On algebraically integrable outer billiards

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Abstract

We prove that if the outer billiard map around a plane oval is algebraically integrable in a certain non-degenerate sense then the oval is an ellipse.

In this note, an outer billiard table is a compact convex domain in the plane bounded by an oval (closed smooth strictly convex curve) $C$. Pick a point $x$ outside of $C$. There are two tangent lines from $x$ to $C$; choose one of them, say, the right one from the viewpoint of $x$, and reflect $x$ in the tangency point. One obtains a new point, $y$, and the transformation $T : x \mapsto y$ is the outer (a.k.a. dual) billiard map. We refer to [3, 4, 5] for surveys of outer billiards.

If $C$ is an ellipse then the map $T$ possesses a 1-parameter family of invariant curves, the homothetic ellipses; these invariant curves foliate the exterior of $C$. Conjecturally, if an outer neighborhood of an oval $C$ is foliated by the invariant curves of the outer billiard map then $C$ is an ellipse – this is an outer version of the famous Birkhoff conjecture concerning the conventional, inner billiards.

In this note we show that ellipses are rigid in a much more restrictive sense of algebraically integrable outer billiards; see [2] for the case of inner billiards.

We make the following assumptions. Let $f(x, y)$ be a (non-homogeneous) real polynomial such that zero is its non-singular value and $C$ is a component
of the zero level curve. Thus \( f \) is the defining polynomial of the curve \( C \), and if a polynomial vanishes on \( C \) then it is a multiple of \( f \). Assume that a neighborhood of \( C \) is foliated by invariant curves of the outer billiard map \( T \), and this foliation is algebraic in the sense that its leaves are components of the level curves of a real polynomial \( F(x, y) \). Since \( C \) itself is an invariant curve, we assume that \( F(x, y) = 0 \) on \( C \), and that \( dF \) is not identically zero on \( C \). Thus \( F(x, y) = g(x, y)f(x, y) \) where \( g(x, y) \) is a polynomial, not identically zero on \( C \). Under these assumptions, our result is as follows.

**Theorem 1** \( C \) is an ellipse.

**Proof.** Consider the tangent vector field \( v = F_y \partial/\partial x - F_x \partial/\partial y \) (the symplectic gradient) along \( C \). This vector field is non-zero (except, possibly, a finite number of points) and tangent to \( C \). The tangent line to \( C \) at point \((x, y)\) is given by \((x + \varepsilon F_y, y - \varepsilon F_x)\), and the condition that \( F \) is \( T \)-invariant means that the function

\[
F(x + \varepsilon F_y, y - \varepsilon F_x)
\]

is even in \( \varepsilon \) for all \((x, y) \in C \). Expand in a series in \( \varepsilon \); the first non-trivial condition is cubic in \( \varepsilon \):

\[
W(F) := F_{xxx}F_y^3 - 3F_{xxy}F_y^2F_x + 3F_{xyy}F_yF_x^2 - F_{yyy}F_x^3 = 0
\]

on \( C \). We claim that this already implies that \( C \) is an ellipse. The idea is that otherwise the complex curve \( f = 0 \) would have an inflection point, in contradiction with identity \((2)\).

Consider the polynomial

\[
H(F) = \det \begin{pmatrix} F_y & -F_x \\ F_{yy}F_x - F_{xy}F_y & F_{xx}F_y - F_{xy}F_x \end{pmatrix}.
\]

**Lemma 2** One has:

1). \( v(H(F)) = W(F) \);
2). \( H(F) = H(gf) = g^3H(f) \) on \( C \);
3). If \( C' \) is a non-singular algebraic curve with a defining polynomial \( f(x, y) \) and \((x, y)\) is an inflection point of \( C' \) then \( H(f)(x, y) = 0 \).
Proof of Lemma 2. The first two claims follow from straightforward computations. To prove the third, note that $H(f)$ is the second order term in $\varepsilon$ of the Taylor expansion of the function $f(x + \varepsilon f_y, y - \varepsilon f_x)$ (cf. (1)), hence $H(f) = 0$ at an inflection point. □

It follows from Lemma 2 and (2) that $H(F) = \text{const}$ on $C$. Since $C$ is convex, $H(F) \neq 0$, and we may assume that $H(F) = 1$ on $C$. It follows that $g^3H(f) - 1$ vanishes on $C$ and hence

$$g^3H(f) - 1 = hf$$

where $h(x, y)$ is some polynomial.

Now consider the situation in $\mathbb{CP}^2$. We continue to use the notation $C$ for the complex algebraic curve given by the homogenized polynomial $\bar{f}(x : y : z) = f(x/z, y/z)$. Unless $C$ is a conic, this curve has inflection points (not necessarily real). Let $d$ be the degree of $C$.

Lemma 3 Not all the inflections of $C$ lie on the line at infinity

Proof of Lemma 3. Consider the Hessian curve given by

$$\det \begin{pmatrix} \bar{f}_{xx} & \bar{f}_{xy} & \bar{f}_{xz} \\ \bar{f}_{yx} & \bar{f}_{yy} & \bar{f}_{yz} \\ \bar{f}_{zx} & \bar{f}_{zy} & \bar{f}_{zz} \end{pmatrix} = 0.$$ 

The intersection points of the curve $C$ with its Hessian curve are the inflection points of $C$ (recall that $C$ is non-singular). The degree of the Hessian curve is $3(d - 2)$, and by the Bezout theorem, the total number of inflections, counted with multiplicities, is $3d(d - 2)$. Furthermore, the order of intersection equals the order of the respective inflection and does not exceed $d - 2$, see, e.g., [6]. The number of intersection points of $C$ with a line equals $d$, hence the inflection points of $C$ that lie on a fixed line contribute, at most, $d(d - 2)$ to the total of $3d(d - 2)$. The remaining inflection points lie off this line. □

To conclude the proof of Theorem 1 consider a finite inflection point of $C$. According to Lemma 2 at such a point point, we have $f = H(f) = 0$ which contradicts (3). This is proves that $C$ is a conic. □
Remarks. 1). It would be interesting to remove the non-degeneracy assumptions in Theorem 1.

2). A more general version of Birkhoff’s integrability conjecture is as follows. Let $C$ be a plane oval whose outer neighborhood is foliated by closed curves. For a tangent line $\ell$ to $C$, the intersections with the leaves of the foliation define a local involution $\sigma$ on $\ell$. Assume that, for every tangent line, the involution $\sigma$ is projective. Conjecturally, then $C$ is an ellipse and the foliation consists of ellipses that form a pencil (that is, share four – real or complex – common points). For a pencil of conics, the respective involutions are projective: this is a Desargues theorem, see [1]. It would be interesting to establish an algebraic version of this conjecture.

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