On Conditional Correlations

Lei Yu *

May 28, 2019

Abstract

The Pearson correlation, correlation ratio, and maximal correlation have been well-studied in the literature. In this paper, we study the conditional versions of these quantities. We extend the most important properties of the unconditional versions to the conditional versions, and also derive some new properties. Based on the conditional maximal correlation, we define an information-correlation function of two arbitrary random variables, and use it to derive an impossibility result for the problem of the non-interactive simulation of random variables.

Index terms— Correlation coefficient, correlation ratio, maximal correlation, information-correlation function, non-interactive simulation

1 Introduction

In the literature, there are various measures available to quantify the strength of the dependence between two random variables. These include the Pearson correlation coefficient, the correlation ratio, the maximal correlation coefficient, etc. The Pearson correlation coefficient is a well-known measure that quantifies the linear dependence between two real-valued random variables. For real-valued random variables $X$ and $Y$, it is defined as

\[
\rho(X; Y) = \begin{cases} \frac{\text{cov}(X,Y)}{\sqrt{\text{var}(X)\text{var}(Y)}}, & \text{var}(X)\text{var}(Y) > 0, \\ 0, & \text{var}(X)\text{var}(Y) = 0. \end{cases}
\]

The correlation ratio was introduced by Pearson (see e.g. [1]), and studied by Rényi [2, 3]. For a real-valued random variable $X$ and a random variable $Y$ defined on an arbitrary Borel-measurable space, the correlation ratio of $X$ on $Y$ is defined by

\[
\theta(X; Y) = \sup_g \rho(X; g(Y)),
\]

where the supremum is taken over all Borel-measurable real-valued functions $g(y)$ such that $\text{var}(g(Y)) < \infty$. Rényi [2, 3] showed that

\[
\theta(X; Y) = \sqrt{\frac{\text{var}(\mathbb{E}[X|Y])}{\text{var}(X)}} = \sqrt{1 - \frac{\mathbb{E}[	ext{var}(X|Y)]}{\text{var}(X)}}.
\]

Another related measure of dependence is the Hirschfeld-Gebelein-Rényi maximal correlation (or simply maximal correlation), which quantifies the maximum possible (Pearson) correlation between square integrable real-valued random variables that are respectively generated by each of two random variables. For two random variables $X$ and $Y$ defined on arbitrary Borel-measurable spaces, the maximal correlation between $X$ and $Y$ is defined by

\[
\rho_m(X; Y) = \sup_{f,g} \rho(f(X); g(Y)),
\]

*Department of Electrical and Computer Engineering, National University of Singapore, Singapore (Email: leiyu@nus.edu.sg).
Remark 1. Where the supremum is taken over all Borel-measurable real-valued functions \( f(x), g(y) \) such that \( \text{var}(f(X)), \text{var}(g(Y)) < \infty \). This measure was first introduced by Hirschfeld [4] and Gebelein [5], then studied by Rényi [2]. Recently, it has been exploited in studying some information-theoretic problems, such as measuring non-local correlations [6], maximal correlation secrecy [7], deriving converse results for distributed communication [8], etc. Furthermore, the maximal correlation is also related to the Gács-Körner or Wyner common information [9, 10]. The Gács-Körner common information is strictly positive, if and only if the maximal correlation is equal to 1. The Wyner common information is strictly positive, if and only if the maximal correlation is strictly positive.

In this paper, we extend the Pearson correlation, the correlation ratio, and the maximal correlation to their conditional versions. We investigate various properties of these correlations. We also introduce an information-correlation function of two arbitrary random variables, and use it to derive an impossibility result for the problem of the non-interactive simulation of random variables.

2 Definition

Let \((\Omega, \Sigma, \mathbb{P})\) be a probability space. Let \((X, Y, Z, U) : (\Omega, \Sigma) \rightarrow (\mathbb{R}^4, \mathcal{B}(\mathbb{R}^4))\) be a real-valued random vector, where \(\mathcal{B}(\mathbb{R}^4)\) denotes the Borel \(\sigma\)-algebra on \(\mathbb{R}^4\). For a random variable (or random vector) \(W\), we denote the probability distribution as \(P_W\), i.e., \(P_W := \mathbb{P} \circ W^{-1}\). If \(W\) is discrete, then we use \(P_W\) to denote the probability mass function (pmf). If \(W\) is absolutely continuous (the distribution is absolutely continuous with respect to the Lebesgue measure), then we use \(p_W\) to denote the probability density function (pdf).

In the following, we define several conditional correlations, including the conditional (Pearson) correlation, the conditional correlation ratio, and the conditional maximal correlation.

**Definition 1.** The conditional (Pearson) correlation\(^1\) of \(X\) and \(Y\) given \(U\) is defined by

\[
\rho(X; Y|U) = \begin{cases} \frac{\mathbb{E}[	ext{cov}(X, Y|U)]}{\sqrt{\mathbb{E}[\text{var}(X|U)]} \sqrt{\mathbb{E}[\text{var}(Y|U)]}}, & \mathbb{E}[\text{var}(X|U)] \mathbb{E}[\text{var}(Y|U)] > 0, \\ 0, & \mathbb{E}[\text{var}(X|U)] \mathbb{E}[\text{var}(Y|U)] = 0. \end{cases}
\]

**Definition 2.** The conditional correlation ratio of \(X\) on \(Y\) given \(U\) is defined by

\[
\theta(X; Y|U) = \sup_g \rho(X; g(Y, U)|U),
\]

where the supremum is taken over all Borel-measurable real-valued functions \(g(y, u)\) such that \(\mathbb{E}[\text{var}(g(Y, U)|U)] < \infty\).

**Definition 3.** The conditional maximal correlation of \(X\) and \(Y\) given \(U\) is defined by

\[
\rho_m(X; Y|U) = \sup_{f,g} \rho(f(X, U); g(Y, U)|U),
\]

where the supremum is taken over all Borel-measurable real-valued functions \(f(x, u), g(y, u)\) such that \(\mathbb{E}[\text{var}(f(X, U)|U)], \mathbb{E}[\text{var}(g(Y, U)|U)] < \infty\).

**Remark 1.** If \(U\) is degenerate, then these three conditional correlations reduce to their unconditional versions.

**Remark 2.** Note that \(\rho(X; Y|U) = \rho(Y; X|U)\) and \(\rho_m(X; Y|U) = \rho_m(Y; X|U)\), but in general \(\theta(X; Y|U) \neq \theta(Y; X|U)\). That is, the conditional correlation and the conditional maximal correlation are symmetric, but the conditional correlation ratio is not.

\(^1\) Here \(U\) does not need to be real-valued. But for brevity, we assume it is. Similarly, in the following, \((Y, U)\) does not need to be real-valued in the definition of conditional correlation ratio, and \((X, Y)\) does not need to be real-valued in the definition of the conditional maximal correlation.
By the definitions, it is easy to verify that
\[ \rho_m(X;Y|U) = \sup_f \theta(f(X,U);Y|U), \]  
where the supremum is taken over all Borel-measurable real-valued functions \( f(x, u) \) such that \( \mathbb{E}[\text{var}(f(X,U)|U)] < \infty \).

Note that the unconditional versions of correlation coefficient, correlation ratio, and maximal correlation have been studied extensively in the literature; see [2, 3]. The conditional version of maximal correlation was first introduced by Ardestanizadeh et al. [11]. They studied the conditional maximal correlation of Gaussian random variables, and showed that for this case, the conditional maximal correlation is equal to the conditional Pearson correlation. They applied this property to derive upper bounds for the sum-capacity of a Gaussian multi-access channel (with linear feedback). Beigi and Gohari [6] applied the conditional maximal correlation to study the problem of non-local correlations in a bipartite quantum system (which is modeled as a no-signaling box). For such a no-signaling box, a sub-tensorization property of the conditional maximal correlation was proven [6, Corollary 6]. The sub-tensorization (or tensorization) property is rather useful in bounding the correlation between two random vectors, especially when the two random vectors consist of a large number of i.i.d. pairs of components. This is because, due to the sub-tensorization (or tensorization) property, the resulting bound is independent of the number of components in the vectors, and hence it is non-trivial even when the number of components tends to infinity. In [14] Beigi and Gohari studied the relationship between the conditional maximal correlation and the conditional hypercontractivity. They also introduced a general principle to obtain new measures from additivity measures such that the new measures have both tensorization and data processing properties. In this paper, we study various properties of the conditional versions of Pearson correlation, correlation ratio, and maximal correlation of arbitrary random variables (not restricted to be Gaussian or discrete). In order to state our results clearly, we define the various correlations conditioned on a given event as follows.

**Definition 4.** Given an event \( A \), denote the conditional distribution of \((X,Y)\) given \( A \) as \( P_{X,Y|A} \). Assume \((X',Y')\) is a pair of random variables satisfying \((X',Y') \sim P_{X,Y|A}\). Then we define \( \kappa(X';Y') := \kappa(X;Y) \) as the *event conditional correlations* of \( X \) and \( Y \) given \( A \), where \( \kappa \in \{\rho, \theta, \rho_m\} \) and \( \kappa(X';Y') \) denotes the corresponding unconditional correlation of \( X' \) and \( Y' \).

Obviously, these event conditional correlations are special cases of the corresponding conditional correlations. Moreover, if the distribution of \((X',Y')\) is the same as the conditional distribution of \((X,Y)\) given \( U = u \), then the unconditional correlations of \((X',Y')\) respectively equal the corresponding event conditional correlations of \((X,Y)\) given \( U = u \), i.e., \( \kappa(X';Y') = \kappa(X;Y|U = u) \) where \( \kappa \in \{\rho, \theta, \rho_m\} \). Moreover, if the distribution of \( U \) satisfies \( P(U = u) = 1 \) for some \( u \), then the conditional correlations of \((X,Y)\) given \( U \) respectively equal the corresponding event conditional correlations of \((X,Y)\) given \( U = u \), i.e., \( \kappa(X;Y|U) = \kappa(X;Y|U = u) \) where \( \kappa \in \{\rho, \theta, \rho_m\} \).

### 3 Properties

#### 3.1 Basic Properties: Other Characterizations, Continuity, and Concavity

In this subsection, we provide other characterizations for the conditional correlation ratio and conditional maximal correlation, and then study continuity (or discontinuity) and concavity of the conditional maximal correlation. First by definition, we have the following basic properties.

**Theorem 1.** For any random variables \( X, Y, Z, U \), we have that
\[
\theta(X;Y,Z|U) \geq \theta(X;Y|U);
\rho_m(X;Y,Z|U) \geq \rho_m(X;Y|U).
\]

Next we characterize the conditional correlation ratio and conditional maximal correlation by ratios of variances.

3
Theorem 2. (Characterization by the ratio of variances). For any random variables $X, Y, Z, U$, we have that

$$\theta(X; Y|U) = \sqrt{\frac{\mathbb{E}[\text{var}(E[X|Y,U]|U)]}{\mathbb{E}[\text{var}(X|U)]}}$$

$$= \sqrt{1 - \frac{\mathbb{E}[\text{var}(X|Y,U)]}{\mathbb{E}[\text{var}(X|U)]}};$$

$$\rho_m(X; Y|U) = \sup_f \sqrt{\frac{\mathbb{E}[\text{var}(E[f(X,U)|Y,U]|U)]}{\mathbb{E}[\text{var}(f(X,U)|U)]}}$$

$$= \sup_f \sqrt{1 - \frac{\mathbb{E}[\text{var}(f(X,U)|Y,U)]}{\mathbb{E}[\text{var}(f(X,U)|U)]}}. \quad (3)$$

Remark 3. The correlation ratio is also closely related to the Minimum Mean Square Error (MMSE). The optimal MMSE estimator is $\mathbb{E}[X|Y, U]$, hence the variance of the MMSE for estimating $X$ given $(Y, U)$ is $\text{mmse}(X|Y, U) = \mathbb{E}(X - \mathbb{E}[X|Y, U])^2 = \mathbb{E}[\text{var}(X|Y, U)] = \mathbb{E}[\text{var}(X|U)](1 - \theta^2(X; Y|U))$. Hence the proof is omitted here. Next we characterize conditional correlations by event conditional correlations.

The unconditional version of Theorem 2 was proven by Rényi [2]. Theorem 2 can be proven similarly as the unconditional versions in [2].

Theorem 3. (Characterization by event conditional correlations). For any random variables $X, Y, U$,

$$\rho(X; Y|U) \leq \text{ess sup}_u \rho(X; Y|U = u), \quad (5)$$

$$\text{ess inf}_u \theta(X; Y|U = u) \leq \theta(X; Y|U) \leq \text{ess sup}_u \theta(X; Y|U = u), \quad (6)$$

$$\rho_m(X; Y|U) = \text{ess sup}_u \rho_m(X; Y|U = u), \quad (7)$$

where $\text{ess sup}_u f(u) := \inf \{ \lambda : \mathbb{P}(f(U) > \lambda) = 0 \}$ and $\text{ess inf}_u f(u) := \sup \{ \lambda : \mathbb{P}(f(U) < \lambda) = 0 \}$ respectively denote the essential supremum and the essential infimum of $f$.

Remark 5. It is worth noting that $\rho(X; Y|U) \geq \text{ess inf}_u \rho(X; Y|U = u)$ does not hold in general. This can be seen from the following example. Assume $(a, b, \eta)$ are three numbers such that $0 < \eta \leq 1, 0 < a < b$. Suppose that $(W, Z)$ is a pair of random variables such that $\text{var}(W) = a, \text{var}(Z) = b$ and $\rho(W; Z) = \eta$. (It is obvious that there are many random variable pairs satisfying the conditions.) Denote the distribution of $(W, Z)$ as $P_{W,Z}$. Now we consider a triple of random variables $(X, Y, U)$ such that $P_U(0) = P_U(1) = \frac{1}{2}$ and $(X,Y)|U = 0 \sim P_{W,Z}$ and $(X,Y)|U = 1 \sim P_{Z,W}$. Then we have $\rho(X; Y|U = 0) = \rho(X; Y|U = 1) = \eta$. Hence $\text{ess inf}_u \rho(X; Y|U = u) = \eta$. However, $\rho(X; Y|U) = \frac{2\sqrt{ab}}{a+b} < \eta$. Hence $\rho(X; Y|U) < \text{ess inf}_u \rho(X; Y|U = u)$ for this example.

Remark 6. If $U$ is a discrete random variable, then

$$\rho_m(X; Y|U) = \sup_{u: P_U(u) > 0} \rho_m(X; Y|U = u), \quad (8)$$

where $P_U$ denotes the pmf of $U$. Beigi and Gohari [6, 14] defined the conditional maximal correlation via (8). Theorem 3 implies the equivalence between the conditional maximal correlation defined by us and that defined by Beigi and Gohari.

Remark 7. If $U$ is an absolutely continuous random variable, then

$$\rho_m(X; Y|U) = \inf_{q_U: p_U = p_U \text{ a.e. } u: q_U(u) > 0} \sup_{u: q_U(u) > 0} \rho_m(X; Y|U = u),$$

where $p_U$ denotes the pdf of $U$ and $q_U$ denotes another pdf on the same space.
Therefore, to show (5), we only need to show
\[
\rho(X; Y|U) = \frac{\mathbb{E}[\text{cov}(X, Y|U)]}{\sqrt{\mathbb{E}[\text{var}(X|U)] \mathbb{E}[\text{var}(Y|U)]}}
\]
(9)
\[
\leq \frac{\mathbb{E}[\text{cov}(X, Y|U)]}{\mathbb{E}[\text{var}(X|U)] \mathbb{E}[\text{var}(Y|U)]}
\]
(10)
\[
= \inf_{\lambda > \lambda^*} \frac{\mathbb{E}[\text{cov}(X, Y|U)]}{\sqrt{\mathbb{E}[\text{var}(X|U)] \mathbb{E}[\text{var}(Y|U)]} \cdot 1 \{U \in \mathbb{R} \backslash A_\lambda\}}
\]
(11)
\[
\leq \inf_{\lambda > \lambda^*} \sup_{u \in \mathbb{R} \backslash A_\lambda} \rho(X; Y|U = u)
\]
\[
\leq \inf_{\lambda > \lambda^*} \lambda = \lambda^*,
\]
(12)
where (10) follows by the Cauchy-Schwarz inequality, and (11) follows from [15, Theorem 15.2 (v)] and the
fact \(P_U(A_\lambda) = 0\) for any \(\lambda > \lambda^*\).

By using the relationship (3) and by derivations similar as (9)-(12), it is easy to obtain (6).

Finally, we prove (7). Similarly as in the proof above, we denote \(A_\lambda := \{u : \rho_m(X; Y|U = u) > \lambda\}\) and \(\lambda^* := \inf \{\lambda : P_U(A_\lambda) = 0\}\). Hence \(P_U(A_\lambda) = 0\) for any \(\lambda > \lambda^*\); \(P_U(A_\lambda) > 0\) for any \(\lambda < \lambda^*\); and \(\lambda^* = \text{ess sup}_u \rho_m(X; Y|U = u)\). Therefore, to prove (7), we only need to show \(\rho_m(X; Y|U) = \lambda^*\). On one hand, by derivations similar as (9)-(12), we can upper bound \(\rho_m(X; Y|U)\) as follows.
\[
\rho_m(X; Y|U) = \sup_f \sqrt{\frac{\mathbb{E}[\text{var}(\mathbb{E}[f(X, U)|Y, U]|U)]}{\mathbb{E}[\text{var}(f(X, U)|U)]}}
\]
\[
= \sup_f \inf_{\lambda > \lambda^*} \sqrt{\frac{\mathbb{E}[\text{var}(\mathbb{E}[f(X, U)|Y, U]|U)]}{\mathbb{E}[\text{var}(f(X, U)|U)]} \cdot 1 \{U \in \mathbb{R} \backslash A_\lambda\}}
\]
\[
\leq \sup_f \inf_{\lambda > \lambda^*} \sup_{u \in \mathbb{R} \backslash A_\lambda} \sqrt{\frac{\mathbb{E}[\text{var}(\mathbb{E}[f(X, U)|Y, U]|U = u)]}{\mathbb{E}[\text{var}(f(X, U)|U = u)]}}
\]
\[
\leq \inf_{\lambda > \lambda^*} \sup_{u \in \mathbb{R} \backslash A_\lambda} \rho_m(X; Y|U = u)
\]
\[
\leq \inf_{\lambda > \lambda^*} \lambda = \lambda^*.
\]

On the other hand, we assume that \(\tilde{f}(x, u)\) is a function such that
\[
\sqrt{\frac{\text{var}(\mathbb{E}[f(X, U)|Y, U = u]|U = u)}{\text{var}(f(X, U)|U = u)}}
\]
\[
\geq \alpha \rho_m(X; Y|U = u)
\]
for each \(u \in A_\lambda\), where \(\lambda < \lambda^*\) and \(0 < \alpha < 1\). The existence of \(\tilde{f}(x, u)\) follows from the
definition of \(\rho_m(X; Y|U = u)\). According to the definition of \(A_\lambda\), we have that \(P_U(A_\lambda) > 0\), and for each \(u \in A_\lambda\),
\[
\frac{\text{var}(\mathbb{E}[f(X, U)|Y, U = u]|U = u)}{\text{var}(f(X, U)|U = u)} \geq (\alpha \lambda)^2.
\]
(13)
Set \(f(x, u) = \tilde{f}(x, u) \cdot 1 \{u \in A_\lambda\}\). Then
\[
\rho_m(X; Y|U) \geq \sqrt{\frac{\mathbb{E}[\text{var}(\mathbb{E}[f(X, U)|Y, U]|U)]}{\mathbb{E}[\text{var}(f(X, U)|U)]}}
\]
and any \( \{0,1\} \), denote \( \rho_m(X;Y|U) \) as the second largest singular value of the bivariate function \( \frac{p(x,y|u)}{\sqrt{p(x|u)p(y|u)}} \), where \( p(x,y|u) \) denotes a conditional pdf of \( (X,Y) \) respect to \( U \). Then we have the following singular value characterization of conditional maximal correlation.

**Theorem 4.** (Singular value characterization). Assume \( X,Y \) are discrete random variables with finite supports, or absolutely-continuous random variables such that \( \int_{x,y} \left( \frac{p(x,y|u)}{\sqrt{p(x|u)p(y|u)}} \right)^2 \, dx dy < \infty \) a.s. Then

\[
\rho_m(X;Y|U) = \text{ess sup}_u \lambda_2(u).
\]

**Remark 8.** This property is consistent with the one of the unconditional version by setting \( U \) to a constant, i.e., \( \rho_m(X;Y) = \lambda_2 \).

**Proof.** The unconditional version of this theorem was proven in [10]. That is, for discrete \( X,Y \) with finite supports, \( \rho_m(X;Y) \) equals the second largest singular value \( \lambda_2 \) of the matrix \( Q \) with entries \( \frac{P(x,y|u)}{\sqrt{P(x|u)p(y|u)}} \); for absolutely-continuous \( X,Y \) such that \( \int_{x,y} \left( \frac{p(x,y|u)}{\sqrt{p(x|u)p(y|u)}} \right)^2 \, dx dy < \infty \) with \( p(x,y) \) denoting a pdf of \( (X,Y) \), \( \rho_m(X;Y) \) equals the second largest singular value \( \lambda_2 \) of the bivariate function \( \frac{p(x,y)}{\sqrt{p(x)p(y)}} \). Combining this with Theorem 3, we have (15).

Note that, \( \rho_m(X;Y|U) \) is a mapping that maps a distribution \( P_{X,Y,U} \) to a real number in \([0,1]\). Now we study the concavity of such a mapping.

**Corollary 1.** (Concavity). Given \( P_{X,Y,U} \), \( \rho_m(X;Y|U) \) is concave in \( P_U \). That is, for any distributions \( P_U \) and \( Q_U \), and any \( \lambda \in [0,1] \), \( \rho_m((1-\lambda)P_U+\lambda Q_U)(X;Y|U)(X;Y|U) \geq (1-\lambda)\rho_m(P_U X,Y|U) + \lambda \rho_m(Q_U X,Y|U) \geq \rho_m(P_{X,Y,U}) \).

**Proof.** This theorem directly follows from the characterization in (7).

For a discrete random variable, the distribution is uniquely determined by its pmf. Therefore, for discrete random variables \( (X,Y,U) \), \( \rho_m(X;Y|U) \) can be also seen as a mapping that maps a pmf \( P_{X,Y,U} \) to a real number in \([0,1]\). Assume \( X,Y,U \subset \mathbb{R} \) are three finite sets. Denote \( \mathcal{P}(X \times Y \times U) \) as the set of all pmfs defined on \( X \times Y \times U \) (i.e., \( |X| |Y| |U| - 1 \) dimensional probability simplex). Consider \( \rho_m(X;Y|U) \) as a mapping \( \rho_m(X;Y|U) : \mathcal{P}(X \times Y \times U) \rightarrow [0,1] \). Now we study the continuity (or discontinuity) of such a mapping.
Corollary 2. (Continuity and discontinuity). For finite sets $X, Y, U \subset \mathbb{R}$, $\rho_m(X; Y|U)$ is continuous (under the total variation distance) on $\{P_{X,Y,U} \in \mathcal{P}(X \times Y \times U) : P_U(u) > 0, \forall u \in U\}$. But in general, $\rho_m(X; Y|U)$ is discontinuous at $P_U$ such that $P_U(u) = 0, \exists u \in U$.

Proof. For a pmf $P_{X,Y,U} \in \mathcal{P}(X \times Y \times U)$, $\rho_m(X; Y|U) = \max_{u:P(u)>0} \lambda_2(u)$. On the other hand, singular values are continuous in the matrix (see [16, Corollary 8.6.2]), hence $\lambda_2(u)$ is continuous in $P_{X,Y,U}$. Furthermore, since $P_U(u) > 0, \forall u \in U$, $Q_{X,Y,U} \rightarrow P_{X,Y,U}$ in the total variation distance sense implies $Q_U \rightarrow P_U$ and $Q_{X,Y,U}=u \rightarrow P_{X,Y,U}=u, \forall u \in U$. Therefore, $\rho_m^Q(X; Y|U) \rightarrow \rho_m(X; Y|U)$, where $\rho_m^Q(X; Y|U)$ denotes the conditional maximal correlation under distribution $Q_{X,Y,U}$. However, if there exists $u_0 \in U$ such that $P_U(u_0) = 0$ and $\lambda_2(u_0) > \max_{u:P(u)>0} \lambda_2(u)$. Then letting $Q_U \rightarrow P_U$ in a direction such that $Q_U(u_0) > 0$ always holds, we have that $\rho_m^Q(X; Y|U) \geq \lambda_2(u_0) > \rho_m(X; Y|U)$ always holds. This implies $\rho_m(X; Y|U)$ is discontinuous at $P_{X,Y,U}$.

For random variables $X, Y, U$, the conditional Gács-Körner common information between $X$ and $Y$ given $U$ is defined as

$$C_{GK}(X; Y|U) = \sup_{(f,g):f(X,U)=g(Y,U) \text{ a.s.}} H(f(X,U)|U),$$

where $H(Z|U) := -\mathbb{E} \log P_Z(Z|U)$ denotes the conditional entropy of $Z$ given $U$. If $U$ is degenerate, then $C_{GK}(X; Y) := C_{GK}(X; Y|U)$ is the unconditional version of Gács-Körner common information between $X$ and $Y$ [9].

Theorem 5. 1) For any random variables $X, Y, U$,

$$0 \leq |\rho(X; Y|U)| \leq \theta(X; Y|U) \leq \rho_m(X; Y|U) \leq 1.$$

2) Moreover, $\rho_m(X; Y|U) = 0$ if and only if $X$ and $Y$ are conditionally independent given $U$. Furthermore, for discrete random variables $X, Y, U$ with finite supports, $\rho_m(X; Y|U) = 1$ if and only if $C_{GK}(X; Y|U) > 0$.

Proof. The statement 1) follows from the definitions of the conditional correlations. The statement 2) with degenerate $U$ (i.e., the unconditional version) was proven by Rényi [2]. The statement 2) with non-degenerate $U$ (i.e., the conditional version) follows by combining Rényi’s result on the unconditional version [2] and the characterization of conditional maximal correlation in (7).

Next we focus on the Gaussian case. It was shown in [11] that the conditional maximal correlation and the conditional Pearson correlation are equal for jointly Gaussian random variables, i.e., $\rho_m(X; Y|U) = |\rho(X; Y|U)|$. However, in [11], the conditional Pearson correlation was defined differently, although it is equal to our definition for the Gaussian case. More specifically, the conditional Pearson correlation in [11] was defined as the expectation of the event conditional correlation, i.e., $\mathbb{E}_u(\rho(X; Y|U = u))$.

Theorem 6. [11] (Gaussian case). For jointly Gaussian random variables $X, Y, U$, we have

$$|\rho(X; Y|U)| = \theta(X; Y|U) = \theta(Y; X|U) = \rho_m(X; Y|U).$$

(17)

For completeness, we provide the following proof of Theorem 6, in which the properties derived above are applied.

Proof. The unconditional version of (17) was proven in [17, Sec. IV, Lem. 10.2]. On the other hand, given $U = u$, $(X, Y)$ also follows jointly Gaussian distribution, and $\rho(X; Y|U = u) = \rho(X; Y|U)$ for any $u$. Hence

$$\rho_m(X; Y|U) = \sup_u \rho_m(X; Y|U = u) \quad (18)$$

$$= \sup_u |\rho(X; Y|U = u)| \quad (19)$$

where (18) follows from Theorem 3, and (19) follows from the unconditional version [17, Sec. IV, Lem. 10.2].

Furthermore, both $\theta(X; Y|U)$ and $\theta(Y; X|U)$ are between $\rho_m(X; Y|U)$ and $|\rho(X; Y|U)|$. Hence (17) holds.
3.2 Other Properties: Tensorization, DPI, Correlation ratio equality, and Conditioning reducing covariance gap

The tensorization property and the data processing inequality for the unconditional maximal correlation were proven in [10, Thm. 1] and [18, Lem. 2.1] respectively. Here we extend them to the conditional case.

**Theorem 7. (Tensorization).** Assume \((X^n, Y^n) = (X_i, Y_i)_{i=1}^n\) and given \(U\), \((X_i, Y_i), 1 \leq i \leq n\) are conditionally independent. Then we have

\[
\rho_m(X^n; Y^n|U) = \max_{1 \leq i \leq n} \rho_m(X_i; Y_i|U).
\]

**Proof.** The unconditional version

\[
\rho_m(X^n; Y^n) = \max_{1 \leq i \leq n} \rho_m(X_i; Y_i),
\]

for a sequence of pairs of independent random variables \((X^n, Y^n)\) is proven in [10, Thm. 1]. Hence the result for the event conditional maximal correlation also holds. Using this result and Theorem 4, we have

\[
\rho_m(X^n; Y^n|U) = \sup_u \rho_m(X^n; Y^n|U = u)
\]

\[
= \sup_u \max_{1 \leq i \leq n} \rho_m(X_i; Y_i|U = u)
\]

\[
= \max_{1 \leq i \leq n} \sup_u \rho_m(X_i; Y_i|U = u)
\]

\[
= \max_{1 \leq i \leq n} \rho_m(X_i; Y_i|U),
\]

(20)

where (20) follows by the following lemma.

**Lemma 1.** Assume \(I\) is a countable set, and \(P_U\) is an arbitrary distribution on \((\mathbb{R}, \mathcal{B}_R)\). Then for any function \(f : I \times \mathbb{R} \to \mathbb{R}\), we have

\[
\sup_i \sup_u f(i, u) = \sup_i \sup_u f(i, u).
\]

This lemma follows from the following two points. For a number \(\epsilon > 0\), assume \(i^* \in I\) satisfies that \(\sup_i f(i, u) \geq \sup_{i \in I} \sup_u f(i, u) - \epsilon\). Then for any function \(f\),

\[
\sup_i \sup_u f(i, u) \geq \sup_i f(i, u) = \sup_i \sup_u f(i, u) - \epsilon.
\]

Since \(\epsilon > 0\) is arbitrary, we have \(\sup_i \sup_u f(i, u) \geq \sup_i \sup_u f(i, u)\)

On the other hand, denote \(\lambda_i^* := \sup_u f(i, u)\). Then by the definition of \(\sup\), we have \(P_U \{u : f(i, u) > \lambda_i^*\} = 0\) for all \(i \in I\). Hence by the union bound, we have \(P_U \{u : \exists i \in I \text{ s.t. } f(i, u) > \lambda_i^*\} = 0\). Furthermore, for a number \(\epsilon > 0\), \(\sup_{i \in I} f(i, u) > \sup_{i \in I} \lambda_i^* + \epsilon\) implies that there exists an \(i' \in I\) such that \(f(i', u) \geq \sup_{i \in I} f(i, u) - \epsilon > \sup_{i \in I} \lambda_i^* \geq \lambda_i^*\). Hence

\[
P_U \left\{u : \sup_{i \in I} f(i, u) > \sup_{i \in I} \lambda_i^* + \epsilon\right\} \leq P_U \left\{u : \exists i \in I \text{ s.t. } f(i, u) > \lambda_i^*\right\} = 0.
\]

By the definition of \(\sup\), we have \(\sup_u \sup_{i \in I} f(i, u) \leq \sup_{i \in I} \lambda_i^* + \epsilon\). Since \(\epsilon > 0\) is arbitrary, we have \(\sup_u \sup_{i \in I} f(i, u) \leq \sup_{i \in I} \sup_u f(i, u)\). \(\Box\)

**Theorem 8. (Data processing inequality).** If random variables \(X, Y, Z, U\) form a Markov chain \(X \rightarrow (Z, U) \rightarrow Y\) (i.e., \(X\) and \(Y\) are conditionally independent given \((Z, U)\)), then

\[
|\rho(X; Y|U)| \leq \theta(X; Z|U)\theta(Y; Z|U),
\]

(21)
\( \theta(X; Y|U) \leq \theta(X; Z|U)\rho_m(Y; Z|U), \) \hfill (22)
\( \rho_m(X; Y|U) \leq \rho_m(X; Z|U)\rho_m(Y; Z|U). \) \hfill (23)

Moreover, equalities hold in (21)-(23) if \((X, Z, U)\) and \((Y, Z, U)\) have the same joint distribution.

**Proof.** Consider

\[
E[\text{cov}(X, Y|U)] = E[(X - E[X|U])(Y - E[Y|U])]
= E[E[(X - E[X|U])(Y - E[Y|U])|Z, U]]
= E[E[X - E[X|U]|Z, U]E[Y - E[Y|U]|Z, U]]
= E[E[X|Z, U] - E[X|U])(E[Y|Z, U] - E[Y|U])]
\leq \sqrt{E[\text{cov}(X, Y|Z, U)^2]}\sqrt{E[\text{cov}(Y|Z, U)^2]}
= \sqrt{E[\text{var}(E[X|Z, U]|U)]E[\text{var}(E[Y|Z, U]|U)]}
\]

where (24) follows by the conditional independence, and (25) follows by the Cauchy-Schwarz inequality. Hence

\[
|\rho(X; Y|U)| = \frac{E[\text{cov}(X, Y|U)]}{\sqrt{E[\text{var}(X|U)]\sqrt{E[\text{var}(Y|U)]}}}
\leq \sqrt{\frac{E[\text{cov}(E[X|Z, U]|U)]E[\text{cov}(E[Y|Z, U]|U)]}{E[\text{var}(X|U)]E[\text{var}(Y|U)]}}
= \theta(X; Z|U)\theta(Y; Z|U).
\]

By the definitions of conditional correlation ratio and conditional maximal correlation and using the relationships (1) and (2), (22) and (23) can be derived from (21). Furthermore, it is easy to verify that equalities in (21)-(23) hold if \((X, Z, U)\) and \((Y, Z, U)\) have the same joint distribution. \(\square\)

**Theorem 9.** (Correlation ratio equality). For any random variables \(X, Y, Z, U,\)

\[
1 - \theta^2(X; Y, Z|U) = (1 - \theta^2(X; Z|U))(1 - \theta^2(X; Y|Z, U));
\] \hfill (26)
\[
1 - \rho_m^2(X; Y, Z|U) \geq (1 - \rho_m^2(X; Z|U))(1 - \rho_m^2(X; Y|Z, U));
\] \hfill (27)
\[
\theta(X; Y, Z|U) \geq \theta(X; Y|Z, U);
\] \hfill (28)
\[
\rho_m(X; Y, Z|U) \geq \rho_m(X; Y|Z, U).
\] \hfill (29)

**Remark 9.** The inequality in (29) is analogue to a similar property for mutual information, i.e., \(I(X; Y, Z|U) \geq I(X; Y|Z, U),\) where\(^2\)

\[
I(X; Y|W) := \mathbb{E}\log \frac{P_{X,Y|W}(X, Y|W)}{P_{X|W}(X|W)P_{Y|W}(Y|W)}
\] \hfill (30)

denotes the conditional mutual information between \(X\) and \(Y\) given \(W\) [24]. If \(W\) is degenerate, then \(I(X; Y) := I(X; Y|W)\) is the (unconditional) mutual information between \(X\) and \(Y\). Furthermore, \(\rho_m(X; Y, Z|U) \geq \rho_m(X; Y|U)\) and \(\rho_m(X; Y, Z|U) \leq \rho_m(X; Y|U)\) do not always hold. This is also analogue to the fact that \(I(X; Y, Z|U) \geq I(X; Y|U)\) and \(I(X; Y, Z|U) \leq I(X; Y|U)\) do not always hold.

**Proof.** From (3), we have

\[
1 - \theta^2(X; Y, Z|U) = \frac{E[\text{var}(X|Y, Z, U)]}{E[\text{var}(X|U)]},
\]

\(^2\)If \(X, Y, W\) are not discrete, then the term \(P_{X,Y|W}(X, Y|W)\) in (30) is replaced by the Radon–Nikodym derivative of \(P_{X,Y|W}(X, Y|W)\) with respect to \(P_{X|W}(X|W)P_{Y|W}(Y|W)\).
Theorem 10. (Conditioning reduces covariance gap). For any random variables $X, Y, Z, U$,

$$
1 - \theta^2(X; Z|U) = \frac{\mathbb{E}[\text{var}(X|Z,U)]}{\mathbb{E}[\text{var}(X|U)]},
$$

$$
1 - \theta^2(X; Y|Z, U) = \frac{\mathbb{E}[\text{var}(X|Y, Z, U)]}{\mathbb{E}[\text{var}(X|Z,U)]}.
$$

Hence (26) follows immediately.

The inequality (27) follows since

$$
1 - \rho^2_m(X; Y, Z|U) = \inf \left\{1 - \theta^2(f(X,U); Y, Z|U) \right\}
$$

$$
= \inf \left\{(1 - \theta^2(f(X,U); Z|U))(1 - \theta^2(f(X,U); Y, Z|U)) \right\}
$$

$$
\geq \inf \left\{(1 - \theta^2(f(X,U); Z|U)) \inf \left\{(1 - \theta^2(f(X,U); Y|Z,U)) \right\} \right\}
$$

$$
= (1 - \rho^2_m(X; Z|U))(1 - \rho^2_m(X; Y|Z,U)).
$$

Furthermore, observe that $\theta^2(X; Z|U) \geq 0$. Hence (28) follows immediately from (26).

By (28), we have

$$
\rho_m(X; Y|Z,U) = \sup_f \theta(f(X,U); Y|Z,U) \leq \sup_f \theta(f(X,U); Y, Z|U) \leq \rho_m(X; Y, Z|U).
$$

\[\square\]

Corollary 3. For any random variables $U, X, Y, V$ such that $U \rightarrow X \rightarrow Y$ and $X \rightarrow Y \rightarrow V$, we have

$$
\rho_m(U, X; Y, V) = \max\{\rho_m(X; Y), \rho_m(U; V|X, Y)\}. \quad (31)
$$

Remark 10. A “dual” (i.e., additivity) property holds for mutual information, i.e., for any random variables $U, X, Y, V$ such that $U \rightarrow X \rightarrow Y$ and $X \rightarrow Y \rightarrow V$,

$$
I(U, X; V, Y) = I(X; Y) + I(U; V|X, Y).
$$

Remark 11. Corollary 3 can be applied to measure the non-local correlations in a bipartite quantum system. Imagine that two parties share (possibly correlated) subsystems of a bipartite physical system. Each party applies a measurement on her subsystem by tuning her measurement device according to some parameter, and obtains a measurement outcome. Denote the measurement parameters by $X$ and $Y$, and the measurement outcomes by $U$ and $V$. Assume the no-signaling condition holds, i.e., the random variables $U, X, Y, V$ satisfy Markov chains $U \rightarrow X \rightarrow Y$ and $X \rightarrow Y \rightarrow V$. For this case, the conditional distribution $P_{U,V|X,Y}$ is termed a no-signaling box and the measurement parameter pair $(X,Y)$ is termed a priori correlation. Hence by Corollary 3, the equality (31) holds for such a system. This means that the maximal correlation between the two input-output pairs of the box, is equal to the maximum of the priori maximal correlation between the two parties and the maximal correlation of the box shared between them. For more details, please refer to [6].

Proof. Beigi and Gobari [6, Eqn. (4)] proved $\rho_m(U, X; V, Y) \leq \max\{\rho_m(X; Y), \rho_m(U; V|X, Y)\}$. Hence we only need to prove that $\rho_m(U, X; V, Y) \geq \max\{\rho_m(X; Y), \rho_m(U; V|X, Y)\}$. According to the definition, $\rho_m(U, X; V, Y) \geq \rho_m(X; Y)$ is straightforward. From (29) of Theorem 9, we have $\rho_m(U, X; V, Y) \geq \rho_m(U, X; V|Y) \geq \rho_m(U; V|X, Y)$. This completes the proof. \[\square\]

We also prove that conditioning reduces covariance gap as shown in the following theorem, the proof of which is given in Appendix A.

Theorem 10. (Conditioning reduces covariance gap). For any random variables $X, Y, Z, U$,

$$
\sqrt{\mathbb{E}\text{var}(X|Z,U)}\mathbb{E}\text{var}(Y|Z,U) - \mathbb{E}\text{cov}(X, Y|Z, U) \leq \sqrt{\mathbb{E}\text{var}(X|Z)\mathbb{E}\text{var}(Y|Z)} - \mathbb{E}\text{cov}(X, Y|Z),
$$

i.e.,

$$
\sqrt{(1 - \theta^2(X; U|Z))(1 - \theta^2(Y; U|Z))} \leq 1 - \rho(X, Y|Z, U)).
$$
Remark 12. The following two inequalities follow immediately.
\[
\sqrt{(1 - \rho_m^2(X|Z))(1 - \theta^2(Y|Z))(1 - \theta(X,Y|Z,U))} \leq 1 - \theta(X,Y|Z),
\]
\[
\sqrt{(1 - \rho_m^2(X|Z))(1 - \rho_m^2(Y|Z))(1 - \rho_m(X,Y|Z,U))} \leq 1 - \rho_m(X,Y|Z).
\]

4 Application to Non-Interactive Simulation

Assume \((X,Y) \sim P_{XY}\) is a pair of random variables on a product measurable space \((\mathcal{X} \times \mathcal{Y}, \mathcal{B}_X \otimes \mathcal{B}_Y)\), and \((X^n,Y^n) \sim P^n_{XY}\) are \(n\) i.i.d. copies of \((X,Y)\). Then we focus on the following non-interactive simulation problem: Given the distribution \(P_{XY}\) and a product measurable space \((U \times V, \mathcal{B}_U \otimes \mathcal{B}_V)\), what is the possible probability distribution \(P_{UV}\) on \((U \times V, \mathcal{B}_U \otimes \mathcal{B}_V)\) such that \(U^n \rightarrow X^n \rightarrow Y^n \rightarrow V^n\) and \((U^n,V^n) \sim P^n_{UV}\)?

Definition 5. The simulation set of \(n\) i.i.d. pairs \((X^n,Y^n) \sim P^n_{XY}\) is defined as
\[
S_n(P_{XY}) := \{P_{UV} : \exists (U^n, V^n) \sim P^n_{UV} \text{ s.t. } U^n \rightarrow X^n \rightarrow Y^n \rightarrow V^n\}.
\]

This problem is termed Non-Interactive Simulation of Random Variables [12]. The case in which \(X,Y,U,V \in \{0,1\}\) and only one-dimensional \((U,V)\) is required to be generated was studied in [19]. The non-interactive simulation problem is motivated naturally by several models in distributed control systems and cryptography. It is also related to the Non-Interactive Correlation Distillation Problem, in which the collision probability \(P(U = V)\) is required to be maximized [20, 21, 22]. Therefore, studying the non-interactive simulation problem is not only of theoretical significance, but is also of tremendous applicabilities. Furthermore, the non-interactive simulation problem or the non-interactive correlation distillation problem can be also interpreted from the perspectives of noise-stability (or noise-sensitivity); see [21].

In this paper, we will provide an impossibility result for the non-interactive simulation problem by using a new quantity named information-correlation function. Before investigating the non-interactive simulation problem, we introduce the information-correlation function first.

4.1 Information-Correlation Function

Definition 6. For \((X,Y) \sim P_{XY}\), the information-correlation function of \(X\) and \(Y\) is defined by
\[
C_\beta(X;Y) := \inf_{P_{W|X,Y} : \rho_m(X;Y|W) \leq \beta} I(X,Y;W), \quad \beta \in [0,1],
\]
where the mutual information \(I(X,Y;W)\) is defined in (30). Furthermore, for \(\beta \in (0,1]\), we define
\[
C_{\beta^-}(X;Y) := \lim_{\alpha \uparrow \beta} C_\alpha(X;Y).
\]

Intuitively, the information-correlation function of \(X\) and \(Y\) quantifies the minimum amount of “common information” that can be extracted from \((X,Y)\) such that the “private information” of \(X\) and \(Y\) is at most \(\beta\)-correlated under the conditional maximal correlation measure, i.e., \(\rho_m(X;Y|W) \leq \beta\). Two closely related quantities are the Gács-Körner common information [9] and Wyner common information [23]. The former is defined in (16), and the latter is defined as follows. The Wyner common information between \(X\) and \(Y\) is
\[
C_W(X;Y) := \inf_{P_{W|X,Y} : X \rightarrow W \rightarrow Y} I(X,Y;W).
\]

Properties of the information-correlation function, as well as its relationship to the Gács-Körner common information and Wyner common information are shown in the following proposition. The proof is given in Appendix B.

---

\(^3\)We use \(P^n_{XY}\) to denote the \(n\)-fold product of distribution \(P_{XY}\) with itself.
Proposition 1. (a) If \((X, Y)\) has finite support \(X \times Y\), then for the infimum in (32), it suffices to restrict the support size of \(W\) such that \(|W| \leq |X||Y|\).

(b) For any random variables \(X, Y\), the information-correlation function \(C_\beta(X; Y)\) is non-increasing in \(\beta\). Moreover,

\[
\begin{align*}
C_\beta(X; Y) &= 0 \text{ for } \rho_m(X; Y) \leq \beta \leq 1, \quad (33) \\
C_\beta(X; Y) &> 0 \text{ for } 0 \leq \beta < \rho_m(X; Y), \quad (34) \\
C_0(X; Y) &= C_W(X; Y), \quad (35) \\
C_1(X; Y) &= C_{\text{GK}}(X; Y). \quad (36)
\end{align*}
\]

\(C_\beta(X; Y)\) is discontinuous at \(\beta = 1\), if the Gács-Körner common information between \(X, Y\) is strictly positive. Lemma 1 implies the Gács-Körner common information and Wyner common information are two special points on the information-correlation function.

\(C_\beta(X; Y)\) is non-increasing in \(\beta\), but it is not necessarily convex or concave; see the Gaussian case in the next subsection.

Another important property of the information-correlation function — the data processing inequality — is provided in the following proposition.

Proposition 2. (Data processing inequality). If random variables \(X, Z, Y\) form a Markov chain \(X \rightarrow Z \rightarrow Y\) (i.e., \(X\) and \(Y\) are conditionally independent given \(Z\)), then

\[
C_\beta(X; Y) \leq C_\beta(X; Z), \forall \beta \in [0, 1].
\]

Proof. Assume random variables \(X, Z, Y\) form a Markov chain \(X \rightarrow Z \rightarrow Y\). For an arbitrary conditional distribution \(P_{W|X,Z}\), we introduce a new random vector \(W\) such that \((X, Z, Y, W) \sim P_{XZ}P_{Y|Z}P_{W|X,Z}\). Hence \(W \rightarrow (X, Z) \rightarrow Y\) and \(X \rightarrow (Z, W) \rightarrow Y\). By the data processing inequality on mutual information [24, Theorem 2.8.1], we have

\[
I(X, Y; W) \leq I(X, Z; W). \quad (38)
\]

By the data processing inequality on maximal correlation (Theorem 8), we have

\[
\rho_m(X; Y|W) \leq \rho_m(X; Z|W). \quad (39)
\]

Combining (38) and (39), we obtain that

\[
C_\beta(X; Y) \leq C_\beta(X; Z), \forall \beta \in [0, 1].
\]

For jointly Gaussian random variables, the information-correlation function is characterized in the following proposition. The proof is given in Appendix C.
Proposition 3. (Gaussian random variables). For jointly Gaussian random variables \((X, Y)\) with correlation coefficient \(\beta_0\),
\[
C_{\beta}(X; Y) = \frac{1}{2} \log^+ \left[ \frac{1 + \beta_0}{1 - \beta_0} \right]. \tag{40}
\]

Remark 14. When specialized to the case \(\beta = 0\), we obtain \(C_{W}(X; Y) = C_{0}(X; Y) = \frac{1}{2} \log^+ \left[ \frac{1}{1 - \beta_0} \right]\), which was first proven in [25].

For the symmetric bivariate random variable, an upper bound on the information-correlation function is given in the following proposition. The proof is given in Appendix D.

Proposition 4. For the symmetric bivariate random variable \((X, Y)\) with distribution
\[
P_{XY} = \begin{bmatrix}
\frac{1}{2} (1 - p_0) & \frac{1}{2} p_0 & \frac{1}{2} (1 - p_0)
\end{bmatrix},
\]
(i.e., the crossover probability being \(p_0\)), we have
\[
C_{\beta}(X; Y) \leq 1 + H_2(p_0) - H_4 \left( \frac{1}{2} \left( 1 - p_0 + \sqrt{\frac{1 - 2p_0 - \beta}{1 - \beta}} \right), \frac{1}{2} \left( 1 - p_0 - \sqrt{\frac{1 - 2p_0 - \beta}{1 - \beta}} \right), \frac{p_0}{2}, \frac{p_0}{2} \right) \tag{41}
\]
for \(0 \leq \beta < 1 - 2p_0\), and \(C_{\beta}(X; Y) = 0\) for \(\beta \geq 1 - 2p_0\), where
\[
H_2(p) = -p \log p - (1 - p) \log(1 - p), \tag{42}
\]
\[
H_4(a, b, c, d) = -a \log a - b \log b - c \log c - d \log d \tag{43}
\]
respectively denote the binary and quaternary entropy functions.

Remark 15. Numerical results show that the upper bound in (41) is tight.

4.2 Impossibility Result

Based on the information-correlation function, we can establish the following impossibility result for the non-interactive simulation problem.

Theorem 11. The simulation set of \(n\) i.i.d. pairs \((X^n, Y^n) \sim P^n_{XY}\) satisfies
\[
\mathcal{S}_n(P_{XY}) \subseteq \{P_{UV} : C_{\beta}(U; V) \leq C_{\beta}(X; Y), \forall \beta \in [0, 1] \}.
\]

Proof. Consider the simulation of \((U^n, V^n) \sim P^n_{UV}\) from \((X^n, Y^n) \sim P^n_{XY}\) such that \(U^n \rightarrow X^n \rightarrow Y^n \rightarrow V^n\). Applying the vector version of the data processing inequality in Proposition 2, we obtain that
\[
C_{\rho}(U^n; V^n) \leq C_{\rho}(X^n; V^n) \leq C_{\rho}(X^n; Y^n).
\]
By the additivity of the information-correlation function (Lemma 1), we obtain
\[
C_{\rho}(U; V) \leq C_{\rho}(X; Y).
\]
Intuitively, since $X^n, Y^n, U^n, V^n$ form a Markov chain $U^n \to X^n \to Y^n \to V^n$, it makes sense that $X^n$ and $Y^n$ possess more “common information” than $U^n$ and $V^n$. Hence the necessary condition given in Theorem 11 holds.

We can also obtain the following simple bounds for the simulation problem. By the data processing inequality on maximal correlation (i.e., the unconditional version of Theorem 8), we obtain the following outer bound.

$$
S_n(P_{XY}) \subseteq \{P_{UV} : \rho_m(U;V) \leq \rho_m(X;Y)\}.
$$

(44)

From (34), we know that our outer bound in Theorem 11 is at least as tight as the outer bound in (44). By the data processing inequality on mutual information [24, Theorem 2.8.1], we obtain the following outer bound.

$$
S_n(P_{XY}) \subseteq \{P_{UV} : I(U;V) \leq I(X;Y)\}.
$$

(45)

Furthermore, we can obtain the following inner bound by using a pair of product conditional distributions $(P^n_{U|X}, P^n_{V|Y})$.

$$
S_n(P_{XY}) \supseteq \{P_{UV} : \exists (P_{U|X}, P_{V|Y}) \text{ s.t. } P_{UV} \text{ is the marginal distribution of } P_{U|X}P_{X,Y}P_{V|Y} \text{ on } (U,V)\}.
$$

(46)

Our outer bound in Theorem 11, the maximal correlation outer bound in (44), the mutual information outer bound in (45), and the inner bound in (46) are plotted in Fig. 1. For this figure, we assume $X, Y, U, V \in \{0, 1\}$, i.e., they are binary random variables. For this case, the joint distribution of $X$ and $Y$ is determined by the triple $(P_X(0), P_Y(0), P_{X,Y}(0,0))$, and so is the joint distribution of $U$ and $V$. For Fig. 1, we assume $P_X(0) = P_Y(0) = \frac{1}{2}$ and $P_{X,Y}(0,0) = p$. Similarly, we assume $P_U(0) = P_V(0) = \frac{1}{2}$ and $P_{U,V}(0,0) = q$. Given $p$, we focus on the possible range of $q$. The end points of the possible range of $q$ are located between our outer bound and the inner bound.

Figure 1: Illustration of our outer bound in Theorem 11, the maximal correlation outer bound in (44), the mutual information outer bound in (45), and the inner bound in (46). The curves are symmetric with respect to the line $q = \frac{1}{4}$. 
5 Concluding Remarks

In this paper, we defined several conditional correlation measures and derived their properties, especially for the conditional maximal correlation. From these properties, one can observe that the maximal correlation and correlation ratio share many similar properties as the mutual information, such as invariance to bijections, chain rule (correlation ratio equality), data processing inequality, etc. On the other hand, the maximal correlation and correlation ratio also have some properties that are different from those of the mutual information. For example, for a sequence of pairs of independent random variables, the mutual information between them is the sum of mutual information over all pairs of components (i.e., additivity); while the maximal correlation is the maximum of the maximal correlations over all pairs of components (i.e., tensorization). Furthermore, we used the conditional maximal correlation to define the information-correlation function, and derived a data processing inequality for such a function. As an application, we applied this data processing inequality to obtain an impossibility result for the non-interactive simulation problem.

The (conditional) maximal correlation also has applications in inference and privacy. In inference and privacy, a fundamental question is that: Given an observation $Y$, how much information can we learn about a hidden random variable $X$? Or equivalently, how much information is leaked from $X$ to $Y$? In [26, 27, 28, 7, 29], the maximal correlation $\rho_m(X;Y)$ was used to measure the information leakage from $X$ to $Y$ (or from $Y$ to $X$). Furthermore, in [8], the present author, together with Li and Chen, applied the conditional maximal correlation to derive converse results for the problem of transmission of correlated sources over a multi-access channel, in which the correlated sources are assumed to have a common part. The tensorization and data processing properties of the conditional maximal correlation (derived in the present paper) play a crucial role in such an application.

A Proof of Theorem 10

For simplicity, we only prove the degenerate $Z$ case, i.e.,

$$\sqrt{\mathbb{E} \text{var}(X|U) \mathbb{E} \text{var}(Y|U)} - \mathbb{E} \text{cov}(X,Y|U) \leq \sqrt{\text{var}(X)\text{var}(Y)} - \text{cov}(X,Y).$$

(47)

For non-degenerate $Z$ case, it can be proven similarly.

By the law of total covariance, we have

$$\text{cov}(X,Y) = \mathbb{E} \text{cov}(X,Y|U) + \text{cov}(\mathbb{E}(X|U), \mathbb{E}(Y|U)).$$

Hence to prove (47), we only need to show

$$\sqrt{\mathbb{E} \text{var}(X|U) \mathbb{E} \text{var}(Y|U)} + \text{cov}(\mathbb{E}(X|U), \mathbb{E}(Y|U)) \leq \sqrt{\text{var}(X)\text{var}(Y)}. \quad (48)$$

To prove this, we consider

$$\mathbb{E} \text{var}(X|U) \mathbb{E} \text{var}(Y|U) = (\text{var}(X) - \text{var}(\mathbb{E}(X|U))) (\text{var}(Y) - \text{var}(\mathbb{E}(Y|U)))$$

$$=\text{var}(X)\text{var}(Y) + \text{var}(\mathbb{E}(X|U))\text{var}(\mathbb{E}(Y|U))$$

$$- \text{var}(X)\text{var}(\mathbb{E}(Y|U)) - \text{var}(Y)\text{var}(\mathbb{E}(X|U))$$

$$\leq \text{var}(X)\text{var}(Y) + \text{var}(\mathbb{E}(X|U))\text{var}(\mathbb{E}(Y|U))$$

$$- 2\sqrt{\text{var}(X)\text{var}(\mathbb{E}(Y|U)) \cdot \text{var}(Y)\text{var}(\mathbb{E}(X|U))}$$

$$= \left(\sqrt{\text{var}(X)\text{var}(Y)} - \sqrt{\text{var}(\mathbb{E}(X|U))\text{var}(\mathbb{E}(Y|U))}\right)^2. \quad (50)$$

where (49) follows from the law of total variance

$$\text{var}(X) = \mathbb{E} \text{var}(X|U) + \text{var}(\mathbb{E}(X|U)). \quad (51)$$
Therefore, by the Cauchy-Schwarz inequality, it holds that another random variable

\[ \text{var}(X) \leq \text{cov}(X,Y) \leq \text{var}(Y). \]

Thus, we have

\[ \text{var}(X) \leq \text{var}(Y). \]

Therefore,

\[ \text{var}(X) \text{var}(Y) \leq \text{var}(X) \text{var}(Y). \]

Combining (50) and (52), we have

\[ \sqrt{\text{var}(X) \text{var}(Y)} \leq \sqrt{\text{var}(X) \text{var}(Y)} - \sqrt{\text{var}(X) \text{var}(Y)}. \]

Furthermore, by the Cauchy-Schwarz inequality, it holds that

\[ |\text{cov}(X,Y)| = |E[(E[X|U] - E(X))(E[Y|U] - E(Y))]| \]

\[ \leq \sqrt{E[(E[X|U] - E(X))^2] \cdot (E[(E[Y|U] - E(Y))^2]}

\[ = \sqrt{\text{var}(X) \text{var}(Y)}. \]

Therefore,

\[ \sqrt{\text{var}(X) \text{var}(Y)} \leq \sqrt{\text{var}(X) \text{var}(Y)} - |\text{cov}(X,Y)| \]

\[ \leq \sqrt{\text{var}(X) \text{var}(Y)} - \text{cov}(X,Y). \]

This is just the inequality (48). Hence the proof is complete.

## B Proof of Lemma 1

### Proof of (a): To show (a), we only need to show that for any random variable $W$, there always exists another random variable $W'$ with support $W$ such that $|W'| \leq |X||Y|$, $\rho_m(X; Y|W') \leq \rho_m(X; Y|W)$, and

\[ I(X,Y; W') = I(X,Y; W). \]

According to the support lemma [30], there exists a random variable $W'$ with $W' \subseteq W$ and $|W'| \leq |X||Y|$ such that

\[ H(X,Y|W') = H(X,Y|W), \]

\[ P_{X,Y} = \sum_{w'} P_{W'}(w') P_{X,Y|W'}(\cdot|w'). \]

Since $W' \subseteq W$, we have $\rho_m(X; Y|W') \leq \rho_m(X; Y|W)$. Furthermore, (54) implies that $H(X,Y)$ is also preserved. Hence $I(X,Y; W') = I(X,Y; W')$. This completes the proof of (a).

Proof of (b): (33) and (34) follow straightforwardly by definition. By definition and Lemma 5 ($\rho_m(X; Y|W) = 0$ if and only if $X \rightarrow W \rightarrow Y$), we obtain (35).

Next we prove (36). Assume $(f^*, g^*)$ attains the Gács-Körner common information between $X$ and $Y$ (see the definition in (16)). Set $W = f^*(X)$, then we have

\[ \rho_m(X; Y|W) < 1, \]

\[ I(X,Y; W) = H(f^*(X)) = C_{\text{GK}}(X; Y). \]

Hence by definition,

\[ C_{1-}(X; Y) \leq C_{\text{GK}}(X; Y). \]

On the other hand, for any $W$ such that $\rho_m(X; Y|W) < 1$, the random variable $f^*(X)$ is a deterministic function of $W$, i.e., $f^*(X) = g(W)$ for some function $g$. This is because, otherwise, we have

\[ \rho_m(X; Y|W) \geq \rho(f^*(X); g^*(Y)|W) \]

\[ = \rho(f^*(X); f^*(X)|W) \]

\[ = 1. \]
Since $f^*(X)$ is a deterministic function of $W$, we have

$$I(X, Y; W) = I(X, Y; W, f^*(X)) \geq H(f^*(X)) = C_{\text{GK}}(X; Y).$$

Hence

$$C_{1-}(X; Y) \geq C_{\text{GK}}(X; Y). \tag{56}$$

Combining (55) and (56) gives us

$$C_{1-}(X; Y) = C_{\text{GK}}(X; Y).$$

Proof of (c): Suppose $P_{W|X,Y}$ achieves the infimum in (32). If $V$ satisfies both $(X, Y) \to W \to V$ and $(X, Y) \to V \to W$, then we have $\rho_m(X: Y|W) = \rho_m(X: Y|W, V) = \rho_m(X: Y|V)$.

If $V$ satisfies $(X, Y) \to W \to V$ but does not satisfy $(X, Y) \to V \to W$, then $I(X, Y; W) = I(X, Y; W, V) > I(X, Y; V)$. Hence $\rho_m(X, Y|W) \leq \rho_m(X, Y|V)$, otherwise it contradicts with that $P_{W|X,Y}$ achieves the infimum in (32).

Proof of (d): For (37) it suffices to prove the case of $n = 2$, i.e.,

$$C_\beta(X^2; Y^2) = C_\beta(X_1; Y_1) + C_\beta(X_2; Y_2). \tag{57}$$

Observe for any $P_{W|X^2,Y^2},$

$$\rho_m(X^2, Y^2|W) \geq \rho_m(X_i; Y_i|W), i = 1, 2,$$

and

$$I(X^2, Y^2; W) \geq I(X_1, Y_1; W) + I(X_2, Y_2; W|X_1, Y_1)
= I(X_1, Y_1; W) + I(X_2, Y_2; W, X_1, Y_1)
\geq I(X_1, Y_1; W) + I(X_2, Y_2; W).$$

Hence we have

$$C_\beta(X^2; Y^2) \geq C_\beta(X_1; Y_1) + C_\beta(X_2; Y_2). \tag{58}$$

Moreover, if we choose $P_{W|X^2,Y^2} = P_{W_i|X_i,Y_i}P_{W_1|X_1,Y_1}P_{W_2|X_2,Y_2}$ in $C_\beta(X^2; Y^2)$, where $P_{W_i|X_i,Y_i}, i = 1, 2$, is a distribution achieving $C_\beta(X_i; Y_i)$, then we have

$$\rho_m(X^2, Y^2|W) = \max_{i \in \{1, 2\}} \rho_m(X_i; Y_i|W) \leq \beta,$$

and

$$I(X^2, Y^2; W) = I(X_1, Y_1; W_1) + I(X_2, Y_2; W_2) = C_\beta(X_1; Y_1) + C_\beta(X_2; Y_2). \tag{59}$$

Therefore,

$$C_\beta(X^2; Y^2) = \inf_{P_{W|X^2,Y^2}: \rho_m(X^2, Y^2|W) \leq \beta} I(X^2, Y^2; W) \leq C_\beta(X_1; Y_1) + C_\beta(X_2; Y_2). \tag{60}$$

Inequalities (58) and (60) imply (37) with $n = 2$.  

17
C Proof of Proposition 3

For continuous random variables, a lower bound on the information-correlation function is given in the following lemma.

**Lemma 2.** (Lower bound on $C_\beta(X;Y)$) For any absolutely continuous random variables $(X,Y)$ with correlation coefficient $\beta_0$, we have

$$C_\beta(X;Y) \geq h(X,Y) - \frac{1}{2} \log \left[ (2\pi e)^2 \det(\Sigma_{XY|W}) \right]$$

for $0 \leq \beta \leq \beta_0$, and $C_\beta(X;Y) = 0$ for $\beta_0 \leq \beta \leq 1$.

**Proof.**

$$I(X,Y;W) = h(X,Y) - h(X,Y|W)$$

$$\geq h(X,Y) - \mathbb{E}_W \frac{1}{2} \log \left[ (2\pi e)^2 \det(\Sigma_{XY|W}) \right]$$

$$\geq h(X,Y) - \frac{1}{2} \log \left[ (2\pi e)^2 \det(\Sigma_{XY|W}) \right]$$

$$= h(X,Y) - \frac{1}{2} \log \left[ (2\pi e)^2 \left[ \mathbb{E}_W(X|W)\mathbb{E}_Y(Y|W) - (\mathbb{E}_W(X|W)\mathbb{E}_Y(Y|W))^2 \right] \right]$$

$$= h(X,Y) - \frac{1}{2} \log \left[ (2\pi e)^2 \mathbb{E}_W(X|W)\mathbb{E}_Y(Y|W)(1 - \rho^2(X;Y|W)) \right]$$

$$\geq h(X,Y) - \frac{1}{2} \log \left[ (2\pi e)^2 \left( \frac{1 - \beta_0}{1 - \rho(X,Y|W)} \right)^2 (1 - \rho^2(X;Y|W)) \right]$$

$$= h(X,Y) - \frac{1}{2} \log \left[ (2\pi e)(1 - \beta_0)^2 \frac{1 + \rho(X,Y|W)}{1 - \rho(X,Y|W)} \right]$$

$$\geq h(X,Y) - \frac{1}{2} \log \left[ (2\pi e(1 - \beta_0))^2 \frac{1 + \beta}{1 - \beta} \right],$$

where (62) follows from the fact that given the covariance matrix $\Sigma_{XY}$ of $(X,Y)$, $h(X,Y) \leq \frac{1}{2} \log \left[ (2\pi e)^2 \det(\Sigma_{XY}) \right]$, (63) follows from the function log DET is concave on the set of symmetric positive definite square matrices \cite[p.73]{31}, (64) follows from Lemma 10, and (65) follows from the constraint $\rho(X;Y|W) \leq \beta$. \hfill \qed

Furthermore, it is easy to verify that equality in Theorem 2 holds if $X,Y$ are jointly Gaussian. (This can be shown by choosing $W$ such that $(W,X,Y)$ are jointly Gaussian.) Hence we complete the proof.

D Proof of Proposition 4

Assume $W \sim \text{Bern}(\frac{1}{2})$. Define two distributions $P_{X,Y|W}(\cdot|0) = \begin{bmatrix} a & p_0/2 \\ p_0/2 & b \end{bmatrix}$, $P_{X,Y|W}(\cdot|1) = \begin{bmatrix} b & p_0/2 \\ p_0/2 & a \end{bmatrix}$ with $a + b = 1 - p_0$. Then $P_{X,Y} = P_W(0)P_{X,Y|W}(\cdot|0) + P_W(1)P_{X,Y|W}(\cdot|1)$. By using the formula \cite{32}

$$\rho_m^2(X;Y) = \sum_{x,y} \frac{P^2(x,y)}{P(x)P(y)} - 1$$

for binary-valued $(X,Y)$, we have

$$\rho_m(X;Y) = 1 - 2p_0,$$

and

$$\rho_m(X;Y|W = 0) = \rho_m(X;Y|W = 1)$$

18
\[ \sqrt{\frac{2 (p_0/2)^2}{(a + p_0/2) (b + p_0/2)} + \frac{a^2}{(a + p_0/2)^2} + \frac{b^2}{(b + p_0/2)^2}} - 1. \]  

Hence \( \rho_m(X; Y|W) \) is also equal to the RHS of (66). By choosing

\[
\begin{align*}
a &= \frac{1}{2} \left( 1 - p_0 + \sqrt{\frac{1 - 2p_0 - \beta}{1 - \beta}} \right), \\
b &= \frac{1}{2} \left( 1 - p_0 - \sqrt{\frac{1 - 2p_0 - \beta}{1 - \beta}} \right),
\end{align*}
\]

we have \( \rho_m(X; Y|W) \leq \beta \). For this case, \( I(X, Y; W) \) is equal to the RHS of (41). Hence, by definition, the RHS of (41) is an upper bound of \( C_\beta(X; Y) \).

Acknowledgments

The authors are supported in part by a Singapore National Research Foundation (NRF) National Cybersecurity R&D Grant (R-263-000-C74-281 and NRF2015NCR-NCR003-006), and in part by a National Natural Science Foundation of China (NSFC) under Grant (61631017).

References

[1] H. Cramér. Mathematical methods of statistics (PMS-9), volume 9. Princeton university press, 2016.
[2] A. Rényi. On measures of dependence. Acta mathematica hungarica, 10(3-4):441–451, 1959.
[3] A. Rényi. New version of the probabilistic generalization of the large sieve. Acta Mathematica Hungarica, 10(1-2):217–226, 1959.
[4] H. O. Hirschfeld. A connection between correlation and contingency. In Mathematical Proceedings of the Cambridge Philosophical Society, volume 31, pages 520–524. Cambridge University Press, 1935.
[5] H. Gebelein. Das statistische problem der korrelation als variations-und eigenwertproblem und sein zusammenhang mit der ausgleichsrechnung. ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik, 21(6):364–379, 1941.
[6] S. Beigi and A. Gohari. Monotone measures for non-local correlations. IEEE Trans. on Inform. Theory, 61(9):5185–5208, 2015.
[7] C. T. Li and A. El Gamal. Maximal correlation secrecy. IEEE Trans. on Inform. Theory, 64(5):3916–3926, 2018.
[8] L. Yu, H. Li, and C. Chen. Distortion bounds for transmitting correlated sources with common part over mac. In Communication, Control, and Computing (Allerton), 2016 54th Annual Allerton Conference on, pages 687–694. IEEE, 2016.
[9] P. Gács and J. Körner. Common information is far less than mutual information. Problems of Control and Information Theory, 2(2):149–162, 1973.
[10] H. S. Witsenhausen. On sequences of pairs of dependent random variables. SIAM Journal on Applied Mathematics, 28(1):100–113, 1975.
[11] E. Ardestanizadeh, M. Wigger, Y.-H. Kim, and T. Javidi. Linear-feedback sum-capacity for gaussian multiple multiple access channels. IEEE Trans. on Inform. Theory, 58(1):224–236, 2012.
[12] S. Kamath and V. Anantharam. On non-interactive simulation of joint distributions. *IEEE Trans. on Inform. Theory*, 62(6):3419–3435, 2016.

[13] W. Kang and S. Ulukus. A new data processing inequality and its applications in distributed source and channel coding. *IEEE Trans. on Inform. Theory*, 57(1):56–69, 2011.

[14] S. Beigi and A. Gohari. On the duality of additivity and tensorization. *arXiv preprint arXiv:1502.00827*, 2016.

[15] P. Billingsley. *Probability and measure*. John Wiley & Sons, 2008.

[16] G. H. Golub and C. F. van Loan. *Matrix computations*, volume 3. JHU Press, 2012.

[17] Y. A. Rozanov. *Stationary random processes*. Holden-Day, 1967.

[18] Y. Yu. On the maximal correlation coefficient. *Statistics & Probability Letters*, 78(9):1072–1075, 2008.

[19] L. Yu and V. Y. F. Tan. Bounds on the average distance and distance enumerator with applications to non-interactive simulation. *arXiv preprint arXiv:1904.03932*, 2019.

[20] K. Yang. On the (im)possibility of non-interactive correlation distillation. *Theoretical Computer Science*, 382(2):157–166, 2007.

[21] E. Mossel and R. O’Donnell. Coin flipping from a cosmic source: On error correction of truly random bits. *Random Structures & Algorithms*, 26(4):418–436, 2005.

[22] E. Mossel, R. O’Donnell, O. Regev, J. E. Steif, and B. Sudakov. Non-interactive correlation distillation, inhomogeneous Markov chains, and the reverse Bonami-Beckner inequality. *Israel Journal of Mathematics*, 154(1):299–336, 2006.

[23] A. Wyner. The common information of two dependent random variables. *IEEE Trans. on Inform. Theory*, 21(2):163–179, 1975.

[24] T. M. Cover and J. A. Thomas. *Elements of Information Theory*. Wiley-Interscience, 2nd edition, 2006.

[25] G. Xu, W. Liu, and B. Chen. Wyner’s common information: Generalizations and a new lossy source coding interpretation. *arXiv preprint arXiv:1301.2237*, 2013.

[26] S. Asoodeh, F. Alajaji, and T. Linder. On maximal correlation, mutual information and data privacy. In *Information Theory (CIIT), 2015 IEEE 14th Canadian Workshop on*, pages 27–31. IEEE, 2015.

[27] F. P. Calmon, M. Varia, M. Medard, M. M. Christiansen, K. R. Duffy, and S. Tessaro. Bounds on inference. In *Communication, Control, and Computing (Allerton), 2013 51st Annual Allerton Conference on*, pages 567–574. IEEE, 2013.

[28] I. Issa, A. B. Wagner, and S. Kamath. An operational approach to information leakage. *arXiv preprint arXiv:1807.07878*, 2018.

[29] B. Rassouli and D. Gunduz. On perfect privacy and maximal correlation. *arXiv preprint arXiv:1712.08500*, 2017.

[30] A. El Gamal and Y.-H. Kim. *Network Information Theory*. Cambridge University Press, 2011.

[31] S. Boyd and L. Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

[32] V. Anantharam, A. Gohari, S. Kamath, and C. Nair. On hypercontractivity and a data processing inequality. In *2014 IEEE International Symposium on Information Theory*, pages 3022–3026. IEEE, 2014.