On general weighted cumulative residual extropy and general weighted negative cumulative extropy

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ABSTRACT
In this paper, we define general weighted cumulative residual extropy (GWCRJ) and general weighted negative cumulative extropy (GWNCJ). We obtain simple estimators for complete and right censored data. We obtain some results on GWCRJ and GWNCJ. We establish its connection to reliability theory and coherent systems. We also propose empirical estimators of GWNCJ. GWCRJ based on first-order statistics has been studied. Proposed estimators are calculated through data analysis. A new test of uniformity is proposed and analysed based on GWNCJ.

1. Introduction
Entropy was introduced by Shannon [1] as a measure of uncertainty involved in a random experiment. The entropy of a non-negative absolutely continuous random variable $X$ with probability density function (pdf) $f$ is defined as

$$H(X) = - \int_0^\infty f(x) \log(f(x)) \, dx.$$  

Many well-developed entropy applications in information theory, economics, communication theory, and physics have been studied in the literature (see [2]). Belis and Guiasu [3] considered a weighted entropy measure as

$$H^x(X) = - \int_0^\infty xf(x) \log(f(x)) \, dx,$$

where by assigning greater importance to larger values of $X$, the weight $x$ in (2) emphasizes the occurrence of the event $X = x$. Di Crescenzo and Longobardi [4] stated the necessity of the existence of the weighted measures of uncertainty. In the Shanon entropy $H(X)$, only the pdf of the random variable $X$ is regarded. Moreover, it is known that this information measure is shift-independent, in the sense that the information content of a random variable $X$ is equal to that of $X + b$. Indeed, some applied fields such as neurobiology do not tend to deal with shift-independent but shift-dependent.

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Rao et al. [5] defined and investigated cumulative residual entropy (CRE). CRE of a non-negative absolutely continuous random variable \( X \) is defined as

\[
CRE(X) = -\int_0^{\infty} \bar{F}(x) \log(\bar{F}(x)) \, dx.
\]

The extropy of a non-negative absolutely continuous random variable \( X \) defined by Lad et al. [6] is defined as

\[
J(X) = -\frac{1}{2} \int_0^{\infty} f^2(x) \, dx.
\]

Different versions of weighted extropy were studied by Balakrishnan et al. [7], Bansal and Gupta [8] and Sathar and Nair [9–11] among others. In their example, [8] showed that while the extropies of the two different random variables \( X \) and \( Y \) are the same, their weighted extropies may differ. As a result, weighted extropies and general weighted extropies are different even if extropies in this example are the same. As a result, weighted extropies and general weighted extropies are useful as measures of uncertainty. Unlike the extropy defined by Lad et al. [6], the shift-dependent measure takes into account the values of the random variable. Gupta and Chaudhary [12] introduced and studied general weighted extropy. Gupta and Chaudhary [13] gave some characterization of symmetry using extropy.

Jahanshahi et al. [14] introduced cumulative residual extropy (CRJ) and studied its properties. CRJ of a non-negative absolutely continuous random variable \( X \) is defined as

\[
CRJ(X) = \eta J(X) = -\frac{1}{2} \int_0^{\infty} \bar{F}^2(x) \, dx
\]

and cumulative past extropy (CPJ) for a non-negative absolutely continuous random variable \( X \) is defined as

\[
CPJ(X) = -\frac{1}{2} \int_0^{\infty} F^2(x) \, dx.
\]

Kattumannil and Sreedevi [15] obtained a simple estimator of cumulative residual extropy for complete and right-censored data and studied their properties. In this paper, we also obtained a simple estimator of weighted cumulative residual extropy for complete and right censored data and studied their properties. Tahmasebi and Toomaj [16] introduced negative cumulative extropy (NCJ) and studied several properties of this concept. NCJ of a non-negative absolutely continuous random variable is defined as

\[
NCJ(X) = \hat{\eta} J(X) = \frac{1}{2} \int_0^{\infty} (1 - F^2(x)) \, dx.
\]

The integrand function on the right-hand-side of any uncertainty measure depends on \( x \) only via \( F(x) \), thus making CRJ or NCJ shift-independent. Hence, CPJ or NCJ stays unchanged if, for instance, \( X \) is uniformly distributed in \((a, b)\) or \((a + h, b + h)\), whatever \( h \in \mathbb{R} \). However, in certain applied contexts, such as reliability or mathematical neurobiology, it is desirable to deal with shift-dependent information measures. Indeed, knowing that a device fails to operate, or a neuron to release spikes in a given time interval, yields relevantly different information from the case when such an event occurs
in a different equally wide interval. In some cases, we are thus led to resort to a shift-dependent information measure that, for instance, assigns different measures to such distributions.

We consider a generalization of cumulative residual extropy introduced by Jahanshahi et al. (2019) by general weight \( w(x) \geq 0 \). Hashempour et al. [17] considered a generalization of cumulative residual extropy (CRJ) with weight in particular \( w(x) = x \), called weighted cumulative residual extropy (WCRJ).

Several researchers recently studied several types of extropy and introduced some new measures. Jose and Sathar [18] and Qiu [19] studied the extropy of order statistics and record values. Qiu and Jia [20] studied the residual extropy of order statistics. Qiu and Jia [21] developed the goodness of fit test of uniform distribution using extropy. Qiu et al. [22] studied the properties of extropy for mixed systems. Jahanshahi et al. [14] and Tahmasebi and Toomaj [16] studied cumulative residual extropy and negative cumulative extropy in detail. Kazemi et al. [23] introduced weighted cumulative past extropy (WCPJ) and developed some characterization results. Several applications of extropy and its generalizations, such as in information theory, economics, computer science, communication theory, and physics, can be found in the literature. Tahmasebi and Toomaj [16] studied the stock market in OECD countries based on a generalization of extropy known as negative cumulative extropy. Balakrishnan et al. [24] applied another version of extropy known as the Tsallis extropy to a pattern recognition problem. Kazemi et al. [25] explored an application of a generalization of extropy known as the fractional Deng extropy to a problem of classification. Tahmasebi et al. [26] used some extropy measures for the problem of compressive sensing.

The rest of this paper is organized as follows. Section 2 discusses general weighted cumulative residual extropy (GWCRJ) and its simple non-parametric estimator for a complete and right censored case. In Section 3, we define general weighted negative cumulative extropy (GWNCJ) and obtain a simple non-parametric estimator for a complete and right censored case. Also, some results related to WNCJ are proposed. Section 4 is devoted to some inequalities which provide bounds for WNCJ. In Section 5, we established some connections of WNCJ to reliability theory. Conditional weighted negative cumulative extropy is defined and studied in Section 6. In Section 7, we studied the WNCJ of a coherent system. Section 8 provides GWCRE based on first-order statistics. Section 9 provides two empirical estimators for WNCJ and GWNCJ. In Section 10, Data is used to evaluating the proposed estimators. Section 11 is devoted to the application in which we proposed a test of uniformity based on GWNCJ. In the end, Section 12 concludes this paper.

2. General weighted cumulative residual extropy (GWCRJ)

In this section, we introduce a new information measure called general weighted cumulative residual extropy (GWCRJ). The cumulative residual extropy can be generalized to GWCRJ. The main objective of the study is to extend general weighted extropy to random variables with continuous distributions.

Let \( X \) be a non-negative absolutely continuous random variable having pdf \( f \), distribution function \( F \) and survival function \( \bar{F} = 1 - F \).
Definition 2.1. The GWCRJ with weight \( w \geq 0 \) is given by
\[
\eta^w J(X) = -\frac{1}{2} \int_0^\infty w(x) \tilde{F}^2(x) \, dx.
\]
If \( w(x) = x^m, m > 0 \), then we represent \( \eta^w J(X) \) as \( \eta^m J(X) \).

Example 2.1. Let \( X \) be an exponential random variable with a density function \( f(x) = \lambda e^{-\lambda x}, x > 0, \lambda > 0 \) then for \( m > 0 \),
\[
\eta^m J(X) = -\frac{1}{2} \int_0^\infty x^m \tilde{F}^2(x) \, dx = -\frac{1}{2} \int_0^\infty x^m e^{-2\lambda x} \, dx = -\frac{\Gamma(m + 1)}{2^m \lambda^{m+1}}.
\]
Here, \( \Gamma \) is a gamma function defined as \( \Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \, dx \).

In the following subsections, we will obtain U-statistics-based estimators of GWCRJ.

2.1. Uncensored case

Let \( X_1 \) and \( X_2 \) be independent and identically distributed random variables with common distribution function \( F \). The survival function of \( \min(X_1, X_2) \) is equal to \( (\tilde{F}(x))^2 \). For a non-negative random variable \( X \), we have
\[
E(X^m) = \int_0^\infty x^m f(x) \, dx = m \int_0^\infty x^{m-1} \tilde{F}(x) \, dx.
\]
Therefore, we have
\[
\eta^m J(X) = -\frac{1}{2} \int_0^\infty x^m \tilde{F}^2(x) \, dx = \frac{1}{2(m+1)} E \left( (\min(X_1, X_2))^{m+1} \right).
\]
Based on U-statistics an estimator of \( \eta^m J(X) \) is given by
\[
T_{1,m} = \frac{-1}{n(n-1)(m+1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^n (\min(X_i, X_j))^{m+1}.
\]
Here, \( T_{1,m} \) is an unbiased estimator of \( \eta^m J(X) \). Note that using order statistics, we have
\[
T_{1,m} = \frac{1}{n(n-1)(m+1)} \sum_{i=1}^n (i-n) X_{(i)}^{m+1}, \quad (4)
\]
where \( X_{(i)} \) be the \( i \)th order statistics from a random sample \( X_1, \ldots, X_n \) from \( F \). Now, using the limit theorems of U-statistics, we investigate the asymptotic characteristics of \( T_{1,m} \). \( T_{1,m} \) is a consistent estimator of \( \eta^m J(X) \), as \( T_{1,m} \) is a U-statistic [27].

Theorem 2.1. As \( n \to \infty \), \( \sqrt{n}(T_{1,m} - \eta^m J(X)) \) is asymptotic normal with mean zero and variance \( 4\sigma_1^2 \), where
\[
\sigma_1^2 = \operatorname{Var} \left( X^{m+1} \tilde{F}(X) + \int_0^X y^{m+1} \, dF(y) \right). \quad (5)
\]
Proof: Asymptotic normality follows from the central limit theorem of U-statistics (see Theorem 2.1, p. 76, [28]). Also, the asymptotic variance is \( 4 \sigma^2_1 \), where

\[
\sigma^2_1 = \text{Var} \left( E \left( \left( \min(X_1, X_2) \right)^{m+1} \big| X_1 \right) \right).
\]

Note that

\[
E \left( \left( \min(X_1, X_2) \right)^{m+1} \big| X_1 = x \right) = E \left( x^{m+1} I(x < X_2) + X_2^{m+1} I(X_2 \leq x) \right)
= x^{m+1} \hat{F}(x) + \int_0^x y^{m+1} \, dF(y).
\]

Hence, the expression for variance is obtained as (5). Hence the result.  

2.2. Right censored case

Consider the case when the censoring times are independent of the lifetimes and the observations are randomly right-censored. Let \( K \) be the survival function of censoring random variable \( C \). We want to find an estimator of the GWCRJ \( \eta^m J(X) \) based on \( n \) independent and identical observations \( \{ (Y_i, \delta_i), 1 \leq i \leq n \} \), where \( \delta_i = I(X_i \leq C_i) \), is the censoring indicator and \( Y_i = \min(X_i, C_i) \). A U-statistic defined for right censored data is given by Datta et al. [29]

\[
T_{1c,m} = -\frac{1}{n(n-1)(m+1)} \sum_{i=1}^{n} \sum_{j<i=1}^{n} \frac{(\min(Y_i, Y_j))^{m+1} \delta_i \delta_j}{\hat{K}(Y_i-) \hat{K}(Y_j-)},
\]

where \( \hat{K}(\cdot) \) is the Kaplan–Meier estimator of \( K(\cdot) \) (see [29]). The proof of the following theorem is similar to the proof of Theorem 2.3 of [30].

Theorem 2.2. As \( n \to \infty \), \( T_{1c,m} \overset{p}{\to} \eta^m J(X) \).

For deriving the asymptotic distribution of \( T_{1c,m} \), let us define \( N^c_i(t) = I(Y_i \leq t, \delta_i = 0) \) as the counting process corresponding to censoring random variable for the \( i \)th subject and \( R_i(u) = I(Y_i \geq u) \). Let \( \lambda_c(t) \) be the hazard rate of the censoring variable \( C \). The martingale associated with the counting process \( N^c_i(t) \) is given by

\[
M^c_i(t) = N^c_i(t) - \int_0^t R_i(u) \lambda_c(u) \, du.
\]

Let \( H_c(x) = P(Y_1 \leq x, \delta = 1) \), \( y(t) = P(Y_1 > t) \) and

\[
w(t) = \frac{1}{y(t)} \int \frac{h_1(x)}{K(x-)} I(x > t) \, dH_c(x),
\]

where \( h_1(y) = E(h(Y_1, Y_2)|Y_1 = y) \). The proof of the following theorem follows from [29] when we choose \( h(Y_1, Y_2) = (\min(Y_1, Y_2))^{m+1} \).
Theorem 2.3. Assume $E((\text{min}(Y_1, Y_2))^{2(m+1)}) < \infty$, $\int \frac{h_1(x)}{K(x)} dH_c(x) < \infty$ and $\int_0^\infty w^2(t)\lambda_c(t) dt < \infty$. As $n \to \infty$, the distribution of $\sqrt{n}(T_{1,m} - \eta^{m}J(X))$ is normal with mean zero and variance $4\sigma_{1,m}^2$, where $\sigma_{1,m}^2$ is given by

$$\sigma_{1,m}^2 = \text{Var}\left(\frac{h_1(X)\delta_1}{K(Y)} + \int w(t) dM'_1(t)\right).$$

3. General weighted negative cumulative extropy (GWNCJ)

In this section, we will define and investigate the general weighted negative cumulative extropy (GWNCJ) estimation.

Definition 3.1. The GWNCJ of a non-negative random variable $X$ is defined as

$$\bar{\eta}^wJ(X) = \frac{1}{2} \int_0^\infty w(x) \left(1 - F^2(x)\right) dx.$$  

If $w(x) = x^m$, $m > 0$, then we represent $\bar{\eta}^wJ(X)$ as $\bar{\eta}^mJ(X)$.

Tahmasebi and Toomaj [16] defined negative cumulative extropy given in (3). Here, we are defining WNCJ which is a particular case of GWNCJ when we choose $w(x) = x \geq 0$.

Definition 3.2. Let $X$ be a non-negative absolutely continuous random variable with cumulative distribution function (cdf) $F$. We define the weighted negative cumulative extropy (WNCJ) of $X$ by

$$\bar{\eta}^1J(X) = \frac{1}{2} \int_0^\infty x \left(1 - F^2(x)\right) dx. \quad (6)$$

We will be investigating the characteristics of WNCJ. Before we get to the main results, let’s see a few examples.

Example 3.1. Let $X$ has $U[a, b]$ distribution where $b > a > 0$. Then NCJ and WNCJ of the $U[a, b]$ are

$$\bar{\eta}J(X) = \frac{1}{2} \int_0^\infty \left(1 - F^2(x)\right) dx = \frac{b-a}{3},$$

and

$$\bar{\eta}^1J(X) = \frac{1}{2} \int_0^\infty x \left(1 - F^2(x)\right) dx = \frac{b-a}{24}(5a + 3b),$$

respectively.

Example 3.2. Let $X$ has power distribution with pdf $f(x) = \lambda x^{\lambda - 1}$, $x \in (0, 1)$, $\lambda > 1$. Then NCJ and WNCJ of the distribution are

$$\bar{\eta}J(X) = \frac{1}{2(2\lambda + 1)} \quad \text{and} \quad \bar{\eta}^1J(X) = \frac{\lambda}{4(\lambda + 1)}.$$
Example 3.3. Let $X$ has exp($\lambda$) distribution with pdf $f(x) = \lambda e^{-\lambda x}$, $x > 0$, $\lambda > 0$. Then NCJ and WNCJ of the distribution are

$$\bar{\eta}J(X) = \infty, \text{ and } \bar{\eta}^1 J(X) = \frac{5}{4\lambda^2}. \quad (7)$$

Now, we see the effect of linear transformation on WNCJ in the following proposition.

Proposition 3.1. Let $X$ be a non-negative random variable with cdf $F$. If $Y = aX + b$, $a > 0$, $b \geq 0$, then

$$\bar{\eta}^1 J(Y) = a^2 \bar{\eta}^1 J(X) + ab \bar{\eta} J(X).$$

Proof: The proof holds using (6) and note that $F_Y(y) = F\left(\frac{y-b}{a}\right)$, $y > b$. \qed

Here, we provide a lower bound for WNCJ in terms of extropy $J(X)$.

Theorem 3.1. Let random variable $X$ has pdf $f$ and extropy $J(X)$, then

$$\bar{\eta}^1 J(X) \geq C^* \exp\{2J(X)\}, \quad (8)$$

where $C^* = \frac{1}{2} \exp\{E(\log(X(1 - F^2(X))))\}$ and $\exp(x) = e^x$.

Proof: From the log-sum inequality, we have

$$\int_0^\infty f(x) \log \left(\frac{f(x)}{x(1 - F^2(x))}\right) \, dx \geq - \log \left(\int_0^\infty x(1 - F^2(x)) \, dx\right).$$

Then, it follows that

$$\int_0^\infty f(x) \log f(x) \, dx - \int_0^\infty f(x) \log \left(x(1 - F^2(x))\right) \, dx \geq - \log \left(\int_0^\infty x(1 - F^2(x)) \, dx\right).$$

Note that $\log(f) < f$, hence,

$$- \int_0^\infty f^2(x) \, dx + \int_0^\infty f(x) \log \left(x(1 - F^2(x))\right) \, dx = 2J(X) + E\left(\log \left(X(1 - F^2(X))\right)\right) \leq \log \left(2\bar{\eta}^1 J(X)\right), \quad (9)$$

taking exponential both sides, (9) reduces to

$$\bar{\eta}^1 J(X) \geq \frac{1}{2} \exp\{2J(X) + E\left(\log \left(X(1 - F^2(X))\right)\right)\}. \quad \text{Hence the result.} \quad \Box$$

Theorem 3.2. $X$ is degenerate if and only if $\bar{\eta}^1 J(X) = 0$. 
Proof: Let $X$ be degenerate at point $k$, then by using the definition of degenerate function and $\tilde{\eta}^1J(X)$, we have $\tilde{\eta}^1J(X) = 0$. Now, consider $\tilde{\eta}^1J(X) = 0$, i.e.

$$\int_0^\infty x(1 - F^2(x)) \, dx = 0.$$  

Noting that the integrand in the above integral is non-negative, we have $1 - F^2(x) = 0$, that is, $F(x) = 1$ for almost all $x \in S$, where $S$ denote the support of random variable $X$. Hence $X$ is degenerate.

3.1. Uncensored case

Let $X_1$ and $X_2$ be independent and identically distributed random variables with common distribution function $F$. The distribution function of $\max(X_1, X_2)$ is equal to $(F(x))^2$. For the non-negative random variable $X$, we have

$$E(X^m) = \int_0^\infty x^m f(x) \, dx = m \int_0^\infty x^{m-1} \tilde{F}(x) \, dx.$$  

Here,

$$\tilde{\eta}^mJ(X) = \frac{1}{2} \int_0^\infty x^m (1 - F^2(x)) \, dx = \frac{1}{2(m+1)} E((\max(X_1, X_2))^{m+1}).$$  

Based on U-statistics, an estimator of $\tilde{\eta}^mJ(X)$ is given by

$$T_{2,m} = \frac{1}{n(n-1)(m+1)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} (\max(X_i, X_j))^{m+1}.$$  

Here, $T_{2,m}$ is an unbiased and consistent estimator of $\tilde{\eta}^mJ(X)$. Note that using order statistics, we have

$$T_{2,m} = \frac{1}{n(n-1)(m+1)} \sum_{i=1}^{n} (i - 1) X_{(i)}^{m+1}. \quad (10)$$  

Now, we discuss the asymptotic distribution of $T_{2,m}$ and the proof follows in a similar fashion as Theorem 2.1.

Theorem 3.3. As $n \to \infty$, $\sqrt{n}(T_{2,m} - \tilde{\eta}^mJ(X))$ is asymptotic normal with mean zero and variance $4\sigma_2^2$, where

$$\sigma_2^2 = \text{Var} \left( X^{m+1} F(X) + \int_X^\infty y^{m+1} \, dF(y) \right). \quad (11)$$

3.2. Right censored case

Here, we obtain a simple estimator of $\tilde{\eta}^mJ(X)$ in the presence of right-censored observations. Working with same notations used in Subsection 2.2, an estimator of $\tilde{\eta}^mJ(X)$ is given
by
\[ T_{2c,m} = \frac{1}{n(n-1)(m+1)} \sum_{i=1}^{n} \sum_{j<i=1}^{n} \frac{(\max(Y_i, Y_j))^{m+1} \delta_i \delta_j}{\hat{K}(Y_i-\hat{K})(Y_j)}, \]
where \( \hat{K}(.) \) is the Kaplan–Meier estimator of \( K(.) \).

**Theorem 3.4.** As \( n \to \infty \), \( T_{2c,m} \to \bar{\eta}_m J(X) \).

**Theorem 3.5.** Let \( h_1(y) = E((\max(Y_1, Y_2))^{m+1}|Y_1 = y) \). Assume \( E((\max(Y_1, Y_2))^{2(m+1)}) < \infty \), \( \int \frac{h_1(x)}{K^2(x-)} \) \( dH_c(x) < \infty \) and \( \int_0^\infty w^2(t) \lambda_c(t) \) \( dt < \infty \). As \( n \to \infty \), the distribution of \( \sqrt{n}(T_{2c,m} - \bar{\eta}^m J(X)) \) is normal with mean zero and variance \( 4\sigma_{2c,m}^2 \), where \( \sigma_{2c,m}^2 \) is given by
\[ \sigma_{2c,m}^2 = \text{Var} \left( \frac{h_1(X)\delta_1}{\hat{K}(Y)} + \int w(t) \, dM^c_1(t) \right). \]

**4. Some inequalities**

This section deals with obtaining the lower and upper bounds for WNCJ.

**Proposition 4.1.** Consider \( X \) to be a non-negative random variable. then
\[ \bar{\eta}^1 J(X) \geq \frac{E(X^2)}{4}. \]  
**Proof:** Using \( E(X^2) = 2 \int_0^\infty xF(x) \, dx \) and inequality \( F^2(x) \leq F(x) \), the result follows. \( \blacksquare \)

**Proposition 4.2.** Consider a non-negative absolutely continuous random variable \( X \) having cdf \( F \) and support \([a, \infty)\), \( a > 0 \). Then
\[ \bar{\eta}^1 J(X) \geq a\bar{\eta} J(X). \]
**Proof:** Note that
\[ \int_a^\infty x(1 - F^2(x)) \, dx \geq a \int_a^\infty (1 - F^2(x)) \, dx. \]
Hence, \( \bar{\eta}^1 J(X) \geq a\bar{\eta} J(X). \) \( \blacksquare \)

Consider two random variables \( X \) and \( Y \) having cdfs \( F \) and \( G \), respectively. Then \( X \preceq_{st} Y \) whenever \( F(x) \geq G(x), \forall x \in \mathbb{R} \); where the notation \( X \preceq_{st} Y \) means that \( X \) is less than or equal to \( Y \) in usual stochastic order. One may refer to Shaked and Shanthikumar [31] for detail of stochastic ordering. In the following proposition, we show the ordering of WNCJ is implied by the usual stochastic order.

**Proposition 4.3.** Let \( X_1 \) and \( X_2 \) be non-negative absolutely continuous random variables. If \( X_1 \preceq_{st} X_2 \), then \( \bar{\eta}^1 J(X_1) \leq \bar{\eta}^1 J(X_2) \).
\textbf{Proof:} Using $X_1 \leq_{st} X_2$ and (6), the result follows.

Now, we obtain the WNCJ of the $n$th order statistic. The WNCJ of the $n$th order statistic is

$$\bar{\eta}^1 J(X_{n:n}) = \frac{1}{2} \int_0^\infty x \left(1 - F_{n:n}^2(x)\right) \, dx,$$

where $F_{n:n}(x) = F_{X}^{2n}(x)$. Taking $u = F(x)$ in (14),

$$\bar{\eta}^1 J(X_{n:n}) = \frac{1}{2} \int_0^1 \frac{(1 - u^{2n}) F^{-1}(u)}{f(F^{-1}(u))} \, du,$$

where $F^{-1}$ is the inverse function of $F$.

\textbf{Example 4.1.} Let $X$ has the uniform distribution on $(0,1)$ with pdf $f(x) = 1, \ x \in (0,1)$. Then $F^{-1}(u) = u, \ u \in (0,1)$ and $f(F^{-1}(u)) = 1, \ u \in (0,1)$, hence $\bar{\eta}^1 J(X_{n:n}) = \frac{n}{4(n+1)}$.

\textbf{Example 4.2.} Let $X$ have power distribution with pdf $f(x) = \lambda x^{\lambda - 1}, \ \lambda > 1, x \in (0,1)$. Then $F^{-1}(u) = u^{\frac{1}{\lambda}}, \ u \in (0,1)$ and $f(F^{-1}(u)) = \lambda u^{\frac{\lambda - 1}{\lambda}}, \ u \in (0,1)$, hence $\bar{\eta}^1 J(X_{n:n}) = \frac{-n\lambda}{4(n\lambda+1)}$.

\textbf{Remark 4.1.} Consider $\Lambda = \bar{\eta}^1 J(X_{n:n}) - \bar{\eta}^1 J(X)$. Since $F_{2n}(x) \leq F^2(x)$, hence $\Lambda \geq 0$.

5. **Connection to reliability theory**

Consider a non-negative absolutely continuous random variable $X$ with cdf $F$ and $E(X) < \infty$. The mean inactivity time (MIT) of $X$ is defined as

$$\text{MIT}(t) = \int_0^t \frac{F(x)}{F(t)} \, dx, \quad t \geq 0.$$ \hfill (16)

The MIT function finds many applications in reliability, forensic science, and so on. In the following theorem, we show the relationship between WNCJ and the second moment of inactivity time (SMIT) function. For detail about SMIT one may refer [32].

\textbf{Definition 5.1.} Let $X$ be a non-negative absolutely continuous random variable. Then SMIT is

$$\text{SMIT}(t) = E((t - X)^2|X \leq t) = 2t\text{MIT}(t) - \int_0^t 2x \frac{F(x)}{F(t)} \, dx, \quad t \geq 0.$$ \hfill (17)

\textbf{Theorem 5.1.} Let $X$ be a non-negative random variable with finite WNCJ. Then we have

$$\bar{\eta}^1 J(X) = \frac{1}{2} \left[ \frac{1}{2} E(X^2) + E(XF(X)\text{MIT}(X)) - \frac{1}{2} E(F(X)\text{SMIT}(X)) \right].$$
Proof: Consider
\[ \bar{\eta} J^{1}(X) = \frac{1}{2} \int_{0}^{\infty} x(1 - F^2(x)) \, dx \]
\[ = \frac{1}{2} \int_{0}^{\infty} x\bar{F}(x)(1 + F(x)) \, dx \]
\[ = \frac{1}{2} \left[ \int_{0}^{\infty} x\bar{F}(x) \, dx + \int_{0}^{\infty} xF(x)\bar{F}(x) \, dx \right] \]
\[ = \frac{1}{2} \left[ \frac{1}{2} E(X^2) + \int_{0}^{\infty} xF(x) \left( \int_{x}^{\infty} f(t) \, dt \right) \, dx \right] \]
\[ = \frac{1}{2} \left[ \frac{1}{2} E(X^2) + \int_{0}^{\infty} f(t) \left( \int_{0}^{t} xF(x) \, dx \right) \, dt \right]. \quad (18) \]

Now, using (17) and (18), we have
\[ \bar{\eta} J^{1}(X) = \frac{1}{2} \left[ \frac{1}{2} E(X^2) + \int_{0}^{\infty} f(t)F(t) \left\{ 2tMIT(t) - SMIT(t) \right\} \, dt \right] \]
\[ = \frac{1}{2} \left[ \frac{1}{2} E(X^2) + E(XF(X)MIT(X)) - \frac{1}{2} E(F(X)SMIT(X)) \right]. \]

A bound for \( \bar{\eta} J^{1}(X) \) is provided below in terms of the hazard rate function.

**Proposition 5.1.** Let \( X \) be a non-negative absolutely continuous random variable with finite hazard rate function \( h \) and WNCJ \( \bar{\eta} J^{1}(X) \). Then,
\[ \bar{\eta} J^{1}(X) \geq \frac{1}{2} E(S(X)), \quad (19) \]
where \( S(t) = \int_{0}^{t} x(f_{0}^{x} h(v) \, dv) \, dx \).

**Proof:** From (18) we have
\[ \bar{\eta} J^{1}(X) = \frac{1}{2} \left[ \frac{1}{2} E(X^2) + \int_{0}^{\infty} f(t) \left( \int_{0}^{t} xF(x) \, dx \right) \, dt \right] \]
\[ \geq \frac{1}{2} \left[ \frac{1}{2} E(X^2) + \int_{0}^{\infty} f(t) \left( \int_{0}^{t} x \log \bar{F}(x) \, dx \right) \, dt \right] \]
\[ = \frac{1}{2} \left[ \frac{1}{2} E(X^2) + \int_{0}^{\infty} f(t) \left( \int_{0}^{t} x \left( \int_{0}^{x} h(v) \, dv \right) \, dx \right) \, dt \right] \]
\[ = \frac{1}{2} \left[ \frac{1}{2} E(X^2) + E(S(X)) \right] \]
\[ \geq \frac{1}{2} E(S(X)), \]
where \( S(t) = \int_{0}^{t} x(f_{0}^{x} h(v) \, dv) \, dx \). Hence the result.
6. Conditional weighted negative cumulative extropy

Now, we consider the conditional weighted negative cumulative extropy. Consider a random variable \( Z \) on probability space \((\Omega, \mathcal{A}, P)\) such that \( E|Z| < \infty \). The conditional expectation of \( Z \) given sub \( \sigma \)-field \( G \), where \( G \subseteq \mathcal{A} \), is denoted by \( E(Z|G) \). For the random variable \( I(Z \leq z) \), we denote \( E(I(Z \leq z)|G) \) by \( F_{Z}(z|G) \).

**Definition 6.1.** For a non-negative random variable \( X \), given \( \sigma \)-field \( G \), the conditional weighted negative cumulative extropy \( \bar{\eta} J(X|G) \) is defined as

\[
\bar{\eta} J(X|G) = \frac{1}{2} \int_{0}^{\infty} x \left( 1 - F^{2}(x|G) \right) \, dx. \tag{20}
\]

Now, we assume that the random variables are absolutely continuous and non-negative.

**Lemma 6.1.** If \( G \) is a trivial \( \sigma \)-field, then \( \bar{\eta} J(X|G) = \bar{\eta} J(X) \).

**Proof:** Since here \( F(x|G) = F(x) \), then the proof follows. \( \square \)

**Proposition 6.1.** If \( X \in L^{p} \) for some \( p > 2 \), then \( E[\bar{\eta} J(X|G)|G^{*}] \leq \bar{\eta} J(X|G^{*}) \), provided that \( G^{*} \subseteq G \).

**Proof:** Consider

\[
E[\bar{\eta} J(X|G)|G^{*}] = \frac{1}{2} \int_{0}^{\infty} x \left[ 1 - E \left( [P(X \leq x|G)]^{2} |G^{*} \right) \right] \, dx
\]
\[
\leq \frac{1}{2} \int_{0}^{\infty} x \left[ 1 - E \left( E(P(X \leq x|G)) |G^{*} \right)^{2} \right] \, dx
\]
\[
= \frac{1}{2} \int_{0}^{\infty} x \left[ 1 - E \left( E(I(X \leq x)|G) |G^{*} \right)^{2} \right] \, dx
\]
\[
= \frac{1}{2} \int_{0}^{\infty} x \left[ 1 - F^{2}(x|G^{*}) \right] \, dx
\]
\[
= \bar{\eta} J(X|G^{*}),
\]

where the second step follows using Jensen’s inequality for a convex function \( \phi(x) = x^2 \). Hence the result. \( \square \)

In the following theorem, we investigate the relationship between conditional extropy \( J(X|G) \) and conditional weighted negative cumulative extropy \( \bar{\eta} J(X|G) \).

**Theorem 6.1.** Let \( \bar{\eta} J(X|G) \) be conditional weighted negative cumulative extropy. Then we have

\[
\bar{\eta} J(X|G) \geq B^{*} \exp \{ 2J(X|G) \}, \tag{21}
\]

where \( B^{*} = \frac{1}{2} \exp \{ E(\log(X(1 - F^{2}(X)))|G) \} \).
Proof: The proof is on the similar lines as of Theorem 3.1, hence omitted. ■

**Theorem 6.2.** Let $X$ be a random variable and $\mathcal{G}$ be $\sigma$-field. Then
\[
E \left( \bar{\eta}^{1} J(X|\mathcal{G}) \right) \leq \bar{\eta}^{1} J(X),
\] (22)
and $E(\bar{\eta}^{1} J(X|\mathcal{G})) = \bar{\eta}^{1} J(X)$ if and only if $X$ is independent of $\mathcal{G}$.

Proof: If in Proposition 6.1, $\mathcal{G}^{*}$ is trivial $\sigma$-field, then (22) can be easily obtained. Now, suppose that $X$ is independent of $\mathcal{G}$, then
\[
F(x|\mathcal{G}) = F(x)
\]
therefore, $\bar{\eta}^{1} J(X|\mathcal{G}) = \bar{\eta}^{1} J(X).$ (23)

On taking expectation to both sides of (23), we get equality in (22). Conversely, assume that equality in (22) holds. Showing $F(x|\mathcal{G}) = F(x)$ is enough to to prove $X$ and $\sigma$-field $\mathcal{G}$ are independent. Take $U = F(x|\mathcal{G})$, and since $\phi(u) = u^{2}$ is convex hence $E(U^{2}) \geq E^{2}(U) = F^{2}(x)$, and also due to equality in (22), we have
\[
\int_{0}^{\infty} x \left( 1 - E(U^{2}) \right) \, dx = \int_{0}^{\infty} x \left( 1 - F^{2}(x) \right) \, dx = \int_{0}^{\infty} x \left( 1 - E^{2}(U) \right) \, dx.
\]
Hence $E(U^{2}) = E^{2}(U)$. Now, using Corollary 8.1 of [17], we have $F(x|\mathcal{G}) = F(x)$. Hence the proof. ■

We have the following proposition for the Markov property for non-negative random variables $X$, $Y$ and $Z$.

**Proposition 6.2.** Let $X \rightarrow Y \rightarrow Z$ is a Markov chain, then
\[
\bar{\eta}^{1} J(Z|X, Y) = \bar{\eta}^{1} J(Z|Y)
\] (24)
and
\[
E \left( \bar{\eta}^{1} J(Z|Y) \right) \leq E \left( \bar{\eta}^{1} J(Z|X) \right).
\] (25)

Proof: We note that (24) holds using Markov property and definition of $\bar{\eta}^{1} J(Z|X, Y)$. Now, letting $\mathcal{G}^{*} = \sigma(X), \mathcal{G} = \sigma(X, Y)$ and $X = Z$ in Proposition 6.1, we have
\[
\bar{\eta}^{1} J(Z|X) \geq E \left( \bar{\eta}^{1} J(Z|X, Y)|X \right).
\] (26)
Taking expectation on both sides of (26), we have
\[
E \left( \bar{\eta}^{1} J(Z|X) \right) \geq E \left( E \left( \bar{\eta}^{1} J(Z|X, Y)|X \right) \right)
= E \left( \bar{\eta}^{1} J(Z|X, Y) \right)
= E \left( \bar{\eta}^{1} J(Z|Y) \right),
\]
where the last equality holds using (24). Hence the result (25) holds. ■
7. WNCJ of coherent system

A system with a monotone structure function and without any irrelevant components is called a coherent system (for more on a coherent system, see [33,34]). Navarro et al. [35] proved

\[ F_T(x) = q(F(x)) \]

for a coherent system with identically distributed components where \( F_T(x) \) is cdf of a coherent system, \( F \) is the common cdf of the components and \( q \) is distortion function. Function \( q \) is an increasing continuous function in \([0,1]\) such that \( q(0) = 0 \) and \( q(1) = 1 \) which only depends on the structure of the system and the copula of the random vector \((X_1, X_2, \ldots, X_n)\). From the WNCJ defined in (6) and the probability integral transformation \( V = F(T) \), we have

\[
\bar{\eta}^1 J(T) = \frac{1}{2} \int_0^\infty x \left(1 - F_T^2(x)\right) \, dx = \int_0^\infty x \phi(F_T(x)) \, dx \\
= \int_0^\infty x \phi(q(F(x))) \, dx = \int_0^1 \frac{F^{-1}(v)\phi(q(v))}{f(F^{-1}(v))} \, dv, \tag{27}
\]

where \( \phi(u) = \frac{1-u^2}{2} \), for \( 0 < u < 1 \). Also, we have

\[
\bar{\eta}^1 J(X) = \frac{1}{2} \int_0^\infty x \left(1 - F^2(x)\right) \, dx = \int_0^1 \frac{F^{-1}(v)\phi(v)}{f(F^{-1}(v))} \, dv. \tag{28}
\]

Example 7.1. For a parallel system with independent and identically distributed components of \( U(0,1) \) and lifetime \( T = \max(X_1, X_2, X_3, \ldots, X_n) \), we have \( q(v) = v^\mu \), \( \bar{\eta}^1 J(T) = \frac{n}{4(n+1)} \) and \( \bar{\eta}^1 J(X) = \frac{1}{4} \). Thus we note that \( \bar{\eta}^1 J(T) \leq \bar{\eta}^1 J(X) \).

Example 7.2. For a coherent system with identically distributed components of exponential distribution with cdf

\[ F(x) = 1 - e^{-\frac{x}{\mu}} \]

for \( \mu > 0 \) and \( x > 0 \) and lifetime \( T = \max\{\min\{X_1, X_2\}, \min\{X_3, X_4\}\} \), \( f(F^{-1}(v)) = \frac{1-v}{\mu} \). The maximal signature of the system is \((0, 4, -4, 1)\) and so, from (27) we obtain

\[
\bar{\eta}^1 J(T) = \int_0^1 \frac{F^{-1}(v)\phi(q(v))}{f(F^{-1}(v))} \, dv \\
= -\frac{\mu^2}{2} \int_0^1 \ln(1-v) \left[1 - (4v^2 - 4v^3 + v^4)^2\right] \frac{1}{1-v} \, dv \\
= 0.3602\mu^2.
\]

Proposition 7.1. Let \( T \) be the lifetime of the coherent system with identically distributed components and distortion function \( q \). If \( \phi(q(v)) \geq (\leq)\phi(v) \) for all \( 0 < v < 1 \), it holds that

\[ \bar{\eta}^1 J(T) \geq (\leq)\bar{\eta}^1 J(X). \]
Proof: Proof follows from (27), (28) and \( \phi(q(v)) \geq (\leq) \phi(v) \).

Theorem 7.1. Let \( T \) denote the lifetime of a coherent system with identically distributed components with a common pdf \( f \) and distortion function \( q \). If \( S \) is the support of \( f \), \( m = \inf_{x \in S} \frac{f(x)}{x} > 0 \) and \( M = \sup_{x \in S} \frac{f(x)}{x} < \infty \), then
\[
\frac{I_q}{M} \leq \bar{\eta}^{-1} J(T) \leq \frac{I_q}{m}
\]
where \( I_q = \int_0^1 \phi(q(v)) \, dv \) and \( \phi(v) = \frac{1-v^2}{2} \) for \( 0 < v < 1 \).

Proof: From (27), we have
\[
\eta J(T) = \int_0^1 \frac{F^{-1}(v)\phi(q(v))}{f(F^{-1}(v))} \, dv.
\]
Proof follows from
\[
\int_0^1 \frac{F^{-1}(v)\phi(q(v))}{f(F^{-1}(v))} \, dv \geq \int_0^1 \phi(q(v)) \, dv \sup_{v \in (0,1)} \frac{f(F^{-1}(v))}{f(F^{-1}(v))}
\]
and
\[
\int_0^1 \frac{F^{-1}(v)\phi(q(v))}{f(F^{-1}(v))} \, dv \leq \int_0^1 \phi(q(v)) \, dv \inf_{v \in (0,1)} \frac{f(F^{-1}(v))}{f(F^{-1}(v))}.
\]

Proposition 7.2. Let \( T \) be the lifetime of a coherent system with identically distributed components and with distortion function \( q \). Then
\[
B_1 \bar{\eta}^{-1} J(X_1) \leq \bar{\eta}^{-1} J(T) \leq B_2 \bar{\eta}^{-1} J(X_1)
\]
where \( B_1 = \inf_{v \in (0,1)} \left( \frac{\phi(q(v))}{\phi(v)} \right) \) and \( B_2 = \sup_{v \in (0,1)} \left( \frac{\phi(q(v))}{\phi(v)} \right) \).

Proof: We have,
\[
\bar{\eta}^{-1} J(T) = \int_0^1 \frac{F^{-1}(v)\phi(q(v))}{f(F^{-1}(v))} \, dv = \int_0^1 \frac{\phi(q(v))F^{-1}(v)\phi(v)}{\phi(v)f(F^{-1}(v))} \, dv.
\]
The proof follows from the fact that
\[
B_1 = \inf_{v \in (0,1)} \left( \frac{\phi(q(v))}{\phi(v)} \right) \leq \left( \frac{\phi(q(v))}{\phi(v)} \right)
\]
and
\[
B_2 = \sup_{v \in (0,1)} \left( \frac{\phi(q(v))}{\phi(v)} \right) \geq \left( \frac{\phi(q(v))}{\phi(v)} \right).
\]
Proposition 7.3. Let \( T_1 \) and \( T_2 \) be the lifetimes of two coherent systems with identically distributed components and with distortion functions \( q_1 \) and \( q_2 \), respectively. Then
\[
D_1 \bar{\eta}^1 J(T_1) \leq \bar{\eta}^1 J(T_2) \leq D_2 \bar{\eta}^1 J(T_1),
\]
where \( D_1 = \inf_{v \in (0,1)} \left( \frac{\phi(q_2(v))}{\phi(q_1(v))} \right) \) and \( D_2 = \sup_{v \in (0,1)} \left( \frac{\phi(q_2(v))}{\phi(q_1(v))} \right) \).

Proof: It is easy to prove
\[
D_1 \leq \frac{\bar{\eta}^1 J(T_2)}{\bar{\eta}^1 J(T_1)} \leq D_2.
\]
Hence result follows.

Proposition 7.4. Let \( T_1 \) and \( T_2 \) be the lifetimes of two coherent systems with the same structure and with identically distributed components having common distributions \( F \) and \( G \), respectively. If \( X \leq_d Y \) then we have
\[
\eta^1 J(T_1) \leq \eta^1 J(T_2).
\]

Corollary 7.1. Let \( T_1 \) and \( T_2 \) be the lifetimes of two coherent systems with the same structure and with identically distributed components having common distributions \( F \) and \( G \), respectively. If \( X \leq_{hr} Y \) and \( X \) or \( Y \) is DFR, then
\[
\eta^1 J(T_1) \leq \eta^1 J(T_2).
\]

8. GWCRJ based on first order statistics

In this part, we obtain the GWCRJ defined in Definition 2.1, of the first-order statistic. The GWCRJ of the first-order statistic is
\[
\eta^w J(X_{1:n}) = \frac{-1}{2} \int_0^\infty w(x) \bar{F}^2_{1,n}(x) \, dx
\]
and
\[
\eta^m J(X_{1:n}) = \frac{-1}{2} \int_0^\infty x^m \bar{F}^2_{1,n}(x) \, dx,
\]
where \( \bar{F}^2_{1,n}(x) = \bar{F}^{2n}(x) \). Using substitution \( u = \bar{F} \) in (30),
\[
\eta^m J(X_{1:n}) = \frac{-1}{2} \int_0^1 u^{2n}(F^{-1}(1-u))^m f(F^{-1}(1-u)) \, du,
\]
where \( F^{-1} \) is the inverse function of \( F \). Note that when \( m = 1, \eta^1 J(X_{1:n}) \) and \( \eta^1 J(X) \) are calculated in Hashempour et al. [17] for Rayleigh distribution, Pareto distribution and exponential distribution. We calculate in \( \eta^m J(X_{1:n}) \) and \( \eta^m J(X) \), for \( m \geq 0 \) in following examples.

Example 8.1. Let random variable \( X \) have the Pareto distribution with cdf \( F(x) = 1 - \left( \frac{\beta}{x} \right)^\alpha, \ x \geq \beta, \ \beta > 0, \ \alpha > 0 \). Then, \( \eta^m J(X_{1:n}) = \frac{\beta^{m+1}}{2(m+1-2\alpha n)} \) for \( \alpha > \frac{1+m}{2n} \) and \( \eta^m J(X) = \)
\[
\frac{\beta_{m+1}}{2(m+1-2\alpha)} \quad \text{for} \quad \alpha > \frac{1+m}{2}. \text{ Also, } \eta^m J(X_{1:n}) = -\infty \text{ for } 0 < \alpha < \frac{1+m}{2n} \text{ and } \eta^m J(X) = -\infty \text{ for } 0 < \alpha < \frac{1+m}{2n}. 
\]

**Example 8.2.** Let random variable \( X \) have the Rayleigh distribution with cdf \( F \). Then, \( \eta^m J(X_{1:n}) = -\infty \text{ for } 0 < \alpha < \frac{1+m}{2n} \) and \( \eta^m J(X) = -\infty \text{ for } 0 < \alpha < \frac{1+m}{2n} \).

**Example 8.3.** Let random variable \( X \) have the exponential distribution with cdf \( F \). Then, \( \eta^m J(X_{1:n}) = -\frac{\Gamma(m+1)}{2^{m+1}} \) and \( \eta^m J(X) = -\frac{\Gamma(m+1)}{2^{m+1}} \).

Let us see the following lemma which is needed to prove Theorem 8.1 and Proposition 11.1.

**Lemma 8.1.** Let \( g \) be a continuous function with support \([0, 1]\), such that \( \int_0^1 g(y) y^m \, dy = 0 \), for \( m \geq 0 \); then \( g(y) = 0 \), for all \( y \in [0, 1] \).

**Theorem 8.1.** Let \( X_1, \ldots, X_n \) and \( Y_1, \ldots, Y_n \) be two non-negative random samples from absolutely continuous cdfs \( F \) and \( G \) and pdfs \( f \) and \( g \), respectively. Then, \( F(x) = G(x) \) if and only if \( \eta^m J(X_{1:n}) = \eta^m J(Y_{1:n}) \), for all \( n \).

**Proof:** If \( F(x) = G(x) \) then obviously, \( \eta^m J(X_{1:n}) = \eta^m J(Y_{1:n}) \), for all \( n \).

Conversely, when \( \eta^m J(X_{1:n}) = \eta^m J(Y_{1:n}) \), then we have

\[
-\frac{1}{2} \int_0^1 u^{2n} \left( \frac{(F^{-1}(1-u))^m}{f(F^{-1}(1-u))} - \frac{(G^{-1}(1-u))^m}{g(G^{-1}(1-u))} \right) \, du = 0.
\]

By Lemma 8.1, we conclude that \( \frac{(F^{-1}(1-u))^m}{f(F^{-1}(1-u))} = \frac{(G^{-1}(1-u))^m}{g(G^{-1}(1-u))} \).

Let \( t = 1-u \). Since \( \frac{d}{dt}(F^{-1}(t)) = \frac{1}{f(F^{-1}(t))} \) then,

\[
(F^{-1}(t))^m \frac{d}{dt}(F^{-1}(t)) = (G^{-1}(t))^m \frac{d}{dt}(G^{-1}(t)), \quad t \in [0, 1],
\]

which implies \( F^{-1}(t) = G^{-1}(t), t \in [0, 1] \). Hence the result. \( \blacksquare \)

**Remark 8.1.** Consider \( \Delta = \eta^m J(X_{1:n}) - \eta^m J(X) \). Since \( \bar{F}^2(x) \leq \bar{F}^2(x) \), hence \( \Delta \geq 0 \).

**9. Empirical estimator**

Let \( X_1, X_2, \ldots, X_n \) be a random sample from an absolutely continuous cdf \( F \). If \( X_{1:n}, X_{2:n}, \ldots, X_{n:n} \) denote order statistics of random samples \( X_1, X_2, \ldots, X_n \), then the empirical
measure of $F$ is given by

\[
\hat{F}_n(X) = \begin{cases} 
0, & x < X_{1:n} \\
\frac{i}{n}, & X_{i:n} \leq x < X_{i+1:n} \quad \text{for } i = 1, 2, \ldots, n-1 \\
1, & x \geq X_{n:n}.
\end{cases}
\]

### 9.1. Empirical of WNCJ

Using the empirical distribution function, we shall find an estimator of WNCJ. The empirical measure of \( \hat{\eta}_1 J(X) \) is obtained as

\[
\hat{\eta}_1 J(\hat{F}_n) = \frac{1}{2} \int_0^\infty x (1 - \hat{F}_n^2(x)) \, dx
\]

\[
= \frac{1}{2} \sum_{i=1}^{n-1} \int_{X_{i:n}}^{X_{i+1:n}} \left( 1 - \left( \frac{i}{n} \right)^2 \right) x \, dx
\]

\[
= \frac{1}{4} \sum_{i=1}^{n-1} (X_{i+1:n}^2 - X_{i:n}^2) \left[ 1 - \left( \frac{i}{n} \right)^2 \right]. \quad (32)
\]

Moreover, from equation (23) of [16], the empirical measure of negative cumulative extropy \( \hat{\eta}_J(X) \) is

\[
\hat{\eta}_J(\hat{F}_n) = \frac{1}{2} \sum_{i=1}^{n-1} (X_{i+1:n} - X_{i:n}) \left[ 1 - \left( \frac{i}{n} \right)^2 \right] \quad (33)
\]

and from equation (16) of Jahanshahi [14], the empirical measure of negative cumulative residual extropy \( \hat{\xi}(X) \) is

\[
\hat{\xi}(\hat{F}_n) = \frac{1}{2} \sum_{i=1}^{n-1} (X_{i+1:n} - X_{i:n}) \left[ 1 - \left( \frac{i}{n} \right)^2 \right]. \quad (34)
\]

The following theorem states that empirical weighted negative cumulative extropy \( \hat{\eta}_1 J(\hat{F}_n) \) converges almost surely to weighted negative cumulative extropy. Since almost sure convergence is a stronger condition than convergence in probability and convergence in distribution. Therefore \( \hat{\eta}_1 J(\hat{F}_n) \) is also a consistent estimator. Proof of the next statement follows from the proof of Theorem 4.1 of [16].

**Theorem 9.1.** \( \hat{\eta}_1 J(\hat{F}_n) \) converges almost surely to \( \eta_1 J(X) \).

We can write \( \hat{\eta}_1 J(X) \) as

\[
\hat{\eta}_1 J(X) = \frac{1}{2} \int_0^\infty x (1 - F^2(x)) \, dx
\]

\[
= \frac{1}{4} \int_0^1 (1 - u^2) \left[ \frac{d(F^{-1}(u))^2}{du} \right] \, du. \quad (35)
\]
One other way to find an estimator of $\bar{\eta}^1 J(X)$ is to use the method suggested by Vasicek [36]. Following the idea of [36], an estimator of $\bar{\eta}^1 J(X)$ will be calculated by replacing the distribution function $F$ with an empirical distribution function $\hat{F}_N$ and using the difference operator in place of a differential operator. The derivative of $F^{-1}(u)$ with respect to $u$, that is, $\frac{dF^{-1}(u)}{du}$ will be estimated as

$$\frac{X_{i+r;n} - X_{i-r;n}}{\hat{F}_N(X_{i+r;n}) - \hat{F}_N(X_{i-r;n})} = \frac{X_{i+r;n} - X_{i-r;n}}{\frac{i+r}{n} - \frac{i-r}{n}} = \frac{X_{i+r;n} - X_{i-r;n}}{2r/n}.$$ 

Here, window size $r$ is a positive integer less than $\frac{n}{2}$ and $X_{r;n}$ denotes $r$th order statistics from sample $X_1, X_2, \ldots, X_n$. If $i + r > n$ then we consider $X_{i+r;n} = X_{n;n}$ and if $i - r < 1$ then we consider $X_{i-r;n} = X_{1;n}$.

Empirical estimator of $\bar{\eta}^1 J(X)$ is obtained as

$$\bar{\eta}^1_{2J}(\hat{F}_n) = \frac{1}{4n} \sum_{i=1}^{n} \frac{(X_{i+r;n}^2 - X_{i-r;n}^2)}{2r/n} \left[ 1 - \left( \frac{i}{n+1} \right)^2 \right]. \quad (36)$$

The following theorem states that empirical WNCJ $\bar{\eta}^1_{2J}(\hat{F}_n)$ converges almost surely to WNCJ. Since almost sure convergence is a stronger condition than convergence in probability and convergence in distribution. Therefore, $\bar{\eta}^1_{2J}(\hat{F}_n)$ is also a consistent estimator.

**Theorem 9.3.** $\bar{\eta}^1_{2J}(\hat{F}_n)$ converges almost surely to $\bar{\eta}^1 J(X)$.

### 9.2. Empirical of GWNCJ

Using the empirical measure of cdf, we shall find an estimator of GWNCJ given in Definition 3.1.

The empirical measure of $\bar{\eta}^m J(X) = \frac{1}{2} \int_0^\infty x^m(1 - F^2(x)) \, dx$ is obtained as

$$\bar{\eta}^m_{1J}(\hat{F}_n) = \frac{1}{2} \int_0^\infty x^m(1 - \hat{F}_n^2(x)) \, dx$$

$$= \frac{1}{2} \sum_{i=1}^{n-1} \int_{X_{i;n}}^{X_{i+1;n}} \left[ 1 - \left( \frac{i}{n} \right)^2 \right] x^m \, dx$$

$$= \frac{1}{2(m+1)} \sum_{i=1}^{n-1} (X_{i+1;n}^m - X_{i;n}^m) \left[ 1 - \left( \frac{i}{n} \right)^2 \right]. \quad (37)$$

The following theorem states that empirical WNCJ $\bar{\eta}^m_{1J}(\hat{F}_n)$ converges almost surely to WNCJ. Since almost sure convergence is a stronger condition than convergence in probability and convergence in distribution. Therefore, $\bar{\eta}^m_{1J}(\hat{F}_n)$ is also a consistent estimator.

**Theorem 9.3.** $\bar{\eta}^m_{1J}(\hat{F}_n)$ converges almost surely to $\bar{\eta}^m J(X)$. 
Table 1. Value of $T_{1,m}$ for different value of $m$.

| $m$  | 1   | 2   | 3   | 4   | 5   |
|------|-----|-----|-----|-----|-----|
| $T_{1,m}$ | -1.1274 | -1.9931 | -4.3937 | -11.2385 | -32.0084 |

We can write $\bar{\eta}^m J(X)$ as

$$\bar{\eta}^m J(X) = \frac{1}{2} \int_0^\infty x^m (1 - F^2(x)) \, dx$$

$$= \frac{1}{2(m+1)} \int_0^1 (1 - u^2) \left[ \frac{d}{du} (F^{-1}(u))^{m+1} \right] \, du$$  

(38)

One other way to find an estimator of $\bar{\eta}^1 J(X)$ is to use the method suggested by Vasicek [36]. Empirical estimator of $\bar{\eta}^1 J(X)$ is obtained as

$$\bar{\eta}_2^m J(\hat{F}_n) = \frac{1}{2n(m+1)} \sum_{i=1}^n \frac{(X_{i+r_n}^{m+1} - X_{i-r_n}^{m+1})}{2r/n} \left[ 1 - \left( \frac{i}{n+1} \right)^2 \right].$$  

(39)

The following theorem says empirical estimator $\bar{\eta}_2^m J(\hat{F}_n)$ is a consistent estimator of the WNCJ.

The proof follows from similar lines of Theorem 2.1 of [36].

Theorem 9.4. $\bar{\eta}_2^m J(\hat{F}_n)$ converges in probability to $\bar{\eta}^m J(X)$.  

10. Data analysis

Using a real dataset for the uncensored case, we use the suggested estimate methodologies. The GWCRJ and the GWNCJ defined in Sections 2 and 3, respectively, are estimated using the U-statistic and empirical distribution function.

Let us consider dataset-1 from [37], concerning the failure times of 84 mechanical components. Tahmasebi and Toomaj [16] and Kattumannil and Sreedevi [15] also used this dataset to calculate the value of estimators.

Dataset-1: 0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.07, 1.914, 32.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.01, 2.688, 3.924, 1.281, 2.038, 2.823, 4.035, 1.281, 2.085, 2.89, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.24, 1.48, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.19, 3, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.57, 1.652, 2.3, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663.

We obtain $T_{1,m}$ for different values of $m$ in Table 1. Note that when $m = 0$, value of the estimator $T_{1,m}$ is -0.9636 as given in Kattumannil and Sridevi (2022). We obtain $\bar{\eta}_1^m J(\hat{F}_n)$ for different value of $m$ in Table 2. Note that when $m = 0$, value of the estimator $\bar{\eta}_1^m J(\hat{F}_n)$ is -0.9494 as given in Tahmasebi and Toomaj [16]. We obtain $\bar{\eta}_2^m J(\hat{F}_n)$ for different value of $m$ and window size $r = 2$ in Table 3.
Table 2. Value of $\eta_{1m}^n J(\hat{F}_n)$ for different value of $m$.

| $m$   | 1     | 2     | 3     | 4     | 5     |
|-------|-------|-------|-------|-------|-------|
| $\eta_{1m}^n J(\hat{F}_n)$ | 9.0157 | 143.9347 | 3379.8231 | 87,824.5989 | 2,387,785.4916 |

Table 3. Value of $\eta_{2m}^n J(\hat{F}_n)$ for different value of $m$.

| $m$   | 1     | 2     | 3     | 4     | 5     |
|-------|-------|-------|-------|-------|-------|
| $\eta_{2m}^n J(\hat{F}_n)$ | 11.9378 | 208.4374 | 4961.4424 | 129,138.6574 | 3,511,758.8850 |

11. Application

In the following, we use $\eta_{1m}^n J(\hat{F}_n)$ obtained in (37) for testing the uniformity of the random sample $X_1, \ldots, X_n$. We first discuss the nice property of uniform distribution among all distributions defined on interval $(0, 1)$ before moving on to a test statistic. For a random variable $X$ with the cdf $F$ and for $p \in (0, 1)$, let $\psi_{p}^{m} J(F)$ be defined as

$$\psi_{p}^{m} J(F) = \frac{1}{2} \int_{0}^{p} x^m (1 - F^2(x)) \, dx, \quad m \geq 0. \quad (40)$$

**Example 11.1.** If $F_0$ is cdf of uniform random variable on interval $(0, 1)$, then $\psi_{p}^{m} J(F_0) = \frac{p^{m+1}}{2} \left( \frac{1}{m+1} - \frac{p^2}{m+3} \right)$ and in particular, when $m = 1$, $\psi_{p}^{1} J(F_0) = \frac{p^2 (2-p^2)}{8}$ and when $m = 0$, $\psi_{p}^{0} J(F_0) = \frac{p (3-p^2)}{6}$.

In view of Lemma 8.1 and Example 11.1, the following proposition is obtained.

**Proposition 11.1.** Let $F_0$ is cdf of uniform random variable on interval $(0, 1)$. Suppose that for a cdf $F$ in the class of cdfs defined on interval $(0, 1)$, $\psi_{p}^{m} J(F) = \psi_{p}^{m} J(F_0)$. Then $F(x) = F_0(x)$, almost everywhere.

**Proof:** Let $\psi_{p}^{m} J(F) = \psi_{p}^{m} J(F_0)$, using (40) we get

$$\int_{0}^{p} x^m \left( F^2(x) - F_0^2(x) \right) \, dx = 0, \quad \forall \ p \in (0, 1).$$

It is known that the $(0, p)$ generate the Borel $\sigma$-algebra of $\Omega = (0, 1]$. Therefore, we can write

$$\int_{B} x^m \left( F^2(x) - F_0^2(x) \right) \, dx = 0, \quad \forall \ B \subseteq (0, 1].$$

By Lemma 8.1, we conclude that $F(x) = F_0(x)$, almost everywhere.

Thus, $\psi_{p}^{m} J(F)$ for $p \in (0, 1)$ is uniquely determined by the uniform distribution in the sense that for some cdfs defined on $(0, 1)$, they take a value less than $\frac{p^{m+1}}{2} \left( \frac{1}{m+1} - \frac{F^2}{m+3} \right)$ and for some of them, they take more than $\frac{p^{m+1}}{2} \left( \frac{1}{m+1} - \frac{p^2}{m+3} \right)$, and only for the standard uniform distribution, we have $\psi_{p}^{m} J(F_0) = \frac{p^{m+1}}{2} \left( \frac{1}{m+1} - \frac{p^2}{m+3} \right)$. ■
11.1. The test of uniformity

A test statistic for the uniform goodness of fit test can be constructed based on the above proposition. One can construct a test statistic based on \( \bar{\eta}_1^m J(\hat{F}_n) \) given in (37), which is the sampling counterpart of the GWCPJ measure. Let \( X_1, X_2, \ldots, X_n \) be a random sample of size \( n \) from a population with unknown cdf \( F \). We consider \( X \) as a non-negative random variable. Here, we are interested in testing whether a distribution is a standard uniform distribution. That is, \( H_0 : F = F_0 \) against \( H_1 : F \neq F_0 \), where \( F_0 \) is the cdf of standard uniform distribution. A new non-parametric test is being developed and

\[
\bar{\eta}_1^m J(\hat{F}_n) = \frac{1}{2(m + 1)} \sum_{i=1}^{n-1} (X_{i+1:n}^{m+1} - X_{i:n}^{m+1}) \left[ 1 - \left( \frac{i}{n} \right)^2 \right]
\]

is our test statistic. For computational purposes, we choose \( m = 1 \). The process will remain the same when we choose any other value of \( m \). We need to calculate the critical region for the uniform goodness of fit test problem. The critical region is then obtained in the sense that \( \bar{\eta}_1^m J(\hat{F}_n) \) is less than \( C_1(\alpha) \) or greater than \( C_2(\alpha) \), where \( \alpha \) is a prespecified level of significance of the test. We needs to determine \( C_1(\alpha) \) and \( C_2(\alpha) \), and whenever \( \bar{\eta}_1^m J(\hat{F}_n) < C_1(\alpha) \) or \( \bar{\eta}_1^m J(\hat{F}_n) > C_2(\alpha) \), then the null hypothesis is rejected in favour of an alternative one. We will fail to reject the null hypothesis at the level of significance \( \alpha \) when \( C_1(\alpha) < \bar{\eta}_1^m J(\hat{F}_n) < C_2(\alpha) \). Since the distribution of \( \bar{\eta}_1^m J(\hat{F}_n) \) the is difficult to derive, therefore, \( C_1(\alpha) \) and \( C_2(\alpha) \) can be estimated using the empirical quantile of the test statistic \( \bar{\eta}_1^m J(\hat{F}_n) \) under the standard uniform distribution. All tests are carried out at a 5% nominal level and all simulations are done using 100,000 replications. We generate a random sample of size \( n \) from the standard uniform distribution and then compute the value of \( \bar{\eta}_1^m J(\hat{F}_n) \) with 100,000 replications. We sort these 100,000 values of \( \bar{\eta}_1^m J(\hat{F}_n) \). Then, \( C_1(\alpha) \) and \( C_2(\alpha) \) are estimated by the 0.025th quantile and 0.975th quantile of the empirical distribution of \( \bar{\eta}_1^m J(\hat{F}_n) \), respectively. We list the values of \( C_1(\alpha) \) and \( C_2(\alpha) \), for different sample sizes at 5% level of significance (\( \alpha = 0.05 \) and \( m = 1 \)) in Table 5.

### Table 5. Values of \( C_1(\alpha) \) and \( C_2(\alpha) \), for \( \alpha = 0.05 \).

| \( n \) | 10 | 20 | 30 | 40 | 50 |
|--------|----|----|----|----|----|
| \( C_1(\alpha) \) | 0.0599 | 0.0804 | 0.0889 | 0.0943 | 0.0976 |
| \( C_2(\alpha) \) | 0.1706 | 0.1611 | 0.1557 | 0.1520 | 0.1495 |

11.2. Power of the test

In this section, the power of the proposed test statistic is calculated and listed in Table 4 against alternative Beta(1.5,1.5) and Beta(1.0,1.0) distribution for different sample sizes at the level of significance \( \alpha = 0.05 \) and \( m = 1 \). We consider beta distribution as an alternative because it has support (0,1) like standard uniform distribution. Kazemi et al. [23] have
also obtained power against Beta(1.5, 1.5) distribution in testing uniformity using WCPJ. Our newly proposed test is more powerful for large sample sizes. We found that power increases when the sample size \( n \) increases. Since Beta(1,1) is identically distributed to a standard uniform distribution, therefore power is equal to the size of the test. The power against Beta (1.5,1.5) and Beta(1.0,1.0) are given in Table 4 for sample size \( n = 40, 50, 100. \)

### 12. Conclusion

Extropy has been studied and generalized by various researchers as discussed in the introduction section. We defined and studied GWCRJ, GWNCJ and conditional WNCJ. Non-Parametric estimators for GWCRJ and GWNCJ have been proposed based on U-statistics and empirical distribution function. We obtained the upper and lower bound of WNCJ. We proposed some results to WNCJ related to a coherent system and reliability theory. GWCRJ based on first-order statistics is calculated. Data analysis of proposed estimators is provided. A new test of uniformity has been given based on GWNCJ.

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