Discrete spectrum of kink velocities in Josephson structures: the nonlocal double sine-Gordon model

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Abstract

We study a model of Josephson layered structure which is characterized by two peculiarities: (i) superconducting layers are thin; (ii) due to suppression of superconducting states in superconducting layers the current-phase relation is non-sinusoidal and is described by two sine harmonics. The governing equation is a nonlocal generalization of double sine-Gordon (NDSG) equation. We argue that the dynamics of fluxons in the NDSG model is unusual. Specifically, we show that there exists a set of particular velocities for non-radiating fluxon propagation. In dynamics the presence of these “privileged” velocities results in phenomenon of quantization of fluxon velocities: in our numerical experiments a travelling kink-like excitation radiates energy and slows down to one of these particular velocities, taking a shape of predicted $2\pi$-kink. This situation differs from both, double sine-Gordon local model and the nonlocal sine-Gordon model, considered before. We conjecture that the set of these velocities is infinite and present an asymptotic formula for them.

Keywords: Josephson junction, double sine-Gordon equation, nonlocal Josephson electrodynamics, Josephson vortex, embedded solitons

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1. Introduction

Since mid-60’s it is known that the description of a long contact between two superconductors (Josephson junction, JJ) is based on the classical sine-Gordon equation:

$$\sin \varphi + \omega_J^2 \varphi_{tt} = \lambda_J^2 \varphi_{xx}. \tag{1}$$

Here $\varphi = \varphi(x,t)$ is the phase difference of the order parameters in the superconducting banks, $\omega_J$ is the Josephson plasma frequency, $\lambda_J$ is the Josephson
length. The first term in the left-hand side of Eq. (1) comes from the formula for supercurrent across the JJ

\[ J(\varphi) = J_c \sin \varphi, \]

where \( J_c \) is the critical Josephson current.

In mid-70's it was recognized that in some situations the second derivative term in the right-hand side of Eq. (1) should be corrected. In particular, this should be done if the London penetration depth \( \lambda_L \) becomes comparable with Josephson length \( \lambda_J \). Then Eq. (1) must be replaced by integral equation of the type

\[ \sin \varphi(x, t) + \omega J^{-2} \varphi_{tt}(x, t) = \lambda_J^2 \frac{\partial}{\partial x} \int dx' G(x, x') \varphi_{xx}'(x', t). \]

In literature Eq. (3) has been called nonlocal sine-Gordon equation [1]. Explicit form of the kernel \( G(x, x') \) depends on physical and geometrical properties of JJ. Some survey of recent results in this field (sometimes called nonlocal Josephson electrodynamics) including a list of kernels which have been used in literature can be found in [1]. It has been found that dynamics of vortex in nonlocal Josephson electrodynamics has essential peculiarities. One of the most exciting of them is forming of bound states of more then one flux quanta and the quantization of velocities of these structures. This issue has been discussed in many papers [2, 3, 4, 5, 6].

The principles of nonlocal Josephson electrodynamics can be also applied to stacked Josephson structures. In particular, in [7] it has been shown that structures consisting of alternating flat superconducting and tunnel layers (see Fig. 1) can be described by Eq. (3) with the Kac-Baker kernel \( G(x, x') \sim e^{-2|x-x'|}, \)

\[ \sin \varphi(x, t) + \omega J^{-2} \varphi_{tt}(x, t) = \Lambda \frac{\partial}{\partial x} \int dx' e^{-|x-x'|/\lambda_J} \varphi_{x'x'}(x', t), \]

where

\[ \Lambda = \frac{(\lambda_L + d)\lambda_J^2}{2\lambda_L \sqrt{(L + d) L}}, \]
λ_L is the London penetration depth, 2d is the thickness of the each tunnel layer, 2L is the thickness of the each superconducting layer and

\[ \lambda_{eff} \equiv \lambda_L \sqrt{\frac{L}{L+d}}. \]

In [7] Eq. (4) has been derived under the following assumptions: (i) superconducting layers are identical, (ii) tunnel layers are identical, (iii) vortex formation is symmetric, i.e. the phase difference and the magnetic field in all the JJs are identical, and (iv) superconducting layers are thin. The model (4) possesses many fascinating features [2]. In particular, it allows to describe travelling vortices of more then one flux quantum. At the same time, the mobility of traditional fluxons in this model is essentially reduced.

Another motivation to modify Eq. (1) is caused by non-sinusoidal character of current-phase relation. Generally speaking, Eq. (2) represents the first term of the relation [8, 9]

\[ J(\phi) = J_c \sin \phi + J_2 \sin 2\phi + J_3 \sin 3\phi + \ldots \] (5)

In many situations \( J_2 \gg J_k, k = 3, 4 \ldots \) Therefore in many studies (see e.g. [10, 11, 12, 13]) the current-phase relation has been assumed to be of the form

\[ J(\phi) = J_c \sin \phi + J_2 \sin 2\phi. \] (6)

The sign of \( J_2 \) depends on a mechanism of suppression of superconducting state in electrodes. In SIS-type junctions \( J_2 > 0 \) due to suppression of superconductivity near the tunnel barrier by a supercurrent [14, 15]. In SNINS and SFIFS junctions there exists another mechanism, associated with proximity effect [14, 15, 16] which may result in negative values of \( J_2 \).

The equation for the phase difference \( \phi \) in the case of current-phase relation (6) reads

\[ \sin \phi(x, t) + 2A \sin 2\phi(x, t) + \omega_j^{-2} \phi_{tt}(x, t) = \lambda_j^2 \phi_{xx}(x, t), \] (7)

where \( A \equiv J_2/2J_c \). This equation is the double sine-Gordon equation which has been widely discussed in both physical and mathematical literature.

In this paper we study an effect of non-sinusoidal current-phase relation on mobility of fluxons in Josephson structures with nonlocal electrodynamics. We present a model of infinite Josephson structure consisting of alternating superconducting and tunnel layers. It will be assumed that current-phase relation is of the form (6). Also we assume that S-layers are thin, and replace the second derivative term in Eq. (1) by nonlocal term. Repeating the reasoning of [7] (the details of derivation can be found in Appendix A) we arrive at the equation

\[ \sin \phi(x, t) + 2A \sin 2\phi + \omega_j^{-2} \phi_{tt} = \lambda_{eff} \frac{\partial}{\partial x} \int dx' e^{-|x-x'|/\lambda_{eff}} \phi_{x'}(x', t). \] (8)

The main output of our study can be formulated as follows. The properties of free propagation of Josephson vortices in the model (8) essentially differ from
ones in (i) the traditional sine-Gordon model, (ii) double sine-Gordon local model, Eq. (7), and (iii) nonlocal sine-Gordon model, Eq. (4). Specifically, it has been found that there exist “privileged” velocities of free fluxon propagation in radiationless regime. Contrary to the traditional sine-Gordon and double sine-Gordon equations, the set of these velocities is discrete and infinite. The nonlocal sine-Gordon equation possesses similar property, but for the vortices of topological charge greater than 1 only (these entities can be regarded as “bound states” of simple fluxons). It is worth mentioning that in the case of nonlocal double sine-Gordon model the shapes of fluxons corresponding to different “privileged” velocities are nearly the same, the difference takes place in the asymptotics of the “tails” of these vortices.

The paper is organized as follows. In Section 2 we transform Eq. (8) into a dimensionless equation which depends on two external parameters, \( \lambda \) and \( A \). We call it nonlocal double sine-Gordon equation and discuss some its features in general. In Section 3 we give arguments for quantization of fluxon velocities described by this model. Specifically, we present the dependencies of these velocities \( v_n(\lambda) \), \( n = 0, 1, \ldots \), on the parameter \( \lambda \) and give a formula for asymptotics of these dependencies. In Section 4 we report on results of numerical simulation of fluxon evolution. We show that the dynamics “feels” the velocities of this spectrum. Section 5 contains summary and discussion. Some physical details, including the derivation of the basic nonlocal equation (9), are postponed in Appendix A.

2. Preliminaries

2.1. Dimensionless form

In the dimensionless variables,

\[
\zeta = \sqrt{\frac{L + d}{\lambda L + d}} \frac{x}{\lambda J}, \quad \tau = \omega_J t
\]

Eq. (8) takes the form

\[
\sin \varphi + 2A \sin 2\varphi + \varphi_{\tau\tau} = \frac{1}{2\lambda} \frac{\partial}{\partial \zeta} \int d\zeta' e^{-|\zeta - \zeta'|/\lambda} \varphi(\zeta', \tau),
\]

where \( \varphi \equiv \varphi(\zeta, \tau) \) and

\[
\lambda \equiv \frac{\lambda_{eff}}{\lambda_J} \sqrt{\frac{L + d}{\lambda L + d}} = \frac{\sqrt{L} \lambda_J}{\lambda_J \sqrt{L_J + d}}.
\]

Eq. (9) has the energy integral, \( dW/d\tau = 0 \), where

\[
W = \int d\zeta \left[ 1 - \cos \varphi + A(1 - \cos 2\varphi) + \frac{1}{2} \left( \frac{\partial \varphi}{\partial \tau} \right)^2 \right] + \]

\[
+ \frac{1}{4\lambda} \int d\zeta_1 d\zeta_2 \exp \left\{ -\frac{\zeta_1 - \zeta_2}{\lambda} \right\} \frac{\partial \varphi(\zeta_1, \tau)}{\partial \zeta_1} \frac{\partial \varphi(\zeta_2, \tau)}{\partial \zeta_2}.
\]
This follows from the formula (A.7) (see Appendix A), which can be written as

\[ W = \frac{\phi_0 j \lambda J}{2\pi c} \sqrt{\frac{\lambda L + d}{L + d}} W. \]

We call Eq. (9) nonlocal double sine-Gordon equation (NDSG).

2.2. Kink solutions

We consider travelling wave solutions, \( \varphi(\xi) = \varphi(\zeta - v\tau) \), of Eq. (9) which satisfy the equation

\[ \sin \varphi + 2A \sin 2\varphi + v^2 \varphi_{\xi\xi} = \frac{1}{2\lambda} \frac{d}{d\xi} \int e^{-|\xi-\xi'|/\lambda} \varphi_{\xi'}(\xi') \, d\xi'. \tag{11} \]

A single Josephson vortex (fluxon) corresponds to 2\( \pi \)-kink solution of Eq. (11). It obeys the boundary conditions

\[ \lim_{\xi \to -\infty} \varphi(\xi) = 0, \quad \lim_{\xi \to +\infty} \varphi(\xi) = 2\pi. \]

2.3. Approximations

The operator \( \mathcal{L} \) defined as

\[ \mathcal{L}\varphi = \frac{1}{2\lambda} \frac{d}{d\xi} \int e^{-|\xi-\xi'|/\lambda} \varphi_{\xi'}(\xi') \, d\xi' \]

is a Fourier multiplier operator, \( \hat{\mathcal{L}}\varphi(k) = \hat{\mathcal{L}}(k) \cdot \hat{\varphi}(k) \), where \( \hat{f}(k) \) means the Fourier transform of function \( f(\xi) \) and \( \hat{\mathcal{L}}(k) \) is the symbol of operator \( \mathcal{L} \),

\[ \hat{\mathcal{L}}(k) = -\frac{k^2}{1 + k^2\lambda^2}. \tag{12} \]

In the limit \( \lambda \ll 1 \) one can replace the symbol \( \hat{\mathcal{L}}(k) \) by its Taylor approximations. One term approximation, \( \hat{\mathcal{L}}(k) \approx -k^2 \), returns us to the double sine-Gordon case

\[ \sin \varphi + 2A \sin 2\varphi = (1 - v^2)\varphi_{\xi\xi}. \tag{13} \]

For \( A > -1/4 \) and \( v^2 < 1 \) Eq. (13) admits exact 2\( \pi \)-kink solution

\[ \hat{\varphi}(\xi) = \pi + 2 \arctan \left( \frac{1}{\sqrt{1 + 4A}} \sinh \left( \frac{\sqrt{1 + 4A}}{\sqrt{1 - v^2}} \xi \right) \right). \tag{14} \]

Simple phase plane analysis shows that this 2\( \pi \)-kink solution is unique. Therefore, in the local limit the model allows for fluxons which can travel with arbitrary velocity \( v^2 < 1 \).
Two term approximation for \((12)\) reads

\[
\hat{L}(k) \approx -k^2 + \lambda^2 k^4.
\]

In this case \(2\pi\)-kink solution obeys the 4-th order ODE

\[
\sin \varphi + 2A \sin 2\varphi + (1 - v^2) \varphi_{\xi\xi} + \lambda^2 \varphi_{\xi\xi\xi\xi} = 0.
\]

(15)

After scaling of independent variable, \(\eta = \xi/\sqrt{1-v^2}\), one arrives at the equation

\[
\sin \varphi + 2A \sin 2\varphi + \varphi_{\eta\eta} + \delta^2 \varphi_{\eta\eta\eta\eta} = 0, \quad \delta = \frac{\lambda}{\sqrt{1-v^2}}.
\]

(16)

The fourth-order term in (15) becomes essential for \(\lambda \sim \sqrt{1-v^2} \ll 1\).

2.4. Embedded solitons

Let \(\lambda \neq 0\). Linearizing Eq. (11) near the equilibrium \(\varphi = 0\) and seeking for small amplitude excitations \(\varphi \propto e^{ik\xi}\) we arrive at the equation for \(k\)

\[
-v^2 k^2 + \frac{k^2}{1 + \lambda^2 k^2} = 1 + 4A.
\]

(17)

Direct calculation shows that if \(A > -1/4\) and \(v^2 < 1\) Eq. (17) has a single pair of real roots \(k = \pm k_0(\lambda)\). In terminology of [17, 18], we are in situation when the resonance prohibits propagation of localized wave for Eq. (9) and so called embedded solitons may appear. In this case, the velocity \(v\) of embedded soliton (i.e. \(2\pi\)-kink), is not arbitrary but should be “adjusted” to avoid oscillatory asymptotics of its tail due to merging with linear modes. Typically, each value \(v\) is isolated and belongs to some discrete set. This set may be empty (i.e. no localized waves propagate) or include either finite or infinite number of values. Note that in the case \(\lambda = 0\) no resonance occurs, since in this case Eq. (17) has no real roots for \(v^2 < 1\).

3. Kink solutions: results

3.1. Numerical results for Eq. (11)

In order to study \(2\pi\)-kink solutions for Eq. (11) we employ an approach described in [2, 19]. Since

\[
q(\xi) = \frac{1}{2\lambda} \int e^{-|\xi - \xi'|/\lambda} \varphi(\xi') \, d\xi'
\]

is a solution of equation \(-\lambda^2 q_{\xi\xi} + q = \varphi_{\xi}\), we replace Eq. (11) with the system

\[
\begin{align*}
v^2 \varphi_{\xi\xi} &= q_{\xi} + \sin \varphi + 2A \sin 2\varphi, \\
-\lambda^2 q_{\xi\xi} + q &= \varphi_{\xi},
\end{align*}
\]

(18) (19)
which is completely equivalent to Eq. (11). Then the study of $2\pi$-kinks for Eq. (11) may be reduced to analysis of heteroclinic separatrices which connect equilibrium states $\varphi = 0$ and $\varphi = 2\pi$ in 4D phase space of the system (18) — (19). Generically, these separatrices may exist not for any choice of parameters $\lambda$ and $v$. For a fixed $\lambda$, numerical computation of allowed velocities $v$ can be done by seeking for zeroes of some function $R_\lambda(v)$ (technical details can be found in [2, 19]). Theoretically, this approach allows to find all possible kink velocities from a given interval $[\tilde{v}_1, \tilde{v}_2]$.

Numerical study confirms the existence of embedded solitons for Eq. (11). Specifically, for a fixed $\lambda$ and $A > 0$ there is a discrete set of $2\pi$-kink solutions of Eq. (9). Each of them corresponds to its own velocity $v_n$, $n = 1, 2, \ldots$. We conjecture (see Sect. 3.3) that the spectrum of velocities is infinite and $v_n$ accumulate to zero velocity when $n \to \infty$. The curves $v_n(\lambda)$ corresponding to the three highest velocities $v_1$, $v_2$ and $v_3$ are shown in Fig. 2 (left panel). Two profiles of the $2\pi$-kinks corresponding to points A and B ($\lambda = 0.2$) are shown in right panel of Fig. 2. It is worth noting that the central parts of kink profiles for different $v$ are very close to the profile of kink (14), corresponding to the local case $\lambda = 0$. The difference between them becomes substantial in the asymptotics of their tails.

All the curves $v_n(\lambda)$ in the diagram of Fig. 2 are originated at the point $\lambda = 0$, $v = 1$. In vicinity of this point the dependence $v_n(\lambda)$ can be described by Eq. (16).

3.2. The limit case $\lambda \sim \sqrt{1-v^2} \ll 1$

If $\lambda \sim \sqrt{1-v^2} \ll 1$ and $\delta = \lambda/\sqrt{1-v^2} \sim 1$ the profile of the kink is described by Eq. (16). The dispersion relation for this case reads

$$k^2 + \delta^2 k^4 = 1 + 4A.$$  \hspace{1cm} (20)
This equation admits a pair of real roots \( k = \pm \hat{k}_0(\lambda) \) for \( A > -1/4 \). Following the “embedded soliton” approach, one can expect that kink solution may exist only for discrete set of values of governing parameter \( \delta \). Numerical study fulfilled for various values of \( A > 0 \) confirms the existence of this discrete set \( \delta_1, \delta_2, \ldots \) (see Table 2 in Sect. 3.3). Therefore, in a vicinity of the point \( \lambda = 0, v = 1 \) in the diagram in Fig. 2 the dependence \( v_n(\lambda) \) for \( n = 1, 2, \ldots \) obeys the asymptotic formula

\[
v_n(\lambda) \simeq \sqrt{1 - \frac{\lambda^2}{\delta_n}}.
\]

### 3.3. Asymptotic formula for \( v(\lambda) \)

Numerical study allows to suppose that for \( \lambda \) fixed the values of \( v_n \) tend to zero as \( n \) grows. Simultaneously, if \( v \) is fixed, the corresponding values of \( \lambda_n \) also tend to zero when \( n \) increases. To describe asymptotical properties of the set of families \( v_n(\lambda) \) one can apply the approach presented in [21]. The main statement of [21] concerns the equation

\[
G_\lambda \varphi = F(\varphi),
\]

where \( \varphi(\xi) \) and \( F(\varphi) \) are real-valued functions and \( \varphi(\xi) \) is defined on whole \( \mathbb{R} \). Assume that \( G_\lambda \) is a Fourier multiplier operator which depends continuously on real parameter \( \lambda \) and \( G_0 = \Omega^2 d^2/d\xi^2 \) where \( \Omega \) is a real number. The symbol \( \hat{G}_\lambda(k) \) of the operator \( G_\lambda \) is supposed to be an even function.

**Conjecture, [21].** Assume that

(a) \( \varphi = \varphi_+ \) and \( \varphi = \varphi_- \) are zeroes of the function \( F(\varphi) \);

(b) \( F'(\varphi_+) = F'(\varphi_-) > 0 \) and the equation

\[
\hat{G}_\lambda(k) - F'(\varphi_1) = 0,
\]

has only one pair of real roots \( k = \pm k(\lambda) \), \( k(\lambda) > 0 \) and \( k(\lambda) \to \infty \) as \( \lambda \to 0 \);

(c) the equation \( \Omega^2 d^2 \varphi/d\xi^2 = F(\varphi) \) has a kink solution \( \hat{\varphi}(\xi) \), such that \( \hat{\varphi}(\xi) \to \varphi_+ \) as \( \xi \to +\infty \) and \( \hat{\varphi}(\xi) \to \varphi_- \) as \( \xi \to -\infty \) and \( \hat{\varphi}'(\xi) \) is an even function;

(d) the solution \( \hat{\varphi}(\xi) \) can be continued into the complex plane and the closest to the real axis singularities of \( \hat{\varphi}(\xi) \) in the upper-half plane form a pair which is symmetric with respect to imaginary axis,

\[
z_\pm = \pm \alpha + i\beta, \quad \alpha, \beta > 0.
\]

Then one can expect an infinite sequence of parameter values \( \lambda = \lambda_n, n = 1, 2, \ldots \), such that for each of them Eq. (21) has a kink solution and this sequence obeys the asymptotics

\[
k(\lambda_n) \sim (n\pi + \theta_0)/\alpha,
\]
where \( k(\lambda) \) is the real root of Eq. (22), \( \alpha \) is the modulus of real part of singularity \( z_\pm \) and \( \theta_0 \) is a constant.

Up to the moment we have no rigorous proof of the Conjecture stated above (some heuristic arguments are presented in [21]). At the same time the asymptotic formula (24) has been strongly confirmed numerically for various examples of operator \( G_\lambda \) and the function \( F(\varphi) \). Note that the type of the singularity of \( \tilde{\varphi}(\xi) \) is not specified, it can be pole, logarithmic or transcendental branching point etc.

Let us show that the Conjecture gives remarkable good results for Eq. (9).

1. Let us check the points (a) — (d) for Eq. (9). The equilibrium states are \( \varphi_- = 0 \) and \( \varphi_+ = 2\pi \). The operator \( G_\lambda \) has the symbol

\[
\hat{G}_\lambda = -v^2 k^2 + \frac{k^2}{1 + \lambda^2 k^2}
\]

and \( G_0 = (1 - v^2)d^2\varphi/d\xi^2 \), i.e \( \Omega = \sqrt{1 - v^2} \). The relation (22) has the form (17). The positive root \( k(\lambda) \) is unique and has the asymptotics

\[
k(\lambda) \simeq \sqrt{1 - v^2 - \lambda v}, \quad \text{as} \quad \lambda \to 0.
\]

The \( 2\pi \)-kink solution of the equation \((1 - v^2)d^2\varphi/d\xi^2 = F(\varphi)\) has the form (14).

If \( A > 0 \) then the closest to the real axis singularities of (14) in the upper complex half-plane are \( z_\pm = \pm \alpha + i\beta \) where

\[
\alpha = \frac{\sqrt{1 - v^2}}{2\sqrt{1 + 4A}} \arccosh(1 + 8A), \quad \beta = \frac{\pi \sqrt{1 - v^2}}{2\sqrt{1 + 4A}}
\]

Then, according to Conjecture one can expect that for \( v \) fixed

\[
\lambda_n \sim \frac{(1 - v^2)\arccosh(1 + 8A)}{v\sqrt{1 + 4A((2n - 1)\pi + \theta_0)}}, \quad n \to \infty,
\]

where \( \tilde{\theta}_0 = 2\theta_0 - \pi \), see (24). The Table 1 shows the values of \( \lambda_n \) computed numerically in comparison with ones calculated by the formula (25) with \( \tilde{\theta}_0 = 0 \) (the arguments about this choice of \( \tilde{\theta}_0 \) can be found in [21]). It follows from Table 1 that the correspondence between numerical and asymptotical values is quite good for large enough \( n \). However, for \( n \) large the function \( R_\lambda(v) \) (mentioned in Sect 3.1) is small, and there appears a difficulty of localizing its zeroes. This explains empty entries in the Table 1 for \( n = 5, 6 \) and \( v = 0.5 \).

2. If \( A > 0 \) for Eq. (16) the points (a) — (d) are also fulfilled. The equilibrium states are \( \varphi_- = 0 \) and \( \varphi_+ = 2\pi \). The operator \( G_\delta \) has the symbol

\[
\hat{G}_\delta = k^2 + \delta^2 k^4
\]

and \( G_0 = d^2\varphi/d\eta^2 \). The relation (22) reads

\[
k^2 + \delta^2 k^4 = 1 + 4A.
\]
Table 1: Comparison of asymptotical and numerical values of $\lambda$ when $2\pi$-kink solutions exist for Eq. (9): $A = 1/8$. In the cases $n = 5, 6$ and $v = 0.5$ the accuracy of the numerical computation is insufficient and the results are not exposed.

| $v$ | $n$ | Asympt. $\lambda_n$ | Calcul. $\lambda_n$ | $v$ | $n$ | Asympt. $\lambda_n$ | Calcul. $\lambda_n$ |
|-----|-----|----------------------|----------------------|-----|-----|----------------------|----------------------|
| 0.1 | 1   | 3.3885               | 0.9116               | 0.5 | 1   | 0.5134               | 0.3751               |
| 2   |     | 1.1295               | 0.6809               | 2   |     | 0.1711               | 0.1698               |
| 3   |     | 0.6777               | 0.5319               | 3   |     | 0.1026               | 0.1043               |
| 4   |     | 0.4840               | 0.4311               | 4   |     | 0.0733               | 0.0745               |
| 5   |     | 0.3765               | 0.3584               | 5   |     | -                    | -                    |
| 6   |     | 0.3080               | 0.3041               | 6   |     | -                    | -                    |

Eq. (26) has unique positive root $k(\delta) \sim 1/\delta$, $\delta \to 0$. The $2\pi$-kink solution of the equation $d^2\varphi/d\eta^2 = F(\varphi)$ is

$$\tilde{\varphi}(\eta) = \pi + 2 \arctan \left( \frac{1}{\sqrt{1 + 4A}} \sinh \left( \sqrt{1 + 4A} \eta \right) \right)$$

and its closest to the real axis singularities in the upper complex half-plane for $A > 0$ are $z_\pm = \pm \alpha + i\beta$ where

$$\alpha = \frac{\text{arccosh}(1 + 8A)}{2\sqrt{1 + 4A}}, \quad \beta = \frac{\pi}{2\sqrt{1 + 4A}}.$$

Then, according to the Conjecture one can expect that

$$\delta_n \sim \frac{\text{arccosh}(1 + 8A)}{\sqrt{1 + 4A}((2n - 1)\pi + \theta_0)}, \quad n = 1, 2, \ldots \quad (27)$$

The Table 2 presents the values of $\delta_n$ computed both, numerically and by the formula (27) with $\theta_0 = 0$. It follows from Table 2 that the correspondence between numerical and asymptotical values is also good for large enough $n$. Also for $n = 6, 7, 8$ and $A = 1$ the accuracy of the numerical computation is insufficient and the results are not exposed.

Since the correspondence between the numerical and asymptotical results is good we assume that the Conjecture is valid and there exists infinitely many branches $v_n(\lambda)$ for $A > 0$. Let us note that in the interval $-1/4 < A < 0$ there also exists $2\pi$-kink solution (13) of Eq. (13) but its closest to the real axis singularities are situated on imaginary axis, therefore the conditions (28) of the Conjecture do not hold. Numerical study does not reveal for $-1/4 < A < 0$ any root of the function $R_\lambda(v)$. Consequently, we conclude that no $2\pi$-kinks with nonzero velocities exist for $A < 0$ in the nonlocal model.

4. Propagation of fluxons: numerical experiments

In this section we consider the evolution governed by the nonlocal equation (9). We show that the model supports the propagation of $2\pi$-kinks of Sect. 3.1.
A \[ \begin{array}{cccc}
| n | \text{Asympt. } \delta_n | \text{Calcul. } \delta_n | A | \text{Asympt. } \delta_n | \text{Calcul. } \delta_n | \hline
1 | 0.4110 & 0.3149 & 10 | 0.2529 & 0.1320 | \\
2 | 0.1370 & 0.1350 & 2 | 0.0843 & 0.0664 | \\
3 | 0.0822 & 0.0823 & 3 | 0.0506 & 0.0486 | \\
4 | 0.0587 & 0.0588 & 4 | 0.0361 & 0.0364 | \\
5 | 0.0456 & 0.0457 & 5 | 0.0281 & 0.0283 | \\
6 | - & - & 6 | 0.0230 & 0.0231 | \\
7 | - & - & 7 | 0.0195 & 0.0195 | \\
8 | - & - & 8 | 0.0169 & 0.0169 | \\
\end{array} \]

Table 2: Comparison of asymptotical and numerical values of \( \delta \) when \( 2\pi \)-kink solutions exist for Eq. (16): \( A = 1 \) and \( A = 10 \). In the cases \( n = 6, 7, 8 \) and \( A = 1 \) the accuracy of the numerical computation is insufficient, and the results are not exposed.

in radiationless regime with the velocities \( v_n(\lambda) \). Moreover, we argue that this regime of propagation is asymptotical for some class of kink-like excitations. Below all the results are shown for \( \lambda = 0.3 \) and \( A = 1/8 \). For this case the first and second discrete velocities for radiationless propagation are \( v_1 \approx 0.5831 \) and \( v_2 \approx 0.3213 \). The corresponding values of energy integral are \( W_1 \approx 10.9490 \) and \( W_2 \approx 7.9399 \). The profiles of \( 2\pi \)-kinks corresponding to the first and second discrete velocities were found by numerical solution of (18) — (19). We denote them by \( \phi_1(\xi) \) and \( \phi_2(\xi) \).

1. First, \( 2\pi \)-kink \( \phi_1(\xi) \) was furnished at the initial moment its natural velocity \( v = v_1 \). The evolution of this entity is shown in Fig. 3 panel A. It follows from this figure that the velocity is conserved and no radiation appears.

Then the same \( 2\pi \)-kink \( \phi_1(\xi) \) was furnished a velocity \( v = 0.9 > v_1 \). The energy integral in this case is \( W \approx 12.9662 \) > \( W_1 \). Fig. 3 panel B and Fig. 4 show that the travelling kink slows down until its natural velocity \( v_1 \). Extra energy has been radiated. The similar phenomena for the model \( \phi^4 - \phi^6 \) was reported on in [19].

2. In the next series of experiments we considered the evolution of kink-like
excitation of the form
\[ \varphi = 4 \arctan \left( \exp \left( \frac{\gamma \xi}{\sqrt{1 - v^2}} \right) \right). \] (28)

Here \( \gamma \) is a coefficient describing the slope of the kink front.

(i) Fig. 5 shows the propagation of this kink-like excitation for \( \gamma = 0.5 \) supplied at the initial moment \( \tau = 0 \) with a velocity \( v = 0.99 \). This velocity is greater than the first discrete velocity \( v_1 \) of radiationless 2\( \pi \)-kink propagation. The energy integral in this case is \( W \approx 26.2718 \) which is also greater than \( W_1 \).

The profiles of the kink-like excitation (28) and 2\( \pi \)-kink solution \( \varphi_1(\xi) \) are depicted in Fig. 5 panel A. Then, the profiles were compared at the moment \( \tau = 120 \) (Fig. 5 panel B). One can observe that they match each other quite well. The decreasing of the velocity for the kink-like excitation is shown in Fig. 5, panel C. Note that the velocity in this case changes smoothly, without large oscillations.

(ii) The kink-like excitation (28) with \( \gamma = 0.5 \) was furnished initially a velocity \( v = 0.8 \). This velocity is greater than the first discrete velocity \( v_1 \) of 2\( \pi \)-kink solution \( \varphi_1(\xi) \). However, the value of energy integral of the excitation \( W \approx 9.4018 \) lies between the first and second discrete energy values \( W_1 \) and \( W_2 \). Numerical simulation shows (see Fig. 6 panel A) that in this case the velocity falls down to the second discrete velocity \( v_2 \). Fig. 6 panels A and B, show the difference between the profiles of kink-like excitation and and 2\( \pi \)-kink solution \( \varphi_1(\xi) \) at the moment \( \tau = 0 \) and \( \tau = 150 \). Again the profiles in Fig. 6 panel B match well each other. In this case the velocity of kink-like excitation exhibits considerable oscillations (Fig. 6 panel C).

So, the numerical experiments show that the phenomenon of quantization of 2\( \pi \)-kink velocities is essential for dynamics of nonlocal model. In the process of evolution a kink-like excitation “adjusts” its shape and velocity approaching...
to one of $2\pi$-kinks which propagates in radiationless regime. The energy $W$ of initial state is crucial for selection of the velocity of such propagation.

5. Discussion

To summarize, we have studied the properties of vortices in stacked Josephson structures. Two assumptions have been made: (i) the electrodynamics of Josephson structure is nonlocal and (ii) the current-phase relation is represented by two sine harmonics, instead of one sine harmonic for the sine-Gordon case. The main equation for the model is the nonlocal double sine-Gordon (NDSG) equation. This equation depends on two governing parameters: $\lambda$ which measures the “strength” of nonlocality and $A$ which is the amplitude of the second harmonic in current-phase relation. The limit $\lambda = 0$ corresponds to traditional (local) double sine-Gordon equation, whereas the limit $A = 0$ corresponds to nonlocal sine-Gordon equation which has been discussed in many studies [1, 2, 3].

The simplest Josephson vortex (fluxon) corresponds to $2\pi$-kink solution of NDSG equation. We have found that the velocity for radiationless motion of this $2\pi$-kink cannot be arbitrary. Our study allows to suppose that there are
infinitely many branches $v_n(\lambda)$ of the velocities of the kink. Some asymptotic properties of families $v_n(\lambda)$ are also described.

Numerical simulations of vortex propagation reveal the following picture. Let the nonlocality parameter $\lambda$ be fixed. Then there exist the set of kink velocities $v_1(\lambda) > v_2(\lambda) > \ldots$ and the set of energy values $W_1(\lambda) > W_2(\lambda) > \ldots$, corresponding to propagation of kink with these velocities. Kink-like excitation launched with velocity greater than $v_1(\lambda)$ with the energy greater than $W_1(\lambda)$ radiates and slows down to the velocity $v_1(\lambda)$. If the initial energy of kink-like excitation lies between $W_1(\lambda)$ and $W_2(\lambda)$, it radiates and slows down to the velocity $v_2(\lambda)$. So, the process of evolution does “feel” the selected values of $2\pi$-kink radiationless propagation.

It is important to stress that there are significant differences of NDSG equation from both, double sine-Gordon equation and nonlocal sine-Gordon equation. Specifically

(i) In the local case $\lambda = 0$ the $2\pi$-kink can have arbitrary velocity $v^2 < 1$. In the case of NDSG equation the spectrum of kink velocities is discrete;

(ii) In the case of nonlocal sine-Gordon equation $2\pi$-kink can have zero velocity only, therefore $2\pi$-kinks cannot propagate. At the same time there exist many kinks with higher topological charge (i.e. $4\pi$-, $6\pi$-kinks etc) and each of them corresponds to its own velocity of propagation. In the case of NDSG model the phenomenon of velocities quantization takes place for $2\pi$-kinks. Moreover, the profiles of $2\pi$-kinks corresponding to different velocities are very similar and differ mainly in asymptotics of the “tails”.

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Appendix A. Josephson layered structure with thin layers: derivation of nonlocal double sine-Gordon equation

Consider a Josephson structure consisting of alternating superconducting layers of thickness $2L$, strips $(2n-1)d + 2(n-1)L < y < (2n-1)d + 2nL$, and nonsuperconducting (tunnel) layers of thickness $2d$, strips $(2n-1)d + 2nL < y < (2n+1)d + 2nL$, $n \in \mathbb{Z}$. Assume that the phase differences for the superconducting order parameters of the electrodes on different sides of the all tunnel layers are the same and are equal to $\varphi(x, t)$. Assume also that the magnetic field $H_J(x, t)$ is also the same in all tunnel layers. Then the magnetic field in $n$-th superconducting layer obeys the London equation

$$\frac{\partial^2}{\partial x^2} H(x, y, t) + \frac{\partial^2}{\partial y^2} H(x, y, t) = \frac{1}{\lambda^2} H(x, y, t), \quad (A.1)$$
where $\lambda_L$ is the London penetration length. The boundary conditions for Eq. (A.1) are

$$H(x, y_n - d - 2L, t) = H(x, y_n - d, t) = H_J(x, t), \quad y_n \equiv 2n(d + L).$$

Assume that the superconducting layers are thin and the following condition holds

$$2L \sqrt{\lambda_L^{-2} + k^2} \ll 1, \quad (A.2)$$

where $k$ is the inverse characteristic spatial scale of the variation of the phase difference along the layered structure. Then the solution of (A.1) can be approximated by the following formula

$$H(x, y, t) \simeq \left[ 1 + \frac{1}{2}(y - y_n + d + 2L)(y - y_n + d) \left( -\frac{\partial^2}{\partial x^2} + \frac{1}{\lambda_L^2} \right) \right] H_J(x, t). \quad (A.3)$$

Taking into account similar relation for $(n + 1)$-th superconducting layer and making use of relation

$$E = \frac{\lambda_e^2}{c} \text{rot} \frac{\partial H}{\partial t}$$

we arrive at the following formula for the jump of tangential component of electric field across $n$-th tunnel layer

$$E_x(x, y_n + d, t) - E_x(x, y_n - d, t) = -\frac{2\lambda^2_L}{c} \left( -\frac{\partial^2}{\partial x^2} + \frac{1}{\lambda_L^2} \right) \frac{\partial H_J(x, t)}{\partial t}, \quad (A.4)$$

where $c$ is the speed of light in vacuum. Taking into account (A.4), the relation between the normal component of electric field $E$ in the tunnel layer with the phase difference

$$E_y(x, t) = \frac{\phi_0}{4\pi d} \frac{\partial \phi}{\partial t}(x, t)$$

and $z$-component of Maxwell equation

$$\text{rot} E = -\frac{1}{c} \frac{\partial H}{\partial t}$$

we arrive at the expression for magnetic field in tunnel layer

$$H_J(x, t) = -\frac{\phi_0 \lambda_{eff}}{8\pi \lambda_L^2 L} \int dx' \exp \left\{ -\frac{|x - x'|}{\lambda_{eff}} \right\} \frac{\partial \phi(x', t)}{\partial x'}, \quad (A.5)$$

Here $\phi_0$ is the magnetic flux quantum and

$$\lambda_{eff} \equiv \frac{\lambda_L}{\sqrt{1 + d/L}}.$$
Then the density of superconducting current on the boundary between superconducting and nonsuperconducting layers is

\[- \frac{c}{4\pi} \frac{\partial H_J(x, t)}{\partial x}.\]

The matching condition for the current on this boundary yields

\[j(\phi) + \frac{c\phi_0}{16\pi^2cd} \frac{\partial^2 \phi}{\partial t^2} = \frac{c\phi_0\lambda_{eff}}{32\pi^2\lambda_L^2} \frac{\partial}{\partial x} \int dx' \exp \left\{ -\frac{|x-x'|}{\lambda_{eff}} \right\} \frac{\partial \varphi(x', t)}{\partial x'}, \]  \tag{A.6}

where \(j(\phi)\) is density of Josephson supercurrent, \(\epsilon\) is the permittivity of the tunnel layer (cf. with Eq. (5.6) of article \[7\]).

For the energy of one period of the layered structure, (i.e. for the area \(y_n - d - 2L < y < y_n + d\)) per unit of \(Oz\) one has

\[W = \frac{\phi_0}{2\pi c} \int dx \left( \int_0^\phi j(\phi') d\phi' + \frac{\phi_0\epsilon}{32\pi^2cd} \left( \frac{\partial \varphi}{\partial t} \right)^2 \right) +
\]

\[+ \frac{\phi_0^2 \lambda_{eff}}{128\pi^3\lambda_L^2} \int \int dx_1 dx_2 \exp \left\{ -\frac{|x_1 - x_2|}{\lambda_{eff}} \right\} \frac{\partial \varphi(x_1, t)}{\partial x_1} \frac{\partial \varphi(x_2, t)}{\partial x_2}.\]

If \(j(\phi)\) is defined by the formula \(j = j_c \sin \varphi + j_2 \sin 2\varphi\) (cf. formula (6) for supercurrent), the energy takes the following form:

\[W = \frac{\phi_0 j_c}{2\pi c} \left[ \int dx \left( 1 - \cos \varphi + A(1 - \cos 2\varphi) + \frac{1}{2\omega_J^2} \left( \frac{\partial \varphi}{\partial t} \right)^2 \right) \right] +
\]

\[+ \frac{(\lambda_L + d)\lambda_J^3}{4\lambda_L \sqrt{(L + d)L}} \int \int dx_1 dx_2 \exp \left\{ -\frac{|x_1 - x_2|}{\lambda_{eff}} \right\} \frac{\partial \varphi(x_1, t)}{\partial x_1} \frac{\partial \varphi(x_2, t)}{\partial x_2}. \]  \tag{A.7}

where \(\omega_J \equiv 4\pi \sqrt{c_j d/\phi_0\epsilon}\) is the Josephson plasma frequency and

\[\lambda_J \equiv \frac{1}{4\pi} \sqrt{\frac{\phi_0\epsilon}{j_c(\lambda_L + d)}}\]

is the Josephson length.

References

[1] A.A. Abdumalikov Jr., G.L. Alfimov, A.S. Malishevskii, Nonlocal electrodynamics of Josephson vortices in superconducting circuits, Supercond. Sci. Technol. 22 (2009) 023001.

[2] G.L. Alfimov, V.M. Eleonsky, N.E. Kulagin, N.V. Mitskevich, Dynamics of topological solitons in models with nonlocal interactions, Chaos 3 (1993) 405-414.
[3] G.L. Alfimov, V.M. Eleonsky, L.M. Lerman, Solitary wave solutions of nonlocal sine-Gordon equations, Chaos 8 (1998) 257-271.

[4] A.S. Malishevskii, V.P. Silin, S.A. Uryupin, $4\pi$-kink vortices in long Josephson junctions, Phys. Lett. A 253 (1999) 333-340.

[5] A.S. Malishevskii, V.P. Silin, S.A. Uryupin, Gluing of Josephson vortices by Cherenkov-trapped Swihart waves, JETP 90 (2000) 671-688.

[6] A.S. Malishevskii, V.P. Silin, S.A. Uryupin, $10\pi$-kink and conception of Cherenkov gluing of the Josephson vortices, Phys. Lett. A 270 (2000) 347-352.

[7] Yu.M. Aliev, K.N. Ovchinnikov, V.P. Silin, S.A. Uryupin, Nonlocal Josephson electrodynamics of layered structures, JETP 80 (3) (1995) 551-559.

[8] K.K. Likharev, Dynamics of Josephson Junctions and Circuits, New York: Gordon and Breach, 1986.

[9] Y. Tanaka, S. Kashiwaya, Theory of Josephson effects in anisotropic superconductors, Phys. Rev. B 56 (1997) 8929-12.

[10] E. Goldobin, D. Koelle, R. Kleiner, A. Buzdin, Josephson junctions with second harmonic in the current-phase relation: Properties of $\phi$ junctions, Phys. Rev. B 76 (2007) 224523.

[11] P. Komissinskiy, G.A. Ovsyannikov, K.Y. Constantinian, Y.V. Kislinski, I.V. Borisenko, I.I. Soloviev, V.K. Kornev, E. Goldobin, D. Winkler, High-frequency dynamics of hybrid oxide Josephson heterostructures, Phys. Rev. B 78 (2008) 024501.

[12] P.Kh. Atanasova, T.L. Boyadjiev, E.V. Zemlyanaya, Yu.M. Shukrinov, Numerical Study of Magnetic Flux in the LJJ Model with Sine-Gordon Equation, Numerical Methods and Applications; Lecture Notes in Computer Science 6046 (2011) 347-352.

[13] S.V. Bakurskiy, N.V. Klenov, T.Yu. Karminskaya, M.Yu. Kupriyanov, A.A. Golubov, Josephson $\varphi$-junctions based on structures with complex normal/ferromagnet bilayer, Supercond. Sci. Technol. 26 (2013) 015005.

[14] M.Y. Kupriyanov, Effect of a finite transmission of the insulating layer on the properties of SIS tunnel junctions, JETP Letters 56 (1992) 399-405.

[15] A.A. Golubov, M.Yu. Kupriyanov, The current phase relation in Josephson tunnel junctions, JETP Letters 81 (7) (2005) 419-425.

[16] A.A. Golubov, M.Yu. Kupriyanov, E. Il’ichev, The current-phase relation in Josephson junctions, Rev. Mod. Phys. 76 (2) (2004) 411-469.

[17] A.R. Champneys, B.A. Malomed, M.J. Friedman, Thirring solitons in the presence of dispersion, Phys. Rev. Lett. 80 (1998) 41694172.
[18] A.R. Champneys, B.A. Malomed, J. Yang, D.J. Kaup, Embedded solitons: solitary waves in resonance with the linear spectrum, Physica D 152 (2001) 340-354.

[19] G.L. Alfimov, E.V. Medvedeva, Moving nonradiating kinks in nonlocal $\phi^4$ and $\phi^4 - \phi^6$ models, Phys. Rev. E. 84 (5) (2011) 056606.

[20] G.L. Alfimov. On the dimension of the set of solutions for nonlocal nonlinear wave equation, Rus. J. Nonlin. Dyn. 7 (2) (2011) 209226 (in Russian).

[21] G.L. Alfimov, E.V. Medvedeva, D.E. Pelinovsky, Hamiltonian systems with an infinite number of localized travelling waves, arXiv (2013) preprint arXiv:1309.0183