MATRICES, CHARACTERS AND DESCENTS

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Abstract. A new family of asymmetric matrices of Walsh-Hadamard type is introduced. We study their properties and, in particular, compute their determinants and discuss their eigenvalues. The invertibility of these matrices implies that certain character formulas are invertible, yielding expressions for the cardinalities of sets of combinatorial objects with prescribed descent sets in terms of character values of the symmetric group.

1. Introduction

Many character formulas involve the descent set of a permutation or of a standard Young tableau. We propose here a general setting for such formulas, involving a new family of asymmetric matrices of Walsh-Hadamard type. These matrices turn out to have fascinating properties, some of which are studied here using a transformation based on Möbius inversion. These include the evaluation of determinants, entries of transformed matrices and their inverses, and eigenvalues.

The inverse matrices lead to formulas expressing the cardinalities of sets of combinatorial objects with prescribed descent sets in terms of character values of the symmetric group. Examples of such objects include permutations of fixed length, involutions, standard Young tableaux, and more. It also follows that certain statements in permutation statistics have equivalent formulations in character theory. For example, the fundamental equi-distribution Theorem of Foata and Schützenberger, independently proved by Garsia and Gessel, is equivalent to a theorem of Lusztig and Stanley in invariant theory.

The organization of this paper is as follows: Section 2 contains the necessary definitions and background material, ending with a statement of the central motivating question. Section 3 introduces the main tool – a family (actually, two “coupled” families) of square matrices – and states some of their properties. Section 4 contains a proof of the invertibility of these matrices, using a transformation corresponding to Möbius inversion. Properties of the transformed matrices

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are described in Section 5. The application to character formulas, involving the concept of fine sets, is described in Section 6.

2. Preliminaries and notation

2.1. Intervals, compositions, partitions and runs.

For positive integers $m, n$ denote 
\[ [m, n] := \begin{cases} \{m, m + 1, \ldots, n\}, & \text{if } m \leq n; \\ \emptyset, & \text{otherwise.} \end{cases} \]

Denote also $[n] := [1, n] = \{1, \ldots, n\}$.

A composition of a positive integer $n$ is a vector $\mu = (\mu_1, \ldots, \mu_t)$ of positive integers such that $\mu_1 + \cdots + \mu_t = n$. A partition of $n$ is a composition with weakly decreasing entries $\mu_1 \geq \cdots \geq \mu_t > 0$. The underlying partition of a composition is obtained by reordering the entries in weakly decreasing order.

For each composition $\mu = (\mu_1, \ldots, \mu_t)$ of $n$ define the set of its partial sums 
\[ S(\mu) := \{\mu_1, \mu_1 + \mu_2, \ldots, \mu_1 + \cdots + \mu_t = n\} \subseteq [n], \]

as well as its complement 
\[ I(\mu) := [n] \setminus S(\mu) \subseteq [n-1]. \]

For example, for the composition $\mu = (3, 4, 2, 5)$ of 14: 
\[ S(\mu) = \{3, 7, 9, 14\} \quad \text{and} \quad I(\mu) = \{1, 2, 4, 5, 6, 8, 10, 11, 12, 13\}. \]

The correspondence $\mu \leftrightarrow I(\mu)$ is a bijection between the set of all compositions of $n$ and the power set (set of all subsets) of $[n-1]$. The runs (maximal consecutive intervals) in $I(\mu)$ correspond to some of the components of $\mu$ – those satisfying $\mu_k > 1$. The length of the run corresponding to $\mu_k$ is $\mu_k - 1$.

2.2. Permutations, Young tableaux and descent sets.

Let $S_n$ be the symmetric group on the letters $1, \ldots, n$. For $1 \leq i \leq n - 1$ denote $s_i := (i, i + 1)$, a simple reflection (adjacent transposition) in $S_n$. For a composition $\mu = (\mu_1, \ldots, \mu_t)$ of $n$ let 
\[ s_{\mu} := (1, 2, \ldots, \mu_1)(\mu_1 + 1, \mu_1 + 2, \ldots, \mu_1 + \mu_2)\cdots \in S_n, \]

a product of $t$ cycles of lengths $\mu_1, \mu_2, \ldots, \mu_t$ consisting of consecutive letters. The permutation $s_{\mu}$ may be obtained from the product $s_1 s_2 \cdots s_{n-1}$ of all simple reflections (in the usual order) by deleting the factors $s_{\mu_1 + \cdots + \mu_k}$ for all $1 \leq k < t$; equivalently, 
\[ s_{\mu} = \prod_{i \in I(\mu)} s_i. \]

The descent set of a permutation $\pi \in S_n$ is $\text{Des}(\pi) := \{i : \pi(i) > \pi(i + 1)\}$.

The descent set of a standard Young tableau $T$ is the set $\text{Des}(T) := \{1 \leq i \leq n - 1 : i + 1 \text{ lies southwest of } i\}$. 

2.3. \(\mu\)-unimodality.

A sequence \((a_1, \ldots, a_n)\) of distinct positive integers is \emph{unimodal} if there exists \(1 \leq m \leq n\) such that \(a_1 > a_2 > \ldots > a_m < a_{m+1} < \ldots < a_n\). (This definition differs slightly from the commonly used one, where all inequalities are reversed.)

Let \(\mu = (\mu_1, \ldots, \mu_t)\) be a composition of \(n\). A sequence of \(n\) positive integers is \(\mu\)-\emph{unimodal} if the first \(\mu_1\) integers form a unimodal sequence, the next \(\mu_2\) integers form a unimodal sequence, and so on. A permutation \(\pi \in S_n\) is \(\mu\)-\emph{unimodal} if the sequence \((\pi(1), \ldots, \pi(n))\) is \(\mu\)-unimodal. For example, \(\pi = 936871254\) is \((4,3,2)\)-unimodal, but not \((5,4)\)-unimodal.

Let \(U_\mu\) be the set of all \(\mu\)-unimodal permutations in \(S_n\).

2.4. A family of character formulas.

Let \(\lambda\) and \(\mu\) be partitions of \(n\). Let \(\chi^\lambda\) be the \(S_n\)-character of the irreducible representation \(S^\lambda\), and let \(\chi^\lambda_\mu\) be its value on a conjugacy class of cycle type \(\mu\). The following formula for the irreducible characters is a special case of [12, Theorem 4]. For a direct combinatorial proof see [11].

\[\text{Theorem 2.1. [12, Theorem 4]} \]
\[\chi^\lambda_\mu = \sum_{\pi \in C \cap U_\mu} (-1)^{|\text{Des}(\pi) \cap I(\mu)|},\]
where \(C\) is any Knuth class of RSK-shape \(\lambda\).

Let \(\chi^{(k)}\) be the \(S_n\)-character defined by the symmetric group action on the \(k\)-th homogeneous component of the coinvariant algebra. Then

\[\text{Theorem 2.2. [1, Theorem 5.1]} \]
\[\chi^{(k)}_\mu = \sum_{\pi \in L(k) \cap U_\mu} (-1)^{|\text{Des}(\pi) \cap I(\mu)|},\]
where \(L(k)\) is the set of all permutations of length \(k\) in \(S_n\).

A complex representation of a group or an algebra \(A\) is called a Gelfand model for \(A\) if it is equivalent to the multiplicity free direct sum of all the irreducible \(A\)-representations. Let \(\chi^G\) be the character of the Gelfand model of \(S_n\) (or of its group algebra).

\[\text{Theorem 2.3. [3, Theorem 1.2.3]} \]
\[\chi^G_\mu = \sum_{\pi \in I_n \cap U_\mu} (-1)^{|\text{Des}(\pi) \cap I(\mu)|},\]
where \(I_n := \{\sigma \in S_n : \sigma^2 = \text{id}\}\) is the set of all involutions in \(S_n\).
More character formulas of this type are described in Subsection 6.4.

In this paper we propose a general setting for all of these results. In particular, we provide an answer to the following question.

**Question 2.4.** Are these character formulas invertible?

3. Two families of matrices

It is well known that partitions of \( n \) are the natural indices for the characters and conjugacy classes of \( S_n \). It turns out that a major step towards an answer to Question 2.4 is to use, instead, compositions of \( n \) (or, equivalently, subsets of \([n-1]\)), in spite of the apparent redundancy. A surprising structure arises, in the form of a certain matrix \( A_{n-1} \). In fact, it is convenient to define two “coupled” families of matrices, \((A_n)\) and \((B_n)\); for each nonnegative integer \( n \), \( A_n \) and \( B_n \) are square matrices of order \( 2^n \), with entries 0, ±1, which may be viewed as asymmetric variants of Walsh-Hadamard matrices. These matrices and some of their properties will be presented in this section.

We shall give two equivalent definitions for these matrices. The explicit definition is closer in spirit to the subsequent applications, but the recursive definition is very simple to describe and easy to use, and will therefore be presented first.

3.1. A recursive definition.

Recall the well known Walsh-Hadamard (Sylvester) matrices, defined by the recursion

\[
H_n = \begin{pmatrix}
H_{n-1} & H_{n-1}
n & -H_{n-1}
\end{pmatrix} \quad (n \geq 1)
\]

with \( H_0 = (1) \).

**Definition 3.1.** Define, recursively,

\[
A_n = \begin{pmatrix}
A_{n-1} & A_{n-1}
n & -B_{n-1}
\end{pmatrix} \quad (n \geq 1)
\]

with \( A_0 = (1) \), and

\[
B_n = \begin{pmatrix}
A_{n-1} & A_{n-1}
0 & -B_{n-1}
\end{pmatrix} \quad (n \geq 1)
\]

with \( B_0 = (1) \).

Each of the matrices \( A_n \) and \( B_n \) may be obtained from the corresponding Walsh-Hadamard matrix \( H_n \), all the entries of which are ±1, by replacing some of the entries by 0.

**Example 3.2.**

\[
A_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad B_1 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}
\]
3.2. An explicit definition.

It will be convenient to index the rows and columns of $A_n$ and $B_n$ by subsets of the set $\{1, \ldots, n\}$.

**Definition 3.3.** Let $P_n$ be the power set (set of all subsets) of $[n] := \{1, \ldots, n\}$. Endow $P_n$ with the anti-lexicographic linear order: for $I, J \in P_n$, $I \neq J$, let $m$ be the largest element in the symmetric difference $I \triangle J := (I \cup J) \setminus (I \cap J)$, and define: $I < J \iff m \in J$.

**Example 3.4.** The linear order on $P_3$ is
\[ \emptyset < \{1\} < \{2\} < \{1, 2\} < \{3\} < \{1, 3\} < \{2, 3\} < \{1, 2, 3\}. \]

**Definition 3.5.** For $I \in P_n$ let $I_1, \ldots, I_t$ be the sequence of *runs* (maximal consecutive intervals) in $I$, namely: $I$ is the disjoint union of the $I_k$ ($1 \leq k \leq t$), and each $I_k$ is a nonempty set of the form $\{m_k + 1, m_k + 2, \ldots, m_k + \ell_k\}$ with $\ell_k \geq 1$ ($\forall k$) and $0 \leq m_1 < m_1 + \ell_1 < m_2 < m_2 + \ell_2 < \ldots < m_t < m_t + \ell_t \leq n$. In particular, $|I| = \ell_1 + \ldots + \ell_t$.

**Example 3.6.** For $I = \{1, 2, 4, 5, 6, 8, 10\} \in P_{10}$: $I_1 = \{1, 2\}$, $I_2 = \{4, 5, 6\}$, $I_3 = \{8\}$, $I_4 = \{10\}$.

Order $P_n$ as in Definition 3.3. The entries of the Walsh-Hadamard matrix $H_n = (h_{I, J})_{I, J \in P_n}$ are explicitly given by the formula
\[ h_{I, J} := (-1)^{|I \cap J|} \quad (\forall I, J \in P_n). \]

**Definition 3.7.** A *prefix* of an interval $I = \{m + 1, \ldots, m + \ell\}$ is an interval of the form $\{m + 1, \ldots, m + p\}$, for $0 \leq p \leq \ell$.

**Lemma 3.8.** (Explicit definition) Order $P_n$ as in Definition 3.6 and let $I_1, \ldots, I_t$ be the runs of $I \in P_n$. Then:

(i) $A_n = (a_{I, J})_{I, J \in P_n}$, where
\[ a_{I, J} = \begin{cases} (-1)^{|I \cap J|}, & \text{if } I_k \cap J \text{ is a prefix of } I_k \text{ for each } k; \\ 0, & \text{otherwise}. \end{cases} \]

(ii) $B_n = (b_{I, J})_{I, J \in P_n}$, where:
\[ b_{I, J} = \begin{cases} (-1)^{|I \cap J|}, & \text{if } I_k \cap J \text{ is a prefix of } I_k \text{ for each } k, \text{ and } \\ 0, & n \notin I \setminus J; \\ 0, & \text{otherwise}. \end{cases} \]
Proof. It will be convenient here to define $A_n$ and $B_n$ explicitly as in the lemma, and then show that they satisfy the recursions in Definition 3.1.

We shall start with $A_0$. Clearly $A_0 = (1)$.

For $I, J \in P_n$ ($n \geq 1$) denote $I' := I \setminus \{n\}$ and $J' := J \setminus \{n\}$.

The “upper left” quarter of $A_n$ corresponds to $I, J \in P_n$ such that $n \not\in I$ and $n \not\in J$. In this case, clearly $a_{I,J}$ in $A_n$ is the same as $a_{I',J'}$ in $A_{n-1}$.

Similarly when $n \not\in I$ and $n \in J$, and also when $n \in I$ and $n \not\in J$: $|I \cap J| = |I' \cap J'|$, and $I_k \cap J$ is a prefix of $I_k$ for all $k$ if and only if $I'_k \cap J$ is a prefix of $I'_k$ for all $k$.

The “lower right” quarter of $A_n$ corresponds to $I, J \in P_n$ such that $n \in I \cap J$. If $n - 1 \not\in I$ then $I_k \cap J$ is a prefix of $I_k$ for all $k$ if and only if $I'_k \cap J$ is a prefix of $I'_k$ for all $k$. Also $|I \cap J| = |I' \cap J'| + 1$, so that $a_{I,J}$ in $A_n$ is equal to $-a_{I',J'}$ in $A_{n-1}$ and also to $-b_{I',J'}$ in $B_{n-1}$ (since $n - 1 \not\in I'$ so $n - 1 \not\in I' \setminus J'$). If $n - 1 \in I \cap J$ then, again, $a_{I,J}$ in $A_n$ is equal to $-a_{I',J'}$ in $A_{n-1}$ and also to $-b_{I',J'}$ in $B_{n-1}$ (since $n - 1 \in J'$ so $n - 1 \not\in I' \setminus J'$). Finally, if $n - 1 \not\in I$ but $n - 1 \not\in J$ then, for the last run $I_k$ of $I$, $I_k \cap J$ is not a prefix of $I_k$, and thus $a_{I,J} = 0$ in $A_n$ as well as $-b_{I',J'} = 0$ in $B_{n-1}$ (since $n - 1 \not\in I' \setminus J'$).

We have proved the recursion for $A_n$. The entries of $B_n$ are equal to the corresponding entries of $A_n$, except for those in the quarter corresponding to $(I, J)$ with $n \in I$ and $n \not\in J$, which are all zeros (since $n \in I \setminus J$). This proves the recursion for $B_n$ as well.

\[ \square \]

3.3. Determinants.

It turns out that the invertibility of $A_n$ is the key factor in an answer to Question 2.4.

Theorem 3.9. $A_n$ and $B_n$ are invertible for all $n \geq 0$. In fact,

\[ \det(A_n) = (n + 1) \cdot \prod_{k=1}^{n} k^{2^{n-1-k}(n+4-k)} \quad (n \geq 2) \]

while $\det(A_0) = 1$ and $\det(A_1) = -2$, and

\[ \det(B_n) = \prod_{k=1}^{n} k^{2^{n-1-k}(n+2-k)} \quad (n \geq 2) \]

while $\det(B_0) = 1$ and $\det(B_1) = -1$.

A proof of Theorem 3.9 will be given in the next section. For comparison,

\[ \det(H_n) = 2^{2^{n-1}n} \quad (n \geq 2) \]

with $\det(H_0) = 1$ and $\det(H_1) = -2$. 
3.4. Eigenvalues.

Consider the matrix

\[
A_2 = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 0 & 1
\end{pmatrix}.
\]

As an asymmetric matrix, it might conceivably have non-real eigenvalues. Surprisingly, computation shows that its characteristic polynomial is

\[(x^2 - 3)(x^2 - 4),\]

and thus all its eigenvalues are (up to sign) square roots of positive integers!

This is not a coincidence. The following combinatorial description of the eigenvalues of \(A_n\) and \(B_n\), which was stated as a conjecture in an earlier version of this paper, has recently been proved by Gil Alon.

**Theorem 3.10.** (G. Alon [3])

1. The roots of the characteristic polynomial of \(A_n\) are in 2 : 1 correspondence with the compositions of \(n\): each composition \(\mu = (\mu_1, \ldots, \mu_t)\) of \(n\) corresponds to a pair of eigenvalues \(\pm \sqrt{\pi_\mu}\) of \(A_n\), where

\[
\pi_\mu := \prod_{i=1}^{t} (\mu_i + 1).
\]

2. The roots of the characteristic polynomial of \(B_n\) are in 2 : 1 correspondence with the compositions of \(n\): each composition \(\mu = (\mu_1, \ldots, \mu_t)\) of \(n\) corresponds to a pair of eigenvalues \(\pm \sqrt{\pi'_\mu}\) of \(B_n\), where

\[
\pi'_\mu := \prod_{i=1}^{t-1} (\mu_i + 1).
\]

Surprising connections between the eigenvalues and the diagonal elements of \(A_2^2\), as well as the column sums of some related matrices, appear in Theorem 5.13 below.

4. Möbius inversion

In this section we prove Theorem 3.9. Our approach is to study certain matrices, with more transparent structure, obtained from \(A_n\) and \(B_n\) by a transformation corresponding to (poset theoretic) Möbius inversion.

4.1. Auxiliary definitions.

Let us define certain auxiliary families of matrices.
**Definition 4.1.** Define, recursively,
\[ Z_n = \begin{pmatrix} Z_{n-1} & Z_{n-1} \\ 0 & Z_{n-1} \end{pmatrix} \quad (n \geq 1) \]
with \( Z_0 = (1) \), as well as
\[ M_n = \begin{pmatrix} M_{n-1} & -M_{n-1} \\ 0 & M_{n-1} \end{pmatrix} \quad (n \geq 1) \]
with \( M_0 = (1) \).

\( Z_n \) is the *zeta matrix* of the poset \( P_n \) with respect to set inclusion (not with respect to its linear extension, described in Definition 3.3). Thus \( Z_n = (z_{I,J})_{I,J \in P_n} \) is a square matrix, with entries satisfying
\[ z_{I,J} = \begin{cases} 1, & \text{if } I \subseteq J; \\ 0, & \text{otherwise}. \end{cases} \]

\( M_n = Z_n^{-1} \) is the corresponding *Möbius matrix*, expressing the Möbius function (see [15]) of the poset \( P_n \). Thus \( M_n = (m_{I,J})_{I,J \in P_n} \) has entries satisfying
\[ m_{I,J} = \begin{cases} (-1)^{|J \setminus I|}, & \text{if } I \subseteq J; \\ 0, & \text{otherwise}. \end{cases} \]

**Definition 4.2.** Denote \( AM_n := A_n M_n \), \( BM_n := B_n M_n \) and \( HM_n := H_n M_n \).

It follows from Definitions 3.1 and 4.1 that
\[ (1) \quad AM_n = \begin{pmatrix} AM_{n-1} & 0 \\ 0 & -(AM_{n-1} + BM_{n-1}) \end{pmatrix} \quad (n \geq 1) \]
with \( AM_0 = (1) \) and
\[ (2) \quad BM_n = \begin{pmatrix} AM_{n-1} & 0 \\ 0 & -BM_{n-1} \end{pmatrix} \quad (n \geq 1) \]
with \( BM_0 = (1) \), as well as
\[ (3) \quad HM_n = \begin{pmatrix} HM_{n-1} & 0 \\ HM_{n-1} & -2HM_{n-1} \end{pmatrix} \quad (n \geq 1) \]
with \( HM_0 = (1) \).

The block triangular form of \( AM_n \) and block diagonal form of \( BM_n \) facilitate a recursive computation of the determinants of \( A_n \) and \( B_n \).

**4.2. A proof of Theorem 3.9**

By recursion (2),
\[ \det(BM_n) = \det(AM_{n-1}) \det(-BM_{n-1}) \quad (n \geq 1). \]

Now \( M_n \) is an upper triangular matrix with 1-s on its diagonal, so that
\[ \det(M_n) = 1. \]
We conclude that
\begin{equation}
\det(B_n) = \delta_{n-1} \det(A_{n-1}) \det(B_{n-1}) \quad (n \geq 1),
\end{equation}
where
\[ \delta_n = (-1)^{2^n} = \begin{cases} 
-1, & \text{if } n = 0; \\
1, & \text{if } n \geq 1.
\end{cases} \]

Similarly, for any scalar \( t \),
\[ AM_n + tBM_n = \begin{pmatrix} (t+1)AM_{n-1} & 0 \\
AM_{n-1} & -AM_{n-1} - (t+1)BM_{n-1} \end{pmatrix} \quad (n \geq 1) \]
and a similar argument yields
\[ \det(A_n + tB_n) = \delta_{n-1} \det((t+1)A_{n-1}) \det(A_{n-1} + (t+1)B_{n-1}) \quad (n \geq 1). \]

It follows that
\[ \det(A_n) = \delta_{n-1} \det(A_{n-1} + B_{n-1}) = \delta_{n-1} \det(A_{n-1}) \delta_{n-2} \det(2A_{n-2}) \det(A_{n-2} + 2B_{n-2}) = \ldots = \prod_{k=1}^{n} \delta_{n-k} \det(kA_{n-k}) \cdot \det(A_0 + nB_0) = - (n+1) \cdot \prod_{k=1}^{n} k^{2^{n-k}} \cdot \prod_{k=1}^{n} \det(A_{n-k}) \quad (n \geq 1). \]

Since \( A_0 = 1 \) it follows that \( \det(A_n) \neq 0 \) for any nonnegative integer \( n \), and therefore
\[ \det(A_n)/\det(A_{n-1}) = -\frac{n+1}{n} \cdot \prod_{k=1}^{n-1} k^{2^{n-1-k}} \cdot \det(A_{n-1}) \quad (n \geq 2). \]

The solution to this recursion, with initial value \( \det(A_1) = -2 \), is
\[ \det(A_n) = (n+1) \cdot \prod_{k=1}^{n} k^{2^{n-1-k}(n+1-k)} \quad (n \geq 2). \]

Recursion (4) above, with initial value \( \det(B_1) = -1 \), now yields
\[ \det(B_n) = \prod_{k=1}^{n} k^{2^{n-1-k}(n+2-k)} \quad (n \geq 2). \]

For comparison,
\[ \det(H_n) = 2^{2^{n-1}} \det(H_{n-1})^2 \quad (n \geq 2) \]
with initial value \( \det(H_1) = -2 \), so that
\[ \det(H_n) = 2^{2^{n-1}} n \quad (n \geq 2). \]
Remark 4.3. We can also write
\[
det(A_n) = \prod_{k=1}^{n+1} k^{a_{n+1-k}} \quad (n \geq 2),
\]
where the sequence \((a_0, a_1, \ldots) = (1, 2, 5, 12, 28, 64, \ldots)\) coincides with [16] sequence A045623.

5. Properties of the transformed matrices

In this section we describe some additional properties of the transformed matrices \(A_M^n\) and \(B_M^n\), namely: Explicit expressions for their entries, for the entries of their inverses, and for their row sums and column sums. The latter are surprisingly related to the eigenvalues of \(A_n\) and \(B_n\) described in Theorem 3.10 above.

The proofs of most of the results in this section follow from recursion formulas (1) and (2), and are therefore indicated only when additional ingredients are present.

5.1. Matrix entries.

We shall now compute explicitly the entries of \(A_M^n\) and \(B_M^n\), starting with \(H_M^n\) as a “baby case”.

Example 5.1.

\[
H_M^3 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & -2 & 4 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & -2 & 4 & 0 & 0 \\
1 & 0 & -2 & 0 & -2 & 0 & 4 & 0 \\
1 & -2 & -2 & 4 & -2 & 4 & 4 & -8
\end{pmatrix}
\]

This generalizes to an explicit description of the entries of \(H_M^n\), which follows easily from recursion (3).

Lemma 5.2.

\((H_M^n)_{I,J} \neq 0 \iff J \subseteq I\)

and

\((H_M^n)_{I,J} \neq 0 \implies (H_M^n)_{I,J} = (-2)^{|J|}\).

The corresponding results for \(A_M^n\) and \(B_M^n\) are much more subtle (and interesting). Their proofs follow, in general, from recursions (1) and (2).

Lemma 5.3. For every \(n \geq 0\), the matrices \(A_M^n\) and \(B_M^n\) are lower triangular.

Corollary 5.4. \((A_M^n) \cdot Z_n\) is an LU factorization of \(A_n\); similarly for \(B_n\).
Example 5.5.

\[
AM_1 = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} \quad BM_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

\[
AM_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & -2 & -1 & 3 \end{pmatrix} \quad BM_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

\[
AM_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & -1 & 3 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & -1 & 0 & 3 & 0 \\ 1 & -2 & -1 & 3 & -1 & 2 & 1 & -4 \end{pmatrix} \quad BM_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & -1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}
\]

Apparently, \( AM_n \) has the same zero pattern and the same sign pattern as \( HM_n \). \( BM_n \) also has the same sign pattern, but has more zero entries. The absolute values of entries in both families are more intricate than in \( HM_n \).

Theorem 5.6. (Entries of \( AM_n \) and \( BM_n \))

1. Zero pattern:

\[
(AM_n)_{I,J} \neq 0 \iff J \subseteq I
\]

and

\[
(BM_n)_{I,J} \neq 0 \iff J \subseteq I \text{ and } \text{maxout}(J) = \text{maxout}(I),
\]

where

\[
\text{maxout}(I) := \max\{0 \leq i \leq n \mid i \notin I\} \quad (\forall I \in P_n).
\]

2. Signs:

\[
(AM_n)_{I,J} \neq 0 \Rightarrow \text{sign}((AM_n)_{I,J}) = (-1)^{|J|}
\]

and

\[
(BM_n)_{I,J} \neq 0 \Rightarrow \text{sign}((BM_n)_{I,J}) = (-1)^{|J|}.
\]
3. **Absolute values:** For $I, J \in P_n$, let $J_1, \ldots, J_t$ be the runs (maximal consecutive intervals) in $J$. For $J_k = \{m_k + 1, \ldots, m_k + \ell_k\}$ ($1 \leq k \leq t$), let

$$c_k(I) = \begin{cases} 0, & \text{if } m_k \in I; \\ 1, & \text{otherwise.} \end{cases}$$

Then

$$|(AM_n)_{I,J}| = \prod_{k=1}^{t} (|J_k| + 1)^{c_k(I)}$$

and

$$|(BM_n)_{I,J}| = \prod_{k=1}^{t'} (|J_k| + 1)^{c_k(I)},$$

where

$$t' = \begin{cases} t - 1, & \text{if } n \in I \text{ (equivalently, } n \in J); \\ t, & \text{otherwise.} \end{cases}$$

**Proof.** It is clear from recursion formulas (1) and (2) that all the entries in column $J$ of $AM_n$ (or $BM_n$) have sign $(-1)^{|J|}$ or are zero, exactly as in $HM_n$.

Comparison of the two recursions shows that wherever $AM_n$ has a zero entry so does $BM_n$, but not conversely. The zero pattern of $AM_n + BM_n$ is therefore the same as that of $AM_n$, and thus recursions (1) and (3) imply that $AM_n$ and $HM_n$ have the same zero pattern. The zero pattern of $BM_n$ now follows from recursion (2).

Finally, the explicit formulas for the absolute values of entries are relevant, of course, only when $J \subseteq I$. They are a little difficult to come up with, but easy to confirm by recursion. \(\square\)

**Corollary 5.7.** Let $I, J \in P_n$ satisfy $J \subseteq I$. Then:

1. 

$$|(AM_n)_{I,J}| \leq |(HM_n)_{I,J}| = 2^{|J|},$$

with equality if and only if $|J_k| = 1$ for each $k$ for which $m_k \notin I$.

2. 

$$|(BM_n)_{I,J}| \leq |(AM_n)_{I,J}|,$$

with equality if and only if either $n \notin I$ or $m_t \in I$.

**Proof.** $|J_k| + 1 \leq 2^{|J_k|}$, with equality if and only if $|J_k| = 1$. \(\square\)

An alternative description of the entries may be given in terms of compositions, using the correspondence $\mu \leftrightarrow I(\mu)$ described in Subsection 2.1 above.

**Theorem 5.8.** (Entries of $AM_n$ and $BM_n$, composition version)

Let $\lambda$ and $\mu$ be compositions of $n+1$. Write $(AM_n)_{\lambda,\mu}$ instead of $(AM_n)_{I(\lambda),I(\mu)}$, and similarly for $BM_n$. 




1. Zero pattern:

\[(AM_n)_{\lambda,\mu} \neq 0 \iff \mu \text{ is a refinement of } \lambda\]

and

\[(BM_n)_{\lambda,\mu} \neq 0 \iff \mu \text{ is a refinement of } \lambda \text{ and the last component of } \lambda \text{ is unrefined in } \mu.\]

2. Signs:

\[(AM_n)_{\lambda,\mu} \neq 0 \iff \text{sign}((AM_n)_{\lambda,\mu}) = (-1)^{n+1-\ell(\mu)}\]

and

\[(BM_n)_{\lambda,\mu} \neq 0 \iff \text{sign}((BM_n)_{\lambda,\mu}) = (-1)^{n+1-\ell(\mu)},\]

where \(\ell(\mu)\) is the number of components of \(\mu\).

3. Absolute values:

\[(AM_n)_{\lambda,\mu} \neq 0 \iff |(AM_n)_{\lambda,\mu}| = \prod_i \mu_\text{init}(\lambda_i)\]

and

\[(BM_n)_{\lambda,\mu} \neq 0 \iff |(BM_n)_{\lambda,\mu}| = \prod_i \mu_\text{init}(\lambda_i),\]

where \(\mu_\text{init}(\lambda_i)\) is the first component in the subdivision (in \(\mu\)) of the component \(\lambda_i\) of \(\lambda\), and \(\prod_i\) is a product over all values of \(i\) except the last one.

5.2. Diagonal entries.

The following corollary of Theorem 5.6(3) and Theorem 5.8(3) is stated simultaneously, in terms of a composition \(\mu\) of \(n+1\) and the corresponding subset \(J = I(\mu)\) of \([n]\). Different indices \((i\text{ and }k)\) are used for the components of \(\mu\) and the runs of \(J\), since runs correspond only to the components satisfying \(\mu_i > 1\).

**Corollary 5.9.** (Diagonal and last row of \(AM_n\))

1. The diagonal entries of \(AM_n\) are

\[|(AM_n)_{J,J}| = \prod_i \mu_i = \prod_k (|J_k| + 1)\]

and the entries in its last row are

\[|(AM_n)_{[n],J}| = \mu_1 = \begin{cases} |J_1| + 1, & \text{if } 1 \in J; \\ 1, & \text{otherwise}. \end{cases}\]

2. Each nonzero entry \((AM_n)_{I,J}\) divides the corresponding diagonal entry \((AM_n)_{J,J}\) and is divisible by the corresponding last row entry \((AM_n)_{[n],J}\).
For any finite set $J$ of positive integers, let $\text{ord}(J) := \sum_{j \in J} 2^{j-1}$. This is the ordinal number of $J$ in the anti-lexicographic order on $P_n$ (see Definition 3.3), where the enumeration starts with 0 for the empty set. The number $\text{ord}(J)$ is independent of $n$, as long as $J \in P_n$. Define

$$a_{\text{ord}(J)} := |(AM_n)_{I,J}|,$$

and consider the resulting sequence $(a_m)_{m \geq 0}$ of absolute values of diagonal entries of the “limit matrix” $AM_\infty = \lim_{n \to \infty} AM_n$.

**Lemma 5.10.** The sequence $(a_m)$ satisfies the recursion

$$a_0 = 0, \quad a_{2m} = a_m, \quad a_{4m+1} = 2a_{2m}, \quad a_{4m+3} = 2a_{2m+1} - a_m \quad (\forall m \geq 0).$$

**Proof.** Consider the formula in Corollary 5.9(1) expressing a diagonal entry of $AM_n$ in terms of the corresponding run lengths. We shall not distinguish a set $J$ from its ordinal number $\text{ord}(J)$.

(The set corresponding to) $2m$ has the same runs as $m$, shifted forward by 1, so that $a_{2m} = a_m$. $4m + 1$ has the same runs as $2m$, shifted forward by 1, plus a singleton run $\{1\}$, so that $a_{4m+1} = 2a_{2m}$.

If $m$ is even then $a_{4m+3} = 3a_m$ and $a_{2m+1} = 2a_m$, so that $a_{4m+3} = 2a_{2m+1} - a_m$. If $m$ is odd, let $\ell$ be the length of the first run in $m$. The corresponding runs in $2m + 1$ and in $4m + 3$ have lengths $\ell + 1$ and $\ell + 2$, respectively, so that again $a_{4m+3} = 2a_{2m+1} - a_m$. \qed

**Corollary 5.11.** The sequence $(a_m)$ coincides with [16] sequence A106737. Thus, in particular,

$$a_m = \sum_{k=0}^{m} \left[ \binom{m+k}{m} \binom{m}{k} \mod 2 \right],$$

where the expression in the square brackets is interpreted as either 0 or 1 and summed as an ordinary integer.

### 5.3. Row sums and column sums.

The following two results, regarding row and column sums of $AM_n$ and $BM_n$, are stated, for simplicity, almost entirely in the language of compositions.

**Lemma 5.12.** (Row sums of $AM_n$, $BM_n$)

Let $\lambda$ be a composition of $n + 1$, and let $I = I(\lambda)$ be the corresponding subset of $[n]$.

1. The sum of all entries in row $I$ of $AM_n$ (or $BM_n$, or $HM_n$) is $(-1)^{|I|}$.
2. The sum of absolute values of all entries in row $I$ of $AM_n$ is

$$\prod_{i}(2^{\lambda_i} - 1).$$
The sum of absolute values of all entries in row $I$ of $BM_n$ is

$$\prod'_i (2^{\lambda_i} - 1),$$

where $\prod'_i$ is a product over all values of $i$ except the last. In $HM_n$ the corresponding sum is $3^{\vert I \vert}$.

**Theorem 5.13.** (Column sums of $AM_n$, $BM_n$ and diagonal entries of $A^2_n$, $B^2_n$)

Let $\mu$ be a composition of $n + 1$, and let $J = I(\mu)$ be the corresponding subset of $[n]$. Let $\mu^*$ be the composition of $n$ obtained from $\mu$ by reducing its first component by 1, without changing the other components: $\mu^*_1 = \mu_1 - 1$, $\mu^*_i = \mu_i$ ($\forall i > 1$).

1. The sum of absolute values (also: absolute value of the sum) of all the entries in column $J$ of $AM_n$ is equal to the diagonal entry $(A^2_n)_{J,J}$, which in turn is equal to

$$\prod_i (\mu^*_i + 1).$$

2. The sum of absolute values (also: absolute value of the sum) of all the entries in column $J$ of $BM_n$ is equal to the diagonal entry $(B^2_n)_{J,J}$, which in turn is equal to

$$\prod'_i (\mu^*_i + 1),$$

where $\prod'_i$ is a product over all values of $i$ except the last.

3. For comparison, the sum of absolute values of all the entries in column $J$ of $HM_n$ is equal to the diagonal entry $(H^2_n)_{J,J}$, which in turn is equal to the constant $2^n$.

**Proof.** The recursions for $A^2_n$ and $B^2_n$ are

$$A^2_n = \begin{pmatrix} 2A^2_{n-1} & A_{n-1}(A_{n-1} - B_{n-1}) \\ (A_{n-1} - B_{n-1})A_{n-1} & A^2_{n-1} + B^2_{n-1} \end{pmatrix} (n \geq 1)$$

and

$$B^2_n = \begin{pmatrix} A^2_{n-1} & A_{n-1}(A_{n-1} - B_{n-1}) \\ 0 & B^2_{n-1} \end{pmatrix} (n \geq 1),$$

with $A^2_0 = B^2_0 = 1$. Denoting $\alpha_n(J) := (A^2_n)_{J,J}$, $\beta_n(J) := (B^2_n)_{J,J}$ and $J' := J \setminus \{n\}$, it follows that

$$\alpha_n(J) = \begin{cases} 2\alpha_{n-1}(J'), & \text{if } n \notin J; \\ \alpha_{n-1}(J') + \beta_{n-1}(J'), & \text{otherwise} \end{cases}$$

and

$$\beta_n(J) = \begin{cases} \alpha_{n-1}(J'), & \text{if } n \notin J; \\ \beta_{n-1}(J'), & \text{otherwise} \end{cases}$$

with $\alpha_0(\emptyset) = \beta_0(\emptyset) = 1$. 

A short look at recursions (1) and (2) shows that the above recursions also hold if \( \alpha_n(J) \) and \( \beta_n(J) \) denote the sum of absolute values of all the entries in column \( J \) of \( AM_n \) and of \( BM_n \), respectively. The same recursions also hold if \( \alpha_n(J) \) and \( \beta_n(J) \) stand for the explicit product formulas in the theorem, since if \( J = I(\mu) \) and \( J' = J \setminus \{n\} = I(\mu') \) then \( n \notin J \) means that \( \mu \) is obtained from \( \mu' \) by appending a new component of size 1, while \( n \in J \) means that \( \mu \) is obtained from \( \mu' \) by increasing the last component by 1.

Comparing Theorem 5.13 with Theorem 3.10 gives a surprising conclusion.

**Corollary 5.14.** The multiset of eigenvalues, counted by algebraic multiplicity, of \( A_n^2 \) (or \( B_n^2 \)) is equal to the multiset of diagonal entries of this matrix.

This is remarkable since, apparently, for \( n \geq 3 \) the matrices \( A_n^2 \) are not even diagonalizable!

### 5.4. Inverse matrix entries.

We would like to have explicit expressions for the entries of \( A_n^{-1} \), for use in Section 6. This turns out to be difficult to do directly, and we shall compute, as an intermediate step, the entries of \( AM_n^{-1} \). Note that \( A_n^{-1} = M_n \cdot AM_n^{-1} \).

**Example 5.15.**

\[
A_3^{-1} = \begin{pmatrix}
1/24 & 1/24 & 1/12 & 1/12 & 1/8 & 1/8 & 1/4 & 1/4 \\
1/8 & -1/24 & 1/12 & -1/12 & 5/24 & -1/8 & 1/12 & -1/4 \\
5/24 & 5/24 & -1/12 & -1/12 & 1/8 & 1/8 & -1/4 & -1/4 \\
1/8 & -5/24 & -1/12 & 1/12 & 1/24 & -1/8 & -1/12 & 1/4 \\
1/8 & 1/8 & 1/4 & 1/4 & -1/8 & -1/8 & -1/4 & -1/4 \\
5/24 & -1/8 & 1/12 & -1/4 & -5/24 & 1/8 & -1/12 & 1/4 \\
1/8 & 1/8 & -1/4 & -1/4 & -1/8 & -1/8 & 1/4 & 1/4 \\
1/24 & -1/8 & -1/12 & 1/4 & -1/24 & 1/8 & 1/12 & -1/4
\end{pmatrix}
\]

\[
AM_3^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/2 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/2 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 \\
1/6 & -1/3 & -1/6 & 1/3 & 0 & 0 & 0 & 0 \\
1/2 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 \\
1/4 & -1/4 & 0 & 0 & -1/4 & 1/4 & 0 & 0 \\
1/6 & 0 & -1/3 & 0 & -1/6 & 0 & 1/3 & 0 \\
1/24 & -1/8 & -1/12 & 1/4 & -1/24 & 1/8 & 1/12 & -1/4
\end{pmatrix}
\]

We shall attempt an inductive computation of \( AM_n^{-1} \). Recursion formulas (1) and (2) yield corresponding recursions for the inverse matrices:

\[
AM_n^{-1} = \begin{pmatrix}
AM_{n-1}^{-1} & 0 \\
(AM_{n-1} + BM_{n-1})^{-1} & -(AM_{n-1} + BM_{n-1})^{-1}
\end{pmatrix} \quad (n \geq 1)
\]
and

\[ BM_n^{-1} = \begin{pmatrix} AM_{n-1}^{-1} & 0 \\ 0 & -BM_{n-1}^{-1} \end{pmatrix} \quad (n \geq 1), \]

with \( AM_0^{-1} = BM_0^{-1} = (1) \); however, the recursion for \( AM_n^{-1} \) involves the inverse of a new matrix, \( AM_{n-1} + BM_{n-1} \), which in turn involves the inverse of \( AM_{n-2} + 2BM_{n-2} \), and so forth. We are thus led to consider a more general situation.

Definition 5.16. For any real number \( x \)

\[ M_n(x) := xAM_n + (1 - x)BM_n. \]

In particular, \( M_n(0) = BM_n \) and \( M_n(1) = AM_n \).

Theorem 5.17. For each \( n \geq 0 \) and \( x > 0 \),

\[ M_n^{-1}(x)_{I,J} \neq 0 \iff J \subseteq I \]

and, for \( J \subseteq I \),

\[ M_n^{-1}(x)_{I,J} = (-1)^{|J|} \prod_{i \in J} d_{I,J,x}(i), \]

where \( I_1, \ldots, I_t \) are the runs of \( I \) and, for \( i \in I_k \):

(i) If \( n \notin I_k \) then

\[ d_{I,J,x}(i) := \begin{cases} \max(I_k) - i + 1, & \text{if } i \in J; \\ 1, & \text{otherwise} \end{cases} \]

and

\[ e_{I,J,x}(i) := \max(I_k) - i + 2. \]

(ii) If \( n \in I_k \) (and thus necessarily \( k = t \)) then

\[ d_{I,J,x}(i) := \begin{cases} (\max(I_k) - i) \cdot x + 1, & \text{if } i \in J; \\ x, & \text{otherwise} \end{cases} \]

and

\[ e_{I,J,x}(i) := (\max(I_k) - i + 1) \cdot x + 1. \]

Proof. Let \( x > 0 \). By Definition 5.16 and recursion formulas (1) and (2),

\[ M_n(x) = \begin{pmatrix} AM_{n-1} & 0 \\ xAM_{n-1} & -(xAM_{n-1} + BM_{n-1}) \end{pmatrix} = \begin{pmatrix} M_{n-1}(1) & 0 \\ xM_{n-1}(1) & -(1 + x)M_{n-1} \left( \frac{x}{1+x} \right) \end{pmatrix} \quad (n \geq 1) \]

with \( M_0(x) = (1) \). Invertibility of \( M_{n-1}(x) \) for all \( x > 0 \) clearly implies the invertibility of \( M_n(x) \) for all \( x > 0 \). The inverse satisfies

\[ M_n^{-1}(x) = \begin{pmatrix} M_{n-1}(1) & 0 \\ \frac{x}{1+x}M_{n-1}(1) & \frac{-1}{1+x}M_{n-1}^{-1}(1) \left( \frac{x}{1+x} \right) \end{pmatrix} \quad (n \geq 1) \]
with $M_n^{-1}(x) = (1)$, for all $x > 0$.

Recursion (5) shows that, indeed, for $x > 0$: $M_n^{-1}(x)_{I,J} \neq 0 \iff J \subseteq I$, and that the sign of this entry is $(-1)^{|J|}$.

Regarding the absolute value of this entry, assume by induction that the prescribed formula holds for $M_n^{-1}(x)$, $\forall x > 0$.

If $n \notin I$ then also $n \notin J$, and clearly $M_n^{-1}(x)_{I,J} = M_n^{-1}(1)_{I,J}$ satisfies the required formula.

If $n \in I$, let $I' := I \setminus \{n\}$, $J' := J \setminus \{n\}$ and $x' := \frac{x}{1+x}$. The assumed formula for $M_{n-1}(x')_{I',J'}$ and the claimed formula for $M_n^{-1}(x)_{I,J}$ have exactly the same factors for all $i \notin I_t$, so we need only consider $i \in I_t$.

If $|I_t| = 1$ (i.e., $n - 1 \notin I$) then there is nothing else in $M_{n-1}(x')_{I',J'}$, but according to (5) there is an extra factor $\frac{1}{1+x}$ or $\frac{x}{1+x}$ in $M_n^{-1}(x)_{I,J}$ (depending on whether or not $n \in J$), and this is exactly the missing $d_{I,J,x}(n)/e_{I,J,x}(n)$.

Finally, assume that $|I_t| > 1$. Again, the extra factor $\frac{1}{1+x}$ or $\frac{x}{1+x}$ is exactly $d_{I,J,x}(n)/e_{I,J,x}(n)$. The other factors in $M_{n-1}(x')_{I',J'}$, corresponding to $i \in I_t'$, are (if $i \in J$)

$$\frac{d_{I',J',x'}(i)}{e_{I',J',x'}(i)} = \frac{(n-1-i)x' + 1}{(n-i)x' + 1} = \frac{(n-1-i)x + 1 + x}{(n-i)x + 1 + x} = \frac{d_{I,J,x}(i)}{e_{I,J,x}(i)},$$

or (if $i \notin J$)

$$\frac{d_{I',J',x'}(i)}{e_{I',J',x'}(i)} = \frac{x'}{(n-i)x' + 1} = \frac{x}{(n-i)x + 1 + x} = \frac{d_{I,J,x}(i)}{e_{I,J,x}(i)},$$

exactly as claimed for $M_n^{-1}(x)_{I,J}$.

We are especially interested, of course, in the special case $x = 1$.

**Corollary 5.18.** (Entries of $AM_n^{-1}$)

For each $n \geq 0$

$$(AM_n^{-1})_{I,J} \neq 0 \iff J \subseteq I$$

and, for $J \subseteq I$,

$$(AM_n^{-1})_{I,J} = (-1)^{|J|} \prod_{i \in I} \frac{d_{I,J}(i)}{e_{I,J}(i)},$$

where $I_1, \ldots, I_t$ are the runs of $I$ and, for $i \in I_k$:

$$d_{I,J}(i) := \begin{cases} \max(I_k) - i + 1, & \text{if } i \in J; \\ 1, & \text{otherwise} \end{cases}$$

and

$$e_{I,J}(i) := \max(I_k) - i + 2.$$ 

Equivalently, for $J \subseteq I$,

$$(AM_n^{-1})_{I,J} = (-1)^{|J|} \prod_{k=1}^{t} \frac{1}{(|I_k| + 1)!} \prod_{i \in I_k \cap J} (\max(I_k) - i + 1).$$
Note that the denominator \( \prod_{k=1}^{t} (|I_k| + 1)! \) is the cardinality of the parabolic subgroup \( \langle I \rangle \) of \( S_{n+1} \) generated by the simple reflections \( \{ s_i : i \in I \} \).

**Corollary 5.19.**

(i) Each nonzero entry of \( AM_n^{-1} \) is the inverse of an integer.

(ii) In each row of \( AM_n^{-1} \), the sum of absolute values of all the entries is 1.

(iii) In each row \( I \) of \( AM_n^{-1} \), the first entry

\[
(AM_n^{-1})_{I, \emptyset} = \prod_{k=1}^{t} \frac{1}{(|I_k| + 1)!}
\]

divides all the other nonzero entries and the diagonal entry

\[
(AM_n^{-1})_{I, I} = (-1)^{|I|} \prod_{k=1}^{t} \frac{1}{|I_k| + 1}
\]

is divisible by all the other nonzero entries, where a rational number \( r \) is said to divide a rational number \( s \) if the quotient \( s/r \) is an integer.

6. Fine sets

6.1. The concept.

A general setting for character formulas is introduced in this section. It will serve as a framework for the answer to Question 2.4.

Recall from Subsection 2.1 the definition of \( I(\mu) \) for a composition \( \mu \).

**Definition 6.1.** Let \( \mu = (\mu_1, \ldots, \mu_t) \) be a composition of \( n \). A subset \( J \subseteq [n-1] \) is \( \mu \)-unimodal if each run of \( J \cap I(\mu) \) is a prefix of the corresponding run of \( I(\mu) \); in other words, if \( J \cap I(\mu) \) is a disjoint union of intervals of the form \( \left[ \sum_{i=1}^{k-1} \mu_i + 1, \sum_{i=1}^{k-1} \mu_i + \ell_k \right] \), where \( 0 \leq \ell_k \leq \mu_k - 1 \) for every \( 1 \leq k \leq t \).

**Observation 6.2.** A permutation \( \pi \in S_n \) is \( \mu \)-unimodal according to the definition in Subsection 2.3 if and only if its descent set \( \text{Des}(\pi) \) is \( \mu \)-unimodal according to Definition 6.1.

**Definition 6.3.** Let \( \mathcal{B} \) be a set of combinatorial objects, and let \( \text{Des} : \mathcal{B} \to P_{n-1} \) be a map which associates with each element \( b \in \mathcal{B} \) a subset \( \text{Des}(b) \subseteq [n-1] \). Denote by \( \mathcal{B}^\mu \) the set of elements in \( \mathcal{B} \) whose “descent set” \( \text{Des}(b) \) is \( \mu \)-unimodal. Let \( \rho \) be a complex \( S_n \)-representation. Then \( \mathcal{B} \) is called a fine set for \( \rho \) if, for each composition \( \mu \) of \( n \), the character of \( \rho \) at a conjugacy class of cycle type \( \mu \) satisfies

\[
\chi_{\mu}^\rho = \sum_{b \in \mathcal{B}^\mu} (-1)^{|\text{Des}(b) \cap I(\mu)|}.
\]

It follows from Theorems 2.1, 2.2, and 2.3 that
Proposition 6.4.

(i) Any Knuth class of RSK-shape \( \lambda \) is a fine set for the Specht module \( S^\lambda \).

(ii) The set of permutations of a fixed Coxeter length \( k \) in \( S_n \) is a fine set for the \( k \)-th homogeneous component of the coinvariant algebra of \( S_n \).

(iii) The set of involutions in \( S_n \) is a fine set for the Gelfand model of \( S_n \).

More examples of fine sets are given in Subsection 6.4.

6.2. Distribution of descent sets.

We are now ready to state our main application.

Theorem 6.5. If \( B \) is a fine set for an \( S_n \)-representation \( \rho \) then the character values of \( \rho \) determine the distribution of descent sets over \( B \). In particular, for every \( I \subseteq [n-1] \), the number of elements in \( B \) whose descent set contains \( I \) satisfies

\[
|\{ b \in B : \text{Des}(b) \supseteq I \}| = \frac{1}{|\langle I \rangle|} \sum_{J \subseteq I} (-1)^{|J|} \chi^\rho(c_J) \prod_{k=1}^t \prod_{i \in I_k \cap J} (\max(I_k) - i + 1),
\]

where \( I_1, \ldots, I_t \) are the runs in \( I \), \(|\langle I \rangle|\) is the cardinality of the parabolic subgroup of \( S_n \) generated by \( \{ s_i : i \in I \} \), and \( c_I \) is any Coxeter element in this subgroup.

Proof. The mapping \( \mu \mapsto I(\mu) \) (see Subsection 2.1) is a bijection between the set of all compositions of \( n \) and the set \( P_{n-1} \) of all subsets of \( [n-1] \). For a subset \( J = \{ j_1, \ldots, j_k \} \subseteq [n-1] \) with \( j_1 < j_2 < \cdots < j_k \) let \( c_J \) be the product \( s_{j_1}s_{j_2}\cdots s_{j_k} \in S_n \). This is a Coxeter element in the parabolic subgroup generated by \( \{ s_i : i \in J \} \), and its cycle type is (the partition corresponding to) the composition \( \mu \), where \( J = I(\mu) \). Let \( x^\mu \) be the vector with entries \( \chi^\rho(c_J) \), where the subsets \( J \in P_{n-1} \) are ordered anti-lexicographically as in Definition 3.3.

Similarly, let \( v^B = (v^B_J)_{J \in P_{n-1}} \) be the vector with entries

\[
v^B_J = |\{ b \in B : \text{Des}(b) = J \}| \quad (\forall J \in P_{n-1}).
\]

By Definition 6.3 and Lemma 3.3, \( B \) is a fine set for \( \rho \) if and only if

\[
x^\rho = A_{n-1}v^B,
\]

where \( x^\rho \) and \( v^B \) are written as column vectors. By Theorem 3.9, \( A_{n-1} \) is an invertible matrix, which proves that \( x^\rho \) uniquely determines \( v^B \).

The explicit formula follows from Corollary 5.18 as soon as equation (7) is written in the form

\[
Z_{n-1}v^B = AM_{n-1}^{-1}x^\rho.
\]

\[\square\]

The Inclusion-Exclusion Principle (namely, multiplication by \( M_{n-1} \)) gives an equivalent form of the explicit formula.
Corollary 6.6. Let $B$ be a fine set for an $S_n$-representation $\rho$. For every $I \subseteq [n-1]$, the number of elements in $B$ with descent set exactly $D$ satisfies
\[
|\{b \in B : \text{Des}(b) = D\}| = \sum_J \chi^\rho(c_J) \sum_{I : D \cup J \subseteq I} (-1)^{|I \setminus D|} (AM_{n-1}^{-1})_{I,J}
\]
where $(AM_{n-1}^{-1})_{I,J}$ is as in Corollary 6.5 and the notation is as in Theorem 6.5.

6.3. Permutation statistics versus character theory.

By Theorem 6.5 certain statements in permutation statistics have equivalent statements in character theory. In particular,

Corollary 6.7. Given two symmetric group modules with fine sets, the isomorphism of these modules is equivalent to equi-distribution of the descent set on their fine sets.

Proof. Combining Theorem 6.5 with Definition 6.3. □

Here is a distinguished example. Recall the major index of a permutation $\pi$,
\[
\text{maj}(\pi) := \sum_{i \in \text{Des}(\pi)} i.
\]

For a subset $I \subseteq [n-1]$ denote $x^I := \prod_{i \in I} x_i$. The following is a fundamental theorem in permutation statistics.

Theorem 6.8. (Foata-Schützenberger) [7]
\[
\sum_{\pi \in S_n} x^{\text{Des}(\pi)} q^{\ell(\pi)} = \sum_{\pi \in S_n} x^{\text{Des}(\pi)} q^{\text{maj}(\pi)^{-1}}.
\]

See also [8].

For $0 \leq k \leq \binom{n}{2}$ denote by $R_k$ the $k$-th homogeneous component of the coinvariant algebra of the symmetric group $S_n$. The following is a classical theorem in invariant theory.

Theorem 6.9. (Lusztig-Stanley) [17, Prop. 4.11] For a partition $\lambda$ denote by $m_{k,\lambda}$ the number of standard Young tableaux of shape $\lambda$ with major index $k$. Then
\[
R_k \cong \bigoplus_{\lambda \vdash n} m_{k,\lambda} S^\lambda,
\]
where the sum is over all partitions of $n$ and $S^\lambda$ denotes the irreducible $S_n$-module indexed by $\lambda$.

It follows from Corollary 6.7 that

Corollary 6.10. The Foata-Schützenberger Theorem is equivalent to the Lusztig-Stanley Theorem.
Proof. First, notice that the set of permutations \( B_k = \{ \pi \in S_n : \text{maj}(\pi^{-1}) = k \} \) is a disjoint union of Knuth classes, where for each partition \( \lambda \vdash n \), there are exactly \( m_{k,\lambda} \) Knuth classes of RSK-shape \( \lambda \) in this disjoint union. Combining this fact with Proposition 6.4(1) implies that \( B_k \) is a fine set for the representation \( \rho_k := \bigoplus_{\lambda \vdash n} m_{k,\lambda} S^\lambda \).

On the other hand, by Proposition 6.4(2), the set of permutations \( L_k = \{ \pi \in S_n : \ell(\pi) = k \} \) is a fine set for \( R_k \).

Combining these facts with Corollary 6.7, \( \rho_k \cong R_k \) if and only if the distributions of the descent set over \( B_k \) and \( L_k \) are equal.

\[ \square \]

Remark 6.11. A combinatorial proof of the Lusztig-Stanley Theorem as an application of Foata-Schützenberger’s Theorem appears in [13]. The opposite implication is new.

6.4. More examples of fine sets.

The following criterion for fine sets is useful.

Proposition 6.12. Let \( \rho \) be an \( S_n \)-representation, let \( \{ C_b : b \in B \} \) be a basis for the representation space, and let \( \text{Des} : B \to P_{n-1} \) be a map. If for every \( 1 \leq i \leq n-1 \) and \( b, v \in B \) there are suitable coefficients \( a_{i}(b, v) \) such that

\[
s_{i}(C_b) = \begin{cases} -C_b, & \text{if } i \in \text{Des}(b); \\ C_b + \sum_{v \in B \text{ s.t. } i \in \text{Des}(v)} a_{i}(b, v)C_v, & \text{otherwise} \end{cases}
\]

then \( B \) is a fine set for \( \rho \).

The proof is a natural extension of the proof of [13, Theorem 1] and is omitted.

Two well known bases which satisfy the assumptions of Proposition 6.12 are the Kazhdan-Lusztig basis for the group algebra [10, (2.3.b), (2.3.d)] and the Schubert polynomial basis for the coinvariant algebra [5, Theorem 3.14(iii)]. Since \( S_n \) embeds naturally in classical Weyl groups of rank \( n \), it follows that Kazhdan-Lusztig cells, as well as subsets of elements of fixed Coxeter length in these groups, are fine sets for the \( S_n \)-action.

Another useful criterion is the following. For a partition \( \nu \vdash n \) let \( \text{SYT}(\nu) \) be the set of standard Young tableaux of shape \( \nu \).

Proposition 6.13. A subset \( B \subseteq S_n \) is a fine set if and only if for every partition \( \nu \vdash n \) there exist a nonnegative integer \( m(\nu, B) \) such that

\[
\sum_{\pi \in B} x^\text{Des}(\pi) = \sum_{\nu \vdash n} m(\nu, B) \sum_{T \in \text{SYT}(\nu)} x^\text{Des}(T).
\]

Proof. Follows from Theorem 6.5. \( \square \)
It follows that a subset $B \subseteq S_n$ is a fine set if and only if the sum of quasi-symmetric functions $\sum_{\pi \in B} F_{\text{Des}(\pi)}$ is Schur positive.

By [9, Thm. 2.1] (as reformulated in [14, Thm. 2.2]), conjugacy classes in the symmetric group satisfy this criterion. It follows that subsets of the symmetric group, which are closed under conjugation, are fine sets.

Another example which satisfies this criterion has been given recently. A permutation $\pi \in S_n$ is called an arc permutation if for every $1 \leq k \leq n$ the set $\{\pi(1), \ldots, \pi(k)\}$ forms an interval in the cyclic group $\mathbb{Z}_n$ (where $n$ is identified with $0$). By [6, Theorem 5], the subset of arc permutations in $S_n$ satisfies the criterion of Proposition 6.13; thus, the subset of arc permutations is a fine set in $S_n$.

We conclude with a list of the known fine subsets of the symmetric group.

**Proposition 6.14.** The following subsets of $S_n$ are fine sets:

1. Subsets closed under Knuth relations. In particular, Knuth classes, inverse descent classes and 321-avoiding permutations.
2. Subsets closed under conjugation.
3. Permutations of fixed Coxeter length.
4. The set of arc permutations.

The first two examples appear in [9, Thm. 5.5]. It is a challenging problem to find a characterization of fine subsets in $S_n$.

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