On the volume of the intersection of two $L^n_p$ balls

by

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1. Introduction

This note deals with the following problem, the case $p = 1, q = 2$ of which was introduced to us by Vitali Milman: What is the volume left in the $L^n_p$ ball after removing a $t$-multiple of the $L^n_q$ ball? Recall that the $L^n_p$ ball is the set \( \{(t_1, t_2, \ldots, t_n); t_i \in \mathbb{R}, n^{-1} \sum_{i=1}^{n} |t_i|^r \leq 1\} \) and note that for $0 < p < q < \infty$ the $L^n_q$ ball is contained in the $L^n_p$ ball.

In Corollary 4 below we show that, after normalizing Lebesgue measure so that the volume of the $L^n_p$ ball is one, the answer to the problem above is of order $e^{-ct^n_p^{p/q}}$ for $T < t < \frac{1}{2}n^{\frac{1}{p}-\frac{1}{q}}$, where $c$ and $T$ depend on $p$ and $q$ but not on $n$.

The main theorem, Theorem 3, deals with the corresponding question for the surface measure of the $L^n_p$ sphere. Theorem 3 and Corollary 4 together with some other remarks form Section 3. In Section 2 we introduce a class of random variables to be used in the proof of the main theorem. These random variables are related to $L^n_p$ in the same way that Gaussian variables are related to $L^2$. 

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2. Preliminaries

Here we introduce a class of random variables to be used in the proof of the main theorem and summarize some of their properties. Fix a $0 < p < \infty$ and let $x, x_1, x_2, \ldots, x_n$ be independent random variables each with density function $c_p e^{-t^p}$, $t > 0$. Note that necessarily $c_p = p/\Gamma(1/p)$. The first claim is known, though we could not locate a reference.

**Lemma 1.** Put $S = \left(\sum_{i=1}^{n} x_i^p\right)^{1/p}$, then $\left(\frac{x_1}{S}, \frac{x_2}{S}, \ldots, \frac{x_n}{S}\right)$ is uniformly distributed over the positive quadrant of the sphere of $l_p^n$, i.e., over the set $\Delta_p = \{(t_1, t_2, \ldots, t_n) ; t_i \geq 0, \sum t_i^p = 1\}$ equipped with the $(n-1)$-dimensional normalized Lebesgue measure. Moreover, $\left(\frac{x_1}{S}, \frac{x_2}{S}, \ldots, \frac{x_n}{S}\right)$ is independent of $S$.

**Proof.** For any Borel subset $A$ of $\Delta_p$,

$$P\left(\left(\frac{x_1}{S}, \frac{x_2}{S}, \ldots, \frac{x_n}{S}\right) \in A \mid S = a\right) =$$

$$\lim_{\epsilon \to 0} \frac{P((x_1, \ldots, x_n) \in \mathbb{R}_+ A \& a - \epsilon \leq S \leq a + \epsilon)}{P(a - \epsilon \leq S \leq a + \epsilon)}$$

$$= \lim_{\epsilon \to 0} \int_{(a-\epsilon)^p \leq \sum t_i^p < (a+\epsilon)^p} e^{-\sum t_i^p} \frac{dt}{\int_{(a-\epsilon)^p \leq \sum t_i^p < (a+\epsilon)^p} e^{-\sum t_i^p} \ dt}$$

$$\leq \limsup_{\epsilon \to 0} e^{-(a-\epsilon)^p + (a-\epsilon)^p} \int_{(a-\epsilon)^p \leq \sum t_i^p < (a+\epsilon)^p} dt / \int_{(a-\epsilon)^p \leq \sum t_i^p < (a+\epsilon)^p} dt$$

$$= \lambda(A),$$

where $\lambda$ is the normalized Lebesgue measure on $\Delta$. Similarly,

$$P\left(\left(\frac{x_1}{S}, \frac{x_2}{S}, \ldots, \frac{x_n}{S}\right) \in A \mid S = a\right) \geq \lambda(A).$$

This proves that $P\left(\left(\frac{x_1}{S}, \frac{x_2}{S}, \ldots, \frac{x_n}{S}\right) \in A\right) = \lambda(A)$ and that $\left(\frac{x_1}{S}, \frac{x_2}{S}, \ldots, \frac{x_n}{S}\right)$ is independent of $S$. 


In the next claim we gather some more properties of the random variables \( x_i \).

**Lemma 2.** Let \( x, x_1, \ldots, x_n \) be as above, then

1. \( c_p \) is bounded away from zero and infinity when \( p \to \infty \).

2. For all \( h > 0 \) and all \( 0 < p < \infty \), \( \mathbb{E} e^{-hx^p} = \left( \frac{1}{1+h} \right)^{1/p} \). In particular,

\[
\mathbb{E} e^{-hx^p} \geq e^{-h/p} \text{ for all } h > 0 \text{ and } \mathbb{E} e^{-hx^p} \leq e^{-h/2p} \text{ for all } 0 < h \leq 1.
\]

3. For all \( 0 < u < \infty \) and all \( 0 < p < \infty \), \( P(x^p > u) \geq \frac{c_p}{2p} e^{-2u} \). If \( p \geq 1 \) and \( u \geq 1 \), then also \( P(x^p > u) \leq \frac{c_p}{p} e^{-u/2} \). In particular, for \( p \geq 1 \) and all \( u \), \( P(x^p > u) \leq C e^{-u/2} \) for some universal \( C \).

4. For all \( 1 \leq p \leq q < \infty \), \( \mathbb{E} \left( \sum_{i=1}^n x_i^q \right)^{1/q} \) is equivalent, with universal constants, to \( q^{1/p} n^{1/q} \), if \( q \leq \log n \), and to \( (\log n)^{1/p} \) otherwise.

**Proof.**

1. Follows easily from the fact that \( c_p = p/\Gamma(1/p) = \Gamma(\frac{1}{p} + 1)^{-1} \).

2. is a simple computation.

3. is also simple, here is a sketch of the proof.

\[
P(x^p > u) = c_p \int_{u^{1/p}}^\infty e^{-t^p} \, dt
\geq c_p \int_{u^{1/p}}^{(u+1)^{1/p}} \frac{pt^{(p-1)}}{p(u+1)^{(p-1)/p}} e^{-t^p} \, dt
= \frac{c_p}{p(u+1)^{(p-1)/p}} \left( 1 - \frac{1}{e} \right) e^{-u}
\geq \frac{c_p}{2p(u+1)} e^{-u}
\geq \frac{c_p}{2p} e^{-2u}.
\]

The other inequality in 3 is proved in a similar way.

4. First note that for all \( 0 < p, q < \infty \)

\[
\mathbb{E}x^q = c_p \int_0^\infty t^q e^{-t^p} \, dt = \frac{c_p}{p} \Gamma\left( \frac{q+1}{p} \right)
\]
so that, by the triangle inequality and 1, if $1 \leq p \leq q < \infty$

$$\mathbb{E}\left(\sum_{i=1}^{n} x_i^q\right)^{1/q} \leq \left(\sum_{i=1}^{n} \mathbb{E} x_i^q\right)^{1/q} = \left(\frac{c_p}{p} \Gamma\left(\frac{q+1}{p}\right)\right)^{1/q} n^{1/q} \leq C q^{1/p} n^{1/q}$$

for some universal $C$. For the lower bound in the case $q \leq \log n$, divide \{1, 2, \ldots, n\} into approximately $n/e^q$ disjoint sets of cardinality approximately $e^q$ each, then

$$\mathbb{E}\left(\sum_{i=1}^{n} x_i^q\right)^{1/q} = \mathbb{E}\left(\sum_{j} \left(\sum_{i \in \sigma_j} x_i^q\right)^{q/q}\right)^{1/q}$$

$$\geq \mathbb{E}\left(\sum_{j} \left(\max_{i \in \sigma_j} x_i\right)^q\right)^{1/q}$$

$$\geq \left(\sum_{j} \left(\mathbb{E}\max_{i \in \sigma_j} x_i\right)^q\right)^{1/q}$$

$$\geq c' (\log e^q)^{1/p} (n/e^q)^{1/q}$$

$$\geq c'' q^{1/p} n^{1/q}.$$

Now, for the case $q > \log n$ we note first that, by 3,

$$P\left(\max_{1 \leq i \leq n} x_i > t\right) \geq 1 - \left(1 - \frac{c_p}{2p} e^{-2t^p}\right)^n.$$

For $n$ smaller than an absolute multiple of $p$, the lower bound follows easily from the fact that $\mathbb{E} x_1$ is larger that a universal positive constant, so assume that $n \geq 20p/c_p$ and put $t = 2^{-1/p} \left(\log \frac{nc_p}{2p}\right)^{1/p}. Then, for some universal $c$,

$$P\left(\max_{1 \leq i \leq n} x_i > c (\log n)^{1/p}\right) \geq 1/2.$$

In particular, $\mathbb{E}\max_{1 \leq i \leq n} x_i \geq c (\log n)^{1/p}$, which implies the lower bound in this case since $\left(\sum_{i=1}^{n} x_i^q\right)^{1/q}$ is universally equivalent to $\max_{1 \leq i \leq n} x_i$. The upper bound in this case, though a bit harder, is also standard and since we don’t use it in the sequel we shall leave it to the reader.
The statement in 4, for the case \( p = 2 \), was noticed by the first named author several years ago while seeking a precise estimate for the dimension of the Euclidean sections of \( l^n_p \) spaces (see [MS] p.145 Remark 5.7). The original proof was more complicated. The proof presented here is an adaptation of a proof of the case \( p = 2 \) shown to us by J. Bourgain.

3. The main result

**Theorem 3.** For all \( 1 \leq p < q < \infty \) there are constants \( c = c(p,q) \), \( C = C(p,q) \) and \( T = T(p,q) \) such that if \( \mu \) denotes the normalized Lebesgue measure on the positive quadrant of the unit sphere of \( L^n_p \) then

\[
\mu(\|u\|_{L^n_q} > t) \leq \exp(-ct^{p}n^{p/q}) \tag{1}
\]

for all \( t > T \), and

\[
\mu(\|u\|_{L^n_q} > t) \geq \exp(-Ct^{p}n^{p/q}) \tag{2}
\]

for all \( 2 \leq t \leq \frac{1}{2}n^{\frac{1}{p}-\frac{1}{q}} \).

Moreover, for \( q > 2p \) (or any other universal positive multiple of \( p \)), one can take \( c(p,q) = \gamma_p \), \( C(p,q) = \Gamma_p \) and \( T(p,q) = \tau \min\{q, \log n\}^{1/p} \leq q^{1/p} \). Here \( \gamma, \Gamma \) and \( \tau \) are universal constants.

**Proof.** By Lemma 1 above,

\[
\mu(\|u\|_{L^n_q} > t) = P\left(n^{\frac{1}{p}-\frac{1}{q}}\left(\sum_{i=1}^{n} x_i^q\right)^{1/q}/\left(\sum_{i=1}^{n} x_i^p\right)^{1/p} > t\right)
\]

where \( x_i \) are independent random variables each with density \( c_p e^{-t^p} \). Assume, for the simplicity of the presentation, that \( n \) is even. Put \( S = (\sum_{i=1}^{n} x_i^p)^{1/p} \) and let \( p_j, \ j = 1, 2, \ldots, n/2 \) be positive numbers with sum \( \leq 1/2 \). Then

\[
P\left(n^{\frac{1}{p}-\frac{1}{q}}\left(\sum_{i=1}^{n} x_i^q\right)^{1/q}/\left(\sum_{i=1}^{n} x_i^p\right)^{1/p} > t\right) =
\]
\[= P\left(\sum_{i=1}^{n} x_i^q > \frac{t^q \left(\sum_{i=1}^{n} x_i^p\right)^{q/p}}{n^{p-1}}\right)\]
\[\leq \sum_{i=1}^{n/2} P\left(x_i^* > t p_j^{1/q} S / n^{\frac{1}{p} - \frac{1}{q}}\right) + P\left(\sum_{i=n/2+1}^{n} x_i^q > t^q S^q / 2n^{\frac{q}{p}-1}\right) \tag{3}\]

where \(\{x_i^*\}\) denotes the nonincreasing rearrangement of \(|x_j|\).

Since
\[\sum_{j=n/2+1}^{n} x_i^q \leq \frac{n}{2} x_{n/2}^q \leq \frac{n}{2} \left(\sum_{i=1}^{n/2} x_i^{*p}\right)^{q/p}\]
\[\leq 2^{\frac{q}{p}-1} S^q / n^{\frac{q}{p}-1},\]
we get that, if \(t \geq 2^{1/p}\), the second term in (3) is zero.

To evaluate the first term in (3), fix \(1 \leq j \leq n/2\). Then,
\[P(x_j^* > t p_j^{1/q} S / n^{\frac{1}{p} - \frac{1}{q}}) \leq \binom{n}{j} P(x_1, \ldots, x_j > t p_j^{1/q} S / n^{^\frac{1}{p} - \frac{1}{q}})\]
\[\leq \binom{n}{j} P\left(x_1^p, \ldots, x_j^p > t p_j^{p/q} \sum_{i=j+1}^{n} x_i^p / n^{1-p/q}\right).\]

From Lemma 2 (first 3 and then 2) we get that the last expression is dominated by
\[\binom{n}{j} C^j \mathbb{E} \exp\left(-j p_j^{p/q} t^{p} \sum_{i=j+1}^{n} x_i^p / n^{1-p/q}\right)\]
\[\leq \binom{n}{j} C^j \exp\left(-j p_j^{p/q} t^{p} (n-j)/ 2pn^{1-p/q}\right)\]
for some universal \(C\). Note that the last inequality holds if \(j n^{p/q-1} p_j^{p/q} t^{p} \leq 1\). If this is not the case the probability we are trying to evaluate is zero. Finally, the last term is dominated by
\[\exp\left(j \left(\log \frac{en}{j} + C - \frac{p_j^{p/q} t^{p} n^{p/q}}{4p}\right)\right). \tag{4}\]
Now, for \( \alpha \) to be chosen momentarily, let \( p_j, j = 1, \ldots, n/2 \), be such that

\[
j \left( \log \frac{en}{j} + C - \frac{p_j^{p/q} q_n^{p/q}}{4p} \right) = -\alpha n^{p/q} t^p
\]

i.e.,

\[
p_j = \left( 4p \frac{\log \frac{en}{j}}{t^p n^{p/q}} + \frac{4Cp}{t^p n^{p/q}} + \frac{\alpha}{j} \right)^{q/p}.
\]

We thus get that, for some universal constant \( C \),

\[
p_j \leq 2^{\frac{q}{p}-1} \frac{(Cp)^{q/p} (\log \frac{en}{j})^{q/p}}{t^q n} + 2^{\frac{q}{p}-1} \alpha^{q/p} \frac{(4p)^{q/p}}{j^{q/p}}.
\]

(5)

It is easy to see that, for \( 1 \leq p < q < \infty \),

\[
\sum_{j=1}^{n/2} (\log \frac{en}{j})^{q/p} \leq An \min \{q^{1/p}, (\log n)^{q/p}\}
\]

for some universal \( A \). Thus the sum over \( j \) of the first terms in (5) is smaller than \( 1/4 \) if, for some universal \( \gamma \), \( t > \gamma \min \{q^{1/p}, (\log n)^{1/p}\} \). The sum over \( j \) of the second terms in (5) is bounded by \( 1/4 \) if \( \alpha < B \frac{1}{p} (\frac{q}{p} - 1)^{p/q} \), for some universal \( B \). Choosing \( \alpha \) to satisfy this inequality and using (3), (4) and (5) we get that, for \( t > \gamma \min \{q^{1/p}, (\log n)^{1/p}\} \),

\[
\mu(\|u\|_{L_q^n} > t) \leq \frac{n}{2} e^{-\alpha n^{p/q} t^p}.
\]

Under the conditions on \( t \), the factor \( n/2 \) can be absorbed in the second term (changing \( \alpha \) to another constant of the same order of magnitude as a function of \( p \)), thus proving (1).

We now turn to the proof of the lower bound (2) which is simpler. Using Claim 1 again,

\[
\mu(\|u\|_{L_q^n} > t) = P(n^{\frac{1}{p} - \frac{1}{q}} \left( \sum_{i=1}^{n} x_i^q \right)^{q/q} / \left( \sum_{i=1}^{n} x_i^p \right)^{1/p} > t) \\
\quad \geq P(x_1 > St/n^{\frac{1}{p} - \frac{1}{q}}) \\
\quad = P \left( x_1 > \frac{t}{(n(1 - p/q) - t) / n} \left( \sum_{i=2}^{n} x_i^p \right)^{1/p} \right).
\]
Since $t^p \leq \frac{1}{2} n^{(1-p/q)}$, this dominates

$$P(x_1 > \frac{2^{1/p} t}{n^{1-p/q}} \left(\sum_{i=2}^{n} x_i^p\right)^{1/p}).$$

Now, by Claim 2.3.,

$$P\left(x_1 > \frac{2^{1/p} t}{n^{1-p/q}} \left(\sum_{i=2}^{n} x_i^p\right)^{1/p}\right) \geq \frac{c_p}{2p} E \exp(-4t^p \sum_{i=2}^{n} x_i^p/n^{(1-p/q)})$$

$$= \frac{c_p}{2p} \left(E \exp(-4t^p x_1^p/n^{(1-p/q)})\right)^{n-1}$$

$$= \frac{c_p}{2p} \left(1 + \frac{4t^p}{n^{1-p/q}}\right)^{(n-1)/p} \quad \text{(by Claim 2.2.)}$$

$$\geq \frac{c_p}{2p} \exp\left(-\frac{4t^p(n-1)}{pn^{1-p/q}}\right)$$

$$\geq \frac{c_p}{2p} e^{4t^p n^{p/q}/p}.$$

Finally observe that, since $c_p$ is bounded away from zero and $t \geq 2$, the factor $\frac{c_p}{2p}$ can be absorbed in the second term (changing 4 to another universal constant).

Remarks:

1. It follows from the proof that, for $n$ large enough and $q$ close to $p$, one can take $c(p, q) = \frac{\alpha}{p} \left(\frac{\alpha}{p} - 1\right)$ for some universal constant $c$.

2. It follows from the statement of the theorem that, for $q = \infty$,

$$\mu(\|u\|_{\infty} > t) \leq e^{-\gamma t^p/p}$$

for all $t > \tau (\log n)^{1/p}$, and

$$\mu(\|u\|_{\infty} > t) \geq e^{-\Gamma t^p/p}$$

for all $2 \leq t \leq \frac{1}{2} n^{\frac{1}{p}}$, where $\gamma$, $\Gamma$ and $\tau$ are universal constants.

3. Note that it follows from Claim 1 and Claim 2.4. that the order of magnitude of $T$ is the correct one.
4. The restriction $p \geq 1$ in Theorem 3 above and in Corollary 4 below can be replaced by $p > 0$ if one replaces the inequality $t \geq 2$ with $t \geq d$, for some $d$ depending only on $p$ and $q$, and removes the “moreover” part. We didn’t check the dependence of the constants on $p$ and $q$ in this case.

The last remark is that one can get a similar statement for the full balls. We state it as a corollary.

**Corollary 4.** For all $1 \leq p < q < \infty$ there are constants $c = c(p, q)$, $C = C(p, q)$ and $T = T(p, q)$ such that if $\nu$ denotes the normalized Lebesgue measure on the ball of $L^n_p$ then, for all $n$ large enough,

$$\nu(\|u\|_{L^n_q} > t) \leq \exp(-ct^n p/n^{p/q})$$  \(6\)

for all $t > T$, and

$$\nu(\|u\|_{L^n_q} > t) \geq \exp(-Ct^n p/n^{p/q})$$  \(7\)

for all $2 \leq t \leq \frac{1}{2}n^{\frac{1}{p}} - \frac{1}{q}$. Moreover, for $q > 2p$ (or any other universal positive multiple of $p$), one can take $c(p, q) = \frac{\gamma}{p}$, $C(p, q) = \Gamma$ and $T(p, q) = \tau \min\{q, \log n\}^{1/p} \leq q^{1/p}$, where $\gamma$, $\Gamma$ and $\tau$ are universal constants.

The proof follows easily from Theorem 3 and the formula

$$\nu(A) = n \int_0^1 r^{n-1} \mu\left(\frac{A}{r}\right) dr$$

which holds for all Borel sets $A$ in the ball of $L^n_p$. 

References

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