1 Introduction

Quantum mechanical probability amplitudes may be multiplied by an arbitrary phase factor without affecting measured quantities which depend on them. We have recently introduced generalized spin quantities whose derivation depends upon probability amplitudes characterizing spin-projection measurements [1–3]. In these derivations, the phase of these probability amplitudes was arbitrarily chosen. In this paper, we investigate the effect of changing the phase. We find that, as expected, when we change the phase of one of the probability amplitudes, both the operators for the generalized spin components and their eigenvectors change form. Focusing our attention on the spin-1/2 case, we obtain alternative forms for the components of the spin operator and for the spin states. We give the eigenvectors of these alternative forms and explain how these can be easily computed for whatever phase choice we make.

This paper is organized as follows. The general theory underlying the method used to obtain the generalized spin quantities is presented in Section 2. Thus, the general implicit formulas for the z component of the spin are given in Subsection 2.1. The method by which the generalized probability amplitudes are derived is sketched in Subsection 2.2. This theory is then illustrated in Subsection 2.3 where, using one phase choice, the explicit formulas for the probability amplitudes, the three spin components and their eigenvalues are given. In Section 3, we change the phase and thus obtain different probability amplitudes. These are presented in Subsection 3.1. The new z component of spin and its eigenvalues resulting from this change in probability amplitudes are given in Subsection 3.2, while the x and y components of spin and their eigenvectors are presented in Subsections 3.3 and 3.4. In Section 4, we relate the new results to standard ones by giving the limit in which the new results reduce to the standard ones. The Discussion and Conclusion in Section 5 close the paper.

2 General Formulas

2.1 z Component of the Spin Operator

In this paper, we are concerned with the case of spin 1/2. We measure spin in units of $\hbar/2$. Let $\hat{a}$ be an arbitrary vector whose polar angles are $(\theta'', \varphi'')$, and let $\hat{c}$ be the vector with polar angles $(\theta', \varphi')$. We wish to measure the spin projection with respect to the vector $\hat{c}$ after having measured it with respect to the vector $\hat{a}$. Thus, the spin projection is initially known to be either up or down with respect to the vector $\hat{a}$. We denote by $\psi$ the probability amplitudes for the spin projection measurement. Thus, if the spin projection is initially up with respect to $\hat{a}$, the probability amplitude for finding it up with respect to $\hat{a}$ after measurement is $\psi((+\frac{1}{2})(\hat{a}); (+\frac{1}{2})(\hat{c}))$. There are three other probability amplitudes: these are $\psi((+\frac{1}{2})(\hat{a}); (-\frac{1}{2})(\hat{c})), \psi((-\frac{1}{2})(\hat{a}); (+\frac{1}{2})(\hat{c}))$ and
\( \psi((-\frac{1}{2})\hat{a};\frac{1}{2})\hat{c}) \): their interpretation is obvious.

The probability amplitudes \( \psi \) can be expanded in terms of the probability amplitudes \( \chi \) and \( \phi \). The probability amplitudes \( \chi \) describe spin-projection measurements from the quantization direction \( \hat{a} \) to the quantization direction \( \hat{b} \), where the vector \( \hat{b} \) has the polar angles \((\theta, \varphi)\). The expansions are

\[
\psi((+\frac{1}{2})\hat{a};(+\frac{1}{2})\hat{c}) = \chi((+\frac{1}{2})\hat{a};(+\frac{1}{2})\hat{b})\phi((+\frac{1}{2})\hat{b};(+\frac{1}{2})\hat{c}) \\
+\chi((+\frac{1}{2})\hat{a};(-\frac{1}{2})\hat{b})\phi((-\frac{1}{2})\hat{b};(+\frac{1}{2})\hat{c}), \quad (1)
\]

\[
\psi((+\frac{1}{2})\hat{a};(-\frac{1}{2})\hat{c}) = \chi((+\frac{1}{2})\hat{a};(+\frac{1}{2})\hat{b})\phi(+\frac{1}{2})\hat{b};(-\frac{1}{2})\hat{c}) \\
+\chi((+\frac{1}{2})\hat{a};(-\frac{1}{2})\hat{b})\phi((-\frac{1}{2})\hat{b};(-\frac{1}{2})\hat{c}), \quad (2)
\]

\[
\psi((-\frac{1}{2})\hat{a};(+\frac{1}{2})\hat{c}) = \chi((-\frac{1}{2})\hat{a};(+\frac{1}{2})\hat{b})\phi((+\frac{1}{2})\hat{b};(+\frac{1}{2})\hat{c}) \\
+\chi((-\frac{1}{2})\hat{a};(-\frac{1}{2})\hat{b})\phi((-\frac{1}{2})\hat{b};(+\frac{1}{2})\hat{c}) \quad (3)
\]

and

\[
\psi((-\frac{1}{2})\hat{a};(-\frac{1}{2})\hat{c}) = \chi((-\frac{1}{2})\hat{a};(+\frac{1}{2})\hat{b})\phi((+\frac{1}{2})\hat{b};(-\frac{1}{2})\hat{c}) \\
+\chi((-\frac{1}{2})\hat{a};(-\frac{1}{2})\hat{b})\phi((-\frac{1}{2})\hat{b};(-\frac{1}{2})\hat{c}). \quad (4)
\]

Let the quantity \( R(\sigma \cdot \hat{c}) \) have the value \( r_1 \) when the spin projection is up (quantum number \( m_1 \) with respect to the vector \( \hat{c} \), and \( r_2 \) when it is down (quantum number \( m_2 \)). Thus, the possible values of \( R \) are \( r_n \) \((n = 1, 2)\). Suppose that the initial state corresponds to the quantum number \( m_i \) with respect to \( \hat{a} \). As the probability amplitude for obtaining \( r_n \) is \( \psi(m_1\hat{a};m_n\hat{c}) \), the expectation value of \( R \) is

\[
\langle R \rangle = \sum_{n=1}^{2} \left| \psi(m_1\hat{a};m_n\hat{c}) \right|^2 r_n. \quad (5)
\]

Since the expansions for \( \psi^*(m_1\hat{a};m_n\hat{c}) \) and \( \psi(m_1\hat{a};m_n\hat{c}) \) are

\[
\psi^*(m_1\hat{a};m_n\hat{c}) = \sum_{j=1}^{2} \chi^*(m_1\hat{a};m_j\hat{b})\phi^*(m_j\hat{b};m_n\hat{c}) \quad (6)
\]

and
\[
\psi(m_i; m_n) = \sum_{j'=1}^{2} \chi(m_i; m_j) \phi(m_j; m_n),
\]

it follows that

\[
\langle R \rangle = \sum_j \sum_{j'} \chi^*(m_i; m_j) R_{jj'} \chi(m_i; m_j)
\]

\[
= [\psi(m_i; m_n)]^\dagger [R] [\psi(m_i; m_n)]
\]

where [1]

\[
[\psi(m_i; m_n)] = \begin{pmatrix}
\chi(m_i; (+\frac{1}{2}) m_n) \\
\chi(m_i; (-\frac{1}{2}) m_n)
\end{pmatrix}
\]

and

\[
[R] = \begin{pmatrix}
R_{11} & R_{12} \\
R_{21} & R_{22}
\end{pmatrix}
\]

with the elements of [R] being given by

\[
R_{jj'} = \sum_{n=1}^{2} \phi^*(m_j; m_n) \phi(m_j; m_n) r_n.
\]

Written out explicitly, the elements of [R] are

\[
R_{11} = \left| \phi((+\frac{1}{2}) m_n; (+\frac{1}{2}) m_n) \right|^2 r_1 + \left| \phi((+\frac{1}{2}) m_n; (-\frac{1}{2}) m_n) \right|^2 r_2,
\]

\[
R_{12} = \phi^*((+\frac{1}{2}) m_n; (+\frac{1}{2}) m_n) \phi((-\frac{1}{2}) m_n; (+\frac{1}{2}) m_n) r_1 + \phi^*((+\frac{1}{2}) m_n; (-\frac{1}{2}) m_n) \phi((-\frac{1}{2}) m_n; (-\frac{1}{2}) m_n) r_2;
\]

\[
R_{21} = \phi^*((-\frac{1}{2}) m_n; (+\frac{1}{2}) m_n) \phi((+\frac{1}{2}) m_n; (+\frac{1}{2}) m_n) r_1 + \phi^*((-\frac{1}{2}) m_n; (-\frac{1}{2}) m_n) \phi((+\frac{1}{2}) m_n; (-\frac{1}{2}) m_n) r_2
\]

and

\[
R_{22} = \left| \phi((-\frac{1}{2}) m_n; (+\frac{1}{2}) m_n) \right|^2 r_1 + \left| \phi((-\frac{1}{2}) m_n; (-\frac{1}{2}) m_n) \right|^2 r_2.
\]
When \( R = \sigma \cdot \hat{c} \), then it is the component of the spin in the direction \( \hat{c} \). Hence, \([R]\) is the matrix form of the operator for the component of the spin along the axis defined by \( \hat{c} \). In its most generalized form, we shall denote this operator by \([\sigma_\hat{c}]\).

When \( R = \sigma^2 \), then \([R]\) is the matrix form of the square of the total spin.

The precise form of both the vector state Eqn. 3 and the operator Eqns. 12 - 13 is evidently a function of the phase adopted for the probability amplitudes \( \psi, \chi \) and \( \phi \). Since the phase does not change the probabilities, we see from Eqns. 12 and 13 that the diagonal elements remain the same no matter what phase we choose. But the form of the off-diagonal elements is a function of the choice of phase. In order to make these observations concrete, we need to obtain explicit expressions for the probability amplitudes.

### 2.2 Probability Amplitudes

Following the method we introduced in Ref. 1 and used in Refs. 2 and 3, we obtain the probability amplitudes in the following manner. Consider the vector \( \hat{a} \) (whose polar angles are \((\theta'', \phi'')\)). The dot product of this vector with the standard spin operator

\[
[\sigma] = \hat{i} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \hat{j} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \hat{k} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

is

\[
[\sigma \cdot \hat{a}] = \begin{pmatrix} \cos \theta'' & \sin \theta'' e^{-i\phi''} \\ \sin \theta'' e^{i\phi''} & -\cos \theta'' \end{pmatrix}
\]

This operator is the spin projection in the direction \( \hat{a} \). As it is the generalized form of the operator for \( z \) component of the spin, we shall call it the "standard generalized \( z \) component" of the spin. Corresponding to it are the "standard generalized eigenvectors". Similarly, the \( x \) and \( y \) components that go with this operator, as well as their eigenvectors are called "standard generalized forms". The reason is that they appear to be the most generalized forms for these quantities appearing in the literature. However, as we have already shown in Refs. 1 - 3, these are in fact specialized forms of even more generalized quantities.

The elements of a matrix eigenvector are probability amplitudes. As proven in Ref. 1, these probability amplitudes are not just constants but have a structure. Thus, in this case the eigenvectors for the eigenvalues \(+1\) and \(-1\) have the respective forms

\[
[\xi_+] = \begin{pmatrix} \chi((+\frac{1}{2})\hat{a}); (+\frac{1}{2})\hat{f}_1) \\ \chi((+\frac{1}{2})\hat{a}); (-\frac{1}{2})\hat{f}_1) \end{pmatrix}
\]

4
and

\[
\begin{pmatrix}
[\xi_-] \\
[\xi_+]
\end{pmatrix} = \begin{pmatrix}
\chi((\frac{1}{2})\hat{a}); (+\frac{1}{2})\hat{f}) \\
\chi((\frac{1}{2})\hat{a}); (-\frac{1}{2})\hat{f})
\end{pmatrix}
\]  

(19)

where \(\hat{f}\) is an unknown direction vector.

On the other hand, for \(\hat{c}\) (whose polar angles are \((\theta', \varphi')\)), we have

\[
[\sigma \cdot \hat{c}] = \begin{pmatrix}
\cos \theta' & \sin \theta' e^{-i\varphi'} \\
\sin \theta' e^{i\varphi'} & -\cos \theta'
\end{pmatrix}
\]  

(20)

The eigenvectors of this matrix for the eigenvalues +1 and −1 are respectively

\[
\begin{pmatrix}
[\xi_+] \\
[\xi_-]
\end{pmatrix} = \begin{pmatrix}
\chi((+\frac{1}{2})\hat{c}); (+\frac{1}{2})\hat{f}) \\
\chi((+\frac{1}{2})\hat{c}); (-\frac{1}{2})\hat{f})
\end{pmatrix}
\]  

(21) and

\[
\begin{pmatrix}
[\xi_+] \\
[\xi_-]
\end{pmatrix} = \begin{pmatrix}
\chi((\frac{1}{2})\hat{c}); (+\frac{1}{2})\hat{f}) \\
\chi((-\frac{1}{2})\hat{c}); (-\frac{1}{2})\hat{f})
\end{pmatrix}
\]  

(22)

We now use the Landé expansion [4−7] for probability amplitudes to obtain the generalized probability amplitudes \(\chi(m_i^\hat{a}; m_n^\hat{c})\), where \(i, n = 1, 2\). This expansion concerns three observables \(A, B\) and \(C\) which belong to the same quantum system. These observables are characterized by the states corresponding to the eigenvalues \(A_i, B_j\) and \(C_k\) respectively. The probability amplitudes connecting states of these observables are inter-related in the following way:

\[
\psi(A_i; C_n) = \sum_j \xi(A_i; B_j)\phi(B_j; C_n) 
\]  

(23)

where \(\psi, \xi\) and \(\phi\) are all probability amplitudes and \(j\) runs over all the states corresponding to \(B\). In this case, all three observables are spin projections; thus, \(A \rightarrow \hat{a}, B \rightarrow \hat{f}\) and \(C \rightarrow \hat{c}\). Owing to the fact that all three probability amplitudes refer to spin projection measurements, they have the same structure; for this reason, we shall for the moment collectively denote them by \(\chi\). Using the Hermiticity property of the probability amplitudes [4]

\[
\chi(A_i; C_n) = \chi^*(C_n; A_i),
\]  

(24)

we are able to apply the Landé expansion, Eqn. (23), in order to eliminate the unknown vector \(\hat{f}\) and so obtain the probability amplitudes \(\chi(m_i^\hat{a}; m_n^\hat{c})\).
2.3 Spin States

The spin states, Eqn. (9), are seen to have a form that depends on the expressions for the probability amplitudes. Their general form is identical to the general form of the eigenvectors of spin operators. A change in the phase of the probability amplitudes alters the form of the spin states. Some of these states are special in the sense that they are eigenvectors of spin operators, but this specialness consists in the arguments which the expressions for the probability amplitudes have.

2.4 Spin Quantities For Old Choice of Phase

As observed above, the form of the generalized probability amplitudes depends on the phase choice we make when we determine the vectors Eqns. (18), (19), (21) and (22). Our phase choice in Refs. 1-3 led to
\[
[\xi_+] = \begin{pmatrix} \cos \theta''/2 \\ e^{i\varphi''} \sin \theta''/2 \end{pmatrix}
\]
(25)
and
\[
[\xi_-] = \begin{pmatrix} \sin \theta''/2 \\ -e^{i\varphi''} \cos \theta''/2 \end{pmatrix}
\]
(26)
with identical expressions for $[\xi_+]$ and $[\xi_-]$, but with argument transformations $\theta'' \to \theta'$ and $\varphi'' \to \varphi'$. The use of the Landé expansion Eqn. (23) together with the condition Eqn. (24) led to the generalized probability amplitudes
\[
\chi\left( (+\frac{1}{2})\hat{a}, (+\frac{1}{2})\hat{c} \right) = \cos \theta''/2 \cos \theta'/2 + e^{i(\varphi''-\varphi')} \sin \theta''/2 \sin \theta'/2 \quad (27)
\]
\[
\chi\left( (+\frac{1}{2})\hat{a}, (-\frac{1}{2})\hat{c} \right) = \cos \theta''/2 \sin \theta'/2 - e^{i(\varphi''-\varphi')} \sin \theta''/2 \cos \theta'/2 \quad (28)
\]
\[
\chi\left( (-\frac{1}{2})\hat{a}, (+\frac{1}{2})\hat{c} \right) = \sin \theta''/2 \cos \theta'/2 - e^{i(\varphi''-\varphi')} \cos \theta''/2 \sin \theta'/2 \quad (29)
\]
\[
\chi\left( (-\frac{1}{2})\hat{a}, (-\frac{1}{2})\hat{c} \right) = \sin \theta''/2 \sin \theta'/2 + e^{i(\varphi''-\varphi')} \cos \theta''/2 \cos \theta'/2 \quad (30)
\]

Using these forms for the probability amplitudes in the general formulas Eqns. (12) - (15), and remembering that the $\phi$’s have the same form as the $\chi$’s, we are able to obtain explicit formulas for the elements of $[\sigma_{\hat{c}}^\dagger]$. We have to remember that as well as changing from the $\chi$’s to the $\phi$’s, we have to change the arguments to $\phi(m_i^\dagger; m_n^\dagger)$, where the polar angles of $\hat{b}$ are $(\theta, \varphi)$. We find that the expressions for the elements of the operator $[\sigma_{\hat{c}}^\dagger]$ are:
\[
\begin{align*}
\sigma_{\mathbf{c}}^{(11)} &= \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\varphi - \varphi') \\
\sigma_{\mathbf{c}}^{(12)} &= \sin \theta \cos \theta' - \sin \theta \cos \theta' - \sin \theta' \cos(\varphi - \varphi') + i \sin(\varphi - \varphi') \\
\sigma_{\mathbf{c}}^{(21)} &= \sin \theta \cos \theta' - \sin \theta \cos \theta' - \sin \theta' \cos(\varphi - \varphi') - i \sin(\varphi - \varphi')
\end{align*}
\]

and
\[
\sigma_{\mathbf{c}}^{(11)} = -\cos \theta \cos \theta' - \sin \theta \sin \theta' \cos(\varphi - \varphi')
\]

In order to obtain the eigenvectors of \([\sigma_{\mathbf{c}}]\), we use the fact that while the probability amplitudes \(\phi(m_{1}; m_{\varphi})\) are needed to obtain the elements of \([\sigma_{\mathbf{c}}]\), the elements of the eigenvectors are of the form \(\chi(m_{1}; m_{\varphi})\), as deduced in Ref. 1. Thus, the eigenvectors of this operator are

\[
\begin{align*}
\xi_{\mathbf{c}}^{(+)} &= \begin{pmatrix}
\chi((+\frac{1}{2})_{\mathbf{c}}, (+\frac{1}{2})_{\mathbf{b}}) \\
\chi((+\frac{1}{2})_{\mathbf{c}}, (-\frac{1}{2})_{\mathbf{b}})
\end{pmatrix}
\begin{pmatrix}
\cos \theta'/2 \cos \theta/2 + e^{i(\varphi' - \varphi)} \sin \theta'/2 \sin \theta/2 \\
\cos \theta'/2 \sin \theta/2 - e^{i(\varphi' - \varphi)} \sin \theta'/2 \cos \theta/2
\end{pmatrix}
\end{align*}
\]

for eigenvalue +1 and

\[
\begin{align*}
\xi_{\mathbf{c}}^{(-)} &= \begin{pmatrix}
\chi((-\frac{1}{2})_{\mathbf{c}}, (+\frac{1}{2})_{\mathbf{b}}) \\
\chi((-\frac{1}{2})_{\mathbf{c}}, (-\frac{1}{2})_{\mathbf{b}})
\end{pmatrix}
\begin{pmatrix}
\sin \theta'/2 \cos \theta/2 - e^{i(\varphi' - \varphi)} \cos \theta'/2 \sin \theta/2 \\
\sin \theta'/2 \sin \theta/2 + e^{i(\varphi' - \varphi)} \cos \theta'/2 \cos \theta/2
\end{pmatrix}
\end{align*}
\]

for eigenvalue −1.

The operator \([\sigma_{\mathbf{c}}]\) is a generalized form of the z component of the spin. As we can see, it is more generalized than the "standard generalized forms” discussed earlier. Corresponding to \([\sigma_{\mathbf{c}}]\) are operators which we may formally obtain through the generalized ladder operators. The generalized ladder operators are obtainable from their actions on the eigenvectors \([\xi_{\mathbf{c}}^{(+)}]\) and \([\xi_{\mathbf{c}}^{(-)}]\) of \([\sigma_{\mathbf{c}}]\). Then we use the definitions of the ladder operators in terms of the x and y components of the spin operator to obtain these quantities. However, a shorter method was introduced in Refs. 2 and 3.

The operator \([\sigma_{x}]\) may be obtained for this case from the operator \([\sigma_{\mathbf{c}}]\) by setting \(\theta' \to \theta' - \pi/2\) and leaving \(\varphi'\) unchanged [2, 3]. The same transformation gives the eigenvectors of \([\sigma_{x}]\) from those of \([\sigma_{\mathbf{c}}]\).

The elements of \([\sigma_{x}]\) are found to be [1, 2]

\[
\begin{align*}
\sigma_{x}^{(11)} &= -\sin \theta \cos \theta' \cos(\varphi' - \varphi) + \sin \theta' \cos \theta
\end{align*}
\]
\[
(\sigma_x)_{12} = \cos \theta \cos \theta' \cos(\varphi' - \varphi) + \sin \theta \sin \theta' - i \cos \theta' \sin(\varphi' - \varphi) \tag{38}
\]
\[
(\sigma_x)_{21} = \cos \theta \cos \theta' \cos(\varphi' - \varphi) + \sin \theta \sin \theta' + i \cos \theta' \sin(\varphi' - \varphi) \tag{39}
\]
and
\[
(\sigma_x)_{22} = \sin \theta \cos \theta' \cos(\varphi' - \varphi) - \sin \theta' \cos \theta \tag{40}
\]
The eigenvectors of \([\sigma_x]\) are \([2]\)

\[
[\xi_x^{(+)}] = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \sin \theta' \cos \theta' \cos(\varphi' - \varphi) + e^{i(\varphi' - \varphi)} \sin \theta' \sin \theta \cos \theta' \\ \sin \theta' \sin \theta - e^{i(\varphi' - \varphi)} \sin \theta' \cos \theta \end{array} \right) \tag{41}
\]
and
\[
[\xi_x^{(-)}] = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \sin \theta' \cos \theta' \cos(\varphi' - \varphi) - e^{i(\varphi' - \varphi)} \sin \theta' \sin \theta \cos \theta' \\ \sin \theta' \sin \theta + e^{i(\varphi' - \varphi)} \sin \theta' \cos \theta \end{array} \right) \tag{42}
\]
for the eigenvalues +1 and −1 respectively.

To obtain the operator \([\sigma_y]\) from \([\sigma_c]\), we transform the arguments thus: \(\theta' \rightarrow \pi/2, \varphi' \rightarrow \varphi' - \pi/2\). We get the eigenvectors of \([\sigma_y]\) by applying the same transformation to those of \([\sigma_c]\). The elements of \([\sigma_y]\) are \([1, 2]\)

\[
(\sigma_y)_{11} = \sin \theta \sin(\varphi' - \varphi) \tag{43}
\]
\[
(\sigma_y)_{12} = -i \cos(\varphi' - \varphi) - \cos \theta \sin(\varphi' - \varphi) \tag{44}
\]
\[
(\sigma_y)_{21} = i \cos(\varphi' - \varphi) - \cos \theta \sin(\varphi' - \varphi) \tag{45}
\]
and
\[
(\sigma_y)_{22} = -\sin \theta \sin(\varphi' - \varphi) \tag{46}
\]
The eigenvectors are \([2]\)

\[
[\chi_y^{(+)}] = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \cos \frac{\theta}{2} - i e^{i(\varphi' - \varphi)} \sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} + i e^{i(\varphi' - \varphi)} \cos \frac{\theta}{2} \end{array} \right) \tag{47}
\]
and
\[
[\chi_y^{(-)}] = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \cos \frac{\theta}{2} + i e^{i(\varphi' - \varphi)} \sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} - i e^{i(\varphi' - \varphi)} \cos \frac{\theta}{2} \end{array} \right) \tag{48}
\]
corresponding to the eigenvalues +1 and −1 respectively.

We emphasize that the spin states have the same forms as the eigenvectors of the $z$ component of spin. The two kinds of quantities differ only in the arguments.

### 3 New Choice of Phase

#### 3.1 Probability Amplitudes

We now choose a different phase for the probability amplitudes in the vectors Eqs. (25) and (26). This will lead to different forms for the generalized probability amplitudes Eqs. (27) and (30). Many alternatives are possible, but one alternative will suffice to illustrate what happens to the spin quantities when we make this change of phase. We effect this change in phase by taking for the solutions of

$$[\sigma \hat{a}] = \begin{pmatrix} \cos \theta'' & \sin \theta'' e^{-i\varphi''} \\ \sin \theta'' e^{i\varphi''} & -\cos \theta'' \end{pmatrix}$$

the vectors

$$[\xi_-] = \begin{pmatrix} \sin \theta''/2 \\ -e^{i\varphi''} \cos \theta''/2 \end{pmatrix}$$

and

$$[\xi_+] = \begin{pmatrix} e^{-i\varphi''} \cos \theta''/2 \\ \sin \theta''/2 \end{pmatrix}$$

in place of Eqn. (23) and Eqn. (26). Thus only Eqn. (23) is changed; Eqn. (23) remains unchanged. We do the same thing when computing the eigenvectors of $[\sigma \hat{c}]$. Using the Landé expansion as before to get $\psi(m_\alpha^a; m_\alpha^c)$, we then obtain the following probability amplitudes

$$\psi((+\frac{1}{2})\hat{a}; (+\frac{1}{2})\hat{c}) = e^{i(\varphi' - \varphi'')} \cos \theta'/2 \cos \theta''/2 + \sin \theta'/2 \sin \theta''/2$$

$$\psi((+\frac{1}{2})\hat{a}; (-\frac{1}{2})\hat{c}) = e^{-i\varphi''} \cos \theta'/2 \sin \theta''/2 - e^{-i\varphi'} \sin \theta''/2 \cos \theta'/2$$

$$\psi((-\frac{1}{2})\hat{a}; (+\frac{1}{2})\hat{c}) = e^{i\varphi'} \cos \theta'/2 \sin \theta''/2 - e^{i\varphi''} \sin \theta'/2 \cos \theta''/2$$

and

$$\psi((-\frac{1}{2})\hat{a}; (-\frac{1}{2})\hat{c}) = e^{-i(\varphi' - \varphi'')} \cos \theta'/2 \cos \theta''/2 + \sin \theta'/2 \sin \theta''/2$$
These probability amplitudes of course lead to the same probabilities as Eqs. (27) - (30), namely $\, |1\rangle = c^+ (\pm) c^-$.

$$\left| \psi\left(\frac{1}{2}\hat{a}(\pm)\frac{1}{2}\hat{c}\right) \right|^2 = \left| \psi\left(-\frac{1}{2}\hat{a}(\mp)\frac{1}{2}\hat{c}\right) \right|^2 = \cos^2(\theta'' - \theta')/2 - \sin\theta'' \sin \theta' \sin^2(\varphi' - \varphi'')/2$$ (56)

and

$$\left| \psi\left(\frac{1}{2}\hat{a}(\mp)\frac{1}{2}\hat{c}\right) \right|^2 = \left| \psi\left(-\frac{1}{2}\hat{a}(\pm)\frac{1}{2}\hat{c}\right) \right|^2 = \sin^2(\theta'' - \theta')/2 + \sin\theta'' \sin \theta' \sin^2(\varphi' - \varphi'')/2$$ (57)

### 3.2 New $z$ Component of Spin

We now plug the new probability amplitudes $\chi(m_\hat{b}, m_\hat{c})$, Eqs. (52) - Eqn. (55), into the expressions Eqs. (12) - (15) for the elements of $[\sigma_z]$, the generalized component of $[\sigma_z]$. We find that

$$\begin{align*}
(\sigma^a_c)_{11} &= \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\varphi - \varphi') \\
(\sigma^a_c)_{12} &= \sin \theta \cos \theta' e^{i\varphi} + \sin \theta' \sin^2 \frac{\theta}{2} e^{i\varphi'} - \sin \theta' \cos^2 \frac{\theta}{2} e^{i(2\varphi - \varphi')} \\
(\sigma^a_c)_{21} &= \sin \theta \cos \theta' e^{-i\varphi} + \sin \theta' \sin^2 \frac{\theta}{2} e^{-i\varphi'} - \sin \theta' \cos^2 \frac{\theta}{2} e^{-i(2\varphi - \varphi')} \\
(\sigma^a_c)_{22} &= -\cos \theta \cos \theta' - \sin \theta \sin \theta' \cos(\varphi - \varphi')
\end{align*}$$

According to the reasoning given in Ref. 1, the eigenvectors of this operator have for their elements generalized probability amplitudes. Thus the form of these elements is given by Eqs. (52) - (55). However, the arguments have to change so that $\hat{a} \to \hat{c}$ and $\hat{c} \to \hat{b}$. Hence the eigenvectors are

$$\begin{bmatrix} c^{(+)} \rangle \\
\langle c^{(-)} \end{bmatrix} = \begin{pmatrix}
\chi((+\frac{1}{2})\hat{c}; (+\frac{1}{2})\hat{b}) \\
\chi((+\frac{1}{2})\hat{c}; (-\frac{1}{2})\hat{b})
\end{pmatrix} = \begin{pmatrix}
\cos \frac{\theta}{2} \cos \frac{\theta'}{2} e^{i(\varphi - \varphi')} + \sin \frac{\theta}{2} \sin \frac{\theta'}{2} \\
\sin \frac{\theta}{2} \cos \frac{\theta'}{2} e^{-i\varphi'} - \cos \frac{\theta}{2} \sin \frac{\theta'}{2} e^{-i\varphi}
\end{pmatrix}$$ (62)

for eigenvalue +1, and

$$\begin{bmatrix} c^{(+) \rangle} \\
\langle c^{(-)} \end{bmatrix} = \begin{pmatrix}
\chi((-\frac{1}{2})\hat{c}; (+\frac{1}{2})\hat{b}) \\
\chi((-\frac{1}{2})\hat{c}; (-\frac{1}{2})\hat{b})
\end{pmatrix} = \begin{pmatrix}
\sin \frac{\theta'}{2} \cos \frac{\theta}{2} e^{i\varphi} - \cos \frac{\theta}{2} \sin \frac{\theta'}{2} e^{i\varphi'} \\
\cos \frac{\theta}{2} \sin \frac{\theta'}{2} e^{-i(\varphi - \varphi')} + \sin \frac{\theta}{2} \cos \frac{\theta'}{2}
\end{pmatrix}$$ (63)
for eigenvalue $-1$. Direct calculation confirms that, indeed,

$$[\sigma_z][\xi^{(\pm)}_c] = \pm[\xi^{(\pm)}_c].$$

(64)

### 3.3 New $x$ and $y$ Components of Spin

As demonstrated in Refs. 2 and 3, the $x$ component of spin and its eigenvectors are obtainable from the $z$ component and its eigenvectors through the transformation $\theta' \rightarrow \theta' - \pi/2$ for the old choice of phase. The $y$ component and its eigenvectors are obtainable through the transformations $\theta' = \pi/2$, $\varphi' \rightarrow \varphi' - \pi/2$. But these transformations do not apply to the new choice of phase. When we use them, the resulting operators together with the $z$ component do not satisfy the commutation relations. We therefore resort to the more laborious but sure method of obtaining the $x$ and $y$ components through the ladder operators. Using the properties

$$[\sigma_+][\xi^{(+)}_c] = 0; \quad [\sigma_+][\xi^{(-)}_c] = 2[\xi^{(+)}_c]$$

(65)

and

$$[\sigma_-][\xi^{(-)}_c] = 0; \quad [\sigma_-][\xi^{(+)}_c] = 2[\xi^{(-)}_c],$$

(66)

we find that the elements of $[\sigma_+]$ are

$$(\sigma_+)^{11} = \cos \theta \sin \theta' e^{-i\varphi'} + \sin \theta \sin^2 \frac{\theta'}{2} e^{-i\varphi} - \sin \theta \cos^2 \frac{\theta'}{2} e^{i(\varphi - 2\varphi')}$$

(67)

$$(\sigma_+)^{12} = 2e^{2i(\varphi - \varphi')} \cos^2 \frac{\theta}{2} \cos^2 \frac{\theta'}{2} + \sin \theta \sin \theta' e^{i(\varphi - \varphi')} + 2 \sin^2 \frac{\theta}{2} \sin^2 \frac{\theta'}{2}$$

(68)

$$(\sigma_+)^{21} = -2e^{-2i\varphi'} \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta'}{2} - 2e^{-2i\varphi} \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta'}{2} + \sin \theta \sin \theta' e^{-i(\varphi + \varphi')}$$

(69)

and

$$(\sigma_+)^{22} = -\cos \theta \sin \theta' e^{-i\varphi'} - \sin \theta \sin^2 \frac{\theta'}{2} e^{-i\varphi} + \sin \theta \cos^2 \frac{\theta}{2} e^{i(\varphi - 2\varphi')}.$$

(70)

The elements of $[\sigma_-]$ are

$$(\sigma_-)^{11} = \sin \theta \sin^2 \frac{\theta'}{2} e^{i\varphi'} + \sin \theta \cos \varphi e^{i\varphi} - \sin \theta \cos^2 \frac{\theta}{2} e^{i(2\varphi' - \varphi)}$$

(71)

$$(\sigma_-)^{12} = -2e^{2i\varphi} \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta'}{2} - 2e^{2i\varphi'} \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta'}{2} + \sin \theta \sin \theta' e^{i(\varphi + \varphi')}$$

(72)

$$(\sigma_-)^{21} = 2e^{2i(\varphi' - \varphi)} \cos^2 \frac{\theta}{2} \cos^2 \frac{\theta'}{2} + \sin \theta \sin \theta' e^{i(\varphi' - \varphi)} + 2 \sin^2 \frac{\theta}{2} \sin^2 \frac{\theta'}{2}$$

(73)
and

$$(\sigma_-)_{22} = -\sin \theta \sin^2 \frac{\theta'}{2} e^{i\varphi} - \sin \theta' \cos e^{i\varphi'} + \sin \theta \cos^2 \frac{\theta'}{2} e^{i(2\varphi' - \varphi)}. \quad (74)$$

Hence, the elements of $[\sigma_x]$, obtained using

$$[\sigma_x] = \frac{1}{2} ([\sigma_+] + [\sigma_-]), \quad (75)$$

are

$$
(\sigma_x)_{11} = \frac{1}{2} \sin \theta' \cos \theta e^{i\varphi'} + \frac{1}{2} \sin \theta' \cos \theta e^{-i\varphi'} + \frac{1}{2} \sin \theta \sin^2 \frac{\theta'}{2} e^{i\varphi} - \frac{1}{2} \sin \theta \sin^2 \frac{\theta'}{2} e^{i(2\varphi' - \varphi)} - \frac{1}{2} \sin \theta \cos^2 \frac{\theta'}{2} e^{i(\varphi - 2\varphi')} \quad (76)
$$

$$
(\sigma_x)_{12} = \cos^2 \frac{\theta}{2} \cos^2 \frac{\theta' e^{2i(\varphi - \varphi')}}{2} + \frac{1}{2} \sin \theta \sin \theta' e^{i(\varphi - \varphi')} - \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta' e^{2i\varphi'}}{2} - \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta' e^{2i\varphi'}}{2} + \frac{1}{2} \sin \theta \sin \theta' e^{i(\varphi + \varphi')} + \sin^2 \frac{\theta}{2} \sin^2 \frac{\theta'}{2} \quad (77)
$$

$$
(\sigma_x)_{21} = \cos^2 \frac{\theta}{2} \cos^2 \frac{\theta' e^{2i(\varphi - \varphi')}}{2} + \frac{1}{2} \sin \theta \sin \theta' e^{i(\varphi - \varphi')} - \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta' e^{2i\varphi'}}{2} - \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta' e^{2i\varphi'}}{2} + \frac{1}{2} \sin \theta \sin \theta' e^{-i(\varphi + \varphi')} + \sin^2 \frac{\theta}{2} \sin^2 \frac{\theta'}{2} \quad (78)
$$

and

$$
(\sigma_x)_{22} = -\frac{1}{2} \sin \theta' \cos \theta e^{i\varphi'} - \frac{1}{2} \sin \theta' \cos \theta e^{-i\varphi'} - \frac{1}{2} \sin \theta \sin^2 \frac{\theta'}{2} e^{i\varphi} - \frac{1}{2} \sin \theta \sin^2 \frac{\theta'}{2} e^{i(2\varphi' - \varphi)} - \frac{1}{2} \sin \theta \cos^2 \frac{\theta'}{2} e^{i(\varphi - 2\varphi')} + \frac{1}{2} \sin \theta \cos^2 \frac{\theta'}{2} e^{i(2\varphi' - \varphi)} \quad (79)
$$

The elements of $[\sigma_y]$, obtained from

$$[\sigma_y] = \frac{1}{2i} ([\sigma_+] - [\sigma_-]), \quad (80)$$

are

$$
(\sigma_y)_{11} = \frac{i}{2} [\sin \theta' \cos \theta e^{i\varphi'} - \sin \theta' \cos \theta e^{-i\varphi'} + \sin \theta \sin^2 \frac{\theta'}{2} e^{i\varphi} - \sin \theta \sin^2 \frac{\theta'}{2} e^{i(2\varphi' - \varphi)} - \sin \theta \cos^2 \frac{\theta'}{2} e^{i(\varphi - 2\varphi')} - \sin \theta \cos^2 \frac{\theta'}{2} e^{i(2\varphi' - \varphi)}] \quad (81)
$$
\[(\sigma_y)_{12} = -i [\cos^2 \frac{\theta}{2} \cos^2 \frac{\theta'}{2} e^{2i(\phi' - \phi)} + \sin^2 \frac{\theta}{2} \sin^2 \frac{\theta'}{2} + \frac{1}{2} \sin \theta \sin \theta' e^{i(\phi - \phi')} \\
+ \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta'}{2} e^{2i\phi} + \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta'}{2} e^{2i\phi'} \\
- \frac{1}{2} \sin \theta \sin \theta' e^{i(\phi + \phi')} ] \]  

(82)

\[(\sigma_y)_{21} = i [\cos^2 \frac{\theta}{2} \cos^2 \frac{\theta'}{2} e^{2i(\phi' - \phi)} + \sin^2 \frac{\theta}{2} \sin^2 \frac{\theta'}{2} + \frac{1}{2} \sin \theta \sin \theta' e^{i(\phi' - \phi')} \\
+ \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta'}{2} e^{-2i\phi} + \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta'}{2} e^{-2i\phi'} \\
- \frac{1}{2} \sin \theta \sin \theta' e^{-i(\phi + \phi')} ] \]  

(83)

and

\[(\sigma_y)_{22} = -\frac{i}{2} [\sin \theta' \cos \theta e^{i\phi'} - \sin \theta' \cos \theta e^{-i\phi'} + \sin \theta \sin^2 \frac{\theta'}{2} e^{i\phi} \\
- \sin \theta \sin^2 \frac{\theta'}{2} e^{-i\phi} + \sin \theta \cos^2 \frac{\theta'}{2} e^{i(\phi - 2\phi')} \\
- \sin \theta \cos^2 \frac{\theta'}{2} e^{-i(2\phi' - \phi)} ] \]  

(84)

The three spin operators possess the usual properties. Thus each one gives the unit matrix when squared.

\[[\sigma_x]^2 = [\sigma_y]^2 = [\sigma_z]^2 = I \]  

(85)

The commutators of these matrices are

\[[[\sigma_i], [\sigma_j]] = 2i[\sigma_k], \]  

(86)

where \(i, j, k = x, y, z\) are taken in cyclic permutation. Finally, the operators anti-commute:

\[[[\sigma_i], [\sigma_j]]_+ = 0 \]  

(87)

for \(i \neq j\).

3.4 Eigenvectors of the \(x\) and \(y\) Components of Spin

The surest way of obtaining the eigenvectors of \([\sigma_x]\) and \([\sigma_y]\) is by means of rotations. Thus, if the \(z\) axis is rotated through the angle \(\pi/2\) in the positive sense about the \(y\) axis, it becomes the \(x\) axis. Its rotated eigenvectors become the eigenvectors of \([\sigma_x]\). The eigenvectors are obtained through the formula [8]

\[[\xi_x^{(\pm)}] = \frac{1}{\sqrt{2}} (I - i[\sigma_y]) [\xi_x^{(\pm)}], \]  

(88)
where $I$ is the unit $2 \times 2$ matrix. Using Eqs. (81) - (84) for the elements of $[\sigma_y]$, we find that the eigenvectors of $[\sigma_y]$ are

$$[\xi_x^+] = \frac{1}{\sqrt{2}} \left( \frac{\cos \frac{\theta}{2} \cos \frac{\theta'}{2} e^{i(\varphi' - \varphi)} + \sin \frac{\theta}{2} \cos \frac{\theta'}{2} e^{i\varphi'} - \sin \frac{\theta}{2} \cos \frac{\theta'}{2} e^{-i\varphi'} + \sin \frac{\theta}{2} \sin \frac{\theta'}{2}}{\cos \frac{\theta}{2} \cos \frac{\theta'}{2} e^{i(\varphi' - \varphi)} - \sin \frac{\theta}{2} \cos \frac{\theta'}{2} e^{i\varphi'} + \sin \frac{\theta}{2} \cos \frac{\theta'}{2} e^{-i\varphi'} + \sin \frac{\theta}{2} \sin \frac{\theta'}{2}} \right)$$

(89)

for eigenvalue $+1$ and

$$[\xi_x^-] = \frac{1}{\sqrt{2}} \left( \frac{- \cos \frac{\theta}{2} \cos \frac{\theta'}{2} e^{i(\varphi' - \varphi)} + \sin \frac{\theta}{2} \cos \frac{\theta'}{2} e^{i\varphi'} - \sin \frac{\theta}{2} \cos \frac{\theta'}{2} e^{-i\varphi'} + \sin \frac{\theta}{2} \sin \frac{\theta'}{2}}{\cos \frac{\theta}{2} \cos \frac{\theta'}{2} e^{i(\varphi' - \varphi)} + \sin \frac{\theta}{2} \cos \frac{\theta'}{2} e^{i\varphi'} - \sin \frac{\theta}{2} \cos \frac{\theta'}{2} e^{-i\varphi'} + \sin \frac{\theta}{2} \sin \frac{\theta'}{2}} \right)$$

(90)

for eigenvalue $-1$.

For the $z$ axis to change into the $y$ axis, we need to rotate it in the negative sense about the $x$ axis through an angle of $\pi/2$. Thus the eigenvectors of $[\sigma_y]$ are given by

$$[\xi_y^{(\pm)}] = \frac{1}{\sqrt{2}}(I + i[\sigma_x])[\xi_c^{(\pm)}].$$

(91)

Hence, using Eqs. (70) - (74) for the elements of $[\sigma_x]$, we find that the eigenvectors are

$$[\xi_y^+] = \left( \frac{\cos \frac{\theta}{2} \cos \frac{\theta'}{2} e^{i(\varphi' - \varphi)} + i \sin \frac{\theta}{2} \cos \frac{\theta'}{2} e^{i\varphi'} - i \sin \frac{\theta}{2} \cos \frac{\theta'}{2} e^{i\varphi'} + \sin \frac{\theta}{2} \sin \frac{\theta'}{2}}{i \cos \frac{\theta}{2} \cos \frac{\theta'}{2} e^{i(\varphi' - \varphi)} - \sin \frac{\theta}{2} \cos \frac{\theta'}{2} e^{i\varphi'} + \sin \frac{\theta}{2} \cos \frac{\theta'}{2} e^{-i\varphi'} + i \sin \frac{\theta}{2} \sin \frac{\theta'}{2}} \right)$$

(92)

for the eigenvalue $+1$, and

$$[\xi_y^-] = \left( \frac{\cos \frac{\theta}{2} \cos \frac{\theta'}{2} e^{i(\varphi' - \varphi)} + \sin \frac{\theta}{2} \cos \frac{\theta'}{2} e^{i\varphi'} - \sin \frac{\theta}{2} \cos \frac{\theta'}{2} e^{i\varphi'} + \sin \frac{\theta}{2} \sin \frac{\theta'}{2}}{\cos \frac{\theta}{2} \cos \frac{\theta'}{2} e^{i(\varphi' - \varphi)} - i \sin \frac{\theta}{2} \cos \frac{\theta'}{2} e^{i\varphi'} + i \sin \frac{\theta}{2} \cos \frac{\theta'}{2} e^{-i\varphi'} + \sin \frac{\theta}{2} \sin \frac{\theta'}{2}} \right)$$

(93)

for the eigenvalue $-1$.

### 4 Reduction to Standard Forms

Since we know the standard forms of the spin operators and their eigenvectors, we can deduce the limit in which the generalized operators and their eigenvectors reduce to these standard forms. To obtain the Pauli matrices and their eigenvectors, we find that we need to set $\theta = \theta'$ and $\varphi = \varphi'$. Thus, we find that in this limit,

$$[\sigma_{c}^{-}] \rightarrow [\sigma_{z}] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

(94)
with the eigenvectors
\[
\xi^{(+)}_c \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \xi^{(-)}_c \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\] (95)
for the eigenvalues +1 and −1 respectively. The \(x\) component becomes
\[
[\sigma_x] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\] (96)
with the eigenvectors
\[
[\xi^{(+)}_x] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad [\xi^{(-)}_x] = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}
\] (97)
for the eigenvalues +1 and −1 respectively. The \(y\) component becomes
\[
[\sigma_y] = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},
\] (98)
with the eigenvectors
\[
[\xi^{(+)}_y] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \quad [\xi^{(-)}_y] = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}
\] (99)
for the eigenvalues +1 and −1 respectively.

To obtain the "standard generalized form" for \([\sigma_c]\), Eqn. (20), we find that we have to set \(\theta = 0, \varphi = \frac{\pi}{2}\). We then recover Eqn. (20); its eigenvectors for the respective eigenvalues +1 and −1 are found to be
\[
[\xi^{(+)}_c] = i \begin{pmatrix} \cos \frac{\theta'}{2} e^{-i\varphi'} \\ \sin \frac{\theta'}{2} \end{pmatrix}
\] (100)
and
\[
[\xi^{(-)}_c] = i \begin{pmatrix} \sin \frac{\theta'}{2} \\ -\cos \frac{\theta'}{2} e^{-i\varphi'} \end{pmatrix},
\] (101)
which differ from Eqns. (50) and (51) through the appearance of the phase factor \(i\) (apart from the change of argument arising from the change in quantization direction from \(\hat{a}\) to \(\hat{c}\)).

In the same limit, we obtain
\[
[\sigma_x] = \begin{pmatrix} 
\sin \theta' \cos \varphi' & \sin^2 \frac{\theta'}{2} - \cos^2 \frac{\theta'}{2} e^{-2i\varphi'} \\
\sin^2 \frac{\theta'}{2} - \cos^2 \frac{\theta'}{2} e^{2i\varphi'} & -\sin \theta' \cos \varphi' \n\end{pmatrix}
\] (102)
with the eigenvectors
\[ [\xi_x^{(+)}] = \frac{i}{\sqrt{2}} \begin{pmatrix} \sin \frac{\theta'}{2} + \cos \frac{\theta'}{2} e^{-i \phi'} \\ \sin \frac{\theta'}{2} - \cos \frac{\theta'}{2} e^{i \phi'} \end{pmatrix} \] (103)

and

\[ [\xi_x^{(-)}] = -\frac{i}{\sqrt{2}} \begin{pmatrix} \cos \frac{\theta'}{2} e^{i \phi'} - \sin \frac{\theta'}{2} \\ \cos \frac{\theta'}{2} e^{i \phi'} + \sin \frac{\theta'}{2} \end{pmatrix} \] (104)

We also obtain in this limit

\[ [\sigma_y] = \begin{pmatrix} -\sin \theta' \cos \phi' & i \left[ \cos^2 \frac{\theta'}{2} e^{-2i \phi'} + \sin^2 \frac{\theta'}{2} \right] \\ -i \left[ \cos^2 \frac{\theta'}{2} e^{2i \phi'} + \sin^2 \frac{\theta'}{2} \right] & \sin \theta' \cos \phi' \end{pmatrix} \] (105)

with the eigenvectors

\[ [\xi_y^{(+)}] = \frac{1}{\sqrt{2}} \begin{pmatrix} i \cos \frac{\theta'}{2} e^{-i \phi'} - \sin \frac{\theta'}{2} \\ \cos \frac{\theta'}{2} e^{i \phi'} + i \sin \frac{\theta'}{2} \end{pmatrix} \] (106)

and

\[ [\xi_y^{(-)}] = -\frac{1}{\sqrt{2}} \begin{pmatrix} -i \sin \frac{\theta'}{2} + \cos \frac{\theta'}{2} e^{-i \phi'} \\ \sin \frac{\theta'}{2} + i \cos \frac{\theta'}{2} e^{i \phi'} \end{pmatrix} \] (107)

We observe that in order to recover the Pauli matrices and their eigenvectors from these "generalized standard forms", we have to set \( \theta' = 0 \) and \( \phi' = \pi/2 \), so as to achieve the conditions \( \theta = \theta' \) and \( \phi = \phi' \).

## 5 Discussion and Conclusion

In this paper, we have derived new forms for the operators and states describing spin 1/2. Thus, we now have two sets of forms for the generalized probability amplitudes and correspondingly for the operators. Since there are other choices of phase for the eigenvectors of Eqn. (20), it follows that there are yet more forms of the probability amplitudes, operators and their eigenvectors. Using the methods illustrated here and in Refs. 1–3, it is straightforward, but perhaps tedious, to obtain the spin quantities for any choice of phase. It is probable that in some cases it will be possible to obtain the \( x \) and \( y \) components of the operators from the \( z \) components by straightforward substitution of angles, as we did for the original choice of phase [2, 3]. It should be mentioned that this choice was purely arbitrary, and it seems a fortunate accident that this choice permitted of obtaining the \( x \) and \( y \) components of spin from the \( z \) component by the procedure of changing the arguments. For any choice of phase, the method employing the ladder operators will always yield the \( x \) and \( y \) components of spin. Since the eigenvectors of \([\sigma_z]\) are easily obtained for any choice of phase, it will always be possible by using rotations to deduce the eigenvectors of these operators.
Though we have illustrated our considerations using the case of spin 1/2, the general theory will work perfectly well for any value of spin [3]. Of course, the total number of phase combinations goes up with value of \( J \), and since in the general case, the \( x \) and \( y \) components of spin have to be obtained via the ladder operators, this will result in a great deal of calculational labour.

It would seem obvious that in any calculations involving these generalized quantities, it is necessary to match the particular probability amplitudes with the correct forms of the operators. The correct forms of the operators are those which result from using the same probability amplitudes when the elements of the operator are being derived. It is not clear at this time what the effect would be of mixing one set of probability amplitudes with operators corresponding to another set.

At this point, it seems accurate to say that whatever form is obtained for the generalized quantities, the Pauli forms are obtained by setting \( \theta = \theta' \) and \( \varphi = \varphi' \). On the other hand, there does not appear to be a uniform prescription for obtaining the ”standard generalized quantities”; while we must set \( \theta \) equal to zero, the required value of \( \varphi \) seems to depend on the exact choice of phase.

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