Implementation of Kalman filter algorithm on reduced models using Linear Matrix Inequality method and its application to measurement water level

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Abstract. This paper presents the model reduction and estimation of the state variables of the water level system using the Linear Matrix Inequality (LMI) method and the Kalman filter algorithm. We assume the system is asymptotic stable, controllable and observable, then we reduce it by LMI method. The reduced system obtained is a system that remains asymptotic stable, controllable, and observable. The reduction error using LMI method is smaller than the reduction error using Balanced Truncated (BT) method and Singular Perturbation Approximation (SPA) method. Next, we implemented the Kalman filter algorithm in the original system and the system was reduced by LMI method.

1. Introduction

Kalman filter (KF) algorithm is a method for estimating stochastic dynamic systems that combine models with measurement data. The latest measurement data is an important part of the KF algorithm because the latest data will correct the prediction results, so the estimation results are always close to the actual conditions [1,2]. Estimation using KF has been carried out, among them are to estimate river water level [3,4], environmental problems [5], heat distribution [6], groundwater pollution [7], and many others.

The order of a system influences computational time. Hence, to reduce computing time, high order systems are replaced by simple systems with smaller orders without significant errors. Therefore to reduce computational time, it can be done by replacing a high-order system with a simple system with smaller orders without significant errors. The system with smaller orders is called a reduced model [8]. There are many model reduction methods, including the Balanced Truncation (BT) method [9,10,11], the Singular Perturbation Approximation (SPA) method [4,12], and the Linear Matrix Inequality (LMI) method [13, 14,15, 8, 16,17,18]. The LMI method produces an error reduction, that is measured by the $\mathcal{H}_\infty$ norm, much smaller than the upper bounds of the error of model reduction [6].

Research that combines model reduction methods with estimation methods to obtain accurate estimation results and short computation time has also been carried out, such as combining KF algorithm and BT methods [9,11], combining KF algorithm and SPA methods [4], also combining KF
algorithm and LMI methods [6]. This paper presents an estimation of the river water level using a combination of KF algorithm and LMI method.

2. Mathematical Model of The Shallow Water

In this section, we present the shallow water equation as follows [4]

\[
\frac{\partial h}{\partial t} + D \frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial t} + g \frac{\partial h}{\partial x} + C_f u = 0 = 0
\]

(1)

with initial conditions:

\[h(x, 0) = 2 + \sin(2\pi x), \quad u(x, 0) = 0\]

(2)

and boundary conditions:

\[h(0, t) = h(x - 1, t), \quad h(L, t) = h(2, t), \quad u(0, t) = u(x - 1, t), \quad u(L, t) = u(2, t),\]

(3)

where \(h(x, t)\) is the water level above the reference plane at position \(x\) and time \(t\), \(u(x, t)\) is the average current velocity, \(D\) is the water depth above the reference point, \(g\) is the gravitational acceleration and \(C_f\) is a friction constant.

Before we apply reduction model to the water level problem, we discretize the water shallow equation [4]. The discrete form of equation (1) is

\[h^{k+1}_i = h^k_i - a(u^k_{i+1} - u^k_{i-1}) + c(h^k_{i+1} - 2h^k_i + h^k_{i-1}), \quad u^{k+1}_i = du^k_i - b(h^k_{i+1} - h^k_{i-1}) + c(u^k_{i+1} - 2u^k_i + u^k_{i-1})\]

(4)

with

\[a = \frac{\Delta t}{\Delta x} (1 - C_f \Delta t), \quad b = \frac{g \Delta t}{2 \Delta x^2}, \quad c = \frac{g \Delta t^2}{2 \Delta x^2}, \quad d = (1 - 2C_f \Delta t).\]

Generate terms (4) for \(i = 0, 1, 2, \ldots, N - 1\). The state space system for equation (4) is

\[x_{k+1} = Ax_k + Bu_k\]

(5)

where

\[
\begin{bmatrix}
h_1 \\
u_1 \\
h_2 \\
u_2 \\
\vdots \\
h_{N-1} \\
u_{N-1}
\end{bmatrix}^{k+1}, \quad
\begin{bmatrix}
h_1 \\
u_1 \\
h_2 \\
u_2 \\
\vdots \\
h_{N-1} \\
u_{N-1}
\end{bmatrix}^k, \quad
\begin{bmatrix}
u_0 \\
h_0 \\
u_0 \\
h_N \\
u_0
\end{bmatrix}
\]

and

\[y_k = Cx_k + Du_k\]

(6)

with

\[
C =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]
From equations (5) and (6), we get the system \((A, B, C, D)\). We assume the system is asymptotic stable, controllable, and observable.

3. **Model Reduction Using Linear Matrix Inequality Method**

The following theorem states the necessary and sufficient conditions for the existence of reduction model with LMI method.

**Theorem 1.**\(^{[14]}\) Let \(G(z) \in \mathcal{RH}_\infty\) be the transfer function of the system \((A, B, C, D)\) and the realization of the space of the system is minimal, i.e.

\[
G(z) = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix},
\]

then there is an reduced system \((A_r, B_r, C_r, D_r)\) with the transfer function \(G_r(z) \in \mathcal{RH}_\infty\) that satisfy \(\| G(z) - G_r(z) \|_\infty < \gamma\) if and only if there is \(X_{11} \in S_n, P_{11} \in S_n, P_{12} \in \mathbb{R}^{n \times r},\) and \(P_{22} \in S_r\) which satisfy the following matrix inequalities:

i. 
\[
-X_{11} + AX_{11}A^T + \frac{1}{\gamma^2} BB^T < 0
\]

ii. 
\[
-P_{11} + A^T P_{11} A + C^T C < 0
\]

with 
\[
X_{11} = (P_{11} - P_{12} P_{22}^{-1} P_{12}^T)^{-1}
\]

and \(S_n\) denotes the set of positive definite matrices of size \(n \times n\).

**Definition 2.**\(^{[14]}\) The controllability gramian of system \((A, B, C, D)\) is 
\[
M := \sum_{k=0}^{\infty} A^k BB^T (A^T)^k.
\]

The observability gramian of system \((A, B, C, D)\) is 
\[
N := \sum_{k=0}^{\infty} (A^T)^k C^T CA^k.
\]

The gramians \(M\) and \(N\) are positive definite matrices and unique solutions from the following Lyapunov equations:

\[
A MA^T + BB^T - M = 0,
\]

and

\[
A^T NA + C^T C - N = 0.
\]

**Definition 3.**\(^{[14]}\) The Hankel singular value of the system \((A, B, C, D)\) with the transfer function \(G(z)\) is defined as 
\[
\sigma_i = \sqrt{\lambda_i(MN)}
\]

with \(\lambda_i(MN)\) declaring the largest eigenvalue of the matrix \(MN\) for \(i = 1, 2, ..., n\).

We assume that \(G(z) = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}\) is balanced, so

\[
A \Sigma A^T + BB^T - \Sigma = 0,
\]

and

\[
A^T \Sigma A + C^T C - \Sigma = 0,
\]

with \(\Sigma = \text{diag}(\sigma_1 l_1, \sigma_2 l_1, \sigma_2 l_1 + l_2 k_2, ..., \sigma_m l_{k_1} + \cdots + l_m k_m),\)

\(\sigma_1 > \cdots > \sigma_i > \sigma_{i+1} > \cdots > \sigma_m > 0\),

\(k_i\) is the multiplicity of \(\sigma_i\) and \(k_1 + \cdots + k_i + k_{i+1} + \cdots + k_m = n\).
Based on Theorem 1, a lower limit of the error of the model order reduction results can be derived. The following theorem states about the lower limit of \( \|G(z) - G_r(z)\|_\infty \).

**Theorem 4.** [14]. Let \( G(z) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{RH}_\infty \) be the transfer function of the system \((A, B, C, D)\) with which has the singular value Hankel \( \sigma_1 \geq \cdots \geq \sigma_r \geq \sigma_{r+1} \geq \cdots \geq \sigma_n > 0 \). Then, for all reduced system \((A_r, B_r, C_r, D_r)\) which has \( r < n \) with the transfer function \( G_r(z) = \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix} \in \mathcal{RH}_\infty \) we have

\[
\|G(z) - G_r(z)\|_\infty \geq \sigma_{r+1}. \tag{8}
\]

Furthermore, the infimum of \( \|G(z) - G_{n-k_m}(z)\|_\infty \) is stated in the following theorem.

**Theorem 5.** [14]. Let \( G(z) \in \mathcal{RH}_\infty \) be the transfer function of the system \((A, B, C, D)\) which has the Hankel singular value \( \sigma_1 > \cdots > \sigma_2 > \sigma_{i+1} > \cdots > \sigma_m > 0 \), with \( \sigma_i \) as many as \( k_1 \), \( \sigma_2 \) as many as \( k_2 \), and so on until \( \sigma_m \) as many as \( k_m \) so that \( k_1 + k_2 + \cdots + k_m = n \). Then, for arbitrary \( \gamma > \sigma_m \), there exists an reduced system \((A_{n-k_m}, B_{n-k_m}, C_{n-k_m}, D_{n-k_m})\) which has the order \( n - k_m \) with the transfer function \( G_{n-k_m}(z) \in \mathcal{RH}_\infty \) that satisfies

\[
\|G(z) - G_{n-k_m}(z)\|_\infty < \gamma. \tag{9}
\]

One important implication of Theorem 5 is that, in the case where \( r = n - k_m \), we can fix the matrix variable \( P_{11} \) and \( P_{22} \) to be constant, with

\[
P_{12} = \begin{bmatrix} l_{n-k_m} \\ 0_{k_m \times (n-k_m)} \end{bmatrix}, \quad P_{22} = \text{diag} \left( (\sigma_1 - \frac{\sigma_2}{\sigma_1})^{-1} I_{k_1}, \ldots, (\sigma_{m-1} - \frac{\sigma_m}{\sigma_{m-1}})^{-1} I_{k_{m-1}} \right) > 0 \tag{10}
\]

The reduced system \((A_r, B_r, C_r, D_r)\), \( r = n - k_m \) which has the order \( n - k_m \) with the transfer function \( G_{n-k_m}(z) \) that minimizes \( \|G(z) - G_{n-k_m}(z)\|_\infty \) can be obtained by the following theorem.

**Theorem 6.** [14]. The reduced system \((A_r, B_r, C_r, D_r)\), \( r = n - k_m \) which has the order \( n - k_m \) with the transfer function \( G_{n-k_m}(z) \) that minimizes \( \|G(z) - G_{n-k_m}(z)\|_\infty \) can be obtained by the following procedure:

1. Minimize \( \gamma^2 \) subject to the LMIs:

\[
\begin{bmatrix}
P_{11} & P_{12}Q_{22} \\ Q_{22}P_{12}^T & Q_{22}
\end{bmatrix} > 0, \quad -(P_{11} - P_{12}Q_{22}P_{12}^T) (P_{11} - P_{12}Q_{22}P_{12}^T)^T \geq 0, \quad A^T(P_{11} - P_{12}Q_{22}P_{12}^T) - (P_{11} - P_{12}Q_{22}P_{12}^T)^T B \geq 0, \quad B^T(P_{11} - P_{12}Q_{22}P_{12}^T) \geq 0, \quad -P_{11} + A^T P_{12} A + C^T C \leq 0,
\]

where \( P_{11} \in \mathcal{S} \) and \( Q_{22} \in \mathcal{S} \) are matrix variables to be determined whereas \( P_{12} \in \mathbb{R}^{n \times (n-k_m)} \) is a constant matrix given by \( P_{12} = \begin{bmatrix} l_{n-k_m} \\ 0_{k_m \times (n-k_m)} \end{bmatrix} \). For the subsequent step, define \( \bar{P} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & Q_{22}^{-1} \end{bmatrix} \) and denote the optimal value of \( \gamma \) by \( \gamma_{\text{opt}} \).
2. Obtained \((\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r, \mathbf{D}_r)\) by solving (12), where \(P\) is fixed to \(\bar{P}\) and \(\gamma\) to \(\gamma_{opt}\):

\[
\begin{bmatrix}
-P_{11} & -P_{12} & 0 & \mathbf{A}^T P_{11} & \mathbf{A}^T P_{12} & \mathbf{C}^T \\
-P_{22} & 0 & \mathbf{A}^T P_{12} & \mathbf{A}^T P_{22} & -\mathbf{C}^T \\
0 & \mathbf{B}^T P_{11} + \mathbf{B}^T P_{12} & \mathbf{B}^T P_{12} + \mathbf{B}^T P_{22} & \mathbf{D}^T - \mathbf{D}_r^T \\
0 & \mathbf{B}^T P_{11} + \mathbf{B}^T P_{12} & \mathbf{B}^T P_{12} + \mathbf{B}^T P_{22} & \mathbf{D}^T - \mathbf{D}_r^T \\
0 & \mathbf{B}^T P_{11} + \mathbf{B}^T P_{12} & \mathbf{B}^T P_{12} + \mathbf{B}^T P_{22} & \mathbf{D}^T - \mathbf{D}_r^T \\
0 & \mathbf{B}^T P_{11} + \mathbf{B}^T P_{12} & \mathbf{B}^T P_{12} + \mathbf{B}^T P_{22} & \mathbf{D}^T - \mathbf{D}_r^T \\
\end{bmatrix} < 0. \tag{12}
\]

By the similarity transformation \(\tilde{A}_r := P_{22} \mathbf{A}_r P_{22}^{-1}, \quad \tilde{B}_r := P_{22} \mathbf{B}_r, \quad \text{and} \quad \tilde{C}_r := \mathbf{C}_r P_{22}^{-1}\), we see that there exist \((\mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r, \mathbf{D}_r)\) that satisfy (11) if and only if

\[
\begin{bmatrix}
-P_{11} & -P_{12} & 0 & \mathbf{A}^T P_{11} & \mathbf{A}^T P_{12} & \mathbf{C}^T \\
-P_{22} & 0 & \mathbf{A}^T P_{12} & \mathbf{A}^T P_{22} & -\mathbf{C}^T \\
0 & \mathbf{B}^T P_{11} + \mathbf{B}^T P_{12} & \mathbf{B}^T P_{12} + \mathbf{B}^T P_{22} & \mathbf{D}^T - \mathbf{D}_r^T \\
0 & \mathbf{B}^T P_{11} + \mathbf{B}^T P_{12} & \mathbf{B}^T P_{12} + \mathbf{B}^T P_{22} & \mathbf{D}^T - \mathbf{D}_r^T \\
0 & \mathbf{B}^T P_{11} + \mathbf{B}^T P_{12} & \mathbf{B}^T P_{12} + \mathbf{B}^T P_{22} & \mathbf{D}^T - \mathbf{D}_r^T \\
0 & \mathbf{B}^T P_{11} + \mathbf{B}^T P_{12} & \mathbf{B}^T P_{12} + \mathbf{B}^T P_{22} & \mathbf{D}^T - \mathbf{D}_r^T \\
\end{bmatrix} < 0. \tag{13}
\]

By the congruence transformation with \(\text{diag}(I, Q_{22}, I, I, Q_{22}, I)\) where \(Q_{22} = P_{22}^{-1}\), the above inequality reduces to

\[
\begin{bmatrix}
-P_{11} & -P_{12} Q_{22} & 0 & \mathbf{A}^T P_{11} & \mathbf{A}^T P_{12} Q_{22} & \mathbf{C}^T \\
0 & -Q_{22} & 0 & \mathbf{A}^T Q_{22} P_{12}^T & \mathbf{A}^T Q_{22} P_{22} & -\mathbf{C}^T \\
0 & \mathbf{B}^T P_{11} + \mathbf{B}^T Q_{22} P_{12} & \mathbf{B}^T P_{12} + \mathbf{B}^T Q_{22} & \mathbf{D}^T - \mathbf{D}_r^T \\
0 & \mathbf{B}^T P_{11} + \mathbf{B}^T Q_{22} P_{12} & \mathbf{B}^T P_{12} + \mathbf{B}^T Q_{22} & \mathbf{D}^T - \mathbf{D}_r^T \\
0 & \mathbf{B}^T P_{11} + \mathbf{B}^T Q_{22} P_{12} & \mathbf{B}^T P_{12} + \mathbf{B}^T Q_{22} & \mathbf{D}^T - \mathbf{D}_r^T \\
0 & \mathbf{B}^T P_{11} + \mathbf{B}^T Q_{22} P_{12} & \mathbf{B}^T P_{12} + \mathbf{B}^T Q_{22} & \mathbf{D}^T - \mathbf{D}_r^T \\
\end{bmatrix} < 0. \tag{14}
\]

Inequality (14) is an LMI with respect to the matrix variables \(P_{11}, Q_{22}\), and \(\tilde{A}_r := Q_{22} \tilde{A}_r, \tilde{B}_r := Q_{22} \tilde{B}_r,\) \(\tilde{C}_r := \tilde{C}_r \tilde{C}_r^T\). Furthermore, the reduced model can be constructed with

\[G_r(z) = \begin{bmatrix} Q_{22} & \tilde{A}_r \\ \tilde{C}_r & \tilde{D}_r \end{bmatrix}. \tag{15}\]

4. Kalman Filter Algorithm

Estimation using KF algorithm is done by predicting state variable in dynamic systems which are then corrected using measurement data [2]. In system modeling there is no mathematical model of a perfect system, because there are noise factors in each system. Therefore, it is necessary to add stochastic factors to the deterministic system \((A, B, C, D)\) in the form of noise system and noise measurement, in order to obtain the following stochastic dynamic system:

\[
x_{k+1} = \mathbf{A} x_k + \mathbf{B} u_k + \mathbf{w}_k, \tag{16}
\]

\[
z_k = \mathbf{C} x_k + \mathbf{D} u_k + \mathbf{v}_k, \tag{17}
\]

with \(w_k\) and \(v_k\) are noise system and noise measurement, and each is a stochastic scale. Noise system and noise measurement are assumed to be normally distributed with zero mean and covariance, respectively \(Q_k\) and \(R_k\).

The KF algorithm for discrete stochastic dynamic systems [1] can be written as follows:

i. Given

\[
x_{k+1} = \mathbf{A} x_k + \mathbf{B} u_k + \mathbf{w}_k, \tag{18}
\]

\[
z_k = \mathbf{C} x_k + \mathbf{D} u_k + \mathbf{v}_k \tag{19}
\]

\[x_0 \sim N(\bar{x}_0, P_{x_0}); \quad w_k \sim N(0, Q); \quad v_k \sim N(0, R).
\]
ii. Initialization
\[ P_0 = P_{x_0} : \hat{x}_0 = \bar{x}_0. \]  

(20)

iii. Prediction Stage (time update)

Error Covariance: \[ P_{k+1}^- = A P_k A^T + GG^T. \]  

(21)

Estimation: \[ \hat{x}_{k+1} = A \hat{x}_k + B u_k. \]  

(22)

iv. Correction Stage (measurement update)

Error Covariance: \[ P_{k+1}^- = (P_{k+1}^-)^{-1} + C^T R^{-1} C \]  

(23)

Estimation: \[ \hat{x}_{k+1} = \hat{x}_{k+1}^- + P_{k+1}^- C^{-1} (z_{k+1} - C \hat{x}_{k+1}^-). \]  

(24)

Kalman gain \[ K_{k+1} = P_{k+1}^- C^T (C P_{k+1}^- C^T + R)^{-1}. \]  

(25)

Error Covariance: \[ P_{k+1}^- = (I - K_{k+1} C) P_{k+1}^- \]  

(26)

Estimation: \[ \hat{x}_{k+1} = \hat{x}_{k+1}^- + K_{k+1} (z_{k+1} - C \hat{x}_{k+1}^-). \]  

(27)

v. Return to the Prediction Stage.

5. Simulation Result

In this simulation we use the parameter values as follows

\[ D = 10 \text{ m}, \quad g = 9.8 \frac{m}{s^2}, \quad C_f = 0.0002, \quad x = 1, \quad t = 0.2, \quad n = 20 \]

The BT method and SPA method guarantee the upper bounds of the error \( \|G(z) - G_r(z)\|_\infty \) is \( 2(\sigma_{r+1} + \cdots + \sigma_n) \), and then using the LMI method, we can find the supremum of the error \( \|G(z) - G_r(z)\|_\infty \), denoted by nonnegative scalar \( \gamma \) which satisfy \( \sigma_{r+1} \leq \|G(z) - G_r(z)\|_\infty < \gamma \leq 2(\sigma_{r+1} + \cdots + \sigma_n) \).

| \( R \) | \( \sigma_{r+1} \) | LMI | BT | SPA | \( 2(\sigma_{r+1} + \cdots + \sigma_n) \) |
|------|---------|------|----|-----|----------------|
| 4    | 1.8864  | 2.1  | 2.1348 | 15.2376 | 56.9248 |
| 5    | 1.8760  | 1.97 | 2.0862 | 9.6388  | 36.0096 |
| 6    | 1.7235  | 1.89 | 1.9612 | 9.6388  | 32.2576 |
| 7    | 1.7202  | 1.89 | 1.9612 | 9.6388  | 28.8106 |
| 8    | 1.5419  | 1.7  | 1.8254 | 9.7475  | 25.3702 |
| 9    | 1.5392  | 1.7  | 1.8254 | 9.7475  | 22.2864 |
| 10   | 1.3386  | 1.4  | 1.5084 | 7.9397  | 19.208  |
| 12   | 1.1746  | 1.2  | 1.5431 | 12.228  | 13.883  |

Table 1 show that the error from the model reduction using the LMI method is the smallest error compared to the error from the model reduction using the BT method and the SPA method. Furthermore, we obtain that system that have been reduced using LMI method, BT method and SPA method are stable, controlled, and observed systems.

We also compare the estimation results on the original system and the system that has been reduced by the LMI method.
From Figure 1, it can be seen that the original system estimation with the Kalman filter algorithm produces a very good estimate because the plot of the estimation result of the variable is very close to the real state variable. This is also strengthened by the results of the plot of the error covariance which show that the results of the error covariance are convergent (Table 2).

Figure 2. Estimation of the reduced system with orders 10, 12, 14, and 16 using the Kalman filter algorithm
From Figures 2, it can be seen that the implementation of the KF algorithm in the reduced system produces a very good estimation too because the plot of the estimated variable results are very close to the real state variable (Table 2).

Table 2. Norm Covariance Error of original system and the reduced system.

| System     | Norm Covariance Error |
|------------|----------------------|
| Original   | 0.043666391527211    |
| 7th Order  | 0.012426454408909    |
| 8th Order  | 0.013108832470840    |
| 9th Order  | 0.012805814055405    |
| 10th Order | 0.053436212714418    |
| 12th Order | 0.231132005856672    |
| 14th Order | 5.530756570303712    |
| 16th Order | 8.920554100709582    |
| 18th Order | 34.321318428766098   |

Table 2 shows the 7th order system has the smallest norm covariance error compared to the others.

6. Conclusion
From the results of the analysis, we obtain that error of the model reduction using LMI method is smaller than error of model reduction using BT method and SPA method. Furthermore, we obtain that system that have been reduced using LMI method, BT method and SPA method are stable, controlled, and observed systems. We also find that the estimation using the Kalman Filter algorithm in the original system and the system that has been reduced by the LMI method is a very good estimate close to the actual state variable.

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