Time-consistent decisions and rational expectation equilibrium existence in DSGE models

By Minseong Kim∗

We demonstrate that if all agents in an economy make time-consistent decisions and policies, then there exists no rational expectation equilibrium in a dynamic stochastic general equilibrium (DSGE) model, unless under very restrictive and special circumstances. Some time-consistent interest rate rules, such as Taylor rule, worsen the equilibrium non-existence issue in general circumstances. Monetary policy needs to be lagged in order to avoid equilibrium non-existence due to agents making time-consistent decisions. We also show that due to the transversality condition issue, either fiscal-monetary coordination may need to be modeled, or it may be necessary to write a model such that bonds or money provides utility as medium of exchange or has liquidity roles.

JEL: C62, C32, E12, E13, E31, E43, E52, C61

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I. Introduction

The central result of this article is the general impossibility result (GIR). If consumers in a multiple-consumer dynamic stochastic general equilibrium (DSGE) model do not have, as in a standard neoclassical model, past consumption inertia, and if agents make time-consistent decisions, then there exists no equilibrium unless under very restrictive circumstances.

To break away from this non-existence issue, central bank needs to provide agents a reason to commit to what would otherwise be time-inconsistent decisions. After this reason is provided, agent decisions do need to become time-consistent. If central bank uses an interest rate rule to set monetary policy, then the necessary condition is to have a rule that is lagged - that is, how central bank sets nominal interest rate $i_t$ depends on past variables, instead of just present-time variables in a standard Taylor rule.

For the basic canonical New Keynesian model, as described in Woodford (2003) and Galí (2015), we demonstrate that unless initial (central bank-issued) bond holding $B_{-1}$ is zero, there is a transversality condition problem that prevents an equilibrium from forming, regardless of monetary policy. We show that the transversality condition issue apply for heterogeneous consumer variants of the canonical New Keynesian model as well.

In order to resolve the transversality condition issue, fiscal-monetary coordination is needed if we are not to impose a restriction on a model. That is, money redeemed from bonds needs to be taxed so that the transversality condition can be satisfied.

Or one can instead consider rewriting a model such that money or bonds provide utility as medium of exchange or liquidity, which allows us to drop or evade the transversality condition. In case only money has utility as medium
of exchange, a part of monetary policy in a model should be about trading off bonds with money such that the transversality condition is satisfied. Effectively, this option amounts to a statement that a cashless economy, as envisioned in Woodford (2003), is not plausible, though note that the above fiscal-monetary coordination alternative allows for a cashless economy. This restricts forms of an economic model, but does have economic justifications and relevance.

II. Review of time consistency

It may be beneficial to discuss time consistency (see also Kydland and Prescott (1977)) firsthand before going straight into the result. It is true that solutions to each consumer optimization problem in a DSGE model with standard neoclassical consumer assumptions satisfies time consistency, since it is equivalent to a recursive dynamic programming problem of the Bellman equation under very mild assumptions - see Stokey, Lucas and Prescott, 1989, Chapter 4) and an appendix section on this equivalence. However, these consumer optimization problems must be provided with a given price vector for each economic condition and state. Only when this price vector is known can we solve consumer optimization problems. This price vector can only be known after solving for an equilibrium model, and it is this interaction process that causes time inconsistency issues.

That is, time consistency applies not only to each consumer optimization problem but also to the combined result of optimization problems agents solve, and it is the latter that the conventional understanding of DSGE models misses.

To summarize in a mathematical notation, if a model can be re-written,
appropriately, in a time-invariant form, then $x_t = f(k_t, u_t, o_t)$, where $x_t$ is the vector of non-predetermined endogenous variables, $k_t$ is the vector of predetermined endogenous variables, $u_t$ is the vector of exogenous variables, $o_t$ is the vector of past variables that have to be explicitly used for agent decisions, is required for a rational expectation equilibrium of DSGE models. $x_t$ cannot have explicit time-dependence, in form of $x_t = f(k_t, u_t, o_t, t)$. Also, $k_{t+1}$ is required to have no explicit time-dependence and needs to be in form of $k_{t+1} = g(k_t, u_t, o_t)$.

One may ask why. The idea is simple. If a model is time-invariant, and state variables, including exogenous variables, predetermined endogenous variables and past variables explicitly used for agent decisions, are same at two different times $t$, then agent decisions of $x_t$ must be exactly same at two different times $t$, because agents are facing exactly the same economic circumstance. Written this way, it may seem that this is so obvious and innocuous such that any rational expectation equilibrium would satisfy this condition. Whether surprisingly or not, many rational expectation equilibria in DSGE models are not time-consistent as described above - that is, solutions feature explicit time-dependence.

What if a model is not time-invariant? That is, optimization decisions that agents face are different at each time, not just in terms of state variables but in terms of form of problems. This can happen if agents have time-varying time preference discount factor $\beta_t$, as an example, and other more complicated examples do exist.

But even there, the time consistency requirement would still hold. $x_t = f(k_t, u_t, o_t, a_t)$, where $a_t$ refers to decision problems agents face at each time $t$. If agents face the same decision problems at time $T$ and $T'$, then $x_T = x_{T'}$. 
and \( k_{T+1} = k_{T'+1} \).

The effect of the time consistency requirement can be pedagogically demonstrated, replicating analysis in McCallum (2004), in a linear rational expectation model of form:

\[
y_t = A\varepsilon_t y_{t+1} + Du_t
\]

\[
u_t = R u_{t-1} + \varepsilon_t
\]

with \( \varepsilon_t \) an iid white noise vector of zero mean, and solutions required to have form of

\[
y_t = \Omega y_{t-1} + \Gamma u_t
\]

where \( y_t \) is the vector of endogenous variables, both predetermined and non-predetermined.

The non-fundamental solutions, or non-MSV solutions, are of form:

\[
y_t = A^{-1} y_{t-1} + \Gamma u_t
\]

where as a unique MSV (minimal state variable) solution is:

\[
y_t = \Gamma u_t
\]

A non-MSV solution is time-inconsistent, because even when \( u_T = u_{T'} \) and \( k_T = k_{T'} \), it is possible that \( x_T \neq x_{T'} \). Only the MSV solution, if a solution is required to be of form \( y_t = \Omega y_{t-1} + \Gamma u_t \), is time-consistent.

From now on, we assume a perfect-foresight deterministic economy with rational expectation throughout the article. However, the conclusions derived apply without loss of generality to stochastic economies as well.
III. General impossibility result

A. Result

MODEL

We present a multiple-consumer DSGE model. We will not specify most details of the model, since we intend to allow as much generalization as possible. Consumer $j = 1, 2, ..$ has utility function $U_j$ that consumer $j$ maximizes by controlling \{\{C_{j,t}, B_{j,t}, \ldots\}\}:

$$\max\ \{C_{j,t}, B_{j,t}, \ldots\} \quad U_j = \max\ \{C_{j,t}, B_{j,t}, \ldots\} \sum_{t=0}^{\infty} (\beta_j)^t u_j(C_{j,t}, \ldots)$$

Caution is required: $U$ and $u$ different, and time-discounted $U$ is utility function for every period. Let the budget constraint of each consumer be:

$$P_t C_{j,t} + (1 + i_t)^{-1} B_{j,t} + \ldots \leq B_{j,t-1} + \ldots$$

where $\ldots$ terms do not contain any of $C_{j,t}$ and $B_{j,t}$. $P_t$ refers to price level, $C_{j,t}$ refers to consumer $j$’s consumption, $i_t$ is nominal interest rate set by central bank, $B_{j,t}$ refers to quantity of central bank bonds agent $j$ holds.

GENERAL IMPOSSIBILITY RESULT (GIR)

The derivation of the result is provided in an appendix section. The general impossibility result refers to:

$$\frac{u_j'(C_{j,t})}{u_k'(C_{k,t})} = \frac{(1 - \gamma_{jk})}{\gamma_{jk}} \left(\frac{\beta_k}{\beta_j}\right)^t$$
where
\[ u'(C) = \frac{\partial u}{\partial C} \]
at particular value \( C \) and \( \gamma_{jk} \) is some time-invariant constant. Equation 3 is a time-inconsistent result - the general impossibility result (GIR). Regardless of whatever happens to state variables, Equation 3 says that the consumption distribution variable \( u'_j(C_{j,t})/u'_k(C_{k,t}) \) has predetermined explicit time-dependence. At this point, we are yet to be able to determine anything about \( B_{j,t} \). Thus a multiple-consumer DSGE model cannot have an equilibrium unless in very restrictive and special circumstances.

If we are to avoid heavily restricting the form of economic models more than what can be justified by economic relevance, then an interest rate rule for \( i_t \) must feature dependence on \( t - 1 \) variables such that the right consumption distribution given by Equation 3 is maintained in a time-consistent way. This allows an equilibrium to form as usual in DSGE models.

\[ B. \quad \text{Discussion of a lagged interest rate rule} \]

Why would an interest rate rule for \( i_t \) featuring explicit dependence on \( t - 1 \) variables, done the right way, resolve the equilibrium non-existence issue, caused by the time consistency requirement?

This is because when consumers make decisions, they do have to form expectations of future \( i_t \), and it is here that an interest rate rule plays a role. Of course it is not sufficient for central bank to simply provide a lagged interest rate rule - it needs to provide a right rule such that the consumption distribution with explicit time-dependence, suggested by Equation 3, can be supported as time-consistent agent decisions.
C. An example of GIR, and corresponding explanations on why GIR is about time consistency

In case we have CRRA utility $u$, then $u_j'(C) = C^{-\sigma_j}$. In such a case, GIR for consumer $j = 1, 2$ is, with $\gamma_{12} = \gamma$:

$$\frac{(C_{1,t})^{-\sigma_1}}{(C_{2,t})^{-\sigma_2}} = \frac{(1 - \gamma)}{\gamma} \left( \frac{\beta_2}{\beta_1} \right)^t$$

and in case all consumers share $\sigma_j = \sigma$, then:

$$\frac{(C_{1,t})^{-\sigma}}{(C_{2,t})^{-\sigma}} = \frac{(1 - \gamma)}{\gamma} \left( \frac{\beta_2}{\beta_1} \right)^t$$

allowing us to see how distribution of consumption has evolved and will evolve over time from past to future using few details of utility function and budget constraints and present-time aggregate consumption and its distribution. No other details of the economy are needed - for example, we do not need to discuss what labor skills each consumer or worker in the past had. This should already make some realize of severity of the general impossibility result in form of Equation 3.

If agent decisions do not feature explicit dependence on past variables, then Equation 5 means that we only need aggregate consumption and its distribution at a faraway future time point to determine distribution of consumption up to the future time point.

One may question severity of the result by arguing that state variables, such as $B_{j,t}$ and others, record past history compactly, which allows avoidance of time inconsistency. This counter-argument cannot work, as one can see from the discussion of linear rational expectation models. Also, in
many contexts, this counter-argument amounts to stating that there exists a unique path to obtaining state variables at time $t$ in a perfect-foresight economy.

Furthermore, Equation 3 can be derived even when there is no state variable such as $B_{j,t}$, and the counter-argument cannot work. We demonstrate that Equation 3 can be derived in much more general contexts by re-adapting the Negishi approach to computing a competitive equilibrium, described in Negishi (1960) - the alternative derivation is provided in the appendix. The requirement for this derivation is that feasible utility vectors of utility of each consumer form a convex set, with an equilibrium utility vector sitting on the boundary of the set. This holds in a price-taking competitive equilibrium, and while extensions to monopolistic competitions require re-defining the word “feasible,” they can be done in some contexts.

**D. A sequential equilibrium is an optimal time-consistent equilibrium**

What if a consumer chooses the best plan (“optimal time-consistent plan”) out of feasible time-consistent decisions, instead of solving for an optimal sequential plan which involves directly controlling present and future control variables to maximize today’s time-discounted utility?

But a consumer in a DSGE model already solves for an optimal time-consistent plan of a consumer optimization problem given a price vector - one can see this from equilibrium conditions in DSGE models without the perfect-foresight assumption. One example is a log-linearized consumption Euler equation:

$$c_t = E_t c_{t+1} - (i_t - E_t \pi_{t+1} - r^n_t)$$

Note that the expectation operator is $E_t$, not $E_0$. In case of perfect foresight,
difference between $E_t$ and $E_0$ disappears, so we did not have to deal with this issue.

In case of perfect foresight, difference between an optimal time-consistent plan and an optimal sequential plan in a consumer optimization problem disappears. A rigorous explanation on this matter is provided in (Stokey, Lucas and Prescott, 1989, Chapter 4). Note that each consumer optimization problem requires a given price vector to solve for. This equivalence seems to have been misunderstood as proving that a sequential equilibrium is always a time-consistent equilibrium, but as we can see in case of linear rational expectation models, a sequential equilibrium can be time-inconsistent.

We leave replication of equivalence of a sequential optimization problem with a recursive dynamic programming problem to a section in the appendix, which simply quotes [Recursive Methods in Economic Dynamics] (Stokey, Lucas and Prescott, 1989, Chapter 4).

IV. A $B_t$ transversality condition issue in the canonical New Keynesian model

We do not need to specify most details of the canonical New Keynesian model (Galí, 2015, Chapter 3) to probe a transversality condition issue - we will only need a representative consumer budget constraint and justification for a transversality condition to see that there is a problem. Thus analysis applies generally for models sharing the same budget constraint and accounting identities.

There is only one representative consumer in the canonical New Keynesian model. The only production factor in the model economy is labor. In the model, budget constraint and accounting identity enforce for feasible
outcomes (and not just equilibrium outcomes):

\[ P_t C_t + \frac{B_t}{1 + i_t} \leq W_t N_t + F_t + B_{t-1} \]  
Consumer budget constraint

\[ P_t C_t = W_t N_t + F_t \]  
Aggregate accounting identity for firms

thus

\[ \frac{B_t}{1 + i_t} \leq B_{t-1} \]

where \(1/(1 + i_t)\) is the price of bond \(B_t\) that unit quantity pays unit money at \(t + 1\), with \(i_t\) set by central bank. \(P_t\) is price level, \(C_t\) is consumption, \(W_t\) is wage level, \(N_t\) is labor utilized for production, \(F_t\) is profits or loss of firms redistributed entirely to the consumer as a shareholder.

If agents are truly optimizing, then the budget constraint inequality becomes equality and thus:

\[ \frac{B_t}{1 + i_t} = B_{t-1} \]

But this is inconsistent with optimal behavior of the representative consumer, unless \(B_{t-1} = 0\), because of a transversality condition issue. And if \(B_{-1} = 0\), then \(B_t = 0\) forever.

Without referring to a transversality condition, one can see why Equation 9 is problematic. The equation says that a consumer uses all of money redeemed from \(B_{t-1}\) to invest back into \(B_t\). Thus there is no consumption smoothing.

One may say this means \(B_t = 0\), but one must recall that \(B_t\) is a state
variable. Thus there must be an equilibrium for all possible values of the initial condition $B_{-1} \geq 0$, assuming present time is set at $t = 0$. But this is not feasible - there is no equilibrium for $B_{-1} > 0$.

What is a transversality condition in this circumstance? It is defined as:

$$\lim_{T \to \infty} \left( \prod_{t=0}^{T} \frac{1}{1 + i_t} \right) B_T = 0 \quad \text{Transversality condition}$$

It means that the consumer, in long-run, should not buy $B_t$ today to again purchase $B_t$ tomorrow.

Note that simple taxation suffices to evade the transversality condition problem. Re-write the budget constraint such that:

$$P_tC_t + \frac{B_t}{1 + i_t} + T_t \leq W_tN_t + F_t + B_{t-1}$$

where $T_t \geq 0$ refers to exogenous taxes that agents cannot optimize against. We know that

$$P_tC_t = W_tN_t + F_t$$

and assuming an optimal decision, the inequality of Equation 11 turns into equality such that:

$$\frac{B_t}{1 + i_t} + T_t = B_{t-1}$$

Given sufficient taxation, any form of a required transversality condition can be evaded. This demonstrates that unless a model is re-written as to provide utility of money or bonds as medium of exchange or to incorporate liquidity, fiscal-monetary coordination may be essential to ensure an equilibrium for some economically relevant models, since conventional monetary policy can
only swap one asset with another.

A. A transversality condition issue in the canonical heterogeneous-agent New Keynesian model

We demonstrate that a simple multiple-consumer variant of the canonical New Keynesian model (we refer to it as a canonical HANK model) may have an equilibrium only for one set of $B_{j,-1}$, where index $j$ refers to each consumer, when an equilibrium outcome is independent of value of $B_{j,t}$, though an equilibrium outcome can depend on $i_t$.

Let the budget constraint for each agent $j$ be:

\[(13) \quad P_t C_{j,t} + \frac{B_{j,t}}{1 + i_t} \leq W_t N_{j,t} + F_{j,t} + B_{j,t-1} \]  

Consumer budget constraint

where $C_{j,t}$ is quantity consumed for $j$, $B_{j,t}$ refers to quantity of central bank bonds consumer $j$ holds, $N_{j,t}$ refers to labor $j$ supplied for production, $F_{j,t}$ refers to profits $j$ received as a shareholder.

We can establish independence of an equilibrium outcome from value of $B_t$ (which refers to all of $B_{j,t}$) when variables other than $B_t$ can be derived from first-order conditions and accounting identities do not reference $B_t$ at all, though $i_t$ may be referenced, and an interest rate rule for $i_t$ does not involve $B_t$. This is the case for models having consumers with the budget constraint of Equation 13 and the utility function of each consumer not involving $B_t$, firm decisions having nothing to do with $B_t$ directly and an interest rate rule is a Taylor rule that references only output gap and inflation. Thus, we say that a canonical HANK model satisfies equilibrium independence from $B_t$.

Given the above, suppose that for some value $B_{j,-1} = B_{a,j,-1}$, a transver-
sality condition of:

\[ \lim_{T \to \infty} \left( \prod_{t=0}^{T} \frac{1}{1 + i_t} \right) B_{j,T} = 0 \] Transversality condition

is satisfied, so

\[ \lim_{T \to \infty} \left( \prod_{t=0}^{T} \frac{1}{1 + i_t} \right) B_{a,j,T} = 0 \]

holds. Let us change the initial condition such that \( B_{j,-1} = B_{a,j,-1} + k_j \). Then,

\[ B_{j,T} = B_{a,j,T} + k_j \prod_{t=0}^{T} (1 + i_t) \]

follows from the budget constraint Equation 13 because of equilibrium independence from \( B_t \). Equation 16 violates the transversality condition in Equation 14.

B. The usual transversality condition

The transversality condition we used so far was the least stringent one. However, the usual transversality condition used is much more stringent.

It is possible to argue that Equation 8 suggests that the transversality condition issue is not robust in that consumers can continuously lose small proportion of their wealth at each time, and the transversality condition in form of Equation 10 or Equation 14 would be satisfied. Thus, we need to consider how far the transversality condition actually is restrictive.
usual form that can be derived is:

\[
\lim_{T \to \infty} \beta^T \frac{u'(C_T)}{P_T} \frac{B_T}{1 + i_T} = 0
\]

We can rewrite the above as, considering the fact that \( u'(C_0)/P_0 > 0 \):

\[
\lim_{T \to \infty} \frac{B_T}{1 + i_T} = 0
\]

Thus, it is not enough for consumers to simply lose small proportion of their wealth at each time to evade the transversality condition issue. The full derivation of the usual transversality condition is provided in the appendix.

\[C. \ B_t = 0 \text{ is problematic for effectiveness of an interest rate rule}\]

In the above, it was stated that because \( B_t \) is effectively a state variable, equilibrium existence should be guaranteed for all \( B_t \geq 0 \).

There actually is one more reason why \( B_t = 0 \) is problematic. Stated in common words, \( B_t = 0 \) means that there is nothing in the market that monetary policy can act on. Thus, why should agents believe that an interest rate rule can ever be effective in controlling an economy?

It may still be argued that as long as agents do have beliefs that an interest rate rule is effective, then it still works in controlling an economy even when \( B_t = 0 \). This is disproved in the appendix. Mainly, the point is that if \( B_t = 0 \), then an additional Lagrange multiplier that was suppressed when \( B_t > 0 \) appears such that we can no longer guarantee equilibrium uniqueness.
V. Conclusions

DSGE models are different from traditional static general equilibrium analysis and present numerous difficulties not encountered in static general equilibrium analysis. This article demonstrates that there are still many aspects not fully uncovered.

In general, we showed that it is quite difficult to ensure rational expectation equilibrium existence in a DSGE model, unless right fiscal and monetary policy are conducted, if agents make time-consistent decisions. The requirements for fiscal and monetary policy are very restrictive and tight.

Time-consistent decisions are not about making no reference to past variables in a decision process. If agent decisions are explicitly required by a model to reference past variables, then agents must do so. Time consistency rather states that in a DSGE model, what agents set are decision functions such that $x_t = f(u_t, k_t, o_t, a_t)$ and $k_{t+1} = g(u_t, k_t, o_t, a_t)$, where $x_t$ is the vector of non-predetermined endogenous variables, $u_t$ is the vector of exogenous variables, $k_t$ is the vector of predetermined endogenous variables, $o_t$ is the vector of past variables agents are explicitly required to use for decisions and $a_t$ refers to decision problems agents face. Time consistency thus states that if $u_T = u_{T'}$, $k_T = k_{T'}$, $o_T = o_{T'}$ and $a_T = a_{T'}$, then one should get $x_T = x_{T'}$ and $k_{T+1} = k_{T'+1}$. For models that can be converted to a time-invariant form, $x_t = f(u_t, k_t, o_t)$ and $k_{t+1} = g(u_t, k_t, o_t)$ with variables are those used in a time-invariant model.

We also demonstrated that even if there is right fiscal and monetary policy are conducted to eliminate the time consistency issue, in some types of models, especially the ones often used for pedagogical purposes, they also may have to fight against the transversality condition issue. To remove the
problem generally, some taxation is necessary in order to ensure equilibrium existence in a DSGE model. Also, one may instead write a model so that liquidity plays a role or such that money or bonds have utility as medium of exchange as to eliminate or evade the transversality condition.

The necessary requirements for equilibrium existence in a DSGE model seem quite staggering, and thus the question of how far the convention understanding of DSGE models can be saved may have to be considered.

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Derivation of the general impossibility result (GIR)

Set up a Lagrangian:

\[ U_j + \sum_{t=0}^{\infty} [\lambda_{j,t} (B_{j,t-1} + .. - [P_tC_{j,t} + (1 + i_t)^{-1}B_{j,t} + ..])] \]

Let us consider the first-order condition associated with \( B_{j,t} \). It goes:

\[ \lambda_{j,t+1} - \lambda_{j,t}(1 + i_t)^{-1} = 0 \]

due to

(A1) \[ \frac{\lambda_{j,t+1}}{\lambda_{j,t}} = \frac{1}{1 + i_t} \]

Consider the first-order condition associated with \( C_{j,t} \). It goes:

\[ (\beta_j)^t u_j'(C_{j,t}) - \lambda_{j,t}P_t = 0 \]

due to

\[ \lambda_{j,t} = \frac{(\beta_j)^t u_j'(C_{j,t})}{P_t} \]

Now pick two consumers \( j \) and \( k \):

\[ \frac{\lambda_{j,t}}{\lambda_{k,t}} = \frac{(\beta_j)^t u_j'(C_{j,t})}{(\beta_k)^t u_k'(C_{k,t})} \]

\[ \frac{u_j'(C_{j,t})}{u_k'(C_{k,t})} = \frac{(\beta_k)^t \lambda_{j,t}}{(\beta_j)^t \lambda_{k,t}} \]
Because of Equation A1

\[ \frac{\lambda_{j,t}}{\lambda_{k,t}} = \frac{\lambda_{j,t+1}}{\lambda_{k,t+1}} = 1 - \frac{\gamma_{jk}}{\gamma_{jk}} \]

for some constant \( \gamma_{jk} \). Thus we get GIR in Equation 3.

**The general-context derivation of GIR via the Negishi procedure**

Because the derivation of GIR relied on existence of a central bank bond \( B_t \), we would prefer the derivation that reproduces Equation 3 without having to rely on existence of \( B_t \). This can be done by re-adapting the idea in Negishi (1960).

Negishi (1960) states that a candidate competitive equilibrium has to be a solution of a social planner problem with the utility function that assigns constant weights to the utility function of individual agent.

But here, we are looking at general DSGE models, so there are cases when firms are not price-taking and et cetera. So can we generalize Negishi (1960)? The answer is yes.

The proof in Negishi (1960) only requires that “feasible” utility vectors form a convex set and equilibria to sit on the boundary of the set.

By a utility vector, it means \( U = (U_1, U_2, \ldots) \), where subscript indices refer to consumers/agents.

In original Negishi (1960), “feasible” is defined as resource-wise feasible. But this does not need to be the case. If we can redefine the word “feasible” - such as considering how firms behave to set price that would prevent additional possible allocations - we may allow feasible utility vectors to form a convex set, while equilibria sit on the boundary of the set.
In such a case, let $U_s$ be:

(B1) \[ U_s = \gamma U_1 + (1 - \gamma)U_2 + \sum_{j=3}^{n} \nu_j U_3 \]

where $\gamma$ and $\nu_j$ are constants, and $n$ refers to the number of consumers. Let some of sequential constraints for the obtained social planner problem at each time $t$ be of form:

(B2) \[ C_{1,t} + C_{2,t} + \ldots \leq \ldots \]

where ... terms do not contain any of $C_{1,t}$ and $C_{2,t}$, and rest of constraints do not contain any of $C_{1,t}$ and $C_{2,t}$.

Then we get Equation (3) as the result of first-order conditions. First-order conditions say:

\[ \gamma U_1'(C_{1,t}) - \lambda_t = 0 \]
\[ (1 - \gamma)U_2'(C_{2,t}) - \lambda_t = 0 \]

where $\lambda_t$ is the sum of Lagrange multipliers attached to constraints that share the form of Equation (B2). We can rewrite the above equations as:

\[ \gamma(\beta_1) u_1'(C_{1,t}) = \lambda_t \]
\[ (1 - \gamma)(\beta_2) u_2'(C_{2,t}) = \lambda_t \]

Thus:

\[ \gamma(\beta_1) u_1'(C_{1,t}) = (1 - \gamma)(\beta_2) u_2'(C_{2,t}) \]

which gives us Equation (3).
While the required form of constraints to derive GIR seem stringent, it actually is not. One way to see this is as follows. From an initially competitive competition economy, we may change firms to be monopolistically competitive. But in such cases, consumer profiles do not change. Unless firms engage in price discrimination against consumers, derived constraints will follow the form of Equation \[B2\] or will not have \(C_{1,t}\) and \(C_{2,t}\) terms.

**Equivalence of a sequential consumer optimization problem with a recursive dynamic programming problem**

We now “replicate” equivalence of a sequential optimization problem with a recursive dynamic programming problem by simply quoting [Recursive Methods in Economic Dynamics] by Nancy Stokey, Robert Lucas with Edward Prescott - the book is often more commonly referred to as SLP after authors, and is considered a canonical macroeconomic textbook. Specifically, we refer to Chapter 4 “Dynamic Programming under Certainty” ([Stokey, Lucas and Prescott, 1989, Chapter 4]).

A “sequence problem” (SP) in SLP corresponds to a consumer optimization problem that leads to a sequential equilibrium in a DSGE model. It is given by:

\[
\begin{align*}
\text{(SP)} & \quad \sup_{\{x_{t+1}\}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \\
\text{(C1)} & \quad \text{s.t. } x_{t+1} \in \Gamma(x_t) \\
& \quad x_0 \in X \text{ given}
\end{align*}
\]

We map \(F\) with \(U\) in a consumer optimization problem, \(\Gamma\) absorbs all given
price variables and budget constraints, with state vector \( x_t \) re-defined to keep \( \Gamma \) time-invariant. SLP shows that under very weak assumptions which we replicate soon, SP is equivalent to a functional equation (FE):

\[
(C2) \quad (FE) \quad v(x) = \sup_{y \in \Gamma(x)} \left[ F(x, y) + \beta v(y) \right]
\]

In such a case, because FE clearly is time-consistent, a consumer optimization problem itself is time-consistent. We now provide assumptions used.

\[
\Xi(x_0) = \{ \{x_t\}_{t=0}^{\infty} | x_{t+1} \in \Gamma(x_t) \}
\]

\( \Xi \) is the set of plans that are feasible from \( x_0 \). Let us denote an element \( \Xi \) as \( \bar{x} \).

Assumption 4.1 in SLP states that \( \Gamma(x) \) is nonempty for all \( x \in X \).

Assumption 4.2 in SLP states that for all \( x_0 \in X \) and \( \bar{x} \in \Xi(x_0) \)

\[
\lim_{n \to \infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})
\]

exists. Both assumptions are standard macroeconomic assumptions, so these are very weak assumptions.

Let \( u_n \), as distinguished from \( u_j \), defined as:

\[
u_n(\bar{x}) = \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})
\]

\[
u(\bar{x}) = \lim_{n \to \infty} u_n(\bar{x})
\]
Then define the supremum function $v^*$:

$$v^*(x_0) = \sup_{\bar{x} \in \Xi(x_0)} u(\bar{x})$$

Theorem 4.4 in SLP establishes that if Assumption 4.1 and 4.2 are satisfied, and a feasible plan $x^* \in \Xi(x_0)$ attains the supremum in (SP) - see Equation C1 - for initial state $x_0$, then (SP) reduces to (FE) in a way that:

(C3) \[ v^*(x_t^*) = F(x_t^*, X_{t+1}^*) + \beta v^*(x_{t+1}^*) \]

Furthermore, Theorem 4.5 in SLP establishes that if Assumption 4.1 and 4.2 are satisfied, and feasible plan $\bar{x}^* \in \Xi(x_0)$ satisfies Equation C3 and

(C4) \[ \lim_{t \to \infty} \sup_{t} \beta^t v^*(x_t^*) \leq 0 \]

then $\bar{x}^*$ attains the supremum in (SP) for initial state $x_0$. Equation C4 is a common assumption in macroeconomics, so is a very weak one.

Theorem 4.4 and 4.5 complete a demonstration of equivalence of (SP) and (FE) under very weak assumptions. Thus, for our purpose, we can think of each consumer optimization problem as granting time consistency of its solution.

**Derivation of the usual transversality condition in the canonical New Keynesian model**

The budget constraint in the canonical New Keynesian model (see Galí (2015)) was:

$$P_tC_t + \frac{B_t}{1 + \delta_t} \leq W_tN_t + F_t + B_{t-1}$$
The utility function was $\sum_{t=0}^{\infty} \beta^t u(C_t, N_t)$. Let $u_t = u(C_t, N_t)$. For simplification, we assume a perfect-foresight economy with rational expectation. Set up the lagrangian:

$$\sum_{t=0}^{\infty} \beta^t u_t + \lambda_t \left[ W_t N_t + F_t + B_{t-1} - \left( P_t C_t + \frac{B_t}{1 + i_t} \right) \right] + \omega_t B_t + \xi_t C_t + \eta_t N_t$$

The usual assumptions are that $C_t > 0$ and $N_t > 0$. Thus, $\xi_t = 0$ and $\eta_t = 0$. By slackness conditions, it is required that:

$$\omega_t B_t = 0$$

The equilibrium condition associated with optimizing $C_t$ is:

$$\beta^t u'(C_t) - \lambda_t P_t = 0$$

thus

$$\lambda_t = \frac{\beta^t u'(C_t)}{P_t}$$

The equilibrium condition associated with optimizing $B_t$ (we safely assume $\omega_t = 0$ until later) is:

$$\lambda_{t+1} - \lambda_t (1 + i_t)^{-1} = 0$$

thus

$$\frac{\lambda_{t+1}}{\lambda_t} = \frac{1}{1 + i_t}$$

Now suppose that an economy ends at time $t = T$ instead of forever continuing. While $\omega_t = 0$ up until $t = T$, at $t = T$, circumstances change. When
optimizing for $B_T$, we get:

$$-\lambda_T (1 + i_T)^{-1} + \omega_T = 0$$

Thus $\omega_T$ can no longer be zero. Now we extend this result to the infinite-duration economy as:

$$\lim_{T \to \infty} \omega_T B_T = \lim_{T \to \infty} \lambda_T (1 + i_T)^{-1} B_T = 0$$

which we rewrite as:

$$\lim_{T \to \infty} \beta^T \frac{u'(C_T)}{P_T} \frac{B_T}{1 + i_T} = 0$$

which is Equation 17. We get Equation 18 using $\lambda_{t+1}/\lambda_t = (1 + i_t)^{-1}$.

$B_t = 0$ MEANS EQUILIBRIUM INDETERMINACY

Let us continue from the discussion above on the derivation of the usual transversality condition. There, we noticed that there actually is an additional Lagrange multiplier factor $\omega_t$ that enforces $B_t \geq 0$. When we safely assume $B_t > 0$, we can suppress $\omega_t = 0$. However, when $B_t = 0$, we must bring back $\omega_t$. This invalidates the consumption Euler equation, and leaves us with undetermined $\omega_t$ that creates equilibrium indeterminacy.