Cosmological solutions in generalized hybrid metric-Palatini gravity

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(Dated: May 14, 2018)

We construct exact solutions representing a Friedmann-Lemaître-Robertson-Walker (FLRW) universe in a generalized hybrid metric-Palatini gravity theory. By writing the gravitational action in a scalar-tensor representation, the new solutions are obtained by either making an ansatz on the scale factor or on the effective potential. Among other relevant results, we show that it is possible to obtain exponentially expanding solutions for flat universes even when the cosmology is not purely vacuum. We then derive the classes of actions for the original theory which generate these solutions.

I. INTRODUCTION

The late-time cosmic accelerated expansion [1,2] has posed important and challenging problems to theoretical cosmology. Although the standard model of cosmology has favored dark energy models [3] as fundamental candidates responsible for the accelerated cosmic expansion, it is also viable that this expansion is due to modifications of general relativity [4,5], which introduce new degrees of freedom to the gravitational sector itself. Indeed, the phenomenology of \( f(R) \) gravity, where \( R \) is the metric Ricci curvature scale and \( f \) a general function, has been scrutinized motivated by the possibility to account for the self-accelerated cosmic expansion without invoking dark energy sources [6,7]. Besides, this kind of modified gravity is capable of addressing the dynamics of several self-gravitating systems alternatively to the presence of dark matter [8,9]. In this approach of \( f(R) \) gravity, and using its equivalent scalar-tensor representation, one can show that in order to satisfy local, i.e., solar system, observational constraints a large mass of the scalar field is required, which scales with the curvature through the chameleon mechanism [10,11]. In turn, this has undesirable effects at cosmological scales. There are other drawbacks in these models, see [6,7]. A Palatini version of the \( f(R) \) gravity theory, where the connection rather than the metric represents the fundamental gravitational field, has been proposed [12], and it has been established that it has interesting features but the manifest deficiencies and downsides of the metric \( f(R) \) gravity also appear [12].

A hybrid combination of the two versions of the \( f(R) \) gravity theories, containing elements from both of the two formalisms, i.e., a hybrid metric-Palatini theory, consists of adding an \( f(R) \) term constructed à la Palatini to the Einstein-Hilbert Lagrangian [13]. It turns out to be very successful in accounting for the observed phenomenology and is able to avoid some of the shortcomings of the original approaches [14]. In the scalar-tensor representation of the hybrid metric-Palatini theory there is a long-range light mass scalar field, which is able to modify, in a way consistent with the observations, the cosmological and galactic dynamics, but leaves the solar system unaffected [15,16]. This light scalar field thus allows to evade the screening chameleon mechanism. In addition, absence of instabilities in perturbations were also verified in this model [17].

The theory was further developed into a generalized hybrid metric-Palatini theory, in which a general function, \( f(R,R) \), was postulated depending on both the metric and Palatini curvature scalars, \( R \) and \( \mathcal{R} \), respectively [18]. Of course, the positive aspects of the initial hybrid metric-Palatini theory are preserved, namely, the consistency with cosmological and galactic dynamics, and correct solar system tests [19,20], see [21] for a review. There are other developments. Considering linear homogeneous perturbations, the stability regions of the Einstein static universe were analyzed, and it was shown that a large class of stable solutions exists [22]. In addition, the full set of linearized evolution equations for the perturbed potentials, in the Newtonian and synchronous gauges, were derived. It was concluded that the main deviations from general relativity arise in the distant past, with an oscillatory signature in the ratio between the Newtonian potentials [23]. Furthermore, using specific models and a combination of cosmic microwave background, supernovae and baryonic acoustic oscillations background data, it was shown that the model’s free parameters are in agreement with the observational constraints [24,25]. It was also shown in this theory that the initial value problem can always be well-formulated and be well-posed depending on the adopted matter sources [26].

PACS numbers: 04.50.Kd,04.20.Cv,
The understanding and the building of solutions in complex theories like the generalized hybrid metric-Palatini gravity, and its scalar-tensor representation, is a difficult task. Among other methods to find solutions, the reconstruction technique method has been often valuable in this search. This method was first employed in [27] in order to select the form for the inflation potential able to resolve the open problems of the inflationary paradigm, e.g., the graceful exit. Also other elegant attempts were made to generalize this approach to nonminimally coupled scalar-tensor theories with a single scalar field [28].

In this work, we aim to find cosmological solutions in the generalized hybrid metric-Palatini gravity proposed in [18] through the use of its scalar-tensor representation. We will then devise a reconstruction technique algorithm for scalar-tensor theories in a Friedmann-Lemaître-Robertson-Walker (FLRW) universe and we will show that the generalized theory through its scalar-tensor representation provides a very rich structure in astrophysical and cosmological applications. In particular, we will find that the cosmology of the generalized hybrid metric-Palatini gravity can differ in subtle ways from both general relativity and $f(R)$ gravity. In principle these differences can be used in combination with observational data to verify the viability of this class of theories.

This paper is organized in the following manner: In Sec. II we present the formalism of the generalized hybrid metric-Palatini gravity, and express the action in the scalar-tensor representation. In Sec. III we consider the cosmological dynamics of FLRW spacetimes, using the scalar-tensor representation of the hybrid theory. In Sec. IV we obtain, from the solutions found in the scalar-tensor representation, the specific forms for $f(R, \mathcal{R})$. In Sec. V we set out our conclusions.

II. GENERALIZED HYBRID GRAVITY: FORMALISM

A. Action and field equations for the generalized hybrid gravity

Consider the action of the generalized hybrid metric-Palatini modified theory of gravity, given by (the velocity of light is set to one)

$$S = \frac{1}{2\kappa} \int \Omega \sqrt{-g} f(R, \mathcal{R}) d^4x + S_m(g_{ab}, \chi),$$

where $\kappa \equiv 8\pi G$, $G$ is the gravitational constant, $g$ is the determinant of the metric $g_{ab}$, $f$ is a function of $R$ and $\mathcal{R}$, and $S_m$ is the matter action, in which matter is minimally coupled to the metric $g_{ab}$, and $\chi$ collectively denotes the matter fields. $R$ is the metric Ricci scalar and $\mathcal{R} \equiv g_{ab} \mathcal{R}^{ab}$ is the Palatini scalar curvature, with $\mathcal{R}_{ab}$ being defined in terms of an independent connection $\hat{\Gamma}_{ab}$ as,

$$\mathcal{R}_{ab} = \partial_a \hat{\Gamma}^c_{cb} - \partial_b \hat{\Gamma}^c_{ac} + \hat{\Gamma}^c_{cd} \hat{\Gamma}^d_{ab} - \hat{\Gamma}^c_{ad} \hat{\Gamma}^d_{cb}. \quad (2)$$

One can also use the additional independent connection as a building block to construct higher order curvature invariants, in the action $S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R, \mathcal{R}, Q_H)$, where the term $Q_H$, can take the following forms $R_{\mu\nu}R_{\mu\nu}$, $R_{\mu\nu}R_{\mu\nu}$, $R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}$, $R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}$, $R\mathcal{R}$, etc. However we will not consider these cases here. Relative to the presence of Ostrogradski instabilities, in the hybrid theory, like in $f(R)$-gravity, it is possible to avoid the problem by allowing a separation of the additional degrees of freedom into a harmless scalar degree of freedom [6] [7]. In fact, it turns out that this feature is a similar exception in the larger space of metric-affine theories, where a generic theory is plagued by ghosts, superluminalities and other unphysical degrees of freedom [17].

Varying the action (1) with respect to the metric $g_{ab}$ and the independent connection, yields the following field equations

$$\frac{\partial f}{\partial R} R_{ab} + \frac{\partial f}{\partial \mathcal{R}} \mathcal{R}_{ab} - \frac{1}{2} g_{ab} f(R, \mathcal{R}) = \kappa^2 T_{ab},$$

and

$$\nabla_c \left( \sqrt{-g} \frac{\partial f}{\partial \mathcal{R}} g^{ab} \right) = 0,$$  

respectively. The equation of motion (4) implies that the independent connection is the Levi-Civita connection of a new metric tensor $h_{ab}$ which is conformally related to $g_{ab}$ by the relation

$$h_{ab} = g_{ab} \frac{f(R, \mathcal{R})}{\kappa^2}.$$  

The independent connection can then be written in terms of the new metric as

$$\hat{\Gamma}_{bc} = \frac{1}{2} h^{ad}(\partial_b h_{dc} + \partial_c h_{bd} - \partial_d h_{bc}).$$

Thus, the new metric $h_{ab}$ is an auxiliary metric related to the independent connection, that was used to define the Palatini tensor, given by Eq. (2). We emphasize that matter is coupled to the physical metric $g_{ab}$, so that only the Levi-Civita connection $\Gamma^{\mu}_{\nu\rho}(g)$ should be used in the geodesic equation applied to the metric-Palatini theory. Note that since the matter action $S_m$ does not depend on the independent connection $\Gamma$, the equation of motion for this connection is independent of the stress-energy tensor, whereas the same does not happen to the equation of motion of the metric $g_{ab}$.

B. Scalar-tensor representation

It is useful to express the action (1) in a scalar-tensor representation. This can be achieved by considering an
action with two auxiliary fields, $\alpha$ and $\beta$, respectively, in the following form

$$ S = \frac{1}{2\kappa^2} \int \sqrt{-g} \left[ f (\alpha, \beta) + \frac{\partial f}{\partial \alpha} (R - \alpha) + \frac{\partial f}{\partial \beta} (\mathcal{R} - \beta) \right] d^4x + S_m. $$

Putting $\alpha = R$ and $\beta = \mathcal{R}$ we recover action $[1]$. To proceed, we define two scalar fields as

$$ \varphi = \frac{\partial f}{\partial \alpha}, \quad \psi = -\frac{\partial f}{\partial \beta}, $$

where the negative sign in Eq. $[9]$ is imposed to guarantee a positive kinetic energy for the scalar field. Defining a potential $V$ as

$$ V (\varphi, \psi) = -f (\alpha, \beta) + \varphi \alpha - \psi \beta, $$

the action equivalent to $[7]$ is of the form

$$ S = \frac{1}{2\kappa^2} \int \sqrt{-g} \left[ \varphi R - \psi \mathcal{R} - V (\varphi, \psi) \right] d^4x + S_m, $$

where the variation of the matter action $S_m$ with respect to the metric $g_{ab}$ yields the stress-energy tensor $T_{ab}$.

It is important to stress that in this representation the scalar fields are not actually matter fields but only alternative representations of the curvatures $R$ and $\mathcal{R}$, and consequently they are allowed to assume values, such as negative ones, which would not be physically valid. Also, since we imposed the signs on the definitions of $\varphi$ and $\psi$ to guarantee that the kinetic energies are positive, then we should not worry about ghost instabilities. This range of values corresponds in the generalized hybrid gravity picture to some specific behavior of the derivatives of the function $f$ in the action. That said, it should also be clear that zeros for the scalar fields are problematic points in terms of the relation between the representations. In the following, the solutions we will obtain are only valid in the interval between the zeros of the scalar fields. In addition, it should be remarked that the scalar field representation will only be meaningful provided that $R$ and $\mathcal{R}$ can be expressed in terms of the scalar fields. This is not the case for all forms of the function $f$. In the next sections we will show some examples in which the consequences of these limitations become important.

Taking into account that $h_{ab} = -\psi g_{ab}$, see Eqs. $[5]$ and $[9]$, we have

$$ \mathcal{R} = R + \frac{3}{\psi^2} \partial^a \psi \partial_a \psi - \frac{3}{\psi} \Box \psi. $$

Thus, we can replace $\mathcal{R}$ in the action $[11]$, to obtain

$$ S = \frac{1}{2\kappa^2} \int \sqrt{-g} \left[ (\varphi - \psi) R - \frac{3}{2\psi} \partial^a \psi \partial_a \psi ight. 
- \left. V (\varphi, \psi) \right] d^4x + S_m. $$

Varying the previous action, Eq. $[13]$, with respect to the metric $g_{ab}$ and the scalar fields $\varphi$ and $\psi$, and rearranging terms, we obtain the following equations of motion

$$ (\varphi - \psi) G_{ab} = \kappa^2 T_{ab} + (\nabla_a \nabla_b - g_{ab} \Box) (\varphi - \psi) $n
+ \frac{3}{2\psi} \partial_a \psi \partial_b \psi + \left( \frac{1}{2} \kappa^2 + \frac{3}{4\psi} \partial^a \psi \partial_a \psi \right) g_{ab}, $$

$$ \Box \varphi + \frac{1}{3} (2V - \psi V_{\psi} - \varphi V_{\varphi}) = \frac{\kappa^2 T}{3}, $$

$$ \Box \psi - \frac{1}{2\psi} \partial^a \psi \partial_a \psi - \frac{\psi}{3} (V_{\varphi} + V_{\psi}) = 0, $$

respectively, where $V_{\varphi}$ and $V_{\psi}$ are defined by

$$ V_{\varphi} \equiv \frac{\partial V}{\partial \varphi}, \quad V_{\psi} \equiv \frac{\partial V}{\partial \psi}, $$

respectively.

The dynamical equation for $\varphi$, Eq. $[15]$, can be obtained in the following manner. The variation of the action with respect to $\varphi$ gives

$$ R = V_{\varphi}. $$

Then, taking the trace of Eq. $[14]$ and inserting $R = V_{\varphi}$ gives Eq. $[15]$. The latter equation shows an important difference between the two scalar fields, namely, $\varphi$ is coupled to matter whereas $\psi$, given through Eq. $[16]$, is not. This fact will have important consequences when we will look for cosmological solutions in which matter is present.

III. COSMOLOGICAL EQUATIONS AND SOLUTIONS FOR THE GENERALIZED HYBRID GRAVITY IN THE SCALAR-TENSOR REPRESENTATION

A. Cosmological equations in the scalar-tensor representation

In this section, we consider the FLRW spacetime with the spatial curvature parameter $k$, where $k$ can assume three values, $k = -1, 0, 1$. In spherical coordinates $(t, r, \theta, \phi)$ the line element can be written as

$$ ds^2 = -dt^2 + a^2 (t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right], $$

where $a(t)$ is the scale factor. We also assume that the cosmological scalar fields are time-dependent, $\varphi = \varphi (t)$, $\psi = \psi (t)$, and that the matter is described by a perfect fluid,

$$ T^a_b = \text{diag} (-\rho, p, p, p), $$

so that the trace is given by

$$ T = T^a_a = -\rho + 3p. $$
with $\rho$ and $p$ being the energy density and the pressure of the fluid, respectively, and both are assumed to depend solely on $t$.

Then the modified cosmological equations for this case can be obtained by computing directly the two independent components of the gravitational field equation, Eq. (14), i.e., the Friedmann equation

$$3 \left( \frac{\dot{a}}{a} \right)^2 + \frac{3k}{a^2} = \frac{1}{\kappa^2} \left[ \dot{\varphi}^2 + \frac{1}{2} \dot{\psi}^2 \right] + \frac{\kappa^2}{a^2} \left[ \rho + \frac{1}{3} V + \frac{1}{4} \kappa^2 T \right],$$

(22)

and the Raychaudhuri equation

$$\frac{2}{a^2} \left( \frac{\dot{a}}{a} \right) - \frac{2k}{a^2} = \frac{1}{\kappa^2} \left[ \dot{\varphi}^2 + \frac{1}{2} \dot{\psi}^2 \right] + \frac{\kappa^2}{a^2} \left( \dot{\varphi}^2 + \frac{1}{2} \dot{\psi}^2 \right),$$

(23)

respectively, where a dot means differentiation with respect to $t$. We will choose $k = 0$ and a scale factor described by

$$a(t) = A e^{\sqrt{\Lambda} (t-t_0)},$$

(30)

where $A$ and $\Lambda$ are free constants, and $\Lambda$ can be seen as some cosmological constant. This scale factor is plotted in Fig. 1.

FIG. 1. Plot of the scale factor $a(t)$ for the de Sitter case, given in Eq. (30), for the values $a_0 = 1$, $\Lambda = 1$.

With these assumptions, Eq. (26) becomes

$$V_\varphi = 12\Lambda.$$

(31)

This equation can be directly integrated with respect to $\varphi$ to obtain a potential of the form

$$V(\varphi, \psi) = 12\Lambda \varphi + b(\psi).$$

(32)
where $b(\psi)$ is an arbitrary function on $\psi$ which arises from the fact that the potential is a function of both fields. These results leave Eqs. (24) and (25) as
\[
\ddot{\phi} + 3\sqrt{\Lambda} \dot{\phi} - \frac{1}{3} [12 \Lambda \varphi + 2b(\psi) - \psi b'(\psi)] = 0, \tag{33}
\]
\[
\ddot{\psi} + 3\sqrt{\Lambda} \dot{\psi} - \frac{1}{2} \dot{\psi}^2 + \frac{1}{3} \psi [12 \Lambda + b'(\psi)] = 0, \tag{34}
\]
respectively. In the particular case where
\[
b(\psi) = -12\Lambda \psi, \tag{35}
\]
the potential becomes
\[
V(\varphi, \psi) = 12\Lambda (\varphi - \psi), \tag{36}
\]
and Eq. (34) can be divided by $\psi$ and directly integrated over $t$ to obtain the solution
\[
\psi(t) = \psi_0 e^{-6\sqrt{\Lambda} t} \left[ e^{3\sqrt{\Lambda}(t-t_0)} - 1 \right]^2, \tag{37}
\]
where $t_0$ and $\psi_0$ are constants of integration. In the same case of Eq. (35), where $b(\psi) = -12\Lambda \psi,$ Eq. (33) becomes
\[
\ddot{\varphi} + 3\sqrt{\Lambda} \dot{\varphi} - 4\Lambda (\varphi - \psi) = 0, \tag{38}
\]
and upon using Eq. (37) it can be integrated to obtain the solution
\[
\varphi(t) = \varphi_0 e^{-6\sqrt{\Lambda} t_0} - \frac{2}{5} \psi_0 e^{-6\sqrt{\Lambda} t} - 2\psi_0 e^{-3\sqrt{\Lambda}(t+t_0)} + \varphi_0 e^{-4\sqrt{\Lambda} t} + \varphi_1 e^{\sqrt{\Lambda} t}, \tag{39}
\]
where $\varphi_0$ and $\varphi_1$ are constants of integration. The solutions for the scalar fields $\psi(t)$ and $\varphi(t)$ are plotted in Figs. 2 and 3, respectively.

The solution is complete since $k, a, \rho, p, V, \varphi,$ and $\psi$ are known.

FIG. 3. Plot of the scalar field $\phi(t)$ for the de Sitter case, given in Eq. (38), for the values $\phi_0 = \varphi_1 = \Lambda = 1,$ and $t_0 = 0.$

2. Solution with a simplified potential equation I

In this section, we set again the matter component to be vacuum, i.e., $\rho = p = 0,$ and we choose again a flat universe, $k = 0.$ Inspecting Eq. (26), we can see that it simplifies if the first two terms within the brackets cancel each other, i.e.,
\[
\dot{H} + 2H^2 = 0, \tag{40}
\]
where we have used the definition for the Hubble function $H.$ Eq. (29). Equation (40) has the following solution for $H$,
\[
H = \frac{1}{2(t-t_0)}, \tag{41}
\]
where $t_0$ is an integration constant. Now, using again Eq. (29), then Eq. (40) can be integrated to obtain
\[
a(t) = a_0 \sqrt{2(t-t_0)}, \tag{42}
\]
where $a_0$ is an integration constant. This solution is plotted in Fig. 4.

The interest in this solution resides in the fact that, for a universe populated by radiation, we expect that the behavior of the scale factor is proportional to $\sqrt{t},$ but in this case the same behavior can be obtained with $\rho = 0.$

With this set of assumptions, the equation for the potential, Eq. (26), becomes
\[
V_\varphi = 0, \tag{43}
\]
which can be directly integrated to obtain
\[
V(\varphi, \psi) = b(\psi), \tag{44}
\]
where $b(\psi)$ is an arbitrary function on $\psi.$ Using Eqs. (41) and (43), Eq. (25) for $\psi$ becomes
\[
\ddot{\psi} + \frac{3\dot{\psi}}{2(t-t_0)} - \frac{\dot{\psi}^2}{2\psi} + \frac{\psi b'(\psi)}{3} = 0. \tag{45}
\]
In the particular case where

\[ b(\psi) = V_0, \]

with \( V_0 \) a constant, the last term on the left-hand side of Eq. (44) vanishes and the equation can be integrated directly after dividing through by \( \dot{\psi} \), yielding the solution

\[ \psi(t) = \frac{\psi_1 \left( \psi_0 \sqrt{2(t-t_0)} - 1 \right)^2}{2(t-t_0)}, \]

where \( \psi_0 \) and \( \psi_1 \) are constants of integration. On the other hand, the equation for \( \varphi \), namely Eq. (24), takes the form

\[ \ddot{\varphi} + \frac{3}{2} \dot{\varphi} \frac{2}{t-t_0} - \frac{1}{3} \left[ 2b(\psi) - \psi b'(\psi) \right] = 0. \]

Under the choice given in Eq. (43) for \( b(\psi) \), Eq. (47) decouples completely from the equation for \( \psi \), yielding

\[ \ddot{\varphi} + \frac{3}{2} \dot{\varphi} \frac{2}{t-t_0} - \frac{2}{3} V_0 = 0, \]

which can be directly integrated to obtain the following solution,

\[ \varphi(t) = \frac{2}{15} V_0 t (t - 2t_0) - \frac{\varphi_0}{\sqrt{2(t-t_0)}} + \varphi_1, \]

where \( \varphi_0 \) and \( \varphi_1 \) are constants of integration. The solutions for the scalar field \( \psi(t) \) and \( \varphi(t) \) are plotted in Figs. 5 and 6 respectively.

The solution is complete since \( k, a, p, V, \varphi, \) and \( \psi \) are known.

3. Solution with a simplified potential equation II

In this section, we set again the matter component to be vacuum, \( \rho = p = 0 \), and we choose again a flat universe, \( k = 0 \). Taking into account Eq. (26), we look for solutions in which the first two terms in the brackets are equal to a constant \( \Omega^2/2 \) say, where the 2 enters for convenience. Then Eq. (26) is now

\[ \dot{H} + 2H^2 + \frac{\Omega^2}{2} = 0, \]

where we have used the definition for the Hubble function \( H \), Eq. (29). This condition yields the solution

\[ H = -\frac{\Omega}{2} \tan \left[ \Omega (t-t_0) \right], \]

where \( t_0 \) is a constant of integration. Now, using again Eq. (29), then Eq. (51) can be integrated to obtain the scale factor,

\[ a(t) = a_0 \sqrt{\cos \left[ \Omega (t-t_0) \right]}, \]

where \( a_0 \) is a positive constant of integration, \( a_0 > 0 \). The behavior of \( a(t) \) in Eq. (52) is given in Fig. 7. This
solution is valid in between times \( t = -\frac{\pi}{2n} + t_0 \) and \( t = \frac{\pi}{2n} + t_0 \), or times that are translated from these ones by \( 2\pi n \) with integer \( n \). In between times \( t = \frac{\pi}{2n} + t_0 \) and \( t = \frac{3\pi}{2n} + t_0 \), or times that are translated from these ones by \( 4\pi n \), there are no physical solutions, since the scale factor in Eq. (52) gets imaginary values. The solution thus represents a universe which starts expanding at \( t = \frac{\pi}{2n} + t_0 \), attains a maximum value at \( t = t_0 \), and then collapses again at \( t = \frac{\pi}{2n} + t_0 \). The importance of this solution for \( a(t) \) stands on the fact that in the later times of the universe expansion, where we might approximate the distribution of matter to be vacuum and the geometry to be flat, it is possible to obtain solutions for the scale factor with a dependence on time different than the usual \( t^{1/2} \) power-law expected in standard general relativity.

Assumption (50) leaves the equation for the potential, Eq. (26), in the form

\[
V \varphi = -3\Omega^2, 
\]

which can be integrated to obtain

\[
V (\varphi, \psi) = -3\Omega^2 \varphi + b (\psi),
\]

where \( b (\psi) \) is an arbitrary function of \( \psi \). With these results, the equation for \( \psi \), Eq. (25), can be written as

\[
\ddot{\psi} - \frac{3}{2} \Omega \tan [\Omega (t - t_0)] \dot{\psi} - \frac{\psi^2}{2\psi} + \frac{\psi}{3} \left[ b' (\psi) - 3\Omega^2 \right] = 0.
\]

In the particular case where

\[
b (\psi) = 3\Omega^2 \psi,
\]

the potential becomes

\[
V (\varphi, \psi) = -3\Omega^2 (\varphi - \psi),
\]

the derivatives are \( V_{\varphi} = -V_{\psi} \) and the last term in the left-hand side of Eq. (55) vanishes. The equation can then be integrated analytically to obtain a solution for \( \psi \) of the form

\[
\psi (t) = \psi_1 \sec \left[ \Omega (t - t_0) \right] \left\{ \sqrt{\cos [\Omega (t - t_0)]} \times \right. \\
\left. \times \left[ \psi_0 \frac{\Omega}{\sqrt{2}} - \sqrt{2} E \left( \frac{\Omega}{2} (t - t_0) \right) \right]^{1/2} + \sqrt{2} \sin [\Omega (t - t_0)] \right\}^2,
\]

where \( \psi_0 \) and \( \psi_1 \) are constants of integration and \( E (a|b) \) is the incomplete elliptic integral of the second kind defined as \( E (a|b) = \int_0^a \sqrt{1 - b \sin^2 \theta} d\theta \). Figure 8 plots \( \psi (t) \) as a function of \( t \). As for the scale factor \( a(t) \), the solution for \( \psi(t) \) is valid in between times \( t = -\frac{\pi}{2n} + t_0 \) and \( t = \frac{\pi}{2n} + t_0 \), or times that are translated from these ones by \( 2\pi n \) with integer \( n \).

Finally, the equation for \( \varphi \), Eq. (24), becomes

\[
\ddot{\varphi} - \frac{3}{2} \Omega \tan [\Omega (t - t_0)] \dot{\varphi} + \frac{\psi}{3} b' (\psi) - \frac{2}{3} b (\psi) + \Omega^2 \varphi = 0.
\]

Choosing the same function \( b (\psi) \), see Eq. (56), Eq. (59) simplifies to

\[
\ddot{\varphi} - \frac{3}{2} \Omega \tan [\Omega (t - t_0)] \dot{\varphi} + \Omega^2 (\varphi - \psi) = 0.
\]

This equation does not have an analytical solution. However, it can be integrated numerically. In Fig. 9 we plot the solution for \( \varphi(t) \) for a specific choice of the parameters and with \( \varphi(0) = \varphi_0 = 0 \) and \( \dot{\varphi}(0) = \dot{\varphi}_0 = 0 \). As for the scale factor \( a(t) \) and for the scalar \( \psi(t) \) the solution for \( \varphi(t) \) is valid in between times \( t = -\frac{\pi}{2n} + t_0 \) and \( t = \frac{\pi}{2n} + t_0 \), or times that are translated from these ones by \( 2\pi n \) with integer \( n \).

The solution is complete since \( k, a, \rho, p, V, \varphi, \) and \( \psi \) are known.
which can be integrated to yield \( \varphi = \frac{2\psi_0}{\varphi_0} \sqrt{\psi + \varphi_1} \), which upon inverting gives,

\[
\psi = \left( \frac{\psi_0}{2\varphi_0} \right)^2 (\varphi - \varphi_1)^2,
\]

where \( \varphi_1 \) is a constant of integration. Inserting the potential \( \psi(\varphi) \) in Eq. (26), we obtain a relation between the Hubble function \( H \), and so the scale factor, and the scalar fields as

\[
\dot{H} + 2H^2 = \frac{V_0}{3} (\varphi - \psi).
\]

Inserting Eq. (67) into Eq. (68) we obtain

\[
\dot{H} + 2H^2 = \frac{V_0}{3} \left[ \varphi - \left( \frac{\psi_0}{2\varphi_0} \right)^2 (\varphi - \varphi_1)^2 \right].
\]

Now, after differentiating Eq. (69) with respect to time, using Eq. (64) to cancel the \( \varphi \) term, and deriving again with respect to time, we can write Eq. (69) as

\[
\ddot{H} + 7HH + 4H^2 + 12\dot{H}H^2 = -\frac{V_0\psi_0^2}{6a^5}.
\]

This is a fourth degree nonlinear differential equation for \( a \). It may not have an analytical general solution, but one can try a particular solution through an ansatz of the form \( a = a_0 t^\alpha \), for \( a_0 \) a constant and \( \alpha \) a number. This ansatz put into Eq. (70) leads to

\[
-12\alpha^3 + 18\alpha^2 - 6\alpha = \frac{V_0\psi_0^2}{6a_0^5} \frac{1}{\alpha}.
\]

Equating the exponents gives \( \alpha = 2/3 \). So the scale factor \( a(t) \) behaves as

\[
a(t) = a_0 t^{2/3}
\]

This solution is plotted in Fig. 10. Moreover, one finds

\[
\varphi(t) = \sqrt{\frac{2\psi_0}{\varphi_0}} \sqrt{\psi(t) + \varphi_1(t)},
\]

\[
\psi(t) = \left( \frac{\psi_0}{2\varphi_0} \right)^2 (\varphi(t) - \varphi_1(t))^2,
\]

\[
\ddot{a} = -\frac{V_0\psi_0^2}{6a^5}.
\]

FIG. 9. Plot of the function \( \varphi(t) \) for the case with a simplified potential equation II, from numerical results for the integration of Eq. (60) for \( \Omega = 1, \psi_0 = 0, \psi_1 = 1, \varphi_0 = 0, \varphi_1 = 0 \), and \( t_0 = 0 \).

FIG. 10. Plot of the scale factor \( a(t) \) for the case with simplified scalar field equations, given in Eq. (71), for the values \( a_0 = 1 \), and \( t_0 = 0 \).
of the equation above yields a constraint for the constants appearing in Eq. \( (70) \), namely,

\[
\frac{V_0 \psi_0^2}{6a_0^6} = -\frac{4}{9}.
\]  

(72)

The solution for the scale factor, Eq. \((52)\), allows us to integrate Eqs. \((64)\) and \((66)\) directly over time, leading to the following solutions for the scalar fields

\[
\varphi(t) = \frac{\varphi_0}{a_0^4} + \varphi_1,
\]

(73)

and

\[
\psi(t) = \left(\frac{\psi_0}{2a_0^2} + \frac{\psi_1}{2}\right)^2,
\]

(74)

respectively. These solutions are plotted in Figs. 11 and 12 respectively. This solution is interesting because, for

\[\varphi(t) = \frac{\varphi_0}{a_0^4} + \varphi_1,\]

and

\[\psi(t) = \left(\frac{\psi_0}{2a_0^2} + \frac{\psi_1}{2}\right)^2,\]

the scale factor to behave as \(t^{\frac{2}{3}}\); in this case we obtain this behavior considering \(\rho = 0\).

The solution is complete since \(k, a, \rho, p, V, \varphi, \) and \(\psi\) are known.

5. Solution with nonflat geometry

In this section, we set again the matter component to be vacuum, \(\rho = p = 0\). Now, we consider a nonflat geometry, i.e.,

\[k = 1, -1.\]

(75)

Looking at Eq. \((26)\), we can try to find a simplified equation for the potential considering that the first three terms inside the brackets cancel each other. Using Eq. \((29)\) this ansatz can be written as

\[\dot{H} + 2H^2 + \frac{k}{a^2} = 0.\]

(76)

Equation \((76)\) is in fact a nonlinear ordinary differential equation of the second order for \(a(t)\). Integrating this equation, we find the solution

\[a(t) = \sqrt{\frac{a_0^2 - k^2(t - t_0)}{k}},\]

(77)

where \(k\) is one of the two values given in Eq. \((75)\), \(a_0\) and \(t_0\) are constants of integration, and we have not considered the solution with negative sign for the scale factor \(a(t)\). Notice that for \(k = 1\), there is an end value of \(t\), given by \(t_e = a_0^2 + t_0\), for which the scale factor drops to zero, ending the solution, as for \(t > t_e\) it becomes a pure imaginary number. The same happens for \(k = -1\) when \(t < t_e\). Therefore, the solution for \(k = 1\) is only defined for \(-\infty < t < t_e\) and the solution for \(k = -1\) is only defined for \(t_e < t < +\infty\), see Fig. 13. We know from observations that \(k \sim 0\) is a very good approximation. However, the field equations \((14)\) depend on \(k\) in such a way that if \(k\) varies even slightly from 0, the structure of the equations changes dramatically. The relevance of the solution we have found, therefore, is to provide a glimpse of the role of spatial curvature in these theories.

Taking into account Eq. \((76)\), the equation for the potential, Eq. \((26)\), reduces to \(V_\varphi = 0\). This can be directly integrated over \(\varphi\) to obtain

\[V(\varphi, \psi) = b(\psi),\]

(78)

where \(b(\psi)\) is an arbitrary function of \(\psi\). With these results, we further assume \(b'(\psi) = 0\), i.e., \(b(\psi) = V_0\), so that

\[V(\varphi, \psi) = V_0,\]

(79)

with \(V_0\) is a constant.

Then, Eq. \((25)\) becomes

\[\psi + \frac{2k^2(t - t_0)}{k^2(t - t_0)^2 - a_0^2} \dot{\psi} - \frac{\dot{\psi}^2}{2\psi} = 0.\]

(80)
This equation can be solved analytically and we obtain
\[
\psi(t) = \frac{\psi_1}{a_0^2 - k^2 (t - t_0)^2} \left[ \left( a_0^2 \psi_0 k^2 + 1 \right) (t - t_0)^2 - a_0^2 \psi_0 \right] e^{2 \tanh^{-1} \left( \frac{t - t_0}{a_0 \sqrt{a_0^2 - k^2 (t - t_0)^2}} \right)},
\]
where \( \psi_0 \) and \( \psi_1 \) are constants of integration.

On the other hand, Eq. (24), simplifies to
\[
\ddot{\varphi} - \frac{3k^2 (t - t_0)}{a_0^2 - k^2 (t - t_0)^2} \dot{\varphi} - \frac{2}{3} V_0 = 0. \tag{82}
\]
This equation is once again decoupled from \( \psi \) and can be integrated directly to obtain
\[
\varphi(t) = \varphi_1 + \frac{t - t_0}{12a_0^2 k \sqrt{a_0^2 - k^2 (t - t_0)^2}} \times \\
\left[ 3a_0^4 V_0 \tanh^{-1} \left( \frac{k (t - t_0)}{\sqrt{a_0^2 - k^2 (t - t_0)^2}} \right) \right. \\
+ \left. k \left( a_0^2 V_0 (t - t_0) \sqrt{a_0^2 - k^2 (t - t_0)^2} + \varphi_0 \right) \right], \tag{83}
\]
where \( \varphi_0 \) and \( \varphi_1 \) are constants of integration. The solutions for \( \psi(t) \) and \( \varphi(t) \) are plotted in Figs. 14 and 15, respectively.

The solution is complete since \( k, a, \rho, p, V, \varphi, \psi \) are known.

6. Solution with perfect fluid matter

In this section we assume the presence of a perfect fluid, Eq. (20), with an equation of state as given in Eq. (27). Then Eq. (28) provides solutions for \( \rho \) and \( p \) as
\[
\rho = \rho_0 a^{-3(1+w)}, \tag{84}
\]
where \( \rho_0 \) and \( p_0 \) are constants of integration that are related to each other by the equation of state (27). We also assume a flat universe, \( k = 0 \).

Now, using Eqs. (84) and (85) we have that the energy-momentum tensor \( T_{ab} \) has the trace \( T \) given by \( T = -(\rho + 3p) = -\rho_0 (1 - 3w) a^{-3(w+1)} \). Then Eq. (24) becomes
\[
\ddot{\varphi} + 3H \dot{\varphi} - \frac{1}{3} \left[ 2V - \psi V_\psi - \varphi V_\varphi \right] = \frac{\kappa^2 \rho_0}{3} (1 - 3w) a^{-3(w+1)}. \tag{86}
\]
In addition, Eq. (25) for \( \psi \) and Eq. (26) for \( V_\psi \) remain the same. So, we just have to solve Eq. (86) for any of the particular cases discussed before, the other functions remaining the same. We consider two cases, namely, the de Sitter expansion of Sec. III B 1 and the simplified equation for the potential \( G \) of Sec. III B 2.

If we assume a de Sitter expansion, see Sec. III B 1, then the results from Eq. (30) to (37) hold, in particular \( a(t) \) behaves as in Fig. 9 and \( \varphi(t) \) as in Fig. 8. However,

---

FIG. 13. Plot of the scale factor (77) for \( a_0 = 1, t_0 = 0 \).

FIG. 14. Plot of the scalar field \( \psi(t) \) for the case with nonflat geometry, given in Eq. (81), for the values \( \psi_0 = \psi_1 = a_0 = 1, \) and \( t_0 = 0 \).

FIG. 15. Plot of the scalar field \( \varphi(t) \) for the case with nonflat geometry, given in Eq. (83), for the values \( \varphi_0 = \varphi_1 = a_0 = V_0 = 1, \) and \( t_0 = 0 \).
instead of Eq. (38) and the corresponding Fig. 2 we have to solve Eq. (86), which can be integrated analytically to yield

$$\varphi(t) = -\frac{\kappa^2 \rho_0 \left(a_0 e^{\sqrt{-\Lambda} t}\right)^{-3w} e^{-3\sqrt{-\Lambda} t}}{3a_0^2 \Lambda (4 + 3w)} + \psi_0 e^{-6\sqrt{\Lambda} t} - \frac{2}{7} \psi_0 e^{-6\sqrt{\Lambda} t} + 2\psi_0 e^{-3\sqrt{\Lambda} (t+t_0)} + \varphi_0 e^{-4\sqrt{\Lambda} t} + \varphi_1 e^{\sqrt{\Lambda} t}.$$  \hspace{1cm} (87)

This solution is plotted in Fig. 16.

If we assume a simplified equation for the potential I, see Sec. III B 2, then the results from Eq. (39) to (46) also hold, in particular $a(t)$ behaves as in Fig. 4 and $\varphi(t)$ as in Fig. 5. Again, instead of Eq. (47) and the corresponding Fig. 6, we have to solve Eq. (86), which can be integrated analytically to yield

$$\varphi(t) = \frac{2}{15} V_0 (t - t_0)^2 - \frac{\varphi_0}{\sqrt{2} (t - t_0)} + \varphi_1$$

$$- \frac{2\kappa^2 \rho_0 (t - t_0)}{3a_0^2 (3w - 2)} \left(a_0 \sqrt{2 (t - t_0)}\right)^{(3w+1)}.$$  \hspace{1cm} (88)

This solution is plotted in Fig. 17.

The solutions are complete since $k, a, \rho, p, V, \varphi,$ and $\psi$ are known.

IV. PULLING THE PREVIOUS SOLUTIONS INTO GENERALIZED HYBRID GRAVITY AND THE FORM OF $f(R, \mathcal{R})$

A. The main equation for generalized hybrid gravity

It is useful to obtain the form of the function $f(R, \mathcal{R})$ in all the previous cases as the scalar-tensor form of the theory is only a derived form and used to provide an easier platform to perform the calculations. To obtain $f(R, \mathcal{R})$, we are going to use the definition of the potential given by Eq. (10). If we replace here the definitions of the scalar fields given in Eqs. (8) and (9), we obtain a partial differential equation for $f$ as

$$V(f_R, f_\mathcal{R}) = -f(R, \mathcal{R}) + f_R R + f_\mathcal{R} \mathcal{R},$$  \hspace{1cm} (89)

where the subscripts $R$ and $\mathcal{R}$ represent derivatives with respect to these variables, respectively.

As we have seen in the previous section the solutions for the scalar field can have zeros and assume a negative sign. While this does not compromise the physical meaning of the solution in the generalized hybrid gravity picture (in this picture the scalar fields are not real matter fields) it can lead to further constraints on the form of the action. One might require that the derivatives of $f$, which correspond to the scalar fields by Eqs. (8) and (9) are never zero. In this case the solution we have found will only be meaningful in the intervals in which the scalar field have a fixed sign.

Let us now analyze this equation for the various forms of the potential obtained so far.

B. The previous solutions pulled into the generalized hybrid gravity

1. The de Sitter solution

The potential obtained in Sec. III B 1 is given by $V = 12\Lambda (\varphi - \psi)$ and therefore Eq. (10) can be written as

$$f(R, \mathcal{R}) = f_R (R - 12\Lambda) + f_\mathcal{R} (\mathcal{R} - 12\Lambda).$$  \hspace{1cm} (90)

Defining two new variables as $\bar{R} = R - 12\Lambda$ and $\bar{\mathcal{R}} = \mathcal{R} - 12\Lambda$, the derivatives of the function $f$ remain the same for the new variables and the equation can be written in a simpler manner as $f(\bar{R}, \bar{\mathcal{R}}) = f_R \bar{R} + f_\mathcal{R} \bar{\mathcal{R}}$, which has...
a general solution of the form \( f \left( \hat{R}, \hat{\mathcal{R}} \right) = g \left( \frac{R - 12\Lambda}{R - 12\Lambda} \right) \left( R - 12\Lambda \right) \), where \( g \) is an arbitrary function. Coming back to the variables \( R \) and \( \mathcal{R} \), we obtain

\[
f \left( R, \mathcal{R} \right) = g \left( \frac{R - 12\Lambda}{R - 12\Lambda} \right) \left( R - 12\Lambda \right), \tag{91}
\]

where \( g \) is still an arbitrary function. As an example, let us consider a function \( g \) of the form

\[
g \left( \frac{R - 12\Lambda}{R - 12\Lambda} \right) = \exp \left( \frac{R - 12\Lambda}{R - 12\Lambda} \right). \tag{92}
\]

Inserting this function \( g \) into Eq. (91) to obtain \( f \), inserting the result into Eq. (89) and using the definitions of the scalar fields given by Eqs. (8) and (9), we recover the result \( V = 12\Lambda \left( \varphi - \psi \right) \) for the potential and, consequently, the solutions for the scalar fields are the ones obtained in Sec. III B 1.

However, not all forms of the function \( g \) can be used to obtain the correct solutions. Take for example a function \( g \) of a power-law form, i.e., \( g \left( \frac{R - 12\Lambda}{R - 12\Lambda} \right) = \left( \frac{R - 12\Lambda}{R - 12\Lambda} \right)^{\gamma} \), where the exponent \( \gamma \) is a number. This form of the function \( g \) is compatible with Eq. (90) but it does not possess the solution we have reconstructed. This is due to the fact that the relationship between the scalar fields and the Ricci scalars given by \( \varphi \left( R, \mathcal{R} \right) \) and \( \psi \left( R, \mathcal{R} \right) \) cannot be inverted to obtain \( R \left( \varphi, \psi \right) \) and \( \mathcal{R} \left( \varphi, \psi \right) \). The equivalence between the metric and the scalar-tensor representations breaks down and one cannot immediately say that the theory will have the solutions reconstructed.

2. Solution with a simplified potential equation I

The potential obtained in Sec. III B 2 is given by \( V = V_0 \), a constant. We can then write Eq. (10) as

\[
f \left( R, \mathcal{R} \right) = f_R R + f_{\mathcal{R}} \mathcal{R} - V_0. \tag{93}
\]

This equation has a general solution of the form

\[
f \left( R, \mathcal{R} \right) = -V_0 + g \left( \frac{\mathcal{R}}{R} \right) R, \tag{94}
\]

where \( g \) is again an arbitrary function. As an example, let us consider a function \( g \) of the form

\[
g \left( \frac{\mathcal{R}}{R} \right) = \exp \left( \frac{\mathcal{R}}{R} \right). \tag{95}
\]

This exponential form allows us to invert the relation between the scalar fields and the Ricci tensors given by \( \varphi \left( R, \mathcal{R} \right) \) and \( \psi \left( R, \mathcal{R} \right) \) to obtain \( R \left( \varphi, \psi \right) \) and \( \mathcal{R} \left( \varphi, \psi \right) \), and therefore we recover the constant potential \( V_0 \) and the solutions for the scalar fields presented in Sec. III B 2.

Interestingly, in this case the power-law form for the function \( g \), i.e., \( g \left( \frac{\mathcal{R}}{R} \right) = \left( \frac{\mathcal{R}}{R} \right)^{\gamma} \), also allows us to invert the relation between the scalar fields and the Ricci tensors.

3. Solution with a simplified potential equation II

The simple potential obtained in Sec. III B 3 is given by \( V = -3\Omega^2 \left( \varphi - \psi \right) \), see Eq. (57). This potential is of the same form of the one obtained in Sec. III B 1 where the only difference is the factor \(-3\Omega^2\) instead of \(12\Lambda\). Therefore, the procedure to obtain the function \( f \) is the same as in Sec. IIIB 1 and the solution is

\[
f \left( R, \mathcal{R} \right) = g \left( \frac{R - 6c}{R + 6c} \right) \left( R + 6c \right). \tag{96}
\]

where \( g \) is an arbitrary function.

4. Solution with simplified scalar field equations

The potential obtained in Sec. III B 4 is given by \( V = V_0 \left( \varphi - \psi \right)^2 \). With this potential, Eq. (10) can be written as

\[
f \left( R, \mathcal{R} \right) = f_R R + f_{\mathcal{R}} \mathcal{R} - V_0 \left( f_R + f_{\mathcal{R}} \right)^2. \tag{97}
\]

This equation is a particular case of the Clairaut equation, with a general solution of the form

\[
f \left( R, \mathcal{R} \right) = RC_1 + RC_2 - V_0 \left( C_1 + C_2 \right)^2, \tag{98}
\]

where \( C_1 \) and \( C_2 \) are arbitrary constants of integration.

5. Solution with nonflat geometry

The potential obtained in Sec. III B 5 is given by \( V = V_0 \), a constant potential. This is the same potential as in Sec. III B 2 and therefore the procedure to find the function \( f \) is the same and the solution is given by Eq. (94).

6. Solution with perfect fluid matter

In Sec. III B 6 we presented two different solutions with perfect fluid matter. These solutions were generalizations of the solutions obtained in Secs. III B 1 and III B 2 where the potentials were given by \( V = 12\Lambda \left( \varphi - \psi \right) \) and \( V = V_0 \), respectively. Therefore, the solutions for the function \( f \) remain the same, given by Eqs. (91) and (94) respectively.

V. CONCLUSIONS

In this paper, we devised a number of reconstruction strategies to obtain new exact cosmological solutions in the context of the generalized hybrid metric-Palatini gravity. This theory can be recast in a scalar-tensor theory with two scalar fields whose solutions can be computed analytically for a convenient choice of the potential or the scale factor. Using the new methods we obtained a number of physically interesting solutions including power-law and exponential scale factors.
These solutions reveal some crucial differences not only with respect to GR but also to $f(R)$ gravity. For example, it becomes evident that a given behavior of the scale factors in vacuum has always a counterpart in the presence of a perfect fluid, as shown in Sec. III B 6.

The impact in terms of the interpretation of the cosmological dark phenomenology via this class of theories is clear: a negligible matter distribution still gives rise to expansion laws which are compatible with the observations without requiring the total baryonic matter to be a relevant percentage of the total energy density. This is particularly important in the case of the de Sitter solutions given in Sec. III B 1, because it implies that in these theories inflation could start even if matter is not negligible, and in the result of Sec. III B 4 which shows that in these theories matter is not central to obtain the classical $\rho^{2/3}$ Friedmann solution.

Another interesting aspect of the solutions we found concern the behavior of vacuum cosmology and the role of the spatial curvature. In the first case, see Sec. III B 3 we obtained that the generalized hybrid metric-Palatini theory can have a surprisingly complex behavior in the flat vacuum case. This has important consequences in terms, for example, of the far future of the cosmological models and the meaning of the cosmic no hair theorem in this framework.

Like in any higher order model, in the generalized hybrid metric-Palatini theory the spatial curvature has an important role. Even small deviations from perfect flatness are able to change dramatically the evolution of the cosmology. The result of Sec. III B 5 gives us a glimpse of these differences and their magnitude, showing that vacuum closed models will collapse, whereas open ones expand forever both with a relatively simple expansion law.

It is also interesting to note that our results imply that generalized hybrid metric-Palatini gravity can generate a de Sitter evolution without the appearance of an explicit cosmological constant in the action. Instead, the presence of such a constant leads, throughout our method, to solutions which are radiationlike, see Sec. III B 2. This result gives us information on the properties of effective dark fluid related to the non Hilbert-Einstein terms present in this theory. It also implies, by the cosmological no-hair theorem, that these solutions should be unstable. The stability of the other solutions we have found cannot be obtained with the same ease. This is a limitation of the reconstruction in general. It allows to obtain some exact solution but it does not offer information on their stability which has to be investigated with different approaches.

Our results suggests that the hybrid theory space is a priori large and requires further investigation not only in terms of the exploration of the solution space, but also in terms of the stability of these solutions. A final thought should be spent on the role of our results for the testability of this class of theory. It is clear that the solutions we have found can be used to perform a number of cosmological tests, e.g., the ones based on distances, for example. However, a more complete use of the accuracy of the current observations, such as cosmic microwave background anisotropies, requires a full analysis of the cosmological perturbations which is well outside of the scope of our contribution.

ACKNOWLEDGMENTS

JLR acknowledges financial support from Fundação para a Ciência e Tecnologia (FCT) - Portugal for an FCT-IDPASC Grant No. PD/BD/114072/2015. SC is supported by an Investigador FCT Research contract through project IF/00250/2013 and acknowledges financial support provided under the European Union’s H2020 ERC Consolidator Grant “Matter and strong-field gravity: New frontiers in Einstein’s theory” Grant Agreement No. MaGRaTh646597, and under the H2020-MSCA-RISE-2015 Grant No. StronGrHEP-690094. JPSL acknowledges financial support from the FCT Project No. UID/FIS/00099/2013, the FCT Grant No. SFRH/BSAB/128455/2017, the Grant No. 88887.068694/2014-00 from Aperfeiçoamento do Pes-soal de Nível Superior (CAPES), Brazil, and thanks Piotr Chruściel and the Gravitational Physics Group at the Faculty of Physics, University of Vienna, for hospitality. FSNL acknowledges financial support of FCT through an Investigador FCT Research contract, with reference IF/00859/2012 and the Grant No. PEst-OE/FIS/UI2751/2014.

[1] S. Perlmutter et al. (Supernova Cosmology Project Collaboration), “Measurements of Omega and Lambda from 42 high redshift supernovae”, Astrophys. J. 517, 565 (1999) [astro-ph/9812133].
[2] A. G. Riess et al. (Supernova Search Team Collaboration), “Observational evidence from supernovae for an accelerating universe and a cosmological constant”, Astron. J. 116, 1009 (1998) [astro-ph/9805201].
[3] E. J. Copeland, M. Sami, and S. Tsujikawa, “Dynamics of dark energy”, Int. J. Mod. Phys. D 15, 1753 (2006) [hep-th/0603057].
[4] T. Clifton, P. G. Ferreira, A. Padilla, and C. Skordis, “Modified gravity and cosmology”, Phys. Rep. 513, 1 (2012) [arXiv:1106.2476 [astro-ph.CO]].
[5] S. Nojiri and S. D. Odintsov, “Unified cosmic history in modified gravity: from $f(R)$ theory to Lorentz non-invariant models”, Phys. Rep. 505, 59 (2011) [arXiv:1011.0544 [gr-qc]].
[6] T. P. Sotiriou and V. Faraoni, “$f(R)$ theories of gravity”, Rev. Mod. Phys. 82, 451 (2010) [arXiv:0805.1726 [gr-qc]].
[7] A. De Felice and S. Tsujikawa, “$f(R)$ theories”, Living Rev. Rel. 13 (2010) arXiv:1002.4928 [gr-qc].

[8] C. G. Böhmer, T. Harko, and F. S. N. Lobo, “Dark matter as a geometric effect in $f(R)$ gravity”, Astropart. Phys. 29, 386 (2008) arXiv:0709.0046 [gr-qc].

[9] C. G. Böhmer, T. Harko, and F. S. N. Lobo, “Generalized virial theorem in $f(R)$ gravity”, J. Cosmol. Astropart. Phys. (JCAP) 03 (2008) 024 arXiv:0710.0966 [gr-qc].

[10] J. Khoury and A. Weltman, “Chameleon fields: Awaiting surprises for tests of gravity in space”, Phys. Rev. Lett. 93, 171104 (2004) astro-ph/0309300.

[11] J. Khoury and A. Weltman, “Chameleon cosmology”, Phys. Rev. D 69, 044026 (2004) astro-ph/0309411.

[12] G. J. Olmo, “Palatini approach to modified gravity: $f(R)$ theories and beyond”, Int. J. Mod. Phys. D 20, 413 (2011) arXiv:1101.3864 [gr-qc].

[13] T. Harko, T. S. Koivisto, F. S. N. Lobo, and G. J. Olmo, “Metric-Palatini gravity unifying local constraints and late-time cosmic acceleration”, Phys. Rev. D 85, 084016 (2012) arXiv:1110.1049 [gr-qc].

[14] S. Capozziello, T. Harko, F. S. N. Lobo, and G. J. Olmo, “Hybrid modified gravity unifying local tests, galactic dynamics and late-time cosmic acceleration”, Int. J. Mod. Phys. D 22, 1342006 (2013) arXiv:1305.3750 [gr-qc].

[15] S. Capozziello, T. Harko, T. S. Koivisto, F. S. N. Lobo, and G. J. Olmo, “Cosmology of hybrid metric-Palatini gravity”, J. Cosmol. Astropart. Phys. (JCAP) 04 (2013) 011 arXiv:1209.2895 [gr-qc].

[16] S. Capozziello, T. Harko, T. S. Koivisto, F. S. N. Lobo, and G. J. Olmo, “The virial theorem and the dark matter problem in hybrid metric-Palatini gravity”, J. Cosmol. Astropart. Phys. (JCAP) 07 (2013) 024 arXiv:1212.5817 [physics.gen-ph].

[17] T. S. Koivisto and N. Tamanini, “Ghost in pure and hybrid formalisms of gravity theories: A unified analysis”, Phys. Rev. D 87, 104030 (2013) arXiv:1304.3607 [gr-qc].

[18] N. Tamanini and C. G. Böhmer, “Generalized hybrid metric-Palatini gravity”, Phys. Rev. D 87, 084031 (2013) arXiv:1302.2355 [gr-qc].

[19] S. Carloni, T. Koivisto, and F. S. N. Lobo, “Dynamical system analysis of hybrid metric-Palatini cosmologies”, Phys. Rev. D 92, 064035 (2015) arXiv:1507.04306 [gr-qc].

[20] S. Capozziello, T. Harko, T. S. Koivisto, F. S. N. Lobo, and G. J. Olmo, “Galactic rotation curves in hybrid metric-Palatini gravity”, Astropart. Phys. 50-52, 65 (2013) arXiv:1307.0752 [gr-qc].

[21] S. Capozziello, T. Harko, T. S. Koivisto, F. S. N. Lobo, and G. J. Olmo, “Hybrid metric-Palatini gravity”, Universe 1, 199 (2015) arXiv:1508.04641 [gr-qc].

[22] C. G. Böhmer, F. S. N. Lobo, and N. Tamanini, “Einstein static Universe in hybrid metric-Palatini gravity”, Phys. Rev. D 88, 104019 (2013) arXiv:1305.0025 [gr-qc].

[23] N. A. Lima, “Dynamics of linear perturbations in the hybrid metric-Palatini gravity”, Phys. Rev. D 89, 083527 (2014) arXiv:1402.4458 [astro-ph.CO].

[24] N. A. Lima and V. S. Barreto, “Constraints on hybrid metric-Palatini gravity from background evolution”, Astrophys. J. 818, 186 (2016) arXiv:1501.05786 [astro-ph.CO].

[25] N. A. Lima, V. Smer-Barreto, and L. Lombriser, “Constraints on decaying early modified gravity from cosmological observations”, Phys. Rev. D 94, 083507 (2016) arXiv:1603.05239 [astro-ph.CO].

[26] S. Capozziello, T. Harko, F. S. N. Lobo, G. J. Olmo, and S. Vignolo, “The Cauchy problem in hybrid metric-Palatini $f(X)$-gravity”, Int. J. Geom. Meth. Mod. Phys. 11, 1450042 (2014) arXiv:1312.1320 [gr-qc].

[27] F. Lucchin and S. Matarrese, “Power-law inflation”, Phys. Rev. D 32, 1316 (1985).

[28] G. F. R. Ellis and M. S. Madsen, “Exact scalar field cosmologies”, Classical Quantum Gravity 8, 667 (1991).