SHRINKING TARGETS ON PRIMITIVE SQUARE-TILED SURFACES

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ABSTRACT. We prove a shrinking target property for the action of subgroups of Veech groups on square-tiled surfaces. Our main tool, of independent interest, is the construction of a Fourier-like transform that we use to show that the $L^2$-norm of the Koopman operator for the action is equivalent to the $L^2$-norm of a Markov operator related to a random walk on a subgroup. This generalizes work of V. Finkelshtein on the flat torus.

1. INTRODUCTION

We study a shrinking target problem on primitive square-tiled surfaces. We show that actions of subgroups of the Veech group of a primitive square-tiled surface satisfy a shrinking target property on the corresponding square-tiled surface. This generalizes the work of Finkelshtein, who studied a similar problem on the flat torus [8].

Definition 1.1 (Square-tiled Surface [15] [24]). A square-tiled surface is a pair $(X, \omega)$, where $X$ is a closed Riemann surface and $\omega$ a holomorphic 1-form, given by a (finite) branched cover over the square torus, $\pi : X \to T^2$, branched over 0. The one-form $\omega$ is given by the pullback of $dz$ under the covering map $\pi$, $\omega = \pi^*(dz)$.

Square-tiled surfaces come with a combinatorial description: $(N, \sigma, \tau)$, where $N$ denotes the degree of the cover and $\sigma, \tau \in S_N$ are permutations that encode gluing information. $\sigma(i) = j$ means that the right edge of the $i^{th}$ square is glued to the left edge of the $j^{th}$ square. $\tau(i) = j$ means that the top edge of the $i^{th}$ square is glued to the bottom edge of the $j^{th}$ square.

Definition 1.2 (Primitive Square-Tiled Surface). A square-tiled surface $(N, \sigma, \tau)$ is called primitive if it is not a proper ramified covering of another square-tiled surface. Equivalently, a square-tiled surface $(N, \sigma, \tau)$ is called primitive if the group $\langle \sigma, \tau \rangle$ is primitive [24].

Square-tiled surfaces are examples of translation surfaces, which are pairs $(Y, \eta)$ where $Y$ is a closed Riemann surface and $\eta$ a holomorphic one-form on $Y$ (see Definition 3.1). Translation surfaces can be represented as Euclidean polygons with parallel sides identified by translation. The zeros of order $k$ of the one-form $\eta$ correspond to points which have total angle $2\pi (k+1)$ under these identifications. For an excellent survey on translation surfaces, see, for example, either of the surveys written by Wright or Zorich [22] [25].

A translation surface $(Y, \eta)$ carries a natural measure, the area form associated to the flat metric coming from $\eta$. For square-tiled surfaces, this is the pullback of Lebesgue measure on the torus.

There is a natural $SL_2(\mathbb{R})$ action on the space of translation surfaces that preserves area, where the action on the polygonal representation is the linear action.
on the plane \[ [25] \]. Square-tiled surfaces have large stabilizer groups under this action. In contrast, a typical (in the sense of the natural Masur-Veech measure on the space of translation surfaces) translation surface has a trivial stabilizer. We call the stabilizer of a translation surface \((X, \omega)\) the Veech group \([12]\), and denote it \(\text{SL}_2(\mathbb{Z})\).

We will use the following fact, which was originally proven by Veech \([23]\).

**Proposition 1.1.** The Veech group of a primitive square-tiled surface is a lattice subgroup of \(\text{SL}_2(\mathbb{Z})\).

**Remark 1.1.** Gutkin and Judge \([9]\) have provided a proof of the converse of Proposition 1.1.

Our goal is to understand the fine-scale dynamical behavior of the action of the Veech group on the surface. The action of the Veech group action on a square-tiled surface preserves Lebesgue measure. In fact, the action is ergodic, since, for example, hyperbolic elements act ergodically on the surface, and every lattice group contains a hyperbolic element. We give a quantitative characterization of the density of orbits of large subgroups of the Veech group by showing their action exhibits a shrinking target property.

**Definition 1.3** (Borel-Cantelli, Shrinking Target Property \([1]\) \([7]\)). Let \(G\) be a group acting on a finite measure space \((X, \mathcal{F}, \mu)\). Let \(g_n\) be a sequence of group elements of a group \(G\). A sequence of measurable sets \(A_n\) is Borel-Cantelli, (BC), if \(\sum_n \mu(A_n) = \infty\) and

\[
\mu(\{ x \in X : g_n x \in A_n \text{ infinitely often} \}) = 1
\]

We say a transformation \(T\) of a metric space exhibits a shrinking target property, (STP), if the family of metric balls is BC.

1.1. **Main Result.** Our main result shows that the action of a subgroup of a Veech group acting on a square-tiled surface exhibits a shrinking target property governed by the critical exponent of the subgroup.
**Theorem 1.1.** Let \((X, \omega)\) be a primitive square tiled surface, and let \(\Gamma\) be a subgroup of the Veech group \(SL(X, \omega)\). For any \(y \in X\), for Lebesgue a.e. \(x \in X\), the set

\[
\{ g \in \Gamma : |g \cdot x - y| < ||g||^{-\alpha} \}
\]

is

1. finite for every \(\alpha > \delta_\Gamma\)
2. infinite for every \(\alpha < \delta_\Gamma\)

where \(\delta_\Gamma\) is the critical exponent of the subgroup \(\Gamma\).

Recall the definition of critical exponent:

**Definition 1.4 (Critical Exponent, \(\delta_\Gamma\)).** Let \(\Gamma\) be a Fuchsian group. The critical exponent, \(\delta_\Gamma\), is

\[
\delta_\Gamma := \limsup_{R \to \infty} \frac{\log(\# \{ g \in \Gamma : d_H(g \cdot x_0, x_0) \leq R \})}{R},
\]

for any \(x_0\), where \(g \cdot x_0\) denotes the action of \(g\) on \(x_0\) by Möbius transformation. \(\delta_\Gamma\) is independent of the basepoint \(x_0\).

The critical exponent \(\delta_\Gamma\) is the exponent required for convergence in the Poincaré series of the group \(\Gamma\) [4] [17], which is equivalent to the exponential growth rate of the number of points in the orbit of \(\Gamma\) acting on the upper half-plane [19] seen in Definition 1.4.

Patterson [17] showed that for a finitely generated Fuchsian group \(\Gamma\), the critical exponent is precisely the Hausdorff dimension of the limit set, \(\Lambda = \Gamma \cap \mathbb{S}^1\), where \(\mathbb{S}^1\) is the circle at infinity. Sullivan [19] showed that in the general case of a Fuchsian group, the critical exponent is the Hausdorff dimension of the radial limit set, \(\Lambda_r \subset \Lambda\) consisting of all points in the limit set such that there exists a sequence \(\lambda_n x \to y\) remaining within a bounded distance of a geodesic ray ending at \(y\).

These various interpretations are particularly relevant to our work since we obtain Theorem 1.1 through spectral estimates of the boundary representation of the subgroup \(\Gamma\), see Example 2.1. The spectral estimates were first observed by Finkelshtein, who used them to prove a similar theorem in the setting of the flat torus [8].

By connecting the \(L^2\)-norm of the Koopman representation, see Definition 2.2, to the \(L^2\)-norm of a Markov operator, see Definition 2.4, we can use similar spectral estimates to exhibit Theorem 1.1 as a consequence of the following theorem.

**Theorem 1.2.** Let \(\Gamma\) be an arbitrary subgroup of the Veech group of a primitive square-tiled surface \((X, \omega)\) with the Lebesgue probability measure \(m\). Let \(\pi_0\) denote the Koopman representation on \(L_0^2(X)\), the subspace of \(L^2(X)\) orthogonal to the set of constant functions. Then, for any probability measure \(\mu\) on \(\Gamma\)

\[
||\pi_0(\mu)||_2 = ||\lambda_\Gamma(\mu)||_2,
\]

where \(\lambda_\Gamma(\mu)\) is the Markov operator associated with a random walk on \(\Gamma\) with law \(\mu\).

To prove Theorem 1.1 using Theorem 1.2 it is crucial that primitive square-tiled surfaces have Veech groups that are subgroups of \(SL_2(\mathbb{Z})\). The proof of Theorem
using Theorem 1.2 follows immediately from an argument developed in the setting of the flat torus in [8].

Our approach to proving Theorem 1.2 is constructive. We relate functions on the square-tiled surface to functions on a direct sum of tori. This opens a door through which we can leverage a modular version of Fourier analysis on the surface.

1.2. Connections. Our results require a formulation of harmonic analysis on square-tiled surfaces: we decompose the surface into “commutative pieces”. Hillairet [11] used similar ideas to generate a spectrum for a Laplacian on square-tiled surfaces.

1.3. Organization. Section 2 includes a short history of shrinking target problems, and a fuller description of Finkelshtein’s work on Diophantine approximations of toral automorphisms. Section 3 develops the noncommutative harmonic analysis necessary to prove the main result. Section 4 is the proof of Theorem 1.2.

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2. Brief History of Shrinking Targets

2.1. Poincaré Recurrence and the Borel-Cantelli Lemma. A fundamental lemma in dynamics is the Poincaré Recurrence theorem:

**Theorem 2.1** (Poincaré Recurrence). Let \( (X, \mathcal{B}, \mu) \) be a finite measure space. Let \( T : X \rightarrow X \) be a measure-preserving transformation. Let \( E \in \mathcal{B} \). Define a semigroup action on \( X \) by the group \( \mathbb{N} \) as follows: \( n \cdot x := T^n(x) \) for any \( n \in \mathbb{N} \), where \( T^0 = \text{Id} \). Then for almost every point \( x \in E \), the set

\[
\{ n \in \mathbb{N} : n \cdot x \in E \}
\]

is infinite.

The set of points in \( E \) that return to \( E \) infinitely often has full measure. In many dynamical settings, maps have more structure than being measure preserving, hence, we would expect to be able to strengthen Poincaré Recurrence. For example, with regards to a translation surface we can ask about the action of the Veech group. Given that the Veech group contains elements that are mixing on the surface, a strengthening of Poincaré Recurrence could help us give quantitative answers to questions about the density of the orbits under the action. One way to do this is to formulate a shrinking target question. Let \( X \) be a lattice surface with Veech group \( SL(X, \omega) \), let \( y \in X \), and let \( B_r(y) \) be a ball of radius \( r \) centered at \( y \). For (almost every) \( x \in X \), how small can we make the radius before only finitely many \( g \in SL(X, \omega) \) have the property that \( g \cdot x \in B_r(y) \)? In other words: how small can we make the target before only finitely many \( g \cdot x \) hit the target?

Philipp was the first to write out this idea of a shrinking target in order to get Diophantine estimates [18]. Hill and Velani are often attributed coining the term “Shrinking Target” in their fundamental work on the subject [10]. Others took hold of this idea and developed it in different settings. In 1982, Sullivan [20] published a logarithm law for the geodesic flow on \( \mathbb{H}^{d+1} / \Gamma \) for a discrete subgroup of isometries of \( \mathbb{H}^{d+1} \) whose quotient has finite volume. In 1999, Kleinbock and Margulis [14] proved a generalization: geodesics on locally symmetric spaces with
finite volume satisfied a similar logarithm law. Here, the authors explicitly use a shrinking target argument. In 2009, Athreya and Margulis \[2\] \[3\] extended Sullivan’s result to unipotent flows on the space of lattices also using a shrinking target argument. For the interested reader, many of these results are put into a more general framework in a survey by Athreya \[1\].

Historically, the workhorse behind proving such theorems are quantitative versions of the Borel-Cantelli lemma.

**Lemma 2.1** (Borel-Cantelli and Partial Converse). Let \((X, \mu)\) be a probability space with \(\sigma\)-algebra \(\mathcal{F}\). Let \(E_n\) be a sequence of measurable sets.

1. (Borel-Cantelli Lemma) If \(\sum_n \mu(E_n) < \infty\), then the set of points \(x \in X\) such that \(x\) occurs infinitely often has measure 0.
2. Conversely, if the \(E_n\) are pairwise independent, and \(\sum_n \mu(E_n) = \infty\), then the set \(x \in X\) such that \(x\) occurs infinitely often has full measure.

Many papers explore just how far one can push a quantitative Borel-Cantelli lemma, including the recent work of Dolgopyat, Fayad, and Liu \[5\] and Dolgopyat and Fayad \[6\]. Additionally, much work has been done classifying shrinking target properties. Athreya and Tseng independently produced particularly nice expositions on these classifications \[1\], \[21\].

Our setting is somewhat different from those considered above, as lattice subgroups of \(\text{SL}_2(\mathbb{R})\) are not amenable groups. In place of traditional mixing estimates which are used to understand converses of Borel-Cantelli type lemmas, we use spectral estimates of a boundary operator to understand the interactions between the events \(|gx - y| \leq ||g||^{-\alpha}\) as \(g \in \Gamma\) varies.

### 2.2. Diophantine Approximation on Tori.

In the context of a torus, Finkelshtein was the first to explain how spectral estimates of a boundary operator could be used to understand shrinking target properties. He replaced the requirement for pairwise independence in the converse of the Borel-Cantelli lemma with properties derived from the induced action of subgroups of \(\text{SL}_2(\mathbb{Z})\) on the boundary of the upper-half plane \[8\]. Finkelshtein achieves the following strict bounds for the 2-dimensional torus.

**Theorem 2.2** (Finkelshtein \[8\]). Let \(\Gamma\) be an arbitrary subgroup of \(\text{SL}_2(\mathbb{Z})\). For any \(y \in \mathbb{T}^2\), for Lebesgue a.e. \(x \in \mathbb{T}^2\), the set

\[
\{g \in \Gamma : |gx - y| < ||g||^{-\alpha}\}
\]

is

1. finite for every \(\alpha > \delta_{\Gamma}\)
2. infinite for every \(\alpha < \delta_{\Gamma}\)

where \(\delta_{\Gamma}\) is the critical exponent of the subgroup.

The proof of this statement relies on a folklore theorem, Theorem 3.7 in \[8\]. We state the theorem below, but first, we need a few definitions.

**Definition 2.1** (Left Regular Representation). Let \(\Gamma\) be a discrete group. \(\Gamma\) acts on itself by left-multiplication which induces the unitary left-regular representation \(\lambda_\Gamma : \Gamma \rightarrow \mathcal{U}(l^2(\Gamma))\) where \(\lambda_\Gamma(g)f(h) = f(g^{-1}h)\) for \(f \in l^2(\Gamma)\) and \(g \in \Gamma\).
Definition 2.2 (Koopman Representation). Let $\Gamma$ act by measure preserving transformations on a probability space $(X, \mu)$. The Koopman Representation $\pi : \Gamma \to U(L^2(X))$ is given by $\pi(g)f(x) = f(g^{-1}x)$.

We denote the Koopman representation restricted to the orthogonal complement of the constant functions by $\pi_0$.

Definition 2.3 (Quasi-regular Representation). Now, assume the group action is not necessarily measure preserving, but does preserve the measure class. Then we can produce a variant of the Koopman representation called the quasi-regular representation $\pi_X : \Gamma \to U(L^2(X, \nu))$ where

$$
\pi_X(g)f(x) = f(g^{-1}x)\sqrt{\frac{dg\nu}{d\nu}(x)}.
$$

Example 2.1 (Boundary Representation). For an example of a quasi-regular representation, consider $\Gamma$ acts on its visual boundary (see [8]) equipped with a Patterson-Sullivan measure.

Definition 2.4 (Markov Operator). Given any finitely supported probability measure $\mu$ on the discrete group $\Gamma$, and given any unitary representation $\pi : \Gamma \to U(H)$ where $H$ is a Hilbert space, we can average the representation to produce a Markov operator $\pi(\mu) : H \to H$ given by

$$
\pi(\mu) = \sum_{g \in \Gamma} \mu(g)\pi(g)
$$

Theorem 2.3. (Folklore) Let $\Gamma \subset SL_2(\mathbb{Z})$ act on the torus $T^2$ equipped with the Lebesgue measure $m$ and let $\pi_0$ be the Koopman representation on $L^2_0(T^2)$. Then, for any probability measure $\mu$ on $\Gamma$

$$
||\pi_0(\mu)||_2 = ||\lambda_\Gamma(\mu)||_2
$$

The $L^2$-norm of the Markov operator above is known to be equivalent to the $L^2$-norm of the boundary representation when $\Gamma$ is a convex cocompact subgroup. This fact coupled with Theorem 2.3 enables Finkelshtein to replace the pairwise independence hypothesis in the converse of the Borel-Cantelli lemma with spectral estimates of the boundary representation for convex cocompact subgroups of $SL_2(\mathbb{Z})$. These arguments extend to the case of a generic subgroup. See Section 5.2 in [8]

3. Harmonic Analysis on Lattice Surfaces

In this section, we develop the tools we will need to prove Theorem 1.2. First, we define a modular version of a Fourier transform for lattice surfaces, a more general class of surfaces than square-tiled surfaces.

Definition 3.1 (Translation Surface, Lattice Surface [12] [22] [25]). An abelian differential $\omega$ on a Riemann surface $X$ is a global section of the cotangent bundle of $X$. A translation surface $(X, \omega)$ is a pair consisting of a closed Riemann surface $X$ and a nonzero abelian differential $\omega$ on that surface. A lattice surface is a translation surface whose stabilizer under the $SL_2(\mathbb{R})$ action is a lattice subgroup.
Second, we show that we can compute the Koopman representation on lattice surfaces by using the dynamical partition matrix, see Definition 3.5, that tracks the movement of points relative to partitions of the surface. Third, we give a generalization of the transpose representation used in the proof of Theorem 2.3 above.

3.1. Modular Fourier Transform on a Lattice Surface. To construct a useful variant of a Fourier transform, we make a few simple observations that enable us to connect the structure of the $L^2$-space of functions on lattice surfaces to the $L^2$-space of functions on a disjoint collection of tori. To do this, we will leverage the fact that lattice surfaces admit a cylinder decomposition \cite{12}. By cutting open the cylinders, we have a decomposition into rectangles.

**Lemma 3.1.** Let $(X, \omega)$ be a lattice surface that can be partitioned into $m$ rectangles $\{R_1, R_2, \cdots, R_m\}$, where each partition contains the interior of a rectangle along with two sides. Then

$$L^2(X, \nu) \cong \bigoplus_{k=1}^{m} L^2(T^2_k, \mu)$$

where $\nu$ is the measure induced by $\omega$, and $\mu$ is the Haar measure on the torus, and $T^2_k$ is the result of identifying opposite edges on the closure of the rectangle $R_k$.

**Proof.** First, pick coordinates on $(X, \omega)$ by setting one of the lower left corners of the surface equal to $(0,0)$. Label each of the lower left corners of the partitioned rectangles $a_i$, the offset vector from $(0,0)$. Note that when partitioning rectangles, only two sides are included.

**Definition 3.2** (Spatial Measurable Isomorphism, $i$). Define a map $i : X \to \bigsqcup_{k=1}^{m} T^2_k$ by $i(x) = (y, a_k)$, where $a_k$ is a label corresponding to which rectangle $x$ resides, and $y = x - a_k$.

We have an inverse map $i^{-1}(y, a_k) = y + a_k$.

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**Figure 2. Spatial Isomorphism**

**Definition 3.3** (Functional Isometric Isomorphism, $h$). Now, define a map $h : L^2(X, \nu) \to \bigoplus_{k=1}^{m} L^2(T^2_k, \mu)$ by $h(f) = f \circ i^{-1}$. 
The map \( h \) has inverse \( h^{-1}(g) = g \circ i \), and \( h \) respects the inner product. Let \( f_1, f_2 \in L^2(X, \nu) \), and consider the following.

\[
\langle h(f_1), h(f_2) \rangle = \int_{\mathbb{T}^2_i}^\nu (f_1 \circ i^{-1}(x))(f_2 \circ i^{-1}(x)) \, d\nu \\
= \int_{\mathbb{T}^2_i}^\nu f_1(i^{-1}(x))f_2(i^{-1}(x)) \, d\mu \\
= \int_X f_1(x)f_2(x) \, d\mu \\
= \langle f_1, f_2 \rangle.
\]

\[\blacksquare\]

**Remark 3.1.** We could have used a direct product in Lemma 3.1; however, we want to emphasize that we are thinking of the underlying set of rectangles on a surface as a set of disjoint tori.

**Remark 3.2.** We will use two notations to describe a function in \( \bigoplus_{i=1}^m L^2(\mathbb{T}^2_i) \) (and \( \bigoplus_{i=1}^m L^2(a_i \mathbb{Z} \times b_i \mathbb{Z}) \)). The first is the usual notation: \( f_1 \oplus f_2 \oplus \cdots \oplus f_m \). The second is a vector notation:

\[
\begin{bmatrix}
  f_1 \\
  f_2 \\
  \vdots \\
  f_m
\end{bmatrix}.
\]

We can use Lemma 3.1 to pullback the following variant of a Fourier transform on the space of \( L^2 \) functions on a lattice surface, or in fact, any surface that can be partitioned into rectangles as described in Lemma 3.1.

**Definition 3.4 (Modular Fourier Transform).** Let \( f \in \bigoplus_{i=1}^m L^2(\mathbb{T}^2_i) \), where each \( \mathbb{T}^2_i \) is a flat torus isomorphic to \([0, a_i] \times [0, b_i] / \sim\) where the equivalence is given by \((x, 0) \sim (x, 1)\) and \((0, y) \sim (1, y)\). Write \( f = f_1 \oplus f_2 \oplus \cdots \oplus f_m \). The modular Fourier transform

\[
\mathcal{F} : \bigoplus_{i=1}^m L^2(\mathbb{T}^2_i) \to \bigoplus_{i=1}^m L^2(a_i \mathbb{Z} \times b_i \mathbb{Z})
\]

of \( f \) is given by

\[
\mathcal{F}(f) = (\hat{f}_1, \hat{f}_2, \cdots, \hat{f}_m)
\]

and \( \hat{f}_i \) denotes the usual Fourier transform on the torus \( \mathbb{T}^2_i \).

With this definition, we can use Lemma 3.1 to pullback the modular Fourier transform to the space of \( L^2 \) functions on a lattice surface, or in fact, any surface that can be partitioned into rectangles as described in Lemma 3.1. However, the transform will be highly dependent on the partitioning.

### 3.2. Koopman Representation on a Partitioned Space and the Dynamical Partition Matrix

Let \( (X, \omega) \) be a lattice surface that can be decomposed into \( m \) rectangles (cylinders), let \( SL(X, \omega) \) be its Veech group, and let \( f \in L^2(X) \). Label the rectangles in the tiling \( R_1, R_2, \cdots, R_m \). Let \( f \in L^2(X) \), and recall that the Koopman representation of \( \Gamma \subset SL(X, \omega) \) is given by \( (\pi(g)f)(x) = f(g^{-1}x) \) for \( g \in \Gamma \).
We can use the spatial isomorphism \( i \) from Lemma 3.1 to induce the Koopman representation on the modularized function space.

\[
\pi(g) (f_1 \oplus f_2 \oplus \cdots \oplus f_m (i(x))) = (f_1 \oplus f_2 \oplus \cdots \oplus f_m) (i(g^{-1}x))
\]

There are multiple ways to go about unraveling this expression. We can define

\[
g^{-1} \cdot i(x) := i(g^{-1}x),
\]

then use the labeling to compute the induced map. This is not hard. Using the notation from the proof of Lemma 3.1, we can compute:

\[
i(g^{-1}x) = g^{-1} \cdot i(x) = g^{-1} \cdot (y, a_k) = (g^{-1}y + g^{-1}a_k - \sum_{i=1}^{m} \chi_{ik} a_j)
\]

where \( \chi_{ij} = \chi_{R_i \cap R_j}(x) \).

However, the Koopman representation is a representation on the \( L^2 \) space, not the underlying topological space. From the above formula, it is difficult to deduce how the modularized function has changed. We want to express how \( g^{-1} \) acts on the functions in \( \bigoplus_{i=1}^{m} L^2(T^2_i) \). The following definition will enable us to compute this action on the function space as well as provide us with some interpretation of the characteristic functions in the prior equation.

**Definition 3.5 (Dynamical Partition Matrix).** Let \( X \) be a Borel measure space partitioned into a finite number of positive measure sets \( \{Y_i\}_{i=1}^{m} \). Let \( T : X \to X \) be a measure preserving transformation. The dynamical partition matrix of degree \( k \) is

\[
\begin{bmatrix}
\chi_{11} & \chi_{12} & \cdots & \chi_{1m} \\
\chi_{21} & \chi_{22} & \cdots & \chi_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
\chi_{m1} & \chi_{m2} & \cdots & \chi_{mm}
\end{bmatrix}
\]

where \( \chi_{ij} = \chi_{Y_i \cap T^k(Y_j)}(x) \).

We can define a dynamical partition matrix for \( T(x) = gx \). In this case, notice \( \chi_{R_i \cap gR_j}(x) = \chi_{R_i}(x) \chi_{R_j}(g^{-1}x) = \chi_{R_i}(x) \chi_{R_j}(g^{-1}x) \), meaning this characteristic function indicates when a point \( x \) is in \( R_i \), and multiplication by \( g^{-1} \) sends it to \( R_j \). Additionally, if we apply this matrix to a single \( x \), all but one entry in this matrix of characteristic functions is zero (it is a unit matrix).

Using the dynamical partition matrix, we can represent the action of \( g^{-1} \) on the function.

\[
\pi(g) f_1 \oplus f_2 \oplus \cdots \oplus f_m (i(x)) = \begin{bmatrix}
\chi_{11} & \chi_{12} & \cdots & \chi_{1m} \\
\chi_{21} & \chi_{22} & \cdots & \chi_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
\chi_{m1} & \chi_{m2} & \cdots & \chi_{mm}
\end{bmatrix} \begin{bmatrix}
f_1 \circ g^{-1} \\
f_2 \circ g^{-1} \\
\vdots \\
f_m \circ g^{-1}
\end{bmatrix} (i(x))
\]

where \( \chi_{ij} = \chi_{R_i \cap gR_j}(x) \). This matrix is the dynamical partition matrix associated to \( T(x) = gx \). The multiplication is the usual matrix multiplication:
$\begin{bmatrix}
\chi_{11} & \chi_{12} & \cdots & \chi_{1m} \\
\chi_{21} & \chi_{22} & \cdots & \chi_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
\chi_{m1} & \chi_{m2} & \cdots & \chi_{mm}
\end{bmatrix}
\begin{bmatrix}
f_1 \circ g^{-1} \\
f_2 \circ g^{-1} \\
\vdots \\
f_m \circ g^{-1}
\end{bmatrix}
(i(x))

= \left[
\begin{array}{c}
\chi_{11} f_1 \circ g^{-1} + \chi_{12} f_2 \circ g^{-1} + \cdots + \chi_{1m} f_m \circ g^{-1} \\
\chi_{21} f_1 \circ g^{-1} + \chi_{22} f_2 \circ g^{-1} + \cdots + \chi_{2m} f_m \circ g^{-1} \\
\vdots \\
\chi_{m1} f_1 \circ g^{-1} + \chi_{m2} f_2 \circ g^{-1} + \cdots + \chi_{mm} f_m \circ g^{-1}
\end{array}
\right] (i(x))$

**Proposition 3.1.** The induced Koopman representation is a representation.

**Proof.** Firstly, the induced Koopman representation is continuous since it is conjugate (by isomorphism) with the Koopman representation on $L^2(X)$.

Secondly, the induced Koopman representation respects the group structure. Let $\pi$ denote the induced Koopman representation, let $h(f) = f_1 \oplus f_2 \oplus \cdots \oplus f_m \in \bigoplus_{i=1}^n L^2(\mathcal{T}_i^2)$, and let $M_{gh}$ denote the dynamical partition matrix associated with $T(x) = (gh)x$. Let $M_x$ and $M_y$ be defined analogously.

Notice that $M_{gh}(x) = M_g(x)M_h(g^{-1}x)$, where the multiplication on the right is the usual matrix multiplication. The $ij^{th}$ entry of $M_{gh}$ is $\chi_{R_i \cap (gh)R_j}(x)$, which identifies the set of points in $R_i$ such that when we apply $g^{-1}$ and subsequently $h^{-1}$, the point lies in $R_j$. The $ij^{th}$ entry of $M_x(g^{-1}x)$ is $\sum_{k=1}^m \chi_{R_i \cap gR_k \cap hR_j}(x)$.

This is the collection of points that starts in $R_i$, passes through some $R_k$ after the application of $g^{-1}$ and ends up in $R_j$ after the application of $h^{-1}$:

$$M_{gh}(x) = \left[ \chi_{R_i \cap (gh)R_j}(x) \right]$$

$$= \chi_{R_i}(x)\chi_{R_j}(h^{-1}g^{-1}x)$$

$$= \sum_{k=1}^m \chi_{R_k}(x)\chi_{R_k}(g^{-1}x)\chi_{R_j}(h^{-1}g^{-1}x)$$

$$= \sum_{k=1}^m \chi_{R_k}(x)\chi_{gR_k \cap hR_j}(x)$$

$$= \sum_{k=1}^m \chi_{R_k \cap gR_k \cap hR_j}(x)$$

$$= \sum_{k=1}^m \chi_{R_k \cap gR_k \cap hR_j}(x)$$

$$= \chi_{R_i \cap (gh)R_j}(x)\chi_{R_i \cap hR_j}(g^{-1}x)$$

$$= M_g(x)M_h(g^{-1}x)$$

Now, we can compute and show that $\pi$ respects the group structure.
\[ \pi(gh)f_1 \oplus f_2 \oplus \cdots \oplus f_m(i(x)) = M_{gh} \begin{bmatrix} f_1 \circ (gh)^{-1} \\ f_2 \circ (gh)^{-1} \\ \vdots \\ f_m \circ (gh)^{-1} \end{bmatrix} (i(x)) = M_g M_h \begin{bmatrix} f_1 \circ h^{-1} \circ g^{-1} \\ f_2 \circ h^{-1} \circ g^{-1} \\ \vdots \\ f_m \circ h^{-1} \circ g^{-1} \end{bmatrix} (i(x)) = \pi(g) M_h \begin{bmatrix} f_1 \circ h^{-1} \\ f_2 \circ h^{-1} \\ \vdots \\ f_m \circ h^{-1} \end{bmatrix} (i(x)) = \pi(g) \pi(h)f_1 \oplus f_2 \oplus \cdots \oplus f_m(i(x)) \]

We could have proven this by appealing to the fact that the induced Koopman representation is conjugate to the Koopman representation. However, the proof above shows that the dynamical partition matrix behaves as we would expect.

### 3.3. A Transpose Representation

In this section, we restrict our functions to be in \( \bigoplus_{i=1}^m L^2_0(T^2_i) \), where \( L^2_0(T^2_i) \) is the subspace of \( L^2(T^2_i) \) orthogonal to the constant functions. Similarly, we restrict to \( \bigoplus_{i=1}^m L^2(Z^2 \setminus \{0\}) \).

**Definition 3.6 (Inverse Fourier Partitioning).** Let \( f_1 \oplus f_2 \oplus \cdots \oplus f_m \in \bigoplus \mathcal{L}^2(a, \mathbb{Z} \times b, \mathbb{Z} \setminus \{0\}) \) and let \( M \) be a matrix whose entries \( M_{ij} \) are characteristic functions in \( \bigoplus_{i=1}^m L^2_0(T^2_i) \), where \( M_{ij} \in L^2_0(T^2_i) \) and for each \( i, \sum \lambda_{ij}(x) = 1 \). (For each \( i \), the measurable sets corresponding the characteristic functions \( M_{ij} \) partition the space.) An inverse Fourier partitioning, \( \mathcal{F}_M^{-1} \), of the function \( f_1 \oplus f_2 \oplus \cdots \oplus f_m \) with respect to the partitioning matrix \( M \) is

\[
\mathcal{F}_M^{-1}(f_1 \oplus f_2 \oplus \cdots \oplus f_m) := \int_{T^2_1} M_{11} \tilde{f}_1(x) e^{-2\pi i \langle n, x \rangle} \, dx + \int_{T^2_2} M_{12} \tilde{f}_1(x) e^{-2\pi i \langle n, x \rangle} \, dx + \cdots + \int_{T^2_1} M_{1m} \tilde{f}_1(x) e^{-2\pi i \langle n, x \rangle} \, dx \\
\quad \oplus \int_{T^2_2} M_{21} \tilde{f}_2(x) e^{-2\pi i \langle n, x \rangle} \, dx + \int_{T^2_2} M_{22} \tilde{f}_2(x) e^{-2\pi i \langle n, x \rangle} \, dx + \cdots + \int_{T^2_2} M_{2m} \tilde{f}_2(x) e^{-2\pi i \langle n, x \rangle} \, dx \\
\quad \oplus \cdots \\
\quad \oplus \int_{T^2_m} M_{1m} \tilde{f}_m(x) e^{-2\pi i \langle n, x \rangle} \, dx + \int_{T^2_m} M_{2m} \tilde{f}_m(x) e^{-2\pi i \langle n, x \rangle} \, dx + \cdots + \int_{T^2_m} M_{mm} \tilde{f}_m(x) e^{-2\pi i \langle n, x \rangle} \, dx
\]

where \( \tilde{f}_i \) is the inverse Fourier transform of \( f_i \).
The inverse Fourier partitioning breaks our function into convenient pieces, very similar to how the dynamical partition matrix partitions a space into convenient pieces.

**Example 3.1.** For example, let \((X, \omega)\) be a lattice surface that can be partitioned into \(m\) rectangles. Let \(M\) be the dynamical partition matrix associated with the map \(T(x) = g^{-1}x\). Note that this matrix tracks forward iterates, whereas the dynamical partition matrix used to express the Koopman representation tracks backward iterates. Let \(f \in L^2_0(X)\) and let \(h(f) = f_1 \oplus f_2 \oplus \cdots \oplus f_m \in \bigoplus_{i=1}^m L^2_0(T^2_i)\). Then

\[
\mathcal{F}^{-1}_M(f_1 \oplus f_2 \oplus \cdots \oplus f_m) = \\
\int_{D_{11}} f_1(x)e^{-2\pi i(n,x)}dx + \int_{D_{12}} f_1(x)e^{-2\pi i(n,x)}dx + \cdots + \int_{D_{1m}} f_1(x)e^{-2\pi i(n,x)}dx \\
\oplus \int_{D_{21}} f_2(x)e^{-2\pi i(n,x)}dx + \int_{D_{22}} f_2(x)e^{-2\pi i(n,x)}dx + \cdots + \int_{D_{2m}} f_2(x)e^{-2\pi i(n,x)}dx \\
\oplus \cdots \\
\oplus \int_{D_{m1}} f_m(x)e^{-2\pi i(n,x)}dx + \int_{D_{m2}} f_m(x)e^{-2\pi i(n,x)}dx + \cdots + \int_{D_{mm}} f_m(x)e^{-2\pi i(n,x)}dx
\]

where \(D_{ij} := i(R_i \cap g^{-1}R_j)\).

**Remark 3.3.** Similar to how we wrote the function \(f_1 \oplus f_2 \oplus \cdots \oplus f_m\) in vector form, we can also impose a matrix-form for a function that is partitioned in this way:

\[
\begin{bmatrix}
\int_{D_{11}} f_1(x)e^{-2\pi i(n,x)}dx \\
\int_{D_{21}} f_2(x)e^{-2\pi i(n,x)}dx \\
\vdots \\
\int_{D_{m1}} f_m(x)e^{-2\pi i(n,x)}dx
\end{bmatrix} = \\
\begin{bmatrix}
\int_{D_{12}} f_1(x)e^{-2\pi i(n,x)}dx \\
\int_{D_{22}} f_2(x)e^{-2\pi i(n,x)}dx \\
\vdots \\
\int_{D_{m2}} f_m(x)e^{-2\pi i(n,x)}dx
\end{bmatrix} = \\
\begin{bmatrix}
\int_{D_{1m}} f_1(x)e^{-2\pi i(n,x)}dx \\
\int_{D_{2m}} f_2(x)e^{-2\pi i(n,x)}dx \\
\vdots \\
\int_{D_{mm}} f_m(x)e^{-2\pi i(n,x)}dx
\end{bmatrix}
\]

Here, we are “summing across rows” of the matrix, and taking the direct sum of the row sum down the resulting column.

With these definitions, we can define a transpose representation.

**Definition 3.7 (Transpose Representation).** Let \(\Gamma \subset SL_2(\mathbb{Z})\) and let \(f_1 \oplus f_2 \oplus \cdots \oplus f_m \in \bigoplus_{i=1}^m L^2(\mathbb{Z}^2 \setminus 0)\). The **transpose representation** denoted \(\hat{\tau}\) is defined as follows.
\[ \hat{\pi}(g)f_1 \oplus f_2 \oplus \cdots \oplus f_m(\vec{n}) \]
\[ := \mathcal{F}_M^{-1} \omega(f_1 \oplus f_2 \oplus \cdots \oplus f_m) \cdot (g^t \oplus g^t \oplus \cdots \oplus g^t)(\vec{n}) \]

where \( M \) is the dynamical partition matrix associated with \( T(x) = g^{-1}x \), and the transpose is taken with respect to the matrix-form of the function. \( g^t \) denotes the usual transpose of the group element \( g \).

**Lemma 3.2.** Let \((X, \omega)\) be a primitive square-tiled surface decomposable into \( m \) squares. Let \( \pi \) be the induced Koopman representation of a subgroup \( \Gamma \) on \( \bigoplus_{i=1}^{m} L^2_0(T_i^2) \), and let \( \hat{\pi} \) be the transpose representation on \( \bigoplus_{i=1}^{m} l^2(Z^2 \setminus 0) \). Then \( \pi \) and \( \hat{\pi} \) are intertwined via the modular Fourier transform.

**Proof.** Let \( f \in L^2_0(X) \), note that \( h \) restricts to an isomorphism from \( L^2_0(X) \rightarrow \bigoplus_{i=1}^{m} \mathcal{L}_0(T_i) \), let \( h(f) = f_1 \oplus f_2 \oplus \cdots \oplus f_m \in \bigoplus_{i=1}^{m} L^2(T_i) \), and let \( \chi_{ij} \) be the \( ij^{th} \) entry in the dynamical partition matrix associated to \( T(x) = gx \). We need to show that \( \mathcal{F}(\pi(g)h(f)(i(x))) = \hat{\pi}(g)\mathcal{F}(h(f)(i(x))) \), i.e. that the following diagram commutes.

\[
\begin{array}{ccc}
\bigoplus_{i=1}^{m} L^2_0(T_i^2) & \xrightarrow{\mathcal{F}} & \bigoplus_{i=1}^{m} l^2_0(Z^2 \setminus 0) \\
\downarrow{\pi(g)} & & \downarrow{\hat{\pi}(g)} \\
\bigoplus_{i=1}^{m} L^2_0(T_i) & \xrightarrow{\mathcal{F}} & \bigoplus_{i=1}^{m} l^2(Z^2 \setminus 0)
\end{array}
\]

Computing \( \mathcal{F}(\pi(g)h(f)(i(x))) \):
\[
\mathcal{F}(\pi(g)h(f))(i(x)) = \mathcal{F} \left( \begin{bmatrix}
\chi_{11} f_1 \circ g^{-1} + \chi_{12} f_2 \circ g^{-1} + \cdots + \chi_{1m} f_m \circ g^{-1} \\
\chi_{21} f_1 \circ g^{-1} + \chi_{22} f_2 \circ g^{-1} + \cdots + \chi_{2m} f_m \circ g^{-1} \\
\vdots \\
\chi_{m1} f_1 \circ g^{-1} + \chi_{m2} f_2 \circ g^{-1} + \cdots + \chi_{mm} f_m \circ g^{-1}
\end{bmatrix} \right)(i(x))
\]

\[
= \int_{\tilde{T}_1^2} [\chi_{11} f_1 (g^{-1}x) + \chi_{12} f_2 (g^{-1}x) + \cdots + \chi_{1m} f_m (g^{-1}x)] e^{-2\pi i(n,x)} dx
\]
\[\oplus \int_{\tilde{T}_2^2} [\chi_{21} f_1 (g^{-1}x) + \chi_{22} f_2 (g^{-1}x) + \cdots + \chi_{2m} f_m (g^{-1}x)] e^{-2\pi i(n,x)} dx
\]
\[\oplus \cdots
\]
\[\oplus \int_{\tilde{T}_m^2} [\chi_{m1} f_1 (g^{-1}x) + \chi_{m2} f_2 (g^{-1}x) + \cdots + \chi_{mm} f_m (g^{-1}x)] e^{-2\pi i(n,x)} dx
\]

where we use the change of variables \( y = g^{-1}x \) in the second to last expression. In the integrals we are suppressing the isomorphism \( i \) that moves the dynamics from \((X, \omega)\) to the disjoint tori, until the last expression.

Computing \( \mathcal{F}(g) \mathcal{F}(h(f)(i(x))) \), where \( \chi_{g^{-1}} \) is the dynamical partition matrix associated to \( T(x) = g^{-1}x \) and \( D_{ij} = i(R_i \cap g^{-1}R_j) \):
Proof. Firstly, the operator \( \hat{\pi}(g) \) is continuous since we can express it as the composition of three continuous operators: \( \hat{\pi}(g) = \mathcal{F}(\pi(g)) \mathcal{F}^{-1} \).

Secondly, since the operator is intertwined with the induced Koopman representation, it inherits the property of respecting the group multiplication.

\[
\hat{\pi}(gh) = \mathcal{F}(\pi(gh)) \mathcal{F}^{-1} = \mathcal{F}(\pi(g) \pi(h)) \mathcal{F}^{-1} = \mathcal{F}(\pi(g)) \mathcal{F}^{-1} \mathcal{F}(\pi(h)) \mathcal{F}^{-1} = \hat{\pi}(g) \hat{\pi}(h)
\]

**Proposition 3.2.** The transpose representation is a representation.

**Proof.** Firstly, the operator \( \hat{\pi}(g) \) is continuous since we can express it as the composition of three continuous operators: \( \hat{\pi}(g) = \mathcal{F}(\pi(g)) \mathcal{F}^{-1} \).

Secondly, since the operator is intertwined with the induced Koopman representation, it inherits the property of respecting the group multiplication.

\[
\hat{\pi}(gh) = \mathcal{F}(\pi(gh)) \mathcal{F}^{-1} = \mathcal{F}(\pi(g) \pi(h)) \mathcal{F}^{-1} = \mathcal{F}(\pi(g)) \mathcal{F}^{-1} \mathcal{F}(\pi(h)) \mathcal{F}^{-1} = \hat{\pi}(g) \hat{\pi}(h)
\]

4. **The \( L^2 \)-Norm of the Koopman Representation**

In this section, we will prove Theorem 1.2.

**Proof.** Let \( \pi_0^i(g) \) denote the induced Koopman representation on \( \bigoplus_{i=1}^m L^2_0(\mathbb{T}^2) \).

Since \( h : L^2_0(X) \to \bigoplus_{i=1}^m L^2(\mathbb{T}^2) \) is an isometric isomorphism and \( \pi(g)f = f \) for constant functions \( f \), \( ||\pi(g)|| = ||\pi_0(g)|| = ||\pi_0^i(g)|| \).
Let $\hat{\pi}(g)$ be the transpose representation. By Lemma 3.2, $\pi^0_0(g)$ and $\hat{\pi}(g)$ are intertwined by the modular Fourier transform, and we can conclude that for each $g$, $||\pi^0_0(g)|| = ||\hat{\pi}(g)||$. Thus, $||\pi_0(g)|| = ||\hat{\pi}_0(g)||$. If we average the representation over $\mu$, the measure on $\Gamma$, we have the following:

$$||\pi_0(\mu)|| = ||\hat{\pi}_0(\mu)||$$

Next, let $\Gamma^t$ denote the group whose elements consists of all the transposed elements of $\Gamma$, and let $\mu^t$ be the measure on $\Gamma^t$ that is the pullback of the measure $\mu$ under this transpose map. Notice that the transpose representation induces an action of $\Gamma^t$ (by left multiplication) on the disjoint set of lattices, so we can decompose the space into orbits. Pick a vector $v_i$ out of each orbit, and we have the following:

$$\bigcup_{i=1}^m \mathbb{Z}^2 \setminus 0 = \bigcup_i \Gamma^t/\text{Stab}(v_i)$$

This allows us to write the representation as a direct sum of representations on each of these orbits:

$$\hat{\pi}_0 = \bigoplus \hat{\pi}_{\Gamma^t/\text{Stab}(v_i)}$$

and we can conclude $||\hat{\pi}(\mu)|| = \sup_i ||\hat{\pi}_{\Gamma^t/\text{Stab}(v_i)}||$.

Notice that the stabilizer subgroups are parabolic subgroups, hence amenable, so we can apply Kesten’s Theorem.

**Theorem 4.1** (Kesten [13]). Let $\mu$ be a uniform measure on some generating set $S$ of $\Gamma$. If $H$ is amenable, then

$$||\lambda_\Gamma(\mu)|| = ||\pi_{\Gamma/H}||.$$

Thus, $||\hat{\pi}(\mu)|| = ||\lambda_\Gamma(\mu^t)||$. Define a continuous operator $\rho : L^2(\Gamma, \mu) \to L^2(\Gamma^t, \mu^t)$ where $\rho(f)(g) = f(t^*g^t)$, where $t$ is the transpose operator on the group. Since $\rho \circ \lambda_\Gamma(\mu) = \lambda_{\Gamma^t}(\mu^t) \circ \rho$, we have $||\lambda_{\Gamma^t}(\mu^t)|| = ||\lambda_\Gamma(\mu)||$, as desired. ■

**References**

[1] J. Athreya, *Logarithm laws and shrinking target properties*, Proc. Indian Acad. Sci. 119, no. 4 (2009), 541-557.
[2] J. Athreya and G. Margulis, *Logarithm laws for unipotent flows, I*, Journal of Modern Dynamics 3, no. 3 (2009), 399-378.
[3] ______, *Logarithm laws for unipotent flows, II*, Journal of Modern Dynamics 11 (2017), 1-16.
[4] R. Beardon, *The exponent of convergence of Poincaré series*, Proceedings of the London Mathematical Society s3-18, Issue 3 (1968), 461-483.
[5] D. Dolgopyat, F. Fayad, and S. Liu, *Multiple Borel Cantelli lemma in dynamics and multilog law for recurrence*, preprint (2020).
[6] D. Dolgopyat and F. Fayad, *Limit theorems for toral translations*, preprint (2020).
[7] B. Fayad, *Mixing in the absence of the shrinking target property*, Bull. London Math. Soc. 38 no. 5 (2006), 829-838.
[8] V. Finkelshtein, *Diophantine properties of groups of toral automorphisms*, preprint (2016).
[9] E. Gutkin and C. Judge, *The geometry and arithmetic of translation surfaces with applications to polygonal billiards*, Mathematical Research Letters 3 (1996), 391-403.
[10] R. Hill and S. Velani, *The ergodic theory of shrinking targets*, Inventiones Mathematicae 119 (1995), 175-198.
[11] L. Hillairet, *Spectral decomposition of square-tiled surfaces*, Mathematische Zeitschrift vol. 260, no. 2 (2008).
[12] P. Hubert and T. Schmidt, *An introduction to Veech surfaces*, Handbook of Dynamical Systems vol. 1B (2006).

[13] H. Kesten, *Symmetric random walks on groups*, Transactions of the American Mathematical Society 92, no. 2 (1959), 336-354.

[14] D. Kleinbock and G. Margulis, *Logarithm laws for flows on homogeneous spaces*, Inventiones Mathematicae 138, no. 3 (1999), 451-494.

[15] C. Matheus, *Three lectures on square-tiled surfaces* (2018).

[16] D. Kleinbock and G. Margulis, *Logarithm laws for flows on homogeneous spaces*, Inventiones Mathematicae 138, no. 3 (1999), 451-494.

[17] C. McMullen, *Hausdorff dimension and conformal dynamics I: strong convergence of Kleinian groups*, J. Differential Geom. 51, no. 3 (1999), 471-515.

[18] S.J. Patterson, *The exponent of convergence for Poincaré series*, Monatshefte für Mathematik 82 (1976), 297-315.

[19] W. Philipp, *Some metrical theorems in number theory*, Pacific Journal of Mathematics vol. 20, no. 1 (1967).

[20] D. Sullivan, *The density at infinity of a discrete group of hyperbolic motions*, Publications mathématiques de l'I.H.É.S. 50 (1979), 171-202.

[21] __________, *Disjoint spheres, approximation by imaginary quadratic numbers, and the logarithm law for geodesics*, Acta Math. 149 (1982), 215-237.

[22] J. Tseng, *More remarks on shrinking target properties* (2008).

[23] A. Wright, *Translation surfaces and their orbit closures: An introduction for a broad audience*, EMS Surv. Math. Sci. (2015).

[24] W.A. Veech, *Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards*, Inventiones Mathematicae 97 (1989), 553-583.

[25] D. Zmiaikou, *Origamis and permutation groups*, PhD Thesis, University Paris-Sud (2011).

A. Zorich, *Flat surfaces*, Frontiers in Number Theory, Physics, and Geometry vol. 1 (2006).

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