NON-MINIMAL BRIDGE POSITION OF 2-CABLE LINKS

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Abstract. Suppose that every non-minimal bridge position of a knot $K$ is perturbed. We show that if $L$ is a $(2, 2q)$-cable link of $K$, then every non-minimal bridge position of $L$ is also perturbed.

1. Introduction

A knot in $S^3$ is said to be in bridge position with respect to a bridge sphere, the original notion introduced by Schubert [12], if the knot intersects each of the 3-balls bounded by the bridge sphere in a collection of $\partial$-parallel arcs. It is generalized to knots (and links) in 3-manifolds with the development of Heegaard splitting theory, and is related to many interesting problems concerning, e.g. bridge number, (Hempel) distance, and incompressible surfaces in 3-manifolds.

From any $n$-bridge position, we can always get an $(n + 1)$-bridge position by creating a new local minimum point and a nearby local maximum point of the knot. A bridge position isotopic to one obtained in this way is said to be perturbed. (It is said to be stabilized in some context.) A bridge position of the unknot is unique in the sense that any $n$-bridge ($n > 1$) position of the unknot is perturbed [7]. The uniqueness also holds for 2-bridge knots [8]. See also [11], where all bridge surfaces for 2-bridge knots are considered. Ozawa [9] showed that non-minimal bridge positions of torus knots are perturbed. Zupan [14] showed such property for iterated torus knots and iterated cables of 2-bridge knots. More generally, he showed that if $K$ is an mp-small knot and every non-minimal bridge position of $K$ is perturbed, then every non-minimal bridge position of a $(p, q)$-cable of $K$ is also perturbed [14]. (Here, a knot is mp-small if its exterior contains no essential meridional planar surface.) We remark that there exist examples of a knot with a non-minimal bridge position that is not perturbed [10] and furthermore, knots with arbitrarily high index bridge positions that are not perturbed [5].

In this paper, we consider non-minimal bridge position of 2-cable links of a knot $K$ without the assumption of mp-smallness of $K$.

Theorem 1.1. Suppose that $K$ is a knot in $S^3$ such that every non-minimal bridge position of $K$ is perturbed. Let $L$ be a $(2, 2q)$-cable link of $K$. Then every non-minimal bridge position of $L$ is also perturbed.

For the proof, we use the notion of t-incompressibility and $t-\partial$-incompressibility of [3]. We isotope an annuls $A$ whose boundary is $L$ to a good position so that it is t-incompressible and $t-\partial$-incompressible in one side, say $B_2$, of the bridge sphere. Then $A \cap B_2$ consists of bridge disks and (possibly) properly embedded disks. By using the idea of changing the order of $t-\partial$-compressions in [2] or [3], we show that in fact $A \cap B_2$ consists of bridge disks only. Then by a further argument, we find a cancelling pair of disks for the bridge position.
2. T-INCOMPRESSIBLE AND T-∂-INCOMPRESSIBLE SURFACES IN A 3-BALL

A trivial tangle $T$ is a union of properly embedded arcs $b_1, \ldots, b_n$ in a 3-ball $B$ such that each $b_i$ cobounds a disk $D_i$ with an arc $s_i$ in $\partial B$, and $D_i \cap (T - b_i) = \emptyset$. By standard argument, $D_i$'s can be taken to be pairwise disjoint.

Let $F$ denote a surface in $B$ satisfying $F \cap (\partial B \cup T) = \partial F \neq \emptyset$. A $t$-compressing disk for $F$ is a disk $D$ in $B - T$ such that $D \cap F = \partial D$ and $\partial D$ is essential in $F$, i.e. $\partial D$ does not bound a disk in $F$. A surface $F$ is $t$-compressible if there is a $t$-compressing disk for $F$, and $F$ is $t$-incompressible if it is not $t$-compressible.

An arc $\alpha$ properly embedded in $F$ with its endpoints on $F \cap \partial B$, is $t$-essential if $\alpha$ does not cobound a disk in $F$ with a subarc of $F \cap \partial B$. In particular, an arc in $F$ parallel to a component of $T$ can be $t$-essential. See Figure 1. (Such an arc will be called bridge-parallel in Section 3.) A $t$-$\partial$-compressing disk for $F$ is a disk $\Delta$ in $B - T$ such that $\partial \Delta$ is an endpoint union of two arcs $\alpha$ and $\beta$, and $\alpha = \Delta \cap F$ is $t$-essential, and $\beta = \Delta \cap \partial B$. A surface $F$ is $t$-$\partial$-compressible if there is a $t$-$\partial$-compressing disk for $F$, and $F$ is $t$-$\partial$-incompressible if it is not $t$-$\partial$-compressible.

![Figure 1. A t-essential arc $\alpha$.](image)

Hayashi and Shimokawa [3] classified t-incompressible and t-$\partial$-incompressible surfaces in a compression body in more general setting. Here we give a simplified version of the theorem.

**Lemma 2.1.** [3] Let $(B, T)$ be a pair of a 3-ball and a trivial tangle in $B$, and let $F \subset B$ be a surface satisfying $F \cap (\partial B \cup T) = \partial F \neq \emptyset$. Suppose that $F$ is both t-incompressible and t-$\partial$-incompressible. Then each component of $F$ is either

1. a disk $D_i$ cobounded by an arc $b_i$ of $T$ and an arc in $\partial B$ with $D_i \cap (T - b_i) = \emptyset$, or
2. a disk $C$ properly embedded in $B$ with $C \cap T = \emptyset$.

3. BRIDGE POSITION

Let $S$ be a 2-sphere decomposing $S^3$ into two 3-balls $B_1$ and $B_2$. Let $K$ denote a knot (or link) in $S^3$. Then $K$ is said to be in bridge position with respect to $S$ if $K \cap B_i$ ($i = 1, 2$) is a trivial tangle. Each arc of $K \cap B_i$ is called a bridge. If the number of bridges of $K \cap B_i$ is $n$, we say that $K$ is in $n$-bridge position. The minimum such number $n$ among all bridge positions of $K$ is called the bridge number $b(K)$ of $K$. A bridge cobounds a bridge disk with an arc in $S$, whose interior is disjoint from $K$. We can take a collection of $n$ pairwise disjoint bridge disks by standard argument, and it is called a complete bridge disk system. For a bridge disk
Take a bridge number, it is known that $b$ is precisely, $R \cap V = \emptyset$. Take a new bridge disk $R \cup V$ a longitude of $b$ and $\alpha$. Lemma 3.1. After possibly changing $R$, the bridges contained in $\Gamma$ is a knot $C$ e.g. [11, Lemma 3.1]). Hence, as a definition, we say that a bridge position is perturbed if it admits a cancelling pair.

Let $V_1$ be a standard solid torus in $S^3$ with core $\alpha$, and $V_2$ be a solid torus in $S^3$ whose core is a knot $C$. A meridian $m_1$ of $V_1$ is uniquely determined up to isotopy. Let $l_1 \subset \partial V_1$ be a longitude of $V_1$ such that the linking number $\text{lk}(l_1, \alpha) = 0$, called the preferred longitude. Similarly, let $m_2$ and $l_2$ be a meridian and a longitude of $V_2$ respectively such that $\text{lk}(l_2, C) = 0$.

Take a $(p,q)$-torse knot (or link) $T_{p,q}$ in $\partial V_1$ that wraps $V_1$ longitudinally $p$ times; more precisely, $|T_{p,q} \cap m_1| = p$ and $|T_{p,q} \cap l_1| = q$. Let $h : V_1 \rightarrow V_2$ be a homeomorphism sending $m_1$ to $m_2$ and $l_1$ to $l_2$. Then $K = h(T_{p,q}) \subset S^3$ is called a $(p,q)$-cable of $C$. Concerning the bridge number, it is known that $b(K) = p \cdot b(C)$ [12], [13].

Let $K$ be a knot (or link) in $n$-bridge position with respect to a decomposition $S^3 = B_1 \cup_S B_2$, so $K \cap B_1$ is a union of bridges $b_1, \ldots, b_n$. Let $R = R_1 \cup \cdots \cup R_n$ be a complete bridge disk system for $\bigcup b_i$, where $R_i$ is a bridge disk for $b_i$. Let $F$ be a surface bounded by $K$ and $F_1 = F \cap B_1$. When we move $K$ to an isotopic bridge position, $F_1$ moves together. We consider $R \cap F_1$. By isotopy we assume that in a small neighborhood of $b_i$, $R \cap F_1 = b_i$.

An arc $\gamma$ in $F_1$ is bridge-parallel ($b$-parallel briefly) if $\gamma$ is parallel, in $F_1$, to some $b_i$ and cuts off a rectangle $P$ from $F_1$ whose four edges are $\gamma$, $b_i$, and two arcs in $S$. Let $\alpha(\neq b_k)$ denote an arc of $R \cap F_1$ which is outermost in some $R_k$ and cuts off the corresponding outermost disk $\Delta$ disjoint from $b_k$. The following lemma will be used in Section 6.

**Lemma 3.1.** After possibly changing $K$ to an isotopic bridge position, there is no $\alpha$ that is $b$-parallel.

**Proof.** We isotope $K$ and $F_1$ and take $R$ so that the minimal number of $|R \cap F_1|$ is realized. Suppose that there is such an arc $\alpha$ which is parallel in $F_1$ to $b_i$ (same or not with $b_k$). Isotope $b_i$ along $P$ to an arc parallel to $\alpha$ so that the changed surface $F'_1$ is disjoint from $\Delta$. See Figure 2. Take a new bridge disk $R'_j$ for $b_i$ to be a parallel copy of $\Delta$. Other bridge disks $R_j$ ($j \neq i$) remain unaltered. They are mutually disjoint. Hence $R' = R - R_i \cup R'_i$ is a new complete bridge disk system. We see that $|R' \cap F'_1| < |R \cap F_1|$ since at least $\alpha$ no longer belongs to the intersection $R' \cap F'_1$. This contradicts the minimality of $|R \cap F_1|$. \hfill \Box

Now we consider a sufficient condition for a bridge position to be perturbed.

**Lemma 3.2.** Suppose a separating arc $\gamma$ of $F \cap S$ cuts off a disk $\Gamma$ from $F$ such that

1. $\Gamma \cap B_1$ is a single disk $\Gamma_1$, and
2. $\Gamma \cap B_2(\neq \emptyset)$ consists of bridge disks $D_1, \ldots, D_k$.

Then the bridge position of $K$ is perturbed.

**Proof.** Let $b_i$ ($i = 1, \ldots, k$) denote the bridge for $D_i$ and $s_i = D_i \cap S$. Let $r_1, \ldots, r_{k+1}$ denote the bridges contained in $\Gamma_1$. We assume that $r_1$ is adjacent to $b_{i-1}$ and $b_i$. See Figure 3. Let $R = R_1 \cup \cdots \cup R_{k+1}$ be a union of disjoint bridge disks, where $R_i$ is a bridge disk for $r_i$. In the following argument, we consider $R \cap \Gamma_1$ except for $r_1 \cup \cdots \cup r_{k+1}$.
Suppose there is a circle component of $R \cap \Gamma_1$. Let $\alpha(\subset R_i \cap \Gamma_1)$ be one which is innermost in $\Gamma_1$ and $\Delta$ be the innermost disk that $\alpha$ bounds. Let $\Delta'$ be the disk that $\alpha$ bounds in $R_i$. 

Figure 2. Sliding $b_i$ along $P$.

Figure 3. The disk $\Gamma_1$ and bridge disks $D_i$'s.
Then by replacing \( \Delta' \) with \( \Delta \), we can reduce \( |R \cap \Gamma_1| \). So we assume that there is no circle component of \( R \cap \Gamma_1 \).

Suppose there is an arc component of \( R \cap \Gamma_1 \) with both endpoints on the same arc of \( \Gamma_1 \cap S \). Let \( \alpha(\subset R_i \cap \Gamma_1) \) be one which is outermost in \( \Gamma_1 \) and \( \Delta \) be the corresponding outermost disk in \( \Gamma_1 \) cut off by \( \alpha \). The arc \( \alpha \) cuts \( R_i \) into two disks and let \( \Delta' \) be one of the two disks that does not contain \( r_i \). By replacing \( \Delta' \) with \( \Delta \), we can reduce \( |R \cap \Gamma_1| \). So we assume that there is no arc component of \( R \cap \Gamma_1 \) with both endpoints on the same arc of \( \Gamma_1 \cap S \).

If \( R \cap \Gamma_1 = \emptyset \), then \((R_i, D_i)\) is a desired cancelling pair and the bridge position of \( K \) is perturbed. So we assume that \( R \cap \Gamma_1 \neq \emptyset \).

Case 1. Every arc of \( R \cap \Gamma_1 \) is b-parallel.

Let \( \alpha \) be an arc of \( R \cap \Gamma_1 \) which is outermost in some \( R_j \) and \( \Delta \) be the outermost disk that \( \alpha \) cuts off from \( R_j \). In addition, let \( \alpha \) be b-parallel to \( r_i \) via a rectangle \( P \). Then \( P \cup \Delta \) is a new bridge disk for \( r_i \), and \((P \cup \Delta, D_i)\) or \((P \cup \Delta, D_{i-1})\) is a cancelling pair.

Case 2. There is a non-b-parallel arc of \( R \cap \Gamma_1 \).

Consider only non-b-parallel arcs of \( R \cap \Gamma_1 \). Let \( \beta \) denote one which is outermost in \( \Gamma_1 \) among them and \( \Gamma_0 \) denote the outermost disk cut off by \( \beta \). Because there are at least two outermost disks, we take \( \Gamma_0 \) such that \( \partial \Gamma_0 \) contains some \( s_i \). Let \( r_1, \ldots, r_m \) be the bridges contained in \( \Gamma_0 \) and \( R' = R_1 \cup \cdots \cup R_m \). In the following, we consider \( R' \cap \Gamma_0 \) except for \( r_1 \cup \cdots \cup r_m \) and \( \beta \). If \( R' \cap \Gamma_0 = \emptyset \), then there exists a cancelling pair \((R_i, D_i)\). Otherwise, every arc of \( R' \cap \Gamma_0 \) is b-parallel. Let \( \alpha \) be an arc of \( R' \cap \Gamma_0 \) which is outermost in some \( R_j \) and \( \Delta \) be the outermost disk that \( \alpha \) cuts off from \( R_j \). In addition, let \( \alpha \) be b-parallel to \( r_i \) via a rectangle \( P \). Then \( P \cup \Delta \) is a new bridge disk for \( r_i \), and \((P \cup \Delta, D_i)\) or \((P \cup \Delta, D_{i-1})\) is a cancelling pair.

\[ \square \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{The disks \( D_0 \) and \( D_1 \) are cancelling disks, whereas \( D_2 \) and \( D_3 \) are not.}
\end{figure}
Remark 3.3. Let \( K \) be an unknot in \( n \)-bridge position, with \( K \cap B_1 = r_0 \cup r_1 \cup \cdots \cup r_{n-1} \) and \( K \cap B_2 = b_0 \cup b_1 \cup \cdots \cup b_{n-1} \). We assume that the bridge \( r_i \) (\( i = 0, 1, \ldots, n-1 \)) is adjacent to \( b_{i+1} \) and \( b_i \), where we consider the index \( i \) modulo \( n \). Let \( R_i \) and \( D_i \) denote bridge disks for \( r_i \) and \( b_i \), respectively. Then \( C = R_0 \cup D_0 \cup \cdots \cup R_{n-1} \cup D_{n-1} \) is called a complete cancelling disk system if each \((R_i, D_i)\) and each \((R_i, D_i)\) is a cancelling pair.

Let \( D \) denote a disk bounded by \( K \). By following the argument of the proof of Theorem \[1.1\] we can assume that \( D \cap B_2 \) consists of bridge disks \( D_0, \ldots, D_{n-1} \) and \( D \cap B_1 \) is a single disk, as in \[3\]. Then if \( n > 1 \), the bridge position of \( K \) admits a cancelling pair by Lemma \[3.2\], giving a proof of the uniqueness of bridge position of the unknot. One may hope that \( D_0 \cup \cdots \cup D_{n-1} \) extends to a complete cancelling disk system. But when \( n \geq 4 \), there exists an example such that \( D_0 \cup \cdots \cup D_{n-1} \) does not extend to a complete cancelling disk system, as expected in \[3\] Remark 1.2]. Some \( D_i \) is even not a cancelling disk. This issue was related to one of the motivations for the present work. In Figure \[4\], \( K \) is an unknot in 4-bridge position bounding a disk \( D \) and \( D \cap B_2 = D_0 \cup D_1 \cup D_2 \cup D_3 \). Each of the disks \( D_0 \) and \( D_1 \) is a cancelling disk. However, \( D_2 \) and \( D_3 \) are not cancelling disks because, say for \( D_2 \), an isotopy of \( b_2 \) along \( D_2 \) and then slightly into \( B_1 \) does not give a 3-bridge position of \( K \) (see \[11\], \[6\]).

4. Proof of Theorem \[1.1\] First step

Let \( K \) be a knot such that every non-minimal bridge position of \( K \) is perturbed. Let \( L \) be a \((2, 2q)\)-cable of \( K \), with components \( K_1 \) and \( K_2 \). Suppose that \( L \) is in non-minimal bridge position with respect to a bridge sphere \( S \) bounding 3-balls \( B_1 \) and \( B_2 \). Each \( L \cap B_i \) (\( i = 1, 2 \)) is a trivial tangle. Since \( L \) is a 2-cable link, \( L \) bounds an annulus, denoted by \( A \). We take \( A \) so that \( |A \cap S| \) is minimal.

Claim 1. One of the following holds.

- \( L \) is the unlink in a non-minimal bridge position, hence perturbed.
- \( A \cap B_2 \) is t-incompressible in \( B_2 \).

Proof. Suppose that \( A \cap B_2 \) is t-compressible. Let \( \Delta \) be a t-compressing disk for \( A \cap B_2 \) and let \( \alpha = \partial \Delta \). Let \( F \) be the component of \( A \cap B_2 \) containing \( \alpha \).

Case 1. \( \alpha \) is essential in \( A \).

A t-compression of \( A \) along \( \Delta \) gives two disjoint disks bounded by \( K_1 \) and \( K_2 \) respectively. Then \( L \) is an unlink. Since the complement of an unlink has a reducing sphere, by \[11\] a bridge position of an unlink is a split union of bridge positions of unknot components. Since a non-minimal bridge position of the unknot is perturbed, we see that \( L \) is perturbed.

Case 2. \( \alpha \) is inessential in \( A \).

Let \( \Delta' \) be the disk that \( \alpha \) bounds in \( A \). Then \( (\text{Int} \Delta') \cap S \neq \emptyset \), since otherwise \( \alpha \) is inessential in \( F \). By replacing \( \Delta' \) of \( A \) with \( \Delta \), we get a new annulus \( A' \) bounded by \( L \) such that \( |A' \cap S| < |A \cap S| \), contrary to the minimality of \( |A \cap S| \). \( \square \)

Since our goal is to show that the bridge position of \( L \) is perturbed, from now on we assume that \( L \) is not the unlink. By Claim \[1\] \( A \cap B_2 \) is t-incompressible in \( B_2 \). If \( A \cap B_2 \) is t-\( \partial \)-compressible in \( B_2 \), we do a t-\( \partial \)-compression.

Claim 2. A t-\( \partial \)-compression preserves the t-incompressibility of \( A \cap B_2 \).

Proof. Let \( \Delta \) be a t-\( \partial \)-compressing disk for \( A \cap B_2 \). Suppose that the surface after the t-\( \partial \)-compression along \( \Delta \) is t-compressible. A t-compressing disk \( D \) can be isotoped to be disjoint
from two copies of $\Delta$ and the product region $\Delta \times I$. Then $D$ would be a $t$-compressing disk for $A \cap B_2$ before the $t$-$\partial$-compression, a contradiction. $\square$

A $t$-$\partial$-compression simplifies a surface because it cuts the surface along a $t$-essential arc. So if we maximally $t$-$\partial$-compress $A \cap B_2$, we obtain a $t$-$\partial$-incompressible $A \cap B_2$. Note that the effect on $A$ of a $t$-$\partial$-compression of $A \cap B_2$ is just pushing a neighborhood of an arc in $A$ into $B_1$, which is called an isotopy of Type $A$ in [4]. After a maximal sequence of $t$-$\partial$-compressions, $A \cap B_2$ is both $t$-incompressible and $t$-$\partial$-incompressible by Claim 2. Then by applying Lemma 2.1

(*) $A \cap B_2$ consists of bridge disks $D_i$'s and properly embedded disks $C_j$'s.

5. Proof of Theorem 1.1: T-$\partial$-compression and its dual operation

Take an annulus $A$ bounded by $L$ so that (*) holds and the number $m$ of properly embedded disks $C_j$ is minimal. In this section, we will show that $m = 0$, i.e. $A \cap B_2$ consists of bridge disks only.

Suppose that $m > 0$. Then $A \cap B_1$ is homeomorphic to an $m$-punctured annulus. A similar argument as in the proof of Claim 1 leads to that $A \cap B_1$ is $t$-incompressible. By Lemma 2.1 again, $A \cap B_1$ is $t$-$\partial$-compressible. We can do a sequence of $t$-$\partial$-compressions on $A \cap B_1$ until it becomes $t$-$\partial$-incompressible. Note that the $t$-incompressibility of $A \cap B_1$ is preserved.

Now we are going to define a $t$-$\partial$-compressing disk $\Delta_i$ $(i = 0, 1, \ldots, s$ for some $s$) and its dual disk $U_{i+1}$ inductively. Let $A_0 = A$. Let $\Delta_0$ be a $t$-$\partial$-compressing disk for $A_0 \cap B_1$ and $\alpha_0 = \Delta_0 \cap A_0$ and $\beta_0 = \Delta_0 \cap S$. By a $t$-$\partial$-compression along $\Delta_0$, a neighborhood of $\alpha_0$ is pushed along $\Delta_0$ into $B_2$ and thus a band $b_1$ is created in $B_2$. Let $A_1$ denote the resulting annulus bounded by $L$. Let $U_1$ be a dual disk for $\Delta_0$, that is, a disk such that an isotopy of Type $A$ along $U_1$ recovers a surface isotopic to $A_0$. For the next step, let $U_1 = U_1$.

Let $\Delta_1$ be a $t$-$\partial$-compressing disk for $A_1 \cap B_1$ and $\alpha_1 = \Delta_1 \cap A_1$ and $\beta_1 = \Delta_1 \cap S$. After a $t$-$\partial$-compression along $\Delta_1$, a band $b_2$ is created in $B_2$. Let $A_2$ denote the resulting annulus bounded by $L$. There are three cases to consider.

Case 1. $\beta_1$ intersects the arc $U_1 \cap S$ more than once.

The band $b_2$ cuts off small disks $U_{2,1}, U_{2,2}, \ldots, U_{2,k_2}$ from $U_1$, which are mutually parallel along the band. We designate any one among the small disks, say $U_{2,1}$, as the dual disk $U_2$.

Let $R_2 = \bigcup_{j=2}^{k_2} U_{2,j}$ be the union of others.

Case 2. $\beta_1$ intersects $U_1 \cap S$ once.

We take the subdisk that $b_2$ cuts off from $U_1$ as the dual disk $U_2$, and let $R_2 = \emptyset$ in this case.

Case 3. $\beta_1$ does not intersect $U_1 \cap S$.

We take a dual disk $U_2$ freely, and let $R_2 = \emptyset$ in this case.

In any case, we get $U_2 = U_1 \cup U_2 - \text{int } R_2$.

In general, assume that $A_i$ and $U_i$ are defined. Let $\Delta_i$ be a $t$-$\partial$-compressing disk for $A_i \cap B_1$ and $\alpha_i = \Delta_i \cap A_i$ and $\beta_i = \Delta_i \cap S$. After a $t$-$\partial$-compression along $\Delta_i$, a band $b_{i+1}$ is created in $B_2$. Let $A_{i+1}$ denote the resulting annulus bounded by $L$.

Case a. $\beta_i$ intersects the collection of arcs $U_i \cap S$ more than once.

The band $b_{i+1}$ cuts off small disks $U_{i+1,1}, U_{i+1,2}, \ldots, U_{i+1,k_{i+1}}$ from $U_i$, which are mutually parallel along the band. We designate any one among the small disks, say $U_{i+1,1}$, as the dual disk $U_{i+1}$. Let $R_{i+1} = \bigcup_{j=2}^{k_{i+1}} U_{i+1,j}$ be the union of others.

Case b. $\beta_i$ intersects $U_i \cap S$ once.
We take the subdisk that $b_{i+1}$ cuts off from $U_i$ as the dual disk $U_{i+1}$, and let $R_{i+1} = \emptyset$ in this case.

**Case c.** $\beta_i$ does not intersect $U_i \cap S$.

We take a dual disk $U_{i+1}$ freely, and let $R_{i+1} = \emptyset$ in this case.

In any case, let $U_{i+1} = U_i \cup U_{i+1} - \text{int} R_{i+1}$.

Later, we do isotopy of Type A, dual to the $t-\partial$-compression, in reverse order along $U_i U_{i+1}, \ldots, U_1$. Let us call it dual operation for our convenience. Let $b_{i+1}$ be the band mentioned above, cutting off $U_{i+1}, U_{i+1,1}, \ldots, U_{i+1,k+1}$ from $U_i$ (in Case a). When the dual operation along $U_{i+1}$ is done, we modify every $U_j$ and $U_j (j \leq i)$ containing any $U_{i+1,s}$ (s > 1) of $R_{i+1}$, by replacing each $U_{i+1,s}$ (s > 1) with the union of a subband of $b_{i+1}$ between $U_{i+1,s}$ and $U_{i+1}$ and a copy of $U_{i+1}$, and doing a slight isotopy. We remark that, although it is not illustrated in Figure 5, some $U_j$'s and $U_j$'s temporarily become immersed when the subband passes through some removed region, say $U_{r,s}$ (s > 1). But the $U_{r,s}$ (s > 1) is also modified as we proceed the dual operations, and the $U_j$'s and $U_j$'s again become embedded. (In Figure 5 and Figure 6, the dual operation along $U_{i+1}$ is done, and the dual operation along $U_i$ is not done yet.) Actually, before the sequence of dual operations, $U_j$'s and $U_j$'s (j ≤ k) are modified in advance so that $U_k$ is disjoint from the union of certain band $b_{k+1}$ and a disk $C_l$ (which will be explained later). See Figure 6.

**Claim 3.** For each $c_j$, there exists an $\alpha_i$ such that $\alpha_i$ connects $c_j$ to other component of $A \cap S$.

**Proof.** Suppose that there exists a $c_j$ which is not connected to other component of $A \cap S$. That is, for such $c_j$, every $\alpha_i$ incident to $c_j$ connects $c_j$ to itself. Then after a maximal
sequence of $t$-$\partial$-compressions on $A$, some non-disk components will remain. This contradicts Lemma 2.1.

Let $k$ be the smallest index such that $\alpha_k$ connects some $c_j$, say $c_l$, to other component (other $c_j$ or $D_i \cap S$). If $k = 0$, then by a $t$-$\partial$-compression along $\Delta_0$, either $C_l$ and other $C_j$ are merged into one properly embedded disk, or $C_l$ and a bridge disk are merged into a new bridge disk. This contradicts the minimality of $m$. So we assume that $k \geq 1$.

Suppose that we performed $t$-$\partial$-compressions along $\Delta_0, \Delta_1, \ldots, \Delta_k$. Consider the small disks that the band $b_{k+1}$ cuts off from $U_k$. They are parallel along $b_{k+1}$. We replace the small disks one by one, the nearest one to $C_l$ first, so that $U_k$ is disjoint from $b_{k+1} \cup C_l$. Let $\Delta$ be the small disk nearest to $C_l$. Let $\Delta'$ be the union of a subband of $b_{k+1}$ and $C_l$ that $\Delta \cap b_{k+1}$ cuts off from $b_{k+1} \cup C_l$. For every $U_j$ and $U_j$ ($j \leq k$) containing $\Delta$, we replace $\Delta$ with $\Delta'$. Then again let $\Delta$ be the (next) small disk nearest to $C_l$ and we repeat the above operation until $U_k$ is disjoint from $b_{k+1} \cup C_l$.

Now we do the dual operation on $A_k$ in reverse order along $U_k, U_{k-1}, \ldots, U_1$. Let $A'_i$ ($i = 1, \ldots, k$) be the resulting annulus after the dual operation along $U_i$. The shape of the dual disk $U_k$ is possibly changed but the number of circle components and arc components of $A'_{k-1} \cap S$ is same with those of $A_{k-1} \cap S$. After the dual operation along $U_k$, it is necessary to modify some $U_j$'s and $U_j$'s ($j \leq k - 1$) further as in Figure 6 so that $U_{k-1}$ is disjoint from $b_{k+1} \cup C_l$. In this way, we do the sequence of dual operations, and the number of circle components of $A'_0 \cap S$ is also $m$. Then because $b_{k+1}$ is disjoint from $U_1, \ldots, U_k$, we can do the $t$-$\partial$-compression of $A'_0$ along $\Delta_k$ first and the number $m$ of properly embedded disks $C_j$ is reduced, contrary to our assumption.

We have shown the following claim.

**Claim 4.** $A \cap B_2$ consists of bridge disks.
6. Proof of Theorem 4.1 Finding a cancelling pair

By Claim 4.2, \( A \cap B_2 \) consists of bridge disks. Let \( d_0, \ldots, d_{k-1} \) and \( e_0, \ldots, e_{l-1} \) be bridges of \( K_1 \cap B_2 \) and \( K_2 \cap B_2 \) respectively that are indexed consecutively along each component. Let \( D_i \subset A \cap B_2 \) \((i = 0, \ldots, k-1)\) be the bridge disk for \( d_i \) and \( s_i = D_i \cap S \), and let \( E_j \subset A \cap B_2 \) \((j = 0, \ldots, l-1)\) be the bridge disk for \( e_j \) and \( t_j = E_j \cap S \). Let \( R = R_1 \cup \cdots \cup R_{k+l} \) be a complete bridge disk system for \( L \cap B_1 \) and let \( F = A \cap B_1 \). We consider \( R \cap F \) except for the bridges \( L \cap B_1 \).

If there is an inessential circle component of \( R \cap F \) in \( F \), it can be removed by standard innermost disk argument. If there is an essential circle component of \( R \cap F \) in \( F \), then \( L \) would be the unlink as in Case 1 of the proof of Claim 4.1. So we assume that there is no circle component of \( R \cap F \). If there is an inessential arc component of \( R \cap F \) in \( F \) with both endpoints on the same \( s_i \) (or \( t_j \)), then the arc can be removed by standard outermost disk argument. So we assume that there is no arc component of \( R \cap F \) with both endpoints on the same \( s_i \) (or \( t_j \)).

If \( R \cap F = \emptyset \), we easily get a cancelling pair, say \((R_m, D_i)\) or \((R_m, E_j)\), so we assume that \( R \cap F \neq \emptyset \). Let \( \alpha \) denote an arc of \( R \cap F \) which is outermost in some \( R_m \) and let \( \Delta \) denote the outermost disk that \( \alpha \) cuts off from \( R_m \). Applying Lemma 3.1, \( \alpha \) is not b-parallel. Suppose that one endpoint of \( \alpha \) is in, say \( s_{i_1} \), and the other is in \( s_{i_2} \) with the cyclic distance \( d(i_1, i_2) = \min \{|i_1 - i_2|, |k - (i_1 - i_2)|\} \) greater than 1. Then after the \( t \)-\( \partial \)-compression along \( \Delta \), we get a subdisk of \( A \) satisfying the assumption of Lemma 3.2, hence \( L \) is perturbed. So without loss of generality, we assume that one endpoint of \( \alpha \) is in \( s_0 \) and the other is in \( t_0 \).

Let \( A_1 \) be the annulus obtained from \( A \) by the \( t \)-\( \partial \)-compression along \( \Delta \) and \( F_1 = A_1 \cap B_1 \). The bridge disks \( D_0 \) and \( E_0 \) are connected by a band, and let \( P_1 \) be the resulting rectangle with four edges \( d_0, e_0 \), and two arcs in \( S \), say \( p_1, p_2 \). Let \( \alpha_1 \) denote an arc of \( R \cap F_1 \) which is outermost in some \( R_m \) and let \( \Delta_1 \) denote the outermost disk cut off by \( \alpha_1 \). If at least one endpoint of \( \alpha_1 \) is contained in \( p_1 \) or \( p_2 \), or one endpoint of \( \alpha_1 \) is in \( s_{i_1} (t_{j_1} \text{ respectively}) \) and the other is in \( s_{i_2} (t_{j_2} \text{ respectively}) \), then similarly as above,

- either \( \alpha_1 \) is inessential with both endpoints on the same component of \( F_1 \cap S \), or
- \( \alpha_1 \) is b-parallel, or
- Lemma 3.2 can be applied.

Hence we may assume that one endpoint of \( \alpha_1 \) is in \( s_i \) \((i \neq 0)\) and the other is in \( t_j \) \((j \neq 0)\). After the \( t \)-\( \partial \)-compression along \( \Delta_1 \), \( D_i \) and \( E_j \) are merged into a rectangle. Arguing in this way, each \( D_i \) \((i = 0, \ldots, k-1)\) is merged with some \( E_j \) because of the fact that \( R \cap s_i = \emptyset \) gives us a cancelling pair. Moreover, we see that \( k = l \). After \( k \) successive \( t \)-\( \partial \)-compressions on \( A \), the new annulus \( A' \) intersects \( B_1 \) and \( B_2 \) alternately, in rectangles.

Note that \( b(L) = 2b(K) \) and \( L \) is in \( 2k \)-bridge position. Since \( L \) is in non-minimal bridge position, \( k > b(K) \). So by the assumption of the theorem, the bridge position of \( K_i \) \((i = 1, 2)\) is perturbed. Let \((D, E)\) be a cancelling pair for \( K_1 \) with \( D \subset B_1 \) and \( E \subset B_2 \). However, \( D \) and \( E \) may intersect \( K_2 \). Let \( P_i \) and \( P_{i+1} \) be any adjacent rectangles of \( A' \) in \( B_1 \) and \( B_2 \) respectively. We remove any unnecessary intersection of \( D \cap P_i \) and \( E \cap P_{i+1} \), the nearest one to \( K_2 \) first, by isotopies along subdisks of \( P_i \) and \( P_{i+1} \) respectively. See Figure 7 for an example. Then \((D, E)\) becomes a cancelling pair for the bridge position of \( L \) as desired.

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Figure 7. Isotoping $D$ and $E$.

REFERENCES

[1] D. Bachman and S. Schleimer, Distance and bridge position, Pacific J. Math. 219 (2005), no. 2, 221–235.
[2] H. Doll, A generalized bridge number for links in 3-manifolds, Math. Ann. 294 (1992), no. 4, 701–717.
[3] C. Hayashi and K. Shimokawa, Heegaard splittings of the trivial knot, J. Knot Theory Ramifications 7 (1998), no. 8, 1073–1085.
[4] W. Jaco, Lectures on three-manifold topology, CBMS Regional Conference Series in Mathematics, 43. American Mathematical Society, Providence, R.I., 1980.
[5] Y. Jang, T. Kobayashi, M. Ozawa, and K. Takao, A knot with destabilized bridge spheres of arbitrarily high bridge number, J. Lond. Math. Soc. (2) 93 (2016), no. 2, 379–396.
[6] J. H. Lee, Reduction of bridge positions along bridge disks, Topology Appl. 223 (2017), 50–59.
[7] J. -P. Otal, Présentations en ponts du nœud trivial, C. R. Acad. Sci. Paris Sér. I Math. 294 (1982), no. 16, 553–556.
[8] J. -P. Otal, Présentations en ponts des nœuds rationnels, Low-dimensional topology (Chelwood Gate, 1982), 143–160, London Math. Soc. Lecture Note Ser., 95, Cambridge Univ. Press, Cambridge, 1985.
[9] M. Ozawa, Nonminimal bridge positions of torus knots are stabilized, Math. Proc. Cambridge Philos. Soc. 151 (2011), no. 2, 307–317.
[10] M. Ozawa and K. Takao, A locally minimal, but not globally minimal, bridge position of a knot, Math. Proc. Cambridge Philos. Soc. 155 (2013), no. 1, 181–190.
[11] M. Scharlemann and M. Tomova, Uniqueness of bridge surfaces for 2-bridge knots, Math. Proc. Cambridge Philos. Soc. 144 (2008), no. 3, 639–650.
[12] H. Schubert, Über eine numerische Knoteninvariante, Math. Z. 61 (1954), 245–288.
[13] J. Schultens, Additivity of bridge numbers of knots, Math. Proc. Cambridge Philos. Soc. 135 (2003), no. 3, 539–544.
[14] A. Zupan, Properties of knots preserved by cabling, Comm. Anal. Geom. 19 (2011), no. 3, 541–562.

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