A Scaling Law for the Energy Levels of a Nonlinear Schrödinger Equation

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Abstract
It is shown that the energy levels of the one-dimensional nonlinear Schrödinger, or Gross-Pitaevskii, equation with the homogeneous trap potential $x^p$, $p \geq 1$, obey an approximate scaling law and as a consequence the energy increases approximately linearly with the quantum number. Moreover, for a quadratic trap, $p = 1$, the rate of increase of energy with the quantum number is independent of the nonlinearity: this prediction is confirmed with numerical calculations. It is also shown that the energy levels computed using a variational approximation do not satisfy this scaling law.

1 Introduction
The Bose-Einstein condensate is described, approximately, by a mean-field approximation, see for example Friedrich (1998), that gives the Gross-Pitaevskii equation. In the one-dimensional problem considered here this equation takes the form,

$$-\frac{\hbar^2}{2\mu} \frac{d^2 y}{dx^2} + \frac{1}{2} \mu \omega^2 x^2 y + A |y|^2 y = Ey,$$  \hspace{1cm} (1)

where $x$ is the spatial coordinate, $\mu$ the atomic mass of the atoms comprising the condensate, $\omega$ the classical frequency of a single atom in the trap potential. The nonlinear parameter $A$ results from the use of a mean-field approximation to describe the particle interactions and is defined in terms of fundamental constants, $A = 4\pi \hbar^2 \alpha_0 N / \mu$ where $\alpha_0$ is the scattering length and $N$ the effective density of atoms along the condensate axis. In most experimental circumstances the nonlinear constant $A$ is large so perturbation methods are of little value. For the ground state, because the wave function varies relatively slowly and because the nonlinearity is large the Thomas-Fermi approximations, equation [1] below, provides a reasonable approximation to both the energy level and the wave function. For excited states no such simple approximation seems to be available. Yabulov et al (1997) have derived
a re-normalised perturbation theory that gives approximate energy levels and wave functions, but we show in section 5 that this method seems to provide a poor estimate of the excited energy levels.

In this paper we show that the energy levels satisfy a simple approximate scaling law and consequently that they are given approximately by the simple formula,

\[ E_n(A) = \frac{1}{2} \left( \frac{3}{2} A \omega \sqrt{\mu} \right)^{2/3} + \frac{7\pi}{32} \omega \hbar n. \] (2)

The first term is just the Thomas-Fermi estimate of the ground state energy, obtained by neglecting the kinetic energy term. The second term is the dominant correction and is linear in \( n \) independent of \( A \). We show also that the latter behaviour is a consequence of the particular form of the trap potential.

2 Theory

The eigenvalues of equation (1), \( E_n(A) \), \( n = 0, 1, 2 \cdots \), are those values of \( E \) for which \( y(x) \) satisfy the boundary conditions \( |y| \to 0 \) as \( |x| \to \infty \) and the normalisation condition

\[ \int_{-\infty}^{\infty} dx |y(x)|^2 = 1. \] (3)

For real eigenvalues we may assume \( y(x) \) to be real.

Two of the four independent parameters in this equation may be removed by rescaling \( x \) and \( y \) and ensuring that the normalisation conditions is invariant,

\[ x = \alpha x', \quad y = \frac{y'}{\sqrt{\alpha}}, \quad \omega = \frac{\omega'}{\hbar}, \quad A = \alpha A', \quad \alpha = \frac{\hbar}{\sqrt{\mu}}, \]

which replaces \( \mu \) and \( \hbar \) by unity. In the following we drop all primes.

By re-writing equation (1) in the form

\[ \frac{d^2 y}{dx^2} + \frac{\partial V}{\partial y} = 0, \quad V(y, x) = E(x)y^2 - \frac{1}{2} A y^4, \quad E(x) = E - \frac{1}{2} \omega^2 x^2 \] (4)

and treating \( x \) as the ‘time’ we may interpret equation (1) as that of a classical particle of unit mass moving in a time-dependent potential, \( V(y, x) \). Conventional methods of classical dynamics provide a means of estimating the eigenvalues.

The potential \( V(y, x) \) is stationary at \( y = 0 \) and this is a minimum for times \( x < x_0 = \sqrt{2E/\omega} \) and for these times there are also maxima at

\[ y^2 = y_n(x)^2 = E(x)/A. \]

For larger times, when \( E(x) < 0 \), there is only a maximum at \( y = 0 \). Hence quasi-periodic motion is possible for small times but for larger times almost all orbits diverge as \( |x| \to \infty \): for every \( E > 0 \), however, there are initial conditions for which \( y(x) \to 0 \) as \( x \to \infty \).

To be specific consider the even solution with initial conditions \( y(0) = a > 0 \) and \( y'(0) = 0 \). For small \( a \) and large enough \( E \) this orbit will oscillate in the potential well
until the barrier at $y = y_m(x)$ is low enough for the orbit to either escape or to ride on the barrier top and eventually to zero: most orbits escape to infinity. Examples of these types of orbit are shown in the following figure. Here $E = 15.0810$, $A = 100$, $\omega = 1$ and $a = a_1 = 0.23975967$ and $a = a_1 \pm 0.0000001$; the converged solution is not normalised.

**Figure 1** Some examples of even solutions of equation 1, with $E = 15.0810$ and $y(0) = a$, given in the text.

This figure shows that the required solutions with $y(x) \to 0$ as $|x| \to \infty$ comprise a quasi-periodic part, for $|x| < x_t$ where $x_t$ is defined in equation 4 below, and a monotonically decreasing segment for $|x| > x_t$. It also shows that the distance between nodes is almost constant: reasons for this are discussed later.

Consider the oscillatory region. When $\overline{E}$ =constant it follows from the definition of the Jacobi elliptic function that the odd and even solutions are, respectively

$$y = a \text{sn}(\tau, k), \quad (y(0) = 0), \quad y = a \text{sn}(K - \tau, k), \quad (y(0) = a),$$  \hspace{1cm} (5)

where $K = K(k)$ is the complete elliptic integral of the first kind and

$$\tau = x \sqrt{2\overline{E} - a^2A}, \quad k^2 = \frac{Aa^2}{2\overline{E} - Aa^2}.$$  

The period of these oscillations is

$$T = \frac{4}{\sqrt{2\overline{E} - Aa^2}} K(k).$$  \hspace{1cm} (6)

When $\overline{E}$ is constant the action of the above oscillatory solution may be written in the form

$$I = \frac{1}{2} a^2 \sqrt{2\overline{E}} F(k), \quad F(k) = \frac{4}{3\pi k^2 \sqrt{1 + k^2}} \left( (1 + k^2) E(k) - (1 - k^2) K(k) \right),$$  \hspace{1cm} (7)

where $E(k)$ is the complete elliptic integral of the second kind, and is not to be confused with the energy. For each $\overline{E}$ there is bound motion if $0 < Aa^2 < \overline{E}$ and as $k$ increases from zero to unity $F(k)$ decreases from 1 to $4\sqrt{2}/(3\pi) \simeq 0.6$. The action is bounded by $0 \leq I \leq I_s$, where $I_s$ is the action of the bound, non-periodic motion on the separatrix, where $Aa^2 = \overline{E}$ ($k = 1$),

$$I_s = \frac{4}{3\pi A} \overline{E}^{3/2}. \hspace{1cm} (8)$$
Now consider the effect of $E$ decreasing, but changing little during one period of the unperturbed motion. The principle of adiabatic invariance (Percival and Richards, 1982, chapter 9) shows that the action is almost invariant. The separatrix action, however, is not constant and decreases to zero at $x = x_0$ where $\omega x_0 = \sqrt{2E}$. All orbits cease to oscillate before this time and if the change in $E$ is sufficiently slow this change occurs when the action equals the separatrix action. If $x_t$ is this time it is given by the solution of

$$\frac{4}{3\pi} \left( E - \frac{1}{2} \omega^2 x_t^2 \right)^{3/2} = AI(E)$$

where the action is evaluated at $E$, the initial value of $E$. Adiabatic invariance shows that the solution oscillates with a local period, $T$, given by equation 6, which depends upon $x$. However, the period although singular at $E(x_0) = Aa^2$, does not change significantly until $E(x)$ is close to $Aa^2$, so the nodes of the wave function are almost equally spaced.

The quantum number, $n$, that labels the state is the number of zeros in the eigenfunction. The ground state, $n = 0$, has no zeros: the first excited state is odd and has one zero at the origin and the second excited state is even and has two zeros. Thus the oscillatory parts of the solution are represented by orbits that encircle the phase-space origin $(n + 1)/4$ times before approaching the origin almost parallel to the $y'$-axis. There are $n/4$ oscillations in the interval $0 \leq x \leq x_t$ so we have the approximate relation $x_t = nT/4$. For later use it is convenient to introduce the scaled variables

$$N = \frac{\pi}{2} \omega n, \quad E = \frac{2E}{N}, \quad \text{and} \quad z = \frac{2E}{Aa^2} \geq 2,$$

in terms of which $k^2 = 1/(z - 1)$ and the quantisation condition becomes

$$\omega x_t(E, z) = \sqrt{\frac{N}{2E}} g(z), \quad g(z) = \frac{2K(k)}{\pi \sqrt{1 - 1/z}}.$$

For large $z$, $g(z) = 1 + \frac{\sqrt{z}}{2z} + O(z^{-2})$.

Finally, we need an approximation to the motion for $x > x_t$. The value of $y(x_t)$ must be close to the barrier height, $y(x_t) \simeq y_m(x_t)$: if $y(x_t) \ll y_m(x_t)$ the orbit would complete another $1/2$ period and if $y(x_t) > y_m(x_t) it would escape. But if $y(x_t) \simeq y_m(x_t)$ the required subsequent orbit is approximated by expanding about the point in phase space that follows the potential maximum, by making the canonical transformation

$$y = Q + y_m(x), \quad \frac{dy}{dx} = P + \frac{dy_m}{dx}$$

and expanding the equations of motion to second-order. Then if $x_0 > 0$ is the time $E(x_0) = 0$ for $x_t < x < x_0$ the equations of motion are

$$\frac{dQ}{dx} = P, \quad \frac{dP}{dx} = E(x)Q - \frac{d^2 y_m}{dx^2}, \quad \frac{d^2 y_m}{dx^2} = -\frac{E\omega^2}{2E(x)\sqrt{AE(x)}}.$$

These equations may be solved numerically and it is seen that $Q(x)$ remains small provided both $|P(x_t)|$ and $|E(x_t)Q(x_t) - y_m''(x_t)|$ are small or zero. As $x \to x_0$ the
solution diverges. However, over the interval of interest this expansion shows that an approximate solution is

\[ y(x) \simeq y_m(x) = \sqrt{\frac{E(x)}{A}}, \quad x_t \leq x < x_0, \quad E(x_0) = 0. \]  

(11)

This is, of course, the standard Thomas-Fermi approximation, obtained from equation [3] by ignoring the kinetic energy term.

Some idea of the accuracy of the approximations [5] and [11] is given in the next figure comparing these with an exact solution. In this case \( E = 15, A = 100 \) which gives \( a = 0.23976 \) and \( x_t = 2.7272 \).

Figure 2  Graphs of the exact solution, the Thomas-Fermi solution [11] for \( x_t \leq x \leq x_0 \) and the adiabatic solution [5] for \( 0 \leq x \leq x_t \) in the case \( E = 15, A = 100 \), for which \( a = 0.23976 \) and \( x_t = 2.7272 \).

In the next section we use equations [3], [8] and [5] to approximate the eigenvalues of equation [9] and to obtain an approximate scaling law.

### 3  An approximate scaling law

Here we show that the approximations described above may be used to derive an approximate scaling law relating the energy, \( E \), quantum number \( n \) and the nonlinearity parameter \( A \) by the single equation,

\[ E = H \left( \omega A N^{-3/2} \right), \quad \mathcal{E} = \frac{2E}{\pi n \omega} = \frac{E}{N} \]  

(12)

for some function \( H \). A consequence of this is that the energy levels behave like those of the linear oscillator in that the difference \( E_{n+1}(A) - E_n(A) \) is almost independent of \( n \) and also of \( A \).

In order to derive this relation we first express \( z \) in terms of \( \mathcal{E} \) using the adiabatic and the quantisation conditions, equations [9] and [10] respectively. These equations may be combined to give

\[ \frac{2\sqrt{2}}{3\pi} \left( 1 - \frac{g(z)^2}{4\mathcal{E}^2} \right)^{3/2} = \frac{F(k)}{z}, \quad k^2 = \frac{1}{z - 1} \]  

(13)

which, in principle gives \( z(\mathcal{E}) \). The behaviour of this function is shown in the next figure where \( 1/z \) is plotted as a function of \( \mathcal{E} = E/N \).
As \( z \to \infty \), \( k \to 0 \), \( g \to 1 \) and \( F \to 1 \), and so \( 2\mathcal{E} \to 1 \); in this limit,
\[
\frac{1}{z} = \frac{2\sqrt{2}}{3\pi} \frac{(4\mathcal{E}^2 - 1)^{3/2}}{(2\mathcal{E})^3}, \quad 2\mathcal{E} \sim 1.
\]

As \( \mathcal{E} \) increases \( 1/z(\mathcal{E}) \) increases monotonically to \( 1/2 \).

The normalisation condition, equation (3), can be written in the form
\[
1 = 2na^2 \int_{0}^{T/4} dx \, \text{sn}(\tau, k)^2 + 2 \int_{x_0}^{x_t} dx \, \frac{E(x)}{A}. \quad (14)
\]

The first of these integrals may be evaluated using relations given in Abramowitz and Stegun (1965, section 16.25), so we have
\[
1 = \frac{2na^2}{\sqrt{2E - Aa^2}} \frac{K(k) - E(k)}{k^2} + \frac{2}{3\omega A} \frac{(2E)^{3/2}}{k^2} - \frac{x_t}{3A} \left( 6E - \omega^2 x_t^2 \right). \quad (15)
\]

In terms of the scaled variables introduced in equation (10) this becomes
\[
\frac{3A\omega}{2N^{3/2}} = (2\mathcal{E})^{3/2} \left\{ 1 + \frac{3}{4\mathcal{E}} \frac{4K(k) - E(k)}{\pi k^2 \sqrt{z(z - 1)}} - g(z) \right\} + \frac{g(z)^3}{16}\mathcal{E} \quad (16)
\]

Since \( k^2 = 1/(z - 1) \) and \( z \) is a function of \( \mathcal{E} \) through equation (13), the right hand side of this equation depends only upon \( \mathcal{E} \). Thus \( \mathcal{E} \) is a function only of the variable \( \omega A N^{-3/2} \), which is the scaling law (12).

This analysis can be carried further with more approximations, but first we show the graph of the ratio
\[
R(\mathcal{E}) = \frac{3A\omega}{2N^{3/2}} \frac{1}{(2\mathcal{E})^{3/2}} \quad (17)
\]

which is seen from equation (16), and the fact that \( z \to 2 \), tends to unity as \( \mathcal{E} \to \infty \).
Expanding equation 16 in powers of $1/z$ gives

$$
\frac{3A\omega}{2N^{3/2}} = (2\mathcal{E})^{3/2} \left\{ 1 - \frac{3}{4\mathcal{E}} \left( 1 - \frac{1}{4z} + \cdots \right) + \frac{1}{16\mathcal{E}^3} \left( 1 + \frac{9}{4z} + \cdots \right) \right\}. \tag{18}
$$

An analysis of $R(\mathcal{E})$ suggests that $1 - R(\mathcal{E}) \sim \mathcal{E}^{-1}$ for large $\mathcal{E}$, that in this range $z \simeq \frac{1}{2}$ and that $z$ changes relatively slowly with $\mathcal{E}$. Thus a simple approximation to this ratio is given by setting $z$ equal to its asymptotic value, $z = 2$, to give

$$
R(\mathcal{E}) \simeq R_1(\mathcal{E}) = 1 - \frac{21}{32\mathcal{E}}.
$$

The graph of $100(R(\mathcal{E}) - R_1(\mathcal{E}))$ is shown in figure 4 and this demonstrates the accuracy of this simple approximation.

On using $R_1$ to approximate $R(\mathcal{E})$ in equation 16 and rearranging the equation we obtain

$$
E_n(A) = \frac{1}{2} \left( \frac{3A\omega}{2} \right)^{2/3} + \frac{7\pi}{32} \omega n + \text{higher order terms}. \tag{19}
$$

The first term in this equation is just the Thomas Fermi approximation, which follows from the normalisation condition, equation 14, by setting $x_t = 0$. The second term increases linearly with $n$ and, because the trap potential quadratic, is independent of $A$. Higher-order corrections come from the expansion about the asymptotic value of $z$ and are complicated and not warranted because of other approximations made.

The scaling law 12 exists because the trap potential is homogeneous in $x$, so the adiabatic condition 9 may be expressed in terms of only two variables. For the quadratic potential these are $\mathcal{E} = \frac{2E}{\pi \omega n} = \frac{E}{N}$ and $z = \frac{2E}{A\omega}$ and it is the form of these variables that gives the scaling law 12 and ultimately the energy level 19. If the trap potential is $(\omega x)^p/2p$, $p \geq 1$, the scaled energy may be taken to be $\mathcal{E} = 2EN^{-2p/(2p+1)}$ and then the scaling law 12 becomes

$$
E = N^{2p/(2p+1)} H \left( \frac{A\omega}{N^{2p/(2p+1)}} \right)
$$

and the energy levels become

$$
E_n(A) = \frac{1}{2p} \left( \left( p + \frac{1}{2} \right) A\omega \right)^{2p/(2p+1)} + \frac{7\pi \omega n}{32\sqrt{p}} \left( \left( p + \frac{1}{2} \right) A\omega \right)^{2p/(2p+1)}. \tag{20}
$$

When $p = 1$ this reduces to equation 19, but when $p \neq 1$ the coefficient of $n$ depends upon the nonlinearity, $A$. 

![Figure 4: Graphs of the ratio $R(\mathcal{E})$, equation 17, and the difference 100($R(\mathcal{E}) - R_1(\mathcal{E})$).]
4 Variational method

Yukalov et al (1997) have used renormalised perturbation theory to obtain analytic approximations to the energy levels of the 3d nonlinear Schrödinger equation. Here we show that this method is equivalent to an Euler-Lagrange variational method and that the resulting energy levels of the excited states do not satisfy the scaling law described in equation 19. Thus this method cannot be as accurate as implied by Yukalov et al (1997).

With the Lagrangian

\[ L(y, y', x) = \frac{1}{2} \left( \frac{dy}{dx} \right)^2 + \frac{1}{2} \omega^2 x^2 y^2 + \frac{1}{2} Ay^4 \]  

(21)

and treating the energy as the Lagrange multiplier we see that the Euler-Lagrange equations with the functional and the constraint

\[ J[y] = \int_{-\infty}^{\infty} dx \left[ L(y, y', x) - Ey^2 \right] + \int_{-\infty}^{\infty} dx y(x)^2 = 1, \]

(22)

gives equation 1, with \( \mu = \bar{\hbar} = 1 \), and that the energy is then given by

\[ E = \int_{-\infty}^{\infty} dx \left[ \frac{1}{2} \left( \frac{dz}{dx} \right)^2 + \frac{1}{2} \omega^2 x^2 z^2 + A z^4 \right] \]

(23)

where \( z(x) \) is a solution of the Euler-Lagrange equation. For trial functions satisfying the normalisation condition we may use the simpler functional

\[ J[y] = \int_{-\infty}^{\infty} dx L(y, y', x). \]

(24)

A natural trial function is

\[ z(x) = \sqrt{\frac{\hbar^2}{2n}} H_n(ax) \exp \left( -\frac{1}{2} a^2 x^2 \right), \quad \hbar^2 = 2^n n! \sqrt{\pi} \]

(25)

where \( a \) is the variational parameter. Then the functional 23 becomes

\[ J(a) = \frac{1}{2} \left( n + \frac{1}{2} \right) \left( a^2 + \frac{\omega^2}{a^2} \right) + \frac{aA}{2h_n^2} I_n, \quad I_n = \int_{-\infty}^{\infty} dw H_n(w)^4 e^{-2w^2}. \]

(26)

This is stationary so the appropriate value of \( a \) is given by the positive root of

\[ \frac{\omega^2}{a^4} = a + \frac{AI_n}{(2n+1)h_n^2} \]

and then

\[ E_n = \frac{1}{2} \left( n + \frac{1}{2} \right) \left( a^2 + \frac{\omega^2}{a^2} \right) + \frac{aA}{h_n^2} I_n. \]

(27)

If \( A = 0 \) these equations give the unperturbed energy levels and if \( A \) is small perturbation theory may be used to obtain the equivalent of Yukalov et al (1997), equation 44. For \( A \gg 1 \) and \( n = 0 \) they give \( E_0 = 0.677(\omega A)^{2/3} \) which is 3.4% larger than the Thomas-Fermi energy, given by the first term in equation 19. In this limit of large \( A \) perturbation theory may be used to give

\[ E_n = \frac{5}{4}(2n+1)\omega^2 B^{2/3} \left( 1 + \frac{\epsilon}{15} + \frac{\epsilon^2}{15} + \cdots \right), \quad B = \frac{AI_n}{(2n+1)\omega^2 h_n^2}, \quad \epsilon = \frac{1}{\omega^2 B^{4/3}}. \]
It is also clear from equations 26 that \( E/N \) depends only upon the variable \( z = AI_n/(2n + 1)\hbar^2/\sqrt{\omega} \), which is different from the scaling law derived in the previous section.

5 Numerical results

In this section we compare the behaviour of the energy levels of equation 1, computed numerically, with the predictions of the above formula, equations 19 and 26.

One method of numerically solving equation 1 is to perform a two-dimensional search in the \((a, E)\) plane, where \( E \) is the energy and for even solutions \( y(0) = a > 0 \) and for odd solutions \( y'(0) = a > 0 \). These solutions must a) satisfy the quantisation condition, b) tend to zero as \( x \to \infty \) and c) satisfy the normalisation condition. Since most solutions are unbounded this calculation is expedited by using a good first approximation, which is given by

\[
\tilde{y}(x) = \begin{cases} 
(a + xy_m(x_t)/x_t) \cos \Omega x, & 0 \leq x \leq x_t \\
y_m(x), & x_t \leq x \leq x_0 \\
0, & x > x_0 = \sqrt{2E}
\end{cases}
\]

where \( y_m(x) \) is the Thomas-Fermi solution defined in equation 11 and \( \Omega = 2\pi/T \) where \( T \) is the period defined in equation 6. In practice the harmonic balance approximation \( \Omega^2 = 2E - 2a^2A/2 \) was used for \( \Omega \). The oscillatory part of this approximation has a slowly increasing amplitude in order that \( \tilde{y}(x) \) is continuous at \( x = x_t \).

This approximation has two free parameters, \( a \) and \( E \), which were varied using the Marquardt algorithm to find values that simultaneously satisfied the normalisation condition and the quantisation condition. For \( A = 200 \) this crude approximation gives a relative error of less than 1% for the ground state and 5% for the 16th energy level.

In the second stage of the calculation we use the energy \( E \) found above and vary \( a \) to find a value at which \(|y(x_f)| < \delta \), for some small \( \delta \) and where \( x_f = 1.25x_0 \). This was achieved using a shooting algorithm that varied \( a \) according to the value of \( y(x_f) \). The solution obtained in this manner is not normalised, but we find that for small changes in \( E \), \( \int_0^{x_f} dx y(x)^2 \) depends approximately linearly on \( E \) so it is possible to interpolate the energy to obtain values of \((a, E)\) that give a correctly normalised solutions.

In the following table are shown energy levels for \( A = 100 \) and 200. The exact numerical values are well approximated by the straight lines \( E_n \approx 14.04 + 0.66n \) and \( E_n \approx 22.40 + 0.74n \), for \( A = 100 \) and 200 respectively, and the gradient of these lines is close to that predicted by equation 13. The energy levels of the variational method do not behave in this manner, particularly for large \( A \), and we conclude that the excited energy levels given by the re-normalised perturbation method used by Yukalov et al (1997) is not accurate for the one-dimensional nonlinear Schrödinger equation.
\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
   & \multicolumn{4}{|c|}{$A = 100$} \\
\hline
   $n$ & \multicolumn{4}{|c|}{Numerical} \\
\hline
$E_n$ (equation 19) & 14.02 & 15.37 & 16.69 & 17.98 \\
$E_n$ (equation 26) & 14.60 & 18.70 & 20.34 & 21.69 \\
\hline
   & \multicolumn{4}{|c|}{$A = 200$} \\
\hline
   $n$ & \multicolumn{4}{|c|}{Numerical} \\
\hline
$E_n$ (equation 19) & 22.42 & 23.87 & 25.34 & 26.86 \\
$E_n$ (equation 26) & 22.41 & 23.77 & 25.13 & 26.49 \\
\hline
\end{tabular}
\end{table}

6 Conclusions

We have shown that the energy levels $E_n$ of the Gross-Pitaevskii equation \cite{1} satisfy the approximate scaling law \cite{12}, which relates the variables $E$, $n$, $\omega$, $A$ in a single equation, which leads to the approximate energy levels \cite{2}. We have shown that other homogeneous trap potentials lead to similar scaling laws but only the energy levels of the quadratic trap have a coefficient of $n$ that is independent of the nonlinear constant, see equation \cite{20}. It is also shown that the energy levels of the renormalised perturbation method of Yukalov et al (1997) are equivalent to a simple variational method and do not satisfy the scaling law derived here.

The method used to derive these results involves interpreting the Gross-Pitaevskii equation as a mechanical system with a slowly varying potential, so that the idea of adiabatic invariance can be used. With this equivalence the spatial coordinate becomes the time, so the generalisation to the 2d- or 3d Gross-Pitaevskii equation is not apparent. For symmetric, many dimensional systems, however a similar approach may be possible though there are some problems with singularities at the origin that need to be resolved.

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