Monotone Grid Drawings of Planar Graphs

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Abstract. A monotone drawing of a planar graph \( G \) is a planar straight-line drawing of \( G \) where a monotone path exists between every pair of vertices of \( G \) in some direction. Recently monotone drawings of planar graphs have been proposed as a new standard for visualizing graphs. A monotone drawing of a planar graph is a monotone grid drawing if every vertex in the drawing is drawn on a grid point. In this paper we study monotone grid drawings of planar graphs in a variable embedding setting. We show that every connected planar graph of \( n \) vertices has a monotone grid drawing on a grid of size \( O(n) \times O(n^2) \), and such a drawing can be found in \( O(n) \) time.

1 Introduction

A straight-line drawing of a planar graph \( G \) is a drawing of \( G \) in which each vertex is drawn as a point and each edge is drawn as a straight-line segment without any edge crossing. A path \( P \) in a straight-line drawing of a planar graph is monotone if there exists a line \( l \) such that the orthogonal projections of the vertices of \( P \) on \( l \) appear along \( l \) in the order induced by \( P \). A straight-line drawing \( \Gamma \) of a planar graph \( G \) is a monotone drawing of \( G \) if \( \Gamma \) contains at least one monotone path between every pair of vertices. In the drawing of a graph in Fig. 1 the path between the vertices \( s \) and \( t \) drawn as a thick line is a monotone path with respect to the direction \( d \), whereas no monotone path exists with respect to any direction between the vertices \( s' \) and \( t' \). We call a monotone drawing of a planar graph a monotone grid drawing if every vertex is drawn on a grid point.

Monotone drawings of graphs are well motivated by human subject experiments by Huang et al. [8], who showed that the “geodesic tendency” (paths following a given direction) is important in comprehending the underlying graph. Upward drawings [5,10,49] are related to monotone drawings where every directed path is monotone with respect to the vertical line, while in a monotone drawing each monotone path, in general, is monotone with respect to a different line. Arkin et al. [3] showed that any strictly convex drawing of a planar graph is monotone and they gave an \( O(n \log n) \) time algorithm for finding such a path between a pair of vertices in a strictly convex drawing of a planar graph of \( n \) vertices. Angelini et al. [1] showed that every biconnected planar graph of \( n \) vertices
has a monotone drawing in real coordinate space. They also showed that every tree of \( n \) vertices admits a monotone grid drawing on a grid of \( O(n) \times O(n^2) \) or \( O(n^{1.6}) \times O(n^{1.6}) \) area. It is known that every outerplane graph of \( n \) vertices admits a monotone grid drawing on a grid of area \( O(n) \times O(n^2) \) \cite{2}. Recently, Hossain and Rahman \cite{7} showed that every series-parallel graph of \( n \) vertices admits a monotone grid drawing on an \( O(n) \times O(n^2) \) grid, and such a drawing can be found in \( O(n \log n) \) time. It is also known that not every plane graph (with fixed embedding) admits a monotone drawing \cite{1}.

In this paper we investigate whether every connected planar graph has a monotone drawing and what are the area requirements for such a drawing on a grid. We show that every connected planar graph of \( n \) vertices has a monotone grid drawing on an \( O(n) \times O(n^2) \) grid, and such a drawing can be computed in \( O(n) \) time. As a byproduct, we introduce a spanning tree of a plane graph with some interesting properties. Such a spanning tree may find applications in other areas of graph algorithms as well.

We now give an outline of our algorithm for constructing a monotone grid drawing of a planar graph \( G \). We first construct a “good spanning tree” \( T \) of \( G \) and find a monotone drawing of \( T \) by the method given in \cite{1}. We then draw each non-tree edge by a straight-line segment by shifting the drawing of some subtree of \( T \), if necessary. Figure 2 illustrates the steps of our algorithm. The input planar graph \( G \) is shown in Fig. 2(a). We first find a planar embedding of \( G \) containing a good spanning tree as illustrated in Fig. 2(b), where the edges of the spanning tree are drawn by thick lines. We then find a monotone drawing of \( T \) on \( O(n) \times O(n^2) \) grid using the algorithm in \cite{1} as illustrated in Fig. 2(c). Finally we elongate the drawing of some edges and draw the non-tree edges of \( G \) using straight-line segments as illustrated in Fig. 2(d).

The rest of the paper is organized as follows. Section 2 describes some of the definitions that we have used in our paper. Section 3 deals with monotone drawings of connected planar graphs. Finally, Section 4 concludes the paper with discussions.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{The path between vertices \( s \) and \( t \) (as shown by thick line) is monotone with respect to direction \( d \).}
\end{figure}
2 Preliminaries

In this section we give some definitions and present a known result. For the graph theoretic terminologies not given here, see [9].

Let \( G = (V, E) \) be a connected graph with vertex set \( V \) and edge set \( E \). A subgraph of \( G \) is a graph \( G' = (V', E') \) such that \( V' \subseteq V \) and \( E' \subseteq E \). The degree of a vertex \( v \) in \( G \) is denoted by \( d(v) \). We denote an edge joining vertices \( u \) and \( v \) of \( G \) by \( (u, v) \). A pair \( \{u, v\} \) of vertices in \( G \) is a split pair if there exist two subgraphs \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) satisfying the following two conditions: 1. \( V = V_1 \cup V_2, V_1 \cap V_2 = \{u, v\} \); and 2. \( E = E_1 \cup E_2, E_1 \cap E_2 = \emptyset, |E_1| \geq 1, |E_2| \geq 1 \). Thus every pair of adjacent vertices is a split pair. A \( \{u, v\} \)-split component of a split pair \( u, v \) in \( G \) is either an edge \( (u, v) \) or a maximal connected subgraph \( H \) of \( G \) such that \( \{u, v\} \) is not a split pair of \( H \). If \( v \) is a vertex in \( G \), then \( G - v \) is the subgraph of \( G \) obtained by deleting the vertex \( v \) and all the edges incident to \( v \). Similarly, if \( e \) is an edge of \( G \), then \( G - e \) is a subgraph of \( G \) obtained by deleting the edge \( e \). Let \( v \) be a cut-vertex in a connected graph \( G \). We call a subgraph \( H \) of \( G \) a \( v \)-component if \( H \) consists of a connected component \( H' \) of \( G - v \) and all edges joining \( v \) to the vertices of \( H' \).

Let \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) be two graphs. The union of \( G_1 \) and \( G_2 \), denoted by \( G_1 \cup G_2 \), is a graph \( G_3 = (V_3, E_3) \) such that \( V_3 = (V_1 \cup V_2) \) and \( E_3 = (E_1 \cup E_2) \).

Fig. 2. Illustration for an outline of our algorithm
Let $G = (V, E)$ be a graph and $T = (V, E')$ be a spanning tree of $G$. An edge $e \in E$ is called a tree edge if $e \in E'$ otherwise $e$ is said to be a non-tree edge. Let $e$ be a non-tree edge with respect to $G$ and $T$. Then by $T \cup e$ we denote the subgraph $G'$ of $G$ obtained adding edge $e$ to $T$. The graph $G' = T \cup e$ always has a single cycle $C$ and we call $C$ the cycle induced by the non-tree edge $e$. If $X$ is a set of edges of $G$ and $T$ is a spanning tree of $G$ then $T \cup X$ denotes the graph obtained by adding the edges in $X$ to $T$ and replacing each multi-edge by a single edge. Let $T$ be a rooted tree and let $u$ be a vertex of $T$. Then by $T_u$ we denote the subtree of $T$ rooted at $u$. By $T - T_u$ we denote the tree obtained from $T$ by deleting the subtree $T_u$.

A graph is planar if it can be embedded in the plane without edge intersections except at the vertices where the edges are incident. A plane graph is a planar graph with a fixed planar embedding. A plane graph divides the plane into some connected regions called faces. The unbounded region is called the outer face and each of the other faces is called an inner face. Let $G$ be a plane graph. The boundary of the outer face of $G$ is called the outer boundary of $G$. We call a simple cycle induced by the outer boundary of $G$ an outer cycle of $G$. We call a vertex $v$ of $G$ an outer vertex of $G$ if $v$ is on the outer boundary of $G$, otherwise $v$ is an inner vertex of $G$.

Let $p$ be a point in the plane and $l$ be a half-line with an end at $p$. The slope of $l$, denoted by $\text{slope}(l)$, is the angle spanned by a counterclockwise rotation that brings a horizontal half-line started at $p$ and directed towards increasing $x$-coordinates to coincide with $l$. Let $\Gamma$ be a drawing of a graph $G$ and let $(u,v)$ be an edge of $G$. We denote the direction of a half-line by $d(u,v)$ which is started at $u$ and passed through $v$. The direction of a drawing of an edge $e$ is denoted by $d(e)$ and the slope of the drawing of $e$ is denoted by $\text{slope}(e)$.

Let $G$ be a planar graph and $\Gamma$ be a straight-line drawing of $G$. A path $u = u_1, \ldots, u_k = v$ between vertices $u$ and $v$ in $G$ is denoted by $P(u,v)$. The drawing of the path $P(u,v)$ in $\Gamma$ is monotone with respect to a direction $d$ if the orthogonal projections of vertices $u_1, \ldots, u_k$ on $d$ appear in the same order as the vertices appear on the path. The drawing $\Gamma$ is a monotone drawing of $G$ if there exists a direction $d$ for every pair of vertices $u$ and $v$ such that $P(u,v)$ is monotone with respect to $d$. A monotone drawing is a monotone grid drawing if every vertex is drawn on a grid point. The following lemma is known from [1].

**Lemma 1.** Let $T$ be a tree of $n$ vertices. Then $T$ admits a monotone grid drawing on a grid of area $O(n) \times O(n^2)$, and such a drawing can be found in $O(n)$ time.

In this paper we use a modified version of the algorithm for monotone grid drawing of a tree in [1], which we call Algorithm **Draw-Monotone-Tree** throughout this paper. Algorithm **Draw-Monotone-Tree** first assigns a slope to each vertex of a planar embedded tree then obtains a slope-disjoint drawing of the tree which is monotone. A brief description of the algorithm is given in the rest of this section. Let $T$ be an embedded rooted tree of $n$ vertices. (Note that in [1] $T$ is not embedded, but here we use $T$ as an embedded tree for the sake of our
algorithm.) Let $S = \{s_1, s_2, \ldots, s_{n-1}\} = \{1/1, 2/1, 3/1, \ldots, (n-1)/1\}$ be an ordered set of $n$ slopes in increasing order, where each slope is represented by the ratio $y/x$. Let $v_1, v_2, \ldots, v_n$ be an ordering of vertices in $T$ in a counterclockwise postorder traversal. (In a counterclockwise postorder traversal of a rooted ordered tree, subtrees rooted at the children of the root are recursively traversed in counterclockwise order and then the root is visited.) Then we assign the slope $s_i$ to vertex $v_i$ ($i \neq n$). Let $u_1, u_2, \ldots, u_k$ be the children of $v$ in $T$. Then the subtree $T_{u_i}$ gets $|T_{u_i}|$ consecutive elements of $S$ from the $(1 + \sum_{j=1}^{i-1} |T_{u_j}|)$-th to the $(\sum_{j=1}^{i} |T_{u_j}|)$-th. Let $v'$ be the parent of $v$. If $v$ is not the root of $T$ then the drawing of the edge $e = (v', v)$ will be a straight-line with slope $s_i$.

We now describe how to find a monotone grid drawing of $T$ using the slope assigned to each vertex of $T$. We first draw the root vertex $r$ at $(0, 0)$, and then use a counter clockwise preorder traversal for drawing each vertex of $T$. (In a counterclockwise preorder traversal of a rooted ordered tree, first the root is visited and then the subtrees rooted at the children of the root are visited recursively in counterclockwise order.) We fix the position of a vertex $u$ when we traverse $u$. Note that when we traverse $u$, the position of the parent $p(u)$ has already been fixed. Let $(p_x(u), p_y(u))$ be the position of $p(u)$. Then we place $u$ at grid point $(p_x(u) + x_b, p_y(u) + y_b)$, where $s_b = y_b/x_b$. Figure 3(b) illustrates a monotone grid drawing of the tree as shown in the Fig. 3(a). Algorithm **Draw-Monotone-Tree** computes a slope-disjoint monotone drawing of a tree on an $O(n) \times O(n^2)$ grid in linear time [1].

![Fig. 3.](image-url) (a) A tree $T$ with assigned slope to each vertex, (b) a monotone drawing $\Gamma$ of $T$ and (c) a monotone drawing $\Gamma'$ of $T$ with elongation of the edge $(j, g)$. 
3 Monotone Grid Drawings

In this section we show that every connected planar graph of \( n \) vertices has a monotone grid drawing on an \( O(n) \times O(n^2) \) grid.

Let \( G \) be a planar graph and let \( G_\phi \) be a plane embedding of \( G \). Let \( T \) be an ordered rooted spanning tree of \( G_\phi \) such that the root \( r \) of \( T \) is an outer vertex \( G_\phi \), and the ordering of the children of each vertex \( v \) in \( T \) is consistent with the ordering of the neighbors of \( v \) in \( G_\phi \). Let \( P(r,v) = (r = u_1, u_2, \ldots, (v = u_k) \) be the path in \( T \) from the root \( r \) to a vertex \( v \neq r \). The path \( P(r,v) \) divides the children of \( u_i \), \( 1 \leq i < k \), except \( u_{i+1} \), into two groups; the left group \( L \) and the right group \( R \). A child \( x \) of \( u_i \) is in group \( L \) and denoted by \( u_i^L \) if the edge \( (u_i, x) \) appears before the edge \( (u_i, u_{i+1}) \) in clockwise ordering of the edges incident to \( u_i \) when the ordering is started from the edge \( (u_i, u_{i+1}) \), as illustrated in the Fig. 4(a). Similarly, a child \( x \) of \( u_i \) is in the group \( R \) and denoted by \( u_i^R \) if the edge \( (u_i, x) \) appears after the edge \( (u_i, u_{i+1}) \) in clockwise order of the edges incident to \( u_i \) when the ordering is started from the edge \( (u_i, u_{i+1}) \). We call \( T \) a \textit{good spanning} tree of \( G_\phi \) if every vertex \( v \) \( (v \neq r) \) of \( G \) satisfies the following two conditions with respect to \( P(r,v) \).

\begin{itemize}
  \item[(Cond1)] \( G \) does not have a non-tree edge \( (v, u_i), i < k \); and
  \item[(Cond2)] the edges of \( G \) incident to the vertex \( v \) excluding \( (u_{k-1}, v) \) can be partitioned into three disjoint (possibly empty) sets \( X_v, Y_v \) and \( Z_v \) satisfying the following conditions (a)-(c) (see Fig. 4(b)):
    \begin{itemize}
      \item[(a)] Each of \( X_v \) and \( Z_v \) is a set of consecutive non-tree edges and \( Y_v \) is a set of consecutive tree edges.
      \item[(b)] Edges of set \( X_v, Y_v \) and \( Z_v \) appear clockwise in this order from the edge \( (u_{k-1}, v) \).
      \item[(c)] For each edge \( (v, v') \in X_v, v' \) is contained in \( T_{u_i^L}, i < k \), and for each edge \( (v, v') \in Z_v, v' \) is contained in \( T_{u_i^R}, i < k \).
    \end{itemize}
\end{itemize}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4.png}
\caption{(a) An illustration for \( P(r,v) \), \( L \) and \( R \) groups, (b) an illustration for \( X_v, Y_v \) and \( Z_v \) sets of edges, and (c) an illustration for a good spanning tree \( T \) on \( G_\phi \) where bold edges are tree edges.}
\end{figure}

Figure 4(c) illustrates a good spanning tree \( T \) in a plane graph. The following two lemmas are based on the properties of a good spanning tree and on Lemma 1.
Let \( T \) be a good spanning tree of \( G \). Let \( \Gamma \) be a monotone drawing of \( T \). Let \( \Gamma' \) be a straight-line drawing of \( T \) obtained from \( \Gamma \) by elongation of the drawing of an edge \( e \) of \( T \) preserving the slope of \( e \). Then \( \Gamma' \) is also a monotone drawing. (See Fig. 3(c), where the edge \((j,g)\) is elongated.)

**Proof.** Let \( e = (u,v) \) be an edge of \( T \) where \( u \) is the parent of \( v \) in \( T \). The elongation of the drawing of \( e \) does not change slope of \( e \) and the drawing of \( T_v \) in \( \Gamma' \) is shifted outwards preserving its drawing in \( \Gamma \). The drawing of \( T - T_v \) is remained same in \( \Gamma' \). Since the elongation does not change the slope of the drawing of any edge, the new drawing \( \Gamma' \) preserves monotone drawing of \( T \).

Let \( T \) be a good spanning tree of \( G \). Let \( v \) be a vertex in \( G \), and let \( u \) be the parent of \( v \) in \( T \). Let \( X \subseteq X_v \) and \( Z \subseteq Z_v \). Assume that \( \Gamma \) is a the monotone drawing of \( T \cup X \cup Z \) where a monotone path exists between every pair of vertices in the drawing of \( T \) in \( \Gamma \). If a straight-line drawing \( \Gamma'' \) of \( T \cup X \cup Z \) is obtained from \( \Gamma \) by elongation of the drawing of the edge \((u,v)\), then \( \Gamma'' \) is a monotone drawing of \( T \cup X \cup Z \) where a monotone path exists between every pair of vertices in the drawing of \( T \) in \( \Gamma'' \).

**Proof.** Let \( r \) be the root of \( T \). Let \( m \) be the slope assigned to the vertex \( v \) in \( T \).

Let \( M_X \) and \( M_Z \) be the sets of slopes assigned to \( T_u \) and \( T_{u,v} \), respectively. According to assignment of slopes, for any \( m_x \in M_X \) and \( m_z \in M_Z \) the relation \( m_x > m > m_z \) holds.

Since each vertex in \( X \) and \( Z \) are visible from the vertex \( v \) in \( \Gamma \) and \( m_x < m < m_z \), \( v \) must be visible from each vertex in \( X \) and \( Z \) even after elongation of edge \((u,v)\) without changing the slope of \((u,v)\). Note that the elongation only changes the slopes of the drawings of non-tree edges in \( X \) and \( Z \). The drawing of \( T_v \) is shifted outwards preserving the slopes of the edges in \( T_v \) and the drawing of \( T - T_v \) is remained same. Let \( \Gamma' \) be the new drawing of \( T \cup X \cup Z \). Then obviously the edges in \( X \) and in \( Z \) does not produce any edge crossing in \( \Gamma'' \).

By Lemma 2 the elongation of the edge \((u,v)\) does not break the monotone property in the drawing of \( T \) in \( \Gamma'' \). Thus a monotone path exists between every pair of vertices in the drawing of \( T \) in \( \Gamma'' \).

We now have the following lemma on monotone grid drawings of a plane graph with a good spanning tree.

**Lemma 4.** Let \( G \) be a planar graph of \( n \) vertices and let \( G_{\phi} \) be a plane embedding of \( G \). Assume that \( G_{\phi} \) has a good spanning tree \( T \). Then \( G_{\phi} \) admits a monotone grid drawing on an \( O(n) \times O(n^2) \) grid.

**Proof.** Let \( T \) be a good spanning tree of \( G_{\phi} \). We prove the claim by induction on the number \( z \) of non-tree edges in \( G \) with respect to \( T \). Algorithm **Draw-Monotone-Tree** uses a counterclockwise postorder traversal for finding a vertex ordering in the tree and assigns slope to each vertex of the tree using that ordering. Note that the ordering of the vertices is fixed once the child of \( r \) that has to be visited first is fixed. Let \( T \) be a good spanning tree of \( G_{\phi} \) and let \( r \) be the root of \( T \). We take a child \( s \) of \( r \) as the first child to be visited in
counter-clockwise post-order traversal such that \( s \) is an outer vertex of \( G_\phi \) and if \( s \) is on an outer cycle \( C \) of \( G_\phi \) then \( s \) is the counterclockwise neighbor of \( r \) on \( C \). We call the edge \((r, s)\) the reference edge of \( T \). (Later in Corollary 1 we show that such a reference edge always exists.) By induction on the number \( z \) of non-tree edges of \( G_\phi \), we now prove the claim that \( G_\phi \) admits a monotone grid drawing on an \( O(n) \times O(n^2) \) grid and a monotone path exists between every pair of vertices of \( G_\phi \) through the edges of \( T \) in the drawing.

We first assume that \( z = 0 \). In this case \( G_\phi = T \). We then find monotone drawing \( \Gamma \) of \( T \) on an \( O(n) \times O(n^2) \) grid using Algorithm \textbf{Draw-Monotone-Tree} taking the reference edge \((r, s)\) as the starting edge for traversals (counter-clockwise post-order traversal for slope assignment and counter-clockwise preorder traversal for drawing vertices). Since \( G_\phi = T \), \( \Gamma \) is a monotone drawing of \( G_\phi \).

That is, a monotone path exists between every pair of vertices of \( T \) in \( \Gamma \).

We thus assume that \( z > 0 \) and the claim holds for any plane graph \( G_\phi \) with number of non-tree edges \( z' \), where \( z' < z \).

Let \( G_\phi \) have \( z \) non-tree edges with respect to \( T \) and let \( e = (u, v) \) be a non-tree edge of the outer boundary of \( G_\phi \).

Let \( m_u \) and \( m_v \) be the slopes assigned to the vertices \( u \) and \( v \), respectively in \( T \) by Algorithm \textbf{Draw-Monotone-Tree}. Without loss of generality let us assume \( m_u > m_v \). Let \( w \) be the common ancestor of \( u \) and \( v \) in \( T \) and let \( u' \) and \( v' \) be the parents of \( u \) and \( v \) in \( T \). According to (Cond1) \( u \) does not lie on the path \( P(v, r) \) and \( v \) does not lie on the path \( P(u, r) \). Let \( C = \{ P(u, w) \cup P(v, w) \cup (u, v) \} \) and \( G'_\phi = \{ P(r, w) \cup C \} \). Clearly \( G'_\phi \) has \( l_1 \) non-tree edges where \( l_1 < z \).

By induction hypothesis, \( G'_\phi - (u, v) \) has a straight-line monotone drawing on \( O(n) \times O(n^2) \) grid where the edges in \( T \) are drawn with the slope assigned to them and a monotone path exists between every pair of vertices through the edges in \( T \). Let \( \Gamma' \) be the drawing of \( G'_\phi - (u, v) \) in \( \Gamma \). Let \( p_x \) be the largest \( x \)-coordinate used for the drawing of \( \Gamma' \). We now shift the drawing of \( T_u \) and \( T_v \) on the line \( x = p_x + 1 \) by preserving slopes of the drawings of the edges \((u', u)\) and \((v', v)\). Since the slopes are integer numbers, it guarantees that all vertices remain on grid points after the shifting operation. According to Lemma 2, elongations of \((u', u)\) and \((v', v)\) do not produce any edge crossing in the drawing. According to (Cond2) in the good spanning tree \( T \), \( e \) belongs to set \( Z_u \) and set \( Z_v \). Then no tree edge incident to \( u \) exists between the edge \((u', u)\) and \((u, v)\) in counterclockwise from the edge \((u', u)\), and no tree edge incident to \( v \) exists between the edge \((v', v)\) and \((v, u)\) in clockwise from the edge \((v', v)\). (Remember that we have used counterclockwise post-order traversal starting from a reference edge for ordering the vertices in algorithm \textbf{Draw-Monotone-Tree}.) Hence we can draw the edge \( e \) on the line \( x = p_x + 1 \) by a straight-line segment without any edge crossings. In worst case, \( y \)-coordinate can be at most \( O(n^2) + O(n^2) \). Hence the drawing takes a grid of size \( O(n) \times O(n^2) \).

We now prove that every connected planar graph \( G \) has an embedding \( G_\phi \) where a good spanning tree \( T \) of \( G_\phi \) exists. We give a constructive proof for our claim. Before giving our formal proof we give an outline of our construction using an illustrative example in Fig. 5. We take an arbitrary plane embedding
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$G_\gamma$ of $G$ and start breath-first-search (BFS) from an arbitrary outer vertex $v$ of $G_\gamma$ and regard $r$ as the root of our desired spanning tree. In Fig. 5(a) BFS is started from vertex $a$, and vertex $b, c$ and $d$ are visited from $a$ in this order by BFS, as illustrated in Fig. 5(b). Next we visit $e$ from $b$ by BFS, as illustrated in Fig. 5(c). When we visit a new vertex $x$ then we check whether there is an edge $(x, y)$ such that $y$ is already visited and there is an $(x, y)$-split component or an $x$-component or a $y$-component inside the cycle induced by the edge $(x, y)$ which does not contain the root $r$. The $\{e, d\}$-split component $H_1$ induced by the vertices $\{d, h, i, j, e\}$ is such a split component in Fig. 5(c) and the subgraph $H_2$ induced by vertices $d, m, n$ is such a $y$-component for $y = d$ which are inside the cycle induced by the edge $(e, d)$. We move the subgraphs $H_1$ and $H_2$ out of the cycle induced by the non-tree edge $(e, d)$, as illustrated in Fig. 5(d). Since $(b, e)$ is a tree edge and $(e, d)$ is a non-tree edge, according to definition of a good spanning tree, the edges $(e, f)$ and $(e, k)$ must be non-tree edges. Similarly since $(a, d)$ is a tree edge and $(e, d)$ a non-tree edge, then the edge $(d, l)$ must be non-tree edge. We mark $(e, f)$, $(e, k)$ and $(d, l)$ non-tree edges as shown in the Fig. 5(e). We then visit vertices $f, l, m, n$ and $g$, as illustrated in Fig. 5(f). When we visit $k$, we find a $k$-component $H$ induced by vertices $\{k, p, o\}$ and we move $H$ out of the cycle induced by $(e, k)$ as shown in Fig. 5(g). Finally, at the end of BFS we find an embedding $G_\phi$ of $G$ and a good spanning tree $T$ as illustrated in Fig. 5(h), where the good spanning tree $T$ is shown by solid edges, and non-tree edges are shown by dashed edges.

![Diagrams showing the construction of a good spanning tree T.](image-url)

**Fig. 5.** Illustration for an outline of construction of a good spanning tree $T$. White vertices are visited vertices. Black vertices are not visited. Solid edges are tree edges. Dashed edges are non-tree edges.

We now formally prove our claim as in the following lemma.

**Lemma 5.** Let $G$ be a planar graph of $n$ vertices. Then $G$ has a plane embedding $G_\phi$ that contains a good spanning tree.
Proof. We give a constructive proof. Let \( G_\gamma \) be any arbitrary embedding of \( G \). We first mark an arbitrary outer vertex \( r \) of \( G_\gamma \) visited, and start clockwise BFS from \( r \). The vertex \( r \) will be the root of the BFS tree \( T \).

Note that after visiting each vertex by BFS, the embedding of \( G \) may be changed by our algorithm. Let \( G_\gamma^1 \) be the embedding of \( G \) after visiting the \( i \)th vertex by BFS. Then \( G_\gamma^1 = G_\gamma \) since we do not change the embedding after visiting \( r \). Let \( T^1 \) be the BFS tree after visiting the \( i \)th vertex. Then \( T^1 \) contains the single vertex \( r \). In counterclockwise BFS, we first visit a neighbor \( s \) of \( r \) such that \( s \) is an outer vertex of \( G_\gamma \) and if \( s \) is on an outer cycle \( C \) of \( G_\gamma \), then \( s \) is the counterclockwise neighbor of \( r \) on \( C \). We call the edge \((r, s)\) the BFS-Start edge.

We now assume that vertices \( w_1 (= r), w_2, \ldots, w_{j-1} \) \((j - 1 < n)\) are visited by BFS and we are visiting \( w_j \) form \( w' \), that is, \( w' \) is the parent of \( w_j \) in \( T^j \). We mark \( w_j \) as visited and mark \((w', w_j)\) as a tree edge. If there is no edge \( e = (w_j, v) \) such that \( v \in V(T^j-1) \) and \( v \neq w' \), then we proceed for the next vertex \( w_{j+1} \). Otherwise, an edge \( e = (w_j, v) \) exists such that \( v \in V(T^j-1) \) and \( v \neq w' \). In such a case we mark \( e \) as a non-tree edge, and change the embedding of \( G_\gamma^j \) to get \( G_\gamma^j \), if necessary, as follows.

We set \( x = w_j \) and \( y = v \) if \( v \) comes earlier in the counterclockwise postorder traversal of \( T^j \) started from the BFS-Start edge; otherwise, we set \( x = v \) and \( y = w_j \). Let \( C \) be the cycle induced by the non-tree edge \( e = (x, y) \). Let \( G_j(C) \) be the plane subgraph of \( G_j \) inside \( C \) (including \( C \)). We check whether there is any \((x, y)\)-split component or \( x \)-component or \( y \)-component in \( G_j(C) \). If there is a \((x, y)\)-split component or a \( x \)-component or a \( y \)-component \( H \) in \( G_j(C) \) such that \( r \notin V(H) \). We move \( H \) out of the cycle \( C \) and obtain embedding \( G_\gamma \). In Fig. (a) \( I_1, \ldots, I_k \) are \((x, y)\)-split components, \( J_1, \ldots, J_l \) are \( x \)-components and \( K_1, \ldots, K_m \) are \( y \)-components, are move out of \( C \) in Fig. (b).

![Diagram](image-url)

**Fig. 6.** Illustration for \((x, y)\)-split components, \( x \)-components and \( y \)-components.

One can observe that unvisited vertices inside cycle \( C \) can be accessed from a visited vertex other than \( x \) and \( y \) in a later BFS step. We thus mark some edges incident to \( x \) and \( y \) as non-tree edges for maintaining the properties of a good spanning tree for \( T^j \), as follows. Let \( y_p \) and \( x_p \) be the parent of \( y \) and \( x \) in \( T^j \), respectively. Let \( E_z = \{e_1, e_2, \ldots, e_k\} \) be the set of edges which are incident to
the vertex \( x \), and between the edges \((x_p, x)\) and \((x, y)\) in counterclockwise order starting from the edge \((x_p, x)\) in \( G_j^\gamma \), as shown in Fig. 7. We mark the edges in \( E_x \) as non-tree edges. Note that the edges set \( E_x \cup \{(x, y)\} \) will be the set \( Z_x \) with respect to the vertex \( x \) in the final good spanning tree. Let \( E_x = \{e_1, e_2, \ldots, e_l\} \) be the edges which are incident to the vertex \( y \) between the edges \((y_p, y)\) and \((y, x)\) in clockwise order started from the edge \((y_p, y)\) in \( G_j^\gamma \) as shown in Fig. 7. We also mark the edges in \( E_x \) as non-tree edges. The edge set \( E_x \cup \{(y, x)\} \) will be the set \( X_y \) with respect to the vertex \( y \) in the final spanning tree.

Finally we get \( G^n_\gamma \) and \( T^n \) after visiting \( n \)th vertex. We now show that \( T^n \) is a good spanning tree in \( G^n_\gamma \).

Clearly \( T^n \) is a spanning tree in \( G^n_\gamma \), since \( T^n \) consists of tree edges identified by BFS in the connected graph \( G^n_\gamma \).

We now show that the embedded tree \( T^n \) is a good spanning tree in the embedded graph \( G^n_\gamma \). Firstly, each vertex \( v \) of \( T^n \) satisfies the condition (Cond1) of good spanning trees, because the tree edges are marked from BFS steps. According to the property of BFS, every edge of \( G^n_\gamma \), whether a tree or a non-tree edge, joins two vertices whose levels differ by at most one. For a non-tree edge \( e = (w_j, v) \), if \( w_j \) lies on the path \( P(r, v) \) in \( T^n \) then \( w_j \) is the parent of \( v \). In this case \( e \) must be a tree edge. Similarly, \( v \) does not lie on the path \( P(r, w_j) \) in \( T^n \) when \( e = (w_j, v) \) is a non-tree edge. Hence, (Cond1) holds.

We next show that \( T^n \) satisfies (Cond2). Consider the situation when we dealt with edge \((x, y)\) while constructing \( G^j_\gamma \) and \( T^j \). For the non-tree edge \((x, y)\), the non-tree edges in \( Z_x \) are consecutive non-tree edges with respect to \( x \) in \( G^j_\gamma \). Let \( Z \) be the set of vertices that contain other end vertices of the edges in \( Z_x \) edges. Clearly each vertex in \( Z \) must be inside the cycle induced by \((x, y)\) in \( T^n \). One can easily observe that if we traverse \( T^n \) as counterclockwise postorder traversal starting form the BFS-Start edge, the vertex \( x \) will be visited after visiting all vertices in \( Z \). Thus each vertex in \( Z \) is contained in \( T_{u,v} \) where \( u_i \) is a vertex that lies on the path \( P(r, x) \) and \( x \neq u_i \) in \( T^n \). Similarly for the vertex \( y \), it can be shown that the other end vertices of the edges in \( X_y \) are contained in \( T_{u^l} \). Thus one can easily observe that all edges incident to a vertex \( v \) in \( T^n \) can be partitioned into three consecutive edge sets \( X_v, Y_v \) and \( Z_v \). Hence, (Cond2) holds. Hence \( T^n \) is a good spanning tree in \( G_\phi = G^n_\gamma \).
We have the following corollary based on the proof of Lemma 5.

**Corollary 1.** Let $T$ be a good spanning tree in the embedding $G_\phi$ of $G$ obtained by the construction given in the proof of Lemma 5. Then $T$ always has an edge with the property of the reference edge, mentioned in the proof of Lemma 4.

**Proof.** The DFS-Start edge can be taken as the reference edge, mentioned in the proof of Lemma 4.

The following theorem is the main result of this paper.

**Theorem 1.** Every connected planar graph of $n$ vertices admits a monotone grid drawing on a grid of area $O(n) \times O(n^2)$, and such a drawing can be found in $O(n)$ time.

**Proof.** Let $G$ be a connected planar graph. By Lemma 5 $G$ has a plane embedding $G_\phi$ such that $G_\phi$ contains a good spanning tree $T$. By Lemma 4 $G_\phi$ admits a monotone grid drawing on a grid of area $O(n) \times O(n^2)$.

We can find a good spanning tree $G_\phi$ of $G$ using the construction in the proof of Lemma 5. After visiting each vertex during construction we need to identify $v$-components and $\{u, v\}$-split components for a non-tree edge $(u, v)$ then we need to check whether these components are inside of $G_j(C)$ in the intermediate step. If any component is found then we need to move out the component.

A $v$-component is introduced by a cut vertex. All cut vertices can be found in $O(n + m)$ time using DFS. $v$-components and $\{u, v\}$-split components can also be found in linear time [6]. We maintain a data structure to store each cut vertex or every pair of vertices with split components. We then use this record in the intermediate steps for finding $G_\phi$ in Lemma 5. Let us assume we are traversing $(u, v)$ non-tree edge in a intermediate step $j$ of our algorithm. We can check whether any $u$-components, $v$-components and $\{u, v\}$-split components of $G$ exist in $G_j(C)$ of Lemma 5 by checking each edges incident to $u$ and $v$. This checking costs $O(d(u) + d(v))$ time. Throughout the algorithm it needs $O(m)$ time. For moving a component outside of cycle $C$ we need to change at most four pointers in the adjacency list of $u$ and $v$, which takes $O(1)$ time. Hence the required time is $O(m)$. Since $G$ is a planar graph, $G_\phi$ can be constructed in $O(n)$ time.

After constructing $G_\phi$ we can construct a monotone drawing of $G$ using a recursive algorithm based on the inducting proof of Lemma 4. It is not difficult to implement the recursive algorithm in $O(n)$ time.

### 4 Conclusion

In this paper we have studied monotone grid drawings of planar graphs. We have shown that a connected planar graph of $n$ vertices has a straight-line planar monotone drawing on an $O(n) \times O(n^2)$ grid and we can find such a drawing in $O(n)$ time. Finding straight-line monotone grid drawings of planar graphs on a smaller grid is our future work.
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