Abstract. In this article, we describe how to compute slopes of $p$-adic $L$-invariants of arbitrary weight and level by means of the Greenberg-Stevens formula. Our method is based on work of Lauder and Vonk on computing the reverse characteristic series of the $U_p$-operator on overconvergent modular forms. Using higher derivatives of this characteristic series, we construct a polynomial whose zeros are precisely the $L$-invariants appearing in the corresponding space of modular forms with fixed sign of the Atkin-Lehner involution at $p$. In addition, we describe how to compute this polynomial efficiently. In the final section, we give computational evidence for relations between slopes of $L$-invariants for small primes.

1. Introduction

Let $f$ be a newform of even weight $k \geq 2$ for $\Gamma_0(pN)$, where $p$ is prime and $(N, p) = 1$. Let $\chi$ be a Dirichlet character of conductor prime to $pN$ with $\chi(p) = 1$. By the work of Mazur and Swinnerton-Dyer, see for example [MTT86], there exists a $p$-adic $L$-function $L_p(f, \chi, s)$ attached to $f$ that interpolates the algebraic parts $L_{alg}(f, \chi, j)$ for $j \in \{1, \ldots, k-1\}$ of the special values of the classical $L$-function attached to $f$. The $p$-adic $L$-invariant $L_p(f) \in \mathbb{C}_p$ attached to $f$ satisfies

$$L_p(f, \chi, \frac{k}{2}) = L_p(f) \cdot L_{alg}(f, \chi, \frac{k}{2})$$

and it depends only on the local Galois representation attached to $f$. Mazur, Tate and Teitelbaum conjectured that such invariant should exist. In the sequel, several possible candidates have been proposed. The fundamental breakthrough in relating these candidates to one another and proving the above formula, due to Greenberg and Stevens [GS93], is the relation between $L_p(f)$ and the (essentially unique) $p$-adic family of eigenforms passing through $f$. If we denote this family by

$$f_\kappa = \sum_{n=1}^\infty a_n(\kappa)q^n \quad \text{with} \quad f_k = f,$$

where the coefficients $a_n(\kappa)$ are rigid analytic functions on a disc containing $k$ in the weight space $\text{Hom}_{\text{cont}}(\mathbb{Z}_p^X, \mathbb{Q}_p)$, then the relation is given as

$$L_p(f) = -2 \text{dlog}(a_p(\kappa))|_{\kappa = k}.$$ 

A more detailed account of the history of $L$-invariants can be found in [Col10].

Recently, there has been lot of interest in computing $p$-adic $L$-invariants of higher weight. In [Gra17], the third author conjectured some relations between (slopes of) $L$-invariants of different levels and weights for $p = 2$. In [Ber17], Bergdall explains relations between slopes of $L$-invariants and the size of the $p$-adic family passing through the given newform. On a different note, a main motivation to study
and compute $L$-invariants arises from the study of coefficient fields of classical newforms. In particular, Buzzard asked whether there exists a bound $B_{N,p}$ for all Hecke eigenforms $f$ of level $\Gamma_1(N) \cap \Gamma_0(p)$ and any weight $k \geq 2$ such that $[Q_{f,p} : \mathbb{Q}_p] \leq B_{N,p}$, where $Q_{f,p}$ is the coefficient field of $f$ completed at a prime dividing $p$, see [Buz05, Question 4.4].

For $N = 1$ and $p \leq 7$ Chenevier has shown ([Che08, Corollaire p.3]) that $Q_{f,p}$ is either $\mathbb{Q}_p(a_p(f))$ if $f$ is old at $p$, or $\mathbb{Q}_p(L_p(f))$ if $f$ is new at $p$, where $a_p(f)$ is the $p$-th coefficient of the $q$-expansion of $f$ at infinity. Therefore, Buzzard’s bound $B_{N,p}$ would constrain the degrees of $L_p(f)$ and $a_p(f)$ over $\mathbb{Q}_p$ for all $f$ as long as $N$ and $p$ are fixed.

In this article, we describe a procedure for computing $L$-invariants via the Greenberg-Stevens formula, building on the work of Lauder and Vonk on the action of the $U_p$-operator on overconvergent modular forms. In [Lau11, Section 3.3.4] Lauder describes how to compute the $L$-invariant from the first derivative of the (inverse) characteristic series of the $U_p$-operator if there is a unique split multiplicative cusp form in the given weight, using a formula of Coleman, Stevens and Teitelbaum [CST98]. The main aim of this article is to extend this method to spaces of higher dimension. For this purpose, we explain how to efficiently compute higher derivatives of the characteristic series and how the $L$-invariants can be read off from these derivatives. In order for this approach to work and not increase too much the precision required to perform the computations, one first needs to decompose the space of $p$-newforms by their Atkin-Lehner eigenvalue at $p$. Since the needed precision grows with the dimension of these subspaces, we derive explicit dimension formulas, analogous to the ones presented in [Mar18].

Let us remark that the procedure presented in [Gra17] computes more than just the slopes of $L$-invariants, but it is naturally more restricted, since it involves the Jacquet-Langlands correspondence and, therefore, it needs an auxiliary prime in the level. Another approach (unpublished) due to Pollack R. is to compute the $L$-invariants directly in terms of $p$-adic and classical $L$-values via modular symbols.

The layout is as follows. In Section 2 we recall briefly Coleman classicality and the main result of [Lau11], which are going to be used in several central parts of this article. In Section 3 we describe how to generalize a formula of Coleman, Stevens and Teitelbaum [CST98] proving the existence of a polynomial $Q_{p,k}$, built from higher derivatives of the reverse characteristic series of the $U_p$ operator on the space of overconvergent $p$-adic cusp forms of level $\Gamma_0(pN)$, and whose roots are precisely the $L$-invariants of level $\Gamma_0(pN)$. In Sections 4 and 5 we show how to compute this polynomial extending a method of Lauder [Lau11] and in addition, we give dimension formulae for the relevant spaces of classical modular forms. In Section 6 we present the data collected together with observations on the slopes.

Acknowledgements. This work was partially supported by the DFG Forschergruppe 1920, the DFG Priority Program SPP 1489 and the Luxembourg FNR. We would also thank Alan Lauder, Robert Pollack and Jan Vonk for helpful remarks and suggestions. Finally, we would like to thank John Cremona for providing access to the servers of the Number Theory Group at the Warwick Mathematics Institute.

2. Classical and overconvergent modular forms

Throughout this article, let $p$ be prime and $N$ be a positive integer coprime to $p$. Let $W$ denote the even $p$-adic weight space, i.e. the space of continuous characters $\kappa : \mathbb{Z}_p^\times \to \mathbb{C}_p$ with $\kappa(-1) = 1$. For each $\kappa \in W$, let $S_{\kappa}^!(\Gamma_0(pN))$ denote the space of
overconvergent $p$-adic cusp forms of weight $\kappa$, see [Col97]. This algebra is equipped with the action of the compact operator $U_p$. In the sequel, the term slope will always refer to the valuation of the $U_p$-eigenvalues.

For each integer $k$ we realize the space of classical cuspforms $S_k(\Gamma_0(pN))$ as a $U_p$-stable subspace of $S^!_k(\Gamma_0(pN))$ considering the associated weight $z \mapsto z^k$ in $W$. Let

\[ P(k, t) := \det(1 - t U_p | S_k(\Gamma_0(pN))), \]
\[ P^!(k, t) := \det(1 - t U_p | S^!_k(\Gamma_0(pN))), \]

be the reverse characteristic polynomial (respectively series) of $U_p$. If we write

\[ P^!(\kappa, t) = 1 + \sum_{n=1}^{\infty} b_n(\kappa) t^n, \]

locally, then each function $\kappa \mapsto b_n(\kappa)$ is defined by a power series with coefficients in $\mathbb{Z}_p$, as shown in [Col97, Appendix I]. The following theorem, due to Coleman, links $P(k, t)$ and $P^!(k, t)$:

**Theorem 2.1** (Coleman classicality, [Col96, Theorem 6.1], [Bel12, Corollary 2.6]).

(i) Let $\alpha < k - 1$. Then the roots of $P^!(k, \cdot)$ of slope $\alpha$ are precisely the roots of $P(k, \cdot)$ of slope $\alpha$.

(ii) Let $m \leq k - 1$. Then

\[ P(k, t) \equiv P^!(k, t) \mod p^m. \]

**Remark 2.2.** For $p \geq 5$, the algorithm of Lauder presented in [Lau11] computes $P^!(k, t) \mod p^m$ for given $k$ and $m$ and runs in polynomial time with respect to $p, N$ and $m$ and linear time in $\log(k)$. This algorithm was extended by Vonk in [Von15] to include the primes $p = 2, 3$. These algorithms give us the input for our subsequent computations.

3. $L$-invariants and derivatives of the characteristic series of $U_p$

For a fixed positive even integer $k$, in [CST98] Coleman, Stevens and Teitelbaum showed that whenever there is a unique split multiplicative cusp form $f$ in $S_k(\Gamma_0(pN))$, it is possible to compute its $L$-invariant as

\[ L_p(f) = -2a_p(f) \left. \frac{\partial P^!(k, t)}{\partial t} \right|_{t=a_p(f)^{-1}}, \]

where $a_p(f)$ is the $p$-th coefficient of the $q$-expansion of the newform. Clearly, the formula above works only for 1-dimensional spaces, and in [Lau11], Lauder used it to compute the $L$-invariant.

The aim of this section is to develop a similar approach for spaces of higher dimension. In order to achieve this, we first split the space into eigenspaces for the Atkin-Lehner operator at $p$ and then use higher derivatives to distinguish among the zeros of $P^!(k, t)$ which correspond to different eigenforms, i.e. to separate the $p$-adic families passing through the different eigenforms.

Let $S_k(\Gamma_0(pN))^{p-\text{new}}$ denote the subspace of cuspforms which are new at $p$. For each $f \in S_k(\Gamma_0(pN))^{p-\text{new}}$, we have

\[ U_p f = -p^{(k-2)/2} W_p f, \]
where $W_p$ denotes the Atkin-Lehner involution acting on $S_k(\Gamma_0(pN))^{p-\text{new}}$.

Let $f \in S_k(\Gamma_0(pN))^{p-\text{new}}$ be an eigenform for the Hecke operators away from $N$. Then $f$ has an associated $\mathcal{L}$-invariant $\mathcal{L}_p(f) \in \mathbb{C}_p$. By a generalization of the Greenberg-Stevens formula, see [Col10], $\mathcal{L}_p(f)$ is given as follows. The eigensystem (away from $N$) attached to $f$ defines a classical point on the eigencurve $\mathcal{C}_N$ of tame level $N$, see [Buz07], and there is a $p$-adic Coleman family through $f$ (i.e., an irreducible component of a small affinoid neighbourhood of the point attached to $f$ in $\mathcal{C}_N$ that maps isomorphically onto an open affinoid subdomain of the weight space) of constant slope. In particular, via this $p$-adic family, the rigid morphism $a_p$ on $\mathcal{C}_N$ defines a morphism on the affinoid subdomain of the weight space. The $\mathcal{L}$-invariant attached to $f$ is then given by

$$\mathcal{L}_p(f) = -2 \text{diag}(a_p(\kappa))|_{\kappa = \kappa}.$$

Let us point out that, for $\kappa$ sufficiently close to $k$, the eigenvalue $a_p(\kappa)$ in level $\Gamma_0(p) \cap \Gamma_1(N)$ appears in fact in level $\Gamma_0(pN)$; i.e., on the space $S_k(\Gamma_0(pN))$. The expositions in [Ber17] and [Bel12] provide a nice summary of the constructions that are relevant in our setting.

**Remark 3.1.** Note that, contrary to the exposition in the introduction, we avoided using a Fourier expansion of the $p$-adic Coleman family though $f$. This is due to the fact that in our setting the form $f$ is not new everywhere. We believe that the existence of a Fourier expansion in this situation follows from extending the inclusion and degeneracy maps used in the definition of oldforms to overconvergent forms and checking various compatibilities. Since we did not find statements along these lines in the literature, we are only working with the eigensystem attached to $f$, which is all we need to define its $\mathcal{L}$-invariant.

Before we can study the relation between $\mathcal{L}$-invariants and the characteristic series of $U_p$, we need some preparations.

**Lemma 3.2.** Let $\varepsilon$ be an eigenvalue of the Atkin-Lehner operator $W_p$ acting on $S_k(\Gamma_0(pN))$. Let $a_{p,k} = -\varepsilon p^{(k-2)/2}$. Then

$$S_k(\Gamma_0(pN))^{p-\text{new},\varepsilon} = S_k(\Gamma_0(pN))^{U_p = a_{p,k}},$$

where $S_k(\Gamma_0(pN))^{p-\text{new},\varepsilon}$ denotes the subspace of $S_k(\Gamma_0(pN))^{p-\text{new}}$ on which $W_p$ has eigenvalue $\varepsilon$, and $S_k(\Gamma_0(pN))^{U_p = a_{p,k}}$ is the subspace of $S_k(\Gamma_0(pN))$ where the $U_p$ operator acts as multiplication by $a_{p,k}$.

**Proof.** By definition $S_k(\Gamma_0(pN))^{p-\text{new},\varepsilon} \subseteq S_k(\Gamma_0(pN))^{U_p = a_{p,k}}$. In order to show the other inclusion, let us first remark that if a cuspform in $S_k(\Gamma_0(pN))^{U_p = a_{p,k}}$ is new at $p$, then clearly its an eigenform for $W_p$ with eigenvalue $\varepsilon$. Let us suppose that some cuspform in $S_k(\Gamma_0(pN))^{U_p = a_{p,k}}$ is $p$-old. Then, by a deep theorem of Deligne, its $U_p$-eigenvalue has complex absolute value $p^{(k-1)/2}$, giving a contradiction.

Let us remark that $S_k(\Gamma_0(pN))^{p-\text{new}}$ has a basis of eigenforms for the Hecke operators away from $N$. These split into two orbits with respect to the eigenvalues of the Atkin-Lehner involution.

From now on we fix an eigenvalue $\varepsilon$ of $W_p$ and set $d_\varepsilon = \dim S_k(\Gamma_0(pN))^{p-\text{new},\varepsilon}$.

Let $f_1, \ldots, f_{d_\varepsilon}$ denote the basis of $S_k(\Gamma_0(pN))^{p-\text{new},\varepsilon}$. For each $i \in \{1, \ldots, d_\varepsilon\}$, we denote by $a_i(\kappa)$ the $p$-th coefficient of the Coleman family passing through $f_i$ as above.
Proposition 3.3. Let $\varepsilon$ be an eigenvalue of the Atkin-Lehner operator $W_p$ acting on $S_k(\Gamma_0(pN))^{p-\text{new}}$, then
\[ P^1(k, t) = (1 - a_{p,k}^\varepsilon t)^{d_k}C(k, t), \]
where $C(k, (a_{p,k}^\varepsilon)^{-1}) \neq 0$.

Proof. For $\alpha < k - 1$, by Coleman classicality, Theorem 2.1[(i)] the set of roots of $P^1(k, t)$ of slope $\alpha$ is the set of roots of $P(k, t)$ of slope $\alpha$. Therefore, Lemma 3.2 implies the claim.

In the sequel, for $\bullet \in \{k, t\}$, let us denote by $\partial^n\bullet P^1(k, t)$ the partial derivative of $P^1(k, t)$ with respect to $\bullet$ at the point $(k, t)$, viewed as an element in $Z_p[t]$.

Corollary 3.4. For $n \leq d_e$ and $\bullet \in \{k, t\}$, we have
\[ \partial^n\bullet P^1(k, t) = (1 - a_{p,k}^\varepsilon t)^{n - d_k} \cdot C^{(n)}(k, t) \]
where
\[ C^{(n)}(k, (a_{p,k}^\varepsilon)^{-1}) = n! \cdot C(k, t) \sum_{1 \leq m_1 < \cdots < m_n \leq d_k} \prod_{j=1}^n \partial^{j_1}(1 - a_{m_j}(k)t) \bigg|_{\kappa = k}. \]

Proof. By definition we have
\[ P^1(k, t) = \prod_{i=1}^{d_k} (1 - a_i(k)t) \cdot C(k, t), \]
where the function $C(k, t)$ specializes to the corresponding function in Proposition 3.3 for $\kappa = k$. Thus, the $n$-th partial derivative of $P^1$ for $\bullet \in \{k, t\}$ satisfies
\[ \partial^n\bullet P^1(k, t) = \sum_{i_1 + i_2 + \cdots + i_{d_k+1} = n} \binom{n}{i_1, i_2, \ldots, i_{d_k+1}} \cdot \prod_{j=1}^{d_k} \partial^{j_1}(1 - a_j(k)t) \cdot \partial^{d_k+1}_{i_1+i_2+\cdots+i_{d_k+1}}C(k, t). \]

For $n \leq d_e$, let us split the sum into $\Sigma_{\leq 1}(k, t)$ and $\Sigma_{> 1}(k, t)$ as follows:
\[ \Sigma_{\leq 1}(k, t) = \sum_{i_1 + i_2 + \cdots + i_{d_k+1} = n, \ i_j \leq 1 \forall j} \binom{n}{i_1, i_2, \ldots, i_{d_k+1}} \cdot \prod_{j=1}^{d_k} \partial^{j_1}(1 - a_j(k)t) \cdot \partial^{d_k+1}_{i_1+i_2+\cdots+i_{d_k+1}}C(k, t), \]
\[ \Sigma_{> 1}(k, t) = \sum_{i_1 + i_2 + \cdots + i_{d_k+1} = n, \ i_j > 1 \text{ for some } j} \binom{n}{i_1, i_2, \ldots, i_{d_k+1}} \cdot \prod_{j=1}^{d_k} \partial^{j_1}(1 - a_j(k)t) \cdot \partial^{d_k+1}_{i_1+i_2+\cdots+i_{d_k+1}}C(k, t). \]

We claim that $\Sigma_{> 1}(k, t) = (1 - a_{p,k}^\varepsilon t)^{d_k-n+1} \cdot C^{(n)}_{> 1}(k, t)$ for some $C^{(n)}_{> 1}(k, t)$.

If $i_\ell > 1$ for some $\ell \in \{1, \ldots, d_k + 1\}$, we must have $i_\ell = 0$ for at least $d_e - n + 1$ many $j$s, since all $i_j$ are positive, their sum equal to $n$ and the cardinality of the range is $d_e + 1$. Specializing to $\kappa = k$, this immediately implies the result. We can proceed in a similar fashion to show that the terms with $i_{d_k+1} = 1$ in $\Sigma_{\leq 1}$ satisfy an analogous formula. Therefore, we are left with the term
\[ n! \cdot C(k, t) \sum_{i_1 + \cdots + i_{d_k} = n, \ i_j \leq 1 \forall j} \prod_{j=1}^{d_k} \partial^{j_1}(1 - a_j(k)t). \]
Let us observe that after separating the terms with \( i_j = 0 \) at \( \kappa = k \) and evaluating at \( t = (a_{p,k}^*)^{-1} \) we obtain

\[
n! \cdot C(k, t) \cdot (1 - a_{p,k}^* t)^{d_n - n} \sum_{1 \leq m_1 < \cdots < m_n \leq d_n} \prod_{j=1}^n \partial_t (1 - a_{m_j}(\kappa) t) \bigg|_{\kappa = k}.
\]

Putting these terms together completes the proof. \( \square \)

**Remark 3.5.** In the spirit of the above proposition, one could also analyse the behaviour of mixed derivatives with respect to \( \kappa \) and \( t \). However, for our applications only the above derivatives are relevant.

Finally, we can state the main result of this section:

**Theorem 3.6.** Let \( \varepsilon \) be an eigenvalue of the Atkin-Lehner operator \( W_p \) acting on \( S_k(\Gamma_0(pN))^{p\text{-new}} \), then the monic polynomial \( Q_{p,k}^\varepsilon(x) = \sum_{n=1}^{d_n} c_n x^n \in \mathbb{Q}_p[x] \), where

\[
c_n := 2^n (a_{p,k}^*)^n \left( \frac{d_x}{n} \right) \frac{\partial^n P^1(\kappa, t)}{\partial_t^n P^1(\kappa, t)} \bigg|_{\kappa = k, t = (a_{p,k}^*)^{-1}},
\]

satisfies

\[
Q_{p,k}^\varepsilon(x) = \prod_{i=1}^{d_n} (x - L_p(f_i)).
\]

**Proof.** Taking the ratio of the \( n \)-th partial derivatives of \( P^1(\kappa, t) \) and applying Corollary 3.3 we obtain

\[
\frac{\partial^n P^1(\kappa, t)}{\partial_t^n P^1(\kappa, t)} \bigg|_{\kappa = k, t = (a_{p,k}^*)^{-1}} = \frac{\sum_{1 \leq m_1 < \cdots < m_n \leq d_n} \prod_{j=1}^n \partial_t (1 - a_{m_j}(\kappa) t) \bigg|_{\kappa = k, t = (a_{p,k}^*)^{-1}}}{\sum_{1 \leq m_1 < \cdots < m_n \leq d_n} \prod_{j=1}^n a_{m_j}(k)^{-n}} \frac{\prod_{j=1}^n a_{m_j}(k)}{(a_{p,k}^*)^{-n}} = \left( \frac{d_x}{n} \right)^{-1} (a_{p,k}^*)^{-n} \sum_{1 \leq m_1 < \cdots < m_n \leq d_n} \prod_{j=1}^n a_{m_j}(k).
\]

Multiplying both sides by \( 2^n (a_{p,k}^*)^n \left( \frac{d_x}{n} \right) \) yields

\[
c_n = 2^n (a_{p,k}^*)^n \left( \frac{d_x}{n} \right) \frac{\partial^n P^1(\kappa, t)}{\partial_t^n P^1(\kappa, t)} \bigg|_{\kappa = k, t = (a_{p,k}^*)^{-1}} = \sum_{1 \leq m_1 < \cdots < m_n \leq d_n} \prod_{j=1}^n (2 \log(a_{m_j}(\kappa)) \big|_{\kappa = k})
\]

The last term is precisely the \((d_x - n)\)-th coefficient of the polynomial

\[
\prod_{i=1}^{d_n} (X - L_p(f_i)),
\]

which completes the proof. \( \square \)

**Remark 3.7.** Note that in the above proof we used that \( C_k^{(n)}(k, (a_{p,k}^*)^{-1}) \neq 0 \).

The analogous statement for \( C_k^{(n)}(k, (a_{p,k}^*)^{-1}) \) is equivalent to all of the \( L \)-invariants...
\( \mathcal{L}(f_i) \) being non-zero. While all our computational results support this claim, we are not aware of a proof of this statement.

4. Dimension formulae

In this section we show how to compute the dimension \( d_\varepsilon \) of the cuspform space \( \mathcal{S}_k(\Gamma_0(pN))^{p\text{-new,}\varepsilon} \) which appears in Section 3. Let us recall that \( N \) is a positive integer coprime to \( p \). As before, we will denote by \( W_p \) the \( p \)-th Atkin-Lehner involution and whenever \( W_p \) is an operator on a vector space \( V \), we will denote by \( \text{tr}_V W_p \) its trace on \( V \).

**Proposition 4.1.** The trace of the Atkin-Lehner operator \( W_p \) satisfies

\[
\text{tr}_{\mathcal{S}_k(\Gamma_0(pN))^{p\text{-new}}} W_p = \text{tr}_{\mathcal{S}_k(\Gamma_0(pN))} W_p.
\]

**Proof.** The cuspform space \( \mathcal{S}_k(\Gamma_0(pN)) \) decomposes into a \( p \)-new and a \( p \)-old component \( \mathcal{S}_k(\Gamma_0(pN)) = \mathcal{S}_k(\Gamma_0(pN))^{p\text{-new}} \oplus \mathcal{S}_k(\Gamma_0(pN))^{p\text{-old}} \). The statement is then equivalent to say that \( \text{tr}_{\mathcal{S}_k(\Gamma_0(pN))^{p\text{-old}}} W_p = 0 \) and this follows from [AL70, Lemma 26]. \( \square \)

In the case of \( N \) squarefree, building on previous works of Yamauchi, Skoruppa and Zagier, Martin gave a formula [Mar18, Equation (1.6)] for the trace of the Atkin-Lehner operator \( W_p \) on \( \mathcal{S}_k(\Gamma_0(pN)) \). In the simple case where \( N = q > 3 \) is a prime and \( p > 3 \), we have

\[
\text{tr}_{\mathcal{S}_k(\Gamma_0(pN))^{p\text{-old}}} W_p = \frac{1}{2} (-1)^k a(p, q) h(\mathbb{Q}(\sqrt{-p}))(1 + \left( \frac{\Delta_p}{q} \right)) + \delta_{k, 2},
\]

where \( a(p, q) \) is 1, 4 or 2 for \( p \) congruent to 1 or 5, 3, 7 modulo 8 respectively, \( h(\mathbb{Q}(\sqrt{-p})) \) denotes the class number of \( \mathbb{Q}(\sqrt{-p}) \) and \( \Delta_p \) its discriminant, the Legendre symbol for \( \Delta_p \) and \( q \) is denoted by \( \left( \frac{\Delta_p}{q} \right) \), and \( \delta_{k, 2} \) is the Kronecker delta function.

By simple algebraic manipulations, we have the following

**Corollary 4.2.** Let \( \varepsilon \) be an eigenvalue of the Atkin-Lehner operator \( W_p \) and set \( d := \dim \mathcal{S}_k(\Gamma_0(pN))^{p\text{-new}} \). Then

\[
d_\varepsilon = \frac{1}{2} (d + \varepsilon \text{tr}_{\mathcal{S}_k(\Gamma_0(pN))} W_p)
\]

Let us remark that the dimension of \( \mathcal{S}_k(\Gamma_0(pN))^{p\text{-new}} \) can be obtained by recursively computing the dimension of \( \mathcal{S}_k(\Gamma_0(pN))^{p\text{-old}} \). Below there is a small table

| \( p \) | \( N \) | \( k \) | \( \text{Tr} \) | \( d \) | \( d_{+1} \) | \( d_{-1} \) |
|---|---|---|---|---|---|---|
| 2 | 1 | 20 | 0 | 2 | 1 | 1 |
| 2 | 1 | 40 | 1 | 3 | 2 | 1 |
| 3 | 1 | 20 | 1 | 3 | 2 | 1 |
| 3 | 1 | 40 | 0 | 6 | 3 | 3 |
| 13 | 1 | 20 | 1 | 19 | 10 | 9 |
| 13 | 1 | 40 | 1 | 39 | 20 | 19 |
| 13 | 2 | 20 | 1 | 57 | 29 | 28 |
| 13 | 2 | 40 | 1 | 117 | 59 | 58 |
| 7 | 11 | 200 | 2 | 1194 | 598 | 596 |
| 7 | 11 | 400 | 2 | 2394 | 1198 | 1196 |
with dimensions of $S_k(\Gamma_0(pN))^{p\text{-new}}$ and $\dim S_k(\Gamma_0(pN))^{p\text{-new},e}$ for some levels and weights, where, for simplicity, we set $\text{Tr} := \text{tr}_{S_k(\Gamma_0(pN))} W_p$.

5. Computing $Q^e_{p,k}$

The main difficulty in computing $Q^e_{p,k}$ is given by the partial derivatives, as shown in Theorem 3.6. For the denominators, this is straightforward by Remark 2.2, since we only use formal differentiation with respect to $t$. However, the computation of the numerators is more involved.

In [Lau11, Lemma 3.10] Lauder explains a method using finite differences for the first derivative. The aim of this section is to generalize this method to derivatives of higher order. The study of finite differences in this context is due to Gouvêa and Mazur, see [GM93].

Let us first observe that it is enough to work only at classical points since, by definition, we have

$$\partial_k^\nu P^j(k, t) = \lim_{m \to \infty} \frac{\partial_k^{n-1} P^j(k + (p - 1)p^m, t) - \partial_k^{n-1} P^j(k, t)}{(p - 1)p^m}.$$  

The finite differences used to approximate $\partial_k^\nu P^j(k, t)$ are defined as follows. Let $s = (p - 1)p^m$ and set

$$\partial_k^\nu P^j(k, t)_m := s^{-n} \sum_{j=0}^n (-1)^j \binom{n}{j} P^j(k + (n - j)t).$$

**Theorem 5.1.** We have

$$\partial_k^\nu P^j(k, t)_{m+1} \equiv \partial_k^\nu P^j(k, t)_m \mod p^{m+1}.$$  

Therefore, $\partial_k^\nu P^j(k, t) \equiv \partial_k^\nu P^j(k, t)_m \mod p^{m+1}.$

**Proof.** Let $P^j(k, t) = \sum_{i=0}^\infty b_i(k)t^i$. In the sequel we fix a choice of $i$ and write $b(k) = b_i(k)$. Then, for $s = (p - 1)p^m$ it is enough to show

$$\sum_{j=0}^n (-1)^j \binom{n}{j} b(k + (n - j)s) = p^m \sum_{j=0}^n (-1)^j \binom{n}{j} b(k + (n - j)s) \mod p^{(\nu+1)(m+1)}$$

We define the difference functions $\delta_\nu(b, k)$ recursively by

$$\delta_1(b, k) := b(k + s) - b(k),$$

$$\delta_\nu(b, k) := \delta_{\nu-1}(b, k + s) - \delta_{\nu-1}(b, k) \quad \text{for } \nu \geq 2.$$  

By [GM93] Theorem 2, we have $\delta_\nu(b, k) \equiv 0 \mod p^{\nu(m+1)}$. Thus, rewriting the above equation, it suffices to show that

$$\sum_{j=0}^n (-1)^j \binom{n}{j} b(k + (n - j)s) - p^m \sum_{j=0}^n (-1)^j \binom{n}{j} b(k + (n - j)s)$$

is a $\mathbb{Z}$-linear combination of $\delta_\nu(b, k)$ for $\nu \geq n + 1$. By definition, we have

$$\delta_\nu(b, k) = \sum_{j=0}^\nu \binom{\nu}{j} (-1)^j b(k + (\nu - j)s).$$

Let $X$ be a variable and write

$$(X - 1)^\nu = \sum_{j=0}^\nu \binom{\nu}{j} (-1)^j X^{\nu-j}.$$
Then, after substituting $X^l$ for $b(k+ls)$, we just need to prove that the polynomial $R(X) = (X^p - 1)^n - p^n(X - 1)^n$ is a $\mathbb{Z}$-linear combination of $(X - 1)^\nu$ for $\nu \geq n + 1$ or equivalently, that $R(X)$ vanishes to order at least $n + 1$ at $X = 1$. Clearly, $R(X)$ vanishes to order at least $n$ at $X = 1$ and we just need to consider the $(n + 1)$-th derivative $R^{(n+1)}(X)$. Similarly to the proof of Theorem 3.6 by applying the product rule we have

$$R^{(n+1)}(1) = n!p^n1^{n(p-1)} - p^n(n!1^n) = 0.$$ 

This concludes the proof of the first part of the theorem. The second part is an immediate consequence, since the higher derivatives of every function given by a power series are approximated by finite differences on a small neighbourhood. □

6. Computations

6.1. Overview of the code. The algorithm implemented computes the polynomial $Q_{p,k}^\varepsilon$, defined in Section 3. The input of the algorithm are a prime $p$, a positive integer $N$ coprime to $p$, an even integer $k$ and a positive integer $m$, the desired precision.

The code works in a straightforward manner following the steps below:

Step 1: The first step consists in computing the reverse characteristic series of the $U_p$ operator on overconvergent modular forms at the given precision $m$. We performed this step using algorithms of Lauder [Lau11] and Vonk [Von15], which are polynomial time algorithms with respect to $p$, $N$ and $m$, and linear time in $\log(k)$.

Step 2: Find the Atkin-Lehner decomposition of the space. We will decompose the space according to the eigenvalues of the Atkin-Lehner operator $W_p$, since we will work on each $S_k(\Gamma_0(pN))^{p-\text{new},\varepsilon}$ separately, where $\varepsilon$ is one of such eigenvalues. In this step, we actually only compute the dimension $d_\varepsilon$ of such eigenspaces. As shown in Section 4, this computation can be done via Corollary 4.2, Equation (1.6) in [Mar18] and by recursion.

Step 3: For each subspace $S_k(\Gamma_0(pN))^{p-\text{new},\varepsilon}$, compute $\partial_\bullet P^1(k, t)$ the partial derivative with respect to $\bullet$ of $P^1(\kappa, t)$ at the point $(k, t)$ viewed as an element in $\mathbb{Z}_p[t]$. Theorem 5.1 gives us the precision to which this computation is performed for the derivative in $\kappa$, which is the relevant piece.

Step 4: Build the quotient of both derivatives, if possible. If the precision is too low, this step cannot be performed, and the algorithm restarts with higher precision $m$.

Step 5: Compute the coefficients $c_n$ as in Theorem 3.6 and return $Q_{p,k}^\varepsilon$.

Some examples of the output are presented in the next subsection. Building from the output of the algorithm, we analyzed the distribution of (the slopes of) $L$-invariants for increasing weight. We tabled the results obtained in Subsection 6.4 and we formulated some speculations, based on the data collected, in Subsection 6.3.

Remark 6.1. It should be pointed out that, once the reverse characteristic series of $U_p$ is obtained, our algorithm is independent of the work of Lauder and Vonk, and can thus be combined with different methods of computing the series. The main bottleneck of our computations is the $p$-adic precision to which the coefficients $c_n$ of $Q_{p,k}^\varepsilon$ have to computed. As shown in Theorem 5.1, this precision depends on the dimension $d_\varepsilon$ of the relevant space of modular forms. This leads to work with very large matrices of $U_p$ on overconvergent modular forms in the algorithms of
Lauder and Vonk, making even the computation of the characteristic series rather time-consuming.

6.2. Examples. All of the following examples were computed using an Intel® Xeon® E5-2650 v4 CPU processor with 512 GB of RAM memory.

Example 6.2. Let us consider the space \( S_4(\Gamma_0(6)) \). We have

\[
\dim S_4(\Gamma_0(6))^{p\text{-new}, \varepsilon} > 0 \quad \text{for} \quad (p, \varepsilon) \in \{(2, 1), (3, 1)\}.
\]

In both cases the space is one-dimensional, i.e. the polynomial \( Q_{p,4}^\varepsilon(X) \in \mathbb{Q}_p[X] \) is given by a linear factor with the \( \mathcal{L} \)-invariant of the unique newform in \( S_4(\Gamma_0(6))^{p\text{-new}} \) as a zero. Modulo 2\(^21\) resp. 3\(^21\), we have

\[
Q_{2,4}^{+1}(X) = X + 94387 \cdot 2
\]

\[
= X - (2 + 2^4 + 2^7 + 2^9 + 2^{10} + 2^{11} + 2^{12} + 2^{18} + 2^{19} + 2^{20}),
\]

\[
Q_{3,4}^{-1}(X) = X - 41502709
\]

\[
= X - (1 + 3^2 + 2 \cdot 3^7 + 3^8 + 2 \cdot 3^9 + 2 \cdot 3^{13} + 2 \cdot 3^{14} + 2 \cdot 3^{15}).
\]

The latter example was also computed in [Tei90] and in [Gra17]. The duration of the above computations was 1.8 and 5.6 seconds respectively.

Example 6.3. Let \( p = 2, N = 7 \) and \( k = 10 \). The space \( S_{10}(\Gamma_0(14))^{2\text{-new}, \varepsilon} \) has dimension 3 for \( \varepsilon = \pm 1 \) and the polynomials \( Q_{2,10}^\varepsilon(X) \) is equal to

\[
Q_{2,10}^{+1}(X) = X^3 + (102587339304770571 \cdot 2^4 + O(2^{63})) \cdot X^2
\]

\[
+ (10546154801 \cdot 2^{-2} + O(2^{33})) \cdot X + 23 \cdot 2^{-2} + O(2^4),
\]

\[
Q_{2,10}^{-1}(X) = X^3 + (210960224044090299 \cdot 2^2 + O(2^{63})) \cdot X^2
\]

\[
- (12204518771 \cdot 2^{-2} + O(2^{33})) \cdot X + 13 \cdot 2^{-2} + O(2^4).
\]

The slopes of both polynomials are \([0, -1_2]\).

Computation time was approximately 49 minutes.

Let us look more closely at the space \( S_{10}(\Gamma_0(14))^{2\text{-new}, -1} \); it is spanned by a rational newform \( f \) together with two oldforms coming from \( S_{10}(\Gamma_0(2)) \). The slope factorization of \( Q_{2,10}^{-1} \) factors the polynomial into a linear and a quadratic part. Thus, we are able to read off the \( \mathcal{L} \)-invariant of slope zero, which is

\[
1 + 2 + 2^4 + 2^5 + O(2^6).
\]

By the methods of [Gra17], which directly take place in \( S_{10}(\Gamma_0(14))^{\text{new}, -1} \), we are able to confirm that this is indeed the \( \mathcal{L} \)-invariant \( \mathcal{L}_2(f) \).

Example 6.4. Let \( p = 5, N = 3 \) and \( k = 10 \). Then we have

\[
\dim S_{10}(\Gamma_0(15))^{5\text{-new}, +1} = 5 \quad \text{and} \quad \dim S_{10}(\Gamma_0(15))^{5\text{-new}, -1} = 7.
\]

In this case, the algorithm took 5, 2 days to complete and used 7, 79 Gigabytes of memory.

The coefficients of the following polynomials have been reduced modulo 5\(^{15} \) in order to allow a readable presentation of the output. The original precision of the output was 5\(^{136} \) for the first coefficient and 5\(^{15} \) for the last coefficient of the second
The computation required approximately 5 hours.

The slopes are two-dimensional for $21$ and three-dimensional for $2$.

Interchanging the roles of $p$ and $N$ we have the following. Let $p = 3, N = 5$ and $k = 10$. Then $S_{10}(15)^{\text{new}, \epsilon}$ has dimension 5 for every choice of $\epsilon$. The polynomials $Q_{3,10}(X)$ are given by

$$Q_{3,10}^{+}(X) = X^{5} - (54755965941799683058786327743853006127 \cdot 3^{-1} + O(3^{83})) \cdot X^{4} - (257461380437518760049330350554 \cdot 3^{-4} + O(3^{83})) \cdot X^{3} + (90851888482397169277 \cdot 3^{-5} + O(3^{42})) \cdot X^{2} + (5706984676643 \cdot 3^{-8} + O(3^{22})) \cdot X - 18013 \cdot 3^{-9} + O(3^{2}),$$

$$Q_{3,10}^{-}(X) = X^{5} - (449588499865942027654475093549758242560 \cdot 3^{-1} + O(3^{83})) \cdot X^{4} - (238552986635971663584874262916 \cdot 3^{-4} + O(3^{83})) \cdot X^{3} - (4398751740570004601518 \cdot 3^{-5} + O(3^{42})) \cdot X^{2} + (6759342611602 \cdot 3^{-8} + O(3^{22})) \cdot X + 25628 \cdot 3^{-9} + O(3^{2}).$$

In this example the slopes of both polynomials coincide again and are $[-1, -2]$. The computation required approximately 5 hours.

**Example 6.5.** Let $p = 7, N = 11$ and $k = 2$. The space $S_{2}(177)^{\text{new}, \epsilon}$ is two-dimensional for $\epsilon = +1$ and three-dimensional for $\epsilon = -1$. The coefficients of the following polynomials have been reduced modulo $7^{21}$.

$$Q_{7,2}^{+}(X) = X^{2} + 526982521374955003 \cdot 7 \cdot X + 192454376115114681 \cdot 7^{2}$$

$$Q_{7,2}^{-}(X) = X^{3} + 2221404449450143105 \cdot 7 \cdot X^{2} - 103688086480269397 \cdot 7^{2} \cdot X - 39596022807935252 \cdot 7^{3}$$

The slopes can be computed to be $[12]$ and $[13]$ respectively. The computation required approximately 2 hours.

In this case, we have $S_{2}(177)^{\text{new}, \epsilon} = S_{2}(177)^{\text{new}, \epsilon}$ for $\epsilon = \pm 1$. The space $S_{2}(177)^{\text{new}, +1}$ is spanned by two rational newforms $f_1$ and $f_2$. These correspond to the isogeny classes of elliptic curves with Cremona labels 77a and 77c and their $L$-invariants are given by

$$L_{f_1} = 2 + 7 + 3 \cdot 7^{2} + 4 \cdot 7^{3} + 7^{4} + 4 \cdot 7^{7} + 7^{9} + O(7^{10}),$$

$$L_{f_2} = 4 + 7 + 2 \cdot 7^{2} + 6 \cdot 7^{3} + 5 \cdot 7^{4} + 6 \cdot 7^{5} + 6 \cdot 7^{6} + 3 \cdot 7^{7} + 3 \cdot 7^{8} + 7^{9} + O(7^{10}).$$

The space $S_{2}(177)^{\text{new}, -1}$ is spanned by a rational newform $f$ and two newforms $g_1$ and $g_2$ with coefficient field $K = \mathbb{Q}(\sqrt{5})$. The newform $f$ corresponds to the isogeny class of elliptic curves with Cremona label 77b, which has split multiplicative reduction at 7. Its $L$-invariant is given by

$$L_{f} = 3 + 7 + 7^{2} + 2 \cdot 7^{3} + 5 \cdot 7^{4} + 4 \cdot 7^{5} + 7^{6} + 4 \cdot 7^{7} + 7^{8} + 3 \cdot 7^{9} + O(7^{10}).$$
If we divide the polynomial $Q_{17}^2(X)$ by $(X - L_7(f))$, we thus obtain (modulo $7^{11}$)

$$(X - L_7(g_1)) \cdot (X - L_7(g_2)) = X^2 + 225931960 \cdot X + 21342634 \cdot 7^2.$$ 

This polynomial is in fact irreducible over $\mathbb{Q}_7$, showing that the $L$-invariant $L_7(g_1)$ generates the quadratic unramified extension of $\mathbb{Q}_7$, and similarly for $L_7(g_2)$. This is a first example of higher level $N$, where Chenevier’s result in [Che08] mentioned in the introduction holds.

### 6.3. Observations collected

Given a positive even integer $k$, we denote by $\nu_\varepsilon(k, p, N)$ the finite sequence of slopes of the $p$-adic $L$-invariants attached to forms in $S_k(\Gamma_0(pN))^{p\text{-new}, \varepsilon}$, where $\varepsilon$ is an eigenvalue of the Atkin-Lehner operator $W_p$ acting on $S_k(\Gamma_0(pN))^{p\text{-new}}$. Note that this differs from the tables in [Gra17], where the space is decomposed with respect to the Atkin-Lehner involution at $N$. When say a slope $\alpha$ is lower than a slope $\beta$ if $\alpha > \beta$ as a picture of the Newton polygon suggests.

The observations collected are based on the data and on the tables presented in the next section.

We begin with the case $p = 2$, that was extensively studied in [Gra17]. There, several relations between slopes of different level and weights were conjectured for levels $N = 3, 5, 7$. Table 1 provides data for the case $N = 1$, which is not accessible by the methods in [Gra17]. The observations lead us to the following analogous conjecture.

**Conjecture 6.6 (p = 2).**

(a) For $k \in 2 + 4\mathbb{Z}$, $k \geq 10$ and $\varepsilon \in \{\pm 1\}$, we have

$$\nu_\varepsilon(k, 2, 1) = \nu_{-\varepsilon}(k + 6, 2, 1).$$

(b) For every even integer $k$, the highest $\min\{d+1, d-1\}$ slopes in $\nu_\varepsilon(k, 2, 1)$ and $\nu_{-\varepsilon}(k, 2, 1)$ agree.

The above table verifies this statement up to $k = 70$ computationally. We also observe that in general almost all slopes are negative. Our observations is in line with similar observations in [Gra17]. The table also provides the missing data for [Gra17, Conjecture 5.6 (ii)] stating that the slopes appearing in level 7 are the union (two copies) of the slopes in levels 1 and 3.

The case $p = 3$ and $N = 1$ is studied in Table 2. Again we observe relations between various different slopes that a priori do not have any theoretical foundations. We collect our observations in the following conjecture.

**Conjecture 6.7 (p = 3).**

(a) For $k \in 2 + 6\mathbb{Z}$, $k \geq 8$ and $\varepsilon \in \{\pm 1\}$, we have

$$\nu_\varepsilon(k, 3, 1) = \nu_{-\varepsilon}(k + 4, 3, 1).$$

(b) For every even integer $k$, the highest $\min\{d+1, d-1\}$ slopes in $\nu_\varepsilon(k, 3, 1)$ and $\nu_{-\varepsilon}(k, 3, 1)$ agree.

Interestingly, the numbers 4 and 6 from Conjecture 6.6 (a) are essentially switched here: We have relations between $k$ and $k + 4$ while considering $k \mod 6$. The same is true for the Atkin-Lehner sign, since here we always have relations between slopes of the same sign. While Conjecture 6.7 is very similar to Conjecture 6.6, we should
point out that here there are slopes that do not have (obvious) relations to other slopes. These are the slopes for $k \in 4 + 6\mathbb{Z}$. We still observe that for example the slopes for $k = 10$ and $k = 16$ only differ by $-1$. Similarly, the slopes for $k = 22$ and $k = 28$ are very similar. However, we are not able to formulate a precise conjecture.

The data in Tables 3, 4 and 5 shows various relations between slopes of different weights supporting conjectural connections between them, analogous to the conjectures presented above. Moreover, after possibly removing oldforms, the analogous observation to part (b) of the Conjectures 6.6 and 6.7 is still present in the data presented in Tables 6 and 7. We are able to say even more, namely that observation (b) is a purely local phenomenon: Let for example $p = 13$ and $k = 10$. The corresponding space of modular forms consists of two Galois orbits of dimensions 4 and 5. Each of the orbits makes up one of the two columns in Table 5. Thus, the pairs in observation (b) arise from different orbits. Moreover, within each orbit (aside from the lowest slope) all slopes are distinct.

In particular the data computed for $p = 2$ and $N = 3$ matches with the one presented in [Gra17] and we observe again a relation between the slopes, see [Gra17, Conjecture 5.2]. In all cases, it is an interesting question to analyze the growth of the slopes with the weight $k$ further. However, at this stage, the collected data is not sufficient to make precise claims. It would also be interesting to separate the slopes with respect to mod $p$ eigensystems. As this is only relevant for larger $p$, this question becomes more interesting once more data in these cases is available.
6.4. **Tables.** In the tables below, for a positive even integer we denote by \( \nu_k^\pm (k, p, N) \) the finite sequence of slopes of the \( p \)-adic \( \mathcal{L} \)-invariants attached to cuspforms in \( S_k(\Gamma_0(pN))^{p-\text{new}, \epsilon} \), where \( \epsilon \) is an eigenvalue of the Atkin-Lehner operator \( W_p \) acting on \( S_k(\Gamma_0(pN))^{p-\text{new}} \). Note that in the case \( N > 1 \) our tables can include oldforms. Their slopes are indicated by blue font. When the multiplicity of a slope is larger than 1, it is indicated with a lower index.

| \( k \) | \( d \) | \( \nu_k^+ (k, 2, 1) \) | \( \nu_k^- (k, 2, 1) \) |
|-------|-------|----------------|----------------|
| 8     | 1     | 0              | -1             |
| 10    | 1     | -4             | -4             |
| 12    | 0     | -1             | -1             |
| 14    | 2     | -2             | -2             |
| 18    | 1     | -2             | -2             |
| 20    | 2     | -6             | -6             |
| 22    | 2     | -6             | -6             |
| 24    | 1     | -2             | -2             |
| 26    | 3     | -7, -7         | -2, -7         |
| 28    | 2     | -6             | -6             |
| 30    | 2     | -5             | -5             |
| 32    | 3     | -2, -7         | -7             |
| 34    | 3     | -3, -7         | -3, -7         |
| 36    | 2     | -5             | -5             |
| 38    | 4     | -6, -10        | -6, -10        |
| 40    | 3     | -3, -7         | -7             |
| 42    | 3     | -9             | -3, -9         |
| 44    | 4     | -6, -10        | -6, -10        |
| 46    | 4     | -4, -11        | -4, -11        |
| 48    | 3     | -3, -9         | -9             |
| 50    | 5     | -9, -12        | -3, -9, -12    |
| 52    | 4     | -4, -11        | -4, -11        |
| 54    | 4     | -7, -10        | -7, -10        |
| 56    | 5     | -3, -9, -12    | -9, -12        |
| 58    | 5     | -8, -12        | -3, -8, -12    |
| 60    | 4     | -7, -10        | -7, -10        |
| 62    | 6     | -6, -12, -12_2 | -6, -12_2     |
| 64    | 5     | -3, -8, -12    | -8, -12        |
| 66    | 5     | -8, -12        | -4, -8, -12    |
| 68    | 6     | -6, -12, -12_2 | -6, -12_2     |
| 70    | 6     | -7, -10, -16   | -7, -10, -16   |

*Table 1. \( p = 2, N = 1 \)*
Table 2. $p = 3, N = 1$

| $k$ | $d$ | $\nu^+_{\mathcal{L}}(k, 3, 1)$ | $\nu^-_{\mathcal{L}}(k, 3, 1)$ |
|-----|-----|-------------------------------|-------------------------------|
| 6   | 1   | 1                             |                               |
| 8   | 1   | $-1$                          |                               |
| 10  | 2   | $-2$                          | $-2$                          |
| 12  | 1   | $-1$                          |                               |
| 14  | 3   | $-4$                          | $0, -4$                       |
| 16  | 2   | $-3$                          | $-3$                          |
| 18  | 3   | $-4$                          | $0, -4$                       |
| 20  | 3   | $-2, -4$                      | $-4$                          |
| 22  | 4   | $-3, -6$                      | $-3, -6$                      |
| 24  | 3   | $-2, -4$                      | $-4$                          |
| 26  | 5   | $-5, -7$                      | $-2, -5, -7$                  |
| 28  | 4   | $-1, -6$                      | $-2, -6$                      |
| 30  | 5   | $-5, -7$                      | $-2, -5, -7$                  |
| 32  | 5   | $-2, -4, -9$                  | $-4, -9$                      |
| 34  | 6   | $-3, -6, -10$                 | $-3, -6, -10$                 |
| 36  | 5   | $-2, -4, -9$                  | $-4, -9$                      |
| 38  | 7   | $-4, -9, -11$                 | $-1, -4, -9, -11$             |
| 40  | 6   | $-3, -7, -10$                 | $-3, -7, -10$                 |
| 42  | 7   | $-4, -9, -11$                 | $-1, -4, -9, -11$             |
| 44  | 7   | 0, $-6, -8, -11$              | $-6, -8, -11$                 |
| 46  | 8   | $-2, -7, -11_2$               | $-2, -7, -11_2$               |
| 48  | 7   | 0, $-6, -8, -11$              | $-6, -8, -11$                 |
| 50  | 9   | $-5, -9, -11, -13$            | $-1, -5, -9, -11, -13$        |

Table 3. $p = 5, N = 1$

| $k$ | $d$ | $\nu^+_{\mathcal{L}}(k, 5, 1)$ | $\nu^-_{\mathcal{L}}(k, 5, 1)$ |
|-----|-----|-------------------------------|-------------------------------|
| 4   | 1   | 0                             |                               |
| 6   | 1   |                               |                               |
| 8   | 3   | 0, $-2$                       | $-2$                          |
| 10  | 3   | $-2$                          | $2, -2$                       |
| 12  | 3   | $-1, -2$                      | $-2$                          |
| 14  | 5   | $-2, -4$                      | $-1, -2, -4$                  |
| 16  | 5   | $-1, -3, -4$                  | $-3, -4$                      |
| 18  | 5   | $-2, -4$                      | $-1, -2, -4$                  |
| 20  | 7   | $-1, -2, -5, -6$              | $-2, -5, -6$                  |
| 22  | 7   | $-2, -5, -6$                  | $-1, -2, -5, -6$              |
| 24  | 7   | $-1, -2, -4, -7$              | $-2, -4, -7$                  |
| 26  | 9   | $-3, -4, -7, -8$              | $-1, -3, -4, -7, -8$          |
| 28  | 9   | $-1, -2, -4, -7, -10$         | $-2, -4, -7, -10$             |
| $k$ | $d$ | $\nu_k^+ (k, 7, 1)$ | $\nu_k^- (k, 7, 1)$ |
|-----|-----|---------------------|---------------------|
| 4   | 1   | 0                   | 0                   |
| 6   | 3   | -1                  | 0, -1               |
| 8   | 3   | 0, -1               | -1                  |
| 10  | 5   | -1, -3              | 0, -1, -3           |
| 12  | 5   | 0, -2, -3           | -2, -3              |
| 14  | 7   | -2, -3, -4          | 2, -2, -3, -4       |
| 16  | 7   | -1, -2, -3, -4      | -2, -3, -4          |
| 18  | 9   | -2, -3, -5, -6      | -1, -2, -3, -5, -6  |
| 20  | 9   | 0, -1, -3, -5, -6   | -1, -3, -5, -6      |
| 22  | 11  | -1, -4, -5, -6, -7  | -1, -4, -5, -6, -7  |

Table 4. $p = 7, N = 1$

| $k$ | $d$ | $\nu_k^+ (k, 13, 1)$ | $\nu_k^- (k, 13, 1)$ |
|-----|-----|----------------------|----------------------|
| 4   | 3   | 0_2                  | 0                    |
| 6   | 5   | 0, -1                | 0_2, -1              |
| 8   | 7   | 0_2, -1, -2          | 0, -1, -2            |
| 10  | 9   | 0, -1, -2, -3        | 0_2, -1, -2, -3      |
| 12  | 11  | 0_2, -1, -2, -3, -4  | 0, -1, -2, -3, -4   |
| 14  | 13  | 0, -1, -2, -3, -4, -5| 0_2, -1, -2, -3, -4 |

Table 5. $p = 13, N = 1$

| $k$ | $d$ | $\nu_k^+ (k, 2, 3)$ | $\nu_k^- (k, 2, 3)$ |
|-----|-----|---------------------|---------------------|
| 4   | 1   | 1                   | 1                   |
| 6   | 1   | 0                   | 0                   |
| 8   | 3   | 0_2                 | -1                  |
| 10  | 3   | 0                   | -1_2                |
| 12  | 3   | -1, -4              | -4                  |
| 14  | 5   | -1_2, -4            | -1, -4_2            |
| 16  | 5   | -1_2, -4            | -2_2, -4            |
| 18  | 5   | -1, -4              | -2_2, -4            |
| 20  | 7   | -2, -4_2, -6        | -4_2, -6            |
| 22  | 7   | -4, -6_2            | -2, -4, -6_2        |
| 24  | 7   | -2_2, -6, -7        | -2, -6, -7          |

Table 6. $p = 2, N = 3$

| $k$ | $d$ | $\nu_k^+ (k, 3, 2)$ | $\nu_k^- (k, 3, 2)$ |
|-----|-----|---------------------|---------------------|
| 4   | 1   | 0                   | 0                   |
| 6   | 3   | -1                  | 1_2                 |
| 8   | 3   | -1_2                | 0                   |
| 10  | 5   | -2_2                | -1, -2_2            |
| 12  | 5   | -1_2, -4            | 0_2, -4_2           |
| 14  | 7   | 1, -3_2, -4         | -3_2, -4            |
| 16  | 7   | 1, -3_2, -4         | -3_2, -4            |
| 18  | 9   | -2, -4, -4_2        | 0_2, -4, -4_2       |
| 20  | 9   | -2_2, -4, -4_2      | 0, -4, -4_2         |
| 22  | 11  | -3_2, -4, -6_2      | -2, -3_2, -4, -6_2  |
| 24  | 11  | -3_2, -4, -6_2      | -2, -3_2, -4, -6_2  |

Table 7. $p = 3, N = 2$
References

[AL70] A. O. L. Atkin and J. Lehner. Hecke operators on $Γ_0(m)$. Math. Ann., 185:134–160, 1970.

[Bel12] Joël Bellaïche. Critical $p$-adic $L$-functions. Inventiones mathematicae, 189(1):1–60, Jul 2012.

[Ber17] J. Bergdall. Upper bounds for constant slope $p$-adic families of modular forms. ArXiv e-prints, August 2017.

[Buz05] Kevin Buzzard. Questions about slopes of modular forms. Astérisque, (298):1–15, 2005. Automorphic forms. I.

[Buz07] Kevin Buzzard. Eigenvarieties. In $L$-functions and Galois representations, volume 320 of London Math. Soc. Lecture Note Ser., pages 59–120. Cambridge Univ. Press, Cambridge, 2007.

[Che08] Gaëtan Chenevier. Quelques courbes de Hecke se plongent dans l’espace de Colmez. J. Number Theory, 128(8):2430–2449, 2008.

[Col96] Robert F. Coleman. Classical and overconvergent modular forms. Invent. Math., 124(1-3):215–241, 1996.

[Col97] Robert F. Coleman. $p$-adic Banach spaces and families of modular forms. Invent. Math., 127(3):417–479, 1997.

[Col10] Pierre Colmez. Invariants $L$ et dérivées de valeurs propres de Frobenius. Astérisque, (331):13–28, 2010.

[CST98] R. Coleman, G. Stevens, and J. Teitelbaum. Numerical experiments on families of $p$-adic modular forms. In Computational perspectives on number theory (Chicago, IL, 1995), volume 7 of AMS/IP Stud. Adv. Math., pages 143–158. Amer. Math. Soc., Providence, RI, 1998.

[GM93] F. Q. Gouvea and B. Mazur. On the characteristic power series of the $U$ operator. Ann. Inst. Fourier (Grenoble), 43(2):301–312, 1993.

[Gra17] P. M. Graef. A control theorem for $p$-adic automorphic forms and Teitelbaum’s $L$-invariant. ArXiv e-prints, May 2017.

[GS93] Ralph Greenberg and Glenn Stevens. $p$-adic $L$-functions and $p$-adic periods of modular forms. Invent. Math., 111(2):407–447, 1993.

[Lau11] Alan G. B. Lauder. Computations with classical and $p$-adic modular forms. LMS J. Comput. Math., 14:214–231, 2011.

[Mar18] Kimball Martin. Refined dimensions of cusp forms, and equidistribution and bias of signs. Journal of Number Theory, 2018.

[MTT86] B. Mazur, J. Tate, and J. Teitelbaum. On $p$-adic analogues of the conjectures of Birch and Swinnerton-Dyer. Invent. Math., 84(1):1–48, 1986.

[Tei90] Jeremy T. Teitelbaum. Values of $p$-adic $L$-functions and a $p$-adic Poisson kernel. Invent. Math., 101(2):395–410, 1990.

[Von15] Jan Vonk. Computing overconvergent forms for small primes. LMS J. Comput. Math., 18(1):250–257, 2015.

Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany
E-mail address: samuele.anni@gmail.com

Universität Heidelberg, IWR, Im Neuenheimer Feld 205, 69120 Heidelberg, Germany
E-mail address: gebhard.boeckle@iwr.uni-heidelberg.de
E-mail address: peter.grae@iwr.uni-heidelberg.de
E-mail address: troya@stud.uni-heidelberg.de