On Cosymplectic Conformal Connections
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(Dedicated to K.Yano)

Abstract. The aim of this paper is to introduce a cosymplectic analouge of conformal connection in a cosymplectic manifold and proved that if cosymplectic manifold $M$ admits a cosymplectic conformal connection which is of zero curvature, then the Bochner curvature tensor of $M$ vanishes.

Mathematics Subject Classification. 53C05, 53C18, 53C20 53D05.

Keywords. Kähler manifold, Sasakian manifold, Cosymplectic manifold.

1 Introduction

In 1958, Libermann [6, 7] introduced the cosymplectic manifolds. He defined it as :The pair $(\eta, \omega)$ of a 1-form $\eta$ and a 2-form $\omega$ such that $\eta \wedge \omega^n$ is a volume form on odd-dimensional manifold $M^{2m+1}$, defines an almost cosymplectic structure on $M$. If both $\eta$ and $\omega$ are closed, the structure is said to be cosymplectic. So, a manifold endowed with a cosymplectic structure $(M, \eta, \omega)$ is called a cosymplectic manifold.

After few years, in 1967, Blair [1] used the term “cosymplectic” to define manifold with almost contact metric structure satisfying a normality condition, although related to the Libermann’s definition.

In differential geometry, one of the most important fields is to study smooth maps which preserve certain geometric properties. In those transforma- tions, the conformal transformation is very interesting, where only angles are preserved both in magnitude and orientation but not necessarily distances. Stereographic projection is the simplest example of conformal transformation. It is believed that, in 1569, the property of conformal transformations is first used by Gerardus Mercator to produce the famous Mercator’s world map (the first angle-preserving (or conformal) world map). For more details on conformal transformations, see [3].

Definition 1.1 A diffeomorphism $f : (M, g) \to (N, \tilde{g})$ is said to be conformal mapping [4] if

$$\tilde{g} = e^{2p} g,$$  \hspace{1cm} (1.1)

where $p$ is a function on $M$. If $p$ is constant, then conformal mapping is called homothetic mapping.

Let $\nabla$ be a linear connection in an $n$-dimen- sional manifold $M$. The torsion tensor $T^h_{ji}$ of $\nabla$ is

$$T^h_{ji} = \Gamma^h_{ji} - \Gamma^h_{ij}.$$

The connection $\nabla$ is symmetric if the torsion tensor $T^h_{ji}$ vanishes, otherwise it is non-symmetric. The connection $\nabla$ is metric if there is a Riemannian metric $g_{ji}$ in $M$ such that $\nabla_k g_{ji} = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection.
Let $D$ be covariant differentiation operator with respect to Christoffel symbols $\Gamma^h_{ji}$ formed with $\tilde{g}$ corresponding to the conformal change (1.1), then we have

$$\Gamma^h_{ji} = \left\{ \begin{array}{c} h \\ j \end{array} \right\} + \delta^h_j p_i + \delta^h_i p_j - g_{ji} p^h,$$

(1.2)

where $p_i$ is the gradient of $p$, that is, $p_i = \partial_i p$ and $p^h = p_t g^{th}$, $g^{th}$ is the contravariant components of the metric tensor $g_{th}$. This affine connection is known as conformal connection.

So, we have

$$D_k(e^{2\psi} g_{ji}) = 0.$$ 

Let $R$ be the curvature tensor of the Riemannian metric $\tilde{g}$. Then

$$R^h_{kji} = K^h_{kji} + \delta^h_k p_{ji} - \delta^h_j p_{ki} + p^h_{k} g_{ji} - p^h_{j} g_{ki},$$

where $K$ be the curvature tensor of the Riemannian metric $g$ and

$$p_{ji} = \nabla_j p_i - p_j p^i + \frac{1}{2} \lambda p_t p^t g_{ji}.$$ 

(1.3)

If the Riemannian metric $g$ is conformally equivalent to the Riemannian metric $\tilde{g}$, which is locally Euclidean, then the Riemannian manifold with the metric $g$ is said to be conformally flat [10]. For which, we are stating the well known theorem given by Weyl [9].

**Theorem 1.2** A necessary and sufficient condition for a Riemannian manifold to be conformally flat is that Weyl conformal curvature tensor $C = 0$ for $n > 3$, or $C^h_{kji} = 0$, where

$$C^h_{kji} = K^h_{kji} + \delta^h_k C_{ji} - \delta^h_j C_{ki} + C^h_{k} g_{ji} - C^h_{j} g_{ki},$$

and

$$C_{ji} = p_{ji}.$$ 

In 1975, K. Yano [11] studied a complex analogue of conformal connection and proved the following remarkable theorems:

**Theorem 1.3** [11] In a Kählerian manifold with Hermitian metric tensor $g_{ji}$ and almost complex structure tensor $F^h_{i}$, the affine connection $D$ with components $\Gamma^h_{ji}$ which satisfies

$$D_k(e^{2\psi} g_{ji}) = 0,$$

$$D_k(e^{2\varphi} \varphi_{ji}) = 0,$$

$$\Gamma^h_{ji} - \Gamma^h_{ij} = -2 \varphi_{ji} q^h,$$

where $p$ is a scalar function, $q^h$ a vector field and $\varphi_{ji} = \varphi^t_j g_{ti}$, is given by

$$\Gamma^h_{ji} = \left\{ \begin{array}{c} h \\ j \end{array} \right\} + \delta^h_j p_i + \delta^h_i p_j - g_{ji} p^h + \varphi^h_j q_i + \varphi^h_i q_j - \varphi_{ji} q^h,$$

where $p_i$ is the gradient of $p$, that is, $p_i = \partial_i p$ and $p^h = p_t g^{th}$, $q_i = -p_t \varphi^t_i$, $q^h = q_t g^{th}$.
He called such an affine connection a complex conformal connection in a Kählerian manifold.

**Theorem 1.4** If in a real \( n \)-dimensional Kaehlerian manifold, \((n \geq 4)\), there exists a scalar function \( p \) such that the complex conformal connection is of zero curvature, then the Bochner curvature tensor of the manifold

\[
B^h_{kji} = R^h_{kji} + \delta^h_k L_{ji} - \delta^h_j L_{ki} + L^h_k g_{ji} - L^h_j g_{ki} \\
+ F^h_k M_{ji} - F^h_j M_{ki} + M^h_k F_{ji} - M^h_j F_{ki} \\
- 2(M_{kj} F^h_i + F_{kj} M^h_i)
\]

vanishes, where

\[
L_{ji} = -\frac{1}{n + 4} K_{ji} + \frac{1}{2(n + 2)(n + 4)} K g_{ji}, \quad L^h_k = L_{ki} g^h, \\
M_{ji} = -L_{ji} F^h_i, \quad M^h_k = M_{ki} g^h.
\]

After that Yano [12] studied a contact analogue of conformal connection and proved the following results:

**Theorem 1.5** In a Sasakian manifold with structure tensors \( (\varphi^h_i, \eta_i, g_{ji}) \), the affine connection \( D \) with components \( \Gamma^h_{ji} \) which satisfies

\[
D_k(e^{2p} g_{ji}) = 2e^{2p} p_k \eta_j \eta_i, \quad D_j \varphi^h_i = 0, \quad D_j \eta^h = 0 \\
\Gamma^h_{ji} - \Gamma^h_{ij} = -2\varphi_{ji} u^h,
\]

where \( p \) is a scalar function and \( u^h \) a vector field, is given by

\[
\Gamma^h_{ji} = \left\{ \begin{array}{c}
\frac{h}{j} \\
\delta^h_j - \eta_j \eta^h \end{array} \right\} + \left( \delta^h_j - \eta_j \eta^h \right) p_i + \left( \delta^h_i - \eta_i \eta^h \right) p_j - (g_{ji} - \eta_j \eta_i) p^h \\
+ \varphi^h_j (q_i - \eta_i) + \varphi^h_i (q_j - \eta_j) - \varphi_{ji} (q^h - \eta^h),
\]

where

\[
p_i = \partial_i p, \quad p^h = p_t g^h, \quad q_i = -p_t \varphi^h_i, \quad q^h = q_t g^h,
\]

and \( p \) satisfies

\[
\mathcal{L} p = 0.
\]

He called such an affine connection a contact conformal connection in a Sasakian manifold.

**Theorem 1.6** If, in a \( n(= 2m + 1) \)-dimensional Sasakian manifold, \((n \geq 4)\), there exists a scalar function \( p \) such that the contact conformal connection is of zero curvature, then the contact Bochner curvature tensor of the manifold

\[
B^h_{kji} = K^h_{kji} + (\delta^h_k - \eta_k \eta^h) L_{ji} - (\delta^h_j - \eta_j \eta^h) L_{ki} \\
+ L^h_k (g_{ji} - \eta_j \eta_i) - L^h_j (g_{ki} - \eta_k \eta_i) \\
+ \varphi^h_k M_{ji} - \varphi^h_j M_{ki} + M^h_k \varphi_{ji} - M^h_j \varphi_{ki} \\
- 2(M_{kj} \varphi^h_i + \varphi_{kj} M^h_i) + (\varphi^h_k \varphi_{ji} - \varphi^h_j \varphi_{ki} - 2 \varphi_{kj} \varphi^h_i),
\]
vanishes, where
\[ L_{ji} = -\frac{1}{2(m+2)}(K_{ji} + (L+3)g_{ji} - (L-1)\eta_j\eta_i), \]
\[ L^h_k = L_{kt}g^{th}, \quad M_{ji} = -L_{jt}\phi^t_j, \quad M^h_k = M_{kt}g^{th}, \quad L = g^{ji}L_{ji}. \]

The main purpose of the present paper is to introduce a cosymplectic analogue of conformal connection in a cosymplectic manifold and study its properties. In Section 2, we state some of preliminaries on cosymplectic manifold and on the cosymplectic Bochner curvature tensor. In section 3, we introduce cosymplectic conformal connection and in the last section 4, we prove that if cosymplectic manifold \( M \) admits a cosymplectic conformal connection which is of zero curvature, then the Bochner curvature tensor of \( M \) vanishes.

2 Preliminaries

2.1 Cosymplectic manifolds

Let \( M \) be a \((2m + 1)\)-dimensional differentiable manifold of class \( C^\infty \) covered by a system of coordinate neighborhoods \( \{ U, y^h \} \). Let \( M \) admit an almost contact structure \((\phi^h_i, \xi^h, \eta_i, g_{ij})\) of a tensor field \( \phi^h_i \) of type \((1, 1)\), a vector field \( \xi^h \), a 1-form \( \eta_i \) and a Riemannian metric \( g_{ij} \) satisfying
\[ \phi^h_i \phi^i_h = -\delta^h_i + \eta_j \xi^h, \quad \phi^h_i \xi^i = 0, \quad \eta_i \phi^i_j = 0, \quad \eta_i \xi^i = 1, \quad (2.1) \]
where the indices \( h, i, j, k, \ldots \in \{1, 2, \ldots, 2n + 1\} \). A manifold \( M \) with an almost contact structure is known as an almost contact manifold \cite{2, 8}.

If the set \((\phi^h_i, \xi^h, \eta_i)\) satisfies
\[ N^h_{ji} + (\partial_j \eta_i - \partial_i \eta_j)\xi^h = 0, \quad (2.2) \]
where
\[ N^h_{ji} = \phi^h_j \partial_i \phi^h_i - \phi^i_j \partial_i \phi^h_h - (\partial_j \phi^h_i - \partial_i \phi^h_j)\phi^h_i \]
is the Nijenhuis tensor formed with \( \phi^h_i \) and \( \partial_j = \partial/\partial x^j \), then the almost contact structure is said to be normal and the manifold is called a normal almost contact manifold.

If, in an almost contact manifold, there is given a Riemannian metric \( g_{ji} \) such that
\[ g_{ts} \phi^t_j \phi^s_i = g_{ji} - \eta_j \eta_i, \quad \eta_i = g_{ih} \xi^h, \quad g_{kk} \xi^k \xi^k = 1, \quad (2.3) \]
then the almost contact manifold is called an almost contact metric manifold \cite{2, 8}.

Comparing the first equations of (2.1) and (2.3), we see that
\[ \phi_{ji} = \phi^i_j g_{ti} \]
is skew-symmetric.

An almost contact metric manifold is said to be cosymplectic manifold \cite{2} if the 2-form \( \phi_{ji} \) and the 1-form \( \eta_j \) are closed. It is well known that the cosymplectic manifold is characterized by
\[ \nabla_j \phi^h_i = 0, \quad \nabla_i \xi^h = 0, \]
where \( \nabla_j \) denotes the operator of covariant differentiation with respect to \( g_{ji} \).
2.2 Cosymplectic Bochner curvature tensor

The cosymplectic Bochner curvature tensor [5] is given by

\[ B^h_{kji} = K^h_{kji} + (\delta^h_k - \eta_k\eta^h) L_{ji} - (\delta^h_j - \eta_j\eta^h) L_{ki} \]
\[ + L^h_k (g_{ji} - \eta_j\eta_i) - L^h_j (g_{ki} - \eta_k\eta_i) \]
\[ + \varphi^h_i M_{ji} - \varphi^h_j M_{ki} + M^h_k \varphi_{ji} - M^h_j \varphi_{ki} \]
\[ - 2(M_{kj} \varphi^h_i + \varphi_{kj} M^h_i), \]  
(2.4)

where

\[ L_{ji} = -\frac{1}{2(m+2)} (K_{ji} + L(g_{ji} - \eta_j\eta_i)) \quad M_{ji} = -L_{ji} \varphi^t_i, \]

\[ L^h_k = L_{kt} g^{th} \quad M^h_k = M_{kt} g^{th} \quad L = g^{ji} L_{ji} = -\frac{K}{4(m+1)} \]

By above equations, we can say that \( L_{ji} \) is symmetric and \( M_{ji} \) is skew-symmetric.

It is easy to verify that

\[ L_{ji} \eta^i = 0 \quad M_{ji} \eta^i = 0. \]

The Bochner curvature tensor satisfies the following identities:

\[ B^h_{kji} = -B^h_{jki}, \]
\[ B^h_{kji} + B^h_{jik} + B^h_{ikj} = 0, \]
\[ B^h_{hkji} = -B^h_{hjki}, \]
\[ B^h_{hkji} = -B^h_{khji}, \]
\[ B^h_{hkji} = B^h_{jikh}, \]
\[ B^h_{kji} \eta^i = 0, \]
\[ B^h_{kji} \varphi^t_i = 0, \]

where \( B^t_{hkji} = B^t_{hjki} g_{ti}. \)

3 Cosymplectic conformal connections

Let \( D \) be an affine connection in a cosymplectic manifold \( M \) and \( \Gamma^h_{ji} \) be the components of the affine connection \( D \). Assume that the affine connection \( D \) satisfies

\[ D_k (e^{2p} g_{ji}) = 2e^{2p} p_k \eta_j \eta_i, \]  
(3.1)

where \( D_k \) the operator of covariant differentiation with respect to \( \Gamma^h_{ji} \) and \( p \) is a certain scalar function, \( p_i = \partial_i p \). The torsion tensor of \( D \) satisfies

\[ \Gamma^h_{ji} - \Gamma^h_{ij} = -2 \varphi_{ji} q^h, \]  
(3.2)

where \( q^h \) is a certain vector field.
By solving (3.1) and (3.2), we get
\[
\Gamma^h_{ji} = \left\{ \begin{array}{c}
h \\
ij \end{array} \right\} + (\delta^h_j - \eta_j \eta^h)p_i + (\delta^h_i - \eta_i \eta^h)p_j - (g_{ji} - \eta_j \eta_i)p^h + \varphi^h_j q_i + \varphi^h_i q_j - \varphi_{ji} q^h,
\]
where
\[p^h = p_t g^ih, \quad q^h = q_t g^{ih}.\]
Using (3.3), we can find that
\[
D_j \varphi^h_i = (\delta^h_j - \eta_j \eta^h)(p_t \varphi^t_i + q_i) + (g_{ji} - \eta_j \eta_i)(\varphi^h_i p^t - q^h) + \varphi_j^h (q_t \varphi^t_i - p_i) + \varphi_{ji} (p^h + \varphi^h_i q^t).
\]
Now assume that affine connection satisfies
\[
D_j \varphi^h_i = 0.
\]
Using (3.5) in (3.4) and contracting with respect to \(h\) and \(j\), we have
\[
n(p_t \varphi^t_i + q_i) + \eta_i \eta_t q^t = 0,
\]
on transvecting with \(\eta^t\), we have
\[
\eta_t q^t = 0 = q_t \eta^t = 0.
\]
Using (3.7) in (3.6), we get
\[
q_i = -p_t \varphi^t_i.
\]
Now assume that the affine connection \(D\) also satisfies
\[
D_j \eta^h_i = 0.
\]
So
\[
(\delta^h_j - \eta_j \eta^h) \eta_t p^t = 0.
\]
On contracting above equation with respect to \(h\) and \(j\), we obtain
\[
\eta_t p^t = 0 = \eta^t p_t.
\]
Therefore
\[
p_t \varphi^t_i, \quad q^h = \varphi^h_i p^t, \quad p^h = -\varphi^h_i q^t.
\]
We can easily calculate
\[
p_t q^t = 0, \quad p_t q^t = q_t q^t.
\]
By using (3.4) and (3.8), we obtain

**Proposition 3.1** Let \(M\) be a cosymplectic manifold with structure tensors \((\varphi^h_i, \xi^h, \eta, g_{ij})\) such that the affine connection \(D\) satisfies
\[
D_k (e^{2p} g_{ji}) = 2 e^{2p} p_k \eta_j \eta_i, \quad D_j \varphi^h_i = 0, \quad D_j \eta^h = 0
\]
and whose torsion tensor satisfies
\[
\Gamma^h_{ji} - \Gamma^h_{ij} = -2 \varphi_{ji} q^h,
\]
where \( p \) is a scalar function and \( q^h \) is a vector field. Then the components of the affine connection \( D \) are given by

\[
\Gamma^h_{ji} = \left\{ \begin{array}{l}
h_{ji} \\
(ji)
\end{array} \right\} + (\delta^h_j - \eta_j \eta^h) p_i + (\delta^h_i - \eta_i \eta^h) p_j - (g_{ji} - \eta_j \eta_i) p^h + \varphi^h_j q_i + \varphi^h_i q_j - \varphi_{ji} q^h,
\]

where \( p^h = p_l g^{th} \), \( u^h = u_l g^{th} \), \( q_i = -p_l \varphi^i_l \), \( p_i = q_l \varphi^l_i \).

We will call such an affine connection \( D \) a cosymplectic conformal connection.

**Proposition 3.2** A cosymplectic conformal connection satisfies

\[
D_k \left( e^{2p}(g_{ji} - \eta_j \eta_i) \right) = 0.
\]

4 Curvature tensor of a cosymplectic conformal connection

In this section, we will derive the curvature tensor of the cosymplectic conformal connection.

Using (3.11), the curvature tensor of the cosymplectic conformal connection is

\[
R^h_{kji} = K^h_{kji} - (\delta^h_k - \eta_k \eta^h) p_{ji} + (\delta^h_j - \eta_j \eta^h) p_{ki} - p^h_k (g_{ji} - \eta_j \eta_i) + p^h_j (g_{ki} - \eta_k \eta_i) - \varphi^h_k q_{ji} + \varphi^h_j q_{ki} - p^h_k \varphi_{ji} + p^h_j \varphi_{ki} + (\nabla_k q_j - \nabla_j q_k) \varphi^h_i + 2 \varphi_{kj} (q^h_i p^h_k - p^h_i q^h_k),
\]

where

\[
\begin{align*}
p_{ji} &= \nabla_j p_i - p_j p_i + q_j q_i + \frac{1}{2} \lambda (g_{ji} - \eta_j \eta_i), \\
q_{ji} &= \nabla_j q_i - p_j q_i - p_i q_j + \frac{1}{2} \lambda \varphi_{ji}, \\
p^h_k &= p_k g^{th}, \quad q^h_k = q_k g^{th}, \quad \beta^h_k = \beta_k g^{th}.
\end{align*}
\]

\( \lambda \) being defined by

\[
\lambda = p_i p^i = q_i q^i.
\]

Let

\[
\alpha_{ji} = - (\nabla_j q_i - \nabla_i q_j),
\]

\[
\beta_{ji} = 2 (p_j q_i - p_i q_j).
\]

Since \( p_i = \partial_i p \), so \( p_{ji} = p_{ij} \). We can easily check that

\[
\begin{align*}
\eta^i p_{ji} &= 0, \quad \eta^i q_{ji} = 0, \quad \alpha_{ji} \eta^i = 0, \\
\beta_{ji} \eta^i &= 0, \quad \alpha_{ji} = - \alpha_{ij}, \quad \beta_{ji} = - \beta_{ij}, \\
q_{ji} &= - p_{ji} \varphi^l_i, \quad p_{ji} = q_{ji} \varphi^l_i, \\
\alpha_{ji} &= -(q_{ji} - q_{ij} - \lambda \varphi_{ji}),
\end{align*}
\]

(4.7)
\[ \alpha = \varphi^{ij} \alpha_{ij} = -2 \nabla_t p^t, \quad \text{(4.8)} \]
\[ \beta = \varphi^{ij} \beta_{ij} = 4 \lambda. \quad \text{(4.9)} \]

By (4.2), we have
\[ p^k_i = \nabla_k p^k + mp_k p^k. \quad \text{(4.10)} \]

We now consider here that the curvature tensor of cosymplectic curvature connection vanishes, that is,
\[ R^h_{kji} = 0. \quad \text{(4.11)} \]
Consequently (4.1) becomes
\[ K^h_{kji} = (\delta^h_k - \eta_k \eta^h)p_{ji} - (\delta^h_j - \eta_j \eta^h)p_{ki} + p^h_k (g_{ji} - \eta_j \eta_i) - p^h_j (g_{ki} - \eta_k \eta_i) + \varphi^h_k q_{ji} - \varphi^h_j q_{ki} - q^h_k \varphi_{ji} - q^h_j \varphi_{ki} + \alpha_{kj} \varphi^h_i - \varphi_{kj} \beta^h_i, \quad \text{(4.12)} \]
in covariant form,
\[ K_{kji} = (g_{kh} - \eta_k \eta_h)p_{ji} - (g_{jh} - \eta_j \eta_h)p_{ki} + p_{kh} (g_{ji} - \eta_j \eta_i) - p_{jh} (g_{ki} - \eta_k \eta_i) + \varphi_{kh} q_{ji} - \varphi_{jh} q_{ki} + q_{kh} \varphi_{ji} = q_{jh} \varphi_{ki} + \alpha_{kj} \varphi_{ih} - \varphi_{kj} \beta_{ih}, \quad \text{(4.12)} \]
Using (4.12) in the identity \( K_{kji} = K_{ihkj} \), we have
\[ 0 = \varphi_{ji} (q_{kh} + q_{hh}) - \varphi_{jh} (q_{ki} + q_{ik}) + \varphi_{kh} (q_{ji} + q_{ij}) - \varphi_{ki} (q_{jh} + q_{hj}) - \alpha_{kj} \varphi_{ih} - \alpha_{ih} \varphi_{kj} - \beta_{ih} \varphi_{kj} + \beta_{kj} \varphi_{ih}, \quad \text{(4.13)} \]
transvecting (4.13) with \( \varphi^{kh} \), we get
\[ q_{ji} + q_{ij} = 0. \quad \text{(4.14)} \]
Using (4.14) in (4.13), we obtain
\[ (\alpha_{kj} + \beta_{kj}) \varphi_{ih} = (\alpha_{ih} + \beta_{ih}) \varphi_{kj}, \]
transvecting the above equation with \( \varphi^{ih} \), we get
\[ \alpha_{kj} + \beta_{kj} = \frac{1}{2m} (\alpha + \beta) \varphi_{kj}, \]
using (4.8), (4.9), we have
\[ \alpha_{kj} + \beta_{kj} = \frac{1}{m} \left( -\nabla_t p^t + 2p_t p^t \right) \varphi_{kj}. \]
Using (4.7) and (4.14) in above equation, we get
\[ \beta_{kj} = \frac{1}{m} \left( -\nabla_t p^t + (2 - m)p_t p^t \right) \varphi_{kj} + 2q_{kj}. \]
By use of (4.10) in above equation, we have

\[ \beta_{kj} = \frac{1}{m} \left( -p_t^t + 2p_t p_t^t \right) \varphi_{kj} + 2q_{kj}. \quad (4.15) \]

By use of (4.12) in the identity \( K_{kjih} + K_{jikh} + K_{ikjh} = 0 \) and using (4.7), (4.14), (4.15), we have

\[ \frac{1}{m} \left( (m - 2)p_t p_t^t + p_t^t \right) (\varphi_{kj} \varphi_{ih} + \varphi_{ji} \varphi_{kh} + \varphi_{ik} \varphi_{jh}) = 0, \]

from which, we obtain

\[ (m - 2)p_t p_t^t + p_t^t = 0. \quad (4.16) \]

Substituting (4.15) in (4.16), we have

\[ \beta_{kj} = p_t p_t^t \varphi_{kj} + 2q_{kj}. \quad (4.17) \]

By (4.6), we have

\[ q_{ji} \varphi_{i}^s \varphi_{k}^j = -q_{sk}. \]

Using (4.17), we get

\[ \beta_{jk} \varphi_{s}^j = -2p_{ks} - p_t p_t^t (g_{ks} - \eta_k \eta_s). \quad (4.18) \]

By use of (4.7) and (4.14), we have

\[ \alpha_{kj} \varphi_{s}^j = 2p_{ks} - p_t p_t^t (g_{ks} - \eta_k \eta_s). \quad (4.19) \]

Contracting (4.12) with respect to \( h \) and \( k \) and using (4.18), (4.19), we obtain

\[ K_{ji} = (2m + 4)p_{ji} + p_t^t (g_{ji} - \eta_j \eta_i). \quad (4.20) \]

Transvecting (4.20) with \( g_{ji}^t \), we find

\[ K = (4m + 4)p_t^t. \quad (4.21) \]

Consequently, we have

\[ p_t^t = \frac{K}{4(m + 1)} = -L \quad (4.22) \]

Using (4.22) in (4.20), we have

\[ K_{ji} = (2m + 4)p_{ji} - L(g_{ji} - \eta_j \eta_i), \]

so

\[ p_{ji} = \frac{1}{(2m + 4)}(K_{ji} + L(g_{ji} - \eta_j \eta_i)) = -L_{ji}. \quad (4.23) \]

Therefore

\[ M_{ji} = -L_{ji} \varphi_{i}^t = p_{jt} \varphi_{i}^t = -q_{ji}. \quad (4.24) \]

From (4.16), we have

\[ p_t p_t^t = \frac{L}{m - 2}. \]
By use of (4.17), (4.7) and (4.14), we obtain

\[
\alpha_{ji} = 2M_{ji} + \frac{L}{(m-2)}\varphi_{ji},
\]

(4.25)

\[
\beta_{ji} = -2M_{ji} + \frac{L}{(m-2)}\varphi_{ji}.
\]

(4.26)

Substituting (4.12), (4.23), (4.24), (4.25), (4.26) in (2.4), we obtain

\[B_{kjih} = 0.\]

Thus, we have the following result:

**Theorem 4.1** Let \(M\) be a real \((2m+1)\)-dimensional \((m > 2)\) cosymplectic manifold. If \(M\) admits a cosymplectic conformal connection which is of zero curvature, then the Bochner curvature tensor of \(M\) vanishes.

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