Degeneration, rigidity and irreducible components of Hopf algebras

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The aim of this work is to discuss the concepts of degeneration, deformation and rigidity of Hopf algebras and to apply them to the geometric study of the varieties of Hopf algebras. The main result is the description of the n-dimensional rigid Hopf algebras and the irreducible components for \( n = p^2 \), \( p \) is prime number, and for \( n < 14 \).

This paper is organized as follows: first we present the basic concepts of degeneration, deformation and rigidity with examples and some useful properties. The second part of this work is devoted to the geometric description of the algebraic varieties of \( n \)-dimensional Hopf algebras. In the last section, we give a necessary and sufficient condition for the existence of a degeneration of a given Hopf algebra.

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1. Introduction

The discovery of quantum groups gives impulse to a strong development in the theory of Hopf algebras. Many articles were devoted to the class of semisimple Hopf algebras; several characterizations were given and some fundamental results were established. See for example the surveys [1] [18]. The complete classification of all Hopf algebras of a fixed dimension \( n \) is known only for \( n = p \) (\( p \) prime) [25], \( n = p^2 \) (\( p \) prime) [24] and for certain small dimension \( n \), \( n < 14 \), \( n = 15, 21, 35 \). Besides this, substantial results are known for some classes, like pointed Hopf algebras (see [3]), and triangular Hopf algebras (see [4]).

This paper is focused on the geometric and local study of the variety of Hopf algebras of fixed dimension \( n \). The only known antecedent is the paper [24], where D. Stefan considers, in a cohomological way, the invariant of the irreducible components of semisimple and cosemisimple Hopf algebras. Here, we approach the varieties of Hopf algebras, in a more geometric way, via the algebraic deformation theory introduced by M. Gerstenhaber [10] and degeneration theory.

The concept of degeneration appeared first in the physics literature. The question was to show in which sense a group can be a limiting case of other groups. Degenerations, called also contractions or specialisations, were introduced for Lie groups by Segal, Inouu and Wigner (1953) [12]. They showed that the Galilei group of classical mechanic is a limiting case of the Lorentz group corresponding to relativistic mechanic. Later, Saletan (1960) [21] generalized the notion and stated a general condition for the existence of degenerations (or contractions) of Lie algebras.

The notions of deformation and degeneration were used by several authors in the studies of associative algebras varieties or Lie algebras varieties (Ancochea-Bermudez, Carles, Gabriel, Gerstenhaber, Goze, Happel, Mazzola, Makhlouf, Schaps,
Schack...). It was also used, in the theory of quantum groups, by Cehegilenti, Gia- 
chetti, Sorace and Tarlini \cite{3} to define Heisenberg and Euclidean quantum groups.

We now discuss the contents of this paper. We first recall the algebraic variety 
structure of the set of bialgebras $Bialg_n$ and Hopf algebras $Hopf_n$. Then, we 
describe the linear group action on these varieties. The section 3 is devoted to the 
degeneration of Hopf algebras, we give some examples and show that the graded 
Hopf algebra associated to a filtered Hopf algebra is always a degeneration of this 
Hopf algebra. The section 4 is devoted to deformation and rigidity of Hopf algebras, 
we give some useful properties and prove that if a finite dimensional Hopf algebra 
is rigid then its dual is rigid. In the section 5, we study the deformations and 
degenerations of Hopf algebras appearing in known classification. We consider 
first the algebraic varieties $Hopf_{p^q}$ where $p$ is any prime number. We prove that 
every Hopf algebra, in these algebraic varieties, is rigid. Therefore these algebraic 
varieties are union of Zariski open orbits. And next we show that, for $n < 14$, every 
n-dimensional Hopf algebra is rigid except $A_{14}$ in $Hopf_8$ and $A_0$, $B_1$ in $Hopf_{12}$.

We prove that $A_{14}$ is a degeneration of $A_{14}$ and $A_0$ is a degeneration of $A_1$ and 
by duality $A_0 \cong B_1$ is a degeneration of $A_1$. Then, we describe the irreducible 
components of $Hopf_n$ for $n < 14$. In the last section, we give a necessary and 
sufficient condition for the existence of a degeneration of a given Hopf algebra.

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2. Structure of Algebraic Varieties

Throughout this paper $K$ will be an algebraically closed field of characteristic 0 
and $V$ be an $n$-dimensional vector space over the field $K$. Let $H = (V, \mu, \eta, \Delta, \varepsilon, S)$ 
be a finite dimensional Hopf algebra, where $\mu : V \otimes V \rightarrow V$ is the multiplication, 
$\eta : K \rightarrow V$ is the unity, $\Delta : V \rightarrow V \otimes V$ is the comultiplication, $\varepsilon : V \rightarrow k$ 
is the counity and the endomorphism $S$ is the antipode. We refer to \cite{12} for the 
definitions.

Setting a basis $\{e_1, ..., e_n\}$ of $V$ where $e_1 = \eta(1)$, we identify the multiplication 
$\mu$ and the comultiplication $\Delta$ with their $n^3$ structure constants $C^i_{j,k}$ and $D^{i\ell}_{jk} \in K$, 
where $\mu (e_i \otimes e_j) = \sum^n_{k=1} C^i_{j,k} e_k$ and $\Delta (e_i) = \sum^n_{j=1} \sum^n_{k=1} D^{i\ell}_{jk} e_j \otimes e_k$. The counity 
$\varepsilon$ is identified with its $n$ structure constants $\xi_i$, where $\varepsilon (e_i) = \xi_i$. The collection 
$(C^i_{j,k}, D^{i\ell}_{jk}, \xi_i : i,j,k = 1, \cdots, n)$ represents a bialgebra if it satisfies the following 
polynomial equations:

\begin{align*}
(1) & \quad \begin{cases}
\sum_{i=1}^{n} C^i_{j,k} C^j_{k,l} - C^i_{j,l} C^j_{k,k} = 0 \\
C^i_{1,l} = C^j_{i,1} = \delta_{ij} \quad \text{the Kronecker symbol}
\end{cases} \\
& \quad \begin{cases}
\sum_{i=1}^{n} D^{i\ell}_{jk} D^{k\ell}_{ji} - D^{i\ell}_{ij} D^{k\ell}_{kj} = 0 \\
\sum_{i=1}^{n} D^{i\ell}_{1k} \xi_l - \sum_{i=1}^{n} D^{i\ell}_{1j} \xi_k = 0 \\
\sum_{i=1}^{n} D^{i\ell}_{ij} \xi_l = \sum_{i=1}^{n} D^{i\ell}_{ij} \xi_k = 0 \quad i \neq j
\end{cases} \quad i,j,k,s \in \{1,\ldots,n\}
\end{align*}

\begin{align*}
(2) & \quad \begin{cases}
\sum_{i=1}^{n} C^i_{j,k} - \sum_{t,p,q=1} C^i_{t,p} D^{p\ell}_{j} D^{q\ell}_{k} C^q_{t,q} = 0 \\
D^{11}_{12} = 1, \quad D^{ij}_{ij} = 0 \quad (i,j) \neq (1,1) \\
\xi_1 = 1, \quad \sum_{i=1}^{n} C^i_{1j} \xi_l = \xi_l \xi_j \quad i,j,k,s \in \{1,\ldots,n\}
\end{cases}
\end{align*}
The polynomial relations (1) (2) (3) endow the set of the \( n \)-dimensional bialgebras, denoted by \( \text{Bialg}_n \), with a structure of an algebraic variety embedded in \( K^{2n^3+n} \) which we do consider here together with its natural structure of an algebraic variety over \( K \).

As a subset of \( K^{2n^3+n} \), \( \text{Bialg}_n \) may be provided with the Zariski topology. If \( K \) is the complex field the set \( \text{Bialg}_n \) may be provided with the metric topology of \( K^{2n^3+n} \). Recall that the metric topology is finer than the Zariski topology.

**Remark 1.** We can also fix the counity as in Stephan’s work but it seems more general keeping it free. The set where the counity is fixed is an open subset in \( \text{Bialg}_n \) (for metric or Zariski topologies).

The elements of \( \text{Bialg}_n \) with antipodes define the set of Hopf algebras, denoted by \( \text{Hopf}_n \). If \( S = (S_{ij}) \) defines the antipode, with respect to the basis \( \{e_1, \ldots, e_n\} \) of \( V \), then, in addition to relations (1) (2) (3), we have the following equations:

\[
\begin{align*}
\sum_{j,k,r=1}^n D_i^{jk} S^1_{jr} C^1_{rk} &= \sum_{j,k,r=1}^n D_i^{jk} S_{rk} C^1_{jr} = \xi_i, \\
\sum_{j,k,r=1}^n D_i^{jk} S^1_{jr} C^1_{rk} &= \sum_{j,k,r=1}^n D_i^{jk} S_{rk} C^1_{jr} = 0 \\
&
\end{align*}
\]

The set \( \text{Hopf}_n \) is a Zariski open subset of \( \text{Bialg}_n \).

**Remark 2.** The sets \( \text{Bialg}_n \) and \( \text{Hopf}_n \) carry also a structure of scheme.

### 2.1. The \( \text{GL}_n(K) \) action, orbits.

In the following we will define \( \text{GL}_n(K) \) action for Hopf algebras. Naturally, we have a similar definition for bialgebras.

Geometrically, a point \( (C_{ij}, D_i^{jk}, \xi_i : i,j,k = 1, \ldots, n) \) of \( K^{2n^3+n} \) satisfying (1) (2) (3) (4) where the matrix \( (S_{ij}) \) defines the antipode, represents an \( n \)-dimensional Hopf algebra \( H \), along with a particular choice of basis. A change of basis in \( H \) may give rise to a different point of \( \text{Hopf}_n \). Let \( H = (V, \mu, \eta, \Delta, \varepsilon, S) \) be a Hopf algebra, The “structure transport” action is defined on \( H \) by the following action of \( \text{GL}_n(K) \)

\[
\begin{align*}
\text{GL}_n(K) \times \text{Hopf}_n & \to \text{Hopf}_n \\
(f, H) & \mapsto f \cdot H
\end{align*}
\]

\( \forall X,Y \in V \)

\[
(f \cdot \mu)(X \otimes Y) = f^{-1}(\mu(f(X) \otimes f(Y)))
\]

\[
(f \cdot \Delta)(X) = f^{-1} \otimes f^{-1}(\Delta(f(X)))
\]

\[
(f \cdot \varepsilon)(X) = \varepsilon(f(X))
\]

The orbit of a Hopf algebra \( H \) is given by \( \vartheta(H) = \{ f \cdot H, \quad f \in \text{GL}_n(K) \} \). The orbits are in 1-1-correspondence with the isomorphism classes of \( n \)-dimensional Hopf algebras. The stabilizer subgroup of \( H \) \( (\text{stab}(H) = \{ f \in \text{GL}_n(K) : H = f \cdot H \}) \) is exactly \( \text{Aut}(H) \), the automorphism group of \( H \). The orbit \( \vartheta(H) \) is identified with the homogeneous space \( \text{GL}_n(K)/\text{Aut}(H) \). Then

\[
\dim \vartheta(H) = n^2 - \dim \text{Aut}(H)
\]

The orbit \( \vartheta(H) \) is provided, when \( K = \mathbb{C} \) (a complex field), with the structure of a differentiable manifold. In fact, \( \vartheta(H) \) is the image through the action of the Lie group \( \text{GL}_n(K) \) of the point \( H \), considered as a point of \( \text{Hom}(V \otimes V, V) \times \text{Hom}(V, V \otimes V) \).

The Zariski open orbits have a special interest in the geometric study of \( \text{Hopf}_n \). It corresponds to a so called rigid Hopf algebras. The orbit’s closure of a rigid
Hopf algebra determines an irreducible component of $Hopf_n$. Many but not all components of $Hopf_n$ are orbits closures of rigid Hopf algebras, for example there are infinitely many isomorphism classes for $dim H = p^4$ ($p$ odd and prime).

This parameter family cannot be in the closure of a rigid Hopf algebra.

3. Degenerations

3.1. Definition. Let $H_0$ and $H$ be two $n$-dimensional Hopf algebra. We say that $H_0$ is a degeneration of $H$ if $H_0$ is in $\mathcal{G}(H)$, the Zariski closure of the orbit of $H$.

Remark 3.

• Let $t$ be a parameter in $K$ and $\{f_t\}_{t \neq 0}$ be a family of continuous invertible linear maps on $V$ over $K$ and $H = (V, \mu, \Delta, \eta, \varepsilon, S)$ be a Hopf algebra over $K$. The limit (when it exists) of a sequence $f_t \cdot H$, $H_0 = \lim_{t \to 0} f_t \cdot H$, is a degeneration of $H$ in the sense that $H_0$ is in the Zariski closure of the set $\{f_t \cdot H\}_{t \neq 0}$.

The multiplication and the comultiplication $\mu_0$ and $\Delta_0$ of $H_0$ are given by

$$
\mu_0 = \lim_{t \to 0} f_t \cdot \mu = \lim_{t \to 0} f_t^{-1} \circ \mu \circ f_t
$$

$$
\Delta_0 = \lim_{t \to 0} f_t \cdot \Delta = \lim_{t \to 0} f_t^{-1} \circ \Delta \circ f_t
$$

• The multiplication $\mu_t = f_t^{-1} \circ \mu \circ f_t$ and the comultiplication $\Delta_t = f_t^{-1} \circ f_t^{-1} \circ \Delta \circ f_t$ satisfy the conditions of bialgebra, then when $t$ tends to $0$ the conditions remain satisfied.

• In the point 1, the linear map $f_t$ is invertible when $t \neq 0$ and may be singular when $t = 0$. Then, one may obtain by degeneration a new Hopf algebra.

• Geometrically, $H_0$ is a degeneration of $H$ means that $H_0$ and $H$ belong to the same irreducible component.

• When $K$ is the complex field, the multiplication and the comultiplication given by the limit, follows from a limit of the structure constants, using the metric topology. In fact, $f_t \cdot \mu$ and $f_t \cdot \Delta$ correspond to a change of basis when $t \neq 0$. when $t = 0$, they give eventually a new point in $Hopf_n \subset K^{2n^3+n}$.

• The same definitions and remarks hold for bialgebras.

3.2. Example. In $Hopf_8$, the Hopf algebra $H_0$ defined by

$$
\Delta(g) = g \otimes g,
$$

$$
\Delta(x) = x \otimes g + 1 \otimes x,
$$

$$
\varepsilon(x) = 0, \quad \varepsilon(g) = 1,
$$

$$
S(x) = -xg^2, S(g) = g^3,
$$

is a degeneration of the Hopf algebra $H_1$ defined by

$$
\Delta_K(x, g) = \left(\frac{K(x, g)}{g^2 - x^2 g + xg}, g^2 - x^2 g + xg \right),
$$

with the same comultiplication, antipode and coalgebra structure. In fact, the family $H_t$ defined by

$$
\Delta_K(x, g) = \left(\frac{K(x, g)}{g^2 - x + t(1 - g^2) + xg}, g^2 - x + t(1 - g^2) + xg \right)
$$

where $t$ is a parameter, is isomorphic to $H_1$ when $t \neq 0$ and tends to $H_0$ when $t$ tends to $0$.

3.3. The graded Hopf algebra.

Theorem 3.1. Let $H$ be a Hopf $K$-algebra and $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n \subseteq \cdots$ a Hopf algebra filtration of $H$, $H = \cup_{n \geq 0} H_n$ and $\Delta H_n = \sum_{m \geq 0} H_m \otimes H_{n-m}$. Then the graded Hopf algebra $gr(H) = \oplus_{n \geq 1} H_n / H_{n-1}$ is a degeneration of $H$.

Proof. Let $H[[t]]$ be a power serie ring in one variable $t \in K$ over $H$, $H[[t]] = H \otimes K[[t]] = \oplus_{n \geq 0} H \otimes t^n$.

We denote by $H_t$ the Rees algebra associated to the filtered Hopf algebra $H$, $H_t = \sum_{n \geq 0} H_n \otimes t^n$. The Rees algebra $H_t$ is contained in the algebra $H[[t]]$. 
For every $\lambda \in K$, we set $H_{(\lambda)} = H_t/(t - \lambda) \cdot H_t$. For $\lambda = 0$, $H_{(0)} = H_t/(t \cdot H_t)$. The Hopf algebra $H_{(0)}$ corresponds to the graded algebra $gr(H)$ and $H_{(1)}$ is isomorphic to $H$. In fact, we suppose that the parameter $t$ commutes with the elements of $H$ then $t \cdot H_t = \oplus_{n \geq 0} H_n \otimes t^{n+1}$. It follows that $H_{(0)} = H_t/(t \cdot H_t) = (\sum_n H_n \otimes t^n)/(\oplus_{n \geq 0} H_n \otimes t^{n+1}) = \oplus_{n \geq 0} H_n/H_{n-1} = gr(H)$. By using the linear map from $H_t$ to $H$ where the image of $a_n \otimes t^n$ is $a_n$, we have also $H_{(1)} = H_t/(t - 1) \cdot H_t \cong H$.

If $\lambda \neq 0$, the change of parameter $t = \lambda T$ shows that $H_{(\lambda)}$ is isomorphic to $H_{(1)}$. This ends the proof that $H_{(0)} = gr(H)$ is a degeneration of $H_{(1)} \cong H$. 

### 4. Deformation and rigidity of Hopf algebras

In this section, we recall the algebraic deformation and the rigidity notions introduced by Gerstenhaber. We show the connection between degeneration and deformation and give some remarks and properties useful in the geometric study of $Hopf_n$. We prove in particular that the rigidity of a finite dimensional Hopf algebra is equivalent to the rigidity of its dual.

#### 4.1. Algebraic deformation. The notion of deformation is in some sense the dual notion of the degeneration. Let $H = (V, \mu, \Delta, \eta, \varepsilon, S)$ be a Hopf algebra over a field $K$. Let $K[[t]]$ be the power series ring in one variable $t$. Let $V[[t]]$ be the extension of $V$ by extending the coefficient domain from $K$ to $K[[t]]$. Then $V[[t]]$ is a $K[[t]]$-module and $V[[t]] = V \otimes_K K[[t]]$. A deformation of $H$ is a one parameter family $H_t = (V[[t]], \mu_t, \Delta_t, \eta_t, \varepsilon_t, S_t)$. Since the unit, counit and the antipode are preserved by deformation \[ K[[t]] \] it follows that a deformation of $H = (V, \mu, \Delta, \eta, \varepsilon, S)$ can be considered as a pair of deformations $(\mu_t, \Delta_t)$ which together give on $V[[t]]$ the structure of bialgebra over $K[[t]]$. By $K[[t]]$-linearity the morphisms $\mu_t, \Delta_t$ are determined by their restrictions to $V \otimes V$:

$$\mu_t : V \otimes V \rightarrow V[[t]]$$
$$x \otimes y \rightarrow \mu_t(x \otimes y) = \sum_{m=0}^{\infty} \mu_m(x \otimes y)t^m \quad \text{with} \quad \mu_m \in Hom(V \otimes V, V)$$

$$\Delta_t : V \otimes V[[t]] = (V \otimes V)[[t]] \otimes V[[t]] = (V \otimes V)[[t]]$$
$$x \rightarrow \Delta_t(x) = \sum_{m=0}^{\infty} \Delta_m(x)t^m \quad \text{with} \quad \Delta_m \in Hom(V \otimes V)$$

and they satisfy:

- $\mu_t$ is associative
- $\Delta_t$ is coassociative
- $(\mu_t \otimes \mu_t) \circ (id \otimes \tau \otimes id) \circ (\Delta_t \otimes \Delta_t) = \Delta_t \circ \mu_t$ where $\tau$ is the twist map.

Two deformations $H_t$ and $H'_t$ of $H$ are said equivalent if there exits a formal isomorphism $\phi_t = \phi_0 + t\phi_1 + \ldots + t^k\phi_k + \ldots$ where $\phi_0 = Id_V$ and $\phi_k \in End(V)$ such that $H'_t = \phi_t \cdot H_t$ (as defined in section 2.). A deformation $H_t$ of $H$ is said trivial if $H_t$ is equivalent to $H$.

We give here some helpful remarks on deformations of Hopf algebras.

**Remark 4.**

- If $H_t$ is a deformation of a Hopf algebra $H$ then the dual of $H_t$ is a deformation of the dual of $H$. This follows from the linearity of the operations.
- The deformation of pointed algebra in not always a pointed Hopf algebra. Consider in $Kp^d$ where $p$ is prime and odd number the Hopf algebra defined by $A_t = (g^p - 1, x, y, \Delta g = g \otimes g, \Delta x = (g^p - 1, x, y, \Delta g = g \otimes g, \Delta x = (g^p - 1, x, y, \Delta g = g \otimes g, \Delta x =$
$x \otimes 1 + g \otimes x$, $\Delta y = y \otimes 1 + g \otimes y$, where $t$ is a parameter in $K$ and $q$ is a primitive $p$-th root of unity. We set $B_t = A_t^*$ (the dual Hopf algebra) and $B_0 = A_0^*$. Since $A_t$ is a deformation of $A_0$ then $B_t$ is a deformation of $B_0$. The Hopf algebra $B_0$ is pointed while its deformation $B_t$ is not pointed.

- The number of grouplike elements of Hopf algebra may decrease by deformation. (see the previous example)
- The number of primitive idempotents of Hopf algebra do not decrease by de-formation (see [11]). More generally, the order of an element do not decrease by deformation.

The following proposition gives a connection between degeneration and deformation.

**Proposition 1.** If $H_0$ is a degeneration of $H_1$ then $H_1$ is a deformation of $H_0$.

In fact, let $H_0 = \lim_{t \to 0} f_t \cdot H$ be a degeneration of $H$ then $H_t = f_t \cdot H$ is a deformation of $H_0$.

**Remark 5.** The converse is, in general, false. We take from [11] the following example: for a fixed primitive $p$-th root of unity $\lambda$ and $t \in K^*$, the family of Hopf algebra $H_t$ generated by $c, x_1, x_2$ and subject to $c^p = 1, x_1^p = c^p - 1, x_2^p = c^p - 1$,

$$\Delta(c) = c \otimes c, \Delta(x_1) = c \otimes x_1 + x_1 \otimes 1, \Delta(x_2) = c \otimes x_2 + x_2 \otimes 1,$$

The Hopf algebras $H_t$ and $H_s$ are isomorphic if and only if $ts^{-1}$ is a $p$-th root of unity. The Hopf algebra $H_t$ exists when $t \neq 0$ but not its limit when $t$ tends to $0$.

### 4.2. Rigid Hopf algebras and irreducible components.

**Definition 1.** A Hopf algebra $H$ is said rigid if and only if every deformation of $H$ is trivial.

**Remark 6.**

- The definition may be rephrased by ”The orbit of $H$ is Zariski open”.
- If the second cohomological group of the Hopf algebra $H$ is trivial then $H$ is rigid.
- Every semisimple Hopf algebra is rigid because its second cohomological groups is trivial [10]. Thus the group Hopf algebras are rigid.
- The Zariski open orbits have a special interest in the geometric study of $Hopf_n$. Their closure determines an irreducible component.
- Two non isomorphic rigid Hopf algebras belong to different irreducible components.
- There is a finite number of open orbits because the algebraic variety $Hopf_n$ decomposes in a finite number of irreducible components.

**Theorem 4.1.** The dual $H^*$, of a finite dimensional rigid Hopf algebra $H$, is rigid.

**Proof.** Suppose that $H$ is rigid and $H^*$ is not rigid. This means that it exists a deformation $H_t^*$ such that $H_t^*$ is not isomorphic to $H^*$. Since $(H_t^*)^*$ is a deformation of $H^{**}$, which is isomorphic to $H$ and $H$ rigid then $(H_t^*)^* \simeq H$. By duality $(H_t^*)^{**} \simeq H^*$, which is equivalent to $(H_t^*)^* \simeq H^*$, contradicting the hypothesis. □
**Corollary 4.2.** The rigidity of a finite-dimensional Hopf algebra is equivalent to the rigidity of its dual.

**Corollary 4.3.** If a Hopf algebra $H_0$ is a degeneration of $H$ then $H_0^*$ is a degeneration of $H^*$.

**Proof.** Since $H_0$ is a degeneration of $H$ then it exists a deformation $H_t$ of $H$ which tends to $H_0$ when $t$ tends to 0. By duality $H_t^*$ is a deformation of $H^*$ and by the linearity of the operations the limit of $H_t^*$ is $H_0^*$. □

The following proposition gives a necessary condition for the rigidity of a graded Hopf algebra.

**Proposition 2.** Let $H$ be a filtered Hopf algebra and $gr(H)$ its associated graded Hopf algebra. If $H$ is not isomorphic to $gr(H)$ then $gr(H)$ is not rigid.

**Proof.** Since $gr(H)$ is a degeneration of $H$ (theorem (3.3.1)) then $H$ is a deformation of $gr(H)$ (proposition 4.1.2). Thus, the graded Hopf algebra is not rigid if $H$ is not isomorphic to $gr(H)$. □

5. **Rigidity and irreducible components in $Hopf_p$**

The general classification, up to isomorphism, of Hopf algebras is not known. However, the complete classification is known for dimension $n$, $n < 14$ and for dimension $p$ and $p^2$ where $p$ is prime, see [20] [23] [14] [13] [24] and [25]. If the dimension is any prime number $p$ there is only one Hopf algebra, the Hopf group algebra $KZ_p$. Then, $Hopf_p$ is formed by a unique irreducible component given by a Zariski open orbit. In the following, we consider the varieties $Hopf_{p^2}$, where $p$ is prime, and $Hopf_n$, where $n < 14$.

5.1. **Rigidity and irreducible components in $Hopf_{p^2}$, $p$ prime.** It is known from [20] that in $Hopf_{p^2}$, where $p$ is any prime number, every Hopf algebra is isomorphic to one of the following Hopf algebras:

1. $K[Z_{p^2}]
2. K[Z_p] \times K[Z_p]
3. T_{p^2}$ a Taft Hopf algebra, which is defined by

$$\frac{K \langle x, y \rangle}{\langle x^p, y^p - 1, xy - qyx \rangle}$$

where $q$ is the primitive root of unity of order $p$.

The coalgebra algebra structure and the antipode are determined by

$\Delta(y) = y \otimes y$, $\Delta(x) = x \otimes y + 1 \otimes x$,

$\varepsilon(x) = 0$, $\varepsilon(y) = 1$,

$S(x) = -xy^{-1}$, $S(y) = y^{-1}$.

**Theorem 5.1.** Every Hopf algebra $H$ in $Hopf_{p^2}$, where $p$ is any prime number, is rigid.

Therefore, the algebraic variety $Hopf_{p^2}$ is an union of $p + 1$ Zariski open orbits.

**Proof.** The Hopf algebra $K[Z_{p^2}]$ and $K[Z_p] \times K[Z_p]$ are semisimple and rigid. A Taft Hopf algebra cannot be deformed in a commutative Hopf algebra, then also rigid.

The number of irreducible components corresponds to the two semisimple Hopf algebras and the $p - 1$ non isomorphic Taft Hopf algebras. □
5.2. Rigidity and irreducible components in $\text{Hopf}_n$ $n < 14$. For $n < 12$, the Hopf algebra was classified by Williams and reconsidered by Stefan. For $n = 12$, the semisimple case was studied by Fukuda and the classification was completed by Natale. In the following, we recall the classification (see for the details [23], [3], [19]).

Let $Z_n$ denote the cyclic group, $D_n$ the dihedral group, $S_n$ the symmetric group, $H_4$ the quaternion group and $Al$ the alternate group. Let $KG$ be the Hopf algebra of the group $G$ and $(KG)^*$ its dual. Let $T_n$ be a Taft Hopf algebra described above.

**Theorem 5.2.** If $H$ is Hopf algebra of dimension $n < 14$, then $H$ is isomorphic with one and only one of the following Hopf algebra

- $n \in \{2, 3, 5, 7, 11, 13\}$ The group Hopf algebra $KZ_n$.
- $n = 4$. The semisimple Hopf algebras $KZ_4$ and $K(Z_2 \times Z_2)$, and the Taft-Sweedler Hopf algebra $T_4$.
- $n = 6$. $KZ_6$, $KS_3$ and $(KS_3)^*$.
- $n = 8$. The semisimple Hopf algebras : $K(Z_2 \times Z_2 \times Z_2)$, $K(Z_2 \times Z_4)$, $KZ_8$, $KD_4$, $(KD_4)^*$, $KH_4$, $(KH_4)^*$ and $A_8$. The nonsemisimple Hopf algebras : $A_{C_{2^2}}$, $A'_{C_4}$, $A''_{C_4}$, $A'_{C_2 \times C_2}$, $A''_{C_2 \times C_2}$ (where $q$ is the primitive root of unity of order 4), $(A''_{C_2})^*$, and $A_{C_2 \times C_2}$.
- $n = 9$. $KZ_9$, $K(Z_3 \times Z_3)$ and the Taft Hopf algebras $T_9$.
- $n = 10$. $KZ_{10}$, $KD_5$ and $(KD_5)^*$.
- $n = 12$. The semisimple Hopf algebras : $KZ_{12}$, $K(Z_6 \times Z_2)$, $K(Z_4 \times Z_3)$, $KD_6$, $(KD_6)^*$, $A_{14}$, $(A_{14})^*$, $A_4$ and $A_-$

The nonsemisimple Hopf algebras : $A_0$, $A_1$ $B_0$ $B_1$ and $A_1^*$. A

**Theorem 5.3.** Every Hopf algebra of dimension $n < 14$ is rigid except $A'_{c_4}$ in $\text{Hopf}_{14}$, and $A_0^* \simeq B_1$ in $\text{Hopf}_{12}$.

Furthermore, in $\text{Hopf}_{15}$, $A'_{c_4}$ is a degeneration of $A''_{c_4}$ and in $\text{Hopf}_{12}$, $A_0$ is a degeneration of $A_1$ and $B_1$ is a degeneration of $A_1^*$.

**Proof.** Following the previous remark, every semisimple Hopf algebra is rigid.

For $n \in \{2, 3, 5, 7, 11, 13\}$, The Hopf algebras are all group Hopf algebras, then rigid.

For $n = 4$ and $n = 9$, see theorem (5.1).

For $n = 6$, all Hopf algebras are semisimple, then rigid.

For $n = 8$, the semisimple Hopf algebras are rigid.

The family $A'_{c_4,t} = \frac{K(x,g)}{(g^t - 1, x^2 + f(1 - y^t), gx + xg)}$ is a deformation of $A'_{c_4}$ and is isomorphic when $t \neq 0$ to $A'_{c_4}$. Then $A'_{c_4}$ is a degeneration of $A''_{c_4}$. The order of $g$ and the $(a,b)$-primitivity of $x$ don’t allow deformation of other pointed nonsemisimple Hopf algebras.

For $n = 10$, all the algebras are semisimple, then rigid.

For $n = 12$, the semisimple Hopf algebras are rigid.

The family $A_t = \frac{K(x,g)}{(g^t - 1, x^2 + f(1 - y^t), gx + xg)}$ is isomorphic to $A_t$ when $t \neq 0$, and tends to $A_0$ when $t$ tends to 0. then $A_0$ is a degeneration of $A_1$.

By duality $A_1^*$ is a deformation of $A_0^*$. Then $A_1^* \simeq A_1^*$ ($t \neq 0$) is a deformation of $B_1$ ($B_1 \simeq A_0^*$) and $B_1$ is a degeneration of $A_1^*$. Therefore $A_0$ and $B_1$ are not rigid, and all the others are rigid. □

**Corollary 5.4.** The following table gives the number of irreducible components of $\text{Hopf}_n$ for $n < 14$
| dimension       | number of irreducible components of Hopf algebra |
|-----------------|-----------------------------------------------|
| $n \in \{2, 3, 5, 7, 11, 13\}$ | 1                                             |
| $n = 4$         | 3                                             |
| $n = 6$         | 3                                             |
| $n = 8$         | 14                                            |
| $n = 9$         | 4                                             |
| $n = 10$        | 3                                             |
| $n = 12$        | 14                                            |

Remark 7.  
- The orbit dimension’s of the $n$-dimensional semisimple Hopf algebra is $n^2$.  
- It is interesting to see whether, for the nonsemisimple rigid Hopf algebras, the second cohomological group is trivial.

6. Degenerations with $f_t = v + t \cdot w$

The aim of this section is to find a necessary and sufficient conditions such that a degeneration of a given Hopf algebra $H = (V, \mu, \Delta, \eta, \varepsilon, S)$ exists. Let $f_t = v + tw$ be a family of linear maps where $v$ is a singular linear map, $w$ is a regular linear map and $t$ is a parameter in $K$. We suppose $v$ singular because when $v$ is regular, the family $f_t$ corresponds to isomorphisms. We can also set $w = id$ because $f_t = v + tw = (v \circ w^{-1} + t) \circ w$ which is isomorphic to $v \circ w^{-1} + t$. Then with no loss of generality we consider the family $f_t = \varphi + t \cdot id$ from $V$ into $V$ where $\varphi$ is a singular map and $t$ is in open set containing 0. The vector space $V$ can be decomposed by $\varphi$ under the form $V_R \oplus V_N$ where $V_R$ and $V_N$ are $\varphi$-invariant defined in a canonical way such that $\varphi$ is surjective on $V_R$ and nilpotent on $V_N$. Let $q$ be the smallest integer such that $\varphi^q(V_N) = 0$. The inverse of $f_t$ exists on $V_R$ and is equal to $\varphi^{-1}(t\varphi^{-1} + id)^{-1}$. But on $V_N$, since $\varphi^q = 0$, it is given by

$$\frac{1}{\varphi + t \cdot id} = \frac{1}{t} \cdot \frac{1}{\varphi/t + id} = \frac{1}{t} \cdot \sum_{i=0}^{\infty} \left( -\frac{\varphi}{t} \right)^i = \frac{1}{t} \cdot \sum_{i=0}^{\infty} \left( -\frac{\varphi}{t} \right)^i$$

It follows:

**Fitting lemma.** Let $f_t = \varphi + t \cdot id$ be a family of linear maps from $V$ into $V$ where $\varphi$ is a singular map. Then, $V = V_R \oplus V_N$ where $V_R$ and $V_N$ are $\varphi$-invariant and $\varphi$ is surjective on $V_R$ and nilpotent on $V_N$. The inverse of $f_t$ is defined by

$$f_t^{-1} = \begin{cases} \varphi^{-1}(t\varphi^{-1} + id)^{-1} & \text{on } V_R \\ \frac{1}{t} \cdot \sum_{i=0}^{q-1} (-\varphi)^i & \text{on } V_N \end{cases}$$

where $q$ is the smallest integer such that $\varphi^q(V_N) = 0$.

6.1. Degeneration of an algebra. Let $f_t = \varphi + t \cdot id$ be a family of linear maps on $V$, where $\varphi$ is a singular map. The action of $f_t$ on $\mu$ is defined by $f_t \cdot \mu = f_t^{-1} \circ \mu \circ f_t \oplus f_t$ then

$$f_t \cdot \mu(x \otimes y) = f_t^{-1} \circ \mu(f_t(x) \otimes f_t(y)) = f_t^{-1} \left( \mu(\varphi(x) \otimes \varphi(y)) + t(\mu(\varphi(x) \otimes y) + \mu(x \otimes \varphi(y))) + t^2\mu(x \otimes y) \right)$$

Since every element $v$ of $V$ decomposes in $v = v_R + v_N$, we set
\[ A = \mu(x \otimes y) = A_R + A_N, \]
\[ B = \mu(\varphi(x) \otimes y) + \mu(x \otimes \varphi(y)) = B_R + B_N \]
\[ C = \mu(\varphi(x) \otimes \varphi(y)) = C_R + C_N \]

Then
\[ f_t \cdot \mu(x \otimes y) = \varphi^{-1}(t\varphi^{-1} + id)^{-1}(t^2A_R + tB_R + C_R) + \frac{1}{t} \sum_{i=0}^{q-1} \left(-\frac{\varphi}{t}\right)^i(t^2A_N + tB_N + C_N) \]

If \( t \) goes to 0, then \( \varphi^{-1}(t\varphi^{-1} + id)^{-1}(t^2A_R + tB_R + C_R) \) goes to \( \varphi^{-1}(C_R) \).

The limit of the second term is:
\[ \lim_{t \to 0} \frac{1}{t} \cdot \sum_{i=0}^{q-1} \left(-\frac{\varphi}{t}\right)^i(t^2A_N + tB_N + C_N) \]
\[ = \lim_{t \to 0} \left( \frac{id}{t} - \frac{\varphi}{t^2} + \frac{\varphi^2}{t^3} - \cdots (-1)^{q-1} \frac{\varphi^{q-1}}{t^q} \right) (t^2A_N + tB_N + C_N) \]
\[ = \lim_{t \to 0} tA_N - \varphi(A_N) + \left( \frac{id}{t} - \frac{\varphi}{t^2} + \frac{\varphi^2}{t^3} - \cdots (-1)^{q-1} \frac{\varphi^{q-1}}{t^q} \right) \varphi^2(A_N) + \left( \frac{id}{t} - \frac{\varphi}{t^2} + \frac{\varphi^2}{t^3} - \cdots (-1)^{q-1} \frac{\varphi^{q-1}}{t^q} \right) C_N \]
\[ = \lim_{t \to 0} tA_N - A_N - \varphi(A_N) + \left( \frac{id}{t} - \frac{\varphi}{t^2} + \frac{\varphi^2}{t^3} - \cdots (-1)^{q-1} \frac{\varphi^{q-1}}{t^q} \right) (\varphi^2(A_N) - \varphi(B_N) + C_N) \]

This limit exists if and only if
\[ \varphi^2(A_N) - \varphi(B_N) + C_N = 0 \]

which is equivalent to
\[ \varphi^2(\mu(x \otimes y)_N) - \varphi(\mu(\varphi(x) \otimes y)_N) - \varphi(\mu(x \otimes \varphi(y))_N) + \mu(\varphi(x) \otimes \varphi(y))_N = 0 \]

And the limit is \( B_N - \varphi(A_N) \).

**Proposition 3.** The degeneration of the algebra \( \mu \) exists if and only if the condition
\[ (11) \quad \varphi^2(\mu(x \otimes y)_N) = \varphi(\mu(\varphi(x) \otimes y)_N) + \varphi(\mu(x \otimes \varphi(y))_N) = 0 \]
holds. And it is defined by
\[ \mu_0 = \varphi^{-1} \circ \mu_R \circ \varphi \circ \mu_N \circ id - \varphi \circ \mu_N \circ id \otimes \varphi + \mu_N \circ \varphi \otimes \varphi = 0 \]

**6.2. Degeneration of a coalgebra.** Let \( f_t = \varphi + t \cdot id \) be a family of linear maps on \( V \), where \( \varphi \) is a singular map. The action of \( f_t \) on \( \Delta \) is defined by
\[ f_t \cdot \Delta = f_t^{-1} \otimes f_t^{-1} \circ \Delta \circ f_t \]

Setting \( \Delta(x) = x^{(1)} \otimes x^{(2)}, \Delta(\varphi(x)) = \varphi(x)^{(1)} \otimes \varphi(x)^{(2)} \), and for \( i = 1, 2 \)
\[ x^{(i)} = x_R^{(i)} + x_N^{(i)}, \varphi(x)^{(i)} = \varphi(x)^{(i)} \otimes \varphi(x)^{(i)} \].

Then
\[ f_t \cdot \Delta(x) = t \cdot \left( (f_t^{-1}(x_R^{(1)}) + f_t^{-1}(x_N^{(1)})) \otimes (f_t^{-1}(x_R^{(2)}) + f_t^{-1}(x_N^{(2)})) \right) + \left( f_t^{-1}(\varphi(x)_R^{(1)}) + f_t^{-1}(\varphi(x)_N^{(1)}) \right) \otimes \left( f_t^{-1}(\varphi(x)_R^{(2)}) + f_t^{-1}(\varphi(x)_N^{(2)}) \right) \]

Setting \( \psi = \varphi^{-1}(t\varphi^{-1} + id)^{-1} \), then when \( t \to 0, \psi = \varphi^{-1} \)
and
\[ f_t \cdot \Delta(x) = t \cdot \psi(x_R^{(1)}) \otimes \psi(x_R^{(2)}) + \psi(x_N^{(1)}) \otimes \psi(x_N^{(2)}) + \]
satisfy the condition
by

Proposition 4. The degeneration of the coalgebra \( \Delta \) exists if and only if the condition (12) holds and it is defined for all \( x \in V \) by

\[
\Delta_0 (x) = \varphi^{-1} \left( \varphi (x)_R^{(1)} \otimes \varphi^{-1} \left( \varphi (x)_R^{(2)} \right) \right) + \varphi^{-1} \left( x_R^{(1)} \otimes x_N^{(2)} \right) + x_N^{(1)} \otimes \varphi^{-1} \left( x_R^{(2)} \right)
\]

6.3. Degeneration of Hopf algebra. Let \( H = (V, \mu, \eta, \Delta, \varepsilon, S) \) be a Hopf algebra and \( f_t = \varphi + t \cdot \text{id} \) be a family of linear maps of \( V \), where \( \varphi \) is a singular map. We suppose also that \( \varphi \) decomposes the vector space \( V \) as \( V = V_R + V_N \). Then the degeneration \( H_0 = \lim_{t \to 0} f_t \cdot H \) exists if and only if the conditions (11) (12) holds, the antipode remains for \( H_0 \). The multiplication and the comultiplication defined by

\[
\mu_0 (x \otimes y) = \varphi^{-1} \left( \mu (\varphi (x) \otimes \varphi (y))_R \right) + \mu (\varphi (x) \otimes y)_N + \mu (x \otimes \varphi (y))_N - \varphi (\mu (x \otimes y)_N)
\]

\[
\Delta_0 (x) = \varphi^{-1} \left( \varphi (x)_R^{(1)} \otimes \varphi^{-1} \left( \varphi (x)_R^{(2)} \right) \right) + \varphi^{-1} \left( x_R^{(1)} \otimes x_N^{(2)} \right) + x_N^{(1)} \otimes \varphi^{-1} \left( x_R^{(2)} \right)
\]
satisfy the condition

\[
\mu_0 \circ (\text{id} \otimes \tau \otimes \text{id}) \circ (\Delta_0 \otimes \Delta_0) = \Delta_0 \circ \mu_0 \quad \text{where} \quad \tau \text{ is the twist map}
\]

\[
\mu_0 \circ (S \otimes \text{Id}) \circ \Delta_0 = \mu_0 \circ (\text{Id} \otimes S) \circ \Delta_0 = \eta \circ \varepsilon
\]
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