A Simple Sum for Simplices

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Consider a planar quadrilateral whose vertices have position vectors $A, B, C, D$, in cyclic order. Let $K_{ABC}$ denote the area of the triangle $ABC$, as shown in Figure 1, and use similar notation for the other triangles. The identity

$$K_{BCD}A - K_{ACD}B + K_{ABD}C - K_{ABC}D = 0$$

was given in [1], where it was proved as a consequence of the Jacobi vector triple product identity in $\mathbb{R}^3$.

One has $2K_{ABC} = \det[B - A, C - A]$, and similarly for the other triangles, so (1) gives

$$\det[C - B, D - B]A - \det[D - C, A - C]B + \det[A - D, B - D]C - \det[B - A, C - A]D = 0. \tag{2}$$

The object of this paper is to generalize this fact to arbitrary dimension $n$. Consider points $A_0, \ldots, A_{n+1}$ in $\mathbb{R}^n$ and think of them as column vectors. For $i = 0, \ldots, n + 1$, consider the $n \times n$ matrix

$$M_i = [A_{i+2} - A_{i+1} | A_{i+3} - A_{i+2} | \cdots | A_{i+n+1} - A_{i+n}],$$

where the indices are computed modulo $n + 2$, and let $\Delta_i = \det M_i$. Here is the main result of this paper.

**Theorem 1.** In the above notation, one has

$$\sum_{i=0}^{n+1} (-1)^i \Delta_i A_i = 0. \tag{3}$$

Before proving this result, let us consider some special cases. For $n = 1$, the above theorem gives

$$(A_2 - A_1)A_0 + (A_0 - A_2)A_1 + (A_1 - A_0)A_2 = 0,$$

which is obvious. For $n = 2$, the theorem gives us (2). For $n = 3$, we have five points in $\mathbb{R}^3$, which we may regard as the vertices of a (possibly degenerate) polyhedron, and the theorem gives

$$\det[A_2 - A_1, A_3 - A_1, A_4 - A_1]A_0 + \det[A_3 - A_2, A_4 - A_2, A_0 - A_2]A_1 + \det[A_4 - A_3, A_0 - A_3, A_1 - A_3]A_2 + \det[A_0 - A_4, A_1 - A_4, A_2 - A_4]A_3 + \det[A_1 - A_0, A_2 - A_0, A_3 - A_0]A_4 = 0.$$

Here for each $i$, the coefficient of $A_i$ is six times the signed volume of the tetrahedron defined by the other four vertices. For example, in Figure 2, for $A_0 = (0, 0, -1), A_1 = (1, 0, 0), A_2 = (0, 1, 0)$, and $A_3 = (-1, -1, 0), A_4 = (0, 0, 1)$, we obtain

$$3A_0 - 2A_1 - 2A_2 - 2A_3 + 3A_4 = 0.$$

In dimension $n$, the convex hull of $n + 1$ points is a (possibly degenerate) $n$-simplex. The coefficient $\Delta_i$ in (3) is the signed volume of the $n$-simplex defined by the points other than $A_i$. In particular, $\Delta_i$ is unchanged by translation. So on translating by a nonzero vector $T$, (3) gives

$$\sum_{i=0}^{n+1} (-1)^i \Delta_i (A_i + T) = 0.$$  

Then subtracting (3) and taking the coefficient of $T$ gives the following scalar identity.

**Corollary 1.** $\sum_{i=0}^{n+1} (-1)^i (n+1) \Delta_i = 0$.

In other words, given $n + 2$ points in $\mathbb{R}^n$, when $n$ is odd the sum of the signed volumes of the $n$-simplices is zero, and when $n$ is even, the alternating sum of the signed volumes is zero.

**Figure 1.** Quadrilateral $ABCD$ showing the area $K_{ABC}$ of triangle $ABC$.

**Figure 2.** Applying (3) to a triangular bipyramid.
Multilinear Algebra

Our proof of the theorem is a simple argument using multilinear algebra. Let us summarize the well-known basic ideas we require. Consider real vector spaces $V$ and $W$. Suppose that $k$ is a positive integer. Recall that a function $f : V^k \to W$ is said to be multilinear if it is linear in each variable with the other variables held constant; for a gentle introduction, see [4, Chapter 3]. A multilinear function $f : V^k \to W$ is said to be alternating if for all elements $A_0, A_1, \ldots, A_{k-1} \in V$ and all permutations $\sigma$ of $\{0, 1, \ldots, k-1\}$, one has

$$f(A_{\sigma(0)}, A_{\sigma(1)}, \ldots, A_{\sigma(k-1)}) = \text{sgn}(\sigma)f(A_0, A_1, \ldots, A_{k-1}),$$

where $\text{sgn}(\sigma)$ denotes the sign of $\sigma$. For example, the determinant function $\det : V^n \to \mathbb{R}$, $(A_0, \ldots, A_{n-1}) \mapsto \det[A_0 \mid \ldots \mid A_{n-1}]$ is an alternating multilinear function of the column vectors; see [3, Chapter XIII].

There are several known sets of generators for the symmetric group $S_k$ of permutations of $\{0, 1, \ldots, k-1\}$; a nice survey is given in [2]. In particular, $S_k$ is generated by the cycle $(0, 1, \ldots, k-1)$ and the transposition $(0, 1)$; see [2, Theorem 2.5]. Thus, in order to show that a multilinear function $f$ is alternating, it suffices to show that for all $A_0, A_1, \ldots, A_{k-1} \in V$:

(a) $f(A_1, A_2, \ldots, A_{k-1}, A_0) = (-1)^{k-1}f(A_0, A_1, \ldots, A_{k-1})$,
(b) $f(A_0, A_1, A_2, \ldots, A_{k-1}) = -f(A_0, A_1, A_2, \ldots, A_{k-1})$.

Note that from (b) we have

\[ f(A_0, A_0, A_2, A_3, \ldots, A_{k-1}) = 0. \]

Conversely, it is easy to see that if (b') holds for all $A_0, A_2, \ldots, A_{k-1} \in V$, then (b) follows. So in order to show that a multilinear function $f$ is alternating, it suffices to verify conditions (a) and (b'). Note that it follows that if $f$ is alternating and if $A_i = A_j$ for some $i \neq j$, then $f(A_0, A_1, \ldots, A_{k-1}) = 0$; indeed, one can just permute $A_i, A_j$ to the extreme left and employ (b').

Finally, a key fact about alternating multilinear functions that we will use below is that if $k > \dim V$, then $f$ is identically zero. In the literature, this fact can be quickly deduced once one has constructed the exterior algebra on $V$, but we don’t do that here, since we will not require the exterior product. Instead, one can use the following straightforward proof. First choose a basis for $V$. Using multilinearity, the image of the alternating function $f$ is determined by its values on the basis elements. But if $A_0, A_1, \ldots, A_{k-1}$ are basis elements and $k > \dim V$, then by the pigeonhole principle, there must be a repetition of one of the basis elements. It follows that since $f$ is alternating, $f(A_0, A_1, \ldots, A_{k-1}) = 0$.

Proof of the Theorem

Let $n$ be an arbitrary positive integer, let $V = \mathbb{R}^n$, and consider the function $f : V^{n+2} \to V$ defined by

\[ f(A_0, A_1, \ldots, A_{n+1}) = \sum_{i=0}^{n+1} (-1)^i \Delta_i A_i. \]

Because $\det$ is multilinear, and because for each $i$, the variable $A_i$ does not occur in $\Delta_i$, it follows that $f$ is multilinear. We will show that $f$ is alternating, and hence identically zero. Condition (a) is immediate from the definition of $f$. So it remains to prove (b'). Using the fact that the determinant is an alternating multilinear function of the column vectors and computing the indices modulo $n + 2$, we have

\[ \Delta_i = \det [A_{i+2} - A_{i+1} | A_{i+1} - A_i | \ldots | A_{i+n+1} - A_i] \]

\[ = \det [A_{i+2} | A_{i+1} | \ldots | A_{i+n+1}] \]

\[ - \sum_{j=0}^{n+1} (-1)^i \det [A_{i+j} | \ldots | A_{i+j+1}| A_{i+j} | A_{i+j+1} | \ldots | A_{i+n+1}]. \]

Moving the column $A_{i+j}$ in the summation $j + 2$ positions to the far left, we have

\[ \Delta_i = \det [A_{i+2} | A_{i+3} | \ldots | A_{i+n+1}] \]

\[ - \sum_{j=0}^{n+1} (-1)^i \det [A_{i+j} | \ldots | A_{i+j+1} | A_{i+j} | A_{i+j+1} | \ldots | A_{i+n+1}] \]

where in the above, the hat symbol indicates that the term has been omitted. In particular,

\[ \Delta_0 = \sum_{j=1}^{n+1} (-1)^{j-1} \det [A_1 | \ldots | \hat{A}_j | \ldots | A_{n+1}] \]

and

\[ \Delta_1 = \sum_{j=1}^{n+1} (-1)^{j-1} \det [A_2 | \ldots | \hat{A}_j+1 | \ldots | A_{n+1} | A_0]. \]

Now suppose $A_1 = A_0$. Moving $A_0$ to the far left and replacing it by $A_1$, and then adjusting $j$, we have

\[ \Delta_1 = \sum_{j=0}^{n} (-1)^{j+1} \det [A_1 | \ldots | \hat{A}_{j+1} | \ldots | A_{n+1}] \]

\[ = \sum_{j=1}^{n+1} (-1)^{j-1} \det [A_1 | \ldots | \hat{A}_j | \ldots | A_{n+1}]. \]

So $\Delta_0 + (1)^{n+1} \Delta_1 = 0$. Since $A_1 = A_0$, we have $\Delta_i = 0$ for all $i \geq 2$. Hence, from the definition of $f$,
f(A_0, A_1, \ldots, A_{n+1}) = (\Delta_0 + (-1)^{n+1} \Delta_1)A_0 = 0,

as required.

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