Brane-like States in Superstring Theory
and the Dynamics of non-Abelian Gauge Theories

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Abstract
We propose a string-theoretic ansatz describing the dynamics of SU(N) Yang-Mills theories in the limit of large N in $D = 4$. The construction uses in a crucial way brane-like states, open-string non-perturbative states described by vertex operators of non-trivial ghost cohomologies. As we have discussed in previous papers, these physical vertex operators do not account for particle states in the open-string spectrum, but rather have to do with the dynamics of extended objects such as Dp-branes. In this paper, we show their relevance to the dynamics of gluons. We show that this string-theoretic ansatz enjoys worldsheet reparametrization invariance and chirality in loop space, satisfies the loop equations and the criterium for quark confinement. According to our proposal, various gauge theories are described by string theories with the same classical action, but with different measures in the functional integral. The choice of measure defines the gauge group, as well as the effective space-time dimension of the resulting gauge theory.

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1. Introduction

Quark confinement in non-Abelian gauge theories is one of the most intriguing problems in modern physics, unsolved up to now. While for gauge theories with compact $U(1)$ gauge group an elegant explanation of the confining mechanism has been found \[^{[1]}\], it is still a mystery how confinement appears in gauge theories with essentially non-Abelian gauge groups. In his pioneering work K.Wilson \[^{[2]}\] has shown that confinement does occur in non-Abelian theories on the lattice, yet unfortunately his method cannot be applied to address the problem directly in the continuum limit. The difficulty has largely to do with our lack of understanding of the dynamics of non-Abelian gauge theories in general. The straightforward analysis of these non-linear theories is extremely difficult. One particularly promising and potentially powerful approach to the problem is to relate it to the dynamics of extended objects, such as strings and p-branes (see, for example, \[^{[3,4,5,6,7,8,9]}\], for the original as well as for the modern context of this idea) As for the string-theoretic approach, there are two basic difficulties with it. The first, and perhaps the less significant one, has to do with the fact that we still do not know how to extract gauge theories with realistic gauge groups from superstring theory. The second, and perhaps the more fundamental one is that the “internal” gauge symmetries of Yang-Mills theories and string theory are significantly different: in string theory there is a worldsheet reparametrization invariance under diffeomorphisms, i.e. under coordinate transformations with a positive Jacobian, while in Yang-Mills theories the reparametrizational invariance of the contour, which defines the Wilson loop, includes the transformations which change the orientation of the contour, i.e. those with a negative Jacobian as well. In other words, the reparametrization invariance in gauge theories is extended and is not restricted to diffeomorphisms \[^{[8]}\]. Therefore any string theory that possibly may describe the dynamics of gauge theories must be a non-standard one, i.e. it must have this special property of extended reparametrization invariance. Examples of such string theories without gravity were proposed in \[^{[3]}\]; however the problem to find a string-theoretic ansatz describing gauge theories with particular realistic gauge groups still remains unsolved. In this paper, we propose and study the string-theoretical ansatz for $SU(N)$ gauge theories in $D = 4$, explore the confining properties in these theories and attempt to explain the interplay between our approach and the idea of elucidating the properties of gauge theories from brane dynamics. The crucial element in our construction will be to observe the existence of a peculiar class of soliton-like physical states in the spectrum of open strings, described by vertex operators of non-trivial ghost cohomologies \[^{[10]}\] and the relevance of some of them to the dynamics
of gluons. Let us recall the criteria necessary for confinement in gauge theories. Let \( A_m, m = 1, \ldots, D \) be the vector potential in some Yang-Mills theory in \( D \) dimensions. Let us define the expectation value of the Wilson loop:

\[
W(C) = \langle \Psi(C) \rangle = \langle e^{\oint_C A_m dX^m} \rangle
\]

where \( C \) is the contour defining the loop, and \( X^m \) is a set of space-time coordinates. The necessary condition for the confinement is that for large enough contours \( W(C) \) vanishes exponentially with the area \( A \) of the contour:

\[
W(C) \sim e^{-\lambda A}
\]

In the approach that we are going to develop in this paper, we aim to present a string-theoretical ansatz for the SU(\( N \)) Yang-Mills theories in the limit of large \( N \) in \( D = 4 \) by including the set of non-perturbative open string states (which we call brane-like states because of their apparent p-brane or Dp-brane nature) in the conventional superstring theory. This peculiar class of states is described by vertex operators of essentially non-zero ghost numbers (which cannot be removed by a picture-changing gauge transformation). Their S-matrix elements between elementary open string states vanish, however these operators may play an important role in the non-perturbative dynamics of branes [10]. Examples of these brane-like vertex operators are given by:

\[
Z_{mn}(k) = e^{-2\phi} \psi_m \psi_n e^{ikX}
\]

and

\[
Z_{m_1 \ldots m_5}(k) = e^{-3\phi} \psi_{m_1} \ldots \psi_{m_5} e^{ikX}
\]

As follows from superalgebraic arguments the zero-momentum parts of these vertex operators are related to the topological Page charges of a membrane and a fivebrane respectively. However, there is another way to understand their origin. Namely, consider the following Ramond-Ramond vertex operator:

\[
V_{RR} = e^{-\frac{3}{2} \phi} \Sigma^\alpha(z) \Gamma_{\alpha\beta}^{m_1 \ldots m_p} F_{m_1 \ldots m_p} e^{-\frac{1}{2} \tilde{\phi} \Sigma^\beta} e^{ikX}
\]

In the presence of a D-brane this vertex operator is defined on the disc (or, equivalently, on the half-plane), therefore its holomorphic and anti-holomorphic parts are no longer independent of each other, because of the boundary. For this reason, when the vertex
operator (5) approaches the "edge" of the D-brane, i.e. the boundary where \( z = \bar{z} \), internal normal ordering must be performed between \( \Sigma \) and \( \bar{\Sigma} \), as well as inside the ghost part. As a result of such an internal normal ordering the vertex operator (3) will appear. This is somewhat similar to the situation in the NS sector where internal normal ordering between \( \partial X \) and \( \bar{\partial} X \) in the vertex operator of the graviton leads to the non-standard dilaton term \( \sim \int d^2 z R^{(2)}(\phi(X)) \) in the \( \sigma \)-model action. The vertex operators (3),(4) have to do with the dynamics of branes, but, as we shall argue in this paper, they are also relevant to the dynamics of gluons. The idea is that the zero momentum part of the membrane vertex operator (3) defines, at the same time, the field strength \( F_{mn} \) in a certain non-Abelian gauge theory with some (yet unknown) gauge group:

\[
F_{mn}(z) \equiv \oint dw \frac{e^{-2\phi}}{2i\pi} \psi_m \psi_n(z + w) \tag{6}
\]

where the dependence on the holomorphic coordinate \( z \) defines the dependence on the parameter of the Wilson line. In order to show that \( F_{mn} \) defines a meaningful gauge theory and to find a gauge group to which it corresponds, we have to carry out the following steps:

1) Propose a suitable definition of the Wilson loop expectation value in the string-theoretic approach;

2) Show that our string-theoretic ansatz has reparametrization invariance and satisfies the loop space analogue of the Yang-Mills equations;

3) Show that it satisfies the loop equation for non-Abelian gauge theories;

4) Find the gauge group analyzing the structure of the loop equation satisfied by proposed superstring ansatz;

5) Discuss the relevance to \( D = 4 \) in view of the fact that our original superstring theory is ten-dimensional.

Let us now begin to implement the above program. First of all, the Wilson line \( \Psi(C) \) is determined by the expression:

\[
\Psi(C) = e^{g \oint_C A_m dX^m} \approx e^{g \int_C dz \bar{z} F_{mn} d\sigma^{mn}} = e^{g \int d^2 z \theta(C) \partial X^m \bar{\partial} X^n(z,\bar{z}) \oint dw \frac{e^{-2\phi}}{2i\pi} \psi_m \psi_n(z + w)} \tag{7}
\]

Here \( \theta(C) \) is a step function which is equal to 1 inside the contour \( C \) and equal to zero outside, and \( g \) is a coupling constant related in a certain way to the bare coupling constant of Yang-Mills theory; the precise form of this relation will be discussed below. Before giving the definition of the expectation value of the Wilson line \( W(C) \), we would like to comment on the reason why such an average should be computed using the standard NSR
superstring action (since usually $\Psi(C)$ should be averaged with respect to the Yang-Mills action $\sim \int F_{mn} F^{mn}$). The justification is that for the $F_{mn} = \oint dz (z - w)^2 i \pi e^{-2\phi} \psi^m \psi^n (z)$ of (6) we have the identity:

$$: \Gamma^4 e^{-2\phi} \psi^m \psi^n e^{-2\phi} \psi^m \psi^n : (z) = T_{NSR}(z)$$

where $T_{NSR}$ is the stress-energy tensor in the NSR superstring theory, $\Gamma^4$ is the normal ordered fourth power of the picture-changing operator:

$$\Gamma = : e^\phi (G_{\text{matter}} + G_{\text{ghost}}) := - \frac{1}{2} e^\phi \psi^m \partial X_m + \text{ghosts}$$

and $G_{\text{matter}}$ and $G_{\text{ghost}}$ are the matter and ghost worldsheet supercurrents. The average of the Wilson loop $W(C)$ is then defined as:

$$W(C) = \langle e^{\int d^2 z (C) \partial X^m \bar{\partial} X^n (z, \bar{z})} \oint dw \frac{dw}{2i\pi} e^{-2\phi} \psi^m \psi^n (z + w) \rangle_{f(\Gamma)}$$

$$= \int DXD\psi D[\text{ghosts}] f(\Gamma) e^{\int d^2 z \partial X^m \bar{\partial} X^n (\eta_{mn}) + g(C) \oint dw \frac{dw}{2i\pi} e^{-2\phi} \psi^m \psi^n (z + w) + \psi^m \bar{\partial} \psi^m + \bar{\psi}^m \partial \bar{\psi}^m + S_{\text{ghost}}}$$

Here $f(\Gamma)$ is some function of the picture-changing operator, regulating the total ghost numbers of the correlation functions that appear in the expansion in the gauge coupling constant $g$. This ghost number regulator is needed to make the expression (10) for the average $W(C)$ meaningful. Without it, because of ghost number conservation, the only non-vanishing correlator on the sphere appearing in the expansion in $g$ would be the one-point function (which is zero anyway). Therefore the presence of the ghost number regulator $f(\Gamma)$ in the measure is crucial. Moreover, as we will find out, it is the precise form of this regulator that determines the gauge group in the corresponding gauge theory. Namely, the proper choice of the measure function $f(\Gamma)$ will ensure that the string-theoretical ansatz (10) for the Wilson loop $W(C)$ satisfies the dynamical equations in the loop space for some gauge theory with a gauge group $G$. The gauge group will be determined from the structure of the loop equations for $W(C)$. At the same time, a suitable choice for the measure function $f(\Gamma)$ will enable us to effectively truncate the expansion in the gauge coupling constant $g$, leaving only a finite number of terms in the series. Therefore the constant $g$ does not have to be small and the expansion is in principle non-perturbative. Consider a measure function of the type

$$f(\Gamma) = \sum_n a(n) : \Gamma^n :$$

(11)
where the coefficients $a(n)$ defining the gauge group will be determined later. Expanding $W(C)$ in $g$ we obtain:

$$W(C)|_{f(\Gamma)} = \sum_n \frac{1}{n!} g^n < \theta(C) \prod_{k=1}^n \int d^2z_k \oint \frac{dw_k}{2i\pi}$$

$$< \partial X^{m_k} \partial X^{n_k}(z_k, \bar{z}_k) : e^{-2\phi} \psi_{m_k} \psi_{n_k} : (z_k + w_k) >_{f(\Gamma)}$$

(12)

where $f(\Gamma)$ implies that the correlation function is computed with the appropriate insertion in the functional integration. On the sphere, due to the ghost number conservation, for every given $n$ in the sum (12) the measure function $f(\Gamma)$ of (11) may be replaced with $f(\Gamma) \to a(n-1) : \Gamma^{4n-2} :$. Then, ignoring $\theta(C)$ in the limit of large $C$, we find the following expression for the average $W(C)$:

$$W(C)|_{f(\Gamma)} = \sum_n \frac{a(n-2)g^n}{n!} \prod_{k=1}^n \int dz_k d\bar{z}_k \oint \frac{dw_k}{2i\pi}$$

$$< : \Gamma^{2n-2} : \partial X^{m_k} \partial X^{n_k}(z_k, \bar{z}_k) : e^{-2\phi} \psi_{m_k} \psi_{n_k} : (z_k + w_k) >$$

(13)

Using the independence of the correlators on the locations of the picture-changing operators as well as the fact that $\lim_{z \to w} : \Gamma^2(z) e^{-2\phi} \psi_{m} \psi_{n} : (w) \sim \psi_{m} \psi_{n}(w) + \text{ghosts}$ the computation of $W(C)$ in the limit of large contours gives

$$W(C)|_{f(\Gamma)} = \sum_n \frac{a(2n-2)g^n}{n!} \prod_{k=1}^n \int dz_k d\bar{z}_k \oint \frac{dw_k}{2i\pi}$$

$$\{ \prod_{i_1, j_1=1}^n \frac{\eta^{m_{i_1} m_{j_1}} \eta^{n_{i_1} n_{j_1}}}{(z_{i_1} - z_{j_1})^2 (\bar{z}_{i_1} - \bar{z}_{j_1})^2 (\bar{w}_{i_1} - \bar{w}_{j_1})^2} + \text{permutations} \}$$

(14)

where $\eta_{mn}$ is the Minkowski metric and $\bar{w}_{i_1} \equiv z_{i_1} + w_{i_1}$. To compute the expression (14) let us first evaluate the integral

$$I = \int dz_i d\bar{z}_i \int dz_j d\bar{z}_j \oint \frac{dw_i}{2i\pi} \oint \frac{dw_j}{2i\pi} \frac{1}{(z_i - z_j)^2 (\bar{z}_i - \bar{z}_j)^2 (\bar{w}_i - \bar{w}_j)^2}$$

(15)

Changing variables according to:

$$u_1 = z_i - z_j + w_i$$

$$u_2 = z_i + z_j$$

$$u_3 = w_i$$

$$u_4 = w_j$$

(16)
for a given $i$ and evaluating the contour integrals in (15) we obtain the following result

$$I \sim 2A \left( \int_{-\infty}^{\infty} du \frac{\delta(u)}{u} - \left[ \int_{-\infty}^{\infty} du (\delta(u))^2 \right]^2 \right)$$  \hspace{1cm} (17)$$

where $A$ is the area of the contour $C$. The divergent integrals in (17) must be regularized. The regularization scheme is the standard one, based on the Fourier representation for the delta-function:

$$\delta(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp e^{ipu}$$  \hspace{1cm} (18)$$

and the identity

$$\int_{0}^{\infty} du e^{ipu} = \lim_{\epsilon \to 0} \int_{0}^{\infty} du e^{(ip-\epsilon)u} = \frac{i}{p}$$  \hspace{1cm} (19)$$

Then

$$\int_{-\infty}^{\infty} du (\delta(u))^2 = \frac{2}{4\pi^2} \int_{0}^{\infty} du \int dp_1 dp_2 e^{i(p_1+p_2)u}$$

$$= \frac{i}{2\pi^2} \int dp_1 dp_2 \frac{1}{p_1 + p_2} \sim \frac{i}{\pi^2} \Lambda(\ln\Lambda - \ln\alpha)$$  \hspace{1cm} (20)$$

where $\Lambda$ is the ultraviolet momentum cutoff and $\alpha$ is the momentum corresponding to the inverse size of the contour $C$. For contours $C$ such that $|ln\alpha| \ll |ln\Lambda|$ that is, for contours small enough compared to the absolute value of the cutoff, but still large enough to ignore the boundary effects in the correlators (12),(13), the term with $ln\alpha$ may be ignored in the last formula. Analogously, one may show that

$$\int_{-\infty}^{\infty} du \frac{\delta(u)}{u} \sim \frac{1}{\pi} \Lambda(\ln\Lambda - 1)$$  \hspace{1cm} (21)$$

We see that for $\Lambda \to \infty$ the second divergent term in the expression (17) dominates, and therefore the result of the regularization is given by:

$$I \sim \frac{2}{\pi^4} A(\ln\Lambda)^2$$  \hspace{1cm} (22)$$

Substituting (22) into the expansion (14) for $W(C)$ and taking the permutations into account we obtain:

$$W(C) \big|_{f(\Gamma)} \sim \sum_{n} \frac{a(n-2)(n-1)!g^n}{n!} [Ah(\Lambda)]^n$$  \hspace{1cm} (23)$$

where $h(\Lambda) \sim (\ln\Lambda)^2$ is the cutoff function resulting from the regularization (22). It is not difficult now to choose the coefficients $a(n)$ in (23) so that $W(C) \big|_{f(\Gamma)}$ will have the
proper behaviour (2) of the confining phase of large $N$ Yang-Mills. The crucial restriction, however, is that these coefficients must be chosen so that $W(C)$ satisfies the loop equations, i.e. the dynamical equations of Yang-Mills theory with some gauge group, written in the space of loops. If this requirement is satisfied, $W(C)|_{f(\Gamma)}$ with the given choice of $a(n)$ can indeed be identified with the desired gauge variable, i.e. it will define the average of the Wilson loop corresponding to some non-Abelian theory. The resulting gauge group will be determined from the structure of loop equations. A priori, it is far from obvious that such a choice is possible. Nevertheless, we shall show that it does really exist. Namely, we shall argue that the following choice of the measure function $f(\Gamma)$:

$$f(\Gamma) \equiv f(\Gamma, \Lambda) = \frac{(1 + \Gamma^2)}{N(\ln(\Lambda))^2} e^{-\frac{\Gamma^4}{N(\ln(\Lambda))^2}} \equiv N e_R^4 (1 + \Gamma^2) e^{-N(e_R \Gamma)^4}$$  \hspace{1cm} (24)$$

in the limit of large $N$ connects the $D = 10$ string theory to a description of the dynamics of $SU(N)$ Yang-Mills theory in $D = 4$. In the last equation $e_R$ stands for the renormalized Yang-Mills coupling constant in the $SU(N)$ theory. The effective change of space-time dimension will be attributed to the insertion of $f(\Gamma)$ in the measure of integration, that corresponds to some configuration of branes (or brane-like states). Let us now return to the choice (24) of the measure function $f(\Gamma)$. First of all, it is not difficult to show that with this choice the average $W(C)$ does have the behaviour (2) corresponding to the confining phase. The substitution of (24) into (23) gives:

$$W(C)|_{f(\Gamma)} = \frac{(1 + \Gamma^2)}{N(\ln(\Lambda))^2} e^{-\frac{\Gamma^4}{N(\ln(\Lambda))^2}} \equiv N e_R^4 (1 + \Gamma^2) e^{-N(e_R \Gamma)^4}$$  \hspace{1cm} (25)$$

Note that the ultraviolet cutoff dependence of this result is the one predicted by the lattice calculations for $SU(N)$ gauge theories in $D = 4$ [3]. Now we have to show that $W(C)$ with the choice (24) of measure function is indeed the dynamical variable of the large $N$ limit of a $SU(N)$ gauge theory, i.e. that it satisfies the loop equations corresponding to the $SU(N)$ gauge group:

$$\frac{\partial^2 W(C)}{\partial X^2(z, \bar{z})} = -e^2 \oint dY e^{\int dY m(\delta(X(z) - Y)[W(C_1, C_2) - \frac{1}{N} W(C)]\delta X_m(z, \bar{z})}$$  \hspace{1cm} (26)$$

Here $C$ is the contour with one self-intersection; the self-intersection point divides into two contours $C_1$ and $C_2$ and $W(C_1, C_2) \equiv < \Psi(C_1) \Psi(C_2) >$ and the second derivative with respect to $X$ is defined as follows:

$$\frac{\partial^2 W(C)}{\partial X^2(z, \bar{z})} \equiv \lim_{\epsilon \to 0} \int_{|\alpha| < \epsilon} d^2 \alpha \frac{\delta W(C)}{X^m(z)} \frac{\delta W(C)}{X_m(z + \alpha)}$$  \hspace{1cm} (27)$$

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Before doing that, however, let us show first that the classical variable $\Psi(C)$ of (7) satisfies the following relations

$$\partial X^m(z) \frac{\delta \Psi(C)}{\delta X^m(z)} = 0,$$

equivalent to the reparametrization invariance, together with

$$\frac{\delta^2 \Psi(C)}{\delta X^m(z) \delta X^n(w)} - \frac{\delta^2 \Psi(C)}{\delta X^n(z) \delta X^m(w)} + \left[ \frac{\delta \Psi(C)}{\delta X^m(z)} \frac{\delta \Psi(C)}{\delta X^n(w)} \right] = 0,$$

chirality in the loop space; and finally

$$\frac{\delta}{\delta X^m(z)} \left( \frac{\delta \Psi(C)}{\delta X_m(z)} \Psi^{-1}(C) \right) = 0,$$

Yang-Mills equations written in the loop space formalism. Let us now prove that the superstring ansatz (7) for $\Psi(C)$ satisfies all these identities.

For convenience, let us denote

$$L_m(z,C) \equiv \frac{\delta \Psi(C)}{\delta X_m(z)}.$$

On the other hand, one may show that on-shell, due to the NSR equations of motion, one has

$$\bar{L}_m(z,\bar{z}) = 0.$$

The proof of (28) follows from

$$\bar{\partial} X^m L_m = g \{ \lim_{u \to z} \frac{\eta_{mn}}{(\bar{z} - \bar{u})^2} + : \bar{\partial} X^m \bar{\partial} X^n : (z, \bar{z}) \} : F_{mn}(X(z)) \Psi(C) : = 0$$

since in the last formula $F_{mn}$ is multiplied by an expression that is symmetric in the indices $m$ and $n$. Next, the condition (29) of chirality in the loop space is fulfilled due to the fact that

$$\frac{\delta L_m(z,C)}{\delta X^n(w)} - \frac{\delta L_n(z,C)}{\delta X^m(w)} = g^2 \bar{\partial} X_\rho(\bar{z}) \bar{\partial} X_\sigma(\bar{w}) : [F_{n\sigma}(X(w)), F_{m\rho}(X(z))] \Psi(C) : = -[L_m, L_n]$$

Finally, the loop space analogue (30) of the Yang-Mills equations is satisfied by the ansatz (7), because

$$\frac{\delta}{\delta X^m(z)} \left( \frac{\delta \Psi(C)}{\delta X_m(z)} \Psi^{-1}(C) \right) = \frac{\delta L_m(z,C)}{\delta X^m(z)} \Psi^{-1}(C) + L^m(z,C) \frac{\delta \Psi^{-1}(C)}{\delta X^m(z)}$$

$$= -g^2 \{ \lim_{\bar{u} \to \bar{z}} \frac{\eta^{\rho\sigma}}{(\bar{z} - \bar{u})^2} + : \bar{\partial} X^\rho \bar{\partial} X^\sigma : (z, \bar{z}) \} \times$$

$$\times : \{ [F_{m\rho}(X(z)), F_{n\sigma}(X(u))] - [F_{m\rho}(X(z)), F_{m\sigma}(X(u))] \} \Psi(C) \Psi^{-1}(C) : = 0.$$
This ends the proof of (28)-(30). We thus see that the expression (7) for $\Psi(C)$ indeed corresponds to the dynamical variable in some classical gauge theory, i.e. to the Wilson loop in some Yang-Mills theory (with a gauge group yet to be identified). Now we have to show that, for the choice (24) of the measure function $f(\Gamma)$ the superstring ansatz (10) describes the gauge theory dynamics in the quantum sense as well; that is, that the average (10) of $\Psi(C)$ corresponds to the expectation value of the Wilson loop in the $D = 4$ $SU(N)$ Yang-Mills theory in the limit of large $N$. To do this we first of all have to show that the expression (10) for $W(C)$ with the choice (24) for the measure function $f(\Gamma)$ satisfies the loop equation (26) for $SU(N)$ gauge theories. Let us compute the second derivative of $W(C)$. According to the definition (27), we have:

$$\lim_{\epsilon \to 0} \int_{|\alpha|<\epsilon} d^2 \alpha < \partial X_\rho(\bar{z}) \partial X_\sigma(\bar{z} + \bar{\alpha}) : F_{n\sigma} F_{m\rho} : (X(z)) \Psi(C) >_{f(\Gamma)}$$

Unfortunately, since the ansatz (6) for the field strength has an essentially non-zero ghost number, differentiation with respect to $X$ inevitably changes the “balance of ghosts” in the correlation functions. In order to compensate this change the differentiation must be accompanied by the appropriate picture-changing transformation. Thus, the correct physical variable to consider is $\Gamma^4 \frac{\partial^2 W(C)}{\partial X^2(z, \bar{z})} :_{f(\Gamma)}$, rather than simply $\frac{\partial^2 W(C)}{\partial X^2(z, \bar{z})} :_{f(\Gamma)}$. Expanding $\Psi(C)$ in $g$ in a way similar to (13), using the identity: $\partial X(z) \bar{\partial} X(\bar{w}) \sim \delta^2(z - w)$ and multiplying both sides of the last equation by $\Gamma^4$ we obtain:

$$\Gamma^4 \frac{\partial^2 W(C)}{\partial X^2(z, \bar{z})} := \lim_{\epsilon \to 0} \int_{|\alpha|<\epsilon} d^2 \alpha \int d^2 u_j \delta^2(z + \alpha - u_j) \eta^{\sigma m_j} \bar{\partial} X_\rho(\bar{z}) \partial X^{n_j}(\bar{u}_j)$$

$$\times \Gamma^4 f(\Gamma) F_{m_jn_j}(X(u_j)) F_{m\sigma}(X(z + \alpha)) F_{m\rho}(X(z))$$

$$\{ \sum_{n} \frac{1}{n!} \prod_{k=1, k \neq j}^{n} \int dz_k d\bar{z}_k \int \frac{dw_k}{2i\pi} \partial X^{m_k} \bar{\partial} X^{n_k}(z_k, \bar{z}_k) e^{-2\phi} \psi_{m_k} \psi_{n_k}(z_k + w_k) \} > + ...$$

where we have dropped terms that vanish as $\epsilon \to 0$, i.e. only the terms with $\alpha$ in the argument of the delta-function are retained. Evaluation of the integral over $\alpha$ then leads to the following expression for the second derivative of $W(C)$ with the measure function.
the expression for the two-point correlation function: 

\( \Gamma^4 \frac{\partial^2 W(C)}{\partial X^2(z)} = \)

\[ g^2 < \sum_j \int d\bar{u}_j \delta(X(\bar{u}_j)) - X(\bar{z})) \partial X_{\rho}(\bar{z}) \partial X_{n_j}(\bar{u}_j) \eta_{n_j, \rho} : F_{m\sigma} F_{m\sigma} : (X(z)) \]

\[ \times \{ \frac{1}{n!} \prod_{k=1}^{n-1} \int d^2 z_k \oint \frac{d\omega_k}{2i\pi} \Gamma^2 f(\Gamma) : \partial X^{m_k} \partial X^{n_k}(z_k, \bar{z}_k) e^{-2\phi} \psi_{m_k} \psi_{n_k}(z_k + w_k) \} = \]

\[ \frac{2Tr(\eta) g^3}{N(ln\Lambda)^2} \int dY \partial X_{\rho}(\bar{z}) \delta(Y - X(\bar{z})) \sum_{a,b=1}^{n-1,n} \int dz_a \bar{d}z_a \int dz_b \bar{d}z_b (\delta(2)(z_a - z) \delta(2) \times [(\delta(2)(z_a - z_b))^2 - \frac{\delta(2)(z_a - z_b)}{z_a - z_b}] \]

\[ \times \frac{g^{n-2}}{[N(ln\Lambda)^2]^{n-2} n!} \prod_{k=1}^{n-2} \int d^2 z_k \oint \frac{d\omega_k}{2i\pi} \partial X_{m_k} \partial X_{n_k} e^{-2\phi} \psi_{m_k} \psi_{n_k} > \}

(37)

where: \( F_{m\sigma} F_{m\sigma} : (X(z)) \equiv \sum \int d\omega \oint \frac{d\omega}{2i\pi} e^{-2\phi} \psi_{m} \psi_{\sigma}(z + w_1) e^{-2\phi} \psi_{m} \psi_{\sigma}(z + w_2) \) and we have used the O.P.E. rule: \( \Gamma^2 F_{m_1 n_1} : (u) F_{m_2 n_2}(z) \sim \delta(2)(u - z) \eta_{m_1 m_2} F_{n_1 n_2}(z) + ... \) and the expression for the two-point correlation function:

\[ < \Gamma^2 \partial X^a \partial X^b F_{ab}(z, \bar{z}) \partial X^c \partial X^d F_{cd}(u, \bar{u}) > = (Tr\eta)^2 (\delta(2)(z - u))^2 - \frac{\delta(2)(z - u)}{z - u} (38) \]

To see that the loop equation (26) is satisfied we have to show that the right-hand side of (37) is identical to the right-hand side of (26). Let us therefore compute \( W(C_1, C_2) \), with the measure function (24). We have:

\[ W(C_1, C_2) = < \Gamma^2 e^{\theta(C_1)} \int d^2 z_1 \partial X^{m_1} \partial X^{n_1} F_{m_1 n_1}(z_1, \bar{z}_1) e^{\theta(C_2)} \int d^2 z \partial X^{m_1} \partial X^{n_1} F_{m_2 n_2}(z_2, \bar{z}_2) > = \]

\[ = \sum \gamma^{n+m} n! m! \Gamma^2 f(\Gamma) \prod_{k,l=1}^{n,m} \int d^2 z_k \partial X^{m_k} \partial X^{n_k} F_{m_k n_k}(z_k, \bar{z}_k) \]

\[ \times \int d^2 z_i \partial X^{m_i} \partial X^{n_i} F_{m_i n_i}(z_l, \bar{z}_l) >= \]

\[ = \frac{2g}{N(ln\Lambda)^2} \sum_m \frac{1}{n!} \prod_{k=1}^{n} \sum_{l=1}^{n} \int d^2 u \int d^2 z_k \delta(2)(z - u)((\delta(2)(u - z))^2 - \frac{\delta(2)(u - z_a)}{u - z_a}) f(\Gamma) \gamma^{n+1} n! \partial X^{m_k} \partial X^{n_k} F_{m_k n_k}(z_k, \bar{z}_k) + ... \]

(39)
where we have dropped the terms which become small in the limit of the large N (with the choice (24) of the measure function $f(\Gamma)$). We have also used of the fact that in the terms proportional to $\sim \theta(C_1)\theta(C_2)$ in (39) the interaction between two given vertices located at the different halves $C_1$ and $C_2$ of the contour $C$ occurs only when one of the vertices passes through the self-intersection point - this is the origin of the delta-function $\delta^{(2)}(z-u)$ in (39). Comparing the expression (39) for $W(C_1, C_2)$ with the right-hand side of (37) we see that, up to a picture-changing transformation that restores the correct ghost balance, the second derivative of $W(C)$ is equal to

$$
\Gamma^4 \frac{\partial^2 W(C)}{\partial X^2(z)} = g^2 \int dY^\rho \bar{\rho} X_\rho \delta(Y - X(z)) W(C_1, C_2)
$$

(40)

The loop equation (26) for the SU(N) Yang-Mills theory is thus satisfied by the superstring ansatz (10) with the measure function (24) in the limit $N \to \infty$. The important question, however, is why the gauge theory described by the ansatz (10) is four-dimensional, while the original string theory is in $D = 10$. Our conjecture is that introducing the function $f(\Gamma)$ in the measure of the functional integration effectively changes the dimension of the space-time in which the theory lives. First of all, note that the cutoff dependence of $W(C)$ in (25) is the same as in four dimensions, obtained from the computation on the lattice [2]. The reason for the change of space-time dimensionality may be understood as follows. The inclusion of $f(\Gamma, \Lambda)$ in the measure is equivalent to introducing boundaries (or a curvature singularity) on the worldsheet which, in turn, leads to brane-like effects. Thus the inclusion of the brane-like states (3), (4) along with some particular choice of $f(\Gamma, \Lambda)$ is equivalent to a certain configuration of branes in space-time. Some of such brane configurations give rise to the gauge theories in lower space-time dimensions. There is another, even more heuristic, way to understand the conjectured reduction of the space-time dimensionality. That is, because of the ghost number anomaly cancelation condition the choice of $f(\Gamma, \Lambda)$ imposes strong restrictions on the possible worldsheet topologies that appear in the expansion in $g_{string}$. For instance, the choice (24) of the measure function in the limit of large N effectively suppresses the contributions from all non-trivial topologies, allowing us to consider the correlation functions on the sphere only. As a result, the set of all possible loop configurations is reduced as well. In turn, as a result of such a reduction, the loop space including the loop configurations in the $D = 10$ space-time is reduced to that of loop configurations in the space-time of a lower dimension. To deduce the effective dimensionality of the space-time for a given $f(\Gamma, \Lambda)$ one has to compute the average of
the Wilson loop $W(C)_{f(\Gamma)}$ provided that it satisfies the loop equation, otherwise the given choice of the measure function cannot be related to any physical gauge theory). Then the cutoff dependence of the result must be compared to the dependence on the ultraviolet cutoff parameter in the corresponding gauge theory in $D$ dimensions, that follows from the computations on the lattice. Roughly speaking, given the measure function $f(\Gamma, \Lambda)$, the dependence on $\Gamma$ determines the gauge group, while the dependence on $\Lambda$ fixes the effective space-time dimensions of the theory. This statement is not quite precise, of course, since the dependence on $\Gamma$ plays a role in fixing the effective space-time dimension as well. According to the behaviour (25) of the expectation value of the Wilson loop $W(C)$, the measure function (24) presumably describes the confining phase of SU(N) Yang-Mills in four dimensions. It is also possible to choose the measure function $f(\Gamma)$ so that $W(C)$, while still satisfying the large $N$ limit of the loop equation (26), would behave like $W(C) \sim e^{-\lambda \Lambda^2 A^2}$, corresponding to either the Coulomb or the magnetic phase of large $N$ Yang-Mills in $D = 4$. Namely with the measure function

$$f(\Gamma, \Lambda) = \frac{1}{N(ln(\Lambda))^2} e^{-\frac{r^2}{N(ln(\Lambda))^2}} \equiv Ne_H^4 e^{-Ne_H^4 \Gamma^2}$$

(41)

a computation similar to the one in (12)-(25) gives

$$W(C)|_{f(\Gamma)} \sim e^{-\frac{2\Lambda^2\sqrt{A}}{N}},$$

(42)

the behaviour which characterizes the Coulomb or the magnetic phase of Yang-Mills theory. The element which is still missing in the construction, and which is necessary to be able to distinguish between Coulomb and magnetic phases as well as to give a full proof of the confinement conjecture for the choice (24) of the measure function, is the expression for the QCD order parameter, the “magnetic moment” $M$. This order parameter is known to satisfy the relation

$$M\Psi(C) = e^\phi \Psi(C)M$$

(43)

where $\phi$ is the phase factor. In the confining phase it should behave as

$$M \sim e^{-\lambda \sqrt{A}}$$

(44)

So far we were unable to find an exact relation of this parameter to any brane-like state (3), (4). The natural suggestion is that it should be related to the magnetic-type fivebrane state (4) according to

$$M \sim \exp\left\{ \int d^2 z \partial X_{m_1} \bar{\partial} X_{m_2} (z, \bar{z}) \left( \oint dw \frac{e^{-3\phi} \psi_{m_3} \cdots \psi_{m_7} F_{RR}^{m_1 \cdots m_7} (z + w) + \text{permute.} \right)\right\}$$

(45)
where $F_{RR}$ is the Ramond-Ramond 7-form field strength; or the similar expression in the +1-picture, with $e^{-3\phi}$ replaced with $e^\phi$. It appears that the order parameter in the +1-picture does satisfy the property

$$L^{(+1)}W(C) = W(C)L^{(+1)}$$

(46)

since the vertices $\oint \frac{dz}{2\pi i} e^{\phi} \psi_{m_1} \ldots \psi_{m_5}(z)$ and $\oint \frac{dw}{2\pi i} e^{-2\phi} \psi_{m} \psi_{n}(w)$ commute trivially; this result is sensible in the $N \to \infty$ limit. It will be important to show that the parameter $M$ has the appropriate behaviour in the confining phase (with the measure function (24)) as well as to study its asymptotic behaviour for other choices of measure functions that we hope may describe the dynamics of Yang-Mills in Coulomb and magnetic phases. We hope to complete such a computation in the future.

**Conclusion**

We have argued in this paper that the brane-like states (3), (4) play a significant role in building a relation between string theory and non-Abelian gauge dynamics. The insertion of vertices (3), (4) is equivalent to introducing branes; the choice of the measure function $f(\Gamma)$ regulates the mutual position of branes. There is no question that our results are still very far from being complete and suffer a number of serious drawbacks. First of all, in the result (25) for the expectation value of the Wilson loop, while the dependence on the cutoff is in agreement with what one would expect from lattice arguments, the dependence on the bare coupling constant does not reproduce the result of the computation on the lattice in $D = 4$, that is, in the formula (25) $g$ appears rather than $ln(g)$. Our arguments regarding the conjectured reduction of the effective space-time dimension due to the insertion of $f(\Gamma)$ in the measure appear to be quite heuristic and intuitive at the present time. In order to make the ground for these arguments more solid one has to present a precise analysis of the relation between a given choice of measure function and the “induced” brane configuration. One also has to explore the mechanism of supersymmetry breaking produced by this insertion (since the resulting gauge theory is not supersymmetric). The supersymmetry-breaking term in the action is proportional to $\sim \int d^2 z ln(f(\Gamma(z)))\delta^{(2)}(z - w)$ where $w$ is a location of picture-changing operator. In spite of the above drawbacks, we feel that the properties satisfied by the gauge-theoretic observable (7) constructed out of brane-like vertices (3), (4), such as the large $N$ limit of the loop equation (26), as well as the confining behaviour (25), give a certain evidence for the role of the brane-like states in QCD. The brane-like states appear as a result of the internal normal ordering inside the Ramond-Ramond vertex operators, often ignored in calculations involving RR states but, in our
opinion, crucial when one studies the properties of RR vertex operators on surfaces with boundaries (such as D-branes). Another possible consequence of this internal normal ordering is the worldsheet interpretation of the gauge symmetry enhancement which occurs when several D-branes are joined together \([11,12]\). That is, when several D-branes, each carrying Ramond-Ramond charge, come together, one may need to perform the normal ordering inside the corresponding Ramond-Ramond vertices, as they approach each other. Another interesting implication for the proposed formalism that relates brane-like states and measure functions to gauge groups would be to try to give a string-theoretic description of the recently constructed six-dimensional string-like theories without dynamical gravity \([13,14,15,16,17]\). These theories arise from either specific configuration of fivebranes in M-theory or from IIB in an ALE background, and are known to reduce in the low-energy limit to six-dimensional gauge theories (with either U(N) or SU(N) gauge groups) decoupled from supergravity. The microscopic description of these theories, however, is still obscure. It will be interesting to find a class of measure functions that generate ALE-type singularities in the background, and then to show by explicit computations that gravitons do not in fact arise or become massive in the spectrum of such superstring theories (originally in \(D = 10\) with \(f(\Gamma)\) effectively reducing them to \(D = 6\).) In view of the fact that gravitons are in general responsible for breaking the extended reparametrization symmetry in usual superstring theories in ten dimensions, one may expect that the recently discovered string-like theories without gravity in \(D = 6\) are in fact invariant under extended reparametrizations and satisfy dynamical equations similar to large N loop equations (in the limit \(N \to \infty\)). This is another place where the interplay between non-Abelian gauge theories and brane-like states may appear. Hopefully, a deeper analysis of these problems may also bring more understanding of U-duality in terms of the worldsheet physics.

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