Abstract

We extend the main result of the companion paper math-ph/0212063 to the case of the pfaffian ensembles.

1 Introduction and Formulation of Results

Let us consider a $2n$-particle pfaffian ensemble introduced by Rains in [9]: Let $(X,\lambda)$ be a measure space, $\phi_1, \phi_2, \ldots, \phi_{2n}$ be complex-valued functions on $X$, and $\epsilon(x, y)$ be an antisymmetric kernel such that
\[
p(x_1, \ldots, x_{2n}) = \frac{1}{Z_{2n}} \det(\phi_j(x_k))_{j,k=1,\ldots,2n} pf(\epsilon(x_j, x_k))_{j,k=1,\ldots,2n}
\] (1)
defines the density of a $2n$-dimensional probability distribution on $X^{2n} = X \times \cdots \times X$ with respect to the product measure $\lambda^\otimes 2n$. Ensembles of this form were introduced in [9] and [11]. We recall (see e.g. [5]) that the pfaffian of a $2n \times 2n$ antisymmetric matrix $A = (a_{jk})$, $j, k = 1, \ldots, 2n$, $a_{jk} = -a_{kj}$, is defined as
\[
pf(A) = \sum_\tau (-1)^{\text{sign}(\tau)} a_{i_1 j_1} \cdots a_{i_n j_n},
\] where the summation is over all partitions of the set $\{1, \ldots, 2m\}$ into disjoint pairs $\{i_1, j_1\}, \ldots, \{i_n, j_n\}$ such that $i_k < j_k$, $k = 1, \ldots, n$, and $\text{sign}(\tau)$ is the sign of the permutation $\tau = (i_1, j_1, \ldots, i_n, j_n)$. The normalization constant in (1) (usually called the partition function)
\[
Z_{2n} = \int_{X^{2n}} \det(\phi_j(x_k))_{j,k=1,\ldots,2n} pf(\epsilon(x_j, x_k))_{j,k=1,\ldots,2n}
\] (2)
can be shown to be equal $(2n)! pf(M)$, where the $2n \times 2n$ antisymmetric matrix $M = (M_{jk})_{j,k=1,\ldots,2n}$ is defined as
\[
M_{jk} = \int_{X^2} \phi_j(x) \epsilon(x, y) \phi_k(y) \lambda(dx) \lambda(dy).
\] (3)

For the pfaffian ensemble (1) one can explicitly calculate k-point correlation functions
\[
\rho_k(x_1, \ldots, x_k) := \frac{(2n)!}{(2n-k)!} \int_{X^{2n-k}} p(x_1, \ldots, x_k, x_{k+1}, \ldots, x_{2n}) d\lambda(x_{k+1}) \cdots d\lambda(x_{2n}), \ k = 1, \ldots, 2n
\] and show that they have the pfaffian form (2)
\[
\rho_k(x_1, \ldots, x_k) = pf(K(x_i, x_j))_{i,j=1,\ldots,k},
\] (4)
where \( K(x,y) \) is the antisymmetric matrix kernel

\[
K(x,y) = \left( \begin{array}{cc}
\sum_{1 \leq j,k \leq 2n} \phi_j(x) M^{-t}_{jk} \phi_k(y) \\
\sum_{1 \leq j,k \leq 2n} (\epsilon \phi_j(x) M^{-t}_{jk} \phi_k(y)) - \epsilon(x,y) + \sum_{1 \leq j,k \leq 2n} (\epsilon \phi_j(x) M^{-t}_{jk} \phi_k(y))
\end{array} \right),
\]

provided the matrix \( M \) is invertible (by definition \((\epsilon \phi)(x) = \int_X \epsilon(x,y) \phi(y) \lambda(dy))\). If \( X \subset \mathbb{R} \) and \( \lambda \) is absolutely continuous with respect to the Lebesgue measure, then the probabilistic meaning of the k-point correlation functions is that of the density of probability to find an eigenvalue in each infinitesimal interval around points \( x_1, x_2, \ldots, x_k \). In other words

\[
\rho_k(x_1, x_2, \ldots, x_k) \lambda(dx_1) \cdots \lambda(dx_k) = \Pr \{ \text{there is a particle in each infinitesimal interval \( (x_i, x_i + dx_i) \)} \}.
\]

On the other hand, if \( \mu \) is supported by a discrete set of points, then

\[
\rho_k(x_1, x_2, \ldots, x_k) \lambda(x_1) \cdots \lambda(x_k) = \Pr \{ \text{there is a particle at each of the points } x_i, i = 1, \ldots, k \}.
\]

In general, random point processes with the k-point correlation functions of the pfaffian form are called pfaffian random point processes. Pfaffian point processes include determinantal point processes as a particular case when the matrix kernel has the form \( \begin{pmatrix} \epsilon & K \\ -K & 0 \end{pmatrix} \) where \( K \) is a scalar kernel and \( \epsilon \) is an antisymmetric kernel.

So-called Janossy densities \( J_{k,I}(x_1, \ldots, x_k) \), \( k = 0, 1, 2, \ldots \), describe the distribution of the eigenvalues in any given interval \( I \). If \( X \subset \mathbb{R} \) and \( \lambda \) is absolutely continuous with respect to the Lebesgue measure then

\[
J_{k,I}(x_1, \ldots, x_k) \lambda(dx_1) \cdots \lambda(dx_k) = \Pr \{ \text{there are exactly } k \text{ particles in } I, \text{ one in each of the } k \text{ distinct infinitesimal intervals}(x_i, x_i + dx_i) \}.
\]

If \( \lambda \) is discrete then

\[
J_{k,I}(x_1, \ldots, x_k) = \Pr \{ \text{there are exactly } k \text{ particles in } I, \text{ one at each of the } k \text{ points } x_i, i = 1, \ldots, k \}.
\]

See \[4\] and \[3\] for details and additional discussion. For pfaffian point processes the Janossy densities also have the pfaffian form (see \[3\], \[8\]) with an antisymmetric matrix kernel \( L_I \):

\[
J_{k,I}(x_1, \ldots, x_k) = \text{const}(I) \text{pf}(L_I(x_i, x_j))_{i,j=1,\ldots,k},
\]

where

\[
L_I = K_I(Id + JK_I)^{-1},
\]

\( J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) and \( \text{const}(I) = \text{pf}(J - K_I) \) is the Fredholm pfaffian of the restriction of the operator \( K \) on the interval \( I \), i.e. \( \text{const}(I) = \text{pf}(J - K_I) = (\text{pf}(J + L_I))^{-1} = (\det(Id + J \times K_I))^{1/2} = (\det(Id - JL_I))^{-1/2} \). (we refer the reader to \[4\], section 8 for the treatment of Fredholm pfaffians).

Let us define three \( 2n \times 2n \) matrices \( G^I, M^I, M^X \):

\[
G^I_{jk} = \int_I \phi_j(x) \int_X \epsilon(x,y) \phi_k(y) \lambda(dy) \lambda(dx),
\]

\[
M^I = \int_I \phi_j(x) \int_X \epsilon(x,y) \phi_k(y) \lambda(dy)
\]

\[
M^X = \int_X \epsilon(x,y) \phi_j(x) \phi_k(y) \lambda(dx)
\]
Theorem 1.1 The kernel $L_I$ has a form similar to the formula (5) for $K$. Namely, $L_I$ is equal to
\[
L_I(x, y) = \left( \sum_{1 \leq j, k \leq 2n} \phi_j(x)(M^{X \setminus I})^{-1}_{jk} \phi_k(y) - \sum_{1 \leq j, k \leq 2n}(\epsilon_{X \setminus I} \phi_j)(x)(M^{X \setminus I})^{-1}_{jk}(\epsilon_{X \setminus I} \phi_k)(y) \right)
\]
where $(\epsilon_{X \setminus I} \phi)(x) = \int_{X \setminus I} \epsilon(x, y)\phi(y)\lambda(dy)$.

Comparing (11) with (5) one can see that the kernel $L_I$ is constructed in the following way: 1) first it is constructed on $X \setminus I$ by the same recipe used to construct the kernel $K$ on the whole space, 2) it is extended then to $I$ (we recall that $L_I$ acts on $L^2(I, d\lambda(x))$, not on $L^2(X \setminus I, d\lambda(x))$).

This result contains as a special case Theorem 1.1 from the companion paper [3]. The rest of the paper is organized as follows. We discuss some interesting special cases of the theorem, namely so-called polynomial ensembles ($\beta = 1, 2$ and 4) in section 2. The proof of the theorem is given in section 3.

2 Random Matrix Ensembles with $\beta = 1, 2, 4$.

We follow the discussion in [9] (see also [11] and [12]).

Biorthogonal Ensembles.

Consider the particle space to be the union of two identical measure spaces $(V, \mu)$ and $(W, \mu)$: $X = V \cup W$, $V = W$. The configuration of $2n$ particles in $X$ will consist of $n$ particles $v_1, \ldots, v_n$ in $V$ and $n$ particles $w_1, \ldots, w_n$ in $W$ in such a way that the configurations of particles in $V$ and $W$ are identical (i.e. $v_j = w_j$, $j = 1, \ldots, n$). Let $\xi_j$, $\psi_j$, $j = 1, \ldots, n$ be some functions on $V$. We define $\{\phi_j\}$ and $\epsilon$ in (11) so that $\phi_j(v) = 0$, $v \in V$, $\phi_j(w) = \xi_j(w)$, $w \in W$, $j = 1, \ldots, n$, $\phi_j(v) = \psi_{j-n-1}(v)$, $v \in V$, $\phi_j(w) = 0$, $w \in W$, $j = n + 1, \ldots, 2n$, and $\epsilon(v_1, v_2) = 0$, $v_1, v_2 \in V$, $\epsilon(w_1, w_2) = 0$, $w_1, w_2 \in W$. The restriction of the measure $\lambda$ on both $V$ and $W$ is defined to be equal to $\mu$. Then (11) specializes into (see Corollary 1.5. in [9])
\[
p(v_1, \ldots, v_n) = const_n \det(\xi(v_i))_{i,j=1,\ldots,n} \det(\psi(v_i))_{i,j=1,\ldots,n}.
\]
Ensembles of the form (12) are known as biorthogonal ensembles (see [4], [1]). The statement of the Theorem 1.1 in the case (12) has been proven in the companion paper [3]. The special case of the biorthogonal ensemble (12) when $V = \mathbb{R}$, $\xi_j(x) = \psi_j(x) = x^{j-1}$, and $V = \{z \mid |z| = 1\}$, $\xi_j(z) = \overline{\psi_j(z)} = z^{j-1}$, such ensembles are well known in Random Matrix Theory as unitary ensembles, see [6] for details. An ensemble of the form (12) which is different from random matrix ensembles was studied in [7]. We specifically want to single out the polynomial ensemble with $\beta = 2$. 

3
Polynomial (\(\beta = 2\)) Ensembles.

Let \(X = \mathbb{R}\) or \(Z\), \(\phi_j(x) = x^j - 1\), \(j = 1, \ldots, 2n\), and \(\lambda(dx)\) has a density \(\omega(x)\) with respect to the reference measure on \(X\) (Lebesgue measure in the continuous case, counting measure in the discrete case). Then the formula (12) specializes into

\[
p(v_1, \ldots, v_n) = \text{const}_n \prod_{1 \leq i < j \leq n} (v_i - v_j)^2 \prod_{1 \leq j \leq n} \omega(v_j).
\]  

(13)

Polynomial (\(\beta = 1\)) Ensembles.

Let \(X = \mathbb{R}\) or \(Z\), \(\phi_j(x) = x^j - 1\), \(j = 1, \ldots, 2n\), \(\epsilon(x, y) = \frac{1}{2} \text{sgn}(y - x)\) and \(\lambda(dx)\) has a density \(\omega(x)\) with respect to the reference measure on \(X\) (Lebesgue measure in the continuous case, counting measure in the discrete case). Then the formula (11) specializes into the formula for the density of the joint distribution of \(2n\) particles in a so-called \(\beta = 1\) polynomial ensemble (see [9], Remark 1):

\[
p(x_1, \ldots, x_{2n}) = \text{const}_n \prod_{1 \leq i < j \leq 2n} |x_i - x_j| \prod_{1 \leq j \leq 2n} \omega(x_j).
\]  

(14)

In Random Matrix Theory the ensembles (14) in the continuous case are known as orthogonal ensembles, see [8].

Polynomial (\(\beta = 4\)) Ensembles.

Similar to the biorthogonal case (\(\beta = 2\)) let us consider the particle space to be the union of two identical measure spaces \((Y, \mu), (Z, \mu)\), \(X = Y \cup Z\), \(Y = Z\), where \(Y = \mathbb{R}\) or \(Y = \mathbb{Z}\). The configuration of \(2n\) particles \(x_1, \ldots, x_{2n}\), in \(X\) will consist of \(n\) particles \(y_1, \ldots, y_n\) in \(Y\) and \(n\) particles \(z_1, \ldots, z_n\), in \(Z\) in such a way that the configurations of particles in \(Y\) and \(Z\) are identical. We define \(\{\phi_j\}\) and \(\epsilon\) so that \(\phi_j(y) = y^j\), \(\epsilon(y, z) = jz^{j-1}\), \(z \in Z\), \(\epsilon(y_1, y_2) = 0\), \(\epsilon(z_1, z_2) = 0\), \(\epsilon(y, z) = -\epsilon(z, y) = \delta_{y,z}\). As above we assume that the measure \(\mu\) has a density \(\omega\) with respect to the reference measure on \(Y\). Then the formula (11) specializes into the formula for the density of the joint distribution of \(n\) particles in a \(\beta = 4\) polynomial ensemble (see Corollary 1.3. in [9])

\[
p(y_1, \ldots, y_n) = \text{const}_n \prod_{1 \leq i < j \leq n} (y_i - y_j)^4 \prod_{1 \leq j \leq n} \omega(y_j).
\]  

(15)

In Random Matrix Theory the ensembles (15) are known as symplectic ensembles, see [8].

3 Proof of the Main Result

Consider matrix kernels

\[
\mathcal{K}_I = -JK_I, \quad \mathcal{L}_I = -JL_I.
\]  

(16)

The the relation (7) simplifies into

\[
\mathcal{L}_I = \mathcal{K}_I(\text{Id} - \mathcal{K}_I)^{-1}
\]  

(17)

which is the same relation that is satisfied by the correlation and Janossy scalar kernels in the determinantal case (14, 2). The consideration of \(\mathcal{K}_I\) and \(\mathcal{L}_I\) is motivated by the fact that the pfaffians of the \(2k \times 2k\) matrices with the antisymmetric matrix kernels \(K_I\) and \(L_I\) are equal to the quaternion determinants (6) of \(2k \times 2k\) matrices with the kernels \(K_I\), \(L_I\) when the latter matrices
are viewed as $k \times k$ quaternion matrices (i.e. each quaternion entry corresponds to a $2 \times 2$ block with complex entries). It follows from (5) and (16) that the kernel $K_I$ is given by the formula
\[
K_I = \sum_{j,k=1,...,2n} M_{jk}^{\epsilon} \left( \begin{array}{cc} -(\epsilon\phi_j) \otimes \phi_k & -(\epsilon\phi_j) \otimes (\epsilon\phi_k) \\ \phi_j \otimes \phi_k & \phi_j \otimes (\epsilon\phi_k) \end{array} \right) + \left( \begin{array}{cc} 0 & \epsilon \\ 0 & 0 \end{array} \right).
\]  
Let us denote by $\tilde{L}_I$ the following kernel
\[
\tilde{L}_I(x,y) = \sum_{1 \leq j,k \leq 2n} (M^X \setminus I)^{\epsilon}_{jk} \left( \begin{array}{cc} -(\epsilon X \setminus I \phi_j) \otimes \phi_k & -(\epsilon X \setminus I \phi_j) \otimes (\epsilon X \setminus I \phi_k) \\ \phi_j \otimes \phi_k & \phi_j \otimes (\epsilon X \setminus I \phi_k) \end{array} \right) + \left( \begin{array}{cc} 0 & \epsilon \\ 0 & 0 \end{array} \right).
\]
As above, $\epsilon \phi$ stands for $\int_X \epsilon(x,y)\phi(y)$. We use the notation $\phi_j \otimes \phi_k$ is a shorthand for $\phi_j(x)\phi_k(y)$. To prove the main result of the paper we will show that $\tilde{L}_I = K_I(Id - K_I)^{-1}$ (in other words we are going to prove that $\tilde{L}_I = L_I$, where $L_I$ is defined in (17)). The proof relies on Lemmas 1 and 2 given below. Let us introduce the notation $(\epsilon_I \phi)(x) = \int_I \epsilon(x,y)\phi(y)d\lambda(y)$. We will show that the finite-dimensional subspace $\mathcal{H} = \text{Span}\left\{ \left( \begin{array}{c} \epsilon \phi_s \\ -\phi_s \end{array} \right), \left( \begin{array}{c} -\epsilon \phi_s \\ -\phi_s \end{array} \right), \left( \begin{array}{c} \epsilon_I \phi_s \\ 0 \end{array} \right) \right\}_{s=1,...,2n}$ is invariant under $K_I$ and $\tilde{L}_I$. The main part of the proof of the theorem is to show that $\tilde{L}_I = K_I(Id - K_I)^{-1}$ holds on $\mathcal{H}$.

**Lemma 1** The operators $K_I$, $\tilde{L}_I$ leave $\mathcal{H}$ invariant and $\tilde{L}_I = K_I(Id - K_I)^{-1}$ holds on $\mathcal{H}$.

Below we give the proof of the lemma. Using the notations introduced above in (8)-(10) one can easily calculate
\[
K_I \left( \begin{array}{c} \epsilon \phi_s \\ 0 \end{array} \right) = \sum_{j=1,...,2n} -(G^I)^{t}M^{-1}_{sj} \left( \begin{array}{c} \epsilon \phi_j \\ -\phi_j \end{array} \right),
\]
\[
K_I \left( \begin{array}{c} 0 \\ -\phi_s \end{array} \right) = \sum_{j=1,...,2n} (G^I M^{-1})_{sj} \left( \begin{array}{c} \epsilon \phi_j \\ -\phi_j \end{array} \right) - \left( \begin{array}{c} \epsilon_I \phi_s \\ 0 \end{array} \right).
\]
Defining the $2n \times 2n$ matrix $T$ as
\[
T_{sk} = \int_I \phi_s(x) \int_{X \setminus I} \epsilon(x,y)\phi_k(y)d\lambda(y)d\lambda(x)
\]
we compute
\[
K_I \left( \begin{array}{c} \epsilon \phi_s \\ 0 \end{array} \right) = \sum_{j=1,...,2n} ((G^I - T)M^{-1})_{sj} \left( \begin{array}{c} \epsilon \phi_j \\ -\phi_j \end{array} \right),
\]
where $(G^I - T)_{sk} = M^I_{sk} = \int_I \phi_s(x)\epsilon(x,y)\phi_k(y)d\lambda(x)d\lambda(dy)$. One can rewrite the equations (20)-(21) as
\[
K_I \left( \begin{array}{c} \epsilon \phi_s \\ -\phi_s \end{array} \right) = \sum_{j=1,...,2n} ((G^I - (G^I)^{t})M^{-1})^{t}_{sj} \left( \begin{array}{c} \epsilon \phi_j \\ -\phi_j \end{array} \right) - \left( \begin{array}{c} \epsilon_I \phi_s \\ 0 \end{array} \right),
\]
\[
K_I \left( \begin{array}{c} -\epsilon \phi_s \\ -\phi_s \end{array} \right) = \sum_{j=1,...,2n} ((G^I + (G^I)^{t})M^{-1})_{sj} \left( \begin{array}{c} \epsilon \phi_j \\ -\phi_j \end{array} \right) - \left( \begin{array}{c} \epsilon_I \phi_s \\ 0 \end{array} \right)
\]
We conclude that that the subspace $\mathcal{H}$ is indeed invariant under $\mathcal{K}_I$ and the matrix of the restriction of $\mathcal{K}_I$ on $\mathcal{H}$ has the following block structure in the basis \(\left\{ \left( \begin{array}{c} \epsilon_\phi s \\ -\phi_s \\ \phi_s \\ -\epsilon_\phi s \\ \epsilon_I \phi_s \\ 0 \end{array} \right) \right\}_{s=1, \ldots, 2n} : \)

\[
\begin{pmatrix}
(G^I - (G^I)^t)M^{-1} & (G^I + (G^I)^t)M^{-1} & (G^I - T)M^{-1} \\
0 & 0 & 0 \\
-I\text{Id} & -I\text{Id} & 0
\end{pmatrix}
\] (26)

(in particular $\text{Ran}(\mathcal{K}_I|_\mathcal{H}) = \text{Span}\left\{ \left( \begin{array}{c} \epsilon_\phi s \\ -\phi_s \\ \phi_s \\ -\epsilon_\phi s \\ \epsilon_I \phi_s \\ 0 \end{array} \right) \right\}_{s=1, \ldots, 2n}$. Let us introduce some additional notations:

\[
A = (G^I - (G^I)^t)M^{-1},
B = (G^I + (G^I)^t)M^{-1},
C = (G^I - T)M^{-1}.
\] (27-29)

When a matrix has a block form $\mathcal{M} = \begin{pmatrix} A & B & C \\ 0 & 0 & 0 \\ -I\text{Id} & -I\text{Id} & 0 \end{pmatrix}$ (as it is in our case) the matrix $\mathcal{M} \times (I\text{Id} - \mathcal{M})^{-1}$ has the block form

\[
\begin{pmatrix}
(I\text{Id} - A + C)^{-1} - I\text{Id} & (B - C)(I\text{Id} - A + C)^{-1} & C(I\text{Id} - A + C)^{-1} \\
0 & 0 & 0 \\
-(I\text{Id} - A + C)^{-1} & -I\text{Id} - (B - C)(I\text{Id} - A + C)^{-1} & -C(I\text{Id} - A + C)^{-1}
\end{pmatrix}
\] (30)

As one can see from the formulas (31)-(33) the invertibility of $I\text{Id} - \mathcal{M}$ follows from the invertibility of $M^{X|I}$ which has been assumed throughout the paper. We have

\[
\begin{align*}
(I\text{Id} - A + C)^{-1} & = M(M + (G^I)^t - T)^{-1} = M(M^{X|I})^{-1} \\
C(I\text{Id} - A + C)^{-1} & = (G^I - T)(M + (G^I)^t - T)^{-1} = (G^I)^t(M^{X|I})^{-1} \\
(B - C)(I\text{Id} - A + C)^{-1} & = ((G^I)^t + T)(M + (G^I)^t - T)^{-1} = ((G^I)^t + T)(M^{X|I})^{-1}.
\end{align*}
\] (31-33)

Let us now compute the matrix of the restriction of $\tilde{\mathcal{L}}_I$ on $\mathcal{H}$. We have

\[
\tilde{\mathcal{L}}_I = \sum_{j,k=1, \ldots, 2n} (M^{X|I})^t \begin{pmatrix} -(\epsilon_X \mathbf{1} | \phi_j) \otimes \phi_k \\ \phi_k \otimes \phi_k \end{pmatrix} + \begin{pmatrix} 0 & \epsilon_X \mathbf{1} | \phi_j \\ 0 & 0 \end{pmatrix}.
\] (34)

Similarly to the computations above one can see that $\mathcal{H}$ is invariant under $\tilde{\mathcal{L}}_I$ and

\[
\tilde{\mathcal{L}}_I \begin{pmatrix} \epsilon_\phi s \\ -\phi_s \end{pmatrix} = \sum_{j=1, \ldots, 2n} ((T - (G^I)^t)(M^{X|I})^{-1})_{sj} \begin{pmatrix} \epsilon_\phi_j \\ -\phi_j \end{pmatrix}
- \sum_{1 \leq j \leq 2n} ((T - (G^I)^t)(M^{X|I})^{-1})_{sj} \begin{pmatrix} \epsilon_I \phi_j \\ 0 \end{pmatrix} - \begin{pmatrix} \epsilon_I \phi_s \\ 0 \end{pmatrix},
\] (35)

\[
\tilde{\mathcal{L}}_I \begin{pmatrix} -\epsilon_\phi s \\ -\phi_s \end{pmatrix} = \sum_{j=1, \ldots, 2n} ((T + (G^I)^t)(M^{X|I})^{-1})_{sj} \begin{pmatrix} \epsilon_\phi_j \\ -\phi_j \end{pmatrix}
- \sum_{1 \leq j \leq 2n} ((T + (G^I)^t)(M^{X|I})^{-1})_{sj} \begin{pmatrix} \epsilon_I \phi_j \\ 0 \end{pmatrix} - \begin{pmatrix} \epsilon_I \phi_s \\ 0 \end{pmatrix},
\] (36)
and
\[ \mathcal{L}_I \left( \begin{array}{c} \epsilon I \phi_s \\ 0 \end{array} \right) = \sum_{j=1 \ldots 2n} ((G^I - T)(M^{X\setminus I})^{-1})_{sj} \left( \begin{array}{c} \epsilon \phi_j \\ -\phi_j \end{array} \right) - \sum_{j=1 \ldots 2n} ((G^I - T)(M^{X\setminus I})^{-1})_{sj} \left( \begin{array}{c} \epsilon I \phi_j \\ 0 \end{array} \right). \] (37)

Therefore the restriction of \( \mathcal{L}_I \) to \( \mathcal{H} \) in the basis \( \left\{ \left( \begin{array}{c} \epsilon \phi_s \\ -\phi_s \end{array} \right), \left( \begin{array}{c} -\epsilon \phi_s \\ -\phi_s \end{array} \right), \left( \begin{array}{c} \epsilon I \phi_s \\ 0 \end{array} \right) \right\} \) has the following block structure
\[ \left( \begin{array}{ccc} (T - (G^I)^t)(M^{X\setminus I})^{-1} & (T + (G^I)^t)(M^{X\setminus I})^{-1} & (G^I - T)(M^{X\setminus I})^{-1} \\ 0 & 0 & 0 \\ -Id - (T - (G^I)^t)(M^{X\setminus I})^{-1} & -Id - (T + (G^I)^t)(M^{X\setminus I})^{-1} & -(G^I - T)(M^{X\setminus I})^{-1} \end{array} \right). \] (38)

Comparing (30), (31)-(33) and (38) we see that \( \mathcal{L}_I = \mathcal{K}_I(Id - \mathcal{K}_I)^{-1} \) on \( \mathcal{H} \). Lemma is proven.

To show that \( \mathcal{L}_I = \mathcal{K}_I(Id - \mathcal{K}_I)^{-1} \) also holds on the complement of \( \mathcal{H} \) it is enough to prove it on the subspaces \( \left( (\mathcal{H}_1)^\perp \right) \), and \( \left( (\mathcal{H}_2)^\perp \right) \), where \( \mathcal{H}_1 = Span(\epsilon I \phi_s)_{k=1 \ldots 2n} \) and \( \mathcal{H}_2 = Span(\overline{\phi}_k)_{k=1 \ldots 2n} \). The inveribility of the matrix \( M_I \) implies that actually it is enough to prove \( \mathcal{L}_I = \mathcal{K}_I(Id - \mathcal{K}_I)^{-1} \) on the subspaces \( \left( (\mathcal{H}_2)^\perp \right) \), and \( \left( (\mathcal{H}_1)^\perp \right) \). Here we use the standard notation \( (\mathcal{H}_k)^\perp \) for the orthogonal complement in \( L^2(I) \) with the standard scalar product \( (f,g)_I = \int_I \times \overline{f(x)} g(x) d\lambda(x) \). We start with the first subspace.

**Lemma 2** The relation \( \mathcal{L}_I = \mathcal{K}_I(Id - \mathcal{K}_I)^{-1} \) holds on \( \left( 0 \right) \).

The proof is a straightforward check. The notations are slightly simplified when the functions \( \left\{ \epsilon I \phi_k, \epsilon \phi_k, k = 1 \ldots 2n \right\} \) are linearly independent in \( L^2(I) \). The degenerate case is left to the reader. Consider \( f_s \in (\mathcal{H}_1)^\perp, s = 1 \ldots 2n \) such that
\[ \left( \epsilon \phi_k, f_s \right)_I = \left( \epsilon \phi_k, \phi_s \right)_I, k = 1 \ldots 2n. \] (39)

We are going to establish the relation for \( \left( 0 \right) \), which then immediately extends by linearity to the linear combinations of \( \left( 0 \right) \). We write
\[ \mathcal{K}_I \left( \begin{array}{c} 0 \\ -f_s \end{array} \right) = \sum_{j,k=1 \ldots 2n} M_{jk}^{-1}(\epsilon \phi_k, -f_s)_I \left( \begin{array}{c} \epsilon \phi_j \\ -\phi_j \end{array} \right) - \left( \begin{array}{c} \epsilon I f_s \\ 0 \end{array} \right) \]
\[ = \sum_{j,k=1 \ldots 2n} M_{jk}^{-1}(\epsilon \phi_k, -\phi_s)_I \left( \begin{array}{c} -\epsilon \phi_j \\ \phi_j \end{array} \right) - \left( \begin{array}{c} \epsilon I f_s \\ 0 \end{array} \right) \]
\[ = \sum_{j=1 \ldots 2n} (G^I M_{sj}^{-1}) \left( \begin{array}{c} \epsilon \phi_j \\ -\phi_j \end{array} \right) - \left( \begin{array}{c} \epsilon I f_s \\ 0 \end{array} \right) \] (40)

(we have used (39) in the second equality) and
\[ \mathcal{K}_I \left( \begin{array}{c} \epsilon I \phi_s \\ -f_s \end{array} \right) = \sum_{j=1 \ldots 2n} ((G^I - (G^I)^t)M^{-1})_{sj} \left( \begin{array}{c} \epsilon \phi_j \\ -\phi_j \end{array} \right) - \left( \begin{array}{c} \epsilon I f_s \\ 0 \end{array} \right). \] (41)
Combining (40) and (41) we get
\[
\mathcal{K}_I \left( \begin{array}{c} -\epsilon_I \phi_s \\ -f_s \end{array} \right) = \sum_{j=1}^{2n} \left( (G^I + (G^I)^t)M^{-1} \right)_{sj} \left( \begin{array}{c} \epsilon_i \phi_j \\ -\phi_j \end{array} \right) - \left( \begin{array}{c} \epsilon_I f_s \\ 0 \end{array} \right),
\]
(42)

Similarly to (23) we compute
\[
\mathcal{K}_I \left( \begin{array}{c} \epsilon_I \phi_s \\ 0 \end{array} \right) = \sum_{j=1}^{2n} \left( (G^I - T)M^{-1} \right)_{sj} \left( \begin{array}{c} \epsilon_i \phi_j \\ -\phi_j \end{array} \right)
\]
(43)

It should be noted that \( \mathcal{K}_I \left( \begin{array}{c} \epsilon_I f_s \\ 0 \end{array} \right) = 0 \) because \( \int_I (\epsilon_I \phi_s)(x)\phi_j(x)d\lambda(x) = -\int_I f_s(x)(\epsilon_I \phi_j)(x) \times d\lambda(x) = 0 \) for all \( j = 1, \ldots, 2n \). This together with (39) allows us to conclude that the calculation of \( \mathcal{K}_I(Id - \mathcal{K}_I)^{-1} \left( \begin{array}{c} 0 \\ f_s \end{array} \right) \) is almost identical to the calculation of \( \mathcal{K}_I(Id - \mathcal{K}_I)^{-1} \left( \begin{array}{c} 0 \\ \phi_s \end{array} \right) \) with the only difference that in the former one we have to replace the term \(- \left( \begin{array}{c} \epsilon_I \phi_s \\ 0 \end{array} \right)\) by \(- \left( \begin{array}{c} \epsilon_I f_s \\ 0 \end{array} \right)\) (see the last equation of (41)). Namely
\[
\mathcal{K}_I(Id - \mathcal{K}_I)^{-1} \left( \begin{array}{c} 0 \\ f_s \end{array} \right) = \mathcal{K}_I(Id - \mathcal{K}_I)^{-1} \left( \begin{array}{c} \epsilon_i \phi_s \\ 0 \end{array} \right) + \left( \begin{array}{c} -\epsilon_i \phi_s \\ f_s \end{array} \right)
\]
\[
= \sum_{j=1}^{2n} \frac{1}{2} \left( (A + B)(Id - A + C)^{-1} \right)_{sj} \left( \begin{array}{c} \epsilon_i \phi_j \\ -\phi_j \end{array} \right) - \sum_{j=1}^{2n} \frac{1}{2} \left( (A + B)(Id - A + C)^{-1} \right)_{sj} \left( \begin{array}{c} \epsilon_I \phi_j \\ 0 \end{array} \right) - \left( \begin{array}{c} \epsilon_I f_s \\ 0 \end{array} \right)
\]
\[
= \sum_{j=1}^{2n} \left[ G^I(M^{X \setminus I})^{-1} \right]_{sj} \left[ \left( \begin{array}{c} \epsilon_i \phi_j \\ -\phi_j \end{array} \right) - \left( \begin{array}{c} \epsilon_I \phi_j \\ 0 \end{array} \right) \right] - \left( \begin{array}{c} \epsilon_I f_s \\ 0 \end{array} \right)
\]
(44)

where \( A, B, C \) are defined in (27)-(29). At the same time
\[
\tilde{\mathcal{L}}_I \left( \begin{array}{c} 0 \\ -f_s \end{array} \right) = \sum_{j,k=1}^{2n} \left( (M^{X \setminus I})^{-t} \right)_{jk} \left( \begin{array}{c} \epsilon_i \phi_j \\ -\phi_j \end{array} \right) e_{X \setminus I} \phi_k, f_s \right) - \left( \begin{array}{c} \epsilon_I f_s \\ 0 \end{array} \right)
\]
\[
= \sum_{j,k=1}^{2n} \left( (M^{X \setminus I})^{-t} \right)_{jk} \left( \begin{array}{c} \epsilon_i \phi_j \\ -\phi_j \end{array} \right) \phi_k, f_s \right) - \left( \begin{array}{c} \epsilon_I f_s \\ 0 \end{array} \right)
\]
\[
= \left[ G^I(M^{X \setminus I})^{-1} \right]_{sj} \left[ \left( \begin{array}{c} \epsilon_i \phi_j \\ -\phi_j \end{array} \right) - \left( \begin{array}{c} \epsilon_I \phi_j \\ 0 \end{array} \right) \right] - \left( \begin{array}{c} \epsilon_I f_s \\ 0 \end{array} \right)
\]
(45)

Therefore \( \tilde{\mathcal{L}}_I \left( \begin{array}{c} 0 \\ -f_s \end{array} \right) = \mathcal{K}_I(Id - \mathcal{K}_I)^{-1} \left( \begin{array}{c} 0 \\ -f_s \end{array} \right), \ s = 1, \ldots, 2n \). By linearity result follows for all \( \left( \begin{array}{c} 0 \\ f \end{array} \right) \) such that \( (\epsilon_i \phi_k, f)I = \int_I (\epsilon_i \phi_k)(x)f(x)d\lambda(x) = 0, \ k, j = 1, \ldots, 2n \). Lemma 2 is proven.

To check (17) on \( (\mathcal{H}_2)^\perp \) we note that \( \mathcal{K}_I(Id - \mathcal{K}_I)^{-1} \left( \begin{array}{c} g \\ 0 \end{array} \right) = \tilde{\mathcal{L}}_I \left( \begin{array}{c} g \\ 0 \end{array} \right) = 0 \) for \( g \) such that \( \int_I g(x)\phi_k(x)d\lambda(x) = 0, \ k = 1, \ldots, 2n \), which together with the invertibility of \( M \) finishes the proof. Theorem is proven.
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