Feynman Rules in terms of the Wigner transformed Green functions

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In the models defined on the inhomogeneous background the propagators depend on the two space-time momenta rather than on one momentum as in the homogeneous systems. Therefore, the conventional Feynman diagrams contain extra integrations over momenta, which complicate calculations. We propose to express all amplitudes through the Wigner transformed propagators. This approach allows us to reduce the number of integrations. As a price for this the ordinary products of functions are replaced by the Moyal products. The corresponding rules of the diagram technique are formulated using an example of the model with the fermions interacting via an exchange by scalar bosons. The extension of these rules to the other models is straightforward. This approach may simplify calculations in certain particular cases. The most evident one is the calculation of various non-dissipative currents.

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I. INTRODUCTION

External fields provide the nontrivial inhomogeneous background to the field theoretical models that describe various physical systems both in the high energy physics and in the condensed matter physics. Strong magnetic fields appear in the description of the early universe [1], of the neutron stars [2] and can be created in heavy-ion collision experiments [3]. In condensed matter physics magnetic fields cause a lot of interesting phenomena. Possibly, the most popular one is the quantum Hall effect (QHE), which includes the mysterious fractional quantum Hall effect (FQHE). In quantum Hall systems, the Hall resistivity $R_H$ as a function of magnetic field $B$ possesses the plateaus in the presence of interactions between the electrons, and impurities [4, 5]. Elastic deformations in materials is another kind of external field experienced by the charged carriers, and may also change the behavior of the electric transport [6–9].

In the presence of external fields, the translational invariance is broken. Therefore, the two-point Green functions $G(x_1, x_2)$ can not be expressed in the form of the function of $(x_1 - x_2)$, and after the Fourier transformation the Green function depends on the initial and the final momenta, that are not equal in general case. An alternative to the Fourier transformation is Wigner transformation [10–12]. The Wigner-transformed Green function $G_W(R, p)$ has certain advantages compared to the ordinary Fourier transform $G(p_1, p_2)$. Expressions in terms of $G_W(R, p)$ are more concise. We will see below, that the corresponding Feynmann diagram technique contains the same amount of integrations over momenta as in the homogeneous theory. The price for this is the appearance of the Moyal products instead of the ordinary multiplications. Sometimes the resulting expressions for the physical quantities are more useful than those that are obtained using the conventional Feynmann diagrams. An example is given by the Hall conductivity, which is expressed through the Wigner transformed Green functions using the Moyal products. The corresponding expression is the topological invariant in phase space, i.e. its value is not changed under the smooth modification of the system (see [15]). The similar (but simpler) constructions were used earlier to consider the intrinsic anomalous Hall effect and chiral magnetic effect [13]. It has been shown that the corresponding currents are proportional to the topological invariants in momentum space. This method allows to reproduce the conventional expressions for Hall conductivity [14], and to prove the absence of the equilibrium chiral magnetic effect. Recently, the electron-electron interactions have been taken into account, and it has been shown to all orders in perturbation theory that the Hall conductivity (averaged over the system area) is proportional to the same topological invariant, as in the presence of interactions [16, 17]. This proof works equally well for the homogeneous systems with intrinsic anomalous Hall effect, and the non-homogeneous systems with quantum Hall effect in the presence of varying magnetic field, and varying electric potential of impurities.

In the theories with interactions, Feynman diagrams and Feynman rules are necessary to evaluate various Green functions [18, 19], and various physical amplitudes. Each particular Feynman diagram $\mathcal{F}(X^{(f)}, X^{(i)})$ with $2N$ external

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fermion lines depends on the set of input coordinates \( X^{(i)} = \{ x_a^{(i)} | a = 1, ..., N \} \) and output coordinates \( X^{(f)} = \{ x_a^{(f)} | a = 1, ..., N \} \). The fermion lines pass through the whole diagram, and connect the input/output coordinates into pairs \( X_a = (x_a^{(i)}, x_a^{(f)}) \) with \( a = 1, ..., N \). If we denote the Wigner-transformed function of \( F(X^{(f)}|X^{(i)}) \) by \( F(R|P) \), where \( R = \{ r_a = (x_a^{(f)} + x_a^{(i)})/2 | a = 1, ..., N \} \) and \( P = \{ p_a | a = 1, ..., N \} \), while \( p_a \) are the conjugate momenta with respect to \( x_a^{(f)} - x_a^{(i)} \), then the relation between \( F(X^{(f)}|X^{(i)}) \) and \( F(R|P) \) is

\[
F(X^{(f)}|X^{(i)}) = \int \frac{d^{ND}P}{(2\pi)^D} e^{\sum_a p_a(x_a^{(f)} - x_a^{(i)})} F_W(R|P),
\]

where \( D \) is the dimension of space - time. The physical Green function \( G(x_a^{(f)}|x_a^{(i)}) \) corresponding to \( N \) incoming fermions, and \( N \) outgoing fermions may be expressed through the Feynman diagrams as the sum over permutations \( \Pi \) of the sequence \( \{ a = 1, ..., N \} \):

\[
G(X^{(f)}|X^{(i)}) = \sum_{\Pi} (-1)^{\Pi(\Pi)} F(X^{(f)}_{\Pi}|X^{(i)}) = \sum_{\Pi} (-1)^{\Pi(\Pi)} \int \frac{d^{ND}P}{(2\pi)^D} e^{-i\sum_a p_a(x_{\Pi(a)}^{(f)} - x_{\Pi(a)}^{(i)})} F_W((X_f^{(f)} + X^{(i)})/2|P).
\]

Here \( (-1)^{\Pi(\Pi)} \) is the parity of the permutation \( \Pi \). \( X_f^{(f)}_{\Pi} \) is the set of permutated output coordinates \( \{ x_{\Pi(a)}^{(f)} | a = 1, ..., N \} \) according to the permutation \( \Pi \). In the present paper we construct the diagram technique for the calculation of the Wigner transformed Feynmann diagrams \( F_W(R|P) \). This technique expresses these quantities through the Wigner transformed bare propagators. For definiteness we consider the model of one Dirac fermion interacting minimally with the scalar field. The inhomogeneity is introduced to the system through the external Abelian gauge fields that interact with the scalar field and the fermion. Those fields may depend arbitrarily on coordinates. Correspondingly, each propagator depends on two momenta rather than one one momentum (as in the case of the homogeneous system). The generalization of our construction to the more general case is straightforward.

The paper is organized as follows: In Sect. 2 the lagrangian of the model under consideration is introduced and certain basic expressions for the Wigner transformed Green functions are presented. In Sect. 3, the Feynman rules are obtained for the diagrams, which contain zero or two external fermion lines. In Sect. 4, the Feynman rules are obtained for the diagrams, which contain two external fermion lines and several internal fermion loops. In Sect. 5, we discuss the extension of our construction to the diagrams with more than two external fermion lines. In Sect. 6 we end with the conclusions.

## II. WIGNER TRANSFORM AND THE MODEL UNDER CONSIDERATION

Wigner transform is also known as Weyl transform [20]. Originally, it was closely intertwined with the development of phase-space formulation of quantum mechanics (also called "deformation quantization") in 1930 - s (see [21, 22] and references therein). Compared with the operator formulation of quantum mechanics, the phase-space formulation deals with the ordinary functions of coordinates and momenta: the so-called Wigner function \( W(q,p) \), which is considered as a quantum counterpart of classical distribution in phase space \( (q,p) \) [23]. In addition to its applications in quantum mechanics [24, 27], the Wigner transform has been extended to quantum field theory and has been adopted to the investigation of the high energy physics and the many-body effects, including various anomalous transport phenomena such as the quantum Hall effect, the chiral separation effect, and the chiral vortical effect [28, 33].

In this section we will go along this line to introduce the Wigner transformation of the Green function. Our starting point is the following lagrangian for the Dirac fermion interacting with the scalar field (in the presence of the inhomogeneous background given by the external hauge fields):

\[
\mathcal{L} = \bar{\psi}((i\partial_\mu - A_\mu)\gamma^\mu - m)\psi + (i\partial_\mu - B_\mu)\phi(i\partial^\mu - B^\mu)\phi - m_\phi^2\phi^2 - g\bar{\psi}\psi\phi
\]

where \( A_\mu \) and \( B_\mu \) are the vector potentials of the external fields, which can be different one from another.

For the fermions the two-point Green function \( G \) satisfies equation \( \hat{Q}(x_1)G(x_1, x_2) = \delta(x_1 - x_2) \), where \( \hat{Q}(x) = (i\partial_\mu - A_\mu(x))\gamma^\mu - m \). The Wigner transformation of \( G \) is defined as

\[
G_W(R, p) = \int dr G(R + r/2, R - r/2)e^{-ipr}.
\]

It satisfies the Groenewold equation \( Q_W(R, p) \star G_W(R, p) = 1 [13, 34] \), where \( Q_W \) is the Weyl symbol of operator \( \hat{Q} \), while \( \star = e^{i(\sqrt{R} \pi \overleftarrow{p} - \sqrt{R} \pi \overrightarrow{p})/2} \) is the Moyal product.
Similarly for the bosonic field \( \phi \) operator \( \hat{U}(x) = (i\partial_x - B_\mu(x))(i\partial^\mu - B^\mu(x)) - m_\phi^2 \) is the inverse bare propagator, and the Wigner - transformed Green function \( D_W \) satisfies \( U_W(R, p) \star D_W(R, p) = 1 \). An important result of Wigner - Weyl calculus is \( C(x_1, x_2) = \int A(x_1, y)B(y, x_2)dy \Rightarrow C_W(R, p) = A_W(R, p) \star B_W(R, p) \).

This result and its consequences will be used frequently in the further text.

### III. FEYNMAN RULES FOR THE SELF ENERGY AND THE FERMION BUBBLES

In this section we construct the Feynman rules for those diagrams corresponding to the fermion self energy \( \langle G^0 \rangle \), in which there are no internal fermion loops. We use the Wigner-transformed bosonic propagators \( D^{(j)} \) and fermion propagators \( G_a \). Here indices \( j \) and \( a \) enumerate the boson and the fermion lines correspondingly entering the Feynman diagram. Before the formulation of the Feynman rules, we introduce several auxiliary mathematical results.

1. Our first auxiliary formula is as follows:

\[
C(x_1, x_2) = \int A(x_1, y)H(y)B(y, x_2)dy \Rightarrow C_W(R, p) = A_W(R, p) \star H(R) \star B(R, p)
\]  

This result may be proven directly using Eq. (3). Notice, that the Moyal product is associative, i.e.

\[
\left( A(R, p) \star A(R, p) \right) \star C(R, p) = A(R, p) \star \left( A(R, p) \star C(R, p) \right)
\]

This allows to omit the brackets in Eq. (4).

2. The application of the Moyal product gives rise to the following expressions

\[
e^{ikR} \star G_W(R, p) = e^{ikR}G_W(R, p - k/2)
\]

\[
G_W(R, p) \star e^{ikR} = e^{ikR}G_W(R, p + k/2)
\]

and

\[
(A(R, p)e^{ikR}) \star B(R, p) = [A(R, p) \star B(R, p - k/2)]e^{ikR}
\]

\[
A(R, p) \star (e^{ikR}B(R, p)) = [A(R, p + k/2) \star B(R, p)]e^{ikR}
\]

3. The above results give rise to the following Lemma

\[
G_1(R, p) \star e^{ik_1R} \star G_2(R, p) \star \ldots \star e^{ik_nR} \star G_{n+1}(R, p)
\]

\[
def = \frac{1}{G_1(R, p) \prod_{i=1}^{n} \star (e^{ik_iR} \star G_{i+1}(R, p))}
\]

\[
= \left[ \prod_{i=1}^{n} \star G_i(R, p + p_i/2) \right] e^{i \sum_{j=1}^{n} kj_R},
\]

where \( p_m = -\sum_{j=1}^{m-1} k_j + \sum_{j=m}^{n} k_j \).

In the Feynmann diagrams of the theory with the lagrangian of Eq. (1) the two-point Green functions (propagators) are typically represented by the solid lines (fermions), and the dashed lines (bosons). After the Fourier transformation the bosonic propagator \( \hat{D}(k_a, k_b) \) in the presence of external field becomes the function of the two momenta, and the two exponential factors appear. Let us take \( \hat{D}^{(j)}(k_{ia}, k_{jb}) \) in Fig. 1 as an example. The propagator \( \hat{D}^{(j)} \) at its left end produces the factor \( e^{ik_{ja}R} \), and at the right end the factor \( e^{ik_{jb}R} \), after the Fourier transform.

The positions of the fermionic propagator \( G_m \) with respect to the given dash (the given bosonic propagator) are divided into three cases:

1. Both ends of the dashed line are right to \( G_m \).
FIG. 1: The schematic representation of the diagrams of the fermionic self-energy without internal fermion loops. The solid line represents the fermion while the dashed line represents the scalar. $G_i$ and $D^{(j)}$ are the fermionic and the bosonic Green functions respectively. Dots stand for the additional ejections and absorbtions of the scalar by the fermion (those that are not shown explicitly).

2. Both ends of the dashed line are left to $G_m$.

3. One end of the dashed line is on the left from $G_m$, and the other end of the dashed line is right to $G_m$.

Within the expression for the Wigner transformed self energy the influence of $D^{(j)}$ on various fermion propagators (better to say - on their Wigner transformations) $G_m$ ($1 \leq m \leq n$) is as follows

$$G_1(R, p + q_j/2) \ast ... \ast G_s(R, p + q_j/2) \ast G_{s+1}(R, p - k_j) \ast ... \ast G_{t-1}(R, p - k_j) \ast G_t(R, p - q_j/2) \ast ... \ast G_n(R, p - q_j/2)$$

(12)

where $q_j = k_{ja} - k_{jb}$, and $k_j = (k_{ja} + k_{jb})/2$. Here symbols of the Wigner transformation are omitted for brevity. The corresponding expression for the Wigner transformation of the given self energy diagram becomes (we represent here only one bosonic propagator, and its influence on the Wigner transformed fermionic Green functions, the exponential factors coming from the other bosonic propagators are hidden inside the dots):

$$\int \int [G_1(R, p) \ast G_s(R, p) \circ_j \ast G_{s+1}(R, p - k_j) \ast ... \ast G_{t-1}(R, p - k_j) \ast j \circ G_t(R, p) \ast ... G_n(R, p)] D^{(j)}(R, k_j) dk_1 ... dk_j.$$  

(13)

where $\circ_j = e^{-i \hat{\mathcal{D}}_p \theta_R^{(j)}/2}$ and $j \circ = e^{i \hat{\mathcal{D}}_p \theta_R^{(j)} /2}$. $\hat{\mathcal{D}}_p$ acts on $D^{(j)}$ only. The right derivative $\hat{\mathcal{D}}_p$ acts on all propagators standing right to the symbol $\circ_j$. The left derivative $\hat{\mathcal{D}}_p$ acts on all propagators standing left to the symbol $\circ_j$. In order to better understand the computation rules, let us consider the specific example, which is shown in Fig. 2. The solid line is separated into 5 segments by the two dashes. The second and the third segments are in ”parallel” with $D^{(1)}$, therefore, their momentum variables include $k_1$. Operators $\circ_1$ and $j \circ$ are inserted before and after these segments. Similarly, the propagator $D^{(2)}$ affects the third and the fourth segments. Finally, the corresponding expression of the Feynmann diagram of Fig. 2 is given by

$$\int \int [G_1(R, p) \circ_1 \ast G_2(R, p - k_1) \circ_2 \ast G_3(R, p - k_1 - k_2) \ast 1 \circ G_4(R, p - k_2) \ast 2 \circ G_5(R, p)] D^{(1)}_W(R, k_1) D^{(2)}_W(R, k_2) dk_1 dk_2.$$  

(14)

FIG. 2: An example of the two-loop Feynmann diagram for the self energy.
The fermionic bubbles like those presented in Figs. 3 do not enter the expressions for the physical scattering amplitudes. However, they enter expressions for the thermodynamical potentials, and, moreover, are used for the proof that the Hall conductivity is not affected by weak interactions (see, e.g. [17]).

Below we represent the Feynman diagrams given on Fig. 3 in terms of the Wigner transformed propagators. The bubble (a) corresponds to expression
\[ \frac{1}{2} \int Tr(G_W(R, p - k) \star_1 G_W(R, p)) D_W^{(1)}(R, R) dk. \]  

Because of the trace, it also can be rewritten as
\[ \frac{1}{2} \int Tr(G_W(R, p) \circ_1 G_W(R, p - k)) D_W^{(1)}(R, R) dk, \]
equivalently. As for bubble (b) the corresponding formula is
\[ \frac{1}{4} \int Tr(G_W(R, p - k_1) \circ_2 G_W(R, p - k_1 - k_2) \star_1 G_W(R, p - k_2) \star_2 G_W(R, p)) D_W^{(1)}(R, k_1) D_W^{(2)}(R, k_2) dk_1 dk_2. \]

FIG. 3: Fermionic bubbles.

We conclude this section with the formulation of the rules for the calculation of the Feynman diagrams with two external fermion lines (the self energy), and without external fermion lines (the fermion bubbles). In both cases there should be no extra fermion loops.

1. Label momenta, \( p, p - k_1, \ldots \) within the graph, according to the "law of momentum conservation". This "law" means that we write down the momenta that would take place in the same diagram of the homogeneous theory. The combinatorial symmetry factor is to be added to each diagram. This factor is also identical to the one of the same diagram of the homogeneous theory.

2. Write down the series \( G_W(R, p) \star G_W(R, p - k_1) \star \ldots \) along the fermion line, according to the labelled momenta of the graph. The inhomogeneity of the theory is now encoded in the dependence on \( R \) while momenta entering these expressions are conserved as if we would deal with the homogeneous theory.

3. Insert \( \circ \) and \( \circ \) to the series at the starting and ending points of \( D^{(j)}(R, k_j) \), according to the Feynman diagram. For the case of the bubble, the trace is introduced, and the first \( \circ \) operator is omitted.

IV. THE CASES WHEN THE INTERNAL FERMION LOOP IS PRESENT

In the previous section, the Feynman rules have been obtained for the diagrams corresponding to the two-point Green functions (and the fermion bubbles), in which only one fermion line is present. (This line passes through the whole diagram in the case of the fermion self-energy, and is closed to form the loop in the case of the fermion bubble.) In the present section, we consider the diagrams for the self-energy/fermion bubbles that include additional fermion loops. An example of such a diagram is presented in Fig. 4. The corresponding Feynman diagram may be evaluated as follows:

\[
\mathcal{F}(x_1|x_2) = \int G(x_1, y_1)G(y_1, y_2)G(y_2, y_3)G(y_3, x_2) Tr\{G(y_4, y_5)G(y_5, y_6)G(y_6, y_4)\} \\
\quad D(y_1, y_4) D(y_2, y_5) D(y_3, y_6) dy_1 \ldots dy_6
\]  

(18)
Using relation $D(x, y) = \int e^{ikx} \tilde{D}(k_a, k_b)e^{-iky}dk$, and applying Wigner transformation, we obtain

$$
\mathcal{F}_W(R_1|p_1) = \int G_W(R_1, p_1 + \frac{k_{1a}}{2} + \frac{k_{2a}}{2} + \frac{k_{3a}}{2}) * G_W(R_1, p_1 - \frac{k_{1a}}{2} - \frac{k_{2a}}{2} - \frac{k_{3a}}{2}) * 
G_W(R_1, p_1 - \frac{k_{1b}}{2} - \frac{k_{2b}}{2} - \frac{k_{3b}}{2}) * G_W(R_1, p_1 - \frac{k_{1b}}{2} - \frac{k_{2b}}{2} - \frac{k_{3b}}{2})
$$

\[ Tr[GW(R_2, p_2 - \frac{k_{1b}}{2} - \frac{k_{2b}}{2} - \frac{k_{3b}}{2}) * GW(R_2, p_2 + \frac{k_{1b}}{2} + \frac{k_{2b}}{2} + \frac{k_{3b}}{2})] \]

$$
e^{i(k_{1a}R_1 + k_{1b}R_2)e^{-ik_{2a}R_1} + i(k_{2a}R_1 + k_{2b}R_2)e^{-ik_{3a}R_1} + i(k_{3a}R_1 + k_{3b}R_2)}
$$

$$
dk_{1a}dk_{2a}dk_{3a}dk_{1b}dk_{2b}dk_{3b}dR_2dp_2
$$

This expression can be rewritten as follows

$$
\mathcal{F}_W(R_1|p_1) = G_W(R_1, p_1) * (1, 1) G_W(R_1, p_1) * (2, 1) G_W(R_1, p_1) * (3, 1) G_W(R_1, p_1)
$$

\[ \int Tr[(1, 2) G_W(R_2, p_2) * (2, 2) G_W(R_2, p_2) * (3, 2) G_W(R_2, p_2)] \]

\[ D(1)(R_1, R_2)D(2)(R_1, R_2)D(3)(R_1, R_2)dp_2 \]

(19)

where $^{(i,j)}_o = \exp(\frac{1}{2} (\partial^{(i)}_{p_i} \overrightarrow{\partial}^{(j)}_{p_j} - \overrightarrow{\partial}^{(j)}_{p_j} \partial^{(i)}_{p_i}))$, in which $\partial^{(i)}_{p_i}$ acts on $D^{(i)}(R_1, R_2)$ only. The right derivative $\overrightarrow{\partial}^{(j)}_{p_j}$ acts on all fermion propagators standing right to the symbol $^{(i,j)}_o$. The left derivative $\overleftarrow{\partial}^{(i)}_{p_i}$ acts on all propagators standing left to the symbol $^{(i,j)}_o$.

\[ (v,i) = \exp(\frac{1}{2} (\overleftarrow{\partial}^{(i)}_{p_i} \overrightarrow{\partial}^{(j)}_{p_j} - \partial^{(j)}_{p_j} \overleftarrow{\partial}^{(i)}_{p_i})) \]

\[ D(1)(R_1, R_2)D(1)(R_1, R_2)D(2)(R_1, R_2)D(3)(R_1, R_2)dp_2 \]

(20)

This example may be easily extended to the diagram of general type with two or zero external fermion lines, and any number of internal fermion loops. The resulting rules of the diagram technique are as follows.

1. **Fermi skeleton.**
   For each fermion line $L_i$ (either closed or open), one needs a spatial coordinate $R_i$, and momenta $p_a$. Write down the series $G \star G \ldots$ according to the rules presented in the previous section (there instead of $R$ we insert $R_i$).

2. **Dashed lines connecting points that belong to the same fermion line.**
   The dashed lines, that start and end at the same fermion line result in the same operators $\phi^j$ and $\lambda^\circ$ as in the previous section. The dashed lines connecting different fermion lines are omitted in this step.

3. **Dashed lines connecting distinct fermion lines.**
   For the boson propagators, whose ends belong to different fermion lines (denote those fermion line $L_i$ and $L_j$), we use the boson propagator $D(R_i, R_j)$ in coordinate space rather than the Wigner transformed propagator $D_W$. Then the circle operators $^{(i,j)}_\circ$ are inserted to the series $G \star G \ldots$ at the positions of the ejection/absorption of the dashed line connecting $L_i$ and $L_j$. If necessary, $D(R_i, R_j)$ may be expressed through $D_W$.

**V. DIAGRAMS WITH MORE THAN TWO LEGS**

Up to now, we only considered the two-point fermionic Green functions, which correspond to the Feynman diagrams with two legs formed by one fermion line, and the fermion bubbles without external legs. Our consideration may
easily be generalized to the case of an arbitrary number of external fermion lines. As an illustration let us consider the simple example shown in Fig. 5. In coordinate space the corresponding Feynman diagram is

\[ F(x_1, x_1', x_2, x_2') = \int G(x_1, y_1) G(y_1, x_1') D(y_1, y_2) G(x_2, y_2) G(y_2, x_2') dy_1 dy_2. \]  

(21)

It has been mentioned in the Introduction, that we define the Wigner transformation of such a diagram that corresponds to the pairs \((x_1, x_1') \rightarrow (R_1, p_1), (x_2, x_2') \rightarrow (R_2, p_2)\). After some tedious algebra, one can get the final result for \(F_W(R_1, R_2 | p_1, p_2)\). Extending the diagram technique of the previous section to this case we are able to write the corresponding formula directly:

\[ F_W(R_1, R_2 | p_1, p_2) = [G_W(R_1, p_1) \circ (1, 1) G_W(R_1, p_1)] [G_W(R_2, p_2) \circ (1, 2) G_W(R_2, p_2)] D^{(1)}(R_1, R_2) \]  

(22)

FIG. 5: The simplest diagram with four external fermion lines.

Another example, which is more complicated, is shown in Fig. 6. The corresponding Wigner-transformed expression is

\[ F_W(R_1, R_2 | p_1, p_2) = \int dk [G_W(R_1, p_1) \circ (1, 1) G_W(R_1, p_1) \circ G_W(R_2, p_2) \circ (3, 2) G_W(R_2, p_2)] D^{(1)}(R_1, R_2) D^{(2)}(R_1, k) D^{(3)}(R_1, R_2) \]  

(23)

FIG. 6: The example of the diagram with four external fermion lines.

VI. CONCLUSIONS

In the present paper we constructed the diagram technique for the non-homogeneous model with Dirac fermion interacting with the scalar field. Both types of excitations propagate in the presence of the inhomogeneous background provided by the external gauge field. We consider arbitrary Green functions in this model corresponding to the external fermion legs. The obtained construction may easily be generalized to the case of the other models existing in the inhomogeneous background. The essential feature of our technique is that all diagrams are expressed through the Wigner transformed propagators. Those depend on both coordinates \(R\) and momenta \(p\). The main advantage of the proposed construction is that the momenta in the diagrams are "conserved", i.e. in each vertex the same rules are
valid as in the homogeneous theory: the sum of the incoming and outgoing momenta is equal to zero. Each propagator carries one momentum. This reveals the one-to-one correspondence with the corresponding homogeneous model. The inhomogeneity of the theory is encoded in the dependence of the propagators on $R$, and in the replacement of ordinary products by the Moyal products of Wigner - Weyl calculus. This construction has been used in our two previous papers [17] and [40]. In those papers the particular case of the diagram technique of the present paper has been developed for the particular cases of the diagrams. It allows to prove the non-renormalization of Hall conductivity by weak interactions, and the non-renormalization of total persistent current by weak interactions for the model of massive fermions. Here we generalize the constructions of [17] and [40] to arbitrary forms of the diagrams.

The other possible applications of our technique may exist to the physics of high-energy heavy-ion collisions, when the two atomic nuclei collide at relativistic energies and generate strong electromagnetic fields [38]. If the collision energy surpasses a certain threshold, the collision produces quark-gluon plasma (see [39] and references therein). The behavior of the charged particles is affected by those strong electromagnetic fields. One can say, that various excitations move in the inhomogeneous background given by the mentioned electromagnetic fields. Various non-dissipative transport effects are expected to be observed in the heavy-ion collisions. Those are, for example, the chiral magnetic effect, the chiral separation effect, the chiral vortical effect. In the presence of the homogeneous magnetic field/homogeneous rotation angular velocity the corresponding conductivities are given by the topological invariants in momentum space (see [39, 41–47]). It is expected that in the presence of the inhomogeneous background (given by the inhomogeneous magnetic field, inhomogeneous rotation, etc) the corresponding conductivities will be given by the topological invariants in phase space composed of the Wigner transformed Green functions. This expectation is based on the recent extension of the topological representation for the Hall conductivity to the case of inhomogeneous magnetic field [17]. The representation of the non-dissipative conductivities in terms of the Wigner transformed propagators may repeat the corresponding homogeneous constructions with the ordinary product replaced by the Moyal product. The topological nature of the quantities simplifies the use of the Moyal product and may allow us to consider relatively easily the interaction corrections to those effects (as in [17] and [40]). Possible applications of the proposed technique may be found also in various problems of condensed matter physics, where the inhomogeneity may be caused by many factors including the elastic deformations. Here as well the consideration of various non-dissipative transport phenomena (fractional quantum Hall effect, spin Hall effect, etc) may benefit from the use of our diagram technique.

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