Some properties of subspaces-hypercyclic operators

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Abstract

In this paper, we answer a question posed in the introduction of [10] positively, i.e., we show that if $T$ is $M$-hypercyclic operator with $M$-hypercyclic vector $x$ in a Hilbert space $H$, then $P(\text{Orb}(T, x))$ is dense in the subspace $M$ where $P$ is the orthogonal projection onto $M$. Furthermore, we give some relations between $M^\perp$-hypercyclicity and the orthogonal projection onto $M^\perp$. We also give sufficient conditions for a bilateral weighted shift operators on a Hilbert space $\ell^2(\mathbb{Z})$ to be subspace-hypercyclic, consequently, there exists an operator $T$ such that both $T$ and $T^*$ are subspace-hypercyclic operators. Finally, we give an $M$-hypercyclic criterion for an operator $T$ in terms of its eigenvalues.

1 Introduction

A bounded linear operator $T$ on a separable Hilbert space $\mathcal{H}$ is hypercyclic if there is a vector $x \in \mathcal{H}$ such that $\text{Orb}(T, x) = \{T^n x : n \geq 0\}$ is dense in $\mathcal{H}$, such a vector $x$ is called hypercyclic for $T$. The first example of a hypercyclic operator on a Banach space was constructed by Rolewicz in 1969 [13]. He showed that if $B$ is the backward shift on $\ell^p(\mathbb{N})$ then $\lambda B$ is hypercyclic if and only if $|\lambda| > 1$. The hypercyclicity concept was probably born with the thesis of Kitai in 1982 [8] who introduced the hypercyclic criterion to ensure the existence of hypercyclic operators. For more information on hypercyclic operators we refer the reader to [1, 5].

In 2011, Madore and Martínez-Avendaño [10] studied the density of the orbit in a non-trivial subspace instead of the whole space, this phenomenon is called the subspace-hypercyclicity. For the series of references on subspaces-hypercyclic operators see [9, 10, 12].

Definition 1.1. [10] Let $T \in B(\mathcal{H})$ and $M$ be a closed subspace of $\mathcal{H}$. Then $T$ is called $M$-hypercyclic or subspace-hypercyclic operator for a subspace $M$ if there exists a vector $x \in \mathcal{H}$ such that $\text{Orb}(T, x) \cap M$ is dense in $M$. We call $x$ an $M$-hypercyclic vector for $T$.

Definition 1.2. [10] Let $T \in B(\mathcal{H})$ and $M$ be a closed subspace of $\mathcal{H}$. Then $T$ is called $M$-transitive or subspace-transitive for a subspace $M$ if for each pair of non-empty open sets $U_1, U_2$ of $M$ there exists an $n \in \mathbb{N}$ such that $T^{-n}U_1 \cap U_2$ contains a non-empty relatively open set in $M$.

Theorem 1.3. [10] Every $M$-transitive operator on $\mathcal{H}$ is $M$-hypercyclic.

Theorem 1.4. [11] Every hypercyclic operator is subspace-hypercyclic for a subspace $M$ of $\mathcal{H}$.
In the present paper, we answer positively a question posed by Madore and Martínez-Avendaño in the introduction of [10]. In particular, Theorem 2.1 shows that if $P$ is the orthogonal projection onto a subspace $M$, then $P(\text{Orb}(T, x))$ will be dense in $M$ whenever $T$ is $M$-hypercyclic. Furthermore, we give some relations between $M^\perp$-hypercyclicity and the orthogonal projection onto $M^\perp$.

Through Theorem 2.4, we give a set of sufficient conditions for a bilateral weighted shift on a Hilbert space $\ell^2(\mathbb{Z})$ to be $M$-hypercyclic operator. Like the hypercyclicity case, Example 2.7 shows that there is an operator $T$ such that both $T$ and its adjoint are subspace-hypercyclic; however, we dont know whether they are subspace-hypercyclic for the same subspace or not. According to which an operator having a large supply of eigenvectors is hypercyclic, Godefroy-Shapiro [4] exhibited a hypercyclic criterion which is called “Godefroy-Shapiro Criterion”. We extend such a criterion to a subspaces and we call it “Spectrum $M^\perp$-Hypercyclic Criterion”. We give the Example 2.9 to show that the $M$-hypercyclic criterion is a stronger result than Spectrum $M^\perp$-Hypercyclic Criterion.

2 Main results

The set of all $M$-hypercyclic operators is denoted by $HC(H, M)$ and the set of all $M$-hypercyclic vectors of $T$ is denoted by $HC(T, M)$.

To prove the following results, the reader should be convenient with the properties of the projection map $P$, see [2].

In the introduction of [10], the authors asks the possibility of $P(\text{Orb}(T, x))$ to be dense in $M$ where $P$ is the orthogonal projection onto $M$.

**Theorem 2.1.** If $x \in HC(T, M)$ and $P : H \rightarrow M$ is the orthogonal projection onto $M$, then $P(\text{Orb}(T, x)) \cap M$ is dense in $M$.

**Proof.** Since $x \in HC(T, M)$, then there exist a sequence $\{n_k\}$ of positive numbers such that $\text{Orb}(T, x) \cap M = \{T^{n_k}x : k \in \mathbb{N}\}$ is dense in $M$. Since

$$\{T^{n_k}x : k \in \mathbb{N}\} \subseteq P(\text{Orb}(T, x)) \cap M,$$

then $P(\text{Orb}(T, x)) \cap M$ is dense in $M$. $\square$

The following examples gives an application to Theorem 2.1

**Example 2.2.** Let $B$ be the backward shift on $\ell^2(\mathbb{N})$ and $M$ be the subspace of $\ell^2(\mathbb{N})$ consisting of all sequences with zeroes on the even entries; that is

$$M = \{\{x_n\}_{n=0}^{\infty} \in \ell^2(\mathbb{N}) : x_{2k} = 0 \text{ for all } k\}.$$ 

Since $2B \in HC(M, H)$ see [10], Example 3.8], then there exists an $x \in HC(2B, M)$. We may, and will, assume that $x = (a_0, 0, a_1, 0, \ldots) \in M$, then for all $n \geq 0$,

$$(2B)^{2n+1}x = 2^{2n+1}(0, a_{2n+2}, 0, a_{2n+4}, \ldots)$$

and

$$(2B)^{2n}x = 2^{2n}(a_{2n}, 0, a_{2n+2}, 0, a_{2n+4}, \ldots).$$

Therefore

$$\text{Orb}(2B, x) \cap M = (2B)^{2n}x \text{ is dense in } M$$

Moreover

(1)
\[ P \left( \text{Orb}(T, x) \right) = \begin{cases} 0 & \text{if } n \text{ is odd}, \\ (2B)^n x & \text{if } n \text{ is even}. \end{cases} \]

Thus, it is clear from eq. (7) that \((2B)^{2n} x \in P \left( \text{Orb}(T, x) \right) \cap M = (2B)^{2n} x \cup \{0\} \) for all \(n \geq 0\) and this gives the proof.

**Proposition 2.3.** Let \(M^\perp\) be an invariant subspace under \(T\), \(P\) be the orthogonal projection onto \(M^\perp\) and \(T \in HC(\mathcal{H}, M^\perp)\), then

1. \(S \in HC(\mathcal{H}, M^\perp)\) where \(S = PT : M^\perp \rightarrow M^\perp\),
2. \(\bar{T} \in HC(\mathcal{H}/M)\) where \(\bar{T} : \mathcal{H}/M \rightarrow \mathcal{H}/M\).

**Proof.** (1): Let \(x \in HC(T, M^\perp)\) where \(x = u + v; u \in M\) and \(v \in M^\perp\), then \(Px = v\). Without loss of generality, we may assume that \(x \in M^\perp\), then

\[
\text{Orb}(PT, v) \cap M^\perp = \{v, PTv, (PT)^2v, \ldots\} \cap M^\perp \\
= \{v, PTv, PT^2v, \ldots\} \cap M^\perp \\
= \{Px, PTPx, PT^2Px, \ldots\} \cap M^\perp \\
= \{Px, PTPx, PT^2Px, \ldots\} \cap M^\perp \\
= P \left( \text{Orb}(T, x) \right) \cap M^\perp \\
= P \left( \text{Orb}(T, x) \right)
\]

Since \(P\) is continuous, then

\[
\overline{\text{Orb}(PT, v) \cap M^\perp} = \overline{P \left( \text{Orb}(T, x) \right)} \\
\supseteq P \left( \overline{\text{Orb}(T, x)} \right) \\
\supseteq M^\perp.
\]

Thus \(PT \in HC(\mathcal{H}, M^\perp)\).

(2): Without loss of generality, we may assume that \(x \in M^\perp\). By part (1), \(PT \in HC(\mathcal{H}, M^\perp)\) where \(PT : M^\perp \rightarrow M^\perp\) and since \((PT)^n x = PT^n x\) for all \(n \geq 0\), then \(P \left( \text{Orb}(T, x) \right)\) is dense in \(M^\perp\). Now, we will show that \(x + M \in HC(T)\),

\[
\left\{ (T^n(x + M)) : n \geq 0 \right\} = \left\{ (T^n x + M) : n \geq 0 \right\} \\
= \left\{ (PT^n x) : n \geq 0 \right\} + M, \text{ since } T^n x \in M^\perp \\
= P(\text{Orb}(T, x)) + M \\
= M^\perp / M = \mathcal{H}/M.
\]

\(\square\)

The next Theorem gives sufficient conditions for a bilateral weighted shift operator on the Hilbert space \(\ell^2(\mathbb{Z})\) to satisfy the \(M\)-hypercyclic criterion. We will suppose that

\[ B = \{e_{m_r} : r \in \mathbb{N}\} \]

is a Schauder basis for \(M\), where \(m_r \in \mathbb{Z}\). Let \(T\) be the bilateral forward weighted shift operator with a weigh sequence \(w_r\), then \(T(e_{m_r}) = w_{m_r}e_{m_{r+1}}\) for all \(r \in \mathbb{N}\). We may define a right inverse (backward shift) \(S\) to \(T\) as follows: \(S(e_{m_r}) = \frac{1}{w_{m_{r+1}}^{-1}}e_{m_{r-1}}\). Observe that \(TSe_{m_r} = e_{m_r}\) for all \(r \in \mathbb{N}\). If \(T\) is invertible then \(T^{-1} = S\). Also note that for all \(r \in \mathbb{N}\) and \(k \geq 0\), we have
Conjecture 1. If the conditions of Theorem 2.5 and Corollary 2.6. It immediately follows that both

and hypercyclic criterion if there exists a sequence of positive integers

From the above theorem, one can construct a weight sequence

Proof. We will verify the

Example 2.7. [6] holds true even on a subspace of a Hilbert space.

By the same way we can characterize the

M-hypercyclic backward weighted shifts since they are unitarily equivalent to forward shifts.

Corollary 2.6. If S is an invertible bilateral backward weighted shift in the Hilbert space \( \ell^2(\mathbb{Z}) \) with a positive weight sequence \( \{ w_n \} \in \mathbb{Z} \) and \( \mathcal{M} \) is a subspace of \( \ell^2(\mathbb{Z}) \), then T satisfies the \( \mathcal{M} \)-hypercyclic criterion if there exists a sequence of positive integers \( \{ n_k \} \) such that \( T^{n_k} \mathcal{M} \subseteq \mathcal{M} \) and \( i \in \mathbb{N} \) such that

\[
\lim_{k \to \infty} \lim_{j=m_i}^{m_i+n_k-1} w_j = 0 \quad \text{and} \quad \lim_{k \to \infty} \lim_{j=m_i}^{m_i+n_k-1} \frac{1}{w_j} = 0
\]

From the above theorem, one can construct a weight sequence \( \{ w_n \} \) such that \( \{ w_n \} \) satisfy the conditions of Theorem 2.5 and Corollary 2.6. It immediately follows that both T and \( T^* \) are subspaces-hypercyclic operators. In other words, we can show that the Herrero question [3] holds true even on a subspace of a Hilbert space.

Example 2.7. Let T be a bilateral shift defined as in the example of Salas [14]. Then, both T and \( T^* \) are \( \mathcal{M} \)-hypercyclic operators.

Proof. The proof follows directly from theorem [3]. In particular, there exist two subspaces \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) such that T is \( \mathcal{M}_1 \)-hypercyclic and \( T^* \) is \( \mathcal{M}_2 \)-hypercyclic.

Question 1. If T is \( \mathcal{M}_1 \)-hypercyclic and \( T^* \) is \( \mathcal{M}_2 \)-hypercyclic. What is the relation between \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \)?

Conjecture 1. If T is \( \mathcal{M}_1 \)-transitive and \( T^* \) is \( \mathcal{M}_2 \)-transitive, then \( \mathcal{M}_2 = \mathcal{M}_1^\perp \).
**Theorem 2.8** (Spectrum \( M \)-Hypercyclic Criterion). Let \( T \in \mathcal{B}(\mathcal{H}) \) and \( M \) be a subspace of \( \mathcal{H} \). If there is a positive integers \( p \) such that the subspace \\
\( X = \text{span} \{ x \in \mathcal{H}; T^p x = \lambda x, \lambda \in \mathbb{C}, |\lambda| < 1 \} \cap M, \) \nY = \text{span} \{ x \in \mathcal{H}; T^p x = \lambda x, \lambda \in \mathbb{C}, |\lambda| > 1 \} \cap M \nare dense in \( M \). Then \( T \) is \( M \)-transitive.

**Proof.** Let \( U, V \) be nonempty open subsets of \( M \). By hypothesis there exist \( x \in X \cap U \) and \( y \in Y \cap V \) such that \\
\[ x = \sum_{k=1}^{m} a_k x_k \quad \text{and} \quad y = \sum_{k=1}^{m} b_k y_k \]
where \( x_k \in X \) and \( y_k \in Y \) which mean that \( T^p x_k = \lambda_k x_k, |\lambda_k| < 1 \) and \( T^p y_k = \mu_k y_k, |\mu_k| > 1 \), for some \( a_k, b_k, \lambda_k, \mu_k \in \mathbb{C}, k = 1, \ldots, m \). Let \( z_n = \sum_{k=1}^{m} b_k \frac{1}{\mu_k} y_k \). Since as \( (n) \to \infty \),
\[ T^{n+p} x = \sum_{k=1}^{m} a_k \lambda_k^n x_k \to 0 \quad \text{and} \quad z_n \to 0 \]
and since \( T^{n+p} z_n = y \) for all \( n \geq 0 \), then there exists \( N \in \mathbb{N} \) such that, for all \( n \geq N \),
\[ x + z_n \in U \quad \text{and} \quad T^{n+p} (x + z_n) = T^{n+p} x + y \in V. \]
It follows that for all \( n \geq N \),
\[ T^{n+p} U \cap V \neq \emptyset. \]
Now, let \( w \in M \), since \( X \) is dense in \( M \), then there exists a sequence \( \{ x_k \} \subset X \) such that \( (x_k) \to w \). Since \( T \in \mathcal{B}(\mathcal{H}) \), then \( T^{n+p}(x_k) = (\lambda_k^n x_k) \to T^{n+p} w \) for all \( n \geq 0 \). Since \( |\lambda_k| < 1 \) for all \( k \geq 0 \), then \( (\lambda_k^n x_k) \to 0 \) as \( n \to \infty \). Thus, \( 0 = T^{n+p} w \in M \) and we get \( T^{n+p} M \subseteq M \) for all \( n \geq N \). This show that \( T \) is \( M \)-transitive.

We give an example showing that the \( M \)-hypercyclic criterion is a stronger result than Spectrum \( M \)-Hypercyclic Criterion.

**Example 2.9.** Consider the bilateral weighted forward shift \( T = F : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \), with the weight sequence
\[ w_n = \begin{cases} \frac{1}{2} & \text{if } n \geq 0, \\ \frac{1}{3} & \text{if } n < 0. \end{cases} \]
Let \( M \) be the subspace of \( \ell^2(\mathbb{Z}) \) consisting of all sequences with zeroes on the even entries; that is,
\[ M = \{ \{ a_n \}_{n=-\infty}^{\infty} \in \ell^2(\mathbb{Z}) : a_{2n} = 0, n \in \mathbb{Z} \} \]
then \( T \) satisfies \( M \)-hypercyclic criterion, but not spectrum \( M \)-hypercyclic criterion.

**Proof.** Applying Theorem 2.3 with \( m_k = 2k \) and \( m_1 = 1 \), we get
\[ \lim_{k \to \infty} \prod_{j=1}^{2k} w_j = \lim_{k \to \infty} \prod_{j=1}^{2k} \frac{1}{2} = 0 \quad \text{and} \quad \lim_{k \to \infty} \prod_{j=1}^{2k} \frac{1}{w_j} = \lim_{k \to \infty} \prod_{j=1}^{2k} \frac{1}{3} = 0. \]
It can be easily deduced from the definition of \( M \) that for each sequence \( x \in M \) and each \( k \in \mathbb{Z} \), the sequence \( T^{2k} x \) will have a zero entry on all even positions. It follows that \( T^{2k} M \subseteq M \). Thus \( T \) satisfies the \( M \)-hypercyclic criterion.
On the other hand, let \( x = (\ldots, 0, x_{-3}, 0, x_{-1}, 0, x_1, 0, x_3, 0, \ldots) \) be a non-zero element in \( M \). Towards a contradiction, assume that there exist \( p \in \mathbb{N} \) such that \( T^p x = \lambda x \), then if \( p \) is odd
number, then we get $x = 0$; otherwise, without loss of generality we will assume that $p = 2$. The equality $T^2x = \lambda x$ implies that

$$x = \left( \ldots, \left( \frac{\lambda}{3^2} \right)^2 x_{-1}, 0, \left( \frac{\lambda}{3^2} \right) x_{-1}, 0, x_{-1}, 0, 0, 3 \frac{1}{2\lambda} x_{-1}, 0, 0, 3 \frac{1}{2^3\lambda^2} x_{-1}, 0, 0, 3 \frac{1}{2^5\lambda^3} x_{-1}, 0, \ldots \right)$$

$$= \left( \ldots, \left( \frac{\lambda}{3^2} \right)^2 x_{-1}, 0, \left( \frac{\lambda}{3^2} \right) x_{-1}, 0, x_{-1}, 0, 6 \left( \frac{1}{2^2\lambda} \right) x_{-1}, 0, 6 \left( \frac{1}{2^2\lambda} \right)^2 x_{-1}, 0, 6 \left( \frac{1}{2^2\lambda} \right)^3 x_{-1}, 0, \ldots \right)$$

where $\lambda \neq 0$ and $x_{-1} \neq 0$. But then we have that

$$||x|| = |x_{-1}| \left( 1 + \sum_{n \in \mathbb{N}} \left( \frac{|\lambda|}{3^2} \right)^n \right) + 6 \sum_{n \in \mathbb{N}} \left( \frac{1}{2^2|\lambda|} \right)^n = \infty$$

whatever the value of $\lambda$. Thus $T^2$ has no eigenvalues and therefore it does not satisfy the spectrum $\mathcal{M}$-hypercyclic criterion.

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