Invariant forms on irreducible modules of simple algebraic groups

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Abstract

Let $G$ be a simple linear algebraic group over an algebraically closed field $K$ of characteristic $p \geq 0$ and let $V$ be an irreducible rational $G$-module with highest weight $\lambda$. When $V$ is self-dual, a basic question to ask is whether $V$ has a non-degenerate $G$-invariant alternating bilinear form or a non-degenerate $G$-invariant quadratic form.

If $p \neq 2$, the answer is well known and easily described in terms of $\lambda$. In the case where $p = 2$, we know that if $V$ is self-dual, it always has a non-degenerate $G$-invariant alternating bilinear form. However, determining when $V$ has a non-degenerate $G$-invariant quadratic form is a classical problem that still remains open. We solve the problem in the case where $G$ is of classical type and $\lambda$ is a fundamental highest weight $\omega_i$, and in the case where $G$ is of type $A_l$ and $\lambda = \omega_r + \omega_s$ for $1 \leq r < s \leq l$. We also give a solution in some specific cases when $G$ is of exceptional type.

As an application of our results, we refine Seitz’s 1987 description of maximal subgroups of simple algebraic groups of classical type. One consequence of this is the following result. If $X < Y < \text{SL}(V)$ are simple algebraic groups and $V \downarrow X$ is irreducible, then one of the following holds: (1) $V \downarrow Y$ is not self-dual; (2) both or neither of the modules $V \downarrow Y$ and $V \downarrow X$ have a non-degenerate invariant quadratic form; (3) $p = 2$, $X = \text{SO}(V)$, and $Y = \text{Sp}(V)$.

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1 Introduction

Let $V$ be a finite-dimensional vector space over an algebraically closed field $K$ of characteristic $p \geq 0$.

A fundamental problem in the study of simple linear algebraic groups over $K$ is the determination of maximal closed connected subgroups of simple groups of classical type (SL($V$), Sp($V$) and SO($V$)). Seitz [Sei87] has shown that up to a known list of examples, these are given by the images of $p$-restricted, tensor-indecomposable irreducible rational representations $\varphi : G \rightarrow \text{GL}(V)$ of simple algebraic groups $G$ over $K$.

Then given such an irreducible representation $\varphi$, one should still determine which of the groups SL($V$), Sp($V$) and SO($V$) contain $\varphi(G)$. In most cases the answer is known.

- If $V$ is not self-dual, then $\varphi(G)$ is only contained in SL($V$). Furthermore, we know when $V$ is self-dual (see Section 2).

- If $p \neq 2$ and $V$ is self-dual, then $\varphi(G)$ is contained in Sp($V$) or SO($V$), but not both [Ste68, Lemma 78, Lemma 79]. Furthermore, we know for which irreducible representations the image is contained in Sp($V$) and for which the image is contained in SO($V$) (see Section 2).

- If $p = 2$ and $V$ is self-dual, then $\varphi(G)$ is contained in Sp($V$) [Fon74, Lemma 1].

Currently what is still missing is a method for determining in characteristic two when exactly $\varphi(G)$ is contained in SO($V$). This problem is the main subject of this paper, and we can state it equivalently as follows.

**Problem 1.1.** Assume that $p = 2$ and let $L_G(\lambda)$ be an irreducible $G$-module with highest weight $\lambda$. When does $L_G(\lambda)$ have a non-degenerate $G$-invariant quadratic form?

This is a nontrivial open problem. There is some literature on the subject [Wil77, SW91, GW95, GW97, GN16], but currently only partial results are known. The main result of this paper is a solution to Problem 1.1 in the following cases:

- $G$ is of classical type ($A_l$, $B_l$, $C_l$ or $D_l$) and $\lambda$ is a fundamental dominant weight $\omega_r$ for some $1 \leq r \leq l$ (Theorem 4.2).

- $G$ is of type $A_l$ and $\lambda = \omega_r + \omega_s$ for $1 \leq r < s \leq l$ (Theorem 5.1).
In the case where $G$ is of exceptional type, we will give some partial results in Section 6. For $G$ of type $G_2$ and $F_4$, we are able to give a complete solution (Proposition 6.1, Proposition 6.3). For types $E_6$, $E_7$, and $E_8$, we give the answer for some specific $\lambda$ (Table 6.1). In the final section of this paper, we will give various applications of our results and describe some open problems motivated by Problem 1.1.

One particular application, given in subsection 7.3, is a refinement of Seitz’s [Sei87] description of maximal subgroups of simple algebraic groups of classical type. In [Sei87], Seitz gives a full list of all non-maximal irreducible subgroups of $\text{SL}(V)$, but the question of which classical groups contain the image of an irreducible representation is not considered. For example, it is possible that we have a proper inclusion $X < Y$ of irreducible subgroups of $\text{SL}(V)$ such that $X$ is a maximal subgroup of $\text{SO}(V)$. In subsection 7.3, we go through the list given by Seitz and describe when exactly such inclusions occur. In particular, our results have the consequence (Theorem 7.9) that if $X < Y < \text{SL}(V)$ are simple algebraic groups and $V \downarrow X$ is irreducible, then one of the following holds:

(i) The module $V \downarrow Y$ is not self-dual;

(ii) Both $V \downarrow X$ and $V \downarrow Y$ have an invariant quadratic form;

(iii) Neither of $V \downarrow X$ or $V \downarrow Y$ has an invariant quadratic form;

(iv) $p = 2$, $X = \text{SO}(V)$ and $Y = \text{Sp}(V)$.

The general approach for the proofs of our main results is as follows. A basic method used throughout is Theorem 9.5. from [GN16] (recorded here in Proposition 2.2), which allows one to determine whether $L_G(\lambda)$ is orthogonal (when $p = 2$) by computing within the Weyl module $V_G(\lambda)$. For $G$ of classical type and $V$ irreducible with fundamental highest weight, we will first prove our result in the case where $G$ is of type $C_l$ (Proposition 3.1). From this the result for other classical types is a fairly straightforward consequence (Theorem 3.2).

In the case where $G$ is of type $C_l$ and $\lambda = \omega_r$, and in the case where $G$ is of type $A_l$ and $\lambda = \omega_r + \omega_s$, the proofs of our results are heavily based on various results from the literature on the representation theory of $G$. We will use results about the submodule structure of the Weyl module $V_G(\lambda)$ found in [PS83], [Ada84], and [Ada86]. We will also need the first cohomology groups of $L_G(\lambda)$ which were computed in [KS99] and [KS01, Corollary 3.6]. One more key ingredient in our proof will be the results of Baranov and Suprunenko in [BS00] and [BS05], which give the structure of the restrictions of $L_G(\lambda)$ to certain subgroups defined in terms of the natural module of $G$. 

4
Notation and terminology

We fix the following notation and terminology. Throughout the whole text, let $K$ be an algebraically closed field of characteristic $p \geq 0$. All groups that we consider are linear algebraic groups over $K$, and by a subgroup we always mean a closed subgroup. All modules and representations will be finite-dimensional and rational.

Unless otherwise mentioned, $G$ denotes a simply connected simple algebraic group over $K$ with $l = \text{rank } G$, and $V$ will be a finite-dimensional vector space over $K$. Throughout we will view $G$ as its group of rational points over $K$, and most of the time $G$ will studied either as a Chevalley group constructed with the usual Chevalley construction (see e.g. [Ste68]), or as a classical group with its natural module (i.e. $G = \text{SL}(V)$, $G = \text{Sp}(V)$ or $G = \text{SO}(V)$). We will occasionally denote $G$ by its type, so notation such as $G = C_l$ means that $G$ is a simply connected simple algebraic group of type $C_l$.

We fix the following notation, as in [Jan03].

- $T$: a maximal torus of $G$, with character group $X(T)$.
- $X(T)^+$: the set of dominant weights for $G$, with respect to some system of positive roots.
- $\text{ch} V$: the character of a $G$-module $V$. Here $\text{ch} V$ is an element of $\mathbb{Z}[X(T)]$.
- $\omega_1, \omega_2, \ldots, \omega_l$: the fundamental dominant weights in $X(T)^+$. We use the standard Bourbaki labeling of the simple roots, as given in [Hum72, 11.4, pg. 58].
- $L(\lambda), L_G(\lambda)$: the irreducible $G$-module with highest weight $\lambda \in X(T)^+$.
- $V(\lambda), V_G(\lambda)$: the Weyl module for $G$ with highest weight $\lambda \in X(T)^+$.
- $\text{rad} V(\lambda)$: unique maximal submodule of $V(\lambda)$.

For a dominant weight $\lambda \in X(T)^+$, we can write $\lambda = \sum_{i=1}^{l} m_i \omega_i$ where $m_i \in \mathbb{Z}_{\geq 0}$. We say that $\lambda$ is $p$-restricted if $p = 0$, or if $p > 0$ and $0 \leq m_i \leq p-1$ for all $1 \leq i \leq l$. The irreducible representation $L_G(\lambda)$ is said to be $p$-restricted if $\lambda$ is $p$-restricted.

A bilinear form $b$ is non-degenerate, if its radical $\text{rad} b = \{v \in V : b(v, w) = 0 \text{ for all } w \in V\}$ is zero. For a quadratic form $Q : V \to K$ on a vector space $V$, its polarization is the bilinear form $b_Q$ defined by
$b_Q(v, w) = Q(v + w) - Q(v) - Q(w)$ for all $v, w \in V$. We say that $Q$ is \textit{non-degenerate}, if its radical $\text{rad } Q = \{ v \in \text{rad } b_Q : Q(v) = 0 \}$ is zero.

For a $KG$-module $V$, a bilinear form $(-, -)$ is $G$-\textit{invariant} if $(gv, gw) = (v, w)$ for all $g \in G$ and $v, w \in V$. A quadratic form $Q : V \to K$ is $G$-\textit{invariant} if $Q(gv) = Q(v)$ for all $g \in G$ and $v \in V$. We say that $V$ is \textit{symplectic} if it has a non-degenerate $G$-invariant alternating bilinear form, and we say that $V$ is \textit{orthogonal} if it has a non-degenerate $G$-invariant quadratic form.

Note that if $V$ has a $G$-invariant bilinear form, then for $\lambda, \mu \in X(T)$ the weight spaces $V_\lambda$ and $V_\mu$ are orthogonal if $\lambda \neq -\mu$. Thus to compute the form on $V$ it is enough to work in the zero weight space of $V$ and $V_\lambda \oplus V_{-\lambda}$ for nonzero $\lambda \in X(T)$. For a $G$-invariant quadratic form $Q$ on $V$, we have $Q(v) = 0$ for any weight vector $v \in V$ with non-zero weight.

If a representation $\phi : G' \to G$ of algebraic groups, we can twist representations of $G$ with $\phi$. That is, if $\rho : G \to \text{GL}(V)$ is a representation of $G$, then $\rho \phi$ is a representation of $G'$. We denote the corresponding $G'$-module by $V^\phi$.

When $p > 0$, we denote by $F : G \to G$ the Frobenius endomorphism induced by the field automorphism $x \mapsto x^p$ of $K$, see for example [Ste68, Lemma 76]. When $G$ is simply connected and $\lambda \in X(T)^+$, we have $L_G(p\lambda) \cong L_G(\lambda)^F$.

If a representation $V$ of $G$ has composition series $V = V_1 \supset V_2 \supset \cdots \supset V_i \supset V_{i+1} = 0$ with composition factors $W_i \cong V_i/V_{i+1}$, we will occasionally denote this by $V = W_1/W_2/\cdots/W_\ell$.

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\section{Invariant forms on irreducible $G$-modules}
Let $L(\lambda)$ be an irreducible representation of a simple algebraic group $G$ with highest weight $\lambda = \sum_{i=1}^l m_i \omega_i$. Write $d(\lambda) = \sum_{\alpha > 0} \langle \lambda, \alpha^\vee \rangle$, where the sum runs over the positive roots $\alpha$, where $\alpha^\vee$ is the coroot corresponding to $\alpha$, and $\langle \cdot, \cdot \rangle$ is the usual dual pairing between $X(T)$ and the cocharacter group.

We know that $L(\lambda)$ is self-dual if and only if $w_0(\lambda) = -\lambda$, where $w_0$ is the longest element in the Weyl group [Ste68, Lemma 78]. Furthermore, if $L(\lambda)$ is self-dual and $p \neq 2$, then $L(\lambda)$ is orthogonal if $d(\lambda)$ is even and
When is \( \lambda = -w_0(\lambda) \)?

\[
d(\lambda) \mod 2 \quad \text{when} \quad \lambda = -w_0(\lambda)
\]

| Root system | When is \( \lambda = -w_0(\lambda) \)? | \( d(\lambda) \mod 2 \) when \( \lambda = -w_0(\lambda) \) |
|-------------|------------------------------------------|--------------------------------------------------|
| \( A_l \) \((l \geq 1)\) | iff \( m_i = m_{l-i+1} \) for all \( i \) | 0, when \( l \) is even \[
\frac{l+1}{2} \cdot m_{\frac{l+1}{2}} \]
when \( l \) is odd |
| \( B_l \) \((l \geq 2)\) | always | 0, when \( l \equiv 0, 3 \mod 4 \) \( m_l \), when \( l \equiv 1, 2 \mod 4 \). |
| \( C_l \) \((l \geq 2)\) | always | \( m_1 + m_3 + m_5 + \cdots \) |
| \( D_l \) \((l \geq 4)\) | \( l \) even: always \( l \) odd: iff \( m_l = m_{l-1} \) | 0, when \( l \not\equiv 2 \mod 4 \) \( m_l + m_{l-1} \), when \( l \equiv 2 \mod 4 \). |
| \( G_2 \) | always | 0 |
| \( F_4 \) | always | 0 |
| \( E_6 \) | iff \( m_1 = m_6 \) and \( m_3 = m_5 \) | 0 |
| \( E_7 \) | always | \( m_2 + m_5 + m_7 \) |
| \( E_8 \) | always | 0 |

Table 2.1: Values of \( d(\lambda) \) modulo 2 for a weight \( \lambda = \sum_{i=1}^{l} m_i \omega_i \)

symplectic if \( d(\lambda) \) is odd [Ste68 Lemma 79]. Hence in characteristic \( p \neq 2 \) deciding whether an irreducible module is symplectic or orthogonal is a straightforward computation with roots and weights. In Table 2.1 we give the value of \( d(\lambda) \mod 2 \) (when \( \lambda = -w_0(\lambda) \)) for each simple type, in terms of the coefficients \( m_i \).

In characteristic 2, it turns out that each nontrivial, irreducible self-dual module is symplectic, as shown by the following lemma found in [Fon74]. We include a proof for convenience.

**Lemma 2.1.** Assume that \( \text{char} \ K = 2 \). Let \( V \) be a nontrivial, irreducible self-dual representation of a group \( G \). Then \( V \) is symplectic for \( G \).

**Proof.** [Fon74] Since \( V \) is self-dual, there exists an isomorphism \( \varphi : V \rightarrow V^* \) of \( G \)-modules, which induces a non-degenerate \( G \)-invariant bilinear form \( (\cdot, \cdot) \) defined by \( (v, w) = \varphi(v)(w) \). Since \( \varphi^t : V \rightarrow V^* \) defined by \( \varphi^t(v)(w) = \varphi(w)(v) \) is also an isomorphism of \( G \)-modules, by Schur’s lemma there exists a scalar \( c \) such that \( (v, w) = c(w, v) \) for all \( v, w \in V \). Then \( (v, w) = c^2(v, w) \), so \( c^2 = 1 \) because \( (\cdot, \cdot) \) is nonzero. Because we are in characteristic two, it follows that \( c = 1 \), so \( (\cdot, \cdot) \) is a symmetric form. Now \( \{ v \in V : (v, v) = 0 \} \)
is a submodule of $G$. Because $V$ is nontrivial and irreducible, this submodule must be all of $V$ and so $(-,-)$ is alternating.

Lemma 2.1 above shows that the image of any irreducible self-dual representation lies in $\text{Sp}(V)$. The following general result reduces determining whether $L(\lambda)$ is orthogonal (in characteristic two) to a computation within the Weyl module $V(\lambda)$.

**Proposition 2.2.** Assume that $\text{char } K = 2$. Let $\lambda \in X(T)^+$ be nonzero, $\lambda = -w_0(\lambda)$ and suppose that $\lambda \neq \omega_1$ if $G$ has type $C_l$. Then

(i) The Weyl module $V(\lambda)$ has a nonzero $G$-invariant quadratic form $Q$, unique up to scalar.

(ii) The unique maximal submodule of $V(\lambda)$ is equal to $\text{rad } b_Q$.

(iii) The irreducible module $L(\lambda)$ has a nonzero, $G$-invariant quadratic form if and only if $\text{rad } Q = \text{rad } b_Q$. If this is not the case, then $\text{rad } Q$ is a submodule of $\text{rad } b_Q$ with codimension 1, and $H^1(G, L(\lambda)) \neq 0$.

(iv) If $V(\lambda)$ has no trivial composition factor, then $L(\lambda)$ is orthogonal.

**Proof.** See Theorem 9.5. and Proposition 10.1. in [GN16] for (i), (ii) and (iii). The claim in (iii) about $H^1(G, L(\lambda))$ can also be deduced from [Wil77, Satz 2.5]. The claim (iv) is a consequence of (iii), since $H^1(G, L(\lambda)) \cong \text{Ext}^1_G(K, L(\lambda)) \cong \text{Hom}_G(\text{rad } V(\lambda), K)$ by [Jan03, II.2.14].

In the case where $G$ is of type $C_l$ and $\lambda = \omega_1$, we have the following result which is well known. We include a proof for completeness.

**Proposition 2.3.** Assume that $\text{char } K = 2$ and that $G$ is of type $C_l$. Then $V = V(\omega_1) = L(\omega_1)$ has no nonzero $G$-invariant quadratic form.

**Proof.** ([GN16, Example 8.4]) The claim follows from a more general result that any $G$-invariant rational map $f : V \to K$ is constant. Indeed, for such $f$ we have $f(gv) = f(v)$ for all $g \in G, v \in V$. Because $G$ acts transitively on nonzero vectors in $V$, it follows that $f(w) = f(v)$ for all $w \in V - \{0\}$. Thus $f(w) = f(v)$ for all $w \in V$ since $f$ is rational.

**Lemma 2.4.** Let $V$ and $W$ be $G$-modules. If $V$ and $W$ are both symplectic for $G$, then $V \otimes W$ is orthogonal for $G$.

**Proof.** See [SW91, Proposition 3.4], [GN16, Proposition 9.2], or [KL90, 4.4, pg. 126-127].
Remark 2.5. Assume that \( \text{char} \, K = 2 \). Then lemmas 2.1 and 2.4 show that if \( V \) is a non-orthogonal irreducible \( G \)-module, then \( V \) must be tensor indecomposable. By Steinberg’s tensor product theorem, this implies that \( V \) is a Frobenius twist of \( L_G(\lambda) \) for some \( 2 \)-restricted weight \( \lambda \in X(T)^+ \). Therefore to determine which irreducible representations of \( G \) are orthogonal, it suffices to consider \( V = L_G(\lambda) \) with \( \lambda \in X(T)^+ \) a \( 2 \)-restricted dominant weight.

3 Fundamental representations for type \( C_l \)

Throughout this section, assume that \( G \) is simply connected of type \( C_l \), \( l \geq 2 \). In this section we determine when in characteristic 2 a fundamental irreducible representation \( L(\omega_r) \), \( 1 \leq r \leq l \), of \( G \) has a nonzero \( G \)-invariant quadratic form. The answer is given by the following proposition, which we will prove in what follows.

**Proposition 3.1.** Assume \( \text{char} \, K = 2 \). Let \( 1 \leq r \leq l \). Then \( L(\omega_r) \) is not orthogonal if and only if \( r = 1 \), or \( r = 2^{i+1} \) for some \( i \geq 0 \) and \( l + 1 \equiv 2^{i+1} + 2^i + t \mod 2^{i+2} \), where \( 0 \leq t < 2^i \).

The following examples are immediate consequences of Proposition 3.1.

**Example 3.2.** If \( \text{char} \, K = 2 \), then \( L(\omega_2) \) is orthogonal if and only if \( l \not\equiv 2 \mod 4 \).

**Example 3.3.** If \( \text{char} \, K = 2 \), then \( L(\omega_4) \) is orthogonal if and only if \( l \not\equiv 5, 6 \mod 8 \).

**Example 3.4.** If \( \text{char} \, K = 2 \), then \( L(\omega_l) \) is orthogonal if and only if \( l \geq 3 \) (this was also proven in [Gow97, Corollary 4.3]) and \( L(\omega_{l-1}) \) is orthogonal if and only if \( l = 3, l = 4 \) or \( l \geq 6 \).

A rough outline for the proof of Proposition 3.1 is as follows. Various results from the literature about the representation theory of \( G \) will reduce the claim to specific \( r \) which must be considered. We will then study \( V(\omega_r) \) by using a standard realization of it in the exterior algebra of the natural module \( V \) of \( G \). Here we can explicitly describe a nonzero \( G \)-invariant quadratic form \( Q \) on \( V(\omega_r) \). We will then find a vector \( \gamma \in \text{rad} \, V(\omega_r) \) such that \( L(\omega_r) \) is orthogonal if and only if \( Q(\gamma) = 0 \). The proof is finished by computing \( Q(\gamma) \).
3.1 Representation theory

The composition factors of $V(\omega_r)$ were determined in odd characteristic by Premet and Suprunenko in [PS83, Theorem 2]. Independently, the composition factors and the submodule structure of $V(\omega_r)$ were found in arbitrary characteristic by Adamovich in [Ada84, Ada86]. Using the results of Adamovich, it was shown in [BS00, Corollary 2.9] that the result of Premet and Suprunenko also holds in characteristic two.

To state the result about composition factors of $V(\omega_r)$, we need to make a few definitions first. Let $a, b \in \mathbb{Z}_{\geq 0}$ and write $a = \sum_{i \geq 0} a_i p^i$ and $b = \sum_{i \geq 0} b_i p^i$ for the expansions of $a$ and $b$ in base $p$. We say that $a$ contains $b$ to base $p$ if for all $i \geq 0$ we have $b_i = a_i$ or $b_i = 0$. For $r \geq 1$, we define $J_p(r)$ to be the set of integers $0 \leq j \leq r$ such that $j \equiv r \mod 2$ and $l + 1 - j$ contains $\frac{r-1}{2}$ to base $p$. The main result of [PS83], also valid in characteristic 2, can be then described as follows. Here we set $\omega_0 = 0$, so that $L(\omega_0)$ is the trivial irreducible module.

**Theorem 3.5.** Let $1 \leq r \leq l$. Then in the Weyl module $V(\omega_r)$, each composition factor has multiplicity 1, and the set of composition factors is $\{L(\omega_j) : j \in J_p(r)\}$.

In view of Proposition 2.2 (iii), it will also be useful to know when the first cohomology group $H^1(G, L(\omega_r))$ is nonzero. This has been determined by Kleshchev and Sheth in [KS99, KS01, Corollary 3.6].

**Theorem 3.6.** Let $1 \leq r \leq l$ and write $l + 1 - r = \sum_{i \geq 0} a_i p^i$ in base $p$. Then $H^1(G, L(\omega_r)) \neq 0$ if and only if $r = 2(p - a_i) p^i$ for some $i$ such that $a_i > 0$, and either $a_{i+1} < p - 1$ or $r < 2p^{i+1}$.

In characteristic 2, the result becomes the following.

**Corollary 3.7.** Assume that $\text{char } K = 2$. Let $1 \leq r \leq l$. Then $H^1(G, L(\omega_r)) \neq 0$ if and only if $r = 2^{i+1}$ for some $i \geq 0$, and $l + 1 \equiv 2^i + t \mod 2^{i+1}$ for some $0 \leq t < 2^i$.

Throughout this section we will consider subgroups $C_l < C_l' = G$, which are embedded into $G$ as follows. Consider $G = \text{Sp}(V)$ and let $(-,-)$ be the non-degenerate $G$-invariant alternating form $(-, -)$ on $V$. Fix a symplectic basis $e_1, \ldots, e_l, e_{-1}, \ldots, e_{-l}$ of $V$, where $(e_i,e_{-i}) = 1 = -(e_{-i},e_i)$ and $(e_i,e_j) = 0$ for $i \neq -j$. Then for $2 \leq l' < l$, the embedding $C_{l'} < C_l$ is

\footnotetext{Note that in [PS83] there is a typo, the definition on pg. 1313, line 9 should say “for every $i = 0, 1, \ldots, n$ . . .”}
$\text{Sp}(V') < \text{Sp}(V)$, where $V' \subseteq V$ has basis $e_{\pm 1}, \ldots, e_{\pm r}$ and $\text{Sp}(V')$ fixes the basis vectors $e_{\pm (r+1)}, \ldots, e_{\pm t}$.

The module structure of the restrictions $L(\omega_r) \downarrow C_{l-1}$ have been determined by Baranov and Suprunenko in [BS00, Theorem 1.1 (i)]. We will only need to know the composition factors which occur in such a restriction, and in this case the result is the following. Below we define $L_{C_{l-1}}(\omega_r) = 0$ for $r < 0$.

**Theorem 3.8.** Let $1 \leq r \leq l$ and assume that $l \geq 3$. Set $d = \nu_2(l + 1 - r)$, and $\varepsilon = 0$ if $l + 1 - r \equiv -p^d \mod p^{d+1}$ and $\varepsilon = 1$ otherwise. Then the character of $L_{C_l}(\omega_r) \downarrow C_{l-1}$ is given by

\[
\text{ch} L_{C_{l-1}}(\omega_r) + 2 \text{ch} L_{C_{l-1}}(\omega_{r-1}) + \left( \sum_{k=0}^{d-1} 2 \text{ch} L_{C_{l-1}}(\omega_{r-2^k}) \right) + \varepsilon \text{ch} L_{C_{l-1}}(\omega_{r-2^d})
\]

where the sum in the brackets is zero if $d = 0$.

Above $\nu_2$ denotes the $p$-adic valuation on $\mathbb{Z}$, so for $a \in \mathbb{Z}^+$ we have $\nu_2(a) = d$, where $d \geq 0$ is maximal such that $p^d$ divides $a$. Note that if $d = \nu_2(l+1-r)$, then $l+1-r \equiv 2^d \equiv -2^d \mod 2^{d+1}$. Therefore if char $K = 2$, we always have $\varepsilon = 0$ in Theorem 3.8. In particular, the composition factors occurring in $L(\omega_r) \downarrow C_{l-1}$ are $L_{C_{l-1}}(\omega_r)$ and $L_{C_{l-1}}(\omega_{r-2^k})$ for $0 \leq k \leq d$.

We will now give some applications of Theorem 3.8 and Theorem 3.5 in characteristic two, which will be needed in our proof of Proposition 3.1.

**Lemma 3.9.** Assume that char $K = 2$, and let $l \geq 2^{i+1}$, where $i \geq 0$. Suppose that $l+1 \equiv 2^i + t \mod 2^{i+1}$, where $0 \leq t < 2^i$. Then for $t+1 \leq j \leq 2^{i+1}$, the following hold:

(i) All composition factors of the restriction $L(\omega_j) \downarrow C_{l-1}$ have the form $L(\omega_{j'})$ for some $l-1 \geq j' \geq t$.

(ii) $L_{C_l}(\omega_j) \downarrow C_{l-1}$ has no trivial composition factors.

**Proof.** If $t = 0$ there is nothing to prove, so suppose that $t \geq 1$. It will be enough to prove (i) as then (ii) will follow by induction on $t$. Let $d = \nu_2(l+1-j)$. Suppose first that $0 \leq d < i+1$. Now $l+1-j \equiv t-j \mod 2^t$, so then $\nu_2(l+1-j) = \nu_2(j-t)$. By Theorem 3.8, the composition factors occurring in $L(\omega_j) \downarrow C_{l-1}$ are $L(\omega_j)$ and $L(\omega_{j-2^l})$ for $0 \leq k \leq d$, so the claim follows since $\nu_2(j-t) = d$ and thus $j-2^d \geq t$.

Consider then the case where $d \geq i+1$. Then $l+1-j \equiv 2^i + (t-j) \equiv 0 \mod 2^{i+1}$, so $j-t \equiv 2^i \mod 2^{i+1}$. On the other hand $0 < j-t < 2^{i+1}$, so $j-t = 2^i$. By Theorem 3.8, the composition factors occurring in $L(\omega_j) \downarrow C_{l-1}$ are $L(\omega_j)$ and $L(\omega_{j-2^k})$ for $0 \leq k \leq i$ (because $j-2^k < 0$ for $i+1 \leq k \leq d$), so again the claim follows. \[\square\]
Lemma 3.10. Let \( x \geq 2^{i+1}, \) where \( i \geq 0. \) Suppose that \( x \equiv 2^i \pmod{2^{i+1}}. \) If \( 0 \leq k \leq 2^i \) and \( x - 2k \) contains \( 2^i - k \) to base 2, then \( k = 0 \) or \( k = 2^i. \)

Proof. If \( i = 0 \) there is nothing to do, so suppose that \( i > 0. \) Replacing \( k \) by \( 2^i - k, \) we see that it is equivalent to prove that if \( x + 2k \) contains \( k \) to base 2, then \( k = 0 \) or \( k = 2^i. \)

Suppose that \( 0 \leq k < 2^i \) and that \( x + 2k \) contains \( k \) to base 2. Consider first the case where \( 0 \leq k < 2^{i-1}. \) Here since \( x + 2k \equiv 2k \pmod{2^i}, \) we have that \( 2k \) contains \( k \) to base 2, which can only happen if \( k = 0. \)

Consider then \( 2^{i-1} \leq k < 2^i \) and write \( k = 2^{i-1} + k', \) where \( 0 \leq k' < 2^{i-1}. \) Then \( x + 2k \equiv 2k' \pmod{2^i}, \) so \( 2k' \) contains \( k = 2^{i-1} + k' \) to base 2. But then \( 2k' \) must also contain \( k' \) to base 2, so \( k' = 0 \) and \( k = 2^{i-1}. \) In this case \( x + 2k \equiv 2^i + 2^{i-1} \equiv 0 \pmod{2^{i+1}}, \) so \( x + 2k \) does not contain \( k \) to base 2, contradiction.

\[ \square \]

Lemma 3.11. Let \( x \geq 2^{i+1}, \) where \( i \geq 0. \) Suppose that \( x \equiv 2^i + t \pmod{2^{i+1}}. \) If \( 0 \leq t < 2^i. \) If \( 0 \leq 2j \leq t \) and \( x - 2j \) contains \( 2^i - j \) to base 2, then \( j = 0. \)

Proof. We prove the claim by induction on \( i. \) If \( i = 0 \) or \( i = 1, \) then the claim is immediate since \( 0 \leq 2j \leq t < 2. \) Suppose then that \( i > 1. \) Assume that \( 0 < 2j \leq t \) and that \( x - 2j \) contains \( 2^i - j \) to base 2. Now \( 2j < 2^i, \) so \( 0 < j < 2^{i-1}. \) Therefore \( 2^{i-1} \) must occur in the binary expansion of \( 2^i - j = 2^{i-1} + (2^{i-1} - j), \) so by our assumption \( 2^{i-1} \) occurs in the binary expansion of \( x - 2j. \) Note that this also means that \( x - 2j \) contains \( 2^{i-1} - j \) to base 2.

Now \( x - 2j \equiv 2^i + (t - 2j) \pmod{2^{i+1}} \) and \( 0 \leq t - 2j < 2^i, \) so it follows that \( 2^{i-1} \) will occur in the binary expansion of \( t - 2j. \) Write \( t = 2^{i-1} + t', \) where \( 0 \leq t' < 2^{i-1}. \) Here \( t' \geq 2j \) because \( t - 2j \geq 2^{i-1}. \) Finally, since \( x - 2j \) contains \( 2^{i-1} - j \) in base 2 and \( x \equiv 2^{i-1} + t' \pmod{2^i}, \) we have \( j = 0 \) by induction.

Now the following corollaries are immediate from Theorem 3.5 and lemmas 3.10 and 3.11.

Corollary 3.12. Assume that \( \text{char } K = 2, \) and let \( l \geq 2^{i+1}, \) where \( i \geq 0. \) Suppose that \( l + 1 \equiv 2^i \pmod{2^{i+1}}. \) Then \( V(\omega_{2^{i+1}}) = L(\omega_{2^{i+1}})/L(0). \)

Proof. For \( 0 \leq j \leq 2^{i+1}, \) by Theorem 3.5 the irreducible \( L(\omega_j) \) is a composition factor of \( V(\omega_{2^{i+1}}) \) if and only if \( j = 2j' \) and \( l + 1 - 2j' \) contains \( 2^i - j' \) to base 2. By Lemma 3.10, this is equivalent to \( j' = 0 \) or \( j' = 2^i. \)

\[ \square \]

Corollary 3.13. Assume that \( \text{char } K = 2, \) and let \( l \geq 2^{i+1}, \) where \( i \geq 0. \) Suppose that \( l + 1 \equiv 2^i + t \pmod{2^{i+1}}, \) where \( 0 \leq t < 2^i. \) Then any nontrivial composition factor of \( V(\omega_{2^{i+1}}) \) has the form \( L(\omega_j), \) where \( 2^{i+1} \geq 2j \geq t + 1. \)
Proof. For $0 \leq j \leq 2^t + 1$, by Theorem 3.5 the irreducible $L(\omega_j)$ is a composition factor of $V(\omega_{2^t+1})$ if and only if $j = 2j'$ and $l + 1 - 2j'$ contains $2^i - j'$ to base 2. If $0 \leq j \leq t$, then by Lemma 3.11 we have $j = 0$.

\[\square\]

### 3.2 Construction of $V(\omega_r)$

We now describe the well known construction of the Weyl modules $V(\omega_r)$ for $G$ using the exterior algebra of the natural module. We will consider our group $G$ as a Chevalley group constructed from a complex simple Lie algebra of type $C_l$. For details of the Chevalley group construction see [Ste68].

Let $e_1, \ldots, e_l, e_{-l}, \ldots, e_{-1}$ be a basis for a complex vector space $V_C$, and let $V_Z$ be the $\mathbb{Z}$-lattice spanned by this basis. We have a non-degenerate alternating form $(-, -)$ on $V_C$ defined by $(e_i, e_{-i}) = 1 = -(e_{-i}, e_i)$ and $(e_i, e_j) = 0$ for $i \neq j$. Let $\mathfrak{sp}(V_C)$ be the Lie algebra formed by the linear endomorphisms of $V_C$ satisfying $(Xv, w) + (v, Xw) = 0$ for all $v, w \in V_C$. Then $\mathfrak{sp}(V_C)$ is a simple Lie algebra of type $C_l$. Let $\mathfrak{h}$ be the Cartan subalgebra formed by the diagonal matrices in $\mathfrak{sp}(V_C)$. Then $\mathfrak{h} = \{\text{diag}(h_1, \ldots, h_l, -h_1, \ldots, -h_1) : h_i \in \mathbb{C}\}$. For $1 \leq i \leq l$, define maps $\varepsilon_i : \mathfrak{h} \to \mathbb{C}$ by $\varepsilon_i(h) = h_i$ where $h$ is a diagonal matrix with diagonal entries $(h_1, \ldots, h_l, -h_1, \ldots, -h_1)$. Now $\Phi = \{\pm(\varepsilon_i \pm \varepsilon_j) : 1 \leq i < j \leq l\} \cup \{\pm 2\varepsilon_i : 1 \leq i \leq l\}$ is the root system for $\mathfrak{sp}(V_C)$, $\Phi^+ = \{\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq l\} \cup \{2\varepsilon_i : 1 \leq i \leq l\}$ is a system of positive roots, and $\Delta = \{\varepsilon_i - \varepsilon_{i+1} : 1 \leq i < l\} \cup \{2\varepsilon_1\}$ is a base for $\Phi$.

For any $i, j$ let $E_{i,j}$ be the linear endomorphism on $V_C$ such that $E_{i,j}(e_j) = e_i$ and $E_{i,j}(e_k) = 0$ for $k \neq j$. Then a Chevalley basis for $\mathfrak{sp}(V_C)$ is given by $X_{\varepsilon_i - \varepsilon_j} = E_{i,j} - E_{j,i}$ for all $i \neq j$, by $X_{\varepsilon_i \pm \varepsilon_j} = E_{i,j} \pm E_{j,i}$ for all $i \neq j$, by $X_{2\varepsilon_i} = E_{i,i}$ for all $i$, and by $H_{\varepsilon_i - \varepsilon_{i+1}} = E_{i,i} - E_{i+1,i}, H_{2\varepsilon_i} = E_{i,i} - E_{i,-i}$. Let $\mathcal{U}_Z$ be the Kostant $\mathbb{Z}$-form with respect to this Chevalley basis of $\mathfrak{sp}(V_C)$. That is, $\mathcal{U}_Z$ is the subring of the universal enveloping algebra of $\mathfrak{sp}(V_C)$ generated by 1 and all $X^\lambda_{\alpha}$ for $\alpha \in \Phi$ and $k \geq 1$.

Now $V_Z$ is a $\mathcal{U}_Z$-invariant lattice in $V_C$. We define $V = V_Z \otimes \mathbb{C} K$. Note that $(-, -)$ also defines a non-degenerate alternating form on $V$. Then the simply connected Chevalley group of type $C_l$ induced by $V$ is equal to the group $G = \text{Sp}(V)$ of invertible linear maps preserving $(-, -)$ [Ree57, pg. 396-397]. By abuse of notation we identify the basis $(e_i \otimes 1)$ of $V$ with $(e_i)$.

Note that for all $1 \leq k \leq 2l$, the Lie algebra $\mathfrak{sp}(V_C)$ acts naturally on $\wedge^k(V_C)$ by

\[X \cdot (v_1 \wedge \cdots \wedge v_k) = \sum_{i=1}^{k} v_1 \wedge \cdots \wedge v_{i-1} \wedge Xv_i \wedge v_{i+1} \wedge \cdots \wedge v_k\]

for all $X \in \mathfrak{sp}(V_C)$ and $v_i \in V_C$. With this action, the $\mathbb{Z}$-lattice $\wedge^k(V_Z)$ is
invariant under $U_Z$ and this induces an action of $G$ on $\Lambda^k(V_Z) \otimes Z K$. One can show that $g \cdot (v_1 \wedge \cdots \wedge v_k) = gv_1 \wedge \cdots \wedge gv_k$ for all $g \in G$ and $v_i \in V$, so we can and will identify $\Lambda^k(V_Z) \otimes Z K$ and $\Lambda^k(V)$ as $G$-modules.

The diagonal matrices in $G$ form a maximal torus $T$. Then a basis of weight vectors of $\Lambda^k(V)$ is given by the elements $e_{i_1} \wedge \cdots \wedge e_{i_k}$, where $-l \leq i_1 < \cdots < i_k \leq l$. The basis vector $e_1 \wedge \cdots \wedge e_k$ has weight $\omega_k$.

The form on $V$ induces a form on the exterior power $\Lambda^k(V)$ by

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle = \det((v_i, w_j))_{1 \leq i,j \leq k}$$

for all $v_i, w_j \in V$ [Bou59, §1, Définition 12, pg. 30]. This form on $\Lambda^k(V)$ is invariant under the action of $G$ since $(-, -)$ is. Furthermore, let $e_{i_1} \wedge \cdots \wedge e_{i_k}$ and $e_{j_1} \wedge \cdots \wedge e_{j_k}$ be two basis elements of $\Lambda^k(V)$. Then

$$\langle e_{i_1} \wedge \cdots \wedge e_{i_k}, e_{j_1} \wedge \cdots \wedge e_{j_k} \rangle = \begin{cases} \pm 1, & \text{if } \{i_1, \ldots, i_k\} = \{-j_1, \ldots, -j_k\}, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore it follows that the form $(-, -)$ on $\Lambda^k(V)$ is nondegenerate if $1 \leq k \leq l$. In precisely the same way we find a basis of weight vectors for $\Lambda^k(V_Z)$ and define a form $\langle -,- \rangle_Z$ on $\Lambda^k(V_Z)$. Note that $\langle -,- \rangle_Z$ are alternating if $k$ is odd and symmetric if $k$ is even.

It is well known that there is a unique submodule of $\Lambda^k(V)$ isomorphic to the Weyl module $V(\omega_k)$ of $G$, as shown by the following lemma. The following lemma is also a consequence of [AJ84, 4.9].

**Lemma 3.14.** Let $1 \leq k \leq l$, and let $W$ be the $G$-submodule of $\Lambda^k(V)$ generated by $e_1 \wedge \cdots \wedge e_k$. Then

(i) $W$ is equal to the subspace of $\Lambda^k(V)$ spanned by all $v_1 \wedge \cdots \wedge v_k$, where $\langle v_1, \ldots, v_k \rangle$ is a $k$-dimensional totally isotropic subspace of $V$. Furthermore, $\dim W = \binom{2l}{k} - \binom{2l}{k-\omega}$.

(ii) $W$ is isomorphic to the Weyl module $V(\omega_k)$.

**Proof.** (i) Since $G$ acts transitively on the set of $k$-dimensional totally isotropic subspaces of $V$, it follows that $W$ is spanned by all $v_1 \wedge \cdots \wedge v_k$, where $\langle v_1, \ldots, v_k \rangle$ is a $k$-dimensional totally isotropic subspace of $V$. Then the claim about the dimension of $W$ follows from a result proven for example in [DB09, Theorem 1.1], [Bro92, Theorem 1.1] or (in odd characteristic) [PS83, pg. 1337].
(ii) Since \( e_1 \wedge \cdots \wedge e_k \) is a maximal vector of weight \( \omega_k \) for \( G \), the submodule \( W \) generated by it is an image of \( V(\omega_k) \) \cite{Jan03} II.2.13. Now \( \dim V(\omega_k) = \binom{2l}{k} - \binom{2l}{k-2} \) \cite{Bou75} Ch. VIII, 13.3, pg. 203, so by (i) \( W \) must be isomorphic to \( V(\omega_k) \).

In what follows we will identify \( V(\omega_k) \) as the submodule \( W \) of \( \wedge^k(V) \) given by Lemma 3.14. Set \( V(\omega_k) = U \otimes Z(e_1 \wedge \cdots \wedge e_k) \). Note that now we can (and will) identify \( V(\omega_k) \) and \( V(\omega_k) \otimes \mathbb{Z} \).

We will denote \( y_i = e_i \wedge e_{-i} \) for all \( 1 \leq i \leq l \). Then if \( k = 2s \) is even, a basis for the zero weight space of \( \wedge^k(V) \) is given by vectors of the form \( y_{i_1} \wedge \cdots \wedge y_{i_s} \), where \( 1 \leq i_1 < \cdots < i_s \leq l \). There is also a description of a basis for the zero weight space of \( V(\omega_k) \) in \cite{Jan73} Lemma 10, pg.43. For our purposes, we will only need a convenient set of generators given by the next lemma.

Lemma 3.15 (\cite{Jan73} pg. 40, Lemma 6). Suppose that \( k \) is even, say \( k = 2s \), where \( 1 \leq k \leq l \). Then the zero weight space of \( V(\omega_k) \) (thus also of \( V(\omega_k) \)) is spanned by vectors of the form

\[
(y_{j_1} - y_{k_1}) \wedge \cdots \wedge (y_{j_s} - y_{k_s}),
\]

where \( 1 \leq k_r < j_r \leq l \) for all \( r \) and \( j_r, k_r \neq j_{r'}, k_{r'} \) for all \( r \neq r' \).

Lemma 3.16. Suppose that \( k \) is even, say \( k = 2s \), where \( 1 \leq k \leq l \). Then the vector

\[
\gamma = \sum_{1 \leq i_1 < \cdots < i_s \leq l} y_{i_1} \wedge \cdots \wedge y_{i_s}
\]

is fixed by the action of \( G \) on \( \wedge^k(V) \). Furthermore, any \( G \)-fixed point in \( \wedge^k(V) \) is a scalar multiple of \( \gamma \).

Proof. To see that \( \gamma \) is fixed by \( G \), see for example \cite{ DB10} 3.4 where it is shown that the definition of \( \gamma \) does not depend on the symplectic basis chosen.

For the other claim, note first that any \( G \)-fixed point must have weight zero. Recall that the zero weight space of \( \wedge^k(V) \) has basis

\[
\mathcal{B} = \{ y_{i_1} \wedge \cdots \wedge y_{i_s} : 1 \leq i_1 < \cdots < i_s \leq l \}.
\]

Now the group \( \Sigma_l \) of permutations of \( \{1, 2, \ldots, l\} \) acts on \( V \) by \( \sigma \cdot e_{\pm i} = e_{\pm \sigma(i)} \) for all \( \sigma \in \Sigma_l \). Clearly this action preserves the form \((-, -)\) on \( V \), so this gives an embedding \( \Sigma_l < G \). Note also that \( \Sigma_l \) acts transitively on \( \mathcal{B} \). Thus any \( \Sigma_l \)-fixed point in the linear span of \( \mathcal{B} \) must be a scalar multiple of \( \sum_{b \in \mathcal{B}} b = \gamma \). \qed
With these preliminary steps done, we now move on to proving Proposition 3.1. For the rest of this section, we will make the following assumption.

Assume that $\text{char } K = 2$.

Let $1 \leq r \leq l$. By Proposition 2.3, we know that $L(\omega_1)$ is not orthogonal. Suppose then that $r \geq 2$ and that $L(\omega_r)$ is not orthogonal. By Proposition 2.2 (iii) we have $H^1(G, L(\omega_r)) \neq 0$, so by Corollary 3.7 we have $r = 2^{i+1}$ for some $i \geq 0$ and $l + 1 \equiv 2^i + t \mod 2^{i+1}$ for some $0 \leq t < 2^i$. What remains is to determine when $L(\omega_r)$ is orthogonal for such $r$. With the lemma below, we reduce this to the evaluation of $Q(v)$ for a single vector $v \in V(\omega_r)$, where $Q$ is a non-zero $G$-invariant quadratic form on $V(\omega_r)$.

Lemma 3.17. Let $l \geq 2^{i+1}$, where $i \geq 0$. Suppose that $l + 1 \equiv 2^i + t \mod 2^{i+1}$, where $0 \leq t < 2^i$. Define the vector $\gamma \in \wedge^{2^{i+1}}(V)$ to be equal to

$$\sum_{1 \leq i_1 < \ldots < i_{2^i} \leq l-t} y_{i_1} \wedge \cdots \wedge y_{i_{2^i}}.$$  

Then

(i) $\gamma$ is in $V(\omega_2^{i+1})$ and is a fixed point for the subgroup $C_{l-t} < G$,

(ii) $\gamma$ is in $\text{rad } V(\omega_2^{i+1})$,

(iii) $L(\omega_2^{i+1})$ is orthogonal if and only if $Q(\gamma) = 0$, where $Q$ is a nonzero $G$-invariant quadratic form on $V(\omega_2^{i+1})$.

Proof. (i) By Lemma 3.16, $\gamma$ is a fixed by the action of $C_{l-t}$. It follows from Lemma 3.14 that the $C_{l-t}$-submodule $W$ generated by $e_1 \wedge \cdots \wedge e_{2^{i+1}}$ is isomorphic to the Weyl module $V_{C_{l-t}}(\omega_2^{i+1})$. By Corollary 3.12, the module $W$ has a trivial $C_{l-t}$-submodule and by Lemma 3.16 it is generated by $\gamma$. Since $W$ is contained in $V(\omega_2^{i+1})$, the $C_l$-submodule generated by $e_1 \wedge \cdots \wedge e_{2^{i+1}}$, the claim follows.

For a different proof, one can also prove $\gamma \in V(\omega_2^{i+1})$ by showing that $\gamma$ is in the kernel of certain linear maps as defined in [DB10, Theorem 3.5] or [Bro92, Theorem 3.1, Proposition 3.3].

(ii) By Proposition 2.2 (ii), it will be enough to show that $\gamma$ is orthogonal to $V(\omega_2^{i+1})$ with respect to the form $\langle -,- \rangle$ on $\wedge^{2^{i+1}}(V)$. Since $\gamma$ has weight 0, it is orthogonal to any vector of weight $\neq 0$. Therefore it suffices to show that $\gamma$ is orthogonal to any vector of weight 0 in $V(\omega_2^{i+1})$. By
Lemma 3.15] this follows once we show that $\gamma$ is orthogonal to any vector

$$\delta = (y_{j_1} + y_{k_1}) \land \cdots \land (y_{j_{2^l}} + y_{k_{2^l}}),$$

where $1 \leq k_r < j_r \leq l$ for all $r$ and $k_r, j_r \neq j_r', k_r'$ for all $r \neq r'$.

Because the set $\{j_1, \ldots, j_{2^l}\}$ contains $2^l$ distinct integers, we cannot have $j_r \geq l - t + 1$ for all $r$. Indeed, otherwise $j_r \geq l - t + 2^l \geq l + 1$ for some $r$, contradicting the fact that $j_r \leq l$. Let $q > 0$ be the number of $j_r$ such that $j_r \leq l - t$.

The vector $\delta$ can be written as

$$\sum_{f_s \in \{j_s, k_s\}} y_{f_1} \land \cdots \land y_{f_{2^l}}.$$

Now

$$\langle y_{f_1} \land \cdots \land y_{f_{2^l}}, y_{g_1} \land \cdots \land y_{g_{2^l}} \rangle = \begin{cases} 1, & \text{if } \{f_1, \ldots, f_{2^l}\} = \{g_1, \ldots, g_{2^l}\}, \\ 0, & \text{otherwise.} \end{cases}$$

and so $\langle \delta, \gamma \rangle$ is an integer, equal to the number of $y_{f_1} \land \cdots \land y_{f_{2^l}}$ in the sum such that $f_s \leq l - t$ for all $1 \leq s \leq 2^l$. Thus if $j_r \geq k_r \geq l - t + 1$ for some $r$, then $\langle \gamma, \delta \rangle = 0$. If $k_r \leq l - t$ for all $r$, then it follows that $\langle \delta, \gamma \rangle = 2^q = 0$ since $q > 0$.

(iii) By Lemma 3.17 we have $H^1(G, L(\omega_{2^{l+1}})) \neq 0$ and so there exists a non-split extension of $L(\omega_{2^{l+1}})$ by the trivial module $K$. We can find this extension as an image of the Weyl module $V(\omega_{2^{l+1}})$ [Jan03, II.2.13, II.2.14], so rad $V(\omega_{2^{l+1}})/M \cong K$ for some submodule $M$ of rad $V(\omega_{2^{l+1}})$. Since each composition factor of $V(\omega_{2^{l+1}})$ occurs with multiplicity one (Theorem 3.5), each composition factor of $M$ is nontrivial. Then by Corollary 3.13 and Lemma 3.9 (ii), the restriction $M \downarrow C_{l-1}$ has no trivial composition factors. But by (i) $\gamma$ is a fixed point for $C_{l-1}$, so it follows that $\gamma \not\in M$ and then rad $V(\omega_{2^{l+1}}) = \langle \gamma \rangle \oplus M$ as $C_{l-1}$-modules.

Now let $Q$ be a nonzero $G$-invariant quadratic form on $V(\omega_{2^{l+1}})$. Since for the polarization $b_Q$ of $Q$ we have rad $b_Q = \text{rad } V(\omega_{2^{l+1}})$ (Proposition 2.2 (ii)), composing $Q$ with the square root map $K \to K$ defines a morphism rad $V(\omega_{2^{l+1}}) \to K$ of $G$-modules. Therefore $Q$ must vanish on $M$, since $M$ has no trivial composition factors. Thus for all $m \in M$ and scalars $c$ we have $Q(c\gamma + m) = c^2 Q(\gamma)$, so $Q$ vanishes on rad $V(\omega_{2^{l+1}})$ if and only if $Q(\gamma) = 0$. Hence by Proposition 2.2 (iii) $L(\omega_{2^{l+1}})$ is orthogonal if and only if $Q(\gamma) = 0$. 

□
3.3 Computation of a quadratic form $Q$ on $V(\omega_r)$

To finish the proof of Proposition 3.1 we still have to compute $Q(\gamma)$ for the vector $\gamma$ from Lemma 3.1.

We retain the notation from the previous subsection and keep the assumption that $\text{char} K = 2$. Let $r$ be even, say $r = 2s$, where $1 \leq r \leq l$.

Now the form $\langle -,- \rangle_Z$ on $\land^r(V_Z)$ induces a quadratic form $q_Z$ on $\land^r(V_Z)$ by $q_Z(x) = \langle x, x \rangle$. We will use this form to find a nonzero $G$-invariant quadratic form on $V(\omega_r) = V(\omega_r)_Z \otimes Z K$.

**Lemma 3.18.** We have $q_Z(V(\omega_r)_Z) \subseteq 2Z$ and $q_Z(V(\omega_r)_Z) \not\subseteq 4Z$.

**Proof.** Let $\alpha = e_1 \land \cdots \land e_r$ and $\beta = e_{-1} \land \cdots \land e_{-r}$. Now $\alpha, \beta \in V(\omega_r)_Z$ and $\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle = 0$ and $\langle \alpha, \beta \rangle = 1$, giving $q_Z(\alpha + \beta) = 2$ and so $q_Z(V(\omega_r)_Z) \not\subseteq 4Z$. If we had $q_Z(V(\omega_r)_Z) \subseteq 2Z$, then $Q = q_Z \otimes Z K$ defines a nonzero $G$-invariant quadratic form on $V(\omega_r)$. But then the polarization of $Q$ is equal to $2 \langle -, - \rangle = 0$, which by Proposition 2.2 (i) and (ii) is not possible. \qed

Therefore $Q = \frac{1}{2} q_Z \otimes Z K$ defines a nonzero $G$-invariant quadratic form on $V(\omega_r)$ with polarization $\langle -,- \rangle$. A similar construction when $r$ is odd is discussed in [GN16, Proposition 8.1].

Now if we consider a zero weight vector of the form

$$\sum_{\{i_1, \ldots, i_s\} \in \mathcal{I}} y_{i_1} \land \cdots \land y_{i_s}$$

in $V(\omega_r)$, the value of $Q$ for this vector is equal to $\frac{|\mathcal{I}|}{2}$ since

$$\langle y_{f_1} \land \cdots \land y_{f_k}, y_{g_1} \land \cdots \land y_{g_k} \rangle = \begin{cases} 1, & \text{if } \{f_1, \ldots, f_k\} = \{g_1, \ldots, g_k\}, \\ 0, & \text{otherwise.} \end{cases}$$

Now we can compute the value of $Q(\gamma)$ from Lemma 3.17. Since there are $\binom{l-t}{t}$ terms occurring in the sum that defines $\gamma$, we have $Q(\gamma) = \frac{1}{2} \binom{l-t}{2}$. Thus $Q(\gamma) = 0$ if and only if $\binom{l-t}{t}$ is divisible by 4. Now the proof of Proposition 3.1 is finished with the following lemma.

**Lemma 3.19.** Let $l + 1 \equiv 2^i + t \mod 2^{i+1}$, where $0 \leq t < 2^i$. The integer $\binom{l-t}{2}$ is divisible by 4 if and only if $l + 1 \equiv 2^i + t \mod 2^{i+2}$.

**Proof.** According to Kummer’s theorem, if $p$ is prime and $d \geq 0$ is maximal such that $p^d$ divides $\binom{x}{y}$ $(x \geq y \geq 0)$, then $d$ is the number of carries that
occur when adding $y$ to $x - y$ in base $p$. Now $(l - t) - 2^i \equiv -1 \equiv 2^i + \cdots + 2 + 1 \mod 2^{i+2}$. If 
\[
l - t - 2^i \equiv 2^i + \cdots + 2 + 1 \mod 2^{i+2},
\]
then adding $2^i$ to $l - t - 2^i$ in binary results in just one carry. The other possibility is that 
\[
l - t - 2^i \equiv 2^i + \cdots + 2 + 1 \mod 2^{i+2},
\]
and in this case there are $\geq 2$ carries. Therefore \(\binom{l-t}{2^i}\) is divisible by 4 if and only if 
\[
l - t - 2^i \equiv 2^i + \cdots + 2 + 1 \equiv -1 \mod 2^{i+2},
\]
which is equivalent to $l + 1 \equiv 2^i + t \mod 2^{i+2}$.

\[\Box\]

4 Fundamental irreducibles for classical types

With a bit more work, we can use Proposition 3.1 to determine for all classical types the fundamental irreducible representations that are orthogonal. In this section assume that $\text{char } K = 2$.

For a groups of type $A_l$ ($l \geq 1$), the only self-dual fundamental irreducible representations are those of form $L(\omega_{l+1})$, where $l$ is odd. Furthermore, all fundamental representations are minuscule, so $V(\omega_{l+1}) = L(\omega_{l+1})$ and thus by Proposition 2.2 (iv) the representation $L(\omega_{l+1})$ is orthogonal.

Now for type $B_l$, there exists an exceptional isogeny $\varphi : B_l \to C_l$ between simply connected groups of type $B_l$ and $C_l$ [Ste68, Theorem 28]. Then irreducible representations of $C_l$ induce irreducible representations $B_l$ by twisting with the isogeny $\varphi$. For fundamental irreducible representations, we have $L_{C_l}(\omega_r)^\varphi \cong L_{B_l}(\omega_r)$ if $1 \leq r \leq l - 1$, and $L_{C_l}(\omega_l)^\varphi$ is a Frobenius twist of $L_{B_l}(\omega_l)$. Therefore for all $1 \leq r \leq l$, the representation $L_{C_l}(\omega_r)$ is orthogonal if and only if $L_{B_l}(\omega_r)$ is orthogonal.

Consider then type $D_l$ ($l \geq 4$). First note that the natural representation $L_{D_l}(\omega_1)$ of $D_l$ is orthogonal. Now since we are working in characteristic two, there is an embedding $D_l < C_l$ as a subsystem subgroup generated by the short root subgroups. Then if $1 \leq r \leq l - 2$, we have $L_{C_l}(\omega_r) \downarrow D_l \cong L_{D_l}(\omega_r)$ for $1 \leq r \leq l - 2$ by [Sei87, Theorem 4.1]. By combining this fact with the lemma below, we see for $2 \leq r \leq l - 2$ that $L_{C_l}(\omega_r)$ is orthogonal if $L_{D_l}(\omega_r)$ is orthogonal.

**Lemma 4.1.** Let $G$ be simple of type $C_l$ and consider $H < G$ of type $D_l$ as the subsystem subgroup generated by short root subgroups. Suppose that $V$ is a nontrivial irreducible 2-restricted representation of $G$ and $V \neq L_G(\omega_1)$.
Then if \( V \Downarrow H \) is 2-restricted irreducible, the representation \( V \) is orthogonal for \( G \) if and only if \( V \) is orthogonal for \( H \).

**Proof (G. Seitz).** If \( V \) is an orthogonal \( G \)-module, it is clear that it is an orthogonal \( H \)-module as well. Suppose then that \( V \) is not orthogonal for \( G \). Since \( V \) is not the natural module for \( G \), by Proposition 2.2 (iii) there exists a nonsplit extension
\[
0 \to \langle w \rangle \to M \to V \to 0
\]
of \( G \)-modules, where \( w \in M \). Furthermore, there exists a nonzero \( G \)-invariant quadratic form \( Q \) on \( M \) such that \( Q(w) \neq 0 \).

We claim that \( M \Downarrow H \) is also a nonsplit extension. If this is not the case, then \( M \Downarrow H = W \oplus \langle w \rangle \) for some \( H \)-submodule \( W \) of \( M \). We will show that \( W \) is invariant under \( G \), which is a contradiction since \( M \) is nonsplit for \( G \). Now \( W \) is 2-restricted irreducible for \( H \), so by a theorem of Curtis [Bor70, Theorem 6.4] the module \( W \) is also an irreducible representation of \( \text{Lie}(H) \). Since \( \text{Lie}(H) \) is an ideal of \( \text{Lie}(G) \) that is invariant under the adjoint action of \( G \), it follows that \( gw \) is \( \text{Lie}(H) \)-invariant for all \( g \in G \). But as a \( \text{Lie}(H) \)-module \( M \) is the sum of a trivial module and \( W \), so we must have \( gw = W \) for all \( g \in G \).

Thus if \( V \Downarrow H = L_H(\lambda) \), then there exists a surjection \( \pi : V_H(\lambda) \to M \) of \( H \)-modules [Jan03, II.2.13]. Now the quadratic form \( Q \) induces via \( \pi \) a nonzero, \( H \)-invariant quadratic form on \( V_H(\lambda) \) which does not vanish on the radical of \( V_H(\lambda) \). By Proposition 2.2 (iii) the representation \( V \) is not orthogonal for \( H \).

Finally, the half-spin representations of \( D_t \) are minuscule representations, so \( L_{D_t}(\omega_l) = V_{D_t}(\omega_l) \) and \( L_{D_t}(\omega_{l-1}) = V_{D_t}(\omega_{l-1}) \). As before, by Proposition 2.2 (iv) it follows that \( L_{D_t}(\omega_l) \) and \( L_{D_t}(\omega_{l-1}) \) are orthogonal if they are self-dual. Therefore we can conclude that for \( l = 4 \) and \( l = 5 \) all self-dual \( L_{D_t}(\omega_l) \) are orthogonal.

Note that if \( l \geq 6 \), then \( L_{C_t}(\omega_l) \) and \( L_{C_t}(\omega_{l-1}) \) are also orthogonal (Example 3.4). Thus for \( l \geq 6 \) we have for all \( 2 \leq r \leq l \) that \( L_{C_t}(\omega_r) \) is orthogonal if \( L_{D_t}(\omega_r) \) is orthogonal.

Taking all of this together, Proposition 3.1 is improved to the following.

**Theorem 4.2.** Assume that \( \text{char } K = 2 \). Let \( G \) be simple of type \( A_t \) \((l \geq 1)\), \( B_t \) \((l \geq 2)\), \( C_t \) \((l \geq 2)\) or \( D_t \) \((l \geq 4)\). Suppose \( 1 \leq r \leq l \) and \( \omega_r = -w_0(\omega_r) \). Then \( L(\omega_r) \) is not orthogonal if and only if one of the following holds:

- \( G \) is of type \( B_t \) \((l \geq 2)\) or \( C_t \) \((l \geq 2)\) and \( r = 1 \).
- \( G \) is of type \( B_t \) \((l \geq 2)\), \( C_t \) \((l \geq 2)\) or \( D_t \) \((l \geq 6)\) and \( r = 2^{i+1} \) for some \( i \geq 0 \) such that \( l + 1 \equiv 2^{i+1} + 2^i + t \mod 2^{i+2} \), where \( 0 \leq t < 2^i \).
5 Representations $L(\omega_r + \omega_s)$ for type $A_l$

Assume that $G$ is simply connected of type $A_l$, $l \geq 2$. Set $n = l + 1$.

In this section, we determine when in characteristic 2 the irreducible representation $L(\omega_r + \omega_s)$, $1 \leq r < s \leq l$, of $G$ has a nonzero $G$-invariant quadratic form. Now $L(\omega_r + \omega_s)$ is not orthogonal if it is not self-dual, so it will be enough to consider $L(\omega_r + \omega_{n-r})$, where $1 \leq r < n - r \leq l$ (see Table 2.1). In this case, the answer and the methods to prove it are very similar to those found in Section 3. The result is the following theorem, which we will prove in what follows.

**Theorem 5.1.** Assume $\text{char } K = 2$. Let $1 \leq r < n - r \leq l$. Then $L(\omega_r + \omega_{n-r})$ is not orthogonal if and only if $r = 2^i$ for some $i \geq 0$ and $n + 1 \equiv 2^i + 2^i + t \mod 2^i + 2$, where $0 \leq t < 2^i$.

The following examples follow easily from Theorem 5.1 (cf. examples 3.2 and 3.3).

**Example 5.2.** If $\text{char } K = 2$, then $L(\omega_1 + \omega_l)$ is orthogonal if and only if $n \not\equiv 2 \mod 4$. This result was also proven in [GW95, Theorem 3.4 (b)].

**Example 5.3.** If $\text{char } K = 2$, then $L(\omega_2 + \omega_{l-1})$ is orthogonal if and only if $n \not\equiv 5, 6 \mod 8$.

5.1 Representation theory

The composition factors and the submodule structure of the Weyl modules $V(\omega_r + \omega_s)$, $1 \leq r < s \leq l$, were determined by Adamovich [Ada92]. Using her result, Baranov and Suprunenko have given in [BS05, Theorem 2.3] a description of the set of composition factors, similarly to Theorem 3.5. For $1 \leq r < s \leq l$, define $J_p(r, s)$ be the set of pairs $(r - k, s + k)$, where $n - s, r \geq k \geq 0$ and $s - r + 2k$ contains $k$ to base $p$. Here we will define $\omega_0 = 0$ and $\omega_n = 0$, so then $L(\omega_0) = L(\omega_n) = L(\omega_0 + \omega_n)$ is the trivial irreducible module and $L(\omega_0 + \omega_r) = L(\omega_r + \omega_n) = L(\omega_r)$. Now [BS05, Theorem 2.3] gives the following.

**Theorem 5.4.** Let $1 \leq r < s \leq l$. Then in the Weyl module $V(\omega_r + \omega_s)$, each composition factor has multiplicity 1, and the set of composition factors is $\{L(\omega_j + \omega_{j'}) : (j, j') \in J_p(r, s)\}$.

---

2Baranov and Suprunenko give the result in terms of $\pi_{i,j} = L(\omega_{j-i+1}/2 + \omega_{i+j-1}/2)$, but from $\pi_{y-x+1,x+y} = L(\omega_x + \omega_y)$ we get the formulation in Theorem 5.3.
The result of Kleshchev and Sheth in [KS99, KS01, Corollary 3.6] about the first cohomology groups for groups of type $A_i$ gives the following (cf. Corollary 3.7).

**Theorem 5.5.** Assume that $\text{char } K = 2$. Let $1 \leq r < s \leq l$. Then $H^1(G, L(\omega_r + \omega_s)) \neq 0$ if and only if $r = 2^i$, $s = n - 2^i$ for some $i \geq 0$, and $n + 1 \equiv 2^i + t \mod 2^{i+1}$ for some $0 \leq t < 2^i$.

Throughout this section we will consider subgroups $A_{l'} < A_l = G$, which are embedded into $G$ as follows. We consider $G = \text{SL}(V)$, where $V$ has basis $e_1, e_2, \ldots, e_{l+1}$. Then for $1 \leq l' < l$, the embedding $A_{l'} < A_l$ is $\text{SL}(V') < \text{SL}(V)$, where $V' \subseteq V$ has basis $e_1, \ldots, e_{l'+1}$ and $\text{SL}(V')$ fixes the basis vectors $e_{l'+2}, \ldots, e_{l+1}$.

Baranov and Suprunenko have determined the submodule structure of the restrictions $L(\omega_r + \omega_s) \downarrow A_{l-1}$ for all $0 \leq r \leq s \leq n$ in their article [BS05, Theorem 1.1]. As in Section 3 for our purposes it will be enough to know which composition factors occur in the restriction. To state the result of Baranov and Suprunenko, we will denote $\pi_{r,s}^l = L_{A_l}(\omega_r + \omega_{l+1-s})$ for all $0 \leq r \leq l+1-s \leq l+1$. We will define $\pi_{r,s}^l = 0$ if $r < 0$, $s < 0$ or $r+s > l+1$. Now the main result of [BS05] gives the following (cf. Theorem 3.3).

**Theorem 5.6.** Let $0 \leq r \leq n-s \leq n$ and assume that $n \geq 3$. Set $d = \nu_p(n+1-(r+s))$, and $\varepsilon = 0$ if $n+1-(r+s) \equiv -p^{d} \mod p^{d+1}$ and $\varepsilon = 1$ otherwise. Then the character of $\pi_{r,s}^l \downarrow A_{l-1}$ is given by

$$\text{ch } \pi_{r,s}^{l-1} + \text{ch } \pi_{r-1,s}^{l-1} + \text{ch } \pi_{r,s}^{l-1} + \left( \sum_{k=0}^{d-1} 2 \text{ch } \pi_{r-p^k,s-p^k}^{l-1} \right) + \varepsilon \text{ch } \pi_{r-p^d,s-p^d}^{l-1}$$

where the sum in the brackets is zero if $d = 0$.

As with Theorem 3.3, note that when $\text{char } K = 2$, we always have $\varepsilon = 0$ in Theorem 5.6. The following applications of theorems 5.4 and 5.5 in characteristic two will be needed later.

**Lemma 5.7.** Assume that $\text{char } K = 2$, and let $n \geq 2^{i+1}$, where $i \geq 0$. Suppose that $n+1 \equiv 2^i + t \mod 2^{i+1}$, where $0 \leq t < 2^i$. Let $0 \leq x \leq 2^i$ and $0 \leq y \leq 2^i$ be such that $2^{i+1} \geq x + y + t$. Then

(i) All composition factors of the restriction $\pi_{x,y}^l \downarrow A_{l-1}$ have the form $\pi_{x',y'}^{l-1}$ for some $0 \leq x', y' \leq 2^i$ such that $x' + y' \geq t$.

---

3Baranov and Suprunenko give their result in terms of $L_{A_i}^l(\omega_i + \omega_j)$ for $0 \leq i \leq j \leq n$, but replacing $j$ by $n-j$ gives the formulation in Theorem 5.6.
(ii) \( \pi_{\nu, t} \downarrow A_{t-1} \) has no trivial composition factors.

Proof. (cf. Lemma 3.9) If \( t = 0 \) there is nothing to prove, so suppose that \( t \geq 1 \). It will be enough to prove (i) as then (ii) will follow by induction on \( t \). Let \( d = \nu_2(n + 1 - (x + y)) \). Suppose first that \( 0 \leq d < i + 1 \). Then \( \nu_2(n + 1 - (x + y)) \equiv t \) mod \( 2^i \), so \( \nu_2(n + 1 - (x + y)) = \nu_2((x + y) - t) \). By Theorem 5.6, the composition factors occurring in \( \pi_{\nu, t} \downarrow A_{t-1} \) are \( \pi_{t-1}^{\nu}, \pi_{x-1, y}^{\nu}, \pi_{x, y}^{\nu-1}, \pi_{x-2^k, y-2^k}^{\nu-1} \) for \( 0 \leq k \leq d - 1 \). Therefore the claim follows since \( \nu_2((x + y) - t) = d \) and thus \( x + y - 2^d \geq t \).

Consider then the case where \( d \geq i + 1 \). Then \( n + 1 - (x + y) \equiv 2^i + t - (x + y) \equiv 0 \) mod \( 2^{i+1} \), so \( (x + y) - t \equiv 2^i \) mod \( 2^{i+1} \). On the other hand \( 0 \leq (x + y) - t < 2^{i+1} \), so \( (x + y) - t = 2^i \). By Theorem 5.6 the composition factors occurring in \( \pi_{\nu, t} \downarrow A_{t-1} \) are \( \pi_{t-1}^{\nu}, \pi_{x-1, y}^{\nu}, \pi_{x, y}^{\nu-1}, \pi_{x-2^k, y-2^k}^{\nu-1} \) for \( 0 \leq k \leq i - 1 \) (since \( x - 2^k < 0 \) or \( y - 2^k < 0 \) for \( i \leq k \leq d - 1 \)), so again the claim follows.

As a consequence of Theorem 5.4 and lemmas 3.10 and 3.11, we get the following (cf. corollaries 3.12 and 3.13).

Corollary 5.8. Assume that \( \text{char } K = 2 \) and let \( n > 2^{i+1} \), where \( i \geq 0 \). Suppose that \( n + 1 \equiv 2^i \) mod \( 2^{i+1} \). Then \( V(\omega_2^i + \omega_{n-2}^i) = L(\omega_2^i + \omega_{n-2}^i)/L(0) \).

Proof. According to Theorem 5.4 the composition factors of \( V(\omega_2^i + \omega_{n-2}^i) \) are \( L(\omega_2^i + \omega_{n-2}^i) \), where \( 0 \leq k \leq 2^i \) and \( n + 1 - 2^{i+1} + 2k \) contains \( k \) to base 2. We can replace \( k \) by \( 2^i - k \), and then the condition is equivalent to \( n + 1 - 2k \) containing \( 2^i - k \) to base 2, which implies \( k = 0 \) or \( k = 2^i \) by Lemma 3.11.

Corollary 5.9. Assume that \( \text{char } K = 2 \) and let \( n > 2^{i+1} \), where \( i \geq 0 \). Suppose that \( n + 1 \equiv 2^i + t \) mod \( 2^{i+1} \), where \( i \geq 0 \) and \( 0 \leq t < 2^i \). Then any nontrivial composition factor of \( V(\omega_2^i + \omega_{n-2}^i) \) has the form \( L(\omega_2^i + \omega_{n-2}^i) \) for some \( 0 \leq x \leq n - y \leq n \) and \( x + y \geq t + 1 \).

Proof. According to Theorem 5.4 the composition factors of \( V(\omega_2^i + \omega_{n-2}^i) \) are \( L(\omega_2^i + \omega_{n-2}^i) \), where \( 0 \leq k \leq 2^i \) and \( n + 1 - 2^{i+1} + 2k \) contains \( k \) to base 2. Setting \( k' = 2^i - k \), the composition factors are \( L(\omega_{2^i} + \omega_{n-2^i}) \), where \( n + 1 - 2k' \) contains \( 2^i - k' \) to base 2. By Lemma 3.11 we have \( k' = 0 \) or \( 2k' \geq t + 1 \), which proves the claim.

5.2 Construction of \( V(\omega_r + \omega_{n-r}) \)

We now describe a construction of \( V(\omega_r + \omega_{n-r}) \), in many ways similar to that of \( V_{C_0}(\omega_r) \) described in Section 3.2. We will consider our group \( G \) as a Chevalley group constructed from a complex simple Lie algebra of type \( A_t \).
Let $e_1, e_2, \ldots, e_n$ be a basis for a complex vector space $V_{\mathbb{C}}$, and let $V_{\mathbb{Z}}$ be the $\mathbb{Z}$-lattice spanned by this basis. Let $\mathfrak{sl}(V_{\mathbb{C}})$ be the Lie algebra formed by the linear endomorphisms of $V_{\mathbb{C}}$ with trace zero. Then $\mathfrak{sl}(V_{\mathbb{C}})$ is a simple Lie algebra of type $A_l$. Let $\mathfrak{h}$ be the Cartan subalgebra formed by the diagonal matrices in $\mathfrak{sl}(V_{\mathbb{C}})$ (with respect to the basis $(e_i)$). For $1 \leq i \leq n$, define maps $\varepsilon_i : \mathbb{C} \to \mathbb{C}$ by $\varepsilon_i(h) = h_i$ where $h$ is a diagonal matrix with diagonal entries $(h_1, h_2, \ldots, h_n)$. Now $\Phi = \{\varepsilon_i - \varepsilon_j : i \neq j\}$ is the root system for $\mathfrak{sl}(V_{\mathbb{C}})$ and $\Phi^+ = \{\varepsilon_i - \varepsilon_j : i < j\}$ is a system of positive roots, and $\Delta = \{\varepsilon_i - \varepsilon_{i+1} : 1 \leq i \leq l\}$ is a base for $\Phi$.

For any $i, j$ let $E_{i,j}$ be the linear endomorphism on $V_{\mathbb{C}}$ such that $E_{i,j}(e_j) = e_i$ and $E_{i,j}(e_k) = 0$ for $k \neq j$. Now a Chevalley basis for $\mathfrak{sl}(V_{\mathbb{C}})$ is given by $X_{\varepsilon_i - \varepsilon_j} = E_{i,j}$ for $i \neq j$ and $H_{\varepsilon_i - \varepsilon_{i+1}} = E_{i,i} - E_{i+1,i+1}$ for $1 \leq i \leq l$. Let $\mathfrak{h}_{\mathbb{Z}}$ be the Kostant $\mathbb{Z}$-form with respect to this Chevalley basis of $\mathfrak{sl}(V_{\mathbb{C}})$. That is, $\mathfrak{h}_{\mathbb{Z}}$ is the subring of the universal enveloping algebra of $\mathfrak{sl}(V_{\mathbb{C}})$ generated by 1 and all $\frac{X_{\alpha}}{k!}$ for $\alpha \in \Phi$ and $k \geq 1$.

Now $V_{\mathbb{Z}}$ is a $\mathfrak{h}_{\mathbb{Z}}$-invariant lattice in $V_{\mathbb{C}}$. We define $V = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} K$. Then the simply connected Chevalley group of type $A_l$ induced by $V$ is equal to $G = \text{SL}(V)$.

Let $e_1^*, e_2^*, \ldots, e_n^*$ be a basis for $V_{\mathbb{C}}^*$, dual to the basis $(e_1, e_2, \ldots, e_n)$ of $V_{\mathbb{C}}$ (so here $e_i^*(e_j) = \delta_{ij}$). Denote the $\mathbb{Z}$-lattice spanned by $e_1^*, e_2^*, \ldots, e_n^*$ by $V_{\mathbb{Z}}^*$. Then $V_{\mathbb{Z}}^*$ is $\mathfrak{h}_{\mathbb{Z}}$-invariant and we can identify $V_{\mathbb{Z}}^* \otimes_{\mathbb{Z}} K$ and $V^*$ as $G$-modules.

Here the action of $G$ on $V^*$ is given by $(g \cdot f)(v) = f(g^{-1}v)$ for all $g \in G$, $f \in V^*$ and $v \in V$.

By abuse of notation we identify the basis $(e_i \otimes 1)$ of $V$ with $(e_i)$, and the basis $(e_i^* \otimes 1)$ of $V^*$ with $(e_i^*)$.

Let $1 \leq k < n - k \leq l$. Now the Lie algebra $\mathfrak{sl}(V_{\mathbb{C}})$ acts naturally on $\wedge^k(V_{\mathbb{C}})$ by

$$X \cdot (v_1 \wedge \cdots \wedge v_k) = \sum_{i=1}^{k} v_1 \wedge \cdots \wedge \hat{v}_{i-1} \wedge X v_i \wedge v_{i+1} \wedge \cdots \wedge v_k$$

for all $X \in \mathfrak{sl}(V_{\mathbb{C}})$ and $v_i \in V_{\mathbb{C}}$. Similarly we have an action of $\mathfrak{sl}(V_{\mathbb{C}})$ on $\wedge^k(V_{\mathbb{C}}^*)$. Furthermore, $\mathfrak{sl}(V_{\mathbb{C}})$ acts on $\wedge^k(V_{\mathbb{C}}) \otimes \wedge^k(V_{\mathbb{C}}^*)$ by $X \cdot (v \otimes w) = Xv \otimes w + v \otimes Xw$ for all $X \in \mathfrak{sl}(V_{\mathbb{C}})$, $v \in \wedge^k(V_{\mathbb{C}})$ and $w \in \wedge^k(V_{\mathbb{C}}^*)$. Here $\wedge^k(V_{\mathbb{Z}}) \otimes \wedge^k(V_{\mathbb{Z}}^*)$ is an $\mathfrak{h}_{\mathbb{Z}}$-invariant lattice in $\wedge^k(V_{\mathbb{C}}) \otimes \wedge^k(V_{\mathbb{C}}^*)$, and we can and will identify $\wedge^k(V_{\mathbb{Z}}) \otimes \wedge^k(V_{\mathbb{Z}}^*) \otimes_{\mathbb{Z}} K$ and $\wedge^k(V) \otimes \wedge^k(V^*)$ as $G$-modules.

The diagonal matrices in $G$ form a maximal torus $T$. Then a basis of weight vectors of $\wedge^k(V) \otimes \wedge^k(V^*)$ is given by the elements $(e_{i_1} \wedge \cdots \wedge e_{i_k}) \otimes (e_{j_1}^* \wedge \cdots \wedge e_{j_k}^*)$, where $1 \leq i_1 < \cdots < i_k \leq n$ and $1 \leq j_1 < \cdots < j_k \leq n$. The basis vector $(e_1^* \wedge \cdots \wedge e_k^*) \otimes (e_n^* \wedge e_{n-1}^* \wedge \cdots \wedge e_{n-k+1}^*)$ has weight $\omega_k + \omega_{n-k}$.
The natural dual pairing between $\wedge^k(V)$ and $\wedge^k(V^*)$ (see for example [FH91 B.3, pg.475-476]) induces a $G$-invariant symmetric form $\langle -,- \rangle$ on $\wedge^k(V) \otimes \wedge^k(V^*)$. If $x = (v_1 \wedge \cdots \wedge v_k) \otimes (f_1 \wedge \cdots \wedge f_k)$ and $y = (w_1 \wedge \cdots \wedge w_k) \otimes (g_1 \wedge \cdots \wedge g_k)$, with $v_i, w_j \in V$ and $f_i, g_j \in V^*$, we define

$$\langle x, y \rangle = \det(f_i(w_j))_{1 \leq i,j \leq k} \det(g_i(v_j))_{1 \leq i,j \leq k}.$$

Let $b = (e_{i_1} \wedge \cdots \wedge e_{i_k}) \otimes (e_{j_1}^* \wedge \cdots \wedge e_{j_k}^*)$ and $b' = (e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*) \otimes (e_{j_1} \wedge \cdots \wedge e_{j_k})$ be two basis elements of $\wedge^k(V) \otimes \wedge^k(V^*)$. Then

$$\langle b, b' \rangle = \begin{cases} \pm 1, & \text{if } \{j_1, \cdots, j_k\} = \{j_1', \cdots, j_k'\} \text{ and } \{i_1, \cdots, i_k\} = \{j_1', \cdots, j_k'\} \\ 0, & \text{otherwise.} \end{cases}$$

Therefore the form $\langle -,- \rangle$ on $\wedge^k(V) \otimes \wedge^k(V^*)$ is non-degenerate. In precisely the same way we can find a basis of weight vectors for $\wedge^k(V) \otimes \wedge^k(V^*)$ and define a symmetric form $\langle -,- \rangle$ on $\wedge^k(V) \otimes \wedge^k(V^*)$.

We can find the Weyl module $V(\omega_k + \omega_{n-k})$ as a submodule of $\wedge^k(V) \otimes \wedge^k(V^*)$, as shown by the following lemma (cf. Lemma 3.14).

**Lemma 5.10.** Let $1 \leq k < n - k \leq l$, and let $W$ be the $G$-submodule of $\wedge^k(V) \otimes \wedge^k(V^*)$ generated by $v^+ = (e_1 \wedge \cdots \wedge e_k) \otimes (e_{n-1}^* \wedge \cdots \wedge e_{n-k+1}^*)$. Then $W$ is isomorphic to the Weyl module $V(\omega_k + \omega_{n-k})$.

**Proof.** It is a general fact about Weyl modules that $V(\lambda) \otimes V(\mu)$ always has $V(\lambda + \mu)$ as a submodule. For simple groups of classical type (in particular, for our $G$ of type $A_l$) this follows from results proven first by Lakshmibai et. al. [LMS79 Theorem 2 (b)] or from a more general result of Wang [Wan82, Theorem B, Lemma 3.1]. For other types, the fact is a consequence of results due to Donkin [Don85] (all types except $E_7$ and $E_8$ in characteristic two) or Mathieu [Mat90] (in general). In any case, now the weight $\lambda + \mu$ occurs with multiplicity 1 in $V(\lambda) \otimes V(\mu)$, so any vector of weight $\lambda + \mu$ in $V(\lambda) \otimes V(\mu)$ will generate a submodule isomorphic to $V(\lambda + \mu)$.

To prove our lemma, note that $\wedge^k(V) = L(\omega_k)$ and $\wedge^k(V^*) = L(\omega_{n-k})$. Furthermore, $\omega_k$ and $\omega_{n-k}$ are minuscule weights, so $L(\omega_k) = V(\omega_k)$ and $L(\omega_{n-k}) = V(\omega_{n-k})$. Here $v^+$ is a vector of weight $\omega_k + \omega_{n-k}$ in $\wedge^k(V) \otimes \wedge^k(V^*)$, so the claim follows from the result in the previous paragraph. \qed

For all $1 \leq k < n - k \leq l$, we will identify $V(\omega_k + \omega_{n-k})$ with the submodule $W$ from Lemma 5.10. Set $V(\omega_k + \omega_{n-k})_Z = \mathbb{Z} v^+$ where $v^+$ is as in Lemma 5.10. Then we can and will identify $V(\omega_k + \omega_{n-k})_Z \otimes Z K$ and $V(\omega_k + \omega_{n-k})_Z$ as $G$-modules.

Note that a basis for the zero weight space of $\wedge^k(V) \otimes \wedge^k(V^*)$ is given by vectors of the form $(e_{i_1} \wedge \cdots \wedge e_{i_k}) \otimes (e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*)$, where $1 \leq i_1 < \cdots < i_k \leq n$. 25
We will need the following lemma, which gives a set of generators for the zero weight space of $V(\omega_k + \omega_{n-k})$ (cf. Lemma 3.15).

**Lemma 5.11.** Suppose that $1 \leq k < n - k \leq l$. Then the zero weight space of $V(\omega_k + \omega_{n-k})$ (thus also of $V(\omega_k + \omega_{n-k})$) is spanned by vectors of the form

$$\sum_{f_r \in \{j_1, k_s\}} (-1)^{|\{s \mid f_s = j_s\}|} (e_{f_1} \wedge \cdots \wedge e_{f_k}) \otimes (e_{f_1}^* \wedge \cdots \wedge e_{f_k}^*),$$

where $(k_1, \ldots, k_k)$ and $(j_1, \ldots, j_k)$ are sequences such that $1 \leq k_r < j_r \leq n$ for all $r$, and $j_r, k_r \neq j_{r'}, k_{r'}$ for all $r \neq r'$.

**Proof.** We give a proof somewhat similar to that of Lemma 3.15 given in [Jan73, pg. 40, Lemma 6]. The zero weight space of $V(\omega_k + \omega_{n-k})$ is generated by elements of the form

$$\prod_{\alpha \in \Phi^+} \frac{X_{-\alpha}^{k_\alpha}}{k_\alpha \alpha^+},$$

where $k_\alpha$ are non-negative integers, $\sum_{\alpha \in \Phi^+} k_\alpha \alpha = \omega_k + \omega_{n-k}$ and the product is taken with respect to some fixed ordering of the positive roots. For $\alpha \in \Phi^+$ such that $X_{-\alpha} v^+ = 0$, we can assume $k_\alpha = 0$ by choosing a suitable ordering of $\Phi^+$. Therefore we will assume that if $k_\alpha > 0$, then $\alpha$ is of one of the following types.

(I) $\alpha = \varepsilon_i - \varepsilon_j$ for $1 \leq i \leq k$ and $k + 1 \leq j \leq n - k$.

(II) $\alpha = \varepsilon_i - \varepsilon_j$ for $k + 1 \leq i \leq n - k$ and $n - k + 1 \leq j \leq n$.

(III) $\alpha = \varepsilon_i - \varepsilon_j$ for $1 \leq i \leq k$ and $n - k + 1 \leq j \leq n$.

Note that the $X_{-\alpha}$ with $\alpha$ of type (I) commute with each other. The same is also true for types (II) and (III).

Writing $\omega_k + \omega_{n-k}$ in terms of the simple roots, we see that $\omega_k + \omega_{n-k}$ is equal to

$$\alpha_1 + 2\alpha_2 + \cdots + (k-1)\alpha_{k-1} + k\alpha_k + \cdots + k\alpha_{n-k} + (k-1)\alpha_{n-k+1} + \cdots + \alpha_{n-1} \quad (*)$$

Then from the fact that $\sum_{\alpha \in \Phi^+} k_\alpha \alpha = \omega_k + \omega_{n-k}$ we will deduce the following.

1. For any $1 \leq i \leq k$, there exists a unique $\alpha \in \Phi^+$ such that $k_\alpha = 1$ and $\alpha = \varepsilon_i - \varepsilon_{j'}$ for some $k + 1 \leq j' \leq n$.

2. For any $n - k + 1 \leq j \leq n$ there exists a unique $\alpha \in \Phi^+$ such that $k_\alpha = 1$ and $\alpha = \varepsilon_{i'} - \varepsilon_j$ for some $1 \leq i' \leq n - k$.
For $i = 1$ and $j = n$ these claims are clear, since $\alpha_1$ and $\alpha_{n-1}$ occur only once in the expression (*) of $\omega_k + \omega_{n-k}$ as a sum of simple roots. For $i > 1$ claim (1) follows by induction, since $\varepsilon_i - \varepsilon_{j'}$ contributes $\alpha_i + \alpha_{i+1} + \ldots + \alpha_k + \ldots + \alpha_{j'-1}$ to the expression (*) of $\omega_k + \omega_{n-k}$ as a sum of simple roots. Claim (2) follows similarly for $j < n$.

In particular, it follows from claims (1) and (2) that $k_\alpha \in \{0, 1\}$ for all $\alpha \in \Phi^+$. Let $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ be the sets of $\alpha \in \Phi^+$ of type (I), (II) and (III) respectively such that $k_\alpha = 1$. It follows from claim (1) that $|\mathcal{A}_1| + |\mathcal{A}_3| = k$ and from claim (2) that $|\mathcal{A}_2| + |\mathcal{A}_3| = k$, so then $|\mathcal{A}_1| = |\mathcal{A}_2| = k'$ for some $0 \leq k' \leq k$. Thus we can write

$$\mathcal{A}_1 = \{\varepsilon_i - \varepsilon_w, \ldots, \varepsilon_{i_{k'}} - \varepsilon_{w_{k'}}\}$$
$$\mathcal{A}_2 = \{\varepsilon_z, \ldots, \varepsilon_{z_{j}}, \ldots, \varepsilon_{z_{j}} - \varepsilon_{j_{k'}}\}$$
$$\mathcal{A}_3 = \{\varepsilon_{i_{k'+1}} - \varepsilon_{j_{k'+1}}, \ldots, \varepsilon_{i_k} - \varepsilon_{j_k}\}$$

where $1 \leq i_r \leq k$ and $n - k + 1 \leq j_r \leq n$ for all $1 \leq r \leq k$, and $k + 1 \leq w_r, z_r \leq n - k$ for all $1 \leq r \leq k'$. Furthermore, $\{i_1, \ldots, i_k\} = \{1, 2, \ldots, k\}$ and $\{j_1, \ldots, j_k\} = \{n - k + 1, \ldots, n - 1, n\}$.

We choose the ordering of $\Phi^+$ so that

$$\prod_{\alpha \in \Phi^+} \frac{X^{k_\alpha}}{k_\alpha!} = \prod_{\alpha \in \mathcal{A}_1} X_{-\alpha} \prod_{\alpha \in \mathcal{A}_2} X_{-\alpha} \prod_{\alpha \in \mathcal{A}_3} X_{-\alpha}.$$

It is another consequence of $\sum_{\alpha \in \Phi^+} k_\alpha \alpha = \omega_k + \omega_{n-k}$ that $\{w_1, \ldots, w_{k'}\} = \{z_1, \ldots, z_{k'}\}$. Indeed, in the expression (*) of $\omega_k + \omega_{n-k}$ as a sum of simple roots, for any $k + 1 \leq r \leq n - k$ the simple root $\alpha_r$ occurs $k$ times. On the other hand, the $\alpha$ of types (I), (II), (III) that contribute to $\alpha_r$ in the sum are precisely those of type (I) or (III) with $j > r$ (total of $k - |\{r' : w_{r'} \leq r\}|$), and those of type (II) with $j \leq r$ (total of $|\{r' : z_{r'} \leq r\}|$).

Therefore in the sum $\sum_{\alpha \in \Phi^+} k_\alpha \alpha$, the contribution to $\alpha_r$ is equal to $k - |\{r' : w_{r'} \leq r\}| + |\{r' : z_{r'} \leq r\}|$. Since this has to be equal to $k$, we get $|\{r' : z_{r'} \leq r\}| = |\{r' : w_{r'} \leq r\}|$ for all $k + 1 \leq r \leq n - k$, which implies $\{w_1, \ldots, w_{k'}\} = \{z_1, \ldots, z_{k'}\}$.

Then since the $X_{-\alpha}$ with $\alpha$ of type (II) commute with each other, we may assume that $z_r = w_r$ for all $1 \leq r \leq k'$. Denote $w = \prod_{r=1}^{k'} E_{w_r,i_r} v^+$. A straightforward computation shows that $w = (e_{\pi(1)} \wedge \cdots \wedge e_{\pi(k)}) \otimes (e_n^* \wedge e_{n-1}^* \wedge \cdots \wedge e_{n-k'}^*)$.
\[ \cdots \wedge e_{n-k+1}^* \), where \( \pi(r) = w_{r'} \) if \( r = i_{r'} \) and \( \pi(r) = r \) otherwise. Now
\[
\prod_{\alpha \in \Phi^+} \frac{X_{\alpha}}{k_{\alpha}!} v^+ = \prod_{r=k'+1}^{k} E_{j_r,i_r} \prod_{r=1}^{k'} E_{j_{r'},w_{r'}} v^+ \\
= \prod_{r=k'+1}^{k} E_{j_r,i_r} \prod_{r=1}^{k'} E_{j_{r'},w_{r'}} w \\
= \prod_{r=1}^{k} E_{j_r,k_r} w
\]
where \( (k_1, \ldots, k_k) = (w_1, \ldots, w_{k'}, i_{k'+1}, \ldots, i_k) \). In the last equality we just combine the terms, and this makes sense since \( X_{-\alpha} \) of type (III) commute with those of type (II).

Computing the expression \( \prod_{r=1}^{k} E_{j_r,k_r} w \), we see that it is equal to a sum of \( 2^k \) distinct elements of \( \wedge^k(V_Z) \otimes \wedge^k(V_Z^*) \), with each summand being equal to \( w \) transformed in the following way:

- For all \( 1 \leq s \leq k' \), replace \( e_{w_s'} \) by \( e_{j_s} \), or replace \( e_{j_s}^* \) by \( -e_{w_s'}^* \).
- For all \( k'+1 \leq s \leq k \), replace \( e_{i_s} \) by \( e_{j_s} \), or replace \( e_{j_s}^* \) by \( -e_{i_s}^* \).

For this we conclude that up to a sign, \( \prod_{\alpha \in \Phi^+} \frac{X_{\alpha}}{k_{\alpha}!} v^+ \) is as in the statement of the lemma, with sequences \( (k_1, \ldots, k_k) \) and \( (j_1, \ldots, j_k) \) as defined here. \( \square \)

**Lemma 5.12.** Suppose that \( 1 \leq k < n - k \leq l \). Then the vector
\[
\sum_{1 \leq i_1 < \cdots < i_k \leq n} (e_{i_1} \wedge \cdots \wedge e_{i_k}) \otimes (e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*)
\]
is fixed by the action of \( G \) on \( \wedge^k(V) \otimes \wedge^k(V^*) \). Furthermore, any \( G \)-fixed point in \( \wedge^k(V) \otimes \wedge^k(V^*) \) is a scalar multiple of \( \gamma \).

**Proof.** (cf. Lemma 3.16) The fact that \( \gamma \) is fixed by \( G \) is an exercise in linear algebra. We will give a proof for convenience of the reader. For this we first need to introduce some notation. Let \( A \) be an \( n \times n \) matrix with entries in \( K \) and denote the entry on \( i \)th row and \( j \)th column of \( A \) by \( A_{i,j} \). For indices \( 1 \leq i_1 < \cdots < i_k \leq n \) and \( 1 \leq j_1 < \cdots < j_k \leq n \), set \( I = \{i_1, \ldots, i_k\} \) and \( J = \{j_1, \ldots, j_k\} \). Then the \( k \times k \) minor of \( A \) defined by \( I \) and \( J \) is the determinant of the \( k \times k \) matrix \( (A_{i_r,j_t}) \). We denote this minor by \( [A]_{I,J} \). The following relation between minors of matrices \( A \), \( B \), and \( AB \) (similar to the matrix multiplication rule) is a special case of the Cauchy-Binet formula. A proof can be found in [BW89, 4.6, pg. 208-214].
Proposition (Cauchy-Binet formula). Let $A$ and $B$ be $n \times n$ matrices. For any $k$-element subsets $I, J$ of $\{1, \ldots, n\}$, we have

$$[AB]_{I,J} = \sum_T [A]_{I,T} [B]_{T,J}$$

where the sum runs over all $k$-element subsets $T$ of $\{1, \ldots, n\}$.

Consider $A \in \text{GL}(V)$ as a matrix with respect to the basis $e_1, \ldots, e_n$ of $V$. Now for any $1 \leq k \leq n$, the matrix $A$ acts on the exterior power $\wedge^k(V)$ by $A \cdot (v_1 \wedge \cdots \wedge v_k) = Av_1 \wedge \cdots \wedge Av_k$ for all $v_i \in V$. For $I = \{i_1, \ldots, i_k\}$ with $1 \leq i_1 < \cdots < i_k \leq n$, denote $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$. A straightforward calculation shows that $A \cdot e_I = \sum_J [A]_{J,I} e_J$, where the sum runs over all $k$-element subsets $J$ of $\{1, \ldots, n\}$.

With respect to the dual basis $e_1^*, \ldots, e_n^*$ of $V^*$, the action of $A$ on $V^*$ has matrix $(A^t)^{-1}$, where $A^t$ is the transpose of $A$. If we denote $e_I^* = e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*$ as before, then $A \cdot e_I^* = \sum_J [(A^t)^{-1}]_{J,I} e_J^* = \sum_J [A^{-1}]_{J,I} e_J^*$ where the sum runs over all $k$-element subsets $J$ of $\{1, \ldots, n\}$.

We are now ready to prove that $A$ fixes the vector $\gamma$. Note that $\gamma = \sum_I e_I \otimes e_I^*$, where the sum runs over all $k$-element subsets $I$ of $\{1, \ldots, n\}$. From the observations before, we see that $A \cdot \gamma$ is equal to

$$\sum_I \sum_{J,J'} [A]_{I,J} [A^{-1}]_{J,J'} e_J \otimes e_J^* = \sum_{J,J'} \sum_I [A]_{I,J} [A^{-1}]_{J,J'} e_J \otimes e_J^*$$

where the sums run over $k$-element subsets $I$, $J$, and $J'$ of $\{1, \ldots, n\}$. From the Cauchy-Binet formula, we have $\sum_I [A]_{I,J} [A^{-1}]_{J,J'} = [1]_{J,J'} = 0$ if $J \neq J'$ and $1$ if $J = J'$. Therefore $A \cdot \gamma = \gamma$.

To show that $\gamma$ is a unique $G$-fixed point up to a scalar, note first that any $G$-fixed point must have weight zero. Recall also that the zero weight space of $\wedge^k(V)$ has basis

$$\mathcal{B} = \{(e_i^* \wedge \cdots \wedge e_i^*) : 1 \leq i_1 < \cdots < i_k \leq n\}.$$

Now the group $\Sigma_n$ of permutations of $\{1, 2, \ldots, n\}$ acts on $V$ by $\sigma \cdot e_i = e_{\sigma(i)}$ for all $\sigma \in \Sigma_n$. This gives an embedding $\Sigma_n \leq \text{GL}(V)$. Note that then $\sigma \cdot e_I^* = e_{\sigma(I)}^*$ for all $\sigma \in \Sigma_n$, so it follows that $\Sigma_n$ acts on $\mathcal{B}$.

For $\sigma \in \Sigma_n$ we have $\det \sigma = 1$ if and only if $\sigma$ is an even permutation, so we get an embedding $\text{Alt}(n) < G$ for the alternating group. It is well known that $\text{Alt}(n)$ is $(n-2)$-transitive, so $\text{Alt}(n)$ acts transitively on $\mathcal{B}$ since $k \leq n - 2$. Thus any $\text{Alt}(n)$-fixed point in the linear span of $\mathcal{B}$ must be a scalar multiple of $\sum_{b \in \mathcal{B}} b = \gamma$.  \[ \square \]
We now begin the proof of Theorem 5.1. For the rest of this section, we will make the following assumption.

Assume that $\text{char } K = 2$.

Let $1 \leq k < n - k \leq l$ and suppose that $L(\omega_k + \omega_{n-k})$ is not orthogonal. Then by Proposition 2.2 (iii) we have $H^1(G, L(\omega_k + \omega_{n-k})) \neq 0$, so by Corollary 5.5 we have $k = 2^i$ for some $i \geq 0$ and $n + 1 \equiv 2^i + t \mod 2^{i+1}$ for some $0 \leq t < 2^i$. What remains is to determine when $L(\omega_k + \omega_{n-k})$ is orthogonal for such $k$. The main argument is the following lemma (cf. Lemma 3.17), which reduces the question to the evaluation of the invariant quadratic form on $V(\omega_k + \omega_{n-k})$ on a particular $v \in V(\omega_k + \omega_{n-k})$.

**Lemma 5.13.** Let $n > 2^{i+1}$, where $i \geq 0$. Suppose that $n + 1 \equiv 2^i + t \mod 2^{i+1}$, where $0 \leq t < 2^i$. Define the vector $\gamma \in \wedge^2 (V) \otimes \wedge^2 (V^*)$ to be equal to

$$\sum_{1 \leq i_1 < \cdots < i_{2^i} \leq n-t} (e_{i_1} \wedge \cdots \wedge e_{i_{2^i}}) \otimes (e_{i_1}^* \wedge \cdots \wedge e_{i_{2^i}}^*).$$

Then

(i) $\gamma$ is in $V(\omega_2i + \omega_{n-2i})$ and is a fixed point for the subgroup $A_{2^i-t} < G$,

(ii) $\gamma$ is in $\text{rad } V(\omega_{2i} + \omega_{n-2i})$,

(iii) $L(\omega_{2i} + \omega_{n-2i})$ is orthogonal if and only if $Q(\gamma) = 0$, where $Q$ is a nonzero $G$-invariant quadratic form on $V(\omega_{2i} + \omega_{n-2i})$.

**Proof.** (i) Same as Lemma 3.17 (i). Apply Lemma 5.12, Lemma 5.10 and Corollary 5.8

(ii) Same as Lemma 3.17 (ii). Apply Lemma 5.11 and note that for $b = (e_{j_1} \wedge \cdots \wedge e_{j_{2^i}}) \otimes (e_{j_1}^* \wedge \cdots \wedge e_{j_{2^i}}^*)$ and $b' = (e_{k_1} \wedge \cdots \wedge e_{k_{2^i}}) \otimes (e_{k_1}^* \wedge \cdots \wedge e_{k_{2^i}}^*)$ we have

$$\langle b, b' \rangle = \begin{cases} 1, & \text{if } \{j_1, \ldots, j_{2^i}\} = \{k_1, \ldots, k_{2^i}\}; \\ 0, & \text{otherwise.} \end{cases}$$

(iii) Same as Lemma 3.17 (iii). Apply Theorem 5.5 to find a submodule $M \subseteq \text{rad } V(\omega_{2i} + \omega_{n-2i})$ such that $\text{rad } V(\omega_{2i+1})/M \cong K$. Each composition factor of $V(\omega_{2i} + \omega_{n-2i})$ occurs with multiplicity one by Theorem 5.4, so $M$ has no nontrivial composition factors. By Lemma 5.7 (i) the restriction $M \downarrow A_{2^i-t}$ has no trivial composition factors, and by (i) the vector $\gamma$ is fixed by $A_{2^i-t}$. Thus $\gamma \notin M$ and then $\text{rad } V(\omega_{2i} + \omega_{n-2i}) = \{0\}$. 

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\( \langle \gamma \rangle \oplus M \) as \( A_{l-r} \)-modules. As in Lemma 3.17 (iii), we see that \( L(\omega_2, + \omega_{n-2}) \) is orthogonal if and only if \( Q(\gamma) = 0 \) for a nonzero \( G \)-invariant quadratic form \( Q \) on \( V(\omega_2, + \omega_{n-2}) \). \(
\)

5.3 Computation of a quadratic form \( Q \) on \( V(\omega_r + \omega_{n-r}) \)

To finish the proof of Theorem 5.1 we still have to compute \( Q(\gamma) \) for the vector \( \gamma \) from Lemma 5.13.

We retain the notation and assumptions from the previous subsection. Let \( 1 \leq k < n - k \leq l \). Now the form \( \langle - , - \rangle_Z \) on \( \Lambda^k(V_Z) \otimes \Lambda^k(V_Z^*) \) induces a quadratic form \( q_Z \) on \( \Lambda^k(V_Z) \otimes \Lambda^k(V_Z^*) \) by \( q_Z(x) = \langle x, x \rangle \). We will use this form to find a nonzero \( G \)-invariant quadratic form on \( V(\omega_k + \omega_{n-k}) = V(\omega_k + \omega_{n-k})_Z \otimes K \).

**Lemma 5.14.** We have \( q_Z(V(\omega_k + \omega_{n-k})_Z) \subseteq 2\mathbb{Z} \) and \( q_Z(V(\omega_k + \omega_{n-k})_Z) \not\subseteq 4\mathbb{Z} \).

**Proof.** Same as Lemma 3.18 but with \( \alpha = (e_1 \wedge \cdots \wedge e_k) \otimes (e^*_n \wedge e^*_{n-1} \wedge \cdots \wedge e^*_{n-k+1}) \) and \( \beta = (e_n \wedge e_{n-1} \wedge \cdots \wedge e_{n-k+1}) \otimes (e^*_1 \wedge \cdots \wedge e^*_k) \).

Therefore \( Q = \frac{1}{2} q_Z \otimes K \) defines a nonzero \( G \)-invariant quadratic form on \( V(\omega_k + \omega_{n-k}) \) with polarization \( \langle - , - \rangle \). As in Section 3.3 we have \( Q(\gamma) = \frac{1}{2} \binom{n-r}{2} \) for the vector \( \gamma \in \Lambda^2(V) \otimes \Lambda^2(V^*) \) from Lemma 5.13. Finally applying Lemma 3.19 completes the proof Theorem 5.1.

6 Simple groups of exceptional type

In this section, let \( G \) be a simple group of exceptional type and assume that \( \text{char } K = 2 \). We will give some results about the orthogonality of irreducible representations of \( G \). For \( G \) of type \( G_2 \) or \( F_4 \) we give a complete answer. For types \( E_6 \), \( E_7 \), and \( E_8 \) we only have results for some specific representations, given in Table 6.1 below and proven at the end of this section. For irreducible representations occurring in the adjoint representation of \( G \), answers were given earlier by Gow and Willems in [GW95] Section 3.

**Proposition 6.1.** Let \( G = G_2 \) and let \( V \) be a non-trivial irreducible representation of \( G \). Then \( V \) is not orthogonal if and only if \( V \) is a Frobenius twist of \( L_G(\omega_1) \).

**Proof.** In view of Remark 2.5 it will be enough to consider \( V = L_G(\lambda) \) with \( \lambda \in X(T)^+ \) a 2-restricted dominant weight. If \( \lambda = \omega_2 \) or \( \lambda = \omega_1 + \omega_2 \), then \( V_G(\lambda) = L_G(\lambda) \) and so \( V \) is orthogonal by Proposition 2.2.
What remains is to show that $V = L_G(\omega_1)$ not orthogonal. There are several ways to see this, for example since $\dim V = 6$ this could be done by a direct computation. Alternatively, note that the composition factors of $\wedge^2(V)$ are $L_G(\omega_2)$ and $L_G(0)$ [LS96 Proposition 2.10], so $H^1(G, \wedge^2(V)) = 0$. Then by [SW91 Proposition 2.7], the module $V$ is not orthogonal. For a third proof, note that the action of a regular unipotent $u \in G$ on $V$ has a single Jordan block [Sup95 Theorem 1.9], but no such element exists in $\text{SO}(V)$ [LS12 Proposition 6.22].

The following lemma will be useful throughout this section to show that certain representations are orthogonal.

**Lemma 6.2.** Let $V$ be a nontrivial, self-dual and irreducible $G$-module. Suppose that one of the following holds:

(i) $\dim V \equiv 2 \mod 4$, and $\wedge^2(V)$ has exactly one trivial composition factor as a $G$-module.

(ii) $\dim V \equiv 0 \mod 8$, and $\wedge^2(V)$ has exactly two trivial composition factors as a $G$-module.

Then any nontrivial composition factor of $\wedge^2(V)$ occurring with odd multiplicity is an orthogonal $G$-module.

**Proof.** Since $V$ is nontrivial, we can assume $G < \text{Sp}(V)$ by Lemma 2.1. If (i) holds, then by applying results in Section 3.2 (or [McN98 Lemma 4.8.2]) we can find a vector $\gamma \in \wedge^2(V)$ such that $\wedge^2(V) = Z \oplus \langle \gamma \rangle$ as an $\text{Sp}(V)$-module. Here $Z$ is irreducible of highest weight $\omega_2$ for $\text{Sp}(V)$, so by Proposition 3.1 (see Example 3.2) the module $Z$ is orthogonal for $\text{Sp}(V)$. Therefore $Z$ is an orthogonal $G$-module with no trivial composition factors. From this [GW95 Lemma 1.3] shows that any composition factor of $Z$ with odd multiplicity is an orthogonal $G$-module.

In case (ii), the assumption on $\dim V$ implies (for example by [McN98 Lemma 4.8.2]) that there exist $\text{Sp}(V)$-submodules $Z' \subseteq Z \subseteq \wedge^2(V)$ such that $\dim Z' = \dim \wedge^2(V)/Z = 1$. Furthermore, $Z/Z'$ is an irreducible $\text{Sp}(V)$-module with highest weight $\omega_2$, so by Proposition 3.1 (see Example 3.2) the module $Z/Z'$ is orthogonal for $\text{Sp}(V)$. Therefore $Z/Z'$ is an orthogonal $G$-module with no trivial composition factors, so by [GW95 Lemma 1.3] any composition factor of $Z/Z'$ with odd multiplicity is an orthogonal $G$-module.

**Proposition 6.3.** Let $G = F_4$ and let $V$ be a non-trivial irreducible representation of $G$. Then $V$ is orthogonal.
Proof. Let \( \tau : G \to G \) be the exceptional isogeny of \( G \) as given in [Ste68, Theorem 28]. Then \( L_G(a_1 \omega_1 + a_2 \omega_2 + a_3 \omega_3 + a_4 \omega_4)^T \cong L_G(a_4 \omega_1 + a_3 \omega_2 + 2a_2 \omega_3 + 2a_1 \omega_4) \), and by Steinberg’s tensor product theorem this is isomorphic to \( L_G(a_4 \omega_1 + a_2 \omega_2) \otimes L(a_2 \omega_3 + a_1 \omega_4)^F \). Thus by lemmas [2.1] and [2.3], it is enough to prove the claim in the case where \( V = L_G(\lambda) \) with \( \lambda = a_3 \omega_3 + a_4 \omega_4 \) a 2-restricted dominant weight. Now for \( \lambda = \omega_4 \) and \( \lambda = \omega_3 + \omega_4 \), we have \( V_G(\lambda) = L_G(\lambda) = V \) and so \( V \) is orthogonal by Proposition [2.2] (iv).

What remains is to show that \( L_G(\omega_3) \) is orthogonal. Let \( W = L_G(\omega_1) \). Now \( \dim W = 26 \), so by Lemma [6.2] (i), it will be enough to prove that \( \wedge^2(W) \) has exactly one trivial composition factor and that \( L_G(\omega_3) \) occurs in \( \wedge^2(W) \) with odd multiplicity.

We have \( V_G(\omega_4) = L_G(\omega_4) \) and then by a computation with Magma [BCP97] (or [Don85, 7.4.3, pg. 98]) the \( G \)-character of \( \wedge^2(W) \) is given by \( \chi \wedge^2(W) = \chi V_G(\omega_4) + \chi V_G(\omega_3) \). Furthermore, from the data in [Lüb17], we can deduce \( V_G(\omega_1) = L_G(\omega_1)/L_G(\omega_4) \) and \( V_G(\omega_3) = L_G(\omega_3)/L_G(\omega_4)/L_G(0) \). Therefore as a \( G \)-module \( \wedge^2(W) \) has composition factors \( L_G(\omega_1), L_G(\omega_4), L_G(\omega_3), \) and \( L_G(0) \). \( \Box \)

| \( G \) | \( \lambda \) | \( L_G(\lambda) \) orthogonal? |
|---|---|---|
| \( E_6 \) | \( \omega_2 \) | yes |
| \( E_6 \) | \( \omega_4 \) | yes |
| \( E_6 \) | \( \omega_1 + \omega_6 \) | yes |
| \( E_7 \) | \( \omega_1 \) | no |
| \( E_7 \) | \( \omega_2 \) | yes |
| \( E_7 \) | \( \omega_5 \) | yes |
| \( E_7 \) | \( \omega_6 \) | yes |
| \( E_7 \) | \( \omega_7 \) | yes |
| \( E_8 \) | \( \omega_1 \) | yes |
| \( E_8 \) | \( \omega_7 \) | yes |
| \( E_8 \) | \( \omega_8 \) | yes |

Table 6.1: Orthogonality of some \( L_G(\lambda) \) for \( G \) of type \( E_6, E_7 \) and \( E_8 \).

We finish this section by verifying the information given in Table 6.1.

Suppose that \( G \) is of type \( E_6 \). We have \( V_G(\omega_2) = L_G(\omega_2) \) and \( V_G(\omega_1 + \omega_6) = L(\omega_1 + \omega_6)/L(\omega_2) \) by [Lüb17], so \( L_G(\omega_2) \) and \( L_G(\omega_1 + \omega_6) \) are orthogonal by Proposition 2.2 (iv). We show next that \( L(\omega_4) \) is orthogonal. Now \( W = ^4One can also construct a non-degenerate \( G \)-invariant quadratic form on \( L_G(\omega_4) \) explicitly by realizing it as the space of trace zero elements in the Albert algebra. The details of this construction can be found in [Wil09, 4.8.4, pg. 151-152].
$L_G(\omega_2)$ is self-dual and \( \dim W = 78 \), so by Lemma \ref{lem:orthogonal} (ii) it will be enough to prove that $\wedge^2(W)$ has exactly one trivial composition factor and that $L_G(\omega_1)$ occurs in $\wedge^2(W)$ with odd multiplicity.

Now $V_G(\omega_2) = L_G(\omega_2)$, and then a computation with Magma \cite{BCP97} (or \cite{Don85} 8.12, pg. 136) shows that $\text{ch} \wedge^2(W) = \text{ch} V_G(\omega_2) + \text{ch} V_G(\omega_1)$. From \cite{Lub17}, we can deduce that the composition factors of $V_G(\omega_1)$ are $L_G(\omega_1)$, $L_G(\omega_1 + \omega_6)$, $L_G(\omega_1 + \omega_8)$, $L_G(\omega_2)$ and $L_G(0)$. Thus $L_G(0)$ and $L_G(\omega_1)$ both occur exactly once as a composition factor of $\wedge^2(W)$.

Consider next $G$ of type $E_7$. We can assume that $G$ is simply connected. Then the Weyl module $V_G(\omega_1)$ is the Lie algebra of $G$, and $L_G(\omega_1)$ is not orthogonal by \cite[Theorem 3.4 (a)]{GW95}. We have $V_G(\omega_2) = L_G(\omega_2)$, $V_G(\omega_5) = L_G(\omega_5)/L_G(\omega_1 + \omega_7)$ and $V_G(\omega_7) = L_G(\omega_7)$ by the data in \cite{Lub17}. Therefore $L_G(\omega_2)$, $L_G(\omega_5)$ and $L_G(\omega_7)$ are orthogonal by Proposition \ref{prop:orthogonal}(iv).

We show that $L_G(\omega_6)$ is orthogonal. Now for $W = L_G(\omega_7)$ we have $\dim W = 56$, so by Lemma \ref{lem:orthogonal} (ii), it will be enough to prove that $\wedge^2(W)$ has exactly two trivial composition factors and that $L_G(\omega_6)$ occurs in $\wedge^2(W)$ with odd multiplicity. Now $V_G(\omega_7) = L_G(\omega_7)$, so by a computation with Magma \cite{BCP97}, we see $\text{ch} \wedge^2(W) = \text{ch} V_G(\omega_6) + \text{ch} V_G(0)$. From \cite{Lub17}, we see that $V_G(\omega_6)$ has composition factors $L_G(\omega_6)$, $L_G(\omega_1)$, $L_G(\omega_1)$ and $L_G(0)$. Therefore $\wedge^2(W)$ has exactly two trivial composition factors and $L_G(\omega_6)$ occurs exactly once as a composition factor.

For $G$ of type $E_8$, we have $V_G(\omega_8) = L_G(\omega_8)$ and so $L_G(\omega_8)$ is orthogonal by \ref{prop:orthogonal}(iv). Finally, we show that $L_G(\omega_1)$ and $L_G(\omega_7)$ are orthogonal. For $W = L_G(\omega_8)$ we have $\dim W = 248$, so by Lemma \ref{lem:orthogonal} (ii), it will be enough to prove that $\wedge^2(W)$ has exactly two trivial composition factors and that $L_G(\omega_1)$ and $L_G(\omega_7)$ occur in $\wedge^2(W)$ with odd multiplicity. By a computation with Magma \cite{BCP97}, we see that $\text{ch} \wedge^2(W) = \text{ch} V_G(\omega_8) + \text{ch} V_G(\omega_7)$. From \cite{Lub17}, we see that $V_G(\omega_7)$ has composition factors $L_G(\omega_1)$, $L_G(\omega_8)$, $L_G(0)$, and $L_G(0)$. Therefore $\wedge^2(W)$ has exactly two trivial composition factors and both $L_G(\omega_1)$ and $L_G(\omega_7)$ occur with multiplicity one.

## 7 Applications and further work

In this section, we describe consequences of some of our findings and propose some questions motivated by Problem \ref{prob:main}. Unless otherwise mentioned, we let $G$ be a simply connected algebraic group over $K$ and we assume that $\text{char } K = 2$. 
7.1 Connection with representations of the symmetric group

Denote the symmetric group on \(n\) letters by \(\Sigma_n\). We will describe a connection between orthogonality of certain irreducible \(K[\Sigma_n]\)-representations and the irreducible representations \(L(\omega_r)\) of \(\text{Sp}_{2l}(K)\). This is done by an application of Proposition 3.1 and various results from the literature. The result is not too surprising, since the representation theory of the symmetric group plays a key role in the representation theory of the modules \(L(\omega_r)\) of \(\text{Sp}_{2l}(K)\). For example, many of the results that we applied in the proof of Proposition 3.1 above are based on studying certain \(K[\Sigma_n]\)-representations associated with \(V(\omega_r)\).

It is well known that there exists an embedding \(\Sigma_{2l+1} < \text{Sp}_{2l}(K) = G\) for all \(l \geq 2\) (see e.g. [GK99] or [Tay92, Theorem 8.9]). Therefore if a representation \(V\) of \(G\) is orthogonal, it is clear that the same is true for the restriction \(V \downarrow \Sigma_{2l+1}\). We will proceed to show that the converse is also true when \(V = L(\omega_r)\) for \(2 \leq r \leq l\), which does not seem to be a priori obvious.

First of all, the following result due to Gow and Kleshchev [GK99, Theorem 1.11] gives the structure of \(L(\omega_r) \downarrow \Sigma_{2l+1}\).

**Theorem 7.1.** Let \(1 \leq r \leq l\). Then the restriction \(L(\omega_r) \downarrow \Sigma_{2l+1}\) is irreducible, and it is isomorphic to the irreducible \(K[\Sigma_n]\)-module \(D(2l+1-r,r)\) labeled by the partition \((2l+1-r,r)\) of \(2l+1\).

Now Gow and Quill have determined in [GQ04] when the irreducible \(K[\Sigma_n]\)-modules \(D(n-r,r)\) are orthogonal. Their result is the following.

**Theorem 7.2.** Let \(0 \leq r \leq n\). Then the \(K[\Sigma_n]\)-module \(D(n-r,r)\) is not orthogonal if and only if \(r = 2^j\), \(j \geq 0\) and \(n \equiv k \mod 2^{j+2}\) for some \(2^{j+1} + 2^j - 1 \leq k \leq 2^{j+2} - 2\).

In the case where \(n = 2l+1\), one can express the result in the following way.

**Corollary 7.3.** Let \(n = 2l+1\) and \(1 \leq r \leq l\). Then \(K[\Sigma_n]\)-module \(D(n-r,r)\) is not orthogonal if and only if \(r = 2^i+1\), \(i \geq 0\) and \(l+1 \equiv 2^{i+1} + 2^i + t\) mod \(2^{i+2}\) for some \(0 \leq t < 2^i\).

**Proof.** By Theorem 7.2, the module \(D(n-1,1)\) is not orthogonal if and only if \(2l+1 \equiv 2 \mod 4\), which never happens. Therefore \(D(n-1,1)\) is always orthogonal, as desired.

Consider then \(r > 1\). According to Theorem 7.2 if \(D(n-r,r)\) is not orthogonal, then \(r = 2^i\) for some \(j > 0\). In this case \(D(n-r,r)\) is not orthogonal if and only if \(2l+1 \equiv k \mod 2^{j+2}\) for some \(2^{j+1} + 2^j - 1 \leq k \leq 2^{j+2} - 2\). This

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is equivalent to saying that $2(l + 1) \equiv k \mod 2^{j+2}$ for some $2^{j+1} + 2^j \leq k \leq 2^{j+2} - 1$. Now this condition is equivalent to $l + 1 \equiv k \mod 2^{j+1}$ for some $2^j + 2^{j-1} \leq k \leq 2^{j+1} - 2$, giving the claim. ∎

Finally combining Theorem 7.1, Proposition 3.1 and Corollary 7.3 gives the following result.

**Proposition 7.4.** Let $2 \leq r \leq l$. Then $L(\omega_r)$ is orthogonal for $Sp_{2l}(k)$ if and only if $L(\omega_r) \downarrow \Sigma_{2l+1}$ is orthogonal for $\Sigma_{2l+1}$.

### 7.2 Reduction for Problem 1.1

To determine which irreducible $G$-modules are orthogonal, it is enough to consider $L_G(\lambda)$ with $\lambda \in X(T)^+ \cap$ a 2-restricted dominant weight (Remark 2.5). For groups of exceptional type, this leaves finitely many $\lambda$ to consider. For groups of classical type, we can further reduce the question to $G$ of type $A_l$ and type $C_l$. This follows from the next two lemmas. Note that in Lemma 7.5, we identify the fundamental dominant weights of $B_l$ and $C_l$ by abuse of notation.

**Lemma 7.5.** Let $\lambda = \sum_{i=1}^l a_i \omega_i$, where $l \geq 2$ and $a_i \in \{0, 1\}$ for all $1 \leq i \leq l$.

(i) Let $G$ be of type $B_l$ or $C_l$. If $a_l = 1$, then $V = L_G(\lambda)$ is orthogonal, except when $l = 2$ and $\lambda = \omega_2$.

(ii) The irreducible $B_l$-representation $L_{B_l}(\lambda)$ is orthogonal if and only if the irreducible $C_l$-representation $L_{C_l}(\lambda)$ is orthogonal.

**Proof.** (i) Let $\varphi : B_l \rightarrow C_l$ be the usual exceptional isogeny between simply connected groups of type $B_l$ and $C_l$ [Ste68, Theorem 28]. Then

$$L_{C_l}(\lambda)^{\varphi} \cong L_{B_l}(\sum_{i=1}^{l-1} a_i \omega_i + 2a_l \omega_l) \cong L_{B_l}(\sum_{i=1}^{l-1} a_i \omega_i) \otimes L_{B_l}(a_l \omega_l)^F$$

where the last equality follows by Steinberg’s tensor product theorem. Assume that $a_l = 1$. Note that a $C_l$-module $V$ is orthogonal if and only if $V^{\varphi}$ is an orthogonal $B_l$-module. Thus it follows from Lemma 2.4 that $L_{C_l}(\lambda)$ is orthogonal, except possibly when $\lambda = \omega_l$. Finally, we know that $L_{C_l}(\omega_l)$ is orthogonal if and only if $l \geq 3$ by Example 3.3. This proves the claim for $G$ of type $C_l$. 

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For $G$ of type $B_l$, let $\tau : C_l \rightarrow B_l$ be the usual exceptional isogeny between simply connected groups of type $C_l$ and $B_l$. Then 

$$L_{B_l}(\lambda)^r \cong L_{C_l}(\sum_{i=1}^{l-1} 2a_i \omega_i + a_l \omega_l) \cong L_{C_l}(\sum_{i=1}^{l-1} a_i \omega_i)^F \otimes L_{C_l}(a_l \omega_l),$$

and now the claim follows as in the type $C_l$ case.

(ii) If $a_l = 1$, the claim follows from (i). If $a_l = 0$, then $L_{C_l}(\lambda)^r \cong L_{B_l}(\lambda)$ and the claim follows since a $C_l$-module $V$ is orthogonal if and only if $V^r$ is an orthogonal $B_l$-module.

Lemma 7.6. Let $\lambda = \sum_{i=1}^{l} a_i \omega_i$, where $l \geq 4$ and $a_i \in \{0, 1\}$ for all $1 \leq i \leq l$.

(i) If $a_{l-1} \neq a_l$, then $L_{D_l}(\lambda)$ is orthogonal if it is self-dual.

(ii) If $a_{l-1} = a_l$, then $L_{D_l}(\lambda)$ is orthogonal if and only if $L_{C_l}(\sum_{i=1}^{l-1} a_i \omega_i)$ is orthogonal, except when $\lambda = \omega_1$.

Proof. (i) If $a_{l-1} \neq a_l$, then for example by [Hum72, Table 13.1] we see that for $G$ of type $D_l$, the weight $\lambda$ is not a sum of roots. Therefore $L_G(0)$ cannot be a composition factor of $V_G(\lambda)$, and thus $L_G(\lambda)$ is orthogonal by Proposition 2.2 (iv).

(ii) Suppose that $a_{l-1} = a_l$. Considering $D_l < C_l$ as the subsystem subgroup generated by long roots, we have $L_{C_l}(\sum_{i=1}^{l-1} a_i \omega_i) \downarrow D_l \cong L_{D_l}(\lambda)$ by [Sei87, Theorem 4.1]. Now the claim follows from Lemma 4.1.

7.3 Application to maximal subgroups of classical groups

In this subsection only, we allow $\text{char } K$ to be arbitrary.

As mentioned in the introduction, one motivation for Problem 1.1 is in the study of maximal closed connected subgroups of classical groups. Let $\text{Cl}(V)$ be a classical simple algebraic group, that is, $\text{Cl}(V) = \text{SL}(V)$, $\text{Cl}(V) = \text{Sp}(V)$, or $\text{Cl}(V) = \text{SO}(V)$. Finding maximal closed connected subgroups of $\text{Cl}(V)$ can be reduced to the representation theory of simple algebraic groups. We proceed to explain how this is done. For more details, see [LS98] and [Sei87].

In [LS98], certain collections $\mathcal{C}_1, \ldots, \mathcal{C}_6$ of geometric subgroups were defined in terms of the natural module $V$ and its geometry. A reduction theorem due to Liebeck and Seitz [LS98, Theorem 4.1] implies that for a positive-dimensional maximal closed subgroup $X$ of $\text{Cl}(V)$ one of the following holds:
(i) $X$ belongs to some $\mathcal{C}_i$,

(ii) The connected component $X^o$ is simple, and $V \downarrow X^o$ is irreducible and tensor-indecomposable.

In particular, the reduction theorem implies the following.

**Theorem 7.7.** Let $X < \text{Cl}(V)$ be a subgroup maximal among the closed connected subgroups of $\text{Cl}(V)$. Then one of the following holds:

(i) $X$ is contained in a member of some $\mathcal{C}_i$,

(ii) $X$ is simple, and $V \downarrow X$ is irreducible and tensor-indecomposable.

The maximal closed connected subgroups in case (i) of Theorem 7.7 are well understood [Sei87, Theorem 3]. Furthermore, the maximal closed connected subgroups occurring in case (ii) of Theorem 7.7 can also be described. These were essentially determined by Seitz [Sei87] and Testerman [Tes88]. The result can be stated in the following theorem, which tells when an irreducible tensor-indecomposable subgroup is not maximal.

**Theorem 7.8.** Let $Y$ be a simple algebraic group and let $V$ be a non-trivial irreducible tensor-indecomposable $p$-restricted and rational $Y$-module. If $X$ is a closed proper connected subgroup of $Y$ such that $X$ is simple and $V \downarrow X$ is irreducible, then $(X, Y, V)$ occurs in [Sei87, Table 1].

To refine the characterization of maximal closed connected subgroups of $\text{Cl}(V)$ given in [Sei87, Theorem 3], one should determine which of $\text{SL}(V)$, $\text{Sp}(V)$ and $\text{SO}(V)$ contain $X$ and $Y$ in Theorem 7.8.

For example, let $Y$ be simple of type $D_5$ and let $X < Y$ be simple of type $B_4$ embedded in the usual way. Then for $V = L_Y(\omega_5)$ we have $V \downarrow X = L_X(\omega_4)$; this situation corresponds to entry $\text{IV}_{1}$ in [Sei87, Table 1]. Here $V$ is not self-dual as a $Y$-module, so $Y < \text{SL}(V)$ only. However, $V \downarrow X$ is self-dual and $X < \text{SO}(V)$ if $p \neq 2$, and $X < \text{SO}(V) < \text{Sp}(V)$ if $p = 2$ (see Table 2.1 and Theorem 4.2). In this situation $Y$ is maximal in $\text{SL}(V)$, while $X$ is maximal in $\text{SO}(V)$.

In fact, the results we have presented in this text allow one to determine for almost all $(X, Y, V)$ occurring in [Sei87, Table 1] whether $V \downarrow X$ and $V \downarrow Y$ are orthogonal, symplectic, both, or neither. If $p \neq 2$, then this is easily done using Table 2.1.

For $p = 2$, we list this information in Table 7.1 which is deduced as follows. Entry IV$_1$ is a consequence of Lemma 7.5, Example 3.4 and Lemma 7.6. In entry $S_3$, we have $V \downarrow Y = L_{C_3}(\omega_2)$ which is orthogonal by Example...
and thus $V \downarrow X$ is also orthogonal. In entry $S_4$, we have $V \downarrow Y = L_{C_3}(\omega_1 + \omega_2)$, which is orthogonal by Proposition 2.2(iv) since $V_{C_3}(\omega_1 + \omega_2)$ is irreducible. Entry $S_6$ follows from Lemma 2.4 and Lemma 7.6. In entries $S_7, S_8$, and $S_9$, we have $V \downarrow X = L_X(\lambda)$ and $V_X(\lambda)$ is orthogonal, so $V \downarrow X$ is orthogonal by Proposition 2.2(iv). Entries $MR_2$ and $MR_3$ follow from Proposition 6.3, which show that $V \downarrow Y$ is orthogonal. Entry $MR_5$ is a consequence of Lemma 7.5(i).

| No. | $X < Y$ | $V \downarrow X$ | $V \downarrow Y$ |
|-----|---------|-------------------|------------------|
| IV₁ | $B_l < D_{l+1}$ | orthogonal | $l + 1$ even: orthogonal $l + 1$ odd: not self-dual |
| $S_3$ | $G_2 < C_3$ | orthogonal | orthogonal |
| $S_4$ | $G_2 < C_3$ | orthogonal | orthogonal |
| $S_6$ | $B_{n_1} \cdots B_{n_k} < D_{l+\sum n_i}$ | orthogonal | $1 + \sum n_i$ even: orthogonal $1 + \sum n_i$ odd: not self-dual |
| $S_7$ | $A_3 < D_7$ | orthogonal | not self-dual |
| $S_8$ | $D_4 < D_{13}$ | orthogonal | not self-dual |
| $S_9$ | $C_4 < D_{13}$ | orthogonal | not self-dual |
| $MR_2$ | $D_4 < F_4$ | orthogonal | orthogonal |
| $MR_3$ | $C_4 < F_4$ | orthogonal | orthogonal |
| $MR_4$ | $D_l < C_l$ | ? | ? |
| $MR_5$ | $B_{n_1} \cdots B_{n_k} < B_{n_{l+\cdots+n_k}}$ | orthogonal | orthogonal |

Table 7.1: Invariant forms on $V \downarrow X$ and $V \downarrow Y$ for $(X, Y, V)$ occurring in [Sei87, Table 1] in the case $p = 2$.

What remains is the entry $MR_4$ from [Sei87, Table 1]. Here $X = D_l$ ($l \geq 4$) embedded in $Y = C_l$ as the subsystem subgroup of long roots, and we have $V = L_Y(\sum_{i=1}^{l-1} a_i \omega_i)$ with $a_i \in \{0, 1\}$, and $V \downarrow X = L_X(\sum_{i=1}^{l-2} a_i \omega_i + a_{l-1}(\omega_{l-1} + \omega_l))$. In this situation we do not know in general whether $V \downarrow Y$ and $V \downarrow X$ are orthogonal, but we do know that except in the case where $\sum_{i=1}^{l-1} a_i \omega_i = \omega_1$, it is true that $V \downarrow Y$ is orthogonal if and only if $V \downarrow X$ is orthogonal (Lemma 7.6). Using this fact and the information in Table 7.1, we can deduce the following result.

**Theorem 7.9.** Let $Y$ be a simple algebraic group and let $V$ be a non-trivial irreducible tensor-indecomposable $p$-restricted $Y$-module. If $X$ is a closed proper connected subgroup of $Y$ such that $X$ is simple and $V \downarrow X$ is irreducible, then one of the following holds.
(i) \( V \downarrow Y \) is not self-dual.

(ii) Both \( V \downarrow Y \) and \( V \downarrow X \) are orthogonal.

(iii) Neither of \( V \downarrow Y \) or \( V \downarrow X \) is orthogonal.

(iv) \( p = 2 \), \( X \) is of type \( D_l \), \( Y \) is of type \( C_l \) and \( V \) is the natural module of \( Y \).

### 7.4 Fundamental self-dual irreducible representations

Among the irreducible self-dual \( G \)-modules that are not orthogonal, so far the only ones that we know of are in some sense minimal among the self-dual irreducible modules of \( G \). We make this more precise in what follows, and pose the question whether any other examples can be found.

Recall that \( L_G(\lambda) \) is self-dual if and only if \( \lambda = -w_0(\lambda) \), where \( w_0 \) is the longest element in the Weyl group. We know that any dominant weight \( \lambda \in X(T)^+ \) can be written uniquely as a sum of fundamental dominant weights, that is, \( \lambda = \sum_{i=1}^l a_i \omega_i \) for unique integers \( a_i \geq 0 \). Now similarly, there exists a collection \( \mu_1, \ldots, \mu_t \in X(T)^+ \) such that \( \mu_i = -w_0(\mu_i) \) for all \( i \), and such that any \( \lambda \in X(T)^+ \) with \( \lambda = -w_0(\lambda) \) can be written uniquely as \( \sum_{i=1}^l a_i \mu_i \) with \( a_i \geq 0 \). For each simple type, these \( \mu_i \) are listed below.

- **Type \( A_l \) (\( l \) odd):** \( \mu_i = \omega_i + \omega_{i+1} - i \) for \( 1 \leq i \leq \frac{l-1}{2} \), and \( \mu_{\frac{l+1}{2}} = \omega_{\frac{l+1}{2}} \).

- **Type \( A_l \) (\( l \) even):** \( \mu_i = \omega_i + \omega_{l+1-i} \) for \( 1 \leq i \leq \frac{l}{2} \).

- **Types \( B_l, C_l, D_l \) (\( l \) even), \( G_2, F_4, E_7 \), and \( E_8 \):** \( \mu_i = \omega_i \) for \( 1 \leq i \leq \text{rank}\ G \).

- **Type \( D_l \) (\( l \) odd):** \( \mu_i = \omega_i \) for \( 1 \leq i \leq l-2 \), and \( \mu_{l-1} = \omega_{l-1} + \omega_l \).

- **Type \( E_6 \):** \( \mu_1 = \omega_1 + \omega_6, \mu_2 = \omega_2, \mu_3 = \omega_3 + \omega_5, \) and \( \mu_4 = \omega_4 \).

Currently the only known examples of non-trivial irreducible modules \( L_G(\lambda) \) that are self-dual and not orthogonal are of the form \( L_G(\mu_i) \). Are there any others?

**Problem 7.10.** Let \( \lambda \in X(T)^+ \) be 2-restricted and suppose that \( \lambda = \lambda_1 + \lambda_2 \), where \( \lambda_i \in X(T)^+ \) are nonzero and \( -w_0(\lambda_i) = \lambda_i \). Is \( L_G(\lambda) \) orthogonal?

If the answer to Problem 7.10 is yes, then our results would settle Problem 1.1 almost completely. Indeed, a positive answer to Problem 7.10 would show that any non-orthogonal self-dual irreducible representation of \( G \) must be
equal to a Frobenius twist of $L_G(\mu_i)$ for some $i$. Our results determine the orthogonality of $L_G(\mu_i)$ when $G$ is of classical type. The non-orthogonal ones for type $A_l$ are the $L_{A_l}(\omega_i + \omega_{i+1})$ described in Theorem 4.1. For $G$ of type $B_l$, $C_l$, or $D_l$, the non-orthogonal ones are $L_G(\omega_i)$ as described in Theorem 4.2, with the unique exception of $L_G(\omega_4 + \omega_5)$ for $G$ of type $D_5$ (arising from restriction of $L_{C_5}(\omega_4)$ to $G$).

Then a handful of $\mu_i$ still remain for exceptional types. For $G$ simple of exceptional type, the irreducibles $L_G(\mu_i)$ whose orthogonality was not decided in Section 6 are as follows.

- $L_G(\omega_3 + \omega_5)$ for $G$ of type $E_6$,
- $L_G(\omega_3)$ and $L_G(\omega_4)$ for $G$ of type $E_7$,
- $L_G(\omega_i)$ for $2 \leq i \leq 6$ for $G$ of type $E_8$.

In any case, a natural next step towards solving Problem 1.1 should be determining an answer to Problem 7.10. The methods we have used in this paper to solve Problem 1.1 for certain families of $L_G(\lambda)$ rely heavily on detailed information about the structure of the Weyl module $V_G(\lambda)$, which is not known in general. For small-dimensional representations the composition factors of $V_G(\lambda)$ can be found using the results of Lübeck given in [Lüb01] and [Lüb17]. However, in general this sort of information is not available, and in characteristic 2 the composition factors of $V_G(\lambda)$ are known only in a relatively few cases. For example, for $G$ of type $E_8$ we do not even know the dimension of $L_G(\omega_i)$ for all $i$ in characteristic 2.

### 7.5 Fixed point spaces of unipotent elements

We finish by a question about a possible orthogonality criterion for irreducible representations. Let $\varphi : G \to \text{SL}(V)$ be a non-trivial irreducible representation of $G$. Assume that $V$ is self-dual, so that $\varphi(G) < \text{Sp}(V)$ (Lemma 2.1). If $V$ is an orthogonal $G$-module, then $\varphi(G) < \text{O}(V)$ and so $\varphi(G) < \text{SO}(V)$ since $G$ is connected. Then for any unipotent element $u \in G$, the number of Jordan blocks of $\varphi(u)$ is even [LS12, Proposition 6.22]. In other words, for all $u \in G$ we have that $\dim V^u$ is even, where $V^u$ is the subspace of fixed points for $u$. Does the converse hold?

**Problem 7.11.** Let $V$ be a non-trivial irreducible self-dual representation of $G$. If $V$ is not orthogonal, does there exist a unipotent element $u \in G$ such that $\dim V^u$ is odd?
In Table 7.2, we have listed examples (without proof) of some non-orthogonal representations \( V \) of \( G \) for which the answer to Problem 7.11 is yes. If the answer to Problem 7.11 turns out to be yes, we would have an interesting criterion for an irreducible representation \( V \) of \( G \) to be orthogonal. A positive answer would show that the orthogonality of an irreducible representation can be decided from the properties of individual elements of \( G \).

| Type of \( G \) | \( V \) | Conjugacy class of \( u \) | \( \dim V^u \) |
|---------------|-------|-----------------|----------|
| \( A_l, l + 1 \equiv 2 \mod 4 \) | \( L_G(\omega_1 + \omega_l) \) | regular | \( 2l + 1 \) |
| \( C_l \) | \( L_G(\omega_1) \) | regular | 1 |
| \( C_l, l \equiv 2 \mod 4 \) | \( L_G(\omega_2) \) | regular | \( l - 1 \) |
| \( A_4 \) | \( L_G(\omega_2 + \omega_3) \) | \( A_3 \) | 19 |
| \( A_5 \) | \( L_G(\omega_2 + \omega_4) \) | regular | 21 |
| \( C_5 \) | \( L_G(\omega_4) \) | regular in \( D_5 \) | 21 |
| \( C_6 \) | \( L_G(\omega_4) \) | regular | 25 |
| \( G_2 \) | \( L_G(\omega_1) \) | regular | 1 |
| \( E_7 \) | \( L_G(\omega_1) \) | regular | 7 |

Table 7.2: Non-orthogonal irreducible representations \( V \) of \( G \) with examples of \( \dim V^u \) odd for some unipotent element \( u \in G \).

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