Semi-analytical computation of displacement in linear viscoelastic materials

S Spinu and D Gradinaru

“Stefan cel Mare” University of Suceava, Department of Mechanics and Technologies, 13th University Street, 720229, Suceava, Romania

E-mail: sergiu.spinu@fim.usv.ro

Abstract. Prediction of mechanical contact performance based on elastic models is not accurate in case of viscoelastic materials; however, a closed–form description of the viscoelastic contact has yet to be found. This paper aims to advance a semi-analytical method for computation of displacement induced in viscoelastic materials by arbitrary surface tractions, as a prerequisite to a semi-analytical solution for the viscoelastic contact problem. The newly advanced model is expected to provide greater generality, allowing for arbitrary contact geometry and / or arbitrary loading history. While time-independent equations in the purely elastic model can be treated numerically by imposing a spatial discretization only, a viscoelastic constitutive law requires supplementary temporal discretization capable of simulating the memory effect specific to viscoelastic materials. By deriving new influence coefficients, computation of displacement induced in a viscoelastic material by a known but otherwise arbitrary history of surface tractions can be achieved via superposition authorized by the Boltzmann superposition theory applicable in the frame of linear viscoelasticity.

1. Introduction

The design of automotive belts and tires, seals or biomedical devices require extensive knowledge on the behavior of materials like elastomers or rubber, which fall into the category of viscoelastic materials. From a mechanical point of view, the tribological performances of the latter are particularly important in assessing the strength of the mechanical contact. When load is transmitted through a contact whose region is small compared to dimensions of the contacting bodies (i.e. a non-conforming contact), important gradients of stress and strain develop in the contact proximity. When the elastic limit of the material is surpassed, irreversible changes in material behavior occur, concluding with the destruction of the integrity of the involved machine element and thus leading to contact failure. The prediction of these stress and strain gradients based on the Hertz model is no longer accurate in case of viscoelastic materials, as the viscoelastic response to load depend explicitly on time and on the loading history.

The classic literature on the viscoelastic contact is based on the so-called correspondence principle between the elastic and the viscoelastic solution of a problem of stress analysis, stating that a viscoelastic problem has an associated elastic problem to which the former reduces after removal of time variable via transfer to Laplace transform domain. Indeed, the basic integral equations of the viscoelastic problem of stress analysis are identical, in form, when transferred to the Laplace domain, with the equations describing stresses in purely elastic materials. Consequently, if the boundary conditions are handled properly, a viscoelastic solution in the frequency domain is identical in form to
its associated elastic solution. This remarkable property of the elastic and viscoelastic equations can thus be used to derive solutions of viscoelastic problems of stress analysis from their elastic counterparts, which are usually obtained more easily, e.g. from the work of Sneddon [1]. The applicability of this technique is however limited because the transient boundary conditions encountered in contact problems cannot be properly handled when transferring to Laplace domain, e.g. in generic contact problems there exist regions for which the boundary conditions are specified only for a short period of time, and thus the integral in the Laplace transform cannot be computed.

The first application of this principle to contact mechanics was reported by Lee and Radok [2], who derived the contact radius and the pressure distribution in a Hertz-type contact between linear viscoelastic materials. Displacement computation was achieved via integration of pressure over the contact area. Lee and Radok’s result [2] is often referred to as a tentative solution, as it covers only the case when the contact radius increases monotonically with time. The generalization of this result was achieved later by Ting [3], who advanced a model in which, depending on the specifics of the loading history, up to five separate cases must be considered, leading to tedious algebraic manipulations of the emerging equations. Moreover, the resulting solutions, although explicit, involve numerical differentiation followed by numerical integration, as well as numerical resolution of transcendental equations. This mandatory numerical treatment often raises convergence issues, as opposed to the semi-analytical methods. The latter are based on manipulation of the so-called influence coefficients derived by closed-form integration of fundamental solutions in the theory of elasticity (i.e. the Green functions), and thus raise fewer convergence problems. Recent applications of this type of methods to viscoelastic contact analysis can be found in [4, 5].

This paper aims to advance a novel semi-analytical solution for the computation of displacement in viscoelastic materials, solution that is expected to assist the development of a semi-analytical method for the contact problem involving viscoelastic bodies.

2. Displacement computation

A surface distribution of normal or shear tractions, such as the ones resulting from a mechanical contact process, induce a displacement field whose knowledge is essential in solving the contact problem and in performing stress analysis in the contacting bodies. Displacement computation can only be accomplished after establishing the constitutive law of the material. While complete description of linear elastic and isotropic materials involves only two constants, e.g. the Young modulus and the Poisson’s ratio $\nu$, viscoelastic models employ various models and parameters. The assumptions described in [6] are adopted in this work to reduce the complexity of the arising equations to a computationally friendly form.

Firstly, the viscoelastic response is assumed to be linear (which is reasonable in the frame of small strain theory), thus allowing for the use of Boltzmann superposition theory. The latter states that that the stress or strain response to various strain or stress histories, acting simultaneously, is identical to the sum of responses when the same histories are applied separately. In mathematical form, the application of the Boltzmann hereditary integral operator is authorized in computation of viscoelastic response to various sequences of stress or strain.

Secondly, the viscoelastic material is assumed incompressible, which is reasonable for polymers, thus limiting the model complexity to a stress-strain law in shear, between the deviatoric stress $s$, the deviatoric strain $e$ and the shear modulus $G : s = 2Ge$. In this framework, the strain response to a unit step change in stress is completely described by the creep compliance function $\Phi(t)$, and the stress response to a unit step change in strain by the relaxation modulus $\Psi(t)$. These two functions are interchangeable, but, unlike the purely elastic case, there is no reciprocity, i.e. $\Psi(t)\Phi(t) \neq 1$. By employing the Boltzmann hereditary integral together with these time-dependent functions, the viscoelastic response to any sequence of stress or strain in a window of observation $[0, t]$ can be expressed as:
Although the limiting boundary of a real solid is intrinsically rough, computational contact mechanics employs the half-space assumption, allowing for the use of fundamental solutions derived in the theory of linear elasticity for a semi-infinite body bounded by a plane surface. In a Cartesian coordinate system \((x_1, x_2, x_3)\) having its origin on the half-space boundary and the \(x_3\)-axis along the direction of the normal to the half-space boundary, the normal displacement field \(u_3\) generated in a linear elastic and isotropic solid by a distribution of normal tractions \(p(x_1, x_2)\) is computed using the Green function \(G_e(x_1, x_2)\) for the elastic half-space derived by Boussinesq \([7]\):

\[
u_e(x_1, x_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_e(x_1 - x'_1, x_2 - x'_2) p(x'_1, x'_2) dx'_1 dx'_2
\]

where \(G_e(x_1, x_2) = (1 - \nu) / (2\pi G\sqrt{x_1^2 + x_2^2})\) is the normal displacement induced at a point of coordinates \((x_1, x_2)\) by a unity concentrated force acting in origin along direction of \(\bar{x}_3\).

Lee and Radok \([2]\) obtained the contact radius in the viscoelastic spherical contact problem by applying the hereditary integral operator of the type described in equation (1) to the Hertz (i.e. purely elastic) solution in which the elastic modulus \(1/(2G)\) was replaced by the viscoelastic creep compliance. We are applying the same technique to equation (2), aiming to obtain the viscoelastic displacement generated by a known history of pressure \(p(x'_1, x'_2, t')\) in a window of observation \([0, t]\):

\[
u_e(x_1, x_2, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\Phi(t - t') \partial}{\partial t'} \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x'_1, x'_2, t') \frac{dx'_1 dx'_2}{\sqrt{(x_1 - x'_1)^2 + (x_2 - x'_2)^2}} \right] dt'
\]

It should be remembered that the viscoelastic material is assumed incompressible, therefore \(\nu = 0.5\). By interchanging differentiation and integration, an equivalent form can be obtained, as shown in Eq. 4. The semi-analytical treatment of these equation to attain a computationally friendly form is detailed in the following section.

3. Semi-analytical method
The integral equations (2) (for the elastic case) and (3) (for the viscoelastic framework) cannot be solved analytically at this point. The principle of the semi-analytical method consist in considering all continuous distributions as piecewise constant on a uniformly spaced rectangular mesh established in the plane \(x_1, x_2\). Control points must be chosen for all the elementary cells of the grid (the centers of the cells are good candidates), and all continuous parameters are evaluated in these representing points, resulting in a digitized counterpart for each continuous distribution.

This discretization encourages a simplified notation taking as arguments the indexes of the cells rather than the continuous coordinates. For example, \(p(i, j)\) denotes the pressure value computed in the center of the cell \((i, j)\), and \(p(i, j) = p(x_1^{(i)}, x_2^{(j)})\), where \(x_1^{(i)}\) and \(x_2^{(j)}\) are coordinates of the
center of the cell \((i, j)\). It follows that pressure is assumed uniform in the rectangular patch centered in \((x_1^{(i)}, x_2^{(j)})\), and therefore can be factored outside the integral operator in equation (2). The integral of the Green function \(G_e(x_1, x_2)\) taken over the elementary patch of side lengths \(\Delta_1\) and \(\Delta_2\) along directions of \(\vec{x}_1\) and \(\vec{x}_2\), respectively, yield the influence coefficient for the elastic displacement \(IC_e\):

\[
IC_e(i - k, j - \ell) = \int_{x_1(k) - \Delta_1/2}^{x_1(k) + \Delta_1/2} \int_{x_2(l) - \Delta_2/2}^{x_2(l) + \Delta_2/2} G_e(x_1(i) - x_1^l, x_2(j) - x_2^l) dx_1^l dx_2^l,
\]

which expresses the normal displacement induced in the observation cell \((i, j)\) by a uniform pressure of magnitude \(1/(\Delta_1\Delta_2)\) [Pa] acting in the cell \((k, \ell)\). The closed-form solution of the double integral in equation (5) was derived by Love [8]:

\[
IC_e(i, j) = \frac{1 - \nu}{2\pi G} \left( f(x_1(i) + \Delta_1/2, x_2(j) + \Delta_2/2) + f(x_1(i) - \Delta_1/2, x_2(j) - \Delta_2/2) - f(x_1(i) - \Delta_1/2, x_2(j) + \Delta_2/2) - f(x_1(i) + \Delta_1/2, x_2(j) - \Delta_2/2) \right),
\]

where

\[
f(x_1, x_2) = x_1 \ln \left( x_1^2 + x_2^2 + \sqrt{x_1^2 + x_2^2} \right) + x_2 \ln \left( x_1^2 + x_2^2 + \sqrt{x_1^2 + x_2^2} \right).
\]

Within this framework, the semi-analytical counterpart of equation (2) results as:

\[
u_e(i, j) = \sum_{k=1}^{N_1} \sum_{\ell=1}^{N_2} IC_e(i - k, j - \ell) p(k, \ell)
\]

where \(N_1\) and \(N_2\) denotes the number of grids along directions of \(\vec{x}_1\) and \(\vec{x}_2\), respectively. The double convolution in equation (7) can be performed for any imposed pressure distribution. Optimum algorithmic efficiency is achieved using the Discrete Convolution Fast Fourier Transform (DCFFT) algorithm advanced by Liu, Wang and Liu [9]. The reduction of the order of computation comes from the convolution theorem, which states that the convolution operation reduces to an element-wise product in the Fourier transform domain. The semi-analytical displacement computation using equation (7) together with the DCFFT technique is now widely used in computational contact mechanics. In this paper, we intend to generalize this equation to the case of viscoelastic behavior.

As equations describing the purely elastic model are intrinsically time-independent, spatial discretization is adequate to circumvent the continuous integration in equation (2). An additional integration over the time span in which the body was loaded is present in equation (3), requiring an additional temporal mesh capable of simulating the memory effect specific to viscoelastic materials (i.e. the property that the current state depends upon all previous states achieved from the first loading). This temporal discretization should be chosen so that at \(t = 0\) the body was undisturbed, and the time increment \(\Delta_t\) should be small enough so that, during each step, the problem parameters can be assumed constant. A piecewise constant law is thus imposed along the temporal axis, adding a third parameter to the notation implemented in the purely elastic model. For example, \(p(i, j, k)\) is the discrete counterpart of \(p(x_1, x_2, t)\), denoting the pressure in the elementary cell \((i, j)\) in the spatial mesh, achieved after \(k\) time increments, where \(t = k\Delta_t\), with \(k = 1, \ldots, N_t\). This assumption regarding the temporal variation of model parameters authorize the substitution of the partial derivative \(\partial p(x_1^l, x_2^l, t) / \partial t^l\) in equation (4) with the finite difference \(p(i, j, k) - p(i, j, k - 1)\). The latter can
be factored outside the spatial integral operator as in the purely elastic model. A viscoelastic influence coefficient can be defined similar to its elastic counterpart in equation (5), by including the viscoelastic material property, i.e. the creep compliance function:

\[ IC_{ve}(i - \ell, j - m, k - n) = 2G\Phi((k - n)\Delta_t)IC_e(i - \ell, j - m) \]  

(8)

This influence coefficient expresses the displacement observed after \( k \) time steps in the elementary patch \((i, j)\) of the spatial mesh, due to a uniform pressure of \( 1/(\Delta_t \Delta_z) \) [Pa] that acted in the patch \((\ell, m)\) in the \( n^{th} \) time step after the reference moment in which the body was undisturbed, with \( n \leq k \). The semi-analytical counterpart of Eq. 4 can thus be expressed as:

\[ u_{ve}^i(i, j, k) = \sum_{n=1}^{N_x} \sum_{\ell=1}^{N_x} \sum_{m=1}^{N_x} IC_{ve}(i - \ell, j - m, k - n)(p(\ell, m, n) - p(\ell, m, n - 1)), \]  

(9)

where \( i = 1..N_x, j = 1..N_x, k = 1..N_x \). This equation clearly shows that the memory effect is considered explicitly in the displacement computation, as pressure distributions in all previous states, i.e. at all previous time increments, together with the current pressure, are needed to evaluate the current displacement. It is noteworthy that the contribution of the historical pressures can be separated from the contribution of the current pressure. This feature that will be used in the following section for validation purposes.

4. Program validation

Most results in the viscoelastic contact literature are given in pressure, not in displacement. Therefore, in order to validate this new displacement computation method, we need to link it to the corresponding pressure. In electrostatics, the one-to-one correspondence between pressure and displacement is guaranteed by the theorem of uniqueness of solutions, and this correspondence is at the base of the semi-analytical method [10] which solves the elastic contact problem by deriving both pressure and displacement from the contact interference equation. Provided displacement is computed according to the viscoelastic material model instead of the elastic one, this contact solver [10] should equally handle viscoelastic contact problems. Indeed, provided the memory effect of the viscoelastic material is properly considered in displacement computation, the same equilibrium and geometrical equations describe both elastic and viscoelastic contacts. However, while the contact solver can only be used to derive the current pressure, all historical pressures are needed in computation of displacement according to equation (9). To overcome this, we are using the Ting’s partially analytical formulas [3] to obtain the historical pressures, and we subsequently use the viscoelastic displacement computed with the newly advanced semi-analytical method to obtain a new pressure. We are then comparing it with the one predicted by the Ting’s formula for the same loading history. Based on the one-to-one correspondence between pressure and displacement, an agreement between the two pressure distributions would indicate that the performed semi-analytical displacement computation is indeed accurate.

A linear viscoelastic half-space described by a Maxwell rheological model having a \( \tau \) relaxation time is indented by a rigid spherical punch in a step loading. Pressure distributions computed for different times \( t \) subsequent to the first contact, depicted in figure 1, match well the Ting’s model, giving confidence in the capability of the newly advanced semi-analytical model to accurately predict viscoelastic displacement. Dimensionless pressure \( p \), radial coordinate \( x_i \) and normal displacement \( u_3 \) are defined as ratio to Hertz (i.e. purely elastic) central pressure \( p_H \), contact radius \( a_H \) and rigid-body approach \( \omega_{3_H} \), respectively, all corresponding to the maximum loading level.
Equation (9) can also be used to predict the post-unloading displacement, by equating the post-unloading pressures to zero. This scenario can only be simulated using the semi-analytical formulation advanced herein. The displacement profiles prior and subsequent to indenter removal are depicted in figure 2. The elastic part due to the elastic spring in the Maxwell model is recovered instantaneously, while the dashpot displacement is predicted to persist indefinitely.

5. Conclusions
A semi-analytical method for the computation of displacement induced in a viscoelastic material by a known, but otherwise arbitrarily chosen history of surface tractions is advanced in this paper. The method is based on the correspondence principle between the elastic and the viscoelastic problems of stress analysis, authorizing the manipulation of the elastic solution in achievement of its viscoelastic counterpart. The linear viscoelastic model employs the Boltzmann hereditary integral to obtain the material response to various sequences of stress, in a directly additive manner. Both spatial and temporal discretizations are imposed in the semi-analytical formulation in order to circumvent analytical integration, which is eventually substituted by a discrete convolution product raising no convergence issues. The purely elastic influence coefficient for displacement is extended to a form that accounts for the time-dependent material property.

Simulations for a viscoelastic half-space described by a Maxwell rheological model, indented by a spherical rigid indenter in a step loading, confirm the validity of the newly advanced formula. Based on the latter, we are anticipating a semi-analytical method for the resolution of the viscoelastic contact problem, in which contact pressure and displacement are obtained for every new time increment in a step-by-step approach, by solving repeatedly the purely elastic contact problem with a modified initial contact geometry which accounts for the memory effect of the viscoelastic material.

References
[1] Sneddon I N 1965 The relation between load and penetration in the axisymmetric Boussinesq problem for a punch of arbitrary profile Int. J. Engng Sci. 3 pp 47-57
[2] Lee E H and Radok J R M 1960 The contact problem for viscoelastic bodies ASME J. Appl. Mech. 27 pp 438-444
[3] Ting T C T 1968 Contact problems in the linear theory of viscoelasticity ASME J. Appl. Mech. 35 pp 248-254
[4] Chen W W, Wang Q J, Huan Z and Luo X 2008 Semi-analytical viscoelastic contact modeling of polymer-based materials ASME J. Tribol. 133 pp 041404-1—041404-10
[5] Kozhevnikov I F, Duhamel D, Yin H P and Feng Z Q 2010 A new algorithm for solving the multi-indentation problem of rigid bodies of arbitrary shapes on a viscoelastic half-space *Int. J. Mech. Sci.* **52** pp 399-409

[6] Johnson K L 1985 Contact mechanics (Cambridge: University Press)

[7] Boussinesq J 1969 Application des potentiels à l'étude de l'équilibre et du mouvement des solides élastiques (Paris: Reed. A. Blanchard)

[8] Love A E H 1929 The stress produced in a semi-infinite solid by pressure on part of the boundary *Phil. Trans. Roy. Soc.* **228** pp 377-420

[9] Liu S B, Wang Q and Liu G 2000 A versatile method of discrete convolution and FFT (DC-FFT) for contact analyses *Wear* **243** pp 101-111

[10] Polonsky I A and Keer L M 1999 A numerical method for solving rough contact problems based on the multi-level multi-summation and conjugate gradient techniques *Wear* **231** pp 206-219