Anomalies and large $N$ limits in matrix string theory

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We study the loop expansion for the low energy effective action for matrix string theory. For long string configurations we find the result depends on the ordering of limits. Taking $g_s \to 0$ before $N \to \infty$ we find free strings. Reversing the order of limits however we find anomalous contributions coming from the large $N$ limit that invalidate the loop expansion. We then embed the classical instanton solution corresponding to a high energy string interaction into a long string configuration. We find the instanton has a loop expansion weighted by fractional positive powers of $N$. Finally we identify the scaling regime for which interacting long string configurations have a loop expansion with a well defined large $N$ limit. The limit corresponds to large “classical” strings and can be identified with the “dual” of the 't Hooft limit, $g_{SYM}^2 \sim N$.

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1. Introduction

One of the most important technical mysteries of the matrix theory \([1]\) approach to non-perturbative description of M-theory/string theory is the meaning of the large \(N\) limit. All concrete calculations to date have essentially been performed at finite \(N\) (see \([2]\)\([3]\)\([4]\) for reviews and further references). Finite \(N\) matrix theory has been identified by Susskind\([5]\) as having a physical meaning: it corresponds to discrete light-cone quantized M theory and the proposal has even been given a concrete “derivation” \([6]\)\([7]\). However various problems with this identification have emerged. In particular it has been argued that three graviton scattering in flat space cannot be reproduced by the finite \(N\) matrix model \([8]\). Recently a number of papers have appeared on this issue \([9]\)\([10]\)\([11]\)\([12]\)\([13]\) claiming and disclaiming the correctness of finite \(N\) matrix theory in this context. It has also been suggested in \([2]\)\([3]\) that the large \(N\) limit might resolve any disparity, further arguments for this being given in \([14]\). Furthermore it appears that in a curved background it is impossible to describe even two body scattering with a finite number of matrix variables \([15]\)\([16]\). Infinite matrices are also required for a correct supergravity correspondence for more general objects such as spherical membranes \([17]\)\([18]\). It is thus crucial to understand the large \(N\) limit. It is not even obvious that the large \(N\) limit is well-defined.

In this article we study the domain of validity of the loop expansion for the low energy effective action of matrix string theory \([19]\)\([20]\)\([21]\). We will focus on long string configurations interacting at one or two points. For string interactions we take as a background the instanton high energy string interactions constructed in \([22]\). We start by studying long string configurations. We find the result depends on the ordering of limits. Taking \(g_s \rightarrow 0\) before \(N \rightarrow \infty\) leads to free strings. Reversing the limits however we find anomalous contributions from neighbouring eigenvalues on the string worldsheet. These lead to \(L\) loop contribution to the effective action being weighted with a factor \(N^{2(L-1)}\) indicating that the loop expansion is no longer valid. We then imbed the instanton two string interaction into two long interacting strings and calculate the effective action about this instanton background. We find that the \(L\) loop contribution is weighted by \(g_s^{\frac{4}{3}} N^{(L-\frac{2}{3})}\). This again indicates that the loop expansion is not valid. Finally we identify a domain in which the effective action is valid for interacting long strings. The limit corresponds to large classical strings of size \(\sqrt{N}\), and can be identified with the “dual” to the ‘t Hooft limit: \(g_{SYM} \sim \sqrt{N}\). Curiously this limit corresponds to the boundary limit found in \([23]\) separating the supergravity description from the orbifold CFT description.
2. Low energy effective action for matrix string theory

The form of the loop expansion for the effective action for matrix theory and its various compactifications has been studied in [24][25]. Below we briefly recall the argument in the context of matrix string theory.

We will concentrate on the higher derivative bosonic terms and for compactness of notation denote symbolically by $F$ both the two dimensional gauge field $F_{\alpha\beta}$ and the derivatives of the 8 transverse coordinates $\partial_\alpha X^I$. We rescale the fields so that the loop counting parameter, $1/\alpha' g^2_{SYM} = \alpha' g^2$, appears as a multiplicative factor in front of the action (it is convenient to set $\alpha' = 1$ in what follows):

$$S \sim g^2_s \int d^2 \sigma \tilde{F}^2 \quad \text{with} \quad \tilde{A} = \frac{A_\alpha}{g_s}, \frac{X^I}{g_s}.$$  \hspace{1cm} (2.1)

The effective action expressed as a sum over loops is then given by

$$W \sim \sum_{L=0}^{\infty} \frac{1}{g_{2(L-1)}^2} \int d^2 \sigma \tilde{L}_L \left( X, \frac{F}{g_s}, \frac{X^4}{g_s}, \cdots \right).$$  \hspace{1cm} (2.2)

where $\tilde{L}_L$ is the contribution to the effective action from $L$ loops and the dots represent higher order derivative terms. Observing that $\tilde{L}_L$ has the dimension of $\text{length}^{2(L-1)}$, and using dimensional analysis we arrive at the effective action

$$W \sim \int d^2 \sigma F^2 + \sum_{L=1}^{\infty} \mathcal{L}_L \quad \text{with} \quad \mathcal{L}_L = \sum_{n=2}^{\infty} g_{2n-2}^{2n-2} \frac{F^{2n}}{X^{4n+2L-4}}.$$  \hspace{1cm} (2.3)

By $F^{2n}$ we simply mean the bosonic terms with $2n$ derivatives.

Since the two derivative term is not renormalized in this theory the $L$ loop terms start at $F^4$ or higher. In fact an important element of the matrix theory conjecture is that the $L$ loop term starts at $F^{2L+2}$. This is a necessary requirement for matrix theory to correctly reproduce supergravity results. This dominant contribution should correspond to classical supergravity. To date this has only been checked up to the two loop level [26][24].

3. Long strings

We now apply the above analysis to long string configurations of [13][21]. We will do this explicitly for the $F^4$ contribution from the one loop calculation. The general form of the higher order contributions will then be easily discussed.
At one loop the calculation of the effective action reduces to the calculation of the determinant from the quadratic fluctuations of the off-diagonal fields (bosonic, fermionic and ghost) in the background of the diagonal elements forming the long string configuration. The quadratic part of the Lagrangian is

\[ \mathcal{L} = \sum_{j=1}^{N-1} w_j^{\mu*} \left( D_j^2 \eta_{\mu\nu} - 2 \frac{i}{g_s} F_j^{\mu\nu} \right) w_j^{\nu'} + \eta_j^{1/2} \partial_j \eta_j + c_j^* D_j^2 c, \]  

(3.1)

where all fields are defined on the interval \( \sigma \in [0, 2\pi N] \). The fields \( w \) and \( \eta \) are the off-diagonal elements of the bosons and fermions and \( c \) are the off diagonal elements of the ghosts. The covariant derivatives \( D_j \) and the field strength \( F_j \) are given by

\[ D_j = \partial - \frac{i}{g_s} (a(\sigma + 2\pi j) - a(\sigma)) \quad \text{and} \quad F_j = f(\sigma + 2\pi j) - f(\sigma) \quad \text{with} \quad f_{\mu\nu}(\sigma) = \partial_\mu a_\nu - \partial_\nu a_\mu. \]  

(3.2)

The fields \( w_j \) and \( a \) are related to the original matrix elements by

\[ w_j(\sigma + 2\pi i) = w_{ij}(\sigma) \quad \text{and} \quad a(\sigma + 2\pi i) = a_i(\sigma) \quad \text{with} \quad \sigma \in [0, 2\pi]. \]  

(3.3)

Integrating over the off-diagonal elements leads to a series of terms in powers of \( F \), the lowest order being

\[ \sum_{j=1}^{N-1} \int_0^{2\pi N} d\tau \int_0^{2\pi N} d\sigma \frac{F_j^4}{X_j} \sum_p (1 + \frac{p^2}{X_j^2 N^2})^{-\frac{1}{2}} \quad \text{with} \quad X_j(\sigma) = |X^I(\sigma) - X^I(\sigma + 2\pi j)|. \]  

(3.4)

The sum over \( p \) is the sum over discrete momenta \( p/(2\pi N) \) around the compact \( \sigma \) direction. For our purposes the precise numerical factors and the tensor structure of the \( F^4 \) term are not important.

To keep a fixed total string length in the limit \( N \to \infty \) we now rescale the coordinates \((\sigma, \tau) \to (\frac{\sigma}{N}, \frac{\tau}{N})\). This leads to an overall factor of \( 1/N^3 \) (\( N^{-4} \) from the \( F^4 \) term, \( N^2 \) from the \( d^2\sigma \) and \( N^{-1} \) from in front of the sum over \( p \)). Naively, taking into account the sum over \( j \), one would conclude that the term (3.4) scales away with a factor of \( 1/N^2 \). Of course this is false since for small \( j \) (mod \( N \)) there is a singular contribution. Indeed for small \( j \) we have

\[ F_j = j \frac{2\pi}{N} \partial_\sigma f \quad \text{and} \quad X_j = j \frac{2\pi}{N} \partial_\sigma X \]  

(3.5)

\( \diamond \) A fuller discussion of why the fields are defined on the interval \([0, 2\pi N]\) rather than \([0, 2\pi]\) can be found in [27].
This leads to an anomalous, \( N \) independent, local contribution to the action:

\[
\int d^2\sigma \left( \frac{\partial_\sigma f}{(\partial_\sigma X)^2} \right)^4 \sum_{j=1}^\infty \frac{1}{j^3} \sum_{p=-\infty}^\infty (1 + \frac{p^2}{j^2(\partial_\sigma X)^2})^{-\frac{7}{2}}.
\] (3.6)

This type of anomaly is familiar from matrix models of 2D gravity [28][29].

The factors of \( N \) can be very simply deduced for the higher order derivative terms to the one loop effective action. For the terms with \( F^{2n} \) the powers of \( N \) are 2 from the \( d^2\sigma \), \( 4n \) from the \( (\partial_\sigma f)^{2n} \) and \(-4n - 2 \) from the \( (\partial_\sigma X)^{-4n-2} \). In other words all the one loop terms give local \( N \) independent contributions to the effective action.

An identical reasoning can be applied to the higher loop contributions. One does not need to know the precise index structure. Since one sums over all indices, there is guaranteed to be a part of the sum which gives the most singular contribution. Picking out these most singular terms one finds that the effective action (2.3) for the long string configurations takes the form

\[
\mathcal{L}_L = N^{2(L-1)} \sum_{n=2}^\infty g_s^{2n-2} \frac{(\partial_\sigma f)^{2n}}{(\partial_\sigma X)^{4n+2L-4}}.
\] (3.7)

In other words the \( L \) loop contribution is weighted by a factor \( N^{2(L-1)} \). All this means is that the calculation of the effective action in term of a perturbative loop expansion is not valid. Notice however that if we first take the limit \( g_s \to 0 \) before we take the large \( N \) limit all the loop contributions disappear. Stated more carefully, one sees that if the matrix theory hypothesis is true, and the \( L \) loops contribution to the effective action starts at \( n = L + 1 \), the loop contributions disappear provided \( g_s \sim 1/N \).

4. String interactions

Recently finite instanton like solutions to the classical equations of motion have been found which correspond to string interactions [22] (see also [30]). A novel property of these solutions is that there is a minimal non-zero distance between the strings. They split and join without touching by stepping off into the noncommutative part of the space. Technically the fact that they have a minimal separation means that the fluctuations about these configurations remain massive throughout the interaction region and leads to the hope that we might have some control over the calculation of the effective action.
Ultimately we are interested in the large $N$ limit so we will embed the solution found in [22] into a configuration of two long strings joining to become a single long string. If the two strings are of the identical length the embedding is particularly simple.

First let us recall the construction of [22] to which we refer the reader for fuller details. Instanton solutions are found by studying the four dimensional self dual YM equations dimensionally reduced to two dimensions:

\[
F_{w,\bar{w}} = -\frac{i}{g_s}[X, \bar{X}],
\]

\[
D_w X = 0
\]

\[
D_{\bar{w}} \bar{X} = 0
\]

(4.1)

where we have defined complex coordinates

\[
X = \frac{1}{\sqrt{2}}(X^1 + iX^2),\quad \bar{X} = \frac{1}{\sqrt{2}}(X^1 - iX^2),
\]

(4.2)

and similar complex coordinates for the gauge fields $A$ and $\bar{A}$.

Single valued matrix configurations corresponding to interacting strings can be generated by gauge transforming the diagonal multivalued matrix with a gauge transform $U$ that creates, by Wilson lines, the correct monodromies around the branch points [27]. This leads to delta function singularities in the field strength at the interaction points. The key observation of [22] was that these singularities can be removed once we are working with complex coordinates $X$, by using a complexified “gauge” transformation $G$ which also has a singularity at the origin tuned in such a way as to leave a singularity free field strength.

\[4.1. \ 2 \times 2 \ \text{matrices}\]

For the case of two eigenvalues we have

\[
X = U G \hat{X} G^{-1} U^\dagger
\]

\[
A = -ig_s \left[ G^{-1} (\partial_w G) + U^\dagger (\partial_{\bar{w}} U) \right]
\]

(4.3)

where the diagonal matrix $\hat{X}$, and the matrices $U$ and $G$ are given by

\[
\hat{X} = B \sqrt{\bar{w} \tau_3}, \quad U = e^{\frac{i}{\tau_1} \ln \frac{w}{\bar{w}}}, \quad \text{and} \quad G = e^{a(w \bar{w}) \tau_1}.
\]

(4.4)

The unitary matrix $U$ generates the monodromy around the branch point so that the matrix $X$ remains single-valued even though its eigenvalues interchange. This ansatz
automatically satisfies the last two equations of (4.1) with the first equation leading to a differential equation for $\alpha$

$$\left(\partial_r^2 + \frac{1}{r} \partial_r\right) \alpha = \frac{8B^2}{g_s^2} r \sinh 2\alpha$$

with $\alpha \rightarrow \begin{cases} 0 & \text{for } r \rightarrow \infty \\ -\frac{1}{4} \ln r & \text{for } r \rightarrow 0 \end{cases}$ (4.5)

where $r = \sqrt{w \bar{w}}$ is the radial distance from the branch point. The boundary conditions are necessary for a finite solution. In particular the second boundary condition ensures that there are no $\frac{1}{w}$ pole terms in the gauge field $A$ and hence no delta function singularity in the field strength $F_{w\bar{w}}$.

The differential equation can be given a dimensionless form by absorbing the coupling constants into a redefinition of the radial coordinate so that

$$\alpha(r, B, g_s) = \alpha'(r') \quad \text{with} \quad r' = \left(\frac{g_s^2}{8B^2}\right)^{\frac{1}{3}} r.$$ (4.6)

Before turning to the effective action let us give here the expressions for $X$ and $F_{w\bar{w}}$:

$$X = B(x_3 \tau_3 + ix_2 \tau_2) \quad \text{where} \quad x_3 = \sqrt{w} \cosh \alpha, \quad x_2 = \sqrt{w} \sinh \alpha$$

$$F_{w\bar{w}} = \frac{2iB^2}{g_s} f_1 \tau_1 \quad \text{where} \quad f_1 = r \sinh 2\alpha.$$ (4.7)

For simplicity we do not include the final gauge transformation $U$ (4.4). We now focus on the $F^4$ contribution from the one loop calculation, and use a formalism that generalizes easily for large $N$. The quadratic part of the fluctuation Lagrangian reads

$$\mathcal{L} = \text{Tr} \left[ V^\mu \left( D^2 \eta_{\mu\nu} - 2 \frac{i}{g_s} F^{\mu\nu} \right) V^\nu + \eta^\dagger \overleftrightarrow{D} \eta + c^* D^2 c \right],$$ (4.8)

where $V$ and $\eta$ are the bosonic and fermionic fluctuations and $c$ are the ghost fluctuations. All background fluctuations couple to the background fields and are massive. It is convenient to express the mass term for $V^\mu$ and the quadratic coupling to the background $F^{\mu\nu}$ in terms of the $SU(2)$ generators. The mass term comes from the double commutator $[X, [\bar{X}, V]] + \text{c.c.}$ in the kinetic $D^2$ term of (4.8).

$$\mathcal{L}_{\text{mass}} = \frac{8 B^2}{g_s^2} \left[ |v_1|^2 (x_3^2 + x_2^2) + |v_2|^2 x_3^2 + |v_3|^2 x_2^2 \right]$$

$$\mathcal{L}_F = \frac{8 B^2}{g_s^2} \left[ v_2^0 v_3^0 - v_2^0 v_3^0 - v_1^0 v_3^2 + v_2^2 v_3^1 \right] f_1,$$ (4.9)

where lower indices are group indices and upper indices correspond to spacetime indices. The indices 0 and 9 correspond to the two dimensional gauge fields. We have written the
terms in this form so that they generalize easily to the case of strings of length \( N \). The \( F^4 \) contribution is (up to numerical factors) given by:

\[
S_{F^4} = \frac{8B^2}{g_s^2} \int d^2 \sigma \, h(x_2, x_3, f_1) = \int d^2 \sigma' \, \tilde{h}(r', \alpha(r'))
\]

(4.10)

where

\[
h(x_2, x_3, f_1) = f_1 \left[ \frac{x_3^2 + x_3^2}{x_3^2 x_3^2 (x_3^2 - x_3^2)} - 2 \frac{\ln x_3^2 - \ln x_2^2}{(x_3^2 - x_3^2)^3} \right]
= r \left[ 2 \sinh^2 2 \alpha \cosh 2 \alpha - \sinh^4 2 \alpha \ln \coth \alpha \right]
= \left( \frac{g_s^2}{8B^2} \right) \tilde{h}(r', \alpha(r')).
\]

(4.11)

The precise form of \( h(x_2, x_3, f_1) \) is not important for our purposes. All that is important is that it consists of a factor of \( r \) multiplied by a function depending only on \( \alpha \) so that on rescaling the coordinates according to (4.6) we arrive at the last line of (4.11) and the right-hand side of (4.10).

A simplifying assumption used in the above is that the size of the instanton is much less than the size of the compact direction \( \sigma \). It is then a good approximation to use the instanton ansatz (4.4), which assumes the space is non-compact. The instanton is then assumed to be glued into a globally defined configuration corresponding to a branched covering of the cylinder \([27]\) with the instanton sitting at the branch point.

We see that the result (4.10) is independent of \( g_s \) and \( B \). This reasoning can be generalized to show that all the terms in the 1-loop expansion are independent of \( g_s \) and \( B \). Each factor of \( F \) in (2.3) contributes \( B^2 r/g_s \) and each factor of \( X^2 \) contributes \( B^2 r \). The net result being that all terms in the 1 loop contribution can be written as

\[
\frac{B^2}{g_s^2} \int d^2 \sigma \, r \, f(\alpha),
\]

(4.12)

where \( f \) is a function of \( \alpha \) alone. After rescaling of the coordinates (4.6) such contributions are independent of \( g_s \) and \( B \). Applying this reasoning to the higher loop terms in (2.3) it is easy to see that the \( L \) loop contribution is weighted with a factor of \( (g_s B^2)^{-\frac{2}{3}(L-1)} \).

4.2. \( 2N \times 2N \) matrices

Let us now imbed this solution, which consists of just two eigenvalues, into a configuration of two long strings of equal length \( N \) joining into a single string. This will permit us to relax the condition on the size of the instanton to be less than the total length of...
the long string rather than less than the length of an individual string. This is necessary physically since the parameter $B$ of the instanton solution is proportional to $1/\sqrt{N}$ and hence leads via (4.6) to an instanton of size $N^{\frac{2}{3}}$, i.e. physically the instanton will spread out over $O(\sqrt{N})$ individual strings.

To do this we simply tensor the $2 \times 2$ solution by diagonal $N \times N$ blocks, with the diagonal elements forming cycles of length $N$. The complex coordinates $w$ and $\bar{w}$ then sit on the cylinder of length $2\pi N$. The matrices for the instanton configuration $X$ and $F_{w\bar{w}}$ and the matrix for the background fluctuations $V$ are given by

$$X = B(x_3 \otimes \tau_3 + ix_2 \otimes \tau_2)$$

$$F_{w\bar{w}} = \frac{2iB^2}{g_s} f_1 \otimes \tau_1$$

$$V = v_1 \otimes \tau_1 + v_2 \otimes \tau_2 + v_3 \otimes \tau_3$$

The matrices $x_2$, $x_3$ and $f_1$ are diagonal matrices with the entries forming cycles of length $N$, i.e.

$$\begin{align*}
(x_3)_{ij} &= \delta_{ij} x_3(\sigma + 2\pi(i - 1), \tau) \\
(x_2)_{ij} &= \delta_{ij} x_2(\sigma + 2\pi(i - 1), \tau) \quad \text{with} \quad x_2(\sigma, \tau) = \sqrt{\bar{w}(\sigma, \tau)} \sinh(\alpha(\sigma, \tau)) \\
(f_1)_{ij} &= \delta_{ij} f_1(\sigma + 2\pi(i - 1), \tau), \quad f_1(\sigma, \tau) = r(\sigma, \tau) \sinh(2\alpha(\sigma, \tau))
\end{align*}$$

This corresponds to long strings of length $N$. The $SU(2)$ structure of the instanton then splits and joins the two long strings. It is trivial to see that the differential equation (4.3) remains unchanged.

The matrices $V$ are general hermitean $N \times N$ matrices. They also have a periodicity condition, since they end and start on the diagonal matrix background, provided by (4.14).

$$v_{ij} = v_i(\sigma + 2\pi(j - i))$$

The calculation of the effective action is a simple generalization of that for $N = 1$. The key difference being that whereas before the commutators give rise to products of scalars $x_3$, $x_2$, $f_1$ they now give rise to anticommutators for the $N \times N$ matrices $x_3$, $x_2$, $f_1$. For example the contribution to the $v_2$ mass term is given by

$$\int_0^{2\pi} d\sigma \frac{1}{2} \text{Tr}[v_2 \otimes \tau_2[X, [X, v_2 \otimes \tau_2]] + \text{c.c.}] = 2 \int \text{Tr}[v_2 \{x_3, \{\bar{x}_3, v_2\}\} + \text{c.c.}]$$

$$= 4 \int \sum_{i,j} \left| (v_2)_{ij} \right|^2 \left| (x_3)_i + (\bar{x}_3)_j \right|^2$$

$$= 8 \int_0^{2\pi N} d\sigma \sum_j \left| (v_2)_j \right|^2 \left| (x_3)_j \right|^2.$$
where
\[(x_3)_j = \frac{1}{2} (x_3(\sigma, \tau) + x_3(\sigma + 2\pi j, \tau)).\] (4.17)

The total mass term and interaction vertex read
\[L_{\text{mass}} = \frac{8 B^2}{g_s^2} \sum_j \left( (|v_1)_j|^2 (x_3)_j^2 + (x_2)_j^2) + (|v_2)_j|^2 (x_3)_j^2 + (|v_3)_j|^2 (x_2)_j^2 \vphantom{\left( \frac{1}{1} \right)} \right) \]
\[L_F = \frac{8 B^2}{g_s^2} \sum_j \left[ (v_2^0)_j (v_3^0)_j^* - (v_2^0)_j (v_3^0)_j^* - (v_2^1)_j (v_3^2)_j + (v_2^2)_j (v_3^1)_j^* \vphantom{\left( \frac{1}{1} \right)} \right] (f_1)_j, \] (4.18)

where
\[(f_1)_j = \frac{1}{2} (f_1(\sigma, \tau) + f_1(\sigma + 2\pi j, \tau)).\] (4.19)

These are simple generalizations of equations \[4.9\] where for each \(j\) one replaces the instanton background fields \(x\) and \(f\) by their average over points separated by a distance \(2\pi j\) in the \(\sigma\) direction.

The effective action likewise generalizes straightforwardly. For the one loop \(F^4\) contribution for example one sums over \(j\) the result \[4.10\] with the masses and fields strengths in \[4.11\] replaced by their average values over points separated by \(2\pi j\).

We are now in a position to analyse the behaviour at large \(N\) of the effective action for the instanton configuration. The crucial difference between the large \(N\) behaviour of the long string effective action and the part of the effective action for the instanton is that for the instanton there is no large \(N\) anomaly. This is due to the fact that one takes the average over background field values separated by \(2\pi j\) in the \(\sigma\) direction, not the difference. There is thus no singular behaviour for small \(j\) and if the size of the instanton is such that it spreads out over a large number of strips, the sums over \(j\) can be replaced by integrals. The \(F^4\) term in the one loop expansion is thus given by
\[S_{F^4} = \frac{8 B^2}{g_s^2} \int d\tau d\sigma \sum_j h((x_2)_j, (x_3)_j, (f_1)_j) \]
\[= \frac{8 B^2}{g_s^2} \int d\tau d\sigma_1 d\sigma_2 h((x_2), (x_3), (f_1)) \]
\[= \left( \frac{g_s^2}{B^2} \right)^{\frac{1}{4}} \int d\tau' d\sigma'_1 d\sigma'_2 \tilde{h}(\tau'_1, \alpha(\sigma'_1, \tau'_1), \tau'_2, \alpha(\sigma'_2, \tau'_2)) \] (4.20)

where in the second line we have defined
\[(x_3) = \frac{1}{2} (x_3(\sigma, \tau) + x_3(\sigma + 2\pi j, \tau))\] (4.21)
and in the final line we have rescaled all coordinates to the natural size of the instanton, using (4.4) and we have also used the last line of (4.11).

Finally the $N$ dependence of the $F^4$ contribution is determined by the $N$ dependence of $B$ which specifies the asymptotic behaviour of the instanton/branch point. For the asymptotic behaviour to have a physically sensible large $N$ limit we see from equation (4.7) and the fact that we rescale coordinates $w \to \sqrt{N}w$ that the constant $B$ must scale as $B \sim 1/\sqrt{N}$. The $F^4$ term thus scales as $g_s^{2/3} N^{1/3}$.

The factors of $g_s$ and $N$ for the higher derivative contributions to the the 1 loop effective action can likewise be easily determined. The difference from the scaling arguments given for the case $N = 1$ at the end of section (4.1) is that there is now an extra $\int d\sigma$ integral from the sum over $j$ and hence an extra $(g_s^2/B^2)^{1/3}$ factor. All terms in the 1 loop effective action are thus seen to scale as $g_s^{2/3} N^{1/3}$.

For the higher loop terms in the effective action the dominant contribution comes from planar diagrams. In terms of the matrix theory conjecture the first such planar diagram at $L$ loops is hoped to be the term $F^{2L+2}$. For our purposes each planar loop brings an extra index and hence an extra $\int d\sigma$ integral in addition to the $\int d^2\sigma$ integral of (2.3). This leads to the $L$ loop weight

$$S_L \sim g_s^{2/3} B^{-2(L-2)} \sim g_s^{2/3} N^{(L-2)/3}. \quad (4.22)$$

As for the long string configurations into which the instanton is embedded we conclude that the loop expansion is not valid.

5. Large $N$ limits

In this section we search for a large $N$ limit in which the loop expansion is well defined both for the long strings and for their interactions (instanton configurations). In other words we allow both the string coupling constant and the size of the string to scale with powers of $N$:

$$X, F \sim N^x \Rightarrow B \sim N^{x - \frac{1}{2}}$$

$$g_s \sim N^g. \quad (5.1)$$

We then look for the region in $x, g$ space where all terms in the loop expansions scale with non-positive powers of $N$.

From the analysis of the long string configurations we have:

$$L - 1 + (n - 1)g - (n + L - 2)x \leq 0 \quad (5.2)$$
If we further assume that \( n \geq L + 1 \) as is required for the matrix theory to be correct the inequality leads to the two inequalities
\[
g \leq x \quad \text{and} \quad g \leq 2x - 1. \tag{5.3}
\]
The analysis of the loop expansion for the instanton/string interaction consists of two cases depending on whether or not the world sheet scale of the instanton (4.6) spreads out over many strips. If \( g > x - 1/2 \) the instanton size will scale as a positive power of \( N \), and the analysis of section 4.2 is then appropriate. Imposing, in this case, that all terms in the loop expansion scale with non positive powers of \( N \) leads to the inequality
\[
g \leq (3L - 2)(x - \frac{1}{2}) \quad \text{which implies} \quad g \leq x - \frac{1}{2} \quad \text{and} \quad x \geq \frac{1}{2}, \tag{5.4}
\]
in contradistinction with the domain of validity of (5.4) itself. We are left with the case \( g \leq x - 1/2 \), in which the size of the instanton is less than an individual strip, and for which the analysis of section 4.1 is appropriate. We thus have the inequality
\[
g \geq 1 - 2x. \tag{5.5}
\]
We plot the inequalities (5.3)(5.5) in the graphic below

![Fig. 1. x, g parameter space for which the loop expansion is valid (shown in white)](image)

For most of the available parameter space all of the terms in the loop expansions are scaled away. At the point \( x = 1/2, \ g = 0 \), however, all terms in the loop expansion for the instanton contribute. This limit corresponds to large classical strings of size \( \sqrt{N} \) in \( \alpha' \) units with \( \sqrt{N} \) individual diagonal elements/strips of the world sheet occupying an interval of length \( \sqrt{\alpha'} \). This resembles the scaling limits studied in the AdS - SYM correspondence (see \[31\] and references thereto). Indeed an alternative way of thinking of this scaling regime is as a fixed string size but a rescaling of \( \alpha' \) by \( 1/N \), which via the relation \( g_{SYM}^2 = 1/g_s^2 \alpha' \) leads to \( g_{SYM}^2 \sim N \). This is the “dual” of the ’t Hooft limit. Interestingly it is also precisely the point found in \[23\] separating the CFT orbifold description from the SUGRA description.
6. Conclusions

We have seen in the previous sections that the loop expansions for the case of physical interest, that of interacting matrix strings, is ill defined in the limit $N \to \infty$. This does not imply that the theory itself is ill defined, it just highlights the fact that it is not justified to use such an expansion. To decide whether the theory does or does not have a well defined large $N$ limit would require finding some way of integrating out, at least partially, some of the non-perturbative contributions. It could be that in the full non-perturbative calculation the “mass” of the off-diagonal elements connecting neighbouring strips is smoothed out in such a way that all the large $N$ anomalies found in section(3) disappear. A hint that this might be the case comes from the study of the effective action for the diagonal elements in the 0 dimensional matrix model. For $N = 2$ it is possible to integrate out the off-diagonal elements explicitly and one finds that the singularity for coinciding diagonal elements is resolved by the appearance of extra massless fields [32].

We have found that there is a non-trivial scaling limit in which the loop expansion is well defined in the $N \to \infty$. It corresponds to the “dual” of the ’t Hooft limit, $g_{SYM}^2 \sim N$.

Finally in the calculation of the fluctuations about the instanton careful attention should be paid to the translation and scale modes. These are not important for our purposes since we are focusing on the large $N$ limit, but could well contain important contributions to the full result. We leave the investigation of this point to future work.

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