On continuity of local epsilon factors of $\ell$-adic sheaves

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Abstract

Let $S$ be a noetherian scheme and $f: X \to S$ be a smooth morphism of relative dimension 1. For a locally constant sheaf on the complement of a divisor in $X$ flat over $S$, Deligne and Laumon proved that the universal local acyclicity is equivalent to the local constancy of Swan conductors. In this article, assuming the universal local acyclicity, we show an analogous result of the continuity of local epsilon factors. We also give a generalization of this result to a family of isolated singularities.

1 Introduction

Let $S$ be a noetherian scheme and $Y$ be a smooth separated curve over $S$. In the paper [15], Deligne-Laumon investigate the relation between local acyclicity and ramification along a boundary.

To be precise, let us fix notations. Let $\Lambda$ be a finite local ring with characteristic invertible in $S$. Let $Z$ be a closed subscheme of $Y$ which is finite flat over $S$. For a locally constant sheaf $\mathcal{F}$ of finite free $\Lambda$-modules on the complement $U = Y \setminus Z$, define a function $\varphi_{dt}: S \to \mathbb{Z}$ by

$$s \mapsto \sum_{\bar{z} \in \bar{Z}} \text{dimtot}_{\bar{z}} \mathcal{F}|_{U_{\bar{z}}}$$

where $\bar{s}$ is a geometric point above $s$ with algebraically closed residue field and $\text{dimtot}_{\bar{z}} = \dim +\text{Sw}_{\bar{z}}$ is the total dimension function with respect to the valuation of the function field of $Y_{\bar{s}}$ defined by the closed point $\bar{z}$. The main result in [15] states the following.

**Theorem 1.1.** ([15, Théorème 2.1.1]) Let $j: U \to Y$ be the inclusion. Assume that $S$ is excellent. Then the structure morphism $Y \to S$ is universally locally acyclic relatively to $j_* \mathcal{F}$ if and only if the function $\varphi_{dt}$ is locally constant, i.e. there exists a unique map $\pi_0(S) \to \mathbb{Z}$ which makes the diagram

$$
\begin{array}{ccc}
S & \xrightarrow{\varphi_{dt}} & \mathbb{Z} \\
\downarrow & & \downarrow \\
\pi_0(S) & & 
\end{array}
$$

commutative. Here $\pi_0(S)$ is the set of connected components and the map $S \to \pi_0(S)$ is the canonical one.

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In particular, the universal local acyclicity implies the local constancy of the function $\varphi_{dt}$. In this article, we would like to investigate an arithmetic analogue of this, replacing total dimensions by local epsilon factors.

Let $k$ be a finite field of characteristic $p$ and $T$ be a henselian trait of equal characteristic with residue field $k$. Fix a non-trivial additive character $\psi : \mathbf{F}_p^\times \to \mathbf{Q}^\times_\ell$, where $\ell$ is a prime number different from $p$. For a smooth $\ell$-adic sheaf $V$ on the generic point $\eta$ of $T$, i.e. an $\ell$-adic representation of the absolute Galois group of $\eta$, and a non-zero rational 1-form $\omega \in \Omega^1_{\eta}$, Langlands-Deligne define a constant $\varepsilon_0(T, V, \omega)$ in $\mathbf{Q}^\times_\ell$, the local epsilon factor, as a candidate which should corresponds via local Langlands correspondence to the local constant appearing in the functional equation of automorphic local $L$-functions [4], [16]. In this article, we use this constant to measure ramifications of sheaves.

In the papers [22] and [21], Yasuda generalizes local epsilon factors to representations in torsion coefficients. His theory also can treat the case where the residue field $k$ is a general perfect field of characteristic $p > 0$. Using his works, our results are valid in this general setting. Although local epsilon factors are defined in mixed characteristic cases, we only treat the equal characteristic cases. See the comment before the summary of the proof below.

We give a precise description of our results. Let $S$ be a normal connected scheme of finite type over $\mathbf{F}_p$. Let $Y$ be a smooth relative curve over $S$ and $Z$ be a closed subscheme of $Y$ which is finite over $S$. Fix a section $\omega \in \Gamma(Y, \Omega^1_{Y/S})$ which generates the relative cotangent sheaf $\Omega^1_{Y/S}$ around $Z$. Let $\Lambda$ be a finite local ring with residue characteristic $\ell$ which is invertible in $\mathbf{F}_p$. For a locally constant sheaf $F$ of finite free $\Lambda$-modules on the complement $U = Y \setminus Z$, consider the function

\[
\varphi_{ep} : |S| \to \Lambda^\times,
\]

from the set of closed points in $S$, defined by

\[
s \mapsto \prod_{z \in Z} (-1)^{a_z} \varepsilon_0,\Lambda(Y_{s(z)}, F|_{U \times_Y Y_{s(z)}}, \omega).
\]

Here $a_z$ is the integer $[k(z) : k(s)] \dim_{\mathbf{Q}_\ell} F|_{U_z}$ and $\varepsilon_0,\Lambda(-)$ is the theory of local epsilon factors in torsion coefficients defined by Yasuda [22]. The symbol $Y_{s(z)}$ means the henselization of $Y_s = Y \times_S s$ at $z$. As an analogue of Theorem 1.1, we prove that this function satisfies the reciprocity law.

**Theorem 1.2.** Further assume that the structure map $Y \to S$ is universally locally acyclic relatively to the $0$-extension $j : \mathcal{F}$ for the open immersion $j : U \to Y$. If, Zariski-locally on $Y$, there exists an étale morphism $f : Y \to \mathbb{A}^1_S$ such that $\omega = df$, there exists a unique character $\rho_{\mathcal{F}} : \pi^a_1(S) \to \Lambda^\times$ which makes the diagram

\[
\begin{array}{ccc}
|S| & \xrightarrow{\varphi_{ep}} & \Lambda^\times \\
& \downarrow{\pi^a_1(S)} & \downarrow{\rho_{\mathcal{F}}} \\
& \pi^a_1(S) &
\end{array}
\]

commutative. Here the arrow $|S| \to \pi^a_1(S)$ sends closed points $s$ to geometric Frobeniuses $\text{Frob}_s$. 


Let $\varphi : S \to \mathbb{Z}$ be the function defined as in (1.1). By the compatibility of $\varepsilon_{0, \Lambda}$ with unramified twists, we have $\rho_{F(1)} \cdot \rho_{F}^{-1} = \chi_{\text{cyc}}^\varphi$. Here $\chi_{\text{cyc}}$ denotes the $\ell$-adic cyclotomic character and $\chi_{\text{cyc}}^\varphi$ is the character sending $\text{Frob}_s \mapsto q_s^{\varphi(s)}$, where $q_s$ is the number of the field $k(s)$. The symbol (1) means the Tate twist.

We require that the differential $\omega$ is Zariski-locally of the form $\omega = df$ for an étale morphism $f : Y \to \mathbb{A}^1_S$. This assumption is necessary to apply Theorem 1.3.

In [19], Saito gives a generalization of Theorem 1.1 to a family of isolated singularities. Similarly our result also has a generalization to a setting analogous to his as follows.

Consider the following commutative diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{f} & X \\
\downarrow{g} & & \downarrow{t} \\
S & \xrightarrow{t} & \mathbb{A}^1_S
\end{array}
\]

of $\mathbb{F}_p$-schemes of finite type. We suppose that $S$ is normal of finite type over $\mathbb{F}_p$, that $Z$ is a closed subscheme of $X$ finite over $S$, and that $t : Y \to \mathbb{A}^1_S$ is étale. Let $K$ be a complex of constructible sheaves of $\Lambda$-modules with finite tor-dimension on $X$. We further assume that $f|_{X\setminus Z}$ and $g$ are universally locally acyclic relatively to $K$. For a closed point $s \in S$, the fibers define a sequence of morphisms

\[
Z_s \to X_s \xrightarrow{f_s} Y_s \xrightarrow{t} \mathbb{A}^1_S.
\]

Then the morphism $f_s$ is universally locally acyclic relatively to $K|_{X_s}$ outside $Z_s$, which is finite. In this sense, we refer to (1.3) as a family of isolated singularities relative to $K$. For a point $z \in Z_s$, the vanishing cycles complex $R\Phi_{f_s}(K)_z$ is defined from (1.4) and is a complex of representations of the absolute Galois group of the generic point of the henselization $Y_s(z)$. Thus we can take its local epsilon factor $\varepsilon_{0, \Lambda}(Y_s(z), R\Phi_{f_s}(K)_z, dt)$. We finally consider a function $\theta_{\text{ep}} : |S| \to \Lambda^\times$ defined by

\[
s \mapsto \prod_{z \in Z_s} (-1)^{a_z} \varepsilon_{0, \Lambda}(Y_s(z), R\Phi_{f_s}(K)_z, dt)
\]

for $a_z = [k(z) : k(s)] \dim_{\text{tot}} R\Phi_{f_s}(K)_z$. The following is our main result.

**Theorem 1.3. (Theorem 4.8)** Assume that $S$ is normal and connected. Then there exists a unique character $\pi^{ab}_{1}(S) \to \Lambda^\times$ which makes the diagram

\[
\begin{array}{ccc}
|S| & \xrightarrow{\theta_{\text{ep}}} & \Lambda^\times \\
\downarrow{\pi^{ab}_{1}(S)} & & \downarrow{} \\
\end{array}
\]

commutative.

Putting $X = Y$, $f = \text{id}_Y$, and $K = j_* F$, Theorem 1.2 is a special case of this theorem. To obtain the desired character, first we construct it on the generic points and prove that it extends to a character of the whole of $S$ (Lemma 2.12). This is the reason why we assume that $S$ is normal. Conversely the assumption of normality is sufficient for our
result; we prove the theorem for a noetherian normal scheme $S$, not necessarily of finite type, over $\mathbb{F}_p$.

The generalization of Theorem 1.1 by Saito [19, Proposition 2.16] is a key ingredient for constructing characteristic cycles. In [20], putting coefficients on the irreducible components of singular supports, we show that we can define refinements of characteristic cycles, called epsilon cycles, and we prove that they give global epsilon factors modulo roots of unity as the intersection numbers with the 0-section in the cotangent bundle.

It may be natural to ask whether Theorem 1.3 also holds in the mixed characteristic setting. One difficulty is to fix a normalization of non-trivial additive characters. Note that fixing an additive character of the ring of adeles seems to have nothing to do with our problem. Suppose that we could fix it and could obtain the desired character. We cannot expect to get this character geometrically. For example, let $S = \text{Spec}(\mathbb{Z}[1/2])$ and consider the map $X = \mathbb{A}^1_S \to Y = \mathbb{A}^1_S$ defined by $x \mapsto x^2$ and the constant sheaf $\Lambda$ on $X$. The corresponding character $\rho$ of $G_{\mathbb{Q}}$ should gives quadratic Gauss sums as the values of Frobeniiuses. Thus $\rho$ must be pure of weight one, which is not geometric in the sense of Fontaine-Mazur.

Let us give a summary of the proof. In the course of the proof of Theorem 1.3, we need to treat a family of vanishing cycles complexes. To deal with it, we use oriented products of topoi, which give a formalism of vanishing cycles complexes over general base schemes. In Section 2, we review the notion of them. In particular we give a criterion for vanishing cycles complexes to be locally constant (Proposition 2.8). After recalling Laumon’s cohomological interpretation of local epsilon factors, which relates local epsilon factors and vanishing cycles complexes, we prove the main result in Section 4. Actually we construct a locally constant constructible complex on some topos whose rank gives $\varphi_{dt}$ and whose determinant gives $\theta_{ep}$. See Theorem 4.5 for the detail.

We give notation which we use throughout this paper.

- We denote by $G_k$ the absolute Galois group of a field $k$.
- We denote by $\chi_{\text{cyc}}: G_k \to \mathbb{Z}_\ell^\times$ the $\ell$-adic cyclotomic character. For a finite local ring with residue characteristic $\ell$, we write the same letter $\chi_{\text{cyc}}$ for the composition $G_k \to \mathbb{Z}_\ell^\times \to \Lambda^\times$.
- For a finite separable extension $k'/k$ of fields, we denote by $\text{tr}_{k'/k}: G_k^{ab} \to G_{k'}^{ab}$ the transfer morphism induced by the inclusion $G_{k'} \hookrightarrow G_k$. The determinant character of the induced representation $\text{Ind}_{G_{k'}}^{G_k} 1_{G_{k'}}$ of the trivial representation is denoted by $\delta_{k'/k}$.
- For a scheme $X$ and its point $x$, $k(x)$ is the residue field of $X$ at $x$.
- For a finite extension $x'/x$ of the spectra of fields, we denote by $\text{deg}(x'/x)$ the degree of the extension. When $x = \text{Spec}(k)$ and $x' = \text{Spec}(k')$, we also denote it by $\text{deg}(k'/k)$.
- Let $x$ be a geometric point on a scheme $X$. We denote the strict henselization of $X$ at $x$ by $X_{(x)}$. On the other hand, we denote the henselization at a point $x \in X$ by $X_{(x)}$. More generally, for a finite separable extension $y$ of $x \in X$, we denote the henselization of $X$ at $y$ by $X_{(y)}$. 


2 Preliminaries on Vanishing topoi ([13])

Let \( f: X \to S \) and \( g: Y \to S \) be morphisms of schemes. By abuse of notation, we also denote the associated étale topoi by \( X, Y, S \). Otherwise stated, geometric points are assumed to be algebraic geometric points.

For the definition of the oriented product \( X \leftarrow \times_S Y \), we refer to [13]. This is the universal object of triples \((T \to X, T \beta \to Y, g \circ \beta \circ \sigma \to f \circ \alpha)\), where \( \alpha \) and \( \beta \) are morphisms of topoi, and \( \sigma \) is a natural transformation. In particular, if \( Y = S \) and \( g \) is the identity, there exists a morphism of topoi \( \Psi_f: X \to X \leftarrow \times_S S \) such that the two triangles in the diagram are 2-commutative.

Here are examples of oriented products.

Example 2.1. 1. Let \( X \to S \) be a morphism of schemes and \( s \) be a geometric point of \( S \). Then, the oriented product \( s \leftarrow \times_S X \) is canonically isomorphic to the étale topos of \( S(\sigma) \times_S X \). Here \( S(\sigma) \) is the strict henselization of \( S \) at \( s \). In particular, we have \( s \leftarrow \times_S S \cong S(\sigma) \).

2. Let \( S \) be a Dedekind scheme. Let \( s \in S \) be a closed point of codimension 1, i.e. the local ring \( O_{S,s} \) is a discrete valuation ring. Denote by \( \eta \) the generic point of the henselization \( S(\sigma) \). For an \( s \)-scheme \( X \), the vanishing topos \( X \leftarrow \times_S (S \setminus s) \) is canonically isomorphic to the topos \( X \times_{S(\eta)} \eta \), appeared in [3]. In particular, \( s \leftarrow \times_S (S \setminus s) \) is canonically isomorphic to the étale topos of \( \eta \).

A point on the topos \( X \leftarrow \times_S Y \) can be considered as a triple denoted by \( x \leftarrow y \) consisting of a geometric point \( x \) of \( X \), a geometric point \( y \) of \( Y \) and a specialization \( s = f(x) \leftarrow t = g(y) \), i.e. an \( S \)-morphism \( t \to S(\sigma) \).

Assume that \( X \) and \( S \) are quasi-compact and quasi-separated. Under this assumption, the topos \( X \leftarrow \times_S S \) is coherent, as is explained in [18 Section 9]. Let \( \Lambda \) be a finite local ring whose residue characteristic is invertible on \( S \). For a sheaf of sets or a sheaf of \( \Lambda \)-modules on \( X \leftarrow \times_S S \), we say that it is constructible if there exist finite partitions \( X = \bigsqcup_\alpha X_\alpha, S = \bigsqcup_\beta S_\beta \) by locally closed constructible subschemes such that its restrictions to \( X_\alpha \leftarrow \times_S S_\beta \) are locally constant constructible sheaves. This notion of constructibility is the same with the one given in [9 1.9.3].
Let $D^b_c(-, \Lambda)$ be the full subcategory of $D^+(-, \Lambda)$ consisting of bounded complexes whose cohomology sheaves are constructible.

Let $f: X \to S$ be a morphism of schemes. Then the morphism of topoi $\Psi_f: X \to X \times_S S$ defines a functor $R\Psi_f: D^+(X, \Lambda) \to D^+(X \times_S S, \Lambda)$, which we call the nearby cycles functor. For an object $K$ of $D^+(X, \Lambda)$ and a point $x \leftarrow t$ of $X \times_S S$, the stalk $R\Psi_f K(x \leftarrow t)$ is canonically identified with $R\Gamma(X \times_S S_t, K)$ ([11, 1.3]).

The relation $id = p_1 \circ \Psi_f$ defines a natural transformation of functors $p_1^* \to R\Psi_f$ by adjunction and by the isomorphism $id \to p_1 \circ \Psi_f$. The cone of this map defines the vanishing cycles functor $R\Phi_f: D^+(X, \Lambda) \to D^+(X \times_S S, \Lambda)$.

We consider a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{p} & & \downarrow{g} \\
S & \leftarrow & T
\end{array}
$$

of schemes. This defines a morphism $\overleftarrow{g}: X \times_Y Y \to X \times_S S$. The canonical isomorphism $\overleftarrow{g} \circ \Psi_f \to \Psi_p$ induces an isomorphism $R\overleftarrow{g}_* \circ R\Psi_f \to R\Psi_p$. Let $K$ be an object of $D^+(X \times_Y Y)$. We can compute the stalk $R\overleftarrow{g}_* K(x \leftarrow t)$ of $R\overleftarrow{g}_* K$ at a point $x \leftarrow t$ of $X \times_S S$ as follows ([11, Proposition 1.13]). Let $s$ and $y$ be the images of $x$ in $S$ and $Y$ respectively. The topoi $x \times_S S$ and $x \times_Y Y$ are canonically isomorphic to $S_s$ and $Y_y$. Under these identifications, $R\overleftarrow{g}_* K(x \leftarrow t)$ can be identified by $Rg(y)_*(K|_{Y_y})$, where $g(y)$ is the map $Y_y \to S_s$ induced from $g$. Hence we have

$$(2.2) \quad R\overleftarrow{g}_* K(x \leftarrow t) \cong R\Gamma(Y_y \times S_s, S_t, K).$$

A cartesian diagram

$$
\begin{array}{ccc}
X_T & \leftarrow & X \\
\downarrow{f_T} & & \downarrow{f} \\
S & \leftarrow & T
\end{array}
$$

of schemes defines a 2-commutative diagram

$$
\begin{array}{ccc}
X_T & \xleftarrow{p_1} & X_T \times_T T \\
\downarrow{i_T} & & \downarrow{i} \\
X & \leftarrow & X \times_S S
\end{array}
\begin{array}{ccc}
& \xleftarrow{\Psi_{f_T}} & X_T \\
\downarrow{i} & & \downarrow{i} \\
& \xleftarrow{\Psi_f} & X
\end{array}
$$

and the base change morphisms define a morphism of distinguished triangles

$$
\begin{array}{cccc}
i^* p_1^* & \longrightarrow & i^* R\Psi_f & \longrightarrow & i^* R\Phi_f \\
\sim & & \downarrow & & \downarrow \\
p_1^! i^* & \longrightarrow & R\Psi_{f_T} i^* & \longrightarrow & R\Phi_{f_T} i^*.
\end{array}
$$
For an object $K$ of $D^+(X, \Lambda)$, we say that the formation of $R\Psi_f K$ commutes with the base change $T \to S$ if the middle (hence all) vertical arrow is an isomorphism. In particular, if the formation of $R\Psi_f(K)$ commutes with any finite base change $T \to S$, $R\Psi_f(K)(z \to t) \cong R\Gamma(X(z) \times S(t), K)$ is canonically isomorphic to $R\Gamma(X(z) \times S(t), t, K)$, taking $T$ as the closure of the image of $t \to S$. Note that, by [19, Proposition 2.7.2], $f : X \to S$ is universally locally acyclic relatively to $K \in D^+(X, \Lambda)$ if and only if the map $p_1^*i^*K \to R\Psi_f i^*K$ is an isomorphism for any $i : T \to S$. Therefore the formation of $R\Psi_f K$ commutes with any base change $T \to S$.

### 2.1 Calculation of vanishing cycles complexes

**Proposition 2.2.** ([18, Proposition 2.8.] Let $f : X \to S$ be a morphism of finite type of noetherian schemes and $Z \subset X$ be a closed subscheme which is quasi-finite over $S$. Let $K$ be an object of $D^b_c(X, \Lambda)$ such that the restriction of $f : X \to S$ to the complement $X \setminus Z \to S$ is universally locally acyclic relatively to the restriction of $K$.

1. $R\Psi_f K$ and $R\Phi_f K$ are constructible. If $K$ is of finite tor-dimension, so are they. Their formations commute with arbitrary base change. $R\Phi_f K$ is supported on $Z \times_S S$.

2. Let $x$ be a geometric point of $X$ and $s = f(x)$ be the geometric point of $S$ defined by the image of $x$ by $f$. Let $t$ and $u$ be geometric points of $S_s$ and $t \leftarrow u$ be a specialization. Then, there exists a distinguished triangle

$$R\Psi_f K(z \to t) \longrightarrow R\Psi_f K(z \to u) \longrightarrow \bigoplus_{z \in Z \times S_s} R\Phi_f K(z \to u) \longrightarrow,$$

where the first map is the cospecialization and the second one is the direct sum of the compositions of the maps $R\Psi_f K(z \to u) \to R\Phi_f K(z \to u)$ and the cospecializations $R\Phi_f K(z \to u) \to R\Phi_f K(z \to u)$.

**Proof.** 1. The commutativity with base change is proved in [18, Proposition 6.1.], taking a compactification and using the proper base change theorem. The last assertion follows from the remark made before this proposition. The constructibility follows from [18, Théorème 8.1.] and the commutativity with base change. The finiteness of tor-dimension follows from the finiteness of cohomological dimension of $R\Psi_f$ [18, Proposition 3.1.].

2. We may assume that $S = S_s$, that $Z$ is local and finite over $S$, and that $X$ is separated over $S$. Let us denote the morphisms obtained by the base change $X \to S$ by the same letter, by abuse of notation. By 1, we may further replace $S$ by the normalization in $u$. Consider the following diagram

\[
\begin{array}{ccc}
  & s & t \\
  i_s & & i_t \\
 S & j & S_t & k & u
\end{array}
\]

and define a complex $\Delta$ on $X \times_S S_t$ fitting in the distinguished triangle $j^* K \to Rk_*(jk)^* K \to \Delta \to$. Note that, since $u$ is the generic point of $S$, the geometric point $t$ can be identified with its image in $S$ and that $S_t$ can be identified with the localization of $S$ at $t$. Hence
the complex $\Delta$ is supported on $Z \times_S S(t)$. Indeed, let $t'$ be a geometric point of $S(t)$. The restriction of $\Delta$ to $X \times_S S(t')$ equals to the complex defined in the same way as $\Delta$ replacing $t$ by $t'$. Thus the restriction $\Delta|_{X_t}$ is supported on $Z_{t'}$. Since the formation of nearby cycles functor commutes with any base change by 1, $(Rj_{s})^*K_{x}$ and $(R(j_{s}))^*(j_{s})^*K_{x}$ are isomorphic to $R\Psi_{f}K_{(x\leftarrow i)}$ and $R\Psi_{f}K_{(x\leftarrow w)}$ respectively. We have isomorphisms

$$(Rj_{s})\Delta_{x} \rightarrow R\Gamma(Z, Rj_{s}\Delta) \rightarrow R\Gamma(Z \times_S S(t), \Delta).$$

Since the last term is isomorphic to $\bigoplus_{z \in Z \times_S t} R\Phi_{f}K_{(z\leftarrow w)}$, the assertion follows.

To deduce corollaries, we need a following lemma. Let $f : X \to S$ be a morphism of schemes. For a (usual) point $x \in X$, we denote by $X_{x}$ the spectrum of the localization $\mathcal{O}_{X,x}$.

To a complex $K \in D^{+}(X, \Lambda)$, we define $ULA(K, f) \subset X$ to be the subset consisting of points $x \in X$ such that the morphism $X_{x} \to S$ is universally locally acyclic relatively to $K|_{X_{x}}$. Note that, for a morphism $g : S' \to S$ of schemes, the inverse image of $ULA(K, f)$ by $g_{X} : X_{S'} := X \times_S S' \to X$ is contained in $ULA(g_{X}^{*}K_{S'}, f_{S'})$, where $f_{S'}$ is the base change $X_{S'} \to S'$.

**Lemma 2.3.** Let the notation be as above.

1. Let $x \to X$ be a geometric point and let $j : X_{(x)} \to X$ be the morphism from the strict henselization. Then, we have $j^{-1}(ULA(K, f)) = ULA(K|_{X_{(x)}}, f_{j})$.

2. Assume that $f : X \to S$ is of finite type and that $S$ is (hence also $X$ is) noetherian. When $K$ is constructible, $ULA(K, f)$ is an open subset of $X$.

**Proof.**

1. Let $y' \in X_{(x)}$ be a point and $y = j(y') \in X$ be its image. Fix a geometric point $\bar{y} \to X_{(y)}$ over $y$. We also regard $\bar{y}$ as a geometric point of $X$ by $j$. The strict henselizations $X_{(y)(y)}$ and $X_{(y)}$ are canonically isomorphic. Since the universal local acyclicity of $X_{(y)(y)} := \text{Spec}(\mathcal{O}_{X_{(y)}, y'}) \to S$ (resp. $X_{y} \to S$) is equivalent to that of the morphism $X_{(y)(y)} \to S$ (resp. $X_{(y)} \to S$), the assertion follows.

2. First we show the assertion assuming that the formation of $R\Phi_{f}(K)$ commutes with arbitrary base change $S' \to S$. Under this assumption, $R\Phi_{f}(K)$ is constructible by [18 Théorème 8.1].

Let $x \in X$ be a point in $ULA(K, f)$. We need to find an open neighborhood $U$ of $x$ contained in $ULA(K, f)$. Since $R\Phi_{f}(K)$ is a constructible complex on $X \times_S S$ and the restriction of $R\Phi_{f}(K)$ to $X_{x} \times_S S$ is acyclic, there is an open neighborhood $U$ of $x$ such that $R\Phi_{f}(K)$ is acyclic on $U \times_S S$. Hence $f_{U} : U \to S$ is universally locally acyclic by the commutativity of the formation of $R\Phi_{f}(K)$. See [19 Proposition 2.7.2] for the detail.

Next we consider the general case. By [18 Théorèmes 2.1, 8.1], there exists a proper surjective morphism $g : T \to S$ such that the vanishing cycles complex $R\Phi_{f_{T}}(K|_{X_{T}})$ of $K|_{X_{T}}$ with respect to the base change $f_{T} : X_{T} := X \times_S T \to T$ is constructible and its formation commutes with arbitrary base change $T' \to T$. Let $g_{X} : X_{T} \to X$ be the projection. We already know that the complement $Z := X_{T} \setminus ULA(g_{X}^{*}K_{T}, f_{T})$ is closed. Let $Z' := g_{X}(Z)$ be the image, which is closed. We show the equality $ULA(K, f) = X \setminus Z' =: U'$. The intersection $ULA(K, f) \cap Z'$ is empty since $g_{X}^{-1}(ULA(K, f)) \subset ULA(g_{X}^{*}K_{T}, f_{T})$. Hence it suffices to show that $U' \to S$ is universally locally acyclic relatively to $K$. Since the base change $U'_{T}$ is contained in $ULA(g_{X}^{*}K_{T}, f_{T})$, the assertion follows from the oriented cohomological descent [17 Lemma 6.1].
Corollary 2.4. Let $f : X \to S$ be a morphism of finite type of noetherian schemes. Let $Z \subset X$ be a closed subscheme quasi-finite over $S$. Let $x \to X$ be a geometric point and let $s := f(x)$ be the geometric point of $S$ induced from $x$.

Let $K \in D_c^b(X, \Lambda)$ be a constructible complex. If $f(x) : X(x) \to S(s)$ is universally locally acyclic relatively to $K$ outside $Z \times_X X(x)$, $R\Psi_{f(x)}(K)$ is constructible and its formation commutes with arbitrary base change $S' \to S(s)$. 

Proof. Replacing $X \to S$ by $X \times_S S(s) \to S(s)$, we may assume that $S = S(s)$. Since $K$ is constructible, after replacing $X$ by an étale neighborhood of $x$, we may assume that there exists a constructible complex $K'$ on $X$ which restricts to $K$. Define $U := \text{ULA}(K', f)$. This is an open subset of $X$ by Lemma 2.3.2. By the assumption and Lemma 2.3.1, we have $(Z \times_X X(x)) \cup (U \times_X X(x)) = X(x)$. Hence, further replacing $X$, we may assume that $Z \cup U = X$. Then, the assertion follows from Proposition 2.2.1.

We give a partial generalization of Corollary 2.4 to the following setting.

Consider a commutative diagram of schemes, to which we refer as a pair $f = (f, g)$. This induces a morphism of topoi $\tilde{f} : Y \times_X X \to T \times_S S$. For a geometric point $y$ of $Y$, let $t$ denote the image $g(y)$. This is a geometric point of $T$. They are also regarded as geometric points of $X$ and $S$ via the vertical arrows of (2.3). Thus, they give a morphism of topoi $f(y) : X(y) \to S(t)$, which is also obtained from $\tilde{f}$ via the identifications $X(y) \cong y \times_X X$ and $S(t) \cong t \times_S S$.

Let $i := (i_T : T' \to T, i_S : S' \to S)$ be a pair of morphisms of schemes such that the diagram is commutative. The pull-backs give a commutative diagram of schemes, which has a projection to the diagram (2.3). Hence we get a commutative diagram of topoi.
Definition 2.5. Let $K \in D^+(Y \times_X X, \Lambda)$ be a bounded below complex.

1. Let $Z \subset X$ be a closed subset with complement $U$. We say that the pair $f = (f, g)$ is (resp. universally) locally acyclic relatively to $K$ outside $Z$ if, for every geometric point $y$ of $Y$ with the image $t = g(y)$, the morphism $f(y)\vert_{X(y) \times_X U} : X(y) \times_X U \to S(t)$ is (resp. universally) locally acyclic relatively to $K\vert_{X(y) \times_X U}$.

2. Let $i := (i_T : T' \to T, i_S : S' \to S)$ be as in (2.4). We say that the formation of $Rf_*K$ commutes with a base change $i$ if the base change map $i Rf_*K \to Rf_*i K$ defined from (2.5) is an isomorphism.

Lemma 2.6. Assume that $g$ is finite and that $f$ is the identity $S \to S$, and consider the morphism of topoi $f : Y \times_S S \to T \times_S S$.

1. The formation of $f_*$ commutes with base change.

2. The cohomological dimension of $f_*$ is zero.

3. Assume that $Y, T, S$ are quasi-separated and quasi-compact. Then the pushforward $f_* : D^+(Y \times_S S, \Lambda) \to D^+(T \times_S S, \Lambda)$ preserves the constructibility.

Proof. 1, 2. Let $K$ be a bounded below complex on $Y \times_S S$. Since $g$ is finite, for a geometric point $t$ of $T$, the restriction of $Rf_*K$ to $t \times_S S$ is canonically isomorphic to $Rg(t)_*K$, where $g(t) : \bigsqcup_{y \in Y \times_T t} S(y) \to S(t)$. Hence the assertions 1, 2 follow.

3. Taking a finite stratification of $T$, we may assume that $g$ is finite étale by 1. Further replacing $T$ by a finite étale cover, we may assume that $Y$ is isomorphic to the disjoint union of finite copies of $T$. Then the assertion is clear.

Corollary 2.7. Consider a commutative diagram (2.3) of schemes. Let $Z \subset X$ be a closed subset which is quasi-finite over $S$. Assume that $S$ is noetherian and that $f$ is of finite type. Further assume that $g$ is finite.

Let $K \in D_c^b(Y \times_X X, \Lambda)$ be a constructible complex. Assume that $f = (f, g)$ is universally locally acyclic relatively to $K$ outside $Z$. Then, the formation of $Rf_*K$ commutes with base change $(T' \to T, S' \to S)$.

Proof. The morphism $f$ decomposes into

$$Y \times_X X \xrightarrow{f} Y \times_S S \xrightarrow{g} T \times_S S.$$ 

The formation of $g_*$ commutes with base change by Lemma 2.6.

Let $i = (i_Y : Y' \to Y, i_S : S' \to S)$ be a pair of morphisms of schemes such that the diagram

$$
\begin{array}{ccc}
Y' & \xrightarrow{i_Y} & Y \\
\downarrow & & \downarrow \\
S' & \xrightarrow{i_S} & S
\end{array}
$$


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is commutative. Let \((y' \leftarrow u')\) be a geometric point of \(Y' \times_S S'\) and \((y \leftarrow u)\) be its image in \(Y \times_S S\). Let \(t\) (resp. \(t'\)) be the geometric point of \(S\) (resp. \(S'\)) defined by \(y \rightarrow Y \rightarrow S\) (resp. \(y' \rightarrow Y' \rightarrow S'\)). By (2.2), we have

\[
(\tilde{Rf}_*K)_{(y \leftarrow u)} \cong R\Gamma(X(y) \times_{S(t)} S(u), K) \cong R\Psi_f(y) K_{(y \leftarrow u)}.
\]

Here \(f(y)\) is the morphism \(X(y) \rightarrow S(t)\) of the strict henselizations. By Corollary 2.4, the pull-back of \(R\Psi_f(y) K\) to \((X(y) \times_{S(t)} S'(t')) \times_{S'(t')} S'(t')\) is isomorphic to \(R\Psi_f(y) (K|_{X(y) \times_{S(t)} S'(t')}\), where \(f(y) : X(y) \times_{S(t)} S'(t') \rightarrow S'(t')\) is the base change of \(f(y)\). The assertion follows.

**Proposition 2.8.** Let \(X, Z, S\) be as in Proposition 2.2. Further assume that \(Z \rightarrow S\) is finite. Let \(Z'\) be the image \(f(Z)\). Let \(g : Z \rightarrow Z'\) be the restriction of \(f\) and \(\tilde{g} : Z \times_S (S \setminus Z') \rightarrow Z' \times_S (S \setminus Z')\) be the induced morphism of topos. Let \(K\) be an object of \(D^b_c(X, \Lambda)\) such that the restriction of \(K\) to \(f^{-1}(Z')\) is acyclic. Assume that \(f : X \rightarrow S\) is universally locally acyclic relatively to \(K\) outside \(Z\). Then, the following hold.

1. The formation of \(\tilde{g}_*(R\Psi_f(K)|_{Z \times_S (S \setminus Z')}\) commutes with arbitrary base change \(S' \rightarrow S\).

2. The complex \(\tilde{g}_*(R\Psi_f(K)|_{Z \times_S (S \setminus Z')}\) is locally constant constructible. If \(K\) is of finite tor-dimension, so is \(\tilde{g}_*(R\Psi_f(K)|_{Z \times_S (S \setminus Z')}\).

**Proof.** 1. The formation of \(R\Psi_f(K)\) commutes with base change by Proposition 2.2.1. The formation of \(\tilde{g}_*\) commutes with base change by Lemma 2.6.1.

2. The constructibility of \(R\Psi_f(K)|_{Z \times_S (S \setminus Z')}\) follows from Proposition 2.2.1. By Lemma 2.6.3, the push-forward \(\tilde{g}_*\) preserves constructibility.

The finiteness of tor-dimension follows from Proposition 2.2.1 and Lemma 2.6.2.

Let \(\mathcal{G} := \tilde{g}_*(R\Psi_f(K))\). By (2.2) and the fact that \(g\) is finite, for a point \((z' \leftarrow t)\) of \(Z' \times_S S\), we have a canonical isomorphism

\[
\mathcal{G}(z' \leftarrow t) \cong \bigoplus_{x \in Z \times z'} R\Phi_f(K)(x \leftarrow t).
\]

We show that the restriction \(\mathcal{G}|_{Z' \times_S (S \setminus Z')} = \tilde{g}_*(R\Psi_f(K)|_{Z \times_S (S \setminus Z')}\) is locally constant. To prove this, it is enough to show that, for points \((z'_1 \leftarrow t_1), (z'_2 \leftarrow t_2)\) of \(Z' \times_S (S \setminus Z')\) and a specialization \((z'_1 \leftarrow t_1) \rightarrow (z'_2 \leftarrow t_2)\), the morphism of stalks \(\mathcal{G}(z'_1 \leftarrow t_1) \rightarrow \mathcal{G}(z'_2 \leftarrow t_2)\) is an isomorphism by Lemma 2.4 below since the topos \(Z' \times_S (S \setminus Z')\) is noetherian ([18 Lemme 9.3.]). We prove this by showing that \(\mathcal{G}(z'_1 \leftarrow t_1) \rightarrow \mathcal{G}(z'_2 \leftarrow t_2)\) and \(\mathcal{G}(z'_1 \leftarrow t_2) \rightarrow \mathcal{G}(z'_2 \leftarrow t_2)\) are both isomorphisms.

By Proposition 2.2.2, we have a distinguished triangle

\[
\mathcal{G}(z'_1 \leftarrow t_1) \rightarrow \mathcal{G}(z'_1 \leftarrow t_2) \rightarrow \bigoplus_{x \in Z \times z'_1 \times z \times z'_1} R\Phi_f(K)(z \leftarrow t_2) \rightarrow.
\]

Since \(t_1\) is a geometric point of \(S \setminus Z'\), \(Z(x) \times_{S(z)} t_1\) is empty. This shows the former isomorphism \(\mathcal{G}(z'_1 \leftarrow t_1) \cong \mathcal{G}(z'_2 \leftarrow t_2)\).
To prove the latter isomorphism, also by Proposition 2.2, we have a distinguished triangle

\[(2.6) \quad \mathcal{G}(z_1 \leftarrow z_2') \rightarrow \mathcal{G}(z_1' \leftarrow t_2) \rightarrow \bigoplus_{z \in Z \times Z'} \bigoplus_{z \in Z \times S_{(t_1', z_2')}} R\Phi_f(K)_{(z \leftarrow t_2)} \rightarrow \]

Since the restriction $K_{|f^{-1}(Z')}$ is acyclic, the third term is naturally identified with

\[
\bigoplus_{z \in Z \times Z'} R\Phi_f(K)_{(z \leftarrow t_2)} \cong \bigoplus_{z \in Z \times Z'} R\Psi_f(K)_{(z \leftarrow t_2)} \cong \mathcal{G}(z_2' \leftarrow t_2).
\]

Note that the restriction of $\mathcal{G}$ to $Z' \times Z', Z' \subset Z' \times S$ is acyclic since $K_{|f^{-1}(Z')}$ is acyclic and the formations of $\mathcal{Z}_t$ and $R\Phi_f(K)$ commute with the base change $Z' \rightarrow S$. In particular, the first term of (2.6) is acyclic. The assertion follows.

To give a proof of Lemma 2.9, we need to recall some facts on noetherian topoi. Let $T$ be a noetherian topos. Recall that a point of $T$ is a morphism $t: (\text{Set}) \rightarrow T$ of topoi from the category of sets. For an object $X$ of $T$ and a point $t$ of $T$, denote $X_t := t^*X$. For points $s$ and $t$, we call a natural transformation $s^* \rightarrow t^*$ a specialization $t \rightarrow s$. Let $|T|$ be the set of isomorphism classes of points of $T$. We denote the final object of $T$ by the same letter $T$. For a subobject $U$ of $T$, denote by $|U|$ the set of points $t \in |T|$ such that $U_t$ is not empty. There is a topology on $|T|$ whose open subsets are precisely $|U|$ for subobjects $U$ of $T$. This makes $|T|$ a noetherian topological space. The assignment $U \mapsto |U|$ gives a bijection between the set of subobjects of $T$ and the set of open subsets of $|T|$, since $T$ has enough points ([9, Proposition 9.0]).

Let $t$ be a point of $T$. The stalk $X_t$ of an object $X$ of $T$ at $t$ can be computed as follows ([8 6.8]). Let $\text{Nbd}(t)$ be the category whose objects are pairs $(U, x)$ where $U$ are quasi-compact objects of $T$ and $x$ are elements of $U_t$. The morphisms $(U_1, x_1) \rightarrow (U_2, x_2)$ are morphisms $U_1 \rightarrow U_2$ which send $x_1$ to $x_2$. There is a functorial isomorphism

\[
\lim_{(U,x) \in \text{Nbd}(t)} \text{Hom}_T(U, X) \cong X_t.
\]

**Lemma 2.9.** Let $T$ be a noetherian topos. Assume that, for a quasi-compact object $U$ of $T$ and a point $t$ of $T$, the stalk $U_t$ is finite.

1. For points $s$ and $t$ of $T$, the following are equivalent.
   - (a) There is a specialization $t \rightarrow s$.
   - (b) $s \in \{t\} \subset |T|$.

2. Let $X$ be an object of $T$ whose stalks at all points are finite sets. The following are equivalent.
   - (a) $X$ is locally constant constructible.
   - (b) For all specializations $t \rightarrow s$ of points of $T$, the canonical maps $X_s \rightarrow X_t$ are bijective.
Proof. Although these are well-known, we include proofs since we cannot find a reference.

1. The implication $(a) \Rightarrow (b)$ is obvious from the definition of the topology of $|T|$. We prove $(b) \Rightarrow (a)$. For all objects $(U, x)$ of $Nbd(s)$, $U_i$ are non-empty finite sets. Hence $\lim_{\leftarrow (U, x) \in Nbd(s)} U_t$ is non-empty since $Nbd(s)^{op}$ is filtered. An element of $\lim_{\leftarrow (U, x) \in Nbd(s)} U_t$ gives a functor $Nbd(s) \rightarrow Nbd(t)$, which defines a specialization $t \rightarrow s$.

2. First we prove $(a) \Rightarrow (b)$. Take a covering $(T_i \rightarrow T)_i$ of $T$ so that $X \times_T T_i$ is a disjoint union of $T_i$. Let $t \rightarrow s$ be a specialization of points. Take $i$ so that $T_{i, s}$ is non-empty. A lift $\bar{s}$ of $s$ to a point of $T_i$ gives a lift of $t$ to a point $\bar{t}$ of $T_i$ and a specialization $i \rightarrow \bar{i}$. Since the stalk $(X \times_T T_i)_{s}$ (resp. $(X \times_T T_i)_{\bar{t}}$) is canonically isomorphic to $X_s$ (resp. $X_{\bar{t}}$), the assertion follows.

Next we show $(b) \Rightarrow (a)$. Fix a point $t$ of $T$. Take a quasi-compact object $U$ of $T$ and $x \in U_t$ so that the natural map $\text{Hom}_T(U, X) \rightarrow X_t$ is surjective. Replacing $(T, t)$ by $(U, x)$, we may assume that there is a morphism $\coprod T \rightarrow X$, from the disjoint union of finitely many copies of $T$, which gives a bijection $\coprod T_t \rightarrow X_t$. Further replacing $T$ by the connected component which contains $t$, we may assume that $|T|$ is connected. Then it follows that the morphism $\coprod T \rightarrow X$ is an isomorphism from 1 and the fact that $T$ has enough points (Proposition 9.0)).

\[ \square \]

2.2 Tame symbols for étale sheaves

Definition 2.10. Let $S$ be a noetherian scheme. Let $C$ be a separated smooth $S$-curve. Let $Z \subset C$ be a closed subscheme finite étale over $S$. For a point $z$ of $Z$, denote by $\eta_z$ the generic point of the henselization of $C_{k(s), (z)}$ where $s \in S$ is the image of $z$. For a locally constant constructible object $X$ of $Z \times_C (C \setminus Z)$, we say that $X$ is a tame object if, for every point $z \in Z$, the restriction of $X$ to $\eta_z := z \times_{C_{k(s)}} (C_{k(s)} \setminus z) \subset Z \times_C (C \setminus Z)$ is tamely ramified.

By Proposition 5.5, a locally constant constructible object $X$ is a tame object if and only if, for every generic point $z \in Z$, the restriction to $\eta_z$ is tamely ramified.

Let $f \in \Gamma(C, O_C)$ be a global section which generates the ideal sheaf of $Z$. We construct, from a tame object of $Z \times_C (C \setminus Z)$ and such a section $f$, a locally constant constructible object $\langle X, f \rangle$ of the étale topos of $Z$ when $S$ is normal. We start with more general setting [5, 1.7.8]. Let $Y$ be a regular scheme (resp. a smooth scheme over a scheme $S$). Let $D \subset Y$ be a regular divisor (resp. a smooth divisor relative to $S$). Denote the complement by $U$. Denote by $i : D \rightarrow Y, j : U \rightarrow Y$ the immersions. Assume that the ideal sheaf of $D$ is globally generated by $z \in \Gamma(Y, O_Y)$. Let $\mathcal{F}$ be a locally constant constructible sheaf of sets on $U$ which is tamely ramified along $D$. For an integer $n \geq 1$ which is invertible in $Y$, let $Y_n := \text{Spec}(O_Y[t]/(t^n + z))$. This is a finite totally tamely ramified covering of $Y$ with a lifting $D \rightarrow Y_n$ of $D \rightarrow Y$. Denote by $U_n := U \times_Y Y_n$ the complement of $D$ in $Y_n$. Zariski locally on $Y$, we can find such an $n$ that $\mathcal{F}|_{U_n}$ extends to a locally constant sheaf on $Y_n$, which we denote by $\mathcal{F}_n$.

Definition-Lemma 2.11. Let the notation be as above.

1. The restriction of $\mathcal{F}_n$ to $D \subset Y_n$ is independent of the choice of $n$. Hence the restrictions glue to a locally constant constructible sheaf on $D$, which we denote by $\langle \mathcal{F}, z \rangle$.
2. Suppose that $Y$ and $D$ are smooth over a scheme $S$. Then, the formation of $(\mathcal{F}, z)$ commutes with base change $S' \to S$.

Proof. 1. Let $n, m \geq 1$ be integers. The assertion follows from the fact that there is an $O_Y$-morphism $O_Y[t]/(t^n + z) \to O_Y[u]/(u^{nm} + z)$ sending $t \mapsto u^m$.

2. It follows since the fibered product $Y_n \times_S S'$ is canonically isomorphic to $(Y \times_S S')_n$.

We go back to the situation of vanishing topoi. Let $S$ be a noetherian normal scheme. Let $f \in \Gamma(C, O_C)$ be a global section which generates the ideal sheaf of $Z$ which is flat over $S$. Let $X$ be a tame object on $Z \times_C (C \setminus Z)$. For a point $z \in Z$, the restriction of $X$ to $C_z \setminus Z_z \cong \hat{z} \times_C (C \setminus Z) \subset Z \times_C (C \setminus Z)$ is a locally constant constructible sheaf tamely ramified along $Z(z)$.

Definition-Lemma 2.12. Assume that $S$ is noetherian normal.

1. Locally constant constructible sheaves $(X|_{C(z) \setminus Z(z)}, f)$ on $Z(z)$ glue to a locally constant constructible sheaf on $Z$. We denote this sheaf by $(X, f)$.

2. The formation of $(X, f)$ commutes with arbitrary base change $S' \to S$.

Proof. 1. Let $t \in Z$ be a generic point and $z \in Z$ be a point which is a specialization of $t$. The restriction of $(X|_{C(t) \setminus t}, f)$ to $Z(t) \times_Z t$ and that of $(X|_{C(z) \setminus Z(z)}, f)$ to $Z(t) \times_Z t$ coincide by Lemma 2.11.2. The assertion follows.

2. It follows from Lemma 2.11.2.

2.3 Flat functions and trace maps

We give a definition of flat functions.

Definition 2.13. ([14, Definition 2.1]) Let $h: Z \to S$ be a quasi-finite morphism of schemes. A function $\varphi: Z \to \mathbb{Z}$ is said to be flat if, for every geometric point $(x \leftarrow t)$ of $\hat{Z} \times_S S$, we have

$$\varphi(x) = \sum_{z \in Z(z) \times_S S(t)} \varphi(z),$$

where $s$ is the image $h(x)$. For a geometric point $z$ of $Z$, we write $\varphi(z)$ for the value of $\varphi$ at the image of $z \to Z$.

Example 2.14. Let $Z$ and $S$ be as in Definition 2.13.

1. Let $i: S' \to S$ be a morphism of schemes and $i_Z: Z' = Z \times_S S' \to Z$ be the projection. If the function $\varphi: Z \to \mathbb{Z}$ is flat over $S$, the composition $\varphi \circ i_Z$ is flat over $S'$.

2. Assume that $O_Z$ is of finite tor-dimension as an $h^{-1}O_S$-module. For a point $z \in Z$, write $\bar{z}$ for a geometric point above $z$ and $\bar{s}$ for the geometric point of $S$ defined by $\bar{z} \to Z \to S$. Then the function $Z \to \mathbb{Z}$ defined by $z \mapsto \sum_i (-1)^i \dim_{k(\bar{s})} \text{Tor}_i O_Z^{\bar{s}(\bar{z})}(O_{Z, z}, k(\bar{s}))$ is flat over $S$.

Lemma 2.15. Let $h: Z \to S$ be a separated quasi-finite morphism of noetherian schemes. Let $\varphi: Z \to \mathbb{Z}$ be a flat function over $S$. 

1. There exists a unique morphism $\text{Tr}_\varphi: h_! \mathbb{Z} \to \mathbb{Z}$ of étale sheaves on $S$ such that, for every geometric point $s$ of $S$, the stalk $\text{Tr}_{\varphi,s}: \bigoplus_{z \in \mathbb{Z}_s} \mathbb{Z} \to \mathbb{Z}$ sends $(n_z)_z$ to $\sum_z n_z \varphi(z)$.

2. Let $i: S' \to S$ be a morphism of noetherian schemes and let $Z \xrightarrow{i_Z} Z' = Z \times_S S' \xrightarrow{h'} S'$ be the projections. Then the diagram

$$
\begin{array}{ccc}
 i^* h_! \mathbb{Z} & \xrightarrow{i^* \text{Tr}_\varphi} & i^* \mathbb{Z} \\
 \cong & & \cong \\
 h'_i i^* Z & \xrightarrow{\text{Tr}_{\varphi \circ i_Z}} & Z,
\end{array}
$$

where the vertical arrows are canonical isomorphisms, is commutative.

Proof. They are done in [2, Proposition 6.2.5].

Let $S$ be a noetherian $\mathbb{F}_p$-scheme. Let $\Lambda$ be a finite local ring in which $p$ is invertible. The Artin-Schreier covering $\mathbb{A}_S^1 \to \mathbb{A}_S^1$ defined by $x \mapsto x^p - x$ is a Galois covering whose Galois group is canonically isomorphic to $\mathbb{F}_p$. For a non-trivial character $\psi: \mathbb{F}_p \to \Lambda^\times$, this covering defines a locally constant sheaf $L_\psi(x)$ of invertible $\Lambda$-modules on $\mathbb{A}_S^1$, which is the Artin-Schreier sheaf. For a section $S \to A_1^1$, we denote by $L_\psi(f \cdot x)$ the pull-back of the Artin-Schreier sheaf by the multiplication-by-$f$ map $A_1^1 \to A_1^1$.

**Lemma 2.16.** Let the notation be as above. Fix a non-trivial character $\psi: \mathbb{F}_p \to \Lambda^\times$. Let $Z$ be a finite $S$-scheme. For a $Z$-morphism $f: Z \to A_1^1$ and a flat function $\varphi: Z \to \mathbb{Z}$ over $S$, there exists an element $L_\psi(\varphi \cdot f) \in H^1(A_1^1, \Lambda^\times)$ with the following properties.

1. For a morphism $i: S' \to S$ of noetherian schemes, $L_\psi(\varphi \circ i \cdot f')$ coincides with the image of $L_\psi(\varphi \cdot f)$ by $H^1(A_1^1, \Lambda^\times) \to H^1(A_1^1, \Lambda^\times)$. Here $i_Z: Z' := Z \times_S S' \to Z$ is the projection and $f': Z' \to A_1^1$ is the base change of $f$.

2. When $S = \text{Spec}(k)$ is the spectrum of a perfect field $k$, $L_\psi(\varphi \cdot f)$ is equal to $L_\psi((\sum_{z \in Z} \varphi(z) \text{Tr}_{k(z)/k}(f(z)) \cdot x)$, where $x$ is the standard coordinate of $A_1^1$.

Proof. Let $\varphi \circ p_Z: \mathbb{A}_Z^1 \to Z \to \mathbb{Z}$ be the composition of $\varphi$ and the projection $p_Z: \mathbb{A}_Z^1 \to Z$. This is a flat function over $A_1^1$. Let $\text{Tr}_{\varphi \circ p_Z}: (h \times \text{id})_* Z \to \mathbb{Z}$ be the trace map constructed in Lemma 2.15 from $\varphi \circ p_Z$ and the base change $h \times \text{id}: \mathbb{A}_Z^1 \to A_1^1$ of $h$. Tensoring $\Lambda^\times$ and taking $H^1$, we get a group homomorphism

$$(2.7) \quad H^1(A_1^1, \Lambda^\times) \to H^1(A_1^1, \Lambda^\times).$$

Let $L_\psi(f \cdot x) \in H^1(A_1^1, \Lambda^\times)$ be the pull-back of the rank 1 locally constant $\Lambda$-sheaf $L_\psi(x)$ on $A_1^1$ by the map $A_1^1 \to A_1^1$ defined by $x \mapsto f \cdot x$. Define $L_\psi(\varphi \cdot f)$ to be the image of $L_\psi(f \cdot x)$ by (2.7). By Lemma 2.15, this satisfies the properties. □

### 3 Local Epsilon Factors (cf. [4], [16], [21])

In this preliminary section, we review theories of local epsilon factors for henselian traits of equal-characteristic. We fix prime numbers $p$ and $\ell$ so that $p \neq \ell$. Let $\Lambda$ be a finite local ring with residual characteristic $\ell$. We also fix a non-trivial character $\psi: \mathbb{F}_p \to \Lambda^\times$. 


3.1 Generalities on local epsilon factors

Let $k$ be a perfect field of characteristic $p$. Let $T$ be a henselian trait which is isomorphic to the henselization of $\mathbb{A}^1_k$ at a closed point. Let $s$ and $\eta$ be the closed point and the generic point of $T$ respectively. When $k$ is a finite field, in [4], a constant $\varepsilon(T, F, \omega)$, is defined for a constructible complex $F \in D^b_c(T, E)$ and a non-zero rational 1-form $\omega \in \Omega^1_{k(\eta)}$. Here $E$ is a finite extension of $\mathbb{Q}_p$ and $\psi: \mathbb{F}_p \to E^\times$ is a fixed non-trivial character.

**Theorem 3.1.** ([3], [16]) Let $T$ be as above. Assume that $k$ is finite. Fix a non-trivial character $\psi: \mathbb{F}_p \to E^\times$. For each complex $F \in D^b_c(T, E)$ and a non-zero rational 1-form $\omega \in \Omega^1_{k(\eta)}$, we can attach, in a canonical way, an element

$$\varepsilon(T, F, \omega) \in E^\times$$

which satisfies the following properties:

1. The element $\varepsilon(T, F, \omega)$ only depends on the isomorphism class of $(T, F, \omega)$.

2. For an distinguished triangle

$$F_1 \to F_2 \to F_3 \to$$

in $D^b_c(T, E)$, we have $\varepsilon(T, F_2, \omega) = \varepsilon(T, F_1, \omega) \cdot \varepsilon(T, F_3, \omega)$.

3. If $F$ is supported on the closed point $s$, we have

$$\varepsilon(T, F, \omega) = \det(-\text{Frob}_s, F)$$

4. Let $\eta_1/\eta$ be a finite separable extension and $f: T_1 \to T$ be the normalization in $\eta_1$. For a constructible complex $F_1 \in D^b_c(T_1, E)$ with generic rank 0, i.e. $\text{rk} F_{1, \eta_1} = 0$, we have

$$\varepsilon(T_1, F_1, f^*\omega) = \varepsilon(T, f_* F_1, \omega).$$

5. Let $G$ be a smooth $E$-sheaf on $\eta$ of rank 1, which induces a character $\chi: k(\eta)^\times \to E^\times$ via the Artin map $k(\eta)^\times \to G^\text{ab}_{\eta}$. Let $j: \eta \to T$ be the immersion. Then, we have

$$\varepsilon(T, j_! G, \omega) = \varepsilon(\chi, \Psi_\omega).$$

Here $\Psi_\omega(a) = \psi \circ \text{Tr}_{k(s)/\mathbb{F}_p}(\text{Res}(a \cdot \omega))$ and $\varepsilon(\chi, \Psi_\omega)$ is the Tate constant ([16, 3.1.3.2]).

Here we fix a normalization of the local class field theory so that the Artin map sends a uniformizer to a geometric Frobenius. We identify a smooth $E$-sheaf on $\eta$ and a finite dimensional $E$-representation of the absolute Galois group $G_\eta$. For a smooth $E$-sheaf $V$ on $\eta$, we also denote by $\varepsilon_0(T, V, \omega)$ the constant $\varepsilon(T, V_i, \omega)$, where $V_i$ is the 0-extension of $V$ to $T$.

The local epsilon factors admit the following properties.

**Proposition 3.2.** 1. ([16, 3.1.5.5]) For a non-zero element $a \in \Gamma(\eta, \mathcal{O}_\eta)$, we have

$$\varepsilon(T, F, a \cdot \omega) = \det(F_\eta)(a) \cdot \varepsilon(T, F, \omega).$$
2. \([16, 3.1.5.6]\) Let \(G\) be a smooth \(E\)-sheaf on \(T\). Then, we have
\[
\varepsilon(T, F \otimes G, \omega) = \det(G)(\text{Frob}_s)^{a(T, F, \omega)} \cdot \varepsilon(T, F, \omega)^{\text{rk} G}.
\]

Here \(a(T, F, \omega)\) is defined to be \(\text{rk}(F_\eta) + \text{Sw}(F_\eta) - \text{rk}(F_s) + \text{rk}(F_\eta) \cdot \text{ord}(\omega)\).

Note that, with a constructible complex \(F \in D^b_c(T, \mathcal{O}_E)\), we can define a constant \(\varepsilon(T, F, \omega)\) to be \(\varepsilon(T, F \otimes \mathcal{O}_E, \omega)\).

Yasuda gives a generalization of this theory \([22]\), \([21]\). In his setting, we can take a finite local ring as the coefficient ring of sheaves and can treat a henselian trait with a general perfect residue field (which further can be of mixed characteristic).

We adjust his result to our setting. Let \(\Lambda\) be a finite local ring of residual characteristic \(\ell\). Take and fix a non-trivial additive character \(\psi: \mathbb{F}_p \to \Lambda^\times\). Let \(k\) be a perfect field of characteristic \(p\) and \(T\) be a henselian trait isomorphic to the henselization of \(A^1_k\) at a closed point. We denote its generic point by \(\eta\). We denote the absolute Galois group of \(k\) (resp. \(\eta\)) by \(G_k\) (resp. \(G_\eta\)).

Theorem 3.3. \([21, 4.12]\) Let the notation be as above. For a triple \((T, (\rho, V), \omega)\) where \(V\) is a finite free \(\Lambda\)-module with a continuous group homomorphism \(\rho: G_\eta \to \text{GL}(V)\) and \(\omega \in \Omega^1_\eta\) is a non-zero rational section, there is a canonical way to attach a continuous character \(\varepsilon_0, \Lambda(T, V, \omega): G_k^{ab} \to \Lambda^\times\) with the following properties.

1. The character only depends on the isomorphism class of \((T, (\rho, V), \omega)\).

2. For a short exact sequence \(0 \to V' \to V \to V'' \to 0\) of representations of \(G_\eta\), we have
\[
\varepsilon_0, \Lambda(T, V, \omega) = \varepsilon_0, \Lambda(T, V', \omega) \cdot \varepsilon_0, \Lambda(T, V'', \omega).
\]

3. For a local ring homomorphism \(f: \Lambda \to \Lambda'\), we have
\[
f \circ \varepsilon_0, \Lambda(T, V, \omega) = \varepsilon_0, \Lambda'(T, V \otimes_\Lambda \Lambda', \omega)
\]
as characters \(G_k \to \Lambda'^\times\).

4. Assume that the residue field \(k\) of \(T\) is finite and that there exists a local ring morphism \(f: \mathcal{O}_E \to \Lambda\) from the ring of integers of a finite extension \(E/\mathbb{Q}_\ell\) such that \(V\) comes from a representation on \(\mathcal{O}_E\), i.e. there is a representation \(V'\) of \(G_\eta\) on a finite free \(\mathcal{O}_E\)-module such that \(V' \otimes_{\mathcal{O}_E} \Lambda \cong V\). Then we have
\[
\varepsilon_0, \Lambda(T, V, \omega)(\text{Frob}_k) = (-1)^{\text{rk} V + \text{Sw} V} f(\varepsilon_0(T, V' \otimes_{\mathcal{O}_E} E, \omega)).
\]

Here the local epsilon factor in the right hand side is the one in Theorem 3.1.

Remark 3.4. In the case that the residue field \(k\) is finite, before the work of \([21]\), Yasuda defines a local epsilon factor \(\varepsilon_0, \Lambda(T, V, \omega)\) as an element of \(\Lambda^\times\) in \([22]\). The relation between them is as follows:
\[
\varepsilon_0, \Lambda(T, V, \omega)(\text{Frob}_k) = (-1)^{\text{rk} V + \text{Sw} V} \varepsilon_{0, \Lambda}(T, V, \omega).
\]
Here the local epsilon factor in the left hand side is given in Theorem 3.3.
For more properties, see [21]. By the multiplicativity, we also define a local epsilon factor \( \varepsilon_{0,A}(T, K, \omega) \) for a complex \( K \in D_{\text{tr}}(\eta, \Lambda) \).

Let us explain precisely the relation between the local epsilon factors defined in [21] and the one in Theorem 3.3.

First let us explain a counterpart of an additive character in the classical setting. Let \( F \) be the completion of the function field of \( T \). Let \( m_F \) be the maximal ideal of the ring of integers of \( F \). For integers \( m \leq n \), define a contravariant functor \( F^{[m,n]} \) from the category of affine \( k \)-schemes to the category of sets to be \( \text{Spec}(A) \mapsto A \otimes_k m_F^m / m_F^{n+1} \). This functor is represented by an affine smooth group \( k \)-scheme. Indeed, fixing a uniformizer \( \pi \in F \), it is isomorphic to the functor \( A = A_\omega \). We have a natural inclusion \( F^{[m,n]} \to F^{[m-1,n]} \) defined by \( m_F^m / m_F^{n+1} \to m - 1 \to m_F^{n+1} \). A non-trivial invertible character sheaf \( \tilde{\psi}^{[m,n]} \) on \( F^{[m,n]} \) is a non-trivial invertible sheaf of \( \Lambda \)-modules on \( F^{[m,n]} \) such that \( m^* \tilde{\psi}^{[m,n]} \) is isomorphic to the external product \( \tilde{\psi}^{[m,n]} \otimes \tilde{\psi}^{[m,n]} \), where \( m: F^{[m,n]} \to F^{[m-1,n]} \) is the addition. Denote the set of isomorphism classes of non-trivial invertible character sheaves on \( F^{[m,n]} \) by \( ACh^0(F^{[m,n]}, \Lambda) \) [21 4.1]. The natural inclusion \( F^{[m,n]} \to F^{[m-1,n]} \) induces a map \( ACh^0(F^{[m-1,n]}, \Lambda) \to ACh^0(F^{m,n], \Lambda) \) by the pull-back. A non-trivial additive character sheaf \( \tilde{\psi} \) on \( F \) is an element of

\[
\prod_{n \in \mathbb{Z}, m \leq n-1} \text{ACh}^0(F^{[m,n-1]}, \Lambda).
\]

When \( \tilde{\psi} \in \bigcap_{m \leq n-1} \text{ACh}^0(F^{[m,n-1]}, \Lambda) \), the integer \( n \) is called the conductor of \( \tilde{\psi} \).

In [21], Yasuda defines a character \( \tilde{\varepsilon}_{0,A}(V, \tilde{\psi}) \) from a representation \((\rho, V)\) as above and a non-trivial additive character sheaf \( \tilde{\psi} \) on \( F \). For a non-zero rational 1-form \( \omega \in \Omega_1 \), we denote the corresponding additive character sheaf by \( \tilde{\psi}_\omega \). Then we define \( \varepsilon_{0,A}(T, V, \omega) = \tilde{\varepsilon}_{0,A}(V, \tilde{\psi}_\omega) \).

The construction of \( \tilde{\psi}_\omega \) goes as follows. When the residue field \( k \) is finite, \( \omega \) defines an additive character \( \psi_\omega: F \to \Lambda^\times \) by \( a \mapsto \psi(\text{Tr}_k/F_p \circ \text{Res}(a\omega)) \). Then \( \tilde{\psi}_\omega \) is the additive character sheaf corresponding to \( \psi_\omega \) [21 Corollary 4.3]. In general, take a uniformizer \( \pi \in F \), which defines an inclusion \( F_p((\tau)) \to F \). When \( \omega = d\tau \), define \( \tilde{\psi}_\omega \) to be the pull-back of the additive character sheaf \( \psi_{d\tau} \) on \( F_p((\tau)) \) [21 Corollary 4.4.2]. When \( \omega = a \pi d\tau \) for a non-zero element \( a \in F \), define \( \tilde{\psi}_\omega \) to be the pull-back of \( \psi_{d\tau} \) by the multiplication-by-\( a \) map \( F \to F \) [21 4.1].

The next aim of this section is to explain a formula of local epsilon factors analogous to Laumon’s formula [16 3.5.1.1].

We slightly generalize Laumon’s local Fourier transform to general base schemes. Let \( S \) be a noetherian scheme over \( F_p \). Fix a finite local ring \( \Lambda \) of residue characteristic \( \ell \). For a non-trivial character \( \psi: F_p \to \Lambda^\times \), let \( \mathcal{L}_\psi(x) \) be the Artin-Schreier sheaf on \( \mathbb{A}^1_S \). Let \( \mathcal{L}_\psi(x_1 \cdot x_2) \) be the pull-back of \( \mathcal{L}_\psi(x) \) by the multiplication \( \mathbb{A}^1_{F_p} \times \mathbb{A}^1_{F_p} \to \mathbb{A}^2_{F_p} \), where \( x_1 \) and \( x_2 \) are the standard coordinates of the first and second factors respectively. We define \( \mathcal{L}_\psi \) to be the 0-extension of \( \mathcal{L}_\psi(x_1 \cdot x_2) \) to \( \mathbb{A}^1_{F_p} \times \mathbb{P}^1_{F_p} \). We also use the same notation \( \mathcal{L}_\psi \) for the pull-back by \( \mathbb{A}^1_S \times \mathbb{P}^1_S \to \mathbb{A}^1_{F_p} \times \mathbb{P}^1_{F_p} \). Denote by \( 0_S \) (resp. \( \infty_S \)) the section \( S \to \mathbb{A}^1_S \) (resp. \( S \to \mathbb{P}^1_S \) at the origin (resp. at the infinity). Let

\[
0_S \times_{\mathbb{A}^1_S} \mathbb{A}^1_{\mathbb{P}^1_S} \xrightarrow{\mathbb{P}^1} (0_S \times S \times S) \xrightarrow{\mathbb{A}^1_S \times \mathbb{P}^1_S} (\mathbb{A}^1_S \times S \times \mathbb{P}^1_S) \xrightarrow{\mathbb{P}^1} \infty_S \times S \times \mathbb{P}^1_S
\]

be the morphisms of topoi induced from the projections \( \mathbb{A}^1_S \xrightarrow{\mathbb{P}^1} \mathbb{A}^1_S \times \mathbb{P}^1_S \xrightarrow{\mathbb{P}^1} \mathbb{P}^1_S \). Let \( g: (0_S \times S \times S) \xrightarrow{\mathbb{P}^1} (\mathbb{A}^1_S \times \mathbb{P}^1_S) \to \mathbb{A}^1_S \times \mathbb{P}^1_S \) be the second projection.
Definition 3.5. Let $K \in D_{ctf}(0_S \times_{\mathbb{A}^1_S} \mathbb{G}_{m,S}, \Lambda)$ be a complex of finite tor-dimension whose cohomology sheaves are locally constant. Define the local Fourier transform $F^{(0,\infty)}(K)$ by

$$F^{(0,\infty)}(K) := Rp^{\ast} \left( p_1^{\ast} K_1 \otimes_{\Lambda} q^{\ast} \mathcal{L}_\psi \right)[1]|_{\mathcal{O}_S \times_{\mathbb{A}^1_S} \mathbb{A}^1_S} \in \mathcal{D}^b(\mathcal{O}_S \times_{\mathbb{A}^1_S} \mathbb{A}^1_S, \Lambda).$$

Here $K_1$ is the 0-extension of $K$ by $0_S \times_{\mathbb{A}^1_S} \mathbb{G}_{m,S} \hookrightarrow 0_S \times_{\mathbb{A}^1_S} \mathbb{A}^1_S$.

Proposition 3.6. (cf. [16 Proposition (2.4.2.2).]) Let the notation be as above. Let $\Lambda_0$ be the residue field of $\Lambda$.

1. The formation of $F^{(0,\infty)}(K)$ commutes with arbitrary base change $S' \to S$.

2. When $K$ has tor-amplitude in $[0,0]$, so is $F^{(0,\infty)}(K)$.

3. The local Fourier transform $F^{(0,\infty)}(K)$ is constructible and of finite tor-dimension. It is locally constant if and only if, for each $i \in \mathbb{Z}$, the function on $S$ defined by $s \mapsto \dim_{\text{tot}} \mathcal{H}^i(K \otimes_{\Lambda} \Lambda_0)|_{s_0}$, where $s_0$ is the spectrum of an algebraic closure of $k(s)$ and $\eta_s \cong 0 \times_{\mathbb{A}^1_{k(s)}} \mathbb{G}_{m,k(s)}$ is the generic point of the henselization $\mathbb{A}^1_{k(s),0}$ at the origin, is locally constant.

Proof. 1. Let $s$ be a geometric point of $S$. Since the projection $\mathbb{A}^1_S \times_S \mathbb{P}^1_S \to \mathbb{P}^1_S$ is universally locally acyclic relatively to $\mathcal{L}_\psi$ ([16 Théorème (1.3.1.2)]), so is the morphism $(\mathbb{A}^1_S \times_S \mathbb{P}^1_S)(0_{(s,\infty)}) \to \mathbb{P}^1_S(0_{(s,\infty)})$ relatively to the restriction of $\mathcal{L}_\psi$. Since $K$ is locally constant, the pair $(\mathbb{A}^1_S \times_S \mathbb{P}^1_S \to \mathbb{P}^1_S, 0_S \times_S \mathcal{O}_S \cong \mathcal{O}_S)$ is universally locally acyclic relatively to $K_1 \otimes q^* \mathcal{L}_\psi$ outside $0_S \times_S \mathbb{P}^1_S \subset \mathbb{A}^1_S \times_S \mathbb{P}^1_S$, in the sense of Definition 2.5.1. The assertion follows from Corollary 2.7 taking $X, Y, T$, and $Z$ in the corollary as $\mathbb{A}^1 \times S \mathbb{P}^1_S, \mathbb{P}^1_S, 0_S \times_S \mathcal{O}_S \times \mathcal{O}_S, \mathcal{O}_S$, and $0_S \times_S \mathbb{P}^1_S$ respectively.

2. Replacing $K$ by $K \otimes_{\Lambda} \Lambda_0$, we may assume that $\Lambda$ is a field. By 1, we may also assume that $S$ is the spectrum of a field. In this case, the assertion follows from the fact that the vanishing cycles functor preserves perversity up to shift [12 Corollaire 4.6].

3. The complex $F^{(0,\infty)}(K)$ is of finite tor-dimension since the cohomological dimension of $p^{\ast}z$ is finite [15 Proposition 3.1]. For the constructibility and the local constancy, replacing $K$ by $K \otimes_{\Lambda} \Lambda_0$, we may assume that $\Lambda$ is a field, and that $K$ is a locally constant sheaf. By 1 and the constructibility of $K$, we may assume that $S$ is of finite type over $\mathbb{F}_p$. Let $S' \to S$ be a proper surjective morphism of $\mathbb{F}_p$-schemes. Note that $F^{(0,\infty)}(K)$ is (resp. locally constant) constructible if and only if so is $F^{(0,\infty)}(K|_{S'})$. Indeed, the equivalence of the constructibility follows from [18 Lemme 10.5.] and 1. By Lemma 2.9, $F^{(0,\infty)}(K)$ is locally constant if and only if, for all specializations of points of $\mathcal{O}_S \times_S \mathbb{A}^1_S$ of types $(x_1 \leftarrow t) \leftarrow (x_2 \leftarrow t)$ or $(x \leftarrow t_1) \leftarrow (x \leftarrow t_2)$, the canonical maps of the stalks of $F^{(0,\infty)}(K)$ are isomorphisms. Hence the assertion follows from 1. Thus we may assume that $S$ is regular by de Jong’s alteration theorem.

Let $\eta \in S$ be a generic point. By the Gabber-Katz extension [14], there exists a locally constant constructible sheaf $K'$ on $\mathbb{G}_{m,\mathbb{K}(\eta)}$ which restricts to $K$ on $0 \times_{\mathbb{A}^1_{\mathbb{K}(\eta)}} \mathbb{G}_{m,\mathbb{K}(\eta)}$. By Zariski-Nagata’s purity theorem, $K'$ extends to a locally constant constructible sheaf on $\mathbb{G}_{m,S}$, which is denoted also by $K'$. The projection $p_2 : \mathbb{A}^1_S \times_S \mathbb{P}^1_S \to \mathbb{P}^1_S$ is universally locally acyclic relatively to $p^{\ast}K' \otimes \mathcal{L}_\psi$ outside $0_S \times_S \mathbb{P}^1_S$ since so is $\mathcal{L}_\psi$ ([16 Théorème (1.3.1.2)]) and $K'$ is locally constant. If the function $s \mapsto \dim_{\text{tot}} K|_{s_0}$ is locally constant, $\mathbb{A}^1_S \to S$ is
universally locally acyclic relatively to \( K' \) by [13 Théorème 2.1.1]. Hence \( \mathbb{A}^1_k \times_S \mathbb{P}^1_S \to \mathbb{P}^1_S \) is universally locally acyclic relatively to \( p_1^* K'_1 \otimes^L \mathcal{L}_{\psi} \) outside \( 0_S \times_S \infty_S \). The assertion except the “only if” part follows from Propositions 2.2.1, 2.8.

The “only if” part can be verified as follows. Assume that \( F^{(0, \infty)}(K) \) is locally constant. Replacing \( K \) by \( K \otimes k_0 \mathbb{A}_0 \), we may assume that \( \Lambda \) is a field. By 2, the function \( \dim \text{tot} \mathcal{H}^i(K)|_{\eta_s} \) coincides with \( \text{rk} \mathcal{H}^i F^{(0, \infty)}(K)|_{\eta_s} \) (Here we use Theorem 3.7, which is proved independently of this proposition), hence the assertion.

Let \( k \) be a perfect field of characteristic \( p > 0 \), and \( T \) and \( T' \) be henselian traits of equal-characteristic with residue field \( k \). Fix uniformizers \( \pi \) and \( \pi' \) of \( T \) and \( T' \) respectively. Assume that the \( k \)-morphism \( T \to \mathbb{A}^1_k \) (resp. \( T' \to \mathbb{P}^1_k \)) defined by \( x \mapsto \pi \) (resp. \( x' \mapsto 1/\pi' \)), where \( x \) (resp. \( x' \)) is the standard coordinate of \( \mathbb{A}^1_k \) (resp. \( \mathbb{A}^1_k \subset \mathbb{P}^1_k \)), induces an isomorphism between \( T \) (resp. \( T' \)) and the henselizations of \( \mathbb{A}^1_k \) (resp. \( \mathbb{P}^1_k \)) at 0 (resp. at \( \infty \)). We identify \( T \) and \( T' \) with the henselizations. Let \( s, \eta \) and \( s', \eta' \) be the closed points and the generic points of \( T \) and \( T' \) respectively. We also denote the pull-back of \( \mathcal{L}_{\psi} \) by the map \( T \times_k T' \to \mathbb{A}^1_k \times_k \mathbb{P}^1_k \) by the same letter \( \mathcal{L}_{\psi} \).

Let \( G_{\eta} \) and \( G_{\eta'} \) be the absolute Galois groups of \( \eta \) and \( \eta' \). We identify the category \( D_{\text{ctf}}(\eta, \Lambda) \) (resp. \( D_{\text{ctf}}(\eta', \Lambda) \)) with the derived category of complexes of continuous \( G_{\eta} \)-representations (resp. continuous \( G_{\eta'} \)-representations) on finite \( \Lambda \)-modules whose tor-dimensions are finite.

There is a canonical upper numbering filtration \( (G^v_{\eta'})_v \) of \( G_{\eta'} \) indexed by \( v \in \mathbb{Q}_{\geq 0} \). \( G^0_{\eta'} \) is the inertia subgroup and \( G^0_{\eta'} := \bigcup_{v > 0} G^v_{\eta'} \) is the wild inertia subgroup. Let \( D_{\text{ctf}}(\eta', \Lambda)[0,1[ \) be the full subcategory of \( D_{\text{ctf}}(\eta', \Lambda) \) consisting of complexes on which \( G^1_{\eta'} \) acts trivially.

**Theorem 3.7.** (cf. [16 Théorème (2.4.3)]) Let the notation be as above.

1. \( F^{(0, \infty)} \) induces an equivalence of categories

   \[ F^{(0, \infty)} : D_{\text{ctf}}(\eta, \Lambda) \xrightarrow{\cong} D_{\text{ctf}}(\eta', \Lambda)[0,1[. \]

2. For \( K \in D_{\text{ctf}}(\eta, \Lambda) \), we have

   \[ \text{rk} F^{(0, \infty)}(K) = \text{rk} K + \text{Sw} K. \]

   \[ \text{Sw} F^{(0, \infty)}(K) = \text{Sw} K. \]

**Proof.** Originally similar results for \( \mathcal{O}_L \)-sheaves are proven in [16 Théorème (2.4.3)]. Since the arguments in loc. cit. work, we only sketch the proof.

1. Similarly as \( F^{(0, \infty)} \), we define a functor \( F^{(\infty, 0)} : D_{\text{ctf}}(\eta', \Lambda) \to D_{\text{ctf}}(\eta, \Lambda) \) by \( F^{(\infty, 0)}(K) := \mathcal{R}p_1^*(\mathcal{P}_2^* K_1 \otimes^L \mathcal{L}_{\psi})[1]|_{\eta} \), where we denote the projections by

   \[ s \times_T T \xleftarrow{p_1} (s \times_k s') \times_{(T \times_k T')} (T \times_k T') \xrightarrow{p_2} s' \times_{T'} T'. \]

   Similarly as [16 Théorème (2.4.3)], we have natural isomorphisms \( F^{(\infty, 0)} \circ F^{(0, \infty)}(K) \cong a^* K(-1) \) for \( K \in D_{\text{ctf}}(\eta, \Lambda) \), and \( F^{(0, \infty)} \circ F^{(\infty, 0)}(K) \cong a^* K(-1) \) for \( K \in D_{\text{ctf}}(\eta', \Lambda)[0,1[ \), where \( a \) denotes the \( k \)-isomorphisms \( T \to T \) and \( T' \to T' \) which send \( \pi \) and \( \pi' \) to \( -\pi \) and \( -\pi' \) respectively. To show the assertion, then, it suffices to prove that the image of \( F^{(0, \infty)} \) is contained in \( D_{\text{ctf}}(\eta', \Lambda)[0,1[. \) To this end, we may assume that \( \Lambda \) is a field and that \( K \in D_{\text{ctf}}(\eta, \Lambda) \) is a sheaf. The local Fourier transform \( V := F^{(0, \infty)}(K) \) is a
sheaf by Proposition 3.6.2. Similarly as [16, Théorème (2.4.3)], we have an isomorphism $F^{(\infty, 0)}(V^{G_{\eta'}}) \cong F^{(\infty, 0)}(V)$. Hence the assertion follows from the isomorphisms

$$a^* V^{G_{\eta'}}(-1) \cong F^{(0,\infty)}(V) = F^{(0,\infty)} \circ F^{(\infty, 0)}(K) \cong F^{(0,\infty)}(a^* K(-1)) \cong a^* V(-1).$$

2. The equalities follow from the existence of the Gabber-Katz extension and the Grothendieck-Ogg-Shafarevich formula.

Corollary 3.8. For a constructible complex $K \in D_{ct}(\eta, \Lambda)$, the determinant $\det(F^{(0,\infty)}(K))$ is tamely ramified.

Proof. Since $\det(F^{(0,\infty)}(K))$ is of rank 1 and $G_{\eta'}^1$ acts trivially on it, the assertion follows from Hasse-Arf Theorem.

Theorem 3.9. (cf. [23, 8.3], [16, Théorème (3.5.1.1)]) Let the notation be as in Theorem 3.7. For a constructible complex $K \in D_{ct}(\eta, \Lambda)$, we have

$$(3.1) \quad \varepsilon_{0,\Lambda}(T, K, d\pi) = \langle \det(F^{(0,\infty)}(K)), \pi' \rangle.$$

Proof. We may assume that $K$ is represented by a locally constant sheaf $V$ of finite free $\Lambda$-modules. When the residue field $k$ is finite, it is proved in [23, Proposition 8.3], and the argument given there is also valid in the general setting.

Remark 3.10. In the sequel, it is harmless to take the formula (3.1) as the definition of local epsilon factors.

To explain more properties on local epsilon factors by Yasuda, we need to recall the construction in [21, 4.2].

Let $\chi: G_{\eta} \to M$ be a continuous group homomorphism into a finite abelian group $M$. Fix a presentation

$$(3.2) \quad 0 \to \mathbb{Z}^{\oplus n} \xrightarrow{\alpha} \mathbb{Z}^{\oplus n} \to M \to 0$$

as an abelian group. This defines an injection $H^1(\eta, M) \to H^2(\eta, \mathbb{Z}^{\oplus n})$ of cohomology groups of $G_{\eta}$-modules with trivial action. Let $k(\eta)^{\text{sep}}$ be a separable closure of $k(\eta)$ and $k(\eta)^{\text{ur}}$ be the maximal unramified extension in it. Then the inflation morphism $H^2(k(\eta)^{\text{ur}}/k(\eta), k(\eta)^{\text{ur}\times}) \to H^2(k(\eta)^{\text{sep}}/k(\eta), k(\eta)^{\text{sep}\times})$ is an isomorphism. Thus we have a sequence of maps

$$(3.3) \quad k(\eta)^{\times} \times H^1(\eta, M) \to k(\eta)^{\times} \times H^2(\eta, \mathbb{Z}^{\oplus n}) \to H^2(k(\eta)^{\text{sep}}/k(\eta), k(\eta)^{\text{sep}\times})^{\oplus n} \cong H^2(k(\eta)^{\text{ur}}/k(\eta), k(\eta)^{\text{ur}\times})^{\oplus n} \to H^2(k, \mathbb{Z}^{\oplus n}).$$

Here the second one is the cup product and the last one is induced from the valuation. Since $H^1(\eta, M)$ (resp. $H^1(k, M)$) is the kernel of $\alpha: H^2(\eta, \mathbb{Z}^{\oplus n}) \to H^2(\eta, \mathbb{Z}^{\oplus n})$ (resp. $H^2(k, \mathbb{Z}^{\oplus n}) \to H^2(k, \mathbb{Z}^{\oplus n})$)), The image of (3.3) lands into $H^1(k, M)$. In this way, we have a pairing $k(\eta)^{\times} \times H^1(\eta, M) \to H^1(k, M)$. This is independent of the choice of (3.2) and natural in $M$.

Definition 3.11. For a pair $(a, \chi) \in k(\eta)^{\times} \times H^1(\eta, M)$, we denote by $\chi[a] \in H^1(k, M)$ the image by this pairing. It is a character $G_k \to M$. 

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Lemma 3.12. Let $T$ be a henselian trait of equal-characteristic $p > 0$ with perfect residue field $k$. Let $\chi: G_\eta \to M$ be a continuous group homomorphism into a finite abelian group $M$ and denote by $\mathcal{F}$ the corresponding locally constant sheaf on $\eta$. Assume that $\chi$ (hence also $\mathcal{F}$) is tamely ramified. Then, for a uniformizer $z \in k(\eta)^\times$, the character $\chi_{[z]}$ corresponds to $\langle \mathcal{F}, z \rangle$ in Definition 2.11.

Proof. First assume that $\chi$ is unramified. Then $\chi$ comes from an element of $H^1(k(\eta)^{ur}/k(\eta), M)$, which is denoted by the same letter. Then the character $\chi_{[z]}$ coincides with the image of the pair $(z, \chi(z)) \in k(\eta)^\times \times H^1(k(\eta)^{ur}/k(\eta), M)$ by the composition of

$$k(\eta)^\times \times H^1(k(\eta)^{ur}/k(\eta), M) \to H^1(k(\eta)^{ur}/k(\eta), k(\eta)^{ur}\otimes M) \xrightarrow{\text{ord} \otimes \text{id}} H^1(k(\eta)^{ur}/k(\eta), M),$$

where the first arrow is the cup product. Since $\text{ord} z = 1$, the assertion follows.

Next we treat the general case. For an integer $m \geq 1$ which is prime to $p$, denote by $\eta_m$ the totally tamely ramified extension $\text{Spec}(k(\eta)[X]/(X^m + z))$ of $\eta$. Note that the norm map $k(\eta_m)^\times \to k(\eta)$ sends $X$ to $z$. We use the same letters $\eta$ and $\eta_m$ for their étale topoi by abuse of notation. Let $\widetilde{\eta}$ and $\eta_m$ denote the topoi associated with the étale sites of unramified extensions of $\eta$ and $\eta_m$ respectively. The cohomological functors $H^1(\eta_m, -), H^1(\eta_m, -)$ are naturally identified with $H^1(k(\eta_m)^{\text{sep}}/k(\eta_m), -), H^1(k(\eta_m)^{\text{ur}}/k(\eta_m), -)$ respectively. We have a natural commutative diagram

$$\begin{array}{ccc}
\eta_m & \xrightarrow{f} & \eta \\
\pi_m \downarrow & & \downarrow \pi \\
\eta_m & \xrightarrow{f} & \tilde{\eta}
\end{array}$$

of topoi. The norm map $N: f_*\mathcal{O}_{\eta_m}^{\times} \to \mathcal{O}_{\tilde{\eta}}^{\times}$ gives the following commutative diagrams

(3.4)

$$\begin{array}{ccc}
k(\eta_m)^\times \times H^1(\eta, M) & \xrightarrow{N \times \text{id}} & k(\eta_m)^\times \times H^2(\eta, \mathbb{Z}^{\otimes n}) \\
\downarrow \text{id} \times f^* & & \downarrow \text{id} \times f^* \\
k(\eta)^\times \times H^1(\eta, M) & \xrightarrow{N \times \text{id}} & k(\eta)^\times \times H^2(\eta, \mathbb{Z}^{\otimes n}) \\
\end{array} \quad \begin{array}{ccc}
H^2(\eta_m, \mathcal{O}_{\eta_m}^{\times}) & \xrightarrow{\cup} & H^2(\eta_m, \mathcal{O}_{\eta_m}^{\times})^{\otimes n} \\
\downarrow N & & \downarrow N \\
H^2(\eta, \mathcal{O}_{\eta}^{\times}) & \xrightarrow{\cup} & H^2(\eta, \mathcal{O}_{\eta}^{\times})^{\otimes n}.
\end{array}$$

The commutative diagram

$$\begin{array}{ccc}
\tilde{f}_*\mathcal{O}_{\tilde{\eta_m}}^{\times} & \xrightarrow{N} & \mathcal{O}_{\tilde{\eta}}^{\times} \\
\downarrow \pi_*f_*\mathcal{O}_{\eta_m}^{\times} & & \downarrow \pi_*\mathcal{O}_{\eta}^{\times} \\
\pi_*\mathcal{O}_{\eta_m}^{\times} & \xrightarrow{\pi_*N} & \pi_*\mathcal{O}_{\eta}^{\times}
\end{array}$$

induces the commutative diagram

(3.5)

$$\begin{array}{ccc}
H^2(\eta_m, \mathcal{O}_{\eta_m}^{\times}) & \xrightarrow{\text{res}} & H^2(\eta_m, \mathcal{O}_{\eta_m}^{\times}) \\
\downarrow N & & \downarrow N \\
H^2(\eta, \mathcal{O}_{\eta}^{\times}) & \xrightarrow{\text{res}} & H^2(\eta, \mathcal{O}_{\eta}^{\times}).
\end{array}$$

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Since $\eta_m/\eta$ is totally ramified, we have a commutative diagram

$$
\begin{align*}
H^2(\eta_m, O_{\eta_m}^\times) & \overset{\text{ord}_{\eta_m}}{\longrightarrow} H^2(\eta_m, \mathbb{Z}) \\
\downarrow^N & \quad \downarrow^{f^*} \\
H^2(\eta, O_\eta^\times) & \overset{\text{ord}_\eta}{\longrightarrow} H^2(\eta, \mathbb{Z})
\end{align*}
$$

Since the pull-back $f^*: H^1(\tilde{\eta}, M) \to H^1(\tilde{\eta}_m, M)$ is canonically identified with the identity map of Hom($G_k, M$), the assertion follows by combining (3.4), (3.5), and (3.6).

**Lemma 3.13.** Let the notation be as above. Let $K \in D_{\text{ctf}}(\eta, \Lambda)$ be a complex.

1. We have

$$
\varepsilon_{0,\Lambda}(T, K, \omega) \cdot \varepsilon_{0,\Lambda}(T, K, \omega')^{-1} = \det(K)_{\chi_{\text{cyc}}(\text{ord}(\omega') - \text{ord}(\omega))}\cdot \varepsilon_{0,\Lambda}(T, K, \omega)^{\text{rk} K}.
$$

2. Let $L$ be a locally constant sheaf of finite free $\Lambda$-modules on $T$. We have

$$
\varepsilon_{0,\Lambda}(T, K \otimes L, \omega) = \det(L)^{a(T,K,\omega)} \cdot \varepsilon_{0,\Lambda}(T, K, \omega)^{\text{rk} L}.
$$

The definition of $a(T,K,\omega)$ is given in Proposition 3.2.

**Proof.** 1. It is given in [21, 4.12.(5)].

2. This follows from the isomorphism $F^{(0,\infty)}(K \otimes L|_\eta) \cong L|_\eta \otimes F^{(0,\infty)}(K)$, Theorem 3.9, and 1.



**4 Main Results**

In this section, we state and prove the main results.

First we give notations. Let $T$ be a trait with perfect residue field and $f: X \to T$ be a morphism of schemes of finite type. Let $\Lambda$ be a finite local ring whose residue characteristic is invertible in $T$. For a constructible complex $K \in D_{\text{ctf}}(X, \Lambda)$ and a closed point $x \in X$ over the closed point $s$ of $T$, we say that $x$ is an at most isolated singularity relative to $K$ if there exists an open neighborhood $U$ of $x$ in $X$ such that the restriction $f|_{U \setminus \{x\}}$ is universally locally acyclic relatively to the restriction of $K$. Let $T(s)$ and $T(x)$ be the henselization of $T$ and its unramified extension with residue field $k(x)$. We denote by $\eta$ and $\eta_x$ their generic points respectively.

Suppose that $x$ is an at most isolated singularity relative to $K$ and $U$ is an open neighborhood of $x$ as above. The restriction of the vanishing cycles complex $R\Phi_f(K)$ to $U \times_T T$ is supported on $x \times_{T(s)} \eta \cong \eta_x$.

**Definition 4.1.** Let the notation be as above. We denote by $R\Phi_f(K)_x$ the pull-back of $R\Phi_f(K)$ by $\eta_x \cong x \times_{T(s)} \eta \to U \times_T T$. This is an object of $D_{\text{ctf}}(\eta_x, \Lambda)$.

For an object $M \in D_{\text{ctf}}(\eta, \Lambda)$, define the total dimension $\dim \text{tot} M$ of $M$ to be $\dim \text{tot} M := \text{rk} M + \text{Sw} M$. 

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Let $\Lambda$ be a finite local ring of residual characteristic $\ell \neq p$. Fix a non-trivial character $\psi: \mathbb{F}_p \to \Lambda^\times$. Let $S$ be a noetherian scheme over $\mathbb{F}_p$. Let

\[
\begin{array}{ccc}
Z' & \xrightarrow{f} & X \\
\downarrow g & & \downarrow t \\
S & \xrightarrow{t} & \mathbb{A}^1_S
\end{array}
\]

be a commutative diagram of $S$-schemes of finite type. Let $K \in D_{ctf}(X, \Lambda)$ be a constructible complex of $\Lambda$-sheaves on $X$. Consider the following conditions on these data.

1. $Y$ is a smooth relative curve over $S$. The morphism $t: Y \to \mathbb{A}^1_S$ is étale.
2. $Z$ is a closed subscheme of $X$ finite over $S$.
3. $g$ is universally locally acyclic relatively to $K$.
4. $f$ is universally locally acyclic relatively to $K$ on $X \setminus Z$.

**Lemma 4.2.** Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of finite type of noetherian schemes. Let $K \in D_{ctf}(X, \Lambda)$ and $L \in D_{ctf}(Y, \Lambda)$ be constructible complexes on $X$ and $Y$. If $f$ (resp. $g$) is universally locally acyclic relatively to $K$ (resp. $L$), so is the composition $gf$ relatively to $K \otimes^L f^*L$.

**Proof.** Let $h_1: Z' \to Z$ be a morphism of noetherian schemes. Let

\[
\begin{array}{ccc}
X' & \xrightarrow{\beta} & Y' \\
\downarrow h_3 & & \downarrow h_2 \\
X & \xrightarrow{f} & Y \\
\downarrow g & & \downarrow t \\
Z & \xrightarrow{h_1} & Z'
\end{array}
\]

be the base change of $(f, g)$ by $h_1$. By [6, Theorem 7.6.9], it is enough to show that, for any $M \in D^+(Z', \Lambda)$, the base change map

\[
(gf)^*Rh_{1*}M \otimes^L (K \otimes^L f^*L) \to Rh_{3*}((\alpha \beta)^*M \otimes^L h_{3*}^*(K \otimes^L f^*L))
\]

defined from the outer square is an isomorphism. Note that, since the morphisms $f$ and $g$ are assumed to be of finite type, the universal local acyclicity is equivalent to the strongly universal local acyclicity. The map (4.2) decomposes as follows.

\[
(LHS) \cong f^*(g^*Rh_{1*}M \otimes^L L) \otimes^L K \to f^*(Rh_{2*}(\alpha^*M \otimes^L h_{2*}^*L)) \otimes^L K \\
\quad \to Rh_{3*}((\alpha^*M \otimes^L h_{2*}^*L) \otimes^L h_{3*}^*K) \cong (RHS)
\]

Then the assertion follows from [6, Theorem 7.6.9].

**Lemma 4.3.** Let the notation be as above. Assume that the conditions from 1 to 4 hold. Then the projection $p_2: X \times_S \mathbb{P}^1_S \to \mathbb{P}^1_S$ is universally locally acyclic relatively to $p_1^*K \otimes^L ((t \circ f) \times \text{id})^*\mathcal{L}_\psi$ on $(X \times_S \mathbb{P}^1_S) \setminus (Z \times_S \infty_S)$ where $p_1: X \times_S \mathbb{P}^1_S \to X$ is the projection and $\infty_S \subset \mathbb{P}^1_S$ is the closed subscheme defined by the section $S \to \mathbb{P}^1_S$ at the infinity.

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Let the commutative diagram
\[
\begin{array}{ccc}
\mathcal{I} := p_1^* K \otimes & (f \times id)^* \mathcal{L}_\psi.
\end{array}
\]
Since \( \mathcal{L}_\psi \) is locally constant on \( \mathbb{A}_S^1 \times S \) and \( g \) is universally locally acyclic relatively to \( K \), \( p_2 : X \times S \mathbb{P}^1_S \to \mathbb{P}^1_S \) is universally locally acyclic relatively to \( \mathcal{I} \) on \( X \times S \mathbb{A}^1_S \). Let \( U = X \setminus Z \) be the complement. We need to show that \( U \times S \mathbb{P}^1_S \to \mathbb{P}^1_S \) is universally locally acyclic relatively to \( \mathcal{I} \). This follows from Lemma 1.2 for the data \( U \times S \mathbb{P}^1_S \to \mathbb{A}^1_S \times S \mathbb{P}^1_S \to \mathbb{P}^1_S, p_1^* K \in D_{ctf}(U \times S \mathbb{P}^1_S, \Lambda), \) and \( \mathcal{L}_\psi \in D_{ctf}(\mathbb{A}^1_S \times S \mathbb{P}^1_S, \Lambda) \). The universal local acyclicity of the projection \( \mathbb{A}^1_p \times S \mathbb{P}^1_p \to \mathbb{P}^1_p \) relative to \( \mathcal{L}_\psi \) is proved in [16, Théorème (1.3.1.2)].

Before stating Theorem 4.5, we prepare some constructions and notation.

Suppose that the conditions from 1 to 4 on (4.1) hold. We replace \( Y \) and \( t \) by \( \mathbb{A}^1_S \) and the identity. Consider the following diagram of topoi

\[
\begin{array}{ccc}
X \times_S \mathbb{P}^1_S & \xrightarrow{\psi f \times id} & (X \times S \mathbb{P}^1_S) \times_{\mathbb{A}^1_S \times S \mathbb{P}^1_S} (A^1_S \times S \mathbb{P}^1_S) \\
p_1 & & \downarrow p' \\
X & \xrightarrow{\psi f} & X \times \mathbb{A}^1_S \\
\downarrow \text{id} & & \downarrow \text{id} \\
X & \xrightarrow{f} & \mathbb{A}^1_S.
\end{array}
\]

Consider the following distinguished triangle
\[
\text{pr}^*_X K \xrightarrow{R\Psi f(K)} R\Phi f(K) \xrightarrow{R \Phi f(K)}
\]
of complexes on \( X \times \mathbb{A}^1_S \). By Proposition 2.2.1, the complexes \( R\Psi f(K) \) and \( R\Phi f(K) \) are objects of \( D_{ctf}(X \times \mathbb{A}^1_S, \Lambda) \) and their formations commute with arbitrary base change \( S' \to S \). Hence we have a distinguished triangle
\[
\begin{array}{ccc}
p^* \text{pr}^*_X K & \xrightarrow{R\Psi f id(p_1^* K)} & p^* \text{pr}^*_X K \\
& \xrightarrow{R \Phi f(K)} & R \Phi f(K).
\end{array}
\]

Let the pull-back of \( \mathcal{L}_\psi \) by \( \mathbb{A}^1_S \times S \mathbb{P}^1_S \to \mathbb{A}^1_p \times S \mathbb{P}^1_p \) be denoted by the same letter \( \mathcal{L}_\psi \). Tensoring \( q^* \mathcal{L}_\psi \), we get the following distinguished triangle
\[
\begin{array}{ccc}
p^* \text{pr}^*_X K \otimes q^* \mathcal{L}_\psi & \xrightarrow{R \Psi f \times id(p_1^* K) \otimes L q^* \mathcal{L}_\psi} & p^* \Phi f(K) \otimes L q^* \mathcal{L}_\psi.
\end{array}
\]

Let \( h : (X \times S \mathbb{P}^1_S) \times_{\mathbb{A}^1_S \times S \mathbb{P}^1_S} (A^1_S \times S \mathbb{P}^1_S) \to (X \times S \mathbb{P}^1_S) \times_{\mathbb{P}^1_S} \mathbb{P}^1_S \) be the morphism defined from the commutative diagram
\[
\begin{array}{ccc}
X \times_S \mathbb{P}^1_S & \xrightarrow{f \times id} & \mathbb{A}^1_S \times S \mathbb{P}^1_S \\
p_2 & & \downarrow h \\
& \downarrow \text{id} \\
& \mathbb{P}^1_S.
\end{array}
\]

Taking the push-forward of the triangle (4.3) through \( h \), we get a distinguished triangle
\[
\begin{array}{ccc}
\tilde{h}_* \left( p^* \text{pr}^*_X K \otimes L q^* \mathcal{L}_\psi \right) & \xrightarrow{\tilde{h}_* \left( R \Psi f \times id(p_1^* K) \otimes L q^* \mathcal{L}_\psi \right)} & \tilde{h}_* \left( R \Phi f(K) \otimes L q^* \mathcal{L}_\psi \right).
\end{array}
\]
on \((X \times_\mathbb{P}^1_S) \times_{\mathbb{P}^1_S} \mathbb{P}^1_S\). Here we denote by \(\hat{h}_*\) the derived push-forward by abuse of notation. Let

\[
\mathcal{K} \to \mathcal{H} \to \mathcal{G} \to \mathcal{G}
\]

be the distinguished triangle which is the restriction of \([4.4]\) to the subtopos \((Z \times_\mathbb{S} \mathbb{S}) \times_{\mathbb{P}^1_S} \mathbb{P}^1_S\). Let \(\bar{g} : (Z \times_\mathbb{S} \mathbb{S}) \times_{\mathbb{P}^1_S} \mathbb{A}^1_S \to \mathbb{A}^1_S \times_{\mathbb{P}^1_S} \mathbb{A}^1_S\) be the morphism of topoi defined from topoi defined from \(g : Z \to S\).

We show that the complex \(\bar{g}_* \mathcal{G}\) is locally constant constructible and of finite tor-dimension and that its formation commutes with arbitrary base change \(S' \to S\). To this end, we prove the following two claims.

**Lemma 4.4.**

1. The complex \(\mathcal{K}\) is acyclic.

2. The complex \(\bar{g}_* \mathcal{H}\) is locally constant constructible and of finite tor-dimension. Its formation commutes with arbitrary base change \(S' \to S\).

**Proof.**

1. Take a point \((z \leftarrow t)\) of the topos \((Z \times_\mathbb{S} \mathbb{S}) \times_{\mathbb{P}^1_S} \mathbb{A}^1_S\). Namely, \(z\) is a geometric point of \(Z\), \(t\) is a geometric point of \(\mathbb{A}^1_S\), and a specialization \(\infty_{g(z)} \leftarrow t\) is given. Here \(\infty_{g(z)}\) is the geometric point of \(\mathbb{P}^1_S\) defined by the compositions

\[
\begin{array}{c}
z \longrightarrow Z \xrightarrow{g} S \longrightarrow \mathbb{P}^1_S.
\end{array}
\]

By \([2.2]\), the stalk \(\mathcal{K}_{(z \leftarrow t)}\) is isomorphic to the following complex:

\[
R\Gamma((\mathbb{A}^1_S \times_\mathbb{S} \mathbb{S}) \times_{\mathbb{P}^1_S} \mathbb{S})_{(f(z), \infty_{g(z)})} \times_{\mathbb{P}^1_S(t)} \mathbb{P}^1_S, K_z \otimes^L \mathcal{L}_\psi)
\]

\[
\cong R\Gamma((\mathbb{A}^1_S \times_\mathbb{S} \mathbb{S}) \times_{\mathbb{P}^1_S} \mathbb{S})_{(f(z), \infty_{g(z)})} \times_{\mathbb{P}^1_S(t)} \mathbb{P}^1_S, \mathcal{L}_\psi \otimes^L K_z,
\]

where \(K_z\) is a constant complex. Since the projection \(\mathbb{A}^1_S \times_\mathbb{S} \mathbb{P}^1_S \to \mathbb{P}^1_S\) is universally locally acyclic relatively to \(\mathcal{L}_\psi\) \([16, Théorème (1.3.1.2)]\) and the restriction of \(\mathcal{L}_\psi\) to \(\mathbb{A}^1_S \times_\mathbb{S} \mathbb{S}\) is zero, we have the assertion.

2. We consider the following cartesian diagrams

\[
X \times_\mathbb{S} \mathbb{A}^1_S \xrightarrow{R\Psi'} (X \times_\mathbb{S} \mathbb{P}^1_S) \times_{\mathbb{A}^1_S \times_\mathbb{S} \mathbb{P}^1_S} (\mathbb{A}^1_S \times_\mathbb{S} \mathbb{A}^1_S) \xrightarrow{\hat{h}_*} (X \times_\mathbb{S} \mathbb{P}^1_S) \times_{\mathbb{P}^1_S} \mathbb{A}^1_S
\]

\[
X \times_\mathbb{S} \mathbb{P}^1_S \xrightarrow{R\Psi \times \text{id}} (X \times_\mathbb{S} \mathbb{P}^1_S) \times_{\mathbb{A}^1_S \times_\mathbb{S} \mathbb{S}} (\mathbb{A}^1_S \times_\mathbb{S} \mathbb{P}^1_S) \xrightarrow{\hat{h}_*} (X \times_\mathbb{S} \mathbb{P}^1_S) \times_{\mathbb{P}^1_S} \mathbb{P}^1_S.
\]

Since \(\mathcal{L}_\psi\) is locally constant on \(\mathbb{A}^1_S \times_\mathbb{S} \mathbb{A}^1_S\), we compute

\[
(4.5)\quad \mathcal{H} \cong \hat{h}_*(R\Psi'(p_1^* K \otimes^L q^* \mathcal{L}_\psi)|_{(Z \times_\mathbb{S} \mathbb{S}) \times_{\mathbb{P}^1_S} \mathbb{A}^1_S})
\]

\[
\cong \hat{h}_*(R\Psi'(p_1^* K \otimes^L (f \times \text{id})^* \mathcal{L}_\psi)|_{(Z \times_\mathbb{S} \mathbb{S}) \times_{\mathbb{P}^1_S} \mathbb{A}^1_S})
\]

\[
\cong R\Psi_{p_2} (p_1^* K \otimes^L (f \times \text{id})^* \mathcal{L}_\psi)|_{(Z \times_\mathbb{S} \mathbb{S}) \times_{\mathbb{P}^1_S} \mathbb{A}^1_S}.
\]

Hence the assertion follows from the conditions 2, Lemma 4.3 and Proposition 2.8. \(\square\)

Let \(\mathcal{C} := \bar{g}_* \mathcal{G}\). This is a constructible complex of locally constant \(\Lambda\)-sheaves of finite tor-dimension on \(\mathbb{A}^1_S \times_{\mathbb{P}^1_S} \mathbb{A}^1_S\).
**Theorem 4.5.** Suppose that the conditions from 1 to 4 hold. Then, the complex $C$ on $\infty_S \times \mathbb{A}_S^1$ constructed above admits the following properties.

1. The formation of $C$ commutes with arbitrary base change $S' \to S$.

2. Assume that $S = \text{Spec}(k)$ is the spectrum of a perfect field $k$. Let $\eta_k$ be the generic point of the henselization $\mathbb{P}^1_{k,(\infty)}$ of $\mathbb{P}^1_k$ at the infinity. Then, by the identification $\eta_k \cong \infty \times \mathbb{P}^1_k$, $C$ is isomorphic to

$$\bigoplus_{z \in \mathbb{Z}} \text{Ind}^{G_{\eta_z}}_{G_{\eta_k}}(F(0,\infty)(R\Phi_f(K)_z) \otimes \mathcal{L}_\psi(f(z) \cdot x'))[-1],$$

where $\eta_z$ is the unramified extension of $\eta_k$ whose residue field is isomorphic to that of $z$, $f(z) \in k(z)$ is the image of the standard coordinate by $z \to \mathbb{A}_k^1$, and $x'$ is the standard coordinate of $\mathbb{A}_k^1 = \mathbb{P}^1_{k(z)} \setminus \infty_z$. The definition of $R\Phi_f(K)_z$ is given in Definition 4.4.

**Proof.** 1. This follows from Lemma 4.4.

2. Define $g(\infty) : \bigsqcup_{z \in \mathbb{Z}} \eta_z \to \eta_k$ to be the disjoint union of the canonical maps. We have

$$(\overline{g_*G})_{\eta_k} \cong g(\infty)_* (G|_{\bigsqcup_{z \in \mathbb{Z}} \eta_z})_{\eta_k} \cong \bigoplus_{z \in \mathbb{Z}} \text{Ind}^{G_{\eta_z}}_{G_{\eta_k}}(G_{\eta_z}).$$

We have

$$G_{\eta_z} \cong R\Gamma((\mathbb{A}_k^1 \times_{k(z)} \mathbb{P}^1_{k(z)}) (f(z),\infty_z) \times_{\mathbb{P}^1_{k(z)},(\infty_z)} \overline{\eta_z}, p^* R\Phi_f(K) \otimes L q^* \mathcal{L}_\psi).$$

Thus we get

$$C \cong \bigoplus_{z \in \mathbb{Z}} \text{Ind}^{G_{\eta_z}}_{G_{\eta_k}}(F(0,\infty)(R\Phi_f(K)_z) \otimes \mathcal{L}_\psi(f(z) \cdot x'))[-1],$$

hence the assertion. \qed

The inverse of the determinant $\det C$ is a locally constant sheaf of rank 1 on $\infty_S \times \mathbb{A}_S^1$, which is the tensor product of the determinants of local Fourier transforms and Artin-Schreier sheaves. We show that the part of Artin-Schreier sheaves itself forms a locally constant sheaf $\mathcal{L}$ of rank 1.

**Lemma 4.6.** Let the notation be as above. Assume that the conditions from 1 to 4 are satisfied. Let $h : Z \to S$ be the structure morphism. Then, the map $\varphi_K : Z \to \mathbb{Z}$ defined by $z \mapsto \dim_{\text{tot}} R\Phi_f(K)_z$, where $s$ is the image $h(z)$ of $z$, is flat in the sense of Definition 2.15.

**Proof.** This is a consequence of the existence of a locally constant complex $C$ as in Theorem 4.5 whose rank at $s \in S$ equals to $\sum_{z \in Z_s} \varphi_K(z)$. It is also proved in [14, Proposition 2.16]. \qed

Let $\varphi_K : Z \to \mathbb{Z}$ be the flat function defined in Lemma 4.6. Applying Lemma 2.16 to $\varphi_K$ and the section $f|Z : Z \to \mathbb{A}_S^1$ defined from the composition $Z \to X \xrightarrow{f} \mathbb{A}_S^1$, we obtain a locally constant $\Lambda$-sheaf $\mathcal{L}_\psi(\varphi_K \cdot f|Z)$ of rank 1 on $\mathbb{A}_S^1$.

**Definition 4.7.** We define the locally constant $\Lambda$-sheaf $\mathcal{L}_{\psi,K,f}$ of rank 1 on $\mathbb{A}_S^1$ to be $\mathcal{L}_\psi(\varphi_K \cdot f|Z)$. 
Theorem 4.8. Let the notation be as in Theorem 4.5. Assume that the conditions from 1 to 4 hold.

1. Let \( \text{pr}: \mathbb{A}_S^1 \times_{\mathbb{P}^1_S} \mathbb{A}_S^1 \to \mathbb{A}_S^1 \) be the second projection. Then, the product \( \det(C)^{-1} \otimes \text{pr}^* \mathcal{L}_{\psi,K,f}^{-1} \in H^1(\mathbb{A}_S^1, \Lambda^x) \) is a tame object in the sense of Definition 2.10.

2. Further assume that \( S \) is connected and normal. Then, the continuous group homomorphism

\[
\rho_t: \pi_1(S)^{ab} \to \Lambda^x
\]

corresponding to \( \langle \det(C)^{-1} \otimes \text{pr}^* \mathcal{L}_{\psi,K,f}^{-1}, 1/x' \rangle \in H^1(S, \Lambda^x) \) defined in Lemma 2.12 has the following properties.

(a) The formation of \( \rho_t \) commutes with arbitrary base change \( S' \to S \).
(b) When \( S = \text{Spec}(k) \) is the spectrum of a perfect field \( k \), \( \rho_t \) coincides with

\[
\prod_{z \in Z} \delta_{k/z}^{\dimtot(R\Phi_{f}(K))} \cdot \varepsilon_{0,\Lambda}(Y(z), R\Phi_f(K)_z, dt) \circ \text{tr}_{k(z)/k}.
\]

Here the definition of \( \delta_{k(z)/k} \) is given at the end of Section 4. When \( S \) is of finite type over \( \mathbb{F}_p \), \( \rho_t \) is uniquely determined by the properties above.

3. Further assume that \( S = \text{Spec}(k) \) is the spectrum of a perfect field \( k \). Let \( t' \in \Gamma(Y, \mathcal{O}_Y) \) be another section which satisfies the conditions 1 and 4. We have

\[
\rho_{t'} \rho_t^{-1} = \prod_{z \in Z} \det(R\Phi_f(F)_z)_{\omega} \circ \text{tr}_{k(z)/k}.
\]

Proof. Let \( \text{Spec}(k) \to S \) be a morphism from the spectrum of a perfect field \( k \). By Theorem 4.5 1 and 2, we have

\[
\text{(4.6)} \quad \det(C)|_{\text{Spec}(k)} \cong \bigotimes_{z \in Z_k} \det(\text{Ind}_{G_{\text{et}}}^{G_k}(F(0,\infty)(R\Phi_{f_k}(K_k)_z) \otimes \mathcal{L}_\psi(f_k(z) \cdot x')))^{-1}
\]

\[
\cong \mathcal{L}_\psi(-\alpha \cdot x') \otimes \bigotimes_{z \in Z_k} \det(\text{Ind}_{G_{\text{et}}}^{G_k}(F(0,\infty)(R\Phi_{f_k}(K_k)_z)))^{-1},
\]

where \( \alpha \) is an element of \( k \) defined by \( \sum_{z \in Z_k} \dimtot R\Phi_{f_k}(K_k)_z \cdot \text{tr}_{k(z)/k}(f_k(z)) \). The subscripts \( (-)_k \) mean the base changes by \( \text{Spec}(k) \to S \). We have the equality (4.4, Proposition 1.2)

\[
\text{(4.7)} \quad \det(\text{Ind}_{G_{\text{et}}}^{G_k}(F(0,\infty)(R\Phi_{f_k}(K_k)_z))) = \delta_{k(z)/k}^{\dimtot(R\Phi_{f_k}(K_k)_z)} \cdot \det(F(0,\infty)(R\Phi_{f_k}(K_k)_z)) \circ \text{tr}_{k(z)/k}.
\]

1. This follows from (4.6), (4.7), and Corollary 3.8.

2. The assertion (a) follows from Theorem 4.5 1. The assertion (b) follows from (4.6) and (4.7). The last assertion follows from the Chebotarev density.

3. This follows from Lemma 3.13 1 and (b).

(Proof of Theorem 1.2) Shrinking \( Y \) and \( S \), we may assume that there is an étale morphism \( t: Y \to \mathbb{A}_S^1 \) such that \( \omega = dt \). The assertion is a special case of Theorem 4.8 2.
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