Generalized Taylor’s Theorem

Garret Sobczyk,
Departamento de Actuaría, Física y Matemáticas
Universidad de Las Américas - Puebla,
72820 Cholula, Mexico

February 2, 2008

Abstract
The Euclidean algorithm makes possible a simple but powerful generalization of Taylor’s theorem. Instead of expanding a function in a series around a single point, one spreads out the spectrum to include any number of points with given multiplicities. Taken together with a simple expression for the remainder, this theorem becomes a powerful tool for approximation and interpolation in numerical analysis. We also have a corresponding theorem for rational approximation.

Generalize Rolle’s Theorem

Let \( h(x) = \prod_{i=1}^{r} (x - x_i)^{m_i} \) for distinct \( x_i \in [a, b] \subset \mathbb{R} \) with multiplicity \( m_i \geq 1 \), and let \( n = \deg(h(x)) \). Given two functions \( f(x) \) and \( g(x) \), we say that \( f(x) = g(x) \mod(h(x)) \) or \( f(x) \equiv g(x) \) if for each \( 1 \leq i \leq r \) and \( 0 \leq k < m_i \)

\[
f^{(k)}(x_i) = g^{(k)}(x_i). \tag{1}\]

If \( f(x) \) and \( g(x) \) are polynomials, then \( f(x) \equiv g(x) \) is equivalent to saying that if \( f(x) \) and \( g(x) \) are divided by the polynomial \( h(x) \) (the Euclidean algorithm), they give the same remainder. We denote the factor ring of polynomials modulo \( h(x) \) over the real numbers \( \mathbb{R} \) by

\[
\mathbb{R}[x]/< h(x) >, \]

see [1, p.266].

Generalized Rolle’s Theorem: Let \( f(x) \in C[a, b] \) and \((n - 1)\)-times differentiable on \((a, b)\). If \( f(x) = 0 \mod(h(x)) \), then there exist a \( c \in (a, b) \) such that \( f^{(n-1)}(c) = 0 \).

Proof: Following [2, p.38], define the function \( \sigma(u, v) := \begin{cases} 1, & u < v \\ 0, & u \geq v \end{cases} \). The function \( \sigma \) is needed to count the simple zeros of the polynomial \( h(x) \) and its derivatives.

Let \( \#h^{(k)} \) denote the number of simple zeros that the polynomial equation \( h^{(k)}(x) = 0 \) has. Clearly \( \#h = r = \sum_{i=1}^{r} \sigma(0, m_i) \). By the Classical Rolle’s theorem

\[
\#h' = \sum_{i=1}^{r} \sigma(1, m_i) + (\#h) - 1 = \sum_{i=1}^{r} \sigma(1, m_i) + \sum_{i=1}^{r} \sigma(0, m_i) - 1.
\]

Continuing this process, we find that

\[
\#h'' = \sum_{i=1}^{r} \sigma(2, m_i) + (\#h') - 1 = \sum_{i=1}^{r} \sigma(2, m_i) + \sum_{i=1}^{r} \sigma(1, m_i) + \sum_{i=1}^{r} \sigma(0, m_i) - 2,
\]

and more generally that

\[
\#h^{(k)} = \sum_{i=1}^{r} \sigma(k, m_i) + (\#h^{(k-1)}) - 1 = \sum_{i=1}^{r} \sum_{j=0}^{k} \sigma(j, m_i) - k
\]
for all integers $k \geq 0$.  
For $k = n - 1$, we have 
\[
\# h^{(n-1)} = \sum_{i=1}^{r} \sum_{j=0}^{n-1} \sigma(j, m_i) - (n - 1) = \sum_{i=1}^{r} m_i - (n - 1) = 1.
\]
The proof is completed by noting that $\# f^{(k)} \geq \# h^{(k)}$ for each $k \geq 0$, and hence $\# f^{(n-1)} \geq 1$.

\[\square\]

**Approximation Theorems:**

We can now prove

**Generalized Taylor’s Theorem:** Let $f(x) \in C[a, b]$ and $n$ times differentiable on $(a, b)$. Suppose that $f(x) = g(x) \mod(h(x))$ for some polynomial $g(x)$ where $\deg(g) < \deg(h)$. Then for every $x \in [a, b]$ there exist a $c \in (a, b)$ such that 
\[
f(x) = g(x) + \frac{f^{(n)}(c)}{n!} h(x).
\]

**Proof:** For a given $x \in [a, b]$, define the function 
\[
p(t) = f(t) - g(t) - [f(x) - g(x)] \frac{h(t)}{h(x)}.
\]
In the case that $x = x_i$ for some $1 \leq i \leq r$, it is shown below that $p(t)$ has a removable singularity and can be redefined accordingly. Noting that 
\[
p(t) = 0 \mod(h(t)) \quad \text{and} \quad p(x) = 0
\]
it follows that $p(t) = 0 \mod(h(t)(t - x))$. Applying the Generalized Rolle’s Theorem to $p(t)$, there exists a $c \in (a, b)$ such that $p^{(n)}(c) = 0$. Using (2), we calculate $p^{(n)}(t)$, getting 
\[
p^{(n)}(t) = f^{(n)}(t) - g^{(n)}(t) - [f(x) - g(x)] \left( \frac{d}{dt} \right)^n \frac{h(t)}{h(x)}
\]
\[
= f^{(n)}(t) - [f(x) - g(x)] \frac{n!}{h(x)},
\]
so that 
\[
0 = p^{(n)}(c) = f^{(n)}(c) - [f(x) - g(x)] \frac{n!}{h(x)}
\]
from which the result follows.

Applying the theorem to the case when $x = x_i$, we find by repeated application of L’Hospital’s rule that 
\[
\lim_{x \to x_i} \frac{f(x) - g(x)}{h(x)} = \frac{f^{(m)}(c)}{h^{(m)}(x)} = \frac{f^{(n)}(c)}{n!}.
\]

There remains the question of how do we calculate the polynomial $g(x)$ with the property that $f(x) = g(x)$ where $\deg(g(x)) < \deg(h(x))$? The brute force method is to impose the conditions \[1\], and solve the resulting system of linear equations for the unique solution known as the osculating polynomial approach to $f(x)$, see \[8\, 11\, p.52\]. A far more powerful method is to make use of the special algebraic properties of the spectral basis of the factor ring $\mathbb{R}[x]_h$, as has been explained in \[5\, 6\]. See also \[7\].

Much time is devoted to explaining the properties of Lagrange, Hermite, and other types of interpolating polynomials in numerical analysis. In teaching this subject, the author has discovered that many of the formulas and theorems
follow directly from the above theorem. For rational approximation, we have the following refinement:

**Rational Approximation Theorem:** Let \( f(x) \in C[a, b] \) and \( n \) times differentiable on \((a, b)\). Let \( u(x) \) and \( v(x) \) be polynomials such that \( v(0) = 1 \) and \( \text{det}(u(x)v(x)) < \text{deg}(h(x)) \), and suppose that \( f(x)v(x) - u(x) = 0 \mod(h(x)) \). Then

\[
f(x) = u(x)v(x) + \frac{1}{n!v(x)}[f(t)v(t)]^{(n)}(c)h(x)
\]

for some \( c \in (a, b) \).

**Proof:** Define the function

\[
p(t) = f(t)v(t) - u(t) - \frac{h(t)}{h(x)}[f(t)v(t) - u(x)]
\]

where \( x \in [a, b] \). Clearly, \( p(t) = 0 \mod(h(t)(t - x)) \). Applying the Generalized Rolle’s Theorem to \( p(t) \), it follows that there exist a \( c \in (a, b) \) such that

\[
f(x)v(x) - u(x) = \frac{1}{n!}\left(\frac{d}{dt}\right)^n[f(t)v(t)]_{t=c}h(x),
\]

from which it follows that

\[
f(x) = u(x)v(x) + \frac{1}{n!v(x)}[f(t)v(t)]^{(n)}(c)h(x).
\]

\[\square\]

**Acknowledgements**

The author is grateful to Dr. Reyla Navarro, Chairwoman, and Dr. Guillermo Romero, Vice Rector, of the Universidad de Las Americas for support for this research. The author is is a member of SNI 14587.

**References**

[1] J. A. Gallian, *Contemporary Abstract Algebra*, 6th ed., Houghton Mifflin Company, Boston, 2006.

[2] P. Linz, *Theoretical Numerical Analysis*, John Wiley & Sons, Inc., 1979.

[3] P. J. Davis, *Interpolation and Approximation*, Dover Publications, New York, 1975.

[4] J. Stoer and R. Bulirsch, *Introduction to Numerical Analysis 2nd Ed.*, Translated by R. Bartels, W. Gautschi, and C. Witzgall, Springer-Verlag, New York, 1993.

[5] G. Sobczyk, *The missing spectral basis in algebra and number theory*, The American Mathematical Monthly 108 April 2001, pp. 336-346.

[6] G. Sobczyk, *Generalized Vandermonde determinants and applications*, Aportaciones Matematicas, Serie Comunicaciones Vol. 30 (2002) 203 - 213.

[7] G. Sobczyk, *The Spectral Basis and Rational Interpolation*, arXiv:math/0602405v1, 2006.

e-mail: garret.sobczyk@udlap.mx

url: http://www.garretstar.com