Polynomial pseudomonads and dependent type theory

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Abstract

We assemble polynomials in a locally cartesian closed category into a tricategory, allowing us to define the notion of a polynomial pseudomonad and polynomial pseudoalgebra. Working in the context of natural models of type theory, we prove that dependent type theories admitting a unit type and dependent sum types give rise to polynomial pseudomonads, and that those admitting dependent product types give rise to polynomial pseudoalgebras.

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1 Review of polynomial monads

The material reviewed in this section can be found entirely in [GK13], with the exception of Lemma 1.14.

Recall that locally cartesian closed categories $\mathcal{E}$ are characterised by the fact that every morphism $f : B \to A$ induces a triple of adjoint functors

\[
\begin{array}{ccc}
\mathcal{E}/A & \xleftarrow{\Sigma_f} & \mathcal{E}/B \\
\downarrow & \Delta_f \downarrow & \downarrow \\
\mathcal{E}/B & \xrightarrow{\Pi_f} & \mathcal{E}/A
\end{array}
\]

where $\Sigma_f$ is given by postcomposition with $f$ and $\Delta_f$ is given by pullback along $f$. This condition is equivalent to the assertion that all slices of $\mathcal{E}$ are cartesian closed. We adopt the convention that locally cartesian closed categories have a terminal object, so in particular they have all finite limits.

We emphasise that locally cartesian closed categories are categories with additional structure. In particular, given a morphism $x : X \to A$, the functor $\Delta_f : \mathcal{E}/A \to \mathcal{E}/B$ gives a choice of pullback $\Delta_f(x) : \Delta_f X \to B$ of $x$ along $f$.

Examples of locally cartesian closed categories include the category $\text{Set}$ of sets, the category $\widehat{\mathcal{C}} = \text{Set}^{\mathcal{C}^{\text{op}}}$ of presheaves on a small category $\mathcal{C}$, and more generally, any topos.

In what follows, unless otherwise specified, $\mathcal{E}$ will be a fixed locally cartesian closed category.

**Definition 1.1.** A polynomial $F = (s, f, t)$ from $I$ to $J$ in $\mathcal{E}$ is a diagram of the form

\[
\begin{array}{ccc}
& B & A \\
I & \xleftarrow{\quad s \quad} & \xrightarrow{\quad f \quad} & \xrightarrow{\quad t \quad} & J
\end{array}
\]

We will write $F : I \to J$ to denote the assertion that $F$ is a polynomial from $I$ to $J$ in $\mathcal{E}$, and if we wish to be more precise, we will write $F : I \xleftarrow{s} B \xrightarrow{f} A \xrightarrow{t} J$ to specify all the data.

**Example 1.2.** Some examples of polynomials include:

(a) Every morphism $f : B \to A$ can be considered as a polynomial $1 \to 1$, by taking $s, t$ to be the unique morphisms to the terminal object. In the following, we will blur the distinction between morphisms of $\mathcal{E}$ and polynomials $1 \to 1$ in $\mathcal{E}$.

(b) Given an object $I$, the identity polynomial on $I$ is given by $i_I : I \xleftarrow{id_I} I \xrightarrow{id_I} I \xrightarrow{id_I} I$. 

(c) Every span \( I \xleftarrow{s} A \xrightarrow{t} J \) induces a polynomial \( I \xleftarrow{s} A \xrightarrow{id_A} A \xrightarrow{t} J \). Polynomials induced by spans are called **linear polynomials**.

**Definition 1.3.** Let \( F : I \xleftarrow{s} B \xrightarrow{f} A \xrightarrow{t} J \) be a polynomial in \( \mathcal{E} \). The **extension** of \( F \) is the functor

\[
P_F = \Sigma_t \Pi_f \Delta_g : \mathcal{E}/I \to \mathcal{E}/J.
\]

A **polynomial functor** is one that is naturally isomorphic to the extension of a polynomial.

We can describe the extension \( P_F \) of a polynomial \( F : I \xleftarrow{s} B \xrightarrow{f} A \xrightarrow{t} J \) in the internal language of \( \mathcal{E} \) by

\[
P_F(X_i \mid i \in I) = \left( \sum_{a \in A} \prod_{b \in B_a} X_{s(b)} \mid j \in J \right).
\]

**Example 1.4.** The extensions of the polynomials in Example 1.2 are given by the following:

(a) If \( f : B \to A \) is a morphism of \( \mathcal{E} \), considered as a polynomial \( 1 \xleftarrow{s} B \xrightarrow{f} A \xrightarrow{t} 1 \), then \( P_f : \mathcal{E} \to \mathcal{E} \) is a **polynomial endofunctor**, defined in the internal language by

\[
P_f(X) = \sum_{a \in A} X_B a.
\]

(b) The extension of the identity polynomial \( i_I : I \xleftarrow{1} I \) is the identity functor \( \mathcal{E}/I \to \mathcal{E}/I \).

(c) The extension of a linear polynomial \( F : I \xleftarrow{s} A \xrightarrow{id_A} A \xrightarrow{t} J \) is the functor \( P_F : \mathcal{E}/I \to \mathcal{E}/J \) defined by \( P_F(X_i \mid i \in I) = \left( \sum_{a \in A} X_{s(a)} \mid j \in J \right) \). This justifies the term ‘linear’.

**Definition 1.5.** Let \( F : I \xleftarrow{s} B \xrightarrow{f} A \xrightarrow{t} J \) and \( G : J \xleftarrow{u} D \xrightarrow{g} C \xrightarrow{v} K \) be polynomials in \( \mathcal{E} \). The **polynomial composite** of \( G \) with \( F \) is the polynomial \( G \cdot F : I \to K \) as in

\[
\begin{array}{c}
\text{I} \\
\downarrow\text{s} \quad \downarrow\text{p} \quad \downarrow\text{q} \quad \quad \downarrow\text{w} \\
N \xrightarrow{q \circ p} M \xrightarrow{v \circ w} K
\end{array}
\]

where \( n, p, q, w \) are defined as in Figure \( \text{I} \) in which (1) is a pullback square, \( w = \Pi_g(h) \), (2) is a pullback square, \( e \) is the component at \( h \) of the counit of the adjunction \( \Delta_g \dashv \Pi_g \), and (3) is a pullback square.

**Remark 1.6.** As explained in [GK13], in the internal language of \( \mathcal{E} \), we have

\[
(M \xrightarrow{q \circ p} K) = \left( \sum_{c \in C_k} \prod_{d \in D_c} A_{u(d)} \mid k \in K \right) \quad \text{and} \quad (N \xrightarrow{v \circ w} M) = \left( \sum_{d \in D_c} B_{m(d)} \mid (c, m) \in M \right).
\]
so that, in full detail, we can write

\[ P_{G,F}(X) = \left( \sum_{(c,m) \in \sum_{c \in C} \prod_{a \in D} A_{u(a)}} \prod_{d,b \in \sum_{d \in D} B_{m(d)}} X_{u(d)} \right) \mid k \in K \].

This definition of composition of polynomials is motivated by the following.

**Proposition 1.7** (*Extension preserves composition of polynomials*). Let \( F : I \to J \) and \( G : J \to K \) be polynomials in \( \mathcal{E} \). There is a natural isomorphism

\[ P_{G,F} \cong P_G \circ P_F : \mathcal{E}/I \to \mathcal{E}/K. \]

**Definition 1.8.** Let \( F : I \leftarrow B f A \quad J \) and \( G : I \leftarrow D g C \rightarrow J \) be polynomials from \( I \) to \( J \) in \( \mathcal{E} \). A **morphism of polynomials** \( \varphi \) from \( F \) to \( G \) consists of an object \( D_\varphi \) of \( \mathcal{E} \) and a triple \( (\varphi_0, \varphi_1, \varphi_2) \) of morphisms in \( \mathcal{E} \) fitting into a commutative diagram of the following form, in which the lower square is a pullback:
We write $\varphi : F \Rightarrow G$ to denote the assertion that $\varphi$ is a morphism of polynomials from $F$ to $G$.

Each morphism $\varphi : F \Rightarrow G$ of polynomials induces a strong natural transformation $P_F \Rightarrow P_G$, which we shall by abuse of notation also call $\varphi$, whose component at $\vec{X} = (X_i | i \in I)$ can be expressed in the internal language of $E$ by

$$(\varphi_{\vec{X}})_j : \sum_{a \in A_j} \prod_{b \in B_a} X_{s(b)} \rightarrow \sum_{c \in C_j} \prod_{d \in D_c} X_{u(d)}; \quad (\varphi_{\vec{X}})_j(a, t) = (\varphi_0(a), t \cdot (\varphi_2)_a \cdot (\varphi_1)_a^{-1}).$$

**Definition 1.9.** A morphism $\varphi : F \Rightarrow G$ is **cartesian** if $\varphi_2$ is invertible.

As the name suggests, if $\varphi : F \Rightarrow G$ is a cartesian morphism, then the induced strong natural transformation $P_F \Rightarrow P_G$ is cartesian.

**Remark 1.10.** Every cartesian morphism of polynomials has a unique representation as a commutative diagram of the following form:

\[ I \xrightarrow{s} B \xleftarrow{u} D \xrightarrow{g} C \xrightarrow{v} J. \] \hfill (1.1)

Indeed, if $(\varphi_0, \varphi_1, \varphi_2)$ is cartesian, replacing $\varphi_1$ in the above diagram by $\varphi_1 \circ \varphi_2^{-1}$ yields the desired diagram. Conversely, if $(\varphi_0, \varphi_1)$ are as in the above diagram, then $(\varphi_0, \varphi_1', \varphi_2')$ is a cartesian morphism of polynomials, where $\varphi_1' : \Delta_{\varphi_0} D \rightarrow D$ is the chosen pullback of $\varphi_0$ along $g$ and $\varphi_2' : \Delta_{\varphi_0} D \rightarrow B$ is the canonical isomorphism induced by the universal property of pullbacks, as illustrated in the following:

\[ I \xrightarrow{s} B \xleftarrow{u} D \xrightarrow{g} C \xrightarrow{v} J. \] \hfill (1.1)

\[ I \xrightarrow{s} B \xleftarrow{u} D \xrightarrow{g} C \xrightarrow{v} J. \] \hfill (1.2)

Note that, in general, for each diagram of the form (1.1), there are possibly many cartesian morphisms inducing it. Conversely, there are many potential ways of turning a diagram

\[ [a] \quad \text{Every polynomial functor has a natural strength, and the natural candidate for morphisms between polynomial functors are those natural transformations which are comappable with the strength. See [GK13] for more on this.} \]
of the form (1.1) into a cartesian morphism. Another possibility would be to take the
induced cartesian morphism to be \((\varphi_0, \varphi_1, \text{id}_B)\). Theorem 3.14 below implies that these are
essentially equivalent.

In particular, when \(I = J = 1\), we can regard pullback squares as cartesian morphisms in a
canonical way.

We are now ready to assemble polynomials into a bicategory (and polynomial functors into
a 2-category). In fact, as proved in [GK13], more is true:

**Theorem 1.11.** Let \(\mathcal{E}\) be a locally cartesian closed category.

(a) There is a bicategory \(\text{Poly}_{\mathcal{E}}\) whose 0-cells are the objects of \(\mathcal{E}\), whose 1-cells are
polynomials in \(\mathcal{E}\), and whose 2-cells are morphisms of polynomials.

(b) There is a 2-category \(\text{PolyFun}_{\mathcal{E}}\) whose 0-cells are the slices \(\mathcal{E}/I\) of \(\mathcal{E}\), whose 1-cells
are polynomial functors, and whose 2-cells are strong natural transformations.

(c) Extension defines a biequivalence \(\text{Ext} : \text{Poly}_{\mathcal{E}} \cong \rightarrow \text{PolyFun}_{\mathcal{E}}\).

(d) Parts (a)–(c) hold true if we restrict the 1-cells to cartesian
morphisms of polynomials in \(\text{Poly}_{\mathcal{E}}\) and cartesian
strong natural transformations in \(\text{PolyFun}_{\mathcal{E}}\); thus there is a
bicategory \(\text{Poly}_{\mathcal{E}}^{\text{cart}}\) and a 2-category \(\text{PolyFun}_{\mathcal{E}}^{\text{cart}}\), which are biequivalent.

**Definition 1.12.** A polynomial monad is a monad in the bicategory \(\text{Poly}_{\mathcal{E}}^{\text{cart}}\). Specif-
ically, a polynomial monad is a quadruple \(P = (I, p, \eta, \mu)\) consisting of an object
\(I\) of \(\mathcal{E}\), a polynomial \(p : I \rightarrow I\) in \(\mathcal{E}\) and cartesian morphisms of polynomials
\(\eta : i_1 \Rightarrow p\) and \(\mu : p \cdot p \Rightarrow p\), satisfying the usual monad axioms, namely

\[
\mu \circ (\mu \cdot p) = \mu \circ (p \cdot \mu) \quad \text{and} \quad \mu \circ (\eta \cdot p) = \text{id}_p = \mu \circ (p \cdot \eta).
\]

**Remark 1.13.** What is usually (e.g. [GK13]) meant by a polynomial monad is a monad
\((P, \eta, \mu)\) on a slice \(\mathcal{E}/I\) of \(\mathcal{E}\), with \(P : \mathcal{E}/I \rightarrow \mathcal{E}/I\) a polynomial functor and \(\eta, \mu\) cartesian
natural transformations; equivalently, this is a monad in the 2-category \(\text{PolyFun}_{\mathcal{E}}^{\text{cart}}\). We
recover this notion from Definition 1.12 by applying the extension bifunctor \(\text{Poly}_{\mathcal{E}}^{\text{cart}} \rightarrow \text{PolyFun}_{\mathcal{E}}^{\text{cart}}\). Furthermore, every polynomial monad in the usual sense is the extension of
a polynomial monad in the sense of Definition 1.12.

Before we continue, the following technical lemma will simplify matters for us greatly down
the road, as it allows us in most instances to prove results about polynomials in the case
when \(I = J = 1\).

**Lemma 1.14.** For fixed objects \(I\) and \(J\) of a locally cartesian closed category \(\mathcal{E}\), there are
isomorphisms of categories

\[
S : \text{Poly}_{\mathcal{E}}(I, J) \cong \text{Poly}_{\mathcal{E}/I \times J}(1, 1) \quad \text{and} \quad S^{\text{cart}} : \text{Poly}_{\mathcal{E}}^{\text{cart}}(I, J) \cong \text{Poly}_{\mathcal{E}/I \times J}^{\text{cart}}(1, 1).
\]

**Proof sketch.** Given a polynomial \(F : I \leftarrow B \rightarrow A \rightarrow J\), define \(S(F) = (s, f) : B \rightarrow I \times A\)
over \(I \times J\) (considered as a polynomial \(1 \rightarrow 1\) in \(\mathcal{E}/I \times J\)) as in
Given a morphism of polynomials $\varphi : F \Rightarrow G$, as in

![Diagram](image)

define $S(\varphi) = (\text{id}_I \times \varphi_0, \varphi_1, \varphi_2) : S(F) \Rightarrow S(G)$, as in the following diagram, where we consider $E$ as an object over $I \times J$ via $\langle s \circ \varphi_2, t \circ f \circ \varphi_2 \rangle : E \to I \times J$.

![Diagram](image)

It is easy to see that $\text{id}_I \times \varphi_0, \varphi_1$ and $\varphi_2$ are morphisms over $I \times J$ and that the lower square of the above diagram truly is cartesian, so that $S(\varphi)$ is a morphism in $\text{Poly}_{E/_{I \times J}}(1, 1)$. Verifying functoriality and invertibility of $S$ is elementary but tedious.

That $S$ restricts to an isomorphism $S^{\text{cart}} : \text{Poly}_{E/_{I \times J}}^{\text{cart}}(I, J) \to \text{Poly}_{E/_{I \times J}}^{\text{cart}}(1, 1)$ is immediate, since $S(\varphi)$ is cartesian if and only if $\varphi_2$ is invertible, which holds if and only if $\varphi$ is cartesian.

\[\square\]
2 Review of natural models

Natural models [Awo16] are a notion equivalent to that of categories with families [Dyb95] which provide a natural setting for interpreting type theory (see also [Fio12]).

Definition 2.1. A natural model is a category $\mathcal{C}$ together with the following data:

- A terminal object $\varnothing$;
- A map of presheaves $p : \hat{\mathcal{U}} \to \mathcal{U}$ over $\mathcal{C}$;
- **Representability data.** For each object $\Gamma$ of $\mathcal{C}$ and each element $A \in \mathcal{U}(\Gamma)$, an object $\Gamma.A$, a morphism $p_A : \Gamma.A \to \Gamma$ in $\mathcal{C}$ and an element $q_A \in \hat{\mathcal{U}}(\Gamma)$, such that for all $\Gamma$ and all $A$, the following square is a pullback:

\[
\begin{array}{ccc}
\mathcal{U} & \xrightarrow{q_A} & \hat{\mathcal{U}} \\
\downarrow \scriptstyle{y(\Gamma.A)} & & \downarrow \scriptstyle{y} \\
\mathcal{U} & \xrightarrow{p} & \hat{\mathcal{U}} \\
\downarrow \scriptstyle{y(p_A)} & & \downarrow \scriptstyle{\pi} \\
\mathcal{U} & \xrightarrow{A} & \mathcal{U}
\end{array}
\]

Here $y : \mathcal{C} \to \hat{\mathcal{C}} = \text{Set}^{\mathcal{C}^{\text{op}}}$ is the Yoneda embedding, and we have identified $A$ and $q_A$ with the corresponding natural transformations by the Yoneda lemma.

Definition 2.1 says essentially that a natural model is a category equipped with a terminal object and a representable natural transformation $p : \hat{\mathcal{U}} \to \mathcal{U}$, but we include in the definition the data witnessing the representability of $p$.

In what follows, we will write $[A]$ to denote the fibre $\hat{\mathcal{U}}_A$ of $p$ over $A : \mathcal{U}$ in the internal language of $\hat{\mathcal{C}}$.

Remark 2.2. Regarding $\mathcal{C}$ as a category of contexts and substitutions of dependent type theory [ML84], we can regard $\mathcal{U}$ as a presheaf of types, $\hat{\mathcal{U}}$ as a presheaf of terms and $p : \hat{\mathcal{U}} \to \mathcal{U}$ as the map sending a term to its unique type. Specifically, the elements of the set $\mathcal{U}(\Gamma)$ are the types $A$ in context $\Gamma$, and the elements $a \in \hat{\mathcal{U}}(\Gamma)$ with $p_T(a) = A$ are the terms $a$ of type $A$ in context $\Gamma$. Representability of $p$ allows us to form the context extension $\Gamma.A$ of a context $\Gamma$ by a type $A$ in context $\Gamma$. This is explained in detail in [Awo16], a treatment of signatures for dependent type theory in a natural model is forthcoming work.

In [Awo16], necessary and sufficient conditions are given for a natural model to support a unit type, dependent sum types and dependent product types, as summarised in the following.

Theorem 2.3. Let $\mathcal{C} = (\mathcal{C}, p)$ be a natural model.

(a) $\mathcal{C}$ supports a unit type if and only if there exist morphisms $\hat{1}$ and $\hat{\star}$ fitting into the following pullback square:
(b) \( C \) supports dependent sum types (\( \Sigma \)-types) if and only if there exist morphisms \( \hat{\Sigma} \) and \( \hat{\text{pair}} \) fitting into the following pullback square:

\[
\begin{array}{ccc}
\sum_{(A,B) : A \times B} \sum_{a : A} \mathcal{U}^{[a]} & \xrightarrow{\hat{\text{pair}}} & \mathcal{U} \\
\pi & \downarrow & \downarrow p \\
\sum_{A : \mathcal{U}} \mathcal{U}^{[A]} & \xrightarrow{\hat{\Sigma}} & \mathcal{U}
\end{array}
\]

where \( \pi \) is the projection morphism;

(c) \( C \) supports dependent product types (\( \Pi \)-types) if and only if there exist morphisms \( \hat{\Pi} \) and \( \hat{\lambda} \) fitting into the following pullback square:

\[
\begin{array}{ccc}
\sum_{A \in \mathcal{U}} \mathcal{U}^{[A]} & \xrightarrow{\hat{\lambda}} & \mathcal{U} \\
p' & \downarrow & \downarrow p \\
\sum_{A \in \mathcal{U}} \mathcal{U}^{[A]} & \xrightarrow{\hat{\Pi}} & \mathcal{U}
\end{array}
\]

where \( p' = \sum_{A \in \mathcal{U}} p^{[A]} \).

Since \( p : \hat{\mathcal{U}} \to \mathcal{U} \) is a morphism in \( \hat{\mathcal{C}} \), which is a locally cartesian closed category, it can be viewed as a polynomial \( 1 \to 1 \), where \( 1 = \mathcal{U}(p) \) is our choice of terminal object in \( \hat{\mathcal{C}} \). Furthermore, observe that the morphism \( \pi \) in (b) is the polynomial composite \( p \cdot p \), and the morphism \( p' \) in (c) is \( P_p(p) \), where \( P_p \) is the extension of \( p \). We can therefore rephrase the statement of Theorem 2.3 in terms of morphisms of polynomials.

**Theorem 2.4.** Let \( C = (\mathcal{C}, p) \) be a natural model.

(a) \( C \) supports unit types if and only if there is a cartesian morphism \( \eta : i_1 \Rightarrow p \) in \( \text{Poly}_{\hat{\mathcal{C}}} \);

(b) \( C \) supports dependent sum types if and only if there is a cartesian morphism \( \mu : p \cdot p \Rightarrow p \) in \( \text{Poly}_{\hat{\mathcal{C}}} \);
(c) \( \mathbb{C} \) supports dependent product types if and only if there is a cartesian morphism \( \zeta : P_p(p) \Rightarrow p \) in \( \text{Poly}_{\mathbb{C}} \):

We originally conjectured that \((1, p, \eta, \mu)\) is, moreover, a polynomial monad in the sense of Definition \[1.12\] and that \((p, \zeta)\) is an algebra for this monad in a suitable sense, but this turned out to be false. For example, consider the monad unit laws \( \mu \circ (\eta \cdot p) = \text{id}_p = \mu \circ (p \cdot \eta) \)—they state precisely that the following equations of pasting diagrams hold:

\[
\begin{array}{c}
\hat{\mathcal{U}} \xrightarrow{(\eta \cdot p)} \sum_{A,B \in \mathbb{C}} [B(a)] \xrightarrow{\mu_1} \hat{\mathcal{U}} \\
\hat{\mathcal{U}} \xrightarrow{(p \cdot \eta)} \sum_{A \in \mathbb{C}} [A] \xrightarrow{\rho_0} \hat{\mathcal{U}}
\end{array}
\]

However, these equations do not hold strictly. Indeed, in the internal language of \( \hat{\mathbb{C}} \), we have

\[
(\mu \circ (\eta \cdot p))_0(A) = \sum_{x:A} 1 = A \times 1 \quad \text{and} \quad (\mu \circ (p \cdot \eta))_0(A) = \sum_{x:A} A = 1 \times A.
\]

But in type theory, the types \( A \times 1, A \) and \( 1 \times A \) are not equal, although there are canonical isomorphisms between them. We therefore cannot, in general, expect the monad laws to hold strictly. However, it is still reasonable to expect this structure to satisfy the laws of a pseudomonad. As such, we require a suitable notion of equivalence between morphisms of polynomials—however, this is not currently available to us, since \( \text{Poly}_{\hat{\mathbb{C}}} \) is merely a bicategory.

In Section \[8\] we will equip \( \text{Poly}_{\mathbb{E}} \) with 3-cells, endowing it with the structure of a \( \text{2Cat} \)-enriched bicategory, which is a kind of tricategory with strict composition in dimension 2. This affords us the ability to show that we have a polynomial pseudomonad and a polynomial pseudoalgebra—proving this is then the content of Section \[4\].
3 Polynomial pseudomonads and pseudoalgebras

Much as monads naturally live in bicategories, pseudomonads naturally live in tricategories. To define the notion of a polynomial pseudomonad, we therefore need to endow the bicategory $\text{Poly}^\text{cart}_\varepsilon$ with 3-cells turning it into a tricategory.

3.1 A tricategory of polynomials

In general, tricategories are fiddly, with lots of coherence data to worry about [Gur13]—fortunately for us, our situation is simplified by the fact that composition of 2-cells of polynomials is strict, so that the 3-cells turn the hom categories $\text{Poly}^\text{cart}_\varepsilon(I, J)$ into 2-categories, rather than bicategories. The emerging structure is that of a 2Cat-enriched bicategory.

**Definition 3.1.** A 2Cat-enriched bicategory $\mathcal{B}$ consists of:

- A set $\mathcal{B}_0$, whose elements we call the 0-cells of $\mathcal{B}$;
- For all 0-cells $I, J$, a 2-category $\mathcal{B}(I, J)$, whose 0-cells, 1-cells and 2-cells we call the 1-cells, 2-cells and 3-cells of $\mathcal{B}$, respectively;
- For all 0-cells $I, J, K$, a 2-functor $\circ_{I, J, K} : \mathcal{B}(J, K) \times \mathcal{B}(I, J) \to \mathcal{B}(I, K)$, which we call the composition 2-functor;
- For all 0-cells $I$, a 2-functor $\iota_I : 1 \to \mathcal{B}(I, I)$, which we call the identity 2-functor, where 1 is the terminal 2-category;
- For all 0-cells $I, J, K, L$, a 2-natural isomorphism

$$
\mathcal{B}(K, L) \times \mathcal{B}(J, K) \times \mathcal{B}(I, J) \xrightarrow{\circ_{J, K, L} \times \text{id}} \mathcal{B}(J, L) \times \mathcal{B}(I, J)
$$

called the associator;

- For all 0-cells $I, J$, 2-natural isomorphisms

$$
\mathcal{B}(I, J) \times 1 \xrightarrow{\text{id} \times \iota_I} \mathcal{B}(I, J) \times \mathcal{B}(I, I) \quad \text{and} \quad \mathcal{B}(J, J) \times 1 \xrightarrow{\iota_J \times \text{id}} 1 \times \mathcal{B}(I, J)
$$

\[\lambda_{I, J} \cong \phi_{I, J, I} \circ_{I, I, J} \phi_{I, J} \quad \text{and} \quad \rho_{I, J} \cong \phi_{I, J} \circ_{I, J, I} \phi_{I, J, J}\]
called the left unitor and right unitor, respectively.

such that for all compatible 1-cells \( I \xrightarrow{f} J \xrightarrow{g} K \xrightarrow{h} L \xrightarrow{k} M \), the following diagrams commute:

\[
\begin{array}{ccc}
((k \circ h) \circ g) \circ f & \xrightarrow{\alpha_{I,J,K,M}} & (k \circ h) \circ (g \circ f) & \xrightarrow{\alpha_{I,J,L,M}} & k \circ (h \circ (g \circ f)) \\
\downarrow & & \downarrow & & \downarrow \kappa_{\alpha_{I,J,K,L}}
\end{array}
\]

\[
(k \circ (h \circ g)) \circ f & \xrightarrow{\alpha_{I,J,L,M}} & k \circ ((h \circ g) \circ f) \\
\downarrow & & \downarrow
\]

\[
(g \circ t_J) \circ f & \xrightarrow{\alpha_{I,J,K}} & g \circ (t_J \circ f) \\
\downarrow & & \downarrow \lambda_{J,K} \circ f & \xrightarrow{g \circ \rho_{I,J}} & g \circ f
\]

**Remark 3.2.** Every 3-category is trivially a \(2\text{Cat}\)-enriched bicategory, and every \(2\text{Cat}\)-enriched bicategory is a tricategory. Every \(2\text{Cat}\)-enriched bicategory has an underlying bicategory, obtained by forgetting the 3-cells, and every bicategory can be equipped with the structure of a \(2\text{Cat}\)-enriched bicategory by taking only identities as 3-cells. An equivalent viewpoint is that \(2\text{Cat}\)-enriched bicategories are tricategories, whose hom-bicategories are 2-categories and whose coherence isomorphisms in the top dimension are identities.

Connections between polynomials and \(2\text{Cat}\)-enriched bicategories have been studied in different but related settings by Tamara von Glehn \[vG15\] and by Mark Weber \[Web15\] (the latter referring to them as ‘2-bicategories’).

In order to motivate our definition of 3-cells, we make an observation relating polynomials with internal categories. First, we recall (e.g. \[Jac01\]) the definition of the internal full subcategory associated with a morphism in a locally cartesian closed category.

**Definition 3.3.** Let \( f : B \to A \) be a morphism in a locally cartesian closed category \( \mathcal{E} \). The internal full subcategory of \( \mathcal{E} \) associated with \( f \) is the internal category \( \mathcal{A}_f \) whose object of objects is \( A \) and whose object of morphisms is \( \sum_{a,a' \in A} B_{a,a'}^{B_{a}} \), with the projections \( \partial_0, \partial_1 \) to \( A \) giving the domain and codomain morphisms, and with identity and composition morphisms defined in the obvious way.

Explicitly, the morphism \( \partial = (\partial_0, \partial_1) : (\mathcal{A}_f)_1 \to A \times A \) is defined to be the exponential object \( f_2^1 \) in \( \mathcal{E}/_{A \times A} \), where \( f_1 \) and \( f_2 \) are the given by pulling back \( f \) along the projections \( \pi_1, \pi_2 : A \times A \Rightarrow A \).

**Theorem 3.4.** Fix objects \( I \) and \( J \) in a locally cartesian closed category \( \mathcal{E} \). There is a functor

\[
\mathcal{A}_(-) : \text{Poly}_{\mathcal{E}}^{\text{cart}}(I,J) \to \text{Cat}(\mathcal{E}_{/I \times J}).
\]
Moreover, every functor of the form \( A_\varphi \) is full and faithful.

**Proof.** We assume \( I = J = 1 \), letting Lemma 1.14 take care of the general case.

Given a morphism \( f : B \to A \) of \( \mathcal{E} \), let \( A_f \) be the internal full subcategory of \( \mathcal{E} \) associated with \( f \) (as in Definition 3.3).

Given a cartesian morphism of polynomials \( \varphi : f \Rightarrow g \), represented by the following pullback square:

\[
\begin{array}{ccc}
B & \xrightarrow{\varphi_1} & D \\
\downarrow f & & \downarrow g \\
A & \xrightarrow{\varphi_0} & C
\end{array}
\]

let \( A_\varphi : A_f \to A_g \) be the internal functor defined as follows. The action of \( A_\varphi \) on objects is given by \( \varphi_0 : A \to C \). Since \( f \cong \Delta_\varphi(g) \) and pullbacks preserve exponentials in locally cartesian closed categories, it follows that \( f_1^1 \cong \Delta_{\varphi_0 \times \varphi_0}(g_2^{g_1}) \). This determines a canonical morphism \( (A_\varphi)_1 : (A_f)_1 \to (A_g)_1 \), as in the following pullback square:

\[
\begin{array}{ccc}
\sum_{a,a' \in A} B_{a'}^{B_a} & \xrightarrow{(A_\varphi)_1} & \sum_{c,c' \in C} D_{c'}^{D_c} \\
\downarrow f_1^{f_1} & & \downarrow g_1^{g_1} \\
A \times A & \xrightarrow{\varphi_0 \times \varphi_0} & C \times C
\end{array}
\]

It is easy to verify that \( A_\varphi \) is an internal functor and that \( A_{\psi \circ \varphi} = A_\psi \circ A_\varphi \) for all composable pairs of cartesian morphisms \( \varphi, \psi \). The fact that \( A_\varphi \) is full and faithful is expressed precisely by the fact that the square defining \( (A_\varphi)_1 \) is cartesian. \( \square \)

**Remark 3.5.** Theorem 3.4 yields a 1-functor between 1-categories. However, \( \text{Cat}(\mathcal{E}/I \times J) \) has the structure of a 2-category, so it is therefore reasonable to expect that when we equip \( \text{Poly}_\mathcal{E} \) with 3-cells, the functor \( A_{(-)} \) should extend to a 2-functor. In particular, any 3-cell between cartesian morphisms of polynomials should induce an internal natural transformation between the induced internal functors. However, since the association of internal functors to morphisms of polynomials works only for cartesian morphisms of polynomials, we cannot simply take internal natural transformations as the 3-cells of \( \text{Poly}_\mathcal{E} \). Lemmas 3.6 and 3.7 provide a correspondence between internal natural transformations \( A_\varphi \Rightarrow A_\psi \) and particular morphisms of \( \mathcal{E} \) in a way that generalises to the case when \( \varphi \) and \( \psi \) are not required to be cartesian.
Lemma 3.6. Let $f : B \to A$ and $g : D \to C$ be polynomials in a locally cartesian closed category $E$ and let $\varphi, \psi : f \Rightarrow g$ be cartesian morphisms of polynomials. There is a bijection between the set of morphisms $\alpha : \Delta_{\varphi_0} D \to \Delta_{\psi_0} D$ in $E/A$ and the set of morphisms $\hat{\alpha} : A \to (A_g)_1$ in $E/C \times C$, as indicated by dashed arrows in the following diagrams:

\[
\begin{array}{ccc}
\Delta_{\varphi} D & \xrightarrow{\varphi_2} & \Delta_{\psi_0} D \\
\downarrow & & \downarrow^{\psi_2} \\
B & \xrightarrow{f} & A
\end{array}
\quad
\begin{array}{ccc}
A & \xrightarrow{\alpha} & (A_g)_1 \\
\downarrow & & \downarrow \partial \\
\langle \varphi_0, \psi_0 \rangle & \xrightarrow{\partial} & C \times C
\end{array}
\]

where $\varphi_2, \psi_2$ are the canonical isomorphisms induced by the universal property of pullbacks, as in Remark 1.10.

Proof. Given $\alpha : \Delta_{\varphi_0} D \to \Delta_{\psi_0} D$ in $E/A$, the exponential transpose of $\alpha$ in $E/A$ is, as a morphism in $E$, a section $\alpha : A \to H$ of the projection $H \to A$, where $H = \sum_{a \in A} D_{\varphi_0(a)}^{D_{\psi_0(a)}}$.

This projection is precisely the pullback of $(A_g)_1 \to C \times C$ along $\langle \varphi_0, \psi_0 \rangle$, as illustrated in the following diagram:

\[
\begin{array}{ccc}
H & \xrightarrow{\pi} & (A_g)_1 \\
\downarrow & & \downarrow \partial \\
A & \xrightarrow{\langle \varphi_0, \psi_0 \rangle} & C \times C
\end{array}
\]

But sections of the pullback correspond with diagonal fillers $\hat{\alpha} : A \to (A_g)_1$ of the pullback square. This is as required, since such a filler making the lower triangle commute makes the upper triangle commute automatically. This concludes the proof of (a). \(\square\)

Lemma 3.7. Let $f : B \to A$ and $g : D \to C$ be polynomials in a locally cartesian closed category $E$, let $\varphi, \psi : f \Rightarrow g$ be cartesian morphisms of polynomials, and let $\alpha, \hat{\alpha}$ be as in Lemma 3.6. The following are equivalent:

(i) $\hat{\alpha}$ is an internal natural transformation $\hat{\alpha}_\varphi \Rightarrow \hat{\alpha}_\psi$;

(ii) In the internal language of $E$, we have $\left( \hat{\alpha}_\psi(k) \circ \alpha_a = \alpha_{a'} \circ \hat{\alpha}_\varphi(k) \right)$ if $a, a' \in A$, $k \in B_a^{B_{a'}}$;

(iii) In the internal language of $E$, we have $\left( \gamma_{a'} \circ k = k \circ \gamma_a \right)$ if $a, a' \in A$, $k \in B_{a'}^{B_a}$, where $\gamma = \psi_2 \circ \alpha \circ \varphi_2^{-1} : B \to B$;

(iv) $\alpha$ is a morphism in $E/B$, i.e. $\psi_2 \circ \alpha = \varphi_2$.  

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Proof. We prove (i) ⇔ (ii) ⇔ (iii) ⇔ (iv).

(i) ⇔ (ii) In light of Lemma 3.6, this is just a translation into the internal language of \( \mathcal{E} \) of the definition of an internal natural transformation.

(ii) ⇔ (iii) Consider the following ‘internal’ diagram, parametrised by \( a, a' \in A \) and \( k \in B_a \):

\[
\begin{array}{ccc}
B_a & \xrightarrow{(\varphi_2)_a} & D\varphi(a) \\
\downarrow k & & \downarrow A_\varphi(k) \\
B_{a'} & \xrightarrow{(\varphi_2)_{a'}} & D\varphi(a')
\end{array}
\quad
\begin{array}{ccc}
D\psi(a) & \xrightarrow{\alpha_a} & D\psi(a) \\
\downarrow A_\psi(k) & & \downarrow A_\psi(k) \\
D\psi(a') & \xrightarrow{(\varphi_2)_a} & D\psi(a')
\end{array}
\quad
\begin{array}{c}
B_a \\
\downarrow k \\
B_{a'}
\end{array}
\]

The left- and right-hand squares commute by functoriality of \( A_\varphi \) and \( A_\psi \). The centre square commutes if and only if (ii) holds, and the outer square commutes if and only if (iii) holds. But the centre square commutes if and only if the outer square commutes.

(iii) ⇔ (iv) Let \( a \in A \) and \( b \in B_a \), and let \( k \in B_a \) be the constant (internal) function with value \( b \). If (iii) holds, then

\[
\gamma_a(b) = \gamma_a(k(b)) = k(\gamma_a(b)) = b
\]

so that \( (\gamma_a = \text{id}_{B_a}) | a \in A \) holds. But this says precisely that \( \gamma = \text{id}_B \), and hence \( \psi_2 \circ \alpha = \varphi_2 \). The converse \( (iv) \Rightarrow (iii) \) is immediate.

\[\square\]

Definition 3.8. Let \( F : I \leftarrow B \xrightarrow{f} A \xrightarrow{t} J \) and \( G : I \leftarrow D \xrightarrow{g} C \xrightarrow{v} J \) be polynomials and let \( \varphi, \psi : F \Rightarrow G \) be morphisms of polynomials, as in:

An adjustment \( \alpha \) from \( \varphi \) to \( \psi \), denoted \( \alpha : \varphi \Rightarrow \psi \), is a morphism \( \alpha : D\varphi \to D\psi \) over \( B \):
Remark 3.9. Lemma 3.7 tells us that, when $\varphi$ and $\psi$ are cartesian, adjustments $\alpha : \varphi \Rightarrow \psi$ can equivalently be described as internal natural transformations $\hat{\alpha} : \varphi \Rightarrow \psi$.

Conjecture 3.10. There is a 2Cat-enriched bicategory $\mathbb{P}ol\mathcal{E}$, whose underlying bicategory is $\mathbf{Poly}_\mathcal{E}$ and whose 3-cells are adjustments.

Unfortunately, the details required to fully prove Conjecture 3.10 turned out to be somewhat cumbersome and, since its full force is not required for our main results, we have left the task of verifying these details for future work. Our progress so far is outlined in Lemma 3.11 and Remark 3.12 and we prove the analogous result with attention restricted to cartesian morphisms of polynomials in Theorem 3.14.

Lemma 3.11. Let $I$ and $J$ be objects in a locally cartesian closed category $\mathcal{E}$. There is a 2-category $\mathbb{P}ol\mathcal{E}(I,J)$ whose underlying category is $\mathbf{Poly}_\mathcal{E}(I,J)$ and whose 2-cells are adjustments.

Proof. Given polynomials $F, G : I \to J$, the category $\mathbb{P}ol\mathcal{E}(I,J)(F,G)$ has morphisms of polynomials $F \Rightarrow G$ as its objects and adjustments as its morphisms, with identity and composition inherited from $\mathcal{E}/B$.

Given a polynomial $F : I \leftarrow B \to A \rightarrow J$, we have an evident functor $1 \to \mathbb{P}ol\mathcal{E}(I,J)(F,F)$ picking out the identity morphism $F$ and the identity adjustment on this morphism.

Let $F, G, H : I \to J$ be polynomials. The composition functor

$$c : \mathbf{Poly}_\mathcal{E}(I,J)(G,H) \times \mathbf{Poly}_\mathcal{E}(I,J)(F,G) \to \mathbf{Poly}_\mathcal{E}(I,J)(F,H)$$

is defined as follows. The composite $c(\varphi, \psi)$ of $\varphi : F \Rightarrow G$ and $\psi : G \Rightarrow H$ is defined using a pullback construction, as defined in [GK13, 3.9]—in particular, the morphism $(\psi \circ \varphi)_2 : D_{\psi \circ \varphi} \to B$ is induced by the universal property of pullbacks. This yields, for each pair of adjustments $\alpha : \varphi \Rightarrow \varphi'$ and $\beta : \psi \Rightarrow \psi'$, a unique morphism $D_{\psi \circ \varphi} \to D_{\psi' \circ \varphi'}$ in $\mathcal{E}$ induced by the universal property of pullbacks, which is an adjustment since it makes the required triangle in $\mathcal{E}/B$ commute. We take this morphism to be $c(\beta, \alpha)$. Functoriality of $c$ is then immediate from the universal property of pullbacks.

It can be easily verified that this data satisfies the required identity and associativity axioms. Thus we have a 2-category.

Remark 3.12. In order to prove Conjecture 3.10 in its entirety, it remains to define the coherence 2-natural isomorphisms $\alpha, \lambda, \rho$, as described in Definition 3.1, and verify that the required diagrams commute.

To give the reader an idea of the flavour of this task, we present some progress towards defining the associator 2-natural transformation $\alpha$. For each quadruple of objects $I, J, K, L$ of $\mathcal{E}$, this must assign to each triple of polynomials $I \xrightarrow{F} J \xrightarrow{G} K \xrightarrow{H} L$ a morphism of polynomials $\alpha_{F,G,H} : (H \cdot G) \cdot F \Rightarrow H \cdot (G \cdot F)$ and, to each triple of morphisms of polynomials

$$\varphi : F \Rightarrow F', \quad \chi : G \Rightarrow G', \quad \psi : H \Rightarrow H',$$
an adjustment
\[ \alpha_{\varphi,\chi,\psi} : \psi \cdot (\chi \cdot \varphi) \circ \alpha_{F,G,H} \Rightarrow \alpha_{F',G',H'} \circ (\psi \cdot \chi) \cdot \varphi : (F \cdot G) \cdot H \Rightarrow F' \cdot (G' \cdot H'), \]
which satisfy naturality laws and behave well with respect to composition and identity.

Restricting to the case \( I = J = K = L = 1 \), let \( f : B \to A \), \( g : D \to C \) and \( h : F \to E \) be
morphisms of \( \mathcal{E} \), considered as polynomials \( 1 \to 1 \) as usual. We will construct an invertible
(and hence cartesian) morphism of polynomials \( \alpha_{f,g,h} : (h \cdot g) \cdot f \Rightarrow h \cdot (g \cdot f) \). Such a
morphism must fit into the following pullback square:

\[
\begin{array}{ccc}
\sum_{e,n,q} \sum_{f \in F} \sum_{d \in D_{n(f)}} B_{q,f,d} & \xrightarrow{(\alpha_{f,g,h})_1} & \sum_{e,n,q} \sum_{f \in F} \sum_{d \in D_{n(f)}} B_{m,f,d} \\
\downarrow{(h \cdot g) : f} & & \downarrow{h \cdot (g : f)} \\
\sum_{e \in E} \sum_{n \in C} \prod_{f \in F} \prod_{d \in D_{n(f)}} A & \xrightarrow{(\alpha_{f,g,h})_0} & \sum_{e \in E} \prod_{f \in F} \prod_{c \in C} \prod_{d \in D_{c}} A
\end{array}
\]

In the above, we have overloaded the letter \( f \), which is ambiguous between the morphism
\( f : B \to A \) of \( \mathcal{E} \) and an internal ‘element’ \( f \in F_e \); and we have written \( p(f) = (c_f,m_f) \) for
\( p \in \prod_{f \in F_e} \sum_{c \in C} \prod_{d \in D_c} A \) and \( f \in F_e \).

The isomorphism \( (\alpha_{f,g,h})_0 \) is given by applying the type theoretic axiom of choice to exchange the middle \( \Sigma \Pi \). Specifically, we have
\[ (\alpha_{f,g,h})_0(e,n,q) = (e, \lambda f. \langle n(f), q(f) \rangle). \]

The isomorphism \( (\alpha_{f,g,h})_1 \) acts trivially; that is, we have
\[ (\alpha_{f,g,h})_1(e,n,q,f,d,b) = ((\alpha_{f,g,h})_0(e,n,q), f,d,b). \]

We suspect that the definition of \( \alpha_{\varphi,\chi,\psi} \) will also be an instance of the type theoretic axiom
of choice. From this, it will be an exercise in symbolic manipulations to check that the ‘Mac Lane pentagon’ will commute.

The situation in which we restrict our attention to cartesian morphisms of polynomials is
greatly simplified by the following lemma, allowing us to prove Conjecture 3.10 for this case
in Theorem 3.14.

**Lemma 3.13.** Let \( \varphi \) and \( \psi \) be morphisms of polynomials. If \( \psi \) is cartesian then there is a
unique adjustment from \( \varphi \) to \( \psi \).

**Proof.** When \( \psi \) is cartesian, the morphism \( \psi_2 \) is invertible, so that \( \alpha = \psi_2^{-1} \circ \varphi_2 \) is the only
morphism making the required triangle commute. \( \square \)
From Theorem 1.11(d) and Lemma 3.13, we immediately obtain the following theorem.

**Theorem 3.14.** There is a $\textbf{2Cat}$-enriched bicategory $\mathcal{P}ol\mathcal{E}_{\text{cart}}$ (which, modulo Conjecture 3.10, is a sub-$\textbf{2Cat}$-enriched bicategory of $\mathcal{P}ol\mathcal{E}$), whose underlying bicategory is $\mathcal{P}ol\mathcal{E}_{\text{cart}}$ and whose hom 2-categories $\mathcal{P}ol\mathcal{E}_{\text{cart}}(I,J)$ are locally codiscrete for all objects $I, J$ of $\mathcal{E}$.

**Proof.** The description of the $\textbf{2Cat}$-enriched bicategory data is given in the work towards a proof of Conjecture 3.10. The coherence data is uniquely defined and satisfies the required equations by Lemma 3.13.

Before moving on, we extend Lemma 1.14 to our tricategorical setting.

**Lemma 3.15.** For fixed objects $I$ and $J$ of a locally cartesian closed category $\mathcal{E}$, there are isomorphisms of 2-categories

$$S : \mathcal{P}ol\mathcal{E}(I,J) \xrightarrow{\cong} \mathcal{P}ol\mathcal{E}/_{I\times J}(1,1) \quad \text{and} \quad S^{\text{cart}} : \mathcal{P}ol\mathcal{E}_{\text{cart}}(I,J) \xrightarrow{\cong} \mathcal{P}ol\mathcal{E}^{\text{cart}}/_{I\times J}(1,1).$$

**Proof.** Let $F : I \leftarrow B \xrightarrow{f} A \xrightarrow{g} J$ and $G : I \leftarrow D \xrightarrow{g} C \xrightarrow{v} J$ be polynomials $I \to J$, and let $\varphi, \psi$ be morphisms of polynomials $F \Rightarrow G$. An adjustment $\alpha : \varphi \Rightarrow \psi$ is simply a morphism $\alpha : \varphi_2 \to \psi_2$ in $\mathcal{E}/B$. Since $S(\varphi)_2 = \varphi_2$ and $S(\psi)_2 = \psi_2$, an adjustment $S(\varphi) \Rightarrow S(\psi)$ is a morphism $\varphi_2 \to \psi_2$ in $(\mathcal{E}/_{I\times J})/_{s,t\circ f} \cong \mathcal{E}/_{I\times J}$. So we can take $S$ to be the identity on adjustments. This trivially extends the functors $S$ and $S^{\text{cart}}$ of Lemma 1.14 to 2-functors.

**Theorem 3.16.** Fix objects $I$ and $J$ in a locally cartesian closed category $\mathcal{E}$. There is a locally full and faithful 2-functor

$$\mathcal{A}_(-) : \mathcal{P}ol\mathcal{E}_{\text{cart}}(I,J) \to \textbf{Cat}(\mathcal{E}/_{I\times J}),$$

whose underlying 1-functor is as in Theorem 3.4.

**Proof.** Let $\varphi, \psi : F \Rightarrow G$ be cartesian morphisms of polynomials $I \to J$. We proved in Lemma 3.7 that adjustments $\alpha : \varphi \Rightarrow \psi$ correspond bijectively with internal natural transformations $\widehat{\alpha} : \mathcal{A}_\varphi \Rightarrow \mathcal{A}_\psi$. Moreover, by Lemma 3.13 there is a unique internal natural transformation $\widehat{\mathcal{A}}_{\varphi} \Rightarrow \widehat{\mathcal{A}}_{\psi}$. As such, defining $\mathcal{A}_\alpha = \widehat{\alpha}$ for all adjustments $\alpha$, we automatically obtain a 2-functor, which is locally full and faithful since the hom-sets $\mathcal{P}ol\mathcal{E}_{\text{cart}}(I,J)(F,G)(\varphi, \psi)$ and $\textbf{Cat}(\mathcal{E}/_{I\times J})(\mathcal{A}_F, \mathcal{A}_G)(\mathcal{A}_\varphi, \mathcal{A}_\psi)$ are both singletons.

### 3.2 Polynomial pseudomonads

We are now ready to define the notion of a polynomial pseudomonad. First, we recall the definition of a pseudomonad in a $\textbf{2Cat}$-enriched bicategory (in fact, the definition works just fine in an arbitrary tricategory).
Definition 3.17. Let \( \mathcal{B} \) be a 2Cat-enriched bicategory. A pseudomonad \( T \) in \( \mathcal{B} \) consists of:

- A 0-cell \( I \) of \( \mathcal{B} \);
- A 1-cell \( t : I \to I \);
- 2-cells \( \eta : \text{id}_I \Rightarrow t \) and \( \mu : t \cdot t \Rightarrow t \), called the unit and multiplication of the pseudomonad, respectively;
- Invertible 3-cells \( \alpha, \lambda, \rho \), called the associator, left unitor and right unitor of the pseudomonad, respectively, as in

\[
\begin{align*}
    t \cdot t \cdot t \xrightarrow{t \cdot \mu} t \cdot t \\
    \mu \cdot t & \quad \quad \eta \otimes \mu \\
    t \cdot t \xrightarrow{\mu} t
\end{align*}
\]

such that the following equations of pasting diagrams hold:

\[
\begin{align*}
    t \cdot t \cdot t \cdot t \xrightarrow{t \cdot t \cdot \mu} t \cdot t \cdot t \\
    \mu \cdot t \cdot t & \quad \quad t \cdot t \cdot \mu \\
    t \cdot t \cdot t \xrightarrow{t \cdot \alpha} t \cdot t \cdot t \\
    \mu \cdot t & \quad \quad \mu \\
    t \cdot t \xrightarrow{\alpha \otimes \mu} t \cdot t \\
    \mu & \quad \quad > t
\end{align*}
\]

\[
\begin{align*}
    t \cdot t \cdot t \cdot t \xrightarrow{t \cdot t \cdot \mu} t \cdot t \cdot t \\
    \mu \cdot t \cdot t & \quad \quad t \cdot \mu \\
    t \cdot t \cdot t \xrightarrow{t \cdot \alpha} t \cdot t \cdot t \\
    \mu \cdot t & \quad \quad \mu \\
    t \cdot t \xrightarrow{\alpha \otimes \mu} t \cdot t \\
    \mu & \quad \quad > t
\end{align*}
\]

\[
\begin{align*}
    t \cdot t \cdot t \xrightarrow{t \cdot \mu} t \cdot t \\
    \mu \cdot t & \quad \quad \mu \\
    t \cdot t \xrightarrow{\alpha \otimes \mu} t \cdot t \\
    \mu & \quad \quad > t
\end{align*}
\]

\[
\begin{align*}
    t \cdot t \cdot t \xrightarrow{t \cdot \mu} t \cdot t \\
    \mu \cdot t & \quad \quad \mu \\
    t \cdot t \xrightarrow{\alpha \otimes \mu} t \cdot t \\
    \mu & \quad \quad > t
\end{align*}
\]

Remark 3.18. We reserve the following terminology for particular cases of pseudomonads in 2Cat-enriched bicategories:

- When the 3-cells \( \alpha, \lambda, \rho \) are identities, we call \( T \) a 2-monad in \( \mathcal{B} \). Note that a 2-monad in \( \mathcal{B} \) restricts to a monad in the underlying bicategory of \( \mathcal{B} \), and that every monad in the underlying bicategory of \( \mathcal{B} \) is automatically a 2-monad in \( \mathcal{B} \).
When $\mathcal{B} = 2\text{Cat}$ is the 3-category of 2-categories, 2-functors, pseudo-natural transformations and modifications, and the underlying 0-cell of $\mathcal{T}$ is a 2-category $\mathcal{K}$, we say that $\mathcal{T}$ is a pseudomonad (or 2-monad) on $\mathcal{K}$.

**Definition 3.19.** A polynomial 2-monad (resp. polynomial pseudomonad) is a 2-monad (resp. pseudomonad) in the $2\text{Cat}$-enriched bicategory $\mathcal{Pol}_{\mathcal{E}}$. Specifically, a polynomial pseudomonad consists of the following data:

- An object $I$ of $\mathcal{E}$;
- A polynomial $p : I \to I$;
- Cartesian morphisms of polynomials $\eta : i_I \Rightarrow p$ and $\mu : p \cdot p \Rightarrow p$;
- Invertible adjustments $\alpha : \mu \circ (p \cdot \mu) \Rightarrow \mu \circ (\mu \cdot p), \lambda : \mu \circ (\eta \cdot p) \Rightarrow \text{id}_p$ and $\rho : \mu \circ (p \cdot \eta) \Rightarrow \text{id}_p$;

such that the adjustments $\alpha, \lambda, \rho$ satisfy the coherence axioms of Definition 3.17.

A consequence of Theorem 3.14 is that all parallel pairs of cartesian morphisms of polynomials are uniquely isomorphic. It follows that, in this case, simply specifying the data for a polynomial monad suffices for defining a polynomial pseudomonad—this is stated precisely in the following lemma, whose proof is immediate.

**Lemma 3.20.** Let $I$ be an object of $\mathcal{E}$, let $p : I \to I$ be a polynomial and let $\eta : i_I \Rightarrow p$ and $\mu : p \cdot p \Rightarrow p$ be cartesian morphisms of polynomials. Then there are unique adjustments $\alpha, \lambda, \rho$ such that the septuple $P = (I, p, \eta, \mu, \alpha, \lambda, \rho)$ is a polynomial pseudomonad in $\mathcal{E}$. □

The next result allows us to lift polynomial 2-monads and polynomial pseudomonads in $\mathcal{E}$ to 2-monads and pseudomonads on the hom 2-categories of $\mathcal{Pol}_{\mathcal{E}}$. This will be key in Section 4 for identifying the sense in which a natural model $p : \mathcal{U} \to \mathcal{U}$ is a pseudoalgebra over the polynomial pseudomonad it induces.

**Theorem 3.21.** Let $P = (p, \eta, \mu, \alpha, \lambda, \rho)$ be a polynomial 2-monad (resp. pseudomonad) on an object $I$ of a locally cartesian closed category $\mathcal{E}$. Then $P$ lifts to a 2-monad (resp. pseudomonad) $P^+ = (P, h, m, \ldots)$ on $\mathcal{Pol}_{\mathcal{E}}(I, I)$.

**Proof.** By Lemma 3.15 we may take $I = 1$ without loss of generality, so that $p$ is just a morphism $p : Y \to X$ in $\mathcal{E}$ and $\eta, \mu$ are pullback squares in $\mathcal{E}$ (cf. Remark 1.10).

For notational simplicity, write $\mathcal{K}$ to denote the 2-category $\mathcal{Pol}_{\mathcal{E}}(1, 1)$. Note $\mathcal{K}$ has as its underlying category the wide subcategory $\mathcal{E}_{\text{cart}}$ of $\mathcal{E}^\to$ whose morphisms are the pullback squares. Thus the 0-cells of $\mathcal{K}$ are the morphisms of $\mathcal{E}$, the 1-cells of $\mathcal{K}$ are pullback squares in $\mathcal{E}$, and between any two 1-cells there is a unique 2-cell by Theorem 3.14.

First we must define a 2-functor $P : \mathcal{K} \to \mathcal{K}$. Define $P$ on the 0-cells of $\mathcal{K}$ by letting $P(f) = P_p(f)$ for all $f : B \to A$ in $\mathcal{E}$. Given a 1-cell $\varphi : f \Rightarrow g$ of $\mathcal{K}$—that is, a pullback
square in $\mathcal{E}$—let $P(\varphi)$ be the result of applying the extension $P_p$ of $p$ to the pullback square defining $\varphi$, as in:

\[
\begin{array}{ccc}
\sum_{x \in X} B^Y \xrightarrow{P_p(\varphi_1)} \sum_{x \in X} D^Y & \xrightarrow{\downarrow} & \sum_{x \in X} C^Y \\
\sum_{x \in X} A^Y & \xrightarrow{P_p(f)} & \sum_{x \in X} C^Y \\
\sum_{x \in X} A^Y & \xrightarrow{P_p(\varphi_0)} & \sum_{x \in X} O^Y \\
\end{array}
\]

Note that $P(\varphi)$ is indeed a pullback square, since polynomial functors preserve all connected limits \cite{GK13}. Thus $P(\varphi)$ is a 1-cell from $P(f)$ to $P(g)$ in $\mathcal{K}$.

Now $P$ respects identity 1-cells in $\mathcal{K}$, since if $f : B \to A$ is a 0-cell then

\[P(\text{id}_f)_0 = P_p(\text{id}_B) = \text{id}_{P_p(B)} = (\text{id}_{P(f)})_0,\]

and likewise $P(\text{id}_f)_1 = (\text{id}_{P(f)})_1$; and $P$ respects composition of 2-cells in $\mathcal{K}$, since for $i \in \{0, 1\}$ we have

\[P(\psi \circ \varphi)_i = P_p((\psi \circ \varphi)_i) = P_p(\psi_i \circ \varphi_i) = P_p(\psi_i) \circ P_p(\varphi_i) = P(\psi)_i \circ P(\varphi)_i = (P(\psi) \circ P(\varphi))_i.\]

Hence the action of $P$ defines a functor on the underlying category of $\mathcal{K}$.

The fact that $P$ extends to a 2-functor is trivial: given an adjustment $\alpha : \varphi \Rightarrow \psi$, there is a unique adjustment $P(\varphi) \Rightarrow P(\psi)$. We take this to be $P(\alpha)$, and note that the axioms governing identity and composition of 2-cells hold trivially by uniqueness of adjustments.

The pseudo-natural transformations $h : \text{id}_{\mathcal{K}} \Rightarrow P$ and $m : P \circ P \Rightarrow P$ giving the unit and multiplication of $\mathbb{P}^+$ are induced by the unit $\eta : i_1 \Rightarrow p$ and $\mu : p \circ p \Rightarrow p$ of $\mathbb{P}$. Specifically, define the components $h_f : f \Rightarrow P(f)$ and $m_f : P(P(f)) \Rightarrow P(f)$ at a 0-cell $f : B \to A$ of $\mathcal{K}$ to be the following squares, respectively:

\[
\begin{array}{ccc}
B & \xrightarrow{(P_\eta)_B} & \sum_{X} B^Y_x \\
\downarrow f & \xrightarrow{P(f)} & \sum_{X} A^Y_x \\
A & \xrightarrow{(P_\mu)_A} & \sum_{X} A^Y_x \\
\end{array}
\]

Note that these squares commute and are cartesian by naturality and cartesianness of the extensions $P_\eta, P_\mu$ of $\eta, \mu$. That $h$ and $m$ extend to pseudo-natural transformations
is immediate from Theorem 3.14: the pseudo-naturality 2-cells in $\mathcal{K}$ are adjustments, so they exist uniquely and satisfy the coherence axioms for pseudo-natural transformations automatically.

If $P$ is a polynomial 2-monad, it is now easy to verify that the 2-monad laws hold for $P^+$. If $P$ is a polynomial pseudomonad, then the pseudomonad laws for $P^+$ concern existence of and equations between adjustments, hence are trivially true by Theorem 3.14.

**Definition 3.22.** Given a polynomial monad (resp. pseudomonad) $P$, the **lift** of $P$ is the 2-monad (resp. pseudomonad) $P^+$ as in Theorem 3.21.

**Definition 3.23.** Let $T = (T, h, m, \alpha, \lambda, \rho)$ be a pseudomonad on a 2-category $\mathcal{K}$. A **pseudoalgebra** over $T$ consists of

- A 0-cell $A$ of $\mathcal{K}$;
- A 1-cell $a : T(A) \to A$ in $\mathcal{K}$;
- Invertible 2-cells $\sigma, \tau$ of $\mathcal{K}$, as in:

\[
\begin{align*}
T(T(A)) & \xrightarrow{T(a)} T(A) & A & \xrightarrow{h_A} T(A) \\
\downarrow m_T & \quad \downarrow a & \downarrow \tau & \quad \downarrow a \\
T(A) & \xrightarrow{a} A & A & \xrightarrow{id_A} A, \\
\end{align*}
\]

such that the following equations of pasting diagrams hold:

\[
\begin{align*}
T^3 A & \xrightarrow{TTa} T^2 A & T^3 A & \xrightarrow{T^2 a} T^2 A \\
\downarrow m_{T^2} & \quad \downarrow Tm_A & \downarrow m_{T^2} & \quad \downarrow m_A \\
T^2 A & \xrightarrow{\alpha_A} T^2 A & T^2 A & \xrightarrow{Ta} TA \\
\downarrow m_A & \quad \downarrow m_A & \downarrow \sigma \circ \eta & \quad \downarrow a \\
TA & \xrightarrow{a} A & TA & \xrightarrow{a} A \\
\end{align*}
\]

\[
\begin{align*}
T^2 A & \xrightarrow{Ta} TA & T^2 A & \xrightarrow{Ta} TA \\
\downarrow m_A & \quad \downarrow m_A & \downarrow m_A & \quad \downarrow m_A \\
TA & \xrightarrow{a} A & TA & \xrightarrow{a} A \\
\end{align*}
\]

\[
\begin{align*}
T^2 A & \xrightarrow{Ta} TA & T^2 A & \xrightarrow{Ta} TA \\
\downarrow m_A & \quad \downarrow m_A & \downarrow m_A & \quad \downarrow m_A \\
TA & \xrightarrow{a} TA & TA & \xrightarrow{a} A \\
\end{align*}
\]

\[
\begin{align*}
T^2 A & \xrightarrow{Ta} TA & T^2 A & \xrightarrow{Ta} TA \\
\downarrow m_A & \quad \downarrow m_A & \downarrow m_A & \quad \downarrow m_A \\
TA & \xrightarrow{a} TA & TA & \xrightarrow{a} A \\
\end{align*}
\]

\[
\begin{align*}
T^2 A & \xrightarrow{Ta} TA & T^2 A & \xrightarrow{Ta} TA \\
\downarrow m_A & \quad \downarrow m_A & \downarrow m_A & \quad \downarrow m_A \\
TA & \xrightarrow{a} TA & TA & \xrightarrow{a} A \\
\end{align*}
\]

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Definition 3.24. Let \( \mathbb{P} = (1, p : Y \to X, \ldots) \) be a polynomial pseudomonad in a locally cartesian closed category \( \mathcal{E} \). A polynomial pseudoalgebra over \( \mathbb{P} \) is a pseudoalgebra over the lift \( \mathbb{P}^+ \). Specifically, it consists of:

- A polynomial \( f : B \to A \);
- A cartesian morphism of polynomials \( \zeta : P(f) \Rightarrow f \);
- Invertible adjustments \( \sigma, \tau \) whose types are as in Definition 3.23

such that the adjustments \( \sigma, \tau \) satisfy the coherence conditions of Definition 3.23.

Much like with polynomial pseudomonads (Lemma 3.20), merely specifying the data for a polynomial pseudoalgebra suffices for the conditions to hold—again, this follows immediately from Theorem 3.14.

Lemma 3.25. Let \( \mathbb{P} = (I, p : Y \to X, \ldots) \) be a polynomial pseudomonad in a locally cartesian closed category \( \mathcal{E} \), let \( f : B \to A \) be a polynomial and let \( \zeta : P(f) \Rightarrow f \) be a morphism of polynomials. Then there are unique adjustments \( \sigma, \tau \) making \( (f, \zeta, \sigma, \tau) \) into a polynomial pseudoalgebra over \( \mathbb{P} \). \( \square \)


## 4 Type theory is a pseudomonad and a pseudoalgebra

The results of Section 3 allow us to precisely formulate and easily prove the conjecture outlined in Section 2.

**Theorem 4.1.** Let $C = (\mathbb{C}, p)$ be a natural model.

(a) $C$ supports a unit type and dependent sum types if and only if $p$ can be equipped with the structure of a polynomial pseudomonad $P$ in $\mathbb{C}$.

(b) $C$ additionally supports dependent product types if and only if $p$ can be equipped with the structure of a polynomial pseudoalgebra over $P$.

**Proof.** By Theorem 2.4, $C$ supports a unit type and dependent sum types if and only if there exist cartesian morphisms of polynomials $\eta : i_1 \Rightarrow p$ and $\mu : p \cdot p \Rightarrow p$, and additionally supports dependent product types if and only if there exists a cartesian morphism of polynomials $\zeta : P_p(p) \Rightarrow p$. By Lemmas 3.20 and 3.25, there are unique adjustments turning $(p, \eta, \mu)$ into a polynomial pseudomonad $P$, and unique adjustments turning $(p, \zeta)$ into a polynomial pseudoalgebra over $P$. □

**Remark 4.2.** Theorem 4.1 makes a connection between logic and algebra by exhibiting a correspondence between laws concerning dependent sums and dependent products in type theory with laws concerning monads in algebra. Specifically, for $\eta : i_1 \Rightarrow p$, $\mu : p \cdot p \Rightarrow p$ and $\zeta : P_p(p) \Rightarrow p$, the (pseudo)monad and (pseudo)algebra equations correspond to certain type isomorphisms as follows:

| Monads and algebras | Type theory |
|----------------------|-------------|
| $\mu \circ (p \cdot \mu) \cong \mu \circ (\mu \cdot p)$ | $\sum_{x:A} \sum_{y:B(x)} C(x, y) \cong \sum_{(x,y):A} B(x)$, $C(x, y)$ |
| $\mu \circ (p \cdot \eta) \cong \text{id}_p$ | $\sum_{x:A} 1 \cong A$ |
| $\mu \circ (\eta \cdot p) \cong \text{id}_p$ | $\sum_{x:A} A \cong A$ |
| $\zeta \circ (p \cdot \zeta) \cong \zeta \circ (\mu \cdot p)$ | $\prod_{x:A} \prod_{y:B(x)} C(x, y) \cong \prod_{(x,y):A} B(x)$, $C(x, y)$ |
| $\zeta \circ (\eta \cdot p) \cong \text{id}_p$ | $\prod_{x:A} A \cong A$ |
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Since our first public presentation of this material in March 2017 [AN17], we have discovered that Thorsten Altenkirch and Gun Pinyo have independently presented related results (under the name monadic containers) at the TYPES conference that took place in May–June 2017 in Budapest [AP17].

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