GRADINGS ON THE ALBERT ALGEBRA AND ON $\mathfrak{f}_4$

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Abstract. We study group gradings on the Albert algebra and on the simple exceptional Lie algebra $\mathfrak{f}_4$ over algebraically closed fields of characteristic zero. The immediate precedent of this work is [13] where we described (up to equivalence) all the gradings on the exceptional simple Lie algebra $\mathfrak{g}_2$. In the cases of the Albert algebra and $\mathfrak{f}_4$, we look for the nontoral gradings finding that there are only eight nontoral nonequivalent gradings on the Albert algebra (three of them being fine) and nine on $\mathfrak{f}_4$ (also three of them fine).

1. Introduction

The interest on gradings on simple Lie and Jordan algebras has been remarkable in the last years. The gradings of finite dimensional simple Lie algebras, ruling out $\mathfrak{g}_2$, $\mathfrak{d}_4$ and the exceptional cases, are described in [7]. The gradings on simple Jordan algebras of type $H_n(F)$ and $H_n(Q)$ are given in the same reference, for an algebraically closed field $F$ of characteristic zero and a quaternion algebra $Q$. In that work, the authors use their previous results in [9] about gradings of associative algebras $M_n(F)$. In [6] all gradings on the simple Jordan algebras of Clifford type have been described. The fine gradings on $\mathfrak{g}_2$ have been determined in [20] solving the related problem of finding maximal abelian groups of diagonalizable automorphisms of the algebra (not only in GL($n, \mathbb{C}$) but also in O($n, \mathbb{C}$) for $n \neq 8$ and SP($2n, \mathbb{C}$)). General notions about Lie gradings are considered in [36], and the real case is treated in [21]. Notice that all the mentioned works make use of techniques related to the associative case. The first studies of gradings on exceptional Lie algebras are [13] and [8], which describe the group gradings on $\mathfrak{g}_2$. To continue the study of exceptional Lie and Jordan algebras, we aboard in this paper the task of describing nontoral group gradings on $\mathfrak{f}_4$ and on the Albert algebra. The notion of group grading is closely related to that of commuting set of semisimple automorphisms (or equivalently abelian subgroup of semisimple automorphisms) of the algebra. But $\mathfrak{f}_4$, the automorphism group of the Albert algebra $J$, is isomorphic to the automorphism group of the Lie algebra $\mathfrak{f}_4 = \text{Der}(J)$. So automorphism information can be transferred from one to the other context. That is the reason to study both algebras jointly.

We have ruled out the study of toral gradings on $\mathfrak{f}_4$ (and then on $J$) because of the overwhelming proliferation of nonequivalent cases, and the fact that, their determination (though tedious) follows from mechanical coarsenings of the Cartan grading. On the contrary, the nontoral gradings on a simple Lie algebra are not compatible with the root system. So they could lead us to new ways of looking at the algebra. This is specially true in the case of fine gradings, a fact that explains

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the activity around this subject (see, for instance, [22] or [34]). In other papers of
Lie gradings ([35]), the aim is to study the Lie gradings without the restriction of a
grading group, but we have preferred not to adopt this approach because of [15]. It
is still an open question the existence of a grading group on any finite dimensional
graded simple Lie algebra over an algebraically closed field of characteristic zero.

The techniques employed to search the gradings have been of very different
nature. It could be said that this is a multidisciplinary field nowadays. There is a
first tool, very intuitive, which is the usage of models of the algebra. But, although
it provides most of the existing gradings (actually all of them), this fact cannot be
proved without a more powerful tool. This is why we exploit the benefit of working
inside the normalizer of a maximal torus of the automorphism group. This turns
out to be quite technical and less intuitive (some computer aided arguments have
been essential). However this approach allows to confirm with full precision all the
hypothesis about how many nontoral gradings appear, the relation among them
and other aspects.

A summary of the contents of the work follows in the next paragraphs.

Section 2 presents some preliminaries, and compiles basic facts on gradings and
related topics. In Subsection 2.1 we introduce some terminology of algebraic groups
needed for our approach. In 2.2 we devote some attention to recall the most known
model of the Albert algebra, \( J = H_3(C) \) for a Cayley algebra \( C \), described in
[38]. Each model of a given algebra has its own advantages (and drawbacks) to
present particular gradings, thus in 2.4 Tits construction of the Albert algebra is
also recalled. In 2.3 we consider the notion of toral grading. In the Lie algebras
case, a grading is toral when its homogeneous components are sum of root spaces.
In general a grading is toral if it is produced by a set of automorphisms contained
in a torus of the automorphism group. At this point we fix a maximal torus of
\( F_4 \),
and characterize the torality of a grading in different terms.

In Section 3 we start inducing gradings on \( J \) from gradings on the related Cayley
algebra \( C \) such that \( J = H_3(C) \). This arises from the well-known possibility of
extending automorphisms of \( C \) to elements in \( F_4 \) \( (G_2 \subset F_4) \). It turns out that the
unique (up to equivalence) nontoral grading of \( g_2 \), induces a nontoral grading on
\( f_4 \) \( (a \mathbb{Z}_2^3 \text{-grading}) \). Thus we are led to the birth of our first nontoral grading on \( J \).
This grading will induce a numerous family of gradings whose relatives come from a
mixing process described in Subsection 3.3. Previously, we consider all the gradings
of the subalgebra \( H_3(F) \), where \( F \) stands for the ground field. But automorphisms
of \( H_3(F) \) can also be nicely extended to automorphisms of \( J \). And curiously all
of these commute with automorphisms of \( J \) coming from \( C \). Thus by crossing the
nontoral \( \mathbb{Z}_2^3 \text{-grading} \) on \( J \) coming from \( C \) with all the gradings detected in \( H_3(F) \)
we obtain a family of six nontoral gradings on \( J \) described in 3.3. But, as it is
pointed out at the end of such subsection, one of these six nontoral gradings admits
a proper nontoral coarsening. Thus we are led to a set of seven nonequivalent
nontoral gradings on \( J \).

Unfortunately we are not done with this set of gradings. To detect the remaining
nontoral gradings on \( J \) we need to invoke some other model of \( J \) different from the
usual \( H_3(C) \). But at this point, Tits construction comes in our help to provide
a \( \mathbb{Z}_3^3 \text{-grading} \) with comes from the natural embedding of automorphisms of the
algebra \( M_3(F) \) into \( F_4 \). The origin of this \( \mathbb{Z}_3^3 \text{-grading} \) is a known nontoral \( \mathbb{Z}_2^3 \text{-grading} \) of \( M_3(F) \) which can be lifted and finally refined to the nontoral \( \mathbb{Z}_3^3 \text{-grading} \).
on $J$. In this way we get a set of eight pairwise nonequivalent nontoral gradings. Furthermore, any nontoral grading is equivalent to some of these. This is one of our main results presented at the end of Section 3 (though the proof will have to be postponed to the final sections of the work).

In Section 4 we focus on $f_4$. How can we present a concrete grading on a Lie algebra? One of the ways in which a grading on an algebra can be given is to provide the set of automorphisms inducing the grading. Specially because the automorphisms can be given in terms of toral elements and the Weyl group. This happens because of the relevant fact that the quasitorus inducing the grading is always contained in the normalizer of a maximal torus. Consequently, our first task is to fix a particular representation of its Weyl group $W$ and provide a set of representatives of conjugacy classes in $W$. Next we present in Subsection 4.1 a maximal torus of $\mathfrak{g}_4 := \text{aut}(f_4)$ nicely related to the maximal torus previously presented in $F_4 := \text{aut}(J)$. Also the action of $W$ on the maximal torus is described.

Section 5 is intended to provide the main results on quasitori which will allow the classification of fine and nontoral gradings. We introduce a family of quasitori $A(j, t) \subset \mathfrak{g}_4$ and immediately proceed with the study of those which are nontoral. Theorem 4 describes the maximal quasitori of $\mathfrak{g}_4$ up to conjugacy which in particular yields the classification of fine gradings on $f_4$ up to equivalence (Corollary 4). Finally in Subsection 5.2 we get a more detailed classification up to conjugacy of nontoral quasitori of $\mathfrak{g}_4$. This result contained in Theorem 5 might be of independent interest. As a byproduct, we get an exhaustive set of representatives of 9 equivalence classes of nontoral gradings on $f_4$.

In Section 6 we revisit the gradings on the Albert algebra to provide a proof that, up to equivalence, the nontoral gradings on this algebra are the eight ones given in Theorem 3.

Having given the quasitori which induce all the nontoral gradings on $f_4$ we look more closely at the fine gradings in Section 7. There, we describe those gradings in a twofold way. First we give the homogeneous components of each grading in a previously fixed basis of the algebra. This description depends on computational methods (simultaneous diagonalization algorithms). In spite of their computational nature, these descriptions may be interesting for applications in which explicit calculations are needed. The second description procedure for the gradings is the exhibition of models, which make the grading appear in a natural setting with no appeal to a particular basis. For instance, the fine $\mathbb{Z}_3^3$-grading on $f_4$ comes from a $\mathbb{Z}_3^3$-grading in $\mathfrak{e}_6$ which can be described in a very convenient way by using the known model based in the $\mathbb{Z}_3$-grading $\mathfrak{e}_6 = \mathfrak{sl}(X_1) \oplus \mathfrak{sl}(X_2) \oplus \mathfrak{sl}(X_3) \oplus X_1 \otimes X_2 \otimes X_3 \oplus X_1^* \otimes X_2^* \otimes X_3^*$ with zero homogeneous component three copies of the algebra of type $a_2$ [1, p. 85]. The fine $\mathbb{Z}_2^5$-grading may be seen directly in $f_4 = \text{Der}(J)$ but there is also an easy way to see it in the Tits unified construction for the exceptional Lie algebras (see, for instance, [38, p. 122]), so as the $\mathbb{Z}_3^2 \times \mathbb{Z}$ fine grading. Finally, in 7.4 we justify why this philosophy can be valid for describing most of the gradings on a simple Lie algebra. To illustrate how this works, we describe the $\mathbb{Z}_2^2 \times \mathbb{Z}_4$ and $\mathbb{Z}_2^2 \times \mathbb{Z}_8$-nontoral gradings on $f_4$, and also the $\mathbb{Z}_2^2 \times \mathbb{Z}_8$-fine grading equivalent to the $\mathbb{Z}_3^2 \times \mathbb{Z}$-one, by using the model of $f_4$ described in [12], which is based in an initial $\mathbb{Z}_4$-grading.
2. Preliminary definitions and results

2.1. Group gradings. Our aim is the study of group gradings on certain nonassociative algebras over fields. If $V$ is such an algebra and $G$ is an abelian group, we shall say that the decomposition $V = \bigoplus_{g \in G} V_g$ is a $G$-grading whenever for all $g, h \in G$, $V_g V_h \subseteq V_{gh}$.

In this work we shall use some notions borrowed from the theory of algebraic groups. Since we only need linear algebraic groups, all concepts must be understood in that context. Notice that the group of automorphisms of the algebra $V$ is an algebraic linear group. The ground field $F$ will be supposed to be algebraically closed and of characteristic zero throughout this work. There is a deep relationship between gradings on $V$ and quasitori of the group of automorphisms $\text{aut}(V)$, according to [33, §3, p. 104]. Following this reference, a commutative algebraic group whose identity component is an algebraic torus is called an algebraic quasitorus. An algebraic linear group is a quasitorus if and only if there is a basis relative to which the elements of the quasitorus are simultaneously diagonalizable. If $S$ is a finitely generated abelian group, then its group of characters $\mathfrak{X}(S) = \text{hom}(S, F^\times)$ is a quasitorus and reciprocally, the group of characters of a quasitorus turns out to be a finitely generated abelian group.

If $V = \bigoplus_{g \in G} V_g$ is a $G$-grading, the map $\psi: \mathfrak{X}(G) \to \text{aut}(V)$ mapping each $\alpha \in \mathfrak{X}(G)$ to the automorphism $\psi_\alpha: V \to V$ given by $V_g \ni x \mapsto \psi_\alpha(x) := \alpha(g)x$ is a group homomorphism. In particular $\psi(\mathfrak{X}(G))$ is a quasitorus. And conversely, if $Q$ is a quasitorus and $\psi: Q \to \text{aut}(V)$ is a homomorphism, $\psi(Q)$ is formed by semisimple automorphisms ([25, p. 99]) and we have a $\mathfrak{X}(Q)$-grading $V = \bigoplus_{g \in \mathfrak{X}(Q)} V_g$ given by

$$V_g = \{ x \in V \mid \psi(g)(x) = g(q)x \ \forall q \in Q \}.$$ 

When we speak in this paper about a $G$-grading $V = \bigoplus_{g \in G} V_g$, we mean that $G$ is generated by the set $\{ g \in G \mid V_g \neq 0 \}$, called the support of the grading and denoted by $\text{Supp}(G)$. In terms of algebraic groups, this is equivalent to the fact that the homomorphism $\psi: Q \to \text{aut}(V)$ is injective. Let us see it. If $G$ is generated by the support and $\psi(q_0) = \text{id}_V$ for some $q_0 \in Q$, $x = \psi(q_0)x = g(q_0)x$ for any $x \in V_g$, so $g(q_0) = 1$ for any $g$ in the support. But since $G$ is generated by the support, we have $g(q_0) = 1$ for any $g \in G$ and this implies $q_0 = 1$.

Reciprocally if $\psi$ is a monomorphism, the subgroup $S$ of $G$ generated by the support is $S = \mathfrak{X}(Q')$ for some quasitorus $Q'$. The inclusion $i: S \to G$ induces by duality an epimorphism $\pi: Q \to Q'$. The grading induced by $S$ comes from a homomorphism $\psi': Q' \to \text{aut}(V)$ making commutative the diagram on the right. Since $\psi$ is a monomorphism, so is it $\pi$. Hence $\pi$ is an isomorphism and by duality the same can be said about $i$. Thus $S = G$.

We say that two gradings $V = \bigoplus_{g \in G} X_g = \bigoplus_{g' \in G} Y_{g'}$ are isomorphic if there is $f \in \text{aut}(V)$ and $\alpha: G \to G'$ a group isomorphism such that $f(X_g) = Y_{\alpha(g)}$ for any $g \in G$. So, if $\psi: \mathfrak{X}(G) \to \text{aut}(V)$ and $\psi': \mathfrak{X}(G') \to \text{aut}(V)$ are the corresponding homomorphisms, and we take $\alpha^*: \mathfrak{X}(G') \to \mathfrak{X}(G)$, $\alpha^*(\beta) = \beta \alpha$, and $\text{Ad}(f): \text{aut}(V) \to \text{aut}(V)$ given by $\text{Ad}(f)(\rho) = f \rho f^{-1}$, the previous condition is equivalent to the commutativity $\text{Ad}(f) \psi' = \psi \alpha^*$.

We say that two gradings $V = \bigoplus_{g \in G} X_g = \bigoplus_{g' \in G} Y_{g'}$ are equivalent if the sets of homogeneous subspaces are the same up to isomorphism, that is, there are an
automorphism \(f \in \text{aut}(V)\) and a bijection between the supports \(\alpha: \text{Supp}(G) \to \text{Supp}(G')\) such that \(f(X_g) = Y_{\alpha(g)}\) for any \(g \in \text{Supp}(G)\). Our objective is to classify gradings up to equivalence. A convenient invariant for equivalence is that of type. Suppose we have a grading on a finite dimensional algebra, then for each positive integer \(i\) we will denote (following [23]) by \(h_i\) the number of homogeneous components of dimension \(i\). In this case we shall say that the grading is of type \((h_1, h_2, \ldots, h_l)\), for \(l\) the greatest index such that \(h_l \neq 0\). Of course the number \(\sum_i i h_i\) agrees with the dimension of the algebra.

Another key notion is that of a coarsening of a given grading. Thus consider an \(F\)-algebra \(V\), a \(G\)-grading \(V = \bigoplus_{g \in G} X_g\) and an \(H\)-grading \(V = \bigoplus_{h \in H} Y_h\). We shall say that the \(H\)-grading is a coarsening of the \(G\)-grading if and only if each nonzero homogeneous component \(Y_h\) with \(h \in H\) is a direct sum of some homogeneous components \(X_g\). In this case we shall also say that the \(G\)-grading is a refinement of the \(H\)-grading. Notice that there is not a relationship between \(G\) and \(H\), but if \(\psi: X(G) \to \text{aut}(V)\) and \(\psi': X(H) \to \text{aut}(V)\) are the homomorphisms producing the above gradings, any automorphism in \(\psi'(X(H))\) acts as a scalar multiple of the identity in \(X_g\) and hence, all of them commute with \(\psi(X(G))\).

The concept of universal grading group is fundamental to obtain the coarsenings of a given grading. Though this notion is given in the context of simple Lie algebras \(G\) to be a group grading, that is, each \(i\)-identity in \(X\) of type \(\psi\) the above gradings, any automorphism in \(\psi'(X(H))\) acts as a scalar multiple of the identity in \(X_g\) and hence, all of them commute with \(\psi(X(G))\).

If \(\psi: X(G) \to \text{aut}(V)\) is a grading with \(G = X(Q)\), then \(G\) is the universal grading group if and only if \(\psi' \circ \psi = \psi'\). This is easily proved applying duality to the above universal property of \(G\). From another viewpoint, \(G\) is the universal group of a grading if and only if \(X(G)\) is a maximal element in the set of quasitori of \(\text{aut}(V)\) producing exactly the same grading.

A group grading is fine if its unique refinement is the given grading. If \(G\) is the universal group of a grading \(\psi: X(G) \to \text{aut}(V)\), the grading is fine if and only if \(\psi(X(G))\) is a maximal abelian subgroup of semisimple elements, which is usually called a MAD ("maximal abelian diagonalizable") in papers about fine gradings, like [19]. Besides, each MAD \(Q \subset \text{aut}(V)\) produces a \(X(Q)\)-fine grading on \(V\) such
that $\mathfrak{X}(Q)$ is the universal group of this grading. The converse is true in the sense just mentioned but notice that there are fine $G$-gradings such that $\psi(\mathfrak{X}(G))$ is not a MAD, for instance the $\mathbb{Z}_2^3 \times \mathbb{Z}_8$-grading on $f_4$ described in 7.4 or the $\mathbb{Z}$-grading on $g_2$ described in [13, Theorem 2,(4)]. In particular from the above, the number of fine gradings on $V$ up to equivalence is the same than the number of MAD’s of aut$(V)$ up to isomorphism (and less than the number of fine gradings up to isomorphism, in general).

Other notations which will be used along this paper are the following. For a linear algebraic group $G$, and a subset $S \subset G$, the centralizer of $S$ in $G$ will be denoted by $\mathcal{C}_G(S)$. Analogously, by $\mathfrak{N}_G(S)$ we shall mean the normalizer of $S$ in $G$.

2.2. About the Albert algebra and $f_4$. Consider the Cayley $F$-algebra $C$, which under our hypothesis on the ground field must be isomorphic to Zorn matrices algebra. Take the standard involution $x \mapsto \bar{x}$, the norm $n : C \to F$ given by $n(x) := xx$ and the trace $\text{tr} : C \to F$ defined as $\text{tr}(x) := x + \bar{x}$. Recall that the polar form $f : C \times C \to F$ of $n$ is the symmetric bilinear form $f(x,y) := \frac{1}{2}(n(x+y) - n(x) - n(y))$. The Albert algebra $J = H_3(C) = \{x = (x_{ij}) \in M_3(C) \mid x_{ij} = x_{ji}\}$ is the exceptional reduced Jordan algebra, that is, the set of matrices of the form

$$
(1)
\begin{pmatrix}
\alpha & o_1 & o_2 \\
o_1 & \beta & o_3 \\
o_2 & o_3 & \gamma
\end{pmatrix}
$$

where $\alpha, \beta, \gamma \in F$ and $o_i \in C (i = 1, 2, 3)$. Since our base field is of characteristic zero, we can shelter on the linear theory of Jordan algebras and the product in $J$ may be defined as $x \cdot y := \frac{1}{2}(xy + yx)$ where juxtaposition stands for the usual matrix product. This simple Jordan algebra is exceptional in the sense that it is not a subalgebra of the symmetrization of any associative algebra. It will be convenient to introduce some notations for further reference. Thus the element in $J$ obtained in (1) for $\alpha = 1$, $\beta = \gamma = o_i = 0$ will be denoted by $E_1$ while the one obtained for $\beta = 1$, $\alpha = \gamma = o_i = 0$ will be denoted by $E_2$. In a similar way we can define $E_3$ so that $1 := \sum_{i=1}^3 E_i$ is the unit of $J$. Next, for any $a \in C$ we define by $a^{(i)}$ the element in $J$ obtained making $\alpha = \beta = \gamma = o_i = o_2 = 0$, $o_3 = a$ in (1). Define by $a^{(2)}$ the one obtained for $\alpha = \beta = \gamma = o_1 = o_3 = 0$ and $o_2 = \bar{a}$. Finally denote by $a^{(3)}$ the element arising for $\alpha = \beta = \gamma = o_2 = o_3 = 0$, and $o_1 = a$. The multiplication table of the commutative algebra $J$ may be summarized in the following relations:

$$
E_i^2 = E_i, \quad E_i a^{(i)} = 0, \quad a^{(i)} b^{(i)} = f(a,b)(E_j + E_k), \\
E_i E_j = 0, \quad E_i a^{(j)} = \frac{1}{2}a^{(j)}, \quad a^{(i)} b^{(j)} = \frac{1}{2}(b\bar{a})^{(k)},
$$

where $(i,j,k)$ is any cyclic permutation of $(1,2,3)$ and $a,b \in C$. Following Schafer ([38, (4.41), p.109]), any $x \in J$ satisfies a cubic equation $x^3 - \text{Tr}(x)x^2 + Q(x)x - N(x)1 = 0$ where $\text{Tr}(x), Q(x), N(x) \in F$. Moreover, the inversibility of $x$ (in Jordan context) is equivalent to the fact that $N(x) \neq 0$.

For further reference we fix first $(e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3)$ the standard basis of the Cayley algebra $C$, defined for instance in [13, Section 3], given by the relations

$$
\begin{align*}
e_1 u_j &= u_j = u_j e_2, & u_i u_j &= v_k = -u_j u_i, & u_i v_i &= e_1, \\
e_2 v_j &= v_j = v_j e_1, & -v_i v_j &= u_k = v_j e_i, & v_i u_i &= e_2,
\end{align*}
$$

where $e_1$ and $e_2$ are orthogonal idempotents, again $(i,j,k)$ is any cyclic permutation of $(1,2,3)$, and the remaining relations are null. Thus we can fix our standard basis
of the Albert algebra:
\[ B = \langle E_1, E_2, E_3, e^{(3)}_1, e^{(3)}_2, u^{(3)}_1, u^{(3)}_2, v^{(3)}_1, v^{(3)}_2, v^{(3)}_3, e^{(2)}_1, e^{(2)}_2, -u^{(2)}_1, -u^{(2)}_2, -u^{(2)}_3, \]
\[ v^{(2)}_1, -v^{(2)}_2, -v^{(2)}_3, e^{(1)}_1, e^{(1)}_2, u^{(1)}_1, u^{(1)}_2, v^{(1)}_1, v^{(1)}_2, v^{(1)}_3 \rangle. \]

Let us define now the group \( F_4 := \text{aut}(J) \) and its Lie algebra \( \mathfrak{f}_4 = \text{Der}(J) \). Recall (see for instance [26, p. 285]) that the automorphism group \( F_4 \) and the automorphism group \( \mathfrak{f}_4 := \text{aut}(\mathfrak{f}_4) \) are isomorphic via the map \( \text{Ad} : F_4 \rightarrow \mathfrak{f}_4 \) such that \( \text{Ad}(f)d := fdf^{-1} \) for any \( f \in F_4 \) and \( d \in \mathfrak{f}_4 \). This isomorphism of algebraic groups provides a tool for translating gradings from the Albert algebra to \( \mathfrak{f}_4 \) and conversely. However, unfortunately this translating tool does not preserve equivalence. This phenomenon is similar to the one explained in [13, Section 4] in the similar context produced by the analogue group isomorphism \( \text{Ad} : G_2 = \text{aut}(C) \rightarrow \text{aut}(\mathfrak{g}_2) = \text{aut}(\text{Der}(C)) \).

2.3. **Maximal torus of \( F_4 \).** Let us fix certain maximal torus of \( F_4 \). We use the standard basis of \( J \) above defined. Define now the maximal torus \( \mathfrak{T}_0 \) of \( F_4 \) whose elements are the automorphisms of \( J \) which are diagonal relative to \( B \). This is isomorphic to \((F^\times)^4\) and it is easy to see that the matrix of any such automorphism relative to \( B \) is
\[
\text{diag}\left(1, 1, 1, \frac{1}{\alpha}, \beta, \gamma, \frac{\delta^2}{\alpha \beta \gamma}, \frac{1}{\alpha \beta \gamma}, \frac{1}{\beta \gamma}, \frac{1}{\delta}, \frac{1}{\delta}, \frac{\alpha \beta \gamma}{\delta}, \frac{\alpha \gamma}{\delta}, \frac{\alpha \beta}{\delta}, \frac{\alpha \gamma}{\delta}, \frac{\alpha \beta \gamma}{\delta}, \frac{\alpha}{\delta}, \frac{\beta}{\delta}, \frac{\gamma}{\delta}, \frac{\delta}{\alpha}, \frac{\delta}{\alpha}, \frac{\delta}{\alpha}, \frac{\delta}{\alpha}\right),
\]
for some \( \alpha, \beta, \gamma, \delta \in F^\times \). Define now \( t_{\alpha, \beta, \gamma, \delta} \) as the automorphism in \( \mathfrak{T}_0 \) whose matrix relative to \( B \) is just the above one.

Consider a grading of an algebra \( A \) given by a group homomorphism \( \rho : \mathfrak{X}(G) \rightarrow \text{aut}(A) \). This grading is said to be **toral** if \( \rho(\mathfrak{X}(G)) \) is contained in some torus of the algebraic group \( \text{aut}(A) \).

Notice that by means of the above isomorphism \( \text{Ad} : F_4 \rightarrow \mathfrak{f}_4 \), a grading on \( J \) is toral if and only if it is applied to a toral grading on \( \mathfrak{f}_4 \). This does not imply that the number of non-toral gradings (up to equivalence) on \( J \) and on \( \mathfrak{f}_4 \) is necessarily the same, because the mechanism of translating gradings does not preserve equivalence.

An useful characterization of the torality of a grading on \( J \) is the following:

**Proposition 1.** A grading on the Albert algebra \( J \) is toral if and only if the elements of the standard basis of \( J \) (or any conjugated basis) are homogeneous.

Proof. Consider a toral grading induced by automorphisms \( \{t_i\}_{i \in I} \) all of them contained in the previous maximal torus \( \mathfrak{T}_0 \) of \( J \) (which supposes no restriction because any other maximal tori is conjugated to \( \mathfrak{T}_0 \)). Then obviously the simultaneous diagonalization of \( J \) relative to the family \( \{t_i\}_{i \in I} \) provides the original grading and the elements in the standard basis are homogeneous. Conversely, if this holds, then any element in this basis is an eigenvector of any of the grading automorphisms. Thus these automorphisms are in the maximal torus specified before and the grading is toral.

In particular, the proof of this proposition implies that if a grading on \( J \) is toral, then there are three orthogonal idempotents contained in the zero component (because up to conjugacy, \( \{E_1, E_2, E_3\} \) is contained in such component). Another
way of checking the torality is to look at the rank of the zero part of the induced grading on \( f_1 \), it will be toral in case this rank is 4 (see [13, Subsection 2.4]).

By using this set of idempotents, we can give more information about the general form of any semisimple automorphism of \( J \), because they are the building blocks of the grading sets. First notice that there is a group monomorphism \( D_4 \to F_4 \).

Indeed, if \( U \in O(C,n) = \{ g \in gl(C) \mid n(x,y) = n(g(x),g(y)) \ \forall x,y \in C \} \), there are \( U', U'' \in O(C,n) \) such that \( U(xy) = U'(x)U''(y) \) for any \( x,y \in C \). This is called the global triality principle in [11, Th. 3, p. 90]. Let \( \Psi_U : J \to J \) given by

\[
\Psi_U(E_i) = E_i, \quad \Psi_U(x^{(i)}) = (f_i(x))^{(i)} \quad \text{for} \quad f_1 = U, f_2 = U'', f_3 = U'.
\]

It is easy to check that \( \Psi_U \) is an automorphism of \( J \) fixing each idempotent. Conversely, if \( \psi \) is an automorphism of \( J \) fixing each idempotent, there exists \( U \in O(C,n) \) such that \( \psi = \Psi_U \). Moreover, any semisimple automorphism in \( F_4 \) is toral, since \( F_4 \) is a connected group, so that up to conjugacy it is contained in \( T_0 \), fixes each \( E_i \) and is of the form \( \Psi_U \) for some \( U \in O(C,n) \).

Besides, there are two outstanding automorphisms which deserve some consideration. The first one \( \theta : J \to J \) applies \( E_i \to E_{i+1} \) and \( x^{(i)} \to x^{(i+1)} \) cyclically. The second one \( \theta^2 : J \to J \) fixes \( E_3 \) permuting \( E_1 \) and \( E_2 \), and acts as the identity on elements \( x^{(3)} \) while \( x^{(1)} \to x^{(2)} \to x^{(1)} \). These automorphisms fix the set \( \{ E_1, E_2, E_3 \} \) and are of order three and two respectively. Our previous argument shows that any semisimple element in \( F_4 \) fixing the set \( \{ E_1, E_2, E_3 \} \) is a composition of one automorphism in \( \{ \text{id}, \theta, \theta^2, \theta \theta, \theta \theta^2 \} \) with one in \( \{ \Psi_U \mid U \in O(C,n) \} \).

2.4. Tits construction of the Albert algebras. There is another way in which the Albert algebra can be constructed. Let us start with the \( F_4 \)-algebra \( A = M_3(F) \) and denote by \( \text{Tr}_A, Q_A, N_A : A \to F \) the coefficients of the generic minimal polynomial such that \( x^3 - \text{Tr}_A(x)x^2 + Q_A(x)x - N_A(x)1 = 0 \) for all \( x \in A \). Recall that if \( x = (x_{ij}) \in A \) then \( \text{Tr}_A(x) = \sum_1^3 x_{ii} \),

\[
Q_A(x) = -x_{12}x_{21} + x_{11}x_{22} - x_{13}x_{31} - x_{23}x_{32} + x_{11}x_{33} + x_{22}x_{33},
\]

and \( N_A(x) = \det(x) \). Define also the quadratic map \( \sharp : A \to A \) given by \( x^\sharp := x^2 - \text{Tr}_A(x)x + Q_A(x)1 \). For any \( x,y \in A \) denote \( x \times y := (x+y)^\sharp - x^\sharp - y^\sharp \), and \( x^* := \frac{1}{2}x \times 1 = \frac{1}{2}\text{Tr}_A(x)1 - \frac{1}{2}x \). Finally consider the Jordan algebra \( A^+ \) whose underlying vector space agrees with that of \( A \) but whose product is \( x \cdot y = \frac{1}{4}(x^y + y^x) \). Next, define in \( A^3 := A \times A \times A \) the product

\[
(a_1, b_1, c_1)(a_2, b_2, c_2) := (a_1 \cdot a_2 + (b_1c_2)^* + (b_2c_1)^*, a_1^*b_2 + a_2^*b_1 + \frac{1}{4}(c_1 \times c_2), c_2a_1^* + c_1a_2^* + \frac{1}{2}(b_1 \times b_2)).
\]

Then \( A^3 \) with this product is isomorphic to \( J = H_3(C) \). This is the so called Tits construction of the Albert algebra. This allows us to identify \( J \) with the algebra \( A^3 \) in the rest of this section. For further reference we shall recall that the norm \( N \) (module the identification of \( J \) with the Tits construction) is given by

\[
N(a, b, c) = N_A(a) + N_A(b) + N_A(c) - \text{Tr}_A(abc)
\]

for any \( a, b, c \in A \) (see [29, p. 525]).

One of the relevant facts on Tits construction from our viewpoint is that it allows to embed \( \text{aut}(A) \) in \( F_4 \) via the map \( \text{aut}(A) \to F_4 \) given by \( f \mapsto f^\star \) where \( f^\star : J \to J \) is the automorphism such that \( f^\star(x,y,z) := (f(x), f(y), f(z)) \). As a further consequence we will be able to get gradings on \( J \) coming from gradings in the associative algebra \( A \) via this monomorphism of algebraic groups.
3. Inducing gradings on the Albert algebra

Some remarkable subalgebras of $J = H_3(C)$ induce gradings on $J$ by means of a particular embedding of its automorphism group in $F_4$. For instance, the automorphism group of $H_3(F)$ can be considered as a subgroup of $F_4$ providing thus a source of gradings on $J$: those whose grading automorphisms come from automorphisms of $H_3(F)$. More generally, if $V$ is an algebra such that $\text{aut}(V)$ is a subgroup of $F_4$, then any grading in $V$ will also induce a grading on $J$. We shall see that this happens for instance for the octonion $F$-algebra. It turns out that this idea for inducing gradings on $J$ provides a great number of the gradings existing on $J$.

3.1. Gradings from octonions. Let $C$ be the Cayley algebra. We consider as in the previous section the standard basis of $C$. We shall also need the maximal torus of $G_2 := \text{aut}(C)$ given by the automorphisms $t_{\alpha,\beta}$ whose matrix relative to the standard basis is

$$\text{diag}(1, 1, \alpha, \beta, (\alpha\beta)^{-1}, \alpha^{-1}, \beta^{-1}, \alpha\beta).$$

It has been first proved in [14], and then in [13, Subsection 3.3], that up to equivalence the unique nontoral grading on $C$ is the $\mathbb{Z}_2^4$-grading whose order-two grading automorphisms are $\{t_{1,-1}, t_{-1,1}, f_0\}$ where $f_0$ is the automorphism whose matrix relative to the standard basis is

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0
\end{pmatrix}.
$$

The construction $J = H_3(C)$ of the Albert algebra is particularly interesting for extending derivations and automorphisms from $C$ to $J$. More precisely if $f \in \text{aut}(C)$ is an automorphism of $C$ then we can construct the automorphism $\hat{f}$ of $J$ fixing the idempotents $E_i$ and such that $\hat{f}(o^{(i)}) := f(o^{(i)})$ for $i = 1, 2, 3$ and any $o \in C$. This provides a monomorphism of algebraic groups $i: G_2 \rightarrow F_4$ such that $f \mapsto \hat{f}$. By differentiating at 1 we get a monomorphism of Lie algebras $d\iota(1): \text{Der}(C) = g_2 \rightarrow \text{Der}(J) = f_4$ mapping each derivation $d \in \text{Der}(C)$ to the derivation $\hat{d} \in \text{Der}(J)$ annihilating the idempotents and making $\hat{d}(o^i) = d(o)^i$, for $i = 1, 2, 3$. This, of course, has an immediate application to gradings: any grading on $C$ induces a grading on $J$. Indeed, a $G$-grading on $C$ comes from an algebraic group homomorphism $\rho: X(G) \rightarrow \text{aut}(C)$, therefore $i\rho: X(G) \rightarrow \text{aut}(J)$ provides a $G$-grading on the Albert algebra. This device gives a first source of gradings on $J$, namely, all those coming from gradings on $C$. Since $i$ maps tori of $\text{aut}(C)$ to tori of $\text{aut}(J)$, toral gradings on $C$ induce toral gradings on $J$. The unique nontoral grading on $C$ up to equivalence provides a $\mathbb{Z}_2^4$-grading on $J$ induced by the automorphisms $\{t_{1,-1}, t_{-1,1}, f_0\}$ whose homogeneous spaces are

\begin{align*}
J_{1,1,1} &= \langle E_1, E_2, E_3, 1^{(3)}, t^{(1)}, t^{(2)} \rangle, \\
J_{1,1,-1} &= \langle -(e_1 + e_2)^{(3)}, -(e_1 + e_2)^{(2)}, (e_1 + e_2)^{(1)} \rangle, \\
J_{1,-1,1} &= \langle (u_2 + v_2)^{(3)}, (u_2 + v_2)^{(2)}, (u_2 + v_2)^{(1)} \rangle.
\end{align*}
Supposing a corresponding element in $G_{\alpha,\beta}$ of a tool for translating gradings from $\text{SO}(3) \to H$ the gradings on $H$ and will provide a more self contained exposition.

This will provide a comfortable landscape for testing geometric tools in a particular setting it is worth to find this description by using elementary algebraic methods of describing group gradings on $f^\prime$. Thus we have a three-dimensional algebraic group isomorphism $\text{In}: \text{SO}(3) \to \text{aut}(H_3(F)) =: G$.

Now consider the algebra $M_3(C)$ with the usual matrix product. This contains, as a subalgebra, $M_3(F)$ and any element in this subalgebra associates with any two other elements in $M_3(C)$. Since $H_3(F)$ is a (Jordan) subalgebra of $J = H_3(C)$, for any $p \in \text{SO}(3)$, the map $\text{In}(p): J \to J$ such that $x \mapsto pxp^{-1}$ (products in $M_3(C)$) is an automorphism of $J$. Thus we have an algebraic group monomorphism $\text{In}: \text{SO}(3) \to F_3$ and thus a monomorphism $\text{SO}(3) \cong G \to F_3 = \text{aut}(J)$ so that we can extend automorphisms from $H_3(F)$ to $J$ (in fact this monomorphism maps $\text{In}(p)$ seen as an element in $G$ to $\text{In}(p)$ as element in $F_3$). So we have constructed a tool for translating gradings from $H_3(F)$ to $J$. This suggests the convenience of describing group gradings on $H_3(F)$. The work \cite{7} contains an exhaustive and deep description of gradings on some simple Jordan and Lie algebras, in particular the gradings on $H_3(F)$ could be obtained along the line of this work. But in our particular setting it is worth to find this description by using elementary algebraic group theory. This will provide a comfortable landscape for testing geometric tools and will provide a more self contained exposition.

It is well known that a maximal torus $P$ of $\text{SO}(3)$ is given by the matrices of the form

\[ p_{\alpha,\beta} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta & \alpha \end{pmatrix} \]

with $\alpha, \beta \in F$ such that $\alpha^2 + \beta^2 = 1$. Then denote by $\tau_{\alpha,\beta} := \text{In}(p_{\alpha,\beta})$ the corresponding element in $G = \text{aut}(H_3(F))$. The set of all $\tau_{\alpha,\beta}$ is a maximal torus $T$ of $G$ and the set of eigenvalues of $\tau_{\alpha,\beta}$ is $S_{\alpha,\beta} = \{ 1, z, z^{-1}, z^2, z^{-2} \}$ for $z := \alpha + i\beta$.

Supposing $|S_{\alpha,\beta}| = 5$, we find for $\tau_{\alpha,\beta}$ the following eigenspaces

\[ H_3(F)_{\tau_{\alpha,\beta}} = \{ iE_2 - iE_3, -iE_2 + iE_3 + 1^{(1)} \} \]

For this grading we have $h_3 = 0$ except $h_3 = 7$, $h_6 = 1$. Thus the algebra is of type $(0, 0, 7, 0, 0, 1)$. It is easy to prove that the subalgebra of $f_3$ whose elements are those $d \in f_3$ such that $[d, t_{-1,1}] = [d, t_{-1,1}] = [d, f_0] = 0$ is three-dimensional. Thus, the grading on $f_3$ induced by $\{ \text{Ad}(t_{1,-1}), \text{Ad}(t_{-1,1}), \text{Ad}(f_0) \}$ is nontoral since its zero homogeneous component has dimension 3, hence its rank is less than $4 = \text{rank}(f_3)$.

We summarize the results in this subsection in the following:

**Proposition 2.** The unique nontoral grading on $J$ coming from a grading on $C$ is the above grading (5) up to equivalence.
where the subindex indicates the eigenvalue of $\tau_{\alpha,\beta}$. This gives a $\mathbb{Z}$-grading of $H_3(F)$, with $n$-th component $H_3(F)_{s_n}$. This is toral and fine (as it is produced by the whole torus $T$). Any other toral grading of $H_3(F)$ is a coarsening of this.

For $|S_{\alpha,\beta}| < 5$ we have the following excluding possibilities:

- $1 = z$ which gives the trivial grading.
- $1 = z^2$ which excluding the previous case implies $z = -1$. This is the $\mathbb{Z}_2$-grading induced by the involutive automorphism $\tau_{-1,0}$ and it is given by

\begin{equation}
H_3(F)_1 = \langle E_1, E_2, E_3, 1^{(1)} \rangle, \quad H_3(F)_{-1} = \langle 1^{(2)}, 1^{(3)} \rangle.
\end{equation}

- $z = z^{-2}$ implying $z^3 = 1$. Ruling out previous cases, this induces a $\mathbb{Z}_3$-grading coming for instance from $\tau_{-1/2, \sqrt{3}/2}$. The grading is

\begin{equation}
\begin{align*}
H_3(F)_1 &= \langle E_1, E_2 + E_3 \rangle, \\
H_3(F)_z &= \langle (1^{(2)} - i^{(3)}), iE_2 - iE_3 + 1^{(1)} \rangle, \\
H_3(F)_{z^2} &= \langle (1^{(2)} + i^{(3)}), -iE_2 + iE_3 + 1^{(1)} \rangle.
\end{align*}
\end{equation}

- $z^2 = z^{-2}$ implying $z^4 = 1$. This gives a $\mathbb{Z}_4$-grading coming from $\tau_{0,1}$. The grading is

\begin{equation}
\begin{align*}
H_3(F)_1 &= \langle E_1, E_2 + E_3 \rangle, \\
H_3(F)_z &= \langle (1^{(2)} + i^{(3)}), E_2 - E_3, 1^{(1)} \rangle, \\
H_3(F)_{z^2} &= \langle (1^{(2)} - i^{(3)}), 1^{(2)} \rangle, \\
H_3(F)_{z^4} &= \langle -i^{(3)} + 1^{(2)} \rangle.
\end{align*}
\end{equation}

The gradings in (6)-(9) are therefore the unique cyclic (hence necessarily toral, see Appendix) gradings. To find the rest of the gradings on $H_3(F)$ we compute the centralizers of the grading automorphisms producing the previous gradings. The computations of the various centralizers are easy taking advantage of the isomorphism $SO(3) \to G$. For any $\tau_{\alpha,\beta}$ with $|S_{\alpha,\beta}| = 5$ we have $CG(\tau_{\alpha,\beta}) = T$ (for this, we only need to prove that the centralizer of $p_{\alpha,\beta}$ in $SO(3)$ is the maximal torus $P$), so that the grading (6) is fine, as mentioned. The centralizer of $\tau_{-1,0}$ has two connected components: the identity component is the maximal torus $T$ and $CG(\tau_{-1,0})/T \cong \mathbb{Z}_2$. Working in $SO(3)$, the identity component of the centralizer of $p_{-1,0}$ is the torus $P$ while its other component is $sP$ where $s = -E_1 + 1^{(1)}$. Taking any $\tau \in T$ the grading induced by $\{\tau_{-1,0}, \tau\}$ is some of (6)-(9). But if we consider the grading $\{\tau_{-1,0}, \text{In}(s)\}$, we get the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading given by $H_{1,1} = \langle E_1, E_2 + E_3, 1^{(1)} \rangle, H_{1,-1} = \langle 1^{(2)} - 1^{(3)} \rangle, H_{-1,1} = \langle E_2 - E_3 \rangle, H_{-1,-1} = \langle 1^{(2)} + 1^{(3)} \rangle$, which is isomorphic to:

\begin{equation}
\begin{align*}
H_{1,1} &= \langle E_1, E_2, E_3 \rangle, \\
H_{1,-1} &= \langle 1^{(1)} \rangle, \\
H_{-1,1} &= \langle 1^{(2)} \rangle, \\
H_{-1,-1} &= \langle 1^{(3)} \rangle.
\end{align*}
\end{equation}

On the other hand $CG(\langle \tau_{-1,0}, \text{In}(s) \rangle) = CG(\tau_{-1,0})$, which implies that (10) is fine (and nontoral taking into account [13, Theorem 1]). The grading (8) is produced by $\tau_{-1/2, \sqrt{3}/2}$ whose centralizer is $T$. The grading (9) is produced by $\tau_{0,1}$ whose centralizer is again $T$.

Summarizing all the above results we claim:

**Theorem 1.** Any nontrivial grading on $H_3(F)$ is equivalent to one of the gradings (6)-(10) above.
The gradings induced on the Albert algebra by the monomorphism \( \text{aut}(H_3(F)) \rightarrow F_4 \) are all of them toral, but these automorphisms of \( J \) coming from \( H_3(F) \) commute with the automorphisms coming from \( C \), fact which will provide larger abelian sets of semisimple automorphisms and hence a source of nontoral gradings on \( J \).

3.3. A family of nontoral gradings on the Albert algebra. We now construct a machinery for building gradings on the Albert algebra \( J = H_3(C) \) by mixing gradings on \( C \) with that on \( H_3(F) \). We must start with the simple observation that \( J = H_3(C) \cong H_3(F) \otimes F \oplus K_3(F) \otimes C_0 \) (tensor product of \( F \)-spaces) where \( K_3(F) \) is the subspace of \( 3 \times 3 \) skewsymmetric matrices with entries in \( F \) and \( C_0 := \{ x \in C \mid \text{tr}(x) = 0 \} \) the subspace of trace zero elements in \( C \). The above isomorphism is given by \( E_i \mapsto E_i \otimes 1, 1^{(i)} \mapsto 1^{(i)} \otimes 1 \) and for \( x \in C_0 \), \( x^{(i)} \mapsto (e_{ijk} - e_{kji}) \otimes x \), being \( (i,j,k) \) any cyclic permutation of \( \{1,2,3\} \) and \( e_{ij} \) the elementary \( (i,j) \)-matrix in \( M_3(F) \). Taking in \( J' = H_3(F) \otimes F \oplus K_3(F) \otimes C_0 \), subspace of \( M_3(F) \otimes C \), the product \( (c \otimes x) \cdot (d \otimes y) = \frac{1}{2}((c \otimes x)(d \otimes y) + (d \otimes y)(c \otimes x)) \) for \( (c \otimes x)(d \otimes y) = cd \otimes xy \), \( J' \) is a Jordan subalgebra of \( M_3(F) \otimes C \) such that the previous vector space isomorphism between \( J \) and \( J' \) is an algebra isomorphism. Module this identification of \( J \) with \( J' \), the embedding of \( G_2 \) in \( F_4 \) described in Subsection 3.1 can be seen in the following way. Given \( f \in G_2 \), take \( \hat{f} \) the restriction of \( \text{id} \otimes f \in \mathfrak{gl}(M_3(F) \otimes C) \) to \( J' \), which is an automorphism of \( J' \) (notice that \( C_0 \) is \( f \)-invariant for any \( f \in G_2 \)). On the other hand given any automorphism \( g \) of \( H_3(F) \), \( g \) is the restriction to \( H_3(F) \) of an automorphism \( g \) of \( M_3(F) \) commuting with the transposition involution. Hence \( g(K_3(F)) \subset K_3(F) \) and we can define \( \hat{g} \) as the restriction of \( g \otimes \text{id} \in \mathfrak{gl}(M_3(F) \otimes C) \) to \( J' \), which is an automorphism of \( J' \). A trivial though remarkable fact is the commutativity \( \hat{f} \hat{g} = \hat{g} \hat{f} \) for any \( f \in G_2 \), \( g \in \text{aut}(H_3(F)) \). Thus we have:

**Theorem 2.** Let \( \{f_1, \ldots, f_n\} \subset G_2 \) and \( \{g_1, \ldots, g_n\} \subset \text{aut}(H_3(F)) \) be commutative sets of diagonalizable automorphisms. Then \( \{\hat{f}_1, \ldots, \hat{f}_k, \hat{g}_1, \ldots, \hat{g}_n\} \subset F_4 \) is a commutative set of diagonalizable automorphisms of \( J \). In particular if \( C \) is graded by a group \( G_1 \) and \( H_3(F) \) is graded by a second group \( G_2 \), then the Albert algebra \( J \) is \( G_1 \times G_2 \)-graded.

It is always the case that the grading induced by \( \{\hat{f}_1, \ldots, \hat{f}_k, \hat{g}_1, \ldots, \hat{g}_n\} \) is a refinement of the one given by \( \{f_1, \ldots, f_k\} \). Besides if one of the gradings \( \{f_1, \ldots, f_k\} \) or \( \{g_1, \ldots, g_n\} \) is nontoral, the refinement is also nontoral. These results allow us to combine gradings on \( C \) with gradings on \( H_3(F) \). Thus, if we pick the (unique up to equivalence) nontoral grading on \( C \) given by \( \{t_{1,-1}, t_{-1,1}, f_0\} \) and any of the gradings (6)-(10) plus the trivial grading, which are given respectively by: \( \{\tau_{\alpha,\beta}\} \) (with \( |S_{\alpha,\beta}| = 5 \), \( \{\tau_{-1,0}\}, \{\tau_{-1/2,\sqrt{3}/2}\}, \{\tau_{0,1}\}, \{\tau_{-1,0}, \text{In}(s)\} \) and \( \{1\} \), we get six nontoral gradings on \( J \) which are given in the next

**Corollary 1.** The following six nonequivalent gradings of the Albert algebra are nontoral:

a) \( \{t_{1,-1}, t_{-1,1}, \hat{f}_0, \tau_{\alpha,\beta}\} \) with \( |S_{\alpha,\beta}| = 5 \). This is a \( \mathbb{Z}_3^2 \times \mathbb{Z} \)-grading.

b) \( \{t_{1,-1}, t_{-1,1}, \hat{f}_0, \tau_{-1,0}\} \). This is a \( \mathbb{Z}_2^4 \)-grading.

c) \( \{t_{1,-1}, t_{-1,1}, f_0, \tau_{-1,0} \} \). This is a \( \mathbb{Z}_2^3 \times \mathbb{Z}_2 \)-grading.

d) \( \{t_{1,-1}, t_{-1,1}, \hat{f}_0, \tau_{0,1}\} \). This is a \( \mathbb{Z}_3^2 \times \mathbb{Z}_2 \)-grading.

e) \( \{t_{1,-1}, t_{-1,1}, \hat{f}_0, \text{In}(s)\} \). This is a \( \mathbb{Z}_2^3 \)-grading.

f) \( \{t_{1,-1}, t_{-1,1}, f_0\} \). This is a \( \mathbb{Z}_3^2 \)-grading.
We now describe explicitly the six previous gradings.

a) If \( \epsilon = \pm 1 \) then the grading is

\[
J_{0000} = (E_1, E_2 + E_3), \quad J_{0000} = (-i \epsilon \langle 3 \rangle + 1 \langle 2 \rangle),
\]

\[
J_{0014} = (i \epsilon (e_1 - e_2) \langle 3 \rangle + (e_1 - e_2) \langle 2 \rangle), \quad J_{0010} = (-i (e_1 + e_2) \langle 1 \rangle),
\]

\[
J_{0106} = (-i (u_2 + v_2) \langle 3 \rangle - (u_2 + v_2) \langle 2 \rangle), \quad J_{0100} = ((u_2 + v_2) \langle 1 \rangle),
\]

\[
J_{000e} = (-i (u_1 + v_1) \langle 3 \rangle - (u_1 + v_1) \langle 2 \rangle), \quad J_{1000} = ((u_3 + v_1) \langle 1 \rangle),
\]

\[
J_{0114} = (i (u_2 - v_2) \langle 3 \rangle + (u_2 - v_2) \langle 2 \rangle), \quad J_{0110} = ((u_2 - v_2) \langle 1 \rangle),
\]

\[
J_{1106} = (i (u_3 - v_3) \langle 3 \rangle + (u_3 - v_3) \langle 2 \rangle), \quad J_{1100} = ((u_3 - v_3) \langle 1 \rangle),
\]

\[
J_{1114} = (-i (u_3 + v_3) \langle 3 \rangle - (u_3 + v_3) \langle 2 \rangle), \quad J_{1110} = ((u_3 + v_3) \langle 1 \rangle),
\]

\[
J_{0002} = (-i (E_2 - E_3) + 1 \langle 1 \rangle), \quad J_{000-2} = (-i (E_2 - E_3) - 1 \langle 1 \rangle).
\]

This grading is a \( \mathbb{Z}_2^3 \times \mathbb{Z} \)-grading of type \((25,1)\).

b) This is the \( \mathbb{Z}_2^4 \)-grading

\[
J_{0000} = (E_1, E_2, E_3, 1 \langle 1 \rangle), \quad J_{0001} = (1 \langle 3 \rangle, 1 \langle 2 \rangle),
\]

\[
J_{0010} = ((e_1 - e_2) \langle 1 \rangle), \quad J_{0100} = ((u_2 + v_2) \langle 1 \rangle),
\]

\[
J_{0000} = ((u_1 + v_1) \langle 1 \rangle), \quad J_{1100} = ((u_3 - v_3) \langle 1 \rangle),
\]

\[
J_{0100} = ((u_1 - v_1) \langle 1 \rangle), \quad J_{1000} = ((u_3 + v_3) \langle 1 \rangle),
\]

\[
J_{0101} = ((u_2 + v_2) \langle 1 \rangle), \quad J_{0101} = ((u_2 + v_2) \langle 1 \rangle),
\]

\[
J_{0011} = ((e_1 - e_2) \langle 1 \rangle, (e_1 - e_2) \langle 2 \rangle), \quad J_{1110} = ((u_3 - v_3) \langle 1 \rangle),
\]

\[
J_{1101} = ((u_1 - v_1) \langle 1 \rangle, (u_1 - v_1) \langle 2 \rangle), \quad J_{0110} = ((u_3 - v_3) \langle 2 \rangle),
\]

\[
J_{0111} = ((u_2 - v_2) \langle 1 \rangle, (u_2 - v_2) \langle 2 \rangle), \quad J_{1111} = ((u_3 - v_3) \langle 2 \rangle),
\]

which is of type \((7,8,0,1)\).

c) This is the \( \mathbb{Z}_2^5 \times \mathbb{Z}_3 \)-grading given by

\[
J_{0000} = (E_1, E_2 + E_3), \quad J_{0000} = ((u_1 + v_1) \langle 1 \rangle),
\]

\[
J_{0100} = ((u_2 + v_2) \langle 1 \rangle), \quad J_{0100} = ((e_1 - e_2) \langle 1 \rangle),
\]

\[
J_{0010} = ((u_1 + v_1) \langle 1 \rangle), \quad J_{1100} = ((u_3 - v_3) \langle 1 \rangle),
\]

\[
J_{0101} = ((u_2 + v_2) \langle 1 \rangle), \quad J_{0101} = ((e_1 - e_2) \langle 1 \rangle),
\]

\[
J_{0011} = ((u_2 - v_2) \langle 1 \rangle, (u_2 - v_2) \langle 2 \rangle), \quad J_{1110} = ((u_3 - v_3) \langle 1 \rangle),
\]

\[
J_{1101} = ((u_1 - v_1) \langle 1 \rangle, (u_1 - v_1) \langle 2 \rangle), \quad J_{0111} = ((u_3 - v_3) \langle 2 \rangle),
\]

\[
J_{0002} = ((i (e_1 - e_2) + 1 \langle 2 \rangle, i E_2 - E_3 + 1 \langle 1 \rangle), \quad J_{0002} = ((u_3 + v_3) \langle 3 \rangle - (u_3 + v_3) \langle 2 \rangle),
\]

\[
J_{0102} = ((i u_2 + v_2) \langle 2 \rangle, (u_2 + v_2) \langle 1 \rangle), \quad J_{0102} = ((e_1 - e_2) \langle 3 \rangle + (e_1 - e_2) \langle 2 \rangle),
\]

\[
J_{0112} = ((i u_2 - v_2) \langle 2 \rangle, (u_2 - v_2) \langle 1 \rangle), \quad J_{1012} = ((u_3 - v_3) \langle 3 \rangle - (u_3 - v_3) \langle 2 \rangle),
\]

\[
J_{1012} = ((i u_2 + v_2) \langle 2 \rangle, (u_2 + v_2) \langle 1 \rangle), \quad J_{1012} = ((u_3 + v_3) \langle 3 \rangle - (u_3 + v_3) \langle 2 \rangle),
\]

which is of type \((21,3)\).

d) This is the \( \mathbb{Z}_2^6 \times \mathbb{Z}_4 \)-grading

\[
J_{0000} = (E_1, E_2 + E_3), \quad J_{1000} = ((u_1 + v_1) \langle 1 \rangle),
\]

\[
J_{0010} = ((e_1 - e_2) \langle 1 \rangle), \quad J_{0010} = ((u_2 + v_2) \langle 1 \rangle),
\]

\[
J_{0100} = ((u_2 - v_2) \langle 1 \rangle), \quad J_{1100} = ((u_3 - v_3) \langle 1 \rangle),
\]

\[
J_{0001} = (-i (e_1 - e_2) + 1 \langle 2 \rangle), \quad J_{1001} = ((u_1 + v_1) \langle 3 \rangle + (u_1 + v_1) \langle 2 \rangle),
\]

\[
J_{0101} = ((i u_2 + v_2) \langle 3 \rangle + (u_2 + v_2) \langle 2 \rangle), \quad J_{0101} = ((i e_1 - e_2) \langle 3 \rangle + (e_1 - e_2) \langle 2 \rangle),
\]

\[
J_{0111} = ((i u_2 - v_2) \langle 3 \rangle + (u_2 - v_2) \langle 2 \rangle), \quad J_{1011} = ((i u_1 - v_1) \langle 3 \rangle + (u_1 - v_1) \langle 2 \rangle),
\]

\[
J_{1101} = ((i u_3 - v_3) \langle 3 \rangle + (u_3 - v_3) \langle 2 \rangle), \quad J_{1111} = ((i u_3 + v_3) \langle 3 \rangle + (u_3 + v_3) \langle 2 \rangle),
\]

which is of type \((21,3)\).
which is of type (23, 2).

e) This is the $\mathbb{Z}_2^5$-grading given by

\begin{align}
J_{00000} &= \langle E_1, E_2 + E_3, (1) \rangle, \\
J_{00001} &= \langle E_3 - E_2, \rangle, \\
J_{00010} &= \langle (u_1 + v_1)(3) + (u_1 + v_1)(2) \rangle, \\
J_{00100} &= \langle (u_2 + v_2)(3) + (u_2 + v_2)(2) \rangle, \\
J_{01000} &= \langle (e_1 - e_2)(3) + (e_1 - e_2)(2) \rangle, \\
J_{00111} &= \langle (e_2 - e_1)(3) + (e_1 - e_2)(2) \rangle, \\
J_{01011} &= \langle (e_2 - e_1)(3) + (e_1 - e_2)(2) \rangle, \\
J_{01101} &= \langle (u_1 + v_1)(3) + (u_1 + v_1)(2) \rangle, \\
J_{01110} &= \langle (u_2 + v_2)(3) + (u_2 + v_2)(2) \rangle, \\
J_{10011} &= \langle (u_3 - v_3)(3) + (u_3 - v_3)(2) \rangle, \\
J_{10101} &= \langle (u_3 - v_3)(3) + (u_3 - v_3)(2) \rangle, \\
J_{11011} &= \langle (u_3 - v_3)(3) + (u_3 - v_3)(2) \rangle,
\end{align}

which is of type (24, 0, 1).

f) This is the $\mathbb{Z}_2^4$-grading (5), which is nontoral of type (0, 0, 7, 0, 0, 1), as mentioned in Proposition 2.

Before finishing this subsection we would like to exhibit another (nontoral) grading obtained as a coarsening of the one in case d) above, by removing $J_0$ in the set of grading automorphisms. Thus consider the grading on the Albert algebra induced by the automorphisms \{$\hat{t}_1$, $\hat{t}_{-1}$, $\hat{t}_{-1}$, $\hat{t}_{-1}$, $\hat{t}_{-1}$\}. This is a $\mathbb{Z}_2^2 \times \mathbb{Z}_4$-grading whose homogeneous component $J_{ijk}$ is just $J_{ijk} \oplus J_{jlk}$ in (d), that is

\begin{align}
J_{000} &= \langle E_1, E_2 + E_3, (e_2 - e_1)(1) \rangle, \\
J_{001} &= \langle -ie_1^3 + e_2^2, -ie_2^3 + e_1^2 \rangle, \\
J_{010} &= \langle u_2^2, v_2^2 \rangle, \\
J_{013} &= \langle iu_2^3 - u_2^2, iv_2^3 - v_2^2 \rangle, \\
J_{101} &= \langle -iu_1^3 - u_1^2, -iv_1^3 - v_1^2 \rangle, \\
J_{110} &= \langle u_3^3, v_3^3 \rangle, \\
J_{113} &= \langle iu_3^3 - u_3^2, iv_3^3 - v_3^2 \rangle.
\end{align}

As the subalgebra of fixed elements of $f_4$ by \{Ad($\hat{t}_{-1}$), Ad($\hat{t}_{-1}$), Ad($\hat{t}_{-1}$)\} has rank 3, the induced grading on $f_4$ is nontoral and our grading on the Albert algebra is also nontoral, of type (0, 12, 1).

3.4. Gradients from $M_3(F)$. Up to the moment we have detected seven equivalence classes of nontoral gradings on $J$, all of them coming from the refinements of the nontoral grading on $C$ by gradings on $H_3(F)$. In order to find new nontoral gradings on the Albert algebra we need to look at $J$ from another point of view, that is, we can use a different model of $J$ which provides a new perspective. For instance, Tits construction of $J$ recalled in Subsection 2.4.
Thus let us consider again the associative algebra \( A := M_3(F) \) and the monomorphism \( \varphi: \text{aut}(A) \to F_4 \) such that \( f \mapsto f^\bullet \) as described in 2.4. If \( \{ f_i \} \) is a finite commutative family of semisimple automorphisms of \( A \), the same is true for the family \( \{ f_i^\bullet \} \). Hence, for a \( G \)-grading on \( A \) given by a group homomorphism \( \rho: \mathcal{X}(G) \to \text{aut}(A) \) we immediately can define the grading on \( J \) given by \( \varphi \circ \rho: \mathcal{X}(G) \to F_4 \). Consider now the \( \mathbb{Z}_3^4 \)-grading on \( A \) produced by the commuting automorphisms \( f := \text{In}(p) \) and \( g := \text{In}(q) \) where \( p = \text{diag}(1, \omega, \omega^2) \) being \( \omega \) a primitive cubic root of the unit and

\[
q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.
\]

These automorphisms of \( A \) are semisimple of order 3. The group they generate, \( \langle f, g \rangle \), is usually called Pauli group. The simultaneous diagonalization of \( A \) relative to \( \{ f, g \} \) yields \( A = \bigoplus_{i,j=0}^2 A_{i,j} \) for

\[
\begin{align*}
A_{00} &= \langle 1_A \rangle, \\
A_{01} &= \langle \omega^2 e_{11} - \omega e_{22} + e_{33} \rangle, \\
A_{02} &= \langle -\omega e_{11} + \omega^2 e_{22} + e_{33} \rangle, \\
A_{10} &= \langle e_{13} + e_{21} + e_{32} \rangle, \\
A_{11} &= \langle \omega^2 e_{13} - \omega e_{21} + e_{32} \rangle, \\
A_{12} &= \langle -\omega e_{13} + \omega^2 e_{21} + e_{32} \rangle, \\
A_{20} &= \langle e_{12} + e_{23} + e_{31} \rangle, \\
A_{21} &= \langle \omega^2 e_{12} - \omega e_{23} + e_{31} \rangle, \\
A_{22} &= \langle -\omega e_{12} + \omega^2 e_{23} + e_{31} \rangle,
\end{align*}
\]

where again \( e_{ij} \) denotes the elementary \((i, j)\)-matrix in \( M_3(F) \). Thus we have a \( \mathbb{Z}_3^4 \)-nontoral grading on \( A \) (since any maximal torus of \( A \) fixes a frame of idempotents and so any toral grading has a zero component of dimension at least 3). Next we can consider the grading induced on \( J \) by \( \{ f^\bullet, g^\bullet \} \). If we make a simultaneous diagonalization of \( J \) relative to these automorphisms we get the \( \mathbb{Z}_3^4 \)-grading \( J = \bigoplus_{i,j=0}^2 J_{i,j} \), which has 9 summands of dimension 3 each one. This \( \mathbb{Z}_3^4 \)-grading on \( J \) is obviously toral according to Lemma 2. Let us consider a third semisimple automorphism \( \phi \) of order 3 in the centralizer of \( \{ f^\bullet, g^\bullet \} \). This will allow us to refine the previous \( \mathbb{Z}_3^4 \)-grading on \( J \) to a \( \mathbb{Z}_3^3 \)-grading. So consider \( \phi \in \text{aut}(J) \) given by \( \phi(a_0, a_1, a_2) = (\omega a_0, \omega a_1, \omega^2 a_2) \) where \( \omega \) is as before a primitive cubic root of the unit. It is clear that \( \{ f^\bullet, g^\bullet, \phi \} \) is a commutative set of semisimple automorphisms of \( J \). Making again a simultaneous diagonalization of \( J \) relative to \( \{ f^\bullet, g^\bullet, \phi \} \) we get \( J = \bigoplus_{i,j,k=0}^2 J_{i,j,k} \) with

\[
J_{i,j,0} = A_{ij} \times 0 \times 0, \\
J_{i,j,1} = 0 \times A_{ij} \times 0, \\
J_{i,j,2} = 0 \times 0 \times A_{ij},
\]

so that we have 27 one-dimensional homogeneous components. In particular this \( \mathbb{Z}_3^3 \)-grading on \( J \) is fine and nontoral (otherwise \( J_{0,0,0} \) would contain three orthogonal idempotents, by Proposition 1). Consequently, the subgroup \( \langle f^\bullet, g^\bullet, \phi \rangle \) of \( F_4 \) is maximal among the abelian subgroups of \( F_4 \) whose elements are semisimple (MAD). Observe also that the generators of the subspaces \( A_{ij} \) of \( A \) are invertible elements in \( A \), hence taking into account (4), the generators of the homogeneous components \( J_{i,j,k} \) are also invertible in \( J \). Thus we have found a basis of invertible homogeneous elements in the Albert algebra.

Since we are describing gradings on the Albert algebra in the usual standard basis and this last \( \mathbb{Z}_3^3 \)-grading has been given in a different one, we are now giving the mentioned grading relative to some standard basis. We take, for instance, the grading:

\[
\begin{align*}
J_{000} &= \langle E_1 + E_2 + E_3 \rangle, \\
J_{001} &= \langle \omega E_1 + \omega^2 E_2 + E_3 \rangle,
\end{align*}
\]
The standard basis of \( C_{\text{nontoral}} \) for odd prime \( t \) are isomorphic.

Theorem 3. Announce the first of our main results: those described in (11), (12), (13), (14), (15), (5), (16) and (17).

In Section 6 we shall be able to give another description of this grading in terms of the Weyl group of \( F_4 \). Besides, the uniqueness of the \( \mathbb{Z}_3^4 \)-grading will also be a consequence.

Once we have described the previous gradings on the Albert algebra, we can announce the first of our main results:

**Theorem 3.** The unique up to equivalence nontoral gradings on the Albert algebra are those described in (11), (12), (13), (14), (15), (5), (16) and (17).

In fact, there are four fine gradings, taking into account that the Cartan decomposition is a toral and fine grading on \( f_4 \) which induces a toral and fine grading on \( J \). The proof of the above theorem will have to be postponed to a forthcoming section.

\[
\begin{align*}
J_{002} &= \langle \omega^2 E_1 + \omega E_2 + E_3 \rangle, \\
J_{011} &= \langle \omega^2 u_{3}^{(3)} + \omega e_{1}^{(2)} + v_{3}^{(1)} \rangle, \\
J_{020} &= \langle \omega^3 v_{3}^{(3)} - e_{3}^{(2)} + u_{3}^{(1)} \rangle, \\
J_{022} &= \langle \omega^2 u_{3}^{(3)} + \omega^2 e_{3}^{(2)} + u_{3}^{(1)} \rangle, \\
J_{101} &= \langle -\omega^2 v_{2}^{(3)} - \omega u_{2}^{(2)} + e_{1}^{(1)} \rangle, \\
J_{110} &= \langle e_{2}^{(3)} - u_{1}^{(2)} + v_{1}^{(1)} \rangle, \\
J_{112} &= \langle \omega e_{3}^{(3)} - \omega^2 u_{3}^{(2)} + e_{3}^{(1)} \rangle, \\
J_{121} &= \langle \omega^2 v_{3}^{(3)} + \omega v_{3}^{(2)} + e_{3}^{(1)} \rangle, \\
J_{200} &= \langle u_{2}^{(3)} + \omega v_{2}^{(2)} + e_{2}^{(1)} \rangle, \\
J_{202} &= \langle \omega u_{2}^{(3)} + \omega^2 v_{2}^{(2)} + e_{2}^{(1)} \rangle, \\
J_{211} &= \langle \omega^{2} u_{1}^{(3)} + \omega u_{3}^{(2)} + u_{3}^{(1)} \rangle, \\
J_{220} &= \langle e_{1}^{(3)} - v_{1}^{(2)} + u_{1}^{(1)} \rangle, \\
J_{222} &= \langle -\omega^2 e_{1}^{(3)} - \omega^2 v_{1}^{(2)} + u_{1}^{(1)} \rangle.
\end{align*}
\]

(17) It is produced by the set of commuting diagonalizable automorphisms \( \{ t_{\omega^2 \omega^2 \omega^2 1}, t_{\omega^2 \omega 1 \omega^2}, \varphi \} \), where \( \varphi = \theta \circ \Psi_U \) for \( U \in O(C, n) \) is given by the matrix relative to the standard basis of \( C \)

\[
\begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
\]
4. Weyl group of $f_4$

In next sections we shall use the Weyl group as an important tool for our purposes. First of all we must invoke a version of the Borel-Serre theorem (Theorem 6) asserting that a supersolvable subgroup of semisimple elements in an algebraic group is contained in the normalizer of some maximal torus. In particular, this can be applied to finitely generated abelian groups. The point of this is that most of our arguments can be carried out within the normalizer of a maximal torus, hence the relevance of the Weyl group, which in our context is isomorphic to the quotient of the normalizer of any maximal torus by the torus itself.

In order to describe the abstract Weyl group of $f_4$, we must begin by fixing a basis $\Delta = \{\alpha_i \mid i = 1, \ldots, 4\}$ of a root system of $f_4$. Its Dynkin diagram is

\[
\begin{array}{cccc}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
\circ & \circ & \circ & \circ
\end{array}
\]

and its Cartan matrix is

\[
\begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -2 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 2
\end{pmatrix}
\]

(18)

Taking the euclidean space $E = \sum_{i=1}^{4} \mathbb{R}\alpha_i$ with the inner product $\langle , \rangle$, the Weyl group of $f_4$ is the subgroup $W$ of $GL(E)$ generated by the (simple) reflections $s_i$ with $i = 1, 2, 3, 4$, given by $s_i(x) := x - \langle x, \alpha_i \rangle \alpha_i$. Identifying $GL(E)$ to $GL(4, \mathbb{R})$ by means of the matrices relative to the $\mathbb{R}$-basis $\Delta$, the reflections $s_i$ are represented by

\[
\begin{align*}
& s_1 = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \\
& s_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \\
& s_3 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 1 & 1
\end{pmatrix}, \\
& s_4 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & -1
\end{pmatrix}
\end{align*}
\]

since the Cartan integers $\langle \alpha_i, \alpha_j \rangle$ are the entries of the Cartan matrix.

We shall consider $W \subset GL(4, \mathbb{R})$ ordered lexicographically, that is, first for any two different couples $(i, j), (k, l)$ such that $i, j, k, l \in \{1, 2, 3, 4\}$ we define $(i, j) < (k, l)$ if and only if either $i < k$ or $i = k$ and $j < l$, and second, for any two different matrices $\sigma = (\sigma_{ij}), \sigma' = (\sigma'_{ij})$ in $W$, $\sigma < \sigma'$ if and only if $\sigma_{ij} < \sigma'_{ij}$ where $(i, j)$ is the least element (with the previous order in the couples) such that $\sigma_{ij} \neq \sigma'_{ij}$. One possible way to compute the Weyl group with this particular enumeration is provided by the following code implemented with Mathematica:

```mathematica
W = Table[s_i, {i, 4}];
a[L_, x_] := Union[L, Table[L[[i]].x, {i, Length[L]}], Table[x.L[[i]], {i, Length[L]}]]
Do[W = a[W, s_i], {i, 4}] (4 times repeated)
```
We get a list of 1152 = 2^73^2 elements in the table W which is nothing but the Weyl group \( \text{W} \) of \( f_4 \). We are denoting by \( \sigma_i \) the \( i \)-th element of \( \text{W} \) lexicographically ordered. The following result comes from a straightforward computation which may be done with any matrix multiplication software.

**Proposition 3.** The 1152 elements of the Weyl group \( \text{W} \) of \( f_4 \) are distributed in 25 orbits (=conjugacy classes) according to the following table

| order | no. of elements | no. of orbits | representatives |
|-------|-----------------|---------------|-----------------|
| 1     | 1               | 1             | \( \sigma_{108} = 1 \) |
| 2     | 139             | 7             | \( \sigma_{28}, \sigma_{42}, \sigma_{55}, \sigma_{103}, \sigma_{105}, \sigma_{142}, \sigma_{405} \) |
| 3     | 80              | 3             | \( \sigma_7, \sigma_{15}, \sigma_{114} \) |
| 4     | 228             | 5             | \( \sigma_{1}, \sigma_{3}, \sigma_{36}, \sigma_{104}, \sigma_{110} \) |
| 6     | 464             | 7             | \( \sigma_{4}, \sigma_{8}, \sigma_{3}, \sigma_{14}, \sigma_{30}, \sigma_{78}, \sigma_{106} \) |
| 8     | 144             | 1             | \( \sigma_2 \) |
| 12    | 96              | 1             | \( \sigma_{10} \) |

The column in the left gives the order of every element in the corresponding orbit. We shall denote by \( I \) the set of indices of representatives in the right column:

\[ (19) \quad 1, 2, 3, 4, 7, 8, 9, 10, 14, 15, 28, 30, 42, 55, 56, 78, 103, 104, 105, 106, 110, 114, 142, 405, 748. \]

4.1. **The maximal torus of** \( \text{aut}(f_4) \). If \( h \) is a Cartan subalgebra of \( L = f_4 \), consider

\( L = h \oplus (\oplus_{\alpha \in h^+} L_{\alpha}) \) the decomposition in root spaces relative to \( h \), that is, \( L_{\alpha} = \{ x \in L \mid [h, x] = \alpha(h)x \forall h \in h \} \) if \( \alpha \in h^+ \), \( \Phi = \{ \alpha \in h^+ \mid L_{\alpha} \neq 0 \} \) the root system, and take a basis \( \Delta = \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4 \} \) of the root system. Identifying the roots to their coordinates relative to the basis \( \Delta \), the 24 positive roots of \( h^+ \) are:

\[
\begin{align*}
(0, 0, 0, 1), & \quad (0, 1, 1, 1), \quad (1, 2, 2, 1), \\
(0, 0, 1, 0), & \quad (0, 1, 2, 0), \quad (1, 1, 2, 2), \\
(0, 1, 0, 0), & \quad (1, 1, 1, 1), \quad (1, 2, 3, 1), \\
(1, 0, 0, 0), & \quad (0, 1, 2, 1), \quad (1, 2, 2), \\
(0, 0, 1, 1), & \quad (1, 1, 2, 0), \quad (1, 2, 3, 2), \\
(0, 1, 1, 0), & \quad (1, 1, 2, 1), \quad (1, 2, 4, 2), \\
(1, 1, 0, 0), & \quad (0, 1, 2, 2), \quad (1, 3, 4, 2), \\
(1, 1, 1, 0), & \quad (1, 2, 2, 0), \quad (2, 3, 4, 2).
\end{align*}
\]  

(20)

As usual, the nondegeneracy of the Killing form \( k \) allows to identify \( h \) to \( h^\ast \), calling \( t_a \) the unique element in \( h \) satisfying \( \alpha(h) = k(t_a, h) \) for all \( h \in h \), as in [24, p. 37].

Any automorphism fixing pointwise \( h \) preserves the root spaces. The set of all such automorphisms is a maximal torus of \( \text{aut}(L) \); more precisely, given \( x, y, z, u \in F^\times \) there is an only automorphism \( \Psi \) such that \( \Psi|_h = \text{id} \), \( \Psi|_{L_{\alpha_1}} = x \text{id} \), \( \Psi|_{L_{\alpha_2}} = y \text{id} \), \( \Psi|_{L_{\alpha_3}} = z \text{id} \), \( \Psi|_{L_{\alpha_4}} = u \text{id} \) (particular case of the isomorphism theorems in [24, p. 75]). Obviously, if \( \alpha = n_1 \alpha_1 + n_2 \alpha_2 + n_3 \alpha_3 + n_4 \alpha_4 \), then \( \Psi|_{L_{\alpha}} = x^{n_1}y^{n_2}z^{n_3}u^{n_4}\text{id} \).

Denote by \( \Psi_{xyzu} \) the above automorphism \( \Psi \), and by \( T_h \) the maximal torus \( \{ \Psi_{xyzu} \mid x, y, z, u \in F^\times \} \) \( (T_h \) depends only on \( h \), and \( \Psi_{xyzu} \) depends on \( h \) and \( \Delta \)).

On the other hand, we have got a concrete maximal torus of \( \mathfrak{g}_4 \), since we have got an algebraic group isomorphism \( \text{Ad} : F_4 \rightarrow \mathfrak{g}_4 \) and we have already introduced
the maximal torus $\mathfrak{T}_0$ of $F_4$ (see (2)). Thus we get a maximal torus $\mathfrak{T} := \text{Ad}(\mathfrak{T}_0)$ in $\mathfrak{F}_4$, whose generic element is $t'_{xyzu} := \text{Ad}(t_{xyzu})$. Let us take as $\mathfrak{h}$ the Cartan subalgebra of the elements fixed by $\mathfrak{T}$. Since $\mathfrak{T}$ is contained in $\mathfrak{T}_0$, they necessarily coincide.

Let us choose a comfortable basis of $\mathfrak{f}_4$ for which we know the matrix representation of $t'_{xyzu}$ relative to it. Let $\omega_i$ ($i \in \{1, \ldots, 27\}$) be the $i$-th element in $B$, the standard basis on $J$ that we chose in Subsection 2.2. Recall that $t_{xyzu}(\omega_i) = \eta_i \omega_i$ where $\eta_i = \eta_i(x,y,z,u)$ is the $i$-th entry of the vector

$$(1, 1, 1, x, \frac{1}{x}, y, z, \frac{u^2}{xy}, \frac{u}{y}, \frac{1}{z}, \frac{1}{u}, \frac{xy}{z}, \frac{xz}{y}, \frac{u}{u^2}, \frac{u}{u}, \frac{u}{u}, \frac{u}{u}, \frac{u}{u}, \frac{u}{u}, \frac{u}{u}, \frac{u}{u}).$$

For any $v \in J$ define the map $R_v : J \to J$ such that $a \mapsto av$. Since $\mathfrak{f}_4 = \text{Der}(J) = \{R_i, R_j\}$ ([38, p. 117]), we can extract a basis of $\mathfrak{f}_4$ from the generators set $\{(R_{\omega_i}, R_{\omega_j})\}_{i,j=1}^{27}$. Taking into account that

$$\text{Ad}(t_{xyzu})[R_{\omega_i}, R_{\omega_j}] = [R_{t_{xyzu}(\omega_i)}, R_{t_{xyzu}(\omega_j)}] = \eta_i \eta_j [R_{\omega_i}, R_{\omega_j}],$$

and defining $S$ as the set of all couples $(i, j) \in \{1, \ldots, 27\}^2$ such that $[R_{\omega_i}, R_{\omega_j}] \neq 0$, we have that the eigenvalues of $t'_{xyzu} = \text{Ad}(t_{xyzu})$ are those of the set $\{\eta_i \eta_j \mid (i, j) \in S\}$, which are precisely:

$$(1, 1, 1, x, \frac{1}{x}, y, z, \frac{u^2}{xy}, \frac{u}{y}, \frac{1}{z}, \frac{1}{u}, \frac{xy}{z}, \frac{xz}{y}, \frac{u}{u^2}, \frac{u}{u}, \frac{u}{u}, \frac{u}{u}, \frac{u}{u}, \frac{u}{u}, \frac{u}{u}, \frac{u}{u}, \frac{u}{u}, \frac{u}{u}, \frac{u}{u}, \frac{u}{u}, \frac{u}{u}, \frac{u}{u}),$$

where each eigenvalue is repeated according to its multiplicity (looking only at $S$, we would not know the multiplicities because the set $\{\eta_i \eta_j \mid (i, j) \in S\}$ has 228 elements, but 1 must appear 4 times, and the remaining values at least once, so by dimensions those are just the multiplicities). On the other hand, recalling that $T_0 = \mathfrak{T}$, there must exist rational fractions $X, Y, Z, U \in F(x, y, z, u)$ in the list (21) such that the whole list agrees with $(1, 1, 1, X_n Y_m Z^n U^m \mid (n_1, n_2, n_3, n_4) \in \Phi)$, with $\Phi = \Phi^+ \cup (-\Phi^+)$ and $\Phi^+$ given by (20). One solution is, for instance

$$(21) \quad X = \frac{u^2}{xy^z}, \quad Y = yz, \quad Z = \frac{1}{u}, \quad U = x.$$

Next we choose as our reference basis of $\mathfrak{f}_4$ anyone extracted of $\{(R_{\omega_i}, R_{\omega_j})\}_{(i,j) \in S}$ such that the matrix of $t'_{xyzu}$ relative to this basis is diagonal with the list (21) as diagonal. One possible choice is

$$b_1 = [R_{c_1}, R_{c_4}], \quad b_2 = [R_{c_2}, R_{c_3}], \quad b_3 = [R_{c_2}, R_{c_7}], \quad b_4 = [R_{c_4}, R_{c_9}],$$

$$b_5 = [R_{u_2}, R_{u_{21}}], \quad b_6 = [R_{c_1}, R_{c_{19}}], \quad b_7 = [R_{c_7}, R_{c_{19}}], \quad b_8 = [R_{c_1}, R_{c_{17}}],$$

$$b_9 = [R_{u_2}, R_{u_{27}}], \quad b_{10} = [R_{u_5}, R_{u_{11}}], \quad b_{11} = [R_{u_2}, R_{u_{25}}], \quad b_{12} = [R_{u_1}, R_{u_{11}}],$$

$$b_{13} = [R_{u_5}, R_{u_{9}}], \quad b_{14} = [R_{u_1}, R_{u_{9}}], \quad b_{15} = [R_{u_4}, R_{u_{11}}], \quad b_{16} = [R_{u_5}, R_{u_{7}}]$$

(22)
where these are root vectors relative to \( h \), but the missing elements spanning the Cartan subalgebra must be taken carefully. If we denote \( \beta_i \in h^* \) for \( i = 1 \ldots 48 \) such that \([ h, b_i ] = \beta_i(h)b_i \) for any \( h \in h \), it is easy to check that \( (\beta_4, \beta_3, \beta_2, \beta_1) \) is a basis of \( \Phi \) with Cartan matrix (18), which will be our election for \( \Delta \) from now on (and respectively for \( \alpha_i \)). Thus, we can denote

\[
\begin{align*}
&b_1 = v_{\alpha_4} & b_{25} = v_{-\alpha_4} \\
&b_2 = v_{\alpha_3} & b_{26} = v_{-\alpha_3} \\
&b_3 = v_{\alpha_2} & b_{27} = v_{-\alpha_2} \\
&b_4 = v_{\alpha_1} & b_{28} = v_{-\alpha_1} \\
&b_5 = v_{\alpha_1 + \alpha_4} & b_{29} = v_{-\alpha_1 - \alpha_4} \\
&b_6 = v_{\alpha_2 + \alpha_3} & b_{30} = v_{-\alpha_2 - \alpha_3} \\
&b_7 = v_{\alpha_1 + \alpha_2} & b_{31} = v_{-\alpha_1 - \alpha_2} \\
&b_8 = v_{\alpha_1 + \alpha_2 + \alpha_3} & b_{32} = v_{-\alpha_1 - \alpha_2 - \alpha_3} \\
&b_9 = v_{\alpha_2 + \alpha_3 + \alpha_4} & b_{33} = v_{-\alpha_2 - \alpha_3 - \alpha_4} \\
&b_{10} = v_{\alpha_2 + 2\alpha_3} & b_{34} = v_{-\alpha_2 - 2\alpha_3} \\
&b_{11} = v_{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4} & b_{35} = v_{-\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4} \\
&b_{12} = v_{\alpha_2 + 2\alpha_3 + \alpha_4} & b_{36} = v_{-\alpha_2 - 2\alpha_3 - \alpha_4} \\
&b_{13} = v_{\alpha_1 + \alpha_2 + 2\alpha_3} & b_{37} = v_{-\alpha_1 - \alpha_2 - 2\alpha_3} \\
&b_{14} = v_{\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4} & b_{38} = v_{-\alpha_1 - \alpha_2 - 2\alpha_3 - \alpha_4} \\
&b_{15} = v_{\alpha_2 + 2\alpha_3 + 2\alpha_4} & b_{39} = v_{-\alpha_2 - 2\alpha_3 - 2\alpha_4} \\
&b_{16} = v_{\alpha_1 + \alpha_2 + 2\alpha_3} & b_{40} = v_{-\alpha_1 - \alpha_2 - 2\alpha_3} \\
&b_{17} = v_{\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4} & b_{41} = v_{-\alpha_1 - \alpha_2 - 2\alpha_3 - \alpha_4} \\
&b_{18} = v_{\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4} & b_{42} = v_{-\alpha_1 - \alpha_2 - 2\alpha_3 - 2\alpha_4} \\
&b_{19} = v_{\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4} & b_{43} = v_{-\alpha_1 - 2\alpha_2 - 3\alpha_3 - \alpha_4} \\
&b_{20} = v_{\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4} & b_{44} = v_{-\alpha_1 - 2\alpha_2 - 3\alpha_3 - 2\alpha_4} \\
&b_{21} = v_{\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4} & b_{45} = v_{\alpha_1 + 2\alpha_2 - 3\alpha_3 - 2\alpha_4} \\
&b_{22} = v_{\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4} & b_{46} = v_{\alpha_1 + 2\alpha_2 - 4\alpha_3 - 2\alpha_4} \\
&b_{23} = v_{\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4} & b_{47} = v_{\alpha_1 - 3\alpha_2 - 4\alpha_3 - 2\alpha_4} \\
&b_{24} = v_{\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4} & b_{48} = v_{-\alpha_1 - 3\alpha_2 - 4\alpha_3 - 2\alpha_4} 
\end{align*}
\]

where each \( v_\alpha \) is a root vector relative to the root \( \alpha \), verifying

\[
\text{Ad}(t_{xyzu})v_{m_1\alpha_1+m_2\alpha_2+m_3\alpha_3+m_4\alpha_4} = X^{m_1}Y^{m_2}Z^{m_3}U^{m_4}v_{m_1\alpha_1+m_2\alpha_2+m_3\alpha_3+m_4\alpha_4}.
\]

At last we choose our standard basis of \( \mathfrak{g}_1 \) as

\[
B' = \{ 4[b_4, b_{28}], 4[b_{27}, b_3], 8[b_{26}, b_2], 8[b_{25}, b_1], b_i \mid i = 1, \ldots, 48 \},
\]

formed by root vectors but besides described with whole precision. This will be needed in the next section to extend elements from \( \mathcal{W} \) to \( \mathfrak{g}_4 \). The first four elements in \( B' \), which of course form a basis of \( \mathfrak{h}_1 \), have not been chosen arbitrarily, but they are respectively \( t_{\alpha_1} \), \( t_{\alpha_2} \), \( t_{\alpha_3} \) and \( t_{\alpha_4} \) for our election of \( \Delta \).

Notice that we have algebraic group isomorphisms \( \alpha: (F^\times)^4 \to \mathfrak{g} \) such that \( \alpha(x, y, z, u) = t'_{xyzu} = \text{Ad}(t_{xyzu}) \) and \( \beta: (F^\times)^4 \to \mathfrak{g} \) acting as \( \beta(X, Y, Z, U) = \).
the maximal torus of aut(\( F^x \)) which is of type \((24, 2)\). Moreover all the toral gradings on the Albert algebra are coarsenings of this and introduce this action as the integration of the action of \( W \) on the Albert algebra.

The map \( \eta: (x, y, z, u) \mapsto (X, Y, Z, U) \) (see (22)) is an automorphism of the algebraic group \((F^x)^4\) and we have a commutative diagram

\[
\begin{array}{ccc}
(F^x)^4 & \xrightarrow{\alpha} & \mathcal{T} \\
\eta \downarrow & & \downarrow \beta \\
(F^x)^4 & \xrightarrow{\beta} & \mathcal{T}
\end{array}
\]

in which all the arrows are isomorphisms. But \( \text{Ad}: \mathcal{T} \to \mathcal{T} \) is also an isomorphism, which allows to write the simultaneous diagonalization of \( J \) relative to \( \mathcal{T} \) as a \( \mathbb{Z}^4 \)-grading in such a way that the induced grading on \( \mathfrak{f}_4 \) by \( \text{Ad} \) is just the root decomposition indexed in the coordinates of \( \Phi \) relative to \( \Delta \), that is, \([R_{f_{m_1,n_2,n_3,n_4}}, R_{f_1',n_4'}] \subset L(\alpha_1+n_1')\alpha_1+(n_2+n_1')\alpha_2+(n_3+n_1')\alpha_3+(n_4+n_1')\alpha_4\). Let us explain this in a practical way: Since the set of eigenvalues of \( t_{xyzu} \) is contained in the set of eigenvalues of \( t_{xyzu} \), any eigenvalue of \( t_{xyzu} \) can be written in the form \( X^{m_1}Y^{m_2}Z^{m_3}U^{m_4} \). This defines a map from the set of eigenvalues of \( t_{xyzu} \) to \( \mathbb{Z}^4 \) such that \( X^{m_1}Y^{m_2}Z^{m_3}U^{m_4} \mapsto (m_1, m_2, m_3, m_4) \), providing a \( \mathbb{Z}^4 \)-grading on the Albert algebra \( J = \oplus J_{m_1,m_2,m_3,m_4} \) such that \((m_1, m_2, m_3, m_4)\) is in the image of the above map. To determine \( J_{m_1,m_2,m_3,m_4} \) we find the eigenvalue of \( t_{xyzu} \) of the form \( X^{m_1}Y^{m_2}Z^{m_3}U^{m_4} \), and then take the element of the standard basis of the Albert algebra which is an eigenvector for that eigenvalue. For instance, to find \( J_{1,2,2,1} \) we compute \( XY^2Z^2U = z \) and write \( J_{1,2,2,1} = \langle u_2^{(3)} \rangle \) since \( u_2^{(3)} \) is the basic element such that \( t_{xyzu}(u_2^{(3)}) = zu_2^{(3)} \). The complete description of the grading is:

\[
\begin{align*}
J_{0,0,0,0} &= \langle E_1, E_2, E_3 \rangle, \\
J_{0,0,0.1} &= \langle e_1^{(3)} \rangle, \quad J_{0,0.1,0} = \langle e_1^{(2)} \rangle, \quad J_{0,0.1,-1} = \langle e_1^{(1)} \rangle, \\
J_{0,0,-1} &= \langle e_2^{(3)} \rangle, \quad J_{0,0,-1,0} = \langle e_2^{(2)} \rangle, \quad J_{0,0,-1,1} = \langle e_2^{(1)} \rangle, \\
J_{-1,-1,-2,-1} &= \langle u_1^{(3)} \rangle, \quad J_{-1,-1,-1,0} = \langle u_1^{(2)} \rangle, \quad J_{-1,-1,-1,-1} = \langle u_1^{(1)} \rangle, \\
J_{1,2,2,1} &= \langle u_2^{(3)} \rangle, \quad J_{1,2,3,2} = \langle u_2^{(2)} \rangle, \quad J_{1,2,3,1} = \langle u_2^{(1)} \rangle, \\
J_{0,-1,-2,-1} &= \langle u_3^{(3)} \rangle, \quad J_{0,-1,-1,0} = \langle u_3^{(2)} \rangle, \quad J_{0,-1,-1,-1} = \langle u_3^{(1)} \rangle, \\
J_{1,1,2,1} &= \langle v_1^{(3)} \rangle, \quad J_{1,1,1,0} = \langle v_1^{(2)} \rangle, \quad J_{1,1,1,1} = \langle v_1^{(1)} \rangle, \\
J_{-1,-2,-2,-1} &= \langle v_2^{(3)} \rangle, \quad J_{-1,-2,-3,-2} = \langle v_2^{(2)} \rangle, \quad J_{-1,-2,-3,-1} = \langle v_2^{(1)} \rangle, \\
J_{0,1,2,1} &= \langle v_3^{(3)} \rangle, \quad J_{0,1,1,0} = \langle v_3^{(2)} \rangle, \quad J_{0,1,1,1} = \langle v_3^{(1)} \rangle,
\end{align*}
\]

(24)

which is of type \((24, 0, 1)\). We remark again that this \( \mathbb{Z}^4 \)-grading on the Albert algebra is toral and fine and the induced grading on \( \mathfrak{f}_4 \) is precisely the Cartan grading. Alternatively we could have got this grading in Section 3 directly from the maximal torus of aut(\( J \)) but in that case its relationship to the Cartan grading would have not been so direct. The group \( \mathbb{Z}^4 \) is the universal group of the grading. Moreover all the toral gradings on the Albert algebra are coarsenings of this and so can be obtained by constructing equivalence classes of epimorphisms \( \mathbb{Z}^4 \to G \) module the relation given as in [13, 4.1]. This is another way of understanding Proposition 1.

To finish this subsection we must devote a few lines to the action of the Weyl group \( W \) on the maximal torus of \( \mathfrak{f}_4 \). Since this is isomorphic to \((F^x)^4\) we can introduce this action as the integration of the action of \( W \) on the Cartan subalgebra \( \mathfrak{h} \).
$\mathcal{W} \times \mathfrak{h} \to \mathfrak{h}$ such that $\sigma \cdot t_{\alpha} = t_{\sigma(\alpha)}$ for any $\sigma = (a_{ij}) \in \mathcal{W}$ and $\alpha \in \Phi$. Since $\sigma$ is an endomorphism of the dual space $\mathfrak{h}^*$, then the transposed matrix $\sigma^t$ represents the dual map $\mathfrak{h} \to \mathfrak{h}$. So identifying the elements in $\mathfrak{h}$ with their coordinates relative to the basis $(t_{\alpha_i})_{i=1}^d$, the action of $\sigma$ on the element $\sum_{i=1}^d x_i t_{\alpha_i} \in \mathfrak{h}$ is given by $(x_1, x_2, x_3, x_4) \mapsto (a_{ij})^t_{i,j=1} \cdot [(a_{ij})^t_{i,j=1}]^t$ where the product $\cdot$ is the usual matrix product. The integration of this, is the desired action $\mathcal{W} \times (F^\times)^4 \to (F^\times)^4$ which consequently acts in the form $\sigma \cdot (X, Y, Z, U) = (X', Y', Z', U')$ where

$$
X' = X^{011} Y^{012} Z^{013} U^{014},
Y' = X^{021} Y^{022} Z^{023} U^{024},
Z' = X^{031} Y^{032} Z^{033} U^{034},
U' = X^{041} Y^{042} Z^{043} U^{044}.
$$

(25)

4.2. Extending Weyl group elements to automorphisms of $f_4$. In this subsection we shall use the isomorphism theorem of [24, p.75] for extending any $\sigma \in \mathcal{W}$ to an automorphism $\tilde{\sigma} \in \mathfrak{g}_4$. In the context of the mentioned theorem we can take $L = L' = f_4$, $\mathfrak{h} = \mathfrak{h}^'$ agreeing with the Cartan subalgebra generated by the four first elements in $B'$ the standard basis of $f_4$, $\Phi = \Phi'$ the root system relative to $\mathfrak{h}$, we choose $\Delta = \Delta'$ the basis of $\Phi$ as in 4.1 ($\alpha_1, \ldots, \alpha_4$ the roots corresponding to $b_1, \ldots, b_1 \in B'$ respectively) and finally, we take as isomorphism $\mathfrak{h} \to \mathfrak{h}$ the induced by $\sigma$ (as above, by means of the identification $\mathfrak{h} \to \mathfrak{h}^*$ through $t \mapsto k(t, -)$). According to that theorem, for any choice $x_{\alpha_i} \in L_{\alpha_i} \setminus \{0\}$ and $x'_{\sigma(\alpha_i)} \in L_{\sigma(\alpha_i)} \setminus \{0\}$ for $i = 1, 2, 3, 4$, there is only one $\tilde{\sigma} \in \mathfrak{g}_4$ such that $\tilde{\sigma}(t_{\alpha_i}) = t_{\sigma(\alpha_i)}$ and $\tilde{\sigma}(x_{\alpha_i}) = x'_{\sigma(\alpha_i)}$ for every $i = 1, 2, 3, 4$. We choose $x_{\alpha_i}$ to be the generator $v_{\alpha_i} \in L_{\alpha_i}$ as in (23) and also $x'_{\sigma(\alpha_i)} = v_{\sigma(\alpha_i)} \in L_{\sigma(\alpha_i)}$. The matrix of $\tilde{\sigma}$ relative to the standard basis is block diagonal \[ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \] where $A$ is just the $4 \times 4$ matrix of $\sigma$ relative to the basis $\{\alpha_i\}_{i=1}^4$ and $D$ is a $48 \times 48$ matrix with only one nonzero element in each row and in each column. Thus we have constructed an injective map $\mathcal{W} \to \mathfrak{g}_4$ such that $\sigma \mapsto \tilde{\sigma}$. It is important to highlight that this is not a group homomorphism but only a map. In fact, there does not exist a group monomorphism $\mathcal{W} \to \mathfrak{g}_4$, as it is proved in [32, p.717].

We shall denote by $\mathfrak{N}$ the normalizer of $\mathfrak{T}$ in $\mathfrak{g}_4$. It is a standard result that $\mathcal{W} \cong \mathfrak{N}/\mathfrak{T}$. It follows easily, by construction of $\tilde{\sigma}$, that $\tilde{\sigma} \in \mathfrak{N}$ for any $\sigma \in \mathcal{W}$. Thus the previous map $\mathcal{W} \to \mathfrak{g}_4$ is actually a map $\mathcal{W} \to \mathfrak{N}$, and composing with the universal epimorphism $\mathfrak{N} \to \mathfrak{N}/\mathfrak{T}$ we get an injective map $\mathcal{W} \to \mathfrak{N}/\mathfrak{T}$ such that $\sigma \mapsto \tilde{\sigma} T$ (the equivalence class of $\tilde{\sigma}$ in the quotient group). Since domain and codomain of this map share the same finite cardinal, the map is a bijection. Even more: it can be proved that $\tilde{\sigma}_1 \tilde{\sigma}_2$ is in the same equivalence class that $\tilde{\sigma}_1 \tilde{\sigma}_2$ (since $\tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\sigma}_2^{-1} \tilde{\sigma}_1^{-1}$ acts in $\mathfrak{h}$ as $\sigma_1 \sigma_2 \sigma_2^{-1} \sigma_1^{-1} = 1$, so that it belongs to $\mathfrak{T}$) which proves that the previous map $\mathcal{W} \to \mathfrak{N}/\mathfrak{T}$ is a group isomorphism. In particular

$$
\mathfrak{N} = \{ \tilde{\sigma} t \mid \sigma \in \mathcal{W}, t \in \mathfrak{T} \}.
$$

We can now revisit the action of the Weyl group $\mathcal{W}$ on the maximal torus $(F^\times)^4 \cong \mathfrak{T}$ from another viewpoint. Identifying $\mathcal{W}$ with $\mathfrak{N}/\mathfrak{T}$ we can define the action $\mathcal{W} \times \mathfrak{T} \to \mathfrak{T}$ given by $\sigma \cdot t := \tilde{\sigma} t \tilde{\sigma}^{-1}$ for $\sigma \in \mathcal{W}$ and $t \in \mathfrak{T}$. Then the isomorphism $\beta : (F^\times)^4 \to \mathfrak{T}$ given by $\beta(X, Y, Z, U) = \Psi_{XYZU}$ is an isomorphism of $\mathcal{W}$-groups in the sense that $\beta(\sigma \cdot t) = \sigma \cdot \beta(t)$. Thus $\sigma \cdot \Psi_{XYZU} = \Psi_{X'Y'Z'U'}$ as in (25). And since $\Psi_{XYZU} = t'_{xyzu}$ for (22), a simple computation proves that
\[ \sigma \cdot t_{x'yz'u'} = t_{x'y'z'u'} \] where now
\[ x' = x^{b_{11}}y^{b_{12}}z^{b_{13}}u^{b_{14}}, \]
\[ y' = x^{b_{21}}y^{b_{22}}z^{b_{23}}u^{b_{24}}, \]
\[ z' = x^{b_{31}}y^{b_{32}}z^{b_{33}}u^{b_{34}}, \] with \( (b_{ij}) = m\sigma m^{-1}, \quad m = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & -1 & -2 & -1 \\ 1 & 2 & 2 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \]

(26)

The action of \( W \) on \( (F^\times)^4 \) given by \( \sigma \cdot (x, y, z, u) = (x', y', z', u') \) as above is essential for our study, specially the study of fixed elements in the torus under the action of certain elements of \( W \). Denote
\[ \mathfrak{T}^{(j)} = \{ t \in \mathfrak{T} \mid \sigma_j \cdot t = t \}. \]

It is easily seen that this is a subgroup of \( \mathfrak{T} \) such that \( \mathfrak{T}^{(i)} \cong \mathfrak{T}^{(j)} \) when \( \sigma_i \) is conjugated to \( \sigma_j \) in \( W \). The information given by these subgroups \( \mathfrak{T}^{(i)} \) is needed for our study, so we are calculating them. For this, it suffices to consider the representatives of conjugacy classes given in the table of Proposition 3. We summarize all this information in the following table. In it, \( \mathfrak{T}^{(j)} \) is the subgroup of all \( t_{x'yz'u'} \in \mathfrak{T} \) such that the element given satisfies the displayed condition. We also write down the abstract group isomorphic to \( \mathfrak{T}^{(j)} \) in the right column.

| \( j \) | Generic element of \( \mathfrak{T}^{(j)} \) | Membership condition | Isomorphic to |
|---|---|---|---|
| 1 | \((u^2y, y, u, u)\) | \( y^2 = 1, u \in F^\times \) | \( F^\times \times \mathbb{Z}_2 \) |
| 2 | \((x, x, x, 1)\) | \( x^2 = 1 \) | \( \mathbb{Z}_2 \) |
| 3 | \((u^2y, y, u^2, u)\) | \( u^4 = 1 = y^2 \) | \( \mathbb{Z}_4 \times \mathbb{Z}_2 \) |
| 4 | \((x, x^{-1}, 1, x^{-1})\) | \( x \in F^\times \) | \( F^\times \) |
| 7 | \((u^2y^{-3}, y, y, u)\) | \( u, y \in F^\times \) | \((F^\times)^2\) |
| 8 | \((x, x, x, x^2)\) | \( x \in F^\times \) | \( F^\times \) |
| 9 | \((1, u^{2/3}, u^{2/3}, u)\) | \( u \in F^\times \) | \( F^\times \) |
| 10 | \((1, 1, 1, 1)\) | \( \{1\} \) | \( F^\times \) |
| 14 | \((x, x, x, y)\) | \( x^2 = y^2 = 1 \) | \( \mathbb{Z}_2^2 \) |
| 15 | \((x, y^2, x, y^2)\) | \( x^3 = y^3 = 1 \) | \( \mathbb{Z}_2^2 \) |
| 28 | \((x, x^{-1}, z, x^{-1})\) | \( x, z \in F^\times \) | \((F^\times)^2\) |
| 30 | \((1, 1, 1, 1)\) | \( z \in F^\times \) | \( F^\times \) |
| 42 | \((x, x, z, u)\) | \( x^2 = u^2 = 1, z \in F^\times \) | \( F^\times \times \mathbb{Z}_2^2 \) |
| 55 | \((x, y, y, u)\) | \( x, y, u \in F^\times \) | \((F^\times)^3\) |
| 56 | \((x, y, y, xy)\) | \( x, y \in F^\times \) | \((F^\times)^2\) |
| 78 | \((1, 1, 1, 1)\) | \( \{1\} \) | \( F^\times \times \mathbb{Z}_2^2 \) |
| 103 | \((x, y, y, u)\) | \( y^2 = 1, x, u \in F^\times \) | \((F^\times)^2 \times \mathbb{Z}_2 \) |
| 104 | \((x, y, y, xy)\) | \( y^2 = 1, x \in F^\times \) | \( F^\times \times \mathbb{Z}_2 \) |
| 105 | \((x, y, xy, u)\) | \( x^2 = y^2 = 1, u \in F^\times \) | \( F^\times \times \mathbb{Z}_2^2 \) |
| 106 | \((x, y, xy, y)\) | \( x^2 = y^2 = 1 \) | \( \mathbb{Z}_2^2 \) |
| 110 | \((x, x, x, u)\) | \( x^2 = u^2 = 1 \) | \( \mathbb{Z}_2^2 \) |
| 114 | \((x, 1, z, 1)\) | \( x, z \in F^\times \) | \((F^\times)^2\) |
| 142 | \((x, 1, z, u)\) | \( x, z, u \in F^\times \) | \((F^\times)^3\) |
| 405 | \((x, y, z, u)\) | \( x^2 = y^2 = z^2 = u^2 = 1 \) | \( \mathbb{Z}_2^4 \) |
| 748 | \((x, y, z, u)\) | \( x, y, z, u \in F^\times \) | \((F^\times)^4\) |

(27)
5. **Quasitori in $F_4$**

Recall that a quasitorus is a commutative algebraic group whose identity component is a torus [33, p. 105]. An algebraic linear group is a quasitorus if and only if in some basis its elements can be expressed simultaneously by diagonal matrices. Such groups are also called *diagonalizable*. Besides, a quasitorus $Q$ in an algebraic group $G$ can be written as a disjoint union $Q = T \cup T_1 \cup \cdots \cup T_k$ where $T$ is a torus and \{1, $a_1$, \ldots, $a_k$\} a finite abelian subgroup of $G$. We remark that, as a consequence of the algebraic version of the Borel-Serre theorem (Theorem 6 in the Appendix), any quasitorus in $G$ normalizes some of the maximal tori of $G$. To see that, define $Z$ as the centralizer in $G$ of $T$. Applying this theorem to $H := \{1, a_1, \ldots, a_k\}$, which is contained in $Z$, there is $T'$ some maximal torus of $Z$ that $H \subset \mathfrak{N}_G(T')$ (the normalizer of $T'$ in $Z$). But $T$ is contained in the center of $Z$ and since all its elements are semisimple, $T$ is contained in the intersection of all maximal tori of $Z$, hence in $T'$. Thus $T \subset T' \subset \mathfrak{N}_G(T')$ and since we had $H \subset \mathfrak{N}_G(T') \subset \mathfrak{N}_G(T')$ then $Q \subset \mathfrak{N}_G(T')$. But actually $T'$ is a maximal torus in $G$, because if $T''$ is a maximal torus of $G$ which contains $T' \supset T$, then $T'' \subset Z$ and $T'' = T'$.

As any grading is given by a quasitorus, the above paragraph gives the reason why we want to work inside $\mathfrak{N} = \mathcal{W}\Sigma$ and we have studied in detail the Weyl group and its action on the torus $\Sigma$ in the previous section.

Next we consider a class of quasitori which is relevant for our study. Define for each $j \in \{1, \ldots, 1152\}$ and each $t \in \Sigma$ the quasitorus $A(j, t)$ as the (closed) subgroup of $\mathfrak{f}_4$ generated by $\Sigma^{(j)}$ and $\sigma_{j,t}$, which of course defines a grading on $\mathfrak{f}_4$ by the group $\mathfrak{X}(A(j, t))$ as in 2.1. But it suffices to consider the gradings induced by the quasitorus $A(j, t)$ with $j \in I$ (the set $I$ defined in (19)), taking into account that if $\sigma_{1}$ and $\sigma_{j}$ are conjugated in $W$, then $A(i, t) \cong A(j, t')$ for a suitable $t' \in \Sigma$.

We also have

**Proposition 4.** If for some $j$ the group $A(j, \text{id})$ is toral then $A(j, t)$ is toral for any $t \in \Sigma$.

**Proof.** Let $Z = \mathfrak{c}_{\mathfrak{f}_4}(\Sigma^{(j)})$ and $Z_0$ its unit component. Since $A(j, \text{id})$ is toral there is some maximal torus $T$ of $\mathfrak{f}_4$ such that $A(j, \text{id}) \subset T$. Then $T \subset Z$ but from $\Sigma^{(j)} \subset \Sigma$ we also get $\Sigma \subset Z$. Of course $\Sigma, T \subset Z_0$ and since $t \in \Sigma$ and $\sigma_j \in A(j, \text{id}) \subset T$ we have $\sigma_{j,t} \in Z_0$ hence $\sigma_{j,t} \in Z_0$. But $\sigma_{j,t}$ is a semisimple element of $Z_0$ and consequently there is some $p \in Z_0$ such that $p \sigma_{j,t} p^{-1} \in \Sigma$. This equality together with the fact that $p \Sigma^{(j)} p^{-1} = \Sigma^{(j)}$ imply that $p A(j, t) p^{-1} \subset \Sigma$. □

The same proof shows that $A(j, t)$ is toral if and only if $A(j, t')$ is toral for all $t' \in \Sigma$. Now let us detect the indices which cause nontorality.

**Proposition 5.** For $j \in I$ the group $A(j, \text{id})$ is nontoral if and only if $j = 3, 15, 105, 106$ or 405.

**Proof.** Let us prove first that the five quasitori are nontoral. For $A(15, \text{id})$ we have $\Sigma^{(15)} \cong \mathbb{Z}_3^3$ (see Table of Section 4.2). The grading induced by this quasitorus is produced by the automorphisms $\{t'_{\omega, \omega, 1}, t'_{1, \omega, \omega}, \sigma_{15}\}$ where $\omega$ is a primitive cubic root of 1. This is a $\mathbb{Z}_3^3$-grading and computing the subalgebra of fixed elements by the three previous automorphisms we find that this is null. This implies that the grading is nontoral since in the toral case, this should be an algebra of
rank four. For $A(405, \text{id})$ we have $\mathfrak{T}^{(405)} \cong \mathbb{Z}_4^4$ and the associated grading agrees with the one produced by $\{t^{\prime}_{-1,1,1}, t^{\prime}_{1,1,1}, t^{\prime}_{1,1,1}, t^{\prime}_{1,1,1}, \sigma^{(405)}_{105}\}$. This is a $\mathbb{Z}_4^2$-grading whose 0-homogeneous component is again null, hence the grading is nontoral. For $A(3, \text{id})$ we have $\mathfrak{T}^{(3)} \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ and the induced grading is the produced by $\{t^{\prime}_{-1,1,1}, t^{\prime}_{1,1,1}, t^{\prime}_{1,1,1}, \sigma_{105}\}$ which is a $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$-grading (see remarks 1 and 2 after this proof) whose 0-homogeneous component has dimension 1. Hence $A(3, \text{id})$ is nontoral. The grading induced by $A(106, \text{id})$ is also nontoral since $\mathfrak{T}^{(106)} \cong \mathbb{Z}_2^2$ and the grading is the one induced by $\{t^{\prime}_{1,1,1}, t^{\prime}_{1,1,1}, t^{\prime}_{1,1,1}, \sigma^{(106)}_{105}\}$, which is a $\mathbb{Z}_2^2 \times \mathbb{Z}_4$-grading whose 0-homogeneous component is one-dimensional. The last grading is the induced by $A(105, \text{id})$. We have $\mathfrak{T}^{(105)} \cong F^\times \times \mathbb{Z}_2^2$, so $A(105, \text{id}) = \langle \mathfrak{T}^{(105)}, \sigma^{(105)}_{105} \rangle \cong F^\times \times \mathbb{Z}_2^2$ which induces a grading over $\mathfrak{X}(F^\times \times \mathbb{Z}_2^2) \cong \mathbb{Z} \times \mathbb{Z}_2^2$. The grading agrees with the one produced for instance by $\{\sigma_{105}, t^{\prime}_{x,y,z}, t^{\prime}_{1,1,1,2} \mid x^2 = y^2 = 1\}$, which can be implemented in a computer. This is a $\mathbb{Z}_2^2 \times \mathbb{Z}_4$-grading whose 0-homogeneous component is one-dimensional and so $A(105, \text{id})$ is nontoral. We include also a table of homogeneous components dimensions for further reference. These types can be computed with any linear algebra software allowing simultaneous diagonalization.

| Quasitorus  | Type  |
|------------|-------|
| $A(3, \text{id})$ | (19, 6, 7) |
| $A(15, \text{id})$ | (0, 26) |
| $A(105, \text{id})$ | (31, 0, 7) |
| $A(106, \text{id})$ | (3, 14, 7) |
| $A(405, \text{id})$ | (24, 0, 0, 7) |

Let us prove now that $A(j, \text{id})$ is toral in the rest of the cases. If $\mathfrak{T}^{(j)} = \text{id}$ then $A(j, \text{id})$ is cyclic and then toral (this applies to the cases $j = 10, 78$). In case $\mathfrak{T}^{(j)}$ is cyclic or $\mathfrak{T}^{(j)} \cong F^\times$, then $A(j, \text{id})$ has two factors and by Lemma 2 (\cite[Lemma 1.1.3, p. 5]{2}) the grading is toral (this applies to $j = 2, 4, 8, 9, 30$). Another trivial case is $j = 748$ since $\sigma_{748} = \text{id}$ and $\mathfrak{T}^{(748)} = \mathfrak{T}$. For $j = 1, 7, 14, 28, 42, 55, 56, 103, 104, 114$ and 142, performing a simultaneous diagonalization of the algebra relative to the set of automorphisms inducing the grading, one finds that the zero homogeneous component of the corresponding grading is an abelian four-dimensional algebra. Thus the grading is toral. Finally, for $j = 110$ we have $A(110, \text{id}) = \langle t^{\prime}_{-1,1,1}, t^{\prime}_{1,1,1,1}, \sigma^{(110)}_{105} \rangle$, which produces a $\mathbb{Z}_2^2 \times \mathbb{Z}_4$-grading. In this case the zero homogeneous component is a six-dimensional (reductive) algebra $L_c = \langle y_1, \ldots, y_6 \rangle$ where

$$
\begin{align*}
y_1 &= b_3 - b_{10} + b_{27} + b_{34}, & y_4 &= -b_4 + b_{22} + b_{28} + b_{46}, \\
y_2 &= -b_7 - b_{18} - b_{21} + b_{42}, & y_5 &= b_1 + b_{23} - b_{37} + b_{47}, \\
y_3 &= b_{16} - b_{20} - b_{40} + b_{44}, & y_6 &= b_{15} - b_{24} - b_{39} + b_{48},
\end{align*}
$$

which has rank 4 because $\{y_1 - y_6, y_2 - y_5, y_3 - y_4\}$ is contained in the center of $L_c$ (there are only two types of six-dimensional reductive subalgebras, $a_1$ plus a three-dimensional center and $2a_1$, of ranks 4 and 2 respectively). Hence the grading is toral. □

**Remark 1.** Notice that the order of $\sigma_j$ does not necessarily coincide with the order of $\tilde{\sigma}_j$, but it is a divisor. That happens, for instance, for $i = 3$, since $\sigma_3$ has order 4 while $\tilde{\sigma}_3$ has order 8. This is not because of a bad choice of $\tilde{\sigma}_3$, since all the possible extensions of $\sigma_3$ have the same order, as the next lemma shows.
Lemma 1. Take \( j \in \{ 1, \ldots, 1152 \} \), and \( m \) the order of \( \sigma_j \in \mathcal{W} \). Then the following conditions are equivalent:

i) \( \mathcal{T}^{(j)} \) is finite,

ii) \( \mathcal{T}^{(j)} \subset \{ t'_{x,y,z,u} | x^m = y^m = z^m = u^m = 1 \} \),

iii) \( (\tilde{\sigma}_j t)^m = \tilde{\sigma}_j^m \) for any \( t \in \mathcal{T} \),

iv) Every element in \( \{ f \in \mathcal{R} | \pi(f) = \sigma_j \} \) \( (\pi: \mathcal{R} \to \mathcal{W} \) the canonical projection\) has the same order.

Proof. Take the element in \( \mathcal{T} \) given by

\[
\begin{align*}
\tilde{s}_{x,y,z,u} & := t'_{x,y,z,u}(\tilde{\sigma}_j t'_{x,y,z,u} \tilde{\sigma}_j^{-1})(\tilde{\sigma}_j t'_{x,y,z,u} \tilde{\sigma}_j^{-2}) \cdots (\tilde{\sigma}_j t'_{x,y,z,u} \tilde{\sigma}_j^{-m})(\tilde{\sigma}_j t'_{x,y,z,u} \tilde{\sigma}_j^{-m+1})
\end{align*}
\]

(product of elements in \( \mathcal{T} \)). Since \( \sigma_j^m = \text{id} \), we have \( \tilde{\sigma}_j^m \in \mathcal{T} \) and thus \( \tilde{\sigma}_j \tilde{s}_{x,y,z,u} \tilde{\sigma}_j^{-1} = \tilde{s}_{x,y,z,u} \), that is, \( \tilde{s}_{x,y,z,u} \in \mathcal{T}^{(j)} \). Besides it verifies that \( (\tilde{\sigma}_j t'_{x,y,z,u})^m = \tilde{s}_{x,y,z,u} \tilde{\sigma}_j^m \).

The implication \( ii) \Rightarrow i) \) is trivial. Now, if we assume \( iii) \), \( \tilde{s}_{x,y,z,u} = \text{id} \) for any \( x, y, z, u \in F^x \). But if \( t'_{x,y,z,u} \in \mathcal{T}^{(j)} \), then \( (t'_{x,y,z,u})^m = \tilde{s}_{x,y,z,u} \), and so \( t^m \) is id for any \( t \in \mathcal{T}^{(j)} \), and we have \( ii) \).

Next suppose \( i) \). We have \( s_{x,y,z,u} = t'_{x,y,z,u}(\sigma_j t'_{x,y,z,u} \sigma_j^{-1})(\sigma_j t'_{x,y,z,u} \sigma_j^{-2}) \cdots (\sigma_j t'_{x,y,z,u} \sigma_j^{-m})(\sigma_j t'_{x,y,z,u} \sigma_j^{-m+1}) \).

Conversely, let \( c \) be such that \( mc \) is the order of \( \tilde{\sigma}_j t'_{x,y,z,u} \) for any \( x, y, z, u \). Thus \( \text{id} = (\tilde{\sigma}_j t'_{x,y,z,u})^{mc} = (s_{x,y,z,u} \tilde{\sigma}_j^m)^c = s_{x,y,z,u}^c \), hence \( f_i(x, y, z, u)^c = 1 \) and so \( s_{x,y,z,u} = \text{id} \), which is equivalent to \( iii) \). \( \square \)

Remark 2. Apparently the grading induced by the quasitorus \( \mathcal{A}(3,\mathbb{I}) = \{ \tilde{s}_3, t'_{1-1-1-1,1}, t'_{-1-1,1-1,1} \} \) is a \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \) grading, since \( 8, 4, 2 \) are respectively the orders of the generators. However, the right group generated by the support is \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) because \( (\tilde{s}_3)^2 t'_{1-1-1,1} \) has order 2.

Returning to our quasitori \( A(j, t) \), the toral element \( t \) plays a secondary role.

Proposition 6. If \( j \in \{ 3, 15, 105, 106, 405 \} \), the quasitorus \( A(j, t) \) is conjugated to \( A(j, t') \) for any \( t, t' \in \mathcal{T} \).

Proof. Take

\[
\mathcal{G}^{(j)} = \{ \tilde{\sigma}_j^{-1} t \tilde{\sigma}_j t^{-1} | t \in \mathcal{T} \}.
\]

Denoting by \( \sigma = \tilde{\sigma}_j \), we have that \( (\sigma^{-1} t \sigma t^{-1})(\sigma^{-1} \sigma s^{-1}) = \sigma^{-1} t \sigma t^{-1} \sigma^{-1} \sigma s^{-1} = \sigma^{-1} t \sigma t^{-1} = \sigma^{-1} \sigma s^{-1} \in \mathcal{T} \) and so it commutes with \( s \). Thus \( \mathcal{G}^{(j)} \) is a subgroup. Besides it has the property that \( A(j, \text{id}) \) is conjugated to \( A(j, s) \) for any \( s \in \mathcal{G}^{(j)} \). Indeed, if \( s = \sigma^{-1} t \sigma t^{-1} \), then \( \text{Ad}(t)(\mathcal{T}^{(j)}) = \mathcal{T}^{(j)} \) and \( \text{Ad}(t)(\sigma) = \sigma s \).

On the other hand, it is clear that \( A(j, s) = A(j, st) \) for any \( t \in \mathcal{T}^{(j)} \), therefore \( A(j, \text{id}) \equiv A(j, st) \) for any \( s \in \mathcal{G}^{(j)}, t \in \mathcal{T}^{(j)} \). Let us see that \( \mathcal{T}^{(j)} / \mathcal{G}^{(j)} = \mathcal{T} \). First we observe that the map

\[
\begin{align*}
\mathcal{T}^{(j)} / \mathcal{G}^{(j)} & \to \mathcal{G}^{(j)} \\
[t] & \to \tilde{\sigma}_j^{-1} t \tilde{\sigma}_j t^{-1}
\end{align*}
\]
is a group isomorphism. It is well defined and injective because \( t \in \mathfrak{T}(j) \) if and only if \( \overline{\sigma}_j^{-1} \overline{\sigma}_j t^{-1} = \text{id} \). In particular \( \dim \mathfrak{T} = \dim \mathfrak{T}(j) + \dim \mathfrak{S}(j) \) (see [30, Proposition 2.26, p. 41]). And we have another isomorphism:

\[ \mathfrak{S}(j)/\mathfrak{S}(j) \cap \mathfrak{T}(j) \to \mathfrak{S}(j)\mathfrak{T}(j)/\mathfrak{S}(j), \]

hence \( \dim \mathfrak{T}(j) + \dim \mathfrak{S}(j) = \dim \mathfrak{S}(j)\mathfrak{T}(j) + \dim \mathfrak{S}(j) \cap \mathfrak{T}(j). \)

But \( \mathfrak{S}(j) \cap \mathfrak{T}(j) \) is a finite group for \( j \) any of our indices. Indeed, for \( j = 3, 15, 106, 405 \) the group \( \mathfrak{T}(j) \) is already finite, and for \( j = 105, \mathfrak{S}(105^3) = \{ t'_{x,y,z,u} | u^2 = xyz \}, \mathfrak{T}(105) = \{ t'_{x,y,z,u} | x^2 = y^2 = 1, z = xy, u \in F^{\times} \} \) and \( \mathfrak{S}(105) \cap \mathfrak{T}(105) = \{ t'_{x,y,z,u} | x^2 = y^2 = u^2 = 1 \} \cong \mathbb{Z}_2^3. \)

Consequently \( \dim \mathfrak{T} = \dim \mathfrak{T}(j) + \dim \mathfrak{S}(j) = \dim \mathfrak{S}(j)\mathfrak{T}(j) \) and so \( \mathfrak{T}(j)\mathfrak{S}(j) = \mathfrak{T}. \)

\[ \square \]

Furthermore, \( \mathfrak{S}(j) \cap \mathfrak{T}(j) \) is a finite group for all \( j \in I \), so also for any \( j \in \{ 1, \ldots, 1152 \} \), and thus \( A(j, t) \cong A(j', t') \) for all \( t, t' \in \mathfrak{T} \), although it is unnecessary for our purposes.

As a consequence of this lemma, if \( \mathfrak{T}(j) \) is finite, then \( \mathfrak{S}(j) = \mathfrak{T} \), and every \( t \in \mathfrak{T} \) is in \( \mathfrak{S}(j) \), that is, there is \( s \in \mathfrak{T} \) such that \( t = \overline{\sigma}_j^{-1}s^{-1}\overline{\sigma}_j s \). Thus we have obtained the following technical result, which will be very useful in some forthcoming proofs.

**Corollary 2.** If \( \mathfrak{T}(j) \) is finite, for any \( t \in \mathfrak{T} \) there is \( s \in \mathfrak{T} \) such that

\[ s\overline{\sigma}_j s^{-1} = \overline{\sigma}_j \]

The relevance of the quasitori \( A(j, t) \) is highlighted by the following result.

**Proposition 7.** Let \( F = \{ f_1, \ldots, f_n, f_{n+1} \} \) be a nontoral commutative family of semisimple elements in a connected algebraic group \( G \) such that \( \{ f_1, \ldots, f_n \} \) is toral. Fix \( T \) any maximal torus of \( G \). Then, the subgroup generated by \( F \) is conjugated to some subgroup of the form \( \langle t_1, \ldots, t_n, \sigma \rangle \) where \( t_i \in T \) and \( \sigma \) is a conjugate of \( f_{n+1} \) in the normalizer of \( T \) in \( G \).

Proof. Define \( Z \) as the centralizer of \( \{ f_1, \ldots, f_n \} \) in \( G \) and let \( T' \) be some maximal torus in \( G \) containing \( \{ f_1, \ldots, f_n \} \). The subgroup \( \langle F \rangle \subset Z \) is a quasitorus of \( Z \), hence is contained in the normalizer \( \mathfrak{N}_Z(T'') \) of some maximal torus \( T'' \) in \( Z \).

Then \( T'' \) is also a maximal torus in \( G \) (since \( T' \subset Z \)) and there is some \( p \in G \) such that \( T''' = pT''p^{-1} \). On the other hand \( \{ f_1, \ldots, f_n \} \) is contained in the center of \( Z \), and since these are semisimple elements, then they are contained in each maximal torus of \( Z \). In particular \( f_i \in T''' \) for \( i \in \{ 1, \ldots, n \} \), and \( f_{n+1} \in \mathfrak{N}_Z(T'') \subset \mathfrak{N}_G(T'''). \)

Thus, \( p^{-1}\langle F \rangle p \) is generated by \( f_i' = p^{-1}f_ip \) for \( i = 1, \ldots, n+1 \), with \( f_i' \in T \) for \( i \leq n \) and \( f_{n+1}' \in p^{-1}\mathfrak{N}_G(T''')p = \mathfrak{N}_G(T) \).

Thus we have proved that any nontoral grading has a coarsening isomorphic to a grading induced by a subquasitorus of \( A(j, t) \) for some \( j \in \{ 3, 15, 105, 106, 405 \} \). Furthermore, by Proposition 6, the element \( t \) can be taken to be the identity. We can even remove two more possibilities for \( j \), as the following corollary shows.

**Corollary 3.** Each of the subgroups \( A(3, \text{id}) \) and \( A(106, \text{id}) \) of \( \mathfrak{S}_4 \) is conjugated to a subgroup of \( A(105, \text{id}) \).

Proof. We know that \( A(3, \text{id}) = \langle \{ t'_{-1,-1,1,1}, t'_{-1,-1,-1,1} \overline{\sigma}_3 \} \rangle \) so that making \( f_1 = t'_{-1,-1,1,1}, f_2 = \overline{\sigma}_3 \) and \( f_3 = t'_{-1,-1,1,1} \) we can apply the previous proposition to \( F = \{ f_1, f_2, f_3 \} \). Of course \( \{ f_1, f_2 \} \) is toral by Lemma 2, while \( A(3, \text{id}) \) is nontoral.
as proved in Proposition 5. Thus \(A(3, \text{id})\) is conjugated to a group of the form \(\langle t_1, t_2, \sigma \rangle\) with \(t_1, t_2 \in \mathfrak{T}\) and \(\sigma \in \mathfrak{N}\). Moreover \(\sigma\) has order 2 since it is conjugated to \(t_{-1,1,1,1,1}\). We also know that \(\sigma = \tilde{\sigma}_i t\) for some \(t \in \mathfrak{T}\) and \(i \in \{1, \ldots, 1152\}\). Since \(\sigma\) has order two, the same can be said about \(\sigma_i\). That is, \(A(3, \text{id})\) is conjugated to some subgroup of \(A(i, t)\) with \(\sigma_i\) of order two. Furthermore \(A(i, t)\) is nontoral so that, applying Proposition 4, the quasitorus \(A(i, t)\) is nontoral. Then Proposition 5 implies that, up to conjugacy, \(i = 3, 15, 105, 106\) or 405. We get, by Proposition 6, that the quasitorus \(A(3, \text{id})\) is conjugated to some subgroup of \((i, t)\). Since \(\sigma_1^2 = 1\) the only possible values of \(i\) are 105 and 405. If we had a copy of \(A(3, \text{id})\) within \(A(405, \text{id})\) then this group (isomorphic to \(Z_2^5\)) should contain an element of order 8.

Consider now \(A(106, \text{id}) = \langle \{t'_{-1,1,1,1,1}, t_{1,-1,1,-1,1}, \sigma_{106}\} \rangle\) and apply the previous proposition to \(F = \{f_1, f_2, f_3\}\) with \(f_1 = \sigma_{106}, f_2 = t'_{-1,1,1,1,1}\) and \(f_3 = t_{1,-1,1,-1,1}\). As before \(A(106, \text{id})\) is conjugated to a subgroup of a nontoral \(A(i, t)\) with \(\sigma_i\) of order two. Again, up to conjugacy, \(i = 105\) or 405 but \(A(405, \text{id})\) contains no order six element. \(\square\)

Our objective in next subsection is to show that in fact any nontoral grading is isomorphic to one produced by a quasitorus contained in \(A(j, \text{id})\) for \(j = 15, 105, 405\).

5.1 Fine gradings. Next we study the maximality of some of the previous quasitori. As a first step, their maximality in \(\mathfrak{N}\), the normalizer of the maximal torus \(\mathfrak{T}\), is clear:

**Proposition 8.** Let \(A = A(j, \text{id})\) for \(j \in \{15, 105, 405\}\). Then \(A\) is its own centralizer in \(\mathfrak{N}\), that is \(\mathfrak{C}_{\mathfrak{N}}(A) = A\).

**Proof.** To prove \(\mathfrak{C}_{\mathfrak{N}}(A) \subset A\) take \(f \in \mathfrak{C}_{\mathfrak{N}}(A)\). Since \(f \in \mathfrak{N}\) there is some \(i\) and some \(t \in \mathfrak{T}\) such that \(f = \tilde{\sigma}_i t\). Consider first the possibility \(j = 405\). Of course \(f\) commutes with each element in \(\mathfrak{T}^{(405)} \subset A\) implying that \(\tilde{\sigma}_i\) does the same. Consequently \(\mathfrak{T}^{(405)} \subset \mathfrak{T}(i)\). According to table (27), this is possible only for \(i = 405\) or \(i = 748\) (take into account that any \(\mathfrak{T}^{(k)}\) is isomorphic to some in the table, that the unique groups in the table which may contain a copy of \(Z_2^5\) are \(\mathfrak{T}^{(405)}\) and \(\mathfrak{T}^{(748)}\), and that the orbits of \(\sigma_{105}\) and \(\sigma_{748}\) have cardinal one). So \(\tilde{\sigma}_i\) (equal to \(\sigma_{105}\) or \(\sigma_{748}\)) commutes with \(\sigma_{405}\), and since \(f = \tilde{\sigma}_i t\) also does, this forces the commutativity of \(\sigma_i t\) and \(\sigma_{405}\) so that \(t \in \mathfrak{T}^{(405)}\) and \(f = \tilde{\sigma}_i t \in A(405, \text{id})\). Let us consider now the case \(j = 15\). From \(f = \tilde{\sigma}_i t\) and the fact that \(f\) commutes with \(\mathfrak{T}^{(15)}\) we get that \(\tilde{\sigma}_i\) commutes with \(\mathfrak{T}^{(15)}\). Therefore \(\mathfrak{T}^{(15)} \subset \mathfrak{T}(15)\) and as \(\mathfrak{T}^{(15)} = \langle \eta'_{1,1,1,1,1} \rangle\), we conclude \(\eta_i \cdot t = t\) for both \(t = \eta_{1,1,1,1,1}\) and \(t = \eta_{1,1,1,1,1}\). But using (26) we see that the unique values of \(i\) satisfying the above relations are 15, 748 and 1075. Hence either \(\eta_1 = \sigma_{15}\) or \(\eta_i = \sigma_{1075} = \sigma_{15}\) or \(\eta_i = \sigma_{748} = \sigma_{15}\). In any case \(\tilde{\sigma}_i\) commutes with \(\sigma_{15}\) (which is not automatic from the commutativity of \(\eta_i\) and \(\sigma_{15}\), but can be proved directly for those indices or checking the equalities \(\sigma_n^{15} = \sigma_{15}^{-n}\) for \(n = 2, 3\)). Thus, recalling that \(f\) commutes with \(\sigma_{15}\) we get that \(t\) commutes with \(\sigma_{15}\), that is, \(t \in \mathfrak{T}^{(15)} \subset A(15, \text{id})\). Therefore \(f = \tilde{\sigma}_i t \in A(15, \text{id})\). Finally, we must investigate the case \(j = 105\). Since \(f = \tilde{\sigma}_i t\) commutes with \(A(105, \text{id})\) and \(\mathfrak{T}^{(105)}\) is generated by \(t'_{-1,1,1,1,1}, t'_{1,-1,1,-1,1}\) and \(t'_{1,1,1,1,1,1}\) with \(u \in F^\times\) we must have that \(\tilde{\sigma}_i\) commutes with any of the mentioned generators. So solving in \(i\) the system \(\sigma_i \cdot t = t\) for \(t \in \{t_{-1,1,1,1,1}, t_{1,-1,1,-1,1}, t_{1,1,1,1,1,1} \mid u \in F^\times\}\), we get that \(i \in \{105, 748\}\).
But $\tilde{\sigma}_{105}$ and $\tilde{\sigma}_{48}$ commute and consequently $\tilde{\sigma}_i$ commutes with $\tilde{\sigma}_{105}$. As before we get that $t \in \mathcal{T}^{(105)}$, implying $f \in A(105, \text{id})$. □

We must emphasize that the fact that the centralizer (in $\mathfrak{M}$) of a subgroup agrees with the subgroup itself does not imply that the centralizer of the group in $\mathfrak{G}$ agrees with the group. However this will be the case for some special relevant subgroups in our work. For the next result we must recall from 2.1 that if we have a maximal abelian subgroup of semisimple elements in aut($L$) for a Lie algebra $L$, then the induced grading on $L$ is fine.

**Proposition 9.** The gradings induced by $A(j, \text{id})$ for $j = 15, 105, 405$ are fine.

Proof. Let $A = A(j, \text{id})$ and suppose that the induced grading is not fine. Then there is some semisimple $f \in \mathfrak{C}_{\mathfrak{G}_t}(A) \setminus A$ such the grading induced by $A \cup \{f\}$ is a proper refinement of the original grading (proper in the sense that it is different from the given grading). Define $Z = \mathfrak{C}_{\mathfrak{G}_t}(\mathcal{T}^{(j)})$. Then the group $\langle A \cup \{f\} \rangle$ is an abelian subgroup of $Z$ and also its closure $\langle A \cup \{f\} \rangle$ in the Zarisky topology. But this is again a quasitorus, whence it is contained in the normalizer of some maximal torus $T$ of $Z$. In particular $\langle A \cup \{f\} \rangle \subset \mathfrak{N}_Z(T)$ and by construction also $\mathfrak{T} \subset Z$ so that there is some $p \in Z$ such that $pfp^{-1} = \mathfrak{T}$. Consequently $p\mathfrak{N}_{\mathfrak{G}_t}(T)p^{-1} = \mathfrak{N}_{\mathfrak{G}_t}(\mathfrak{T}) = \mathfrak{M}$. Thus $p\langle A \cup \{f\} \rangle p^{-1} \subset \mathfrak{M}$ and

$$pfp^{-1}, p\tilde{\sigma}_j p^{-1} \in \mathfrak{M} \cap \mathfrak{C}_{\mathfrak{G}_t}(\mathcal{T}^{(j)})$$

with $pfp^{-1} = t$ for any $t \in \mathcal{T}^{(j)}$.

For $j = 105$ we have $\mathfrak{M} \cap \mathfrak{C}_{\mathfrak{G}_t}(\mathcal{T}^{(105)}) = \mathfrak{T} \cup \tilde{\sigma}_{105} \mathfrak{T}$, taking into account the previous Proposition. Now we must analyze different possibilities:

- If $p\tilde{\sigma}_{105} p^{-1} \in \mathfrak{T}$, then $pAp^{-1} \subset \mathfrak{T}$ and the grading induced by $A$ would be toral, which is a contradiction.
- If $pfp^{-1} \in \mathfrak{T}$ and $p\tilde{\sigma}_{105} p^{-1} = \tilde{\sigma}_{105} t$ for some $t \in \mathfrak{T}$, it is clear that $pfp^{-1} \in \mathcal{T}^{(105)}$ (it commutes with $\tilde{\sigma}_{105}$ and $pAp^{-1} \subset A(105, t)$). Hence $\langle A \cup \{f\} \rangle \subset p^{-1}A(105, t)p$, and it cannot produce a proper refinement. This is another contradiction.
- Thus we have $pfp^{-1} = \tilde{\sigma}_{105} t_1$ and $p\tilde{\sigma}_{105} p^{-1} = \tilde{\sigma}_{105} t_2$ for some $t_1, t_2 \in \mathfrak{T}$.

Since both elements commute we get easily that $t_1 t_2^{-1} \in \mathcal{T}^{(105)}$. Therefore $pfp^{-1} = p\tilde{\sigma}_{105} p^{-1} \mathcal{T}^{(105)} = p\sigma_{105} \mathcal{T}^{(105)} p^{-1} \subset pAp^{-1}$ implying $f \in A$, a contradiction again.

The case $j = 405$ is proved analogously since $\mathfrak{M} \cap \mathfrak{C}_{\mathfrak{G}_t}(\mathcal{T}^{(405)}) = \mathfrak{T} \cup \tilde{\sigma}_{405} \mathfrak{T}$. However for $j = 15$ we need some slight modifications since $\mathfrak{M} \cap \mathfrak{C}_{\mathfrak{G}_t}(\mathcal{T}^{(15)}) = \mathfrak{T} \cup \tilde{\sigma}_{15} \mathfrak{T} \cup (\tilde{\sigma}_{15})^2 \mathfrak{T}$. Let use the notation $\sigma := \tilde{\sigma}_{15}$ for simplicity. Since $pfp^{-1}, p\sigma p^{-1} \in \mathfrak{M} \cap \mathfrak{C}_{\mathfrak{G}_t}(\mathcal{T}^{(15)})$, we have several possibilities:

- If $pfp^{-1}$ or $p\sigma p^{-1}$ is of the form $\sigma t$ for some $t \in \mathfrak{T}$, applying Corollary 2, there is some $s \in \mathfrak{T}$ such that $st \sigma s^{-1} = \sigma$. This implies that $\mathcal{T}^{(15)} \cup \{\sigma\} \subset sp(A \cup \{f\})(sp)^{-1} \subset \mathfrak{M}$ and by Proposition 8 we have $\mathfrak{C}_{\mathfrak{M}}(A(15, \text{id})) = A(15, \text{id})$ implying $sp(A \cup \{f\})(sp)^{-1} = A(15, \text{id}) = A$. Thus the grading induced by $sp(A \cup \{f\})(sp)^{-1}$, which is equivalent to a proper refinement of the induced by $A$, is also equivalent to $A$, a contradiction.
- If $pfp^{-1}$ or $p\sigma p^{-1}$ is of the form $\sigma t^2$ for some $t \in \mathfrak{T}$, applying again Corollary 2, there is some $s \in \mathfrak{T}$ such that $s \sigma t s^{-1} = \sigma^2 = \sigma_{1075}$. This implies
that \( \mathfrak{T}^{(15)} \cup \{ \sigma^2 \} \subset \text{sp}(A \cup \{ f \})(\text{sp})^{-1} \subset \mathfrak{R} \) and since \( \mathfrak{T}^{(15)} \cup \{ \sigma^2 \} \) generates \( A(15, \text{id}) = A(1075, \text{id}) \), we conclude as before.

- And if \( pf \sigma^{-1} \) and \( p \sigma^{-1} \) are both in \( \mathfrak{T} \), the grading is obviously toral. \( \square \)

We finish this section with one of our main results. Before proceeding with its precise statement and proof we must realize that any quasitorus \( A \) in an algebraic group \( G \) has a (not necessarily unique) maximal toral part, a toral subgroup \( B \) which is not contained in another toral subgroup of \( A \). This is trivial if the quasitorus \( A \) has finite cardinal. Otherwise consider the family \( F \) of all the toral subgroups of \( A \). The elements in \( F \) are toral in the sense that each of them is contained in some torus of \( G \). To see that \( F \) has maximal elements we consider a maximal subtorus \( T \) of \( A \). For any \( B \in F \) containing \( T \) (for instance, \( T \) verifies this) we have \( T \subset B_0 \) (the unit component of \( B \)) and \( B_0 \subset A_0 = T \). Thus \( B_0 = T \) and the quotient group \( B/T \) is finite and its cardinal is bounded by that of \( A/T \). Hence any \( B \in F \) with \( T \subset B \) and such that \( B/T \) has maximal cardinal can be proved to be a maximal element in \( F \). Such subgroups are what we shall understand as maximal toral subgroups of the quasitorus \( A \).

**Theorem 4.** Let \( A \subset \mathfrak{G}_4 \) be a quasitorus, then up to conjugacy, \( A \) falls in one of the following cases:

- \( A \subset \mathfrak{T} \) (maximal torus).
- \( A \subset A(105, \text{id}) \).
- \( A \subset A(405, \text{id}) \).
- \( A = A(15, \text{id}) \).

Proof. The precise meaning of this theorem is that there is some \( \phi \in \mathfrak{G}_4 \) such that \( A' := \phi A \phi^{-1} \) is in some of the cases above. So we can replace at any moment \( A \) with some of its conjugated subgroups in the group \( \mathfrak{G}_4 \). From the beginning we suppose that \( A \) is nontoral. Consider a maximal toral subgroup \( B \) of \( A \) which may be supposed to be a subgroup of the maximal torus \( \mathfrak{T} \) defined in Subsection 4.2. Define now the group \( Z := \mathcal{C}_{\mathfrak{G}_4}(B) \) which contains to \( A \) and to \( \mathfrak{T} \). Since \( A \) is a quasitorus of \( Z \) there is a maximal torus \( T \subset Z \) such that \( A \subset \mathfrak{N}_Z(T) \) (see the first paragraph in Section 5). But \( \mathfrak{T}, T \subset Z \) are maximal tori in \( Z \) so that there exists \( p \in Z \) such that \( \mathfrak{T} = p \mathfrak{T} p^{-1} \). Thus \( p \mathfrak{A} p^{-1} \subset \mathfrak{N}_Z(\mathfrak{T}) \subset \mathfrak{N}_{\mathfrak{G}_4}(\mathfrak{T}) = \mathfrak{R} \) and since \( p \) centralizes \( B \), the subgroup \( B = p \mathfrak{A} p^{-1} \) is still a toral maximal subgroup of \( p \mathfrak{A} p^{-1} \).

In this way, replacing \( A \) by \( p \mathfrak{A} p^{-1} \) we can suppose that \( A \subset \mathfrak{R} \) with \( B \subset \mathfrak{T} \) a maximal toral subgroup of \( A \). Furthermore \( A = \langle B \cup \{ f_1, \ldots, f_k \} \rangle \) with \( f_i \in \mathfrak{R} \) so that each \( f_i \) is of the form \( \tilde{\sigma}_j t \) with \( j \in \{ 1, \ldots, 1152 \} \) and \( t \in \mathfrak{T} \). Moreover, for any \( \tilde{\sigma}_j t \in A \) we have \( B \subset \mathfrak{T}^{(j)} \) since any element in \( B \) centralizes \( A \) and \( \mathfrak{T} \) and so it commutes with \( \tilde{\sigma}_j \). On the other hand, \( \langle B \cup \{ \tilde{\sigma}_j t \} \rangle \) being nontoral and contained in \( A(j, t) \), implies that \( \sigma_j \) is in the orbit (under conjugation) of \( \sigma_3, \sigma_{15}, \sigma_{105}, \sigma_{106} \) or \( \sigma_{405} \). Next we analyze the different possibilities arising.

- If some of the \( \sigma_j \)’s is in the orbit of \( \sigma_{15} \) we can suppose that \( f_1 = \tilde{\sigma}_j t \). Then \( f_1 \) is conjugated in \( \mathfrak{R} \) to \( \tilde{\sigma}_{15} t' \) for some \( t' \in \mathfrak{T} \) and without loss in generality we can take \( f_1 = \tilde{\sigma}_{15} \), by Corollary 2 (the element \( s \in \mathfrak{T} \) does not change neither \( \mathfrak{T} \) nor \( B \)). Thus \( B \subset \mathfrak{T}^{(15)} \equiv \mathbb{Z}_2^3 \). Since \( \langle B \cup \{ f_1 \} \rangle \) is nontoral, then it has at least three generators (see Lemma 2). Consequently \( B \) has at least two generators, and this implies \( B = \mathfrak{T}^{(15)} \). So \( A(15, \text{id}) \subset A \subset \mathfrak{R} \) and applying Proposition 8, we have \( \mathcal{C}_{\mathfrak{R}}(A(15, \text{id})) = A(15, \text{id}) \), hence \( A = A(15, \text{id}) \).
• Some of the $\sigma_j$'s is conjugated to $\sigma_{106}$. As before $f_1$ can be taken to be $f_1 = \sigma_{106}$ and $B \subset T^{(106)} \cong \mathbb{Z}_2^2$. Since $B$ must have at least two generators we have $B = T^{(106)}$. Now, for any other $f_i = \sigma_k t$ we must have $T^{(106)} \subset T^{(k)}$. The commutativity of $f_1$ and $f_i$ implies that of $\sigma_{106}$ and $\sigma_k$. But a computation of the number of $k$'s satisfying both conditions

\[
\begin{align*}
\{T^{(106)} \subset T^{(k)}, \\
\sigma_{106} \sigma_k = \sigma_k \sigma_{106},
\end{align*}
\]

reveals that there are only six possible values of $k$. Since $\sigma_{106}$ has order 6, obviously the six powers $\sigma_{106}^n$ with $n \in \{0, \ldots, 5\}$ satisfy the conditions so that there is $n \in \mathbb{N}$ and $t_1 \in \mathfrak{T}$ with $f_i = \sigma_{106}^n t_1$. But $t_1 \in T^{(106)}$ again by the commutativity of $f_1$ with $f_i$, so $f_i \in A(106, \text{id})$. From here $A = A(106, \text{id})$ which up to conjugacy is a subgroup of $A(105, \text{id})$ by Corollary 3.

• Suppose now that some of the $\sigma_j$'s is in the orbit of $\sigma_3$. So we can suppose $f_1 = \sigma_3$ without loss of generality, and $B \subset T^{(3)} = \langle \{t'_{-1,-1,1,1}, t'_{1,1,1,-1}\} \rangle$, which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$. But again by Lemma 2, the subgroup $B$ must have at least two generators since $\langle B \cup \{f_1\} \rangle$ is nontoral. Hence either $B = T^{(3)}$ or $B = \langle \{t'_{-1,-1,1,1}, t'_{1,1,1,-1}\} \rangle$. But this last possibility can not occur because then $\langle B \cup \{f_1\} \rangle = \langle \{t'_{-1,-1,1,1}, t'_{1,1,1,-1}, \tilde{\sigma}_3\} \rangle$ would be toral because $\tilde{\sigma}_3^4 = t_{1,1,1,1}$, hence $\langle B \cup \{f_1\} \rangle$ would have two generators. So necessarily $B = T^{(3)}$ and for any other $f_i = \sigma_k t$ we must have $T^{(3)} \subset T^{(k)}$ and $\sigma_3 \sigma_k = \sigma_k \sigma_3$. We obtain only four possible $k$'s satisfying the previous conditions. Hence $\sigma_k \in \{\sigma_3^n \mid n = 0, 1, 2, 3\}$ and $f_i = \sigma_{3}^n t_1$ for some $t_1 \in \mathfrak{T}$ (in fact $t_1 = t$ because $\sigma_3^n = \sigma_3^3$). Then $t_1 \in T^{(3)}$ and as in previous cases $A = A(3, \text{id})$. But this is conjugated to a subgroup of $A(105, \text{id})$ by Corollary 3.

• We may suppose now that each $\sigma_j$ is in the orbit of $\sigma_{105}$ or of $\sigma_{405}$. We prove next that either all the $\sigma_j$ are in the orbit of $\sigma_{105}$, or all of them are in the orbit of $\sigma_{405}$. Otherwise, after re-ordering and applying Corollary 2 we can take $f_1 = \sigma_{405}$ and $f_2 = \sigma_{105} t$ for some $t \in \mathfrak{T}$. Thus $\langle B \subset T^{(105)} \cap T^{(405)} \rangle$. Moreover $B := \langle B \cup \{f_1, f_2\} \rangle \subset A$ and $\mathfrak{B} \subset \langle T^{(105)} \cap T^{(405)} \rangle \cup \langle \sigma_{405} \sigma_{105} t \rangle \subset A(1048, t_1)$ for some $t_1 \in \mathfrak{T}$, taking into account that $\sigma_{405} \sigma_{105} = \sigma_{1048}$. But $A(1048, t_1)$ is toral since $\sigma_{1048}$ is in the orbit of $\sigma_{142}$ (see Propositions 4 and 5). This contradicts the maximal toral nature of $B$ within $A$.

• If all the $\sigma_j$'s are in the orbit of $\sigma_{105}$, since this orbit has cardinal one, then we can take $f_1 = \sigma_{405}$ and any other $f_i$ of the form $\sigma_{405} t$ with $t \in T^{(405)}$. So $A \subset A(405, \text{id})$.

• If all the $\sigma_j$'s are in the orbit of $\sigma_{105}$. The unique elements in this orbit which commute with $\sigma_{105}$ are $\sigma_{105}, \sigma_{405}, \sigma_{429}$ and $\sigma_{1011}$. So we can take $f_1 = \sigma_{105} t_1$ and any other $f_i$ of the form $\sigma_k t_2$ for some $t_1, t_2 \in \mathfrak{T}$ and $k \in \{105, 403, 429, 1011\}$. Then $\mathfrak{B} \subset B := \langle B \cup \{f_1, f_1\} \rangle \subset A$ and depending on the values of $k$ we have:

- For $k = 403$, since $\sigma_{105} \sigma_{405} = \sigma_{1050}$, we have $f_1 f_i = \sigma_{105} t_1 \sigma_{405} t_2 = \sigma_{1050} t$ for some $t \in \mathfrak{T}$. So $\mathfrak{B} \subset A(1050, t)$ which is toral because $\sigma_{1050}$ is in the orbit of $\sigma_{103}$ (see again Propositions 4 and 5). This contradicts the maximality of $B$ among the toral subgroups of $A$. 

•
Corollary 4. Up to equivalence the unique fine gradings on \( f_4 \) are: (1) the Cartan grading, (2) the induced by \( A(105, \text{id}) \), (3) the induced by \( A(405, \text{id}) \), and (4) the induced by \( A(15, \text{id}) \).

We summarize the information about these gradings in the following table

| Quasitorus \( G \) | Universal grading group | Type |
|------------------|------------------------|------|
| \( \mathfrak{f} \) | \( \mathbb{Z}^4 \) | \((48, 0, 0, 1)\) |
| \( A(15, \text{id}) \) | \( \mathbb{Z}^3 \) | \((0, 26)\) |
| \( A(105, \text{id}) \) | \( \mathbb{Z}_2^3 \times \mathbb{Z} \) | \((31, 0, 7)\) |
| \( A(405, \text{id}) \) | \( \mathbb{Z}_2^5 \) | \((24, 0, 0, 7)\) |

Fine gradings on \( f_4 \)

Let us observe that the groups given in the table above are the universal grading groups. Because for any of the quasitorus \( Q \) in the table, we have proved in Proposition 9 that \( Q \) is its own centralizer in \( \mathfrak{g}_4 \), that is, \( Q \) is a MAD, and according to Subsection 2.1, \( \mathfrak{X}(Q) \) is the universal group of the related grading.

Remark 3. The four fine gradings on \( f_4 \) are also fine when are considered as Lie gradings (as in \([36]\)) instead of being considered as group gradings. This observation is pertinent because the question about the existence of a grading group is still on the table.

On the other hand, we observe that every homogeneous element in the Cartan grading is either semisimple or nilpotent (according to its membership to the zero component) and that all the homogeneous elements in the three remaining fine gradings are semisimple, as a consequence of the nullity of their zero components (\([33, \text{Corollary after Th. 3.4}]\)). This happens not only in \( f_4 \).

Proposition 10. In a fine grading on a simple Lie algebra, every homogeneous element is either semisimple or nilpotent.

Proof. If \( L = \oplus_{g \in G} L_g \) is a fine grading, the group \( Q \) of the automorphisms of \( L \) for which all the homogeneous components are eigensubspaces is just the MAD which produces the grading.

First of all, if \( x \in L_g \) then \( x \) is uniquely represented as \( x = x_s + x_n \), where \( x_s \) and \( x_n \) are respectively semisimple and nilpotent homogeneous elements in \( L_g \) verifying \([x_s, x_n] = 0\) \((33, \text{Th. 3.3})\). Thus, either every element in \( L_g \) is semisimple or there is a nilpotent element \( x \in L_g \). Let us see that in this case every element in \( L_g \) is nilpotent. According to \([33, \text{Th. 3.4}]\), there is a semisimple element \( h \in L_e \) and a nilpotent element \( y \in L_{-g} \) such that \([h, x] = 2x, [h, y] = -2y \) and \([x, y] = h \). Notice then that \( \expad(h) \in \text{aut}(L) \) is an automorphism which leaves invariant all
the homogeneous components. Moreover, \( \exp \text{ad}(h) \) commutes with every \( f \in Q \), because if \( v \) is a homogeneous element with \( f(v) = \alpha v \), then \( \exp \text{ad}(h)(f(v)) = \exp \text{ad}(h)(v) = f(\exp \text{ad}(h)(v)) \) since \( \exp \text{ad}(h)(v) \) belongs to the same homogeneous component. By maximality, \( \exp \text{ad}(h) \in Q \). Since \( \exp \text{ad}(h)(x) = e^{x} x \), there are some automorphisms \( f_{1}, \ldots, f_{s} \in Q \) and scalars \( \alpha_{1}, \ldots, \alpha_{s} \in F^{\times} \) such that

\[
L_{g} = \{ v \in L \mid f_{1}(v) = \alpha_{1} v, \ldots, f_{s}(v) = \alpha_{s} v, \exp \text{ad}(h)(v) = e^{x} v \}.
\]

Therefore \( g \in G \) is an element of infinite order, and, by [33, Prop. 3.5], all the elements in \( L_{g} \) are nilpotent. \( \square \)

In particular, the zero homogeneous component of a fine grading is an abelian (toral) subalgebra, with dimension equal to the dimension of the quasitorus producing the grading. More information on homogeneous semisimple and nilpotent elements of gradings on semisimple Lie algebras can be found in [40].

### 5.2. Nontoral gradings

Up to the moment we have described the fine gradings on \( \mathfrak{f}_{4} \). If we want to find all the nontoral ones, it suffices to describe all the nontoral coarsenings of the fine ones. For this, we must find all the (nontoral) subquasitori \( Q \) of the tori producing fine gradings up to conjugacy.

Consider first the fine grading provided by \( A(105, \text{id}) = \langle \mathfrak{z}^{(105)} \cup \{ \sigma_{105} \} \rangle \). Let us use the following notations \( g_{1} := l'_{-1,1,-1,1}, g_{2} := l'_{1,-1,-1,1}, g_{3} := \sigma_{105} \) and \( g_{4} := l'_{1,1,1,-1} \). Since the number of generators of the group \( H = X(Q) \) must be at least three, we know that \( H \) has the following possibilities: either \( H \cong \mathbb{Z}_{2}^{3} \times \mathbb{Z}_{m} \), or \( H \cong \mathbb{Z}_{2}^{2} \times \mathbb{Z}_{2m} \), or \( H \cong \mathbb{Z}_{2}^{5} \times \mathbb{Z} \). Let us analyze first the case \( H \cong \mathbb{Z}_{2}^{5} \). In this case the quasitorus \( Q \) providing the coarsening has three order-two generators \( \varphi_{i} \) with \( i = 1, 2, 3 \). Thus \( \varphi_{i} = g_{1}^{n_{i}} g_{2}^{m_{i}} g_{3}^{l_{i}} \) and \( (\varphi_{1}, \varphi_{2}, \varphi_{3}) = (g_{1}, g_{2}, g_{3}) \cdot M \) where \( M \) is the 3 \( \times \) 3 integer matrix

\[
\begin{pmatrix}
n_{1} & n_{2} & n_{3} \\
m_{1} & m_{2} & m_{3} \\
l_{1} & l_{2} & l_{3} \\
s_{1} & s_{2} & s_{3}
\end{pmatrix}.
\]

We are using here the action of \( M_{k \times l}(\mathbb{Z}) \) on any power group \( G^{k} \) given by \( G^{k} \times M_{k \times l}(\mathbb{Z}) \to G^{k} \) such that \( (g_{1}, \ldots, g_{k}) \cdot (n_{ij}) := (g_{1}^{n_{1}}, \ldots, g_{k}^{n_{k}}) \) where \( g_{i}^{n_{i}} = \prod_{j} g_{ij}^{n_{ij}} \). Now, there are two actions that we can use to simplify the matrix \( M \) without changing the quasitorus \( Q \) (in the worst case \( Q \) changes to some of its conjugated ones). First we can act on the columns of the matrix by elementary operations (exchanging columns or adding one column to another). All this can be made module two. This reduces the possible matrices \( M \) to a few, but there is a second action on the matrix coming from conjugacy of elements in \( \mathfrak{f}_{4} \). Consider the subgroup \( G \) of \( \mathfrak{f}_{4} \) fixing \( A(105, \text{id}) \) by conjugation, that is, \( f \in G \) if and only if \( f A(105, \text{id}) f^{-1} = A(105, \text{id}) \). The appendix contains relevant information on certain elements in \( G \). By passing from \( Q \) to \( fQf^{-1} \subset A(105, \text{id}) \) the matrix \( M \) transforms into a new matrix \( M' \). We are proving that the joint action by elementary operations on the columns together with the action of \( G \) by conjugation, either moves \( M \) to the matrix

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix},
\]

or the induced grading is toral. First we observe that if this grading is nontoral there must be some nonzero entry in the first two rows of \( M \). Second, \( \sigma_{468} \in G \)
acts on $M$ permuting its first two rows (it permutes $g_1$ and $g_2$ fixing $g_i$ for $i = 3, 4$). So we can suppose that the first row in $M$ is $(1\ 0\ 0)$ (after operations on columns). Hence the first two rows are \( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \) or \( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \) also after operations on columns. However $\sigma_{34} \in G$ acting on the second matrix produces the third one. The grading induced by the third matrix is \( \langle g_1 g_2 g_3^{m_1} g_4^{m_2}, g_3^{m_1} g_4^{m_2}, g_1^{l_1} g_2^{l_2} \rangle \) where the couples $(m_1, m_2)$ and $(l_1, l_2)$ are linearly independent in the vector space $\mathbb{Z}_2^2$ (otherwise the grading has two generators and so is toral). Thus $(n_1, n_2)$ is a linear combination of them and the grading can be written as \( \langle g_1 g_2 f^k g^h, f, g \rangle \) where $f, g \in \langle g_3, g_4 \rangle$. But \( \langle g_1 g_2 f^k g^h, f, g \rangle = \langle g_1 g_2, f, g \rangle \) hence we have the (agreeing) possibilities \( \langle g_1 g_2, g_3, g_4 \rangle = \langle g_1 g_2 g_3 g_4, g_3 \rangle = \langle g_1 g_2, g_3 g_4, g_4 \rangle \). So in this case we must only worry about the grading \( \langle g_1 g_2, g_3, g_4 \rangle \), which can be proved to be toral (seen as a grading on the Albert algebra, this comes from a necessarily toral $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading on the octonions refined with the $\mathbb{Z}_2$-grading on $H_3(F)$, see Subsection 3.1, but alternatively we next prove that is conjugated to $\langle g_1, g_2, g_4 \rangle$). The first matrix above produces the grading \( \langle g_1 g_3^{m_1} g_4^{m_2}, g_2 g_3^{m_1} g_4^{m_2}, g_3^{l_1} g_4^{l_2} \rangle \). But $\sigma_{34}$ permutes $g_1$ and $g_2$ while $\sigma_{34} g_3 g_4 \sigma_{34}^{-1} = g_3 g_4$ and $\sigma_{34} g_2 g_3 \sigma_{34}^{-1} = g_4$, so we can suppose $l_1 = 1, l_2 = 0$ or $l_1 = 0, l_2 = 1$. This reduces the possibilities to \( \langle g_1 g_3^{m_1}, g_2 g_3^{m_1}, g_4 \rangle \) or \( \langle g_1 g_3^{m_1}, g_2 g_3^{m_1}, g_4 \rangle \). The four different possibilities for the first case are \( \langle g_1, g_2, g_3 \rangle, \langle g_1 g_2, g_3 g_4, g_3 \rangle, \langle g_1, g_2 g_3 g_4, g_4 \rangle \), but conjugating by $\sigma_{34} \in G$, the first and second gradings turn out to be isomorphic, so as the third and the fourth ones; while conjugation by $\sigma_{468} \in G$ proves the isomorphism between the second and the third gradings. On the other hand the four possibilities for \( \langle g_1 g_3^{m_1}, g_2 g_3^{m_1}, g_4 \rangle \) are \( \langle g_1, g_2, g_4 \rangle \) (which is obviously toral), \( \langle g_1 g_3, g_2, g_4 \rangle \), \( \langle g_1, g_2 g_3, g_4 \rangle \) and \( \langle g_1 g_3, g_2 g_3, g_4 \rangle \). But conjugation by $\sigma_{468}$ makes evident the isomorphism between the second and the third gradings while \( \langle g_1 g_3, g_2 g_3, g_4 \rangle = \langle g_1 g_2, g_2 g_3, g_4 \rangle = \langle g_1 g_2, g_1 g_3, g_4 \rangle \) and conjugation by $\sigma_{34} \in G$ proves that this is isomorphic to $\langle g_1 g_3, g_2 g_3, g_4 \rangle$. Now it is possible to show that there exists an element $\psi \in G$ such that $\psi g_i \psi^{-1} = g_i$ for $i = 1, 4$ while $\psi g_2 \psi^{-1} = g_3$ and $\psi g_3 \psi^{-1} = g_2$ (the element $\psi$ can be taken of order 4). So $\psi g_1 \psi^{-1} = g_4$ whose torality has been previously stated. Moreover, $\langle g_1 g_2, g_3, g_4 \rangle$ is conjugated to $\langle g_1, g_3, g_4 \rangle$ by $\sigma_{34}$ and this is conjugated to $\langle g_1, g_2, g_4 \rangle$ by $\psi$. This last one is obviously toral. Summarizing the results in this paragraph we have:

**Proposition 11.** If $Q \subset A(105, id)$ is a quasitorus with $X(Q) \cong \mathbb{Z}_2^3$ then it is conjugated to $\langle g_1, g_2, g_3 \rangle$ if $Q$ is nontoral, and to $\langle g_1, g_2, g_4 \rangle$ if $Q$ is toral. Moreover the conjugating element can be taken to fix the subgroup $A(105, id)$. The $\mathbb{Z}_2^3$-grading induced by $\langle g_1, g_2, g_3 \rangle$ is of type $(0, 0, 0, 0, 0, 0)$.

Proof. We have proved in the previous paragraph that $Q$ is conjugated to either $\langle g_1, g_2, g_3 \rangle$ or $\langle g_1, g_2, g_4 \rangle$. Obviously $\langle g_1, g_2, g_4 \rangle$ is toral. On the other hand, the grading induced by $\langle g_1, g_2, g_3 \rangle$ is nontoral, of type $(0, 0, 7, 0, 0, 0)$ in the Albert algebra and of type $(0, 0, 1, 0, 0, 0)$ in $f_4$. Its nontoral character is obvious noticing that the dimension of its 0-homogeneous component in $f_4$ is 3. \( \square \)

**Proposition 12.** If $Q \subset A(105, id)$ is a nontoral quasitorus such that $X(Q) \cong \mathbb{Z}_2^3 \times \mathbb{Z}_m$, with $m > 1$, then up to conjugacy there is $v \in F^\times$ a primitive $m$-root of the unit such that $Q = \langle g_1, g_2, g_3, t_1^{v,1,1,1,v} \rangle$.

Proof. The quasitorus $Q = \langle \phi_i \rangle_{i=1}^4$ is generated by $\phi_1, \phi_2, \phi_3$ order-two elements in $G_4$ and $\phi_4$ an order-$m$ element in $G_4$. We can apply the previous proposition
to \( Q' := \langle \phi_1, \phi_2, \phi_3 \rangle \). So we can assume (by conjugation) that either (1) \( Q' = \langle g_1, g_2, g_3 \rangle \), or (2) \( Q' = \langle g_1, g_2, g_4 \rangle \), with \( \phi_4 = g_4^u g_5^v g_6^{t'_1,1,1,u} \) for some \( n, k, l \in \{0, 1\} \) and some \( u \in F^\times \). From \( \phi_4^m = \text{id} \), it follows that \( lm \) is even (either \( l \) or \( m \)) and \( t'_1 = (1)^n, (1)^k, (1)^l, u \) has order \( m \), in particular \( u^m = 1 \).

In case (1), \( Q = \langle g_1, g_2, g_3, g_4^k g_5^{t_1,1,1,u} \rangle = \langle g_1, g_2, g_3, t'_1,1,1,u \rangle \), but \( t'_1,1,1,u \notin Q' \), so if \( u \) would have order \( m' \) (divisor of \( m \)), \( \mathcal{X}(Q) \) would be isomorphic to \( \mathbb{Z}_2^3 \times \mathbb{Z}_{m'} \). Hence \( u \) is a primitive \( m \)-root of the unit.

In case (2), \( Q = \langle g_1, g_2, g_3, g_4^k g_5^{t'_1,1,1,u} \rangle = \langle g_1, g_2, t'_1,1,1,1,1,u \rangle \). Since \( Q \) is nontoral, \( l = 1 \). Hence \( m = 2m' \) must be even, with \( u^{m'} \in \{ \pm 1 \} \). If \( u^{m'} = -1 \), \((g_3 t'_1,1,1,1)^{m'} \) would be \( t'_1,1,1,1 \) if \( m' \) is even, and \( g_3 t'_1,1,1,1 \) if \( m' \) is odd. In the first case \( Q = \langle g_1, g_2, g_3 t'_1,1,1,1,u \rangle \), a contradiction with the number of generators of \( \mathcal{X}(Q) \). In the other case, \( g_3 \in Q \), hence \( Q = \langle g_1, g_2, t'_1,1,1,1,1,1,u \rangle = \langle g_1, g_2, t'_1,1,1,1,1,1,u \rangle = \langle g_1, g_2, g_3, t'_1,1,1,1,1,u \rangle \), and so \( \mathcal{X}(Q) \) would be a subgroup of \( \mathbb{Z}_2^3 \times \mathbb{Z}_{m'} \), a contradiction. Thus \( u^{m'} = 1 \) and \((g_3 t'_1,1,1,1)^{m'} \) would be \( t'_1,1,1,1,1 \) if \( m' \) is even, and \( g_3 \) if \( m' \) is odd. In the first case again \( \mathcal{X}(Q) \) would be a subgroup of \( \mathbb{Z}_2^3 \times \mathbb{Z}_{m'} \). So that we have the case \( m' \) odd, in which \( g_3 \in Q \). In this way \( Q = \langle g_1, g_2, t'_1,1,1,1,1,u \rangle = \langle g_1, g_2, g_3, t'_1,1,1,1,1,u \rangle \), since \((t'_1,1,1,1,1)^{m'} = t'_1,1,1,1,1,u \). But now \( v = -u \) is the required primitive \( m \)-root of the unit.

Notice that the obtained quasitorus \( Q = \langle g_1, g_2, g_3, t'_1,1,1,1,v \rangle \) are obviously nontoral because they contain \( \langle g_1, g_2, g_3 \rangle \).

The induced gradings by the previous quasitorus depend on \( m \). For \( m = 2 \) we are talking about a \( \mathbb{Z}_2^3 \)-grading of type \( (1, 8, 0, 0, 0, 7) \), for \( m = 3 \) it is a \( \mathbb{Z}_2^3 \times \mathbb{Z}_3 \)-grading of type \( (3, 14, 7) \), for \( m = 4 \) it is a \( \mathbb{Z}_3^3 \times \mathbb{Z}_4 \)-grading of type \( (17, 7, 7) \), but for \( m \geq 5 \) all the gradings are equivalent to the one produced by \( A(105, \text{id}) \), since they are all of type \( (31, 0, 7) \). In general two gradings having the same type are not necessarily equivalent, but of course if the quasitorus producing one of the gradings is contained in the other one, since the homogeneous components of the former are pieces of the homogeneous components of the latter.

**Remark 4.** If \( Q' \subset Q \) are quasitorus whose induced gradings are of the same type, then these gradings are equivalent.

To continue the study of nontoral gradings coming from subquasitorus of \( A(105, \text{id}) \) we must analyze those quasitorus \( Q \subset A(105, \text{id}) \) with \( \mathcal{X}(Q) \cong \mathbb{Z}_2^3 \times \mathbb{Z}_{m'} \), where \( m' \) must be even (otherwise \( Q \) would have two generators and the grading would be toral). Moreover \( m \) must be a multiple of \( 4 \), because we have already studied the cases \( m = 2 \) in Proposition 11 and \( m = 2m' \) with \( m' \) odd in Proposition 12.

**Proposition 13.** If \( Q \subset A(105, \text{id}) \) is a nontoral quasitorus such that \( \mathcal{X}(Q) \cong \mathbb{Z}_2^3 \times \mathbb{Z}_{4m} \) with \( m \geq 1 \), then up to conjugacy there is \( v \in F^\times \) a primitive \( 4m \)-root of the unit such that \( Q = \langle g_1, g_2, g_3 t'_1,1,1,v \rangle \).

Proof. Suppose that \( \{ \phi_1, \phi_2, \phi_3 \} \) is a set of generators of \( Q \) with \( \phi_1 \) and \( \phi_2 \) of order two and \( \phi_3 \) of order \( 2k = 4m \). Then \( \phi_3^k \) is an order-two element and we can suppose (by conjugation) that the subgroup \( \langle \phi_1, \phi_2, \phi_3^k \rangle \) is either \( \langle g_1, g_2, g_3 \rangle \) or \( \langle g_1, g_2, g_4 \rangle \), according to Proposition 11. Besides, \( \phi_3 = g_3^u g_3^{t'_1,1,1,u} \) for some \( n, k, l \in \{0, 1\} \) and \( u \in F^\times \). As \( k \) is even, \( \phi_3^k = t'_1,1,1,1,1,u \). Since \( \phi_3^k \) has exactly order two, \( u^k = -1 \) and \( \phi_3^k = g_4 \).

But \( \langle \phi_1, \phi_2, \phi_3^k \rangle = \langle g_1, g_2, g_3 \rangle \) is not possible, since \( g_4 \notin \langle g_1, g_2, g_3 \rangle \), so necessarily \( \langle \phi_1, \phi_2, \phi_3^k \rangle = \langle g_1, g_2, g_4 \rangle \). Thus \( Q = \langle \phi_1, \phi_2, \phi_3^k \rangle = \langle g_1, g_2, g_4, g_3 t'_1,1,1,u \rangle \) and
$l = 1$ since $Q$ is nontoral. But $(g_3t_{1,1,1,1}^u)^k = g_4$ so that $Q = \langle g_1, g_2, g_3t_{1,1,1,1}^u \rangle$, where $u$ has order $2k = 4m$. $\Box$

Notice that in this case all the $\mathbb{Z}^2_2$-coarsenings of the gradings are toral. In spite of that, these gradings are nontoral, because the dimension of the zero homogeneous component is 3 in case $m = 1$ and 1 in the remaining cases. If $m = 1$, we obtain a $\mathbb{Z}^2_2 \times \mathbb{Z}_4$-grading of type $(0, 8, 2, 0, 6)$, if $m = 2$ we have a $\mathbb{Z}^2_2 \times \mathbb{Z}_8$-grading of type $(19, 6, 7)$, but for $m \geq 3$ we obtain gradings of type $(31, 0, 7)$, so equivalent to the fine $\mathbb{Z}^2_2 \times \mathbb{Z}$-grading, again by the previous remark.

Finally we analyze the case $\mathfrak{X}(Q) = \mathbb{Z}^2_2 \times \mathbb{Z}$. The quasitorus $Q$ must be the direct product of a one-dimensional subtorus $P$ of $A(105, \text{id})$ times $\langle \phi_1, \phi_2 \rangle$, with $\phi_i$ two elements of order two. But necessarily $P = \{ t_{1,1,1,1,1}^u | u \in F^\times \}$ (it is the unique nontrivial subtorus of $A(105, \text{id})$). Changing the generators if necessary (to remove $g_1$), we can write $\langle \phi_1, \phi_2 \rangle = \langle g_3, g_2, g_1 \rangle M$ where $M$ is a $3 \times 2$ matrix with entries in $\mathbb{Z}_2$. The first row is nonzero because otherwise the grading would be toral, and making column operations we can suppose that it is $(1, 0)$. Besides there must be some 1 in each column. As $\sigma_{468}$ interchanges the second and third, and by doing column operations, the first two rows in $M$ can be taken to be $\left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)$. Thus either $Q = P\langle g_3g_1, g_2 \rangle$, or $Q = P\langle g_3, g_1g_2 \rangle$, or $Q = P\langle g_3g_1, g_2g_1 \rangle$. By applying the element $\sigma_{491}$ to the third one, we obtain $P\langle g_1g_3g_2g_4, g_2g_4 \rangle = P\langle g_1g_3, g_2g_4 \rangle = P\langle g_1g_3, g_2 \rangle$, the first quasitorus. This one is conjugated to the second one by means of $\psi$ (see Appendix). Besides we can replace $g_1g_2$ by $g_1$ by using $\sigma_{94}$. Summarizing these facts we have:

**Proposition 14.** If $Q \subset A(105, \text{id})$ is nontoral and $\mathfrak{X}(Q) \cong \mathbb{Z}^2_2 \times \mathbb{Z}$ then $Q$ is conjugated to $\langle g_1, g_3, t_{1111}^u | u \in F^\times \rangle$.

Notice that the induced grading on $f_4$ is of type $(31, 0, 7)$, so it is again equivalent to the produced by the whole $A(105, \text{id})$.

The following step is to examine the proper subquasitori of $A(15, \text{id})$, but all of them are toral since $A(15, \text{id})$ is isomorphic to $\mathbb{Z}^4_2$ hence any proper subquasitorus has a system of generators of cardinal $\leq 2$ (Lemma 2).

Thus, to finish our classification of subquasitori of the maximal quasitori, we must analyze $A(405, \text{id})$. This group is isomorphic to $\mathbb{Z}^4_2$. Indeed we have $A(405, \text{id}) = \langle \mathfrak{T}^{(405)} \cup \{ \sigma_{405} \} \rangle$ where $\mathfrak{T}^{(405)} = \{ t_{xyzu}^u | x^2 = y^2 = z^2 = u^2 = 1 \} \cong \mathbb{Z}^4_2$ and $\sigma := \sigma_{405}$ is an order two element. Consider a subquasitorus $Q \subset A(405, \text{id})$ with three order-two generators. If $Q$ is nontoral, then $\sigma t \in Q$ for some order-two $t \in \mathfrak{T}$. But applying Corollary 2 we can conjugate $Q$ to some new quasitorus $Q'$ with $\sigma \in Q'$. Thus we can suppose since the beginning that $\sigma \in Q$, and by elementary operations we can take $Q = \langle \sigma, t_1, t_2 \rangle$ where $t_i \in \mathfrak{T}$ are order-two elements.

It is known that there are two conjugacy classes of elements of order 2 in $f_4$, related to the diagrams obtained by removing the nodes marked with the number 2 in the affine Dynkin diagram [28, Ch. 8]. The first automorphism, related to

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![Diagram](https://via.placeholder.com/150)

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fixes a subalgebra of type $\mathfrak{b}_4$, of dimension 36, and the second one, related to
fixes a subalgebra of type $c_3 \oplus a_1$, of dimension 24. But if two elements in a torus are conjugated, they are conjugated inside the normalizer. This means that the Weyl group acts on $T(405)$ producing two orbits, apart from the trivial one, characterized by the fact that the dimensions of their fixed parts (the number of 1’s in the list (21)) are 36 and 24, respectively. This could have been checked directly making act $W$ by computer. In the first orbit there are three elements, $t'_{1,1,1,1,1}, t'_{1,1,1,1,1}$ and $t'_{1,1,1,1,1}$, and in the second orbit the remaining 12 elements. Thus we can move the element $t_1$ in $Q$, since not only $\sigma_{405} = \text{id}$ clearly commutes with any element in $W$, but $\sigma \tilde{\sigma}_j = \tilde{\sigma}_j \sigma$ for any $j \in \{1, \ldots, 1152\}$. Therefore the possibilities for $Q$ are (1) $Q = \langle \sigma, t'_{1,1,1,1,1}, \rangle$ and (2) $Q = \langle \sigma, t'_{1,1,1,1,1}, \rangle$, as $t'_{1,1,1,1,1}$ and $t'_{1,1,1,1,1}$ are representatives of the two orbits. In the first case, the third element can be computed by considering the subgroup of $W$ fixing $t_{1,1,1,1,1}$ and the orbits it produces on the set of order-two elements different from $t_{1,1,1,1,1}$. Indeed, that subgroup produces three orbits such that any order-two element different from $t_{1,1,1,1,1}$ is conjugated to either $t'_{1,1,1,1,1}$ or $t'_{1,1,1,1,1}$ or $t'_{1,1,1,1,1}$ by an element in $W$ fixing $t'_{1,1,1,1,1}$. As a consequence the possibilities for $Q$ are $\langle \sigma, t_{1,1,1,1,1}, t_{1,1,1,1,1}, \rangle$, $\langle \sigma, t_{1,1,1,1,1}, t_{1,1,1,1,1}, \rangle$, and $\langle \sigma, t_{1,1,1,1,1}, t_{1,1,1,1,1}, \rangle$. But the second and third of these are conjugated and toral ($\tilde{\sigma}_{405}$ relates both of them), hence the unique nontoral $Q$ is, up to conjugacy, $Q = \langle \sigma, t_{1,1,1,1,1}, t_{1,1,1,1,1} \rangle$.

In the case (2) we can take the third element $t_2$ also in the orbit of $t'_{1,1,1,1,1}$ and so $Q = \langle \sigma, t'_{1,1,1,1,1}, t'_{1,1,1,1,1} \rangle$, which produces a toral grading again. It is obviously not conjugated to the previous toral one, because the number of order-two elements in a determined orbit is preserved by conjugation (besides the zero homogeneous components have different dimensions).

**Proposition 15.** The unique proper subquasitorus of $A(405, \text{id})$ of order 8 (up to conjugacy) are:

- $\langle \sigma, t'_{1,1,1,1,1}, t'_{1,1,1,1,1} \rangle$, a nontoral $Z_2^3$-grading of type $<0,0,1,0,0,7>$.
- $\langle \sigma, t'_{1,1,1,1,1}, t'_{1,1,1,1,1} \rangle$ which is toral.
- $\langle \sigma, t'_{1,1,1,1,1}, t'_{1,1,1,1,1} \rangle$ which is also toral.

The last two gradings are not isomorphic.

To finish this subsection we should now describe the nontoral subquasitori $Q$ of $A(405, \text{id})$ isomorphic to $Z_2^3$. For this we take any subquasitori $Q'$ of $Q$ of cardinal 8, that is, isomorphic to $Z_2^3$ and apply the previous study to it. Thus we should study the possible refinements of the quasitori given in Proposition 15 which give nontoral gradings. The techniques already used in the previous paragraph give that any such quasitori is conjugated to $\langle \sigma, t'_{1,1,1,1,1}, t'_{1,1,1,1,1}, t'_{1,1,1,1,1} \rangle$, which gives a $Z_2^3$-grading of type $<1,8,0,0,7>$. The agreement with the type of the $Z_2^3$-quasitorus contained in $A(105, \text{id})$ suggests that they could be conjugated and the corresponding gradings isomorphic. Indeed, up to conjugacy there is only one abelian nontoral subgroup of $F_4$ isomorphic to $Z_2^3$ and only one to $Z_4^3$ (see for instance [39, Prop. 3.2]). For completeness and self-containedness we prove it in our context.

**Proposition 16.** Any nontoral proper subquasitorus of $A(405, \text{id})$ is conjugated to a subquasitorus of $A(105, \text{id})$.

Proof. Notice that for $f_1 = t'_{1,1,1,1,1}, f_2 = t'_{1,1,1,1,1}$ and $f_3 = t'_{1,1,1,1,1}$, both $\sigma_{105}$ and $\sigma_{405}$ commute with the subgroup $T(405) \cap T(105) = \langle f_1, f_2, f_3 \rangle \cong Z_2^3$. In fact both gradings induced by $\langle f_1, f_2, f_3, \sigma_{105} \rangle$ and $\langle f_1, f_2, f_3, \sigma_{405} \rangle$ are nontoral.
The key fact is that \( \langle f_1, f_2, f_3, \overline{\sigma_{105}} \overline{\sigma_{105}} \rangle \) is toral (since the fixed part is a four-dimensional abelian subalgebra), so we can apply Proposition 7 to \( F = \{ f_i \}_{i=1}^3 \), for \( f_1 = \overline{\sigma_{105}} \overline{\sigma_{105}} \) and \( f_3 = \overline{\sigma_{105}} \). The group \( F \) is obviously nontoral since it is isomorphic to \( \mathbb{Z}_2^5 \). As in the proof of Proposition 7, there is \( p \in \mathfrak{g}_4 \) such that \( pf_i p^{-1} \in \mathfrak{z} \) for \( i = 1, \ldots, 4 \) and \( pf_5 p^{-1} \in \mathfrak{n} \), that is, there are \( j \in \{1, \ldots, 1152\} \) and \( t \in \mathfrak{z} \) such that \( pf_5 p^{-1} = \overline{\sigma_j} t \). As \( \mathbb{Z}_2^4 \cong \langle pf_i p^{-1} \mid i = 1 \ldots 4 \rangle \subset \mathfrak{z}^{(4)} \), we have either \( j = 405 \) or \( j = 748 \). But if we had this last possibility, the grading \( \langle pf_i p^{-1} \mid i = 1 \ldots 5 \rangle \) would be toral. Hence \( pf_5 p^{-1} = \overline{\sigma_{405}} t \). Moreover, \( p \) can be taken such that \( pf_5 p^{-1} = \overline{\sigma_{405}} \), by Corollary 2, so that \( p(F)p^{-1} \subset A(405, id) \).

Thus, the unique (up to conjugation) nontoral subquasitorus of \( A(105, id) \) isomorphic to \( \mathbb{Z}_2^4 \), which is \( \langle f_1, f_2, f_3, f_5 \rangle \) according to Proposition 12, is conjugated by means of \( p \) to a subquasitorus of \( A(405, id) \), and consequently, the same can be said about the \( \mathbb{Z}_2^4 \)-nontoral quasitorus \( \langle f_1, f_2, f_3, f_5 \rangle \). The proof is finished because there are only two nontoral proper subquasitorus of \( A(405, id) \), by Proposition 15 and the paragraph above.

Summarizing the previous propositions, we have proved the following theorem, which describes all the nontoral quasitorus of \( \mathfrak{g}_4 \).

**Theorem 5.** Any nontoral subquasitorus of \( \mathfrak{g}_4 \) is conjugated to some of the following:

1. \( A(15, id) = \langle \overline{\sigma_{15}} t_{\omega,1,1,\omega}, t_{1,\omega,1,\omega} \rangle \cong \mathbb{Z}_2^3 \), where \( \omega \) is a primitive cubic root of 1.
2. \( A(105, id) = \langle \{ \overline{\sigma_{105}} t_{1,-1,1,-1,1} \} \cup \{ t_{111u} \in F^\times \} \cong F^\times \times \mathbb{Z}_2^4 \), and its nontoral (proper) coarsenings which up to conjugacy are:
   - \( A \langle \sigma_{105}, t_{-1,1,-1,1}, t_{1,-1,1,1} \rangle \cong \mathbb{Z}_2^4 \).
   - \( A \langle \sigma_{105}, t_{1,-1,1,-1,1} \rangle \cong \mathbb{Z}_2^3 \).
   - \( A \langle \sigma_{105}, t_{1,-1,1,-1,1}, t_{1,1,1,1,1} \rangle \cong \mathbb{Z}_2^3 \times \mathbb{Z}_m \), where \( m > 2 \).
3. \( A(405, id) = \langle \sigma_{405}, t_{-1,1,1,1}, t_{-1,1,1,1}, t_{1,1,1,1,1} \rangle \cong \mathbb{Z}_2^5 \). Its nontoral coarsenings are conjugated to the quasi-tori in \( 2.1 \) and \( 2.2 \).

Therefore, the following table gives all the nontoral gradings on \( f_4 \) up to equivalence. In it, we have taken into consideration that no repeated gradings arise produced by the infinite families of quasitorus in some of the cases of the theorem (computed after Propositions 12, 13 and 14). We give the quasitorus, the universal grading groups and the types:

| Grading | Group | Type |
|---------|-------|------|
| I       | \( \mathbb{Z}_2^3 \) | (0,0,6) | Fine |
| II      | \( \mathbb{Z}_2^3 \times \mathbb{Z}_2 \) | (31,0,7) | Fine |
| II.1    | \( \mathbb{Z}_2^3 \times \mathbb{Z}_2 \) | (0,0,1,0,0,0,7) | |
| II.2    | \( \mathbb{Z}_2^3 \) | (1,0,8,0,7) | |
| II.3.1  | \( \mathbb{Z}_2^3 \times \mathbb{Z}_3 \) | (3,14,7) | |
| II.3.2  | \( \mathbb{Z}_2^3 \times \mathbb{Z}_4 \) | (7,17,7) | |
| II.4.1  | \( \mathbb{Z}_2^3 \times \mathbb{Z}_4 \) | (0,8,2,0,6) | |
| II.4.2  | \( \mathbb{Z}_2^3 \times \mathbb{Z}_8 \) | (19,6,7) | |
| III     | \( \mathbb{Z}_2^5 \) | (24,0,0,7) | Fine |

**Remark 5.** We should notice that there are 9 different equivalence classes of nontoral gradings on \( f_4 \). But there are only 8 different ones on the Albert algebra.
GRADINGS ON ALBERT ALGEBRA AND ON $f_4$

J. The device for translating gradings from J to $f_4$ (and conversely) has this deficiency. However it works well when applied to fine gradings, because MAD’s are preserved by the adjoint map. This explains the agreement in the number of fine gradings (up to equivalence).

Remark 6. Theorem 4 states that every quasitorus in $\mathfrak{g}_4$ is contained in some $A(j, \text{id})$. Thus every nontoral grading is induced by a set of automorphisms formed by one element in the Weyl group together with several elements in $T$. This result is also true in nontoral gradings on $\mathfrak{g}_2$, but it is false when applied to other Lie algebras, for instance $\mathfrak{d}_4 = \mathfrak{o}(8, F)$. Then, an alternative approach is to find “the first steps”, that is, the minimal nontoral quasitori instead of the maximal ones. Independently of the simple Lie algebra under study, they are contained in some $A(j, \text{id})$ and any nontoral grading can be obtained by refining one of the gradings produced by them. In the case of $f_4$, we have found three nontoral minimal quasitori, namely, I, II.1 and II.4.1 in Theorem 5, which provide $Z^3_3$, $Z^2_2$ and $Z^2_2 \times Z^4_4$-gradings respectively.

6. Gradings on the Albert Algebra revisited

In this section we prove Theorem 3. We start by considering the quasitori $Q$ inducing a nontoral grading on the Albert algebra $J$. The key tool here is the previously mentioned fact that the automorphism group $F_4 = \text{aut}(J)$ and the automorphism group $\mathfrak{f}_4 = \text{aut}(f_4)$ are isomorphic via the map $\text{Ad}: F_4 \rightarrow \mathfrak{f}_4$ such that $\text{Ad}(f)d := fd(f)^{-1}$ for any $f \in F_4$ and $d \in f_4$ ([26]). Then we can apply Theorem 5 to have an immediate view (up to conjugacy) of the quasitory $Q$. So we study the different possibilities I)-III) provided by Theorem 5 and collect them in the following table

| Grading | Group | Type |
|---------|-------|------|
| I       | $Z^3_3$ | (27) Fine |
| II      | $Z^2_2 \times Z_2$ | (25, 1) Fine |
| II.1    | $Z^2_2$ | (0, 0, 7, 0, 0, 1) |
| II.2    | $Z^2_2$ | (7, 8, 0, 1) |
| II.3.1  | $Z^2_2 \times Z_4$ | (21, 3) |
| II.3.2  | $Z^2_2 \times Z_4$ | (23, 2) |
| II.4    | $Z^2_2 \times Z_4$ | (0, 12, 1) |
| III     | $Z^5_5$ | (24, 0, 1) Fine |

It is remarkable the fact that all the quasitori in case II.4 of Theorem 5 induce the same grading on $J$ up to equivalence (by Remark 4). This was not the case in $f_4$ where these quasitori produced two nonequivalent gradings. This is the reason why we only have eight equivalence classes of nontoral gradings on $J$ while in $f_4$ we have detected nine. Now, any nontoral grading on $J$ must be equivalent to any of these, and these are nonequivalent. But the gradings (11), (12), (13), (14), (15), (5), (16) and (17) described in Theorem 3 are nonequivalent since their types are different:

| Grading | Type |
|---------|------|
| (11)    | (25, 1) |
| (12)    | (7, 8, 0, 1) |
| (13)    | (23, 2) |
| (14)    | (25, 1) |
| (15)    | (0, 12, 1) |
| (16)    | (27) |
| (17)    | (0, 0, 7, 0, 0, 1) |

Remark 6. Theorem 4 states that every quasitorus in $\mathfrak{g}_4$ is contained in some $A(j, \text{id})$. Thus every nontoral grading is induced by a set of automorphisms formed by one element in the Weyl group together with several elements in $T$. This result is also true in nontoral gradings on $\mathfrak{g}_2$, but it is false when applied to other simple Lie algebras, for instance $\mathfrak{d}_4 = \mathfrak{o}(8, F)$. Then, an alternative approach is to find “the first steps”, that is, the minimal nontoral quasitori instead of the maximal ones. Independently of the simple Lie algebra under study, they are contained in some $A(j, \text{id})$ and any nontoral grading can be obtained by refining one of the gradings produced by them. In the case of $f_4$, we have found three nontoral minimal quasitori, namely, I, II.1 and II.4.1 in Theorem 5, which provide $Z^3_3$, $Z^2_2$ and $Z^2_2 \times Z^4_4$-gradings respectively.

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| Grading | Group | Type |
|---------|-------|------|
| I       | $Z^3_3$ | (27) Fine |
| II      | $Z^2_2 \times Z_2$ | (25, 1) Fine |
| II.1    | $Z^2_2$ | (0, 0, 7, 0, 0, 1) |
| II.2    | $Z^2_2$ | (7, 8, 0, 1) |
| II.3.1  | $Z^2_2 \times Z_4$ | (21, 3) |
| II.3.2  | $Z^2_2 \times Z_4$ | (23, 2) |
| II.4    | $Z^2_2 \times Z_4$ | (0, 12, 1) |
| III     | $Z^5_5$ | (24, 0, 1) Fine |

It is remarkable the fact that all the quasitori in case II.4 of Theorem 5 induce the same grading on $J$ up to equivalence (by Remark 4). This was not the case in $f_4$ where these quasitori produced two nonequivalent gradings. This is the reason why we only have eight equivalence classes of nontoral gradings on $J$ while in $f_4$ we have detected nine. Now, any nontoral grading on $J$ must be equivalent to any of these, and these are nonequivalent. But the gradings (11), (12), (13), (14), (15), (5), (16) and (17) described in Theorem 3 are nonequivalent since their types are different:

| Grading | Type |
|---------|------|
| (11)    | (25, 1) |
| (12)    | (7, 8, 0, 1) |
| (13)    | (23, 2) |
| (14)    | (25, 1) |
| (15)    | (0, 12, 1) |
| (16)    | (27) |
| (17)    | (0, 0, 7, 0, 0, 1) |
and so they give a complete system of pairwise nonequivalent nontoral gradings on the Albert algebra.

Remark 7. There is only one equivalence class of group gradings on $J$ with every nonzero component spanned by an invertible element, namely the $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$-grading (19). This condition is equivalent to be a Jordan $\Lambda$-torus, according to [5, Remark 9.2.1]. Although Jordan tori have been classified by [41], Theorem 3 provides an alternative proof of the uniqueness.

7. Description of the nontoral gradings on $\mathfrak{f}_4$

In this section we would like to give a more detailed description of the fine gradings on $\mathfrak{f}_4$. This description is going to be twofold. On the one hand by using any software allowing simultaneous diagonalization we can get the homogeneous spaces of the grading under consideration in terms of the basis introduced in 4.1. We include this description for its possible use in applications requiring explicit computations. For instance, the subject of gradings is closely related to the graded contractions [31]. In the latter, new Lie algebras are obtained by modifying the commuting relations respecting the grading.

But on the other hand we would like to highlight the fact that the whole algebraic group stuff used in this work has been needed to prove that we have captured all the gradings. However this is not necessary at all to describe these gradings. This can be made in an independent way and this is why in this section we are giving natural descriptions of all the fine gradings. Of course the term natural is used here in a subjective manner meaning that the gradings are given with no reference to computer methods neither Weyl group nor maximal torus. Thus any mathematician could check the gradings ignoring such tools.

7.1. $\mathbb{Z}_3^3$-grading on $\mathfrak{f}_4$. This grading has some referring in the literature though mostly in geometry than in algebra. Geometers have studied elementary p-groups with different purposes other than the study of gradings. The fine $\mathbb{Z}_3^3$-grading appears for instance in [18, THEOREM 11.13] from the viewpoint of compact Lie groups but [3, 8.1] gives results showing how to translate the arguments to the algebraic groups setting. On the other hand this $\mathbb{Z}_3^3$-grading has been studied from the viewpoint of Jordan groups (see [33, p. 127]). It was Alekseevskij, in [4, Table 1], who classified Jordan subgroups in the exceptional case.

The description of the gradings in terms of root vectors can be obtained instantaneously by performing a simultaneous diagonalization of $\mathfrak{f}_4$ relative to the set of automorphisms $\{\sigma_{15}, t'_{\omega, \omega, 1, \omega^2}, t'_{1, \omega, \omega, 1}\}$. Thus we obtain the following fine $\mathbb{Z}_3^3$-grading of type $(0, 26)$ on $L = \mathfrak{f}_4$:

$$L_{000} = 0,$$

$$L_{001} = \langle b_{17} + b_{32} + b_{33}, -b_{22} + b_{34} + b_{42} \rangle,$$

$$L_{002} = \langle b_{8} + b_{9} + b_{41}, b_{10} + b_{18} + b_{46} \rangle,$$

$$L_{010} = \langle b_{1} + b_{2} + b_{29}, b_{4} - b_{23} + b_{45} \rangle,$$

$$L_{011} = \langle -b_{19} + b_{30} + b_{38}, -b_{20} + b_{31} + b_{39} \rangle,$$

$$L_{012} = \langle b_{1} + b_{13} + b_{40}, b_{11} - b_{12} + b_{45} \rangle,$$

$$L_{020} = \langle b_{5} + b_{25} + b_{26}, -b_{24} - b_{28} + b_{47} \rangle,$$

$$L_{021} = \langle -b_{23} - b_{35} + b_{36}, b_{16} - b_{27} + b_{37} \rangle,$$

$$L_{022} = \langle -b_{6} - b_{14} + b_{43}, b_{7} - b_{15} + b_{44} \rangle,$$
we can identify the exterior product with the dual space by the map such that
\[ Z \to a \text{ particular basis or computer methods, may be considering Lie alg ebra models} \]
\[ X(28) \text{ endowed with a Lie algebra structure with the product} \]
\[ \text{spread for its nice 3-symmetry. Once the automorphisms have bee n given in} \]
\[ \text{could hopefully restrict them to} \]
\[ \text{of this algebra from three copies of} \]
\[ \text{for any} \]
\[ \text{the natural ones (the} \]
\[ \text{root of the unit.} \]
\[ \text{grading with the nonzero homogeneous components of the same dim ension (78} \]
\[ \text{induces the grading, that is,} \]
\[ \text{e} \]
\[ \bar{\text{e}} \]
\[ \land \]
\[ * \]
\[ \text{The easiest way to visualize this grading intrinsically, that is, with no reference} \]
\[ \text{to a particular basis or computer methods, may be considering Lie algebra models} \]
\[ \text{based upon} \mathbb{Z}_3 \text{-gradings. Perhaps the most natural place where to look at the au to-} \]
\[ \text{morphisms inducing the grading is the Lie algebra} \epsilon_6. \text{ Adams gave a construction of this algebra from three copies of} \alpha_2 \text{ ([1, p. 85]). This model has been widely} \]
\[ \text{spread for its nice 3-symmetry. Once the automorphisms have been given in} \epsilon_6 \text{ we could} \]
\[ \text{hopefully restrict them to} \mathfrak{l}_4. \text{ Given a three-dimensional} \ F \text{-vector space} X \]
\[ \text{in which a nonzero alternate trilinear map} \det: X \times X \times \to F \text{ has been fixed,} \]
\[ \text{we can identify the exterior product with the dual space by the map} X \land X \mapsto X^* \]
\[ \text{such that} \ x \land y \mapsto \det(x, y, -) \in \text{hom}(X, F) . \text{ And in a dual way we can} \]
\[ \text{identify} X^* \land X^* \text{ with} X \text{ through} \det^*, \text{ the dual map of} \det. \text{ Consider three tridimensional vector spaces} X_i \ (i = 1, 2, 3), \text{ and define:} \]
\[ \mathcal{L} = \text{sl}(X_1) \oplus \text{sl}(X_2) \oplus \text{sl}(X_3) \oplus X_1 \otimes X_2 \otimes X_3 \oplus X_1^* \otimes X_2^* \otimes X_3^*, \]
\[ \text{endowed with a Lie algebra structure with the product} \]
\[ \mathbf{f}_i \cdot \mathbf{x}_i = \sum_{1 \leq i, j \leq 3} f_i(x_i) f_j(x_j) \left( f_k(-) x_k - \frac{1}{4} f_k(x_k) i\mathbf{d}x_k \right) \]
\[ (28) \]
\[ \mathbf{x}_i \cdot \mathbf{y}_i = \mathbf{x}_i \mathbf{y}_i \]
\[ \mathbf{f}_i \cdot \mathbf{g}_i = \mathbf{f}_i \mathbf{g}_i \]
\[ \text{for any} \ x_i, y_i, \mathbf{f}_i, \mathbf{g}_i \in X^*_i, \text{ with the wedge products as above, and where the} \]
\[ \text{actions of the Lie subalgebra} \sum \text{sl}(X_i) \text{ on} \ X_1 \otimes X_2 \otimes X_3 \text{ and} \ X_1^* \otimes X_2^* \otimes X_3^* \text{ are} \]
\[ \text{the natural ones (the} i \text{-th simple ideal acts on the} i \text{-th slot). The Lie algebra} \mathcal{L} \text{ is} \]
\[ \text{isomorphic to} \epsilon_6. \]
\[ \text{Following Hesselink ([23]), we say that a grading is} \text{special} \text{ if and only if its} \]
\[ 0 \text{-homogeneous component is zero. We can observe that} \epsilon_6 \text{ admits a} \text{special} \mathbb{Z}_3^3 \]
\[ \text{-grading with the nonzero homogeneous components of the same dimension (78/26 =} \]
\[ 3). \text{ Since} \mathcal{L} = L_0 \oplus L_1 \oplus L_2 \text{ is a} \mathbb{Z}_3 \text{-grading for} L_0 = \text{sl}(X_1) \oplus \text{sl}(X_2) \oplus \text{sl}(X_3), \]
\[ L_1 = X_1 \otimes X_2 \otimes X_3 \text{ and} \ L_2 = X_1^* \otimes X_2^* \otimes X_3^*, \text{ take} \phi_1 \text{ the automorphism which} \]
\[ \text{induces the grading, that is,} \phi_1|_{L_1} = \omega \mathbf{id}_{L_1} \text{ where} \ \omega = e^{\frac{2\pi i}{3}} \text{ is a primitive cubic} \]
\[ \text{root of the unit.} \]
In order to provide the other automorphisms, take into account the following observation. If \( \rho_i: X_i \to X_i, \ i = 1, 2, 3 \), are linear maps preserving \( \det: X_i^3 \to F \) (that is, \( \det(x_i, y_i, z_i) = \det(\rho_i(x_i), \rho_i(y_i), \rho_i(z_i)) \)), or equivalently, \( \det(\rho_i) = 1 \), the linear map \( \rho_1 \otimes \rho_2 \otimes \rho_3: L \to L \) can be uniquely extended to an automorphism of \( L \) such that its restriction to \( sl(V_i) \subset L_0 \) is the conjugation map \( g \mapsto \rho_i g \rho_i^{-1} \).

Next fix basis \( \{ u_0, u_1, u_2 \} \) of \( X_1 \), \( \{ v_0, v_1, v_2 \} \) of \( X_2 \), and \( \{ w_0, w_1, w_2 \} \) of \( X_3 \) with \( \det(u_0, u_1, u_2) = \det(v_0, v_1, v_2) = \det(w_0, w_1, w_2) = 1 \). Consider now \( \phi_2 \) the unique automorphism of \( \mathfrak{e}_6 \) extending the map

\[
u_i \otimes v_j \otimes w_k \mapsto u_{i+1} \otimes v_{j+1} \otimes w_{k+1}.
\]

(indices 3). This is of course an order three semisimple automorphism commuting with \( \phi_1 \). Finally let \( \phi_3 \) be the unique automorphism of \( \mathfrak{e}_6 \) extending the map

\[
u_i \otimes v_j \otimes w_k \mapsto \omega^i \nu_i \otimes \omega^j v_j \otimes \omega^k w_k = \omega^{i+j+k} \nu_i \otimes v_j \otimes w_k.
\]

This is also semisimple and commutes with \( \phi_1 \) and \( \phi_2 \). Thus the set \( \{ \phi_i \}_{i=1}^3 \) induces a \( \mathbb{Z}_3 \)-grading on \( \mathfrak{e}_6 \). A computation of its 0-homogeneous component will suffice to prove that this grading is nontoral. A first calculation reveals that the subalgebra of elements fixed by \( \phi_1 \) and \( \phi_2 \) is the linear span of

\[
\{ u_1 \otimes u_2 + u_2 \otimes u_3 + u_3 \otimes u_1, \quad u_1 \otimes u_3 + u_2 \otimes u_1 + u_3 \otimes u_2, \\
v_1 \otimes v_2 + v_2 \otimes v_3 + v_3 \otimes v_1, \quad v_1 \otimes v_3 + v_2 \otimes v_1 + v_3 \otimes v_2, \\
w_1 \otimes w_2 + w_2 \otimes w_3 + w_3 \otimes w_1, \quad w_1 \otimes w_3 + w_2 \otimes w_1 + w_3 \otimes w_2 \},
\]

where we have taken \( u_i^* = u_{i+1} \wedge u_{i+2}, v_i^* = v_{i+1} \wedge v_{i+2} \) and \( w_i^* = w_{i+1} \wedge w_{i+2} \) the dual basis of \( X_1^* \), \( X_2^* \) and \( X_3^* \) respectively. While the corresponding fixed subalgebra for \( \phi_1 \) and \( \phi_3 \) is the linear span of

\[
\{ u_1 \otimes u_1^* - u_2 \otimes u_2^*, \quad u_1 \otimes u_1^* - u_3 \otimes u_3^*, \\
v_1 \otimes v_1^* - v_2 \otimes v_2^*, \quad v_1 \otimes v_1^* - v_3 \otimes v_3^*, \\
w_1 \otimes w_1^* - w_2 \otimes w_2^*, \quad w_1 \otimes w_1^* - w_3 \otimes w_3^* \},
\]

again a six-dimensional abelian subalgebra. The intersection of both Cartan subalgebras is obviously zero and therefore our grading is special and nontoral. Similar computations prove that the rest of the homogeneous components are three-dimensional. The grading produced by any of the automorphisms \( \phi_i \) is of type (24, 27), since they are in the same conjugacy class. The grading produced by any of the three couples \( \{ \phi_i, \phi_j \} \) has one six-dimensional component and eight nine-dimensional ones. The grading induced by the three automorphisms together is of type (0, 0, 26). Now we must go down to see this grading in \( f_4 \).

To a certain extend, the nice 3-symmetry described in \( \mathfrak{e}_6 \) is inherited by \( f_4 \). Indeed graphically speaking, \( f_4 \) arises by folding \( \mathfrak{e}_6 \). More precisely, taking \( X_2 = X_3 \) we can consider on \( \mathfrak{e}_6 \) the unique automorphism \( \tau: \mathfrak{e}_6 \to \mathfrak{e}_6 \) extension of \( u \otimes v \otimes w \mapsto u \otimes w \otimes v \). This is an order two automorphism commuting with the previous \( \phi_i \) for \( i = 1, 2, 3 \). The subalgebra of elements fixed by \( \tau \) is

\[
\text{sl}(X_1) \oplus \text{sl}(X_2) \oplus X_1 \otimes \text{Sym}^2(X_2) \oplus X_1^* \otimes \text{Sym}^2(X_2^*),
\]

where \( \text{Sym}^nX_i \) denotes the symmetric powers (as in [16, p. 473]). This is a simple Lie algebra of dimension 52, hence \( f_4 \). Furthermore, denoting also by \( \phi_i: f_4 \to f_4 \) the restriction of the corresponding automorphisms of \( \mathfrak{e}_6 \) to fix \( \tau \), the set \( \{ \phi_i \}_{i=1}^3 \) is also a set of commuting semisimple order three automorphisms of \( f_4 \) with no fixed
points other than 0. So it induces a special nontoral $\mathbb{Z}_2^3$-grading on $f_4$ with all its homogeneous components of the same dimension ($52/26 = 2$), of type $(0, 26)$.

7.2. $\mathbb{Z}_2^3$-grading on $f_4$. The group of automorphisms inducing this grading is also an elementary $p$-group so that it is described in Gries work [18, Th. 7.3, p. 277]. Moreover, the grading is pure (that is, there is some homogeneous component which contains a Cartan subalgebra), therefore it also appears in Hesselink paper [23, Table 1, p. 146]. As before an instantaneous computer calculation provides its type, $(24, 0, 0, 7)$, as well as the description of its homogeneous components in terms of the fixed basis:

$$L_{0,0,0,0,0} = 0,$$
$$L_{0,0,0,0,1} = \langle b_2 + b_{26} \rangle,$$
$$L_{0,0,0,1,0} = \langle b_{41} - b_{17} \rangle,$$
$$L_{0,0,0,1,1} = \langle b_{43} - b_{19} \rangle,$$
$$L_{0,0,1,0,0} = \langle b_{38} - b_{14} \rangle,$$
$$L_{0,0,1,0,1} = \langle b_{35} - b_{11} \rangle,$$
$$L_{0,0,1,1,0} = \langle b_3 + b_{27}, b_{34} - b_{10}, b_{39} - b_{15}, b_{48} - b_{24} \rangle,$$
$$L_{0,0,1,1,1} = \langle b_{30} - b_6 \rangle,$$
$$L_{0,1,0,0,0} = \langle b_1 + b_{25} \rangle,$$
$$L_{0,1,0,0,1} = \langle b_5 + b_{29} \rangle,$$
$$L_{0,1,0,1,0} = \langle b_{28} - b_4, b_{30} - b_{16}, b_{44} - b_{20}, b_{22} + b_{46} \rangle,$$
$$L_{0,1,0,1,1} = \langle b_{15} - b_{21} \rangle,$$
$$L_{0,1,1,0,0} = \langle b_7 + b_{31}, b_{37} - b_{13}, b_{42} - b_{18}, b_{47} - b_{23} \rangle,$$
$$L_{0,1,1,0,1} = \langle b_3 + b_{28} \rangle,$$
$$L_{0,1,1,1,0} = \langle b_{36} - b_{12} \rangle,$$
$$L_{0,1,1,1,1} = \langle b_{33} - b_9 \rangle,$$
$$L_{1,0,0,0,0} = \langle t_{\alpha_1}, t_{\alpha_2}, t_{\alpha_3}, t_{\alpha_4} \rangle,$$
$$L_{1,0,0,0,1} = \langle b_{26} - b_2 \rangle,$$
$$L_{1,0,0,1,0} = \langle b_{17} + b_{41} \rangle,$$
$$L_{1,0,0,1,1} = \langle b_{19} + b_{43} \rangle,$$
$$L_{1,0,1,0,0} = \langle b_4 + b_{28}, b_{40} + b_{20} + b_{44}, b_{46} - b_{22} \rangle,$$
$$L_{1,0,1,0,1} = \langle b_{21} + b_{45} \rangle,$$
$$L_{1,0,1,1,0} = \langle b_7 - b_3, b_{10} - b_{34}, b_{15} + b_{39}, b_{24} + b_{48} \rangle,$$
$$L_{1,0,1,1,1} = \langle b_6 + b_{30} \rangle,$$
$$L_{1,1,0,0,0} = \langle b_{25} - b_1 \rangle,$$
$$L_{1,1,0,0,1} = \langle b_{29} - b_5 \rangle,$$
$$L_{1,1,0,1,0} = \langle b_4 + b_{28}, b_{16} + b_{40}, b_{20} + b_{44}, b_{46} - b_{22} \rangle,$$
$$L_{1,1,0,1,1} = \langle b_{21} + b_{45} \rangle,$$
$$L_{1,1,1,0,0} = \langle b_{31} - b_7, b_{13} + b_{37}, b_{18} + b_{42}, b_{23} + b_{47} \rangle,$$
$$L_{1,1,1,0,1} = \langle b_8 + b_{32} \rangle,$$
$$L_{1,1,1,1,0} = \langle b_{12} + b_{36} \rangle,$$
$$L_{1,1,1,1,1} = \langle b_9 + b_{33} \rangle.$$
given by
\[ J_{e,0,0} = \langle E_1, E_2, E_3 \rangle, \quad J_{g,0,0} = 0 (g \neq e), \]
\[ J_{g,0,1} = C^2_g, \quad J_{g,1,1} = C^2_{g^2}, \quad J_{g,1,0} = C_{g^3}, \]
with \( g, e = (0, 0, 0) \in \mathbb{Z}_2^3 \). Obviously the grading induced in \( L = \text{Der}(J) = [R_J, R_J] \)
has as homogeneous components \( L_a = \{ d \in \text{Der}(J) \mid d(J_b) \subset J_{a+b} \forall b \in \mathbb{Z}_2^5 \} \),
therefore \( L_{e,0,0} = 0 \) and
\[ L_{g,0,1} = \{ [R_{x^{(1)}}, R_{E_2-E_3}] \mid x \in C_g \} \]
\[ L_{g,1,1} = \{ [R_{x^{(2)}}, R_{E_3-E_1}] \mid x \in C^2_g \} \]
\[ L_{g,1,0} = \{ [R_{x^{(3)}}, R_{E_1-E_2}] \mid x \in C^2_g \} \quad \forall g \in \mathbb{Z}_2^3 \]
\[ L_{g,0,0} = \{ D_{U} \mid U \in N_g \} \oplus \{ D_{r_x} \mid x \in (C_0)_{g} \} \oplus \{ D_{l_x} \mid x \in (C_0)_{g} \} \quad \forall e \neq g \in \mathbb{Z}_2^3 \]
where
- \( N_g = \{ d \in \text{Der}(C) \mid d(C_b) \subset C_{g+b} \forall b \in \mathbb{Z}_2^3 \} \) are the components of the
  grading induced on \( g_2 \), all of them two-dimensional and Cartan subalgebras,
  except for \( N_e = 0 \),
- \( r_x \) and \( l_x \) are the right and left multiplication operators on \( C \),
- if \( U \in \text{o}(C, n) \), \( D_{U} \in \text{Der}(J) \) is the derivation given by
\[ E_i \mapsto 0, \quad x^{(1)} \mapsto U(x)^{(1)}, \quad x^{(2)} \mapsto U'(x)^{(2)}, \quad x^{(3)} \mapsto U''(x)^{(3)}, \]
where \( U' \) and \( U'' \) are the elements in \( \text{o}(C, n) \) given by the local trilality
principle [38, p. 88], that is, \( U(xy) = U'(xy) + xU''(y) \) for all \( x, y \in C \).

Then clearly \( h_1 = 7 \) (dim \( L_{g,0,0} = 2 + 1 + 1 = 4 \)), also \( h_1 = 8 \cdot 3 = 24 \) and the
grading is of type \((24, 0, 0, 7)\).

Anyway, we think that there is a more intuitive way of looking at this grading,
as well as at the gradings obtained by crossing gradings on the Cayley algebra \( H_3(F) \) with
gradings on \( H_3(F) \). Recall from 3.3 that if \( H \equiv H_3(F) = \{ x \in M_3(F) \mid x = x^t \} \)
and \( K \equiv K_3(F) = \{ x \in M_3(F) \mid x = -x^t \} \), we could write
\[ J = H \oplus K \oplus C_0. \]

Since \( f_4 = \text{Der}(J) \) there must exist some model of \( f_4 \) in these terms. In fact we can
see \( f_4 \) as
\[ L = \text{Der}(C) \oplus K \oplus H_0 \oplus C_0 \]
identifying the Lie algebra \( K \) (subalgebra of \( M_3(F) \) ) to \( \text{Der}(H_3(F)) \) in the known
Tits unified construction for the Lie exceptional algebras (for instance, see [38, p. 122]).

Consider a \( G_1 \)-grading on the Jordan algebra \( H = \oplus_{g \in G_1} H_g \). This grading will
come from a grading on \( M_3(F) \) so that the Lie algebra \( K \) will also have an induced
grading. Take now the \( \mathbb{Z}_2^3 \)-grading on the Cayley algebra \( C = \oplus_{g \in G_2 = \mathbb{Z}_2^3} C_g \) and the
induced grading \( \text{Der}(C) = \oplus_{g \in G_2 = \mathbb{Z}_2^3} N_g \). All this material induces a \( G_1 \times G_2 \)-grading
on \( J \) and also on \( L \) given by
\[ J_{g_1, e} = H_{g_1}, \quad J_{g_1, g_2} = K_{g_1} \otimes (C_0)_{g_2}, \]
\[ L_{g_1, e} = K_{g_1}, \quad L_{g_1, g_2} = N_{g_2} \oplus (H_0)_{g_1} \otimes (C_0)_{g_2}, \quad L_{g_1, g_2} = (H_0)_{g_1} \otimes (C_0)_{g_2}. \]

In the case of the \( \mathbb{Z}_2^3 \)-grading we have \( G_1 = \mathbb{Z}_2^3 \), with the gradings on \( H \) and \( K \)
given by
\[ H_{0,0} = \{ E_1, E_2, E_3 \} \quad H_{0,1} = \{ e_{12} + e_{21} \} \quad H_{1,1} = \{ e_{23} + e_{32} \} \quad H_{1,0} = \{ e_{13} + e_{31} \} \]
\[ K_{0,0} = 0 \quad K_{0,1} = \{ e_{12} - e_{21} \} \quad K_{1,1} = \{ e_{23} - e_{32} \} \quad K_{1,0} = \{ e_{13} - e_{31} \} \]
and \( \dim(C_0)_g = 1 \), \( \dim N_g = 2 \) for all \( g \in \mathbb{Z}_2^3 \setminus \{(0,0,0)\} \). Therefore

\[
\begin{align*}
\dim J_{e,e} &= \dim H_e = 3 & \dim L_{e,e} &= 0 \\
\dim J_{e,g_2} &= 0 & \dim L_{e,g_2} &= \dim(N_{g_2} + (H_0)_e \otimes (C_0)_{g_2}) = 4 \\
\dim J_{g_1,e} &= \dim H_{g_1} = 1 & \dim L_{g_1,e} &= \dim K_{g_1} = 1 \\
\dim J_{g_1,g_2} &= \dim K_{g_1} \otimes (C_0)_{g_2} = 1 & \dim L_{g_1,g_2} &= \dim(H_{g_1})_0 \otimes (C_0)_{g_2} = 1
\end{align*}
\]

and so the grading is of type \((24,0,1)\) on \( J \), and \((24,0,0,7)\) on \( L = f_4 \), as we knew from previous sections.

### 7.3. \( \mathbb{Z}_2^3 \times \mathbb{Z} \)-grading of \( f_4 \)

By contrast with previous gradings, as long as we know this one does not appear in the mathematical literature. Again a simple computer aided calculation reveals that the fine \( \mathbb{Z}_2^3 \times \mathbb{Z} \)-grading of type \((31,0,7)\) is

\[
\begin{align*}
L_{0,0,0,-2} &= 0, & L_{0,0,0,-1} &= (b_2 + b_3), \\
L_{0,0,0,0} &= (t_{a_2} + 2t_{a_3} + t_{a_4}), & L_{0,0,0,1} &= (b_{13} - b_{26}), \\
L_{0,0,0,2} &= 0, & L_{0,0,1,-2} &= (b_{22} + b_8), \\
L_{0,0,1,-1} &= (b_{25} - b_{12}), & L_{0,0,1,0} &= (b_{38} - b_{14}, b_{40} - b_{16}, b_{44} - b_{20}), \\
L_{0,0,1,1} &= (b_{15} - b_{11}), & L_{0,0,1,2} &= (b_{46} - b_4), \\
L_{0,1,0,-2} &= (b_{19} - b_{32}), & L_{0,1,0,1} &= (b_{29} + b_3), \\
L_{0,1,0,0} &= (b_{17} - b_{13}, b_{41} - b_{17}, b_{42} - b_{18}), & L_{0,1,1,-2} &= (b_{23} + b_{31}), \\
L_{0,1,1,-1} &= (b_{39} - b_{34}), & L_{0,1,1,0} &= (b_{37} - b_{13}, b_{41} - b_{17}, b_{42} - b_{18}), \\
L_{0,1,1,1} &= (b_8 + b_{43}), & L_{0,1,1,2} &= (b_{47} - b_7), \\
L_{1,0,0,-2} &= (b_{12}), & L_{1,0,0,1} &= (b_{26} + b_{33}), \\
L_{1,0,0,0} &= (t_{a_2} + \frac{t_{a_2}}{2} + t_{a_3} + t_{a_4}), & L_{1,0,0,1} &= (b_{28} - b_{22}), \\
L_{1,0,0,2} &= (b_{36}), & L_{1,0,1,-2} &= (b_{13} + b_{35}), \\
L_{1,0,1,-1} &= (b_{38} - b_{14}, b_{16} + b_{40}, b_{20} + b_{44}), & L_{1,0,1,0} &= (b_{14} + b_{38}, b_{16} + b_{40}, b_{20} + b_{44}), \\
L_{1,0,1,1} &= (b_{11} + b_{45}), & L_{1,0,1,2} &= (b_{4} + b_{46}), \\
L_{1,1,0,-2} &= (b_{16} + b_{1}), & L_{1,1,0,1} &= (b_{30} - b_{29}), \\
L_{1,1,0,0} &= (b_{25} - b_{1} - b_{27} - b_{3} + b_{24} - b_{48}), & L_{1,1,0,1} &= (b_{14} + b_{38}, b_{16} + b_{40}, b_{20} + b_{44}), \\
L_{1,1,0,2} &= (b_{34} + b_{39}), & L_{1,1,1,-2} &= (b_{31} - b_{23}), \\
L_{1,1,1,-1} &= (b_{32} - b_{19}), & L_{1,1,1,0} &= (b_{13} + b_{37}, b_{17} + b_{41}, b_{18} + b_{42}), \\
L_{1,1,1,1} &= (b_{43} - b_{8}), & L_{1,1,1,2} &= (b_{7} + b_{47}).
\end{align*}
\]

But we can detect this grading without reference to explicit computations using again the model \( f_4 = L = \text{Der}(C) \oplus H_0 \otimes C_0 \). We can write the \( \mathbb{Z} \)-grading on \( H \) given in (6), jointly with the induced one in \( K \) by its extension to \( M_3(F) \), in the following equivalent form:

\[
\begin{align*}
H_2 &= \langle e_{23} \rangle, & K_2 &= 0, \\
H_1 &= \langle e_{13} + e_{21} \rangle, & K_1 &= \langle e_{13} - e_{21} \rangle, \\
H_e &= \langle E_1, E_2 + E_3 \rangle, & K_e &= \langle E_2 - E_3 \rangle, \\
H_{-1} &= \langle e_{12} + e_{31} \rangle, & K_{-1} &= \langle e_{12} - e_{31} \rangle, \\
H_{-2} &= \langle e_{32} \rangle, & K_{-2} &= 0.
\end{align*}
\]
Thus, by crossing it with the \( \mathbb{Z}_3 \)-grading on \( C \) we obtain

\[
\begin{align*}
\dim J_{2,e} &= \dim H_2 = 1 & \dim L_{2,e} &= 0 \\
\dim J_{1,e} &= \dim H_1 = 1 & \dim L_{1,e} &= \dim K_1 = 1 \\
\dim J_{e,e} &= \dim H_e = 2 & \dim L_{e,e} &= \dim K_e = 1 \\
\dim J_{2,g} &= 0 & \dim L_{2,g} &= \dim H_2 \otimes (C_0)_g = 1 \\
\dim J_{1,g} &= \dim K_1 \otimes (C_0)_g = 1 & \dim L_{1,g} &= \dim H_1 \otimes (C_0)_g = 1 \\
\dim J_{e,g} &= \dim K_e \otimes (C_0)_g = 1 & \dim L_{e,g} &= \dim N_g \oplus (H_0)_e \otimes (C_0)_g = 3
\end{align*}
\]

and the same dimensions of the homogeneous components with opposite indices. So the grading is of type \((25, 1)\) on \( J \) and \((31, 0, 7)\) on \( f_4 \).

### 7.4. The remaining nontoral gradings

Any of the detected gradings on \( f_4 \) can be obtained by projections given by epimorphisms from the universal grading groups corresponding to the fine gradings. However, it is worth to point out that all of them can also be described by using different models of the algebra. More precisely, consider a \( G \)-grading on a simple Lie algebra \( L \) without outer automorphisms (equivalently, without automorphisms of the Dynkin diagram). Then \( G \) is a product of cyclic groups \( G_i \), each of which produces certain cyclic \( G_i \)-grading. The zero component of most of these \( G_i \)-gradings is a direct sum of Lie subalgebras of type either \( \text{sl}(V) \) or \( \text{so}(V) \). As modules for the zero component, the remaining components are isomorphic to either tensor of natural modules or spin ones, respectively (see [12] for more details). In the first case, it is possible to describe the \( G \)-grading in a similar way that just illustrated with the \( \mathbb{Z}_4 \)-grading on \( f_4 \). We now give some sketches of how this basis-free method works for the other gradings found in \( f_4 \), for instance the \( \mathbb{Z}_2 \times \mathbb{Z}_4 \) and \( \mathbb{Z}_2 \times \mathbb{Z}_8 \)-gradings.

Let \( V \) and \( W \) be \( F \)-vector spaces of dimensions 2 and 4 respectively. According to [12], \( f_4 \) can be seen in the way

\[
\mathcal{L} = \text{sl}(V) \oplus \text{sl}(W) \oplus V \otimes W \oplus \text{Sym}^2 V \otimes \bigwedge^2 W \oplus \bigwedge^3 W
\]

and its product given in a similar way that (28).

One of the advantages of this model is that given linear maps \( \varphi : V \to V \) and \( \tilde{\varphi} : W \to W \) so that the first one preserves \( \text{det} : \bigwedge^2 V \to F \) and the second preserves \( \text{det} : \bigwedge^4 W \to F \), the map \( \varphi \otimes \tilde{\varphi} : V \otimes W \to V \otimes W \) can be uniquely extended to an automorphism of the algebra \( \mathcal{L} \cong f_4 \), in such a way that the restriction of this automorphism to \( \text{sl}(V) \) is the conjugation \( g \mapsto \varphi g \tilde{\varphi}^{-1} \), and to \( \text{sl}(W) \) is the conjugation \( g \mapsto g\varphi g^{-1} \). Now, denote by \( \mathcal{L}_0 = \text{sl}(V) \oplus \text{sl}(W) \), \( \mathcal{L}_1 = V \otimes W \), \( \mathcal{L}_2 = \text{Sym}^2 V \otimes \bigwedge^2 W \) and \( \mathcal{L}_3 = V \otimes \bigwedge^3 W \). As \( \mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3 \) is a \( \mathbb{Z}_4 \)-grading, take \( \phi_1 \) the automorphism which gives the grading, that is, \( \phi_1|_{\mathcal{L}_i} = I_{\text{id}\mathcal{L}_i} \) where \( I \) is a primitive fourth root of the unit. Let us fix \( \{u_0, u_1\} \) a basis of \( V \) with \( \text{det}(u_0, u_1) = 1 \), and \( \{w_0, w_1, w_2, w_3\} \) a basis of \( W \) with \( \text{det}(w_0, w_1, w_2, w_3) = 1 \). Take \( \varphi_2 : V \to V, u_0 \mapsto Iu_0, u_1 \mapsto -Iu_1 \) and \( \tilde{\varphi}_2 : W \to W, w_0 \mapsto w_0, w_1 \mapsto w_1, w_2 \mapsto -w_2, w_3 \mapsto -w_3 \). Define \( \phi_2 \in \mathfrak{g}_4 \) as the extension of \( \varphi_2 \otimes \tilde{\varphi}_2 \). Take now \( \varphi_3 : V \to V, u_0 \mapsto u_1, u_1 \mapsto -u_0 \) and \( \tilde{\varphi}_3 : W \to W, w_0 \mapsto w_2, w_1 \mapsto w_3, w_2 \mapsto w_0, w_3 \mapsto w_1 \). Define then \( \phi_3 \in \mathfrak{g}_4 \) as the extension of \( \varphi_3 \otimes \tilde{\varphi}_3 \). Take finally \( \xi^8 = 1 \) a primitive eighth root, and define \( \tilde{\varphi}_4 : W \to W \) by \( \tilde{\varphi}_4(w_0) = \xi^3 w_2, \tilde{\varphi}_4(w_1) = \xi^3 w_3, \tilde{\varphi}_4(w_2) = \xi^3 w_0, \tilde{\varphi}_4(w_3) = \xi^8 w_1 \). Consider \( \phi_4 \in \mathfrak{g}_4 \) the extension of \( \varphi_3 \otimes \tilde{\varphi}_4 \). The set \( \{\phi_i\}_{i=1}^4 \) is a commutative set of semisimple automorphisms and one can see with some easy though boring hand computations that the grading induced
by \{\phi_1, \phi_2, \phi_3\} is a \mathbb{Z}_2^3 \times \mathbb{Z}_4\text{-grading of type } (0, 8, 2, 0, 6),\text{ the grading produced by } \{\phi_1, \phi_2, \phi_4\} \text{ is a } \mathbb{Z}_2^3 \times \mathbb{Z}_6\text{-grading of type } (19, 6, 7) \text{ and the grading produced by } \{\phi_1, \phi_2, \phi_3, \phi_4\} \text{ is a } \mathbb{Z}_2^3 \times \mathbb{Z}_8\text{-grading of type } (31, 0, 7), \text{ equivalent to the fine } \mathbb{Z}_2^3 \times \mathbb{Z}_2\text{-grading. We display their homogeneous components.}

The \(\mathbb{Z}_2^3 \times \mathbb{Z}_4\text{-grading induced by } \{\phi_1, \phi_2, \phi_3\}\text{ is:}

\[
\mathcal{L}_0 = \begin{cases} 
L_{1,1,1} &= \text{diag}(a, a)_W \quad \text{dim } 3 \\
L_{1,1,-1} &= \text{diag}(1, -1)_V \oplus \text{diag}(b, -b)_W \quad \text{dim } 5 \\
L_{1,-1,1} &= \text{antidiag}(1, -1)_V \oplus \text{antidiag}(b, b)_W \quad \text{dim } 5 \\
L_{1,-1,-1} &= \text{antidiag}(1, 1)_V \oplus \text{antidiag}(b, -b)_W \quad \text{dim } 5 
\end{cases}
\]

with \(a \in \text{sl}(2)\) and \(b \in M_2(F),\)

\[
\mathcal{L}_1 = \begin{cases} 
L_{1,1,1} &= (u_0 \otimes w_0 - I u_1 \otimes w_2, u_0 \otimes w_1 - I u_1 \otimes w_3) \quad \text{dim } 2 \\
L_{1,1,-1} &= (u_0 \otimes w_0 + I u_1 \otimes w_2, u_0 \otimes w_1 + I u_1 \otimes w_3) \quad \text{dim } 2 \\
L_{1,-1,1} &= (u_0 \otimes w_2 - I u_1 \otimes w_0, u_0 \otimes w_3 - I u_1 \otimes w_1) \quad \text{dim } 2 \\
L_{1,-1,-1} &= (u_0 \otimes w_2 + I u_1 \otimes w_0, u_0 \otimes w_3 + I u_1 \otimes w_1) \quad \text{dim } 2 
\end{cases}
\]

\[
\mathcal{L}_2 = \begin{cases} 
L_{-1,1,1} &= ((u_0 \cdot u_0 + u_1 \cdot u_1) \otimes (w_0 \wedge w_3 - w_1 \wedge w_2), \quad \text{dim } 5 \\
(u_0 \cdot u_0 - u_1 \cdot u_1) \otimes (w_0 \wedge w_3 + w_1 \wedge w_2, w_0 \wedge w_2, w_1 \wedge w_3), \quad \text{dim } 5 \\
u_0 \cdot u_1 \otimes (w_0 \wedge w_1 + w_2 \wedge w_3) \quad \text{dim } 5 
\end{cases}
\]

\[
\mathcal{L}_3 = \begin{cases} 
L_{-1,-1,1} &= ((u_0 \cdot u_0 + u_1 \cdot u_1) \otimes (w_0 \wedge w_3 + w_1 \wedge w_2, w_0 \wedge w_2, w_1 \wedge w_3), \quad \text{dim } 5 \\
(u_0 \cdot u_0 - u_1 \cdot u_1) \otimes (w_0 \wedge w_1 - w_2 \wedge w_3), \quad \text{dim } 5 \\
u_0 \cdot u_1 \otimes (w_0 \wedge w_1 + w_2 \wedge w_3) \quad \text{dim } 3 
\end{cases}
\]

and \(\mathcal{L}_3\) dual to \(\mathcal{L}_1,\) hence \(h_5 = 6, h_3 = 2\) and \(h_6 = 8,\) as we wondered. The notation \(\text{antidiag}(x_1, \ldots, x_n)\) stands for the \(n \times n\) matrix \((a_{ij})\) with all entries zero except for \(a_{i,n-i+1} = x_i.\) The subindices in \(L_{ijk}\) indicate that \(\phi_1, \phi_2, \phi_3\) act with eigenvalues \(i, j, k\) respectively. Notice that, although \(\phi_i\) has order 4 for \(i = 1, 2, 3, \phi_1 \phi_2\) and \(\phi_1 \phi_3\) have order 2.

The \(\mathbb{Z}_2^3 \times \mathbb{Z}_8\text{-grading induced by } \{\phi_1, \phi_4, \phi_2\}\) is:

\[
\mathcal{L}_0 = \begin{cases} 
L_{1,1,1} &= \langle \text{diag}(1, -1, 1, -1) \rangle_W \quad \text{dim } 1 \\
L_{1,1,-1} &= \langle \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right) \rangle_V \oplus \langle \text{antidiag}(a, b, a, b) \rangle_W \quad \text{dim } 3 \\
L_{1,-1,1} &= \langle \text{antidiag}(a, a, b, b) \rangle_W \quad \text{dim } 2 \\
L_{1,-1,-1} &= \langle \text{antidiag}(a, b, a, b) \rangle_W \quad \text{dim } 3 
\end{cases}
\]

\[
\mathcal{L}_1 = \begin{cases} 
L_{1,1,1} &= \langle \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right) \rangle_V \oplus \langle \text{antidiag}(a, a, b, b) \rangle_W \quad \text{dim } 3 \\
L_{1,-1,1} &= \langle \left(\begin{array}{cc} a & 0 \\ 0 & -a \end{array}\right) \rangle_W \quad \text{dim } 3 \\
L_{1,-1,-1} &= \langle \left(\begin{array}{cc} 0 & a \\ b & 0 \end{array}\right) \rangle_W \quad \text{dim } 2 
\end{cases}
\]

\[
\mathcal{L}_2 = \begin{cases} 
L_{1,1,1} &= \langle \left(\begin{array}{cc} 0 & a \\ b & 0 \end{array}\right) \rangle_W \quad \text{dim } 2 \\
L_{1,-1,1} &= \langle \left(\begin{array}{cc} 0 & b \\ a & 0 \end{array}\right) \rangle_W \quad \text{dim } 2 
\end{cases}
\]
with \(a, b \in F\),

\[
\mathcal{L}_1 = \begin{cases} 
L_{1,\xi^0,-1} &= \langle u_0 \otimes w_3 - I u_1 \otimes w_1 \rangle \tag{dim 1} \\
L_{1,\xi^0,1} &= \langle u_0 \otimes w_2 - I u_1 \otimes w_0 \rangle \tag{dim 1} \\
L_{1,\xi^1,-1} &= \langle u_0 \otimes w_1 + I u_1 \otimes w_2 \rangle \tag{dim 1} \\
L_{1,\xi^1,1} &= \langle u_0 \otimes w_2 + I u_1 \otimes w_0 \rangle \tag{dim 1} \\
L_{1,\xi^2,-1} &= \langle u_0 \otimes w_1 - I u_1 \otimes w_3 \rangle \tag{dim 1} \\
L_{1,\xi^2,1} &= \langle u_0 \otimes w_3 - I u_1 \otimes w_1 \rangle \tag{dim 1} \\
L_{1,\xi^3,-1} &= \langle u_0 \otimes w_1 + I u_1 \otimes w_3 \rangle \tag{dim 1} \\
L_{1,\xi^3,1} &= \langle u_0 \otimes w_3 + I u_1 \otimes w_1 \rangle \tag{dim 1} \\
\end{cases}
\]

\[
\mathcal{L}_2 = \begin{cases} 
L_{-1,1,1} &= \langle u_0 \cdot u_0 + u_1 \cdot u_1 \rangle \otimes (w_0 \wedge w_3 - w_1 \wedge w_2), \tag{dim 1} \\
& \quad \langle u_0 \cdot u_1 \otimes (w_0 \wedge w_1 - w_2 \wedge w_3), \langle u_0 \cdot u_0 - u_1 \cdot u_1 \rangle \otimes (w_0 \wedge w_3 + w_1 \wedge w_2) \rangle \tag{dim 3} \\
L_{-1,1,-1} &= \langle u_0 \cdot u_0 + u_1 \cdot u_1 \rangle \otimes (w_0 \wedge w_1 + w_2 \wedge w_3), \tag{dim 3} \\
& \quad \langle u_0 \cdot u_1 \otimes (w_0 \wedge w_3 + w_1 \wedge w_2), \langle u_0 \cdot u_0 - u_1 \cdot u_1 \rangle \otimes (w_0 \wedge w_1 - w_2 \wedge w_3) \rangle \tag{dim 3} \\
L_{-1,1,1} &= \langle u_0 \cdot u_0 + u_1 \cdot u_1 \rangle \otimes (w_0 \wedge w_3 + w_1 \wedge w_2), \langle u_0 \cdot u_1 \otimes (w_0 \wedge w_1 + w_2 \wedge w_3), \langle u_0 \cdot u_0 - u_1 \cdot u_1 \rangle \otimes (w_0 \wedge w_3 - w_1 \wedge w_2) \rangle \tag{dim 3} \\
& \quad \langle u_0 \cdot u_1 \otimes (w_0 \wedge w_3 - w_1 \wedge w_2), \langle u_0 \cdot u_0 - u_1 \cdot u_1 \rangle \otimes (w_0 \wedge w_1 - w_2 \wedge w_3) \rangle \tag{dim 3} \\
L_{-1,1,-1} &= \langle u_0 \cdot u_0 + u_1 \cdot u_1 \rangle \otimes (w_0 \wedge w_3 + w_1 \wedge w_2), \tag{dim 2} \\
& \quad \langle u_0 \cdot u_1 \otimes (w_0 \wedge w_1 + w_2 \wedge w_3), \langle u_0 \cdot u_0 - u_1 \cdot u_1 \rangle \otimes (w_0 \wedge w_3 + w_1 \wedge w_2) \rangle \tag{dim 3} \\
& \quad \langle u_0 \cdot u_1 \otimes (w_0 \wedge w_3 - w_1 \wedge w_2), \langle u_0 \cdot u_0 - u_1 \cdot u_1 \rangle \otimes (w_0 \wedge w_1 - w_2 \wedge w_3) \rangle \tag{dim 3} \\
L_{-1,1,1} &= \langle u_0 \cdot u_0 + u_1 \cdot u_1 \rangle \otimes (w_0 \wedge w_3 + w_1 \wedge w_2), \tag{dim 2} \\
& \quad \langle u_0 \cdot u_1 \otimes (w_0 \wedge w_1 + w_2 \wedge w_3), \langle u_0 \cdot u_0 - u_1 \cdot u_1 \rangle \otimes (w_0 \wedge w_3 + w_1 \wedge w_2) \rangle \tag{dim 3} \\
& \quad \langle u_0 \cdot u_1 \otimes (w_0 \wedge w_3 - w_1 \wedge w_2), \langle u_0 \cdot u_0 - u_1 \cdot u_1 \rangle \otimes (w_0 \wedge w_1 - w_2 \wedge w_3) \rangle \tag{dim 3} \\
L_{-1,1,-1} &= \langle u_0 \cdot u_0 + u_1 \cdot u_1 \rangle \otimes (w_0 \wedge w_3 + w_1 \wedge w_2), \tag{dim 1} \\
& \quad \langle u_0 \cdot u_1 \otimes (w_0 \wedge w_1 + w_2 \wedge w_3), \langle u_0 \cdot u_0 - u_1 \cdot u_1 \rangle \otimes (w_0 \wedge w_3 + w_1 \wedge w_2) \rangle \tag{dim 1} \\
& \quad \langle u_0 \cdot u_1 \otimes (w_0 \wedge w_3 - w_1 \wedge w_2), \langle u_0 \cdot u_0 - u_1 \cdot u_1 \rangle \otimes (w_0 \wedge w_1 - w_2 \wedge w_3) \rangle \tag{dim 1} \\
\end{cases}
\]

and \(\mathcal{L}_3\) dual to \(\mathcal{L}_1\), hence \(b_3 = 7, b_2 = 6\) and \(b_1 = 19\), and so this is the grading we are looking for.

Along this lines all the gradings can be located by using models of \(\mathfrak{f}_4\).

**Appendix.**

Some results which play a fundamental roll in our work are included here. Their references may not be so easily accessible and so we state the results (without proof) for the seek of self-containedness. So for instance the well known Borel-Serre theorem for Lie groups has a version for algebraic groups which is owed to V. P. Platonov in the following terms:

**Theorem 6.** ([37, Theorem 3.15, p. 92]) A supersoluble subgroup of semisimple elements of an algebraic group \(G\) is contained in the normalizer of a maximal torus.

Here we must recall that a group is called supersolvable (or supersoluble) if it has an invariant normal series whose factors are all cyclic. Any finitely generated abelian group is supersolvable.

Another result that we are applying since the beginning is related to the number of generators of the quasitorus inducing a grading on \(\mathfrak{f}_4\). By abuse of notation we speak of the number of generators of a quasitorus \(Q\) instead of the number of
generators of the related finitely generated abelian group \( \mathfrak{X}(Q) \). When this number is \( \leq 2 \) we can say for sure that the grading is toral. A first approach to this is the fact that every cyclic grading on \( f_4 \) is toral. Indeed, a grading is cyclic if it is induced by a diagonalizable automorphism \( f \) of \( \mathfrak{g}_4 \). This is a semisimple element and since \( \mathfrak{g}_4 \) is a connected algebraic group, \( f \) is in a maximal torus of \( \mathfrak{g}_4 \) [10, Theorem 11.10, p.151]. It is possible to strengthen the previous result to the case in which the grading group has two generators (stated, for instance, in [2, Lemma 1.1.3, p.5]).

**Lemma 2.** Every subquasitorus \( Q \) of \( \mathfrak{g}_4 \) such that \( \mathfrak{X}(Q) \) has two generators is toral.

Proof. It is known ([17, Th3.5.6,p.93]) that if \( G \) is a connected reductive group whose derivated subgroup is simply connected, then the centralizer of every semisimple element in \( G \) is connected. Let \( Q = \langle f_1, f_2 \rangle \) be a (closed) abelian subgroup of semisimple elements in \( \mathfrak{g}_4 \). As any semisimple element belongs to a torus, we replace \( f_1 \) by \( t_1 \in T \) by conjugation. Now take \( Z = C_{\mathfrak{g}_4}(t_1) \), which is a connected group. Applying [13, Theorem 1, p.94] for \( n = 1 \) we finish the proof. But without using this fact, we can go on considering that the element \( f_2 \in Z \) must be in some maximal torus of \( Z \), say \( T \). But \( t_1 \) is in the center of \( Z \) and hence in all the maximal torus of \( Z \). We have finished since \( \langle t_1, f_2 \rangle \subset T \), implying \( Q \subset T \). □

We finish this appendix describing the action by conjugation of some elements in \( \mathfrak{g}_4 \) on the automorphisms \( g_1 := t'_{1,-1,1,-1}; g_2 := t'_{1,-1,1,1}; g_3 := \sigma_{105} \) and \( g_4 := t'_{1,1,1,-1} \).

| Element \( f \) | \( fg_1f^{-1} \) | \( fg_2f^{-1} \) | \( fg_3f^{-1} \) |
|----------------|----------------|----------------|----------------|
| \( \sigma_{94} \) | \( g_1g_2 \) | \( g_2g_4 \) | \( g_3 \) |
| \( \sigma_{103} \) | \( g_1g_4 \) | \( g_2 \) | \( g_3 \) |
| \( \sigma_{468} \) | \( g_2 \) | \( g_1 \) | \( g_3 \) |
| \( \sigma_{385} \) | \( g_2 \) | \( g_1 \) | \( g_3g_4 \) |
| \( \sigma_{391} \) | \( g_1g_2 \) | \( g_1g_4 \) | \( g_3g_4 \) |
| \( \psi \) | \( g_1 \) | \( g_3 \) | \( g_2 \) |

Moreover \( ft'_{1,1,1,u}f^{-1} = t'_{1,1,1,u} \) for any \( u \in F^\times \) and \( f \) in the first column on the left of the table. We have employed this information to find the nontoral subquasitori of \( A(105, \text{id}) \).

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