THE HIRZEBRUCH $\chi_y$-GENUS AND POINCARÉ POLYNOMIAL REVISITED

PING LI

Abstract. The Hirzebruch $\chi_y$-genus and Poincaré polynomial share some similar features. In this article we investigate two of their similar features simultaneously. Through this process we shall derive several new results as well as reprove and improve some known results.

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1. Introduction

Let $X$ be a $k$-dimensional closed orientable manifold and $b_i(X)$ its $i$-th Betti number. The Poincaré polynomial of $X$, denoted by $P_y(X)$, is by definition the generating function of its Betti numbers:

$$P_y(X) := \sum_{i=0}^{k} b_i(X) \cdot y^i.$$ 

$P_y(X)$ satisfies a basic relation $P_y(X) = y^k \cdot P_{y^{-1}}(X)$ which is nothing but a reformulation of the Poincaré dualities $b_i = b_{k-i}$ ($0 \leq i \leq k$). When evaluated at $y = -1$, $P_y(X)|_{y=-1}$ gives the
most important (combinatorial) invariant: the Euler characteristic. In principle, to determine the Poincaré polynomial of a manifold is equivalent to knowing its all Betti numbers.

Now we turn to the definition of the Hirzebruch $\chi_y$-genus, which was first introduced by Hirzebruch in his seminal book [7] for projective manifolds, and can be computed by the celebrated Hirzebruch-Riemann-Roch formula also established in [7]. Later on the discovery of the Atiyah-Singer index theorem tells us that it still holds for almost-complex manifolds. To be more precise, let $(M^{2n}, J)$ be a compact almost-complex manifold with complex dimension $n$ and an almost-complex structure $J$. As usual we use $\bar{\partial}$ to denote the $d$-bar operator which acts on the complex vector spaces $\Omega^{p,q}(M)$ ($0 \leq p, q \leq n$) of $(p, q)$-type differential forms on $(M^{2n}, J)$ in the sense of $J$ ([29, p. 27]). The choice of an almost Hermitian metric on $(M^{2n}, J)$ enables us to define the Hodge star operator $\star$ and the formal adjoint $\bar{\partial}^* = - \ast \bar{\partial} \ast$ of the $\bar{\partial}$-operator. Then for each $0 \leq p \leq n$, we have the following Dolbeault-type elliptic operator

\[
\bigoplus_{q \text{ even}} \Omega^{p,q}(M) \xrightarrow{\bar{\partial} + \bar{\partial}^*} \bigoplus_{q \text{ odd}} \Omega^{p,q}(M),
\]

whose index is denoted by $\chi^p(M)$ in the notation of Hirzebruch in [7]. The Hirzebruch $\chi_y$-genus, denoted by $\chi_y(M)$, is the generating function of these indices $\chi^p(M)$:

\[
\chi_y(M) := \sum_{p=0}^{n} \chi^p(M) \cdot y^p.
\]

The general form of the Hirzebruch-Riemann-Roch theorem, which is a corollary of the Atiyah-Singer index theorem, allows us to compute $\chi_y(M)$ in terms of the Chern numbers of $M$ as follows

\[
\chi_y(M) = \int_M \prod_{i=1}^{n} \frac{x_i(1 + ye^{-x_i})}{1 - e^{-x_i}},
\]

where $x_1, \ldots, x_n$ are formal Chern roots of $(M, J)$, i.e., the $i$-th elementary symmetric polynomial of $x_1, \ldots, x_n$ is the $i$-th Chern class of $(M, J)$. Similar to that of the Poincaré polynomial, $\chi_y(M)$ also satisfies $\chi_y(M) = (-y)^n \cdot \chi_{y-1}(M)$ which are equivalent to the relations $\chi^p = (-1)^p \chi^{n-p}$ and can be derived from (1.2). For three values of $y$, this $\chi_y$-genus is an important invariant: $\chi_y(M)|_{y = -1}$ is the Euler characteristic of $M$, $\chi_y(M)|_{y = 0}$ is the Todd genus of $M$, and $\chi_y(M)|_{y = 1}$ is the signature of $M$.

When $J$ is integrable, i.e., $M$ is an $n$-dimensional compact complex manifold, which is equivalent to the condition that $\bar{\partial}^2 \equiv 0$, the two-step elliptic complex (1.1) above has the following resolution, which is the well-known Dolbeault complex:

\[
0 \rightarrow \Omega^{p,0}(M) \xrightarrow{\bar{\partial}} \Omega^{p,1}(M) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \Omega^{p,n}(M) \rightarrow 0
\]

and hence

\[
\chi^p(M) = \sum_{q=0}^{n} (-1)^{q} \dim C H^{p,q}_\bar{\partial}(M) =: \sum_{q=0}^{n} (-1)^{q} h^{p,q}(M).
\]

Here $h^{p,q}(M)$ are the corresponding Hodge numbers of $M$, which are the complex dimensions of the corresponding Dolbeault cohomology groups $H^{p,q}_\bar{\partial}(M)$. The famous Serre duality ([6, p. 102]) gives the relation $h^{p,q} = h^{n-p,n-q}$, which can be used to give an alternative proof of the
fact \( \chi^p = (-1)^n \chi^{n-p} \) in the case of \( J \) being integrable:

\[
\chi^p = \sum_{q=0}^{n} (-1)^q h^{p,q} = \sum_{q=0}^{n} (-1)^q h^{n-p,n-q} = (-1)^n \sum_{q=0}^{n} (-1)^q h^{n-p,n-q} = (-1)^n \chi^{n-p}.
\]

Poincaré polynomial and the Hirzebruch \( \chi_y \)-genus are two fundamental mathematical objects and have been studied intensively from various aspects. However, as far as the author knows, there is no explicit investigation in the existing literature towards direct connections between Poincaré polynomial and the Hirzebruch \( \chi_y \)-genus. Indeed they do share some similarities. For instance, as we have mentioned, their coefficients satisfy similar duality relations: \( b_i = b_{k-i} \) and \( \chi^p = (-1)^n \chi^{n-p} \), and when evaluated at \( y = -1 \), both \( P_y(X) \big|_{y=-1} \) and \( \chi_y(M) \big|_{y=-1} \) give the Euler characteristic.

Over the past several years, the author gradually realizes that these two mathematical objects should share some more interesting and deeper similarities in various senses. For instance, Thompson noticed in [28] that the character of the natural \( SL(2, \mathbb{C}) \)-representation on any hyperKähler manifold induced by the holomorphic two form is essentially its \( \chi_y \)-genus. Inspired by this interesting observation and Keeping the similarity between the \( \chi_y \)-genus and Poincaré polynomial in mind, the author showed in [18] that there exists an analogous result for any compact Kähler manifold: the character of the natural \( SL(2, \mathbb{C}) \)-representation on any compact Kähler manifold induced by the Kähler form is essentially its Poincaré polynomial.

The main purpose of the current article is to strengthen this belief from two aspects by investigating some properties simultaneously for manifolds with some extra structures (Kähler structure, hyperKähler structure, symplectic structure etc.). Through this process we can derive a number of nontrivial results. Among these results, some have been known for some time by using somewhat different methods while some should be new, at least to the author’s best knowledge. The author believes that the relationship between Poincaré polynomial and the Hirzebruch \( \chi_y \)-genus deserves more attention and there should exist deeper interactions between them.

The rest of this article is arranged as follows. In Section 2 we introduce and investigate the slightly modified coefficients of the Taylor expansions of Poincaré polynomial and the \( \chi_y \)-genus at \( y = -1 \). Section 3 is devoted to some related applications of these coefficients to Kähler and hyperKähler manifolds. In the first two subsections of Section 4 we recall two residue formulas related to the \( \chi_y \)-genus and Poincaré polynomial. Then in the third subsection, Section 4.3, we provide some geometric and topological obstructions to the existence of Hamiltonian torus actions with isolated fixed points on compact symplectic manifolds.

2. “−1”-PHENOMENON OF THE \( \chi_y \)-GENUS

The material in this section is inspired by an interesting phenomenon of the \( \chi_y \)-genus, which the author calls the “−1”-phenomenon and has been observed, implicitly or explicitly, in several independent articles.

2.1. “−1”-Phenomena. The purpose of this subsection is to recall this “−1”-phenomenon for the Hirzebruch \( \chi_y \)-genus.

As we have mentioned in the introduction, when evaluated at \( y = -1 \), \( \chi_y(M) \big|_{y=-1} \) gives the Euler characteristic, which is equal to the top Chern number \( c_n \) of \( M \). Note that \( \chi_y(M) \big|_{y=-1} \)
is exactly the constant term in the Taylor expansion of $\chi_y(M)$ at $y = -1$. In fact, several independent articles ([24], [19], [26]), with different backgrounds, have observed that, when expanding the right-hand side of (1.2) at $y = -1$, its corresponding coefficients have explicit expressions. More precisely, if we denote

$$\int_M \prod_{i=1}^n \frac{x_i(1 + ye^{-x_i})}{1 - e^{-x_i}} =: \sum_{i=0}^n a_i(M) \cdot (y + 1)^i,$$

then we have

$$a_0 = c_n, \quad a_1 = -\frac{1}{2} nc_n,$$

$$a_2 = \frac{1}{12} [\frac{n(n-1)}{2} c_n + c_1 c_{n-1}],$$

$$a_3 = -\frac{1}{24} [\frac{n(n-2)(n-3)}{2} c_n + (n-2)c_1 c_{n-1}],$$

$$a_4 = \frac{1}{5760} [n(15n^3 - 150n^2 + 485n - 502)c_n + 4(15n^2 - 85n + 108)c_1 c_{n-1} + 8(c_1^2 + 3c_2)c_{n-2} - 8(c_1^3 - 3c_1 c_2 + 3c_3)c_{n-3}],$$

$$\ldots$$

The derivations of $a_0$ and $a_1$ are easy. The calculation of $a_2$ appears implicitly in [24, p. 18] and [5, Corollary 5.3.12] and explicitly in [19, p. 141-143]. Narasimhan and Ramanan used $a_2$ to give a topological restriction on some moduli spaces of stable vector bundles over Riemann surfaces. The primary interest of [5, Ch. 5] is to interpret the Futaki invariant on Fano manifolds as a special case of a family of integral invariants. But in Corollary 5.3.12 Futaki also implicitly computed the expression $a_2$. Libgober and Wood used $a_2$ to prove the uniqueness of the complex structure on Kähler manifolds of certain homotopy types [19, Theorems 1 and 2]. Inspired by [24], Salamon applied $a_2$ ([26, Corollary 3.4]) to obtain a restriction on the Betti numbers of hyperKähler manifolds ([26, Theorem 4.1]). In [8], Hirzebruch applied $a_1$, $a_2$ and $a_3$ to deduce a divisibility result on the Euler number of almost-complex manifolds with $c_1 = 0$. The expressions $a_3$ and $a_4$ are also included in [26, p. 145].

2.2. Technical preliminaries. The purpose of this subsection is to introduce and investigate some numerical values $h(p^i)$ and $f(i)$ related to the $\chi_y$-genus and Poincaré polynomial, which are slightly modified from the coefficients of the Taylor expansion of them at $y = -1$.

For any $n$-dimensional compact complex manifold $M$, this “$-1$”-phenomenon tells us that, via the H-R-R formula (1.2) and the relation (1.4), some linear combinations of the Hodge numbers of $M$ can be expressed in terms of its Chern numbers:

$$a_0 = \sum_{p=i}^n (-1)^p \cdot \chi^p,$$
and for $i \geq 1$,
\[
    a_i = \frac{1}{i!} \sum_{p=1}^{n} (-1)^p \cdot \chi^p \cdot p(p - 1) \cdots (p - i + 1)
\]
(2.1)
\[
    = \frac{1}{i!} \sum_{p=0}^{n} (-1)^p \cdot \chi^p \cdot p(p - 1) \cdots (p - i + 1)
\]
\[
    = \frac{1}{i!} \sum_{p,q=0}^{n} (-1)^{p+q} \cdot h^{p,q} \cdot p(p - 1) \cdots (p - i + 1)
\]

For our later convenience, we define, for any polynomial $x = x(p, q)$,
\[
    (2.2) \quad h(x) := \sum_{p,q=0}^{n} (-1)^{p+q} \cdot h^{p,q} \cdot x.
\]

Using this symbol we know that
\[
a_0 = h(1) = h(p^0)
\]
and
\[
a_i = \frac{1}{i!} h(p(p - 1) \cdots (p - i + 1)) \quad \text{for} \quad i \geq 1.
\]

However, in order to reveal this “$-1$”-phenomenon more efficiently, we would like to investigate the slightly modified coefficients
\[
h(p^i) = \sum_{p,q=0}^{n} (-1)^{p+q} \cdot h^{p,q} \cdot p^i
\]
originating from $a_i$.

The following technical lemma tells us that the three sets $\{a_i\}$, $\{h(p^i)\}$ and $\{h(p^{2i})\}$ contain the same information.

**Lemma 2.1.**

1. Any element in the set $\{a_0, a_1, \ldots, a_n\}$ can be expressed in terms of the elements in the set $\{h(p^i)\}$, $0 \leq i \leq n$ and vice versa.
2. The first several explicit expressions of $h(p^i)$ in terms of Chern numbers are given below:

\[
h(1) = c_n, \quad h(p^1) = \frac{n}{2} c_n, \quad h(p^2) = \frac{n(n+1)}{12} c_n + \frac{1}{6} c_1 c_{n-1}, \quad h(p^3) = \frac{n^2(n+1)}{8} c_n + \frac{n}{4} c_1 c_{n-1},
\]
\[
h(p^4) = \frac{n(15n^3 + 30n^2 + 5n - 2)}{240} c_n + \frac{15n^2 + 5n - 2}{60} c_1 c_{n-1} + \frac{(c_1^2 + 3c_2)c_{n-2}}{30} - \frac{(c_1^3 - 3c_1 c_2 + 3c_3)c_{n-3}}{30}, \ldots
\]

3. Each $h(p^{2i+1})$ can be expressed by the even powers $h(1), h(p^2), \ldots, h(p^{2i})$. This means the set $\{h(p^i)\}$, $0 \leq i \leq n$, as well as $\{a_i\}$ contains the same information as that in the set $\{h(p^{2i})\}$, $0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$. So this “$-1$”-phenomenon gives us $\left\lfloor \frac{n}{2} \right\rfloor + 1$ independent relations on the Chern numbers.
Proof. (1) is an elementary linear algebra exercise. The derivations of \( h(p^i) \) are also direct via the concrete expressions of \( a_i \) in (2.1). For example,

\[
h(p^2) = h[p(p - 1) + p] = 2! \cdot a_2 + a_1 = \frac{n(3n + 1)}{12} c_n + \frac{1}{6} c_1 c_{n-1},
\]

\[
h(p^3) = h[p(p - 1)(p - 2) + 3p(p - 1) + p] = 6a_3 + 6a_2 + a_1 = \cdots,
\]

and so on.

(3) is an application of the Serre dualities \( h^{p,q} = h^{n-p,n-q} \). Indeed,

\[
h(p^{2i+1}) = \sum_{p,q} (-1)^{p+q} \cdot h^{p,q} \cdot p^{2i+1}
\]

\[
= \frac{1}{2} \sum_{p,q} (-1)^{p+q} \cdot h^{p,q} \cdot [p^{2i+1} + (n-p)^{2i+1}] \quad \text{(by } h^{p,q} = h^{n-p,n-q} \text{)}
\]

\[
= \frac{1}{2} \sum_{p,q} (-1)^{p+q} \cdot h^{p,q} \cdot \left[ \sum_{j=1}^{2i} \binom{2i+1}{j} n^j (-p)^{2i+1-j} \right]
\]

\[
= \frac{1}{2} \sum_{p,q} (-1)^{p+q} \cdot h^{p,q} \cdot [(2i+1)n p^{2i+1} + \cdots]
\]

\[
= (i + \frac{1}{2}) \cdot n \cdot h(p^{2i}) + (\cdots),
\]

where \((\cdots)\) is a sum of the terms \( h(p^j) \) with \( j < 2i \). Using this formula repeatedly yields the fact that \( h(p^{2i+1}) \) can be expressed by \( h(p^2) \), \( 0 \leq j \leq i \). \( \square \)

We have an analogous result to Lemma 2.1 for the Poincaré polynomial, which we record in the following lemma.

Lemma 2.2.

(1) The coefficients of the Taylor expansion of \( P_y(X) \) at \( y = -1 \) can be expressed in terms of the elements in the set

\[
\{ f(i) := \sum_{p=0}^{k} (-1)^p \cdot b_p \cdot p^i \mid 0 \leq i \leq k \}.
\]

(2) The information contained in the set above is the same as that in its subset

\[
\{ f(2i) \mid 0 \leq i \leq \frac{k}{2} \}.
\]

Proof. The proof of this lemma is almost identically the same as that in Lemma 2.1. We only need to note that, in the proof of (2) in this lemma, Poincaré dualities play the role of the Serre dualities as in the proof of (3) in Lemma 2.1. \( \square \)

The expressions \( f(2i) \) and \( h(p^i) \) highlighted in Lemmas 2.1 and 2.2 shall be investigated intensively in the next section.
3. Applications of “−1”-phenomenon to Kähler and hyperKähler manifolds

With the preliminaries presented in the last section at hand, we now give some applications via some suitable manipulations on the above-introduced modified coefficients \( h(p^i) \) and \( f(2i) \).

In order to achieve our purpose, we need to bridge a link between the Betti numbers \( b_i \) and the Hodge numbers \( h^{p,q} \) involving in \( f(2i) \) and \( h(p^i) \) respectively. Recall that the Hodge theory imposes intimate relations among the Betti numbers and the Hodge numbers for compact Kähler manifolds. So from now on we assume throughout this section that \( M \) be a complex \( n \)-dimensional compact connected Kähler manifold. Then its Betti numbers \( b_i \) and Hodge numbers \( h^{p,q} \) satisfy the following well-known relations ([6, p. 116]):

\[
(3.1) \quad b_i = \sum_{p+q=i} h^{p,q}, \quad h^{p,p} \geq 1, \quad h^{p,q} = h^{n-p,n-q} = h^{q,p}, \quad (0 \leq i \leq 2n, \; 0 \leq p, q \leq n).
\]

The first equality is a consequence of the Hodge decomposition theorem, the second one is due to the fact that any \( p \)-th power of the Kähler form represents a nonzero cohomology class in \( H^{p,p}(M) \), and the third one comes from the Serre duality which we have mentioned in the introduction and the complex conjugation.

Now it is time for us to illustrate some applications via comparing the modified coefficients \( f(2i) \) and \( h(p^i) \) simultaneously.

Comparing \( f(0) \) and \( h(1) \) leads to the well-known fact for the Euler characteristic:

\[
f(0) = \sum_{i=0}^{2n} (-1)^i \cdot b_i \quad \text{(by (3.1))} \quad = \sum_{p,q=0}^{n} (-1)^{p+q} h^{p,q} = h(1) = c_n.
\]

Now we consider \( f(2) \):

\[
f(2) = \sum_{i=0}^{2n} (-1)^i \cdot b_i \cdot i^2 = \sum_{p,q=0}^{n} (-1)^{p+q} \cdot h^{p,q} \cdot (p + q)^2 \quad \text{(by (3.1))}
\]

\[
= h(p^2) + 2h(pq) + h(q^2) \quad \text{(by (2.2))}
\]

\[
= 2h(p^2) + 2h(pq). \quad \text{(by } h^{p,q} = h^{q,p})
\]

We know through Lemma 2.1 that \( h(p^2) \) can be expressed in terms of the Chern numbers \( c_n \) and \( c_1c_{n-1} \). Thus in order to obtain some nontrivial relations from (3.2), we need to impose more restrictions on \( h^{p,q} \) in order to deal with another term \( h(pq) \). Here we impose two different restrictions. The first one leads to the following result due to Salamon which gives a restriction on the Betti numbers of hyperKähler manifolds ([26, Theorem 4.1, Corollary 4.2]).

**Theorem 3.1** (Salamon). Suppose \( M \) is a compact Kähler manifold whose complex dimension \( n \) is even and Hodge numbers are “invariant by mirror symmetry” in the sense that \( h^{p,q} = h^{p,n-q} \). Then the Chern number \( c_1c_{n-1} \) of \( M \) can be expressed in terms of its Betti numbers:

\[
(3.3) \quad c_1c_{n-1} = \sum_{i=0}^{2n} (-1)^i \cdot b_i \cdot [3i^2 - n(3n + 1)\frac{1}{2}].
\]
In particular, this gives a restriction on the Betti numbers of a compact hyper-Kähler manifold whose complex dimension is $n$:

$$
\sum_{i=0}^{2n} (-1)^i \cdot b_i \cdot [3i^2 - n(3n + \frac{1}{2})] = 0.
$$

**Proof.** Under our assumptions we have

$$
h(pq) = \sum_{p,q=0}^n (-1)^{p+q} \cdot h^{p,q} \cdot pq
$$

$$
= \sum_{p,q=0}^n (-1)^{p+n-q} \cdot h^{p,q} \cdot p(n-q) \quad \text{(by } h^{p,q} = h^{p,n-q})
$$

$$
= \sum_{p,q=0}^n (-1)^{p+q} \cdot h^{p,q} \cdot p(n-q) \quad \text{(by } n \text{ is even)}
$$

$$
= n \cdot h(p) - h(pq).
$$

Thus $h(pq) = \frac{n}{2} h(p)$, which, together with (3.2), yields

$$
\sum_{i=0}^{2n} (-1)^i \cdot b_i \cdot i^2 = 2h(p^2) + n \cdot h(p)
$$

$$
= \frac{n(3n+1)}{6} c_n + \frac{1}{3} c_1 c_{n-1} + \frac{n^2}{2} c_n \quad \text{(by Lemma 2.1)}
$$

$$
= \frac{1}{3} c_1 c_{n-1} + n \left( n + \frac{1}{6} \right) \sum_{i=0}^{2n} (-1)^i \cdot b_i. \quad \text{(by } c_n = \sum_{i=0}^{2n} (-1)^i \cdot b_i)
$$

Singling out the term $c_1 c_{n-1}$ in the equality above leads to (3.3).

Recall that a hyperKähler manifold is a compact Riemannian manifold whose real dimension is divisible by 4, say $4m$, and holonomy group is contained in $Sp(m)$, which is a higher-dimensional analogue to $K3$-surfaces. It is well-known that a hyperKähler manifold possesses a family of Kähler structures parameterized by a 2-dimensional sphere. Moreover, its Hodge numbers are “invariant by mirror symmetry” in the sense that $h^{p,q} = h^{p,2m-q}$ and its all odd Chern classes $c_{2i+1} \ (0 \leq i \leq m - 1)$ vanish in $H^*(M, \mathbb{R})$. HyperKähler manifolds form an important subclass in compact Kähler manifolds. We refer the reader to [10] and [11] for an account for them and their basic properties. So hyperKähler manifolds satisfy the conditions assumed in this theorem, which gives the required restriction.

We call a compact Kähler manifold $M$ are of pure type if its Hodge numbers satisfy $h^{p,q} = 0$ whenever $p \neq q$. Many important compact Kähler manifolds are of pure type (cf. Remark 3.3). Our second restriction on $h^{p,q}$ involving in the notion of pure type leads to the following result, which is an improvement of the author’s previous result ([16, Theorem 1.3]).

**Theorem 3.2.** The Chern number $c_1 c_{n-1}$ of any $n$-dimensional compact Kähler manifold $M$ has a lower bound in terms of its Betti numbers as follows:

$$
c_1 c_{n-1} \geq \frac{1}{2} \left\{ \sum_{i \text{ even}} b_i [3i^2 - n(3n + 1)] - \sum_{i \text{ odd}} b_i [9i^2 - n(3n + 1)] \right\},
$$

where the equality holds if and only if $M$ is of pure type.
Proof. We first claim that for any \( n \)-dimensional compact Kähler manifold we have

\[
4h(pq) \leq \sum_{i=0}^{2n} b_i \cdot i^2,
\]

where the equality holds if and only if \( M \) is of pure type. Indeed,

\[
h(pq) = \sum_{p,q=0}^{n} (-1)^{p+q} \cdot h^{p,q} \cdot pq \leq \sum_{p,q=0}^{n} h^{p,q} \cdot pq \leq \sum_{p,q=0}^{n} h^{p,q} \frac{(p+q)^2}{2} \quad (3.1) \quad \frac{1}{4} \sum_{i=0}^{2n} b_i \cdot i^2.
\]

Note that in the formula above the equality in the second “\( \leq \)” holds if and only if \( h^{p,q} = 0 \) whenever \( p \neq q \), which also implies the validity of the equality in the first “\( \leq \)”.

Thus (3.4), together with (3.2), yields

\[
\sum_{i=0}^{2n} (-1)^i \cdot b_i \cdot i^2 = 2h(p^2) + 2 \cdot h(pq)
\]

\[
\leq \frac{n(3n+1)}{6} c_n + \frac{1}{3} c_1 c_{n-1} + \frac{1}{2} \sum_{i=0}^{2n} b_i \cdot i^2
\]

\[
= \frac{n(3n+1)}{6} \sum_{i=0}^{2n} (-1)^i \cdot b_i + \frac{1}{3} c_1 c_{n-1} + \frac{1}{2} \sum_{i=0}^{2n} b_i \cdot i^2.
\]

Rewriting this inequality by singling out the term \( c_1 c_{n-1} \) leads to the desired one in Theorem 3.2.

\[\square\]

Remark 3.3.

(1) An additional assumption in [16, Theorem 1.3] that the odd Betti numbers \( b_{2i+1} \) all vanish has been removed.

(2) As we have remarked in [16, Remark 1.2], many important compact Kähler manifolds are of pure type. For instance, the flag manifold \( G/P \) ([2, §14.10]), where \( G \) is a complex semisimple linear algebraic group and \( P \) is a parabolic subgroup, the Fano contact manifolds ([14, p. 118]), and the nonsingular projective toric varieties ([4, p. 106]).

(3) It is well-known that the simplest Chern number \( c_n \) of any \( n \)-dimensional compact almost-complex manifold is equal to the alternating sum of its Betti numbers. Theorem 3.2 illustrates an interesting phenomenon that, for any \( n \)-dimensional compact Kähler manifold, the next-to-simplest Chern number \( c_1 c_{n-1} \) can also be related to Betti numbers.

Recall that a Calabi-Yau manifold (in the weak sense) is a compact Kähler manifold whose first Chern class \( c_1 = 0 \) in \( H^2(M, \mathbb{R}) \). Clearly all hyperKähler manifolds are Calabi-Yau manifolds. Theorem 3.2 has the following interesting consequence on the Betti numbers of Calabi-Yau manifolds, which, to the author’s best knowledge, should be new.

Corollary 3.4. The Betti numbers of any complex \( n \)-dimensional compact Kähler manifold whose Chern number \( c_1 c_{n-1} = 0 \) satisfy

\[
\sum_{i \text{ odd}} b_i [9i^2 - n(3n + 1)] \geq \sum_{i \text{ even}} b_i [3i^2 - n(3n + 1)],
\]

\[
\sum_{i \text{ odd}} b_i [9i^2 - n(3n + 1)] \geq \sum_{i \text{ even}} b_i [3i^2 - n(3n + 1)].
\]
where the equality holds if and only if it is of pure type. In particular, this inequality holds for Calabi-Yau manifolds.

**Remark 3.5.** This inequality is sharp in the sense that its equality can be attained for some manifolds. For \( n = 2 \) the most famous examples are Enriques surfaces, which are of pure type and \( c_1 = 0 \) in \( H^2(M, \mathbb{R}) \). For general \( n \), recall that in Remark 3.3 we commented that nonsingular projective toric varieties are of pure type. Their Chern classes can be explicitly described by their irreducible \( T \)-divisors ([5, p. 109]). So we can choose appropriately a fan \( \Delta \) such that the sum of the irreducible \( T \)-divisors is zero. This means by [5, p. 109, Lemma] that the first Chern class of the corresponding nonsingular projective toric variety \( X(\Delta) \) is zero.

In the discussions above the simultaneous investigations on \( f(2) \) and \( h(p^2) \) have produced plentiful results. In particular, it provides a lower bound for the next-to-simplest Chern number \( c_1c_{n-1} \) of \( n \)-dimensional compact Kähler manifolds in terms of their Betti numbers in Theorem 3.2. However, as we have discussed in the proof of Theorem 3.1, \( n \)-dimensional hyperKähler manifolds have vanishing odd Chern classes and so \( c_1c_{n-1} = 0 \) automatically. This means that their next-to-simplest Chern numbers are \( c_2c_{n-2} \). A natural question is whether or not we have a lower bound in terms of Betti numbers for the Chern numbers \( c_2c_{n-2} \) of hyperKähler manifolds. The answer is yes and the method is to continue our employment of \( f(4) \) and \( h(p^4) \) simultaneously. To be more precise, we have the following result.

**Theorem 3.6.** Suppose \( M \) is a hyperKähler manifold of complex dimension \( n \), which is necessarily even by definition. Then \( c_2c_{n-2} \), the next-to-simplest Chern number of \( M \), has the following lower bound in terms of Betti numbers:

\[
(3.5) \quad c_2c_{n-2} \geq \frac{1}{24} \left\{ \sum_{i \text{ even}} b_i [75i^4 - n(75n^3 + 90n^2 + 5n - 2)] - \sum_{i \text{ odd}} b_i [165i^4 - n(75n^3 + 90n^2 + 5n - 2)] \right\}.
\]

However, this lower bound is not sharp in the sense that the equality cannot be attained. The reason will be clear in the process of the following proof.

**Proof.** First we have

\[
(3.6) \quad f(4) = \sum_i (-1)^i \cdot b_i \cdot i^4 = \sum_{p,q} (-1)^{p+q} \cdot h^{p,q} \cdot (p + q)^4 = 2h(p^4) + 8h(p^3q) + 6h(p^2q^2).
\]

Since the odd Chern classes of \( M \) vanish, the expression \( h(p^4) \) in Lemma 2.1 can be simplified to the following form

\[
(3.7) \quad h(p^4) = \frac{n(15n^3 + 30n^2 + 5n - 2)}{240} c_n + \frac{1}{10} c_2c_{n-2}.
\]
The conditions that $n$ be even and $h^{p,q} = h^{p,n-q}$ can be employed to deal with the term $h(p^3q)$:

$$h(p^3q) = \sum_{p,q} (-1)^{p+q} \cdot h^{p,q} \cdot p^3q = \sum_{p,q} (-1)^{p+n-q} \cdot h^{p,q} \cdot p^3(n - q)$$

$$= \sum_{p,q} (-1)^{p+q} \cdot h^{p,q} \cdot p^3(n - q) = n \cdot h(p^3) - h(p^3q).$$

We thereby obtain $h(p^3q) = \frac{n}{2} h(p^3)$. Combining this with the expression $h(p^3)$ in Lemma 2.1 we have

$$h(p^3q) = \frac{n^3(n+1)}{16} c_n. \quad (3.8)$$

At last we derive an inequality for $h(p^2q^2)$, whose method is the same as that in (3.4).

$$h(p^2q^2) = \sum_{p,q} (-1)^{p+q} \cdot h^{p,q} \cdot (pq)^2 \leq \sum_{p,q} h^{p,q} \cdot (pq)^2 \leq \sum_{p,q} h^{p,q} \cdot \left(\frac{p+q}{2}\right)^4 = \frac{1}{16} \sum_i b_i \cdot i^4. \quad (3.9)$$

Putting (3.6)-(3.9) together and doing some calculations we can obtain (3.5).

Now we explain that why the equality in (3.5) cannot be attained. Indeed, similar to the reason in (3.4), (3.9) is an equality if and only if $M$ is of pure type: $h^{p,q} = 0$ whenever $p \neq q$. But this is not compatible with the additional symmetry $h^{p,q} = h^{p,n-q}$ for hyperKähler manifolds as, for instance, $h^{0,n} = h^{0,0} = 1 \neq 0$. \hfill $\blacksquare$

A few more remarks are in order before we end this section. We also know the explicit expressions for $h(p^5)$ and $h(p^6)$. Indeed Libgober and Wood described a concrete algorithm to compute $h(p^i)$ for general $i$ ([19, p. 144]). So in principle we can employ these to deal with $f(6)$, $f(8)$ and so on. But when $i$ increases, the expressions for $h(p^i)$ become more and more complicated. This means their expressions are too complicated to formulate some geometrically interesting consequences.

4. Residue formulas and their applications

The material in this section is inspired by the residue formulas for the $\chi_y$-genus on almost-complex manifolds and for the Poincaré polynomial on symplectic manifolds. We recall the residue formulas for the $\chi_y$-genus and Poincaré polynomial in Sections 4.1 and 4.2 respectively, and give some related applications to symplectic geometry in Section 4.3.

When a smooth or an almost-complex manifold admits a compatible vector field or a compact Lie group action, the philosophy of residue formula is to reduce the investigation of some global invariants on this manifold to the consideration of the local information around the zero point set of this vector field or the fixed point set of this group action. Here by “compatible” we mean that the one-parameter group action induced by the vector field or the Lie group action preserves the smooth or almost-complex structure. What we are concerned with in this section are vector fields on almost-complex manifolds with isolated zero points and Hamiltonian torus actions on symplectic manifolds with isolated fixed points, which we shall discuss respectively in what follows.
4.1. **Residue formula for the $\chi_y$-genus.** The purpose of this subsection is to review a residue formula for the $\chi_y$-genus.

Suppose $(M, g, J)$ is a compact connected almost-Hermitian manifold with complex dimension $n$. This means $J$ is an almost-complex structure and $g$ an almost-Hermitian metric, i.e., a Riemannian metric which is $J$-invariant. Now suppose we have a smooth vector field $A$ on $(M, g, J)$ preserving the metric $g$ and the almost-complex structure $J$ such that zero($A$), the zero point set of $A$, is isolated (but nonempty). Let $P \in$ zero($A$) be an arbitrary isolated zero point. Then $T_P$, the tangent space to $(M, J)$ at $P$, is an $n$-dimensional complex vector space equipped with an Hermitian inner product induced by $J$. Since $A$ preserves the Hermitian metric and so $A$ induces a skew-Hermitian transformation on $T_P$. This means $T_P$ can be decomposed into a sum of $n$ 1-dimensional complex vector spaces:

$$T_P = \bigoplus_{i=1}^{n} L_P(\lambda_i), \quad (L_P(\lambda_i) \cong \mathbb{C}, \ \lambda_i \in \mathbb{R} - \{0\}),$$

such that the eigenvalue of the skew-Hermitian transformation on $L_P(\lambda_i)$ is $\sqrt{-1}\lambda_i$. Or equivalently, the eigenvalue of the action induced by the one-parameter group $\exp(tA)$ on $L_P(\lambda_i)$ is $\exp(\sqrt{-1}\lambda_i t)$. Note that these nonzero real numbers $\lambda_1, \ldots, \lambda_n$ are counted with multiplicities and thus not necessarily mutually distinct. Of course they depend on the choice of $P$ in zero($A$) but are independent of the choice of the almost-Hermitian metric which $A$ preserves.

The following residue formula for the $\chi_y$-genus of $(M^n, g, J)$, which is a beautiful application of the Atiyah-Bott fixed point formula, is essentially due to Kosniowski ([13, Theorem 1]).

**Theorem 4.1** (Residue formula for the $\chi_y$-genus). *With the above notation and symbols understood, we have*

$$\chi_y(M) = \sum_{P \in \text{zero}(A)} (-y)^{d_P} = \sum_{P \in \text{zero}(A)} (-y)^{n-d_P}$$

*where $d_P$ denote the number of negative numbers among $\lambda_1, \ldots, \lambda_n$ and the sum is over all the points in zero($A$).*

**Remark 4.2.**

(1) For complex manifolds this result was discovered by Kosniowski in [13, Theorem 3], whose proof is an application of the Atiyah-Bott fixed point formula to the Dolbeault complex (1.3) originating from [20]. Indeed, if we replace the Dolbeault complex (1.3) by the two-step elliptic complex (1.1), this result still holds for almost-complex manifolds, which has been carried out by the author in [15] and used to give some related applications to symplectic geometry. In [17] this idea was further extended.

(2) In the theorem above, the condition that $A$ preserve some almost-Hermitian metric on $(M, J)$ can be relaxed to assume only that the endomorphism induced by $A$ on $T_P$ is nonsingular for any $P \in$ zero($A$), which is what [13] treated and the condition of which is called “isolated simple zero points” in [13]. However, if the zero point set zero($A$) is not necessarily isolated, there is a similar residue formula for $\chi_y(M)$ which has also been treated in [13, Theorem 3]. But in this case we need to use the general Lefschetz fixed point formula of Atiyah-Singer and it needs the additional condition that $A$ be compact. This means the one-parameter group of $A$ lies in a compact group,
which is equivalent to the condition that $A$ preserve an almost-Hermitian metric on $(M, J)$ as we have required in Theorem 4.1.

4.2. Morse identity for Hamiltonian torus actions. We now turn to the discussion on the Poincaré polynomial. Unlike the case of the $\chi_y$-genus in Theorem 4.1, even if a compact almost-complex manifold $(M, J)$ admits a compatible vector field $A$, in general there is no residue formula expressing the Betti numbers of $(M, J)$ in terms of the local information around zero($A$). The reason for the existence of Theorem 4.1 is that the coefficients $\chi^p$ in $\chi_y(M)$ are indices of some natural elliptic operators (1.1) and so we can apply the Lefschetz-type fixed point formula of elliptic complexes developed by Atiyah, Bott and Singer to the vector field $A$ to obtain Theorem 4.1. So the lack of an analogue to Theorem 4.1 for the Poincaré polynomial lies in the fact that in general the Betti numbers can not be realized as some natural elliptic operators. Nevertheless, if we impose more structures on $(M, J)$ and $A$, we can reduce the calculations of the Betti numbers of $(M, J)$ to those of zero($A$) via the Morse equality for perfect Morse functions.

Suppose in this subsection that $(M, \omega)$ is a compact connected symplectic manifold of real dimension $2n$ and with a symplectic form $\omega$. All relevant facts mentioned in what follows can be found in three excellent books: [1, Ch. 4], [23, §5.5] or [25, §3.6]. We call a torus action (T-action) on $(M^n, \omega)$ symplectic if this action preserves the symplectic form $\omega$. We choose a generating vector field $A$ for this T-action, i.e., the closure of the one-parameter group of $A$ is exactly this $T$. Note that in this case the T-action on $(M^n, \omega)$ is symplectic if and only if the one form $i_A(\omega) := \omega(A, \cdot)$ is closed, where $i_A(\cdot)$ denotes the contraction operator with respect to $A$. Indeed, we know from the definition of $A$ that the T-action on $(M^n, \omega)$ is symplectic if and only if the Lie derivative $L_A(\omega) = 0$. The Cartan formula $L_A = d \circ i_A + i_A \circ d$ and the closedness of $\omega$ tell us that $L_A(\omega) = 0$ is equivalent to $d(i_A(\omega)) = 0$. We call this T-action Hamiltonian if the one form $i_A(\omega)$ is exact. This means there exists a function $f$ on $M$, which is called the moment map of this T-action and is unique up to an additive constant, such that $i_A(\omega) = df$. It is well-known that this $f$ is a perfect Morse-Bott function and Crit($f$), the critical point set of $f$, coincides with zero($A$). The latter also coincides with the fixed point set of the T-action.

Note also that we can choose an almost-complex structure $J$ on $M$ such that it is both compatible with $\omega$ and preserved by this T-action ([23, Lemma 5.52]). The compatibility between $\omega$ and $J$ tells us that the bilinear form $g(v, w) := \omega(v, Jw)$ is an almost-Hermitian metric on $M$. The facts that T-action preserve $\omega$ and $J$ imply that this T-action preserves the metric $g$:

$$\forall t \in T, \ t^*(g)(v, w) = g(t_*v, t_*w) = \omega(t_*v, Jt_*w) = \omega(t_*v, t_*Jw) = \omega(v, Jw) = g(v, w).$$

This means the vector field $A$ preserves both the almost-complex structure $J$ and the almost-Hermitian metric $g$. Now we assume further that the fixed points of this T-action, which coincides with zero($A$) = Crit($f$), are all isolated. In this case the above-mentioned function $f$ degenerates to a perfect Morse function and, at each isolated point $P \in \text{zero}(A) = \text{Crit}(f)$, the Morse index of $f$ is $2d_P$, twice of the number of negative numbers among $\lambda_1, \ldots, \lambda_n$. Here we use the notation and symbols introduced in the last subsection. Then the Morse-type equality for this perfect Morse function $f$ yields the following result.

**Theorem 4.3** (Residue formula for the Poincaré polynomial). Suppose a compact connected symplectic manifold $(M^n, \omega)$ admits a Hamiltonian torus action with isolated fixed point set.
With the above-discussed notions and symbols understood, the Morse-type equality for the perfect Morse function $f$ gives us

\begin{equation}
P_y(M) = \sum_{P \in \text{zero}(A)} y^{2d_P} = \sum_{P \in \text{zero}(A)} y^{2(n-d_P)},
\end{equation}

where the sum is over all the isolated points in $\text{Crit}(f) = \text{zero}(A)$. Moreover, (4.2) still holds if we replace $P_y(M)$ with the Poincaré polynomial with respect to any coefficient field $K$, $P_y(M; K)$, where

$$P_y(M; K) := \sum_i b_i(M; K)y^i$$

and $b_i(M; K)$ is the $i$-th Betti number with respect to the field $K$. Consequently, the integral homology $H_*(M, \mathbb{Z})$ of $M$ has no odd-dimensional homology and is torsion-free.

Remark 4.4. When the almost-complex structure $J$ is integral, i.e., $M$ is Kähler, Theorem 4.3 is a classical result due to Frankel in [3, §4, Corollary 2].

4.3. Applications. In this subsection we give an application via the two residue formulae discussed above.

Our application here is concerned with variously geometric and topological obstructions to the existence of Hamiltonian torus actions on compact connected symplectic manifold with isolated fixed points, and is motivated by a famous conjecture in symplectic geometry, which was raised by McDuff in her seminal paper [22] and now is commonly called the McDuff conjecture or Frankel-McDuff conjecture as [22] is inspired by Frankel’s another seminal paper [3]. Suppose we have a compact connected symplectic manifold equipped with a symplectic torus action. In symplectic geometry it is an important topic to detect whether or not this given symplectic torus action is Hamiltonian ([23, Ch 5]). The famous McDuff conjecture, which is still open in its generality, says that any symplectic circle action with isolated fixed points must be Hamiltonian ([22]). Many partial results towards this conjecture have been obtained over the past two decades (see [15, Introduction] and the references therein). In [15], the author applies the rigidity property of the elliptic operators (1.1) to give a criterion to detect if a given symplectic circle action with isolated fixed points is Hamiltonian. By using this criterion we can both recover all the previously known results towards this conjecture and simplify their proofs.

Roughly speaking, our next application, Theorem 4.5, attempts to explain that the existence of compact symplectic manifolds equipped with Hamiltonian torus actions with isolated fixed points is a very “rare” phenomenon via finding out as many geometric and topological obstructions as possible imposed on these symplectic manifolds. The main strategy of this application is to employ the two residue formulas (4.1) and (4.2) simultaneously.

**Theorem 4.5.** If a compact connected symplectic manifold $(M^{2n}, \omega)$ admits a Hamiltonian torus action with isolated fixed points, then

1. The integral homology $H_*(M, \mathbb{Z})$ of $M$ is torsion-free and has no odd-dimensional homology.
2. \[ \chi_{-y^2}(M) = P_y(M). \]

This means that the $\chi_y$-genus and the Poincaré polynomial are essentially the same.
(3) The signature of $M$ is equal to
$$\sum_{i=0}^{n} (-1)^i b_{2i}(M),$$
the alternating sum of its even-dimensional Betti numbers.

(4) The characteristic numbers
$$c_n, \quad c_1c_{n-1}, \quad (c_1^2 + 3c_2)c_{n-2} - (c_1^3 - 3c_1c_2 + 3c_3)c_{n-3}, \quad \cdots$$
of $M$ can be completely determined by Betti numbers of $M$ in a very explicit manner:
$$c_n = \sum_i b_{2i}, \quad c_1c_{n-1} = 6 \sum_i i(i-1)b_{2i} - \frac{n(3n-5)}{2} \sum_i b_{2i}, \quad \cdots.$$

**Proof.** (1) has been mentioned in Theorem 4.3. (2) comes from the two residue formulas (4.1) and (4.2). Indeed under our condition (4.1) and (4.2) read
$$\chi_y(M) = \sum_{P \in \text{zero}(A)} (-y)^{d_P} \quad \text{and} \quad P_y(M) = \sum_{P \in \text{Crit}(f)} y^{2d_P},$$
which yield (2). (3) is a corollary of (2) as $\chi_y(M) \big|_{y=1}$ is nothing but the signature of $M$. (4) is a corollary of (2) and the “$-1$”-phenomenon described in Section 2.1. □

**Remark 4.6.**

(1) In the theorem above, property (1) should be quite well-known to experts. But to the author’s best knowledge, nobody states it as explicitly as ours in the previous literature for compact connected symplectic manifolds with Hamiltonian torus actions.

(2) In the case of circle actions, property (3) has been obtained by Jones-Rawnsley ([12]) via the Atiyah-Bott fixed point formula. Recall that the signature is by definition the index of the intersection pairing on the middle dimensional cohomology of $M$ and thus a priori depends on the ring structure of $H^*(M; \mathbb{R})$. However, property (3) reveals an interesting phenomenon for $M$: its signature depends only on the additive structure of $H^*(M; \mathbb{R})$.

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