ON EMBEDDINGS OF MORI DREAM SPACES

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Abstract. General criteria are given for when an embedding of a Mori dream space into another satisfies certain nice combinatorial conditions on some of their associated cones. An explicit example of such an embedding is studied.

1. Introduction

Mori dream spaces were introduced by Hu and Keel in [6] as varieties for which the minimal model program can be run in a very general way. They also are the varieties for which the Cox ring is finitely generated. Using a presentation of this ring, they showed that a Mori dream space \( X \) always embeds into a toric variety \( W \) in such a way that the combinatorics of \( W \) describes much of the birational geometry of \( X \). In this paper we give simple criteria for when embeddings between Mori dream spaces behave this way, and an explicit example of a nontrivial such embedding for a nontoric del Pezzo surface. Further background on the connection between Mori dream spaces and their Cox rings can be found in the excellent surveys [7] and [10], which also give much more extensive references to the current literature.

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2. Mori dream spaces

In this section we define our main objects of study, and discuss certain embeddings.

Definition 2.1. A normal projective variety \( X \) is a Mori Dream Space if

1. \( X \) is \( \mathbb{Q} \)-factorial and \( \text{Pic}(X) \subset \mathbb{Q} = N^1(X) \),
2. \( \text{Nef}(X) \) is the affine hull of finitely many semiample divisors,
3. there is a finite collection of birational maps \( f_i : X \dasharrow X_i \) which are isomorphisms in codimension one, such that when \( D \) is a movable divisor on \( X \), then there is an \( i \) such that \( D = f_i^* D_i \) for some semiample divisor \( D_i \) on \( X_i \).

Examples of such spaces are all Fano varieties [1] and all toric varieties [6].

These spaces have a number of nice properties, and some interesting related cone structures. Perhaps the most important property and hence their name is that the minimal model program can be run and will terminate using not just the canonical divisor to test for nefness, but any divisor [6]. Turning to the cone structures, there is a chamber structure on the pseudo-effective of divisors of a Mori dream space, which parametrizes the rational maps from it.
Definition 2.2. Given a line bundle $L$ on a scheme $X$, the section ring is
$$R_L := \bigoplus_{n \in \mathbb{N}} H^0(X, L^\otimes n).$$

Note that when $D$ is effective and $R_D$ is finitely generated, there is an induced rational map
$$f_D : X \dasharrow \text{Proj}(R_D).$$

We say that two such divisors, $D_1$ and $D_2$, are Mori equivalent if the rational maps $f_{D_i}$ have isomorphic images making the natural triangular diagram commute. The equivalence classes, which we call Mori chambers, impose a natural chamber structure on $\overline{\text{NE}^1}(X)$. For Mori dream spaces, these chambers are the same as the GIT chambers, equivalence classes of effective divisors whose linearizations give the same GIT quotient. These chamber structures are tied to a single ring:

Definition 2.3. Suppose that the classes of the line bundles $L_1, ..., L_n$ are a basis of $\text{Pic}(X)$. The Cox ring of $X$ is defined:
$$\text{Cox}(X, L_1, ..., L_n) := \bigoplus_{(m_1, ..., m_n) \in \mathbb{Z}^n} H^0(X, L_1^{\otimes m_1} \otimes ... \otimes L_n^{\otimes m_n}).$$

Note that this ring is graded naturally by $\text{Pic}(X)$. Furthermore, when the choice of basis does not matter, such as for determining finite generation, we use the notation $\text{Cox}(X)$. We now get the following criterion for determining when we have a Mori dream space:

Theorem 2.4. [6] Given a $\mathbb{Q}$-factorial projective variety $X$ with $\text{Pic}(X)_\mathbb{Q} = \mathbb{N}^1(X)$, $X$ is a Mori dream space iff $\text{Cox}(X)$ is finitely generated.

In the case of a Mori dream space the Cox ring, along with a line bundle, tells us about the small birational type of $X$. To see this, note that the Cox ring has an ideal that plays the role of the irrelevant ideal. For a line bundle $L$, let $J_L := \sqrt{(R_L)}$. This ideal determines the non-semistable points in $\text{Spec}(\text{Cox}(X))$.

Theorem 2.5. [6, 7]. A Mori dream space $X$ is a good geometric quotient of $\text{Spec}(\text{Cox}(X)) - J_L$ by the torus $\text{Hom}(\text{Pic}(X), k^*)$, where $L$ is any ample line bundle on $X$.

As we vary the line bundle across the different top-dimensional GIT chambers of $X$, we get varieties with the same Cox ring, which are isomorphisms in codimension one, such as flips. Thus, we can recover $X$ and its flips by understanding $\text{Cox}(X)$ and $\text{Pic}(X)$. An interesting consequence of this is that we can identify embeddings of $X$ that respect the chamber structure.

Definition 2.6. Suppose $X$ and $Y$ are Mori dream spaces. An embedding $X \subset Y$ is called a Mori embedding if it satisfies the following:

1. The restriction of $\text{Pic}(W)_\mathbb{Q}$ to $\text{Pic}(X)_\mathbb{Q}$ is an isomorphism.
2. There is an induced morphism $\overline{\text{NE}^1}(W) \to \overline{\text{NE}^1}(X)$.
3. Every Mori chamber of $X$ is a finite union of Mori chambers of $W$.
4. Given a contraction $f : X \dasharrow X'$, there is a rational contraction $\hat{f} : W \dasharrow X'$ which restricts to $f$. 
Nontrivial such embeddings always exist for non-toric Mori dream spaces:

**Proposition 2.7.** [6] Given a Mori dream space $X$, there exists quasi-smooth projective toric variety $W$ and a Mori embedding $X \subset W$.

Lastly, we can give alternate criteria for determining when an embedding is a Mori embedding.

**Theorem 2.8.** Suppose $X$ and $W$ are Mori Dream Spaces and $\Phi : X \to W$ is an embedding such that

1. the restriction of $\text{Pic} (W)_\mathbb{Q}$ to $\text{Pic} (X)_\mathbb{Q}$ is an isomorphism, and
2. the induced map $\Phi^* : \text{Cox} (W) \to \text{Cox} (X)$ is surjective.

Then $\Phi$ is a Mori embedding.

*Proof.* First, we show the existence of an isomorphism $\overline{\text{NE}}^1 (W) \to \overline{\text{NE}}^1 (X)$ after restriction. Injectivity is an immediate consequence of (1). To show surjectivity, suppose $D$ is a representative of a class in $\overline{\text{NE}}^1 (X)$. By definition, there exists some $m$ large enough such that $H^0(X,mD) \neq 0$. Since the induced map between Cox rings is surjective by (2), there must exist a $D_W$ on $W$ which restricts to $D$ and such that $H^0(W,kD_W) \neq 0$ for some large enough $k$. Thus, the class of $D_W$ is in $\overline{\text{NE}}^1 (W)$.

Second, we show that a Mori chamber of $X$ is a finite union of Mori chambers of $W$. By definition of a Mori Dream Space, there are only finitely many chambers for $W$, thus it suffices to show that any chamber for $W$ is contained in a chamber for $X$. Suppose that $D$ is an effective divisor in a Mori chamber $C$ of $W$. Since its restriction $D_X$ is effective, finite generation of the corresponding section rings implies the following diagram commutes, where vertical arrows represent inclusions:

\[
\begin{array}{ccc}
W & \xrightarrow{\text{Proj} (R(W,D))} & \text{Proj} (R(W,D)) \\
\downarrow & & \downarrow \\
X & \xrightarrow{\text{Proj} (R(X,D_X))} & \text{Proj} (R(X,D_X))
\end{array}
\]

Note that (2) implies that the rightmost vertical arrow is a closed embedding. Now consider an effective divisor $D'$ on $W$, Mori equivalent to $D$, and restricting to $D_X$. Since by definition $\text{Proj} (R(W,D)) \cong \text{Proj} (R(W,D'))$, and the closed embeddings given by the right-hand side of the diagram are induced by the restriction of global sections of the corresponding section rings, then up to isomorphism the models given by $D_X$ and $D'_X$ are the same.

Lastly, the restriction of contractions property of a Mori embedding is an immediate consequence of the definition of Mori Dream spaces, and the containment property of Mori Cones above.

\[\square\]

### 3. Example

In this section we construct a nontrivial example of a Mori embedding. We consider the case of embedding the simplest smooth, non-toric del Pezzo surface in a toric variety found by considering a presentation of the Cox ring of the surface.
Fixing notation, let $\pi : X_4 \to \mathbb{P}^2$ be the blow-up of four points in general linear position, and denote by $h := \pi^*l$ the pullback of a line, and by $l_i$ the exceptional divisor corresponding to a blown up point. Recall that these divisors form a basis for $\text{Pic} (X_4) = \mathbb{Z}^5$. In [2] and [3] the generators for the Cox rings of smooth del Pezzo surfaces are given. With this choice of basis the generators of Cox ($X$) surjection $R$ for Pic ($X$) gives us a torus action on $\text{Spec} (R)$ compatible with the torus action on Cox ($X$), and hence the compatible torus action can be understood by considering the exact sequence of abelian groups:

$$0 \to M \to \mathbb{Z}^{\Delta(1)} \to A_1 (W) \to 0$$

where $\mathbb{Z}^{\Delta(1)} = \mathbb{Z}^{10}$ has a canonical basis given by the ten exceptional divisors $E_1, E_2, \ldots, E_{10}$. Thus, $\mathbb{Z}^{10} \to A_1 (W)$ is represented by the matrix

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

Computing the kernel of this matrix gives us the columns of the matrix $A$ representing $M \to \mathbb{Z}^{\Delta(1)}$. Taking the $\mathbb{Z}$-dual of the above sequence describes the torus action on lattices via:

$$0 \to A_1 (W)^* \to \mathbb{Z}^{10} \to N \to 0$$

To work on the level of fans we tensor the above sequence by $\mathbb{R}$, giving us maps between real vector spaces where the injective map $f : \mathbb{R}^5 \to \mathbb{R}^{10}$ given in the exact sequence is still represented by
Each column above represents the coordinates of a $T$-invariant divisor on $W$. Unfortunately, the 1-skeleton is not sufficient to describe $\Delta(W)$; Cox rings do not uniquely determine the fan of a variety. To proceed, we select a natural toric variety with this 1-skeleton, and show that we can obtain a Mori embedding for this choice.

Following Example 2.11 of [7], consider the ample divisor $D = 11h - 5l_1 - 3l_2 - 2l_3 - l_4$ on $X_4$. We find the irrelevant ideal $J_D$ by taking the radical of the monomial ideal generated by a basis of $\text{Cox}(X_4)_D$. This can be done in Macaulay 2:

```plaintext
t1 : S=QQ[a,b,c,d,e,f,x,y,z,w,Degrees=>{{1,-1,-1,0,0},{1,-1,0,-1,0},{1,-1,0,0,-1},{1,0,-1,-1,0},{1,0,-1,0,-1},{1,0,0,-1,-1},{0,1,0,0,0},{0,0,1,0,0},{0,0,0,1,0},{0,0,0,0,1}},Heft=>{3,1,1,1,1}]
t2 : radical monomialIdeal basis({11,-5,-3,-2,-1},S)
```

This gives us 42 monomials, each consisting of 5 variables out of the 10. These relations determine the cone structure of a $\mathbb{Q}$-factorial toric variety $W$ with the above 1-skeleton. Since all the cones must be simplicial in this case, we conclude from the fan structure that $W$ must be $\mathbb{P}^5$ blown up along four $T$-invariant copies of $\mathbb{P}^2$.

Remark 3.1. Choosing $D$ to be the anti-canonical divisor, we can modify the above script to get a different set of cones. We can check the combinatorics in Magma via:

```plaintext
R<a,b,c,d,e,f,x,y,z,w> := PolynomialRing(Rationals(),10);
I := [ ideal<R|a*b*c*x, a*d*e*y, a*c*d*x*y, a*f*x*y, b*d*f*z, b*c*d*x*z, b*e*x*z, a*b*f*x*z, c*d*y*z, b*d*e*y*z, a*d*f*y*z, c*e*f*w, c*d*x*w, b*c*e*x*w, a*c*f*x*w, b*e*y*w, c*d*e*y*w, a*e*f*y*w, a*f*z*w, c*d*f*z*w, b*e*f*z*w>];
Z := [[[1,1,1,1,1,0,0,0,0,0],[1,1,1,1,1,0,0,0,0,0],[1,1,1,1,1,0,0,0,0,0],[1,1,1,1,1,0,0,0,0,0],[1,1,1,1,1,0,0,0,0,0],[1,1,1,1,1,0,0,0,0,0],[1,1,1,1,1,0,0,0,0,0],[1,1,1,1,1,0,0,0,0,0],[1,1,1,1,1,0,0,0,0,0],[1,1,1,1,1,0,0,0,0,0]],
[0,0,0,0,0,0,0,0,0,0],[0,0,0,0,0,0,0,0,0,0],[0,0,0,0,0,0,0,0,0,0],[0,0,0,0,0,0,0,0,0,0],[0,0,0,0,0,0,0,0,0,0],[0,0,0,0,0,0,0,0,0,0],[0,0,0,0,0,0,0,0,0,0],[0,0,0,0,0,0,0,0,0,0],[0,0,0,0,0,0,0,0,0,0],[0,0,0,0,0,0,0,0,0,0]],
Q := [];
C := CoxRing(R,I,Z,Q);
X := ToricVariety(C);
IsQFactorial(X);
IsProjective(X);
IsComplete(X);
```

This gives us a projective toric variety with the same 1-skeleton as before, but has 22 cones in its fan, and is not $\mathbb{Q}$-factorial. Since projectivity depends solely on the 1-skeleton, this shows that all the toric varieties with the same Cox ring as $W$ are projective, correcting the example in [7].

Let $x_0,...,x_5$ be coordinates on $\mathbb{P}^5$. Under the standard torus action of scalar multiplication, the $T$-invariant planes are given by those planes which have three nonzero coordinates. Without loss of generality consider:

$$\Sigma_1 = \{ x_0 = x_3 = x_5 = 0 \}$$
\[ \Sigma_2 = \{x_0 = x_2 = x_4 = 0\} \]
\[ \Sigma_3 = \{x_1 = x_2 = x_3 = 0\} \]
\[ \Sigma_4 = \{x_1 = x_4 = x_5 = 0\} \]

**Lemma 3.2.** There exists a 2-plane in \( \mathbb{P}^5 \) that intersects the four 2-planes \( \Sigma_i \) each in a point not lying on the others.

**Proof.** Consider the subvariety of the Grassmannian \( G(2, 5) \), \( \Gamma = \{ P \in G(2, 5) : P \cap \Sigma_i \neq \emptyset, \forall i \} \). Since \( \dim G(2, 5) = 9 \) and each intersection with a \( \Sigma_i \) imposes one condition, then as long as \( \Gamma \) is nonempty, \( \dim \Gamma \geq 5 \). But \( \Gamma \neq \emptyset \), as any plane containing the line through \( \Sigma_j \) and \( \Sigma_k \) is in this variety.

Next, let \( U \) be the subset of \( \Gamma \) containing planes of the desired type. To show this is a non-empty, open subvariety of \( \Gamma \), we claim that its complement is a finite union of closed subvarieties of dimension no greater than four. There are two general cases to consider:

- **Planes** \( P \) **which intersect** \( \Sigma_i \) **in a line** \( l \) (for a fixed \( i \).) Note \( P \) is given by \( l \) and a point \( p \) not on this line. Now no more than one point of any \( \Sigma_j \), \( j \neq i \), can be on \( l \), else \( l \subset \Sigma_i \cap \Sigma_j \). Hence, \( p \) has at most two degrees of freedom to vary over \( \Sigma_j \), and \( l \) has two degrees to vary over \( \Sigma_i \). Since our choice of \( j \) is irrelevant to this upper bound, the set of all such \( P \) can have at most a dimension of four.

- **Planes** \( P \) **which intersect each** \( \Sigma_i \) **in a point**, but without loss of generality, \( \Sigma_1 \cap P = \Sigma_2 \cap P = p \). Since \( p \) is fixed by our choice of the special planes, \( P \) is then determined by two other points, \( q \) and \( r \), such that \( p, q, \) and \( r \) are not collinear. There are two special cases to consider: First, suppose that \( q = \Sigma_3 \cap P \), and \( r = \Sigma_4 \cap P \), such that \( p, q, \) and \( r \) are not collinear. Since there are two degrees of freedom in which to vary each of \( q \) and \( r \), the collection of planes they span has at most dimension 4. Secondly, if \( p, q, \) and \( r \) are as above, but collinear, then \( P \) is determined by an additional point, not in any of the special planes. Since the ambient space is \( \mathbb{P}^5 \), we have 5-2=3 degrees in which to vary this point. The ways to get lines \( \overline{pq} \) which intersect \( \Sigma_4 \) is bounded above by the degrees of freedom we have to vary \( q \) in its plane, i.e. two. However, not every line \( \overline{pq} \) will intersect \( \Sigma_4 \). Consider a general line \( l_3 \) in \( \Sigma_3 \). Denote by \( Q_l \) the plane spanned by \( p \) and \( l \). Note \( Q_l \) cannot intersect \( \Sigma_4 \) in more than a point, as if it were to intersect in a line, say \( l_4 \), then \( l_3 \cap l_4 = \Sigma_3 \cap \Sigma_4 \). Whence, for any line \( l_3 \), there is at most one point \( q \) such that \( \overline{pq} \) intersects \( \Sigma_4 \). Thus, the planes spanned by \( \overline{pq} \), and a point outside the planes has at most dimension 3+1=4.

\[ \square \]

In fact, for this example such a plane can be found explicitly. Let \( \Sigma \subset \mathbb{P}^5 \) be the plane given by the system of equations:

\[
\begin{align*}
x_2 + x_4 &= 0 \\
x_0 + x_1 + x_3 &= 0 \\
x_0 + x_3 + x_5 &= 0
\end{align*}
\]

One can check that \( \Sigma \) intersects the \( \Sigma_i \) in the following points, where each \( P_i \) is disjoint from \( \Sigma_j \) for \( j \neq i \). Furthermore, by inspection it is easily seen that the points are in general linear position.

\[ P_i = [0, 0, 1, 0, -1, 0] \]
\[ P_2 = [0,1,0,-1,0,1] \]
\[ P_3 = [1,0,0,0,0,-1] \]
\[ P_4 = [1,0,0,-1,0,0] \]

**Proposition 3.3.** The map \( \Phi : X_4 \to W \), induced by sending the base \( P_2 \) of \( X_4 \) to a general plane in \( \mathbb{P}^5 \) intersecting \( \Sigma_1, \ldots, \Sigma_4 \) in single points, is a Mori embedding.

**Proof.** To start, we show that \( \Phi \) is an embedding. First, by the lemma we know there exists a plane \( \Sigma \) in \( \mathbb{P}^5 \) intersecting the \( \Sigma_i \) in the desired manner. Thus, we have an embedding of the base \( P_2 \) into the base \( P_5 \). By \([5, II.7.15]\), \( \Phi \) is also an embedding. By construction, \( \text{Pic}(X) = \text{Pic}(W) \), and \( \Phi \) induces a surjection between the finitely generated Cox rings. Thus, \( \Phi : X_4 \to W \) is a Mori embedding by Theorem 2.8. \( \square \)

To further illustrate the geometry of the situation, we now show explicitly that \( \text{NE}^1(W) \) restricts to \( \text{NE}^1(X) \) for the above embedding. To this end, consider the six \( T \)-invariant hyperplanes in \( \mathbb{P}^5 \) given by \( P_i : \{ x_i = 0 \} \).

Letting \( E_i \) denote the exceptional divisors corresponding to the respective \( \Sigma_i \), we have found 10 \( T \)-invariant divisors:

\[
\begin{align*}
D_0 &:= \pi^*P_0 - E_1 - E_4 \\
D_1 &:= \pi^*P_1 - E_1 - E_2 \\
D_2 &:= \pi^*P_2 - E_1 - E_3 \\
D_3 &:= \pi^*P_3 - E_2 - E_4 \\
D_4 &:= \pi^*P_4 - E_2 - E_3 \\
D_5 &:= \pi^*P_5 - E_3 - E_4 \\
E_1, E_2, E_3, E_4
\end{align*}
\]

The effective cone of divisors of a toric variety is generated by classes of \( T \)-invariant divisors, and from \( \Delta^1(W) \) we know there are ten in this case. Now let \( d_i \) represent the restriction of \( D_i \) to \( X_4 \). Then \( d_i = \pi^*l - l_j - l_k \), where \( l \) is a line and \( l_j, l_k \) are the exceptional divisors in \( X \) corresponding to the points \( p_j, p_k \). The \( E_i \) restrict to the \( l_i \). Since these are the ten exceptional curves of \( X \), by Batyrev and Popov’s result \([2]\), these are exactly the generators of \( \text{NE}^1(X) \).

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