Quantum renormalization group approach to geometric phases in spin chains

R. Jafari

Research Department, Nanosolar System Company (NSS), Zanjan 45158-65911, Iran
Department of Physics, Institute for Advanced Studies in Basic Sciences (IASBS),
Zanjan 45137-66731, Iran

Abstract

A relation between geometric phases and criticality of spin chains are studied using the quantum renormalization-group approach. I have shown how the geometric phase evolve as the size of the system becomes large, i.e., the finite size scaling is obtained. The renormalization scheme demonstrates how the first derivative of the geometric phase with respect to the field strength diverges at the critical point and maximum value of the first derivative, and its position, scales with the exponent of the system size.

Keywords: Quantum Renormalization Group, Geometric Phase

Email address: jafari@iasbs.ac.ir, rohollah.jafari@gmail.com
Tel: (+98) 241-4152118, Fax: (+98) 241 4244949

1. Introduction

Quantum phase transition (QPT) has been one of the most interesting topics in the area of strongly correlated systems [1]. At zero temperature, the properties of the ground state may change drastically showing a non-analytic behavior of a physical quantity by reaching the quantum critical point. This
can be done by tuning a parameter in the Hamiltonian, for instance, the magnetic field or the amount of disorder. Traditionally such a problem is addressed by resorting to notions such as order-parameter and symmetry breaking i.e., the Landau-Ginzburg paradigm [2]. In the last few years a big effort has been devoted to the analysis of QPTs from the Quantum Information perspective [3, 4, 5, 6, 7, 8, 9], the main tool being the study of different entanglement measures [10]. In the view of some difficulties [11], attention has shifted to include other, potentially related, means of characterizing QPTs [12]. One such approach centers around the notion of geometric phase (GP). GP has been offered as a typical mechanism for a quantum system to keep the memory of its evolution in Hilbert space. Such phases were introduced in quantum mechanics by Berry in 1984 [13]. Since then, geometric phases became objects of theoretical and experimental researches [14] uncovering that they are related to a number of important physical phenomena [15] such as Aharonov-Bohm [16] and quantum Hall effects [17]. In recent years, this interest is increased due to their applicability in quantum-information processing [18]. In other words, GP has become extendable to product states of composite systems since the uncorrelated subsystems pick up independent geometric phase factors. However, GP could be induced by quantum entanglement, if the full state is pure. On the other hand, classical correlations and quantum entanglement can coexist in mixed quantum states, which means the forms of the mixed state of the geometric phases [19], applied to the path of the relative states, may contain portions from both types of correlations. Nevertheless, their connection to the quantum phase transitions has been manifested recently in Ref. [20, 21, 22], where it is shown
that the geometric phase could be used to investigate the critical properties of the spin chains [20]. On the other hand, the critical exponents can be evaluated from the scaling behavior of the geometric phases [21]. Therefore, the geometric phase could be considered as a topological test for manifestation of quantum phase transitions [22]. These general relations originate from topological property of the geometric phase. It describes the curvature of the Hilbert space and is directly related to the degeneracy property in the quantum systems. The degeneracy in the many-body systems plays crucial role in our understanding of the quantum phase transition. Thus the geometric phase can be considered as another powerful tool for detecting the QPT.

Our main purpose in this work is to hire quantum renormalization group (QRG) [23] to study the evolution of the geometric phase of spin models. To have a concrete discussion, the one dimensional $S = \frac{1}{2}$ Ising model in transverse field (ITF) is considered by implementing the quantum renormalization group approach [4, 5, 6, 8, 7, 9, 24, 25, 26]. To the best of my knowledge, the GP properties study has only been done for exactly solvable models and this is the first report which addresses how to get GP properties of the models which are not exactly solvable using QRG. I also show that QRG-based investigation of the GP of the models is more convenient and also accurate than that of entanglement (concurrence).
2. Theoretical Model

Consider the ITF model on a periodic chain of $N$ sites with Hamiltonian

$$H = -J \sum_{i=1}^{N} (\sigma^x_i \sigma^x_{i+1} + \lambda \sigma^z_i),$$

where $J > 0$ and $\lambda$ are the exchange coupling and the transverse field, respectively. From the exact solution [26, 27] it can be seen that a second order phase transition occurs for $\lambda_c = 1$ where the behavior of the order parameter or magnetization is given by $\langle \sigma^x \rangle = (1 - \lambda)^{1/8}$ for $\lambda < 1$ and $\langle \sigma^x \rangle = 0$ for $\lambda > 1$.

3. Quantum Renormalization Group

The main idea of the RG method is the mode elimination or thinning of the degrees of freedom followed by an iteration which reduces the number of variables step by step till reaching a fixed point. In Kadanoff’s approach, the first step of the QRG method consists of assembling a set of lattice points into disconnected blocks of $n_B$ sites. In this fashion, the total number of blocks in the whole chain would be $N' = N/n_B$. This partitioning of the lattice into blocks induces a decomposition of the Hamiltonian into two parts: intra-block ($H_B$) and inter-block ($H_{BB}$) Hamiltonians. The block Hamiltonian $H_B$ is a sum of commuting Hamiltonians ($h_B$) acting on individual blocks. The diagonalization of $h_B$ for small $n_B$ is achieved analytically and then intra-block Hamiltonian and inter-block Hamiltonian is projected into the low energy subspace of $H_B$. Afterwards, the original Hamiltonian is mapped into an effective Hamiltonian ($H_{eff}$) which acts on the renormalized subspace [24, 25, 26].
In this paper, to implement QRG, the Hamiltonian is divided into two-site blocks

\[ H^B = \sum_{l=1}^{N/2} h^B_l, \quad h^B_l = -J(\sigma_{1,l}^x \sigma_{2,l}^x + \lambda \sigma_{1,l}^z), \]

and the remaining part of the Hamiltonian is included in the inter-block part

\[ H^{BB} = -J \sum_{l=1}^{N/2} (\sigma_{2,l}^x \sigma_{1,l+1}^x + \lambda \sigma_{2,l}^z), \]

where \( \sigma_{j,l}^\alpha \) refers to the \( \alpha \)-component of the Pauli matrix at site \( j \) of the block labeled by \( I \). The Hamiltonian of each block \( (h^B_l) \) is diagonalized exactly and the projection operator

\[ P_0 = |\psi_0\rangle \langle \psi_0| + |\psi_1\rangle \langle \psi_1|, \quad (2) \]

is constructed from the two lowest eigenstates in which \( |\psi_0\rangle \) is the ground state and \( |\psi_1\rangle \) is the first excited state. In this respect the effective Hamiltonian

\[ H^{eff} = P_0[H^B + H^{BB}]P_0 \]

is matched to the original one (Eq. (1)) replacing the couplings with the following renormalized coupling constants.

\[ J' = J \frac{2q}{1 + q^2}, \quad q = \lambda + \sqrt{\lambda^2 + 1}, \quad \lambda' = \lambda^2. \quad (3) \]
4. Geometric Phase and Renormalization Group Application

To investigate the geometric phase in systems, a new family of Hamiltonians are introduced that can be described by applying a rotation of $\phi$ around the $z$ direction to each spin [20], i.e.,

$$H_\phi = g_\phi^\dagger H g_\phi; \quad g_\phi = \prod_{j=1}^{N} \exp(-i\phi\sigma_j^z/2).$$

(4)

The critical behavior is independent of $\phi$ as the spectrum of the system is $\phi$ independent [21]. The geometric phase of the ground state, accumulated by varying the angle $\phi$ from 0 to $\pi$, is described by

$$\beta_y = -i \int_0^{\pi} \langle \psi_\phi | \frac{\partial}{\partial \phi} | \psi_\phi \rangle d\phi,$$

(5)

here $|\psi_\phi\rangle$ is the ground state of $H_\phi$ [20].
Figure 2: (Color online) The first derivative of geometric phase for different system size. For the limit of large system (high RG step), the non-analytic behavior of the first derivative of GP is obtained through the diverging.

The eigenvalues of the Hamiltonian $H$ will not affected by this unitary transformation. So the eigenvectors of new Hamiltonians $H_\phi$ can be obtained by acting the rotation operator on the eigenvectors of the former Hamiltonian $(H)$. In other words, $|\psi_\phi\rangle = g_\phi |\psi\rangle$ where $|\psi\rangle$ and $|\psi_\phi\rangle$ are the eigenvectors of $H$ and $H_\phi$, respectively. However, the projection operators of new Hamiltonian $H_\phi((P_0(\phi))$ and the unrotated Hamiltonian (Eq. (2)) are related by

$$P_0(\phi) = g_\phi^\dagger P_0 g_\phi.$$

On the other hand, the ground state of the renormalized chain $|\psi'\rangle$ will be related to that of the original one by the transformation $|\psi\rangle = P_0 |\psi'\rangle$. It is straightforward to show that the geometric phase in the renormalized chain is described by
Figure 3: (Color online) Scaling of the position ($\lambda_{Max}$) of $\frac{d\beta}{d\lambda}$ for different length chains. $\lambda_{Max}$ goes to $\lambda_c$ as the size of the system increase as $\lambda_c = \lambda_{Max} + N^{-1/0.957}$.

$$\beta_{g}^{n+1} = \beta_{g}^{n} - \frac{\pi}{2} \frac{\gamma^{n}}{2}$$

(6)

where $\beta_{g}^{n}$ is geometric phase at the $n$th step of RG and $\gamma^0$ is defined by $\frac{q^2-1}{q^2+1}$. The expression for $\gamma^n$ is similar to $\gamma^0$ where the coupling constants should be replaced by the renormalized ones at the corresponding RG iteration ($n$). In this approach, geometric phase at each iteration of RG is connected to its value after a RG iteration by Eq. (6). This will be continued till reaching a controllable fixed point where the value of the geometric phase could be obtained \cite{6,25}.
5. Numerical Results: Scaling properties of Geometric Phase

In this section the numerical results of the model would be discussed. The evolution of $\beta_g$ under RG steps versus $\lambda$ is presented in Fig. 1. In the $n$th RG step the expression given in Eq. (6) is evaluated at the renormalized coupling given by the $n$ iteration of $\lambda$ given in Eq. (3). The zero RG step means a bare two-site model, while in the first RG step the effective two-site model represents a four-site chain. Generally, in the $n$th step of RG, a chain of $2^{n+1}$ sites is represented effectively by the two sites with renormalized couplings. All curves in Fig. 1 have a kink at the critical point, $\lambda_c = 1$, for large systems. At the critical point, correlation length is infinite and fluctuations occur on all length scales which means that the system is scale-invariant. The non-analytic behavior is a feature of second-order quantum phase transition. It is also accompanied by a scaling behavior since the correlation length diverges and there is no characteristic length in the system at the critical point.

Zhu has verified that the GP of ground state in the XY model in the transverse field obeys scaling behavior in the vicinity of a quantum phase transition [21]. In particular he has shown that the geometric phase is non-analytical and its derivative with respect to the magnetic field diverges at the critical point. As it is previously stated, a large system, i.e. $N = 2^{n+1}$, can be effectively describe by two sites with the renormalized coupling in the $n$th RG step. The first derivative of GP is analyzed as a function of magnetic field at different RG steps which manifest the size of the system. In Fig. 2 the derivative of GP with respect to the coupling constant $d\beta_g/d\lambda$ is presented which, shows a singular behavior at the critical point as the size
of system becomes large. This singular behavior is the result of the kink in GP at $\lambda = \lambda_c$ (Fig. (1)).

It is found that the position of the maximum of $d\beta_g/d\lambda$ ($\lambda_{Max}$) tends towards the critical point like

$$\lambda_c = \lambda_{Max} + N^{-1/\theta},$$

where $\theta \simeq 0.957$ (Fig. (3)). Moreover, the scaling behavior of $\frac{d\beta_g}{d\lambda}|_{\lambda_{Max}}$ versus $\ln N$ is derived. This quantity is shown in Fig. (4), which behaves linearly and the scaling behavior is obtained as

$$\frac{d\beta_g}{d\lambda}|_{\lambda_{Max}} \approx \kappa \ln N$$
with $\kappa = 0.393$.

The exponent $\theta$ is directly related to the correlation length exponent ($\nu$) close to the critical point. The correlation length exponent gives the behavior of the correlation length in the vicinity of $\lambda_c$, i.e., $\xi \sim (\lambda - \lambda_c)^{-\nu}$. Under the RG transformation of Eq. (3), the correlation length scales in the $n$th RG step as $\xi^{(n)} = (\lambda_n - \lambda_c)^{-\nu} = \xi/n_B^n$, which immediately leads to an expression for $|\frac{d\beta}{d\lambda}|_{\lambda_c}$ in terms of $\nu$ and $n_B$. Dividing the last equation into $\xi \sim (\lambda - \lambda_c)^{-\nu}$ gives rise to $|\frac{d\beta}{d\lambda}|_{\lambda_c} \sim N^{1/\nu}$, which implies that $\theta = 1/\nu$, since $|\frac{d\beta}{d\lambda}|_{\lambda_{Max}} \sim N^{1/\nu}$ at the critical point. We should note that the scaling of in the the position of $\lambda_{Max}$ (Fig. 3), comes from the divergence of the correlation length near the critical point. In the large system size limit, when approaching to the critical point the correlation length almost covers the size...

Figure 5: (Color online) Finite-size scaling for different lattice sizes through the RG treatment. The curves which correspond to different system sizes clearly collapse on a single curve.
of the system $\xi \sim N$ which results in the following scaling form

$$\lambda_c = \lambda_{Max} + N^{-1/\nu}.$$  

To obtain the finite-size scaling behavior of $\frac{d\lambda}{dN}|_{\lambda_{Max}}$, we look for a scaling function when all graphs tend to collapse on each other under RG evolution which results in a large system. This is also a manifestation of the existences of the finite size scaling for the GP. Fig. (5) shows the plot of $1 - exp\left(\frac{d\lambda}{dN} - \frac{d\lambda}{dN}|_{\lambda_{Max}}\right)$ versus $N(\lambda - \lambda_{Max})$. The lower curves, which are for large system sizes, clearly show that all plots fall on each other.

The similar scaling behaviors as well as their relation to correlation length exponent have been reported in our previous works [4, 8], where the static properties of the ground state entanglement and low energy state dynamics of entanglement of ITF model by RG method were studied. These facts strongly imply the important relation between quantum entanglement and geometric phase, and provides a possible understanding of entanglement from the topological structure of the systems. This point can be understood by noting that both of the mentioned methods are connected to the correlation functions, and also are connected directly to each other by the inequality [28].

6. Summary

To summarize, the idea of renormalization group (RG) to study the geometric phase of Ising model in transverse field is implemented. In order to explore the critical behavior of the ITF model the evolution of geometric
phase through the renormalization of the lattice were examined. In this respect I have shown that the RG procedure can be implemented to obtain the GP of a system and its finite size scaling in terms of the effective Hamiltonian which is described by the renormalized coupling constants. The phase transition becomes significant which shows a diverging behavior in the first derivative of the geometric phase. This divergence of GP are accompanied by a scaling behavior near the critical point where the size of the system becomes large. The scaling behavior characterizes how the critical point of the model is touched as the system size is increased. It is also shown that the non-analytic behavior of GP is originated from the correlation length exponent in the vicinity of the critical point. This shows that the behavior of the GP near the critical point is directly connected to the quantum critical properties of the model. We get the properties of GP for a large system dealing with a small block which make it possible to get analytic results. However, the numerical results of QRG show that the application of QRG to manifest the GP properties, is quantitatively more accurate than its application on quantum information resource [4, 5, 6, 8, 7].

7. acknowledgments

The author would like to thank S. N. S. Reihani, A. Akbari and A. Langari for reading the manuscript, fruitful discussions and comments.

References

[1] M. Vojta, Rep. Prog. Phys. 66, (2003) 2069 and references therein.
[2] N. Goldenfeld, *Lectures on phase transitions and the renormalization group*, Westview Press, Boulder, 1992.

[3] A. Osterloh, Luigi Amico, G. Falci and Rosario Fazio, Nature **416**, 608 (2002).

[4] M. Kargarian, R. Jafari and A. Langari, Phys. Rev. A **76**, 060304(R)(2007).

[5] M. Kargarian, R. Jafari and A. Langari, Phys. Rev. A **77**, 032346 (2008).

[6] R. Jafari, M. Kargarian, A. Langari, and M. Siahatgar, Phys. Rev. B **78**, 214414 (2008).

[7] M. Kargarian, R. Jafari and A. Langari, Phys. Rev. A **79**, 042319 (2009).

[8] R. Jafari, Phys. Rev. A **82**, 052317 (2010).

[9] A. Langari, A. T. Rezakhani, New J. Phys. **14**, 053014 (2012); N. Amiri, A. Langari, physica status solidi (b). **250**, 417 (2013).

[10] T.J. Osborne, M.A. Nielsen, Phys. Rev. A. **66**, 032110 (2002); G. Vidal, J.I. Latorre, E. Rico, and A. Kitaev, Phys. Rev. Lett. **90**, 227902 (2003); Y. Chen, P. Zanardi, Z. D. Wang, F. C. Zhang, New J. Phys. **8**, **97** (2006); L.-A. Wu, M.S. Sarandy, D. A. Lidar, Phys. Rev. Lett. **93**, 250404 (2004)

[11] M.E. Reuter, M.J. Hartmann, M.B. Plenio, Proc. Roy. Soc. Lond. A **463**, 1271 (2007).
[12] P. Zanardi and N. Paunkovic, Phys. Rev. E 74, 031123 (2006); A. T. Rezakhani, P. Zanardi, Phys. Rev. A 73, 012107 (2006); 73, 052117 (2006).

[13] M. V. Berry, Proc. R. Soc. London 392, 45 (1984).

[14] A. Shapere and F. Wilczek (Eds.), Geometric Phases in Physics, World Scientific, Singapore, 1989.

[15] A. Bohm, A. Mostafazadeh, H. Koizumi, Q. Niu, and J. Zwanziger, The Geometric Phase in Quantum Systems, Springer, Berlin, 2003.

[16] Y. Aharonov and D. Bohm, Phys. Rev. 115, 485 (1959).

[17] K. v. Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. 45, 494 (1980).

[18] P. Zanardi and M. Rasetti, Phys. Lett. A 264, 94 (1999); J. A. Jones, V. Vedral, A. Ekert, and G. Castagnoli, Nature (London) 403, 869 (2000).

[19] A. Uhlmann, Rep. Math. Phys. 24, 229 (1986); E. Sjoqvist, A.K. Pati, A. Ekert, J.S. Anandan, M. Ericsson, D.K.L. Oi, V. Vedral, Phys. Rev. Lett. 85 2845 (2000).

[20] A. Carollo, and J. K. Pachos, Phys. Rev. Lett. 95, 157203(2005).

[21] S. L. Zhu, Phys. Rev. Lett. 96, 077206(2006).

[22] A. Hamma, e-print: quant-ph/0602091.
[23] P. Pfeuty, R. Jullian, K. L. Penson, in *Real-Space Renormalization*, edited by T. W. Burkhardt and J. M. J. van Leeuwen (Springer, Berlin, 1982), Chap. 5.

[24] M. A. Martin-Delgado and G. Sierra, Int. J. Mod. Phys. A 11, 3145 (1996).

[25] R. Jafari and A. Langari, Phys. Rev. B 76, 014412 (2007); Physica A 364, 213 (2006); A. Langari, Phys. Rev. B 69, 100402(R) (2004).

[26] M. A. Martin-Delgado and G. Sierra, Phys. Rev. Lett. 76, 1146 (1996).

[27] P. Pfeuty, ANNALS of Physics, 57, 79 (1970).

[28] H.T. Cui, Y.F. Zhang, Eur. Phys. J. D, 51, 393 (2009).