The Strong Decays of $X(3940)$ and $X(4160)$

Zhi-Hui Wang$^{[1],[2] *}$, Yi Zhang$^{[1]}$, Li-Bo Jiang$^{[3]}$
Tian-hong Wang$^{[2]}$, Yue Jiang$^{[2]}$, Guo-Li Wang$^{[2]}

$^{1}$School of Electrical & Information Engineering,
Beifang University of Nationalities,
Yinchuan, 750021, Peoples Republic of China
$^{2}$Department of Physics, Harbin Institute of Technology,
Harbin, 150001, Peoples Republic of China
$^{3}$Department of physics and astronomy,
University of Pittsburgh, PA 15260, USA

Abstract

The new mesons $X(3940)$ and $X(4160)$ have been found by Belle Collaboration in the processes $e^+e^- \to J/\psi D^{(*)}\bar{D}^{(*)}$. Considering $X(3940)$ and $X(4160)$ as $\eta_c(3S)$ and $\eta_c(4S)$ states, the two-body open charm OZI-allowed strong decay of $\eta_c(3S)$ and $\eta_c(4S)$ are studied by the improved Bethe-Salpeter method combine with the $^3P_0$ model. The strong decay width of $\eta_c(3S)$ is $\Gamma_{\eta_c(3S)} = (33.5^{+18.4}_{-15.3})$ MeV, which is closed to the result of $X(3940)$, therefore, $\eta_c(3S)$ is a good candidate of $X(3940)$. The strong decay width of $\eta_c(4S)$ is $\Gamma_{\eta_c(4S)} = (69.9^{+22.4}_{-21.1})$ MeV, considering the errors of the results, it’s closed to the lower limit of $X(4160)$. But the ratio of the decay width $\frac{\Gamma(D\bar{D}^*)}{\Gamma(D^*\bar{D}^*)}$ of $\eta_c(4S)$ is larger than the experimental data of $X(4160)$. According to the above analysis, $\eta_c(4S)$ is not the candidate of $X(4160)$, and more investigations of $X(4160)$ is needed.

Keywords: $X(3940)$; $X(4160)$; Strong Decay; Improved Bethe-Salpeter Method.

* 2013086@nun.edu.cn
I. INTRODUCTION

In the past few years, many more new charmonium-like states, so-called $XYZ$ states, have been observed by the Belle, BABAR and BESIII Collaborations. The discovery of these states not only enriched the spectroscopy of charmonium-like states but also provided us an opportunity to research the properties of charmonium-like states. For example, the $X(3940)$ state was observed from the inclusive process $e^+e^- \rightarrow J/\psi X(3940)$ and had the decay mode $X(3940) \rightarrow D^*\bar{D}$ by the Belle Collaboration at a mass of $(3943 \pm 6 \pm 6) \text{ MeV}$. The decay width of this state was less than 52 MeV at the 90% C.L. which has taken into the systematics. Later Belle Collaboration confirmed the observation of $X(3940)$ with a significance of $5.7\sigma$, they got the mass and width of $X(3940)$ were $M = (3942^{+7}_{-6} \pm 6) \text{ MeV}$, $\Gamma = (37^{+26}_{-15} \pm 15) \text{ MeV}$. At the same time, they also observed a new charmonium-like state $X(4160)$ in the process $e^+e^- \rightarrow J/\psi D^*\bar{D}$, the mass and width of $X(4160)$ were $M = (4156^{+25}_{-20} \pm 15) \text{ MeV}$, $\Gamma = (139^{+111}_{-61} \pm 21) \text{ MeV}$. The observations of these $XYZ$ states inspire many interests about their physical natures.

There are already many theoretical approaches which have been used to study the properties of these $XYZ$ states. In this paper, we mainly discuss the properties and decays of two $X$ states: $X(3940)$ and $X(4160)$. Ref. [4] and Ref. [5] have assigned that the $C$ parity of $X(3940)$ and $X(4160)$ should be even, $C = +$. Assuming $X(3940)$ as $3^1S_0$ or one of $2^3P_J$ charmonium-like states, Ref. [6] studied $e^+e^- \rightarrow J/\psi X(3940)$ process by the light-cone formalism, and they considered that $X(3940)$ is $3^1S_0(\eta_c(3S))$. Ref. [7, 8] investigated the properties of $X(3940)$ and $X(4160)$ which were $\eta_c(3S)$ and $\chi_{c0}(3P)$, respectively. Ref. [13] calculated the strong decays of $\eta_c(nS)$, they found that the explanation of $X(3940)$ as $\eta_c(3S)$ is possible and the assignment of $X(4160)$ as $\eta_c(4S)$ can not be excluded. In Ref. [14], the authors studied vector-vector interaction of $X(4160)$ which was basically a $D_s^*\bar{D}_s^*$ molecular state with $J^{PC} = 2^{++}$. Ref. [16] had studied the inclusive production of $X(3940)$ in the decay of ground bottomonium state $\eta_b$ by the NRQCD factorization formula, and they also considered $X(3940)$ as the excited $\eta_c(3S)$ state. Ref. [17] calculated the strong decay of $X(4160)$ which was assumed as $\chi_{c0}(3P)$, $\chi_{c1}(3P)$, $\eta_{c2}(2D)$ or $\eta_c(4S)$ by the $^3P_0$ model. In Ref. [18], they also explored the properties of $X(3940)$ and $X(4160)$ as the $\eta_c(3S)$ and $\eta_c(4S)$, respectively. But their results suggested that $X(3940)$ as $\eta_c(3S)$ was established and the explanation of $X(4160)$ to be $\eta_c(4S)$ is fully excluded. Using the NRQCD factorization approach, Ref. [19] calculated the branching fractions of $\Upsilon(nS) \rightarrow J/\psi + X$ with $X = X(3940)$ or $X = X(4160)$, they thought that the $X(3940)$ and $X(4160)$ can be explained
as $3^1S_0$ and $4^1S_0$ charmonium-like states, respectively. Up to now, it is very difficult to confirm the constructions of $X(3940)$ and $X(4160)$, because lack of the enough experimental data. Many more theoretical prediction and experimental data are needed for $X(3940)$ and $X(4160)$.

The mesons can be described by the B-S equation. Ref. [21] took the B-S equation to describe the light mesons $\pi$ and $K$, then they calculated the mass and decay constant of $\pi$ by the B-S amplitudes [22], they also studied the weak decays [23] and the strong decays [24] combine with the Dyson-Schwinger equation.

We will use the B-S equation to study the properties of heavy mesons. In Ref. [25], we had calculate the Spectrum of heavy quarkonia the improved Bethe-Salpeter (B-S) method, for the the charmonium state with the quantum numbers $J^{PC} = 0^{-+}$, the mass of $3^1S_0$ ($\eta_c(3S)$) is $M=3948.8$ MeV which is closed to the mass of $X(3940)$ with error, the mass of $4^1S_0$ ($\eta_c(4S)$) was $M=4224.6$ MeV which was larger than the center mass of $X(4160)$ about 70 MeV. In this paper, to check if the $X(4160)$ is the charmonium $\eta_c(4S)$, we calculate the strong decay of $\eta_c(4S)$, but assign the mass of $\eta_c(4S)$ as 4156 MeV by varying the parameter $V_0$ in interaction potential, where in potential model the parameter $V_0$ is added to move the theoretical mass spectra parallel to match the experimental data.

Using the the improved B-S method, we calculated the weak decay of $B_c$ to $\eta_c(1S)$ and $\eta_c(2S)$ [26], and the weak decay of $B_c$ to $\eta_c(3S)$ and $\eta_c(4S)$ [27]. There is nobody to calculate $B_c$ to $\eta_c(4S)$, but the results of $B_c$ to $\eta_c(1S)$, $\eta_c(2S)$, $\eta_c(3S)$ were close to the other theoretical results. We also studied the properties of some XYZ states, such as radiative E1 decay of $X(3872)$ [9, 10], two-body strong decay of $Z(3930)$ which was $\chi_{c2}(2P)$ state combine with the $^3P_0$ model [11], the strong decay of $X(3915)$ as $\chi_{c0}(2P)$ state [28], and the strong decay of $\Upsilon$ [29]. All the theoretical results consist with experimental data or other theoretical results. Because the higher excited states have larger relativistic correction than the corresponding ground state, a relativistic model is needed in a careful study. The improved B-S method is a relativistic model that describe bound states with definite quantum number, the corresponding relativistic form of wavefunctions are solutions of the full Salpeter equations. So the improved B-S method is good method to describe the properties and decays of the radial high excited states, In this paper, we focus on the strong decays of $X(3940)$ and $X(4160)$ as radial high excited states $\eta_c(3S)$ and $\eta_c(4S)$ by the improved B-S method.

In our method, we study the natures of heavy mesons by the coupling of $L + S$ for the
quark and anti-quark in mesons. According to the $L+S$ coupling, we show the wavefunctions of the heavy mesons in terms of the quantum number $J^P$ (or $J^{PC}$) which are very good to describe the equal mass systems in heavy mesons. The quantum numbers $J^{PC}$ of $\eta_c(3S)$ and $\eta_c(4S)$ both are $0^{-+}$, the $C$ parities are even which agree with the results of Ref. [4] and Ref. [5]. The corresponding Okubo-Zweig-Iizuka (OZI) rule allowed two-body open charm strong decay modes are: $0^- \to 0^- 1^-$ and $0^- \to 1^- 1^-$, while other strong decays in the final state are ruled out by the kinematic possible mass region. In order to calculate the two-body open charm strong decay, we adopt the $^3P_0$ model which assumes that a quark-antiquark pair is created with vacuum quantum numbers, $J^{PC} = 0^{-+}$ [33–35]. The $^3P_0$ model was proposed in Ref. [33], then Ref. [34] and Ref. [35] applied the $^3P_0$ model to study the open-flavor strong decays of the light mesons. Now, People have extended this model to study the natures of heavy-light mesons [36, 37] and heavy quarkonia [29, 38, 39]. In Ref. [11] and Ref. [29], we have calculated the OZI allowed two-body strong decays of charmonium and bottomonium in the $^3P_0$ model with the relativistic B-S wavefunctions. The results were good according with experimental data and the other theoretical results. Furthermore, the strong decay widths are related to the parameter $\gamma$, but the ratio of the decay width $\Gamma(\eta_c(4S) \to D^* D^*)$ and $\Gamma(\eta_c(4S) \to D^* D^*)$ were independent of the parameter $\gamma$, so the results of the ratios are more reliable than the decay widths. In this paper, we take the same method as Ref. [11] and Ref. [29] to study strong decays of $\eta_c(3S)$ and $\eta_c(4S)$ states.

The paper is organized as follows. In Sec. II, we introduce the instantaneous B-S equation; We show the relativistic wavefunctions of initial mesons and final mesons in Section. III; In Sec. IV, we give the formulation of two-body open charm strong decays; The corresponding results and conclusions are present in Sec. V.

II. INSTANTANEOUS BETHE-SALPETER EQUATION

In this section, we briefly review the Bethe-Salpeter equation and its instantaneous one, the Salpeter equation.

The BS equation is read as [40]:

$$ (\not{p}_1 - m_1) \chi(q)(\not{p}_2 + m_2) = i \int \frac{d^4k}{(2\pi)^4} V(P, k, q) \chi(k) ,$$

where $\chi(q)$ is the B-S wave function, $P$ is the total momentum of the meson, $q$ is relative quantum between quark and anti-quark, $V(P, k, q)$ is the interaction kernel between the
quark and anti-quark, $p_1, p_2$ and $m_1, m_2$ are the momentum and mass of the quark 1 and anti-quark 2, respectively.

We divide the relative momentum $q$ into two parts, $q_{\parallel}$ and $q_{\perp}$.

$$q^\mu = q_{\parallel}^\mu + q_{\perp}^\mu,$$

$$q_{\parallel}^\mu \equiv (P \cdot q / M^2) P^\mu, \quad q_{\perp}^\mu \equiv q^\mu - q_{\parallel}^\mu.$$

Correspondingly, we have two Lorentz invariants:

$$q_p = \frac{(P \cdot q)}{M}, \quad q_T = \sqrt{q_p^2 - q^2} = \sqrt{-q_{\perp}^2}.$$

When $\vec{P} = 0$, $q_p = q_0$ and $q_T = |\vec{q}|$, respectively.

In instantaneous approach, the kernel $V(P, k, q)$ takes the simple form [41]:

$$V(P, k, q) \Rightarrow V(|\vec{k} - \vec{q}|).$$

Let us introduce the notations $\varphi_p(q_{\perp}^\mu)$ and $\eta(q_{\perp}^\mu)$ for three dimensional wave function as follows:

$$\varphi_p(q_{\perp}^\mu) \equiv i \int \frac{dq_p}{2\pi} \chi(q_{\parallel}^\mu, q_{\perp}^\mu),$$

$$\eta(q_{\perp}^\mu) \equiv \int \frac{dk_{\perp}}{(2\pi)^3} V(k_{\perp}, q_{\perp}) \varphi_p(k_{\perp}).$$

Then the BS equation can be rewritten as:

$$\chi(q_{\parallel}, q_{\perp}) = S_1(p_1) \eta(q_{\perp}) S_2(p_2).$$

(3)

The propagators of the two constituents can be decomposed as:

$$S_i(p_i) = \frac{\Lambda_{ip}^+(q_{\perp})}{J(i)q_p + \alpha_i M - \omega_i + i\epsilon} + \frac{\Lambda_{ip}^-(q_{\perp})}{J(i)q_p + \alpha_i M + \omega_i - i\epsilon},$$

(4)

with

$$\omega_i = \sqrt{m_i^2 + q_{\perp}^2}, \quad \Lambda_{ip}^\pm(q_{\perp}) = \frac{1}{2\omega_{ip}} \left[ \frac{P}{M} \omega_i \pm J(i)(m_i + q_{\perp}) \right],$$

(5)

where $i = 1, 2$ for quark and anti-quark, respectively, and $J(i) = (-1)^{i+1}$.

Introducing the notations $\varphi_p^{\pm\pm}(q_{\perp})$ as:

$$\varphi_p^{\pm\pm}(q_{\perp}) \equiv \Lambda_{ip}^\pm(q_{\perp}) \frac{P}{M} \varphi_p(q_{\perp}) \frac{P}{M} \Lambda_{2p}^\pm(q_{\perp}).$$

(6)

With contour integration over $q_p$ on both sides of Eq. (3), we obtain:

$$\varphi_p(q_{\perp}) = \frac{\Lambda_{ip}^+(q_{\perp})\eta_{ip}(q_{\perp})\Lambda_{2p}^+(q_{\perp})}{(M - \omega_1 - \omega_2)} - \frac{\Lambda_{ip}^-(q_{\perp})\eta_{ip}(q_{\perp})\Lambda_{2p}^-(q_{\perp})}{(M + \omega_1 + \omega_2)},$$

(7)
and the full Salpeter equation:

\[(M - \omega_1 - \omega_2)\varphi_p^{++}(q_\perp) = \Lambda_{1p}(q_\perp)\eta_p(q_\perp)\Lambda_{2p}(q_\perp),\]
\[(M + \omega_1 + \omega_2)\varphi_p^{--}(q_\perp) = -\Lambda_{1p}(q_\perp)\eta_p(q_\perp)\Lambda_{2p}(q_\perp),\]
\[\varphi_p^{-+}(q_\perp) = \varphi_p^{+-}(q_\perp) = 0.\]  \(\text{(7)}\)

For the different \(J^PC\) (or \(JP\)) states, we give the general form of wave functions. Reducing the wave functions by the last equation of Eq. (7), then solving the first and second equations in Eq. (7) to get the wave functions and mass spectrum. We have discussed the solution of the Salpeter equation in detail in Ref. \([25, 42]\).

The normalization condition for BS wave function is:

\[\int \frac{q^2 dq}{2\pi^2} Tr \left[ \frac{P}{M} \varphi^{++} \frac{P}{M} - \frac{P}{M} \varphi^{--} \frac{P}{M} \right] = 2P_0.\]  \(\text{(8)}\)

In our model, the instantaneous interaction kernel \(V\) is Cornell potential, which is the sum of a linear scalar interaction and a vector interaction:

\[V(r) = V_s(r) + V_0 + \gamma_0 \otimes \gamma^0 V_v(r) = \lambda r + V_0 - \gamma_0 \otimes \gamma^0 \frac{4\alpha_s}{3r},\]  \(\text{(9)}\)

where \(\lambda\) is the string constant and \(\alpha_s(q)\) is the running coupling constant. In order to fit the data of heavy quarkonia, a constant \(V_0\) is often added to confine potential. To avoid the infrared divergence \(V_v(q)\) at \(q = 0\) in the momentum space, we introduce a factor \(e^{-\alpha r}\) to avoid the divergence:

\[V_s(r) = \frac{\lambda}{\alpha}(1 - e^{-\alpha r}), \quad V_v(r) = -\frac{4\alpha_s}{3r} e^{-\alpha r}.\]  \(\text{(10)}\)

It is easy to know that when \(\alpha r \ll 1\), the potential becomes to Eq. (9). In the momentum space and the C.M.S of the bound state, the potential reads:

\[V(q) = V_s(q) + \gamma_0 \otimes \gamma^0 V_v(q),\]
\[V_s(q) = -\frac{\lambda}{\alpha} V_0 \delta^3(q) + \frac{\lambda}{\pi^2} \frac{1}{(q^2 + \alpha^2)^2}, \quad V_v(q) = -\frac{2\alpha_s(q)}{3\pi^2} \frac{1}{(q^2 + \alpha^2)^2},\]  \(\text{(11)}\)

where the running coupling constant \(\alpha_s(q)\) is:

\[\alpha_s(q) = \frac{12\pi}{33 - 2N_f} \log(a + \frac{q^2}{\Lambda_{QCD}^2}).\]

We introduce a small parameter \(a\) to avoid the divergence in the denominator. The constants \(\lambda, \alpha, V_0\) and \(\Lambda_{QCD}\) are the parameters that characterize the potential. \(N_f = 3\) for \(\bar{b}q\) (and \(\bar{c}q\)) system.
III. THE RELATIVISTIC WAVEFUNCTIONS

In this paper, we focus on the two-body open charm strong decay of $X(3940)$ and $X(4160)$ which are considered as $\eta_c(3S)$ $\eta_c(4S)$ states. $\eta_c(3S)$ $\eta_c(4S)$ states have two decay modes: $0^- \rightarrow 0^- 1^-$ and $0^- \rightarrow 1^- 1^-$. So we only discuss the relativistic wavefunctions of $J^P$ equal to $0^-(^1S_0)$ and $1^- (^3S_1)$ states.

A. For pseudoscalar meson with quantum numbers $J^P = 0^-$

The general form for the relativistic wavefunction of pseudoscalar meson can be written as [42]:

$$\varphi_{0^-}(\vec{q}) = \left[f_1(\vec{q}) P + f_2(\vec{q}) M + f_3(\vec{q}) \not{q}_\perp + f_4(\vec{q}) \frac{P \not{q}_\perp}{M}\right] \gamma_5,$$  \hspace{1cm} (12)

where $M$ is the mass of the pseudoscalar meson, and $f_i(\vec{q})$ are functions of $|\vec{q}|^2$. Due to the last two equations of Eq. (7): $\varphi_{0^-}^+ = \varphi_{0^-}^- = 0$, we have:

$$f_3(\vec{q}) = \frac{f_2(\vec{q}) M (\omega_1 + \omega_2)}{m_2 \omega_1 + m_1 \omega_2}, \quad f_4(\vec{q}) = -\frac{f_1(\vec{q}) M (\omega_1 + \omega_2)}{m_2 \omega_1 + m_1 \omega_2}. \hspace{1cm} (13)$$

where $m_1, m_2$ and $\omega_1 = \sqrt{m_1^2 + q_\perp^2}, \omega_2 = \sqrt{m_2^2 + q_\perp^2}$ are the masses and the energies of quark and anti-quark in mesons, $q_\perp^2 = -|\vec{q}|^2$.

The numerical values of radial wavefunctions $f_1, f_2$ and eigenvalue $M$ can be obtained by solving the first two Salpeter equations in Eq. (7). In Ref. [27], we have plot the wavefunctions of $X(3940)$ and $X(4160)$ which are considered as $\eta_c(3S)$ and $\eta_c(4S)$, respectively.

According to the Eq. (6), the relativistic positive wavefunction of pseudoscalar meson in C.M.S can be written as [42]:

$$\varphi_{0^-}^{++}(\vec{q}) = b_1 \left[b_2 + \frac{P}{M} + b_3 \not{q}_\perp + b_4 \frac{\not{q}_\perp P}{M}\right] \gamma_5,$$ \hspace{1cm} (14)

where the $b_i$s ($i = 1, 2, 3, 4$) are related to the original radial wavefunctions $f_1, f_2$, quark masses $m_1, m_2$, quark energy $w_1, w_2$, and meson mass $M$:

$$b_1 = \frac{M}{2} \left(f_1(\vec{q}) + f_2(\vec{q}) \frac{m_1 + m_2}{\omega_1 + \omega_2}\right), b_2 = \frac{\omega_1 + \omega_2}{m_1 + m_2}, b_3 = -\frac{(m_1 - m_2)}{m_1 \omega_2 + m_2 \omega_1}, b_4 = \frac{(\omega_1 + \omega_2)}{(m_1 \omega_2 + m_2 \omega_1)}.$$

B. For vector meson with quantum numbers $J^P = 1^-$

The general form for the relativistic wavefunctions of vector state $J^P = 1^-$ (or $J^{PC} = 1^{--}$ for quarkonium) can be written as eight terms, which are constructed by $P_{f_1}, q_{f_1 \perp}, \epsilon_1$ and...
gamma matrices \[43],

\[
\varphi^- (\bar{q} f_1) = q_{f1\perp} \cdot \epsilon_1 \left[ f'_1 + \frac{P_{f1}}{M_{f1}} f'_2 + \frac{q_{f1\perp}}{M_{f1}} f'_3 + \frac{P_{f1} q_{f1\perp}}{M_{f1}^2} f'_4 \right] + M_{f1} f'_5,
\]

where \(\epsilon_1\) is the polarization vector of the vector meson in the final state.

Due to the last two equations of Eq. (15): \(\varphi_0^{+} = \varphi_0^{-} = 0\), we have \[44]\:

\[
f'_1 = \left[ q_{f1\perp} f'_3 + M_{f1}^2 f'_5 \right] \frac{(m_1' m_2' - w'_1 w'_2 + q_{f1\perp}^2)}{M_{f1}(m_1' + m_2')q_{f1\perp}^2}, \quad f'_7 = \frac{f'_5 M_{f1}(-w'_1 + w'_2)}{(m_1' w'_2 + m_2' w'_1)},
\]

\[
f'_2 = \left[ -q_{f1\perp} f'_3 + M_{f1}^2 f'_6 \right] \frac{(m_1' w'_2 - m_2' w'_1)}{M_{f1}(w'_1 + w'_2)q_{f1\perp}^2}, \quad f'_8 = \frac{f'_6 M_{f1}(w'_1 w'_2 - m_1' m_2' - q_{f1\perp}^2)}{(m_1' + m_2')q_{f1\perp}^2}.
\]

The relativistic positive wavefunctions of \(^3S_1\) state can be written as \[45]\:

\[
\varphi_{1^+} (\bar{q} f_1) = b_1 \varphi_1 + b_2 \varphi_1 P_{f1} + b_3 (q_{f1\perp} \varphi_1 - q_{f1\perp} \cdot \epsilon_1) + b_4 (P_{f1} \varphi_1 \varphi_1 P_{f1} q_{f1\perp} \cdot \epsilon_1) + q_{f1\perp} \cdot \epsilon_1 (b_5 + b_6 P_{f1} + b_7 q_{f1\perp} + b_8 q_{f1\perp} P_{f1}),
\]

(16)

where we first define the parameter \(n_i\) which are the functions of \(f'_i\) \(^3S_1\) wave functions:

\[n_1 = f'_5 - f'_6 \left( \frac{w'_1 + w'_2}{m'_1 + m'_2} \right), n_2 = f'_5 - f'_6 \left( \frac{(m'_1 + m'_2)}{(w'_1 + w'_2)} \right), n_3 = f'_3 + f'_4 \left( \frac{(m'_1 + m'_2)}{(w'_1 + w'_2)} \right),\]

then we define the parameters \(b_i\) which are the functions of \(f'_i\) and \(n_i\):

\[b_1 = \frac{M_{f1}}{2} n_1, b_2 = \frac{(m'_1 + m'_2)}{2(w'_1 + w'_2)} n_1, b_3 = \frac{M_{f1}(w'_1 - w'_2)}{2(m'_1 w'_2 + m'_2 w'_1)} n_1, b_4 = \frac{(w'_1 + w'_2)}{2(w'_1 w'_2 + m'_1 m'_2 - q_{f1\perp}^2)} n_1,\]

\[b_5 = \frac{1}{2M_{f1}} \left( \frac{(m'_1 + m'_2)(M_{f1}^2 n_2 + q_{f1\perp}^2 n_3)}{(w'_1 w'_2 + m'_1 m'_2 + q_{f1\perp}^2)} \right), b_6 = \frac{1}{2M_{f1}^2} \frac{(w'_1 - w'_2)(M_{f1}^2 n_2 + q_{f1\perp}^2 n_3)}{(w'_1 w'_2 + m'_1 m'_2 + q_{f1\perp}^2)},\]

\[b_7 = \frac{n_3}{2M_{f1}} - \frac{f'_6 M_{f1}}{(m'_1 w'_2 + m'_2 w'_1)}, b_8 = \frac{1}{2M_{f1}} \left( \frac{w'_1 + w'_2 n_3 - f'_3 (m'_1 + m'_2)(w'_1 w'_2 + m'_1 m'_2 - q_{f1\perp}^2)}{(m'_1 + m'_2)(w'_1 w'_2 + m'_1 m'_2 - q_{f1\perp}^2)} \right).\]

**IV. THE FORMULATION OF TWO-BODY OPEN CHARM STRONG DECAYS**

For the two-body OZI-allowed open charm strong decays, such as \(\eta_c(3S) \rightarrow DD^*\), we adopt the \(^3P_0\) model to calculate the strong decay amplitude. The non-relativistic \(^3P_0\) model describe the decay matrix elements by the \(q\bar{q}\) pair-production Hamiltonian: \(H = g \int d^3x \bar{v} \psi \) [38]. According to the improved B-S method which is a relativistic model, we can extend the non-relativistic \(^3P_0\) model to the relativistic form: \(H = -ig \int d^3x \bar{v} \psi \) [11, 29].
Here $\psi$ is the Dirac quark field, $g = 2m_q\gamma$, $m_q$ is the quark mass of the light quark-pairs, $\gamma$ is a dimensionless constant which describe the pair-production strength and can be obtained by fitting the experimental data. In this paper, we choose $\gamma = 0.483$ which give reasonable calculation of $\eta_c(3S)$, then we use the same value to $\eta_c(4S)$.

\begin{equation}
\langle BC \mid H \mid A \rangle = -ig \int \frac{d^4 \bar{q}}{(2\pi)^4} \text{Tr} \left[ \chi_P(q) S_2^{-1}(p_2) \bar{\chi}_{P_{f_2}}(q_{f_2}) \bar{\chi}_{P_{f_1}}(q_{f_1}) S_1^{-1}(p_1) \right] = g \int \frac{d^3 \bar{q}}{(2\pi)^3} \text{Tr} \left[ \frac{P}{M} \varphi_P^+(\bar{q}) \frac{P}{M} \varphi_{P_{f_2}}^+(\bar{q}_{f_2}) \varphi_{P_{f_1}}^+(\bar{q}_{f_1}) \left( 1 - \frac{M - w_1 - w_2}{2w_{12}} \right) \right] \tag{17}
\end{equation}

where $\varphi_P^+(\bar{q})$, $\varphi_{P_{f_2}}^+(\bar{q}_{f_2})$ and $\varphi_{P_{f_1}}^+(\bar{q}_{f_1})$ are the relativistic positive wavefunctions of initial meson $A$, finial meson $B$ and $C$, respectively. $\varphi = \gamma^0 \varphi^+ \gamma^0$. We have given the detailed form of wavefunctions in Sec. III. $P$, $P_{f_1}$, $P_{f_2}$ and $\bar{q}$, $\bar{q}_{f_1}$, $\bar{q}_{f_2}$ are the momentum and three dimension relative momentum between quark and anti-quark of initial meson $A$, finial meson $B$ and $C$, respectively. $\bar{q}_{f_1} = \bar{q} - \frac{m}{m + m_{u,d,s}} \bar{P}_{f_1}$, $\bar{q}_{f_2} = \bar{q} + \frac{m}{m + m_{u,d,s}} \bar{P}_{f_2}$. $\bar{P}_{f_1}$ and $\bar{P}_{f_2}$ are the three momentum of finial mesons $B$ and $C$. $w_{12} = \sqrt{m_{u,d,s}^2 + \bar{q}_{f_1}^2}$.

Using the B-S wavefunctions in Sec. III and the formula of amplitude Eq. (17), the two-body open charm strong decay amplitude can be defined as,

\begin{align*}
\mathcal{M}(A \to DD^*) &= \epsilon_{1\mu} P^\mu t_1, \\
\mathcal{M}(A \to D^* \bar{D}^*) &= \epsilon_{\mu\nu\alpha\beta} P^\mu P_{f_1}^\nu \epsilon_1^\alpha \epsilon_2^\beta t_2, \tag{18}
\end{align*}

where $A$ denote $\eta_c(3S)$ or $\eta_c(4S)$, $\epsilon_1$ and $\epsilon_2$ are the polarization vector of the final mesons $B$ and $C$. $t_1$ and $t_2$ are the strong decay coupling constants which are related to the B-S wavefunctions.
Finally, using Eq. (18) the two-body open charm strong decay width can be written as,

$$\Gamma = \frac{|\vec{P}_{f1}|}{8\pi M^2} \sum_{\lambda} |\mathcal{M}|^2,$$

where $$|\vec{P}_{f1}| = \sqrt{[M^2 - (M_{f1} - M_{f2})^2][M^2 - (M_{f1} + M_{f2})^2]/(2M)}$$ which is the three momentum of the final mesons.

V. NUMBER RESULTS AND DISCUSSIONS

In order to fix Cornell potential in Eq. (11) and masses of quarks, we take these parameters: 
$$a = c = 2.7183, \lambda = 0.210 \text{ GeV}^2, \Lambda_{QCD} = 0.270 \text{ GeV}, \alpha = 0.060 \text{ GeV}, m_u = 0.305 \text{ GeV},$$
$$m_d=0.311 \text{ GeV}, m_s=0.500 \text{ GeV}, m_b = 4.96 \text{ GeV}, m_c = 1.62 \text{ GeV}, \text{etc} \quad [25],$$
which are best to fit the mass spectra of ground states B, D mesons and other heavy mesons. And we get the masses: 
$$M_{D_{s\pm}} = 1.869 \text{ GeV}, M_{D^{\pm}} = 1.865 \text{ GeV}, M_{D_{s0}} = 1.968 \text{ GeV}, M_{D^{*\pm}} = 2.007 \text{ GeV},$$
$$M_{D^{*\pm}} = 2.010 \text{ GeV}, M_{D_{s0}} = 2.112 \text{ GeV}, M_{\eta_c(3S)}=3.942 \text{ GeV}, M_{\eta_c(4S)}=4.156 \text{ GeV}.$$

**TABLE I: The exclusive strong decay widths of \( \eta_c(3S) \) and \( \eta_c(4S) \) (unit in MeV).**

| Mode | \( D^0 D^{*0} \) | \( D^+ D^{*-} \) | \( D D^{*} \) | \( D^+ D^{*0} \) | \( D^{*-} D^{*+} \) | \( D^{*+} D^{*} \) |
|------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( \eta_c(3S) \) | 18.0^{+9.9}_{-7.7} | 15.5^{+9.4}_{-7.6} | 33.5^{+18.4}_{-15.3} | – | – | – |
| \( \eta_c(4S) \) | 27.7^{+4.7}_{-5.6} | 27.0^{+5.7}_{-6.0} | 54.7^{+10.4}_{-11.6} | 0.28^{+0.24}_{-0.15} | 7.7^{+0.1}_{-0.1} | 7.2^{+5.7}_{-4.4} | 14.9^{+11.8}_{-10.3} |

Considering \( X(3940) \) as \( \eta_c(3S) \) state, there is only one decay mode: \( 0^- \to 1^-0^- \), According to the kinematic ranges, the corresponding final states are: \( D^0 D^{*0}, \bar{D}^0 D^{*0}, D^+ D^{*-} \) and \( D^{*-} D^{*+} \). Considering \( X(4160) \) as \( \eta_c(4S) \) state, there are two decay mode: \( 0^- \to 1^-0^- \) and \( 0^- \to 1^-1^- \), within the kinematic ranges, the corresponding decay channels include: \( D^0 D^{*0}, D^0 D^{*0}, D^+ D^{*-}, D^{*-} D^{-}, D^+ D^{*-}, D^+ D^{*-}, D^{*0} D^{*0} \) and \( D^{*-} D^{*+} \). We have shown the exclusive two-body open charm strong decay widths of \( \eta_c(3S) \) and \( \eta_c(4S) \) in Table. II where \( D \bar{D}^{*} \) means \( D^0 \bar{D}^{*0} + D^{*0} \bar{D}^{-} \), and \( D^* \bar{D}^{*} \) means \( D^{*0} \bar{D}^{*0} + D^{*0} \bar{D}^{*+} \). for \( D^0 \bar{D}^{*0}, D^+ D^{*-} \) and \( D^{*-} D^{*+} \), we have considered the isospin conservation of the final mesons. In Table. III we have presented the total widths with different theoretical model and the experimental data for convenience. We also consider the uncertainties by varying all the input parameters simultaneously within \( \pm 5\% \) of the central values in Table. II and Table. III.
FIG. 2: The relation of decay width to the mass of $\eta_c(3S)$.

In Table. II we find that the dominant strong decay channels of $\eta_c(3S)$ is $D \bar{D}^*$, and agree with the experimental observation by Belle collaboration [2, 3]. The total two-body open charm strong decay widths of $\eta_c(3S)$ is $\Gamma_{\eta_c(3S)} = (33.5^{+18.4}_{-15.3})$ MeV, which is smaller than the result of Ref. [20], but it is in accordance with experimental results. So $\eta_c(3S)$ could be a good candidate of the $X(3940)$. Because of the mass of $X(3940)$ has the errors, we plot the relations of decay widths of $\eta_c(3S)$ to the masses of $\eta_c(3S)$ in Fig. 2, the relations of decay widths to the masses of $\eta_c(3S)$ are linear. The decay widths increase with the increase of the masses of $\eta_c(3S)$.

TABLE II: The total strong decay widths of $\eta_c(3S)$ and $\eta_c(4S)$ (unit in MeV). ‘Ex.’ means the experimental data of $X(3940)$ and $X(4160)$ from PDG [1].

| Mode        | Ours [17] | [20] | Ex                  |
|-------------|-----------|------|---------------------|
| $\Gamma_{\eta_c(3S)}$ | $33.5^{+18.4}_{-15.3}$ | $99.8 \pm 12.0$ | $37^{+26}_{-15} \pm 8$ |
| $\Gamma_{\eta_c(4S)}$ | $69.9^{+22.4}_{-21.1}$ | 25.0 | $139^{+111}_{-61} \pm 21$ |

For $\eta_c(4S)$ state, the main strong decay channels are $D \bar{D}^*$ and $D^* \bar{D}$, $\eta_c(4S) \rightarrow D^- \bar{D}^{*+}$ is very small with the small phase space, and the decay $\eta_c(4S) \rightarrow D \bar{D}$ is forbidden. In Table. III the total two-body open charm strong decay widths of $\eta_c(4S)$ is $\Gamma_{\eta_c(4S)} = (69.9^{+22.4}_{-21.1})$ MeV. Our result is larger than the result of Ref. [17], but considering the uncertainties of the results, our result is closed to the lower limit of $X(4160)$ for experimental data [3]. In our calculation, the ratio of the decay width $\frac{\Gamma(\eta_c(4S) \rightarrow D \bar{D})}{\Gamma(\eta_c(4S) \rightarrow D^* \bar{D}^*)} = 0$, which is consistent with the
experimental data $\frac{\Gamma(X(4160)\rightarrow D\bar{D}}{\Gamma(X(4160)\rightarrow D^{*}\bar{D}^{*})} < 0.09$ [3]. There is another ratio of the decay width: $\frac{\Gamma(\eta_c(4S)\rightarrow D\bar{D}^{*})}{\Gamma(\eta_c(4S)\rightarrow D^{*}\bar{D}^{*})} = 3.67$, which is much larger than the upper limit of the experimental data $\frac{\Gamma(X(4160)\rightarrow D\bar{D}^{*})}{\Gamma(X(4160)\rightarrow D^{*}\bar{D}^{*})} < 0.22$ which is reported by Belle [3]. In order to find out the relation of the decay width to the mass of $\eta_c(4S)$, we plot the relation of different decay width and decay ratio to the mass of $\eta_c(4S)$ in Fig. 3 and Fig. 4. Especially in Fig. 4, the decay ratio is decreased with the increased mass of $\eta_c(4S)$, but the decay ratio is larger than the experimental data at large mass, so $\eta_c(4S)$ is not the candidate of $X(4160)$, and more investigations of $X(4160)$ is needed in future.

FIG. 3: The relation of different decay width to the mass of $\eta_c(4S)$.

FIG. 4: The relation of $\Gamma_{\eta_c(4S)\rightarrow DD^{*}}/\Gamma_{\eta_c(4S)\rightarrow D^{*}\bar{D}^{*}}$ to the mass of $\eta_c(4S)$.

In summary, considering $X(3940)$ and $X(4160)$ as $\eta_c(3S)$ and $\eta_c(4S)$ states, we study the two-body open charm OZI-allowed strong decay of $\eta_c(3S)$ and $\eta_c(4S)$ by the improved B-S method combine with the $^3P_0$ model. For the strong decay of $\eta_c(3S)$, the dominant
strong decay is $\eta_c(3S) \rightarrow D\bar{D}^*$, the corresponding strong decay width is $\Gamma_{\eta_c(3S)} = (33.5^{+18.4}_{-15.3})$ MeV, which is close to the experimental data, therefore, $\eta_c(3S)$ is a good candidate of $X(3940)$. For $\eta_c(4S)$ state, the main strong decay channels are $D\bar{D}^*$ and $D^*\bar{D}^*$, $\eta_c(4S)$ cannot decay to $D\bar{D}$, which have not been observed for $X(4160)$ in experiment. $\Gamma(D^*\bar{D}^*)$ is smaller than $\Gamma(D\bar{D}^*)$, the ratio of the decay width $\frac{\Gamma(D\bar{D}^*)}{\Gamma(D^*\bar{D}^*)}$ is larger than the experimental data by Belle. We also find that the ratio of the decay width $\frac{\Gamma(D\bar{D}^*)}{\Gamma(D^*\bar{D}^*)}$ is dependent on the mass of $\eta_c(4S)$. Finally, we calculate the strong decay width of $\eta_c(4S)$: $\Gamma_{\eta_c(4S)} = (69.9^{+22.4}_{-21.1})$ MeV, considering the errors of the results, it’s closed to the lower limit of $X(4160)$. With large errors of full decay width, it’s hard to confirm that $\eta_c(4S)$ is the candidate of $X(4160)$. But the ratio of the decay width $\frac{\Gamma(D\bar{D}^*)}{\Gamma(D^*\bar{D}^*)}$ is not consistent with the experimental data, so taking the $\eta_c(4S)$ as an assignment of $X(4160)$ can be excluded and more investigations is needed in future.

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