ATTRACTIONS AND ORBIT-FLIP HOMOCLINIC ORBITS FOR
STAR FLOWS

C. A. MORALES

ABSTRACT. We study star flows on closed 3-manifolds and prove that they either have a finite number of attractors or can be $C^1$ approximated by vector fields with orbit-flip homoclinic orbits.

1. INTRODUCTION

The notion of attractor deserves a fundamental place in the modern theory of dynamical systems. This assertion, supported by the nowadays classical theory of turbulence [27], is enlightened by the recent Palis conjecture [24] about the abundance of dynamical systems with finitely many attractors absorbing most positive trajectories. If confirmed such a conjecture means the understanding of a great part of dynamical systems in the sense of their long-term behaviour.

Here we attack a problem which goes straight to the Palis conjecture: The finitude of the number of attractors for a given dynamical system. Such a problem have been solved positively under certain circumstances. For instance we have the work by Lopes [16] who, based upon early works by Mañé [15] and extending previous ones by Liao [15] and Pliss [23], studied the structure of the $C^1$ structural stable diffeomorphisms and proved the finitude of attractors for such diffeomorphisms. His work was largely extended by Mañé himself in the celebrated solution of the $C^1$ stability conjecture [17]. On the other hand, the japanese researchers S. Hayashi [11] and N. Aoki [2] studied the star diffeomorphisms, i.e., diffeomorphisms which cannot be $C^1$ approximated by ones with nonhyperbolic periodic points, and proved that they are Axiom A and so with only a finite number of attractors. Their investigation triggered the study of the star flows, i.e., vector fields which cannot be $C^1$ approximated by ones with nonhyperbolic closed orbits. Indeed, although it was known from the very beginning that these flows are not necessarily Axiom A [1, 7, 8], the aforementioned works by Liao and Pliss proved that they display finitely many attracting closed orbits.

A progress toward understanding star flows was tackled in 2003 by the author in collaboration with Pacifico [20]. Indeed, these authors proved on closed 3-manifolds that, except in a meager set, all such flows are singular-Axiom A and so with only a finite number of attractors. Soon later the chinesse authors Gan and Wen [5] extended the Aoki-Hayashi’s conclusion to nonsingular star flows on closed manifolds implying that these flows has a finite number of attractors too. In light of these works it seems quite promising to prove the finiteness of the number of attractors for star flows in any closed manifold.

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In this paper we shall provide a result which though partial provides an insight for a positive solution of this problem. Basically, we present the so-called orbit-flip homoclinic orbits as obstruction for the finiteness of attractors of star flows on closed 3-manifolds. More precisely, we show that a star flow on a closed 3-manifold either has a finite number of attractors or can be $C^1$ approximated by vector fields exhibiting orbit-flip homoclinic orbits. Orbit-flip homoclinic orbits are very rich dynamical structures which have been studied during decades [3], [13], [14], [22], [23], [28], [29]. Let us state this result in a precise way.

Hereafter $M$ will denote a compact connected boundaryless Riemannian manifold of dimension $n \geq 2$ (a closed $n$-manifold for short). We shall consider a $C^1$ vector field $X$ in $M$ together with its induced one-parameter group $X_t$, $t \in \mathbb{R}$, the so-called flow of $X$. The space of $C^1$ vector fields in $M$ comes equipped with the $C^1$-topology which, roughly speaking, measures the distance between vector fields and their corresponding derivatives.

The long-time behavior of a point $x \in M$ is often analyzed through its omega-limit set

$$\omega(x) = \{ y \in M : y = \lim_{n \to \infty} X_{t_n}(y) \text{ for some sequence } t_n \to \infty \}.$$ 

A compact invariant set is transitive if it coincides with the omega-limit set of one of its points, whereas, in this work, an attractor will be a transitive set of the form

$$A = \bigcap_{t > 0} X_t(U)$$

for some neighborhood $U$ of it. The most representative example of attractors are the sinks, that is, hyperbolic closed orbits of maximal Morse index. Sometimes we use the term source referring to a sink for the time reversal vector field $-X$.

A homoclinic orbit is a regular (i.e. nonsingular) trajectory $\Gamma = \{X_t(q) : t \in \mathbb{R}\}$ which is biasymptotic to a singularity $\sigma$, namely,

$$\lim_{t \to \pm \infty} X_t(q) = \sigma.$$

We call it orbit-flip as soon as the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of $\sigma$ are real, satisfy the eigenvalue inequalities $\lambda_2 < 0 < \lambda_3 < \lambda_1$ and $\Gamma \subset W^{uu}(\sigma)$ where $W^{uu}(\sigma)$, the strong unstable manifold [12], is the unique invariant manifold of $X$ which is tangent at $\sigma$ to the eigenspace associated to the eigenvalue $\lambda_1$ (c.f. Figure 1).

![An orbit-flip homoclinic orbit](image)

**Figure 1.**
With such definitions and notations we can state our result.

**Main Theorem.** A star flow on a closed 3-manifold either has a finite number of attractors or can be $C^1$ approximated by vector fields exhibiting orbit-flip homoclinic orbits.

The proof relies on recent results in the theory of star flows [5] together with some techniques resembling those in [19].

The Main Theorem motivates the obvious question if star flows which can be $C^1$ approximated by vector fields with orbit-flip homoclinic orbits exist on any closed 3-manifold. Actually this is true but, as the reader can see by himself [15], [19], [22], [25], the set of such flows constitute a meager subset of star flows. We therefore conclude that every closed 3-manifold comes equipped with an open and dense subset of star flows, all of whose elements have a finite number of attractors. However, it is worth noting that we can obtain exactly the same conclusion by making use of [19] and [20].

Another question is if, in the statement of the Main Theorem, we can replace the finitely many attractor’s option by the stronger property of being singular-Axiom A (in the sense of [20]). Unfortunately, such a question has negative answer as we can easily find star flows in the 3-sphere which neither are singular-Axiom A nor can be $C^1$ approximated by vector fields with orbit-flip homoclinic loops. Finally let us mention that, in the statement of the Main Theorem, we can replace the term attractor by that of Lyapunov stable omega-limit set (in the spirit of [21]).

2. Proof

We denote by $\|\cdot\|$ the norm induced by a Riemannian metric in $M$ and by $m(\cdot)$ its corresponding minimum norm. Given a $C^1$ vector field $X$ with flow $X_t$ in $M$ we denote by $\text{Sing}(X,U)$ the set of singularities of $X$ in $U$ and we write $\text{Sing}(X) = \text{Sing}(X,M)$. Likewise, the union of the periodic orbits of $X$ is denoted by $\text{Per}(X)$. The elements of $\text{Per}(X)$ will be called periodic points. A subset $\Lambda \subset M$ is called invariant if $X_t(\Lambda) = \Lambda$ for all $t \in \mathbb{R}$. A compact invariant set $\Lambda$ is hyperbolic if there are a tangent bundle decomposition $T_{\Lambda}M = \hat{E}^s_{\Lambda} \oplus E^u_{\Lambda} \oplus \hat{E}^u_{\Lambda}$ and positive constants $K, \lambda$ such that

- $\hat{E}^s_{\Lambda}$ is contracting, i.e.,
  \[ \|DX_t(x)/\hat{E}^s_{\Lambda}\| \leq Ke^{-\lambda t}, \quad \forall x \in \Lambda, t \geq 0. \]

- $\hat{E}^u_{\Lambda}$ is expanding, i.e.,
  \[ m(DX_t(x)/\hat{E}^u_{\Lambda}) \geq K^{-1}e^{\lambda t}, \quad \forall x \in \Lambda, t \geq 0. \]

- $E^u_{\Lambda}$ is the subbundle generated by $X$ in $T_{\Lambda}M$.

Recall that a closed orbit $O$ (i.e. a periodic orbit or singularity) is hyperbolic if it does as a compact invariant set. A periodic point is hyperbolic if its corresponding orbit is. We say that a compact invariant set is nontrivial if it does not reduce to a single closed orbit.

For any given index $i$ in between 0 and $n - 1$ we denote by $\text{Per}_i(X)$ the union of the hyperbolic periodic orbits $O$ of $X$ with $\text{dim}(\hat{E}^s_x) = i$ for some (and hence for all) $x \in O$. As pointed out earlier by Wen [30] we can extended this set to include periodic orbits for nearby vector fields. More precisely, we denote by $\text{Per}^*_i(X)$ the $i$-preperiodic set of $X$ consisting of those $x \in M$ for which there are sequences $X_k$
and $x_k$ of vector fields and points in $\text{Per}_i(X_k)$ such that $X_k \to X$ and $x_k \to x$. We shall also use the notion of fundamental $i$-limit which are limits (in the Hausdorff metric) of sequences of hyperbolic periodic orbits $O_n \subset \text{Per}_i(X_n)$ of vector fields $X_n \to X$.

Now we state four technical lemmas the first of which is Lemma 3.4 in [5]:

**Lemma 2.1.** If $X$ is a star flow on a closed $n$-manifold and $\Lambda$ is a fundamental $i$-limit of $X$ with $\text{Sing}(X, \Lambda) = \emptyset$, then $\Lambda$ is a sink or a source depending on whether $i = n - 1$ or $i = 0$.

Denote by $N \to M \setminus \text{Sing}(X)$ the vector bundle with fiber $N_x = \{ v \in T_xM : v \perp X(x) \}$. We define the *linear* Poincaré flow $P_t : N \to N$ by

$$P_t = \pi \circ DX_t,$$

where $\pi : TM \to N$ stands for orthogonal projection. A $P_t$-invariant splitting over an invariant set $\Lambda \subset M \setminus \text{Sing}(X)$ is a direct sum $N_\Lambda = \Delta^-_\Lambda \oplus \Delta^+_\Lambda$ such that $P_t(\Delta^-_\Lambda) = \Delta^-_{X_t(x)}$ and $P_t(\Delta^+_\Lambda) = \Delta^+_{X_t(x)}$ for all $x \in \Lambda$ and $t \in \mathbb{R}$.

We shall use the following Doering’s criterion for hyperbolicity [4]: A compact invariant set $\Lambda$ with $\text{Sing}(X, \Lambda) = \emptyset$ is hyperbolic if and only if there is a $P_t$-invariant splitting $N_\Lambda = \Delta^-_\Lambda \oplus \Delta^+_\Lambda$ over $\Lambda$ such that $\Delta^-_\Lambda$ is contracting and $\Delta^+_\Lambda$ is expanding, i.e., there are positive constants $K, \lambda$ satisfying

$$\frac{\|P_t(x)/\Delta^-_\Lambda\|}{m(P_t(x)/\Delta^+_\Lambda)} \leq Ke^{-\lambda t} \quad \text{and} \quad m(P_t(x)/\Delta^+_\Lambda) \geq K^{-1}e^{\lambda t} \quad \forall x \in \Lambda, \forall t \geq 0.
$$

A dominated splitting for $P_t$ over $\Lambda$ is a $P_t$-invariant splitting $N_\Lambda = \Delta^-_\Lambda \oplus \Delta^+_\Lambda$ for which there are positive constants $K, \lambda$ satisfying

$$\frac{\|P_t(x)/\Delta^-_\Lambda\|}{m(P_t(x)/\Delta^+_\Lambda)} \leq Ke^{-\lambda t}, \quad \forall (x,t) \in \Lambda \times \mathbb{R}^+.
$$

A dominated $p$-splitting for $P_t$ over $\Lambda$ is a dominated splitting $N_\Lambda = \Delta^-_\Lambda \oplus \Delta^+_\Lambda$ such that $\dim(\Delta^-_\Lambda) = \rho \quad \forall x \in \Lambda$.

The following is Lemma 3.10 in [5].

**Lemma 2.2.** Let $X$ a star flow on a closed $n$-manifold and $\Lambda$ be a compact invariant set with $\text{Sing}(X, \Lambda) = \emptyset$ for which there is a dominated $p$-splitting $N_\Lambda = \Delta^-_\Lambda \oplus \Delta^+_\Lambda$ for $P_t$ over $\Lambda$ with $1 \leq \rho \leq n - 2$. If $\Delta^-_\Lambda$ is not contracting there is a fundamental $r$-limit contained in $\Lambda$ with $r < \rho$. Likewise, if $\Delta^+_\Lambda$ is not expanding there is a fundamental $r$-limit contained in $\Lambda$ with $r > \rho$.

The proof of the following result can be obtained as in Theorem 3.8 of [51] (see also the proof of Lemma 2.8 in [6]).

**Lemma 2.3.** If $X$ is a star flow, then for every index $1 \leq i \leq n - 2$ there is a dominated $i$-splitting

$$N_{\text{Per}_i^r(X) \setminus \text{Sing}(X)} = \Delta^-_{\text{Per}_i^r(X) \setminus \text{Sing}(X)} \oplus \Delta^+_{\text{Per}_i^r(X) \setminus \text{Sing}(X)}$$

for $P_t$ over $\text{Per}_i^r(X) \setminus \text{Sing}(X)$ such that

$$\Delta^-_x = \pi_x(\hat{E}^-_x) \quad \text{and} \quad \Delta^+_x = \pi_x(\hat{E}^+_x), \quad \forall x \in \text{Per}_i^r(X),$$

where $T_xM = \hat{E}^s_x \oplus \hat{E}^X_x \oplus \hat{E}^u_x$ is the corresponding hyperbolic splitting along the orbit of $x$.

The following lemma is Theorem B in [5]. Denote by $\text{Cl}(\cdot)$ the closure operation.
Lemma 2.4. If \( X \) is a star flow on a closed manifold, then every compact invariant set \( \Lambda \subset \text{Cl}(\text{Per}(X)) \) with \( \text{Sing}(X, \Lambda) = \emptyset \) is hyperbolic.

These lemmas will be used to analyze attractors for star flows on closed 3-manifolds. To start with we extend the conclusion of Lemma 2.4 to all such attractors.

Proposition 2.5. If \( X \) is a star flow on a closed 3-manifold, then every attractor \( A \) of \( X \) with \( \text{Sing}(X, A) = \emptyset \) is hyperbolic.

Proof. First we show that \( A \subset \text{Per}_1^+(X) \) unless \( A \) is a sink or a source. Indeed, if \( A \not\subset \text{Per}_1^+(X) \) we can select \( y \in A \setminus \text{Per}_1^+(X) \). As \( A \) has no singularities and \( y \in A \) we have that \( y \) is a regular point (i.e. \( X(y) \neq 0 \)). Then, it follows from the Pugh’s closing lemma [26] that \( y \in \text{Per}_0^+(X) \cup \text{Per}_-^+(X) \) and so there exist a fundamental i-limit with \( i = 0, 2 \) intersecting \( A \). As \( A \) is an attractor we conclude that such a fundamental i-limit is contained in \( A \). Therefore, by Lemma 2.1 it would exist a sink or a source contained in \( A \). In such a case \( A \) is a sink or a source. Then, we can assume that \( A \subset \text{Per}_1^+(X) \). So, Lemma 2.3 implies that there is a dominated 1-splitting \( N_A = \Delta_A^- \oplus \Delta_A^+ \) for \( P_t \) over \( A \). If subbundle \( \Delta_A^- \) were not contracting it would exist a fundamental 0-limit in \( A \) in virtue of Lemma 2.2. Therefore \( A \) is a source and so hyperbolic. Hence we can assume that \( \Delta_A^- \) is contracting and analogously that \( \Delta_A^+ \) is expanding. Then, \( A \) is hyperbolic by the Doering’s criterium.

The following elementary lemma will be used to prove Proposition 2.7.

Lemma 2.6. For every \( \epsilon > 0 \) there is \( \delta > 0 \) such that if \( c : [a, b] \subset [-\epsilon, \epsilon] \rightarrow [-\epsilon, \epsilon] \) is a \( C^1 \) map satisfying

(i) \( |c'(t)| \leq \frac{\delta}{6} \) for all \( t \in [a, b] \);
(ii) \( |c(t_0)| \leq \delta \) for some \( t_0 \in [a, b] \);
(iii) \( (a, c(a)), (b, c(b)) \in \partial([-\epsilon, \epsilon]^2) \),

then \( a = -\epsilon, b = \epsilon \) and \( |c(\pm \epsilon)| < \epsilon \).

Proof. We take \( \delta = \frac{\epsilon}{3} \). Without loss of generality we can assume that \( t_0 \) in (ii) belongs to \( [a, b] \). If \( -\epsilon < a \) then condition (iii) implies \( c(a) = \pm \epsilon \). On the other hand, condition (i) and the mean value theorem imply \( |c(t_0) - c(a)| \leq \frac{\delta}{6}|t_0 - a| \leq \frac{\epsilon}{3} \) thus (ii) yields \( \epsilon \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3} \) which is absurd. Therefore \( a = -\epsilon \) and analogously \( b = \epsilon \). The same computation shows \( |c(\pm \epsilon)| \leq \frac{\epsilon}{3} \).

Hereafter we will use the standard stable and unstable manifold notation \( W^s(\cdot), W^u(\cdot) \) (c.f. [12]).

Proposition 2.7. Let \( X \) be a star flow on a closed 3-manifold and \( \sigma \in \text{Sing}(X) \) be such that either

(iv) \( \dim(W^s(\sigma)) = 2 \) or
(v) \( \sigma \) has three real eigenvalues \( \lambda_2 < 0 < \lambda_3 < \lambda_1 \) and \( X \) cannot be \( C^1 \) approximated by vector fields with orbit-flip homoclinic orbits.

Then, for every \( x \in (W^s(\sigma) \setminus \{\sigma\}) \cap \text{Per}_1^+(X) \) there is \( \delta > 0 \) such that \( d(A, x) > \delta \) for every nontrivial hyperbolic attractor \( A \) of \( X \).

Proof. Clearly \( x \in \text{Per}_1^+(X) \setminus \text{Sing}(X) \), and so, by Lemma 2.3 there is a dominated 1-splitting \( N_x = \Delta_x^- \oplus \Delta_x^+ \) for \( P_t \). It turns out that \( \Delta_x^- = L_x \cap N_x \) where \( L_x \) is either
The cross-section $\Sigma = \{x \in \mathbb{R}^n \mid f(x) = \lambda x\}$ corresponds to the eigenvalues $\lambda_2, \lambda_3$ (c.f. [12], [28]). Using this we can fix a cross-section $\Sigma = [-\epsilon, \epsilon]^2$ through $x = (0, 0)$ of $X$ so that:

- If (iv) holds then $W^s(\sigma) \cap \Sigma$ contains the graph $\gamma = \{(u(y), y) : y \in [-\epsilon, \epsilon]\}$ of a $C^1$ map $u : [-\epsilon, \epsilon] \to [-\epsilon, \epsilon]$ with $u(0) = 0$ (c.f. Figure 2(a)).

- If (v) holds then there are $C^1$ maps $u_1 \leq u_2 : [-\epsilon, \epsilon] \to [-\epsilon, \epsilon]$ with $u_i(0) = u'_i(0) = 0$ (for $i = 1, 2$) such that $\text{Per}_\epsilon^*(X) \cap R = \emptyset$ where $R \subset \Sigma$ is the complement of the region of $\Sigma$ in between the graphs of $\gamma_1$ and $\gamma_2$ of $u_1$ and $u_2$ respectively (i.e. the complement of the shadowed region in Figure 2(b)).

![Figure 2](image-link)

Shrinking $\epsilon$ if necessary we can assume that any $C^1$ map $c : [a, b] \subset [-\epsilon, \epsilon] \to [-\epsilon, \epsilon]$ whose graph $\gamma = \{(t, c(t)) : t \in [a, b]\}$ is tangent to $\Delta^+_{\text{Per}_\epsilon^*(X) \cap \Sigma}$ satisfies hypothesis (i) in Lemma 2.6 i.e., $|c'(t)| \leq \frac{1}{\delta}$ for all $t \in [a, b]$.

Now, take $\delta > 0$ as in Lemma 2.6 for such an $\epsilon$ and a nontrivial hyperbolic attractor $A$ with $d(A, x) \leq \delta$. Then, using tubular flow box around $x$ we have that there is $y \in A \cap \Sigma$ such that $d(x, y) \leq \delta$.

Let $\beta$ be the connected component of $W^u(y) \cap \Sigma$ containing $y$. Since $A$ is a nontrivial hyperbolic attractor standard facts about hyperbolic sets (e.g. the local product structure [9]) imply that the end points of $\beta$ belong to $\partial \Sigma$. Moreover, $A \subset \text{Per}_\epsilon^*(X) \setminus \text{Sing}(X)$ (by the shadowing lemma for flows [9]) and, since $\Delta^+_i = \pi_x(E^u_{\nu})$ for $x \in \text{Per}_1(X)$ (by Lemma 2.3) and the periodic orbits in $A$ are dense in $A$, we obtain that $\beta$ is tangent to $\Delta^+_{\text{Per}_\epsilon^*(X) \cap \Sigma}$. Then, $\beta$ is the graph of a $C^1$ map $c : [a, b] \to [-\epsilon, \epsilon]$ with $c(t_0) = y$ for some $t_0 \in (a, b)$, and so, $c$ satisfies hypotheses (i) and (ii) of Lemma 2.6. Additionally, since the end points of $\beta$ belong to $\partial \Sigma$ we also have $c(a), c(b) \in \partial \Sigma$ and so $c$ also satisfies hypothesis (iii) of Lemma 2.6. Then, Lemma 2.6 implies $a = -\epsilon$ and $b = \epsilon$ and $|c(\pm \epsilon)| < \epsilon$. Consequently, $\beta$ joins $-\epsilon \times [-1, 1]$ to $\epsilon \times [-1, 1]$ as indicated in Figure 2.

If (iv) holds, then $\beta$ (which is contained in $A$) intersects $\gamma$ (which is contained in $W^s(\sigma)$) whence $\sigma \in A$ which is absurd since $A$ is a nontrivial hyperbolic attractor. Therefore, (v) holds and so $\beta \cap R \neq \emptyset$ yielding $\text{Per}_\epsilon^*(X) \cap R \neq \emptyset$ again an absurdity. These contradictions prove the result. \qed
Now we prove the following key result.

**Proposition 2.8.** Let $X$ be a star flow with singularities on a closed 3-manifold which cannot be $C^1$ approximated by vector fields with orbit-flip homoclinic orbits. Then, there is a neighborhood $U$ of $\text{Sing}(X)$ such that if $A$ is an attractor of $X$ then $A \cap U \neq \emptyset$ if and only if $\text{Sing}(X, A) \neq \emptyset$.

**Proof.** Otherwise there is a sequence of attractors $A_n$ with $\text{Sing}(X, A_n) = \emptyset$ and $\sigma \in \Lambda \cap \text{Sing}(X)$, where

$$\Lambda = \text{Cl}\left(\bigcup_{n} A_n\right).$$

Since star flows have finitely many sinks (15, 25) we can assume that each $A_n$ is nontrivial and they are all hyperbolic by Proposition 2.5. It follows that every $A_n$ is a nontrivial hyperbolic attractor and so $\Lambda \subset \text{Per}_n(X)$.

We clearly have that $\sigma$ is neither a sink nor a source (otherwise it could not be accumulated by periodic orbits which is the case for $\sigma$). So, we can order its eigenvalues $\lambda_1, \lambda_2, \lambda_3$ in a way that either $\lambda_2$ or $\lambda_1$ is real and, in each case,

$$\text{Re}(\lambda_2) \leq \text{Re}(\lambda_3) < 0 < \lambda_1 \quad \text{or} \quad \lambda_2 < 0 < \text{Re}(\lambda_3) \leq \text{Re}(\lambda_1)$$

with $\text{Re}(\cdot)$ denoting real part.

In the first case $\sigma$ clearly satisfies hypothesis (iv) of Proposition 2.7. In the second we must have that both $\lambda_3$ and $\lambda_1$ are real (otherwise the dominated 1-splitting claimed to exist in Lemma 2.3 would not exist) and, since $X$ cannot be approximated by vector fields with orbit-flip homoclinic loops, we still have $\lambda_3 < \lambda_1$. In other words, in such a case $\sigma$ satisfies hypothesis (v) of Proposition 2.7. On the other hand, in both cases it is certainly possible to find $x \in (W^u(\sigma) \setminus \{\sigma\}) \cap \Lambda$. In particular, $x \in \text{Per}_n(X) \setminus \text{Sing}(X)$ and so, by Proposition 2.7, there is $\delta > 0$ such that $d(A_n, x) \geq \delta$ for all $n$. But this is clearly impossible due to the definition of $\Lambda$ so the result is true. \hfill \Box

**Proof of the Main Theorem.** Let $X$ be a star flow on a closed 3-manifold which cannot be $C^1$ approximated by vector fields exhibiting orbit-flip homoclinic orbits. We can assume without any loss of generality that $X$ has singularities (if not we apply [5]). Suppose by contradiction that it has infinitely many distinct attractors $A_n$, $n \in \mathbb{N}$. Since $X$ has finitely many singularities and sinks, and, since the attractors are pairwise disjoint, we can assume that each $A_n$ is not a sink and satisfies $\text{Sing}(X, A_n) = \emptyset$. In particular, each $A_n$ is a nontrivial hyperbolic attractor by Proposition 2.5. Moreover, by Proposition 2.8 there is a neighborhood $U$ of $\text{Sing}(X)$ such that $A_n \cap U = \emptyset$ for all $n$. It follows that the closure $\text{Cl}(\bigcup_n A_n)$ has no singularities. Since each $A_n$ is a nontrivial hyperbolic attractor we have $A_n \subset \text{Cl}(\text{Per}(X))$ and so $\text{Cl}(\bigcup_n A_n)$ is also a compact invariant set in $\text{Cl}(\text{Per}(X))$. Applying Lemma 2.4 we conclude that $\text{Cl}(\bigcup_n A_n)$ is a hyperbolic set. However, as is well known, hyperbolic sets contains only a finite number of attractors which is certainly not the case for $\text{Cl}(\bigcup_n A_n)$. We obtain so a contradiction which proves the result. \hfill \Box

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Instituto de Matemática, Universidade Federal do Rio de Janeiro, P. O. Box 68530, 21945-970 Rio de Janeiro, Brazil

E-mail address: morales@impa.br