AVERAGE OF L-FUNCTIONS OF ARTIN-SCHREIER EXTENSIONS

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Abstract. Let $k = \mathbb{F}_q(t)$ be a rational function field over the finite field $\mathbb{F}_q$. In this paper we obtain formulas of average values of $L$-functions of some family of Artin-Schreier extensions over $k$.

1. Introduction

The average of a family of $L$-functions has been studied by many authors. This problem was initiated by Gauss who made two famous conjectures on average values of class numbers of of orders in quadratic fields. These conjectures were proved by Lipschitz in imaginary quadratic fields case and by Siegel [7] in real quadratic fields case. By the Dirichlet’s class number formula, these conjectures can be stated as an average of $L$-functions at $s = 1$ associated to orders in quadratic fields. Takhtadzjan and Vinogradov [8] obtained an average formula for the $L$-functions of quadratic fields which holds for all $s \in \mathbb{C}$ with $\text{Re}(s) \geq 1$. They also gave an average formula for the $L$-functions of quadratic fields with prime discriminants [9]. Let $k = \mathbb{F}_q(t)$ be a rational function field over the finite field $\mathbb{F}_q$, where $q$ is a power of a prime number $p$, and $\mathcal{A} = \mathbb{F}_q[t]$ be the polynomial ring. The formulas of average values of the $L$-functions associated to orders in quadratic extensions of $k$ are obtained by Hoffstein and Rosen [3] when $q$ is odd and by Chen [2] when $q$ is even. Hofstein and Rosen [3] also gave average formulas for the $L$-functions associated to maximal orders in quadratic extensions of $k$. Prime [5] obtained an average of the $L$-functions associated to maximal orders in ramified imaginary quadratic extensions of $k$ with prime fundamental discriminants. Bae, Jung and Kang [1] obtained averages

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of the $L$-functions associated to maximal orders of Kummer extensions $K = k(\sqrt[\ell]{P})$ of $k$, where where $\ell$ is a prime divisor of $q - 1$ and $P$ runs over monic irreducible polynomials in $\mathfrak{A}$. Rosen [6] gave averages of the $L$-functions associated to orders of Kummer extensions of $k$ of degree $\ell$. Bae, Jung and Kang [1] obtained averages of the $L$-functions associated to orders of Artin-Schreier extensions of $k$. Let $K_u = k(\alpha_u)$ be the Artin-Schreier extension of $k$ generated by a root $\alpha_u$ of $x^p - x = u$, where $u = B/A \in k$ is normalized (see §2.1). Then $G(K_u) = A$ which is a monic polynomial in $\mathfrak{A}$ is uniquely determined by the field $K_u$. In [1], Bae, Jung and Kang gave an average of the $L$-functions associated to maximal orders of Artin-Schreier extensions $K_u$ of $k$ of degree 2 with monic irreducible $G(K_u)$. In this paper we study the average of the $L$-functions associated to maximal orders of Artin-Schreier extensions $K_u$ of $k$ of general degree $p$ with monic irreducible $G(K_u)$. In §2, we recall some basic facts on the Artin-Schreier extensions of $k$ and $L$-functions associated to maximal orders of Artin-Schreier extensions. We also give two key lemmas and their corollaries which play important roles in the computations of average of $L$-functions. The proofs of these lemmas are given in [1] for $p = 2$. In §3, we give averages of the $L$-functions associated to maximal orders of real/inert imaginary/ramified imaginary Artin-Schreier extensions $K_u$ of $k$ with monic irreducible $G(K_u)$. In ramified imaginary case, for a given monic irreducible polynomial $P \in \mathfrak{A}$, there are infinitely many ramified imaginary Artin-Schreier extensions $K_u$ of $k$ with $G(K_u) = P$, so we also need to fix the degree of numerators of $u$ in the computation of average of $L$-functions.

2. Preliminaries

2.1. Artin-Schreier extensions

Let $k = \mathbb{F}_q(t)$ and $\mathfrak{A} = \mathbb{F}_q[t]$, where $q$ is a power of a prime $p$. Let $\infty_k = (1/t)$ be the infinite prime of $k$. We denote by $\mathfrak{A}^+$ the set of monic polynomials in $\mathfrak{A}$ and by $\mathcal{P}(\mathfrak{A})$ the set of monic irreducible polynomials in $\mathfrak{A}$. Write $\mathfrak{A}_n = \{N \in \mathfrak{A} : \deg(N) = n\}$, $\mathfrak{A}^+_n = \mathfrak{A}^+ \cap \mathfrak{A}_n$ and $\mathcal{P}_n(\mathfrak{A}) = \mathcal{P}(\mathfrak{A}) \cap \mathfrak{A}_n \ (n \geq 0)$. For any $0 \neq N \in \mathfrak{A}$, let $|N| = \#(\mathfrak{A}/N\mathfrak{A}) = q^{\deg(N)}$, $\Phi(N) = \#(\mathfrak{A}/N\mathfrak{A})^\times$, where $\#X$ denotes the cardinality of a set $X$, and sgn($N$) denote the leading coefficient of $N$. Let $\wp(x) = x^p - x$ be the Artin-Schreier operator. For $u = B/A \in k$ with $A \in \mathfrak{A}^+$, $B \in \mathfrak{A}$ and $\gcd(A, B) = 1$, we say that $u$ is normalized if it satisfies the following conditions: (i) if $A = \prod_{i=1}^r P_i^{e_i}$, then $pe_i$ for each $1 \leq i \leq r$, (ii) if
deg(B) > deg(A), then \( p(\deg(B) - \deg(A)) \), and (iii) if \( \deg(B) = \deg(A), \) then \( \text{sgn}(B) \notin \mathcal{F}_q. \) Let \( K_u = k(\alpha_u) \) be the Artin-Schreier extension of \( k \) generated by a root \( \alpha_u \) of \( \mathcal{F}(\mathbb{F}_q) = u. \) Let \( \mathcal{O}_u \) be the integral closure of \( \mathcal{A} \) in \( K_u. \) If we write \( u = f(T) + B \) with \( f(T) \in \mathcal{A} \) and \( \deg(B) < \deg(A), \) then one can show that \( f(T) \) and \( A \) are uniquely determined by the field \( K_u. \) Also, if \( K \) is an Artin-Schreier extension of \( k, \) then there exists such a normalized \( u \in k \) such that \( K = K_u. \) Let \( G(K) = A \) be the denominator of \( u \) as above. The discriminant \( d_u \) of \( \mathcal{O}_u \) over \( \mathcal{A} \) is \( (A \cdot \text{rad}(A))^{p-1}, \) where \( \text{rad}(A) \) denotes the product of the distinct monic irreducible divisors of \( A \) (see [1, Corollary 2.7]). The local discriminant \( d_{\infty_k} \) at \( \infty_k \) is \( \infty_k^{p-1)(\deg(f(T))+1) \) if \( \deg(f(T)) > 0 \) and 1 otherwise. The discriminant \( d_{K_u} \) of \( K_u \) is defined to be \( d_u \cdot d_{\infty_k}. \) We say that the Artin-Schreier extension \( K/k \) is real, inert imaginary or ramified imaginary according as \( \infty_k \) splits completely, is inert or ramifies in \( K. \) Then, the extension \( K_u/k \) is real, inert imaginary or ramified imaginary according as \( \deg(B) < \deg(A), \) \( \deg(A) = \deg(B) \) or \( \deg(A) < \deg(B). \) (See [1, 4] for details.)

### 2.2. \( L \)-functions of Artin-Schreier extensions

Fix an isomorphism \( \psi : \mathbb{F}_p \to \mu_p \) sending 1 to a primitive \( p \)-th root \( \zeta_p \) of unity, where \( \mu_p \) is the group of \( p \)-th roots of unity in \( \mathbb{C}. \) For \( u \in k \) and \( P \in \mathcal{P}(\mathcal{A}) \) which is prime to the denominator of \( u, \) define \( [u, P] \in \mathbb{F}_p \) by \( (P, K_u/k)(\alpha_u) = \alpha_u+[u, P], \) where \( (P, K_u/k) \) is the Artin automorphism at \( P. \) Extend this to \( N \in \mathcal{A}^+, \) which is prime to the denominator of \( u, \) by multiplicativity. For any \( N \in \mathcal{A}^+, \) define \( \{ \frac{N}{N} \} \) to be \( \psi([u, N]) \) if \( N \) is prime to the denominator of \( u \) and 0 otherwise. The \( L \)-function \( L(s, \chi_u^i) \) associated to \( \chi_u^i(\cdot) = \{ \frac{N}{N} \} \) \( (0 \leq i \leq p-1) \) is defined by

\[
L(s, \chi_u^i) = \sum_{N \in \mathcal{A}^+} \chi_u^i(N)|N|^{-s}.
\]

We can write

\[
L(s, \chi_u^i) = \sum_{n=0}^{\infty} \sigma_n^{(i)}(u)q^{-ns} \quad \text{with} \quad \sigma_n^{(i)}(u) = \sum_{N \in \mathcal{A}^+} \chi_u^i(N).
\]

It is well known that \( L(s, \chi_u^i) \) is a polynomial in \( q^{-s} \) of degree \( \deg(\text{rad}(A)) + \deg(B) - 1 \) or \( \deg(A) + \deg(\text{rad}(A)) - 1 \) according as \( \infty_k \) ramifies in \( K_u \) or otherwise for \( 1 \leq i \leq p-1. \)
2.3. Two key lemmas

For $M, N \in \mathbb{A}^+$, two sums $T_{M,N}^{(i)}$ and $\Gamma_{M,N}^{(i)}$ are defined by

$T_{M,N}^{(i)} = \sum_{\substack{\deg(D) < \deg(M) \\gcd(D,M) = 1}} \left\{ \frac{D}{N} \right\}^i,$

$\Gamma_{M,N}^{(i)} = \sum_{\substack{\deg(D) < \deg(M) \\gcd(D,M) = 1}} \left\{ \frac{D}{N} \right\}^i$ for $1 \leq i \leq p - 1$. Note that $\Gamma_{M,N}^{(i)} = \sum_{\bar{M} \in \mathbb{A}^+, \bar{M} \mid M} T_{N,\bar{M}}^{(i)}$.

By definition, we have that $T_{M,N}^{(i)} = 0$ if $\gcd(N,M) \neq 1$ and $\Gamma_{M,N}^{(i)} = \Phi(M)$ if $\gcd(N,M) = 1$ and $N$ is a $p$-th power.

**Lemma 2.1.** Let $P \in \mathcal{P}_m(\mathbb{A})$ and $N \in \mathbb{A}_n^+$ with $PN$. If $N$ is not a $p$-th power, then $\Gamma_{N,P}^{(i)} = 0$ for $1 \leq i \leq p - 1$.

**Proof.** We first consider the case $n \leq m$. The set $\{ B/P : B \in \mathbb{A}, \deg(B) < m \}$ contains a complete residue system modulo $N$. So the map $B \mapsto \{ B/P \}$ is a surjective additive character from $\{ B \in \mathbb{A} : \deg(B) < m \}$ onto $\mu_p$. Hence $\Gamma_{N,P}^{(i)}$ is a multiple of $1 + \zeta_p + \cdots + \zeta_p^{p-1} = 0$, i.e., $\Gamma_{N,P}^{(i)} = 0$. Now assume $n > m$, say $n = m + h$ for some positive integer $h$. Since $\{ B/P : B \in \mathbb{A}, \deg(B) < n \}$ contains a complete residue system modulo $N$, we have

$$0 = \sum_{\deg(B) < n} \left\{ \frac{B/P}{N} \right\}^i = \Gamma_{N,P}^{(i)} + \sum_{l=0}^{h-1} \sum_{B \in \mathbb{A}_{m+l}} \left\{ \frac{B/P}{N} \right\}^i$$

as above. For any $B \in \mathbb{A}_{m+l}$, we can write $B = QP + R$ with $Q \in \mathbb{A}_l$ and $\deg(R) < m$. Then,

$$\sum_{B \in \mathbb{A}_{m+l}} \left\{ \frac{B/P}{N} \right\}^i = \left( \sum_{Q \in \mathbb{A}_l} \left\{ \frac{Q}{N} \right\}^i \right) \left( \sum_{\deg(R) < m} \left\{ \frac{R/P}{N} \right\}^i \right) = \Gamma_{N,P}^{(i)} \sum_{Q \in \mathbb{A}_l} \left\{ \frac{Q}{N} \right\}^i.$$  

Hence, we get

$$0 = \Gamma_{N,P}^{(i)} + \sum_{l=0}^{h-1} \sum_{Q \in \mathbb{A}_l} \left\{ \frac{Q}{N} \right\}^i = \Gamma_{N,P}^{(i)} \left( 1 + \sum_{\deg(Q) < h} \left\{ \frac{Q}{N} \right\}^i \right).$$
Assume that $\Gamma_{N,P}^{(i)} \neq 0$. Then $\sum_{\deg(Q) < h} \{ \frac{Q}{N} \}^i = -1$. If there exists $Q \in \mathbb{A}$ with $\deg(Q) < h$ such that $\{ \frac{Q}{N} \} \neq 1$, then $\sum_{\deg(Q) < h} \{ \frac{Q}{N} \}^i = 0$, which is a contradiction. But, if $\{ \frac{Q}{N} \} = 1$ for all $Q \in \mathbb{A}$ with $\deg(Q) < h$, then $\sum_{\deg(Q) < h} \{ \frac{Q}{N} \}^i = q^h$, which is also a contradiction. Therefore, we have $\Gamma_{N,P}^{(i)} = 0$.

**Corollary 2.2.** Let $P \in \mathcal{P}_m(\mathbb{A})$ and $N \in \mathbb{A}^+$. If $N$ is not a $p$-th power, then $T_{N,P}^{(i)} = -1$ for $1 \leq i \leq p - 1$.

**Proof.** By Lemma 2.1 and (2.2), we have $T_{N,P}^{(i)} = -1$. □

For $M, N \in \mathbb{A}^+$ and positive integer $c$, two sums $\tilde{T}_{N,M,c}^{(i)}$ and $\tilde{\Gamma}_{N,M,c}^{(i)}$ are defined by

$$
\tilde{T}_{N,M,c}^{(i)} = \sum_{\deg(B) = \deg(M) + c \mod \gcd(B, M) = 1} \left\{ \frac{B/M}{N} \right\}^i,
\tilde{\Gamma}_{N,M,c}^{(i)} = \sum_{\deg(B) = \deg(M) + c} \left\{ \frac{B/M}{N} \right\}^i
$$

for $1 \leq i \leq p - 1$. Note that

$$
\tilde{\Gamma}_{N,M,c}^{(i)} = \sum_{M \in \mathbb{A}^+, \tilde{M}|M} \tilde{T}_{N,\tilde{M},c}^{(i)}
$$

and by Möbius inversion formula,

$$
(2.2) \quad \tilde{T}_{N,M,c}^{(i)} = \sum_{\tilde{M} \in \mathbb{A}^+, \tilde{M}|M} \mu(\tilde{M}) \tilde{\Gamma}_{N,M/\tilde{M},c}^{(i)}.
$$

By definition, we have that $\tilde{T}_{N,M,c}^{(i)} = 0$ if $\gcd(N, M) \neq 1$ and $\tilde{T}_{N,M,c}^{(i)} = (q - 1)q^h \Phi(M)$ if $\gcd(N, M) = 1$ and $N$ is $p$-th power.

**Lemma 2.3.** Let $P \in \mathcal{P}_m(\mathbb{A})$ and $N \in \mathbb{A}^+$. If $N$ is not a $p$-th power, then $\tilde{\Gamma}_{N,P,c}^{(i)} = 0$ for $1 \leq i \leq p - 1$.

**Proof.** Since $\Gamma_{N,P}^{(i)} = 0$, there exists $B_0 \in \mathbb{A}$ with $\deg(B_0) < m$ such that $\{ \frac{B_0/P}{N} \} \neq 1$, say $\{ \frac{B_0/P}{N} \} = \zeta_j^{j_0}$ for some $1 \leq j_0 \leq p - 1$. Let $X_0$ be the set of $B \in \mathbb{A}_{m+c}$ such that $\{ \frac{B/P}{N} \}^i = \zeta_a^a (0 \leq a \leq p - 1)$. Let $j_a$ be an integer such that $(i j_0)j_a \equiv a \mod p$. Then the map $B \mapsto B + j_a B_0$ is a bijection from $X_0$ onto $X_a$. Hence, we have

$$
\tilde{\Gamma}_{N,P,c}^{(i)} = \sum_{a=0}^{p-1} \sum_{B \in X_a} \left\{ \frac{B/P}{N} \right\}^i = |X_0|(1 + \zeta_p + \cdots + \zeta_p^{p-1}) = 0,
$$
which completes the proof. \hfill \square

**Corollary 2.4.** Let $P \in \mathcal{P}_m(\mathbb{A})$ and $N \in \mathbb{A}_n^+$ with $PN$. If $N$ is not a $p$-th power, then

$$T^{(i)}_{N,P,c} = - \sum_{B \in \mathbb{A}_c} \{B/N\}^i$$

for $1 \leq i \leq p - 1$.

**Proof.** It follows from Lemma 2.3 and (2.2). \hfill \square

3. Average of $L$-functions of Artin-Schreier extensions

Let $\zeta_A(s) = \sum_{N \in \mathbb{A}^+} |N|^{-s}$ be the zeta function of $A$. It is easy to see that $\zeta_A(s) = \frac{1}{1-q^{-s}}$. In this section we study the averages of $L$-functions associated to maximal orders of real/inert imaginary/ramified imaginary Artin-Schreier extensions $K_u$ of $k$ with monic irreducible $G(K_u)$, respectively.

3.1. Real case

For $P \in \mathcal{P}(\mathbb{A})$, let $\mathfrak{F}_P = \{B \in \mathbb{A} : B \neq 0, \deg(B) < \deg(P)\}$ and $\mathfrak{F}_P$ be the set of real Artin-Schreier extensions $K$ of $k$ with $G(K) = P$. It is easy to show that for any $B_1, B_2 \in \mathfrak{F}_P$, $K_{B_1/P} = K_{B_2/P}$ if and only if $B_1 = B_2$. Hence, the map $B \mapsto K_{B/P}$ is a bijection from $\mathfrak{F}_P$ onto $\mathfrak{F}_P$.

**Theorem 3.1.** For $s \in \mathbb{C}$ with $\text{Re}(s) > \frac{1}{2}$ and $1 \leq i \leq p - 1$, we have

$$\lim_{m \to \infty} \frac{\sum_{P \in \mathcal{P}_m(\mathbb{A})} \sum_{B \in \mathfrak{F}_P} L(s, \chi_{B/P}^i)}{(q^m - 1) \# \mathcal{P}_m(\mathbb{A})} = \zeta_A(ps).$$

**Proof.** Let

$$f_m(s) = \frac{\sum_{P \in \mathcal{P}_m(\mathbb{A})} \sum_{B \in \mathfrak{F}_P} L(s, \chi_{B/P}^i)}{(q^m - 1) \# \mathcal{P}_m(\mathbb{A})} - \zeta_A(ps).$$

Since $L(s, \chi_{B/P}^i)$ is a polynomial in $q^{-s}$ of degree $2m - 1$ for $P \in \mathcal{P}_m(\mathbb{A})$ and $B \in \mathfrak{F}_P$, we have

$$f_m(s) = \sum_{n=0}^{2m-1} \sum_{P \in \mathcal{P}_m(\mathbb{A})} \sum_{B \in \mathfrak{F}_P} \sigma_n^{(i)}(B/P) q^{-ns} \frac{1}{(q^m - 1) \# \mathcal{P}_m(\mathbb{A})} - \sum_{n=0}^{\infty} q^{(1-ps)n}.

Put

$$f_{m,n} = \sum_{P \in \mathcal{P}_m(\mathbb{A})} \sum_{B \in \mathfrak{F}_P} \sigma_n^{(i)}(B/P).$$
Then, we have

\[ f_{m,n} = \sum_{P \in \mathcal{P}_m(\mathbb{A}_q^+)} \sum_{N \in \mathcal{A}_p^+} \sum_{B \in \mathfrak{d}_P} \left\{ \frac{B/P}{N} \right\}^i = \sum_{P \in \mathcal{P}_m(\mathbb{A}_q^+)} \sum_{N \in \mathcal{A}_p^+} T_{N,P}^{(i)}. \]

By definition, we have that \( T_{N,P}^{(i)} = 0 \) if \( P \mid N \) and \( T_{N,P}^{(i)} = q^m - 1 \) if \( PN \) and \( N \) is a \( p \)-th power. If \( PN \) and \( N \) is not a \( p \)-th power, by Corollary 2.2, we have \( T_{N,P}^{(i)} = -1 \). For \( pn \), since any \( N \in \mathcal{A}_p^+ \) will never be a \( p \)-th power, we have

(3.2) \[ f_{m,n} = \begin{cases} -q^n \#\mathcal{P}_m(\mathbb{A}_q^+) & \text{if } n < m, \\ -(q^n - q^{n-m}) \#\mathcal{P}_m(\mathbb{A}_q^+) & \text{if } m \leq n \leq 2m - 1. \end{cases} \]

For \( p \mid n \), we have

(3.3) \[ f_{m,n} = (q^m - 1) \sum_{P \in \mathcal{P}_m(\mathbb{A}_q^+)} \sum_{N \in \mathcal{A}_p^+, PN} 1 - \sum_{P \in \mathcal{P}_m(\mathbb{A}_q^+)} \sum_{N \in \mathcal{A}_p^+, PN} 1. \]

For \( P \in \mathcal{P}_m(\mathbb{A}_q^+) \) and \( N \in \mathcal{A}_p^+ \), since \( n \leq 2m - 1 \), if \( N \) is a \( p \)-th power, then \( N \) is not divisible by \( P \). Hence we have

(3.4) \[ \sum_{N \in \mathcal{A}_p^+, PN, N: p\text{-th power}} 1 = \sum_{N \in \mathcal{A}_p^+, N: p\text{-th power}} 1 = q^\frac{n}{p}, \]

and

(3.5) \[ \sum_{N \in \mathcal{A}_p^+, PN, N: not p\text{-th power}} 1 = \begin{cases} q^n - q^\frac{n}{p} & \text{if } n < m, \\ q^n - q^\frac{n}{p} - q^{n-m} & \text{if } m \leq n \leq 2m - 1. \end{cases} \]

For \( p \mid n \), by inserting (3.4) and (3.5) into (3.3), we have

(3.6) \[ f_{m,n} = q^\frac{n}{p}(q^m - 1) \#\mathcal{P}_m(\mathbb{A}_q^+) - \begin{cases} (q^n - q^\frac{n}{p}) \#\mathcal{P}_m(\mathbb{A}_q^+) & \text{if } n < m, \\ (q^n - q^\frac{n}{p} - q^{n-m}) \#\mathcal{P}_m(\mathbb{A}_q^+) & \text{if } m \leq n \leq 2m - 1. \end{cases} \]

By inserting (3.2) and (3.6) into (3.1) and rearranging the terms, we have

\[ f_m(s) = -\sum_{n=\left\lceil \frac{2m-1}{p} \right\rceil + 1}^{\infty} q^{(1-ps)n} = -\frac{1}{q^m - 1} \left( \sum_{n=0}^{2m-1} q^{(1-s)n} - \sum_{n=0}^{\left\lfloor \frac{2m-1}{p} \right\rfloor} q^{(1-ps)n} - q^{-m} \sum_{n=m}^{2m-1} q^{(1-s)n} \right). \]
For $s \in \mathbb{C}$ with $\sigma = \text{Re}(s) > \frac{1}{p}$, since $1 - p\sigma < 0$, we have
\[
\left| \sum_{n=\left\lfloor \frac{2m-1}{p} \right\rfloor + 1}^{\infty} q^{(1-ps)n} \right| \leq \sum_{n=\left\lfloor \frac{2m-1}{p} \right\rfloor + 1}^{\infty} q^{(1-ps)n} = \frac{q^{(1-ps)(\left\lfloor \frac{2m-1}{p} \right\rfloor + 1)}}{1-q^{1-ps}} \to 0
\]
and
\[
\left| \sum_{n=0}^{\frac{2m-1}{p}} q^{(1-ps)n} \right| \leq \sum_{n=0}^{\frac{2m-1}{p}} q^{(1-ps)n} < \frac{\left\lfloor \frac{2m-1}{p} \right\rfloor + 1}{q^{m-1}} \to 0
\]
as $m \to \infty$. Now, for $s \in \mathbb{C}$ with $\sigma = \text{Re}(s) > \frac{1}{2}$, we have
\[
\left| \sum_{n=0}^{\frac{2m-1}{p}} q^{(1-s)n} \right| \leq \sum_{n=0}^{\frac{2m-1}{p}} q^{(1-s)n}. \quad \text{If } \sigma \neq 1,
\]
and if $\sigma = 1$,
\[
\left| \sum_{n=m}^{2m-1} q^{(1-s)n} \right| \leq \sum_{n=m}^{2m-1} q^{(1-s)n} = \frac{2m}{q^{m-1}} \to 0
\]
as $m \to \infty$. Finally, for $s \in \mathbb{C}$ with $\sigma = \text{Re}(s) > \frac{1}{2}$, we have
\[
\frac{\sum_{n=m}^{2m-1} q^{(1-s)n}}{q^{m}(q^{m-1} - 1)} \leq \frac{\sum_{n=0}^{\frac{2m-1}{p}} q^{(1-s)n}}{q^{m}(q^{m-1} - 1)} \leq \frac{\sum_{n=0}^{\frac{2m-1}{p}} q^{(1-s)n}}{q^{m-1}} \to 0
\]
as $m \to \infty$. Therefore, for $s \in \mathbb{C}$ with $\text{Re}(s) > \frac{1}{2}$, we get $f_m(s) \to 0$ as $m \to \infty$, which completes the proof.

### 3.2. Inert imaginary case

Let $\{0, \xi_1, \ldots, \xi_{p-1}\}$ be a set of representatives of $\mathbb{F}_q/\varphi(\mathbb{F}_q)$. For $P \in \mathcal{P}(\mathbb{A})$, let $\mathfrak{S}_P = \{\xi_aP + B : B \in \mathfrak{F}_P, 1 \leq a \leq p - 1\}$ and $\mathcal{G}_P$ be the set of inert imaginary Artin-Schreier extensions $K$ of $k$ with $G(K) = P$.

It is easy to show that, for any $B_1, B_2 \in \mathfrak{F}_P$ and $1 \leq a, b \leq p - 1$, $K(\xi_aP + B_1)/P = K(\xi_bP + B_2)/P$ if and only if $a = b$ and $B_1 = B_2$. Thus, the map $\xi_aP + B \mapsto K(\xi_aP + B)/P$ is a bijection from $\mathfrak{S}_P$ onto $\mathcal{G}_P$.

**Theorem 3.2.** For $s \in \mathbb{C}$ with $\text{Re}(s) > \frac{1}{2}$ and $1 \leq i \leq p - 1$, we have
\[
\lim_{m \to \infty} \frac{\sum_{B \in \mathfrak{S}_P} \sum_{a=1}^{p-1} L(s, \chi_{\xi_a+B/P}^i)}{(p-1)(q^m-1)\#P_m(\mathbb{A})} = \zeta_A(ps).
\]

**Proof.** Let
\[
g_m(s) = \frac{\sum_{B \in \mathfrak{S}_P} \sum_{a=1}^{p-1} L(s, \chi_{\xi_a+B/P}^i)}{(p-1)(q^m-1)\#P_m(\mathbb{A})} \sum_{P \in P_m(\mathbb{A})} -\zeta_A(ps).
\]
Since \( L(s, \chi_{\xi}^i + B/P) \) is a polynomial in \( q^{-s} \) of degree \( 2m - 1 \) for any \( P \in \mathcal{P}_m(\mathbb{A}) \) and \( \xi aP + B \in \mathfrak{S}_P \), we have
\[
g_m(s) = \frac{\sum_{n=0}^{2m-1} \sum_{P \in \mathcal{P}_m(\mathbb{A})} \sum_{B \in \mathfrak{S}_P} \sum_{a=1}^{p-1} \sigma_n^{(i)}(\xi a + B/P)q^{-ns}}{(p-1)(q^m-1)\#\mathcal{P}_m(\mathbb{A})} - \sum_{n=0}^{\infty} q^{(1-ps)n}.
\]

Put
\[
g_{m,n} = \sum_{P \in \mathcal{P}_m(\mathbb{A})} \sum_{B \in \mathfrak{S}_P} \sum_{a=1}^{p-1} \sigma_n^{(i)}(\xi a + B/P).
\]

Then we have
\[
g_{m,n} = \sum_{P \in \mathcal{P}_m(\mathbb{A})} \sum_{a=1}^{p-1} \sum_{N \in \mathbb{A}^+} \sum_{\xi} \left\{ \frac{(\xi a + B/P)}{N} \right\}^i
\]
\[
= \sum_{P \in \mathcal{P}_m(\mathbb{A})} \sum_{\xi} \sum_{\xi} \sum_{\xi} \left\{ \frac{N}{N} \right\}^i T_{N,P}^{(i)}.
\]

Since \( \varphi(F_q) \) is contained in the kernel of \( Tr_{F_q/F_p} : F_q \to F_p \), we have \( \{Tr_{F_q/F_p}(\xi) : 1 \leq a \leq p-1\} = F_p \{0\} \). Note that \( \{\frac{\xi}{N}\} = \psi(Tr_{F_q/F_p}(\xi)) \) for \( N \in \mathbb{A}^+ \). If \( pn \), then \( \sum_{a=1}^{p-1} \left( \frac{\xi a}{N} \right)^i = \xi_p + \cdots + \xi_p^{p-1} = -1 \), so \( g_{m,n} = -f_{m,n} \). If \( p | n \), then \( \{\frac{\xi}{N}\} = 1 \), so \( g_{m,n} = (p-1)f_{m,n} \). For the rest of the proof, we can now follow a similar procedure in the proof of Theorem 3.1 to show \( g_m(s) \to 0 \) as \( m \to \infty \).

3.3. Ramified imaginary case

For \( P \in \mathcal{P}(\mathbb{A}) \) and positive integer \( c \) with \( pc \), let \( \mathfrak{S}_{P,c} = \{B \in \mathbb{A} : PB, \deg(B) = \deg(P) + c\} \) and \( \mathcal{H}_{P,c} \) be the set of ramified Artin-Schreier extensions \( K \) of \( K \) with \( G(K) = P \) and whose discriminant \( d_K \sim P^{2(p-1)\infty(c)} \). It is easy to show that, for any \( B, B' \in \mathfrak{S}_{P,c} \), we have that \( K_{B/P} = K_{B'/P} \) if and only if \( B' = B + P(D^p - D) \) for some \( D \in \mathbb{A} \). We say that \( B, B' \in \mathfrak{S}_{P,c} \) are equivalent, denoted by \( B \sim B' \), if \( B' = B + P(D^p - D) \) for some \( D \in \mathbb{A} \). Let \( [B] \) be the equivalence class of \( B \in \mathfrak{S}_{P,c} \) with respect to \( \sim \), and \( \mathfrak{S}_{P,c} = \{[B] : B \in \mathfrak{S}_{P,c} \} \). Then, the map \( [B] \mapsto K_{B/P} \) is a bijection from \( \mathfrak{S}_{P,c} \) onto \( \mathcal{H}_{P,c} \). For \( B \in \mathfrak{S}_{P,c} \), we have a surjective map
\[
\{D \in \mathbb{A} : \deg(D) \leq [c/p]\} \to [B], \ D \mapsto B + P(D^p - D).
\]

For \( D, E \in \mathbb{A} \) with \( \deg(D), \deg(E) \leq [c/p] \), we have that \( B + P(D^p - D) = B + P(E^p - E) \) if and only if \( D - E \in \mathbb{F}_p \). Hence, the map in (3.7)
is $p$ to $1$, so we have $\#[B] = \frac{q^{c/p}}{p}$. Since $\#\mathcal{H}_{P,c} = \#\mathcal{H}_{\deg(P)+c} - \#\mathcal{H}_c = q^c(q-1)(q^{\deg(P)}-1)$, we have

$$\#\tilde{\mathcal{H}}_{P,c} = pq^{c-\frac{c}{p}}(q-1)(q^{\deg(P)}-1).$$

**Theorem 3.3.** For $s \in \mathbb{C}$ with $\Re(s) > \frac{1}{2}$, positive integer $c$ with $pc$ and $1 \leq i \leq p-1$, we have

$$\lim_{m \to \infty} \frac{\sum_{P \in \mathcal{P}_m(A)} \sum_{|B| \in \tilde{\mathcal{H}}_{P,c}} L(s, \chi^i_B/B)}{I_q(m,c)} = \zeta_A(p^s),$$

where $I_q(m,c) = pq^{c-\frac{c}{p}}(q-1)(q^m-1)\#\mathcal{P}_m(A)$.

**Proof.** Let

$$h_{m,c}(s) = \frac{\sum_{P \in \mathcal{P}_m(A)} \sum_{|B| \in \tilde{\mathcal{H}}_{P,c}} L(s, \chi^i_B/B)}{I_q(m,c)} - \zeta_A(p^s).$$

Since $L(s, \chi^i_B/B)$ is a polynomial in $q^s$ of degree $2m + c - 1$ for $P \in \mathcal{P}_m(A)$ and $B \in \tilde{\mathcal{H}}_{P,c}$, we have

$$h_{m,c}(s) = \sum_{n=0}^{2m+c-1} \sum_{P \in \mathcal{P}_m(A)} \sum_{|B| \in \tilde{\mathcal{H}}_{P,c}} \sigma_n^{(i)}(B/P) q^{-ns}.$$

Put

$$h_{m,n,c} = \sum_{P \in \mathcal{P}_m(A)} \sum_{|B| \in \tilde{\mathcal{H}}_{P,c}} \sigma_n^{(i)}(B/P).$$

Then, we have

$$h_{m,n,c} = \frac{p}{q^{c/p}} \sum_{P \in \mathcal{P}_m(A)} \sum_{N \in \mathbb{A}_n^{+}} \sum_{B \in \mathcal{H}_{P,c}} \{ B/P_N \}^i = \frac{p}{q^{c/p}} \sum_{P \in \mathcal{P}_m(A)} \sum_{N \in \mathbb{A}_n^{+}} \tilde{T}_{N,P,c}^{(i)}.$$

If $P|N$, then $\tilde{T}_{N,P,c}^{(i)} = 0$ by definition. If $PN$ and $N$ is a $p$-th power, then $\tilde{T}_{N,P,c}^{(i)} = q^c(q-1)(q^m-1)$. If $PN$ and $N$ is not a $p$-th power, by Corollary 2.4, we have $\tilde{T}_{N,P,c}^{(i)} = -\sum_{B \in \mathcal{H}_c} \{ B/P_N \}^i$. Put $\alpha_{N,c} = \sum_{B \in \mathcal{H}_c} \{ B/P_N \}^i$. If $pn$, since any $N \in \mathbb{A}_n^{+}$ will never be a $p$-th power, we have

$$h_{m,n,c} = -\frac{p}{q^{c/p}} \sum_{P \in \mathcal{P}_m(A)} \sum_{N \in \mathbb{A}_n^{+}} \alpha_{N,c}.$$
If \( p \mid n \), we have
\[
\begin{align*}
\theta_{m,n,c} &= -\frac{p}{q^{[c/p]}} \sum_{P \in \mathcal{P}_m(k)} \sum_{N \in \mathcal{A}^+_n, PN \atop N: \text{not } p\text{-th power}} \alpha_{N,c} \\
&\quad + pq^{-[c/p]}(q - 1)(q^m - 1) \sum_{P \in \mathcal{P}_m(k)} \sum_{N \in \mathcal{A}^+_n, PN \atop N: \text{not } p\text{-th power}} 1.
\end{align*}
\]

Since
\[
\sum_{N \in \mathcal{A}^+_n, PN \atop N: \text{not } p\text{-th power}} 1 = \sum_{N_1 \in \mathcal{A}^+_n / p, P N_1} 1 = \begin{cases} q^{n/p} & \text{if } \frac{n}{p} < m, \\ q^{n/p}(1 - q^{-m}) & \text{if } \frac{n}{p} \geq m, \end{cases}
\]

we have (3.10)
\[
\theta_{m,n,c} = -\frac{p}{q^{[c/p]}} \sum_{P \in \mathcal{P}_m(k)} \sum_{N \in \mathcal{A}^+_n, PN \atop N: \text{not } p\text{-th power}} \alpha_{N,c} + \begin{cases} q^{n/p} \tilde{I}_q(m, c) & \text{if } \frac{n}{p} < m, \\ q^{n/p}(1 - q^{-m}) \tilde{I}_q(m, c) & \text{if } \frac{n}{p} \geq m. \end{cases}
\]

By inserting (3.9) and (3.10) into (3.8) and rearranging the terms, we have
\[
\theta_{m,n,c}(s) = -q^{-m} \sum_{n=m}^{\frac{2m+c-1}{p}} q^{n(1-ps)} - \sum_{n=\frac{2m+c-1}{p}+1}^{\infty} q^{n(1-ps)} \\
- \frac{1}{\tilde{I}_q(m, c)} \sum_{n=0}^{2m+c-1} p \frac{q^{[c/p]}}{q^{[c/p]}} \sum_{N \in \mathcal{A}^+_n, PN \atop N: \text{not } p\text{-th power}} \alpha_{N,c} q^{-ns}.
\]

For \( s \in \mathbb{C} \) with \( \sigma = \text{Re}(s) > \frac{1}{p} \), as \( m \to \infty \), we have
\[
\left| q^{-m} \sum_{n=m}^{\frac{2m+c-1}{p}} q^{n(1-ps)} \right| \leq q^{-m} \sum_{n=m}^{\frac{2m+c-1}{p}} q^{n(1-ps)} \\
\leq q^{-mp\sigma} \left( \frac{2m + c - 1}{p} - m + 1 \right) \to 0
\]
and
$$\left| \sum_{n=\left[\frac{2m+c-1}{p}\right]+1}^{\infty} q^{n(1-ps)} \right| \leq \sum_{n=\left[\frac{2m+c-1}{p}\right]+1}^{\infty} q^{n(1-ps)} = q^{(1-ps)\left(\frac{2m+c-1}{p}\right)+1} \frac{1}{1-q^{(1-ps)}} \to 0.$$  

Note that

$$\sum_{N \in \mathbb{A}^+_n, P|N} 1 = \#\mathbb{A}^+_n - \sum_{N \in \mathbb{A}^+_n, P|N} 1 - \sum_{N \in \mathbb{A}^+_n, P|N} 1,$$

where

$$\sum_{N \in \mathbb{A}^+_n, P|N} 1 = \begin{cases} 0 & \text{if } n < m, \\
q^{n-m} & \text{if } m \leq n \leq 2m + c - 1. \end{cases}$$

Since $c$ will be fixed and we will take $m \to \infty$, without loss of generality, we may assume $m > c$, so that $n \leq 2m + c - 1 < 3m - 1$. Since the proof of theorem for $p = 2$ is already given in [1], we will assume that $p$ is odd, so that $n < pm$. If $N$ is a $p$-th power and $P|N$, then $P^p|N$, so $pm \leq n$. Then, we have

$$\sum_{N \in \mathbb{A}^+_n, P|N} 1 = \sum_{N \in \mathbb{A}^+_n, P|N} 1 = \begin{cases} 0 & \text{if } pn, \\
q^\frac{n}{p} & \text{if } p|n. \end{cases}$$

Hence, we have

$$\text{(3.11)} \quad \sum_{N \in \mathbb{A}^+_n, P|N} 1 = \begin{cases} q^n, & \text{if } n < m, pn, \\
q^n - q^\frac{n}{p}, & \text{if } n < m, p|n, \\
q^n - q^{n-m}, & \text{if } n \geq m, pn, \\
q^n - q^{n-m} - q^\frac{n}{p}, & \text{if } n \geq m, p|n. \end{cases}$$

For $\sigma = \text{Re}(s) > \frac{1}{2}$, by using the fact that $|\alpha_{N,c}| \leq \#\mathbb{A}_c = (q-1)q^c$ and (3.11), we have

$$\left| \frac{1}{I_q(m,c)} \sum_{n=0}^{2m+c-1} \frac{p}{q^{[c/p]}} \sum_{P \in P_m(\mathbb{A})} \sum_{N \in \mathbb{A}^+_n, P|N} \alpha_{N,c} q^{-ns} \right|$$

$$\leq \frac{1}{I_q(m,c)} \sum_{n=0}^{2m+c-1} \frac{p}{q^{[c/p]}} \sum_{P \in P_m(\mathbb{A})} \sum_{N \in \mathbb{A}^+_n, P|N} |\alpha_{N,c}| q^{-ns}$$
Average of $L$-functions of Artin-Schreier extensions \[ \leq \frac{1}{q^m - 1} \left( \sum_{n=0}^{2m+c-1} q^n(1-\sigma) - \sum_{n=0}^{m+c-1} q^{n\left(\frac{1}{p^m}\right)-\sigma} - q^{-m} \sum_{n=m}^{2m+c-1} q^n(1-\sigma) \right) \]

\[ < \frac{1}{q^m - 1} \sum_{n=0}^{2m+c-1} q^n(1-\sigma). \]

If $\sigma = 1$, we have

\[ \sum_{n=0}^{2m+c-1} q^n(1-\sigma) \] \[ q^m - 1 = \frac{2m + c}{q^m - 1} \to 0 \]

and if $\sigma \neq 1$, we have

\[ \sum_{n=0}^{2m+c-1} q^n(1-\sigma) \] \[ q^m - 1 = \frac{1}{1 - q^{-m}} \cdot \frac{q^{-m} - q^{m(1-2\sigma)+c(1-\sigma)}}{1 - q(1-\sigma)} \to 0 \]

as $m \to \infty$. This completes the proof. \[ \square \]

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