Quantum Zeno paradox: survival and decay

Charles B. Chiu
Department of Physics, University of Texas at Austin, Austin, Texas

Abstract. Quantum mechanics enables the calculation of the time evolution of a dynamical system provided the Hamiltonian is defined and the initial state is specified. Due to the time reversal invariance, the time evolution of a given initial state at \( t=0 \) does not have a time-arrow. These problems have been extensively studied by E. C. G. Sudarshan and collaborators, including myself. Here the survival probability, which is the square of the survival amplitude, is an even function of \( t \). Using the survival amplitude as a common framework, we discuss Zeno effects in several processes: the decay of an unstable particle, pion productions in proton-nucleus collisions and the Aharonov-Vardi effect which is another manifestation of the Zeno effect.

Introduction: The quantum Zeno effect is the slowing down of the time evolution of a quantum system due to repeated measurements. I would like to discuss three examples in connection with this effect. My talk is based on several pieces of work of E. C. G. Sudarshan in collaboration with Misra, Valanju and myself.

Example 1. Unstable quantum system Consider an unstable quantum system where the Hamiltonian is defined. We prepare the unstable quantum state \( |M\rangle \), at \( t = 0 \). The survival amplitude at \( t \) is given by the overlap between this state and the time evolved state, i.e.

\[
a(t) = \langle M | e^{-iHt} | M \rangle = \int_{\lambda_h}^{\infty} d\lambda \rho(\lambda) e^{-i\lambda t}
\]

In the last step, this amplitude is expressed as the Fourier transform of the energy spectrum continuum of \( H \), i.e.

\[
\rho(\lambda) = \langle M | \lambda \rangle \langle \lambda | M \rangle.
\]

The survival probability is by definition the square of the survival amplitude,

\[
Q(t) = |a(t)|^2.
\]

Its time derivative at \( t = 0 \) is given by

\[
\dot{Q}(0) = \dot{a}^*(0)a(0) + \dot{a}(0)a^*(0) = 0.
\]

where the identity

\[
\dot{a}^*(0) = -\dot{a}(0),
\]

which is a consequence of the Fourier representation of (1), together \( a(0) = 1 \) (with by definition,)and thus \( a^*(0) = 1 \) were used.

In ref. [1], Misra and Sudarshan show that eq. (4), along with some general assumptions and a rigorous analysis, leads to the quantum Zeno paradox. In particular, if the unstable quantum
system is monitored over time $T$ in the $N$ equal time intervals, in the large $N$ limit the survival probability at $T$ is given by

$$ Q(T) = Q(T/N)^N \rightarrow 1. \quad (6) $$

So repeated measurement will prolong the survival. And in the large $N$ limit there will be 100% survival, or nondecay.

**Lee model:** In ref. [2], we investigated the survival amplitude based on a Lee model. In the Lee model we considered, the basis states contain a discrete state $|V\rangle$, and the continuum states $|N\theta(E)\rangle$, where the energy $E$, ranges from the threshold energy to infinity. The interaction Hamiltonian specifies that $V$ is coupled to $N\theta$ through a coupling function

$$ f(E) = \langle N\theta(E)|H|V\rangle. \quad (7) $$

In this model the survival amplitude of $V$ at $t$ is given by

$$ a(t) \equiv \langle V|e^{-iHt}|V\rangle = \frac{i}{2\pi} \int_C e^{-izt} \gamma(z) \, dz. \quad (8) $$

For this talk we will be mainly interested in the situation near $t = 0$. The last step of eq.(8) gives the contour integral representation of the survival amplitude where the contour $C$ wraps around the right-hand cut of the denominator function $\gamma(z)$ in a clockwise manner in the complex $z$-plane, where $z = E - E_{th}$, with $z$ being the displaced complex energy variable $E$ with a displacement $-E_{th}$ (see Fig. 1). It is helpful to discuss contour deformation using the new conformal variable

$$ k = [ze^{i\pi/2}]^{1/2}. \quad (9) $$

The complex $k$-plane is shown in Fig. 2. Here the right hand cut contour $C$ of Fig. 1 corresponds to a line passing through the origin $O$, and runs along the $45^\circ$-line. It is labeled as the contour $S$. The region above the line corresponds to the physical sheet and below the line, the second sheet. We associate an unstable state with a pole on the second sheet. Hermiticity of the Hamiltonian requires that the second sheet poles are present in pairs. For our case, there is a pole in the first quadrant and the companion pole in the third quadrant. They are indicated in Fig. 2. We are interested in the temporal evolution in the positive time domain, $t \geq 0$. The appropriate deformed contours for this time domain are shown in Fig. 2. The contours of concern satisfy the relationship:

$$ S = S_+ + S_1 + S_2 = S_+ + S_1. \quad (10) $$

In the last step, the fact that for $t \geq 0$, $S_2 = 0$ at $|k| = \infty$, was used. Thus the survival amplitude can be written as

$$ a(t) = a_{\text{pole}}(t) + a_{\text{bg}}(t), \text{ with } a_{\text{pole}}(t) \sim e^{-i(E_0 - i\Gamma/2)t}. \quad (11) $$

**Cancellation near $t = 0$:** Notice that the pole term gives

$$ \text{Re} \, \dot{a}_{\text{pole}}(0) = -\Gamma/2, \quad (12) $$

which is the characteristic slope of the survival amplitude at $t = 0$ associated with an exponential decay. On the other hand, eq. (5) implies that the real part of the survival amplitude at $t = 0$ vanishes, since

$$ \text{Re} \, \dot{a}(0) = \frac{1}{2}[\dot{a}^*(0) + \dot{a}(0)] = 0. \quad (13) $$
Figure 1. The right hand cut in the complex energy plane.

Figure 2. The $k$-plane: $S = S_+ + S_1 + S_2$. $S_2 = 0$, for $t \geq 0$.

For a sufficiently small region in $t$, say $t \leq T_1$, eqs. (11), (12) and (13) together imply that there must be a cancellation between the pole-term and the background term. So eq. (13) is satisfied for $t \leq T_1$.

In ref. [2], we have shown that this cancellation does take place in the region $0 < t < T_1$, where

$$T_1 \sim 1/(E_0 - E_{th}),$$

(14)

Beyond $t = T_1$, the background term falls of with $t^{-3/2}$ behavior and the pole term begins to dominate. We also found that in the small $t$ region, the survival amplitude takes on the general form

$$Q(t) \to 1 - const \times t^\beta/\beta,$$

thus

$$\dot{Q}(t) \to -const \times t^{\beta-1}.$$  

(15)

For present case of interest, to ensure $\dot{Q}(0) = 0$, $\beta > 1$. In large $n$ limit, with $t/n << T_1$,

$$Q_n(t) \to \left[1 - \frac{const}{\beta} \times \left(\frac{t}{n}\right)^\beta\right]^n \to 1.$$  

(16)

So the $\beta > 1$ case, demonstrates the Zeno paradox.

**Zeno time:** We see that there is a delay near $t = 0$ before the pole contribution, which is characterized by an exponential decay, becomes dominate. One may understand qualitatively the value of $T_1$ given in eq. (14) in the following way. From eq. (1) one sees that in the survival
amplitude at \( t = 0 \) the pole spectrum is integrated over. The presence of the specific pole contribution becomes important only when the exponent in the fourier integral, \( \Delta \lambda t \), is of the order of unity, where \( \Delta \lambda \) represents some significant energy scale in the spectrum. Eq. (14) indicates that this scale should be associated with the distance from the pole to the threshold energy. Since the energy scale given here is general, we suggest that the presence of Zeno time should be a common feature among unstable quantum systems.

In the small time domain, there are other physics considerations that may prevent Zeno effect from manifesting. Take for example the decay of a charged pion \( \pi \to \mu \nu \). \( \Gamma \sim (3 \times 10^{-8} \text{sec})^{-1} \) and \( T^{-1} = m_\pi - m_\mu = 34 \text{MeV} = (2 \times 10^{-23} \text{sec})^{-1} \). Here the natural check points are spaced at the inter-atomic distance \( 10^{-8}/(3 \times 10^{10}) = 3 \times 10^{-19} \text{sec} \). There is no way to monitor the natural evolution of a system having time scale of \( 10^{-23} \text{sec} \) by inter-atomic time scale of \( 10^{-19} \text{sec} \). On the other hand inter-nucleonic distance within a nucleus may provide a more promising arena for Zeno to perform. This leads us to consider the second example.

**Example 2: Evidence of “Equilibrium time” in hadron-nucleus collisions.** Our first example suggests that when a quantum state is created, it is a non-equilibrium state. There is a delay time for the state to reach the equilibrium state. We refer to this delay time as the equilibrium time or the Zeno time. We mentioned that the smallness of this time prevents the direct demonstration of the Zeno effect in unstable particle decay if the inter-atomic distance is to be used as the distance between the adjacent checkpoints. In ref. [9], which is based on Dr. Valanju’s dissertation work, we followed the approach of Feinberg[6] in estimating the equilibrium time. We found that inter-nucleon collisions provide a promising arena in demonstrating the Zeno effect.

**Bremsstrahlung process:** In ref. [6], Feinberg considered the traversal of an electron through a matter medium. Here a bremsstrahlung process is typically preceded by a precursor collision where there is a large momentum change. There is a time delay between the precursor collision and the radiation. In particular, he observed that after the collision, the electron becomes “bare”. It takes the order of the equilibrium time before the electron can radiate again. This time is given by

\[
T_k = \gamma/k_0,
\]

where \( k_0 \) is the photon energy defined in the rest frame of the electron and \( \gamma \) the Lorentz factor of the electron in the medium. So \( T_k \) is the equilibrium time of the electron in the medium for the emission of the next \( k_0 \) quantum.

Equilibrium time is Zeno time. We suggest that this equilibrium time is a Zeno time, since this is the time scale for the bare particle to continue to “survive” in the bare state. We explain below how one may look for the effect of the equilibrium time in pion production from pp and p-nucleus collisions at high energy.

**Equilibrium time effect in p-nucleus collisions.** Fig. 3 are schematic plots illustrating the “equilibrium time” effect for an ideal scenario.

**pp-plot.** Fig. 3a shows an ideal pion rapidity spectrum in pp collision. The spectrum is symmetric about the “midpoint” which is indicated in the plot. Pions to the right of the midpoint, which will be referred to as being in the forward range, are to be associated with pion-production due to the projectile, and those to the left, the backward range, associated with pion-production due to the target-proton.

**pA-plot.** Now we turn to the corresponding pion rapidity spectrum in proton-nucleus production. The situation is schematically depicted in Fig. 3b. It involves successive collisions between the projectile nucleon and \( \nu \) target-nucleons.

- Forward range: We envision that after the first collision, the projectile nucleon turns into a
Figure 3. Schematic illustrations of pion production spectra (a) Pion inclusive distribution for pp collision. Forward pions produced due to projectile p, and backward pions due to target p. (b) pA case. Forward production independent of collision number $\nu$, and backward $\propto \nu$. (c) $\nu$-dependence in pA collision for the case (b) shown. Forward case-flat, and backward case $\propto \nu$. (d) Comparison between $\nu$ dependence of our model prediction (lines) and the data (points, triangles) at a typical $y$, i.e. 4.8, in the forward range, and the circles at $y = 2$, in the backward range.

bare state. For an ideal scenario, we assume that the equilibrium time of the bare projectile is much longer than the time taken for it to traverse through the nucleus. Consequently during the time when the bare-projectile traverses through the entire nucleus, no pions are produced. The pion production due to the projectile takes place only after the projectile has reached its equilibrium state, which takes place outside of the nucleus. The pion spectrum produced by this projectile proton is assumed to be comparable to the pion-spectrum produced by the projectile proton in pp collision.

• Backward range: On the other hand, at the target-nucleus side, for each collided target-nucleon after its equilibrium time is reached there will be the production of a pion-spectrum, which is assumed to be comparable to that produced by the target-proton in a pp collision. Thus below the midpoint, i.e. in the backward region, for the ideal scenario the pion-spectrum from a p-nucleus collision should be $\nu$ times the pion-spectrum of the pp case. Fig. 3b illustrates the ideal spectra for $\nu=1$, 2, 3 and 4 both for the forward range as well as the backward range.

Signal of equilibrium time effect. We now proceed to display the effect of equilibrium time by comparing the $\nu$ dependence of the heights of the forward spectra to that of the backward spectra. Fig. 3c illustrates the situation. For the heights in the forward range of the present ideal case there is no $\nu$ dependence, so the plot of height vs $\nu$ is flat. On the other hand, for the spectra in the backward region, the height is proportional to $\nu$. This would be the characteristic
signal which would demonstrate the presence of the equilibrium time effect.

**A quantitative model.** So far we have considered an ideal scenario. In the work of Valanju et al[9], a quantitative model was constructed, which is an extension of our earlier phenomenological model for p-A multiple scattering[8]. For the quantitative model, analogous to the electron bremsstrahlung process, the equilibrium time of pion production is taken to be

\[ T_\pi \sim \frac{\gamma}{M_{\text{eff}}} \], with \( M_{\text{eff}} = \sqrt{M_c^2 + p_T^2} \) \hspace{1cm} (18)

where \( M_{\text{eff}} \) is the average transverse mass of a mean hadronic cluster, which has typically \( \sim 3 \) pions. We have also taken into account the internal motion of pions within the cluster. The experimental shape of pion rapidity distribution in pp collision is used as an input. Also we use the mean free path in the description of the multiple scattering process and take into account the effect of conservation of energy.

Comparison with the data.

- The comparison between the data (histograms) and our model predictions (lines) is shown in fig. 4. The agreement is reasonable.
- The \( \nu \)-dependence of the pion-spectrum at a typical forward rapidity, \( y = 4.8 \), and that at a typical backward rapidity, \( y = 2 \) are shown in Fig. 3d. The data are indicated by points, and our model predictions by lines. Notice there is indeed the expected pattern from the equilibrium time signal shown in the plot, i.e. the proportional-to-\( \nu \) dependence associated with the backward region and the relatively flat \( \nu \) dependence associated with the forward region.

Thus our quantitative model supports the equilibrium-time interpretation of the data and contributes to the evidence of the Zeno effect in proton-nucleus collision.

**Figure 4.** A comparison between a quantitative model considered by Valanju et al[9] (lines) and the data (histogram). Here pseudorapidity \( \eta \)-distributions are compared. Each pseudorapidity \( \eta \)-distribution here is approximately the same as the corresponding rapidity \( y \)-distribution.

**Example 3: Zeno’s effect & moving observer.**
This example is based on recent discussions between E. C. G. Sudarshan and myself on the quantum measurement related effect discussed by Aharonov and Vardi[10] over two decades ago.

**AV effect:** Let me first recall their observation. Consider the following 3 sets of spinor states:

- A stationary spinor state: \( |\psi_0> = |\sigma_x = +1> \),
- A rotating spinor state: \( |\psi(t)> = R_z(\theta(t))|\psi_0> \), where \( \theta(t) = \omega t \), with \( \omega \) being some constant frequency.
- A set of fiducial spinor states \( |\mu_i> \equiv |\psi(t_i)> \). In words, the \( |\mu_i> \)'s are a set of discrete states, judiciously chosen to follow the rotating spinor state \( |\psi(t)> \). At the discrete points, the fiducial states are identical to the corresponding rotated states.

Now consider the effect due to invoking \( N \) equal-time consecutive measurements on the successive measured quantum-states, beginning with the initial state \( |\psi_0> \). Here measurements are carried out in the time sequence \( \{ t = t_i \} \), with \( i = 1, 2, \ldots, N \). The corresponding measuring fiducial states are \( \mu_i \)'s. In the large \( N \) limit, the “survival probability” at \( T \) is given by,

\[
Q(T) = \left| \mu_1|\psi_0>^2 \right| \left| \mu_2|\psi_0>^2 \right| \cdots \left| \mu_N|\psi_0>^2 \right| \to 1. \tag{19}
\]

In other words, quantum measurements on \( |\psi_0> \), using a judiciously chosen set of fiducial states which follows the rotating state, causes \( |\psi_0> \) to rotate in the same manner as the rotating state. We will refer to quantum measurements leading the motion of a stationary state as the AV effect.

Our objective and approach: We would like to show that the AV effect may be regarded as a Zeno effect as seen by a moving observer. To avoid considerations involving a rotating frame (which is a non-initial frame), we will first construct a toy model, which involves only inertial frames. Using this toy model, we will simulate the AV effect in an inertial frame. We will then re-interpret the simulated effect in terms of the Zeno effect.

**A toy model**

- A box-state:

  ![Figure 5](image)

  **Figure 5.** Schematic illustration of the box-state.

Consider a one-dimensional box state as shown in Fig. 5. The box is moving in the x-direction with a velocity \( v \). The wave function inside of the box is given by

\[
\psi(x, vt) = f(x, vt), \quad \text{where} \quad f(x, vt) = \sqrt{\frac{\sigma}{L}} \sin \frac{\pi(x - vt)}{L}. \tag{20}
\]

Outside of the box, the wave function is zero.

- Fiducial states: The measuring device box is moving in the x-direction with a velocity \( u \). The wave function of the \( i \)th fiducial state of the device is defined in terms of the corresponding wave function of the box-state at \( t = t_i \), i.e.

\[
\mu(x, ut_i) = f(x, ut_i). \tag{21}
\]

where \( t_i = iT/N \), with \( i = 1, 2, \cdots, N \).
Survival amplitude: The survival amplitude at \( t = t_i \) is given as the overlap between the \( i \)th fiducial state and the box-state evaluated at \( t = t_i \), which is given by

\[
a(t_i) = \int_{u t_i}^{L+u t_i} \mu_i(x, u t_i) \psi(x, v t_i) dx = \cos \left[ \frac{\pi (v-u)}{L} t_i \right] + R(u, v, t_i).
\] (22)

The wave function of the box-state is constrained to be non-vanishing only inside of the box. The same constraint is also applicable for the fiducial state. It follows that, for the integrand of the \( x \)-integral in eq. (22), when \( v > u \), a step function \( \theta(x - v t_i) \) is present, and for \( v < u \), \( \theta(L + v t_i - x) \) is present. On the right hand side of eq. (22), the first term is the contribution, when the step function constraint is not included, while the second term is the correction term after the inclusion of the step-function. In the large \( N \) limit, it turned out that \( R(u, v, T/N) \propto (T/N)^2 \).

Quantum measurement: We invoke the orthodox theory of quantum measurement, that at the \( i \)-th measurement the box-state is collapsed to the \( \{|\mu_i(x, u t_i)\rangle\} \) state. Then with this new state-identity, the box-state further evolves.

This completes our definition of the toy model.

Zeno effect case & AV effect case

Zeno effect case: \( v > 0 \) and \( u = 0 \). Here the box state is moving with a velocity \( v \) and the measuring device is stationary. Now invoke the standard \( N \) equal-time-interval measurement over the time \( T \). From eq. (22) the survival amplitude in the large \( N \) limit at \( t = T \) is given by

\[
Q(T) = \left[ \cos \left( \frac{\pi v}{L} \cdot \frac{T}{N} \right) + R(0, v, T/N) \right]^N \sim \left[ 1 - \frac{\text{const}}{N^2} \right]^N \to \exp \left[ -\frac{\text{const}}{N} \right] \to 1.
\] (23)

In other words, in the large \( N \) limit, the box state is stationary with respect to the device which is stationary. In other words, due to large-\( N \) quantum measurements, the original moving box-state, is now stationary. This is the Zeno effect.

AV effect case: \( v = 0 \) and \( u > 0 \). Here the box-state is stationary and the measuring device is moving with a velocity \( u \). Again perform the standard set of \( N \) equal-time measurements on the box state over a time interval \( T \). From eq.(22), in the large \( N \) limit the survival probability at \( T \) is given by,

\[
Q(T) = \left[ \cos \left( \frac{\pi u}{L} \cdot \frac{T}{N} \right) + R(u, 0, T/N) \right]^N \sim \left[ 1 - \frac{\text{const}}{N^2} \right]^N \to \exp \left[ -\frac{\text{const}}{N} \right] \to 1.
\] (24)

So with the measurements, the original stationary box-state is now moving along with the measuring device, which is the AV effect.

Reinterpretation: Now consider an observer co-moving with the measuring device. Without measurements, the observer sees that the box-state is moving backward with a velocity \( -u \). On the other hand, with measurements, in large \( N \) limit the survival probability is again 1. This implies that the box-state is now stationary relative to the measuring device, or to the co-moving observer. Thus the large-\( N \) measurements interrupt the observed backward motion of the box-state. This is Zeno effect, more specifically the Zeno effect as seen by the moving observer and the wave moves with the same velocity.

Summary: We have looked at 3 examples of Zeno effect. We would like to make some comments in a somewhat broader context.
Unstable quantum system. Our Lee model study suggests that in general there should be a Zeno-time $T_1$ associated with the decay of an unstable particle. In all cases we have considered, the time scale turned out to be very small.

The equilibrium time: Our investigation of solvable models agrees with Feinberg’s conclusion that when a quantum system is created, it is in a non-equilibrium state. Some finite time is required for the system to reach equilibrium. We identify this equilibrium time as the Zeno time. In this quantum Zeno paradox session of this symposium, you have already learned about experimental efforts to demonstrate the Zeno effect in the talks [12, 13, 14]. We would like to add to the list of evidence for Zeno effect the pion-production spectrum in proton-nucleus collisions at high energies.

AV effect: According to the orthodox theory of quantum measurement, each measurement leads to the collapse of the wave function of the measured state into that of the fiducial state. Thus when successive fiducial states are varying slowly, the measured quantum-state is expected to be guided along by the successive fiducial states. This is the AV effect, which is the Zeno effect seen by an observer co-moving with the measuring device. In this context one may regard the Zeno effect hitherto considered as a special case of our example 3, since in earlier cases considered, the observer is co-moving with measuring device with a zero velocity.

To conclude, let us reiterate that the Quantum Zeno effect is a manifestation of general properties of nature (i.e quantum systems where the Hamiltonian is defined). We have already seen that Zeno effect shows up quite convincingly in proton-nucleus collision. We anticipate Zeno effect may also play an important role in relativistic heavy ion production.[16]

Acknowledgement

Special thanks to Professor Rodger Walser and the organizing committee for inviting me to give this talk. George has been an important mentor in my research career. I am indebted to him for his guidance and invaluable discussions throughout these years, including during my preparation of this talk. As he continues to be active in physics, Zeno effect should continue to be operative. The survival probability of his brilliant physics mind should remain at unity. My best wishes to George’s health, and to many active and productive years to come.

Thanks to Dr. Preshant Valanju and my students Man-Fung Cheung and Matt Haley for discussions.

References

[1] Misra B and Sudarshan E C G 1977 J. Math. Phys. 10, 756
[2] Chiu C B, Misra B, and Sudarshan E C G 1977 Phys. Rev. D16, 520
[3] Lee T D 1954 Phys. Rev. 95, 1329
[4] Fredrichs K O 1948 Commun. Pure Appl. Math. 1, 361
[5] Dirac P A M 1935 The Principles of quantum Mechanics (clarendon Press, London), 2nd edition, Chap. 9
[6] Feinberg E L 1966 Sov. Phys. JETP 23, 132
[7] Elias J et. al. 1977 Phys. Rev. Lett. 39, 1499
[8] Bialkowski G, Chiu C B and Tow D 1978 Phys. Rev. D 17, 862
[9] Valanju P, Sudarshan E C G, and Chiu C B 1980 Phys. Rev. D, 21, 1304
[10] Aharonov Y and Vardi M 1980 Phys. Rev. D, 21, 2235
[11] Jenkins F A and White H E 1957 Fundamentals of Optics, McGraw-Hill, Chapters 27, 28
[12] Marmo G 2006 Zeno Dynamics, talk at this symposium
[13] Pascazio S 2006 Quantum Zeno Dynamics, talk at this symposium
[14] Itano W M 2006 Perspectives on the Quantum Zeno Paradox, talk at this symposium
[15] RHIC data: See for example Adler S S et. al. 2006 (PHENIX collaboration), Phys. Rev. Lett. 97, 052301
[16] At the end of my talk, Professor Sid Meshkov commented that it is worthwhile to investigate the implication of Zeno effect in relativistic heavy ion collisions (RHIC)[15]. His comment is very appropriate. Indeed, we have just begun our investigation on this problem.