RESEARCH ARTICLE

Judiciously 3-partitioning 3-uniform hypergraphs

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Abstract
Bollobás, Reed, and Thomason proved every 3-uniform hypergraph $H$ with $m$ edges has a vertex-partition $V(H) = V_1 \sqcup V_2 \sqcup V_3$ such that each part meets at least $\frac{1}{3} \left( 1 - \frac{1}{\epsilon} \right)m$ edges, later improved to 0.6$m$ by Halsegrave and improved asymptotically to 0.65$m + o(m)$ by Ma and Yu. We improve this asymptotic bound to $\frac{19}{27}m + o(m)$, which is best possible up to the error term, resolving a special case of a conjecture of Bollobás and Scott.

KEYWORDS
hypergraph, judicious partition, concentration inequality, azuma-hoeffding, optimization

1 | INTRODUCTION

Judicious partitioning problems seek to partition the vertices of a hypergraph $H$ with $m$ edges such that various quantities are simultaneously maximized. The first such problem was treated by Bollobás and Scott [2], where they proved that one can partition the vertices of a graph with $m$ edges into $k$ parts such that each part contains at most $\frac{m}{k} + o(m)$ edges. They later proved [3] that the vertices of a 3-uniform hypergraph can be partitioned into $k$ parts each of which contains at most $\frac{m}{k} + o(m)$ edges. Recently Hou, Wu, Zeng, and Zhu [6] claimed to have shown the result for 4-uniform hypergraphs with $\frac{m}{k} + o(m)$ edges, but their key technical lemma is incorrect.

The judicious partitioning problem we consider involves partitioning the vertices of an $r$-uniform hypergraph with $m$ edges into $k$ parts so that the minimum number of edges touched by any part is maximized. Bollobás and Scott [4] conjectured that this maximum is $\left( 1 - \left( 1 - \frac{1}{k} \right)^r \right) m + o(m)$, which if true is optimal up to the error term by considering the complete $r$-uniform hypergraph. Special cases of this problem have garnered considerable interest, particularly the case $r = k = 3$. Bollobás, Reed, and Thomason [1] proved every 3-uniform hypergraph $H$ has a vertex-partition $V(H) = V_1 \sqcup V_2 \sqcup V_3$ such that each part meets at least $\frac{1}{3} \left( 1 - \frac{1}{\epsilon} \right)m$ edges, which was claimed to have been improved to $\frac{5m - 1}{9}$.
by Bollobás and Scott [4] (although their proof apparently contains a subtle error, see Haslegrave [5]), improved to 0.6m by Halsegrave [5] and improved asymptotically to 0.65m + o(m) by Ma and Yu [7].

Here we resolve the conjecture of Bollobás and Scott in this special case.

**Theorem 1.1.** Every 3-uniform hypergraph \( H \) has a vertex-partition \( V(H) = V_1 \sqcup V_2 \sqcup V_3 \) such that each part meets at least \( \frac{19}{27}m - O(m^{9/7}) \) edges.

Our proof follows a strategy of Bollobás and Scott [3] and Ma and Yu [7]. The key technical lemma in [7] cannot be optimized sufficiently to attain \( \frac{19}{27} \), in fact cannot be pushed past \( \frac{1}{3}m + o(m) \). We make an additional observation which imposes an additional inequality in this technical lemma.

We now outline the strategy. It would be nice if for a given 3-uniform hypergraph, the uniformly random 3-partition of the vertices worked. Each part would touch in expectation \( \frac{19}{27}m + o(m) \) edges, but the presence of high degree vertices prevents these three quantities from simultaneously concentrating around the mean. For example, if there is a pair of vertices which belong to every edge, then a part which contains neither vertex will touch with high probability only \( \frac{1}{3}m + o(m) \) edges. To avoid this situation, we partition the vertices according to a two stage process.

We first partition “high” degree vertices \( V_{\text{high}} \subseteq V(H) \) into three parts \( V_1, V_2, V_3 \), and then take the remaining “low” degree vertices \( V_{\text{low}} = V(H) \setminus V_{\text{high}} \) and assign each of these vertices independently to \( V_i \) with some probability \( p_i \) with \( p_1 + p_2 + p_3 = 1 \), so that the expected number of edges touched in each part is at least \( \frac{19}{27}m + o(m) \). Because we exclude the high degree vertices from the random process, we have a tight concentration around the mean in each part by the Azuma-Hoeffding inequality (or more specifically the version of the inequality due to McDiarmid [8]), and we conclude.

The threshold for “high” degree vertices and the application of the Azuma-Hoeffding inequality are essentially automatic (if we only desire an \( o(m) \) error, we only have to ensure that the threshold for “high” degree is \( o(m) \)). The challenge is to appropriately partition the “high” degree vertices so that the \( p_i \) can be chosen to make each part touch the correct number of edges in expectation.

As in [7], after choosing the high degree threshold, we partition \( V_{\text{high}} \) as follows. Denote by \( E(H)^i \) for \( i \in \{0, 1, 2, 3\} \) to be the edges in \( E(H) \) which touch \( i \) vertices in \( V_{\text{high}} \) and \( 3 - i \) vertices in \( V_{\text{low}} \). The edges in \( E(H)^2 \) induce a multigraph \( G \) with vertex set \( V_{\text{high}} \) obtained by replacing each edge \( e \) with \( e \cap V_{\text{high}} \). Denote by \( x_i \) the number of edges of \( G \) within \( V_i \), and \( b_{ij} \) the number of edges of \( G \) with endpoints in \( V_i \) and \( V_j \). We choose our partition \( V_{\text{high}} = V_{\text{high}}^1 \sqcup V_{\text{high}}^2 \sqcup V_{\text{high}}^3 \) so that \( b_{23} + b_{13} + b_{12} \) is maximal (equivalently \( x_1 + x_2 + x_3 \) is minimal). The only facts used in [7] about this partition were that \( b_{ij} \geq \max(2x_i, 2x_j) \). These inequalities follow by noting that the multigraph between \( V_{\text{high}}^i \) and \( V_{\text{high}}^j \) maximizes the number of edges between the two parts for fixed \( V_{\text{high}}^i \sqcup V_{\text{high}}^j \).
However, these numerical constraints fail to rule out the possibility that $G$ is (close to) a configuration which is complete in $V_{k}^{\text{high}}$, and complete bipartite between $V_{k}^{\text{high}}$ and the other two parts, which prevents the method from proving a bound exceeding 0.7$m$. We identify the additional inequality that if $G$ maximizes $b_{23} + b_{13} + b_{12}$, then $b_{ij} \geq \frac{1}{2}x_{k}$, where $k$ is the index $\neq i, j$. This arises from considering the effect of moving all vertices from $V_{i}^{\text{high}}$ to $V_{j}^{\text{high}}$, decreasing $b_{23} + b_{13} + b_{12}$ by $b_{ij}$, and dividing $V_{k}^{\text{high}}$ between $V_{i}^{\text{high}}$ and $V_{j}^{\text{high}}$ in such a way that $b_{ik}$ is maximized, increasing $b_{23} + b_{13} + b_{12}$ by at least $\frac{1}{2}x_{k}$.

The core of this paper is a rather technical lemma. This technical lemma is used to show that if $\text{there does not exist } q_{i} \in [0, 1]$ such that the left-hand sides of the inequalities in the system are nondecreasing in the $q_{i}$ and are zero when $q_{i} = 0$, by symmetry the contrapositive of the lemma is equivalent to the following.

There does not exist $q_{1}, q_{2}, q_{3} \in [0, 1]$ with $q_{1} + q_{2} + q_{3} = 2$ and nonnegative variables

$$x_{1} + x_{2} + x_{3} + a_{1} + a_{2} + a_{3} + b_{12} + b_{13} + b_{23} + c = 1$$

with

$$b_{ij} \geq \max \left(2x_{i}, 2x_{j}, \frac{1}{2}x_{k}\right),$$

there exists $q_{1}, q_{2}, q_{3} \in [0, 1]$ with $q_{1} + q_{2} + q_{3} = 2$ such that the following inequalities are satisfied:

$$q_{1}(b_{23} + x_{2} + x_{3}) + q_{2}^{2}(a_{2} + a_{3}) + q_{3}^{2}c \leq \frac{8}{27}$$

$$q_{2}(b_{13} + x_{1} + x_{3}) + q_{2}^{2}(a_{1} + a_{3}) + q_{3}^{2}c \leq \frac{8}{27}$$

$$q_{3}(b_{12} + x_{1} + x_{2}) + q_{2}^{2}(a_{1} + a_{2}) + q_{3}^{2}c \leq \frac{8}{27}.$$  

This lemma improves the lemma in [7], which does not impose the inequality $b_{ij} \geq \frac{1}{2}x_{k}$, and has the constant $\frac{8}{27}$ relaxed to 0.35. The following three sections are devoted to the proof of this lemma.

As the left-hand sides of the inequalities in the system are nondecreasing in the $q_{i}$ and are zero when $q_{i} = 0$, by symmetry the contrapositive of the lemma is equivalent to the following.

There does not exist $q_{1}, q_{2}, q_{3} \in [0, 1]$ with $q_{1} + q_{2} + q_{3} = 2$ and nonnegative variables

$$x_{1} + x_{2} + x_{3} + b_{23} + b_{13} + b_{12} + a_{1} + a_{2} + a_{3} + c = 1$$

with

$$b_{ij} \geq \max \left(2x_{i}, 2x_{j}, \frac{1}{2}x_{k}\right)$$

such that one of the following two systems holds.

2 | THE TECHNICAL LEMMA, ITS CONTRAPOSEITIVE, AND OUR STRATEGY

The following technical lemma is the crux of our proof.

**Lemma 2.1.** For any nonnegative variables

$$x_{1} + x_{2} + x_{3} + a_{1} + a_{2} + a_{3} + b_{12} + b_{13} + b_{23} + c = 1$$

with

$$b_{ij} \geq \max \left(2x_{i}, 2x_{j}, \frac{1}{2}x_{k}\right),$$

there exists $q_{1}, q_{2}, q_{3} \in [0, 1]$ with $q_{1} + q_{2} + q_{3} = 2$ such that the following inequalities are satisfied:

$$q_{1}(b_{23} + x_{2} + x_{3}) + q_{2}^{2}(a_{2} + a_{3}) + q_{3}^{2}c \leq \frac{8}{27}$$

$$q_{2}(b_{13} + x_{1} + x_{3}) + q_{2}^{2}(a_{1} + a_{3}) + q_{3}^{2}c \leq \frac{8}{27}$$

$$q_{3}(b_{12} + x_{1} + x_{2}) + q_{2}^{2}(a_{1} + a_{2}) + q_{3}^{2}c \leq \frac{8}{27}.$$
System 1.

\[ q_1(b_{23} + x_2 + x_3) + q_1^2(a_1 + a_3) + q_1^3c > \frac{8}{27} \]
\[ q_2(b_{13} + x_1 + x_3) + q_2^2(a_1 + a_3) + q_2^3c > \frac{8}{27} \]
\[ q_3(b_{12} + x_1 + x_2) + q_3^2(a_1 + a_2) + q_3^3c > \frac{8}{27}. \]

System 2. \( q_1 = 1 \) (so \( q_2 + q_3 = 1 \)) and

\[ q_2(b_{13} + x_1 + x_3) + q_2^2(a_1 + a_3) + q_2^3c > \frac{8}{27} \]
\[ q_3(b_{12} + x_1 + x_2) + q_3^2(a_1 + a_2) + q_3^3c > \frac{8}{27}. \]

We will arrive at a contradiction assuming one of the two systems holds.

We may replace \((a_1, a_2, a_3, c) \mapsto (a_1 + (1 - q_1)c, a_2 + (1 - q_2)c, a_3 + (1 - q_3)c, 0)\) and the corresponding system still holds, so we may assume from now on that \( c = 0 \). By working with the contrapositive we may assume \( q_1, q_2, q_3 \) are fixed and hence we are able to eliminate \( c \) in this fashion, which was not exploited previously, but drastically reduces the algebraic complexity of all future systems of inequalities.

Denoting \( L = x_1 + x_2 + x_3 + b_{23} + b_{13} + b_{12} + a_1 + a_2 + a_3 + c \) and \( L_4 = q_i(b_{jk} + x_j + x_k) + q_i^2(x_j + x_k) + q_i^3c \), our strategy for System 1 is as follows. Keeping the \( q_i \) fixed, we repeatedly linearly perturb the remaining variables in such a way that \( L \) stays constant and \( L_1, L_2, L_3 \) do not decrease. We then perturb in this direction until one of the constraints attains equality. We then merge the associated variables (or set the corresponding variable to 0 if a variable hits zero), and repeat.

By linear algebra, it is clear we can do this (e.g., by choosing any perturbation fixing \( L, L_1, L_2 \) and choosing the sign so that \( L_3 \) does not decrease), until the number of variables besides \( q_1, q_2, q_3 \) is 3. However, we do some manual reductions first to reduce the final number of cases and simplify the exposition.

We handle System 2 in Section 3. System 1 will be dealt with in Sections 4-6, completing the proof of Lemma 2.1.

3 | SYSTEM 2

Recall we may assume \( c = 0 \). We may now replace

\[
(x_1, x_2, x_3, a_1, a_2, a_3) \mapsto (x_1, 0, 0, 0, 0, 0)
\]
\[
(b_{23}, b_{13}, b_{12}) \mapsto \left( \frac{1}{2}x_1, b_{13} + x_3 + q_2(a_1 + a_2 + a_3), \right.
\]
\[
\left. b_{12} + x_2 + q_3(a_1 + a_2 + a_3) + b_{23} - \frac{1}{2}x_1 \right)
\]

so we may assume that \( x_2 = x_3 = a_1 = a_2 = a_3 = 0 \) and \( b_{23} = \frac{1}{2}x_1 \). The system is now \( b_{12}, b_{13} \geq 2x_1 \geq 0, \)

\[ b_{12} + b_{13} + \frac{3}{2}x_1 = 1 \]
and
\[ q_2(b_{13} + x_1) > \frac{8}{27} \]
\[ q_3(b_{12} + x_1) > \frac{8}{27}. \]

As \( b_{12}, b_{13} \geq 2x_1 \), we have \( x_1 \leq 2 \frac{1}{11} \). Hence,
\[
1 > \frac{8}{27} \left( \frac{1}{b_{13} + x_1} + \frac{1}{b_{12} + x_1} \right)
\geq \frac{8}{27} \frac{1}{b_{13} + b_{12} + 2x_1}
= \frac{32}{27} \frac{1}{1 + \frac{1}{2}x_1} \geq \frac{32}{27} \frac{1}{1 + \frac{1}{11}} = \frac{88}{81},
\]
where the second inequality is by AM-HM, a contradiction.

4 | SYSTEM 1 ASSUMING \( X_1 \geq 4X_2 \) AND \( X_2 \geq X_3 \)

In this section, we prove the result assuming \( x_1 \geq 4x_2 \) and \( x_2 \geq x_3 \). The inequalities governing the \( b_{ij} \) are thus \( b_{12}, b_{13} \geq 2x_1 \) and \( b_{23} \geq \frac{1}{2}x_1 \). Our plan is to vary the variables, with the promise that we never break any of the constraints of System 1. Recall we may assume \( c = 0 \). Then we have
\[ x_1 + x_2 + x_3 + b_{23} + b_{13} + b_{12} + a_1 + a_2 + a_3 = 1 \]

and
\[ q_1(b_{23} + x_2 + x_3) + q_1^2(a_2 + a_3) > \frac{8}{27} \]
\[ q_2(b_{13} + x_1 + x_3) + q_2^2(a_1 + a_3) > \frac{8}{27} \]
\[ q_3(b_{12} + x_1 + x_2) + q_3^2(a_1 + a_2) > \frac{8}{27}. \]

Cases 1a and 1b. First, suppose that \( x_2 = x_3 \). Call this common value \( x_{23} \). Then we have
\[ x_1 + 2x_{23} + b_{23} + b_{13} + b_{12} + a_1 + a_2 + a_3 = 1 \]

and
\[ q_1(b_{23} + 2x_{23}) + q_1^2(a_2 + a_3) > \frac{8}{27} \]
\[ q_2(b_{13} + x_1 + x_{23}) + q_2^2(a_1 + a_3) > \frac{8}{27} \]
\[ q_3(b_{12} + x_1 + x_{23}) + q_3^2(a_1 + a_2) > \frac{8}{27}. \]

Now, replace \( (x_{23}, b_{23}) \mapsto (x_{23}, b_{23}) + \epsilon(1, -2) \) starting with \( \epsilon = 0 \), and increase \( \epsilon \) until either \( x_1 = 4x_{23} \) or \( b_{23} = \frac{1}{2}x_1 \).
Case 1a. Suppose first that $x_1 = 4x_{23}$, so that $b_{12}, b_{13} \geq 8x_{23}$, $b_{23} \geq 2x_{23}$, and $x_{23} \geq 0$. Then we have

$$6x_{23} + b_{23} + b_{13} + b_{12} + a_1 + a_2 + a_3 = 1$$

and

$$q_1(b_{23} + 2x_{23}) + q_1^2(a_2 + a_3) > \frac{8}{27}$$

$$q_2(b_{13} + 5x_{23}) + q_2^2(a_1 + a_3) > \frac{8}{27}$$

$$q_3(b_{12} + 5x_{23}) + q_3^2(a_1 + a_2) > \frac{8}{27}.$$ 

Now, replace $(x_{23}, b_{23}, b_{13}, b_{12}) \mapsto (x_{23}, b_{23}, b_{13}, b_{12}) + (1, 2, -4, -4)$ starting with $\epsilon = 0$, and increase $\epsilon$ until either $b_{12} = 8x_{23}$ or $b_{13} = 8x_{23}$. Without loss of generality assume $b_{12} = 8x_{23}$. Then we have

System 1a. $b_{13} \geq 8x_{23}$ and $b_{23} \geq 2x_{23}$ and $x_{23} \geq 0$ and

$$14x_{23} + b_{23} + b_{13} + a_1 + a_2 + a_3 = 1$$

and

$$q_1(b_{23} + 2x_{23}) + q_1^2(a_2 + a_3) > \frac{8}{27}$$

$$q_2(b_{13} + 5x_{23}) + q_2^2(a_1 + a_3) > \frac{8}{27}$$

$$q_3(13x_{23}) + q_3^2(a_1 + a_2) > \frac{8}{27}.$$

Case 1b. Suppose now instead that $b_{23} = \frac{1}{2}x_1$. Then we have

$$\frac{3}{2}x_1 + 2x_{23} + b_{12} + b_{13} + a_1 + a_2 + a_3 = 1$$

and

$$q_1(\frac{1}{2}x_1 + 2x_{23}) + q_1^2(a_2 + a_3) > \frac{8}{27}$$

$$q_2(b_{13} + x_1 + x_{23}) + q_2^2(a_1 + a_3) > \frac{8}{27}$$

$$q_3(b_{12} + x_1 + x_{23}) + q_3^2(a_1 + a_2) > \frac{8}{27}.$$ 

Now, replace $(x_1, b_{12}, b_{13}) \mapsto (x_1, b_{12}, b_{13}) + (1, -\frac{3}{4}, -\frac{3}{4})$ starting with $\epsilon = 0$, and increase $\epsilon$ until either $b_{12} = 2x_1$ or $b_{13} = 2x_1$. Without loss of generality assume $b_{12} = 2x_1$. Then we have

System 1b. $b_{13} \geq 2x_1 \geq 8x_{23} \geq 0$ and

$$\frac{7}{2}x_1 + 2x_{23} + b_{13} + a_1 + a_2 + a_3 = 1$$

and

$$q_1(\frac{1}{2}x_1 + 2x_{23}) + q_1^2(a_2 + a_3) > \frac{8}{27}.$$
\[q_2(b_{13} + x_1 + x_{23}) + q_2^2(a_1 + a_3) > \frac{8}{27}\]
\[q_3(3x_1 + x_{23}) + q_3(a_1 + a_2) > \frac{8}{27}\].

**Cases 1c and 1d.** Now, we suppose that \(x_2 \geq x_3\) and equality is not attained. We will do similar replacements of variables, but if we ever have \(x_2 = x_3\) then we can reduce to either System 1a or System 1b, so we may suppose this never happens.

Replace \((x_3, b_{23}) \mapsto (x_3, b_{23}) + \epsilon(1, -1)\) starting with \(\epsilon = 0\), and increase \(\epsilon\) until \(b_{23} = \frac{1}{2}x_1\). Now, replace \((x_3, b_{13}) \mapsto (x_3, b_{13}) + \epsilon(1, -1)\) starting with \(\epsilon = 0\), and increase \(\epsilon\) until \(b_{13} = 2x_1\). Then we have

\[\frac{7}{2} x_1 + x_2 + x_3 + b_{12} + a_1 + a_2 + a_3 = 1\]

and

\[q_1\left(\frac{1}{2} x_1 + x_2 + x_3\right) + q_1^2(a_2 + a_3) > \frac{8}{27}\]
\[q_2(3x_1 + x_3) + q_2^2(a_1 + a_3) > \frac{8}{27}\]
\[q_3(b_{12} + x_1 + x_2) + q_3^2(a_1 + a_2) > \frac{8}{27}\].

Replace \((x_2, b_{12}) \mapsto (x_2, b_{12}) + \epsilon(1, -1)\) starting with \(\epsilon = 0\), and increase \(\epsilon\) until either \(b_{12} = 2x_1\) or \(x_1 = 4x_2\).

**Case 1c.** Suppose first that \(b_{12} = 2x_1\). Then we have

**System 1c.** \(x_1 \geq 4x_2 \geq 4x_3 \geq 0\) and

\[\frac{11}{2} x_1 + x_2 + x_3 + a_1 + a_2 + a_3 = 1\]

and

\[q_1\left(\frac{1}{2} x_1 + x_2 + x_3\right) + q_1^2(a_2 + a_3) > \frac{8}{27}\]
\[q_2(3x_1 + x_3) + q_2^2(a_1 + a_3) > \frac{8}{27}\]
\[q_3(3x_1 + x_2) + q_3^2(a_1 + a_2) > \frac{8}{27}\].

**Case 1d.** Finally, suppose that \(x_1 = 4x_2\). Then we have

**System 1d.** \(b_{12} \geq 8x_2 \geq 8x_3 \geq 0\) and

\[15x_2 + x_3 + b_{12} + a_1 + a_2 + a_3 = 1\]

and

\[q_1(3x_2 + x_3) + q_1^2(a_2 + a_3) > \frac{8}{27}\]
\[q_2(12x_2 + x_3) + q_2^2(a_1 + a_3) > \frac{8}{27}\]
\[q_3(b_{12} + 5x_2) + q_3^2(a_1 + a_3) > \frac{8}{27}\].
4.1 Solving System 1a, b, c, d

For each of these systems, we carry out the procedure described at the end of Section 2. This process terminates when we have 3 variables apart from \(q, \tilde{q} \) and for each system there are \( \binom{6}{3} = 20 \) resulting cases. For each of System 1a, b, c, d one of the 20 resulting cases will be when \( a_1 = a_2 = a_3 = 0 \), and one of the resulting cases will be when the remaining 3 variables are equal to zero.

In the former case, when \( a_1 = a_2 = a_3 = 0 \), we have

\[
2 > \frac{8}{27} \left( \frac{1}{b_{23} + x_2 + x_3} + \frac{1}{b_{13} + x_1 + x_3} + \frac{1}{b_{12} + x_1 + x_2} \right) \\
\geq \frac{8}{27} b_{23} + x_2 + x_3 + b_{13} + x_1 + x_3 + b_{12} + x_1 + x_2 \\
= \frac{8}{3(1 + x_1 + x_2 + x_3)} \geq \frac{8}{3(1 + \frac{1}{3})} = 2,
\]

where the second inequality is by AM-HM, a contradiction.

In the latter case, we have \( a_1 + a_2 + a_3 = 1 \) and

\[
2 > \sqrt[3]{\frac{8}{27} \left( \frac{1}{\sqrt{a_2 + a_3}} + \frac{1}{\sqrt{a_1 + a_3}} + \frac{1}{\sqrt{a_1 + a_2}} \right)}.
\]

The right-hand side is at least 2 by Jensen’s inequality applied to \( f(x) = \frac{1}{\sqrt{x}} \), a contradiction.

Aside from these two cases for each system, we handle all other cases by computer as follows. Let \( \tilde{q}_i \) be the minimum of 1 and the solution to the quadratic equation \( L_i = \frac{8}{27} \) (recall that \( c = 0 \)). Then we may equivalently check that \( \tilde{q}_1 + \tilde{q}_2 + \tilde{q}_3 \geq 2 \) for any choice of the remaining variables. As it turns out, the infimum of \( \tilde{q}_1 + \tilde{q}_2 + \tilde{q}_3 \) is strictly larger than 2 in all remaining cases, which we can check by subdividing the space of the remaining variables very finely (boxes of dimension \( 0.001 \times 0.001 \times 0.001 \) suffices in all cases), and crudely bounding in each box the smallest possible value of \( \tilde{q}_1 + \tilde{q}_2 + \tilde{q}_3 \). We report the results of these computations in the Appendix.

5 SYSTEM 1 ASSUMING \( 4x_2 \geq x_1 \geq x_2 \geq x_3 \)

We assume from now on that \( 4x_2 \geq x_1 \geq x_2 \geq x_3 \), and so the inequalities with the \( b_{ij} \) become \( b_{12}, b_{13} \geq 2x_1, b_{23} \geq 2x_2 \). The symmetry of System 1, along with the previous two sections, completes the verification of the technical lemma. As in the previous section, we will use the method described at the end of Section 2. However, if we ever attain equality in \( 4x_2 \geq x_1 \), then we may stop and use Section 4, so we may assume from now on that we never attain equality in \( 4x_2 \geq x_1 \). Recall we may assume \( c = 0 \). Then we have

\[
x_1 + x_2 + x_3 + b_{23} + b_{13} + b_{12} + a_1 + a_2 + a_3 = 1
\]

and

\[
q_1(b_{23} + x_2 + x_3) + q_1^2(a_2 + a_3) > \frac{8}{27} \\
q_2(b_{13} + x_1 + x_3) + q_2^2(a_1 + a_3) > \frac{8}{27} \\
q_3(b_{12} + x_1 + x_2) + q_3^2(a_1 + a_2) > \frac{8}{27}.
\]
Replace \((x_1, x_2, x_3, b_{23}, b_{13}, b_{12}) \mapsto (x_1, x_2, x_3, b_{23}, b_{13}, b_{12}) + \epsilon(-1, -1, 1, 0, 0, 0)\) starting with \(\epsilon = 0\), and increase \(\epsilon\) until \(x_2 = x_3\). Call this common value \(x_{23}\), so that \(b_{12}, b_{13} \geq 2x_1, b_{23} \geq 2x_{23}\) and \(4x_{23} \geq x_1 \geq x_{23}\). Then we have

\[x_1 + 2x_{23} + b_{23} + b_{13} + b_{12} + a_1 + a_2 + a_3 = 1\]

and

\[q_1(b_{23} + 2x_{23}) + q_1^2(a_2 + a_3) > \frac{8}{27}\]
\[q_2(b_{13} + x_1 + x_{23}) + q_2^2(a_1 + a_3) > \frac{8}{27}\]
\[q_3(b_{12} + x_1 + x_{23}) + q_3^2(a_1 + a_2) > \frac{8}{27}\]

We now replace \((x_1, b_{13}) \mapsto (x_1, b_{13}) + \epsilon(1, -1)\) starting with \(\epsilon = 0\), and increase \(\epsilon\) until either \(b_{13} = 2x_1\) or \(b_{12} = 2x_1\). Without loss of generality, assume that \(b_{13} = 2x_1\). Finally, replace \((x_{23}, b_{23}) \mapsto (x_{23}, b_{23}) + \epsilon(1, -2)\) starting with \(\epsilon = 0\), and increase \(\epsilon\) until either \(x_{23} = x_1\) or \(b_{23} = 2x_{23}\).

In the former case, let \(x\) be the common value of \(x_{23}\) and \(x_1\). Set \(A = 4x, B = b_{23} + 2x, C = b_{12} + 2x\). Then (assuming \(C \geq B\) without loss of generality) we have

**System 1e.** \(C \geq B \geq A \geq 0\) and

\[\frac{1}{4}A + B + C + a_1 + a_2 + a_3 = 1\]

and

\[q_1B + q_1^2(a_2 + a_3) > \frac{8}{27}\]
\[q_2A + q_2^2(a_1 + a_3) > \frac{8}{27}\]
\[q_3C + q_3^2(a_1 + a_2) > \frac{8}{27}\]

In the latter case, set \(A = 4x_{23}, B = 3x_1 + x_{23},\) and \(C = b_{12} + x_1 + x_{23}\). Then we have

**System 1f.** \(C \geq B \geq A \geq 0\) and \(B \leq \frac{13}{4}A\) and

\[\frac{7}{12}A + \frac{2}{3}B + C + a_1 + a_2 + a_3 = 1\]

and

\[q_1A + q_1^2(a_2 + a_3) > \frac{8}{27}\]
\[q_2B + q_2^2(a_1 + a_3) > \frac{8}{27}\]
\[q_3C + q_3^2(a_1 + a_2) > \frac{8}{27}\]

For each of these systems, we carry out the procedure described at the end of Section 2. This process terminates when we have 3 variables apart from \(q_1, q_2, q_3\). Note in System 1f that if we have
B = \frac{13}{4}A \text{ then } x_1 = 4x_{23} \text{ so this is handled by Section 4 and we can avoid analyzing all of these cases. Hence we are left with 20 cases for each system. The case } A = B = C = 0 \text{ is handled as in Section 4.1. Every case when } A = 0 \text{ other than } A = B = C = 0 \text{ we check by computer as before. Note that System 1e is identical to System 1f when } A = B \text{ as } 1 + \frac{1}{4} = \frac{2}{3} + \frac{7}{12}, \text{ so for these cases we only need to check one of the two systems. We report the results of these computations in the Appendix.}

5.1 Remaining cases of Systems 1e, f

The only cases remaining are the 20 cases, 10 from System 1e and 10 from System 1f, when three of \( A = B, B = C, a_1 = 0, a_2 = 0, a_3 = 0 \) occur. Note that \( \frac{7}{12}A + \frac{2}{3}B + C + a_1 + a_2 + a_3 \leq \frac{1}{4}A + B + C + a_1 + a_2 + a_3 \) when \( A \leq B \leq C \), so it suffices to check the 10 cases as above in the following system.

System 1f' C \geq B \geq A \geq 0 \text{ and }

\[
\frac{7}{12}A + \frac{2}{3}B + C + a_1 + a_2 + a_3 = 1
\]

and

\[
q_1A + q_1^2(a_2 + a_3) > \frac{8}{27}
\]

\[
q_2B + q_2^2(a_1 + a_3) > \frac{8}{27}
\]

\[
q_3C + q_3^2(a_1 + a_2) > \frac{8}{27}
\]

It is vitally important to note that there are solutions to System 1f', but we claim there are no solutions in the 10 cases mentioned above. Suppose there is a solution in one of the cases above, we will derive a contradiction. We have

\[
2 = q_1 + q_2 + q_3
\]

\[
\geq \frac{16}{27}A + \frac{1}{A + \sqrt{A^2 + \frac{32}{27}(a_2 + a_3)}} + \frac{1}{B + \sqrt{B^2 + \frac{32}{27}(a_1 + a_3)}} + \frac{1}{C + \sqrt{C^2 + \frac{32}{27}(a_1 + a_2)}}
\]

\[
\geq \frac{16}{3}A + B + C + \sqrt{A^2 + \frac{32}{27}(a_2 + a_3)} + \sqrt{B^2 + \frac{32}{27}(a_1 + a_3)} + \sqrt{C^2 + \frac{32}{27}(a_1 + a_2)}
\]

where the second inequality is by AM-HM, and therefore

\[
A + B + C + \sqrt{A^2 + \frac{32}{27}(a_2 + a_3)} + \sqrt{B^2 + \frac{32}{27}(a_1 + a_3)} + \sqrt{C^2 + \frac{32}{27}(a_1 + a_2)} > \frac{8}{3}
\]

We will in fact show that

\[
A + B + C + \sqrt{A^2 + \frac{32}{27}(a_2 + a_3)} + \sqrt{B^2 + \frac{32}{27}(a_1 + a_3)} + \sqrt{C^2 + \frac{32}{27}(a_1 + a_2)} \leq \frac{8}{3}
\]

in each of the 10 cases, yielding a contradiction.
Suppose first we are in one of the three cases where \( A = B = C \). Then \( \frac{9}{4} A + a_1 + a_2 + a_3 = 1 \), and by Jensen’s inequality applied to \( \sqrt{x} \), it suffices to show that

\[
3A + \sqrt{3} \sqrt{3A^2 + \frac{64}{27}(a_1 + a_2 + a_3)} \leq \frac{8}{3}.
\]

As \( A \leq \frac{4}{9} \), we have \( \frac{8}{3} \geq 3A \). As \( a_1 + a_2 + a_3 = 1 - \frac{9}{4} A \), this inequality is

\[
3A + \left| \frac{8}{3} - 3A \right| \leq \frac{8}{3},
\]

and as \( \frac{8}{3} \geq 3A \), both sides are equal.

If \( a_1 = a_2 = a_3 = 0 \), then we have \( 2(A + B + C) \leq \frac{8}{3} \left( \frac{7}{12} A + \frac{2}{3} B + C \right) = \frac{8}{3} \) by the Chebyshev sum inequality.

In the remaining cases, we have one of \( A = B \) and \( B = C \). We have dealt with all cases where the maximum occurs on the boundary except when \( A = 0 \), so we can assume the maximum is attained in the interior of the domain or \( A = 0 \).

When \( A = B \), we have by symmetry two distinct cases, when \( a_1 = a_2 = 0 \) and when \( a_1 = a_3 = 0 \).

When \( a_1 = a_2 = 0 \) we have \( 0 \leq A \leq C \), \( \frac{5}{4} A + C + a_3 = 1 \) and we want to show

\[
2A + 2C + 2 \sqrt{A^2 + \frac{32}{27} a_3} \leq \frac{8}{3}.
\]

By Lagrange multipliers, if the triple \((A, C, a_3)\) attains a maximum on the interior of the domain then the gradient of the constraint \((\frac{5}{4}, 1, 1)\) is parallel to the gradient \(2 + \frac{2A}{\sqrt{A^2 + \frac{32}{27} a_3}}, 2, \frac{32}{27} \sqrt{A^2 + \frac{32}{27} a_3}\) of the objective function, so we have

\[
\frac{4}{5} \left( 2 + \frac{2A}{\sqrt{A^2 + \frac{32}{27} a_3}} \right) = 2
\]

\[
\frac{32}{27} \frac{1}{\sqrt{A^2 + \frac{32}{27} a_3}} = 2.
\]

But then \( \sqrt{A^2 + \frac{32}{27} a_3} = \frac{16}{27} \), so \( A = \frac{4}{27}, a_3 = \frac{5}{18}, \) and so \( C = \frac{29}{54} \). These values make the left-hand side \( \frac{23}{9} < \frac{8}{3} \). If the maximum occurs when \( A = 0 \), then the inequality is equivalent to \( \left( \frac{4}{3} - C \right)^2 \geq \frac{32}{27}(1 - C) \), which is true.

When \( a_1 = a_3 = 0 \) we have \( 0 \leq A \leq C \), \( \frac{5}{4} A + C + a_2 = 1 \), and we want to show

\[
3A + C + \sqrt{A^2 + \frac{32}{27} a_2} + \sqrt{C^2 + \frac{32}{27} a_2} \leq \frac{8}{3}.
\]

If the maximum is attained in the interior of the domain, by Lagrange multipliers

\[
3 + \frac{A}{\sqrt{A^2 + \frac{32}{27} a_2}} = \frac{5}{4} + \frac{5}{4} \frac{C}{\sqrt{C^2 + \frac{32}{27} a_2}}.
\]
which cannot happen as \( \frac{c}{\sqrt{c^2 + \frac{128}{27}a_2}} \leq 1 \). If the maximum occurs when \( A = 0 \), then the left-hand side is at most \( C + \sqrt{2C^2 + \frac{128}{27}a_2} \), and the inequality then becomes \( \left( \frac{8}{3} - C \right)^2 \geq 2C^2 + \frac{128}{27}(1 - C) \), which is true as \( C \leq 1 \).

Now, when \( B = C \), we have by symmetry two distinct cases, when \( a_2 = a_3 = 0 \) and when \( a_1 = a_3 = 0 \).

When \( a_2 = a_3 = 0 \), we have \( 0 \leq A \leq C \), \( \frac{7}{12}A + \frac{5}{2}C + a_1 = 1 \), and we want to show

\[
2A + 2C + 2\sqrt{C^2 + \frac{32}{27}a_1} \leq \frac{8}{3}.
\]

If the maximum is attained in the interior of the domain, by Lagrange multipliers

\[
-\frac{120}{21} + 2 + \frac{2C}{\sqrt[3]{C^2 + \frac{32}{27}a_1}} = 0
\]

which cannot happen as \( \frac{2C}{\sqrt[3]{C^2 + \frac{32}{27}a_1}} \leq 2 \). If the maximum occurs when \( A = 0 \), then the inequality is equivalent to \( \left( \frac{4}{3} - C \right)^2 \geq C^2 + \frac{32}{27}(1 - \frac{5}{3}C) \), which is true as \( C \leq \frac{3}{5} \).

Finally, when \( a_1 = a_3 = 0 \), we have \( 0 \leq A \leq C \), \( \frac{7}{12}A + \frac{5}{2}C + a_2 = 1 \), and we want to show

\[
A + 3C + \sqrt{A^2 + \frac{32}{27}a_2} + \sqrt{C^2 + \frac{32}{27}a_2} \leq \frac{8}{3}.
\]

By Jensen’s inequality applied to \( \sqrt{x} \), it suffices to show that

\[
A + 3C + \sqrt{2A^2 + 2C^2 + \frac{128}{27}a_2} \leq \frac{8}{3}.
\]

As \( A + 3C \leq \frac{8}{3}(\frac{7}{12}A + \frac{5}{3}C) \leq \frac{8}{3} \), we can move \( A + 3C \) to the right-hand side, square the equation, and then replace \( a_2 \) with \( 1 - \frac{7}{12}A - \frac{5}{3}C \) to get

\[
\left( A + 3C - \frac{8}{3} \right)^2 - \left( 2A^2 + 2C^2 + \frac{128}{27}\left( 1 - \frac{7}{12}A - \frac{5}{3}C \right) \right) \geq 0.
\]

If the minimum is attained in the interior of the domain, then we have the partial derivatives with respect to \( A \) and \( C \) vanishing. This yields

\[
2\left( A + 3C - \frac{8}{3} \right) - 4A + \frac{224}{81} = 0
\]

\[
6\left( A + 3C - \frac{8}{3} \right) - 4C + \frac{640}{81} = 0,
\]

so \( A = \frac{16}{81}, C = \frac{40}{81} \). Evaluating at this point yields \( \frac{256}{2187} \geq 0 \). The boundary case \( A = C \) is identical to the case \( A = B = C \) when \( a_1 = a_3 = 0 \). The boundary case \( A = 0 \) is equivalent to \( \left( \frac{8}{3} - 3C \right)^2 \geq 2C^2 + \frac{128}{27}(1 - \frac{5}{3}C) \) for \( 0 \leq C \leq \frac{3}{5} \), and the boundary case \( a_2 = 0 \) is equivalent to \( \left( \frac{8}{3} - 3C - \left( \frac{12}{7} - \frac{20}{7}C \right) \right)^2 \geq 2C^2 + 2\left( \frac{12}{7} - \frac{20}{7}C \right)^2 \) for \( \frac{4}{9} \leq C \leq \frac{3}{5} \) (the lower bound on \( C \) when \( a_2 = 0 \) follows from \( 1 = \frac{7}{12}A + \frac{5}{3}C \leq \left( \frac{7}{12} + \frac{5}{3} \right)C = \frac{9}{4}C \)).
6 | PROOF OF THEOREM 1.1

In this section, we prove Theorem 1.1 using Lemma 2.1.

We can assume every vertex has positive degree. Order the vertices $v_1, \ldots, v_n$ such that $d(v_1) \geq d(v_2) \geq \cdots \geq d(v_n)$. Let $V_{\text{high}}$ consist of the $t = m^a$ vertices of highest degree with some fixed $0 < a < \frac{1}{3}$. Then

$$m^a d(v_{i+1}) < \sum_{v \in H} d(v) = 3m,$$

so

$$d(v_{i+1}) \leq 3m^{1-a}.$$

Hence in particular,

$$\sum_{i=i+1}^n d(v_i)^2 < 3m^{1-a} \sum_{i=i+1}^n d(v_i) \leq 9m^{2-a}.$$

Recall $x_1, x_2, x_3, b_{12}, b_{13}, b_{23}$ were defined in the introduction. Further define $a_i$ for $i \in \{1, 2, 3\}$ to be the number of edges in $E(H)^1$ such that the high degree vertex lies in $V_{\text{high}}$, and let $c$ be the number of edges in $E(H)^0$.

Apply Lemma 2.1 to all of our variables scaled by $\frac{1}{m - E(V_{\text{high}})}$ to get probabilities $p_i = 1 - q_i$. Place the low degree vertices independently into $V_i$ with probability $p_i$. Then with $d(V_i)$ the number of edges with at least one vertex in $V_i$, we have $\mathbb{E}(d(V_i)) \geq \frac{19}{27} (m - E(V_{\text{high}}))$.

We recall that the Azuma-Hoeffding inequality [8] asserts that if $X_1, \ldots, X_n$ are independent random variables with values in $\{1, \ldots, k\}$, and $f : \{1, \ldots, k\}^n \to \mathbb{N}$ such that $|f(Y) - f(Y')| \leq d_i$ if $Y$ and $Y'$ differ only on the $i$'th coordinate, then for $t \geq 0$ we have $\mathbb{P}(f(X_1, \ldots, X_n) - \mathbb{E}(f(X_1, \ldots, X_n)) < t) \leq e^{-\frac{t^2}{2E^2}}$.

By the Azuma-Hoeffding inequality with $k = 3$, we have

$$\mathbb{P}(d(V_i) < \mathbb{E}(d(V_i)) - z) \leq e^{-\frac{z^2}{2(3m - E(V_{\text{high}}))}} < e^{-\frac{z^2}{9m^{2-a}}}.$$

Taking $z = \sqrt{(9/2) \ln 3m^{1-a}}$ yields $\mathbb{P}(d(V_i) < \mathbb{E}(d(V_i)) - z) < \frac{1}{3}$. Hence there is a partition which has $d(V_i) > \frac{19}{27} (m - E(V_{\text{high}})) - z$ simultaneously for $i = 1, 2, 3$. Because $E(V_{\text{high}}) = O(m^{3a})$ and $z = O(m^{1-a})$, taking $\alpha = \frac{2}{7}$ yields $d(V_i) > \frac{19}{27} m - O(m^{6/7})$ simultaneously for $i = 1, 2, 3$ as desired.

ACKNOWLEDGMENTS

We would like to thank our advisor Béla Bollobás for all of his support, and the anonymous referees for helpful comments. The first author would like to thank Harvard University for travel funding, and the second author would like to thank the Trinity Hall Research Studentship.

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How to cite this article: Spink H, Tiba M. Judiciously 3-partitioning 3-uniform hypergraphs. Random Struct Alg. 2020;56:1205–1221. https://doi.org/10.1002/rsa.20908

A | RESULTS OF COMPUTATION

Tables A1–A6 show the results of computation. We denote by $\epsilon$ the side length of the cubes used in the interval method (when applicable). We use $*$ to indicate that the minimum value was proved in the body of the paper. We record a proven lower bound in the “Lower Bound” column. Finally, we note that when $B = A$ in System 1f, the case is identical to the corresponding case in System 1e.

| $x_{23} = 0$ | $b_{13} = 8x_{23}$ | $b_{23} = 2x_{23}$ | $a_1 = 0$ | $a_2 = 0$ | $a_3 = 0$ | $\epsilon$ | Lower bound |
|---|---|---|---|---|---|---|---|
| ✓ | ✓ | ✓ | 0.002 | 2.078 |
| ✓ | ✓ | ✓ | 0.002 | 2.077 |
| ✓ | ✓ | ✓ | 0.002 | 2.077 |
| ✓ | ✓ | ✓ | 0.002 | 2.077 |
| ✓ | ✓ | ✓ | 0.002 | 2.077 |
| ✓ | ✓ | ✓ | 0.002 | 2.077 |
| ✓ | ✓ | ✓ | 0.002 | 2.077 |
| ✓ | ✓ | ✓ | 0.002 | 2.077 |
| ✓ | ✓ | ✓ | 0.002 | 2.085 |
| ✓ | ✓ | ✓ | 0.002 | 2.086 |
| ✓ | ✓ | ✓ | 0.002 | 2.086 |

| $x_{23} = 0$ | $b_{13} = 8x_{23}$ | $b_{23} = 2x_{23}$ | $a_1 = 0$ | $a_2 = 0$ | $a_3 = 0$ | $\epsilon$ | Lower bound |
|---|---|---|---|---|---|---|---|
| ✓ | ✓ | ✓ | 0.001 | 2.005 |
| ✓ | ✓ | ✓ | 0.002 | 2.033 |
| ✓ | ✓ | ✓ | 0.002 | 2.033 |
| ✓ | ✓ | ✓ | 0.002 | 2.057 |
| ✓ | ✓ | ✓ | 0.002 | 2.057 |
| ✓ | ✓ | ✓ | 0.002 | 2.043 |
| ✓ | ✓ | ✓ | 0.002 | 2.045 |
| ✓ | ✓ | ✓ | 0.002 | 2.044 |
| ✓ | ✓ | ✓ | 0.002 | 2.041 |
| ✓ | ✓ | ✓ | 0.002 | 2+ or 2.046 |
### TABLE A2  System 1b

| $x_{23} = 0$ | $b_{13} = 2x_1$ | $x_1 = 4x_{23}$ | $a_1 = 0$ | $a_2 = 0$ | $a_3 = 0$ | $\epsilon$ | Lower bound |
|--------------|----------------|----------------|----------|----------|----------|---------|-------------|
| ✓            | ✓              | ✓              |          |          |          |         | 2*         |
| ✓            | ✓              | ✓              |          |          |          | 0.002   | 2.042      |
| ✓            | ✓              | ✓              |          |          |          | 0.002   | 2.069      |
| ✓            | ✓              | ✓              |          |          |          | 0.002   | 2.069      |
| ✓            | ✓              | ✓              |          |          |          | 0.002   | 2.077      |
| ✓            | ✓              | ✓              |          |          |          | 0.002   | 2.078      |
| ✓            | ✓              | ✓              |          |          |          | 0.002   | 2.077      |
| ✓            | ✓              | ✓              |          |          |          | 0.002   | 2.072      |
| ✓            | ✓              | ✓              |          |          |          | 0.002   | 2.070      |
| ✓            | ✓              | ✓              |          |          |          | 0.001   | 2.005      |
| ✓            | ✓              | ✓              |          |          |          | 0.002   | 2.033      |
| ✓            | ✓              | ✓              |          |          |          | 0.002   | 2.033      |
| ✓            | ✓              | ✓              |          |          |          | 0.002   | 2.026      |
| ✓            | ✓              | ✓              |          |          |          | 0.002   | 2.026      |
| ✓            | ✓              | ✓              |          |          |          | 0.002   | 2.024      |
| ✓            | ✓              | ✓              |          |          |          | 0.002   | 2.045      |
| ✓            | ✓              | ✓              |          |          |          | 0.002   | 2.044      |
| ✓            | ✓              | ✓              |          |          |          | 0.002   | 2.041      |
| ✓            | ✓              | ✓              |          |          |          | 2+ or 2.025 |

### TABLE A3  System 1c

| $x_3 = 0$ | $x_2 = x_3$ | $4x_2 = x_1$ | $a_1 = 0$ | $a_2 = 0$ | $a_3 = 0$ | $\epsilon$ | Lower bound |
|-----------|-------------|--------------|----------|----------|----------|---------|-------------|
| ✓         | ✓           | ✓            |          |          |          |         | 2*         |
| ✓         | ✓           | ✓            |          |          |          | 0.002   | 2.042      |
| ✓         | ✓           | ✓            |          |          |          | 0.002   | 2.069      |
| ✓         | ✓           | ✓            |          |          |          | 0.002   | 2.069      |
| ✓         | ✓           | ✓            |          |          |          | 0.001   | 2.019      |
| ✓         | ✓           | ✓            |          |          |          | 0.002   | 2.036      |
| ✓         | ✓           | ✓            |          |          |          | 0.002   | 2.037      |
| ✓         | ✓           | ✓            |          |          |          | 0.002   | 2.027      |
| ✓         | ✓           | ✓            |          |          |          | 0.002   | 2.028      |
| ✓         | ✓           | ✓            |          |          |          | 0.002   | 2.026      |
| ✓         | ✓           | ✓            |          |          |          | 0.001   | 2.005      |
| ✓         | ✓           | ✓            |          |          |          | 0.002   | 2.033      |
| ✓         | ✓           | ✓            |          |          |          | 0.002   | 2.033      |
| ✓         | ✓           | ✓            |          |          |          | 0.002   | 2.026      |
| ✓         | ✓           | ✓            |          |          |          | 0.002   | 2.026      |
| ✓         | ✓           | ✓            |          |          |          | 0.002   | 2.024      |
| ✓         | ✓           | ✓            |          |          |          | 0.001   | 2.042      |
| ✓         | ✓           | ✓            |          |          |          | 0.001   | 2.042      |
| ✓         | ✓           | ✓            |          |          |          | 0.001   | 2+ or 2.033 |
| ✓         | ✓           | ✓            |          |          |          | 2+ or 2.033 |



### Table A4  System 1d

| \( b_{12} = 8x_2 \) | \( x_2 = x_3 \) | \( x_3 = 0 \) | \( a_1 = 0 \) | \( a_2 = 0 \) | \( a_3 = 0 \) | \( \epsilon \) | Lower bound |
|----------------------|-----------------|-------------|-------------|-------------|-------------|-------|-----------|
| ✓                    | ✓               | ✓           |             |             |             | ✓     | 2*        |
| ✓                    | ✓               | ✓           |             |             |             | ✓     | 0.002     | 2.077    |
| ✓                    | ✓               | ✓           |             |             |             | ✓     | 0.002     | 2.078    |
| ✓                    | ✓               | ✓           |             |             |             | ✓     | 0.001     | 2.019    |
| ✓                    | ✓               | ✓           |             |             |             | ✓     | 0.002     | 2.036    |
| ✓                    | ✓               | ✓           |             |             |             | ✓     | 0.002     | 2.037    |
| ✓                    | ✓               | ✓           |             |             |             | ✓     | 0.002     | 2.047    |
| ✓                    | ✓               | ✓           |             |             |             | ✓     | 0.002     | 2.047    |
| ✓                    | ✓               | ✓           |             |             |             | ✓     | 0.002     | 2.047    |
| ✓                    | ✓               | ✓           |             |             |             | ✓     | 0.002     | 2.047    |
| ✓                    | ✓               | ✓           |             |             |             | ✓     | 0.002     | 2.047    |
| ✓                    | ✓               | ✓           |             |             |             | ✓     | 0.002     | 2.047    |
| ✓                    | ✓               | ✓           |             |             |             | ✓     | 0.002     | 2.047    |

### Table A5  System 1e

| \( A = 0 \) | \( B = A \) | \( C = B \) | \( a_1 = 0 \) | \( a_2 = 0 \) | \( a_3 = 0 \) | \( \epsilon \) | Lower bound |
|-------------|-------------|-------------|-------------|-------------|-------------|-------|-----------|
| ✓           | ✓           | ✓           |             |             |             | ✓     | 2*        |
| ✓           | ✓           | ✓           |             |             |             | ✓     | 0.002     | 2.077    |
| ✓           | ✓           | ✓           |             |             |             | ✓     | 0.002     | 2.077    |
| ✓           | ✓           | ✓           |             |             |             | ✓     | 0.002     | 2.078    |
| ✓           | ✓           | ✓           |             |             |             | ✓     | 0.002     | 2.075    |
| ✓           | ✓           | ✓           |             |             |             | ✓     | 0.002     | 2.076    |
| ✓           | ✓           | ✓           |             |             |             | ✓     | 0.002     | 2.076    |
| ✓           | ✓           | ✓           |             |             |             | ✓     | 0.002     | 2.086    |
| ✓           | ✓           | ✓           |             |             |             | ✓     | 0.002     | 2.085    |
| ✓           | ✓           | ✓           |             |             |             | ✓     | 0.002     | 2.084    |
| ✓           | ✓           | ✓           |             |             |             | ✓     | 2*        |
| ✓           | ✓           | ✓           |             |             |             | ✓     | 2*        |
| ✓           | ✓           | ✓           |             |             |             | ✓     | 2*        |
| ✓           | ✓           | ✓           |             |             |             | ✓     | 2*        |
| ✓           | ✓           | ✓           |             |             |             | ✓     | 2*        |
| ✓           | ✓           | ✓           |             |             |             | ✓     | 2*        |
| ✓           | ✓           | ✓           |             |             |             | ✓     | 2*        |

Note: Lower bound varies.有时使用2*表示。
**TABLE A6** System 1f

| $A = 0$ | $B = A$ | $C = B$ | $a_1 = 0$ | $a_2 = 0$ | $a_3 = 0$ | $\epsilon$ | Lower bound |
|---------|---------|---------|-----------|-----------|-----------|-----------|-------------|
| ✓       | ✓       | ✓       | ✓         | 0.002     | 2.077     |           | 2*          |
| ✓       | ✓       | ✓       | ✓         | 0.002     | 2.077     |           | 2*          |
| ✓       | ✓       | ✓       | ✓         | 0.002     | 2.077     |           | 2*          |
| ✓       | ✓       | ✓       | ✓         | 0.002     | 2.076     |           | 2*          |
| ✓       | ✓       | ✓       | ✓         | 0.002     | 2.076     |           | 2*          |
| ✓       | ✓       | ✓       | ✓         | 0.002     | 2.086     |           | 2*          |
| ✓       | ✓       | ✓       | ✓         | 0.002     | 2.085     |           | 2*          |
| ✓       | ✓       | ✓       | ✓         | 0.002     | 2.084     |           | 2*          |
| ✓       | ✓       | ✓       | ✓         |           |           |           | 2*          |
| ✓       | ✓       | ✓       | ✓         |           |           |           | 2*          |
| ✓       | ✓       | ✓       | ✓         |           |           |           | 2*          |
| ✓       | ✓       | ✓       | ✓         |           |           |           | 2*          |
| ✓       | ✓       | ✓       | ✓         |           |           |           | 2*          |
| ✓       | ✓       | ✓       | ✓         |           |           |           | 2*          |
| ✓       | ✓       | ✓       | ✓         |           |           |           | 2*          |