The Noether charge entropy in anti-deSitter space and its field theory dual

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Abstract

We express the Noether charge entropy density of a black brane in anti-deSitter space in terms of local operators in the anti-deSitter space bulk. We find that Wald’s expression for the Noether charge entropy needs to be modified away from the horizon by an additional term that vanishes on the horizon. We then determine the field theory dual of the Noether charge entropy for theories that asymptote to Einstein theory. We do so by calculating the value of the entropy density at the anti-deSitter space boundary and applying the standard rules of the AdS/CFT correspondence. We interpret the variation of the entropy density operator from the horizon to the boundary as due to the renormalization of the effective gravitational couplings as they flow from the ultra-violet to the infra-red. We discuss the cases of Einstein-Hilbert theory and $f(R)$ theories in detail and make general comments about more complicated cases.
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1 Introduction

The Noether charge entropy (NCE) of black holes has been proposed by Wald ([1, 2]) and was
expressed as

$$S_W = -2\pi \oint_{\Sigma} \left( \frac{\delta \mathcal{L}}{\delta R_{abcd}} \right)^{(0)} \hat{\epsilon}_{ab} \epsilon_{cd}$$

in [3]. The NCE will be discussed in detail in Section 2.

It was shown in [4] that the kinetic terms for metric perturbations $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ for the
general action ([8]) are given by

$$\delta I^{(2)} = \int d^{d+1}x \sqrt{-\bar{g}} \left( \frac{\delta \mathcal{L}}{\delta R_{\rho\mu\lambda\nu}} \right)^{(0)} \left( \nabla_{\delta} h_{\lambda\mu} \nabla^{\delta} h_{\nu\rho} + 2\nabla^{\delta} h_{\lambda\rho} \nabla_{\mu} h_{\nu\delta} \right).$$
The background covariant derivative is denoted by $\overline{\nabla}$ and the superscript $(0)$ indicates that the partial derivative $\left(\frac{\delta L}{\delta R_{abcd}}\right)^{(0)}$ is evaluated on the solution of the equations of motion. In this expansion we keep only terms that contain two factors of the metric perturbation and two background covariant derivatives. The coefficient tensor $\left(\frac{\delta L}{\delta R_{abcd}}\right)^{(0)}$ determines the various effective gravitational coupling constants for the different polarizations. For example, the effective coupling relevant to the entropy is defined as

$$\frac{1}{(\kappa_{eff})^2} = -\frac{1}{4} \left(\frac{\delta L}{\delta R_{abcd}}\right)^{(0)} \hat{\epsilon}_{ab} \hat{\epsilon}_{cd},$$

so that Wald’s entropy (1) can be written as

$$S_W = \frac{1}{4} \oint_{\Sigma} \frac{8\pi}{(\kappa_{eff})^2} dA,$$

$dA$ being the surface element. The binormal vectors $\hat{\epsilon}_{ab}$, in this case, pick a specific polarization of the metric fluctuations that corresponds to fluctuations of the area of the bifurcation surface $\Sigma$.

The effective gravitational coupling constant $\kappa_{eff}$ appears to have two roles – it determines the coupling constant for a specific polarization and it also determines the entropy density per unit area on the bifurcate surface $\Sigma$. Considering $\kappa_{eff}$ as a coupling constant leads us to look at the coefficients tensor $\left(\frac{\delta L}{\delta R_{abcd}}\right)^{(0)}$ away from the horizon.

The gravitational coupling constant gives the bulk entropy density evaluated on the horizon. Our interpretation of the entropy density as a coupling constant highlights the fact that, from this point of view, the entropy is given in terms of a local quantity in the bulk. Yet, there is another description of the entropy density in the context of AdS/CFT as the entropy density of the field theory on the boundary. According to the AdS/CFT correspondence the two descriptions should agree. We propose that the coupling constant should be promoted to a field in the bulk by allowing it to vary in the radial direction. Doing so will allow us to use the standard tools of the AdS/CFT correspondence in order to relate it to a dual operator in the field theory. We will then be able to relate the entropy of the black brane to the entropy of the boundary field theory. In addition we would like to look at the renormalization group (RG) flow of the effective coupling constant from the horizon (UV) to the boundary (IR) in asymptotically AdS spacetimes.
In this paper we consider the following ansatz for the metric of a d+1-dimensional black brane in AdS
\[ ds^2 = -g_{tt}dt^2 + g_{rr}dr^2 + g_{xx}dx^i dx_i. \] (5)

The black brane event horizon is at \( r = r_h \), where \( g_{tt} \) has a first order zero and \( g_{rr} \) has a first order pole. We assume that all other metric components are finite at the horizon. All the metric components are taken to depend only on \( r \) and therefore the metric is Poincaré invariant in the \((t, x_i)\) subspace. We assume that the AdS boundary is at \( r = \infty \). We also assume that asymptotic form of the metric approaches the AdS metric. Any asymptotically AdS metric can be brought to the Fefferman-Graham form near the boundary \([5]\)
\[ ds^2 = l^2 \left( \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{ij}(x, \rho) dx^i dx^j \right), \] (6)

where
\[ g(x, \rho) = g(0) + \ldots + \rho^d g(d) + h(d) \rho^d \log \rho + \ldots, \] (7)

\( l \) is related to the cosmological constant as \( \Lambda = -\frac{d(d-1)}{2l^2} \) and the boundary is at \( \rho = 0 \) (The coefficient \( h(d) \) is present only when \( d \) is even).

We consider a general theory of gravity whose action depends on the metric \( g_{\mu \nu} \), the curvature (through the Riemann tensor) and on matter fields \( \phi \) and their covariant derivatives
\[ I = \int d^{d+1}x \sqrt{-g} \mathcal{L} \left( R_{\rho \mu \lambda \nu}, g_{\mu \nu}, \nabla_\sigma R_{\rho \mu \lambda \nu}, \phi, \nabla \phi, \ldots \right). \] (8)

We assume that the Lagrangian \([8]\) has stationary black brane solutions of the form \([5]\) with a bifurcate Killing horizon. In this paper we will consider only higher-derivative actions such that the effective gravitational coupling at the boundary approaches asymptotically the Newton’s constant of the Einstein-Hilbert Lagrangian. We will discuss this requirement in section \([4]\).

2 The local form of the Noether charge entropy density

The NCE is given by
\[ S_W = -2\pi \oint_\Sigma \left( \frac{\delta \mathcal{L}}{\delta R_{abcd}} \right)^{(0)} \hat{e}_{ab} e_{cd}. \] (9)
The variation of the Lagrangian with respect to $R_{abcd}$ is performed as if $R_{abcd}$ and the metric $g_{mn}$ are independent.

The surface element $(d-1)$-form $\epsilon_{cd}$ is defined on the space-like bifurcation surface $\Sigma$. The hatted variable $\hat{\epsilon}_{ab}$ is the binormal vector to the bifurcation surface defined as $\hat{\epsilon}_{ab} = \nabla_a \tilde{\chi}_b$, the binormal is antisymmetric under the exchange $a \leftrightarrow b$. The vector $\tilde{\chi}^b$ is the normalized Killing vector which generates the Killing horizon. The Killing vector $\chi^b$ satisfies the Killing equation

$$\nabla_a \chi_b \nabla^a \chi^b = -2\kappa^2,$$

(10) $\kappa$ being is the surface gravity. The normalized Killing vector is defined so that $\chi^b = \kappa \tilde{\chi}^b$. Due to this normalization the entropy is computed in units such that the brane temperature is equal to $\frac{1}{2\pi}$. If this normalization is not enforced then the entropy formula reads

$$S_W = -\frac{1}{T} \int_\Sigma \left( \frac{\delta L}{\delta R_{abcd}} \right)^{(0)} \nabla_a \chi_b \epsilon_{cd}.$$

(11)

Wald’s entropy for a black brane solution of the form (5) is proportional to $A_h$, the area of the horizon at $r = r_h$,

$$S_W = -8\pi A_h \left. \frac{\partial L}{\partial R_{r t r t}} \right|_{r=r_h}.$$

(12)

We can regard $\Theta^t_t(r) \equiv -2 \frac{\partial L}{\partial R_{r t}}$ as a function of the radial coordinate $r$ whose value at the horizon determines the entropy density. Let us define another quantity $\Theta^x_x(r)$,

$$\Theta^t_t(r) \equiv -2 \frac{\partial L}{\partial R^t_{r t}},$$

(13)

$$\Theta^x_x(r) \equiv 2 \frac{\partial L}{\partial R_{r x}}.$$

(14)

The motivations for these definitions will be revealed and explained later.

With the newly defined operators we may define quantities that will be closely related to thermodynamic quantities in the field theory. First we define

$$\sigma_t \equiv 2 \Theta^t_t(r)(g_{xx})^{\frac{1}{2}}.$$

(15)

The entropy per unit area on the horizon is then related to $\sigma_t$

$$sT = \sigma_t|_{r=r_h}.$$

(16)
Additionally, the metric is translation invariant in the non-radial spatial directions. Hence, using the Killing vectors in these directions one can define an analogous quantity associated with $\Theta^x_x(r)$,

$$\sigma_x \equiv 2 \Theta^x_x(r) (g_{xx})^{\frac{d-2}{2}} \sqrt{-g_{tt}}. \tag{17}$$

In Eq. (17) we have used the binormal $\hat{e}_{rx}$ with respect to the radial direction and one of the spatial orthogonal directions. Clearly, $\sigma_x$ vanishes at the horizon due to the vanishing of $g_{tt}$ there. Nevertheless, we will show that $\sigma_x$ has a definite meaning from the field theory point of view. The vanishing of $\sigma_x$ at the horizon can be viewed as due to being infinitely red-shifted there. As a precursor to the later discussion let us notice that on the horizon $\sigma_t + \sigma_x = sT$. For later reference we will also define a linear combination of $\Theta^t_t$ and $\Theta^x_x$,

$$S \equiv -\frac{1}{2} (\Theta^t_t + \Theta^x_x). \tag{18}$$

As stated previously, we will consider only higher derivative actions with asymptotically AdS solutions, for which the effective gravitational couplings approach their values in the Einstein-Hilbert Lagrangian, and its definition will be given in section 4. In particular,

$$\lim_{r \to \infty} \Theta^t_t(r) = \frac{1}{16 \pi G_{d+1}} \),$$

$$\lim_{r \to \infty} \Theta^x_x(r) = \frac{1}{16 \pi G_{d+1}} \). \tag{19}$$

Here $G_{d+1}$ is the $d + 1$-dimensional Newton’s constant.

3 The Noether charge entropy density in the bulk and on the boundary

3.1 Conjugate variables and Hamiltonian holography

We would like to find the expectation value of the operator dual to $S$ defined in Eq. (18). For this purpose let us recall some general principles of the AdS/CFT correspondence.

Let us look at a general action in the bulk of the form

$$S = \int_{r_h}^{\infty} dr \int dt \int d^{d-1}x \mathcal{L}(\Phi_A(r, x)) \tag{20}$$

6
where $\Phi_A(r, x)$ represents collectively all the fields of the theory.

According to the AdS/CFT correspondence and the prescription for holographic renormalization, the expectation value of the operator $O_A(x^i)$ on the boundary that is dual to the field $\Phi_A(r, x^i)$ in the bulk is given by

$$\langle O_A(x^i) \rangle \equiv \frac{1}{\sqrt{-h}} \frac{\delta S_{\text{on-shell, ren}}}{\delta \Phi^{(0)}_A}. \quad (21)$$

$S_{\text{on-shell, ren}}$ is the renormalized on-shell action, $\Phi^{(0)}_A(x^i)$ is the value of the bulk field on the boundary (which is also the source for the generating function of the field theory) and $\sqrt{-h}$ is the determinant of boundary metric.

Let us consider a variation of the bulk action

$$\delta S = \left. \frac{\partial L}{\partial \dot{\Phi}_A} \delta \Phi_A \right|_{r_h} + \int_{r_h}^{\infty} dr \left. \frac{\partial L}{\partial \dot{\Phi}_A} \delta \Phi_A \right|_{r_h} - \frac{\partial}{\partial r} \left[ \frac{\partial L}{\partial \Phi_A} \right] \left. \delta \Phi_A \right|_{r_h} \quad (22)$$

(a dot denotes the differentiation with respect to the radial coordinate). Recall that the AdS boundary is at $r \to \infty$ and the horizon is at $r = r_h$.

Introducing the conjugate momentum in the bulk

$$\Pi_{\Phi_A}(r, x^i) \equiv \left. \frac{\partial L}{\partial \dot{\Phi}_A(r, x^i)} \right|_{r_h} \quad (23)$$

we can express the variation of the on-shell action as

$$\delta S_{\text{on-shell}} = \lim_{r \to \infty} \Pi_{\Phi_A}(r, x^i) \delta \Phi_A^{(0)} - \lim_{r \to r_h} \Pi_{\Phi_A}(r, x^i) \delta \Phi_A(r_h). \quad (24)$$

Then we can use Eq. (21) to obtain the expectation value of the dual operator from the value of the conjugate momentum on the boundary after an appropriate renormalization (for details see [6]),

$$\langle O_A(x^i) \rangle = \lim_{r \to \infty} \frac{1}{\sqrt{-h}} \Pi_{\Phi_A}(r, x^i)_{\text{ren}}. \quad (25)$$

### 3.2 The decomposition of the action with respect to a radial hypersurface

In this section we decompose a general action with respect to the hypersurface $r = \text{constant}$. The action is of the form

$$S = \frac{1}{16 \pi G_{d+1}} \int dr \int dt \int d^{d-1} x \sqrt{-g} \mathcal{L}(g_{ab}, R_{ab} e, \Phi_A, \nabla \Phi_A, \cdots ). \quad (26)$$
We wish to find the functional derivative of the action with respect to the Riemann tensor \( \delta S/\delta R_{rbcr} \). If derivatives of the Riemann tensor appear in \( \mathcal{L} \) then one has to perform integrations by parts first and then take the derivative. The procedure is similar to finding the Euler-Lagrange equations in a theory with higher derivatives of the canonical variables. If matter fields (collectively denoted by \( \Phi_A \)) and their covariant derivatives appear in \( \mathcal{L} \), the covariant derivatives have to be expressed in terms of the Riemann tensor prior to evaluation of the functional derivative [2]. The remaining terms that contain matter fields that do not depend on the Riemann tensor will be decomposed in a procedure similar to the one given below for the Riemann tensor. Such terms do not affect any of the results below. Therefore we can safely ignore them for the sake of simplicity.

The decomposition that we will describe follows a procedure similar to that introduced in [7] for the decomposition with respect to a constant \( t \) hypersurface. The decomposition with respect to \( r = \text{const.} \) has been extensively used in the AdS/CFT context since the radial direction in many ways plays the role of time (see, for example, [6]).

We define a hypersurface \( r = \text{const.} \) for the geometry given by (5) by its normal \( n_r = -\frac{1}{\sqrt{g_{rr}}} \). Let \( e^a_\alpha \) be a basis of tangent vectors to the hypersurface, and \( h_{\alpha\beta} = g_{ab} e^a_\alpha e^b_\beta \) the induced metric of the hypersurface. We use Greek indices to denote the induced coordinates on the hypersurface. The decomposition of the metric \( g_{ab} \) is then

\[
g_{ab} = e^a_\alpha e^b_\beta h_{\alpha\beta} - n_a n_b.
\]

The covariant derivative on the hypersurface is defined as a projection of the general covariant derivative \( D_a = e^a_\alpha \nabla_a \).

Next we wish to decompose the Riemann tensor with respect to the hypersurface as the first step towards a decomposition of a general action. For this purpose we introduce the Lie derivative in the direction of \( n^a \), \( L_n = \frac{1}{N} \partial_r \) where \( N = \sqrt{g_{rr}} \) is the lapse function. We fix the shift to be zero, so that the bulk metric has the following form

\[
ds^2 = N^2 dr^2 + h_{\alpha\beta}(r) dx^\alpha dx^\beta.
\]
The Gauss-Codazzi-Ricci decomposition of $R_{abcd}$ is given then by

\[
e^d_\delta e^c_\gamma e^b_\beta e^a_\alpha R_{abcd} = R_{\alpha\beta\gamma\delta} + K_{\alpha\gamma}K_{\delta\beta} - K_{\alpha\delta}K_{\gamma\beta} \tag{28}
\]

\[
n^a e^d_\delta e^c_\gamma e^b_\beta R_{abcd} = D_\gamma K_{\delta\beta} - D_\delta K_{\beta\gamma} \tag{29}
\]

\[
n^a n^d e^b_\beta e^c_\gamma R_{abcd} = L_n K_{\beta\gamma} + K_{\beta\eta}K^{\eta\gamma} + \frac{D_\beta D_\gamma N}{N} \tag{30}
\]

Now for the decomposition of the action let us introduce two auxiliary non-dynamical tensors $V_{abcd}$ and $U_{abcd}$ and look at the following action

\[
\tilde{S} = \frac{1}{16 \pi G_{d+1}} \int dr \int dt \int d^{d-1}x \sqrt{-g} \times \left[ \mathcal{L}(g_{ab}, V_{abcd}) + U_{abcd} R_{abcd} - U_{abcd} V_{abcd} \right], \tag{31}
\]

which is equivalent to the original action \((26)\) when we substitute the equations of motion

\[
\frac{\partial L}{\partial V_{abcd}} = U_{abcd}, \tag{32}
\]

\[
\frac{\partial L}{\partial V_{abcd}} = R_{abcd}. \tag{33}
\]

### 3.3 The dual of $\frac{\partial L}{\partial R_{rbcr}}$

With the auxiliary fields only one term in the action $\tilde{S}$ depends explicitly on $R_{abcd}$. We have to decompose this term according to Eqs. \((28)-(30)\),

\[
\tilde{S} = \frac{1}{16 \pi G_{d+1}} \int dr \int dt \int d^{d-1}x \sqrt{-h} N \times \left[ U^{\alpha\beta\gamma\delta}(R_{\alpha\beta\gamma\delta} + 2 K_{\alpha\gamma}K_{\beta\delta}) + 8U^{\gamma\beta\delta}D_\gamma K_{\beta\delta} \right.
\]

\[
+4U^{\gamma\beta\tau}N^{-1}(\dot{K}_{\beta\gamma} + N K_{\beta\eta}K^{\eta\gamma} + D_\beta D_\gamma N) + \cdots \right]. \tag{34}
\]

The dots denote the decomposition of the rest of the fields.

Since $K_{\beta\gamma} = \dot{h}_{\beta\gamma}/(2N)$, the term with $\dot{K}_{\beta\gamma}$ in Eq. \((34)\) contains second derivatives with respect to $r$. Following \([7]\) we introduce an additional auxiliary field $P^{\alpha\beta}$ and treat $K_{\alpha\beta}$ as an
independent variable,
\[
\bar{S} = \frac{1}{16 \pi G_{d+1}} \int dr \int dt \int d^{d-1}x \left\{ \mathcal{P}^{\alpha \beta} \left( 2 \, N \, K_{\alpha \beta} - \dot{h}_{\alpha \beta} \right) \right. \\
\left. + \sqrt{-h} \, N \left[ U^{\alpha \beta \gamma \delta} \left( R_{\alpha \beta \gamma \delta} + 2 \, K_{\alpha \gamma} \dot{K}_{\beta \delta} + 8 U^{\tau \beta \gamma \delta} D_{\tau} K_{\beta \delta} ight) + 4 U^{r \beta \gamma r} N^{-1} \left( \dot{K}_{\beta \gamma} + N \, K_{\beta \eta} K_{\eta \gamma} + D_{\beta} D_{\gamma} N \right) + \cdots \right] \right\}.
\] (35)

Integrating by parts the term \(1/4 \pi G_{d+1} \sqrt{-h} U^{r \beta \gamma r} \dot{K}_{\beta \gamma}\) in Eq. (35) we find that the conjugate variables to
\[
\Theta^{tt} \equiv 2 U^{r r t t}(r, t, x^i) \\
\Theta^{xx} \equiv -2 U^{r x x x}(r, t, x^i),
\] (36)
are
\[
\Pi_{\Theta^{tt}}(r, t, x^i) = -\frac{\sqrt{-h}}{8 \pi G_{d+1}} K_{tt}(r, t, x^i) \\
\Pi_{\Theta^{xx}}(r, t, x^i) = \frac{\sqrt{-h}}{8 \pi G_{d+1}} K_{xx}(r, t, x^i).
\] (37)

Then according to Eq. (25) the one-point functions of the dual operators to \(\Theta^t_t\) and \(\Theta^x_x\) are given by
\[
\langle O_{\Theta^t_t} \rangle = -\lim_{r \to \infty} \frac{(K^t_t)_{\text{ren}}}{8 \pi G_{d+1}} \\
\langle O_{\Theta^x_x} \rangle = \lim_{r \to \infty} \frac{(K^x_x)_{\text{ren}}}{8 \pi G_{d+1}}.
\] (38)

4 The field theory dual of the Noether charge entropy density

To complete the process of identifying the dual of the NCE density we need to express the extrinsic curvature in terms of field theory operators. In this paper we consider only higher-derivative actions with solutions whose asymptotic form coincides with a solution to the Einstein-Hilbert Lagrangian with a negative cosmological constant. In this case, near the boundary of AdS we may use the Einstein-Hilbert action in order to relate quantities in the bulk to quantities in the field theory.
4.1 The dual of the NCE

Let us recall the relation between the extrinsic curvature and the energy-momentum tensor in the case of the Einstein-Hilbert action:

\[
T_{\mu \nu} = -\frac{1}{8\pi G_{d+1}} \left( K_{\mu \nu} - K h_{\mu \nu} \right). \tag{39}
\]

As usual, \( h_{\mu \nu} \) is the induced metric on \( r = \text{const.} \) and \( K_{\mu \nu} \) is its extrinsic curvature. In order to find the field theory dual of such quantities a holographic renormalization procedure is implemented. The renormalization procedure of the energy-momentum tensor for the Einstein-Hilbert action is well-known (see for example [6]).

Holographic renormalization requires to formally perform a transformation to Euclidean signature in order to define the relation between the dual field theory and the on-shell gravity action. Here we use all the relations after transforming back to Lorenzian signature, assuming that such transformations can be performed without obstructions. The induced energy-momentum tensor in the field theory is given by

\[
\langle (T_{\mu \nu})_{FT} \rangle = \lim_{r \to \infty} \left( r^{d-2} T_{\mu \nu} \right)_{\text{ren}}. \tag{40}
\]

Trace-reversing Eq. (39) we obtain

\[
K_{\mu \nu} = -8\pi G_{d+1} \left( T_{\mu \nu} - \frac{T}{d-1} h_{\mu \nu} \right). \tag{41}
\]

Then, we can transform this relation to the field theory and get a relation in terms of the corresponding one-point functions:

\[
\langle O_{\Theta^t} \rangle = \langle (T^t_{\mu \nu})_{FT} \rangle - \frac{\langle T_{FFT} \rangle}{d-1}, \\
\langle O_{\Theta^x} \rangle = -\langle (T^x_{\mu \nu})_{FT} \rangle + \frac{\langle T_{FFT} \rangle}{d-1}. \tag{42}
\]

We assume that the energy-momentum of the field theory is of the perfect fluid form \( (T_{\mu \nu})_{FT} = \text{diag}(\varepsilon, P, P, ...) \) to describe the thermodynamics of the field theory. Then

\[
\lim_{r \to \infty} \left( K^t_{\mu \nu} \right)_{\text{ren}} = 8\pi G_{d+1} \left( \varepsilon + \frac{\langle T_{FFT} \rangle}{d-1} \right),
\]

\[
\lim_{r \to \infty} \left( K^x_{\mu \nu} \right)_{\text{ren}} = 8\pi G_{d+1} \left( -P + \frac{\langle T_{FFT} \rangle}{d-1} \right). \tag{43}
\]

\footnote{In the case that the theory is conformal \( \langle T_{FFT} \rangle = 0 \) and Eqs. (42) simplify to \( \langle O_{\Theta^t} \rangle = -\varepsilon \) and \( \langle O_{\Theta^x} \rangle = -P \).}
Recalling the linear combination in Eq. (18) we find that

\[ \langle O_S \rangle = \varepsilon + P = \lim_{r \to \infty} \frac{(K^t)^{\text{ren}} - (K^x)^{\text{ren}}}{8\pi G_{d+1}}. \]  

(44)

The entropy density of the field theory (which we denote by \( s \)) is identified with the dual operator of \( S \) so that

\[ sT = \varepsilon + P = \langle O_S \rangle. \]  

(45)

The corresponding field in the bulk is \( \sigma_t + \sigma_x \) and in section 2 we have seen that it gives the entropy density at the horizon. Here we have established the duality of the two descriptions of entropy: The NCE in bulk and the field theory entropy on the boundary.

4.2 Asymptotic conditions

We have required that asymptotically, when \( r \to \infty \), Eq. (39) holds without corrections to the energy-momentum one-point function. The condition that this requirement holds depends on the type of the higher order corrections to the Einstein-Hilbert Lagrangian and on the number of space-time dimensions. We wish to establish a criterion for the validity of the asymptotic condition. If the condition is not satisfied, we classify the higher derivative correction to the action as a correction which modifies the theory on the boundary. These interesting cases are left for a future study since their analysis is more complicated.

In general, the energy-momentum tensor on a hypersurface is derived from the action

\[ T_{\mu\nu} = \frac{2}{\sqrt{-h}} \frac{\partial K^{\alpha\beta}}{\partial h^{\mu\nu}} \frac{\delta S}{\delta K^{\alpha\beta}}. \]  

(46)

Since \( K_{\beta\gamma} = \hat{h}_{\beta\gamma}/(2N) \) we can express the field theory energy-momentum tensor as follows

\[ \langle (T_{\mu\nu})_{FT} \rangle = \lim_{r \to \infty} \left( \frac{r^{d-2} \delta S}{\sqrt{-g} \delta K^{\mu\nu}} \right)^{\text{ren}}. \]  

(47)

In order to obtain one-point function of the energy-momentum tensor we have to expand \( \delta S/\delta K^{\mu\nu} \) in the radial coordinate \( r \) in the neighborhood of the boundary \( r \to \infty \) and according to Eq. (47) the relevant terms are only those that decay asymptotically as \( 1/r^{d-2} \). The expansion of the metric for asymptotically AdS spacetimes (for the components which are orthogonal to the radial direction) is given by

\[ g_{\mu\nu} = g^{(0)}_{\mu\nu} r^2 + g^{(2)}_{\mu\nu} \frac{r^2}{r^2} + \cdots + g^{(d)}_{\mu\nu} \frac{r^{d-2}}{r^{d-2}} + \cdots \]  

(48)
Let us expand a general action, whose leading term is the Einstein-Hilbert action, in a parameter \( \ell \) which has length dimensions as following:

\[
\mathcal{L} = R + \sum_{n=2} \ell^{2n} \mathcal{L}_{2n}.
\]

(49)

The correction term \( \mathcal{L}_{2n} \) has \( 2n \) derivatives of the metric. Each \( \mathcal{L}_{2n} \) has to be decomposed on the hypersurface \( r = \text{const.} \) according to the Gauss-Codazzi-Ricci decomposition in Eq. (28) and can be expressed as a sum of products of components of the extrinsic and intrinsic curvatures and their contractions. In this decomposition it is important to include the appropriate (generalized) Gibbons-Hawking surface terms so that the action is written as a sum of bulk and surface contributions \( S = S_{\text{bulk}} + S_{\text{GH}} \). The surface contributions are “absorbed” after the above rewriting of the action on a hypersurface. The details of the action and its surface terms are not important for the kind of dimensional analysis that we will present here.

Now we would like to determine the leading radial dependence of the contribution of \( \mathcal{L}_{2n} \) to \( \langle T_{\mu\nu} \rangle_{FT} \) and decide when this contribution is relevant, namely, whether it contributes to terms of order \( 1/r^{d-2} \).

The asymptotic behavior of the extrinsic curvature is

\[
K_{\mu\nu} \sim r^2.
\]

Any Riemann tensor that appears in the decomposition has no derivatives with respect to \( r \) since it describes the intrinsic curvature of the hypersurface. Therefore

\[
R_{\alpha\beta\gamma\delta} \sim r^4,
\]

and due to our interest in the dependence of the leading term on \( r \) we can count each appearance of the Riemann tensor as two factors of the extrinsic curvature. Since \( \langle T_{\mu\nu} \rangle_{FT} \) is obtained as a variation of the action with respect to \( K^{\mu\nu} \), we are left with \( 2n - 1 \) factors of the extrinsic curvature \( K_{\mu\nu} \). (Factors of the Riemann tensor are counted as explained above).

Since each contribution of \( \mathcal{L}_{2n} \) to \( \langle T_{\mu\nu} \rangle \) has to be a rank two covariant tensor, the \( 2n - 1 \) factors of the extrinsic curvature have to be contracted with \( 2n - 2 \) inverse metrics whose leading terms scale as

\[
g^{\alpha\beta} \sim \frac{g^{(0)}_{\alpha\beta}}{r^2}.
\]

(50)
Consequently, the leading terms in the contribution of \( L_{2n} \) to \( \delta S/\delta K^{\alpha\beta} \) all scale as \( r^2 \). We are interested in higher derivative actions for which Eq. (39) is satisfied. This requires that all contributions to order \( 1/r^{d-2} \) to this equation for a specific higher derivative action cancel. Therefore this condition determines some algebraic equations that the coefficients of the higher derivative terms must satisfy. For example, if the higher derivative action is \( f(R) \) we require that \( \lim_{r \to \infty} f'(R) = 1 \) (see section 5.2). The algebraic equations determine which higher derivative terms are “relevant” and which are “irrelevant” in the language of RG flow equations.

There are no higher derivative corrections to the one-point function if all the corrections satisfy the algebraic equations. In this case we can use the relation in Eq. (39) for the one-point function of the energy-momentum tensor. If the algebraic conditions are not satisfied then Eq. (39) should be modified in order to obtain the correct one-point function.

5 Renormalization group flow of effective couplings

We wish to discuss the variation of the entropy density operator from the horizon to the boundary and interpret this variation as a RG flow of the effective gravitational couplings.

We have shown that

\[
sT = \frac{1}{\kappa_{\text{eff}}^2} \sqrt{-h} \left( K^t_t - K^x_x \right) \bigg|_{r \to \infty}
\]

when renormalized properly on the boundary at \( r \to \infty \). The quantity \( \sqrt{-h} (K^t_t - K^x_x) \) is also proportional to the entropy density operator on the “second” boundary, namely, the horizon. This was already realized in [8, 9] for the Einstein-Hilbert action case (when \( s \) is the Bekenstein-Hawking entropy density).

For a black brane ansatz of the form (5) we have

\[
\sqrt{-h} K^t_t = -\frac{g_{tt,r}}{2 \sqrt{-g_{tt}g_{rr}}} (g_{xx}) \frac{d-1}{2}, \quad (52)
\]

\[
\sqrt{-h} K^x_x = \frac{g_{xx,r}}{2 \sqrt{g_{xx}g_{rr}}} (g_{xx}) \frac{d-2}{2} \sqrt{-g_{tt}}. \quad (53)
\]

The temperature at the horizon is given by

\[
T = \frac{g_{tt,r}}{4 \pi \sqrt{-g_{tt}g_{rr}}} \bigg|_{r=r_h} , \quad (54)
\]

and the entropy density is

\[
s = \frac{2 \pi}{(\kappa_{\text{eff}})^2} (g_{xx}) \frac{d-1}{2} \bigg|_{r=r_h} . \quad (55)
\]
The $tt$ component of the metric $g_{tt}$ vanishes at the horizon and thus $K^x_x$ vanishes there. So at the horizon all the contribution to (51) comes from $K^t_t$:

$$
\left. \frac{1}{(\kappa_{\text{eff}})^2} \sqrt{-h} K^t_t \right|_{r=r_h} = s T. \quad (56)
$$

In conclusion, it turns out that the difference $\sqrt{-h} \left( K^t_t - K^x_x \right)$ gives $sT$ up to $(\kappa_{\text{eff}})^2$ on both boundaries - the AdS boundary and the horizon. Our interpretation is, as we will see below in more detail, that the effective coupling that flows from the horizon (UV) to the boundary (IR) is related to the dependence of the entropy density operator on the radial direction. According to the AdS/CFT correspondence the entropy density of the black brane is equal to the entropy density of the gauge theory on the boundary. This is a direct consequence of the identification of the temperature of the thermal state in the gauge theory with the Hawking temperature of the black brane, and as a result the corresponding free energies on both sides (see, for example, [10]).

Thus $sT$ is equal on both boundaries, so that the change in $\kappa_{\text{eff}}$ between the two boundaries exactly cancels the change in the entropy density operator $A$,

$$
A \equiv \sqrt{-h} \left( K^t_t - K^x_x \right). \quad (57)
$$

Using the RG flow terminology, we consider rescaling with respect to $r$. The effective coupling $\kappa_{\text{eff}}$ is a running coupling between two scales. Its running is equal to the accumulated rescaling of the corresponding operator $A$. Let us denote the variation of the quantities between the two boundaries (scales) by $\Delta$. Then we can write the RG flow equation

$$
2 \frac{\Delta \kappa_{\text{eff}}}{\kappa_{\text{eff}}} = \frac{\Delta A}{A}. \quad (58)
$$

In the rest of this section we give two examples to the RG flow of $\kappa_{\text{eff}}$ - no RG flow for the Einstein-Hilbert action and a simple RG flow for $f(R)$ Lagrangians. An additional non-trivial example is the RG flow in the context of the leading order 8-derivative correction proportional to the fourth power of the Weyl tensor to the type IIB black hole in AdS$_5$ [11] which will be presented elsewhere.

### 5.1 No RG flow for the Einstein-Hilbert action

In the case of Einstein-Hilbert action the effective coupling is a radial constant. From the gravity point of view it corresponds to having a purely Einstein-Hilbert action in the UV. In
this case we do not expect any renormalization of Newton’s constant in the IR. From the AdS bulk perspective, no RG flow means that the entropy density operator should not change with \(r\), namely, that it is a constant of the equations of motion. Here we show it explicitly following [8].

The Poincare symmetry of the brane geometry implies that the sources of the brane should satisfy

\[ T_{tt} + T_{xx} = 0. \]  (59)

From Einstein’s equations it follows that

\[ R_{tt} + R_{xx} = 0, \]  (60)

which is equivalent to

\[ R^t_t = R^x_x. \]  (61)

(See [12] for more details).

For a brane ansatz of the form (5) we can rewrite equation (61) as

\[
\frac{1}{\sqrt{-g}} \frac{d}{dr} \left( \sqrt{-h} K^t_t - \sqrt{-h} K^x_x \right) = 0,
\]  (62)

so that \(sT\) is conserved as we move from the horizon (UV) to the boundary (IR).

### 5.2 The flow in \(f(R)\) Lagrangians

As a second example we would like to discuss the NCE dual for Lagrangians that depend only on the Ricci scalar, i.e. of the form \(\mathcal{L} = f(R)\).

In order to find the relation of the extrinsic curvature to the energy-momentum tensor let us introduce an auxiliary scalar field \(\phi\) and consider the following action:

\[
\bar{S} = \frac{1}{16 \pi G_d} \int dr \int dt \int d^{d-1}x \sqrt{-\bar{g}} \left[ f(\phi) + f'(\phi) (R - \phi) \right],
\]  (63)

which is equivalent on-shell to the original action for \(f(R)\) (provided \(f''(R) \neq 0\). For more details see, for example, [13]). Then we can substitute the decomposition of the Ricci scalar with respect to a hypersurface \(r = \text{const}\).

\[
^{(d+1)}R = ^{(d)}R + K^{\alpha \beta} K_{\alpha \beta} - K^2 - 2 \nabla_a \left( n^b \nabla_b n^a - n^a \nabla_b n^b \right).
\]  (64)
The last term is canceled with the appropriate boundary term of the Gibbons-Hawking type. The boundary term in \( f(R) \) theories turns out to be \( [13, 14] \)

\[
-2 \int dt \int d^d x \sqrt{-h} f'(R) K.
\]

The energy momentum tensor for the action \( [63] \) is given by

\[
T^\alpha \beta = \frac{2}{\sqrt{-h}} \frac{\delta \bar{S}}{\delta h^\alpha \beta} = \frac{2}{\sqrt{-h}} \frac{\partial K^\alpha \beta}{\partial h^\alpha \beta} \frac{\delta \bar{S}}{\delta K^\alpha \beta},
\]

so that we get

\[
T^\alpha \beta = \frac{f'(R)}{8 \pi G_{d+1}} \left( K^\alpha \beta - K h^\alpha \beta \right).
\]

Trace-reversing Eq. \( [66] \) we obtain

\[
K^\alpha \beta = -\frac{8 \pi G_{d+1}}{f'(R)} \left( T^\alpha \beta - \frac{T}{d-1} h^\alpha \beta \right).
\]

The entropy of the black brane is given by Wald’s formula (substituting, for example, in Eq. \( [12] \))

\[
S_W = \frac{A_h}{4 G_{d+1}} f'(R) \bigg|_{r=r_h}.
\]

The effective coupling in this case is

\[
(k_{eff})^2 = \frac{G_{d+1}}{8 \pi f'(R)}.
\]

This effective coupling is in general \( r \) dependent. In this example the effective coupling changes as it flows from the horizon to the boundary, which corresponds to a flow from the UV to the IR. This is consistent with the way that we have defined the NCE as a function of the radial coordinate. In order for \( [39] \) to hold on the boundary we require

\[
\lim_{r \to \infty} f'(R) = 1
\]

so that

\[
\lim_{r \to \infty} (k_{eff})^2 = \frac{G_{d+1}}{8 \pi},
\]

which is its the value for the Einstein-Hilbert case.

The conservation law along the radial direction \( [62] \) will no longer hold and, in contrast to the Einstein-Hilbert case, the equation will acquire a source term. The source term corresponds
to the fact that $\kappa_{\text{eff}}$ changes in the radial direction and it is not a constant any more. For each value of $r$ it has a different value.

This example illustrates how the RG flow equation (58) is satisfied. From Eq. (67) and the definition of $A$ (57) we observe that

$$A = \frac{1}{f'(R)} A_{EH},$$

(72)

where $A_{EH}$ is the value of $A$ in the case of Einstein-Hilbert action (no RG flow). Therefore

$$\frac{\Delta A}{A} = \frac{1}{f'(R)} \bigg|_{r=r_h} - 1.$$ (73)

The same value is obtained also from $2\Delta \kappa_{\text{eff}}/\kappa_{\text{eff}}$ when $\kappa_{\text{eff}}$ was given in Eq. (69). To conclude, in $f(R)$ Lagrangians we already observe a non-trivial RG flow from the UV to the IR.

### 5.3 Some comments on the general case

For theories for which the algebraic equations discussed in section 4.2 are not satisfied it is more difficult to find the explicit expression for the extrinsic curvature $K_{\mu\nu}$ in terms of the energy-momentum tensor $T_{\mu\nu}$ on the boundary. For instance, this relation was obtained only recently for the case of a Gauss-Bonnet correction in [15] and only perturbatively in $\alpha'$. We expect that the a mapping of the entropy from the bulk to the boundary will exist also for any general Lagrangian of the form (26). The correction terms could be analyzed according to their relevance or irrelevance as in the standard RG analysis. We expect relations between the conformal invariance (or covariance) properties of operators in the field theory and the nature of the RG flow of their duals in the bulk.

### 6 Summary and conclusions

In this paper we have expressed the Noether charge entropy density of a black brane in AdS space in terms of local operators in the AdS space bulk. This allowed us to determine the field theory dual of the Noether charge entropy for theories that asymptote to Einstein theory. We

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2Any black hole solution of the Einstein equations with a cosmological constant is also a solution of the $f(R)$ equations of motion [13].
have defined such theories as theories in which the relation of the extrinsic curvature to the energy-momentum tensor on the boundary is the same as for the Einstein-Hilbert action for relevant terms, terms that decay asymptotically no faster than $1/r^{d-2}$. Such asymptotic dependence is dictated by the holographic renormalization procedure that determines the one-point functions of the dual operators. The latter are the energy density $\varepsilon$ and pressure $P$ of the field theory, so that the entropy density of the field theory is obtained as the combination

$$sT = \varepsilon + P.$$ 

We have interpreted the variation of the entropy density operator from the horizon to the boundary as due to the renormalization of the effective gravitational couplings when they flow from the UV to the IR. We have found that the RG flow equation for the effective coupling is related to the scaling of the entropy density operator:

$$2 \frac{\Delta \kappa_{\text{eff}}}{\kappa_{\text{eff}}} = \frac{\Delta A}{A},$$

where

$$A \equiv \sqrt{-h} \left( K^t_t - K^x_x \right).$$

For the Einstein-Hilbert theory we found that there is no RG flow. In the case of $f(R)$ theories we have found the RG flow explicitly as

$$2 \frac{\Delta \kappa_{\text{eff}}}{\kappa_{\text{eff}}} = \frac{1}{f'(R)} \Big|_{r=r_h} - 1.$$ 

When corrections introduce new relevant terms and Eq. (39) is not satisfied, the situation is more complicated and interpreted as a modification of the theory on the boundary. We leave such cases for future study.

Our results support the association of the NCE density operator with an effective gravitational coupling constant, an association that was introduced for the first time in [4].

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