Stability for QED in d=3 : an overview

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Abstract

We report on a result on quantum electrodynamics on a three dimensional Euclidean spacetime. The model is formulated on a toroidal lattice with unit volume and variable lattice spacing. The result is that the renormalized partition function is bounded above and below uniformly in the lattice spacing. This is a first step toward showing that the partition function and correlation functions have limits as the lattice spacing goes to zero.

1 Introduction

The topic is quantum electrodynamics (QED) on a three dimensional Euclidean spacetime, one short of the physical dimension. It is a quantum field theory for relativistic electrons and photons. The model is defined initially in finite volume and with a short distance regularization. A basic problem in the construction of the model is to remove the regularization and take the infinite volume limit. In this note we report on some recent progress toward the first of these problems.

The model is formulated on a 3-dimensional toroidal lattice. These will have the general form

$$T_N^{-N} = (L^{-N}\mathbb{Z}/L^{-N'}\mathbb{Z})^3$$

(1)

where $L$ is a fixed large positive number. We start with $T_0^{-N}$ which has lattice spacing $\epsilon \equiv L^{-N}$ and unit volume. The theory is defined by the action

$$S(\epsilon, A, \psi) = \frac{1}{2}\|dA\|^2 + <\bar{\psi}, (\mathcal{D}_\epsilon(A) + \bar{m})\psi > + m^N <\bar{\psi}, \psi > + \epsilon^N$$

(2)

Here $A(b)$ is an abelian gauge field defined on bonds $b$ (nearest neighbor pairs) in the lattice and $dA(p) = \epsilon^{-1}\sum_{b \in \partial p} A(b)$ is the field strength defined on plaquettes (squares) in the lattice. The notation is $\|dA\|^2 = \sum_p \epsilon^3|dA(p)|^2$. The $\bar{\psi}_\alpha(x), \psi_\alpha(x)$ are fermi fields indexed by lattice points and spin indices and $<\bar{\psi}, \psi >= \sum_{x,\alpha} \epsilon^3\bar{\psi}_\alpha(x)\psi_\alpha(x)$. The fermi fields are

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anti-commuting generators of a Grassmann algebra. The $\mathcal{D}_e(A)$ is the Wilson form of the covariant lattice Dirac operator with charge $e$ defined by

$$\mathcal{D}_e(A) = \gamma \cdot \nabla_{eA} - \frac{1}{2} e \Delta_{eA}$$

(3)

Here $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ are Dirac matrices, the forward covariant derivative is

$$\left(\partial_{eA,\mu}\psi\right)(x) = e^{-1}\left(e^{ieA(x,x+e\mu)}\psi(x+e\mu) - \psi(x)\right),$$

(4)

$\nabla_{eA,\mu}$ is a symmetric version of this, and $\Delta_{eA}$ is the associated Laplacian. The $\bar{m}$ is a background mass and $m^N, \varepsilon^N$ are respectively mass and energy density counterterms which will be chosen to depend on the lattice spacing, thereby renormalizing these quantities. The model is super-renormalizable and no renormalization is required for charge or field strength.

A basic item of interest is the partition function which is the exponential of the action integrated over all fields

$$Z_N(e) = \int \exp(-S(e,A,\psi)) \, D\psi \, DA$$

(5)

Here $DA = \prod_b d(A(b))$ and $D\psi$ stands for Grassmann integral over all fermi fields $\bar{\psi}, \psi$, the projection onto the maximal element. Actually the integral over the gauge fields does not converge because the integrand is invariant under gauge transformations. One must fix a gauge, that is pick a representative for each gauge orbit and integrate over them. Once this choice is made the partition function $Z_N(e)$ is well-defined. The main result is a bound on the relative partition function uniform in the lattice spacing:

**Theorem 1.** *(Ultraviolet Stability)* Assume $\bar{m} > 0$ and $e$ is sufficiently small. Then one can fix the gauge and choose counterterms $m^N, \varepsilon^N$ so that for all $N$

$$K_- \leq \frac{Z_N(e)}{Z_N(0)} \leq K_+$$

for some constants $K_\pm$.

The proof can be found in the series of papers [21], [22], [23] (see also [18], [19], [20]). It uses a block averaging renormalization group technique due to Balaban, a rigorous version of earlier ideas of Wilson. The method was introduced by Balaban [1]-[4] for scalar electrodynamics in $d=3$. It was further developed by Balaban, Imbrie, and Jaffe [14], [15], [24] for the abelian Higgs model in $d=3$ and by Balaban [5]-[13] for Yang Mills in $d=3,4$. There is also relevant work on block averaged Dirac Green’s functions by Balaban, O’Carroll, and Shor [16], [17].

The proof should allow one to add local gauge invariant perturbations to the action like $<dA, J>$ or $<\bar{\psi}\gamma_\mu\psi, j_\mu>$. Then the partition function $Z_N(e)$ becomes a generating function $Z_N(e, J, j)$, derivatives of log $Z_N(e, J, j)$ in $J, j$ at $J = 0, j = 0$ give correlation functions. Uniform bounds on the generating function and analyticity in $J, j$ would give uniform bounds on correlation functions. The continuum limit $N \to \infty$ for the partition function, generating function, and correlation functions should also be feasible. The method works equally well
for any finite volume, not just unit volume. But the infinite volume limit would pose new difficulties.

In the remainder of the paper we explain some of the ideas that go into the proof. Before proceeding it is convenient to first reformulate the problem by scaling up to a unit lattice. If \( \Psi_0, A_0 \) are fields on the large unit lattice \( \mathbb{T}_N \), we define a new action by \( S_0(e_0, \Psi_0, A_0) = S(e, \Psi_{0,L-N}, A_{0,L-N}) \) where \( \Psi_{0,L-N}(x) = L^N \tilde{\Psi}_0(L^N x) \) and \( A_{0,L-N}(b) = L^{N/2} A_0(L^N b) \) are functions on the original lattice \( \mathbb{T}_{0}^{-N} \). We find that

\[
S_0(e_0, A_0, \Psi_0) = \frac{1}{2} \| dA_0 \|^2 + \langle \tilde{\Psi}_0, (\mathcal{D}_{e_0}(A_0) + \bar{m}_0) \Psi_0 \rangle + m_0 < \tilde{\Psi}_0, \Psi_0 > + \varepsilon_0 |m_0| \tag{6}
\]

Here norms, inner products, and operators are now on the unit lattice, and the parameters have become tiny

\[
e_0 = e_0^N = L^{-N/2} e \quad \bar{m}_0 = \bar{m}_0^N = L^{-N} \bar{m} \tag{7}
\]

as well as \( m_0 = m_0^N = L^{-N} \bar{m} \) and \( \varepsilon_0 = \varepsilon_0^N = L^{-3N} \bar{\varepsilon} \). Our \( N \rightarrow \infty \) problem now has become an infinite volume problem with scaled parameters.

## 2 Renormalization group transformations

The difficulty with the problem is that we have an unbounded number of variables. The renormalization group method consists of integrating out a few variables at a time, and keeping careful track of the effective actions at each stage. (The transformations on the actions are not invertible and do not actually form a group. Nevertheless the terminology "renormalization group" is ubiquitous.)

In the block averaging method one integrates out all fields which have a fixed average over \( L \)-cubes. Let \( y \) be a point in the \( L \)-lattice \( \mathbb{T}_N^L \) and let \( B(y) \) be the \( L \)-cube in the unit lattice \( \mathbb{T}_N \) centered on \( y \). For fermion fields \( \Psi_0 \) the averaging operator in the presence of the gauge field \( A_0 \) is

\[
(Q(A_0)\Psi_0)(y) = L^{-3} \sum_{x \in B(y)} e^{i e_0 \delta(A_0(x)} \Psi_0(x) \tag{8}
\]

Here \( A_0(\Gamma) = \sum_{b \in \Gamma} A_0(b) \) and \( \Gamma(y, x) \) is a path from \( x \) to \( y \). We use it to parallel translate to the center of the cube before averaging, so that gauge covariance is preserved. For the gauge field let \( (y, y + Le_\mu) \) be a bond in the \( L \)-lattice, and define the averaged gauge field on this bond to be

\[
(QA_0)(y, y + Le_\mu) = \sum_{x \in B(y)} L^{-4} A_0(\Gamma_{x,x+Le_\mu}) \tag{9}
\]

This also preserves gauge covariance.

With these definitions the renormalization group transformation sends the initial density \( \rho_0 = e^{-S_0} \) to a new density \( \rho_1 \) defined for fields \( A_1, \Psi_1 \) defined on the \( L \)-lattice \( \mathbb{T}_N^L \) by

\[
\rho_1(A_1, \Psi_1) = N_0 \int DA_0 D\Psi_0 \delta(A_1 - QA_0) \delta_\varepsilon(A_0) \exp \left( -b/L < \tilde{\Psi}_1 - Q(-A_0)\Psi_0, \Psi_1 - Q(A_0)\Psi_0 > \right) \rho_0(A_0, \Psi_0) \tag{10}
\]
Here $b > 0$ is a constant and $N_0$ normalizes the fermion integral. We have avoided using a delta function for the fermion fields (which would not work for Grassmann elements) and instead used a Gaussian factor. We have also begun to fix the gauge by taking a certain maximal tree in each $L$-cube, and inserting a delta function $\delta(A_0)$ that sets $A_0 = 0$ on the bonds in each tree. With this axial gauge, and the fixed averages, the integral over the remaining fields converges and $\tilde{\rho}_1$ is well-defined.

After the block averaging we scale back down defining for $A_1, \Psi_1$ on the unit lattice $T_{N-1}^0$

$$\rho_1(A_1, \Psi_1) = \tilde{\rho}_1(A_{1,L}, \Psi_{1,L}) \quad (11)$$

where $A_{1,L}(b) = L^{-\frac{1}{2}}A_1(b/L)$ and $\Psi_{1,L}(x) = L^{-1}\Psi_1(x/L)$ are fields on $T_1^N$.

We would like the density $\rho_1$ to give the same partition function as $\rho_0$. Formally this is true since it satisfies, up to an overall constant,

$$\int \delta_x(A_1)\rho_1(A_1, \Psi_1)DA_1D\Psi_1 = \int \delta_x(QA_0)\delta_x(A_0)\rho_0(A_0, \Psi_0)DA_0D\Psi_0 \quad (12)$$

However these integrals do not converge since we do not as yet have a complete gauge fixing. Nevertheless one can repeat the transformation generating a sequence $\rho_0, \rho_1, \rho_2, \ldots$. After $k$ steps this yields a density $\rho_k(A_k, \Psi_k)$ where the fields are defined on the unit lattice $T_{N-k}^0$ with shrunken volume. We continue to iterate for $K$ steps with $K$ close to $N$. Then we have a density $\rho_K$ on the lattice $T_{N-K}^0$ with bounded volume. Instead of (12) we get something like

$$\int \delta_x(A_K)\rho_K(A_K, \Psi_K)DA_KD\Psi_K = \int \left( \prod_{j=0}^{K} \delta_x(Q^jA_0) \right)\rho_0(A_0, \Psi_0)DA_0D\Psi_0 \quad (13)$$

The right side of (13) does have a more or less complete gauge fixing [23]. It is an axial gauge with a hierarchical structure, and it provides the gauge fixed definition of the partition function $Z_N(e)$. Thus the identity (13) shows that the partition function can be computed in principle from $\rho_K$ which has a bounded number of variables. Thus the issue is to keep control over the sequence $\rho_0, \rho_1, \rho_2, \ldots$.

### 3 Bounded gauge fields

In analyzing the flow of the renormalization group it is useful to have bounds on the gauge field, the fermion field is effectively already bounded. We discuss how to arrange this in a simple case. The actual implementation is rather more complicated, but the following gives the basic idea.

In the initial density we have a factor $e^{-\frac{1}{2}\|dA_0\|^2}$ which is going to suppress large values of $dA_0$. To translate this fact into hard bounds on $dA_0$ we divide the torus into cubes of size $M = L^m$ larger than $L$. Then we introduce the characteristic functions for each $M$-cube $\Box$ by

$$\chi_0(\Box, A_0) = \chi\left( \text{supp}_{p \in \Box}|dA_0(p)| \leq p(e_0) \right) \quad (14)$$

Here

$$p(e_0) = (-\log e_0)^p \quad (15)$$
for some positive integer $p$. Since $\epsilon_0$ is tiny $p(\epsilon_0)$ is large, but not too large in the sense that $e_0 p(\epsilon_0)$ is still very small. Then with $\zeta_0(\square) = 1 - \chi_0(\square)$ we write

$$1 = \prod_\square \zeta_0(\square) + \chi_0(\square) = \sum_{\Omega} \prod_{\square \subset \Omega^c} \zeta_0(\square) \prod_{\square \subset \Omega} \chi_0(\square) \equiv \sum_{\Omega} \zeta_0(\Omega^c) \chi_0(\Omega) \quad (16)$$

where $\Omega$ is an arbitrary union of $M$-cubes. The identity could be inserted under the integral sign in (52) and then the sum taken outside the integral. For the term $\Omega$ the characteristic function $\chi_0(\Omega)$ enforces that $|dA_0(p)| \leq p(\epsilon_0)$ everywhere in $\Omega$. The characteristic function $\zeta_0(\Omega^c)$ enforces that in each $M$-cube $\square$ in $\Omega^c$ there is at least one plaquette $p$ such that $|dA_0(p)| > p(\epsilon_0)$. It follow that for $\square \subset \Omega^c$

$$\zeta_0(\square) e^{-\frac{1}{4} |dA_0(\square)|^2} \leq e^{-\frac{1}{4} p(\epsilon_0)^2} \quad (17)$$

The quantity $e^{-\frac{1}{4} p(\epsilon_0)^2}$ is in fact a very small number; it satisfies $e^{-\frac{1}{4} p(\epsilon_0)^2} = O(\epsilon_0^n)$ for any positive integer $n$. The bound (17) gives that

$$\zeta_0(\Omega^c))(e^{-\frac{1}{4} |dA_0|_{\Omega^c}^2} \leq e^{-\frac{1}{4} p(\epsilon_0)^2}|\Omega^c|_M \quad (18)$$

where $|\Omega^c|_M$ is the number of $M$ cubes in $\Omega^c$. This is enough for the convergence of the sum over $\Omega$ for we have

$$\sum_{\Omega} e^{-\frac{1}{4} p(\epsilon_0)^2}|\Omega^c|_M = \prod_{\square}(1 + e^{-\frac{1}{4} p(\epsilon_0)^2}) \leq \prod_{\square} \exp(e^{-\frac{1}{4} p(\epsilon_0)^2}) \leq \exp\left(e^{-\frac{1}{4} p(\epsilon_0)^2}|\Omega^c|_M\right) \quad (19)$$

The last expression is $\exp(O(\epsilon_0^n)) \approx 1$ since $e^{-\frac{1}{4} p(\epsilon_0)^2} = O(L^{-nN/2} \epsilon_0^n)$ beats $|\Omega^c|_M = L^{3N} M^{-3}$.

Tiny factors like $e^{-\frac{1}{4} p(\epsilon_0)^2}$ also ensure that contributions from large field regions $\Omega^c$ are negligible relative to the contribution of the small field region $\Omega$. They do not require renormalization and can be bounded without further processing. The contribution of a small field region does require careful analysis and renormalization as we will explain.

### 4 The first step

Starting with $\rho_0 = e^{-S_0}$ defined in (5), the density after the first step defined in (10) has the form

$$\tilde{\rho}_1(A_1, \Psi_1) = \int DA_0 \delta(A_1 - QA_0) \delta_\varepsilon(A_0) \exp\left(-\frac{1}{2} |dA_0|^2\right) \rho_f(A_0, \Psi_1) \quad (20)$$

where the fermion integral is

$$\rho_f(A_0, \Psi_1) = \mathcal{N}_0 \int D\Psi_0 \exp\left(-bL^{-1} < \tilde{\Psi}_1 - Q(-A_0)\tilde{\Psi}_0, \Psi_1 - Q(A_0)\Psi_k > - < \tilde{\Psi}_0, (\mathcal{D}_{<\epsilon_0} A_0 + \tilde{m}_0)\Psi_0 > - m_0 < \tilde{\Psi}_0, \Psi_0 > - \varepsilon_0 |\Omega^c|_M \right) \quad (21)$$

Our strategy is to make some large/small field splits, and then in the small field region write the new density as the exponential of an effective action. In the following for simplicity we only discuss the most important case where the small field region is the full torus.
Furthermore we do not always explicitly record the characteristic functions bounding the fields.

First for the gauge field we define \( A_0^{\text{min}} \) to be minimizer in \( A_0 \) of the quadratic term \( \frac{1}{2} \| dA_0 \|^2 \) subject to the constraints that \( QA_0 = A_1 \) and that \( A_0 \) is in the axial gauge. This is a linear function of \( A_1 \) and is written \( A_0^{\text{min}} = H_0^* A_1 \). We now substitute \( A_0 = A_0^{\text{min}} + Z_0 \) and integrate over \( Z_0 \) rather than \( A_0 \). The free gauge action splits as

\[
\frac{1}{2} \| dA_0 \|^2 = \frac{1}{2} \| dA_0^{\text{min}} \|^2 + \frac{1}{2} \| dZ_0 \|^2 \quad (22)
\]

The delta functions in (20) become \( \delta(QZ_0) \delta(x(Z_0)) \). We identify a Gaussian measure \( d\mu_{C_0} \) with covariance \( C_0 \) and a normalization factor \( Z_0 \) by

\[
Z_0 \ d\mu_{C_0}(Z_0) = \delta(QZ_0) \delta(x(Z_0)) \exp\left(-\frac{1}{2} \| dZ_0 \|^2\right) DZ_0 \quad (23)
\]

The conditions \( QZ_0 = 0 \) and \( Z_0 \) axial imply \( \| dZ_0 \|^2 \geq \text{const} \| Z_0 \|^2 \) so this is non-degenerate. The exact meaning of (23) requires parametrizing the hyplane selected by the delta function \([8], [21]\).

Now the expression (20) becomes

\[
\tilde{\rho}_1(A_1, \Psi_1) = Z_0 \exp\left(-\frac{1}{2} \| dA_0^{\text{min}} \|^2\right) \int \rho_1^f \left(A_0^{\text{min}} + Z_0, \Psi_1\right) d\mu_{C_0}(Z_0) \quad (24)
\]

The integral over \( Z_0 \) is a fluctuation integral, that is an integral over the deviations from the minimizer.

Before proceeding we change gauges in the minimizer. Instead of the minimizer \( A_0^{\text{min}} \) in axial gauge we consider the minimizer \( A_0^{\text{min}} \) in the Landau gauge. It is defined to be the minimizer of \( \frac{1}{2} \| dA_0 \|^2 \) subject to the constraints that \( QA_0 = A_1 \) and a version of the condition that the divergence of \( A_0 \) vanishes. It has the form \( A_0^{\text{min}} = H_0 A_1 \). These two minimizers are gauge equivalent: \( A_0^{\text{min}} = A_0^{\text{min}} + d\omega \). Thus in gauge invariant positions like \( \| dA_0^{\text{min}} \|^2 \) and \( \rho_0^f \left(A_0^{\text{min}} + Z_0, \Psi_1\right) \) we can replace \( A_0^{\text{min}} \) by \( A_0^{\text{min}} \). The advantage of the Landau gauge is that we have better control over the minimizing operator \( H_0 \). In particular one can establish that it has an exponentially decaying kernel and that it has good bounds on low order derivatives. The remark about derivatives is not so important in this first step where we are on a unit lattice. But after many steps the minimizers will be on increasingly fine lattices and control on derivatives is important.

The Landau gauge minimizer \( A_0^{\text{min}} \) is only a function of \( A_1 \) and we introduce the more suggestive notation \( \tilde{A}_1 = A_0^{\text{min}} = H_0 A_1 \). So now we have

\[
\tilde{\rho}_1(A_1, \Psi_1) = Z_0 \exp\left(-\frac{1}{2} \| dA \|^2\right) \int \rho_1^f \left(A + Z_0, \Psi_1\right) d\mu_{C_0}(Z_0) \bigg|_{A = \tilde{A}_1} \quad (25)
\]

Note that the fluctuation field stays in axial gauge which is better for convergence of the integrals.

Next in \( \rho_1^f \left(A + Z_0, \Psi_1\right) \) we have a term \( <\tilde{\Psi}_0, \mathcal{D}_{e_0}(A + Z_0)\Psi_0> \) and we write

\[
<\tilde{\Psi}_0, \mathcal{D}_{e_0}(A + Z_0)\Psi_0> = <\tilde{\Psi}_0, \mathcal{D}_{e_0}(A)\Psi_0> + V_0(A, Z_0, \Psi_0) \quad (26)
\]
Then $V_0$ depends on $Z_0$ through
\[ e^{ie_0(A+Z_0)} - e^{ie_0A} = e^{ie_0A}(e^{ie_0Z_0} - 1) \] (27)

Introducing a large/small field split in $Z_0$ (thanks to $d\mu_{C_0}(Z_0)$) one can arrange $|Z_0| \leq p(e_0)$ and so $|e^{ie_0Z_0} - 1| \leq O(e_0 p(e_0))$ with the same bound for the kernel of $V_0$. So $V_0$ is very small. Similar remarks apply to the term $<\bar{\Psi}_1 - Q(-A - Z_0)\Psi_0, \Psi_1 - Q(A + Z_0)\Psi_k>$; subtracting off the expression at $Z_0 = 0$ gives another small term which we include in the definition of $V_0$.

Now we have
\[ \rho_1^f(A + Z_0, \Psi_1) = N_0 \int D\Psi_0 \exp \left( -\tilde{S}_1(A, \Psi_1, 0) - V_0(A, Z_0, \Psi_0) - m_0 <\bar{\Psi}_0, \Psi_0 > -\varepsilon_0|T_N^0| \right) \] (28)

Here we have isolated the quadratic form
\[ \tilde{S}_1(A, \Psi_1, 0) = bL^{-1} <\bar{\Psi}_1 - Q(-A)\Psi_0, \Psi_1 - Q(A)\Psi_0 > + <\bar{\Psi}_0, (\Theta^{\varepsilon_0}(A) + m_0)\Psi_0 > \] (29)

Let $\Psi_0^{\varepsilon_0}(A)$ be the critical point of this form in $\Psi_0$, and similarly define $\bar{\Psi}_0^{\varepsilon_0}(A)$. Explicitly we have
\[ \Psi_0^{\varepsilon_0}(A) = H_0(A)\Psi_1 = bL^{-1} \Gamma_0(A)Q^T(-A)\Psi_1 \] (30)

where
\[ \Gamma_0(A) = \left( \Theta^{\varepsilon_0}(A) + m_0 + bL^{-1}Q^T(-A)Q(A) \right)^{-1} \] (31)

There is a similar expression for $\bar{\Psi}_0^{\varepsilon_0}(A)$.

We expand around the critical point. With the more suggestive notation $\tilde{\Psi}_1(A) = \Psi_0^{\varepsilon_0}(A) = H_0(A)\Psi_1$ we substitute $\Psi_0 = \tilde{\Psi}_1(A) + W_0$ and integrate over $W_0$ rather than $\Psi_0$. The quadratic form splits as
\[ \tilde{S}_1(A, \Psi_1, \tilde{\Psi}_1(A) + W_0) = \tilde{S}_1(A, \Psi_1, \tilde{\Psi}_1(A)) + <W_0, \Gamma_0(A)^{-1}W_0> \] (32)

We also identify a Grassmann Gaussian integral with covariance $\Gamma_0(A)$ and formal measure $d\mu_{\Gamma_0(A)}(W_0)$ given by
\[ \mathcal{Z}_0(A) d\mu_{\Gamma_0(A)}(W_0) = \exp \left( -<W_0, \Gamma_0(A)^{-1}W_0> \right) DW_0. \] (33)

where $\mathcal{Z}_0(A) = \det(\Gamma_0(A)^{-1})$. The interaction is now $V_0(A, Z_0, \tilde{\Psi}_1(A) + W_0)$. We absorb into it the difference between the mass counterterm at $\tilde{\Psi}_1(A) + W_0$ and at $\tilde{\Psi}_1(A)$, and call the result $V_0(A, Z_0, \tilde{\Psi}_1(A), W_0)$. So now (33) has become
\[ \rho_1^f(A_1 + Z_0, \Psi_1) = N_0 \mathcal{Z}_0(A) \exp \left( -\tilde{S}_1(A, \Psi_1, \psi) - m_0 <\tilde{\psi}, \psi > -\varepsilon_0|T_N^0| \right) \int d\mu_{\Gamma_0(A)}(W_0) \exp \left( V_0(A, Z_0, \Psi, W_0) \right) \bigg|_{A = \tilde{A}_1, \Psi = \tilde{\Psi}_1(A_1)} \] (34)

We add some comments about the operators $\Gamma_0(A), H_0(A)$. Controlling the inverse in the definition (33) of $\Gamma_0(A)$ is difficult unless one has a bound on the field strength $dA$. 

7
to the factor $e^{-\frac{1}{2} ||dA||^2}$ in (25) we can supply such a bound with a large/small field split. In addition it is useful to establish that the kernels of $\Gamma_0(A), H_0(A)$ have exponential decay. The mass $\bar{m}_0$ in $\Gamma_0(A)$ is too tiny to get any useful estimate, but the term $bL^{-1} Q^T(-A) Q(A)$ provides an effective mass of order $O(L^{-1})$ to do the job for $\Gamma_0(A)$ and hence for $H_0(A)$. [17], [21] The Landau gauge minimizer $H_0$ also has exponential decay. [12], [21]. Since $V_0(\hat{A}_1, Z_0, \psi_1(\hat{A}_1), W_0)$ has a local structure, some approximate locality in the fundamental fields $A_1, \Psi_1$ is preserved.

We insert the fermion integral (34) back into the full density (25) and find

$$\tilde{\rho}_1(A_1, \Psi_1) = N_0 \delta_0 \delta_0(A) \Xi_0(A, \psi)$$

$$\exp \left( -\frac{1}{2} ||dA||^2 - c_1(A, \Psi_1, \psi) - \bar{m}_0 < \tilde{\psi}, \psi > -\bar{\varepsilon}_0 ||\Xi^0_N|| \right)_{|A=\hat{A}_1, \psi=\tilde{\psi}_1(\hat{A}_1)}$$

where

$$\Xi_0(A, \psi) = \int d\mu_{\gamma_0}(A) W_0 d\mu_c(0) \chi_0(0) \exp \left( V_0(\hat{A}_1, Z_0, \psi, W_0) \right)$$

is the complete fluctuation integral. Here we have now made explicit a characteristic function $\chi_0(0)$ enforcing $|Z_0| \leq \rho(e_0)$.

We need to write the fluctuation integral as the exponential of an expression with good local properties. First we need norms on elements of the Grassman algebra. Let $\{\Psi(x)\}$ be the generators of a Grassman algebra indexed by a unit lattice. A general element has the form

$$E(\Psi) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{x_1, \ldots, x_n} E_n(x_1, \ldots, x_n) \Psi(x_1) \ldots \Psi(x_n)$$

Then for $h > 0$ a norm is defined by

$$||E||_h = \sum_{n=0}^{\infty} \frac{h^n}{n!} \sum_{x_1, \ldots, x_n} |E_n(x_1, \ldots, x_n)|$$

This generalizes to families of generators like our $\{\tilde{\Psi}_{0,\alpha}(x), \Psi_{0,\alpha}(x)\}$ as well as fields defined on lattices with spacing $\varepsilon$ in which case the sum over $x_i$ would be weighted by $\varepsilon^3$ approximating integrals.

Next we write the interaction $V_0$ in a form that can reproduce itself. These are polymer expansions. Divide the torus $T^0_N$ into cubes $\square$ of width $M = L^m$ much larger than $L$. A polymer is defined to be a connected union of $M$-cubes. A polymer expansion is a function of the form $f = \sum_X f(X)$ where the sum is over polymers $X$ and $f(X)$ only depends on fields in $X$. We generally assume also that $f(X)$ is exponentially decaying in the size of $X$ in the sense that $||f(X)||_h = O(e^{-\kappa d_M(X)})$ for a sufficiently large constant $\kappa$. The quantity $d_M(X)$ is defined by

$$Md_M(X) = \text{length of the shortest continuum tree joining the cubes in } X$$

and the factor $e^{-\kappa d_M(X)}$ gives tree decay on the scale $M$. This is sufficient for the convergence of the sum over $X$.
We express \( V_0 \) as polymer functions by

\[
V_0 = \sum_{\square} E_0(\square) = \sum_X E_0(X)
\]

where first we break the integrals up into integrals \( E_0(\square) \) over \( M \)-cubes \( \square \) and then we define \( E_0(X) \) to be \( E_0(\square) \) if \( X = \square \) and zero otherwise. We have \( \|E_0(\square)\|_h \leq O(e_0 p(e_0)) \) and then trivially

\[
\|E_0(X, A, Z_0)\|_h \leq O(e_0 p(e_0)) e^{-\kappa d_M(X)}
\]

The key ingredients in the cluster expansion are that the initial functions \( E_0(\square) \) are efficiently small, true by our assumption that \( e_0 \) and \( \kappa \) are small, and the exponential decay of the covariances \( C_0, \Gamma_0(\mathcal{A}_0) \) of the Gaussian measures. In carrying out the proof it is useful to use the identities

\[
\int f(Z_0) \, d\mu_{C_0}(Z_0) = \int f\left(C_0^2 Z_0\right) \, d\mu_L(Z_0)
\]

\[
\int f(W_0, W_0) \, d\mu_{\Gamma_0}(W_0, W_0) = \int f\left(W_0, \Gamma_0(\mathcal{A})W_0\right) \, d\mu_L(W_0, W_0)
\]

which put the non-locality in the function rather than the measure. This compromises the polymer expansion since \( E_0(X, A, C_0^2 Z_0, \psi, W_0, \Gamma_0(\mathcal{A})W_0) \) now depends on \( Z_0, W_0 \) outside of \( X \). But one can restore it by a further decomposition using random walk expansions for \( C_0^2 \) and \( \Gamma_0(\mathcal{A}) \). With a new polymer expansion in the exponent and the ultra-local measures \( d\mu_L(Z_0) \) and \( d\mu_L(W_0, W_0) \) the cluster expansion is just a combinatoric problem.

With bounds on \( d\mathcal{A} \) one can also establish a polymer expansion for the normalizing factor

\[
\mathcal{Z}_0(\mathcal{A}) = \det\left(D_{e_0}(\mathcal{A}) + m_0 + bL^{-1} Q^T(-\mathcal{A})Q(\mathcal{A})\right)
\]

of the form [21]

\[
\mathcal{Z}_0(\mathcal{A}) = \mathcal{Z}_0(0) \exp\left(\sum_X E_0^{\text{det}}(X, \mathcal{A})\right)
\]

and \( E_0^{\text{det}}(X, \mathcal{A}) \) has a bound similar to [18].

Now we have with \( E_0^\# = \sum_X E_0^\#(X) \), etc.

\[
\tilde{\rho}_1(\mathcal{A}_1, \Psi_1) = N_0 \mathcal{Z}_0(0) \exp\left(-\frac{1}{2} \|d\mathcal{A}\|^2 - \tilde{\mathcal{G}}_1(\mathcal{A}, \Psi_1, \psi)
\right.

\[
- m_0 < \tilde{\psi}, \psi > - \varepsilon_0 |\Psi_N^0| + E_0^\#(\mathcal{A}, \psi) + E_0^{\text{det}}(\mathcal{A})\bigg|_{\mathcal{A}=\tilde{\mathcal{A}}, \psi=\tilde{\psi}_1(\tilde{\mathcal{A}})}
\]

9
Next we scale this expression defining \( \rho_1(A_1, \Psi_1) = \tilde{\rho}_1(A_{1,L}, \Psi_{1,L}) \) for \( A_1, \Psi_1 \) on \( \mathbb{T}_{N-1}^0 \). Scaled fields \( A_1, \psi_1(A_1) \) on the \( L^{-1} \) lattice \( \mathbb{T}_{N-1}^{-1} \) are given by \( A_{1,L} = H_0 A_{1,L} \) and \( [\psi_1(A_1)]_L = H_0(A_{1,L})\Psi_{1,L} \). Thanks to our choice of scaling factors the \( \|dA\|^2 \) is invariant and the quadratic fermion term becomes

\[
\mathcal{S}_1(A_1, \Psi_1, \psi_1(A_1)) \equiv b < \bar{\Psi}_1 - Q(-A_1)\bar{\psi}_1(A_1), \Psi_1 - Q(A_1)\psi_1(A_1) > + < \bar{\psi}_1(A_1), \left( \mathcal{D}_{\varepsilon_1}(A_1) + \bar{m}_1 \right) \psi_1(A_1) >
\]

(47)

Parameters change by

\[
\varepsilon_1 = L^3 \varepsilon_0 \quad \bar{m}_1 = L\bar{m}_0 \quad m_1 = Lm_0 \quad \bar{\varepsilon}_1 = L^3 \bar{\varepsilon}_0
\]

(48)

and we have up to a multiplicative constant

\[
\rho_1(A_1, \Psi_1) = \exp \left( -\frac{1}{2}\|dA_1\|^2 - \mathcal{S}_1(A_1, \Psi_1, \psi_1(A_1)) ight)
\]

\[
- m_1 < \bar{\psi}_1(A_1), \psi_1(A_1) > - \bar{\varepsilon}_1 \| \mathbb{T}_{N-1}^0 \| + E_1(A_1, \psi_1(A_1))
\]

(49)

The function \( E_1(A, \psi) \) is defined as follows. It has a polymer expansion \( E_1 = \sum_X E_1(X) \) where

\[
E_1(X, A, \psi) = (\mathcal{L}E_0^\#)(X, A, \psi) + (\mathcal{L}E_0^\text{det})(X, A, \psi)
\]

(50)

Here the linear operator \( \mathcal{L} \) reblocks and scales. It is defined for an \( M \)-polymer \( X \) by

\[
(\mathcal{L}E)(X, A, \psi) = \sum_{Y : Y = LX} E(Y, A_L, \psi_L)
\]

(51)

The sum is over all \( M \)-polymers \( Y \) such that \( \bar{Y} = LX \), where \( \bar{Y} \) is the union of all \( LM \) polymers intersecting \( Y \).

The sum in (51) tends to increase norms by a factor \( L^3 \). This is not a problem in this first step, but is a key issue to be dealt with when we repeat the procedure. The remedy will involve modifying the simple scaling in (48).

### 5 Iteration

Now we repeat the first step and generate the sequence of densities \( \rho_1, \rho_2, \ldots \). With \( A_k, \Psi_k \) on \( \mathbb{T}_{N-k}^0 \) we pass from \( \rho_k \) to \( \rho_{k+1} \) by block averaging

\[
\tilde{\rho}_{k+1}(A_{k+1}, \Psi_{k+1}) = N_k \int DA_k D\Psi_k \delta(A_{k+1} - QA_k) \delta_k(A_k)
\]

\[
\exp \left( -bL^{-1} < \tilde{\Psi}_{k+1} - Q(-\tilde{A}_{k+1})\tilde{\psi}_{k+1}, \Psi_{k+1} - Q(\tilde{A}_{k+1})\psi_k > \right) \rho_k(A_k, \Psi_k)
\]

(52)

and then scaling \( \rho_{k+1}(A_{k+1}, \Psi_{k+1}) = \tilde{\rho}_{k+1}(A_{k+1,L}, \Psi_{k+1,L}) \). The field \( \tilde{A}_{k+1} \) is yet to be specified.
We continue to consider the case where the small field region is the whole torus. If this is the only contribution the claim is that up to a multiplicative constant

$$\rho_k(A_k, \Psi_k) = \exp \left( -\frac{1}{2} \|dA_k\|^2 - \mathcal{S}_k(A_k, \Psi_k, \psi_k(A_k)) \right)$$

(53)

Here $A_k = H_k A_k$ is a $k$-step Landau gauge minimizer. The operator $H_k$ is generated as the composition of $k$ single step operators. Similarly the $\psi_k(A_k) = H_k(A_k) \Psi_k$ and $H_k(A)$ is generated as the composition of $k$ single step operators. The fields $A_k, \psi_k(A_k)$ are defined on the finer lattice $T^{-k}_{N-k}$, so we are heading back toward the original problem on $T^{-N}_0$, even though the number of fundamental variables $A_k, \Psi_k$ is decreasing. The quadratic fermion term is

$$\mathcal{S}_k(A_k, \Psi_k, \psi_k(A_k)) \equiv b_k < \bar{\Psi}_k - Q_k(-A_k) \bar{\psi}_k(A_k), \Psi_k - Q_k(A_k) \psi_k(A_k) >$$

$$+ < \bar{\psi}_k(A_k), \left( \mathcal{D}_{e_k}(A_k) + \bar{m}_k \right) \psi_k(A_k) >$$

(54)

The scaled coupling constant is $e_k = L^{k/2} e_0 = L^{-(N-k)/2} e$ and $Q_k(A_k)$ is a $k$-fold averaging operator. The function $E_k$ has a polymer expansion $E_k = \sum_X E_k(X)$. The $E_k(X, A_k, \psi_k(A_k))$ are restrictions of functions $E_k(X, A, \psi)$ which are invariant under lattice symmetries and gauge transformations. But note that the gauge invariance only holds in positions corresponding to the smeared fields $A_k, \psi_k(A_k)$, not in the fundamental variables $A_k, \Psi_k$.

We have discussed why $\rho_k$ has the form (53). We now assume $\rho_k$ has the form (53) and discuss why $\rho_{k+1}$ has this form. We insert (53) into (52) and follow the treatment of the first step. However there are new features. Let $A_{k+1}^{\text{mix}, x} = H_k A_{k+1}$ be the minimizer in $A_k$ of the term $\|dA_k\|^2$ subject to the constraints $A_{k+1} = QA_k$ and $A_k$ axial. Expand the action around the minimizer by $A_k = A_{k+1}^{\text{mix}, x} + Z_k$. Then $A_k = H_k A_k$ becomes $H_k A_{k+1}^{\text{mix}, x} + H_k Z_k$. The $H_k A_{k+1}^{\text{mix}, x} = H_k H_k^T A_{k+1}$ is mixed Landau and axial gauge, but it is gauge equivalent to the all Landau $\tilde{A}_{k+1} \equiv H_k H_k A_{k+1}^{\text{mix}, x}$ which will scale to $A_{k+1} = H_k A_{k+1}^{\text{mix}, x}$. We make this change for better estimates. Also defining $Z_k = H_k Z_k$ we can now write

$$A_k = \tilde{A}_{k+1} + Z_k$$

(55)

and in the action we have the split

$$\frac{1}{2} \|dA_k\|^2 = \frac{1}{2} \|d\tilde{A}_{k+1}\|^2 + \frac{1}{2} \|dZ_k\|^2$$

(56)

The translation $A_k = \tilde{A}_{k+1} + Z_k$ induces changes in the quadratic fermion action and now also in the existing polymer functions. As in (26) we separate out a leading term and a fluctuating term by

$$\mathcal{S}_k \left( \tilde{A}_{k+1} + Z_k, \Psi_k, \psi_k(\tilde{A}_{k+1} + Z_k) \right) = \mathcal{S}_k \left( \tilde{A}_{k+1}, \Psi_k, \psi_k(\tilde{A}_{k+1}) \right) + [\cdots]$$

$$E_k \left( \tilde{A}_{k+1} + Z_k, \psi_k(\tilde{A}_{k+1} + Z_k) \right) = E_k \left( \tilde{A}_{k+1}, \psi_k(\tilde{A}_{k+1}) \right) + [\cdots]$$

(57)

The fluctuation terms $[\cdots]$ will be discussed further.
The quadratic fermion term now has the form at $\mathcal{A} = \tilde{A}_{k+1}$
\[
b L^{-1} < \Psi_{k+1} - Q(-\mathcal{A})\Psi_k, \Psi_{k+1} - Q(\mathcal{A})\Psi_k > + \mathcal{G}_k(\mathcal{A}, \Psi_k, \psi_k(\mathcal{A})) \quad (58)
\]
Let $\Psi_{k}^{\text{crit}}(\mathcal{A}) = H_k(\mathcal{A})\Psi_{k+1}$ be the critical point in $\Psi_k$ of this expression. We expand around this field by by $\Psi_k = \Psi_{k}^{\text{crit}}(\mathcal{A}) + W_k$. The field $\psi_k(\mathcal{A})$ becomes
\[
\psi_k(\mathcal{A}) = \tilde{\psi}_{k+1}(\mathcal{A}) + W_k(\mathcal{A}) \quad (59)
\]
where $\tilde{\psi}_{k+1}(\mathcal{A}) = H_k(\mathcal{A})H_k(\mathcal{A})\Psi_{k+1}$ will scale to $\psi_{k+1}(\mathcal{A})$ and $W_k(\mathcal{A}) = H_k(\mathcal{A})W_k$. The quadratic fermion term (58) splits as
\[
\tilde{\mathcal{G}}_{k+1}(\mathcal{A}, \Psi_{k+1}, \tilde{\psi}_{k+1}(\mathcal{A})) + < \tilde{W}_k, \Gamma_k(\mathcal{A})^{-1}W_k > \quad (60)
\]
Here $\Gamma_k(\mathcal{A})$ is a certain invertible operator similar to (31) and
\[
\tilde{\mathcal{G}}_{k+1}(\mathcal{A}, \Psi_{k+1}, \psi) = b_k L^{-1} < \Psi_{k+1} - Q_{k+1}(\mathcal{A})\tilde{\psi}, \Psi_{k+1} - Q_{k+1}(\mathcal{A})\psi > + < \tilde{\psi}, (\mathcal{D}_{e_k}(\mathcal{A}) + \tilde{m}_k)\psi >
\]
where $Q_{k+1}(\mathcal{A}) = Q(\mathcal{A})Q_k(\mathcal{A})$. The term $E_{k}(\mathcal{A}, \tilde{\psi}_{k+1}(\mathcal{A}))$ in (57) becomes
\[
E_k(\mathcal{A}, \tilde{\psi}_{k+1}(\mathcal{A}) + W_k(\mathcal{A})) = E_k(\mathcal{A}, \tilde{\psi}_{k+1}(\mathcal{A})) + [\ldots] \quad (62)
\]
The fluctuation part $[\ldots]$ will be discussed further.

Continuing as in the first step we identify Gaussian integrals by
\[
3_k \, d\mu_{C_k}(Z_k) = \delta(QZ_k)\delta_k(Z_k) \exp \left( - \frac{1}{2} \|Z_k\|^2 \right) DZ_k
\]
\[
3_k(A) d\mu_{\Gamma_k(\mathcal{A})}(W_k) = \exp \left( - < \tilde{W}_k, \Gamma_k(\mathcal{A})^{-1}W_k > \right) DW_k \quad (63)
\]
Then we find
\[
\tilde{\rho}_{k+1}(A_{k+1}, \Psi_{k+1}) = N_k 3_k(A) \Xi_k \left( A, \psi \right) \exp \left( - \frac{1}{2} \|A\|^2 - \tilde{\mathcal{G}}_{k+1}(A, \Psi_{k+1}, \psi) \right.
\]
\[
- m_k < \tilde{\psi}, \psi > - \varepsilon_k[T^0_{N-k} + E_k(A, \psi) \big|_{A=\tilde{a}_{k+1}, \psi=\tilde{\psi}_{k+1}(\tilde{a}_{k+1})}] \quad (64)
\]
where the fluctuation integral is
\[
\Xi_k \left( A, \psi \right) = \int d\mu_{\Gamma_k(\mathcal{A})}(W_k) d\mu_{C_k}(Z_k) \chi_k(Z_k) \exp \left( E_k'(A, Z_k, \psi, W_k(\mathcal{A})) \right) \exp \left( E_k'(A, Z_k, \psi, W_k(\mathcal{A})) \right) \quad (65)
\]
Here again we have implicitly made a large/small field split to introduce a characteristic function $\chi_k(Z_k)$ enforcing a bound like $|Z_k| \leq p(e_k) = (- \log e_k)^p$.

The $E'_k$ are the terms $[\ldots]$ in (57), (62). They inherit an expansion $E'_k = \sum X E'_k(X)$. However the $E'_k(X, A, Z_k, \psi, W_k(\mathcal{A}))$ depend on the fundamental variables $Z_k, W_k$ outside $X$ since $Z_k = H_k Z_k$ and $W_k(\mathcal{A}) = H_k(\mathcal{A}) W_k$ and the kernels of $H_k, H_k(\mathcal{A})$ are only exponentially decaying. When we shift to ultralocal Gaussian measures with unit covariance it gets worse.
with \( Z_k = \mathcal{H}_k C^k Z_k \) and \( \mathcal{W}_k(A) = \mathcal{H}_k(A) \Gamma_k(A) W_k \). The remedy as before is to break \( Z_k, \mathcal{W}_k(A) \) and hence \( E^k_k(X) \) into local pieces using random walk expansions, now for each the operators of \( \mathcal{H}_k, C^k, \mathcal{H}_k(A), \Gamma_k(A) \). This leads to a polymer expansion with \( Z_k, W_k \) strictly localized. Then one can do a cluster expansion as before. The result is that the fluctuation integral has the form

\[
\Xi_k(A, \psi) = \exp \left( E^\#_k(A, \psi) \right) = \exp \left( \sum_X E^\#_k(X, A, \psi) \right)
\]

with good bounds on \( E^\#_k(X) \).

We also need a polymer expansion for \( 3_k(A) = \det(\Gamma_k(A)^{-1}) \), the Gaussian normalization factor in (63). With restrictions on the size of \( dA \) one can show

\[
3_k(A) = 3_k(0) \exp \left( E^\text{det}_k(A) \right) = 3_k(0) \exp \left( \sum_X E^\text{det}_k(X, A) \right)
\]

with good bounds on \( E^\text{det}_k(X) \).

At this point we have the generalization of the \( k = 0 \) case (66)

\[
\tilde{\rho}_{k+1}(A_{k+1}, \Psi_{k+1}) = N_k 3_k(0) \exp \left( -\frac{1}{2} ||dA||^2 - \mathcal{G}_{k+1}(A, \Psi_{k+1}, \psi) \right)
- m_k < \bar{\psi}, \psi > -\varepsilon_k [T^0_{N-k} + E_k(A, \psi) + E^\#_k(A, \psi) + E^\text{det}_k(A)]|_{A=A_{k+1}, \psi=\bar{\psi}_{k+1}(A_{k+1})}
\]

Before scaling we make some further adjustments. Our goal is to show that things do not grow too rapidly as we iterate the procedure. There is a potential problem with the polymer functions \( E_k(X) \) which tend to grow like \( L^3 \) in each step as noted at the end of section 4. However this only occurs for terms with a few fields, since scaling also supplies a factor \( L^{-\frac{1}{2}} \) for gauge fields, and \( L^{-1} \) for fermion fields. Furthermore we gain powers of \( L^{-\frac{1}{2}} \) from \( e_k = L^{-\frac{1}{2}} \varepsilon_{k+1} \). Taking account the symmetries of the \( E_k(X) \) the only relevant terms are the energy density (constants) and the fermion mass ( \( \int \bar{\psi}\psi \) ). Boson mass terms \( \int |A|^2 \) are forbidden by gauge invariance. Other terms shrink and are said to be irrelevant; for example four fermion terms shrink like \( L^{-1} \). There are no marginal terms.

Accordingly we can write

\[
E_k(X, A, \psi) = -\varepsilon^*_k(X) \text{Vol}(X) - \int \bar{\psi} [m_k^*(X)] \psi + (\mathcal{R}E_k)(X, A, \psi)
\]

and with a suitable choice of \( \varepsilon_k^*(X) \), \( m_k^*(X) \) the remainder \( (\mathcal{R}E_k)(X) \) shrinks under scaling. Summing over \( X \) gives

\[
\sum_X E_k(X, A, \psi) = -\varepsilon^*_k [T^0_{N-k}] - m_k^* \int \bar{\psi}\psi + \sum_X (\mathcal{R}E_k)(X, A, \psi)
\]

where \( \varepsilon^*_k = \sum_{X \supset \square} \varepsilon^*_k(X) \) and \( m_k^* = \sum_{X \supset \square} m_k^*(X) \) for any \( M \)-cube \( \square \).

We make these adjustments in (68). After scaling we find that up to a constant \( \rho_{k+1} \) has the claimed form:

\[
\rho_{k+1}(A_{k+1}, \Psi_{k+1}) = \exp \left( -\frac{1}{2} ||dA||^2 - \mathcal{G}_{k+1}(A, \Psi_{k+1}, \psi) \right)
- m_{k+1} < \bar{\psi}, \psi > -\varepsilon_{k+1} [T^0_{N-k+1} - E_{k+1}(A, \psi)]|_{A=A_{k+1}, \psi=\bar{\psi}_{k+1}(A_{k+1})}
\]
The coupling constant in $\mathcal{G}_{k+1}$ is now $\varepsilon_{k+1} = L^2 \varepsilon_k$ and we have the new parameters:

$$
\begin{align*}
\varepsilon_{k+1} &= L^2 (\varepsilon_k + \varepsilon_k^*) \\
\varepsilon_k^* &= \varepsilon_k^*(E_k) \\
m_{k+1} &= L(m_k + m_k^*) \\
E_{k+1} &= L (RE_k + E_{k}^\# + E_{k}^\text{det})
\end{align*}
$$

(72)

Here $\varepsilon_k^* = \varepsilon_k^*(E_k)$, $m_k^* = m_k^*(E_k)$, $E_{k}^\# = E_{k}^\#(m_k,E_k)$, and $L$ is the reblocking and scaling operation [51]. The possibility of dangerous growth is now isolated in $m_k, \varepsilon_k$.

We need to study the flow of these equations, a problem in discrete dynamical systems. We start with $E_0 = 0$ and the renormalization problem is to choose initial values $\varepsilon_0 = \varepsilon_0^N$, $m_0 = m_0^N$ such that $\varepsilon_k$ and $m_k$ take specified final values and $E_k$ stays small. We stop after $K < N$ steps with $N - K = O(1)$, and arbitrarily ask for final values $\varepsilon_K = 0$, $m_K = 0$. (This does not give massless fermions since the background mass is $\bar{m}_K = L^{-(N-K)\bar{m}} > 0$.) The existence of a unique solution can be formulated as a fixed point problem in a Banach space of sequences $\{\varepsilon_k, m_k, E_{k}\}_{k=0}^K$ satisfying the boundary conditions. The fixed point exists if $e$ is sufficiently small. This leads to a result that looks something like the following [21].

**Theorem 2.** Let $m > 0$, and $e$ be sufficiently small. Then for any $N$ and $N - K = O(1)$ there exists a unique sequence $\{\varepsilon_k, m_k, E_k\}_{k=0}^K$ satisfying (72) with the initial condition $E_0 = 0$ and the final condition $\varepsilon_K = 0$, $m_K = 0$. With $e_k = e^{-(N-k)/2} e$ and $h_k = e_k^{-\frac{3}{4}}$, the solution satisfies

$$
|\varepsilon_k| \leq e_k^{\frac{1}{4}} \quad |m_k| \leq e_k^{\frac{3}{4}} \quad \|E_k(X)\|_{h_k} \leq e_k^{\frac{3}{4}} e^{-\kappa d_M(X)}
$$

(73)

With this sequence the density $\rho_k$ has the form [53] for all $0 \leq k \leq K$.

The fractional exponents here are an artifact of the proof and have no fundamental significance.

Theorem 2 gives good bounds for the partition function $Z_N(e)$ if one takes it to be

$$
\int \delta \chi(A_K) \rho_K(A_K, \Psi_K) D\!A_k D\!\Psi_K.
$$

But this density $\rho_K$ is only the global small field contribution. For the true partition function we have to include the contribution from terms which have large field regions.

The history of small field regions is given by a non-increasing sequence $\Omega = (\Omega_1, \ldots, \Omega_k)$ with $\Omega_j$ a union of $L^{-(k-j)} M$ cubes in $\mathbb{T}_{N-k}$. It is created as follows. In passing from $k$ to $k+1$ a new small field region $\Omega_{k+1}$ is created. It is a union of $L M$ cubes (or larger) in $\mathbb{T}_{N-k}$, and is defined with bounds depending on $p(e_{k+1}) = (-\log e_{k+1})^p$ to keep pace with the running coupling constant $e_{k+1}$. The $p(e_k)$ are decreasing in $k$ so the small field constraints are becoming tighter as we proceed. The new small field regions are introduced only where needed, namely inside the old small field region: $\Omega_{k+1} \subset \Omega_k$. At the end of step $k+1$ the $\Omega_{k+1}$ is scaled down to a union of $M$ cubes in $\mathbb{T}_{N-k-1}$.

The complete expression for $\rho_k$ involves a sum over histories $\Omega$:

$$
\rho_k(A_k, \Psi_k) = \sum_{\Omega} \rho_{k,\Omega}(A_k, \Psi_k)
$$

(74)

In the small field region $\Omega_k$ for $\rho_{k,\Omega}$ the action density is identical with the global small field case we have discussed at length. The contribution of the large field region $\Omega_k^L$ is much smaller is estimated along the lines discussed in section 3.
There are difficulties with carrying out this program. The density $\rho_{k,\Omega}(A_k, \Psi_k)$ is naturally expressed in terms of smeared fields $A_{k,\Omega}, \psi_{k,\Omega}$ which depend on the history $\Omega$. These depend on the fundamental fields $A_k, \Psi_k$ through operators $\mathcal{H}_{k,\Omega}$ and $\mathcal{H}_{k,\Omega}(A)$. Estimates and local expansions for these operators are needed and the tool is again random walk expansions. But because the history $\Omega$ has a multiscale structure these are multi-scale random walk expansions. These operators also embody coupling between large and small field regions, which is abundant and needs to be controlled. Another area of difficulty is the presence of localized characteristic functions which can be a source of non-locality when subjected to expansions around critical points. Finally at each step there is not a single large/small split as our notation would indicate, but several such. Things get rather complicated.

These difficulties can be overcome, and when we divide by the free partition function we get the stability bound of Theorem 1 which we recall is the bound uniform in $N$

$$K_- \leq \frac{Z_N(e)}{Z_N(0)} \leq K_+ \quad (75)$$

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