Functional calculus on Venturi for groups with finite propagation speed

Gordon Blower
g.blower@lancaster.ac.uk
Corresponding author:
Department of Mathematics and Statistics,
Lancaster University,
Lancaster LA1 4YF, United Kingdom

Ian Doust
i.doust@unsw.edu.au
School of Mathematics and Statistics,
University of New South Wales,
Sydney, NSW 2052, Australia

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ABSTRACT Let $\mathcal{M}$ be a complete Riemannian manifold with Ricci curvature bounded below and Laplace operator $\Delta$. The paper develops a functional calculus for the cosine family $\cos(t\sqrt{\Delta})$ which is associated with waves that travel at unit speed. If $f$ is holomorphic on a Venturi shaped region, and $z^k f(z)$ is bounded for some positive integer $k$, then $f(\sqrt{\Delta})$ defines a bounded linear operator on $L^p(\mathcal{M})$ for some $p > 2$. For Jacobi hypergroups with invariant measure $m$ the generalized Fourier transform of $f \in L^1(m)$ gives $\hat{f} \in H^\infty(\Sigma_\omega)$ for some strip $\Sigma_\omega$. Hence one defines $\hat{f}(A)$ for operators $A$ in some Banach space that have a $H^\infty(\Sigma_\omega)$ functional calculus. The paper introduces an operational calculus for the Mehler–Fock transform of order zero. By transference methods, one defines $\hat{f}(A)$ when $\hat{f}$ is a $s$-Marcinkiewicz multiplier and $e^{itA}$ is a strongly continuous operator group on a $L^p$ space for $|1/2 - 1/p| < 1/s$.

1. Introduction

This paper develops a functional calculus for groups generated by differential operators which have finite propagation speed, in the following sense. Let $\mathcal{M}$ be a complete Riemannian manifold of dimension $n$ with volume measure $\mu$ and metric $\rho$, and suppose that $\mathcal{M}$ has Ricci curvature bounded below by $-\kappa(n-1)$ so that a geodesic ball $B(x_0; r) = \{x \in \mathcal{M} : \rho(x, x_0) \leq r\}$ has volume $m(r) = \mu(B(x_0; r))$ such that $m(r) \leq C e^{\kappa r}$
for some $C, \kappa' > 0$ and all $r > 0$ and $x_0 \in \mathcal{M}$; see [8]. Let $\Delta$ be the Laplace–Beltrami operator, so that $\Delta$ is essentially self-adjoint on $C_c^\infty(\mathcal{M})$ in $L^2(\mathcal{M}; \mu)$ by Chernoff’s theorem [10], and the cosine family $\cos(\xi \sqrt{\Delta})$ gives a bounded and strongly continuous family of operators on $L^2(\mathcal{M}; \mu)$. In cases of interest, the spectrum of $\Delta$ as an operator in $L^2(\mathcal{M}; \mu)$ will be contained in $[\omega_0^2, \infty)$ for some $0 < \omega_0 \leq \kappa'/2$. The spectral theorem then allows us to define first $\sqrt{\Delta}$ as a positive operator and then $f(\sqrt{\Delta})$ as a bounded linear operator on $L^2(\mathcal{M}; \mu)$ for continuous and bounded complex functions $f$ on the $L^2$ spectrum. It is of interest to determine classes of complex functions $f$ such that $f(\sqrt{\Delta})$ also defines a bounded linear operator on $L^\nu(\mathcal{M}; \mu)$. We are particularly concerned with the functional calculus associated with the fundamental formula

$$f(\sqrt{\Delta}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Ff(\xi) \cos(\xi \sqrt{\Delta}) \, d\xi$$

which is defined for even functions $f \in C_c^\infty(\mathbb{R})$ in terms of the Fourier transform $Ff(\xi) = \int_{-\infty}^{\infty} f(x) e^{-ix\xi} \, dx$. Then $\cos(\xi \sqrt{\Delta}) \delta_{x_0}$ gives a wave emanating from $x_0 \in \mathcal{M}$ that travels at unit speed across $\mathcal{M}$ with respect to the geodesic distance as in [9] page 297. This enables us to use the notion of finite propagation speed to deal with the exponential growth of the volume of geodesic balls and thus obtain a functional calculus suitable for this context.

On $L^\nu(\mathcal{M})$, the spectrum of $\Delta$ will typically depend upon $\nu$ [28], and the cosine family $\cos(\xi \sqrt{\Delta})$ does not generally give a family of bounded operators on $L^\nu(\mathcal{M})$ for $2 < \nu < \infty$. To address these issues, we introduce a family of Riesz–Liouville fractional integration operators $W_\alpha$ and work with

$$W_\alpha(\cos t \sqrt{\Delta}) = \frac{1}{\Gamma(\alpha)} \int_0^t (\cosh t - \cosh s)^{\alpha-1} \sinh s \cos(s \sqrt{\Delta}) \, ds \quad (\alpha, t > 0),$$

and develop a functional calculus which can accommodate fractional integrals are of exponential growth. Thus we pursue ideas used in [14] in the case of hyperbolic space.

The key idea is to work with holomorphic functions on Venturi regions. For $0 < \theta < \pi$, we introduce the open sector $S_\theta^\circ = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \theta\}$ and its reflection $-S_\theta^\circ = \{z : -z \in S_\theta^\circ\}$. For $\omega > 0$ we let $\Sigma_\omega$ denote the strip $\{z : |z| < \omega\}$. Then

$$V_{\theta,\omega} = \Sigma_\omega \cup S_\theta^\circ \cup (-S_\theta^\circ)$$

defines a Venturi region. Note that for all $s > 0$, the dilatation $z \mapsto sz$ takes $V_{\phi,\omega} \rightarrow V_{\phi,s\omega}$. For any domain $\Omega$, let $H^\infty(\Omega)$ be the space of bounded analytic functions $f : \Omega \rightarrow \mathbb{C}$ with the norm $\|f\|_\infty = \sup_{z \in \Omega} |f(z)|$; we also use the topology $\tau$ associated with uniform convergence on compact subsets. Furthermore, for $k \geq 0$, let $H^\infty,k(\Omega)$ be the subspace
\( \{ z \in H^\infty(\Omega) : z^k f(z) \in H^\infty(\Omega) \} \) with the norm \( \| f \|_{\infty,k} = \sup_{z \in \Omega} |f(z)| + \sup_{z \in \Omega} |z^k f(z)| \). When \( \Omega \) is symmetrical with respect to reflection \( z \mapsto -z \), the subscript \( e \) will be used to denote the subspace of even functions, as in \( H^\infty_e(\Omega) \).

**Definition 1.1.** (Functional calculus) Throughout this paper, \( E \) denotes a complex and separable Banach space and \( B(E) \) the space of bounded linear operators on \( E \).

(i) A functional calculus consists of a locally convex topological algebra \( A \) with the Mackey topology \( \tau \) and a homomorphism \( T : A \to B(E) \) that is continuous from \( \tau \) to the strong operator topology.

(ii) The functional calculus is associated with the region \( \Omega \) if the formal inclusion map \( \iota : A \to H^\infty(\Omega) \) is injective and continuous from \( \tau \) to the topology of uniform convergence on compact subsets of \( \Omega \).

A densely defined and closed linear operator \( A \) in \( E \) is said to have an \( H^\infty \) functional calculus on \( \Omega \) if there is a bounded linear homomorphism \( T : H^\infty(\Omega) \to B(E) \) given by \( Tf = f(A) \); see [13]. Our main results relate to the case of \( \Omega = V_{\theta,\omega} \) for some \( 0 < \theta < \pi/2 \) and \( \omega > 0 \) which have the following property.

(iii) The formal inclusion map \( \iota_k : H^\infty_{e,k}(V_{\theta,\omega}) \to A \) is continuous from the norm topology to \( \tau \) for some \( k \in \mathbb{N} \).

In section 2 we consider a commutative convolution operation \( * \) on \( L^1(m) \) and a generalized Fourier transform \( \hat{f}(\lambda) \) for \( f \in L^1(m) \) with the following property. The space of bounded characters on \( (L^1(m), *) \) is parameterized by \( \lambda \in \Sigma_\omega \), and \( A = \{ \hat{f} : f \in L^1(m) \} \) satisfies (ii) with \( \Omega = \Sigma_\omega \). In this context, (ii) is related to the Kunze–Stein phenomenon, as in Proposition 2.1 below. Let \( \gamma \geq 1/2 \) and \( \omega_0 \geq 0 \), and suppose that:

(i) \( m(x) = x^{2\gamma+1} q(x) \) where \( q \in C^2(\mathbb{R}) \) is even and positive, and such that \( m(x)/x^{2\gamma+1} \to q(0) > 0 \) as \( x \to 0^+ \);

(ii) \( m(x) \) increases to infinity as \( x \to \infty \), and \( m'(x)/m(x) \to 2\omega_0 \) as \( x \to \infty \);

(iii) \( Q(x) \) is positive and integrable over \((0, \infty)\), where

\[
Q(x) = \frac{1}{2} \left( \frac{q'}{q} \right)' + \frac{1}{4} \left( \frac{q'}{q} \right)^2 + \frac{2\gamma + 1}{2x} \left( \frac{q'}{q} \right) - \omega_0^2. \tag{1.4}
\]

Then we introduce the solutions of

\[
-\phi''_\lambda(x) - \frac{m'(x)}{m(x)} \phi'_\lambda(x) = (\lambda^2 + \omega_0^2) \phi_\lambda(x); \quad \phi_\lambda(0) = 1, \quad \phi'_\lambda(0) = 0. \tag{1.5}
\]

The functions \( \lambda \mapsto \phi_\lambda(x) \) are entire and belong to \( H^\infty(\Sigma_{\omega_0}) \). Moreover, a Laplace representation expresses \( \phi_\lambda(x) = \int_{[-x,x]} \cos(\lambda t) \tau_x(\lambda t) \, dt \) for some bounded positive measure \( \tau_x \).
on $[-x, x]$, as we review in Lemma 2.3. The generalized Fourier transform associated with the differential equation is

$$\hat{f}(\lambda) = \int_{0}^{\infty} \phi_{\lambda}(x) f(x) m(x) \, dx \quad (f \in L^1(m)), \quad (1.6)$$

and each $\hat{f}(\lambda)$ belongs to $H^\infty_{\Sigma}(\omega_0)$. Such expressions arise frequently in the classical theory of integral transforms, where $\phi_{\lambda}(x)$ is a standard special function; see [27].

Transference methods involve comparing the functional calculus for a general class of operator groups from the functional calculus associated with a particular group, see [11]. We develop a functional calculus $f \mapsto \hat{f}(A)$ with a suitable operator $A$ in a Banach space $E$, and develop an analogue of transference in which our model family of operators is $\cos(t\sqrt{L})$, where $L$ is the differential operator in $C^\infty_c((0, \infty))$ given by $Lf = -f'' - m' f'/m$ with $m$ as above.

Let $t \mapsto \cos(tA)\xi$ is continuous on $\mathbb{R}$ for all $\xi \in E$; then there exist $\kappa, \omega \geq 0$ such that $\| \cos(tA)\|_{B(E)} \leq \kappa \cosh(\omega t)$ for all $t \in \mathbb{R}$. Suppose that $0 < \omega_0$ and $\omega \leq \omega_0$. Then in section 2 we use the Laplace representation to define $\phi_{\lambda}(x) = \int_{[-x,x]} \cos(tA) \tau_x(dt)$, and hence define

$$T_A(f) = \hat{f}(A) = \int_{0}^{\infty} f(x) \phi_{A}(x) m(x) \, dx \quad (f \in L^1(m)). \quad (1.7)$$

Theorem 2.5 provides conditions under which $\hat{f}(A)$ is bounded, and thus gives an operational calculus $T_A : L^1(m) \to B(E)$ for various classical integral transforms.

In section 3, we introduce hypergroups on $[0, \infty)$ with specially chosen invariant measures. Let $m_{\alpha,\beta}(x) = 2^{2(\alpha+\beta+1)} \sinh^{2\alpha+1} x \cosh^{2\beta+1} x$, and let $L_{\alpha,\beta}$ be the differential operators

$$L_{\alpha,\beta} = -\frac{d^2}{dx^2} - \frac{m'_{\alpha,\beta}(x)}{m_{\alpha,\beta}(x)} \frac{d}{dx}, \quad (1.8)$$

which were considered by Trimèche and Chebli [29]. We are particularly interested in the cases where either $\beta = -1/2$ and $2\alpha + 1 \in \mathbb{N}$; or $\alpha = -1/2$ and $2\beta + 1 \in \mathbb{N}$; these were previously considered by Mehler and Jacobi [24]. We show that $\cos(t\sqrt{L_{\alpha,\beta}})$ on $L^p((0, \infty); m_{\alpha,\beta})$ is a model for general classes of cosine families ($\cos(tA)$) on suitable Banach spaces. In particular, we obtain a functional calculus related to the Mehler–Fock transform of order zero, as in [27] page 290. Gigante also considered a form of transference in this context [16].

Then we introduce geometrical examples. Let $\mathcal{M}$ be a complete Riemannian manifold of dimension $n$ with bounded geometry and Ricci curvature bounded below by $-(n-1)\kappa$. on
Fixing some \( q \in \mathcal{M} \), we can consider the Riemannian distance \( x = \rho(p, q) \) to a variable \( p \in \mathcal{M} \) as the radius in a system of polar coordinates, and seek to apply the results of section 3 to \( m(x) = \mu(B(p; x)) \). For hyperbolic space \( \mathcal{H}^n \), we can carry this out explicitly and (1.5) emerges as an associated Legendre equation. In this example, the volume satisfies 

\[
\mu(B(p; x))/e^{x(n-1)} \to c_n \quad \text{as} \quad x \to \infty
\]

for some \( c_n > 0 \), and \( \mathcal{H}^n \) acts as a model for more general manifolds in which the volume grows exponentially with the radius.

In section 5 we show that for \( \Delta \) the Laplace operator \( \mathcal{H}^{\nu+1} \) the operators \( W_{\nu/2}(\cos t\sqrt{\Delta}) \) are bounded on \( L^r \) and \( \|W_{\nu/2}(\cos t\sqrt{\Delta})\|_{B(L^r)} \leq \kappa(\nu, r) \sinh^\nu t \) for \( \nu/(\nu - 1) < r < \infty \).

In section 6, we use a transference theorem for Marcinkiewicz multipliers and unbounded strongly continuous groups of operators to obtain a functional calculus for Laplace operators \( \Delta \) on \( L^r \) spaces. The intended applications of this paper are to unbounded groups, since a wide range of transference results are already available for bounded groups.

**2. Necessary and sufficient conditions for a functional calculus**

Our aim in this section to to prove a converse of Haase’s form of the transference theorem for bounded groups of operators [18]. This is closely related to the Kunze–Stein phenomenon [11]. The close connection between specific examples and general theorems is evident from the special functions that appear in the proof of [18, Corollary 4.3] and the proof below.

**Proposition 2.1.** Let \( f \in L^1(\mathbb{R}; C; dx) \) be an even function with the following property.

If \( E \) any complex Banach space and \( \cos(tA) \) any strongly continuous cosine family on \( E \) such that \( \|\cos(At)\|_{B(E)} \leq Ke^{\omega_0|t|} \) for all \( t \in \mathbb{R} \), for some \( K \geq 1 \) and \( 0 < \omega_0 < 1 \), then the limit

\[
T(f) = \lim_{R \to \infty} \int_{-R}^R \hat{f}(t) \cos(At) \frac{dt}{2\pi}
\]

exists in the strong operator topology and defines a bounded operator on \( E \).

Then \( f \) belongs to \( H^\infty_c(\Sigma_\omega) \) for all \( 0 < \omega < \omega_0 \).

The proof requires some tools from the theory of harmonic analysis on hypergroups. We shall only require a rather special case which we shall now briefly summarise. A full account may be found in [4] or [22].

Let \( \mathbf{X} \) denote the half-line \([0, \infty)\), and \( C_c(\mathbf{X}) \) the space of compactly supported continuous functions \( f : \mathbf{X} \to \mathbb{C} \). The set \( M^b(\mathbf{X}) \) of bounded Radon measures on \( \mathbf{X} \) with the weak topology forms a complex vector space. When equipped with a suitable associative multiplication or ‘generalized convolution’ operation \( * \) on \( M^b(\mathbf{X}) \) this convolution measure algebra is called a hypergroup or ‘convo’. We shall usually denote this as \((\mathbf{X}, *)\)
although one needs to remember that the operations are defined on $M^b(X)$ rather than the underlying base space $X$.

Denote the Dirac point mass at $x$ by $\delta_x \in M^b(X)$. It is a hypergroup axiom that for all $x,y \in X$, $\delta_x * \delta_y$ is a compactly supported probability measure. The action of $*$ in a hypergroup is in fact completely determined by the convolutions $\delta_x * \delta_y$. When the base space is $X = [0, \infty)$, the convolution $*$ is necessarily commutative and $\delta_0$ is a multiplicative identity element (see [4, Section 3.4]).

For $x \in X$, the left translation operator $T^x$ is defined, initially on $C_c(X)$ by

$$T^x f(y) = \int_X f(t) (\delta_x * \delta_y)(dt) \quad (x, y \in X).$$

Since $*$ is commutative, there exists an essentially unique nontrivial positive invariant measure $\mu_X$ on $[0, \infty)$ satisfying

$$\int_X T^x f(y) \mu_X(dy) = \int_X f(y) \mu_X(dy) \quad (x \in X).$$

for all $f \in C_c(X)$; see [4, Section 1.3]. In the cases of interest to us, $\mu_X(dx) = m(x) dx$ for some locally integrable and positive function $m$.

For a continuous $\phi : X \to \mathbb{C}$ and $x, y \in X$, we shall write $\phi(x * y)$ for the quantity $\int_X \phi(t) (\delta_x * \delta_y)(dt)$. The function $\phi$ is said to be multiplicative if $\phi(x * y) = \phi(x)\phi(y)$ for all $x, y \in X$. A character on $X$ is a real-valued bounded multiplicative function on $X$, and $\hat{X}$ denotes the set of all characters. We shall use the following example in the proof of Proposition 2.2.

**Example 2.2** Zeuner [30] considered the cosh hypergroup $Z$ on $[0, \infty)$ with the convolution

$$\delta_x * \delta_y = \frac{\cosh(x - y)}{2 \cosh x \cosh y} \delta_{|x-y|} + \frac{\cosh(x + y)}{2 \cosh x \cosh y} \delta_{x+y}, \quad (x, y \geq 0) \quad (2.2)$$

for which the invariant measure is $\cosh^2 x dx$. Zeuner [30] shows that $Z$ does not arise as the double coset space $G//K$ of a locally compact group $G$ with maximal compact subgroup $K$. Likewise, $Z$ does not come under the scope of the pseudo-Riemannian symmetric spaces which Flemsted-Jensen considers in [15]. A small calculation shows that for every $\lambda \in \mathbb{C}$ the function

$$\phi_\lambda(x) = \frac{\cos \lambda x}{\cosh x} \quad (x \geq 0),$$

\[(2.3)\]
is multiplicative in the above sense, and that $\phi_\lambda$ is bounded if $|\Re \lambda| \leq 1$. These multiplicative functions arise as eigenfunctions of a differential operator. In this case $L \phi_\lambda = (1 + \lambda^2) \phi_\lambda$ where $L$ is the differential operator

$$Lf(x) = -f''(x) - 2(\tanh x)f'(x)$$

acting in $L^2([0, \infty), \cosh^2 x \, dx)$.

The character space $\hat{X}$ is always sufficiently large in our context to enable one to do harmonic analysis. We can define the Fourier transform of $f \in L^1(X; m)$ by setting

$$\hat{f}(\phi) = \int_X f(x) \phi(x)m(x) \, dx, \quad (\phi \in \hat{X}).$$

By a theorem of Levitan [22], there exists a unique Plancherel measure $\pi_0$ supported on a closed subset $S$ of $\hat{X}$ such that $f \mapsto \hat{f}$ for $f \in L^2(m) \cap L^1(m)$ extends to a unitary isomorphism $L^2(m) \to L^2(\pi_0)$. By [4, Theorem 2.3.19], there exists a unique positive character $\phi_0 \in S$, and $\phi_0$ can be different from the trivial character $1$. In many cases, indeed all the cases that we wish to consider, the space of multiplicative functions can be parametrized by a domain in $\mathbb{C}$, the $\varphi_\lambda(x)$ are holomorphic functions of $\lambda$, and $\hat{X}$ can be identified with unions of line segments in $\mathbb{C}$. Indeed, for Sturm–Liouville hypergroups, $\varphi_\lambda(x)$ are eigenfunctions of a differential operator and $\lambda$ is a spectral parameter as in [4] and [29].

**Lemma 2.3.** Suppose that the multiplicative functions $\{\phi_\lambda\}_{\lambda \in \mathbb{C}}$ for the hypergroup $(X, \ast)$ satisfy the following conditions:

(i) $\lambda \mapsto \phi_\lambda(x)$ is entire and of exponential type, with

$$x = \limsup_{s \to \infty} \frac{\log |\phi_{is}(x)|}{s};$$

(ii) $t \mapsto \phi_t(x)$ is real-valued and even on $\mathbb{R}$, with $\phi_0(x) \leq 1$;

(iii) $t \mapsto \phi_t(x)$ is positive definite on $\mathbb{R}$.

Then $\phi_t \in \hat{X}$ for all $t \in \mathbb{R}$ and there exists a positive Radon measure $\tau_x$ on $[-x, x]$ such that $\tau_x([-x, x]) = \phi_0(x)$ and

$$\phi_\lambda(x) = \int_{-x}^{x} \cos(\lambda t) \tau_x(dt).$$

**Proof.** For each $h > 0$, the function

$$g_h(\lambda) = \frac{\sin(h \lambda)}{h \lambda} \phi_\lambda(x), \quad (\lambda \in \mathbb{C})$$
is entire and of exponential type and
\[ h + x = \limsup_{s \to \pm\infty} \frac{\log |g_h(is)|}{|s|}. \]  \hfill (2.7)

By (iii) \( f \) is bounded on \( \mathbb{R} \) and so the function \( t \mapsto g_h(t) \) is in \( L^2(\mathbb{R}) \). Hence by the Paley–Wiener theorem
\[ g_h(\lambda) = \int_{-x-h}^{x+h} \cos(\lambda t) \tau^h_x(t) \, dt \]  \hfill (2.8)
for some \( \tau^h_x \in L^2[-x-h, x+h] \). From (iii), we deduce that \( \tau^h_x(t) \geq 0 \), and
\[ g_h(t) \leq \phi_0(x) = g_h(0) = \int_{-x-h}^{x+h} \tau^h_x(t) \, dt. \]  \hfill (2.9)

We can therefore introduce \( \tau_x(dt) \) as the weak* limit of the measures \( \tau^h_x(dt) \), and obtain the representation (2.6) since \( g_h(\lambda) \to \phi_\lambda(\lambda) \) uniformly on compact sets as \( h \to 0^+ \).

**Definition 2.4.** A hypergroup \((X, \ast)\) is said to have a Laplace representation if every character \( \phi_\lambda \) in the support of the Plancherel measure has a representation of the form (2.6). Given a strongly continuous cosine family \((\cos(tA))_{t \in \mathbb{R}}\) on \( E \), we define a family of bounded linear operators on \( E \) by the strong operator convergent integral
\[ \phi_A(x) = \int_{[-x,x]} \cos(At) \tau_x(dt) \quad (x \geq 0). \]  \hfill (2.10)

**Theorem 2.5** Suppose that there exist \( M_0, \omega_0 > 0 \) such that
\[ \int_{-x}^{x} \cosh(u\omega_0) \tau_x(du) \leq M_0 \quad (x \geq 0). \]  \hfill (2.11)
This holds in particular when \( \phi_{i\omega_0} \) is the trivial character.

(i) Then \( \phi_\lambda(x) \) is a bounded multiplicative function on \((X, \ast)\) for all \( \lambda \in \Sigma_{\omega_0} \), and the Fourier transform \( f \mapsto \hat{f}(\lambda) \) is bounded \( L^1(m) \to H^\infty(\Sigma_{\omega_0}) \).

(ii) Suppose furthermore that \( \phi_0 \in L^q(m) \) for some \( 2 < q < \infty \). Then \( \phi_{t+i\omega_0} \in L^p(m) \) for \( p = q\omega_0/(\omega_0 - |s|) \) and \( |s| < \omega_0 \).

(iii) Suppose that
\[ \| \cos(tA) \|_{B(E)} \leq \kappa \cosh(t\omega_0) \quad (t \geq 0) \]  \hfill (2.12)
and that \( f, g \in L^1(m) \). Then \((\phi_A(x))_{x>0}\) is a bounded family of operators and
\[ T_A(f) = \int_0^\infty f(x)\phi_A(x)m(x)dx \]  \hfill (2.13)
defines a bounded linear operator on $E$, and $T_A(f * g) = T_A(f)T_A(g)$.

(iv) Suppose in particular that $A$ has a bounded $H^\infty$ functional calculus on $\Sigma_{\omega_0}$. Then $T_A(f)$ defines a bounded linear operator on $E$ for all $f \in L^1(m)$.

**Proof.** (i) From the identity (2.6), we have

$$|\phi_{t+is}(x)| \leq \int_{-\infty}^{\infty} \cosh(su)\tau_x(du) \quad (|s| \leq \omega_0) \quad (2.14)$$

so $\phi_{t+is}$ is bounded; so $\phi_{t+is} \in \mathcal{X}$ if either $t \in \mathbb{R}$ and $s = 0$, or $t = 0$ and $s \in [-\omega_0, \omega_0]$. The identity $\phi_\lambda(x * y) = \phi_\lambda(x)\phi_\lambda(y)$ follows from analytic continuation, and $\int f(x)\phi_\lambda(x)m(x) \, dx$ is holomorphic on the strip $\{ \lambda : |\Im\lambda| < \omega_0 \}$, and bounded by $M_0 \int |f(x)|m(x) \, dx$. Thus $\mathcal{X} = \mathcal{R} \cup [-i\omega_0, i\omega_0]$, and in the particular case in which $\varphi_{i\omega_0}$ is the trivial character, we can take $M_0 = 1$.

(ii) For $|s| < \omega_0$, by Hölder’s inequality, we have

$$|\phi_{t+is}(x)| \leq \left( \int_{-\infty}^{\infty} \cosh(\omega_0 u)\tau_x(du) \right)^{|s|/\omega_0} \phi_0(x)^{(\omega_0-|s|)/\omega_0} \quad (2.15)$$

and one can deduce that $\phi_{t+is}$ belongs to $L^p(m)$.

(iii) We observe that by convexity $\phi_A(x)$ defines a bounded linear operator on $E$, and $\|\phi_A(x)\|_{B(E)} \leq \kappa M_0$. By the addition rule for the cosine family, the identity $\phi_A(x * y) = \phi_A(x)\phi_A(y)$ is unambiguous. We also have

$$\int_0^\infty \int_0^\infty \phi_\lambda(x * y)f(x)g(y)m(x)m(y) \, dx \, dy = \int_0^\infty \phi_\lambda(z)(f * g)(z)m(z) \, dz \quad (2.16)$$

by a standard identity for the convolution, so $f \mapsto T_A(f)$ is multiplicative.

(iv) We observe that $\lambda \mapsto \phi_\lambda(x)$ is in $H^\infty(\Sigma_{\omega_0})$, with uniform bounds on the norm for $x \geq 0$. Hence we can define $\phi_A(x) \in B(E)$ such that $\|\phi_A(x)\|_{B(E)} \leq C(E)\sup\{|\phi_\lambda(x)| : \lambda \in \Sigma_{\omega_0}\}$ independent of $x \geq 0$. By integrating (2.9), we obtain $T_A(f) \in B(E)$.

**Proof of Proposition 2.1.** The cosh hypergroup $Z$ on $[0, \infty)$ has the multiplicative functions

$$\varphi_\lambda(s) = \frac{\cos \lambda s}{\cosh s}, \quad (\lambda \in \mathbb{C}, s \geq 0) \quad (2.17)$$

which are bounded for $|\Im\lambda| \leq 1$; whereas the Plancherel measure $(2/\pi)d\lambda$ is supported on $S = \{ \phi_\lambda : \lambda \in [0, \infty) \}$, which contains $\phi_0$, but not the trivial character. Observe that $Z$ is a Laplace hypergroup for $\tau_x = (\delta_x + \delta_{-x})/(2 \cosh x)$. If $f \in L^1(\cosh^2 x \, dx)$ and $(\cos(tA))_{t \in \mathbb{R}}$
is a strongly continuous cosine family such that \( \| \cos(tA) \|_{B(E)} \leq \kappa \cosh t \) for all \( t \geq 0 \), then by Theorem 2.3

\[
T_A(f) = \int_0^\infty \cos(tA)f(t) \cosh t \, dt \tag{2.18}
\]
defines a bounded linear operator on \( E \). In particular, we consider the differential operator \( L \) given by

\[
Lf = -f''(s) - 2(tanh s)f'(s) \tag{2.19}
\]
is symmetric and bounded below on \( L^2(\cosh^2 s \, ds) \), and \( L\varphi_\lambda = (\lambda^2 + 1)\varphi_\lambda \). Let \( A \) be the positive and self-adjoint operator such that \( A^2 = L - I \); then

\[
\cos(tA) h(s) = \frac{h(s + t) \cosh(s + t) + h(s - t) \cosh(s - t)}{2 \cosh s}, \tag{2.20}
\]
as one can verify by solving the initial value problem \( (\frac{\partial^2}{\partial t^2} + L - I)u(s,t) = 0 \), with initial conditions \( (\partial u/\partial t)(s,0) = 0 \), and \( u(s,0) = h(s) \). We deduce that for \( p \geq 2 \), \( \cos(tA) \) defines a bounded operator on \( E = L^p(\cosh^2 s \, ds) \) and

\[
\| \cos(tA) \|_{B(E)} \leq c_p(\cosh t)^{(p-2)/p} \quad (t > 0). \tag{2.21}
\]

Suppose that \( T_A(f) = \int_{-\infty}^\infty \hat{f}(\xi) \cos(\xi A) \, d\xi \) defines a bounded linear operator on \( L^p(\cosh^2 s \, ds) \) for some \( p > 2 \) so we are in the context of the Proposition. Then \( f \) extends to define a bounded and holomorphic function on \( \Sigma_{(p-2)/2} \). Observe that \( \varphi_\lambda(s) \) belongs to \( L^p(\cosh^2 s \, ds) \) for \( |\Im \lambda| < (p - 2)/p \) and that

\[
T_A(f)\varphi_\lambda = \frac{1}{2\pi} \int_{-\infty}^\infty \hat{f}(\xi) \cos\xi\lambda d\xi \varphi_\lambda(s). \tag{2.22}
\]

By Morera’s theorem, \( \lambda \mapsto T_A(f)\varphi_\lambda \) defines a holomorphic function on \( \Sigma_{(p-2)/2} \) and hence \( T_A(f)\varphi_\lambda(s)/\varphi_\lambda(s) \) is holomorphic on \( \Sigma_{(p-2)/2} \) since the quotient is independent of \( s \). We deduce that \( f(\lambda) = \int_{-\infty}^\infty \hat{f}(\xi) \cos\xi\lambda d\xi/2\pi \) extends to define a holomorphic function on \( \Sigma_{(p-2)/2} \) which is bounded by \( \| T_A(f) \| \).

**Remarks 2.6** (i) An example in [13] and [7] shows that for each \( p \neq 2 \) and \( 0 < \theta < \pi \), there exists \( f \in H^\infty(\Sigma_\theta) \) that is not a bounded Fourier multiplier on \( L^p(\mathbb{R}; \, dx) \).

(ii) Hytönen, McIntosh and Portal [19] develop a functional calculus for second order differential operators \( L \) on \( \mathbb{R}^n \) and give sufficient conditions on \( A(x) \in M_{n \times n}(L^\infty(\mathbb{R}^n; \mathbb{C})) \) for \( Lf = -\text{div} A(x) \nabla f \) to have a bounded \( H^\infty(V_{\phi,0}) \) functional calculus.
Examples 2.7 (i) Suppose that \(-iA\) generates a \(C_0\) semigroup on Hilbert space \(H\) such that \(\|e^{-itA}\|_{B(H)} \leq Me^{\omega_1 t}\) for all \(t \geq 0\), for some \(M, \omega_1 \geq 0\). Then by Corollary 5.1 of [17], \(A\) has a bounded \(H^\infty(\Sigma_{\omega_0})\) functional calculus on \(H\) for all \(\omega_0 > \omega_1\).

3. Representations of hypergroups arising from second order differential operators on the positive axis

In this section we focus upon second order differential operators \(L\) on \([0, \infty)\), and the solutions of \(L\phi_\lambda(x) = (\omega_0^2 + \lambda^2)\phi_\lambda(x)\), where \(\lambda \mapsto \phi_\lambda(x)\) is the cosine transform of a bounded positive measure on \([-x, x]\). We consider the operator \(\sqrt{L - \omega_0^2I}\) and the cosine family that it generates, and use it to form a functional calculus. In this paper, we are mainly concerned with cases in which \(\omega_0 > 0\). The following is a variant of results of Chebli and Trimèche [29].

**Lemma 3.1** Let \(\gamma \geq 1/2\) and \(\omega_0 \geq 0\), and suppose that:

(i) \(m(x) = x^{2\gamma+1}q(x)\) where \(q \in C^2(\mathbb{R})\) is even, positive and such that \(m(x)/x^{2\gamma+1} \to q(0) > 0\) as \(x \to 0+\);

(ii) \(m(x)\) increases to infinity as \(x \to \infty\), and \(m'/m(x) \to 2\omega_0\) as \(x \to \infty\);

(iii) \(Q(x)\) is positive and integrable over \((0, \infty)\), where

\[
Q(x) = \frac{1}{2} \left(\frac{q'}{q}\right)' + \frac{1}{4} \left(\frac{q'}{q}\right)^2 + \frac{2\gamma + 1}{2x} \left(\frac{q'}{q}\right) - \omega_0^2. \tag{3.1}
\]

Then there exists a hypergroup on \([0, \infty)\) such that the solutions of

\[
-d^2\varphi_\lambda \over dx^2 - \frac{m'(x)}{m(x)}\varphi_\lambda = (\omega_0^2 + \lambda^2)\varphi_\lambda \tag{3.2}
\]

such that \(\varphi_\lambda(0) = 1\), and \(\varphi_\lambda'(0) = 0\) for \(\lambda \geq 0\) are characters, and \(\lambda \mapsto \varphi_\lambda(x)\) is even and positive definite on \(\mathbb{R}\).

**Proof.** Let \(\varphi_\lambda(x) = \psi_\lambda(x)/\sqrt{m(x)}\). Then \(\psi_\lambda\) satisfies

\[
-\psi''_\lambda(x) + \left(\frac{m''}{2m} - \left(\frac{m'}{2m}\right)^2 - \omega_0^2\right)\psi_\lambda(x) = \lambda^2\psi_\lambda(x). \tag{3.3}
\]

We let \(J_\gamma\) be Bessel’s function of the first kind of order \(\gamma\) and introduce the function

\[
j_\lambda(x) = \lambda^{-\gamma}x^{1/2}2^{\gamma}J_\gamma(\lambda x) = \frac{\Gamma(\gamma + 1)x^{\gamma + 1/2}}{\Gamma(1/2)\Gamma(\gamma + 1/2)} \int_{-x}^x \left(1 - \frac{s^2}{x^2}\right)^\gamma \frac{\cos s\lambda}{\sqrt{x^2 - s^2}} ds, \tag{3.4}
\]

so that \(\lambda \mapsto j_\lambda(x)\) is positive definite for \(x \geq 0\) and \(\lambda \in \mathbb{R}\), \(\lambda \mapsto j_\lambda(x)\) is entire and of exponential type and

\[
-j''_\lambda(x) + \frac{4\gamma^2 - 1}{4x^2}j_\lambda(x) = \lambda^2 j_\lambda(x). \tag{3.5}
\]
Note that \( Q(x) \) is even and continuous on \( \mathbb{R} \). We then introduce \( \rho_\lambda(x) \) as the solution of the integral equation
\[
\rho_\lambda(x) = \int_0^x \frac{\sin \lambda(x - y)}{\lambda} Q(y) j_\lambda(y) \, dy + \int_0^x \frac{\sin \lambda(x - y)}{\lambda} \left(Q(y) + \frac{4\gamma^2 - 1}{4y^2}\right) \rho_\lambda(y) \, dy. \tag{3.6}
\]
The function \( j_\lambda \) belongs to the space
\[
E_\gamma = \{ f \in C^1(0, 1) : t^{-\gamma-1/2}|f(t)| \leq M_0 \text{ and } t^{-\gamma+1/2}|f'(t)| \leq M_1 \text{ for some } M_0, M_1 \}, \tag{3.7}
\]
and the operator
\[
V : f(x) \mapsto \int_0^x \frac{\sin \lambda(x - y)}{\lambda} \left(Q(y) + \frac{4\gamma^2 - 1}{4y^2}\right) f(y) \, dy \tag{3.8}
\]
is bounded on \( E_\gamma \) since
\[
\left| \frac{(4\gamma^2 - 1)}{4\lambda} \int_0^x \frac{\sin \lambda(x - y)}{y^{3/2-\gamma}} \, dy \right| \leq \frac{(4\gamma^2 - 1)\Gamma(\gamma - 1/2)}{4\Gamma(\gamma + 3/2)x^{\gamma+1/2}}. \tag{3.9}
\]
Hence \( \rho_\lambda \) may be expressed as a Volterra series, and gives a solution such that \( \lambda \mapsto \rho_\lambda(x) \) is positive definite since \( Q(y) \geq 0 \) and \( \gamma \geq 1/2 \). From the integral equation we deduce that
\[
-\rho_\lambda''(x) + Q(x)(j_\lambda(x) + \rho_\lambda(x)) + \frac{4\gamma^2 - 1}{4x^2} \rho_\lambda(x) = \lambda^2 \rho_\lambda(x), \tag{3.10}
\]
and hence \( \psi_\lambda(x) = j_\lambda(x) + \rho_\lambda(x) \) satisfies the differential equation (3.3), so \( \phi_\lambda(x) = m(x)^{-1/2}\psi_\lambda(x) \) is even and satisfies (3.2); multiplying by a suitable constant, we can ensure that \( \phi_\lambda(0) = 1 \). By Sturm’s oscillation theorem, \( \varphi_0(x) \) has no zeros on \((0, \infty)\) whereas \( \varphi_\lambda(x) \) has infinitely many zeros on \((0, \infty)\) for all \( \lambda > 0 \).

**Theorem 3.2** Suppose that \( m \) is as in Lemma 3.1 with \( \omega_0 > 0 \).

(i) Then there exists a commutative hypergroup \((X, \ast)\) on \([0, \infty)\) such that \( \phi_\lambda \) is a bounded multiplicative function on \((X, \ast)\) for all \( \lambda \in \Sigma_{\omega_0} \), and the Fourier transform \( f \mapsto \hat{f}(\lambda) \) is bounded \( L^1(m) \to H^\infty(\Sigma_{\omega_0}) \).

(ii) \( \phi_\lambda \in L^p(m) \) for all \( 2 < p \leq \infty \) and \( \lambda \in \Sigma_{\omega_0} \).

(iii) Suppose that \( (\cos(tA))_{t \in \mathbb{R}} \) is a strongly continuous cosine family on \( E \) such that
\[
\| \cos(tA) \|_{B(E)} \leq \kappa \cosh(\omega_0 t) \quad (t \in \mathbb{R}) \tag{3.11}
\]
and some \( \kappa < \infty \). Then \( (\phi_\lambda(x))_{x \geq 0} \) as in (2.10) gives a bounded family of linear operators on \( E \), and \( T_A(f) = \int_0^\infty f(x) \phi_\lambda(x) m(x) \, dx \) as in (2.13) defines a bounded linear operator on \( E \) such that \( T_A(f \ast g) = T_A(f) T_A(g) \) for all \( f, g \in L^1(m) \).
Proof. (i) By Lemma 3.1, there exists such a hypergroup, and the multiplicative functions on the hypergroup have a Laplace representation. Indeed, the function \( \lambda \mapsto \varphi_\lambda(x) \) is entire, and by Lemma 3.1, there exists a family of positive measures such that \( \varphi_\lambda(x) = \int_{-x}^{x} \cos(\lambda t) \tau_x(dt) \). In particular, \( \lambda = \pm i \omega_0 \) gives the trivial character.

(ii) Suppose now that \( \omega_0 > 0 \). From the solution of the integral equation, one obtains a bound \( \psi_\lambda(x) = O(e^{\eta x}) \) as \( x \to \infty \) where \( \eta = |\Im \lambda| \), and hence \( \varphi_\lambda = O(e^{(\eta-\omega_0)x}) \) as \( x \to \infty \). By Lemma 3.1, \( \varphi_\lambda \) is a bounded multiplicative function for \( |\Im \lambda| \leq \omega_0 \), and \( \varphi_\lambda \in L^p(m(x)dx) \) for \( |\Im \lambda| < (p-2)\omega_0/p \) and \( 2 < p < \infty \).

(iii) Thus all the hypotheses of Theorem 2.5 apply.

Definition 3.3 (i) Legendre’s functions are defined by

\[
\phi_\lambda(x) = R_{i\lambda -(1/2)}^{(0,0)}(\cosh x) = \frac{1}{\pi \sqrt{2}} \int_{-x}^{x} \frac{\cos \lambda y}{\sqrt{\cosh x - \cosh y}} dy \quad (\lambda \in \mathbb{C});
\] (3.12)

these are associated with Laplace’s equation in toroidal coordinates, and sometimes called toroidal functions; see [29]. In particular, \( \phi_0(x) \) can be expressed in terms of Jacobi’s complete elliptic integral of the first kind with modulus \( i \sinh(x/2) \).

(ii) As in [27, page 390], the Mehler–Fock transform of order zero is

\[
\hat{f}(\lambda) = \int_{0}^{\infty} f(x) \phi_\lambda(x) \sinh x dx \quad (f \in L^1(\sinh x dx)).
\] (3.13)

One can compute the transforms of polynomials in \( \text{sech}(x/2) \) by contour integration. For example, one can adapt the formulae in [27] to obtain the array

\[
\begin{array}{cccc}
  f(x) & \hat{f}(\lambda) \\
  \text{sech}(x/2) & (2/\lambda) \co\csc\pi \lambda \\
  (\text{sech}(x/2))^3 & 8\lambda \co\csc\pi \lambda \\
  (\text{sech}(x/2))^5 & (16/3)\lambda^3 \co\csc\pi \lambda
\end{array}
\] (3.14)

in which the last two transforms are bounded and holomorphic on \( V_{\phi,1} \) for all \( 0 < \phi < \pi/2 \). Likewise, any positive even power \( (\text{sech}(x/2))^\nu \) transforms to a constant multiple of \( \lambda^{\nu-2} \co\csc\pi \lambda \).

Corollary 3.4 Let \( (\cos(tA))_{t \in \mathbb{R}} \) be a cosine family on \( E \) such that \( \| \cos(sA) \|_{B(E)} \leq \kappa \cosh(s/2) \) for some \( \kappa \) and all \( s \geq 0 \). Then there exists a hypergroup \( ([0, \infty), *) \) with Laplace representation (2.10) such that \( f \mapsto \hat{f} \) is the Mehler–Fock transform of order zero.
\((\phi_A(x))_{x>0}\) is a bounded family of operators, and for all \(f, g \in L^1((0, \infty); \sinh x \, dx)\), the integral
\[
T_A(f) = \int_0^\infty \phi_A(x) f(x) \sinh x \, dx
\]
defines a bounded linear operator, such that \(T_A(f * g) = T_A(f)T_A(g)\).

**Proof.** Mehler [24] showed in (8b) of page 184 that
\[
-\phi''_{\lambda}(x) - \coth x \phi'_{\lambda}(x) = (\lambda^2 + (1/4))\phi_{\lambda}(x).
\]
Trimèche [29] introduces a hypergroup structure on \((0, \infty)\) such that the \(\phi_{\lambda}\) for \(\{\lambda \in \mathbb{C} : \vert \Im \lambda \vert \leq 1/2\}\) are bounded and multiplicative for this hypergroup, and he shows that the invariant measure and the Plancherel measure are supported on \([0, \infty)\), and satisfy
\[
m(x) \, dx = \sinh x \, dx,
\]
\[
\pi_0(d\lambda) = \frac{2\vert \Gamma((1/4) + (i\lambda/2))\Gamma((3/4) + (i\lambda/2))\vert^2}{\vert \Gamma(i\lambda/2)\Gamma(1 + (i\lambda/2))\vert^2} \, d\lambda.
\]
By a computation involving \(\Gamma\) functions, particularly the identity
\[
-z\Gamma(-z)\Gamma(z) = \pi \cosec(\pi z),
\]
one can reduce (3.17) to \(\pi_0(d\lambda) = \lambda \tanh(\pi \lambda) d\lambda\), so the generalized Fourier transform \(\hat{f}(\lambda) = \int_0^\infty f(x) \phi_{\lambda}(x) m(x) \, dx\) reduces to the Mehler–Fock transform of order zero. Note that \(\lambda = i/2\) gives the trivial character, which is not in the support of \(\pi_0\). We now observe that \(\|\phi_A(x)\|_{B(E)} \leq \kappa\), and hence the integral (3.15) is absolutely convergent. Given that the hypergroup convolution \(*\) exists, we can compute the convolution of \(f, g \in L^1(m)\) by \(f * g(x) = \int_0^\infty \phi_{\lambda}(x) \hat{f}(\lambda) \hat{g}(\lambda) \pi_0(d\lambda)\). As in Theorem 3.2(iii), we deduce from this expression that \(T_A(f * g) = T_A(f)T_A(g)\).

4 Geometrical applications

**Example 4.1** Hyperbolic space \(\mathcal{H}^n\) provides the basic example of a manifold on which the volume of balls grow exponentially with increasing radius, hence \(\mathcal{H}^n\) is used in subsequent volume comparison estimates. As a model for \(\mathcal{H}^n\) of dimension \(n \geq 2\), we use the upper half-space
\[
\mathcal{H}^n = \{p = (u, t) : u \in \mathbb{R}^{n-1}, t > 0\}
\]
with metric \(dp^2 = t^{-2}(du^2 + dt^2)\) and volume measure \(\mu(dp) = t^{-n} dt du\). The Riemannian distance \(\rho(p, q)\) between \(p = (u, t)\) and \(q = (v, s)\) satisfies
\[
\sinh^2(\rho/2) = \frac{\|u - v\|^2_{\mathbb{R}^{n-1}} + (t - s)^2}{4st},
\]

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and it is convenient to use $\rho$ as the radius in a system of polar coordinates.

The Laplace operator is

$$\Delta g = -t^2 \left( \frac{\partial^2 g}{\partial t^2} + \sum_{j=1}^{n-1} \frac{\partial^2 g}{\partial u_j^2} \right) - (n-2)t \frac{\partial g}{\partial t},$$

(4.3)

where $u = (u_j)_{j=1}^{n-1} \in \mathbb{R}^{n-1}$, and $\Delta$ is essentially self-adjoint on the dense core $C_0^\infty (\mathcal{H}^n; C)$ in $L^2(\mathcal{H}^n; \mu)$. The Ricci curvature is constant $-(n-1)$, which is reflected in the lower bound of the spectrum $[(n-1)^2/4, \infty)$. For fixed $q \in \mathcal{H}^n$, and $g$ depending only upon $\rho(p,q)$, we can write

$$\Delta g = -g''(\rho) - (n-1)(\coth \rho) g'(\rho).$$

(4.4)

Theorem 3.2 applies to this differential operator for all integers $n \geq 2$. For $n = 2$ we recognise (3.16). For $n \geq 2$, Lax and Phillips [8] observe that the Green’s function $G(p,q)$ of $-k^2 I + \Delta$ on $\mathcal{H}^n$ is rotationally symmetric and satisfies $G(p,q) = g(\rho)$ with $\rho = \rho(p,q)$ for some $g$ that satisfies

$$-g''(\rho) - (n-1)(\coth \rho) g'(\rho) = (4^{-1}(n-1)^2 - k^2) g(\rho) \quad (\rho > 0).$$

(4.5)

Let $\mathcal{M}$ be a complete Riemannian manifold of dimension $n$ and metric $\rho$ that has bounded geometry, so that the Ricci curvature is bounded below by $-(n-1)\kappa$ and the injectivity radius is bounded below by some $r_0 > 0$. This means that the exponential map is injective on the tangent space above $B(p, r_0) = \{ q \in \mathcal{M} : \rho(q, p) \leq r_0 \}$ for all $p \in \mathcal{M}$; see [8]. For fixed $p_0 \in \mathcal{M}$, we can use $\rho(p, p_0)$ as the radius in a system of polar coordinates with centre $p_0$, noting that $\rho$ is not differentiable on the cut locus. We assume that $\mathcal{M}$ is noncompact, so $0 \leq \rho < \infty$. Let $\mu$ be the Riemannian volume measure, let $A(r; \theta)$ the area element in polar coordinates on the complement of the cut locus of a normal ball $B(p_0; r)$, and let $A(r) = \int A(r; \theta) d\theta$ be the surface area of $B(p_0; r)$.

The Laplace operator is essentially self-adjoint on $C_0^\infty (\mathcal{M}; C)$ by Chernoff’s theorem [10], so we can define functions of $\sqrt{\Delta}$ via the spectral theorem in $L^2(\mathcal{M}; \mu)$. The following result gives conditions under which we can define $T_{\sqrt{\Delta}}(f)$ for suitable $f$ via Theorem 3.2, and ensure that such operators are bounded on $L^2(\mathcal{M}; \mu)$. Let $T_{\sqrt{\Delta}}(f)$ have kernel $F(p,q)$ and let $\phi_{\sqrt{\Delta}}(x)$ have kernel $\Phi_x(p,q)$ as integral operators on $L^2(\mathcal{M}; \mu)$. The following result extends ideas from Taylor’s paper [28] and [9].

**Theorem 4.2** (i) Suppose that $A(r)/A'(r)$ increases to infinity as $r \to \infty$. Then Theorem 3.2 applies to $m(r) = \mu(B(p_0, r))$ for all $p_0 \in \mathcal{M}$ with $\gamma = (n-1)/2$ and $0 \leq \omega_0 \leq (n-1)\kappa/2$. 

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(ii) Suppose that \( \psi : (0, \infty) \to (0, \infty) \) is a continuous function such that

\[
\left( \int_{B(p; x)} (|\Phi_x(p, q)|^2 + |\Phi_x(q, p)|^2) \mu(dq) \right)^{1/2} \leq \psi(x) \quad (p \in \mathcal{M}; x \geq 0)
\]

where for some \( \kappa' > \kappa \)

\[
\int_0^{\infty} e^{\kappa'x/2} |f(x)| \psi(x) m(x) \, dx < \infty.
\]

Then \( T_\sqrt{\Delta}(f) \) defines a bounded linear operator on \( L^\nu(\mathcal{M}; \mu) \) for all \( 1 < \nu < \infty \).

**Proof.** (i) By the isoperimetric inequality, there exists \( s_\mathcal{M} > 0 \) such that \( A(x) \geq s_\mathcal{M} m(x)^{(n-1)/n} \), so by integrating this inequality we obtain \( m(x) \geq (s_\mathcal{M}x/n)^n \). Croke gave an estimate on the isoperimetric constant \( s_\mathcal{M} \) in terms of other geometric parameters, as in [8]. Hence \( m(r) \) has the required behaviour for Lemma 3.1(i) as \( r \to 0^+ \).

Let \( M_{\kappa, n}(r) = \kappa^{-(n-1)} \int_0^r (\sinh \kappa s)^{n-1} ds \). Then by standard volume comparison estimates in section 4 of [8], the function \( m(r)/M_{\kappa, n}(r) \) decreases with increasing \( r > 0 \). By an elementary calculation, one shows that \( M'_{\kappa, n}(r)/M_{\kappa, n}(r) \) decreases monotonically with limit \( (n-1)\kappa \) as \( r \to \infty \). Since by hypothesis \( m'/A' \) is increasing, a standard application of the mean value theorem shows that \( m(r)/A(r) \) is also increasing, so \( m'(r)/m(r) \) is decreasing as \( r \) increases to infinity, hence \( m'(r)/m(r) \to 2\omega_0 \) as \( r \to \infty \), where \( 0 \leq 2\omega_0 \leq (n-1)\kappa \) by the volume comparison estimates.

(ii) We aim to verify Schur’s condition

\[
\sup_{p \in \mathcal{M}} \int_{\mathcal{M}} |F(p, q)|\mu(dq) + \sup_{q \in \mathcal{M}} \int_{\mathcal{M}} |F(p, q)|\mu(dp) < \infty,
\]

which implies the result by standard theory of integral operators. By finite propagation speed, the distributional kernel of the operator \( \phi_\sqrt{\Delta}(x) \) is supported on the diagonal strip \( \{(p, q) \in \mathcal{M} \times \mathcal{M} : \rho(p, q) \leq x \} \), so \( \Phi_x(p, q) = 0 \) whenever \( \rho(p, q) > x \). Hence the definition (2.13) gives

\[
F(p, q) = \int_0^\infty \Phi_x(p, q)f(x)m(x) \, dx,
\]

which gives

\[
\int_{\mathcal{M}} |F(p, q)|\mu(dq) \leq \int_0^\infty \int_{B(p; x)} |\Phi_x(p, q)|\mu(dq)|f(x)|m(x) \, dx
\]

and so by the Cauchy–Schwarz inequality, we have

\[
\int_{\mathcal{M}} |F(p, q)|\mu(dq) \leq \int_0^\infty \left( \int_{B(p; x)} |\Phi_x(p, q)|^2\mu(dq) \right)^{1/2} \mu(B(p; x))^{1/2}|f(x)|m(x) \, dx
\]
and by estimates on the volume growth, for each $\kappa' > \kappa$ we have a constant $C$ such that

$$\int_{\mathcal{M}} |F(p,q)|\mu(dq) \leq C \int_{\mathcal{M}} e^{\kappa'x/2}\psi(x)|f(x)|m(x)\,dx \quad (p \in \mathcal{M}).$$

(4.12)

By repeating this argument with obvious adaptation for $p$ interchanged with $q$, we obtain Schur’s condition and hence the result.

**Example 4.3.** Let $m(x) = 2^2\sinh^2 x$, so that $\omega_0 = 1$ and $\gamma = 1/2$, and one can check that $Q(x) = 0$; thus the hypotheses of Lemma 3.1 are satisfied. We have

$$\varphi_\lambda(x) = \frac{\sin \lambda x}{\lambda \sinh x} = \int_{-x}^x \frac{\cos \lambda t}{2\sinh x} dt \quad (\lambda \in \mathbb{C})$$

(4.13)

so that $\varphi_\lambda$ is a bounded multiplicative function for $|\Im \lambda| \leq 1$, and $\varphi_{\pm i}$ is the trivial character. The Plancherel measure is

$$\pi_0(d\lambda) = \frac{\lambda^2}{4\pi} I_{(0,\infty)}(\lambda) d\lambda.$$  

(4.14)

This hypergroup can otherwise be obtained as a double coset space. The Lie group $G = SL(2, \mathbb{C})$ has a maximal compact subgroup $K = SU(2, \mathbb{C})$ such that $K \times K$ acts upon $G$ via $(h,k) : g \mapsto h^{-1}gk$ for $h,k \in K$ and $g \in G$, producing a space of orbits $G//K = \{KgK : g \in G\}$. The double coset space $G//K$ inherits the structure of a commutative hypergroup modelled on $[0, \infty)$, as in [22].

**5. Fractional integration of cosine families**

In the Cauchy problem for the wave equation in dimension three, the solution can have one order of differentiability fewer than the initial data, due to the possible formation of caustics [9]. Hence it is natural to apply Riemann–Liouville fractional integration operators to the cosine families which address this possible loss of smoothness, and the order of the fractional integration required can depend directly upon the dimension. The operators that we require are described in the following lemma.

**Definition 5.1.** The fractional integration operators $W_\alpha$ and $U_\beta$ are defined on $C^\infty(\mathbb{R})$ by

$$W_\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (\cosh x - \cosh t)^{\alpha-1} \sinh t f(t)\,dt,$$

$$U_\beta f(x) = \frac{1}{\Gamma(\beta)} \int_0^x (\cosh x - \cosh t)^{\beta-1} f(t)\,dt,$$

(5.1)

where $\alpha$ and $\beta$ are the orders of $W_\alpha$ and $U_\beta$, such that $\Re \alpha > 0$ and $\Re \beta > 0$. 

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Lemma 5.2 (i) Let $Df = f'$. Then the operators satisfy

$$W_\alpha W_\beta = W_{\alpha + \beta}, \quad W_\alpha U_\beta = U_{\alpha + \beta}, \quad \text{cosech}x DW_1 = I, \quad DU_1 = I.$$  \hspace{1cm} (5.2)

(ii) For $\nu \in \mathbb{Z}$ such that $\nu \geq 0$ and $\lambda \in \mathbb{R}$, the associated Legendre function satisfies

$$U_{\nu+1/2}(\cos(x\lambda)) = \sqrt{\frac{\pi}{2\Gamma(1/2 + i\lambda - \nu)}} \frac{\Gamma(1/2 + i\lambda + \nu)}{2} P_{i\lambda-1/2}^{\nu}(\cosh x),$$  \hspace{1cm} (5.3)

$$W_{\nu-1/2}(\cos(x\lambda)) = \frac{d}{dx} U_{\nu+1/2}(\cos(x\lambda)) \quad (\nu \in \mathbb{N}).$$  \hspace{1cm} (5.4)

Proof. (i) This is essentially contained in the statement and proof of Lemma 5.2 of [23]. See also Theorem 5.2 of [29].

(ii) The first is known as the Mehler–Dirichlet formula [27] page 373, from which we obtain the second by differentiating.

In particular, the trigonometric and Legendre functions of Definition 3.3 are thus related by

$$\phi_{\lambda}(x) = \sqrt{\frac{3}{\pi}} U_{1/2}(\cos \lambda x), \quad \cos \lambda x = \frac{d}{dx} \sqrt{\frac{3}{2}} W_{1/2}(\phi_{\lambda}(x)).$$  \hspace{1cm} (5.5)

For operator families, we have the following theorem.

Theorem 5.3 Suppose that $(\cos(tA))_{t \in \mathbb{R}}$ is a densely defined cosine family on $E$.

(i) Suppose that $(\cos(tA))_{t \in \mathbb{R}}$ is strongly continuous and there exists $M > 0$ such that $\|\cos(tA)\|_{B(E)} \leq M \cosh(t/2)$ for all $t \in \mathbb{R}$. Then $(U_{1/2}(\cos(tA)))_{t \in \mathbb{R}}$ is a bounded family of operators.

(ii) More generally, suppose that $I + A^2$ is invertible and there exist $M_0 > 0$ and $1 > \omega_0 > 0$ such that

$$\|(I + A^2)^{-1} \cos(tA)\|_{B(E)} \leq M_0 \cosh(\omega_0 t) \quad (t \geq 0),$$  \hspace{1cm} (5.6)

and $(tA)^{-1} \sin(tA)$ is a bounded family of operators for $0 \leq t \leq R$ for some $R > 0$. Then $W_k(\cos tA)$ is a bounded operator on $E$ for all $t > 0$ and integers $k \geq 1$, and there exist $M_k, \omega_k > 0$ such that

$$\|W_k(\cos(tA))\|_{B(E)} \leq M_k \cosh(\omega_k t) \quad (t \in \mathbb{R}).$$  \hspace{1cm} (5.7)

Moreover, for all $\omega > \omega_1$ and $\theta > 0$ there is a bounded functional calculus map $\Lambda_A : H^{\infty,1}_e(V_{\theta,\omega}) \to B(E)$ given by

$$\Lambda_A : f \mapsto f(A) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos(tA) \mathcal{F}f(t) \, dt.$$  \hspace{1cm} (5.8)
(iii) Suppose that (5.7) holds for some \( k \in \mathbb{N} \), and \( A \) has spectrum contained in \( V_{\phi, \omega_k} \), for some \( \phi > 0 \) and that there exists \( M > 0 \) and \( s > 1 \) such that \( \| (\zeta I + A)^{-1} \|_{B(E)} \leq M \) for all \( \zeta \) on the boundary of \( V_{s, \phi, s, \omega_k} \). Then there is a bounded functional calculus map \( \Lambda_A : H_c^\infty, 2k + 2(V_{s, \phi, s, \omega_k}) \to B(E) \).

**Proof.** (i) In the notation of Corollary 3.4, we have \( \phi_A(t) = U_{1/2}(\cos(tA)) \), whence the result.

(ii) By standard results about cosine families, the hypotheses of (i) imply those of (ii).

By integration by parts, we obtain

\[
W_1(\cos(tA)) = (I + A^2)^{-1}(\cos(tA) \cosh t - I + A \sin(tA) \sinh t), \quad (5.9)
\]

so \( W_1(\cos(tA)) \) is the sum of operators which are themselves bounded. Next we establish the bound

\[
\frac{\| W_n(\cos(tA)) \|_{B(E)}}{\sinh^n t} \leq \frac{M_0 \Gamma(\omega_0 + 1)}{\Gamma(\omega_0 + n + 1)} \cosh(\omega_0 t) \quad (t \in \mathbb{R}). \quad (5.10)
\]

By Lemma 5.2, we have

\[
W_n(M_0 \cosh(\omega_0 t)) = \frac{M_0}{(n-1)!} \int_0^t (\cosh t - \cosh s)^{n-1} \sinh s \cosh(\omega_0 s) \, ds. \quad (5.11)
\]

Recall that \( 0 < \omega_0 < 1 \). Then by substituting \( \cosh t = 1 + \xi \) and \( \cosh s = 1 + \eta \), we obtain an upper bound on (5.11) of the form

\[
W_n(M_0 \cosh(\omega_0 t)) \leq \frac{M_0 \Gamma(\omega_0 + 1)}{\Gamma(n + \omega_0 + 1)} \xi^{n+\omega_0} \quad (5.12)
\]

where \( \xi^n \leq \sinh^n t \) and \( \xi^{\omega_0} \leq (1 + o(1)) \cosh(\omega_0 t) \) as \( t \to \infty \). Hence we obtain (5.7) from

\[
\| W_n(\cos(tA)) \|_{B(E)} \leq \frac{M_0 \Gamma(\omega_0 + 1)}{\Gamma(n + \omega_0 + 1)} (\sinh^n t \cosh(\omega_0 t))(1 + o(1)) \quad (t \to \infty). \quad (5.13)
\]

Let \( R > 0 \) and choose \( \varphi_0 \) and \( \varphi_1 \in C^\infty(\mathbb{R}) \) such that \( 0 \leq \varphi_0, \varphi_1 \leq 1, \varphi(x) + \varphi_1(x) = 1 \) for all \( x \geq 0 \), \( \varphi_0 \) is supported on \((-1, R + 1)\) and \( \varphi_1 \) is supported on \((R, \infty)\). We then decompose the integral in (5.8) as

\[
f(A) = \int_0^{R+1} \varphi_0(\xi) F f(\xi) \cos(\xi A) \frac{d\xi}{\pi} + \int_0^\infty \varphi_1(\xi) F f(\xi) \cos(\xi A) \frac{d\xi}{\pi}. \quad (5.14)
\]

The first integral is

\[
- \int_0^{R+1} \xi \frac{d}{d\xi} (\varphi_0(\xi) F f(\xi)) \frac{\sin(\xi A)}{\xi A} \frac{d\xi}{\pi}. \quad (5.15)
\]
where \( \sin(\xi A)/(\xi A) \) gives a uniformly bounded family of operators on \( E \) for \( 0 < \xi < R \), and by Plancherel’s formula

\[
\int_0^{R+1} \frac{d}{d\xi} (\varphi_0(\xi) \hat{f}(\xi)) d\xi \leq CR^{1/2} \left( \int_{-\infty}^{\infty} (x^2|f'(x)|^2 + |f(x)|^2) dx \right)^{1/2}
\]

\[
\leq CR^{1/2} \left( \int_{-\infty}^{\infty} \frac{C||f||^2_{H^{\infty,1}(V_0,\omega)}}{1 + x^2} dx \right)^{1/2} \quad (5.16)
\]

since \( x^2|f'(x)| + |xf(x)| \leq C||f||_{H^{\infty,1}(V_0,\omega)} \) by Cauchy’s estimates and hypotheses on \( f \).

By Cauchy’s theorem and a simple integration by parts, we obtain the formula

\[
\mathcal{F}f(\xi) = \lim_{R \to \infty} \int_{-R}^{R} f(x) e^{-ix \xi} dx = \frac{e^{-\omega \xi}}{i \xi} \int_{-\infty}^{\infty} f'(x - i \omega) e^{-ix \xi} dx \quad (\xi > 0), \quad (5.17)
\]

and a corresponding formula for \( \xi < 0 \), where the final integral converges absolutely by Cauchy’s estimates. We still obtain a convergent integral if we differentiate both sides \( 2\ell \) times with respect to \( \xi \), as we now use to advantage.

By integrating by parts we have

\[
\int_{R}^{\infty} \varphi_1(\xi) \mathcal{F}f(\xi) \cos(\xi A) d\xi
\]

\[
= \int_{R}^{\infty} \left( \varphi_1(\xi) \mathcal{F}f(\xi) \right) \left( I - \frac{d^2}{d\xi^2} \right)^{\ell} (I + A)^{-\ell} \cos(\xi A) d\xi
\]

\[
= \int_{R}^{\infty} \left( I - \frac{d^2}{d\xi^2} \right)^{\ell} \left( \varphi_1(\xi) \mathcal{F}f(\xi) \right) (I + A)^{-\ell} \cos(\xi A) d\xi \quad (5.18)
\]

where the integrand is continuous in the strong operator topology and the integral is absolutely norm convergent by the estimates (5.6) and (5.16).

(iii) The hypotheses refer to \( W_k \), where the range of \( W_k \) is contained in an auxiliary algebra of functions which is defined on the Fourier transform side. Let \( \omega' > 0 \) and let \( k \in \mathbb{N} \). Then

\[
\mathcal{A}_{2k,\omega'} = \{ g \in C^{2k}(\mathbb{R}) : |g^{(j)}(x)| e^{\omega' |x|} \leq C_j, \forall x \in \mathbb{R}, \text{ for some } C_j, j = 0, \ldots, 2k \} \quad (5.19)
\]

is an algebra under pointwise multiplication. Note also that

\[
\mathcal{A}_{2k,\omega'}^0 = \{ g \in \mathcal{A}_{2k,\omega'} : g^{(j)}(0) = 0, j = 0, \ldots, 2k \} \quad (5.20)
\]

is a subalgebra under pointwise multiplication. Given \( \omega_k < \omega' < \omega \), the Fourier transform maps \( H^{\infty,2k+2}(\Sigma_\omega) \to \mathcal{A}_{2k,\omega'} \), so there exist \( C_j(\omega') \) such that

\[
\left| \frac{d^j}{dt^j} \mathcal{F}f(t) \right| \leq C_j(\omega') ||f||_{\infty} e^{-\omega' t} \quad (t > 0; j = 0, 1, \ldots). \quad (5.21)
\]
Let \( D_s g(t) = (d/dt)(g(t)\cosh t) \). Suppose that \( f \in H^{\infty, 2k+2}(\Sigma_\omega) \) satisfies

\[
\int_{-\infty}^{\infty} x^j f(x) \, dx = 0 \quad (j = 0, \ldots, 2k);
\]

then we observe that \( \mathcal{F} f \in \mathcal{A}_{2k,\omega}^0 \) and we deduce that \( D_s^j \mathcal{F} f(t) \) is bounded in a neighbourhood of \( t = 0 \) for all \( j = 0, \ldots, 2k \). Indeed one can prove by induction that for integers \( 0 \leq j \leq \ell \) there exist polynomials \( p_{j,\ell} \) in two indeterminates such that

\[
D_s^j g(t) = \frac{1}{\sinh^{2\ell} t} \sum_{j=0}^\ell p_{j,\ell}(\cosh t, \sinh t) \sinh^j t g^{(j)}(t);
\]

hence, by the general mean value theorem and the choice of \( g \in \mathcal{A}_{2k,\omega}^0 \), the function \( D_s^k g(t) \) is bounded in a neighbourhood of \( t = 0 \).

To deal with large \( t \), we use a similar identity. By induction one can prove that for all integers \( 0 \leq j \leq n \), there exist polynomials \( q_{j,n} \) in two indeterminates, with total degree less than or equal to \( n \), such that

\[
D_s^j g(t) = \frac{1}{\sinh^{2n} t} \sum_{j=0}^n q_{j,n}(\cosh t, \sinh t) g^{(j)}(t).
\]

Hence \( \cosh(\omega_k t) \sinh^n t D_s^{n+k} \mathcal{F} f(t) \in L^1(0, \infty) \) for some \( n = 0, 1, 2, \ldots \). For \( h \in C(0, \infty) \) and \( g \in C^\infty(\delta, L) \) with \( 0 < \delta < L < \infty \), we obtain by integration by parts the formula

\[
\int_0^\infty g(t) h(t) \, dt = (-1)^{n+k} \int_0^\infty D_s^{n+k} g(t) W_{n+k} h(t) \, dt,
\]

so in particular we have

\[
\int_0^\infty \mathcal{F} f(t) \cos(tA) \, dt = (-1)^{n+k} \int_0^\infty (\sinh^n t) D_s^{n+k} (\mathcal{F} f)(t) \frac{W_{n+k}(\cos(tA))}{\sinh^n t} \, dt,
\]

where the integral is absolutely convergent, hence defines a bounded linear operator \( f(A) \).

Finally we extend the functional calculus to all of \( H^{\infty, 2k+2}(V_{s^2, \phi, s^2 \omega_k}) \) by adding on a complementary subspace of rational functions, as suggested by Theorem 2.5 of [1]. Let \( f \in H^{\infty, 2k+2}(V_{s^2, \phi, s^2 \omega_k}) \). Then by Cauchy’s integral formula with \( \lambda \in V_{\phi, \omega_k} \), we have the identity

\[
f(\lambda) = \frac{1}{2\pi i} \int_{V_+} \frac{(z-i)^{2k+1} f(z) \, dz}{(\lambda - i)^{2k+1}(\lambda - z)} - \frac{1}{2\pi i} \int_{V_-} \frac{(z+i)^{2k+1} f(z) \, dz}{(\lambda + i)^{2k+1}(\lambda - z)}
\]

\[+ \sum_{j=0}^{2k} \frac{1}{2\pi i} \int_{V_+} \frac{(z-i)^j f(z) \, dz}{(\lambda - i)^{j+1}} - \sum_{j=0}^{2k} \frac{1}{2\pi i} \int_{V_-} \frac{(z+i)^j f(z) \, dz}{(\lambda + i)^{j+1}},\]

(5.27)
where $V_+$ is the curve made of the ray $s\omega_k(-\cot s\phi + i) + te^{-is\phi}$ for $-\infty < t < 0$ followed by $[s\omega_k(-\cot s\phi + i), s\omega_k(\cot s\phi + i)]$ and followed by the ray $s\omega_k(\cot s\phi + i) + te^{is\phi}$ for $0 < t < \infty$ along the top boundary of $V_{s\phi,s\omega_k}$, and $V_-$ is the bottom boundary of $V_{s\phi,s\omega_k}$, obtained by reflecting $V_+$ in the origin. For even $f$, this reduces to the identity

$$f(\lambda) = \frac{1}{2\pi i} \int_{V_+} \left( \frac{1}{(\lambda - i)^{2k+1}(\lambda - z)} + \frac{1}{(\lambda + i)^{2k+1}(\lambda + z)} \right) (z - i)^{2k+1} f(z) \, dz$$

$$+ \sum_{j=0}^{2k} \frac{1}{2\pi i} \int_{-\infty}^\infty \frac{(x - i)^j f(x) \, dx}{(\lambda - i)^{j+1}} - \sum_{j=0}^{2k} \frac{1}{2\pi i} \int_{-\infty}^\infty \frac{(x + i)^j f(x) \, dx}{(\lambda + i)^{j+1}}$$

(5.28)

by Cauchy’s Theorem. This expresses the decomposition of $f$ into the sum of a rational function with poles at $\pm i$ of order less than or equal to $2k+1$, plus a function as in (5.22).

Observe that for $z$ on $V_+$,

$$\int_{-\infty}^\infty \left( \frac{1}{(\lambda - i)^{2k+1}(\lambda - z)} + \frac{1}{(\lambda + i)^{2k+1}(\lambda + z)} \right) \lambda^j \, d\lambda = 0 \quad (j = 0, \ldots, 2k)$$

(5.29)

by Cauchy’s theorem, and that the term in parentheses is an even function of $\lambda$. Furthermore, if $f$ is as in (5.22), then the integrals in the sums in (5.28) are all zero. Hence we can define $f(A)$ for any $f \in H^{\infty,2k+2}(\Sigma_\omega)$ consistently by replacing $\lambda$ by $A$ in (5.28). Observe that $(zI - A)^{-1} = z^{-1} \xi + z^{-1}(zI - A)^{-1}A \xi$, and hence $\| (zI - A)^{-1} \xi \| = O(1/|z|)$ as $|z| \to \infty$ along the boundary of $V_{s\phi,s\omega_k}$ for all $\xi$ in the dense domain of $A$, so the integrals are absolutely convergent for such $\xi$ and define bounded linear operators by the preceding discussion. Hence the algebra homomorphism $f \mapsto f(A)$ is bounded, and consistent with the functional calculus for rational functions that are holomorphic on $V_{s^2\phi,s^2\omega_k}$.

**Remark 5.4** (i) Let $f(z) = H_{2\nu}(z) e^{-z^2}$, where $\nu = k + 2, k + 3, \ldots$ and $H_{2\nu}$ is the Hermite polynomial of degree $2\nu$. Then $f$ satisfies (5.22) for $0 < \phi < \pi/4$ and all $0 < \omega < \infty$.

(ii) To compare the conclusions of Theorem 5.3 with the results of [8] and [28], we note that

$$\{ f \in H^{\infty}(\Sigma_\omega) : (1 + x^2)^j/2 |f^{(j)}(x)| \leq C j!(\sin \phi)^j \quad \forall x \in \mathbb{R}, j = 0, 1, 2, \ldots; \text{for some } C \}$$

(5.30)

is a subset of $H^{\infty}(V_{\phi,\omega})$.

Hyperbolic space $H^{\nu+1}$ gives a significant example of Theorem 5.3(iii) in which the order of fractional integration $k$ and the exponent of growth $\omega > \omega_k$ depends directly upon the dimension of the underlying manifold. While this result is essentially known, we include a proof for completeness.
Proposition 5.5 The operator $W_{\nu/2}(\cos t\sqrt{\Delta})$ is bounded on $L^p(\mathcal{H}^{\nu+1})$ for all $\nu \geq 1$ and $\nu/(\nu - 1) < p < \infty$, and there exists $\kappa(\nu, p) > 0$ such that

$$
\|W_{\nu/2}(\cos t\sqrt{\Delta})f\|_{L^p(\mathcal{H}^{\nu+1})} \leq \kappa(\nu, p)(\sinh t)\nu \|f\|_{L^p(\mathcal{H}^{\nu+1})} \quad (t > 0, f \in L^p(\mathcal{H}^{\nu+1})).
$$

(5.31)

Proof. For fixed $w = (u, v) \in \mathcal{H}^{\nu+1}$, the hyperbolic distance to $z = (x, y) \in \mathcal{H}^{\nu+1}$ is $\rho$ as in Example 4.1. Hence the hyperbolic sphere $\{(x, y) : \rho((x, y), (u, v)) = t\}$ is also the Euclidean sphere with centre $(u, v \cosh t)$ and radius $v \sinh t$. Let $\sigma_t$ be the measure on this sphere induced by $\mu$. Let $\nu$ be even. Then for any smooth and compactly supported function $f : \mathcal{H}^{\nu+1} \to \mathbb{R}$, the spherically symmetric function

$$
F(z, t) = \frac{\Gamma((\nu + 1)/2)}{2\pi^{(\nu+1)/2} \sinh t} D_s^{(\nu-2)/2} \int_{\rho(z, w) = t} f(w)\sigma_t(dw)
$$

satisfies the Cauchy problem for the wave equation

$$
\frac{\partial^2 F}{\partial t^2} = -\Delta F
$$

and the initial conditions

$$
F(z, 0) = 0, \quad \frac{\partial F}{\partial t}(z, 0) = f(z),
$$

(5.34)

so that $F(z, t) = (\sin(t\sqrt{\Delta})/\sqrt{\Delta})f(z)$, or $F(z, t) = U_1(\cos t\sqrt{\Delta})f(z)$. See [9] page 296 for explicit formulas relating the hyperbolic and Euclidean wave equation. By the strict form of Huygens’s principle, the solution $F(z, t)$ depends only upon the values of $f$ in thin shell surrounding the sphere of radius $t$ with centre $z$. The spherical average operator is

$$
A_t f(z) = \frac{\Gamma((\nu + 1)/2)}{2\pi^{(\nu+1)/2}(\sinh t)^\nu} \int_{\rho(z, w) = t} f(w)\sigma_t(dw).
$$

(5.35)

As in Lemmas 5.2 and 5.3 of [23], we have

$$
W_{\nu/2}(\cos t\sqrt{\Delta})f = (\sinh t)\nu A_t f.
$$

(5.36)

Nevo and Stein [25] considered spherical maximal functions

$$
M^* f(z) = \sup_{t>0} |A_t f(z)|
$$

(5.37)
for simple Lie groups of real rank one, and Ionescu [21] refined their results to show that $M^*$

is bounded on $L^p(\mathcal{H}^{\nu+1}) \to L^p(\mathcal{H}^{\nu+1})$ for all $\nu/(\nu-1) < p < \infty$. Hence for $k \geq (\nu-2)/2$, the operator

$$W_k \left( \frac{\sin(t\sqrt{\Delta})}{\sqrt{\Delta}} \right)$$

is bounded on $L^p(\mathcal{H}^{\nu+1}) \to L^p(\mathcal{H}^{\nu+1})$ for $2 < p < \infty$ and has a kernel that is supported on $\{(x,y) \in \mathcal{H}^{\nu+1} \times \mathcal{H}^{\nu+1} : \rho(x,y) \leq t\}$. The calculation for odd $\nu$ is similar.

Remarks 5.6

(i) One can continue this analysis and show that $\sqrt{\Delta}$ satisfies (iii) of Theorem 5.3.

(ii) Suppose momentarily that we strengthen (iii) of Theorem 5.3 slightly, so that $A$ is a densely defined operator in $E$ with spectrum contained in $\Sigma_{\omega_0}$ and $\|(tI \pm i\omega I + A)^{-1}\|_{B(E)} \leq 1/(\omega - \omega_0)$ for all $t \in \mathbb{R}$ and $\omega > \omega_0$. Then by a classical result of Hille and Bade, [1, Theorem 5.1], $iA$ generates a strongly continuous group $(e^{itA})_{t \in \mathbb{R}}$ such that $\|e^{itA}\|_{B(E)} \leq e^{\omega_0|t|}$ for all $t \in \mathbb{R}$. So in the next section, we work with groups of operators on $E$. See also [5,6, 18].

6. Transference of Marcinkiewicz multipliers

When $E$ satisfies certain special geometrical properties, we can improve upon Theorem 4.3 by replacing $H^\infty(V_{\theta,\omega})$ by an algebra of Fourier multipliers. The idea is to replace the condition of holomorphy by assumptions about the variation of functions.

Definition 6.1. Suppose that $1 \leq s < \infty$ and that $h : \mathbb{R} \to \mathbb{C}$. Then

(i) $h$ has finite $s$-variation over a bounded interval $J$ if

$$\text{var}_s(h; J) = \sup \left\{ \left( \sum_{j=1}^{n-1} |h(\xi_{j+1}) - h(\xi_j)|^s \right)^{1/s} : \xi_j \in J, \xi_1 < \xi_2 < \ldots < \xi_n \right\}$$

is finite, where the supremum is over all finite partitions.

(ii) $h$ is a $s$-Marcinkiewicz multiplier if $h$ is bounded and has uniformly bounded $s$ variation over all the dyadic intervals in $\mathcal{D} = \{[2^j, 2^{j+1}), (-2^{j+1}, -2^j] : j \in \mathbb{Z} \}$.

The set $M_s$ of all $s$-Marcinkiewicz multipliers forms a Banach algebra under pointwise multiplication and norm

$$\|h\|_{M_s} = \sup_{x \in \mathbb{R}} |h(x)| + \sup_{J \in \mathcal{D}} \text{var}_s(h; J).$$

This terminology alludes to the Fourier multiplier $f \mapsto F^{-1}(hFf)$ given by $h$, as in [12]. Here $\text{var}_1(h; J)$ gives the standard notion of variation, and $s \mapsto \text{var}_s(h; J)$ is a decreasing
function, so \( M_1 \subset M_s \subset L^\infty \) for \( s > 1 \). We also introduce the algebra \( M^2_s = M_s \cap L^2(0, \infty) \) with the norm \( \|h\|_{M^2_s} = \|h\|_{L^2(0, \infty)} + \|h\|_{M_s} \).

Let \( E = L^p(\mathcal{M}; F_0) \) where \( F_0 \) is a finite-dimensional complex vector space over a manifold \( \mathcal{M} \). Suppose further that \( (e^{itA})_{t \in \mathbb{R}} \) is a \( C_0 \) group of operators on \( E \) such that

\[
\|e^{itA}\|_{B(E)} \leq K_p e^{\omega_p|t|} \quad (t \in \mathbb{R})
\]

for some \( \omega_p > 0 \) and \( K_p > 0 \); see [26].

**Theorem 6.2** Let \( 1 < p < \infty \), \( \omega_p < \omega \) and let \( s \in [1, 2] \) satisfy \( |(1/p) - (1/2)| < 1/s \). Suppose that \( \hat{g} \in L^2(d\lambda) \) and \( \hat{g} \in M_s \). Then

\[
g(x) = \frac{2}{\pi} \int_0^\infty \cos(\lambda x) \frac{\hat{g}(\lambda)}{\cosh \lambda} d\lambda
\]

has \( g \in L^2(\cosh^2 x dx) \) and

\[
\Lambda_{A/\omega}(g) = \frac{1}{\pi} \int_0^\infty \cos(tA/\omega) g(t) dt
\]

defines a bounded linear operator on \( E \) such that

\[
\|\Lambda_{A/\omega}(g)\|_{B(E)} \leq K(E, \omega) \|g\|_{M_s}
\]

for some \( K(E, \omega) < \infty \). Thus \( \hat{g} \mapsto \Lambda_{A/\omega}(g) \) gives a bounded functional calculus map \( M^2_s \to B(E) \).

**Proof.** The Banach space \( E \) belongs to the class \( I \) introduced by Berkson and Gillespie in [3]. There exist a Hilbert space \( E_0 \), a unconditional martingale difference (UMD) Banach space \( E_1 \) and \( 0 < \theta < 1 \) such that \( E \) is linearly homeomorphic to Calderon’s complex interpolation space \( [E_0, E_1]_\theta \). Let \( E' \) be the dual space of \( E \) and note that \( E \) is reflexive.

Suppose that \( 2 \leq p < \infty \); the case of \( 1 < p < 2 \) follows by considering dual spaces. Let \( R_\xi : h(t) \mapsto h(t - \xi) \) be the operator of translation on \( L^p(\mathbb{R}; E) \) and extend this to the convolution operator \( R(k) = \int_{-\infty}^{\infty} k(\xi) R_\xi d\xi \) given by

\[
R(k) : h(t) \mapsto \int_{-\infty}^{\infty} k(\xi) h(t - \xi) d\xi \quad (h \in L^p(\mathbb{R}; E)).
\]

Evidently \( R(k) \) is a bounded linear operator on \( L^p(\mathbb{R}; E) \) for all \( k \in L^1(\mathbb{R}) \), and we seek to extend this to a larger set of \( k \) by imposing conditions upon \( \mathcal{F}(k)(x) = \int_{-\infty}^{\infty} e^{-ix\xi} k(\xi) d\xi \). Suppose that \( \mathcal{F}(k) \) belongs to \( M_s \) for some \( 1 \leq s < 2p/(p - 2) \); then by the multiplier theorem of [20], \( \mathcal{F}(k) \) gives a bounded Fourier multiplier \( f \mapsto \mathcal{F}^{-1}(\mathcal{F}(k)\mathcal{F} f) \) from \( L^p(\mathbb{R}) \to \).
\( L^p(\mathbb{R}) \) and hence by Fubini’s theorem from \( L^p(\mathbb{R}; L^p(\mathcal{M})) \rightarrow L^p(\mathbb{R}; L^p(\mathcal{M})) \) and hence \( L^p(\mathbb{R}; E) \rightarrow L^p(\mathbb{R}; E) \). By the Plancherel theorem for the cosh hypergroup \( Z \) of Example 2.2, we have \( g \in L^2(\cosh^2 x) \) for \( \hat{g} \in L^2(d\lambda) \).

Hence we obtain a bounded convolution operator \( \tilde{R}(k) : L^p(\mathbb{R}; E) \rightarrow L^p(\mathbb{R}; E) \) in the case \( k(t) = g(|t|) \cosh t \), where \( \mathcal{F}k(\lambda) = 2\hat{g}(\lambda) \). Given \( \hat{g} \in \mathcal{M}_s \) we introduce the indicator function \( I_{(0,r)} \) of \((0,r)\) and \( \hat{g}_r(\lambda) = I_{(0,r)}(\lambda)\hat{g}(\lambda) \) such that \( \hat{g}_r \in \mathcal{M}_s \), \( \hat{g}_r \in L^2(d\lambda) \). Indeed, letting \( (r_j)_{j=1}^{\infty} \) be an increasing sequence such that \( g(x) \) is the almost sure limit of the \( (2/\pi) \int_0^{r_j} \cos x\lambda\hat{g}(\lambda)d\lambda/\cosh x \), we deduce from Lemma 2.1 of [3] that there is a bounded Fourier multiplier defined as \( \hat{g} = \lim_{r_j \to \infty} \hat{g}_{r_j} \), which we can transfer to a bounded linear operator on \( E \). There is a group representation of \( \mathbb{R} \) as operators on \( E \) given by \( t \mapsto e^{-itA/\omega} \) such that \( \|e^{-itA/\omega}\|_{B(E)} \leq K \cosh(t\omega_p/\omega) \). By Haase’s [18] form of the transference theorem for group representations

\[
\mathcal{F}g_{r_j}(A/\omega) = 2\int_0^{\infty} g_{r_j}(t) \cos(tA/\omega) \, dt
\]

defines a bounded linear operator on \( E \) such that

\[
\|\mathcal{F}g(A/\omega)\|_{B(E)} \leq K^2K(p)\|R(k)\|_{L^p(\mathbb{R}; E)} \leq K^2K(p, E, s)\|\mathcal{F}k\|_{\mathcal{M}_s}
\]

where \( K(p) \) and \( K(p, E, s) \) are constants independent of \( k \). We then define \( \Lambda_{A/\omega}(g) \) as the weak limit of the bounded family of operators \( \Lambda_{A/\omega}(g_r) \) as \( r \to \infty \). By the Eberlein–Smulian theorem, every bounded sequence in \( E \) has a weakly convergent subsequence. So we can select a countable dense subset \( (\xi_j)_{j=1}^{\infty} \subset E \) and a countable dense subset \( (\eta_j)_{j=1}^{\infty} \subset E' \), then choose a diagonal subsequence \( (r_\ell)_{\ell=1}^{\infty} \) such that \( \langle \Lambda_{A/\omega}g_{r_\ell}\xi_j, \eta_k \rangle \) converges as \( r_\ell \to \infty \) for all \( j, k \).

We observe that

\[
\Lambda_{A/\omega}(g) = T_{A/\omega}(g(x)/\cosh x),
\]

where \( T_{A/\omega} \) is the operator involved in Theorem 2.5, hence the operator product \( \Lambda_{A/\omega}(g)\Lambda_{A/\omega}(h) \) arises from the convolution in \( Z \) of \( g(x)/\cosh x \) and \( h(x)/\cosh x \). We deduce that \( \Lambda_{A/\omega} \) is a functional calculus map.

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