Manifestly Covariant Canonical Quantization of Gravity and Diffeomorphism Anomalies in Four Dimensions

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Abstract

Canonical quantization of gravity requires knowledge about the representation theory of its constraint algebra, which is physically equivalent to the algebra of arbitrary 4-diffeomorphisms. All interesting lowest-energy representations are projective, making the relevant algebra into a four-dimensional generalization of the Virasoro algebra. Such diffeomorphism anomalies are invisible in field theory, because the relevant cocycles are functionals of the observer’s trajectory in spacetime. The multi-dimensional Virasoro algebra acts naturally in the phase space of arbitrary histories, with dynamics playing the role of first-class constraints. General relativity is regularized by expanding all fields in Taylor series around the observer’s trajectory, and truncating at some fixed order. This regularized but manifestly general-covariant theory is quantized in the history phase space, and dynamics is imposed afterwards, in analogy with BRST quantization. Infinites arise when the regularization is removed; it is presently unclear how these should be dealt with.
1 Introduction

After its invention by Dirac in 1927, local relativistic quantum field theory (QFT) has undergone three major crises.

The first crisis occurred in the 1930s, when it became clear that quantum mechanics (QM) was naïvely incompatible with special relativity, and in particular that quantization of electromagnetism gave infinite answers. This led many people to believe that radical new physics was needed. Alas, the problems were solved by renormalization, which involves no new physics, but merely a reinterpretation (albeit a radical one) of old physics.

The second crisis occurred in the 1960s, when QFT appeared to be incapable of describing the strong and weak interactions, in particular asymptotic freedom. Therefore many people turned to other, allegedly more fundamental ideas, such as analyticity of the S-matrix or string theory. However, it again turned out that straightforward QFT was the correct answer once it was applied to non-abelian Yang-Mills theory.

The third crisis is of course the apparent incompatibility between QM and gravity. This problem was realized already around 1960, when it was found that perturbative quantum gravity is not renormalizable and hence not a predictive theory, but it only became acute when the standard model was completed 25 years ago, and gravity remained the only interaction not described by a consistent QFT. As during the previous crises, the difficulties with gravity has led many people to abandon QFT in favor of more fashionable ideas.

The philosophy of this author is that quantum gravity should also be described by straightforward QFT, but that a small but profound fix is needed. The fix should be small, in order to preserve the experimental successes of the standard model and general relativity, but it must also be profound, because otherwise it would already have been found. This philosophy is in many ways similar to renormalization, which can be regarded as a small but profound fix which made QM compatible with locality.

The key idea in the present work is that we need to quantize not only the fields, but also the observer’s trajectory in spacetime. In non-relativistic QM, observation is a complicated, non-local process which assigns numbers to experiments. However, in a relativistic theory, a process must be localized; it happens somewhere. In order to maintain locality, we must assign the process of observation to some definite event in spacetime. As time proceeds, the observer (or detector or test particle) traces out a curve $q(t)$ in spacetime. Like the quantum fields, the observer’s trajectory should be treated as a material, quantized object; it has a conjugate momentum, it is represented on the Hilbert space, etc.

To consider the observer’s trajectory as a material object is certainly a very small fix, which does not introduce any new physical ideas. One may wonder why such a small, almost trivial modification should be important. The reason is that it makes it possible for new types of anomalies to arise. If the field theory has a
gauge symmetry of Yang-Mills type, there is a gauge anomaly proportional to the quadratic Casimir, and the gauge (or current) algebra becomes a higher-dimensional generalization of affine Kac-Moody algebras. Similarly, a general-covariant theory, in any number of dimensions, acquires a diffeomorphism anomaly, which is described by a higher-dimensional generalization of the Virasoro algebra. The reason why these anomalies can not be seen in conventional QFT, without explicit reference to the observer, is that the relevant cocycles are functionals of the observer’s trajectory. If this trajectory has not been introduced, it is of course impossible to write down the relevant anomalies.

At this point it is necessary to discuss the issue of gauge anomalies and consistency, in particular unitarity. Gauge and diffeomorphism anomalies are usually considered as a sign of inconsistency and should therefore be cancelled [1, 21]. There is ample evidence that this is the correct prescription for conventional gauge anomalies, arising from chiral fermions coupled to gauge fields. However, the anomalies discussed in this paper are of a completely different type, depending on the observer’s trajectory, and intuition derived from conventional anomalies needs not apply. Instead, the situation is similar to conformal field theory applied to two-dimensional statistical physics, where it is well known that infinite spacetime symmetry (gauge or not) is compatible with locality (in the sense of correlation functions depending on separation) only in the presence of a conformal anomaly. Hence gauge anomalies are viewed as a means to gauge symmetry breaking.

Be that as it is. Even if we would want to mod out the new gauge and diff anomalies (which this author believes is wrong), we still need to know about them. A good analogy is bosonic string theory, whose conformal anomaly cancels precisely in 26 dimensions. However, if we did not know about conformal anomalies, there would be no condition that singled out the number 26. Similarly, we need to know about the new anomalies that arise when we quantize gauge-invariant or diff-invariant theories in the presence of the observer’s trajectory, even if all we wanted to do with these anomalies were to cancel them.

Anomalies manifest themselves as extensions of the constraint algebras. Extensions of the diffeomorphism algebra were classified by Dzhumadildaev [7], and their representation theory was developed in [2, 3, 4, 5, 14, 15, 16, 23]; see [17] for a recent review. Unfortunately, the canonical formalism is not very well suited for quantization of relativistic theories, because the foliation of spacetime into fixed time slices breaks manifest covariance. This problem becomes especially serious in general-covariant theories, where the very constraint algebra is modified, from the algebra of arbitrary 4-diffeomorphisms into the Dirac algebra of constraints. However, this modification is a consequence of the chosen formalism rather than a true physical effect, and in covariant approaches the constraint algebra of general relativity is indeed the 4-diffeomorphism algebra.

It is well known that phase space is a covariant concept; it is the space of histories which solve the dynamics. Each phase space point \((q, p)\) generates a unique history
(q(t), p(t)) under Hamiltonian evolution, and thus we may view (q, p) = (q(0), p(0)) as a particular coordinatization of phase space. We can now understand how the Dirac algebra arises. A canonical transformation in the history phase space maps the history (q(t), p(t)) → (q'(t'), p'(t')). If in particular

\[(q, p) = (q(0), p(0)) \xrightarrow{\text{symm}} (q'(0), p'(0)) = (q', p'),\]

the transformation preserves the standard coordinatization; this is the situation with spatial diffeomorphisms in general relativity. However, temporal diffeomorphisms break the standard coordinatization, and if we insist on keeping it, we must add a compensating transformation to move back to the t = 0 surface. Schematically,

\[(q, p) = (q(0), p(0)) \xrightarrow{\text{symm}} (q'(t), p'(t)) \xrightarrow{\text{comp}} (q'(0), p'(0)) = (q', p').\]

The combination of symmetry and compensating transformations generate the Dirac algebra.

It is clear that we can avoid this complication if we work in the history phase space directly, because then we do not need to worry about compensating transformations. To this end, a novel quantization scheme has recently been proposed, called manifest covariant canonical quantization (MCCQ) [18, 19]. The idea is to make the space of arbitrary histories (q(t), p(t)) into a phase space \(\mathcal{P}\) by defining the Poisson brackets

\[
[p(t), q(t')] = \delta(t - t'), \quad [p(t), p(t')] = [q(t), q(t')] = 0.
\]

The Euler-Lagrange equations now define a constraint \(\mathcal{E}(t) \approx 0\) in \(\mathcal{P}\); since \(\mathcal{E}(t)\) only depends on \(q(t)\) this constraint is first class. This observation allows us to apply powerful cohomological methods from BRST quantization of theories with first class constraints. In other words, the idea in MCCQ is to quantize in the history phase space first and to impose dynamics afterwards, by passing to cohomology. Since dynamics is regarded as a constraint, the cohomology is nontrivial even for systems without gauge symmetries, like the harmonic oscillator and the free scalar field.

It is important to understand which ideas are crucial and which are merely convenient. The formulation of QFT using MCCQ is “just formalism”. It is very convenient to have a canonical quantization scheme which respects general covariance, but it is probably possible to reexpress the results in this paper using non-covariant canonical quantization, at the price of great complications. In contrast, adding the observer’s trajectory to the quantum fields is absolutely essential. Without it, the new observer-dependent anomalies can not be formulated, and this is a hard obstruction to quantization.

History methods have recently been advocated by Savvidou and Isham [11, 24, 25]; in particular, the last reference contains a very good summary of the conceptual problems involved in non-covariant canonical quantization. Their formalism differs in details from MCCQ, e.g. because they do not use cohomological methods.
There is also a substantial difference, namely that the observer’s trajectory is not introduced, and hence no diffeomorphism anomalies are seen.

Finally, let us emphasize the philosophical motivation for introducing MCCQ. The representations considered here are of lowest-energy type, i.e. there is a natural Hamiltonian whose eigenvalues are bounded from below. This is the kind of representations expected to be relevant to quantum theory. However, they do not look like standard formulations of QFT, because their natural habitat is in history space.

To apply them to physics, we must first recast physics in a suitable, history-oriented form. It is the same argument that leads us to use tensor calculus in general relativity; the classical irreps of the diffeomorphism group act on modules of tensor fields, so we should formulate physics in terms of those.

This article is organized as follows.

In the next section, the relation between anomalies, consistency, locality and unitarity is further discussed, using the infinite conformal symmetry in two-dimensional spacetime as a paradigm. The new anomalies, i.e. the higher-dimensional generalizations of the affine and Virasoro algebras, are reviewed in Section 3. For easy comparison to the one-dimensional case, we describe these extensions in a Fourier basis on the $N$-dimensional torus. However, the geometrical content is clearer in a real-space basis, which is introduced in Section 4. It turns out that in addition to the diffeomorphism algebra $\text{vect}(N)$ (algebra of vector fields in $N$ dimensions) and the gauge algebra $\text{map}(N, g)$ (algebra of maps from $N$-dimensional spacetime to the finite-dimensional Lie algebra $g$), we must also introduce the observer’s trajectory and the algebra of reparametrizations; the full algebra is called the DGRO (Diffeomorphism, Gauge, Reparametrization, Observer) algebra $DGRO(N, g)$.

Its representation theory is developed in Section 5. Rather than starting from the fields themselves, as is done in one dimension, the right approach is to first expand all fields in a Taylor series around the observer’s trajectory and truncate at some finite order $p$ before quantization. This gives us a non-linear realization of the diffeomorphism algebra on finitely many functions of a single variable, which is precisely the situation where normal ordering works. We also get an action of a $\text{vect}(1)$ algebra describing reparametrizations for free, i.e. without enlarging the realization.

In Section 6 we introduce the manifestly covariant canonical quantization scheme mentioned above, and apply it to the free scalar field in Section 7. However, the Hamiltonian still singles out a preferred time direction. To remedy this, we introduce in Section 8 the observer and define the Hamiltonian covariantly as the operator which translates the fields relative to the observer. The scalar field is again used as an example in Section 9. The formalism is then extended to theories with gauge symmetries in Section 10. As examples we treat the free Maxwell field in Section 11 and pure Einstein gravity in Section 12. The $u(1)$ gauge symmetry of the Maxwell field is anomaly free in the absence of matter, but there are diffeomorphism anomalies already in pure gravity, because it is an interacting theory.
Truncating the Taylor expansion at order $p$ is a regularization, and at the end we want to remove the regulator by taking the limit $p \to \infty$. This limit is problematic and poorly understood, as is discussed in Section 13. We conclude with a conceptual discussion in Sections 14 and 15.

2 Anomalies, consistency, locality, and unitarity

A quantum theory is defined by a Hilbert space and a Hamiltonian which generates time evolution. The main conditions for consistency are unitarity and lack of infinities. If the theory has some symmetries, these must be realized as unitary operators acting on the Hilbert space as well. In particular, if time translation is included among the symmetries, which is the case for the Poincaré and diffeomorphism algebras, a unitary representation of the symmetry algebra is usually enough for consistency. From this viewpoint, there is a 1-1 correspondence between general-covariant QFTs and unitary representations of the diffeomorphism group on a conventional Hilbert space. Namely, given the QFT, its Hilbert space carries a unitary representation of the diffeomorphism group. Conversely, if we have a unitary representation of the diffeomorphism group, the Hilbert space on which it acts can be interpreted as the Hilbert space of some general-covariant QFT.

Unfortunately, this observation is not yet so powerful, because no non-trivial, unitary, lowest-energy irreps of the diffeomorphism algebra are known except in one dimension. Nevertheless, we are able to make some very general observations. Assume that some algebra (or group) $\mathfrak{g}$ has a unitary representation $\mathcal{R}$ and a subalgebra $\mathfrak{h}$. Then the restriction of $\mathcal{R}$ to $\mathfrak{h}$ is still unitary, and this must hold for every subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. In particular, let $\mathfrak{g} = \text{vect}(N)$ be the diffeomorphism algebra in $N$ dimensions and $\mathfrak{h} = \text{vect}(1)$ the diffeomorphism algebra in one dimension. There are infinitely many such subalgebras, and the restriction of $\mathcal{R}$ to each and every one of them must be unitary. Fortunately, the unitary irreps of the diffeomorphism algebra in one dimension are known. The result is that the only proper unitary irrep is the trivial one, but there are many unitary irreps with a diffeomorphism anomaly. From this it follows that the trivial representation is the only unitary representation also in $N$ dimensions. “There are no local observables in quantum gravity”.

However, there is one well-known case where we know how to combine locality and infinite spacetime symmetry with quantum theory: conformal field theory (CFT). This is usually thought of as a theory of conformally invariant quantum fields in two dimensions, but since the local conformal group is the same as (twice) the diffeomorphism group in one dimension, it is also about diffeomorphism invariant QFT in one dimension. Locality means that the correlation functions depend on separation. For two points $z$ and $w$ in $\mathbb{R}$ or $\mathbb{C}$, the correlator is

$$\langle \phi(z)\phi(w) \rangle \sim \frac{1}{(z-w)^{2n}} + \text{more},$$

(2.1)
where \textit{more} stands for less singular terms when \( z \to w \). That the correlation function has this form is a diffeomorphism-invariant statement. The \textit{more} terms will change under an arbitrary diffeomorphism, but the leading singularity will always have the same form, and in particular the anomalous dimension \( h \) is well defined.

We can phrase this slightly differently. The short-distance singularity only depends on two points being infinitesimally close. This is good, because we cannot determine the finite distance between two points without knowing about the metric. General relativity does not have a background metric structure, but it does have a background differentiable structure (locally at least), and that is enough for defining anomalous dimensions. Diffeomorphisms move points around, but they do not separate two points which are infinitesimally close.

The relevant algebra in CFT is not really the one-dimensional diffeomorphism algebra (or the two-dimensional conformal algebra), but rather its central extension known as the Virasoro algebra:

\[
[L_m, L_n] = (n - m)L_{m+n} - \frac{c}{12}(m^3 - m)\delta_{m+n}. 
\]

(2.2)

A lowest-energy representation is characterized by a vacuum satisfying

\[
L_0|0\rangle = h|0\rangle, \quad L_m|0\rangle = 0 \text{ for all } m < 0. 
\]

(2.3)

In particular, the lowest \( L_0 \) eigenvalue can be identified with the anomalous dimension \( h \) in the correlation function (2.1). This means that locality, in the sense of correlation functions depending on separation, requires that \( h > 0 \). It is well known that unitarity either implies that \( c \geq 1 \), \( h \geq 0 \).

\[
c = 1 - \frac{6}{m(m+1)}, \quad h = h_{rs}(m) = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)}, 
\]

(2.4)

where \( m \geq 2 \) and \( 1 \leq r < m, 1 \leq s < r \) are positive integers, or that \( c \geq 1, h \geq 0 \). In particular, the central extension \( c \) is non-zero for any non-trivial, unitary irrep with \( h \neq 0 \). This leads to the important observation:

\begin{center}
\textbf{Locality and unitarity are compatible with diffeomorphism (and local conformal) symmetry only in the presence of an anomaly.}
\end{center}

This is true in higher dimensions as well. Consider the correlator \( \langle \phi(x)\phi(y) \rangle \), where \( x \) and \( y \) are points in \( \mathbb{R}^N \). We could take some one-dimensional curve \( q(t) \) passing through \( x \) and \( y \), such that \( x = q(t) \) and \( y = q(t') \). Then the short-distance behaviour is of the form

\[
\langle \phi(x)\phi(y) \rangle \sim \frac{1}{(t - t')^{2h}} + \text{more}, 
\]

(2.5)

and \( h \) is independent of the choice of curve, provided that it is sufficiently regular. The subalgebra of \( \text{vect}(N) \) which preserves \( q(t) \) is a Virasoro algebra, so \( h > 0 \) implies that \( c > 0 \).
This observation is completely standard in the application of CFT to statistical physics in two dimensions. The simplest example of a unitary model is the Ising model, which consists of three irreps, with \( c = \frac{1}{2} \) and \( h = 0, \ h = \frac{1}{16}, \) and \( h = \frac{1}{2}. \) The Ising model is perfectly consistent despite the anomaly, both mathematically (unitarity), and more importantly physically (it is realized in nature, in soft condensed matter systems). The standard counter-argument is that infinite conformal symmetry in condensed matter is not a gauge symmetry, but rather an anomalous global symmetry. However, if we could take the classical limit of such a system, the conformal symmetry would seem to be a gauge symmetry. Namely, the anomaly vanishes in the classical limit, and we can write down a classical BRST operator which is nilpotent, and the symmetry is gauge on the classical level. There is no classical way to distinguish between such a “fake” gauge symmetry and a genuine gauge symmetry which extends to the quantum level.

More generally, let us assume that we have some phase space, and a Lie algebra \( \mathfrak{g} \) with generators \( J_a, \) satisfying

\[
[J_a, J_b] = f_{ab}^\ c J_c, \tag{2.6}
\]

acts on this phase space. The Einstein convention is used; repeated indices, one up and one down, are implicitly summed over. If the bracket with the Hamiltonian gives us a new element in \( \mathfrak{g}, \)

\[
[J_a, H] = C_a^b J_b, \tag{2.7}
\]

we say that \( \mathfrak{g} \) is a symmetry of the Hamiltonian system. If \( \mathfrak{g} \) in addition contains arbitary functions of time, the symmetry is a gauge symmetry. In this case, a solution to Hamilton’s equations depends on arbitrary functions of time and is thus not fully specified by the positions and momenta at time \( t = 0. \) The standard example is electromagnetism, where the zeroth component \( A_0 \) of the vector potential is arbitrary, because its canonical momentum \( F_{0}^0 = 0. \) An arbitrary time evolution is of course not acceptable. The reason why this seems to happen is that a gauge symmetry is a redundancy of the description; the true dynamical degrees of freedom are fewer than what one naively expects. In electromagnetism, the gauge potential has four components but the photon has only two polarizations.

There are various ways to handle quantization of gauge systems. One is to eliminate the gauge degrees of freedom first and then quantize. This is cumbersome and it is usually preferable to quantize first and eliminate the gauge symmetries afterwards. The simplest way is to require that the gauge generators annihilate physical states,

\[
J_a |\text{phys}\rangle = 0, \tag{2.8}
\]

and also that two physical states are equivalent if the differ by some gauge state, \( J_a |> \). This procedure produces a Hilbert space of physical states.
However, one thing may go wrong. Upon quantization, a symmetry may acquire some quantum corrections, so that $g$ is replaced by

$$[J_a, J_b] = f_{a b}^c J_c + \hbar D_{a b} + O(\hbar^2). \quad (2.9)$$

The operator $D_{a b}$ is called an anomaly. We can also have anomalies of the type

$$[J_a, H] = C^b_a J_b + \hbar E_a + O(\hbar^2). \quad (2.10)$$

If we now try to keep the definition of a physical state, we see that we must also demand that

$$D_{a b} |\text{phys}\rangle = 0. \quad (2.11)$$

This implies further reduction of the Hilbert space. In the case that $D_{a b}$ is invertible, there are no physical states at all, so the Hilbert space is empty. However, this does not necessarily mean that the anomaly by itself is inconsistent, only that our definition of physical states is. In the presence of an anomaly, additional states become physical. So our Hilbert space becomes larger, containing some, or even all, of the previous gauge degrees of freedom. A gauge anomaly implies that the gauge symmetry is broken on the quantum level.

Such a “fake” gauge symmetry may well be consistent. The Virasoro algebra is obviously anomalous, with the central charge playing the role of the $D_{a b}$, and still it has unitary representations with non-zero $c$. Of course, a gauge anomaly may be inconsistent, if the anomalous algebra does not possess any unitary representations. This is apparently what happens for the chiral-fermion type anomaly which is relevant e.g. in the standard model.

### 3 Multi-dimensional Virasoro algebra

All non-trivial, unitary, lowest-energy irreps of the diffeomorphism algebra are anomalous, in any number of dimensions. This is well-known in one dimension, where the diffeomorphism algebra acquires an extension known as the Virasoro algebra. It is also true in several dimensions, which one proves by considering the restriction to the many Virasoro subalgebras living on lines in spacetime. This is perhaps rather surprising, in view of the following two no-go theorems:

- The diffeomorphism algebra has no central extension except in one dimension.
- In field theory, there are no pure gravitational anomalies in four dimension.

However, the assumptions in these no-go theorems are too strong; the Virasoro extension is not central except in one dimension, and one needs to go slightly beyond field theory by explicitly specifying where observation takes place.
To make contact with the Virasoro algebra in its most familiar form, we describe its multi-dimensional sibling in a Fourier basis on the $N$-dimensional torus. Recall first that the algebra of diffeomorphisms on the circle, $\text{vect}(1)$, has generators

$$ L_m = -i \exp(imx) \frac{d}{dx}, \quad (3.1) $$

where $x \in S^1$. $\text{vect}(1)$ has a central extension, known as the Virasoro algebra:

$$ [L_m, L_n] = (n - m)L_{m+n} - \frac{c}{12}(m^3 - m)\delta_{m+n}, \quad (3.2) $$

where $c$ is a c-number known as the central charge or conformal anomaly. This means that the Virasoro algebra is a Lie algebra; anti-symmetry and the Jacobi identities still hold. The term linear in $m$ is unimportant, because it can be removed by a redefinition of $L_0$. The cubic term $m^3$ is a non-trivial extension which cannot be removed by any redefinition.

The generators (3.1) immediately generalize to vector fields on the $N$-dimensional torus:

$$ L_\mu(m) = -i \exp(im\rho x^\rho) \partial_\mu, \quad (3.3) $$

where $x = (x^\mu)$, $\mu = 1, 2, ..., N$ is a point in $N$-dimensional space and $m = (m_\mu) \in \mathbb{Z}^N$. These operators generate the algebra $\text{vect}(N)$:

$$ [L_\mu(m), L_\nu(n)] = n_\mu L_\nu(m + n) - m_\nu L_\mu(m + n). \quad (3.4) $$

The question is now whether the Virasoro extension, i.e. the $m^3$ term in (3.2), also generalizes to higher dimensions.

Rewrite the ordinary Virasoro algebra (3.2) as

$$ [L_m, L_n] = (n - m)L_{m+n} + cm^2 nS_{m+n}, $$

$$ [L_m, S_n] = (n + m)S_{m+n}, $$

$$ [S_m, S_n] = 0, $$

$$ mS_m \equiv 0. \quad (3.5) $$

It is easy to see that the two formulations of the Virasoro algebra are equivalent (the linear cocycle has been absorbed into a redefinition of $L_0$). The second formulation immediately generalizes to $N$ dimensions. The defining relations are

$$ [L_\mu(m), L_\nu(n)] = n_\mu L_\nu(m + n) - m_\nu L_\mu(m + n) $$

$$ + (c_1 m_\nu n_\mu + c_2 m_\mu n_\nu)m_\rho S^\rho(m + n), $$

$$ [L_\mu(m), S_\nu^\rho(n)] = n_\mu S_\nu^\rho(m + n) + \delta_\nu^\rho m_\rho S^\rho(m + n), $$

$$ [S_\mu^\rho(m), S_\nu^\sigma(n)] = 0, $$

$$ m_\mu S_\nu^\rho(m) \equiv 0. \quad (3.6) $$
This is an extension of \( \text{vect}(N) \) by the abelian ideal with basis \( S^\mu(m) \). Geometrically, we can think of \( L_\mu(m) \) as a vector field and \( S^\mu(m) = \epsilon^{\mu\nu_1...\nu_N} S_{\nu_1...\nu_N}(m) \) as a dual one-form (and \( S_{\nu_1...\nu_N}(m) \) as an \((N - 1)\)-form); the last condition expresses closedness. The cocycle proportional to \( c_1 \) was discovered by Rao and Moody [23], and the one proportional to \( c_2 \) by this author [13].

There is also a similar multi-dimensional generalization of affine Kac-Moody algebras, presumably first written down by Kassel [12]. It is sometimes called the central extension, but this term is somewhat misleading because the extension does not commute with diffeomorphisms, although it does commute with all gauge transformations.

Let \( \mathfrak{g} \) be a finite-dimensional Lie algebra with structure constants \( f_{abc} \) and Killing metric \( \delta_{ab} \). The Kassel extension of the current algebra \( \text{map}(N, \mathfrak{g}) \) is defined by the brackets

\[
\begin{align*}
    [J_a(m), J_b(n)] &= f_{abc} J_c(m + n) + k \delta_{ab} m_\mu S^\mu(m + n), \\
    [J_a(m), S^\mu(n)] &= [S^\mu(m), S^\nu(n)] = 0, \\
    m_\mu S^\mu(m) &\equiv 0.
\end{align*}
\]

This algebra admits an intertwining action of the \( N \)-dimensional Virasoro algebra \( (3.9) \):

\[
[L_\mu(m), J_a(n)] = n_\mu J_a(m + n). \quad (3.8)
\]

The current algebra \( \text{map}(N, \mathfrak{g}) \) also admits another type of extension in some dimensions. The best known example is the Mickelsson-Faddeev algebra, relevant for the conventional anomalies in field theory, which arise when chiral fermions are coupled to gauge fields in three spatial dimensions. Let \( d_{abc} = \text{tr}\{T_a T_b T_c\} \) be the totally symmetric third Casimir operator, and let \( \epsilon^{\mu\nu\rho} \) be the totally anti-symmetric epsilon tensor in three dimensions. The Mickelsson-Faddeev algebra [20] reads in a Fourier basis:

\[
\begin{align*}
    [J_a(m), J_b(n)] &= f_{abc} J_c(m + n) + d_{abc} \epsilon^{\mu\nu\rho} m_\mu n_\nu A^\rho_c(m + n), \\
    [J_a(m), A^b_\nu(n)] &= -f_{ac}^b A^c_\nu(m + n) + \delta^b_\nu m_\mu \delta(m + n), \\
    [A^a_\mu(m), A^b_\nu(n)] &= 0.
\end{align*}
\]

\( A^a_\mu(m) \) are the Fourier components of the gauge connection.

Note that \( Q_a = J_a(0) \) generates a Lie algebra isomorphic to \( \mathfrak{g} \), whose Cartan subalgebra is identified with the charges. Moreover, the subalgebra of \( (3.7) \) spanned by \( J_a(m_0) \equiv J_a(m) \), where \( m = (m_0, 0, ..., 0) \in \mathbb{Z} \), reads

\[
[J_a(m_0), J_b(n_0)] = f_{abc} J_c(m_0 + n_0) + k \delta_{ab} m_0 \delta(m_0 + n_0), \quad (3.10)
\]

which we recognize as the affine algebra \( \hat{\mathfrak{g}} \). Since all non-trivial unitary irreps of \( \hat{\mathfrak{g}} \) has \( k > 0 \) [19], it is impossible to combine unitary and non-zero \( \mathfrak{g} \) charges also for
DGRO algebra

The Fourier formalism in the previous section makes the analogy with the usual Virasoro algebra manifest, but it is neither illuminating nor a useful starting point for representation theory. To bring out the geometrical content, we introduce the DGRO (Diffeomorphism, Gauge, Reparametrization, Observer) algebra \(DGRO(N, \mathfrak{g})\), whose ingredients are spacetime diffeomorphisms which generate \(\text{vect}(N)\), reparametrizations of the observer’s trajectory which form an additional \(\text{vect}(1)\) algebra, and gauge transformations which generate a current algebra. Classically, the algebra is \(\text{vect}(N) \ltimes \text{map}(N, \mathfrak{g}) \oplus \text{vect}(1)\).

Let \(\xi = \xi^\mu(x)\partial_\mu, \ x \in \mathbb{R}^N, \ \partial_\mu = \partial/\partial x^\mu\), be a vector field, with commutator \([\xi, \eta] \equiv \xi^\mu \partial_\mu \eta^\nu - \eta^\mu \partial_\mu \xi^\nu\), and greek indices \(\mu, \nu = 1, 2, \ldots, N\) label the spacetime coordinates. The Lie derivatives \(\mathcal{L}_\xi\) are the generators of \(\text{vect}(N)\).

Let \(f = f(t)d/dt, \ t \in S^1\), be a vector field in one dimension. The commutator reads \([f, g] = (fg - gf)d/dt\), where the dot denotes the \(t\) derivative: \(\dot{f} \equiv df/dt\). We will also use \(\partial_t = \partial/\partial t\) for the partial \(t\) derivative. The choice that \(t\) lies on the circle is physically unnatural and is made for technical simplicity only (quantities can be expanded in Fourier series). However, this seems to be a minor problem at the present level of understanding. Denote the reparametrization generators \(L_f\).

Let \(\text{map}(N, \mathfrak{g})\) be the current algebra corresponding to the finite-dimensional semisimple Lie algebra \(\mathfrak{g}\) with basis \(J_a\), structure constants \(f_{ab}^c\), and Killing metric \(\delta_{ab}\). The brackets in \(\mathfrak{g}\) are given by \((2.6)\). A basis for \(\text{map}(N, \mathfrak{g})\) is given by \(\mathfrak{g}\)-valued functions \(X = X^a(x)J_a\) with commutator \([X, Y] = f_{ab}^c X^a Y^b J_c\). The intertwining \(\text{vect}(N)\) action is given by \(\xi X = \xi^\mu \partial_\mu X^a J_a\). Denote the \(\text{map}(N, \mathfrak{g})\) generators by \(J_X\).

Finally, let \(\text{Obs}(N)\) be the space of local functionals of the observer’s trajectory \(q^\mu(t)\), i.e. polynomial functions of \(q^\mu(t), \dot{q}^\mu(t), \ldots, d^k q^\mu(t)/dt^k, \ k\) finite, regarded as a commutative algebra. \(\text{Obs}(N)\) is a \(\text{vect}(N)\) module in a natural manner.

\(DGRO(N, \mathfrak{g})\) is an abelian but non-central Lie algebra extension of \(\text{vect}(N) \ltimes \text{map}(N, \mathfrak{g}) \oplus \text{vect}(1)\) by \(\text{Obs}(N)\):

\[0 \longrightarrow \text{Obs}(N) \longrightarrow DGRO(N, \mathfrak{g}) \longrightarrow \text{vect}(N) \ltimes \text{map}(N, \mathfrak{g}) \oplus \text{vect}(1) \longrightarrow 0.\]
The brackets are given by

\[
\begin{align*}
[L_\xi, L_\eta] &= L_{[\xi, \eta]} + \frac{1}{2\pi i} \int dt \, \dot{q}^\rho(t) \left\{ c_1 \partial_\rho \partial_\sigma \xi^\mu(q(t)) \partial_\mu \eta^\nu(q(t)) + c_2 \partial_\rho \partial_\mu \xi^\mu(q(t)) \partial_\nu \eta^\nu(q(t)) \right\}, \\
[L_\xi, J_X] &= J_{\xi X}, \\
[J_X, J_Y] &= J_{[X,Y]} - \frac{c_5}{2\pi i} \delta_{ab} \int dt \, \dot{q}^a(t) \partial_\rho X^a(q(t)) Y^b(q(t)), \\
[L_f, L_\xi] &= \frac{c_3}{4\pi i} \int dt \, (\dddot{f}(t) - i\dot{f}(t)) \partial_\sigma \xi^\mu(q(t)), \\
[L_f, J_X] &= 0, \\
[L_f, L_g] &= L_{[f,g]} + \frac{c_4}{24\pi i} \int dt (\dddot{f}(t) \dot{g}(t) - \dddot{f}(t) g(t)), \\
[L_\xi, q^\mu(t)] &= \xi^\mu(q(t)), \\
[L_f, q^\mu(t)] &= -f(t) \dot{q}^\mu(t), \\
[J_X, q^\mu(t)] &= [q^a(s), q^\mu(t)] = 0,
\end{align*}
\]

extended to all of \(\text{Obs}(N)\) by Leibniz’ rule and linearity. The numbers \(c_1 - c_5\) are called \textit{abelian charges}, in analogy with the central charge of the Virasoro algebra. In [14, 15] slightly more complicated extensions were considered, which depend on three additional abelian charges \(c_6 - c_8\). However, these vanish automatically when \(\mathfrak{g}\) is semisimple.

5 Representations of the DGRO algebra

To construct Fock representations of the ordinary Virasoro algebra is straightforward:

- Start from classical modules, i.e. primary fields = scalar densities.
- Introduce canonical momenta.
- Normal order.

The first two steps of this procedure generalize nicely to higher dimensions. The classical representations of the DGRO algebra are tensor fields over \(\mathbb{R}^N \times S^1\) valued in \(\mathfrak{g}\) modules. The basis of a classical DGRO module \(Q\) is thus a field \(\phi^\alpha(x, t), x \in \mathbb{R}^N, t \in S^1\), where \(\alpha\) is a collection of all kinds of indices. The \(\text{DGRO}(N, \mathfrak{g})\) action on \(Q\) can be succinctly summarized as

\[
\begin{align*}
[L_\xi, \phi^\alpha(x, t)] &= -\xi^\mu(x) \partial_\mu \phi^\alpha(x, t) - \partial_\sigma \xi^\mu(x) T^\alpha_{\beta \mu} \phi^\beta(x, t), \\
[J_X, \phi^\alpha(x, t)] &= -X^a(x) J^\alpha_{\beta a} \phi^\beta(x, t), \\
[L_f, \phi^\alpha(x, t)] &= -f(t) \partial_\sigma \phi^\alpha(x, t) - \lambda(\dddot{f}(t) - i\dot{f}(t)) \phi^\alpha(x, t).
\end{align*}
\]
Here \( J_\alpha = (J^\alpha_{\beta\alpha}) \) and \( T^\mu_\nu = (T^{\alpha\mu}_{\beta\nu}) \) are matrices satisfying \( g \) and \( gl(N) \), respectively:

\[
[T^\mu_\nu, T^\sigma_\tau] = \delta^\sigma_\tau T^\mu_\nu - \delta^\mu_\nu T^\sigma_\tau. \quad (5.2)
\]

The tensor field representations of the DGRO algebra can thus be expressed in matrix form as

\[
L_\xi = - \int d^N x \int dt \left( \xi^\mu(x) \partial_\mu \phi^\alpha(x, t) + \partial_\nu \xi^\nu(x) T^{\alpha\nu}_{\beta\mu} \phi^\beta(x, t) \right) \pi_\alpha(x, t),
\]

\[
J_X = - \int d^N x \int dt \left( X^a(x) J^a_{\alpha\beta} \phi^\beta(x, t) \pi_\alpha(x, t) \right), \quad (5.3)
\]

\[
L_f = - \int d^N x \int dt \left( f(t) \partial_\mu \phi^\alpha(x, t) - \lambda(\dot{f}(t) - i f(t) \phi^\alpha(x, t)) \pi_\alpha(x, t) \right),
\]

where the conjugate momentum \( \pi_\alpha(x, t) = \delta / \delta \phi^\alpha(x, t) \) satisfies

\[
[\pi_\alpha(x, t), \phi^\beta(x', t')] = \delta^\beta_\alpha \delta(x - x') \delta(t - t'). \quad (5.4)
\]

However, the normal-ordering step simply does not work in several dimensions, because

- It requires that a foliation of spacetime into space and time has been introduced, which runs against the idea of diffeomorphism invariance.
- Normal ordering of bilinear expressions always results in a central extension, but the Virasoro cocycle is non-central when \( N \geq 2 \).
- It is ill defined. Formally, attempts to normal order result in an infinite central extension, which of course makes no sense.

To avoid this problem, the crucial idea in [14] was to expand all fields in a Taylor series around the observer’s trajectory and truncate at order \( p \), before introducing canonical momenta. Hence we expand e.g.,

\[
\phi^\alpha(x, t) = \sum_{|\mathbf{m}| \leq p} \frac{1}{\mathbf{m}!} \phi^\alpha_{\mathbf{m}}(t)(x-q(t))^\mathbf{m}, \quad (5.5)
\]

where \( \mathbf{m} = (m_1, m_2, \ldots, m_N) \), all \( m_\mu \geq 0 \), is a multi-index of length \( |\mathbf{m}| = \sum_{\mu=1}^N m_\mu \), \( \mathbf{m}! = m_1! m_2! \ldots m_N! \), and

\[
(x - q(t))^\mathbf{m} = (x^1 - q^1(t))^{m_1} (x^2 - q^2(t))^{m_2} \ldots (x^N - q^N(t))^{m_N}. \quad (5.6)
\]

Denote by \( \mu \) a unit vector in the \( \mu \)-th direction, so that \( \mathbf{m} + \mu = (m_1, \ldots, m_\mu + 1, \ldots, m_N) \), and let

\[
\phi^\alpha_{\mathbf{m}}(t) = \partial_{m_1} \partial_{m_2} \ldots \partial_{m_N} \phi^\alpha(q(t), t) \quad (5.7)
\]
be the $|m|$:th order derivative of $\phi^\alpha(x,t)$ evaluated on the observer’s trajectory $q^\mu(t)$.

Given two jets $\phi, m(t)$ and $\psi, m(t')$, we define their product

$$(\phi(t)\psi(t'))_m = \sum_n \binom{m}{n} \phi, n(t)\psi, m-n(t').$$

(5.8)

It is clear that $(\phi(t)\psi(t'))_m$ is the jet corresponding to the field $\phi(x,t)\psi(x,t')$. For brevity, we also denote $(\phi\psi)_m(t) = (\phi(t)\psi(t))_m$.

$p$-jets transform under $DGRO(N, g)$ as

$$[L_\xi, \phi^\alpha_m(t)] = \partial_m ([L_\xi, \phi^\alpha(q(t), t)]) + [L_\xi, q^\mu(t)]\partial_m \phi^\alpha(q(t), t)$$

$$= - \sum_{|n| \leq |m| \leq p} T_{3m}^m (\xi(q(t))) \phi^\beta_n (t),$$

$$[J_X, \phi^\alpha_m(t)] = \partial_m ([J_X, \phi^\alpha(q(t), t)])$$

$$= - \sum_{|n| \leq |m| \leq p} J_{3m}^m (X(q(t))) \phi^\beta_n (t),$$

$$[L_f, \phi^\alpha_m(t)] = -f(t)\phi^\alpha_m(t) - \lambda(\dot{f}(t) - if(t))\phi^\alpha_m(t),$$

where

$$T_{n}^m (\xi) \equiv (T_{3m}^m (\xi)) = \sum_{\mu} \binom{m}{n} \partial_{\xi^\mu} T_{\mu}$$

$$+ \sum_{\mu} \binom{m}{n} \partial_{\xi^\mu} \phi^\beta_n (t),$$

(5.9)

$$J_{n}^m (X) \equiv (J_{3m}^m (X)) = \binom{m}{n} \partial_{\xi^\mu} \phi^\beta_n (t),$$

and

$$\binom{m}{n} = \frac{m!}{n!(m-n)!} = \binom{m_1}{n_1} \binom{m_2}{n_2} \cdots \binom{m_N}{n_N}. $$

(5.11)

We thus obtain a non-linear realization of $\text{vect}(N)$ on the space of trajectories in the space of tensor-valued $p$-jets$^\dagger$ denote this space by $J^p Q$. Note that $J^p Q$ is spanned by $q^\mu(t)$ and $\{\phi^\alpha_m(t)\}_{|m| \leq p}$ and thus not a $DGRO(N, g)$ module by itself, because diffeomorphisms act non-linearly on $q^\mu(t)$, as can be seen in (4.11). However, the space $C(J^p Q)$ of functionals on $J^p Q$ (local in $t$) is a module, because the action

$^\dagger$p-jets are usually defined as an equivalence class of functions: two functions are equivalent if all derivatives up to order $p$, evaluated at $q^\mu$, agree. However, each class has a unique representative which is a polynomial of order at most $p$, namely the Taylor expansion around $q^\mu$, so we may canonically identify jets with truncated Taylor series. Since $q^\mu(t)$ depends on a parameter $t$, we deal in fact with trajectories in jet space, but these will also be called jets for brevity.
on a $p$-jet can never produce a jet of order higher than $p$. The space $C(q) \otimes_q J^p Q$, where only the trajectory itself appears non-linearly, is a submodule.

The crucial observation is that the jet space $J^p Q$ consists of finitely many functions of a single variable $t$, which is precisely the situation where the normal ordering prescription works. After normal ordering, denoted by double dots $::$, we obtain a Fock representation of the DGRO algebra:

\[
\mathcal{L}_\xi = \int dt \left\{ :\xi^\mu(q(t)) p_\mu(t) : - \sum_{|n| \leq |m| \leq p} T_{\beta \gamma}^{\alpha \mu}(\xi(q(t))) : \phi_\beta(t) \pi^\gamma_m(t) : \right\},
\]

\[
\mathcal{J}_X = - \int dt \left\{ \sum_{|n| \leq |m| \leq p} J_{\beta \gamma}^{\alpha \mu}(\xi(q(t))) : \phi_\beta(t) \pi^\gamma_m(t) : \right\}, \tag{5.12}
\]

\[
L_f = \int dt \left\{ -f(t) : \phi_\alpha(t) \pi^\gamma_m(t) : -\lambda(f(t) - i f(t)) : \phi_\alpha(t) \pi^\gamma_m(t) : \right\},
\]

where we have introduced canonical momenta $p_\mu(t) = \delta/\delta q^\mu(t)$ and $\pi^\gamma_m(t) = \delta/\delta \phi_\alpha_m(t)$. The field $\phi_\alpha(x, t)$ can be either bosonic or fermionic but the trajectory $q^\mu(t)$ is of course always bosonic.

Normal ordering is defined with respect to frequency; any function of $t \in S^1$ can be expanded in a Fourier series, e.g.

\[
p_\mu(t) = \sum_{m=\pm \infty}^\infty \hat{p}_\mu(m) e^{-imt} \equiv p_\mu^\alpha(t) + \hat{p}_\mu(0) + p_\mu^\beta(t), \tag{5.13}
\]

where $p_\mu^\alpha(t)$ ($p_\mu^\beta(t)$) is the sum over negative (positive) frequency modes only. Then

\[
:\xi^\mu(q(t)) p_\mu(t) : \equiv \xi^\mu(q(t)) p_\mu^\alpha(t) + p_\mu^\beta(t) \xi^\mu(q(t)), \tag{5.14}
\]

where the zero mode has been included in $p_\mu^\alpha(t)$.

It is clear that (5.12) defines a Fock representation for every $gl(N)$ irrep $\mathfrak{g}$ and every $\mathfrak{g}$ irrep $M$; denote this Fock space by $J^p \mathcal{F}$, which indicates that it also depends on the truncation order $p$. Namely, introduce a Fock vacuum $|0\rangle$ which is annihilated by half of the oscillators, i.e.

\[
\phi_\alpha^\beta_m(t)|0\rangle = \pi^\gamma_{\alpha \beta} m(t)|0\rangle = q_\beta^\alpha(t)|0\rangle = p_\beta^\gamma(t)|0\rangle = 0. \tag{5.15}
\]

Then DGRO($N, \mathfrak{g}$) acts on the space of functionals $C(q^\alpha^\gamma_m, p^\beta_\mu, \phi_\alpha^\beta_m, \pi^\gamma_m)$ of the remaining oscillators; this is the Fock module. Define numbers $k_0(\mathfrak{g})$, $k_1(\mathfrak{g})$, $k_2(\mathfrak{g})$ and $y_M$ by

\[
\text{tr}_\alpha T_\nu^\mu = k_0(\mathfrak{g}) \delta_\nu^\mu, \\
\text{tr}_\nu T_\mu^\nu T_\tau^\sigma = k_1(\mathfrak{g}) \delta_\mu^\nu \delta_\sigma^\tau + k_2(\mathfrak{g}) \delta_\mu^\nu \delta_\nu^\tau, \\
\text{tr}_M J_\alpha J_\beta = y_M \delta_{ab}. \tag{5.16}
\]
For an unconstrained tensor with \(p\) upper and \(q\) lower indices and weight \(\kappa\), we have

\[
dim(\mathcal{g}) = N^{p+q}, \quad k_0(\mathcal{g}) = -(p - q - \kappa N) N^{p+q-1},
\]
\[
k_1(\mathcal{g}) = (p + q) N^{p+q-1}, \quad k_2(\mathcal{g}) = ((p - q - \kappa N)^2 - p - q) N^{p+q-2}.
\]

Note that if \(\kappa = (p - q)/N\), \(\mathcal{g}\) is an \(sl(N)\) representation. For the symmetric representations on \(\ell\) lower indices, \(S_\ell\), and on \(\ell\) upper indices, \(S^\ell\), we have

\[
dim(S_\ell) = \dim(S^\ell) = \binom{N - 1 + \ell}{\ell},
\]
\[
k_0(S_\ell) = -k_0(S^\ell) = \binom{N - 1 + \ell}{\ell - 1},
\]
\[
k_1(S_\ell) = k_1(S^\ell) = \binom{N + \ell}{\ell - 1},
\]
\[
k_2(S_\ell) = k_2(S^\ell) = \binom{N - 1 + \ell}{\ell - 2}.
\]

The values of the abelian charges \(c_1 - c_5\) were calculated in [14], Theorems 1 and 3, and in [15], Theorem 1:

\[
c_1 = 1 - u \binom{N + p}{N} - x \binom{N + p + 1}{N + 2},
\]
\[
c_2 = -v \binom{N + p}{N} - 2w \binom{N + p}{N + 1} - x \binom{N + p}{N + 2},
\]
\[
c_3 = 1 + (1 - 2\lambda)(w \binom{N + p}{N} + x \binom{N + p}{N + 1}),
\]
\[
c_4 = 2N - x(1 - 6\lambda + 6\lambda^2) \binom{N + p}{N},
\]
\[
c_5 = y \binom{N + p}{N}.
\]

where

\[
u = \mp k_1(\mathcal{g}) \dim M, \quad x = \mp \dim \mathcal{g} \dim M,
\]
\[
v = \mp k_2(\mathcal{g}) \dim M, \quad y = \mp \dim \mathcal{g} \dim \mathcal{M},
\]
\[
w = \mp k_0(\mathcal{g}) \dim M,
\]

and the sign factor depends on the Grassmann parity of \(\phi^\alpha\); the upper sign holds for bosons and the lower for fermions, respectively. The \(p\)-independent contributions to \(c_1\), \(c_3\) and \(c_4\) come from the trajectory \(q^\mu(t)\) itself.
6 MCCQ: Manifestly Covariant Canonical Quantization

In the previous sections we constructed representations of gauge and diffeomorphism algebras. These representations are of lowest-energy type, i.e. there is a Hamiltonian whose eigenvalues are bounded from below; this is the kind of representations relevant to quantum theory. Now it is time to apply them to physics. To this end, we first reformulate quantum physics in the history phase space, which is the representation theory’s natural habitat.

Consider a classical dynamical system with action $S$ and degrees of freedom $\phi^\alpha$. As is customary in the antifield literature, we use an abbreviated notation where the index $\alpha$ stands for both discrete indices and spacetime coordinates. Dynamics is governed by the Euler-Lagrange (EL) equations,

$$E^\alpha = \partial_\alpha S \equiv \frac{\delta S}{\delta \phi^\alpha} = 0.$$  \hspace{1cm} (6.1)

An important role is also played by the Hessian, i.e. the symmetric second functional-derivative matrix

$$K_{\alpha\beta} = K_{\beta\alpha} = \partial_\beta E^\alpha = \frac{\delta E^\alpha}{\delta \phi^\beta} = \frac{\delta^2 S}{\delta \phi^\alpha \delta \phi^\beta}.$$  \hspace{1cm} (6.2)

The Hessian is assumed non-singular, so it has an inverse $M^{\alpha\beta}$ satisfying

$$K_{\beta\gamma} M^{\gamma\alpha} = M^{\alpha\gamma} K_{\gamma\beta} = \delta^\alpha_\beta.$$  \hspace{1cm} (6.3)

Introduce an antifield $\phi^*_\alpha$ for each EL equation (6.1), and replace the space of $\phi$-histories $Q$ by the extended history space $Q^*$, spanned by both $\phi$ and $\phi^*$. In $Q^*$ we define the Koszul-Tate (KT) differential $\delta$ by

$$\delta \phi^\alpha = 0, \hspace{1cm} \delta \phi^*_\alpha = E^\alpha.$$  \hspace{1cm} (6.4)

One checks that $\delta$ is nilpotent, $\delta^2 = 0$. Define the antifield number $\text{afn} \phi^\alpha = 0$, $\text{afn} \phi^*_\alpha = 1$. The KT differential clearly has antifield number $\text{afn} \delta = -1$.

The space $C(Q^*)$ decomposes into subspaces $C^k(Q^*)$ of fixed antifield number

$$C(Q^*) = \sum_{k=0}^{\infty} C^k(Q^*)$$  \hspace{1cm} (6.5)

The KT complex is

$$0 \overset{\delta}{\longleftrightarrow} C^0 \overset{\delta}{\longleftrightarrow} C^1 \overset{\delta}{\longleftrightarrow} C^2 \overset{\delta}{\longleftrightarrow} \ldots$$  \hspace{1cm} (6.6)
The cohomology spaces are defined as usual by
\[ H^k_{\text{cl}}(\delta) = \ker \delta / \operatorname{im} \delta, \]
i.e. \( H^k_{\text{cl}}(\delta) = (\ker \delta_k) / (\operatorname{im} \delta_k), \) where the subscript \( \text{cl} \) indicates that we deal with a classical phase space. It is easy to see that
\[ (\ker \delta)_0 = C(Q), \]
\[ (\operatorname{im} \delta)_0 = C(Q)E_\alpha \equiv N. \] (6.7)
Thus \( H^0_{\text{cl}}(\delta) = C(Q)/N = C(\Sigma). \) Since we assume that there are no non-trivial relations among the \( E_\alpha, \) the higher cohomology groups vanish. This is a standard result [10].

The complex (6.6) thus gives us a resolution of the covariant phase space \( C(\Sigma), \) which by definition means that
\[ H^0_{\text{cl}}(\delta) = C(\Sigma), \]
\[ H^k_{\text{cl}}(\delta) = 0, \quad \text{for all } k > 0. \]

Alas, the antifield formalism is not suited for canonical quantization. We can define an antibracket in \( Q^*, \) but in order to do canonical quantization we need an honest Poisson bracket. To this end, we introduce canonical momenta conjugate to the history and its antifield, and obtain an even larger space \( P^*, \) which may be thought of as the phase space corresponding to the extended history space \( Q^*. \)

Introduce canonical momenta \( \pi_\alpha = \delta / \delta \phi_\alpha \) and \( \pi^*_\alpha = \delta / \delta \phi^*_\alpha \) for both the fields and antifields. The momenta satisfy by definition the graded canonical commutation relations (\( \phi^\alpha \) is assumed bosonic),
\[ [\pi_\beta, \phi^\alpha] = \delta^\alpha_\beta, \quad [\phi^\alpha, \phi^\beta] = [\pi_\alpha, \pi_\beta] = 0, \]
\[ \{\pi^\alpha, \phi^\alpha\} = \delta^\alpha_\beta, \quad \{\phi^\alpha, \phi^\beta\} = \{\pi^\alpha, \pi^\beta\} = 0, \] (6.8)
where \( \{\cdot, \cdot\} \) is the symmetric bracket. Let \( P \) be the phase space of histories with basis \( (\phi^\alpha, \pi_\beta), \) and let \( P^* \) be the extended phase space with basis \( (\phi^\alpha, \pi_\beta, \phi^*_\alpha, \pi^*_\alpha) \).

The definition of the KT differential extends to \( P^* \) by requiring that \( \delta F = [Q_{KT}, F] \) for every \( F \in C(P^*), \) where the KT operator is
\[ Q_{KT} = E_\alpha \pi^\alpha. \] (6.9)

It acts on the various fields as
\[ \delta \phi^\alpha = 0, \]
\[ \delta \phi^*_\alpha = E_\alpha, \]
\[ \delta \pi_\alpha = -\frac{\delta E_\beta}{\delta \phi^\alpha} \pi^\beta = -K_{\alpha\beta} \pi^\beta, \]
\[ \delta \pi^*_\alpha = 0, \] (6.10)
where \( K_{\alpha\beta} \) is the Hessian (6.2). We check that \( \delta \) is still nilpotent: \( \delta^2 = \{Q_{KT}, Q_{KT}\} = 0. \)

Like \( C(Q), \) the function space \( C(P^*) \) decomposes into subspaces of fixed antifield number, \( C(P^*) = \sum_{k=-\infty}^{\infty} C^k(P^*). \) We can therefore define a KT complex in \( C(P^*) \)
\[ \ldots \delta \leftarrow C^{-2} \delta \leftarrow C^{-1} \delta \leftarrow C^0 \delta \leftarrow C^1 \delta \leftarrow C^2 \delta \leftarrow \ldots \] (6.11)
Because the Hessian \([6.2]\) is non-singular by assumption with inverse \(M^{\alpha \beta}\), we can invert the relation \(\delta \pi_\alpha = -K_{\alpha \beta} \pi^\beta\) and get

\[
\pi^\alpha = -M^{\alpha \beta} \delta \pi_\beta = \delta (-M^{\alpha \beta} \pi_\beta),
\]

since \(M^{\alpha \beta}\) depends on \(\phi\) alone.

Let us now compute the cohomology. Any function which contains \(\pi_\alpha\) is not closed, so \(\ker \delta = C(\phi, \phi^*, \pi_\alpha)\). Moreover, \(\ker \delta\) is generated by the two ideals \(E_\alpha\) and \(\pi_\alpha^*\). The momenta \(\pi_\alpha\) and \(\pi_\alpha^*\) thus vanish in cohomology, and the part with zero antifield number is thus still \(H^0_{\alpha}(\delta) = C(Q)/N = C(\Sigma)\). The higher cohomology groups \(H^k_{\alpha}(\delta) = 0\) by the same argument as above. Hence the complex \([6.11]\) yields a different resolution of the function space \(C(\Sigma)\).

It is important that the spaces \(C^k\) in \([6.11]\) are phase spaces, equipped with the Poisson bracket \([6.8]\). Unlike the resolution \([6.6]\), the new resolution \([6.11]\) therefore allows us to do canonical quantization: replace Poisson brackets by commutators and represent the graded Heisenberg algebra \([6.8]\) on a Hilbert space. However, the Heisenberg algebra can be represented on different Hilbert spaces; there is no Stone-von Neumann theorem in infinite dimension. To pick the correct one, we must impose the physical condition that there is an energy which is bounded on below.

To define the Hamiltonian, we must single out a privileged variable \(t\) among the \(\alpha\)'s, and declare it to be time. Thus replace \(\alpha = (i, t)\), so e.g. \(\phi^\alpha = \phi^i(t)\), \(E_\alpha = E_i(t)\), etc. This step means of course that we sacrifice covariance. The Hamiltonian reads

\[
H = -i \int dt \, \dot{\phi}^i(t) \pi_i(t) + \dot{\phi}^*_i(t) \pi^*_i(t).
\]

It satisfies

\[
[H, \phi^i(t)] = -i \dot{\phi}^i(t), \quad [H, \pi_i(t)] = -i \dot{\pi}_i(t),
\]

\[
[H, \phi^*_i(t)] = -i \dot{\phi}^*_i(t), \quad [H, \pi^*_i(t)] = -i \dot{\pi}^*_i(t).
\]

(6.14)

Expand all fields in a Fourier series with respect to time, e.g.,

\[
\phi^i(t) = \sum_{m=-\infty}^{\infty} \phi^i(m)e^{imt}.
\]

(6.15)

The Fourier modes \(\pi_i(m)\), \(\phi^*_i(m)\) and \(\pi^*_i(m)\) are defined analogously. The Hamiltonian acts on the Fourier modes as

\[
[H, \phi^i(m)] = m \phi^i(m), \quad [H, \pi_i(m)] = m \pi_i(m),
\]

\[
[H, \phi^*_i(m)] = m \phi^*_i(m), \quad [H, \pi^*_i(m)] = m \pi^*_i(m).
\]

(6.16)

Now quantize. In the spirit of BRST quantization, our strategy is to quantize first and impose dynamics afterwards. In the extended history phase space \(P^*\), we define a Fock vacuum \(|0\rangle\) which is annihilated by all negative frequency modes, i.e.

\[
\phi^i(-m)|0\rangle = \pi_i(-m)|0\rangle = \phi^*_i(-m)|0\rangle = \pi^*_i(-m)|0\rangle = 0,
\]

(6.17)
for all \(-m < 0\). We must also decide which of the zero modes that annihilate the vacuum, but the decision is not important unless zero-momentum modes will survive in cohomology, and even then it will not affect the eigenvalues of the Hamiltonian.

The Hamiltonian (6.13) does not act in a well-defined manner, because it assigns an infinite energy to the Fock vacuum. To correct for that, we replace the Hamiltonian by

\[
H = -i \int dt \, :\dot{\phi}^i(t)\pi_i(t): + :\phi^*_i(t)\pi^*_i(t):,
\]

where normal ordering \(\cdot\) moves negative frequency modes to the right and positive frequency modes to the left. The vacuum has zero energy as measured by the normal-ordered Hamiltonian,

\[
H\lvert 0 \rangle = 0.
\]

The Hilbert space can be identified with

\[
\mathcal{H}(P^*) = C(\phi^i(m > 0), \pi_i(m > 0), \phi^*_i(m > 0), \pi^*_i(m > 0)).
\]

The energy of a state in \(\mathcal{H}(P^*)\) follows from

\[
H\phi^i_1(...\pi^*_i(n)\lvert 0 \rangle = (m_1 + ... + m_n)\phi^i_1(...\pi^*_i(n)\lvert 0 \rangle.
\]

It is important that the KT operator

\[
Q_{KT} = \mathcal{E}_\alpha\pi^\alpha = \int dt \, \mathcal{E}_i(t)\pi^i(t) = \sum_{m=-\infty}^{\infty} \mathcal{E}_i(m)\pi^i(-m)
\]

is already normal ordered, because \(\mathcal{E}_\alpha\) and \(\pi^\alpha\) commute. This means that \(Q^2_{KT} = 0\) also quantum mechanically; there are no anomalies. Moreover, \(Q_{KT}\) still commutes with the Hamiltonian, \([Q_{KT}, H] = 0\), and this property is not destroyed by normal ordering. Hence the Hilbert space \(\mathcal{H}(P^*)\) has also a well-defined decomposition into subspaces of definite antifield number,

\[
\mathcal{H}(P^*) = \ldots + \mathcal{H}^{-2} + \mathcal{H}^{-1} + \mathcal{H}^0 + \mathcal{H}^1 + \mathcal{H}^2 + \ldots
\]

There is a KT complex in \(\mathcal{H}(P^*)\)

\[
\ldots \xleftarrow{\delta} \mathcal{H}^{-2} \xleftarrow{\delta} \mathcal{H}^{-1} \xleftarrow{\delta} \mathcal{H}^0 \xleftarrow{\delta} \mathcal{H}^1 \xleftarrow{\delta} \mathcal{H}^2 \xleftarrow{\delta} \ldots
\]

The physical Hilbert space is identified with \(\mathcal{H}(\Sigma) = \mathcal{H}^0_{gm}(Q_{KT}) = (\ker Q_{KT})_0/(\im Q_{KT})_0\). The action of the Hamiltonian on the physical Hilbert space is still given by (6.20), restricted to \(\mathcal{H}(\Sigma) \subset \mathcal{H}(P^*)\), and that coincides with the conventional action of the Hamiltonian.

Hence we have quantized the theory given by the EL equation (6.1) by first quantizing the space of phase space histories \(P^*\), and then imposing dynamics through KT cohomology.
7 Scalar field I: non-covariant quantization

The action, Euler-Lagrange equations, and Hessian read
\[ S = \frac{1}{2} \int d^N x \left( \partial_\mu \phi(x) \partial^\mu \phi(x) - \omega^2 \phi^2(x) \right), \]
\[ \mathcal{E}(x) \equiv -\frac{\delta S}{\delta \phi(x)} = \Box \phi(x) + \omega^2 \phi(x) = 0, \]
\[ K(x, x') \equiv -\frac{\delta^2 S}{\delta \phi(x) \delta \phi(x')} = \Box \delta(x - x') + \omega^2 \delta(x - x'), \quad (7.1) \]
where \( \Box = \partial_\mu \partial^\mu. \)

Introduce antifields \( \phi^*(x) \) and canonical momenta \( \pi(x) = \delta / \delta \phi(x) \) and \( \pi^*(x) = \delta / \delta \phi^*(x) \). The non-zero brackets are
\[ \{\pi(x), \phi^*(x')\} = \{\pi^*(x), \phi(x')\} = \delta(x - x'). \quad (7.2) \]

The KT differential reads
\[ Q_{KT} = \int d^N x \left( \Box \phi(x) + \omega^2 \phi(x) \right) \pi^*(x). \quad (7.3) \]

\( Q_{KT} \) acts as \( \delta F = [Q_{KT}, F] \), where
\[ \begin{align*}
\delta \phi(x) & = 0, \\
\delta \phi^*(x) & = \Box \phi(x) + \omega^2 \phi(x), \\
\delta \pi(x) & = -\left( \Box \pi(x) + \omega^2 \pi(x) \right), \\
\delta \pi^*(x) & = 0. 
\end{align*} \quad (7.4) \]

Now we do a Fourier transformation. The extended phase space \( \mathcal{P}^* \) is spanned by modes \( \phi(k), \phi^*(k), \pi(k) \) and \( \pi^*(k) \), and the EL equation becomes
\[ \mathcal{E}(k) = -(k^2 - \omega^2) \phi(k) = 0. \quad (7.5) \]

The non-zero brackets are
\[ \{\pi(k), \phi(k')\} = \{\pi^*(k), \phi^*(k')\} = \delta(k + k'). \quad (7.6) \]

The KT differential is
\[ Q_{KT} = \int d^N k \left( k^2 - \omega^2 \right) \phi(k) \pi^*(-k). \quad (7.7) \]

\( Q_{KT} \) acts as \( \delta F = [Q_{KT}, F] \), where
\[ \begin{align*}
\delta \phi(k) & = 0, \\
\delta \phi^*(k) & = (k^2 - \omega^2) \phi(k), \\
\delta \pi(k) & = -(k^2 - \omega^2) \pi^*(k), \\
\delta \pi^*(k) & = 0. 
\end{align*} \quad (7.8) \]
The cohomology is computed as follows. Since the equations (7.8) decouple, we can consider each value of $k$ separately. First assume that $k^2 \neq \omega^2$. $\phi(k)$ and $\pi_*(k)$ are closed for all $k$, but $\phi^*(k)$ and $\pi(k)$ are not closed since $\delta \phi^*(k) \neq 0$, etc. We can invert the second and third equations to read

$$\phi(k) = \frac{1}{k^2 - \omega^2} \delta \phi^*(k),$$

$$\pi_*(k) = -\frac{1}{k^2 - \omega^2} \delta \pi(k).$$

(7.9)

Hence $\phi(k)$ and $\pi_*(k)$ lie in the image of $\delta$, and the cohomology vanishes completely: only $\phi(k)$ and $\pi_*(k)$ lie in the kernel, but they also lie in the image.

Now turn to the case $k^2 = \omega^2$, say $k = (\omega, 0, 0, 0)$. Clearly, $\delta \phi(k) = \delta \pi(k) = \delta \phi^*(k) = \delta \pi_*(k) = 0$, so all four variables lie in the kernel but not in the image. Thus the cohomology spaces are too big; the classical cohomology spaces can be identified with $H^*_{cl}(\delta) = C(\phi(k), \pi(k), \phi^*(k), \pi_*(k))$. The zeroth cohomology space consists of such functions with total antifield number zero, i.e. $H^0_{cl}(\delta) = C(\phi(k), \pi(k), (\phi^*(k)\pi_*(k')))$. In [18] it was proposed that this problem could be handled by adding a small perturbation to make the Hessian non-singular, so the momenta can be killed in cohomology. This unwanted cohomology is an embarrassment, especially since it reappears in Maxwell theory, but we have nothing more to say about it.

To quantize the theory we must specify a Hamiltonian. Let it be

$$H = -i \int d^N x \left( \partial_0 \phi(x) \pi(x) + \partial_0 \phi^*(x) \pi_*(x) \right)$$

$$= \int d^N k \ k_0 (\phi(k) \pi(-k) + \phi^*(k) \pi_*(-k)).$$

(7.10)

Note that at this stage we break Poincaré invariance, since the Hamiltonian treats the $x^0$ coordinate differently from the other $x^\mu$. Quantize by introducing a Fock vacuum $|0\rangle$ satisfying

$$\phi(k)|0\rangle = \pi(k)|0\rangle = \phi^*(k)|0\rangle = \pi_*(k)|0\rangle = 0,$$

(7.11)

for all $k$ such that $k_0 < 0$. After adding a small perturbation to make the Hessian invertible, $\pi(k)$ and $\pi_*(k)$ vanish in cohomology, as do the off-shell components of $\phi(k)$ and $\phi^*(k)$. The classical cohomology $H^*_{cl}(Q_{KT}) = C(\phi(k; k^2 = \omega^2), \phi^*(k; k^2 = \omega^2))$ consists of functions of the on-shell components of $\phi$ and $\phi^*$, and $H^0_{cl}(Q_{KT}) = C(\phi(k; k^2 = \omega^2))$ is the classical phase space. The quantization step eliminates the components $\phi(k)$ with $k_0 < 0$, which leaves us with the physical Hilbert space $\mathcal{H} = H^0_{gm}(Q_{KT}) = C(\phi(k; k^2 = \omega^2 \text{ and } k_0 > 0))$. A basis for $\mathcal{H}$ consists of multi-quanta states

$$|k, k', ..., k^{(n)}\rangle = \phi(k) \phi(k') ... \phi(k^{(n)}) |0\rangle$$

(7.12)

with energy $H = k + k' + ... + k^{(n)}$. 

Manifestly Covariant Canonical Quantization of Gravity
8 MCCQ: Jets and covariant quantization

A covariant definition of the phase space was given in the Section 6, but the Hamiltonian and thus the quantum Hilbert space broke covariance, due to the selection of a privileged time coordinate. In this section we correct this defect.

The compact notation is not very useful here, because the notion of covariance does not make sense unless some indices are identified with spacetime coordinates. So we assume that we have some fields $\phi^\alpha(x)$, where $x = (x^\mu) \in \mathbb{R}^N$ is the spacetime coordinate. The EL equations read

$$E^\alpha(x) \equiv \frac{\delta S}{\delta \phi^\alpha(x)} = 0. \quad (8.1)$$

We also need the Hessian

$$K_{\alpha\beta}(x,x') = K_{\beta\alpha}(x',x) = \frac{\delta E^\alpha(x)}{\delta \phi^\beta(x')} = \frac{\delta^2 S}{\delta \phi^\alpha(x) \delta \phi^\beta(x')} . \quad (8.2)$$

which we assume is non-singular.

Now let all fields depend on an additional parameter $t$. It will eventually be identified with time, but so far it is completely unrelated to the $x^\mu$. Upon the substitution $\phi^\alpha(x) \rightarrow \phi^\alpha(x,t)$, the EL equations are replaced by

$$E^\alpha(x,t) = 0. \quad (8.3)$$

The Hessian (8.2) becomes

$$K_{\alpha\beta}(x,t,x',t') = K_{\beta\alpha}(x',t',x,t) = \frac{\delta E^\alpha(x,t)}{\delta \phi^\beta(x',t')}, \quad (8.4)$$

which has the inverse $M^{\alpha\beta}(x,t,x',t')$ satisfying

$$\int d^Nx'' \int dt'' \ K_{\beta\gamma}(x,t,x'',t'')M^{\gamma\alpha}(x'',t'',x',t') = \delta^\alpha_{\beta}\delta(x-x')\delta(t-t').$$

To remove the condition (8.3) in cohomology we introduce antifields $\phi^*_\alpha(x,t)$. But the fields in the physical phase space do not depend on the parameter $t$, which gives rise to the extra condition

$$\partial_t \phi^\alpha(x,t) \equiv \frac{\partial \phi^\alpha(x,t)}{\partial t} = 0. \quad (8.5)$$

We can implement this condition by introducing new antifields $\tilde{\phi}^\alpha(x,t)$. However, the identities $\partial_t E^\alpha(x,t) \equiv 0$ give rise to unwanted cohomology. To kill this condition, we must introduce yet another antifield $\tilde{\phi}_\alpha^*(x,t)$. The KT differential $\delta$ is defined by

$$\begin{align*}
\delta \phi^\alpha(x,t) &= 0, \\
\delta \phi^*_\alpha(x,t) &= E^\alpha(x,t), \\
\delta \tilde{\phi}^\alpha(x,t) &= \partial_t \phi^\alpha(x,t), \\
\delta \tilde{\phi}_\alpha^*(x,t) &= \partial_t \phi^*_\alpha(x,t) - \int d^N x'' \int dt'' \ K_{\alpha\beta}(x,t,x'',t'') \tilde{\phi}_\beta^*(x'',t''). \quad (8.6)
\end{align*}$$
The zeroth cohomology group $H^0_{cl}(\delta)$ equals $C(\phi)$, modulo the ideals generated by $\mathcal{E}_\alpha(x,t)$ and $\partial_t \phi^\alpha(x,t)$. Moreover, the wouldbe cohomology related to the identity

$$\delta \left\{ \partial_t \phi^*_\alpha(x,t) - \int d^N x' \int dt' \frac{\delta \mathcal{E}_\alpha(x,t)}{\delta \phi^\beta(x',t')} \dot{\phi}^\beta(x',t') \right\} = 0 \quad (8.7)$$

is killed because the expression equals $\delta \dot{\phi}^*_\alpha(x,t)$.

Introduce canonical momenta for all fields and antifields: $\pi_\alpha(x,t) = \delta / \delta \phi^\alpha(x,t)$, $\pi^*_\alpha(x,t) = \delta / \delta \phi^*_\alpha(x,t)$, $\pi_\alpha(x,t) = \delta / \delta \phi^\alpha(x,t)$, and $\pi^*_\alpha(x,t) = \delta / \delta \phi^*_\alpha(x,t)$. The KT operator takes the explicit form

$$Q_{KT} = \int d^N x \int dt \left\{ \mathcal{E}_\alpha(x,t) \pi^*_\alpha(x,t) + \partial_t \phi^\alpha(x,t) \pi_\alpha(x,t) \right.$$ 

$$\left. + (\partial_t \phi^*_\alpha(x,t) - \int d^N x' \int dt' K_{\alpha\beta}(x,t,x',t') \dot{\phi}^\beta(t')) \pi^*_\alpha(x,t) \right\}. \quad (8.8)$$

From this we can read off the action of $\delta$ on the momenta. As in the previous section, the zeroth cohomology group consists of functions $\phi^\alpha(x,t)$ which satisfy $\mathcal{E}_\alpha(x,t) = 0$ and $\partial_t \phi^\alpha(x,t) = 0$. Hence $H^0_{cl}(\delta) = C(\Sigma)$, as desired.

At this point, we must define a Hamiltonian. The candidate

$$H_0 = -i \int dt \left\{ \partial_t \phi^\alpha(x,t) \pi_\alpha(x,t) + \partial_t \phi^*_\alpha(x,t) \pi^*_\alpha(x,t) \right. \quad (8.9)$$

$$\left. + \partial_t \dot{\phi}^\alpha(x,t) \pi_\alpha(x,t) + \partial_t \dot{\phi}^*_\alpha(x,t) \pi^*_\alpha(x,t) \right\}$$

might seem natural, but it is not acceptable. The action of the Hamiltonian is KT exact, e.g.

$$[H_0, \phi^\alpha(x,t)] = \partial_t \phi^\alpha(x,t) = \delta \dot{\phi}^\alpha(x,t), \quad (8.10)$$

and thus $H_0 \approx 0$. $H_0$ is not a genuine Hamiltonian, but rather a Hamiltonian constraint $H_0 \approx 0$, familiar from canonical quantization of general relativity.

However, we can construct a well-defined and physical Hamiltonian with some extra work. The crucial idea is to introduce the observer’s trajectory $q^\mu(t) \in \mathbb{R}^N$, and then expand all fields in a Taylor series around this trajectory as in (5.5). Expand also the Euler-Lagrange equations and the antifields in a similar Taylor series, e.g. $\mathcal{E}_\alpha(x,t) = \sum_m \frac{1}{m!} \mathcal{E}_{\alpha,m}(t)(x - q(t))^m$. Such relations define the jets $\mathcal{E}_{\alpha,m}(t)$, $\phi^\alpha_{\alpha,m}(t)$, $\phi^\alpha_{\alpha,m}(t)$ and $\phi^*_{\alpha,m}(t)$. Jets of antifields will sometimes be called antijets.

The equation of motion and the time-independence condition translate into

$$\mathcal{E}_{\alpha,m}(t) = 0,$$

$$D_t \phi^\alpha_{\alpha,m}(t) \equiv \dot{\phi}^\alpha_{\alpha,m}(t) - \sum_\mu \dot{q}^\mu(t) \phi^\alpha_{\alpha,m+\mu}(t) = 0. \quad (8.11)$$
The KT differential $\delta$ which implements these conditions is

\begin{align}
\delta \phi^\alpha_m(t) &= 0, \\
\delta \phi^*_\alpha_m(t) &= \mathcal{E}_{\alpha,m}(t), \\
\delta \tilde{\phi}^\alpha_m(t) &= D_t \phi^\alpha_m(t), \\
\delta \bar{\phi}^*_\alpha_m(t) &= D_t \phi^*_\alpha_m(t) - \sum_n \int dt' K^\alpha_{m,\alpha\beta}(t,t') \bar{\phi}^\beta_n(t').
\end{align}

The cohomology group $H^0_{cl}(\delta)$ consists of linear combinations of jets $\phi^\alpha_m(t)$ satisfying $\mathcal{E}_{\alpha,m}(t) = 0$ and $D_t \phi^\alpha_m(t) = 0$.

The Taylor expansion requires that we introduce the observer’s trajectory as a physical field, but what equation of motion does it obey? The obvious answer is the geodesic equation, which we compactly write as $\mathcal{G}_\mu(t) = 0$. The geodesic operator $\mathcal{G}_\mu(t)$ is a function of the metric $g_{\mu\nu}(q(t),t)$ and its derivatives on the curve $q^\mu(t)$.

To eliminate this ideal in cohomology we introduce the trajectory antifield $q^*_\mu(t)$, and extend the KT differential to it:

\begin{align}
\delta q^\mu(t) &= 0, \\
\delta q^*_\mu(t) &= \mathcal{G}_\mu(t).
\end{align}

For models defined over Minkowski spacetime, the geodesic equation simply becomes $\ddot{q}^\mu(t) = 0$, and the KT differential reads

\begin{align}
\delta q^*_\mu(t) &= \eta_{\mu\nu} q^\nu(t).
\end{align}

$H^0_{cl}(\delta)$ only contains trajectories which are straight lines,

\begin{align}
q^\mu(t) = u^\mu t + a^\mu,
\end{align}

where $u^\mu$ and $a^\mu$ are constant vectors. We may also require that $u^\mu$ has unit length, $u_\mu u^\mu = 1$. This condition fixes the scale of the parameter $t$ in terms of the Minkowski metric, so we may regard it as proper time rather than as an arbitrary parameter.

Now introduce the canonical momenta $\pi^\alpha_m(t) = \delta / \delta \phi^\alpha_m(t)$, $\pi^\alpha_*(t) = \delta / \delta \phi^*_\alpha_m(t)$, $\pi^*_\alpha_m(t) = \delta / \delta \bar{\phi}^\alpha_m(t)$, $\pi^*_{\alpha*}(t) = \delta / \delta \bar{\phi}^*_\alpha_m(t)$ for the jets and antijets (jet and antijet momenta), and momenta $p_\mu(t) = \delta / \delta q^\mu(t)$ and $p^*_\mu(t) = \delta / \delta q^*_\mu(t)$ for the observer’s trajectory and its antifield. We can now define a genuine Hamiltonian $H$, which translates the fields relative to the observer or vice versa. Since the formulas are shortest when $H$ acts on the trajectory but not on the jets, we make that choice, and define

\begin{align}
H = i \int dt \ (\dot{q}^\mu(t)p_\mu(t) + q^*_\mu(t)p^*_\mu(t)).
\end{align}
Note the sign; moving the fields forward in $t$ is equivalent to moving the observer backwards. From (5.5), we get the energy of the fields:

$$[H, \phi^\alpha(x, t)] = -i \dot{q}^\mu(t) \partial_\mu \phi^\alpha(x, t). \quad (8.17)$$

This a crucial result, because it allows us to define a genuine energy operator in a covariant way. In Minkowski space, the trajectory is a straight line (8.15), and $\dot{q}^\mu(t) = u^\mu$. If we take $u^\mu$ to be the constant four-vector $u^\mu = (1, 0, 0, 0)$, then (8.17) reduces to

$$[H, \phi^\alpha(x, t)] = -i \frac{\partial}{\partial x^0} \phi^\alpha(x, t). \quad (8.18)$$

Equation (8.16) is thus a genuine covariant generalization of the energy operator.

Now we quantize the theory. Since all operators depend on the parameter $t$, we can define the Fourier components as in (5.13). The Fock vacuum $\lvert 0 \rangle$ is defined to be annihilated by all negative frequency modes, $\phi^\alpha_m(-m)$, $q^\mu(-m)$, etc. with $m < 0$. The normal-ordered form of the Hamiltonian (8.16) reads, in Fourier space,

$$H = - \sum_{m=-\infty}^{\infty} m \langle q^\mu(m)p_\mu(-m): + :q^*_\mu(m)p_\mu(-m): \rangle, \quad (8.19)$$

where double dots indicate normal ordering with respect to frequency. This ensures that $H\lvert 0 \rangle = 0$. The classical phase space $H^0_\text{cl}(\delta)$ is thus the the space of fields $\phi^\alpha(x)$ which solve $E_\alpha(x) = 0$, and trajectories $q^\mu(t) = u^\mu t + a^\mu$, where $u^2 = 1$. After quantization, the fields and trajectories become operators which act on the physical Hilbert space $H = H^0_{qm}(Q_K T)$, which is the space of functions of the positive-energy modes of the classical phase space variables.

This construction differs technically from conventional canonical quantization, but there is also a physical difference. Consider the state $\lvert \phi^\alpha(x) \rangle = \phi^\alpha(x)\lvert 0 \rangle$ which excites one $\phi$ quantum from the vacuum. The Hamiltonian yields

$$H \lvert \phi^\alpha(x) \rangle = -i \dot{q}^\mu(t) \partial_\mu \phi^\alpha(x) \lvert 0 \rangle = -i \dot{q}^\mu(t) \partial_\mu \phi^\alpha(x) \rangle = -i u^\mu \partial_\mu \phi^\alpha(x). \quad (8.20)$$

If $u^\mu$ were a classical variable, the state $\lvert \phi^\alpha(x) \rangle$ would be a superposition of energy eigenstates:

$$H \lvert \phi^\alpha(x) \rangle = -iu^\mu \partial_\mu \phi^\alpha(x). \quad (8.21)$$

In particular, let $u^\mu = (1, 0, 0, 0)$ be a unit vector in the $x^0$ direction and $\phi^\alpha(x) = \exp(ik \cdot x)$ be a plane wave. We then define the state $\lvert 0; u, a \rangle$ by

$$q^\mu(t) \lvert 0; u, a \rangle = (u^\mu t + a^\mu) \lvert 0; u, a \rangle. \quad (8.22)$$
Now write $|k; u, a\rangle = \exp(ik \cdot x)|0; u, a\rangle$ for the single-quantum energy eigenstate.

$$H|k; u, a\rangle = k_\mu u^\mu |k; u, a\rangle,$$

(8.23)

so the eigenvalue of the Hamiltonian is $k_\mu u^\mu = k_0$, as expected. Moreover, the lowest-energy condition ensures that only quanta with positive energy will be excited; if $k_\mu u^\mu < 0$ then $|k; u, a\rangle = 0$.

However, the present analysis shows that it is in principle wrong to consider $u^\mu$ and $a^\mu$ as classical variables. The definition (8.22) means that the reference state $|0; u, a\rangle$ is a very complicated, mixed, macroscopic state where the observer moves along a well-defined, classical trajectory. This is of course an excellent approximation in practice, but in principle wrong.

9 Scalar field II: covariant quantization

Following the prescription in Section 8, we make the replacement $\phi(x) \to \phi(x, t)$, where $t \in \mathbb{R}$ is a parameter. The EL equation (7.1) becomes

$$\mathcal{E}(x, t) \equiv \Box \phi(x, t) + \omega^2 \phi(x, t) = 0.\quad (9.1)$$

To remove this condition in cohomology we introduce antifields $\phi^*(x, t)$. But there is an extra condition

$$\partial_t \phi(x, t) \equiv \frac{\partial \phi(x, t)}{\partial t} = 0.\quad (9.2)$$

We can implement this condition by introducing new antifields $\bar{\phi}(x, t)$. However, the identities $\partial_t \mathcal{E}(x, t) \equiv 0$ give rise to unwanted cohomology. To kill this condition, we must introduce a second-order antifield $\bar{\phi}^*(x, t)$. After passage to jet space, the equation of motion and the time-independence condition translate into

$$\sum_\mu \phi_{,m+2\mu}(t) + \omega^2 \phi_{,m}(t) = 0,$$

$$D_t \phi_{,m}(t) \equiv \dot{\phi}_{,m}(t) - \sum_\mu \dot{\phi}^\mu(t)\phi_{,m+\mu}(t) = 0.\quad (9.3)$$

We introduce anti-jets $\phi^*_{,m}(t), \bar{\phi}_{,m}(t)$ and $\bar{\phi}^*_{,m}(t)$ and the KT differential $\delta$ to implement these conditions:

$$\delta \phi_{,m}(t) = 0,$$

$$\delta \phi^*_{,m}(t) = \sum_\mu \phi_{,m+2\mu}(t) + \omega^2 \phi_{,m}(t),$$

$$\delta \bar{\phi}_{,m}(t) = D_t \phi_{,m}(t),$$

$$\delta \bar{\phi}^*_{,m}(t) = D_t \phi^*(t) - \sum_\mu \bar{\phi}_{,m+2\mu}(t) + \omega^2 \bar{\phi}_{,m}(t).\quad (9.4)$$
The classical cohomology group $H^0_{cl}(\delta)$ is spanned by of linear combinations of jets satisfying

$$\phi_m(t) = e^{ik\cdot q(t)}(ik)^m$$  \hspace{1cm} (9.5)

where $k^2 = \omega^2$, $k \cdot q = k_\mu q^\mu$ and the power $k^m$ is defined in analogy with (5.6). It is hardly surprising that the Taylor series can be summed, giving

$$\phi(x,t) = e^{ik\cdot q(t)}\sum_m \frac{1}{m!}(ik)^m(x - q(t))^m$$

$$= e^{ik\cdot q(t)}e^{ik\cdot(x-q(t))}$$

$$= e^{ik\cdot x}. \hspace{1cm} (9.6)$$

The physical Hamiltonian $H$ is defined as in Equation (8.19). The classical phase space $H^0_{cl}(\delta)$ is thus the space of plane waves $e^{ik\cdot x}$, cf (9.6), and trajectories $q^\mu(t) = u^\mu t + a^\mu$. The energy is given by

$$[H, e^{ik\cdot x}] = k_\mu \dot{q}^\mu(t)e^{ik\cdot x} = k_\mu u^\mu e^{ik\cdot x},$$

$$[H, q^\mu(t)] = i\dot{q}^\mu(t). \hspace{1cm} (9.7)$$

This is a covariant description of phase space, because the energy $k_\mu u^\mu$ is Poincaré invariant.

We now quantize the theory before imposing dynamics. To this end, we introduce the canonical momenta $\pi^m(t)$, $\pi^m_*(t)$, $\bar{\pi}^m(t)$, $\bar{\pi}^m_*(t)$ for the jets and antijets, and $p_\mu(t)$ and $p^\mu_*(t)$ for the observer’s trajectory and its antifield. Since the jets also depend on the parameter $t$, we can define their Fourier components as in (5.13). The Fock vacuum $|0\rangle$ is defined to be annihilated by the negative frequency modes of the jets and antijets, and the quantum Hamiltonian is still defined by (8.19), where double dots indicate normal ordering with respect to frequency, ensuring that $H|0\rangle = 0$.

The rest proceeds as in the end of Section 8. We can consider the one-quantum state with momentum $k$ over the true Fock vacuum, $|k\rangle = \exp(ik\cdot x)|0\rangle$. This state is not an energy eigenstate, because the Hamiltonian excites a quantum of the observers trajectory: $H|k\rangle = k_\mu u^\mu|k\rangle$. We may think of the observer’s trajectory as a classical variable and introduce the macroscopic reference state $|0; u, a\rangle$, on which $q^\mu(t)|0; u, a\rangle = (u^\mu t + a^\mu)|0; u, a\rangle$. We can then consider a state $|k; u, a\rangle = \exp(ik\cdot x)|k; u, a\rangle$ with one quantum over the reference state. The Hamiltonian gives $H|k; u, a\rangle = k_\mu u^\mu|k; u, a\rangle$. In particular, if $u^\mu = (1, 0, 0, 0)$, then the eigenvalue of the Hamiltonian is $k_\mu u^\mu = k_0$, as expected. Moreover, the lowest-energy condition (5.15) ensures that only quanta with positive energy will be excited; if $k_\mu u^\mu < 0$ then $|k; u, a\rangle = 0$. 

10 MCCQ: Gauge symmetries

In the previous sections MCCQ was applied to the free scalar field. However, it is mainly useful for theories with gauge symmetries, due to its connection with the representation theory of gauge algebras developed earlier. We now come to this case, and assume that there are some relations between the EL equations (6.1). In other words, let there be identities of the form

\[ r^\alpha_a \mathcal{E}_\alpha \equiv 0, \]  

where the \( r^\alpha_a \) are some functionals of \( \phi^\alpha \). The zeroth cohomology group \( H^0_{cl}(\delta) = C(\mathcal{Q})/\mathcal{N} = C(\Sigma) \) is not changed, but the higher cohomology groups no longer vanish, since \( \delta(r^\alpha_a \phi^\alpha_\alpha) = r^\alpha_a \mathcal{E}_\alpha \equiv 0 \). The standard method to kill this unwanted cohomology is to introduce a bosonic second-order antifield \( \zeta_a \), so that \( r^\alpha_a \phi^\alpha_\alpha = \delta \zeta_a \) is KT exact. The differential (6.4) is thus modified to read

\[ \delta \phi^\alpha = 0, \]
\[ \delta \phi^*_\alpha = \mathcal{E}_\alpha, \]
\[ \delta \zeta_a = r^\alpha_a \phi^*_\alpha. \]  

By introducing canonical momenta \( \chi^a = \delta / \delta \zeta_a \) for the second-order antifields, we can write the KT differential as a bracket, \( \delta F = [Q_{KT}, F] \), where the full KT operator is

\[ Q_{KT} = \mathcal{E}_\alpha \pi^\alpha_s + r^\alpha_a \phi^*_\alpha \chi^a. \]  

\( Q_{KT} \) is an operator in the extended phase space \( \mathcal{P}^* \) with basis \( (\phi^\alpha, \pi^\alpha, \phi^*_\alpha, \pi^\beta, \zeta_a, \chi^b) \), and \( \{Q_{KT}, Q_{KT}\} = 0 \).

The identity (10.1) implies that \( J_a = r^\alpha_a \pi^\alpha \) generate a Lie algebra under the Poisson bracket. Namely, all \( J_a \)'s preserve the action, because

\[ [J_a, S] = r^\alpha_a [\pi^\alpha, S] = r^\alpha_a \mathcal{E}_\alpha \equiv 0, \]  

and the bracket of two operators which preserve some structure also preserves the same structure. We will only consider the case that the \( J_a \)'s generate a proper Lie algebra \( g \) as in (2.6). The formalism extends without too much extra work to the more general case of structure functions \( f_{abc}(\phi) \), but we will not need this complication here. It follows that the functions \( r^\alpha_a \) satisfy the identity

\[ \partial_\beta r^\alpha_b r^\beta\alpha - \partial_\beta r^\alpha_a r^\beta\alpha = f_{abc} r^\alpha_c. \]  

The Lie algebra \( g \) also acts on the antifields:

\[ [J_a, \phi^\alpha] = r^\alpha_a, \]
\[ [J_a, \phi^*_\alpha] = - \partial_\alpha r^\beta a \phi^*_\beta \]
\[ [J_a, \zeta_b] = f_{abc} \zeta_c. \]
In particular, it follows that $\phi^*_\alpha$ carries a $\mathfrak{g}$ representation because it transforms in the same way as $\pi_\alpha$ does.

Classically, it is always possible to reduce the phase space further, by identifying points on $\mathfrak{g}$ orbits. To implement this additional reduction, we introduce ghosts $c^a$ with anti-field number $\text{afn}(c^a) = -1$, and ghost momenta $b_a$ satisfying $\{b_a, c^b\} = \delta^b_a$. The Lie algebra $\mathfrak{g}$ acts on the ghosts as $[J_a, c^b] = -f_{abc}^c$. The full extended phase space, still denoted by $\mathcal{P}^*$, is spanned by $(\phi^\alpha, \pi^\beta, \phi^*_\alpha, \pi^*_\beta, \zeta^a, \chi^b, c^a, b_a)$. The generators of $\mathfrak{g}$ are thus identified with the following vector fields in $\mathcal{P}^*$:

$$J_a = r^a_{\alpha} \pi_\alpha - \partial_\alpha r^\beta_a \phi^*_\alpha \pi^\beta_a + f_{ab}^c \zeta^b c^b - f_{abc}^d b_d c^d$$

$$= J^\text{field}_a + J^\text{ghost}_a,$$

where $J^\text{ghost}_a = -f_{abc}^d b_d c^d$ and $J^\text{field}_a$ is the rest.

Now define the longitudinal derivative $d$ by

$$dc^a = -\frac{1}{2} f_{bc}^a c^b c^c,$$

$$d\phi^\alpha = r^\alpha_a c^a,$$

$$d\phi^*_\alpha = \partial_\alpha r^\beta_a \phi^*_\beta c^a,$$

$$d\zeta^a = -f_{ab}^c \zeta^b c^c.$$  (10.8)

The longitudinal derivative can be written as $dF = [Q_{\text{Long}}, F]$ for every $F \in C(Q^*)$, where

$$Q_{\text{Long}} = J^\text{field}_a c^a - \frac{1}{2} f_{ab}^c c^a c^b c^c = J^\text{field}_a c^a + \frac{1}{2} J^\text{ghost}_a c^a.$$  (10.9)

We note that $Q_{\text{Long}}$ can be considered as smeared gauge generators, $J_X = X^a J_a$, where the smearing function $X^a$ is the fermionic ghost $c^a$:

$$Q_{\text{Long}} = J^\text{field}_c + \frac{1}{2} J^\text{ghost}_c.$$  (10.10)

One verifies that $d^2 = 0$ when acting on the fields and antifields by means of the identify (10.5) and the Jacobi identities for $\mathfrak{g}$. Moreover, it is straightforward to show that $d$ anticommutes with the KT differential, $d\delta = -\delta d$; the proof is again done by checking the action on the fields. Hence we may define the nilpotent BRST derivative $s = \delta + d$,

$$sc^a = -\frac{1}{2} f_{bc}^a c^b c^c,$$

$$s\phi^\alpha = r^\alpha_a c^a,$$

$$s\phi^*_\alpha = \mathcal{E}_\alpha + \partial_\alpha r^\beta_a \phi^*_\beta c^a,$$

$$s\zeta^a = r^\alpha_a \phi^*_\alpha - f_{ab}^c \zeta^b c^b.$$  (10.11)
Nilpotency immediately follows because $s^2 = \delta^2 + d\delta + d^2 = 0$. The BRST operator can be written in the form $sF = [Q_{BRST}, F]$ with

$$Q_{BRST} = Q_{KT} + Q_{Long}$$
$$= \mathcal{E}_\alpha \pi^\alpha + r_a^\alpha \phi^* \chi^a + J^\text{field}_a c^a + \frac{1}{2} J^\text{ghost}_a c^a$$
$$= -\frac{1}{2} f_{ab}^c c^a \epsilon^b c^b + r_a^\alpha c^a \pi_\alpha + (\mathcal{E}_\alpha + \partial_\alpha \delta^a \phi^* c^a)\pi^\alpha$$
$$+ (r_a^\alpha \phi^*_\alpha - f_{ab}^c \epsilon^b \chi^a)\chi^a.$$  

(10.12)

In non-covariant quantization, we single out a privileged variable $t$ among the $\alpha$'s, and declare it to be time. In the absence of gauge symmetries, the BRST operator reduces to the KT operator (10.3), which is already normal ordered and hence nilpotent on the quantum level. The question is whether the full BRST operator also has this property. The dangerous part is the longitudinal operator

$$Q_{Long} = \int dt \left\{ J^\text{field}_a(t) : c^a(t) + \frac{1}{2} J^\text{ghost}_a(t) c^a(t) : \right\}.  

(10.13)$$

which ceases to be nilpotent unless the normal-ordered gauge generators $J_a(t) = :J^\text{field}_a(t): + :J^\text{ghost}_a(t):$ generate the algebra (2.6) without additional quantum corrections. If such an extension arises, the BRST operator ceases to be nilpotent. However, the situation is even worse. Not only do quantum effects generically ruin nilpotency of the BRST operator, but they make the gauge generators ill defined. However, it is possible to regularize the theory formulated in terms of Taylor data, in such a way that the full gauge symmetry of the original model is preserved, and the regularized gauge generators are well-defined operators. The price to pay is the appearance of an anomaly.

The next step in Section 8 was to introduce the observer’s trajectory, expand all fields in a Taylor series around it, and quantize in the space of Taylor data histories. The motivation was mainly aesthetic; by adding the observer’s trajectory, it is possible to write down a covariant expression (8.16) for the Hamiltonian, namely as the operator which translates the fields relative to the observer. However, it is in the presence of gauge symmetries that this construction becomes indispensable.

Thus, we reformulate the classical theory in jet coordinates. To the fields $c^a(x)$, $\phi^\alpha(x)$, $\phi^*_\alpha(x)$ and $\zeta_a(x)$ we associates $p$-jets $c^a_{m}(t)$, $\phi^a_{m}(t)$, $\phi^*_a_{m}(t)$ and $\zeta_a_{m}(t)$, with canonical momenta $b^a_{m}(t)$, $\pi^a_{m}(t)$, $\pi^*_a_{m}(t)$ and $\chi^a_{m}(t)$. We also introduce extra antifields $\hat{\phi}^a_{m}(t)$ etc. to eliminate the $t$-dependence, but as in [19], they will not be written down explicitly.

To be concrete, consider the case that the symmetry is the DGRO algebra (4.1).
To each symmetry, we assign ghosts as in the following table:

| Gen        | Smear        | Ghost           | Momentum        | $Q_{Long}$ |
|------------|--------------|-----------------|-----------------|-----------|
| Diffeomorphisms | $\xi^\mu(x)$ | $c^\mu_{diff}(x,t)$ | $b^\mu_{diff}(x,t)$ | $Q^\mu_{diff}$ |
| Gauge      | $J_xX^a(x)$  | $c^a_{gauge}(x,t)$ | $b^a_{gauge}(x,t)$ | $Q^a_{gauge}$ |
| Reparametrizations | $L_f f(t)$  | $c_{rep}(t)$    | $b^{rep}(t)$    | $Q^{rep}_{Long}$ |

The BRST operator is $Q_{BRST} = Q_{Long} + Q_{KT}$, where the longitudinal operator is given by the prescription \((10.10)\). For brevity, we only write down the formulas for the fields $\phi^\alpha(x,t)$ and the ghosts; the antifields do of course give rise to additional terms.

\[
Q^\text{diff}_{Long} = - \int d^N x \int dt \left\{ (c^\mu_{diff}(x,t)) \partial_\mu \phi^\alpha(x,t) \\
+ \partial_\nu c^\mu_{diff}(x,t) T^{\alpha\nu}_{\beta\mu} \phi^\beta(x,t) \pi_\alpha(x,t) \\
+ c^\mu_{diff}(x,t) \partial_\mu c^\nu_{diff}(x,t) b^\nu_{diff}(x,t) \right\}, \\
Q^\text{gauge}_{Long} = - \int d^N x \int dt \left\{ c^a_{gauge}(x,t) J^a_{\beta a} \phi^\beta(x,t) \pi_\alpha(x,t) \\
+ \frac{1}{2} f_{ab} c^a_{gauge}(x,t) c^b_{gauge}(x,t) b^a_{gauge}(x,t) \right\}, \\
Q^{\text{rep}}_{Long} = - \int d^N x \int dt \left\{ (c_{rep}(t)) \partial_t \phi^\alpha(x,t) \\
+ \lambda (c_{rep}(t) - i c_{rep}(t)) \phi^\alpha(x,t) \pi_\alpha(x,t) \\
- \int dt c_{rep}(t) \dot{c}_{rep}(t) b^{rep}(t). \right\}
\]

These formulas assume that the field $\phi^\alpha(x)$ transforms as a tensor field. There is an additional term if the field is a connection, but this terms does not lead to any complications. After passage to jet space and normal ordering, we use the prescription \((10.10)\) to find the longitudinal derivative, i.e. $\partial_m^N \xi^\mu \rightarrow c^\mu_{diff,m}$, $\partial_m X^a \rightarrow c^a_{gauge,m}$.
and $f \to c_{\text{rep}}$:

\[
Q_{\text{Long}}^{\text{diff}} = \int dt \left\{ c_{\text{diff},0\mu}(t) - \sum_{|n| \leq |m| \leq p} T_{\beta \alpha}^{\gamma \delta}(c_{\text{diff}}(t)) : \phi_{\beta}^{\gamma}(t) \pi_{\alpha}^{\delta}(t) : \\
- \sum_{|n| \leq |m| \leq p} : T_{\nu \mu}^{\gamma \delta}(c_{\text{diff}}(t)) c_{\text{diff},n}(t) b_{\beta}^{\mu}(m)(t) : \right\},
\]

\[
Q_{\text{Long}}^{\text{gauge}} = - \int dt \left\{ \sum_{|n| \leq |m| \leq p} J_{\beta \alpha}^{\gamma \delta}(c_{\text{gauge}}(t)) : \phi_{\beta}^{\gamma}(t) \pi_{\alpha}^{\delta}(t) : \\
- \frac{1}{2} \sum_{|n| \leq |m| \leq p} : J_{\beta \alpha}^{\gamma \delta}(c_{\text{gauge}}(t)) c_{\text{gauge},n}(t) b_{\beta}^{\mu}(m)(t) : \right\},
\]

\[
Q_{\text{Long}}^{\text{rep}} = - \int dt \left\{ \sum_{|m| \leq p} c_{\text{rep}}(t) : \phi_{\beta}^{\gamma}(t) \pi_{\alpha}^{\delta}(t) : \\
+ \lambda \sum_{|m| \leq p} (c_{\text{rep}}^{\gamma})(t) - i c_{\text{rep}}(t) : \phi_{\beta}^{\gamma}(t) \pi_{\alpha}^{\delta}(t) : \\
+ : c_{\text{rep}}(t) c_{\text{rep}}^{\gamma}(t) b_{\beta}^{\mu}(m)(t) : \right\}.
\]

The matrices are given by (cf. (5.10))

\[
T_{n}^{m}(c_{\text{diff}}(t)) \equiv (T_{\beta \alpha}^{\gamma \delta}(c_{\text{diff}}(t))) = \sum_{\mu \nu} \left( \begin{array}{c} n \\ m \end{array} \right) c_{\text{diff},n-m+\nu}(t) T_{\mu}^{\nu} \\
+ \sum_{\mu} \left( \begin{array}{c} n \\ m - \mu \end{array} \right) c_{\text{diff},n-m+\mu}(t) - \sum_{\mu} \delta_{n-m,\mu} c_{\text{diff},0}(t),
\]

\[
J_{n}^{m}(c_{\text{gauge}}(t)) \equiv (J_{\beta \alpha}^{\gamma \delta}(c_{\text{gauge}}(t))) = \left( \begin{array}{c} n \\ m \end{array} \right) c_{\text{gauge},n-m}(t) J_{\alpha}.
\]

$T_{\nu \mu}^{\gamma \delta}$ and $J_{\nu \mu}^{\gamma \delta}$ denote the specializations of $T_{\beta \alpha}^{\gamma \delta}$ and $J_{\beta \alpha}^{\gamma \delta}$ to the adjoint representations; $\sum_{m} T_{\nu \mu}^{\gamma \delta}(c) c_{\nu}^{\beta} = (c' c_{\mu}^{\beta})_{\nu}$ and $\sum_{m} J_{\nu \mu}^{\gamma \delta}(c) c_{\nu}^{\beta} = (c' c_{\beta}^{\gamma})_{\nu}$.

The condition for $Q_{\text{Long}}^{\text{diff}} = 0$, and thus $Q_{\text{BRST}}^{2} = 0$, is that the algebra generated by the normal-ordered gauge generators is anomaly free. However, even if this condition fails, which is the typical situation, everything is not lost. The KT operator is still nilpotent, and we can implement dynamics as the KT cohomology in the extended phase space without ghosts. The physical phase space now grows, because some gauge degrees of freedom become physical upon quantization.

In the next two sections, we apply this formalism to some well-known theories.

11 The free Maxwell field

The Maxwell field $A_{\mu}(x)$ transforms as a vector field under the Poincaré group and as a connection under the gauge algebra $\text{map}(N, u(1))$, whose smeared generators
are denoted by \( J = \int d^4 x \ X(x) J(x) \):

\[
[J, A_\mu(x)] = \partial_\mu X(x).
\]

(11.1)

We use the Minkowski metric \( \eta_{\mu\nu} \) and its inverse \( \eta^{\mu\nu} \) to freely raise and lower indices, e.g. \( F^{\mu\nu} = \eta^{\mu\rho} \eta^{\nu\sigma} F_{\rho\sigma} \). The field strength \( F^{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \) transforms in the adjoint representation, i.e. trivially. The action

\[
S = \frac{1}{4} \int d^4 x \ F^{\mu\nu}(x) F_{\mu\nu}(x)
\]

(11.2)

leads to the equations of motion

\[
E_\mu(x) \equiv - \frac{\delta S}{\delta A_\mu(x)} = \partial_\mu F^{\mu\nu}(x) = 0.
\]

(11.3)

The Maxwell equations are not all independent, because of the constraints

\[
\partial_\mu E_\mu(x) = 0.
\]

(11.4)

We are thus instructed to introduce the following fields: the first-order antifield \( A_\mu^*(x) \) for the EL equation \( \partial_\nu F^{\mu\nu}(x) = 0 \); the second-order antifield \( \zeta(x) \) for the identity \( \partial_\mu \partial_\nu F^{\mu\nu}(x) \equiv 0 \); and the ghost \( c(x) \) to identify fields related by a gauge transformation of the form (11.1).

The BRST operator \( s \) acts as

\[
sc(x) = 0,
\]

\[
sA_\mu(x) = \partial_\mu c(x),
\]

\[
sA_\nu^*(x) = \partial_\nu F^{\mu\nu}(x),
\]

\[
s\zeta(x) = \partial_\mu A_\mu^*(x),
\]

(11.5)

We check that \( s^2 = 0 \) and \( sF^{\mu\nu} = s\partial_\mu A_\nu^* = 0 \), so the kernel of \( s \) is spanned by \( c \), the field strengths \( F^{\mu\nu} \), and \( \partial_\mu A_\nu^* \). \( s \) is generated by the ideals \( \partial_\mu c, \partial_\nu F^{\mu\nu}, \) and \( \partial_\mu A_\nu^* \). Hence \( H^0_\text{cl}(s) \) consists of the gauge-invariant parts of \( A_\mu \) (i.e. \( F^{\mu\nu} \)) which solve the Maxwell equations, as expected.

Introduce canonical momenta \( E^\mu(x), \ E_\nu^*(x), \chi(x) \) and \( b(x) \), defined by the following non-zero brackets:

\[
[E^\mu(x), A_\nu(x')] = \delta^\mu_\nu \delta(x - x'),
\]

\[
\{E_\nu^*(x), A_\mu^*(x')\} = \delta^\nu_\mu \delta(x - x'),
\]

\[
[\chi(x), \zeta(x')] = \delta(x - x'),
\]

\[
\{b(x), c(x')\} = \delta(x - x').
\]

(11.6)

It should be emphasized that \( E^\mu = \delta/\delta A_\mu \) is the conjugate of the gauge potential in history space, and not yet related to the electric field \( F^{\mu\nu} \). We could introduce
the condition $E^\mu \approx F^{\mu 0}$ as a constraint in the history phase space, turning the Maxwell equations into second class constraints. By keeping dynamics as a first-class constraint no such condition, which would ruin covariance, is necessary. The BRST operator can explicitly be written as

$$Q_{BRST} = \int d^4x \left\{ \partial_\mu c(x)E^\mu(x) + \partial_\nu F^{\mu \nu}(x)E^\nu_\mu(x) + \partial_\mu A^\mu_\alpha(x)\chi(x) \right\}. \tag{11.7}$$

The physical content of the theory is clearer in Fourier space. The BRST operator

$$Q_{BRST} = \int d^Nk \left\{ k_\mu c(k)E^\mu(-k) + (k^\mu k_\nu A^\nu(k) - k^\nu k_\nu A^\mu(k))E^\nu_\mu(-k) + k_\mu A^\mu_\alpha(k)\chi(-k) \right\}, \tag{11.8}$$

acts on the Fourier modes as

$$sc(k) = 0,$$
$$sA_\mu(k) = k_\mu c(k),$$
$$sA^\mu_\alpha(k) = k^\mu k_\nu A^\nu(k) - k^\nu k_\nu A^\mu(k),$$
$$s\zeta(k) = k_\mu A^\mu_\alpha(k). \tag{11.9}$$

We distinguish between two cases:

1. $k^2 = \omega^2 \neq 0$, say $k = (\omega, 0, 0, 0)$. Then $sc = 0$, $sA_0 = \omega c$, $sA_1 = sA_2 = sA_3 = 0$, $sA^0_\alpha = \omega^2 A_0 - \omega \omega A_0 = 0$, $sA^1_\alpha = \omega^2 A_1$, $sA^2_\alpha = \omega^2 A_2$, $sA^3_\alpha = \omega^2 A_3$ and $s\zeta = \omega A^0_\alpha$. The kernel is thus spanned by $c$, $A_1$, $A_2$, $A_3$ and $A^0_\alpha$. Since $\ker s = \text{im } s$ there is no cohomology.

2. $k^2 = 0$, say $k = (k_0, 0, 0, k_0)$. Then $sc = 0$, $sA_0 = sA_3 = k_0 c$, $sA_1 = sA_2 = 0$, $sA^0_\alpha = sA^3_\alpha = k_0 k_\nu A^\nu$, $sA^1_\alpha = sA^2_\alpha = 0$ and $s\zeta = k_\mu A^\mu_\alpha$. The kernel is thus spanned by $c$, $A_1$, $A_2$, $k^\mu A_\mu = A_0 - A_3$, $A^1_\alpha$, $A^2_\alpha$, and $k_\mu A^\mu_\alpha = A^0_\alpha - A^3_\alpha$. The image is spanned by $c$, $k^\mu A_\mu$ and $k_\mu A^\mu_\alpha$, which factor out in cohomology. We are left with two physical polarizations $A_1$ and $A_2$.

We here assumed that the momenta factor out in cohomology. As for the scalar field in Section 7 and the harmonic oscillator in [18], this is not quite true. There is unwanted cohomology because the Hessian is singular. However, this problem has nothing to do with gauge invariance.

We now quantize in the history phase space before introducing dynamics by passing to the BRST cohomology. We single out one direction $x^0$ as time, and take
the Hamiltonian to be the generator of rigid time translations,
\[ H = -i \int d^4x \left\{ \partial_\mu A_\mu(x) E^\mu(x) + \partial_\mu A^\mu_\mu(x) E^{\ast}_\mu(x) \right\} \]
\[ + \partial_0 \zeta(x) \chi(x) + \partial_0 c(x)b(x) \}
\[ = \int d^N k \frac{k_0}{k} \left\{ A_\mu(k)E^\mu(-k) + A^{\ast}_\mu(k)E^{\ast}_\mu(-k) \right\} + \zeta(k)\chi(-k) + c(k)b(-k) . \]

Note that at this stage we break Poincaré invariance, since the Hamiltonian treats the \( x^0 \) coordinate differently from the other \( x^\mu \). Quantize by introducing a Fock vacuum \( |0\rangle \) satisfying
\[ A_\mu(k)|0\rangle = E^\mu(k)|0\rangle = A^{\ast}_\mu(k)|0\rangle = E^{\ast}_\mu(k)|0\rangle = \]
\[ \zeta(k)|0\rangle = \chi(k)|0\rangle = c(k)|0\rangle = b(k)|0\rangle = 0 , \]
for all \( k \) such that \( k_0 < 0 \).

At this point we want to pass to BRST cohomology. There might be problems with normal ordering, but in fact the BRST operator (11.8) is already normal ordered. This is because the generator of \( u(1) \) gauge transformations
\[ J_X = - \int d^4x X(x) \partial_\mu E^\mu(x) \]
is itself already normal ordered. There are thus no anomalies, and the BRST operator (11.8) remains nilpotent. We define the BRST state cohomology as the space of physical states, where a state is physical if it is BRST closed, \( Q_{BRST} |phys\rangle = 0 \), and two physical states are equivalent if they differ by a BRST exact state, \( |phys\rangle \sim |phys\rangle \) if \( |phys\rangle - |phys\rangle' = Q_{BRST} |\rangle \).

The rest proceeds as for the harmonic oscillator \[18\] or the free scalar field in Section 7. After adding a small perturbation to make the Hessian invertible, all momenta vanish in cohomology, and only the transverse polarizations \( e^\mu A_\mu(k) = 0 \) with \( e^\mu k_\mu = 0 \) and \( e^0 = 0 \) survive. A basis for the history Hilbert space consists of multi-quanta states
\[ e_1^\mu A_\mu_1(k(1)) \ldots e_n^\mu A_\mu_n(k(n)) |0\rangle \]
where \( k^{(j)}_\mu k^{(j)\mu} = 0 \) and \( k_0^{(j)} > 0 \). The energy is given by \( H = k^{(1)}_0 + \ldots + k^{(n)}_0 \).

The gauge generators (11.12) act in a well-defined manner, in fact trivially, on the Hilbert space, because \( e^\mu k_\mu = 0 \).

As in Section 8 we want to give a completely covariant description of the Hamiltonian. Therefore we pass to jet data, e.g.
\[ A_\mu(x) = \sum_{|m| \leq p} \frac{1}{m!} A_{\mu,m}(t)(x - q(t))^m. \]
The equations of motion (11.3) translate into

$$\sum_\nu F_{\mu\nu,m\nu}(t) = 0,$$  \hfill (11.15)

and the constraint (11.4) becomes

$$\sum_\mu \mathcal{E}_{\mu,m}(t) = \sum_\mu F_{\mu,m+\mu\nu}(t) \equiv 0,$$  \hfill (11.16)

where the field strength is

$$F_{\mu\nu,m}(t) = A_{\mu,m+\nu}(t) - A_{\nu,m+\mu}(t) = 0.$$  \hfill (11.17)

We introduce jets also for the antifields and for the ghost, denoted by $A_{\mu,m}(t)$, $\zeta_{m}(t)$, and $c_{m}(t) \equiv c^{auge}_{m}(t)$. The BRST differential $s$ which implements all these conditions is defined by

$$s c_{m}(t) = 0,$$

$$s A_{\mu,m}(t) = c_{m+\mu}(t),$$

$$s A_{\mu,m}(t) = \sum_\nu F_{\mu,n,m+\nu}(t),$$

$$s \zeta_{m}(t) = \sum_\mu A^\mu_{*,m+\mu}(t).$$  \hfill (11.18)

Moreover, we demand that the Taylor series does not depend on the parameter $t$, which gives rise to conditions of the type

$$D_{t}A_{\mu,m}(t) \equiv \dot{A}_{\mu,m}(t) - \sum_\nu \ddot{q}_{\nu}(t)A_{\mu,n+\nu}(t) = 0.$$  \hfill (11.19)

As in (8.6), we need to double the number of antifields and introduce an additional differential $\sigma$ to remove these conditions in cohomology. Thus we introduce antifields $c_{m}(t)$, $\tilde{A}_{\mu,m}(t)$, $\tilde{A}_{\mu,m}(t)$, $\tilde{\zeta}_{m}(t)$ and set

$$\sigma c_{m}(t) = \dot{c}_{m}(t),$$

$$\sigma \tilde{A}_{\mu,m}(t) = \dot{\tilde{A}}_{\mu,m}(t),$$

$$\sigma \tilde{A}_{\mu,m}(t) = \dot{\tilde{A}}_{\mu,m}(t),$$

$$\sigma \tilde{\zeta}_{m}(t) = \dot{\tilde{\zeta}}_{m}(t),$$

$$\sigma c_{m}(t) = \sigma A_{\mu,m}(t) = \sigma A_{\mu,m}(t) = \sigma \zeta_{m}(t) = 0.$$  \hfill (11.20)

Clearly, $\sigma^2 = 0$. We also extend the definition of the BRST differential $s$ to the
barred antifields:

\[ s \bar{c}, m(t) = 0, \]
\[ s \bar{A}_\mu, m(t) = -\bar{c}, m(t) + \sum_\nu (\bar{A}_\nu, m(t) - \bar{A}_\mu, m + \nu(t)), \]
\[ s \bar{A}_{*, m}(t) = -\sum_\mu \bar{A}_\mu, m + \mu(t). \] (11.21)

That \( s^2 = 0 \) follows in the same way as for (11.18). Moreover, we verify that \( s \sigma = -\sigma s \), and hence \( s + \sigma \) is nilpotent.

The classical cohomology group \( H_0^{cl}(s + \sigma) \) consists of linear combinations of jets

\[ A_{\mu, m}(t) = \epsilon_\mu(t) e^{ik \cdot q(t)} (ik)^m \] (11.22)

where \( k^2 = 0 \) and the polarization vector \( \epsilon_\mu(t) \) is perpendicular both to the photon momentum and the observer’s trajectory:

\[ \epsilon_\mu(t) k^\mu = \epsilon_\mu(t) \dot{q}^\mu(t) = 0. \] (11.23)

The latter is evidently equivalent to the non-covariant condition \( \epsilon^0 = 0 \). Moreover, \( k \cdot q = k_\mu q^\mu \). The Taylor series (11.14) can be summed in the same way as for the scalar field (9.6).

We now quantize the theory before imposing dynamics. To this end, we introduce the canonical momenta for all jets and antijets, and \( p_\nu(t) \) and \( p_{\nu}^*(t) \) for the observer’s trajectory and its antifield. The defining relations are

\[ [E^\mu, m(t), A_{\nu, n}(t')] = \delta_\nu^\mu \delta_n^m \delta(t - t'), \]
\[ \{E^\mu, m(t), A^*_{\nu, n}(t')\} = \delta_\nu^\mu \delta_n^m \delta(t - t'), \]
\[ [\chi, m(t), \zeta, n(t')] = \delta_n^m \delta(t - t'), \]
\[ \{b, m(t), c, n(t')\} = \delta_n^m \delta(t - t'), \]
\[ [p_\nu(t), q^\mu(t')] = \delta_\nu^\mu \delta(t - t'). \] (11.24)

Since the jets also depend on the parameter \( t \), we can define their Fourier components as in (5.13). The Fock vacuum (11.11) is replaced by a new vacuum, also denoted by \( |0 \rangle \), which is defined to be annihilated by the negative frequency modes. The quantum Hamiltonian is still defined by (8.19), where double dots indicate normal ordering with respect to frequency, ensuring that \( H |0 \rangle = 0. \)

It remains to check that the algebra of \( u(1) \) gauge transformations acts in a well-defined manner before we can pass to the BRST cohomology. Since a gauge potential transforms as

\[ [J_X, A_{\mu, m}(t)] = \partial_{m+\mu} X(q(t)) \] (11.25)
we have

$$\mathcal{J}_X = \sum_{|m| \leq p} \sum_{\mu} \int dt \, \partial_{m+\mu} X(q(t))E^{\mu,m}(t).$$

(11.26)

There are no contributions from the antifields, since $A^{\mu}_s$, $\zeta$ and $c$ all transform trivially under $\text{map}(N,u(1))$. The prescription (10.10) gives

$$Q_{\text{Long}} = \sum_{|m| \leq p} \sum_{\mu} \int dt \, c_{m+\mu}(t)E^{\mu,m}(t).$$

(11.27)

The expressions (11.26) and (11.27) are evidently normal ordered as they stand, and consequently there are no gauge anomalies.

The rest proceeds as for the scalar field.

12 Gravity

Finally we are ready to apply the MCCQ formalism to general relativity. For simplicity we consider only pure gravity. The only field is the symmetric metric $g_{\mu\nu}(x)$. The inverse $g^{\mu\nu}$, the determinant $g = \det(g_{\mu\nu})$, the Levi-Civita connection $\Gamma^\rho_{\nu\rho}$, Riemann’s curvature tensor $R^\rho_{\sigma\mu\nu}$, the Ricci tensor $R_{\mu\nu}$, the scalar curvature $R = g^{\mu\nu}R_{\mu\nu}$ and the Einstein tensor $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$ are defined as usual. The covariant derivative is

$$\nabla_{\mu} = \partial_{\mu} + \Gamma^\rho_{\nu\mu}T^\nu_{\rho},$$

(12.1)

where $T^\mu_{\rho}$ are finite-dimensional matrices satisfying $gl(N)$ (5.2).

The Einstein action

$$S_E = \frac{1}{16\pi} \int d^4x \, \sqrt{g(x)}R(x).$$

(12.2)

leads to Einstein’s equation of motion

$$G^{\mu\nu}(x) = 0,$$

(12.3)

which is subject to the identity

$$\nabla_{\nu}G^{\mu\nu}(x) \equiv 0$$

(12.4)

We introduce a fermionic antifield $g^{\mu\nu}_s(x)$ for (12.3), a bosonic second-order antifield $\zeta^\mu(x)$ for (12.4), and a ghost $c^{\mu}_{\text{diff}}(x)$ to eliminate diffeomorphisms. The total field content in the extended history phase space is thus

| afn | Field | Momentum | Parity |
|-----|-------|----------|--------|
| $-1$ | $\zeta^\mu_{\text{diff}}(x)$ | $\pi_{\mu\nu}(x)$ | $F$ |
| $0$ | $g_{\mu\nu}(x)$ | $\pi_{\mu\nu}(x)$ | $B$ |
| $1$ | $g^{\mu\nu}_s(x)$ | $\pi^{\mu\nu}(x)$ | $F$ |
| $2$ | $\zeta^\mu(x)$ | $\chi_{\mu}(x)$ | $B$ |
The KT differential $\delta$ is defined by
\begin{align}
\delta c^\mu_{\text{diff}}(x) &= 0, \\
\delta g_{\mu\nu}(x) &= 0, \\
\delta g^\mu_{\nu}(x) &= G^{\mu\nu}(x), \\
\delta \zeta^\mu(\mathbf{x}) &= \nabla_\nu G^{\mu\nu}(x),
\end{align}
(12.5)
i.e. the KT operator is
\[
Q_{KT} = \int d^4x \left( G^{\mu\nu}(x) \pi^*_{\mu\nu}(x) + \nabla_\nu G^{\mu\nu}(x) \chi_\mu(\mathbf{x}) \right).
\]
(12.6)

The longitudinal operator was written down in (10.14), $Q_{\text{Long}} = Q_{\text{diff}}^\text{Long}$, and the BRST operator is the sum of the KT and the longitudinal operators, as usual. Of the originally ten degrees of freedom $g_{\mu\nu}(x)$, the antifield $g^\mu_{\nu}(x)$ eliminates four and the ghost $c^\mu_{\text{diff}}(x)$ another four, leaving two graviton polarizations in the BRST cohomology.

We now quantize as usual by passing to jet space, introducing a Fock vacuum that is annihilated by the negative frequency modes of all fields and antifields, and normal ordering. The fields are symmetric tensor fields, i.e. they correspond to the symmetric $gl(N)$ modules $S_\ell$ and $S^\ell$ in (5.18). The values of the parameters in (5.20) are
\begin{align*}
\text{Field} & | \rho | u | v | w | x | p \\
c^\mu_{\text{diff}}(x) & | S^1 | 1 | 0 | -1 | N | p + 1 \\
g_{\mu\nu}(x) & | S_2 | -(N + 2) | -1 | -(N + 1) | -N(N + 1)/2 | p \\
g^\mu_{\nu}(x) & | S^2 | (N + 2) | 1 | -(N + 1) | N(N + 1)/2 | p - 2 \\
\zeta^\mu(x) & | S^1 | -1 | 0 | 1 | -N | p - 3
\end{align*}
The parameters were written down in arbitrary dimension $N$ for generality, although we are primarily interested in the physical case $N = 4$. The last column is the truncation order for the corresponding jets.

The diffeomorphism anomalies are now read off from (5.19); for definiteness, we only consider $c_1$. Depending on whether we exclude the ghost $c^\mu_{\text{diff}}(x)$ or not, the abelian charge $c_1 = c_1^{\text{tot}}$ becomes $c_1^{\text{tot}} = c_1^{\text{field}} \equiv 1 + c^g_1 + c^{g^*}_1 + c^\zeta_1$ or $c_1^{\text{tot}} = c_1^{\text{field}} + c_1^{\text{ghost}}$, where
\begin{align}
c_1^{\text{ghost}} &= - \left( \frac{N + p + 1}{N} \right) - N \left( \frac{N + p + 2}{N + 2} \right), \\
c_1^g &= (N + 2) \left( \frac{N + p}{N} \right) + \frac{N(N + 1)}{2} \left( \frac{N + p + 1}{N + 2} \right), \\
c_1^{g^*} &= -(N + 2) \left( \frac{N + p - 2}{N} \right) - \frac{N(N + 1)}{2} \left( \frac{N + p - 1}{N + 2} \right), \\
c_1^\zeta &= \left( \frac{N + p - 3}{N} \right) + N \left( \frac{N + p - 2}{N + 2} \right). \tag{12.7}
\end{align}
It is clear that $c_{\text{tot}}^{\mu}$ does not vanish for generic $p$, with or without the ghost contribution (the leading term proportional to $p^{N+2}/(N+2)!$ does however vanish in the $p \to \infty$ limit). The longitudinal operator (10.15) thus acquires an anomaly, and we can only implement the KT cohomology. Hence the ghost plays no role and should be discarded.

The quantum KT operator becomes

$$Q_{\text{KT}} = \int dt \left\{ \sum_{|m| \leq p-2} : G_{\mu m}^{\mu}(t) \pi_{m}(t): ight\} + \sum_{|m| \leq p-3} : (\nabla_{\nu} G^{\mu\nu})_{m}(t) \chi_{m}(t):,$$

(12.8)

where $G_{\mu m}^{\mu}(t)$ and $(\nabla_{\nu} G^{\mu\nu})_{m}(t)$ are the corresponding jets. Note that the sums run up to $|m| = p-2$ and $p-3$, respectively, because Einstein’s equation is second order and the identity (12.4) is third order.

One difference compared to Minkowski space is that the geodesic equation depends on a dynamical field. In order to make the geodesic operator $G_{\mu}(t)$ transform tensorially under reparametrizations as well, we need to add an extra term. We can construct the following quantities from the metric and the observer’s trajectory:

1. The Levi-Civita connection $\Gamma^\nu_{\sigma\tau}(x,t) = \frac{1}{2} g^\nu_{\rho\sigma}(x,t) (\partial_{\tau} g_{\rho\sigma}(x,t) - \partial_{\sigma} g_{\rho\tau}(x,t)).$

2. The einbein $e(t) = \sqrt{g_{\mu\nu,0}(t) \dot{q}^\mu(t) \dot{q}^\nu(t)}.$

3. The reparametrization connection $\Gamma(t) = -e^{-1}(t) \dot{\epsilon}(t).$

The geodesic operator reads [16]

$$G_{\mu}(t) = e^{-1}(t) g_{\mu,0}(t) (\dot{q}^\mu(t) + \Gamma(t) \dot{q}^\nu(t) + \Gamma^\nu_{\sigma\tau,0}(t) \dot{q}^\sigma(t) \dot{q}^\tau(t)),$$

(12.9)

where $g_{\mu,0}(t)$ and $\Gamma^\nu_{\sigma\tau,0}(t)$ are the zero-jets corresponding to the metric and Levi-Civita connection, respectively. It is straightforward to check that (12.9) transforms nicely under the full DGRO algebra,

$$[L_\xi, G_{\nu}(t)] = -\partial_{\nu} \xi^\mu(q(t)) G_{\mu}(t),$$

$$[L_\xi, G_{\nu}(t)] = -\partial_{\nu} \xi^\mu(q(t)) G_{\mu}(t),$$

(12.10)

The contribution to the KT operator is thus $Q_{\text{KT}} = \int dt G_{\mu}(t) p^\mu(t)$, which eliminates the observer’s trajectory $q^\mu(t)$ in cohomology.

## 13 Finiteness conditions

In the previous section we applied the MCCQ formalism to gravity, and found a well-defined but anomalous action of the DGRO algebra. However, the passage to
the space of \( p \)-jets amounts to a regularization. The regularization is unique in that it preserves the full constraint algebra, but it must nevertheless be removed in the end. In order to reconstruct the original field by means of the Taylor series (5.5), we must take the limit \( p \to \infty \). A necessary condition for taking this limit is that the abelian charges have a finite limit.

Taken at face value, the prospects for succeeding appear bleak. When \( p \) is large, \( \binom{m+p}{p} \approx p^n/n! \), so the abelian charges (5.19) diverge; the worst case is \( c_1 \approx c_2 \approx p^{N+2}/(N+2)! \), which diverges in all dimensions \( N > -2 \). In [15] a way out of this problem was devised: consider a more general realization by taking the direct sum of operators corresponding to different values of the jet order \( p \). Take the sum of \( r+1 \) terms like those in (5.12), with \( p \) replaced by \( p, p-1, \ldots, p-r \), respectively, and with \( q \) and \( M \) replaced by \( q^{(i)} \) and \( M^{(i)} \) in the \( p-i \) term.

Such a sum of contributions arises naturally from the KT complex, because the antifields are only defined up to an order smaller than \( p \) (e.g. \( p - o_\alpha \) or \( p - c_\alpha \)). Denote the numbers \( u, v, w, x, y \) in the modules \( q^{(i)} \) and \( M^{(i)} \), defined as in (5.20), by \( u_i, v_i, w_i, x_i, y_i \), respectively. Of course, there is only one contribution from the observer’s trajectory. Then it was shown in [15], Theorem 3, that

\[
\begin{align*}
c_1 &= -U \binom{N + p - r}{N - r}, & c_2 &= -V \binom{N + p - r}{N - r}, \\
c_3 &= W \binom{N + p - r}{N - r}, & c_4 &= -X \binom{N + p - r}{N - r}, \\
c_5 &= Y \binom{N + p - r}{N - r},
\end{align*}
\]

where \( u_0 = U, v_0 = V, w_0 = W, x_0 = X \) and \( y_0 = Y \), provided that the following conditions hold:

\[
\begin{align*}
u_i - (-r) \binom{r - 1}{i - 1} w - (-r) \binom{r - 2}{i - 2} x &= (-r) \binom{r}{i} U, \\
v_i - 2(-r) \binom{r - 1}{i - 1} w - (-r) \binom{r - 2}{i - 2} x &= (-r) \binom{r}{i} V, \\
w_i - (-r) \binom{r - 1}{i - 1} x &= (-r) \binom{r}{i} W, \\
x_i &= (-r) \binom{r}{i} X, \\
y_i &= (-r) \binom{r}{i} Y.
\end{align*}
\]

The contributions from the observer’s trajectory have also been eliminated by antifields coming from the geodesic equation; this is not important in the sequel because these contributions were finite anyway.
Let us now consider the solutions to (13.2) for the numbers $x_i$, which can be interpreted as the number of fields and anti-fields. First assume that the field $\phi^\alpha_{m}(t)$ is fermionic with $x_F$ components, which gives $x_0 = x_F$. We may assume, by the spin-statistics theorem, that the EL equations are first order, so the bosonic antifields $\phi^*_{m}(t)$ contribute $-x_F$ to $x_1$. The barred antifields $\bar{\phi}^\alpha_{m}(t)$ are also defined up to order $p-1$, and so give $x_1 = -x_F$, and the barred second-order antifields $\bar{\phi}^*_{m}(t)$ give $x_2 = x_F$. Further assume that the fermionic EL equations have $x_S$ gauge symmetries, i.e. the second-order antifields $\zeta_{m}(t)$ give $x_2 = x_S$. In established theories, $x_S = 0$, but we will need a non-zero value for $x_S$. Finally, the corresponding barred antifields give $x_3 = -x_S$.

For bosons the situation is analogous, with two exceptions: all signs are reversed, and the EL equations are assumed to be second order. Hence $\phi^*_{m}(t)$ yields $x_2 = x_B$ and the gauge antifields $\bar{\zeta}_{m}(t)$ give $x_3 = -x_G$. Accordingly, the barred antifields are one order higher.

The situation is summarized in the following tables, where the upper half is valid if the original field is fermionic and the lower half if it is bosonic:

| $n$ | $\phi^\alpha_{m}(t)$ | $\phi^*_{m}(t)$ | $\bar{\phi}^\alpha_{m}(t)$ | $\bar{\phi}^*_{m}(t)$ | $\bar{\zeta}_{m}(t)$ | $\zeta_{m}(t)$ | Order | $x$ |
|-----|---------------------|-------------------|-----------------------------|------------------------|----------------------|----------------|-------|-----|
| 0   | $\phi^\alpha_{m}(t)$ | $\phi^*_{m}(t)$   | $\bar{\phi}^\alpha_{m}(t)$ | $\bar{\phi}^*_{m}(t)$ | $\bar{\zeta}_{m}(t)$ | $\zeta_{m}(t)$ | $p$   | $x_F$ |
| 1   | $\phi^\alpha_{m}(t)$ | $\phi^*_{m}(t)$   | $\bar{\phi}^\alpha_{m}(t)$ | $\bar{\phi}^*_{m}(t)$ | $\bar{\zeta}_{m}(t)$ | $\zeta_{m}(t)$ | $p-1$ | $-x_F$|
| 1   | $\phi^\alpha_{m}(t)$ | $\phi^*_{m}(t)$   | $\bar{\phi}^\alpha_{m}(t)$ | $\bar{\phi}^*_{m}(t)$ | $\bar{\zeta}_{m}(t)$ | $\zeta_{m}(t)$ | $p-1$ | $-x_F$|
| 2   | $\phi^\alpha_{m}(t)$ | $\phi^*_{m}(t)$   | $\bar{\phi}^\alpha_{m}(t)$ | $\bar{\phi}^*_{m}(t)$ | $\bar{\zeta}_{m}(t)$ | $\zeta_{m}(t)$ | $p-2$ | $x_F$ |
| 2   | $\phi^\alpha_{m}(t)$ | $\phi^*_{m}(t)$   | $\bar{\phi}^\alpha_{m}(t)$ | $\bar{\phi}^*_{m}(t)$ | $\bar{\zeta}_{m}(t)$ | $\zeta_{m}(t)$ | $p-2$ | $x_S$ |
| 3   | $\phi^\alpha_{m}(t)$ | $\phi^*_{m}(t)$   | $\bar{\phi}^\alpha_{m}(t)$ | $\bar{\phi}^*_{m}(t)$ | $\bar{\zeta}_{m}(t)$ | $\zeta_{m}(t)$ | $p-3$ | $-x_S$|

If we add all contributions of the same order, we see that fourth relation in (13.2)
can only be satisfied provided that

\begin{align*}
p : & \quad x_F - x_B = X \\
p - 1 : & \quad -2x_F + x_B = -rX, \\
p - 2 : & \quad x_B + x_F + x_S = \left(\frac{r}{2}\right)X, \\
p - 3 : & \quad -x_B - x_S - x_G = -\left(\frac{r}{3}\right)X, \quad (13.4) \\
p - 4 : & \quad x_G = \left(\frac{r}{4}\right)X, \\
p - 5 : & \quad 0 = -\left(\frac{r}{5}\right)X, ...
\end{align*}

The last equation holds only if \( r \leq 4 \) (or trivially if \( X = 0 \)). On the other hand, if we demand that there is at least one bosonic gauge condition, the \( p - 4 \) equation yields \( r \geq 4 \). Such a demand is natural, because both the Maxwell/Yang-Mills and the Einstein equations have this property. Therefore, we are unambiguously guided to consider \( r = 4 \) (and thus \( N = 4 \)). The specialization of (13.4) to four dimensions reads

\begin{align*}
p : & \quad x_F - x_B = X \\
p - 1 : & \quad -2x_F + x_B = -4X, \\
p - 2 : & \quad x_B + x_F + x_S = 6X, \quad (13.5) \\
p - 3 : & \quad -x_B - x_S - x_G = -4X, \\
p - 4 : & \quad x_G = X.
\end{align*}

Clearly, the unique solution to these equations is

\[ x_F = 3X, \quad x_B = 2X, \quad x_S = X, \quad x_G = X. \quad (13.6) \]

The solutions to the remaining equations in (13.2) are found by analogous reasoning. The result is

\begin{align*}
u_B &= 2U \\
u_F &= 3U \\
u_S &= U - X \\
u_G &= U - X \\
u &= 2W + X \\
u &= 3W + X \\
u &= W + X \\
u &= W + X \\
u &= 2Y \\
u &= 3Y \\
u &= Y \\
u &= Y \\
u &= Y.
\end{align*}
This result expresses the twenty parameters $x_B - w_G$ in terms of the five parameters $X, Y, U, V, W$. For this particular choice of parameters, the abelian charges in (13.1) are given by

$$c_1 = -U, \quad c_2 = -V, \quad c_3 = W, \quad c_4 = -X, \quad c_5 = Y, \quad (13.8)$$

independent of $p$. Hence there is no manifest obstruction to the limit $p \to \infty$.

The prediction that spacetime has $N = 4$ dimensions is of course very nice. Unfortunately, at closer scrutiny the situation appears less appealing. In particular, the need for fermionic gauge symmetries ($z_S \neq 0$) is apparently in disagreement with observation. It was also found in [19] that the gauge anomaly $c_5$ does not have a finite $p \to \infty$ limit for reasonable choices of field content.

Hence it is presently unclear how to remove the regulator and take the field limit, and this is of course a major unsolved problem. Nevertheless, it should be emphasized that already the regularized theories carry representations of the full gauge and diffeomorphism algebras.

14 Conceptual issues

One of the most important tasks of any putative quantum theory of gravity is to shed light on the various conceptual difficulties which arise when the principles of quantum mechanics are combined with general covariance [6, 25]. These issues include:

1. In conventional canonical quantization, the canonical commutation relations are defined on a “spacelike” surface. However, a surface is spacelike w.r.t. some particular spacetime metric $g_{\mu\nu}$, which is itself a quantum operator.

2. Microcausality requires that the field variables defined in spacelike separated regions commute. Again, it is unclear what this means when the notion of spacelikeness is dynamical.

3. Different choices of foliation lead to a priori different quantum theories, and it by no means clear that these are unitarily equivalent.

4. The problem of time: The Hamiltonian of general relativity is a first class constraint, hence it vanishes on the reduced phase space. This means that there is no notion of time evolution among diffeomorphism-invariant degrees of freedom.

5. The notion of time as a causal order is lost. This is not really a problem in the classical theory, where one can solve the equations of motion first, but in quantum theory causality is needed from the outset.
6. QFT rests on two pillars: quantum mechanics and locality. However, locality is at odds with diffeomorphism invariance underlying gravity; “there are no local observables in quantum gravity”.

Let us see how MCCQ addresses these conceptual issues.

1. The canonical commutation relations are defined throughout the history phase space $P$, and hence not restricted to variables living on a spacelike surface. Dynamics is implemented as a first class constraint in $P$. Only if we solve this constraint prior to quantization need we restrict quantization to a spacelike surface.

2. By passing to $p$-jet space, we eliminate the notion of spacelikeness altogether. The $p$-jets live on the observer’s trajectory, and the observer moves along a timelike curve. It might seem strange to dismiss the notion of spacelike separation, but distant events can never be directly observed, and a physical theory only needs to describe directly observable events. What can be observed are indirect effects of distant events. E.g., a terrestrial detector does not directly observe the sun, but only photons emanating from the sun eight light-minutes ago. The detector signals are of course compatible with the existence of the sun, but a physical theory only needs to deal with directly observed events, i.e. the absorption of photons in the detector.

3. In MCCQ there is no foliation, but rather an explicit observer, or detector. The theory is unique since the observer’s trajectory is a quantum object; we do not deal with a family of theories parametrized by the choice of observer, but instead the observer’s trajectory is represented on the Hilbert space in the same way as the quantum fields.

4. By introducing an explicit observer, we can define a genuine energy operator which translates the fields relative to the observer, or vice versa. In contrast, there is also a Hamiltonian constraint, which translates both the observer and the fields the same amount. This constraint is killed in KT cohomology and is thus identically zero on physical observables.

5. The $p$-jets live on the observer’s trajectory $q(t)$ and are thus causally related; causal order is defined by the parameter $t$. The relation between this order and the fields is encoded in the geodesic equation (12.9).

6. As we saw in Section 2, locality is compatible with infinite-dimensional spacetime symmetries, but only in the presence of an anomaly. This is the key lesson from CFT.

It is gratifying that the MCCQ formalism yields natural explanations of many of the conceptual problems that plague quantum gravity.
15 Conclusion

The key insight underlying the present work is that the process of observation must be localized in spacetime in order to be compatible with the philosophy of QFT. The innocent-looking introduction of the observer’s trajectory leads to dramatic consequences, because new gauge and diffeomorphism anomalies arise. On the mathematical side, this construction leads to well-defined realizations of the constraint algebra generators as operators on a linear space, at least for the regularized theory.

We have also developed a manifestly covariant canonical quantization method, based on the form of the DGRO algebra modules. This formalism is convenient due to its relation to representation theory, but it is presumably possible to repeat the analysis in any sensible quantization scheme, at the cost of additional work. In contrast, the introduction of the observer’s trajectory is absolutely crucial, because the new anomalies cannot be formulated without it. Anomalies matter!

Four critical problems remain to be solved. As was discussed in Section 13, the original fields must be reconstructed from the \( p \)-jets, i.e. we must take the limit \( p \to \infty \). This limit is problematic because the abelian charges diverge. Second, the issue of unitarity needs to be understood. So far we only noted that an extension is necessary for unitarity by restriction to Virasoro subalgebras, and then we proceeded to construct anomalous representations. The main problem is to find an invariant inner product. Third, perturbation theory and renormalization must be transcribed to the formalism, to make contact with numerical predictions of ordinary QFT. Finally, we know from CFT that reducibility conditions analogous to Kac’ formula \[8\] are needed in physically interesting situations. Unfortunately, none of these problems appears to be easy.

References

[1] L. Bonora, P. Pasti and M. Tonin, The anomaly structure of theories with external gravity, J. Math. Phys. 27 (1986) 2259–2270.

[2] S. Berman and Y. Billig, Irreducible representations for toroidal Lie algebras, J. Algebra 221 (1999) 188–231.

[3] S. Berman, Y. Billig and J. Szmigielski; Vertex operator algebras and the representation theory of toroidal algebras, math.QA/0101094 (2001)

[4] Y. Billig, Principal vertex operator representations for toroidal Lie algebras, J. Math. Phys. 7 (1998) 3844–3864.

[5] Y. Billig and K. Zhao, Weight modules over exp-polynomial Lie algebras, math.RT/0305293 (2003)

[6] S. Carlip, Quantum gravity: a progress report, gr-qc/0108040
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[7] A. Dzhumadildaev, *Virasoro type Lie algebras and deformations*, Z. Phys. C **72** (1996) 509–517.

[8] P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal field theory*, New York: Springer-Verlag, 1996.

[9] P. Goddard and D. Olive, *Kac-Moody and Virasoro algebras in relation to quantum physics*, Int. J. Mod. Phys. **1** (1986) 303–414.

[10] M. Henneaux and C. Teitelboim, *Quantization of gauge systems*, Princeton Univ. Press (1992).

[11] C. J. Isham and N. Linden, *Continuous histories and the history group in generalised quantum theory*, J. Math. Phys. **36** (1995) 5392.

[12] C. Kassel, *Kahler differentials and coverings of complex simple Lie algebras extended over a commutative algebra*, J. Pure and Appl. Algebra **34** (1985) 256–275.

[13] T.A. Larsson, *Central and non-central extensions of multi-graded Lie algebras*, J. Phys. A. **25** (1992) 1177–1184.

[14] T.A. Larsson, *Extended diffeomorphism algebras and trajectories in jet space*, Comm. Math. Phys. **214** (2000) 469–491. [math-ph/9810003](#)

[15] T.A. Larsson, *Multi-dimensional diffeomorphism and current algebras from Virasoro and Kac-Moody Currents*, [math-ph/0101007](#) (2001).

[16] T.A. Larsson, *Koszul-Tate cohomology as lowest-energy modules of non-centrally extended diffeomorphism algebras*, [math-ph/0210023](#) (2002).

[17] T.A. Larsson, *Multi-dimensional Virasoro algebra and quantum gravity*, in *Mathematical physics research on the leading edge*, ed. C. V. Benton, New York: Nova Science Publishers, 2003.

[18] T.A. Larsson, *Manifestly covariant canonical quantization I: the free scalar field*, [hep-th/0411028](#) (2004).

[19] T.A. Larsson, *Manifestly covariant canonical quantization II: Gauge theory and anomalies*, [hep-th/0501043](#) (2005).

[20] J. Mickelsson, *Current algebras and groups*, Plenum Monographs in Nonlinear Physics, London: Plenum Press, 1989.

[21] P. Nelson and L. Alvarez-Gaumé, *Hamiltonian interpretation of anomalies*, Comm. Math. Phys. **99** (1985) 103–114.
[22] D. Pickrell, *On the Mickelsson-Faddeev extensions and unitary representations*, Comm. Math. Phys. 123 (1989) 617.

[23] S.E. Rao and R.V. Moody, *Vertex representations for N-toroidal Lie algebras and a generalization of the Virasoro algebra*. Comm. Math. Phys. 159 (1994) 239–264.

[24] K. Savvidou, *The action operator in continuous time histories*, J. Math. Phys. 40 (1999) 5657.

[25] N. Savvidou, *General relativity histories theory*, gr-qc/0412059 (2004)