \( \mu_* = w_{s_j}, y_t = x_t(a_t), \tilde{\mu}_t = [\tilde{W}_t(a_t)]_j \) (the \( j^{th} \) column of \( \tilde{W}_t(a_t) \)), to bound \( \|\sum_t ([\tilde{W}_t(a_t)]_j - w_{s_j})^T x_t(a_t)\| \leq \sum_j \|([\tilde{W}_t(a_t)]_j - w_{s_j})^T x_t(a_t)\| \) for every \( j \), and then take the norm.

\[\square\]

### E Appendix for Section 3.2

**Proof of Theorem 2.** We will use \( \mathcal{R}' \) to denote the main term in the regret bound.

\[
\mathcal{R}'(T) := O\left( m\sqrt{\ln(mdT/\delta) \ln(T)T} \right)
\]

Let \( \tau \) be the stopping time of the algorithm. Let \( H_{t-1} \) be the history of plays and observations before time \( t \), i.e. \( H_{t-1} := \{\theta_\tau, X_\tau, a_\tau, r_\tau(a_\tau), v_\tau(a_\tau) : \tau = 1, \ldots, t-1\} \). Note that \( H_{t-1} \) determines \( \theta_t, \tilde{\mu}_t, \tilde{W}_t, G_t \), but it does not determine \( X_t, a_t, \tilde{W}_t \) (since \( a_t \) and \( \tilde{W}_t(a) \) depend on the context \( X_t \) at time \( t \)). The proof is in 3 steps:
We use the shorthand notation of $\tilde{\theta}$.

Recall that the proof of Theorem 2.

Substituting the inequality from Lemma 9 in Lemma 8, we get

$$\left\| \sum_{t=1}^\tau \nu_t(a_t) - W_\ast^T x_t(a_t) \right\|_{\infty} \leq \mathcal{R}'(T).$$

(14)

Similarly, a lower bound on the sum of $\mu_t^T x_t(a_t)$ is sufficient.

**Step 2:** From Corollary 2 with probability $1 - \delta$,

$$\left\| \sum_{t=1}^\tau (W_\ast - \tilde{W}_t(a_t))^T x_t(a_t) \right\|_{\infty} \leq \mathcal{R}'(T).$$

(15)

It is therefore sufficient to bound the sum of the vectors $\tilde{W}_t(a_t)^T x_t(a_t)$, and similarly for $\mu_t^T x_t(a_t)$.

We use the shorthand notation of $\tilde{r}_t := \mu_t^T x_t(a_t)$, $\tilde{r}_t := \sum_{t=1}^\tau \tilde{r}_t$, $	ilde{\nu}_t := \tilde{W}_t(a_t)^T x_t(a_t)$ and $\tilde{\nu}_t := \sum_{t=1}^\tau \tilde{\nu}_t$ for the rest of this proof.

**Step 3:** The proof is completed by showing that

$$\mathbb{E}[\tilde{r}_t] \geq \text{OPT} - Z\mathcal{R}'(T).$$

**Lemma 8.**

$$\sum_{t=1}^\tau \mathbb{E}[\tilde{r}_t - \tilde{\nu}_t] \geq \frac{T}{T} \text{OPT} + Z \sum_{t=1}^\tau \theta_t \cdot \mathbb{E}[\tilde{\nu}_t - 1B_{H_{t-1}}].$$

**Proof.** Let $r_t^* := \mu_t(a_t)^T X_t \pi^*(X_t)$ and $\tilde{r}_t^* := \tilde{W}_t(a_t)^T X_t \pi^*(X_t)$. By Corollary 2 with probability $1 - \delta$, we have that $T\mathbb{E}[r_t^*|H_{t-1}] \geq \text{OPT}$, and $\mathbb{E}[\tilde{r}_t|H_{t-1}] \leq \frac{T}{T}$. By the choice made by the algorithm,

$$\tilde{r}_t - Z(\theta_t \cdot \tilde{\nu}_t) \geq r_t^* - Z(\theta_t \cdot \tilde{r}_t^*)$$

$$\mathbb{E}[\tilde{r}_t - Z(\theta_t \cdot \tilde{\nu}_t)|H_{t-1}] \geq \mathbb{E}[\tilde{r}_t|H_{t-1}] - Z(\theta_t \cdot \mathbb{E}[\tilde{\nu}_t|H_{t-1}]) \geq \frac{T}{T} \text{OPT} - Z\theta_t \cdot \frac{B_1}{B}.$$

Summing above inequality for $t = 1$ to $\tau$ gives the lemma statement.

**Lemma 9.**

$$\sum_{t=1}^\tau \theta_t \cdot (\tilde{\nu}_t - \frac{B}{T}1) \geq B - \frac{T}{T} \mathcal{R}'(T).$$

**Proof.** Recall that $g_t(\theta_t) = \theta_t \cdot (\tilde{\nu}_t - \frac{B}{T}1)$, therefore the LHS in the required inequality is $\sum_{t=1}^\tau g_t(\theta_t)$. Let $\theta^* := \arg \max_{||\theta||, \theta \geq 0} \sum_{t=1}^\tau g_t(\theta)$. We use the regret definition for the OCO algorithm to get that $\sum_{t=1}^\tau g_t(\theta_t) \geq \sum_{t=1}^\tau g_t(\theta^*) - \mathcal{R}(T)$. Note that from the regret bound given in Lemma 4, $\mathcal{R}(T) \leq \mathcal{R}'(T)$.

**Case 1:** $\tau < T$. This means that $\sum_{t=1}^\tau (\tilde{\nu}_t \cdot e_j) \geq B$ for some $j$. Then from (14) and (15), it must be that $\sum_{t=1}^\tau (\tilde{\nu}_t \cdot e_j) \geq B - \mathcal{R}'(T)$ so that $\sum_{t=1}^\tau g_t(\theta^*) \geq \sum_{t=1}^\tau g_t(\theta^*) + \tau B_T - \mathcal{R}'(T)$.

**Case 2:** $\tau = T$. In this case, $B - \tau B = 0 = \sum_{t=1}^\tau g_t(\theta^*) \leq \sum_{t=1}^\tau g_t(\theta^*)$, which completes the proof of the lemma.

Now, we are ready to prove Theorem 2, which states that Algorithm 1 achieves a regret of $Z\mathcal{R}'(T)$.

**Proof of Theorem 2** Substituting the inequality from Lemma 9 in Lemma 8, we get

$$\sum_{t=1}^\tau \mathbb{E}[\tilde{r}_t|H_{t-1}] \geq \frac{T}{T} \text{OPT} + ZB \left(1 - \frac{T}{T}\right) - \mathcal{R}'(T).$$

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Also, $Z \geq \frac{\text{OPT}}{T}$. Substituting in above,

$$E[\bar{r}_{\text{sum}}] = \sum_{t=1}^T E[\bar{r}_t | H_{t-1}] \geq \frac{T}{T} \text{OPT} + \text{OPT} (1 - \frac{T}{T}) - ZR(T) \geq \text{OPT} - ZR'(T)$$

From Steps 1 and 2, this implies a lower bound on $E[\sum_{t=1}^T \tau t(a_t)]$. The proof is now completed by using Azuma-Hoeffding to bound the actual total reward with high probability. \qed

F Appendix for Section 3.3

Proof of Lemma 5. Let us define an “intermediate sample optimal” as:

$$\overline{\text{OPT}}^\gamma := \max_q \frac{T}{T_0} \sum_{i=1}^{T_0} \mu^T x_i \pi(X_i)$$

such that \( \frac{T}{T_0} \sum_{i=1}^{T_0} W^T X_i \pi(X_i) \leq B + \gamma \) \hspace{1cm} (16)

Above sample optimal knows the parameters $\mu^*$, $W^*$, the error comes only from approximating the expected value over context distribution by average over the observed contexts. We do not actually compute $\overline{\text{OPT}}^\gamma$, but will use it for the convenience of proof exposition. The proof involves two steps.

Step 1: Bound $|\overline{\text{OPT}}^\gamma - \text{OPT}|$.

Step 2: Bound $|\hat{\text{OPT}}^{2\gamma} - \text{OPT}^\gamma|$.

**Step 1** bound can be borrowed from the work on Online Stochastic Convex Programming in [4]: since $\mu^*$, $W^*$ is known, so there is effectively full information before making the decision, i.e., consider the vectors \([\mu^T x_t(a), W^T x_t(a)], a \in [K]\) as outcome vectors which can be observed for all arms $a$ before choosing the distribution over arms to be played at time $t$, therefore, the setting in [4] applies. In fact, $\hat{\text{OPT}}^\gamma$ as defined by Equation (F.10) in [4] when $A_t = \{[\mu^T x_t(a), W^T x_t(a)], a \in [K]\}$, $f$ identity, and $S = \{v \leq \frac{B}{\tau}\}$, is same as $\frac{1}{T}$ times $\overline{\text{OPT}}^\gamma$ defined here. And using Lemma F.4 and Lemma F.6 in [4] (using $L = 1$, $Z^* = \text{OPT} / B$), we obtain that for any $\gamma \geq \left(\frac{T}{T_0}\right) 2m \sqrt{T_0 \ln(T_0) \ln(T_0 d / \delta)}$, with probability $1 - O(\delta)$,

$$\text{OPT} - \gamma \leq \overline{\text{OPT}}^\gamma \leq \text{OPT} + 2\gamma \left(\frac{\text{OPT}}{B} + 1\right). \hspace{1cm} (17)$$

For **Step 2**, we show that with probability $1 - \delta$, for all $\pi$, $\gamma \geq \left(\frac{T}{T_0}\right) 2m \sqrt{T_0 \ln(T_0) \ln(T_0 d / \delta)}$

$$\sum_{i=1}^{T_n} (\hat{\mu}_i - \mu^T) X_i \pi(X_i) \leq \gamma \hspace{1cm} (18)$$

$$\left\| \frac{T}{T_0} \sum_{i=1}^{T_0} (\hat{W}_i - W^T) X_i \pi(X_i) \right\|_{\text{\infty}} \leq \gamma \hspace{1cm} (19)$$

This is sufficient to prove both lower and upper bound on $\hat{\text{OPT}}^{2\gamma}$ for $\gamma \geq \left(\frac{T}{T_0}\right) 2m \sqrt{T_0 \ln(T_0) \ln(T_0 d / \delta)}$. For lower bound, we can simply use (19) for optimal policy for $\hat{\text{OPT}}^\gamma$, denoted by $\hat{\pi}$. This implies that (because of relaxation of distance constraint by $\gamma$) $\hat{\pi}$ is a feasible primal solution for $\hat{\text{OPT}}^{2\gamma}$, and therefore using (17) and (18),

$$\hat{\text{OPT}}^{2\gamma} + \gamma \geq \overline{\text{OPT}}^\gamma \geq \text{OPT} - \gamma.$$
For the upper bound, we can use (19) for the optimal policy \( \hat{\pi} \) for \( \hat{\text{OPT}}^{2\gamma} \). Then, using (17) and (18),

\[
\hat{\text{OPT}}^{2\gamma} \leq \text{OPT}^{3\gamma} + \gamma \leq \text{OPT} + 6\gamma\left(\frac{\text{OPT}}{B} + 1\right) + \gamma.
\]

Combining, this proves the desired lemma statement:

\[
\text{OPT} - 2\gamma \leq \hat{\text{OPT}}^{2\gamma} \leq \text{OPT} + 7\gamma\left(\frac{\text{OPT}}{B} + 1\right)
\]

(20)

What remains is to prove the claim in (18) and (19). We show the proof for (19), the proof for (18) is similar. Observe that for any \( \pi \),

\[
\| \sum_{t=1}^{T_0} (\hat{W}_t - W_*)^\top X_t \pi(X_t) \|_\infty \leq \sum_{t=1}^{T_0} \| (\hat{W}_t - W_*)^\top X_t \pi(X_t) \|_\infty \\
\leq \sum_{t=1}^{T_0} \| \hat{W}_t - W_* \|_{M_\pi} \| X_t \pi(X_t) \|_{M_{\hat{M}}^{-1}}
\]

where \( \| \hat{W}_t - W_* \|_{M_\pi} = \max_j \| \hat{w}_{tj} - w_{*j} \|_{M_\pi} \).

Now, applying Lemma 2 to every column \( \hat{w}_{tj} \) of \( \hat{W}_t \), we have that with probability \( 1 - \delta \) for all \( t \),

\[
\| \hat{W}_t - W_* \|_{M_\pi} \leq 2\sqrt{m \log(T_0d/\delta)} \leq 2\sqrt{m \log(T_0d/\delta)}
\]

And, by choice of \( p_t \)

\[
\| X_t \pi(X_t) \|_{M_{\hat{M}}^{-1}} \leq \| X_t p_t \|_{M_{\hat{M}}^{-1}}
\]

Also, by Lemma 3

\[
\sum_{t=1}^{T_0} \| X_t p_t \|_{M_{\hat{M}}^{-1}} \leq \sqrt{m T_0 \ln(T_0)}
\]

Therefore, substituting,

\[
\| \sum_{t=1}^{T_0} (\hat{W}_t - W_*)^\top X_t \pi(X_t) \|_\infty \leq (2\sqrt{m \log(T_0d/\delta)}) \sum_{t=1}^{T_0} \| X_t p_t \|_{M_{\hat{M}}^{-1}} \\
\leq (2\sqrt{m \log(T_0d/\delta)}) \sqrt{m T_0 \ln(T_0)} \\
\leq \frac{T_0}{T^{\gamma}}
\]

\( \square \)