Anomalous Proximity Effect and Theoretical Design for its Realization

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We discuss the stability of zero-energy states appearing in a dirty normal metal attached to a superconducting thin film with Dresselhaus [110] spin-orbit coupling under the in-plane Zeeman field. The Dresselhaus superconductor preserves an additional chiral symmetry and traps more than one zero-energy state at its edges. All the zero-energy states at an edge belong to the same chirality in large Zeeman field due to the effective p-wave pairing symmetry. The pure chiral nature in the wave function enables the penetration of the zero-energy states into the dirty normal metal with keeping their high degree of degeneracy. By applying a theorem, we prove the the perfect Andreev reflection into the dirty normal metal at the zero-energy. This paper gives a microscopic understanding of the anomalous proximity effect.

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I. INTRODUCTION

The proximity effect has been an important issue in physics of superconductivity. In a normal metal attached to a metallic superconductor, penetrating Cooper pairs form the gap structure in the quasiparticle density of states (DOS) at the fermi level (zero-energy) and modify the low energy properties there. In the spin-triplet superconductor junctions, however, the penetrating Cooper pairs form a zero-energy peak in DOS. This brings various anomalous electromagnetic properties in the normal metal. The such effect is called anomalous proximity effect. For instance, the perfect Andreev reflection from a p-wave superconductor into a dirty normal metal causes anomalous low energy transport in the x-direction such as the zero-bias conductance quantization in normal-metal/superconductor (NS) junctions and the fractional current-phase relationship in superconductor/normal-metal/superconductor (SNS) junctions. Recently, these characteristic transport phenomena have been investigated as a part of the Majorana physics based on the topological classification. In fact, using dimensional reduction, i.e., by fixing the wave vector in the transverse direction (say kₚ), the spin-triplet pₓ-wave superconductivity is topologically characterized by one-dimensional winging number. The number of the zero-energy states (ZESs) at an edge is equal to the number of propagating channel Nₓ. As a consequence, the dispersion of edge states becomes flat as a function of kₚ. The anomalous proximity effect is originated from the penetration of the ZESs into the dirty normal metal with keeping their high degree of degeneracy. Theoretically, it has been unclear what symmetry protects the high degeneracy of ZESs and why the perfect Andreev reflection persists at the zero-energy. Although it is difficult to fabricate spin-triplet superconducting junctions using existing materials, the rapid progress in Majorana physics on artificial superconductors and that in spintronics for controlling the spin-orbit interaction may diffuse the situation.

To have topologically nontrivial artificial superconductors, a set of three potentials is necessary: the spin-orbit coupling, the Zeeman field and the pair potential. Among them, the spin-orbit interaction mainly affects the spectra of the edge states. In InSb or GaAs, for example, the Dresselhaus [110] spin-orbit interaction is large on their films growing along the [110] direction. Theoretical studies have shown that such artificial superconductor hosts the ZESs with flat dispersion similar to those of the pₓ-wave superconductor. We also confirm that a proximitized spin helix thin film also traps the flat ZESs under appropriate tuning of Zeeman field. The Dresselhaus superconductors may be classified into BDI symmetry class in a sense that the chiral symmetry and the particle-hole one can be defined independently. Recent theoretical studies have shown that the chiral symmetry is responsible for the stability of more than one Majorana fermion at the edge of the BDI superconductor. On the basis of the novel insight, we solve an outstanding problem of the anomalous proximity effect.

In this paper, we first demonstrate that the Dresselhaus superconductors indicate the anomalous proximity effect in large Zeeman field. After showing the unitary equivalence between the Hamiltonian of the Dresselhaus superconductor and that of spin-triplet pₓ-wave one, we analyze the chiral property of ZESs both at the edge of the superconductor and at the normal metal attached to it. The analysis shows that all the ZESs in the normal metal belong to the same chirality due to the pₓ-wave pairing symmetry. We prove that the pure chiral nature of the wave function protects the high degeneracy at the zero-energy and causes the perfect Andreev reflection into the dirty normal metal. This paper provides a microscopic understanding of the anomalous proximity effect and a design of an artificial pₓ-wave superconductor.
II. ANOMALOUS PROXIMITY EFFECT

At first, we numerically demonstrate the anomalous proximity effect of the Dresselhaus superconductor. Let us consider a NS junction on the two-dimensional tight-binding model as shown in Fig. 1. A lattice site is pointed by a vector \( \mathbf{r} = jx + my \), where \( x \) and \( y \) are the unit vectors in the \( x \) and the \( y \) direction, respectively. In the \( y \) direction, the number of the lattice site is \( M \) and the hard-wall boundary condition is applied. The present junction consists of three segments: an ideal lead wire (\( \infty \leq j \leq 0 \)), a normal disordered segment (\( 1 \leq j \leq L \)) and a superconducting segment (\( L + 1 \leq j \leq \infty \)). The Hamiltonian reads,

\[
\hat{H}_0 = -t \sum_{(r,r')\sigma} \left( c_{r,\sigma}^\dagger c_{r',\sigma} + c_{r',\sigma}^\dagger c_{r,\sigma} \right) + \sum_{r,\sigma} \left[ 4t - \mu + V_{\text{imp}}(r) \right] c_{r,\sigma}^\dagger c_{r,\sigma} - \sum_{r,\sigma,\sigma'} V_{\text{ex}}(\sigma_1)_{\sigma,\sigma'} c_{r,\sigma}^\dagger c_{r,\sigma'} + \sum_r \Delta_0 \left( c_{r,\uparrow}^\dagger c_{r,\downarrow}^\dagger + H.c. \right) - \frac{i\lambda_D}{2} \sum_{r,\sigma,\sigma'} (\sigma_3)_{\sigma,\sigma'} \left( c_{r+x,\sigma}^\dagger c_{r,\sigma'} - c_{r,\sigma}^\dagger c_{r+x,\sigma'} \right),
\]

where \( c_{r,\sigma}^\dagger (c_{r,\sigma}) \) is the creation (annihilation) operator of an electron at the site \( r \) with spin \( \sigma = (\uparrow \text{ or } \downarrow) \), \( t \) denotes the hopping integral among the nearest neighbor sites denoted by \( (r , r') \), \( \mu \) is the chemical potential, and \( \lambda_D \) represents the strength of the Dresselhaus [110] spin-orbit interaction. We consider the impurity potential given randomly in the range of \(-W/2 \leq V_{\text{imp}}(r) \leq W/2\) in the normal segment and the \( s \)-wave pair potential \( \Delta_0 \) in the superconducting segment. The Pauli’s matrices in spin space are represented by \( \hat{\sigma}_j \) for \( j = 1 \sim 3 \) and the unit matrix in spin space is \( \hat{\sigma}_0 \). By tuning the magnetic field \( B \) in the \( x \) direction, it is possible to introduce the external Zeeman potential \( V_{\text{ex}} \). We calculate the differential conductance \( G_{\text{NS}} \) of the NS junctions based on a formula \( ^{26} \)

\[
G_{\text{NS}}(eV) = \frac{e^2}{h} \sum_{\zeta,\eta} \left[ \delta_{\zeta,\eta} - \left| r_{\zeta,\eta}^{ee} \right|^2 + \left| r_{\zeta,\eta}^{he} \right|^2 \right]_{eV = E},
\]

where \( r_{\zeta,\eta}^{ee} \) and \( r_{\zeta,\eta}^{he} \) denote the normal and Andreev reflection coefficients at the energy \( E \), respectively. The indices \( \zeta \) and \( \eta \) label the outgoing channel and the incoming one, respectively. These reflection coefficients are calculated by using the lattice Green’s function method \( ^{27,28} \).

In Fig. 2 we present the differential conductance of the Dresselhaus superconductors as a function of the bias voltage for several choices of the length of the disordered segments \( L \), where we choose parameters as \( \mu = 1.0t \), \( \lambda_D = 0.2t \), \( W = 2.0t \), \( M = 10 \) and \( \Delta_0 = 0.1t \). The results are the normalized to \( G_0 = 2e^2/h \). In Fig. 2(a), we choose \( V_{\text{ex}} = 1.2t \) leading to the number of propagating channel \( N_c = 5 \). The differential conductance decreases with increasing \( L \) for the finite bias voltage. However, the zero-bias conductance is quantized at \( G_0 N_c \) irrespective of \( L \). We have confirmed that the zero-bias conductance is always quantized at \( G_0 N_c \) even when we change the wire width \( M \). The results suggest that the perfect transmission channels exist in the disordered normal segment \( ^{4} \) and their number is equal to \( N_c \). The conductance quantization at the zero-bias is an aspect of the anomalous proximity effect. We have also confirmed the fractional current-phase relationship in SNS junctions \( ^{3,29} \). Such anomalous behavior can be seen when the Zeeman field is larger than a critical value \( V_{\text{ex}} > V_c \) with \( V_c = 0.92t \) at the present parameter choice. For \( V_{\text{ex}} < V_c \), on the other hand, the conductance quantization in absent as shown in Fig. 2(b), where we choose \( V_{\text{ex}} = 0.5t < V_c \).

III. MORE THAN ONE MAJORANA FERMION IN NORMAL METAL

A. Chiral Symmetry

In what follows, we consider the Dresselhaus superconductor in the continuous space for simplicity. The BdG Hamiltonian is represented by

\[
\hat{H}_0 = \begin{pmatrix}
\hat{h} & i\Delta_0 \hat{\sigma}_2 \\
-i\Delta_0 \hat{\sigma}_2 & -\hat{h}^*
\end{pmatrix},
\]

FIG. 1. (Color online) Schematic picture of a NS junction of Dresselhaus superconductor. The superconductor proximitizes the semiconductor thin film growing along the [110] \( \parallel \) \( z \) direction.

FIG. 2. (Color online) The differential conductance is plotted as a function of the bias voltage for several choices of the length of disordered segment \( L \). In (a), the Zeeman potential \( V_{\text{ex}} = 1.2t \) is chosen to be larger than a critical value of \( V_c = 0.92t \). The number of propagating channels \( N_c \) is 5. In (b), we choose \( V_{\text{ex}} = 0.5t < V_c \) leading to \( N_c = 6 \).
\[ \hat{h} = \xi_v \sigma_0 - V_{xx} \sigma_1 + i \lambda_D \partial_x \sigma_3 + V_{\text{imp}}, \]

where \( \xi_v = \frac{\hbar^2}{2m} \nabla^2 - \mu \), and \( m \) denotes the effective mass of an electron. The pair potential \( \Delta_0 \) and the impurity potential \( V_{\text{imp}} \) are introduced in the superconducting segment and in the normal one, respectively. We assume large enough Zeeman potential so that \( \alpha_D = \lambda_D k_F / V_{xx} \ll 1 \) is satisfied with \( k_F = \sqrt{2m \mu_f / \hbar^2} \). By applying the unitary transformations as shown in Appendix B, \( H_0 \) is transformed into \( \hat{H}_1 = \hat{H}_P + \hat{V}_\Delta \) within the first order of \( \alpha_D \), where

\[ \hat{H}_P = \begin{bmatrix} \hat{H}_1 & 0 \\ 0 & \hat{H}_1 \end{bmatrix}, \quad \hat{V}_\Delta = \begin{bmatrix} 0 & i \Delta_0 \sigma_2 \\ -i \Delta_0 \sigma_2 & 0 \end{bmatrix}, \]

and \( s_x = 1 \) (-1) for \( \sigma = \uparrow \) (\( \sigma = \downarrow \)). The Hamiltonian \( \hat{H}_P \) is equivalent to that of the spin-triplet \( \psi_x \)-wave superconductor and \( \hat{V}_\Delta \) mixes the two spin sectors. In the anomalous phase \( V_{xx} > V_c \), all the spin-\( \uparrow \) states pinch off from the Fermi level and only the spin-\( \downarrow \) states remain at the Fermi level. Therefore the spin-mixing term \( \hat{V}_\Delta \) does not affect the remaining spin-\( \downarrow \) states at all. In this way, we can shrink down the 4 \times 4 Hamiltonian \( \hat{H}_1 \) to the 2 \times 2 Hamiltonian \( \hat{H}_1 \). The 2 \times 2 BdG Hamiltonian preserves a chiral symmetry

\[ \hat{\tau}_1 \hat{H}_1 \hat{\tau}_1 = -\hat{H}_1, \quad \hat{\tau}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \]

with \( \hat{\tau}_j \) for \( j = 1 - 3 \) are the Pauli matrices in Nambu space. Here we summarize two important features of the eigen states of \( \hat{H}_1 \) proved in Ref. [10]. (See also Appendix A for details.)

(i) The eigen states of \( \hat{H}_1 \) at the zero-energy are the eigen states of \( \hat{\tau}_1 \) at the same time. Namely, the even vectors at the zero energy \( \varphi_{v_{\lambda}}(r) \) satisfies

\[ \hat{H}_1 \varphi_{v_0,\lambda}(r) = 0, \quad \hat{\tau}_1 \varphi_{v_0,\lambda}(r) = \lambda \varphi_{v_0,\lambda}(r), \]

where \( \lambda = \pm 1 \) represents the eigen value of \( \hat{\tau}_1 \). We have omitted the spin index from the subscripts of \( \varphi_{v_0,\lambda} \) because spin is always \( \downarrow \).

(ii) In contrast to the zero-energy states, the nonzero-energy states are not the eigen states of \( \hat{\tau}_1 \). They are described by the linear combination of the two states: one belongs to \( \lambda = 1 \) and the other belongs to \( \lambda = -1 \). We prove the robustness of the highly degenerate ZESs in the dirty normal segment and the perfect Andreev reflection by taking these features into account.

Next we analyze the edge states of an isolating Dresselhaus superconductor. From the second equation in Eq. (8), we can describe the wave function of the zero-energy states

\[ \varphi_{v_0,\lambda}(r) = \sum_n Y_n(y) \varphi_{n,\lambda}(x) \begin{bmatrix} 1 \\ \lambda \end{bmatrix}, \]

where \( Y_n(y) = \sqrt{2/M} \sin(n \pi y / M) \) is the wave function in the \( y \) direction and \( n \) indicates the transmission channel. In the \( x \) direction, we assume that the length of the superconductor is \( 2L \) (i.e., \( -L \leq x \leq L \)) and apply the hard-wall boundary conditions at its edges, \( \varphi_{n,\lambda}(-L) = \varphi_{n,\lambda}(L) = 0 \). By substituting Eq. (9) into the first equation in Eq. (8), we obtain

\[ \left[ \partial_x^2 + 2 \frac{2 \lambda_D}{\xi \Delta} \partial_x + k_n^2 \right] \varphi_{n,\lambda}(x) = 0, \]

where \( \xi_D = \xi_0 / \alpha_D, \xi_0 = \hbar v_F / \Delta_0, \lambda_n = \sqrt{2m(\mu_n + V_{xx}) / \hbar^2}, \) and \( \mu_n = \mu - (\hbar n \pi / M)^2 / (2m) \). The length of the superconductor must be long enough so that \( L / \xi_D \gg 1 \) is satisfied. For \( V_{xx} > V_c \), we find two solutions as

\[ \varphi_{n,-}(x) = C_L \begin{bmatrix} 1 \\ -1 \end{bmatrix} \sin[q_n(x + L)] e^{-x / \xi_D}, \]

\[ \varphi_{n,+}(x) = C_R \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sin[q_n(x - L)] e^{x / \xi_D}, \]

with \( q_n^2 = k_n^2 - \xi_D^2 \), where \( C_L \) and \( C_R \) are the normalization coefficients. It is easy to confirm that \( \varphi_{n,-}(x) \) localizing at the left edge belongs to \( \lambda = -1 \) and \( \varphi_{n,+}(x) \) localizing at the right edge belongs to \( \lambda = 1 \) as schematically illustrated in Fig. [3].

The field operator of an electron with spin-\( \downarrow \) is described as

\[ \Psi(r) = \sum_{\nu} \left[ \varphi_{\nu}(r) \gamma_\nu + \Xi \varphi_{\nu}(r) \gamma_\nu^\dagger \right], \]

\[ \Xi = \hat{\tau}_1 K, \quad \Xi H_1 \Xi^{-1} = -H_1, \]

where \( \gamma_\nu^\dagger \) (\( \gamma_\nu \)) is the creation (annihilation) operator of the Bogoliubov quasiparticle belonging to \( e_\nu \) and \( \Xi \) is the charge conjugation operator with \( K \) indicating the complex conjugation. Eq. (14) represents the particle-hole symmetry of the BdG Hamiltonian. The wave function of the zero-energy states are described by

\[ \varphi_{n,-}(r) = \varphi_{n,-}(x) Y_n(y), \quad \varphi_{n,+}(r) = \varphi_{n,+}(x) Y_n(y), \]

with Eqs. (11) and (12). We can extract the electron field operator of a ZES for each propagating channel \( n \) as

\[ \Psi_n(r) = i \gamma_{n,-}(r) + \gamma_{n,+}(r), \]

\[ \gamma_{n,-}(r) = -i \left[ \varphi_{n,-}(r) \gamma_{n,-} - (\varphi_{n,-}(r))^\dagger \gamma_{n,+} \right], \]

\[ \gamma_{n,+}(r) = \left[ \varphi_{n,+}(r) \gamma_{n,+} + (\varphi_{n,+}(r))^\dagger \gamma_{n,-}^\dagger \right]. \]

The operator \( \gamma_{n,-}(r) \) is pure imaginary while \( \gamma_{n,+}(r) \) is real in the present gauge choice. It is easy to show that they satisfy the Majorana relation \( \gamma_{n,L,R}(r)^\dagger = \gamma_{n,L,R}(r) \). This relation holds for all propagating channel \( n \). Therefore the number of Majorana fermions at each edge is equal to the number of propagating channel at the spin-\( \downarrow \) sector \( N_1 \). Since the spin-\( \uparrow \) channels are absent for
The spin-triplet $k$ with two different chirality coexist at the same edge are fragile under the potential disorder because the ZESs chiral symmetry. However, the highly degenerate ZESs Hamiltonian of these superconductors also preserve the sufficient condition for the anomalous proximity effect. Therefore the presence of the chiral symmetry is not a necessary one. But only a necessary one.

In the ballistic limit, the perfect conductance quantization at the zero-bias is a common property of unconventional superconductors hosting ZESs with flat dispersion. For instance, the spin-singlet $d_{xy}$-wave and the spin-triplet $f$-wave with the pair potential proportional to $k_x(1-2k_y^2)$ also show such drastic effect. The Hamiltonian of these superconductors also preserve the chiral symmetry. However, the highly degenerate ZESs are fragile under the potential disorder because the ZESs with two different chirality coexist at the same edge. Therefore the presence of the chiral symmetry is not a sufficient condition for the anomalous proximity effect but only a necessary one.

**B. Perfect Andreev Reflection**

Finally and most importantly, we prove the stability of the highly degenerate ZESs in the dirty normal segment which is attached to the left edge of the superconductor as shown in Fig. 1(a). In the absence of impurity potentials, the wave function in the normal segment at $E = 0$ is described by

$$\varphi_N(r) = \sum_n \begin{bmatrix} 1 \\ r_n^{ee} \\ r_n^{he} \end{bmatrix} e^{ik_nx} + \begin{bmatrix} r_n^{ee} \\ 0 \\ r_n^{he} \end{bmatrix} e^{-ik_nx} Y_n(y),$$

(19)

where $r_n^{ee}$ ($r_n^{he}$) is the normal (Andreev) reflection coefficient at the channel $n$. The current conservation law implies $|r_n^{ee}|^2 + |r_n^{he}|^2 = 1$ at $E = 0$ for each channel. From the boundary conditions at the NS interface, the reflection coefficients are calculated to be

$$r_n^{ee} = 0, \quad r_n^{he} = -1,$$

(20)

for all $n$. The wave function in Eq. (19) is turn out to be the eigen state of $\hat{\tau}_1$ belonging to $\lambda = -1$, (i.e., $\varphi_N \propto [1, -1]^T$). This fact is unique to the $p_x$-wave pairing symmetry. For $d_{xy}$ and $f$-wave cases, the ZESs of two different chirality coexist in the normal metal. In the present junction, all the ZESs in the normal metal have the same chirality of $\lambda = -1$ as well as the ZESs at the left edge of the superconductor. According to the property (ii), they cannot form the nonzero-energy states. Therefore, the ZESs can penetrate into the normal segment with keeping their high degree of degeneracy. This conclusion is also valid under the potential disorder because the impurity potential preserves the chiral symmetry and does not damage the pure chiral feature of the ZESs. In addition, it is possible to show that the pure chiral feature of the ZESs protects the perfect Andreev reflection into the dirty normal segment. According to the property (i), the ZESs must be the eigen state of $\hat{\tau}_1$. We emphasize that the wave function in Eq. (19) can be the eigen state of $\hat{\tau}_1$ belonging to $\lambda = -1$ when and only when Eq. (20) is satisfied. Although the channel index $n$ is no longer a good quantum number under the potential disorder, all the wave functions in the normal segment have the same vector structure reflecting the pure chiral nature. This is the mathematical requirement from the chiral symmetry. Physical consequence of the vector structure is the perfect Andreev reflection in the disordered junction at $E = 0$. This explains the perfect quantization of the zero-bias conductance at $2e^2/\hbar$.

**IV. CONCLUSION**

In conclusion, we have discussed the stability of highly degenerate zero-energy states (ZESs) appearing in disordered junctions consisting of a superconducting thin film with Dresselhaus [110] spin-orbit coupling. The Dresselhaus superconductor hosts more than one ZES at its edges. When we make a normal-metal/superconductor junction of the Dresselhaus superconductor, such highly degenerate ZESs can penetrate in to the dirty normal segment and form the resonant transmission channels there. The analysis of the wave function in the normal segment shows that all the ZESs have the same chirality due to the effective $p_x$-wave pairing symmetry. The perfect Andreev reflection into the dirty normal metal is a direct consequence of the the pure chiral feature of the ZESs. Our paper provides a microscopic understanding the anomalous proximity effect of spin-triplet superconductors.

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Appendix A: Zero energy states under a chiral symmetry

Here, we briefly summarize the argument in Ref. 10 which shows the important properties of zero-energy states under a chiral symmetry. We consider the BdG Hamiltonian $H$ which preserves the chiral symmetry

$$\Gamma H \Gamma^{-1} = -H, \quad \Gamma^2 = 1.$$  \hspace{1cm} (A1)

The relation is equivalent to

$$[H^2, \Gamma] = 0.$$  \hspace{1cm} (A2)

The BdG equation is given by

$$H \varphi_E(r) = E \varphi_E(r).$$  \hspace{1cm} (A3)

When we consider the eigen equation of $H^2$,

$$H^2 \chi_{E^2}(r) = E^2 \chi_{E^2}(r),$$  \hspace{1cm} (A4)

Eq. (A2) suggests that the eigen state $\chi_{E^2}(r)$ is also the eigen state of $\Gamma$ at the same time. Since $\Gamma^2 = 1$, we find that the eigen value of $\Gamma$ is $+1$ or $-1$. Namely the eigen equation

$$\Gamma \chi_{E^2}(r) = \lambda \chi_{E^2}(r),$$  \hspace{1cm} (A5)

holds for $\lambda = \pm 1$. By multiplying $H$ to Eq. (A5) from the left side and by using Eq. (A1), we obtain the equation

$$\Gamma H \chi_{E^2}(r) = -\lambda H \chi_{E^2}(r).$$  \hspace{1cm} (A6)

We find that $H \chi_{E^2}(r)$ is the eigen state of $\Gamma$ belonging to $-\lambda$. Thus we can connect $\chi_{E^2+}(r)$ and $\chi_{E^2-}(r)$ as

$$H \chi_{E^2\lambda}(r) = c_{E^2\lambda} \chi_{E^2-\lambda}(r),$$  \hspace{1cm} (A7)

where $c_{E^2\lambda}$ is a constant.

The one-to-one correspondence exists between $\varphi_E(r)$ and $\chi_{E^2}(r)$. At first, we consider zero-energy states $\chi_{0\lambda}(r)$ which satisfies

$$H^2 \chi_{0\lambda}(r) = 0,$$  \hspace{1cm} (A8)

in Eq. (A4). The integration of $r$ after multiplying $\chi_{0\lambda}^\dagger(r)$ from the left results in

$$\int dr |H \chi_{0\lambda}(r)|^2 = 0.$$  \hspace{1cm} (A9)

This means that the norm of $H \chi_{0\lambda}(r)$ is zero. Therefore we conclude that

$$H \chi_{0\lambda}(r) = 0.$$  \hspace{1cm} (A10)

As a result, we find the relation

$$\varphi_{0\lambda}(r) = \chi_{0\lambda}(r).$$  \hspace{1cm} (A11)

When a zero energy state is described by $\varphi_{0\lambda}(r) = \chi_{0+}(r)$, the relations in Eqs. (A7) and (A10) suggest that $\chi_{0-}(r) = 0$. Therefore the zero-energy states are always the eigen states of $\Gamma$.

For $E \neq 0$, it is possible to represent $\varphi_E(r)$ by $\chi_{E^2\pm}(r)$. By calculating the norm of $H \chi_{E^2\lambda}(r)$, we obtain

$$E^2 = |c_{E^2\lambda}|^2.$$  \hspace{1cm} (A12)

Multiplying $H$ to Eq. (A7) from the left alternatively gives a relation

$$c_{E^2\lambda} c_{E^2-\lambda} = 1.$$  \hspace{1cm} (A13)

Therefore, we find the relation

$$H \chi_{E^2\lambda}(r) = E e^{i\lambda \theta_{E^2}} \chi_{E^2-\lambda}(r).$$  \hspace{1cm} (A14)

Although we cannot fix the phase factor $\theta_{E^2}$, it is possible to express the states $\varphi_E(r)$ for $E \neq 0$ as

$$\varphi_E(r) = \frac{1}{\sqrt{2}} \left( e^{-i\theta_{E^2}/2} \chi_{E^2+}(r) + e^{i\theta_{E^2}/2} \chi_{E^2-}(r) \right).$$  \hspace{1cm} (A15)

$$s_E = \begin{cases} 1 & \text{for } E > 0 \\ -1 & \text{for } E < 0. \end{cases}$$  \hspace{1cm} (A16)

The nonzero-energy states are constructed by a pair of eigen states of $\Gamma$; one belongs to $\lambda = 1$ and the other belongs $\lambda = -1$. Therefore, the states with $E \neq 0$ are not the eigen states of $\Gamma$.

Appendix B: Unitary Transformation

The BdG Hamiltonian of the Dresselhaus nanowire represented by

$$\hat{H}_0 = \begin{bmatrix} \hat{h} & i\Delta_0 \hat{\sigma}_2 \\ -i\Delta_0 \hat{\sigma}_2 & -\hat{h}^* \end{bmatrix},$$  \hspace{1cm} (B1)

$$\hat{h} = \xi \hat{\sigma}_0 - V_{ee} \hat{\sigma}_1 + i\lambda_D \hat{\sigma}_3,$$  \hspace{1cm} (B2)

is transformed as follows. By using the unitary matrix

$$\hat{R} = \begin{bmatrix} \hat{r} & 0 \\ 0 & \hat{r}^* \end{bmatrix}, \quad \hat{r} = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\pi/4} & -e^{-i\pi/4} \\ e^{i\pi/4} & e^{i\pi/4} \end{bmatrix},$$  \hspace{1cm} (B3)

the BdG Hamiltonian $\hat{H}_0$ is first transformed to

$$\hat{H}' = \hat{R} \hat{H}_0 \hat{R}^\dagger = \begin{bmatrix} \hat{h}' & i\Delta_0 \hat{\sigma}_2 \\ -i\Delta_0 \hat{\sigma}_2 & -\hat{h}' \end{bmatrix},$$  \hspace{1cm} (B4)

$$\hat{h}' = \xi_0 \hat{\sigma}_0 + V_{ee} \hat{\sigma}_3 + i\lambda_D \hat{\sigma}_2,$$  \hspace{1cm} (B5)
The Hamiltonian in this basis is represented only by real numbers. Next we apply a transformation which is similar to the Foldy-Wouthusen transformation to the BdG Hamiltonian in Eq. (B1). Using a unitary matrix

\[ \tilde{U} = \begin{bmatrix} \hat{u} & 0 \\ 0 & \hat{\bar{u}} \end{bmatrix}, \]  

(B6)

where \( \hat{u} = \exp[i\tilde{S}], \tilde{S} = \frac{\lambda_D}{2hV_{ex}}p_x\sigma_1, \)

(B7)

with \( p_x = -i\hbar \partial_x \), we transform \( H' \) into

\[ \tilde{U}H'\tilde{U}^\dagger = \begin{bmatrix} e^{i\tilde{S}}\hat{h}'e^{-i\tilde{S}} & e^{i\tilde{S}}(i\Delta_0\sigma_2)e^{-i\tilde{S}} \\ -e^{i\tilde{S}}(i\Delta_0\sigma_2)e^{-i\tilde{S}} & -e^{i\tilde{S}}\hat{h}'e^{-i\tilde{S}} \end{bmatrix}. \]  

(B8)

The diagonal term of Eq. (B4) can be expanded as

\[ e^{i\tilde{S}}\hat{h}'e^{-i\tilde{S}} = \hat{h}' + i[\tilde{S}, \hat{h}'] + \frac{i^2}{2!}[\tilde{S}, [\tilde{S}, \hat{h}']] + \cdots, \]  

(B9)

with using the Baker-Housdorff formula. We assume large enough Zeeman potential so that \( \alpha_D = \lambda_D/k_FV_{ex} \ll 1 \) is satisfied where \( k_F = \sqrt{2m\rho}/\hbar \) denotes Fermi wave number. From this assumption, we obtain

\[ e^{i\tilde{S}}\hat{h}'e^{-i\tilde{S}} = \xi_0\hat{V}_{ex} + O(\alpha_D^2), \]  

(B10)

within the first order of \( \alpha_D \). The off-diagonal term corresponding to the pair potential is transformed to

\[ e^{i\tilde{S}}(i\Delta_0\sigma_2)e^{-i\tilde{S}} = i\Delta_0\sigma_2 + i[\tilde{S}, i\Delta_0\sigma_2] + \cdots = i\Delta_0\sigma_2 - \frac{\lambda_D\Delta_0}{hV_{ex}}p_x\sigma_3 + O(\alpha_D^2), \]  

(B11)

where we assume the uniform pair potential (i.e., \( [p_x, \Delta_0] = 0 \)). As a result, the BdG Hamiltonian can be written as

\[ \tilde{U}H'\tilde{U}^\dagger = \begin{bmatrix} \xi_{\hat{r}} + V_{ex} & 0 & -\frac{\lambda_D\Delta_0}{hV_{ex}}p_x & \Delta_0 \\ 0 & \xi_{\hat{r}} - V_{ex} & -\Delta_0 & \frac{\lambda_D\Delta_0}{hV_{ex}}p_x \\ i\frac{\lambda_D\Delta_0}{hV_{ex}}p_x & -\Delta_0 & -\xi_{\hat{r}} & 0 \\ -i\frac{\lambda_D\Delta_0}{hV_{ex}}p_x & 0 & 0 & -\xi_{\hat{r}} + V \end{bmatrix} + O(\alpha_D^2). \]  

(B12)

By interchanging the second column and the third one, and by interchanging the second row and the third one, the Hamiltonian can be deformed as

\[ \tilde{H}_1 = \tilde{H}_P + \tilde{V}_{\Delta}, \]  

(B13)

\[ \tilde{H}_P = \begin{bmatrix} \hat{\tilde{H}} & 0 \\ 0 & \hat{\tilde{H}} \end{bmatrix}, \]  

(B14)

\[ \tilde{H}_{\sigma} = \begin{bmatrix} \xi_{\hat{r}} + s_sV_{ex} & -s_s\frac{\lambda_D\Delta_0}{hV_{ex}}p_x \\ s_s\frac{\lambda_D\Delta_0}{hV_{ex}}p_x & -\xi_{\hat{r}} + s_sV_{ex} \end{bmatrix}, \]  

(B15)

\[ \tilde{V}_{\Delta} = \begin{bmatrix} 0 & i\Delta_0\sigma_2 \\ -i\Delta_0\sigma_2 & 0 \end{bmatrix}, \]  

(B16)

\[ s_s = \begin{cases} 1 & \text{for } \sigma = \uparrow \\ -1 & \text{for } \sigma = \downarrow \end{cases}. \]  

(B17)

These are the starting Hamiltonian in the analytical calculation.

We find that \( \tilde{H}_1 \) preserves chiral symmetry

\[ \Gamma\tilde{H}_1\Gamma^{-1} = -\tilde{H}_1, \quad \Gamma = \begin{bmatrix} \hat{\sigma}_1 & 0 \\ 0 & \hat{\sigma}_1 \end{bmatrix}. \]  

(B18)

Finally, we discuss the symmetry property of \( H_0 \) in Eq. (B11) in its original basis. It is easy to show that \( \tilde{H}_0 \) satisfies the relations,

\[ \hat{\Gamma}_0\tilde{H}_0\hat{\Gamma}_0^{-1} = -\tilde{H}_0, \quad \hat{\Gamma}_0 = \begin{bmatrix} 0 & -i\hat{\sigma}_1 \\ i\hat{\sigma}_1 & 0 \end{bmatrix}, \]  

(B19)

which represents the chiral symmetry. The Hamiltonian \( \tilde{H}_0 \) also satisfies,

\[ \tilde{\Xi}_0\tilde{H}_0\tilde{\Xi}_0^{-1} = -\tilde{H}_0, \quad \tilde{\Xi}_0 = \begin{bmatrix} 0 & \hat{K}\hat{\sigma}_0 \\ \hat{K}\hat{\sigma}_0 & 0 \end{bmatrix}, \]  

(B20)

where \( \tilde{\Xi}_0 \) represents the charge conjugation with \( \hat{K} \) meaning the complex conjugation. The first equation in Eq. (B20) represents the particle-hole symmetry.
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