Holonomic Gates in Pseudo-Hermitian Quantum Systems

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The time-dependent pseudo-Hermitian formulation of quantum mechanics allows to study open system dynamics in analogy to Hermitian quantum systems. In this setting, we show that the notion of holonomic quantum computation can equally be formulated for pseudo-Hermitian systems. Starting from a degenerate pseudo-Hermitian Hamiltonian we show that, in the adiabatic limit, a non-Abelian geometric phase emerges which realizes a pseudounitary quantum gate. We illustrate our findings by studying a pseudo-Hermitian gain/loss system which can be written in the form of a tripod Hamiltonian by using the biorthogonal representation. It is shown that this system allows for arbitrary pseudo-U(2) transformations acting on the dark subspace of the system.

I. INTRODUCTION

In the standard formulation of quantum mechanics (QM), observables are associated with Hermitian operators. This Hermiticity condition ensures that the spectrum of the observable is real-valued, thus making a physical interpretation possible. It was first shown by Bender and Boettcher [1] that also non-Hermitian systems, obeying $\mathcal{PT}$-symmetry (parity-time-reversal symmetry), can also show real spectra. This observation revived serious investigations into unconventional quantum mechanics. In particular, pseudo-Hermitian QM [2] (and the related biorthogonal QM [3]) have received special attention. This theory investigates pseudo-Hermitian systems, in which the Hamiltonian of the quantum system is non-Hermitian but can still be associated with a Hermitian counterpart. Such peculiar behaviour leads to a whole new class of new Hamiltonians that could reveal interesting new physics.

In this work, we are particularly interested in the paradigm of holonomic quantum computation (HQC) [4-5], which is based on the emergence of a (non-Abelian) geometric phase (holonomy) during a cyclic time evolution of a quantum system [6-7]. Corresponding to a holonomy there is a non-Abelian gauge field mediating the computation in form of a parallel transport. These types of gauge fields are realized in systems where the demand for degeneracy can be satisfied. Examples of such systems are cold atomic samples [8] or artificial atoms in superconducting circuits [9]. Recently, the implementation of such gauge fields were realized in systems of coupled waveguides [10]. Another successful scheme utilized the spin-orbit coupling of polarized light in asymmetric microcavities [11].

Holonomic quantum computing is a purely geometric approach to quantum computational problems. Unitary gates are implemented by generating a suitable holonomy from a Hamiltonian system. The transformation that a quantum state undergoes is the shadow (horizontal lift) of a loop in a parameter space (manifold) $\mathcal{M}$. In this context, the question of computational universality can be understood as the capability of generating a set of closed paths such that the holonomy spans up the entire unitary group $\mathcal{U}$. Universality is typically reached only in a subspace of the whole Hilbert space $\mathcal{H}$, the so-called quantum code $\mathcal{C}$. The most common choice is to take $\mathcal{C}$ as the ground state of the Hamiltonian. This results in a type of ground-state computation in the lowest energy eigenvalue manifold [5]. The elements of the (quantum) code $\mathcal{C}$ are called the (quantum) code words, as the gates act on them and, in that way, perform the computation.

Holonomy groups often appear in the context of gauge theories. This stems from their intrinsic connection to gauge fields, which can be elucidated by studying the theory of fiber and vector bundles [13-14]. Physical implementations of holonomic gates were considered in non-linear Kerr media [15], superconducting quantum dots [12] or quantum electrodynamical circuits [16], but to the best of our knowledge only for Hermitian systems. This broad range of possible implementations, together with the fault tolerance of HQC [17], make it desirable to generalize the concept of holonomic gates beyond Hermitian QM.

It has been pointed out that, in order to generate a non-Abelian geometric phase (holonomy), the Hermiticity of the Hamiltonian is not a necessary condition [18]. Indeed, an explicit calculation of an Abelian geometric phase for a $\mathcal{PT}$-symmetric system has been provided in Ref. [19]. However, because degeneracy plays such a crucial role in the theory of HQC, we will extend the theory from Refs. [19-21] to the non-Abelian case. With this, one is in principle able to implement quantum computational gates by means of pseudo-Hermitian systems. The conservation of the norm of quantum states is of utmost importance and will be discussed in this work, referring to time-dependent models for pseudo-Hermitian QM. The occurrence of new physical effects from these types of holonomic gates is deeply connected to the question of measurable consequences of the underlying Hilbert space metric [22]. The idea of a pseudo-Hermitian representation of geometric phases could also be of interest in the theory of open quantum systems. The latter subject showed, by studying lossy systems, deep relations to pseudo-Hermitian and $\mathcal{PT}$-symmetric QM.
This article is organized as follows. In Sec. II we briefly review the dynamics of pseudo-Hermitian systems \[14\] [23], and emphasize the change of the Hilbert space metric associated with a pseudo-Hermitian quantum system \[23\]. In this framework, we will show in Sec. III that it is possible to derive a non-Abelian gauge field arising from an adiabatic mapping onto a degenerate subspace of the system, by extending the ideas of Ref. [22] to the degenerate case. Sec. IV contains additional remarks and theoretical considerations on the construction of pseudo-Hermitian Hamiltonians from a gain/loss system or a biorthogonal basis. Following that, we will discuss the example of a degenerate interaction Hamiltonian, whose Hermitian analogue can be found in the area of light-matter coupling. The gauge field is explicitly calculated and properties of the system are discussed in detail in Sec. V. Finally, we summarize our results with some concluding remarks in Sec. VI. In Appendix A, we derive the transformation law for the gauge field. A more sophisticated treatment of the geometry of pseudo-Hermitian quantum systems involves Grassmann and Stiefel manifolds, which can be found in Appendix B.

II. DYNAMICS OF PSEUDO-HERMITIAN SYSTEMS

We begin by briefly recalling the time-dependent dynamics of pseudo-Hermitian quantum systems, following mainly Refs. [14] [20] [21] [23]. We consider a time-dependent \( N \)-dimensional (\( N < \infty \)) pseudo-Hermitian Hamiltonian \( H(t) \not= H^\dagger(t) \), that is, \( \mathcal{H} \cong \mathbb{C}^N \). The generalization to infinite dimensional systems might be well possible, but is of marginal interest for HQC. Such a pseudo-Hermitian system can be viewed as being Hermitian with respect to a similarity transformation

\[
H^\dagger(t) = \eta(t)H(t)\eta^{-1}(t),
\]

where \( \eta(t) \) (we sometimes suppress the time argument for brevity) is the so-called Hilbert-space metric \[21\] [23]. The latter induces a new inner product

\[
\langle \varphi, \psi \rangle_\eta = \langle \varphi | \eta \psi \rangle,
\]

for all vectors \( \varphi, \psi \) in the new Hilbert space \( \mathcal{H}_\eta(t) \). Note that Hermitian operators in \( \mathcal{H} \) do not have to be Hermitian in \( \mathcal{H}_\eta(t) \).

A different point of view can be taken by investigating the eigenvalue problem of \( H \) \[3\]. For a Hermitian operator over \( \mathcal{H}_\eta(t) \), all its eigenvalues are real and its instantaneous eigenstates

\[
H(t) |\phi_n(t)\rangle = \epsilon_n |\phi_n(t)\rangle, \\
H^\dagger(t) |\phi_n(t)\rangle = \epsilon_n |\phi_n(t)\rangle,
\]

form a biorthogonal basis \( \{|\phi_n\rangle, |\phi_m\rangle\} \) with \( \langle \phi_n | \phi_m \rangle = \delta_{nm} \) \[14\]. Combining Eqs. (1)-(3), we find that \( |\phi_n\rangle = \eta |\phi_n\rangle \).

The time evolution \( U: \mathcal{H} \rightarrow \mathcal{H} \) of a quantum system differs from conventional QM in that \( U \) is no longer unitary, \( UU^\dagger \not= I \). However, as it was shown in Ref. [23], a generalized unitarity condition can be established. For any two physical states \( |\Phi(t)\rangle = U(t,t_0) |\Phi(t_0)\rangle \) and \( |\Psi(t)\rangle = U(t,t_0) |\Psi(t_0)\rangle \) in \( \mathcal{H} \) one demands that

\[
\frac{d}{dt} \langle \Phi | \eta | \Psi \rangle = 0.
\]

Equation (1), together with Eq. (1), implies a generalized time-dependent Schrödinger-like equation \[14\] [23]

\[
i \frac{d}{dt} |\Psi(t)\rangle = \Lambda(t) |\Psi(t)\rangle,
\]

where \( \Lambda(t) \) is the generator of time-displacement given by

\[
\Lambda(t) = H(t) + iK(t),
\]

with \( K(t) = -\eta^{-1}(t)\eta_t/t \). Replacing the state vectors in Eq. (4) by their time evolution \( U(t,t_0) \) = \( \mathcal{T} \exp \left( -i \int_{t_0}^t \Lambda(\tau) d\tau \right) \) (\( \mathcal{T} \) denotes time ordering) and using Eq. (5) one obtains

\[
i\eta = \Lambda^\dagger \eta - \eta \Lambda,
\]

where the dot denotes the time derivative. Equation (7) can be rewritten conveniently by introducing a covariant derivative \( D_t |\Psi(t)\rangle = d/dt - K(t) \). We thus find

\[
i D_t |\Psi(t)\rangle = H(t) |\Psi(t)\rangle.
\]

We conclude this section by highlighting the physical consequences of the dynamical model presented here. Note that we imposed the Hermiticity condition (under the metric \( \eta \)) for all times \( t \) [cf. Eq. (1)]. For this to be true, the Schrödinger equation of conventional QM has to be replaced by the Schrödinger-like equation (1) to satisfy the unitarity condition, Eq. (1) [23]. If one wants to retain the original Schrödinger equation \( i \langle \Psi | \dot{\Psi} \rangle = H \langle \Psi | \Psi \rangle \), then Eq. (1) is violated whenever the metric becomes time-dependent. This can be seen by replacing \( \Lambda \) by \( H \) in Eq. (6). In this case, \( H \) would no longer be an observable for times \( t > t_0 \) \[14\]. Up until now, this seems to be not fully understood, and a number of different approaches to handle this problem were made. However, the model presented here does not produce any contradiction with conventional quantum mechanics and, as it was shown in Ref. [23], a proper mapping to conventional QM is possible. We expect that, however the final formulation of pseudo-Hermitian QM might look like, it will embody the physical demands made in this model up to a matter of notation.

III. DERIVATION OF THE HOLOMONY

The occurrence of non-Abelian geometric phases (holonomies) is, in terms of differential geometry, associated with a connection, i.e. a unique separation of the
(tangent) Hilbert space $\mathcal{H} = \mathcal{H}_{\text{exc}} \oplus \mathcal{H}_0$ into an $n_0$-fold degenerate ground-state subspace $\mathcal{H}_0$ and the space $\mathcal{H}_{\text{exc}}$ containing all excited states. Such a separation can be technically realized by a gauge field (a local connection one-form). Because a dynamical systems leads in general to a time-dependent Hilbert space, we demand that this separation holds while the quantum states undergo a time evolution during the period $T$. Thus, any initial preparation $|\Psi(0)\rangle \in \mathcal{H}_0$ is mapped onto a final state $|\Psi(T)\rangle = U(T) |\Psi(0)\rangle$ lying also in $\mathcal{H}_0$. Such an iso-degenerate mapping is nothing but the adiabatic condition [17].

Returning to the question of time evolution, we now seek an explicit representation of the final state $|\Psi(T)\rangle$. By applying the adiabatic condition, Eq. (7) takes the form

$$iD_t |\Psi(t)\rangle = E_0(t) \Pi_0(t) |\Psi(t)\rangle,$$  
(8)

where $\Pi_0(t) = \sum_{a=1}^{n_0} |\phi_a^0(t)\rangle \langle \phi_a^0(t)|$ for times $t \in [0, T]$ is the (pseudo-Hermitian) ground-state projector and $E_0$ denotes the lowest eigenvalue of $H$. As the state is initially prepared in $\mathcal{H}_0$ and will stay there while the evolution takes place, we can expand it in terms of the basis $\{|\phi_a^0(t)\rangle\}_{a=1}^{n_0}$, i.e.

$$|\Psi(t)\rangle = \sum_{a=1}^{n_0} c_a(t) |\phi_a^0(t)\rangle,$$  
(9)

with complex expansion coefficients $c_a(t)$. Inserting the expansion (9) into Eq. (8) it is readily shown that

$$i \sum_{a=1}^{n_0} (\dot{c}_a |\phi_a^0\rangle + c_a |\dot{\phi}_a^0\rangle) = \sum_{a=1}^{n_0} c_a (E_0 |\phi_a^0\rangle + i K |\phi_a^0\rangle),$$  
(10)

where we used the definition of the covariant derivative $D_t$.

Contracting both sides of Eq. (10) with $\langle \phi_b^0 |$ and noting that $\langle \phi_b^0 |\phi_a^0\rangle = \delta_{ba}$, one obtains

$$ic_b + \sum_{a=1}^{n_0} c_a \langle \phi_b^0 |\dot{\phi}_a^0\rangle = E_0 c_b + \sum_{a=1}^{n_0} c_a \langle \phi_b^0 | K |\phi_a^0\rangle,$$

which can be rearranged as

$$\dot{c}_b + i E_0 c_b + \sum_{a=1}^{n_0} c_a \langle \phi_b^0 | D_t |\phi_a^0\rangle = 0.$$  
(11)

A formal solution to Eq. (11) can be given in terms of a time-ordered integral [13]. By introducing $(A_t)^{ba} = \langle \phi_b^0 | iD_t |\phi_a^0\rangle$, a solution to Eq. (11) is

$$c_b(T) = \sum_{a=1}^{n_0} \left[ \text{Te}^{(T)} \int_{0}^{T} [-iE_0(t) \mathbb{1} + iA_t(t)] \, dt \right]^{ba} c_a(0).$$  
(12)

An evolution in time is associated with a path $\gamma : [0, T] \to \mathcal{M}$ in a control manifold of the underlying quantum system. The $d$-dimensional manifold $\mathcal{M}$ is (locally) parametrized by a set of coordinates $\lambda = \{\lambda^\mu\}_{\mu=1}^d$. These are the so-called control fields which drive the evolution of the Hamiltonian, i.e. $H(\lambda) = H_0(\gamma(t))$. In this framework, the time ordering for the integral over $A_t$ can be replaced by a path ordering $\tilde{\mathcal{P}}$ with respect to the parametrization by the coordinate chart $\{\lambda^\mu\}_{\mu=1}^d$.

Inserting the solution for the coefficients (12) into the expansion (9), we find an explicit form for the quantum state after its evolution

$$|\Psi(T)\rangle = \sum_{a,b=1}^{n_0} |\phi_a^0(0)\rangle \exp \left[ -i \int_{0}^{T} E_0(t) \, dt \right]$$

$$\times \left[ \tilde{\mathcal{P}} \exp \left( i \int_{\lambda(0)}^{\lambda(T)} A \right) \right]^{ba} c_a(0),$$  
(13)

where we introduced the gauge field (local connection one-form) $A = \sum_{\mu=1}^d A_\mu d\lambda^\mu$. Its matrix-valued components $A_\mu$ are given by

$$(A_\mu)^{ba} = i \langle \phi_b^0(\lambda) | (\partial / \partial \lambda^\mu - K_\mu(\lambda)) |\phi_a^0(\lambda)\rangle,$$  
(14)

with $K_\mu(\lambda) = -\eta^{-1}(\lambda) \partial_\mu \eta(\lambda)/2$ (with $\partial_\mu = \partial / \partial \lambda^\mu$). Note that the components in Eq. (14) contain a part that can be found in conventional QM and a metric-dependent term $K_\mu$. This has already been observed in Refs. [20, 21] for the Abelian case. One recovers the Abelian result by setting $a = b$ and simplifying Eq. (14) using $\langle \phi_b^0 | K_\mu |\phi_a^0\rangle = \langle \phi_b^0 | \partial_\mu |\phi_a^0\rangle + \partial_\mu (\langle \phi_b^0 |) |\phi_a^0\rangle)/2$. In this notation $(A_\mu)^{ba} = -\mathcal{J} |\phi_b^0 | \partial_\mu |\phi_a^0\rangle$.

It can be straightforwardly shown that under a pseudo-unitary transformation $|\phi_a^0\rangle = \sum_{\mu=1}^{n_0} U_{ca} |\phi_0^c\rangle$, the components of $A_\mu$ transform like a proper gauge field (cf. Appendix A). Furthermore the term $iA_\mu$ obeys a generalized anti-Hermiticity condition, that is $|i(A_\mu)^{ba}|^* = -i(A_\mu)^{ab}$, where $\phi \leftrightarrow \phi$ means an interchange of $|\phi_0^c\rangle$ and $|\phi_b^0\rangle$ by $|\phi_b^0\rangle$ and $|\phi_a^0\rangle$ respectively. The condition was derived by noting that $\langle \phi_a^0 | \partial_\mu |\phi_b^0\rangle = -\partial_\mu (\langle \phi_b^0 |) |\phi_a^0\rangle$.

The appearance of a gauge field in non-Hermitian QM was of course expected, as we started from a connection $\mathcal{H} = \mathcal{H}_{\text{exc}} \oplus \mathcal{H}_0$. Note that our gauge field differs from the one derived in [14], not only by a Lie-Algebra factor $i$ but also by the term $|\phi_b^0 | \partial_\mu |\phi_a^0\rangle$.

Turning back to Eq. (13) and assuming that the state $|\Psi\rangle$ returns after a full period into its initial state up to a pseudo-unitary rotation, $|\Psi(0)\rangle \to |\Psi(T)\rangle$, where the initial state is assumed to be one of the eigenstates
\(|\phi_0^0(0)\rangle\) rather than a superposition of them, we find

\[
|\Psi(T)\rangle = |\phi_0^0(0)\rangle \exp \left[ -i \int_0^T E_0(t) dt \right] \times \sum_{b=1}^{n_\eta} [U_A(\gamma)]^{ba} c_a(0),
\]

where the cyclic time evolution corresponds to a loop \(\gamma(0) = \gamma(T)\) in the parameter space \(\mathcal{M}\). The mapping of the initial state \(|\phi_0^0(0)\rangle\) described by Eq. (15) is nothing but a unitary transformation with respect to the modified inner product \(\langle \cdot, \cdot \rangle_q\). The exponential factor in Eq. (15) is a dynamical phase factor, while the second term

\[
U_A(\gamma) = \hat{P} \exp \left( i \oint A \right) \tag{16}
\]

has purely geometric origin and is indeed a holonomy.

### IV. CONSTRUCTION OF PSEUDO-HERMITIAN SYSTEMS

For the purpose of illustration we shall consider a benchmark Hamiltonian on which the previously developed theory can be studied.

There are mainly two approaches to construct artificial pseudo-Hermitian systems. The first route is to implement pseudo-Hermiticity via a top-down approach in a gain/loss system. For that one usually starts with an effective non-Hermitian Hamiltonian \(H\) describing an open system phenomenologically. The eigenvectors of this non-Hermitian Hamiltonian result directly in a biorthogonal basis as used in the previous sections. This approach has the advantage that it is directly connected to a physical system. For example, typical experimental realizations exist in the realm of optics, where the similarity of the paraxial Helmholtz equation with the Schrödinger equation allows one to design non-Hermitian characteristics with lossy waveguide systems [26,27]. An approach using parity-time-symmetric lasing in an optical fibre network has been pursued in Ref. [28], and in parity-time synthetic photonic lattices in Ref. [29]. The second approach to non-Hermitian quantum theory is provided by bioriental quantum mechanics [3]. Given any biorthogonal basis, one can construct different pseudo-Hermitian systems from a bottom up approach [22]. Let us investigate the relation between these two approaches in more detail by considering a benchmark system. In the following, \(H = L - \Gamma I\) is a complex Hamiltonian, with \(L\) and \(\Gamma\) being Hermitian operators given by

\[
L = \frac{1}{2\Omega} \begin{pmatrix} 0 & (\Omega - \Delta)\kappa_0 & (\Omega + \Delta)\kappa_- + (\Delta - \Omega)\kappa_+ & (\Delta - \Omega)\kappa_+^* & (\Delta + \Omega)\kappa_-^* \\ (\Omega + \Delta)\kappa_- & 0 & (\Delta + \Omega)\kappa_+ & (\Delta - \Omega)\kappa_-^* & (\Delta - \Omega)\kappa_+^* \\ (\Delta + \Omega)\kappa_+ & (\Delta - \Omega)\kappa_- & 0 & (\Delta + \Omega)\kappa_-^* & (\Delta - \Omega)\kappa_+^* \\ (\Delta - \Omega)\kappa_- & (\Omega + \Delta)\kappa_+^* & (\Delta - \Omega)\kappa_-^* & 0 & (\Delta + \Omega)\kappa_- \\ (\Omega - \Delta)\kappa_0 & (\Omega - \Delta)\kappa_- & (\Omega - \Delta)\kappa_+ & (\Omega - \Delta)\kappa_0 & 0 \end{pmatrix},
\]

\[
\Gamma = \frac{\alpha}{2\Omega} \begin{pmatrix} |\kappa_-|^2 & \kappa_- & \kappa_+ & \kappa_0^* & 0 \\ \kappa_- & 0 & \kappa_0 & \kappa_+^* & 0 \\ \kappa_+ & \kappa_-^* & 0 & \kappa_0 & \kappa_+^* \\ \kappa_0 & \kappa_+^* & \kappa_0 & 0 & \kappa_+^* \\ \kappa_0^* & 0 & \kappa_0 & \kappa_+^* & 0 \end{pmatrix},
\]

where \(\Omega(\alpha) = \sqrt{\Delta^2 - \alpha^2}\) with \(\Delta\) being a real constant and time-dependent parameters \(\alpha(t) \in \mathbb{R}, \kappa_c(t) \in \mathbb{C}\). We assume that \(0 < \alpha^2 < \Delta^2\) so that \(\Omega\) stays real-valued. We can decompose \(H\) as

\[
H = \sum_{c=0, \pm} \left( \kappa_c |G^c(\alpha)\rangle \langle \tilde{E}(\alpha)| + \kappa_+^* |E(\alpha)\rangle \langle \tilde{G}^c(\alpha)| \right).
\]

where

\[
|E\rangle = N_1 \begin{pmatrix} i(\Omega - \Delta) \\ 0 \\ \alpha \end{pmatrix}, \quad |G^0\rangle = N_1 \begin{pmatrix} i(\Omega - \Delta) \\ 0 \\ \alpha \end{pmatrix}, \quad |G^-\rangle = N_2 \begin{pmatrix} -i(\Omega + \Delta) \\ 0 \\ \alpha \end{pmatrix}, \quad |G^+\rangle = N_2 \begin{pmatrix} -i(\Omega + \Delta) \\ 0 \\ \alpha \end{pmatrix},
\]

with normalization factors \(N_1 = 1/\sqrt{2\Omega(\Delta - \Omega)}\) and \(N_2 = i/\sqrt{2\Omega(\Delta + \Omega)}\). Together with the associated states \(|E\rangle = |E^*\rangle\) and \(|G^c\rangle = |G^c^*\rangle\) for \(c = 0, \pm\), they form a biorthogonal basis. The Hamiltonian \(H\) in Eq. (17) possesses a two-fold degenerate dark subspace (zero eigenvalue eigenspace) and is therefore suitable for generating a pseudo-unitary, holonomic gate. The Hamiltonian \(H\) is the pseudo-Hermitian analogue of a typical light-matter coupling Hamiltonian that can be found in a variety of physical applications. For instance, in semiconductor quantum dots [12], trapped ions [30], or neutral atoms [31]. They all fall into the class of tripod systems. By considering a controlled driving of the coupling pa-
parameters $\kappa_c = \kappa_c(t)$ a generalized STIRAP (Stimulated Raman adiabatic passage) process is induced \cite{32}. The system described by $H$ can therefore be seen as such a process taking place in a Hilbert space with a varying inner product structure $\langle \cdot, \cdot \rangle_{\eta(t)}$ (cf. Fig. 1).

FIG. 1. Representation of the level scheme of the pseudo-Hermitian Hamiltonian from Eq. (17) in the time-varying Hilbert space $\mathcal{H}_{\eta(t)}$. In $\mathcal{H}_{\eta(t)}$ the Hamiltonian describes a tripod system.

In the following we show that the Hamiltonian in Eq. (17) can indeed be traced back to a Hermitian system. In general a Hermitian Hamiltonian $h$ can be expanded in an orthonormal basis $\{|g^c\rangle, |e\rangle\}$, that is, $\langle g^d | g^c \rangle = \delta_{cd}$, $\langle g^c | e \rangle = 0$ and $\langle e | e \rangle = 1$. This basis is related to the non-orthogonal states $\{|E\rangle, |G^c\rangle\}$ by a generally non-unitary matrix $u$, i.e. $|g^c\rangle = u|G^c\rangle$ and $|e\rangle = u|E\rangle$. Similarly, for the associated states we have $|g^c\rangle = v|G^c\rangle$ and $|e\rangle = v|E\rangle$, where $v$ is some non-singular matrix. By construction, we have

$$
\begin{align*}
\delta_{cd} &= \langle g^d | g^c \rangle = \langle \tilde{G}^c | v^\dagger u | G^d \rangle, \\
0 &= \langle g^c | e \rangle = \langle \tilde{G}^c | v^\dagger u | E \rangle, \\
1 &= \langle e | e \rangle = \langle \tilde{E} | v^\dagger u | E \rangle,
\end{align*}
$$

only if $v^\dagger u = 1$.

We shall assume that $v \neq u$ to ensure that the problem is non-trivial. A relation of $u$ and $v$ to the metric operator $\eta$ is readily obtained. For example, starting from the state $|E\rangle$ we find that

$$1 = \langle \tilde{E} | E \rangle = \langle E | \eta | E \rangle = \langle e | v^\dagger u v^{-1} | e \rangle,$$

hence, $\eta = v^{-1} u = u^\dagger v$. Finally, we observe that the Hermitian counterpart $h$ to $H$ is given by $h = u H u^\dagger$. In the particular case of the Hamiltonian \cite{17} we find

$$h = \sum_{c=0, \pm} (\kappa_c |g^c\rangle \langle e| + \kappa^*_e |e\rangle \langle g^c|).$$

There is an additional remark to be made about the Hermitian system $h$. As pointed out in Ref. [2], there are a number of different representations of Hermitian counterparts of pseudo-Hermitian Hamiltonians. Especially, the Hamiltonian $\tilde{h} = \eta H$ is Hermitian as long as the spectrum of $H$ is real-valued, which can be seen from Eq. [1]. However, $\tilde{h}$ is then represented in a non-orthogonal basis.

A transformation to $h$, which is expanded in an orthonormal basis, is given by $h = (u^\dagger u)(u^\dagger h u) = u^\dagger h u$. Which Hermitian analogue is suited to a specific system depends crucially on the basis in which one measures physical observables.

V. EVOLUTION IN DARK SUBSPACES

We now turn to the Hamiltonian $H$ from Eq. (17) to investigate its dynamics under an adiabatic evolution. At this point, one should recall that the metric operator of a pseudo-Hermitian system is in general not unique. It is well possible that a whole class of pseudo-Hermitian Hamiltonians is Hermitian under a certain metric operator. There might be even a time-independent metric under which $H$ is Hermitian. In order to resolve this ambiguity, we demand that the metric under which the observable $\hat{V}$ is Hermitian, is the proper metric $\eta$ given by the dyadic products of the left-handed eigenstates of $H$ \cite{25}.

We now investigate the dynamics induced by the Hamiltonian in Eq. (17) with the aim to compute a holonomy. To do so, we have to consider a cyclic time evolution $\gamma$ or, equivalently, a closed loop $\gamma$ in the parameter space $\mathcal{M}$. The evolution is assumed to be driven adiabatically by the time-dependence of the parameters $\kappa_c = \kappa_c(t)$. The holonomy will be generated in the degenerate dark subspace $\mathcal{H}_D = \text{span}\{|D^\pm\rangle\}$. This is suitable for our computational purposes, as it neglects the uncontrollable dynamical phase $\langle E_D(t) \rangle = 0$ for all $t$. Throughout the dynamical process the parameter $\alpha$ will be assumed to be constant. As we will see this will reduce the computational effort by a lot.

We seek a complete set of single qubit gates, thus ensuring that any pseudo-unitary gate with respect to the metric $\eta$ can be implemented over the dark subspace. It is sufficient to design a pair of non-commuting single-qubit gates $U_1$ and $U_2$. For the gate $U_1$ we choose the parametrization $\kappa_- = 0$, $\kappa_+ = -\kappa \sin(\vartheta/2)e^{i \varphi}$ and $\kappa_0 = \kappa \cos(\vartheta/2)$. In this case, the dark states are

$$|D^1\rangle = |G^+\rangle,$$

$$|D^2\rangle = \cos(\vartheta/2) |G^+\rangle + \sin(\vartheta/2)e^{i \varphi} |G^0\rangle.$$ (18)

The remaining bright states (with eigenvalues $\pm \kappa$) read

$$|B^+\rangle = \frac{1}{\sqrt{2}} \left( \sin(\vartheta/2) |G^+\rangle - e^{i \varphi} \cos(\vartheta/2) |G^0\rangle + e^{i \varphi} |E\rangle \right),$$

$$|B^-\rangle = \frac{1}{\sqrt{2}} \left( \sin(\vartheta/2) |G^+\rangle - e^{i \varphi} \cos(\vartheta/2) |G^0\rangle - e^{i \varphi} |E\rangle \right).$$ (19)

Using the left-handed eigenstates associated to Eqs. (18) and (19) we compute the full metric operator
\[ \eta = \sum_{a=1,2} |D^a\rangle \langle D^a| + |\bar{B}^\dagger\rangle \langle \bar{B}^\dagger| + |\bar{B}^-\rangle \langle \bar{B}^-|, \]

\[ = |\tilde{E}\rangle \langle \tilde{E}| + \sum_{c=0,\pm} |\tilde{G}^c\rangle \langle \tilde{G}^c|. \quad (20) \]

We recognize that as long as \( \alpha \) stays constant, the metric operator \( \eta \) does not depend on the parametrization of \( \mathcal{M} \). In terms of the geometry of the underlying Hilbert space, a change of the parameter \( \alpha \) leads to a contribution of the connection \( K_\mu \). Hence, for \( \alpha = \text{const} \), we have \( K_\mu = 0 \). Thus, the gauge field reduces to

\[ (A_\mu)^{ab} = i \langle \bar{D}^\nu | \partial_\mu |D^\nu \rangle. \]

Evaluating the gauge field with respect to the coordinates \( \lambda^\nu \in \{ \theta, \varphi \} \) of \( \mathcal{M} \), we get \( (A_\gamma)^{\theta \varphi} = -\sin^2(\varphi/2) \) as the only non-vanishing component of \( A \). With this, we can compute the associated holonomy, and express the gate \( U_1(\gamma) \) in terms of the Pauli matrices \( \{ \sigma^x, \sigma^y, \sigma^z \} \) with respect to the basis of dark states \( \{ |D^1\rangle, |D^2\rangle \} \), viz.

\[ U_1(\gamma) = e^{i\beta_1(\gamma)|1\rangle\langle 1|}, \quad (21) \]

where \( \beta_1(\gamma) = -\oint \sin^2(\varphi/2) \theta \, d\theta \, d\varphi \). Note that our computational basis is \( |0\rangle = |D^1(0)\rangle = |G^-\rangle \) and \( |1\rangle = |D^2(0)\rangle = |G^+\rangle \). In Eq. \( (21) \), path-ordering can be neglected, as the chosen parametrization effectively generates an Abelian geometric phase, i.e. the matrix-valued components \( A_\theta \) and \( A_\varphi \) commute.

For the second gate \( U_2 \), we choose the parametrization \( \kappa_0 = \kappa \cos(\varphi), \kappa_- = \kappa \sin(\varphi) \cos(\varphi) \) and \( \kappa_+ = \kappa \sin(\varphi) \sin(\varphi) \), and repeat the previous calculation in a similar fashion, starting with the new dark states

\[ |D^1\rangle = \cos(\varphi) \left[ \cos(\varphi) |G^-\rangle + \sin(\varphi) |G^+\rangle \right] - \sin(\varphi) |G^0\rangle, \]

\[ |D^2\rangle = \cos(\varphi) |G^+\rangle - \sin(\varphi) |G^-\rangle. \]

Together with the associated bright states we obtain in this case the same metric operator as in Eq. \( (20) \). Hence, \( K_\mu = 0 \) as long as \( \alpha \neq \alpha(t) \). We find the components of the gauge field in \( \mathcal{H}_D \) to be

\[ A_\theta = 0, \quad A_\varphi = \cos(\varphi) \sigma^y, \quad (22) \]

so that path-ordering can be neglected again.

The associated holonomy \( U_2 \) to \( A \) is thus given by inserting Eq. \( (22) \) into Eq. \( (16) \). Explicitly we have

\[ U_2(\gamma') = e^{i\beta_2(\gamma') \sigma^y}, \]

where \( \beta_2(\gamma') = \oint \sigma(\varphi) \cos(\theta) d\theta d\varphi \) for a path \( \gamma' \) in \( \mathcal{M} \). From here, one is able to compute the commutator of \( U_1 \) and \( U_2 \) that is

\[ [U_1, U_2] = \sin(\beta_2) \left( 1 - e^{i\beta_1} \right) \sigma^x. \quad (23) \]

In general, Eq. \( (23) \) does not vanish for generic loops \( \gamma \) and \( \gamma' \). Hence, we have found a universal set of pseudo-unitary single-qubit gates on which HQC could be based.

This is the key result of this work. The presented procedure shows how a lossy system, which generates the Hamiltonian for a generic holonomic computation, can be described effectively in the pseudo-Hermitian picture. The range of new applications that could stem from this extension of the theory needs further investigations and is out of the scope of this work.

VI. DISCUSSION AND CONCLUDING REMARKS

In this article we have shown how the holonomic approach to quantum computation can be extended to pseudo-Hermitian systems. We derived a non-Abelian geometric phase generating a pseudo-unitary holonomy over the degenerate eigenspace. The gauge field associated with the non-Abelian phase contains an additional term due to the modified inner product structure induced by a pseudo-Hermitian quantum system, which is absent in conventional quantum mechanics.

This general framework was applied to a benchmark Hamiltonian that can be implemented in terms of a gain/loss system. By choice of a suitable biorthogonal basis the system has the form of a tripod Hamiltonian. An explicit calculation showed that the considered system allows for the implementation of an arbitrary pseudo-unitary transformations over the two-dimensional dark subspace.

Furthermore, we investigated the underlying geometry of this Hamiltonian. In particular, we have shown that the inner product structure could be held constant throughout an adiabatic evolution. This can be done by choosing a suitable loop in the parameter space such that the additional term, appearing in the geometric phase, vanishes. Therefore this loop only changed the coupling between certain tripod levels but does not involve the biorthogonal basis, i.e. the inner product structure, in which the Hamiltonian is represented. Generalized to arbitrary pseudo-Hermitian systems, this enables clear analysis of pseudounitary holonomies and their dependence on the changing inner product structure.

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Appendix A: Transformation law for the gauge field

Here we show that \( A \) indeed transforms like a proper gauge field \( \bar{A} \) under a change of basis \( |\psi^0\rangle = \sum_{i=1}^\alpha U_{ia} |\phi^i\rangle \), where \( U_{ia} \in \mathbb{C} \). The transformation is
mediated by a pseudo-unitary matrix
\[ \mathcal{U}(\lambda) = \sum_{i,j=1}^{n} U_{ij}(\lambda) \langle \phi^i(\lambda) | \phi^j(\lambda) \rangle \in U_\eta(n_0). \]

Here, \( U_\eta(n) \) is the group of \( n \)-dimensional \( \eta \)-pseudo-unitary matrices \cite{34}. We find the usual transformation law
\[ (A_\mu')^{ab} = i \langle \tilde{\psi} | (\partial_\mu - K_\mu) | \psi \rangle = i \sum_{i,j=1}^{n} \langle \phi^i | U_{ia}^* \eta (\partial_\mu - K_\mu) U_{jb} | \phi^j \rangle \]
\[ = \sum_{i,j=1}^{n} \langle \phi^i | U_{ia}^* \eta (\partial_\mu U_{bj}) | \phi^j \rangle + \langle \phi^i | U_{ia} \eta U_{bj} \partial_\mu | \phi^j \rangle - \langle \phi^i | U_{ia} \eta K_\mu U_{bj} | \phi^j \rangle \]
\[ = \sum_{i,j=1}^{n} U_{ia}^* \eta U_{bj} \partial_\mu | \phi^j \rangle + \sum_{i,j=1}^{n} U_{ia} \eta U_{bj} (A_\mu)_{ij}^\dagger \]
\[ \rightarrow \mathcal{U}^{-1} A_\mu \mathcal{U} + \mathcal{U}^{-1} i \partial_\mu \mathcal{U}. \]

**Appendix B: Natural geometric picture of pseudo-Hermitian Hamiltonians**

So far, our treatment of pseudo-Hermitian Hamiltonians did not involve the language of fiber bundles. In conventional QM it is well known that the projector formalism used in HQC involves more advanced concepts such as Grassmann and Stiefel manifolds \cite{35}. To the best of our knowledge, these notions have not been established for pseudo-Hermitian systems yet.

Let us consider a pseudo-Hermitian Hamiltonian \( H \in \text{End}(\mathcal{H}) \), with \( R + 1 \) different eigenvalues, defined over the \( N \)-dimensional Hilbert space \( \mathcal{H} \). Suppose \( H \) has a real spectrum so that its spectral decomposition reads
\[ H = \sum_{i=0}^{R} E_i \Pi_i, \]
where \( \{E_i\}_{i=0}^{R} \) are the eigenvalues corresponding to the pseudo-Hermitian projector \( \Pi_i = \sum_{k=1}^{n_i} |\phi^k_i\rangle \langle \phi^k_i| \) with \( n_i \) being the degeneracy of the \( l \)-th level. The states \( \{|\phi^k_i\rangle\}_{k=1}^{n_i} \) of the \( l \)-th eigenspace of \( H \) form a biorthogonal frame
\[ V^\dagger_l = \sum_{k=1}^{n_i} |\tilde{\phi}^k_i\rangle \langle \tilde{\phi}^k_i| \equiv \sum_{k=1}^{n_i} |\phi^k_i\rangle \langle \phi^k_i|. \]

By construction, we have \( V^\dagger_l V_l = \mathds{1}_{n_l} \), which verifies that the set \( \{|\phi^k_i\rangle\}_{k=1}^{n_i} \) constitutes a biorthogonal basis for the ground-state eigenspace. The set of all biorthogonal frames is called the Stiefel manifold defined by
\[ S_{N,n_l,\eta} = \{ V_l \in \mathbb{C}^{N \times n_l} | V^\dagger_l V_l = \mathds{1}_{n_l} \}. \]

It is noteworthy that the projector \( \Pi_l \) can be expressed in terms of a biorthogonal frame in \( S_{N,n_l,\eta} \) (we have dropped \( \eta \) for ease of notation), i.e. \( \Pi_l = V^\dagger_l V_l \). It is easily checked that the so-defined projector belongs to the Grassmann manifold
\[ G_{N,n_l} = \{ \Pi_l \in \mathbb{C}^{N \times n_l} | \Pi^\dagger_l \Pi_l = \Pi_l, \}
\[ \Pi^\dagger_l = \Pi_l, \text{ tr } (\Pi_l) = n_l. \]

Because the projector is a square matrix, its pseudo-adjoint is defined in the usual sense \cite{2}, \( \Pi^\dagger_l = \eta^{-1} \Pi^\dagger_l \eta \).

We are now in a position to illuminate the gauge freedom within the projector \( \Pi_l \). More precisely, we can define a projection \( \pi \) from the Stiefel manifold to the Grassmann manifold by \( V_l \mapsto V_l^\dagger \). It is not hard to show that the image of this map stays invariant under a group action by a pseudo-unitary matrix \( \mathcal{U} \in U_\eta(n_l) \),
\[ \pi(V_l \mathcal{U}) = (V \mathcal{U})(V \mathcal{U})^\dagger = V \mathcal{U} \mathcal{U}^\dagger V^\dagger = V V_l^\dagger, \]
where we applied the useful relation

\[(VU)^\dagger = \eta^{-1}(VU)^\dagger \eta = \eta^{-1}U^\dagger \eta U \eta^{-1} V^\dagger U^\dagger V^\dagger.\]

In conclusion we have constructed a \(U_n(n_l)-\)principal bundle, that is,

\[S_{N,n_l} \xrightarrow{\pi} G_{N,n_l}. \tag{B2}\]

The bundle structure, Eq. (B2), is a direct generalization of the one found in conventional QM (for a review, see e.g. [34]). The standard theory is recovered for \(\eta = \mathbb{1}_N\). It is therefore not surprising that the Stiefel manifold can be written, in analogy to their counterparts in conventional QM, as a coset space, i.e.

\[G_{N,n_l} \cong S_{N,n_l}/U_{n_l}(n_l).\]

Note how, for \(n_l = 1\) (i.e. a non-degenerate situation), the Grassmann manifold reduces to the projective Hilbert space containing the pseudo-Hermitian density operators for a pure state, i.e. \(|\phi\rangle \langle \phi| \in G_{N,1} \cong CP^{N-1}\). The structure group of this principal bundle is \(U_n(1)\) which is identical to the conventional unitary group \(U(1)\).

We conclude this section by recalling that it is rather demanding for a parameter space \(\mathcal{M}\) to be mapped one-to-one (bijectively) onto \(G_{N,n_l}\). In other words, a realistic quantum system, given by a family \(\{H(\lambda)\}_{\lambda \in \mathcal{M}}\) of iso-spectral pseudo-Hermitian Hamiltonians, may have a smaller control manifold than the whole Grassmann manifold. Nevertheless, there is a map \(\Phi\) from \(\mathcal{M}\) onto \(G_{N,n_l}\) defined by \(\Phi(\lambda) = \Pi_{\lambda}\). A natural way to study the geometry of such systems is given in terms of the pullback bundle of the Stiefel manifold

\[\Phi^* S_{N,n_l} \xrightarrow{\pi_{\Phi}} \mathcal{M}, \tag{B3}\]

In order to construct the rest of the bundle structure of Eq. (B3), we can establish the fiber \(\mathcal{F}_\lambda\) of \(\Phi^* S_{N,n_l}\) over the point \(\lambda\) in \(\mathcal{M}\). This fiber is just a copy of the fiber \(\mathcal{F}_{\Phi(\lambda)}\) defined over the projector (point in \(G_{N,n_l}\)) \(\Pi_{\lambda}\). The latter is formally defined as the preimage of the projection \(\pi(V_l) = \Pi_{\lambda}\), that is, \(\mathcal{F}_{\Phi(\lambda)} \cong \pi^{-1}(\Pi_{\lambda})\). Then

\[\Phi^* S_{N,n_l} \xrightarrow{\pi_{\Phi}} \mathcal{M},\]

where \(\pi_{\Phi} : (\lambda, V_l) \mapsto \lambda \in \mathcal{M},\) constitutes a \(U_{n_l}(\lambda)(n_l)\)-principal fiber bundle. By construction, the sections of this bundle are just \(\lambda \mapsto (\lambda, V_l)\).

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