THE CONICAL KÄHLER-RICCI FLOW ON FANO MANIFOLDS

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Abstract. In this paper, we obtain a long time solution for the conical Kähler-Ricci flow on Fano manifold by limiting a sequence of the generalized Kähler-Ricci flows. Under a restriction on the cone angle, by obtain uniform Perelman’s estimates for generalized Kähler-Ricci flows, we prove that the conical Kähler-Ricci flow must converge to a conical Kähler-Einstein metric if there exists one.

1. Introduction

Let $M$ be a compact complex manifold with Kähler metric $\omega_0$. Finding a Kähler-Einstein metric in a given Kähler class $[\omega_0]$ is an important problem in Kähler geometry, that is, when $2\pi c_1(M) = \lambda [\omega_0]$, establishing whether there exists a unique Kähler metric $\omega \in [\omega_0]$, such that $Ric(\omega) = \lambda \omega$. One approach to this problem is the continuous method, see the works of T. Aubin and S.T. Yau (1), (4-4). The other approach is the Kähler-Ricci flow, which was first used by H.D. Cao in [5] to give a parabolic proof of the Calabi-Yau theorem. In recent years, the convergence of Kähler-Ricci flow has become one main object of geometry analysis. H.D. Cao, B.L. Chen, X.X. Chen, S. Donaldson, G. Perelman, D.H. Phong, J. Song, G. Tian, X.P. Zhu, X.H. Zhu and others have done substantial work on this problem (see [6], [10], [11], [16], [30], [32], [33], [37], [42] etc).

Recently the conical Kähler-Einstein metric has received considerable attention. The Riemann surfaces situation was well studied by M. Troyanov [41] and R. McOwen [29]. The high dimensional was first considered by G. Tian in [39]. The renewed interest has been sparked by S. Donaldson’s project of using conical Kähler-Einstein metrics as a continuity method of solving smooth Kähler-Einstein problem on Fano manifolds in [17]. The year before last, Yau-Tian-Donaldson’s conjecture in Fano case was proved by X.X. Chen, S. Donaldson and S. Sun in [7], [8], [9], G. Tian in [40] respectively. The existence of conical Kähler-Einstein metrics still has its own interest, there is by now a large body of work on such conical Kähler-Einstein metrics, see for instance, R. Berman [2], S. Brendle [3], F. Campana, H. Guenancia and M. Păun [4], P. Eyssidieux, V. Guedj and A. Zeriahi [18], H. Guenancia and M. Păun [19], T. Jeffres, R. Mazzeo and Y. Rubinstein [21], C. Li and S. Sun [24], J. Song and X.W. Wang [38], and many others. In [12], X.X. Chen and Y.Q. Wang introduced the strong conical Kähler-Ricci flow and established the short time existence for it. When $n = 1$, H. Yin in [16], [17], R. Mazzeo, Y. Rubinstein and N. Sesum in [28] did it with different function space.
Let $M$ be a Fano manifold of complex dimension $n$ and $D \in \left| -\lambda K_M \right|$ be a smooth divisor. A conical Kähler metric on $M$ with angle $2\pi\beta$ ($0 < \beta \leq 1$) along $D$ is a closed positive $(1, 1)$ current in $2\pi c_1(M)$ and is a smooth Kähler metric in $M \setminus D$ that is asymptotically equivalent along $D$ to the model conic metric
\[ \sqrt{-1}z^n 2^\beta - 2 dz^n \wedge d\bar{z}^n + \sqrt{-1}\sum_{j=1}^{n-1} dz^j \wedge d\bar{z}^j, \]
where $(z^1, \ldots, z^{n-1}, z^n)$ are local holomorphic coordinates such that $D = \{z^n = 0\}$.

We call $\omega$ a conic Kähler-Einstein metric with conic angle $2\pi\beta$ along $D$ if it is a conic Kähler metric and satisfies
\[ \text{Ric}(\omega) = \mu\omega + 2\pi(1 - \beta)[D], \tag{1.1} \]
where $[D]$ is the current of integration along $D$. Here the equation is classical outside $D$ and it is in the sense of currents globally on $M$. There are other definitions of metrics with cone singularities (see [14], [21], etc), which for conical Kähler-Einstein metrics, turn out to be equivalent (see Theorem 2 in [21]).

In this paper, we study the following conical Kähler-Ricci flow on $M$,
\[ \frac{\partial \omega}{\partial t} = -\text{Ric}(\omega) + \beta\omega + (1 - \beta)[D], \tag{1.2} \]
starting with a conical Kähler metric with cone angle $2\pi\beta$ along the divisor $D$. We say $\omega(t)$ ($t \in [0, +\infty)$) is a long time solution of the above conical Kähler-Ricci flow means that every $\omega(t)$ is a conical Kähler metric with conic angle $2\pi\beta$ along $D$, it satisfies (1.2) in sense of currents globally on $M$ and reduces exactly to the usual Kähler-Ricci flow outside $D$, i.e.
\[ \frac{\partial \omega}{\partial t} = -\text{Ric}(\omega) + \beta\omega. \]

For our research, we will combine the conical Kähler-Ricci flow with the generalized Kähler-Ricci flow. Assume the Kähler class and the first Chern class satisfy $2\pi c_1(M) - k[\omega_0] = [\alpha] \neq 0$, fixing a closed $(1, 1)$-form $\theta \in [\alpha]$. The generalized Kähler-Einstein metric
\[ \text{Ric}(\omega) = \beta\omega + \theta, \tag{1.3} \]
and the generalized Kähler-Ricci flow
\[ \frac{\partial \omega}{\partial t} = -\text{Ric}(\omega) + \beta\omega + \theta, \tag{1.4} \]
are studied respectively by X. Zhang and X.W. Zhang in [49], the first author in [26] [27], T. Collins and G. Székelyhidi in [14].

In this paper, we assume that $D \in \left| -K_M \right|$. Let $\omega_0$ be a smooth Kähler metric in $2\pi c_1(M)$, $h$ be a smooth Hermitian metric on the line bundle $-K_M$ with curvature $\omega_0$ and $s$ be the defining section of $D$. It is well know that, for small $k$,
\[ \omega^* = \omega_0 + k\sqrt{-1}\partial\bar{\partial}|s|^2_h \tag{1.5} \]
is a conic Kähler metric with cone angle $2\pi\beta$ along $D$. As in [4], We also denote
\[ \omega_\varepsilon = \omega_0 + \sqrt{-1}k\partial\bar{\partial}\chi(\varepsilon^2 + |s|^2_h), \tag{1.6} \]
where
\[ \chi(\varepsilon^2 + t) = \frac{1}{\beta}\int_0^t \frac{(\varepsilon^2 + r^2)^{\beta} - \varepsilon^{2\beta}r}{r}dr, \tag{1.7} \]
$k$ is a sufficiently small number such that $\omega_\varepsilon$ is a Kähler form for each $\varepsilon > 0$. It is easy to see that $\omega_\varepsilon \to \omega^*$ in the sense of currents globally on $M$ and in $C^\infty_{\text{loc}}$ topology outside $D$. From [4], we know that the function $\chi(\varepsilon^2 + t)$ is smooth for each $\varepsilon > 0$, and there exists constants $C > 0$ and $\gamma > 0$ independent of $\varepsilon$ such that

\begin{equation}
0 \leq \chi(\varepsilon^2 + t) \leq C
\end{equation}

provided that $t$ belongs to a bounded interval and

\begin{equation}
\omega_\varepsilon \geq \gamma \omega_0.
\end{equation}

We consider the following generalized Kähler-Ricci flow:

\begin{equation}
\begin{cases}
\frac{\partial}{\partial t} \omega_{\phi_*} = -\text{Ric}(\omega_{\phi_*}) + \beta \omega_{\phi_*} + (1 - \beta)(\omega_0 + \sqrt{-1} \partial \bar{\partial} \log(\varepsilon^2 + |s_h^2|)) \\
\omega_{\phi_*}|_{t=0} = \omega_\varepsilon
\end{cases}
\end{equation}

where $\omega_{\phi_*} = \omega_\varepsilon + \sqrt{-1} \partial \bar{\partial} \phi_\varepsilon$. We can see that $(1 - \beta)(\omega_0 + \sqrt{-1} \partial \bar{\partial} \log(\varepsilon^2 + |s_h^2|))$ is a smooth closed semi-positive $(1,1)$-form. Since the generalized Kähler-Ricci flow preserves the Kähler class, we can write this flow as the parabolic Monge-Ampère equation on potentials:

\begin{equation}
\begin{cases}
\frac{\partial \phi_\varepsilon}{\partial t} = \log \frac{\omega_n^0}{\omega_\varepsilon} + F_0 + \beta(k \chi + \phi_\varepsilon) + \log(\varepsilon^2 + |s_h^2|)^{1 - \beta} \\
\phi_\varepsilon|_{t=0} = c_{\varepsilon_0}
\end{cases}
\end{equation}

where the constant $c_{\varepsilon_0}$ is uniformly bounded for $\varepsilon$, we will give its representation in section 5, $F_0$ satisfies $-\text{Ric}(\omega_0) + \omega_0 = \sqrt{-1} \partial \bar{\partial} F_0$ and $\frac{1}{M} \int_M e^{-F_0} dV_0 = 1$, $\chi$ denotes the function $\chi(\varepsilon^2 + |s_h^2|)$. We will rewrite the flow (1.11) as follows:

\begin{equation}
\begin{cases}
\frac{\partial \phi_\varepsilon}{\partial t} = \log \frac{\omega_n^0}{\omega_\varepsilon} + F_\varepsilon + \beta(k \chi + \phi_\varepsilon) \\
\phi_\varepsilon|_{t=0} = c_{\varepsilon_0}
\end{cases}
\end{equation}

where $F_\varepsilon = F_0 + \log(\frac{\omega_n^0}{\omega_\varepsilon} \cdot (\varepsilon^2 + |s_h^2|)^{1 - \beta})$.

In our paper, we obtain a long time solution for the conical Kähler-Ricci flow (1.2) on Fano manifold by limiting a sequence of the generalized Kähler-Ricci flows (1.10) for $\beta \in (0, 1)$ as $\varepsilon \to 0$. When $\beta \in (0, \frac{1}{2}]$, we can obtain uniform Perelman’s estimates and uniform Sobolev inequality along the generalized Kähler-Ricci flows (1.10), here the uniform we mean that the constants in the estimates and inequality are independent of $\varepsilon$ and $t$. Using these estimates, we prove the conical Kähler-Ricci flow (1.2) must converge to a conical Kähler-Einstein metric in $C^\infty_{\text{loc}}$ topology outside the divisor $D$ if there exists one conical Kähler-Einstein metric. In fact, we prove the following theorem:

**Theorem 1.1** Let $\omega_{\phi_*}$ be a long time solution of the generalized Kähler-Ricci flow (1.10), then there must exist a sequence $\varepsilon_i \to 0$ such that $\omega_{\phi_{\varepsilon_i}}$ converge in $C^\infty_{\text{loc}}$ topology outside the divisor $D$ to a solution of the conical Kähler-Ricci flow

\begin{equation}
\begin{cases}
\frac{\partial \omega_{\phi_*}}{\partial t} = -\text{Ric}(\omega_{\phi_*}) + \beta \omega_{\phi_*} + (1 - \beta) |D| \\
\omega_{\phi_*}|_{t=0} = \omega^*
\end{cases}
\end{equation}
on $M \times [0, +\infty)$ in the sense of currents, where $\omega^*$ is the limit of $\omega_{\epsilon}$ when $\epsilon \to 0$, that is, $\omega^* = \omega_0 + \frac{k}{\sqrt{-1} \partial \bar{\partial}} |s_{h \epsilon}|^2$. Here the potential $\varphi(t)$ is Hölder continuous with respect to the smooth metric $\omega_0$ on $M$.

When $\beta \in (0, \frac{1}{2}]$, if there exists a conical Kähler-Einstein metric with cone angle $2\pi \beta$ along $D$, then the long time solution $\omega_{\varphi}(\cdot, t)$ must converge in $C^\infty_{\text{loc}}$ topology outside the divisor $D$ to a conical Kähler-Einstein metric.

Very recently, in [43], Y.Q. Wang using the same limiting method to prove the long time solution of conical Kähler-Ricci flow (1.2). In [13], X.X. Chen and Y.Q. Wang prove the existence of long time solution of the strong conical Kähler-Ricci flow, and obtain the convergence result when $\mu = 1 - (1 - \beta) \lambda \leq 0$, i.e. the twisted first Chern class is negative or zero.

This paper is organized as follows. In section 2, we obtain uniform Laplacian estimate and local $C^\infty$ estimates for the generalized Kähler-Ricci flow (1.10). Then we get a long time solution of the conical Kähler-Ricci flow (1.13) by limiting a sequence of generalized Kähler-Ricci flow in section 3. In section 4, assume $\beta \in (0, \frac{1}{2}]$, we obtain uniform Perelman’s estimates which independent of $\epsilon$ and $t$ for generalized Kähler-Ricci flows (1.10). Making use of these estimates, we prove uniform $C^0$ estimate for the metric potential with the uniform properness of twisted Mabuchi $K$-energy functional and the uniform Sobolev inequality along the flow (1.10) in section 5. At last section, we show that the properness of Log Mabuchi $K$-energy functional implies that the twisted Mabuchi $K$-energy functional $\mathcal{M}_{\omega_0, \theta_\epsilon}$ is uniformly proper. Then, we prove the conical Kähler-Ricci flow (1.13) must converge to a conical Kähler-Einstein metric in $C^\infty_{\text{loc}}$ topology outside the divisor $D$ if there exists one, where we will use the uniqueness of the conical Kähler-Einstein metric proved in [2].

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2. The local estimates for the generalized Kähler-Ricci flow

In this section, we will give the Laplacian estimate and local higher order estimates for the parabolic Monge-Ampère equation (1.12). In the following sections, a uniform constant means that it independent of $\epsilon$ and $t$. We shall use the letter $C$ for a uniform constant which may differ from line to line. We use Guenancia-Paun’s trick in [19] to obtain the Laplacian estimate, we have:

**Lemma 2.1** Let $\varphi_{\epsilon}$ be a solution of equation (1.12). Assume that there exists uniform constant $C > 0$ such that

1. $\sup_{M \times [0, T]} |\varphi_{\epsilon}| \leq C;
2. $\sup_{M \times [0, T]} |\dot{\varphi}_{\epsilon}| \leq C.$

Then there exists a uniform constant $A$ depending only on $\omega_0$, $n$, $\beta$ and $C$, such that

$$A^{-1} \omega_{\epsilon} \leq \omega_\epsilon + \sqrt{-1} \partial \bar{\partial} \varphi_{\epsilon} \leq A \omega_\epsilon$$

on $M \times [0, T]$.

Note that the estimates independent of the time $T$ and so the results holds also for time intervals $[0, T)$ or $[0, +\infty)$. The notion in the above relations is as follows:
in local coordinates, we write

\[ \omega = \sqrt{-1} g_{ij} dz^i \wedge dz^j, \]

the corresponding components of the curvature tensor are

\[ R_{ijkl} = \frac{\partial^2 g_{ij}}{\partial z^k \partial \bar{z}^l} - g_{ik} \frac{\partial g_{lj}}{\partial z^k} - g_{il} \frac{\partial g_{kj}}{\partial \bar{z}^l} - g_{kl} \frac{\partial g_{ij}}{\partial \bar{z}^l}, \]

and the Ricci curvature are

\[ R_{ij} = g^{kl} R_{ijkl}. \]

**Proof of Lemma 2.1:** We let \( \varphi_\varepsilon \) evolve by the parabolic Monge-Ampère equation \( (1) \). Through computing, we have

\[
\frac{d}{dt} \left( \Delta \omega_{\varphi_\varepsilon} \right) \log tr_{\omega_\varphi} \omega_{\varphi_\varepsilon} = \frac{1}{tr_{\omega_\varphi} \omega_{\varphi_\varepsilon}} (\Delta \omega_\varphi (\varphi_\varepsilon - \log \frac{\omega_\varepsilon^n}{\omega_0^n}) + R_{\omega_\varphi}) \\
- \frac{1}{tr_{\omega_\varphi} \omega_{\varphi_\varepsilon}} (g_{\varphi_\varepsilon}^{\partial \delta} g_{\varphi_\varepsilon}^{\partial \delta} R_{\omega_\varphi}^{\partial \gamma \gamma} + \frac{g_{\partial \delta}^{\beta \gamma} \partial \delta \omega_\varphi \partial \delta \omega_\varphi \omega_\varphi_\varepsilon}{(tr_{\omega_\varphi} \omega_{\varphi_\varepsilon})^2} - \frac{g_{\varphi_\varepsilon}^{\partial \delta} \varphi_\varepsilon^{\partial \delta} \varphi_\varepsilon^{\partial \delta}}{tr_{\omega_\varphi} \omega_{\varphi_\varepsilon}}). 
\]

When we choose a locally holomorphic basis, which such that \( (g_{\varphi_\varepsilon}) \) be identity and \( (g_{\varphi_\varepsilon}^{\partial \delta}) \) be diagonal matrix. Since \( (g_{\varphi_\varepsilon}^{\partial \delta}) \) is positive definite, so we have \( g_{\varphi_\varepsilon}^{\partial \delta} \geq 1 + \varphi_\varepsilon^{\partial \delta} > 0 \). Through computing, we have

\[
\frac{g_{\varphi_\varepsilon}^{\partial \delta} \partial \delta \omega_\varphi \partial \delta \omega_\varphi \omega_\varphi_\varepsilon}{(tr_{\omega_\varphi} \omega_{\varphi_\varepsilon})^2} - \frac{g_{\varphi_\varepsilon}^{\partial \delta} \varphi_\varepsilon^{\partial \delta} \varphi_\varepsilon^{\partial \delta}}{tr_{\omega_\varphi} \omega_{\varphi_\varepsilon}} \leq 0, 
\]

On the other hand,

\[
n = tr_{\omega_\varphi} \omega_0 + k \Delta \omega_\varphi \chi \geq k \Delta \omega_\varphi \chi. 
\]

Then we put \((1)\), \((2)\) and \((3)\) into \((1)\), so

\[
\left( \frac{d}{dt} - \Delta \omega_{\varphi_\varepsilon} \right) \log tr_{\omega_\varphi} \omega_{\varphi_\varepsilon} \leq \frac{1}{tr_{\omega_\varphi} \omega_{\varphi_\varepsilon}} \sum_{i \leq j} \frac{1 + \varphi_{\varepsilon^{\partial \delta}}}{1 + \varphi_{\varepsilon^{\partial \delta}}} + \frac{1 + \varphi_{\varepsilon^{\partial \delta}}}{1 + \varphi_{\varepsilon^{\partial \delta}}} - 2)R_{\omega_\varphi} \omega_{\varphi_\varepsilon} \omega_{\varphi_\varepsilon} \\
+ \frac{1}{tr_{\omega_\varphi} \omega_{\varphi_\varepsilon}} (\Delta \omega_\varphi (F_\varepsilon + \beta \varphi + k \beta \chi)) \\
= \frac{1}{tr_{\omega_\varphi} \omega_{\varphi_\varepsilon}} \sum_{i \leq j} \frac{1 + \varphi_{\varepsilon^{\partial \delta}}}{1 + \varphi_{\varepsilon^{\partial \delta}}} + \frac{1 + \varphi_{\varepsilon^{\partial \delta}}}{1 + \varphi_{\varepsilon^{\partial \delta}}} - 2)R_{\omega_\varphi} \omega_{\varphi_\varepsilon} \omega_{\varphi_\varepsilon} \\
+ \frac{1}{tr_{\omega_\varphi} \omega_{\varphi_\varepsilon}} (\Delta \omega_\varphi (F_\varepsilon + k \beta \chi)) + \beta \sum_{i}(1 + \varphi_{\varepsilon^{\partial \delta}}) \\
\leq \frac{1}{tr_{\omega_\varphi} \omega_{\varphi_\varepsilon}} \sum_{i \leq j} \frac{1 + \varphi_{\varepsilon^{\partial \delta}}}{1 + \varphi_{\varepsilon^{\partial \delta}}} + \frac{1 + \varphi_{\varepsilon^{\partial \delta}}}{1 + \varphi_{\varepsilon^{\partial \delta}}} - 2)R_{\omega_\varphi} \omega_{\varphi_\varepsilon} \omega_{\varphi_\varepsilon} \\
+ \frac{1}{tr_{\omega_\varphi} \omega_{\varphi_\varepsilon}} (\Delta \omega_\varphi F_\varepsilon) + \frac{\beta n}{tr_{\omega_\varphi} \omega_{\varphi_\varepsilon}} + \beta - \beta \sum_{i}(1 + \varphi_{\varepsilon^{\partial \delta}}) 
\]
we have

\[ (2.5) \]

which shows that

\[ (2.4) \]

and thus

\[ (2.3) \]

where we use the fact that

\[ (2.2) \]

After taking suitable uniform constants \( \tilde{\gamma} \), we have

\[ 0 \leq \frac{1}{\text{tr}_{\omega_s}(\sqrt{-1}\partial\bar{\partial}F_0 + C\omega_0)} \leq \gamma^{-1}(Cn + \Delta_{\omega_0}F_0) \]

and thus

\[ (2.4) \]

which shows that \( \Delta_{\omega_0}F_0 \) is uniformly bounded. We denote \( \Psi_{\varepsilon,\rho} = \tilde{\xi}_\rho(\varepsilon^2 + |s|_h^2) \), where

\[ \chi_\rho(\varepsilon^2 + |s|_h^2) = \frac{1}{\rho} \int_0^{s^2} \frac{(s^2 + r^2)^{\rho} - r^{2\rho}}{r} dr. \]

After taking suitable uniform constants \( \tilde{C} \) and \( \rho \), H. Guenancia and M. Păun have proved the following inequality (see (21) in [19])

\[ (2.5) \]

Combine (2.4) and (2.5), we have

\[ \frac{d}{dt} - \Delta_{\omega_\phi}(\log \text{tr}_{\omega_s}(\omega_\phi + \Psi_{\varepsilon,\rho})) \]

\[ \leq \frac{C}{\text{tr}_{\omega_s}(\omega_\phi)} \sum_{i,j}(\frac{1}{1 + \varphi_{i,j}} + \frac{1}{1 + \varphi_{e_{ij}}}) + \frac{C}{\text{tr}_{\omega_s}(\omega_\phi)} \]

\[ + C\text{tr}_{\omega_s}(\omega_\phi) + C \]

\[ = \frac{C}{\text{tr}_{\omega_s}(\omega_\phi)}((\sum_i(\frac{1}{1 + \varphi_{e_i}}))(\sum_j(1 + \varphi_{e_{ij}}) + n)) \]

\[ + C\text{tr}_{\omega_s}(\omega_\phi) + C \]

where we use the fact \( n \leq \text{tr}_{\omega_s}(\omega_\phi + \text{tr}_{\omega_s}(\omega_\phi)) \) in the last inequality. Hence we have the following inequality

\[ \frac{d}{dt} - \Delta_{\omega_\phi}(\log \text{tr}_{\omega_s}(\omega_\phi + \Psi_{\varepsilon,\rho} - B\varphi_\varepsilon)) \]

\[ \leq C\text{tr}_{\omega_s}(\omega_\phi - B\varphi_\varepsilon) + B\Delta_{\omega_\phi}(-\varphi_\varepsilon + C) \]

\[ \leq -\text{tr}_{\omega_s}(\omega_\phi) + C, \]

where we take \( B = C + 1 \).

By the maximum principle, at the maximum point \( p \) of \( \log \text{tr}_{\omega_s}(\omega_\phi + \Psi_{\varepsilon,\rho} - B\varphi_\varepsilon) \), we have

\[ \text{tr}_{\omega_s}(\omega_\phi)(p) \leq C. \]
Combining with the fact that $F_\varepsilon$ is uniformly bounded (see (25) [4]), we obtain

$$tr_{\omega_\varepsilon} \omega_{\varphi_\varepsilon}(p) \leq \frac{1}{(n-1)!}(tr_{\omega_\varepsilon} \omega_\varepsilon)^{n-1}(p) \frac{\omega^n_\varepsilon(p)}{\omega^n_0(p)} \leq C \exp(\varphi_\varepsilon - F_\varepsilon - \beta \varphi_\varepsilon - k\beta \chi)(p) \leq C,$$

hence we have

$$tr_{\omega_\varepsilon} \omega_{\varphi_\varepsilon} \leq \exp(C + B \varphi_\varepsilon - B \varphi_\varepsilon(p)) \leq C.$$

On the other hand, by the conditions on $\varphi_\varepsilon$ and $\varphi_\varepsilon$, we conclude

$$C^{-1} \leq \frac{(\omega_\varepsilon + \sqrt{-1} \partial \bar{\partial} \varphi_\varepsilon)^n}{\omega^n_\varepsilon} = \exp(\varphi_\varepsilon - F_\varepsilon - \beta \varphi_\varepsilon - k\beta \chi) \leq C.$$

So there exists a uniform constant $A$ such that

$$(2.6) \quad A^{-1} \omega_\varepsilon \leq \omega_\varepsilon + \sqrt{-1} \partial \bar{\partial} \varphi_\varepsilon \leq A \omega_\varepsilon$$

for any $\varepsilon$ and $t$. \hfill $\square$

Now we consider the local Calabi’s $C^3$ estimate and higher order estimates to the generalized Kähler-Ricci flow:

$$(2.7) \quad \frac{\partial \omega_\varepsilon}{\partial t} = -\text{Ric}(\omega_\varepsilon) + \lambda \omega_\varepsilon + \theta,$$

where $c_1(M) = \lambda[\omega_0] + [\theta]$, $\omega_\varepsilon = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi$ and $\theta$ is a smooth semi-positive closed (1,1) form. The above flow is equivalent to the following parabolic Monge-Ampère equation for metric potential $\varphi$

$$(2.8) \quad \frac{\partial \varphi}{\partial t} = \log \frac{\omega^n_\varepsilon}{\omega^n_0} + f + \lambda \varphi,$$

where $f$ is the twisted Ricci potential, i.e. $\sqrt{-1} \partial \bar{\partial} f = -\text{Ric}(\omega_0) + \lambda \omega_0 + \theta$. Let

$$S = |\nabla_0 g_{\varphi}|^2_{\omega_\varepsilon} = g_{\varphi}^{ij} k^{il} \nabla_0 g_{\varphi}^{pq} \nabla_0 g_{\varphi}^{kj} \nabla_0 g_{\varphi}^{pl},$$

where $\nabla_0$ denotes the covariant derivative with respect to the metric $\omega_0$. Define $h^i_k = g_0^{il} g_{\varphi}^{lk}$ and $X^k_0 = (\nabla_0 h^{-1})^k_l$, by direct computation, we have

$$X^k_0 = \Gamma^k_{il} - \Gamma^k_{0l}$$

$$S = |X|^2_{\omega_\varepsilon},$$

$$\nabla_{\varphi^n} V^k_l - \nabla_{0m} V^k_l = X^k_{ms} V^s_l - X^k_{ml} V^s_l.$$

Here we let $\nabla_{\varphi}$ and $\Gamma_{\varphi}$ be the covariant derivatives and Christoffel symbols respectively under the metric $\omega_\varepsilon$, and $\Gamma_0$ be the Christoffel symbols with respect to the metric $\omega_0$. In the following, the norms $\|\cdot\|_{C^k}$ and $\|\cdot\|_{C^{k,\alpha}}$ are all with respect to the fixed metric $\omega_0$ unless there is a special statement. We also denote the curvature tensor of $\omega_\varepsilon$ by $Rm_{\varphi}$ for simplicity.

**Proposition 2.2** Let $\varphi(\cdot, t)$ be a solution of the equation (2.8) and satisfy

$$N^{-1} \omega_0 \leq \omega_\varepsilon \leq N \omega_0 \quad \text{on} \quad B_r(p) \times [0, T].$$

Then there exist constant $C'$ and $C''$ such that

$$S \leq \frac{C'}{r^s},$$

$$|Rm_{\varphi}|^2_{\omega_\varepsilon} \leq \frac{C''}{r^t}.$$
on $B_r(p) \times [0, T]$. The constant $C'$ depends only on $\omega_0$, $N$, $\lambda$, $\|\varphi(\cdot, 0)\|_{C^3(B_r(p))}$ and $\|\theta\|_{C^1(B_r(p))}$; constant $C''$ depends only on $\omega_0$, $N$, $\lambda$, $\|\varphi(\cdot, 0)\|_{C^4(B_r(p))}$ and $\|\theta\|_{C^2(B_r(p))}$.

Furthermore, there exist constants $C_1$, $C_2$, and $C_3$ such that

$$|D^k Rm \varphi|^2 \leq C_1, \quad \|\varphi\|_{C^{k+1, 0}} \leq C_2, \quad \|\varphi\|_{C^{k+3, 0}} \leq C_3,$$

for any $k \geq 0$ on $B_r(p) \times [0, T]$. Here constants $C_1$, $C_2$, and $C_3$ depend only on $\omega_0$, $N$, $\lambda$, $\|\varphi(\cdot, 0)\|_{C^{k+2}(B_r(p))}$, $\|\theta\|_{C^{k+2}(B_r(p))}$, $\|\varphi\|_{C^{0}(B_r(p) \times [0, T])}$ and $\|f\|_{C^{0}(B_r(p))}$.

**Proof:** By direct calculation, we have

$$\left( \frac{d}{dt} - \Delta \varphi \right) S = g^\mu \nabla \varphi \theta_{\mu \gamma} g^{\gamma \beta} \left( (g^\alpha \nabla \varphi \theta_{\alpha \beta} + \nabla^\beta R_{\beta 0} \varphi \gamma \omega) \nabla^\mu + X_{\mu i} (g^\mu \nabla \varphi \theta_{i \beta} + \nabla^\beta R_{\beta 0} \varphi \gamma \omega) \right)$$

$$- X_{\mu i} \nabla^\mu (g^\alpha \nabla \varphi \theta_{\alpha \beta} g^{\beta \gamma} \theta_{\mu \gamma} + g^\beta \nabla \varphi \theta_{\mu \gamma} g^{\beta \gamma} \theta_{\gamma \omega} - g^\beta \nabla \varphi \theta_{\mu \gamma} g^{\beta \gamma} \theta_{\omega \gamma} + g^\beta \nabla \varphi \theta_{\mu \gamma} g^{\beta \gamma} \theta_{\gamma \omega})$$

$$- |\nabla \varphi X|^2 \omega - |\nabla \varphi X|^2 \omega - \lambda S.$$

Since

$$\nabla \varphi \theta_{\mu i} = \nabla_{0 \mu} \theta_{\mu i} - X_{\mu i} \theta_{\mu i},$$

$$\nabla \varphi R_{0 i} = \nabla_{0 \mu} R_{0 i} + X_{\mu i} R_{0 \mu} - X_{\mu i} R_{0 \mu} - X_{\mu i} R_{0 \mu}.$$

We have

$$\left( \frac{d}{dt} - \Delta \varphi \right) S \leq C(S + 1) - |\nabla \varphi X|^2 \omega - |\nabla \varphi X|^2 \omega.$$  \hspace{1cm} \text{(2.9)}

where $C$ depends only on $N$, $\lambda$, $\|Rm(o_0)\|_{C^1(B_r(p))}$ and $\|\theta\|_{C^1(B_r(p))}$. Write $r_0 = r$ and $\psi$ be a nonnegative $C^\infty$ cut-off function that is identically equal to 1 on $\overline{B_r(p)}$ and vanishes outside $B_r(p)$, where $r = r_0 > r_1 > \frac{r_0}{2}$, we may assume that

$$|\nabla \varphi X|^2 \omega, \quad \|\nabla \varphi X|^2 \omega \leq C \frac{r}{r}.$$  \hspace{1cm} \text{Through computation, we have}$

\begin{align*}
\frac{d}{dt} - \Delta \varphi) (\psi^2 S) & \leq C \frac{r}{r^2} S + C, \\
\frac{d}{dt} - \Delta \varphi) trh & = \lambda trh + g_{00}^\gamma \theta_{\gamma \omega} + g_{00}^\gamma \nabla \varphi \theta_{\gamma \omega} + g_{00}^\gamma R_{0 \gamma \omega} - g_{00}^\gamma \nabla \varphi \theta_{\gamma \omega} + g_{00}^\gamma \nabla \varphi \theta_{\gamma \omega}
\end{align*}$

\hspace{1cm} \text{From (2.10) and (2.11), we obtain}$

\begin{align*}
\frac{d}{dt} - \Delta \varphi) (\psi^2 S + B trh) & \leq \left( C \frac{r}{r^2} - \frac{B}{N} \right) S + (B + 1) C.
\end{align*}$

\hspace{1cm} \text{Let } (x_0, t_0) \text{ be the maximum point of } \psi^2 S + B trh \text{ on } \overline{B_r(p)} \times [0, T]. \text{ If } t_0 = 0, \text{ then } S \text{ is bounded by the initial data } \|\varphi(\cdot, 0)\|_{C^3(B_r(p))}. \text{ So, we assume for the}
moment that \( t_0 > 0 \) and that \( x_0 \) doesn’t lie in the boundary of \( B_\epsilon(p) \), by maximum
principle,
\begin{equation}
0 \leq \left( \frac{C}{r^2} - \frac{B}{N} \right) S(x_0, t_0) + (B + 1) C.
\end{equation}
Taking \( B = \frac{N(C + 1)}{r^2} \), we conclude \( S(x_0, t_0) \leq C \), where \( C \) independent of \( T \). Since
\( 0 \leq trh \leq nN \), we have
\begin{equation}
S \leq C + BnN \leq \frac{C}{r^2}, \quad \text{on } \overline{B_r(p)} \times [0, T],
\end{equation}
where the constant \( C \) depends only on \( N \), \( \lambda \), \( \|\varphi(0, 0)\|_{C^3(B_r(p))} \), \( \|\varphi\|_{C^1(B_r(p))} \) and \( \omega_0 \).

Direct calculation, we have
\begin{equation}
\left( \frac{d}{dt} - \Delta \omega_\varphi \right) R_{\varphi jik} = -R_{\varphi jil} p^q R_{\varphi lkp} - R_{\varphi jil} p^q R_{\varphi lkp} + R_{\varphi jil} p^q R_{\varphi lkp} - R_{\varphi jil} p^q R_{\varphi lkp}
\end{equation}
\begin{equation}
- R_{\varphi jih} R_{\varphi ilk} + \nabla_{\varphi j} \nabla_{\varphi k} \theta_{ij} + \lambda R_{\varphi jil}^h R_{\varphi ilk}^h - \theta_{jih} R_{\varphi ilk}^h R_{\varphi ilk}^h.
\end{equation}
\begin{equation}
\nabla_{\varphi j} \nabla_{\varphi j} \theta_{ij} = \nabla_{\varphi j} \nabla_{\varphi j} \theta_{ij} - X_{ij}^k \nabla_{\varphi j} \theta_{ik} - \nabla_{\varphi j} X_{ij}^k \theta_{kj}
\end{equation}
\begin{equation}
- X_{ij}^k \nabla_{\varphi j} \theta_{kj} + X_{ij}^k \nabla_{\varphi j} \theta_{kj}.
\end{equation}
\begin{equation}
\nabla_{\varphi j} X_{ij}^k = \partial_{ikj} R_{\varphi j}^i - R_{\varphi j}^i.
\end{equation}
By (2.15), (2.16) and (2.17), we have
\begin{equation}
\left( \frac{d}{dt} - \Delta \omega_\varphi \right) |Rm_\varphi|^2_{\omega_\varphi} \leq C |Rm_\varphi|^3_{\omega_\varphi} + C |Rm_\varphi|^2_{\omega_\varphi} + C |Rm_\varphi|_{\omega_\varphi} + CS^2 |Rm_\varphi|_{\omega_\varphi}
\end{equation}
\begin{equation}
+ CS |Rm_\varphi|_{\omega_\varphi} - |\nabla_\varphi Rm_\varphi|^2_{\omega_\varphi} - |\nabla_\varphi Rm_\varphi|^2_{\omega_\varphi}
\end{equation}
\begin{equation}
\leq C (|Rm_\varphi|^3_{\omega_\varphi} + 1 + |Rm_\varphi|_{\omega_\varphi}^2) - |\nabla_\varphi Rm_\varphi|^2_{\omega_\varphi} - |\nabla_\varphi Rm_\varphi|^2_{\omega_\varphi}.
\end{equation}

To show \( |Rm_\varphi|^2_{\omega_\varphi} \) is uniformly bounded, we will use an argument similar to the
previous part. We fix a smaller radius \( r_2 \) satisfying \( r_1 > r_2 > \frac{r}{2} \). Let \( \rho \) be a cut-off
function, identically 1 on \( \overline{B_{r_2}(p)} \) and identically 0 outside \( B_{r_1} \). As before we may assume
\begin{equation}
|\partial \rho|_{\omega_\varphi}, |\sqrt{-1} \partial \bar{\partial} \rho|_{\omega_\varphi} \leq \frac{C}{r^2}
\end{equation}
for some uniform constant \( C \). From the former part we know that \( S \) is bounded by
\( \frac{C}{r^2} \) on \( B_{r_1}(p) \). Let
\begin{equation}
K = \frac{\hat{C}}{r^2},
\end{equation}
where \( \hat{C} \) is a constant to be determined later, and it is large enough so that \( \frac{K}{2} \leq K - S \leq K \). We consider
\begin{equation}
F = \rho^2 \frac{|Rm_\varphi|^2_{\omega_\varphi}}{K - S} + AS.
\end{equation}
By computing, we have
\[
\left(\frac{d}{dt} - \triangle_{\omega_x}\right) F = (-\triangle_{\omega_x}\rho) \left( \frac{Rm_{\phi}|_{\omega_x}^2}{K - S} + \rho^2 \left( \frac{d}{dt} - \triangle_{\omega_x}\right) S \right) \\
+ \rho^2 \frac{1}{K - S} \left( \frac{d}{dt} - \triangle_{\omega_x}\right) |Rm_{\phi}|_{\omega_x}^2 - 4Re\langle \rho^{\frac{\nabla_{\phi}\rho}{K - S}}, \nabla_{\phi} |Rm_{\phi}|_{\omega_x}^2 \rangle_{\omega_x} \\
- 4Re\langle \rho^2 \nabla_{\phi} S, \nabla_{\phi} |Rm_{\phi}|_{\omega_x}^2 \rangle_{\omega_x} - 2\rho^2 |Rm_{\phi}|_{\omega_x}^2 \left( \frac{\nabla_{\phi} S}{(K - S)^3} \right) |\nabla_{\phi} \omega_x|_{\omega_x}^2 \\
- 2Re\langle \rho^2 \nabla_{\phi} S, \nabla_{\phi} |Rm_{\phi}|_{\omega_x}^2 \rangle_{\omega_x} + A \left( \frac{d}{dt} - \triangle_{\omega_x}\right) S.
\]

(2.21)

As in the previous part, we only consider at a inner point \((x_0, t_0)\) which is a maximum of \(F\) achieved on \(B_{r_1}(p) \times [0, T]\). We use the fact that \(\nabla F = 0\) at this point, then give us

\[
\frac{\partial}{\partial t} \rho \nabla_{\phi} |Rm_{\phi}|_{\omega_x}^2 = 0.
\]

(2.22)

Putting (2.22) into (2.21), we have

\[
\left(\frac{d}{dt} - \triangle_{\omega_x}\right) F = (-\triangle_{\omega_x}\rho) \left( \frac{Rm_{\phi}|_{\omega_x}^2}{K - S} + \rho^2 \left( \frac{d}{dt} - \triangle_{\omega_x}\right) S \right) \\
- 4Re\langle \rho^{\frac{\nabla_{\phi}\rho}{K - S}}, \nabla_{\phi} |Rm_{\phi}|_{\omega_x}^2 \rangle_{\omega_x} + \rho^2 \frac{1}{K - S} \left( \frac{d}{dt} - \triangle_{\omega_x}\right) |Rm_{\phi}|_{\omega_x}^2 \\
- 2A |\nabla_{\phi} S|^2 \left( \frac{1}{K - S} \right) + A \left( \frac{d}{dt} - \triangle_{\omega_x}\right) S.
\]

(2.23)

Our goal is to show that at \((x_0, t_0)\), we have \(|Rm_{\phi}|_{\omega_x}^2 \leq \frac{C}{r^2}\). Hence without loss of generality, we may assume that \(|Rm_{\phi}|_{\omega_x}^3 \geq 1 + \frac{|Rm_{\phi}|_{\omega_x}^2}{r^2}\). So

\[
\frac{\partial}{\partial t} - \triangle_{\omega_x}|Rm_{\phi}|_{\omega_x}^2 \leq C |Rm_{\phi}|_{\omega_x}^3 - |\nabla_{\phi} |Rm_{\phi}|_{\omega_x}^2 - |\nabla_{\phi} Rm_{\phi}|_{\omega_x}^2.
\]

(2.24)

Also note that

\[
|\nabla_{\phi} |Rm_{\phi}|_{\omega_x}^2 |_{\omega_x} \leq |Rm_{\phi}|_{\omega_x} (|\nabla_{\phi} Rm_{\phi}|_{\omega_x} + |\nabla_{\phi} Rm_{\phi}|_{\omega_x}),
\]

(2.25)

\[
|\nabla_{\phi} S|^2 \leq 2S (|\nabla_{\phi} X|_{\omega_x}^2 + |\nabla_{\phi} X|_{\omega_x}^2).
\]

(2.26)

By (2.23), we find that on \(B_{r_1}(p)\) we have

\[
\left(\frac{d}{dt} - \triangle_{\omega_x}\right) S \leq \frac{C}{r^2} - |\nabla_{\phi} X|_{\omega_x}^2 - |\nabla_{\phi} X|_{\omega_x}^2.
\]

(2.27)

Hence at \((x_0, t_0)\)

\[
\left(\frac{d}{dt} - \triangle_{\omega_x}\right) F \leq -A (|\nabla_{\phi} X|_{\omega_x}^2 + |\nabla_{\phi} X|_{\omega_x}^2) + \frac{AC}{K^2} \frac{|Rm_{\phi}|_{\omega_x}^2}{K^2} + \frac{C\rho^2 |Rm_{\phi}|_{\omega_x}^2}{K^2} \\
- \frac{\rho^2 |Rm_{\phi}|_{\omega_x}^2 (|\nabla_{\phi} X|_{\omega_x}^2 + |\nabla_{\phi} X|_{\omega_x}^2)}{K} + \frac{K^2}{K} \\
- \frac{\rho^2 (|\nabla_{\phi} Rm_{\phi}|_{\omega_x}^2 + |\nabla_{\phi} Rm_{\phi}|_{\omega_x}^2)}{K} + \frac{C |Rm_{\phi}|_{\omega_x}^2}{K^2} \\
+ \frac{\rho^2 (|\nabla_{\phi} |Rm_{\phi}|_{\omega_x}^2 + |\nabla_{\phi} Rm_{\phi}|_{\omega_x}^2)}{K} + \frac{8AS (|\nabla_{\phi} X|_{\omega_x}^2 + |\nabla_{\phi} X|_{\omega_x}^2)}{K}.
\]

(2.28)
First choose $\hat{C}$ in the definition of $K$ to be sufficiently large so that $\frac{8AQ}{K} \leq A_Q$, where we denote $Q = |\nabla_\varphi X|^2_{\omega_\varphi} + |\nabla_\varphi X|^2_{\omega_\varphi}$. By (2.17), we have
\[
\frac{C\rho^2 |Rm_\varphi|^3_{\omega_\varphi}}{K} \leq \frac{\rho^2 |Rm_\varphi|^2_{\omega_\varphi}}{2K^2} + C\rho^2 |Rm_\varphi|^2_{\omega_\varphi}
\]
(2.29)
So we have
\[
(\frac{d}{dt} - \triangle_{\omega_\varphi})F \leq -\frac{AQ}{2} + \frac{AC}{r^2} + C|Rm_\varphi|^2_{\omega_\varphi}
\]
(2.30)
Now we choose $A$ sufficiently large so that $A \geq 2(\hat{C} + 1)$ and we obtain at $(x_0, t_0)$
\[
Q \leq \frac{C}{r^2},
\]
which implies that $|Rm_\varphi|^2_{\omega_\varphi} \leq \frac{C}{r^2}$ at this point, where $C$ depends only on $N, \lambda, S, ||\vartheta||_{C^2(B_r(p))}$ and $\omega_0$. It follows that at $(x_0, t_0)$, $F$ is bounded from above by $\frac{C}{r^2}$, where the constant $C$ independent of $T$. Hence on $B_{r_2}(p) \times [0, T]$ we obtain
\[
|Rm_\varphi|^2_{\omega_\varphi} \leq \frac{C}{r^4},
\]
(2.31)
where $C$ depends only on $N, \lambda, ||\varphi(\cdot, 0)||_{C^4(B_r(p))}, ||\vartheta||_{C^2(B_r(p))}$ and $\omega_0$.

Now, we prove the $C^\infty$ estimates of the metric potential $\varphi$ on $B_{\frac{r_2}{2}}(p)$ combining with the high order derivative estimates of the Riemann curvature tensors. Here, when we say $\varphi$ is $C^{k, \alpha}$, we mean it is uniformly bounded in $C^{k, \alpha}$ norm, that is, the norm controlled by a constant depends only on $\omega_0, N, \lambda, r, ||\vartheta||_{C^{k-1}(B_r(p))}, ||\varphi(\cdot, 0)||_{C^{k+1}(B_r(p))}$ and $||\varphi||_{C^0(B_r(p) \times [0, T])}$. Replace $\varphi$ by $\hat{\varphi}$, then the $C^{k, \alpha}$ norm of $\hat{\varphi}$ controlled by constant depends only on $\omega_0, N, \lambda, r, ||\vartheta||_{C^{k+1}(B_r(p))}, ||\hat{\varphi}(\cdot, 0)||_{C^{k+1}(B_r(p))}$ and $||\hat{\varphi}||_{C^0(B_r(p) \times [0, T])}$. Since $|Rm_{\omega_\varphi}|_{\omega_\varphi} \leq C$ on $B_{r_2}(p)$ along the flow (2.7), we know that $\hat{\varphi}$ is $C^{1, \alpha}$. Differentiating the equation (2.8) with respect to $z^k$, we get
\[
(\frac{d}{dt} - \triangle_{\omega_\varphi})\frac{\partial \varphi}{\partial z^k} = \Delta_{\omega_\varphi} \frac{\partial \varphi}{\partial z^k} + g^i_\varphi \frac{\partial g_{ij}}{\partial z^k} - g^i_\varphi \frac{\partial g_{ij}}{\partial z^k} + \frac{\partial f}{\partial z^k} + \lambda \frac{\partial \varphi}{\partial z^k}.
\]
(2.32)
From the above Calabi’s $C^3$ estimate, we know that $\varphi$ is $C^{2, \alpha}$ and then the coefficients of $\Delta_{\omega_\varphi}$ are $C^{0, \alpha}$. Since $f$ is the twisted Ricci potential, then
\[
\Delta_{\omega_\varphi} f = -tr_{\omega_\varphi} Ric(\omega_\varphi) + \lambda n + tr_{\omega_\varphi} \vartheta.
\]
(2.33)
So, the $C^{1, \alpha}$-norm of $f$ on $B_{r_2}(p)$ only depends on $\omega_0, ||\vartheta||_{C^0(B_r(p))}$ and $||f||_{C^0(B_r(p))}$. By the standard elliptic Schauder estimates, we have $\varphi$ is $C^{3, \alpha}$ on $B_{r_3}(p) \times [0, T]$, where $\frac{r_2}{3} < r_3 < r_2$. By computing, we have
\[
(\frac{d}{dt} - \triangle_{\omega_\varphi})|\nabla_\varphi Rm_\varphi|^2_{\omega_\varphi} \leq -|\nabla_\varphi \nabla_\varphi Rm_\varphi|^2_{\omega_\varphi} - |\nabla_\varphi \nabla_\varphi Rm_\varphi|^2_{\omega_\varphi}
\]
(2.34)
\[+ C|\nabla_\varphi Rm_\varphi|^2_{\omega_\varphi} + C|\nabla_\varphi \theta||_{\omega_\varphi}|\nabla_\varphi Rm_\varphi|^2_{\omega_\varphi} + C|\nabla_\varphi \nabla_\varphi Rm_\varphi|^2_{\omega_\varphi},\]
where $C$ depends only on $N$, $\lambda$, $\|\theta\|_{C^0(B_r(p))}$ and $|Rm_\varphi|_{\omega_\varphi}^2$. By (2.16) and (2.17), we know

\begin{align}
(2.35) & \quad |\nabla_\varphi \theta|_{\omega_\varphi} \leq C, \\
(2.36) & \quad |\nabla_\varphi \nabla_\varphi \nabla_\varphi \theta|_{\omega_\varphi} \leq C(1 + |\nabla_\varphi Rm_\varphi|_{\omega_\varphi} + |\nabla_\varphi X|_{\omega_\varphi}).
\end{align}

So we have

\begin{align}
(2.37) & \quad \left( \frac{d}{dt} - \Delta_{\omega_\varphi} \right) |\nabla_\varphi Rm_\varphi|_{\omega_\varphi}^2 \leq -|\nabla_\varphi \nabla_\varphi Rm_\varphi|_{\omega_\varphi}^2 - |\nabla_\varphi \nabla_\varphi \nabla_\varphi Rm_\varphi|_{\omega_\varphi}^2 + C|\nabla_\varphi Rm_\varphi|_{\omega_\varphi}^2 + |\nabla_\varphi X|_{\omega_\varphi}^2 + C.
\end{align}

Let $\varrho$ be a cut-off function, identically 1 on $\overline{B_{r_0^2}}(p)$ and identically 0 outside $B_{r_2}$. As before we may assume

\[ |\partial_\varrho|_{\omega_\varphi}^2, \sqrt{-1} \partial_\varrho \bar{\varrho} \leq C \]

for some uniform constant $C$ depends only on $\omega_0$, $N$ and $r$. From the former part we know that $S$ and $|Rm_\varphi|_{\omega_\varphi}^2$ are bounded by a uniform constant on $B_{r_2}(p)$. Define $H = \varrho^2 |\nabla_\varphi Rm_\varphi|_{\omega_\varphi}^2 + S + B|Rm_\varphi|_{\omega_\varphi}^2$, where $B$ will be determined later,

\begin{align}
(2.38) & \quad \left( \frac{d}{dt} - \Delta_{\omega_\varphi} \right) H \leq -\varrho^2 |\nabla_\varphi \nabla_\varphi Rm_\varphi|_{\omega_\varphi}^2 - \varrho^2 |\nabla_\varphi \nabla_\varphi \nabla_\varphi Rm_\varphi|_{\omega_\varphi}^2 + C|\nabla_\varphi Rm_\varphi|_{\omega_\varphi}^2 + |\nabla_\varphi X|_{\omega_\varphi}^2 \leq -B|\nabla_\varphi Rm_\varphi|_{\omega_\varphi}^2 - B|\nabla_\varphi Rm_\varphi|_{\omega_\varphi}^2 + C.
\end{align}

Let $(x_0, t_0)$ be the maximum point of $H$ on $\overline{B_{r_2}(p)} \times [0, T]$. We assume for the moment that $t_0 > 0$ and that $x_0$ doesn’t lie in the boundary of $B_{r_2}(p)$, we choose $2B = C + 1$, by maximum principle, at this point, we have

\begin{align}
(2.39) & \quad |\nabla_\varphi Rm_\varphi|_{\omega_\varphi}^2 \leq C,
\end{align}

where $C$ depends only on $N$, $\lambda$, $r$, $\|\theta\|_{C^3(B_r(p))}$, $|Rm_\varphi|_{\omega_\varphi}^2$ and $\omega_0$. It follows that at $(x_0, t_0)$, $H$ is bounded from above by $C$ independents of $T$. Hence on $\overline{B_{r_3}(p)} \times [0, T]$ we obtain

\begin{align}
(2.40) & \quad |\nabla_\varphi Rm_\varphi|_{\omega_\varphi}^2 \leq C
\end{align}

where $C$ depends only on $N$, $\lambda$, $r$, $\|\varphi(\cdot, 0)\|_{C^3(B_r(p))}$, $\|\theta\|_{C^3(B_r(p))}$ and $\omega_0$.

Differentiating equation (2.7), we have

\[ D\sqrt{-1} \partial_\varrho \bar{\varrho} = DRic(\omega_\varphi) + D\theta, \]

where $D$ denotes the covariant derivative with respect to the metric $\omega_\varphi$, taking trace on both side with the metric $\omega_\varphi$, we have

\begin{align}
(2.41) & \quad |\Delta_{\omega_\varphi} D\varrho| \leq |Rm_\varphi|_{\omega_\varphi} |\nabla_\varphi \varrho + |DRm_\varphi|_{\omega_\varphi} + C|X|_{\omega_\varphi} + C.
\end{align}

Since $\varrho$ is $C^{1, \alpha}$, $|Rm_\varphi|_{\omega_\varphi}$, $|DRm_\varphi|_{\omega_\varphi}$ and $|X|_{\omega_\varphi}$ is uniformly bounded, hence we have $D\varrho$ is $C^{1, \alpha}$, hence $\varrho$ is $C^{2, \alpha}$. Derivating equation (2.38) two times and using the elliptic Schauder estimates, we have $\varphi$ is $C^{4, \alpha}$ on $B_{r_3}(p) \times [0, T]$, where $\frac{3}{2} < r_4 < r_3$. 

Then we claim that $|D^k Rm_\varphi|_{\omega_\varphi}^2 \leq C$, $\varphi$ is $C^{k+1,\alpha}$, $\varphi$ is $C^{k+3,\alpha}$ are also established for the same $k$ on $B_{r_{k+3}}(p) \times [0,\tau]$, where $C$ depends only on $N$, $\lambda$, $r$, $\|\varphi(\cdot,0)\|_{c^{k+1}(B_{r_{k+3}}(p))}$, $\|\varphi\|_{C^{k+2}(B_{r_{k+3}}(p))}$, $\|\theta\|_{C^{k+2}(B_{r_{k+3}}(p))}$ and $\omega_0$, $r_{k+1} > r_{k+3} > \frac{\tau}{2}$ for any $k \geq 0$. We argue now by induction. First, when $k = 0, 1$, this claim is established. Assume that

\[(2.42) \quad |D^j Rm_\varphi|_{\omega_\varphi}^2 \leq C, \quad \|\varphi\| \text{ is } C^{j+1,\alpha}, \quad \|\varphi\| \text{ is } C^{j+3,\alpha}\]

hold on $B_{r_{j+3}}(p) \times [0,\tau]$ for all $j \leq k$.

Now we estimate $|D^k+1 Rm_\varphi|_{\omega_\varphi}^2$, since any covariant derivative of $Rm_\varphi$ of order $k + 1$ differs from covariant derivatives of the form $\nabla_\varphi \nabla_\varphi Rm_\varphi$ by $D^k Rm_\varphi \ast D^{r+s-2-k} Rm_\varphi$ with $i \geq 0$ and $r+s = k+1$, we should only estimate $|\nabla_\varphi \nabla_\varphi^2 Rm_\varphi|_{\omega_\varphi}^2$.

\[
\begin{align*}
&= \frac{d}{dt} - \Delta_{\omega_\varphi} \mid \nabla_\varphi^2 \nabla_\varphi Rm_\varphi \mid^2 \\
&= - \mid \nabla_\varphi^{i+j} \nabla_\varphi Rm_\varphi \mid^2 - \mid \nabla_\varphi \nabla_\varphi \nabla_\varphi Rm_\varphi \mid^2 - (r+s+2) \mid \nabla_\varphi \nabla_\varphi Rm_\varphi \mid^2 \\
&+ \sum_{i+j=s} \nabla_\varphi^i \nabla_\varphi^j (Rm_\varphi + \theta) \ast \nabla_\varphi^i \nabla_\varphi^j Rm_\varphi \ast \nabla_\varphi \nabla_\varphi Rm_\varphi \\
&+ \sum_{i+j=s} \nabla_\varphi \nabla_\varphi^i \nabla_\varphi^j \nabla_\varphi \nabla_\varphi^i \nabla_\varphi^j Rm_\varphi \ast \nabla_\varphi \nabla_\varphi \nabla_\varphi \nabla_\varphi Rm_\varphi \ast \nabla_\varphi \nabla_\varphi Rm_\varphi \ast \nabla_\varphi \nabla_\varphi Rm_\varphi \\
&+ (\nabla_\varphi \nabla_\varphi^{-1} \nabla_\varphi \varphi \nabla_\varphi \nabla_\varphi Rm_\varphi \ast \nabla_\varphi \nabla_\varphi \nabla_\varphi \nabla_\varphi Rm_\varphi \ast \nabla_\varphi \nabla_\varphi \nabla_\varphi \nabla_\varphi Rm_\varphi \ast \nabla_\varphi \nabla_\varphi \nabla_\varphi \nabla_\varphi Rm_\varphi),
\end{align*}
\]

where we are writing $A \ast B$ to denote a linear combination of the tensors $A$ and $B$ contracted with respect to the metric $\omega_\varphi$. Since $\varphi$ is $C^{k+3,\alpha}$ on $B_{r_{k+3}}(p) \times [0,\tau]$, we have

\[(2.44) \quad |D^{k+1} \theta|_{\omega_\varphi} \leq C \sum_{i=1}^{k} |D^{i} X|_{\omega_\varphi} + C \leq C.
\]

In the case of $r, s \neq 0$, combined with (2.16) and (2.17), then

\[(2.45) \quad |\nabla_\varphi \nabla_\varphi^{-1} \nabla_\varphi \varphi \nabla_\varphi \nabla_\varphi \nabla_\varphi Rm_\varphi|_{\omega_\varphi} \leq C |\nabla_\varphi \nabla_\varphi \nabla_\varphi \nabla_\varphi Rm_\varphi|_{\omega_\varphi} + C.
\]

When $r = 0$ or $s = 0$, without loss of generality, we assume $s = 0$, we have

\[(2.46) \quad |\nabla_\varphi \nabla_\varphi^2 \nabla_\varphi \varphi \nabla_\varphi \nabla_\varphi \nabla_\varphi \nabla_\varphi \nabla_\varphi Rm_\varphi|_{\omega_\varphi} \leq C |\nabla_\varphi \nabla_\varphi^2 \nabla_\varphi \varphi \nabla_\varphi \nabla_\varphi \nabla_\varphi \nabla_\varphi \nabla_\varphi \nabla_\varphi Rm_\varphi|_{\omega_\varphi} + C.
\]

The corresponding evolution equation

\[(2.47) \quad \frac{d}{dt} - \Delta_{\omega_\varphi} \mid \nabla_\varphi^2 \nabla_\varphi Rm_\varphi \mid^2 \leq - \mid \nabla_\varphi \nabla_\varphi \nabla_\varphi Rm_\varphi \mid^2 + \mid \nabla_\varphi \nabla_\varphi \nabla_\varphi \nabla_\varphi \nabla_\varphi Rm_\varphi \mid^2 + C\mid \nabla_\varphi \nabla_\varphi \nabla_\varphi \nabla_\varphi Rm_\varphi \mid^2
\]

Let $\varphi$ be a cut-off function, identically 1 on $B_{r_{k+3}}(p)$ and identically 0 outside $B_{r_{k+3}}$, where $\frac{\tau}{2} < r_{k+3} < r_{k+3}$. As before we may assume

\[|\partial \varphi|^2_{\omega_0}, \mid \sqrt{-1} \partial \varphi \mid_{\omega_0} \leq C\]
Then we argue in the following two case:

1. When $r, s \neq 0$, we define $G_1 = \vartheta^2|\nabla_{\varphi}^r \nabla_{\varphi}^s Rm_{\varphi}|^2_{\omega_{\varphi}} + A_1 |\nabla_{\varphi}^{-1} \nabla_{\varphi}^r Rm_{\varphi}|^2_{\omega_{\varphi}}$.

2. When $s = 0$, we define $G_2 = \vartheta^2|\nabla_{\varphi}^{r+1} Rm_{\varphi}|^2_{\omega_{\varphi}} + A_2 |\nabla_{\varphi}^r Rm_{\varphi}|^2_{\omega_{\varphi}} + |\nabla_{\varphi}^r X|^2_{\omega_{\varphi}}$.

We first analyse the evolution of $|\nabla_{\varphi}^k X|^2_{\omega_{\varphi}}$. By direct computation, we have

\begin{equation}
\frac{d}{dt} - \Delta_{\omega_{\varphi}} X_{ml}^\beta = \nabla_{\varphi m} \theta^\beta_l + \nabla_{\varphi}^l R_0^{\beta l}, \tag{2.49}
\end{equation}

\begin{equation}
\frac{d}{dt} \Gamma_{\varphi mn}^\beta = -g^{\beta i} \nabla_{\varphi m} (R_{\varphi i t} - \theta_{it}). \tag{2.50}
\end{equation}

Since in the evolution equation of $X$ there exists no $Rm_{\varphi}$, and there only exists derivative of $Rm_{\varphi}$ of order 1 in the evolution equation of Christoffel $\Gamma_{\varphi}$ with respect to the metric $\omega_{\varphi}$, hence there exists derivative of $Rm_{\varphi}$ no more than of order $k$ in the evolution equation of $\nabla_{\varphi}^k X$, combining $\varphi$ is $C^{k+3,2}$, we obtain

\begin{equation}
\frac{d}{dt} - \Delta_{\omega_{\varphi}} |\nabla_{\varphi}^k X|^2_{\omega_{\varphi}} \leq -|\nabla_{\varphi}^{k+1} X|^2_{\omega_{\varphi}} - |\nabla_{\varphi}^k X|^2_{\omega_{\varphi}} + C. \tag{2.51}
\end{equation}

Then by computing, when we choose suitable $A_1$ and $A_2$, we have

\begin{equation}
\frac{d}{dt} - \Delta_{\omega_{\varphi}} G_1 \leq -\vartheta^2 |\nabla_{\varphi}^{r+1} \nabla_{\varphi}^s Rm_{\varphi}|^2_{\omega_{\varphi}} - \vartheta^2 |\nabla_{\varphi}^r \nabla_{\varphi}^s Rm_{\varphi}|^2_{\omega_{\varphi}} + C |\nabla_{\varphi}^r \nabla_{\varphi}^s Rm_{\varphi}|^2_{\omega_{\varphi}} - 2Re(\nabla \vartheta^2, \nabla |\nabla_{\varphi}^r \nabla_{\varphi}^s Rm_{\varphi}|^2_{\omega_{\varphi}})
\end{equation}

\begin{equation}
- A_1 |\nabla_{\varphi}^r \nabla_{\varphi}^s Rm_{\varphi}|^2_{\omega_{\varphi}} + C. \tag{2.52}
\end{equation}

\begin{equation}
\frac{d}{dt} - \Delta_{\omega_{\varphi}} G_2 \leq -\vartheta^2 |\nabla_{\varphi}^{k+2} Rm_{\varphi}|^2_{\omega_{\varphi}} - \vartheta^2 |\nabla_{\varphi}^k \nabla_{\varphi}^{k+1} Rm_{\varphi}|^2_{\omega_{\varphi}} + C |\nabla_{\varphi}^{k+1} Rm_{\varphi}|^2_{\omega_{\varphi}} + |\nabla_{\varphi}^{k+1} X|^2_{\omega_{\varphi}} - |\nabla_{\varphi}^{k+1} X|^2_{\omega_{\varphi}} + C
\end{equation}

\begin{equation}
- 2Re(\nabla \vartheta^2, \nabla |\nabla_{\varphi}^{k+1} Rm_{\varphi}|^2_{\omega_{\varphi}}) - A_2 |\nabla_{\varphi}^{k+2} Rm_{\varphi}|^2_{\omega_{\varphi}} \tag{2.53}
\end{equation}

Let $(x_1, t_1)$ and $(x_2, t_2)$ be the maximum point of $G_1$ and $G_2$ on $B_{r_{x+3}}(p) \times [0, T]$ respectively. We assume for the moment that $t_i > 0$ and that $x_i$ doesn’t lie in the boundary of $B_{r_{x+3}}(p)$ for $i = 1, 2$, by maximum principle, we have

\begin{equation}
|\nabla_{\varphi}^r \nabla_{\varphi}^s Rm_{\varphi}|^2_{\omega_{\varphi}}(x_1, t_1) \leq C, \quad |\nabla_{\varphi}^{k+1} Rm_{\varphi}|^2_{\omega_{\varphi}}(x_2, t_2) \leq C. \tag{2.54}
\end{equation}
where $C$ depends only on $N$, $\lambda$, $r$, $\|\theta\|_{C^{k+3}(B_r(p))}$, $\|\varphi(\cdot,0)\|_{C^{k+4}(B_r(p))}$, $\|\varphi\|_{C^0(B_r(p) \times [0,T])}$, $\|f\|_{C^0(B_r(p))}$ and $\omega_0$. It follows that at $(x_i,t_i)$, $G_t$ is bounded from above uniformly. Hence on $B_{r_{k+3}}(p) \times [0,T]$ we obtain

\begin{equation}
\|\nabla^i \nabla^j Rm_{\varphi}\|_{\omega_\varphi}^2 \leq C, \quad \|\nabla^{k+1} Rm_{\varphi}\|_{\omega_\varphi}^2 \leq C,
\end{equation}

where $C$ depends only on $N$, $\lambda$, $r$, $\|\varphi(\cdot,0)\|_{C^{k+3}(B_r(p))}$, $\|\varphi\|_{C^0(B_r(p) \times [0,T])}$, $\|\theta\|_{C^{k+3}(B_r(p))}$, $\|f\|_{C^0(B_r(p))}$ and $\omega_0$. Then we claim that $|D^j Rm_{\varphi}|_{\omega_\varphi}^2 \leq C$ established for $k+1$ on $B_{r_{k+3}}(p) \times [0,T]$.

From equation (2.51), we know that

\begin{equation}
\Delta_{\omega_\varphi} D^{k+1} \varphi \leq C \left( \sum_{i=1}^{k+2} |D^{i-1} Rm_{\varphi}|_{\omega_\varphi} |D^{k-i+2} \varphi|_{\omega_\varphi} + \sum_{i=1}^{k} |D^i X|_{\omega_\varphi} + 1 \right).
\end{equation}

By (2.42), we know that $|\Delta_{\omega_\varphi} D^{k+1} \varphi| \leq C$, so $D^{k+1} \varphi$ is $C^{1+\alpha}$, and then by the assumption, it is easy to see that $\varphi$ is $C^{k+2,\alpha}$. By deriving the parabolic Monge-Ampère equation (2.52) $k+2$ times, and using the elliptic Schauder estimates again, we have $\varphi$ is $C^{k+4,\alpha}$ on $B_{r_{k+3}}(p) \times [0,T]$, where $r_{k+3} > r_0 + \frac{\beta}{2}$. So we complete the proof of the claim. From the claim, we get $C^\infty$ estimates of $\varphi$ on $B_{r_2}(p)$ along the flow (2.7).

**Remark 2.3** If we only consider the regularity estimates for a single flow (2.8), when we get the Calabi's $C^3$ estimate and the curvature estimate, we can get the uniform local $C^\infty$ estimates of $\varphi$ by the standard Schauder estimate of the parabolic equation (see [20]). Since we want to get the conical Kähler-Ricci flow by limiting a sequence of the generalized Kähler-Ricci flow (1.11) as $\varepsilon \rightarrow 0$, so we need to get the uniform $C^\infty$ estimates of $\varphi_\varepsilon(\cdot, t)$ on $\overline{B_r} \times [0,T]$, where $\overline{B_r} \subset \subset M \setminus D$. But apply the Schauder estimates of the parabolic equation we can only get the uniform $C^\infty$ estimates of $\varphi_\varepsilon(\cdot, t)$ on $\overline{B_r} \times [\delta,T]$, where $\delta > 0$ and the uniform bound depends on $\delta$. This is the reason we apply the elliptic estimates in the proof of Proposition 2.2. We can also note that the estimates are independent of the time $T$ and so the results hold also for time intervals $[0,T)$ or $[0,\infty)$.

### 3. The long time existence of the conical Kähler-Ricci flow

In this section, we will use the above estimates to prove the long time existence of the conical Kähler-Ricci flow, we obtain the following theorem:

**Theorem 3.1** Assume $\beta \in (0,1)$. Then there exists a sequence $\{\varepsilon_i\}$ satisfies $\varepsilon_i \rightarrow 0$ as $i \rightarrow +\infty$, such that the flow (1.11) converges in $C^\infty_{\text{loc}}$ topology outside divisor $D$ to the following equation

\begin{equation}
\begin{cases}
\frac{\partial \varphi}{\partial t} = \log \frac{\omega}{\omega_\varepsilon} + F_0 + \beta (k|s|_h^2 + \varphi) + \log |s|_h^{2(1-\beta)} \\
|\varphi|_{t=0} = c_0,
\end{cases}
\end{equation}

and $\omega_\varphi$ is a long time solution of the conical Kähler-Ricci flow (1.12) with initial metric $\omega^*$.

**Proof:** By differentiating equation (1.11), we have

\[
\frac{d}{dt} \varphi_\varepsilon(t) = \Delta_{\omega_\varepsilon(t)} \varphi_\varepsilon(t) + \beta \varphi_\varepsilon(t)
\]
Hence, by maximum principle, we have
\[
\sup_M |\dot{\varphi}_\varepsilon(t)| \leq \sup_M |e^{\beta t} \dot{\varphi}_\varepsilon(0)|
\]
where \(\dot{\varphi}_\varepsilon(0) = \log \frac{\omega^n_\varepsilon(\varepsilon^2 + |s|^2_0)^{1-\beta}}{\omega^n_0} + F_0 + \beta(k\chi + c_0)\), so \(\sup M |\dot{\varphi}_\varepsilon(t)| \leq C e^{\beta t}\) by a uniform constant \(C\). Then on \(M \times [0, T]\), we have \(\|\varphi_\varepsilon(t)\| \leq C e^{\beta T}\). By Lemma 2.1, there exists constant \(C(T)\) satisfying
\[
C^{-1}(T) \omega_\varepsilon \leq \omega_\varphi \leq C(T) \omega_\varepsilon.
\]
on \(M \times [0, T]\). For any \(K \subset M \setminus D\), we have
\[
\frac{1}{N} \omega_0 \leq \omega_\varphi \leq N \omega_0,
\]
where the uniform constant \(N\) depends only on \(K\) and \(C(T)\). Since the initial data \(k\chi + c_\varepsilon(0)\) of the flow \(1.1\), the twisted Ricci potential \(F_0 + \log(\varepsilon^2 + |s|^2_0)^{1-\beta}\) and the twist form \(\theta_\varepsilon\) are \(C^\infty_{loc}\) uniformly bounded away from divisor \(D\). From Proposition 2.2, \(\varphi_\varepsilon + k\chi\) is \(C^\infty\) bounded uniformly (independent of \(\varepsilon\)) on \(K \times [0, T]\). Let \(K\) approximate to \(M \setminus D\) and \(T \to +\infty\), by diagonal rule we have a sequence which we denote \(\{\varepsilon_i\}\), such that \(\varphi_\varepsilon(t)\) converge in \(C^\infty_{loc}\) topology outside divisor \(D\) to a function \(\varphi(t)\), which is smooth on \(M \setminus D\). From \(3.2\), we know that every \(\omega_\varphi(t)\) are conical Kähler metric with cone angle \(2\pi \beta\) along the divisor \(D\).

Now we prove that the limit \(\varphi(t)\) satisfying the conical Kähler-Ricci flow \(3.1\) globally on \(M \times [0, +\infty)\) in the sense of currents. For any \((n-1, n-1)\)-form \(\eta\), since \(\omega_\varphi_\varepsilon(\varepsilon^2 + |s|^2_0)^{1-\beta}\), \(k\chi(\varepsilon^2 + |s|^2_0)\) and \(\varphi_\varepsilon\) are bounded independent of \(\varepsilon\), we have
\[
[\sqrt{-1} \partial \bar{\partial} \frac{\partial \varphi_\varepsilon}{\partial t}, \eta] = [\sqrt{-1} \partial \bar{\partial} (\log \frac{\omega^n_\varepsilon(\varepsilon^2 + |s|^2_0)^{1-\beta}}{\omega^n_0} + F_0 + \beta(k\chi(\varepsilon^2 + |s|^2_0) + \varphi_\varepsilon)), \eta] = [\log \frac{\omega^n_\varepsilon(\varepsilon^2 + |s|^2_0)^{1-\beta}}{\omega^n_0} + F_0 + \beta(k\chi(\varepsilon^2 + |s|^2_0) + \varphi_\varepsilon)), \sqrt{-1} \partial \bar{\partial} \eta]\]
\[
= \int_M (\log \frac{\omega^n_\varepsilon(\varepsilon^2 + |s|^2_0)^{1-\beta}}{\omega^n_0} + F_0 + \beta(k\chi(\varepsilon^2 + |s|^2_0) + \varphi_\varepsilon)) \sqrt{-1} \partial \bar{\partial} \eta)
\]
\[
= [\sqrt{-1} \partial \bar{\partial} (\log \frac{\omega^n_\varepsilon}{\omega^n_0} + F_0 + \beta(k|s|^{2\beta}_h + \varphi) + \log |s|^{2(1-\beta)}_h, \sqrt{-1} \partial \bar{\partial} \eta]
\]
\[
= [\sqrt{-1} \partial \bar{\partial} (\log \frac{\omega^n_\varepsilon}{\omega^n_0} + F_0 + \beta(k|s|^{2\beta}_h + \varphi) + \log |s|^{2(1-\beta)}_h, \sqrt{-1} \partial \bar{\partial} \eta]
\]

On the other hand, let \(K \subset M \setminus D\) be a compact subset, and \(\int_{M \setminus K} \sqrt{-1} \partial \bar{\partial} \eta = \delta\) with \(\delta \to 0\) when \(K \to M \setminus D\), by the facts that \(\frac{\partial \varphi_\varepsilon}{\partial t}\) and \(\frac{\partial \varphi_\varepsilon}{\partial t}\) are also bounded independent of \(\varepsilon\),
\[
| \int_M (\frac{\partial \varphi_\varepsilon}{\partial t} - \frac{\partial \varphi_\varepsilon}{\partial t}) \sqrt{-1} \partial \bar{\partial} \eta | = | \int_K (\frac{\partial \varphi_\varepsilon}{\partial t} - \frac{\partial \varphi_\varepsilon}{\partial t}) \sqrt{-1} \partial \bar{\partial} \eta + \int_{M \setminus K} (\frac{\partial \varphi_\varepsilon}{\partial t} - \frac{\partial \varphi_\varepsilon}{\partial t}) \sqrt{-1} \partial \bar{\partial} \eta |\]
\[ \left( \omega_0 + \sqrt{-1} \partial \overline{\partial} \phi \right)^n = e^{\hat{\phi} - \beta \phi} \frac{\omega^n}{|s_h^{2(1-\beta)}|} \]

Proposition 3.2 For any \( t \in [0, +\infty) \), the potential \( \varphi(t) \) is Hölder continuous with respect to the metric \( \omega_0 \) on \( M \).

Proof: Define \( \phi = \varphi + k|s|^{2\beta} \), for any \( t \), we fix \( T > t \). From Theorem 3.1, we have \( \|\dot{\phi}(t)\|_{C^0} \leq C(T) \) and \( \|\phi(t)\|_{C^0} \leq C(T) \) on \( M \setminus D \times [0, T] \). The flow \( (3.4) \) can be written as

\[ (\omega_0 + \sqrt{-1} \partial \overline{\partial} \phi)^n = e^{\hat{\phi} - \beta \phi} \frac{\omega^n}{|s_h^{2(1-\beta)}|} \]

on \( M \setminus D \). Since \( \beta \in (0, 1) \), there exists \( \delta \) such that \( 2(1-\beta)(1+\delta) < 2 \).

Then by the \( L^p \) estimate of S. Kolodziej [23], we conclude that the potential \( \varphi(t) \) is Hölder continuous with respect to the metric \( \omega_0 \) on \( M \).

Remark 3.3 From Theorem 3.1 and Proposition 3.2, we have

\[ \|\dot{\phi}\|_{C^0} \leq C(T), \quad \|\varphi\|_{C^0} \leq C(T), \quad C^{-1}(T) \varphi \leq \omega_0 \leq C(T) \omega. \]

on \( M \setminus D \times [0, T] \). By the uniqueness theorem of the weak conical Kähler-Ricci flow (see Lemma 3.2 in [13]) and the long time existence of the strong conical Kähler-Ricci flow in [23], we conclude that the conical Kähler-Ricci flow constructed in the above theorem must be the strong conical Kähler-Ricci flow.

4. Uniform Perelman’s estimates along the generalized Kähler-Ricci flow

In this section, we will prove the uniform Perelman’s estimates of the flows (1.10), i.e. Theorem 4.1. We may follow the steps in the generalized Kähler-Ricci flow in [20], see also the case of Kähler-Ricci flow in [35].

Theorem 4.1 Let \( g_\varepsilon(t) \) be a solution of the generalized Kähler Ricci flow, i.e. the corresponding form \( \omega_\varepsilon(t) \) satisfies the equation (1.10) with initial metric \( \omega_\varepsilon \), \( u_\varepsilon(t) \in C^\infty(M) \) is the generalized Ricci potential satisfying

\[ \text{Ric}(\omega_\varepsilon(t)) + \beta \omega_\varepsilon(t) + \theta_\varepsilon = \sqrt{-1} \partial \overline{\partial} u_\varepsilon(t) \]

and \( \frac{1}{V} \int_M e^{-u_\varepsilon(t)} dV_{\varepsilon t} = 1 \), where \( \theta_\varepsilon = (1 - \beta)(\omega_0 + \sqrt{-1} \partial \overline{\partial} \log(\varepsilon^2 + |s_h^n|)) \). If \( \beta \in (0, \frac{1}{2}] \), then there exists a uniform constant \( C \) independent of \( \varepsilon \) and \( t \), such that

\[ |R(g_\varepsilon(t)) - tr_{g_\varepsilon(t)} \theta_\varepsilon| \leq C, \]

\[ \|u_\varepsilon\|_{C^1(g_\varepsilon(t))} \leq C, \]

\[ \text{diam}(M, g_\varepsilon(t)) \leq C, \]
where \( R(g_\varepsilon(t)) \) and \( \text{diam}(M, g_\varepsilon(t)) \) are the scalar curvature and diameter of the manifold respectively with respect to the metric \( g_\varepsilon(t) \).

From equation (4.1), and the flow (1.10), we have

\[
(4.2) \quad u_\varepsilon(t) = \dot{\varphi}_\varepsilon(t) + c_\varepsilon(t),
\]

where \( c_\varepsilon(t) \) depending on \( t \). Now we recall the generalized \( W_\theta \) functional and \( \mu_\theta \) functional. Let

\[
W_\theta(g, f, \tau) = \int_M e^{-f} \tau^{-n}(\tau(R - tr_g \theta + |\nabla f|^2_g) + \beta f)dV_g,
\]

where \( g \) is a Kähler metric, \( f \) is a smooth function on \( M \), \( \tau \) is a positive scale parameter and \( n \) is the complex dimension of the Kähler manifold. Let

\[
\mu_\theta(g, \tau) = \inf\{W_\theta(g, f, \tau) | f \in C^\infty(M), \frac{1}{V} \int_M e^{-f} \tau^{-n} dV = 1\}
\]

be the \( \mu_\theta \) functional with respect to the metric \( g \).

From [26], we have the monotonicity of the generalized \( W_\theta \) and \( \mu_\theta \) functional along the generalized Kähler-Ricci flow.

**Lemma 4.2** Along the evolution equation

\[
\begin{cases}
\frac{\partial g_{ij}}{\partial t} = -R_{ij} + \beta g_{ij} + \theta_{ij} \\
\frac{\partial f}{\partial t} = \beta n \tau^{-1} - R + tr_{g(t)} \theta - \Delta f + |\nabla f|^2 \\
\frac{\partial \tau}{\partial t} = \beta(\tau - 1)
\end{cases}
\]

we know \( W_\theta(g(t), f(t), \tau(t)) \) is nondecreasing.

**Lemma 4.3** If \( \tau \) satisfies the following equality

\[
\frac{\partial \tau}{\partial t} = \beta(\tau - 1).
\]

Then \( \mu_\theta(g, \tau) \) is nondecreasing along the generalized Kähler-Ricci flow.

When we prove Theorem 4.1, we will use \( \mu_\theta(g_\varepsilon, \tau) \) functional’s lower bound, which depends on the Sobolev constant \( C_S(M, g_\varepsilon) \) and \( \max(R(g_\varepsilon) - tr_{g_\varepsilon} \theta_\varepsilon) \) with respect to the metric \( g_\varepsilon \), where \( g_\varepsilon = g_\varepsilon(0) \) and \( \tau \) is a constant.

Let’s first recall the appropriate coordinate system (see Lemma 4.1 in [4]).

**Lemma 4.4** Let \((L, h)\) be the hermitian line bundle associated to a smooth divisor \( D \), and \( s \) be a section of \( L \) such that

\[
D := \{s = 0\}.
\]

Let \( p_0 \in D \), then there exists a constant \( C > 0 \) and an open set \( \Omega \subset M \) centered at \( p_0 \), such that for any point \( p \in \Omega \) there exists a coordinate system \( z = (z^1, \cdots, z^n) \) and a trivialization \( \eta \) for \( L \) such that:

1. \( D \cap \Omega = \{z^n = 0\}; \)
2. With respect to the trivialization \( \eta \), the metric \( h \) has the weight \( \varphi \), such that

\[
(4.4) \quad \varphi(p) = 0, \quad d\varphi(p) = 0, \quad \left| \frac{\partial^{\alpha+\beta} \varphi}{\partial z^\alpha \bar{z}^\beta}(p) \right| \leq C_{\alpha, \beta}
\]

for some constant \( C_{\alpha, \beta} \) depending only on the multi indexes \( \alpha, \beta \).
Under this appropriate coordinate system, metric $\omega_\varepsilon$ can be written as follows:

\[
\omega_\varepsilon = \omega_0 + ke^{-\varphi}(\varepsilon^2 + |z^n|^2)\varepsilon^{-1} \sqrt{-1}dz^n \wedge d\bar{z}^n \\
-ke^{-\varphi}z^n \frac{\partial \varphi}{\partial z^n}(\varepsilon^2 + |z^n|^2)\varepsilon^{-1} \sqrt{-1}dz^n \wedge d\bar{z}^n \\
-ke^{-\varphi}z^n \frac{\partial \varphi}{\partial \bar{z}^n}(\varepsilon^2 + |z^n|^2)\varepsilon^{-1} \sqrt{-1}dz^n \wedge d\bar{z}^n \\
+ke^{-\varphi}|z^n|^2 \frac{\partial \varphi}{\partial z^n} \frac{\partial \varphi}{\partial \bar{z}^n}(\varepsilon^2 + |z^n|^2)\varepsilon^{-1} \sqrt{-1}dz^n \wedge d\bar{z}^n \\
-\frac{k}{\beta}(\varepsilon^2 + |z^n|^2)\varepsilon^{-2} \frac{\partial^2 \varphi}{\partial z^n \partial \bar{z}^n} \sqrt{-1}dz^n \wedge d\bar{z}^n.
\]  

We consider the map

\[
\Psi_\varepsilon: (z^1, z^2, \ldots, z^{n-1}, \xi) \mapsto (z^1, z^2, \ldots, z^{n-1}, z^n),
\]

where $z^n = (\varepsilon^2 + |z^n|^2)^{\frac{1}{\beta}} \xi$. Now, we want to show that $\Psi_\varepsilon^*(g_\varepsilon)$ is uniformly equivalent to the Euclidean metric in small neighborhoods along the divisor $D$.

By a direct calculation, we see that we only need to handle the following term

\[
\Psi_\varepsilon^*(ke^{-\varphi}(\varepsilon^2 + |z^n|^2)e^{-\varphi})^{-1}dz^n \cdot d\bar{z}^n.
\]

We will show that \[147\] is uniformly equivalent to the Euclidean metric on $\mathbb{C}$.

Now we estimate it by the polar coordinates transformation. Let $z^n = x + \sqrt{-1}y$, $x = r \cos \theta$ and $y = r \sin \theta$, we have

\[
dz^n \cdot d\bar{z}^n = dz^n \otimes d\bar{z}^n + dz^n \otimes dz^n = 2(dr^2 + r^2 d\theta^2).
\]

We let $\xi = u + \sqrt{-1}v$, $u = \rho \cos \theta_1$ and $v = \rho \sin \theta_1$, by the definition of $\Psi_\varepsilon$, we know that $\theta_1 = \theta$ and $r = (\varepsilon^2 + \rho^2)^{\frac{1}{\beta}} \rho$. Hence we have

\[
\Psi_\varepsilon^*(ke^{-\varphi}(\varepsilon^2 + |z^n|^2)e^{-\varphi})^{-1}dz^n \cdot d\bar{z}^n = 2ke^{-\varphi \Psi_\varepsilon}(\varepsilon^2 + (\varepsilon^2 + \rho^2))^{\frac{1}{\beta} - 1}(\varepsilon^2 + \rho^2)^{\frac{1}{\beta} - 1} \frac{1}{2}(\varepsilon^2 + \rho^2)^{\frac{1}{\beta} - 1}.
\]

By the fact $1 \leq (1 + (\frac{1}{\beta} - 1)(\varepsilon^2 + \rho^2)\frac{1}{\beta} - 1)^2 \leq \frac{1}{\beta}$, we only need to prove that the term $(\varepsilon^2 + (\varepsilon^2 + \rho^2))^{\frac{1}{\beta} - 1}(\varepsilon^2 + \rho^2)^{\frac{1}{\beta} - 1}$ can be uniformly bounded, and the lower uniform bound away from 0. Firstly, we bounded it from below,

\[
(\varepsilon^2 + (\varepsilon^2 + \rho^2))^{\frac{1}{\beta} - 1}(\varepsilon^2 + \rho^2)^{\frac{1}{\beta} - 1} \geq ((\varepsilon^2 + \rho^2))^{\frac{1}{\beta} - 1}(\varepsilon^2 + \rho^2)^{\frac{1}{\beta} - 1}(\varepsilon^2 + \rho^2)^{\frac{1}{\beta} - 1} \\
\geq (\varepsilon^2 + \rho^2)\left(1 + e^{-\varphi \Psi_\varepsilon}\right) \geq 1 + e^{-\varphi \Psi_\varepsilon} \geq c > 0,
\]
where $c$ independents of $\varepsilon$. Secondly, we prove it can be bounded from above. Let

$\varepsilon^\beta = l \cos \vartheta$ and $\rho = l \sin \vartheta$, where $\vartheta \in [0, \frac{\pi}{2}]$,

\[
(\varepsilon^2 + (\varepsilon^{2\beta} + \rho^2)^{\frac{1}{\beta} - 1}) \rho^2 e^{-\varphi \Psi_\varepsilon} = (l^2 \cos^2 \vartheta + l^2(\frac{1}{\beta} - 1)l^2 \sin^2 \vartheta e^{-\varphi \Psi_\varepsilon})^{\frac{1}{\beta} - 1}2^{(\frac{1}{\beta} - 1)}
\]

\[
= \left( \frac{1}{\cos^2 \vartheta + \sin^2 \vartheta e^{-\varphi \Psi_\varepsilon}} \right)^{1-\beta} \leq \left( \frac{1}{\cos^2 \vartheta + \sin^2 \vartheta \varepsilon^{1-\beta}} \right)^{1-\beta} e^\varepsilon (1-\beta)
\]

In conclusion, this shows that

$C_1(d\rho^2 + \rho^2 dv^2) \leq \Psi_\varepsilon^\ast(k e^{-\varphi}(\varepsilon^2 + |z|^2 e^{-\varphi\beta-1} d\zeta + \cdot d\bar{\zeta}) \leq C_2(d\rho^2 + \rho^2 dv^2)$

for some uniform constant $C_1$ and $C_2$ independent of $\varepsilon$. So it is easy to see that the pull-back of the metric $g_\varepsilon$ under the map $\Psi_\varepsilon$ is uniformly equivalent to the Euclidean metric in a small neighborhood of the divisor $D$, where the uniform equivalence do not depend on $\varepsilon$. Therefore, the Sobolev inequality holds if the function $v$ is supported in the above coordinate charts. The global case follows in the standard way by using a partition of unity. Following this arguments, we conclude:

**Lemma 4.5** Let $v$ be a smooth function on $M$. Then

\[
(4.10) \quad \left( \int_M v^2 dV_\varepsilon \right)^{\frac{1}{2}} \leq C \left( \int_M |dv|^2 g_\varepsilon dV_\varepsilon + \int_M |v|^2 dV_\varepsilon \right)
\]

for some uniform constant $C$ independent of $\varepsilon$.

Under the above appropriate coordinate system $(z^1, z^2, \cdots, z^{n-1}, z^n)$, by direct calculation, we have

\[
(4.11) \quad tr_{g_\varepsilon}(\sqrt{-1} \partial \bar{\partial} \log(\varepsilon^2 + |s|^2)) = k^{-1} \varepsilon^2 (\varepsilon^2 + |z^n|^2)^{-1-\beta} + \mathcal{O}(\varepsilon^2 + |z^n|^2)^{1-2\beta},
\]

The $\mathcal{O}$ being with respect to $\varepsilon^2 + |z|^2$ going to 0.

By further computation based on the computations in [4], we have

\[
(4.12) \quad R(g_\varepsilon) = k^{-1}(1-\beta)\varepsilon^2 (\varepsilon^2 + |z^n|^2)^{-1-\beta} + \mathcal{O}(\varepsilon^2 + |z^n|^2)^{1-2\beta}.
\]

From the above two estimates, we have

\[
(4.13) |R(g_\varepsilon) - tr_{g_\varepsilon} \theta_{\varepsilon}| = |R(g_\varepsilon) - (1-\beta)tr_{g_\varepsilon}(\omega_0 + \sqrt{-1} \partial \bar{\partial} \log(\varepsilon^2 + |s|^2))| = \mathcal{O}(\varepsilon^2 + |z^n|^2)^{1-2\beta} - (1-\beta)tr_{g_\varepsilon}(\omega_0)
\]

Combining (1.9), when $\beta \in (0, \frac{3}{2}]$, we obtain

\[
(4.14) \quad |R(g_\varepsilon) - tr_{g_\varepsilon} \theta_{\varepsilon}| \leq C,
\]

where $C$ is a constant independent of $\varepsilon$.

**Remark 4.6** The estimate (4.14) can also be deduced from the estimate in [4].

By the definition of the generalized Ricci potential, we have

\[
R(g_\varepsilon) - tr_{g_\varepsilon} \theta_{\varepsilon} = \beta n - \Delta u_{\varepsilon},
\]

\[
u_{\varepsilon} = \log \frac{\omega_\varepsilon^n (\varepsilon^2 + |s|^2)^{1-\beta}}{\omega_0^n} + k\beta \chi + F_0.
\]
Since \( \omega_\varepsilon = \omega_0 + \sqrt{-1} \partial \overline{\partial} \chi \), taking trace with respect to \( \omega_\varepsilon \) on both sides,
\begin{equation}
(4.15)
\end{equation}
\[ n = tr_{\omega_\varepsilon} \omega_0 + k \Delta_{\omega_\varepsilon} \chi \geq k \Delta_{\omega_\varepsilon} \chi, \]
by (4.18), we have
\begin{equation}
(4.16)
\end{equation}
\[ k \Delta_{\omega_\varepsilon} \chi = n - tr_{\omega_\varepsilon} \omega_0 \geq n - \gamma^{-1} n. \]
In [4] (see section 4.5), the authors have proved that \( \Delta_{\omega_\varepsilon} (\log \frac{1}{\omega_0})^{(\varepsilon^2 + |s|^2)^{1-\beta}} + F_0 \) is uniformly bounded when \( \beta \in (0, \frac{1}{2}] \). So we conclude that \( R(g_\varepsilon) - tr_{g_\varepsilon} \theta_\varepsilon \) can be uniformly bounded when \( \beta \in (0, \frac{1}{2}] \).

In [26] (see Theorem 2.2), we have that
\begin{equation}
(4.17)
\end{equation}
\[ \mu_{\theta_\varepsilon}(g_\varepsilon, \tau) \geq -\tau V \max_M (R(g_\varepsilon) - tr_{g_\varepsilon} \theta_\varepsilon) - n \beta V (\log 2 + \alpha V^{-\frac{1}{n}} - 1) + \beta nV \log \alpha - \beta V \log V + \beta nV \log \tau \]
where \( V \) is the volume of \( (M, g_\varepsilon) \) and \( \alpha \) satisfies \( 4\tau \geq \beta nC_\mathcal{S}(M, g_\varepsilon) \). Since \( Vol(M, g_\varepsilon) \) is fixed, \( max_M (R(g_\varepsilon) - tr_{g_\varepsilon} \theta_\varepsilon) - C_\mathcal{S}(M, g_\varepsilon) \) are uniformly bounded by (4.14) and Lemma 4.5 respectively. By choosing a suitable \( \alpha \), we know that there exists a uniform constant \( C \) independent of \( \varepsilon \), such that
\begin{equation}
(4.18)
\end{equation}
\[ \mu_{\theta_\varepsilon}(g_\varepsilon, \tau) \geq -C. \]

Now we start to prove Theorem 4.1. Firstly, through differentiating equation (4.1), we get
\begin{equation}
(4.19)
\end{equation}
\[ \frac{d}{dt} u_\varepsilon(t) = \Delta u_\varepsilon(t) + \beta u_\varepsilon(t) - a_\varepsilon(t). \]
By differentiating \( \frac{1}{V} \int_M e^{-u_\varepsilon(t)} dV e-t = 1 \), we conclude
\begin{equation}
(4.20)
\end{equation}
\[ a_\varepsilon(t) = \frac{\beta}{V} \int_M u_\varepsilon(t) e^{-u_\varepsilon(t)} dV e-t. \]

It is obvious that \( a_\varepsilon(t) \leq 0 \) by Jensen’s inequality. Since the functional \( \mu_{\theta_\varepsilon}(g_\varepsilon(t), 1) \) is nondecreasing and \( \mu_{\theta_\varepsilon}(g_\varepsilon, 1) \) is uniformly bounded from below, then following the analogous arguments in [33], we have the following lemma. For readers’ convenience, we give the proof here.

**Lemma 4.7** There exists a uniform constant \( C \), such that
\begin{equation}
(4.21)
\end{equation}
\[ |a_\varepsilon(t)| \leq C \]
for every \( t \) and \( \varepsilon \).

**Proof:** We need only prove that \( a_\varepsilon(t) \) can be uniformly bounded from below.
\begin{equation}
(4.22)
\end{equation}
\[ \mu_{\theta_\varepsilon}(g_\varepsilon, 1) \leq \mu_{\theta_\varepsilon}(g_\varepsilon(t), 1) \]
\begin{align*}
&= \int_M (R(g_\varepsilon(t)) - tr_{g_\varepsilon(t)} \theta_\varepsilon + \beta u_\varepsilon(t) + |\nabla u_\varepsilon(t)|_{g_\varepsilon(t)}^2) e^{-u_\varepsilon(t)} dV e-t \\
&= \int_M (\beta n - \Delta_\varepsilon(t) u_\varepsilon(t) + \beta u_\varepsilon(t) + |\nabla u_\varepsilon(t)|_{g_\varepsilon(t)}^2) e^{-u_\varepsilon(t)} dV e-t \\
&= \int_M (\beta n - \Delta_\varepsilon(t) u_\varepsilon(t) + \beta u_\varepsilon(t) + \int_M u_\varepsilon(t) e^{-u_\varepsilon(t)} dV e-t + \beta nV \\
&= \int_M \Delta e^{-u_\varepsilon(t)} dV e-t + V a_\varepsilon(t) + \beta nV
\end{align*}
Hence, by (4.18), we have
\[ a_\varepsilon(t) \geq \frac{1}{V} \mu_{\theta_\varepsilon}(g_\varepsilon, 1) - \beta n \geq -C \]
for a uniform constant. \(\square\)

**Lemma 4.8** \( R(g_\varepsilon(t)) - tr_{g_\varepsilon(t)}\theta_\varepsilon \) is uniformly bounded from below along the flow \(1.10\), i.e. there exists a constant \( C \) such that
\[ (4.22) \quad R(g_\varepsilon(t)) - tr_{g_\varepsilon(t)}\theta_\varepsilon \geq -C \]
for every \( t \) and \( \varepsilon \).

**Proof:** By direct computation, we have the evolution equation of \( R(g_\varepsilon(t)) - tr_{g_\varepsilon(t)}\theta_\varepsilon \) as follow,
\[ \frac{d}{dt} - \Delta_{g_\varepsilon} (R(g_\varepsilon(t)) - tr_{g_\varepsilon(t)}\theta_\varepsilon) = |R_{\varepsilon ij} - \theta_{\varepsilon ij}|^2 - \beta (R(g_\varepsilon(t)) - tr_{g_\varepsilon(t)}\theta_\varepsilon) \geq -\beta (R(g_\varepsilon(t)) - tr_{g_\varepsilon(t)}\theta_\varepsilon). \]
Applying the maximal principle and (4.14), we imply that \( R(g_\varepsilon(t)) - tr_{g_\varepsilon(t)}\theta_\varepsilon \) is uniformly bounded from below along the flow \(1.10\). \(\square\)

**Lemma 4.9** The generalized Ricci potential \( \phi_\varepsilon(t) \) is uniformly bounded from below along the flow \(1.10\).

**Proof:** By equation (4.11), we have \( \Delta_{g_\varepsilon(t)} u_\varepsilon(t) = -R(g_\varepsilon(t)) + \beta n + tr_{g_\varepsilon(t)}\theta_\varepsilon \).

From Lemma 4.7 and Lemma 4.8, there exists a uniform constant \( C_1 \) satisfies
\[ (4.23) \quad \Delta_{g_\varepsilon(t)} u_\varepsilon(t) - a_\varepsilon(t) \leq C_1. \]

We conjecture \( u_\varepsilon(t) \geq -2C_1 \beta \) for every \( t \) and \( \varepsilon \). If not, then there exists \((\varepsilon_0, y_0, t_0)\) such that \( u_{\varepsilon_0}(y_0, t_0) < -2C_1 \beta \), by (4.19)
\[ \left. \frac{du_{\varepsilon_0}(t)}{dt} \right|_{(y_0, t_0)} \leq \beta u_{\varepsilon_0}(y_0, t_0) + C_1 < -C_1. \]
So there exists \( U(y_0) \times [t_0, t_0 + \delta) \), where \( u_{\varepsilon_0}(t) \) satisfies
\[ \left\{ \begin{array}{l}
\left. \frac{du_{\varepsilon_0}(t)}{dt} \right|_{(y, t)} < 0 \quad (y, t) \in U(y_0) \times [t_0, t_0 + \delta). \\
u_{\varepsilon_0}(y, t) < -\frac{2C_1}{\beta} 
\end{array} \right. \]
By the continuity of \( u_{\varepsilon_0}(t) \) with respect to time \( t \), \( u_{\varepsilon_0}(y) \ll 0 \) on \( U(y_0) \) when \( t \geq t_0 \).
Now we denote \( U(y_0) \) by \( U \) for simplicity.

For every \( z \in U, t \geq t_0, \left. \frac{du_{\varepsilon_0}(t)}{dt} \right|_{(z, t)} \leq \beta u_{\varepsilon_0}(z, t) + C_1 \), so
\[ (4.24) \quad u_{\varepsilon_0}(z, t) \leq e^{\beta t} (u_{\varepsilon_0}(z, t_0) e^{-\beta t_0} - \frac{C_1}{\beta} e^{-\beta t} + \frac{C_1}{\beta} e^{-\beta t_0} \leq -C_2 e^{\beta t}, \]
where \( C_2 \) depending only on \( C_1, \beta \) and \( t_0 \).

\[ u_{\varepsilon_0}(t) = \varphi_{\varepsilon_0}(t) + c_{\varepsilon_0}(t) = (\varphi_{\varepsilon_0}(t) + \int_{t_0}^{t} c_{\varepsilon_0}(s) ds) = \dot{\varphi}_{\varepsilon_0}(t). \]
When \( z \in U \) and \( t \) sufficiently large,
\[ (4.25) \quad \phi_{\varepsilon_0}(z, t) \leq \phi_{\varepsilon_0}(z, t_0) - \frac{C_2}{\beta} e^{\beta t} + \frac{C_2}{\beta} e^{\beta t_0} \leq -C_3 e^{\beta t}, \]
where \( C_3 \) depending only on \( C_1, \beta, t_0 \) and \( \varepsilon_0 \).
On the other hand,
\[ 1 = \frac{1}{V} \int_M e^{-u_{\varepsilon_0}(t)} dV \geq e^{-\sup_M u_{\varepsilon_0}(t)} \]
for every \( t \), hence we have
\[ \sup_M u_{\varepsilon_0}(t) \geq 0. \]
(4.26)

Let \( u_{\varepsilon_0}(x,t) = \sup_M u_{\varepsilon_0}(t) \), combining with
\[ \frac{d}{dt} (u_{\varepsilon_0}(t) - \beta \phi_{\varepsilon_0}(t)) = \triangle_{g_{\varepsilon_0}(t)} u_{\varepsilon_0}(t) - a_{\varepsilon_0}(t) \leq C_1, \]
we have
\[ u_{\varepsilon_0}(x,t) - \beta \phi_{\varepsilon_0}(x,t) - (u_{\varepsilon_0}(x,t) - \beta \phi_{\varepsilon_0}(x,0)) \leq C_1 t, \]
(4.27)
\[ \sup_M \phi_{\varepsilon_0}(\cdot,t) \geq -C_4 - C_1 t, \]
where \( C_4 \) depending only on \( \varepsilon_0 \) and \( t = 0 \). Combined the Green formula with respect to metric \( g_0 \), for sufficiently large \( t \),
\[ \phi_{\varepsilon_0}(x,t) + \chi (\varepsilon_0^2 + |s|^2) = \frac{1}{Vol_0(M)} \int_M \phi_{\varepsilon_0}(y,t) + \chi (\varepsilon_0^2 + |s|^2) dV \]
\[ - \frac{1}{Vol_0(M)} \int_M \triangle_{g_0} (\phi_{\varepsilon_0}(y,t) + \chi (\varepsilon_0^2 + |s|^2)) G_0(x,y) dV \]
\[ \leq V ol_0(M \setminus U) \frac{1}{Vol_0(M)} \sup_M \phi_{\varepsilon_0}(\cdot,t) + \frac{1}{Vol_0(M)} \int_U \phi_{\varepsilon_0}(y,t) dV + C_5 \]
\[ \leq \frac{Vol_0(M \setminus U)}{Vol_0(M)} \sup_M \phi_{\varepsilon_0}(\cdot,t) - C_6 e^{\beta t} + C_5. \]
(4.28)

Then we have
\[ \sup_M \phi_{\varepsilon_0}(\cdot,t) \leq -C_7 e^{\beta t} + C_8, \]
where \( C_7 \) depends only on \( C_1, \beta, \varepsilon_0, t_0 \) and \( t = 0 \). \( C_8 \) depends only on \( \varepsilon_0 \) and \( t = 0 \). From (4.24) and (4.28), we obtain
\[ -C_7 e^{\beta t} + C_8 \geq -C_4 - C_1 t. \]
(4.29)

When \( t \) sufficiently large, the inequality (4.29) is not true, so \( u_{\varepsilon}(t) \) is bounded uniformly from below along the flow (1.10).

Denote \( \Box = \frac{d^2}{dt^2} - \triangle_{g_{\varepsilon_0}(t)} \), as computations in [26], we have
\[ \Box (\triangle_{g_{\varepsilon_0}(t)} u_{\varepsilon}(t)) = -|\nabla \nabla u_{\varepsilon}(t)|^2_{g_{\varepsilon_0}(t)} + \beta \triangle_{g_{\varepsilon_0}(t)} u_{\varepsilon}(t), \]
(4.30)
\[ \Box (|\nabla u_{\varepsilon}(t)|^2_{g_{\varepsilon_0}(t)}) = -|\nabla \nabla u_{\varepsilon}(t)|^2_{g_{\varepsilon_0}(t)} - |\nabla \nabla u_{\varepsilon}(t)|^2_{g_{\varepsilon_0}(t)} \]
\[ + \beta |\nabla u_{\varepsilon}(t)|^2_{g_{\varepsilon_0}(t)} - \frac{1}{2} \theta_{\varepsilon}(\text{grad } u_{\varepsilon}(t), \mathcal{J}(\text{grad } u_{\varepsilon}(t))), \]
(4.31)
where \( \mathcal{J} \) is the complex structure on \( M \).

**Lemma 4.10** There exist a constant \( C \) independent of time \( t \) and \( \varepsilon \), such that
\[ |\nabla u_{\varepsilon}(t)|^2_{g_{\varepsilon_0}(t)} \leq C(u_{\varepsilon}(t) + C), \]
(4.32)
\[ R(g_{\varepsilon}(t)) - tr_{g_{\varepsilon}(t)} \theta \leq C(u_{\varepsilon}(t) + C). \]
(4.33)
Proof: It follows from Lemma 4.9 that there exists a uniform constant \( B > 1 \) such that \( u_\varepsilon(t) > -B \). Define

\[
H_\varepsilon(t) = \frac{|\nabla u_\varepsilon(t)|^2}{u_\varepsilon(t) + 2B},
\]

the same arguments as that in [20], we have

\[
\Box H_\varepsilon(t) \leq -\frac{|\nabla \nabla u_\varepsilon(t)|^2 - |\nabla u_\varepsilon(t)|^2}{u_\varepsilon(t) + 2B} + \frac{|\nabla u_\varepsilon(t)|^2 (2B^2 + a_\varepsilon(t)) + (2 - \delta) \frac{\nabla u_\varepsilon(t) \cdot \nabla H_\varepsilon(t)}{u_\varepsilon(t) + 2B}}{u_\varepsilon(t) + 2B} + \delta \frac{|\nabla u_\varepsilon(t)|^4}{u_\varepsilon(t) + 2B}
\]

where \( C_0 \) independents of \( \varepsilon \) and \( t \). Take \( \delta \) satisfying \( C_0^2 \delta < 1 \). Combining with Lemma 4.7 and \( \theta(\text{grad } u_\varepsilon(t), J(\text{grad } u_\varepsilon(t))) \geq 0 \), we obtain

\[
\Box H_\varepsilon(t) \leq \frac{|\nabla u_\varepsilon(t)|^2 (2B\beta + C_1)(u_\varepsilon(t) + 2B)}{(u_\varepsilon(t) + 2B)^2} + (2 - \delta) \frac{\nabla u_\varepsilon(t) \cdot \nabla H_\varepsilon(t)}{u_\varepsilon(t) + 2B} \delta \frac{|\nabla u_\varepsilon(t)|^4}{2(u_\varepsilon(t) + 2B)^3}.
\]

Let \( u_\varepsilon \) be the generalized Ricci potential with respect to the metric \( g_\varepsilon \). By adjusting \( \chi(\varepsilon^2 + |s|_h^2) \) with a constant (whose variation with respect to \( \varepsilon \) is bounded), we can assume that

\[
\frac{1}{V} \int_M e^{-\beta \varepsilon_0 - k \beta \chi} dV_0 (\varepsilon^2 + |s|_h^2)^{1-\beta} = 1.
\]

Then from (41), we can conclude

\[
u_\varepsilon = \log \frac{\omega^n(\varepsilon^2 + |s|_h^2)^{1-\beta}}{\omega^n_0} + k \beta \chi + F_0.
\]

From [1], we know that \( |\nabla \log \frac{\omega^n(\varepsilon^2 + |s|_h^2)^{1-\beta}}{\omega^n_0} + F_0|_{g_\varepsilon} \leq C \). By computing, we have

\[
|\nabla \chi|_{g_\varepsilon}^2 \leq C \left( \frac{\varepsilon^2 + |s|_h^2}{\beta |s|_h^2} - \varepsilon^{2\beta} \frac{3}{\beta |s|_h^2} \right)^2 |s|_h^2 (\varepsilon^2 + |s|_h^2)^{1-\beta} |D' s|_h^2 + C,
\]

where \( C \) is uniform. Since function

\[
F(x, y) = \frac{(x^2 + y^2)^{\beta} - x^{2\beta}}{y^2} (x^2 + y^2)^{1-\beta}
\]

is continuous on \([0, 1] \times [0, 1]\) if we define \( F(0, 0) = 0 \), so \( |\nabla \chi|_{g_\varepsilon}^2 \) is uniformly bounded by a constant \( C_2 \). Hence we have

\[
\sup_M \frac{|\nabla u_\varepsilon|_{g_\varepsilon}^2}{u_\varepsilon + 2B} \leq C_3,
\]

where \( C_3 \) is a uniform constant.

Next, we prove that \( H_\varepsilon(t) \leq \max\{C_3, 2(2B + C_1)\delta^{-1}\} \) for any \( \varepsilon \) and \( t \). If not, there exists \( \varepsilon_0 \) and time \( T \) such that

\[
\sup_{M \times [0, T]} H_{\varepsilon_0}(t) > \max\{C_3, 2(2B + C_1)\delta^{-1}\},
\]

and \( H_{\varepsilon_0}(t) \) achieves its maximal value at \((p, t_0) \in M \times [0, T] \).
If $t_0 > 0$

$$0 \leq \Box H_{c_0}|_{(p,t_0)} \leq \frac{|\nabla u_{c_0}|^2}{(u_{c_0} + 2B)^2}|_{(p,t_0)}(2B + C_1 - \frac{\delta}{2} H_{c_0}|_{(p,t_0)}).$$

Hence

$$H_{c_0}|_{(p,t_0)} \leq 2(2B + C_1)\delta^{-1},$$

which gives a contradiction.

If $t_0 = 0$, then $\sup_{M \times [0,T]} H_{\epsilon} \leq \sup_{M} H_{\epsilon}(.,0) = \sup_{M} \frac{|\nabla u_{\epsilon}|^2}{u_{\epsilon} + 2B} \leq C_3$, which also gives a contradiction. So there exists a constant $C$, such that $H_{\epsilon}(t) \leq C$ for every $t$ and $\epsilon$.

Now we prove the second inequality. Since $\triangle_{g_{\epsilon}(t)} u_{\epsilon}(t) = 0 - R(g_{\epsilon}(t)) + tr_{g_{\epsilon}(t)} \theta_{\epsilon}$, we need only prove the existence of the uniform constant $C$ such that $-\triangle_{g_{\epsilon}(t)} u_{\epsilon}(t)$ can be controlled by $C(\epsilon u_{\epsilon}(t) + C)$.

Let $G_{\epsilon} = -\triangle_{g_{\epsilon}(t)} u_{\epsilon}(t) + 2H_{\epsilon}$, so

$$\Box G_{\epsilon} = -2|\nabla \nabla u_{\epsilon}(t)|^2 - |\nabla u_{\epsilon}(t)|^2 u_{\epsilon}(t) + 2|\nabla u_{\epsilon}(t)|^2 (2B + a_{\epsilon}(t)) \frac{(u_{\epsilon}(t) + 2B)^2}{u_{\epsilon}(t) + 2B} + 2 \frac{\nabla u_{\epsilon}(t) \cdot \nabla G_{\epsilon}}{u_{\epsilon}(t) + 2B} - \frac{1}{2} \theta_{\epsilon} (\partial \nabla u_{\epsilon}(t), J(\nabla u_{\epsilon}(t))).$$

Since $\theta_{\epsilon}$ is semi-positive, we obtain

$$\Box G_{\epsilon} \leq -2|\nabla \nabla u_{\epsilon}(t)|^2 + \frac{(-\triangle_{g_{\epsilon}(t)} u_{\epsilon}(t) + 2|\nabla u_{\epsilon}(t)|^2 (2B + a_{\epsilon}(t)) \frac{(u_{\epsilon}(t) + 2B)^2}{u_{\epsilon}(t) + 2B} + 2 \frac{\nabla u_{\epsilon}(t) \cdot \nabla G_{\epsilon}}{u_{\epsilon}(t) + 2B}}{n(u_{\epsilon}(t) + 2B)}.$$

In local coordinates,

$$(\triangle u_{\epsilon}(t))^2 = \left( \sum_i u_{\epsilon i} \right)^2 \leq n \sum_i u_{\epsilon i}^2 = n|\nabla \nabla u_{\epsilon}(t)|^2.$$

So

$$\Box G_{\epsilon} \leq \frac{-\triangle u_{\epsilon}(t)}{u_{\epsilon}(t) + 2B} \left( 2B + a_{\epsilon}(t) \right) + \frac{(-\triangle_{g_{\epsilon}(t)} u_{\epsilon}(t) + 2|\nabla u_{\epsilon}(t)|^2 (2B + a_{\epsilon}(t)) \frac{(u_{\epsilon}(t) + 2B)^2}{u_{\epsilon}(t) + 2B} + 2 \frac{\nabla u_{\epsilon}(t) \cdot \nabla G_{\epsilon}}{u_{\epsilon}(t) + 2B}}{n(u_{\epsilon}(t) + 2B)}.$$

Since $-\triangle_{g_{\epsilon}(t)} u_{\epsilon}/u_{\epsilon} + 2B$ is bounded uniformly when $0 < \beta \leq \frac{1}{2}$ from the arguments in [4], as the arguments of the former part, by the maximum principle, there also exists a uniform constant $C > 0$ such that $G_{\epsilon} \leq C$ for every $t$ and $\epsilon$. Then we get $\frac{-\triangle u_{\epsilon}(t)}{u_{\epsilon}(t) + 2B} \leq G_{\epsilon} \leq C$ by the fact $H_{\epsilon} \geq 0$.

From (4.32) in Lemma 4.10 and the same discussion in [35] (see Claim 8), we have the following lemma:

**Lemma 4.11** There exists a uniform constant $C$, such that

\begin{align*}
(4.40) & \quad u_{\epsilon}(y,t) \leq C \text{ dist}_{\mathcal{E}}(x,y) + C, \\
(4.41) & \quad R(g_{\epsilon}(t)) - tr_{g_{\epsilon}(t)} \theta_{\epsilon} \leq C \text{ dist}_{\mathcal{E}}(x,y) + C, \\
(4.42) & \quad |\nabla u_{\epsilon}(t)|_{g_{\epsilon}(t)} \leq C \text{ dist}_{\mathcal{E}}(x,y) + C,
\end{align*}

where $u_{\epsilon}(x,t) = \inf_{y \in M} u_{\epsilon}(y,t)$.

By Lemma 4.11, the statements in Theorem 4.1 will hold if the $diam(M, g_{\epsilon}(t))$ is uniformly bounded. Firstly, we following the arguments in [22] and [36] to prove a generalized version of Perelman’s noncollapsing theorem.
Lemma 4.12 Let $g_c(t)$ be a solution of the flow \[1.10\], then there exists a uniform constant $C$, such that

$$\text{Vol}_{g_c(t)}(B_{g_c(t)}(x, r)) \geq Cr^{2n}$$

for every $g_c(t)$ satisfying $R(g_c(t)) - \text{tr}_{g_c(t)} \theta_c \leq \frac{m}{k}$ on $B_{g_c(t)}(x, r)$, where $\partial B_{g_c(t)}(x, r) \neq \emptyset$ and $0 < r < 1$.

Proof: Comparing \[22\] and \[36\]. We argue by contradiction, that is, there exist $\varepsilon_k$, $p_k$, $t_k$, $r_k$ satisfying $R_{g_{e_k}(t_k)} - \text{tr}_{g_{e_k}(t_k)} \theta_{e_k} \leq \frac{m}{k}$ on $B_{g_{e_k}(t_k)}(p_k, r_k)$, but $\varepsilon_k \to 0$ and $\text{Vol}_{g_{e_k}(t_k)}(B_{g_{e_k}(t_k)}(p_k, r_k)) : r_k^{-2n} \to 0$ when $k \to +\infty$. Define $B_{g_{e_k}(t_k)}(p_k, r_k) = B_k$, $\text{Vol}_{g_{e_k}(t_k)}(B_{g_{e_k}(t_k)}(p_k, r_k)) = V(r_k)$ in the following.

Set $\tau = r_k^2$ at $t_k$, we define function

$$\text{(4.43)}$$

$$u_k(x) = e^{C_k} \phi(r_k^{-1} \text{dist}_{g_{e_k}(t_k)}(x, p_k)),$$

where $\phi$ is smooth function on $\mathbb{R}$, equal 1 on $[0, \frac{1}{2}]$, decreasing on $[\frac{1}{2}, 1]$ and equal 0 on $[1, +\infty)$, $C_k$ is a constant making $u_k$ satisfy the constraint

$$\frac{1}{V} \int_M r_k^{-2n} u_k^2 d\text{Vol}_{\varepsilon_k} = 1.$$

Hence we obtain

$$1 = \frac{1}{V} e^{2C_k} r_k^{-2n} \int_{B_k} \phi^2 d\text{Vol}_{\varepsilon_k} \leq \frac{1}{V} e^{2C_k} r_k^{-2n} V(r_k).$$

By assumption, $V(r_k) r_k^{-2n} \to 0$ when $k \to +\infty$, this shows that $C_k \to +\infty$ when $k \to +\infty$. So we claim:

(a) $V(r_k) r_k^{-2n} \to 0$ as $k \to +\infty$;

(b) $(R_{g_{e_k}(t_k)} - \text{tr}_{g_{e_k}(t_k)} \theta_{e_k}) r_k^2 \leq m$;

(c) $\frac{V(r_k)}{V(\frac{r}{4})}$ is uniformly bounded.

We should only prove (c). If $\frac{V(r_k)}{V(\frac{r}{4})} < 5^n$ for every $k$, we are done. If not, for a given $k$, we have $\frac{V(r_k)}{V(\frac{r}{4})} \geq 5^n$. Let $r_k' = \frac{r_k}{2}$, we have $(r_k')^{-2n} V(r_k') \leq (r_k')^{-2n} \frac{1}{V(r_k)} V(r_k) = (\frac{4}{5})^n r_k^{-2n} V(r_k)$, $(r_k')^2 (R(g_{e_k}(t_k)) - \text{tr}_{g_{e_k}(t_k)} \theta_{e_k}) = \frac{r_k^2}{r_k'^2} (R(g_{e_k}(t_k)) - \text{tr}_{g_{e_k}(t_k)} \theta_{e_k})$, combining (a) and (b), we obtain $(r_k')^2 (R(g_{e_k}(t_k)) - \text{tr}_{g_{e_k}(t_k)} \theta_{e_k}) \leq m$ and $(r_k')^{-2n} V(r_k') \to 0$ when $k \to +\infty$. Replace $r_k$ by $r_k'$, if $\frac{V(r_k')}{V(\frac{r}{4})} < 5^n$, we stop, if not, then we repeat the above process, by the identity $\lim_{r \to 0} \frac{V(r)}{V(\frac{r}{4})} = 4^n$ proved in \[20\] (see (6.9)), we will get $\frac{V(r_k')}{V(\frac{r}{4})} < 5^n$ at some step. Then we consider $\{p_k, r_k\}$ obtained from above.

Considering the function $\frac{1}{r_k' + 1}(u_k^2 + r_k'^{2n+2})$, its integral average

$$\frac{1}{V} \int_M r_k'^{-2n} \frac{1}{r_k'^2 + 1}(u_k^2 + r_k'^{2n+2}) d\text{Vol}_{\varepsilon_k} = 1.$$
Computing the $W_{\theta_{\epsilon_k}}(g_{\epsilon_k}(t_k), -\log \frac{1}{r_k^{2n}}(u_k^2 + r_k^{2n+2}), r_k^2)$ functional, we have

$$W_{\theta_{\epsilon_k}}(g_{\epsilon_k}(t_k), -\log \frac{1}{r_k^{2n}}(u_k^2 + r_k^{2n+2}), r_k^2)$$

$$= \frac{1}{r_k^2 + 1} \int_M r_k^{-2n}(u_k^2 + r_k^{2n+2})(R(g_{\epsilon_k}(t_k)) - tr_{g_{\epsilon_k}(t_k)}\theta_{\epsilon_k}) r_k^2 dV_{\epsilon_k t_k} \quad (1)$$

$$+ \frac{1}{r_k^2 + 1} \int_M r_k^{-2n}(u_k^2 + r_k^{2n+2}) 4u_k^2 |\nabla u_k|^2_{g_{\epsilon_k}(t_k)} (u_k^2 + r_k^{2n+2})^2 r_k^2 dV_{\epsilon_k t_k} \quad (2)$$

$$- \frac{\beta}{r_k^2 + 1} \int_M r_k^{-2n}(u_k^2 + r_k^{2n+2}) \log \frac{1}{r_k^{2n+2}} dV_{\epsilon_k t_k} \quad (3)$$

$$- \frac{\beta}{r_k^2 + 1} \int_M r_k^{-2n}(u_k^2 + r_k^{2n+2}) \log(u_k^2 + r_k^{2n+2}) dV_{\epsilon_k t_k}. \quad (4)$$

$$\leq \frac{m}{r_k^2 + 1} \int_M r_k^{-2n}(u_k^2 + r_k^{2n+2}) dV_{\epsilon_k t_k} \leq mV, \quad (2)$$

$$\leq \frac{1}{r_k^2 + 1} \int_M r_k^{-2n}(u_k^2 + r_k^{2n+2}) 4u_k^2 |\nabla u_k|^2_{g_{\epsilon_k}(t_k)} (u_k^2 + r_k^{2n+2})^2 r_k^2 dV_{\epsilon_k t_k} \quad (2)$$

$$\leq C r_k^{-2n} e^{2C_k} (V(r_k) - V(\frac{r_k^2}{2})), \quad (3)$$

$$= -\frac{\beta}{r_k^2 + 1} \int_M r_k^{-2n}(u_k^2 + r_k^{2n+2}) \log \frac{1}{r_k^{2n+2}} dV_{\epsilon_k t_k} \quad (3)$$

$$= \beta V \log(r_k^2 + 1) \leq \beta V \log 2, \quad (3)$$

$$= -\frac{\beta}{r_k^2 + 1} \int_M r_k^{-2n}(u_k^2 + r_k^{2n+2}) \log(u_k^2 + r_k^{2n+2}) dV_{\epsilon_k t_k} \quad (4)$$

$$\leq \frac{\beta C_k}{r_k^2 + 1} \int_M r_k^{-2n} u_k^2 dV_{\epsilon_k t_k} - \frac{\beta(n + 1)}{r_k^2 + 1} \int_M r_k^2 \log r_k^2 dV_{\epsilon_k t_k} \quad (4)$$

$$\leq -\frac{2\beta C_k}{r_k^2 + 1} \int_M r_k^{-2n} u_k^2 dV_{\epsilon_k t_k} - \frac{\beta}{r_k^2 + 1} \int_M r_k^{-2n} e^{2C_k} \phi^2 \log \phi^2 dV_{\epsilon_k t_k} + \frac{C\beta(n + 1)V}{r_k^2 + 1} \quad (4)$$

$$\leq -\beta V C_k + C r_k^{-2n} e^{2C_k} (V(r_k) - V(\frac{r_k^2}{2})) + C,$$
\[ C - C_k \beta V + C \int_{B_{r_k}(u_k)(p_k, t_k)} r_k^{-2n} e^{2C_k \phi^2} dV_{\varepsilon t_k} \]
\[ \leq C - C_k \beta V \]

where \( C \) are uniform constants independent of time \( t_k \) and \( \varepsilon_k \).
Consider \( \tau = 1 - (1 - r_k^2)e^{-\beta t_k} \), by Lemma 4.3, we conclude
\[ \mu_{\varepsilon_k}(g_{\varepsilon_k}, 1 - (1 - r_k^2)e^{-\beta t_k}) \leq \mu_{\varepsilon_k}(g_{\varepsilon_k}(t_k), r_k^2) \]
\[ \leq \mathcal{W}_{\theta_k}(g_{\varepsilon_k}(t_k), -\log \frac{1}{r_k^2} + 1(u_k^2 + 1)r_k^2) \]
\[ \leq C - 2C_k \beta V. \]

Since \( 0 < 1 - (1 - r_k^2)e^{-\beta t_k} < 1 \), by (4.45), we conclude that
\[ \mu_{\varepsilon_k}(g_{\varepsilon_k}, 1 - (1 - r_k^2)e^{-\beta t_k}) \geq -C, \]
where \( C \) independent of \( \varepsilon_k \) and \( t_k \). We get \( -C \leq C - C_k \beta V \). When \( k \to +\infty \), which is impossible. So we prove the lemma. \( \square \)

Denote \( d_{\varepsilon t}(z) = \text{dist}_{\varepsilon t}(x, z) \), \( B_{\varepsilon t}(k_1, k_2) = \{ z | 2^{k_1} \leq d_{\varepsilon t}(z) \leq 2^{k_2} \} \), where \( u_{\varepsilon}(x, t) = \inf_{M} u_{\varepsilon}(y, t) \). Consider an annulus \( B_{\varepsilon t}(k, k + 1) \). By Lemma 4.11, we have that \( R(g_{\varepsilon t}(t)) - tr_{g_{\varepsilon t}(t)} \theta \leq C 2^{2k} \) on \( B_{\varepsilon t}(k, k + 1) \). Interval \([2^k, 2^{k+1}]\) fits \( 2^k \) balls of radii \( \frac{1}{2} \). By Lemma 4.12 we have that
\[ (4.45) \quad \text{Vol}_{g_{\varepsilon t}}(B_{\varepsilon t}(k, k + 1)) \geq \sum_i \text{Vol}_{g_{\varepsilon t}}(B_{\varepsilon t}(x_i, \frac{1}{2e})) \geq C 2^{2k-2nk}. \]

**Lemma 4.13** For every \( \delta > 0 \), there exists \( B_{\varepsilon t}(k_1, k_2) \), such that if \( \text{diam}(M, g_{\varepsilon t}(t)) \) is large enough, then
\[ (a) \text{Vol}_{g_{\varepsilon t}}(B_{\varepsilon t}(k_1, k_2)) < \delta, \]
\[ (b) \text{Vol}_{g_{\varepsilon t}}(B_{\varepsilon t}(k_1, k_2)) \leq 2^{2n} \text{Vol}_{g_{\varepsilon t}}(B_{\varepsilon t}(k_1 + 2, k_2 - 2)). \]

**Proof:** We first fix any \( \delta > 0 \). Since \( \text{Vol}_{g_{\varepsilon t}}(M) \) is a constant \( V \) along the generalized Kähler-Ricci flow and therefore uniformly bounded. Let \( k \gg 1 \), then
\[ V = \text{Vol}_{g_{\varepsilon t}}(B_{\varepsilon t}(0, k)) + \text{Vol}_{g_{\varepsilon t}}(B_{\varepsilon t}(k, 3k)) + \cdots + \text{Vol}_{g_{\varepsilon t}}(B_{\varepsilon t}(3^{\alpha - 1}k, 3^{\alpha}k)) + \cdots, \]
where \( \alpha > m\frac{\sqrt{V}}{2} + 1 \), \( m \) will be determined later and \( \text{diam}(M, g_{\varepsilon t}(t)) > 2^{\alpha k + 1} \). We claim that there must exists a \( 0 < i \leq \alpha - 1 \), such that \( \text{Vol}_{g_{\varepsilon t}}(B_{\varepsilon t}(3^i k, 3^{i+1}k)) < \delta \). If not, then we have
\[ V > \text{Vol}_{g_{\varepsilon t}}(B_{\varepsilon t}(k, 3k)) + \cdots + \text{Vol}_{g_{\varepsilon t}}(B_{\varepsilon t}(3^{\alpha - 1}k, 3^{\alpha}k)) \geq \alpha \delta > m\delta \frac{V}{\delta} + \delta. \]
We take \( m \) satisfying \( m\delta \frac{V}{\delta} + \delta > V \), then it is a contradiction. So we prove the claim.

Then we determine \( k_1 \) and \( k_2 \). If our estimate \( (b) \) did not hold, that is,
\[ \text{Vol}_{g_{\varepsilon t}}(B_{\varepsilon t}(3^i k, 3^{i+1}k)) > 2^{2n} \text{Vol}_{g_{\varepsilon t}}(B_{\varepsilon t}(3^i k + 2, 3^{i+1}k - 2)). \]
We would consider \( \text{Vol}_{g_{\varepsilon t}}(B_{\varepsilon t}(3^i k + 2, 3^{i+1}k - 2)) \) instead and ask whether \( (b) \) holds for that ball. Assume that for every \( p \), at the \( p \)-th step we are still not able
to find our radii so that (a) and (b) are satisfied. In that case, at the p-th step we would have

\[ \text{Vol}_{g_\varepsilon(t)}(B_{ct}(3^i k, 3^{i+1} k)) > 2^{20np} \text{Vol}_{g_\varepsilon(t)}(B_{ct}(3^i k + 2p, 3^{i+1} k - 2p)). \]

In particular, assume we have the above estimate at the p-th step so that \(3^i k + 2p = \frac{3}{2} 3^i k\), then we have \(3^{i+1} k - 2p = \frac{3}{2} 3^i k\). By inequality (4.45),

\[ \delta > \text{Vol}_{g_\varepsilon(t)}(B_{ct}(3^i k, 3^{i+1} k)) > 2^{5n-3k} \text{Vol}_{g_\varepsilon(t)}(B_{ct}(\frac{3}{2} 3^i k, \frac{5}{2} 3^i k)) \]

\[ > 2^{5n-3k} \text{Vol}_{g_\varepsilon(t)}(B_{ct}(\frac{3}{2} 3^i k, \frac{3}{2} 3^i k + 1)) \]

\[ \geq C 2^{(2n+3)-3^i k}. \]

This leads to contradiction if we let \(k \gg 1\). So there exists some \(1 \leq j \leq p - 1\), such that

\[ \text{Vol}_{g_\varepsilon(t)}(B_{ct}(3^i k + 2j, 3^{i+1} k - 2j)) \leq 2^{20n} \text{Vol}_{g_\varepsilon(t)}(B_{ct}(3^i k + 2(j+1), 3^{i+1} k - 2(j+1))). \]

Let \(k_1 = 3^i k + 2j, k_2 = 3^{i+1} k - 2j\) and we have \(k_2 - k_1 = 2 \cdot 3^i k - 4j \geq 3^j k \gg 1\). This finishes the proof of the lemma. \(\square\)

As the arguments in [35] (see Lemma 11), we have the following lemma:

**Lemma 4.14** There must exist \(r_1 \in [2^{k_1}, 2^{k_1+1}], r_2 \in [2^{k_2-1}, 2^{k_2}]\) and a uniform constant \(C\), such that

\[ \int_{B(r_1, r_2)} (R(g_\varepsilon(t)) - tr g_\varepsilon(t) \theta) dV_{ct} \leq CV < C\delta, \]

where \(\delta > 0\), \(V = \text{Vol}_{g_\varepsilon(t)}(B_{ct}(k_1, k_2))\) that is obtained in Lemma 4.13, and \(B_{ct}(r_1, r_2) = \{z \in M | 2^{r_1} \leq d_{ct}(z) \leq 2^{r_2}\}\).

Finally, we prove that \(\text{diam}(M, g_\varepsilon(t))\) can be uniformly bounded along the flow (1.10) follows from Perelman’s arguments.

**Lemma 4.15** \(\text{diam}(M, g_\varepsilon(t))\) is uniformly bounded along the flow (1.10).

**Proof:** If \(\text{diam}(M, g_\varepsilon(t))\) is not uniformly bounded, there exists \(\{t_i\}\) and \(\varepsilon_i \to 0\) such that \(\text{diam}(M, g_\varepsilon(t_i)) \to +\infty\). Let \(\delta_i \to 0\) be a sequence consists of positive numbers, which corresponding to \(\{t_i\}\) and \(\{\varepsilon_i\}\). By Lemma 4.13, we can find sequences \(\{k_1^i\}\) and \(\{k_2^i\}\), such that

\[ (4.46) \quad \text{Vol}_{g_\varepsilon(t_i)}(B_{ct_i}(k_1^i, k_2^i)) < \delta_i, \]

\[ (4.47) \quad \text{Vol}_{g_\varepsilon(t_i)}(B_{ct_i}(k_1^i, k_2^i)) \leq 2^{20n} \text{Vol}_{g_\varepsilon(t_i)}(B_{ct_i}(k_1^i + 2, k_2^i - 2)). \]

Let \(r_1^i \in [2^{k_1^i}, 2^{k_1^i+1}]\) and \(r_2^i \in [2^{k_2^i-1}, 2^{k_2^i}]\) be given in Lemma 4.14 for each \(i\), \(\phi_i\) be cut off functions such that \(\phi_i = 1\) on \([2^{k_1^i+2}, 2^{k_1^i+2}]\) and \(\phi_i = 0\) on \((-\infty, r_1^i] \bigcup [r_2^i, +\infty)\). Define

\[ u_i(x) = e^{C_1 \phi_i \text{dist}_{ct_i}(x, p_i)}, \]

where \(u_{ct_i}(p_i, t_i) = \inf_M u_{ct_i}(y, t_i)\), \(C_i\) is a constant such that \(u_i(x)\) satisfies \(\frac{1}{V} \int_M u_i^2 dV_{ct_i} = 1\). Then

\[ 1 = \frac{1}{V} \int_M e^{2C_1 \phi_i^2} dV_{ct_i} \leq \frac{1}{V} e^{2C_1} \text{Vol}(B_{ct_i}(k_1^i, k_2^i)) \leq \frac{1}{V} e^{2C_1} \delta_i. \]
Let $i \to +\infty$, since $\delta_i \to 0$, we conclude that $C_i \to +\infty$. Considering the function $\frac{1}{2}(u_i^2 + 1)$, its integral average

$$\frac{1}{V} \int_M \frac{1}{2}(u_i^2 + 1)dV_{\varepsilon_i,t_i} = 1.$$ 

Computing the $W_{\theta_{\varepsilon_i}}(g_{\varepsilon_i}(t_i), - \log \frac{1}{2}(u_i^2 + 1), 1)$ functional, we have

$$W_{\theta_{\varepsilon_i}}(g_{\varepsilon_i}(t_i), - \log \frac{1}{2}(u_i^2 + 1), 1) = \frac{1}{2} \int_M (u_i^2 + 1)(R(g_{\varepsilon_i}(t_i)) - tr_{g_{\varepsilon_i}(t_i)}\theta_{\varepsilon_i} + L)dV_{\varepsilon_i,t_i}$$

$$+ \frac{1}{2} \int_M (u_i^2 + 1)\frac{4u_i^2|\nabla u_i|^2_{g_{\varepsilon_i}(t_i)}}{(u_i^2 + 1)^2}dV_{\varepsilon_i,t_i} - LV$$

$$+ \frac{\beta}{2} \log 2 \int_M (u_i^2 + 1)dV_{\varepsilon_i,t_i} - \frac{\beta}{2} \int_M (u_i^2 + 1)\log(u_i^2 + 1)dV_{\varepsilon_i,t_i},$$

where $L$ satisfies $R(g_{\varepsilon_i}(t_i)) - tr_{g_{\varepsilon_i}(t_i)}\theta_{\varepsilon_i} + L \geq 0$ for every $i$ uniformly.

$$\frac{1}{2} \int_M (u_i^2 + 1)(R(g_{\varepsilon_i}(t_i)) - tr_{g_{\varepsilon_i}(t_i)}\theta_{\varepsilon_i} + L)dV_{\varepsilon_i,t_i}$$

$$\leq \frac{1}{2} \int_{B_{\varepsilon_i}(r_1^i, r_2^i)} e^{2C_i}(R(g_{\varepsilon_i}(t_i)) - tr_{g_{\varepsilon_i}(t_i)}\theta_{\varepsilon_i} + L)dV_{\varepsilon_i,t_i}$$

$$+ \frac{1}{2} \int_M (\beta n - \Delta_{g_{\varepsilon_i}(t_i)}u_{\varepsilon_i}(t_i) + L)dV_{\varepsilon_i,t_i}$$

$$\leq C e^{2C_i} Vol_{g_{\varepsilon_i}(t_i)}(B_{\varepsilon_i,t_i}(k_1^i, k_2^i)) + \frac{1}{2}(\beta n + L)V,$$

$$\frac{1}{2} \int_M (u_i^2 + 1)\frac{4u_i^2|\nabla u_i|^2_{g_{\varepsilon_i}(t_i)}}{(u_i^2 + 1)^2}dV_{\varepsilon_i,t_i} \leq 2 \int_M e^{2C_i}|\phi_i'|^2dV_{\varepsilon_i,t_i}$$

$$\leq C e^{2C_i} Vol_{g_{\varepsilon_i}(t_i)}(B_{\varepsilon_i,t_i}(k_1^i, k_2^i)),$$

$$\frac{\beta}{2} \int_M (u_i^2 + 1)\log(u_i^2 + 1)dV_{\varepsilon_i,t_i}$$

$$= -\frac{\beta}{2} \int_M u_i^2 \log(u_i^2 + 1)dV_{\varepsilon_i,t_i} - \frac{\beta}{2} \int_M \log(u_i^2 + 1)dV_{\varepsilon_i,t_i}$$

$$\leq -\frac{\beta}{2} \int_M u_i^2 \log u_i^2 dV_{\varepsilon_i,t_i}$$

$$= -\beta C_i \int_M u_i^2 dV_{\varepsilon_i,t_i} - \frac{\beta}{2} \int_M e^{2C_i}\phi_i^2 \log \phi_i^2 dV_{\varepsilon_i,t_i}$$

$$\leq -\beta VC_i + C e^{2C_i} Vol_{g_{\varepsilon_i}(t_i)}(B_{\varepsilon_i,t_i}(k_1^i, k_2^i)),$$

where the above constants $C$ are uniform. Combining all these inequalities together, we have

$$W_{\theta_{\varepsilon_k}}(g_{\varepsilon_k}(t_k), - \log \frac{1}{2}(u_k^2 + 1), 1) \leq C - \beta VC_i + C e^{2C_i} Vol_{g_{\varepsilon_i}(t_i)}(B_{\varepsilon_i,t_i}(k_1^i, k_2^i))$$

$$\leq C - \beta VC_i + C 2^{20n} e^{2C_i} Vol_{g_{\varepsilon_i}(t_i)}(B_{\varepsilon_i,t_i}(k_1^i + 2, k_2^i - 2))$$
Then by the uniform Perelman’s estimates for the flow (1.10), we have

\[ \int_M v^{m-1} dV_{ct} \leq C \int_M |\nabla v|_{g_\epsilon(t)}^2 dV_{ct} + C \int_M v^2 dV_{ct}, \]

where \( C \) is a uniform constant.

**Proof:** Since the proof is almost the same as that in [48] by Q.S. Zhang, the only difference is that here we have constants do not depend in addition on \( \varepsilon \), so we give the proof briefly here.
Step 1. By the monotonicity of the functional $\mu_{\theta_\varepsilon}(g_\varepsilon(s), \tau(s))$ (see Lemma 4.3) and taking $\tau(s) = 1 - e^{-\beta t}(1 - \delta^2)e^{\beta s}$, where $\delta \in (0, 1)$. We conclude that

$$\int_M v^2 \log v^2 dV_{\varepsilon t} \leq \int_M \delta^2 \left( R(g_\varepsilon(t)) - tr_{g_\varepsilon(t)} \theta_\varepsilon \right) v^2 + 4|\nabla v|^2_{g_\varepsilon(t)} dV_{\varepsilon t}$$

(5.3)

$$-2n \log \delta + L_0 + \max_M \left( R(g_\varepsilon) - tr_{g_\varepsilon} \theta_\varepsilon \right)^-, \tag{5.3}$$

where $L_0 = n \log(nC_3(M, g_\varepsilon)^2) - n \log 2 - n + \frac{4}{C_3(M, g_\varepsilon)^2} V^{-\frac{1}{2}}$

Step 2. Fixing a time $t_0$ during the generalized Kähler-Ricci flow, we show the upper bound of short time heat kernel for the fundamental solution of equation

$$\Delta_{g_\varepsilon(t_0)} u(x, t) - \frac{1}{4} \left( R(g_\varepsilon(t_0)) - tr_{g_\varepsilon(t_0)} \theta_\varepsilon \right) u(x, t) - \frac{\partial}{\partial t} u(x, t) = 0$$

under the fixed metric $g_\varepsilon(t_0)$.

Let $u$ be a positive solution of equation (5.4). Given $T \in (0, 1]$ and $t \in (0, T]$, we take $p(t) = \frac{T}{T - t}$. Through direct computation and taking $\delta$ satisfying $\delta^2 = \frac{p(t) - 1}{p'(t)} \leq T \leq \frac{1}{T}$, by (5.3) we have

$$\log \frac{\|u(t, T)\|_\infty}{\|u(t, 0)\|_1} \leq -n \log T + L + 2 \max_M \left( R(g_\varepsilon) - tr_{g_\varepsilon} \theta_\varepsilon \right)^-, \tag{5.5}$$

where $L = L_0 + 2n$.

Since $u(x, T) = \int_M P_t(x, y, T) u(y, 0) dV_{\varepsilon t_0}$, where $P_t$ is the heat kernel of equation (5.4), so

$$P_t(x, y, T) \leq \frac{\exp(L + 2 \max_M \left( R(g_\varepsilon) - tr_{g_\varepsilon} \theta_\varepsilon \right)^-)}{T^n} \quad : = \frac{\Lambda}{T^n}. \tag{5.6}$$

Step 3. Let $F_\varepsilon = \max_M \left( R(g_\varepsilon) - tr_{g_\varepsilon} \theta_\varepsilon \right)^-$ and $P_{F_\varepsilon}$ be the heat kernel of operator

$$\Delta_{g_\varepsilon(t_0)} - \frac{1}{4} \left( R(g_\varepsilon(t_0)) - tr_{g_\varepsilon(t_0)} \theta_\varepsilon \right) - F_\varepsilon - 1. \tag{5.7}$$

Since $\max_M \left( R(g_\varepsilon(t_0)) - tr_{g_\varepsilon(t_0)} \theta_\varepsilon \right) \leq F_\varepsilon$, from the short time upper bound for $P_t$, we know that $P_{F_\varepsilon}$ obeys the global upper bound

$$P_{F_\varepsilon}(x, y, t) \leq \tilde{C} t^{-n}, \quad t > 0,$$

where $\tilde{C}$ depends only on $\Lambda$ and $n$. Moreover, by Hölder inequality, for any $f \in L^2(M)$, we have

$$\int_M P_{F_\varepsilon}(x, y, t) f(y) dV_{\varepsilon t_0} \leq \left( \int_M P_{F_\varepsilon}(x, y, t) dV_{\varepsilon t_0} \right)^{\frac{1}{2}} \|f\|_{L^2} \leq \tilde{C}^{\frac{1}{2}} t^{-\frac{n}{2}} \|f\|_{L^2}.$$ Then the Sobolev inequality now follows from Theorem 2.4.2 in [15] and the constants in inequality depend only on $\tilde{C}$, $2n$ and $\frac{1}{2n - 2}$. By the expression of $\Lambda$, (4.14) and Lemma 4.5, we know that the constants independent of $\varepsilon$ and $t$.\hfill\Box

Now, we argue the uniform $C^0$ estimate for metric potential $\varphi_\varepsilon(t)$. We will denote $\phi_\varepsilon(t) = \varphi_\varepsilon(t) + k_\varepsilon \varepsilon^2 + \|s_\varepsilon^2\|_h^2$ and only discuss the $C^0$ estimates for $\phi_\varepsilon(t)$. From flow (1.11), $\phi_\varepsilon(t)$ evolves along the following equation:

$$\begin{aligned}
\frac{\partial \phi_\varepsilon(t)}{\partial t} &= \log \frac{\omega_\varepsilon^n}{\omega_\varepsilon^n(0)} + F_0 + \beta \phi_\varepsilon(t) + \log(\varepsilon^2 + \|s_\varepsilon^2\|_h^2)^{\frac{1}{\beta}} - \frac{1}{2} \left( R(g_\varepsilon) - tr_{g_\varepsilon} \theta_\varepsilon \right) + \beta \phi_\varepsilon(t)
\end{aligned}$$

(5.8)

$$\phi_\varepsilon(t)|_{t=0} = c_0 + k_\varepsilon \varepsilon^2 + \|s_\varepsilon^2\|_h^2,$$
where \( c_{\varepsilon} = \frac{1}{\beta} \int_0^{\infty} e^{-\beta t} \|
abla \phi_{\varepsilon}(t)\|_{L^2}^2 dt - \frac{1}{V} \int_M F_{\varepsilon} dV_{\varepsilon} - \frac{1}{V} \int_M k \beta \chi (\varepsilon^2 + |s|^2) dV_{\varepsilon} \)
and \( F_{\varepsilon} = \log \left( \frac{\omega_0^n}{\omega_{\varepsilon}^n} (\varepsilon^2 + |s|^2 \varepsilon)^{1-\beta} \right) + F_0. \)

**Lemma 5.2** There exists a uniform constant \( C \) such that
\[
\| \phi_{\varepsilon}(t) \|_{C^0} \leq C
\]
for any \( \varepsilon \) and \( t \).

**Proof:** As in [31], we let
\[
\alpha_{\varepsilon}(t) = \frac{1}{V} \int_M \phi_{\varepsilon}(t) dV_{\phi_{\varepsilon}} = \frac{1}{V} \int_M u_{\varepsilon}(t) dV_{\phi_{\varepsilon}} - c_{\varepsilon}(t)
\]
Through computing, we have
\[
\frac{d}{dt} \alpha_{\varepsilon}(t) = \beta \alpha_{\varepsilon}(t) - \| \nabla \phi_{\varepsilon} \|_{L^2}^2
\]
\[
e^{-\beta t} \alpha_{\varepsilon}(t) = \alpha_{\varepsilon}(0) - \int_0^t e^{-\beta s} \| \nabla \phi_{\varepsilon} \|_{L^2}^2 ds
\]
(5.10)
\[
= \frac{1}{V} \int_M u_{\varepsilon} dV_{\varepsilon} - c_{\varepsilon}(0) - \int_0^t e^{-\beta s} \| \nabla \phi_{\varepsilon} \|_{L^2}^2 ds
\]
Putting \( u_{\varepsilon} = F_{\varepsilon} + k \beta \chi \) and \( -c_{\varepsilon}(0) = \beta \phi_{\varepsilon}(0) \) into (5.10), we have
\[
e^{-\beta t} \alpha_{\varepsilon}(t) = \frac{1}{V} \int_M F_{\varepsilon} dV_{\varepsilon} + \frac{1}{V} \int_M k \beta \chi dV_{\varepsilon} - c_{\varepsilon}(0) - \int_0^t e^{-\beta s} \| \nabla \phi_{\varepsilon} \|_{L^2}^2 ds
\]
\[
= \frac{1}{V} \int_M F_{\varepsilon} dV_{\varepsilon} + \frac{1}{V} \int_M k \beta \chi dV_{\varepsilon} + \beta \phi_{\varepsilon}(0) - \int_0^t e^{-\beta s} \| \nabla \phi_{\varepsilon} \|_{L^2}^2 ds
\]
\[
= \frac{1}{V} \int_M F_{\varepsilon} dV_{\varepsilon} + \frac{1}{V} \int_M k \beta \chi dV_{\varepsilon} - \int_0^t e^{-\beta s} \| \nabla \phi_{\varepsilon} \|_{L^2}^2 ds
\]
\[
+ \int_0^{+\infty} e^{-\beta t} \| \nabla \phi_{\varepsilon} \|_{L^2}^2 dt
\]
\[
= \frac{1}{V} \int_M F_{\varepsilon} dV_{\varepsilon} - \frac{1}{V} \int_M k \beta \chi dV_{\varepsilon}
\]
From Theorem 4.1, we conclude that
\[
0 \leq \alpha_{\varepsilon}(t) = \int_t^{+\infty} e^{\beta (t-s)} \| \nabla \phi_{\varepsilon} \|_{L^2}^2 ds \leq C.
\]
By (4.2) and (5.9), we uniformly bound \( \phi_{\varepsilon}(t) \) combining with theorem 4.1. \( \square \)

Now, we recall Aubin’s functionals, Ding functional and the twisted Mabuchi \( \mathcal{K} \)-energy functional.

(5.12)
\[
I_{\omega_0}(\phi) = \frac{n!}{V} \int_M \phi (dV_0 - dV_{\phi})
\]
\[
J_{\omega_0}(\phi) = \frac{n!}{V} \int_0^1 \int_M \phi_t (dV_0 - dV_{\phi_t}) dt
\]
(5.13)
\[
= \frac{1}{n-1} \sum_{i=0}^{n-1} \int_M \partial \phi \wedge \bar{\partial} \phi \wedge \omega_0^i \wedge w_{\phi}^{n-i-1},
\]
where \( \phi_t \) is a path with \( \phi_0 = c, \phi_1 = \phi \).

\[
\begin{align*}
(5.14) \quad F^0_{\omega_0}(\phi) &= J_{\omega_0}(\phi) - \frac{n!}{V} \int_M \phi dV_0, \\
(5.15) \quad F_{\omega_0}(\phi) &= J_{\omega_0}(\phi) - \frac{n!}{V} \int_M \phi dV_0 - \log(\frac{n!}{V}) \int_M e^{-\omega_0} dV_0, \\
(5.16) \quad M_{\omega_0, \theta}(\phi) &= -\beta(I_{\omega_0}(\phi) - J_{\omega_0}(\phi)) - \frac{n!}{V} \int_M u_{\omega_0}(dV_0 - d\phi).
\end{align*}
\]

where \( -\text{Ric}(\omega_0) + \beta \omega_0 + \theta = \sqrt{-1} \partial \bar{\partial} u_{\omega_0} \) and \( \frac{1}{n} \int_M e^{-\omega_0} dV_{\omega_0} = 1 \). Through computing, we conclude \( \frac{1}{n} J_{\omega_0} \leq \frac{1}{n+1} I_{\omega_0} \leq J_{\omega_0} \). The time derivatives of \( I_{\omega_0}, J_{\omega_0} \) and \( M_{\omega_0, \theta} \) along any path \( \phi_t \) can be written as follows:

\[
\begin{align*}
\frac{\partial}{\partial t} I_{\omega_0}(\phi_t) &= \frac{n!}{V} \int_M \phi(dV_0 - d\phi_t) - \frac{n!}{V} \int_M \phi_t \Delta \phi_t dV_0, \\
\frac{\partial}{\partial t} J_{\omega_0}(\phi_t) &= \frac{n!}{V} \int_M \phi_t (dV_0 - d\phi_t), \\
\frac{\partial}{\partial t} M_{\omega_0, \theta}(\phi_t) &= -\frac{n!}{V} \int_M \phi_t (R(\omega_{\phi_t}) - \beta n - tr_{\omega_{\phi_t}} \theta) dV_0.
\end{align*}
\]

Now we establish some relations between the above functionals along the flow.

**Lemma 5.3** There exists uniform constant \( C \), such that \( \phi_t(t) \) which evolves along the flow \([5.8]\) satisfies:

\[
\begin{align*}
(i) \quad M_{\omega_0, \theta}(\phi_t(t)) - \beta F^0_{\omega_0}(\phi_t(t)) - \frac{n!}{V} \int_M \dot{\phi}_t(t) dV_0 &= C, \\
(ii) \quad |\beta F_{\omega_0}(\phi_t(t)) - M_{\omega_0, \theta}(\phi_t(t))| + |\beta F^0_{\omega_0}(\phi_t(t)) - M_{\omega_0, \theta}(\phi_t(t))| &\leq C, \\
(iii) \quad \frac{(n-1)!}{V} \int_M (-\phi_t(t)) dV_0 - C &\leq J_{\omega_0}(\phi_t(t)) \leq \frac{n!}{V} \int_M \phi_t(t) dV_0 + C, \\
(iv) \quad \frac{n!}{V} \int_M \dot{\phi}_t(t) dV_0 &\leq \frac{n \cdot n!}{V} \int_M (-\phi_t(t)) dV_0 - (n + 1) M_{\omega_0, \theta}(\phi_t(t)) + C,
\end{align*}
\]

where \( C \) in (i) can be bounded by a uniform constant \( C \).

**Proof:** We follow the arguments in [34]. Now we only need to prove the following two facts:

1. the constant \( C \) in (i) can be bounded by a uniform constant \( C \);
2. \( M_{\omega_0, \theta}(\phi_t(0)) \) is uniformly bound.

Note that \( \dot{\phi}_t \) is uniformly bounded. (i), (ii), (iii) and (iv) can be easily deduced from the above two facts. Since

\[
(5.17) \quad \frac{d}{dt} M_{\omega_0, \theta}(\phi_t(t)) - \beta F^0_{\omega_0}(\phi_t(t)) - \frac{n!}{V} \int_M \dot{\phi}_t(t) dV_0 = 0.
\]
So we obtain

\[ \mathcal{M}_{\omega_0}, \theta_\ast (\phi_\varepsilon(t)) - \beta \beta F_{\omega_0}^0 (\phi_\varepsilon(t)) - \frac{n!}{V} \int_M \phi_\varepsilon(t) dV_\varepsilon \]

\[ = \mathcal{M}_{\omega_0}, \theta_\ast (\phi_\varepsilon(0)) - \beta F_{\omega_0}^0 (\phi_\varepsilon(0)) - \frac{n!}{V} \int_M \phi_\varepsilon(0) dV_\varepsilon \]

\[ = \frac{n!}{V} \int_M \log \frac{\omega_\varepsilon^n \left( |s|^2_h + \varepsilon^2 \right)}{e^{-F_0 \omega_0^n}} dV_\varepsilon + \frac{\beta n!}{V} \int_M \phi_\varepsilon(0) dV_\varepsilon \]

\[ - \frac{n!}{V} \int_M F_0 + \log(\varepsilon^2 + \varepsilon^2) dV_0 - \frac{n!}{V} \int_M \phi_\varepsilon(0) dV_\varepsilon. \]

where the last equality can be bounded by a uniform constant. Then we prove fact (1). By the definition of \( \mathcal{M}_{\omega_0}, \theta_\ast \), we have

\[ \mathcal{M}_{\omega_0}, \theta_\ast (\phi_\varepsilon(0)) = \frac{n!}{V} \int_M \log \frac{\omega_\varepsilon^n \left( |s|^2_h + \varepsilon^2 \right)}{e^{-F_0 \omega_0^n}} dV_\varepsilon - \beta I_{\omega_0}(\phi_\varepsilon(0)) + \beta J_{\omega_0}(\phi_\varepsilon(0)) \]

\[ - \frac{n!}{V} \int_M F_0 + \log(\varepsilon^2 + \varepsilon^2) dV_0. \]

Since \( I_{\omega_0}(\phi_\varepsilon(0)) \) is uniformly bounded and \( \frac{1}{\pi} J_{\omega_0} \leq \frac{1}{\pi + 1} I_{\omega_0} \leq J_{\omega_0} \), so we prove the second fact.

**Lemma 5.4** Let \( u_\varepsilon(t) \) satisfy (4.1), then for every \( f \in C^\infty(M) \), we have inequality

\[ \frac{1}{V} \int_M f^2 e^{-u_\varepsilon(t)} dV_\varepsilon \leq \frac{1}{\beta V} \int_M (\nabla f)^2 e^{-u_\varepsilon(t)} dV_\varepsilon + \left( \frac{1}{V} \int_M f e^{-u_\varepsilon(t)} dV_\varepsilon \right)^2. \]

**Proof:** It suffices to show the lowest strictly positive eigenvalue \( \mu \) of operator \( L \) satisfying \( \mu \geq \beta \), where

\[ Lf = -g_\varepsilon^j(t) \nabla_i \nabla_j f + g_\varepsilon^j(t) \nabla_i u_\varepsilon(t) \nabla_j f. \]

Note \( L \) is self-adjoint with respect to the inner product

\[ (f, g) = \frac{1}{V} \int_M f g e^{-u_\varepsilon(t)} dV_\varepsilon, \]

and \( \text{Ker} L = \mathbb{C} \).

Suppose \( f \) is the eigenfunction of eigenvalue \( \mu, f \neq \text{Constant} \).

\[ -g_\varepsilon^j(t) \nabla_i \nabla_j f + g_\varepsilon^j(t) \nabla_i u_\varepsilon(t) \nabla_j f = \mu f. \]

Applying \( \nabla_k \) on both sides, combining Ricci identity we have

\[ \mu \nabla_k f = -g_\varepsilon^j(t) \nabla_i \nabla_k \nabla_j f - g_\varepsilon^j(t) R_{\gamma_k(t) \gamma_j \gamma_i \gamma_l} \nabla_k \nabla_j f + g_\varepsilon^j(t) \nabla_i u_\varepsilon(t) \nabla_j \nabla_k \nabla_l f + g_\varepsilon^j(t) \nabla_j f \nabla_k \nabla_i u_\varepsilon(t). \]

Integration after we multiplying \( g_\varepsilon^{ik}(t) \nabla_k f e^{-u_\varepsilon} dV_\varepsilon \) on both sides, using the facts that \( -\text{Ric} (\omega_\varepsilon(t)) + \beta \omega_\varepsilon(t) + \theta_\varepsilon = \sqrt{-1} \theta \partial \partial u_\varepsilon(t) \) and \( \theta_\varepsilon \) is semi-positive, we get

\[ \beta \int_M |\nabla f|^2 e^{-u_\varepsilon(t)} dV_\varepsilon \leq \int_M (\nabla \nabla f)^2 e^{-u_\varepsilon(t)} dV_\varepsilon + \beta \int_M |\nabla f|^2 e^{-u_\varepsilon(t)} dV_\varepsilon \]

\[ + \frac{1}{2} \int_M \theta_\varepsilon (\text{grad } f, \mathcal{J}(\text{grad } f)) e^{-u_\varepsilon(t)} dV_\varepsilon = \mu \int_M |\nabla f|^2 e^{-u_\varepsilon(t)} dV_\varepsilon. \]

Hence \( \mu \geq \beta \).
Since we have the uniform Sobolev inequality \((5.2)\) and the Poincaré inequality \((5.18)\) along the generalized Kähler-Ricci flows \((5.8)\), we can follow the arguments in \([34]\) (see Lemma 10) to obtain the following lemma. The proof is completely similar, so we omit it.

**Lemma 5.5** We have the following estimate along the generalized Kähler-Ricci flow \((5.8)\)

\[
\text{osc}(\phi_\varepsilon(t)) \leq \frac{A}{V} \int_M \phi_\varepsilon(t) dV_0 + B,
\]

where the constants \(A\) and \(B\) independent of \(\varepsilon\) and \(t\).

We define the space of smooth Kähler potentials

\[
\mathcal{H}(\omega_0) = \{ \phi \in C^\infty(M) | \omega_0 + \sqrt{-1} \partial \bar{\partial} \phi > 0 \}.
\]

**Theorem 5.6** Let \(\phi_\varepsilon(t)\) be a solution of the flow \((5.8)\), and \(\theta_\varepsilon = (1 - \beta)(\omega_0 + \sqrt{-1} \partial \bar{\partial} \log(\varepsilon^2 + |s|^2_h))\) is a smooth closed semi-positive \((1,1)\)-form, where \(s\) is the defining section of divisor \(D\) and \(h\) is a smooth Hermitian metric on the line bundle associated to \(D\). If the twisted Mabuchi K-energy functional \(\mathcal{M}_{\omega_0, \theta_\varepsilon}\) is uniformly proper on \(\mathcal{H}(\omega_0)\), i.e. there exists a uniform function \(f\) such that

\[
\mathcal{M}_{\omega_0, \theta_\varepsilon} (\phi_\varepsilon(t)) \geq f(J_{\omega_0}(\phi))
\]

for any \(\varepsilon\) and \(\phi \in \mathcal{H}(\omega_0)\), where \(f(t) : \mathbb{R}^+ \to \mathbb{R}\) is some monotone increasing function satisfying \(\lim_{t \to +\infty} f(t) = +\infty\). Then there exists a uniform constant \(C\) such that

\[
\|\phi_\varepsilon(t)\|_{C^0} \leq C.
\]

**Proof:** Since \(\mathcal{M}_{\omega_0, \theta_\varepsilon} (\phi_\varepsilon(t))\) decreases along the flow \((5.8)\) and \(\mathcal{M}_{\omega_0, \theta_\varepsilon} (\phi_\varepsilon(0))\) is uniformly bounded proved in Lemma 5.3. It follows that \(J_{\omega_0}(\phi_\varepsilon(t))\) is uniformly bounded from above. Thus by Lemma 5.3 \((iii)\), we have

\[
\int_M (-\phi_\varepsilon(t)) dV_\varepsilon \leq C.
\]

Since \(J_{\omega_0} \geq 0\), applying \((5.23)\) we know that the twisted Mabuchi K-energy \(\mathcal{M}_{\omega_0, \theta_\varepsilon} (\phi_\varepsilon(t))\) is uniformly bounded from below. Then by Lemma 5.3 \((iv)\)

\[
\int_M \phi_\varepsilon(t) dV_0 \leq C,
\]

where \(C\) is a uniform constant. By this inequality and Green’s formula with respect to the metric \(g_0\), we get uniformly upper bound of \(\sup_M \phi_\varepsilon(t)\).

By the normalization

\[
1 = \frac{1}{V} \int_M dV_{\phi_\varepsilon(t)} = \frac{1}{V} \int_M e^{\dot{\phi}_\varepsilon(t) - \beta \phi_\varepsilon(t) - F_\varepsilon} dV_\varepsilon
\]

and \(\|\dot{\phi}_\varepsilon(t)\|_{C^0}\) is uniformly controlled along the flow \((5.8)\), we have

\[
0 < C_1 \leq \int_M e^{-\beta \phi_\varepsilon(t)} dV_\varepsilon \leq C_2
\]
where $C_1$ and $C_2$ are uniform constant, and this inequality easily implies a uniformly lower bound for $\sup_M \phi_\varepsilon(t)$. Combined with (5.21) and (5.26), we obtain a uniform bound for $\|\phi_\varepsilon(t)\|_{C^0}$. We also conclude

$$\|\phi_\varepsilon(t)\|_{C^0} \leq C$$

for a uniform constant. \[\square\]

### 6. The convergence of the conical Kähler-Ricci flow

In this section, we will argue the convergence of conical Kähler-Ricci flow. Let’s recall the properness of the Log Mabuchi $K$-energy functional introduced first by C. Li and S. Sun in [24].

For any $\phi \in H(\omega_0)$,

$$M_{\omega_0, (1-\beta)D}(\phi) = -\frac{n!}{V} \int_M H_{\omega_0, (1-\beta)D}(dV_0 - dV_\phi)$$

$$+ \frac{n!}{V} \int_M \log \frac{\omega_0}{\omega_\phi} dV_\phi - \beta (I_{\omega_0}(\phi) - J_{\omega_0}(\phi)).$$

where $H_{\omega_0, (1-\beta)D}$ satisfies $-\text{Ric}(\omega_0) + \beta \omega_0 + (1-\beta) \{ D \} = \sqrt{-1} \partial \bar{\partial} H_{\omega_0, (1-\beta)D}$ and $\frac{1}{V} \int_M e^{-H_{\omega_0, (1-\beta)D}} dV_0 = 1$. It is easy to see that up to a constant $H_{\omega_0, (1-\beta)D} = F_0 + (1 - \beta) \log |s|^2_\phi$.

The Log Mabuchi $K$-energy functional $M_{\omega_0, (1-\beta)D} : H(\omega_0) \to \mathbb{R}$ is called proper if there is an inequality of the type

$$M_{\omega_0, (1-\beta)D}(\phi) \geq f(J_{\omega_0}(\phi))$$

for any $\phi \in H(\omega_0)$, where $f(t) : \mathbb{R}^+ \to \mathbb{R}$ is some monotone increasing function satisfying $\lim_{t \to +\infty} f(t) = +\infty$. C. Li and S. Sun have proved the following lemma:

**Lemma 6.1** (see Corollary 1.4 in [24]) If there is a conical Kähler-Einstein metric for $\beta \in (0, 1)$, then the Log Mabuchi $K$-energy functional $M_{\omega_0, (1-\beta)D}$ is proper.

J. Song and X.W. Wang also proved the similar result in [38] by defining paired Mabuchi $K$-energy functional and using the openness proved by S.K. Donaldson in [17].

**Lemma 6.2** If the Log Mabuchi $K$-energy functional $M_{\omega_0, (1-\beta)D}$ is proper on $H(\omega_0)$, then the twisted Mabuchi $K$-energy functional $M_{\omega_0, \theta_\varepsilon}$ is uniformly proper on $H(\omega_0)$, i.e. there exists a uniform function $f$ such that

$$M_{\omega_0, \theta_\varepsilon}(\phi) \geq f(J_{\omega_0}(\phi))$$

for any $\varepsilon$ and $\phi \in H(\omega_0)$, where $f(t) : \mathbb{R}^+ \to \mathbb{R}$ is some monotone increasing function satisfying $\lim_{t \to +\infty} f(t) = +\infty$.

**Proof:** By assumption, we have

$$M_{\omega_0, (1-\beta)D}(\phi) \geq C f(J_{\omega_0}(\phi)) - C.$$
From the definition of Log Mabuchi $K$-energy functional and twisted Mabuchi $K$-energy functional, we have

$$
M_{\omega_0, \theta_\epsilon} - M_{\omega_0, (1-\beta)(D)} - C \geq 0,
$$

where $C$ independent of $\epsilon$. Hence we obtain

$$
M_{\omega_0, \theta_\epsilon} - M_{\omega_0, (1-\beta)(D)} - C \geq C \tilde{f}(J_{\omega_0}(\phi)) - C.
$$

By setting $f = C \tilde{f} - C$, we have proved (6.3). □

Next, we prove the convergence of the conical Kähler-Ricci flow.

**Theorem 6.3** Assume there exists a conical Kähler-Einstein metric $\omega_{\beta,D}$, then the flow (1.13) converges $C^\infty_{\text{loc}}$ to the conical Kähler-Einstein metric $\omega_{\beta,D}$ globally in the sense of currents.

**Proof:** First, by computing, we have

$$
dt M_{\omega_0, \theta_\epsilon} = -\frac{n!}{V} \int_M |\partial \phi_\epsilon|^2_{\omega_\epsilon} dV_e.
$$

Let $Y_\epsilon(t) = \frac{n!}{V} \int_M |\partial \phi_\epsilon|^2_{\omega_\epsilon} dV_e$. From Lemma 6.1, we know Log Mabuchi $K$-energy functional is proper, so the twisted Mabuchi $K$-energy functional $M_{\omega_0, \theta_\epsilon}$ is uniformly proper by Lemma 6.2, hence it is bounded form below uniformly, so for any $T$, we have

$$
\int_0^T Y_\epsilon(t) dt = M_{\omega_0, \theta_\epsilon(0)} - M_{\omega_0, \theta_\epsilon(\phi_\epsilon(t))) \leq C.
$$

where constant $C$ is uniform. Define

$$
Y(t) = \frac{n!}{V} \int_M |\partial \phi|^2_{\omega_\epsilon} dV_e.
$$

From Theorem 4.1, we know $|\partial \phi|^2_{\omega_\epsilon} \leq C$ for a uniform constant $C$, so

$$
\int_0^T Y_\epsilon(t) dt \xrightarrow{\epsilon \rightarrow 0} \int_0^T Y(t) dt,
$$

where $\{\epsilon_\nu\}$ is obtained in Theorem 3.1. Hence we obtain

$$
\int_0^T Y(t) dt \leq C.
$$

When we let $T \rightarrow +\infty$, we get

$$
\int_0^{+\infty} Y(t) dt < \infty.
$$

Hence there exists a time sequence $\{t_m\}$, where $t_m \in [m, m+1)$ such that $Y(t_m) \rightarrow 0$ as $m \rightarrow +\infty$. 

Next, $Y_\varepsilon(t)$ satisfies the following differential identity,

\[ \dot{Y}_\varepsilon(t) = \beta(n+1)Y_\varepsilon(t) - \int_M |\nabla \dot{\phi}_\varepsilon|^2_{\omega_\varepsilon} (R(g_\varepsilon(t)) - tr_{g_\varepsilon(t)} \theta_\varepsilon) dV_\varepsilon t - \int_M |\nabla \nabla \phi_\varepsilon|^2_{\omega_\varepsilon} dV_\varepsilon t - \frac{1}{2} \int_M \theta(\nabla \dot{\phi}_\varepsilon, \mathcal{J} \nabla \phi_\varepsilon) dV_\varepsilon t \]

By Theorem 4.1, we have $|R(g_\varepsilon(t)) - tr_{g_\varepsilon(t)} \theta_\varepsilon| \leq C$ for a uniform constant. Hence we have

\[ (6.9) \quad \dot{Y}_\varepsilon(t) \leq CY_\varepsilon(t). \]

So $Y_\varepsilon(t) \leq e^{C(t-s)}Y_\varepsilon(s)$ for any $t > s$. Let $\varepsilon_i \to 0$ again, we have

\[ (6.10) \quad Y(t) \leq e^{C(t-s)}Y(s). \]

In particular,

\[ Y(t) \leq e^{2C}Y(t_m) \]

for all $t \in [m+1, m+2)$, and hence $Y(t) \to 0$ as $t \to +\infty$.

Since the twisted Mabuchi $\mathcal{K}$-energy functional $\mathcal{M}_{\omega_0, \theta_s}$ is uniformly proper, we have $\|\phi_\varepsilon\|_{C^0}$ is uniformly bounded. From Lemma 5.2, we have $\|\phi_\varepsilon\|_{C^0}$ is also uniformly bounded. By Theorem 3.1, we have

\[ (6.11) \quad \|\phi\|_{C^0} \leq C, \quad \|\hat{\phi}\|_{C^0} \leq C \]

for some uniform constant $C$ on $M \setminus D \times [0, +\infty)$. Then for any $K \subset K \setminus D$, by the local estimates in Lemma 2.1 and Proposition 2.2, there exists a time sequence $\{t_i\}$ such that $\phi(t_i)$ converge in $C^\infty$ topology to a smooth function $\phi_\infty$ on $K$.

\[
\int_K |\partial (\log \frac{\omega_0^n}{\omega_\phi(t_i)}) + F_0 + \beta(k|s_h^2 + \phi(t_i)) + \log |s_h|^{1-\beta})|^2_{\omega_\phi} dV_0 \\
\leq C \int_K |\partial (\log \frac{\omega_0^n}{\omega_\phi(t_i)}) + F_0 + \beta(k|s_h^2 + \phi(t_i)) + \log |s_h|^{1-\beta})|^2_{\omega_\phi} dV_0 \\
\leq C \int_M |\partial (\log \frac{\omega_0^n}{\omega_\phi(t_i)}) + F_0 + \beta(k|s_h^2 + \phi(t_i)) + \log |s_h|^{1-\beta})|^2_{\omega_\phi} dV_0 \\
= C \int_M |\partial \dot{\phi}(t_i)|^2_{\omega_\phi} dV_0 \to 0
\]

On the other hand, we have

\[
\int_K |\partial (\log \frac{\omega_0^n}{\omega_\phi(t_i)}) + F_0 + \beta(k|s_h^2 + \phi(t_i)) + \log |s_h|^{1-\beta})|^2_{\omega_\phi} dV_0 \\
\to \int_K |\partial (\log \frac{\omega_0^n}{\omega_\phi(t_i)}) + F_0 + \beta(k|s_h^2 + \phi_\infty) + \log |s_h|^{1-\beta})|^2_{\omega_\phi} dV_0
\]

By the uniqueness of the limit,

\[
\int_K |\partial (\log \frac{\omega_0^n}{\omega_\phi}) + F_0 + \beta(k|s_h^2 + \phi_\infty) + \log |s_h|^{1-\beta})|^2_{\omega_\phi} dV_0 = 0
\]

Hence

\[ (6.12) \quad Ric(\omega_\phi) = \beta \omega_\phi, \quad \text{on } K. \]
At the same time, there exists a time subsequence denoted also by \( \{t_i\} \) such that 
\( \varphi(t_i) \) converge in \( C^\infty_{\text{loc}} \) topology to a function \( \varphi_{\infty} \), where \( \varphi_{\infty} \) is smooth on \( M \setminus D \). We also have \( \dot{\varphi}(t_i) \) converge to some constant \( C \) in \( C^\infty_{\text{loc}} \) topology outside \( D \). For any \( (n-1, n-1) \)-form \( \eta \), since 
\[ \log \frac{\omega^n_{\varphi(t_i)}|s_{\eta}^{2(1-\beta)}}{\omega_0^n} \] and \( \|\varphi\|_{C^0} \) are uniformly bounded, in the sense of currents, we have
\[
 [\sqrt{-1}\partial\bar{\partial}\frac{\partial\varphi(t_i)}{\partial t}, \eta] = [\sqrt{-1}\partial\bar{\partial}(\log \frac{\omega^n_{\varphi(t_i)}}{\omega_0^n} + F_0 + \beta(k|s|^2_{\partial\bar{\partial}} + \varphi(t_i)) + \log|s_{\eta}^{2(1-\beta)}|, \eta] 
\]
\[= \int_M (\log \frac{\omega^n_{\varphi(t_i)}}{\omega_0^n} + F_0 + \beta(k|s|^2_{\partial\bar{\partial}} + \varphi(t_i)) + \log|s_{\eta}^{2(1-\beta)}|) - \sqrt{-1}\partial\bar{\partial}\eta \]
\[
\xrightarrow{t_i \to \infty} \int_M (\log \frac{\omega^n_{\varphi_{\infty}}}{\omega_0^n} + F_0 + \beta(k|s|^2_{\partial\bar{\partial}} + \varphi_{\infty}) + \log|s_{\eta}^{2(1-\beta)}|) - \sqrt{-1}\partial\bar{\partial}\eta \]
\[= \int [\sqrt{-1}\partial\bar{\partial}(\log \frac{\omega^n_{\varphi_{\infty}}}{\omega_0^n} + F_0 + \beta(k|s|^2_{\partial\bar{\partial}} + \varphi_{\infty}) + \log|s_{\eta}^{2(1-\beta)}|, \eta] \]
\[= [\pm \text{Ric}(\omega_{\varphi_{\infty}}) + \beta \omega_{\varphi_{\infty}} + (1 - \beta)[D], \eta]. \]

Let \( K \subset M \setminus D \) be a compact subset, and \( \int_{M \setminus K} \sqrt{-1}\partial\bar{\partial}\eta = \delta \) with \( \delta \to 0 \) when \( K \to M \setminus D \).

\[
|\int_M (\frac{\partial\varphi(t_i)}{\partial t} - C)\sqrt{-1}\partial\bar{\partial}\eta| \leq |\int_K (\frac{\partial\varphi(t_i)}{\partial t} - C)\sqrt{-1}\partial\bar{\partial}\eta| + \int_M (\frac{\partial\varphi(t_i)}{\partial t} - C)\sqrt{-1}\partial\bar{\partial}\eta| \xrightarrow{t_i \to \infty} \tilde{C}\delta. \]

When let \( K \to M \setminus D \), we have
\[
\int_M \frac{\partial\varphi(t_i)}{\partial t}\sqrt{-1}\partial\bar{\partial}\eta \xrightarrow{t_i \to \infty} 0. \]

Hence, we conclude
\[
[\sqrt{-1}\partial\bar{\partial}\frac{\partial\varphi(t_i)}{\partial t}, \eta] = [\frac{\partial\varphi(t_i)}{\partial t}, \sqrt{-1}\partial\bar{\partial}\eta] = \int_M \frac{\partial\varphi(t_i)}{\partial t}\sqrt{-1}\partial\bar{\partial}\eta \xrightarrow{t_i \to \infty} 0 \]

Hence, we have
\[
(6.13) \quad \text{Ric}(\omega_{\varphi_{\infty}}) = \beta \omega_{\varphi_{\infty}} + (1 - \beta)[D] \]
in the current sense. Since \( C^{-1}\omega \leq \omega_{\varphi_{\infty}} \leq C\omega \), we also have
\[
(6.14) \quad C^{-1}\omega \leq \omega_{\varphi_{\infty}} \leq C\omega. \]
By estimates \((6.11)\), from the proof in Proposition 3.2, we know that \(\|\varphi\|_{C^\alpha}\) is uniformly bounded for some \(\alpha \in (0, 1)\), so the limit \(\varphi_\infty\) is also Hölder continuous on \(M\). Since we have the properness of the Log Mabuchi K-energy functional, R. Berman (\([2]\)) has proved the uniqueness of the conical Kähler-Einstein metric with Hölder continuous potential, hence \(\omega_\infty = \omega_{\beta,D}\).

Now we use the uniqueness to prove that the flow \((1.13)\) must converge in \(C^\infty_{loc}\) topology outside \(D\) to the conical Kähler-Einstein metric \(\omega_{\varphi_\infty}\) in current sense as \(t \to +\infty\). If not, there exists \(K \subset \subset M\), an integer \(k > 0\), \(\epsilon > 0\), and a time subsequence \(\{t'_i\}\) such that

\[
\|\sqrt{-1} \partial \overline{\partial} (\varphi(t'_i) - \varphi_\infty)\|_{C^k(K)} \geq \epsilon.
\]

But since \(\varphi(t'_i)\) is \(C^\infty_{loc}\) bounded, there exists a subsequence which we also denote it by \(\{t'_i\}\), such \(\varphi(t'_i)\) converge in \(C^\infty_{loc}\) topology to a function \(\tilde{\varphi}_\infty\) and

\[
\|\sqrt{-1} \partial \overline{\partial} (\tilde{\varphi}_\infty - \varphi_\infty)\|_{C^k(K)} \geq \epsilon.
\]

By the same arguments above we know that \(\omega_{\tilde{\varphi}_\infty}\) is also a conical Kähler-Einstein metric and \(\tilde{\varphi}_\infty\) is Hölder continuous, but \(\omega_{\tilde{\varphi}_\infty} \neq \omega_{\varphi_\infty}\), which is impossible by R. Berman’s uniqueness results. Hence we get the convergence of the conical Kähler-Ricci flow.

□

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