1 INTRODUCTION

Modern mathematics begins with \textit{symbolic manipulation}. The central role of signs and symbols \textit{per se} is one of the main achievements of the Medieval culture \cite{88} leading, among others, to the development of elementary or \textit{symbolic algebra}. Starting from the latter, the syntactic manipulation of symbols more or less independently of their meaning — i.e. to what symbols stand for — has become an essential part of mathematical reasoning, not to say of reasoning \textit{in general}. Today, symbolic manipulation is not just a pillar of mathematics, but it is at the very hearth of \textit{computation}. Indeed, the symbolic manipulations of elementary algebra carry a computational content and, vice versa, computational processes can be fully described symbolically.

Rewriting theory \cite{94,25} is the discipline that studies (the computational content of) symbolic manipulation in general. As such, rewriting has its origin both in symbolic algebra as the study of the algorithmic properties of equational reasoning, and in computability and programming language theory, where rewriting systems have been used to define symbolic models of computation — such as the \(\lambda\)-calculus \cite{17} and combinatory logic \cite{34,74} — as well as the (operational) semantics and implementation of programming languages \cite{2}. In both cases, rewriting is motivated by the need to define \textit{operational} notions of equality revealing the computational content of equational deductions. Remarkably, operationality is ultimately achieved by making equality asymmetric, so that the aforementioned computational content can be fully uncovered by orienting equations. Nowadays, these oriented equations (and the evolution thereof) are known as \textit{rewriting} — or \textit{reduction} — relations. All of that highlights a crucial trait of rewriting theory, namely its deep connection with equational reasoning. In fact, rewriting does not actually focus on arbitrary symbolic transformations, but with \textit{equality-preserving} ones: a rewriting relation \textit{refines} equality by making the latter operational, and it is thus contained in it.

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Recent advances in theoretical computer science, however, have questioned the central role played by equality in semantics, arguing for more quantitative and approximated forms of equivalence. For instance, equality is a too strong notion for reasoning about probabilistic computations, where even small perturbations break the equivalence between probabilistic processes. To overcome this problem, researchers have thus refined equality to *distances* between probabilistic processes, this way replacing equivalences with *metrics*. Similarly, metric-based and approximated equivalences have been used to reason about privacy and security of systems [98, 45], not to mention reasoning about intensional aspects of computation, such as resource consumption [41, 40].

Prompted by that, several theories of semantic equality have been refined giving rise to *quantitative theories of semantic differences*, prime examples being general theories of program [6, 16, 52, 45, 13, 61, 32, 33, 62, 41, 38, 42, 43, 35] and system distances [51, 14, 15, 52, 56, 64] and the theory of quantitative algebras and quantitative equational reasoning [84, 85, 10, 86, 87, 12, 90]. The latter, in particular, aim to provide a common foundation for general quantitative reasoning by refining traditional, set-based algebraic structures to metric-like ones and by replacing traditional equations with *quantitative equations* bounding the difference (or distance) between the equated elements. Accordingly, classic equations \( t = s \) are replaced by expressions of the form \( t \approx s \), with the informal reading that \( t \) and \( s \) are at most \( \varepsilon \) apart, or that they are equal up to an error \( \varepsilon \). Thus, quantitative algebraic theories are not theories about *equality* between objects, but about *distances* between them, and can thus be seen as the quintessence of quantitative and metric reasoning.

But what about the *computational content* of quantitative equational reasoning? What is an *operational* notion of quantitative equality or distance allowing us to effectively compute distances by means of quantitative equations? And, more generally, what is the theory of *quantitative symbolic manipulation*, where symbolic transformations can break semantic equivalence? The development of such a theory, which is the main topic of this work, is of paramount importance not only to make quantitative equational deduction effective, but also to develop a general quantitative theory of programming language semantics.

In this paper, we introduce the theory of *quantitative and metric rewriting systems* as a first step towards a general theory of the computational content of quantitative symbolic manipulations. Such a theory is rich and subsumes and largely (and nontrivially) extends traditional rewriting. The goal of this paper is to lay the foundation of quantitative rewriting systems, this way opening the door to a larger research program. More specifically, in this work we introduce the notion of a quantitative abstract rewriting system and its general theory. We define quantitative notions of confluence and termination, refining cornerstone results such as the Newman [94] and Hindley-Rosen [72, 99] Lemma to a quantitative and metric setting. Such notions are crucial for the *operational* study of what we shall call *metric word problems*, the quantitative refinement of traditional word problems. We then introduce *quantitative linear and non-expansive term rewriting systems* and apply the general theory previously developed to such systems. Linearity and the related notion of non-expansiveness will be crucial to avoid distance trivialisation phenomena [33, 62] and the failure of major confluence theorems. Concerning the latter, in fact, we shall prove general quantitative critical pair-like lemmas ensuring confluence of large families of linear and non-expansive systems. Finally, we go beyond linearity and non-expansiveness by introducing *graded quantitative term rewriting systems*. In such systems, rewriting is not only quantitative but also *modal* and *context-sensitive*, meaning that contexts are not required to non-expansively propagate distances — as in linear systems — but they directly act on them, this way behaving as generalised Lipschitz continuous functions. We will extend the confluence results proved for
non-expansive systems to graded ones, as well as prove an additional confluence result for orthogonal systems.

All our theory is developed following the abstract relational theory of distances initiated by Lawvere \[82\], whereby we work with relations taking values in arbitrary quantales \[100\]. Such an approach has been successfully applied to the study of general theories of program and process distances \[117, 61, 62, 69\]. Moreover, since abstract metric and modal reasoning are essentially equivalent \[41, 40\], our theory can be seen both as a general theory of metric and quantitative rewriting systems and as a theory of modal and substructural rewriting, this way suggesting possible connections with modal and coeffectful systems \[95, 66, 97, 60, 1\].

In addition to the just outlined theoretical development, in this paper we deal with several examples of quantitative (and modal) systems. Such examples come from the field of algorithms (notably edit distances on strings), quantitative algebras and algebraic effects (e.g. quantitative barycentric algebras), programming language theory (quantitative and graded combinatory logic), and combinations thereof.

**From Equality to Distances: A Gap** Before outlining the main contents and contributions of the present work in more detail, it is instructive to shortly stress the gap between traditional, equality-based reasoning and quantitative one (this gap will be the main theme of the first example in section 2). When it comes to reason about equality between objects, we usually have at our disposal a heterogeneous arsenal of techniques, ranging from semantic and denotational characterisations of equality to symbolic and operational ones. Think about equality between (natural) numbers: we can approach it foundationally using set-theory or Peano arithmetic — depending on whether we prefer a semantic or syntactic approach — but we can also study it using algebra, category theory, or type theory, this way relying on its inductive nature. And that is not the end of the story, as we can also use plain number theory, this way building upon numerical and analytical techniques, rather than symbolic ones.

When we move to quantitative equality and metric reasoning, the situation vastly changes and only a few of the aforementioned techniques are available, with a strong orientation towards numerical and analytical ones. On natural numbers, we can consider the Euclidean distance, which is ultimately defined numerically. And when it comes to reason about it, numerical and analytical techniques are largely used, other techniques being simply not available. However, the Euclidean distance (between natural numbers) has an embarrassingly simple inductive definition (and thus an associated induction principle) in terms quantitative equations, as well as a well-behaved associated notion of operational equality (i.e. rewriting). The same story can be told for many other (more challenging) distances, ranging from edit to transportation distances \[49\], in all cases obtaining elegant quantitative equational characterisations and well-behaved quantitative rewriting systems. As already mentioned, in the last decade researchers have started to realise that the mathematical heterogeneity characterising equality pertains to quantitative equality and distances too, although this new awareness it still in its infancy. This paper has the ambitious goal to contribute to all of that by beginning the exploration of the computational content of quantitative equality.

**Structure of the Paper** We dedicate \[section 2\] to gently introduce the reader to the theory of quantitative and metric rewriting systems by means of concrete examples. After that, we move to the technical development of our theory, which is divided in three sections. Once recalled the necessary mathematical preliminaries \[section 3\], in \[section 4\] we outline a theory of abstract quantitative rewriting systems, focusing on quantitative notions of confluence and termination. The main results proved are
quantitative refinements of Newman’s Lemma, Church-Rosser Theorem, and Hindley-Rosen Lemma. In section 7, we specialise the theory of section 4 to quantitative term rewriting systems. We prove several quantitative critical pair-like lemmas for linear and non-expansive systems, and use them to infer non-trivial properties of the systems introduced in section 2. Finally, in section 8, we go beyond linearity and introduce the theory of graded (modal) quantitative rewriting systems. We extend quantitative critical pair lemmas to graded systems and prove that (graded) orthogonal systems are always confluent. Using such results, we obtain a confluence result for a system of graded combinators extending bounded combinatory logic.

2 BEYOND TRADITIONAL REWRITING: SHAPING A THEORY

In this section, we gently introduce the reader to quantitative rewriting systems by looking at some simple examples of quantitative systems coming from diverse fields (e.g. algebra, programming language theory, and biology). For pedagogical reasons, we shall focus on non-expansive systems (i.e. systems where rewriting inside terms non-expansively propagates distances) only, postponing graded systems to section 8.

2.1 From Equality to Differences: Warming-Up

Let us begin with one of the simplest possible example: the system of natural numbers with the addition operation. Such a system can be defined in several ways (algebraically, set-theoretically, numerically, etc) each giving a specific perspective on equality between numbers. In this example, we model natural numbers symbolically using the signature \( \Sigma \) containing a constant \( Z \) for zero, a unary function symbol \( S \) for the successor function, and a binary function symbol \( A \) for addition. Fixed a set \( \mathcal{V} \) of variables, we use the set \( \Sigma^{\mathcal{V}} \) to study natural numbers syntactically. Equality between terms in \( \Sigma^{\mathcal{V}} \) is given by the relation \( =_N \) inductively defined by the following rules:

\[
\begin{align*}
Z &=_N Z \\
A(x, Z) &=_N x \\
A(x, S(y)) &=_N S(A(x, y)) \\
&
\begin{array}{ll}
x =_N x \\
y =_N y \\
x =_N y \\
x =_N y \\
x =_N y \\
S(x) =_N S(y) \\
A(x, y) =_N A(x', y') \\
t =_N s \\
t =_N s'
\end{array}
\]

The first three rules are the defining axioms of \( =_N \), whereas the last three rules close \( =_N \) under substitutions and function symbols in \( \Sigma_N \). The remaining three rules, finally, makes \( =_N \) reflexive, symmetric, and transitive, and thus an equivalence.

The equational system \( (\Sigma_N, =_N) \) gives a symbolic approach to numerical equality. In fact, given two numbers (or numerical expressions defined as sums of natural numbers) \( m \) and \( n \), we can check whether \( m \) and \( n \) are equal in (at least) two ways: either we compute \( m \) and \( n \) numerically (assuming to have ways to
perform calculations) or we look at \( m, n \) as expressions \( t, s \) in \( \Sigma_N(X) \) and manipulate them symbolically to produce a formal derivation of \( t =_N s \). For instance, we see that \( 1 + 2 \) is equal to \( 2 + 1 \) because we numerically compute them — obtaining \( 3 \) in both cases — or because we observe that \( A(x, y) =_N A(y, x) \) is provable in \( (\Sigma_N, =_N) \), and thus \( A(S(Z), S(S(Z))) =_N A(S(S(Z)), S(Z)) \) is derivable. Furthermore — and most importantly — we can uncover the computational content of \( =_N \) by orienting its defining equational axioms, this way obtaining a rewriting (or reduction) relation \( \rightarrow_N \) defined as follows:

\[
\begin{align*}
A(x, Z) & \rightarrow_N x \\
A(x, S(y)) & \rightarrow_N S(A(x, y)) \\
t & \rightarrow_N s \\
C[t^\sigma] & \rightarrow_N C[s^\sigma]
\end{align*}
\]

The first two axioms define the relation \( \rightarrow_N \), whereas the last rule extends to \( \rightarrow_N \) to the (full) rewriting relation \( \rightarrow_N \). To define the latter, we have denoted by \( C[\cdot] \) a context in \( \Sigma_N(X) \) — i.e. a term with a single occurrence of a hole \( \Box \) — and by \( C[t] \) the term obtained by replacing the hole \( \Box \) with \( t \) in \( C[\cdot] \). Accordingly, we see that \( \rightarrow_N \) is obtained by applying (substitution) instances of \( \rightarrow_N \) inside arbitrary terms.

The rewriting system \( (\Sigma_N, \rightarrow_N) \) fully describes equality in \( (\Sigma_N, =_N) \) operationally, in the sense that an equality \( t =_N s \) is provable if and only if \( t \) and \( s \) are \( \rightarrow_N \)-convertible, meaning that there is a rewriting path from \( t \) to \( s \) obtained by performing a finite number of bidirectional rewriting steps (i.e. from left to right as well as from right to left). Additionally, \( (\Sigma_N, \rightarrow_N) \) enjoys several nice properties. In particular, it is confluent and terminating, meaning that convertibility (and thus \( =_N \) ) is decidable and coincides with having the same normal form.

**The Computational Content of a Distance** What we have seen so far shows that equality between natural numbers can not only be defined symbolically as \( =_N \), but also operationally via \( \rightarrow_N \), this way making explicit its computational content. All of that is no more than a classic introductory example to rewriting theory. Let us make a step further and ask the following question: what happens if we move from equality to distances between numbers? That is, what happens if instead of determining whether two numbers are equal or not, we ask the finer question about how much different they are? Answering these questions numerically is not a problem at all: we consider the Euclidean metric and define the distance between two numbers \( n \) and \( m \) as \( |n - m| \). But what about symbolic approaches? And what is the computational content, if any, of the Euclidean distance?

Answering these questions in the affirmative ultimately means finding systems like \( (\Sigma_N, =_N) \) and \( (\Sigma_N, \rightarrow_N) \) describing, however, the Euclidean distance between numbers, rather than their equality. To define such systems, we refine \( =_N \) and \( \rightarrow_N \) quantitatively. Let us begin with \( =_N \). Following the methodology of quantitative equational theories \([81,82]\), we move from traditional equations \( t = s \) to quantitative equations, that is ternary relations \( \varepsilon \vdash t = s \) relating pairs of terms \( t, s \) with non-negative numbers \( \varepsilon \), the informal reading of a quantitative equation \( \varepsilon \vdash t = s \) being that \( t \) and \( s \) are at most \( \varepsilon \)-apart.

**Notation 1.** To improve readability, we oftentimes abbreviate \( \varepsilon \vdash t = s \) as \( t =^\varepsilon s \).

---

3For the moment, whether we work with natural, rational, or real numbers is not relevant.

4Other possible readings come from the world of metric spaces (the distance between \( t \) and \( s \) is at most \( \varepsilon \)), intensional and resource analysis (given resource \( \varepsilon \), the terms \( t \) and \( s \) can be proved equal), and fuzzy and graded logic(s) (\( t \) is equal to \( s \) with degree \( \varepsilon \)).
This shift from traditional to quantitative equality leads to a change in the classic rules of equational deduction which now have a quantitative flavour: transitivity, for instance, now describes the usual triangular inequality axiom of metric spaces.

\[
\frac{\varepsilon \vdash t = u \quad \delta \vdash u = s}{\varepsilon + \delta \vdash t = s}
\]

We will see this kind of rules in detail throughout this paper, but for the moment we can leave them aside. Traditional equality now corresponds to the null (zero) distance, whereas congruence rules give non-expansiveness of syntactic constructs. Non-expansiveness is a crucial feature of quantitative systems and we will say more about that in subsection 2.3 and section 8.

Finally, in order to deal with natural numbers, we add a single distance-producing, quantitative equation:

\[
\varepsilon \vdash S(x) = N x.
\]

This equation simply states that a number and its successor are at distance one. Overall, we obtain the following quantitative refinement of system \((\Sigma_N, =_N)\) which, overloading the notation, we still denote by \((\Sigma_N, =_N)\) (this will not create confusion, since from now on we shall deal quantitative systems only).

\[
\begin{array}{c|c|c}
S(x) & y & S(x) = y \\
\hline
\delta & \varepsilon & x = y \\
\hline
A(x, y) & z & A(x, y) = z \\
\hline
A(x, S(y)) & z & A(x, S(y)) = z \\
\hline
x =_N y & S(x) =_N S(y) & A(x, y) =_N A(x', y') \\
\hline
1 & 0 & x =_N y \\
\hline
0 & 1 & z =_N x \\
\hline
\end{array}
\]

Notice that \(_N\) is an inductive notion and that it defines a distance \(E\) on \(\Sigma_N(X)\) as

\[
E(t, s) \triangleq \inf \{\varepsilon \mid t \varepsilon \vdash N s\}.
\]

Such a distance is a pseudometric and when it is applied to terms representing natural numbers it indeed gives the Euclidean distance between such numbers, hence showing that the latter distance has an inductively-defined algebraic characterisation.

Let us now uncover the computational content of the Euclidean distance by giving an operational account of \(_N\). We do so by refining the rewriting relation previously introduced to the (ternary) quantitative rewriting relation \(\rightarrow_N\) giving information on the distance produced when rewriting terms. We thus read \(\varepsilon \vdash t \rightarrow_N s\) as stating that reducing \(t\) to \(s\) produces a difference \(\varepsilon\) between the former and the latter.

**Notation 2.** As before, we often abbreviate \(\varepsilon \vdash t \rightarrow_N s\) as \(t \varepsilon \rightarrow_N s\) (and similarly for the other rewriting relations we are going to introduce).

---

5 The complete definition of \(_N\) actually requires the addition of all rules of quantitative equational deduction previously mentioned. Such rules (which are formally described in [section 7]) include the quantitative refinements of reflexivity, symmetry, and transitivity — which essentially correspond to the usual identity of indiscernibles, symmetry, and triangle inequality axioms of metric spaces, respectively — as well as structural rules for \(_N\) (for instance, there is a weakening rule stating that whenever \(\varepsilon \vdash t =_N s\) is derivable, then so is \(\delta \vdash t =_N s\), for any \(\delta \geq \varepsilon\).
We first define $\rightarrow_N$ by stipulating that actual distances between terms are produced by deleting successor functions and extends it to the full quantitative rewriting relations $\rightarrow_N$ by non-expansively propagating distances produced by substitution instances of $\rightarrow_N$ throughout arbitrary contexts of the language.

$$
\begin{align*}
A(x, Z) \rightarrow_N^0 x & \quad A(x, S(y)) \rightarrow_N^0 S(A(x, y)) & \quad S(x) \rightarrow_N^1 x \\
\quad \quad t \xrightarrow{\varepsilon} s & \quad C[t^\sigma] \xrightarrow{\varepsilon} C[s^\sigma]
\end{align*}
$$

The relation $\rightarrow_N$ induces a (rewriting) distance $N$ between terms defined by

$$
N(t, s) \doteq \inf\{\varepsilon \mid t \xrightarrow{\varepsilon} N s\}
$$

so that we obtain a quantitative relational system $\mathcal{N} = (\Sigma_N(X), N)$, which is our first example of an abstract quantitative rewriting system. We will study such systems in section 4. For the moment, we simply say that as an abstract rewriting system consists of a set $A$ of objects together with a relation $R \subseteq A \times A$ on it, a quantitative abstract rewriting system is given by a set $A$ together with a quantitative relation $R : A \times A \rightarrow [0, \infty]$ on it. Quantitative term rewriting systems are a special class of quantitative abstract rewriting systems where objects are terms and the quantitative relation $R$ is canonically defined as $R(t, s) \doteq \inf\{\varepsilon \mid t \xrightarrow{\varepsilon} R s\}$ starting from a ternary relation $\rightarrow_R$ (then extended to $\rightarrow_R$) like those we have seen so far as.

**Remark 1.** Notice that in a quantitative abstract rewriting system $(A, R)$, the quantitative relation $R$ gives the rewriting distance between elements of $A$. Such a distance, however, is not required to obey the usual (psuedo)metric axioms, nor a subset thereof. Such a requirement, in fact, would be morally the same as requiring a traditional rewriting relation to be an equivalence, which is clearly undesirable.

Let us expand on quantitative relations. As pointed out by Lawvere [82], quantitative relations (or distances) are governed by an algebra close to the one of ordinary relations [105, 106, 26] so that a large part of the calculus of relations [111, 105, 26] can be refined to give rise to a calculus of quantitative relations. In fact, by viewing binary relations as maps $R : A \times B \rightarrow \{\bot, \top\}$, we see that a quantitative relation simply refines the structure $\{\bot, \top\}, \leq, \wedge$ by replacing it with $([0, \infty], \geq, +)$, so that we can use this similarity to generalise many relational constructions and their properties to a quantitative setting. For instance, by refining the existential quantifier $\exists$ as the infimum $\inf$ and the Boolean meet $\wedge$ as addition $+$, we can define the composition between quantitative relations $R, S$ by

$$(R; S)(a, c) \doteq \inf_b R(a, b) + S(b, c).$$

Consequently, we will say that a quantitative relation $R$ is transitive if $R; R \geq R$, i.e. if

$$\inf_b R(a, b) + R(b, c) \geq R(a, c),$$

Actually, we will consider a more general form of quantitative relations (see section 4), but for the moment it is more convenient to restrict to $[0, \infty]$-valued relations.

We could think about such an algebra as a monoidal algebra of relations.
which is nothing but the usual triangle inequality law. In the same way, we can refine the notions of
reflexivity, symmetry, and transitivity to quantitative relations, this way obtaining exactly the defining
axioms of a pseudometric. In particular, as any rewriting relation induces — by taking its reflexive,
symmetric, and transitive closure — an equality between terms as convertibility, any rewriting distance
\( R \) defines a \textit{convertibility distance} (pseudometric, acutally) \( R^\equiv \) by means of its reflexive, symmetric, and
transitive closure. We shall see in detail the general theory of abstract quantitative relations \( \text{à la Lawvere} \)
section{3} What is relevant, for the moment, is that by modelling traditional rewriting relationally \[ 50 \],
we can then rely on such a general theory for quantitatively refining them.

Let us apply the ideas seen so far to the rewriting distance \( N^\equiv \). The pseudometric \( N^\equiv \) gives the
convertibility distance between terms, which is nothing but the distance \( E \) induced by \( \equiv_N \), i.e. the
Euclidean distance. Consequently, the Euclidean distance is not only obtained symbolically via \( \equiv_N \), but
it is also completely described operationally as the convertibility distance induced by the quantitative
rewriting system \( N \) (and thus by \( \rightarrow_N \)).

At this point, it is natural to ask whether \( N^\equiv \) (and thus \( E \)) has nice computational properties. Without
much of a surprise, the ‘nice computational properties’ we have in mind are the quantitative refinements
of well-known rewriting notions, such as confluence and termination. We postpone precise definitions of
these notions until section 3 and content ourselves with some intuitions behind them for now. Suppose
we are approximating the distance \( N^\equiv (t, s) \) with a bidirectional reduction path of the form

\[
t \overset{\delta_1}{\longrightarrow} \cdots \overset{\delta_n}{\longrightarrow} u \leftarrow \cdots \leftarrow s
\]

so that the convertibility distance between \( t \) and \( s \) given by this path is \( \sum_{i=1}^{n} \epsilon_i \). When asked to compute
or approximate such a distance, it is desirable to have a term \( u \) such that

\[
t \overset{\delta_1}{\longrightarrow} \cdots \overset{\delta_n}{\longrightarrow} u \overset{\eta_m}{\leftarrow} \cdots \overset{\eta_1}{\leftarrow} s \quad \text{and} \quad \sum_{i=1}^{n} \epsilon_i \geq \sum_{j=1}^{m} \delta_i + \sum_{k=1}^{p} \eta_k.
\]

Moving to distances, that means that to approximate \( N^\equiv (t, s) \) (and thus \( E \)), we can restrict ourselves to
proper rewriting rather than convertibility. Formally:

\[
N^\equiv (t, s) = \inf_u N^\ast (t, u) + N^\ast (s, u),
\]

where \( N^\ast \) denotes the reflexive and transitive closure of \( N \) (which is precisely the distance induced by the
reflexive and transitive closure of \( \rightarrow_N \)). This is nothing but the quantitative refinement of the well-known
Church-Rosser property. In a similar fashion, we obtain the quantitative refinement of confluence; and
since \( N \) is confluent — as we will see in subsection 7.2 — it also has the aforementioned (quantitative) Church-Rosser property. Additionally, \( N \) is terminating (in a suitable sense that we will see in section 4),
so that not only we can approximate \( N^\equiv (t, s) \) by measuring the distance between the common reducts
of \( t \) and \( s \), but we can also reduce the search space to normal forms.

Now that the reader is warmed-up, we can move to slightly more involved (and interesting) examples:
those will also give us the chance to introduce, still at an informal level, a few more notions related to
quantitative rewriting systems.

2.2 Quantitative String Rewriting Systems

Historically, rewriting systems have first appeared in the form of \textit{string rewriting systems} \[ 28, 112 \]; we thus find appropriate to include examples of \textit{quantitative string rewriting system} in this motivational
section. Recall that given an alphabet $\Sigma$, a string rewriting system is given by a relation $\mapsto_R$ on strings $\Sigma^*$ over $\Sigma$. The relation $\mapsto_R$ induces a rewriting relation $\rightsquigarrow_R$ that rewrites substrings according $\mapsto_R$. That is, whenever we have $t \mapsto_R s$, then we have $utv \rightsquigarrow_R usv$ too, where $u, v$ are strings and we write string concatenation as juxtaposition.

In this example, we consider a family of classic examples of string rewriting systems: DNA-based systems. Let us fix the alphabet $\Sigma_M \triangleq \{A, C, G, T\}$ of DNA bases (the latter $M$ stands form molecule). We view strings over this alphabet as representing DNA molecules or DNA sequences, so that, for instance, we view a string such as $\text{TAGCTAGCTAGCT}$ as describing a DNA molecule. A string rewriting system over $\Sigma$ specifies how DNA molecules can be transformed into one another, and thus it is a crucial tool to deal with word-problems, i.e. problems asking whether two DNA molecules are equal. In fact, once we know that equality coincides with convertibility in a string rewriting system, it is sufficient to prove confluence of the latter to obtain semi-decidability of equality (and thus of its associated word problem); and if, additionally, the system is terminating, then equality is decidable.

**Quantitative String Rewriting Systems** When we transform a DNA molecule into another one, however, we usually obtain different molecules, so that reasoning about DNA sequences in terms of equality or convertibility is often too restrictive. And in fact, researchers are more interested in measuring distances between molecules rather than studying their equivalence. For instance, if we modify a DNA molecule to cure or prevent a disease, we obviously do not want our modification to make the involved molecules equivalent. Similarly, to measure DNA compatibility and similarity it is not realistic to look at exact equivalence between molecules: instead, one should look for metrics and distances between them.

To cope with these problems, we move from traditional string rewriting systems to quantitative string rewriting systems. Following the same ideas of subsection 2.1, we refine string rewriting relations to ternary quantitative (rewriting) relations relating pairs of strings with non-negative extended real numbers. Here is an example of quantitative string rewriting system — called $\mathcal{M}$ — over the DNA alphabet, where $\lambda$ denotes the empty string, $b, c \in \{A, C, G, T\}$ and $b \neq c$ in the last rule.

\[
\begin{array}{c}
  b \mapsto_M \lambda \\
  \lambda \mapsto_M b \\
  b \mapsto_M c
\end{array}
\]

Ignoring its quantitative dimension, system $\mathcal{M} = (\Sigma_M, \mapsto_M)$ allows us to substitute bases with one another inside any molecule — so that, for instance, we can always replace $A$ with $G$ — as well as to arbitrarily erase and insert bases inside a molecule. This results in an inconsistent (equational) system, in the sense that any two molecules are convertible. The situation changes when we take the quantitative information into account. A rewriting step $t \mapsto^e s$ gives the distance between $t$ and $s$ and a rewriting sequence

\[
s_1 \xrightarrow{e_1} s_2 \xrightarrow{e_2} \cdots \xrightarrow{e_{n-1}} s_n
\]

produces the distance $\varepsilon_i$ when rewriting $s_i$ into $s_{i+1}$, so that the overall distance between $s_1$ and $s_n$ is bounded by $\sum_i \varepsilon_i$. For this example, we have stipulated the distance produced by substitution, inserting, and deleting a base to be 1, although we could have chosen any non-negative extended real number.

---

8We apply the same notational conventions of previous section, hence using the notations $e \vdash t \mapsto_R s$ and $t \mapsto_R s$ interchangeably.
For instance, the rewriting relation defined below measures mutations between a purine (A, G) and a pyrimidine (C, T) only:

| Rule   | Relation |
|--------|----------|
| A → C | G → T    |
| A → A | G → C    |
| A → G | C → T    |

But what is the meaning of such distances? As before, the rewriting relation \( \rightarrow_M \) induces a distance \( \varnothing \) on molecules defined by
\[
\varnothing((\Sigma^*, M)) = \inf \{ t \mid t \rightarrow_M s \},
\]
so that \((\Sigma^*, M)\) is a quantitative abstract rewriting system. Consequently, we can consider the convertibility pseudometric \( \varnothing \) induced by \( M \) and realise that the latter is nothing but the Levenshtein distance \[70, 49\] between DNA molecules. This means that system \( M \) is a way to formalise the computational content of the Levenshtein distance, and that \( M \) is an operational definition of the latter. In particular, any bidirectional rewriting sequence between molecules \( t \) and \( s \) approximates the Levenshtein distance between them.

At this point, we can study properties of the Levenshtein distance and, most importantly, of its computation relying on the theory of quantitative rewriting systems that we will introduce later in this paper. As we shall see, \( M \) is confluent, so that we can approximate the convertibility distance (i.e. the Levenshtein distance) between molecules as the sum of the rewriting distances into their common reducts:
\[
\varnothing(t, s) = \inf_{u} M^\ast(t, u) + M^\ast(s, u).
\]

Additionally, even if system \( M \) is not terminating, we can extract a terminating system out of it. All of that holds not only for \( M \), but also for its variations. For instance, allowing \( \rightarrow_M \) to perform substitutions only (so that we simply have the rule \( b \rightarrow_M c \), for \( b, c \) different bases), we see that \( \varnothing \) measures the number of mutations between DNA sequences, and thus gives the Hamming distance between molecules \[70\]. Similarly, the distance induced by the previously defined quantitative relation measuring mutations between purines (A, G) and pyrimidines (C, T) gives the so-called Eigen–McCaskill–Schuster distance \[49\] between molecules.

**Metric Word Problems** Other interesting properties of the Levenshtein distance, as well as of the other aforementioned distances, can be described operationally in terms of *metric word problems*. Contrary to traditional (string) rewriting systems, in the quantitative world a word problem can take several forms to which we shall generically refer to as *metric word problems*. Here are some of those.

**Reachability** The reachability problem is the quantitative refinement of the traditional word problem. Given a quantitative (abstract) rewriting system \( R = (A, \rightarrow) \), the reachability problem asks whether \( \varnothing(a, b) < \infty \), for elements \( a, b \in A \). If \( R \) is confluent, then the reachability problem is semi-decidable; if, additionally, \( R \) is terminating, then we obtain decidability of the reachability problem. The reachability problem for \( M \) — i.e. for its associated abstract system \((\Sigma^*_M, M)\) — as well as for variations thereof, is indeed decidable. In fact, in this paper we will introduce several techniques to prove confluence of \( M \). Moreover, even if \( M \) is not terminating, we can easily extract a terminating
rewriting relation out of it (for instance, we force substitution rules to be asymmetric and stipulate that molecules can be deleted, but not inserted).

**ε-Reachability** More interesting metric word problems are obtained by strengthening the reachability problem to what we shall call ε-reachability problems. Fixing a number ε, the ε-reachability problem asks whether \( R^\varepsilon(a, b) < \varepsilon \) holds. Equivalently, the ε-reachability problem asks whether there exists a bidirectional \( R \)-rewriting sequence between \( a \) and \( b \) producing distance \( \varepsilon \). Contrary to reachability (which is nothing but ∞-reachability), confluence and termination are in general not enough to solve the ε-reachability problem. In fact, looking at the rewriting paths leading to the common normal form (if any) of two objects \( a, b \) can give too coarse (over)approximations of \( R^\varepsilon(a, b) \) only, as illustrated by the following rewriting diagram:

Assuming the system to be confluent, one can try to obtain better approximations by enlarging the state space and looking at arbitrary common reducts (and this strategy is indeed sound and complete, since confluence of \( R \) entails \( R^\varepsilon(a, b) = \inf \varepsilon R^\varepsilon(a, c) + R^\varepsilon(b, c) \)). That, however, does not solve the ε-reachability problem either, as there may be infinitely many such reducts (see, for instance, Example 5).

**Shortest Path** The shortest path is a problem specific to quantitative string and term rewriting systems. Given such a system \( \mathcal{R} = (\Sigma, \vdash_R) \), the shortest path problem for \( \mathcal{R} \) asks to determine whether

\[
\min \{ \varepsilon | \varepsilon \vdash_R t \equiv_R s \},
\]

i.e. whether the infimum \( \inf \{ \varepsilon | \varepsilon \vdash_R t \equiv_R s \} \) is achieved by an actual conversion \( \varepsilon \vdash_R t \equiv_R s \) (we write \( \equiv_R \) for the conversion ternary relation induced by \( \to_R \)). All the systems seen in this section have a shortest path, although finding such a path is usually difficult. Indeed, in all these cases, shortest paths are usually found relying on optimisation techniques and dynamic programming \[70\], and it is thus an interesting question whether solutions to this problem can be given in terms of quantitative rewriting. The shortest path problem, additionally, is particularly interesting from a rewriting perspective because it opens the door to another problem: the optimal strategy problem.

**Optimal Strategy** Assuming a term or string rewriting system \( \mathcal{R} = (\Sigma, \vdash_R) \) to have shortest paths, the optimal strategy problem asks whether there exists a quantitative rewriting strategy \( \to_{\mathcal{R}} \) such that\[72\]

\[
\varepsilon \vdash_R t \equiv_R s \iff \varepsilon \vdash_R t \to_{\mathcal{R}}^* s.
\]

An optimal strategy or an approximation thereof for \( \mathcal{M} \) can be then used to efficiently compute distances distance between DNA molecules. To the best of the authors’ knowledge, optimal strategy problems for the systems considered so far are still open.

---

11Actually, several variations of this problem can be given simply by replacing \( \to_{\mathcal{R}}^* \) with other relations related to \( \to_{\mathcal{R}} \).
2.3 Beyond Traditional Rewriting: Quantitative Term Rewriting Systems

We now go beyond string rewriting systems and take a closer look at examples of quantitative term rewriting systems. Even if we have already seen an example of a quantitative term rewriting system — the system of natural numbers of subsection 2.1 — we now consider more interesting examples of such systems and take the chance to illustrate further features of quantitative rewriting. Among the many examples available, we focus on those coming from the field of (quantitative) algebras and programming language theory.

Affine Combinatory Logic

As a first example, we consider a basic system of affine combinators that we shall enrich with effectful and quantitative primitives in subsequent sections: system $K$ of affine combinatory logic [17, 73, 74]. System $K$ has three constants (known as basic combinators) — $B$, $C$, and $K$ — and a single binary operation symbol $\cdot$ for application. We denote by $\Sigma_K$ the signature thus obtained. As usual, we assume application to associate to the left and omit unnecessary parentheses. We refer to terms written by means of variables, basic combinators, and application as combinators. Even if $K$ is historically defined as an equational theory (from which a rewriting system is then extracted), we directly define $K$ by means of rewriting rules as follows, with $\leftrightarrow$ being the (ground) reduction relation and $\rightarrow$ being defined by applying substitution instances of $\leftrightarrow$ inside arbitrary context.

$$
\begin{align*}
B \cdot x \cdot y \cdot z & \rightarrow x \cdot (y \cdot z) \\
C \cdot x \cdot y \cdot z & \rightarrow x \cdot z \cdot y \\
K \cdot x \cdot y & \rightarrow x
\end{align*}
$$

\[ t \leftrightarrow s \]
\[ C[t^\sigma] \rightarrow C[s^\sigma] \]

To obtain a quantitative refinement of system $K$, we assign distances in $[0, \infty]$ to basic rewriting rules, this way obtaining the quantitative rewriting relation $\leftrightarrow_K$ defined by\(^\text{12}\)

$$
\begin{align*}
B \cdot x \cdot y \cdot z & \leftrightarrow_K x \cdot (y \cdot z) \\
C \cdot x \cdot y \cdot z & \leftrightarrow_K x \cdot z \cdot y \\
K \cdot x \cdot y & \leftrightarrow_K x
\end{align*}
$$

and then extending $\leftrightarrow_K$ to $\rightarrow_K$ by non-expansively propagating distances produced by substitution instances of $\leftrightarrow_K$ throughout arbitrary contexts of the language.

$$
\begin{align*}
\Gamma \quad t \leftrightarrow_K s \\
\rightarrow_K C[t^\sigma] & \rightarrow_K C[s^\sigma]
\end{align*}
$$

Although quantitative, the system thus obtained can only produce trivial distances (i.e. either 0 or $\infty$), since no rewriting rule creates non-zero distances. In the next section, we will introduce effectful quantitative extensions of $K$. For the moment, we simply extend $K$ with (the combinatory counterpart of) system $N = (\Sigma_N, \leftrightarrow_N)$ of subsection 2.1. That is, we add to $K$ natural numbers and addition. Even if the resulting system is not particularly interesting from a programming language perspective,

\(^{12}\)We use the same notational conventions introduced for string rewriting systems.
it gives us the chance to illustrate the role played by linearity in quantitative and metric reasoning. We thus consider the three additional basic combinators: \( Z, S, \) and \( A \) for zero, successor, and addition, respectively. The system \( \mathcal{K}_N = (\Sigma_{\mathcal{K}_N}, \rightarrow_{\mathcal{K}_N}) \) of affine combinators with natural numbers is given by the signature \( \Sigma_{\mathcal{K}_N} \triangleq \Sigma_\mathcal{K} \cup \{Z, S, A\} \) and the quantitative rewriting relation defined thus:

\[
\begin{align*}
B \cdot x \cdot y \cdot z & \rightarrow_{\mathcal{K}_N} x \cdot (y \cdot z) \\
C \cdot x \cdot y \cdot z & \rightarrow_{\mathcal{K}_N} x \cdot z \cdot y \\
K \cdot x \cdot y & \rightarrow_{\mathcal{K}_N} x \\
A \cdot x \cdot Z & \rightarrow_{\mathcal{K}_N} x \\
A \cdot x \cdot (S \cdot y) & \rightarrow_{\mathcal{K}_N} S \cdot (A \cdot x \cdot y) \\
S \cdot x & \rightarrow_{\mathcal{K}_N} 1 \cdot x \\
\end{align*}
\]

As usual \( \rightarrow_{\mathcal{K}_N} \) induces a distance \( K_N \) on combinators defined by \( K_N(t, s) \triangleq \inf\{\epsilon \mid t \xrightarrow{\epsilon}_{\mathcal{K}_N} s\} \), from which we obtain a pseudometric \( K_N^\infty \). Let us now turn our attention to the definition of \( \rightarrow_{\mathcal{K}_N} \). Given a quantitative rewriting relation \( \rightarrow_{\mathcal{K}} \), all the systems considered so far define \( \rightarrow_{\mathcal{K}} \) by forcing non-expansiveness of contexts and substitution (cf. quantitative equational theories). System \( \mathcal{K}_N \) is no exception. The defining rules of \( \rightarrow_{\mathcal{K}_N} \) ensures that the application operation is non-expansive with respect to \( K_N^\infty \). Formally:

\[
K_N^\infty(t, t') + K_N^\infty(s, s') \geq K_N^\infty(t \cdot s, t' \cdot s').
\]

In particular, if \( \epsilon \vdash t \rightarrow_{\mathcal{K}_N} t' \) and \( \delta \vdash s \rightarrow_{\mathcal{K}_N} s' \), then \( \epsilon + \delta \vdash t \cdot s \rightarrow_{\mathcal{K}_N} t' \cdot s' \).

Non-expansiveness, however, does not come for free: it is a direct consequence of linearity\(^{13}\) of \( \mathcal{K}_N \). In fact, the addition of a non-linear combinator such as \( W \) directly leads to breaking non-expansiveness (in the sense that forcing non-expansiveness leads to undesired results, such as distance trivialisation and non-confluence). To see that, let us add the basic combinator \( W \) together with the following rewriting rule to our system.

\[
W \cdot x \cdot y \rightarrow_{\mathcal{K}_N} x \cdot y \cdot y
\]

We now show that the presence of \( W \) makes quantitative reasoning trivial.

**Notation 3.** Let us write \( c_n \) for the combinator \( S \cdot (\cdots (S \cdot Z)) \), with \( n \) applications of \( S \), so that \( K_N^\infty(c_n, c_m) = |n - m| \) and \( K_N^\infty(A \cdot c_n \cdot c_m, c_{n+m}) = 0 \).

**Proposition 1** (Distance Trivialisation). *In presence of the combinator \( W \), the convertibility distance \( K_N^\infty \) trivialises, meaning that the distance \( K_N^\infty(t, s) \) is either 0 or \( \infty \), for all combinators \( t, s \).*

**Proof.** Given combinators \( t, s \), let \( K_N^\infty(t, s) = \epsilon \). If \( \epsilon \) is \( \infty \), we are done. Otherwise, \( \epsilon \) is a natural number, since the defining rule of \( \rightarrow_{\mathcal{K}_N} \) ensures the codomain of \( K_N^\infty \) to actually be \( \mathbb{N}^\infty \). Consequently, we have combinators \( c_m, c_n \) such that \( K_N^\infty(c_m, c_n) = \epsilon \). Notice also that whenever we have combinators \( t, t', s, s' \) such that \( K_N^\infty(t, t') = 0 \) and \( K_N^\infty(s, s') = 0 \), then \( K_N^\infty(t, s) = K_N^\infty(t', s') \). Thus, for instance, we see that

\[
K_N^\infty(W \cdot t \cdot s, W \cdot t' \cdot s') = K_N^\infty(t \cdot s \cdot t', t' \cdot s' \cdot s').
\]

---

\(^{13}\)The word *linearity* is used both in rewriting and in logic with different, although similar, meaning. For the moment, we use it informally to indicate the absence of variable duplication, leaving formal definitions to the technical part of this paper.
Non-expansiveness of application then gives (using $K_N^\pi(W, W) = 0$ and $K_N^\pi(A, A) = 0$):

$$K_N^\pi(c_n, c_m) \geq K_N^\pi(W \cdot A \cdot c_n, W \cdot A \cdot c_m) = K_N^\pi(A \cdot c_n \cdot c_n, A \cdot c_m \cdot c_m) = K_N^\pi(c_{n+m}, c_{m+m})$$

meaning that we have $\varepsilon \geq \varepsilon + \varepsilon$. In $\mathbb{N}^\infty$ this is possible only for 0 and $\infty$, and thus we conclude $K_N^\pi(t, s) = 0$.

Proposition 1 is known as *distance trivialisation* or *distance amplification* [62, 33] and it has been deeply investigated studying (effectful) program distancing. Linearity, and variations thereof, are a way to avoid trivialisation of quantitative reasoning. Additionally, we shall see in subsection 7.2 that linearity is also crucial to ensure quantitative forms of confluence, the latter being, together with distance trivialisation, the reason why in section 7 we will focus on linear non-expansive rewriting systems. In section 8, we will see how moving to graded (modal) systems gives us a way to go beyond the linearity assumption and refine non-expansive systems to Lipschitz continuous ones.

### 2.3.1 Effectful Combinatory Logic

System $\mathcal{K}_N$ of affine combinators and arithmetic allowed us to highlight the role of linearity and non-expansiveness in quantitative reasoning. Apart from that, system $\mathcal{K}_N$ is not particularly interesting. Here, we extend affine combinators with quantitative algebraic theories modelling computational effects [84, 85, 10, 86, 87, 12, 90].

Probabilistic [18] and, more generally, effectful programming languages have been extensively studied in the last decade using, among others, probabilistic [30, 29, 53, 54, 9] and effectful [63] rewriting systems. Such systems come in two flavours, depending on whether effects are considered internally or externally to the system. In the latter case, one obtains probabilistic [30, 29, 9] and monadic rewriting systems [63]. In the former case, instead, one models (equational theories defining) computational effects themselves as rewriting systems and then combines the latter with the actual calculus or programming language at hand, which is modelled as a rewriting system itself. Here, we follow the latter approach and look at computational effects as defined by quantitative equational theories [84, 85].

**Barycentric Algebras** Let us begin with one of the main examples of a quantitative equational theory: *barycentric algebras*. Barycentric algebras have been introduced by Stone [108] as an equational axiomatisation of finite distributions, and they have recently refined as a *quantitative equational theory* by Mardare et al. [84, 85]. Here, we present such a quantitative refinement directly as a quantitative term rewriting system. Let us consider a signature $\Sigma$ containing a family of binary probabilistic choice operations $+_\epsilon$ indexed by rational numbers $\epsilon \in \mathbb{Q} \cap [0, 1]$. The quantitative term rewriting system $\mathcal{B} = (\Sigma, \rightarrow_{\mathcal{B}})$ of Barycentric algebras is defined thus:
combine systems between multi-distributions (see section 8 for another example of a probabilistic distance). We can then model on multi-distributions. The operation or semi-decidable?) and for its operational semantics (is reduction confluent? Do we have an optimal quantitative rewriting properties and metric word problem s become interesting both for the (quantita-

tive choice, this way giving a quantitative theory of probabilistic (affine) computation. In light of that, weighted by probability of variables, a term in \( \Sigma \) can be seen as a finite formal sum, i.e. a syntactic representation of a finitely supported distribution.

As usual, starting from \( \rightarrow_B \), we obtain the rewriting relation \( \rightarrow_B \), and the convertibility distance \( B^\equiv \). Remarkably, the latter is precisely the total variation distance \( \| \) between multi-distributions (see section 8 for another example of a probabilistic distance). We can then combine systems \( B \) and \( K \) (or even \( K \)), this way obtaining the quantitative term rewriting system \( K_B = (\Sigma_K \cup \Sigma_B, \rightarrow_{KB}) \) for probabilistic affine combinatory logic as summarised in figure 1. In particular, the convertibility distance induced by \( \rightarrow_{KB} \) is essentially \( (K \min B)^\equiv \), where \( (K \min B)(t, s) \equiv \min(K(t, s), B(t, s)) \), which gives the total variation distance between probabilistic combinators. That puts together the usual equational theory of combinators with the quantitative analysis of probabilistic choice, this way giving a quantitative theory of probabilistic (affine) computation. In light of that, quantitative rewriting properties and metric word problems become interesting both for the (quantitative) equational theory of probabilistic affine computations (is the theory consistent? Is it decidable or semi-decidable?) and for its operational semantics (is reduction confluent? Do we have an optimal strategy?).

In this paper, we shall prove confluence of \( (\Sigma_K \cup \Sigma_B, \rightarrow_{KB}) \). Consequently, we will obtain consistency of its (quantitative) equational theory and semi-decidability of the reachability problem. Achieving such

\[
\begin{align*}
B \cdot x \cdot y \cdot z & \rightarrow^0_{KB} x \cdot (y \cdot z) \\
C \cdot x \cdot y \cdot z & \rightarrow^0_{KB} x \cdot z \cdot y \\
K \cdot x \cdot y & \rightarrow^0_{KB} x \\
(x + \epsilon_1, y) + \epsilon_2, z & \rightarrow^0_{KB} x + \epsilon_1 \epsilon_2, (y + \epsilon_1 - \epsilon_2) z \\
\epsilon_1, \epsilon_2 & \in (0, 1) \\
x + \epsilon & \rightarrow_B y + 1 - \epsilon x \\
(x + \epsilon_1, y) + \epsilon_2, z & \rightarrow_B x + \epsilon_1 \epsilon_2, (y + \epsilon_1 - \epsilon_2) z \\
\epsilon & \leq \epsilon \in \mathbb{Q} \cap [0, 1] \\
x + \epsilon & \rightarrow_B y + \epsilon \\
\epsilon & \leq \epsilon \in \mathbb{Q} \cap [0, 1]
\end{align*}
\]
a result is nontrivial and requires the introduction of several new results on quantitative rewriting system. In particular, we will prove confluence in a modular fashion relying on a suitable quantitative refinement of the Hindley-Rosen Lemma \[72, 99\] (Proposition 3) and proving confluence of \((\Sigma_{\mathcal{K}}, \rightarrow_{\mathcal{K}})\) and \((\Sigma_{\mathcal{BA}}, \rightarrow_{\mathcal{BA}})\) separately (subsection 7.3), the latter requiring the extension of critical pair-like lemmas \[77\] to quantitative rewriting systems (subsection 7.2 and subsection 7.3).

Ticking Barycentric algebras are just one example of a quantitative algebraic theory used to model computational effects. Other examples include the theory of quantitative semilattices \[84\] (whose associated distance is the Hausdorff distance), quantitative global states, and quantitative output \[11\]. Here, we introduce the quantitative theory of ticking, a specific instance of quantitative output used in improvement theory and cost analysis \[104\] to study intensional aspects of programs.

Let us consider the monoid \((\mathbb{N}, +, 0)\) of natural numbers with addition endowed with the Euclidean distance.\(^{15}\) The (quantitative) term rewriting system \(T = (\Sigma_T, \rightarrow_T)\) of ticking is defined by the signature \(\Sigma_T\) of unary operation symbols \(\cdot\) indexed by elements \(\mathbb{N}\) and the following rewriting rules:

\[
\begin{align*}
0.x & \rightarrow_T x \\
n.(m.x) & \rightarrow_T (n + m).x \\
n.x & \rightarrow_T m.x \\
\epsilon & \geq E(n, m)
\end{align*}
\]

The operation \(n \cdot t\) can be informally read as count \(n\) unit of cost, then continue as \(t\). Oftentimes, one writes terms of the form \(1 \cdot t\) as \(\sqrt{t}\) and decorates programs with \(\sqrt{\cdot}\) annotations to count computation steps (for instance, in systems based on the \(\lambda\)-calculus or combinatory logic, applications \(t \cdot s\) are decorated as \(\sqrt{t \cdot s}\): this way, one measures the cost of a computation as the numbers of applications performed).\(^{16}\) In this case, we actually obtain a simplified system \(T_{\sqrt{\cdot}}\) whose signature contains the unary function symbol \(\sqrt{\cdot}\) only and whose (unique) rewriting rule is the following

\(\sqrt{x} \overset{1}{\rightarrow} x\)

Let us now come back to system \(T\). The first two rewriting rules of system \(T\) model null cost production and cost sequencing, whereas the last rule allows us to measure differences between cost traces. Accordingly, variations of \(T\) are obtained by changing the way we measure cost differences. For instance, having in mind program refinement, one may want to to replace the Euclidean distance with its asymmetric counterpart. Finally, we can combine systems \(T\) and \(K\) — or even \(K_{\mathcal{E}}\) — together, this way obtaining systems for the quantitative cost analysis of affine and probabilistic computations. We can then (and we will) prove confluence of the resulting systems compositionally relying on the quantitative Hindley-Rosen lemma \(\text{Proposition 3}\) and proving confluence of each system separately.

\(^{15}\)A more general definition can be given by fixing a quantitative output monoid, that is a monoid endowed with a generalised distance \[82\] making monoid multiplication non-expansive. Besides the monoid of natural numbers with the Euclidean distance, another classic example of quantitative output monoid is given by the monoid of words over an alphabet endowed with the least common prefix distance.

\(^{16}\)The ticking operation can be seen as a particular instance of an output operation, where the output produced is the cost of computation. Bacci et al. \[11\] have shown that the quantitative equational theory associated to this reading of ticking is exactly the theory of the quantitative writer monad, which specialises to the one of the cost or ticking monad \[37, 104\].
2.4 Further Examples and Where to Find Them

The systems we have seen so far are just some of the many examples of quantitative rewriting systems one can either find in the literature or design independently. For instance, several quantitative equational theories of computational effects have been recently developed in addition to the ones introduced in this motivational section. Examples of those include quantitative nondeterminism (describing the Hausdorff distance between sets) \[84\], global stores \[11\], and combined pure-probabilistic nondeterminism \[90\]. All these theories can be analysed operationally as quantitative rewriting systems and combined with the systems introduced so far.

A further source of examples follows the line of subsection 2.2, where we have provided operational descriptions of several edit distances (e.g. the Hamming and Levenshtein distance) on (DNA) strings. Indeed, quantitative rewriting systems are particularly well-suited to model edit distances — not necessarily on strings — operationally. The *Encyclopedia of Distances* by Deza and Deza \[49\] is a great source of potential examples of (edit) distances that could be approached operationally. Potential applications of quantitative rewriting systems are given by optimisation theory \[83\], where one naturally deals with weighted graphs and searches optimal paths. By characterising such weighted graphs as the reduction graphs of quantitative rewriting systems, one may give a more symbolic account to optimisation.

Numeric and approximated computation provide interesting examples of quantitative rewriting systems too. In fact, several computer algebra systems allow the user to combine symbolic and numeric computation.\(^\text{17}\) It then seems natural to consider exact rewriting to model the symbolic part of a computation and quantitative rewriting to model the numeric one, as the latter naturally involves numerical approximations and precision errors. For instance, the numerical evaluation of the symbolic constant \(\pi\) to, e.g., the numerical approximation \(3.14\) could be modelled as the reduction \(\pi \rightarrow 3.14\), with \(\varepsilon\) the error produced in the approximation.

The reader should now be sufficiently familiar with examples of quantitative rewriting systems and basic ideas behind them. The rest of the paper is devoted to introduce the general theory of quantitative and metric rewriting in full detail, starting from abstract and then moving to quantitative term rewriting systems. In doing so, we also analyse the examples seen in this introductory section (as well as new ones) formally.

3 PRELIMINARIES: QUANTITATIVE RELATIONAL CALCULUS, à la LAWVERE

We begin our analysis of quantitative rewriting by developing a theory of quantitative abstract rewriting systems. To do so, we first recall some mathematical preliminaries.

QUANTALE We begin our analysis of quantitative rewriting by developing a theory of quantitative abstract rewriting systems. To do so, we first recall some mathematical preliminaries.

Quantales Traditional abstract rewriting systems can be naturally defined and studied relationally. To define a theory of quantitative rewriting, it thus seems natural to rely on quantitative relational calculi. Here, we follow the analysis of generalised metric spaces as enriched categories by Lawvere \[82\] and work with relations taking values in a quantale \[100\]. Quantale-valued relations are extensively used in monoidal topology \[70\] and have been successfully applied to define metric and quantitative semantics of higher-order languages \[61, 62\], as well as behavioural metrics \[117, 65\]. Let us begin recalling the definition of a quantale, which we view as modelling abstract quantities.

\(^{17}\)See, for instance, the library SymPy [https://www.sympy.org/en/index.html].
Definition 1. A (unital) quantale $\Omega = (\Omega, \leq, k, \otimes)$ consists of a monoid $(\Omega, k, \otimes)$ and a sup-semilattice $(\Omega, \leq)$ satisfying the following distributivity laws:

$$\delta \otimes \bigvee_{i \in I} \epsilon_i = \bigvee_{i \in I} (\delta \otimes \epsilon_i)$$

$$\left(\bigvee_{i \in I} \epsilon_i\right) \otimes \delta = \bigvee_{i \in I} (\epsilon_i \otimes \delta).$$

The element $k$ is called unit of the quantale, whereas $\otimes$ is called the tensor (or multiplication) of the quantale.

It is easy to see that $\otimes$ is monotone in both arguments. We denote the top and bottom element of a quantale by and $\top, \bot$ respectively. Quantales having unit $k$ coinciding with the top element are called integral quantales. Moreover, we say that a quantale is commutative if its underlying monoid is, and that it is non-trivial if $k \neq \bot$. Integral quantales are particularly well-behaved: for instance, in an integral quantale $\epsilon_1 \otimes \epsilon_2$ is a lower bound of each $\epsilon_i$. Additionally, in an integral quantale we have $\epsilon \otimes \bot = \bot$, for any $\epsilon \in \Omega$. If the opposite direction holds, i.e. whenever $\epsilon \otimes \delta = \bot$, either $\epsilon = \bot$ or $\delta = \bot$ holds, we say that the quantale is cointegral.

From now on, we assume quantales to be commutative, (co)integral, and non-trivial. We refer to such quantales as Lawverian. Finally, we say that a quantale is idempotent if $\epsilon \otimes \epsilon = \epsilon$. Notice that any quantale $(\Omega, \leq, k, \otimes)$ induces an idempotent quantale as $(\Omega, \leq, \top)$ and that in any integral idempotent quantale $\wedge$ and $\otimes$ coincide.

Example 1. 1. The boolean quantale $2 = (2, \leq, \wedge, \top)$, where $2 = \{\top, \bot\}$ and $\bot \leq \top$, is an idempotent Lawverian quantale.

2. Any frame is an idempotent integral quantale. If the frame is cointegral, then we obtain a Lawverian quantale.

3. The Lawver quantale $\mathbb{L} = ([0, \infty], \geq, +, 0)$ consisting of the extended real half-line ordered by the “greater or equal” relation $\geq$ and extended addition as tensor product is a Lawverian quantale. Notice that we use the opposite of the natural ordering, so that, e.g., 0 is the top element of $\mathbb{L}$.

4. The Strong Lawver quantale $\mathbb{L}^{\max} = ([0, \infty], \geq, \max, 0)$ obtained by replacing addition with maximum in the Lawvere quantale is an idempotent Lawverian quantale. Notice that in the strong Lawver quantale tensor and meet coincide, and thus the quantale is idempotent.

5. The unit interval $\mathbb{I} = ([0, 1], \leq, *)$ endowed with a left continuous triangular norm (t-norm for short) $\sqcap$ is an integral quantale. Examples of t-norms are:

\textsuperscript{18} By monotonicity of $\otimes$, we have: $\epsilon_1 \otimes \epsilon_2 \leq \epsilon_1 \otimes \top = \epsilon_1 \otimes k = \epsilon_1$.

\textsuperscript{19} Recall that a frame consists of a sup lattice $V, \leq, \lor$ satisfying the following distributivity laws:

$$x \land \bigvee_{i \in I} x_i = \bigvee_{i \in I} (x \land x_i)$$

$$\left(\bigvee_{i \in I} x_i\right) \land y = \bigvee_{i \in I} (x_i \land y).$$

A main concrete example of a frame is the structure $(\tau, \subseteq, \cap, X)$ given by the open sets $\tau$ of a topological space.

\textsuperscript{20} We extend ordinary addition as follows: $x + \infty = \infty = \infty + x$.

\textsuperscript{21} Recall that a t-norm is a binary operator $* : [0, 1] \times [0, 1] \to [0, 1]$ that induces a quantale structure over the complete lattice $([0, 1], \leq)$ in such a way that the quantale is commutative.
(a) The product t-norm: \( x \ast_p y \equiv x \cdot y \).
(b) The Łukasiewicz t-norm: \( x \ast_q y \equiv \max\{x + y - 1, 0\} \).
(c) The Gödel t-norm: \( x \ast_g y \equiv \min\{x, y\} \).

If, additionally, \( x \ast y = 0 \) implies \( x = 0 \) or \( y = 0 \), then we obtain a Lawverian quantale. In particular, both the product and Gödel t-norms give Lawverian quantales. Such quantales are used to model Fuzzy reasoning, and thus we refer to \( \mathcal{B} = ([0, 1], \leq, 0, 1) \) as the Fuzzy quantale(s).

6. The set of monotone modal predicates \( 2^W \) on a preorder monoid with top element \( (W, \leq, +, 0, \top) \) of possible worlds, endowed with the tensor product defined below, is a Lawverian quantale.

\[(p \otimes q)(w) \iff \exists u, v. \ w \geq u + v \land p(u) \land q(v)\]

Such a quantale is used to study modal and coeffectful properties of programs [40, 41].

7. If in the previous example of modal predicates we replace \( 2^W \) with a (Lawverian) fuzzy quantale \( ([0, 1], \leq, 0, 1) \), then we obtain the Lawverian quantale of Fuzzy modal predicates.

8. The set \( \mathcal{F} \equiv \{ f \in [0, 1]^{[0, \infty]} | f \ \text{monotone and } f(a) = \bigvee_{b \preceq a} f(b) \} \) used by Hofmann and Reis [75] to model probabilistic metric spaces is a quantale. Notice that elements of \( \mathcal{F} \) can be seen as Fuzzy modal predicates over the set of possible worlds \([0, \infty]\).

To help the reader working with quantales, we summarise the correspondence between the Boolean (\( 2 \)), Lawvere (\( L \)), and Strong Lawvere (\( L^{\text{max}} \)) quantale — our main running examples — as well as a generic quantale \( \Omega = (\Omega, \leq, k, \otimes) \), in Table 1.

|     | 2 (Boolean) | L (Lawvere) | \( L^{\text{max}} \) (strong Lawvere) | \( \Omega \) (quantale) |
|-----|-------------|-------------|--------------------------------------|-----------------------|
| Carrier | 2 | \([0, \infty]\) | \([0, \infty]\) | \( \Omega \) |
| Order | \( \leq \) | \( \geq \) | \( \geq \) | \( \leq \) |
| Join | \( \exists \) | inf | inf | \( \bigvee \) |
| Meet | \( \forall \) | sup | sup | \( \land \) |
| Tensor | \( \wedge \) | + | max | \( \otimes \) |
| Unit | \( \top \) | 0 | 0 | \( k \) |

Table 1: Correspondence 2-[0, \infty]-\( \Omega \).

Since any quantale is, in particular, a complete lattice and tensor product is monotone in both arguments, the latter has both left and right adjoints (which coincide in our case, as we assume \( \otimes \) to be commutative):

\[\epsilon \otimes \delta \leq \eta \iff \delta \leq \epsilon \otimes \eta.\]

Explicitly, we have \( \epsilon \rightarrow \delta \equiv \bigvee\{\eta \mid \epsilon \otimes \eta \leq \delta\} \). For instance, in the Boolean quantale \( \rightarrow \) is ordinary implication, whereas in the Lawvere quantale \( \rightarrow \) is truncated subtraction.

This is all the reader has to know about quantales to understand quantitative abstract rewriting systems. When it comes to move to quantitative term rewriting systems, a little more notions about (and a little more conditions on) quantales are needed. In particular, the addition of structural rules akin to quantitative equational theories requires us to work with continuous quantales [23] (but see Remark 7).
Definition 2. Given a quantale $\mathcal{Q}$ and elements $\varepsilon, \delta \in \mathcal{Q}$, the way-below relation $\ll$ is defined thus: $\delta \ll \varepsilon$ if and only if for every subset $A \subseteq \mathcal{Q}$, whenever $\varepsilon \leq \bigvee A$, there exists a finite subset $A_0 \subseteq A$ such that $\delta \leq \bigvee A_0$. We say that $\mathcal{Q}$ is continuous if and only if

$$\varepsilon = \bigvee_{\delta \ll \varepsilon} \delta.$$  

Example 2. The Boolean quantale $\mathcal{B}$ being finite it is trivially continuous. Both the Lawvere and the strong Lawvere quantales are continuous with $\bowtie$ (i.e., greater than) as the way below relation. There, we extend $\bowtie$ to $[0, \infty]$ by stipulating $\infty > \infty$. In the same way, one obtains continuity of Fuzzy quantales. There, the way below relation is given by $\bowtie$ (i.e., less than) extended by stipulating $0 < \infty$. 

Quantale-valued Relations We now move to quantale-valued relations, our main tool to model quantitative rewriting. As quantales model abstract quantities, quantale-valued relations provide abstract notions of distances.

Definition 3. Given a quantale $\mathcal{Q} = (\Omega, \leq, k, \otimes)$, a $\mathcal{Q}$-relation $R : A \to B$ between sets $A$ and $B$ is a function $R : A \times B \to \mathcal{Q}$. For any set $A$, we define the identity (or diagonal) $\mathcal{Q}$-relation $\Delta_A : A \to A$ mapping diagonal elements $(a, a)$ to $k$, and all other elements to $\perp$. Moreover, the composition $R; S : A \to C$ of $\mathcal{Q}$-relations $R : A \to B$ and $S : B \to C$ is defined by the so-called matrix multiplication formula $[76]$:

$$(R; S)(a, c) \doteq \bigvee_{b \in B} R(a, b) \otimes S(b, c).$$

In general, we think about a $\mathcal{Q}$-relation as giving the distance or the degree of relatedness of two elements $[52, 57, 59, 76]$. For instance, when the quantale is Boolean, elements are either related or not, whereas for Fuzzy quantales $\mathcal{Q}$-relations coincide with Fuzzy relations $[19]$, and thus they give the degree to which elements are related, as well as proximity and similarity relations. When we move to the Lawvere quantale (and quantales alike), $\mathcal{Q}$-relations give general notions of distances $[82]$, and thus act as a foundation for metric reasoning $[27]$. Coarser forms of metric reasoning are obtained by considering interval-based $[65]$ and probabilistic quantales $[73]$, where instead of establishing the distance between elements exactly, one obtains only an interval to which such a distance belongs, or a probability of the accuracy of its measurement. Finally, considering the quantale of (fuzzy) modal predicates, we obtain (fuzzy) modal and coeffectful relations $[41, 101, 102, 103, 113]$, whereby (the degree of) relatedness of elements is given with respect a possible world (such as the available resources).

Example 3. We summarise composition on the Boolean, Lawvere, and Strong Lawvere quantale in Table 2.

| $(R; S)(a, c)$ | $\mathcal{B}$ | $\mathcal{L}$ | $\mathcal{L}_{\text{max}}$ |
|---------------|---------------|---------------|----------------------|
| $\exists g. R(a, b) \wedge S(b, c)$ | $\inf_g R(a, b) + S(b, c)$ | $\inf_g \max(R(a, b), S(b, c))$ |

Table 2: Composition on $\mathcal{B}$, $\mathcal{L}$, and $\mathcal{L}_{\text{max}}$

Since $\mathcal{Q}$-relation composition is associative and has $\perp$ as unit element, for any quantale $\mathcal{Q}$ we have a category, denoted by $\mathcal{Q}-\text{Rel}$, with sets as objects and $\mathcal{Q}$-relations as arrows. Moreover, the complete
lattice structure of $\Omega$ lifts to $\Omega$-relations pointwise, so that we can say that a $\Omega$-relation $R : A \to A$ is reflexive if $A \leq R$; transitive if $R : R \leq R$; and symmetric if $R^{-} \leq R$, where the transpose of $R : A \to B$ is the $\Omega$-relation $R^{-} : B \to A$ defined by $R^{-}(b, a) \doteq R(a, b)$. When read pointwise, reflexivity, transitivity, and symmetry give the following inequalities:

\[
\begin{align*}
k &\leq R(a, a) \\
R(a, b) \otimes R(b, c) &\leq R(a, c) \\
R(a, b) &\leq R(b, a).
\end{align*}
\]

Altogether, we obtain the notion of a preorder (i.e. reflexive and transitive) and equivalence (i.e. reflexive, transitive, and symmetric) $\Omega$-relation.

**Notation 4.** Fixed a quantale $\Theta$, we oftentimes refer to $\Omega$-relation on $\Theta$ as $\Theta$-relations. Thus, for example, $\mathbb{L}$-relations are just $\Omega$-relations on the Lawvere quantale $\mathbb{L}$.

**Example 4.**
1. On the Boolean quantale, $\mathbb{2}$-relations are ordinary (binary) relations, and preorder and equivalence $\mathbb{2}$-relations coincide with traditional preorders and equivalences.
2. On the Lawvere quantale, $\mathbb{L}$-relations are distances. Instantiating transitivity on $\mathbb{L}$, we obtain the usual triangle inequality formula:

\[
\inf_{b} R(a, b) + R(b, c) \geq R(a, c)
\]

Similarly, reflexivity gives the identity of indiscernibles inequality:

\[
0 \geq R(a, a).
\]

Altogether, we see that preorder $\mathbb{L}$-relations coincide with generalised metrics [82, 27] and equivalence $\mathbb{L}$-relations with pseudometrics [107].

3. Moving from the Lawvere to the Strong Lawvere quantale, we replace addition with binary maximum, so that transitivity now gives the strong triangle inequality formula:

\[
\inf_{b} \max(R(a, b), R(b, c)) \geq R(a, c)
\]

Consequently, equivalence $\mathbb{L}^{\text{max}}$-relations coincide with ultra-pseudometrics.

4. On the quantale $\mathbb{F}$, equivalence $\mathbb{F}$-relations give probabilistic metric spaces [75]. The informal reading of a $\mathbb{F}$-relation $R$ is that $R(a, b)(\varepsilon)$ gives the probability that $a$ and $2$ are at most $\varepsilon$-far.

5. On the unit interval (fuzzy) quantale(s), $\mathbb{I}$-relations coincide with fuzzy relations [19]. Equivalence $\mathbb{I}$-relations are often called similarity or proximity relations.

\[\text{We summarise how reflexivity, symmetry, and transitivity instantiate on the Boolean, Lawvere, and Strong Lawvere quantale in Table 3.}\]

Finally, we notice that the “algebra” of $\Omega$-relations is close to the on ordinary relations\footnote{As the category of traditional relations, the category $\Omega$-Rel is a quantaloid [74, 110].} so that we can refine a large part of calculi of relations [105] to a quantale-based setting. In fact, we can even think
about \( \Omega \)-relations as “monoidal relations”. Since many notions of traditional rewriting can be given in purely relational terms, we can take advantage of that and rephrase them in terms of \( \Omega \)-relations. For the moment, we simply recall the following useful closure operations.

**Definition 4.** Let \( R : A \rightarrow A \) be a \( \Omega \)-relation. For \( n \in \mathbb{N} \), we define the \( n \)-th iterate of \( R \), notation \( R^n \) by \( R^0 \triangleq \Delta \) and \( R^{n+1} = R; R^n \). We define:

1. The **reflexive closure** of \( R \) as \( R^\ast \triangleq R \lor \Delta \).
2. The **transitive and reflexive closure** of \( R \) as \( R^* \triangleq \lor_{n \geq 0} R^n \).
3. The **equivalence closure** of \( R \) as \( R^e \triangleq (R \lor R^*)^* \).

As already remarked, our approach to quantitative abstract rewriting systems will be algebraic and relational. Accordingly, we shall prove several nontrivial rewriting properties relying on the algebra of \( \Omega \)-relations. To do so, it is useful to exploit fixed point characterisations of relational constructions, as well as their adjunction properties \([13, 20]\). Recall that \( \Omega\text{-Rel}(A, B) \) carries a complete lattice structure, so that any monotone map \( F : \Omega\text{-Rel}(A, B) \rightarrow \Omega\text{-Rel}(A, B) \) has least and greatest fixed points, denoted by \( \mu X. F(X) \) and \( \nu X. F(X) \), respectively. Consequently, we can define \( \Omega \)-relations both inductively and conductively. That gives us the following (fixed point) induction and (fixed point) coinduction proof principles:

\[
\begin{align*}
F(R) & \leq R \\
\mu X. F(X) & \leq R \\
R & \leq F(R) \\
R & \leq \nu X. F(X)
\end{align*}
\]

In particular, we notice that \( R^* \) is the least solution to the equation \( X = \Delta \lor R; X \), so that \( R^* \) can be equivalently defined the least fixed point \( \mu X. \Delta \lor R; X \), and thus as the least pre-fixed point of the map \( F(X) \triangleq \Delta \lor R; X \). Consequently, we obtain the following least fixed point induction rule:

\[
\begin{align*}
\Delta \lor R; S & \leq S \\
R^* & \leq S
\end{align*}
\]

**Notation 5.** We denote by \( \sqcup \) the \( \Omega \)-relation \( \mu X. X \) assigning distance \( \bot \) to all elements, and by \( \sqcap \) the \( \Omega \) relation \( \nu X. X \), i.e. the indiscrete \( \Omega \)-relation assigning distance \( k \) to all elements. Explicitly, we have \( \sqcup (a, b) = \bot \) and \( \sqcap (a, b) = k \), for all \( a, b \).

Finally, we mention that \( \Omega\text{-Rel}(A, A) \) being not only a complete lattice, but a quantale, \( \Omega \)-relation composition has both left and right adjoints, often referred to as left and right division \([20]\):

\[
R; S \leq P \iff S \leq R \setminus P \quad R; S \leq P \iff R \leq P \setminus S.
\]

| \( \sqcup \) \( \sqcap \) \( \sqcup \text{ max} \) | 2 | \( \sqcap \) | \( \sqcup \text{ max} \) |
|---|---|---|---|
| \( \sqcup \leq R(a, a) \) | \( 0 \geq R(a, a) \) | \( 0 \geq R(a, a) \) |
| \( R(a, b) \leq R(b, a) \) | \( R(a, b) \geq R(b, a) \) | \( R(a, b) \geq R(b, a) \) |
| \( R(a, b) \land R(b, c) \leq R(a, c) \) | \( R(a, b) + R(b, c) \geq R(a, c) \) | \( \max(R(a, b), R(b, c)) \geq R(a, c) \) |

Table 3: Correspondences reflexivity-symmetry-transitivity.
Ternary Relations

Even if we model quantitative rewriting relations as $\Omega$-relations, in section 2 we have defined rewriting systems by means of suitable ternary relations from which we have then extracted a $\Omega$-relations. This process, known as strata extension [70], is an instance of a more general correspondence [40] between abstract distances and suitable ternary relations akin to substructural Kripke relations [101, 102, 103, 113] as used in the relational analysis of coeffects [41, 41]. Since we will extensively switch between $\Omega$-relations and ternary relations, we recall the notion of a $\Omega$-ternary relation.

Definition 5. Given a quantale $\Omega$, a $\Omega$-ternary relation over $A \times B$ is a ternary relation $R \subseteq A \times \Omega \times B$ antitone in its second argument (meaning that $R(a, \varepsilon, b)$ implies $R(a, \delta, b)$, for any $\delta \leq \varepsilon$).

Any ternary $\Omega$-relation $R$ induces a $\Omega$-relation $R^*$ thus:

$$R^*(a, b) \triangleq \bigvee_{R(a, \varepsilon, b)} \varepsilon.$$  

Vice versa, any $\Omega$-relation $R$ induces a $\Omega$-ternary relation $R^\circ$ defined by

$$R^\circ(a, \varepsilon, b) \iff \varepsilon \leq R(a, b).$$

This two processes are each other inverses, meaning that $R^\circ \circ R = R$ and $R \circ R^* = R$, so that we can freely switch between $\Omega$-ternary relations and $\Omega$-relations.

Notation 6. Oftentimes, we will use modal relations to define rewriting systems. In those cases, we will use notations of the form $\rightarrow_R$ and write $\varepsilon \vdash a \rightarrow_R b$ in place of $\rightarrow_R (a, \varepsilon, b)$. Moreover, we shall denote by $R$ the $\Omega$-relation associated to $\rightarrow_R$. That is, $R(a, b) \triangleq \bigvee_{\varepsilon \vdash a \rightarrow_R b} \varepsilon$.

Remark 2. To make definitions computationally lighter, the literature on quantitative algebraic theories usually considers ternary relations over a base of $\Omega$, the carrier of such a base usually being considerably smaller than $\Omega$. For instance, quantitative equational theories are often defined using ternary relations over non-negative rationals, the latter being a base for $[0, \infty]$. Since our results are independent of working with bases or with full quantales, we keep the necessary mathematical preliminaries as minimal as possible, this way defining (in section 7) quantitative term rewriting systems relying on quantales rather than on their bases (the interested reader can consult the recent work by Dahlqvist and Neves [36] to convince herself that our theory is invariant with respect to such a design choice).

4 QUANTITATIVE ABSTRACT REWRITING SYSTEMS

In this section, we introduce quantitative abstract rewriting systems and their theory. These systems constitute the foundation of quantitative rewriting, and all other notions of quantitative rewriting system, such as string- and term-based systems, can be ultimately regarded as quantitative abstract rewriting systems. Moreover, we shall use the latter to define crucial notions and properties of rewriting, such as confluence and termination, that we will later specialise to term-based systems. Throughout this and later sections, we fix a (Lawverian) quantale $\Omega = (\Omega, \leq, k, \otimes)$.

Definition 6. A Quantitative Abstract Rewriting Systems ($\Omega$-ARS, for short) is a pair $(A, R : A \rightarrow A)$.

Definition 6 is extremely simple: as a traditional abstract rewriting system is defined as a set of objects together with a binary (rewriting) relation on it, a $\Omega$-ARS is defined as a set of objects together
with a binary (rewriting) ≃-relation on it. Given elements \( a, b \in A \), we say that \( a \) rewrites into (or reduces to) \( b \) if \( R(a, b) \neq \perp \); in that case, we say that the (rewriting) distance or difference between \( a \) and \( b \) is \( R(a, b) \). Further possible informal reading (possibly depending on the quantale considered) refer to \( R(a, b) \) as the degree of the reduction, the cost of the reduction, or as the resource required for the reduction.\(^{23}\) Rewriting paths are obtained by iterating \( R \). In particular, we say that:

1. \( a \) reduces to \( b \) in finitely many-steps if \( R^n(a, b) \neq \perp \);
2. \( a \) reduces to \( b \) in one or zero steps if \( R^0(a, b) \neq \perp \);
3. \( a \) is convertible with \( b \) if \( R^n(a, b) \neq \perp \).

Notice that given a \( \oplus \)-ARS \( (A, R : A \rightarrow A) \), the convertibility \( \oplus \)-relation \( \equiv \) generated by \( R \) is a \( \oplus \)-equivalence and thus endows \( A \) with a metric-like structure.

Sometimes, we will need to explicitly consider reduction sequences. We thus say that a finite sequence \((a_0, \ldots, a_n)\) is a \( R \)-reduction sequence if

\[
R(a_0, a_1) \otimes \cdots \otimes R(a_{n-1}, a_n) \neq \perp
\]

and that an infinite sequence \((a_0, \ldots, a_n, \ldots)\) is a \( R \)-reduction sequence if \( R(a_0, a_1) \otimes \cdots \otimes R(a_{n-1}, a_n) \neq \perp \), for any \( n \geq 0 \). Every reduction sequence\(^{24}\) has a first element: if a reduction sequence starts from \( a \), then we refer to it as a reduction sequence of \( a \). Notice that since \( \oplus \) is Lawverian, for any reduction sequence \((a_0, \ldots, a_n)\), we have \( R(a_i, a_{i+1}) \neq \perp \), for any \( i \)\(^{25}\).

### 4.1 Confluence

Given a set \( A \) of objects together with an equivalence \( \equiv \) on it, traditional rewriting systems are often introduced as ways to give computational content to \( \equiv \). Accordingly, one considers a rewriting relation \( \rightarrow \subseteq A \times A \) on \( A \) such that \( \rightarrow \)-convertibility coincides with \( \equiv \). At this point, properties of \( \rightarrow \) are proved so to ensure \( \equiv \) to be computationally well-behaved. Among those, the so-called Church-Rosser property states that whenever \( a \equiv b \), there exists an object \( c \) such that both \( a \) and \( b \) can be reduced to \( c \) in a finite number of steps. Formally, \( \equiv \) coincides with \( \rightarrow^* \); \( \leftarrow \) (where \( \leftarrow \) stands for \( \rightarrow^\ast \)). The Church-Rosser property thus implies that to study \( \equiv \) it is enough to study directional rewriting. Moreover, if \( \equiv \) is defined axiomatically, then the Church-Rosser property gives a powerful tool to test consistency of \( \equiv \): if \( a \) and \( b \) have no common reduct, then they cannot be equivalent.

In a quantitative setting, the relation \( \equiv \) is replaced by a \( \oplus \)-equivalence \( E \), and the rewriting relation \( \rightarrow \) is replaced by a \( \oplus \)-rewriting relation \( R \) such that \( R^0 = E \). On then looks for properties \( R \) ensuring \( E \) to be computationally well-behaved. In this section, we explore some of these properties, viz. (quantitative) confluence, (metric) Church-Rosser, and termination. We begin with confluence, which states that two reductions originating from the same element can be joined into a common element, as in the classical case, but with the additional property that the merging reduction is achieved without increasing distances.

\(^{23}\)This is the case, in particular, for quantales of modal predicates, where rewriting is ultimately performed in a possible world describing intensional aspects of the rewriting process, such as the available resource.

\(^{24}\)If the underlying \( \oplus \)-ARS \( (A, R) \) is clear from the context, we simply refer to reduction sequences for \( R \)-reduction sequences.

\(^{25}\)In case the underlying quantale is not Lawverian, then one should take this condition as part of the definition of a reduction sequence.
Definition 7. Let $R, S : A \to B$ be $\sqcap$-relations.

1. We say that $R$ commutes with $S$ if $R^– ; S \leq S ; R^–$.

2. We say that $R$ satisfies the diamond property if $R$ commutes with itself, i.e. $R^– ; R \leq R ; R^–$.

Let us comment on Definition 7 by analysing the diamond property. On the Boolean quantale, we recover the usual diamond property as defined for traditional rewriting systems. More interesting is the case of the Lawvere quantale, which we use as vehicle to move to the general case. Pointwise, the diamond property reads as follows:

$$
\inf_c R(c, a) + R(c, b) \geq \inf_d R(a, d) + R(b, d).
$$

The left-hand-side of the inequality, namely $\inf_c R(c, a) + R(c, b)$, gives the minimal peak distance between $a$ and $b$, that is the shortest connection between $a$ and $b$ obtained through a pick $c$ reducing to both $a$ and $b$. The right-hand-side, instead, gives the minimal valley distance between $a$ and $b$, i.e. elements $d$ such that $R(a, d) + R(b, d) \leq \infty$.

The diamond property ensures that whenever we have a peak $c$ over $a$ and $b$, then we also have a collection of valleys under $a$ and $b$, i.e. elements $d$ such that $R(a, d) + R(b, d) \neq \infty$, such that the infimum of such valleys is smaller or equal than $R(c, a) + R(c, b)$. In fact, the diamond property gives:

$$
\infty > R(c, a) + R(c, b) \geq \inf_c R(c, a) + R(c, b) \geq \inf_d R(a, d) + R(b, d).
$$

And if there is no element $d$ such that $R(a, d) + R(b, d) \neq \infty$, then $\inf_d R(a, d) + R(b, d) = \infty$, which gives a contradiction. Notice, however, that there is no guarantee that there is an actual valley $d$ such that $R(c, a) + R(c, b) \geq R(a, d) + R(b, d)$.

Example 5. Consider the Lawvere quantale and the $\mathbb{L}$-ARS over the set $A \subseteq \mathbb{R}^+ \cup \{a, b_1, b_2\}$ with $R(a, b_1) \doteq R(a, b_2) \doteq 0$, $R(b_1, \epsilon) \doteq R(b_2, \epsilon) \doteq \frac{2}{\epsilon}$, for each $\epsilon \in \mathbb{R}^+$, and $R(x, y) \doteq \infty$ otherwise.

Then, there is no $c$ such that $R(b_1, c) + R(b_2, c) = 0$, although $\inf_\epsilon R(b_1, \epsilon) + R(b_2, \epsilon) = \inf_{\epsilon > 0} \frac{2}{\epsilon} + \frac{2}{\epsilon} = 0$.

In the general setting of an arbitrary quantale $\sqcap$, we see that the diamond property has the following pointwise reading:

$$
\bigvee_c R(c, a) \otimes R(c, b) \leq \bigvee_d R(a, d) \otimes R(b, d).
$$

---

\[26\]In the general case, we say that $c$ is a peak over $a$ and $b$ if $R(c, a) \otimes R(c, b) \neq \bot$, so that $R(c, a) \neq \bot$ and $R(c, b) \neq \bot$ follow since the quantale is Lawverian.
The abstract formulation suggests further non-distance-based readings of the diamond property (and properties alike); and among those, noticeable ones are obtained in terms of graded properties and degree of reductions. Accordingly, we read $\bigvee_c R(c, a) \otimes R(c, b)$ as the divergence degree of $a$ and $b$, and $\bigvee_d R(a, d) \otimes R(b, d)$ as the convergence degree of $a$ and $b$. The diamond property then states that the divergence degree between any two elements is always smaller or equal than their convergence degree; that is, the system tends more to converge than to diverge. Instantiating $\mathcal{D}$ with the Boolean element (and thus recovering the traditional diamond property and properties alike), we stipulate degrees of convergence and divergence to be absolute values. On the other hand, taking the unit interval quantale, we let convergence and divergence be fuzzy notions.

Finally, we mention that we also have a modal and coeffectful reading of the diamond property along the lines coeffectful relational calculi [40, 41]. Accordingly, we parametrise the latter property with graded properties — i.e. non-Boolean properties taking values in a quantale.

Remark 3. We have seen that both confluence and the diamond property involve graded properties — namely degrees of divergence and convergence — i.e. non-Boolean properties taking values in a quantale. Nonetheless, both confluence and the diamond property are Boolean properties of $\mathcal{D}$-relations, as they are essentially of the form $\varepsilon \leq \delta$. It is natural to push the quantitative perspective one step further and consider a graded version of, e.g., commutation. In fact, by exploiting the adjunction property of $\mathcal{D}$-relation composition, we see that requiring $R$ to commute with $S$, i.e. $R^{-1}; S \leq S; R^{-1}$, means requiring

$$\Lambda \leq R^{-1}(S; R^{-1})/S.$$  

Forgetting about $\Lambda$, we can think about the $\mathcal{D}$-relation $R^{-1}(S; R^{-1})/S$ as assigning to elements the degree of commutation of $R$ and $S$ on them, i.e. their divergence-convergence distance. Pointwise, we thus obtain:

$$(R^{-1}(S; R^{-1})/S)(a, b) = \bigvee_c R(c, a) \otimes S(c, b) \rightarrow \bigvee_d S(a, d) \otimes R(b, d).$$

Notice that the latter is an element of $\Omega$ rather than a (Boolean) truth value, and thus it indicates how much $R$ commutes with $S$. For instance, on the Lawvere quantale, $(R^{-1}(R; R^{-1})/R)(a, b)$ gives the
difference between the divergence and convergence distance on \(a\) and \(b\), and thus a measure of how much \(R\) has the diamond property on \(a\) and \(b\). Using the vocabulary of (enriched) category theory \([79, 82]\), one may say that enrichment is given not only at the level of relations, but also at the level of equality and refinement of relations (that is, not only relations \(R\) take values in \(\Omega\), but also statements such as \(R = S\) and \(R \leq S\) do). We leave the exploration of this further form of enrichment for future investigation.

As for the traditional case, we are interesting in rewriting paths rather than in single rewriting steps.

**Definition 8.** Let \((A, R)\) be a \(\Omega\)-ARS.

1. We say that \(R\) is **confluent** if \(R^*\) has the diamond property.
2. We say that \(R\) is **locally confluent**\(^{27}\) if \(R^-; R \leq R^*; R^*\).  
3. We say that \(R\) is **Church-Rosser** (CR, for short) if \(R^\equiv = R^*; R^*\)

If a rewriting \(\Omega\)-relation is confluent, then we can characterise the convertibility distance \(R^\equiv\) in terms convergent sequences of rewriting steps.

**Proposition 2.** Let \((A, R)\) be a \(\Omega\)-ARS. Then \(R\) is confluent if and only if it is CR.

**Proof.** Clearly, if \(R\) is CR, then it is confluent. Suppose now \(R\) to be confluent and recall that \(R^\equiv = (R \lor R^*)^*\). First, we notice that since \(R^- = R^*\), we have:

\[
R^*; R^* \leq R^*; R^* \leq (R \lor R^-)^*; (R \lor R^-)^* = (R \lor R^-)^*.
\]

It thus remains to show \((R \lor R^-)^* \leq R^*; R^*\). We proceed by fixed point induction, showing that

\[
\Delta \lor ((R \lor R^-); R^*; R^*) \leq R^*; R^*.
\]

Clearly, \(\Delta \leq R^*; R^*\), so that it remains to show \((R \lor R^-); R^*; R^* \leq R^*; R^*\). We have\(^{28}\)

\[
(R \lor R^-); R^*; R^* = R; R^*; R^* \lor R^*; R^*; R^* \\
\leq R^*; R^* \lor R^*; R^*; R^* \\
\leq R^*; R^* \lor R^*; R^*; R^* \\
= R^*; R^* \lor R^*; R^*; R^* \\
\leq R^*; R^* \lor R^*; R^*; R^* \leq R^*; R^*; R^* \\
= R^*; R^*
\]

\(\Box\)

**Notation 7.** Given a \(\Omega\)-ARS \(\mathcal{A} = (A, R)\) and a property \(\varphi\) on \(\Omega\)-relations, such as being confluent, we say that \(\mathcal{A}\) has property \(\varphi\) if \(R\) has \(\varphi\). Thus, for instance, we say that \(\mathcal{A}\) is confluent if \(R\) is.

\(^{27}\)Notice that \(R^\equiv = R^*\).

\(^{28}\)Recall that for all \(R_i : A \to B\) and \(S : B \to C\), we have \((\lor_i R_i); S = \lor_i R_i; S\).
Thanks to Proposition 2, we see that confluence is a crucial property in quantitative and metric reasoning. Proving confluence of quantitative systems, however, can be cumbersome: indeed, quantitative systems are often built compositionally by joining systems together. Section 3 has already shown us several examples of systems obtained that way. Consequently, it is desirable to design modular techniques to prove confluence of such systems compositionally, i.e. relying on confluence of their component subsystems, rather than proceed monolithically from scratchs. Among such modular techniques, Hindley-Rosen Lemma \[72, 99\] is arguably the most well-known one in traditional rewriting. Proposition 3 generalises such a result to quantitative systems. Before proving it, we recall a few basic properties of \(\Omega\)-relations.

Lemma 1. Given \(\Omega\)-relations \(R, S,\) and \(P\), we have:

1. If \(R; S \leq S; R\) and \(R; P \leq P; R\), then \(R; (S \lor P)^* \leq (S \lor P)^*; R\).
2. \((R^* \lor S^*)^* = (R \lor S)^*.\)

Proof. For the first item, we observe that proving the thesis amounts to prove \((S \lor P)^* \leq R \setminus ((S \lor P)^*; R)\), so that we can use fixed point induction. Proving \(A \leq R \setminus ((S \lor P)^*; R)\) is straightforward. It remains to prove

\[(S \lor P); R \setminus ((S \lor P)^*; R) \leq R \setminus ((S \lor P)^*; R).\]

Since \((S \lor P); R \setminus ((S \lor P)^*; R) = (S; R \setminus ((S \lor P)^*; R)) \lor (P; R \setminus ((S \lor P)^*; R))\), it is sufficient to prove \(S; R \setminus ((S \lor P)^*; R) \leq R \setminus ((S \lor P)^*; R)\) and \(P; R \setminus ((S \lor P)^*; R) \leq R \setminus ((S \lor P)^*; R)\). We prove the former which, by adjunction, is equivalent to

\[R; S; R \setminus ((S \lor P)^*; R) \leq (S \lor P)^*; R.\]

By commutation of \(R\) with \(S\), we obtain:

\[R; S; R \setminus ((S \lor P)^*; R) \leq S; R; R \setminus ((S \lor P)^*; R)\]
\[\leq S; (S \lor P)^*; R\]
\[\leq (S \lor P); (S \lor P)^*; R\]
\[\leq (S \lor P)^*; R.\]

Let us now move to the second item, which essentially amounts to prove \((R^* \lor S^*)^* \leq (R \lor S)^*\). We use fixed point induction and show \((R^* \lor S^*)^* \leq (R \lor S)^*\). We have:

\[(R^* \lor S^*)^*; (R \lor S)^* = R^*; (R \lor S)^* \lor S^*; (R \lor S)^*\]
\[\leq (R \lor S)^*; (R \lor S)^* \lor (R \lor S)^*; (R \lor S)^*\]
\[= (R \lor S)^* \lor (R \lor S)^*\]
\[= (R \lor S)^*.\]

\[\Box\]

Proposition 3 (Hindley-Rosen Lemma). If \(R^*\) commutes with \(S^*\) and \(R, S\) are confluent, then \(R \lor S\) is confluent.
Lemma to a quantitative setting. To do so, we first define the notion of a terminating \( \Delta \)-relation composition, we obtain:

\[
P^\rightarrow; P^r \leq P^s; P^r \quad \iff \quad P^\rightarrow \leq (P^s; P^r)/P^s \quad \iff \quad P^\rightarrow \leq (P^r; P^s)/P^s
\]

Therefore, it is sufficient to prove \( P^\rightarrow \leq (P^r; P^s)/P^s \), which we do using fixed point induction. That amounts to prove \( \Delta \leq (P^s; P^r)/P^s \) and \( P^\rightarrow; (P^s; P^r)/P^s \leq (P^s; P^r)/P^s \). The former holds since

\[
\Delta \leq (P^s; P^r)/P^s \quad \iff \quad \Delta; P^s \leq P^r; P^r
\]

and \( \Delta; P^s = P^s; \Delta \leq P^s; P^r \). For the latter, we have

\[
P^\rightarrow; (P^s; P^r)/P^s \leq (P^r; P^s)/P^s \quad \iff \quad P^\rightarrow; ((P^s; P^r)/P^s); P^s \leq P^r; P^r \quad \iff \quad P^\rightarrow; P^r; P^s \leq P^r; P^r
\]

since \( ((P^s; P^r)/P^s); P^s \leq P^r; P^r \). To prove \( P^\rightarrow; P^r; P^s \leq P^r; P^s \), we notice that

\[
P^\rightarrow; P^r; P^s = (R \lor S)^-; R^\rightarrow; P^r; P^s = (R \lor S)^-; R^\rightarrow; (R \lor S)^-; P^r; P^s = (R \lor S)^-; P^r; P^s \leq P^r; P^s
\]

so that it is sufficient to prove \( R^-; (R \lor S)^-; P^-; P^r \leq P^r; P^-; P^- \). We prove the first inequality, as the second one is similar. Since \( R \) commute both with itself and with \( S \), by Lemma 1, we have:

\[
R^-; (R \lor S)^-; (R \lor S)^-; (R \lor S)^- = (R \lor S)^-; (R \lor S)^-; (R \lor S)^-; (R \lor S)^- \leq (R \lor S)^-; (R \lor S)^-; (R \lor S)^-; (R \lor S)^- \leq (R \lor S)^-; (R \lor S)^-; (R \lor S)^-; (R \lor S)^- \leq (R \lor S)^-; (R \lor S)^-; (R \lor S)^-; (R \lor S)^- \leq P^-; P^r
\]

\[\]

4.2 Locality and Termination

By Proposition 2, we know that nice operational properties of a \( \Omega \)-equivalence \( E \) can be obtained by characterising \( E \) as the convertibility \( \Omega \)-relation of a confluent rewriting \( \Omega \)-relation \( R \). Even if Proposition 3 gives a technique to prove confluence of composed systems compositionally, proving confluence of “atomic” systems may still be not easy. In fact, by its very definition, confluence is a global property of a system, in the sense that it refers to rewriting sequences, rather than to single rewriting steps. Newman’s Lemma 91 is a well-known result in the theory of abstract rewriting stating that if a system is terminating, then confluence follows from local confluence. The rest of this section is dedicated to refining Newman’s Lemma to a quantitative setting. To do so, we first define the notion of a terminating \( \Omega \)-relation and prove that terminating \( \Omega \)-relations satisfy a suitable induction principle. We then use the latter to extend Newman’s Lemma to \( \Omega \)-relations.

Before proceeding any further, we observe that up to this point our analysis of \( \Omega \)-ARSs has been relational, proceeding in an algebraic and pointfree fashion. To make this paper as accessible as possible, we now take a (temporary) break from that methodology and first give a pointwise analysis of quantitative
termination and a \textit{pointwise} proof of (the quantitative refinement of) Newman’s Lemma similar to the one by Belohlávek et al. \cite{22} (see section 9 for a precise comparison). After that, we go back on our choice and review termination and Newman’s Lemma in a novel way, following the relational and algebraic paradigm and extending the relational theory of induction and abstract rewriting by Doornbos et al. \cite{50} to a quantitative, $\Omega$-enriched setting. Such an extension is nontrivial and requires the introduction of suitable relational modalities akin to corelators \cite{41}. The outcome (which the authors believe is worth the effort) is interesting not only because it gives a clean analysis of induction in a $\Omega$-enriched setting — as well as a (slightly) more general version of (quantitative) Newman’s Lemma — but also for the methodology employed, which constitutes a nice example of quantitative relational methods.

\textbf{4.2.1 Quantitative Termination, Induction, and Newman’s Lemma}

As a first step towards a quantitative refinement of Newman’s Lemma, we extend the notion of termination to $\Omega$-relations. Contrary to traditional rewriting, in a quantitative setting the notion of termination may be defined in many, non-equivalent ways. We could define, for instance, a terminal element as one having no \textit{nontrivial} and \textit{non-null} reductions, so that we allow a terminal element $a$ to be reduced to another one $b$, only if the distance between $a$ and $b$ is either $k$ or $\bot$. This notion, which is meaningful in a genuine quantitative setting (especially if interested in ‘metric reasoning modulo equality’), trivialises when instantiated to the Boolean quantale, as elements are always reducible. To stay closer to traditional rewriting, we may exclude null distances too, so that terminal elements can be reduced only to those element that are $\bot$-apart from them. Finally, we may also go beyond finitary notions of reduction \cite{47, 80, 81} and think about termination in the limit \cite{53}, i.e. as possibly infinite reductions converging to $k$, in the limit.

All these proposals are legitimate, and they all deserve to be investigated. A complete analysis of notions of quantitative termination, however, is beyond the scope of this paper and we should thus fix a conceptually minimal notion of termination and focus on that one only. To do that, we take an operational approach and stipulate that a (rewriting) $\Omega$-relation is terminating if it supports an induction principle. But what could the latter possibly be?

Let us recall \cite{50} that, given a binary relation $R \subseteq A \times A$, a property $p$ on $A$ is $R$-\textit{inductive} if it satisfies the law

\[(\forall x. \ x R y \rightarrow p(x)) \rightarrow p(y),\]

for any $y \in A$. We then say that the relation $R$ \textit{admits induction} if for any $R$-inductive property $p$, $p(a)$ holds for any $a \in A$. Consequently, if $R$ admits induction and we want to prove that each element of $A$ has a given property $p$, it is enough to prove $p$ to be $R$-inductive. We thus recover the familiar formulation of well-founded induction:

$$\forall y. (\forall x. \ x R y \rightarrow p(x)) \rightarrow p(y)$$

\[\Rightarrow \forall y. p(y)\]

We now generalise this idea to $\Omega$-relations and $\Omega$-properties.

\textbf{Definition 9.} Let $(A, R)$ be a $\Omega$-ARS.

1. A $\Omega$-property $p : A \rightarrow \Omega$ is $R$-\textit{inductive} if the following holds for any $b \in A$:

\[\bigwedge_a R(a, b) \rightarrow p(a) \leq p(b).\]
2. We say that $R$ admits induction if for any $a \in A$, $p(a) = k$, for any $R$-inductive predicate $p$.

Notice that if $R$ admits induction and $p$ is a $R$-inductive predicate, then by Definition 9 we obtain $p(a) = k$, for any $a \in A$.

**Remark 4.** Being $R$-inductive (as well as admitting induction) is a Boolean property. As already seen in Remark 3 for confluence, we could obtain a finer, quantitative analysis of induction by $\Omega$-enriching (i.e. grading) the properties of Definition 9 in $\Omega$, this way replacing, e.g., $\land_a R(a, b) \dashv p(a) \leq p(a)$ with $\land_a (R(a, b) \dashv p(a)) \dashv p(a)$. The latter formula, intuitively, gives the degree of inductiveness of $R$.

Armed with Definition 9, we can now operationally identify terminating $\Omega$-relations with those admitting induction. Nonetheless, the reader may wonder whether there is an explicit characterisation of terminating relations in terms of familiar conditions akin to the equivalence between inductive and well-founded relations in the traditional case. The answer is in the affirmative and shows that $\Omega$-relations admitting induction are precisely those that terminates in the strongest sense among those discussed at the beginning of this section.

**Definition 10.** Let $(A, R)$ be a $\Omega$-ARS.

1. We say that $a \in A$ is a normal form if $\bigvee_b R(a, b) = \bot$.
2. We say that a reduction sequence $(a_0, ..., a_n)$ terminates if $a_n$ is a normal form.
3. We say that $R$ is weakly normalizing (WN) if for any $a \in A$ there exists a normal form $b$ such that $R^*(a, b) \neq \bot$.
4. We say that $R$ is strongly normalizing (SN) if for each $a \in A$, all reduction sequences starting from $a$ terminate.

Notice that if a reduction sequence terminates, the sequence must be finite. Moreover, if $R$ is WN and $a \in A$, then there must exists an element $b$ such that $R^*(a, b) \neq \bot$. That means $\bigvee_n R^n(a, b) \neq \bot$, which in turn means that there exists an actual index $n$ and elements $a = a_0, a_1, ..., a_{n-1} = b$ such that $(a_0, ..., a_{n-1})$ is a reduction sequence from $a$ (in particular, $R(a_i, a_{i+1}) \neq \bot$, for any index $i$).

The next result shows that terminating and inductive $\Omega$-relations are indeed one and the same.

**Proposition 4.** Let $(A, R)$ be a $\Omega$-ARS. Then $R^-$ admits induction if and only if $R$ is SN.

**Proof.** We prove the two implications separately.

($\implies$) Suppose $R^-$ admits induction. We prove that $R$ is SN. Let $p(a) = k$ if all reduction sequences from $a$ terminates, and $p(a) = \bot$, otherwise. We prove that $p$ is inductive, from which the thesis follows. We have to show $\lor_b R(a, b) \dashv p(b) \leq p(a)$. If $p(a) = k$, then we are trivially done. Otherwise, $p(a) = \bot$ and we have a sequence $a = a_0, a_1, ..., a_n = b$ such that $R(a_n, a_{n+1}) \neq \bot$, for any $n$. To prove $\lor_b R(a, b) \dashv p(b) \leq p(a)$, we show $R(a, a_1) \dashv p(a_1) \leq p(a)$. By very definition of $\dashv$, we have $R(a, a_1) \dashv p(a_1) = \lor \{\varepsilon | \varepsilon \otimes R(a, a_1) \leq p(a_1)\}$, so that it is sufficient to show that for any $\varepsilon$ such that $\varepsilon \otimes R(a, a_1) \leq p(a_1)$, we have $\varepsilon \leq \bot$. Now, obviously $p(a_1) = \bot$, so that $\varepsilon \otimes R(a, a_1) = \bot$ too. Since $\bot$ is Lawverian, we then have that either $\varepsilon = \bot$ or $R(a, a_1) = \bot$. Since $R(a, a_1) \neq \bot$, we thus conclude $\varepsilon = \bot$, and we are done.

Another (more liberal) option that we do not explore in this work is to require $\lor_a p(a) = k$ in place of $(\forall a) p(a) = k$.

It is an interesting question to determine if weaker and quantitative refinements of Definition 9 correspond to weaker and quantitative notions of termination.
(\iffalse) Suppose \( R \) is SN and let \( p \) be \( R \)-inductive. We prove \( p(a) = k \), for any \( a \in A \). We proceed by contradiction showing that if there exists \( a \in A \) such that \( p(a) < k \), then there also exists \( b \in A \) such that \( R(a, b) \neq \bot \) and \( p(b) < k \). Therefore, if there is an \( a \) such that \( p(a) < k \), we also have a non-terminating reduction sequence from \( a \), this way contradicting SN. So suppose to have an element \( a \) such that \( p(a) < k \). Suppose also, for the sake of a contradiction, that for any \( b \) either \( R(a, b) = \bot \) or \( p(b) < k \). In both cases we obtain \( R(a, b) \not\rightarrow p(b) = k \), and thus \( \setminus_b R(a, b) \not\rightarrow p(a) = k \). Since \( p \) is inductive, we also have \( \setminus_b R(a, b) \not\rightarrow p(b) \leq p(a) \) and thus \( k = \setminus_b R(a, b) \not\rightarrow p(b) \leq p(a) < k \). Contradiction.

\qed

We now have all the ingredients to quantitatively refining Newman’s Lemma.

**Proposition 5.** Let \((A, R)\) be a strongly normalising \( \Omega \)-ARS. Then, \( R \) is confluent if and only if it is locally confluent.

**Proof.** Obviously, if \( R \) is confluent, then it is locally confluent too. We prove the converse. Suppose that \( R \) is locally confluent and let us define the \( \Omega \)-property \( p \) as follows:

\[
p(a) = \begin{cases} 
  k & \text{if } \forall b_1, b_2 \in A. \ R^*(a, b_1) \otimes R^*(a, b_2) \leq \vee_b R(b_1, b) \otimes R(b_2, b) \\
  \bot & \text{otherwise.}
\end{cases}
\]

Therefore, \( p \) is a Boolean property, in the sense that for any \( a \in A \), \( p(a) \) is either \( k \) or \( \bot \). Moreover, we see that \( p(a) = k \) if and only if \( R \) is confluent on \( a \). Since \( R \) is SN, it admits induction, and thus to prove the thesis it is sufficient to show that \( p \) is inductive. Let us first notice that since \( p \) is Boolean, we can simplify the proof of its inductiveness.

**Claim.** To prove that \( p \) is inductive it is enough to show:

\[
\forall a. \ (\forall b. \ R(a, b) \neq \bot \implies p(b) = k) \implies p(a) = k.
\]  

**Proof of the claim.** To prove that \( p \) is inductive, we have to show that for any \( a \), \( \setminus_b R(a, b) \not\rightarrow p(b) \leq p(a) \). Now, since \( p \) is Boolean, either \( p(a) = k \) or \( p(a) = \bot \). In the former case, we trivially have \( \setminus_b R(a, b) \not\rightarrow p(b) \leq k = p(a) \) in the latter case (i.e. \( p(a) = \bot \)), by \( \Box \) there exists an element in \( A \), call it \( c \), such that \( R(a, c) \neq \bot \) and \( p(c) = k \). We then have

\[
\bigwedge_b R(a, b) \not\rightarrow p(b) \leq R(a, c) \not\rightarrow p(c) = \bot = p(a),
\]

since \( \Omega \) is Lawverian.

\(\Diamond\)

Coming back to the main proof, we have seen that we can conclude the main thesis by proving \( \Box \). So let us fix \( a \in A \) and assume

\[
\forall b. \ R(a, b) \neq \bot \implies p(b) = k.
\]  

We prove \( p(a) = k \), i.e. \( R^*(a, b_1) \otimes R^*(a, b_2) \leq \vee_b R^*(b_1, b) \otimes R^*(b_2, b) \), for arbitrary \( b_1, b_2 \). Let \( \epsilon \doteq \vee_b R^*(b_1, b) \otimes R^*(b_2, b) \). Since \( \epsilon = \Delta \vee R; R^* \), it is sufficient to prove \( \Delta(a, b_1) \vee (R; R^*)(a, b_1) \leq R^*(a, b_2) \not\rightarrow \epsilon \), which is itself implied by \( \Delta(a, b_1) \leq R^*(a, b_2) \not\rightarrow \epsilon \) and \( (R; R^*)(a, b_1) \leq R^*(a, b_2) \not\rightarrow \epsilon \). For the former, we
assume \( a = b_1 \) (the case for \( a \neq b_1 \) is trivial) and notice that \( R^*(a, b_2) \leq \varepsilon \) follows by taking \( b = b_2 \). For the former, we see that \( (R; R^*)(a, b_1) \leq R^*(a, b_2) \rightarrow \varepsilon \) is equivalent to \( (R; R^*)(a, b_2) \leq R^*(a, b_1) \rightarrow \varepsilon \). We repeat the above argument, this time on \((R; R^*)(a, b_2))\), so that it is sufficient to prove \( \Delta(a, b_2) \leq (R; R^*)(a, b_1) \rightarrow \varepsilon \) and \( (R; R^*)(a, b_2) \leq (R; R^*)(a, b_1) \rightarrow \varepsilon \). For the former, we proceed as usual. The real interesting case is the latter, which is equivalent to

\[
(R; R^*)(a, b_1) \otimes (R; R^*)(a, b_2) \leq \varepsilon.
\]

Since we have

\[
(R; R^*)(a, b_1) \otimes (R; R^*)(a, b_2) = \bigvee_c R(a, c) \otimes R^*(c, b_1) \otimes \bigvee_d R(a, d) \otimes R^*(d, b_2),
\]

it is enough to prove that for all \( c, d \), we have \( R(a, c) \otimes R^*(c, b_1) \otimes R(a, d) \otimes R^*(d, b_2) \leq \varepsilon \). Now, if either \( R(a, c) = \bot \) or \( R(a, d) = \bot \), we are trivially done, since the quantale is integral. Suppose then they are all different from \( \bot \), so that we can apply \( \mathbb{Z} \) on both of them, this way obtaining \( p(d) = p(c) = k \). Since \( R \) is locally confluent, we obtain:

\[
R(a, c) \otimes R^*(c, b_1) \otimes R(a, d) \otimes R^*(d, b_2) = R(a, c) \otimes R(a, d) \otimes R^*(c, b_1) \otimes R^*(d, b_2)
\]

\[
\leq \bigvee_e R^*(c, e) \otimes R^*(d, e) \otimes R^*(c, b_1) \otimes R^*(d, b_2)
\]

It is then sufficient to prove \( R^*(c, e) \otimes R^*(d, e) \otimes R^*(c, b_1) \otimes R^*(d, b_2) \leq \varepsilon \), for any \( e \in A \). From \( p(d) = p(c) = k \), we obtain:

\[
R^*(c, e) \otimes R^*(d, e) \otimes R^*(c, b_1) \otimes R^*(d, b_2) = R^*(c, e) \otimes R^*(c, b_1) \otimes R^*(d, e) \otimes R^*(d, b_2)
\]

\[
\leq \bigvee_f R^*(e, f) \otimes R^*(b_1, f) \otimes R^*(d, e) \otimes R^*(d, b_2)
\]

\[
= \bigvee_f R^*(d, e) \otimes R^*(e, f) \otimes R^*(b_1, f) \otimes R^*(d, b_2)
\]

\[
= \bigvee_f R^*(d, f) \otimes R^*(b_1, f) \otimes R^*(d, b_2)
\]

\[
= \bigvee_f R^*(d, f) \otimes R^*(d, b_2) \otimes R^*(b_1, f)
\]

\[
\leq \bigvee_g \bigvee_f R^*(f, g) \otimes R^*(b_2, g) \otimes R^*(b_1, f)
\]

\[
= \bigvee_g \bigvee_f R^*(b_1, f) \otimes R^*(f, g) \otimes R^*(b_2, g)
\]

\[
\leq \bigvee_g R^*(b_1, g) \otimes R^*(b_2, g)
\]

\[
= \varepsilon
\]

\[\square\]
5 Quantitative Induction and Newman’s Lemma, $\Omega$-Relationally

Even if mathematically fine, the previous section does not follow the relational style we have used to define quantitative rewriting so far. Doornbos et al. [50] have developed a relational theory of induction that allowed them to give an elegant, algebraic proof of Newman’s Lemma. In this section, we extend their proof to $\Omega$-ARSs. As already remarked, such an extension is nontrivial and builds upon the crucial notion of a relational modality (also known as a corelator [41, 40]) to define Boolean properties relationally.

Let us begin by reviewing how Doornbos et al. [50] deal with induction, algebraically. In a nutshell, the (Boolean) calculus of classes is first embedded into the calculus of relations, this way defining (Boolean) relationally. Crucially, we define a predicate over $\Omega$-relation — predicate. There are several ways to define predicates relationally. For instance, thinking about a $\Omega$-relation, predicates is by means of $\Omega$-inductive if $R \cap p \subseteq p$ and that $R$ admits induction if $R \cap p \subseteq p \implies \Lambda \subseteq p$.

We now generalise this construction to the setting of $\Omega$-relations. First, we define the notion of a $\Omega$-predicate. There are several ways to define predicates relationally. For instance, thinking about a $\Omega$-relation $R : A \rightarrow B$ as a $\Omega$-valued matrix, we can see a predicate as a — row or column — vector $[105]$. Accordingly, we define a predicate over $A$ as a $\Omega$-relation $p : A \rightarrow 1$, with $1 \equiv \{\ast\}$ one-element set $[32]$. Notice that since $\pi : 1 \rightarrow 1$ coincides with $\Lambda$, any predicate $p$ satisfies $p ; \pi = p$. Another way to define predicates is by means of coreflexive $\Omega$-relations (also known as monotypes), whereby a predicate on $\Omega$-relation $p : A \rightarrow A$ such that $p \subseteq \Lambda$. Vectors and coreflexives are equivalent notions, in the sense that there is an isomorphism between (column) vectors and coreflexives: any vector $p : A \rightarrow 1$ gives the coreflexive $\pi \triangleright \Lambda ; A \rightarrow A$; vice versa, any coreflexive $p : A \rightarrow A$ gives the vector $p ; \pi$.

Given a column $\Omega$-vector $p$ and a $\Omega$-relation $R$, we notice that

$$\left(R \setminus p\right)(b, \ast) = \bigcup_a R(a, b) \ni p(a, \ast)$$

gives exactly the formula we have used to define inductive predicates. Moreover, the same formula can be obtained if $p$ is a coreflexive by considering $R \setminus (p ; \pi)$. Since we will extensively work with coreflexives, we introduce the notation $R \setminus p$ for $R \setminus (p ; \pi)$.

**Definition 11.** Let $R : A \rightarrow A$ be a $\Omega$-relation.

1. A $\Omega$-vector $p : A \rightarrow 1$ is $\Omega$-inductive if $R \setminus p \subseteq p$. We say that $R$ admits vector induction if, for any $\Omega$-vector $p : A \rightarrow 1$, we have:

$$R \setminus p \subseteq p \implies \pi \subseteq p.$$ 

2. A coreflexive $p : A \rightarrow A$ is $\Omega$-inductive if $R \setminus p \subseteq p$. We say that $R$ admits coreflexive induction if, for any coreflexive $p : A \rightarrow A$, we have:

$$R \setminus p \subseteq p \implies \Lambda \subseteq p.$$ 

---

$33$ $R \setminus p$ is actually defined relying on the axioms of the calculus of relations only, rather than on their set-theoretic semantics.

$32$ We thus view predicates as column vector. Equivalently, we may define predicates as row vectors, i.e. as $\Omega$ relations $p : 1 \rightarrow A$.

$33$ Notice that $R \otimes \Lambda = R \land \Delta$. 

34
Remark 5. Thanks to the correspondence between vectors and coreflexives, it is easy to see that the
notions of vector and coreflexive induction are equivalent, and that they indeed correspond to the point-
wise notion of an inductive \( \omega \)-relation as given in Definition 9. Moreover, we can abstract
from vectors and monotypes, and say, in full generality, that a \( \omega \)-relation \( R \) admits induction if
\[
R \backslash S \leq S \implies \top \leq S
\]
for any \( \omega \)-relation \( S \). Obviously, this definition subsumes those in Definition 11. One can also show that
the vice versa holds too, and that the above three definitions of an inductive \( \omega \)-relation are equivalent.

To prove (the quantitative refinement of) Newman’s Lemma, we will not do induction on an arbitrary
\( \omega \)-property, but on a Boolean one. In previous section, we have modelled Boolean predicates as \( \omega \)-
properties \( p \) that are either equal to \( \top \) or to \( \bot \). Even if correct, such a definition is operationally weak
since it does not readily come with useful algebraic laws and proof techniques. We overcome the problem
by giving a modality-based definition of Boolean \( \omega \)-properties in the spirit of the exponential modality
of linear logic [67]. To do so, we proceed as follows: first, we define a way to extract a Boolean property
out of an \( \omega \)-enriched one. Since we can inject Boolean properties into \( \omega \)-enriched ones, we can then pick
a \( \omega \)-property, extract a Boolean predicate out of it, and the (re)enriched it in \( \omega \). We say that a property
is Boolean if it is invariant under the above procedure.

Given a quantale \( \omega \), there is a canonical adjunction between \( \omega \) and \( 2 \) given by the maps \( \varphi : \omega \to 2 \)
and \( \psi : 2 \to \omega \) defined thus:
\[
\varphi(\varepsilon) \triangleq \begin{cases} 
\top & \text{if } \varepsilon = k \\
\bot & \text{otherwise}
\end{cases} \quad \psi(x) \triangleq \begin{cases} 
k & \text{if } x = \top \\
\bot & \text{otherwise}
\end{cases}
\]
Both \( \varphi \) and \( \psi \) form a so-called change of base functor [76, 79], and their composition \( \psi \circ \varphi : \omega \to \omega \)
is a change of base endofunctor. Since we define Boolean \( \omega \)-relations (and thus \( \omega \)-properties) as those that
are invariant under the map \( \psi \circ \varphi \), we introduce a special notation for the latter.

Definition 12. Define the (set-indexed family of) map(s) \( \Box : \omega\text{-Rel}(A, B) \to \omega\text{-Rel}(A, B) \) by \( \Box \equiv \psi \circ \varphi \circ R \).
We say that a \( \omega \)-relation is Boolean if \( \Box R = R \).

The following result (whose proof is straightforward) simply states that \( \Box \) satisfies (some of) the
axioms of a corelator [41, 40]. We will extensively use this fact in the proof of Proposition 7.

Proposition 6. The map \( \Box \) obeys the following laws, where \( R \otimes S : A \times B \to A' \times B' \) is defined pointwise,
for \( R : A \to A' \) and \( S : B \to B' \).

\[
\begin{align*}
\Delta \leq \Box \Delta & \quad \text{(rel-id)} \\
\Box R : \Box S \leq \Box (R ; S) & \quad \text{(rel-comp)} \\
\Box R \leq R & \quad \text{(rel-der)} \\
\Box R \otimes \Box S \leq \Box (R \otimes S) & \quad \text{(rel-tensor)} \\
\Box \Box R & \leq \Box R \quad \text{(rel-contraction)} \\
R \leq S \implies \Box R \leq \Box S & \quad \text{(rel-mon)}
\end{align*}
\]

Change of base functors will play a crucial role in section 8.
Using the map \( \Box \) we can specialise the notion of coreflexive (and of a vector) to Boolean properties.

**Definition 13.** A Boolean property on \( A \) is a coreflexive \( p : A \to A \) (i.e. \( p \leq \Delta \)) such that \( p = \Box p \).

Before stating our quantitative version of Newman’s Lemma, let us spell out some useful facts about Boolean properties.

**Lemma 2.** Given \( \sqcap \)-relations \( R, S : A \to A \) and a Boolean property \( p : A \to A \), we have:

1. \( R \setminus p \) is a Boolean property.
2. \( (R \setminus p) ; (S \setminus p) = (R \lor S) \setminus p \).
3. \( R \setminus p = p \lor R^\neg \).
4. \( R; (R \setminus p) \leq p; R \) and \( (p \lor S); S \leq S; p \).

We are now ready to state and prove the quantitative refinement of the abstract Newman’s Lemma by \( \Box \).

**Proposition 7.** Let \( R, S : A \to A \) be \( \sqcap \)-relations such that \( R \lor S^\neg \) admits induction. Then \( R; S \leq S^*; R^* \) implies \( R^*; S^* \leq S^*; R^* \).

Before proving Proposition 7, let us observe the following elementary fact.

**Lemma 3.** \( R^*; S^* \leq (R^*; S) \lor (R^*; R; S; P^* \lor (S; P^*)) \).

**Proof of Proposition 7.** Since \( R \lor S^\neg \) admits induction, for any coreflexive \( p \) we have

\[
(R \lor S^\neg) \setminus p \leq p \implies \Delta \leq p.
\]

By [Lemma 2] \( (R \lor S^\neg) \setminus p = (R \setminus p); (p \lor S) \), so that we obtain the following induction principle:

\[
(R \setminus p); (p \lor S) \leq p \implies \Delta \leq p.
\]

We have to prove \( R^*; S^* \leq S^*; R^* \) which, by adjunction, is equivalent to

\[
\Delta \leq R^* \setminus (S^*; R^*)/S^*.
\]

We notice that for any \( \sqcap \)-relation \( P \), we have \( \Delta \leq P \) if and only if \( \Delta \leq \Box (P \land \Delta) \). In fact, the left to right direction follows since \( \Box \Delta = \Delta \), whereas the right to left direction follows from \( \text{rel-der} \) \( (\Delta \leq \Box (P \land \Delta) \leq P \land \Delta \leq P) \). Therefore, to prove \( \Delta \leq R^* \setminus (S^*; R^*)/S^* \), it is enough to show

\[
\Delta \leq (R^* \setminus (S^*; R^*)/S^*) \land \Delta.
\]

Notice that \( p \) is a Boolean coreflexive, and thus we can rely on inductiveness of \( R \lor S^\neg \) and obtain the proof obligation \( (R \setminus p); (p \lor S) \leq p \). Since \( p \) is Boolean, then so are \( R \setminus p \) and \( p \lor S \), so that \( \text{rel-comp} \) gives us:

\[
(R \setminus p); (p \lor S) = \Box (R \setminus p); \Box (p \lor S) \leq \Box ((R \setminus p); (p \lor S)).
\]

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Therefore, our thesis becomes
\[ \Box((R \setminus p); (p \not\succ S)) \leq p = \Box((R^* \setminus (S^*; R^*)/S^*) \land \Delta). \]

By \textit{rel-mon}, it is sufficient to prove \((R \setminus p); (p \not\succ S) \leq (R^* \setminus (S^*; R^*)/S^*) \land \Delta\) which amounts to show
\[
\begin{align*}
(R \setminus p); (p \not\succ S) & \leq \Delta \\
(R \setminus p); (p \not\succ S) & \leq R^* \setminus (S^*; R^*)/S^*.
\end{align*}
\]

The former inequation is straightforward as both \(R \setminus p\) and \(p \not\succ S\) are coreflexives (and thus \(R \setminus p \leq \Delta\) and \(p \not\succ S \leq \Delta\)), since \(p\) is. Let us now move to the second inequation. By adjunction, we have to show
\[
R^*; (R \setminus p); (p \not\succ S); S^* \leq S^*; R^*.
\]

By \textbf{Lemma 3} we reduce the proof to the following three inequations:
\[
\begin{align*}
R^*; (R \setminus p); (p \not\succ S) & \leq S^*; R^* \\
R^*; R; (R \setminus p); (p \not\succ S); S; S^* & \leq S^*; R^* \\
(R \setminus p); (p \not\succ S); S & \leq S^*; R^*.
\end{align*}
\]

For the first one, since both \(R \setminus p\) and \(p \not\succ S\) are coreflexives, we have:
\[
R^*; (R \setminus p); (p \not\succ S) \leq R^*; \Delta; \Delta \leq R^*; S^*.
\]

We prove the third inequation in a similar fashion. Let us now move to the second one. For readability, let \(P\) be \(R^* \setminus (S^*; R^*)/S^*\), so that \(p = \Box(P \land \Delta)\). We have:
\[
\begin{align*}
R^*; R; (R \setminus p); (p \not\succ S); S; S^* & \leq R^*; p; R; S; p; S^* \\
& \leq R^*; p; S^*; R^*; p; S^* \\
& = R^*; \Box(P \land \Delta); S^*; R^*; \Box(P \land \Delta); S^* \\
& \leq R^*; (P \land \Delta); S^*; R^*; (P \land \Delta); S^* \\
& \leq R^*; P; S^*; R^*; P; S^* \\
& = R^*; R^* \setminus (S^*; R^*)/S^*; S^*; R^* \setminus (S^*; R^*)/S^* \\
& \leq S^*; R^*; R^* \setminus (S^*; R^*)/S^*; S^* \\
& \leq S^*; R^* \\
& \leq S^*; R^*.
\end{align*}
\]

\textbf{Corollary 1} (Newman’s Lemma). \textit{Let \((A, R)\) be a \(\cap\)-ARS. If \(R\) is SN, then \(R\) is confluent if and only if it is locally confluent.}

\textit{Proof.} We instantiate \(R\) and \(S\) in \textbf{Proposition 7} as \(R^\sim\) and \(R\), respectively. Consequently, the hypothesis that \(R \lor S^\sim\) admits induction collapses to \(R^\sim\) admitting induction, which is equivalent to \(R\) being SN. \textbf{Proposition 7} then precisely gives confluence of \(R\) (assuming its local confluence). \hfill \Box
Remark 6. In the proof of Corollary 1, we have actually used the equivalence between \( \Omega \)-relations admitting induction and terminating (well-founded) ones, as proved in Proposition 4. We can indeed safely do so as we have seen that the relational pointfree definition of an inductive relation (Definition 11) coincides with its pointwise counterpart (Definition 9). Nonetheless, a complete relational analysis of (quantitative) Newman’s Lemma requires a relational account of termination too. Doing that is beyond the scope of this paper, although it can be done with a reasonable effort. As a guideline, we simply say that a \( \Omega \)-relation \( R \) is well-founded if
\[
S \leq S; R \implies S \leq \bot
\]
for any \( S \) (similar definitions can be obtained restricting to vectors and monotypes, as in Remark 5), and that \( R \) is SN if \( R^- \) is well-founded.

6 QUANTITATIVE TERM REWRITING SYSTEMS: A SHORT PHENOMENOLOGY

Having introduced the general theory of quantitative abstract rewriting systems, in the remaining sections of this paper we shall introduce quantitative term rewriting systems and their connection with quantitative algebras. Contrary to traditional term rewriting systems, there are several notions of a quantitative term rewriting system (and of their associated notion of a quantitative equational theory), each of which is associated with a suitable notion of non-expansiveness of functions. In the next section, we shall deal with non-expansive quantitative term rewriting systems, leaving to section 8 the analysis of graded quantitative term rewriting systems, the most general class of quantitative term-based systems we will study in this work. Before diving into the theory of non-expansive systems, however, it is instructive to anticipate a bit of term-systems phenomenology.

Non-Expansive Systems Non-expansive term rewriting systems (\( \Omega \)-TRSs, for short) are quantitative systems in which reducing terms inside contexts non-expansively propagates distances. Therefore, if \( t \) reduces to \( s \) with distance \( \varepsilon \), then \( C[t] \) reduces to \( C[s] \) with distance \( \varepsilon \), too. That is, by thinking about the context \( C \) as a function on terms, then \( C \) is non-expansive with respect to the rewriting distance. To make this semantic choice coherent at a rewriting level, systems have to be linear, as non-linearity of terms breaks non-expansiveness (cf. distance amplification in subsection 2.3).

Additive Systems Additive (term rewriting) systems constitute the subclass of \( \Omega \)-TRSs whose quantale is idempotent. Even if quantitative, the monoidal structure of additive systems collapses to a cartesian one, as the tensor product of an idempotent quantale coincides with the meet of its underlying lattice. The main consequence of that is that non-expansiveness of rewriting is semantically coherent even with non-linearity of systems, so that we can have non-linear additive systems that do not suffer neither confluence nor distance amplification issues. The theory of additive systems is essentially the same as the one of traditional rewriting systems, the latter being the prime examples of additive systems.

Graded Systems Graded systems constitute the largest class of term-based quantitative rewriting systems. Contrary to non-expansive systems, in a graded system the distance generated by a reduction \( t \to s \) can be amplified (or reduced\(^{35} \)) when performed in a context. Thus for instance, we may have that \( t \) reduces to \( s \) with distance \( \varepsilon \), but \( C[t] \) reduces to \( C[s] \) with distance \( \phi_C(\varepsilon) \). The map \( \phi_C \) is known

\(^{35}\)In which case we may talk of contractive systems.
as the grade or sensitivity of the context C, and it gives the law determining how much distances are amplified by C. For instance, if we work with the Lawvere quantale, \( \phi_C \) is usually a multiplication by a constant map, the intended semantic meaning of such a map being a generalised Lipschitz constant associated to C when regarded as a function. Graded term rewriting systems are an example of modal and coeffectful systems; and because of their modal nature, they allow us to drop the linearity constraint of non-expansive systems without incurring in (semantic and rewriting) (in)consistency issues. The price to pay for that is the need for a more sophisticated (meta)theory than the one of non-expansive systems. The latter, in fact, can be seen as trivial graded systems in which all contexts have grade given by the identity function (i.e. no amplification).

7 Quantitative Term Rewriting: Non-Expansive Systems

Let us now formally introduce non-expansive systems. Through this section, let \( \Omega = (\Omega, \leq, \otimes, k) \) be a fixed continuous quantale. Before going any further, we shortly recall some of the (standard) notions and notation we will use in the rest of the paper.

Terms For a signature \( \Sigma \) and a countable set of variables \( X \), we write \( \Sigma(X) \) for the collection of (\( \Sigma \)-)terms over \( X \). We use small Latin letters \( t, s, u, \ldots \) to range over terms, sometimes using letters \( a, b, c, \ldots \) too.

Positions Recall that a position \( p \) is a finite string of positive integers. We denote by \( \lambda \) the empty string and by \( pq \) the concatenation of positions \( p, q \); we write \( p \leq q \) if \( p \) is a prefix of \( q \), i.e. if there is \( r \) such that \( q = pr \). We write \( p \parallel q \) if \( p \not\leq q \) and \( q \not\leq p \). Finally, we denote by \( t[p] \) the subterm of \( t \) at position \( p \). If \( t[p] = s \), we will also write \( t[s] \).

Context A context is a term over the signature \( \Sigma \cup \{\square\} \). We write \( \mathcal{C}[\cdot] \) for a context containing a single occurrence of \( \square \) and use the notation \( \mathcal{C}[t] \) to denote the term obtained by replacing the (single) occurrence of \( \square \) with \( t \) in \( \mathcal{C}[\cdot] \).

Substitution We denote substitutions by \( \sigma, \tau, \ldots \) and write \( t\sigma \) in place of \( \sigma(t) \). Furthermore, given two substitutions \( \sigma, \tau \), we write \( \sigma \leq \tau \) if there exists \( \rho \) such that \( \tau = \sigma \rho \), where \( (\sigma \rho)(t) \equiv \sigma(\rho(t)) \). Given two term \( t \) and \( s \), if \( t\sigma = s\sigma \), then \( \sigma \) is a unifier of \( t \) and \( s \), while \( t \) and \( s \) are said to be unifiable. Finally, recall that the most general unifier (mgu) of two unifiable terms is their minimal unifier with respect to \( \leq \).

Linearity We say that a term \( t \) is linear if it has no multiple occurrences of the same variable. We say that a mathematical expression (such as a relation or a predicate) involving terms is linear if all terms appearing in it are linear.

We are now ready to define non-expansive quantitative term rewriting systems, which we simply refer to as \( \Omega \)-term rewriting systems.

Definition 14. A \( \Omega \)-term rewriting system (\( \Omega \)-TRS, for short) is a pair \( \mathcal{R} = (\Sigma, \rightarrow_R) \) consisting of a signature \( \Sigma \) and a \( \Omega \)-ternary relation \( \rightarrow_R \) over \( \Sigma \)-terms. The (rewriting) \( \Omega \)-ternary relation \( \rightarrow_R \) generated by \( \rightarrow_R \) is defined by the rules in figure 2.
Base of working with ternary relations over elements of the quantale or over a base thereof makes no difference. Consequently, we will continue working with full rationals in place of those over the quantale itself [36]. Thus, for instance, in the case of the Lawvere quantale, we should take relations over non-negative rationals; and that rewriting satisfies the Archimedean property [84]: to prove \( t \rightarrow s \), it is enough to prove \( t \preceq s \), then \( \forall \varepsilon, \delta \triangleleft \varepsilon, \delta \vDash t \rightarrow s \).

**Remark 7.** Definition 14 stipulates that \( \rightarrow_R \) must be closed under suitable structural rules, viz. weakening, closure under finite joins, and the so-called (infinitary) Archimedean rule [84]. We have included such rules to stay as close as possible to the literature on quantitative equational theories, where structural rules are used to ensure completeness of equational proof systems. From a rewriting perspective, however, such rules can be safely (and maybe naturally) avoided, arguably with the exception of weakening. In fact, not only having weakening as the only structural rule is a natural design choice at a semantic level (making, e.g., the defining rules of \( \rightarrow_R \) finitary), but it also strengthen the theory of Q-TRS we are going to develop. In particular, the presence of the Archimedean rule forces us to formulate,

\[
\begin{array}{cccccccccc}
\varepsilon \vDash a \rightarrow_R b & \varepsilon \vDash t \rightarrow_R s & \delta \leq \varepsilon & \varepsilon_1 \vDash t \rightarrow_R s & \ldots & \varepsilon_n \vDash t \rightarrow_R s & \forall \varepsilon, \delta \vDash t \rightarrow_R s & \varepsilon \vDash t \rightarrow_R s
\end{array}
\]

**Figure 2: Definition of \( \rightarrow_R \)**

**Notation 8.** We refer to a triple \((a, \varepsilon, b) \in \mapsto_R \), i.e., such that \( a \rightarrow_R b \) as a (reduction) rule. We call \( a \) the **redex**, and \( b \) the **contractum**. Moreover, when \( R \) is irrelevant or clear from the context, we shall write \( \rightarrow \) in place of \( \rightarrow_R \) and use the notation \( t \overset{\varepsilon}{\rightarrow} s \) in place of \( \varepsilon \vDash t \rightarrow_R s \).

The first defining rule of the relation \( \rightarrow_R \) in Definition 14 is the main rewriting rule: it states that rewriting can be performed inside any context and on any instance of reductions in \( \mapsto_R \). This rule — which is standard in traditional rewriting — reflects the (semantic) assumption that operation symbols in \( \Sigma \) behave as **non-expansive** functions: accordingly, contexts do not amplify rewriting distances. The reaming rules encode structural properties of quantitative rewriting: the first giving a form of quantitative weakening, the second stating that rewriting is closed under finite join, and the third stating a generalised continuity property. On the Lawvere quantale, for instance, we can read \( t \rightarrow s \) as stating that \( t \) reduces to \( s \) within an error of at most \( \varepsilon \). Equivalently, we can reduce \( t \) to the non-semantically equivalent term \( s \), which differs from \( t \) of at most \( \varepsilon \). Accordingly, the structural rules in Definition 14 respectively state that if \( t \overset{\varepsilon}{\rightarrow} s \) and \( \varepsilon \leq \delta \), then we also have \( t \overset{\delta}{\rightarrow} s \); that rewriting is closed under (necessarily finite) minima; and that rewriting satisfies the Archimedean property [84]: to prove \( t \rightarrow s \), it is enough to prove \( t \overset{\delta}{\rightarrow} s \) for any \( \delta \) strictly bigger than \( \varepsilon \) (i.e., \( \delta > \varepsilon \)).

Any \( \Omega \)-TRS (\( \Sigma, \rightarrow_R \)) induces a \( \Omega \)-ARS whose objects are \( \Sigma \)-terms and whose rewriting \( \Omega \)-relation \( R : \Sigma(X) \rightarrow \Sigma(X) \) defined by

\[
R(t, s) \triangleq \forall \varepsilon \{ \varepsilon \mid \varepsilon \vDash t \rightarrow_R s \}.
\]

Consequently, all definitions and results seen so far extend to \( \Omega \)-TRSs. For that reason, we oftentimes say that a \( \Omega \)-TRS (\( \Sigma, \rightarrow_R \)) has a given property when we actually mean that its associated \( \Omega \)-ARS (\( \Sigma(X), R \)) has it.
e.g., confluence results at the level of the \( \Omega \)-relation \( R \), and one may wonder whether confluence holds also at the level of the ternary relation \( \rightarrow_R \). The answer is, in general, in the negative. Nonetheless, an affirmative answer can be given if we drop all structural rules but weakening. This suggests that it is worth considering an alternative, structurally-free definition of \( \Omega \)-TRSs. We will follow this path in section 8 where we shall define graded systems using weakening as the only structural rule.

Let us now see some examples of \( \Omega \)-TRSs, focusing, in particular, to the systems presented in section 2.

**Example 7.** Traditional term rewriting systems are nothing but \( \mathbb{2} \)-TRSs.

**Example 8.** All the examples seen in section 2 are \( \mathbb{L} \)-TRSs. In particular, systems

\[
\mathcal{N} = (\Sigma_N, \rightarrow_N) \quad \mathcal{B} = (\Sigma_B, \rightarrow_B) \quad \mathcal{K} = (\Sigma_K, \rightarrow_K) \quad \mathcal{T} = (\Sigma_T, \rightarrow_T)
\]

as well combinations thereof (e.g. system \( \mathcal{K}_B = (\Sigma_K \cup \Sigma_B, \rightarrow_K_B) \)) are all \( \mathbb{L} \)-TRSs.

**Example 9.** Any quantitative string rewriting system can be modelled as \( \Omega \)-TRSs. In particular, all quantitative string rewriting systems of subsection 2.2 can be gives as \( \mathbb{L} \)-TRS. To do so, we consider modify the signature seen in subsection 2.2 by taking \( \Sigma_M \triangleq \{ A, C, G, T, \text{nil} \} \), where \( A, C, G, T \) are unary function symbol and \( \text{nil} \) is a constant acting as the empty string. Thus, for instance, we model the string \( \text{AGTC} \) as the term \( A(G(C(T(\text{nil})))) \). Next, we adapt the rewriting relation previously introduced to act on terms (rather than strings). We thus obtain the rewrite \( \mathbb{L} \)-relation \( \rightarrow_M \) defined as follows, where \( b, c \in \{ A, C, G, T \} \) and \( b \neq c \) in the third rule.

\[
\begin{align*}
x \overset{1}{\rightarrow_M} b(x) & \quad b(x) \overset{1}{\rightarrow_M} x & \quad b(x) \overset{1}{\rightarrow_M} c(x)
\end{align*}
\]

As seen in subsection 2.2 (the \( \Omega \)-TRS version of) system \( \mathcal{M} = (\Sigma_M, \rightarrow_M) \) operationally describes the Levenshtein distance \( [70] \) between DNA sequences. Further edit distances on DNA molecules can be easily obtained modifying system \( \mathcal{M} \). For instance, considering the third defining rule of \( \rightarrow_M \) (i.e. \( b(x) \overset{1}{\rightarrow_M} c(x) \)) only, we obtain an operational description of the Hamming distance \( [70] \), whereas the following system gives the Eigen–McCaskill–Schuster distance (one obtains the Watson–Crick distance similarly) \( [49] \).

\[
\begin{align*}
A(x) \overset{1}{\rightarrow_C} (x) & \quad G(x) \overset{1}{\rightarrow_T} (x) & \quad A(x) \overset{1}{\rightarrow T} (x) \\
A(x) \overset{0}{\rightarrow G(x)} & \quad G(x) \overset{1}{\rightarrow C(x)} & \quad C(x) \overset{0}{\rightarrow T(x)}
\end{align*}
\]

**Example 9.** Consider the signature \( \Sigma_L \) containing a single binary operation \( \cup \) for nondeterministic choice and let \( \rightarrow_L \) be the following rewriting relation.

\[
\begin{align*}
x \overset{0}{\rightarrow_L} x \cup x & \quad (x \cup y) \cup z \overset{0}{\rightarrow_L} x \cup (y \cup z) & \quad (x \cup y) \overset{0}{\rightarrow_L} (y \cup x)
\end{align*}
\]

As it is, this rewriting system is not that interesting. The key point here is the choice of the quantale used for distances. Contrary to previous examples, here we consider the **strong** Lawvere quantale \( \mathbb{L}^{\text{max}} \).
The choice of this quantale largely impacts on the definition of \( \rightarrow_L \), which now gives a form of non-expansiveness of \( \cup \) reflecting the ultrametric structure of \( L_{\text{max}} \):

\[
\begin{align*}
    x & \xrightarrow{\varepsilon} x' \quad y \xrightarrow{\delta} y' \\
    x \cup y & \xrightarrow{\text{max}(\varepsilon, \delta)} x' \cup y'
\end{align*}
\]

The convertibility distance \( L^\varepsilon \) gives the so-called theory of quantitative semilattices \([84]\) and axiomatises the Hausdorff distance between sets \([92]\). Ultrametricity of \( L_{\text{max}} \) ultimately relies on its tensor product being idempotent, i.e. satisfying the law \( \varepsilon \otimes \varepsilon = \varepsilon \). In the case of \( L_{\text{max}} \), this law trivially holds as the tensor coincides with the meet. When the underlying quantale is idempotent, then quantitative and metric reasoning becomes similar to traditional, Boolean reasoning, up to the point that the linearity assumption mentioned in subsection 2.3 (which we shall formally rely on it the next section) is not necessary to avoid distance trivialisation and to ensure confluence properties of systems. Finally, we can combine system \( L = (\Sigma_L, \rightarrow_L) \) with, e.g., system \( \mathcal{K} \), this way obtaining a quantitative system for nondeterministic affine combinators.

We summarise the examples of \( \Omega \)-TRSs seen so far in Table 5 (we will see further examples of \( \Omega \)-TRSs in section 8). The rest of this section is dedicated to the development of a general theory of \( \Omega \)-TRSs and to instantiate it to infer nice computational properties of systems in Table 5. In particular, we shall prove (by means of general techniques) confluence of all of them. Before diving into that, however, it is useful to spend few words on quantitative equational theories.

| System | Objects/Name | Distance Induced |
|--------|--------------|------------------|
| \( N = (\Sigma_N, \rightarrow_N) \) | Natural Numbers | Euclidean Distance |
| \( B = (\Sigma_B, \rightarrow_B) \) | Multi-distributions/Barycentric algebras | Total Variation distance |
| \( \mathcal{K}_N = (\Sigma_{\mathcal{K}_N}, \rightarrow_{\mathcal{K}_N}) \) | Affine combinators with Arithmetic | Higher-order Euclidean Distance |
| \( T = (\Sigma_T, \rightarrow_T) \) | Ticking | Cost distance |
| \( M = (\Sigma_M, \rightarrow_M) \) | DNA molecules | Edit distances |
| \( L = (\Sigma_L, \rightarrow_L) \) | Quantitative (semi)lattices | Hausdorff distance |

Table 5: Main Examples of non-expansive \( \Omega \)-TRSs.

7.1 Quantitative Equational Theories

In this section, we formally introduce quantitative equational theories and their connection with \( \Omega \)-TRSs. Approaching the former in light of the latter allows us to highlights some operationally questionable design choices in the definition of a quantitative equational theories

**Definition 15.** A quantitative equational theory is a pair \( \mathcal{E} = (\Sigma, \approx_E) \), where \( \Sigma \) is a signature and \( \approx_E \) is a \( \Omega \)-ternary relation over \( \Sigma \)-terms. The \( \Omega \)-ternary (equality) relation \( =_E \) generated by \( \approx_E \) is defined by the rules in Figure 3.

The first block of rules in Figure 3 states that \( =_E \) is a quantitative equivalence relation containing \( \approx_E \), whereas the last block contains essentially the same structural rules defining a \( \Omega \)-TRS. The second block
of rules, instead, states that function symbols and substitution behave as \textit{non-expansive functions}. In fact, defining the $\Omega$-relation $E$ by

\[
E(t, s) \triangleq \bigvee \{ \epsilon \mid \epsilon \vdash t =_E s \}
\]

we see that $E$ is reflexive, symmetric, and transitive. By regarding any $n$-ary function symbol $f$ as a function $f : \Sigma(X)^n \to \Sigma(X)$, we also see that

\[
E(t_1, s_1) \otimes \cdots \otimes E(t_n, s_n) \leq E(f(t_1, \ldots, t_n), f(s_1, \ldots, s_n)),
\]

meaning that function symbols indeed behave as non-expansive functions.

\textbf{Remark 8.} Sometimes \cite{s4, s5}, quantitative equational theories are defined using the following rule to deal with function symbols.

\[
\frac{\epsilon \vdash t_1 =_E s_1 \quad \cdots \quad \epsilon \vdash t_n =_E s_n}{\bigwedge_i \epsilon_i \vdash f(t_1, \ldots, t_n) =_E f(s_1, \ldots, s_n)}
\]

Semantically, that means requiring function symbols to behave as \textit{strongly non-expansive} maps:

\[
E(t_1, s_1) \wedge \cdots \wedge E(t_n, s_n) \leq E(f(t_1, \ldots, t_n), f(s_1, \ldots, s_n)).
\]

In the case of the Lawvere quantale, for instance, we require the distance between two function applications to bound the \textit{maximum} distance between their arguments, rather than by the \textit{sum} of such distances. Strong non-expansiveness, however, does not properly interact with transitivity, which is based on the tensor, rather than the meet, of the quantale. Consider terms $t_1, t_2, s_1, s_2$ with $\epsilon \vdash t_1 =_E s_1$ and $\delta \vdash t_2 =_E s_2$. For a binary function symbol $f$, we can consider the following two derivations:

\[
\frac{\epsilon \vdash t_1 =_E s_1 \quad k \vdash t_2 =_E t_2}{\epsilon \vdash f(t_1, t_2) =_E f(s_1, t_2)} \quad \frac{\epsilon \vdash t_1 =_E s_1 \quad \delta \vdash t_2 =_E s_2}{\delta \vdash f(s_1, t_2) =_E f(s_1, s_2)}
\]

\[
\frac{\epsilon \wedge \delta \vdash f(t_1, t_2) =_E f(s_1, s_2)}{\epsilon \wedge \delta \vdash f(t_1, t_2) =_E f(s_1, s_2)}
\]

From a rewriting perspective, these two derivations show that rewriting $t_1$ into $s_1$ and $t_2$ into $s_2$ inside $f$ \textit{sequentially} gives a different distance than performing the same rewriting \textit{in parallel}. This is not surprising: the non-expansiveness rule for function symbols is defined ultimately relying on the idempotent quantale $(\Omega, \leq, \wedge, \top)$, whereas transitivity relies on $(\Omega, \leq, \otimes, \Pi)$. Harmony is restored by
taking the following transitivity rule, which amounts to instantiate Definition 15 with the idempotent quantale \((\Omega, \leq, \land, \pi)\).

\[
\frac{\varepsilon \vdash t \approx E s \quad \delta \vdash s \approx E u}{\varepsilon \land \delta \vdash t \approx E u}
\]

Following Remark 8, we introduce some further terminology and refer to equational theories over idempotent quantales as additive or idempotent quantitative equational theories. We employ a similar terminology for quantitative rewriting systems.

**Example 10.**

1. Since the Boolean quantale is obviously idempotent, traditional rewriting systems and equational theories are additive (quantitative) systems.

2. Since the strong Lawvere quantale is idempotent, system \(L\) of Example 9 is additive. Its associated quantitative equational theory is the one of quantitative semilattices [84, 85], which is additive too.

3. Consider the powerset quantale \(\mathcal{P}(\{A, C, G, T\})\) and the system \(b \xrightarrow{(b)} c\), for \(b, c \in \{A, C, G, T\}\) and \(b \neq c\). This way, we obtain a qualitative distance between molecules giving which bases change between two molecules.

4. The open sets of a topological space form a frame [115] and thus an idempotent quantale. Taking open sets as distances between objects, we can model effectively measurable differences or approximated ones. This way, we stipulate that measuring can be done with a limited precision only. Such systems are indeed additive.

Any quantitative equational theory \((\Sigma, \approx_E)\) induces a \(\Omega\)-TRS whose rewriting rules are given by the equations of \(\approx_E\) (actually, it is preferable to consider a subset thereof obtained by giving equations an appropriate orientation), so that we can use \(\Omega\)-TRSs to study properties of quantitative equational reasoning, the main ones being related to confluence, termination, and (therefore) metric word problems. In particular, given a quantitative equational theory \((\Sigma, \approx_E)\), we can define a \(\Omega\)-TRS \((\Sigma, \rightarrow_R)\) such that \(E = R^2\). Consequently, by Proposition 2, if \(R\) is confluent, then we recover the equational distance between terms by looking at their common reducts. If, additionally, the system is terminating (i.e. SN), then we can approximate such a distance by looking at normal forms only: this way, we also obtain decidability of the reachability metric word problem for \((\Sigma, \approx_E)\). It is thus desirable to develop handy techniques to prove confluence and termination of \(\Omega\)-TRSs.

### 7.2 Confluence and Critical Pairs, Part I

From (quantitative) Newman’s Lemma [Proposition 7], we know that to prove confluence of a terminating \(\Omega\)-TRS we only need to verify its local confluence. Proving local confluence of a \(\Omega\)-TRS, however, can be difficult, as reductions may happen inside arbitrary contexts and on arbitrary instances of reduction rules. It is thus natural to ask whether we can prove local confluence locally, i.e. by looking at ground rewriting only.

In this section, we show that local confluence of a linear \(\Omega\)-TRS follows directly from local confluence of its critical pairs [77, 25]. Linearity, as we have already discussed in section 2, is a crucial property in quantitative and metric reasoning: forcing non-expansiveness on non-linear systems often let distance trivialise [31, 33, 62, 61], this way collapsing quantitative equational deduction to traditional, Boolean
reasoning. On rewriting systems, non-linearity leads to further undesired consequences, as shown by the following example.

**Example 11.** Consider the signature $\Sigma \doteq \{f, e, i\}$ with $f$ a binary function symbol and $e, i$ constants. Let $\rightarrow_r$ be the reduction rule over the Lawvere quantale:

$$f(x, x) \rightarrow^0 x$$

$$e \rightarrow^1 i.$$ 

It is easy to see that the system is confluent in the traditional, non-quantitative sense. Taking quantitative information into account, however, we have $e \xleftarrow{\sigma} f(e, e) \rightarrow f(i, e)$, and thus $R(f(e, e), e) = 0$, $R(f(e, e), f(i, e)) = 1$. To close the diagram given by $e \xleftarrow{\sigma} f(e, e) \rightarrow f(i, e)$, we need to reduce $e$ twice:

$$e \rightarrow^1 i \rightarrow^0 f(i, i) \rightarrow f(i, e).$$

This gives $R'(e, i) = 1$ and $R'(f(i, e), f(i, i)) = 1$, this way breaking (local) confluence.

Let us now recall the notion of a critical pair and refine the well-known critical pair lemma [17] to a quantitative setting.

**Definition 16.** Let $(\Sigma, \rightarrow_r)$ be a $\Omega$-TRS.

1. Let $c_1 \xrightarrow{\sigma} d_1, c_2 \xrightarrow{\delta} d_2$ be renamings of rewrite rules without common variables. Then $c_1 \xrightarrow{\sigma} d_1, c_2 \xrightarrow{\delta} d_2$ overlap at position $p$ if:
   - $p$ is a function symbol position of $c_2$;
   - $c_1$ and $c_2[p]$ are unifiable;
   - If $p = \lambda$, then the two rules are not variants (i.e. they cannot be obtained one from the other by variables renaming).

2. Let $c_1 \xrightarrow{\sigma} d_1, c_2 \xrightarrow{\delta} d_2$ overlapping at position $p$ and $\sigma$ be the mgu of $c_2[p]$ and $c_1$. Then the term $c_2^\sigma$ can be rewritten in two ways:

$$d_2^\delta \xleftarrow{\sigma} c_2^\sigma \rightarrow c_2^\sigma[d_1^\sigma]_p$$

We call the triple $(c_1 \xrightarrow{\sigma} d_1, p, c_2 \xrightarrow{\delta} d_2)$ a critical overlap and the pair $(c_2^\sigma[d_1^\sigma]_p, d_2^\delta)$ a critical pair.

**Example 12.**

1. The $L$-TRS of Example 11 has no critical pair since there is no function symbol position at which (renamings of) the rules $f(x, x) \rightarrow^0 x$ and $e \rightarrow^1 i$ overlap.

2. Consider system $\mathcal{B}$ of Barycentric algebras and the (no-common-variable renamings of) commutativity and associativity rules:

$$(x' +_{\epsilon_1} y') \xrightarrow{\delta} y' +_{1-\epsilon_1} x'$$

$$(x' +_{\epsilon_1} y') +_{\epsilon_2} z \xrightarrow{\delta} x' +_{\epsilon_1\epsilon_2} (y' +_{1-\epsilon_1\epsilon_2} z)$$
with \( \epsilon_1, \epsilon_2 \in (0, 1) \). Then, we see that the substitution \( \sigma \) mapping the variable \( x' \) to \( x \) and \( y' \) to \( y \) is the \( \text{mgu} \) of \( ((x + \epsilon_1 y) + \epsilon_2 z)) \) and \( x' + \epsilon y' \). Consequently, the triple and the pair

\[
(x' + \epsilon_1 y' \xrightarrow{0} y' +_{1-\epsilon} x', 1, x + \epsilon_1 \epsilon_2 (y + y') z))
\]

\[
((y + 1 - \epsilon_1 x) + \epsilon_2 z, x + \epsilon_1 \epsilon_2 (y + y') z))
\]

form a critical overlap and a critical pair, respectively. Diagrammatically, we have a critical pair defined by the following peak:

\[
\begin{array}{c}
(x + \epsilon_1 y) + \epsilon_2 z \\
(y + 1 - \epsilon_1 x) + \epsilon_2 z \\
x + \epsilon_1 \epsilon_2 (y + y') z
\end{array}
\]

**Notation 9.** Given a \( \bowtie \text{-TRS} \, \mathcal{R} = (\Sigma, \rightarrow_{\mathcal{R}}) \), we denote by \( CP(\mathcal{R}) \) the collection of its critical pairs. Moreover, we write \( s_1 \xleftarrow{\epsilon_1} t \xrightarrow{\epsilon_2} s_2 \) if \( t \xrightarrow{\epsilon_1} s_1 \) and \( t \xrightarrow{\epsilon_2} s_2 \).

We are now ready to prove the quantitative refinement of the so-called Critical Pair Lemma [77]. In its traditional version, such a lemma states that to prove local confluence of a rewriting relation it is enough to prove its local confluence on critical pairs only. From Newman’s Lemma it thus follows that if a rewriting relation is terminating and locally confluent on critical pairs, then it is confluent. When we move to quantitative rewriting, the Critical Pair Lemma needs a further assumption, namely linearity.

**Lemma 4** (Critical Pair). Let \( \mathcal{R} = (\Sigma, \rightarrow_{\mathcal{R}}) \) be a linear \( \bowtie \text{-TRS} \). If \( \mathcal{R} \) is locally confluent on all critical pairs of \( \mathcal{R} \), then it is locally confluent.

**Proof.** We have to show \( R^-; R \leq R^*; R^- \) given that \( (R^-; R)(t, s) \leq (R^*; R^-)(t, s) \), for any \( (t, s) \in CP(\mathcal{R}) \). Pointwise, we need to prove

\[
\bigvee_t R(t, s_1) \odot R(t, s_2) \leq \bigvee_u R^*(s_1, u) \odot R^*(s_2, u)
\]

for arbitrary terms \( s_1 \) and \( s_2 \). Since \( R(t, s) = \bigvee \{ \epsilon \mid \epsilon \vdash t \rightarrow s \} \) and the join distributes over the tensor, it is enough to show that for any local peak \( s_1 \xleftarrow{\epsilon_1} t \xrightarrow{\epsilon_2} s_2 \), we have

\[
\epsilon_1 \odot \epsilon_2 \leq \bigvee_u R^*(s_1, u) \odot R^*(s_2, u).
\]

We proceed by structural induction on \( \epsilon_1 \vdash t \rightarrow s_1 \) and \( \epsilon_2 \vdash t \rightarrow s_2 \) (see Figure 2). We begin with the structural rules, as those are easy. Suppose that one between \( \epsilon_1 \vdash t \rightarrow s_1 \) and \( \epsilon_2 \vdash t \rightarrow s_2 \) is obtained by one of the structural rules in Figure 2. Without loss of generality, we shall assume that this is the case for \( \epsilon_1 \vdash t \rightarrow s_1 \).
• Suppose that \( \varepsilon_1 \vdash t \rightarrow s_1 \) is obtained by quantitative weakening, so that we have:

\[
\begin{align*}
\delta \vdash t \rightarrow s_1 \quad \varepsilon_1 & \leq \delta \\
\therefore \varepsilon_1 \vdash t \rightarrow s_1
\end{align*}
\]

By induction hypothesis, we know that \( \delta \otimes \varepsilon_2 \leq \bigvee_u R^*(s_1, u) \otimes R^*(s_2, u) \), so that we conclude (recall that \( \vdash \) is integral)

\[
\varepsilon_1 \otimes \varepsilon_2 \leq \delta \otimes \varepsilon_2 \leq \bigvee_u R^*(s_1, u) \otimes R^*(s_2, u)
\]

• Suppose that \( \varepsilon_1 \vdash t \rightarrow s_1 \) is obtained by closure under finite join, so that we have \( \varepsilon_1 = \bigvee_i \delta_i \) for some \( \delta_1, \ldots, \delta_n \) and

\[
\begin{align*}
\delta_1 \vdash t \rightarrow s_1 \quad \ldots \quad \delta_n \vdash t \rightarrow s_1 \\
\therefore \bigvee_i \delta_i \vdash t \rightarrow s_1
\end{align*}
\]

By induction hypothesis, we know that \( \forall i \leq n. \delta_i \otimes \varepsilon_2 \leq \bigvee_u R^*(s_1, u) \otimes R^*(s_2, u) \) which gives, by the universal property of joins,

\[
\bigvee_i (\delta_i \otimes \varepsilon_2) \leq \bigvee_u R^*(s_1, u) \otimes R^*(s_2, u)
\]

and thus the desired thesis, since \( \bigvee_i (\delta_i \otimes \varepsilon_2) = (\bigvee_i \delta_i) \otimes \varepsilon_2 \).

• Suppose that \( \varepsilon_1 \vdash t \rightarrow s_1 \) is obtained by the Archimedean property, so that we have:

\[
\forall \delta_i \leq \varepsilon_1. \delta_i \vdash t \rightarrow s_1
\]

By induction hypothesis, we have \( \delta_i \otimes \varepsilon_2 \leq \bigvee_u R^*(s_1, u) \otimes R^*(s_2, u) \), for any \( \delta_i \leq \varepsilon_1 \), and thus

\[
\bigvee_{\delta_i \leq \varepsilon_1} \delta_i \otimes \varepsilon_2 \leq \bigvee_u R^*(s_1, u) \otimes R^*(s_2, u).
\]

We conclude the thesis since \( \varepsilon_1 = \bigvee_{\delta_i \leq \varepsilon_1} \delta_i \)

Let us now move to the main case, namely the one in which \( s_1 \) and \( s_2 \) are obtained from \( t \) by closure under substitution and context of two rewriting rules. More precisely, suppose that \( \varepsilon_1 \vdash t \rightarrow s_1 \) is obtained by applying rule \( \varepsilon_1 \vdash a_1 \mapsto b_1 \) at position \( p_1 \) and \( \varepsilon_2 \vdash t \rightarrow s_2 \) is obtained by applying rule \( \varepsilon_2 \vdash a_2 \mapsto b_2 \) at position \( p_2 \). We also assume that the two rules have disjoint variables. Thus, we have that \( t[p_1 = a_1^p_1, t[p_2 = a_2^p_2, s_1 = t[b_1^p_1], p_1, s_2 = t[b_2^p_2], p_2. \) We proceed by cases, depending on the relationship between \( p_1 \) and \( p_2 \).

• Suppose \( p_1 \parallel p_2 \). Then, \( t = t[a_1^p_1, [a_2^p_2]_{p_2}, s_1 = t[b_1^p_1, [a_2^p_2]_{p_2} \) and \( s_2 = t[a_1^p_1, [b_2^p_2]_{p_2}. \) Applying \( \varepsilon_2 \vdash a_2 \mapsto b_2 \) to \( s_1[p_2 \) and \( \varepsilon_1 \vdash a_1 \mapsto b_1 \) to \( s_2[p_1 \), we have that \( s_1 \overset{\varepsilon_2}{\longrightarrow} t[b_1^p_1, [a_2^p_2]_{p_2} \) and \( s_2 \overset{\varepsilon_1}{\longrightarrow} t[b_1^p_1, [b_2^p_2]_{p_2}. \) Consequently,

\[
\varepsilon_1 \otimes \varepsilon_2 \leq R(s_1, t[b_1^p_1, [b_2^p_2]_{p_2}) \otimes R(s_2, t[b_1^p_1, [a_2^p_2]_{p_2} \leq \bigvee_u R^*(s_1, u) \otimes R^*(s_2, u).
\]
• Without loss of generality, suppose that \( p_1 \leq p_2 \). Then, there is \( q \) such that \( p_2 = p_1q \). We distinguish two cases:

1. If \( \epsilon \models a_1 \rightarrow b_1 \) and \( \delta \models a_2 \rightarrow b_2 \) do not overlap at position \( q \), then we have that either \( \epsilon \models a_1 \rightarrow b_1 \) and \( \delta \models a_2 \rightarrow b_2 \) are variants and \( p = \lambda \), or \( p_2 = p_1q_1q_2 \) with \( a_1q_1 \) a variable, say \( x \). In the latter case, we have that \( s_1 = t[b\sigma_1\gamma_1]\psi_1 = t[b\sigma_1\gamma_2\gamma_2]\psi_1 \), whereas \( s_2 = t[a\sigma_1\gamma_2\gamma_2]\psi_1 \). Consider the substitution \( \tau \) mapping \( x \) to \( b\sigma_1\gamma_2\gamma_2\psi_1 \). Since \( a_1 \) is linear, applying \( \delta \models a_2 \rightarrow b_2 \) to \( s_1p_1 \) and \( \epsilon \models a_1 \rightarrow b_1 \) to \( s_2p_1 \), with substitution \( \tau \), we have that \( \delta \models s_1 \rightarrow u \) and \( \epsilon \models s_2 \rightarrow u \), with \( u = t[b\sigma_1\gamma_2\gamma_2\gamma_2]\psi_1 \). Distance analysis proceeds as in the first case.

If instead, \( \epsilon \models a_1 \rightarrow b_1 \) and \( \delta \models a_2 \rightarrow b_2 \) are variants with \( q = \lambda \), i.e. \( p_1 = p_2 \), then necessarily \( s_1 = s_2 \), that is, \( k \models s_1 \rightarrow s_2 \) and \( k \models s_2 \rightarrow s_2 \). In this case, we have that

\[
\epsilon \otimes \delta \leq k \otimes k \leq R(s_1, s_2) \otimes R(s_2, s_2) \leq \bigvee_u R^*(s_1, u) \otimes R^*(s_2, u).
\]

2. If \( \epsilon \models a_1 \rightarrow b_1 \) and \( \delta \models a_2 \rightarrow a_2 \) overlap at position \( q \). Then, \( a\sigma_1\gamma_1 = (t\psi_1)\gamma_1 = t\psi_2 = a\sigma_2 \). That is, \( \sigma \) is an unifier of \( a_1q_1 \) and \( a_2 \). Let \( \tau \) be their most general unifier, so that \( \sigma = \rho \circ \tau \), for some substitution \( \rho \). Then \( (a\sigma_1\gamma_2\gamma_2\gamma_2)\psi_1 \) is a critical pair. By hypothesis, we know that

\[
(R^*; R)(a\sigma_1\gamma_2\gamma_2\gamma_2\psi_1, b\gamma_1\psi_1) \leq \bigvee_u R^*(a\sigma_1\gamma_2\gamma_2\gamma_2\psi_1, u) \otimes R^*(b\gamma_1\psi_1, u),
\]

and thus \( \epsilon \otimes \delta \leq \bigvee_u R^*(a\sigma_1\gamma_2\gamma_2\gamma_2\psi_1, u) \otimes R^*(b\gamma_1\psi_1, u) \). To prove the thesis, it is sufficient to prove

\[
\bigvee_u R^*(a\sigma_1\gamma_2\gamma_2\gamma_2\psi_1, u) \otimes R^*(b\gamma_1\psi_1, u) \leq \bigvee_v R^*(s_1, v) \otimes R^*(s_2, v).
\]

As usual, it is enough to show that for any \( u \) such that \( \eta \models a\sigma_1\gamma_2\gamma_2\gamma_2\psi_1 \rightarrow u \) and \( \iota \models b\gamma_1\psi_1 \rightarrow u \) we have \( \eta \otimes \iota \leq \bigvee_v R^*(s_1, v) \otimes R^*(s_2, v) \). Fixed \( u \) as above, we have:

\[
\begin{align*}
  s_1 &= t[b\sigma_1\gamma_1]\psi_1 = t[b\sigma_2\gamma_2]\psi_1, \\
  s_2 &= t[b\sigma_2\gamma_2\gamma_2\psi_1] = t[a\sigma_1\gamma_2\gamma_2\gamma_2\psi_1] = t[a\sigma_1\gamma_2\gamma_2\gamma_2\psi_1] = t[a\sigma_1\gamma_2\gamma_2\gamma_2\psi_1] = s_1,
\end{align*}
\]

and we are done.

\[ \square \]

**Theorem 1.** Any linear and terminating \( \Omega \)-TRS locally confluent on its critical pairs is confluent.

**Proof.** It directly follows from Proposition 7 and Lemma 4. \[ \square \]

**Remark 9.** Example 11 shows that, contrary to what happens in the traditional case, the linearity assumption is indeed necessary in Theorem 1, in fact, it is an easy exercise to prove that if \( \Omega \) is idempotent, then Lemma 4 extends to non-linear \( \Omega \)-TRSs.
Example 13. It is easy to see that system $N$ of natural numbers is terminating and locally confluent on all its critical pairs. By Theorem 1, we conclude that $N$ is confluent. \hspace{1em} \blacksquare

Example 14. Most of the quantitative string rewriting systems introduced so far are not terminating, and thus we cannot rely on Theorem 1 to prove their confluence. One easy way to overcome the problem is to rephrase them as terminating systems. For instance, we can stipulate that molecules can only be deleted and that substitution of molecules is directed, in the sense that, e.g., $A$ can become $C$, $G$, and $T$, but not vice-versa (similarly, $C$ can become $G$ and $T$, and $G$ can only become a $T$). This way, we indeed obtain a terminating system locally confluent on its critical pairs, and thus a confluent system. Although this approach works fine as long as we are interested in reachability problems (and alike), some care is needed when dealing with optimal distances, as forcing termination may lead to increasing minimal distances between molecules. \hspace{1em} \blacksquare

Example 15. System $T$ is terminating and locally confluent, and thus confluent. System $T$, instead, is not terminating, due to the rule $\ldots$. We can easily fix that by imposing $m < n$, this way obtaining a terminating and locally confluent — and thus confluent — system. \hspace{1em} \blacksquare

### 7.3 Confluence and Critical Pairs, Part II

Theorem 1 constitutes a powerful tool to prove confluence of terminating $\Omega$-TRSs. In a quantitative setting, however, termination might be a too strong condition, and interesting $\Omega$-TRSs may not satisfy it. As an example, consider system $\mathcal{B}$ of Barycentric algebras. Due to commutativity and left invariance, the system is obviously non-terminating. We may handle the former point by considering quantitative notions of rewriting modulo \cite{16, 96, 77}, but the former one is a genuine quantitative reduction (and actually the quantitative essence of the total variation distance!). That makes simply not possible to prove confluence of $\mathcal{B}$ via critical pairs and Newmann’s Lemma. And yet, by simply working out examples, we have the feeling that $\mathcal{B}$ is indeed confluent. To solve this issue, we modify Lemma 4 replacing local confluence with a stronger condition, namely strong confluence \cite{77}.

**Definition 17.** We say that a $\Omega$-relation $R : A \rightarrow A$ is **strongly confluent** if it satisfies inequality (strong-1) below and that it is **strongly closed** if it satisfies inequalities (strong-1) and (strong-2). As usual, we say that a $\Omega$-ARS is strongly confluent (resp. strongly closed) if its rewriting $\Omega$-relation is.

\[
R^-; R \leq R^+; R^-
\]  \hspace{1em} (strong-1)

\[
R^+; R \leq R^-; R^+.
\]  \hspace{1em} (strong-2)

The next result states that if a $\Omega$-TRS is linear, then to prove that it is strongly closed, it is enough to look at its critical pairs.

**Lemma 5.** A linear $\Omega$-TRS $\mathcal{R} = (\Sigma, \rightarrow_{\mathcal{R}})$ is strongly closed if and only if $R$ is strongly closed on all the critical pairs of $\mathcal{R}$.

\footnote{This observation generalises to a collection of interesting research problems asking whether completion algorithms on traditional rewriting systems can be extended to quantitative systems. Notice that this question may not have a Boolean answer: in fact, some completion procedures may be correct from the point of view of reachability problems, but not so when it comes to deal with optimal distances.}

\footnote{We leave the detailed development of such a theory for future work.}
Proof. The proof is essentially the same of the one of Lemma 3, the only difference being that in the main case we replace local confluence with strong closure.

Strong confluence (resp. closure) by itself is not immediately informative for our purposes. Its relevance is given by the following result stating that strong confluence (and thus strong closure) entails confluence.

Lemma 6. Strong confluence implies confluence.

Sketch. Let $R : A \to A$ be a $\Omega$-relation and $S = R^*$ (our proof works for an arbitrary $S : A \to A$, actually). Assume (strong-1), i.e. $S; R \leq R^*; S^*$. We prove $S^*; R^* \leq R^*; S^*$. Pointwise proofs rely on lexicographic induction. A lightweight, pointfree proof is obtained by observing that since $S^* = \mu X. \Delta \lor S; X$, we have $S^*; R^* = (\mu X. \Delta \lor S; X); R^* = \mu X. R^* \lor S; X$.

Consequently, to prove $S^*; R^* \leq R^*; S^*$ it is sufficient to prove $\mu X. R^* \lor S; X \leq R^*; S^*$, which can be done using fixed point induction.

Theorem 2. If a linear $\Omega$-TRS is strongly closed on all its critical pairs, then it is confluent.

Proof. Directly from Lemma 5 and Lemma 6.

We can now rely on Theorem 2 to prove confluence of $B$.

Proposition 8. System $B$ is strongly closed, and thus confluent.

Proof. By Theorem 2 it is enough to prove that $B$ is strongly closed on all critical pairs. To do so, we first notice that the associativity rule is ‘reversible’ in the following sense: given $\epsilon_1, \epsilon_2 \in (0, 1)$, the reduction

$$0 \vdash (x +_{\epsilon_1} y) +_{\epsilon_2} z \to x +_{\epsilon_1 \epsilon_2} (y +_{\epsilon_1 \epsilon_2} z).$$

has an inverse reduction

$$0 \vdash x +_{\epsilon_1 \epsilon_2} (y +_{\epsilon_1 \epsilon_2} z) \to^* (x +_{\epsilon_1} y) +_{\epsilon_2} z$$

obtained by alternatively applying commutativity and associativity. We then verify that $B$ is strongly closed. This is a routine case analysis on critical pairs.

As for Lemma 4 and Theorem 1, even for Lemma 5 and Theorem 2 we can drop the linearity assumption if the underlying quantale is idempotent. This gives confluence of the Hausdorff distance.

Example 16. Mimicking Proposition 8 we see that system $L$ is confluent too.
7.4 Confluence and Critical Pairs, Part III

In previous sections, we have proved confluence of most of the systems introduced in section 2 systems \( \mathcal{N}, \mathcal{M}, \mathcal{B}, \mathcal{T}, \mathcal{L} \) are all confluent. An important class of systems not present in this list is the one of quantitative extensions\(^{39}\) of affine combinators. This class includes system \( \mathcal{K}_N \) (combinators plus arithmetic), \( \mathcal{K}_B \) (probabilistic combinators), \( \mathcal{K}_T \) (combinators with cost), and similar systems. Even if different, all these systems are obtained in essentially the same way, namely by joining systems together \(^{40}\). For instance, system \( \mathcal{K}_B \) is obtained by joining systems \( \mathcal{K} \) of pure affine combinators and \( \mathcal{B} \).\(^{41}\) It is then natural to ask whether confluence of such systems can be proved compositionally in terms of confluence of their component subsystems. In our case, for instance, we know that both \( \mathcal{B} \) and \( \mathcal{K} \) are confluent \(^{42}\) and we would like to infer confluence of \( \mathcal{K}_B \). Indeed, we can do so relying on the quantitative refinement of the Hindley-Rosen Lemma \(^{43}\). Let us begin by formalising the idea of joining \( \Omega \)-TRSs.

**Definition 18.** Given \( \Omega \)-TRSs \( \mathcal{R} = (\Sigma_R, \rightarrow_R) \) and \( \mathcal{S} = (\Sigma_S, \rightarrow_S) \) with disjoint signatures, we define their sum as the \( \Omega \)-TRS \( \mathcal{R} + \mathcal{S} = (\Sigma_{RS}, \rightarrow_{RS}) \) defined thus: \( \Sigma_{RS} = \Sigma_R \cup \Sigma_S \) and \( \rightarrow_{RS} = \rightarrow_R \cup \rightarrow_S \).

**Example 17.** We immediately see that \( \mathcal{K}_B = \mathcal{K} + \mathcal{B} \). Extensions of \( \mathcal{K} \) with ticking are obtained as \( \mathcal{K} + \mathcal{T} \) and \( \mathcal{K} + \mathcal{T}_\Omega \), whereas \( \mathcal{K} + \mathcal{L} \) gives nondeterministic affine combinators. Finally, notice that even formally different, system \( \mathcal{K}_N \) is essentially \( \mathcal{K} + \mathcal{N} \).

To relate the sum of \( \Omega \)-TRSs \( \mathcal{R}, \mathcal{S} \) as above with Newman’s Lemma, we first observe that the \( \Omega \)-relation \( RS \) associated to \( \mathcal{R} + \mathcal{S} \) coincides with \( R \vDash S \).

**Lemma 7.** For all \( \Omega \)-TRSs \( \mathcal{R} = (\Sigma_R, \rightarrow_R), \mathcal{S} = (\Sigma_S, \rightarrow_S) \), we have \( RS = R \vDash S \).

*Proof.* Straightforward.

**Lemma 7** puts ourselves in the condition to rely on the quantitative Hindley-Rosen Lemma to prove confluence of \( \mathcal{R} + \mathcal{S} \). Accordingly, to infer confluence of \( R \vDash S \) we need to have confluence of \( R \) as well as commutation of \( R \) with \( S \). Whereas the former usually is our starting hypothesis, the latter requires a specific analysis. For the cases we are interested in, such an analysis is smooth, as systems \( \mathcal{R} \) and \( \mathcal{S} \) are essentially independent, in the sense that \( \mathcal{R} \) does not create new critical pairs.

**Lemma 8.** Given two linear \( \Omega \)-TRS \( \mathcal{R}, \mathcal{S} \) as above, if the collection of critical pairs obtained by overlapping of a rule of \( \mathcal{R} \) and a rule of \( \mathcal{S} \) is empty, then \( R \) strongly commutes with \( S \), and thus \( R \) commutes with \( S \).

*Proof.* The proof that \( R \) strongly commutes with \( S \) is a simplified instance of the proof of **Lemma 4** and **Lemma 5** from that, commutation of \( R \) with \( S \) follows by **Lemma 6**.

Using **Lemma 8** we obtain the necessary hypotheses to apply **Proposition 3** and conclude confluence of \( \mathcal{K}_B = \mathcal{K} + \mathcal{B} \).

\(^{39}\)Affine combinators having no nontrivial quantitative reductions, they are essentially identical to their traditional counterpart (which indeed gives a confluent and terminating system).

\(^{40}\)Algebraically, this operation corresponds to the *sum* of algebraic theories \(^{78}\).

\(^{41}\)Further systems can be obtained either by modifying the ‘effectful’ layer (hence adding to \( \mathcal{K} \), for instance, quantitative output, nondeterminism, etc) or by replacing \( \mathcal{K} \) itself with other rewriting systems modelling programming languages, such as concurrent ones \(^{80},^{24}\).

\(^{42}\)Since \( \mathcal{K} \) alone does not have truly quantitative behaviours, its confluence can be proved as for its traditional counterpart.
Proposition 9. System $\mathcal{K} + \mathcal{B}$ is confluent.

Proof. Confluence of $\mathcal{K} + \mathcal{B}$ immediately follows [Proposition 3] provided that $K$ commute with $B$. That is indeed the case, since $\mathcal{K}$ and $\mathcal{B}$ have no common critical pair, and thus they (strongly) commute, by [Lemma 8].

In a similar fashion, one proves that, e.g., $\mathcal{K} + \mathcal{N}$ and $\mathcal{K} + \mathcal{T}$ are confluent, as well as combinations thereof. Moreover, as usual, if the underlying quantale is idempotent, we can drop the linearity assumption in [Lemma 8] so that by enriching $\mathcal{K}$ over $L^{\max}$ (rather than on $L$), we see that $\mathcal{K} + \mathcal{L}$ is confluent too.

These results complete the (confluence) analysis of the examples introduced in section 2 as well as our general results on linear (or non-expansive) $\Omega$-TRSs. The next — last — section of this work outlines a possible way to go beyond linearity: we are going to move from non-expansive to Lipschitz continuous $\Omega$-TRSs.

8 BEYOND NON-EXPANSIVENESS: GRADES, MODALITIES, AND LIPSCHITZ CONTINUITY

In this section, we introduce a new class of quantitative term rewriting systems — namely graded rewriting systems — that allows us to model non-linear systems avoiding, at the same time, distance trivialisation and lack of confluence issues. So far, in fact, we have focused on linear, non-expansive functions and term constructors. This is reflected almost everywhere in our definitions: for instance, the rule

\[
\varepsilon \vdash t \mapsto_R s \\
\varepsilon \vDash C[t^\omega] \mapsto_R C[s^\omega]
\]

states that differences produced by $\mapsto_R$ are non-expansively propagated by $\rightarrow_R$ through contexts (hence term constructors) and substitution. This can be seen even more clearly in quantitative algebraic theories, where the rule

\[
\varepsilon_1 \vdash t_1 =_E s_1 \quad \cdots \quad \varepsilon_1 \vdash t_1 =_E s_1 \\
\varepsilon_1 \otimes \cdots \otimes \varepsilon_n \vdash f(t_1, \ldots, t_n) =_E f(s_1, \ldots, s_n)
\]

precisely tells us that the function symbol $f$ behaves as a non-expansive function. We have also seen stronger forms of non-expansiveness, namely strong (or ultra) non-expansiveness:

\[
\varepsilon_1 \vdash t_1 =_E s_1 \quad \cdots \quad \varepsilon_1 \vdash t_1 =_E s_1 \\
\varepsilon_1 \wedge \cdots \wedge \varepsilon_n \vdash f(t_1, \ldots, t_n) =_E f(s_1, \ldots, s_n)
\]

As already remarked, strong non-expansiveness is, in its essence, just ordinary non-expansiveness on an idempotent quantale. Even if we can view all of that from a more semantic point of view in terms of arrows and constructions in suitable categories of $\Omega$-spaces [70], such a level of abstraction is not necessary for our goals: it is sufficient to notice that, given a $\Omega$-relation $R : A \to A$, $f : A^n \to A$ is non-expansive (with respect to $R$) if:

\[
\bigotimes_i R(a_i, b_i) \leq R(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n))
\]

Non-expansive maps, however, are not the only maps one is interested in when working with metric spaces. Another interesting class of transformation is the one of contractions and, more generally, the
one of Lipschitz continuous functions \[106\]. Such maps have been extensively studied in the context of metric (program) semantics, due to their link with differential privacy \[48\] and (bounded) linear and coeffectful types \[92\], \[40\], \[60\].

Moving from an original observation by Lawvere \[82\], generalisations of non-expansive maps to \(\Omega\)-relations have been given in terms of change of base functors \[61\], \[62\], and corelators (viz. comonadic lax extension) \[41\]. In a nutshell, we allow functions \(f : A \to A\) to amplify distances, but in a controlled way. Such a way is given by a (family of suitable) function(s) \(\phi : \Omega \to \Omega\), so that we require:

\[
\phi(R(a, b)) \leq R(f(a), f(b)).
\]

The map \(\phi\) is sometimes called the sensitivity of \(f\) and gives the law describing how much differences between outputs are affected by differences between inputs. Accordingly, we think about sensitivity as generalising Lipschitz constants; and indeed, multiplication by a constant is a typical example of a map \(\phi\) on the Lawvere quantale.

Technically speaking, we shall define function sensitivity by means of change of base functors \[79\] which, in our simplified setting, take the form of quantale homomorphisms \[70\]. Any change of base functor \(\phi : \Omega \to \Omega\) induces a map on \(\Omega\)-relations sending a \(\Omega\)-relation \(R\) to the \(\Omega\)-relation \([\phi] R\) mapping \((a, b)\) to \(\phi(R(a, b))\). Maps \([\phi]\) are examples of (graded) relational modalities known as corelators \[41\], \[43\]. Even if the theory of graded rewriting systems we are going to define can be given in full generality in terms of corelators, we shall work with change of base functors only. The authors hope this will choice will help the reader understanding this last section.

When we move from unary to \(n\)-ary functions, it does make sense to talk about the sensitivity of \(f\); instead, we should talk about the sensitivity of \(f\) on a given argument. Assuming \(f\) to have sensitivity \(\phi_i\) on the \(i\)th argument, we then obtain the following new, finer notion of non-expansiveness:

\[
\bigotimes_i [\phi_i] R(a_i, b_i) \leq R(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)).
\]

Armed with this new notion of non-expansiveness, let us see how to make rewriting systems non-expansive in this new sense. The resulting notion, the one of a graded rewriting system, is the main subject of this last section. As usual, before introducing such systems in full generality, let us warm up with a concrete example.

**Graded Combinatory Logic** The main example of a graded system we will deal with is graded combinatory logic \[8\], \[35\], a generalisation of Abramsky’s bounded combinatory logic \[2\], \[4\].

Recall that system \(\mathcal{K}\) (as well as its extensions) is based on affine combinators only. In particular, we have seen how adding the (cartesian) combinator \(W\) leads to distance trivialisation and non-confluent behaviours. The reason is that the reduction rule \(W \cdot x \cdot y \mapsto x \cdot y \cdot y\) duplicates the variable \(y\), and thus the distance between combinators \(W \cdot t \cdot s\) and \(W \cdot t \cdot s'\) is duplicated when reducing \(W\). One way to overcome this problem is to refine system \(\mathcal{K}\) by introducing graded exponential modalities \(!_n\) constraining the usage of terms. The function symbol \(!_n\) is an example of a coeffectful modality \[95\], \[66\], \[97\], \[60\] and can be thought as providing \(n\) copies of its argument, so that we can break linearity up to usage \(n\). From a metric point of view, \(!_n\) is a function symbol whose sensitivity is given by the multiplication by \(n\) function, meaning that whenever we have terms \(t, s\) that are \(\varepsilon\)-apart, \(!_n t\) and \(!_n s\) are stipulated to be \(n \varepsilon\) apart.

\[43\]They actually provided a canonical example of a corelator.
According to this strategy, the reduction \( \mathcal{W} \cdot x \cdot y \mapsto 0 \) is replaced by

\[
\mathcal{W} \cdot x \cdot !_{n+m} y \mapsto x \cdot !_{n} y \cdot !_{m} y.
\]

But that is not the end of the story. The introduction of \( !_{n} \) affects not only ground reductions; it also hugely impacts on the definition of \( \mapsto \), which now becomes modal. In fact, suppose to have combinators \( t, s \) that are \( \epsilon \)-apart. If we now want to reduce \( t \) to \( s \) under the scope of \( !_{n} \), we cannot non-expansively propagate \( \epsilon \) through \( !_{n} t \) and \( !_{n} s \), as we usually do in linear system. Instead, we have to amplify \( \epsilon \) by \( n \), this way obtaining the rule:

\[
\frac{t \mapsto s}{!_{n} t \mapsto !_{n} s}.
\]

All of that extends to reductions inside arbitrary combinator (context) \( C \). When reducing \( C[t] \) to \( C[s] \), we have to amplify the distance \( \epsilon \) between \( t \) and \( s \) according to \( \text{how (much)} \) \( C \) uses its argument, i.e. according to the sensitivity of \( C \) regarded as a term function. Writing \( \partial_{C} \) for such a sensitivity, we obtain the rule

\[
\frac{t \mapsto s}{C[t] \cong_{\partial_{C}(\epsilon)} C[s]}
\]

showing us that contexts now act not only on terms, but also on distances between them (or, taking a modal perspective \[40, 41\], contexts act also on possible worlds).

Before giving a complete definition of the system of graded combinators, there is one last point we need to clarify: how do we determine the sensitivity of a context? A natural solution often employed in the literature on modal and graded calculi is to rely on a type system tracking variable usage in terms. Following such a proposal, we would work with judgements of the form \( \phi_{1}, \ldots, \phi_{n} : t \) stating that \( x_{i} \) has sensitivity \( \phi_{i} \) in \( t \). Introducing type systems, however, would require unnecessary work in large measure independent of rewriting. We shall avoid that by recursively computing the grade of a variable (and even of a variable position) in a term \[33\]. Nonetheless, the reader may find useful to think in terms of type systems at first since trying to design typing rules helps to isolate the compositional properties a good notion of sensitivity should satisfy.

First, we need to be able to add and multiply sensitivity functions, so to model nested and parallel use of terms. For instance, if a variable \( x \) is used \( n \) times by \( t \) and \( m \) times by \( s \), it will be used \( n + m \) times by \( f(t,s) \), provided that \( f \) is non-expansive (e.g. \( x \) is used \( n + m \) times by \( B \cdot t \cdot s \cdot y \)). Similarly, if \( g \) is a function symbol with sensitivity \( j \) (e.g. take \( !_{j} \) as \( g \)), then \( x \) will be used \( jn \) times in \( g(t) \). Such operations are obviously available in the concrete example of graded combinators, where sensitivity is given by multiplication by a constant: in the general setting of quantale homomorphisms, we shall model multiplication as function composition and addition as pointwise tensor product. Secondly, we need to have distinguished functions modelling linear and zero sensitivity, i.e. neutral elements for multiplication and addition, respectively. With no surprise, those will be the identity and the constant \( k \) quantale homomorphisms\[44\].

Let us now formally introduce system \( \mathcal{W} \) of bounded combinators. The signature of the system is defined thus

\[
\Sigma_{\mathcal{W}} \doteq \{ B, C, K, \mathcal{W}_{n,m}, \Omega, \delta_{n,m}, \Gamma, !_{n}, \cdot : | n, m \in [0, \infty]\}.
\]

\[44\] Another approach is to model term sensitivity axiomatically \[62, 63, 64\] by means of suitable semi-ring like structures \( \mathcal{G} \) and then define \( \mathcal{G} \)-indexed relational extensions (i.e. corelators) to model the action of grades on \( \Omega \)-relations \[41, 44\].
$\Sigma_W$ contains basic combinators $B$, $C$, $K$, as well as the (family of) combinator(s) $W_{n,m}$ and (families of) combinators $f_{n}$, $D$, and $\delta_{n,m}$ manipulating the function symbol(s) $!_{n}$. In fact, in addition to the usual binary function symbol for application, we have a $[0,\infty]$-family of exponential modalities $!_{n}$ \footnote{63, 66}. Each function $!_{n}$ has sensitivity (constant multiplication by) $n$, which allows us to leave application non-expansive (meaning that it has sensitivity one on each argument). Notice that the signature $\Sigma_W$ specifies for each function symbol not only its arity, but also the sensitivity of all its arguments. We shall refer to such signatures as **graded signatures**. In general, we will employ the notation $f : (\phi_{1},\ldots,\phi_{n})$ to state that $f$ is an $n$-ary function symbol with sensitivity $\phi_{i}$ on its $i$th argument.

To define rewriting $\Omega$-relations for $W$, we first need to define the sensitivity of a variable in a term. Actually, we look at the sensitivity of a variable position in a term. We define the grade $\partial_{p}(t)$ of a variable position $p$ in $t$ (i.e. $t_{p}$ is a variable) as follows, where $X$ ranges over basic combinators:

\begin{align*}
\partial_{1}(x) & \triangleq 1 \\
\partial_{p}(X) & \triangleq 0 \\
\partial_{p}(t_{1} \cdot t_{2}) & \triangleq \partial_{p}(t_{i}) \\
\partial_{i,p}(!_{n}t) & \triangleq n \cdot \partial_{p}(t).
\end{align*}

We then define the grade of a variable $x$ in a term $t$ as $\partial_{x}(t) \triangleq \sum \{ \partial_{p}(t) \mid t_{p} = x \}$. For instance, the variable $x$ has sensitivity $9 = 3 + (3 \cdot 2)$ (regarded as the multiplication-by-three function) in $t \triangleq !_{3}(x \cdot !_{2}(I \cdot x))$, as it is under the scope both of $!_{3}$ and of $!_{2}$, and the latter, in turn, is itself under the scope of $!_{3}$ (and morally, we can think about a nested $!_{n}!_{m}$ as a unique $!_{nm}$). Indeed, the sensitivity of $x$ at position $1$ is $3$, whereas at position $1212$ is $6$.

**Notation 10.** Given a context $C$, we write $\partial_{C}$ for the sensitivity of the (unique occurrence of the) hole in $C$.

We now have all the ingredients to define system $W$, whose (ground) rewriting $\Omega$-relation $\rightarrow_{W}$ and its extension $\rightarrow_{W}$ are defined in \footnote{Figure 4} As usual, the $L$-relation $W$ is defined by $W(t,s) \triangleq \inf \{ \varepsilon \mid t \xrightarrow{\varepsilon} W s \}$.

### 8.1 Modal and Graded Rewriting: $(\Omega, \Phi)$-Systems

Now that the reader has familiarised with informal ideas behind modal and graded rewriting systems, we introduce such systems formally. To do so, we first recall the notion of a quantale homomorphism.
Figure 4: System $\mathcal{W}$ of graded combinators

**Definition 19.** Given quantales $\Omega = (\Omega, \leq, \otimes, k)$, $\Theta = (\Theta, \sqsubseteq, \circledast, j)$ a lax quantale homomorphism is a monotone map $h : \Omega \to \Theta$ such that

$$j \sqsubseteq h(k)$$

$$h(\varepsilon) \circledast h(\delta) \sqsubseteq h(\varepsilon \otimes \delta).$$

If we replace the above inequalities with full equalities and require $h$ to be continuous (i.e. $h(\bigsqcup_i \varepsilon_i) = \bigsqcup_i h(\varepsilon_i)$), then we say that $h$ is a quantale homomorphism.

From now on, we shall work with quantale homomorphisms on the same quantale $\Omega$. We denote such maps by $\circledast$, $\circledast$, ..., and refer to them as change of base (endo)functors (CBEs, for short) [82, 76].

**Remark 10.** CBEs have been successfully employed to study general program distances [61, 62] and modal coeffectful reasoning [40, 41]. In such setting, actually, one works with lax quantale homomorphisms rather than with full homomorphisms. Although most (but not all) the results presented in this section can be given in terms of lax homomorphisms, important theorems such as confluence of orthogonal systems seem to require full homomorphisms. For that reason, we shall directly work with the latter maps.

**Example 18.**

1. The main example of CBEs we consider is multiplication by a constant on the Lawvere quantale (and variations thereof). Given $\kappa \in \mathbb{R}_{\geq 0}$, we regard $\kappa$ as mapping $\varepsilon \in [0, \infty]$ to $\kappa \varepsilon \in [0, \infty]$. Notice that we do not allow multiplication by infinity. That is because, from a rewriting perspective, multiplying by $\infty$ is semantically meaningless. Nonetheless, we could even include multiplication by infinity in our analysis, provided that we restrict our definition to finitely continuous CBEs [61, 62] and that we carefully define multiplication between zero and infinity (as first observed by de Amorim et al. [45], algebra forces multiplication to become non-commutative, so that $0 \cdot \infty \neq \infty \cdot 0$). [45]

2. Recall that in section 4 we have introduced a relational box modality relying on the map $\psi : 2 \to \Omega$ and its right adjoint $\varphi : \Omega \to 2$. The map $\psi \circ \varphi$ is a CBE.

---

[45] Extended multiplication is defined thus, for $y \neq 0$: $x \cdot \infty \equiv \infty$, $\infty \cdot 0 \equiv 0$, $\infty \cdot y \equiv \infty$. 56
3. Other examples of CBEs, especially on quantales of modal predicates, can be found in the literature on relational reasoning about coeffects [41].

CBEs are closed under composition and the identity function $1 : \Omega \to \Omega$ is a CBE. Moreover, we can extend the order $\leq$ and the multiplication $\otimes$ of $\Omega$ to CBEs pointwise. Finally, we denote by $k^*$ the constant $k$ CBE.

**Remark 11.** Any CBE $\phi$ induces an action $[\phi]$ on $\Omega$-relations defined by $[\phi]R(a, b) = \phi(R(a, b))$. The map $[\phi]$ is an example of a correlator [41].

We now introduce a new class of rewriting systems, which we dub $(\Omega, \Phi)$-systems (or $\Phi$-systems for short). Let us fix a quantale $\Omega$ and a structure $\Phi = (\Phi, \leq, \circ, 1, k^*)$, where $\Phi$ is a set of CBEs containing the identity and constant $k$-functions, and closed under function composition and tensor.

**Definition 20.** 1. The modal arity of an $n$-ary function symbol $f$ is a tuple $(\phi_1, \ldots, \phi_n)$ with $\phi_i \in \Phi$. Given a function symbol $f$ with modal arity $(\phi_1, \ldots, \phi_n)$ (notation $f : (\phi_1, \ldots, \phi_n)$), we say that $f$ has sensitivity (or modal grade) $\phi_i$ on its $i$th argument.

2. A $\Phi$-graded signature is a set $\Sigma$ containing function symbols with their modal arity. Given a $\Phi$-graded signature $\Sigma$ and a set $X$ of variables, the collection of $\Sigma(X)$ is defined as usual.

3. Given a term $t$ and a position $p$ for a variable in $t$, we define the grade $\partial_p(t)$ of $p$ in $t$ as follows:

$$\partial_p(t) \equiv 1$$

$$\partial_p(f(t_1, \ldots, t_n)) \equiv \phi_i \circ \partial_p(t_i) \quad (f : (\phi_1, \ldots, \phi_n)) \in \Sigma.$$

Given a term $t$ and a variable $x$, we can compute the grade $\partial_x(t)$ of $x$ in $t$ by ‘summing’ the grades of all position $p$ such that $t_p = x$. Formally, $\partial_x(t) \equiv \bigotimes \{\partial_p(t) \mid t_p = x\}$, where $\bigotimes \emptyset \equiv k^*$. Notice that we can equivalently define $\partial_x(t)$ recursively as follows:

$$\partial_x(x) \equiv 1$$

$$\partial_x(y) \equiv k^*$$

$$\partial_x(f(t_1, \ldots, t_n)) \equiv \bigotimes \phi_i \circ \partial_x(t_i) \quad (f : (\phi_1, \ldots, \phi_n)) \in \Sigma.$$

We are now ready to define $(\Phi)$-graded term rewriting systems.

**Definition 21.** A $(\Omega, \Phi)$-term rewriting system $((\Omega, \Phi)$-TRS, for short) is a pair $\mathcal{R} = (\Sigma, \to_R)$ consisting of a $\Phi$-signature $\Sigma$ and a $\Omega$-ternary relation. The (rewriting) $\Omega$-ternary relation $\to_R$ generated by $\to_R$ is defined thus:

$$\varepsilon \vdash a \to_R b$$

$$\Delta C(\varepsilon) \vdash C[a^\varepsilon] \to_R C[b^\varepsilon]$$

We say that the system is balanced if for any rule $\varepsilon \vdash a \to_R b$ we have $\partial_x(a) = \partial_x(b)$, for any variable $x$. From now on, we assume all $(\Omega, \Phi)$-TRSs to be balanced.

Compared with the definition of (linear) $\Omega$-TRSs, **Definition 21** has two main differences: first, the definition of the full rewriting relation $\to$ now takes into account the grade of the context; secondly,
we omit all structural rules besides weakening. The omission of such rules (cf. Remark 7) allows us to strengthen our results: in particular, whereas a critical pair lemma can be given for \((\sqcap, \Phi)\)-TRSs extended with all structural rules, our proof of confluence of orthogonal \((\sqcap, \Phi)\)-TRSs seems not to scale to \((\sqcap, \Phi)\)-TRSs extended with the Archimedean (infinitary) rule.

**Notation 11.** We extend to \((\sqcap, \Phi)\)-TRSs all notational conventions introduced for \(\sqcap\)-TRSs.

Let us now have a closed look at the definition of \(\Rightarrow\). First, it is instructive to characterise it inductively as follows:

\[
\begin{align*}
\epsilon \vdash a \Rightarrow b \\
\epsilon \vdash t \Rightarrow s \\
\epsilon \vdash t \Rightarrow s' \\
\epsilon \vdash t \Rightarrow s \\
\phi_i(\epsilon) \vdash f(u_1, \ldots, t, \ldots, u_n) \Rightarrow f(u_1, \ldots, s, \ldots, u_n)
\end{align*}
\]

This characterisation clearly shows that performing reductions inside function symbols amplify distances, whereas applying substitutions does not. This is because in the definition of \(\Rightarrow\) we apply the same substitution on terms. Intuitively, this reflects the fact that passing identical (i.e. at a null distance) arguments to a Lipschitz continuous function produces identical results, and thus there is no distance amplification. Indeed, rephrasing Definition 21 to (balanced) equational theories, we obtain the following substitution rule

\[
\begin{align*}
\epsilon \vdash t \Rightarrow s \\
\delta \vdash u \Rightarrow \nu
\end{align*}
\]

When \(u\) and \(v\) coincide — so that \(\delta = k\) — we obtain \(\epsilon \otimes (\epsilon \otimes (t)(\delta)) \Rightarrow t[u/x] = \epsilon \otimes (\epsilon \otimes k) = \epsilon\), meaning that the distance \(\epsilon\) is non-expansively propagated.

As usual, any \((\sqcap, \Phi)\)-TRS \((\Sigma, \Rightarrow)\) induces a \(\sqcap\)-ARS whose objects are \(\Sigma\)-terms and whose rewriting \(\sqcap\)-relation is defined by

\[
R(t, s) \equiv \bigvee \{\epsilon \mid \epsilon \vdash t \Rightarrow s\}.
\]

**Example 19.** The main example of a \((\sqcap, \Phi)\)-TRS we consider is system \(W\), as introduced at the beginning of this section. As for system \(K\) of affine combinators, we can also consider extensions of \(W\). For instance, we can consider functions \(f : \mathbb{N}^m \rightarrow \mathbb{N}\) with Lipschitz constants \((\phi_1, \ldots, \phi_m)\) and add to \(\Sigma_W\) (properly extended with constants \(0, 1, \ldots\) for natural numbers) a combinator \(f\) for any such a function, together with rules

\[
\begin{align*}
f : \phi_1 \cdot \phi_2 \cdot \ldots \cdot \phi_m \Rightarrow \overline{f(n_1, \ldots, n_m)}.
\end{align*}
\]

Another interesting example of a \((\sqcap, \Phi)\)-TRS is obtained by grading the signature of system \(B\): any function symbol \(\ast_\epsilon\) now takes signature \((\epsilon, 1 - \epsilon)\), where by \(\epsilon\) (resp. \(1 - \epsilon\)) we mean multiplication by \(\epsilon\) (resp. \(1 - \epsilon\)). Notice that such a signature actually makes \(+_\epsilon\) a contraction [100]. Mardare et al. [84] have shown that the system obtained from such a signature together with the (equational) rules of idempotency, commutativity, and associativity provides a (quantitative) equational axiomatisation of the (finitary) Wasserstein-Kantorovich distance [110].

**Modal and Graded Equational Theories.** Before moving to the metatheory of \((\sqcap, \Phi)\)-TRSs, we extend quantitative equational theories to a graded setting [35].

**Definition 22.** A graded (quantitative) equational theory is a pair \(E = (\Sigma, \approx_E)\), where \(\Sigma\) is a \(\Phi\)-signature and \(\approx_E\) is a \(\sqcap\)-ternary relation over \(\Sigma\)-terms. The \(\sqcap\)-ternary (equality) relation \(\approx_E\) generated by \(\approx_E\) is

58
defined by the rules in Figure 5. We say that a graded equational theory is balanced if whenever \( \varepsilon \vdash t \approx_E s \), we have \( \partial_x(t) = \partial_x(s) \), for any variable \( x \). Notice that if equations in \( \approx_E \) are balanced, then so are equations in \( \approx \).

As for quantitative equational theories, we see that \( E \) is reflexive, symmetric, and transitive; and that by regarding any \( n \)-ary function symbol \( f \) as a function \( f : \Sigma(X)^n \rightarrow \Sigma(X) \), we have

\[
\phi_1|E(t_1, s_1) \otimes \cdots \otimes \phi_n|E(t_n, s_n) \leq E(f(t_1, \ldots, t_n), f(s_1, \ldots, s_n)),
\]

meaning that function symbols behave as (generalised) Lipschitz continuous functions. Moreover, it is an easy exercise to prove that for any balanced graded theory the substitution rule

\[
\varepsilon \vdash t =_E s \quad \delta \vdash u =_E v \quad \frac{\varepsilon \otimes \partial_x(t)(\delta) \vdash t[u/x] =_E s[v/x]}{\varepsilon \otimes \delta \vdash t =_E s}
\]

is valid, from which we obtain the following substitution inequality:

\[
E(t, s) \otimes [\partial_x(t)|E(u, v) \leq E(t[u/x], s[v/x]).
\]

Finally, at this point of the work it should be clear that the connection between quantitative equational theories and \( \Omega \)-TRSs extend mutatis mutandis to graded equational theories and \( (\Omega, \Phi) \)-TRSs, modulo the addition of structural rules in the definition of the latter.

### 8.2 Confluence and Critical Pairs, Part IV

We now progressively extend the theory of \( \Omega \)-TRSs to \( (\Omega, \Phi) \)-TRSs, beginning with Theorem 1. The notion of an overlap and of a critical pair straightforwardly extend to \( (\Omega, \Phi) \)-TRSs. Notice, however, that if \( a_1 \overset{\varepsilon_1}{\rightarrow} b_1 \) and \( a_2 \overset{\varepsilon_2}{\rightarrow} b_2 \) overlap at position \( p \), so that there is a substitution \( \sigma \) such that \( a_2^\sigma = C[a_1^\sigma] \) (with \( C = a_2^\sigma[-]_p \)), then the critical pick is

\[
b_2^\sigma \overset{\varepsilon_2}{\rightarrow} a_1^\sigma \overset{\partial_v(\varepsilon_2)}{\rightarrow} C[b_2^\sigma].
\]

We thus have all the ingredients to extend Lemma 3 to \( (\Omega, \Phi) \)-TRSs. Compared to its \( \Omega \)-TRS counterpart, however, the critical pair lemma for \( (\Omega, \Phi) \)-TRSs presents a major difference: we can relax the linearity assumption and require rewriting rules to be left-linear only.
**Lemma 9** (Critical Pair, Graded). Let $\mathcal{R} = (\Sigma, \rightarrow_R)$ be a left-linear (balanced) $(\Omega, \Phi)$-TRS. If $\mathcal{R}$ is locally confluent on all critical pairs of $\mathcal{R}$, then it is locally confluent.

**Remark 12.** Due to the absence of structural rules (besides weakening) in [Definition 21], Lemma 9 can be strengthened to prove local confluence of $\rightarrow_R$ given its local confluence on critical pairs of $\mathcal{R}$.

**Proof of Lemma 9** Following [Remark 12], we prove confluence of $\rightarrow$. The proof proceeds as for Lemma 4, the main difference being the case of nested, non-critical redexes. We analyse this case in detail, and then extend it to the full case of a general peak. Suppose to have rules $a_1 \overset{\xi_1}{\rightarrow} b_1$, $a_2 \overset{\xi_2}{\rightarrow} b_2$. Without loss of generality, we consider the case in which the second reduction happens inside (an instance) of the first one. So there is a variable $x$ in $a_1$ and a substitution instance such that $x^\sigma$ contains $a_2^\sigma$. Since $\mathcal{R}$ is left linear, we know that there is just one occurrence of $x$ in $a_1$. Say it is at position $p$, so that we have $a_1[x]_p$.

Say also that the relevant occurrence of $a_2^\sigma$ in $x^\sigma$ is at position $q$, so that $x^\sigma[a_2^\sigma]_q$ and, consequently, $a_2^\sigma[x^\sigma[a_2^\sigma]_q]$ and $a_2^\sigma[a_2^\sigma]_pq$. The rule $a_1 \overset{\xi_1}{\rightarrow} b_1$ may, in general, duplicate the single occurrence of $x$ in $a_1$. Say we have $b_1[x]_{p_1,...,p_n}$, meaning that $b_1$ has $n$ occurrences of $x$, each at position $p_i$. Therefore, reducing $a_1^\sigma$ gives $b_1^\sigma[x^\sigma]_{p_1,...,p_n}$, and thus $b_1^\sigma[a_2^\sigma]_{p_1q,...,p_nq}$. We can now reduce each of the $n$ occurrences of $a_2^\sigma$ in $b_1^\sigma$. The distance obtained for each reduction, however, is not $\epsilon_2$ but $\partial_{pq}(b_2^\sigma)(\epsilon_2)$. Putting things together, we obtain the following reduction diagram:

![Reduction Diagram](image)

To obtain local confluence, we claim

$$\epsilon_1 \otimes \partial_{pq}(a_1^\sigma)(\epsilon_2) \leq \bigotimes_i \partial_{pq}(b_1^\sigma)(\epsilon_2) \otimes \epsilon_1.$$ 

Since the system is (left-linear and) balanced, we have

$$\partial_p(a_1) = \partial_\sigma(a_1) = \partial_\sigma(b_1) = \bigotimes_i \partial_{p_i}(b_1).$$

Moreover, we notice that

$$\bigotimes_i \partial_{pq}(a_1^\sigma) = \partial_p(a_1) \circ \partial_\sigma(x^\sigma)$$

$$\bigotimes_i \partial_{pq}(b_1^\sigma) = \bigotimes_i \partial_{p_i}(b_1) \circ \partial_\sigma(x^\sigma)$$

Writing $\delta$ for $\partial_\sigma(x^\sigma)$, we obtain the desired inequality as follows:

$$\epsilon_1 \otimes \partial_{pq}(a_1^\sigma)(\epsilon_2) = \partial_p(a_1)(\delta) \otimes \epsilon_1 = \bigotimes_i \partial_{p_i}(b_1)(\delta) \otimes \epsilon_1 = \bigotimes_i \partial_{pq}(b_1^\sigma)(\epsilon_2) \otimes \epsilon_1.$$
This shows how to deal with nested non-critical redexes in isolation. In the general case, we have to consider all of that happening inside a larger term \( t \). This means nothing by considering cases of the form \( C[a_1^1][a_2^2]_{pq} \). The proof proceeds exactly as in the isolated case, with the main difference that distances should now be scaled by \( \partial_C \). But, due to the structural properties of CBEs, this creates no problem at all.

**Theorem 3.** Any left-linear and terminating (balanced) \((\Omega, \Phi)\)-TRS locally confluent on its critical pairs is confluent.

**Proof.** It directly follows from Proposition 7 and Lemma 9.

### 8.3 Orthogonality

Even if useful on many \((\Omega, \Phi)\)-TRSs, Theorem 3 can only be used to infer local confluence of non-terminating systems, system \( \mathcal{W} \) being a prime example of such a system. Yet, examples seem to suggest \( \mathcal{W} \) to be indeed a confluent system. To make the latter intuition into a proved mathematical result, we notice that the hypotheses of Theorem 3 are trivially satisfied in the case of system \( \mathcal{W} \), as the latter simply has no critical pair. Taking advantage of this observation, we now generalise the well-known result [99] that orthogonality implies confluence to a quantitative and graded setting.

**Definition 23.** A \((\Omega, \Phi)\)-TRS is orthogonal if it is left-linear and has no critical pair.

As already remarked, our prime example of an orthogonal \((\Omega, \Phi)\)-TRS is system \( \mathcal{W} \) of graded combinators. To prove confluence of orthogonal systems we employ Tait and Martin-Löf technique [17], properly instantiated to our rewriting setting (see also Aczel’s technique [5]). We extend \( \rightarrow \) to a ternary relation \( \Rightarrow \) allowing us to perform arbitrary (even nested) reductions in a term at once.

**Definition 24.** Given a \((\Omega, \Phi)\)-TRS \( \mathcal{R} = (\Sigma, \rightarrow_R) \), we inductively define the multi-step reduction \( \Rightarrow \) by the rules in Figure 6. We define the \( \Omega \)-relation \( \odot \) by

\[
\odot(t, s) \triangleq \bigvee \{ \varepsilon \mid \varepsilon \vdash t \rightarrow_R s \}.
\]

**Notation 12.** We extend the usual notational convention to \( \Rightarrow \). Moreover, in what follows we often employ the vector notation \( \bar{\varphi} \) for finite sequences \( \varphi_1, \ldots, \varphi_n \) of symbols.

We immediately notice that since \( \Rightarrow \) allows us to reduce several redexes in a term simultaneously, it gives a substitution property similar to the one of graded quantitative equational theory.

**Lemma 10** (Substitution Lemma). The following inference is valid

\[
\varepsilon \vdash t \rightarrow_R s \quad \delta_1 \vdash v_1 \rightarrow_R w_1 \quad \cdots \quad \delta_n \vdash v_n \rightarrow_R w_n
\]

\[
\varepsilon \otimes \bigotimes_i \delta_i(t)(\delta_i) \vdash t[\bar{v}/\bar{x}] \rightarrow_R s[\bar{w}/\bar{x}]
\]

Consequently, we also obtain the following substitution inequality:

\[
\odot(t, s) \otimes \bigotimes_i [\partial_{\delta_i}(t)] \odot(v_i, w_i) \leq \odot(t[\bar{v}/\bar{x}], s[\bar{w}/\bar{x}]).
\]
Proof sketch. By induction on the definition of $\rightsquigarrow_R$ and following the pattern of graded and quantitative substitution lemmas [61, 62, 41].

Given a $(\Omega, \Phi)$-TRS $R = (\Sigma, \rightsquigarrow_R)$, we are going to prove confluence of $R$ by actually proving a stronger result, namely confluence of $\rightsquigarrow_R$. This is possible thanks to the absence of (infinitary) structural rules in the definition of a $(\Omega, \Phi)$-TRS [Definition 21]. To achieve such a result, we shall prove that $\rightsquigarrow_R$ has the diamond property. Since $\rightsquigarrow_R \subseteq \leftarrow \rightarrow \subseteq \rightarrow^*$ (and thus $R \leq \rightarrow^*$), confluence of $\rightarrow_R$ follows. Before proving the diamond property for $\rightarrow_R$, let us remark a useful property of orthogonal systems, namely that if we have a (necessarily unique) rule $a \mathrel{\rightsquigarrow} b$ and we reduce a term of the form $a[\bar{u}/\bar{x}]$, then either we reduce the (instance) of redex $a$, or the term obtained is itself an instance of the redex $a$, i.e. it is of the form $a[\bar{w}/\bar{x}]$, for some terms $\bar{w}$.

Remark 13. Given an orthogonal $(\Omega, \Phi)$-TRS, suppose to have a reduction $a[\bar{u}/\bar{x}] \mathrel{\vec{\epsilon}} t$ not an instance of weakening. Then, either:

1. $t = b[\bar{w}/\bar{x}]$ with $v_i \mathrel{\delta_i} w_i$, for any $i$; $a \mathrel{\eta} b$; and $\epsilon = \eta \otimes \bigotimes_i \partial x_i(a)(\delta_i)$. Or

2. $t = a[\bar{w}/\bar{x}]$ with $v_i \mathrel{\delta_i} w_i$, for any $i$, and $\epsilon = \bigotimes_i \partial x_i(a)(\delta_i)$.

Proposition 10. Let $R = (\Sigma, \rightsquigarrow_R)$ be an orthogonal $(\Omega, \Phi)$-TRS. Then, the relation $\rightarrow_R$ has the diamond property. That is, if $s_1 \mathrel{\epsilon_1} t \mathrel{\epsilon_2} s_2$, there exists a term $s$ such that $s_1 \mathrel{\delta_1} s \mathrel{\delta_2} s_2$ and $\epsilon_1 \otimes \epsilon_2 \leq \delta_1 \otimes \delta_2$.

Proof. The proof is by induction on $t$ with a case analysis on the defining clauses of $\rightarrow_R$. The interesting case is for $t = a[\bar{u}/\bar{x}]$. By previous remark, there are two possibilities for the reduction $a[\bar{u}/\bar{x}] \mathrel{\vec{\epsilon}} s_1$ (the case for weakening is straightforward).

1. $s_1 = b[\bar{w}/\bar{x}]$ with $v_i \mathrel{\delta_i} w_i$, $a \mathrel{\eta} b$, and $\epsilon_1 = \epsilon \otimes \bigotimes_i \partial x_i(a)(\delta_i)$. Since $R$ is orthogonal, the rule $a \mathrel{\eta} b$ is unique and $s_2$ must be of the form $a[\bar{u}/\bar{x}]$ with $v_i \mathrel{\eta_i} u_i$ and $\epsilon_2 = \bigotimes_i \partial x_i(a)(\eta_i)$ (otherwise, we
would have a critical pair). That is, we have the peak:

\[
\begin{array}{ccc}
\varepsilon \otimes \bigotimes_i \partial_{x_i}(a)(\delta_i) & \varepsilon \otimes \bigotimes_i \partial_{x_i}(a)(\eta_i) \\
\downarrow & \downarrow \\
b[\bar{w}/\bar{x}] & a[\bar{u}/\bar{x}]
\end{array}
\]

By induction hypothesis on each \(v_i\), we close the diagram

\[
\begin{array}{ccc}
\delta_i & \varepsilon \otimes \bigotimes_i \partial_{x_i}(a)(\delta_i) & \varepsilon \otimes \bigotimes_i \partial_{x_i}(a)(\eta_i) \\
\downarrow & \downarrow & \downarrow \\
v_i & b[\bar{w}/\bar{x}] & a[\bar{u}/\bar{x}] \\
\downarrow & \downarrow & \downarrow \\
\eta_i & b[\bar{z}/\bar{x}] & a[\bar{v}/\bar{x}]
\end{array}
\]

so that we obtain

\[
\begin{array}{ccc}
\varepsilon \otimes \bigotimes_i \partial_{x_i}(a)(\delta_i) & \varepsilon \otimes \bigotimes_i \partial_{x_i}(a)(\eta_i) \\
\downarrow & \downarrow \\
b[\bar{w}/\bar{x}] & a[\bar{u}/\bar{x}]
\end{array}
\]

Since the system is balanced, \(\delta_i \otimes \eta_i \leq \hat{\eta}_i \otimes \hat{\delta}_i\), infer

\[
\bigotimes_i \partial_{x_i}(a)(\delta_i) \otimes \bigotimes_i \partial_{x_i}(a)(\delta_i)(\eta_i) \leq \bigotimes_i \partial_{x_i}(b)(\eta_i) \otimes \bigotimes_i \partial_{x_i}(b)(\hat{\delta}_i)
\]

and thus

\[
\varepsilon \otimes \bigotimes_i \partial_{x_i}(a)(\delta_i) \otimes \bigotimes_i \partial_{x_i}(a)(\delta_i)(\eta_i) \leq \bigotimes_i \partial_{x_i}(b)(\eta_i) \otimes \varepsilon \otimes \bigotimes_i \partial_{x_i}(b)(\hat{\delta}_i).
\]

2. \(s_1 = a[\bar{w}/\bar{x}]\) with \(v_i \xrightarrow{\delta_i} w_i\) and \(\varepsilon_1 = \bigotimes_i \partial_{x_i}(a)(\delta_i)\). Now, if \(t \xrightarrow{\varepsilon_2} s_2\) is an instance of a rule \(a \xrightarrow{\varepsilon} b\) (i.e. the ‘base’ case in the definition of \(\xrightarrow{\varepsilon}\)), then we proceed as in the previous case. Otherwise, we must have \(s_2 = a[\bar{u}/\bar{x}]\) with \(v_i \xrightarrow{\eta_i} u_i\) and \(\varepsilon_2 = \bigotimes_i \partial_{x_i}(a)(\eta_i)\). That is, we have the peak:

\[
\begin{array}{ccc}
\varepsilon \otimes \bigotimes_i \partial_{x_i}(a)(\delta_i) & \varepsilon \otimes \bigotimes_i \partial_{x_i}(a)(\eta_i) \\
\downarrow & \downarrow \\
a[\bar{u}/\bar{x}] & a[\bar{v}/\bar{x}]
\end{array}
\]

We proceed applying the induction hypothesis as in previous point and close the diagram as follows,
relying on the substitution lemma:

\[
\begin{array}{c}
\otimes_i \partial_{x_i}(a)(\delta_i) \\
\otimes_i \partial_{x_i}(a)(\eta_i) \\
\end{array}
\begin{array}{c}
a[\bar{\alpha}/\bar{x}] \\
[\bar{w}/\bar{x}] \\
[\bar{\beta}/\bar{x}] \\
\otimes_i \partial_{x_i}(a)(\eta_i) \\
\otimes_i \partial_{x_i}(a)(\delta_i) \\
\end{array}
\begin{array}{c}
a[\bar{\alpha}/\bar{x}] \\
[\bar{u}/\bar{x}] \\
[\bar{\eta}/\bar{x}] \\
\otimes_i \partial_{x_i}(a)(\eta_i) \\
\otimes_i \partial_{x_i}(a)(\delta_i) \\
\end{array}
\]

Indeed, \( \delta_i \otimes \eta_i \leq \hat{\eta}_i \otimes \hat{\delta}_i \) implies

\[\bigotimes_i \partial_{x_i}(a)(\delta_i) \otimes \bigotimes_i \partial_{x_i}(a)(\eta_i) \leq \bigotimes_i \partial_{x_i}(a)(\hat{\eta}_i) \otimes \bigotimes_i \partial_{x_i}(a)(\hat{\delta}_i).\]

\[\square\]

**Corollary 2.** Let \( \mathcal{R} = (\Sigma, \rightarrow_R) \) be an orthogonal \((\Omega, \Phi)\)-TRS. Then \( \hat{R} \) has the diamond property.

We can finally prove confluence of orthogonal systems.

**Theorem 4.** Let \( \mathcal{R} = (\Sigma, \rightarrow_R) \) be an orthogonal \((\Omega, \Phi)\)-TRS. Then, \( R \) is confluent.

**Proof.** Let \( S \equiv R^- \). We prove \( S^*; R^* \leq R^*; S^* \). By adjunction, it is sufficient to prove \( S^* \leq (R^*; S^*)/R^* \).

We proceed by fixed point induction, proving \( \Delta \leq (R^*; S^*)/R^* \) and \( S; (R^*; S^*)/R^* \leq (R^*; S^*)/R^* \). The former is straightforward, whereas for the latter it is sufficient to prove \( S; (R^*; S^*)/R^* \leq R^*; S^* \), i.e. \( S; R^*; S^* \leq R^*; S^* \).

Since \( S \leq \hat{S} \), it is enough to show \( \hat{S}; R^*; S^* \leq R^*; S^* \) and thus \( R^* \leq \hat{S} \backslash (R^*; S^*)/S^* \). We do a second fixed point induction, hence proving \( \Delta \leq \hat{S} \backslash (R^*; S^*)/S^* \) and \( \hat{S}; \hat{S} \backslash (R^*; S^*)/S^* \leq \hat{S} \backslash (R^*; S^*)/S^* \). The former obviously holds since \( \hat{S} \leq S^* \), whereas for the latter we first use adjunction and reduce the proof obligation to

\[\hat{S}; \hat{R}; \hat{S} \backslash (R^*; S^*)/S^* \leq R^*; S^*,\]

i.e. \( \hat{S}; \hat{R}; \hat{S} \backslash (R^*; S^*) \leq R^*; S^* \). Since \( R \leq \hat{R} \), it is sufficient to prove \( \hat{S}; \hat{R}; \hat{S} \backslash (R^*; S^*) \leq R^*; S^* \). Now, by **Corollary 2** we have \( \hat{S}; \hat{R} \leq \hat{R}; \hat{S} \), and thus:

\[\hat{S}; \hat{R}; \hat{S} \backslash (R^*; S^*) \leq \hat{R}; \hat{S} \backslash (R^*; S^*) \leq R^*; S^* \leq R^*; S^* \leq R^*; S^* \leq \hat{R}; \hat{S} \backslash (R^*; S^*) / S^* \leq R^*; S^* \backslash (R^*; S^*) / S^* \].

\[\square\]

**Remark 14.** By replacing \( R \) with \( \rightarrow_R \) (and thus replacing algebraic operations on \( \Omega \)-relations with their ternary relation counterparts) in the proof of **Theorem 4**, we obtain confluence of \( \rightarrow_R \).

We conclude this section by observing that system \( \mathcal{W} \) is orthogonal, and thus by **Theorem 4** it is confluent.

**Theorem 5.** System \( \mathcal{W} \) of graded combinatory logic is confluent.

To the best of the authors' knowledge, this is the first confluence result for a system of graded combinators endowed with a quantitative and modal operational (reduction) semantics. Such a result can seen as a first step towards a foundational study of operational properties of graded and coeffectful calculi.
9 CONCLUSION, RELATED, AND FUTURE WORK

In this paper, we have started the development of a systematic theory of metric and quantitative rewriting systems. The abstract nature of the notion of a distance employed makes our framework robust and allows for several conceptual interpretations of our rewriting systems. The latter, in fact, can be thought not only as metric and quantitative systems, but also as substructural (e.g. fuzzy or monoidal) and modal or coeffectful systems, this way suggesting possible applications of our theory to the development of quantitative and modal operational semantics of coeffectful programming languages. We have shortly hinted at that at the end of previous section (the authors are currently working on applications of quantitative rewriting to study operational properties of foundational graded $\lambda$-calculus).

We have focused on fundamental definitions and confluence properties of abstract systems, as well as linear and graded term rewriting systems. Developing a general theory of quantitative rewriting systems is an ambitious project that cannot be exhausted in a single paper. Among the many possible extensions of the theory presented in this paper, we mention the development of a theory of reduction strategies and their application to metric word problems; the design of completion algorithms for quantitative term rewriting systems (both linear and graded), as well as their quantitative correctness with respect to families of metric word problems; and the study of inductive and termination properties of quantitative systems. The latter, in particular, seem to suggest that new rewriting properties can be discovered by pushing the quantitative enrichment one step forward, this way making the notions of termination, induction, confluence, etc quantitative themselves. Finally, we plan to investigate applications of quantitative systems in the spirit of those outlined in section 2.

RELATED WORK  To the best of the authors’ knowledge, this is the first systematic analysis of quantitative and metric rewriting systems. This, of course, does not mean that isolated forms of quantitative rewriting have not been proposed in the literature. For instance, specific forms of weighted reductions have been employed [91, 93] in the study of cost analysis of rewriting systems. Measured abstract rewriting systems, i.e. abstract rewriting systems with a reduction relation enriched in a monoid, have been introduced by van Oostrom and Toyama [114] to study normalisation properties by random descent. In that context, a quantitative notion of confluence is introduced which, however, differs from ours in the way it compares distances between objects. In fact, given a peak $b_1 \xrightarrow{\varepsilon_1} a \xrightarrow{\varepsilon_2} b_2$ and a valley $b_1 \xleftarrow{\delta_1} b \xleftarrow{\delta_2} b_2$, it is required $\varepsilon_1 \otimes \delta_1 \leq \varepsilon_2 \otimes \delta_2$. Even if this requirement has a natural reading when it comes to study normalisation properties of rewriting, it does not fit the algebra of quantitative relations and seems ineffective when applied to the study of distances. We also remark that measured rewriting systems have been studied in the context of abstract rewriting only, whereas our theory of quantitative rewriting systems covers both abstract and (graded) term-based systems.

At the time of writing, the authors have discovered that abstract fuzzy rewriting systems have been studied by Belohlávek et al. [22, 21] relying on the theory of fuzzy relations [19]. Even if the aforementioned theory of fuzzy rewriting systems does not cover term-based systems (neither non-expansive nor graded), the development of fuzzy abstract rewriting systems is in line with our section 4. In particular, Belohlávek et al. [22, 21] define fuzzy notions of confluence and prove a quantitative Newman’s lemma similar to (the pointwise version of) ours. In fact, instantiating the theory of section 4 to Fuzzy quantales, we obtain an extension of the theory of abstract Fuzzy rewriting systems. Remarkably, our
pointwise analysis of quantitative Newman’s Lemma is close to the one by Belohlávek et al. [22]. Besides
the absence of a theory term-based systems, major differences between our work and the one on Fuzzy
rewriting can be found even at the level of abstract systems. First, as already remarked, our approach is
more general and subsumes (and extends) Fuzzy rewriting. Moreover, our theory of abstract \( \Omega \)-systems
is largely pointfree and builds upon general relational techniques nontrivially extending the relational
theory of abstract rewriting by Doornbos et al. [50], as well as other pointfree theories of rewriting
systems [108, 48]. In addition to all of that, we mention that, curiously, Fuzzy rewriting systems have
not been applied to metric reasoning. This is an interesting observation, as it turns out that general
quantitative Fuzzy equational and algebraic theories [20] have been developed before the quantitative
algebraic theories by Mardare et al. [84]. However, even if mathematically sophisticated, such fuzzy
theories have not been applied (to the best of the authors’ knowledge) neither to metric reasoning nor
to the semantics of programming languages.

Contrary to the case of rewriting systems, general theories of quantitative equational reasoning have
been developed. In addition to the aforementioned Fuzzy equational theories, we mention the rich
research line on quantitative algebras and equational theories [84, 83, 10, 86, 87, 12, 90]. With the
exception of the recent work by Dagnino and Pasquali [35], such theories are usually not graded and, to
the best of the authors’ knowledge, are not capable to describe non-linear systems, such as system \( \mathcal{W} \) of
graded combinators. From that point of view, our definition of a graded quantitative equational theory
can be seen as a first extension of quantitative equational theories to graded systems.

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A  Critical Pairs of \( \mathcal{B} \)

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\begin{align*}
& x + 1 \ y \\
& \downarrow \quad \downarrow \\
& x \quad x
\end{align*}
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\begin{align*}
& y + 0 \ x \\
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\begin{align*}
& (\sigma : x \mapsto y + 0 \ x) \\
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& y + 0 \ x
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\begin{align*}
& x + 1 \ y \\
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& x \quad z + 1 \ y
\end{align*}
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\begin{align*}
& y + 1 \ x \\
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& y + 1 \ x
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& (\sigma : x \mapsto y + 1 \ x) \\
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\begin{align*}
& (x + e_1 \ y) + e_2 \ z \\
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& (x + e_1 \ y) + e_2 \ z
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& x + e_1 e_2 (y + \frac{x + e_1}{x + e_2} \ z) \\
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