Closure Hyperdoctrines, with paths

Davide Castelnovo  
Department of Mathematics, Computer Science and Physics, University of Udine, Italy  
ecastelnovo.davide@spes.uniud.it

Marino Miculan  
Department of Mathematics, Computer Science and Physics, University of Udine, Italy  
mario.miculan@uniud.it

Abstract
Spatial logics are modal logics whose modalities are interpreted using topological concepts of neighbourhood and connectivity. Recently, these logics have been extended to (pre)closure spaces, a generalization of topological spaces covering also the notion of neighbourhood in discrete structures.

In this paper we introduce an abstract theoretical framework for the systematic investigation of the logical aspects of closure spaces. To this end we define the categorical notion of closure (hyper)doctrine, which are doctrines endowed with inflationary operators (and subject to suitable conditions). The generality and effectiveness of this notion is demonstrated by many examples arising naturally from topological spaces, fuzzy sets, algebraic structures, coalgebras, and covering at once also known cases such as Kripke frames and probabilistic frames (i.e., Markov chains). In order to model also surroundedness, closure hyperdoctrines are then endowed with paths; this construction allows us to cover all the logical constructs of the Spatial Logic for Closure Spaces. By leveraging general categorical constructions, we provide a first axiomatisation and sound and complete semantics for propositional/regular/first order logics for closure operators.

Therefore, closure hyperdoctrines are useful both for refining and improving the theory of existing spatial logics, but especially for the definition of new spatial logics for various applications.

2012 ACM Subject Classification Theory of computation → Logic

Keywords and phrases categorical logic, topological semantics, closure operators, spatial logic.

Funding Marino Miculan: Supported by the Italian MIUR project PRIN 2017FTXR7S IT MATTERS (Methods and Tools for Trustworthy Smart Systems).

1 Introduction

Recently, much attention has been devoted in Computer Science to systems distributed in physical space; a typical example is provided by the so called collective adaptive systems, such as drone swarms, sensor networks, autonomic vehicles, etc. This arises the problem of how to model and reason formally about spatial aspects of distributed systems. To this end, several researchers have advocated the use of spatial logics, i.e. modal logics whose modalities are interpreted using topological concepts of neighbourhood and connectivity. In fact, the interpretation of modal logics in topological spaces goes back to Tarski; we refer to [1] for a comprehensive discussion of variants and computability and complexity aspects.

More recently, Ciancia et al. [7, 8] extended this approach to preclosure spaces, also called Čech closure spaces, which generalise topological spaces by not requiring idempotence of closure operator. This generalization unifies the notions of neighbourhood arising from topological spaces and from quasi-discrete closure spaces, like those induced by graphs and images. Building on this generalization, [7] introduced Spatial Logic for Closure Spaces (SLCS), a modal logic for the specification and verification on spatial concepts over preclosure

---

1 Not to be confused with spatial logics for reasoning on the structure of agents, such as the Ambient Logic [6] or the Brane Logic [20].
Closure Hyperdoctrines, with paths

spaces. This logic features a closure modality and a spatial until modality: intuitively $\phi \mathcal{U} \psi$ holds in an area where $\phi$ holds and it is not possible to “escape” from it unless passing through an area where $\psi$ holds. There is also a surrounded constructor, to represent a notion of (un)reachability. Actually, SLCS has been proved to be quite effective and expressive, as it has been applied to reachability problems, vehicular movement, digital image analysis (e.g., street maps, radiological images [5]), etc. The model checking problem for this logic over finite quasi-discrete structures is decidable in linear time, making it suitable for the verification of spatial properties [7].

Despite these successful applications, a sound and complete axiomatisation for SLCS is still missing. Moreover, it is not obvious how to extend this logic to other spaces with other closure operators, such as probabilistic automata (e.g., Markov chains). Also, it is not immediate generalizing current definitions of reachability to other cases, e.g., within a given number of steps, or non-deterministic, or probabilistic, etc.

More generally, we miss an abstract theoretical framework for investigating the logical aspects of closure spaces. Such a framework would be the basis for analysing the logic SLCS, but also for developing further extensions and applications thereof.

This is the main aim of this paper. We introduce the new notion of closure (hyper)doctrine as the theoretical basis for studying the logical aspects of closure spaces. Doctrines were introduced by Lawvere [17] as a general way for endowing (the objects of) a category with logical notions from a suitable 2-category $E$, which can be the category of Heyting algebras in the case of intuitionistic logic, of Boolean algebras in the case of classical logic, etc.. Along this line, in order to capture the logical aspects of closure spaces we introduce the notion of closure operators on doctrines, that is, families of inflationary morphisms over objects of $E$ (subject to suitable conditions); a closure (hyper)doctrine is a (hyper)doctrine endowed with a closure operators. These structures arise from many common situations; in order to show its generality, we provide many examples ranging from topology to algebraic structures, from coalgebras to fuzzy sets. These examples cover the usual cases from literature (e.g., graphs, quasi-discrete spaces, (pre)topological spaces) but include also new settings, such as categories of coalgebras and probabilistic frames (i.e., Markov chains).

Then, leveraging general machinery from categorical logic, we introduce a first order logic for closure spaces for which we provide an axiomatisation and a sound and complete categorical semantics. The propositional fragment corresponds to the SLCS from [7].

Within this framework, we can accommodate also the notion of surroundedness of properties, in order to model spatial operators like SLCS’s $S$ [8]. Actually, surroundedness is not a structural property of the logical domain (differently from closure operators); rather, it depends on the kind of paths we choose to explore the space. To this end, we introduce the notion of closure doctrine with paths. Again, the foundational approach we follow allows for many kinds of paths, and hence many notions of surroundedness.

Overall, closure hyperdoctrines (with paths) are useful both for analysing and improving the theory of existing spatial logics, but especially for the definition of new logics for various situations where we have to deal with closure operators, connectivity, surroundedness, etc.

The rest of the paper is organized as follows. In Section 2 we recall the basic definitions about (hyper)doctrines, and introduce the key notion of closure doctrine. Many examples of closure doctrines are provided in Section 3. In Section 4 we introduce logics for closure operators, together with a sound and complete semantics in closure hyperdoctrines. Then, in order to cover the notion of surroundedness, we introduce the notion of closure doctrine with paths (Section 5), and the corresponding logics with the “surrounded” operator (Section 6). Conclusions and directions for future work are in Section 7.
2 Closure (hyper)doctrines

2.1 Kinds of doctrines

In this section we recall the notion of elementary hyperdoctrine, due to Lawvere [17, 18].

The development of semantics of logics in this context or in the equivalent fibrational context is well established; we refer the reader to, e.g., [14, 19, 21].

Definition 2.1 (Existential Doctrine). A primary doctrine or simply a doctrine on a category $C$ is a functor $\mathcal{P} : C^{\text{op}} \to \text{InfSL}$ where $\text{InfSL}$ is the category of meet semilattices.

$\mathcal{P}$ is elementary if $C$ has finite products and for each object $C$ there exists a fibered equality $\delta_C \in \mathcal{P}(C \times C)$ such that $\mathcal{P}(\pi_1, \pi_2)(- \land \mathcal{P}(\pi_2, \pi_3)(\delta_C)) \vdash \mathcal{P}_{1 \times \Delta C}$, where $\pi_1, \pi_2$ and $\pi_3$ are projections $D \times C \times C \to D \times C$. This left adjoint will be denoted by $\exists_{1 \times \Delta C}$.

An (elementary) primary doctrine is existential if

- the image $\mathcal{P}_{\pi_C}$ of any projection $\pi_C : C \times D \to C$ admits a left adjoint $\exists_{\pi_C}$:
- for each pullback

$$
\begin{array}{ccc}
D \times C' & \xrightarrow{\pi_C} & C' \\
1_D \times f \downarrow & & \downarrow f \\
D \times C & \xrightarrow{\pi_C} & C
\end{array}
$$

the Beck-Chevalley condition $\exists_{\pi_C} \circ \mathcal{P}_{1 \times f} = \mathcal{P}_f \circ \exists_{\pi_C}$ holds;

- for any $\alpha \in \mathcal{P}(C)$ and $\beta \in \mathcal{P}(D \times C)$ the Frobenius reciprocity $\exists_{\pi_C}(\mathcal{P}_{\pi_C}(\alpha) \land \beta) = \alpha \land \exists_{\pi_C}(\beta)$ holds.

Definition 2.2 (Hyperdoctrine). An (elementary) hyperdoctrine is an (elementary) existential doctrine $\mathcal{P}$ such that:

- $\mathcal{P}$ factors through the category $\text{HA}$ of Heyting algebras and Heyting algebras morphisms;
- for all projections $\pi_C : D \times C \to C$, $\mathcal{P}_{\pi_C}$ has a right adjoint $\forall_{\pi_C} : \mathcal{P}(D \times C) \to \mathcal{P}(C)$ which must satisfy the Beck-Chevalley condition: $\forall_{\pi_C} \circ \mathcal{P}_{1 \times f} = \mathcal{P}_f \circ \forall_{\pi_C}$ for any $f : C' \to C$.

Remark 2.3. Since $C$ has a terminal object it follows that $\mathcal{P}_{\pi_1}(- \land \delta_C) \vdash \mathcal{P}_{\Delta C}$. This left adjoint will be denoted by $\exists_{\Delta C}$.

Remark 2.4. In this paper, we work with hyperdoctrines over $\text{HA}$, the category of Heyting algebras and their morphisms; hence the resulting logic is inherently intuitionistic. Clearly, all the development still holds if we restrict ourselves to the subcategory of Boolean algebras $\text{BA}$, yielding a classical version of the logic.

Proposition 2.5. Let $\mathcal{P} : C^{\text{op}} \to \text{InfSL}$ be an existential doctrine, $D$ a category with finite products and $\mathcal{F} : D \to C$ a product preserving functor. Then, $\mathcal{P} \circ \mathcal{F}$ is an existential doctrine. If $\mathcal{P}$ is elementary (resp., a hyperdoctrine) then $\mathcal{P} \circ \mathcal{F}$ is elementary (resp., a hyperdoctrine).

Proposition 2.6. Let $\mathcal{P} : C^{\text{op}} \to \text{HA}$ be an elementary existential doctrine. For every arrow $f : C \to D$, the functor $\mathcal{P}_f$ has a left adjoint $\exists_f$ that satisfies the Frobenius reciprocity: $\exists_f(\mathcal{P}_f(\beta) \land \alpha) = \beta \land \exists_f(\alpha)$. If $\mathcal{P}$ is a hyperdoctrine then $\mathcal{P}_f$ has a right adjoint $\forall_f$ too.

Definition 2.7. Let $\mathcal{P} : C^{\text{op}} \to \text{InfSL}$, $\mathcal{S} : D^{\text{op}} \to \text{InfSL}$ be primary doctrines. A morphism $\mathcal{P} \to \mathcal{S}$ is a pair $(\mathcal{F}, \eta)$ where $\mathcal{F} : C \to D$ is a functor and $\eta : \mathcal{P} \to \mathcal{S} \circ \mathcal{F}^{\text{op}}$ is a natural transformation.
(\(F, \eta\)) is a morphism of elementary doctrines, or elementary, if \(F\) preserves products and for any object \(C\) of \(C\), \(\eta_{C \times C}(\delta_C) = S_{(F(\pi_1), F(\pi_2))}(\delta_{F(C)})\).

\(\eta_{D \times C}\)

In this section we introduce the key notion of closure operators on doctrines.

**Definition 2.8.** Let \(\mathcal{P}\) be a doctrine. A closure operator on \(\mathcal{P}\) is a (possibly large) family \(\epsilon = \{\epsilon_C\}_{C \in \text{Obj}(\mathcal{C})}\) of functions \(\epsilon_C : \mathcal{P}(C) \rightarrow \mathcal{P}(C)\) such that:

- for any object \(C\), \(\epsilon_C\) is monotone and inflationary, i.e., \(1_{\mathcal{P}(C)} \leq \epsilon_C\).
- any arrow \(f : C \rightarrow D\) is continuous, i.e., \(\epsilon_C \circ \mathcal{P}_f \leq \mathcal{P}_f \circ \epsilon_D\).

A closure operator \(\epsilon\) is said to be

- grounded if \(\epsilon_C(\bot) = \bot\) for all objects \(C\) such that \(\mathcal{P}(C)\) has a minimum;
- additive if \(\epsilon_C(\alpha \vee \beta) = \epsilon_C(\alpha) \vee \epsilon_C(\beta)\) for all objects \(C\) such that \(\mathcal{P}(C)\) has binary suprema;
- finitely additive if it is grounded and additive;
- full additive if \(\epsilon_C(\bigvee_{i \in I} \alpha_i) = \bigvee_{i \in I} \epsilon_C(\alpha_i)\) for all \(I \neq \emptyset\) and \(C\) such that \(\mathcal{P}(C)\) has \(I\)-indexed suprema;
- idempotent if \(\epsilon_C \circ \epsilon_C = \epsilon_C\) for all object \(C\).

A closure doctrine is a pair \((\mathcal{P}, \epsilon)\) where \(\mathcal{P}\) is a primary doctrine and \(\epsilon\) a closure operator on it. We say that \((\mathcal{P}, \epsilon)\) is elementary, existential, or a hyperdoctrine, if \(\mathcal{P}\) is.

**Remark 2.9.** Full additivity does not imply groundedness since we explicitly ask for preservation of suprema indexed on non-empty set.

**Proposition 2.10.** Let \(\mathcal{P} \in \text{EED}\) be an elementary existential doctrine and \(\epsilon\) a closure operator on it; then, for any \(f : C \rightarrow D\), continuity of \(f\) is equivalent to \(\exists f \circ \epsilon_C \leq \epsilon_D \circ \exists f\).

**Proof.** Let’s compute:

\[\epsilon_C \circ \mathcal{P}_f \leq \mathcal{P}_f \circ \exists f \circ \epsilon_C \circ \mathcal{P}_f \leq \mathcal{P}_f \circ \mathcal{P}_f \circ \epsilon_D \circ \exists f \circ \mathcal{P}_f \leq \mathcal{P}_f \circ \epsilon_D \]

\[\exists f \circ \epsilon_C \leq \exists f \circ \epsilon_C \circ \mathcal{P}_f \circ \exists f \leq \exists f \circ \mathcal{P}_f \circ \epsilon_D \circ \exists f \leq \epsilon_D \circ \exists f\]
Definition 2.11. A morphism of closure (elementary, existential, hyper)doctrines \((\mathcal{F}, \eta) : (\mathcal{P}, \epsilon) \to (\mathcal{Q}, \theta)\) is a morphism of (elementary, existential, hyper)doctrines \(\mathcal{F} : \mathcal{P} \to \mathcal{Q}\) such that \(\eta\) is continuous, i.e., for all \(C : \mathcal{F}(C) \circ \eta_C \leq \eta_C \circ C\).

A 2-cell \(\theta : (\mathcal{F}, \eta) \to (\mathcal{G}, \epsilon)\) is defined as in the case of doctrines. In this way we get the 2-categories \(c\mathcal{PD}, c\mathcal{ED}, c\mathcal{HD}\) of closure doctrines, closure existential doctrines, closure hyperdoctrines and the subcategories \(c\mathcal{EPD}, c\mathcal{EED}, c\mathcal{EHD}\) of their elementary variants.

3 Examples of closure hyperdoctrines

3.1 Topological examples

As a first class of examples, we introduce three closure hyperdoctrines starting from the usual category \(\text{Top}\) of topological spaces and continuous maps. The first one corresponds to the closure spaces used, e.g., in [7, 8, 11].

Definition 3.1. The category \(\text{PrTop}\) of pretopological spaces (or closure spaces) is the category in which:

- objects are pairs \((X, c)\) of a set \(X\) and a monotone function \(c : \mathcal{P}(X) \to \mathcal{P}(X)\) such that
  \(1_{\mathcal{P}(X)} \leq c\) and \(c\) preserves finite (even empty) suprema;
- an arrow \(f : (X, c_X) \to (Y, c_Y)\) is a function \(f : X \to Y\) such that \(f^{-1} : (\mathcal{P}(Y), c_Y) \to (\mathcal{P}(X), c_X)\) is continuous.

The next example comes from topology [9].

Definition 3.2. For any set \(X\) let \(\text{Fil}(X)\) be the set of proper filters (i.e., \(\emptyset\) is not among them) on it. The category \(\text{FC}\) of filter convergence spaces is the category in which:

- an object is a pair \((X, q_X)\) given by a set \(X\) and a function \(q_X : X \to \mathcal{P}(\text{Fil}(X))\) such that, for any \(x \in X\), \(q_X(x)\) is upward closed and \(\natural := \{A \subset X \mid x \in A\}\) belongs to \(q_X(x)\);
- an arrow \(f : (X, q_X) \to (Y, q_Y)\) is a function \(f : X \to Y\) such that the filter \(f(F)\) generated by the images of \(F\)'s elements belongs to \(q_Y(f(x))\) whenever \(F \in q_X(x)\).

Proposition 3.3. The obvious forgetful functors from \(\text{Top}, \text{PrTop}\) and \(\text{FC}\) to \(\text{Set}\) preserves finite products.

Proof. For \(\text{Top}\) it is clear, for the other two categories see [9, Ch.3].

By Proposition 2.5 and the previous one, we have three elementary hyperdoctrines

\[
\begin{align*}
\mathcal{P} : \text{Top}^{\text{op}} & \to \mathcal{HA} \\
\mathcal{P}^{\text{op}} : \text{PrTop}^{\text{op}} & \to \mathcal{HA} \\
\mathcal{P}^{\text{f}} : \text{FC}^{\text{op}} & \to \mathcal{HA}
\end{align*}
\]

which we now endow with closure operators.

Definition 3.4. We define the following closure operators:

1. the Kuratowski closure operator \(k = \{k_{(X, q)}\}_{(X, q) \in \text{Ob}(\text{Top})}\) on \(\mathcal{P}\) where \(k_{(X, q)}\) is the closure operator associated with the topology \(q\);
2. the Čech closure operator \(c = \{c_{(X, q)}\}_{(X, q) \in \text{Ob}(\text{PrTop})}\) on \(\mathcal{P}^{\text{op}}\) where \(c_{(X, q)}\) is just \(c\);
3. the Katětov closure operator \(\ell = \{\ell_{(X, q_X)}\}_{(X, q_X) \in \text{Ob}(\text{FC})}\) on \(\mathcal{P}^{\text{f}}\) where

\[
\begin{align*}
\ell_{(X, q_X)} : \mathcal{P}(X) & \to \mathcal{P}(X) \\
A & \mapsto \{x \in X \mid \exists F \in q_X(x). A \in F\}
\end{align*}
\]

Proposition 3.5. 1. \(k, c\) and \(\ell\) are grounded and additive closure operators, moreover \(k\) is idempotent.
2. There exists a sequence of inclusion functors $\text{Top} \xrightarrow{i} \text{PrTop} \xrightarrow{j} \text{FC}$ each of which has a left adjoint.

3. We have a sequence $(\mathcal{P}, k) \xrightarrow{(i, \eta)} (\mathcal{P}, c) \xrightarrow{(j, \epsilon)} (\mathcal{P}, t)$ of morphisms in $\text{cEHD}$ where $\eta$ and $\epsilon$ have identities as components.

Proof. 1. For $k$ and $c$ the proposition is obvious, let us examine $\eta$: since $\hat{x} \in q_X(x)$ then $A \subset q_X(A)$, if $A \subset B$ then any filters that contains the former contains the latter too and this implies monotonicity, groundedness follows from the fact that $\emptyset$ does not belong to any proper filter, for additivity we can complete any filter $\mathcal{F}$ to which $A \cup B$ belong to an ultrafilter $\mathcal{U}$ that belongs to $q_X(x)$ since the latter is upward closed, either $A$ or $B$ must belong to $\mathcal{U}$ and we are done.

2. $i$ sends a topological space to the pretopological space given by the closure operator associate to its topology, $j$ sends $(X, c)$ to $(X, q_c X)$ where $q_c X : X \rightarrow \mathcal{P}(\text{Fil}(X))$

   $x \mapsto \{ \mathcal{F} \in \text{Fil}(X) \mid V_x \subset \mathcal{F} \}$

   where $V_x := \{ S \subset X \mid x \notin c(X \setminus S) \}$. For the left adjoints see [9].

3. This is obvious. $\triangleleft$

For many other examples of closure operators on topological spaces we refer the interested reader to [9].

3.2 Algebraic examples

Proposition 3.6. Let $\text{Grp}$ be the category of groups and $\text{CRing}$ that of commutative, unital rings (where we require that $f(1_A) = 1_B$ for any $f : A \rightarrow B$). Then, $\text{Sub}_{\text{Grp}}$ and $\text{Sub}_{\text{CRing}}$ are elementary existential doctrines.

Proof. The existence of products in any of the two categories is clear, $f : G \rightarrow H$ is a morphism of groups then $f^{-1}(K)$ is a subgroup for any $K \leq H$; if $g : A \rightarrow B$ is an arrow in $\text{CRing}$ and $C$ a subring of $B$ then $0_A, 1_A \in f^{-1}(C)$ and it is closed under sums and products. So $\text{Sub}_{\text{Grp}}$ and $\text{Sub}_{\text{CRing}}$ are functors, now, the intersection of any two subgroups or subrings is again a subgroups or a subring and for any $G$ with unit $e_G$, $\{ e_G \}$ is the minimal subgroup, while for any ring $A$ its characteristic subring $A^l$ is the minimal subring, so the codomain of this two functors is $\text{InfSL}$. Since the image of a subgroup or a subring is a subgroup or a subring we can define the left adjoint $\exists f$ as images.

Remark 3.7. Notice that, even if $\text{Sub}_{\text{Grp}}(G)$ and $\text{Sub}_{\text{CRing}}(A)$ admit finite suprema for any group $G$ or commutative ring $A$ with unity, preimages do not preserve them in general: for instance they do not preserve the bottom subobject. Then $\text{Sub}_{\text{Grp}}$ or $\text{Sub}_{\text{CRing}}$ cannot be universal doctrines.

The following examples are taken from [9].

Definition 3.8 (Groups). The normal closure on a group $G$ is given by

$$\nu_G : \text{Sub}_{\text{Grp}}(G) \rightarrow \text{Sub}_{\text{Grp}}(G)$$

$$H \mapsto \bigcap\{ N \leq G \mid H \leq N \leq G \}$$

where we have chosen the image of a monomorphism as a canonical representative of it.
Proposition 3.9. The family previously defined forms a closure operators \( \nu \) on \( \text{Sub}_{\text{Grp}} \) that is idempotent, fully additive and grounded.

Proof. Since the preimage of a normal subgroup is normal we have that the \( \nu \) actually exists as a closure operator. The three properties of it follow immediately by the fact that \( \{0\} \) is normal and so are the arbitrary intersections or sums of normal subgroups.

Definition 3.10 (Rings). Let \( A \) be a unital commutative ring and \( B \) a subring, we define \( \text{int}_A(B) \) to be the integral closure of \( B \):

\[
\text{int}_A(B) := \{ a \in A \mid p(a) = 0 \text{ for some } p \in B[x] \}
\]

Again we are denoting a subobject by the image of any representative of it.

Proposition 3.11. For any \( A \) \( \text{int}_A \) is a function \( \text{Sub}_{\text{CRing}}(A) \to \text{Sub}_{\text{CRing}}(A) \), moreover the family of this functions forms an idempotent closure operator \( \text{int} \).

Proof. To show that \( \text{int}_A(B) \) is a subring of \( A \) and idempotency we refer to [2, Cor. 5.3, 5.5]. Let us show that \( \text{int} \) is actually a closure operator. Consider \( f : A \to B \) and \( C \) a subring of \( B \), let \( a \in A \) such that \( p(a) = 0 \) for some \( p \in f^{-1}(C)[X] \) with coefficients \( \{ p_i \}_{i=0}^{\deg(p)} \), then \( q(f(a)) = 0 \) where \( q \in C[X] \) has coefficients \( \{ q_i \}_{i=0}^{\deg(p)} \) and we are done.

3.3 A representable example

Proposition 3.12. Set\((-,[0,1]) : \text{Set}^{op} \to \text{HA} \) is an elementary hyperdoctrine on \( \text{Set} \).

Proof. \([0,1] \), with the usual ordering, is a boolean algebra, hence a Heyting algebra and so, by Yoneda lemma, \( \text{Set}(-,[0,1]) \) factors through \( \text{HA} \). For \( f : X \times Y \to [0,1] \) we can define

\[
\exists_{\pi_X} (f) : X \to [0,1] \quad \forall_{\pi_X} (f) : X \to [0,1]
\]

\[
x \mapsto \bigvee_{y \in Y} f(x,y) \\
\quad \mapsto \bigwedge_{y \in Y} f(x,y)
\]

Frobenius reciprocity comes for free, for the Beck-Chevalley conditions fix \( f : Y \to Z \), another set \( X \) and \( g : X \times Z \to [0,1] \) and compute:

\[
\exists_{\pi_Y} (g \circ (1 \times f))(y) = \bigvee_{x \in X} g(x,f(y)) \\
\forall_{\pi_Y} (g \circ (1 \times f))(y) = \bigwedge_{x \in X} g(x,f(y))
\]

\[
= \exists_{\pi_Y} (g)(f(y)) \\
= \forall_{\pi_Y} (g)(f(y))
\]

\[
= (\exists_{\pi_Y} (g) \circ f)(y) \\
= (\forall_{\pi_Y} (g) \circ f)(y)
\]

The fibered equality \( \delta_X : X \times X \to [0,1] \) is defined as usual \((x,y) \mapsto \begin{cases} 0 & x \neq y \\ 1 & x = y \end{cases} \)

Definition 3.13. For any fixed real \( \epsilon \geq 0 \), and any set \( X \) we define, for an \( f : X \to [0,1] \) we define

\[
\epsilon_{X,\epsilon}(f) : X \to [0,1] \quad \text{where} \\
x \mapsto f(x) + \epsilon
\]

\[
+ : [0,1] \times [0,1] \to [0,1] \\
(t,s) \mapsto \max(t+s,1)
\]

In this way we get a function

\[
\epsilon_{X,\epsilon} : \text{Set}(X,[0,1]) \to \text{Set}(X,[0,1]) \\
f \mapsto \epsilon_{X,\epsilon}(f)
\]
Proposition 3.14. For any \( \epsilon \geq 0 \), the collection \( c_\epsilon \) of all the functions \( c_{X,\epsilon} \) is a closure operator.

Proof. Clearly \( f \leq c_{X,\epsilon}(f) \) for any \( f : X \to [0,1] \), monotonicity is clear, let’s check continuity of any function \( g : X \to Y \):

\[
c_{X,\epsilon}(f \circ g)(x) = (f \circ g)(x) + \epsilon \\
= f(g(x)) + \epsilon \\
= c_{X,\epsilon}(f)(g(x)) \\
= (c_{X,\epsilon}(f) \circ g)(x)
\]

Remark 3.15. \( c_\epsilon \) is not grounded if \( \epsilon \neq 0 \) (in that case it reduces to the discrete closure operator) but it is additive.

3.4 Fuzzy sets

We can refine the previous example considering fuzzy sets.

Definition 3.16. The category \( \mathbf{Fzs} \) of fuzzy sets has:

- pairs \( (A, \alpha) \) with \( \alpha : A \to [0,1] \) as objects;
- as arrows \( f : (A, \alpha) \to (B, \beta) \) functions \( f : A \to B \) such that \( \alpha(x) \leq \beta(f(x)) \).

Definition 3.17. A fuzzy subset of \( (A, \alpha) \) is a function \( \xi : A \to [0,1] \) such that \( \xi(x) \leq \alpha(x) \) for all \( x \in A \).

Let us summarize some results about \( \mathbf{Fzs} \).

Theorem 3.18. 1. \( \mathbf{Fzs} \) is a quasitopos;
2. there exists a proper and stable factorization system given by strong monomorphisms and epimorphisms;
3. fuzzy subsets of \( (A, \alpha) \) correspond to equivalence of strong monomorphisms of codomain \( (A, \alpha) \);
4. the functor

\[
\mathbf{Fzs}^{\text{op}} \to \mathbf{HA} \\
(A, \alpha) \mapsto \mathcal{FzSub}(A, \alpha) \\
f \downarrow \uparrow f^* \\
(B, \beta) \mapsto \mathcal{FzSub}(B, \beta)
\]

where \( \mathcal{FzSub}(A, \alpha) \) is the set of fuzzy subsets of \( (A, \alpha) \) and

\[
f^*(\xi) : A \to [0,1] \\
x \mapsto \alpha(x) \land \xi(f(x))
\]

for any \( \xi \in \mathcal{FzSub}(B, \beta) \), is an elementary hyperdoctrine.

Proof. See [22, Ch. 8]. Explicitly the hyperdoctrine structure is given by:

\[
\exists_f(\xi) : B \to [0,1] \\
y \mapsto \bigvee_{x \in f^{-1}(y)} \xi(x) \\
\forall_f(\xi) : B \to [0,1] \\
y \mapsto \beta(y) \land \bigwedge_{x \in f^{-1}(y)} (\alpha(x) \Rightarrow \xi(x))
\]

for any \( f : (A, \alpha) \to (B, \beta) \) and \( \xi \in \mathcal{FzSub}(A, \alpha) \).
Proof. We have to show continuity of all arrows \( f \). Moreover the fibered equality for a fuzzy set \((A, \alpha)\) must be \( \exists \Delta_{(A, \alpha)}(\alpha) \), i.e.:

\[
\delta_{(A, \alpha)} : A \times A \rightarrow [0, 1]
\]

\[
(x, y) \mapsto \begin{cases} 
\alpha(x) & x = y \\
0 & x \neq y 
\end{cases}
\]

Notice that in \( Fzs \), \((A, \alpha) \times (B, \beta)\) is \((A \times B, \alpha \land \beta)\).

Proposition 3.20. Let \( \mathcal{E} = \{\epsilon_{(A, \alpha)}\}_{(A, \alpha) \in \text{Ob}(Fzs)} \) be a family of functions \( \epsilon_{(A, \alpha)} : (A, \alpha) \rightarrow [0, 1] \). Then, we get an additive closure operator on \( Fzs \) defined as follows:

\[
\epsilon^E_{(A, \alpha)} : Fzs(A, \alpha) \rightarrow Fzs(A, \alpha)
\]

\[
\xi \mapsto (\xi + \epsilon_{(A, \alpha)}) \land \alpha
\]

Proof. We have to show continuity of all arrows \( f : (A, \alpha) \rightarrow (B, \beta) \). Let \( \xi \in (B, \beta) \) and \( x \in A \), we have four cases:

1. \( f^*(\xi)(x) + \epsilon_{(A, \alpha)}(x) < \alpha(x) \) and \( \xi(x) + \epsilon_{(B, \beta)}(x) < \beta(x) \).

\[
(\epsilon^E_{(A, \alpha)}(f^*(\xi)))(x) = (f^*(\xi) + \epsilon_{(A, \alpha)})(x)
\]

\[
= (\alpha(x) \land \xi(f(x))) + \epsilon_{(A, \alpha)}(x)
\]

\[
= \alpha(x) \land (\xi(f(x)) + \epsilon_{(A, \alpha)}(x))
\]

\[
\leq \alpha(x) \land (\xi(f(x)) + \epsilon_{(B, \beta)}(f(x)))
\]

\[
= f^*(\epsilon^E_{(B, \beta)}(\xi))(x)
\]

2. \( f^*(\xi)(x) + \epsilon_{(A, \alpha)}(x) < \alpha(x) \) and \( \xi(f(x)) + \epsilon_{(B, \beta)}(f(x)) \geq \beta(f(x)) \). Notice that \( \alpha(x) \leq \beta(f(x)) \) so

\[
f^*(\epsilon^E_{(B, \beta)}(\xi))(x) = \alpha(x)
\]

from which:

\[
(\epsilon^E_{(A, \alpha)}(f^*(\xi)))(x) = (f^*(\xi) + \epsilon_{(A, \alpha)})(x)
\]

\[
= (\alpha(x) \land \xi(f(x))) + \epsilon_{(A, \alpha)}(x)
\]

\[
= \alpha(x) \land (\xi(f(x)) + \epsilon_{(A, \alpha)}(x))
\]

\[
= \alpha(x)
\]

\[
= f^*(\epsilon^E_{(B, \beta)}(\xi))(x)
\]

3. \( f^*(\xi)(x) + \epsilon_{(A, \alpha)}(x) \geq \alpha(x) \) and \( \xi(x) + \epsilon_{(B, \beta)}(x) < \beta(x) \).

\[
(\epsilon^E_{(A, \alpha)}(f^*(\xi)))(x) = \alpha(x)
\]

\[
= \alpha(x) \land (\xi(f(x)) + \epsilon_{(A, \alpha)}(x))
\]

\[
\leq \alpha(x) \land (\xi(f(x)) + \epsilon_{(B, \beta)}(f(x)))
\]

\[
= f^*(\epsilon^E_{(B, \beta)}(\xi))(x)
\]
Closure Hyperdoctrines, with paths

4. \( f^*(\xi)(x) + \epsilon_{(A,\alpha)}(x) \geq \alpha(x) \) and \( \xi(x) + \epsilon_{(B,\beta)}(x) \geq \beta(x) \).

\[
(c^x_{(A,\alpha)}(f^*(\xi)))(x) = \alpha(x) \\
= \alpha(x) \land \beta(f(x)) \\
= f^*(c^x_{(B,\beta)}(\xi))(x)
\]

We are left with additivity, but this follows immediately since, for \( \xi \) and \( \zeta \in \text{FzSub}(A,\alpha) \) and \( x \in A \) \((\xi \lor \zeta)(x) = \xi(x) \lor \zeta(x)\).

\( △ \)

Remark 3.21. \( c^x \) is not grounded in general.

The condition on the elements of \( E \) is very restrictive. In fact, it can be eased restricting to a suitable subclass of arrows and using the following lemma.

Lemma 3.22. Let \( P : C'^{op} \to \text{InfSL} \) be a doctrine, and \( \epsilon = \{ \epsilon_C : P(C) \to P(C) \}_{c \in \text{Ob}(C)} \) be a family of monotone and inflationary operators. Let \( A \) be a (possibly large) family of \( C \)-arrows such that:

- \( A \) is closed under composition;
- if \( f \in A \) then \( 1_{\text{dom}(A)} \) and \( 1_{\text{cod}(A)} \) are in \( A \);
- \( f : C \to D \) in \( A \) implies \( \epsilon_C \circ P_f \leq P_f \circ \epsilon_D \).

Then \( P \) induces a doctrine \( P^A \) on the subcategory \( C_A \) induced by \( A \) for which \( \epsilon = \{ \epsilon_C \}_{c \in \text{Ob}(C_A)} \) is a closure operator. Moreover, if for all \( f, g \) in \( A \) also \( (f, g) \) and the projections from \( \text{cod}(f) \times \text{cod}(g) \) are in \( A \), then \( P^A \) is existential, elementary or an hyperdoctrine if \( P \) is.

Proof. This is almost tautological since the condition on \( A \) guarantee that the inclusion functor \( C_A \) preserves limits and we can use Proposition 2.5.  

3.5 Coalgebraic examples

Definition 3.23 ([16, 15]). Let \( C \) be a category with finite products and \( F : C \to C \) an endofunctor. The category \( \text{CoAlg}(F) \) of coalgebras for \( F \) has

- arrows \( \gamma_C : C \to F(C) \) as objects;
- arrows \( f : C \to D \) such that \( \gamma_D \circ f = F(f) \circ \gamma_C \) as morphisms \( f : \gamma_C \to \gamma_D \).

Notice that in general \( \text{CoAlg}(F) \) is not complete and products in it can be very different from products in \( C \) [13], so it does not make much sense to look for an existential doctrine on it. However, for Set-based coalgebras we get a primary doctrine \( P^e : \text{CoAlg}(F)^{op} \to \text{InfSL} \) composing the contravariant power object \( P : \text{Set}^{op} \to \text{InfSL} \) with the opposite of the obvious forgetful functor \( \text{CoAlg}(F) \to \text{Set} \).

Definition 3.24. Let \( F : C \to C \) be a functor and \( P \) a primary doctrine on \( C \). A predicate lifting is a natural transformation \( \square : U \circ P \to U \circ P \circ F^{op} \) where \( U \) is the forgetful functor \( \text{InfSL} \to \text{Poset} \).

Let \( \square \) be a predicate lifting. We are going to define two closure operators on \( P^e \).

1. For any coalgebra \( \gamma_X : X \to F(X) \), notice that \( P^e(\gamma_X) = P(X) \); hence we can define \( \text{pre}_{\gamma_X} : P(X) \to P(X) \)

\[
\alpha \mapsto \alpha \lor P_{\gamma_X}(\square_X(\alpha))
\]

2. Suppose that \( P \) admits arbitrary meets; for \( \gamma_X : X \to F(X) \) and \( \alpha \in P(X) \) we define \( s_{\gamma_X}(\alpha) := \bigwedge_{\beta \in N_{\gamma_X}(\alpha)} \beta \) where \( N_{\gamma_X}(\alpha) : = \{ \beta \in P(X) \mid \alpha \leq P_{\gamma_X}(\square_X(\beta)) \} \)
Now we set:

\[ \text{suc}_{\gamma X} : \mathcal{P}(X) \to \mathcal{P}(X) \]

\[ \alpha \mapsto \alpha \lor \text{suc}_{\gamma X}(\alpha) \]

**Lemma 3.25.** Let \( \mathcal{F} : \mathcal{C} \to \mathcal{C} \) be a functor and \( \Box \) a predicate lifting, then:

1. \( \{ \text{pre}_{\gamma X} \}_{\gamma X \in \text{Ob}(\text{CoAlg}(\mathcal{F}))} \) defines a closure operator \( \text{pre} \) on \( \mathcal{P} \).
2. \( \text{suc}_{\gamma X}(\alpha) \) is the minimum of \( \mathcal{N}_{\gamma X}(\alpha) \) whenever \( \mathcal{P} \) has arbitrary meets and, for any coalgebra \( \gamma_X : X \to \mathcal{F}(X) \), \( \mathcal{P}_{\gamma X} \) and \( \Box_X \) commute with them;
3. in the hypothesis above if \( \mathcal{P}f \) commutes with arbitrary meets for all arrows \( f \) then \( \{ \text{suc}_{\gamma X} \}_{\gamma X \in \text{Ob}(\text{CoAlg}(\mathcal{F}))} \) defines a closure operators \( \text{suc} \) on \( \mathcal{P} \).

**Proof.**

1. Clearly \( \alpha \leq \text{pre}_{\gamma X}(\alpha) \); if \( \alpha \leq \beta \) we have that

\[ \mathcal{P}_{\gamma X}(\Box_X(\alpha)) \leq \mathcal{P}_{\gamma X}(\Box_X(\beta)) \]

from which monotonicity follows; for \( f \) an arrow between \( \gamma_X : X \to \mathcal{F}(X) \) and \( \gamma_Y : Y \to \mathcal{F}(Y) \), we have a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\gamma_X \downarrow & & \downarrow \gamma_Y \\
\mathcal{F}(X) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(Y)
\end{array}
\]

and computing we get the thesis:

\[
\text{pre}_{\gamma X}(\mathcal{P}_f(\alpha)) = \mathcal{P}_f(\alpha) \lor \mathcal{P}_{\gamma X}(\Box_X(\mathcal{P}_f(\alpha)))
\]

\[
= \mathcal{P}_f(\alpha) \lor \mathcal{P}_{\gamma X}(\mathcal{P}_{\mathcal{F}(f)}(\Box_Y(\alpha)))
\]

\[
= \mathcal{P}_f(\alpha) \lor \mathcal{P}_{\gamma Y}(\Box_Y(\alpha))
\]

\[
= \mathcal{P}_f(\text{pre}_{\gamma Y}(\alpha))
\]

2. By hypothesis:

\[
\alpha \leq \bigwedge_{\beta \in \mathcal{N}_{\gamma X}(\alpha)} \mathcal{P}_{\gamma X}(\Box_X(\beta))
\]

\[
= \mathcal{P}_{\gamma X}(\bigwedge_{\beta \in \mathcal{N}_{\gamma X}(\alpha)} \Box_X(\beta))
\]

\[
= \mathcal{P}_{\gamma X}(\Box_X(\bigwedge_{\beta \in \mathcal{N}_{\gamma X}(\alpha)} \beta))
\]

\[
= \mathcal{P}_{\gamma X}(\Box_X(\text{suc}_{\gamma X}(\alpha)))
\]

3. The inequality \( \alpha \leq \text{suc}_{\gamma X}(\alpha) \) follows at once, if \( \alpha \leq \beta \) we have \( \mathcal{P}_{\gamma X}(\Box_X(\alpha)) \) as in the first point but this implies that \( \mathcal{N}_{\gamma X}(\beta) \subset \mathcal{N}_{\gamma X}(\alpha) \). Hence, \( \bigwedge_{\beta \in \mathcal{N}_{\gamma X}(\alpha)} \beta \leq \bigwedge_{\beta \in \mathcal{N}_{\gamma X}(\beta)} \beta \), from which we deduce the monotonicity of \( \text{suc}_{\gamma X} \). Let now \( f : X \to Y \) be an arrow such that

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\mathcal{F}(X) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(Y)
\end{array}
\]
commutes, and notice that for all \( \theta \in \mathcal{N}_Y(\alpha) \) then

\[
\mathcal{P}_f(\alpha) \leq \mathcal{P}_f(\mathcal{P}_{\gamma_X}(\square_Y(\theta)))
\]

\[
= \mathcal{P}_{\gamma_X}(\mathcal{P}_f(\mathcal{P}_f(\square_Y(\theta))))
\]

\[
= \mathcal{P}_{\gamma_X}(\square_X(\mathcal{P}_f(\theta)))
\]

hence \( \mathcal{P}_f(\theta) \in \mathcal{N}_X(\mathcal{P}_f(\alpha)) \) and thus

\[
suc_{\gamma_X}(\mathcal{P}_f(\alpha)) = \mathcal{P}_f(\alpha) \vee \mathcal{P}_f(s_{\gamma_X}(\mathcal{P}_f(\alpha)))
\]

\[
= \mathcal{P}_f(\alpha) \vee \bigwedge_{\beta \in \mathcal{N}_X(\mathcal{P}_f(\alpha))} \beta
\]

\[
\leq \mathcal{P}_f(\alpha) \vee \bigwedge_{\beta \in \mathcal{N}_X(\mathcal{P}_f(\alpha))} \mathcal{P}_f(\beta)
\]

\[
\leq \mathcal{P}_f(\alpha) \vee \mathcal{P}_f(\bigwedge_{\beta \in \mathcal{N}_X(\alpha)} \beta)
\]

\[
= \mathcal{P}_f(\alpha \vee \mathcal{P}_f(s_{\gamma_X}(\alpha)))
\]

\[
= \mathcal{P}_f(suc_{\gamma_X}(\alpha))
\]

and we are done. \( \blacktriangleleft \)

The previous proposition provides us with many examples with practical applications.

**Example 3.26 (Kripke frames).** Let \( \mathcal{P} : \text{Set} \to \text{Set} \) be the covariant powerset functor, and \( \mathcal{P} : \text{Set}^{op} \to \text{InfSL} \) be the controvariant one, seen as primary doctrine. We can define a predicate lifting \( \square \) taking as components:

\[
\square_X : \mathcal{P}(X) \to \mathcal{P}(\mathcal{P}(X))
\]

\[
A \mapsto \downarrow A
\]

where \( \downarrow A \) denotes the set of downward-closed subsets of \( A \). In this case for any coalgebra \( \gamma_X : X \to \mathcal{P}(X) \) we have

\[
x \in \gamma_X^{-1}(\downarrow X(A)) \iff \gamma_X(x) \subseteq A
\]

\[
B \in \mathcal{N}_{\gamma_X}(A) \iff \gamma_X(a) \subseteq B \text{ for any } a \in A
\]

so \( s_{\gamma_X}(A) = \bigcup_{a \in A} \gamma_X(a) \) and \( suc_{\gamma_X}(A) = A \cup \bigcup_{a \in A} \gamma_X(a) \).

By this description it is clear that \( suc \) is grounded and fully additive. \( pre \) is grounded too but it is not even finitely additive: take \( 4 := \{0, 1, 2, 3\} \) with structural map \( \gamma_4 \) given by

\[
0 \mapsto \{3\} \quad 1 \mapsto \{2, 3\}
\]

\[
2 \mapsto \{2\} \quad 3 \mapsto \{3\}
\]

Now take \( A := \{2, 3\} \), it is immediate to see that \( \text{pre}_{\gamma_4}(A) = 4 \), on the other hand \( \text{pre}_{\gamma_4}(\{2\}) = \{2\} \) and \( \text{pre}_{\gamma_4}(\{3\}) = \{0, 3\} \).

**Remark 3.27.** In this case the meaning of (and the notation for) \( pre \) and \( suc \) becomes clearer: if we think to the value of \( \gamma_X(x) \) as the family of points accessible from \( x \in X \) then \( pre_{\gamma_X} \) add to a subset \( A \) the set of its predecessors, i.e. points from which some \( a \in A \) is accessible, while \( suc_{\gamma_X} \) add the set of point successors, i.e. points which are accessible from some point of \( A \).
Example 3.28 (Probabilistic frames [12, 3, 4]). Let $\text{Meas}$ be the category of measurable space and measurable functions, we can take as primary doctrine $P$ the functor

$$(X, \Omega_X) \mapsto \Omega_X$$

$$f \downarrow \downarrow f^{-1}$$

$$(Y, \Omega_Y) \mapsto \Omega_Y$$

As endofunctor we can take the Giry monad $G : \text{Meas} \to \text{Meas}$:

- given an object $(X, \Omega_X)$, $G(X, \Omega_X)$ is the set

$\{ \mu : \Omega_X \to [0, 1] \mid \mu \text{ is a probability measure on } \Omega_X \}$

equipped with the smallest $\sigma$-algebra for which all the evaluation functions

$\text{ev}_A : G(X, \Omega_X) \to [0, 1]$

$\mu \mapsto \mu(A)$

with $A \in \Omega_X$, are Borel-measurable.

- for a measurable $f : (X, \Omega_X) \to (Y, \Omega_Y)$,

$G(f) : G(X, \Omega_X) \to G(Y, \Omega_Y)$

$\mu \mapsto \mu \circ f^{-1}$

(For the measurability of $G(f)$ notice that given a Borel subset $L$ of $[0, 1]$ and $A \in \Omega_Y$ we have that $\mu \in G(f)^{-1}(\text{ev}_A(L)) \iff \mu \in \text{ev}_{f^{-1}(A)}(L)$)

For a coalgebra $\gamma(X, \Omega_X)$ and $p \in [0, 1]$ we define

$\square_{(X, \Omega_X), p} : \Omega_X \to \mathcal{P}(G(X))$

$A \mapsto \{ \mu \in G(X, \Omega_X) \mid \mu(A) \geq p \}$

notice that the set on the right is $\text{ev}_A^{-1}([p, 1])$ and so $\square_{(X, \Omega_X), p}$ is well defined. In this situation we have

$\text{pre}_{\gamma(X, \Omega_X)}(A) := A \cup \{ x \in X \mid p \leq \gamma(X, \Omega_X)(x)(A) \}$

Remark 3.29. If we think of a coalgebra $\gamma(X, \Omega_X)$ as describing how likely is a transition from a state to the various $A \in \Omega_X$ then, given a $p \in [0, 1]$, $\text{pre}_{\gamma(X, \Omega_X)}(A)$ is the set of points which access $A$ with probability at least $p$.

4 Logics for Closure Operators

In this section, we provide a sound and complete logic for closure hyperdoctrines. This logic is a (first order) version of Spatial Logic for Closure Spaces (SLCS) [8], although with a slightly different presentation.

4.1 Syntax and derivation rules

We briefly recall the categorical presentation of signatures, as in [14].

Definition 4.1. A signature $\Sigma$ is a triple $(|\Sigma|, \Gamma, \Pi)$ where
Closure Hyperdoctrines, with paths

- $|\Sigma|$ is a set, called the set of basic types;
- $\Gamma$ is a function $|\Sigma|^* \times |\Sigma|$ \rightarrow \text{Sets}$ is a function symbol, we will call function symbol an element $f$ of $\Gamma((\sigma_1, \ldots, \sigma_n), \sigma_{n+1})$ and we will write $f : \sigma_1, \ldots, \sigma_n \rightarrow \sigma_{n+1};$
- $\Pi$ is a functor $|\Sigma|^* \rightarrow \text{Set}$, we will call predicate symbol an element $P$ of $\Pi(\sigma_1, \ldots, \sigma_n)$ and we will write $P : \sigma_1, \ldots, \sigma_n$. A morphism of signature $\phi : \Sigma_1 \rightarrow \Sigma_2$ is a triple $(\phi_1, \phi_2, \phi_3)$ such that
  - $\phi_1$ is a function $|\Sigma_1| \rightarrow |\Sigma_2|$;
  - $\phi_2$ is a natural transformation $\Gamma_1 \rightarrow \Gamma_2 \circ (\phi_1 \times \phi_1)$;
  - $\phi_3$ is a natural transformation $\Pi_1 \rightarrow \Pi_2 \circ \phi_1^*.$

For any $\sigma \in |\Sigma|$ we fix an countably infinite set $X_\sigma$ of variables; definition of terms is straightforward [14].

> **Definition 4.2.** Given a signature $\Sigma$, its classifying category is the category $\text{Cl}(\Sigma)$ in which

- objects are contexts;
- Given $\Gamma := [x_i : \sigma_i]_{i=1}^n$ and $\Delta = [y_i : \tau_i]_{i=1}^m$ an arrow $\Gamma \rightarrow \Delta$ is a $m$-uple of terms $(T_1, \ldots, T_n)$ such that $\Gamma \vdash T_i : \tau_i$ for any $i$;
- composition is given by substitution.

> **Proposition 4.3.** $\text{Cl}(\Sigma)$ is a category with finite products for any signature $\Sigma$.

**Proof.** Associativity of composition and the fact that $(x_1, \ldots, x_n)$ is the identity for $[x_i : \sigma_i]_{i=1}^n$ follows from a straightforward computation. The empty context is clearly terminal while, given two contexts $\Gamma := [x_i : \sigma_i]_{i=1}^n$ and $\Delta = [y_i : \tau_i]_{i=1}^m$ we can take their concatenation as a product $\Gamma \times \Delta$, the universal property follows immediately. ▲

Now we can introduce the rules for context and closure operators of the Spatial Logic for Closure Spaces, over any given signature.

As usual, we denote by $\Gamma \vdash t : \tau$ the judgment “$t$ has type $\tau$ in context $\Gamma$”, and by $\Gamma \vdash \phi : \text{Prop}$ the judgment “$\phi$ is a well-formed formula in context $\Gamma$”.

> **Definition 4.4.** The rules for contexts and well-formed formulae for the closure operators for a signature $\Sigma$ are the usual ones for a first order signature (see [14]) plus:

$$
\Gamma \vdash : \text{Prop} \quad \Gamma \vdash C(\phi) : \text{Prop} \quad \Gamma \vdash \phi : \text{Prop} \quad \Gamma \vdash : \text{Prop} \quad \Gamma \vdash \psi : \text{Prop} \quad \Gamma \vdash : \text{Prop} \quad \Gamma \vdash U\psi : \text{Prop} \quad \Gamma \vdash : \text{Prop} \quad \Gamma \vdash : \text{Prop} \quad \Gamma \vdash : \text{Prop}
$$

For any context $\Gamma$ we define $\text{Form}_\Sigma(\Gamma)$ to be the set of formulae $\phi$ such that $\Gamma \vdash \phi : \text{Prop}$.

Then, we can introduce the rules for the logical judgments of the form $\Gamma | \Phi \vdash \phi$, where $\Phi$ is a finite set of propositions well-formed in $\Gamma$.

> **Definition 4.5.** We define four rules for the well-formed formulae previously defined:

- $C$’s rules:
  $$
  \frac{\Gamma | \Phi \vdash \psi}{\Gamma | \Phi \vdash C(\psi)} \quad \text{Cl-1} \\
  \frac{\Gamma | \Phi, C(\psi) \vdash \phi}{\Gamma | \Phi, C(\psi) \vdash C(\phi)} \quad \text{Cl-2}
  $$

- $U$’s rules:
  $$
  \frac{\Gamma | \Phi, \varphi \vdash \phi}{\Gamma | \Phi, \varphi \vdash \phi \cup \psi} \quad \text{U-1} \\
  \frac{\Gamma | \Phi, \varphi \vdash \phi \cup \psi}{\Gamma | \Phi, \varphi \vdash \psi, \varphi \vdash \theta} \quad \text{U-E}
  $$

for all $\phi$ such that $\Gamma \vdash \varphi : \text{Prop}$.

**Proof.** Associativity of composition and the fact that $(x_1, \ldots, x_n)$ is the identity for $[x_i : \sigma_i]_{i=1}^n$ follows from a straightforward computation. The empty context is clearly terminal while, given two contexts $\Gamma := [x_i : \sigma_i]_{i=1}^n$ and $\Delta = [y_i : \tau_i]_{i=1}^m$ we can take their concatenation as a product $\Gamma \times \Delta$, the universal property follows immediately. ▲

Now we can introduce the rules for context and closure operators of the Spatial Logic for Closure Spaces, over any given signature.

As usual, we denote by $\Gamma \vdash t : \tau$ the judgment “$t$ has type $\tau$ in context $\Gamma$”, and by $\Gamma \vdash \phi : \text{Prop}$ the judgment “$\phi$ is a well-formed formula in context $\Gamma$”.

> **Definition 4.4.** The rules for contexts and well-formed formulae for the closure operators

for a signature $\Sigma$ are the usual ones for a first order signature (see [14]) plus:

$$
\Gamma \vdash : \text{Prop} \quad \Gamma \vdash C(\phi) : \text{Prop} \quad \Gamma \vdash \phi : \text{Prop} \quad \Gamma \vdash : \text{Prop} \quad \Gamma \vdash \psi : \text{Prop} \quad \Gamma \vdash : \text{Prop} \quad \Gamma \vdash U\psi : \text{Prop} \quad \Gamma \vdash : \text{Prop} \quad \Gamma \vdash : \text{Prop} \quad \Gamma \vdash : \text{Prop}
$$

For any context $\Gamma$ we define $\text{Form}_\Sigma(\Gamma)$ to be the set of formulae $\phi$ such that $\Gamma \vdash \phi : \text{Prop}$.

Then, we can introduce the rules for the logical judgments of the form $\Gamma | \Phi \vdash \phi$, where $\Phi$ is a finite set of propositions well-formed in $\Gamma$.

> **Definition 4.5.** We define four rules for the well-formed formulae previously defined:

- $C$’s rules:
  $$
  \frac{\Gamma | \Phi \vdash \psi}{\Gamma | \Phi \vdash C(\psi)} \quad \text{Cl-1} \\
  \frac{\Gamma | \Phi, C(\psi) \vdash \phi}{\Gamma | \Phi, C(\psi) \vdash C(\phi)} \quad \text{Cl-2}
  $$

- $U$’s rules:
  $$
  \frac{\Gamma | \Phi, \varphi \vdash \phi}{\Gamma | \Phi, \varphi \vdash \phi \cup \psi} \quad \text{U-1} \\
  \frac{\Gamma | \Phi, \varphi \vdash \phi \cup \psi}{\Gamma | \Phi, \varphi \vdash \psi, \varphi \vdash \theta} \quad \text{U-E}
  $$

for all $\phi$ such that $\Gamma \vdash \varphi : \text{Prop}$.
The Propositional Logic for Closure Operators on $\Sigma$ (PLCO) is given by the usual propositional rules (i.e., without the quantifiers) for the typed (intuitionistic) sequent calculus (see e.g. [14]), extended with the four rules above.

The Regular Logic for Closure Operators on $\Sigma$ (RLCO) is given by the four rules above, plus the rules for conjunction, $\top$ and the existential quantifier only.

Finally, the First Order Logic for Closure Operators on $\Sigma$ (FOLCO) is given by the four rules above added to the usual rules for first order logic. Similarly with equality.

Formal derivations are defined in the usual way ([21]).

Remark 4.6. PLCO corresponds to the Spatial Logic for Closure Spaces considered in [7].

Remark 4.7. Notice that $U$-E is an infinitary rule saying that a formula $\theta$ can be derived from $\phi U \psi$ if it can be derived from all the formulae $\phi$ satisfying precise conditions. Thus, this rule shows the second-order nature of the $U$ operator.

4.2 Categorical semantics of closure logics

In this section we provide a sound and complete categorical semantics of the logics for the closure operators defined above.

Definition 4.8. Two formulae $\phi, \psi \in \text{Form}_\Sigma(\Gamma)$ are provably equivalent if $\Gamma | \psi \vdash \phi$ and $\Gamma | \phi \vdash \psi$. We will denote the quotient of $\text{Form}_\Sigma(\Gamma)$ by this relation with $L(\Sigma)(\Gamma)$, $[\phi]$ will denote the class of $\phi$ in it.

Proposition 4.9. For any signature $\Sigma$ the following are true:

1. $L(\Sigma)(\Gamma)$ equipped with the order $[\phi] \leq [\psi]$ if and only if $\Gamma | \phi \vdash \psi$ is derivable is:
   - a meet semilattice in the case we are considering regular logic;
   - a Heyting algebra if we are considering propositional or first order logic;

2. $[\phi U \psi]$ is the supremum of the set
   
   \[ u_T(\phi, \psi) := \{ [\varphi] \in L(\Sigma)(\Gamma) \text{ such that } \Gamma | \varphi \vdash \phi, \Gamma | C(\varphi), \neg \varphi \vdash \psi \} \]

3. there exists a (elementary) closure or existential doctrine or a (elementary) hyperdoctrine $(L(\Sigma), c_\Sigma)$ on $\text{Cl}(\Sigma)$ sending $\Gamma$ to $L(\Sigma)(\Gamma)$.

Proof. 1. The logical connectives induce a Heyting algebra or a meet semilattice structure on $L(\Sigma)(\Gamma)$ which has precisely $\leq$ as associated order.

2. From $U$-I follows that $[\phi U \psi]$ is an upper bound for $u_T$ while $U$-E implies that $[\phi U \psi]$ is the least of them.

3. For any morphism $(T_1, \ldots, T_n) : \Gamma \rightarrow \Delta$ substitution of terms gives us a morphism of Heyting algebras/meet semilattices $L(\Sigma)(\Delta) \rightarrow L(\Sigma)(\Gamma)$; quantifiers gives us the existential doctrine/hyperdoctrine structure (cfr. [21] for the details). In any case have to define a preclosure operator $c_{\Sigma, \Gamma}$ on each $L(\Sigma)(\Gamma)$ but this is easily done defining

\[ c_{\Sigma, \Gamma} : L(\Sigma)(\Gamma) \rightarrow L(\Sigma)(\Gamma) \]

\[ [\phi] \mapsto [C(\phi)] \]

The $C$’s rules assure us that $c_{\Sigma}$ is well defined, inflationary and monotone, while an easy induction shows that

\[ L(\Sigma)(T_1, \ldots, T_n)([C(\phi)]) = c_{\Sigma, \Gamma}(L(\Sigma)(T_1, \ldots, T_n)(\phi)) \]
for any \((T_1, \ldots, T_n) : \Gamma \to \Delta\). We can add fibered equalities, given \(\Gamma := [x_i : \sigma_i]\) putting:

\[
\delta_{\Gamma \times \Gamma} := \bigwedge_{i=1}^n [x_i = \sigma_i, y_i]
\]

where \(\{y_i\}_{i=1}^n\) is a set of fresh variables such that \(y_i : \sigma_i\) for any \(i\).

Let us prove the soundness and completeness of the categorical semantics wrt. the various logical fragments.

\textbf{Definition 4.10.} Let \((\mathcal{P}, c) : \text{C} \to \text{InfSL}\) be an (elementary) closure doctrine (existential doctrine/hyperdoctrine) then a morphism of \(c\text{PD}\) (\(c\text{ED}, c\text{EED}, c\text{EHD}, c\text{HD}\)) \((\mathcal{M}, \mu) : \mathcal{L}(\Sigma) \to \mathcal{P}\) is a model of the propositional (regular, first-order) logic (with equality) of closure operators in \((\mathcal{P}, c)\) if it is open.

A sequent \(\Gamma | \Phi \vdash \psi\) is satisfied by \((\mathcal{M}, \mu)\) if \(\bigwedge_{\phi \in \Phi} \mu_{\Gamma}(\phi) \leq \mu_{\Gamma}(\psi)\).

\textbf{Theorem 4.11.} A sequent \(\Gamma | \Phi \vdash \psi\) is satisfied by the generic model \((1_{\text{Cl}(\Sigma)}, 1_{\mathcal{L}(\Sigma)})\) if and only if it is derivable.

\textbf{Proof.} By definition, \(\Gamma | \Phi \vdash \psi\) is satisfied if and only if

\[
\bigwedge_{\phi \in \Phi} [\phi] \leq [\psi]
\]

in \(\mathcal{L}(\Sigma)(\Gamma)\) but this is equivalent to the derivability of

\[
\Gamma | \bigwedge_{\phi \in \Phi} \phi \vdash \psi
\]

whose derivability is equivalent (applying the conjunction rules a finite number of times) to

\[
\Gamma | \Phi \vdash \psi
\]

and we are done.

\textbf{Corollary 4.12.} The above defined categorical semantics for PLCO/RLCO/FOLCO (with or without equality) is sound and complete.

\textbf{Proof.} The only thing left to show is soundness for an arbitrary \((\mathcal{P}, c)\) but this follows at once since each component \(\mu_{\Gamma}\) of \(\mu\) is monotone.

\section{4.3 About the semantics of \(\mathcal{U}\)}

As we have remarked before, the rule \(\mathcal{U}\text{-E}\) for the operator \(\mathcal{U}\) is infinitary. Although in general this is needed, in this section we will define a class of hyperdoctrines in which the semantics of \(\mathcal{U}\) can be given as a supremum of approximants.

\textbf{Definition 4.13.} Let \((\mathcal{P}, c) : \text{C} \to \text{InfSL}\) be a closure doctrine that factors through the category of Heyting algebras. For any object \(C\) define the external boundary:

\[
\partial_C^+: \mathcal{P}(C) \to \mathcal{P}(C)
\]

\[
\alpha \mapsto c_C(\alpha) \land \neg \alpha
\]

For \(\phi\) and \(\psi\) in \(\mathcal{P}(C)\), we define \(\phi \Delta_C \psi \in \mathcal{P}(C)\) as the supremum, if it exists, of the set

\[
u_C(\phi, \psi) := \{\varphi \in \mathcal{P}(C) \mid \varphi \leq \phi \text{ and } \partial_C^+(\varphi) \leq \psi\}\]
Remark 4.14. If \( \mathcal{P} \) is \( \mathcal{L}(\Sigma) \) then \([\phi] u_{\mathcal{U}} \psi = [\phi \mathcal{U} \psi] \) for any \([\phi] \) and \([\psi] \in \mathcal{L}(\Sigma)(\Gamma)\).

Remark 4.15. If \((\mathcal{M}, \mu)\) is a model then \(\mu_\Gamma(u_\mathcal{U}(\phi, \psi)) \subseteq u_{\mathcal{M}(\Gamma)}(\mu_\Gamma([\phi]), \mu_\Gamma([\psi]))\) for any \(\Gamma\).

Example 4.16. Let \((X, c)\) be a pretopological space and \(S, T \in \mathcal{P}^p(X, c)\), then

\[
S \cup (X, c)T = \bigcup \{ W \subset S \mid \partial^+_X(W) \subset T \}
\]

i.e. \(x \in S \cup (X, c)T\) if and only if there exists \(W \subset S\) such that \(X \in W\) and \(\partial^+_X(W) \subset T\).

Example 4.17. Let us consider the closure operator \(c_\xi\) on \(\text{Set}(-, [0, 1])\) (see Section 3.3).

For any \(f : X \to [0, 1]\), it is \((\neg f)(x) = 1\) if and only if \(f(x) = 0\). So,

\[
(\epsilon_X, \xi(f) \land \neg f)(x) = \begin{cases} 
\epsilon & f(x) = 0 \\
0 & f(x) \neq 0
\end{cases}
\]

hence, given \(g, h : X \to [0, 1]\), \(f \in u_T(g, h)\) if and only if \(f \leq g\) and \(h(x) \geq \epsilon\) for any \(x \in f^{-1}(0)\).

Definition 4.18. Let \((\mathcal{P}, c)\) be as in Definition 4.13. A model \((\mathcal{M}, \mu) : \mathcal{L}(\Sigma) \to (\mathcal{P}, c)\) is said continuous if the equality

\[
\mu_\Gamma([\phi \mathcal{U} \psi]) = \mu_\Gamma([\phi]) u_{\mathcal{M}(\Gamma)}(\mu_\Gamma([\psi]))
\]

holds for any context \(\Gamma\) and \([\phi], [\psi] \in \mathcal{L}(\Sigma)(\Gamma)\).

Remark 4.19. If \((\mathcal{M}, \mu)\) is a model (not necessarily continuous) then for any \([\varphi] \in \mathcal{L}(\Sigma)(\Gamma)\) such that \(\varphi \in u_T(\phi, \psi)\) we have \(\mu_\Gamma([\varphi]) \leq \mu_\Gamma([\phi \mathcal{U} \psi])\).

Proposition 4.20. Let \(\Sigma\) be a signature and \((\mathcal{P}, c)\) a complete (elementary, existential, or hyper)doctrine, i.e. \(\mathcal{P}(C)\) is complete for any object \(C\) of \(\mathcal{C}\); then, for any product preserving functor: \(\mathcal{M} : \text{Cl}(\Sigma) \to \mathcal{C}\) and functions

\[
\mu_\Gamma^\mathcal{M} : \Pi(\sigma_1, \ldots, \sigma_n) \to \mathcal{P}(\mathcal{M}(\Gamma))
\]

for all \(\Gamma = [\sigma_i : \sigma_i]_{i=1}^n\), there exists a unique continuous model \((\mathcal{M}, \mu)\) in \((\mathcal{P}, c)\) such that

\[
\mu_\Gamma([P(x_1, \ldots, x_n)]) = \mu_\Gamma^\mathcal{M}(P)
\]

Proof. This follows immediately by induction.

Example 4.21. Let \(\mathcal{X} = \{(X_i, c_i)\}_{i \in I}\) be a small family of pretopological spaces and let us define \(\Sigma\) as follows:

\[
|\Sigma| := \mathcal{X} \quad \Gamma((X_{i_1}, c_{i_1}), \ldots, (X_{i_n}, c_{i_n}), (X_j, c_j)) := \text{PrTop}(\prod_{k=1}^n (X_{i_k}, c_{i_k}), (X_j, c_j))
\]

\[
\Pi(X_{i_1}, c_{i_1}), \ldots, (X_{i_n}, c_{i_n}) := \mathcal{P}(\prod_{k=1}^n X_{i_k})
\]

We can take as \(\mathcal{M}\) the unique product preserving functor \(\text{Cl}(\Sigma) \to \text{PrTop}\) such that

\[
(X_i, c_i) \mapsto (X_i, c_i)
\]

\[
f \downarrow \quad f
\]

\[
(X_i, c_i) \mapsto (X_i, c_i)
\]
18 Closure Hyperdoctrines, with paths

i.e., \( M \) sends contexts to products and list of terms to the corresponding product arrow. We can define \( \mu^* \) sending each predicate \( P : (X_i, \epsilon_i), \ldots, (X_n, \epsilon_n) \) to corresponding subset of \( \prod_{i=1}^n (X_i, \epsilon_i) \). Example 4.16 guarantees that this semantics is the same as the one developed in [7].

**Proposition 4.22.** For any signature \( \Sigma \) a sequent is derivable if and only if it is satisfied by any continuous model.

**Proof.** This follows immediately by the fact that the generic model is continuous. \( \triangleright \)

5 Paths in closure doctrines

Often, in spatial logics we are interested also on reachability of some property. Differently from closure and the “until” operator, reachability is not a structural property of the logical domain; rather, it depends on the kind of paths we choose to explore the space. In this section we formalise this idea, and show how to interpret also the \( \mathcal{S} \) operator from SLCS.

5.1 The reachability closure operator

**Definition 5.1.** Let \( \mathcal{P} : C^{op} \rightarrow HA \) be an hyperdoctrine, an internal preorder in \( \mathcal{P} \) is a pair \( (I, \rho) \) where \( I \) is an object of \( C \) and \( \rho \in \mathcal{P}(I \times I) \) such that is reflexive \( (\delta_1 \leq \rho) \) and transitive \( (\mathcal{P}_{(\pi_1, \pi_2)}(\rho) \land \mathcal{P}_{(\pi_2, \pi_3)}(\rho) \leq \mathcal{P}_{(\pi_1, \pi_3)}(\rho)) \).

\( (I, \rho) \) is called an internal order if in addition \( \rho \) is antisymmetric, i.e. \( \rho \land \mathcal{P}_{(\pi_2, \pi_1)}(\rho) \leq \delta_1 \). Moreover \( (I, \rho) \) is total if \( \rho \lor \mathcal{P}_{(\pi_2, \pi_1)}(\rho) = \top \).

An internal monotone arrow \( f : (I, \rho) \rightarrow (J, \sigma) \) is an arrow of \( C \) such that \( \rho \leq \mathcal{P}_{I \times J}(\sigma) \).

**Definition 5.2.** Let \( (\mathcal{P}, c) : C^{op} \rightarrow HA \) be an elementary existential closure doctrine, we say that \( \phi \in \mathcal{P}(C) \) is connected if \( \phi \lor \psi = \phi \) and \( c(\phi) \land \psi = \bot \) imply \( \phi = \bot \).

An object \( C \) is \( \mathcal{P} \)-connected if \( \top \in \mathcal{P}(C) \) is connected.

**Definition 5.3.** Given a preorder \( (I, \rho) \) in an elementary existential doctrine \( \mathcal{P} \) and \( \alpha \in \mathcal{P}(I) \) we define the downward and upward closure of \( \alpha \) as

\[ \downarrow \alpha := \exists_{\pi_1} (\mathcal{P}_{\pi_2}(\alpha) \land \rho) \quad \uparrow \alpha := \exists_{\pi_2} (\mathcal{P}_{\pi_1}(\alpha) \land \rho) \]

We define the reachability operator \( \text{reach} \) as the family of functions, indexed over the objects:

\[ \text{reach}_C : \mathcal{P}(C) \rightarrow \mathcal{P}(C) \]

\[ \phi \mapsto \phi \lor \bigvee_{p \in C(I,C)} \exists_p(\uparrow P_p(\phi)) \]

**Proposition 5.4.** Given \( \mathcal{P} : C^{op} \rightarrow \text{InfSL} \) in \( \text{EED} \) and \( (I, \rho) \) an internal preorder in it, \( \text{reach} = \{ \text{reach}_C \}_C \in \text{Ob}(C) \) is a grounded closure operator on \( \mathcal{P} \). If, moreover, \( \mathcal{P} \) is an hyperdoctrine, \( \text{reach} \) is fully additive.

**Proof.** Monotonicity and inflationarity comes at once, take an arrow \( f : C \rightarrow D \), for any
$\alpha \in \mathcal{P}(D)$ we have:

$$\exists f (\text{reach}_C(\mathcal{P}_f(\varphi))) = \exists f (\mathcal{P}_f(\varphi)) \vee \bigvee_{p \in C(I,C)} \exists f (\exists \pi_2 (\rho \land \mathcal{P}_{\pi_1}(\mathcal{P}_p(\mathcal{P}_f(\varphi)))))$$

$$= \exists f (\mathcal{P}_f(\varphi)) \vee \bigvee_{p \in C(I,C)} \exists f (\exists \pi_2 (\rho \land \mathcal{P}_{\pi_1}(\mathcal{P}_f(\varphi))))$$

$$\leq \varphi \vee \bigvee_{p \in C(I,C)} \exists f (\exists \pi_2 (\rho \land \mathcal{P}_{\pi_1}(\mathcal{P}_f(\varphi))))$$

$$\leq \varphi \vee \bigvee_{q \in C(I,D)} \exists f (\exists \pi_2 (\rho \land \mathcal{P}_{\pi_1}(\mathcal{P}_q(\varphi))))$$

$$= \varphi \vee \bigvee_{q \in C(I,D)} \exists f (\exists \pi_2 (\rho \land \mathcal{P}_{\pi_1}(\mathcal{P}_q(\varphi))))$$

$$= \text{reach}_D(\varphi)$$

Groundedness is immediate; suppose now that $\mathcal{P}$ is an hyperdoctrine, then $\mathcal{P}_f$ commutes with suprema for any arrow $f$ and, since $\mathcal{P}(C)$ is an Heyting algebra, infima distribute over them, so:

$$\text{reach}_C(\bigvee_{k \in K} \varphi_k) = (\bigvee_{k \in K} \varphi_k) \vee \bigvee_{p \in C(I,C)} \exists f (\exists \pi_2 (\rho \land \mathcal{P}_{\pi_1}(\mathcal{P}_p(\bigvee_{k \in K} \varphi_k))))$$

$$= (\bigvee_{k \in K} \varphi_k) \vee \bigvee_{p \in C(I,C)} \exists f (\exists \pi_2 (\rho \land \mathcal{P}_{\pi_1}(\varphi_k))))$$

$$= \bigvee_{k \in K} (\varphi_k \vee \bigvee_{p \in C(I,C)} \exists f (\exists \pi_2 (\rho \land \mathcal{P}_{\pi_1}(\varphi_k))))$$

$$= \bigvee_{k \in K} \text{reach}_C(\varphi_k)$$

\begin{itemize}
  \item **Example 5.5.** In $\text{Set}$, for any non empty $I$ we have that, for any set $X$:

  $$\text{reach}_X(S) = \begin{cases} X & S \neq \emptyset \\ \emptyset & S = \emptyset \end{cases}$$

  \item **Example 5.6.** Take the elementary hyperdoctrine $\mathcal{P}^p$ on $\text{PrTop}$ (Definition 3.4) and fix an $n \in \mathbb{N}$, as an internal order we can take $n = \{0, 1, \ldots, n-1\}$ with the closure operator

  $$n : \mathcal{P}(n) \rightarrow \mathcal{P}(n)$$

  $$S \mapsto \{ i \in n \mid n = s + 1 \text{ for some } s \in S \}$$

  and the usual ordering $\leq$ as $\rho$. An arrow $p : (I, n) \rightarrow (X, \epsilon)$ is just a function such that

  $$n(p^{-1}(S)) \leq p^{-1}(\epsilon(S))$$

  that is, $p(i+1) \in \epsilon(\{p(i)\})$. So, for instance

  $$\text{reach}_{(n,n)}(S) = \{ k \in \mathbb{N} \mid k = s + n \text{ for some } s \in S \}$$

  where $n : \mathcal{P}(\mathbb{N}) \rightarrow \mathcal{P}(\mathbb{N})$ is defined as for $n$. 
\end{itemize}
5.2 Surroundedness

In this section we will introduce a surrounded operator (similar to the “until” operator of temporal logic) in order to generalize the analogous operator introduced in [8].

**Definition 5.7.** Let \( (I, \rho) \) be an internal order in an elementary existential closure doctrine \((\mathcal{P}, c)\), let \( \phi \) and \( \psi \) \in \mathcal{P}(C)\). We say that an arrow \( p : I \to C \) is an escape route from \( \phi \) avoiding \( \psi \) if

1. at some point in \( p, \phi \) holds: \( \exists_t (\mathcal{P}_p(\phi)) = T; \)
2. from the points where \( \phi \) holds we can reach a point where \( \neg \phi \) holds: \( \mathcal{P}_p(\phi) \leq \downarrow \mathcal{P}_p(\neg \phi); \)
3. there is no point reachable from \( \phi \) and which reaches \( \neg \phi \) along the route, where \( \psi \) holds: \( \nabla \mathcal{P}_p(\phi) \land \neg \mathcal{P}_p(\neg \phi) \land \mathcal{P}_p(\psi) = \bot. \)

We will denote with \( \text{Esc}_C(\phi, \psi) \) the set of such arrows. We also define

\[
\phi \text{Esc}_C := \bigvee_{p \in \text{Esc}_C(\phi, \psi)} \exists_p(T) \quad \phi \text{Esc}_C := \bigwedge_{p \in \text{Esc}_C(\phi, \psi)} \phi \land \neg (\exists_p(T))
\]

Intuitively, \( \phi \text{Esc}_C \) (read “\( \phi \) escapes \( \psi \)”) holds where \( \phi \) holds and it is possible to escape avoiding \( \psi \); conversely, \( \phi \text{Esc}_C \) (read “\( \phi \) is surrounded by \( \psi \)”) holds where \( \phi \) holds and it is not possible to escape from it without avoiding \( \psi \). Notice that these notions depend on the specific choice of the internal order \((I, \rho)\), hence we can deal with different reachability, with different shapes of escape routes, by choosing the adequate internal order.

**Example 5.8** (cfr. [8]). Let us consider the closure hyperdoctrine on pretopological spaces \((\mathcal{P}^p, c)\) as in Definition 5.4. In this case an internal order is just an ordered set \((X, \leq)\) equipped with a closure operator. Given \( S \) and \( T \) subsets of a chosen \((X, \leq)\), then

\[
p \in \text{Esc}_{(X, \leq)}(S, T) \text{ if and only if }
\]

1. \( p^{-1}(S) \neq \emptyset; \)
2. for any \( t \) such that \( p(t) \in S \) there exists an \( s \geq t \) with \( p(s) \notin S; \)
3. \( p(t) \notin T \) for any \( t \in I \) for which there exist \( s \) and \( v \in I \) such that \( p(s) \in A, p(v) \in T \) and \( s \leq t \) \leq v.

\[
x \in S \text{Esc}_{(X, \leq)}(S, T) \text{ if and only if there exists a continuous } p : I \to X, t, s \in I \text{ such that } t \leq s, p(t) = x, p(s) \notin S \text{ and for any pair } (u, v) \leq s \text{ with } p(u) \in S \text{ and } p(v) \in T \text{ there are no } w \text{ between } u \text{ and } v \text{ such that } p(w) \in T.
\]

\[
x \in S \text{Esc}_{(X, \leq)}(S, T) \text{ if and only if } x \in S \text{ and for any continuous } p : I \to X \text{ such that } p(t) = x \text{ for some } t \in I, p \notin \text{Esc}_{(X, \leq)}(S, T).
\]

Therefore, this situation corresponds to the surround operator defined in [8].

**Theorem 5.9.** Let \((\mathcal{P}, c)\) be a boolean elementary closure hyperdoctrine, \((I, \rho)\) a preorder in it with \( \mathcal{P} \)-connected and such that, for all \( \gamma \in \mathcal{P}(I), \varepsilon_I(\gamma) \land \neg \gamma \leq \uparrow \gamma. \) Then, for any \( \phi \) and \( \psi \in \mathcal{P}(C)\):

1. if \( \alpha \in u_c(\phi, \psi) \) and \( p \in \text{Esc}_C(\phi, \psi) \) then \( \mathcal{P}_p(\alpha) = \bot; \)
2. \( \phi \text{Esc}_C \psi \leq \phi \text{Esc}_C \psi. \)

**Proof.** 1. By continuity we have

\[
\varepsilon_I(\mathcal{P}_p(\alpha)) \land \mathcal{P}_p(\neg \alpha) \leq \mathcal{P}_p(\varepsilon_C(\alpha)) \land \mathcal{P}_p(\neg \alpha) \\
\leq \mathcal{P}_p(\varepsilon_C(\alpha) \land \neg \alpha) \\
\leq \mathcal{P}_p(\psi)
\]
By hypothesis,
\[ c_I(P_p(\alpha)) \land P_p(\neg\alpha) \leq \uparrow P_p(\phi) \leq P_p(\phi) \]
and
\[ c_I(P_p(\alpha)) \land P_p(\neg\alpha) = c_I(P_p(\alpha)) \land P_p(\neg\alpha) \land \top \]
\[ = (c_I(P_p(\alpha)) \land P_p(\neg\alpha) \land P_p(\phi)) \lor (c_I(P_p(\alpha)) \land P_p(\neg\alpha) \land P_p(\neg\phi)) \]
\[ \leq \downarrow P_p(\neg\phi) \]
\[ \text{hence, since } p \in \text{Esc}_{C}(\phi, \psi) : \]
\[ c_I(P_p(\alpha)) \land P_p(\neg\alpha) \leq \uparrow P_p(\phi) \land \downarrow P_p(\neg\phi) \land P_p(\psi) \]
\[ = \bot \]
\[ \text{and we conclude by connectedness.} \]

2. By the previous point \( P_p(\alpha) = \bot \) for any \( p \in \text{Esc}_{C}(\phi, \psi) \) so \( P_p(\neg\alpha) = \top \) that implies \( \alpha \leq \neg \exists p(\top) \) from which the thesis follows. \( \textcircled{\triangleright} \)

6 Logics for closure hyperdoctrines with paths

In this section we extend the logics for closure hyperdoctrines we have introduced in Section 4 with formulae constructor for reasoning about surroundedness and reachability.

6.1 Syntax and derivation rules

- **Definition 6.1.** A signature with paths is a triple \( \Sigma = (\Sigma, \iota, R) \) where
  - \( \Sigma \) is a signature as per Definition 4.7
  - \( \iota \in |\Sigma| \) is called the interval type;
  - \( R : \iota, \iota \) is called the preorder of \( \iota \)

  A morphism \( \phi : (\Sigma_1, \iota_1, R_1) \to (\Sigma_2, \iota_2, R_2) \) is a morphism of signature \( (\phi_1, \phi_2, \phi_3) \) such that \( \phi_1(\iota_1) = \iota_2 \) and \( \phi_3(\iota, \iota)(R_1) = R_2. \)

- **Remark 6.2.** Signatures with paths and their morphisms with componentwise composition form a category \( \text{SignPath} \).

- **Definition 6.3.** We add the following rule of well formation to the logic for the closure operators (Definition 4.4):

\[ \Gamma \vdash \phi : \text{Prop} \quad \Gamma \vdash \psi : \text{Prop} \]
\[ \Gamma \vdash \phi S\psi : \text{Prop} \]

\( S-F \)

- **Definition 6.4.** Given a signature \( (\Sigma, \iota, R) \), its classifying category is the category \( \text{Cl}(\Sigma, \iota, R) \) is just \( \text{Cl}(\Sigma) \).

- **Definition 6.5.** We define the following rules for the well-formed formulae previously defined:

  - \( R \)’s rules:

\[ \Gamma, x : \iota, y : \iota \mid \Phi \vdash x = y \quad R-\text{REFL} \]
\[ \Gamma, x : \iota, y : \iota \mid \Phi \vdash R(x, y) \quad R-\text{REFL} \]
\[ \Gamma, x : \iota, y : \iota \mid \Phi \vdash R(x, y) \quad \Gamma, y : \iota, z : \iota \mid \Phi \vdash R(y, z) \quad R-\text{TRANS} \]
Closure Hyperdoctrines, with paths

We say that $\text{COwP}$ (with or without equality) is sound and complete.

\[ \text{Proposition 6.7.} \quad \text{For any signature } (\Sigma, \iota, R), (\mathcal{L}(\Sigma), (\iota, R)) \text{ is a path hyperdoctrine.} \]

\[ \text{Definition 6.8.} \quad \text{Let } ((\mathcal{P}, \iota)(I, \rho)) \text{ be a path hyperdoctrine. Then, a model of closure logic with paths in it is just an open morphism} \]

\[ (\mathcal{M}, \mu) : (\mathcal{L}(\Sigma), \iota_{\Sigma}, (I, R)) \rightarrow ((\mathcal{P}, \iota), (I, \rho)) \]

Satisfiability of sequents is defined as in the case of closure logics (Definition 4.10).

\[ \text{Remark 6.9. As for } U \text{ we have not put any requirement on the interpretation of } S, \text{ but, in} \]

\[ (\mathcal{L}(\Sigma), \iota_{\Sigma}), \text{ for } \Gamma \vdash \phi : \text{Prop} \text{ and } \Gamma \vdash \psi : \text{Prop} \text{ we have } \]

\[ [\phi S\psi] = [\phi]S_{\Gamma} [\psi] \]

so we can again ask for continuous models, i.e. models that preserves this equality.

\[ \text{Theorem 6.10. A sequent } \Gamma \mid \Phi \vdash \psi \text{ is satisfied by the generic model } (1_{\mathcal{U}(\Sigma)}, 1_{\mathcal{L}(\Sigma)}) \text{ if and only if it is derivable.} \]

\[ \text{Proof. The proof is the same as for Theorem 4.11.} \]

\[ \text{Corollary 6.11. The above defined categorical semantics for PLCOwP/RLCOwP/FOLCOwP (with or without equality) is sound and complete.} \]
7 Conclusions and future work

In this paper we have introduced closure (hyper)doctrines as a theoretical framework for studying the logical aspects of closure spaces. First we have proved the generality of this notion by means of a wide range of examples arising naturally from topological spaces, fuzzy sets, algebraic structures, coalgebras, and covering at once also known cases such as Kripke frames and probabilistic frames. Then, we have applied this framework to provide the first axiomatisation and sound and complete categorical semantics for various fragments of a logics for closure doctrines. In particular, the propositional fragment corresponds to the Spatial Logic for Closure Spaces [7], a modal logic for the specification and verification on spatial properties over preclosure spaces. But the flexibility of our approach allows us to readily obtain closure logics for a wide range of cases (like all the examples presented above).

Finally, we have extended closure hyperdoctrines with a notion of paths. This allows us to provide sound and complete logical derivation rules also for the “surroundedness” operator, thus covering all the logical constructs of SLCS.

Albeit already quite general, the theory presented in this paper pave the way for several extensions. First, we can enrich the logic with other spatial modalities, e.g., the spatial counterparts of the various temporal modalities of CTL* [10]. It could be interesting to investigate a spatial logic with fixed points a la µ-calculus; to interpret such a logic, we could consider closure hyperdoctrines over Löb algebras. Moreover, it would be interesting to develop some “generic” model checking algorithm for spatial logics. The abstraction provided by the categorical approach can guide the generalization of existing algorithms, such as [7].

On a different direction, we are interested to investigate the type theory induced by closure hyperdoctrines. In particular, a Curry-Howard isomorphism would yield a functional programming language with constructors for spatial aspects. Such a language would be very useful in collective spatial programming, e.g., for collective adaptive systems.

References

1. Marco Aiello, Ian Pratt-Hartmann, and Johan van Benthem, editors. Handbook of Spatial Logics. Springer, 2007.
2. Michael Atiyah. Introduction to commutative algebra. CRC Press, 2018.
3. Tom Avery. Codensity and the Giry monad. Journal of Pure and Applied Algebra, 220(3):1229–1251, 2016.
4. Giorgio Bacci and Marino Miculan. Structural operational semantics for continuous state stochastic transition systems. J. Comput. Syst. Sci., 81(5):834–858, 2015.
5. Gina Belmonte, Vincenzo Ciancia, Diego Latella, and Mieke Massink. Innovating medical image analysis via spatial logics. In From Software Engineering to Formal Methods and Tools, and Back, volume 11865 of Lecture Notes in Computer Science, pages 85–109. Springer, 2019.
6. Luca Cardelli and Andrew D. Gordon. Anytime, anywhere: Modal logics for mobile ambients. In Proc. POPL, pages 365–377. ACM, 2000.
7. Vincenzo Ciancia, Diego Latella, Michele Loreti, and Mieke Massink. Specifying and verifying properties of space. In IFIP International Conference on Theoretical Computer Science, pages 222–235. Springer, 2014.
8. Vincenzo Ciancia, Diego Latella, Michele Loreti, and Mieke Massink. Spatial logic and spatial model checking for closure spaces. In International School on Formal Methods for the Design of Computer, Communication and Software Systems, pages 156–201. Springer, 2016.
9. Dikran Dikranjan and Walter Tholen. Categorical structure of closure operators: with applications to topology, algebra and discrete mathematics, volume 346. Springer Science & Business Media, 2013.
10 E. Allen Emerson and Joseph Y. Halpern. “sometimes” and “not never” revisited: on branching versus linear time temporal logic. *Journal of the ACM*, 33(1):151–178, 1986.

11 Antony Galton. A generalized topological view of motion in discrete space. *Theoretical Computer Science*, 305(1-3):111–134, 2003.

12 Michèle Giry. A categorical approach to probability theory. In *Categorical aspects of topology and analysis*, pages 68–85. Springer, 1982.

13 H. Peter Gumm and Tobias Schröder. Products of coalgebras. *Algebra Universalis*, 46(1-2):163–185, 2001.

14 Bart Jacobs. *Categorical logic and type theory*, volume 141. Elsevier, 1999.

15 Bart Jacobs. *Introduction to Coalgebra*, volume 59. Cambridge University Press, 2017.

16 Clemens Kupke and Dirk Pattinson. Coalgebraic semantics of modal logics: an overview. *Theoretical Computer Science*, 412(38):5070–5094, 2011.

17 F. William Lawvere. Adjointness in foundations. *Dialectica*, 23(3-4):281–296, 1969.

18 F. William Lawvere. Equality in hyperdoctrines and comprehension schema as an adjoint functor. *Applications of Categorical Algebra*, 17:1–14, 1970.

19 Michael Makkai and Gonzalo E. Reyes. *First order categorical logic: model-theoretical methods in the theory of topoi and related categories*, volume 611. Springer, 2006.

20 Marino Miculan and Giorgio Bacci. Modal logics for brane calculus. In *Proc. CMSB*, volume 4210 of *Lecture Notes in Computer Science*, pages 1–16. Springer, 2006.

21 Andrew M. Pitts. Categorical logic. Technical report, University of Cambridge, Computer Laboratory, 1995.

22 Oswald Wyler. *Lecture notes on topoi and quasitopoi*. World Scientific, 1991.