Network-Growth Rule Dependency of Percolation Criticality
– Generalized Explosive Percolation –

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Abstract – To consider relation between percolation criticality and network-growth rule in an evolving network model, we propose a general network-growth rule which includes the conventional network-growth rule and the rule proposed by Achlioptas et al. very recently [Science 323 (2009) 1453]. We introduce the generalized parameter $q$ which characterizes how to evolve a network. We obtain size dependency of percolation step and fractal dimension of percolated cluster at the percolation step for several $q$’s. Strong size dependency of percolation step and sudden change of fractal dimension are observed for intermediate $q$. From these facts, we find that the percolation criticality changes between at the conventional network-growth rule and at the rule proposed by Achlioptas et al.

Percolation transition has been a hot topic not only in statistical physics but also in a wide area of science\cite{1–3}. In materials science, theory of percolation transition is used to investigate relation between connectivity and properties of materials such as electric conductivity in alloys, magnetic phase transition in diluted ferromagnets, and sol-gel transition in polymers. In natural science, percolation transition relates to dynamical features in nature such as spreading wildfire and spreading epidemic. Although these phenomena are non-linear non-equilibrium behaviors, we can consider these effects by just focusing on connectivity of composing elements in the network. In information science and engineering, theory of percolation transition can also adopt for dynamic nature of evolving network systems such as complex network. A curious example is to investigate relationship between percolation transition and PageRank which is an algorithm for deciding importance of website and is used in search engine\cite{4}. Applicable scope of percolation theory has been extended day by day. On the other hand, it seems that nature of percolation transition such as relationship between the criticality and the spatial dimension has been well-established in terms of statistical physics.

In 2009, Achlioptas et al. proposed a novel type of network-growth rule and studied nature of percolation transition\cite{5}. In the conventional network-growth rule, we randomly select a bond between elements which do not belong to the same cluster and connect the bond definitely. On the other hand, in the rule proposed by Achlioptas et al., we randomly select two bonds between elements where the elements adjoining each bond do not belong to the same cluster. After that we connect the bond where the product of sizes of two clusters which are contacted by the bond is smaller than the other. We refer to this rule as Achlioptas product rule. They calculated time evolution of the maximum size of clusters on system without spatial structure. Nature of time evolution resulting from the Achlioptas product rule is completely different from that obtained by the conventional network-growth rule. In fact, Achlioptas et al. claimed that a discontinuous phase transition takes place in network-growth model adopted the Achlioptas product rule, whereas a network-growth model using conventional rule exhibits a continuous phase transition. Since the size of the maximum clusters increases explosively against time in the Achlioptas product rule, the percolation transition is called explosive percolation transition. Some researchers have confirmed occurrence of explosive percolation transition\cite{6,7}, whereas a num-
We connect an edge by comparing sums of the size of clusters. When we use the same cluster and connect a bond by comparing sums, we select two bonds between elements which do not belong to criticality and network-growth rule. To study this problem in a clear manner, we adopt the sum rule as network-growth rule which is referred to as “random rule” and the Achlioptas sum rule. The state of the edge between the i-th and the j-th vertices is expressed by $\tau_{ij} (\tau_{ij} \in \{0, 1\})$. We assume that all edges are non-directed.

To control a network-growth rule, we introduce a new parameter $q$ which characterizes how to evolve a network. It is defined as $q = n(\sigma_i) = 1$ for $\forall i$ and $\tau_{ij} = 0$ for $\forall i, j$.

Step 1 We randomly choose two different edges $e_{ij} \neq e_{kl}$ satisfying conditions such that $\tau_{ij} = \tau_{kl} = 0$, $\sigma_i \neq \sigma_j$ and $\sigma_k \neq \sigma_l$.

Step 2 We connect $e_{ij}$ with the probability $w_{ij}$ defined by

$$w_{ij} := \frac{e^{-q[n(\sigma_i)+n(\sigma_j)]}}{e^{-q[n(\sigma_i)+n(\sigma_j)]} + e^{-q[n(\sigma_k)+n(\sigma_l)]} + e^{-q[n(\sigma_j)+n(\sigma_k)]}}.$$  \hspace{1cm} (1)

whereas we connect $e_{kl}$ with the probability $w_{kl} := 1 - w_{ij}$. After we connect $e_{ij}(e_{kl})$, the states of one of the clusters where $v_i(\sigma_i)$ belongs to are changed to $\sigma_i(\sigma_k)$. When we connect an edge, time advances from $T$ to $T + 1$.

Step 3 We repeat step 2 and step 3 until all of the elements belong to the same cluster.
Here we assume that all clusters are never separated. In this network-growth rule, the number of clusters decreases one by one in each time. In fact, the number of clusters at time $T$ is $N - T$ for $0 \leq T \leq N - 1$. The graphical representation of the above procedure on the square lattice is summarized in Fig. 1.

The introduced parameter $q$ in Eq. (1) is a generalized parameter and characterizes the network-growth rule. Our proposed rule for $q = 0$ is equivalent to the random rule as follows. In the random rule we randomly choose an edge $e_{ij}$ such that $\epsilon_i \neq \epsilon_j$ and $\tau_{ij} = 0$ and connect the edge $e_{ij}$. In Eq. (1) for $q = 0$, the probabilities are the same $w_{ij} = w_{kl} = 1/2$, and thus it is equivalent to select the connecting edge randomly. On the other hand, our rule for $q = +\infty$ realizes the Achlioptas sum rule. In this rule, we randomly choose two edges $e_{ij}$ and $e_{kl}$ such that the conditions which stated in step 2 are satisfied. We compare the sums $n(\epsilon_i) + n(\epsilon_j)$ and $n(\epsilon_k) + n(\epsilon_l)$ and connect the edge where sum is smaller than the other. Actually, for $q = +\infty$ in Eq. (1), $w_{ij} = 1$ and $w_{kl} = 0$ when $n(\epsilon_i) + n(\epsilon_j) < n(\epsilon_k) + n(\epsilon_l)$ and vice versa. Hereafter we refer to our rule for $q = -\infty$ as “inverse Achlioptas sum rule”. In this case we select the connecting edge where the sum of elements is larger than the other. In this way, network-growth rule can be controlled by the generalized parameter $q$. Thus, we can explore relationship between network-growth rule and nature of percolation transition through the generalized parameter $q$. Notice that because of the conditions imposed in step 2, a cluster has no loop as well as “loopless” percolation [21]. This treatment is the same as the Achlioptas product rule process considered by Ziff [15].

To investigate the network-growth dependencies of nature of percolation transition, its criticality, and fractal dimension of the percolated cluster, we study the network-growth model on $L \times L (= N)$ two-dimensional square lattice with open boundary condition. The coordinate of $v_i$ is represented by the position vector $r_i := (x_i, y_i)$ for $1 \leq x_i, y_i \leq L$.

In order to understand $q$-dependency of network evolution quantitatively, we calculate the dynamics of the number of elements in the maximum cluster $n_{\text{max}}$ defined by $n_{\text{max}} := \max \{ n(\alpha) | 1 \leq \alpha \leq N \}$. Here $n(\alpha)$ represents the number of elements such that $\epsilon_i = \alpha$ $(1 \leq i \leq N)$. The value of $n_{\text{max}}$ characterizes the connectivity of network and is often calculated in studies of conventional percolation. As stated above, since we assume that clusters are never divided, $n_{\text{max}}$ monotonically increases with time. Figure 2 shows time development of the density of elements in the maximum cluster $n_{\text{max}}/N$ for $q = -\infty$ (inverse Achlioptas sum rule), $-1, -10^{-1}, -10^{-2}, -10^{-3}, -10^{-4}, 0$ (random rule), $2.5 \times 10^{-5}, 5 \times 10^{-5}, 10^{-4}, 2 \times 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}$, and $+\infty$ (Achlioptas sum rule) on $256 \times 256$ square lattice. These results are obtained by averaging out the results calculated over 1024 samples. Here we define $t$ as the normalized time $t := T/N$. The step of rise monotonically increases as $q$ increases as shown in Fig. 2. Thus, we can control network-growth speed by tuning network-growth rule. In contrast, the gradient at the step when $n_{\text{max}}/N$ reaches near 1/2 behaves non-monotonically against the parameter $q$. This fact relates to the size dependency of the percolation step as explained later.

So far, we constructed a general network-growth rule by introducing the generalized parameter $q$ and showed the $q$-dependency of dynamical nature of network evolution. Next we concentrate on the $q$-dependency of percolation point. In this study we define percolation as follows. If there is a cluster that spreads from left-side to right-side or from top to bottom, we call it percolated cluster. This is a typical definition in the percolation theory. Here we consider size dependency of the percolation step $t_p(L)$ for several $q$’s. The percolation step is defined by the step when a cluster becomes first percolated. The value of $t_p(L)$ is obtained by averaging out the obtained individual percolation step from 1024 samples. Figure 3 (a) shows size dependency of percolation step $t_p(L)$ for several $q$’s which are the same values using Fig. 2. As $q$ increases, the percolation step increases monotonically. This fact is consistent with the result shown in Fig. 2. The percolation step $t_p(L)$ strongly depends on the lattice size around $q \approx 10^{-4}$, which is the reason why the gradient of $n_{\text{max}}/N$ at the step when $n_{\text{max}}/N$ reaches near 1/2 behaves non-monotonically against $q$ in Fig. 2. Figure 3 (b) shows $q$-dependency of the percolation step for $L = 256$. At $q \leq 10^{-4}$, this value suddenly increases as $q$ increases. Next we study $q$-dependency of geometric aspect of the percolated cluster at the percolation step. To consider it, we calculate the ratio between the number of elements in the percolated cluster $n_p$ and that of surface elements in the percolated cluster $n_s$. The surface element means the element that adjoins the other clusters or an isolated ele-
Fig. 3: (Color online) (a) Size dependency of the percolation step $t_p(L)$ for $q = -\infty$ (inverse Achlioptas sum rule), $-1$, $-10^{-1}$, $-10^{-2}$, $-10^{-3}$, $-10^{-4}$, $0$ (random rule), $2.5 \times 10^{-5}$, $5 \times 10^{-5}$, $10^{-4}$, $2 \times 10^{-4}$, $3 \times 10^{-4}$, $3 \times 10^{-3}$, $10^{-1}$, and $+\infty$ (Achlioptas sum rule) from bottom to top. (b) $q$-dependency of the percolation step $t_p(L)$ for $L = 256$. (c) $q$-dependency of the ratio $n_s/n_p$ for $L = 256$. These results are obtained by averaging 1024 independent samples. Since the error bars of these results are smaller than the symbol sizes, they are omitted.

In order to investigate relation between the geometry of percolated cluster at the percolation step and the network-growth rule more quantitatively, we consider $q$-dependency of fractal dimension. The fractal dimension $D$ can be calculated by the radius gyration $R$ and the number of elements in the percolated cluster $n_p$. The radius gyration of the $\alpha$-th cluster $R(\alpha)$ is defined as

$$ R(\alpha) := \sqrt{\frac{1}{n(\alpha)} \sum_{i \text{ s.t. } \sigma_i = \alpha} |r_i - r_0(\alpha)|^2}, \quad (2) $$

$$ r_0(\alpha) := \frac{1}{n(\alpha)} \sum_{i \text{ s.t. } \sigma_i = \alpha} r_i, \quad (3) $$

where the summation takes over vertices such that $\sigma_i = \alpha$ and $r_0(\alpha)$ represents the position vector of the gravity point of the $\alpha$-th cluster. In this study the definition of the fractal dimension $D$ is adopted as follows:

$$ n_p \propto R_p^D, \quad (4) $$

where $R_p$ denotes the radius gyration of the percolated cluster at the percolation step. The fractal dimension quantitatively characterizes geometric properties of fractal systems and is often used in analysis of the fractal geometries. The fractal dimension displays a criticality in percolation transition. When $D = d$, where $d$ is a spatial dimension, the geometry has no fractality. If $0 \leq D \leq d$, on the other hand, the geometry of the percolated cluster has fractality as well as conventional self-similar structure. The upper panels of Fig. 4 show snapshots of the percolated cluster and those of the second-largest cluster for $q = -\infty$ (inverse Achlioptas sum rule), $-10^{-2}$, $0$ (random rule), $10^{-5}$, $10^{-2}$, and $+\infty$ (Achlioptas sum rule) from left to right. The corresponding radius gyration dependency of the number of elements in the percolated cluster is shown in the lower panels of Fig. 4 which are obtained by calculation from $L = 64$ to $L = 1280$. The dotted lines are obtained by least-squares estimation using Eq. (4). Since these graphs are double logarithmic plots, the gradients of these fitting curves express the fractal dimension. The values of fractal dimensions are denoted in Fig. 4. The obtained fractal dimensions for all negative $q$ are the same as that for $q = 0$ (random rule) $D \approx 1.88$ within the error bar. Moreover, for small $q(\leq 10^{-5})$, the fractal dimension is also the same as that for $q = 0$ within the error bar. It should be noted that the fractal dimension for this region is the same value as the conventional percolation transition on square lattice [3]. On the other hand, the fractal dimensions for positive value $q(\geq 10^{-2})$ are the same as that for the Achlioptas sum rule $D \approx 1.97$ within the error bar. The fractal dimensions of the random and Achlioptas sum rule are clearly distinguishable. Here we should notice that the fractal dimension for Achlioptas product rule is the same value as in the case of Achlioptas sum rule. Then qualitative similar behaviors are expected even when we adopt the rule based on product rule as most preceding studies. Thus we conclude there are at least two regions of $q$ in terms of fractal dimension, and between at these regions the percolation criticality changes. It is an open problem to investigate the change of the fractal dimension in the intermediate $q$ in detail.

In this study we proposed a novel network-growth rule in which the introduced parameter $q$ assigns the rule. Since our rule includes the Achlioptas rule which was introduced in 2009 [5] and the conventional network-growth rule, our rule can be regarded as a general network-growth rule. We studied time evolution of the number of maximum cluster. We concentrated on the rule dependency of percolation criticality which is characterized by size-dependency of the percolation step and the fractal dimension. Strong size-dependency of percolation step in the case of $q \approx 10^{-4}$ were observed. Fractal dimensions for several $q$ were also calculated. The fractal dimensions at all negative $q$ and the small positive value ($q \leq 10^{-5}$)
In this study we focused on the case for two-dimensional square lattice. To investigate the relation between the spatial dimension and the percolation criticality for our proposed rule is a remaining problem. Moreover, our rule is a general rule for many network-growth problem and enables us to design percolation criticality. Then we strongly believe that our rule will provide a greater understanding of percolation transition and will be applied for network-growth phenomena in nature and information technology.

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