Toward quantum-like modeling of financial processes

Olga Choustova
International Center for Mathematical Modeling in Physics and Cognitive Sciences,
University of Växjö, S-35195, Sweden
E-mail: Olga.Choustova@vxu.se

Abstract. We apply methods of quantum mechanics for mathematical modeling of price dynamics at the financial market. We propose to describe behavioral financial factors (e.g., expectations of traders) by using the pilot wave (Bohmian) model of quantum mechanics. Trajectories of prices are determined by two financial potentials: classical-like $V(q)$ ("hard" market conditions, e.g., natural resources) and quantum-like $U(q)$ (behavioral market conditions). On the one hand, our Bohmian model is a quantum-like model for the financial market, cf. with works of W. Segal, I. E. Segal, E. Haven, E. W. Piotrowski, J. Sladkowski. On the other hand, (since Bohmian mechanics provides the possibility to describe individual price trajectories) it belongs to the domain of extended research on deterministic dynamics for financial assets (C.W. J. Granger, W.A. Barnett, A. J. Benhabib, W.A. Brock, C. Sayers, J. Y. Campbell, A. W. Lo, A. C. MacKinlay, A. Serletis, S. Kuchta, M. Frank, R. Gencay, T. Stengos, M. J. Hinich, D. Patterson, D. A. Hsieh, D. T. Caplan, J.A. Scheinkman, B. LeBaron and many others).

1. Introduction
In economics and financial theory, analysts use random walk and more general martingale techniques to model behavior of asset prices, in particular share prices on stock markets, currency exchange rates and commodity prices. This practice has its basis in the presumption that investors act rationally and without bias, and that at any moment they estimate the value of an asset based on future expectations. Under these conditions, all existing information affects the price, which changes only when new information comes out. By definition, new information appears randomly and influences the asset price randomly. Corresponding continuous time models are based on stochastic processes (this approach was initiated in the thesis of L. Bachelier [1] in 1890), see, e.g., the books of R. N. Mantegna and H. E. Stanley [2] and A. Shiryaev [3] for historical and mathematical details.

This practice was formalized through the efficient market hypothesis which was formulated in the sixties, see P. A. Samuelson [4] and E. F. Fama [5] for details:

A market is said to be efficient in the determination of the most rational price if all the available information is instantly processed when it reaches the market and it is immediately reflected in a new value of prices of the assets traded.

Mathematically the efficient market hypothesis was supported by investigations of Samuelson [4]. Using the hypothesis of rational behavior and market efficiency he was able to demonstrate...
how \( q_{t+1} \), the expected value of price of a given asset at time \( t + 1 \), is related to the previous values of prices \( q_0, q_1, ..., q_t \) through the relation

\[
E(q_{t+1}|q_0, q_1, ..., q_t) = q_t.
\]  

Typically there is introduced the \( \sigma \)-algebra \( \mathcal{F}_t \) generated by random variables \( q_0, q_1, ..., q_t \). The condition (1) is written in the form:

\[
E(q_{t+1}|\mathcal{F}_t) = q_t.
\]  

Stochastic processes of such a type are called martingales [3]. Alternatively, the martingale model for the financial market implies that the \( (q_{t+1} - q_t) \) is a “fair game” (a game which is neither in your favor nor your opponent’s):

\[
E(q_{t+1} - q_t|\mathcal{F}_t) = 0.
\]

On the basis of information, \( \mathcal{F}_t \), which is available at the moment \( t \), one cannot expect neither \( E(q_{t+1} - q_t|\mathcal{F}_t) > 0 \) nor \( E(q_{t+1} - q_t|\mathcal{F}_t) < 0 \).

First we remark that empirical studies have demonstrated that prices do not completely follow random walk. Low serial correlations (around 0.05) exist in the short term; and slightly stronger correlations over the longer term. Their sign and the strength depend on a variety of factors, but transaction costs and bid-ask spreads generally make it impossible to earn excess returns. Interestingly, researchers have found that some of the biggest price deviations from random walk result from seasonal and temporal patterns, see the book [2].

There are also a variety of arguments, both theoretical and obtained on the basis of statistical analysis of data, which question the general martingale model (and hence the efficient market hypothesis), see, e.g., [6]–[13]. It is important to note that efficient markets imply there are no exploitable profit opportunities. If this is true then trading on the stock market is a game of chance and not of any skill, but traders buy assets they think are unvalued at the hope of selling them at their true price for a profit. If market prices already reflect all information available, then where does the trader draw this privileged information from? Since there are thousands of very well informed, well educated asset traders, backed by many data researchers, buying and selling securities quickly, logically assets markets should be very efficient and profit opportunities should be minimal. On the other hand, we see that there are many traders whom successfully use their opportunities and perform continuously very successful financial operations, see the book of G. Soros [14] for discussions.³ There were also performed intensive investigations on testing that the real financial data can be really described by the martingale model, see [6]–[13]. Roughly speaking people try to understand on the basis of available financial data:

Do financial asset returns behave randomly (and hence they are unpredictable) or deterministically (and in the latter case one may hope to predict them and even to construct a deterministic dynamical system which would at least mimic dynamics of the financial market)?

Predictability of financial asset returns is a broad and very active research topic and a complete survey of the vast literature is beyond the scope of this work. We shall note, however, that there is a rather general opinion that financial asset returns are predictable, see [6]–[13].

On the other hand, there is no general consensus on the validity of the efficient market hypothesis. As it was pointed out in [11]: “... econometric advances and empirical evidence seem to suggest that financial asset returns are predictable to some degree. Thirty years ago this would have been tantamount to an outright rejection of market efficiency. However,

¹ It seems that G. Soros is sure he does not work at efficient markets.
modern financial economics teaches us that others, perfectly rational factors may account for such predictability. The fine structure of securities markets and frictions in trading process can generate predictability. Time-varying expected returns due to changing business conditions can generate predictability. A certain degree of predictability may be necessary to reward investors for bearing certain dynamic risks.”

Therefore it would be natural to develop approaches which are not based on the assumption that investors act rationally and without bias and that, consequently, new information appears randomly and influences the asset price randomly. In particular, there are two well established (and closely related) fields of research behavioral finance and behavioral economics which apply scientific research on human and social cognitive and emotional biases\(^2\) to better understand economic decisions and how they affect market prices, returns and the allocation of resources. The fields are primarily concerned with the rationality, or lack thereof, of economic agents. Behavioral models typically integrate insights from psychology with neo-classical economic theory. Behavioral analysis are mostly concerned with the effects of market decisions, but also those of public choice, another source of economic decisions with some similar biases.

Since the 1970s, the intensive exchange of information in the world of finances has become one of the main sources determining dynamics of prices. Electronic trading (that became the most important part of the environment of the major stock exchanges) induces huge information flows between traders (including foreign exchange market). Financial contracts are performed at a new time scale that differs essentially from the old ”hard” time scale that was determined by the development of the economic basis of the financial market. Prices at which traders are willing to buy (bid quotes) or sell (ask quotes) a financial asset are not only determined by the continuous development of industry, trade, services, situation at the market of natural resources and so on. Information (mental, market-psychological) factors play a very important (and in some situations crucial) role in price dynamics. Traders performing financial operations work as a huge collective cognitive system. Roughly speaking classical-like dynamics of prices (determined) by ”hard” economic factors are permanently perturbed by additional financial forces, mental (or market-psychological) forces, see the book of J. Soros\(^14\).

In this paper we develop a new approach that is not based on the assumption that investors act rationally and without bias and that, consequently, new information appears randomly and influences the asset price randomly. Our approach can be considered as a special econophysical\(^2\) model in the domain of behavioral finance. In our approach information about the financial market (including expectations of agents of the financial market) is described by an information field \(ψ(q)\) – financial wave. This field evolves deterministically\(^3\) perturbing the dynamics of prices of stocks and options. Since the psychology of agents of the financial market gives an important contribution into the financial wave \(ψ(q)\), our model can be considered as a special psycho-financial model.

This paper can be also considered as a contribution into applications of quantum mechanics outside microworld, see\(^{15},^{16},^{17}\). This paper is fundamentally based on investigations of D. Bohm, B. Hiley, and P. Pylkkänen\(^{18},^{19}\) on the active information interpretation of Bohmian mechanics\(^20,^{21}\) and its applications to cognitive sciences, see also Khrennikov\(^17\).

In this paper we use methods of Bohmian mechanics to simulate dynamics of prices at the financial market. We start with the development of the classical Hamiltonian formalism on the

\(^{2}\) Cognitive bias is any of a wide range of observer effects identified in cognitive science, including very basic statistical and memory errors that are common to all human beings and drastically skew the reliability of anecdotal and legal evidence. They also significantly affect the scientific method which is deliberately designed to minimize such bias from any one observer. They were first identified by Amos Tversky and Daniel Kahneman as a foundation of behavioral economics. Bias arises from various life, loyalty and local risk and attention concerns that are difficult to separate or codify. Tversky and Kahneman claim that they are at least partially the result of problem-solving using heuristics, including the availability heuristic and the representativeness.

\(^{3}\) Dynamics is given by Schrödinger’s equation on the space of prices of shares.
price/price-change phase space to describe the classical-like evolution of prices. This classical dynamics of prices is determined by "hard" financial conditions (natural resources, industrial production, services and so on). These conditions as well as "hard" relations between traders at the financial market are mathematically described by the classical financial potential. As we have already remarked, at the real financial market "hard” conditions are not the only source of price changes. The information and market psychology play important (and sometimes determining) role in price dynamics.

We propose to describe those “soft” financial factors by using the pilot wave (Bohmian) model of quantum mechanics. The theory of financial mental (or psychological) waves is used to take into account market psychology. The real trajectories of prices are determined (by the financial analogue of the second Newton law) by two financial potentials: classical-like (“hard” market conditions) and quantum-like (“soft” market conditions).

Our quantum-like model of financial processes was strongly motivated by consideration by J. Soros [14] of the financial market as a complex cognitive system. Such an approach he called the theory of reflexivity. In this theory there is a large difference between market that is "ruled" by only "hard" economical factors and a market where mental factors play the crucial role (even changing the evolution of the "hard" basis, see [14]).

J. Soros rightly remarked that the “non mental” market evolves due to classical random fluctuations. However, such fluctuations do not provide an adequate description of mental market. He proposed to use an analogy with quantum theory. However, it was noticed that directly quantum formalism could not be applied to the financial market [14]. Traders differ essentially from elementary particles. Elementary particles behave stochastically due to perturbation effects provided by measurement devices, cf. [23], [24].

According to J. Soros, traders at the financial market behave stochastically due to free will of individuals. Combinations of a huge number of free wills of traders produce additional stochasticity at the financial market that could not be reduced to classical random fluctuations (determined by non mental factors). Here J. Soros followed to the conventional (Heisenberg, Bohr, Dirac, see, e.g., [23], [24]) viewpoint to the origin of quantum stochasticity. However, in the Bohmian approach (that is nonconventional one) quantum statistics is induced by the action of an additional potential, quantum potential, that changes classical trajectories of elementary particles. Such an approach gives the possibility to apply quantum formalism to the financial market.

There were performed numerous investigations on applying quantum methods to financial market, see, e.g., E. Haven [25]–[27], that were not directly coupled to behavioral modeling, but based on the general concept that randomness of the financial market can be better described by the quantum mechanics, see, e.g., [28]: “A natural explanation for extreme irregularities in the evolution of prices in financial markets is provided by quantum effects.”

Non-Bohmian quantum model for the financial market was developed by E. W. Piotrowski, J. Sladowski [30]. This model can be also considered as a kind of behavioral quantum-like model.

2. A brief introduction to Bohmian mechanics

In this section we present the basic notions of Bohmian mechanics. This is a special model of quantum mechanics in that, in the opposition to the conventional Copenhagen interpretation, quantum particles (e.g., electrons) have well defined trajectories in physical space.

By the conventional Copenhagen interpretation (that was created by N. Bohr and W. Heisenberg) quantum particles do not have trajectories in physical space. Bohr and Heisenberg motivated such a viewpoint to quantum physical reality by the Heisenberg uncertainty relation:

\[ \Delta q \Delta p \geq \frac{h}{2} \]  

(4)
where \( h \) is the Planck constant, \( q \) and \( p \) are the position and momentum, respectively, and \( \Delta q \) and \( \Delta p \) are uncertainties in determination of \( q \) and \( p \). Nevertheless, David Bohm demonstrated [20], see also [21], that, in spite of Heisenberg’s uncertainty relation (4), it is possible to construct a quantum model in that trajectories of quantum particles are well defined. Since this paper is devoted to mathematical models in economy and not to physics, we would not go deeper into details. We just mention that the root of the problem lies in different interpretations of Heisenberg’s uncertainty relation (4). If one interpret \( \Delta q \) and \( \Delta p \) as uncertainties for the position and momentum of an individual quantum particle (e.g., one concrete electron) then (4), of course implies that it is impossible to create a model in that the trajectory \( q(t), p(t) \) is well defined. On the other hand, if one interpret \( \Delta q \) and \( \Delta p \) as statistical deviations

\[
\Delta q = \sqrt{E(q - E_q)^2}, \quad \Delta p = \sqrt{E(p - E_p)^2},
\]

then there is no direct contradiction between Heisenberg’s uncertainty relation (4) and the possibility to consider trajectories. There is a place to such models as Bohmian mechanics. Finally, we remark (but without comments) that in real experiments with quantum systems, one always uses the statistical interpretation (5) of \( \Delta q \) and \( \Delta p \).

We emphasize that the conventional quantum formalism cannot say anything about the individual quantum particle. This formalism provides only statistical predictions on huge ensembles of particles. Thus Bohmian mechanics provides a better description of quantum reality, since there is the possibility to describe trajectories of individual particles. However, this great advantage of Bohmian mechanics was not explored so much in physics. Up to now there have not been done experiments that would distinguish predictions of Bohmian mechanics and conventional quantum mechanics.

In this paper we shall show that the mentioned advantages of Bohmian mechanics can be explored in applications to the financial market. In the latter case it is really possible to observe the trajectory of the price or price-change dynamics. Such a trajectory is described by equations of mathematical formalism of Bohmian mechanics.

We now present the detailed derivation of the equations of motion of a quantum particle in the Bohmian model of quantum mechanics. Typically in physical books it is presented very briefly. But, since this paper is oriented to economists and mathematicians, who are not so much aware about quantum physics, we shall present all calculations. The dynamics of the wave function \( \psi(t, q) \) is described by Schrödinger’s equation

\[
i \hbar \frac{\partial \psi}{\partial t}(t, q) = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial q^2}(t, q) + V(q)\psi(t, q)
\]

(6)

Here \( \psi(t, q) \) is a complex valued function. At the moment we prefer not to discuss the conventional probabilistic interpretation of \( \psi(t, q) \). We consider \( \psi(t, q) \) as just a field.\(^4\)

We consider the one-dimensional case, but the generalization to the multidimensional case, \( q = (q_1, \ldots, q_n) \), is straightforward. Let us write the wave function \( \psi(t, q) \) in the following form:

\[
\psi(t, q) = R(t, q)e^{i\theta(t, q)}/\hbar
\]

(7)

where \( R(t, q) = |\psi(t, q)| \) and \( \theta(t, q) = S(t, q)/\hbar \) is the argument of the complex number \( \psi(t, q) \).

We put (7) into Schrödinger’s equation (6). We have

\[
i \hbar \frac{\partial \psi}{\partial t} = i\hbar \left( \frac{\partial R}{\partial t} \frac{\psi}{R} + \frac{iR}{\hbar} \frac{\partial S}{\partial t} \frac{\psi}{R} \right) = \frac{i\hbar}{\partial t} \frac{\psi}{R} - \frac{R}{\partial t} \frac{\psi}{R}
\]

\(^4\) We recall that by the probability interpretation of \( \psi(t, q) \) (which was proposed by Max Born) the quantity \( |\psi(t, q)|^2 \) gives the probability to find a quantum particle at the point \( q \) at the moment \( t \).
and
\[ \frac{\partial \psi}{\partial q} = \frac{\partial R}{\partial q} e^{i \frac{\psi}{\hbar}} + i R \frac{\partial S}{\partial q} e^{i \frac{\psi}{\hbar}} \]

and hence:
\[ \frac{\partial^2 \psi}{\partial q^2} = \frac{\partial^2 R}{\partial q^2} e^{i \frac{\psi}{\hbar}} + \frac{2i}{\hbar} \frac{\partial R}{\partial q} \frac{\partial S}{\partial q} e^{i \frac{\psi}{\hbar}} + i R \frac{\partial^2 S}{\partial q^2} e^{i \frac{\psi}{\hbar}} - \frac{R}{\hbar^2} \left( \frac{\partial S}{\partial q} \right)^2 e^{i \frac{\psi}{\hbar}} \]

We obtain the differential equations:
\[ \frac{\partial R}{\partial t} = -\frac{1}{2m} \left( 2 \frac{\partial R}{\partial q} \frac{\partial S}{\partial q} + R \frac{\partial^2 S}{\partial q^2} \right), \quad (8) \]
\[ -R \frac{\partial S}{\partial t} = -\frac{\hbar^2}{2m} \left( \frac{\partial^2 R}{\partial q^2} - \frac{R}{\hbar^2} \left( \frac{\partial S}{\partial q} \right)^2 \right) + VR. \quad (9) \]

By multiplying the right and left-hand sides of the equation (8) by 2R and using: \( \frac{\partial R^2}{\partial t} = 2R \frac{\partial R}{\partial t} \)
and \( \frac{\partial}{\partial q} (R^2 \frac{\partial S}{\partial q}) = 2R \frac{\partial R}{\partial q} \frac{\partial S}{\partial q} + R^2 \frac{\partial^2 S}{\partial q^2} \),
we get the equation for \( R^2 \):
\[ \frac{\partial R^2}{\partial t} + \frac{1}{m} \frac{\partial}{\partial q} (R^2 \frac{\partial S}{\partial q}) = 0. \quad (10) \]

We remark that if one uses the Born’s probabilistic interpretation of the wave function, then \( R^2 = |\psi|^2 \) gives the probability. Thus the equation (10) is the equation describing the dynamics of the probability distribution (in physics it is called the continuity equation).

The second equation can be written in the form:
\[ \frac{\partial S}{\partial t} + \frac{1}{2m} \left( \frac{\partial S}{\partial q} \right)^2 + \left( V - \frac{\hbar^2}{2mR} \frac{\partial^2 R}{\partial q^2} \right) = 0. \quad (11) \]

Suppose that
\[ \frac{\hbar^2}{2m} \ll 1 \]
and that the contribution of the term
\[ \frac{\hbar^2}{2mR} \frac{\partial^2 R}{\partial q^2} \]
can be neglected. Then we obtain the equation:
\[ \frac{\partial S}{\partial t} + \frac{1}{2m} \left( \frac{\partial S}{\partial q} \right)^2 + V = 0. \quad (12) \]

From the classical mechanics, we know that this is the classical Hamilton-Jacobi equation which corresponds to the dynamics of particles:
\[ p = \frac{\partial S}{\partial q} \text{ or } m \dot{q} = \frac{\partial S}{\partial q}, \quad (13) \]

where particles moves normal to the surface \( S = \text{const.} \)

David Bohm proposed to interpret the equation (11) in the same way. But we see that in this equation the classical potential \( V \) is perturbed by an additional ”quantum potential”
\[ U = \frac{\hbar^2}{2mR} \frac{\partial^2 R}{\partial q^2}. \]
Thus in the Bohmian mechanics the motion of a particle is described by the usual Newton equation, but with the force corresponding to the combination of the classical potential $V$ and the quantum one $U$:

$$\frac{m}{dt}\frac{dv}{dt} = -\left(\frac{\partial V}{\partial q} - \frac{\partial U}{\partial q}\right)$$

(14)

The crucial point is that the potential $U$ is by itself driven by a field equation - Schrödinger’s equation (6). Thus the equation (14) cannot be considered as just the Newton classical dynamics (because the potential $U$ depends on $\psi$ as a field parameter). We shall call (14) the Bohm-Newton equation.

We remark that typically in books on Bohmian mechanics [20], [21] it is emphasized that the equation (14) is nothing else than the ordinary Newton equation. This makes the impression that the Bohmian approach gives the possibility to reduce quantum mechanics to ordinary classical mechanics. However, this is not the case. The equation (14) does not provide the complete description of dynamics of a system. Since, as was pointed out, the quantum potential $U$ is determined through the wave function $\psi$ and the latter evolves according to the Schrödinger equation, the dynamics given by Bohm-Newton equation cannot be considered independent of the Schrödinger’s dynamics.

3. Financial phase-space

Let us consider a mathematical model in which a huge number of agents of the financial market interact with one another and take into account external economic (as well as political, social and even meteorological) conditions in order to determine the price to buy or sell financial assets. We consider the trade with shares of some corporations (e.g., VOLVO, SAAB, IKEA,...).

We consider a price system of coordinates. We enumerate corporations which did emissions of shares at the financial market under consideration: $j = 1, 2, ..., n$ (e.g., VOLVO: $j = 1$, SAAB: $j = 2$, IKEA: $j = 3, ...$). There can be introduced the $n$-dimensional configuration space $Q = R^n$ of prices,

$$q = (q_1, ..., q_n),$$

where $q_j$ is the price of a share of the $j$th corporation. Here $R$ is the real line. Dynamics of prices is described by the trajectory

$$q(t) = (q_1(t), ..., q_n(t))$$

in the configuration price space $Q$.

Another variable under the consideration is the price change variable:

$$v_j(t) = \dot{q}_j(t) = \lim_{\Delta t \to 0} \frac{q_j(t + \Delta t) - q_j(t)}{\Delta t},$$

see, for example, the book [2] on the role of the price change description. In real models we consider the discrete time scale $\Delta t, 2\Delta t, ...$. Here we should use a discrete price change variable

$$\delta q_j(t) = q_j(t + \Delta t) - q_j(t).$$

We denote the space of price changes by the symbol $V(\equiv R^n), v = (v_1, ..., v_n)$. As in classical physics, it is useful to introduce the phase space $Q \times V = R^{2n}$, namely the price phase space.

$^5$ Similar models can be developed for trade with options, see E. Haven [29] for the Bohmian financial wave model for portfolio.
A pair \((q,v)\) = (price, price change) is called a state of the financial market. (Later we shall consider quantum-like states of the financial market. A state \((q,v)\) is a classical state.)

We now introduce an analogue \(m\) of mass as the number of items (in our case shares) that a trader emitted to the market.\(^6\) We call \(m\) the financial mass. Thus each trader \(j\) (e.g., VOLVO) has its own financial mass \(m_j\) (the size of the emission of its shares). The total price of the emission performed by the \(j\)th trader is equal to \(T_j = m_j q_j\) (this is nothing else than market capitalization). Of course, it depends on time: \(T_j(t) = m_j q_j(t)\). To simplify considerations, we consider a market at that any emission of shares is of the fixed size, so \(m_j\) does not depend on time. In principle, our model can be generalized to describe a market with time-dependent financial masses, \(m_j = m_j(t)\).

We also introduce financial energy of the market as a function \(H : Q \times V \to R\). If we use the analogue with classical mechanics. (Why not? In principle, there is not so much difference between motions in "physical space" and "price space".), then we could consider (at least for mathematical modeling) the financial energy of the form:

\[
H(q,v) = \frac{1}{2} \sum_{j=1}^{n} m_j v_j^2 + V(q_1, \ldots, q_n).
\]

Here \(K = \frac{1}{2} \sum_{j=1}^{n} m_j v_j^2\) is the kinetic financial energy and \(V(q_1, \ldots, q_n)\) is the potential financial energy, \(m_j\) is the financial mass of \(j\)th trader.

The kinetic financial energy represents efforts of agents of financial market to change prices: higher price changes induce higher kinetic financial energies. If the corporation \(j_1\) has higher financial mass than the corporation \(j_2\), so \(m_{j_1} > m_{j_2}\), then the same change of price, i.e., the same financial velocity \(v_{j_1} = v_{j_2}\), is characterized by higher kinetic financial energy: \(K_{j_1} > K_{j_2}\). We also remark that high kinetic financial energy characterizes rapid changes of the financial situation at market. However, the kinetic financial energy does not give the attitude of these changes. It could be rapid economic growth as well as recession.

The potential financial energy \(V\) describes the interactions between traders \(j = 1, \ldots, n\) (e.g., competition between NOKIA and ERICSSON) as well as external economic conditions (e.g., the price of oil and gas) and even meteorological conditions (e.g., the weather conditions in Louisiana and Florida). For example, we can consider the simplest interaction potential:

\[
V(q_1, \ldots, q_n) = \sum_{j=1}^{n} (q_i - q_j)^2.
\]

The difference \(|q_i - q_j|\) between prices is the most important condition for arbitrage.

We could never take into account all economic and other conditions that have influences to the market. Therefore by using some concrete potential \(V(q)\) we consider the very idealized model of financial processes. However, such an approach is standard for physical modeling where we also consider idealized mathematical models of real physical processes. We apply the Hamiltonian dynamics on the price phase space. As in classical mechanics for material objects, we introduce a new variable \(p = mv\), the price momentum variable. Instead of the price change vector \(v = (v_1, \ldots, v_n)\), we consider the price momentum vector \(p = (p_1, \ldots, p_n)\), where \(p_j = m_j v_j\). The space of price momentums is denoted by the symbol \(P\). The space \(\Omega = Q \times P\) will be also called the price phase space. Hamiltonian equations of motion on the price phase space have the form:

\[
\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}, \quad j = 1, \ldots, n.
\]

\(^6\) ‘Number’ is a natural number \(m = 0, 1, \ldots\) – the price of share, e.g., in the US-dollars. However, in a mathematical model it can be convenient to consider real \(m\). This can be useful for transitions from one currency to another.
If the financial energy has form (15) then the Hamiltonian equations have the form

\[ \dot{q}_j = \frac{p_j}{m_j} = v_j, \quad \dot{p}_j = -\frac{\partial V}{\partial q_j}. \]

The latter equation can be written in the form:

\[ m_j \dot{v}_j = -\frac{\partial V}{\partial q_j}. \]

The quantity

\[ \dot{v}_j = \lim_{\Delta t \to 0} \frac{v_j(t + \Delta t) - v_j(t)}{\Delta t} \]

is natural to call the price acceleration (change of price change). The quantity \( f_j(q) = -\frac{\partial V}{\partial q_j} \) is called the (potential) financial force. We get the financial variant of the second Newton law:

\[ m \ddot{v} = f \]  

(16)

"The product of the financial mass and the price acceleration is equal to the financial force."

In fact, the Hamiltonian evolution is determined by the following fundamental property of the financial energy: The financial energy is not changed in the process of Hamiltonian evolution: \( H(q_1(t), \ldots, q_n(t), p_1(t), \ldots, p_n(t) = H(q_1(0), \ldots, q_n(0), p_1(0), \ldots, p_n(0)) \). We need not restrict our considerations to financial energies of form (15). First of all external (e.g. economic) conditions as well as the character of interactions between traders at the market depend strongly on time. This must be taken into account by considering time dependent potentials: \( V = V(t, q) \). Moreover, the assumption that the financial potential depends only on prices, \( V = V(t, q) \), is not so natural for the modern financial market. Financial agents have the complete information on price changes. This information is taken into account by traders for acts of arbitrage, see [2] for the details. Therefore, it can be useful to consider potentials that depend not only on prices, but also on price changes: \( V = V(t, q, v) \) or in the Hamiltonian framework: \( V = V(t, q, p) \). In such a case the financial force is not potential. Therefore, it is also useful to consider the financial second Newton law for general financial forces: \( m \ddot{v} = f(t, q, p) \).

Remark 1. (On the form of the kinetic financial energy) We copied the form of kinetic energy from classical mechanics for material objects. It may be that such a form of kinetic financial energy is not justified by real financial market. It might be better to consider our choice of the kinetic financial energy as just the basis for mathematical modeling (and looking for other possibilities).

Remark 2. (Domain of price-dynamics) It is natural to consider a model in that all prices are nonnegative, \( q_j(t) \geq 0 \). Therefore financial Hamiltonian dynamics should be considered in the phase space \( \Omega_+ = R_+^n \times R^n \), where \( R_+ \) is the set of nonnegative real numbers. We shall not study this problem in details, because our aim is the study of the corresponding quantum dynamics. But in the quantum case this problem is solved easily. One should just consider the corresponding Hamiltonian in the space of square integrable functions \( L^2(\Omega_+) \). Another possibility in the classical case is to consider centered dynamics of prices: \( z_j(t) = q_j(t) - q(0) \). The centered price \( z_j(t) \) evolves in the configuration space \( R^n \).

4. Classical dynamics does not describe the real stock market

The model of Hamiltonian price dynamics on the price phase space can be useful to describe a market that depends only on “hard” economic conditions: natural resources, volumes of
production, human resources and so on. However, the classical price dynamics could not be applied (at least directly) to modern financial markets. It is clear that the stock market is not based only on these “hard” factors. There are other factors, soft ones (behavioral), that play the important and (sometimes even determining) role in forming of prices at the financial market. Market’s psychology should be taken into account. Negligibly small amounts of information (due to the rapid exchange of information) imply large changes of prices at the financial market. We can consider a model in that financial (psychological) waves are permanently present at the market. Sometimes these waves produce uncontrollable changes of prices disturbing the whole market (financial crashes). Of course, financial waves also depend on “hard economic factors.” However, these factors do not play the crucial role in forming of financial waves. Financial waves are merely waves of information.  

5. Financial pilot waves

If we interpret the pilot wave as a field, then we should pay attention that this is a rather strange field. It differs crucially from “ordinary physical fields,” i.e., the electromagnetic field. We mention some of the pathological features of the pilot wave field, see [20], [18], [21] for the detailed analysis. In particular, the force induced by this pilot wave field does not depend on the amplitude of wave. Thus small waves and large waves equally disturb the trajectory of an elementary particle. Such features of the pilot wave give the possibility to speculate, see [18], [19], that this is just a wave of information (active information). Hence, the pilot wave field describes the propagation of information. The pilot wave is more similar to a radio signal that guides a ship. Of course, this is just an analogy (because a radio signal is related to an ordinary physical field, namely, the electromagnetic field). The more precise analogy is to compare the pilot wave with information contained in the radio signal.

We remark that the pilot wave (Bohmian) interpretation of quantum mechanics is not the conventional one. As we have already noted, there are a few critical arguments against Bohmian quantum formalism:

1. Bohmian theory gives the possibility to provide the mathematical description of the trajectory \( q(t) \) of an elementary particle. However, such a trajectory does not exist according to the conventional quantum formalism.

2. Bohmian theory is not local, namely, via the pilot wave field one particle “feels” another on large distances.

We say that these disadvantages of theory will become advantages in our applications of Bohmian theory to financial market. We also recall that already Bohm and Hiley [18] and Hiley and Pilkänen [19] discussed the possibility to interpret the pilot wave field as a kind of information field. This information interpretation was essentially developed in works of Khrennikov [17] that were devoted to pilot wave cognitive models.

Our fundamental assumption is that agents at the modern financial market are not just “classical-like agents.” Their actions are ruled not only by classical-like financial potentials
V(t, q_1, \ldots, q_n), but also (in the same way as in the pilot wave theory for quantum systems) by an additional information (or psychological) potential induced by a financial pilot wave.

Therefore we could not use the classical financial dynamics (Hamiltonian formalism) on the financial phase space to describe the real price trajectories. Information (psychological) perturbation of Hamiltonian equations for price and price change must be taken into account. To describe such a model mathematically, it is convenient to use such an object as a fundamental postulate of the pilot wave theory that rules the financial market.

In some sense \( \psi(q) \) describes the psychological influence of the price configuration \( q \) to behavior of agents of the financial market. In particular, the \( \psi(q) \) contains expectations of agents.\(^8\)

We underline two important features of the financial pilot wave model:

1. All shares are coupled on the information level. The general formalism \([20], [18], [21]\) of the pilot wave theory says that if the function \( \psi(q_1, \ldots, q_n) \) is not factorized, i.e.,

\[
\psi(q_1, \ldots, q_n) \neq \psi_1(q_1) \ldots \psi_n(q_n),
\]

then any changing the price \( q_i \) will automatically change behavior of all agents of the financial market (even those who have no direct coupling with \( i \)-shares). This will imply changing of prices of \( j \)-shares for \( i \neq j \). At the same time the “hard” economic potential \( V(q_1, \ldots, q_n) \) need not contain any interaction term: for example, \( V(q_1, \ldots, q_n) = q_1^2 + \ldots + q_n^2 \). The Hamiltonian equations in the absence of the financial pilot wave have the form: \( \dot{q}_j = p_j, \ p_j = -2q_j, \ j = 1, 2, \ldots, n. \) Thus the classical price trajectory \( q_i(t) \), does not depend on dynamics of prices of shares for other traders \( i \neq j \) (for example, the price of shares of ERIKSSON does not depend on the price of shares of NOKIA and vice versa).\(^9\)

However, if, e.g.,

\[
\psi(q_1, \ldots, q_n) = ce^{i(q_1q_2 + \ldots + q_{n-1}q_n)}e^{-(q_1^2 + \ldots + q_n^2)},
\]

where \( c \in C \) is some normalization constant, then financial behavior of agents at the financial market is nonlocal (see further considerations).

2. Reactions of the market do not depend on the amplitude of the financial pilot wave: waves \( \psi, 2\psi, 100000\psi \) will produce the same reaction. Such a behavior at the market is quite natural (if the financial pilot wave is interpreted as an information wave, the wave of financial information). The amplitude of an information signal does not play so large role in the information exchange. The most important is the context of such a signal. The context is given by the shape of the signal, the form of the financial pilot wave function.

6. The Dynamics of Prices Guided by the Financial Pilot Wave

In fact, we need not develop a new mathematical formalism. We will just apply the standard pilot wave formalism to the financial market. The fundamental postulate of the pilot wave theory is that the pilot wave (field) \( \psi(q_1, \ldots, q_n) \) induces a new (quantum) potential \( U(q_1, \ldots, q_n) \) which perturbs the classical equations of motion. A modified Newton equation has the form:

\[
\dot{p} = f + g,
\]

\(^8\) The reader may be surprised that there appeared complex numbers \( C \). However, the use of these numbers is just a mathematical trick that provides the simple mathematical description of dynamics of the financial pilot wave.

\(^9\) Such a dynamics would be natural if these corporations operate on independent markets, e.g., ERIKSSON in Sweden and NOKIA in Finland. Prices of their shares would depend only on local market conditions, e.g., on capacities of markets or consuming activity.
where
\[ f = -\frac{\partial V}{\partial q} \]
and
\[ g = -\frac{\partial U}{\partial q}. \]

We call the additional financial force \( g \) a financial mental force. This force \( g(q_1, \ldots, q_n) \) determines a kind of collective consciousness of the financial market. Of course, the \( g \) depends on economic and other ‘hard’ conditions given by the financial potential \( V(q_1, \ldots, q_n) \). However, this is not a direct dependence. In principle, a nonzero financial mental force can be induced by the financial pilot wave \( \psi \) in the case of zero financial potential, \( V \equiv 0 \). So \( V \equiv 0 \) does not imply that \( U \equiv 0 \). Market psychology is not totally determined by economic factors. Financial (psychological) waves of information need not be generated by some changes in a real economic situation. They are mixtures of mental and economic waves. Even in the absence of economic waves, mental financial waves can have a large influence to the market.

By using the standard pilot wave formalism we obtain the following rule for computing the financial mental force. We represent the financial pilot wave \( \psi(q) \) in the form:
\[ \psi(q) = R(q)e^{iS(q)} \]
where \( R(q) = |\psi(q)| \) is the amplitude of \( \psi(q) \), (the absolute value of the complex number \( c = \psi(q) \)) and \( S(q) \) is the phase of \( \psi(q) \) (the argument of the complex number \( c = \psi(q) \)). Then the financial mental potential is computed as
\[ U(q_1, \ldots, q_n) = -\frac{1}{R} \sum_{i=1}^{n} \frac{\partial^2 R}{\partial q_i^2} \]
and the financial mental force as
\[ g_j(q_1, \ldots, q_n) = -\frac{\partial U}{\partial q_j}(q_1, \ldots, q_n). \]

These formulas imply that strong financial effects are produced by financial waves having essential variations of amplitudes.

**Example 1.** (Financial waves with small variation have no effect). Let \( R(q) = c \) \( \equiv \) const. Then the financial (behavioral) force \( g \equiv 0 \). As \( R \equiv \) const, it is impossible to change expectations of the whole financial market by varying the price \( q_j \) of one fixed type of shares, \( j \). The constant information field does not induce psychological financial effects at all. As we have already remarked the absolute value of this constant does not play any role. Waves of constant amplitude \( R = 1 \), as well as \( R = 10^{100} \), produce no effect.

Let \( R(q) = cq, c > 0 \). This is a linear function; variation is not so large. As the result \( g \equiv 0 \) here also. No financial behavioral effects.

**Example 2.** (Speculation) Let
\[ R(q) = c(q^2 + d), \ c, d > 0. \]

Here
\[ U(q) = -\frac{2}{q^2 + d}. \]
(it does not depend on the amplitude $c$) and

$$g(q) = \frac{-4q}{(q^2 + d)^2}.$$  

The quadratic function varies essentially more strongly than the linear function, and, as a result, such a financial pilot wave induces a nontrivial financial force.

We analyze financial drives induced by such a force. We consider the situation: (the starting price) $q > 0$ and $g < 0$. The financial force $g$ stimulates the market (which works as a huge cognitive system) to decrease the price. For small prices,

$$g(q) \approx -4q/d^2.$$  

If the financial market increases the price $q$ for shares of this type, then the negative reaction of the financial force becomes stronger and stronger. The market is pressed (by the financial force) to stop increasing of the price $q$. However, for large prices,

$$g(q) \approx -4/q^3.$$  

If the market can approach this range of prices (despite the negative pressure of the financial force for relatively small $q$) then the market will feel decreasing of the negative pressure (we recall that we consider the financial market as a huge cognitive system). This model explains well the speculative behavior of the financial market.

Let

$$R(q) = c(q^4 + b), \ c, b > 0.$$  

Thus

$$g(q) = \frac{bq - q^5}{(q^4 + b)^2}.$$  

Here the behavior of the market is more complicated. Set $d = \sqrt{b}$. If the price $q$ is changing from $q = 0$ to $q = d$ then the market is motivated (by the financial force $g(q)$) to increase the price. The price $q = d$ is critical for his financial activity. By psychological reasons (of course, indirectly based on the whole information available at the market) the market "understands" that it would be dangerous to continue to increase the price. After approaching the price $q = d$, the market has the psychological stimuli to decrease the price.

Financial pilot waves $\psi(q)$ with $R(q)$ that are polynomials of higher order can induce very complex behavior. The interval $[0, \infty)$ is split into a collection of subintervals $0 < d_1 < d_2 < \ldots < d_n < \infty$ such that at each price level $q = d_j$ the trader changes his attitude to increase or to decrease the price.

In fact, we have considered just a one-dimensional model. In the real case we have to consider multidimensional models of huge dimension. A financial pilot wave $\psi(q_1, \ldots, q_n)$ on such a price space $Q$ induces splitting of $Q$ into a large number of domains $Q = O_1 \cup \ldots \cup O_N$.

The only problem which we have still to solve is the description of the time-dynamics of the financial pilot wave, $\psi(t, q)$. We follow the standard pilot wave theory. Here $\psi(t, q)$ is found as the solution of Schrödinger’s equation. The Schrödinger equation for the energy $H(q, p) = \frac{1}{2} \sum_{j=1}^{n} \frac{p_j^2}{m_j} + V(q_1, \ldots, q_n)$ has the form:

$$i\hbar \frac{\partial \psi}{\partial t}(t, q_1, \ldots, q_n) =$$

$$- \sum_{j=1}^{n} \frac{\hbar^2}{2m_j} \frac{\partial^2 \psi(t, q_1, \ldots, q_n)}{\partial q_j^2} + V(q_1, \ldots, q_n)\psi(t, q_1, \ldots, q_n),$$  

(18)
with the initial condition $\psi(0, q_1, \ldots, q_n) = \psi(q_1, \ldots, q_n)$. Thus if we know $\psi(0, q)$ then by using Schrödinger’s equation we can find the pilot wave at any instant of time $t$, $\psi(t, q)$. Then we compute the corresponding mental potential $U(t, q)$ and mental force $g(t, q)$ and solve Newton’s equation.

We shall use the same equation to find the evolution of the financial pilot wave. We have only to make one remark, namely, on the role of the constant $h$ in Schrödinger’s equation, cf. E. Haven [26], [27], [29]. In quantum mechanics (which deals with microscopic objects) $h$ is the Planck constant. This constant is assumed to play the fundamental role in all quantum considerations. However, originally $h$ appeared as just a scaling numerical parameter for processes of energy exchange. Therefore in our financial model we can consider $h$ as a price scaling parameter, namely, the unit in which we would like to measure price change. We do not present any special value for $h$. There are numerous investigations into price scaling. It may be that there can be recommended some special value for $h$ related to the modern financial market, a fundamental financial constant. However, it seems that $h = h(t)$ evolves depending on economic development.

We suppose that the financial pilot wave evolves via the financial Schrödinger equation (an analogue of Schrödinger’s equation) on the price space. In the general case this equation has the form:

$$ih \frac{\partial \psi}{\partial t}(t, q) = \hat{H}\psi(t, q), \psi(0, q) = \psi(q),$$

where $\hat{H}$ is self-adjoint operator corresponding to the financial energy given by a function $H(q, p)$ on the financial phase space. Here we proceed in the same way as in ordinary quantum theory for elementary particles.

### 7. On the choice of a measure of classical fluctuations

As the mathematical basis of the model we use the space $L_2(Q)$ of square integrable functions $\psi : Q \to \mathbb{C}$, where $Q$ is the configuration price space, $Q = \mathbb{R}^n$, or some domain $Q \subset \mathbb{R}^n$ (for example, $Q = \mathbb{R}^n_+$):

$$||\psi||^2 = \int_Q |\psi(x)|^2 dx < \infty.$$

Here $dx$ is the Lebesque measure, a uniform probability distribution, on the configuration price space. Of course, the uniform distribution $dx$ is not the unique choice of the normalization measure on the configuration price space. By choosing $dx$ we assume that in the absence of the pilot wave influence, all prices “have equal rights.” In general, this is not true. If there is no financial (psychological) waves the financial market still strongly depends on “hard” economic conditions. In general, the choice of the normalization measure $M$ must be justified by a real relation between prices. So in general the financial pilot wave $\psi$ belongs to the space $L_2(Q, dM)$ of square integrable functions with respect to some measure $M$ on the configuration price space:

$$||\psi||^2 = \int_Q |\psi(x)|^2 dM(x) < \infty.$$

In particular, $M$ can be a Gaussian measure:

$$dM(x) = \frac{1}{(2\pi detB)^{n/2}} e^{-\frac{1}{2}(x - \alpha)^T B^{-1}(x - \alpha)} dx,$$

where $B = (b_{ij})_{i,j=1}^n$ is the covariance matrix and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is the average vector. The measure $M$ describes classical random fluctuations in the financial market that are not related to ‘quantum’ (behavioral) effects. The latter effects are described in our model by the financial pilot wave. If the influence of this wave is very small we can use classical probabilistic models; in particular, based on the Gaussian distribution.
8. Momentum representation
The Gaussian model for price change fluctuations was the first financial probabilistic model — Bachelier’s model [1]. In fact, Bachelier described price dynamics by Brownian motion. Therefore it would be even more natural to consider the Gaussian distribution of price changes.

Thus it is useful to study the momentum representation for the pilot wave theory which was recently developed by B. Hiley [22]. Instead of financial waves on the configuration financial space \( Q \) we can consider financial waves \( \psi : P \rightarrow \mathbb{C} \)

on the momentum space. We recall that the momentum \( p = mv \), where \( v \), the velocity, describes price changes. Therefore by following Bachelier we have to consider Gaussian distribution generated by \( \psi \in L_2(P, dM) \), where \( P \) is the price change space and \( M \) is a Gaussian measure on \( P \).

However, further investigations demonstrated that it seems that the Gaussian model for price changes is not the best one for describing the financial market. One of alternative models is based on the Levy process. Therefore it can be useful to investigate the ‘quantum’ financial model that is based on the Cauchy (Lorentzian) measure:

\[
dM(p) = \frac{\gamma}{\pi \gamma^2 + p^2} dp
\]

on the momentum financial space. The Cauchy measure has some advantages. By using this measure we exclude from consideration financial waves which increase at infinity as polynomial function. So \( \psi(p) \approx p^k, k \geq 1, p \rightarrow \infty \), are excluded from such a model.\(^\text{10}\)

It looks very natural since prices (at least in the real financial market) can not change arbitrarily quickly.

9. Application of the general quantum formalism to the financial market
We now turn back to the general scheme, concentrating on the configuration representation, \( \psi : Q \rightarrow \mathbb{C}; \psi \in L_2(Q) \equiv L_2(Q, dx) \). This is the general quantum-like statistical formalism on the price space.

As in ordinary quantum mechanics, we consider a representation of financial quantities, observables, by symmetric operators in \( L_2(Q) \). By using Schrödinger’s representation we define price and price change operators by setting:

\[
\hat{q}_j \psi(q) = q_j \psi(q),
\]

the operator of multiplication by the \( q_j \)-price;

\[
\hat{p}_j = \frac{\hbar}{i} \frac{\partial}{\partial q_j},
\]

the operator of differentiation with respect to the \( q_j \)-price, normalized by the scaling constant \( \hbar \) (and \( -i = \frac{1}{\hbar} \) which provides the symmetry of \( \hat{p}_j \)). Operators of price and price change satisfy the canonical commutation relations:

\[
[\hat{q}, \hat{p}] = \hat{q}\hat{p} - \hat{p}\hat{q} = i\hbar.
\]

\(^\text{10}\)On the other hand, if we use the model with the Gaussian distribution of price changes, then such wave functions should be considered (because any polynomial function is integrable with respect to a Gaussian measure).
By using this operator representation of price and price changes we can represent every function \( H(q, p) \) on the financial phase-space as an operator \( \hat{H}(\hat{q}, \hat{p}) \) in \( L_2(Q) \). In particular, the financial energy operator is represented by the operator:

\[
\hat{H} = \sum_{j=1}^{n} \frac{\hat{p}_j^2}{2m_j} + V(\hat{q}_1, ..., \hat{q}_n) = -\sum_{j=1}^{n} \frac{\hbar^2}{2m_j} \frac{\partial^2}{\partial q_j^2} + V(q_1, ..., q_n).
\]

Here \( V(\hat{q}_1, ..., \hat{q}_n) \) is the multiplication operator by the function \( V(q_1, ..., q_n) \).

In this general quantum-like formalism for the financial market we do not consider individual evolution of prices (in the opposition to the Bohmian approach). The theory is purely statistical. We can only determine the average of a financial observable \( A \) for some fixed state \( \phi \) of the financial market:

\[
\langle A \rangle_{\phi} = \int_Q A(\phi)(x) \hat{\phi}(x) dx.
\]

The use of the Bohmian model gives the additional possibility of determining individual trajectories.

10. Standard deviation of price

We are interested in the standard deviation of the price \( q_t \). Let \( \psi \) be the mental state of the financial market. The quantum formalism gives us the following formula for the price-dispersion:

\[
\sigma^2_{\psi}(q_t) = E_{\psi}q_t^2 - (E_{\psi}q_t)^2,
\]

where for an observable \( a \) the quantum average (with respect to the state \( \psi \)) is given by \( E_{\psi}a = (a\psi, \psi) \).

Since, for any observable \( a_t \),

\[
E_{\psi}a_t = E_{\psi(t)}a_0,
\]

we have

\[
\sigma^2_{\psi}(q_t) = E_{\psi(t)}q^2 - (E_{\psi(t)}q)^2.
\]

So:

\[
\sigma^2_{\psi}(q_t) = (q^2\psi(t), \psi(t)) - (q\psi(t), \psi(t))^2.
\]

Suppose that at the initial instant of time the wave function has the form of a Gaussian packet:

\[
\psi_0(q) \approx \int_{-\infty}^{+\infty} \exp\{-k^2(\Delta q)^2 + ikq\} dk,
\]

where \( \Delta q \) is the width of packet in the price space. Here the mean value of price is equal to zero. It is well known, see, e.g., [18], p. 46, that

\[
\psi(t, q) \approx \int_{-\infty}^{+\infty} \exp\{-k^2(\Delta q)^2 + ikq - (ik^2t)/2m\} dk.
\]

Here the mean value of price is equal to zero for any instance of time. By calculating this integral, see again [18], p. 46, we see that

\[
\sigma_{\psi}(q_t) = \sqrt{(q^2\psi(t), \psi(t)) - (q\psi(t), \psi(t))^2} = \sqrt{(q^2\psi(t), \psi(t))} \approx \hbar t/m \Delta q
\]

for large \( t \).

Thus for a Gaussian packet of prices its standard deviation evolves as a linear function with respect to \( t \). Large financial mass (i.e., the higher level of emission of shares) induces smaller standard deviation – so the price does not fluctuate far from the mean value. If the level of emission was very small, then there can be expected large deviations from the mean value.
11. Comparing with conventional models for the financial market

Our model of the stocks market differs crucially from the main conventional models. Therefore we should perform an extended comparative analysis of our model and known models. This is not a simple task and it takes a lot of efforts.

11.1. The stochastic model

Since the pioneer paper of L. Bachelier [1], there was actively developed various models of the financial market based on stochastic processes. We recall that Bachelier determined the probability of price changes \( P(v(t) \leq v) \) by writing down what is now called the Chapman-Kolmogorov equation. If we introduce the density of this probability distribution: \( p(t, x) \), so \( P(x_t \leq x) = \int_{-\infty}^{x} p(t, x) \, dx \), then it satisfies to the Cauchy problem of the partial differential equation of the second order. This equation is known in physics as Chapman’s equation and in probability theory as the direct Kolmogorov equation. In the simplest case the underlying diffusion process is the Wiener process (Brownian motion), this equation has the form (the heat conduction equation):

\[
\frac{\partial p(t, x)}{\partial t} = \frac{1}{2} \frac{\partial^2 p(t, x)}{\partial x^2}.
\]

We recall again that in the Bachelier paper [1], \( x = v \) was the price change variable.

For a general diffusion process we have the direct Kolmogorov equation:

\[
\frac{\partial p(t, x)}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial x^2} (\sigma^2(t, x)p(t, x)) - \frac{\partial}{\partial x} (\mu(t, x)p(t, x)).
\]

This equation is based on the diffusion process

\[
dx_t = \mu(t, x_t) \, dt + \sigma(t, x_t) \, dw_t,
\]

where \( w(t) \) is the Wiener process. This equation should be interpreted as a slightly colloquial way of expressing the corresponding integral equation

\[
x_t = x_{t_0} + \int_{t_0}^{t} \mu(s, x_s) \, ds + \int_{t_0}^{t} \sigma(s, x_s) \, dw_s.
\]

We pay attention that Bachelier original proposal of Gaussian distributed price changes was soon replaced by a model of in which prices of stocks are log-normal distributed, i.e., stocks prices \( q(t) \) are performing a geometric Brownian motion. In a geometric Brownian motion, the difference of the logarithms of prices are Gaussian distributed.

We recall that a stochastic process \( S_t \) is said to follow a geometric Brownian motion if it satisfies the following stochastic differential equation:

\[
dS_t = u \, S_t \, dt + v \, S \, dw_t
\]

where \( w_t \) is a Wiener process (=Brownian motion) and \( u \) (“the percentage drift”) and \( v \) (“the percentage volatility”) are constants. The equation has an analytic solution:

\[
S_t = S_0 \exp \left( (u - v^2/2) t + vw_t \right)
\]

The \( S_t = S_t(\omega) \) depends on a random parameter \( \omega \); this parameter will be typically omitted. The crucial property of the stochastic process \( S_t \) is that the random variable

\[
\log(S_t/S_0) = \log(S_t) - \log(S_0)
\]
is normally distributed.

In the opposition to such stochastic models our Bohmian model of the stocks market is not based on the theory stochastic differential equations. In our model the randomness of the stocks market cannot be represented in the form of some transformation of the Wiener process.

We recall that the stochastic process model was intensively criticized by many reasons, see, e.g., [2].

First of all there is a number of difficult problems that could be interpreted as technical problems. The most important among them is the problem of the choice of an adequate stochastic process $\xi(t)$ describing price or price change. Nowadays it is widely accepted that the GBM-model provides only a first approximation of what is observed in real data. One should try to find new classes of stochastic processes. In particular, they would provide the explanation to the empirical evidence that the tails of measured distributions are fatter than expected for a geometric Brownian motion. To solve this problem, Mandelbrot proposed to consider the price changes that follow a Levy distribution [31]. However, the Levy distribution has a rather pathological property: its variance is infinite. Therefore, as was emphasized in the book of R. N. Mantegna and H. E. Stanley [2], the problem of finding a stochastic process providing the adequate description of the stocks market is still unsolved.

However, our critique of the conventional stochastic processes approach to the stocks market has no direct relation to this discussion on the choice of an underlying stochastic process. We are more close to scientific groups which criticize this conventional model by questioning the possibility of describing of price dynamics by stochastic processes at all.

11.2. The deterministic dynamical model

In particular, there was done a lot in applying of deterministic nonlinear dynamical systems to simulate financial time series, see [2] for details. This approach is typically criticized through the following general argument: “the time evolution of an asset price depends on all information affecting the investigated asset and it seems unlikely to us that all this information can be essentially described by a small number of nonlinear equations,” [2]. We support such a viewpoint.

We shall use only critical arguments against the hypothesis of the stochastic stocks market which were provided by adherents of the hypothesis of deterministic (but essentially nonlinear) stocks market.

Only at the first sight is the Bohmian financial model is a kind of deterministic model. Of course, dynamics of prices (as well as price changes) are deterministic. It is described by the Newton second law, see the ordinary differential equation (17). It seems that randomness can be incorporated into such a model only through the initial conditions:

$$\dot{p}(t, \omega) = f(t, q(t, \omega)) + g(t, q(t, \omega)), q(0) = q_0(\omega), p(0) = p_0(\omega),$$

(29)

where $q(0) = q_0(\omega), p(0) = p_0(\omega)$ are random variables (initial distribution of prices and momenta) and here $\omega$ is a chance parameter.

However, the situation is not so simple. Bohmian randomness is not reduced to randomness of initial conditions or chaotic behavior of the equation (17) for some nonlinear classical and quantum forces. These are classical impacts to randomness. But a really new impact is given by the essentially quantum randomness which is encoded in the $\psi$-function (=pilot wave=wave function). As we know, the evolution of the $\psi$-function is described by an additional equation – Schrödinger’s equation – and hence the $\psi$-randomness could be extracted neither from the initial conditions for (29) nor from possible chaotic behavior.

In our model the $\psi$-function gives the dynamics of expectations at the financial market. These expectations are a huge source of randomness at the market – mental (psychological)
randomness. However, this randomness is not classical (so it is a non-Kolmogorov probability model).

Finally, we pay attention that in quantum mechanics the wave function is not a measurable quantity. It seems that a similar situation we have for the financial market. We are not able to measure the financial $\psi$-field (which is an infinite dimensional object, since the Hilbert space has the infinite dimension). This field contains thoughts and expectations of millions agents and of course it could not be “recorded” (in the opposition to prices or price changes).

11.3. The stochastic model and expectations of the agents of the financial market

Let us consider again the model of the stocks market based on the geometric Brownian motion:

$$dS_t = u S_t dt + v S dw_t.$$  

We pay attention that in this equation there is no term describing the behavior of agents of the market. Coefficients $u$ and $v$ do not have any direct relation to expectations and the market psychology. Moreover, if we even introduce some additional stochastic processes

$$\eta(t, \omega) = (\eta_1(t, \omega), ..., \eta_N(t, \omega)).$$

describing behavior of agents and additional coefficients (in stochastic differential equations for such processes) we would be not able to simulate the real market. A finite dimensional vector $\eta(t, \omega)$ cannot describe the “mental state of the market” which is of the infinite complexity. One can consider the Bohmian model as the introduction of the infinite-dimensional chance parameter $\psi$. And this chance parameter cannot be described by the classical probability theory.

11.4. The efficient market hypothesis and the Bohmian approach to financial market

The efficient market hypothesis was formulated in sixties, see [4] and [5] for details:

A market is said to be efficient in the determination of the most rational price if all the available information is instantly processed when it reaches the market and it is immediately reflected in a new value of prices of the assets traded.

The efficient market hypothesis is closely related to the stochastic market hypothesis. Mathematically the efficient market hypothesis was supported by investigations of Samuelson [4]. Using the hypothesis of rational behavior and market efficiency he was able to demonstrate how $q_{t+1}$, the expected value of price of a given asset at time $t + 1$, is related to the previous values of prices $q_0, q_1, ..., q_t$ through the relation

$$E(q_{t+1}|q_0, q_1, ..., q_t) = q_t. \quad (30)$$

Stochastic processes of such a type are called martingales [3].

Thus the efficient market hypothesis implies that the financial market is described by a special class of stochastic processes - martingales, see A. Shiryaev [3].

Since the Bohmian quantum model for the financial market is not based on the the stochastic market hypothesis, the efficient market hypothesis can be neither used as the basis of the Bohmian quantum model. The relation between the efficient market model and the Bohmian quantum market model is very delicate. There is no direct contradiction between these models. Since classical randomness is also incorporated into the Bohmian quantum market model (through randomness of initial conditions), we should agree that “the available information is instantly processed when it reaches the market and it is immediately reflected in a new value of prices of the assets traded.” However, besides the available information there is information encoded through the $\psi$-function describing the market psychology. As was already mentioned, $\psi$-function is not measurable, so the complete information encoded in this function is not...
available. Nevertheless, some parts of this information can be extracted from the \( \psi \)-function by some agents of the financial market (e.g., by those who “feel better the market psychology”). Therefore classical forming of prices based on the available information is permanently disturbed by quantum contributions to prices of assets. Finally, we should conclude that the real financial market (and not its idealization based on the mentioned hypotheses) is not efficient. In particular, it determine not the most rational price. It may even induce completely irrational prices through quantum effects.

12. On views of G. Soros: alchemy of finances or quantum mechanics of finances

G. Soros is unquestionably the most powerful and profitable investor in the world today. He has made a billion dollars going against the British pound.

Soros is not merely a man of finance, but a thinker and philosopher as well. Surprisingly he was able to apply his general philosophic ideas to financial market. In particular, the project Quantum Fund inside Soros Fund Management gained hundreds millions dollars and has 6 billion dollars in net assets.

The book “Alchemy of Finance” [14] is a kind of economic-philosophic investigation. In my paper I would like to analyze philosophic aspects of this investigation. The book consists of five chapters. In fact, only the first chapter - “Theory of Reflexivity” - is devoted to pure theoretical considerations.

J. Soros studied economics in college, but found that economic theory was highly unsatisfactory. He says that economics seeks to be a science, but science is supposed to be objective. And it is difficult to be scientific when the subject matter, the participant in the economic process, lacks objectivity. The author also was greatly influenced by Karl Popper’s ideas on scientific method, but he did not agree with Popper’s “unity method.” By this Karl Popper meant that methods and criteria which can be applied to the study of natural phenomena also can be applied to the study of social events. George Soros underlined a fundamental difference between natural and social sciences:

The events studied by social sciences have thinking participants and natural phenomena do not. The participants’ thinking creates problems that have no counterpart in natural science. There is a close analogy with QUANTUM PHYSICS, where the effects of scientific observations give rise to Heisenberg uncertainty relations and Bohr’s complementarity principle.

But in social events the participants’ thinking is responsible for the element of uncertainty, and not an external observer. In natural science investigation of events goes from fact to fact. In social events the chain of causation does not lead directly from fact to fact, but from fact to participants’ perceptions and from perceptions to fact.

This would not create any serious difficulties if there were some kind of correspondence or equivalence between facts and perceptions.

Unfortunately, that is impossible, because the participants perceptions do not relate to facts, but to a situation that is contingent on their own perceptions and therefore cannot be treated as a fact.

In order to appreciate the problem posed by thinking participants, Soros takes a closer look at the way scientific method operates. He takes Popper’s scheme of scientific method, described in technical terms as “deductive-nomological” or “D-N” model. The model is built on three kinds of statements: specific initial conditions, specific final conditions, and generalizations of universal validity. Combining a set of generalizations with known initial conditions yields predictions, combining them with known final conditions provides explanations; and matching known initial with known final conditions serves as testing for generalizations involved. Scientific theories can only be falsified, never verified.

The asymmetry between verification and falsification and the symmetry between prediction and explanation are two crucial features of Popper’s scheme.
The model works only if certain conditions are fulfilled. It is the requirement of universality. That is, if a given set of conditions recurred, it would have to be followed or predicted by the same set of conditions as before. The initial and final conditions must consist of observable facts governed by universal laws. It is this requirement that is so difficult to meet when a situation has thinking participants. Clearly, a single observation by a single scientist is not admissible. Exactly because the correspondence between facts and statements is so difficult to establish, science is a collective enterprise where the work of each scientist has to be open to control and criticism by others. Individual scientists often find the conventions quite onerous and try various shortcuts in order to obtain a desired result. The most outstanding example of the observer trying to impose his will on his subject matter is the attempt to convert base metal into gold. Alchemists struggled long and hard until they were finally persuaded to abandon their enterprise by their lack of success. The failure was inevitable because the behavior of base metals is governed by laws of universal validity which cannot be modified by any statements, incantations, or rituals.

And now, we can at least understand why J. Soros called his book *Alchemy of Finance*, see [14].

Soros considers the behavior of human beings. Do they obey universally valid laws that can be formulated in accordance with “D-N” model? Undoubtedly, there are many aspects of human behavior, from birth to death and in between, which are amenable to the same treatment as other natural phenomena. But there is one aspect of human behavior which seems to exhibit characteristics which are different from those of the phenomena from the subject matter of natural science: the decision-making process. An imperfect understanding of the situation destroys the universal validity of scientific generalizations: given a set of conditions is not necessary preceded or succeeded by the same set every time, because the sequence of events is influenced by participants’ thinking. The “D-N” model breaks down. But social scientists try to maintain the unity of method but with little success.

In a sense, the attempt to impose the methods of natural science on social phenomena is comparable to efforts of alchemists who sought to apply the methods of magic to the field of natural science. And here J. Soros presents (in my opinion) a very interesting idea about the “alchemy method in social science.” He says, that while the failure of the alchemists was total, social scientists have managed to make a considerable impact on their subject matter. Situations which have thinking participants may be impervious to the methods of natural science, but they are susceptible to the methods of alchemy.

The thinking of participants, exactly because it is not governed by reality, is easily influenced by theories. In the field of natural phenomena, scientific method is effective only when its theories are valid, but in social, political, and economic matters, theories can be effective without being valid. Whereas alchemy has failed in natural sciences, social science can succeed as alchemy.

The relationship between the scientist and his subject matter is quite different in natural science as opposed to social science. In natural science the scientist’s thinking is, in fact, distinct from its subject matter. The scientists can influence the subject matter only by actions, not by thoughts, and the scientists’ actions are guided by the same laws as all other natural phenomena. Specifically, a scientist can do nothing do when she wants to turn base metals into gold.

Social phenomena are different. The imperfect understanding of the participant interferes with the proper functioning of the “D-N” model. There is much to be gained by pretending to abide by conventions of scientific method without actually doing so. Natural science is held in great esteem: the theory that claims to be scientific can influence the gullible public much better than one which frankly admits its political or ideological bias.

Soros mentions here Marxism, psychoanalysis and laissez-faire capitalism with its reliance on the theory of perfect competition as typical examples.

Soros underlined that Marx and Freud were vocal in protesting their scientific status and
based many of their conclusions on authority they derived from being "scientific." And Soros says, that once this point sinks in, the very expression “social science” became suspect.

He compares the expression “social science” with a magic word employed by social alchemists in their effort to impose their will on their subject matter by incantation. And it seems to Soros there is only one way for the “true” practitioners of scientific method to protect themselves against such malpractice - to deprive social science of the status it enjoys on account of natural science. Social science ought to be recognized as a false metaphor.

I cannot agree with this Soros statement. First of all there are a lot of boundary sciences. For example, psychoanalysis can be considered as a part of medicine. But medicine accompanied with biology and chemistry are of course the natural sciences. And S. Freud was famous and like many doctors succeeding him they helped many patients.

And J. Soros explains by himself his unusual statement. He says that it does not mean that we must give up the pursuit of truth in exploring social phenomena. It means only that the pursuit of truth requires to recognize that the “D-N” model can not be applied to situations with thinking participants. He asks us to abandon the doctrine of the unity of method and to cease the slavish imitation of natural sciences. He says that there are some new scientific methods in all kinds of science as quantum physics has shown.

Scientific method is not necessarily confined to the “D-N” model: statistical, probabilistic generalizations may be more fruitful. Nor should we ignore the possibility of developing novel approaches which have no counterpart in natural science. Given the differences in subject matter, there ought to be differences in the method of study.

Soros shows us the main distinction between the “D-N” model and his own approach. He says that the world of imperfect understanding does not land itself to generalizations which can be used to explain and to predict specific events. The symmetry between explanation and prediction prevails only in the absence of thinking participants. On the other hand, past events are just as final as in the “D-N” model; thus explanations turn out to be an easier task than prediction.

In another part of his book, see [14], Soros abandons the constraint that predictions and explanations are logically reversible. He builds his own theoretical framework. He says that his theory can not be tested in the same way as these theories which fit into Popper’s logical structure, but that is not to say that testing must be abandoned. And he does tests in a real-time experiment, chapter 3, for his model. He uses the theory of perfect competition for the investigation of financial market, but he takes into account thinking of participants of this market.

G. Soros proved his theory by becoming one of the most powerful and profitable investors in the world today. In my own work I use methods of classical and quantum mechanics for mathematical modeling of price dynamics at financial market and I use Soros’ statement about cognitive phenomena at the financial market.

13. The problem of smoothness of price trajectories

In the Bohmian model for price dynamics the price trajectory \( q(t) \) can be found as the solution of the equation

\[
m \frac{d^2 q(t)}{dt^2} = f(t, q(t)) + g(t, q(t))
\]

with the initial condition

\[
q(t_0) = q_0, \quad q'(t_0) = q'_0.
\]

Here we consider a ”classical” (time dependent) force \( f(t, q) = -\frac{\partial V(t, q)}{\partial q} \) and ”quantum” force \( g(t, q) = -\frac{\partial U(t, q)}{\partial q} \), where \( U(t, q) \) is the quantum potential, induced by the Schrödinger dynamics. In Bohmian mechanics for physical systems the equation (31) is considered as an ordinary
differential equation and \( q(t) \) as the unique solution (corresponding to the initial conditions \( q(t_0) = q_0, q'(t_0) = q'_0 \)) of the class \( C^2 : q(t) \) is assumed to be twice differentiable with continuous \( q''(t) \).

One of possible objections to apply the Bohmian quantum model to describe dynamics of prices (of e.g. shares) at the financial market is smoothness of trajectories. In financial mathematics it is commonly assumed that the price-trajectory is not differentiable, see, e.g., [2], [3].

14. Mathematical model and reality
Of course, one could simply reply that there are no smooth trajectories in nature. Smooth trajectories belong neither to physical nor financial reality. They appear in mathematical models which can be used to describe reality. It is clear that the possibility to apply a mathematical model with smooth trajectories depends on a chosen time scale. Trajectories that can be considered as smooth (or continuous) at one time scale might be nonsmooth (or discontinuous) at a finer time scale.

We illustrate this general philosophic thesis by the history of development of financial models. We recall that at the first stage of development of financial mathematics, in the Bachelier model and the Black and Scholes model, there were considered processes with continuous trajectories: the Wiener process and more general diffusion processes. However, recently it was claimed that such stochastic models (with continuous processes) are not completely adequate to real financial data, see, e.g., [2], [3] for the detailed analysis. It was observed that at finer time scales some Levy-processes with jump-trajectories are more adequate to data from the financial market.

Therefore one could say that the Bohmian model provides a rough description of price dynamics and describes not the real price trajectories by their smoothed versions. However, it would be interesting to keep the interpretation of Bohmian trajectories as the real price trajectories. In such an approach one should obtain nonsmooth Bohmian trajectories. The following section is devoted to theorems providing existing of nonsmooth solutions.

15. Picard’s theorem and its generalization
We recall the standard uniqueness and existence theorem for ordinary differential equations, Picard’s theorem, that gives the guarantee of smoothness of trajectories, see, e.g., [32].

**Theorem 1.** Let \( F : [0, T] \times \mathbb{R} \to \mathbb{R} \) be a continuous function and let \( F \) satisfy the Lipschitz condition with respect to the variable \( x \):

\[
|F(t, x) - F(t, y)| \leq c|x - y|, \ c > 0. \tag{32}
\]

Then, for any point \( (t_0, x_0) \in [0, T] \times \mathbb{R} \) there exists the unique \( C^1 \)-solution of the Cauchy problem:

\[
\frac{dx}{dt} = F(t, x(t)), \quad x(t_0) = x_0, \tag{33}
\]

on the segment \( \Delta = [t_0, a] \), where \( a > 0 \) depends \( t_0, x_0, \) and \( F \).

We recall the standard proof of this theorem, because the scheme of this proof will be used later. Let us consider the space of continuous functions \( x : [t_0, a] \to \mathbb{R} \), where \( a > 0 \) is a number which will be determined. Denote this space by the symbol \( C[t_0, a] \). The Cauchy problem (33) for the ordinary differential equation can be written as the integral equation:

\[
x(t) = x(t_0) + \int_{t_0}^{t} F(s, x(s))ds \tag{34}
\]

The crucial point for our further considerations is that continuity of the function \( F \) with respect to the pair of variables \( (t, x) \) implies continuity of \( y(s) = F(s, x(s)) \) for any continuous \( x(s) \).
But the integral \( z(t) = \int_0^t y(s)ds \) is differentiable for any continuous \( y(s) \) and \( z'(t) = y(t) \) is also continuous. The basic point of the standard proof is that, for a sufficiently small \( a > 0 \), the operator

\[
G(x)(t) = x_0 + \int_{t_0}^t F(s, x(s))ds
\]

maps the functional space \( C[t_0, a] \) into \( C[t_0, a] \) and it is a contraction in this space:

\[
\rho_\infty(G(x_1), G(x_2)) \leq \alpha \rho_\infty(x_{10}, x_{20}), \quad \alpha < 1,
\]

for any two trajectories \( x_1(t), x_2(t) \in C[t_0, a] \) such that \( x_1(t_0) = x_{10} \) and \( x_2(t_0) = x_{20} \). Here, to obtain \( \alpha < 1 \), the interval \([t_0, a]\) should be chosen sufficiently small, see further considerations. Here \( \rho_\infty(u_1, u_2) = ||u_1 - u_2||_\infty \) and

\[
||u||_\infty = \sup_{t_0 \leq t \leq a} |u(s)|.
\]

The contraction condition, \( \alpha < 1 \), implies that the iterations

\[
x_1(t) = x_0 + \int_{t_0}^t F(S, x_0)ds,
\]

\[
x_2(t) = x_0 + \int_{t_0}^t F(S, x_1(s))ds, \ldots,
\]

\[
x_n(t) = x_0 + \int_{t_0}^t F(S, x_{n-1}(s))ds, \ldots
\]

converge to a solution \( x(t) \) of the integral equation (34). Finally, we remark that the contraction condition (36) implies that the solution is unique in the space \( C[t_0, a] \).

Roughly speaking in Theorem 1 the Lipschits condition is ”responsible” for uniqueness of solution and continuity of \( F(t, x) \) for existence. We also recall the well known Peano theorem, [32]:

**Theorem 2.** Let \( F : [0, T] \times \mathbb{R} \) be a continuous function. Then, for any point \((t_0, x_0) \in [0, T] \times \mathbb{R}\) there exists locally a \( C^1 \)-solution of the Cauchy problem (33).

We remark that Peano’s theorem does not imply uniqueness of solution.

It is clear that discontinuous financial forces can induce price trajectories \( q(t) \) which are not smooth: more over, price trajectories can even be discontinuous! From this point of view the main problem is not smoothness of price trajectories \( q(t) \) (and in particular the zero covariation for such trajectories), but the absence of an existence and uniqueness theorem for discontinuous financial forces. We shall formulate and prove such a theorem. Of course, outside the class of smooth solutions one could not study the original Cauchy problem for an ordinary differential equation (33). Instead of this one should consider the integral equation (34).

We shall generalize Theorem 1 to discontinuous \( F \). Let us consider the space \( BM[t_0, a] \) consisting of bounded measurable functions \( x : [t_0, a] \to \mathbb{R} \). Thus:

a) \( \sup_{t_0 \leq t \leq a} |x(t)| \equiv ||x||_\infty < \infty \);

b) for any Borel subset \( A \subset \mathbb{R} \), its preimage \( x^{-1}(A) = \{ s \in [t_0, a] : x(s) \in A \} \) is again a Borel subset in \([t_0, a]\).

**Lemma 1.** The space of trajectories \( BM[t_0, a] \) is a Banach space.
Proof. Let \( \{x_n(t)\} \) be a sequence of trajectories that is a Cauchy sequence in \( BM[t_0, a] \): 
\[
||x_n - x_m||_{\infty} \to 0, n, m \to \infty. 
\]
Thus
\[
\lim_{n,m \to \infty} \sup_{t_0 \leq t \leq a} |x_n(t) - x_m(t)| \to 0. \tag{37}
\]

Thus, for any \( t \in [t_0, a] \), \( |x_n(t) - x_m(t)| \to 0, n, m \to \infty. \) Hence, for any \( t \), the sequence of real numbers \( \{x_n(t)\}_{n=1}^{\infty} \) is a Cauchy sequence in \( \mathbb{R} \).

But the space \( \mathbb{R} \) is complete. Thus, for any \( t \in [t_0, a] \), there exists \( \lim_{n \to \infty} x_n(t) \), which we denote by \( x(t) \). In this way we constructed a new function \( x(t), t \in [t_0, a] \). We now write the condition (37) through the \( \epsilon \)-language: \( \forall \epsilon > 0 \exists N : \forall n, m \geq N : |x_n(t) - x(t)| \leq \epsilon \) for any \( t \in [t_0, a] \).

We now fix \( n \geq N \) and take the limit \( m \to \infty \) in the inequality (38). We obtain:
\[
|x_n(t) - x(t)| \leq \epsilon, \text{ for any } t \in [t_0, a]. \tag{39}
\]

Thus
\[
\sup_{t_0 \leq t \leq a} |x_n(t) - x(t)| \leq \epsilon \tag{40}
\]

This is nothing else than the condition: \( \forall n \geq N : ||x_n - x||_{\infty} \leq \epsilon. \) Therefore \( x_n \to x \) in the space \( BM[t_0, a] \). We remark that the trajectory \( x(t) \) is bounded, because:
\[
||x||_{\infty} \leq ||x_n - x||_{\infty} + ||x_n||_{\infty} \leq \epsilon + ||x_n||_{\infty}
\]

for any fixed \( n_0 \geq N \), and since \( ||x_n||_{\infty} < \infty \), we finally get \( ||x||_{\infty} < \infty \). We also remark that \( x(t) \) is a measurable function as the uniform limit of measurable functions, see [3]. Thus the space \( BM[t_0, a] \) is a complete normed space - a Banach space.

Theorem 3. Let \( F : [0, T] \times \mathbb{R} \to \mathbb{R} \) be a measurable bounded function and let \( F \) satisfy the Lipschitz condition with respect to the \( x \)-variable, see (32). Then, for any point \( (t_0, x_0) \in [0, T] \times \mathbb{R} \), there exists the unique solution of the integral equation (34) of the class \( BM[t_0, a] \), where \( a > 0 \) depends on \( x_0, t_0 \), and \( F \).

Proof. We shall determine \( a > 0 \) later. Let \( u(s) \) be any function of the class \( BM[t_0, a] \). Then the function \( y(s) = F(s, u(s)) \) is measurable (since \( F \) and \( u \) are measurable) and it is bounded (since \( F \) is bounded). Thus \( y \in BM[t_0, a] \). Any bounded and measurable function is integrable with respect to the Lebesque measure \( dt \) on \( [t_0, a] \), see [3]. Therefore
\[
z(t) = \int_{t_0}^{t} y(s)ds \equiv \int_{t_0}^{t} F(s, u(s))ds
\]
is well defined for any \( t \). This function is again measurable with respect to \( t \), see [5], and bounded, because:
\[
\left| \int_{t_0}^{t} y(a)ds \right| \leq \sup_{t_0 \leq s \leq t} |y(s)||t - t_0| \leq ||y||_{\infty}(a - t_0) < \infty.
\]

Thus the operator \( G \) which was defined by (35) maps \( BM[t_0, a] \) into \( BM[t_0, a] \). We now show that, for a sufficiently small \( a > 0, G \) is a contraction in \( BM[t_0, a] \). By using the Lipschitz condition we get:
\[
\sup_{t_0 \leq s \leq a} |G(x_1)(t) - G(x_2)(t)| = \left| \int_{t_0}^{t} (F(s, x_1(s)) - F(s, x_2(s))ds \right|
\]

25
Theorem 3 holds. Then solutions are continuous functions \(y(x)\) in metric spaces (in particular in Banach spaces), see [3], the map \(x\) of trajectories also belongs to the class \(\mathcal{L}\) where \(a > \frac{1}{2}\). Here we estimated later. Then \(G(x) = x\), or \(x(t) = x_0 + \int_{t_0}^{t} F(s, x(s))ds\).

Proposition 1. (Continuity of the solution of the integral equation). Let conditions of Theorem 3 holds. Then solutions are continuous functions \(x : [t_0, a] \rightarrow \mathbb{R}\).

Proof. We use the same notations as in proof of Theorem 3. Let \(u \in BM[t_0, a], y(s) = F(s, u(s))\). As we have shown, this is a bounded measurable function. We prove that \(u(t) = \int_{t_0}^{t} y(s)ds\) is a continuous function. Let \(\tau \in [t_0, t]\) and let \(\Delta\) be a small real number. Then

\[
|\xi(\tau + \Delta) - \xi(\tau)| = \left| \int_{\tau}^{\tau + \Delta} y(s)ds \right| \leq |\Delta||y||_{\infty} \rightarrow 0, \Delta \rightarrow 0.
\]

Here we have used simple properties of the Lebesque integral: \(\int_{\tau}^{\tau + \Delta} y(s)ds \leq \int_{\tau}^{\tau + \Delta} |y(s)|ds\) and, if \(|y(s)| \leq const\), then \(\int_{\tau}^{\tau + \Delta} |y(s)|ds \leq const(b - a)\) (in our case \(const = ||y||_{\infty} = \sup_{t_0 \leq t \leq a} |y(\varepsilon)|\)).

Thus Theorem 3 gives a sufficient condition of the existence of the unique continuous trajectory-solution \(x(t)\) for the integral equation (34). But, of course, in general \(x(t)\) is not continuously differentiable!

Theorem 4. Let \(f\) satisfy the Lipschitz condition (32). Then for any point \((t_0, x_0) \in [0, T] \times \mathbb{R}\) there exists the unique solution of the integral equation (34) of the class \(L_2[t_0, a]\), where \(a > 0\) depends on \(x_0, t_0\), and \(F\).

Proof. Let \(u \in L_2[t_0, a]\) (as always we shall determine \(a > 0\) later). Then \(y(s) = F(s, u(s))\) also belongs to the class \(L_2[t_0, a]\):

\[
\int_{t_0}^{a} y^2(s)ds = \int_{t_0}^{a} F^2(s, u(s))ds \\
\leq \int_{t_0}^{a} (m_1|u(s)| + m_2)^2ds = \\
m_1^2 \int_{t_0}^{a} u^2(s)ds + m_2^2(a - t_0) + 2m_1m_2 \int_{t_0}^{a} |u(s)|ds.
\]

Here we estimated \(F(t, u)\) through the inequality (38).

Now we recall the well known Cauchy-Bunyakovsky inequality in the \(L_2\) space. For any pair of trajectories \(u_1, u_2 \in L_2\), we have

\[
\int_{t_0}^{a} |u_1(s)u_2(s)|ds \leq \sqrt{\int_{t_0}^{a} u_1^2(s)ds} \sqrt{\int_{t_0}^{a} u_2^2(s)ds}.
\]

We would like to estimate the integral

\[
\int_{t_0}^{a} |u(s)|ds
\]

by using the Cauchy-Bunaykovsky inequality. We choose \(u_2(s) = u(s)\) and \(u_1(s) \equiv 1\). We have
\[
\int_{t_0}^{a} |u(s)|ds \leq \sqrt{\int_{t_0}^{a} ds} \sqrt{\int_{t_0}^{a} u^2(s)ds} = \sqrt{a - t_0} \ |u|_2.
\]

Finally, we get
\[
\int_{t_0}^{a} y^2(s)ds \leq m_1^2 |u|_2^2 + m_2^2 (a - t_0) + 2m_1m_2 \sqrt{a - t_0} |u|_2 < \infty.
\]

Thus the function \( y \in L_2[t_0, a] \). Therefore the integral operator given by
\[
G(u)(t) = x_0 + \int_{t_0}^{t} F(s, u(s))ds
\]
maps the space of trajectories \( L_2[t_0, a] \) into \( L_2[t_0, a] \). We recall that \( L_2 \)-spaces are Banach spaces. Hence, these are complete metric spaces. Here we can apply the fixed point theorem for compression-maps. Finally, we shall prove that the integral operator \( G : L_2[t_0, a] \to L_2[t_0, a] \) is compression for a sufficiently small \( a > 0 \).

As always, we use the Lipschitz condition with respect to \( x \). For any pair of trajectories \( x_1(s), x_2(s) \in L_2[t_0, a] \):
\[
\|G(x_1) - G(x_2)\|_2^2 = \int_{t_0}^{a} \left( \int_{t_0}^{t} (F(S, x_1(S)) - F(s, x_2(s)))ds \right)^2 dt
\]
\[
\leq c^2 \int_{t_0}^{a} \left( \int_{t_0}^{t} |x_1(s) - x_2(s)|ds \right)^2 dt.
\]

We now introduce the characteristic function of the interval \([t_0, t] \):
\[
\phi_t(s) = \begin{cases} 
1, s \in [t_0, t] \\
0, s \notin [t_0, t]
\end{cases}
\]

The last integral can be written as
\[
\int_{t_0}^{a} \left( \int_{t_0}^{t} |x_1(s) - x_2(s)|ds \right)^2 dt = \int_{t_0}^{a} \left( \int_{t_0}^{a} \phi_t(s)|x_1(s) - x_2(s)|ds \right)^2 dt.
\]

We now apply the Cauchy-Bunaykovsky inequality for the integral with respect to \( ds \). We choose \( u_1(s) = \phi_t(s) \) and \( u_2(s) = |x_1(s) - x_2(s)| \). We have:
\[
\int_{t_0}^{a} \phi_t(s)|x_1(s) - x_2(s)|ds
\]
\[
\leq \sqrt{\int_{t_0}^{a} \phi_t^2(s)ds} \sqrt{\int_{t_0}^{a} |x_1(s) - x_2(s)|^2ds}
\]
\[
= \sqrt{\int_{t_0}^{t} ds \ |x_1(s) - x_2|_2} = \sqrt{t - t_0} \ |x_1 - x_2|_2 \leq \sqrt{a - t_0} \ |x_1 - x_2|_2.
\]

We have, finally,
\[
\|G(x_1) - G(x_2)\|_2^2 \leq c^2 \int_{t_0}^{a} (a - t_0)|x_1 - x_2|_2^2 dt \leq c^2 (a - t_0)^2 \ |x_1 - x_2|_2^2.
\]
Thus
\[ \rho_2(G(x_1), G(x_2)) = \|G(x_1) - G(x_2)\|_2 \leq c(a - t_0)\rho_2(x_1, x_2). \]
We set
\[ \alpha = c(a - t_0). \]
Hence, if \( \alpha < 1 \), then
\[ G : L_2[t_0, a] \to L_2[t_0, a] \]
is a compression. It has a fixed point which is the unique solution of our integral equation. Thus the proof is finished.

We remark that in the same way as in the case \( BM[t_0, a] \)-space, we can show that solutions existing due to Theorem 4 are continuous functions.

**Proposition 2.** (Continuity) Let conditions of Theorem 4 hold. Then solutions \( x : [t_0, a] \to \mathbb{R} \) are continuous functions.

**Proof.** As we have seen in the proof of Theorem 4, for any trajectory \( u \in L_2[t_0, a] \), the function \( y(s) = F(s, u(s)) \) also belongs to \( L_2[t_0, a] \). We shall prove that
\[ \xi(s) = \int_{t_0}^{s} y(s)ds \]
is continuous. Let us take \( \Delta \geq 0 \) (the case \( \Delta < 0 \) is considered in the same way). We have
\[ |\xi(\tau + \Delta) - \xi(\tau)| \leq \int_{\tau}^{\tau+\Delta} |y(s)|ds. \]
We introduce the characteristic functions
\[ \phi_{[\tau, \tau+\Delta]}(s) = \begin{cases} 1, & s \in [\tau, \tau+\Delta] \\ 0, & s \notin [\tau, \tau+\Delta] \end{cases} \]
We have:
\[ \int_{\tau}^{\tau+\Delta} |y(s)|ds = \int_{t_0}^{\alpha} \phi_{[\tau, \tau+\Delta]}(s)|y(s)|ds \]
\[ \leq \sqrt{\int_{t_0}^{\alpha} \phi_{[\tau, \tau+\Delta]}^2(s)ds} \sqrt{\int_{t_0}^{\alpha} |y(s)|^2ds} = \sqrt{\Delta} \|y\|_2 \to 0, \Delta \to 0. \]
Here we have used the Cauchy-Bunyakovsky inequality for functions \( u_1(s) = \phi_{[\tau, \tau+\Delta]}(s) \) and \( u_2(s) = |y(s)| \). The proof is completed.

Thus we again obtained continuous, but in general non-smooth \( (x \notin C^1) \) solutions of the basic integral equation.

The theory can be naturally generalized to \( L_p \) spaces, \( p \geq 1 \):
\[ L_p[t_0, a] = \{ x : [t_0, a] \to \mathbb{R} : ||x||_p^p = \int_{t_0}^{a} |x(t)|^p dt < \infty \}. \]
We shall not do this, because our aim was just to show that the integral equation (34) with discontinuous \( F \) is well posed (i.e., it has the unique solution) in some classes of (nonsmooth) trajectories.
It is more important for us to remark that Theorems 3, 4 are valid in the multidimensional case:

\[ x_0 = (x_{01}, \ldots, x_{0n}), x(t) = (x_1(t), \ldots, x_n(t)), \]

and

\[ F : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n. \]

To show this, we should change in all previous considerations the absolute value \(|x|\) to be norm on the Euclidean space \(\mathbb{R}^n\):

\[ ||x|| = \sqrt{\sum_{j=1}^{n} x_j^2}. \]

We now use the standard trick to apply our theory to the Newton equation (31) which is a second order differential equation. We rewrite this equation as a system of equations of the first order with respect to

\[ x = (x_1, \ldots, x_n, x_{n+1}, x_{2n}), \]

where

\[ x_1 = q_1, \ldots, x_n = q_n, \]
\[ x_{n+1} = p_1, \ldots, x_{2n} = p_n. \]

In fact, this is nothing else than the phase space representation. The Newton equation (31) will be written as the Hamilton equation, see section? However, the Hamiltonian structure is not important for us in this context. In any event we obtain the following system of the first order equations:

\[ \frac{dx}{dt} = F(t, x(t)), \]  

where

\[ F(t, x) = \begin{pmatrix} x_{n+1} \\ \vdots \\ x_{2n} \\ f_1(t, x_1, \ldots, x_n) + g_1(t, x_1, \ldots, x_n) \\ \vdots \\ f_n(t, x_1, \ldots, x_n) + g_n(t, x_1, \ldots, x_n) \end{pmatrix}, \]

Here

\[ f_j(t, x_1, x_n) = \frac{\partial V}{\partial x_j}(t, x_1, \ldots, x_n) \]

and

\[ g_j(t, x_1, \ldots, x_n) = \frac{\partial U}{\partial x_j}(t, x_1, \ldots, x_n). \]

Therefore if

\[ \nabla V = \left( \frac{\partial V}{\partial x_n}, \ldots, \frac{\partial V}{\partial x_n} \right) \]

or

\[ \nabla U = \left( \frac{\partial U}{\partial x_n}, \ldots, \frac{\partial U}{\partial x_n} \right) \]
are not continuous, then the standard existence and uniqueness theorems, see Theorems 1, 2, could not be applied. But, instead of the ordinary differential equation (41), we can consider the integral equation:

\[ x(t) = x_0 + \int_{t_0}^{t} F(s, x(s))ds \]  

and apply Theorems 3,4 to this equation. We note that due to the structure of \( F(t, x) \), we have in fact

\[
\begin{align*}
p_1(t) &= p_{01} + \int_{t_0}^{t} F_1(s, q(s))ds \\
p_n(t) &= p_{0n} + \int_{t_0}^{t} F_n(s, q(s))ds \\
q_1(t) &= q_{01} + \frac{1}{m} \int_{t_0}^{t} p_1(s, q(s))ds \\
q_n(t) &= q_{0n} + \frac{1}{m} \int_{t_0}^{t} p_n(s)ds.
\end{align*}
\]

By Propositions 1,2, \( p_j(t) \) are continuous functions. Therefore integrals \( \int_{t_0}^{t} p_j(s)ds \) are continuous differentiable functions. Thus under conditions of Theorem 3 or Theorem 4 we obtain the following price dynamics:

*Price trajectories are of the class \( C^1 \) (so \( \frac{dq}{dt}(t) \) exists and continuous), but price velocity*

\[ v(t) = \frac{p(t)}{m} \]

*is in general non-differentiable.*

16. **The problem of quadratic variation**

The quadratic variation of a function \( u \) on an interval \([0, T]\) is defined as

\[ \langle u \rangle(T) = \lim_{\|P\| \to 0} \sum_{k=0}^{n-1} (u(t_{k+1}) - u(t_k))^2, \]

where \( P = \{0 = t_0 < t_1 < ... < t_n = T\} \) is a partition of \([0, T]\) and \( \|P\| = \max \{t_{k+1} - t_k\} \).

We recall the well known result:

**Theorem** If \( u \) is differentiable, then \( \langle f \rangle(T) = 0 \).

Therefore, for any smooth Bohmian trajectory its quadratic variation is equal to zero. On the other hand, it is well known that real price trajectories have nonzero quadratic variation, [2], [3]. This is a strong objection for consideration of smooth Bohmian price-trajectories.

In the previous section there were derived existence theorems which provide nonsmooth trajectories. One may hope that solutions given by those theorems would have nonzero quadratic variation. But this is not the case.

**Theorem.** Assume that,

\[ x(t) = x_0 + \int_{t_0}^{t} F(s, x(s))ds \]

where \( F \) is bounded, i.e., \( |F(t, x)| \leq K \), and measurable. Then the quadratic variation \( \langle F \rangle(t) = 0 \).
Proof We have:

\[ |x(t_k) - x(t_{k-1})|^2 = \int_{t_{k-1}}^{t_k} |F(s, x(s))| ds \leq K^2 (t_k - t_{k-1})^2. \]

Hence, with a partition of \([0, t]\), say, \(0 = t_0 < t_1 < \ldots < t_n = t\), we get

\[ \sum_{k=1}^{n} |x(t_k) - x(t_{k-1})|^2 \leq K^2 \sum_{k=1}^{n} (t_k - t_{k-1})^2 \]

\[ \leq K^2 \max_{1 \leq k \leq n} (t_k - t_{k-1}) \sum_{k=1}^{n} (t_k - t_{k-1}) = K^2 \max_{1 \leq k \leq n} (t_k - t_{k-1}), \]

which converges to zero as the partition gets finer, i.e. the quadratic variation of \(t \mapsto x(t)\) is zero.

Thus the objection related to the nonzero quadratic variation is essentially stronger than the smoothness objection. One possibility to escape this problem is to consider unbounded quantum potentials or even potentials which are given by distributions.

17. Singular potentials and forces
We present some examples of discontinuous quantum forces \(g\) (induced by discontinuous quantum potential \(U\)).

17.1. Example of singularity
Let us consider the wave function

\[ \psi(x) = c(x + 1)^2 e^{-x^2/2} dx, \]

where \(c\) is the normalization constant providing

\[ \int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1. \]

Here \(\psi(x) \equiv R(x) = |\psi(x)|\). We have:

\[ R'(x) = c[2(x + 1) - x(x + 1)^2] e^{-x^2/2} = -c(x^3 + 2x^2 - x - 2) e^{-x^2/2}, \]

and

\[ R''(x) = c(x^4 + 2x^3 - 4x^2 - 6x + 1) e^{-x^2/2}. \]

Hence

\[ U(x) = -\frac{R''(x)}{R(x)} = \frac{x^4 + 2x^3 - 4x^2 - 6x + 1}{(x + 1)^2}. \]

Thus potential has singularity at the point \(x = -1\).

In this example a singularity in the quantum potential \(U(t, x)\) is a consequence of division by the amplitude of the wave function \(R(t, x)\). If \(|\psi(t, x_0)| = 0\), then there can appear a singularity at the point \(x_0\).
17.2. General scheme to produce singular quantum potential for an arbitrary Hamiltonian

Let $\hat{H}$ be a self-adjoint operator, $\hat{H} \geq 0$, in $L_2(\mathbb{R}^n)$ (Hamiltonian – an operator representing the financial energy). Let us consider the corresponding Schrödinger equation

$$\frac{\partial \psi}{\partial t} = \hat{H} \psi,$$

$$\psi(0) = \psi_0,$$

in $L_2(\mathbb{R}^n)$. Then its solution has the form:

$$u_t(\psi_0) = e^{-\frac{it\hat{H}}{\hbar}} \psi_0.$$

If the operator $\hat{H}$ is continuous, then its exponent is defined with aid of the usual exponential power series:

$$e^{-\frac{it\hat{H}}{\hbar}} = \sum_{n=0}^{\infty} \left( -\frac{it\hat{H}}{\hbar} \right)^n / n! = \sum_{n=0}^{\infty} \left( -\frac{it}{\hbar} \right)^n \hat{H}^n / n!.$$

If the operator $\hat{H}$ is not continuous, then this exponent can be defined by using the spectral theorem for self-adjoint operators.

We recall that, for any $t \geq 0$, the map

$$u_t : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$$

is a unitary operator:

(a) it is one-to-one;
(b) it maps $L_2(\mathbb{R}^n)$ onto $L_2(\mathbb{R}^n)$
(c) it preserves the scalar product:

$$(u_t \psi, u_t \phi) = (\psi, \phi), \quad \psi, \phi \in L_2.$$

We pay attention to the (b). By (b), for any $\phi \in L_2(\mathbb{R}^n)$, we can find a $\psi_0 \in L_2(\mathbb{R}^n)$ such that

$$\phi = u_t(\psi_0).$$

It is sufficient to choose

$$\psi_0 = u_t^{-1}(\phi)$$

(any unitary operator is invertible). Thus,

$$\psi(t) = u_t(\psi_0) = \phi.$$

In general a function $\phi \in L_2(\mathbb{R}^n)$ is not a smooth or even continuous function! Therefore in the case under consideration (so we created the wave function $\psi$ such that $\psi(t) = \phi$, where $\phi$ was an arbitrary chosen square integrable function),

$$U(t, x) = -\frac{|\psi(t, x)|''}{|\psi(t, x)|} = -\frac{|\phi(x)|''}{\phi(x)}$$

is in general a generalized function (distribution)! For example, let us choose

$$\phi(x) = \begin{cases} \frac{1}{2b}, & -b \leq x \leq b \\ 0, & x \notin [-b, b] \end{cases}$$
Here $R(t, x) = |\phi(x)| = \phi(x)$ and 
\[ R'(t, x) = \frac{\delta(x + b) - \delta(x - b)}{2b}, \]
\[ R''(t, x) = \frac{\delta'(x + b) - \delta'(x - b)}{2b}. \]

**Conclusion.** In general, the quantum potential $U(t, x)$ is a generalized function (distribution). Therefore the price (as well as price change) trajectory is a generalized function (distribution) of the time variable $t$. Moreover, since the dynamical equation is nonlinear, one cannot guarantee even the existence of a solution.

18. Classical and quantum financial randomness

By considering singular quantum potentials we can model the Bohmian price dynamics *with trajectories having nonzero quadratic variation.* The main problem is that there are no existence theorems for such forces. Derivation of such theorems is an interesting mathematical problem, but it is completely outside of the author’s expertise.

Another possibility to obtain a more realistic quantum-like model for the financial market is to consider additional stochastic terms in the Newton equation for the price dynamics.

18.1. Randomness from initial conditions

Let us consider the financial Newton equation (31) with random initial conditions:

\[
\frac{md^2q(t, \omega)}{dt^2} = f(t, q(t, \omega)) + g(t, q(t, \omega)), \quad (43)
\]
\[ q(0, \omega) = q_0(\omega), \quad \dot{q}(0, \omega) = \dot{q}_0(\omega), \quad (44) \]

where $q_0(\omega)$ and $\dot{q}_0(\omega)$ are two random variables giving the initial distribution of prices and price changes, respectively. This is the Cauchy problem for ordinary differential equation depending on a parameter $\omega$. If $f$ satisfy conditions of Theorem 1, i.e., both classical and quantum (behavioral) financial forces $f(t, q)$ and $g(t, q)$ are continuous and satisfy the Lipschitz condition with respect to the price variable $q$, then, for any $\omega$, there exists the solution $q(t, \omega)$ having the class $C^2$ with respect to the time variable $t$. But through initial conditions the price depends on the random parameter $\omega$ so $q(t, \omega)$ is a stochastic process. In the same way the price change $\nu(t, \omega) = \dot{q}(t, \omega)$ is also a stochastic process. These processes can be extremely complicated (through nonlinearity of coefficients $f$ and $g$). In general, these are *non-stationary processes.*

For example, the mathematical expectation

\[ <q(t)> = Eq(t, \omega) \]

and dispersion ("volatility")

\[ \sigma^2(q(t)) = Eq^2(t, \omega) - <q(t)>^2 \]

can depend on $t$.

If at least one of financial forces, $f(t, x)$ or $g(t, x)$, is not continuous, then we consider the corresponding integral equations:

\[
p(t, \omega) = p_0(\omega) + \int_{t_0}^{t} f(s, q(s, \omega))ds + \int_{t_0}^{t} g(s, q(s, \omega))ds, \quad (45)
\]
$q(t, \omega) = q_0(\omega) + \frac{1}{m} \int_{t_0}^{t} p(s, \omega)ds \quad (46)$

Under assumptions of Theorem 3 or Theorem 4, there exists the unique stochastic process with continuous trajectories, $q(t, \omega), p(t, \omega)$, giving the solution of the system of integral equations (45), (46) with random initial conditions.

However, trajectories still have zero quadratic variation. Therefore this model is not satisfactory.

18.2. Random financial mass

There parameter $m$, ”financial mass”, was considered as a constant of the model. At the real financial market $m$ depends on $t$:

$m \equiv m(t) = (m_1(t), \ldots, m_n(t))$.

Here $m_j(t)$ is the volume of emission (the number of items) of shares of $j$th corporation. Therefore the corresponding market capitalization is given by

$T_j(t) = m_j(t)q_j(t)$.

In this way we modify the financial Newton equation (43):

$m_j(t)\ddot{q}_j = f_j(t, q(t)) + g_j(t, q(t))$.

We set $F_j(t, q) = \frac{f_j(t, q) + g_j(t, q)}{m_j(t)}$.

If these functions are continuous (e.g., $m_j(t) \geq \epsilon_j > 0$ and continuous)\textsuperscript{11}, and satisfy the Lipschitz condition, then by Theorem 1 there exists the unique $C^2$-solution. If components $F_j(t, q)$ are discontinuous, but they satisfy conditions of Theorem 3 or 4, then there exists the unique continuous solution of the corresponding integral equation with time dependent financial masses. By considering the Bohmian model of the financial market with random initial conditions it is natural to assume that even the financial masses $m_j(t)$ are random variables, $m_j(t, \omega)$.

Thus the level of emission of $j$th share $m_j$ depends on the classical state $\omega$ of the financial market: $m_j \equiv m_j(t, \omega)$. In this way we obtain the simplest stochastic modification of Bohmian dynamics:

$\ddot{q}_j(t, \omega) = \frac{f_j(t, q(t, \omega)) + g_j(t, q(t, \omega))}{m_j(t, \omega)}$

or in the integral version:

$q_j(t, \omega) = q_{0j}(\omega) + \int_{t_0}^{t} v(s, \omega)ds \quad (47)$

$v_j(t, \omega) = \int_{t_0}^{t} [f_j(s, q(s, \omega)) + g_j(s, q(s, \omega))]/m_j(s, \omega)ds \quad (48)$

If the financial mass can become zero at some moments of time, then the price can have nonzero quadratic variation. However, under such conditions we do not have an existence theorem.

\textsuperscript{11}The condition $m_j(t) \geq \epsilon_j > 0$ is very natural. To be accounted at the financial market, the volume of emission of any share should not be negligibly small.
19. Bohm-Vigier stochastic mechanics

The quadratic variation objection motivates consideration of the Bohm-Vigier stochastic model, instead of the completely deterministic Bohmian model. We follow here [18]. We recall that in the original Bohmian model the velocity of an individual particle is given by

\[ v = \frac{\nabla S(q)}{m}. \] (49)

If \( \psi = Re^{iS/h} \), then Schrödinger’s equation implies that

\[ \frac{dv}{dt} = -\nabla(V + U), \] (50)

where \( V \) and \( U \) are classical and quantum potentials respectively. In principle one can work only with the basic equation (49).

The basic assumption of Bohm and Vigier was that the velocity of an individual particle is given by

\[ v = \frac{\nabla S(q)}{m} + \eta(t), \] (51)

where \( \eta(t) \) represents a random contribution to the velocity of that particle which fluctuates in a way that may be represented as a random process but with zero average. In Bohm-Vigier the stochastic mechanics quantum potential comes in through the average velocity and not the actual one.

We now shall apply the Bohm-Vigier model to financial market, see also E. Haven [29]. The equation (51) is considered as the basic equation for the price velocity. Thus the real price becomes a random process (as well as in classical financial mathematics [3]). We can write the stochastic differential equation, SDE, for the price:

\[ dq(t) = \frac{\nabla S(q)}{m}dt + \eta(t)dt. \] (52)

To give the rigorous mathematical meaning to the stochastic differential we assume that

\[ \eta(t) = \frac{d\xi(t)}{dt}, \] (53)

for some stochastic process \( \xi(t) \). Thus formally:

\[ \eta(t)dt = \frac{d\xi(t)}{dt}dt = d\xi(t), \] (54)

and the rigorous mathematical form of the equation (52) is

\[ dq(t) = \frac{\nabla S(q)}{m}dt + d\xi(t). \] (55)

The expression (53) one can consider either formally or in the sense of distribution theory (we recall that for basic stochastic processes, e.g., the Wiener process, trajectories are not differentiable in the ordinary sense almost every where).

Suppose, for example, that the random contribution into the price dynamics is given by \textit{white noise}, \( \eta_{\text{white noise}}(t) \). It can be defined as the derivative (in sense of distribution theory) of the Wiener process:

\[ \eta_{\text{white noise}}(t) = \frac{dw(t)}{dt}, \]
thus:

\[ v = \frac{\nabla S(q)}{m} + \eta_{\text{white noise}}(t), \]  

(56)

In this case the price dynamics is given by the SDE:

\[ dq(t) = \frac{\nabla S(q)}{m}dt + dw(t). \]  

(57)

What is the main difference from the classical SDE-description of the financial market? This is the presence of the pilot wave \( \psi(t, q) \), mental field of the financial market, which determines the coefficient of drift \( \frac{\nabla S(q)}{m} \). Here \( S \equiv S_\psi \). And the \( \psi \)-function is driven by a special field equation – Schrödinger’s equation. The latter equation is not determined by the SDE (57). Thus, instead of one SDE, in the quantum-like model, we have the system of two equations:

\[ dq(t) = \frac{\nabla S_\psi(q)}{m}dt + d\xi(t). \]  

(58)

\[ i\hbar \frac{\partial \psi}{\partial t}(t, q) = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial q^2}(t, q) + V(q)\psi(t, q). \]  

(59)

Finally we come back to the problem of the quadratic variation of the price. In the Bohm-Vigier stochastic model (for, e.g., the white noise fluctuations of the price velocity) is nonzero.

20. Comparison of the Bohmian model with models with stochastic volatility

Some authors, see, e.g., [3] for details and references, consider the parameters of volatility \( \sigma(t) \) as representing the market behaviors. From such a point of view our financial wave \( \psi(t, q) \) plays in the Bohmian financial model the role similar to the role of volatility \( \sigma(t) \) in the standard stochastic financial models. We recall that dynamics of \( \psi(t, q) \) is driven by the independent equation, namely the Schrödinger equation, and \( \psi(t, q) \) plays the role of a parameter of the dynamical equation for the price \( q(t) \).

We recall the functioning of this scheme:

a) we find the financial wave \( \psi(t, q) \) from the Schrödinger equation;

b) we find the corresponding quantum financial potential

\[ U(t, q) \equiv U(t, q; \psi) \]

(it depends on \( \psi \) as a parameter);

c) we put \( U(t, q; \psi) \) into the financial Newton equation through the quantum (behavioral) force \( g(t, q; \psi) = -\frac{\partial U(t, q; \psi)}{\partial q} \).

We remark that conventional models with stochastic volatility work in the same way, see [3]. Here the price \( q_t \) is a solution of the stochastic differential equation:

\[ dq_t = q_t(\mu(t, q_t, \sigma_t)dt + \sigma_t dw^\mu_t, \]  

(60)

where \( w^\mu_t \) is the Wiener process, \( \sigma_t \) is the coefficient depending on time, price and volatility. And (this is a crucial point) volatility satisfies the following stochastic differential equation:

\[ d\Delta_t = \alpha(t, \Delta_t)dt + b(t, \Delta_t)dw^\Delta_t, \]  

(61)

where \( \Delta_t = ln\sigma_t^2 \) and \( w^\Delta_t \) is a Wiener process which is independent of \( w^\mu_t \).

One should first solve the equation for the volatility (61), then put \( \sigma_t \) into (60) and, finally, find the price \( q_t \).
21. Classical and quantum contributions to financial randomness

As in conventional stochastic financial mathematics, see, e.g., [2], [3], we can interpret \( \omega \) as representing a state of financial market. The only difference is that in our model such an \( \omega \) should be related to ”classical state” of the financial market. Thus we interpret conventional randomness of the financial market as ”classical randomness”, i.e., randomness that is not determined by expectations of trades and other behavioral factors. Besides this ”classical states” \( \omega \) our model contains also ”quantum states” \( \psi \) of the financial market describing market’s psychology. In fact all processes under consideration depend not only the classical state \( \omega \), but also on the quantum state \( \psi \):

\[
d v_j(t, \omega, \psi) = \frac{f_j(t, q(t, \omega, \psi), v(t, \omega, \psi), \omega)}{m_j(t, \omega)} dt + \frac{g_j(t, q(t, \omega, \psi), \omega, \psi)}{m_j(t, \omega)} dt + \sigma_j(t, \omega) dW_j(t, \omega).
\]

(62)

We remark that the quantum force depends on the \( \psi \)-parameter even directly:

\[
g_j = g_j(t, q, \omega, \psi).
\]

The initial condition for the stochastic differential equation (62) depends only on \( \omega \):

\[
q_j(0, \omega) = q_{j0}(\omega), \quad v_j(0, \omega) = v_{j0}(\omega).
\]

But in general the quantum state of the financial market is given not by the pure state \( \psi \), but by the von Neumann density operator \( \rho \). Therefore \( \psi \) in (62) is a quantum random parameter with the initial quantum probability distribution given by the density operator at the initial moment:

\[
\rho(0) = \rho_0.
\]

We recall that the Schrödinger equation for the pure state implies the von Neumann equation for the density operator:

\[
i \dot{\rho}(t) = [\hat{H}, \rho].
\]

(63)

References

[1] Bachelier L 1890 Ann. Sc. l’Ecole Normale Superiere 111-17
[2] Mantegna R N and Stanley H E 2000 Introduction to econophysics (Cambridge: Cambridge Univ. Press)
[3] Shiryaev A N 1999 Essentials of Stochastic Finance: Facts, Models, Theory (Singapore: World Scientific Publishing Company)
[4] Sammelson P A 1965 Industrial Management Rev. 6 41
[5] Fama E F 1970 J. Finance 25 383
[6] Barnett W A and Serletis A 2000 Martingales, nonlinearity, and chaos J. Economic Dynamics and Control 24 703
[7] Benhabib J 1992 Cycles and Chaos in Economic Equilibrium (Princeton: Princeton University Press)
[8] Granger C W J 1994 Is chaotic theory relevant for economics? A review essay, J. of International and Comparative Economics 3 139-145
[9] Arthur W B, Holland J H, LeBaron B, Palmer R, and Tayler P, 1997, Asset pricing under endogenous expectations in an artificial stock market, in W. A. Arthur, D. Lane, and S. N. Durlauf, eds., The economy as evolving, complex system-2 (Redwood City, CA: Addison-Wesley).
[10] Brock W A and Sayers C 1988 Is business cycle characterized by deterministic chaos? Journal of Monetary Economics 22 71-90
[11] Campbell J Y Lo A W and MacKinlay A C 1997 The econometrics of financial markets (Princeton: Princeton University Press, Princeton)
[12] DeCoste G P and D W Mitchell 1991 J. of Business and Economic Statistics 9 455-462
[13] Hsieh D A 1991 J. of Finance 46 1839
[14] Soros J 1987 The alchemy of finance. Reading of mind of the market (New-York: J. Wiley and Sons, Inc.)
[15] Aerts D and Aerts S 1995 Foundations of Science 1 1
[16] Accardi L 1997 Urne e Camaleoni: Dialogo sulla realtà, le leggi del caso e la teoria quantistica (Rome: Il Saggiatore)
[17] Khrennikov A Yu 2004 Information dynamics in cognitive, psychological and anomalous phenomena (Dordrecht: Kluwer)
[18] Bohm D and Hiley B 1993 The undivided universe: an ontological interpretation of quantum mechanics (London: Routledge and Kegan Paul)
[19] Hiley B and Pylkkänen P 1997 Active information and cognitive science – A reply to Kieseppä, Brain, mind and physics, eds P Pylkkänen, P Pylkkö and A Hautamäki (Amsterdam: IOS Press) p 123
[20] Bohm D 1951 Quantum theory (Englewood Cliffs, New-Jersey: Prentice-Hall)
[21] Holland P 1993 The quantum theory of motion (Cambridge: Cambridge Univ. Press)
[22] Hiley B 2001 From the Heisenberg picture to Bohm: a new perspective on active information and its relation to Shannon information Quantum Theory: Reconsideration of Foundations ser. Math. Modelling vol. 10, ed A Yu Khrennikov (Växjö: Växjö University Press) p 234
[23] Heisenberg W 1930 Physical principles of quantum theory (Chicago: Chicago Univ. Press)
[24] Dirac P A M 1995 The Principles of Quantum Mechanics (Oxford: Claredon Press)
[25] Haven E 2002 Physica A 304 507
[26] Haven E 2003 Physica A 324 201
[27] Haven E 2003 Physica A 344 151
[28] Segal W and Segal I E 1998 Proc. Nat. Acad. Sc. USA 95 4072
[29] Haven E 2006 Bohmian mechanics in a macroscopic quantum system Foundations of Probability and Physics-3 vol. 810, ed A Yu Khrennikov (Melville, New York: AIP) p 330
[30] Piotrowski E W Sladkowski J 2001 Quantum-like approach to financial risk: quantum anthropic principle Preprint quant-ph/0110046
[31] Mandelbrot B B 1963 J. Business 36 394
[32] Kolmogorov A N Fomin S V 1975 Introductory Real Analysis (New York: Dover Publications)