RAPIDLY GROWING ENTIRE FUNCTIONS WITH THREE SINGULAR VALUES

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ABSTRACT. We settle the problem of finding an entire function with three singular values whose Nevanlinna characteristic dominates an arbitrarily prescribed function.

1. Introduction

Let \( f \) be a transcendental meromorphic function in the plane \( \mathbb{C} \). A critical point of \( f \) is a point at which the spherical derivative of \( f \) vanishes. The value of \( f \) at a critical point is called a critical value. A point \( a \) in the sphere \( \overline{\mathbb{C}} \) is called an asymptotic value of \( f \) if there exists a curve \( \gamma: [0,1) \to \mathbb{C} \) such that

\[
\gamma(t) \to \infty \quad \text{and} \quad f(\gamma(t)) \to a \quad \text{as} \quad t \to 1.
\]

A point \( a \) in \( \overline{\mathbb{C}} \) is a singular value of \( f \) if it is either a critical or an asymptotic value. In this paper we study the growth behavior of entire and meromorphic functions which have finitely many singular values. The class of such functions is usually denoted by \( S \), after Speiser [19], [20].

If \( f \) is an arbitrary meromorphic function in the plane, the Nevanlinna characteristic of \( f \) is defined as (see [9], [17])

\[
T(r,f) = N(r,f) + m(r,f),
\]

where

\[
N(r,f) = \int_0^r \frac{n(t,f)}{t} \; dt, \quad m(r,f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\varphi})| \; d\varphi,
\]

and \( n(t,f) \) is the number of poles of \( f \) in \( \{|z| < t\} \). Here we assumed that 0 is not a pole of \( f \). If \( f \) is a rational function of degree \( d \), then its Nevanlinna characteristic

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characteristic $T(r, f)$ grows like $d \log r$, as $r \to \infty$. If $f$ is a transcendental meromorphic function, then $T(r, f)$ grows faster than any multiple of $\log r$, but it is easy to see that for any $a > 1$ one can find a transcendental $f$ for which $T(r, f)$ grows slower than $\log^a r$, as $r \to \infty$.

The question of slowest possible growth of the Nevanlinna characteristic for meromorphic functions with finitely many singular values has been studied in recent years, notably by Eremenko [6], and Langley [14], [15]. In particular, it was proved that if $f$ is a transcendental meromorphic function with three singular values, then

$$\liminf_{r \to \infty} \frac{T(r, f)}{\log^2 r} \geq \frac{\sqrt{3}}{\pi},$$

and the constant on the right-hand side is sharp. Langley established the existence of an absolute constant for the right-hand side, and Eremenko found the exact value for this constant. If $f$ is a transcendental entire function with three singular values, then $\liminf_{r \to \infty} T(r, f)/\log^2 r$ is infinite. In fact, the Nevanlinna characteristic $T(r, f)$ of such a function dominates a positive multiple of $\sqrt{r}$.

In general, if $f$ is a transcendental meromorphic function which has finitely many singular values, then Langley showed that

$$\liminf_{r \to \infty} \frac{T(r, f)}{\log^2 r} > 0,$$

but the left-hand side can be as small as one wishes if the number of singular values is greater than three.

Here we investigate the question of arbitrarily rapid growth.

**Theorem 1.** For every $\mathbb{R}$-valued function $M(r)$, $r \geq 0$, there exists an entire function $f$ with three singular values $0, 1,$ and $\infty$ such that

$$T(r, f) \geq M(r), \text{ for } r \geq r_0,$$

and some $r_0 > 0$.

Our proof of this theorem is based on a combinatorial construction of a Riemann surface spread over the sphere which branches over three points. The desired map is obtained as a composition of a uniformizing map of this Riemann surface and the projection map to the sphere. One of the key steps in proving Theorem 1 is to establish a quantitative control on the volume growth of a graph in terms of the combinatorial modulus. This is done in Lemma 1.

A meromorphic function whose Nevanlinna characteristic dominates an arbitrarily prescribed function is easier to produce. Indeed, there is more flexibility in constructing surfaces spread over the sphere that correspond to meromorphic functions, since one does not need to worry about $\infty$ being an omitted value. The construction is outlined in Section 6.
2. Graphs

A graph $G$ is a pair $(V_G, E_G)$, where $V_G$ is a set of vertices and $E_G$ is a subset of unordered pairs of elements in $V_G$, called edges. If $v_1, v_2 \in V_G$, and $\{v_1, v_2\} \in E_G$, we say that $v_1$ and $v_2$ are connected by an edge, and write $v_1 \sim v_2$. We assume that no vertex is connected to itself by an edge. A graph is called bipartite if the vertices can be subdivided into two disjoint sets, say $A$ and $B$, and every edge connects a vertex from $A$ to one from $B$. A subgraph $G'$ of a graph $G$ is a graph whose vertex set forms a subset of $V_G$, and if two vertices of $G'$ are connected by an edge in $G'$, then they are connected by an edge in $G$. If $A$ is a subset in $V_G$, we denote by $|A|$ the cardinality of $A$, where $|A| = \infty$ if the set $A$ is infinite.

The valence of $v \in V_G$ is $|\{u \in V_G : u \sim v\}|$. The valence of $G$ is the supremum of the valences over all vertices of $G$. A graph $G$ is called locally finite, if the valence of each vertex is finite. A graph is said to have a finite valence, if there is a uniform bound on the valence at each vertex. A graph is called homogeneous of valence $q$ if every vertex has the same valence $q$.

A chain in $G$ is a sequence $(\ldots, x_{-1}, x_0, x_1, \ldots)$ of vertices, finite or infinite in one or both directions, such that $\ldots \sim x_{-1} \sim x_0 \sim x_1 \sim \ldots$. We also refer to a chain as a sequence of vertices along with the edges connecting them. A chain $(\ldots, y_{-1}, y_0, y_1, \ldots)$ is a subchain of a chain $(\ldots, x_{-1}, x_0, x_1, \ldots)$, if $y_j = x_{k(j)}$ for some monotone increasing sequence $(k(j))$. We say that a chain $(x_1, \ldots, x_n)$ connects two subsets $A$ and $B$ of $V_G$, if $x_1 \in A$ and $x_n \in B$. A chain $(x_1, x_2, \ldots)$ connects a finite set $A$ to $\infty$, if $x_1 \in A$ and it eventually leaves every finite set, i.e., for every finite subset $K$ of $V_G$ there exists $k \in \mathbb{N}$ such that $x_j \notin K$ for $j \geq k$. A set $B$ in $V_G$ is said to separate a set $A \subset V_G$ from $\infty$ if every chain connecting $A$ to $\infty$ has a vertex in $B$.

A loop in a graph is a finite chain $(x_1, \ldots, x_n)$, such that $x_1 = x_n$ and all other vertices of the chain are distinct. A tree is a graph that does not contain any loops $(x_1, \ldots, x_n)$ with $n > 3$. A subtree is a subgraph of a tree.

If $G$ is a graph and $V$ is a subset of the vertex set $V_G$, we consider the subgraph $G'$ of $G$ determined by the vertex set $V$ to be the graph whose vertex set is $V$, and two vertices $v_1$ and $v_2$ are connected by an edge in $G'$ if and only if they are connected by an edge in $G$.

A domain $D$ in a graph $G$ is a subset of the vertex set $V_G$ which is connected in the sense that every two vertices in $D$ can be connected by a chain all of whose vertices are in $D$. The boundary of $D$ in $G$, denoted by $\partial_D D$, or $\partial D$ if the graph is understood, is the set of all vertices in $V_G$ that are not in $D$, and each of which is connected by an edge in $E_G$ to a vertex in $D$. An annulus in a graph $G$ is a subset of $V_G$ whose complement in $V_G$ consists of two disjoint domains. Not every graph contains an annulus. A sequence of annuli $(A_k)$ is called nested if the annuli are pairwise disjoint, and $A_{k+1}$ separates $A_k$ from $\infty$. 

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In this paper we only consider planar graphs, i.e., graphs embedded in the plane $\mathbb{R}^2$. If we fix an embedding of a graph into $\mathbb{R}^2$, then we can speak of faces of the graph. These are complementary components of the image of the graph in the plane. A side of a face is a part of its boundary that is the image of an edge under the embedding. If $G$ is a planar graph, one can also define its dual $G^*$. The vertices of $G^*$ are in one to one correspondence with the faces of $G$. Two vertices of $G^*$ are connected by an edge if and only if the boundaries of the corresponding faces of $G$ share a side.

A connected graph can be viewed as a metric space if one declares that every edge is isometric to a unit interval on the real line. This metric restricts to the space whose elements are vertices of the graph, in which case it is said that the graph is endowed with the word metric. Thus, we can speak of geodesics in a graph, i.e., chains connecting two vertices or two sets and having the smallest lengths among all such chains. If $A$ and $B$ are two subsets of $V_G$, we denote by $\delta(A, B)$ the word distance between $A$ and $B$, i.e., the number of edges in a geodesic connecting $A$ and $B$. Similarly, $\delta(v, w)$ stands for $\delta(\{v\}, \{w\})$.

3. Surfaces of Speiser class

A surface spread over the sphere is an equivalence class of pairs $[(X, \pi)]$, where $X$ is an open, i.e., non-compact, simply connected topological surface and $\pi : X \to \overline{\mathbb{C}}$ is a continuous, open, and discrete map. Two pairs $(X_1, \pi_1)$ and $(X_2, \pi_2)$ are equivalent if there exists a homeomorphism $h : X_1 \to X_2$ such that $\pi_1 = \pi_2 \circ h$. The map $\pi$ is called the projection.

In a neighborhood of each point $x$ in $X$, the map $\pi$ is given in some local coordinates (for neighborhoods of $x$ and $\pi(x)$) by $z \mapsto z^k$, where $k = k(x) \in \mathbb{N}$ is called the local degree of $f$ at $x$. If $k \geq 2$, then $x$ is called a critical point of $f$, and in this case the value $f(x)$ is called a critical value. As in the case $X = \mathbb{C}$ and $\pi$ a meromorphic function, $a \in \overline{\mathbb{C}}$ is called an asymptotic value if there exists a curve $\gamma : [0, 1) \to X$, such that

$$\gamma(t) \to \infty \quad \text{and} \quad \pi(\gamma(t)) \to a \quad \text{as} \quad t \to 1.$$  

Here, $\gamma(t) \to \infty$ means that $\gamma(t)$ leaves every compact set of $X$ as $t \to 1$. A point $a$ in $\overline{\mathbb{C}}$ is a singular value of $\pi$ if it is either a critical or an asymptotic value.

According to Stoïlow [21], $X$ supports a complex structure, the pullback structure, in which the map $\pi$ is holomorphic. A surface spread over the sphere is said to have parabolic type, or is called parabolic, if $X$ endowed with the pullback structure is conformally equivalent to the plane. Otherwise it is said to have hyperbolic type. The homeomorphism $h$ in the definition of equivalence is a conformal map in these pullback structures, and therefore the conformal type of a surface spread over the sphere is well defined. For simplicity, below we refer to a pair $(X, \pi)$, rather than an equivalence class, as
a surface spread over the sphere. If \( g \) is a uniformizing map for \( X \) defined in the complex plane or the unit disc, then \( f = \pi \circ g \) is a meromorphic function. If \( \pi \) omits the value \( \infty \), then \( f \) is holomorphic. The surface spread over the sphere \((X, \pi)\) is classically referred to as the “surface of \( f^{-1} \).

A surface spread over the sphere belongs to Speiser class \( S \) if \( \pi \) has only finitely many singular values. If \( \{a_1, \ldots, a_q\} \) is the set of singular values of \( \pi \), then \( \pi \) restricted to \( \pi^{-1}(\mathbb{C} \setminus \{a_1, \ldots, a_q\}) \) is a covering map. Surfaces spread over the sphere of class \( S \) have combinatorial representations in terms of Speiser graphs.

Assuming that \((X, \pi) \in S\) and \( \pi \) has \( q \) singular values \( a_1, \ldots, a_q \), we fix an oriented Jordan curve \( L \) in \( \mathbb{C} \), visiting the points \( a_1, \ldots, a_q \) in cyclic order of increasing indices. This curve decomposes the sphere into two simply connected regions. Let \( L_i, i \in \{1, 2, \ldots, q\} \), be the arc of \( L \) from \( a_i \) to \( a_{i+1} \) (with indices taken modulo \( q \)). Let us fix points \( p_1 \) and \( p_2 \) in the two complementary components of \( L \), and choose \( q \) Jordan arcs \( \gamma_1, \ldots, \gamma_q \) in \( \mathbb{C} \), such that each arc \( \gamma_i \) has \( p_1 \) and \( p_2 \) as its endpoints, and has a unique point of intersection with \( L \), which is in \( L_i \). These arcs are chosen to be interiorwise disjoint, that is, \( \gamma_i \cap \gamma_j = \{p_1, p_2\} \) when \( i \not= j \). Let \( \Gamma' \) denote the graph embedded in \( \mathbb{C} \), whose vertices are \( p_1 \) and \( p_2 \), and whose edges are \( \gamma_i \), \( i = 1, \ldots, q \), and let \( \Gamma = \pi^{-1}(\Gamma') \). We identify \( \Gamma \) with its image in \( \mathbb{R}^2 \) under an orientation-preserving homeomorphism of \( X \) onto \( \mathbb{R}^2 \). The graph \( \Gamma \) is infinite, connected, homogeneous of valence \( q \), and bipartite. The vertices that project to \( p_1 \) are labelled \( \times \) and the ones that project to \( p_2 \) are labelled \( \circ \). A graph, properly embedded in the plane and having these properties is called a Speiser graph. Two Speiser graphs \( \Gamma_1, \Gamma_2 \) are said to be equivalent, if there is an orientation-preserving homeomorphism of the plane which takes \( \Gamma_1 \) to \( \Gamma_2 \). Each face of the Speiser graph \( \Gamma \) is labelled by the corresponding element of the set \( \{a_1, \ldots, a_q\} \).

The above construction of a Speiser graph from a surface spread over the sphere of class \( S \) is reversible. Suppose we are given a Speiser graph \( \Gamma \) whose faces are labelled by \( a_1, \ldots, a_q \). A necessary condition for existence of a surface spread over the sphere of class \( S \) with singular values \( a_1, \ldots, a_q \) and whose Speiser graph is \( \Gamma \) is that the labels should satisfy a certain compatibility condition. Namely, when going counterclockwise around a vertex \( \times \), the indices are encountered in their cyclic order, and around \( \circ \) in the reversed cyclic order. We fix a simple closed curve \( L \subset \mathbb{C} \) passing through \( a_1, \ldots, a_q \). Let \( H_1, H_2 \) be the complementary regions whose common boundary is \( L \), and let \( L_1, \ldots, L_q \) be as above. Let \( \Gamma^* \) be the planar dual of \( \Gamma \). The vertices of \( \Gamma^* \) are naturally labelled by \( a_1, \ldots, a_q \). If \( e \) is an edge of \( \Gamma^* \) connecting \( a_j \) and \( a_{j+1} \), let \( \pi \) map \( e \) homeomorphically onto the corresponding arc \( L_j \) of \( L \). This defines a map \( \pi \) on the edges and vertices of \( \Gamma^* \). We then extend \( \pi \) to map the faces of \( \Gamma^* \) homeomorphically to \( H_1 \) or \( H_2 \), depending on the orientation
of the boundaries. This defines a surface spread over the sphere \((\mathbb{R}^2, \pi) \in \mathcal{S}\). The corresponding labelled Speiser graph is the graph \(\Gamma\) with the prescribed labels. Thus, up to a choice of the curve \(L\), we have a one to one correspondence between surfaces spread over the sphere of class \(\mathcal{S}\) and equivalence classes of labelled Speiser graphs. See [7] and [17] for further details.

4. Type problem

A long studied problem is the one of recognizing the conformal type of a surface spread over the sphere of class \(\mathcal{S}\) from its Speiser graph. An infinite locally finite connected graph is called \textit{parabolic} if the simple random walk on it is recurrent. Otherwise it is called \textit{hyperbolic}. Doyle [3] gave a criterion of type for a surface spread over the sphere of class \(\mathcal{S}\) in terms of a so-called extended Speiser graph.

Let \(\mathbb{Z}_+\) denote the set of non-negative integers. A \textit{half-plane lattice} \(\Lambda\) is the graph embedded in \(\mathbb{R}^2\) whose vertices form the set \(\mathbb{Z} \times \mathbb{Z}_+\), and \((x', y') \sim (x'', y'')\) if and only if \((x'' - x', y'' - y') = (\pm 1, 0)\) or \((0, \pm 1)\). The \textit{boundary} of the half-plane lattice \(\Lambda\) is the infinite connected subgraph of \(\Lambda\) determined by the vertex set \(\mathbb{Z} \times \{0\}\). There is an action of \(\mathbb{Z}\) on \(\Lambda\) by horizontal shifts. A \textit{half-cylinder lattice} \(\Lambda_n\) is \(\Lambda/n\mathbb{Z}\). The boundary of \(\Lambda_n\) is the induced boundary from \(\Lambda\).

Suppose that \(\Gamma\) is a Speiser graph and let \(n \in \mathbb{N}\) be given. If we replace each face of \(\Gamma\) with \(2k\) edges on the boundary, \(k \geq n\), by the half-cylinder lattice \(\Lambda_{2k}\), and each face with infinitely many edges on the boundary by the half-plane lattice \(\Lambda\), identifying the boundaries of the faces with the boundaries of the corresponding lattices along the edges and vertices, we obtain an \textit{extended Speiser graph} \(\Gamma_n\). The graph \(\Gamma_n\) is an infinite connected graph embedded in the plane, containing \(\Gamma\) as a subgraph. It has a finite valence, and all faces of \(\Gamma_n\) have no more than \(\max\{2(n - 1), 4\}\) sides.

**Theorem A ([3]).** A surface spread over the sphere \((X, \pi) \in \mathcal{S}\) is parabolic if and only if \(\Gamma_1\) is parabolic.

In [16] we proved a slight modification of Theorem A.

**Theorem B ([16]).** Let \(n \in \mathbb{N}\) be fixed. A surface spread over the sphere \((X, \pi) \in \mathcal{S}\) is parabolic if and only if \(\Gamma_n\) is parabolic.

Doyle’s arguments are probabilistic and electrical, whereas [16] employs geometric methods, using results of Kanai [12], [13]. Below we derive Theorem B from Theorem A using results from [4].

Kanai shows the invariance of type under quasi-isometries for spaces with bounded geometry. A map \(\Phi: (X_1, d_1) \to (X_2, d_2)\) between two metric spaces is called a \textit{quasi-isometry}, if the following conditions are satisfied:

1. for some \(\varepsilon > 0\), the \(\varepsilon\)-neighborhood of the image of \(\Phi\) in \(X_2\) covers \(X_2\);
2. there are constants \( k \geq 1, \ C \geq 0 \), such that for all \( x_1, \ x_2 \in X_1 \),
\[
k^{-1}d_1(x_1, x_2) - C \leq d_2(\Phi(x_1), \Phi(x_2)) \leq kd_1(x_1, x_2) + C.
\]
The metric space \((X_1, d_1)\) is quasi-isometric to the metric space \((X_2, d_2)\) if there exists a quasi-isometry from \(X_1\) to \(X_2\). This is an equivalence relation. The notion of quasi-isometry, or rough isometry, was introduced by Gromov [8].

A Riemannian surface has bounded geometry if it is complete, the Gaussian curvature is bounded from below, and the radius of injectivity is positive. The latter means that there exists \( \delta > 0 \) such that every open ball whose radius is at most \( \delta \) is homeomorphic to a Euclidean ball. Kanai proves that if a Riemannian surface has bounded geometry and is quasi-isometric to a finite valence graph with the word metric, then the surface and the graph have the same type. Likewise, two quasi-isometric graphs with finite valence have the same type.

**Proof of Theorem B.** By Theorem A one needs to show that \( \Gamma_n \) is parabolic if and only if \( \Gamma_1 \) is. Assume first that \( \Gamma_1 \) is parabolic. The graph \( \Gamma_n \) is obtained from \( \Gamma_1 \) by cutting the edges that connect the vertices of \( \Gamma \), viewed as a subgraph of \( \Gamma_1 \) using the obvious embedding, on the boundary of faces of \( \Gamma \) with \( 2k \) edges, \( k < n \), to the vertices of \( \Lambda_{2k} \). Therefore, this direction follows from the Cutting Law [4], page 100. For the other direction, assume that \( \Gamma_1 \) is hyperbolic. We consider a new graph \( \tilde{\Gamma}_1 \), obtained from \( \Gamma_1 \) by shorting all nonboundary vertices of every half-cylinder lattice \( \Lambda_{2k} \), \( k < n \), that have replaced a face of \( \Gamma \). Here, shorting a set of vertices means identifying them. By the Shorting Law [4], page 100, \( \tilde{\Gamma}_1 \) is also hyperbolic. But \( \tilde{\Gamma}_1 \) has finite valence and is quasi-isometric to \( \Gamma_n \). The quasi-isometry is given by an embedding of \( \Gamma_n \) into \( \tilde{\Gamma}_1 \) induced from the obvious embedding of \( \Gamma_n \) into \( \Gamma_1 \). Therefore, \( \Gamma_n \) is hyperbolic. \( \square \)

Due to the nature of a construction, as in our case below, it is often easier to establish the type for the dual graph \( \Gamma_n^* \) to the extended Speiser graph \( \Gamma_n \).

**Theorem C.** Let \( n \in \mathbb{N} \) be fixed. A surface spread over the sphere \((X, \pi) \in S\) is parabolic if and only if \( \Gamma_n^* \) is parabolic.

**Proof.** The graph \( \Gamma_n \) in question and its dual have finite valences. A map \( \Phi \) that sends every vertex \( v \) of \( \Gamma_n^* \) to any vertex on the boundary of the face of \( \Gamma_n \) corresponding to \( v \) is a quasi-isometry. Indeed, the first condition for quasi-isometry follows since every vertex of \( \Gamma_n \) is on the boundary of a face and there is a uniform bound on the number of sides of each face since \( \Gamma_n^* \) has finite valence. Therefore, every vertex of \( \Gamma_n \) is within a uniformly bounded distance from an image of a vertex in \( \Gamma_n^* \) under \( \Phi \).

The second condition follows since both graphs have finite valence. Let \( \gamma^* \) be a geodesic chain in \( \Gamma_n^* \) connecting two vertices \( v_1 \) and \( v_2 \). By tracing the
boundaries of faces corresponding to the vertices of $\gamma^*$, one can find a chain in $\Gamma_n$ connecting $\Phi(v_1)$ and $\Phi(v_2)$, and whose length is at most $C_1$ times the length of $\gamma$, where $C_1$ depends only on the valences of $\Gamma_n$ and $\Gamma_n^*$. Conversely, for every geodesic chain $\gamma$ in $\Gamma_n$ connecting two vertices $\Phi(v_1)$ and $\Phi(v_2)$, one can find a chain in $\Gamma_n^*$ that connects $v_1$ and $v_2$ by following the faces that contain $\gamma$ on their boundaries, such that the length of this new chain is at most $C_2$ times the length of $\gamma$. The constant $C_2$ depends only on the valences of $\Gamma_n$ and $\Gamma_n^*$.

Since the graphs $\Gamma_n$ and $\Gamma_n^*$ are quasi-isometric and have finite valences, they have the same type. Now the result follows from Theorem B. □

5. Combinatorial modulus

In 1962, Duffin [5] introduced a combinatorial modulus for chain families in graphs. In his setting, the masses are assigned to the edges of the graph. Parabolicity of a locally finite graph is equivalent to the condition that the modulus of the family of chains connecting a fixed vertex to infinity is zero. For our purposes, it is more convenient to use a different notion of modulus, introduced more recently by Cannon [2], where masses are assigned to vertices rather than edges. This approach leads to certain combinatorial uniformization results, see e.g., [18]. If a graph has finite valence, as in our case below, it does not matter which definition of combinatorial modulus one uses when establishing parabolicity. This can be seen by distributing masses from vertices to edges and vice versa.

A mass distribution for a graph $G$ is a non-negative function on $V_G$. Let $C$ be a family of chains in $G$. We say that a mass distribution $m$ is admissible for $C$, if for each chain $(\ldots, x_{-1}, x_0, x_1, \ldots) \in C$, its weighted length $\sum m(x_j) \geq 1$. We denote by $\text{mod}_G C$ the combinatorial modulus of the chain family $C$, namely

$$\text{mod}_G C = \inf \left\{ \sum m(v)^2 \right\},$$

where the infimum is taken with respect to all admissible mass distributions, and the sum is over all vertices in $V_G$. We write $\text{mod} C$ if the graph is understood. To distinguish, the conformal modulus of a curve family on a surface will be denoted by $\text{Mod}$. If $C$ is the family of all chains connecting sets $A$ and $B$, or a set $A$ to $\infty$, we denote $\text{mod} C$ by $\text{mod}(A, B)$ or $\text{mod}(A, \infty)$, respectively. If $A$ is an annulus in a graph $G$, then $\text{mod} A$ denotes the modulus of the family of all chains that connect the complementary components of $A$ in $V_G$.

As for the classical conformal modulus, if $C$ and $C'$ are two families of chains, such that every chain in $C$ contains a subchain which is in $C'$, then $\text{mod} C \leq \text{mod} C'$. Also, if $(A_k)$ is a sequence of (disjoint) nested annuli, then

$$\text{mod}(\{v_0\}, \infty) \leq \frac{1}{\sum 1/\text{mod} A_k},$$
for any vertex $v_0$ that is separated from $\infty$ by every $A_k$. The first property follows immediately from the definition. A proof of the inequality mimics that for the classical conformal modulus. Now, as in the classical case, to show parabolicity of a finite valence graph, it is enough to exhibit a sequence $(A_k)$ of (disjoint) nested annuli, such that

$$\sum \frac{1}{\text{mod } A_k} = \infty.$$  

This will be used in the proof of Lemma 1.

6. Meromorphic example

Since later we prove that the Nevanlinna characteristic of an entire function dominates an arbitrarily prescribed function, here we only give an outline that such a meromorphic function exists.

Consider the infinite half-strip in the plane

$$S = \{z = x + iy : 0 \leq x \leq 2, 0 \leq y < \infty\},$$

subdivided into squares

$$\{z : j \leq x \leq j + 1, n \leq y \leq n + 1\}, \quad j = 0, 1, \ n = 0, 1, 2, \ldots.$$  

For each even $n = 2k$, we attach $N(k)$ Euclidean squares with side length 1 to the edge

$$e_k = \{z : x = 1, 2k \leq y \leq 2k + 1\},$$

so that all of these squares share the side $e_k$, and are otherwise disjoint. More specifically, we cut the strip $S$ along $e_k$, take a two-sided unit square cut along one of its edges, and glue the square to the strip along a cut. We repeat this operation if necessary, attaching more squares to $e_k$. What results can be thought of as a book spread open along its spine.

The result of the gluing of all the squares is a simply connected Riemann surface $Y$ with boundary, which corresponds to the boundary of $S$. Now we consider the double $X$ of $Y$ across the boundary. This means that $X$ is obtained from two copies of $Y$ by identifying every boundary point of one copy with the point of the other copy that corresponds to the same point of $Y$. This is a simply connected Riemann surface without boundary. For each $n = 0, 1, 2, \ldots$, let $A_n$ denote the annulus in $X$ that consists of all points corresponding to the points of the horizontal rectangle $\{n \leq y \leq n + 1\}$ of $S$ and all points of squares attached to $e_{n/2}$ if $n$ is even. Each surface $X$ is parabolic since it contains a sequence of annuli $(A_n)$, where $n$ is odd, of fixed modulus. Using a modulus estimate, one can show that if $F$ is a uniformizing map of $\mathbb{C}$ onto $X$, then the image $I_r$ under $F$ of the disc $D_r$ centered at 0 of radius $r$ contains a ball (in the intrinsic metric of $X$) of radius

$$L(r) \geq C \log r,$$
where $C$ is a constant not depending on the sequence $(N(k))$. Indeed, let $s$ denote the set in $X$ that corresponds to the segment in $S$ connecting $(0,0)$ to $(2,0)$, and let $s_F$ be the preimage of $s$ under $F$. The set $s_F$ is homeomorphic to a line segment. Suppose that $n(r)$ is the smallest natural number so that the annulus $A_{n(r)}$ is not contained in $I_r$. The conformal modulus of the curve family consisting of curves in $D_r$ that separate $s_F$ from the boundary of $D_r$ grows like $\log r/(2\pi)$ as $r \to \infty$. On the other hand, the conformal modulus of the image family in $X$ is bounded above by $C'n(r)$, where $C'$ is a constant independent of $(N(k))$. This can be seen by choosing a weight function equal $1/2$ at all points of the annuli $A_0, A_1, \ldots, A_{n(r)+1}$ that correspond to points of $S$, and equal 0 at all other points of these annuli. From the invariance of modulus under conformal maps we obtain that
\[ n(r) \geq \log r/(2\pi C'), \]
which immediately gives the desired estimate for $L(r)$.

Now, by choosing $N(k)$ to grow sufficiently rapidly, one can arrange arbitrarily rapid growth of the areas, with respect to the radii, of the intrinsic balls of $X$ centered at some point. Arbitrarily rapid growth of the areas implies arbitrarily rapid growth of the Nevanlinna characteristic (see Ahlfors–Shimizu characteristic in [9]). A similar fact is based on the First Main Theorem of Nevanlinna and it will be discussed in Section 8.

By subdividing each square of the surface $X$ into four triangles using diagonals, and considering the Speiser graph which is dual to such a triangulation, we obtain a meromorphic function with three singular values that has the desired properties.

7. Entire functions with three singular values

If $f$ is a transcendental entire function with three singular values 0, 1, and $\infty$, then $f^{-1}([0,1])$ forms a locally finite, infinite tree $T$ embedded in $\mathbb{R}^2$. The vertices are the preimages of 0 and 1, and the edges are the preimages of $[0,1]$. Indeed, the graph is connected since $f$ restricted to $f^{-1}(\mathbb{C} \setminus \{0,1\})$ is a covering map. The valence of each vertex is the local degree of $f$ at the corresponding point. The graph is infinite since $f$ is transcendental. Finally, it is a tree because otherwise there would exist a complementary component of $f^{-1}([0,1])$ that is compactly contained in $\mathbb{C}$. This is impossible since such a component would have to contain a preimage of $\infty$, but $f$ is assumed to be entire.

Conversely, suppose we are given an arbitrary locally finite, infinite, embedded tree $T$, whose vertices are labelled 0 and 1, and each edge connects 0 and 1. We construct a surface spread over the sphere $(X, \pi)$ with three singular values as follows. For every vertex $v$ in $V_T$ of valence $k$, we consider $k$ non-homotopic, non-intersecting Jordan arcs in $\mathbb{R}^2 \setminus T$ that originate at $v$ and escape to infinity. We can choose the arcs corresponding to different vertices
to be disjoint. This gives a triangulation $T'$ of $\mathbb{R}^2$, with each triangle having an ideal vertex at infinity. Every triangle of $T'$ has an edge of $T$ and two arcs escaping to infinity as its sides. Each vertex of $T'$ has an even valence, and it receives a label 0 or 1 from the corresponding label of $T$. The ideal vertices at infinity are labelled by $\infty$.

Consider the dual graph to $T'$, denoted $\Gamma$. The graph $\Gamma$ is an infinite connected graph, properly embedded in the plane. It has valence three at each vertex, and every face of $\Gamma$ has an even (or infinite) number of vertices on its boundary, so $\Gamma$ is bipartite. Therefore $\Gamma$ is a Speiser graph. Let $(X, \pi)$ denote a surface spread over the sphere that corresponds to $\Gamma$ with the induced labels from $T'$, which are 0, 1, and $\infty$. These are the singular values of $\pi$, and $\pi$ omits the value $\infty$. Thus, the composition of a uniformizing map of $X$ with $\pi$ is a holomorphic function. We proceed by explicitly describing $(X, \pi)$ up to conformal equivalence.

Let $\alpha = \{(x, y) : 0 \leq x, 0 \leq y \leq 1\}$ be a half-strip in the plane. To each triangle $t$ of $T'$ we associate a copy of $\alpha$, which we denote by $\alpha(t)$, so that under an orientation-preserving homeomorphism of the plane the side of $t$ contained in $T$ corresponds to the segment joining $(0,0)$ and $(0,1)$, and the sides of $t$ that are in $T' \setminus T$ correspond to two horizontal rays. If $t_1$ and $t_2$ are adjacent triangles, we glue $\alpha(t_1)$ and $\alpha(t_2)$ along the corresponding sides using the identity map. The result of the gluing is a simply connected open Riemann surface, which we denote by $S(T)$.

A tree isomorphic to $T$ embeds in $S(T)$, and we identify this tree with $T$. Now we consider the conformal map, continuously extended to the boundary, from the half-strip

$$\alpha^o = \{(x, y) : 0 < x, 0 < y < 1\}$$

to the lower half-plane that takes $(0,0), (0,1),$ and $\infty$ to $0, 1,$ and $\infty$, respectively. This map extends by reflection to a conformal map from the Riemann surface $S(T)$ to the surface spread over the sphere $(X, \pi)$ with the pullback complex structure. The tree $T$ is isomorphic to $\pi^{-1}([0,1])$ with the natural graph structure.

Since we need to consider an extended Speiser graph in deciding the type of a surface spread over the sphere, the following subdivision of $S(T)$ is useful. We subdivide $\alpha$ into squares

$$\alpha_k = \{(x+k, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}, \quad k = 0, 1, 2, \ldots$$

The subdivision of $\alpha$ by $\alpha_k$, $k = 0, 1, 2, \ldots$, induces a subdivision of $S(T)$ into squares, a square subdivision. The 1-skeleton of this subdivision considered as a graph will be denoted by $\sigma = \sigma(T)$. The tree $T$ is a subgraph of $\sigma$. In the case when the tree $T$ has valence $n$, as in our example below with $n = 3$, the graph $\sigma(T)$ is the dual graph $\Gamma_n^*$ to the extended Speiser graph $\Gamma_n$. According
to Theorem C, the surface spread over the sphere \((X, \pi)\) is parabolic if and only if \(\sigma\) is.

## 8. Volume growth

The First Main Theorem of Nevanlinna (see [9], [17]) asserts that for every \(a \in \mathbb{C}\),

\[
T(r, f) = N\left( r, \frac{1}{f - a}\right) + m\left( r, \frac{1}{f - a}\right) + O(1), \quad r \to \infty.
\]

Therefore, by choosing \(a\) to be either 0 or 1, we conclude that in order to find \(f\) with \(T(r, f)\) growing arbitrarily rapidly, it is sufficient to find an embedded tree \(T\) with the following properties. The corresponding surface \(S(T)\) is parabolic, and if \(M(r), \ r \geq 0\), is a prescribed function, and \(g\) a uniformizing map from \(\mathbb{C}\) to \(S(T)\), then the number of vertices of \(g^{-1}(T)\) in the disc of radius \(r\) about 0 is greater than \(M(r)\), for all \(r \geq r_0 > 0\). In this case, the first term \(N(r, 1/(f - a))\) alone dominates \(M(r)\).

Assuming that \(S(T)\) is parabolic and \(g\) is a uniformizing map from \(\mathbb{C}\) to \(S(T)\), we denote by \(n(r, T, g)\) the number of vertices of \(g^{-1}(T)\) contained in the disc of radius \(r\) centered at 0. This is an analog of the counting function \(n(r, f)\) in the definition of Nevanlinna characteristic \(T(r, f)\). Theorem 1 follows from the above theorem, proved in Section 10.

**Theorem 2.** Given any \(\mathbb{R}\)-valued function \(M(r), \ r \geq 0\), there exists a locally finite, infinite tree \(T\), embedded in the plane, such that \(S(T)\) is parabolic, and \(n(r, T, g) \geq M(r), \ r \geq r_0\), for any uniformizing map \(g\) and \(r_0 = r_0(g) > 0\). Moreover, we can choose \(T\) to be a subtree of the regular tree of valence three, denoted \(T_3\).

The tree \(T_3\) is homogeneous of valence 3, and we think of \(T_3\) as being embedded in the plane. Let \(v_0\) be a fixed vertex in \(V_{T_3}\), and \(\varepsilon_0\) denote the combinatorial modulus \(\text{mod}_{T_3}(\{v_0\}, \infty)\), which is a positive number because \(T_3\) is hyperbolic, as is well known. The complement of \(T_3\) in the plane has infinitely many components, three of which have \(v_0\) on their boundaries. We consider one of these three components, denoted \(D\), and let \(c = (\ldots, v_{-1}, v_0, v_1, \ldots)\) be the chain in \(T_3\) such that \(v_j \neq v_k\) for \(j \neq k\), and \(c\) together with the edges that connect its vertices bounds \(D\).

If \(k \in \mathbb{N}\), then \(T_3 \setminus \{v_k, v_{-k}\}\) is a union of five disjoint domains, one of which contains \(v_0\), and each of the four others is bounded by either \(v_k\) or \(v_{-k}\). For each \(k \in \mathbb{N} \cup \{0\}\), let \(C_k\) be the family of all chains \((x_1, x_2, \ldots)\) in \(T_3\) that connect \(\{v_0\}\) to \(\infty\), and such that all but finitely many of \(x_j\)'s are contained in one of the domains into which \(T_3 \setminus \{v_k, v_{-k}\}\) splits, that does not contain \(v_0\). The family \(C_0\) consists of all chains connecting \(v_0\) to \(\infty\). If \(k > 0\), each chain of \(C_k\) should have all but finitely many of its vertices to lie in one of the four domains of \(T_3 \setminus \{v_k, v_{-k}\}\) that does not contain \(v_0\). In other words, every
chain in \( C_k \) should escape to infinity through either \( v_k \) or \( v_{-k} \). It is easy to see that the sequence \( (\varepsilon_k) \) defined by \( \varepsilon_k = \text{mod}_{T_3} C_k \) decreases, \( 0 < \varepsilon_k \leq \varepsilon_0 \) for every \( k \in \mathbb{N} \), and \( \lim \varepsilon_k = 0 \).

For two quantities \( a \) and \( b \), we use the notation \( a \lesssim b \) if there exists a constant \( C > 0 \) which depends only on the data of an underlying space, such that \( a \leq Cb \). The key step in the proof of Theorem 2 is the following lemma.

**Lemma 1.** Let \( c, C_k \), and \( \varepsilon_k \) be as above, \( k \in \mathbb{N} \cup \{0\} \). Let \( L(\varepsilon) \), \( 0 < \varepsilon \leq \varepsilon_0 \), be a positive decreasing function, \( L(\varepsilon_0) \geq 1 \). Let \( B'_k \) be the subset of vertices of \( T_3 \) defined by

\[
B'_k = \{ v \in V_{T_3} : \delta(v, v_0) = \delta(v, c) + k \},
\]

and let \( B_k \) be the subset of \( B'_k \) given by

\[
B_k = \{ v \in B'_k : \delta(v, c) \leq L(\varepsilon_{k+1}) \}.
\]

Then the subtree \( T \) of \( T_3 \), determined by the vertex set

\[
V_T = \bigcup_{k=0}^{\infty} B_k,
\]

satisfies the property that for every \( \varepsilon \in (0, \varepsilon_0] \), and every domain \( D \) in \( T \) with \( v_0 \in D \), we have

\[
\text{mod}_T(\{v_0\}, \partial D) < \varepsilon \quad \Rightarrow \quad |D| > L(\varepsilon).
\]

Moreover, if

\[
2^{L(\varepsilon_1)} + \cdots + 2^{L(\varepsilon_k)} \leq C2^{L(\varepsilon_{k+1})}, \quad k = 1, 2, \ldots,
\]

where \( C \) is a positive constant, then \( S(T) \) is parabolic.

**Proof.** It follows from the definition that \( B'_k \), \( k = 0, 1, 2, \ldots \), are disjoint, \( \bigcup_{k=0}^{\infty} B'_k = V_{T_3} \), and every chain in \( C_k \) has all but finitely many of its vertices in \( \bigcup_{l \geq k} B'_l \).

Suppose that \( \varepsilon \in (0, \varepsilon_0] \), and let \( D \) be a domain in \( T \) with \( v_0 \in D \), and such that \( \text{mod}_T(\{v_0\}, \partial D) < \varepsilon \). There exists \( k \in \mathbb{N} \cup \{0\} \) such that \( \varepsilon_{k+1} < \varepsilon \leq \varepsilon_k \). Assume for contradiction that \( |D| \leq L(\varepsilon) \). Since \( L \) is decreasing, \( |D| \leq L(\varepsilon_{k+1}) \), and therefore every chain in \( C_k \) contains a subchain in \( T \) that connects \( \{v_0\} \) to \( \partial_T D \), the boundary of \( D \) in \( T \). Indeed, \( D \) can also be considered as a domain in \( T_3 \), and it cannot contain vertices of \( B'_l \), \( l \geq k \), that are more than distance \( [L(\varepsilon_{k+1})] - 1 \) away from \( c \) because \( |D| \leq L(\varepsilon_{k+1}) \).

Thus every boundary vertex of \( D \subset T_3 \) contained in \( \bigcup_{l \geq k} B'_l \) is a boundary vertex of \( D \subset T \). Since \( v_0 \in D \), every chain \( c' \) in \( C_k \) has a subchain connecting \( v_0 \) to some boundary vertex \( v' \) of \( D \) in \( T_3 \). Furthermore, \( c' \) contains a subchain connecting \( v_0 \) to \( v' \in \partial_T D \). If not, let \( v'' \) be the last vertex of \( c' \) that belongs to the boundary of \( D \) in \( T_3 \). Since \( D \) is a domain, and hence is connected, and \( T_3 \) is a tree, \( v'' \) either belongs to \( c \) or is contained in \( \bigcup_{l \geq k} B'_l \). But \( c \) is contained in \( T \), and in the latter case \( v'' \) belongs to \( T \) as a boundary vertex of
$D \subset T_3$ contained in $\bigcup_{l \geq k} B'_l$. The desired subchain is obtained by removing edges of $c'$ that connect vertices outside of $V_T$.

Now, we have

$$\varepsilon_k = \text{mod}_{T_3} C_k \leq \text{mod}_T(\{v_0\}, \partial D) < \varepsilon.$$ 

This last estimate contradicts our understanding that $\varepsilon \leq \varepsilon_k$, and proves (1).

It remains to prove that under condition (2), $S(T)$ is parabolic. The tree $T$ has an axis of symmetry passing through $v_0$ so that under the symmetry transformation the vertex $v_k$ is mapped to $v_{-k}$ and vice versa, and each $B_k$ as well as the chain $c$ are invariant. One should think of this axis of symmetry as being orthogonal to $c$. Let $\sigma = \sigma(T)$ be the 1-skeleton of the square subdivision of $S(T)$ that was created using the $\alpha_k$’s. The graph $\sigma$ has also an axis of symmetry, denoted $a$, induced by the axis of symmetry of $T$. We claim that $\sigma$ is parabolic. For that purpose, we exhibit a sequence of nested annuli $(A_k)$ and verify that $\sum 1/\text{mod} A_k = \infty$.

For each $k = 1, 2, \ldots$, we consider an annulus $A_k$ in $\sigma$ obtained as follows. The vertices of $T$ separate those of $\sigma$ into two groups, which we call $V_+$ and $V_-$. The sets $V_+$ and $V_-$ form the sets of vertices of the upper and lower square grids $\{(m, n) : m \in \mathbb{Z}, n \in \mathbb{N}\}$ and $\{(m, n) : m \in \mathbb{Z}, -n \in \mathbb{N}\}$, respectively, so that for each of these sets the vertices with coordinates $(0, n)$ are located on the symmetry axis $a$. Each $A_k$ consists of the vertices of the set $B_k \subset V_T$ defined above, vertices $(m, n)$ in $V_+$ such that $\max\{|m|, |n|\} = k$, and vertices $(m, n)$ in $V_-$ such that $a_k \leq \max\{|m|, |n|\} \leq b_k$, where $a_k$ and $b_k$ are chosen as follows. The number $a_k$ is the least one such that the vertex $(a_k, -1)$ of $V_-$ is connected by an edge to $v_k$, and $b_k$ is the largest number such that $(b_k, -1)$ is connected by an edge to $v_k$. A direct calculation gives

\[
a_k = 2^{[L(\varepsilon_1)]} + 2^{[L(\varepsilon_2)]} + \cdots + 2^{[L(\varepsilon_k)]} - k + 1,
\]

\[
b_k = 2^{[L(\varepsilon_1)]} + 2^{[L(\varepsilon_2)]} + \cdots + 2^{[L(\varepsilon_{k+1})]} - k - 1.
\]

Indeed, for each $l > 0$, the number of vertices of $B_l$ lying to one side of the axis of symmetry $a$ is $2^{[L(\varepsilon_{l+1})]}$, and the total number of vertices $v$ of $V_-$ to one side of $a$, such that $v$ is connected to a vertex in $B_l$, is $2^{[L(\varepsilon_{l+1})]+1} - 1$. Adding the latter terms for $l = 1, 2, \ldots, k - 1$ and for $l = 1, 2, \ldots, k$ together, each along with $2^{[L(\varepsilon_1)]}$, contributed by $B_0$, we obtain the quantities $a_k$ and $b_k + 1$, respectively.

Now, we assign mass 1 to all vertices in $A_k \cap V_+$, mass $1/2^{l-1}$ to vertices $v$ in $B_k$ such that $\delta(v, c) = l$, $l = 1, 2, \ldots, [L(\varepsilon_{k+1})]$, and mass $1/2^{[L(\varepsilon_{k+1})]-1}$ to the vertices in $A_k \cap V_-$. This is an admissible mass distribution for the family of chains that connect the two components of $V_\sigma \setminus A_k$. Indeed, if a chain contains a vertex in $A_k \cap V_+$, we are done. If a chain only contains vertices of
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$A_k \cap V_-$, then its weighted length is at least

$$\frac{b_k - a_k}{2^{[L(\varepsilon_{k+1})]-1}} = 4 \left( 1 - \frac{1}{2^{[L(\varepsilon_{k+1})]}} \right) \geq 1,$$

since we assumed that $L(\varepsilon_0) \geq 1$, and $L$ is decreasing. A chain that contains only vertices of $B_k$ has weighted length at least 1, because the subgraph of $\sigma$ determined by the vertex set $B_k$ is a tree, and hence such a chain has to contain the vertex $v_k$. The remaining case is when a chain $\gamma$ contains vertices of $A_k \cap V_-$ as well as vertices in $B_k$. It is easy to see that then there is a chain that contains only vertices of $A_k \cap V_-$, and whose weighted length is comparable to that of $\gamma$, with absolute constants. Such a chain is obtained by replacing each vertex $v$ of $\gamma$ that belongs to $B_k$ by a chain of vertices in $A_k \cap V_-$ of the form $(m, -1)$, so that the first and the last vertices of this chain are connected by edges in $\sigma$ to $v$. Multiplying the mass distribution by an appropriate constant produces an admissible mass distribution.

The mass bound is

$$\lesssim k + \sum_{l=1}^{[L(\varepsilon_{k+1})]} \frac{2^l}{2^{2l(l-1)}} + \frac{(2b_k)^2 - (2a_k)^2}{2^{2([L(\varepsilon_{k+1})]-1)}}$$

$$\lesssim k + 1 + \left( 1 + 2^{2L(\varepsilon_{k+1})} \right) \lesssim k, \quad k = 1, 2, \ldots.$$

Since $\sum_{k=1}^{\infty} 1/k = \infty$, we conclude that $\sigma$ is parabolic.

9. Comparison of moduli

The results of this section are essentially contained in [1], Section 8. A pathwise connected metric measure space $(X, d, \mu)$ is an $n$-Loewner space if

$$\inf \{ \text{Mod}_n(E, F) : \Delta(E, F) \leq t \}$$

is a positive function for all $t > 0$, where $\text{Mod}_n(E, F)$ denotes the $n$-modulus of a curve family connecting two disjoint continua $E$ and $F$ in $X$, and

$$\Delta(E, F) = \frac{\text{dist}(E, F)}{\min\{\text{diam} E, \text{diam} F\}}$$

is called the relative distance between $E$ and $F$. Loewner spaces were introduced in [11], see also [10].

Recall that $\sigma = \sigma(T)$ is the 1-skeleton of the square subdivision of $S(T)$. Let $\mathcal{U} = \{ U_v : v \in V_\sigma \}$ be an open cover of $S(T)$, where $U_v$ is the interior of the union of all squares in $\sigma$ that have a vertex at $v \in V_\sigma$. If $J > 0$, we define the $J$-star of $v \in V_\sigma$ as

$$\text{St}_J(v) = \bigcup \{ U_u : u \in V_\sigma, \delta(u, v) < J \}.$$
Note that $St_1(v) = U_v$. Since $T$ is a tree, it is easy to see that $St_J(v)$ is an open, connected, and simply connected subset of $S(T)$. For a set $A$ in $S(T)$ we denote by $V_A$ the set of vertices $v$ such that $U_v \cap A \neq \emptyset$.

**Lemma 2.** Assume that the valence $k$ of $T$ is finite. Let $v$ be a vertex of $\sigma$, and $\rho$ be an arbitrary Borel measurable non-negative function on $St_2(v)$. If $Y_1, Y_2 \subset S(T)$ are continua with $Y_i \cap U_v \neq \emptyset$, and $\text{diam}(Y_i) \geq c_0 > 0$, $i = 1, 2$, then there is a rectifiable curve $\eta$ in $St_2(v)$ connecting $Y_1$ and $Y_2$, such that

$$\int_\eta \rho \, ds \leq C_0 \left( \int_{St_2(v)} \rho^2 \, d\mu \right)^{1/2},$$

where $C_0 > 0$ depends only on $c_0$ and $k$.

**Proof.** The result follows from the observation that there are only finitely many, depending on $k$, different possibilities for $St_2(v)$ that can occur, and from Theorem 6.13 in [11], which implies that $St_2(v)$ is a 2-Loewner space. Indeed, the Loewner property gives that the conformal modulus $\text{Mod}(Y_1, Y_2) \geq c > 0$, where $c$ depends on $c_0$ and $k$ only. This means that for every Borel measurable non-negative function $\rho$ on $St_2(v)$ we have

$$\int_{St_2(v)} \rho^2 \, d\mu \geq c \inf_\gamma \left( \int_\gamma \rho \, ds \right)^2,$$

where the infimum is taken over all curves $\gamma$ in $St_2(v)$ that connect $Y_1$ and $Y_2$. Thus, for every $\varepsilon > 0$ there exists a rectifiable curve $\eta \subset St_2(v)$ connecting $Y_1$ and $Y_2$ such that

$$\left( \int_\eta \rho \, ds \right)^2 \leq \frac{1}{c} \int_{St_2(v)} \rho^2 \, d\mu + \varepsilon.$$

Choosing $\varepsilon = \frac{1}{c} \int_{St_2(v)} \rho^2 \, d\mu$ completes the proof in the case when $\rho$ is not zero almost everywhere on $St_2(v)$. The latter case is trivial. \[ \square \]

**Lemma 3.** If $T$ is an infinite embedded tree of valence $k$, then there exists a constant $C_1 \geq 1$, depending only on $k$, such that if $A, B \subset S(T)$ are two continua not contained in any set $St_2(v)$ for $v$ a vertex of $\sigma$, then

$$\text{mod}_\sigma(V_A, V_B) \leq C_1 \text{Mod}(A, B).$$

**Proof.** Let $\rho : S(T) \to [0, \infty]$ be an admissible Borel function for the pair $(A, B)$, i.e.,

$$\int_\gamma \rho \, ds \geq 1,$$

for every rectifiable curve $\gamma$ that connects $A$ and $B$. We consider the mass distribution on $\sigma$ defined by

$$m(v) = \left( \int_{St_2(v)} \rho^2 \, d\mu \right)^{1/2}. $$
To prove (3) we need to establish a mass bound and verify admissibility. The mass bound is

$$\sum_{v \in V} m(v)^2 \leq \sum_{v \in V} \left( \sum_{u : \delta(u,v) < 2} \int_{U_u} \rho^2 \, d\mu \right) \lesssim \sum_{v \in V} \int_{U_v} \rho^2 \, d\mu \lesssim \int_{S(T)} \rho^2 \, d\mu,$$

where the constants understood depend only on $k$.

To show admissibility, we let $v_1, v_2, \ldots, v_k$ be vertices of a chain in $\sigma$ that connect $V_A$ and $V_B$. Then $U_{v_1} \cap A \neq \emptyset$, $U_{v_k} \cap B \neq \emptyset$, and $U_{v_i-1} \cap U_{v_i} \neq \emptyset$. We set $\lambda_1 = A$, $\lambda_{k+1} = B$, and for $i = 2, \ldots, k$, let $\lambda_i$ be a square in the square subdivision $\sigma$ with two of the vertices being $v_{i-1}$ and $v_i$. Then for $i = 2, \ldots, k$, we have $\lambda_i \subseteq U_{v_{i-1}} \cap U_{v_i}$, and $\text{diam} \lambda_i = \sqrt{2}$. Also, since $A$ and $B$ are not contained in any $St_2(v)$, there exists an absolute constant $c_0 > 0$ such that $\text{diam} A \geq c_0$ and $\text{diam} B \geq c_0$. Using Lemma 2 we can inductively find rectifiable curves $\eta_1, \ldots, \eta_k$, satisfying the condition

$$\int_{\eta_i} \rho \, ds \leq C_0 \sum_{i=1}^{k} m(v_i),$$

and such that $\eta_i$ connects $\lambda_1 \cup \eta_1 \cup \cdots \cup \eta_{i-1}$ and $\lambda_{i+1}$. The constant $C_0$ depends only on $c_0$ and $k$. The union $\eta_1 \cup \cdots \cup \eta_k$ contains a rectifiable curve $\eta$ connecting $A$ and $B$, and having the property

$$1 \leq \int_{\eta} \rho \, ds \leq C_0 \sum_{i=1}^{k} m(v_i).$$

Thus $C_0 m$ is an admissible mass distribution for the pair $(V_A, V_B)$, and the proof is complete. \hfill \Box

### 10. Proof of Theorem 2

Let $M(r)$, $r \geq 0$, be an arbitrary $\mathbb{R}$-valued function, and $L(\varepsilon)$ be a function that satisfies the conditions of Lemma 1, and such that $L(4\pi C_1 / \log r) \geq M(r)$, where $C_1$ is the constant from Lemma 3 when $k = 3$. Let $T$ be the subtree of $T_3$ given by Lemma 1. Then $S(T)$ is parabolic, and let $g$ be a uniformizing map from $\mathbb{C}$ to $S(T)$. Let $A_{r'}$ and $B_r$ be the images under $g$ of circles $C_{r'}$ and $C_r$ centered at 0 of radii $r'$ and $r$, respectively, $1 < r' < r$. We choose $r'$ such that $A_{r'}$ is not contained in any set $St_2(v)$, $v \in V_{\sigma}$. Using Lemma 3 and the conformal invariance of Mod, we obtain that

$$\text{mod}_{\sigma}(V_{A_{r'}}, V_{B_{r}}) \leq C_1 \text{Mod}(C_{r'}, C_r) < \frac{4\pi C_1}{\log r}, \quad r \geq r_0 = (r')^3.$$
Since $T$ is a subgraph of $\sigma$, from monotonicity we have
\[ \text{mod}_T(V_{A,r}, V_{B,r}) < \frac{4\pi C_1}{\log r}, \quad r \geq r_0. \]
If $D$ is the domain in $T$ which is the connected component of $V_T \setminus V_{B,r}$ containing $v_0$, then \( \text{mod}_T(\{v_0\}, \partial D) < 4\pi C_1 / \log r \). Therefore, by Lemma 1, \( |D| > L(4\pi C_1 / \log r) \geq M(r) \). The proof is complete.

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