Byzantine-Resilient High-Dimensional SGD with Local Iterations on Heterogeneous Data

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Abstract

We study stochastic gradient descent (SGD) with local iterations in the presence of malicious/Byzantine clients, motivated by the federated learning. The clients, instead of communicating with the central server in every iteration, maintain their local models, which they update by taking several SGD iterations based on their own datasets and then communicate the net update with the server, thereby achieving communication-efficiency. Furthermore, only a subset of clients communicate with the server, and this subset may be different at different synchronization times. The Byzantine clients may collaborate and send arbitrary vectors to the server to disrupt the learning process. To combat the adversary, we employ an efficient high-dimensional robust mean estimation algorithm at the server to filter-out corrupt vectors; and to analyze the outlier-filtering procedure, we develop a novel matrix concentration result that may be of independent interest.

We provide convergence analyses for both strongly-convex and non-convex smooth objectives in the heterogeneous data setting, where different clients may have different local datasets, and we do not make any probabilistic assumptions on data generation. We believe that ours is the first Byzantine-resilient algorithm and analysis with local iterations. We derive our convergence results under minimal assumptions of bounded variance for SGD and bounded gradient dissimilarity (which captures heterogeneity among local datasets); and we provide bounds on these quantities in the statistical heterogeneous data setting. We also extend our results to the case when clients compute full-batch gradients.

1 Introduction

In the federated learning (FL) paradigm [Kon17,KMRR16,MMR+17,MSS19], several clients (e.g., mobile devices, organizations, etc.) collaboratively learn a machine learning model, where the training process is facilitated by the data held by the participating clients (without data centralization) and is coordinated by a central server (e.g., the service provider). Due to its many advantages over the traditional centralized learning [DCM+12] (e.g., training a machine learning model without collecting the clients’ data, which, in addition to reducing the communication load on the network, provides privacy to clients’ data), FL has emerged as an active area of research recently; see [K+19] for a detailed survey. Stochastic gradient descent (SGD) has become a de facto standard in optimization for training machine learning models at such a large scale [Bot10,MMR+17,K+19], where clients iteratively communicate the gradient updates with the central server, which aggregates the gradients, updates the learning model, and sends the aggregated gradient back to the clients. The promise of FL comes with its own set of challenges [K+19]: (i) optimizing with heterogeneous data at different clients, who may have different local datasets, which may be “non-i.i.d.”, i.e., can be thought of as being generated from different underlying distributions; (ii) slow and unreliable network connections between the server and the clients, so communication in every iteration may not be feasible; (iii) availability of only a subset of clients for training at a given time (maybe due to low connectivity, as clients may be located in different geographic locations); and (iv) robustness against the malicious/Byzantine clients who may send incorrect gradient updates to the central server to disrupt the training process. In this paper, we
propose and analyze a single SGD algorithm that address all these challenges together. First we setup the problem, put our work in context with the related work, and then summarize our contributions.

We consider an empirical risk minimization problem, where data is stored at $R$ clients, each having a different dataset (with no probabilistic assumption on data generation); client $r \in [R]$ has dataset $D_r$. Let $F_r : \mathbb{R}^d \rightarrow \mathbb{R}$ denote the local loss function associated with the dataset $D_r$, which is defined as $F_r(x) \triangleq \mathbb{E}_{i \in [n_r]}[F_{r,i}(x)]$, where $n_r = |D_r|$, $i$ is uniformly distributed over $[n_r] \triangleq \{1, 2, \ldots, n_r\}$, and $F_{r,i}(x)$ is the loss associated with the $i$’th data point at client $r$ with respect to (w.r.t.) $x$. Our goal is to solve the following minimization problem:

$$\arg\min_{x \in \mathbb{R}^d} \left( F(x) \triangleq \frac{1}{R} \sum_{r=1}^{R} \mathbb{E}_{i \in [n_r]}[F_{r,i}(x)] \right).$$

Here, $\mathcal{C} \subseteq \mathbb{R}^d$ denotes the parameter space that is either equal to $\mathbb{R}^d$ or a compact and convex set.

In absence of the above-mentioned FL challenges, we can minimize (1) using distributed vanilla SGD, where in any iteration, server broadcasts the current model parameters to all the clients, each of them then samples a stochastic gradient from its local dataset and sends it back to the server, who aggregates the received gradients and updates the global model parameters. However, this simple solution does not satisfy the FL challenges, as every client communicates with the server (i.e., no sampling of clients) in every SGD iteration (i.e., no local iterations), and furthermore, this solution breaks down even with a single malicious client [BMGS17].

Recent work have proposed variants of the above-described vanilla SGD that address some of the FL challenges. The algorithms in [HKMC19, HM19, KKM+19, KMR19, LHY+20, SLS+20, YYYZ19, BDKD19] work under different heterogeneity assumptions but do not provide any robustness to malicious clients. On the other hand, [CSX17, BMGS17, YCRB18, AAL18, SX19, XKG19b, YCRB19] provide robustness, but with no local iterations or sampling of clients; furthermore, they assume homogeneous (either same or i.i.d.) data across all clients. A different line of work [CWCP18, RWCP19, DSD19b, DD19, DSD19a, LXC+19, GHYR19] provide robustness with heterogeneous data, but without local iterations or sampling of clients; [CWCP18, RWCP19, DSD19b, DD19, DSD19a] use coding across datasets, which is hard to implement in FL; [LXC+19] changes the objective function and adds a regularizer term to combat the adversary; and [HYR19] effectively reduces the heterogeneous problem to a homogeneous problem by clustering, and then learning happens within each cluster having homogeneous data.

We believe that ours is the first work that combines local iterations with Byzantine-resilience for SGD.\(^1\)

Not only that, we also analyze our algorithm on heterogeneous data and allow sampling of clients. Note that the earlier work that provide robustness (without local iterations or sampling of clients) either assume homogeneous data across clients [CSX17, BMGS17, YCRB18, AAL18, SX19, YCRB19] or require strong assumptions, such as the bounded gradient assumption on local functions (i.e., $\|\nabla F_r(x)\| \leq G$ for some finite $G$) [XKG19b]. Note that even without robustness, assuming bounded gradients is a common way to make the analysis on heterogeneous data simple [YYZ19, LHY+20], as under this assumption, we can trivially bound the heterogeneity among local datasets by $\|\nabla F_r(x) - \nabla F_s(x)\| \leq 2G$, which makes handling heterogeneity vacuous.

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\(^1\)At the completion of our work, we found that [XKG19a] also analyzed SGD in the FL setting, but with the following major differences: Not only do they make bounded gradient assumption, the approximation error (even in the Byzantine-free setting) of their solution could be as large as $O(D^2 + G^2)$, where $G$ is the gradient bound and $D$ is the diameter of the parameter space that contains the optimal parameters $x^*$ and all the local parameters $x^*_r$ ever emerged at any client $r \in [R]$ in any iteration $t \in [T]$; this, in our opinion, makes the bound vacuous. In optimization, one would ideally like to have the convergence rates depend on diameter of the parameter space with a factor that decays with the number of iterations, e.g., with $\frac{1}{\sqrt{T}}$ or $\frac{1}{T}$, and also see Theorem 1.

\(^2\)See [KMR19] for a detailed discussion on the inappropriateness of making bounded gradient assumption in heterogeneous data settings and examine the effect of heterogeneity on convergence rates (even without robustness).
1.1 Our contributions

In this paper, we tackle heterogeneity assuming only that the gradient dissimilarity among local datasets is bounded (see (6)), and propose and analyze a Byzantine-resilient SGD algorithm with local iterations and sampling of clients under the bounded variance assumption for SGD (see (2)); see Algorithm 1. We provide convergence guarantees for both strongly-convex and non-convex smooth objectives, and our algorithm can find approximate optimal parameters in the strongly-convex case and reach to a stationary point in the non-convex case both within an error of $O\left(\frac{H^2\kappa^2}{b\epsilon R} (1 + \frac{d}{R}) (\epsilon + \epsilon') + H^2\kappa^2\right)$. Here, $b$ is the mini-batch size for stochastic gradients, $\sigma^2$ is the variance bound, $\kappa^2$ captures the gradient dissimilarity, $H$ is the number of local iterations in between any two consecutive synchronization indices, $K$ is the number of clients sampled at synchronization times, $\epsilon < \frac{4}{R}$ is the fraction of Byzantine clients, and $\epsilon'$ is any constant such that $(\epsilon + \epsilon') \leq \frac{4K}{R}$. The first error term arises because of the stochasticity in gradients due to SGD and is equal to zero if we work with full-batch gradients (which gives $\sigma = 0$), and the second error term arises because of heterogeneity in local datasets; see the discussion on the approximation error analysis after we state our main results in Section 2.2. We provide concrete bounds on the variance $\sigma^2$ and the gradient dissimilarity $\kappa^2$ in a statistical heterogeneous data setting; see Theorem 4. We also give a simplified analysis of our algorithm with full-batch gradients. See Theorem 1 and Theorem 2 for our mini-batch SGD and full-batch GD convergence results, respectively.

To tackle the malicious behavior of Byzantine clients, we borrow tools from recent advances in high-dimensional robust statistics [LRV16, SCV18, DKK19]; in particular, we use the polynomial-time outlier-filtering procedure from [SCV18], which was developed for robust mean estimation in high dimensions. In order to use this algorithm, we develop a novel matrix concentration result (see Theorem 3) which may be of independent interest.

**Paper organization.** We describe our algorithm and state our main convergence results in Section 2. We describe the core part of our algorithm, the robust accumulated gradient estimation (RAGE), and our new matrix concentration result in Section 3. We instantiate our assumptions in the statistical heterogeneous data model in Section 4. Omitted details and proofs are provided in appendices.

## 2 Problem Setup and Our Results

In this section, we state our assumptions, describe the adversary model and our algorithm, and state our main convergence results.

**Assumption 1 (Bounded local variances).** The stochastic gradients sampled from any local dataset have uniformly bounded variance over $C$ for all clients, i.e., there exists a finite $\sigma$, such that

$$
\mathbb{E}_{i \in U} \|\nabla F_{r,i}(x) - \nabla F_r(x)\|^2 \leq \sigma^2, \quad \forall x \in C, r \in [R].
$$

(2)

It will be helpful to formally define mini-batch stochastic gradients, where instead of computing stochastic gradients based on just one data point, each client samples $b > 1$ data points (without replacement) from its local dataset and computes the average of $b$ gradients. For any $x \in \mathbb{R}^d, r \in [R], \ell \in [n_r]$, consider the following set

$$
\mathcal{F}_{r}^{b}(x) := \left\{ \frac{1}{b} \sum_{i \in \mathcal{H}_b} \nabla F_{r,i}(x) : \mathcal{H}_b \in \binom{[n_r]}{b} \right\}.
$$

(3)

Note that $g_r(x) \in \ell \mathcal{F}_{r}^{b}(x)$ is a mini-batch stochastic gradient with batch size $b$ at client $r$. It is not hard to see the following: \(^3\)

$$
\mathbb{E}[g_r(x)] = \nabla F_r(x), \quad \forall x \in C, r \in [R]
$$

(4)

\(^3\)Since clients sample data points without replacement, we can in fact show a stronger variance bound of $\mathbb{E}\|g_r(x) - \nabla F_r(x)\|^2 \leq \frac{(n_r-b)}{b(n_r-1)} \sigma^2$. However, for simplicity, we only use the weaker bound (5) in this paper.
and the global gradient \( \nabla F_r(x) \) is uniformly bounded over \( C \) for all clients, i.e., there exists a finite \( \kappa \), such that

\[
\|\nabla F_r(x) - \nabla F(x)\| \leq \kappa^2, \quad \forall x \in C, r \in [R].
\] (6)

Assumption 1 is standard in the SGD literature. In Assumption 2, \( \kappa \) quantifies the bounded deviation between the local loss functions \( F_r, r \in [R] \) and the global loss function \( F \); see also [YJY19, LYWZ19], where this assumption has been used in heterogeneous data settings in decentralized SGD without Byzantine clients. The gradient dissimilarity bound in (6) can be seen as a deterministic condition on local datasets, under which we derive our results. Since all results (matrix concentration and convergence) in this paper are given a finite \( \kappa \), such that

\[
\|\nabla F_r(x) - \nabla F(x)\| \leq \kappa^2, \quad \forall x \in C, r \in [R].
\] (6)

2.1 Adversary model

We assume that an \( \epsilon \) fraction of \( R \) clients are corrupt; as we see later, we can tolerate \( \epsilon < \frac{K^4}{4H} \) where \( K \leq R \) is the number of clients sampled at synchronization indices. The corrupt clients can collaborate and arbitrarily deviate from their pre-specified programs: In any SGD iteration, instead of sending the true stochastic gradients, corrupt clients may send adversarially chosen vectors (they may not even send anything if they wish, in which case, the server can treat them as erasures and replace them with a fixed value). Note that, in the erasure case, server knows which clients are corrupt; whereas, in the Byzantine problem, server does not have this information.

2.2 Main results

Let \( I_T = \{t_1, t_2, \ldots, t_k, \ldots\} \), with \( t_1 = 0 \), denote the set of synchronization indices (where \( \max_{i \geq 1} |t_{i+1} - t_i| = H \)) when the server samples a subset of \( R \) clients (denoted by \( K \subseteq [R] \)) and sends the global model (denoted by \( x \)) to them; each client \( r \in K \) updates its local model \( x_r \) by taking SGD steps based on its local dataset until the next synchronization time, when all clients in \( K \) send their local models to the server. Note that some of these clients may be corrupt and may send arbitrary vectors.\(^{\text{5}}\) Server employs a decoding RAGE and update the global model \( x \) based on that. We present our Byzantine-resilient SGD algorithm with local iterations in Algorithm 1.

Before we present our results, we need some definitions.

- **L-smoothness:** A function \( F : C \to \mathbb{R} \) is called \( L \)-smooth over \( C \subseteq \mathbb{R}^d \), if for every \( x, y \in C \), we have
  \[
  \|\nabla F(x) - \nabla F(y)\| \leq L\|x - y\| \quad (\text{this property is also known as } L \text{-Lipschitz gradients}).
  \]
  This is also equivalent to \( F(y) \leq F(x) + \langle \nabla F(x), y - x \rangle + \frac{L}{2}\|x - y\|^2 \).

- **\( \mu \)-strong convexity:** A function \( F : C \to \mathbb{R} \) is called \( \mu \)-strongly convex over \( C \subseteq \mathbb{R}^d \), if for every \( x, y \in C \), we have
  \[
  F(y) \geq F(x) + \langle \nabla F(x), y - x \rangle + \frac{\mu}{2}\|x - y\|^2.
  \]

Our convergence results are for both strongly-convex and non-convex smooth functions.

\(^{\text{4}}\)Actually, we can tolerate \( \epsilon < \frac{1}{4} \) fraction of malicious clients from the \( K \) clients that we select; so, \( \epsilon < \frac{K^4}{4H} \) is a worst case bound in case we sample all the malicious clients in a selection, which is an unlikely event.

\(^{\text{5}}\)Note that the only disruption that the corrupt clients can cause in the training process is during the gradient aggregation at synchronization indices by sending adversarially chosen vectors to the server, and we give unlimited power to the adversary for that. Because of this and for the purpose of analysis, we can assume, without loss of generality, that in between the synchronization indices, the corrupt clients sample stochastic gradients and update their local parameters honestly.
Algorithm 1 Byzantine-Resilient SGD with Local Iterations

1: Initialize. Set \( t := 0, \ x^0_r := 0, \forall r \in [R], \) and \( x := 0. \) Here \( x \) denote the global model and \( x^0_r \) denote the local model at client \( r \) at time 0. Fix a constant step-size \( \eta \) and a mini-batch size \( b. \)
2: while \( (t \leq T) \) do
3: Server selects a subset of clients \( K \subseteq [R] \) of size \( |K| = K \) and sends \( x \) to all clients in \( K. \)
4: All clients \( r \in K \) do in parallel:
5: Set \( x^t_r = x. \)
6: while (true) do
7: Take a mini-batch stochastic gradient \( g_r(x^t_r) \in \mathcal{F}^b(x^t_r) \) and update the local model:
   \[
   x^{t+1}_r \leftarrow x^t_r - \eta g_r(x^t_r); \quad t \leftarrow (t + 1).
   \]
8: if \( (t \in I_T) \) then
9: Let \( \bar{x}^t_r = x^t_r, \) if client \( r \) is honest, otherwise can be an arbitrary vector in \( \mathbb{R}^d. \)
10: Send \( \bar{x}^t_r \) to the server and break the inner while loop.
11: end if
12: end while
13: At Server:
14: Receive \( \{\bar{x}_r, r \in K\} \) from the clients in \( K. \)
15: For every \( r \in K, \) let \( \tilde{g}_{r,\text{accu}} := (\bar{x}_r - x)/\eta. \)
16: Apply the decoding algorithm RAGE (see Section 3 for more details) on \( \{\tilde{g}_{r,\text{accu}}, r \in K\}. \) Let
   \[
   \hat{g}_{\text{accu}} := \text{RAGE}(\tilde{g}_{r,\text{accu}}, r \in K).
   \]
17: Update the global model \( x \leftarrow \Pi_C (x - \eta \hat{g}_{\text{accu}}), \) where \( \Pi_C \) is the projection operator onto \( C. \)
18: end while

Theorem 1 (Mini-Batch Local Stochastic Gradient Descent). Suppose an \( \epsilon > 0 \) fraction of clients are adversarially corrupt. Let \( K_t \) denote the set of \( K \) clients that are active at any given time \( t \in [0 : T] := \{0, 1, \ldots, T\}. \) For an \( L\)-smooth global objective function \( F : C \rightarrow \mathbb{R} \) for \( L \geq 0, \) let Algorithm 1 generate a sequence of iterates \( \{x^t_r : t \in [0 : T], r \in K_t\} \) when run with a fixed step-size \( \eta = \frac{1}{8HL}. \) Fix an arbitrary constant \( \epsilon' > 0. \) If \( \epsilon \leq \frac{K}{4\epsilon'}, \) then with probability at least 1 - \( T \exp(-\frac{2\epsilon(1-\epsilon)K}{16}) \), the sequence of average iterates \( \{x^t : = \frac{1}{K_t} \sum_{r \in K_t} x^t_r, t \in [0 : T]\} \) satisfy the following convergence guarantees (where in (7), \( F \) is also \( \mu\)-strongly convex for \( \mu > 0)\):

\[
\text{Strongly-convex:} \quad \mathbb{E}\|x^T - x^*\|^2 \leq \left(1 - \frac{\mu}{16HL}\right)^T \|x^0 - x^*\|^2 + \frac{12}{\mu^2} \Gamma, \quad (7)
\]

\[
\text{Non-convex:} \quad \frac{1}{T} \sum_{t=0}^T \mathbb{E}\left\|\nabla F(x^t)\right\|^2 \leq \frac{32HL^2}{3T} \|x^0 - x^*\|^2 + \frac{5}{6} \Gamma. \quad (8)
\]

Here \( x^* \in C \) is the minimizer of the global loss function \( F(x); \) see (1). In both (7) and (8), \( \Gamma = 3T^2 + \frac{9H^2\epsilon^2}{b} + 33H^2\kappa^2 \) with \( \kappa^2 = O\left(\sigma_0^2(\epsilon + \epsilon')\right), \) where \( \sigma_0^2 = \frac{28H^2\epsilon^2}{30r} \left(1 + \frac{4b}{3r}\right) + 28H^2\kappa^2, \) and expectation is taken over the sampling of mini-batch stochastic gradients.

We prove the strongly-convex part of Theorem 1 in Appendix B and the non-convex part in Appendix C. In addition to other complications arising due to handling Byzantine clients together with local iterations, our proof deviates from the standard proofs for local SGD: We need to show two recurrences, which arise because

\[\text{All convergence results in this paper only require properties of the global loss function } F; \text{ the local loss functions } F_r \text{'s may be arbitrary. For example, in the smooth strongly-convex case, we only require } F \text{ to be smooth and strongly-convex, and } F_r \text{'s can be arbitrary. Similarly for the smooth non-convex case.}\]
at synchronization indices, server performs decoding to filter-out the corrupt clients, while at other indices there is no decoding, as there is no communication.

The failure probability of our algorithm is at most $T \exp(-\frac{C^2(1-\epsilon)K}{16})$, which though scales linearly with $T$, also goes down exponentially with $K$. As a result, in settings such as federated learning, where number of clients could be very large (e.g., in millions) and server samples a few thousand clients, we can get a very small probability of error, even if run our algorithm for a very long time. Note that the error probability is due to the stochastic sampling of gradients, and if we want a “zero” probability of error, we can run full-batch gradient descent, for which we get the following result, which we prove in Appendix D with a much simplified analysis than that of Theorem 1.

**Theorem 2** (Full-Batch Local Gradient Descent). In the same setting as that of Theorem 1, except for that we run Algorithm 1 with a fixed step-size $\eta = \frac{1}{\kappa H L}$, and in any iteration, instead of sampling mini-batch stochastic gradients, every honest client takes full-batch gradients from their local datasets. If $\epsilon \leq \frac{1}{4}$, then with probability 1, the sequence of average iterates $\{x^t = \frac{1}{K} \sum_{r \in K_t} x^t_r : t \in [0 : T]\}$ satisfy the following convergence guarantees:

**Strongly-convex:**
$$\|x^T - x^*\|^2 \leq \left(1 - \frac{\mu}{10HL}\right)^T \|x^0 - x^*\|^2 + \frac{12}{\mu^2} \Gamma_{GD},$$  
(9)

**Non-convex:**
$$\frac{1}{T} \sum_{t=0}^{T} \|\nabla F(x^t)\|^2 \leq \frac{25HL^2}{3T} \|x^0 - x^*\|^2 + \frac{14}{3} \Gamma_{GD}.$$  
(10)

In both (9) and (10), $\Gamma_{GD} = 2\Gamma_{GD}^2 + 23H^2\kappa^2$ where $\Gamma_{GD} = O(H\kappa\sqrt{\epsilon}).$

We prove our convergence results assuming that the parameter space $\mathcal{C}$ is the whole of $\mathbb{R}^d$. This streamlines our analyses, as our focus in this paper is on combining Byzantine-resilient and local iterations. We can easily incorporate projection in our proofs: The strongly-convex proofs trivially extends, and the non-convex proofs hold under a mild technical assumption on the size of the parameter space $\mathcal{C}$; see also [YCRB18, DD20].

**Analysis of the approximation error.** In Theorem 1, the approximation error $\Gamma$ essentially consists of two types of error terms: $\Gamma_1 = O\left(\frac{H^2\sigma^2}{L^2} (1 + \frac{L}{16}) (\epsilon + \epsilon')\right)$ and $\Gamma_2 = O(H^2\kappa^2)$, where $\Gamma_1$ arises due to stochastic sampling of gradients and $\Gamma_2$ arises due to dissimilarity in the local datasets. Note that both $\Gamma_1$ and $\Gamma_2$ have quadratic dependence on the number of local iterations $H$, which is unavoidable because the sum of $H$ gradients (due to local iterations) will blow up both the variance and the gradient dissimilarity by a factor of $H^2$. Note that $\Gamma_1$ decreases as we increase the batch size $b$ of stochastic gradients and becomes zero if we take full-batch gradients (which implies $\sigma = 0$), as is the case in Theorem 2. Note that the presence of the gradient dissimilarity bound $\kappa^2$ in the approximation error is inevitable, and will always show up when bounding the deviation of the true “global” gradient from the decoded one in the presence of Byzantine clients, even when $H = 1$.

**Convergence rates.** In the strongly-convex case, Algorithm 1 approximately finds the optimal parameters $x^*$ (within $\Gamma$ error) “exponentially fast”, which matches the convergence rate of vanilla SGD in the Byzantine-free setting. However, in the non-convex case, our convergence rate is $\frac{H}{T}$ (as opposed to $\frac{1}{T}$), which is affected by the factor of $H$. The reason for this is precisely because, under standard SGD assumptions we need $\eta \leq \frac{1}{\sqrt{16L}}$ to bound the drift of local parameters across different clients; see Lemma 2. Instead, if we had assumed a stronger bounded gradient assumption, then Lemma 2 would hold for a constant step-size that does not depend on $H$ (e.g., $\eta = \frac{1}{2T}$ would suffice), which would lead to a $\frac{1}{T}$ convergence rate for non-convex functions. See also [KMR19, Theorem 5], which obtains a similar rate of $\frac{H}{T}$ even when optimizing a smooth convex function with full-batch gradient descent on heterogeneous data.

### 3 Robust Accumulated Gradient Estimation (RAGE)

In this section, we describe the core part of Algorithm 1 on robust accumulated gradient estimation (RAGE), which is the subroutine for robustly estimating the average of uncorrupted accumulated gradients at every
synchronization index. First we setup the notation. Let Algorithm 1 generate a sequence of iterates \( \{x_t^r : t \in [0 : T], r \in K_t\} \) when run with a fixed step-size \( \eta \) satisfying \( \eta \leq \frac{25H^2}{3Kd} \), where \( K_t \) denotes the set of \( K \) clients that are active at time \( t \in [0 : T] \). Take any two consecutive synchronization indices \( t_k, t_{k+1} \in \mathcal{T}_r \). Note that \( |t_{k+1} - t_k| \leq H \). For an honest client \( r \in K_t \), let \( g_{r,\text{accu}}^{t_k,t_{k+1}} := \sum_{t=t_k}^{t_{k+1}-1} g_r(x_t^r) \) denote the sum of local mini-batch stochastic gradients sampled by client \( r \) between time \( t_k \) and \( t_{k+1} \), where \( g_r(x_t^r) \in U \mathcal{F}_r(x_t^r) \) satisfies (4), (5). At iteration \( t_{k+1} \), every honest client \( r \in K_t \) reports its local model \( x_{t_{k+1}}^r \) to the server, from which server computes \( g_{r,\text{accu}}^{t_k,t_{k+1}} \) (see line 15 of Algorithm 1), whereas, the corrupt clients may report arbitrary and adversarially chosen vectors in \( \mathbb{R}^d \). Server does not know the identity of the corrupt clients, and its goal is to produce an estimate \( \tilde{g}_{\text{accu}}^{t_k,t_{k+1}} \) of the average accumulated gradients from honest clients as best as possible.

To this end, first we show that there exists a large subset \( S \subseteq K_t \) of accumulated gradients from honest clients that are concentrated around their average, i.e., have bounded empirical covariance. Once we have shown that, then we will use the polynomial-time outlier-filtering algorithm from [SCV18] to estimate the average of the accumulated gradients in \( S \). Our main result on RAGE is as follows:

**Theorem 3 (Robust Accumulated Gradient Estimation).** Suppose an \( \epsilon \) fraction of \( K \) clients that communicate with the server are corrupt. In the setting described above, suppose we are given \( K \leq R \) accumulated gradients \( g_{r,\text{accu}}^{t_k,t_{k+1}}, r \in K_t \) in \( \mathbb{R}^d \), where \( g_{r,\text{accu}}^{t_k,t_{k+1}} = g_{r,\text{accu}}^{t_k,t_{k+1}} \) if the \( r \)’th client is honest, otherwise can be arbitrary. For any constant \( \epsilon' > 0 \), if \( (\epsilon + \epsilon') \leq \frac{1}{4} \), then we have:

1. **Matrix concentration:** With probability \( 1 - \exp(-\epsilon^2(1-\epsilon)K) \), there exists a subset \( S \subseteq K_t \) of uncorrupted gradients of size \( (1 - (\epsilon + \epsilon'))K \geq \frac{3K}{4} \), such that

   \[
   \lambda_{\text{max}} \left( \frac{1}{|S|} \sum_{i \in S} (g_i - g_S)(g_i - g_S)^T \right) \leq \frac{25H^2\sigma^2}{bc'} \left( 1 + \frac{4d}{3K} \right) + 28H^2\kappa^2, \tag{11}
   \]

   where, for \( i \in S \), \( g_i = g_{r,\text{accu}}^{t_k,t_{k+1}} \), \( g_S = \frac{1}{|S|} \sum_{i \in S} g_{r,\text{accu}}^{t_k,t_{k+1}} \); and \( \lambda_{\text{max}} \) denotes the largest eigenvalue.

2. **Outlier-filtering algorithm:** We can find an estimate \( \tilde{g} \) of \( g_S \) in polynomial-time with probability 1, such that \( \|\tilde{g} - g_S\| \leq O \left( \sigma_0 \sqrt{\epsilon + \epsilon'} \right) \), where \( \sigma_0^2 = \frac{25H^2\sigma^2}{bc'} \left( 1 + \frac{4d}{3K} \right) + 28H^2\kappa^2. \)

Proving the matrix concentration bound stated in the first part of Theorem 3 is non-trivial and we prove it separately in Section 3.1. For the second part, we use the polynomial-time outlier-filtering procedure of [SCV18], which is a robust mean estimation algorithm, that takes a collection of vectors as input, out of which an unknown large subset (at least a \( \frac{3}{4} \)-fraction) is promised to be well-concentrated around its sample mean, and outputs an estimate of the sample mean of the vectors in that subset. For completeness, we describe this procedure in Appendix E and refer the reader to [DD20, Appendices E, F] for more details.

Note that the same filtering procedure has also been used in [SX19, YCRB19] in the context of Byzantine-robust full batch gradient descent without local iterations for minimizing the population risk, assuming homogeneous i.i.d. data. Our setting is very different from theirs, as we minimize the empirical risk by mini-batch stochastic gradient descent with local iterations on heterogeneous data. They also derived a matrix-concentration result, whose need arises because they minimize the population risk, whereas, we need a matrix concentration bound because we use SGD. On top of that our setting is much more complicated than theirs, as clients have heterogeneous data and do not communicate with the server in every iteration. As a result, as opposed to their matrix concentration bound (which they proved using sub-exponential/sub-Gaussian distributional assumption, which inherently requires i.i.d. data across clients), our matrix concentration result is of a very different nature, and we use entirely different tools to derive that.

### 3.1 Matrix concentration

Now we prove the first part of Theorem 3. For that, we need to show an existence of a subset \( S \) of the \( K \) accumulated gradients (out of which an \( \epsilon < \frac{1}{4} \) fraction is corrupted) that has good concentration, as
quantified by the matrix concentration bound in (11). To prove this, we use a separate matrix concentration result stated in the following lemma from [DD20].

**Lemma 1** (Lemma 1 in [DD20]). Suppose there are m independent distributions \( p_1, p_2, \ldots, p_m \) in \( \mathbb{R}^d \) such that \( \mathbb{E}_{y \sim p_i}[y] = \mu_i, i \in [m] \) and each \( p_i \) has a bounded variance in all directions, i.e., \( \mathbb{E}_{y \sim p_i}[(y - \mu_i, v)^2] \leq \sigma^2_{p_i}, \forall v \in \mathbb{R}^d, \|v\| = 1 \). Take any \( \epsilon' > 0 \). Then, given m independent samples \( y_1, y_2, \ldots, y_m \), where \( y_i \sim p_i \), with probability \( 1 - \exp(-\epsilon^2 m/16) \), there is a subset \( S \) of \( (1 - \epsilon')m \) points such that

\[
\lambda_{\max} \left( \frac{1}{|S|} \sum_{i \in S} (y_i - \mu_i)(y_i - \mu_i)^T \right) \leq \frac{4\sigma^2_{\max}}{\epsilon'} \left( 1 + \frac{d}{(1 - \epsilon')m} \right), \quad \text{where } \sigma^2_{\max} = \max_{i \in [m]} \sigma^2_{p_i}.
\]

Now we prove the first part of Theorem 3 with the help of Lemma 1.

Let \( t_k, t_{k+1} \in \mathcal{T} \) be any two consecutive synchronization indices. For \( i \in \mathcal{K}_{t_k} \) corresponding to an honest client, let \( Y_{t_k}, Y_{t_{k+1}}, \ldots, Y_{t_{k+1} - 1} \) be a sequence of \((t_{k+1} - t_k)\) (dependent) random variables, where, for any \( t \in [t_k : t_{k+1} - 1] \), the random variable \( Y_i \) is distributed as

\[
Y_i \sim \text{Unif} \left( \mathcal{F}^{\otimes b} \left( x_i^t, Y_i^t, \ldots, Y_i^{t-1} \right) \right) \tag{12}
\]

Here, \( Y_i \) corresponds to the stochastic sampling of mini-batch gradients from the set \( \mathcal{F}^{\otimes b} \left( x_i^t, Y_i^t, \ldots, Y_i^{t-1} \right) \), which itself depends on the local parameters \( x_i^t \) (which is a deterministic quantity) at the last synchronization index and the past realizations of \( Y_i^t, \ldots, Y_i^{t-1} \). This is because the evolution of local parameters \( x_i^t \) depends on \( x_i^t \) and the choice of gradients in between time indices \( t_k \) and \( t-1 \). Now define \( Y_i := \sum_{k=t_k}^{t_{k+1}} Y_i \); and let \( p_i \) be the distribution of \( Y_i \). This is the distribution \( p_i \) we will take when using Lemma 1.

**Claim 1.** For any honest client \( i \in \mathcal{K}_{t_k} \), we have \( \mathbb{E}[Y_i] = \mathbb{E}[Y_i], \sigma^2_{p_i} = \frac{H^2\sigma^2}{b} \), where expectation is taken over sampling stochastic gradients by client \( i \) between the synchronization indices \( t_k \) and \( t_{k+1} \).

Claim 1 is proved in Appendix A.

It is easy to see that the hypothesis of Lemma 1 is satisfied with \( \mu_i = \mathbb{E}[y_i], \sigma^2_{p_i} = \frac{H^2\sigma^2}{b} \) for all honest clients \( i \in \mathcal{K}_{t_k} \) (note that \( p_i \) is the distribution of \( Y_i \)):

\[
\mathbb{E}_{y_i \sim p_i}[(y_i - \mathbb{E}[y_i], v)^2] \overset{(d)}{\leq} \mathbb{E}[(y_i - \mathbb{E}[y_i])^2] \cdot \|v\|^2 \overset{(e)}{\leq} \frac{H^2\sigma^2}{b},
\]

where \((d)\) follows from the Cauchy-Schwarz inequality and \((e)\) follows from Claim 1 and \( \|v\| \leq 1 \).

We are given \( K \) different (summations of \( H \)) gradients, out of which at least \( (1 - \epsilon')K \) are according to the correct distribution. By considering only the uncorrupted gradients (i.e., taking \( m = (1 - \epsilon)K \)), we have from Lemma 1 that there exists a subset \( S \subseteq \mathcal{K}_{t_k} \) of \( K \) gradients of size \( (1 - \epsilon')(1 - \epsilon)K \geq (1 - (\epsilon + \epsilon'))K \geq \frac{3K}{4} \) (where in the last inequality we used \( \epsilon + \epsilon' \leq \frac{1}{4} \)) that satisfies

\[
\lambda_{\max} \left( \frac{1}{|S|} \sum_{i \in S} (y_i - \mathbb{E}[y_i])(y_i - \mathbb{E}[y_i])^T \right) \leq \frac{4H^2\sigma^2}{be'} \left( 1 + \frac{4d}{3K} \right). \tag{13}
\]

Note that (13) bounds the deviation of the points in \( S \) from their respective means \( \mathbb{E}[y_i] \). However, in (11), we need to bound the deviation of the points in \( S \) from their sample mean \( \frac{1}{|S|} \sum_{i \in S} y_i \). As it turns out, due to our use of local iterations, this will require a substantial amount of technical work. From the alternate definition of the largest eigenvalue of symmetric matrices \( A \in \mathbb{R}^{d \times d} \), we have

\[
\lambda_{\max}(A) = \sup_{v \in \mathbb{R}^d; \|v\|=1} v^T A v. \tag{14}
\]

Applying this with \( A = \frac{1}{|S|} \sum_{i \in S} (y_i - \mathbb{E}[y_i])(y_i - \mathbb{E}[y_i])^T \), we can equivalently write (13) as

\[
\sup_{v \in \mathbb{R}^d; \|v\|=1} \left( \frac{1}{|S|} \sum_{i \in S} (y_i - \mathbb{E}[y_i], v)^2 \right) \leq \sigma^2_0 := \frac{4H^2\sigma^2}{be'} \left( 1 + \frac{4d}{3K} \right). \tag{15}
\]
Define $y_S := \frac{1}{|S|} \sum_{i \in S} y_i$ to be the sample mean of the points in $S$. Take an arbitrary $v \in \mathbb{R}^d$ such that $\|v\| = 1$.

\[
\frac{1}{|S|} \sum_{i \in S} (y_i - y_S, v)^2 = \frac{1}{|S|} \sum_{i \in S} \left[ (y_i - \mathbb{E}[y_i], v) + \langle \mathbb{E}[y_i] - y_S, v \rangle \right]^2 \\
\leq \frac{2}{|S|} \sum_{i \in S} (y_i - \mathbb{E}[y_i], v)^2 + \frac{2}{|S|} \sum_{i \in S} \langle \mathbb{E}[y_i] - y_S, v \rangle^2 \quad \text{(using $(a+b)^2 \leq 2a^2 + 2b^2$)}
\]

Using (15) to bound the first term, we get

\[
\leq 2\hat{\sigma}_0^2 + \frac{2}{|S|} \sum_{i \in S} \langle \mathbb{E}[y_i] - \frac{1}{|S|} \sum_{j \in S} y_j, v \rangle^2 \\
= 2\hat{\sigma}_0^2 + \frac{2}{|S|} \sum_{i \in S} \left[ \frac{1}{|S|} \sum_{j \in S} \langle y_j - \mathbb{E}[y_i], v \rangle \right]^2 \\
\leq 2\hat{\sigma}_0^2 + \frac{2}{|S|} \sum_{i \in S} \frac{1}{|S|} \sum_{j \in S} \langle y_j - \mathbb{E}[y_i], v \rangle^2 \quad \text{(using the Jensen’s inequality)} \\
= 2\hat{\sigma}_0^2 + \frac{2}{|S|} \sum_{i \in S} \frac{1}{|S|} \sum_{j \in S} \left[ \langle y_j - \mathbb{E}[y_i], v \rangle + \langle \mathbb{E}[y_j] - \mathbb{E}[y_i], v \rangle \right]^2 \\
\leq 2\hat{\sigma}_0^2 + \frac{2}{|S|} \sum_{i \in S} \frac{2}{|S|} \sum_{j \in S} \langle y_j - \mathbb{E}[y_i], v \rangle^2 + \frac{2}{|S|} \sum_{i \in S} \frac{2}{|S|} \sum_{j \in S} \langle \mathbb{E}[y_j] - \mathbb{E}[y_i], v \rangle^2 \quad \text{(using $(a+b)^2 \leq 2a^2 + 2b^2$)} \\
\leq 2\hat{\sigma}_0^2 + \frac{4}{|S|} \sum_{j \in S} \langle y_j - \mathbb{E}[y_j], v \rangle^2 + \frac{4}{|S|} \sum_{i \in S} \frac{1}{|S|} \sum_{j \in S} \|\mathbb{E}[y_j] - \mathbb{E}[y_i]\|^2 \quad \text{(using the Cauchy-Schwarz inequality and that $\|v\| \leq 1$)} \\
\leq 6\hat{\sigma}_0^2 + \frac{4}{|S|} \sum_{i \in S} \frac{1}{|S|} \sum_{j \in S} \|\mathbb{E}[y_j] - \mathbb{E}[y_i]\|^2 \quad \text{(16)}
\]

Claim 2. For any $r, s \in K_{t_k}$, we have

\[
\|\mathbb{E}[y_r] - \mathbb{E}[y_s]\|^2 \leq H \sum_{t=t_k}^{t_{k+1}-1} (6\kappa^2 + 3L^2\|x^t_r - x^t_s\|^2), \quad \text{(17)}
\]

where expectations in $\mathbb{E}[y_r]$ and $\mathbb{E}[y_s]$ are taken over sampling stochastic gradients between the synchronization indices $t_k, \ldots, t_{k+1}$ by client $r$ and client $s$, respectively.

Claim 2 is proved in Appendix A.

Using the bound from (17) in (16) gives

\[
\frac{1}{|S|} \sum_{i \in S} (y_i - y_S, v)^2 \leq 6\hat{\sigma}_0^2 + \frac{4}{|S|} \sum_{i \in S} \frac{1}{|S|} \sum_{j \in S} \sum_{t=t_k}^{t_{k+1}-1} (6\kappa^2 + 3L^2\|x^t_r - x^t_s\|^2) \\
= 6\hat{\sigma}_0^2 + 24H^2\kappa^2 + \frac{12HL^2}{|S|} \sum_{i \in S} \frac{1}{|S|} \sum_{j \in S} \sum_{t=t_k}^{t_{k+1}-1} \mathbb{E} \|x^t_r - x^t_s\|^2 \quad \text{(18)}
\]

Now we bound the last term of (18), which is the drift in local parameters at different clients in between any two synchronization indices.
Lemma 2. For any \( r, s \in \mathcal{K}_{t_k} \), if \( \eta \leq \frac{1}{8HL} \), we have
\[
\sum_{t = t_k}^{t_{k+1} - 1} \mathbb{E} \left\| x^r_t - x^r_s \right\|^2 \leq 7H^3\eta^2 \left( \frac{\sigma^2}{b} + 3\kappa^2 \right),
\]
where expectation is taken over sampling stochastic gradients at clients \( r, s \) between the synchronization indices \( t_k \) and \( t_{k+1} \).

Lemma 2 is proved in Appendix A.

Substituting the bound from (19) for the last term in (18) gives
\[
\frac{1}{|S|} \sum_{i \in S} \langle y_i - y_S, v \rangle^2 \leq 6\tilde{\sigma}_0^2 + 24H^2\kappa^2 + \frac{12HL^2}{|S|} \sum_{i \in S} \frac{1}{|S|} \sum_{j \in S} \left( 7H^3\eta^2 \left( \frac{\sigma^2}{b} + 3\kappa^2 \right) \right) 
\leq 6\tilde{\sigma}_0^2 + 28H^2\kappa^2 + \frac{21H^2\sigma^2}{16b} \left( 1 + \frac{4d}{3K} \right) + 84HL^2\eta^2 \left( \frac{\sigma^2}{b} + 3\kappa^2 \right) 
\leq \frac{24HL^2\sigma^2}{b\epsilon'} \left( 1 + \frac{4d}{3K} \right) + 21H^2\kappa^2 
\leq \frac{25H^2\sigma^2}{b\epsilon'} \left( 1 + \frac{4d}{3K} \right) + 28H^2\kappa^2.
\]
In the last inequality we used \( \frac{24}{16b} \leq \frac{1}{\eta} \leq \frac{1}{\eta} \left( 1 + \frac{4d}{3K} \right) \), where the first inequality follows because \( \epsilon' \leq \frac{1}{4} \). Note that (20) holds for every unit vector \( v \in \mathbb{R}^d \). Using this and substituting \( g^{t_k,t_{k+1}}_{t,\text{accu}} = y_i, g^{t_k,t_{k+1}}_{S,\text{accu}} = y_S \) in (20), we get
\[
\sup_{v \in \mathbb{R}^d; \|v\| = 1} \frac{1}{|S|} \sum_{i \in S} \langle g^{t_k,t_{k+1}}_{t,\text{accu}} - g^{t_k,t_{k+1}}_{S,\text{accu}}, v \rangle^2 \leq \frac{25H^2\sigma^2}{b\epsilon'} \left( 1 + \frac{4d}{3K} \right) + 28H^2\kappa^2.
\]
This, in view of the alternate definition of the largest eigenvalue given in (14), is equivalent to (11), which proves the first part of Theorem 3.

4 Bounding the Variance and the Gradient Dissimilarity in the Statistical Heterogeneous Data Model

In this section, we provide concrete bounds on the local variances \( \sigma^2 \) (from (2)) and the gradient dissimilarity \( \kappa^2 \) (from (6)) in the statistical model in heterogeneous setting, where different clients may have local data generated from potentially different distributions. The results in this section are taken from [DD20] – we briefly describe the setting and state the main results, and refer the reader to [DD20, Section 6] for more details and the proofs.

For simplicity, we assume that all workers have the same number \( n \) of data points. Let \( q_1, q_2, \ldots, q_R \) denote the \( R \) probability distributions, and we are given \( n \) i.i.d. samples \( z_{r,1}, z_{r,2}, \ldots, z_{r,n} \) at the \( r \)th client from \( q_r \). Fix an arbitrary \( x \in \mathcal{C} \). Let \( f_r(z, x) \) denote the local loss function at client \( r \), and we assume that for any fixed \( z, f_r(z, x) \) is \( L \)-smooth in \( x \), i.e., for any \( z \in \mathcal{Q}_r \), we have \( \| \nabla f_r(z, x) - \nabla f_r(z, y) \| \leq L \| x - y \|, \forall x, y \in \mathcal{C} \). Define \( \bar{f}_r(x) := \frac{1}{R} \sum_{r=1}^{R} f_r(x) \). The analogues of (2) and (6) in this statistical heterogeneous model are the following:
\[
\mathbb{E}_{i \in U[n]} \left\| \nabla f_r(z_{r,i}, x) - \nabla f_r(x) \right\|^2 \leq \sigma^2, \quad \forall x \in \mathcal{C},
\]
\[
\left\| \nabla \bar{f}_r(x) - \nabla \bar{f}(x) \right\|^2 \leq \kappa^2, \quad \forall x \in \mathcal{C},
\]
We need to find good upper bounds on \( \kappa \) and \( \sigma \) that hold for all \( r \in [R] \), \( x \in C \) with high probability. We provide two bounds on \( \kappa \), one when the local gradients at workers are assumed to be sub-exponential random vectors, and other when they are sub-Gaussian random vectors. We provide a bound on \( \sigma \) assuming that the local gradients are sub-Gaussian random vectors. These are standard assumptions on gradients in statistical models [CSX17, SX19, YCRB19].

Let \( \mu_r(x) := \mathbb{E}_{z \sim q_r} [f_r(z, x)] \) and \( \mu(x) := \frac{1}{R} \sum_{r=1}^{R} \mu_r(x) \). Note that, for any \( r \in [R] \), \( \nabla \mu_r(x) \) is a property of the distribution \( q_r \), and \( \| \nabla \mu_r(x) - \nabla \mu(x) \| \) captures heterogeneity among distributions through their expected values. So, in order to get a meaningful bound for \( \kappa \), it is reasonable to assume that this heterogeneity is bounded. We assume that \( \| \nabla \mu_r(x) - \nabla \mu(x) \| \leq \kappa_{\text{mean}}, \forall x \in C \).

**Definition 1** (Sub-exponential and sub-Gaussian local gradients). We say that the local gradients have sub-exponential distribution, if for any \( x \in C \), there exists non-negative parameters \((\nu, \alpha)\), such that for every unit vector \( v \in \mathbb{R}^d \), we have

\[
\sup_{v \in \mathbb{R}^d : \|v\|^2 = 1} \mathbb{E}_{z \sim q_r} [\exp(\nu (\nabla f_r(z, x) - \nabla \mu_r(x), v))] \leq \exp(\frac{\nu^2}{2}), \quad \forall |\nu| < \frac{1}{\alpha}.
\]

We say that the local gradients have sub-Gaussian distribution, if for any \( x \in C \), there exists a non-negative parameter \( \sigma \), such that we have

\[
\sup_{v \in \mathbb{R}^d : \|v\|^2 = 1} \mathbb{E}_{z \sim q_r} [\exp(\nu (\nabla f_r(z, x) - \nabla \mu_r(x), v))] \leq \exp(\frac{\nu^2\sigma^2}{2}), \quad \forall \nu \in \mathbb{R}.
\]

Now we state bounds on the variance and the gradient dissimilarity under these distributional assumptions on local gradients.

**Theorem 4.** With probability at least \( 1 - \frac{R}{(1+nLD)^2} \), we have the following bounds for all \( r \in [R] \):

1. **Variance bound.** [DD20, Theorem 7]: If the local gradients have sub-Gaussian distribution and \( n \in \mathbb{N} \) is arbitrary, we have

\[
\mathbb{E}_{i \in \mathcal{U}^n} \left[ \| \nabla f_r(z_{r,i}, x) - \nabla \bar{f}_r(x) \|^2 \right] \leq O(d \log(d)), \quad \forall x \in C.
\]

2. **Gradient dissimilarity bound.** [DD20, Theorem 6]: If, either the local gradients have sub-exponential distribution and \( n = \Omega(d \log(nd)) \), or the local gradients have sub-Gaussian distribution and \( n \in \mathbb{N} \) is arbitrary, we have

\[
\| \nabla \bar{f}_r(x) - \nabla \bar{f}(x) \| \leq \kappa_{\text{mean}} + O\left(\sqrt{\frac{d \log(nd)}{n}}\right), \quad \forall x \in C.
\]

Theorem 4 can be proven using standard tools, such as concentration results for sums of independent sub-Gaussian/sub-exponential random variables and \( \epsilon \)-net arguments, and we refer the reader to [DD20] for a detailed proof.

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A Omitted Details from Section 3.1

In this section, we prove the omitted proofs from Section 3.1, namely, we prove Claim 1, Claim 2, and Lemma 2.

Claim (Restating Claim 1). For any honest client \( i \in K_{t_k} \), we have \( \mathbb{E}[\|Y_i - \mathbb{E}[Y_i]\|^2] \leq \frac{H^2\sigma^2}{b} \), where expectation is taken over sampling stochastic gradients by client \( i \) between the synchronization indices \( t_k \) and \( t_{k+1} \).

Proof. Take an arbitrary honest client \( i \in K_{t_k} \).

\[
\mathbb{E}[\|Y_i - \mathbb{E}[Y_i]\|^2] = \mathbb{E}\left[ \left\| \sum_{\ell=t_k}^{t_{k+1}-1} \left(Y_i^\ell - \mathbb{E}[Y_i^\ell]\right) \right\|^2 \right] \\
\leq (t_{k+1} - t_k) \mathbb{E}\left[ \sum_{\ell=t_k}^{t_{k+1}-1} \|Y_i^\ell - \mathbb{E}[Y_i^\ell]\|^2 \right] \leq \frac{H^2\sigma^2}{b},
\]

where (a) follows from the Jensen’s inequality; in (b) we used \( (t_{k+1} - t_k) \leq H \) and that \( \mathbb{E}[\|Y_i^\ell - \mathbb{E}[Y_i^\ell]\|^2] \leq \frac{\sigma^2}{b} \) for all \( j \in [H] \), which follows from the explanation below:

\[
\mathbb{E}[\|Y_i^\ell - \mathbb{E}[Y_i^\ell]\|^2] = \sum_{y_i^{t_k}, \ldots, y_i^{t_{k-1}}} \mathbb{P}[Y_i^\ell = y_i^\ell, j \in [t_k : t-1]] \\
\times \mathbb{E}\left[ \|Y_i^\ell - \mathbb{E}[Y_i^\ell]\|^2 \mid Y_i^\ell = y_i^\ell, j \in [t_k : t-1] \right] \\
\leq \sum_{y_i^{t_k}, \ldots, y_i^{t_{k-1}}} \mathbb{P}[Y_i^\ell = y_i^\ell, j \in [t_k : t-1]] \cdot \frac{\sigma^2}{b} \\
= \frac{\sigma^2}{b}.
\]

Note that \( Y_i^\ell \sim \text{Unif}(\mathcal{F}_i \land \{x_i^{t_k}, Y_i^{t_k}, \ldots, Y_i^{t_{k-1}}\}) \). So, when we fix the values \( Y_i^{t_k} = y_i^{t_k}, \ldots, Y_i^{t-1} = y_i^{t-1} \), the parameter vector \( x_i^{t_k}(x_i^{t_k}, Y_i^{t_k}, \ldots, Y_i^{t-1}) \) becomes a deterministic quantity. Now we can use the variance bound (5) in order to bound \( \mathbb{E}\left[ \|Y_i^\ell - \mathbb{E}[Y_i^\ell]\|^2 \mid Y_i^\ell = y_i^\ell, j \in [t_k : t-1] \right] \leq \frac{\sigma^2}{b} \). This is what we used in (c).

\[\square\]

Claim (Restating Claim 2). For any \( r, s \in K_{t_k} \), we have

\[
\|\mathbb{E}[y_r] - \mathbb{E}[y_s]\|^2 \leq \frac{H}{t_{k+1} - t_k} \left( 6\kappa^2 + 3L^2\mathbb{E}[\|x_r - x_s\|^2] \right),
\]

where expectations in \( \mathbb{E}[y_r] \) and \( \mathbb{E}[y_s] \) are taken over sampling stochastic gradients between the synchronization indices \( t_k, \ldots, t_{k+1} \) by client \( r \) and client \( s \), respectively.

Proof. Note that we can equivalently write \( \mathbb{E}[y_r] = \mathbb{E}[Y_r] \) and \( \mathbb{E}[y_s] = \mathbb{E}[Y_s] \).

\[
\|\mathbb{E}[Y_r] - \mathbb{E}[Y_s]\|^2 = \|\mathbb{E}[Y_r] - \mathbb{E}[Y_s]\|^2 \\
= \left\| \sum_{\ell=t_k}^{t_{k+1}-1} \left( \mathbb{E}[Y_r^\ell] - \mathbb{E}[Y_s^\ell] \right) \right\|^2
\]

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\begin{align}
& \leq (t_{k+1} - t_k) \sum_{t=t_k}^{t_{k+1}-1} \| \mathbb{E}[Y^t_r] - \mathbb{E}[Y^t_s] \|^2 \\
& \quad \text{(23)}
\end{align}

By definition of \( Y^t_r \) from (12), we have \( Y^t_r \sim \text{Unif}\left( \mathcal{F}_{s}^{b}(x^t_r(x^{t_k}_s, Y^{t_k}_{s-1}, \ldots, Y^{t-1}_{s-1})) \right) \), which implies using (4) that \( \mathbb{E}[Y^t_r] = \mathbb{E}\left[ \nabla F_s(x^t_r(x^{t_k}_s, Y^{t_k}_{s-1}, \ldots, Y^{t-1}_{s-1})) \right] \), where on the RHS, expectation is taken over \((Y^{t_k}_{s-1}, \ldots, Y^{t-1}_{s-1})\). To make the notation less cluttered, in the following, for any \( s \in \mathcal{K}_{t_k} \), we write \( x^t_s \) to denote \( x^t_r(x^{t_k}_s, Y^{t_k}_{s-1}, \ldots, Y^{t-1}_{s-1}) \) with the understanding that expectation is always taken over the sampling of stochastic gradients between \( t_k \) and \( t_{k+1} \). With these substitutions, the \( t \)’th term from (24) can be written as:

\[
\begin{align}
& \| \mathbb{E}[Y^t_r] - \mathbb{E}[Y^t_s] \|^2 = \| \mathbb{E}\left[ \nabla F_s(x^t_r) - \nabla F_s(x^t_s) \right] \|^2 \\
& \quad \leq \mathbb{E}\| \nabla F_r(x^t_r) - \nabla F_s(x^t_s) \|^2 \\
& \quad \leq 3\mathbb{E}\| \nabla F_r(x^t_r) - \nabla F(x^t_r) \|^2 + 3\mathbb{E}\| \nabla F_s(x^t_s) - \nabla F(x^t_s) \|^2 \\
& \quad + 3\mathbb{E}\| \nabla F(x^t_r) - \nabla F(x^t_s) \|^2 \\
& \quad \leq 6\kappa^2 + 3L^2\mathbb{E}\| x^t_r - x^t_s \|^2. \\
& \quad \text{(25)}
\end{align}
\]

Here, (a) and (b) both follow from the Jensen’s inequality. (c) used the gradient dissimilarity bound from (6) to bound the first two terms\(^7\) and \( L \)-Lipschitzness of \( \nabla F \) to bound the last term.

Substituting the bound from (25) back in (24) and using \((t_{k+1} - t_k) \leq H\) proves Claim 2. \( \Box \)

**Lemma** (Restating Lemma 2). For any \( r, s \in \mathcal{K}_{t_k} \), if \( \eta \leq \frac{1}{8HL} \), we have

\[
\sum_{t=t_k}^{t_{k+1}-1} \mathbb{E}\| x^t_r - x^t_s \|^2 \leq 7H^2\eta^2 \left( \frac{\sigma^2}{b} + 3\kappa^2 \right),
\]

where expectation is taken over sampling stochastic gradients at clients \( r, s \) between the synchronization indices \( t_k \) and \( t_{k+1} \).

**Proof.** For any \( t \in [t_k : t_{k+1} - 1] \) and \( r, s \in \mathcal{K}_{t_k} \), define \( D^t_{r,s} = \mathbb{E}\| x^t_r - x^t_s \|^2 \). Note that at synchronization time \( t_k \), all clients in the active set \( \mathcal{K}_{t_k} \) have the same parameters, i.e., \( x^t_{r^k} = x^t_{s^k} \) for every \( r \in \mathcal{K}_{t_k} \).

\[
D^t_{r,s} = \mathbb{E}\| x^t_r - x^t_s \|^2 = \mathbb{E}\left\| x^{t_k}_r - \eta \sum_{j=t_k}^{t-1} \mathbf{g}_r(x^j_r) - x^{t_k}_s - \eta \sum_{j=t_k}^{t-1} \mathbf{g}_s(x^j_s) \right\|^2
\]

\[
= \eta^2 \mathbb{E}\left\| \sum_{j=t_k}^{t-1} \left( \mathbf{g}_r(x^j_r) - \mathbf{g}_s(x^j_s) \right) \right\|^2
\]

\[
\leq \eta^2(t - t_k) \sum_{j=t_k}^{t-1} \mathbb{E}\| \mathbf{g}_r(x^j_r) - \mathbf{g}_s(x^j_s) \|^2
\]

\[
\leq \eta^2H \sum_{j=t_k}^{t-1} \left( 3\mathbb{E}\| \mathbf{g}_r(x^j_r) - \nabla F_r(x^j_r) \|^2 + 3\mathbb{E}\| \mathbf{g}_s(x^j_s) - \nabla F_s(x^j_s) \|^2 \\
+ 3\mathbb{E}\| \nabla F_r(x^j_r) - \nabla F_s(x^j_s) \|^2 \right)
\]

\[
\text{(26)}
\]

\(^7\text{Note that though } x^t_{r^k} \text{’s are random quantities, we can still bound } \mathbb{E}\| \nabla F_r(x^j_r) - \nabla F_s(x^j_s) \|^2 \leq \kappa^2 \text{ because the gradient dissimilarity bound (6) holds uniformly over the entire domain.}\)
To bound the first and the second terms we use the variance bound from (5).\footnote{Note that $x^t_r$'s are random quantities, however, since the variance bound (5) holds uniformly over the entire domain, we can bound $\mathbb{E} \left\| g_r(x^t_r) - \nabla F_r(x^t_r) \right\|^2 \leq \frac{\sigma^2}{b}$.} We can bound the third term in the same way as we bounded it in (24) and obtained (25). This gives

$$D^t_{r,s} \leq \eta^2 H \sum_{j=t_k}^{t-1} \left( \frac{6\sigma^2}{b} + 18H^2 \eta^2 + 9L^2 \mathbb{E} \| x^j - x^k_s \|^2 \right)$$

$$\leq \frac{6H^2 \sigma^2 \eta^2}{b} + 18H^2 \eta^2 \kappa^2 + 9L^2 H \eta^2 \sum_{j=t_k}^{t-1} D^j_{r,s}$$

(Since $D^j_{r,s} = \mathbb{E} \| x^j - x^k_s \|^2$)

Taking summation from $t = t_k$ to $t_{k+1} - 1$ gives

$$\sum_{t=t_k}^{t_{k+1} - 1} D^t_{r,s} \leq \sum_{t=t_k}^{t_{k+1} - 1} \left( \frac{6H^2 \sigma^2 \eta^2}{b} + 18H^2 \eta^2 \kappa^2 + 9L^2 H \eta^2 \sum_{j=t_k}^{t-1} D^j_{r,s} \right)$$

$$\leq \frac{6H^3 \sigma^2 \eta^2}{b} + 18H^3 \eta^2 \kappa^2 + 9L^2 H^2 \eta^2 \sum_{t=t_k}^{t_{k+1} - 1} D^t_{r,s}.$$ 

After rearranging terms, we get

$$(1 - 9L^2 H^2 \eta^2) \sum_{t=t_k}^{t_{k+1} - 1} D^t_{r,s} \leq \frac{6H^3 \sigma^2 \eta^2}{b} + 18H^3 \eta^2 \kappa^2. \quad (27)$$

If we take $\eta \leq \frac{1}{\pi H^2}$, we get $(1 - 9\eta^2 L^2 H^2) \geq \frac{9}{7}$. Substituting this in the LHS of (27) yields $\sum_{t=t_k}^{t_{k+1} - 1} D^t_{r,s} \leq \frac{7H^3 \sigma^2 \eta^2}{b} + 21H^3 \eta^2 \kappa^2$, which proves Lemma 2.

\[ \square \]

**B Convergence Proof of the Strongly-Convex Part of Theorem 1**

Let $I_T := \{t_1, t_2, \ldots, t_k, \ldots \}$ with $t_1 = 0$ be the set of synchronization indices at which server selects a subset $K \subseteq [R]$ of $K$ clients and sends the current global model parameters to them. Upon receiving that, clients in $K$ perform local SGD steps based on their own local datasets until the next synchronization index, at which they send their local model parameters to the server. When server has received the updates from clients, it applies the outlier-filtering procedure RAGE (see Algorithm 1) to robustly estimate the average of the uncorrupted accumulated gradients and then updates the global model parameters. We assume that $H = \max_{i \geq 1} (t_{i+1} - t_i)$.

At any iteration $t \in [T]$, let $K_t \subseteq [R]$ denote the set of clients that are active at time $t$. Let $x^t := \frac{1}{K} \sum_{r \in K} x^t_r$ denote the average parameter vector of the clients in the active set $K_t$. Note that, for any $t_i$ in $I_T$, the clients in $K_{t_i}$ remain active at all time indices $t$ such that $t \in [t_i : t_{i+1} - 1]$.

In the following, we denote the decoded gradient at the server at any synchronization time $t_{i+1}$ by $\hat{g}_{\text{accu}}^{t_i:t_{i+1}}$, which is an estimate of the average of the accumulated gradients between time $t_i$ and $t_{i+1}$ of the honest clients in $K_{t_i}$ as in Theorem 3. From Algorithm 1, we can write the parameter update rule for the global model at the synchronization indices as:

$$x^{t_{i+1}} = x^{t_i} - \eta \hat{g}_{\text{accu}}^{t_i:t_{i+1}}.$$ 

Note that at any synchronization index $t_i \in I_T$, when server selects a subset $K_{t_i}$ of clients and sends the global parameter vector $x^{t_i}$, all clients in $K_{t_i}$ set their local model parameters to be equal to the global model parameters, i.e., $x^{t_i}_r = x^{t_i}$ holds for every $r \in K_{t_i}$. 

\[ \square \]
Now we proceed with proving the strongly-convex part of Theorem 1.
First we derive a recurrence relation for the synchronization indices and then later we extend the proof to all indices. Consider the \((i+1)\)st synchronization index \(t_{i+1} \in \mathcal{I}_t\).

\[
x_{t_{i+1}} = x_{t_i} - \eta g_{\text{accu}}^{t_{i+1}} + 1 \\
= x_{t_i} - \frac{1}{K} \sum_{r \in \mathcal{K}_{t_i}} \sum_{t = t_i}^{t_{i+1} - 1} \nabla F_r(x_r^t) - \eta \left(g_{\text{accu}}^{t_{i+1}} - \frac{1}{K} \sum_{r \in \mathcal{K}_{t_i}} \sum_{t = t_i}^{t_{i+1} - 1} \nabla F_r(x_r^t)\right)
\]

For simplicity of notation, define \(E \triangleq \left(g_{\text{accu}}^{t_{i+1}} - \frac{1}{K} \sum_{r \in \mathcal{K}_{t_i}} \sum_{t = t_i}^{t_{i+1} - 1} \nabla F_r(x_r^t)\). Substituting this in the above and using \(x^t_i = \frac{1}{K} \sum_{r \in \mathcal{K}_{t_i}} x_r^t\) gives:

\[
x_{t_{i+1}} = \frac{1}{K} \sum_{r \in \mathcal{K}_{t_i}} x_r^{t_i} - \frac{1}{K} \sum_{r \in \mathcal{K}_{t_i}} \sum_{t = t_i}^{t_{i+1} - 1} \nabla F_r(x_r^t) - \eta E
\]

\[
= \frac{1}{K} \sum_{r \in \mathcal{K}_{t_i}} \left(x_r^{t_i} - \eta \sum_{t = t_i}^{t_{i+1} - 1} \nabla F_r(x_r^t)\right) - \eta E
\]

\[
= \frac{1}{K} \sum_{r \in \mathcal{K}_{t_i}} (x_r^{t_{i+1} - 1} - \eta \nabla F_r(x_r^{t_{i+1} - 1}))) - \eta E
\]

\[
= x^{t_{i+1} - 1} - \eta \frac{1}{K} \sum_{r \in \mathcal{K}_{t_i}} \nabla F_r(x_r^{t_{i+1} - 1}) - \eta E
\]

\[
= x^{t_{i+1} - 1} - \eta \nabla F(x^{t_{i+1} - 1}) + \frac{1}{K} \sum_{r \in \mathcal{K}_{t_i}} (\nabla F(x_r^{t_{i+1} - 1}) - \nabla F_r(x_r^{t_{i+1} - 1})) - \eta E
\]

(28)

Subtracting \(x^*\) from both sides gives:

\[
x^{t_{i+1}} - x^* = x^{t_{i+1} - 1} - x^* - \eta \nabla F(x^{t_{i+1} - 1}) + \frac{1}{K} \sum_{r \in \mathcal{K}_{t_i}} (\nabla F(x_r^{t_{i+1} - 1}) - \nabla F_r(x_r^{t_{i+1} - 1})) - \eta E
\]

(29)

This gives \(x^{t_{i+1}} - x^* = u + \eta (v - E)\). Taking norm on both sides and then squaring gives:

\[
\|x^{t_{i+1}} - x^*\|^2 = \|u\|^2 + \eta^2 \|v - E\|^2 + 2\eta \langle u, v - E \rangle
\]

(30)

Now we use a simple but powerful trick on inner-products together with the inequality \(2\langle a, b \rangle \leq \|a\|^2 + \|b\|^2\) and get:

\[
2\eta \langle u, v - E \rangle = 2 \left\langle \sqrt{\frac{\eta \mu}{2}} u, \sqrt{\frac{\eta \mu}{2}} (v - E) \right\rangle \leq \frac{\eta \mu}{2} \|u\|^2 + \frac{2\eta}{\mu} \|v - E\|^2
\]

(31)

Substituting this back in (30) gives:

\[
\|x^{t_{i+1}} - x^*\|^2 \leq \left(1 + \frac{\eta \mu}{2}\right) \|u\|^2 + \eta \left(\eta + \frac{2}{\mu}\right) \|v - E\|^2
\]

\[
\leq \left(1 + \frac{\eta \mu}{2}\right) \|u\|^2 + 2\eta \left(\eta + \frac{2}{\mu}\right) \|v\|^2 + 2\eta \left(\eta + \frac{2}{\mu}\right) \|E\|^2
\]

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Substituting the values of \( u, v, E \) and taking expectation w.r.t. the stochastic sampling of gradients by clients in \( K_t \), between iterations \( t_i \) and \( t_{i+1} \) (while conditioning on the past) gives:

\[
\mathbb{E} \left\| x^{t_{i+1}} - x^* \right\|^2 \leq \left( 1 + \frac{\mu \eta}{2} \right) \mathbb{E} \left\| x^{t_{i+1}} - \eta \nabla F(x^{t_{i+1}}) - x^* \right\|^2 \\
+ 2\eta \left( \eta + \frac{2}{\mu} \right) \mathbb{E} \left\| \frac{1}{K} \sum_{r \in K_{t_i}} \left( \nabla F(x^{t_{i+1}}) - \nabla F_r(x^{t_{i+1}}) \right) \right\|^2 \\
+ 2\eta \left( \eta + \frac{2}{\mu} \right) \mathbb{E} \left\| \hat{g}_{\text{accu}}^{t_i, t_{i+1}} - \frac{1}{K} \sum_{r \in K_{t_i}} \sum_{i=t_i}^{t_{i+1}-1} \nabla F_r(x^i) \right\|^2
\]

Now we bound each of the three terms on the RHS of (32) separately in Claim 3, Claim 4, and Claim 5, respectively.

**Claim 3.** For \( \eta < \frac{1}{L} \), we have

\[
\mathbb{E} \left\| x^{t_{i+1}} - \eta \nabla F(x^{t_{i+1}}) - x^* \right\|^2 \leq (1 - \mu \eta) \mathbb{E} \left\| x^{t_{i+1}} - x^* \right\|^2.
\]

**Proof.** Expand the LHS.

\[
\mathbb{E} \left\| x^{t_{i+1}} - x^* - \eta \nabla F(x^{t_{i+1}}) \right\|^2 = \mathbb{E} \left\| x^{t_{i+1}} - x^* \right\|^2 + \eta^2 \mathbb{E} \left\| \nabla F(x^{t_{i+1}}) \right\|^2 \\
+ 2\eta \mathbb{E} \left\langle x^* - x^{t_{i+1}}, \nabla F(x^{t_{i+1}}) \right\rangle
\]

We can bound the second term on the RHS using \( L \)-smoothness of \( F \), which implies that \( \left\| \nabla F(x) \right\|^2 \leq 2L(F(x) - F(x^*)) \) holds for every \( x \in \mathbb{R}^d \); see Fact 1 on page 23. We can bound the third term on the RHS using \( \mu \)-strong convexity of \( F \) as follows: \( \left\langle x^* - x^{t_{i+1}}, \nabla F(x^{t_{i+1}}) \right\rangle \leq F(x^*) - F(x^{t_{i+1}}) - \frac{\mu}{2} \left\| x^{t_{i+1}} - x^* \right\|^2 \).

Substituting these back in (34) gives:

\[
\mathbb{E} \left\| x^{t_{i+1}} - x^* - \eta \nabla F(x^{t_{i+1}}) \right\|^2 \leq (1 - \mu \eta) \mathbb{E} \left\| x^{t_{i+1}} - x^* \right\|^2 \\
- 2\eta(1 - \eta L) \mathbb{E} \left( F(x^{t_{i+1}}) - F(x^*) \right)
\]

Since \( \eta < \frac{1}{L} \), we have \( 1 - \eta L > 0 \). We also have \( F(x^{t_{i+1}}) \geq F(x^*) \). Using these together, we can ignore the last term in the RHS of (35). This proves Claim 3.

**Claim 4.** For \( \eta \leq \frac{1}{8L} \), we have

\[
\mathbb{E} \left\| \frac{1}{K} \sum_{r \in K_{t_i}} \left( \nabla F_r(x^{t_{i+1}}) - \nabla F(x^{t_{i+1}}) \right) \right\|^2 \leq 2\kappa^2 + \frac{7H}{32} \left( \frac{\sigma^2}{b} + 3\kappa^2 \right).
\]

**Proof.** By definition, we have \( x^{t_{i+1}} = \frac{1}{K} \sum_{r \in K_{t_i}} x^{t_{i+1}} \).

\[
\mathbb{E} \left\| \frac{1}{K} \sum_{r \in K_{t_i}} \left( \nabla F_r(x^{t_{i+1}}) - \nabla F(x^{t_{i+1}}) \right) \right\|^2 \leq \frac{1}{K} \sum_{r \in K_{t_i}} \mathbb{E} \left\| \nabla F_r(x^{t_{i+1}}) - \nabla F(x^{t_{i+1}}) \right\|^2 \\
\leq \frac{2}{K} \sum_{r \in K_{t_i}} \left( \mathbb{E} \left\| \nabla F_r(x^{t_{i+1}}) - \nabla F(x^{t_{i+1}}) \right\|^2 + \mathbb{E} \left\| \nabla F(x^{t_{i+1}}) - \nabla F(x^{t_{i+1}}) \right\|^2 \right) \\
\leq \frac{2}{K} \sum_{r \in K_{t_i}} \left( \kappa^2 + L^2 \mathbb{E} \left\| x^{t_{i+1}} - x^{t_{i+1}} \right\|^2 \right)
\]
\[
\begin{align*}
&= 2\kappa^2 + \frac{2L^2}{K} \sum_{r \in K_{t_i}} \mathbb{E} \left\| x_r^{t_{i+1}-1} - \frac{1}{K} \sum_{s \in K_{t_i}} x_s^{t_{i+1}-1} \right\|^2 \\
&\leq 2\kappa^2 + \frac{2L^2}{K} \sum_{r \in K_{t_i}} \frac{1}{K} \sum_{s \in K_{t_i}} \mathbb{E} \left\| x_r^{t_{i+1}-1} - x_s^{t_{i+1}-1} \right\|^2 \\
&\leq 2\kappa^2 + \frac{2L^2}{K} \sum_{r \in K_{t_i}} \frac{1}{K} \sum_{s \in K_{t_i}} \left( 7H^3\eta^2 \left( \frac{\sigma^2}{b} + 3\kappa^2 \right) \right) \\
&= 2\kappa^2 + 14L^2H^3\eta^2 \left( \frac{\sigma^2}{b} + 3\kappa^2 \right) \overset{\text{(c)}}{\leq} 2\kappa^2 + \frac{7H}{32} \left( \frac{\sigma^2}{b} + 3\kappa^2 \right)
\end{align*}
\]

In (a) we used the gradient dissimilarity bound from (6) to bound the first term and \(L\)-Lipschitz gradient property of \(F\) to bound the second term. For (b), note that we have already bounded \(\sum_{t=1}^{t_{i+1}-1} \mathbb{E} \| x_r^t - x_s^t \|^2 \leq 7H^3\eta^2 \left( \frac{\sigma^2}{b} + 3\kappa^2 \right)\) in (19) in Lemma 2. Since each term in the summation is trivially bounded by the same quantity, which we used in (b) to bound \(\mathbb{E} \| x_r^{t_{i+1}-1} - x_s^{t_{i+1}-1} \|^2 \leq 7H^3\eta^2 \left( \frac{\sigma^2}{b} + 3\kappa^2 \right)\). In (c) we used \(\eta \leq \frac{1}{7H}\).

\[\text{Claim 5. If } \eta \leq \frac{1}{7H}, \text{ then with probability at least } 1 - \exp \left( -\frac{\epsilon^2(1-\epsilon)K}{16} \right), \text{ we have}
\]

\[\mathbb{E} \left\| \hat{g}_{\text{accu}}^{t_{i+1}} - \frac{1}{K} \sum_{r \in K_{t_i}} \sum_{t=t_i}^{t_{i+1}-1} \nabla F_r(x_r^t) \right\|^2 \leq 3T^2 + \frac{8H^2\kappa^2}{b} + 30H^2\kappa^2,
\]

\[\text{where } T^2 = \mathcal{O} \left( \sigma_0^2(\epsilon + \epsilon') \right) \text{ and } \sigma_0^2 = \frac{25H^2\kappa^2}{4b} \left( 1 + \frac{4\kappa}{5\kappa} \right) + 28H^2\kappa^2.
\]

\[\text{Proof. Let } S \subseteq K_{t_i} \text{ denote the subset of honest clients of size } (1 - (\epsilon + \epsilon'))K, \text{ whose average accumulated gradient between time } t_i \text{ and } t_{i+1} \text{ that server approximates at time } t_{i+1} \text{ in Theorem 3. Let the average accumulated gradient be denoted by } \hat{g}_{S,\text{accu}}^{t_i,t_{i+1}} = \frac{1}{|S|} \sum_{r \in S} g_r^{t_i,t_{i+1}} \text{, where } g_r^{t_i,t_{i+1}} = \sum_{t=t_i}^{t_{i+1}-1} g_r(x_r^t) \text{, and server approximates it by } \hat{g}_{\text{accu}}^{t_i,t_{i+1}}. \text{ Note that } S \text{ exists with probability at least } 1 - \exp \left( -\frac{\epsilon^2(1-\epsilon)K}{16} \right). \text{ To make the notation less cluttered, for every } r \in K_{t_i}, \text{ define } \nabla F_r^{t_i,t_{i+1}} := \sum_{t=t_i}^{t_{i+1}-1} \nabla F_r(x_r^t).
\]

\[\mathbb{E} \left\| \hat{g}_{\text{accu}}^{t_i,t_{i+1}} - \frac{1}{K} \sum_{r \in K_{t_i}} \nabla F_r^{t_i,t_{i+1}} \right\|^2 \leq 3\mathbb{E} \left\| \hat{g}_{\text{accu}}^{t_i,t_{i+1}} - \frac{1}{|S|} \sum_{r \in S} g_r^{t_i,t_{i+1}} \right\|^2 \\
+ 3\mathbb{E} \left\| \frac{1}{|S|} \sum_{r \in S} g_r^{t_i,t_{i+1}} - \frac{1}{|S|} \sum_{r \in S} \nabla F_r^{t_i,t_{i+1}} \right\|^2 \leq T^2,
\]

Now we bound each term on the RHS of (39).

**Bounding the first term on the RHS of (39).** We can bound this using the second part of Theorem 3 as follows (note that given the first part of Theorem 3 is satisfied, the second part provides deterministic approximation guarantees, which implies that it also holds in expectation):

\[\mathbb{E} \left\| \hat{g}_{\text{accu}}^{t_i,t_{i+1}} - \frac{1}{|S|} \sum_{r \in S} g_r^{t_i,t_{i+1}} \right\|^2 \leq T^2,
\]
where $T^2 = O\left(\sigma_0^2(\epsilon + \epsilon')\right)$ and $\sigma^2 = \frac{2H^2\kappa^2}{\eta^2} (1 + \frac{4\kappa}{\eta^2}) + 28H^2\kappa^2$.

Bounding the second term on the RHS of (39). We can bound this using the variance bound (5).

$$E \left\| \frac{1}{|S|} \sum_{r \in S} (g_{r, t_{i+1}} - \nabla F_{r, t_{i+1}}) \right\|^2 = E \left\| \sum_{t=1}^{t_{i+1}-1} \frac{1}{|S|} \sum_{r \in S} (g_{r} - \nabla F_r) \right\|^2$$

(a) $\leq (t_{i+1} - t_i) \sum_{t=1}^{t_{i+1}-1} E \left\| \frac{1}{|S|} \sum_{r \in S} (g_{r} - \nabla F_r) \right\|^2$

(b) $\leq H \sum_{t=1}^{t_{i+1}-1} \frac{1}{|S|^2} \sum_{r \in S} \left\| g_{r} - \nabla F_r \right\|^2$

(c) $\leq H \sum_{t=1}^{t_{i+1}-1} \frac{1}{|S|^2} \sum_{r \in S} \left\| g_{r} - \nabla F_r \right\|^2$

(d) $\leq H \sum_{t=1}^{t_{i+1}-1} \frac{1}{|S|^2} \sum_{r \in S} \left\| g_{r} - \nabla F_r \right\|^2$

$$\leq \frac{4H^2\sigma^2}{b} \leq \frac{3bK}{\eta^2}.$$ (41)

In (a) we used the Jensen’s inequality. In (b) used $|t_{i+1} - t_i| \leq H$. In (c) we used (4) (which states that $E[g_r(x)] = \nabla F_r(x)$ holds for every honest client $r \in [R]$ and $x \in \mathbb{R}^d$) together with the fact that the stochastic gradients at different clients are sampled independently, and then we used the fact that the variance of independent random variables is equal to the sum of the variances. Note that $\text{Var}(g_r(x^t)) = E \left\| g_r(x^t) - \nabla F_r(x^t) \right\|^2$. In (d) we used the variance bound (5). In (e) we used $|S| \geq (1 - (\epsilon + \epsilon'))K \geq \frac{3bK}{\eta^2}$, where the last inequality uses $(\epsilon + \epsilon') \leq \frac{1}{4}$.

Bounding the third term on the RHS of (39).

$$E \left\| \frac{1}{|S|} \sum_{r \in S} \nabla F_r^{t_{i+1}} - \frac{1}{K} \sum_{s \in K_{t_i}} \nabla F_s^{t_{i+1}} \right\|^2 = E \left\| \sum_{t=1}^{t_{i+1}-1} \frac{1}{|S|} \sum_{r \in S} \nabla F_r(x^t) - \frac{1}{K} \sum_{s \in K_{t_i}} \nabla F_s(x^t) \right\|^2$$

(a) $\leq H \sum_{t=1}^{t_{i+1}-1} E \left\| \sum_{r \in S} \nabla F_r(x^t) - \frac{1}{K} \sum_{s \in K_{t_i}} \nabla F_s(x^t) \right\|^2$ (42)

In (a), first we used the Jensen’s inequality and then substituted $|t_{i+1} - t_i| \leq H$. In order to bound (42), it suffices to bound $E \left\| \frac{1}{|S|} \sum_{r \in S} \nabla F_r(x^t) - \frac{1}{K} \sum_{s \in K_{t_i}} \nabla F_s(x^t) \right\|^2$ for every $t \in [t_i : t_{i+1} - 1]$. We bound this in the following. Take an arbitrary time $t \in [t_i : t_{i+1} - 1]$.

$$E \left\| \frac{1}{|S|} \sum_{r \in S} \nabla F_r(x^t) - \frac{1}{K} \sum_{s \in K_{t_i}} \nabla F_s(x^t) \right\|^2 \leq 3E \left\| \frac{1}{|S|} \sum_{r \in S} \nabla F_r(x^t) - \nabla F(x^t) \right\|^2$$

$$+ 3E \left\| \frac{1}{|S|} \sum_{r \in S} \nabla F_r(x^t) - \frac{1}{K} \sum_{s \in K_{t_i}} \nabla F_s(x^t) \right\|^2$$

$$\leq \frac{3}{|S|} \sum_{r \in S} \left\| \nabla F_r(x^t) - \nabla F(x^t) \right\|^2 + \frac{3}{|S|} \sum_{r \in S} \left\| \nabla F_r(x^t) - \nabla F_s(x^t) \right\|^2$$

$$+ 3E \left\| \frac{1}{|S|} \sum_{r \in S} \nabla F_r(x^t) - \nabla F(x^t) \right\|^2 + \frac{3}{K} \sum_{s \in K_{t_i}} \left\| \nabla F(x^t) - \nabla F_s(x^t) \right\|^2$$

$$\leq \frac{3}{|S|} \sum_{r \in S} \left\| \nabla F_r(x^t) - \nabla F(x^t) \right\|^2 + \frac{3}{|S|} \sum_{r \in S} \left\| \nabla F_r(x^t) - \nabla F_s(x^t) \right\|^2$$

$$+ 3E \left\| \frac{1}{|S|} \sum_{r \in S} \nabla F_r(x^t) - \nabla F(x^t) \right\|^2 + \frac{3}{K} \sum_{s \in K_{t_i}} \left\| \nabla F(x^t) - \nabla F_s(x^t) \right\|^2$$

$$\leq \frac{3}{|S|} \sum_{r \in S} \left\| \nabla F_r(x^t) - \nabla F(x^t) \right\|^2 + \frac{3}{|S|} \sum_{r \in S} \left\| \nabla F_r(x^t) - \nabla F_s(x^t) \right\|^2$$

$$+ 3E \left\| \frac{1}{|S|} \sum_{r \in S} \nabla F_r(x^t) - \nabla F(x^t) \right\|^2 + \frac{3}{K} \sum_{s \in K_{t_i}} \left\| \nabla F(x^t) - \nabla F_s(x^t) \right\|^2$$

$$\leq \frac{3}{|S|} \sum_{r \in S} \left\| \nabla F_r(x^t) - \nabla F(x^t) \right\|^2 + \frac{3}{|S|} \sum_{r \in S} \left\| \nabla F_r(x^t) - \nabla F_s(x^t) \right\|^2$$

$$+ 3E \left\| \frac{1}{|S|} \sum_{r \in S} \nabla F_r(x^t) - \nabla F(x^t) \right\|^2 + \frac{3}{K} \sum_{s \in K_{t_i}} \left\| \nabla F(x^t) - \nabla F_s(x^t) \right\|^2$$

\[20\]
\[ \eta \in (a) \text{ we used } 1 \leq \frac{\mu \eta}{2} (1 + \frac{d}{2r}) + 28H^2 \kappa^2. \]

This completes the proof of Claim 5. \qed

Using the bounds from (33), (36), (38) in (32) and using \((1 + \frac{\mu \eta}{2}) (1 - \mu \eta) \leq (1 - \frac{\mu \eta}{2})\) for the first term gives

\[ \mathbb{E} \| x^{t+1} - x^* \|^2 \leq \left( 1 - \frac{\mu \eta}{2} \right) \mathbb{E} \| x^{t+1} - x^* \|^2 + 2\eta \left( \eta + \frac{2}{\mu} \right) \left( 2\kappa^2 + \frac{7H}{32} \left( \frac{\sigma^2}{b} + 3\kappa^2 \right) \right) \]
\[ x^{t+1} = x^t - \eta \frac{1}{K} \sum_{r \in \mathcal{K}_t} g_r(x^t_r) \]

\[ = x^t - \eta \frac{1}{K} \sum_{r \in \mathcal{K}_t} \nabla F_r(x^t_r) - \eta \left( \frac{1}{K} \sum_{r \in \mathcal{K}_t} g_r(x^t_r) - \frac{1}{K} \sum_{r \in \mathcal{K}_t} \nabla F_r(x^t_r) \right) \]

\[ = x^t - \eta \nabla F(x^t) + \eta \frac{1}{K} \sum_{r \in \mathcal{K}_t} \left( \nabla F(x^t) - \nabla F_r(x^t_r) \right) \]

Now, subtracting \( x^* \) from both sides and following the same steps as in from (29) to (32), we get (in the following, expectation is taken w.r.t. the stochastic sampling of gradients at the \( t \)'th iteration while conditioning on the past):

\[ \mathbb{E} \left\| x^{t+1} - x^* \right\|^2 \leq \left( 1 + \frac{\mu H}{2} \right) \mathbb{E} \left\| x^t - x^* - \eta \nabla F(x^t) \right\|^2 + 2\eta \left( \frac{\eta}{\mu} \right) \mathbb{E} \left\| \frac{1}{K} \sum_{r \in \mathcal{K}_t} \left( \nabla F(x^t) - \nabla F_r(x^t_r) \right) \right\|^2 \]

\[ + 2\eta \left( \frac{\eta}{\mu} \right) \mathbb{E} \left\| \frac{1}{K} \sum_{r \in \mathcal{K}_t} \left( g_r(x^t_r) - \nabla F_r(x^t_r) \right) \right\|^2 \]  

(46)
The value of $\eta$ that minimizes the RHS of (a) is $t = -\frac{L}{2}(\nabla F(x), v)$, this implies (b); (c) follows from the Cauchy-Schwarz inequality: $(u, v) \leq \|u\|\|v\|$, where equality is achieved whenever $u = v$. Now, substituting $\inf_y F(y) = F(x^*)$ yields the result. \hfill $\square$
With this substitution, (50) becomes

\[
-\eta \left( \sum_{r \in \mathcal{K}_{t_i}} \sum_{t = t_i}^{t_{i+1}-1} \nabla F_r(x_r^t) \right)
\]

(50)

Let

\[
C := \frac{1}{K} \sum_{r \in \mathcal{K}_{t_i}} \left( \nabla F(x_{t_i+1}^t) - \nabla F_r(x_{t_i+1}^t) \right) - \left( \frac{1}{K} \sum_{r \in \mathcal{K}_{t_i}} \sum_{t = t_i}^{t_{i+1}-1} \nabla F_r(x_r^t) \right).
\]

(51)

With this substitution, (50) becomes

\[
x_{t_i+1} = x_{t_i+1}^t - \eta \nabla F(x_{t_i+1}^t) + \eta C
\]

(52)

Now, using the definition of \( L \)-smoothness, we have

\[
F(x_{t_i+1}^t) \leq F(x_{t_i+1}^t) + \langle \nabla F(x_{t_i+1}^t), x_{t_i+1}^t - x_{t_i+1}^t \rangle + \frac{L}{2} \| x_{t_i+1}^t - x_{t_i+1}^t \|^2
\]

\[
= F(x_{t_i+1}^t) - \eta \langle \nabla F(x_{t_i+1}^t), \nabla F(x_{t_i+1}^t) - C \rangle + \frac{\eta^2 L}{2} \| \nabla F(x_{t_i+1}^t) - C \|^2
\]

\[
= F(x_{t_i+1}^t) - \eta \| \nabla F(x_{t_i+1}^t) \|^2 + \eta \langle \nabla F(x_{t_i+1}^t), C \rangle + \frac{\eta^2 L}{2} \| \nabla F(x_{t_i+1}^t) - C \|^2
\]

(a) \leq F(x_{t_i+1}^t) - \eta \| \nabla F(x_{t_i+1}^t) \|^2 + \frac{\eta}{2} \left( \| \nabla F(x_{t_i+1}^t) \|^2 + \| C \|^2 \right)

\[+ \frac{\eta^2 L}{2} \| \nabla F(x_{t_i+1}^t) - C \|^2\]

(b) \leq F(x_{t_i+1}^t) - \eta \| \nabla F(x_{t_i+1}^t) \|^2 + \frac{\eta}{2} \left( \| \nabla F(x_{t_i+1}^t) \|^2 + \| C \|^2 \right)

\[+ \eta^2 L \left( \| \nabla F(x_{t_i+1}^t) \|^2 + \| C \|^2 \right)\]

(53)

In (a) and (b), we used the inequalities \( \langle a, b \rangle \leq \frac{1}{2} (\| a \|^2 + \| b \|^2) \) and \( \| a + b \|^2 \leq 2(\| a \|^2 + \| b \|^2) \), respectively. Substituting the value of \( C \) from (51) and using \( \| a + b \|^2 \leq 2(\| a \|^2 + \| b \|^2) \) and taking expectation w.r.t. the stochastic sampling of gradients between iterations \( t_i \) and \( t_{i+1} \) (while conditioning on the past) gives:

\[
\mathbb{E}[F(x_{t_i+1}^t)] \leq \mathbb{E}[F(x_{t_i+1}^t)] - \eta \left( \frac{1}{2} - \eta L \right) \mathbb{E} \| \nabla F(x_{t_i+1}^t) \|^2
\]

\[+ 2\eta \left( \frac{1}{2} + \eta L \right) \mathbb{E} \left\| \frac{1}{K} \sum_{r \in \mathcal{K}_{t_i}} (\nabla F(x_{t_i+1}^t) - \nabla F_r(x_{t_i+1}^t)) \right\|^2\]

\[+ 2\eta \left( \frac{1}{2} + \eta L \right) \mathbb{E} \left\| g_{\text{accu}}^{t_i, t_{i+1}} - \frac{1}{K} \sum_{r \in \mathcal{K}_{t_i}} \sum_{t = t_i}^{t_{i+1}-1} \nabla F_r(x_r^t) \right\|^2\]

(54)

We have already bounded the second and the third terms on the RHS of (54) in (36) and (38), respectively. Substituting those bounds in (54) gives:

\[
\mathbb{E}[F(x_{t_i+1}^t)] \leq \mathbb{E}[F(x_{t_i+1}^t)] - \eta \left( \frac{1}{2} - \eta L \right) \mathbb{E} \| \nabla F(x_{t_i+1}^t) \|^2
\]

\[+ 2\eta \left( \frac{1}{2} + \eta L \right) \left( 2\kappa^2 + \frac{7H}{32} \left( \frac{\sigma^2}{b} + 3\kappa^2 \right) \right)\]

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\[ + 2\eta \left( \frac{1}{2} + \eta L \right) \left( 3 \bar{Y}^2 + \frac{8 \sigma^2}{b} + 30 H^2 \kappa^2 \right), \]  

(55)

where \( \bar{Y}^2 = \mathcal{O} \left( \sigma_0^2 (\epsilon + \epsilon') \right) \) and \( \sigma_0^2 = \frac{25 H^2 \sigma^2}{b} \left( 1 + \frac{4H}{3M} \right) + 28 H^2 \kappa^2 \). Note that (55) holds with probability at least \( 1 - \exp \left( -\frac{\epsilon^2 (1-\epsilon) K}{16} \right) \).

Since \( \eta \leq \frac{1}{8 \bar{Y} L} \leq \frac{1}{8 \bar{Y} L} \), we have \( \left( \frac{1}{2} - \eta L \right) \geq \frac{3}{8} \) and \( \left( \frac{1}{2} + \eta L \right) \leq \frac{5}{8} \). Using \( \eta \leq \frac{1}{8 \bar{Y} L} \), we can also bound the last two terms of (55) in the same way as we bounded them to arrive at (44). Substituting these in (55) gives

\[ \mathbb{E}[F(x_{t+1}^i)] \leq \mathbb{E}[F(x_{t+1}^i)] - \frac{3\eta}{8} \mathbb{E} \| \nabla F(x_{t+1}^i) \|^2 + \frac{5\eta}{4} \left( 3 \bar{Y}^2 + \frac{9H^2 \sigma^2}{b} + 33 H^2 \kappa^2 \right). \]  

(56)

Note that above recurrence in (56) holds only at the synchronization indices \( t_i \in \mathcal{I}_T \) for \( i = 1, 2, 3, \ldots \). However, in order to establish a recurrence that we can use to prove convergence, we need to show a recurrence relation for all \( t \). Note that (56) gives a recurrence at the synchronization indices. Now we give a recurrence at non-synchronization indices.

Take an arbitrary \( t \in [T] \) and let \( t_i \in \mathcal{I}_T \) be such that \( t \in \{ t_i : t_{i+1} - 1 \} \); when \( H \geq 2 \), such \( t_i \)'s exist. Note that \( x^t = \sum_{r \in K_i} x^t_i \).

From (45), we have \( x_{t+1}^i = x^t - \eta \nabla F(x^t) + \eta D \), where

\[ D := \frac{1}{K} \sum_{r \in K_i} (\nabla F(x^t_i) - \nabla F_r(x^t_i)) - \frac{1}{K} \sum_{r \in K_i} (g_r(x^t_i) - \nabla F_r(x^t_i)). \]

Using \( L \)-smoothness of \( F \), and then performing similar algebraic manipulations that we used in order to arrive at (53), we get

\[ F(x_{t+1}^i) \leq F(x^t_i) - \eta \left( \frac{1}{2} - \eta L \right) \| \nabla F(x^t_i) \|^2 + \eta \left( \frac{1}{2} + \eta L \right) \| D \|^2. \]  

(57)

Substituting the value of \( D \) and using the inequality \( \| a + b \|^2 \leq 2(\| a \|^2 + \| b \|^2) \) and taking expectation w.r.t. the stochastic sampling of gradients at the \( t \)'th iteration (while conditioning on the past) gives:

\[ \mathbb{E}[F(x_{t+1}^i)] \leq \mathbb{E}[F(x^t_i)] - \eta \left( \frac{1}{2} - \eta L \right) \mathbb{E} \| \nabla F(x^t_i) \|^2 \\
+ 2\eta \left( \frac{1}{2} + \eta L \right) \mathbb{E} \left( \frac{1}{K} \sum_{r \in K_i} (\nabla F(x^t_i) - \nabla F_r(x^t_i)) \right)^2 \\
+ 2\eta \left( \frac{1}{2} + \eta L \right) \mathbb{E} \left( \frac{1}{K} \sum_{r \in K_i} (g_r(x^t_i) - \nabla F_r(x^t_i)) \right)^2 \\
\leq \mathbb{E}[F(x^t_i)] - \frac{3\eta}{8} \mathbb{E} \| \nabla F(x^t_i) \|^2 + \frac{5\eta}{4} \left( 2 \kappa^2 + \frac{7H}{32} \left( \frac{\sigma^2}{b} + 3 \kappa^2 \right) + \frac{\sigma^2}{K} \right) \]  

(58)

In (a) we used \( \left( \frac{1}{2} - \eta L \right) \geq \frac{3}{8} \) and \( \left( \frac{1}{2} + \eta L \right) \leq \frac{5}{8} \). We bounded the last two expectation terms in the LHS of (a) in the same way as we bounded them to arrive at (47).

Note that the RHS of (58) is smaller than the RHS of (56). Therefore, we can conclude that the following recurrence (which we obtain by proceeding with (54)) holds at every iteration \( t \):

\[ \mathbb{E}[F(x_{t+1}^i)] \leq \mathbb{E}[F(x^t_i)] - \frac{3\eta}{8} \mathbb{E} \| \nabla F(x^t_i) \|^2 + \frac{5\eta}{4} \left( 3 \bar{Y}^2 + \frac{9H^2 \sigma^2}{b} + 33 H^2 \kappa^2 \right). \]  

(59)

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By taking a telescopic sum, and then averaging gives
\[
\frac{1}{T} \sum_{t=0}^{T} \mathbb{E} \left\| \nabla F(x^t) \right\|^2 \leq \frac{8}{3T} \mathbb{E} \left[ F(x^0) - F(x^{T+1}) \right] + \frac{5}{6} \left( 3\gamma^2 + \frac{9H^2\sigma^2}{b} + 33H^2\kappa^2 \right).
\]

Using \( F(x^{T+1}) \geq F(x^*) \) and substituting \( F(x^0) - F(x^*) \leq \frac{L}{2} \| x^0 - x^* \|^2 \) (which follows from \( L \)-smoothness of \( F \)) gives
\[
\frac{1}{T} \sum_{t=0}^{T} \mathbb{E} \left\| \nabla F(x^t) \right\|^2 \leq \frac{4L}{3T} \| x^0 - x^* \|^2 + \frac{5}{6} \left( 3\gamma^2 + \frac{9H^2\sigma^2}{b} + 33H^2\kappa^2 \right).
\]

Note that the last term in (61) is a constant. So, it would be best to take the step-size \( \eta \) to be as large as possible such that it satisfies \( \eta \leq \frac{1}{8HL} \). We take \( \eta = \frac{1}{8HL} \). Substituting this in (61) gives
\[
\frac{1}{T} \sum_{t=0}^{T} \mathbb{E} \left\| \nabla F(x^t) \right\|^2 \leq \frac{32HL^2}{3T} \| x^0 - x^* \|^2 + \frac{5}{6} \left( 3\gamma^2 + \frac{9H^2\sigma^2}{b} + 33H^2\kappa^2 \right),
\]
where \( \gamma^2 = O \left( \frac{\sigma^2}{\epsilon} (1 + \epsilon') \right) \) and \( \sigma^2_0 = \frac{25H^2\sigma^2}{36} \left( 1 + \frac{4L}{hk} \right) + 28H^2\kappa^2 \).

**Error probability analysis.** Note that for any fixed \( t \), (59) holds with probability at least \( 1 - \exp \left( -e^{2(1-\epsilon)K/16} \right) \). Since to get (60), we used (59) \( T \) times; as a consequence, by union bound, we have that (60) holds with probability at least \( 1 - T \exp \left( -e^{2(1-\epsilon)K/16} \right) \), which is at least \( (1 - \delta) \), for any \( \delta > 0 \), provided we run our algorithm for \( T \leq \delta \exp(\frac{e^{2(1-\epsilon)K}}{16}) \) iterations.

## D Convergence Proof of Theorem 2

In this section, we focus on the case when in each local iteration clients compute full-batch gradients (instead of computing mini-batch stochastic gradients) in Algorithm 1 and prove Theorem 2. Note that the robust accumulated gradient estimation (RAGE) result of Theorem 3 (which is for stochastic gradients) is the main ingredient behind the convergence analyses in Theorem 1. So, in order to prove Theorem 2, first we need to show a RAGE result for full-batch gradients. Note that we can obtain such a result by substituting \( \sigma = 0 \) in both the parts of Theorem 3; however, this would give a loose bound on the approximation error in the second part. In the following, we get a tighter bound (both for RAGE and the convergence rates in Theorem 2) by working directly with full-batch gradients. To get a RAGE result for full-batch gradients, we do a much simplified analysis than what we did before to prove Theorem 3, and the resulting result is stated and proved below in Theorem 5.

Note that, in order to prove Theorem 3, we showed an existence of a subset \( S \) of honest clients (from the set \( K \) of clients who communicate with the server) from whom the accumulated stochastic gradients are well-concentrated, as stated in form of a matrix concentration bound (11) in the first part of Theorem 3. It turns out that for full-batch gradients, an analogous result can be proven directly (as there is no randomness due to stochastic gradients); and below we provide such a result. Note that Theorem 3 is a probabilistic statement, where we show that with high probability, there exists a large subset \( S \subseteq K \) of honest clients whose stochastic accumulated gradients are well-concentrated. In contrast, in the following result, we can deterministically take the set of all honest clients in \( K \) to be that subset for which we can directly show the concentration.

First we setup the notation to state our main result on RAGE for full-batch gradients. Let \( K_t \subseteq [R] \) denote the subset of clients of size \( K \) that are active at any time \( t \in [0 : T] \). Let Algorithm 1 generate a sequence of iterates \( \{ x^t_r : t \in [0 : T], r \in K_t \} \) when run with a fixed step-size \( \eta \) satisfying \( \eta \leq \frac{1}{4HL} \) while minimizing a global objective function \( F : \mathcal{C} \rightarrow \mathbb{R} \), where in any iteration, instead of sampling mini-batch
stochastic gradients, every honest client takes full-batch gradients from their local datasets. Take any two consecutive synchronization indices \( t_k, t_{k+1} \in \mathcal{I}_F \). Note that \( |t_{k+1} - t_k| \leq H \). For an honest client \( r \in \mathcal{K}_{t_k} \), let \( \nabla F_{r,\text{accu}}^{t_k,t_{k+1}} := \sum_{t=t_k}^{t_{k+1}-1} \nabla F_r(x^r_t) \) denote the sum of local full-batch gradients taken by client \( r \) between time \( t_k \) and \( t_{k+1} \). Note that at iteration \( t_{k+1} \), every honest client \( r \in \mathcal{K}_{t_k} \) reports its local parameters \( x_r^{t_{k+1}} \) to the server, from which server can compute \( \nabla F_{r,\text{accu}}^{t_k,t_{k+1}} \), whereas, corrupt clients may report arbitrary and adversarially chosen vectors in \( \mathbb{R}^d \). The goal of the server is to produce an estimate \( \nabla F_{\text{accu}}^{t_k,t_{k+1}} \) of the average accumulated gradients from honest clients as best as possible.

**Theorem 5** (Robust Accumulated Gradient Estimation for Full-Batch Gradient Descent). Suppose an \( \epsilon \) fraction of clients who communicate with the server are corrupt. In the setting and notation described above, suppose we are given \( K \leq R \) accumulated full-batch gradients \( \nabla F_{r,\text{accu}}^{t_k,t_{k+1}} \), \( r \in \mathcal{K}_{t_k} \) in \( \mathbb{R}^d \), where \( \nabla F_{r,\text{accu}}^{t_k,t_{k+1}} = \nabla F_{r,\text{accu}}^{t_k,t_{k+1}} \) if the \( r \)th client is honest, otherwise can be arbitrary. Let \( S \subseteq \mathcal{K}_{t_k} \) be the subset of all honest clients in \( \mathcal{K}_{t_k} \) and \( \nabla F_{S,\text{accu}}^{t_k,t_{k+1}} := \frac{1}{|S|} \sum_{i \in S} \nabla F_{i,\text{accu}}^{t_k,t_{k+1}} \) be the sample average of uncorrupted full-batch gradients. If \( \epsilon \leq \frac{1}{4} \), then we can find an estimate \( \nabla F_{\text{accu}}^{t_k,t_{k+1}} \) of \( \nabla F_{S,\text{accu}}^{t_k,t_{k+1}} \) in polynomial-time, such that

\[
\left\| \nabla F_{\text{accu}}^{t_k,t_{k+1}} - \nabla F_{S,\text{accu}}^{t_k,t_{k+1}} \right\| \leq O(H\sqrt{n}) \text{ holds with probability } 1.
\]

**Proof.** First we prove that

\[
\lambda_{\text{max}} \left( \frac{1}{|S|} \sum_{i \in S} \left( \nabla F_{i,\text{accu}}^{t_k,t_{k+1}} - \nabla F_{S,\text{accu}}^{t_k,t_{k+1}} \right) \left( \nabla F_{i,\text{accu}}^{t_k,t_{k+1}} - \nabla F_{S,\text{accu}}^{t_k,t_{k+1}} \right)^T \right) \leq 11 H^2 \kappa^2.
\]

In view of the alternate characterization the largest eigenvalue given in (14), this is equivalent to showing

\[
\sup_{v \in \mathbb{R}^d : \|v\| = 1} \frac{1}{|S|} \sum_{i \in S} \left\langle \nabla F_{i,\text{accu}}^{t_k,t_{k+1}} - \nabla F_{S,\text{accu}}^{t_k,t_{k+1}}, v \right\rangle^2 \leq 11 H^2 \kappa^2,
\]

which we prove below. Define \( F_{i,\text{accu}}^{t_k,t_{k+1}} := \sum_{t=t_k}^{t_{k+1}-1} F(x^r_t) \), where \( x^r_t = \frac{1}{R} \sum_{r \in \mathcal{K}_{t_k}} x^r_t \) for any \( t \in [t_k : t_{k+1} - 1] \). Take an arbitrary unit vector \( v \in \mathbb{R}^d \).

\[
\frac{1}{|S|} \sum_{i \in S} \left\langle \nabla F_{i,\text{accu}}^{t_k,t_{k+1}} - \nabla F_{S,\text{accu}}^{t_k,t_{k+1}}, v \right\rangle^2
\]

\[
= \frac{1}{|S|} \sum_{i \in S} \left\langle \left( \nabla F_{i,\text{accu}}^{t_k,t_{k+1}} - \nabla F_{S,\text{accu}}^{t_k,t_{k+1}} \right) + \nabla F_{i,\text{accu}}^{t_k,t_{k+1}} - \nabla F_{S,\text{accu}}^{t_k,t_{k+1}}, v \right\rangle^2
\]

\[
\leq \frac{2}{|S|} \sum_{i \in S} \left\langle \nabla F_{i,\text{accu}}^{t_k,t_{k+1}} - \nabla F_{S,\text{accu}}^{t_k,t_{k+1}}, v \right\rangle^2 + \frac{2}{|S|} \sum_{i \in S} \left\langle \nabla F_{i,\text{accu}}^{t_k,t_{k+1}} - \nabla F_{S,\text{accu}}^{t_k,t_{k+1}}, v \right\rangle^2
\]

\[
= \frac{2}{|S|} \sum_{i \in S} \left\langle \nabla F_{i,\text{accu}}^{t_k,t_{k+1}} - \nabla F_{S,\text{accu}}^{t_k,t_{k+1}}, v \right\rangle^2 + 2 \left( \nabla F_{S,\text{accu}}^{t_k,t_{k+1}} - \nabla F_{S,\text{accu}}^{t_k,t_{k+1}}, v \right)^2
\]

\[
= \frac{2}{|S|} \sum_{i \in S} \left\langle \nabla F_{i,\text{accu}}^{t_k,t_{k+1}} - \nabla F_{S,\text{accu}}^{t_k,t_{k+1}}, v \right\rangle^2 + 2 \left[ \frac{1}{|S|} \sum_{i \in S} \left\langle \nabla F_{i,\text{accu}}^{t_k,t_{k+1}} - \nabla F_{S,\text{accu}}^{t_k,t_{k+1}}, v \right\rangle \right]^2
\]

\[
\leq \frac{2}{|S|} \sum_{i \in S} \left\langle \nabla F_{i,\text{accu}}^{t_k,t_{k+1}} - \nabla F_{S,\text{accu}}^{t_k,t_{k+1}}, v \right\rangle^2 + 2 \left( \nabla F_{S,\text{accu}}^{t_k,t_{k+1}} - \nabla F_{S,\text{accu}}^{t_k,t_{k+1}}, v \right)^2
\]

\[
= \frac{4}{|S|} \sum_{i \in S} \left\langle \nabla F_{i,\text{accu}}^{t_k,t_{k+1}} - \nabla F_{S,\text{accu}}^{t_k,t_{k+1}}, v \right\rangle^2
\]

\[
\leq \frac{4}{|S|} \sum_{i \in S} \left\| \nabla F_{i,\text{accu}}^{t_k,t_{k+1}} - \nabla F_{S,\text{accu}}^{t_k,t_{k+1}} \right\|^2
\]

(Using Cauchy-Schwarz inequality \( \langle u, v \rangle \leq \|u\| \|v\| \) and that \( \|v\| = 1 \))
\[
\frac{4}{|S|} \sum_{i \in S} \left( \frac{1}{|S|} \sum_{t = t_k}^{t_{k+1} - 1} (\nabla F_i(x'_t) - \nabla F(x_t')) \right)^2 \quad \text{(Since } F_{\text{accu}}^{t_k, t_{k+1}} = \sum_{t = t_k}^{t_{k+1} - 1} F(x_t'))
\]

\[
\leq \frac{4}{|S|} \sum_{i \in S} \sum_{t = t_k}^{t_{k+1} - 1} (t_{k+1} - t_k) \left\| \nabla F_i(x'_t) - \nabla F(x_t') \right\|^2 \leq 4H^2 \kappa^2 + 8HL^2 \sum_{t = t_k}^{t_{k+1} - 1} \left\| \nabla F(x'_t) - \nabla F(x_t') \right\|^2
\]

\[
\leq 4H^2 \kappa^2 + 8HL^2 \sum_{t = t_k}^{t_{k+1} - 1} \frac{1}{|S|} \sum_{i \in S} \left\| x'_t - \frac{1}{K} \sum_{j \in K_t} x'_j \right\|^2 \quad \text{(Using Jensen’s inequality)}
\]

The last inequality follows from the Jensen’s inequality. In (a) we used (6) to bound \( \|\nabla F_i(x'_t) - \nabla F(x_t')\|^2 \leq \kappa^2 \) and \( L\)-Lipschitz gradient property of \( F \) to bound \( \|\nabla F(x'_t) - \nabla F(x_t')\| \leq L\|x'_t - x_t'\| \).

Now we bound the last term of (65).

**Lemma 3.** For any \( r, s \in K_t \), if \( \eta \leq \frac{1}{5H^2} \), we have

\[
\sum_{t = t_k}^{t_{k+1} - 1} \left\| x'_r - x'_s \right\|^2 \leq 7\eta^2 H^3 \kappa^2.
\]

**Proof.** Note that we have shown a similar result (but, in expectation) in Lemma 2 (on page 10), which is for stochastic gradients. We will simplify that proof to prove Lemma 3, which is for full-batch deterministic gradients.

Take an arbitrary \( t \in [t_k : t_{k+1} - 1] \). Following the proof of Lemma 2 until (26), and substituting \( \sigma = 0 \) and removing the factor of 3 inside the summation (the factor of 3 appeared because we applied the Jensen’s inequality earlier to separate the deterministic gradient term and the stochastic gradient terms) would give

\[
\left\| x'_r - x'_s \right\|^2 \leq \eta^2 H \sum_{j = t_k}^{t_{k+1} - 1} \left\| \nabla F(x'_r) - \nabla F(x'_s) \right\|^2.
\]

Following the remaining proof of Lemma 2 from (26) until the end gives the desired result. \( \square \)

Substituting the bound from (66) to (65) gives

\[
\frac{1}{|S|} \sum_{i \in S} \left( \nabla F_{t, \text{accu}}^{t_k, t_{k+1}} - \nabla F_{\text{accu}}^{t_k, t_{k+1}} \right, v \left) \right|^2 \leq 8H^2 \kappa^2 + 56H^2 \eta^2 \kappa^2
\]

\[
\leq 8H^2 \kappa^2 + \frac{56}{25} H^2 \kappa^2 \quad \text{(Substituting } \eta \leq \frac{1}{5H^2} \text{)}
\]

\[
\leq 11H^2 \kappa^2.
\]

Note that (68) holds for an arbitrary unit vector \( v \in \mathbb{R}^d \), implying that (64) holds true. Since (64) and (63) are equivalent, we have thus shown (63).

Now apply the second part of Theorem 3 with \( S \) being the set of all honest clients, and \( F_{t, \text{accu}}^{t_k, t_{k+1}} = \nabla F_{t, \text{accu}}^{t_k, t_{k+1}}, g_{\text{accu}}^{t_k, t_{k+1}} = \nabla F_{\text{accu}}^{t_k, t_{k+1}}, \epsilon' = 0, \) and \( \sigma_0^2 = 11H^2 \kappa^2 \). We would get that we can find an
estimate $\nabla F_{\text{accu}}^{t_k,t_{k+1}}$ of $\nabla F_{S,\text{accu}}^{t_k,t_{k+1}}$ in polynomial-time, such that $\| \nabla F_{\text{accu}}^{t_k,t_{k+1}} - \nabla F_{S,\text{accu}}^{t_k,t_{k+1}} \| \leq O(H\kappa\sqrt{r})$ holds with probability 1.

Now we proceed towards proving Theorem 2.

Proof of Theorem 2. This can be proved along the lines of the proof of Theorem 1. Here we only write what changes in those proofs.

D.1 Strongly-convex

Let $K_t \subseteq [R]$ denote the subset of clients of size $|K_t| = K$ sampled at the $t$'th iteration. For any $t \in [t_i : t_{i+1} - 1]$, let $x^t = \frac{1}{K} \sum_{k \in K_t} x_k^t$ denote the average of the local parameters of clients in the sampling set $K_t$.

Following the proof of the strongly-convex part of Theorem 1 given in Appendix B until (32) gives

$$
\left\| x^{t_i+1} - x^* \right\|^2 \leq \left( 1 + \frac{\mu\eta}{2} \right) \left\| x^{t_i+1-1} - \eta \nabla F(x^{t_i+1-1}) - x^* \right\|^2 
+ 2\eta \left( \eta + \frac{2}{\mu} \right) \left( \frac{1}{K} \sum_{r \in K_t} (\nabla F(x^{t_i+1-1}) - \nabla F_r(x^{t_i+1-1})) \right)^2 
+ 2\eta \left( \eta + \frac{2}{\mu} \right) \left( \frac{1}{K} \sum_{r \in K_t} \sum_{t=t_i}^{t_{i+1}-1} \nabla F_r(x^t_r) \right)^2
$$

(69)

We have already bounded the first term in Claim 3 (on page 18) by

$$
\left\| x^{t_i+1} - \eta \nabla F(x^{t_i+1-1}) - x^* \right\|^2 \leq (1 - \eta\mu) \left\| x^{t_i+1-1} - x^* \right\|^2.
$$

(70)

In order to bound the second term, we follow the proof of Claim 4 (on page 18) exactly until (37), and then to bound $\left\| x^{t_i+1-1} - x^{t_i+1-1} \right\|^2$ for every $r, s \in K_t$, we use the bound from (66) in Lemma 3 and $\eta \leq \frac{1}{\sqrt{\mu H}}$, which would give

$$
\left\| \frac{1}{R} \sum_{r=1}^R (\nabla F_r(x^{t_i+1-1}) - \nabla F_r(x^{t_i+1-1})) \right\|^2 \leq 3H\kappa^2.
$$

(71)

To bound the third term, we can simplify the proof of Claim 5: Firstly, note that, with full-batch gradients, the variance $\sigma^2$ becomes zero; secondly, as shown in Theorem 5, the robust estimation of accumulated gradients holds with probability 1. Following the proof of Claim 5 with these changes and using $\eta \leq \frac{1}{\sqrt{\mu H}}$, we get

$$
\left\| \frac{1}{K} \sum_{r \in K_t} \sum_{t=t_i}^{t_{i+1}-1} \nabla F_r(x^t_r) \right\|^2 \leq 2\mathcal{T}_{\text{GD}}^2 + 20H^2\kappa^2,
$$

(72)

where $\mathcal{T}_{\text{GD}} = O(H\kappa\sqrt{r})$. Substituting all these bounds from (70)-(72) in (69) and simplifying further using $(1 + \frac{\mu\eta}{2}) (1 - \mu\eta) \leq (1 - \frac{\mu\eta}{2})$ and $(\eta + \frac{2}{\mu}) \leq \frac{2}{\mu}$ gives

$$
\left\| x^{t_i+1} - x^* \right\|^2 \leq \left( 1 - \frac{\mu\eta}{2} \right) \left\| x^{t_i+1-1} - x^* \right\|^2 + \frac{6\eta}{\mu} \left( 3H\kappa^2 + 2\mathcal{T}_{\text{GD}}^2 + 20H^2\kappa^2 \right)
$$

(73)

Note that (73) gives a recurrence at the synchronization indices. Similar to how we showed in the proof of the strongly-convex part of Theorem 1 in Appendix B, we can show that in fact (73) holds for all iterations $t \in [T]$. Now, solving the recurrence (73) gives

$$
\left\| x^T - x^* \right\|^2 \leq \left( 1 - \frac{\mu\eta}{2} \right)^T \left\| x^0 - x^* \right\|^2 + \frac{12}{\mu^2} \left( 2\mathcal{T}_{\text{GD}}^2 + 23H^2\kappa^2 \right).
$$

(74)
Substituting the value of $\eta = \frac{1}{5HL}$ yields the convergence rate (9) in the strongly-convex part of Theorem 2. Note that (74) holds with probability 1.

D.2 Non-convex

Following the proof of the non-convex part of Theorem 1 given in Appendix C until (59) with the following changes: In order to bound the last two terms of (54), use the bounds from (71) and (72), and use the step-size $\eta \leq \frac{1}{5HL}$. This gives the following relation at every iteration $t \in [T]$:

$$F(x^{t+1}) \leq F(x^t) - \frac{3\eta}{10} \| \nabla F(x^t) \|^2 + \frac{7\eta}{5} \left( 2\Upsilon_{GD}^2 + 23H^2\kappa^2 \right),$$

where $\Upsilon_{GD} = O(H\kappa\sqrt{\epsilon})$.

Then following the proof from (59) until (62) exactly (except for that here we use $\eta = \frac{1}{5HL}$). This would give

$$\frac{1}{T} \sum_{t=0}^{T} \| \nabla F(x^t) \|^2 \leq \frac{25HL^2}{3T} \| x^0 - x^* \|^2 + \frac{14}{3} \left( 2\Upsilon_{GD}^2 + 23H^2\kappa^2 \right),$$

This yields the convergence rate (10) in the non-convex part of Theorem 2. Note that (76) holds with probability 1.

This concludes the proof of Theorem 2.

E Proof of the second part of Theorem 3

In this section, we describe the procedure for robust mean estimation in high dimensions from [SCV18] that we use in the second part of Theorem 3 to filter-out corrupt vectors and compute an estimate of the average of uncorrupted accumulated gradients. We refer the reader to [DD20, Section 4] to get the intuition on why filtering-out corrupt gradients (even when $H = 1$, i.e., without local iterations) is difficult in high dimensions.

We describe the procedure in Algorithm 2 and refer the reader to [DD20, Appendix E] to get an intuition behind Algorithm 2 and its running-time analysis. For simplicity, we reorder the received gradient indices from $1, 2, \ldots, K$. Now, the proof of the second part of Theorem 3 follows from [SCV18, Proposition 16], which we state below for completeness.

**Lemma 4** (Proposition 16 in [SCV18]). Suppose we are given $K$ arbitrary vectors $g_1, \ldots, g_K \in \mathbb{R}^d$ with the promise that there exists a subset $S$ of these $K$ vectors such that $|S| = (1 - \tilde{\epsilon})K$ for some $\tilde{\epsilon} > 0$ and $S$ satisfies $\lambda_{\max} \left( \frac{1}{|S|} \sum_{i \in S} (g_i - g_S)(g_i - g_S)^T \right) \leq \sigma_0^2$, where $g_S = \frac{1}{|S|} \sum_{i \in S} g_i$ denotes the sample mean of the vectors in $S$. Then, if $\tilde{\epsilon} \leq \frac{1}{4}$, Algorithm 2 can find an estimate $\hat{g}$ of $g_S$ in polynomial-time, such that $\| \hat{g} - g_S \| \leq O(\sigma_0\sqrt{\tilde{\epsilon}})$.

We refer the reader to [DD20, Appendix F] for a comprehensive proof of Lemma 4.

Note that Lemma 4 takes arbitrary vectors as inputs, which are not required to have been generated from a probability distribution. This completes the proof of the second part of Theorem 3.
Algorithm 2 Robust Gradient Estimation (RGE) [SCV18]

1: Initialize. \( c_i := 1, i \in [K], \alpha := (1 - \tilde{\epsilon}) \geq 3/4, A := \{1, 2, \ldots, K\}; G := [g_1, g_2, \ldots, g_K] \in \mathbb{R}^{d \times K}. \)
2: while true do
3:   Let \( W^* \in \mathbb{R}^{|A| \times |A|} \) and \( Y^* \in \mathbb{R}^{d \times d} \) be the minimizer/maximizer of the saddle point problem:
\[
\max_{Y \succeq 0, \text{tr}(Y) \leq 1} \min_{0 \leq W_{ji} \leq \frac{4 - \alpha}{\sigma_0}, \sum_{j \in A} W_{ji} = 1, \forall i \in A} \Phi(W, Y),
\]  
where the cost function \( \Phi(W, Y) \) is defined as
\[
\Phi(W, Y) := \sum_{i \in A} c_i (g_i - G_A w_i)^T Y (g_i - G_A w_i),
\]
To avoid cluttered notation, we index the \(|A|\) rows/columns of \( W \) by the elements of \( A \); \( G_A \) denotes the restriction of \( G \) to the columns in \( A \); for \( i \in A \), \( w_i \) denotes the column of \( W \) indexed by \( i \).
4: For \( i \in A \), let
\[
\tau_i = (g_i - G_A w_i^* )^T Y^* (g_i - G_A w_i^* )
\]
5: if \( \sum_{i \in A} c_i \tau_i > 4 R \sigma_0^2 \) then
6:   For \( i \in A \), \( c_i \leftarrow \left( 1 - \frac{\tau_i}{\tau_{\max}} \right) c_i \), where \( \tau_{\max} = \max_{j \in A} \tau_j \).
7:   For all \( i \) with \( c_i < \frac{1}{2} \), remove \( i \) from \( A \).
8: else
9:   Break while-loop
10: end if
11: end while
12: return \( \hat{g} = \frac{1}{|A|} \sum_{i \in A} g_i \).

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