COSMOLOGICAL MAGNETIC FIELD AMPLIFICATION

AROUND Z \sim 100

Esteban A. Calzetta and Alejandra Kandus
Departamento de Física, Fac.Cs. Ex. y Nat. UBA
Ciudad Universitaria, (1428) Buenos Aires, Argentina
and
Instituto de Astronomía y Física del Espacio
c.c. 67, Suc. 28 (1428) Buenos Aires, Argentina.

Received ________________; accepted ________________
We study the corrections to the conformal evolution of primordial magnetic fields after recombination, produced by the fall of the ionized fraction of matter into the dark matter gravitational wells. This effect enhances the field around the protostructures being formed, and might help to understand the fields observed in galaxy clusters and hydrogen clouds.

*subject headings:* gravitation - magnetic fields - cosmology: theory.
1. Introduction

The goal of this paper is to study the evolution of cosmic magnetic fields interacting with charged matter immediately after recombination. At this time most ordinary matter combines in neutral hydrogen, but still a non negligible ionized fraction remains. On the other hand, as we shall show, the cosmic microwave background radiation (CMBR) is already too cold to provide substantial dissipation, and so the ionized matter, as well as the neutral one, falls within the gravitational potential wells generated by dark matter overdensities. The infall of charged matter drags the field lines, thus resulting in a field amplification in and around the proto structures being formed. Thus a mechanism results which may provide a preamplification of a primordial magnetic field, before other best known processes, such as the galactic dynamo (Zel’dovich et al. 1990), become operational. The primeval motivation for the search of these corrections is to try to understand the presence of magnetic fields of intensity and coherence similar to the galactic ones, in galaxy clusters and much less evolved gaseous systems such as the damped Lyman-α clouds (Oren & Wolfe 1995). Although the existence and ultimate characteristics of those fields can hardly be explained by gravitational collapse alone, it is important to determine if this process may help to relax the requirements on the mechanisms for primordial field generation.

The existence of relatively intense and coherent magnetic fields in galaxies, clusters of galaxies and hydrogen clouds is at present theme of intense research. There is a variety of proposals aimed at explaining their existence and characteristics, but none is completely successful. They can be divided in two main sets: one takes the point of view of local generation, and the other of the amplification of a primordially generated field (Zel’dovich et al. 1990, Harrison 1973, Pudritz & Silk 1989, Howard & Kulsrud 1996). While both mechanisms compete on equal footing in explaining the presence of magnetic fields in low
redshift objects, like our own galaxies, the second approach seems more appropriate for highly redshifted, less developed systems, like the Lyman-α clouds. A model based on amplification mechanisms requires two stages, one to create the field and another to amplify it. For the first process there are several proposals (Dolgov 1993, Dolgov & Silk 1993, Ratra 1992, Mazzitelli & Spedalieri 1993, Calzetta et al. 1998, Cheng & Olinto 1994, Sigl et al. 1997, Enqvist & Olesen 1994, Vachaspati 1991, Hogan 1983, Martin & Davies 1995), some of which create fields strong enough but of coherence length much smaller than observed, while others generate coherent fields but of very weak strength. As for the second stage, the “turbulent dynamo” was a preferred amplifying mechanism for a long time (Zel’dovich et al. 1990). But once again we must question it after the detection in damped Lyman-α clouds (Oren & Wolfe 1995, Wolfe et al. 1992) of magnetic fields of the same intensity and coherence as the galactic ones. Another promising mechanism are inverse cascades of magnetic helicity (Olesen 1997, Brandenburg et al. 1997), but, while this effect is quite well understood in ordinary magnetohydrodynamics (Moffat 1961), its generalization to cosmology is still an open problem.

It is well known from the theory of Large Scale Structure Formation, that galaxies are formed due to the infall of barionic matter into the gravitational potential wells of dark matter, the formation of those last beginning at the moment of equality between matter and radiation ($z \approx 20000$). Although they are not “deep” wells by the decoupling of barionic matter and radiation ($z \approx 1100$), as we know from the anisotropy of the CMBR, their existence cannot be disregarded. Despite this fact, current estimates in the literature often assume that, between decoupling and the starting of the dynamo mechanism or non linear gravitational collapse, their propagation takes place in a homogeneous, FRW Universe, and consequently that their evolution law is $B \propto B_0/a^2$, where $a$ is the scale factor of the Universe and $B_0$ an initial field intensity. In this work we study in more detail the evolution of a primordial magnetic field in a matter dominated, inhomogeneous Universe, with the
aim of estimating the corrections to this conformal evolution due to the evolution of the background perturbations.

Long time ago Ya. B. Zel’dovich et al (1983) investigated the effects of a magnetic field over anisotropic self gravitational collapse of matter. They also estimated the amplification that a primordial field would undergo by that collapse, finding a factor of $\mathcal{O}(10^3 - 10^4)$. Working backwards from present estimates, the inferred intensity for a primordial field results then $B_0 \simeq 10^{-9} - 10^{-10}$ Gauss, much larger than the ones predicted by the available mechanisms of primordial generation, for the required coherence scale.

Our proposal differs from Zel’dovich’s in that we do not study self gravitational collapse, but rather the infall of charged matter into preexistent potential wells created by dark matter. Consider a spatially flat Friedmann - Robertson - Walker Universe endowed with adiabatic perturbations. The main matter component is cold dark matter, which for our purposes only means that the smallest structures to be formed are galaxies. At the moment of equilibrium between matter and radiation, the matter dominated era of the Universe begins and the adiabatic perturbations in the dark matter component can grow, creating an inhomogeneous Universe. On the other side, the rest of the matter component, the baryons, remain coupled to photons until the temperature of the Universe is low enough to let electrons and protons combine. Then neutral hydrogen emerges while a small ionized fraction of $\mathcal{O}(10^{-5} - 10^{-7})$ remains.

Any magnetic field present in the Universe will evolve coupled to the charged matter plasma. This means that before recombination, the field will be tied to the electron-photon plasma, and will remain coupled to the remaining ionized matter after that moment. Before recombination the plasma is hot enough to be considered relativistic and therefore its evolution can be followed without taking into consideration the background inhomogeneities. After recombination, however, the situation is the reverse: the temperature is too low to
neglect the effect of the perturbations and therefore we must study the evolution of the magnetic field coupled to the remaining ionized matter, in an inhomogeneous Universe. Of course the ionized plasma will be coupled to the CMBR as well as to any primordial field; one of our objectives is to show that, in the era under consideration, the interaction with the CMBR does not affect substantially the collapse process, and that our plasma can be consider as a perfect fluid with infinite conductivity.

Several authors have studied related physical situations. For example Ryu et al (1998) and Blasi et al (1999) studied the effect of inhomogeneities in the matter distribution of the Universe on the Faraday rotation of radio signals and light from quasars respectively. The inhomogeneities considered by the former authors are large scale filaments and sheets (generated by a hydrodynamical code), to which the magnetic field is glued. They assumed that a seed field is locally generated by a battery mechanism and further amplified by streaming and turbulent motions. Blasi et al, on the other side, considered redshifts for which the inhomogeneities are well described by the Ly-α forest. The magnetic field is also glued to this structure and they assume it scales with the electron density as $B \propto n_e^{2/3}$. In both works an upper limit for the magnetic field strength in the structures is derived.

Our work differs substantially from the one by Blasi et al (1999) in that we do not assume a given relationship between magnetic field and ionized matter density, rather we derive it. The difference with the work by Ryu et al (1998) is that we do not assume a locally generated magnetic field, but a primordially generated one. Besides, we study the evolution of the magnetic field in a universe permeated by primordial inhomogeneities that are still in their linear regime of evolution, which means higher redshifts than in the mentioned works.

We have therefore the following scenario: a weak primordial magnetic field is coupled to the matter that remains ionized after the recombination of hydrogen, and the whole
system propagates in an inhomogeneous universe. The matter component falls into the background overdensities due to gravitational attraction, dragging with it the magnetic field lines. This drag produces a local enhancement of the magnetic field in the overdense regions, deviating from the evolution to be expected in a pure FRW Universe. Our purpose is to calculate this enhancement and to see if it is intense enough to explain (up to certain level) the detected field intensities.

The paper is organized as follows: In section 2 we describe the geometry of the background Universe, in section 3 we describe the matter component the magnetic field is tied to, discussing in some detail the possible interactions with the CMBR. In section 4 we study the evolution of the magnetic field and summarize our main conclusions in Section 5.

2. Baryon free fall

2.1. The geometry of the background

Consider a matter dominated flat FRW Universe, endowed with adiabatic perturbations. In the longitudinal gauge, the metric tensor for this Universe reads (Brandenberger et al. 1992)

\[ g_{\mu\nu} = a^2(\eta) \begin{pmatrix} - (1 + 2\psi) & 0 \\ 0 & (1 - 2\psi) \gamma_{ij} \end{pmatrix} \]

where \( a(\eta) = (\eta/\eta_b)^2 \) is the scale factor, \( \eta \) the conformal time and \( \gamma_{ij} \) the spatial flat three metric, \( \gamma_{ij} = diag(1,1,1) \). We have normalized the conformal factor to unity at the time of decoupling, given by \( \eta = \eta_b \approx 0.4h^{-1} \times 10^{38}GeV^{-1} \). Greek indices run from 0 to 3, while latin ones take the values 1, 2, 3. The function \( \psi \) depends on \( r \) and \( \eta \) and can be interpreted as a generalized newtonian potential. Its amplitude can be determined by the level of anisotropies in the background radiation caused by the Sachs-Wolfe effect at decoupling. If
\( \psi \) describes a growing perturbation then it must depend only on \( r \) \cite{brandenberger1992}. In the linear regime, this mode will grow as \( \sim a(\eta)\psi(r) \).

From this expression we can find the four velocity of the background fluid flow and construct a 3 + 1 description. This is needed because we will consider the presence of a magnetic field, and its definition relies on the specification of the spatial surfaces of the spacetime.

In a coordinate system where peculiar velocities are zero we have

\[
U^\mu = \frac{dx^\mu}{ds} = \left( \frac{1 - \psi}{a}, 0 \right)
\]

This is normalized by

\[
U^\mu U_\mu = -1
\]

Finally we build a projection tensor for the spatial three surfaces as

\[
h^{\mu\nu} = g^{\mu\nu} + U^\mu U^\nu
\]

As this fluid has zero vorticity, the spatial three surfaces are orthogonal to the fluid four velocity \cite{ellis1998}.

\[.
\]

2.2. The plasma component of the Universe

We want to study the evolution in the matter dominated era of the Universe of a primordially generated magnetic field, considering that in that period of time the process of structure formation is taking place as well. At the beginning of matter dominance \((z \simeq 20000)\), the background perturbations begin to grow, while the baryons remain coupled to the cosmic background radiation. When the temperature of the Universe is low enough, electrons and protons recombine \((z \simeq 1100)\) and most of the baryonic matter
becomes electrically neutral. It can then start falling into the already formed dark matter overdensities, which are described (in the linear regime) by the function $\psi(r)$. There remains a ionized fraction of barionic matter of order $\sim \mathcal{O}(10^{-5} - 10^{-7})$.

We assume that any magnetic field present in the Universe interacts only with the barionic component of the matter. With the electrically neutral fraction, the interaction is very weak and is described by $B \cdot m$ where $\vec{m}$ is the atomic magnetic moment. With the ionized fraction, the interaction is given by the Lorentz force, being therefore stronger than the former. We will study the system of ionized barionic matter coupled to a preexisting magnetic field, in the matter dominated epoch, by means of the equation

$$T_{\mu\nu} = F^{\mu}$$

where $T_{\mu\nu}$ is the stress energy tensor for the matter and $F^{\mu}$ is the Lorentz force density, exerted by the magnetic field on the charged particles.

We shall investigate the effects on the preexisting magnetic field of the free fall of matter into the newtonian potential wells described by $\psi(x)$. To this end, we shall consider a geometry where $\psi = \psi(z)$, and correspondingly the motion of matter is along the $z$ direction only. In this first Section, moreover, we shall make two simplifying assumptions, namely, we shall consider the plasma as a perfect fluid, and we shall neglect the right hand side of Eq. (5). We shall show the validity of the first approximation towards the end of this Section; the second one will put limits on how far the model can be relied upon as a description of collapse, and will be discussed later on.

We shall treat the energy density $\delta \epsilon$, pressure $\delta p$ and four velocity $\delta u^\mu$ of the ionized plasma as perturbations on the corresponding quantities for the dark matter, namely $\epsilon = \epsilon + \delta \epsilon$, $P = p + \delta p$, and $U^\mu = U^\mu + \delta u^\mu$. We shall call $\delta N$ any conserved quantity proper of the ionized fraction, for example the leptonic or barionic numbers, and we will consider that this quantity does not vary with time, i.e. the ionisation fraction remains constant.
Since the gravitational effect of this perturbation is negligible, we shall not consider the perturbations to the geometry that it could produce. We are therefore describing the free fall of the plasma into the potential wells created by the dark matter. The full energy-momentum tensor for the plasma is

\[ T_{\mu\nu} = T_{\mu\nu}^I + \delta T_{\mu\nu}^I \]  \hspace{1cm} (6)

where

\[ T_{\mu\nu}^I = \varepsilon U_{\mu} U_{\nu} + ph_{\mu\nu} \]  \hspace{1cm} (7)

\[ \delta T_{\mu\nu}^I = \delta \varepsilon U_{\mu} U_{\nu} + (\varepsilon + p) (\delta u_{\mu} U_{\nu} + U_{\mu} \delta u_{\nu}) + \delta ph_{\mu\nu} \]  \hspace{1cm} (8)

As stated before, we have

\[ T_{\mu\nu}^I ; \nu = 0 \]  \hspace{1cm} (9)

We will also need the equation for the conservation of the number of matter particles, namely

\[ I_{\mu} = 0 \]  \hspace{1cm} (10)

where

\[ I_{\mu} = \delta N (U_{\mu} + \delta u_{\mu}) \]  \hspace{1cm} (11)

2.3. Timelike and spacelike projections of \( T_{\mu\nu}^I \)

Our ultimate goal is to study the evolution of the matter component and the magnetic field under the gravitational perturbations created by dark matter. As the specification of what magnetic and electric fields are is tied to the definition of spacelike surfaces and their corresponding timelike curves, we must work in a 3 + 1 formalism. For this purpose we project equation (9) onto the spatial surfaces, by \( h^\mu_{\nu} \) (eq.(4)) and along the worldlines given by \( U^\mu \) (eq. (2)). Performing the covariant derivatives with the Christoffel symbols
corresponding to the metric tensor Eq. (1) (see Appendix) and replacing the explicit expressions for the four velocity we obtain the conservation laws to leading and first order as follows:

**zeroth order**

\[
\dot{\varepsilon} + 3\frac{\dot{a}}{a} (\varepsilon + p) = 0 \tag{12}
\]

For the dark matter component, moreover, \( p = 0 \), and we get the expected behavior \( \varepsilon \sim a^{-3} \).

**first order**

\[
\frac{\psi}{a^2} \dot{\varepsilon} - \frac{1}{a} \delta \varepsilon + 3 \left[ \frac{\dot{a}}{a^2} \psi + \frac{\dot{\psi}}{a} \right] (\varepsilon + p) - 3\frac{\dot{a}}{a^2} (\delta \varepsilon + \delta p) - (\varepsilon + p) \delta u^i \cdot j = 0 \tag{13}
\]

\[
\frac{\psi}{a^2} (\varepsilon + p) + \frac{\dot{\psi}}{a^2} (\dot{\varepsilon} + \dot{p}) \delta u^i + (\varepsilon + p) \left\{ \frac{1}{a} \delta \dot{u}^i + 5 \frac{\dot{a}}{a^2} \delta u^i \right\} = 0 \tag{14}
\]

We now replace the zeroth order equation and write \( \delta \varepsilon = \rho \varepsilon, \delta u^i = u^i / a \) and \( p = c_s^2 \varepsilon \), where \( c_s \) is the speed of the sound in the medium. We will also neglect the terms with \( \dot{\psi} \) as they correspond to decaying modes of the background perturbations, leading to

\[
\dot{\rho} + \left( 1 + c_s^2 \right) u^i \cdot j = 0 \tag{15}
\]

\[
\psi^i + \frac{c_s^2}{(1 + c_s^2)} \rho^i + \left( 1 - 3c_s^2 \right) \frac{\dot{a}}{a} u^i + \dot{u}^i = 0 \tag{16}
\]

In what follows, we neglect the effect of the pressure. This is justified in view of the low temperature of the matter.
2.4. Particle number conservation

Developing the covariant derivative of eq. (10) we obtain

\[
\delta \dot{N} (1 - \psi) + 3 \delta N \frac{\dot{a}}{a} (1 - \psi) + \left( \delta N u^j \right)_{,j} - 2 \delta N \psi_{,j} u^j = 0
\]  
(17)

Taking \( \delta N = \delta n/a^3 \) the previous equation reads

\[
(1 - \psi) \delta \dot{n} + \left[ u^j \delta n \right]_{,j} - 2 \delta n \psi_{,j} u^j = 0
\]  
(18)

2.5. Study of the evolution of \( \rho \) and \( u^j \) for a dustlike gas

We are now ready to analyze the effects that the background perturbations may have on the density and velocity of the plasma. Current numerical investigations of large scale structure formation show that gravitational collapse of matter is highly anisotropic, giving rise to a complicated network of rich and poor clusters connected by filaments with void regions in between (Bond et al 1996). This is a difficult process to study analytically so we will retain one simple feature, namely that gravitational collapse is not symmetric, but takes place mainly along a preferred direction, the resulting structures being pancake or cigar-like shaped. This fact allows us to introduce another simplifying hypothesis: we will consider that all quantities depend only on the coordinate along which gravity acts, let it be the \( z \) coordinate. This simplification contains a very important physical assumption about the perturbations: they are vorticity free. Were we to consider two or three dimensional gravitational collapse, vorticity must be taken into account.

When we make the mentioned approximations we are left with

\[
\dot{\rho} + u^j_{,j} = 0
\]  
(19)

\[
\psi^i + \frac{\dot{a}}{a} u^i + \dot{u}^i = 0
\]  
(20)
We write $\mathbf{u} = (u_x, u_y, u_z)$ and obtain

$$\dot{\rho} + \partial_z u_z = 0 \quad (21)$$

$$\dot{u}_x + \frac{a}{\dot{a}} u_x = \dot{u}_y + \frac{a}{\dot{a}} u_y = 0 \quad (22)$$

$$\partial_z \psi + \dot{u}_z = \partial_z \psi = 0 \quad (23)$$

For the initial conditions, $\mathbf{u}(z, \eta_0) = 0$, and $\rho(z, \eta_0) = \rho_0$, the solutions to equations (21) and (23) read

$$\rho = \rho_0 + \frac{\eta_0^2}{3} \partial_z^2 \psi(z) \left[ \frac{1}{2} \left( \frac{\eta}{\eta_0} \right)^2 + \frac{\eta_0}{\eta} - \frac{3}{2} \right] \quad (24)$$

$$u_z(z, \eta) = \frac{\eta_0}{3} \partial_z \psi(z) \left[ \left( \frac{\eta_0}{\eta} \right)^2 - \frac{\eta}{\eta_0} \right] \quad (25)$$

$$u_x = u_y = 0 \quad (26)$$

These are the equations which describe the free fall of the plasma into the potential wells.

### 2.6. Discussion

Equations (18) and (20) are the goal of this Section, namely, they describe the evolution of the number density and velocity of the plasma as it falls on the dark matter overdensities. Since the magnetic field lines are tied to the charges, we expect these will be dragged along with matter, producing an amplification of the field in the overdense regions. The study of this effect is the subject of the rest of this paper.

However, before we go on it is right that we discuss the first of the approximations under which eqs. (19) and (20) were derived, namely, the assumption that the plasma could be described as a perfect fluid. In reality, the plasma interacts not only with the
gravitational field of the dark matter, but also with the photons in the CMBR, and this interaction could in principle result in dissipative behavior.

In the epoch we are interested in, the plasma and the CMBR are not in equilibrium, and therefore the usual estimates of the dissipative effects induced on the plasma do not apply (Weinberg 1971). Indeed, the mean free path for the photons is \( l_{mfp} = (n_e \sigma_T)^{-1} \), where \( n_e \) is the number density of the electrons in the ionized fraction of matter and \( \sigma_T = \frac{8 \pi e^2}{3 m_e^2} \) is the Thompson cross section, (due to the low temperature of the background photons, the main interactive process is Thompson scattering). Their values are \( \sigma_T \simeq 1.2 \times 10^5 \text{GeV}^{-2} \), \( n_e \simeq 0.4 \times 10^{-45} a(\eta)^{-3} \text{GeV}^{-3} \) and therefore \( l_{mfp} \simeq 2 \times 10^{40} a(\eta)^3 \text{GeV}^{-1} \). On the other side, the size of the particle horizon is \( d_h = \eta^3 / \eta_0^2 \). We estimate the number of collisions that a photon suffers per Hubble time as \( n_{coll} = d_h / l_{mfp} \sim 20 a(\eta)^{-3/2} \), which is too low to sustain equilibrium.

Still, the plasma must feel some kind of friction as it moves across the CMBR. Since the plasma is electrically neutral, any baryon motion is matched by a corresponding electron flux. The Thompson cross section is \( 10^{-6} \) orders of magnitude smaller for protons than for electrons, then the interactions of photons with electrons dominate, and we can consider that the electrons suffer a friction due to the photons. Therefore, instead of the conservation equation (9) we should write

\[
\mathcal{T}^{\mu\nu};_\nu = \zeta^{\mu}; \quad \mathcal{T}^{\mu};_\mu = 0 \tag{27}
\]

where \( \mathcal{I}^{\mu} \) is the particle four current. We also have the corresponding equation for the photons

\[
T^{\mu\nu}_{\gamma,\nu} = -\zeta^{\mu} \tag{28}
\]
From the first principle of thermodynamics we have

\[ S^\mu = \Phi^\mu - \beta_{\text{ce}} T^{\mu\nu} - \beta_{\gamma\nu} T^{\mu\nu}_\gamma - \alpha I^\mu \]  

(29)

where \( S^\mu \) is the entropy flux, \( \beta_{\text{ce}} = U^\mu / T_c \), \( \beta_{\gamma\nu} = U^\mu / T_\gamma \), the subindex \( c \) corresponding to the “charged” matter and \( \gamma \) to the CMBR. \( \Phi^\mu \) is the thermodynamic potential (Israel 1988), whose derivatives are

\[ \frac{\partial \Phi^\mu}{\partial \beta_{\text{ce}\nu}} = T^{\mu\nu}, \quad \frac{\partial \Phi^\mu}{\partial \beta_{\gamma\nu}} = T^{\mu\nu}_\gamma; \quad \frac{\partial \Phi^\mu}{\partial \alpha} = I^\mu. \]  

(30)

By the second law of thermodynamics we have

\[ S^\mu_{;\mu} = (\beta_{\gamma\nu} - \beta_{\text{ce}\nu}) \zeta^\nu > 0 \]  

(31)

which is satisfied only if

\[ \zeta^\mu = C_1 \beta^\mu_{\gamma} \left( -\beta^2_{\gamma} - \beta_{\gamma\mu} \beta^\mu_{\text{ce}} \right) + C_2 \left( \beta^\nu_{\gamma} - \beta^\nu_{\text{ce}} \right) \left[ \delta^\mu_{\nu} + \frac{\beta^\mu_{\gamma} \beta_{\gamma\nu}}{\beta^2_{\gamma}} \right] \]  

(32)

where \( \beta^2_{\gamma} = -\beta_{\gamma\mu} \beta^\mu_{\gamma} \) and \( C_1, C_2 \geq 0 \). In terms of the four velocities and temperatures eq. (32) reads

\[ \zeta^\mu = -\frac{C_1 U^\mu}{T_\gamma T_c} \left[ T_c + T_\gamma U_\gamma U^\nu \right] - \frac{C_2}{T_c} \left[ U^\mu + U^\nu U_\nu U^\mu \right] \]  

(33)

We see that the first term in the r.h.s. of eq. (33) can be interpreted as the heat interchange between the photons and the plasma, while the second one as the momentum transfer between them.

To estimate the \( C_1 \) and \( C_2 \) coefficients, let us place ourselves in the rest frame of the CMBR. Since in this frame the photon bath is isotropic, the bombardment of electrons "at rest" by photons averages out, and any net effect is solely due to the motion of the electrons. Each time an electron strikes a photon, the later gains a momentum \( \Delta p_\gamma \sim 1/\lambda \sim T_\gamma \). The electron loses a momentum \( \Delta p_e \sim -T_\gamma \) and changes its energy by \( \Delta U = -v_e T_\gamma \), where \( v_e \) is the velocity of the electrons. The number of collisions per unit time that an electron suffers
is \(dn = n_\gamma \sigma_T v_e\). If the electron number density is given by \(n_e\), then the net force density exerted by the photons on the electrons is given by \(F_{\text{coll}}^e \simeq n_e n_\gamma \sigma_T T_\gamma v_e\), and the mean energy loss is \(Q = n_e n_\gamma \sigma_T T_\gamma v_e^2\). Performing the average over all electrons, and comparing with eq. (33), we conclude

\[
\frac{C_1}{T_\gamma T_c} = \frac{n_e n_\gamma \sigma_T T_\gamma}{m_e} 
\]

\[
\frac{C_2}{T_c} = n_e n_\gamma \sigma_T T_\gamma 
\]  

We should now project eq. (33) along the fluid flow lines and onto the orthogonal spatial surfaces and replace the expressions in the r.h.s. of eqs. (13) and (14) respectively. For the timelike projection only the first term of eq. (33) contributes while for the spacelike projection we need the second term of that equation. Considering \(\varepsilon\) as the critical density of the Universe, using also that \(\varepsilon_{\text{crit}} = \frac{\varepsilon^{(0)}}{a^3(\eta)}, n = \frac{n^{(0)}}{a^3(\eta)}\) and that \(T = \frac{T^{(0)}}{a(\eta)}\), where with the supraindex (0) we refer to quantities at the epoch of recombination, we have that the equations (19) and (20) now read

\[
\dot{\rho} + u_j \rho_{,j} = - \frac{n^{(0)}_e n^{(0)}_\gamma \sigma_T T^{(0)}_\gamma}{m_e \varepsilon^{(0)}_{\text{crit}} a^3(\eta)} (T_c - T_\gamma) 
\]

\[
\psi^i + \frac{\dot{a}}{a} u^i + \dot{u}^i = - \frac{\frac{n^{(0)}_e n^{(0)}_\gamma \sigma_T T^{(0)}_\gamma}{\varepsilon^{(0)}_{\text{crit}} a^5(\eta)}}{\frac{\sigma_T T^{(0)}_\gamma}{\varepsilon^{(0)}_{\text{crit}} a^3(\eta)}} u^i 
\]  

Replacing the figures, \(\varepsilon^{(0)}_{\text{crit}} \sim 10^{-38} \text{ GeV}^4\), \(n^{(0)}_\gamma \sim 10^{-33} \text{ GeV}^3\), \(\sigma_T \sim 10^5 \text{ GeV}^{-2}\), \(T^{(0)}_\gamma \sim 10^{-10} \text{ GeV}\), we obtain for the r.h.s. of eq. (36)

\[
\frac{n^{(0)}_e n^{(0)}_\gamma \sigma_T T^{(0)}_\gamma}{m_e \varepsilon^{(0)}_{\text{crit}} a^4(\eta)} \simeq \frac{10^{-49}}{a^3(\eta)} \text{GeV} 
\]  

where we have assumed \(T_c \ll T_\gamma\). We compare this quantity with the divergence of the velocity in the l.h.s. of eq. (13), using eq. (24). Assuming \(\partial_z \sim z_0^{-1}\) where \(z_0\) is a characteristic scale of the background inhomogeneity, which for a galaxy is \(z^G_0 \sim 10^{35}\).
GeV$^{-1}$, we have $\eta_0/\xi_0^G \sim 10^2$. Recalling that initially the peculiar velocity is zero, we have that the effect of the heat transfer becomes negligible immediately after the matter starts falling into the dark matter overdensities.

For equation (20) we compare the second term in the l.h.s. with the r.h.s. For the first we have that the coefficient of $u^i$ reads $\dot{a}(\eta)/a(\eta) = 2/\eta = 2\eta_0^{-1}a^{-1/2}(\eta)$. For the coefficient of $u^i$ in the r.h.s. we have

$$\frac{n_e^{(0)}n_T^{(0)}\sigma_T}{\varepsilon_{crd}a^5(\eta)} \simeq \frac{10^{-45}}{a^5(\eta)} GeV$$

(39)

This factor will be smaller than the corresponding one in the l.h.s. of eq. (20) when $a(\eta)^{9/2} \geq 1$, and this constraint is always satisfied, because we have $a(\eta) \geq 1$. We therefore have that in the whole time interval considered in the paper, the equations for a perfect fluid apply.

3. Evolution equation for the magnetic field

As we have seen in the previous Section, after decoupling the partially ionized plasma falls into the newtonian potential wells created by dark matter. In its infall, it will drag the magnetic field lines, producing an enhancement of the field intensity, proportional to the depth of the potential well. Our goal in this Section is to study this effect. In the last subsection, we will discuss the limitations on our analysis.

3.1. Maxwell Equations

Let us begin by deriving the Maxwell Equations that satisfy the electromagnetic field in the spacetime we are working in. These were put in the $3 + 1$ form for the first time by G. F. R. Ellis (1973) and latter by K. Thorne & D. MacDonald (1982).
We begin by defining the field strength tensor $F_{\mu \nu}$ as usual

$$F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

(40)

where $A^m$ is the electromagnetic four potential. The electric and magnetic fields are given by

$$\mathcal{E}_\mu = F_{\mu \nu} U^\nu = -F_{\nu \mu} U^\nu$$

(41)

$$\mathcal{B}_\mu = \frac{1}{2} \eta_{\mu \alpha \beta} U^\nu F_{\nu \alpha \beta}$$

(42)

We can rewrite the field strength tensor in terms of these fields as

$$F_{\mu \nu} = U^\mu \mathcal{E}_\nu - \mathcal{E}_\mu U_\nu - \eta_{\mu \alpha \beta} U^\alpha \mathcal{B}_\beta$$

(43)

and with the indices up

$$F^{\mu \nu} = U^\mu \mathcal{E}^\nu - \mathcal{E}^{\mu} U^{\nu\prime} - \eta^{\mu \nu}_{\alpha \beta} U^\alpha \mathcal{B}^\beta$$

(44)

where $U^\mu$ is the four velocity of fiducial observers, which in our case is given by equation (2). The Maxwell equations can be written in covariant form as

$$F^{\mu \nu} ; \nu = J^\mu$$

(45)

$$\eta^{\mu \nu \alpha \beta} F_{\nu \alpha \beta} = 0$$

(46)

We will need the projections of these equations onto spatial surfaces and along the four velocity. We also transform the electromagnetic field and current as $\mathcal{E}^i \to E^i / a^2$, $\mathcal{B}^i \to B^i / a^2$, $J^i \to J^i / a^3$. The previous set of equations then reads

$$- E^{k} ; k + 3 \psi ; k E^{k} = - (1 + \psi) J^{0}$$

(47)

$$- \dot{E}^i (1 - \psi) + \eta^{ij} k_0 \left[ (1 - \psi) B^{k} ; j - 3 \psi ; j B^{k} \right] = J^i$$

(48)
\[ B^j_{\ i} - B^i_{\ j} = 0 \]  

\[ \eta^{ij}_{\ k0} \left[ (1 - \psi) E^k_{\ i,j} - 3 \psi E^k_{\ j} \right] + (1 - \psi) \dot{B}^i = 0 \]  

These are the equations that we will use in the main part of this work.

### 3.2. Plasma conductivity and the magnetic field

The currents and fields appearing in Maxwell's equations above are further related by Ohm's law

\[ \mathcal{J}^\mu + U^\mu U^\nu \mathcal{J}_\nu = \sigma F^{\mu\nu} U_\nu \]  

where \( U^\mu \) is the fluid four velocity given by \( U^\mu = U^\mu + \delta u^\mu \). Keeping only first order terms we have

\[ J^i - u^i \rho_e = \sigma \left[ E^i + \eta^{i0}_{\ pr} u^p B^r \right] \]  

In our case, it is appropriate to consider the plasma as perfectly conducting. This is essentially due to the fact that any charge separation would lead to an electrostatic attraction much stronger than the gravitational forces considered so far, and thus it cannot be sustained over any macroscopic lapse. For an infinite conductivity, the electric field in the rest frame of the plasma must vanish, and this yields an extra relation among the electric and magnetic fields, and the plasma velocity in that frame, namely

\[ E^i = -\eta^{i0}_{\ pr} u^p B^r \]  

For equation (50) we have

\[ (1 - \psi) \nabla \times (u \times B) - 3 \nabla \psi \times (u \times B) = (1 - \psi) \frac{\partial B}{\partial \eta} \]
This is the evolution equation for the magnetic field.

Besides the arguments above, it is possible to give a direct estimate of the plasma conductivity that confirms it may be considered as infinite. Suppose an electric field were imposed on the plasma, generating an electron flux with velocity $v_e$ and a current $j = en_e v_e$. The field yields a Joule power $P = Ej$, which must be equal to the power dissipated into the CMB by the friction force (recall the discussion in the previous Section)

$$P = 2n_e n_\gamma \sigma_T v_e^2 T = \frac{2n_e^0 n_\gamma^0 \sigma_T v_e^2 T^0}{a(\eta)^9}$$

Therefore

$$v_e = \frac{eE}{T n_\gamma \sigma_T}$$

from where we read the conductivity

$$\sigma_c = \frac{e^2 n_e}{T n_\gamma \sigma_T}$$

Estimating its value with the same figures as before we obtain

$$\sigma_c \sim 10^{-11} a(\eta) \, GeV$$

In order to determine if this conductivity is large enough to be considered as infinite, we may compare the conduction current with the displacement current generated by the collapse, $\partial_\eta E \sim E/\eta_0$, where $\eta_0^{-1} \sim 10^{-38}GeV$. We see that we can consider the conductivity of the plasma as infinite for all practical purposes.

4. Evaluation of the magnetic field

The evolution equation for the magnetic field can be simplified as follows: Write $u = (0, 0, u_z)$ and $B = B(\eta, z)$. Then

$$\nabla \times \mathbf{B} = (-\partial_z B_y, \partial_z B_x, 0)$$
\[ \mathbf{u} \times \mathbf{B} = (-u_z B_y, u_z B_x, 0) \]  
\[ \nabla \times (\mathbf{u} \times \mathbf{B}) = (-\partial_z [u_z B_y], -\partial_z [u_z B_y], 0) \]  
\[ \nabla \psi \times (\mathbf{u} \times \mathbf{B}) = (-\psi_z u_z B_x, -\psi_z u_z B_y, 0) \]

We see that \( B_z \) is not affected by the collapse. For \( B_x \) and \( B_y \) we can rewrite equation (54) as (ommiting the subindex)

\[ \partial_\eta \left[(1 - \psi)^3 B\right] + \partial_z \left[(1 - \psi)^3 Bu^z\right] = 0 \]  

which expresses the conservation of the quantity \((1 - \psi)^3 B\).

At this point it is important to note that we cannot recast equation (18), which expresses the conservation of the number density that characterizes the ionized fraction, in a form similar to eq. (63). This means that the relation between \( \delta n \) and \( B \) might not be a simple one. Nevertheless since \( \psi \) is a small deviation from inhomogeneity, we can obtain a conservation law from eq. (18) if we neglect the last term and the newtonian potential in the first term. We obtain

\[ \delta \dot{n} + (u^z \delta n)_{,z} = 0 \]  

where

\[ u^z = f(z) \left[ \left( \frac{\eta_0}{\eta} \right)^2 - \frac{\eta}{\eta_0} \right] \]

with \( f(z) = \eta_0 \partial_z \psi(z)/3 \). Performing the same approximations in equation (63), this equation reads

\[ \dot{B} + [Bu^z]_{,z} = 0 \]

Since both \( B \) and \( \delta n \) are assumed to be homogeneous at \( \eta_0 \), from eqs. (54) and (63) we may conclude that \( B \propto \delta n \), at all times, where the constant of proportionality is given by the value of \( B/\delta n \) at \( \eta = \eta_0 \), i.e. we have

\[ B(\eta, z) = \frac{B_0}{\delta n_0} \delta n(\eta, z) \]
Observe that this is not the behaviour expected for a homogeneous universe, where \( u^j = 0 \) and consequently the solution to eq. (66) would be \( B = \text{const.} \). The solution to eq. (64) reads

\[
\delta n = \frac{1}{f(z)} H \left[ \int^z \frac{dz'}{f(z')} + \eta_0 \left( \frac{1}{2} \frac{\eta^2}{\eta_0^2} + \frac{\eta_0}{\eta} - \frac{3}{2} \right) \right]
\]

(68)

where \( H (\zeta) \) is an arbitrary function obtained from the value of \( \delta n \) at \( \eta = \eta_0 \). The hypothesis that the magnetic field \( B_0 \) is initially uniform over the horizon size, although simplistic, is valid for weak magnetic fields (Calzetta et al. 1998) and also helps to simplify the calculations. The real initial large scale structure of magnetic fields is still under study. Jedamzik et al (1998) have shown that linear magnetohydrodynamic modes suffer damping during recombination and neutrino decoupling, while Subramanian and Barrow (1998) analyzed the non-linear case. They both find scales below which those modes are dissipated. Nevertheless their treatment refers to perturbations to a background magnetic field, while here we aim to study the evolution of that background field in an inhomogeneous universe.

In order to achieve our purpose, we only need the newtonian potential profile \( \psi(z) \) and to our ends, it is enough to consider that the dark matter is still in the linear regime (see below), the shape of the potential being then essentially given by the theory of primordial density generation (Bardeen et al. 1983, Guth & Pi 1982, Starobinski 1982, Hawking 1982). However, this theory does not yield a deterministic prediction, but only the relative probabilities of different density contrast profiles. Therefore we shall restrict ourselves to considering a few simple situations, namely, a power law density profile, a uniform slab of finite height, and a harmonic density contrast.

**Power law density contrast** Let us consider dark matter density profiles of the form

\[
\delta(z) = \begin{cases} 
\xi \left( \frac{z}{z_0} \right)^{p-2} & \text{for } z < z_0 \\
0 & \text{for } z > z_0 
\end{cases}
\]

(69)
They give rise to power law (generalized) Newtonian potentials, i.e.:

\[
\psi(z) = \begin{cases} 
\alpha \left( \frac{z}{z_0} \right)^p & \text{for } z < z_0 \\
\alpha \left( p \frac{z}{z_0} - p + 1 \right) & \text{for } z > z_0
\end{cases}
\]

(70)

where \( \alpha \) has dimensions of \text{length}^{-1} and its amplitude is \( \alpha \sim 10^{-5} \), and \( z_0 \) determines the spatial extension of the perturbation, which can be the one of a galaxy \( (z_0 \approx 1.42 \times 10^{35} \text{GeV}^{-1}) \), or cluster of galaxies \( (z_0 \approx 10^{36} \text{GeV}^{-1}) \) for example.

Physically meaningful profiles require \( 1 \leq p \leq 2 \); \( p < 1 \) implies an infinite force towards the center of the structure and \( p > 2 \) means that the density increases towards the outer edges of it. The function \( f(z) \) reads

\[
f(z) = \begin{cases} 
\frac{pn_0}{3} \alpha \frac{z^{p-1}}{z_0^p} & \text{for } z < z_0 \\
\frac{pn_0}{3z_0} \alpha & \text{for } z > z_0
\end{cases}
\]

(71)

and

\[
\int^z d\zeta' f(\zeta') = \begin{cases} 
\frac{3z_0^p}{pn_0} \alpha \frac{1}{(2-p)} z^{2-p} & \text{for } z < z_0 \\
\frac{3z_0^2}{pn_0} \alpha z & \text{for } z > z_0
\end{cases}
\]

(72)

As stated above, we evaluate the function \( H(\zeta) \) by fixing the functional form of \( \delta n(z, \eta) \) at \( \eta = \eta_0 \). In view of the lack of any prescription for it, we also choose it as a constant, i.e. \( \delta n(z, \eta) = \delta n_0 \). We then have

\[
H[\zeta] = \delta n_0 \frac{pn_0}{3z_0} \alpha \left[ \frac{(2-p)pn_0\alpha \zeta}{3z_0^p} \right]^{(p-1)/(2-p)} ; \quad \zeta \leq \frac{3z_0^2}{pn_0 \alpha (2-p)}
\]

(73)

\[
H[\zeta] = \delta n_0 \frac{pn_0\alpha}{3z_0} ; \quad \text{otherwise}
\]

(74)

Finally the density profile \( \delta n \) reads

For \( z > z_0 \)

\[
H[z, \eta] = \delta n_0 \frac{pn_0\alpha}{3z_0}
\]

(75)
\[
\begin{align*}
    f[z] &= \frac{p\eta_0}{3z_0} \tag{76} \\
    \delta n &= \delta n_0 \tag{77}
\end{align*}
\]

For \( z_0^{-p} - \frac{\alpha p(2-p)\eta_0^2}{3z_0^p} \left( \frac{\eta_0}{\eta} + \frac{\eta^2}{\eta_0^2} - \frac{3}{2} \right) < z^{-p} < z_0^{-p} \)

\[
\begin{align*}
    H[z, \eta] &= \delta n_0 \frac{\alpha p\eta_0}{3z_0} \tag{78} \\
    f[z] &= \frac{p\eta_0\alpha}{3} \frac{z^{p-1}}{z_0^p} \tag{79} \\
    \delta n &= \delta n_0 \frac{z_0^{p-1}}{z^{p-1}} \tag{80}
\end{align*}
\]

For \( 0 < z^{-p} < z_0^{-p} - \frac{\alpha p(2-p)\eta_0^2}{3z_0^p} \left( \frac{\eta_0}{\eta} + \frac{\eta^2}{\eta_0^2} - \frac{3}{2} \right) \)

\[
\begin{align*}
    H[z, \eta] &= \delta n_0 \frac{p\eta_0\alpha}{3z_0} \left[ z^{-p} + \frac{(2-p)p\alpha}{3z_0^p} \eta_0 \left( \frac{\eta_0}{\eta} + \frac{1}{2} \frac{\eta^2}{\eta_0^2} - \frac{3}{2} \right) \right]^{(p-1)/(2-p)} \tag{81} \\
    f[z] &= \frac{p\eta_0\alpha}{3} \frac{z^{p-1}}{z_0^p} \tag{82} \\
    \delta n &= \delta n_0 \frac{1}{z^{p-1}} \left[ z^{-p} + \frac{(2-p)p\alpha}{3z_0^p} \eta_0 \left( \frac{\eta_0}{\eta} + \frac{1}{2} \frac{\eta^2}{\eta_0^2} - \frac{3}{2} \right) \right]^{(p-1)/(2-p)} \tag{83}
\end{align*}
\]

We can see from eq. (77) that the magnetic field will grow with time for all exponents in the physical range \( 1 < p < 2 \). It remains unperturbed outside the dark matter distribution, and inside it freezes for long times with a profile given by eq. (83). For \( p = 1 \) we obtain \( B = B_0 \) for the whole range of \( z \). For example, for \( p = 3/2 \) and the density profile grows like

\[
\begin{align*}
    \delta n &= \delta n_0 \frac{1}{\sqrt{z}} \left\{ \sqrt{z} + \frac{\alpha \eta_0^2}{4z_0^{3/2}} \left[ \frac{\eta_0}{\eta} + \frac{1}{2} \frac{\eta^2}{\eta_0^2} - \frac{3}{2} \right] \right\} \tag{84} \\
    \text{for} \quad 0 < \sqrt{z} < \sqrt{z_0} - \frac{\alpha \eta_0^2}{4z_0^{3/2}} \left[ \frac{\eta_0}{\eta} + \frac{1}{2} \frac{\eta^2}{\eta_0^2} - \frac{3}{2} \right] \tag{85}
\end{align*}
\]

\[
\begin{align*}
    \delta n &= \delta n_0 \frac{\sqrt{z_0}}{z} \\
    \text{for} \quad \sqrt{z_0} - \frac{\alpha \eta_0^2}{4z_0^{3/2}} \left[ \frac{\eta_0}{\eta} + \frac{1}{2} \frac{\eta^2}{\eta_0^2} - \frac{3}{2} \right] < \sqrt{z} < \sqrt{z_0} \tag{85}
\end{align*}
\]
and

\[ \delta n = \delta n_0; \quad for \quad z > z_0 \]  \hspace{1cm} (86)

We now analyze the case \( p = 2 \).

**Homogeneous density contrast.** Let us consider a dark matter density contrast profile given by

\[ \delta(z) = \begin{cases} 
\xi & \text{for } z < z_0 \\
0 & \text{for } z > z_0 
\end{cases} \]  \hspace{1cm} (87)

The generalized Newtonian potential can be written as

\[ \psi(z) = \begin{cases} 
\alpha \left( \frac{z}{z_0} \right)^2 & \text{for } z < z_0 \\
\alpha \left( 2 \frac{z}{z_0} - 1 \right) & \text{for } z > z_0 
\end{cases} \]  \hspace{1cm} (88)

The function \( f(z) \) then reads

\[ f(z) = \begin{cases} 
\frac{2 \eta_0 z}{3 z_0} & \text{for } z < z_0 \\
\frac{2 \eta_0}{3 z_0} & \text{for } z > z_0 
\end{cases} \]  \hspace{1cm} (89)

and

\[ \int^{z'}_{z} \frac{dz'}{f(z')} = \begin{cases} 
\frac{3 z_0^2}{2 \eta_0 \alpha} \ln \left( z \right) & \text{for } z < z_0 \\
\frac{3 z_0}{2 \eta_0 \alpha} & \text{for } z > z_0 
\end{cases} \]  \hspace{1cm} (90)

With the same initial conditions as before, the function \( H(\zeta) \) is given by

\[ H[\zeta] = \delta n_0 \frac{2 \alpha \eta_0 B_0}{3 z_0^2} \exp \left\{ \frac{2 \eta_0 \alpha}{3 z_0^2} \zeta \right\}; \quad \text{if } \zeta \leq \frac{3 z_0^2}{2 \eta_0 \alpha} \ln z_0 \]  \hspace{1cm} (91)

\[ H[\zeta] = \delta n_0 \frac{2 \alpha \eta_0}{3 z_0}; \quad \text{otherwise} \]  \hspace{1cm} (92)

The density contrast reads

\[ \delta n = \delta n_0 \exp \left\{ \frac{2 \eta_0^2 \alpha}{3 z_0^2} \left( \frac{\eta_0}{\eta} + \frac{1}{2 \eta_0^2} - \frac{3}{2} \right) \right\} \]  \hspace{1cm} (93)

\[ \text{for } 0 < z < z_0 \exp \left[-\frac{2 \alpha \eta_0^2}{3 z_0^2} \left( \frac{\eta_0}{\eta} + \frac{1}{2 \eta_0^2} - \frac{3}{2} \right) \right] \]
\[ \delta n = \delta n_0 \frac{z_0}{z}; \quad (94) \]

for \( z_0 \exp \left[ -\frac{2\alpha\eta_0^2}{3z_0^2} \left( \frac{\eta_0}{\eta} + \frac{1}{2} \frac{\eta^2}{\eta_0} - \frac{3}{2} \right) \right] < z < z_0 \]

and

\[ \delta n = \delta n_0; \quad for \quad z > z_0 \quad (95) \]

**Harmonic density profile**  Assume that we have a density distribution whose newtonian potential is given by

\[ \psi(z) = \alpha \left[ \frac{1}{2} - \cos \left( \frac{2\pi z}{z_0} \right) \right] \quad (96) \]

This function is defined over the whole particle horizon, \( z_0 \) is a characteristic scale of the background perturbations, that could be interpreted as the distance between two adjacent galaxies. The function \( f(z) \) reads

\[ f(z) = \left( \frac{2\pi \alpha\eta_0}{3z_0} \right) \sin \left( \frac{2\pi z}{z_0} \right) \quad (97) \]

and

\[ \frac{3z_0}{2\pi\alpha\eta_0} \int z \frac{dz}{\sin (2\pi z/z_0)} = \frac{3z_0^2}{4\pi^2\alpha\eta_0} \ln \tan \left( \frac{\pi z}{z_0} \right) \quad (98) \]

According to what we did before, we have to fix the function \( H[\zeta] \) by specifying the density contrast at \( \eta_0 \). We therefore have

\[ \delta n_0 = \frac{3z_0}{2\pi\alpha\eta_0} H \left[ \frac{3z_0^2}{4\pi^2\alpha\eta_0} \ln \left( \tan \frac{\pi z}{z_0} \right) \right] \quad (99) \]

from where we have

\[ H[\zeta] = \delta n_0 \frac{4\pi\alpha\eta_0}{3z_0} \frac{1}{1 + \exp \left[ \frac{2\pi\alpha\eta_0}{z_0} \zeta \right]} \quad (100) \]
And the density profile is
\[
\delta n = \delta n_0 \left[ 1 + \tan^2 \left( \frac{\pi z}{z_0} \right) \right] \exp \left\{ \frac{2\pi \alpha \eta_0^2}{3z_0^2} \left( \frac{\eta_0}{\eta} + \frac{1}{2} \eta_0^2 - \frac{3}{2} \right) \right\} \exp \left\{ \frac{4\pi \alpha \eta_0^2}{3z_0^2} \left( \frac{\eta_0}{\eta} + \frac{1}{2} \eta_0^2 - \frac{3}{2} \right) \right\}
\]
(101)

We find once again that the density contrast freezes for long times. This time, the final shape is given by

\[
\delta n = \frac{\delta n_0}{\sin^2 \left( \frac{\pi z}{z_0} \right)}
\]
(102)

Obviously, the smaller the value of the sin function, the longer the time it takes to reach the value given by equation (102).

4.1. Interval of validity of the calculations

In the case of the power law density profile we see that the magnetic field diverges for \( z \to 0 \) for all times. This is due to the fact that the density profiles diverges at that point. We will therefore analyze the limits of validity of the calculations for the case of uniform dark matter density contrast and of harmonic density.

The first point is to specify until which moment the background perturbations can be considered in their linear regime. With the value \( \alpha \sim 10^{-5} \) fixed by COBE, we have that the galactic scale entered its nonlinear regime at a redshift \( Z_{nl}^G \sim 5 \), which corresponds to \( \eta \simeq 15\eta_0 \), where \( \eta_0 \simeq 0.4h^{-1} \times 10^{38}GeV^{-1} \) is the conformal time at decoupling. For a galaxy cluster, the dark matter is still in its linear regime (Peebles 1993).

The other assumption we are going to check is neglecting the effect of the magnetic field on the evolution of the perturbations, namely the the r.h.s in equation (5). The field affects the motion of the plasma through the Lorentz force

\[
\mathcal{F}^m = J_n F^{mn}
\]
(103)
$F^{mn}$ is the e.m. field strength tensor and $J^m$ the induced electric four current. Since $F^{mn}$ is spacelike to lowest order, the projection along the four velocity is negligible, and the orthogonal projection, written in terms of the conformal magnetic field, yields $[J \times B]^i$.

The current is given by Maxwell’s equations (47) and (48); after substituting the electric field eq. (53), we obtain $J^0 = 0$ (thus no charge separation) and

$$J = \partial_\eta u \times B + u \times \partial_\eta B + (1 - \psi) \nabla \times B - 3 \nabla \psi \times B$$

(it is interesting to note that $\vec{J}$ is actually orthogonal to the macroscopic mass flow). For the symmetry of our problem, and neglecting pressure effects, the equation of motion for matter is changed into

$$\partial_z \psi + \dot{u}_z + \frac{\dot{a}}{a} u_z = -\frac{B_y \partial_z B_y + B_x \partial_z B_x}{a \varepsilon_0}$$

We have to find at what value of $\eta$ the inequality

$$|\partial_z \psi| \gg \left| \frac{B_y \partial_z B_y + B_x \partial_z B_x}{a \varepsilon_0} \right|$$

breaks down. Let us consider the uniform slab. Using equations (88) and (94) we obtain

$$2 \alpha \frac{z}{z_0} \gg \frac{B_0^2 z_0^2}{z^3 \varepsilon_0} \left( \frac{\eta_0}{\eta} \right)^2 \rightarrow \left( \frac{\eta}{\eta_0} \right)^2 \gg \frac{B_0^2 z_0^4}{2 \alpha z^4 \varepsilon_0}$$

The correct bound is found by replacing the smallest value of the $z/z_0$ coordinate from equation (93) to obtain

$$2 \alpha \frac{z}{z_0} \gg \frac{B_0^2 z_0^2}{z^3 \varepsilon_0} \left( \frac{\eta_0}{\eta} \right)^2 \rightarrow \left( \frac{\eta}{\eta_0} \right)^2 \gg \frac{B_0^2}{2 \alpha z^4 \varepsilon_0} \exp \left[ \frac{8 \alpha \eta_0^2}{3 z_0^2} \left( \frac{\eta_0}{\eta} + \frac{1}{2 \eta_0^2} - \frac{3}{2} \right) \right]$$

As stated before, $\varepsilon_0$ is the critical density of the Universe at recombination, $\varepsilon_0 \sim 8h^2 \times 10^{-38}$ GeV$^4$. We take the value of the primordial field at decoupling as being given by the mechanisms proposed in the literature for a coherent field at galactic scale (Dolgov 1993, Ratra 1992, Mazzitelli & Spedalieri 1995, Calzetta et al. 1998, Cheng &
We estimate its value at decoupling as $B_0 \sim (1 + z_{dec})^2 B_{\text{today}} \sim 10^{-39} \text{GeV}^2$ and we therefore have $B_0^2/\varepsilon_0 \alpha \simeq 10^{-36}$. Replacing in equation (108) we can check that the inequality is satisfied for $\eta/\eta_0 < 2.22$ ($Z \sim 200$) for a galactic scale $z_G^0 \simeq 10^{35} \text{GeV}^{-1}$ obtaining at that moment an amplification factor of the order $10^4$. For a cluster scale $z_C^0 \simeq 10^{36} \text{GeV}^{-1}$ and equation (93) is valid up to $\eta/\eta_0 \leq 17.5$ for the same value of $\alpha$, obtaining a similar amplification factor.

It is also immediate to check that the neglecting of the last term in eq. (18) is a valid approximation for all times.

We therefore conclude that the validity of neglecting the backreaction of the magnetic field on the evolution of the perturbation fixes the time interval of validity of our calculations.

### 5. Final Remarks

We have studied the evolution of a preexisting cosmological magnetic field after recombination of hydrogen, in a background Universe described by a FRW geometry plus adiabatic perturbations, and considered that the field is coupled to the remaining ionized fraction. This matter falls into the background overdensities, dragging with it the magnetic field. This dragging results in a deviation of the behaviour of the magnetic field from the law $B \propto \delta^2/3$ where $\delta$ is a conserved number density of the plasma, even when the background perturbations are in their linear regime of evolution, as can be seen from equation (87) together with (84), (93) and (101). The background density profiles used, although simplistic, retain an important physical characteristic, namely the asymmetry of
the gravitational collapse. In the case of the harmonic potential, we could also think that it describes an alternating pattern of walls and voids (Broadhurst et al. 1990), despite the difference with the scales used in this work. The rate of growth of the field depends on the characteristics of the background inhomogeneities and on the initial distribution of magnetic field intensities. The hypothesis of homogeneity of the field over the horizon scale at recombination can be sustained provided that the intensity of the field is sufficiently low, a fact achieved in most of the mechanisms of primordial field generation (Dolgov 1993, Ratra 1992, Mazzitelli & Spedalieri 1995, Calzetta et al. 1998, Cheng & Olinto 1994, Sigl et al. 1997, Enqvist & Olesen 1994, Vachaspati 1991, Dolgov & Silk 1993, Hogan 1983, Martin & Davies 1995): the larger the coherent scale, the weaker the field intensity.

The level of amplification depends mainly on the scale of the background inhomogeneity, as stated in the last subsection, and have its maximum at the center of the structure. For the $x - y$ plane, the initial symmetry remains because gravitational collapse is along the $z$ axis only. As stated in the previous subsection, our calculations are valid up to $\eta \simeq 2.22\eta_0$ for galaxies and for the whole time interval for cluster scales.

Our main result is that it is not correct to consider the magnetic field evolves as $B \propto B_0/a(\eta)^2$ after recombination. We found that even in the linear regime of gravitational collapse, there can be nontrivial corrections to that behaviour. Even if the growth law were a power law, we could have that the field remains constant between recombination and the beginning of an amplifying mechanism, over the scale of interest. Clearly we do not claim to have solved the problem of the origin of magnetic fields in high redshifted systems, like the damped Lyman-$\alpha$ clouds, but if the scenario of primordial field generation plus further amplification is accepted, our results might help to relax the constraints on mechanisms for primordial field generation, because now the input fields for any further amplifying mechanism could be orders of magnitude stronger than the values that those fields would have if they had simply decayed as $a^{-2}(\eta)$. For instance, it is stated (Zel’dovich et al. 1990).
that the present intensity of primordial magnetic field that can seed a galactic dynamo is $B \sim 10^{-21}$ Gauss over a scale of 1 Mpc. Most of the proposed mechanism for creation of the field give a present value $B \sim \mathcal{O}(10^{-24})$, this means that with the usual conformal evolution of $B$, the value of the field at recombination would be $B_{\text{rec}} \sim 10^{-18}$ Gauss, and at $\eta \sim 2\eta_0$, $B \sim 10^{-19}$, while with our calculations, the intensity of the field at that moment would be $B \sim 10^{-16}$ Gauss.

The analysis in this paper may be improved in several particulars, the most important ones being the inclusion of vorticity in the plasma and the extension of our calculations into the nonlinear regime. The others are assuming a more realistic density profile for the dark matter component, and avoiding treating the ionized matter as a fluid by going directly to a kinetic description, introducing explicitly the dynamical balance between the recombined and ionized fractions of ordinary matter. However, as it stands it already shows that there are rich magnetohydrodynamical processes occurring as early as $Z \sim 100$, which must be properly understood before our picture of the physics of cosmic magnetic fields is complete.

6. Acknowledgments

This work has been supported by University of Buenos Aires, CONICET and Fundación Antorchas. A. K. wishes to acknowledge Dr. H. Rubinstein and the hospitality of the University of Stockholm, where part of this work was completed. E. C. and A. K. thank Oscar Reula for his valuable comments on an earlier version of this manuscript.
7. Appendix

The Christoffel symbols needed to calculate the covariant derivatives are

\[\Gamma^0_{00} = \frac{\dot{a}}{a} + \dot{\psi}\]
\[\Gamma^j_{k0} = \delta^j_k \left( \frac{\dot{a}}{a} - \dot{\psi} \right)\]
\[\Gamma^i_{jk} = \psi^i/j \gamma_{jk} - \psi/j \delta^i_k \delta^j_k\]
\[\Gamma^0_{00} = \gamma_{ik} \frac{\dot{a}}{a} \psi/k\]

In the context of the paper, these are further simplified, since \(\dot{\psi} = 0\).
REFERENCES

Bardeen J. M., Steinhardt P. J. and Turner M. S. 1983, Phys. Rev. D28, 679.

Blasi P, Burles S and Olinto A. V. 1999, ApJ514.

Bond J. R., Kofman L. & Pogosyan D. 1996, Nature380, 603.

Brandenberger R. H., Feldman H. A. and Mukhanov V. F. 1992, Phys. Rep.215, 203.

Brandenburg A. , Enqvist K. and Olesen P. 1997, Phys. Lett. B392, 395.

Broadhurst T. J. , Ellis R. S., Koo D. C. & Szalay A. S. 1990, Nature 343, 726.

Calzetta E. A. , Kandus A. and Mazzitelli F. D. 1998, Phys. Rev. D57, 7139.

Cheng B. and Olinto A. 1994, Phys. Rev. D50, 2421.

Dolgov A. D. 1993, Phys. Rev. D48, 2499.

Dolgov A. and Silk J. 1993, Phys. Rev. D47, 3144.

Ellis G. F. R. 1973, in Cargèse Lectures in Physics, Vol 6, ed. E. Schatzman (Gordon and Breach, New York), 1.

Ellis G.F.R. and van Elst H.1998 , Cosmological Models, in Cargèse Lectures 1998, preprint gr-qc/9812046.

Enqvist K. and Olesen P. 1994, Phys. Lett. B 329, 195.

Guth A. H. and Pi S.-Y. 1982, Phys. Rev. Lett.49, 1110.

Harrison E. R. 1973, MNRAS165, 185.

Hawking S. W. 1982, Phys. Lett. B115, 295.
Hogan C. 1983, Phys. Rev. Lett. 51, 1488.

Howard A. M. and Kulsrud R. M. 1996, preprint [astro-ph/9609031].

Israel W. 1988, in Relativistic fluid dynamics, ed. A. Anile and Y. Choquet - Bruhat (Springer, New York).

Coles, P. & Lucchin F. 1995, Cosmology: The Origin and Evolution of Cosmic Structure, (John Wiley & Sons, New York).

Jedamzik K., Katalinic V. and Olinto A. 1998, Phys. Rev. D 57, 3264.

Martin A. P. and Davies A. C. 1995, Phys. Lett. B 360, 71.

Mazzitelli F. D. and Spedalieri F. M. 1995, Phys. Rev. D 52, 6694.

Moffatt K. 1961, J. Fluid Mech. 11, 625.

Olesen P. 1997, Phys. Lett. B 398, 321.

Oren A. L. and Wolfe A. M. 1995, ApJ 445, 624.

Peebles P. J. E. 1993, Principles of Physical Cosmology, (NJ, Princeton).

Pudritz R. E. and Silk J. 1989, ApJ 342, 650.

Ratra B. 1992, ApJ 391, L1.

Ryu D., Kang H. and Biermann P. L. 1998, A&A 335

Sigl G., Olinto A. & Jedamzik K. 1997, 55, 4582.

Starobinskii A. A. 1982, Phys. Lett. B 117, 175 (1982).

Subramanian K. and Barrow J. D. 1998, Phys. Rev. D 58, 083502.
Thorne K. S. and MacDonald D. 1982, MNRAS198, Microfiche MN 198/1, 339.

Vachaspati T. 1991, Phys. Lett. B265, 258.

Weinberg S. 1971, ApJ168, 175.

Wolfe A. M., Lanzetta K. and Oren A. L. 1992, ApJ388, 17.

Zel’dovich Ya. B. and Novikov I. D. 1983, Relativistic Astrophysics, Vol. 2: The Structure and Evolution of the Universe, (The University of Chicago Press, Illinois).

Zel’dovich Ya. B., Ruzmaikin A. A. and Sokoloff D. D. 1990, Magnetic Fields in Astrophysics, 2nd. ed., (Gordon and Breach, Montreux).