MULTILAYER SPHERICAL STELLAR CLUSTERS WITH UNIFORM DENSITY

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Various solutions of the kinetic equation for the equilibrium of a gravitating sphere of uniform density with a quadratic gravitational potential and a linear dependence of gravitational force on radius are examined. New analytic solutions are obtained for a uniform sphere with a hollow spherical volume and central mass inside the sphere. Solutions are also obtained for an arbitrary number of spherical layers with the same density, but with different equilibrium distribution functions.

Keywords: spherical clusters: quadratic potential: equilibrium

1. Introduction

Because of their simplicity, uniform clusters with a quadratic gravitational potential have been examined analytically by many authors [1-7]. Solutions have been obtained for uniform cylindrical models with an elliptical cross section and ellipsoidal figures of uniform density, as well as elliptical plane disks with a surface density distribution $\sigma(x,y)$ of the form

\[ \sigma(x,y) = \sigma_0 \sqrt{1 - \frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2}}. \]  

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The equilibrium of collisionless, self-gravitating clusters is determined by a solution that follows from the collisionless Boltzmann kinetic equation with a self-consistent gravitational field determined by the Poisson equation. This approach was first formulated by J. Jeans for studies of stellar clusters [8,9] and was subsequently used by A. A. Vlasov to study collisionless plasmas in electric and magnetic fields [10]. Analytic equilibrium solutions for circular and elliptical disks were obtained in Refs. 3, 11, and 12. Solutions were obtained for equilibrium cylinders with circular and elliptical cross sections in Refs. 1, 12, and 13. Solutions were obtained in Refs. 2 and 12 for uniform ellipsoids and spheroids (ellipsoids of rotation). In the latter case the solution was obtained for the case where gravitational attraction in the directions perpendicular to the axis of symmetry of the spheroid is balanced by a centrifugal force and the resulting solution for a uniform sphere is not spherically symmetric in velocity space. The first equilibrium model of a cluster consisting of stars rotating in circular orbits that was spherically symmetric in velocity space was constructed by Einstein [14] in the framework of the general theory of relativity for an arbitrary spherically symmetric density distribution, including a uniform one. The stability of a Newtonian analog of the uniform model of circular gravitating particles was studied in Refs. 15 and 16. A uniform model, that was spherically symmetric in velocity space, was constructed in Ref. 17, while a model with radial orbits was examined in Ref. 18. In this paper solutions are obtained for a uniform sphere with noncircular particle trajectories that conserve spherical symmetry in a phase space with a hollow sphere located at its center. New solutions are also obtained which consist of an arbitrary number of spherical layers with equal density but with different equilibrium distribution functions.

2. Integrals of motion and particle trajectories

The kinetic equation for the distribution function $f(r, v_r, v_\phi)$ of stars in a stationary, spherically symmetric cluster with constant density $\rho$ in a spherical coordinate system $(r, \theta, \phi, v_r, v_\theta, v_\phi)$ is written in the form [19]

$$v_r \frac{\partial f}{\partial r} + \left( \frac{v_r^2}{r} - \frac{Gm}{r^2} \right) \frac{\partial f}{\partial v_r} - \frac{v_r v_\phi}{r} \frac{\partial f}{\partial v_\phi} = 0,$$

$$2\pi \iint f d v_r d v_\phi \frac{d}{d r} = \frac{\rho}{m}, \quad m = \frac{4\pi}{3} \rho r^3. \tag{2}$$

Here the integral is taken over a limited region of phase space and the following conditions are used:

$$\frac{\partial}{\partial \theta} = \frac{\partial}{\partial \phi} = 0, \quad v_r^2 = v_\theta^2 + v_\phi^2. \tag{3}$$

The solution of Eq. (2) is a function of two integrals of motion, the angular momentum $L$ and the total energy $E$: 

220
here the notation \( M, \rho, \) and \( r_0, \) respectively, denote the mass, density, and radius of the spherical cluster, of which two are specified arbitrarily. In determining the energy \( E \) the arbitrary constant is chosen from the continuity condition for the energy at the boundary with the vacuum, where only the gravitational energy \( E_g = -\frac{GM}{r} \) remains, that goes to zero at infinity, and equals \( E_{s0} = -\frac{GM}{r_0} \) at the boundary of the cluster. The radial dependences of the stellar velocities for different values of \( E \) and \( L \) are given by

\[
v_r^2 = 2 \left( E + \frac{3GM}{2r_0} - \frac{L^2}{2r^2} - \frac{2\pi}{3} G \rho r^2 \right) = 2 \left( E' + \frac{L^2}{2r^2} - \frac{2\pi}{3} G \rho r^2 \right),
\]

\[
v_r = \frac{L}{r}, \quad E' = E + \frac{3GM}{2r_0}.
\] (5)

the solution for a particle motion trajectory in a gravitational potential \( \Phi = kr^2/2 \) has been found in Ref. 20. The closed trajectories of the motion are ellipses with a center at the coordinate origin. The maximum and minimum values of the radius of the trajectory \( r_+ \) are reached for \( v_r = 0 \) and are found from Eq. (5) in the form

\[
r_+^2 = \frac{3E'}{4\pi G \rho} \pm \sqrt{\left( \frac{3E'}{4\pi G \rho} \right)^2 - \frac{3L^2}{4\pi G \rho}}.
\] (6)

The radius of maximum separation of a particle from the center \( r_+ \), equal to the major semiaxis of the ellipse of the trajectory corresponds to zero radial velocity \( v_r \) and the minimum tangential velocity \( v_t, \) so that

\[
r_+^2 = \frac{3E'}{4\pi G \rho} \pm \sqrt{\left( \frac{3E'}{4\pi G \rho} \right)^2 - \frac{3L^2}{4\pi G \rho}}, \quad v_r = 0, \quad v_t = \frac{L}{r_+}.
\] (7)

Thus, for the minimum radius of the particle separation from the center equal to the minor semiaxis of the ellipse of the trajectory, the radial velocity passes through zero and the tangential velocity, through its maximum,

\[
r_+^2 = \frac{3E'}{4\pi G \rho} \pm \sqrt{\left( \frac{3E'}{4\pi G \rho} \right)^2 - \frac{3L^2}{4\pi G \rho}}, \quad v_r = 0, \quad v_t = \frac{L}{r_+}.
\] (8)
The maximum radial velocity \( v_{rm} \) is reached at radius \( r_{rm} \), when the derivative \( dv^2/r \) equals zero. From Eq. (5) we have

\[
r_{rm}^2 = \left( \frac{3L^2}{4\pi G \rho} \right)^{1/2}, \quad v_{rm}^2 = 2 \left[ E' \left( \frac{4\pi G \rho}{3} \right)^2 \right]^{1/2}.
\] (9)

When the expression under the root in Eqs. (6)-(8) equals zero, we obtain the equality \( r_v = r_\rho \), which determines a circular orbit, for which

\[
r^2 = r_v^2 = r_\rho^2 = \frac{3E'}{4\pi G \rho}, \quad v_v = 0, \quad v_\rho = v_\rho, = v_\rho = \frac{L}{r}, \quad E' = L \sqrt{\frac{4\pi G \rho}{3}}.
\] (10)

With Eq. (4), this yields the velocity of the motion along a circle

\[
v_\rho = \pm r \sqrt{\frac{4\pi G \rho}{3}},
\] (11)

when the centrifugal force balances the gravitation of the sphere inside radius \( r \). For \( L = 0 \) all the particles move along radii, so that

\[
v_\rho = 0, \quad v_\rho = \sqrt{2 \left( E' - \frac{2\pi G \rho r^2}{3} \right)}; \quad v_v = 0 \quad \text{for}
\]

\[
r = r_0, \quad E' = \frac{2\pi G \rho r^2}{3} = \frac{GM}{2r_0}, \quad E = -\frac{GM}{r_0}.
\] (12)

3. Analytic spherically symmetric solutions for a sphere

3.1. Uniform sphere. The distribution function for a uniform Einstein sphere with radius \( r_0 \) is given by [16]

\[
f = \frac{\rho}{\pi m} \delta(v_\rho) \delta \left( v_\rho^2 - \frac{4\pi G \rho}{3} r^2 \right), \quad r \leq r_0.
\] (13)

Using the integrals of motion (4) and (5) for a sphere with radius \( r_\rho \), the solution can be written in the form
A distribution function of the following form for a uniform sphere that is spherically symmetric in phase space is examined in Ref. 17:

\[
f = \frac{\rho}{\pi m} \delta \left( 2 E' \pm 2 L \sqrt{\frac{4\pi G \rho}{3}} \right) = \frac{\rho}{\pi m} \delta \left[ v_r^2 + \left( v_r \pm \sqrt{\frac{4\pi G \rho}{3}} r \right)^2 \right].
\]  

(14)

Here \( \theta(x) \) is the Heaviside function, equal to unity for a positive and zero for a negative value of the argument. In the latter expression (15), it is assumed that \( \Omega^2 = 4\pi G \rho / 3 = 1 \) and \( 2 E' = v_r^2 + v_t^2 + \Omega^2 r^2 = v_r^2 + v_t^2 + r^2 \). The equation for the density is written in the form

\[
\rho = \pi m \int d v_t \, d v_r^2 = \frac{\rho}{\pi} \int_0^\infty \frac{0 \left( (1-r^2)(1-v_r^2) - v_r^2 \right)}{\sqrt{(1-r^2)(1-v_r^2) - v_r^2}} d v_r \, d v_r^2.
\]  

(16)

The limits of integration over phase space are determined by the condition that the argument of the function \( \theta(x) \) be positive. For this condition, after integration with respect to \( d v_r^2 \) and \( d v_r \) we obtain

\[
\frac{1 - v_r^2}{(1-r^2)} \int_0^1 d v_r^2 \frac{d v_r^2}{\sqrt{(1-r^2)(1-v_r^2) - v_r^2}} = 2 \sqrt{1-r^2 - v_r^2} \left(1-r^2\right),
\]

\[
\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} d v_r \sqrt{\frac{1-r^2 - v_r^2}{1-r^2}} = \int \sqrt{1-x^2} \, dx = \frac{\pi}{2}.
\]  

(17)

Thus, in the expression for \( \rho \) we obtain an identity in Eq. (15), which confirms the self consistency of the choice of the distribution function (15).

The distribution function for a uniform sphere with purely radial orbits can be written in the form [18]

\[
f = \frac{\rho}{\pi^2 m} \frac{\delta(v_r^2)}{\sqrt{-2 E' + \frac{GM}{r_0}}} = \frac{\rho}{\pi^2 m} \frac{\delta(v_r^2)}{\sqrt{\frac{4\pi G \rho}{3} \left(v_0^{-2} - r^2\right) - v_r^2}}.
\]  

(18)
3.2. Sphere with a cut-out core. If all the particles have angular momentum $L$, then the minimum distance of a particle from the center is $r_{\text{min}} = L/v_\perp$, so an empty sphere of this radius develops inside a spherical region of this sort. If the density of matter around this sphere is uniform, it is easy to find analytically the trajectory of the particles flying outside the empty sphere. The presence of an empty sphere, however, leads to deviations from a quadratic dependence of the gravitational potential in the material, where the trajectories of the particles are no longer closed. An analytic solution cannot be found under these conditions. In order to conserve the quadratic dependence and obtain an analytic solution for an equilibrium spherically symmetric cluster, in the center of the cluster we place a body formed by compression into a point of mass $M_c = 4\pi \rho r_{\text{min}}^3/3$ equal to the mass of the uniform sphere with the radius of the void and with the density of the cluster. In this case, gravitation in the substance surrounding the hollow sphere will be the same as in the case of the completely uniform sphere and Eqs. (4)-(13) remain valid for $r \geq r_{\text{min}}$.

Let us consider a uniform sphere of radius $r_o$ with a hollow sphere of radius $r_1$ at its center. This configuration arises from a cluster of bodies moving along identical ellipses around the center of a sphere for which the minor and major semiaxes of the ellipse, $r_1$ and $r_0$, are equal, respectively, to $r$ and $r_0$ as defined in Eqs. (7) and (8). For specified $r_1$ and $r_0$ we find the integrals of motion $E'$ and $L$ of the particles to be

$$E'_0 = \frac{2\pi G \rho}{3} \left( \frac{r^2}{r_0^2 + r_1^2} \right), \quad L_0 = \frac{4\pi G \rho}{3} r_0^2 r_1^2.$$

(19)

The solution (20) is a generalization of the solution (18) for purely radial orbits to the case of particles with nonzero angular momentum: for $L_0 = r_1 = 0$, Eq. (20) yields the solution (18).
3.3. **Multilayer sphere with uniform density.** The solutions (18) and (20) can be used to construct a more complicated equilibrium model for a sphere with uniform density consisting of several layers with different distribution functions. At the center of the sphere with a hollow core, instead of a central gravitating mass it is possible to place a sphere with radial orbits and to obtain a solution of the form

\[
f = \frac{\rho}{\pi^2 m} \frac{\delta \left( \frac{v_r^2}{3} \right) - v_r^2}{\sqrt{\frac{4\pi G \rho}{3} \left( \frac{r^3 - r_1^3}{r_1^3} \right) - v_r^2}} \quad \text{for} \quad r \leq r_1; \tag{21}
\]

with

\[
f = \frac{\rho}{\pi^2 m} \frac{\delta \left( \frac{v_r^2}{3} - \frac{4\pi G \rho}{3} \frac{r_1^3 r_2^3}{r^3} \right) - v_r^2}{\sqrt{\frac{4\pi G \rho}{3} \left( \frac{r_2^3 - r_1^3}{r_1^3} \right) - v_r^2}} \quad \text{for} \quad r_1 \leq r \leq r_2. \tag{22}
\]

\[\text{This figure was drawn by O. Yu. Tsupko, to whom the author is sincerely grateful.}\]
This sphere can be surrounded by yet another layer of matter with the distribution function (22), in which instead of the boundaries of the layers \((r_1, r_2)\) the boundaries \((r_{i+1}, r_i)\) are specified. In the same way it is possible to obtain a multilayer sphere model where each layer is determined by a distribution function of the type (22) and a specified thickness \((r_{i+1} - r_i)\), where \(0 \leq r_1 \leq r_2 \leq \ldots \leq r_0\) and \(r_0\) is the outer boundary of the sphere with uniform density.

In the central region of a multilayer sphere, besides a configuration with radial orbits (18), there may be a sphere with circular orbits (13), as well as a sphere with a more complicated distribution function (17). Instead of an arbitrary spherical layer at \(r_i < r < r_{i+1}\) with the distribution function (20), there may be a layer with circular orbits and the distribution function (13).

4. Discussion

The solutions of the kinetic equation for a uniform sphere of gravitating particles (stars) have been obtained by various authors, many of whom examined distribution functions that are not spherically symmetric [2,5,6]. Three types of models with a spherically symmetric distribution function have also been examined: a model with circular orbits [14], a model with radial orbits [18], and a model of a more general type [17]. In this paper new solutions have been obtained for a uniform sphere that has a hollow sphere with a central gravitating mass at its center which may simulate a region of a galactic core with a supermassive black hole at its center. The resulting solutions describe a model with elliptical trajectories that are intermediate between radial \((L=0)\) and circular \((u_r = 0)\) trajectories. Based on these solutions we obtain a multilayer spherical model with a uniform density for both hollow and filled cores, with different distribution functions in the core and in an separate layer which must belong to one of the four basic solutions indicated above.

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