Regular rotating electrically charged black holes and solitons in non-linear electrodynamics minimally coupled to gravity

Irina Dymnikova¹,² and Evgeny Galaktionov¹

¹ A.F. Ioffe Physico-Technical Institute, Politekhnicheskaja 26, St. Petersburg, 194021 Russia
² Department of Mathematics and Computer Science, University of Warmia and Mazury, Słoneczna 54, 10-710 Olsztyn, Poland

E-mail: irina@uwm.edu.pl

Received 7 May 2015
Accepted for publication 12 June 2015
Published 30 July 2015

Abstract
In non-linear electrodynamics coupled to gravity, regular spherically symmetric electrically charged solutions satisfy the weak energy condition and have an obligatory de Sitter center. By the Gürses–Gürsey algorithm they are transformed to spinning electrically charged solutions that are asymptotically Kerr–Newman for a distant observer. Rotation transforms the de Sitter center into a de Sitter vacuum surface which contains the equatorial disk \( r = 0 \) as a bridge. We present a general analysis of the horizons, ergoregions and de Sitter surfaces, as well as the conditions of the existence of regular solutions to the field equations. We find asymptotic solutions and show that de Sitter vacuum surfaces have properties of a perfect conductor and ideal diamagnetic, violation of the weak energy condition is prevented by the basic requirement of electrodynamics of continued media, and the Kerr ring singularity is replaced with the superconducting current.

Keywords: regular rotating charged black hole, electromagnetic spinning soliton, nonlinear electrodynamics coupled to gravity

1. Introduction
The Kerr–Newman electrically charged rotating solution to the Maxwell–Einstein equations [1]
\[
\begin{align*}
\text{ds}^2 &= \left(\frac{2mr - e^2}{\Sigma} - \frac{\Sigma}{\Delta} + \frac{\Sigma}{\Delta} \right) dr^2 + \frac{\Sigma}{\Delta} d\theta^2 - \frac{2a (2mr - e^2) \sin^2 \theta}{\Sigma} dr d\phi \\
&+ \left( r^2 + a^2 + \frac{(2mr - e^2) a^2 \sin^2 \theta}{\Sigma} \right) \sin^2 \theta d\phi^2;
\end{align*}
\]
\[
\Delta = r^2 - 2mr + a^2 + e^2,
\]

where the \(\Sigma\)-function and the associated electromagnetic potential read
\[
\Sigma = r^2 + a^2 \cos^2 \theta; \quad A_i = -\frac{e_i}{\Sigma} \left[ 1; 0, 0, -a \sin^2 \theta \right].
\]
was obtained from the Reissner–Nordström electrovacuum solution using the complex coordinate transformation discovered by Newman and Janis [2].

The Kerr–Newman geometry has at most two horizons \(r_{\pm} = m \pm \sqrt{m^2 - (a^2 + e^2)}\), and implies the same value of the gyromagnetic ratio \(g = 2\) as predicted for a spinning particle by the Dirac equation [3]. In the case appropriate for particles, \(a^2 + e^2 > m^2\), there are no Killing horizons, the manifold is geodesically complete (except for geodesics which reach the singularity), and any point can be connected to any other point by both a future and a past directed time-like curve. Closed time-like curves originate in the region where \(g_{\phi\phi} < 0\), can extend over the whole manifold and cannot be removed by taking a covering space [3].

The Kerr–Newman solution belongs to the Kerr family of solutions to the source-free Maxwell–Einstein equations, and represents the exterior fields of a rotating charged body [3]. The source models for the Kerr–Newman exterior can be roughly divided into disk-like [4–6], shell-like [7–9], bag-like [10–16] and string-like ([17] and references therein).

The problem of matching the Kerr–Newman exterior to a rotating material source does not have a unique solution, since one is free to choose arbitrarily the boundary between the exterior and the interior [4], not to mention the freedom in choosing the interior model itself.

On the other hand, one can study equations describing a dynamical system to get information about its basic properties rather than postulating properties of a source which could give rise to the Kerr–Newman fields. In the case of an electromagnetically interacting structure the appropriate equations come from non-linear electrodynamics coupled to gravity (NED-GR)\(^3\).

Non-linear electrodynamics was proposed by Born and Infeld as founded on two basic points: to consider an electromagnetic field and particles within the frame of one physical entity, which is an electromagnetic field; to avoid letting physical quantities become infinite [21]. This program can be realized in non-linear electrodynamics coupled to gravity. Source-free NED-GR equations admit regular causally safe axially symmetric asymptotically Kerr–Newman solutions [22], which describe regular rotating charged black holes and electromagnetic spinning solitons.

The key point is that, for any gauge-invariant Lagrangian \(\mathcal{L}(F)\), the stress–energy tensor of electromagnetic field in the spherically symmetric case has the algebraic structure
\[
T^\gamma_i = T_i^\gamma \quad (\rho = -\rho).
\]

Regular spherically symmetric solutions with stress–energy tensors specified by (3) satisfying the weak energy condition (WEC; non-negativity of density as measured by any local observer) have obligatory de Sitter center with \(p = -\rho\) [23–26]. The mass \(m\) of an object is generically related to the de Sitter vacuum and breaking of spacetime symmetry from the de

\(^3\) NED theories appear as low-energy effective limits in certain models of string/M-theories [18–20].
Sitter group in the origin [25]. In the NED-GR regular solutions interior the de Sitter vacuum provides a proper cutoff on self-interaction divergent for a point charge [22, 27].

The regular spherical solutions generated by (3) belong to the Kerr–Schild class [15, 28, 29] and can be transformed by the Gürses–Gürsey algorithm [30] into regular axially symmetric solutions which describe regular rotating electrically charged objects, asymptotically Kerr–Newman for a distant observer [22, 31]. Rotation transforms the de Sitter center into a de Sitter vacuum disk, which has properties of a perfect conductor and ideal diamagnetic and displays superconducting behavior within a single spinning object [22, 32].

In this paper we address the question of the existence and asymptotic behavior of solutions to the dynamical field equations which define the basic generic features of regular rotating electrically charged objects. In the paper [15] it was noted that rotation leads to violation of the WEC in interior regions of neutral regular rotating configurations. Violation of the WEC was found for regular solutions obtained with the Newman–Janis algorithm from the Hayward and Bardeen metrics [33], for rotating regular solutions obtained by postulating a metric $g_{\mu\nu}$ and calculating $T_{\mu\nu}$ from the Einstein equations [34], and for the Ayón-Beato–García solution transformed to rotational form with the Newman–Janis algorithm [35]. Here we study the WEC for NED-GR electrically charged rotating regular structures with an arbitrary Lagrangian $\mathcal{L}(F)$, and generic behavior of electromagnetic fields on the vacuum surfaces ($\rho + \rho = 0$), beyond which the WEC could be violated.

The paper is organized as follows. In section 2 we present and analyze the basic equations. In section 3 we address the question of the existence of horizons and ergoregions, and present interior de Sitter vacuum surfaces. In section 4 we study electromagnetic fields and in section 5 behavior of fields in regular interiors. In section 6 we summarize and discuss the results.

2. Basic equations

Non-linear electrodynamics minimally coupled to gravity is described by the action

$$S = \frac{1}{16\pi G} \int d^4x\sqrt{-g} \left[ R - \mathcal{L}(F) \right]; \quad F = F_{\mu\nu} F^{\mu\nu},$$

where $R$ is the scalar curvature, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field. The gauge-invariant electromagnetic Lagrangian $\mathcal{L}(F)$ is an arbitrary function of $F$ which should have the Maxwell limit, $\mathcal{L} \rightarrow F$, $\mathcal{L}_F \rightarrow 1$ in the weak field regime.

Variation with respect to $A^\mu$ and $g_{\mu\nu}$ yields the dynamic field equations

$$V_\mu(\mathcal{L}_F F^{\mu\nu}) = 0;$$

$$V_\mu \ast F^{\mu\nu} = 0; \ast F^{\mu\nu} = \frac{1}{2} \eta^{\mu\nu\rho\sigma} F_{\rho\sigma}; \eta^{0123} = -\frac{1}{\sqrt{-g}},$$

where Greek indices run from 0 to 3 and $\mathcal{L}_F = d\mathcal{L}/dF$, and the Einstein equations $G_{\mu\nu} = -\kappa T_{\mu\nu}$ with the stress–energy tensor

$$\kappa T^\mu_\nu = -2\mathcal{L}_F F_{\nu\alpha} F^{\mu\alpha} + \frac{1}{2} \delta^\mu_\nu \mathcal{L}; \quad \kappa = 8\pi G.$$

NED-GR equations do not admit regular spherically symmetric solutions with the Maxwell center [36], but they admit regular solutions with the de Sitter center in which field tension goes to zero, while the energy density of the electromagnetic vacuum $T^t_\nu$ achieves its maximal finite value, which represents the de Sitter cutoff on the self-energy density [27]. The question
of correct description of NED-GR regular electrically charged structures by the Lagrange dynamics is clarified in [37]. Regular spherical solutions satisfying (3) are described by the metric

\[ ds^2 = g(r)dt^2 - \frac{dr^2}{g(r)} - r^2d\Omega^2; \quad g(r) = 1 - \frac{2M(r)}{r}; \quad M(r) = 4\pi \int_0^r \rho(x)x^2dx, \quad (8) \]

with the electromagnetic density \( \rho(r) = T^t_t(r) \) from (7). This metric has the de Sitter asymptotic as \( r \to 0 \) and the Reissner–Nordström asymptotic as \( r \to \infty \) [27].

Spherically symmetric solutions of the Einstein equations specified by (3) belong to the Kerr–Schild class [15, 29]. By the Gürses–Gürsey algorithm they can be transformed into axially symmetric regular solutions describing rotating objects [30]. The Kerr–Schild metric has the form

\[ g_{\mu\nu} = \eta_{\mu\nu} + \frac{2f(r)}{\Sigma} k_\mu k_\nu, \quad (9) \]

where \( \eta_{\mu\nu} \) is the Minkowski metric and \( k_\mu \) is a vector field tangent to the Kerr principal null congruence. Metric (9) involves a function \( f(r) = rM(r) \) which comes from a spherically symmetric solution [30]. For the Kerr–Newman geometry \( f(r) = mr - e^2/2r \). The parameter \( r \) is defined as an affine parameter along either of two principal null congruences. The surfaces of constant \( r \) are the oblate confocal ellipsoids of revolution given by [38]

\[ r^4 - \left(x^2 + y^2 + z^2 - a^2\right)r^2 - a^2z^2 = 0 \quad (10) \]

which degenerate, for \( r = 0 \), to the equatorial disk

\[ x^2 + y^2 \leq a^2, \quad z = 0 \quad (11) \]

centered on the symmetry axis and bounded by the ring

\[ x^2 + y^2 = a^2, \quad z = 0. \quad (12) \]

In the Kerr–Newman metric (1) the ring (12) comprises the Kerr singularity of the Kerr–Newman geometry [38].

The Cartesian coordinates \( x, y, z \) are related to the Boyer–Lindquist coordinates \( r, \theta, \phi \) by

\[ x^2 + y^2 = (r^2 + a^2)\sin^2 \theta; \quad z = r \cos \theta. \quad (13) \]

In the Boyer–Lindquist coordinates the Gürses–Gürsey metric reads

\[ ds^2 = \frac{2f - \Sigma}{\Delta} dr^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 - \frac{4af}{\Sigma} \sin \theta d\theta d\phi + \left(r^2 + a^2 + \frac{2f a^2 \sin^2 \theta}{\Sigma}\right) \Sigma^2 \theta d\phi^2, \quad (14) \]

where the Lorentz signature is \([- + + +]\), and

\[ \Delta(r) = r^2 + a^2 - 2f(r); \quad \Sigma = r^2 + a^2 \cos^2 \theta. \quad (15) \]

For the Kerr–Newman geometry \( 2f(r) = 2mr - e^2 \) can change sign, which leads to causality violation related to regions where \( g_{\phi \phi} < 0 \). For regular spherical solutions satisfying the WEC, \( f(r) = rM(r) \) is a non-negative function since \( M(r) \) is monotonically growing from \( 4\pi \rho(0)r^2/3 \) as \( r \to 0 \) to \( m - e^2/2r \) as \( r \to \infty \) [27]. This guarantees the causal safety on the whole manifold due to \( f(r) \geq 0 \) and \( g_{\phi \phi} > 0 \) in (14). For regular spherical configurations \( \kappa(p_\perp + \rho) = -L_f F \), the field invariant \( F \) is non-positive function evolving from \( F = -0 \) as
$r \to 0$ to $F = -0$ for $r \to \infty$ [27]. $\mathcal{L}_F$ plays the role of the electric permeability [27, 31], and electrodynamics of the continued media requires $\mathcal{L}_F > 0$ [39], so WEC is always satisfied. The mass parameter $m = 4\pi \int_0^\infty \! \rho(r) r^2 \mathrm{d}r$ appearing in a spinning solution, is the finite positive electromagnetic mass [27], generically related to the interior de Sitter vacuum for any solution from the class specified by (3) [25].

The anisotropic stress–energy tensor responsible for (14) can be written in the form [30]

$$T_{\mu \nu} = (\rho + p_\bot)(u_\mu u_\nu - l_\mu l_\nu) + p_\bot g_{\mu \nu}$$

(16)

in the orthonormal tetrad

$$u^\mu = \frac{1}{\sqrt{\pm \Delta}}\left[ (r^2 + a^2)\delta^\theta_\phi + a\delta^\phi_\theta \right], \quad l^\mu = \frac{\pm \Delta}{\sqrt{\Sigma}} \delta^\mu_\phi, \quad n^\mu = \frac{1}{\sqrt{\Sigma}} \delta^\mu_\theta,$$

$$m^\mu = -\frac{1}{\sqrt{\Sigma} \sin \theta} \left[ a \sin^2 \theta \delta^\phi_\theta + \delta^\phi_\phi \right].$$

(17)

The plus sign refers to the regions outside the event horizon and inside the Cauchy horizon where the vector $u^\mu$ is time-like, and the minus sign refers to the regions between the horizons where the vector $l^\mu$ is time-like. The vectors $m^\mu$ and $n^\mu$ are space-like in all regions.

The eigenvalues of the stress–energy tensor (7) in the co-rotating frame where each of ellipsoidal layers rotates with the angular velocity $\omega(r) = u^\Phi/u^t = a/(r^2 + a^2)$ [15] are defined by

$$T_{\mu \nu} u^\mu u^\nu = \rho(r, \theta); \quad T_{\mu \nu} l^\mu l^\nu = p_\|; \quad T_{\mu \nu} n^\mu n^\nu = T_{\mu \nu} m^\mu m^\nu = p_\bot(r, \theta)$$

(18)

in the regions outside the event horizon and inside the Cauchy horizon, where density is defined as the eigenvalue of the time-like eigenvector $u^\mu$. They are related to the function $f(r)$ as [15]

$$\kappa \Sigma^2 \rho = 2(f' r - f); \quad \kappa \Sigma^2 p_\bot = 2(f' r - f) - f'' \Sigma.$$

(19)

It follows that

$$\kappa \rho = \frac{r^4}{\Sigma^2} \tilde{\rho}(r); \quad \kappa p_\bot = \left( \frac{r^4}{\Sigma^2} - \frac{2r^2}{\Sigma} \right) \tilde{\rho}(r) - \frac{r^3}{2\Sigma} \tilde{\rho}'(r).$$

(20)

The prime denotes the derivative with respect to $r$. In the co-rotating frame we thus have

$$\kappa (p_\bot + \rho) = 2 \left( \frac{r^4}{\Sigma^2} - \frac{r^2}{\Sigma} \right) \tilde{\rho}(r) - \frac{r^3}{2\Sigma} \tilde{\rho}'(r).$$

(21)

The basic features of regular rotating objects follow from generic behavior of related regular spherical solutions without specifying the particular form of the NED Lagrangian $\mathcal{L}(F)$.

3. Geometry

3.1. Horizons and ergospheres

Horizons are defined by zeros of the function $\Delta(r)$ in (15), which can be written as

$$\Delta(r) = r^2 + a^2 - 2f(r) = a^2 + r^2 g(r).$$

(22)
\[ \Delta = a^2 \] at zero points of the metric function \( g(r_0) = 0 \), and evolves from \( \Delta = a^2 \) as \( r = 0 \) to \( \Delta \to \infty \) as \( r \to \infty \).

The number of horizons depends on the generic properties of the metric function \( g(r) \), which has at most two zero points and one minimum [25].

**H1. The case of two zero points of the metric function \( g(r) \).** Derivatives of \( \Delta \) are

\[
\Delta' = 2rg(r) + r^2g'(r); \quad \Delta'' = 2g(r) + 4rg'(r) + r^2g''(r). \tag{23}
\]

At \( r = 0 \) derivatives take the values \( \Delta' = 0; \ \Delta'' = 2 \) and the function \( \Delta \) has the minimum, \( \Delta = a^2 \). Next it grows and can have a maximum at a certain value \( r_0 < r_a \) where \( r_a \) is the first zero of \( g(r) \). At the maximum \( \Delta'(r_a) = 0 \) and hence \( g'(r_a) = -2g(r_a)/r_a \), second derivative \( \Delta''(r_a) = -6g(r_a) + r_a^2g''(r_a) < 0 \). After passing the maximum \( \Delta \) achieves \( \Delta = a^2 \) at the first zero of \( g(r) \), then it will achieve this value at the second zero point \( r_0 \) of \( g(r) \). Between \( r_a \) and \( r_0 \) it has at least one extremum, which is the minimum, because in the region \( r_a < r < r_0 \) the metric function \( g(r) \) is negative and has the minimum. In this region \( g' \) is first negative, then passes zero and becomes positive, hence \( g'' \geq 0 \) everywhere between zeros of \( g(r) \), while \( g \leq 0 \); as a result \( \Delta'' \geq 0 \) everywhere in the considered region. It is evident that in this region the function \( \Delta(r) \) can have only a minimum and only one.

At \( r > r_a \) we have \( g(r) > 0 \) and \( g'' > 0 \) so that \( \Delta \) cannot vanish. We proved that in the case of two zero points of the metric function \( g(r) \) the number of zero points of the function \( \Delta(r) \) is a maximum of two, i.e. axially symmetric spacetime can have at most two horizons.

**H2. The case of the double root of the function \( g(r) \).** In this case \( g(r) \geq 0 \) everywhere.

\( \Delta' = 0 \) at \( r_a = r_a = r_0 \). At this point \( g'' > 0 \) and \( \Delta'' > 0 \), hence \( \Delta \) has the minimum \( \Delta = a^2 \). In this case \( \Delta(r) \) is an everywhere positive function which has one maximum at \( r = 0 \) and one minimum at \( r = r_a \). Axially symmetric spacetime does not have horizons.

**H3. The case when \( g(r) > 0 \) everywhere does not differ essentially from case H2.** The function \( \Delta(r) \) is positive everywhere. The extremum of \( \Delta \) can only be in the region where \( g' < 0 \), but in this case it can have an inflection point instead of an extremum.

Ergospheres and ergoregions. The ergosphere is a surface of a static limit \( g_o = 0 \) given by

\[
g_o(r, \theta) = r^2 + a^2 \cos^2 \theta - 2f(r) = 0. \tag{24}
\]

It follows that \( z^2 = (2r^2f(r) - r^4)/a^2 \). Each point of the ergosphere belongs to some of the confocal ellipsoids (10) covering the whole space as the coordinate surfaces \( r = \text{const} \). The width of the ergosphere at a certain \( z \) is \( x^2 + y^2 = (a^2 + r^2)(1 - z^2/r^2) \). In the equatorial plane \( x^2 + y^2 = a^2 + r^2 \) provided that \( z^2 = 0 = 2f(r) - r^2 \). For any regular density profile the function \( f(r) \) is everywhere positive and monotonically grows from \( Dr^2 \) as \( r \to 0 \), where \( D \) is constant, to \( m \rho \), where \( m \) is the mass parameter. An ergosphere exists when the curve \( u = 2f(r) \) intersects or touches the parabola \( u = r^2 \) (curve 2 in figure 1). It is evident that in this case the curve \( u = 2f(r) \) also intersects the (situated above) parabola \( u = r^2 + a^2 \cos^2 \theta \) for a given \( \theta \). There are four cases of the existence of ergospheres and ergoregions (the regions where \( g_o < 0 \)).

**E1. Black hole case.** In this case, ergospheres and ergoregions exist for any density profile. At the \( z \)-axis equations (22) and (24) are identical, so the minor axis of the ergosphere is equal to \( r_a \). In accordance with (22), the function \( u = 2f(r) \) intersects or touches the parabola \( u = r^2 + a^2 \) (curve 1 in figure 1) and hence intersects the (situated below) parabola \( u = r^2 \), since near \( r = 0 \) a function \( f(r) \) goes to zero as \( r^2 \), faster than \( r^2 \). In the case of two horizons the curve \( u = 2f(r) \) intersects the parabola \( u = r^2 + a^2 \) on the internal horizon \( r = r_- \) and finally at \( r = r_+ \) which defines the
width of the ergosphere in the equatorial plane $x^2 + y^2 = a^2 + r_e^2$. In this case an ergoregion exists between the event horizon $r = r_e$ and the ergosphere (curve 3a in figure 1). In the case of a double horizon, an ergoregion exists between $r = r_e$ and the ergosphere (curve 3b in figure 1).

**E2. Soliton case.** In the case of absence of horizons, there are three options. The curve $u = 2f(r)$ can intersect the parabola $u = r^2$ twice, and there exists an ergoregion between two surfaces of intersection (curve 4a). The second option is that the curve $u = 2f(r)$ touches the parabola $u = r^2$ at a certain point $r = r_e$ (curve 4b). In this case an ergoregion exists beyond the ergosphere whose width in the equatorial plane is $x^2 + y^2 = a^2 + r_e^2$ and includes the whole interior. The third option is no intersections or touching, and hence no ergospheres.

### 3.2. De Sitter vacuum surfaces

Near the disk (11) the function $f(r)$ in (14) approaches the de Sitter asymptotic [27]

$$2f(r) \to \frac{k\tilde{\rho}(0)}{3} r^4 = \frac{r^4}{r_0^2} = \frac{3}{k\tilde{\rho}(0)}.$$  

(25)

Taking into account $\Sigma = (r^4 + a^2 z^2)r^{-2}$, we get

$$\frac{2f(r)}{\Sigma} = \frac{r^4}{r_0^2} \left( \frac{r^2}{r^4 + a^2 z^2} \right).$$  

(26)

In the equatorial plane this gives $2f(r)/\Sigma = k\tilde{\rho}(0)r^2/3$, so the disk $r = 0$ is intrinsically flat. However, the vacuum density is non-zero throughout the whole disk. In the equatorial plane $r^2/\Sigma \to 1$ as $z \to 0$ [22], and equation (21) reduces to

$$\kappa \left( p_\perp + \rho \right) = -r\tilde{\rho}^\prime(r)/2.$$  

(27)

For the regular spherical solutions $\tilde{\rho}^\prime \leq 0$ [25], and the WEC is satisfied for axially symmetric solutions in the equatorial plane.

By (20), the density in the equatorial plane is $\rho(r, \theta) = \tilde{\rho}(r)$. When $r \to 0$, $\tilde{\rho}(r) \to \tilde{\rho}(0)$, so on the disk $\rho(r, \theta) = \tilde{\rho}(0)$. For the spherically symmetric solutions regularity requires $r\tilde{\rho}^\prime(r) \to 0$ as $r \to 0$ [27]. As a result equation (27) gives on the disk the equation of state

$$p_\perp = \rho = -\rho$$  

(28)

which represents the rotating de Sitter vacuum.
Equation (21) can be written as
\[
\kappa \left( p_\perp + \rho \right) = \frac{2\rho^2}{\Sigma^2} \left( \frac{\Sigma r}{4} |\rho'| - \rho a^2 \cos^2 \theta \right).
\] (29)

This implies a possibility of generic violation of the WEC. The WEC requires \( T_{\mu\nu} w^\mu w^\nu \geq 0 \) for any time-like vector \( w^\mu \). Representing vector \( w^\mu \) in the tetrad (17) we find that the WEC is valid if \( \rho \geq 0 \) and \( p_\perp + \rho \geq 0 \). The first condition is satisfied according to (20). The WEC can be thus violated beyond the vacuum surface \( \mathcal{E}(r, z) = 0 \) on which \( p_\perp + \rho = 0 \) and the right-hand side in (29) can change its sign. It can be expressed through the pressure of a related spherical solution, \( \rho_{\perp, \rho} = -\rho - \rho^2 / 2 \) [25], which gives
\[
\kappa \left( p_\perp + \rho \right) = \frac{r |\rho'|}{2\Sigma^2} \mathcal{E}(r, z) = 0; \quad \mathcal{E}(r, z) = r^4 - z^2 P(r); \quad P(r) = \frac{2a^2}{r |\rho'|} \left( \hat{\rho} - \hat{\rho}_\perp \right).
\] (30)

As we see, the existence of vacuum surfaces and hence possible violation of the WEC is possible only for the mass functions originating from spherical solutions satisfying the dominant energy condition (\( \rho \geq \hat{\rho}_\perp \)).

Each point of the \( \mathcal{E} \)-surface belongs to some of the confocal ellipsoids (10). In the Cartesian coordinates (13) the equation of the ellipsoid (10) reads \( (r^2 - z^2)(a^2 + r^2) = r^2(\chi^2 + \gamma^2) \). On the \( \mathcal{E} \)-surface we have \( z^2 = r^4 / P(r) \). The squared width of the \( \mathcal{E} \)-surface, \( \mathcal{W}_E^2 = (\chi^2 + \gamma^2)_E = (a^2 + |z| \sqrt{P(r)} (1 - |z| / \sqrt{P(r)}) \). It is easily to show that the \( \mathcal{E} \)-surface is entirely confined within the \( r_* \)-ellipsoid whose minor axis coincides with \( z \in \text{max} \) for the \( \mathcal{E} \)-surface [40]. For regular solutions \( r\tilde{\rho} \to 0, \quad p_\perp \to -\rho \) as \( r \to 0 \) [27], and \( P(r) \to A^2 r^{-(n+1)} \) as \( r \to 0 \), with the integer \( n \geq 0 \). As a result the derivative of \( \mathcal{W}_E(z) \) near \( z \to 0 \) behaves as \( z^{-(\alpha + 1)/(\alpha + 5)} \) and goes to \( \pm \infty \) as \( z \to 0 \), so the function \( \mathcal{W}_E(z) \) has a cusp on approaching the disk and at least two symmetric (with respect to the equatorial plane) maxima between \( z = \pm r_* \) and \( z = 0 \) [40].

The \( \mathcal{E} \)-surface is plotted below for the regularized Coulomb profile [27]
\[
\hat{\rho} = \frac{q^2}{(r^2 + r_*^2)^2}; \quad r_q = \frac{\pi q^2}{8 m}.
\] (31)

For this profile \( P(r_\infty) = r_*^2 = r_q a \), and the \( \mathcal{E} \)-surface is given by \( \mathcal{E}(r, z) = r^6 - r_*^4 z^2 = 0 \).

The width of the \( \mathcal{E} \)-surface \( \mathcal{W}_E = (\chi^2 + \gamma^2)_E \) as a function of \( z \) has two maxima at \( z_m = \pm r_m / r_*^2 \). The relation between the width of the \( \mathcal{E} \)-surface in the equatorial plane \( \mathcal{W}_E = a \) and its height \( |z|_{\infty} = \sqrt{a q / r_q^2} \) defines the explicit form of the \( \mathcal{E} \)-surface. It depends on two parameters: \( \alpha = a / m \) and specific charge \( \beta = q / m \). In terms of these parameters
\[
H_E = \sqrt{\frac{\alpha}{r_q}} = \sqrt{2 a / \sqrt{\pi \beta}}; \quad W_E = \frac{\alpha}{r_q} = \frac{8 a}{\pi \beta^2}. \] (32)

For \( \alpha < \pi \beta^2 / 8 \) we have \( H_E / W_E > 1 \) and the \( \mathcal{E} \)-surface is prolate (figure 2). For black holes the parameter \( \beta \) changes within the range \( 0 < \beta < 0.99 \) [41, 42]. This can be the case of a slowly rotating moderately charged black hole.

In the case \( \alpha > \pi \beta^2 / 8 \), the \( \mathcal{E} \)-surface is oblate (figure 3). This is the case for an electromagnetic spinning soliton. It can be also the case of a slightly charged rotating black hole and of an extreme black hole.
4. Electromagnetic fields

4.1. Field equations

Non-zero field components compatible with the axial symmetry are $F_{01}, F_{02}, F_{31}, F_{23}$. In geometry with the metric (14) they are related by

$$F_{31} = a \sin^2 \theta F_{10}; \quad aF_{23} = (r^2 + a^2)F_{02}. \quad (33)$$

The field invariant $F = F_{\mu\nu}F^{\mu\nu}$ in the axially symmetric case reduces to

$$F = 2 \left( \frac{F_{20}^2}{a^2 \sin^2 \theta} - F_{10}^2 \right). \quad (34)$$
In terms of the 3-vectors, denoted by italic indices running from 1 to 3 and defined as
\[ E_j = \{ F_{j0} \}; \quad D^j = \{ \mathcal{L}_F F^0 \}; \quad B^j = \{ * F^0 \}; \quad H_j = \{ \mathcal{L}_F * F^0 \}, \]
the field equations (5) and (6) take the form of the source-free Maxwell equations
\[ \nabla \mathbf{D} = 0; \quad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}; \quad \nabla \mathbf{B} = 0; \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \]

The electric induction \( \mathbf{D} \) and the magnetic induction \( \mathbf{B} \) are related to the electric and magnetic field intensities by
\[ D^j = \epsilon_j^k E^k; \quad B^j = \mu_j^k H^k, \]
where \( \epsilon_j^k \) and \( \mu_j^k \) are the tensors of the electric and magnetic permeability given by [22]
\[ \epsilon_j^r = \frac{(r^2 + a^2)}{\Delta} \mathcal{L}_F; \quad \epsilon_j^\theta = \mathcal{L}_F; \quad \mu_j^r = \frac{(r^2 + a^2)}{\Delta \mathcal{L}_F}; \quad \mu_j^\theta = \frac{1}{\mathcal{L}_F}. \]

Equations (5) and (6) form the system
\[ \frac{\partial}{\partial r} \left[ \left( r^2 + a^2 \right) \sin \theta \mathcal{L}_F F_{10} \right] + \frac{\partial}{\partial \theta} \left[ \sin \theta \mathcal{L}_F F_{20} \right] = 0 \]
\[ \frac{\partial}{\partial r} \left[ \frac{1}{\sin \theta} \mathcal{L}_F F_{31} \right] + \frac{\partial}{\partial \theta} \left[ \frac{1}{\left( r^2 + a^2 \right) \sin \theta} \mathcal{L}_F F_{32} \right] = 0 \]
\[ \frac{\partial F_{23}}{\partial r} + \frac{\partial F_{31}}{\partial \theta} = 0 \]
\[ \frac{\partial F_{01}}{\partial r} + \frac{\partial F_{20}}{\partial \theta} = 0. \]

The four field components are related by two equations (33), so only two are independent and we can apply (33) to transform this system to the form
\[ \frac{\partial}{\partial r} \left[ \left( r^2 + a^2 \right) \sin \theta \mathcal{L}_F F_{10} \right] + \frac{\partial}{\partial \theta} \left[ \sin \theta \mathcal{L}_F F_{20} \right] = 0 \]
\[ \frac{\partial}{\partial r} \left[ a \sin \theta \mathcal{L}_F F_{10} \right] + \frac{\partial}{\partial \theta} \left[ \frac{1}{a \sin \theta} \mathcal{L}_F F_{20} \right] = 0 \]
\[ \frac{\partial}{\partial r} F_{20} - \frac{\partial}{\partial \theta} F_{10} = 0 \]
\[ \frac{\partial}{\partial \theta} \left[ a^2 \sin^2 \theta \mathcal{L}_F F_{10} \right] - \frac{\partial}{\partial r} \left[ \left( r^2 + a^2 \right) F_{20} \right] = 0. \]

For calculation of derivatives in (43) and (44) we need derivatives of the invariant \( \mathcal{F} \) which read
\[ \frac{\partial F}{\partial r} = 4 \left( \frac{F_{20}}{a^2 \sin^2 \theta} \frac{\partial F_{20}}{\partial r} - F_{10} \frac{\partial F_{10}}{\partial r} \right) \]
\[ \frac{\partial F}{\partial \theta} = 4 \left( \frac{F_{20}}{a^2 \sin^2 \theta} \frac{\partial F_{20}}{\partial \theta} - \frac{\cot \theta}{a^2 \sin^2 \theta} \frac{F_{20}^2}{\partial \theta} - F_{10} \frac{\partial F_{10}}{\partial \theta} \right). \]
Taking into account (47), (48) and (45), we reduce the system (43)–(46) to

\[
\begin{align*}
(r^2 + \alpha^2) \left( \mathcal{L}_F - 4\mathcal{L}_{FF} F_{10}^2 \right) \frac{\partial F_{10}}{\partial r} + \left( \mathcal{L}_F + 4\mathcal{L}_{FF} \frac{F_{20}^2}{a^2 \sin^2 \theta} \right) \frac{\partial F_{20}}{\partial \theta} \\
+ 4\mathcal{L}_{FF} \sum F_{10} F_{20} \frac{\partial F_{20}}{\partial r} + 2r \mathcal{L}_F F_{10} + F_{20} \cot \theta \left( \mathcal{L}_F - 4\mathcal{L}_{FF} \frac{F_{20}^2}{a^2 \sin^2 \theta} \right) = 0 \quad (49)
\end{align*}
\]

\[
\left( \mathcal{L}_F - 4\mathcal{L}_{FF} F_{10}^2 \right) \frac{\partial F_{10}}{\partial r} + \frac{1}{a^2 \sin^2 \theta} \left( \mathcal{L}_F + 4\mathcal{L}_{FF} \frac{F_{20}^2}{a^2 \sin^2 \theta} \right) \frac{\partial F_{20}}{\partial \theta} \\
- \frac{\cot \theta}{a^2 \sin^2 \theta} F_{20} \left( \mathcal{L}_F + 4\mathcal{L}_{FF} \frac{F_{20}^2}{a^2 \sin^2 \theta} \right) = 0 \quad (50)
\]

\[
\sum \frac{\partial F_{20}}{\partial r} + 2r F_{20} - a^2 \sin 2\theta F_{10} = 0 \quad (51)
\]

\[
\frac{\partial}{\partial r} F_{20} - \frac{\partial}{\partial \theta} F_{10} = 0. \quad (52)
\]

Two of these equations are strongly non-linear.

### 4.2. Asymptotic solutions

Dynamical equations \( V_\mu (\mathcal{L}_F F^{\mu \nu}) = 0 \) are satisfied by the functions \([22]\)

\[
\Sigma^2 (\mathcal{L}_F F_{01}) = -q \left( r^2 - a^2 \cos^2 \theta \right); \quad \Sigma^2 (\mathcal{L}_F F_{02}) = qa^2 r \sin 2\theta; \quad (53)
\]

\[
\Sigma^2 (\mathcal{L}_F F_{31}) = aq \sin^2 \theta \left( r^2 - a^2 \cos^2 \theta \right); \quad \Sigma^2 (\mathcal{L}_F F_{33}) = aq\left( r^2 + a^2 \right) \sin 2\theta. \quad (54)
\]

In the case \( \mathcal{L}_F = 1 \), they also satisfy the dynamical equations (6) and coincide with the solutions to the Maxwell–Einstein equations \([3, 12]\).

The field functions (54) and (53) satisfy the equations

\[
\frac{\partial (\mathcal{L}_F F_{23})}{\partial r} + \frac{\partial (\mathcal{L}_F F_{31})}{\partial \theta} = 0 \quad (55)
\]

\[
\frac{\partial (\mathcal{L}_F F_{01})}{\partial \theta} + \frac{\partial (\mathcal{L}_F F_{20})}{\partial r} = 0. \quad (56)
\]

It follows that

\[
\frac{\partial F_{23}}{\partial r} + \frac{\partial F_{31}}{\partial \theta} = -\frac{\mathcal{L}_{FF}}{\mathcal{L}_F} \left[ F_{31} \frac{\partial F}{\partial \theta} + F_{23} \frac{\partial F}{\partial r} \right] \quad (57)
\]

\[
\frac{\partial F_{01}}{\partial \theta} + \frac{\partial F_{20}}{\partial r} = -\frac{\mathcal{L}_{FF}}{\mathcal{L}_F} \left[ F_{01} \frac{\partial F}{\partial \theta} + F_{20} \frac{\partial F}{\partial r} \right]. \quad (58)
\]

Left-hand sides vanish identically when right-hand sides are zero. This defines the following cases when the functions (54) and (53) satisfy the dynamical system (39)–(42):

(A) \( \mathcal{L}_{FF} = 0, \mathcal{L}_F \neq 0 \), the case of the linear electromagnetic field;

(B) \( \mathcal{L}_F = \infty, \mathcal{L}_{FF} \neq 0 \), the strongly non-linear regime. The density of the electromagnetic energy given by (20) grows towards the interior disk \( r = 0 \). Applying (54) and (53) we obtain
For the regular solutions on the interior disk, geometry requires \( p_\perp + \rho = 0 \). It follows that \( \mathcal{L}_F \Sigma^2 \to \infty \) and hence \( \mathcal{L}_F \to \infty \), since \( \Sigma \to 0 \) on the disk. Case (B) thus represents realization of the underlying hypothesis of non-linearity replacing a singularity.

The third possibility is vanishing of the expression in the square brackets in the right-hand sides of equations (57) and (58)

\[
\begin{align*}
F_{31} \frac{\partial F}{\partial \theta} + F_{23} \frac{\partial F}{\partial r} &= 0, \\
F_{01} \frac{\partial F}{\partial \theta} + F_{20} \frac{\partial F}{\partial r} &= 0.
\end{align*}
\]

Taking into account (33), this system reduces to

\[
\begin{align*}
a^2 \sin^2 \theta F_{01} \frac{\partial F}{\partial \theta} + \left( r^2 + a^2 \right) F_{20} \frac{\partial F}{\partial r} &= 0, \\
F_{01} \frac{\partial F}{\partial \theta} + F_{20} \frac{\partial F}{\partial r} &= 0.
\end{align*}
\]

This is the system of two algebraic equations for \( F_{01}(\partial F/\partial \theta) \) and \( F_{20}(\partial F/\partial r) \). Its determinant is \( \text{Det} = -\Sigma \). The case \( \Sigma \neq 0 \) corresponds to the trivial solutions \( F_{01}(\partial F/\partial \theta) = 0, F_{20}(\partial F/\partial r) = 0 \), and includes the case

(C) \( F_{10} = F_{20} = 0 \), zero field regime.

4.3. Necessary condition for the existence of solutions

The basic question in the case of the system of four equations (39)–(42) for three functions \( F_{10}, F_{20} \) and \( \mathcal{L}_F \) is the question of its compatibility, i.e. compatibility of equations (5) and (6).

The system (39)–(42) for \( F_{10}, F_{20} \) and \( \mathcal{L}_F \), taking into account (33), reduces to

\[
\begin{align*}
\left( r^2 + a^2 \right) \frac{\partial F_{10}}{\partial r} + \frac{\partial F_{20}}{\partial \theta} &= A_1 F_{10} + A_2 F_{20}, \\
\frac{\partial F_{10}}{\partial r} + \frac{1}{a^2 \sin^2 \theta} \frac{\partial F_{20}}{\partial \theta} &= B_1 F_{10} + B_2 F_{20}, \\
\sin^2 \theta \frac{\partial F_{10}}{\partial \theta} - \frac{\left( r^2 + a^2 \right) \partial F_{20}}{a^2} &= C_1 F_{10} + C_2 F_{20}, \\
-\frac{\partial F_{10}}{\partial \theta} + \frac{\partial F_{20}}{\partial r} &= 0,
\end{align*}
\]

where

\[
\begin{align*}
A_1 &= -2r - \left( r^2 + a^2 \right) \frac{1}{\mathcal{L}_F} \frac{\partial \mathcal{L}_F}{\partial r}; \\
A_2 &= -\cot \theta - \frac{1}{\mathcal{L}_F} \frac{\partial \mathcal{L}_F}{\partial \theta}; \\
B_1 &= -\frac{1}{\mathcal{L}_F} \frac{\partial \mathcal{L}_F}{\partial r}; \\
B_2 &= \frac{\cos \theta}{a^2 \sin^2 \theta} - \frac{1}{a^2 \sin^2 \theta} \frac{\partial \mathcal{L}_F}{\partial \theta}; \\
C_1 &= -\sin 2\theta; \\
C_2 &= \frac{2r}{a^2}.
\end{align*}
\]

The system (64)–(67) can be resolved with respect to the derivatives of \( F_{10} \) and \( F_{20} \). This gives
\[ \frac{\partial F_{10}}{\partial r} = \hat{A}_1 F_{10} + \hat{A}_2 F_{20} \]  
(70)

\[ \frac{\partial F_{20}}{\partial \theta} = \hat{B}_1 F_{10} + \hat{B}_2 F_{20} \]  
(71)

\[ \frac{\partial F_{10}}{\partial \theta} = \frac{\partial F_{20}}{\partial r} = \hat{C}_1 F_{10} + \hat{C}_2 F_{20}, \]  
(72)

where

\[ \hat{A}_1 = -\frac{2r}{\Sigma} - \frac{1}{L_F} \frac{\partial L_F}{\partial r}; \quad \hat{A}_2 = -\frac{2 \cot \theta}{\Sigma}; \quad \hat{B}_1 = \frac{2 a^2 \sin^2 \theta}{\Sigma}; \]  
(73)

\[ \hat{B}_2 = \frac{1}{\Sigma} \left[ \cot \theta \left( r^2 + a^2 + a^2 \sin^2 \theta \right) \right] - \frac{1}{L_F} \frac{\partial L_F}{\partial \theta}; \quad \hat{C}_1 = \frac{a^2 \sin 2\theta}{\Sigma}; \quad \hat{C}_2 = -\frac{2r}{\Sigma}. \]  
(74)

Equality of the mixed second derivatives of \( F_{10} \) and \( F_{20} \) gives, respectively,

\[ D_1 F_{10} + D_2 F_{20} = 0, \quad D_3 F_{10} + D_4 F_{20} = 0, \]  
(75)

where

\[ D_1 = -\frac{\partial}{\partial \theta} \left( \frac{1}{L_F} \frac{\partial L_F}{\partial r} \right); \quad D_2 = \frac{2}{\Sigma L_F} \left[ \frac{\partial L_F}{\partial r} + \cot \theta \frac{\partial L_F}{\partial \theta} \right]; \]; \quad D_3 = \frac{2 a^2 \sin^2 \theta}{\Sigma L_F} \left[ \frac{\partial L_F}{\partial r} + \cot \theta \frac{\partial L_F}{\partial \theta} \right]; \quad D_4 = \frac{\partial}{\partial r} \left( \frac{1}{L_F} \frac{\partial L_F}{\partial \theta} \right). \]  
(76)

We obtained the uniform system of two algebraic equations (75) with respect to field tensions \( F_{10} \) and \( F_{20} \). A necessary and sufficient condition of the existence of a non-trivial solution of this system is vanishing of its determinant. Hence, the necessary and sufficient condition of compatibility of equations (64)–(67) is

\[ D_2 D_4 - D_1 D_3 = 0. \]  
(78)

In the explicit form this reads

\[ \frac{\partial}{\partial r} \left( \frac{1}{L_F} \frac{\partial L_F}{\partial \theta} \right) \frac{\partial}{\partial \theta} \left( \frac{1}{L_F} \frac{\partial L_F}{\partial r} \right) + \frac{4 a^2 \sin^2(\theta)}{\Sigma^2 L_F^2} \left[ \frac{\partial L_F}{\partial r} + \cot(\theta) \frac{\partial L_F}{\partial \theta} \right]^2 = 0. \]  
(79)

This is the condition on a function \( L_F \), which is the necessary and sufficient condition of compatibility of equations (5) and (6) and hence the necessary condition for the existence of solutions.

Condition (79) is evidently satisfied for \( L_F = \text{const} \), which can be normalized to \( L_F = 1 \) and corresponds to the Maxwell weak field limit, and in the case of trivial zero field solutions.

Equation (79) can be written as

\[ \frac{1}{L_F^2} \left( \frac{\partial^2 L_F}{\partial r \partial \theta} \right)^2 = \frac{L_{ff}}{L_F^2} \left( \frac{\partial^2 L_F}{\partial r \partial \theta} \right)^2 \left[ \frac{\partial F}{\partial \theta} \frac{\partial L_F}{\partial r} + \frac{\partial F}{\partial r} \frac{\partial L_F}{\partial \theta} \right] + \frac{4 a^2 \sin^2(\theta)}{\Sigma^2 L_F^2} \left[ \frac{\partial L_F}{\partial r} \right]^2 \]  
(80)

This condition should be analyzed carefully since it implies that \( L_F(r, \theta) \subset C^2 \). However, this may not be the case on the whole manifold. For regular solutions, invariant \( F \) vanishes on
the disk, where it is given by [22]

\[ F = -\frac{\kappa^2 (p_\perp + \rho) \Sigma^2}{2q^2}, \]  

(81)

and vanishes at infinity in the Maxwell weak field limit. Lagrangian \( \mathcal{L}(F) \) is a function of a non-monotonic function \( F \) with equal limiting values and should suffer branching and have a cusp at a certain value of \( F \). Correct description of Lagrange dynamics requires a non-uniform variational problem [37]. On the boundary hypersurface which divides two regions of the manifold with different Lagrangians, the function \( \mathcal{L}_F \) does not belong to the class \( C^2 \) as a function of \( r \) and \( \theta \). This point has been studied separately and the results will be reported elsewhere [43]. Here we point out that \( \mathcal{L}_F \) is finite in the boundary region and is \( C^2 \) in the region described by the internal Lagrangian including the interior region where \( \mathcal{L}_F \to \infty \). In this region asymptotic solutions (54) and (53) satisfy dynamical Equations (5) and (6); regularity requires \( \mathcal{L}_F \Sigma^2 \to \infty \) and \( \mathcal{L}_F \to \infty \), so that the condition (80) is satisfied and gives formal confirmation of the compatibility of the system (5) and (6) in this limit.

5. Vacuum interiors

The relation connecting density and pressure with the electromagnetic fields reads [22]

\[ \kappa (p_\perp + \rho) = 2\mathcal{L}_F \left( F_{\alpha \beta}^2 + \frac{F_{\theta \theta}}{a^2 \sin^2 \theta} \right). \]

(82)

The vacuum \( \mathcal{E} \)-surface is defined by \( p_\perp + \rho = 0 \). By virtue of (59) this leads to \( \mathcal{L}_F \to \infty \). As a result the magnetic permeability vanishes and the electric permeability goes to infinity, so the \( \mathcal{E} \)-surface and the disk display the properties of a perfect conductor and ideal diamagnetic. In the limit \( \mathcal{L}_F \to \infty \) the magnetic induction vanishes on the vacuum \( \mathcal{E} \)-surface and on the disk by virtue of the asymptotic solutions (53), which satisfy the dynamical equations (5) and (6) in this limit.

On the de Sitter disk \( r = 0 \) we obtain from (37)–(38) \( \epsilon' = \epsilon_\theta = \mathcal{L}_F \) and \( \mu' = \mu_\theta = \mathcal{L}_F^{-1} = \mu \to 0 \). The magnetic induction \( B \) also vanishes on the disk. In the electrodynamics of continued media the transition to a superconducting state corresponds to the limits \( B \to 0 \) and \( \mu \to 0 \) in a surface current \( j_s = \frac{(1-p)}{4\pi n} [nB] \), where \( n \) is the normal to the surface. The right-hand side then becomes indeterminate, and there is no condition which would restrict the possible values of the current [39]. On the de Sitter disk we can apply definition of a surface current for a charged surface layer, \( 4\pi j_k = [\epsilon_\alpha \delta^{\alpha}_{(k)} F_{\alpha \beta} n^\beta \] [4], where \( \ldots \) denotes a jump across the layer; \( \epsilon_\alpha^{(k)} \) are the tangential base vectors associated with the intrinsic coordinates on the disk \( t, \phi, 0 \leq \xi \leq \pi/2 \); \( n_\alpha = (1 + \mu^2/a^2)^{-1/2} \cos \xi \) \( \delta^k_\alpha \) is the unit normal directed upwards [4]. Using asymptotic solutions (53) and magnetic permeability \( \mu = 1/\mathcal{L}_F \), we obtain the surface current [32]

\[ j_s = -\frac{q}{2\pi a} \sqrt{1 + q^2/a^2} \sin^2 \xi \frac{\mu}{\cos^2 \xi}. \]

(83)

On approaching the ring \( r = 0 \), \( \xi = \pi/2 \), both terms in the second fraction go to zero quite independently. As a result the surface currents on the ring can take any value and amount to a non-zero total sum. Superconducting currents flowing (forever) on the de Sitter vacuum ring
can be considered as a source of Kerr–Newman fields. This kind of source is non-dissipative so the lifetime of the electromagnetic spinning structure can be unlimited.

We find that the interior de Sitter vacuum \( E \)-surface, which contains the de Sitter disk as the bridge, with zero magnetic induction on the whole surface, exists. The next question is what is going on within the \( E \)-surface, in cavities between its upper and lower boundaries and the bridge. A negative value of \( (p_\perp + \rho) \) in (82) would mean negative values for the electric and magnetic permeabilities, inadmissible in electrodynamics of continued media [39].

One possibility to satisfy the basic requirement of electrodynamics of continued media can be a zero value of \( (p_\perp + \rho) \) inside the \( E \)-surface. This can be the case for the shell-like models ([9] and references therein) with flat vacuum interior, zero fields and in consequence zero density and pressures, and no violation of the WEC.

The other possibility, favored by the underlying idea of non-linearity replacing a singularity and suggested by vanishing of magnetic induction on the surrounding \( E \)-surface, is the extension of \( \mathcal{L}_F \to \infty \) to its interiors. Then we have a de Sitter vacuum core, \( p = -\rho \), with the properties of a perfect conductor and ideal diamagnetic, zero magnetic induction, and valid WEC throughout the whole core.

6. Summary and discussion

We have studied regular rotating electrically charged rotating black holes and solitons in non-linear electrodynamics non-minimally coupled to gravity with a model-independent approach based on generic information following from the source-free dynamical equations, in which non-linear electromagnetic fields provide a source in the Einstein equations.

Regular spherically symmetric NED-GR solutions are asymptotically de Sitter for \( r \to 0 \) and asymptotically Reissner–Nordström for \( r \to \infty \). They always satisfy the WEC since \( \kappa (p_\perp + \rho) = -\mathcal{L}_F \), the invariant \( F \) is generically non-positive and \( \mathcal{L}_F \) represents the electric permeability, which cannot be negative in electrodynamics of continued media.

The spherical solutions give rise, by the Gürses–Gürsey algorithm, to regular axially symmetric solutions describing rotating electrically charged black holes and solitons, asymptotically Kerr–Newman for a distant observer. Black holes have at most two horizons and ergoregions. Solitons can have two ergoregions.

We formulated the necessary and sufficient conditions of compatibility of dynamical equations governing the electromagnetic fields dynamics, necessary for the existence of solutions, and found asymptotic solutions needed to study field dynamics in the interior regions.

Rotation transforms the de Sitter center into the de Sitter disk \( r = 0 \) with \( p_\perp + \rho = 0 \) and with superconducting current flowing on the surrounding ring (in place of the Kerr singularity). Superconducting current can be regarded as a non-dissipative source of the Kerr–Newman exterior fields, providing unlimited lifetime of NED-GR spinning regular objects [32].

The WEC could be violated in the case when the related spherical solution satisfies the dominant energy condition. In this case there exists the vacuum \( E \)-surface defined by \( p_\perp + \rho = 0 \) with the de Sitter disk \( r = 0 \) as a bridge. Both the \( E \)-surface and the disk have properties of the perfect conductor and ideal diamagnetic. Violation of the WEC inside the vacuum surface would need negative values of electric and magnetic permeability, inadmissible in electrodynamics of continued media. We can conclude that the WEC is not violated in the interiors of regular rotating charged black holes and solitons in NED-GR
theories compatible with the basic requirement of electrodynamics of continued media on the electric and magnetic permeability.

An alternative favored by the underlying idea of non-linearity regularizing a singularity, and suggested by vanishing magnetic induction on the vacuum $E$-surface, is extension of its basic properties to its interior. Then the regular interior is presented by a de Sitter vacuum core with properties of a perfect conductor and ideal diamagnetic.

This work was motivated by the search for an image of the electron, inspired by the Dirac paper on the extended electron [44], and by Carter’s remarkable discovery of the ability of the Kerr–Newman solution to represent the electron for a distant observer. The electromagnetic spinning soliton can be considered as the Coleman lump, a non-singular, non-dissipative solution of finite energy holding itself together by its own self-interaction [45], which can be applied as the model for the extended electron.

For the electron $q = -e$ and $a = \lambda/2$, where $\lambda = \hbar/m$ is its Compton wavelength [3]. In the observer region $r \gg \lambda_c$

\[
E_r = \frac{e}{r^2} \left(1 - \frac{\hbar^2}{m_e^2 c^2} \frac{3 \cos^2 \theta}{4r^2}\right); \quad E_\theta = \frac{e\hbar^2}{m_e^2 c^2} \frac{\sin 2\theta}{4r^3};
\]

\[
B_r = -\frac{e\hbar \cos \theta}{m_e c} \frac{\cos \theta}{r^3}; \quad B_\theta = -\frac{e\hbar \sin \theta}{m_e c} \frac{\sin \theta}{r^4}.
\]  \hspace{1cm} (84)

The Planck constant appears due to the ability discovered by Carter of the Kerr–Newman solution to present the electron as seen by a distant observer. In terms of the Coleman lump, (84) describes the following situation: the leading term in $E_r$ gives the Coulomb law as the classical limit $\hbar = 0$, and the higher terms represent the quantum corrections.

The purely electromagnetic reaction of annihilation $e^+ e^- \rightarrow \gamma \gamma$ reveals, at the $5\sigma$ confidence level, the existence of the minimal length $l_\xi = 1.57 \times 10^{-17}$ cm at the scale $E = 1.253$ TeV, which can be explained as the distance of closest approach of annihilating particles where electromagnetic attraction is stopped by the gravitational repulsion of the interior de Sitter vacuum of the electromagnetic spinning soliton [46].

References

[1] Newman E T, Cough E, Chinnapared K, Exton A, Prakash A and Torrence R 1965 J. Math. Phys. 6 918
[2] Newman E T and Janis A J 1965 J. Math. Phys. 6 915
[3] Carter B 1968 Phys. Rev. 174 1559
[4] Israel W 1970 Phys. Rev. D 2 641
[5] Burinskii A Y 1974 Sov. Phys. JETP 39 193
[6] López C A 1983 Nuovo Cim. 76 B 9
[7] de la Cruz V, Chase J E and Israel W 1970 Phys. Rev. Lett. 24 423
[8] Cohen J M 1967 J. Math. Phys. 8 1477
[9] López C A 1984 Phys. Rev. D 30 313
[10] Boyer R H 1965 Proc. Cambridge Phil. Soc. 61 527
[11] Boyer R H 1966 Proc. Cambridge Phil. Soc. 62 495
[12] Trümper M 1967 Z. Naturforschung 22a 1347
[13] Tiomno J 1973 Phys. Rev. D 7 992
[14] Burinskii A Ya 1989 Phys. Lett. B 216 123
[15] Burinskii A 2002 Grav. Cosmology 8 261
[16] Burinskii A, Elizalde E, Hildebrandt S R and Magli G 2002 Phys. Rev. D 65 064039
[17] Burinskii A 2014 Int. J. Mod. Phys. A 29 1450133
[18] Fradkin E S and Tseytlin A A 1985 Phys. Lett. B 163 123
[19] Tseytlin A A 1986 *Nucl. Phys.* B **276** 391
[20] Siberg N and Witten E 1999 *J. High Energy Phys.* JHEP99(1999)032
[21] Born M and Infeld L 1934 *Proc. R. Soc. London* A **144** 425
[22] Dymnikova I 2006 *Phys. Lett.* B **639** 368
[23] Dymnikova I 1992 *Gen. Rel. Grav.* **24** 235
[24] Dymnikova I 2000 *Phys. Lett.* B **472** 33
[25] Dymnikova I 2002 *Class Quantum Grav.* **19** 725
[26] Dymnikova I 2003 *Int. J. Mod. Phys.* D **12** 1015
[27] Dymnikova I 2004 *Class Quantum Grav.* **21** 4417
[28] Kerr R P and Schild A 1965 *Proc. Symp. Appl. Math.* **17** 199
[29] Elizalde E and Hildebrandt S R 2002 *Phys. Rev.* D **65** 124024
[30] Gürses M and Güryüz F 1975 *J. Math. Phys.* **16** 2385
[31] Burinskii A and Hildebrandt S R 2002 *Phys. Rev.* D **65** 104017
[32] Dymnikova I 2015 *J. Grav.* **2015** 904171
[33] Bambi C and Modesto L 2013 *Phys. Lett.* B **721** 329
[34] Neves J C S and Saa A 2014 *Phys. Lett.* B **734** 44
[35] Toshmatov B, Ahmedov B, Abduljabarov A and Stuchlik Z 2014 *Phys. Rev.* D **89** 104017
[36] Bronnikov K A 2001 *Phys. Rev.* D **63** 044005
[37] Dymnikova I, Galaktionov E and Tropp E 2015 *Adv. Math. Phys.* **2015** 496475
[38] Chandrasekhar S 1983 *The Mathematical Theory of Black Holes* (Oxford, UK: Clarendon Press)
[39] Landau L D and Lifshitz E M 1993 *Electrodynamics of Continous Media* (Oxford, UK: Pergamon Press)
[40] Dymnikova I and Galaktionov E 2015 *Phys. Lett.* B submitted
[41] Zilhão M, Cardoso V, Herdeiro C, Lehner L and Sperhake U 2014 *Phys. Rev.* D **89** 044008
[42] Hamilton A J S 2011 *Phys. Rev.* D **84** 124057
[43] Dymnikova I, Galaktionov E and Tropp E 2015 *Adv. Math. Phys.* **2015** 496475
[44] Dirac P A M 1962 *Proc. R. Soc. Lond.* A **268** 57
[45] Coleman S 1977 *New Phenomena Subnuclear Physics* ed A Zichichi (New York: Plenum) 297
[46] Dymnikova I, Sakharov A and Ulbricht J 2014 *AHEP* **2014** 707812