A HEYTING ALGEBRA ON DYCK PATHS OF TYPE A AND B

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ABSTRACT. In this article, we investigate the lattice of Dyck paths under dominance order, and explicitly exhibit the structure of a Heyting algebra, namely a lattice structure where each Dyck path has a relative pseudocomplement with respect to some other Dyck path. While the proof that this lattice forms a Heyting algebra is quite straightforward, the explicit computation of the relative pseudocomplements using the lattice-theoretic definition is quite tedious. We give explicit formulas that allow for the construction of relative pseudocomplements in linear time. Moreover, we give formulas for the construction of pseudocomplements, and we characterize the regular elements in these lattices.

1. Introduction

Classical logic is fundamentally based on Aristotle’s three classical laws of thought: (i) the law of identity, (ii) the law of non-contradiction, and (iii) the law of excluded middle. If we suppose that $B$ is a collection of provable statements, and $\neg$ denotes the negation of a statement, then (i) says that no two elements of $B$ are the same, (ii) says that for every statement $x \in B$ the conjunction $x \land \neg x$ is always false, and (iii) says that the disjunction $x \lor \neg x$ is always true. A well-known way to model such a logical system is by means of a Boolean algebra, namely an algebra $(B, \land, \lor, \neg, 0, 1)$, where $B$ is a set of provable statements, $\land$ and $\lor$ are binary operations on $B$ (interpreted as conjunction and disjunction), $\neg$ is a unary operation on $B$ (interpreted as negation), and $0, 1 \in B$ are constants (interpreted as “false” and “true”), with the additional properties that $\lor$ and $\land$ are distributive, 0 and 1 are neutral elements with respect to $\lor$ and $\land$, respectively, and $\neg$ satisfies the laws of non-contradiction and of excluded middle, see for instance [4, Section 4.16]. A simple way to visualize a Boolean algebra is by its associated Boolean lattice $B = (B, \leq)$, where the partial order is defined as $x \leq y$ if and only if $x \land y = x$ (or equivalently $x \lor y = y$). Then, $B$ is a distributive lattice with least element 0, greatest element 1 such that for every element $x \in B$ there exists a unique element $y \in B$ such that $x \land y = 0$ and $x \lor y = 1$, (then, clearly $y = \neg x$).

In a Boolean algebra, and likewise in classical logic, we can consider the operation $\Rightarrow$ defined by saying that for all $x, y \in B$, the statement $x \Rightarrow y$ is true if and only if $\neg x \lor y$ is true. In the presence of such an operator, the modus ponens—saying that if $x \Rightarrow y$ is true, and $x$ is true, then $y$ must be necessarily true—is always sound. However, we can observe that there is a weaker condition for the
soundness of modus ponens, namely the existence of an operation \( \rightarrow \) such that \( x \rightarrow y \) is the greatest element \( z \) satisfying the inequality \( z \land x \leq y \). Now, we can define a pseudo-negation, denoted by \( \sim \), by saying that for all \( x \in B \), the statement \( \sim x \) is true if and only if \( x \rightarrow 0 \) is true. Then, we can quickly check that \( x \land \sim x = 0 \), but we cannot deduce \( x \lor \sim x = 1 \) any longer. Hence, in such a logical system, the law of excluded middle does not necessarily hold, and we thus enter the realm of intuitionistic logic introduced by Brouwer in [3]. A first formalization of Brouwer’s intuitionistic logic, was given by Heyting in [7] and leads to a Heyting algebra \((B, \land, \lor, \rightarrow, 0, 1)\)—where \( \rightarrow \) is the operation from before—as a way to model intuitionistic logic. In particular, Boolean algebras are special instances of Heyting algebras.

In this article, we explicitly exhibit a Heyting algebra structure on the set of Dyck paths of semilength \( n \), namely lattice paths from \((0,0)\), consisting of \( 2n \) steps which are either horizontal or vertical, and which never go below the diagonal \( x = y \). We show that these paths form a Heyting algebra, when equipped with the so-called dominance order, i.e. two Dyck paths are comparable under this order if they do not cross. This result follows from the fact that there are only finitely many of those paths (for fixed \( n \)), and that the dominance order is distributive, as well as from the fact that every finite distributive lattice is a Heyting algebra. The distributivity of the dominance order was previously investigated in [6] for Dyck paths of type \( A \), namely such Dyck paths ending at \((n,n)\). Analogously, we use the name Dyck paths of type \( B \) if we drop this additional requirement. In particular, we prove the following theorem.

**Theorem 1.1.** For every \( n \in \mathbb{N} \), the lattice \((D_B^n, \leq_D)\) of Dyck paths of semilength \( n \) under dominance order forms a Heyting algebra. The sublattice \((D_A^n, \leq_D)\) of Dyck paths of semilength \( n \) ending at \((n,n)\) under dominance order forms a Heyting subalgebra of the former.

The proof of Theorem 1.1 is quite straightforward, once the right realization of Dyck paths, in terms of height sequences, is established. The main contribution of this article, however, lies in the explicit construction of the relative pseudo-complements for two given Dyck paths which are Theorems 3.1 and 3.7 below. We remark that this construction can be carried out in linear time with respect to the semilength \( n \). In addition, we characterize the pseudocomplement of a given Dyck path, as well as the so-called regular elements in the Heyting algebra of Dyck paths.

This article is organized as follows: in Section 2, we recall the necessary lattice-theoretic notions of distributive lattices and Heyting algebras, as well as formally define Dyck paths, realize them in terms of height sequences, and prove Theorem 1.1. In Section 3, we then explicitly construct the height sequences of the relative pseudocomplements, starting with Dyck paths of type \( A \), and then generalizing this construction to Dyck paths of type \( B \).

2. Preliminaries

2.1. Heyting Algebras and Distributive Lattices. In this section, we recall the notion of Heyting algebras and distributive lattices. The results stated in this
section are well-known to lattice theorists, and the proofs given here solely serve the self-containedness of the article.

2.1.1. Heyting Algebras. Let $\mathcal{L} = (L, \leq_L)$ be a lattice with least element $0_L$ and greatest element $1_L$. Given $x, y \in L$, we say that the greatest element $z \in L$ satisfying

(1) $x \land_L z \leq_L y$

is called—if it exists—the \textit{relative pseudocomplement of $x$ with respect to $y$}, and we usually write $x \rightarrow_L y$. If relative pseudocomplements exist for all $x, y \in L$, then we call $\mathcal{L}$ a \textit{Heyting algebra}. Moreover, if $\mathcal{L}$ is a Heyting algebra, then for $x \in L$ we call the element $x \rightarrow_L 0_L$ the \textit{pseudocomplement of $x$}, and we usually write $x^\perp$. An element $x \in L$ is called \textit{regular} if $(x^\perp)^\perp = x$. It is straightforward to verify that the poset $B = (B, \leq_L)$, where $B = \{x \in L \mid x \text{ is regular}\}$, is a Boolean lattice.

\textbf{Lemma 2.1.} Let $\mathcal{L} = (L, \leq_L)$ be a Heyting algebra. Then, for all $x, y, z \in L$ holds

$$x \land_L z \leq_L y \text{ if and only if } z \leq_L x \rightarrow_L y.$$  

\textit{Proof.} First of all, since $x \land_L (x \rightarrow_L y) \leq_L x, y$, we have $x \land_L (x \rightarrow_L y) \leq_L x \land_L y$. On the other hand, we have $y \leq_L x \rightarrow_L y$, which implies $x \land_L y \leq_L x \land (x \rightarrow_L y)$, and thus $x \land_L y = x \land_L (x \rightarrow_L y)$.

Now, suppose that $x \land_L z \leq_L y$. By definition, $x \rightarrow_L y$ is the greatest element of $\mathcal{L}$ satisfying (1). Hence, $z \leq_L x \rightarrow_L y$.

Conversely, let $z \leq_L x \rightarrow_L y$. Hence, with the reasoning in the beginning of this proof, we obtain $x \land_L z \leq_L x \land_L (x \rightarrow_L y) = x \land_L y \leq_L y$, as desired. \qed

\textbf{Lemma 2.2.} Let $\mathcal{L} = (L, \leq_L)$ be a Heyting algebra. If $x \leq_L y$, then $x \rightarrow_L y = 1_L$.

\textit{Proof.} By assumption, we have $x \land_L 1_L = x \leq_L y$, and hence $1_L$ satisfies (1). Since $1_L$ is the greatest element of $\mathcal{L}$, there can be no larger element satisfying (1), which implies the claim. \qed

2.1.2. Distributive Lattices. If every $x, y, z \in L$ satisfy the two laws

(2) $x \land_L (y \lor_L z) = (x \land_L y) \lor_L (x \land_L z)$, and

(3) $x \lor_L (y \land_L z) = (x \lor_L y) \land_L (x \lor_L z)$,

then we say that $\mathcal{L}$ is a \textit{distributive lattice}. Recall that given two posets $(P, \leq_P)$ and $(Q, \leq_Q)$ two maps $\varphi : P \rightarrow Q$, and $\psi : Q \rightarrow P$ are said to form a \textit{Galois connection} if they satisfy the following condition for all $p \in P$ and $q \in Q$:

(4) $\varphi(p) \leq_Q q \text{ if and only if } p \leq_P \psi(q)$.

\textbf{Proposition 2.3} ([4, Proposition 7.31]). Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be two posets, and let $\varphi : P \rightarrow Q, \psi : Q \rightarrow P$ form a Galois connection. Then, $\varphi$ preserves existing joins in $(P, \leq_P)$ and $\psi$ preserves existing meets in $(Q, \leq_Q)$.

\textbf{Proposition 2.4.} Every Heyting algebra is distributive. Conversely, every finite distributive lattice is a Heyting algebra.
Proof. Let \( \mathcal{L} = (L, \leq_L) \) be a Heyting algebra. Define two maps
\[
\phi_x : L \to L, \quad y \mapsto (x \land_L y), \quad \text{and} \quad \psi_x : L \to L, \quad y \mapsto (x \lor_L y).
\]
In view of Lemma 2.1, we have the following equivalence
\[
\phi(x) \leq_L z \quad \text{if and only if} \quad x \leq_L \psi(z),
\]
which implies that \( \phi \) and \( \psi \) form a Galois connection. Now, Proposition 2.3 implies that \( \phi_x(y \lor_L z) = \phi_x(y) \lor_L \phi_x(z) \), which is precisely (2). It is an easy exercise to show that (2), together with the absorption law of lattices, implies (3). Hence, \( \mathcal{L} \) is distributive.

Now, let \( \mathcal{L} = (L, \leq_L) \) be a finite distributive lattice, and let \( x, y \in L \), and let \( Z \subseteq L \) be defined as \( Z = \{ z \in L \mid x \land_L z \leq_L y \} \). We have \( x \land_L 0_L = 0_L \leq_L y \), and hence \( 0_L \in Z \). Now suppose that there are two mutually incomparable elements \( z, z' \in Z \). Then, \( y \) is an upper bound for both \( x \land_L z \) and \( x \land_L z' \), and we conclude using the distributivity of \( \mathcal{L} \)
\[
x \land_L (z \lor_L z') = (x \land_L z) \lor_L (x \land_L z') \leq_L y.
\]
Hence, \( z \lor_L z' \in Z \). Now, since \( \mathcal{L} \) is finite, we conclude that \( Z \) is finite, and since \( Z \) has a least element and is closed under joins, we conclude that \( Z \) has a greatest element. \( \square \)

2.2. Dyck Paths of Type A and B. A Dyck path of semilength \( n \) is a lattice path which starts at \((0,0)\), which consists of \( 2n \) steps either of the form \((0,1)\), so-called up-steps, or of the form \((1,0)\), so-called right-steps, and which stays strictly above the diagonal \( x = y \). We say that a Dyck path of semilength \( n \) is of type \( A \) if it ends at \((n, n)\). If we do not pose any restrictions on the Dyck path, then we say that it is of type \( B \). Let \( D_n^A \) denote the set of Dyck paths of semilength \( n \) being of type \( A \), and let \( D_n^B \) denote the set of Dyck paths of semilength \( n \) being of type \( B \). Clearly, we have \( D_n^A \subseteq D_n^B \).

It is standard, see for instance [5, Section 2] or [8, Section 3.2], that a Dyck path \( p \in D_n^B \) can be encoded by a so-called Dyck word, namely a word \( w_p \) of length \( 2n \) over the alphabet \( \{u, r\} \) in which every prefix of this word contains at least as many \( u \)'s as it contains \( r \)'s. If \( p \in D_n^A \), then \( w_p \) is additionally required to contain exactly \( n \)-times the letter \( u \), and \( n \)-times the letter \( r \).

2.2.1. Type A. For \( p \in D_n^A \) define a sequence \( h_p = (h_1, h_2, \ldots, h_n) \), where \( h_i \) is the number of \( u \)'s occurring in \( w_p \) before the \( i \)-th occurrence of the letter \( r \), and call \( h_p \) the height sequence of \( p \).

Lemma 2.5. If \( p \in D_n^A \), then \( h_p = (h_1, h_2, \ldots, h_n) \) satisfies \( h_1 \leq h_2 \leq \cdots \leq h_n \), and \( i \leq h_i \leq n \), for all \( i \in \{1, 2, \ldots, n\} \). Conversely, each such sequence uniquely determines a Dyck path in \( D_n^A \).

Proof. First suppose that there exists some \( i \in \{1, 2, \ldots, n - 1\} \) with \( h_i > h_{i+1} \). By definition, this means that there are more \( u \)'s occurring before the \( i \)-th occurrence of \( r \) in \( w_p \), than there are \( u \)'s occurring before the \( (i+1) \)-st occurrence of \( r \) in \( w_p \). Since the \( (i+1) \)-st \( u \) occurs in \( w_p \) after the \( i \)-th \( u \), this is clearly a contradiction.

Since the letter \( u \) is contained in \( w_p \) exactly \( n \) times, it follows immediately that \( h_i \leq n \) for all \( i \in \{1, 2, \ldots, n\} \). Now suppose that there is some \( i \in \{1, 2, \ldots, n\} \)
with \( h_i = c < i \). Hence, there is a prefix of \( w_p \) which ends at the \( i \)-th occurrence of the letter \( r \), and which consists of \( i + c \) letters. Since, \( c < i \) this is a contradiction to the definition of \( w_p \) which claims that every prefix of \( w_p \) has to contain at least as many letters \( u \) as it contains letters \( r \).

Conversely, let \( h = (h_1, h_2, \ldots, h_n) \) have the desired properties. Define a word

\[
\bar{w}_h = uu \cdots urru \cdots urru \cdots uru \cdots u.r.
\]

Hence, \( w_h \) contains precisely \( n \) times the letter \( r \) and \( h_n = n \) times the letter \( u \).

Moreover, since \( h \) is weakly increasing, there is a non-negative number of \( u \)'s between two occurrences of the letter \( r \), and \( h_i - h_{i-1} \)-many \( u \)'s occur before the \( i \)-th occurrence of the letter \( r \). Since \( i \leq h_i \), every prefix of \( w_h \) contains at least as many \( u \)'s as it contains \( r \)’s. Thus, \( w_h \) is a Dyck word of type \( A \).

\[ \square \]

Let \( p, p' \in D_n^A \) with associated height sequences \( h_p = (h_1, h_2, \ldots, h_n) \) and \( h_{p'} = (h'_1, h'_2, \ldots, h'_n) \). Define \( p \leq_D p' \) if and only if \( h_i \leq h'_i \) for \( i \in \{1, 2, \ldots, n\} \), and call this partial order the dominance order on \( D_n^A \). The following result is [6, Corollary 2.2].

**Theorem 2.6.** For every \( n \in \mathbb{N} \), the poset \((D_n^A, \leq_D)\) is a distributive lattice.

**Proof.** It is straightforward to verify that given two Dyck paths \( p, p' \in D_n^A \), with height sequences \( h_p = (h_1, h_2, \ldots, h_n) \) and \( h_{p'} = (h'_1, h'_2, \ldots, h'_n) \), their meet is defined via the height sequence

\[
h_{p \land p'} = (\min\{h_1, h'_1\}, \min\{h_2, h'_2\}, \ldots, \min\{h_n, h'_n\}),
\]

and their join is defined via the height sequence

\[
h_{p \lor p'} = (\max\{h_1, h'_1\}, \max\{h_2, h'_2\}, \ldots, \max\{h_n, h'_n\}).
\]

Since \( \min \) and \( \max \) are distributive, the result follows. \( \square \)

Figure 1 shows the lattice \((D_n^A, \leq_D)\).

2.2.2. Type B. We would like to define a height sequence for \( p \in D_n^B \) analogously to type \( A \). However, we notice that there must not necessarily exist a letter \( r \) in \( w_p \). To overcome this issue, we construct the word \( \bar{w}_p \) from \( w_p \) by appending a letter \( r \) if \( w_p \) ends with \( u \). (If \( w_p \) ends with \( r \), then \( w_p = \bar{w}_p \).) Now, define a sequence \( h_p = (h_1, h_2, \ldots, h_k) \), where \( k \) is the number of \( r \)'s occurring in \( \bar{w}_p \) and for \( i \in \{1, 2, \ldots, k\} \), the number \( h_i \) is the number of \( u \)'s occurring before the \( i \)-th occurrence of the letter \( r \) in \( \bar{w}_p \), and call \( h_p \) the height sequence of \( p \).

**Lemma 2.7.** If \( p \in D_n^B \), then \( h_p = (h_1, h_2, \ldots, h_k) \) for \( k \in \{1, 2, \ldots, n\} \) satisfies

\[
h_k = \begin{cases} 
2n - k + 1, & \text{if } w_p \text{ ends with } u, \\
2n - k, & \text{if } w_p \text{ ends with } r 
\end{cases}
\]

as well as \( h_1 \leq h_2 \leq \cdots \leq h_{k-1} \leq 2n - k \) and \( h_i \geq i \) for \( i \in \{1, 2, \ldots, k\} \). Conversely, each such sequence uniquely determines a Dyck path in \( D_n^B \).
Proof. For the first part of the proof, we proceed by induction on $n$. If $n = 1$, then we notice that $D^1$ has precisely two elements $p_1$ and $p_2$, which can be encoded by the Dyck words $w_{p_1} = ur$ and $w_{p_2} = uu$. The corresponding height sequences are then $h_{p_1} = (1)$ and $h_{p_2} = (2)$ which clearly satisfy the above conditions.

Now assume that the claim is true for all Dyck paths of length $< n$, let $p \in D^n$, and let $h_p = (h_1, h_2, \ldots, h_k)$ be the corresponding height sequence. Let $w_p = a_1 a_2 \cdots a_{2n}$ be the corresponding Dyck word. Consider the prefix $w'_p = a_1 a_2 \cdots a_{2n-2}$. By definition, $w'_p$ is a Dyck word in its own right, and thus it corresponds to a Dyck path $p' \in D^{n-1}$. By induction hypothesis, $p'$ can be described uniquely by its height sequence $h_{p'} = (h'_1, h'_2, \ldots, h'_{k'})$, for $k' \leq n - 1$ satisfying $h'_1 \leq h'_2 \leq \cdots \leq h'_{k'-1} \leq 2n - 2 - k'$ and $h'_i \geq i$ for $i \in \{1, 2, \ldots, k' - 1\}$.

First, suppose that $a_{2n-2} = u$. Thus, by induction hypothesis, we conclude that $h'_{k'} = 2n - 1 - k'$. Moreover, we have $h'_i = h_i$ for $i < k'$. We distinguish four cases:

(i) $a_{2n-1} = a_{2n} = u$. By construction, we have $k = k'$, and in particular follows that $h_k = h_{k'} = h'_{k'} + 2 = 2n - k + 1$ as desired.

(ii) $a_{2n-1} = u, a_{2n} = r$. By construction, we have $k = k'$ again. It follows now that $h_k = h_{k'} = h'_{k'} + 1 = 2n - k$ as desired.

(iii) $a_{2n-1} = r, a_{2n} = u$. By construction, we have $k = k' + 1$, and since $k' \leq n - 1$, it follows that $k \leq n$. Moreover, we have $h_{k-1} = h_{k'} = h'_{k'} = 2n - k$. Since $k \leq n$, it follows also that $h_{k-1} \geq k - 1$. Finally, we have $h_k = h_{k-1} + 1 = 2n - k + 1$ as desired.

(iv) $a_{2n-1} = a_{2n} = r$. By construction, we have $k = k' + 1$, and thus $k \leq n$. Again, we have $h_k = h_{k-1} = h_{k'} = h'_{k'} = 2n - k$ as desired.
Now, suppose that $a_{2n-2} = r$. By induction hypothesis, we conclude that $h'_k = 2n - 2 - k'$, and we have $h_i = h'_i$ for $i \leq k'$. Again, we distinguish four cases:

(i) $a_{2n-1} = a_{2n} = u$. By construction, we have $k = k' + 1$, and $h_k = h_{k-1} + 2 = h'_k + 2 = 2n - k + 1$ as desired.

(ii) $a_{2n-1} = u, a_{2n} = r$. Again, by construction, we have $k = k' + 1$, and $h_k = h_{k-1} + 1 = h'_k + 1 = 2n - k$ as desired.

(iii) $a_{2n-1} = r, a_{2n} = u$. By construction, we have $k = k' + 2$. If $k' = n - 1$, then the prefix $w_p'$ contains the letter $r$ precisely $n - 1$ times. Hence, $w_p$ contains the letter $r$ precisely $n + 1$ times. Since $w_p$ has $2n$ letters, it follows that $w_p$ contains the letter $r$ more often than the letter $u$, which contradicts the assumption that $w_p$ is a Dyck word. Hence, $k' < n - 1$, and thus $k \leq n$. We have $h_{k-1} = h_{k-2} = h_{k'} = h'_{k'} = 2n - k$, and $h_k = h'_{k'} + 1 = 2n - k + 1$ as desired.

(iv) $a_{2n-1} = a_{2n} = r$. By construction, we have $k = k' + 2$. Analogously to (iii), we see that $k \leq n$. Moreover, we have $h_k = h_{k-1} = h_{k-2} = h_{k'} = h'_{k'} = 2n - k$ as desired.

Conversely, let $h = (h_1, h_2, \ldots, h_k)$ satisfy the given conditions. If $k = 1$, then we set $h_0 = 0$. If $h_k = 2n - k + 1$, then we define

$$w_h = \overbrace{u u \cdots u}^{h_1} \overbrace{u u \cdots u}^{h_2-h_1} \overbrace{\cdots u u}^{h_k-h_{k-1}}$$

and if $h_k = 2n - k$, then we define

$$w_h = \overbrace{u u \cdots u}^{h_1} \overbrace{u u \cdots u}^{h_2-h_1} \overbrace{\cdots u u}^{h_k-h_{k-1}} u r.$$

In the first case, the letter $r$ occurs precisely $k - 1$ times, and in the second case the letter $r$ occurs precisely $k$ times, so both words have length $2n$. Since $h_i \geq i$, it is guaranteed that every prefix of each word contains at least as many letters $u$ as it contains letters $r$, and since $h$ is weakly increasing, there is a nonnegative number of $u$’s between any two letters $r$. Hence, $w_h$ is indeed a Dyck word.

Thus, the $i$-th entry of $h_p$ equals the height of the path $p$ at coordinate $x = i - 1$.

**Remark 2.8.** If $p \in D^A_n \subseteq D^B_n$, then the associated Dyck word $w_p$ ends with the letter $r$, and its height sequences has precisely $n$ entries. In this case, the conditions in Lemma 2.7 coincide with those in Lemma 2.5.

Let $p, p' \in D^B_n$ with associated height sequences $h_p = (h_1, h_2, \ldots, h_k)$ and $h_{p'} = (h'_1, h'_2, \ldots, h'_{k'})$ for $k, k' \in \{1, 2, \ldots, n\}$. Define $p \leq_D p'$ if and only if $k \geq k'$ and $h_i \leq h'_i$ for $i \in \{1, 2, \ldots, k'\}$, and call this partial order the dominance order on $D^B_n$.

The following result extends Theorem 2.6.

**Theorem 2.9.** For every $n \in \mathbb{N}$, the poset $(D^B_n, \leq_D)$ is a distributive lattice.

**Proof.** Let $p, p' \in D^B_n$ have height sequences $h_p = (h_1, h_2, \ldots, h_k)$ and $h_{p'} = (h'_1, h'_2, \ldots, h'_{k'})$, and assume without loss of generality that $k \geq k'$. It is straightforward to show that their meet is defined via the height sequence

$$h_{p \wedge p'} = \{\min\{h_1, h'_1\}, \min\{h_2, h'_2\}, \ldots, \min\{h_{k'}, h'_{k'}\}, h_{k'+1}, h_{k'+2}, \ldots, h_k\},$$

and
and their join is defined via the height sequence
\[ h_{p \land p'} = (\max\{h_1, h'_1\}, \max\{h_2, h'_2\}, \ldots, \max\{h_k, h'_k\}) \].

Since min and max are distributive, the result follows.  \(\square\)

Figure 2 shows the lattice \((\mathcal{D}_3^8, \leq_D)\).
Theorems 2.6 and 2.9 state that the dominance order on the set of Dyck paths of type $A$ and $B$, respectively, is a distributive lattice, and hence in view of Proposition 2.4 forms a Heyting algebra. In this section, given two Dyck paths $p$ and $p'$, we explicitly construct the relative pseudocomplement of $p$ with respect to $p'$ from the associated height sequences. Hence, the computation of these relative pseudocomplements can be carried out directly, and much faster, than using the definition (1). Moreover, we conclude the construction of pseudocomplements and characterize the regular elements.

3. The Heyting Algebras on Dyck Paths of Type $A$ and $B$

Let $p_1, p_2 \in D^A_n$ with height sequences $h_{p_1} = (h^{(1)}_1, h^{(1)}_2, \ldots, h^{(1)}_n)$ and $h_{p_2} = (h^{(2)}_1, h^{(2)}_2, \ldots, h^{(2)}_n)$. The relative pseudocomplement $p_1 \rightarrow_D p_2$ is the Dyck path $p$ determined by the height sequence $h_p = (h_1, h_2, \ldots, h_n)$ with

$$h_i = \begin{cases} \begin{array}{ll} n, & \text{if } i = n, \\ h_{i+1}, & \text{if } i < n \text{ and } h^{(1)}_i \leq h^{(2)}_i, \\ \min\{h_{i+1}, h^{(2)}_i\}, & \text{if } i < n \text{ and } h^{(1)}_i > h^{(2)}_i. \end{array} \end{cases}$$

**Proof.** First of all, we need to show that $h_p$ satisfies the conditions from Lemma 2.5, and thus, that $p \in D^A_n$. The conditions $h_1 \leq h_2 \leq \cdots \leq h_n$ and $h_n \geq n$ are satisfied by construction. Suppose that there is a maximal index $i \in \{1, 2, \ldots, n-1\}$ with $h_i < i$. We have two choices: if $h_i = h_{i+1}$, then it follows from the maximality of $i$ that $i > h_i = h_{i+1} \geq i + 1$, which is a contradiction, and if $h_i = h^{(2)}_i$, then it follows that $i > h_i = h^{(2)}_i \geq i$ (since $p_2$ is a Dyck path) which is again a contradiction.

Now we show that $p$ satisfies $p_1 \land_D p \leq_D p_2$. Since the meet in $(D^A_n, \leq_D)$ is given by componentwise minimum of the height sequences, it suffices to show that $\min\{h^{(1)}_i, h_i\} \leq h^{(2)}_i$ for all $i \in \{1, 2, \ldots, n\}$. If $i = n$, then we have $\min\{n, n\} = n$, which is true. If $i < n$, then we have to distinguish two cases: if $h^{(1)}_i \leq h^{(2)}_i$, then we have $\min\{h^{(1)}_i, h_i\} = \min\{h^{(1)}_i, h_{i+1}\} \leq h^{(1)}_i \leq h^{(2)}_i$, and if $h^{(1)}_i > h^{(2)}_i$, then we have $\min\{h^{(1)}_i, h_i\} = \min\{h^{(1)}_i, h_{i+1}, h^{(2)}_i\} \leq h^{(2)}_i$.

Now suppose that there is some $p' \in D^A_n$ with $p_1 \land_D p' \leq_D p_2$ and $p \leq_D p'$. Let $h_{p'} = (h'_1, h'_2, \ldots, h'_n)$. By definition, there must be a maximal index $i \in \{1, 2, \ldots, n\}$ with $h_i < h'_i$. Since $h_n = n$, we conclude that $i < n$. If $h^{(1)}_i \leq h^{(2)}_i$, then we conclude that $h_i = h_{i+1}$, and hence $h'_i > h_{i+1}$. By the maximality of $i$, we conclude that $h'_i > h_{i+1} = h'_{i+1}$, which contradicts Lemma 2.5. If $h^{(1)}_i > h^{(2)}_i$, then with the same argument as before, it suffices to consider the case $h_i = h^{(2)}_i$. Then, however, $h'_i > h_i = h^{(2)}_i$ implies that $\min\{h^{(1)}_i, h'_i\} > h^{(2)}_i$, which contradicts the assumption that $p_1 \land_D p' \leq_D p_2$. Hence, $p$ is indeed the relative pseudocomplement of $p_1$ with respect to $p_2$. 

We obtain the following corollary immediately.
Corollary 3.2. Let \( p \in D_n \) with height sequence \( h_p = (h_1, h_2, \ldots, h_n) \). The pseudocomplement of \( p \) is the Dyck path \( p^c \) determined by the height sequence \( h_{p^c} = (h_1^c, h_2^c, \ldots, h_n^c) \) with

\[
h_i^c = \begin{cases} 
  n, & \text{if } i = n, \\
  h_{i+1}^c, & \text{if } i < n \text{ and } h_i = i, \\
  i, & \text{if } i < n \text{ and } h_i > i.
\end{cases}
\]

Proof. By definition, the least element of \( (D_n, \leq_D) \) is the Dyck path \( \sigma \in D_n \) determined by the height sequence \( h_\sigma = (1, 2, \ldots, n) \). By definition, for \( p \in D_n \), the pseudocomplement of \( p \) is the Dyck path \( p \rightarrow_D \sigma \). Now, the result follows by applying Theorem 3.1. \( \Box \)

Now we can characterize the regular elements of \( (D_n, \leq_D) \).

Proposition 3.3. A Dyck path \( p \in D_n \) is regular if and only if its height sequence \( h_p = (h_1, h_2, \ldots, h_n) \) satisfies for all \( i \in \{1, 2, \ldots, n\} \) either \( h_i = i \) or if \( h_i = c > i \), then \( h_i = h_{i+1} = \cdots = h_c = c \).

Proof. Let \( p \in D_n \) have height sequence \( h_p = (h_1, h_2, \ldots, h_n) \). By construction, we always have \( h_n = n \). Suppose that \( p \) is regular. Then, by definition, we have \( (p^c)^c = p \). Let \( p^c \) have height sequence \( h_{p^c} = (h_1^c, h_2^c, \ldots, h_n^c) \), and let \( (p^c)^c \) have height sequence \( h_{(p^c)^c} = (h_1^{c^c}, h_2^{c^c}, \ldots, h_n^{c^c}) \). Let \( i \in \{1, 2, \ldots, n-1\} \). Suppose first that \( h_i = i \). In view of Lemma 2.5 and Corollary 3.2, we obtain \( h_i^c = h_{i+1}^c \geq i+1 > i \), and thus \( h_i^{c^c} = i = h_i \). Suppose now that \( h_i = j > i \). In view of Corollary 3.2, we obtain \( h_i^c = i \), and thus by assumption \( h_i = h_{i+1}^{c^c} = h_{i+1}^{c^c} = h_{i+1} \). Now, we can continue with \( h_{i+1} \), and again have two choices, either \( h_{i+1} = i + 1 \) or \( h_{i+1} > n \). Since \( h_n = n \), this iteration finally stops, and we obtain the desired properties for \( h_p \).

Conversely, let \( h_p \) have the given properties. Then, either \( h_i = i \) for all \( i \in \{1, 2, \ldots, n\} \), and \( p \) is thus the least element of \( (D_n, \leq_D) \) which is clearly regular, or there is a some \( i \in \{1, 2, \ldots, n-1\} \) with \( h_i = c > i \). Then, by assumption, we have \( h_i = h_{i+1} = \cdots = h_c = c \), and with Corollary 3.2, it follows that \( h_i^c = j \) for \( j < c \) and \( h_c^c = h_{c+1}^c > c \). This implies, again with Corollary 3.2 that \( h_i^{c^c} = h_{i+1}^{c^c} = \cdots = h_c^{c^c} = c \) as desired. \( \Box \)

Example 3.4. The highlighted elements in Figure 1 are the regular elements of \( (D_4, \leq_D) \). The Dyck path \( p \in D_4 \) given by the height sequence \( h_p = (2, 3, 3, 4) \) is for instance not regular, since \( p^c \) has height sequence \( h_{p^c} = (1, 2, 4, 3) \), and \( (p^c)^c \) has height sequence \( h_{(p^c)^c} = (3, 3, 3, 4) \), and thus \( p \neq (p^c)^c \).

We say that a Dyck path \( p \in D_n \) has a touch point at \( i \) if the coordinate \( (i, i) \) belongs to the path. We can reformulate the previous proposition as follows. We say that two touch points of \( p \), say at \( i \) and \( j \) with \( i < j \), are consecutive if \( p \) does not have a touch point at \( k \) for \( i < k < j \). The touch points at 0 and \( n \) are called trivial.

Corollary 3.5. A Dyck path \( p \in D_n \) is regular if and only if for every pair \( (i, j) \) with \( 1 \leq i < j \leq n \) the following is satisfied: if \( p \) has consecutive touch points at \( i \) and \( j \), then the coordinate \( (i, j) \) belongs to the path.
Proof. Let $p \in D_n^A$ have height sequence $h_p = (h_1, h_2, \ldots, h_n)$, and suppose that $p$ has two consecutive touch points at $i$ and $j$. If $j = i + 1$, then the coordinate $(i, i + 1)$ clearly belongs to $p$, and we have $h_i = i$ and $h_{i+1} = i + 1$. If $j > i + 1$, then, since the path goes through $(i, j)$, we conclude that $h_i = i$ and $h_{i+1} = h_{i+2} = \cdots = h_j = j$. Hence, $h_p$ satisfies the conditions of Proposition 3.3 which implies that $p$ is regular.

Conversely, let $p$ be regular. Suppose that $p$ does not contain $(i, j)$. Hence, it necessarily contains the coordinate $(i', j')$ with $i' > i$ and $i' < j' < j$. (If $i' = j'$, then this would be a touch point, which would contradict the assumption that the touch points at $i$ and $j$ are consecutive.) In particular, we have $h_{i+1} > i + 1$ and $h_{i'} = j' < i$ which contradicts Proposition 3.3.

Since the regular elements of a Heyting algebra form a Boolean subalgebra, it is immediately clear that the number of regular elements equals $2^k$ for some $k \in \mathbb{N}$. We can say a bit more.

**Corollary 3.6.** For $n \in \mathbb{N}$, the number of regular elements of $(D_n^A, \leq_D)$ is $2^{n-1}$.

Proof. Corollary 3.5 implies that for every subset $\{i_1, i_2, \ldots, i_k\}$ of $\{1, 2, \ldots, n-1\}$ there exists a unique Dyck path $p \in D_n^A$ such that $i_j$ is a non-trivial touch point of $p$ for $j \in \{1, 2, \ldots, k\}$, which yields the claim.

3.2. Type $B$. The main result of this section is the following theorem.

**Theorem 3.7.** Let $p_1, p_2 \in D_n^B$ with height sequences $h_{p_1} = (h_1^{(1)}, h_2^{(1)}, \ldots, h_k^{(1)})$ and $h_{p_2} = (h_1^{(2)}, h_2^{(2)}, \ldots, h_k^{(2)})$. The relative pseudocomplement $p_1 \rightarrow_D p_2$ is the Dyck path $p$ determined by the height sequence $h_p = (h_1, h_2, \ldots, h_k)$ with

$$k = \begin{cases} 1, & \text{if } p_1 \leq_D p_2, \\ \max\{k_1, k_2\}, & \text{if } h_{\min\{k_1, k_2\}}^{(1)} > h_{\min\{k_1, k_2\}}^{(2)}, \\ \max\{i + 1 \mid h_i^{(1)} > h_i^{(2)}\}, & \text{otherwise}. \end{cases}$$

and if $k_1 < k$, then

$$h_i = \begin{cases} h_i^{(2)}, & \text{if } i > k_1, \\ h_{i+1}, & \text{if } i \leq k_1 \text{ and } h_i^{(1)} \leq h_i^{(2)}, \\ \min\{h_{i+1}, h_i^{(2)}\}, & \text{if } i \leq k_1 \text{ and } h_i^{(1)} > h_i^{(2)} \end{cases}$$

and if $k_1 \geq k$, then

$$h_i = \begin{cases} 2n - k + 1, & \text{if } i = k \text{ and } h_k^{(1)} \leq h_k^{(2)}, \\ \min\{h_k^{(1)}, h_k^{(2)}\}, & \text{if } i = k \text{ and } h_k^{(1)} > h_k^{(2)}, \\ h_{i+1}, & \text{if } i < k \text{ and } h_i^{(1)} \leq h_i^{(2)}, \\ \min\{h_{i+1}, h_i^{(2)}\}, & \text{if } i < k \text{ and } h_i^{(1)} > h_i^{(2)} \end{cases}$$

Proof. First, suppose that $p_1 \leq_D p_2$. In view of Lemma 2.2, it follows that $p_1 \rightarrow_D p_2$ is the maximal element of $(D_n^B, \leq_D)$, which is the Dyck path $p$ given by the height sequence $h_p = (2n)$.
Now, suppose that $p_1 \not\leq_D p_2$, and let $s = \min\{k_1, k_2\}$. Hence, by definition, there is a maximal index $i \in \{1, 2, \ldots, s\}$ with $h_i^{(1)} > h_i^{(2)}$. (It follows directly from Lemma 2.7 that the case $k_1 < k_2$ and $h_i^{(1)} \leq h_i^{(2)}$ for $i \in \{1, 2, \ldots, k_1\}$ is impossible.) We notice further that $h_p$ is indeed the height sequence of a Dyck path of type $B$ since $h_p$ and $h_{p_2}$ are. We distinguish two cases:

(i) $i = s$. Then, by construction, we have $k = \max\{k_1, k_2\}$.

(ia) If $k_1 < k$, then necessarily $k = k_2$. We first show that $p_1 \wedge_D p \leq_D p_2$. Let $h_{p_1 \wedge_D p} = (\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_k)$. By definition of the meet, we have $\bar{h}_i = h_i^{(2)}$ for $i > k_1$, and $\bar{h}_i = \min\{h_i^{(1)}, h_i\} \leq h_i^{(2)}$ for $i \leq k_1$. Thus, $p$ is indeed a candidate for the relative pseudocomplement. Suppose there is some $p' \in D^n_B$ with $p \leq_D p'$ and $p_1 \wedge_D p' \leq_D p_2$ with height sequence $h_{p'} = (h_1', h_2', \ldots, h_k')$. If $k = k'$, then there must be a maximal index $i \in \{1, 2, \ldots, k\}$ with $h_i < h_i'$. By construction, this cannot happen if $i > k_1$. By the maximality of $i$, we conclude that $h_i^{(1)} > h_i^{(2)}$, and then necessarily $h_i = h_i^{(2)}$. However, if $h_i' > h_i$, then $\min\{h_i^{(1)}, h_i'\} > h_i^{(2)}$ and $p_1 \wedge_D p' \not\leq_D p_2$, which is a contradiction. Hence, $p = p_1 \rightarrow_D p_2$.

(ia) If $k_1 \geq k$, then necessarily $k = k_1$. Analogously to (ia), we see that $p = p_1 \rightarrow_D p_2$.

(ii) $i < s$. Then, by construction, we have $k = \max\{i + 1 \mid h_i^{(1)} > h_i^{(2)}\}$. This means in particular that $h_k^{(1)} \leq h_k^{(2)}$, and $k_1, k_2 \geq k$. (Suppose for instance that $k_1 < k$. Then, since $h_{k-1}^{(1)}$ exists by assumption, it follows that $k_1 = k - 1$, and thus $s = k - 1$. Since $h_{k-1}^{(1)} > h_{k-1}^{(2)}$ by assumption, it follows that $i = k - 1 = s$ which contradicts the assumption.) We see immediately that $\min\{h_i^{(1)}, h_i\} \leq h_i^{(2)}$ for all $i \in \{1, 2, \ldots, k\}$ and thus $p_1 \wedge_D p \leq_D p_2$, and $p$ is a candidate for the relative pseudocomplement. The fact that $p = p_1 \rightarrow_D p_2$ can be shown analogously to (i). Thus, the proof is finished. \qed

Again, the following corollary is immediate.

**Corollary 3.8.** Let $p \in D_B^n$ with height sequence $h_p = (h_1, h_2, \ldots, h_k)$. The pseudocomplement of $p$ is the Dyck path $p^c$ determined by the height sequence $h_{p^c} = (h_1^c, h_2^c, \ldots, h_k^c)$ with

\[
k' = \begin{cases} 
1, & \text{if } p = o, \\
n, & \text{if } h_k > k, \\
\max\{i + 1 \mid h_i > i\}, & \text{if } h_k = k,
\end{cases}
\]

where $o$ denotes the least element in $(D_B^n, \leq_D)$, and if $k < k'$, then

\[
h_i^c = \begin{cases} 
i, & \text{if } i > k \text{ or } i \leq k \text{ and } h_i > i, \\
h_i^c, & \text{if } i \leq k \text{ and } h_i = i, \\
h_{i+1}^c, & \text{if } i \leq k \text{ and } h_i = i,
\end{cases}
\]
and if $k \geq k'$, then

$$h^c_i = \begin{cases} 2n - k' + 1, & \text{if } i = k' \text{ and } h_k = k', \\ k', & \text{if } i = k' \text{ and } h_k > k', \\ h_{i+1}, & \text{if } i < k' \text{ and } h_i = i, \\ i, & \text{if } i < k' \text{ and } h_i > i. \end{cases}$$

**Proof.** By definition, the least element of $(D^B_n, \leq_D)$ is the Dyck path $\mathbf{p} \in D^B_n$ determined by the height sequence $h_p = (1, 2, \ldots, n)$. By definition, for $\mathbf{p} \in D^B_n$, the pseudocomplement of $\mathbf{p}$ is the Dyck path $\mathbf{p} \rightarrow_D \mathbf{o}$. Now, the result follows by applying Theorem 3.7. \( \square \)

Again, we can characterize the regular elements of $(D^B_n, \leq_D)$.

**Proposition 3.9.** A Dyck path $\mathbf{p} \in D^B_1$ is regular if and only if its height sequence $h_p = (h_1, h_2, \ldots, h_k)$ satisfies one of the following conditions:

1. $h_k = 2n - k + 1$, $h_{k-1} = k - 1$, and for every $i \in \{1, 2, \ldots, k - 2\}$, we have either $h_i = i$ or if $h_i = c > i$, then $h_i = h_{i+1} = \cdots = h_c = c$; or
2. $h_k = n$, and for every $i \in \{1, 2, \ldots, k - 1\}$, we have either $h_i = i$ or if $h_i = c > i$, then $h_i = h_{i+1} = \cdots = h_c = c$.

**Proof.** Suppose that $\mathbf{p}$ is regular. If $\mathbf{p} = o$, then $h_p = (1, 2, \ldots, n)$ satisfies condition (2). So, suppose that $\mathbf{p} \neq o$, and let $\mathbf{p}^c$ have height sequence $h_{\mathbf{p}^c} = (h^c_1, h^c_2, \ldots, h^c_k)$, and let $(\mathbf{p}^c)^{cc}$ have height sequence $h_{(\mathbf{p}^c)^{cc}} = (h^{cc}_1, h^{cc}_2, \ldots, h^{cc}_k)$ with $k'' > k''$ and $h^{cc}_i = h_i$ for $i \in \{1, 2, \ldots, k\}$. We distinguish two cases:

(i) $h_k > k$. Then, $k' = n$, and since $\mathbf{p} = (\mathbf{p}^c)^{cc}$, we conclude that $h^c_i = i$ for $i \in \{k, k+1, \ldots, n\}$, and $h^{cc}_{k-1} > k - 1$, because otherwise $k'' > k$.

(ii) $h_k = k$. Let first $k < n$. By assumption, we have $h_k > k$, and hence $h^c_k = k$. Thus, $h^{cc}_{k-1} > k - 1$ and since $\mathbf{p}$ is regular, we obtain $h_k = 2n - k + 1$ as desired. Now, consider some $i < k - 1$, and suppose that $h_i = c > i$. Then, $h^c_i = i$, and thus $h^{cc}_{i-1} = h^{cc}_i$. Now, there are two more cases, $h^{cc}_{i+1} = i + 1$ or $h^{cc}_{i+1} > i + 1$. In both cases, since $h^{cc}_{k-1} = h_{k-1} = k - 1$, we obtain the desired condition for $\mathbf{p}$.

(ii) $h_k = k$. If it follows immediately that $k' \geq k$, since the maximal index $i$ with $h_i > i$ is $k = k - 1$. Moreover, we have $h_k' = k'$, and hence $h^{cc}_k = 2n - k' + 1 > k'$, since $k' \leq n$. This implies that $k = k'' = n$, and thus $n = h^{cc}_n = h_n$. Now, consider some $i < n$ with $h_i = c > i$. We have $h^c_i = i$, and thus $h^{cc}_i = h^{cc}_i$. As before, the claim follows.

For the converse, we distinguish two cases.

(i) Suppose first $h_p$ satisfies condition (1). Hence, $h_p^{cc} = (h^c_1, h^c_2, \ldots, h^c_k)$.

(ii) If $k < n$, then $h^c_i = i$ for $i \in \{k, k+1, \ldots, n\}$. Further, $h^{cc}_{k-1} = k$, which implies
that $h_{\{p^c\}^c} = (h_{1}^{cc}, h_{2}^{cc}, \ldots, h_{k}^{cc})$ with $h_{1}^{cc} = 2n - k + 1$, and $h_{k}^{cc} = k - 1$. Now, consider some $i \in \{1, 2, \ldots, k - 2\}$. If $h_{1} = i$, then $h_{1}^{c} = h_{i+1}^{cc} > i$, and thus $h_{i}^{cc} = i$ as desired. If $h_{1} = c > i$, then $h_{1}^{c} = i$, and thus $h_{1}^{cc} = h_{c+1}^{cc}$. By assumption, we have $h_{i} = h_{i+1} = \cdots = h_{c} = c$, and with the previous reasoning, we conclude that $h_{i}^{cc} = h_{i+1}^{cc} = \cdots = h_{c}^{cc} = c$. Hence, $p$ is regular.

(ii) Suppose now that $h_{p}$ satisfies condition (2). We conclude from Lemma 2.7 that $k = n$. Let $l \in \{1, 2, \ldots, n\}$ be maximal such that $h_{l-1} = l - 1$, in particular, the maximality implies $h_{l-1} = l$. Then, $h_{p}^{c} = (h_{1}^{c}, h_{2}^{c}, \ldots, h_{l}^{c})$. We have $h_{l}^{c} = 2n - l + 1$, and thus $h_{p}^{c} = (h_{1}^{c}, h_{2}^{c}, \ldots, h_{n}^{c})$, and it follows from Corollary 3.8 that $h_{i}^{cc} = i$ for $i \in \{l, l+1, \ldots, n\}$. Since $h_{l-1} = l - 1$, we conclude that $h_{l-1}^{cc} = l - 1$, and thus $h_{l-1}^{cc} = h_{c}^{cc} = l$. Let $s$ be the minimal index such that $h_{s} = l$, then, by assumption, we have $h_{s} = h_{s+1} = \cdots = h_{l} = l$. Analogously to before, we obtain $h_{s}^{cc} = h_{s+1}^{cc} = \cdots = h_{l}^{cc} = l$, and for every $i < l$, we obtain $h_{i} = h_{i}^{cc}$ which implies that $p$ is regular.

Example 3.10. The highlighted elements in Figure 2 are the regular elements of $(D_{2}^{B}, \leq_D)$. The Dyck path $p \in D_{2}^{B}$ given by the height sequence $h_{p} = (2, 4)$ is for instance not regular, since $p^{c}$ has height sequence $h_{p}^{c} = (1, 2, 3)$, and $(p^{c})^{c}$ has height sequence $h_{(p^{c})^{c}} = (6)$, and thus $p \neq (p^{c})^{c}$.

We say that a Dyck path $p \in D_{n}^{B}$ has a lower touch point at $i$ if the coordinate $(i, i)$ belongs to the path and $i \leq n$. Moreover, we say that $p$ has an upper touch point at $i$ if the coordinate $(i, 2n - i)$ belongs to the path and $0 \leq i < n$. We notice that the endpoint of $p$ is a lower touch point only if $p \in D_{n}^{A}$. Moreover, we notice that if $p$ has a lower touch point at $i$, then $h_{i+1} = i + 1$ for $i < n$ and $h_{n} = n$ if $i = n$. If $p$ has an upper touch point at $i$, then $h_{i+1} = 2n - i$. (That means, that in comparison to type $A$, we have a shift in the indices here.)

Again, we say that two lower touch points of $p$ at $i$ and $j$ are consecutive if $p$ does not have a lower touch point at $k$ for $i < k < j$. Similarly, we say that a lower touch point of $p$ at $i$ and an upper touch point of $p$ at $j$ are consecutive if $p$ does not have a lower touch point at $k$ for $k > i$ (and then necessarily $i = j$).

Corollary 3.11. A Dyck path $p \in D_{n}^{B}$ is regular if and only if for every pair $(i, j)$ with $0 \leq i < j \leq n$ the following is satisfied: if $p$ has consecutive lower touch points at $i$ and $j$, then the coordinate $(i, j)$ belongs to the path, and if $p$ has a lower touch point at $i$ and an upper touch point at $j$ which are consecutive, then $i = j < n$.

Proof. Let $p \in D_{n}^{B}$ have height sequence $h_{p} = (h_{1}, h_{2}, \ldots, h_{k})$, and suppose that $p$ has two consecutive lower touch points at $i$ and $j$. First, suppose that $j < n$. If $j = i + 1$, there is nothing to show, so let $j > i + 1$. Since $(i, j)$ belongs to the path, we have $h_{i+1} = j > i + 1$, and it follows that $h_{i+1} = h_{i+2} = \cdots = h_{j} = j$. If $j = n$, then $k = n$, and thus $h_{k} = n$. Analogously to before, we obtain $h_{i+1} = h_{i+2} = \cdots = h_{n} = n$. Then, $p$ satisfies Condition (2) in Proposition 3.9,
and is thus regular. Suppose now that \( p \) has a lower and an upper touch point at \( j < n \). Then, \( k = j + 1 \), and thus, \( h_k = 2n - j + 1 \). If these are the only two touch points of \( p \), then we conclude that \( j = 0 \), and \( h_0 = (2n) \) which is the greatest element of \( (\mathcal{D}_n^B, \leq_D) \) and thus clearly regular. If \( p \) has more than the two touch points at \( j \), then there must be a lower touch point at \( i \) with \( i < j \) such that the lower touch points at \( i \) and \( j \) are consecutive. By assumption, we have \( h_{i+1} = h_{i+2} = \cdots = h_j = j \), and \( p \) satisfies Condition (1) of Proposition 3.9, and is thus regular.

Conversely, suppose that \( p \) is regular, and suppose that \( p \) has two consecutive lower touch points at \( i \) and \( j \), but \( (i, j) \) does not belong to the path. Then, there must be indices \( i' > i \) and \( j' < j \) with \( i' \neq j' \) such that \( (i', j') \) belongs to the path, and say that \( i' \) is minimal with this property. Then, however, we have \( h_{i'+1} = j' > j' + 1 \) which violates Proposition 3.9. Suppose that \( p \) has a lower touch point at \( i \), and an upper touch point at \( j \), and both are consecutive, but \( i \neq j \). If \( j = n \), the reasoning goes analogous to the previous case. If \( j < n \), then \( k = j + 1 \), and \( h_k = 2n - k + 1 \), so \( p \) satisfies Condition (2) in Proposition 3.9. However, since \( i \neq j \), we obtain \( h_{k-1} > j = k - 1 \) which is a contradiction. \( \square \)

**Corollary 3.12.** For \( n \in \mathbb{N} \), the number of regular elements in \( (\mathcal{D}_n^B, \leq_D) \) is \( 2^n \).

**Proof.** First we notice that each \( p \in \mathcal{D}_n^B \) has at least two touch points. If \( p \) does not have an upper touch point, then \( p \in \mathcal{D}_n^A \), and thus by Corollary 3.6, we obtain \( 2^{n-1} \) such elements. On the other hand, suppose that \( p \) has lower touch points at \( i_1, i_2, \ldots, i_k \). Then, it is immediate that replacing the lower touch point at \( i_k \) by an upper touch point at \( i_{k-1} \) yields another regular Dyck path \( p' \in \mathcal{D}_n^B \). This clearly is a bijection, and since Corollary 3.11 implies that each regular element of \( \mathcal{D}_n^B \) is of one of the two forms, the result follows. \( \square \)

**Remark 3.13.** There exist explicit bijections between Dyck paths of type \( A \) and type \( B \), and \( \gamma \)-sortable elements of the symmetric group and the hyperoctahedral group, respectively, see for instance [2, 9], for a certain choice of Coxeter element \( \gamma \). Moreover, it is straightforward to verify that these bijections turn the dominance order on Dyck paths into the Bruhat order on these sortable elements. Thus, the Bruhat order on these sortable elements constitutes a distributive lattice. In type \( A \), this has already been observed by Armstrong, see [1, Section 6]. Analogously, there is an explicit bijection between Dyck paths of type \( A \) and type \( B \), and antichains in the root poset of the symmetric group and the hyperoctahedral group, respectively. Hence, the lattices investigated in this article coincide with the lattice of order ideals of these root posets, which again was already noted in [1, Section 6] for type \( A \).

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