The von Karman equations, the stress function, and elastic ridges in high dimensions.

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Abstract

The elastic energy functional of a thin elastic rod or sheet is generalized to the case of an $M$-dimensional manifold in $N$-dimensional space. We derive potentials for the stress field and curvatures and find the generalized von Karman equations for a manifold in elastic equilibrium. We perform a scaling analysis of an $M-1$ dimensional ridge in an $M = N - 1$ dimensional manifold. A ridge of linear size $X$ in a manifold with thickness $h \ll X$ has a width $w \sim h^{1/3}X^{2/3}$ and a total energy $E \sim \mu h^M(X/h)^{M-5/3}$, where $\mu$ is a stretching modulus. We also prove that the total bending energy of the ridge is exactly five times the total stretching energy. These results match those of A. Lobkovsky [Phys. Rev. E 53, 3750 (1996)] for the case of a bent plate in three dimensions.
I. INTRODUCTION

The crumpling of a thin elastic sheet is mediated by the formation of a network of narrow ridges \[1\]. Plastic deformation of the material in the neighborhood of these ridges leads to the ubiquitous linear scars in crushed paper, aluminum foil, and car bodies \[2, 3\]. It was recently discovered by Witten, Lobkovsky, and others that this phenomenon can be accounted for using linear elasticity theory \[4–7\]. The scaling laws for a ridge were first derived by Witten and Li using a Flory type argument. A ridge of length $X$ in a sheet of thickness $h$ and Young’s modulus $Y$ was found to have a total elastic energy $E \sim Y h^3 (X/h)^{1/3}$ and a width $w \sim h^{1/3} X^{2/3}$. Lobkovsky confirmed this result with a scaling analysis of the von Karman equations describing a thin, semi-infinite strip with a single ridge. He also verified these results with detailed simulations.

One important result of this analysis is the discovery that the stresses and curvatures decay rapidly to zero in the direction transverse to the ridge. The length scale of this decay is the ridge width. We therefore interpret ridge formation as a confinement \[8\] of the elastic stress field. Although there is a qualitative appreciation that confinement is the result of the competition between the in-plane strains and the curvatures of the plate, a deeper theoretical understanding is still lacking. In particular there is no proof of confinement under generic boundary conditions. We have been working towards this goal. In a companion paper we prove that a thin plate must have regions of nonzero strain if it is to fit into a small sphere \[9\]. It remains to be shown that these regions necessarily assemble themselves into a network of ridges.

To understand the causes and consequences of stress confinement, it is useful to examine the higher dimensional analogs of a crumpled sheet. For example, a thin plate crumpled in four dimensions doesn’t have to stretch, and we expect there is no stress confinement in this case. In this paper we present the simplest field theory describing the strains and curvatures of a deformed elastic manifold in higher dimensions. We also perform a scaling analysis for a ridge in these systems.
There is considerable precedent for examining higher dimensional systems for insights into membrane elasticity. Most notable are studies of the so-called “crumpling transition” of a thin elastic membrane in thermal equilibrium [10–12]. That work focuses on the way thermal fluctuations and self-avoidance renormalize the elastic constants of a thin elastic sheet. The field theory is often developed in perturbation theory around a convenient, higher dimension. In this paper we work exclusively at zero temperature and the word “crumpling” refers to compression by external forces.

In elasticity theory it is common to approximate a thin plate by its center surface, or centroid [13,14]. A three-dimensional plate is thereby described using a two-dimensional manifold. The elastic energy functional for the centroid is found by integrating out the components of the stress and strain tensors which are transverse to the long directions. The mathematical analysis of these approximations and their range of validity is the subject of the theory of thin elastic shells [15–17]. In this paper we use the standard methods of shell theory to derive the elastic energy of an M-dimensional manifold embedded in N-dimensional space. We treat the manifold as the centroid of an N-dimensional elastic solid with an infinitesimal thickness $h$ in $N - M$ directions. The resulting energy functional has pieces quadratic in the strains, curvatures, and torsions of the manifold. We take a functional derivative of the energy to find the equations of static equilibrium. For a plate in three dimensions these equations are called the von Karman equations, first written down by Theodore von Karman in 1910 [18]. We will refer to our general result by the same name. Our derivation has several new elements.

It is noteworthy that the elastic energy of a thin plate may be written in terms of two scalar potentials. The stress function $\chi$, introduced by Airy in 1863, is the source of in-plane stresses [19,20]. The bending potential $f$ is the source of curvatures. Similarly, studies of a deformed solid often use the tensor stress function $\chi_{\alpha\beta}$, introduced by Maxwell in 1870 [21,22]. In this paper we present the generalization of these potentials for arbitrary $M$ and $N$. Our derivation of the von Karman equations reveals a role for the stress function as the Lagrange multiplier of a geometric constraint in the energy functional. For $M > 2$ the stress
function is a gauge field.

Next we turn our attention to the ridge structure. The width of the ridge in a plate scales like $w \sim h^{1/3}X^{2/3}$, so in the limit $h \ll X$ the ridge is approximately one-dimensional and straight. Analogously, the ridges in an $M$-dimensional manifold are expected to be approximately $(M-1)$-dimensional and to have no curvature. Indeed, simulations of a solid elastic ball crushed by a sphere in four dimensions show that the elastic energy is concentrated into flat, planar structures [1]. As mentioned above, Lobkovsky has done a thorough analysis of an isolated ridge for the case of a semi-infinite sheet in three dimensions [6]. We repeat his analysis for the case of an $M$-dimensional, semi-infinite manifold bent into a ridge in $M+1$-dimensional space. We find that a ridge with linear size $X$ has a width $w \sim h^{1/3}X^{2/3}$ and a total energy $E \sim Yh^N(X/h)^{M-5/3}$. Lastly, we prove that the total energy due to the curvature of the ridge is exactly five times the energy due to the strains. These results match correctly onto the solution for a bent plate in three dimensions [0,23].

In Sec. II we review the differential geometry of a weakly strained $M$-dimensional manifold embedded in $N$-dimensional space. In Sec. III we derive the elastic energy of this manifold as the thin limit of an $N$-dimensional elastic solid. In Sec. IV we present the generalization of the stress function and the bending potential. Then we make a variational derivation of the von Karman equations. In Sec. V we generalize the scaling analysis of Lobkovsky to a ridge in $M > 2$. In Sec. VI we summarize our conclusions.

II. DIFFERENTIAL GEOMETRY REVIEW

In this section we review the differential geometry of a weakly strained $M$-dimensional manifold $\mathcal{M}$ embedded in $N$-dimensional space $\mathbb{R}^N$. By weak strains we mean (1) the strains are small compared to unity and (2) the derivatives of the strains are small compared to the other relevant inverse lengths (curvatures and torsions). These are the usual assumptions of thin plate theory [14]. Note that they do not prohibit arbitrarily large deformations of the manifold. With these assumptions the math simplifies considerably. A treatment of the
topics in this section using the full apparatus of differential geometry may be found in Refs. [24,25].

The manifold is flat in the absence of external forces, so it can be parameterized by the Euclidean coordinate patch \( \{ \vec{x} = x_i \hat{e}_i \in \mathcal{M} \text{ for } i \in [1,M] ; x_i = 0 \text{ for } i \in [M+1,N] \} \) where \( \{ \hat{e}_i \} \) are the Euclidean basis vectors. We will refer to \( (x_\alpha; \alpha \in [1,M]) \) as the manifold coordinate patch and denote it with Greek subscripts. Any deformation of \( \mathcal{M} \) can then be represented as a continuous map \( \vec{r}(x_\alpha) \) from the manifold coordinates to \( \mathbb{R}^N \). At each point on the deformed manifold there is an \( M \)-dimensional tangent space spanned by the tangent vectors \( \vec{t}_\alpha = \partial_\alpha \vec{r} \). The metric on the manifold is then \( g_{\alpha\beta} = \vec{t}_\alpha \cdot \vec{t}_\beta \) and the strain tensor is \( u_{\alpha\beta} = (1/2)(g_{\alpha\beta} - \delta_{\alpha\beta}) \).

We treat all relevant quantities to lowest order in the strains. This immediately gives us \( g_{\alpha\beta} = \delta_{\alpha\beta} + O(u) \), so there is no need to distinguish the covariant components of a tensor from the contravariant components. The Christoffel symbols are \( \Gamma^\gamma_{\alpha\beta} = \partial_\beta u_{\alpha\gamma} + \partial_\alpha u_{\beta\gamma} - \partial_\gamma u_{\alpha\beta} + O(u^2) \) [25]. Thus, \( D_\alpha = \partial_\alpha + O(u) \) and covariant derivatives are just partial derivatives to leading order. Geodesics are approximately straight lines in the manifold coordinates.

The extrinsic curvature tensor for the manifold is defined \( \vec{K}_{\alpha\beta} = D_\alpha \vec{t}_\beta \approx \partial_\alpha \vec{t}_\beta \). It is straightforward to show that the components of this tensor are normal to the tangent space. We start with

\[
\vec{t}_\alpha \cdot \vec{K}_{\beta\gamma} = \vec{t}_\alpha \cdot \partial_\beta \vec{t}_\gamma \\
= \partial_\beta (\vec{t}_\alpha \cdot \vec{t}_\gamma) - \partial_\gamma (\vec{t}_\alpha \cdot \vec{t}_\gamma) \\
= \partial_\beta (\delta_{\alpha\gamma} - \vec{K}_{\alpha\beta} \cdot \vec{t}_\gamma) \\
= -\vec{t}_\gamma \cdot \vec{K}_{\alpha\beta}
\]

This quantity is therefore odd under a cyclic permutation of the indices. Three such permutations gives \( \vec{t}_\alpha \cdot \vec{K}_{\beta\gamma} = -\vec{t}_\alpha \cdot \vec{K}_{\gamma\beta} = 0 \).

We choose a set of orthonormal basis vectors \( \{ \hat{n}^{(\alpha)}(x_\beta) \} \) to span the \( (N-M) \)-dimensional normal space at each point on the manifold. Note that we use Greek-in-parenthesis for the normal index \( (\alpha) \in [M+1,N] \). In this basis the extrinsic curvature tensor becomes
\[ \vec{K}_{\alpha\beta} = C_{\alpha\beta}^{(\gamma)} \hat{n}^{(\gamma)} \] where \( C_{\alpha\beta}^{(\gamma)} = \hat{n}^{(\gamma)} \cdot \vec{K}_{\alpha\beta} \) and summation over repeated indices is implied. We will refer to the \( N - M \) tensors \( C_{\alpha\beta}^{(\gamma)} \) as the normal components of the extrinsic curvature tensor. Note that \( C_{\alpha\beta}^{(\gamma)} = C_{\beta\alpha}^{(\gamma)} \) and \( \vec{K}_{\alpha\beta} = \vec{K}_{\beta\alpha} \) since \( \vec{K}_{\alpha\beta} \approx \partial_{\alpha} \partial_{\beta} \vec{r} \).

It is useful to expand the derivatives of the normal vectors in the full basis \( \{ \vec{t}_{\alpha}, \hat{n}_{(\alpha)} \} \)

\[ \partial_{\alpha} \hat{n}^{(\beta)} = -C_{\alpha\gamma}^{(\beta)} \vec{t}_{\gamma} - \tau_{\alpha}^{(\beta)(\gamma)} \hat{n}^{(\gamma)} \tag{2} \]

where we have defined the torsions \( \tau_{\alpha}^{(\beta)(\gamma)} = -\hat{n}^{(\gamma)} \cdot \partial_{\alpha} \hat{n}^{(\beta)} \) and where we have used \( 0 = \partial_{\alpha} (\vec{t}_{\gamma} \cdot \hat{n}^{(\beta)}) = \vec{t}_{\gamma} \cdot \partial_{\alpha} \hat{n}^{(\beta)} + C_{\alpha\gamma}^{(\beta)} \). Eq. (2) is the generalization of the Weingarten map for a plate [23]. Taking one derivative of \( \hat{n}^{(\alpha)} \cdot \hat{n}^{(\beta)} = \delta^{(\alpha)(\beta)} \) gives us the antisymmetry property for the torsions \( \tau_{\alpha}^{(\beta)(\gamma)} = -\tau_{\alpha}^{(\gamma)(\beta)} \).

The last quantity we will need is the intrinsic curvature tensor \( R_{\alpha\beta\mu\nu} = \vec{K}_{\alpha\mu} \cdot \vec{K}_{\beta\nu} - \vec{K}_{\alpha\nu} \cdot \vec{K}_{\beta\mu} \tag{3} \]. In the normal basis, this becomes

\[ R_{\alpha\beta\mu\nu}[C] = C_{\alpha\mu}^{(\gamma)} C_{\beta\nu}^{(\gamma)} - C_{\alpha\nu}^{(\gamma)} C_{\beta\mu}^{(\gamma)} \]

The intrinsic curvature tensor is related to the strain tensor via the generalization of Gauss’ Theorema Egregium

\[ R_{\alpha\beta\mu\nu}[u] = -u_{\alpha\mu,\beta\nu} + u_{\alpha\nu,\beta\mu} - u_{\beta\nu,\alpha\mu} + u_{\beta\mu,\alpha\nu} + O(u^2) \tag{4} \]

where indices to the right of a comma indicate partial derivatives. Eqs. (3) and (4) together give \( R_{\alpha\beta\mu\nu}[C] = R_{\alpha\beta\mu\nu}[u] \). This is one version of the geometric von Karman equation, so called because it expresses the geometric constraint relating the extrinsic curvature and the strain. It is straightforward to verify this equation by substituting the definition of the strain tensor into \( R_{\alpha\beta\mu\nu}[u] \) and differentiating.

The intrinsic curvature tensor will be most useful to us in the linear combination

\[ G_{\alpha\beta} = R_{\alpha\nu\beta\nu} - \frac{1}{2} \delta_{\alpha\beta} R_{\mu\nu\mu\nu} \tag{5} \]

This is the Einstein curvature tensor, familiar from general relativity [24]. It is symmetric \( G_{\alpha\beta} = G_{\beta\alpha} \) and satisfies the conservation law \( \partial_{\alpha} G_{\alpha\beta} = 0 \). Taking the appropriate contractions of the geometric von Karman equation gives us
III. THE ELASTIC ENERGY FUNCTIONAL

In this section we obtain an expression for the elastic energy of an $M$-dimensional manifold $\mathcal{M}$ via the thin limit of an $N$-dimensional solid $\mathcal{N}$. Versions of this calculation for a rod and plate may be found in Refs. [13] and [14].

To begin we consider the elastic energy functional for an arbitrary $N$-dimensional solid. We keep the assumption of small strains used in the last section, but we relax the condition on the derivatives. As before, there is a Euclidean coordinate patch $(x_i)$ covering the undeformed manifold $\{\vec{x} = x_i \hat{e}_i \in \mathcal{N} \text{ for } i \in [1, N]\}$. This is the *material* coordinate patch, denoted by Latin indices. Under the application of external forces the solid assumes an embedding $\vec{r}(x)$. The tangent space of the solid is the full $\mathbb{R}^N$ and the tangent vectors are $\vec{t}_i(x) = \partial_i \vec{r}$. The metric is $g_{ij}(x) = \vec{t}_i \cdot \vec{t}_j$ and the strain tensor is $u_{ij} = (1/2)(g_{ij} - \delta_{ij})$.

Two consequences of $u_{ij} \ll 1$ are (1) the volume element of the deformed solid $d\vec{r}^N(x)$ is well-approximated by the volume element of the undeformed solid $dx^N$ and (2) the elastic energy of the material only needs to be calculated to second order in the strains. The most general energy functional quadratic in the strains and consistent with an isotropic material is

$$E[u] = \int_{\mathcal{N}} dx^N \left( \mu u_{ij}^2 + \frac{\lambda}{2} u_{ii}^2 \right)$$

(7)

where $\mu$ and $\lambda$ are the Lamé coefficients [14]. It is useful to rewrite this equation $E = (1/2) \int dx^N \sigma_{ij} u_{ij}$ where

$$\sigma_{ij}(x) = 2\mu u_{ij} + \lambda \delta_{ij} u_{kk}$$

(8)

is the stress field conjugate to $u_{ij}$. The stress field satisfies the conservation law $D_i \sigma_{ij} = 0$.

By analogy with the treatments of a rod and sheet, we assume that $\mathcal{N} = \mathcal{M} \times B^{N-M}(h)$ where $B^{N-M}(h)$ is an $(N-M)$-ball of infinitesimal radius $h$. The choice of a spherical “cross-section” is important to preserve the full rotational symmetry in the normal space of $\mathcal{M}$. The internal energy generated by the stress is

$$E_{\text{int}} = \int_{\mathcal{M}} \frac{1}{2} \sigma_{ij} n^i n^j$$

(9)

where $\sigma_{ij}$ is the stress tensor, $n^i$ is the outward normal to $\mathcal{M}$, and $\mathcal{M}$ is the boundary of $\mathcal{N}$. The external work $W_{\text{ext}}$ is the work done by external forces on the solid, and the total energy $E_{\text{total}}$ is

$$E_{\text{total}} = E_{\text{int}} + W_{\text{ext}}.$$

The fundamental problem is to determine the internal energy $E_{\text{int}}$ and the work done by external forces $W_{\text{ext}}$.
The material coordinate patch becomes \( \{ \vec{x} = x_i \hat{e}_i \in \mathcal{M} \, \forall \, i \in [1, M] \, \colon \sum_{i=M+1}^{N} x_i^2 \leq h^2 \} \).

We refer to the long directions as the manifold coordinates and denote them with Greek indices. We refer to the short directions as the normal or transverse coordinates and denote them with Greek-in-parenthesis. For clarity we relabel the transverse coordinates \( \zeta_{(\alpha)} = x_{(\alpha)} \), so that \( (x_i) = (x_1, x_2, \ldots x_M, \zeta_{(M+1)}, \zeta_{(M+2)}, \ldots \zeta_{(N)}) \).

The M-dimensional surface satisfying \( \zeta_{(\alpha)} = 0 \) is the centroid of \( \mathcal{N} \). When the transverse degrees of freedom are integrated out it is the centroid which becomes the manifold \( \mathcal{M} \).

Under the application of external forces the centroid deforms to some equilibrium embedding \( \vec{r}(x, \zeta = 0) \). All the quantities discussed in the previous section are well-defined with respect to this embedding. With the exception of \( C_{\alpha \beta}^{(\lambda)} \) and \( \tau_{(\sigma)(\lambda)}^{(\alpha)} \), we will denote quantities calculated on the centroid with a superscript \( c \).

The first step in deriving the elastic energy functional for the centroid is to make a Taylor expansion of the embedding \( \vec{r}(x, \zeta) \) in \( \zeta \)

\[
\vec{r}(x, \zeta) = \vec{r}^c(x) + \zeta_{(\mu)} \vec{a}^{(\mu)}(x) + \frac{1}{2} \zeta_{(\mu)} \zeta_{(\nu)} \vec{b}^{(\mu)(\nu)}(x) + \cdots
\]

where \( \vec{a}^{(\mu)}(x) = \partial_{(\mu)} \vec{r}(x, \zeta)|_{\zeta = 0} \) and \( \vec{b}^{(\mu)(\nu)}(x) = \partial_{(\mu)} \partial_{(\nu)} \vec{r}(x, \zeta)|_{\zeta = 0} \). Recall that in Sec. II we had the freedom to choose an arbitrary set of torsions due to the rotational symmetry of the normal space. Here we make the natural assignment \( \hat{n}^{(\mu)}(x) = \vec{a}^{(\mu)}/|\vec{a}^{(\mu)}| \). With this identification the torsions of the normal basis are the torsions of the deformed solid.

To calculate the energy we need to solve for \( \vec{a}^{(\mu)} \) and \( \vec{b}^{(\mu)(\nu)} \) in terms of \( u_{\alpha \beta}^{c}, C_{\alpha \beta}^{(\mu)}, \tau_{(\sigma)(\lambda)}^{(\alpha)} \), and their derivatives. We make the following assumptions: (1) \( u_{\alpha \beta}^{c} \ll 1, C_{\alpha \beta}^{(\lambda)} \ll 1/h \), and \( \tau_{(\sigma)(\lambda)}^{(\alpha)} \ll 1/h \). We will see that these are necessary to satisfy the small strain condition \( u_{ij} \ll 1 \). (2) The smallest length scale \( \ell \) over which the strains, curvatures, and torsions vary satisfies \( \ell \gg h \). We therefore write the most general expressions consistent with the rotational and reflection symmetries of the problem to first nontrivial order

\[
\vec{a}^{(\mu)} = (1 + a_1 u_{\alpha \alpha}^{c}) \hat{n}^{(\mu)}
\]

\[
\vec{b}^{(\mu)(\nu)} = b_1 \left( C_{\alpha \alpha}^{(\mu)} \hat{n}^{(\nu)} + C_{\alpha \alpha}^{(\nu)} \hat{n}^{(\mu)} \right) + b_2 \delta^{(\mu)(\nu)} \left( C_{\alpha \alpha}^{(\lambda)} \hat{n}^{(\lambda)} \right)
\]
where $a_1$, $b_1$, and $b_2$ are dimensionless constants to be determined. The corrections are of relative order $O(u, hC, h\tau, \text{and}h/\ell)$. A more detailed account of the derivation of Eq. \(\text{(10)}\) may be found in Appendix A. Note that $a^{(\mu)}$ and $b^{(\mu)(\nu)}$ are independent of $t_\alpha$ and $\tau^{(\mu)(\nu)}$, primarily because the torsions are antisymmetric under $\mu \leftrightarrow \nu$. If the cross-section of $N$ isn’t rotationally symmetric, then there are additional possibilities in Eq. \(\text{(10)}\). This may couple the torsions to the curvatures in a nontrivial way and complicate the resulting theory considerably.

The tangent vectors to first nontrivial order are

$$\partial_\alpha \vec{r} = \tilde{t}_\alpha + \zeta^{(\mu)} C^{(\mu)}_{\alpha\gamma} \hat{n}^{(\gamma)} - \tau^{(\mu)(\gamma)} \hat{n}^{(\gamma)}$$

and

$$\partial_{(\alpha)} \vec{r} = (1 + a_1 u^{c}_{\gamma\gamma}) \hat{n}^{(\alpha)} + b_1 \zeta^{(\mu)} C_{\alpha\alpha}^{(\mu)} \hat{n}^{(\mu)} + C^{(\mu)}_{\gamma\gamma} \hat{n}^{(\mu)} + b_2 \zeta^{(\alpha)} C^{(\lambda)}_{\gamma\gamma} \hat{n}^{(\lambda)}$$

From this it is clear that our assumption \(\text{(1)}\) above is equivalent to the small strain condition $u_{ij} \ll 1$. Also note that the transverse derivatives of the strain tensor $\partial_{(\alpha)} u_{ij}$ are of $O(C, \tau)$ and are not negligible. Only the manifold derivatives $\partial_\alpha u_{ij}$ can be safely neglected to leading order, as assumed in Sec. II.

We derive the energy for a portion of the manifold far from the regions where external forces are applied. We therefore have the boundary condition $\sigma_{(\alpha)(\beta)}|_{\zeta=h} = 0$. Combined with the conservation law $D_i \sigma_{ij} = 0$, we have $\sigma_{(\alpha)(\beta)} = 0$ everywhere. This condition specifies $u_{(\alpha)(\beta)}$ uniquely. Referring to Eq. \(\text{(8)}\)

$$\sigma_{(\alpha)(\beta)} = 2\mu u_{(\alpha)(\beta)} + \lambda \delta_{(\alpha)(\beta)} u_{kk} = 0$$

so $u_{(\alpha)(\beta)} \sim \delta_{(\alpha)(\beta)}$ and $b_1 + b_2 = 0$. Substituting the trace of the strain tensor

$$u_{kk} = u^{c}_{\alpha\alpha} - \zeta^{(\mu)} C^{(\mu)}_{\alpha\alpha} + (N - M)(a_1 u^{c}_{\alpha\alpha} + b_1 \zeta^{(\mu)} C^{(\mu)}_{\alpha\alpha})$$

\(\text{(14)}\)
into Eq. (13) gives us \( a_1 = -b_1 = -c_0 \) where \( c_0 = \lambda/(2\mu + (N - M)\lambda) \). Thus

\[
u_{(\alpha)(\beta)} = -c_0 \delta_{(\alpha)(\beta)} (u^c_{\gamma\gamma} - \zeta(\mu)C^{(\mu)}_{\gamma\gamma}) \tag{15}
\]

\[
u_{kk} = c_1 (u^c_{\gamma\gamma} - \zeta(\mu)C^{(\mu)}_{\gamma\gamma}) \tag{16}
\]

where \( c_1 = 2\mu/(2\mu + (N - M)\lambda) \).

Substituting Eqs. (12) and (16) into the expression for the strain Eq. (8) gives

\[
\sigma_{\alpha\beta} = 2\mu (u^c_{\alpha\beta} - \zeta(\mu)C^{(\mu)}_{\alpha\beta}) + c_1 \lambda \delta_{\alpha\beta} (u^c_{\gamma\gamma} - \zeta(\mu)C^{(\mu)}_{\gamma\gamma}) \\
\sigma_{\alpha(\beta)} = -\mu \zeta(\mu) \tau^{(\mu)(\beta)} \\
\sigma_{(\alpha)(\beta)} = 0 \tag{17}
\]

The elastic energy Eq. (7) becomes

\[
E = \int_{\mathcal{M}} dx^M \int_{B^{N-M}(h)} d\zeta^{N-M} \left\{ \mu (u^c_{\alpha\beta} - \zeta(\mu)C^{(\mu)}_{\alpha\beta})^2 \\
+ c_1 \frac{\lambda}{2} (u^c_{\gamma\gamma} - \zeta(\mu)C^{(\mu)}_{\gamma\gamma})^2 + \frac{\mu}{4} (\zeta(\mu) \tau^{(\mu)(\beta)}\tau^{(\mu)(\alpha)})^2 \right\} \tag{18}
\]

Since the transverse coordinates are being integrated over \( B^{N-M} \), terms odd in \( \zeta(\mu) \) vanish and we have finally

\[
E[u, C, \tau] = \int_{\mathcal{M}} dx^M \left\{ \mu^c ((u^c_{\alpha\beta})^2 + c_0 (u^c_{\alpha\alpha})^2) \\
+ \kappa \left( C^{(\mu)}_{\alpha\beta} C^{(\mu)}_{\alpha\beta} + c_0 C^{(\mu)}_{\alpha\alpha} C^{(\mu)}_{\beta\beta} \right) + \frac{\kappa}{4} (\tau^{(\mu)(\nu)} \tau^{(\mu)(\nu)} \right) \right\} \tag{19}
\]

where

\[
\mu^c = \mu \int_{B^{N-M}(h)} d\zeta^{N-M} \tag{20}
\]

is the effective stretching modulus and

\[
\kappa = \mu \int_{B^{N-M}(h)} d\zeta^{N-M} \zeta^{2(1)} \tag{21}
\]

is the effective bending modulus of the thin manifold \( \mathcal{M} \). The integral in Eq. (20) is just the volume of a sphere with radius \( h \)

\[
\int_{B^d(h)} d\zeta^{N-M} = \frac{1}{d} h^d S_d \tag{22}
\]
where \( S_d = 2\pi^{d/2}/\Gamma(d/2) \) is the area of a unit sphere in \( d \) dimensions. The integral in Eq. (21) is

\[
\int_{B_d(h)} \, d\zeta_2 = \begin{cases} 
\frac{2}{3} h^3 & d = 1 \\
\frac{\pi}{4} h^4 & d = 2 \\
\frac{1}{d+2} h^{d+2} B(3/2, d-2) S_{d-1} & d > 2 
\end{cases}
\]  

(23)

where \( B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b) \) is the beta function [27].

We can rewrite the elastic energy using conjugate fields

\[
E[u, C, \tau] = \frac{1}{2} \int_M \, dx^M \left\{ \sigma_{\alpha\beta}^c u_{\alpha\beta} + M_{\alpha\beta}^{(\mu)} C_{\alpha\beta}^{(\mu)} + T_{\alpha}^{(\mu)(\nu)} \tau_{\alpha}^{(\mu)(\nu)} \right\} 
\]

(24)

where

\[
\sigma_{\alpha\beta}^c(x) = 2\mu^c (u_{\alpha\beta}^c + c_0 \delta_{\alpha\beta} u_{\gamma\gamma}^c) 
\]

(25)
is the resultant strain field,

\[
M_{\alpha\beta}^{(\mu)}(x) = 2\kappa (C_{\alpha\beta}^{(\mu)} + c_0 \delta_{\alpha\beta} C_{\gamma\gamma}^{(\mu)}) 
\]

(26)
is the bending moment field, and

\[
T_{\alpha}^{(\mu)(\nu)}(x) = \frac{1}{2} \kappa_{\alpha}^{(\mu)(\nu)} 
\]

(27)
is the torsional moment field. Eq. (24) is the full elastic energy functional for a thin elastic manifold. We will frequently refer to the term quadratic in the strains as the stretching energy and the term quadratic in the curvatures as the bending energy.

We henceforth drop the superscript \( c \) and assume that all quantities refer to the centroid manifold \( \mathcal{M} \).

**IV. THE POTENTIALS AND THE VON KARMAN EQUATIONS**

**A. The case \( M > 2 \)**

Note that Eq. (24) does not explicitly couple the strains to the curvatures of the manifold. The strains and the curvatures are implicitly coupled because they are both defined via
derivatives of the embedding $\vec{r}(x)$. However, a naive functional derivative of Eq. (24) with respect to $u_{\alpha\beta}$ gives the trivial and incorrect result $\sigma_{\alpha\beta} = 0$. Previous authors, working with a thin plate in three dimensions, have solved this problem by working in a special coordinate system which is approximately tangent to one point on the manifold. In this frame, known as the Monge representation, the embedding is $\vec{r}(x) = [x_\alpha + u_\alpha(x), w(x)]$ and the derivatives of $\vec{u}(x)$ and $w(x)$ are assumed to be small everywhere. To leading nontrivial order the strain tensor is $u_{\alpha\beta} = (1/2)(u_{\alpha\beta} + u_{\beta\alpha} + w_\alpha^{\;\prime}w_\beta^{\;\prime})$ and the extrinsic curvature tensor is $C_{\alpha\beta} = w_{\alpha\beta}$.

Functional derivatives are then taken with respect to $u_\alpha$ and $w$ and the correct equations are obtained [28].

We choose to work instead with the field variables $u_{\alpha\beta}$, $C_{\alpha\beta}^{(\mu)}$, and their potentials. The advantages are (1) we work exclusively in the manifold coordinates, so there is no need for an approximately tangent frame, (2) it is easier to treat the boundary conditions, and (3) we discover a new interpretation for the stress functions of Airy and Maxwell [19,21,22]. Due to some small differences, we focus here on the case $M > 2$ and return to the case $M = 2$ in the next subsection. The coupling between the strain and the curvature is completely accounted for by the geometric von Karman equation Eq. (6). We therefore add the Lagrange multiplier term

$$ E_\chi[u, C, \chi] = \frac{1}{2} \int_M dx^M \chi_{\alpha\beta}(G_{\alpha\beta}[C] - G_{\alpha\beta}[u]) $$

(28)

to the total elastic energy Eq. (24). We will see that the Lagrange multiplier $\chi_{\alpha\beta}(x)$ is the tensor stress function.

One may ask why it is sufficient to use the Einstein curvature tensor $G_{\alpha\beta}$ rather than the full intrinsic curvature tensor $R_{\alpha\beta\mu\nu}$. $G_{\alpha\beta}$ is symmetric in $\alpha \leftrightarrow \beta$ and constrained by the conservation law $G_{\alpha\beta,\alpha} = 0$, so a naive count of the independent degrees of freedom gives $M(M - 1)/2$. $R_{\alpha\beta\mu\nu}$ is symmetric in $\alpha \leftrightarrow \beta$ and $\mu \leftrightarrow \nu$, and antisymmetric in $(\alpha\beta) \leftrightarrow (\mu\nu)$, so in principle it has $M(M - 1)(M^2 - M + 2)/8$ independent components. However, Eq. (4) shows that for small strains the intrinsic curvature tensor is linear in the strain tensor $u_{\alpha\beta}$. The strain tensor is symmetric and constrained by the conservation of the resultant
stress tensor, so it has $M(M-1)/2$ independent components. The Einstein curvature tensor is therefore the most economical choice. The alternative forms for the Lagrange multiplier term $\chi_{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu}$ and $\chi_{\alpha\beta} R_{\alpha\beta\mu}$ both yield the correct von Karman equations for the fields $\sigma_{\alpha\beta}$ and $C_{(\lambda)}^{(\alpha)}$, but the Lagrange multiplier is not identical to the stress function.

If the normal basis has zero torsion we can define a bending potential for each normal component of the extrinsic curvature tensor. In the remainder of this paper we assume there are no external torsional moments acting on the manifold. Because the torsions are not coupled to the strains or to the curvatures in Eqs. (24) and (28), the solution is simply $\tau^{(\mu)(\nu)}(x) = 0$. With this we can prove the Codazzi-Mainardi relation

$$\partial_\alpha C_{(\lambda)}^{(\beta)} = \hat{n}^{(\lambda)} \cdot (\partial_\alpha \hat{K}_{(\beta)}) + (\partial_\alpha \hat{n}_{(\lambda)}) \cdot \hat{K}_{(\beta)}$$

$$= \hat{n}^{(\lambda)} \cdot (\partial_\beta \hat{K}_{(\lambda)})$$

$$= \partial_\beta C_{(\alpha)}^{(\lambda)} \quad (29)$$

We have used the simplified Weingarten map $\partial_\alpha \hat{n}^{(\lambda)} = -C_{(\alpha)(\beta)} \hat{t}_{(\beta)}$ in the second line and the orthogonality condition $\hat{t}_{(\alpha)} \cdot \hat{n}^{(\beta)} = 0$ in the third line. The Codazzi-Mainardi relation is analogous to the zero-curl condition on a vector field. It allows the definition of a scalar potential $f^{(\lambda)}(x)$ via $C_{(\alpha)}^{(\lambda)} = \partial_\alpha \partial_\beta f^{(\lambda)}$.

There is a novel form for the Einstein curvature tensor which greatly simplifies the variational derivatives taken below. We begin by defining the double curl operator, valid for $M > 2$,

$$(d.c.)_{\alpha\beta\mu\nu} = \frac{1}{(M-3)!} \epsilon_{\alpha\gamma\mu_1\cdots\mu_{M-3}} \epsilon_{\beta\delta\nu_1\cdots\nu_{M-3}} \partial_\gamma \partial_\delta$$

$$= \delta_{\alpha\beta} \delta_{\mu\nu} \nabla^2 - \delta_{\alpha\mu} \delta_{\beta\nu} \nabla^2 - \delta_{\alpha\beta} \partial_\mu \partial_\nu$$

$$+ \delta_{\alpha\nu} \partial_\beta \partial_\mu + \delta_{\beta\mu} \partial_\alpha \partial_\nu - \delta_{\mu\nu} \partial_\alpha \partial_\beta \quad (30)$$

where $\epsilon_{\tau_1\cdots\tau_M}$ is the Levi-Civita tensor.
The double curl is antisymmetric in $\alpha \leftrightarrow \mu$ and $\beta \leftrightarrow \nu$ and symmetric in $(\alpha\mu) \leftrightarrow (\beta\nu)$ (compare to $R_{\alpha\beta\mu\nu}$). It satisfies $\partial_\alpha (d.c.)_{\alpha\beta\mu\nu} = 0$ by construction. The Einstein curvature tensor may be written $G[\alpha]_{\alpha\beta} = (d.c.)_{\alpha\beta\mu\nu} u_{\mu\nu}$ or $G[\alpha]_{\alpha\beta} = (d.c.)_{\alpha\beta\mu\nu} [(1/2) f^{(\lambda)}_{\mu} f_{\lambda\nu}^\ast].$ These expressions are easily verified by substitution.

Now we can write the full expression for the energy functional, including the Lagrange multiplier

$$E[u, f, \chi] = \int_M dx \frac{1}{2} \left\{ \sigma_{\alpha\beta} [u_{\alpha\beta} + \kappa (f^{(\lambda)}_{\mu\nu} f^{(\lambda)}_{\mu\nu} + c_0 f^{(\lambda)}_{\mu\mu} f^{(\lambda)}_{\nu\nu})] + \chi_{\alpha\beta} (d.c.)_{\alpha\beta\mu\nu} (\frac{1}{2} f^{(\lambda)}_{\mu} f^{(\lambda)}_{\nu}) - u_{\mu\nu} \right\}$$

Taking a functional variation with respect to $u_{\alpha\beta}$ and integrating the Lagrange multiplier term twice by parts gives

$$\delta E = \int_M dx \frac{1}{2} \delta u_{\mu\nu} \left\{ \sigma_{\mu\nu} - (d.c.)_{\mu\nu\alpha\beta} \chi_{\alpha\beta} \right\} + \frac{1}{(M-3)!} \epsilon_{\alpha\gamma\mu\cdot\cdot\cdot\tau_{M-3}} \epsilon_{\beta\delta\nu\cdot\cdot\cdot\tau_{M-3}} \partial_\gamma (\delta u_{\mu\nu} \chi_{\alpha\beta,\delta} - \delta u_{\mu\nu,\delta} \chi_{\alpha\beta})$$

where we have used the symmetry of $u_{\alpha\beta}$ and $\chi_{\alpha\beta}$. The first term in Eq. \ref{32} gives a conservation law for the resultant stress tensor. Taking $\delta E/\delta u_{\alpha\beta} = 0,$

$$\sigma_{\alpha\beta} = (d.c.)_{\alpha\beta\mu\nu} \chi_{\mu\nu}$$

This is a restatement of the conservation law $\partial_\alpha \sigma_{\alpha\beta} = 0.$ Eq. \ref{34} is the defining equation for Maxwell’s stress function in $M = 3$ \cite{21,22}. We see that the stress function is a Lagrange multiplier. This interpretation persists even when $M = N = 3$ and the extrinsic curvature tensor is identically zero [take $f^{(\lambda)} = 0$ in Eq. (32)]. Eq. \ref{34} also provides a natural generalization of the stress function to higher dimensions.

One can verify by substitution that the stress tensor remains unchanged under the local gauge transformations $\chi_{\alpha\beta} \rightarrow \chi_{\alpha\beta} + (1/2)(\xi_{\alpha,\beta} + \xi_{\beta,\alpha})$ where $\xi_{\alpha}(x)$ is an arbitrary vector
field [23]. The tensor stress function is therefore a gauge field with $M(M - 1)/2$ physical degrees of freedom. This agrees with the fact that the stress tensor itself has $M(M - 1)/2$ independent components.

The second term in Eq. (33) is a perfect differential. Using Gauss’ Law to rewrite it as an integral over the $M - 1$ dimensional boundary of the manifold $\partial M$ gives

$$\frac{1}{(M - 3)!} \epsilon_{\alpha \gamma \mu_1 \cdots \tau_{M-3}} \epsilon_{\beta \delta \nu_1 \cdots \tau_{M-3}} \int_{\partial M} dx^{M-1} \hat{n}_\gamma (\delta u_{\mu \nu} \chi_{\alpha \beta, \delta} - \delta u_{\mu \nu, \delta} \chi_{\alpha \beta})$$

where $\hat{n}$ is the unit outward normal defined in the tangent space of $\mathcal{M}$. The application of this term to a specific problem depends on the boundary conditions imposed at $\partial \mathcal{M}$.

Taking the functional variation of $E$ with respect to $f^{(\lambda)}$ and integrating by parts gives

$$\delta E = \int_{\mathcal{M}} dx^{M} \left( 2\kappa (1 + c_0) \nabla^4 f^{(\lambda)} - f^{(\lambda)}_{\alpha \beta} (d.c.)_{\alpha \beta \mu \nu} \chi_{\mu \nu} \right)$$

$$+ 2\kappa \partial_\gamma \left( \delta f^{(\lambda)}_{\gamma \mu \nu} + c_0 \delta f^{(\lambda)}_{\gamma \mu} \right) - (1 + c_0) \delta f^{(\lambda)}_{\gamma \mu \nu}$$

$$+ \frac{1}{(M - 3)!} \epsilon_{\alpha \gamma \mu_1 \cdots \tau_{M-3}} \epsilon_{\beta \delta \nu_1 \cdots \tau_{M-3}} \partial_\gamma \left( \delta f^{(\lambda)}_{\mu \nu} \chi_{\alpha \beta, \delta} - \delta f^{(\lambda)}_{\mu \nu, \delta} \chi_{\alpha \beta} \right)$$

where there is no sum on $(\lambda)$. The second and third terms are perfect differentials and may be written as a condition on $\partial \mathcal{M}$. Taking $\delta E/\delta f^{(\lambda)} = 0$, the first term gives

$$2\kappa (1 + c_0) \nabla^4 f^{(\lambda)} = f^{(\lambda)}_{\alpha \beta} (d.c.)_{\alpha \beta \mu \nu} \chi_{\mu \nu}$$

which may be rewritten in the more familiar form

$$M^{(\lambda)}_{\alpha \beta, \alpha \beta} = \sigma_{\alpha \beta} C^{(\lambda)}_{\alpha \beta}$$

This is the force von-Karman equation, which expresses the balance of forces on a transverse section of the thin manifold.

To complete our discussion of the von Karman equations, we write the Einstein curvature tensor in terms of the stress function. First invert Eq. (23) to get

$$u_{\alpha \beta} = \frac{1}{2\mu} (\sigma_{\alpha \beta} - c_2 \delta_{\alpha \beta} \sigma_{\gamma \gamma})$$

where $c_2 = \lambda/(2\mu + N\lambda)$. Then
\[ G_{\alpha\beta}[\chi] = \frac{1}{2\mu} \left\{ (d.c.)_{\alpha\beta\mu\nu} (d.c.)_{\mu\nu\sigma\tau} \chi_{\sigma\tau} - c_2 (d.c.)_{\alpha\beta\mu\nu} (d.c.)_{\nu\sigma\tau} \chi_{\sigma\tau} \right\} \]
\[ = \frac{1}{2\mu} \left\{ (c_3 - 1) \left( \delta_{\alpha\beta} \nabla^4 \chi_{\sigma\tau} - \delta_{\alpha\beta} \nabla^2 \chi_{\sigma\tau,\sigma\tau} - \nabla^2 \chi_{\sigma\tau,\alpha\beta} \right) + c_3 \chi_{\sigma\tau,\alpha\beta\sigma\tau} + \nabla^4 \chi_{\alpha\beta} - \nabla^2 \chi_{\alpha\beta,\beta\sigma} - \nabla^2 \chi_{\beta\sigma,\alpha\sigma} \right\} \]  

(40)

where \( c_3 = (M - 2)|2\mu + (N - M + 2)\lambda|/(2\mu + N\lambda) \).

We thus have several alternative expressions for the von Karman equations, depending on which fields are most convenient. In terms of the extrinsic curvature and strain tensors, we have Eqs. (6) and (38). In terms of the bending potentials and the stress function we have \( G_{\alpha\beta}[f] = G_{\alpha\beta}[\chi] \) and Eq. (37).

B. The case \( M = 2 \)

In this section we rederive the von Karman equations for a thin plate. Although the equations of the three-dimensional problem have been discussed in detail by several authors, our variational derivation of the force von Karman equation is particularly transparent [13,14,20,6]. The only change from the previous section is that the tensor double curl operator is not defined for \( M = 2 \). Instead we use the scalar operator

\[(d.c.)_{\alpha\beta} = \epsilon_{\alpha\mu} \epsilon_{\beta\nu} \partial_{\mu} \partial_{\nu} = \delta_{\alpha\beta} \nabla^2 - \partial_{\alpha} \partial_{\beta} \]  

(41)

When \( M = 2 \), the intrinsic curvature tensor has only one independent component, which we take as the generalization of the Gaussian curvature

\[ \kappa_G[C] = \frac{1}{2} R_{\alpha\beta\alpha\beta}[C] = C^{(\lambda)}_{11} C^{(\lambda)}_{22} - C^{(\lambda)}_{12} C^{(\lambda)}_{12} \]  

(42)

It is straightforward to verify that \( \kappa_G[u] = -(d.c.)_{\alpha\beta} u_{\alpha\beta} \) and \( \kappa_G[f] = -(d.c.)_{\alpha\beta} (\frac{1}{2} f^{(\lambda)}_{,\alpha} f^{(\lambda)}_{,\beta}) \).

The elastic energy is

\[ E[u, C, \chi] = \int_M dx M \frac{1}{2} \left\{ \sigma_{\alpha\beta}[u] u_{\alpha\beta} + M^{(\lambda)}_{\alpha\beta} C^{(\lambda)}_{\alpha\beta} \right\} + \chi (-\kappa_G[C] + \kappa_G[u]) \]  

(43)

\[ E[u, f, \chi] = \int_M dx M \frac{1}{2} \left\{ \sigma_{\alpha\beta}[u] u_{\alpha\beta} + \kappa (f^{(\lambda)}_{,\alpha\beta} f^{(\lambda)}_{,\alpha\beta} + c_0 f^{(\lambda)}_{,\beta\beta} f^{(\lambda)}_{,\beta\beta}) \right\} + \chi (d.c.)_{\alpha\beta} \left( \frac{1}{2} f^{(\lambda)}_{,\alpha} f^{(\lambda)}_{,\beta} - u_{\alpha\beta} \right) \]  

(44)
where the Lagrange multiplier $\chi(x)$ is a scalar field. Taking the functional derivative $\delta E/\delta u_{\alpha\beta} = 0$ gives $\sigma_{\alpha\beta} = (d.c.)_{\alpha\beta} \chi$. Thus $\partial_\alpha \sigma_{\alpha\beta} = 0$ and $\chi$ is the scalar stress function of Airy $^{20}$.

In terms of the stress function $\kappa_G[\chi] = [(1 - c_2)/2\mu]\nabla^4 \chi$ and the geometric von Karman equation is

$$\frac{1}{\mu}(1 - c_2)\nabla^4 \chi = - (d.c.)_{\alpha\beta}(f^{(\lambda)}_{,\alpha} f^{(\lambda)}_{,\beta})$$

The force von Karman equation is found via $\delta E/\delta f^{(\lambda)} = 0$ to be

$$2\kappa(1 + c_0)\nabla^4 f^{(\lambda)} = f^{(\lambda)}_{,\alpha\beta}(d.c.)_{\alpha\beta} \chi$$

which reproduces Eq. (38).

V. THE RIDGE

A. Boundary Conditions

In this section we discuss the picture of a ridge as a boundary layer and find a simple boundary condition which yields a ridge for general $M$. Previous analytic studies of stress confinement have been limited to the case of a thin plate ($M = 2$) in $\mathbb{R}^3$. In Ref. [6], Lobkovsky treated the case of a single ridge in isolation and analyzed the resulting von Karman equations to lowest order in the thickness. He found that a ridge of length $X$ has an elastic energy $E \sim \mu h^2 (X/h)^{1/3}$ and a width $w \sim h^{1/3} X^{2/3}$.

We develop the concept of the ridge as a boundary layer by treating the thickness $h$ as a tunable parameter. We start with $h = 0$. Since the width of a ridge scales like $h^{1/3}$, the zero thickness limit of a ridge is a straight line of zero width. The geometry of the ridge, shown in Fig. 1, is two flat plates which meet at a nonzero angle $D$. The curvatures are obviously singular on this line and zero elsewhere. The intuitive reason for this behavior is that $\kappa/\mu \sim h^2$, so there is no energy cost for curvatures when $h = 0$. When $h$ is made nonzero, the plate can achieve a lower total energy by smoothing out the singularity and trading
stretching energy for bending. The resulting balance generates a new length scale, which is the width of the ridge. This picture will remain essentially unchanged when \( M > 2 \). The important point is that the ridge is a boundary layer which regularizes the \( h = 0 \) singularity.

Lobkovsky began with the semi-infinite strip \( \{ -\infty < x_1 < \infty, -X/2 \leq x_2 \leq X/2 \} \). Then he assumed the presence of (unspecified) normal forces acting at the boundary \( x_2 = \pm X/2 \) sufficient to deform the strip into the ridge shown in Fig. 1. The boundary conditions are \( C_{\alpha\beta}(x_1, \pm X/2) = 0, \sigma_{\alpha\beta}(x_1, \pm X/2) = 0, \) and \( f(x_1, \pm X/2) = \alpha|x_1| \). The first two conditions are chosen for convenience. It is the third condition which determines the shape of the ridge. The potential \( f \) plays the role of the normal coordinate. The dihedral angle of the resulting ridge is \( D = \pi - 2\alpha \).

The generalization of this geometry to \( M > 2 \) is straightforward. We limit our discussion to the hypersurface \( N = M + 1 \), since this captures the most important features of the general case. We take for our \( h = 0 \) ridge the singular boundary between two \( M \)-dimensional regions with zero curvature. As discussed in Ref. [9], such a boundary can have no curvature in the material coordinates or in \( \mathbb{R}^N \). Our boundary conditions must be consistent with this ridge.

We take for our undeformed manifold the semi-infinite domain \( \{ -\infty < x_1 < \infty, x_\bar{\alpha} \hat{e}_\bar{\alpha} \in \mathcal{C} \text{ for } \bar{\alpha} \in [2, M] \} \) (see Fig. 2). The material coordinate \( x_1 \) is perpendicular to the ridge and \( \mathcal{C} \) is an arbitrary, simply connected cross-section. The ridge is imposed by some (unspecified) normal forces sufficient to create the “kinked” potential \( f|_{\partial M} = \alpha|x_1| \). For simplicity we take \( C_{\alpha\beta}|_{\partial M} = 0 \) and \( \sigma_{\alpha\beta}|_{\partial M} = 0 \). The solution to the von Karman equations when \( h = 0 \) is the singular ridge \( f(x) = \alpha|x_1| \).

**B. Scaling analysis**

To begin a scaling analysis of the von Karman equations we need to decide which fields to work with. Since the stress function \( \chi_{\alpha\beta} \) is a gauge field in \( M > 2 \), it is convenient to work directly with the stress tensor \( \sigma_{\alpha\beta} \). We use the bending potential \( f^{(1)} \) instead of the
curvature tensor $C^{(3)}_{\alpha\beta}$ because of the obvious advantages of a scalar. Since $N = M + 1$ we can drop the normal index. The geometric and force von Karman equations are then

$$
(d.c.)_{\alpha\beta\mu\nu}(f_{\mu\nu}) = \frac{1}{\mu}(d.c.)_{\alpha\beta\mu\nu}(\sigma_{\mu\nu} - c_2\delta_{\mu\nu}\sigma_{\gamma\gamma}) \tag{47}
$$

$$
2\kappa(1 + c_0)\nabla^4 f = \sigma_{\alpha\beta} f_{\alpha\beta} \tag{48}
$$

Now convert to the dimensionless quantities

$$
\tilde{f} = f/X, \quad \tilde{\sigma}_{\alpha\beta} = X^2 \frac{1}{2\kappa(1 + c_0)}\sigma_{\alpha\beta} \quad \text{and} \quad \tilde{x}_\alpha = x_\alpha/X \tag{49}
$$

where $X$ is a length scale characterizing the cross-section $\mathcal{C}$. Including the conservation law, the dimensionless equations are

$$
\partial_\alpha \tilde{\sigma}_{\alpha\beta} = 0 \tag{50}
$$

$$
(d.c.)_{\alpha\beta\mu\nu}(\tilde{f}_{\mu\nu}) = \epsilon^2(d.c.)_{\alpha\beta\mu\nu}(\tilde{\sigma}_{\mu\nu} - c_2\delta_{\mu\nu}\tilde{\sigma}_{\gamma\gamma}) \tag{51}
$$

$$
\nabla^4 \tilde{f} = \tilde{\sigma}_{\alpha\beta}\tilde{f}_{\alpha\beta} \tag{52}
$$

where $\epsilon^2 = 2\kappa(1 + c_0)/(\mu X^2) \sim (h/X)^2$. As for a bent plate in three dimensions, we expect the ridge solution to be valid when $0 < \epsilon \ll 1$. Note that a naive count of Eqs. (50)-(52) gives $M(M + 1)/2 + N$ constraints on $M(M + 1)/2 + N - M$ field variables. This set is not overdetermined because the argument of the double curl has the local gauge freedom $(\eta)_{\mu\nu} \rightarrow (\eta)_{\mu\nu} + (1/2)(\xi_{\mu,\nu} + \xi_{\nu,\mu})$ where $\xi(x)$ is an arbitrary vector field. The geometric von Karman equation therefore represents only $M(M - 1)/2$ independent constraints.

The $\epsilon = 0$ limit is the $h = 0$ limit. As discussed in the previous section, the solution to this reduced problem is the singular ridge $\tilde{f} = \alpha|\tilde{x}_1|$. We might hope to solve for the $\epsilon > 0$ ridge as a perturbation about this solution. However $\epsilon$ multiplies the highest derivative of $\tilde{\sigma}_{\alpha\beta}$. It is therefore a singular perturbation and naive approaches fail.

To find the exponents which characterize the ridge, we make the following rescalings

$$
\tilde{\sigma}_{11} = \epsilon^\delta \tilde{\sigma}_{11}, \quad \tilde{\sigma}_{1\bar{\alpha}} = \epsilon^s \tilde{\sigma}_{1\bar{\alpha}}, \quad \tilde{\sigma}_{\alpha\bar{\beta}} = \epsilon^t \tilde{\sigma}_{\alpha\bar{\beta}}
$$

$$
\tilde{f} = \epsilon^\beta \tilde{f}, \quad \tilde{x}_1 = \epsilon^\beta \tilde{x}_1 \quad \text{and} \quad \tilde{x}_{\bar{\alpha}} = \epsilon^0 \tilde{x}_{\bar{\alpha}} \tag{53}
$$
where we distinguish the coordinates parallel to the ridge with a barred Greek index \( \bar{\alpha} \in [2, M] \). Note that \( \bar{f} \sim \bar{x}_1 \) is required by the boundary condition, so they scale with the same power of \( \epsilon \).

To solve for the exponents we start with the conservation law Eq. (50). Grouping terms of like order in \( \epsilon \) gives

\[
\epsilon^{\beta - \delta} \partial_1 \bar{\sigma}_{11} + \epsilon^{-s} \partial_{\bar{\alpha}} \bar{\sigma}_{\bar{\alpha}1} = 0 \tag{54}
\]

\[
\epsilon^{\beta - s} \partial_1 \bar{\sigma}_{1\bar{\alpha}} + \epsilon^{-t} \partial_{\bar{\alpha}} \bar{\sigma}_{\bar{\alpha}\bar{\beta}} = 0 \tag{55}
\]

Assuming none of the rescaled quantities vanish, this implies \( s = \delta - \beta \) and \( t = \delta - 2\beta \).

The rescaled force von Karman equation is

\[
(\epsilon^{2\beta} \partial_1^2 + \epsilon^0 \partial_{\bar{\alpha}}^2) \bar{f} = \epsilon^{2\beta - \delta} (\bar{\sigma}_{\bar{\alpha}\bar{\beta}} \bar{f}_{\bar{\alpha}\bar{\beta}}) \tag{56}
\]

If \( \beta \geq 0 \) this equation is dominated by the lowest order terms in \( \epsilon^\beta \) as \( \epsilon \to 0 \) and \( 0 = 2\beta - \delta \).

If \( \beta < 0 \) then \( 4\beta = 2\beta - \delta \).

The rescaled geometric von Karman equation has \( M(M+1)/2 \) components, one for each component of \( G_{\alpha\beta} \). For the scaling analysis they can be grouped into four classes \( G_{11}, G_{1\bar{\alpha}}, G_{\bar{\alpha}\bar{\alpha}} \) (no sum), and \( G_{\bar{\alpha}\bar{\beta}} \). Assuming none of the relevant terms vanish, we only need to consider one example from each class. The \( G_{11} \) component is

\[
\epsilon^{-2\beta} (\tilde{f}_{\bar{\alpha}\bar{\beta}} \tilde{f}_{\bar{\alpha}\bar{\beta}} - \tilde{f}_{\bar{\alpha}\bar{\alpha}} \tilde{f}_{\bar{\alpha}\bar{\beta}}) = -\epsilon^{2-\delta} \left\{ \epsilon^{2\beta} \bar{\sigma}_{\bar{\alpha}\bar{\beta},\bar{\alpha}\bar{\beta}} + \epsilon^{2\beta} c_4 \bar{\sigma}_{\bar{\alpha}\bar{\alpha},\bar{\beta}\bar{\beta}} + \epsilon^0 c_2 (M - 2) \bar{\sigma}_{11,\bar{\beta}\bar{\beta}} \right\} \tag{57}
\]

where \( c_4 = 1 - (M - 2)c_2 \). We have made extensive use of the conservation law \( \bar{\sigma}_{\bar{\alpha}\bar{\beta},\alpha} = 0 \) and the symmetries of the problem to simplify the expression. The other components are derived in Appendix B. A careful analysis shows that all components yield the same constraint on the exponents. If this were not the case, the equations for the exponents would be overdetermined. If \( \beta \geq 0 \) then Eq. (57) gives \( -2\beta = 2 - \delta \). If \( \beta < 0 \) then \( -2\beta = 2 - \delta + 2\beta \).

The constraints on the exponents are only solvable if \( \beta < 0 \). The unique solution is

\[
\beta = -\frac{1}{3}, \quad \delta = +\frac{2}{3} \tag{58}
\]
These are identical to the exponents found by Lobkovsky for the ridge in \((M = 2, N = 3)\) \[6\]. There are a variety of geometric intuitions associated with these exponents. The most important is that the ridge width is characterized by the transverse curvature via \(w \sim 1/C_{11}\).

Thus

\[ w \sim \epsilon^{-\beta} X \sim h^{1/3} X^{2/3} \]  \hspace{1cm} (59)

We refer the reader to Ref. \[6\] for a fuller discussion of the ridge geometry.

It is instructive to consider the rescaled ridge energy to leading order in \(\epsilon\)

\[ E = \int_M (X^M \epsilon^{-\beta} d\tilde{x}^M) \left\{ \frac{1}{2\mu} \left( \frac{2\kappa(1 + c_0)}{X^2} \epsilon^{-\delta + 2\beta} \right)^2 (\tilde{\sigma}_{\alpha\beta}^2 - c_2 \tilde{\sigma}_{\alpha\alpha}^2) + \kappa \left( \frac{\epsilon^\beta}{X} \right)^2 (1 + c_0) \tilde{C}_{11}^2 \right\} \]  \hspace{1cm} (60)

Gathering terms and using the geometric von Karman constraint \(\delta - 4\beta = 2\) gives

\[ E = \mu h^M (c_s \epsilon^{-M - 5\beta} + c_b \epsilon^{-M + 2\beta}) \]  \hspace{1cm} (61)

where \(c_s\) and \(c_b\) are dimensionless constants due to the stretching and bending energy respectively. In this form it is clear how the value of \(\beta\), and hence the width, is generated via the balance between strains and curvatures. As the width \(w \sim \epsilon^{-\beta}\) is increased, the bending energy decreases and the stretching energy increases. The ridge chooses the value \(\beta = -1/3\) which minimizes the total energy. Thus

\[ E \sim \mu h^M \epsilon^{-M + 5/3} \sim \mu h^M (X/h)^{M - 5/3} \]  \hspace{1cm} (62)

Furthermore, Eq. (61) fixes exactly the ratio \(c_b/c_s\). At the minimum,

\[ \left. \frac{\partial E}{\partial \beta} \right|_{\beta = -1/3} = \mu h^M \ln(\epsilon) \left( -5c_s \epsilon^{-M - 5\beta} + c_b \epsilon^{-M + 2\beta} \right) \bigg|_{\beta = -1/3} = 0 \]  \hspace{1cm} (63)

This reduces to \(c_b/c_s = 5\), which means the bending energy is exactly five times the stretching energy in an asymptotic ridge. As noted in the introduction, this ratio also holds for \((M = 2, N = 3)\) \[23\].
VI. CONCLUSIONS

In this paper we derive the equations of static equilibrium for an $M$-dimensional elastic manifold embedded in $N$-dimensional space. We define the potentials $\chi_{\alpha\beta}$ and $f^{(\lambda)}$ on the manifold. These are the higher-dimensional analogs of the stress function $\chi$ and the bending potential $f$ of a thin plate in three dimensions. We find a novel interpretation for the stress function as the Lagrange multiplier of the geometric von Karman equation in the elastic energy functional of the manifold.

We go on to consider the properties of an $M-1$ dimensional ridge in an $M > 2$ dimensional manifold. The scaling is essentially identical to that found by Lobkovsky for a ridge in $M = 2$ [5]. We find that a ridge of linear size $X$ in a manifold of thickness $h$ has a width $w \sim h^{1/3} X^{2/3}$ and a total elastic energy $E \sim \mu h^M (X/h)^{M-5/3}$, where $\mu$ is a stretching modulus. The scaling analysis also fixes exactly the ratio of bending energy to the stretching energy in a ridge $E_{\text{bend}} / E_{\text{stretch}} = 5$. These results are valid in the thin limit $h \ll X$. Although our calculations are explicitly for a bent hypersurface $N = M + 1$, unpublished theory and simulations lead us to expect no change in the ridge exponents when $N > M + 1$.

The purpose of this work was primarily as an aid to future studies of crumpling in high dimensional systems. In particular, in future papers we will discuss the phenomenon of spontaneous ridge formation as a mechanism of stress confinement [3,4]. The elastic energy and ridge exponents derived here are an essential foundation for that work. We should point out that the scaling analysis in this paper is in no way a guarantee that these ridges will form in a crumpled manifold. The question of how the elastic energy is distributed is best resolved in combination with computer simulations and, for $(M = 2, N = 3)$, experiments. As an example of the way ridge formation can fail, we note that when $N > 2M$ a manifold with a free boundary can make its stretching energy zero everywhere. Since the ridge structure depends on the competition between bending and stretching energy, no ridge formation is possible.
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APPENDIX A: COMMENTS ON THE TAYLOR EXPANSION

Begin by expanding $\vec{a}^{(\mu)}$ and $\vec{b}^{(\mu)(\nu)}$ in the full basis $\{\vec{t}_\alpha, \hat{n}_\alpha\}$

$$
\vec{a}^{(\mu)} = a_\alpha^{(\mu)} \vec{t}_\alpha + a^{(\mu)(\nu)} \hat{n}^{(\nu)}
$$

$$
\vec{b}^{(\mu)(\nu)} = b_\alpha^{(\mu)(\nu)} \vec{t}_\alpha + b^{(\mu)(\nu)(\lambda)} \hat{n}^{(\lambda)}
$$

(A1)

The coefficients $(a_\alpha^{(\mu)}, a^{(\mu)(\nu)}, b_\alpha^{(\mu)(\nu)}, b^{(\mu)(\nu)(\lambda)})$ are functions of $u_{\alpha\beta}^c$, $C_{\alpha\beta}^{(\mu)}$, $\tau_\alpha^{(\mu)(\nu)}$, and their derivatives. We require that the expressions for the coefficients in terms of these quantities have the correct number of free manifold and normal indices (there may be an arbitrary number of contracted indices). This is necessary and sufficient for their correct behavior under reflections and rotations of the manifold and normal coordinates.

Note that $b_\alpha^{(\mu)(\nu)}$ and $b^{(\mu)(\nu)(\lambda)}$ must be even under $\mu \leftrightarrow \nu$ since $\vec{b}^{(\mu)(\nu)}(x) = \partial_{(\mu)}\partial_{(\nu)}\vec{r}(x, \zeta)|_{\zeta=0}$. This is why $b_3^{(\mu)(\nu)} \vec{t}_\alpha$ is not a valid term. Note also that $\tau_\alpha^{(\mu)(\mu)} = 0$.

We require that the coefficients have the correct units. Consider the following additions to the expression for $\vec{a}^{(\mu)}$

$$
\vec{a}^{(\mu)} = (\tilde{a}_2 C_{\alpha\beta}^{(\mu)} \tilde{C}_{\alpha\beta}^{(\nu)} + \tilde{a}_3 \tau_\alpha^{(\mu)(\nu)} \tau_\alpha^{(\lambda)(\nu)}) \hat{n}^{(\nu)} + (\tilde{a}_4 \tau_\alpha^{(\mu)(\nu)} \tilde{C}_{\alpha\beta}^{(\nu)} + \tilde{a}_5 \tilde{C}_{\alpha\beta}^{(\mu)}) \tilde{t}_\beta
$$

(A2)

Since $\vec{a}^{(\mu)}$ is dimensionless the constants $\tilde{a}_j$ must have units of (length)$^2$. The only length scale available is the thickness $h$, so we write $\tilde{a}_j = a_j h^2$. The expressions are therefore of $O(h^2C^2, h^2\tau^2, h^2C\tau, \ h^2C/\ell)$ respectively. Although these terms are assumed small, they are not necessarily negligible compared to $u_{\alpha\beta}^c$. One can verify, however, that their
contribution to the energy is negligible compared to the curvature and torsion terms in Eq. (24). Similar arguments lead to the form for $b^{(\mu)(\nu)}$.

In the theory of thin shells it is known that the centroid deformations alone are not sufficient to describe the behavior near the boundary of the shell. The full three-dimensional problem must be solved there. As a consequence, any energy functional derived via a Taylor series expansion in the thickness is not uniformly convergent near the boundary. For a detailed discussion of these considerations, we refer the reader to Ref. [16] and the references therein.

APPENDIX B: SCALING OF THE FORCE VON KARMAN EQUATION

Assuming none of the relevant terms vanish, we only need to consider one example from each of the three classes $G_{1\bar{\alpha}}$, $G_{\bar{\alpha}\bar{\alpha}}$ (no sum), and $G_{\bar{\alpha}\bar{\beta}}$. Dropping the tilde notation, the $G_{12}$ equation is

$$2\epsilon^{-\beta}(f_{,12}f_{,\bar{\alpha}\bar{\alpha}} - f_{,1\bar{\alpha}}f_{,2\bar{\alpha}}) = -\epsilon^2 - \delta(\epsilon^3\sigma_{12,11} + \epsilon^3\sigma_{12,\bar{\alpha}\bar{\alpha}} + \epsilon^3\sigma_{11,12} + \epsilon^3\sigma_{\bar{\alpha}\bar{\alpha},12})$$

Defining dotted Greek indices $\dot{\alpha} \in [3, M]$, the $G_{22}$ equation is

$$2\epsilon^0(f_{,1\dot{\beta}}f_{,1\dot{\beta}} - f_{,11}f_{,\dot{\beta}\dot{\beta}}) + \epsilon^{-2\beta}(f_{,\dot{\alpha}\dot{\beta}}f_{,\dot{\alpha}\dot{\beta}} - f_{,\dot{\alpha}\dot{\alpha}}f_{,\dot{\beta}\dot{\beta}}) = -\epsilon^{2-\delta}\{c_4(\epsilon^2\sigma_{11,11} + \epsilon^0\sigma_{11,\bar{\beta}\bar{\beta}} + \epsilon^4\sigma_{\bar{\alpha}\bar{\alpha},11} + \epsilon^2\sigma_{\bar{\alpha}\bar{\alpha},\bar{\beta}\bar{\beta}}) + \epsilon^{4\beta}\sigma_{22,11} + \epsilon^{2\beta}\sigma_{22,\bar{\alpha}\bar{\alpha}}\}$$

and the $G_{23}$ equation is

$$2\epsilon^0(f_{,23}f_{,11} - f_{,21}f_{,31}) + 2\epsilon^{-2\beta}(f_{,23}f_{,\dot{\beta}\dot{\beta}} - f_{,2\dot{\beta}}f_{,3\dot{\beta}}) = -\epsilon^{2-\delta}(\epsilon^0c_4\sigma_{11,23} + \epsilon^2c_4\sigma_{\bar{\alpha}\bar{\alpha},23} + \epsilon^{4\beta}\sigma_{23,11} + \epsilon^{2\beta}\sigma_{23,\bar{\alpha}\bar{\alpha}})$$

If $\beta \geq 0$, then all three equations give the constraint $-2\beta = 2 - \delta$ in agreement with the $G_{11}$ component. If $\beta < 0$, then all three give $0 = 2 - \delta + 4\beta$, again in agreement with $G_{11}$. 

24
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by parts moves the derivative onto $G_{\alpha\beta}$. 

26
FIG. 1. The ridge in a semi-infinite strip when $h = 0$ and $h > 0$. We have labeled the $h = 0$ ridge to show the ridge length $X$, the dihedral angle $D$, and the manifold coordinate system $(x_1, x_2)$.

FIG. 2. The manifold coordinate system $(x_1, x_2, x_3)$ for the semi-infinite rod $\mathbb{R}^1 \times \mathcal{C}$. The location of the planar ridge is indicated in grey.