MULTIPOLE FORMULAE FOR GRAVITATIONAL LENSING SHEAR AND FLEXION

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ABSTRACT

The gravitational lensing equations for convergence, potential, shear, and flexion are simple in polar coordinates and separate under a multipole expansion once the shear and flexion spinors are rotated into a “tangential” basis. We use this to investigate whether the useful monopole aperture–mass shear formulae generalize to all multipoles and to flexions. We re-derive the result of Schneider and Bartelmann that the shear multipole \( m \) at radius \( R \) is completely determined by the mass multipole at \( R \), plus specific moments \( Q^m \) and \( Q^m_{out} \) of the mass multipoles internal and external, respectively, to \( R \). The \( m \geq 0 \) multipoles are independent of \( Q^m_{out} \). But in contrast to the monopole, the \( m < 0 \) multipoles are independent of \( Q^m_{in} \). These internal and external mass moments can be determined by shear (and/or flexion) data on the complementary portion of the plane, which has practical implications for lens modeling. We find that the ease of \( E/B \) separation in the monopole aperture moments does not generalize to \( m \neq 0 \); the internal monopole moment is the only nonlocal \( E/B \) discriminant available from lensing observations. We have also not found practical local \( E/B \) discriminants beyond the monopole, though they could exist. We show also that the use of weak-lensing data to constrain a constant shear term near a strong-lensing system is impractical without strong prior constraints on the neighboring mass distribution.

Key words: gravitational lensing – methods: analytical

1. INTRODUCTION

Weak gravitational lensing measurements of the shear \( \gamma \) are often used to constrain the mass distributions in galaxies, cluster of galaxies, or even larger scale objects. An exceptionally useful set of aperture–mass formulae gives nonparametric relations between the monopole moments of the lensing shear and the lensing mass. The aperture–mass formulae have the following interesting aspects.

1. A relation between the mean tangential shear component \( \gamma_t \) on a circle of radius \( R \) and the mean convergence at and within \( R \) (Kaiser 1995):

\[
\langle \gamma_t \rangle_R = \langle \kappa \rangle_R - \langle \kappa \rangle_{\leq R}.
\]

(1)

Recall that the convergence \( \kappa \) is the surface mass density in units of the lensing critical density. The monopole of the tangent-shear component at \( R \) is dependent upon the mass at or interior to \( R \) in a simple way, and is independent of the mass exterior to \( R \).

2. A rearrangement of the aperture–mass formula is (Fahlman et al. 1994)

\[
\langle \kappa \rangle_{\leq R} = 2 \int_{R}^{\infty} dr \, r^{-1} \langle \gamma_t \rangle_{r}.
\]

(2)

This relation allows a model-independent determination of the mass monopole within \( R \) using only shear measures at \( r \geq R \). A generalization to radially weighted aperture masses is given by Kaiser et al. (1994) and Schneider (1996), which can allow the shear integral to have finite support.

3. The shear monopole admits an instantaneous test for the presence of “\( B \)-mode” deflections. If we allow the lensing potential to be sourced by the normal scalar mass distribution \( \kappa_E \) plus a pseudoscalar mass \( \kappa_B \), then we find that the tangent-shear formula (1) applies only to the \( E \)-mode mass. The \( B \)-mode mass produces a monopole of the “skew”-shear component \( \gamma_s \) rotated 45° from the tangent direction.\(^1\) The skew shear is a direct test for \( B \)-mode sources within the aperture:

\[
\langle \gamma_s \rangle_R = \langle \kappa_B \rangle_{\leq R} - \langle \kappa_B \rangle_R.
\]

(3)

Hence in real observations the monopole skew shear should be null. This is a special case of the general rule that any \( E \)-mode mass measurement should be nulled when all shears are rotated by 45° (Stebbins et al. 1996; Luppino & Kaiser 1997).

In this paper, we ask: can these three useful formulae relating monopole shear moments to monopole mass moments be extended to multipole moments? Schneider & Bartelmann (1997)[SB97] offer a division of the shear multipoles into internal and external terms (their Appendix B), generalizing property (1). They further extend property (2), the ability to determine the mass moment from a closed-form integral of the shear, to the general multipole case. We offer here a simpler re-derivation of their results, extending them to the case where \( B \)-mode lensing may be present. In the process, we also inquire whether property (3) can be extended: is there a simple test for \( B \)-mode mass in the shear multipole signals?

We further ask: are there equivalent properties for the higher order lensing distortions, i.e., “flexions” (Bacon et al. 2006)? The expected answer is yes, since these are sourced by the same two scalar degrees of freedom (dof) \( \kappa_{E,B} \) that produce the shear field.

In Section 2, we derive a simple differential relation between the moments of convergence (mass), shear, and flexion, using a very compact notation for the standard lensing equations. This derivation will allow for both \( E \)- and \( B \)-mode source terms. In

\(^1\) The skew-shear component is frequently designated by the misnomer “radial shear.”
Section 3, we will examine the multipole generalizations of Equations (1)–(3). In Section 4, we give an application of these multipole formulae to a common problem in lens modeling: a galaxy-scale strong lens is embedded in a more extended group or cluster potential. We show how shear weakens could be used to constrain the cluster potential without assuming a particular geometry for the cluster mass.

2. Lensing Formulae in Polar Coordinates

First we recast the familiar lensing equations into polar coordinates. In the flat-limit sky, lensing is compactly described by the differential operators of Castro et al. (2005)

\[ \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \equiv e^{i\theta} \left( \frac{\partial}{\partial r} + i \frac{\partial}{r \partial \theta} \right). \]

\[ \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \equiv e^{-i\theta} \left( \frac{\partial}{\partial r} - i \frac{\partial}{r \partial \theta} \right). \]

For an arbitrary deflection field \((\alpha_x, \alpha_y)\) defined on the plane of the sky, we define a complex deflection \(\alpha \equiv \alpha_x + i \alpha_y\). The deflection field can be decomposed into a curl-free \(E\)-mode part and a divergence-free \(B\)-mode part by defining a complex potential \(\psi = \psi_E + i \psi_B\) from the scalar potential \(\psi_E\) and pseudoscalar potential \(\psi_B\):

\[ \alpha \equiv \partial \psi. \]

The shear imposed on the background sources is given by the derivatives of the deflection:

\[ \gamma \equiv \gamma_1 + i \gamma_2 = \frac{1}{2} \partial \alpha. \]

Furthermore, the complex convergence \(\kappa \equiv \kappa_E + i \kappa_B\), which is the source term for the complex potential, can be written as \(2\kappa = \partial \alpha\). Then the convergence and shear can be expressed in terms of the projected potential \(\psi\) as

\[ 2\kappa = \nabla^2 \psi = \partial \partial \psi \]

\[ 2\gamma = \partial \partial \psi. \]

From this, we immediately derive the Kaiser (1995) relation between convergence and shear, which holds even when \(B\) modes are present:

\[ \partial \kappa = \partial \gamma. \]

The shear components defined with respect to the radius vector can be expressed as

\[ \Gamma \equiv \gamma_r + i \gamma_\theta = -\gamma e^{-2i\theta}. \]

Substituting this into the Kaiser relation (Equation 10) yields

\[ \kappa_r - i \kappa_\theta = -\Gamma_r + i \Gamma_\theta - \frac{2}{r} \Gamma, \]

where the subscripts after the comma denote differentiation, as usual. From Equation (9), the shear can also be expressed as

\[ -2\Gamma = \left[ \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + 2i \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial \theta} \right) \right] \psi. \]

Both of these equations clearly separate into radial and azimuthal parts under a multipole decomposition of the relevant quantities. For any complex quantity \(z\), we define the multipoles as

\[ z(r) = \frac{1}{2} \sum_{m=-\infty}^{\infty} (1 + \delta_m) z^{(m)}(r)e^{im\theta} \]

\[ z^{(m)}(r) = \frac{1}{(1 + b_m)r} \int d\theta z(r)e^{-im\theta}. \]

The Kaiser relation can be rewritten for each multipole as

\[ \kappa^{(m)}_r - \frac{m}{r} \kappa^{(m)}_\theta = -\Gamma^{(m)}_r - \frac{m + 2}{r} \Gamma^{(m)} \]

\[ = r^m \frac{\partial}{\partial r} (r^{-m} \kappa^{(m)}_r) = -r^{-m-2} \frac{\partial}{\partial r} (r^{m+2} \Gamma^{(m)}). \]

2.1. Flexions

The third derivatives of the lensing potential can be described by two complex-valued “flexion” fields (Castro et al. 2005; Bacon et al. 2006): the spin-1 field \(F \equiv \partial \partial \partial \phi/2\), and the spin-3 field \(G \equiv \partial \partial \partial \phi/2\). These satisfy

\[ F = \partial \kappa, \]

\[ G = \partial \gamma. \]

In analogy with the shear, we define flexions in a tangential basis:

\[ F \equiv e^{-i\theta} F, \]

\[ G \equiv e^{-3i\theta} G. \]

Casting Equation (18) into polar coordinates, performing a multipole decomposition of \(F\) and \(G\), and defining \(H \equiv G - F\), we obtain formulae for flexion multipoles in terms of convergence:

\[ r^m \frac{\partial}{\partial r} (r^{-m} \kappa^{(m)}_r) = F^{(m)} \]

\[ = -r^{-m-2} \frac{\partial}{\partial r} \left( \frac{r^{m+3} H^{(m)}}{2(m + 2)} \right). \]

Note the \(\kappa\) dependence on the left-hand side is identical to Equation (17); not surprisingly, the flexion multipoles are very close to the shear multipoles. In particular,

\[ H^{(m)} = \frac{2(m + 2)}{r} \Gamma^{(m)}. \]

A consequence of this simple relation is that we can produce aperture formulae for the flexion \(H\) whenever we can do so for shear. The flexion \(F\) is local in \(\kappa\), as has been pointed out before.

In summary, the Kaiser relation shows that convergence \(\kappa\), shear \(\gamma\), and flexion are directly related by differentiation of a common source, the projected two-dimensional gravitational potential. Expressing the derivatives in polar coordinates allows for decomposition of this relation (which involves powers of and derivatives with respect to the radius) into independent multipoles. In the following section, we integrate the multipole Kaiser relation by parts in two different ways, and combine them to obtain a relation between the integrated \(\kappa\), \(\gamma\), and flexion multipole quantities. We will see that the integrated \(\kappa^{(m)}\) within a circular region of radius \(R\) is related to the integrated \(\Gamma^{(m)}\) (or flexion \(H^{(m)}\) outside of the circular region, and vice versa.
3. MULTIPOLE FORMULAE

3.1. Interior and Exterior Shears

Several useful results may be obtained by integrating Equation (17) by parts. First, we obtain a closed-form expression for the shear multipoles:

$$\kappa^{(m+2)}_{\text{in}}(R_2) - 2(m+1) \int_{R_1}^{R_2} dr \, r^{m+1} \kappa^{(m)}(r) = -\kappa^{(m+2)}_{\text{in}}(R_2)_{|_{R_1}}. \quad (22)$$

For $m \geq 0$, we can assume $R^{m+2}_1\kappa^{(m)}(R_1) \rightarrow 0$ and $R^{m+2}_2\Gamma^{(m)}(R_1) \rightarrow 0$ as $R_1 \rightarrow 0$ for any mass distribution which remains finite and differentiable at the origin. In this case,

$$\Gamma^{(m)}(r) = -\kappa^{(m)}(r) + \frac{2(m+1)}{R^{m+2}} \int_0^R r \, dr \, r^m \kappa^{(m)}(r) \quad (m \geq 0). \quad (23)$$

Since $\Gamma^{(m)}$ is completely determined by the mass distribution at and interior to $R$ for $m \geq 0$, this is the desired generalization of the monopole formula (1) $\Gamma(r)$ is, however, a complex quantity, so it is not fully specified by the $m \geq 0$ multipoles. For $m < 0$, a bounded mass distribution will have $R^{m+2}_2\kappa^{(m)}(R_2) \rightarrow 0$ and $R^{m+2}_2\Gamma^{(m)}(R_2) \rightarrow 0$ as $R_2 \rightarrow \infty$. We then obtain

$$\Gamma^{(m)}(r) = -\kappa^{(m)}(r) - \frac{2(m+1)}{R^{m+2}} \int_{R}^\infty r \, dr \, r^m \kappa^{(m)}(r) \quad (m < 0). \quad (24)$$

The negative-$m$ multipoles of $\Gamma$ are hence dependent only on mass at or exterior to $R$. Note that $m = -1$ is a special case. The relation between shear and mass is purely local, $\Gamma^{(-1)}(R) = -\kappa^{(-1)}(R)$. An external dipole mass at $r > R$ generates only a constant displacement at $r < R$.

This formula and the previous one fully specify the shear field. So the shear in a region $R_1 < r < R_2$ is determined completely by the mass distribution in this region, plus these multipole moments of the mass interior and exterior to the annulus:

$$Q^{(m)}_{\text{in}}(R) \equiv \int_{r<R} d^2 r \, r^m e^{-im\theta} \kappa(r) \quad (25)$$

$$= (1 + \delta_m)\pi \int_0^R r \, dr \, r^m \kappa^{(m)}(r) \quad (26)$$

$$Q^{(m)}_{\text{out}}(R) \equiv \int_{r>R} d^2 r \, r^m e^{-im\theta} \kappa(r) \quad (27)$$

$$= (1 + \delta_m)\pi \int_{R}^\infty r \, dr \, r^m \kappa^{(m)}(r). \quad (28)$$

These definitions are normalized to agree with Equations (B5) of SB97, who similarly demonstrated that shear at $R$ depends upon interior and exterior masses only through these quantities. We have altered the phase conventions, however, in order to work successfully with the complex (E and B) convergence and potentials. The brackets in Equation (27) indicate a mass-weighted average inside radius $R$.

Figure 1 illustrates the shear patterns produced by the interior and exterior mass moments at the lowest order multipoles.

The present derivation shows that the division into interior and exterior shears holds even when there is an imaginary (B-mode) component to the potential and convergence. The shear components $\gamma_i$ and $\gamma_e$ are always real valued, as is $\kappa$ when there is no B-mode lensing: $\kappa^{(m)}(r) = \kappa^{(m)}$, etc. But this relation need not hold for $\Gamma^{(m)}$. It remains true, however, that the $m \geq 0$ multipoles are produced by mass internal to $R$, while $m < 0$ are produced by external mass.

Equations (23) and (24) can be restated as

$$\Gamma^{(m)}(r) = -\kappa^{(m)}(r) + \frac{2(m+1)}{(1+\delta_m)\pi} \int_{R}^\infty r \, dr \, r^m \kappa^{(m)}(r) \quad (m \geq 0). \quad (30)$$

$$\Gamma^{(m)}(r) = -\kappa^{(m)}(r) - \frac{2(m+1)}{(1+\delta_m)\pi} \int_{R}^\infty r \, dr \, r^m \kappa^{(m)}(r) \quad (m < 0). \quad (31)$$

For the flexions, Equation (21) makes it clear that $H^{(m)}$ depends on local, internal, and external mass exactly as $\Gamma^{(m)}$ does. Equation (20) shows that the $\mathcal{F}$ flexion depends only upon the local value of $\kappa^{(m)}$ and its first derivative, independent of both the internal and external moments $Q^{(m)}$.

3.2. Mass Multipoles from Shear

Multiplying Equation (17) by $r^{-m}$, integrating by parts, and taking the upper integration limit to infinity yields

$$\kappa^{(m)}(R) = -\Gamma^{(m)}(R) + 2(m+1)R^m \int_R^\infty r \, dr \, \Gamma^{(m)}(r)r^{-m-2} \quad (m \geq 0). \quad (32)$$

Comparison to Equation (23) yields

$$\frac{Q^{(m)}_{\text{in}}(R)}{(1+\delta_m)\pi} = \int_0^R r \, dr \, r^m \kappa^{(m)}(r) \quad (33)$$

This is the desired generalization of the monopole aperture–mass formula (Equation 2). Multiplication by a weight function before the integration by parts would yield the full weighted aperture–multipole formulae of SB97. The mass interior to $R$ affects the shear exterior to $R$ through the multipole moments $Q^{(m)}_{\text{in}}(R)$. Here we see that these moments are completely recoverable from the shear field $\Gamma^{(m)}$ exterior to $R$. The $Q^{(m)}_{\text{in}}$ are thus a complete description of the information that lensing data exterior to $R$ can offer on the mass distribution interior to $R$. This holds even in the presence of $B$-mode lensing.

Taking $m \leq -2$ in this integration by parts yields an analogous formula by which the $Q^{(m)}_{\text{out}}(R)$ values may be determined from shear data at $r < R$:

$$\frac{Q^{(-m)}_{\text{out}}(R)}{\pi} = \int_R^\infty r \, dr \, r^m \kappa^{(m)}(r) \quad (m < -2). \quad (34)$$

When the shear data are available in a finite annulus $R_1 < r < R_2$, we can take differences of the above two formulae:

$$\int_{R_1<r<R_2} d^2 r \, [\gamma_i(r) + i \gamma_e(r)] r^{-m-2} e^{-im\theta} = R_1^{-2m-2} Q^{(m)}_{\text{in}}(R_1) - R_2^{-2m-2} Q^{(m)}_{\text{in}}(R_2) \quad (m \geq 0) \quad (35)$$

$$\int_{R_1<r<R_2} d^2 r \, [\gamma_i(r) + i \gamma_e(r)] r^{-m-2} e^{im\theta} = -R_1^{-2m-2} Q^{(m)}_{\text{in}}(R_1) + R_2^{-2m-2} Q^{(m)}_{\text{in}}(R_2) \quad (m \geq 2). \quad (36)$$
3.3. Mass Multipoles from Flexion

Equation (21) implies that mass multipoles can be retrieved from the \( H^{(m)} \) using the preceding shear formulae. Mass reconstruction from the \( \vec{F} \) flexion differs. Equation (20) integrates by parts to yield

\[
k^{(m)}(R) = \begin{cases} 
- \int_0^R \frac{dr}{r} \left( \frac{\kappa^{(m)}(r)}{r} \right) F^{(m)}(r) dr & m \geq 0 \\
\int_0^R \frac{dr}{r} \left( \frac{\kappa^{(m)}(r)}{r} \right) F^{(m)}(r) dr & m < 0. 
\end{cases}
\]  

(37)

3.4. E-B Decomposition

The real part of the monopole shear \( \Gamma^{(0)}(R) \) at \( R \) determines the enclosed \( E \)-mode mass, while the imaginary part gives the enclosed \( B \)-mode mass. This result does not generalize to other multipoles. Consider first the case where \( \kappa^{(m)}(R) = 0 \), i.e., we are in a mass-free zone. Then for \( m \geq 0 \), \( \Gamma^{(m)}(R) \) is fully specified by the complex number \( Q^{(m)}_m(R) \). But for \( m > 0 \), any chosen \( Q^{(m)}_m \) amplitude and phase produced by an \( E \)-mode source \( \kappa_E \) can also be produced by a pseudosymmetric source \( \kappa_B \) that is just the \( \kappa_E \) rotated about the origin by \( (90/m) \circ \). There is hence no way to distinguish an internal \( m > 0 \) \( E \)-mode mass distribution from an internal \( B \)-mode mass distribution. Similarly, we can produce any desired \( Q^{(m)}_m(R) \) with either \( \kappa_E \) or \( \kappa_B \) source terms, so there is no test that can distinguish \( E \)-mode mass from \( B \)-mode mass distributions external to the shear measurement zone. These conclusions hold for flexion data as well as for shear data.

The monopole turns out to be a special case. The \( E/B \) diagnosis is possible because the monopole \( E \) and \( B \)-mode moments each has only 1 degree of freedom, while the observable \( Q^{(0)}_m \) is complex. But for \( m \neq 0 \), the \( E \) and \( B \) mass moments each has 2 degrees of freedom, so cannot be independently retrieved from a single \( Q \).

Thus if we have shear data on the \( R_1 < r < R_2 \) annulus, we have hope only of testing \( \kappa \) for \( E \) and \( B \) components at \( m \neq 0 \) only within the annulus, not interior or exterior to it. Ideally this can be done by noting

\[
2\kappa_E^{(m)} = \kappa^{(m)} + \kappa^{(1-m)} \\
2i\kappa_B^{(m)} = \kappa^{(m)} - \kappa^{(1-m)}. 
\]  

(38) 

(39)

This can be combined with Equation (20), for example, to yield a pure-\( E \) quantity:

\[
2 \left( r \kappa^{(m)}_{E,rr} + \kappa^{(m)}_E + m^2 \kappa^{(m)}_{E,rr} \right) = r^{-m} \frac{\partial}{\partial r} \left( r^{m+1} F^{(m)} \right) + r^{m} \frac{\partial}{\partial r} \left( r^{1-m} \vec{F}^{(m)} \right)
\]  

(40)

Sending \( F \rightarrow iF \) gives a pure-B quantity. These equations are not practical null tests for \( B \)-modes, however, because they involve derivatives of \( F \), which have divergent noise in the presence of shot noise from finite sampling. A practical null test for \( E/B \) modes in an annular region would require an integral of \( F \) (or \( \Gamma \) or \( H \)) over the annulus which could be approximated by a sum over source galaxies. We have not been able to derive such a form.

4. APPLICATION TO STRONG-LENSING MODELS

In modeling a lensing system around a galaxy, one has strong-lensing constraints from multiply imaged sources. The lens-mass model often contains a galaxy-mass distribution \( \kappa_g(r) \),
but it is essential in most cases to consider the influence of the larger scale mass distribution on the system. Call this the “cluster” mass, which generates potential \( \psi_c(r) \). On the assumption that the cluster mass has little structure on the scale of the strong-lensing system, \( \psi_c(r) \) can be approximated by a few terms of a Taylor expansion about the galaxy center within some radius \( R_1 \) that contains all of the strongly lensed features (Kochanek 1991). The constant and linear terms of the Taylor expansion are immaterial to the strong-lensing model. The potential at \( r < R_1 \), to cubic order in the Taylor expansion of the cluster, is

\[
\psi(r) = (1 - \kappa_c) \left[ \psi_g(r) + \text{Re} \left( \frac{\gamma}{2} r^2 e^{-2i\theta} + \frac{\sigma^2 - 3i}{4} r^3 e^{-i\theta} - \frac{\delta}{6} r^3 e^{-3i\theta} \right) + \frac{\kappa_c}{2} r^2 \right],
\]

(41)

with \( \nabla^2 \psi_g = 2\kappa_g \). The strong-lens data produce a likelihood distribution over the (complex) parameters \( \{\gamma, \sigma, \delta\} \) and the galaxy-mass parameters. The mass-sheet degeneracy leaves \( \kappa_c \) unconstrained by strong-lensing data.

We now ask what additional constraints on these model parameters are available from the shear field at \( r > R_1 \). We do not want to assume that the Taylor expansion offers an adequate description of the cluster mass at \( r > R_1 \), but from the previous discussion we know that only the multipole moments \( Q^{(n)}_{\text{out}}(R_1) \) affect the parameters in the strong-lensing potential. In particular, the terms of the form \( r^m e^{i2m\phi} \) in the Taylor expansion of \( \psi_c \) can only be generated by mass outside \( R_1 \), while the other terms can only be generated by mass inside \( R_1 \). Specifically:

1. The \( \kappa_c \) term is a monopole (constant) mass distribution, and \( Q^{(0)}_{\text{in},c} = \pi R^2_1 \kappa_c \).
2. The \( \gamma \) term is a constant shear, producing \( \Gamma^{(2)} = -2\gamma \).
3. The \( \sigma \) term is a dipole mass distribution, \( \kappa = \text{Re}(\sigma r e^{-i\theta}) \).
4. The \( \delta \) term is a \( n = 3 \) external shear, \( \Gamma^{(3)} = -2\delta r \).

Equations (35) and (36) can now be applied to a shear measurement that extends to radius \( R_2 \) from the galaxy center. The multipole moments \( Q \) are split into galaxy and cluster contributions. Those from the galaxy are calculable from the parametric form adopted for \( \kappa_g \). The cluster contributions at \( R_1 \) are parameterized by the Taylor expansion coefficients as above. The cluster contributions at \( R_2 \) are formally unconstrained, but if \( R_2 \) is large enough then these may be bounded by even a rough estimate of the total mass and extent of the cluster. We obtain

\[
I^{(0)} = \int_{R_1 < r < R_2} d^2r \gamma_1(r) r^{-2} = \pi \kappa_c - R^2_2 Q^{(0)}_{\text{in},c}(R_2) + (1 - \kappa_c) \left[ R^{-2} Q^{(0)}_{\text{in},g}(R_1) - R^{-2} Q^{(0)}_{\text{in},g}(R_2) \right]
\]

(42)

\[
I^{(1)} = \int_{R_1 < r < R_2} d^2r \left[ i \gamma_1(r) + i \gamma_2(r) \right] r^{-3} e^{-i\theta} = \pi (1 - \kappa_c) \sigma / 4 - R^{-4} Q^{(1)}_{\text{in},c}(R_2)
\]

(43)

\[
I^{(2)} = \int_{R_1 < r < R_2} d^2r \left[ i \gamma_1(r) + i \gamma_2(r) \right] e^{i\theta} = \pi R^2_1 (1 - \kappa_c) \gamma + R^2_2 Q^{(2)}_{\text{out},c}(R_2)
\]

(44)

\[
+ (1 - \kappa_c) \left[\begin{array}{c}
-R^2_1 Q^{(2)}_{\text{out},g}(R_1) + R^2_2 Q^{(2)}_{\text{out},g}(R_2)
\end{array}\right]
\]

Note that if the galaxy-mass distribution has inversion symmetry about the coordinate origin, then all of the \( Q^{(1)} \) and \( Q^{(3)} \) moments of the galaxy vanish. In each equation, the left-hand side is an observable quantity and the right-hand side is a function of the galaxy parameters, the Taylor-expansion parameters, and some (presumably small) terms for the multipole moments of the cluster potential. Unfortunately, for \( R_2 \rightarrow \infty \), the \( Q^{(1)} \) and \( Q^{(3)} \) would constrain four of these. The \( \psi \propto r^4 \) term, however, would be an internal monopole mass distribution with \( r^2 \) radial dependence. It would contribute to \( Q^{(0)}_{\text{in}} \) and would be degenerate with \( \kappa_c \) within the shear annulus. The degeneracy would have to be broken by the strong-lensing data.

4.1. Measurement Noise

The observable terms in Equations (42)–(45) are each integrals assuming perfect knowledge of shear \( \gamma_i \) at all points within the annulus. In practice the integrals are estimated by summing over shear estimates \( \gamma_i \) at the locations of source galaxies uniformly distributed with density \( n \) on the sky:

\[
\int \gamma \, d^2r \rightarrow \frac{1}{n} \sum_j \gamma_j.
\]

(46)

The intrinsic shapes of galaxies are the dominant source of noise in the measurement of the integrals. If we assume that each galaxy shape leads to an independent uncertainty \( \sigma_i \) on each component of the estimated shear, then we find that the estimates of the integrals have shape-noise uncertainties of

\[
\text{Var}[\text{Re}(I^{(m)})] = \frac{2\pi \sigma^2_i}{n} \left( R_1^{2m-2} - R_2^{2m-2} \right) \frac{2m + 2}{2m + 2}
\]

(47)

in each phase. (Note that the \( m = -1 \) integral would never be used, since external dipole causes no shear.)

The \( Q^{(m)}_{\text{in}}(R_2) \) and \( Q^{(m)}_{\text{out}}(R_2) \) in the formulæ for \( \kappa_c, \sigma, \gamma, \) and \( \delta \) of the cluster potential requires that we make some a priori estimate of their value. Since these terms shrink as \( R_2 \rightarrow \infty \), the weak-lensing estimates of the cluster potential become independent of prior assumptions as we extend the shear integrals to larger outer radii.

For \( m \geq 0 \), the shape noise in the measurement converges as \( R_2 \rightarrow \infty \), and we have a useful tool for constraining the \( \kappa_c \) and \( \sigma \) terms of the cluster potential. Unfortunately, for \( m \leq -2 \), the shape noise in the measurement grows without bound as \( R_2 \rightarrow \infty \), which means that weak-lensing estimates of the external potential terms \( \gamma \) and \( \delta \) must either be model dependent, or noisy. Hence these expressions cannot be expected to provide useful constraints on the cluster quadrupole and octupole mass moments \( \gamma \) and \( \delta \).
Alas the use of flexions does not improve the situation. Following Equation (21) we could substitute $r H^{(m)}/2(m + 2)$ for the shear in each of the integrals $I^{(m)}$, but this would only increase the power of radius in the integral, worsening the divergence as $R_2 \to \infty$.

4.2. (Constant) External Shear

Common practice in analysis of galaxy-scale strong-lensing systems is to limit modeling of external mass to a constant "external" shear across the strong-lensing system, i.e., the $\gamma$ term of Equation (41). If we simplify Equation (44) by setting $\kappa = 0$ and ignoring the shear induced by the galaxy mass, we find that the observable quantity on the left-hand side is just the mean shear inside the annulus, and

$$\gamma = \frac{R_2^2}{\pi R_1^2} \frac{Q_{\text{out}}^{(2)}(R_2)}{Q_{\text{out}}^{(2)}(R_1)} - \frac{R_2^2 - R_1^2}{R_1^2} \langle \gamma \rangle_{\text{ann}}.$$  (48)

One approach is to use a priori knowledge of the mass fluctuation spectrum to find a radius $R_2$ beyond which we can expect the $Q_{\text{out}}^{(2)}$ term to become negligibly small, $\lesssim 0.01$. Note that in this case the desired shear $\gamma$ is opposite to the mean shear in the annulus. Two problems arise however: first, it is not clear that any such radius exists, since the large-scale "cosmic shear" is typically $\sim 0.01$ even before amplification by the $R_2^2/R_1^2$ factor in this term. Second, the shape-noise variance of the measured $(R_2^2 - R_1^2)/(\langle \gamma \rangle_{\text{ann}}/R_1^2)$ contribution will grow as $(R_2^2/R_1)^2$ for fixed $R_1$, rendering the measurement uninteresting. It thus appears problematic to use weak-lensing information to infer the "external shear" in galaxy-scale lenses.

In a different limit, one might assume that the mass within the $R_1 < r < R_2$ annulus has negligible quadrupole moment, perhaps because one does not see any galaxy groups or clusters projected within this annulus. In this case, $Q_{\text{out}}^{(2)}(R_2) = Q_{\text{out}}^{(2)}(R_1) = -\pi \gamma$, and our estimate of the shear parameter becomes simply equal to the mean shear in the annulus

$$\gamma = \langle \gamma \rangle_{\text{ann}}.$$  (49)

In this case the shape noise on $\gamma$ decreases as $R_2$ is increased and can become usefully small.

It thus appears that the use of weak-lensing information to constrain the external shear on galaxy lenses is practical only if one has a priori knowledge that an annulus in the vicinity of the lens is free of mass that would generate a significant shear on the system. We note that these problems are exacerbated for the higher order external moments because of the positive powers of radius that appear within the shear integrals, e.g., Equation (45).

5. CONCLUSIONS

Polar coordinate expressions for the relations between convergence, shear, and flexion are separable under multipole expansions once we rotate the shear and flexion spinors into "tangential" bases. Two well known monopole aperture–mass properties are extensible to all $m \geq 0$ multipoles: first, the shear multipole $\Gamma^{(m)}(R)$ is determined solely by the convergence (mass) multipole $\kappa^{(m)}$ at or interior to radius $R$. The effect of the interior mass is fully described by the moments $Q_{\text{in}}^{(m)}(R)$, $\propto \int_0^R d^2 r \ r^m \kappa^{(m)}$. Second, we find that the value of the interior mass moment $Q_{\text{in}}^{(m)}(R)$ can be exactly recovered by an integral of the shear multipole $\Gamma^{(m)}$ from $R$ to $\infty$.

The multipoles $m < 0$, however, behave oppositely to the monopole: the shear at $R$ is determined exclusively by the mass at or exterior to $R$. And the relevant exterior mass moment $Q_{\text{out}}^{(m)}(R)$ can be determined by an integral of the shear interior to $R$.

The tangential flexion component $H \equiv e^{-i\theta} G - e^{-i\theta} F$ behaves exactly as the tangential shear $\Gamma$. In fact they differ only by a factor $(2(m + 2)/r)$. The vector flexion component $F$ depends purely on the local behavior of $\kappa$, as is well known.

The simple $E/B$ decomposition of the monopole mass distribution does not generalize to $m \neq 0$. We show that shear or flexion data in an annulus $R_1 < r < R_2$ cannot discriminate between $E$- and $B$-mode sources outside this region—except for the monopole case. Shear or flexion data may be able to distinguish $E$ from $B$ sources inside the annulus, but we have not been able to derive a practical estimator which does this.

The multipole formulae presented by SB97 and extended here find application in using weak-lensing data to constrain the large-scale characteristics of mass distributions in the vicinity of strong-lensing systems. Unfortunately, a complete characterization of the exterior mass distributions is not practical because some of the aperture–multipole formulae have divergent shape-noise behavior. In particular, the estimation of the constant "external shear" term often found in strong-lens models is problematic without strong prior constraints on the neighboring mass distributions. We can expect the aperture–multipole formulae to find further use in generating model-independent measures of the shapes of dark-matter halos.

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