Introduction to the ADM Formalism

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Abstract

The authors present concepts and mathematical developments which give rise to the Hamiltonian formulation of Einstein’s general relativity, first introduced by Arnowitt, Deser, and Misner. All the geometrical quantities needed for the construction are explicitly obtained, and examples of some of the application of the formalism are given.
1 Introduction

The formulation of the theory of general relativity established by Einstein is centred around the concept of Principle of Equivalence which states: “The laws of physics are independent of the state of motion of the observer”. This principle implies that these laws must be described covariantly i.e. independent of the coordinate system. Thus, it becomes clear that the covariant formulation is natural, and it allows expressing the interaction of matter with geometry in a simple and elegant manner:

$$G_{\mu\nu} = \kappa T_{\mu\nu},$$  \hspace{1cm} (1.1)

where $G_{\mu\nu}$ is Einstein’s tensor and $T_{\mu\nu}$ is the energy-momentum tensor.

A feature of the covariance of Einstein’s equations is that some of these equations represent constraints on the dynamic variables. These constraints are closely related to Bianchi identities:

$$G_{\mu\nu}^{;\nu} = 0,$$  \hspace{1cm} (1.2)

which also express that these constraints are preserved when the dynamic equations are satisfied.

Despite the high degree of aesthetics in the covariant formulation of Einstein’s equations, in practice, a precise isolation of the equations with dynamic content from those that are constraints is unclear from this covariant way of expressing them.

One way to unravel the dynamics of general relativity is to view it as a Cauchy problem, i.e. to analyze the dynamics as the evolution of a three-dimensional hypersurface where the fields are defined. This way of reformulating general relativity was developed by R. Arnowitt, S. Deser and C.W. Misner and is known as the ADM formulation of general relativity which took its full form in the early 1960’s [1].

This formalism has gone through a continuous development throughout its almost 60 years of existence. However, interest from the part of the community of theoretical physicists, in conducting research on applications of the ADM formalism has been up and down. Since the mid-1980’s, there has been a further growing interest both in the formalism itself and in its applications to different areas. To mention a few examples: it has served as a basis for the development of Ashtekar variables [2]; it has been used in problems related to the quantization of gravity by means of path integrals [3]; it has served as a means to determine the solutions to Einstein’s equations numerically [4]; it has been the basis for the development of the theory of space of spaces [5].

Given the significance of the formalism, coupled with the fact that there are not enough (as far as the authors know) clear and complete introductory expositions of the formalism and some of its applications, the authors have written this manuscript that tries to heal this deficiency. Therefore, in these pages, authors try to give an introduction (as accessible as possible) to the formalism of the ADM—as a Hamiltonian formulation. The present work has the following form: Sec. 2 exposes the ideas and concepts necessary for the construction of ADM formalism with minimum prerequisites—in order to offer a global vistas of the method. In Sec. 3, the formal development is presented, introducing the equations explicitly, and finally in Sec. 4, we present some examples of the application of the formalism and mention some areas that have been developed using the formalism.
2 Conceptual Development

In this section, the steps to follow for the construction of the ADM formalism are explained in broad strokes. The purpose of this section is to establish a relatively clear idea of what the procedures to follow will be, without having gone into details.

As it was mentioned in the introduction, the ADM variables should explicitly show the dynamic content of Einstein’s equations and allow us to build a Hamiltonian in which all the terms have a relatively clear interpretation. If the Hamiltonian can be defined, it is in principle possible to proceed to the quantization of the theory, once the true degrees of freedom have been isolated. Obtaining Einstein’s equations through the ADM formulation can then be achieved by making the use of variational principle.

The variational principle, being one of the possible paths to reach Einstein’s field equations, consists of finding the extremal of the action functional—in this case, of the gravitational action. The action functional must be expressed as an integral over space-time (with the invariant volume element) of a scalar function.

Parallel to the works in which Einstein unveiled his formulation of gravity, Hilbert postulated the variational principle for gravitational theory by expressing the action in the form:

\[ S = \int d^4x \sqrt{-g} R , \]  

(2.1)

where the Lagrangian density is given by \( L = \sqrt{-g} R \), \( g \) is the determinant of the metric tensor and \( R \) is the curvature scalar.

The integral in (2.1) is over the entirety of the 4-dimensional space-time. The action is a functional of the metric tensor i.e. \( S = S[g_{\mu\nu}] \), so the acceptable metric tensor must be the one for which the action is an extremal. In other words, the action must be varied with respect to the tensor components \( g_{\mu\nu} \) to obtain the equations of motion. By making such a variation (being careful about the boundary terms) and setting it equal to zero, we arrive at the Einstein’s equations in vacuum:

\[ R^{\mu\nu} - \frac{1}{2}R g^{\mu\nu} = 0 . \]  

(2.2)

The construction of a classical Hamiltonian theory from this point implies the construction of canonically conjugated momenta for the dynamic variables (i.e. for the dynamic components of the metric tensor), by taking the partial derivatives of the Lagrangian density with respect to the generalized velocities. But how do we define such generalized velocities? To do that, we realize that it is necessary to grant privilege to one of the coordinates such as \( \text{time} \) to be able to define those velocities. This breaks the covariance, but it also slices the space-time into \( x_0 = c \) (constant) slices. The algebraic complication arising from such a procedure is enormous. We can get an idea
of this complexity with the fact that P.A.M. Dirac spent more than a decade in such an attempt without reaching satisfactory results.

It is relatively clear that there is a necessity to separate some of the coordinates (in a different sense), or alternatively a direction in space-time for the construction of a Hamiltonian. One way that was designed to do this, was to consider slices of space-time so that each slice is a 3-dimensional hypersurface with a positive definite metric defined on it. If we consider space-time as formed by the collection of such non-intersecting slices, where each of these is labeled by a number $t$ (which is not necessarily time, as it is a label), then we can think of evolution as the change of these hypersurfaces in the parameter $t$ and thus they cover the entire space-time. By providing each hypersurface with a three-dimensional metric $\gamma_{ij}$ determined by the way we cut the space-time, it is possible to consider the metric $\gamma_{ij}(t)$ of the hypersurface that evolves with $t$ as the dynamic variable.

In addition to the six components of $\gamma_{ij}$ that form dynamic variables, another 4 variables must be defined to have a total of ten—which is the number of components of our previous $g_{\mu\nu}$. These four variables are defined in a natural way by considering the foliation of hypersurfaces in space-time. These new variables are *Lapse* function of interval denoted by $N$, which is related to the separation between each hypersurface and *Shift* functions $N^i$ of displacement, that are related to the movement of a point when it passes to the next hypersurface. These four functions describe how to glue the hypersurfaces to form the foliation.

Subsequently, we proceed to rewrite the line element in terms of the new variables as $ds^2 = ds^2[\gamma_{ij}, N, N^i]$, thus relating the new variables to the old ones.

The next step is to rewrite the gravitational action in terms of these new variables. That is, we need to express the curvature scalar of space-time $4R$ and the invariant volume element $d^4x \sqrt{-g}$ as a function of geometrical objects on the hypersurface and of the new variables $N, N^i$. This is necessary to study the embedding of hypersurfaces and the way in which the hypersurface inherits geometric structure both of space-time and of the particular embedding carried out (i.e. how the slicings are produced). Likewise, it is also necessary to study the way in which space-time tensors are projected on the hypersurface and in the direction orthogonal to it.

In this step, the *extrinsic curvature* $K_{ij}$ plays a very significant role on the hypersurface: $K_{ij} = K_{ij}[N, N^k, \gamma_{ij}]$, with “$\cdot$” denoting the derivative with respect to the parameter $t$ and “$|$” being the covariant derivative on the hypersurface. With the help of this object, it is possible to rewrite the Riemann curvature tensor and therefore the scalar of curvature $4R$ as functions of $\gamma_{ij}, \gamma_{ij}, N$ and $N^i_{ij}$. It is to be emphasized that the new Lagrangian density will not depend on the derivatives of the *Lapse* and *Shift* functions with respect to $t$. Finally, the action is re-expressed in the form:

$$S[\gamma_{ij}, N, N^i] = \int dt \int d^3x \sqrt{\gamma}N (K_{ij}K^{ij} - K^2 + 3R + (\Delta^2)\lambda) ,$$  \hspace{1cm} (2.3)

where $\sqrt{-\tilde{g}} = \sqrt{\gamma}N$ is the new invariant volume element with respect to transformations on the hypersurface.

The non-dependence of Lagrangian density on the velocities of $N$ and $N^i$ allows us to consider them as non-dynamic variables. This fact also suggests that the true dynamic variables are the six components of $\gamma_{ij}(t)$. 

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In this way, we can obtain the reformulation of variational principle of general relativity with a clearer dynamic content—which is what was intended. Now, we can go further and construct the Hamiltonian density $\mathcal{H}$ from this new Lagrangian density in the usual way by defining canonically conjugated momenta to the $\gamma_{ij}$ of the form:

$$\pi^{ij} = \frac{\partial L}{\partial \dot{\gamma}_{ij}},$$

(2.4)

and by performing the Legendre transformation only on the relevant dynamic variables and their momenta:

$$\mathcal{H} = \pi^{ij} \dot{\gamma}_{ij} - \mathcal{L}.$$  

(2.5)

This Hamiltonian density has the form:

$$\mathcal{H} = \mathcal{H}_0 + N^i \mathcal{H}_i.$$  

(2.6)

The scalar densities $\mathcal{H}_0$ and $\mathcal{H}_i$ will play a very important role while analyzing the dynamics of the theory.

As mentioned at the beginning of this section—in general relativity there are constraints between the variables (caused by the covariance of the theory). Now, with the introduction of ADM formalism, the form of the constraints and the variables that are involved becomes clear. In this way we have four primary constraints that are a direct consequence of the form of Lagrangian density, i.e. the momenta conjugated to the four variables $N$ and $N^i$. These momenta are defined as:

$$P^\mu = \frac{\partial L}{\partial \dot{N}_\mu},$$

(2.7)

and are weakly zero since $\mathcal{L}$ does not depend on $\dot{N}_\mu$ (here $N^\mu$ denotes the set of variables $N$ and $N^i$ although they are not the components of a four vector field). From the Hamiltonian density, now we can rewrite the Lagrangian density in terms of the dynamic variables $\gamma_{ab}$ and their canonically conjugate momenta $\pi^{ab}$ thus having a first-order variational principle. The action is then written as:

$$S[\gamma_{ij}, N, N^i] = \int dt \int d^3 x \left( \pi^{ij} \dot{\gamma}_{ij} - N \mathcal{H}_0 + N^i \mathcal{H}_i \right).$$

(2.8)

This expression for action is now in a parametric form as a consequence of the covariance. The equations are obtained by varying with respect to different variables. If the $N^\mu$ are varied, it is obtained that the four components of $\mathcal{H}_\mu$ go to zero. By these expressions, one can relate the dynamic variables ($\gamma_{ij}$) and their momenta ($\pi^{ij}$) to each other. They are known as Hamiltonian constraints. In this way, we have presented the fundamental ideas for the construction of ADM formalism as the Hamiltonian formulation. In the next section, we will present the formal, explicit development of the ideas presented here.

3 Formal Development

3.1 Fitting a Hypersurface into a Manifold

Let us consider the space-time given by a 4-dimensional manifold $\mathbf{M}$ with a metric $g_{\mu\nu}$ defined on it with signature $(-, +, +, +)$. Let us denote the coordinates on this
manifold by $x^\lambda$. We now define the embedding of a 3-dimensional hypersurface $m$ as follows:

$$x^\mu = X^\mu(\xi^a),$$

(3.1)

where $\mu = 0, 1, 2, 3$ and $a = 1, 2, 3$. These four functions are what determine the embedding. For it to be an embedding we must demand that the hypersurface, with intrinsic coordinates $\xi^a$ does not intersect itself, i.e. the map $X : m \rightarrow M$ must be one to one \cite{6}.

The metric on $M$ induces a metric on $m$ if we consider the line element restricted to the manifold $m$:

$$\begin{align*}
  ds^2 &= g_{\mu\nu}(x^\lambda) \left. \right|_{m} \, dx^\mu dx^\nu \\
  &= g_{\mu\nu}(X^\lambda) \left. \frac{\partial X^\mu}{\partial \xi^a} \right. \left. \frac{\partial X^\nu}{\partial \xi^b} \right|_{m} \, d\xi^a d\xi^b \\
  &= \left[ g_{\mu\nu}(X^\lambda) \left. \frac{\partial X^\mu}{\partial \xi^a} \right. \left. \frac{\partial X^\nu}{\partial \xi^b} \right|_{m} \right] \, d\xi^a d\xi^b \\
  &\equiv \gamma_{ab} \, d\xi^a d\xi^b,
\end{align*}$$

(3.2)

where the induced metric is then

$$\gamma_{ab} = g_{\mu\nu}(X^\lambda) \left. \frac{\partial X^\mu}{\partial \xi^a} \right. \left. \frac{\partial X^\nu}{\partial \xi^b} \right|_{m}. $$

Let’s define

$$X^\mu_a = \frac{\partial X^\mu}{\partial \xi^a};$$

(3.3)

note that $X^\mu_a$ is the $\mu$-th component (along the $x^\mu$ coordinate) of the $a$-th vector in natural basis over $m$ given by,

$$e_a = \frac{\partial}{\partial \xi^a}$$

therefore, the vector must be written formally as:

$$X^\mu_a \partial_\mu = \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial}{\partial x^\mu}$$

(3.5)

We then have a natural definition of the components of metric $\gamma_{ab}$ on $m$ given by:

$$\gamma_{ab} = g_{\mu\nu} \cdot X^\mu_a X^\nu_b$$

(3.6)

For the study of dynamics of space-time in the framework of interest here, the embedded hypersurface must be space-type (space-like), so we demand that the metric defined on $m$ i.e. $\gamma_{ij}$ be positive definite ($+$ $+$ $+$).

The 3 vectors $e_a$ form a basis for the tangent space to the manifold $m$ at the point $p$ denoted by $T_p m$. This space is in fact a subspace of the tangent space to $M$ i.e. $T_p M$. To complete the foundation of this space, we build the orthogonal complement to the $T_p m$ defined by the metric $g_{\mu\nu}$. This subspace will be generated by the vector
orthogonal to $e_a$, which we will denote by $n$. This vector with components $\eta^\mu$ in the $\partial_\mu$ basis satisfies:

$$g_{\mu\nu} X^\mu_a \eta^\nu = 0 \quad (3.7)$$

We also demand the following normalization,

$$g_{\mu\nu} \eta^\mu \eta^\nu = -1 \quad (3.8)$$

These two conditions fully determine the vector $n$.

We then have the set of vectors $(e_a, n)$ forming a basis of $T_pM$ for each point $p$. With them, we can express any vector of $T_pM$ as a linear combination of the basis in hypersurface $m$ and the vector normal to it in the following way:

$$(A)^\mu = (A^a e_a + A^\perp n)^\mu$$

where,

$$A^\perp = -(A \cdot n) \quad (3.10)$$

If the vector $A$ is on the hypersurface, it will be of the form $A = A^a e_a$.

The objects $A^a$ behave like scalars under the coordinate transformation of space-time $X^\mu \to X'^\mu$, since for such changes of coordinates, both the vectors $e_a$ and the vector $A$ remain fixed, so the $A^a$ - s do not change. However, we can view them as components of vectors on $T_p m$, since they are transformed accordingly when the coordinates change $(\xi^a \to \xi'^a)$ on the hypersurface:

$$A^a \frac{\partial}{\partial \xi^a} = A^a \frac{\partial \xi'^b}{\partial \xi^a} \frac{\partial}{\partial \xi'^b}$$

$$= A'^a e_a \quad (3.11)$$

so the new components will be:

$$A'^a = \frac{\partial \xi'^a}{\partial \xi^b} A^b \quad (3.12)$$

We define the inverse metric on $m$ as the inverse of $\gamma_{ab}$ that is, $\gamma^{ab} \equiv (\gamma_{ab})^{-1}$, so that we can use the metric to lower and raise indices of the components:

$$A_a = \gamma_{ab} A^b$$

$$A^a = \gamma^{ab} A_b \quad (3.13)$$

The components of the vector can be rewritten as the following:

$$A^a = \gamma^{ab} A_b$$

$$= \gamma^{ab} (A \cdot e_b)$$

$$= \gamma^{ab} g_{\mu\nu} X^\mu_b A^\nu$$

$$= X^a_{\nu} A^\nu \quad (3.14)$$

where we define the object:

$$X^a_{\nu} \equiv \gamma^{ab} g_{\mu\nu} X^\mu_b \quad (3.15)$$
which allows us to find the $a$-th component of vector $\mathbf{A}$.

Given an arbitrary space-time vector $\mathbf{A}$ with components $A^\mu$, we now know how to find its projections on the hypersurface $[A_a = (\mathbf{A} \cdot e_a)]$ in the natural basis of $T_p \mathbf{m}$. We now want to find a relationship that expresses this same projection as components in the $\partial_\mu$ basis. For this purpose, let us observe the following equality:

$$\mathbf{A}^\mu = A^\mu X_a^\mu - (\mathbf{A} \cdot \mathbf{n}) \eta^\mu . \quad (3.16)$$

Thus, we have the $\mu$-th component of the vector projected onto the hypersurface:

$$A^\mu X_a^\mu = A^\mu + A^\nu \eta_\nu \eta^\mu = A^\nu (\delta^\mu_\nu + \eta_\nu \eta^\mu) . \quad (3.17)$$

Therefore, we have found the projection operator that commands a vector of space-time to one on the hypersurface. We will denote this operator by:

$$h^\mu_\nu = \delta^\mu_\nu + \eta^\mu \eta_\nu . \quad (3.18)$$

It is easy to see that this operator satisfies the following properties by making use of (3.7) and (3.8):

$$\eta^\nu h^\mu_\nu = 0$$
$$h^\mu_\nu h^\nu_\rho = h^\mu_\rho$$
$$h^\mu_\mu = 3 . \quad (3.19)$$

### 3.2 Induced Covariant Derivatives

The covariant derivative, defined by the metric $g_{\mu\nu}$, of a vector over $\mathbf{m}$, in the direction of another vector that is also in $\mathbf{m}$ will not be in general a tangent vector to the hypersurface. The natural way to define a covariant derivative on $\mathbf{m}$ would be demand that it also be a vector of $T_p \mathbf{M}$ [7]. At this point it is convenient to consider the case where the first 3 coordinates $x^\mu$ coincide with $\xi^a$, say— $x^a = \xi^a$. These coordinates are called adapted coordinates. Note that in this case, the vectors $e_a$ coincide with the three vectors of the coordinate basis of $T_p \mathbf{M}$, hence its components are trivial: $X^\mu_a = \delta^\mu_a$. Taking this coordinate system for $\mathbf{m}$ generally simplifies the analysis, since the expressions obtained are simpler, without losing generality. This is because it is always possible to take the three intrinsic coordinates of $\mathbf{m}$ as three of the space-time coordinates. From now on, throughout the analysis we shall use these adapted coordinates. We can then consider the covariant derivative (of space-time) of a vector $\mathbf{A} = A^a e_a$ on the hypersurface in the direction of the basis vectors as the following:

$$4 \nabla_{e_a} \mathbf{A} \equiv 4 \nabla_a \mathbf{A} = 4 \nabla_a (e_b A^b)$$
$$= e_b \frac{\partial A^b}{\partial X_a} + (4 \Gamma^b_{\mu \nu} e_{\nu}) A^b$$
$$= e_b \frac{\partial A^b}{\partial X_a} + (4 \Gamma_{ab} e_c + 4 \Gamma^0_{ab} e_0) A^b . \quad (3.20)$$

As a special case, the covariant derivative of the basis vectors is given by:

$$4 \nabla_a e_b = 4 \Gamma^\mu_{ba} e_{\mu}$$
$$= 4 \Gamma^c_{ba} e_c + 4 \Gamma^0_{ba} e_0 . \quad (3.21)$$
In both Eqs. (3.20) and (3.21), there is a term outside the hypersurface:

\[ (4 \Gamma_{ba}^0 A^b)(\mathbf{e}_0 \cdot \mathbf{n}) \]  

(3.22)

which, in general does not cancel. We can get rid of that term by projecting the vector on the hypersurface, so as to have a vector in \( T_p \mathbf{m} \). Thus, we get the intrinsic covariant derivative to the 3-geometry of the hypersurface. Henceforth, we will denote the operator associated with this derivative by \( ^3\nabla \equiv \nabla \). Let us again take a vector on the hypersurface \( A = A^a \mathbf{e}_a \) In terms of the vector components \( A^a \) in \( T_p \mathbf{m} \), components of its covariant derivative is expressed as:

\[ A^c_{\mid a} = ^3 \nabla_a A^c = A^c_{\; \; a} + ^3 \Gamma_{ba}^c A^b , \]  

(3.23)

with \( ^3 \Gamma_{ba}^c = 4 \Gamma_{ba}^c \).

If we now consider the \( c \)-th component of the covariant derivative of vector \( A \) in the direction of \( \mathbf{e}_a \), it will be of the form:

\[ (^3 \nabla_a A)_c = \mathbf{e}_c \cdot ^3 \nabla_{\mathbf{e}_a} A \]  

(3.24)

which can also be expressed as:

\[ A_{c\mid a} = \mathbf{e}_c \cdot \nabla_a A \]  

\[ = A_{c\; a} - ^3 \Gamma_{bca} A^b . \]  

(3.25)

Note that the form of the expressions the covariant derivative takes, both in its 1-form as well as vector components is the same as that of the covariant derivatives on the space-time. This is due to the fact that the covariant derivative on the hypersurface is given by the affine connection on the manifold \( \mathbf{m} \) defined in terms of the metric in the usual form ([5] and [A. Corichi, degree thesis, FCUNAM (1991)]),

\[ ^3 \Gamma_{abc} \equiv \frac{1}{2} (\gamma_{ab,c} + \gamma_{ac,b} - \gamma_{bc,a}) \]  

(3.26)

and with the following interpretation:

\[ ^3 \Gamma_{abc} \equiv \Gamma_{abc} = \mathbf{e}_a \cdot \nabla_c \mathbf{e}_b . \]  

(3.27)

In order to facilitate the calculations, let us note that the covariant derivative can also be expressed (for \( A^\mu = A^b X^\mu_b \)) in terms of components as follows,

\[ A^b_{\mid c} = (A^\mu_{\; \; \nu} X_c^\nu)^b \]  

\[ = X^b_\mu (A^\mu_{\; \; \nu} X_c^\nu) \]  

\[ = X^b_\mu X_c^\nu A^\mu_{\; \; \nu} . \]  

(3.28)

This expression may be of more practical use. However, we consider the way we initially presented it as it is conceptually clearer.
3.3 Extrinsic Curvature

We now define the extrinsic curve to the hypersurface as:

\[ K_{ab} \equiv -e_b \cdot 4\nabla_a n, \]  \hspace{1cm} (3.29)

i.e. the component \((ab)\) of the extrinsic curvature is equal to the projection on the direction \(b\) of the covariant derivative of the normal vector in direction \(a\) (except the sign). Note that the extrinsic curvature is well defined on the hypersurface, since the vector \(4\nabla_a n\) is a vector in \(T_p m\):

\[ 4\nabla_a (n \cdot n) = 0, \]  \hspace{1cm} (3.30)

where,

\[ 4\nabla_a n \cdot n = 0, \]  \hspace{1cm} (3.31)

so \(K_{ab}\) is a geometric object of the manifold \(m\). The notion of extrinsic curvature has no meaning for a manifold itself as it can only take on a meaning when the said manifold is embedded in one of higher dimension, since by the same definition, the curvature \(K_{ab}\) depends on the geometry of the larger manifold (via the space-time covariant derivative of the normal vector to the hypersurface).

A geometric interpretation of the extrinsic curvature is that it gives a measure of how much the hypersurface is curved with respect to the manifold \(M\), or in other words, tells us that how the normal vectors for two nearby points in \(m\) move away from being parallel (see Fig. 1). There exists another way of rewriting the curvature that follows from the orthogonality of \(n\) with respect to the hypersurface, which means, \((e_a \cdot n) = 0\). We then have that,

\[ K_{ab} = n \cdot 4\nabla_a e_b \]  \hspace{1cm} (3.32)

This last expression and Eq. (3.27) tell us how to write the vector \(4\nabla_a e_b\) in terms of its components on the hypersurface and the orthogonal direction:

\[ 4\nabla_a e_b = -K_{ab}n + 3 \Gamma^c_{ba} e_c. \]  \hspace{1cm} (3.33)
This equation is known as the Gauss' equation. Knowledge of the space-time covariant derivative of basis vectors $e_a$ then allows us to write the derivative of an arbitrary vector over $T_p m$:

$$4\nabla_A A^b = A^b_{\mid a} e_b - K_{ab} A^b n.$$ (3.33)

To finish this section, we write the extrinsic curvature in components,

$$K_{ab} = -X^\mu_a X^\nu_b \eta_{\mu\nu},$$ (3.34)

so that in adapted coordinates, it takes the form:

$$K_{ab} = -\eta_{ab}.$$ (3.35)

### 3.4 Decomposition of Space-time into $3 + 1$

So far we have precisely defined how to fit a 3-geometry in space-time, after introducing the metric $\gamma_{ab}$ on this hypersurface along with the covariant derivative defined on it (hence the notion of parallel transport). Thus, we now have a precise knowledge of the decomposition of a space-time vector into projected and orthogonal components.

The next step is to assert that the entirety of space-time can be generated by these hypersurfaces (each of them corresponds to an embedding) without them intersecting with each other. The totality of these generating hypersurfaces is called foliation. As an example, the Euclidean space $\mathbb{R}^3$ can be seen as generated by spheres of radius $a$; where each sphere is a hypersurface ($dim = 2$) embedded in $\mathbb{R}^3$ ($dim = 3$) and the totality of these spheres constitutes a foliation of $\mathbb{R}^3$.

The foliation will be given by a function $t$ in space-time such that a constant value of $t$ corresponds to each hypersurface, i.e. the hypersurfaces are defined by $t = cnst$. 

Figure 2: Two spheres of the foliation of $\mathbb{R}^3$ are represented in which the points $p$ and $p'$ are identical. It is noted that the point $p'$ is not necessarily on the north pole of the exterior sphere.
3.4 Decomposition of Space-time into 3 + 1

(Recall that $t$ is simply a label). In the above example of the spheres, such a function can be the radius function $r = r(x, y, z)$ that assigns $r = a$ to each sphere. Given a hypersurface with label $t = t_0$, a point $p$ on it is characterized by its three coordinates $\xi^a$—the point represents an event. This point would correspond to a point $p'$ on the hypersurface of label $t_0 + dt$, so that the two points are identical. The direction in which the point $p'$ lies with respect to the hypersurface $t_0$ depends on $t$ (function of $t$): the point $p'$ will have the same intrinsic coordinates on the hypersurface labelled by $t_0 + dt$ as those of the point $p$ on the original hypersurface, but its space-time coordinates will be different as it would be a function of $t$.

To illustrate this idea let us take the point $p$ on the sphere of radius $a$ as the north pole (See Fig. 2). This point will have the coordinates $\theta = 0$ on the sphere. The point $p'$ on the sphere of radius $a + dr$, that corresponds to $p$ does not necessarily have to be the north pole of that sphere (which is seen as the point on the z-axis of $\mathbb{R}^3$), although it corresponds to the same intrinsic coordinates ($\theta = 0$) on its sphere. There will then be a vector connecting points $p$ and $p'$ based at point $p$ (we are considering that $dt$ is infinitesimal) which is not necessarily orthogonal to the hypersurface. For the two spheres considered in the figure above, the vector will be orthogonal only if $p'$ is also the north pole.

The foliation is given analytically by the functions:

$$x^\mu = X^\mu(\xi^a, t),$$  \hspace{1cm} (3.36)

where the vector connecting the points of two hypersurfaces is given by $\partial_t$, with components in the $\partial_\mu$ basis given by

$$\frac{\partial}{\partial t} = \frac{dX^\mu}{dt} \frac{\partial}{\partial x^\mu},$$

its components are therefore $t^\mu \equiv \frac{dX^\mu}{dt}$. From Sec 3.1, we know how to decompose a vector into its projection on the hypersurface and its normal ((3.9) and (3.10)), so we can write the components of vector $\partial_t$, in the following form:

$$t^\mu = -(t \cdot n) \eta^\mu + \gamma^{ab}(t \cdot e_b) X^\mu_a = -(t \cdot n) \eta^\mu + \gamma^{ab} g_{\nu \lambda} X^\lambda_b t^\nu X^\mu_a = -(t \cdot n) \eta^\mu + (X^\nu_a t^\nu) X^\mu_a$$

$$\equiv N \eta^\mu + N^a X^\mu_a.$$  \hspace{1cm} (3.38)

The scalar $N$ is called a lapse function, and the vector on the hypersurface $N^a$ is known as the shift vector. These, together with the metric $\gamma_{ab}$ constitute the so called ADM variables. As it can be clearly seen from the example of spheres, the lapse function represents how far apart the new sphere is; the separation in this case is the same for every point on the sphere since all hypersurfaces are spheres, but in the general case the separation will not be a constant. On the other hand, the shift vector indicates how far the sphere has been rotated, or in other words, how much it has been deformed. The interpretation for space-like hypersurfaces is similar and the names of the new variables $N$ and $N^a$ therefore makes sense: the lapse function gives
information of the time lapse between the events $p$ and $p'$ (for an observer it would be the proper time elapsed), since the direction normal to the hypersurface is in some sense the temporal direction (recall that space-time has signature $(-, +, +, +)$); the shift vector represents how much the point $p$ on the same hypersurface is displaced and therefore, (as seen globally) how much it deforms.

It is clear from Eq. (3.38) that specifying the four $N^\mu$ functions completely determines the foliation of the hypersurfaces that gives rise to the space-time. For this to be fully determined, it is necessary to find the metric induced on each hypersurface, which will be given by the dynamic equations that are derived later.

Now let’s see how to rewrite the space-time line element (i.e. the distance between two events) in terms of the new variables, i.e. $\gamma_{ab}$, $N$ and $N^a$. The line element between two space-time events $p(x^a, t)$ and $p'(x^a + dx^a, t + dt)$, can now be decomposed into two parts: the square of the distance over the hypersurface containing $p$ minus the square of the proper time between the hypersurfaces:

$$ ds^2 = \gamma_{ab} (dx^a + N^a dt)(dx^b + N^b dt) - (N dt)^2 $$

$$ = (N^a N_a - N^2) dt^2 + 2N^a dx^a dt + \gamma_{ab} dx^a dx^b $$

(3.39)

Figure 3: The above figure depicts the projections of the vector $\partial_\mu$ on the hypersurface $t$ and on the normal vector $n$, which are identified with the functions $N^a$ and $N$ respectively.

where $(dx^a + N^a dt)$ is the displacement over the base hypersurface and $N dt$ is the proper time between them (see Fig. 3). This can be seen more clearly if we note that in the case where $N^a = 0$, the line element will have contributions from: i) the distance on the base hypersurface and ii) the proper time, since the evolution of hypersurfaces are merely temporal ($t^\mu$ in this case is normal to the hypersurface). From the last expression (3.39) we deduce the relationship between the components of the metric $g_{\mu\nu}$ in adapted coordinates and the variables $\gamma_{ab}$, $N$ and $N^a$:

$$ g_{00} = (N^a N_a - N^2) $$

$$ g_{0a} = N_a $$

$$ g_{ab} = \gamma_{ab} $$

(3.40)
Therefore, the components of inverse of the metric $g_{\mu\nu}$ takes the following form:

$$g^{\mu\nu} = \begin{pmatrix} \frac{1}{N^2} & \frac{N^b}{N^2} \\ \frac{N^a}{N^2} & \gamma_{ab} - \frac{N^a N^b}{N^2} \end{pmatrix}$$  \hspace{1cm} (3.41)

Now, the volume element would be given by:

$$\sqrt{-g} \, d^4x = N \sqrt{\gamma} \, d^3x \, dt$$ , \hspace{1cm} (3.42)

The first two equations of (3.40) tell us that the $g_{0\mu}$ components of the space-time metric and $N^\mu$ variables are directly related, so we can replace the former with the latter. It turns out that the same form of the components $g_{0\mu}$ are non-dynamic variables, and hence $N^\mu$ must be imposed from the outside to build the foliation.

### 3.5 Curvature Scalar

Having introduced the new set of ADM variables, we are now in a possible position of rewriting the curvature scalar $- \,^4 R$ in terms of these variables, with the objective of rewriting the variational principle. As a first step, we shall relate the extrinsic curvature defined in Sec. 3.3 with the ADM variables. From Eq. (3.35) we can write

$$K_{ab} = -\eta_{a;b} = -\eta_{a,b} + \Gamma_{ab}^\mu \eta_\mu$$ \hspace{1cm} (3.43)

in adapted coordinates $\eta_\mu = -N(1,0,0,0)$, so using Eqs. (3.41),(3.26) and (3.25) we have

$$K_{ab} = -N \,^4 \Gamma_{ab}^0 \quad = -N(\eta_{00} \,^4 \Gamma_{0ab} + \eta_{0c} \,^4 \Gamma_{cba})$$

$$= -N \left[ -\frac{1}{2N^2}(N_{a,b} + N_{b,a} - \dot{\gamma}_{ab}) + \frac{1}{N^2} N^c \Gamma_{cba} \right] \hspace{1cm} (3.44)$$

$$= \frac{1}{2N} (N_{a|b} + N_{b|a} - \dot{\gamma}_{ab}) .$$

It is to be noted that $K_{ab}$ does not depend on the derivatives with respect to $t$ of $N^\mu$.

Next, we shall rewrite the Riemann tensor $\,^4 R^\mu_{\nu\lambda\rho}$ restricted to the hypersurface, i.e. the components $\,^4 R_{abcd}$ in terms of the intrinsic Riemann tensor (constructed from the covariant derivative intrinsic to it) on the hypersurface $\,^3 R_{abcd}$ as well as the extrinsic curvature [C. Soto, Thesis, FCUNAM (1990)]. For this, we shall change the basis of space-time; instead of taking the vector $e_0 = \partial_t$ (as before), we shall take the normal vector $n$.

The Riemann tensor is defined by the following expression:

$$[\nabla_\alpha, \nabla_\beta] \, A^\mu = R^\mu_{\nu\alpha\beta} \, A^\nu , \hspace{1cm} (3.45)$$

if the vector $A$ is taken on the hypersurface (in a particular basis vector) and the derivatives are also in the direction of the vectors $e_0$, using the Gauss equation (3.32),
we obtain the second covariant derivative as:

\[ 4 \nabla_{e_a} 4 \nabla_{e_b} e_c = 4 \nabla_{e_a} \left[ -K_{cb} n + \Gamma^d_{cb} e_d \right] \]
\[ = -K_{cb,a} n - K_{cb} 4 \nabla_{e_a} n + 3 \Gamma^d_{cb,a} e_d + 3 \Gamma^d_{cb} \nabla_{e_a} e_d \]
\[ = -K_{cb,a} n + K_{cb} K^d_a e_d + 3 \Gamma^d_{cb,a} e_d \]
\[ + 3 \Gamma^d_{cb} \left[ -K_{da} n + 3 \Gamma^d_{da} e_d \right]. \]  

(3.46)

In the same way, by calculating \( 4 \nabla_{e_b} 4 \nabla_{e_a} e_c \) and taking the difference we have

\[ 4 \left[ \nabla_{a}, \nabla_{b} \right] e_c = -n \left[ K_{cb,a} - K_{db} 3 \Gamma^d_{ca} - K_{dc} 3 \Gamma^d_{ba} - K_{ca,b} + K_{da} 3 \Gamma^d_{cb} + K_{dc} 3 \Gamma^d_{ab} \right] \]
\[ + \left[ K_{cb} K^d_a - K_{ca} K^d_b + 3 R^d_{cab} \right] e_d \]
\[ = n \left[ K_{ca}b - K_{cb}a \right] + \left[ 3 R^d_{cab} + K_{cb} K^d_a - K_{ca} K^d_b \right] e_d. \]  

(3.47)

From Eq. (3.47), we directly have the projections on the hypersurface and the normal, so we can write

\[ 4 R^d_{cab} = 3 R^d_{cab} + K_{cb} K^d_a - K_{ca} K^d_b, \]  

(3.48)

which is known as the Gauss-Codazzi equation and

\[ 4 R^\perp_{cab} = K_{ca}b - K_{cb}a, \]  

(3.49)

which is the Codazzi-Mainardi equation.

The curvature scalar \( 4 R \) is given by,

\[ 4 R \equiv 4 R^\mu_{\nu \mu} = 4 R^\mu_{ab} + 2 4 R^\perp_{\mu \perp}. \]  

(3.50)

We already know the first term of the last expression, but we need to find the second. For that, we shall use the language of components since they render simplified calculations.

The commutator of the covariant derivatives of the normal vector is given in terms of the Riemann tensor as,

\[ \left[ \nabla_{\alpha}, \nabla_{\beta} \right] \eta^\nu = 4 R^\nu_{\rho \alpha \beta} \eta^\rho \equiv R^\nu_{\perp_{\alpha \beta}} \]  

(3.51)

which in components is written as:

\[ R^\perp_{\nu \alpha \beta} = \eta^\nu_{\nu \alpha \beta} - \eta^\nu_{\nu \beta \alpha}. \]  

(3.52)

If we now project the third index of the tensor onto the normal, and contract the second and fourth indices, we get,

\[ R^\perp_{\perp \mu} \equiv \eta^\nu R^\perp_{\nu \mu} \]
\[ = \eta^\nu (\eta^\lambda ; \lambda \mu - \eta^\lambda ; \nu \lambda ) \]
\[ = (\eta^\nu \eta^\lambda ; \lambda \nu) - (\eta^\nu \eta^\lambda ; \nu \lambda ) - \eta^\nu \eta^\lambda ; \lambda \nu + \eta^\nu \eta^\lambda ; \nu \lambda \]
\[ = (\eta^\nu \eta^\lambda ; \nu - \eta^\nu \eta^\lambda ; \mu ) \lambda - \eta^\nu ; \nu \eta^\lambda ; \mu + \eta^\nu ; \mu \eta^\lambda ; \nu. \]  

(3.53)
nevertheless,
\[ \eta_{\mu}^{\nu} = -K_{a}^{a} \equiv -K \],
\[ \eta_{\mu}^{\nu} \eta_{\nu}^{\mu} = K_{ab} K^{ab} \],
(3.54)

with that we can write,
\[ R^{\perp}_{\perp\mu\mu} = K_{ab} K^{ab} - K^{2} + (\Delta^{\lambda})_{;\lambda} \]
(3.55)

with \( \Delta^{\lambda} = \eta^{\lambda} \eta^{\nu}_{\nu} - \eta^{\nu} \eta^{\lambda}_{\nu} \). Using equations (3.50), (3.48) and (3.55) we can then express the curvature scalar as:
\[ 4R = 3R + K^{2} - K_{ab} K^{ab} + 2 \left( K_{ab} K^{ab} - K^{2} + (\Delta^{\lambda})_{;\lambda} \right) \]
(3.56)

In this section we have expressed both the Riemann tensor and the curvature scalar as functions of the new variables, with which we can rewrite the Einstein tensor: \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \), and thus the Einstein’s equations in vacuum: \( G_{\mu\nu} = 0 \). This procedure is used to solve the equations in numerical form (see [4]). We will not follow this path, since we are more interested in the construction of a Hamiltonian for which we need to consider a variational principle.

### 3.6 Variational Principle

Having defined the new set of ADM variables and rewritten the scalar of curvature, as well as the invariant volume element in terms of them, we can then reformulate the variational principle that gives rise to the field equations. The action of the gravitational field is given by the expression:
\[ S = \int d^{4}x \sqrt{-g} R \],
(3.57)

which can then be re-expressed using the results of the previous sections as:
\[ S[\gamma_{ij}, N, N^{i}] = \int dt \int d^{3}x \sqrt{\gamma} N \left( K_{ab} K^{ab} - K^{2} + 3 R + (\Delta^{\lambda})_{;\lambda} \right) \]
(3.58)

It is necessary to emphasize the fact that not only the expression for the Lagrangian has been modified with the new formulation, but it has also become necessary to make a reinterpretation of the action integral. In Eq. (3.57), the integral is over the complete manifold, that is, over the totality of space-time. This implies that when the variation is performed, the terms that are evaluated at the boundary are cancelled by the requirement that the variation of the metric \( \delta g_{\mu\nu} \) and its derivatives are zero at the boundary of the manifold (that is, the values of \( g_{\mu\nu} \) and its derivatives are determined on the boundary). In general, one does not have an apriori topology for the manifold \( M \), so it is not determined from beforehand if it has a boundary or not.

Expression (3.58) has a different interpretation, since in it we find two integrations of different nature: the first integral is over the hypersurface \( m \), so that the Lagrangian function is given by an integral on the hypersurface of the form
\[ L = \int d^{3}x \sqrt{\gamma} N \left( K_{ab} K^{ab} - K^{2} + 3 R + (\Delta^{\lambda})_{;\lambda} \right) \],
(3.59)
so the action is given by the integral of the Lagrangian function between initial parameter \( t_0 \) and final parameter \( t_1 \):

\[
S = \int_{t_0}^{t_1} dt \ L.
\]  

(3.60)

In other words, one is specifying the metric over the initial hypersurface \((t_0)\) and on the final \((t_1)\), so that the variation is over all the possible hypersurfaces (their metrics) that connect the two extreme hypersurfaces. The divergence that appears in the integral that defines the Lagrangian becomes an integral over the boundary of the manifold \( m \). These terms do not provide any information about the dynamics of the system, and therefore can be neglected.

It is convenient to specify that this expression for the gravitational action is limiting the topology of space-time, since it will then be of the form \((m \times \mathbb{R})\), which is maintained throughout the evolution of hypersurfaces.

That is, changes in the topology of the hypersurfaces are not allowed (for example, a universe will always be closed or always open).

### 3.7 Hamiltonian Formulation

So far, we have rewritten the action functional so that we can find the field equations in vacuum by taking the variation of the action and setting it equal to zero \((\delta S = 0)\). The Lagrangian density contains second-order spatial derivatives of \( \gamma_{ab} \), so in principle, the Euler-Lagrange equations should be of fourth order. However, when performing the variation (see [7] and [8]), the terms of higher derivatives vanish as boundary terms, so the resulting field equations are of second order (as one would expect the dynamic equations to be).

One can reduce the order of the field equations in order to have only first-order equations, in several possible ways. The first consists of taking the Palatini action [7], which amounts to considering both—the 10 components of \( g_{\mu \nu} \) and the 40 components of the affine connection \( \Gamma^\mu_{\nu \rho} \) as independent variables and making the variations with respect to all of them. With this procedure, the same Einstein’s equations are obtained as well as the relationship between the metric and the connections.

The other path consists of defining new variables, the momenta, starting from the Lagrangian, and trying to rewrite the Lagrangian in terms of the original variables and momenta, taking them as independent variables (the number of independent variables is doubled), where one of the intermediate steps involves the construction of the Hamiltonian function. This last step is not straightforward in general, when there are constraints that bind the variables. For that, one has to develop a whole new theory for dealing with such systems, known as singular or constrained systems (see [9], [10] and [2]).

With the formalism that we have developed so far, the construction of the Hamiltonian can be done directly, since, as it has been previously explained, the Lagrangian density does not depend on the time derivatives of \( N^\mu \), making these variables non-dynamic.

We then define the canonical conjugate momenta of six \( \gamma_{ab} \) by the following expression,

\[
\pi^{ab} = \frac{\partial L}{\partial \dot{\gamma}_{ab}},
\]  

(3.61)
The momenta associated with the variables $N^\mu$ will be clearly equal to zero,

$$ P^\mu \equiv \frac{\partial L}{\partial \dot{N}_\mu} = 0 \ . $$

(3.62)

In order to calculate the momenta defined by (3.61), we introduce an object with 4 indices of the form

$$ G^{abcd} \equiv \sqrt{\gamma} \left[ \frac{1}{2} (\gamma^{ac}\gamma^{bd} + \gamma^{ad}\gamma^{bc}) - \gamma^{ab}\gamma^{cd} \right] \ . $$

(3.63)

This object is called a supermetric. We can then rewrite the Lagrangian in the following form

$$ L = N (G^{abcd} K_{ab} K_{cd} + \sqrt{\gamma} 3 R) \ . $$

(3.64)

We can then construct the momenta from their usual definition and from the previous expression:

$$ \pi^{ab} = \frac{\partial L}{\partial \dot{\gamma}^{ab}} = N \left( 2 G^{efcd} K_{cd} \frac{\partial K_{ef}}{\partial \dot{\gamma}^{ab}} \right) $$

from Eq. (3.44) it follows that

$$ \frac{\partial K_{ab}}{\partial \dot{\gamma}^{ef}} = - \frac{1}{2N} \left[ \frac{1}{2} (\delta_a^e \delta_b^f + \delta_b^e \delta_a^f) \right] \equiv - \frac{1}{2N} \delta^{ef}_{ab} \ , $$

(3.65)

for that we must have

$$ \pi^{ab} = - G^{abcd} K_{cd} \ . $$

(3.66)

As it can be seen from the definition of the supermetric and from the last expression, the momenta $\pi^{ab}$ are the tensor densities of weight one on the hypersurface.

To perform the Legendre transform of the Lagrangian and to define the Hamiltonian, we need to isolate the velocities as functions of the momenta. The velocities $\dot{N}^\mu$ are arbitrary functions and therefore cannot be expressed in terms of the coordinates and momenta. The velocities of the metric $\gamma_{ab}$, can be expressed in such a way that it is possible to invert Eq. (3.65), i.e. one can express $K_{ab}$ in terms of momenta. For that, we need to find the inverse of the supermetric, i.e. the object $G_{abcd}$ which when contracted with the supermetric forms the identity

$$ G_{abcd} \sigma^{ef} = \delta^{ef}_{ab} \ . $$

(3.67)

The most general form of the inverse would be

$$ G_{abcd} = \frac{1}{\sqrt{\gamma}} \left[ \frac{A}{2} (\gamma_{ac}\gamma_{bd} + \gamma_{ad}\gamma_{bc}) + B \gamma_{ab}\gamma_{cd} \right] \ . $$

(3.68)

Inserting (3.67) into (3.66), it is found that the values taken by the coefficients $A$ and $B$ are $A = 1$, $B = -\frac{1}{2}$. The extrinsic curvature can now be expressed as

$$ K_{ab} = - G_{abcd} \pi^{cd} \ . $$

(3.69)
From Eqs. (3.44) and (3.68), we can write the velocities $\dot{\gamma}_{ab}$ in the form

$$
\dot{\gamma}_{ab} = N_{a|b} + N_{b|a} - 2NK_{ab} \\
= N_{a|b} + N_{b|a} + 2NG_{abcd}\pi^{cd}.
$$

(3.70)

We are now in a position to write the Hamiltonian where the Legendre transform will be done only on the variables $\dot{\gamma}_{ab}$ by taking the following form of the Hamiltonian,

$$
H = \int d^3x \left( \pi^{ab} \dot{\gamma}_{ab} - L \right).
$$

(3.71)

The Hamiltonian density can now be expressed as

$$
(\pi^{ab} \dot{\gamma}_{ab} - L) = -G^{abcd}K_{cd}(N_{a|b} + N_{b|a} - 2NK_{ab}) - N(G^{abcd}K_{cd} + \sqrt{\gamma} \, 3R) \\
= N\sqrt{\gamma}(K_{ab}K^{ab} - K^2 - R) - 2\pi^{ab}N_{a|b}.
$$

(3.72)

The second term can be integrated by parts (taking into account the tensor densities) after which, we are left with the Hamiltonian of the form

$$
H = \int d^3x \left[ N\sqrt{\gamma}(K_{ab}K^{ab} - K^2 - R) + N^a(2\pi^{b}_{a|b}) \right] \\
\equiv \int d^3x \left[ N\mathcal{H}_0 + N^a\mathcal{H}_a \right].
$$

(3.73)

where

$$
\mathcal{H}_0 = N\sqrt{\gamma}(K_{ab}K^{ab} - K^2 - R) \\
\mathcal{H}_a = 2\pi^{b}_{a|b}.
$$

We can then write the first-order Lagrangian density from (3.70) and (3.72)

$$
\mathcal{L}(\gamma_{ab}, \dot{\gamma}_{ab}, \pi^{ab}, N, N^a) = \pi^{ab}\dot{\gamma}_{ab} - NH_0 + N^aH_a
$$

(3.74)

The action integral from this Lagrangian will be a functional of the spatial metric $\gamma_{ab}$ as well as of the momenta $\pi^{ab}$ and the functions $N^\mu$. From the form of the Lagrangian (3.74) two points are to be noted: the functions $N^\mu$ play the role of Lagrange multipliers and the action is written in already parameterized form. We shall clarify these points later. If we vary the action with respect to the $N^\mu$ functions, we find that both $\mathcal{H}_0$ and $\mathcal{H}_a$ are zero so the $N^\mu$ when multiplying functions that take the value zero are the functions of the Lagrange multipliers and they are, in principle, arbitrary functions. The equations of motion for the other dynamic variables will be given from the variations of the action with respect to those. The variables $\gamma_{ab}$ and $\pi^{ab}$ are not independent of each other, since are linked by the four relations $\mathcal{H}_\mu = 0$. These relationships are known as Hamiltonian constraints. The theory is in parameterized form since the first-order Lagrangian is not in canonical form, a form that we would have if the terms of the Hamiltonian exclusively contained variables in terms of $(\pi^{ab}, \dot{\gamma}_{ab})$. Since, in this case, the terms of the Hamiltonian contain $N^\mu$, the action is said not to be in a canonical form. This occurs when the theory is in parameterized form.

To illustrate this distinction between canonical and parameterized form, let us take a non-relativistic particle with generalized coordinates $q^i$ and velocities with respect to time $q^i$. The first-order action is written as:

$$
S = \int_{t_0}^{t_1} dt \, L = \int_{t_0}^{t_1} dt \left( q^i \dot{p}_i - H(p_i, q^i) \right).
$$

(3.75)
which is in canonical form. If we now take time as another coordinate \( q_0 \), and we consider all the other coordinates as functions of a parameter \( \tau \), then we can identify the Hamiltonian \( H \) as a canonically conjugate momentum to the coordinate \( q_0 \) in the following form

\[ p_0 = -H(q^i, p_i) \, , \tag{3.76} \]

From this equality, it follows that the momentum \( p_0 \) is not independent of the other variables; as Eq. (3.76) is a constraint between the variables. The action (3.75) can now be rewritten after having eliminated the coordinate category of time in the following way

\[ S = \int_{\tau_0}^{\tau_1} d\tau \, \dot{q}^\mu p_\mu \, , \tag{3.77} \]

However, there is the constraint (3.76) between the variables, which can be written as \( \mathcal{H} \equiv p_0 + H(q^i, p_i) = 0 \). This term must be added to the Lagrangian together with an arbitrary multiplier \( N \), to have the full Lagrangian. So, let us consider the action to be of the form

\[ L = \dot{q}^\mu p_\mu - N\mathcal{H} \, . \tag{3.78} \]

This Lagrangian is in a parameterized form. To get to the canonical form, it would be necessary to do the steps in reverse. The first step is to get rid of \( \mathcal{H} \) or resolve the constraint by making \( p_0 = -H \). Thus the action becomes

\[ S = \int_{\tau_0}^{\tau_1} d\tau \, (\dot{q}^i p_i - Hq^0) \, , \tag{3.79} \]

which can be rewritten in such a way that the dependency on the parameter \( \tau \) disappears,

\[ S = \int dq^0 \left( \frac{dq^i}{dq^0} p_i - H \right) \, . \tag{3.80} \]

This form of the action is independent of the parameter \( \tau \) (since it does not appear in the expression) and therefore will not get modified by a parameter change.

In practice, what is done to go from (3.79) to (3.80) is specification of a relation between the parameter and the coordinate \( q_0 \). One can achieve this since there is a freedom to arbitrarily choose the parameter (gauge freedom). If one imposes a coordinate condition of the form e.g. \( q_0 = \tau \), the action takes the form (3.80) with the change of notation \( q_0 \rightarrow \tau \). With this example it is therefore possible, to assert that in general terms, the way to reduce a parameterized action to the canonical form is to insert the solution of the constraints and impose coordinates conditions.

The gravitational action is expressed in the form of Eq. (3.78). To solve the equations, in the first step, it would be necessary to transform to the canonical form i.e. to the equivalent of Eq. (3.75), or in other words, it is required to solve the constraints and impose the coordinate conditions. This procedure has only been possible in very particular cases (see the applications), but in general it has only been achieved formally [10].

4 Applications

In this section, we briefly present some of the fields that have been developed using the ADM formalism.
4.1 Quantum Gravity

Until recently, one of the main motivations for building a Hamiltonian formulation (i.e. the identification of canonically conjugate variables from a Lagrangian function, and obtaining the Hamiltonian as the Legendre transform in the velocities of the Lagrangian function) consisted of the possibility for quantizing the theory. That is, associating with each canonical variable an operator on a vector space and imposing rules of switching between pairs of conjugate variables. The Hamiltonian operator now takes the role of the generator for time translations, and it governs the dynamics of the system.

However, for general relativity, the process of quantization is not direct from the constructed Hamiltonian, since there are 4 constraints (3.73) among the canonical variables. To try to overcome this difficulty, one can follow two different paths, which are roughly:

1) Trying to isolate the true degrees of freedom of gravitation (which, from the linearization of the equations is known to be two degrees of freedom per point)—which is the idea followed in ADM formalism. Once the degrees of freedom have been isolated, one proceeds to quantization. This procedure as mentioned above has been formally accomplished.

2) Considering all variables, but imposing conditions on the state vector given by the four constraints, by demanding that the constraints (now as operators) when acting on the state vectors annul it. This is the view taken by Dirac and De-Witt.

In recent years, attempts have been made to quantize the gravitational field using the idea of—sum over trajectories, i.e. all possible evolutions of the 3-geometries (between two fixed 3-geometries) are considered by giving each of those a weight associated to the action evaluated along that trajectory [3].

Likewise, canonical formulations of relativity have been constructed as an alternatives to ADM, such as the Ashtekar formalism [11].

4.2 Geometrodynamics

Geometrodynamics is the name given to general relativity when it is seen as a dynamical theory [12]. As mentioned in the introduction, the covariant formulation of general relativity is not the most appropriate for appreciating its dynamic character. The introduction of new variables and consideration of space-time formed by the evolution of metric $\gamma_{ij}(t)$ suggests which is the dynamic variable that evolves. However, by analyzing the expression that was obtained for the action (3.58), it is observed that the action is invariant considering the diffeomorphism on the hypersurface $m(t)$, as it does not depend on the coordinate system used on this one. This fact suggests that the truly relevant property for each hypersurface $m(t)$ is not the metric defined on it, but the intrinsic geometry of the manifold itself.

This conclusion is relatively clear if one considers the example presented in Sec.3 with the spheres generating Euclidean space. If one wants to visualize $\mathbb{R}^3$ formed by the set of infinite spheres glued together, no matter what intrinsic coordinate system is given to each sphere, it always remains a sphere.
Space-time is, then, the evolution (or the set if one prefers to see it that way) of 3-
geometries. The object that evolves is the geometry of a 3-manifold $m$ and this is the
dynamic variable of general relativity, hence the name geometrodynamics.

If the 3-geometries are what evolves, it is natural to ask: in what space do these ge-
ometries evolve? The answer is almost silly— “In the space of 3-geometries”. This is
the set in which every point is a 3-geometry and in it are all possible 3-geometries for
the manifold $m$. This space is known as Superspace [12] and is denoted by $G$.

There is an analogy between the evolution of a particle and that of a geometry. The
particle lives in space-time and its evolution is a trajectory in it. In the same way, a
3-geometry lives in superspace and its evolution is a trajectory in it. This analogy is
not however, totally literal, since the trajectories in superspace, which define space-
time, are not one-dimensional curves (parameterized by real entities/numbers), but are
submanifolds of the superspace. Let us briefly go through the reason why they are not
one-dimensional curves.

Suppose we have a known space-time. The foliations that can be built for such a space-
time are infinite, since there will be a foliation for each function $t(x^\mu)$ that is defined in
space-time such that the hypersurfaces given by $t = \text{cnst.}$ do not intersect. It is clear
that the number of such functions that can be defined is uncountable. When such a
function is chosen, i.e. when one chooses a particular foliation, space-time will thus
be formed by a curve of hypersurfaces parameterized by the value of $t$. Space-time
is then a curve, once the foliation has been chosen. It is also clear from the above
argument, that the set of all foliations for such a space-time represents a submanifold
of the superspace. This submanifold is known as hyperspace [13].

4.3 Cosmology

In the late 1960s and early 1970s, the study of cosmological models was developed,
(in particular the homogeneous ones) with the help of the Hamiltonian formalism [14].
This method is particularly useful since in some cases, it is possible to find an explicit
solution of Einstein’s equations and in general it allows for a richer qualitative analysis
of the dynamics of the models. Among the most studied models are those in which
the lapse and shift functions take the values $N = 1$ and $N_i = 0$, the hypersurfaces of
t = \text{cnst.} are those that accept “a group of motions” defined in them (there are a total
of nine) and are known as Bianchi – type universes.

The general metric for homogeneous models can be written in the following way:

$$ds^2 = -dt^2 + \gamma_{ij}(t) \, dx^i dx^j . \quad (4.1)$$

The procedure described at the end of Sec. 3, which consists of solving the constraints
and choosing coordinate conditions in order to pass the action to the canonical form
(thus having a true Hamiltonian) has been possible in some cases where the coordinates
are defined as a function of $\gamma_{ij}$ and $\pi^{ij}$, arriving at an expression of the Hamiltonian
in terms of these coordinates (see [14]).
of the problem depend only on the parameter time and not of all the variables as in
the case of a more general space-time. In the language of superspace, the 3-geometries
of cosmological models are a subset of that—baptized by Misner with the name of
minisuperspace [15].

In minisuperspace, where the number of degrees of freedom is at most five, it is possible
to ask whether there is any relationship between Hamilton’s equations derived from
the Hamiltonian, and the geodesic equation associated with the supermetric.

From the constraint \( H_0 = 0 \) defined in (3.37), we can define a Hamiltonian (equal to
zero) from such an equation and rewrite:

\[
H = \frac{1}{2} (\pi_A \pi_B G^{AB} - R) , \tag{4.2}
\]

where two indices \((ij)\) were replaced by \(A\). Note that the form of the Hamiltonian
in (4.2) is similar to the Hamiltonian of the free particle in space-time with metric \(G^{AB}\).
The only difference is that in (4.2), the mass term may depend on the coordinates \(g^A\)
in general.

The Hamilton’s equations for the Hamiltonian will then be:

\[
\begin{align*}
\frac{dg^A}{d\lambda} &= \frac{\partial H}{\partial \pi_A} = G^{AB} \pi_B , \\
\frac{d\pi_A}{d\lambda} &= -\frac{\partial H}{\partial g^A} = -\frac{1}{2} \pi_C \pi_D G^{CD} :A + \frac{1}{2} R :A , \tag{4.3}
\end{align*}
\]

From these equations we derive:

\[
\frac{d^2 g^A}{d\lambda^2} = G^{AB} :C \pi_B G^{CD} \pi_D + G^{AB} \left( -\frac{1}{2} \pi_D \pi_E G^{DE} :B + \frac{1}{2} R :B \right) ,
\]

which can be rewritten as:

\[
\frac{d^2 g^A}{d\lambda^2} + \left[ \frac{1}{2} G^{AB} (G_{BD,C} + G_{BC,D} - G_{DC,B}) \frac{dg^C}{d\lambda} \frac{dg^D}{d\lambda} \right] = \frac{1}{2} R :A ,
\]

Therefore,

\[
\frac{d^2 g^A}{d\lambda^2} + \Gamma^A_{CD} \frac{dg^C}{d\lambda} \frac{dg^D}{d\lambda} = \frac{1}{2} R :A . \tag{4.4}
\]

That is, we obtain the geodesic equation for the metric \(G^{AB}\) with an extra term. This
equation allows (in some way), to relate the Hamilton’s equations with that of geodesics
to which (as if) an \(R\)-potential was added and apparently is responsible for an extra
force. If somehow this term were negligible, we would practically have the behavior of
the cosmological model as if it were of a free particle in curved space-time.

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References

[1] R.L. Arnowitt, S. Deser, and C.W. Misner. The Dynamics of general relativity. Republished in: Gen. Rel. Grav., 40:1997–2027, 2008.
[2] A. Hanson, T. Regge, and C. Teitelboim. Constrained hamiltonian systems. Accademia Nazionale dei Lincei, 1976.
[3] J. Guven and M.P. Ryan Jr. Functional integrals and canonical quantum gravity. Physical Review D, 45(10):3559, 1992.
[4] R. A. Matzner. Numerical relativity. In Relativity and Gravitation, page 98, January 1991.
[5] Bryce S DeWitt. Quantum theory of gravity. i. the canonical theory. Physical Review, 160(5):1113, 1967.
[6] M. Spivak. A comprehensive introduction to differential geometry. Publish or Perish, Boston, 1973.
[7] K.S. Thorne, C.W. Misner, and J.A. Wheeler. Gravitation. Freeman San Francisco, CA, 1973.
[8] L.D. Landau. The classical theory of fields, volume 2. Elsevier, 1973.
[9] P.A.M. Dirac. Lectures on quantum mechanics, volume 2. Courier Corporation, 2001.
[10] K. Sundermeyer. Constrained dynamics with applications to Yang-Mills theory, general relativity, classical spin, dual string model. Springer, 1982.
[11] A. Ashtekar. Lectures on non-perturbative canonical gravity, volume 6. World scientific, 1991.
[12] J.A. Wheeler. Superspace. In Analytic Methods in Mathematical Physics, page 335, 1970.
[13] K. Kuchař. Geometry of hyperspace. i. Journal of Mathematical Physics, 17(5):777–791, 1976.
[14] M. Ryan. Hamiltonian cosmology. Springer, 1972.
[15] C.W. Misner. Minisuperspace. in Magic Without Magic: John Archibald Wheeler, 1972.