Critical velocity for a toroidal Bose–Einstein condensate flowing through a barrier

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Abstract
We consider the setup employed in a recent experiment (Ramanathan et al 2011 Phys. Rev. Lett. \textbf{106} 130401) devoted to the study of the instability of the superfluid flow of a toroidal Bose–Einstein condensate in the presence of a repulsive optical barrier. Using the Gross–Pitaevskii mean-field equation, we observe, consistently with what we found in Piazza et al (2009 Phys. Rev. A \textbf{80} 021601), that the superflow with one unit of angular momentum becomes unstable at a critical strength of the barrier and decays through the mechanism of phase slippage performed by pairs of vortex–antivortex lines annihilating. While this picture qualitatively agrees with the experimental findings, the measured critical barrier height is not very well reproduced by the Gross–Pitaevskii equation, indicating that thermal fluctuations can play an important role (Mathey et al 2012 arXiv:1207.0501). As an alternative explanation of the discrepancy, we consider the effect of the finite resolution of the imaging system. At the critical point, the superfluid velocity in the vicinity of the obstacle is always of the order of the sound speed in that region, $v_{\text{bar}} = c_l$. In particular, in the hydrodynamic regime (not reached in the above experiment), the critical point is determined by applying the Landau criterion inside the barrier region. On the other hand, the Feynman critical velocity $v_f$ is much lower than the observed critical velocity. We argue that this is a general feature of the Gross–Pitaevskii equation, where we have $v_f = \epsilon c_l$ with $\epsilon$ being a small parameter of the model. Given these observations, the question still remains open about the nature of the superfluid instability.

(Some figures may appear in colour only in the online journal)

1. Introduction
Recent experiments performed at NIST [1, 2] and at the Cavendish Laboratory [3] have demonstrated the existence of persistent currents in a toroidally trapped Bose–Einstein condensate (BEC). Such a setup, apart from showing this hallmark manifestation of superfluidity [4], has also allowed, with the addition of a repulsive optical barrier, the construction of a closed-loop atom circuit with a weak link [2]. This circuit constitutes the basic building block for the realization of an ultracold atomic analogue of superconducting/superfluid quantum interference devices such as the sensitive magnetic sensors already realized with superconductors [5] or rotational sensors with superfluid helium [6]. A proof-of-principle BEC rotation sensor has been very recently realized at NIST [7], where the rotation of the reference frame is mimicked by sweeping the barrier around the torus. Aside from the exciting technological applications, the toroidal setup proves to be the most suitable for the investigation of the critical velocity of a superfluid moving across an obstacle, since, compared to other setups [8–10], it has no inhomogeneities along the direction of flow and does not require moving the obstacle given that the condensate can move in the lab frame. The study of superfluid critical velocity and its decay mechanism is a long-standing problem in condensed matter physics.
theoretical problem from the early experiments with superfluid helium [11] and has yet to be fully understood.

In this paper, we model the NIST ultracold atom circuit experiment [2] using the Gross–Pitaevskii (GP) equation. As already predicted in [12], we find that the superflow with one unit of angular momentum becomes unstable at a critical strength of the barrier and decays through the mechanism of phase slippage through pairs of vortex–antivortex lines parallel to the torus axis. While this picture qualitatively agrees with the experimental findings, the measured critical barrier height is not well reproduced by the GP equation. This may indicate that thermal fluctuations play an important role by triggering the vortex nucleation before the zero temperature critical velocity predicted by the GP equation is reached. Using a weak link inside a linear waveguide with periodic boundary conditions, the superfluid decay through thermally nucleated vortices is studied in [13] by means of the truncated Wigner method.

As an alternative explanation of the discrepancy between the GP equation and the experimental data, we consider the effect of the finite resolution of the imaging system. The comparison indeed depends strongly on the precision with which local density and velocity of the fluid inside the barrier region are experimentally determined and is therefore particularly affected by the finite imaging resolution. Taking into account the fact that the experimental data already contain a correction due to finite imaging resolution, we estimate the minimum size of a possible further resolution error which could produce agreement.

The ability to describe this setup using the GP equation can prove to be very useful since, as motivated above, the closed-loop superfluid circuit is of deep technological and foundational interest. Firstly, the GP equation can be employed as an efficient tool to model quantum interference devices, exploiting the versatility of ultracold atomic gases such as the fine tunability of the trapping potential, barrier potential and the atom–atom interactions. For instance, as demonstrated in [7], the ability to tune the barrier height is a means to dynamically change the critical velocity, which constitutes an advantage of the atom circuit over the superconducting/superfluid counterparts. Secondly, the GP equation allows for a simple theoretical understanding of open questions regarding superfluid instability, which will be the subject of this paper.

Indeed, after discussing the comparison with the experiment, we then focus on the existence of a general instability criterion determining the critical velocity and on the mechanism underlying the superfluid decay. At the critical point, we observe that the superfluid velocity in the vicinity of the obstacle is always of the order of the local sound speed, \( v_{\text{barr}} = c_1 \), which is a typical feature of the GP equation, at least close to the hydrodynamic regime [14–17]. The local sound speed \( c_1 \) is the propagation speed of phonons inside the barrier region, assuming a local density approximation along the flow direction.

On the other hand, we find that the Feynman critical velocity \( v_{\text{f}} \) [18, 11] is instead much lower than the true critical velocity. The Feynman critical velocity is calculated as the velocity at which the energy of the state with a vortex (in the present case a vortex ring) becomes smaller than the energy of the stationary state without vortices. Most importantly, we show that in general \( v_{\text{f}} = \epsilon c_1 \), where \( \epsilon \) is a small parameter of the model, that is, the Feynman prediction can never work in cases where the critical velocity is of the order of the sound speed.

In particular, in the hydrodynamic regime (not reached in the above experiment) with the fluid locally homogeneous along the flow, we argue that in general the instability criterion corresponds to that of Landau applied in the region where the fluid velocity is the highest, which in the present setup occurs inside the barrier region. The corresponding excitations can thus be the seeds from which vortices grow [19]. At the same time, we have indications that the instability might be of a dynamical nature, since (i) we do not have any defect inside the barrier region and (ii) the timescale of vortex nucleation is fast. We note that the nature of the phase slip instability, energetic or dynamical, as well as its relation with Feynman’s criterion is not well understood. We conclude by summarizing the current level of understanding and emphasizing the open problems.

2. Model

We model the experiment performed at NIST [2] using the GP equation in three spatial dimensions. In the dimensionless form, using radial harmonic trap units of length \( d_r = 2 \) \( \mu \)m, time \( \omega_\perp^{-1} = 1.45 \) ms and energy \( \hbar \omega_\parallel = h \times 110 \) Hz, the GP equation reads

\[
\frac{i}{\hbar} \frac{\partial \psi(r, t)}{\partial t} = \left[ -\frac{1}{2} \nabla^2 + V(r, t) + g|\psi|^2 \right] \psi(r, t),
\]

where the external potential \( V(r, t) = V_0(r) + V_{\text{barr}}(r, t) \), with \( g = 4\pi a_0 \) (\( a_0 \) is the s-wave scattering length), and \( V_0(x, y, z) = (\omega_x^2/2)z^2 + (1/2)(\sqrt{x^2 + y^2} - R_0)^2 \), where the torus radius \( R_0 = 10 \) and the vertical trap frequency \( \omega_\parallel = 5 \). The results presented in the following are obtained by starting with an initial superfluid flow with angular momentum per atom \( \ell = 1 \) and \( V_{\text{barr}}(r, 0) = 0 \), and then raising the \( x-y \) Gaussian barrier \( V_{\text{barr}}(x, y, t) \) up to a height \( V_0 \). The latter is centred at \( (x = R_0, y = 0) \) with \( 1/e^2 \) radius \( \sigma_x = 7.55 \) and \( \sigma_y = 2.15 \).

The initial state at \( t = 0 \) is obtained by finding the ground state of the GP equation by imaginary time propagation with \( \ell = 0 \) and without a barrier, and then imprinting one quantum of circulation on the wavefunction by multiplying \( \exp(i\phi) \) with \( \phi \) being the azimuthal coordinate. Then, the barrier is linearly ramped up reaching \( V_0 \) at \( t = 40 \) (\( ~60 \) ms). The system is then typically evolved up to \( t = 60 \).

We numerically integrate the 3D GP equation using a finite-difference real-space product formula approach described in [20].

3. Critical barrier height

Upon repeating the real-time propagation of the GP equation with increasing \( V_0 \), we reach the point where, immediately after the ramping stops, the angular momentum \( \ell \) abruptly

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drops from a value close to 1 and then stabilizes around 0. The abrupt drop in \( \ell \) coincides with a vortex–antivortex pair crossing the annulus and takes place over a very short time (about 0.5 ms). The lowest \( V_5 \) sufficient to observe the drop in \( \ell \) defines the critical point.

In figure 1, we show the comparison between the GP prediction for the critical point determining the superfluid instability and the experimental measurement performed at NIST. In order to allow for a comparison, we have calculated the chemical potential, both global and local (inside the barrier region) with the same prescription\(^4\) described in \([2]\).

Given a total atom number \( N \), we calculated the global Thomas–Fermi chemical potential \( \mu_{\text{TF}} \) and subsequently obtained the chemical potential decrease \( \beta = \mu_{\text{TF}} - \mu_1 \), where \( \mu_1 \) is the local chemical potential inside the barrier region. Instead of taking the actual local chemical potential by using \( n(x, y, z) \) directly from our numerical results, we inferred it following the NIST protocol, that is, taking the \( z \)-integrated column density \( v(x, y) \) at the point \((R_0, 0)\), and assuming either a Gaussian or Thomas–Fermi profile along \( z \), which provides a relation between \( v(R_0, 0) \) and \( \mu_1 \). This procedure produces the blue solid line shown in figure 1, which does not very well reproduce the experimental results.

This discrepancy might indicate that the thermal fluctuations induced by the finite temperature can play an important role in the experiment. These fluctuations trigger the vortex nucleation before the critical velocity predicted from the GP equation is reached. In \([13]\), the same experimental data were compared with the theoretical prediction of both the GP equation and the truncated Wigner approximation. The model setup was a waveguide with a periodic boundary condition, and the barrier was taken to be constant along the direction transverse to the flow. The inclusion of thermal fluctuations through the Wigner approach provides a systematic effect which shifts the theoretical predictions closer to the experimental data with respect to the GP results. With an appropriate temperature compatible with the experiment, the Wigner approach was seen to agree slightly better with the experimental data than the GP equation.

An alternative explanation for the discrepancy between theory and experiment, which remains within the framework of GP at zero temperature, can be found in the finite resolution of the imaging system. Indeed, if instead of the column density \( v(R_0, 0) \) we use its average \( \bar{v} \) over a square \( x-y \) region \( A \) centred around \((R_0, 0)\) (see the purple dashed and yellow dotted lines), we obtain a better agreement with the experimental data. This averaging process models the effect of a finite imaging resolution. We emphasize that the value of \( \beta \) strongly depends on the size of \( A \), that is, it is strongly affected by any limitation to the resolution of the imaging system. In figure 1, a square area whose side length ranges from 2.5 to 5 \( \mu \)m provides good agreement. It must anyway be noted that the experimental data reported in figure 1 already contain a 15% correction to the local density due to some limitations to the imaging resolution. Thus, the finite area correction to our theoretical data should account for possible further limiting factors, not contained in this 15%.

4. The instability criterion

We now turn to the discussion of the instability criterion determining the critical point within the GP description. Putting together previous findings involving several different setups and dimensionalities, we begin by proposing a general criterion, which should be valid in the hydrodynamic regime of flow. In this case, the system can be considered locally homogeneous along the flow direction, that is, the Thomas–Fermi or local density approximation can be employed. We argue that the critical point can be determined using the following criterion: given the excitation spectra calculated in the system without anisotropy along the flow direction, apply the Landau criterion to the anisotropic case locally inside the region where the superfluid velocity is the highest. In the present case, one should take the spectrum in a torus without a barrier \([21]\) and calculate the critical velocity, which corresponds to surface-mode excitation. Upon treating the barrier in the Thomas–Fermi approximation, one can obtain the spectrum inside the barrier and then apply the Landau criterion. This provides a critical velocity to be compared with the highest fluid velocity, which in this case (see figure 2) is reached at the surface of the cloud.

For instance, it has been verified by simulating the time-dependent GP equation, both in toroidal and waveguide geometries \([17]\), that the critical point for superfluid instability in the Thomas–Fermi regime is determined by the following condition: the fluid velocity at the Thomas–Fermi surface is approximately equal to the effective sound speed \( c_l \), that is, the Landau critical velocity for phonon excitation calculated inside the barrier region. Actually, as mentioned above, due to
Green solid line: sound velocity inside the barrier region \(c_1\), corresponding to the above radial line. The effective sound speed is lower than the sound velocity for surface modes (excitations with larger momentum for phonons propagating azimuthally in a toroidal geometry, only the decrease like \(1/\ell\) remains due to the quantization of circulation.

5. The failure of the Feynman criterion

The red solid line in figure 2 corresponds instead to the Feynman prediction for the critical velocity \(v_f = \hbar/(mD) \ln(D/a)\), where \(D\) is the annulus width and \(a\) is the vortex core size. For the latter, we took the healing length \(a = \xi_s = (1/\sqrt{\lambda})\hbar/mc_0\) corresponding to the effective sound speed \(c_1\) inside the barrier region. While the width of the annulus is \(D \approx 8d_s\) outside the barrier region, it is reduced inside \(D \approx 5d_s\) (we took the Thomas–Fermi radial width of the torus inside the barrier region). We see that the Feynman prediction \(v_f\) is significantly lower than the fluid velocity inside the barrier region and is even lower than the fluid velocity outside the barrier region \(v_{bulk} \approx 0.7d_s\omega_s\). In general, independent of the geometry, and at least for angular momenta per atom \(\ell < 10\), we always found with GP that the Feynman prediction is typically much lower than the fluid velocity in the barrier region. The generality of this fact becomes apparent if we rewrite the Feynman velocity as

\[
v_f = c_1 \sqrt{2} \frac{\xi_s}{D} \ln \left( \frac{D}{\xi_s} \right) = \epsilon c_1.
\]

As stated before, the superfluid instability, within the GP equation, corresponds typically to a situation where the local fluid velocity inside the barrier region \(v_{barr}\) is of the order of the effective sound speed \(c_1\). Now, since the expression for the Feynman velocity given above is only valid when the channel width (in our case the annulus width) is much larger than the vortex core size, i.e. \(\ell \approx g/(\xi_s)\), we have \(v_f = \epsilon c_1\), where \(\epsilon = \sqrt{2} \ln(D)/D \ll 1\) is a small parameter.
Equation (2) indicates that the Feynman prediction is not simply a very rough estimate of the true GP critical velocity, but is instead intrinsically different, since it cannot be applied to instabilities whose critical point corresponds to a superfluid velocity being of the order of the sound speed.

On the other hand, the measurements performed at NIST [2] seemed to suggest that the Feynman prediction may apply. As discussed above, this is not compatible with the GP equation, and, if we still want to interpret the experiment with the latter, we have to assume that, due to the limitations to the imaging resolution, the density inside the barrier is overestimated from the experimental procedure (see the discussion above regarding the comparison of the critical barrier height). This in turn implies an overestimation of the local chemical potential, which in turn overestimates $v_f$. One can understand this by using equation (2) and assuming the Thomas–Fermi description to hold, even though this is not a very good approximation as discussed above. A larger chemical potential implies a smaller healing length and, if we approximately neglect the change in the effective annulus width, an increased $v_f$.\(^6\) Secondly, the local fluid velocity inside the barrier is underestimated by solving the equivalent 1D flow problem obtained by integrating radially the measured column density, as done in [2].

6 Since both the healing length and the effective annulus width are modified by a change in the chemical potential, it is not true in general that an increase in the chemical potential makes the Feynman velocity increase, but rather holds only for sufficiently small chemical potentials, as is the case here.

### 6. Phase slip dynamics

In figure 3, we show an example of the superfluid decay dynamics resulting from the numerical solution of the GP equation in real time. The positions of the vortex cores, determined using a plaquette method [17], are depicted as black dots, while the grey surface corresponds to the Thomas–Fermi surface of the cloud. Two different views, referring to the same instant after the barrier has reached the critical height, show two straight vortex lines with opposite winding numbers moving towards the centre of the annulus. The two lines eventually meet close to the outer edge of the annulus and annihilate, thereby producing a full phase slip that brings the angular momentum to zero. One line comes from the low density region at the torus centre, while the other moves inwards from the system boundary. As can be observed also from the drop in the angular momentum, the phase slip event is very fast, taking about 0.5 ms for the lines to cross the annulus and annihilate. The inner line enters the bulk first, since the instability is reached first at the inner edge of the annulus due to the velocity asymmetry. The latter is caused by the radial decrease of the velocity due to the quantization of circulation. In [12], we have shown that this asymmetry can give rise to two separate critical barrier heights, so that below the highest of the two only a vortex enters the annulus from the inner edge. However, in the present case, due to the low angular momentum, the difference between the two critical heights is so small that, in order to resolve these two critical points, we would need to fine tune our barrier height. However, even if we do not resolve the two barrier heights, we see that the inner line enters much before the line coming from the outer edge, so that they meet close to the outer edge of the annulus. If we would further decrease the barrier height, we can imagine the point at which the two vortices annihilate to move further and further towards the outer region so that at some point, the annihilation would take place far enough outside the cloud such that the outer vortex would not play any role. This asymmetry characteristic of a torus has been observed to influence the Landau criterion for a BEC in a torus without a barrier [21], where the surface modes become unstable first at the inner edge.

Since the decay dynamics involves straight vortex lines, it is reasonable to ask ourselves what prediction we would obtain if we modify the above Feynman velocity, related to vortex rings, to the case of nucleation of vortex lines. As discussed in [23], the Feynman critical velocity for the nucleation of a vortex line crossing the channel is smaller by a prefactor $2/\pi$ than the critical velocity for ring nucleation, which brings the Feynman prediction even further away from the GP results.

### 7. Conclusions

Our numerical simulations have shown a discrepancy between the critical barrier measured in [2] and the theoretical prediction of the zero temperature GP equation. This might be due to the finite temperature which affects the phase slippage dynamics by allowing for the thermal nucleation of
vortices. As an alternative explanation of the discrepancy, we considered the finite resolution of the imaging system, since the latter strongly influences the comparison between theory and experiment.

We have then turned our attention to the instability criterion determining the critical velocity. Considering the results of our analysis and the one made in the previous literature, some questions arise: (1) why is the Feynman criterion incompatible with the GP, and what is its relation, if any exists, with the local criterion for the instability of surface modes? (2) What is the nature of the GP instability? Is it an energetic instability, a dynamical one or something else? In the conclusion of this work, we will address these questions and argue that while well posed, no clear answers have yet emerged despite their fundamental relevance for the understanding of superfluidity.

(1) The physical reason underlying the failure of the Feynman prediction, observed here with the GP equation and experimentally in [3], is not understood. It can be argued that the Feynman critical velocity is actually only an order of magnitude estimate and its failure does not mean much. However, we showed that the incompatibility is more fundamental (see equation (2)) and that the Feynman criterion cannot provide even the correct order of magnitude. This is true every time the critical velocity is of the order of the sound speed, like it is for the GP equation. On the other hand, one might suppose that the Feynman criterion could work well when additional defects are present inside the barrier region. In general, a useful insight might be provided by relating the Feynman criterion, based on vortices, to the instability of surface modes.

(2) The first observation is that we have a quantitatively viable criterion for predicting the critical velocity, as we discussed in section 4. This criterion has also been previously theoretically verified to a reasonable degree for the case of a BEC flowing past an obstacle in [14–17], and for a rotating elliptical trap [25], and also for a fermionic superfluid using the Bogoliubov–de Gennes equations [26]. Remarkably, it has even been experimentally verified at the Cavendish Laboratory [3] where the instability is attributed to the roughness of the trap creating a region where the flow is constricted. As described above, in general, this criterion is based on the excitation spectra calculated in the system without anisotropy along the flow direction, for instance in a transversally inhomogeneous waveguide [27, 28, 19, 22], a rotationally symmetric trap [29–32], or a torus without barrier [21]. Given the spectrum, one applies the Landau criterion to the anisotropic case locally inside the region where the superfluid velocity is the highest, assuming the local density approximation to hold along the flow direction. Therefore, since the fluid is locally homogeneous, it has been suggested in the literature that the instability is energetic.

However, the second observation is that time-dependent GP simulations give indications that the instability might be dynamical, since (i) no defects exist inside the barrier region or in general the ‘high velocity region’ which would be required in order to trigger the energetic instability [33] and (ii) the nucleation dynamics is generally fast, which means that excitations grow very quickly as for typical dynamical instabilities, whose onset can indeed be observed in the absence of defects.

We note that this kind of apparent ambiguity appears also in the studies of the critical angular velocity for a BEC in a rotating harmonic trap. In particular, using a linear stability analysis in the presence of anisotropy, the surface modes have been shown to become dynamically unstable [34], and the critical angular frequency to agree with the real-time propagation of the GP equation [35]. At the same time, however, the real-time GP results in the anisotropic trap have been observed [25] to be also well predicted by the ‘local Landau criterion’ described above. To our knowledge, a conclusive statement about the relation between the local Landau criterion and dynamical instability is still missing in this context.

A particular setup which could help understanding this issue is the one-dimensional condensate flow crossing the speed of sound twice, thereby forming two sonic horizons [36]. In this configuration, a linear stability analysis predicts the existence of a global dynamical instability when the size of the supersonic region becomes larger than a critical value. This hints to a possible link with the Landau criterion applied inside the supersonic region. The correspondence between the local Landau criterion and the global dynamical instability seems crucial to understand the mechanism underlying vortex nucleation and deserves anyway further study.

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7 For a homogeneous condensate flowing past a very large and impenetrable obstacle, a linear stability analysis provided a critical velocity in agreement with the Feynman criterion [24]. However, time-dependent simulations in this setup [14, 15] have verified that the instability corresponds instead to the local fluid velocity reaching the sound speed. This might be explained by the fact that the linear stability analysis of [24] signals an energetic instability which is however not triggered in the time-dependent simulations of [14, 15], due to the absence of defects.

8 The finite resolution in the computation generates numerical noise which however should not be sufficient to trigger an energetic instability within our computational times.

9 Note that, within the local Landau criterion, the macroscopic obstacle ‘disappears’ as its effect is included as a modification of the parameters of the locally homogeneous system. Therefore, we need additional defects inside the obstacle region in order to trigger the energetic instability in a non-perturbed homogeneous system.
References

[1] Ryu C et al 2007 Phys. Rev. Lett. 99 260401
[2] Ramanathan A et al 2011 Phys. Rev. Lett. 106 130401
[3] Moulder S et al 2012 Phys. Rev. A 86 013629
[4] Leggett A J 1999 Rev. Mod. Phys. 71 318
[5] Clarke J and Braginski A 2004 The SQUID Handbook vols 1 and 2 (Weinheim: Wiley)
[6] Sato Y and Packard R E 2010 Rep. Prog. Phys. 75 016401
[7] Wright K C et al 2012 arXiv:1208.3608v1
[8] Raman C et al 1999 Phys. Rev. Lett. 83 2502
[9] Onofrio R et al 2000 Phys. Rev. Lett. 85 2228
[10] Engels P and Atherton C 2007 Phys. Rev. Lett. 99 160405
[11] Varoquaux E 2006 C. R. Phys. 7 1101
[12] Piazza F, Collins L A and Smerzi A 2009 Phys. Rev. A 80 021601
[13] Mathey A C, Clark C W and Mathey L 2012 arXiv:1207.0501
[14] Frisch T, Pomeau Y and Rica S 1992 Phys. Rev. Lett. 69 1644
[15] Watanabe G et al 2009 Phys. Rev. A 80 053602
[16] Piazza F, Collins L A and Smerzi A 2011 New J. Phys. 13 043008
[17] Feynman R 1955 Prog. Low Temp. Phys. 1 17
[18] Anglin J R 2001 Phys. Rev. Lett. 87 240401
[19] Schneider B I, Collins L A and Hu S X 2006 Phys. Rev. E 73 036708
[20] Dubessy R et al 2012 Phys. Rev. A 86 011602
[21] Fedichev P O and Shlyapnikov G V 2001 Phys. Rev. A 63 045601
[22] Anderson P W 1966 Rev. Mod. Phys. 38 298
[23] Stiessberger J S and Zwerger W 2000 Phys. Rev. A 62 061601
[24] Feder D L, Clark C W and Schneider B I 1999 Phys. Rev. A 61 011601
[25] Spuntarelli A, Pieri P and Strinati G C 2010 Phys. Rep. 488 111
[26] Zarembo E 1998 Phys. Rev. A 57 518
[27] Stringari S 1998 Phys. Rev. A 58 2385
[28] Dalfovo F et al 1997 Phys. Rev. A 56 3840
[29] Lundh E et al 1997 Phys. Rev. A 55 2126
[30] Isoshima T and Machida K 1999 Phys. Rev. A 60 3313
[31] Feder D L et al 2001 Phys. Rev. Lett. 86 564
[32] See Leggett A 2001 Rev. Mod. Phys. 73 307
[33] Carusotto I et al 2006 Phys. Rev. Lett. 97 260403
[34] Sinha S and Castin Y 2001 Phys. Rev. Lett. 87 190402
[35] Parker N G, van Bijnen R M W and Martin A M 2006 Phys. Rev. A 73 061603
[36] Finazzi S and Parentani R 2010 New J. Phys. 12 095015