On a conjectural series for $\pi$ and its $q$-analogue

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Abstract. In terms of the operator method, we prove a conjectural series for $\pi$ of Sun involving harmonic numbers of order two and find a similar result. Furthermore, we also give $q$-analogues of six $\pi$-formulas including the two ones just mentioned.

Keywords: a conjectural series for $\pi$; the partial derivative operator; $q$-analogue

AMS Subject Classifications: 33D15; 05A15

1 Introduction

For a complex variable $x$, define the well-known Gamma function to be

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \quad \text{with} \quad Re(x) > 0.$$ 

Three important properties of it can be stated as follows:

$$\Gamma(x + 1) = x\Gamma(x), \quad \Gamma(x)\Gamma(1 - x) = \frac{\pi}{\sin(\pi x)}, \quad \lim_{n \to \infty} \frac{\Gamma(x + n)}{\Gamma(y + n)} n^{y - x} = 1,$$

which will often be used directly in this paper. For a nonnegative integer $n$, define the shifted-factorial as

$$(x)_n = \frac{\Gamma(x + n)}{\Gamma(x)}.$$

Then we can give the definition of the hypergeometric series

$$\,_{r}F_{s} \left[a_1, a_2, \ldots, a_r ; b_1, b_2, \ldots, b_s ; z \right] = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k \cdots (a_r)_k}{(b_1)_k(b_2)_k \cdots (b_s)_k} \frac{z^k}{k!}.$$ 

In 1914, Ramanujan [20] listed 17 series for $1/\pi$ without proof. Decades later, Borweins [3] proved all of them firstly. Three of Ramanujan’s formulas are expressed as

$$\sum_{k=0}^{\infty} (6k + 1) \frac{(1/2)_k^3}{k!^3 4^k} = \frac{4}{\pi} \quad (1.1)$$

The work is supported by the National Natural Science Foundation of China (No. 12071103).
\[
\sum_{k=0}^{\infty} (8k + 1) \left( \frac{\frac{1}{2} k \left( \frac{1}{2} k \right) k \left( \frac{3}{4} k \right) k}{k! 9^k} \right) = \frac{2\sqrt{3}}{\pi}, \quad (1.2)
\]
\[
\sum_{k=0}^{\infty} (42k + 5) \left( \frac{\frac{3}{2} k^3}{k! 64^k} \right) = \frac{16}{\pi}. \quad (1.3)
\]

There are a lot of different \( \pi \)-formulas in the literature. Two of them (cf. [27, Equation (23)] and [11, P. 221]) read

\[
\sum_{k=0}^{\infty} \frac{k!}{(2k + 1)!!} = \frac{\pi}{2}, \quad (1.4)
\]
\[
\sum_{k=0}^{\infty} (3k + 2) \left( \frac{1}{2} \right)^3 k^3 4^k = \frac{\pi^2}{4}, \quad (1.5)
\]

where the double factorial has been defined by

\((2k)!! = 2^k k!, \quad (1 + 2k)!! = \frac{(2k + 1)!}{2^k k!}\).

In 2021, Guo and Lian [14] conjectured the interesting double series for \( \pi \) related to (1.1):

\[
\sum_{k=0}^{\infty} (6k + 1) \left( \frac{1}{2} \right)^3 k^3 4^k \sum_{j=1}^{k} \left\{ \frac{1}{(2j - 1)^2} - \frac{1}{16j^2} \right\} = \frac{\pi}{12}, \quad (1.6)
\]

which has been proved by the first author [25]. Moreover, the first author and Ruan [26] discovered the following double series for \( \pi \) associated with (1.2):

\[
\sum_{k=0}^{\infty} (8k + 1) \left( \frac{\frac{1}{2} k \left( \frac{1}{2} k \right) k \left( \frac{3}{4} k \right) k}{k! 9^k} \right) \sum_{i=1}^{k} \left\{ \frac{1}{(2i - 1)^2} - \frac{1}{36i^2} \right\} = \frac{\sqrt{3} \pi}{54}, \quad (1.7)
\]

For more known series on \( \pi \), we refer the reader to the papers [2, 4, 12, 19, 24, 28].

For a complex variable \( x \) and two positive integers \( \ell, n \), define the generalized harmonic number of order \( \ell \) to be

\[
H^{(\ell)}_n(x) = \sum_{k=1}^{n} \frac{1}{(x + k)^\ell}.
\]

When \( x = 0 \), it becomes the harmonic number of order \( \ell \):

\[
H^{(\ell)}_n = \sum_{k=1}^{n} \frac{1}{k^\ell}.
\]
Taking $\ell = 1$ in $H^{(\ell)}_n(x)$, we have the generalized harmonic number:

$$H_n(x) = \sum_{k=1}^{n} \frac{1}{x+k}.$$ 

The $x = 0$ case of it is the classical harmonic number:

$$H_n = \sum_{k=1}^{n} \frac{1}{k}.$$ 

In 2015, Sun [21] proved a nice series for $\pi^3$ containing harmonic number of order two related to (1.4):

$$\sum_{k=0}^{\infty} \frac{k!}{(2k+1)!!} H^{(2)}_k = \frac{\pi^3}{48}. \quad (1.8)$$

In a recent paper [23], he rewrote (1.6) and (1.7) as

$$\sum_{k=0}^{\infty} (6k+1) \left(\frac{1}{4}\right)^k \left\{ H^{(2)}_{2k} - \frac{5}{16} H^{(2)}_k \right\} = \frac{\pi}{12},$$

$$\sum_{k=0}^{\infty} (8k+1) \left(\frac{1}{4}\right)^k \left\{ H^{(2)}_{2k} - \frac{5}{18} H^{(2)}_k \right\} = \frac{\sqrt{3} \pi}{54},$$

and proposed the following conjecture associated with (1.3) (cf. [23, Equation (2.17)]).

**Theorem 1.1.**

$$\sum_{k=0}^{\infty} (42k+5) \left(\frac{1}{4}\right)^k \left\{ H^{(2)}_{2k} - \frac{25}{92} H^{(2)}_k \right\} = \frac{2\pi}{69}. \quad (1.9)$$

We shall also deduce the following series for $\pi^4$ containing harmonic numbers of order two on (1.5).

**Theorem 1.2.**

$$\sum_{k=0}^{\infty} (3k+2) \left(\frac{1}{4}\right)^k \left\{ 4 H^{(2)}_{2k+2} - H^{(2)}_{k+1} - 4 H^{(2)}_k \right\} = \frac{\pi^4}{12}. \quad (1.10)$$

For an integer $n$ and two complex numbers $x, q$ with $|q| < 1$, define the $q$-shifted factorial as

$$(x; q)_\infty = \prod_{i=0}^{\infty} (1 - xq^i), \quad (x; q)_n = \frac{(x; q)_\infty}{(xq^n; q)_\infty}.$$
For convenience, we sometimes utilize the compact notation:

\[(x_1, x_2, \ldots, x_r; q)_m = (x_1; q)_m(x_2; q)_m \cdots (x_r; q)_m,\]

where \(r \in \mathbb{Z}^+\) and \(m \in \mathbb{Z}^+ \cup \{0, \infty\}\). Then following Gasper and Rahman [10], the basic hypergeometric series can be defined by

\[r\phi_s \left[ \begin{array}{c} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{array} ; q, z \right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_k}{(b_1, b_2, \ldots, b_s; q)_k} \left\{ (-1)^k q^{\binom{k}{2}} \right\} \frac{1}{z^k}. \]

Let \([n] = 1 + q + \cdots + q^{n-1}\) be the \(q\)-integer. Recently, Guo and Liu [15] and Guo and Zudilin [16] obtained the following \(q\)-analogues of (1.1) and (1.2):

\[\sum_{k=0}^{\infty} q^{k^2}[6k + 1] \frac{(q; q^2)_k^2(q^2; q^4)_k^2}{(q^4; q^4)_k^3} \sum_{j=1}^{k} \frac{q^{2j-1}}{[2j - 1]^2 - [4j]^2} = \frac{(q^2; q^4)_\infty^2(q^5; q^4)_\infty}{(q; q^4)_\infty^3(q^4; q^4)_\infty} \sum_{i=1}^{\infty} (-1)^{i-1} \frac{q^{2i}}{[2i]^2}, \]

\[= \frac{(q^3; q^6)_\infty^2(q^3; q^6)_\infty}{(q^2; q^6)_\infty^3(q^6; q^6)_\infty} \sum_{j=1}^{\infty} (-1)^{j-1} \frac{q^{3j}}{[3j]^2}. \]

More \(q\)-analogues of \(\pi\)-formulas can be seen in the papers [13, 17, 18, 22].

Inspired by the works just mentioned, we shall establish \(q\)-analogues of (1.3)-(1.5) in the following theorem.

**Theorem 1.3.**

\[\sum_{k=0}^{\infty} q^{6k^2} \frac{(q; q^2)_k^3(1 + q^{1+2k})^3(1 - q^{4+6k}) - q^{1+6k}(1 - q^{3+6k})}{(q^2; q^2)_k^2(1 - q^3)(1 + q)(-q^2; q^2)_k^3} = \frac{(q^3; q^5; q^2)_\infty^2}{(q^4; q^2)_\infty^3}, \] (1.11)

\[\sum_{k=0}^{\infty} q^{\binom{k+1}{2}} \frac{(q; q)_k^2(q^2; q)_k^2}{(q^3; q^2)_k^2} = \frac{(q^2; q^2)_\infty^2}{(q; q^3; q^2)_\infty}, \] (1.12)

\[\sum_{k=0}^{\infty} q^{\binom{k+1}{2}} [3k + 2] \frac{(q; q)_k^2(q^2; q)_k^2}{(q^3; q^2)_k^3} = \frac{(q^2; q^2)_\infty^4}{(q^2; q^2)_\infty^3(q^3; q^2)_\infty^2}. \] (1.13)
Further, we shall give $q$-analogues of (1.9), (1.8), (1.10) in the following three theorems.

**Theorem 1.4.**

\[
\sum_{k=0}^{\infty} q^{6k^2} \frac{(q; q^2)_k^6}{(q^2; q^2)_k^3} \left\{ \lambda_q(k) \sum_{i=1}^{2k} \frac{q^{2i}}{[2i]^2} - \mu_q(k) \sum_{i=1}^{k} \frac{q^{2i-1}}{[2i-1]^2} - \nu_q(k)(1 - q)q^{1+6k} \right\}
\]

\[
= \frac{(q, q^3; q^2)_{\infty}}{(q^2; q^2)_\infty^2} \left\{ \sum_{j=1}^{\infty} \frac{q^{2j}}{[2j]^2} - \frac{3(1 + q)^3}{64} \sum_{j=1}^{\infty} \frac{q^{2j-1}}{[2j-1]^2} \right\},
\]

where

\[
\lambda_q(k) = \frac{1 + 2q^{1+2k} - q^{1+6k}(2 + 2q^2 + q^{1+2k} + q^{3+2k} - 3q^{3+6k})}{(1 - q)(1 - q^{1+2k})(1 + q^{1+2k})^3},
\]

\[
\mu_q(k) = \frac{1 + q^{1+2k} + 3q^{2+4k} - 2q^{1+6k} + q^{3+6k} - 3q^{2+6k} - 3q^{3+10k}}{64(1 - q)(1 + q^{1+2k})^3(137 + 27q + 27q^2 + 9q^3)^{-1}},
\]

\[
\nu_q(k) = \frac{3(1 + q)^3(1 + 2q^{1+2k} + 3q^{2+4k})}{64(1 - q^{1+2k})(1 + q^{1+2k})^3} - \frac{q^{1+2k}(1 - q^{1+2k} + q^{2+4k})^2}{(1 - q^{1+2k})(1 + q^{1+2k})^3}.
\]

**Theorem 1.5.**

\[
\sum_{k=0}^{\infty} q^{(k+1)^2} \frac{(q; q)_k^2 (q^3; q^3)_k^2}{(q^2; q^2)_k^4} \sum_{i=1}^{k} \frac{q^i}{[i]^2} = \frac{(q^2; q^2)_\infty^2 (q^3; q^3)_\infty^2}{(q; q^2)_\infty^2 (q^2; q^2)_\infty} \sum_{j=1}^{\infty} \frac{q^j}{[j]^2}.
\]

**Theorem 1.6.**

\[
\sum_{k=0}^{\infty} q^{(k+1)^2} \frac{(q; q)_k^2 (q^2; q^2)_k^2}{(q^3; q^3)_k^4} \left\{ \sum_{i=1}^{k} \frac{q^i}{[i]^2} - \sum_{i=1}^{k+1} \frac{q^{2i-1}}{[2i-1]^2} \right\}
\]

\[
= \frac{(q^2; q^2)_\infty^2 (q; q^2)_\infty^2 (q^3; q^3)_\infty^2}{(q; q^2)_\infty^2 (q^2; q^2)_\infty} \sum_{j=1}^{\infty} (-1)^j \frac{q^j}{[j]^2}.
\]

For a multivariable function $f(x_1, x_2, \ldots, x_m)$, define the partial derivative operator $D_x$ by

\[
D_x f(x_1, x_2, \ldots, x_m) = \frac{d}{dx_i} f(x_1, x_2, \ldots, x_m) \quad \text{with} \quad 1 \leq i \leq m.
\]

Then there are the following two relations:

\[
D_x (x + y)_n = (x + y)_n H_n(x + y - 1),
\]

\[
D_x (xy; q)_n = -(xy; q)_n \sum_{i=1}^{n} \frac{y^{i-1}}{1 - xyq^{i-1}}.
\]

The rest of the paper is arranged as follows. We shall verify Theorems 1.1 and 1.2 via the partial derivative operator and some summation and transformation formulas for hypergeometric series in Section 2. We shall certify Theorems 1.3-1.6 through the partial derivative operator and several summation and transformation formulas for basic hypergeometric series in Section 3.
2 Proof of Theorems 1.1-1.2

Above all, we shall prove Theorem 1.1.

Proof of Theorem 1.1. Recall a known transformation formula (cf. [8, Theorem 31]):

\[ \sum_{k=0}^{\infty} (-1)^k \frac{(b)_k (c)_k (d)_k (e)_k}{(1 + a - b - c)_k (1 + a - b - d)_k (1 + a - b - e)_k} \]
\[ = \sum_{k=0}^{\infty} \frac{(a + 2k)(b)_k (c)_k (d)_k (e)_k}{(1 + a - b)_k (1 + a - c)_k (1 + a - d)_k (1 + a - e)_k}, \]

where

\[ \sigma_k(a, b, c, d, e) = (1 + 2a - b - c - d + 3k)(a - e + 2k) + \frac{(e + k)(1 + a - b - c + k)}{(1 + a - b + 2k)(1 + a - d + 2k)} \]
\[ \times \frac{(1 + a - b - d + k)(1 + a - c - d + k)(2 + 2a - b - d - e + 3k)}{(1 + a - b - c - d - e + 2k)(2 + 2a - b - c - d - e + 2k)} \]
\[ + \frac{(c + k)(e + k)(1 + a - b - c + k)(1 + a - b - d + k)}{(1 + a - b + 2k)(1 + a - c + 2k)(1 + a - d + 2k)(1 + a - e + 2k)} \]
\[ \times \frac{(1 + a - b - e + k)(1 + a - c - d + k)(1 + a - d - e + k)}{(1 + 2a - b - c - d - e + 2k)(2 + 2a - b - c - d - e + 2k)}. \]

Choosing \((a, b, c, d, e) = (\frac{1}{2}, \frac{1}{2}, x, 1 - x, -n)\) in the last equation and calculating the series on the right-hand side by Douchall’s \(_5F_4\) summation formula (cf. [11, P. 71]):

\[ _5F_4 \left[ \frac{a, 1 + \frac{a}{2}, b, c, -n}{\frac{a}{2}, 1 + a - b, 1 + a - c, 1 + a + n}; 1 \right] = \frac{(1 + a)_n (1 + a - b - c)_n}{(1 + a - b)_n (1 + a - c)_n}, \]

we arrive at

\[ \sum_{k=0}^{n} (-1)^k \frac{\left(\frac{1}{2}\right)_k (1 + n)_k (-n)_k}{(1 + 2k)\left(\frac{1}{2} + n\right)_{2k} (\frac{3}{2} + n)_{2k}} \frac{(x)_k (1 - x)_k (\frac{1}{2} + x + n)_k (\frac{3}{2} - x + n)_k}{(\frac{1}{2} + x)_{2k} (\frac{1}{2} - x)_{2k}} \]
\[ \times \Omega_k(x; n) = \frac{\left(\frac{1}{2}\right)_n (\frac{3}{2})_n}{\left(\frac{1}{2} + x\right)_n (\frac{3}{2} - x)_n}, \quad (2.1) \]

where

\[ \Omega_k(x; n) = (1 + 6k) + \frac{4(x + k)(1 - x + k)(3 + 2x + 2n + 6k)(k - n)}{(1 + 2x + 4k)(1 + 2n + 4k)(3 + 2n + 4k)} \]
\[ + \frac{16(x + k)^2 (1 - x + k)(1 + 2x + 2n + 2k)(1 + n + k)(k - n)}{(1 + 2x + 4k)(3 - 2x + 4k)(1 + 2n + 4k)(3 + 2n + 4k)^2}. \]
Apply the operator \( \mathcal{D}_x \) on both sides of (2.1) to obtain

\[
\sum_{k=0}^{n} (-1)^k \frac{\left(\frac{1}{2}\right)_k^2 (1+n)_k (-n)_k}{(1)_{2k} \left(\frac{1}{2} + n\right)_{2k} \left(\frac{3}{2} + n\right)_{2k}} \left(x\right)_k^2 (1-x)_k^2 \left(\frac{1}{2} + x + n\right)_k (\frac{3}{2} - x + n)_k
\]

\[
\times \left\{ 2H_k(x-1) - 2H_k(-x) - H_{2k}(x - \frac{1}{2}) + H_{2k}(\frac{1}{2} - x) + H_k(x - \frac{1}{2} + n) \right. \\
- H_k(\frac{1}{2} - x + n) \right\} \Omega_k(x; n)
\]

\[
+ \sum_{k=0}^{n} (-1)^k \frac{\left(\frac{1}{2}\right)_k^2 (1+n)_k (-n)_k}{(1)_{2k} \left(\frac{1}{2} + n\right)_{2k} \left(\frac{3}{2} + n\right)_{2k}} \left(x\right)_k^2 (1-x)_k^2 \left(\frac{1}{2} + x + n\right)_k (\frac{3}{2} - x + n)_k
\]

\[
\times \mathcal{D}_x \Omega_k(x; n) = \frac{\left(\frac{1}{2}\right)_n (\frac{3}{2})_n}{(\frac{1}{2} + x)_n (\frac{3}{2} - x)_n} \left\{ H_n(\frac{1}{2} - x) - H_n(x - \frac{1}{2}) \right\}. \quad (2.2)
\]

Dividing both sides by \( 1 - 2x \) and employing the relation

\[
\frac{1}{v-u-2x} \left\{ H_m(x+u) - H_m(v-x) \right\} = \sum_{j=1}^{m} \frac{1}{(x+u+i)(v-x+i)}, \quad (2.3)
\]

Equation (2.2) can be manipulated as

\[
\sum_{k=0}^{n} (-1)^k \frac{\left(\frac{1}{2}\right)_k^2 (1+n)_k (-n)_k}{(1)_{2k} \left(\frac{1}{2} + n\right)_{2k} \left(\frac{3}{2} + n\right)_{2k}} \left(x\right)_k^2 (1-x)_k^2 \left(\frac{1}{2} + x + n\right)_k (\frac{3}{2} - x + n)_k
\]

\[
\times \left\{ 2 \sum_{i=1}^{k} \frac{1}{(x-1+i)(-x+i)} - \sum_{i=1}^{2k} \frac{1}{(x-\frac{1}{2}+i)(\frac{1}{2} - x + i)} \right. \\
+ \sum_{i=1}^{k} \frac{1}{(x-\frac{1}{2}+n+i)(\frac{1}{2} - x + n+i)} \right\} \Omega_k(x; n)
\]

\[
+ \sum_{k=0}^{n} (-1)^k \frac{\left(\frac{1}{2}\right)_k^2 (1+n)_k (-n)_k}{(1)_{2k} \left(\frac{1}{2} + n\right)_{2k} \left(\frac{3}{2} + n\right)_{2k}} \left(x\right)_k^2 (1-x)_k^2 \left(\frac{1}{2} + x + n\right)_k (\frac{3}{2} - x + n)_k
\]

\[
\times \frac{\mathcal{D}_x \Omega_k(x; n)}{1 - 2x} = -\frac{\left(\frac{1}{2}\right)_n (\frac{3}{2})_n}{(\frac{1}{2} + x)_n (\frac{3}{2} - x)_n} \sum_{j=1}^{n} \frac{1}{(x-\frac{3}{2}+j)(\frac{1}{2} - x + j)}.
\]

Letting \( (x, n) \rightarrow (\frac{1}{2}, \infty) \) and making use of Euler’s formula:

\[
\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}, \quad (2.4)
\]
we have
\[
\sum_{k=0}^{\infty} \frac{(1/2)_k^3}{k!^3 64^k} \left\{ (42k + 5) \left[ 2H_k^{(2)} - 7H_{2k}^{(2)} \right] + \frac{9}{1 + 2k} \right\} = \frac{8\pi}{3}.
\] (2.5)

Recall a known summation formula (cf. [7, Corollary 2.33]):
\[
\sum_{k=0}^{\infty} \frac{(x)_k^3 (1-x)_k^3}{(1)k^3 (3/2)_k^3} \frac{k(1+3k)(3+9k+7k^2) + x(1-x)(1+6k+6k^2+x-x^2)}{64^k} = \frac{\sin(\pi x)}{\pi},
\] (2.6)
where we have replaced \(\sin(\pi x)/\pi x\) by \(\sin(\pi x)/\pi\) for correction. When \(0 < x < 1\), it is obvious that the series on the left-hand side is uniformly convergent. Applying the operator \(D_x\) on both sides of (2.6) and utilizing (2.3), there holds
\[
3 \sum_{k=0}^{\infty} \frac{(x)_k^3 (1-x)_k^3}{(1)k^3 (3/2)_k^3} \frac{1}{(x-1+i)(-x+i)} \times \frac{k(1+3k)(3+9k+7k^2) + x(1-x)(1+6k+6k^2+x-x^2)}{64^k}
\]
\[
+ \sum_{k=0}^{\infty} \frac{(x)_k^3 (1-x)_k^3}{(1)k^3 (3/2)_k^3} \frac{1 + 6k + 6k^2 + 2x(1-x)}{64^k} = \frac{\cos(\pi x)}{1 - 2x}.
\]
The \(x \to \frac{1}{2}\) case of the upper identity provides
\[
\sum_{k=0}^{\infty} \frac{(1/2)_k^3}{k!^3 64^k} \left\{ (42k + 5) \left[ 4H_{2k}^{(2)} - H_k^{(2)} \right] + \frac{8}{1 + 2k} \right\} = \frac{8\pi}{3}.
\] (2.7)
Hence we deduce (1.9) from the linear combination of (2.5) and (2.7).

Subsequently, we shall display the proof of Theorem 1.2.

**Proof of Theorem 1.2** In order to achieve the goal, we need the summation formula for hypergeometric series (cf. [5, Equation (5.1e)]):
\[
\binom{a-\frac{1}{2}, \frac{2a+2}{3}, 2b-1, 2c-1, 2+2a-2b-2c, a+n,-n}{\frac{2a-1}{3}, 1+a-b, 1+a-c, 1-a-c, 2b+2n,-2n} ; 1
\]
\[
= \frac{(1/2+a)_n(b)_n(c)_n(a-b+c+3/2)_n}{(1/2)_n(1+a-b)_n(1+a-c)_n(b+c-1/2)_n}.
\] (2.8)
Apply the operator $D_b$ on both sides of the $c = 2 - b$ case of (2.8) to get
\[
\sum_{k=0}^{n} \frac{a - \frac{1}{2}}{k}(\frac{2a+2}{3})_k(2a - 2)_k(2b - 1)_k(3 - 2b)_k(a + n)_k(-n)_k}{(1)_k(\frac{2a-1}{3})_k(\frac{3}{2})_k(1 + a - b)_k(a + b - 1)_k(2a + 2n)_k(-2n)_k}
\times \left\{2H_k(2b - 2) - 2H_k(2 - 2b) + H_k(a - b) - H_k(a + b - 2) \right\}
= \frac{(a + \frac{1}{2})_n(a - \frac{1}{2})_n(b)_n(2 - b)_n}{(\frac{1}{2})_n(\frac{3}{2})_n(1 + a - b)_n(a + b - 1)_n}
\times \left\{H_n(b - 1) - H_n(1 - b) + H_n(a - b) - H_n(a + b - 2) \right\}.
\]
Applying the operator $D_b$ on both sides of the above identity, it is routine to understand that
\[
\sum_{k=0}^{n} \frac{a - \frac{1}{2}}{k}(\frac{2a+2}{3})_k(2a - 2)_k(2b - 1)_k(3 - 2b)_k(a + n)_k(-n)_k}{(1)_k(\frac{2a-1}{3})_k(\frac{3}{2})_k(1 + a - b)_k(a + b - 1)_k(2a + 2n)_k(-2n)_k}
\times \left\{2H_k(2b - 2) - 2H_k(2 - 2b) + H_k(a - b) - H_k(a + b - 2) \right\}^2
- \left\{4H_k^2(2b - 2) + 4H_k^2(2 - 2b) - H_k^2(a - b) - H_k^2(a + b - 2) \right\}
= \frac{(a + \frac{1}{2})_n(a - \frac{1}{2})_n(b)_n(2 - b)_n}{(\frac{1}{2})_n(\frac{3}{2})_n(1 + a - b)_n(a + b - 1)_n}
\times \left\{H_n(b - 1) - H_n(1 - b) + H_n(a - b) - H_n(a + b - 2) \right\}^2
- \left\{H_n^2(b - 1) + H_n^2(1 - b) - H_n^2(a - b) - H_n^2(a + b - 2) \right\}. \tag{2.9}
\]
The $(a, b) = (\frac{3}{2}, 1)$ case of (2.9) engenders
\[
\sum_{k=0}^{n} \frac{a - \frac{1}{2}}{k}(\frac{3}{2})_k(\frac{3}{2} + n)_k(-n)_k}{(\frac{3}{2})_k(\frac{3}{2})_k(3 + 2n)_k(-2n)_k}
\times \left\{4H_{2k+2}^2 - H_{k+1}^2 - 4H_k^2 - 4 \right\}
= \frac{\Gamma(2 + n)\Gamma(1 + n)^3 \Gamma(\frac{1}{2})\Gamma(\frac{3}{2})^3}{\Gamma(\frac{1}{2} + n)\Gamma(\frac{3}{2} + n)^3 \Gamma(2)\Gamma(1)^3}
\times \left\{4H_{2n+2}^2 - H_{n+1}^2 - H_n^2 - 4 \right\}. \tag{2.10}
\]
Since the $(a, b, c) = (\frac{3}{2}, 1, 1)$ case of (2.8) reads
\[
\sum_{k=0}^{n} \frac{a - \frac{1}{2}}{k}(\frac{3}{2})_k(\frac{3}{2} + n)_k(-n)_k}{(\frac{3}{2})_k(\frac{3}{2})_k(3 + 2n)_k(-2n)_k}
= \frac{\Gamma(2 + n)\Gamma(1 + n)^3 \Gamma(\frac{1}{2})\Gamma(\frac{3}{2})^3}{\Gamma(\frac{1}{2} + n)\Gamma(\frac{3}{2} + n)^3 \Gamma(2)\Gamma(1)^3}, \tag{2.11}
\]
the linear combination of (2.10) and (2.11) gives
\[
\sum_{k=0}^{n} \frac{a - \frac{1}{2}}{k}(\frac{3}{2})_k(\frac{3}{2} + n)_k(-n)_k}{(\frac{3}{2})_k(\frac{3}{2})_k(3 + 2n)_k(-2n)_k}
\times \left\{4H_{2k+2}^2 - H_{k+1}^2 - 4H_k^2 \right\}
= \frac{\Gamma(2 + n)\Gamma(1 + n)^3 \Gamma(\frac{1}{2})\Gamma(\frac{3}{2})^3}{\Gamma(\frac{1}{2} + n)\Gamma(\frac{3}{2} + n)^3 \Gamma(2)\Gamma(1)^3}
\times \left\{4H_{2n+2}^2 - H_{n+1}^2 - H_n^2 \right\}. \tag{2.9}
\]
Letting \( n \to \infty \) and employing Euler’s formula (2.4), we are led to (1.10).

3 Proof of Theorems 1.3-1.6

Firstly, we are ready to prove Theorem 1.3.

**Proof of Theorem 1.3.** For achieving the purpose, we require two identities for basic hypergeometric series (cf. [8, Theorem 17] and [6, Equation (5.1d)]):

\[
\sum_{k=0}^{\infty} \frac{(b, c, d, e, aq/bc, aq/bd, aq/bc/aq/cd, aq/cd, aq/cd/aq/ce, aq/ce, aq/de; q)_{k}}{(aq/b, aq/c, aq/d, aq/e, a^{2}q/bcde; q)_{2k}} \times q^{k(2k+1)}/2 \left( -\frac{a^{3}}{bcde} \right)^{k} A_{k}(a, b, c, d, e; q) = \sum_{k=0}^{\infty} \frac{1-aq^{2k}}{1-a} \frac{(b, c, d, e; q)_{k}}{(aq/b, aq/c, aq/d, aq/e; q)_{k}} \left( \frac{aq}{bcde} \right)^{k},
\]

where

\[
A_{k}(a, b, c, d, e; q) = \frac{(1-q^{2k}a/e)(1-q^{1+3k}a^{2}/bcd)}{(1-a)(1-q^{1+2k}a^{2}/bcd)} + \frac{q^{2k}a(1-q^{k}e)}{e(1-a)} \\
\times \frac{(1-q^{1+k}a/bc)(1-q^{1+k}a/bd)(1-q^{1+k}a/cd)(1-q^{2+3k}a^{2}/bcd)}{(1-q^{1+2k}a/b)(1-q^{1+2k}a/d)(1-q^{1+2k}a^{2}/bcd)(1-q^{2+2k}a^{2}/bcd)} \\
+ \frac{q^{1+4k}a^{2}}{ce} \frac{(1-q^{k}c)(1-q^{k}e)(1-q^{1+k}a/bc)(1-q^{1+k}a/bd)(1-q^{1+k}a/be)}{(1-a)(1-q^{1+2k}a/b)(1-q^{1+2k}a/c)(1-q^{1+2k}a/d)(1-q^{1+2k}a/e)} \times \frac{(1-q^{1+k}a/cd)(1-q^{1+k}a/de)}{(1-q^{1+2k}a^{2}/bcd)(1-q^{2+2k}a^{2}/bcd)}.
\]

\[
\sum_{k=0}^{n} \frac{1-aq^{3k-1}((q^{-2n}, aq^{2n}, aq^{2n}; q^{2})_{k})(b/q, c/q, aq^{2}/bc; q)_{k}}{1-aq^{-1}(aq^{2}/b, aq^{2}/c, bc/q; q^{2})_{k}} \left( \frac{aq^{2n}}{q, aq^{2n}, q^{-2n}; q} \right)_{k} q^{k} = \sum_{k=0}^{n} \frac{(aq, b, c, aq^{3}/bc; q^{2})_{k}}{(q, aq^{2}/b, aq^{2}/c, bc/q; q^{2})_{k}}(aq^{2n}q^{-2n}).
\]

Performing the replacements \((a, b, c, d, e) \to (x, a, b, xq/c, xq/d)\) in (3.1) and then letting \( x \to 0 \), we find

\[
\sum_{k=0}^{\infty} \frac{(a, b, c/a, c/b, d/a, d/b; q)_{k}}{(c, d; q)_{1+2k}(cd/abq; q)_{2+2k}} q^{6(k)} \left( \frac{c^{2}q^{2}}{ab} \right)^{k} B_{k}(a, b, c, d; q) = \sum_{k=0}^{\infty} \frac{(a, b; q)_{k}}{(c, d; q)_{k}} \left( \frac{cd}{ab} \right)^{k},
\]
where

\[ B_k(a, b, c, d; q) = (1 - q^{2k}c)(1 - q^{2k}d)(1 - q^{2k}cd/ab)(1 - q^{3k-1}cd/b) \]

\[ - \frac{q^{3k-1}cd}{a}(1 - q^k)(1 - q^k c/b)(1 - q^k d/b)(1 - q^{3k}cd/a). \]

When \( d = q \), the series on the right-hand side of (3.3) can be evaluated by the \( q \)-Gauss summation formula (cf. [10, Appendix II. 8]):

\[ \sum_{k=0}^{\infty} \frac{(a, b, q/a, q/b; c/a, c/b; q)_k}{(q; q)_{1+2k}(cq, cq^2/ab; q)_{2k}} q^{3k^2-k} \left( \frac{c^2}{ab} \right)^k C_k(a, b, c; q) = \frac{(c/a, c/b; q)_{\infty}}{(cq, cq^2/ab; q)_{\infty}}. \] (3.4)

So we have

\[ \sum_{k=0}^{\infty} \frac{(a, b, q/a, q/b; c/a, c/b; q)_k}{(q; q)_{1+2k}(cq, cq^2/ab; q)_{2k}} q^{3k^2-k} \left( \frac{c^2}{ab} \right)^k C_k(a, b, c; q) = \frac{(c/a, c/b; q)_{\infty}}{(cq, cq^2/ab; q)_{\infty}}. \]

where

\[ C_k(a, b, c; q) = (1 - q^{2k}c)(1 - q^{1+2k})(1 - q^{1+2k}c/ab)(1 - q^{3k}c/b) \]

\[ - \frac{q^{3k}c}{a}(1 - q^k)(1 - q^k c/b)(1 - q^{1+k} / b)(1 - q^{1+3k}c/a). \]

Letting \((a, b, c, q) \to (q, q, q^2, q^3)\) in (3.4), we obtain (1.11).

Notice that the \((a, b, c) = (0, q^2, q^3)\) case of (3.2) is

\[ \sum_{k=0}^{n} \frac{(q; q)_k (q^{-2n}; q^2)_k q^{3k}}{(q^3; q^2)_k (q^{-2n}; q)_k} q^k = \frac{(q^2; q^2)^2_n}{(q, q^3; q^2)_n}. \]

Letting \( n \to \infty \) in the above identity, we derive (1.12).

The \((a, b, c) = (q^3, q^2, q^2)\) case of (3.2) reads

\[ \sum_{k=0}^{n} \frac{1 - q^{3k+2}}{1 - q^2} \frac{(q, q^2)_k (q^{-2n}; q^2)_k (q^{2n+3}; q^{-2n}; q^2)_k q^k}{(q^3; q^2)_k (q^2; q^2)_k (q^{-2n}; q)_k} q^k = \frac{(q^2; q^2)^3_n (q^4; q^2)_n}{(q; q^2)_n (q^3; q^2)_n (q^2; q^2)_n}. \]

Letting \( n \to \infty \) in the above identity, we catch hold of (1.13).

Secondly, we start to prove Theorem 1.4.

**Proof of Theorem 1.4.** Setting \((a, b, c, d, e) = (q^{1/2}, q^{1/2}, x, q/x, q^{-n})\) in (3.1) and calculating the series on the right-hand side by the \( q \)-analogue of Dougall’s \( _5F_4 \) summation formula (cf. [10, Appendix II. 21]):

\[ _6\phi_5 \left[ \begin{array}{c} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, q^{-n} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, qa/b, qa/c, qa^{n+1} \end{array} ; q, \frac{aq^{n+1}}{bc} \right] = \frac{(aq, aq/bc; q)_n}{(aq/b, aq/c; q)_n}. \]
we get
\[
\sum_{k=0}^{n} \frac{(q^{1/2}; q)_{2k}^{2}(q^{1+n}, q^{-n}; q)_{k}}{(q, q^{1/2+n}, q^{1/2+n}; q)_{2k}} \frac{x^{2}(x q^{1/2+n}, q^{3/2+n}/x; q)_{2k}}{(x q^{2/5}, q^{2}/x; q)_{2k}}
\]

\[
\times (-1)^{k} q^{\frac{3k}{2} + kn} E_{k}(x, n; q) = \frac{(q^{1/2}, q^{3/2}; q)_{n}}{(x q^{2/5}, q^{2}/x; q)_{n}},
\]

where
\[
E_{k}(x, n; q) = \frac{(1 - q^{1/2+3k})}{(1 - q^{2})},
\]

\[
+ \frac{q^{1/2+2k+n}(1 - q^{-k-n})(1 - xq^{k})(1 - q^{1+k}/x)(1 - xq^{1/2+3k+n})}{(1 - q^{2})(1 + q^{2+k})(1 - q^{1/2+2k+n})(1 - xq^{1/2+2k})}
\]

\[
+ \frac{q^{2+4k+n}(1 - q^{1+k+n})(1 - q^{-k-n})(1 - xq^{k})^{2}(1 - q^{1+k}/x)(1 - xq^{1/2+k+n})}{x(1-q^{1/2})(1 + q^{1/2+k})(1 - q^{1/2+2k+n})(1 - xq^{1/2+2k})^{2}(1 - xq^{1/2+2k})(1 - q^{3/2+2k}/x)}
\]

Via the operator \(\mathcal{D}_{x}\) and the last equation, it is clear that
\[
\sum_{k=0}^{n} \frac{(q^{1/2}; q)_{2k}^{2}(q^{1+n}, q^{-n}; q)_{k}}{(q, q^{1/2+n}, q^{1/2+n}; q)_{2k}} \frac{x^{3}(x q^{1/2+n}, q^{3/2+n}/x; q)_{2k}}{(x q^{2/5}, q^{2}/x; q)_{2k}}
\]

\[
\times (-1)^{k} q^{\frac{3k}{2} + kn} E_{k}(x, n; q) \mathcal{D}_{x} E_{k}(x, n; q)
\]

\[
= \frac{(q^{1/2}, q^{3/2}; q)_{n}}{(x q^{2/5}, q^{2}/x; q)_{n}} \left\{ \sum_{j=1}^{n} \frac{q^{j-1/2}/x^{2}}{1 - xq^{j-1/2}} - \sum_{j=1}^{n} \frac{q^{j+1/2}/x^{2}}{1 - xq^{j+1/2}} \right\}, \tag{3.5}
\]

where
\[
F_{k}(x, n; q) = 2 \sum_{i=1}^{k} \frac{q^{i}/x^{2}}{1 - q^{i}/x} - 2 \sum_{i=1}^{k} \frac{q^{i-1}}{1 - xq^{i-1}} + 2k \frac{q^{i-1}}{1 - xq^{i-1}}
\]

\[
- \sum_{i=1}^{2k} \frac{q^{i+1/2}/x^{2}}{1 - q^{i+1/2}/x} + \sum_{i=1}^{k} \frac{q^{i+1/2+n}/x^{2}}{1 - q^{i+1/2+n}/x} - \sum_{i=1}^{k} \frac{q^{i-1/2+n}}{1 - xq^{i-1/2+n}}.
\]

Dividing both sides of (3.5) by \(1 - q/x^{2}\) and then letting \((x, q, n) \to (q, q^{2}, \infty)\), we arrive at

12
\[
\sum_{k=0}^{\infty} q^{6k^2} \frac{\left(q; q^2\right)_k^6}{\left(q^2; q^2\right)_k^3} \left\{ \sum_{i=1}^{2k} \frac{q^{2i}}{[2i]^2} - 2 \sum_{i=1}^{k} \frac{q^{2i-1}}{[2i - 1]^2} \right\} \\
\times \frac{1 + 2q^{1+2k} - q^{1+6k}(2 + 2q^2 + q^{1+2k} + q^{3+2k} - 3q^{3+6k})}{(1-q)(1-q^{2+4k})(1+q^{1+2k})^2} \\
+ \sum_{k=0}^{\infty} q^{6k+8k+2} \frac{\left(q; q^2\right)_k^6}{\left(q^2; q^2\right)_k^3} \frac{(1-q)(1+q^{1+2k} + q^{2+4k})^2}{(1-q^{2+4k})(1+q^{1+2k})^4} \\
= \frac{(q, q^2; q^2)_{\infty}}{(q^2; q^2)_{\infty}^2} \sum_{j=1}^{\infty} \frac{q^{2j}}{[2j]^2}. \quad (3.6)
\]

The \((a, b, c) = (x, q/x, q)\) case of (3.4) can be stated as

\[
\sum_{k=0}^{\infty} \frac{(x, q/x, q)^3}{(q^2; q^2)_k^3} q^{3k} \left\{ (1-q^{1+2k})^3(1-xq^{3k}) - \frac{q^{1+3k}}{x}(1-xq^k)^3(1-q^{2+3k}/x) \right\} \\
= (1-q) \frac{(x, q/x, q)_{\infty}}{(q^2; q^2)_{\infty}^3}. \quad (3.7)
\]

When \(0 < q, x < 1\), it is obvious that the series on the left-hand side of (3.7) is uniformly convergent. Through the operator \(D_x\) and (3.4), it is not difficult to see that

\[
3 \sum_{k=0}^{\infty} \frac{(x, q/x, q)^3}{(q^2; q^2)_k^3} q^{3k^2} \left\{ (1-q^{1+2k})^3(1-xq^{3k}) - \frac{q^{1+3k}}{x}(1-xq^k)^3(1-q^{2+3k}/x) \right\} \\
\times \left\{ \sum_{i=1}^{k} \frac{q^i/x^2}{1-q^i/x} - \sum_{i=1}^{k} \frac{q^{i-1}}{1-xq^{i-1}} \right\} \\
+ \sum_{k=0}^{\infty} \frac{(x, q/x, q)^3}{(q^2; q^2)_k^3} q^{3k^2+3k} \frac{(q-x^2)(x + 3xq^{2+4k} - 2x^2q^{1+3k} - 2q^{2+3k})}{x^3} \\
= (1-q) \frac{(x, q/x, q)_{\infty}}{(q^2; q^2)_{\infty}^3} \left\{ \sum_{j=1}^{\infty} \frac{q^j/x^2}{1-q^j/x} - \sum_{j=1}^{\infty} \frac{q^{j-1}}{1-xq^{j-1}} \right\}. \quad (3.8)
\]

It is valid for \(|q| < 1\) by analytic continuation. Dividing both sides of (3.8) by \(1 - q/x^2\) and then letting \((x, q) \to (q, q^2)\), there holds

\[
3 \sum_{k=0}^{\infty} q^{6k^2} \frac{\left(q; q^2\right)_k^6}{\left(q^4; q^2\right)_k^3} (1-q^{2+4k})(1-q^{1+6k}) - q^{1+6k}(1-q^{1+2k})^3(1-q^{3+6k}) \\
\times \sum_{i=1}^{k} \frac{q^{2i-1}}{[2i - 1]^2} + \sum_{k=0}^{\infty} q^{6k^2+6k} \frac{\left(q; q^2\right)_k^6}{\left(q^4; q^2\right)_k^3} q - 4q^{4+6k} + 3q^{5+8k} \\
\times \frac{1}{(1-q)^2}.
\]
Employing the operator $D_b$ on both sides of the $c \to q^4/b$ case of (3.2) to discover

$$
\sum_{k=0}^{n} \frac{1 - aq^{3k-1}}{1 - aq^{-1}} \left( \frac{q^{-2n}, aq^{2n}, a/q, q^2}{q^3, aq^2/b, ab/q^2; q^2} \right)_k \frac{q^2}{(a/q^2, b/q, q^3/b; q)_k} q^k G_k(a, b)
$$

$$
= \frac{(aq, a/q, b, q^4/b; q^2)_n}{(q, q^3, aq^2/b, ab/q^2; q^2)_n} H_n(a, b),
$$

where

$$
G_k(a, b) = \sum_{i=1}^{k} \frac{q^{i-2}}{1 - bq^{i-2}} - \sum_{i=1}^{k} \frac{q^{i+2}/b^2}{1 - q^{i+2}/b} + \sum_{i=1}^{k} \frac{aq^{2i}/b^2}{1 - aq^{2i}/b} - \sum_{i=1}^{k} \frac{aq^{2i-4}}{1 - abq^{2i-4}},
$$

$$
H_n(a, b) = \sum_{j=1}^{n} \frac{q^{2j-2}}{1 - bq^{2j-2}} - \sum_{j=1}^{n} \frac{q^{2j+2}/b^2}{1 - q^{2j+2}/b} + \sum_{j=1}^{n} \frac{aq^{2j}/b^2}{1 - aq^{2j}/b} - \sum_{j=1}^{n} \frac{aq^{2j-4}}{1 - abq^{2j-4}}.
$$

Employing the operator $D_b$ on both sides of the last equation, it is easy to show that

$$
\sum_{k=0}^{n} \frac{1 - aq^{3k-1}}{1 - aq^{-1}} \left( \frac{q^{-2n}, aq^{2n}, a/q, q^2}{q^3, aq^2/b, ab/q^2; q^2} \right)_k \frac{q^2}{(a/q^2, b/q, q^3/b; q)_k} q^k \left\{ G_k(a, b)^2 - U_k(a, b) \right\}
$$

$$
= \frac{(aq, a/q, b, q^4/b; q^2)_n}{(q, q^3, aq^2/b, ab/q^2; q^2)_n} \left\{ H_n(a, b)^2 - V_n(a, b) \right\},
$$

where

$$
U_k(a, b) = \sum_{i=1}^{k} \frac{q^{4i-4}}{(1 - bq^{i-2})^2} - \sum_{i=1}^{k} \frac{(q^{i+2}/b - 2)q^{i+2}/b^3}{(1 - q^{i+2}/b)^2} + \sum_{i=1}^{k} \frac{(aq^{2i}/b - 2)aq^{2i}/b^3}{(1 - aq^{2i}/b)^2} - \sum_{i=1}^{k} \frac{a^2 q^{4i-8}}{(1 - abq^{2i-4})^2},
$$

$$
V_n(a, b) = \sum_{j=1}^{n} \frac{q^{4j-4}}{(1 - bq^{2j-2})^2} - \sum_{j=1}^{n} \frac{(q^{2j+2}/b - 2)q^{2j+2}/b^3}{(1 - q^{2j+2}/b)^2} + \sum_{j=1}^{n} \frac{(aq^{2j}/b - 2)aq^{2j}/b^3}{(1 - aq^{2j}/b)^2} - \sum_{j=1}^{n} \frac{a^2 q^{4j-8}}{(1 - abq^{2j-4})^2}.
$$
The \((a,b) = (0,q^2)\) case of (3.10) produces
\[
\sum_{k=0}^{n} q^k \frac{1- q^{3k+2}}{1-q^2} \frac{(q; q)_k (q^2; q^2)_k (q^{2n+3}, q^{-2n}; q)_k}{(q^3; q^3)_k (q^{2n+3}, q^{-2n}; q)_k} \left\{ \sum_{i=1}^{k} q^i \frac{1}{[i]^2} - \sum_{i=2}^{k+1} \frac{q^{2i-1}}{[2i-1]^2} \right\} = \frac{(q; q^2)_n (q^4, q^2)}{(q; q^2)_n (q^3, q^2)^2} \frac{1}{n} \left\{ \sum_{i=1}^{n} q^{2j} \frac{1}{[2j]^2} - \sum_{j=2}^{n+1} \frac{q^{2j-1}}{[2j-1]^2} \right\}.
\] (3.11)

The \((a,b,c) = (q^3, q^2, q^2)\) case of (3.12) can be expressed as
\[
\sum_{k=0}^{n} q^k \frac{1- q^{3k+2}}{1-q^2} \frac{(q; q)_k (q^2; q^2)_k (q^{2n+3}, q^{-2n}; q)_k}{(q^3; q^3)_k (q^{2n+3}, q^{-2n}; q)_k} \left\{ \sum_{i=1}^{k} q^i \frac{1}{[i]^2} - \sum_{i=2}^{k+1} \frac{q^{2i-1}}{[2i-1]^2} \right\} = \frac{(q; q^2)_n (q^4, q^2)}{(q; q^2)_n (q^3, q^2)^3} \frac{1}{n} \left\{ \sum_{i=1}^{n} q^{2j} \frac{1}{[2j]^2} - \sum_{j=2}^{n+1} \frac{q^{2j-1}}{[2j-1]^2} \right\}.
\] (3.12)

According to the linear combination of (3.11) and (3.12), we have
\[
\sum_{k=0}^{n} q^k \frac{1- q^{3k+2}}{1-q^2} \frac{(q; q)_k (q^2; q^2)_k (q^{2n+3}, q^{-2n}; q)_k}{(q^3; q^3)_k (q^{2n+3}, q^{-2n}; q)_k} \left\{ \sum_{i=1}^{k} q^i \frac{1}{[i]^2} - \sum_{i=2}^{k+1} \frac{q^{2i-1}}{[2i-1]^2} \right\} = \frac{(q; q^2)_n (q^4, q^2)}{(q; q^2)_n (q^3, q^2)^3} \frac{1}{n} \left\{ \sum_{i=1}^{n} q^{2j} \frac{1}{[2j]^2} - \sum_{j=2}^{n+1} \frac{q^{2j-1}}{[2j-1]^2} \right\}.
\]

Letting \(n \to \infty\) in this identity, we get (1.16). \(\Box\)

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