Moduli for decorated tuples of sheaves and representation spaces for quivers

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Abstract. We extend the scope of a former paper to vector bundle problems involving more than one vector bundle. As the main application, we obtain the solution of the well-known moduli problems of vector bundles associated with general quivers.

Keywords. Quiver; representation; decoration; split sheaf; moduli space; Hitchin map.

1. Introduction

If we are given a projective manifold \( X \), a reductive linear algebraic group \( G \), and a representation \( \rho: G \to \text{GL}(V) \), we may associate to every principal \( G \)-bundle \( \mathcal{P} \) over \( X \) a vector bundle \( \mathcal{P}(V) \) with fibre \( V \). The objects of interest are pairs \((\mathcal{P}, \tau)\) where \( \mathcal{P} \) is an algebraic principal \( G \)-bundle and \( \tau: X \to \mathcal{P}(V) \) is a section of the associated vector bundle. Motivated, e.g., by the quest for differentiable invariants of 4-manifolds, one associates to the data of \( G, \rho, \) and a fixed differentiable principal \( G \)-bundle \( \mathcal{P} \) certain vortex equations. By the so-called Kobayashi–Hitchin correspondence, the solutions of these vortex equations have an interpretation as pairs \((\mathcal{P}, \tau)\) as above, satisfying certain stability conditions which may be understood in purely algebraic terms. Here, \( \mathcal{P} \) is an algebraic structure on the bundle \( \mathcal{P} \). The Kobayashi–Hitchin correspondence first arose in the context of vector bundles, i.e., when no representation is given (see [14]), and was then considered in various special cases before Banfield [2] gave it a unified treatise. It was afterwards widely extended to more general contexts [1, 3, 15, 19, 21]. In order to apply the machinery of algebraic geometry to the gauge theoretic moduli space for the pairs \((\mathcal{P}, \tau)\) with \( \mathcal{P} \) of topological type \( \mathcal{P} \) satisfying the stability conditions, one must equip it with an algebraic structure and find a suitable compactification. One is therefore led to a purely algebro–geometric moduli problem. Another motivation to study this kind of moduli problems that comes from within algebraic geometry is the fact that many interesting classification problems for projective manifolds may be encoded by data of the above type. We will give an example below and refer the reader to [24] for further discussions. A first sufficiently general solution of this kind of moduli problems was given by the author in the case that \( X \) is a projective curve, \( G = \text{GL}(r) \), and \( \rho \) is a homogeneous representation [24]. Later, Gómez and Sols [8] established this case on higher dimensional base manifolds \( X \).

The aim of the present paper is to extend these results to the case when the reductive group is a product of general linear groups, \( G = \text{GL}(r_1) \times \cdots \times \text{GL}(r_l) \), \( \rho \) belongs to the
class of homogeneous representations (which comprises all irreducible representations), and $X$ is a base manifold of arbitrary dimension. However, we will not repeat the detailed constructions of [24], but rather introduce several non-trivial ‘tricks’ which will enable us to adapt the proofs in that paper to the more general situation studied here.

A nice example of a classification problem which can be formulated in our context is provided by the work of Casnati and Ekedahl [4]. Let $X$ be a projective manifold. Then, any integral Gorenstein cover $\pi : Y \to X$ of degree 4 can be obtained from locally free $O_X$-modules $\mathcal{E}$ and $\mathcal{F}$ of rank 3 and 2, respectively, such that $\det(\mathcal{E}) \cong \det(\mathcal{F})$, and a section $s \in H^0(X, \mathcal{F}^\vee \otimes S^2\mathcal{E}) = \text{Hom}(\mathcal{F}, S^2\mathcal{E})$. The construction is as follows: If $\pi : \mathbb{P}(\mathcal{E}^\vee) := \text{Proj}(S^*\mathcal{E}) \to X$ is the projection, then

$$\text{Hom}(\pi^*(\det \mathcal{E})(-4), \pi^*(\mathcal{F})(-2)) = \text{Hom}(\pi^*(\mathcal{F}^\vee \otimes \det \mathcal{E}), O_{\mathbb{P}(\mathcal{E}^\vee)}(2))$$

$$= \text{Hom}(\mathcal{F}^\vee \otimes \det \mathcal{E}, S^2\mathcal{E})$$

$$= \text{Hom}(\mathcal{F}, S^2\mathcal{E}).$$

Here, $\mathcal{F}^\vee \otimes \det \mathcal{E} \cong \mathcal{F}^\vee \otimes \det \mathcal{F} \cong \mathcal{F}$, because $\mathcal{F}$ has rank 2. Thus, any section $s \in H^0(X, \mathcal{F}^\vee \otimes S^2\mathcal{E})$ yields an exact sequence

$$0 \to \pi^* \det \mathcal{E}(-4) \to \pi^*(\mathcal{F})(-2) \to O_Y \to 0.$$ 

Hence, the moduli problem for degree 4 covers of $X$ is included in the moduli problem associated with the group $GL(3) \times GL(2)$ and its representation on $\text{Hom}(\mathbb{C}^2, S^2\mathbb{C}^3)$. Similarly, degree five covers $\rho : Y \to X$ are determined by locally free sheaves $\mathcal{E}$ and $\mathcal{F}$ of rank 5 and 4, respectively, and a homomorphism $\varphi : \mathcal{E} \to \bigwedge^2 \mathcal{F} \otimes \det \mathcal{E}$ [5].

Another interesting moduli problem which we will treat with our methods comes from the representation theory of finite dimensional algebras (see [12] and [22] for introductions to this topic). Let $Q = (V, A, t, h)$ be a quiver with vertex set $V = \{v_1, \ldots, v_t\}$ and $\tilde{Q} = (\tilde{G}_a, a \in A)$ a collection of coherent $O_X$-modules on the projective manifold $X$. This defines a twisted path algebra $B = B(Q, \tilde{Q})$ (see [1] and [7]). Modules over $B$ can now be described by representations of $Q$, i.e., tuples $(\mathcal{E}_v, v \in V; f_a, a \in A)$ composed of $O_X$-modules $\mathcal{E}_v$, $v \in V$, and twisted homomorphisms $f_a : \mathcal{G}_a \otimes \mathcal{E}_{t(a)} \to \mathcal{E}_{h(a)}$, $a \in A$. Numerous famous special cases of this construction have been studied in the literature, such as the Higgs bundles. Recent research has focussed on more general aspects of this theory. Gothen and King [7] have developed the homological algebra of these representations and Álvarez–Cónsul and García-Prada [1] formulated a semistability concept for the representations of $Q$ and proved a Kobayashi–Hitchin correspondence. The semistability concept depends on additional parameters $\sigma = (\sigma_v \in \mathbb{Z}_{>0}, v \in V)$ and $\chi = (\chi_v \in \mathbb{Q}[x], v \in V)$ where the polynomials $\chi_v$ have degree at most $\text{dim } X - 1$, $v \in V$, and an ample line bundle $O_X(1)$ on $X$. For any coherent sheaf $\mathcal{A}$ on $X$, the Hilbert polynomial with respect to $O_X(1)$ is denoted by $P(\mathcal{A})$. We set

$$P_{\sigma, \chi}(\mathcal{E}_v, v \in V) := \sum_{v \in V} (\sigma_v P(\mathcal{E}_v) - \chi_v \text{rk } \mathcal{E}_v)$$

and

$$\text{rk}_{\sigma}(\mathcal{E}_v, v \in V) := \sum_{v \in V} \sigma_v \text{rk } \mathcal{E}_v.$$
A representation \((E_v, v \in V; f_a, a \in A)\) is then called (semi)stable, if (a) the sheaves \(E_v, v \in V\), are torsion free and (b) for any collection of saturated subsheaves \(F_v \subset E_v, v \in V\), (i.e., \(E_v/F_v\) is again torsion free, \(v \in V\)) not all trivial and not all equal to \(E_v\), such that \(f_a(G_a \otimes F_{i(a)}) \subset F_{h(a)}\) for all arrows \(a\), one has

\[
\frac{P_{\sigma, \tau}(F_v, v \in V)}{\text{rk}_\sigma(F_v, v \in V)} \preceq \frac{P_{\sigma, \tau}(E_v, v \in V)}{\text{rk}_\sigma(E_v, v \in V)}.
\]

The notation \((\preceq)\) means that \(\preceq\) is used for defining ‘stable’ and \(\succeq\) for defining ‘semistable’, and \(\prec\) and \(\succeq\) refer to the lexicographic ordering of polynomials. Finally, \((E_v, v \in V; f_a, a \in A)\) is called polystable, if it is a direct sum of stable representations \((E_v, v \in V; f_a^i, a \in A), i = 1, \ldots, s\), with

\[
\frac{P_{\sigma, \tau}(E_v^i, v \in V)}{\text{rk}_\sigma(E_v^i, v \in V)} = \frac{P_{\sigma, \tau}(E_v^j, v \in V)}{\text{rk}_\sigma(E_v^j, v \in V)}, \quad \text{for all } i, j = 1, \ldots, s.
\]

As one of the interesting and important applications of the main result of this paper, we will prove the following theorem.

**Theorem.** Fix Hilbert polynomials \(P = (P_v, v \in V)\), the sheaves \(G\), as well as the parameters \(\sigma\) and \(\chi\) with \(\chi_v = \eta_v \cdot \delta\) for some polynomial \(\delta \in \mathbb{Q}[x]\) and rational numbers \(\eta_v, v \in V\).

i) There exists a quasi-projective moduli space \(\mathcal{D} := \mathcal{D}(Q, P_G, \delta)\) for polystable representations \((E_v, v \in V; f_a, a \in A)\) with \(P(E_v) = P_v, v \in V\). The points corresponding to stable representations form an open subset \(\mathcal{D}^s\).

ii) There are a vector space \(\mathbb{D}\) and a projective morphism \(H: \mathcal{D} \longrightarrow \mathbb{D}\), the generalized Hitchin map.

**Remark.** Álvarez-Cónsul and García-Prada define semistability with respect to parameters \(\sigma = (\sigma_v \in \mathbb{Q}, v \in V)\) and \(\tau = (\tau_v \in \mathbb{Q}, v \in V)\). To be precise, for a representation \((E_v, v \in V; f_a, a \in A)\), they set

\[
\deg_{\sigma, \tau}(E_v, v \in V) := \sum_{v \in V} (\sigma_v \deg(E_v) - \tau_v \text{rk}(E_v))
\]

\[
\mu_{\sigma, \tau}(E_v, v \in V) := \frac{\deg_{\sigma, \tau}(E_v, v \in V)}{\text{rk}_\sigma(E_v, v \in V)}.
\]

Álvarez-Cónsul and García-Prada say that \((E_v, v \in V; f_a, a \in A)\) is (semi)stable, if (a) the sheaves \(E_v, v \in V\), are torsion free and (b) for any collection of saturated subsheaves \(F_v \subset E_v, v \in V\), such that \(f_a(G_a \otimes F_{i(a)}) \subset F_{h(a)}\) for all arrows \(a\), one has

\[
\mu_{\sigma, \tau}(F_v, v \in V) \preceq \mu_{\sigma, \tau}(E_v, v \in V).
\]

Since multiplying all the parameters with a common positive factor does not alter the semistability condition, we may assume that the \(\sigma_v, v \in V\), be positive integers. We may choose \(\delta := x^{\dim X - 1}\) and \(\chi_v := \tau_v \cdot \delta, v \in V\). Then, \(\deg_{\sigma, \tau}(E_v, v \in V)\) is essentially given as the coefficient of \(x^{\dim X - 1}/(\dim X - 1)!\) in \(P_{\sigma, \tau}\) (see [10], Definition 1.2.11,
p. 13, for the precise statement). In particular, we have the following implications: \((E_v, v \in V; f_a, a \in A)\) stable in the sense Álvarez-Cónsul and García-Prada \(\implies\) stable in our sense \(\implies\) semistable in our sense \(\implies\) semistable in the sense Álvarez-Cónsul and García-Prada.

Therefore, the above theorem gives a quasi-projective moduli space for the stable objects (in either sense) together with a Gieseker-type compactification.

In future, we hope to extend the techniques introduced in this paper to treat the case of other reductive groups. This provides another motivation for studying the more general and abstract moduli problems introduced here.

2. Notation

\(X\) will be a fixed projective manifold over the complex numbers, and \(\mathcal{O}_X(1)\) a fixed ample line bundle on \(X\). For any coherent sheaf \(\mathcal{E}\), \(\deg \mathcal{E}\) is the degree of \(\mathcal{E}\) with respect to \(\mathcal{O}_X(1)\), and \(P(\mathcal{E})\) with \(P(\mathcal{E})(l) := \chi(\mathcal{E}(l))\), for all \(l \in \mathbb{N}\), is the Hilbert polynomial of \(\mathcal{E}\) with respect to \(\mathcal{O}_X(1)\). In order to avoid excessive occurrences of the symbol \(\vee\), we define \(\mathbb{P}(V)\) as the projective bundle of lines in the fibres of the vector bundle \(V\). For any scheme \(S\), \(h_S\) denotes its functor of points \(T \to \text{Mor}(T, S)\).

In the appendix, we have stated two auxiliary results which will be used on several occasions.

3. Background, formal set-up and statement of the main results

3.1 Homogeneous representations

Let \(V = \{v_1, \ldots, v_l\}\) be a finite index set, \(r := (r_v, v \in V)\) a tuple of positive integers, and define

\[
\text{GL}(V, r) := \prod_{v \in V} \text{GL}(\mathbb{C}^{r_v}).
\]

A (finite dimensional, rational) representation \(\rho : \text{GL}(V, r) \to \text{GL}(A)\) is said to be homogeneous (of degree \(\alpha\)), if there is an integer \(\alpha\), such that

\[
\rho(z, \ldots, z) = z^\alpha \cdot \text{id}_A, \text{ for all } z \in \mathbb{C}^\times.
\]

Example 3.1.1. Every irreducible representation is homogeneous. For any tuple \(\sigma = (\sigma_1, \ldots, \sigma_l)\) of positive integers and \(a, b, c \in \mathbb{Z}_{\geq 0}\), we define

\[
W(\sigma, r) := \mathbb{C}^{\sum_{i=1}^l \sigma_i r_i}
\]

and the \(\text{GL}(V, r)\)-module

\[
W(\sigma, r)_{a,b,c} := (W(\sigma, r)^{\otimes a})^{\oplus b} \otimes \left(\bigwedge^{\sum_{i=1}^l \sigma_i r_i} W(\sigma, r)\right)^{\otimes -c}.
\]

The corresponding representation

\[
\varrho_{a,b,c} : \text{GL}(V, r) \to \text{GL}(W(\sigma, r)_{a,b,c})
\]

is homogeneous.
PROPOSITION 3.1.2

Let \( \sigma = (\sigma_1, \ldots, \sigma_l) \) be a tuple of positive integers and \( \rho : \text{GL}(V, r) \to \text{GL}(A) \) a homogeneous representation. Then, there are non-negative integers \( a, b \) and \( c \), such that the module \( A \) is a direct summand of the module \( W(\sigma, r)_{a,b,c} \).

Proof. First, there are integers \( s_i, t_i, i = 1, \ldots, k \), such that the module \( A \) is a direct summand of the \( \text{GL}(V, r) \)-module

\[
\bigoplus_{i=1}^{k} W(\sigma, r)^{\otimes s_i} \otimes \text{Hom}(W(\sigma, r), \mathbb{C})^{\otimes t_i}.
\]

This is a consequence of ([6], Proposition 3.1(a), p. 40). Since \( \rho \) is assumed to be homogeneous, we have

\[
s_i - t_i = s_j - t_j, \quad \text{for all } 1 \leq i, j \leq k.
\]

The assertion now follows from Corollary 1.2 in [24]. \( \square \)

Remark 3.1.3. The tuple \( \sigma \) will be a natural parameter in our theory.

Example 3.1.4. Let \( Q = (V, A, t, h) \) be a quiver with vertex set \( V = \{v_1, \ldots, v_l\} \), arrow set \( A = \{a_1, \ldots, a_n\} \), tail map \( t : A \to V \), and head map \( h : A \to V \). Fix a dimension vector \( r = (r_v, v \in V) \) as well as another tuple \( \alpha = (\alpha_v, v \in V) \) of positive integers with \( \alpha_h(a) - \alpha_t(a) = \alpha_h(a') - \alpha_t(a') = \alpha \) for all \( a, a' \in A \). Then, the \( \text{GL}(V, r) \)-module

\[
\bigoplus_{a \in A} \text{Hom}((\mathbb{C}^{r(a)}_{t(a)})^{\otimes \alpha_t(a)}, (\mathbb{C}^{r(a)}_{h(a)})^{\otimes \alpha_h(a)})
\]

is homogeneous of degree \( \alpha \).

3.2 \( V \)-split vector spaces

First, let \( W \) be a finite dimensional \( \mathbb{C} \)-vector space. A weighted flag in \( W \) is a pair \((W^*, \gamma)\) with

\[
W^* : 0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_s 
\]

— not necessarily complete — flag in \( W \) and \( \gamma = (\gamma_1, \ldots, \gamma_{s+1}) \) a vector of integers with \( \gamma_1 < \cdots < \gamma_{s+1} \).

Remark 3.2.1. In our context, weighted flags arise in the following way: Let \( \lambda : \mathbb{C}^s \to \text{GL}(W) \) be a one-parameter subgroup and \( \chi_1, \ldots, \chi_{s+1}, s \geq 0 \), the characters of \( \mathbb{C}^s \) with non-trivial eigenspace in \( W \). Then, \( \chi_i(z) = z^\gamma_i \) with \( \gamma_i \in \mathbb{Z} \). Let \( W^{\chi_i} \subset W \) be the corresponding eigenspace. We number the characters in such a way that \( \gamma_1 < \cdots < \gamma_{s+1} \). This yields the weight vector \( \gamma \). The flag \( W^* \) is obtained by setting

\[
W_i := \bigoplus_{j=1}^{i} W^{\chi_j}, \quad i = 1, \ldots, s.
\]
Let $V = \{v_1, \ldots, v_t\}$ be an index set. A $V$-split vector space is a collection $(W_v, v \in V)$ of vector spaces indexed by $V$. Note that $V$-split vector spaces form in a natural way an Abelian category. A weighted flag $(W_\bullet, \gamma)$ in the $V$-split vector space $(W_v, v \in V)$ is a pair $(W_\bullet, \gamma)$ with

$$W_\bullet: \emptyset \subseteq (W_{v_1}^1, v \in V) \subseteq \cdots \subseteq (W_{v_s}^s, v \in V) \subseteq (W_v, v \in W)$$

a filtration of $(W_v, v \in V)$ by $V$-split subspaces and

$$\gamma = (\gamma_1, \ldots, \gamma_{s+1})$$

a vector of integers. We then have the equivalent notions of

(a) Tuples $((\hat{W}_v^v, \gamma^v), v \in V)$ of weighted flags in the $W_v$, $v \in V$,

$$\hat{W}_v^v: \emptyset \subseteq \hat{W}_1^v \subseteq \cdots \subseteq \hat{W}_s^v \subseteq W_v,$$

$$\gamma^v = (\gamma_1^v, \ldots, \gamma_{s+1}^v), \quad v \in V.$$

Here, $s_v = 0$ is permitted.

(b) Weighted flags $(W_\bullet, \gamma)$ in the $V$-split vector space $(W_v, v \in V)$.

Indeed, suppose we are given a tuple as in (a). Let $\gamma_1 < \cdots < \gamma_{s+1}$ be the different weights occurring among the $\gamma^v_j$, $v \in V$, $j = 1, \ldots, s_v + 1$. Then, we define $(W_j^v, v \in V)$ by $W_j^v := \hat{W}_{i_v(j)}^v$ with

$$i_v(j) := \max\{t = 1, \ldots, s_v + 1 \mid \gamma_t \leq \gamma_j\},$$

$v \in V$, $j = 1, \ldots, s + 1$. Conversely, given a weighted flag $(W_\bullet, \gamma)$ in the $V$-split vector space $(W_v, v \in V)$, we get a weighted flag $(\hat{W}_\bullet^v, \gamma^v)$ in $W_v$ by just projecting onto $W_v$, $v \in V$. These two operations are clearly inverse to each other.

A weight formula

Next, we fix $\sigma = (\sigma_v \in \mathbb{Z}_{>0}, v \in V)$ and set $M := \bigoplus_{v \in V} W_v^\oplus \sigma_v$. Suppose we are given a $V$-split vector space $(W_v, v \in V)$ and a weighted flag $(W_\bullet, \gamma)$. Define $M_j := \bigoplus_{v \in V} (W_j^v)^\oplus \sigma_v$ in order to obtain a weighted flag $(M_\bullet, \gamma)$ in $M$. Assume, furthermore, that we are given quotients $k_v: W_v \rightarrow \mathbb{C}^t_v$, $v \in V$, and set $k := \bigoplus_{v \in V} k_v^\oplus \sigma_v: M \rightarrow \mathbb{C}^t$, $t := \sum_{v \in V} \sigma_v \cdot t_v$. The central formula that we need later is contained in the following.

**PROPOSITION 3.2.2**

Suppose that, in the above situation, we are given a tuple $((\hat{W}_\bullet^v, \gamma^v), v \in V)$ of weighted flags in the $W_v$, $v \in V$. Let $(W_\bullet, \gamma)$ be the corresponding weighted flag in $(W_v, v \in V)$ and $(M_\bullet, \gamma)$ the resulting weighted flag in $M$. Then, the following identity holds
true:

$$\sum_{j=1}^{s} \frac{Y_{j+1} - Y_j}{r} (r \cdot \dim k(M_j) - t \cdot \dim M_j)$$

$$= \sum_{v \in V} \sigma_v \left( \sum_{j=1}^{s_v} \frac{y_{j+1}^v - y_j^v}{r_v} (r_v \cdot \dim k_v(\hat{W}_j^v) - t_v \cdot \dim \hat{W}_j^v) \right)$$

$$- \sum_{v \in V} \sigma_v \cdot \left( \frac{t_v}{r_v} - \frac{t}{r} \right) \cdot \left( \sum_{j=1}^{s_v+1} y_j^v (\dim \hat{W}_j^v - \dim \hat{W}_{j-1}^v) \right).$$

Here, $r_v := \dim W_v, v \in V,$ and $r = \sum_{v \in V} \sigma_v \cdot r_v.$

**Proof.** From the definitions, the formula

$$\sum_{v \in V} \sigma_v \left( \sum_{j=1}^{s_v+1} y_j^v (\dim \hat{W}_j^v - \dim \hat{W}_{j-1}^v) \right) = \sum_{j=1}^{s+1} y_j (\dim M_j - \dim M_{j-1})$$

follows immediately. Therefore, the assertion is equivalent to the following equation:

$$\sum_{j=1}^{s} \frac{Y_{j+1} - Y_j}{r} (r \cdot \dim k(M_j) - t \cdot \dim M_j)$$

$$- \frac{t}{r} \cdot \sum_{j=1}^{s+1} Y_j \cdot (\dim M_j - \dim M_{j-1})$$

$$= \sum_{v \in V} \sigma_v \left( \sum_{j=1}^{s_v} \frac{y_{j+1}^v - y_j^v}{r_v} (r_v \cdot \dim k_v(\hat{W}_j^v) - t_v \cdot \dim \hat{W}_j^v) \right)$$

$$- \sum_{v \in V} \sigma_v \cdot \frac{t_v}{r_v} \cdot \left( \sum_{j=1}^{s_v+1} y_j^v (\dim \hat{W}_j^v - \dim \hat{W}_{j-1}^v) \right).$$

Now,

$$- \frac{t}{r} \cdot \sum_{j=1}^{s} (y_{j+1} - y_j) \dim M_j$$

$$= \left( \frac{t}{r} \sum_{j=1}^{s} y_j \dim M_j \right) - \left( \frac{t}{r} \sum_{j=1}^{s} y_{j+1} \dim M_j \right)$$

$M_0 = [0] \left( \frac{t}{r} \sum_{j=1}^{s} y_j \dim M_j \right) - \left( \frac{t}{r} \sum_{j=1}^{s+1} y_j \dim M_{j-1} \right)$

$$= - \frac{t}{r} \cdot s+1 \dim M + \sum_{j=1}^{s+1} y_j (\dim M_j - \dim M_{j-1}).$$
Therefore, the left-hand side simplifies to
\[-\frac{t}{r} \gamma_{s+1} \dim M + \sum_{j=1}^{s}(\gamma_{j+1} - \gamma_j) \cdot \dim k(M_j).\]

With the same argument as before, we see
\[\sum_{j=1}^{s}(\gamma_{j+1} - \gamma_j) \cdot \dim k(M_j) = \gamma_{s+1} \dim M - \sum_{j=1}^{s+1} \gamma_j (\dim k(M_j) - \dim k(M_{j-1})).\]

Since $\dim M = r$ and $\dim k(M) = t$, the left-hand side finally takes the form
\[\sum_{j=1}^{s+1} \gamma_j (\dim k(M_j) - \dim k(M_{j-1})). \quad (1)\]

Likewise, the right-hand side becomes
\[-\sum_{v \in V} \sigma_v \cdot \left( \sum_{j=1}^{s+1} \gamma_j^v (\dim k_v(\hat{W}_j^v) - \dim k(\hat{W}_{j-1}^v)) \right). \quad (2)\]

The equality of (1) and (2) is now clear from the definitions. \(\square\)

Remark 3.2.3. The conceptual way to see the above formula which explains how it will arise later is the following: Denote by $G_v$ the Grassmannian of $t_v$-dimensional quotients of $W_v$, $v \in V$. Let $\mathfrak{t}_v : W_v \otimes O_{G_v} \rightarrow Q_v$ be the universal quotient and $O_{G_v}(1) := \det(Q_v)$, $v \in V$. Likewise, we let $G$ be the Grassmannian of $t$-dimensional quotients of $M$, $\mathfrak{t} : M \otimes O_G \rightarrow Q$ the universal quotient, and $O_G(1) := \det(Q)$. On $X_v \in V G_v$, we have the quotient
\[\mathfrak{t}' := \bigoplus_{v \in V} \pi_{G_v}^* \otimes_{G_v} O_{G_v}(1) \rightarrow \bigoplus_{v \in V} \pi_{G_v}^* Q_v.\]

This quotient defines a $(X_v \in V \text{GL}(W_v))$-equivariant embedding
\[h : \bigoplus_{v \in V} X_v G_v \hookrightarrow G,\]
such that
\[h^* O_{G}(1) = O_{X_v \in V G_v}(\sigma_{v_1}, \ldots, \sigma_{v_t}).\]

Let $\lambda : \mathbb{C}^* \rightarrow \text{GL}(W_v)$ be a one-parameter subgroup which induces the weighted flag $(\hat{W}_v^\bullet, \gamma^v)$ in $W_v$, $v \in V$, see Remark 3.2.1. Then, the expression (2) is just
\[\mu(\lambda, (k_{v_1}, \ldots, k_{v_t})), \quad \lambda := (\lambda_{v_1}, \ldots, \lambda_{v_t}).\]
with respect to the linearization of the \( GL(V, r) \)-action in \( \mathcal{O}_{X_{v \in V}} \mathbb{G}_r(\sigma_{v_1}, \ldots, \sigma_{v_t}) \). Now, we can view \( \lambda \) as a one-parameter subgroup of \( GL(M) \). The induced weighted filtration of \( M \) is then \( (M_\bullet, \nu) \), and (1) agrees with
\[
\mu(\lambda, k)
\]
with respect to the linearization in \( \mathcal{O}_G(1) \). Obviously, we must have
\[
\mu(\lambda, k) = \mu(\lambda, (k_{v_1}, \ldots, k_{v_t})).
\]

### 3.3 \( V \)-split sheaves

We fix a finite index set \( V = \{v_1, \ldots, v_t\} \). A \( V \)-split sheaf is simply a tuple \( (E_v, v \in V) \) of coherent sheaves on \( X \). Likewise, a homomorphism between \( V \)-split sheaves \( (E_v, v \in V) \) and \( (E'_v, v \in V) \) is a collection \( (f_v, v \in V) \) of homomorphisms \( f_v: E_v \to E'_v, v \in V \).

In this way, the \( V \)-split sheaves on \( X \) form an Abelian category. The type of the \( V \)-split sheaf \( (E_v, v \in V) \) is the tuple \( P(E_v, v \in V) := (P(E_{v_1}), \ldots, P(E_{v_t})) \).

Remark 3.3.1. The datum of a \( V \)-split vector bundle is equivalent to the datum of a principal \( GL(V, r) \)-bundle. Thus, a \( V \)-split sheaf can be seen as the natural ‘singular’ version of a principal \( GL(V, r) \)-bundle.

Now, we fix additional parameters \( \sigma = (\sigma_v \in \mathbb{Z}_{>0}, v \in V) \) and \( \chi = (\chi_v \in \mathbb{Q}[x], v \in V) \) where the polynomials \( \chi_v \) have degree at most \( \dim X - 1 \), \( v \in V \). We denote by \( \overline{\chi}_v \) the coefficient of \( x^{\dim X - 1} \) in \( \chi_v, v \in V \). Then, we define for any \( V \)-split sheaf \( (E_v, v \in V) \) the following quantities.

The \( (\sigma, \chi) \)-degree:
\[
\deg_{(\sigma, \chi)}(E_v, v \in V) := \sum_{v \in V} (\sigma_v \deg E_v - \overline{\chi}_v \mathrm{rk} E_v).
\]

The \( (\sigma, \chi) \)-Hilbert polynomial:
\[
P_{(\sigma, \chi)}(E_v, v \in V) := \sum_{v \in V} (\sigma_v P(E_v) - \chi_v \mathrm{rk} E_v).
\]

The \( \sigma \)-rank:
\[
\mathrm{rk}_{(\sigma)}(E_v, v \in V) := \sum_{v \in V} \sigma_v \mathrm{rk} E_v.
\]

The \( (\sigma, \chi) \)-slope:
\[
\mu_{(\sigma, \chi)}(E_v, v \in V) := \frac{\deg_{(\sigma, \chi)}(E_v, v \in V)}{\mathrm{rk}_{(\sigma)}(E_v, v \in V)}.
\]

Note that the \( (\sigma, \chi) \)-degree, \( (\sigma, \chi) \)-Hilbert polynomial, and \( \sigma \)-rank all behave additively on short exact sequences. Thus, the \( (\sigma, \chi) \)-slope will have all the formal properties of the usual slope. More specifically, we call a \( V \)-split sheaf \( (E_v, v \in V) \) \( (\sigma, \chi) \)-slope (semi)stable, if \( E_v \) is torsion free, \( v \in V \), and, for all non-trivial, proper \( V \)-split subsheaves \( (F_v, v \in V) \), the inequality
\[
\mu_{(\sigma, \chi)}(F_v, v \in V) (\leq) \mu_{(\sigma, \chi)}(E_v, v \in V)
\]
is satisfied.
Remark 3.3.2. Let \((\mathcal{E}_v, v \in V)\) be a \((\sigma, \chi)\)-slope semistable \(V\)-split sheaf. Let \(v_0\) be a vertex with \(\mathcal{E}_{v_0} \neq 0\) and \(0 \subsetneq \mathcal{F} \subsetneq \mathcal{E}_{v_0}\) a subsheaf. Set \(\mathcal{F}_v := 0\) for \(v \neq v_0\) and \(\mathcal{F}_{v_0} := \mathcal{F}\). Then, we get
\[
\mu(\mathcal{F}) \leq \mu_{\sigma, \chi}(\mathcal{E}_v, v \in V) + \frac{\chi_{v_0}}{\sigma_{v_0}}.
\]
Likewise, we find for every non-trivial quotient \(Q\) of \(\mathcal{E}_{v_0}\),
\[
\mu(Q) \geq \mu_{\sigma, \chi}(\mathcal{E}_v, v \in V) + \frac{\chi_{v_0}}{\sigma_{v_0}}.
\]
If we apply this to \(\mathcal{F} = \mathcal{E}_{v_0} = Q\), we see that \(\mathcal{E}_{v_0}\) must be a semistable sheaf with slope \(\mu_{\sigma, \chi}(\mathcal{E}_v, v \in V) + \frac{\chi_{v_0}}{\sigma_{v_0}}\). Note that this forces \(\sum_{v \in V} r_v \cdot \chi_v = 0\).

Recall that any homomorphism \(f : \mathcal{E} \rightarrow \mathcal{E}'\) between the semistable sheaves \(\mathcal{E}\) and \(\mathcal{E}'\) will be zero, if \(\mu(\mathcal{E}) > \mu(\mathcal{E}')\). Therefore, we deduce the following.

**PROPOSITION 3.3.3**

Let \((\mathcal{E}_v, v \in V)\) and \((\mathcal{E}'_v, v \in V)\) be \((\sigma, \chi)\)-slope semistable \(V\)-split sheaves. Assume
\[
\mu_{\sigma, \chi}(\mathcal{E}_v, v \in V) + \min_{v \in V} \left\{ \frac{\chi_v}{\sigma_v} \right\} > \mu_{\sigma, \chi}(\mathcal{E}'_v, v \in V) + \max_{v \in V} \left\{ \frac{\chi_v}{\sigma_v} \right\}.
\]
Then, for any choice of exponents \(s_v, s'_v > 0, v \in V\), any homomorphism \(f : \bigoplus_{v \in V} \mathcal{E}_v^{s_v} \rightarrow \bigoplus_{v \in V} \mathcal{E}'_v^{s'_v}\) is zero.

Finally, we have the following Proposition.

**PROPOSITION 3.3.4** (Harder–Narasimhan filtration)

Let \((\mathcal{E}_v, v \in V)\) be any torsion-free \(V\)-split sheaf. Then, \((\mathcal{E}_v, v \in V)\) possesses a unique filtration by \((\sigma, \chi)\)-destabilizing \(V\)-split subsheaves
\[
0 = (\mathcal{F}_0^v, v \in V) \subsetneq (\mathcal{F}_1^v, v \in V) \subsetneq \cdots \subsetneq (\mathcal{F}_s^v, v \in V) \subsetneq (\mathcal{F}_{s+1}^v, v \in V) = (\mathcal{E}_v, v \in V),
\]
such that
1. the \(V\)-split sheaf \((\mathcal{F}_i^v/\mathcal{F}_{i-1}^v, v \in V)\) is \((\sigma, \chi)\)-slope semistable, \(i = 1, \ldots, s + 1\).
2. \(\mu_{\sigma, \chi}(\mathcal{F}_i^v/\mathcal{F}_{i-1}^v, v \in V) > \mu_{\sigma, \chi}(\mathcal{F}_{i+1}^v/\mathcal{F}_i^v, v \in V), i = 1, \ldots, s\).

The weight formula for sheaves

Let \((\mathcal{E}_v, v \in V)\) be a \(V\)-split sheaf. As before, the following data are equivalent:
(a) Tuples \((\mathcal{E}_v^v, \gamma^v), v \in V\) of weighted filtrations of the \(\mathcal{E}_v, v \in V\),
\[
\mathcal{E}_v^v : 0 \subsetneq \tilde{\mathcal{E}}_1^v \subsetneq \cdots \subsetneq \tilde{\mathcal{E}}_{s_v}^v \subsetneq \mathcal{E}_v,
\]
\[
\gamma^v = (\gamma_1^v, \ldots, \gamma_{s_v+1}^v), \quad v \in V.
\]
(b) Weighted filtrations \((\mathcal{E}_v, \gamma^v)\) of the \(V\)-split sheaf \((\mathcal{E}_v, v \in V)\).
Moreover, given \( \sigma = (\sigma_v \in \mathbb{Z}_{\geq 0}, v \in V) \), a \( V \)-split sheaf \( (\mathcal{E}_v, v \in V) \), and a weighted filtration \( (\mathcal{E}_v, \gamma) \), we set \( \mathcal{E}^\text{total} := \bigoplus_{v \in V} \mathcal{E}^v \) and

\[
\mathcal{E}_v^\text{total} := 0 \subset \mathcal{E}_1^\text{total} := \bigoplus_{v \in V} \mathcal{E}_v^v \oplus \cdots \subset \mathcal{E}_s^\text{total} := \bigoplus_{v \in V} \mathcal{E}_v^{v, \oplus \sigma_v} \subset \mathcal{E}^\text{total}.
\]

**Proposition 3.3.5**

Suppose that, in the above situation, we are given a tuple \( ((\mathcal{E}_v^v, \gamma^v), v \in V) \) of weighted filtrations of the \( \mathcal{E}_v, v \in V \). Let \( (\mathcal{E}^\text{total}^v, \gamma^v) \) be the resulting weighted filtration of \( \mathcal{E}^\text{total} \). Then, for all \( l \gg 0 \),

\[
\sum_{j=1}^{s} \frac{\gamma_j^v + 1 - \gamma_j^v}{P(\mathcal{E}_v^\text{total}^v)(l)} \left( P(\mathcal{E}^\text{total}^v)(l) \cdot \text{rk} \mathcal{E}_v^\text{total}^v - P(\mathcal{E}_v^\text{total}^v)(l) \cdot \text{rk} \mathcal{E}^\text{total}^v \right) = \sum_{v \in V} \sigma_v \left( \sum_{j=1}^{s_v} \gamma_j^v + 1 - \gamma_j^v \right) \left( P(\mathcal{E}_v^v)(l) \cdot \text{rk} \mathcal{E}_v^v - P(\mathcal{E}_v^v)(l) \cdot \text{rk} \mathcal{E}_v^v \right) - \sum_{v \in V} \sigma_v \cdot \left( \text{rk} \mathcal{E}_v^v \cdot \frac{P(\mathcal{E}_v^v)(l)}{P(\mathcal{E}_v^\text{total}^v)(l)} - \frac{\text{rk} \mathcal{E}_v^\text{total}^v}{P(\mathcal{E}_v^\text{total}^v)(l)} \right) \cdot \left( \sum_{j=1}^{s_v} \gamma_j^v \left( P(\mathcal{E}_v^v)(l) - P(\mathcal{E}^\text{total}_v^v)(l) \right) \right).
\]

**Proof.** For \( l \gg 0 \), we have

- \( \mathcal{E}_j^v(l) \) is globally generated, \( v \in V, j = 1, \ldots, s_v + 1 \),
- \( H^i(\mathcal{E}_j^v(l)) = 0, i > 0, v \in V, \) and \( j = 1, \ldots, s_v + 1 \).

Then, we may write \( \mathcal{E}_v(l) \) as a quotient \( q_v : \mathcal{O}_X^{\oplus P(\mathcal{E}_v^v)(l)} \longrightarrow \mathcal{E}_v(l) \), such that \( H^0(q_v) \) is an isomorphism, \( v \in V \). Restricting this to a general point \( x \in X \) yields \( k_v : \mathcal{O}^{\oplus P(\mathcal{E}_v^v)(l)} \longrightarrow \mathcal{O}^{\text{rk} \mathcal{E}_v^v}, v \in V \). Now, apply Proposition 3.2.2 to the tuple \( ((H^0(\mathcal{E}_v^v(l)), \gamma^v), v \in V) \) under the identification of \( \mathcal{O}^{\oplus P(\mathcal{E}_v^v)(l)} \) with \( H^0(\mathcal{E}_v(l)) \).

### 3.4 ρ-pairs

In this section, we will fix a dimension vector \( r = (r_v, v \in V) \) and a homogeneous representation \( \rho : \text{GL}(V, r) \longrightarrow \text{GL}(A) \). By slightly deviating from the conventions in the introduction (see Remark 3.4.1), the objects we would like to consider are pairs \( (\mathcal{E}_v, v \in V; \Psi) \) where \( (\mathcal{E}_v, v \in V) \) is a \( V \)-split vector bundle, such that \( \text{rk} \mathcal{E}_v = r_v, v \in V \), and \( \Psi : X \longrightarrow \mathbb{P}(\mathcal{F}_\rho^v) \) is a section. Here, \( \mathcal{F}_\rho \) is the vector bundle with fibre \( A \) associated to \( (\mathcal{E}_v, v \in V) \) via the representation \( \rho \). Now, the section \( \Psi \) is specified by a line bundle \( \mathcal{N} \) and a surjective homomorphism \( \phi : \mathcal{F}_\rho \longrightarrow \mathcal{N} \), and two such homomorphisms will yield the same section, if and only if they differ by a constant \( z \in \mathbb{C}^\times \). Thus, in order to find projective moduli spaces (at least over curves), we consider tuples \( (\mathcal{E}_v, v \in V; \mathcal{N}; \phi) \) where \( \mathcal{N} \) is a line bundle and \( \phi : \mathcal{F}_\rho \longrightarrow \mathcal{N} \) is a non-trivial homomorphism. Such an object will be referred to as a \( \rho \)-pair, and the tuple \( (\mathcal{P}(\mathcal{E}_v, v \in V), \mathcal{N}) \) is called the type.
Two \( \rho \)-pairs \((E_v, v \in V; N; \phi)\) and \((E'_v, v \in V; N'; \phi')\) will be considered *equivalent*, if there are isomorphisms \( \psi_v: E_v \rightarrow E'_v, v \in V, \) and \( z: N \rightarrow N' \), such that

\[
\phi' = z \circ \phi \circ \psi_{\rho}^{-1},
\]

with \( \psi_{\rho}: F_{\rho} \rightarrow F'_{\rho} \) the isomorphism induced by \( \psi_v, v \in V \).

Given a tuple \( P = (P_v, v \in V) \) of Hilbert polynomials and a line bundle \( N \), a *family of \( \rho \)-pairs of type \((P, N)\)* parametrized by the scheme \( S \) is the datum of a tuple \((E_{S,v}, v \in V; L_S, \phi_S)\) with vector bundles \( E_{S,v}, v \in V, \) on \( S \times X, \) such that \( P(E_{S,v}|_{s \times X}) = P_v \) for all \( s \in S, v \in V, L_S \) a line bundle on \( S, \) and \( \phi_S: F_{S,\rho} \rightarrow \pi_S^* L_S \otimes \pi_{X,S}^* N \) a homomorphism with non-trivial restriction to every fibre \([s] \times X, s \in S \). Two such families \((E_{S,v}, v \in V; L_S, \phi_S)\) and \((E'_{S,v}, v \in V; L'_S, \phi'_S)\) will be considered *equivalent*, if there are isomorphisms \( \psi_{S,v}: E_{S,v} \rightarrow E'_{S,v}, v \in V \) and \( z_S: L_S \rightarrow L'_S \), such that

\[
\phi'_S = (\pi_S^*(z_S) \otimes \text{id}_{\pi_X^* N}) \circ \phi_S \circ \psi_{\rho}^{-1}.
\]

**Remark 3.4.1.** First, we note that using \( F'_{\rho} \) instead of \( F_{\rho} \) is for notational convenience only. Then, the right analogue to the problems mentioned in the introduction would be the study of tuples \((E_v, v \in V; \phi)\) where \( \phi: F_{\rho} \rightarrow O_X \) is a surjective homomorphism, and \((E_v, v \in V; \phi)\) and \((E'_v, v \in V; \phi')\) should be identified, if and only if there are isomorphisms \( \psi_v: E_v \rightarrow E'_v \) with \( \phi' = \phi \circ \psi_{\rho}^{-1} \). In the case of a homogeneous representation of non-zero degree, this equivalence relation will identify \((E_v, v \in V; \phi)\) and \((E_v, v \in V; z \cdot \phi), z \in \mathbb{C}^*, \) anyway. Otherwise, one may add the trivial representation to \( \rho \). This means that we consider tuples \((E_v, v \in V; \phi, \varepsilon)\) with \((E_v, v \in V; \phi)\) as before and \( \varepsilon \in \mathbb{C}, \) but the equivalence relation becomes \((E_v, v \in V; \phi, \varepsilon) \sim (E'_v, v \in V; \phi', \varepsilon')\), if there are isomorphisms \( \psi_v: E_v \rightarrow E'_v, v \in V, \) and a \( z \in \mathbb{C}^* \), such that

\[
z \cdot \phi' = \phi \circ \psi_{\rho}^{-1}, \quad \text{and} \quad z \cdot \varepsilon' = \varepsilon.
\]

Then, we may recover the original objects in the form \((E_v, v \in V; \phi, 1)\). Thus, our concept is more flexible rather than more restrictive than the one presented in the introduction.

In order to define the semistability concept we introduce additional parameters:

- a tuple \( \sigma = (\sigma_v, v \in V) \) of positive integers.
- a tuple \( \eta = (\eta_v, v \in V) \) of rational numbers, such that \( \sum_{v \in V} \eta_v r_v = 0 \).
- a positive polynomial \( \delta \in \mathbb{Q}[[x]] \). Then, \( \chi_v := \eta_v \cdot \delta, v \in V \).

Given any torsion-free \( O_X \)-module \( E \), we call a submodule \( F \subset E \) *saturated*, if the quotient \( E/F \) is still torsion free. The test objects for the semistability concept will be weighted filtrations \((E_\bullet, \gamma)\) of the \( V \)-split vector bundle \((E_v, v \in V)\) where each \((E_j^v, v \in V)\) consists of saturated subsheaves \( E_j^v \subset E_v, v \in V, j = 1, \ldots, s \). For such a weighted filtration, we define \( \alpha = (\alpha_1, \ldots, \alpha_{s+1}) \) by \( \alpha_i := (\gamma_{i+1} - \gamma_i)/R, \) \( R := \text{rk}(\bigoplus_{v \in V} E_j^v), \) \( i = 1, \ldots, s \). We now set

\[
M_{\sigma, \chi}(E_\bullet, \alpha) := \sum_{j=1}^{s} \alpha_j \cdot (P_{\sigma, \chi}(E_v, v \in V) \cdot \text{rk}_{\sigma}(E_j^v, v \in V) - P_{\sigma, \chi}(E_j^v, v \in V) \cdot \text{rk}_{\sigma}(E_v, v \in V)).
\]
If we are also given a non-trivial homomorphism $\varphi: F_\rho \to N$, we have to define the quantity

$$\mu(\mathcal{E}_*, \gamma; \varphi).$$

The natural, though complicated, definition for $\# V = 1$ was explained in [24]. We adapt it to our setting. Let $M := \bigoplus_{v \in V} C^{\sigma_v \cdot r_v}$. Then, for appropriate $a, b, c$, the module $A$ will be a submodule of $M_{a \oplus b} \otimes (\bigwedge \dim W W)^{-c}$. We introduce $E_j^\text{total} := \bigoplus_{v \in V} (E_j^v)^{\oplus \sigma_v}$, $j = 1, \ldots, s$. There is the weighted filtration $(E_i^\text{total}, \alpha)$

$$0 \subsetneq E_1^\text{total} \subsetneq \cdots \subsetneq E_s^\text{total} \subsetneq E^\text{total}$$

of $E^\text{total} = \bigoplus_{v \in V} E_v^\oplus \sigma_v$. We choose a flag

$$0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_s \subsetneq M$$

with $\dim M_i = \text{rk} E_i^\text{total}$, $i = 1, \ldots, s$. Over a suitable open subset $\mathcal{U}$, the homomorphism $\varphi$ will be surjective, and there will be a trivialization $\psi: M \otimes O_{\mathcal{U}} \cong E^\text{total}$ with $\psi(M_i \otimes O_{\mathcal{U}}) = E_i^\text{total}$, $i = 1, \ldots, s$. We may write

$$F_{a, b, c} := E^\text{total}_{a \oplus b} \otimes \text{det}(E^\text{total})^{-c} = F_\rho \oplus F',$$

so that $\varphi$ and the trivialization $\psi$ yield a morphism

$$\sigma: \mathcal{U} \to \mathbb{P}(F'_\rho) \subsetneq \mathbb{P}(F'_{a, b, c}) \cong \mathbb{P}(M_{a \oplus b}^\vee) \times \mathcal{U} \to \mathbb{P}(M_{a \oplus b}^\vee).$$

After the choice of a one-parameter subgroup $\lambda: \mathbb{C}^* \to \text{SL}(M)$ which induces the weighted filtration $(M_*, \gamma')$ with

$$\gamma' = (\gamma'_1, \ldots, \gamma'_{s+1}) = \sum_{i=1}^s \alpha_i \left( \frac{R_i - R, \ldots, R_i - R, R_i, \ldots, R_i}{R_i \times (R-R_i) \times} \right),$$

$$R_i := \text{rk} E_i^\text{total}, \ i = 1, \ldots, s,$$

we set

$$\mu(\mathcal{E}_*, \gamma'; \varphi) := \max\{\mu(\lambda, \sigma(x)) \mid x \in \mathcal{U}\}.$$

As in [24], one verifies that this is, in fact, well-defined.

Remark 3.4.2.

(i) One might define $\mu(\mathcal{E}_*, \gamma'; \varphi)$ without reference to the embedding of $A$ into $M_{a \oplus b} \otimes (\bigwedge \dim W W)^{-c}$, by working with one-parameter subgroups of $\text{GL}(V, r)$. (ii) An easier, more elegant, and equivalent definition [8] is

$$\mu(\mathcal{E}_*, \gamma'; \varphi) := -\min_{(i_1, \ldots, i_d) \in \{1, \ldots, s+1\}^a} \{\gamma'_1 + \cdots + \gamma'_d \mid \tau \text{ restricted to} (E_i^\text{total} \otimes \cdots \otimes E_i^\text{total})_{a \oplus b} \text{ is non-trivial}\}.$$
Convention. Since the quantities introduced above depend only on \(q\), we will refer to a pair \((\mathcal{E}_\bullet, \alpha)\), composed of a filtration

\[
\mathcal{E}_\bullet: \ 0 \subset (\mathcal{E}_1^v, v \in V) \subset \cdots \subset (\mathcal{E}_s^v, v \in V) \subset (\mathcal{E}_s^v, v \in V)
\]

of \((\mathcal{E}_v, v \in V)\) by non-trivial, proper, and saturated \(V\)-split subsheaves and a tuple \(\alpha = (\alpha_1, \ldots, \alpha_s)\) of positive rational numbers, as a weighted filtration in the future.

A \(\rho\)-pair \((\mathcal{E}_v, v \in V; \mathcal{N}, \phi)\) will be called \((\sigma, \eta, \delta\)-(semi)stable or just (semi)stable, if for every weighted filtration \((\mathcal{E}_\bullet, \alpha)\) of \((\mathcal{E}_v, v \in V)\),

\[
M_{\sigma, \xi}(\mathcal{E}_\bullet, \alpha) + \delta \cdot \mu(\mathcal{E}_\bullet, \alpha; \phi)(\geq)0.
\]

A few comments are in order.

Remark 3.4.3. Since \(\sum_{v \in V} \chi_v \cdot r_v \equiv 0\), we may write

\[
M_{\sigma, \xi}(\mathcal{E}_\bullet, \alpha) = \text{rk} \left( \bigoplus_{v \in V} \mathcal{E}_v^\oplus \sigma_v \right) \cdot \sum_j \alpha_j \left( \sum_{v \in V} \chi_v \cdot \text{rk} \mathcal{E}_j^v \right)
\]

\[
+ \sum_j \alpha_j \left( \left( \text{rk} \left( \bigoplus_{v \in V} \mathcal{E}_v^\oplus \sigma_v \right) \cdot \left( \text{rk} \left( \bigoplus_{v \in V} \mathcal{E}_j^v \right) - \left( \text{rk} \left( \bigoplus_{v \in V} \mathcal{E}_j^v \right) \cdot \text{rk} \left( \bigoplus_{v \in V} \mathcal{E}_j^v \right) \right) \right) \right)
\]
Quiver problems

- \( M_{\sigma, \chi}(E, \alpha) \) is defined the same way,
- \( \mu(E, \gamma; \phi) \) does not depend on the \( \eta_v, v \in V \).

In particular, we may define (semi)stability with respect to these parameters. Suppose we are given arbitrary rational numbers \( \eta_v, v \in V \), and \( d \in \mathbb{Q} \). Define \( \eta'_v := \eta_v - d \cdot \sigma_v, v \in V \). Then,

\[
P_{\sigma, \chi}(F_v, v \in V) = P_{\sigma, \chi}(F_v, v \in V) + d \cdot \delta \cdot \left( \sum_{v \in V} \sigma_v \text{rk}(F_v) \right).
\]

It follows that

\[
P_{\sigma, \chi}(E, v \in V) \cdot \text{rk}_E(F_v, v \in V) - P_{\sigma, \chi}(F_v, v \in V) \cdot \text{rk}_F(E_v, v \in V)
\]

\[
= P_{\sigma, \chi}'(E, v \in V) \cdot \text{rk}_E(F_v, v \in V) - P_{\sigma, \chi}'(F_v, v \in V) \cdot \text{rk}_F(E_v, v \in V),
\]

so that always

\[
M_{\sigma, \chi}(E, \alpha) = M_{\sigma, \chi}'(E, \alpha),
\]

and the concept of (semi)stability defined with respect to the parameters \( \sigma_v, \eta_v, v \in V \), and \( \delta \) equals the one defined with respect to the parameters \( \sigma_v, \eta'_v, v \in V \), and \( \delta \). If we now set

\[
d := \sum_{v \in V} \eta_v r_v \sum_{v \in V} \sigma_v r_v,
\]

then

\[
\sum_{v \in V} \eta'_v r_v = \left( \sum_{v \in V} \eta_v r_v - d \cdot \left( \sum_{v \in V} \sigma_v r_v \right) \right) = 0.
\]

**Theorem 3.4.4.** (i) If \( \dim(X) = 1 \), there exist a projective scheme \( M := \mathcal{M}(\rho)^{(\sigma, \eta, \delta) - \text{ss}} / \mathcal{P}/\mathcal{N} \) and a natural transformation \( \theta : \mathcal{M}(\rho)^{(\sigma, \eta, \delta) - \text{ss}} / \mathcal{P}/\mathcal{N} \to h_M \), such that for any other scheme \( M' \) and any other natural transformation \( \theta' : \mathcal{M}(\rho)^{(\sigma, \eta, \delta) - \text{ss}} / \mathcal{P}/\mathcal{N} \to h_{M'} \), there is a unique morphism \( \chi : M \to M' \) with \( \theta' = h_\chi \circ \theta \). The space \( M \) contains an open subscheme \( M^\text{ss} \) which is a coarse moduli scheme for the functor \( \mathcal{M}(\rho)^{(\sigma, \eta, \delta) - \text{ss}} / \mathcal{P}/\mathcal{N} \). (ii) If \( \dim(X) \geq 1 \), there exists a quasi-projective coarse moduli scheme \( \mathcal{M}(\rho)^{(\sigma, \eta, \delta) - \text{ss}} / \mathcal{P}/\mathcal{N} \) for the functor \( \mathcal{M}(\rho)^{(\sigma, \eta, \delta) - \text{ss}} / \mathcal{P}/\mathcal{N} \).

**Example 3.4.5.** The above results yield a semistability concept for coverings in the description of Casnati and Ekedahal (as reviewed in the introduction) and provides moduli spaces.

It is obvious that the theorem has to be proved only for representations of the type \( \rho_{a,b,c} \). If \( \dim(X) > 1 \), then in order to compactify the moduli spaces, one also needs torsion free sheaves. It is however not clear how to associate a sheaf \( \mathcal{F}_\rho \) to a \( V \)-split torsion free sheaf \( (E_v, v \in V) \) via an arbitrary representation \( \rho \). However, for representations of the form \( \rho_{a,b,c} \), this is obvious and one obtains natural compactifications. In the setting of quiver representations, we will exhibit another natural method to reduce a moduli problem to one for \( \rho_{a,b,c} \)-pairs. This illustrates the importance and the usefulness of the theory which we will outline in the next section.
3.5 Decorated $V$-split sheaves

We fix the following data

- a tuple of Hilbert polynomials $\mathcal{P} = (P_v, v \in V)$;
- a positive polynomial $\delta \in \mathbb{Q}[x]$ of degree at most $\dim X - 1$;
- a tuple of rational numbers $\eta = (\eta_v, v \in V)$ with $\sum_{v \in V} \eta_v \cdot r_v = 0$. Here, $r_v$ is the rank dictated by the Hilbert polynomial $P_v$,
- a tuple $\sigma = (\sigma_v, v \in V)$ of positive integers.

Given a $V$-split sheaf $(\mathcal{E}_v, v \in V)$ of type $\mathcal{P}$ and non-negative integers $a, b, c,$ and $m$, a decoration of type $(a, b, c, m)$ on $(\mathcal{E}_v, v \in V)$ is a homomorphism

$$\tau : \left( \bigoplus_{v \in V} E_v^{(\sigma_v)} \right) \otimes \mathcal{O}_X(m).$$

Two tuples $(\mathcal{E}_v, v \in V; \tau)$ and $(\mathcal{E}_v', v \in V; \tau')$ are called equivalent, if there are $z \in \mathbb{C}^*$ and isomorphisms $\psi_v : \mathcal{E}_v \rightarrow \mathcal{E}_v'$, such that

$$z \cdot \left( \left( \det \left( \bigoplus_{v \in V} \mathcal{E}_v^{(\sigma_v)} \right) \otimes \mathcal{O}_X(m) \right) \right)^{\otimes a} \otimes \mathcal{O}_X(m) = \tau'.$$

A family of $V$-split sheaves of type $\mathcal{P}$ with a decoration of type $(a, b, c, m)$ parametrized by the scheme $S$ consists of

- a tuple $(\mathcal{E}_S,v, v \in V)$ on $S \times X$ of $S$-flat families $\mathcal{E}_{S,v}$ of torsion free sheaves on $X$ with Hilbert polynomial $P_v$, $v \in V$;
- a line bundle $\mathcal{L}_S$ on $S$;
- a homomorphism $\tau_S : \left( \bigoplus_{v \in V} E_v^{(\sigma_v)} \right) \otimes \mathcal{L}_S \otimes \mathcal{O}_X(m).$

Two such families will be called equivalent, if there are isomorphisms $\psi_{S,v} : \mathcal{E}_{S,v} \rightarrow \mathcal{E}_{S,v}'$, $v \in V$, and $z_S : \mathcal{L}_S \rightarrow \mathcal{L}_S'$, such that

$$\left( \left( \det \left( \bigoplus_{v \in V} \mathcal{E}_{S,v}^{(\sigma_v)} \right) \otimes \pi_X^*(\mathcal{L}_S) \otimes \mathcal{O}_X(m) \right) \right)^{\otimes a} \otimes \mathcal{O}_X(m) = \tau' .$$

The semistability condition

Let $(\mathcal{E}_v, v \in V)$ be a $V$-split sheaf. Then — as agreed upon before — a weighted filtration of $(\mathcal{E}_v, v \in V)$ is a pair $(\mathcal{E}_\bullet, \mathcal{G})$, composed of a filtration

$$\mathcal{E}_\bullet : 0 \subsetneq (\mathcal{E}_1^v, v \in V) \subsetneq \cdots \subsetneq (\mathcal{E}_s^v, v \in V) \subsetneq (\mathcal{E}_v, v \in V)$$
of \((\mathcal{E}_v, v \in V)\) by non-trivial, proper, and saturated \(V\)-split subsheaves and a tuple \(\alpha = (\alpha_1, \ldots, \alpha_s)\) of positive rational numbers. Recall that

\[
M_{\alpha, \mathcal{F}}(\mathcal{E}_v, \alpha) = \sum_{j=1}^{s} \alpha_j \cdot (P_{\alpha, \mathcal{F}}(\mathcal{E}_v, v \in V) \cdot \text{rk}_{\mathcal{F}}(\mathcal{E}_v, v \in V)) - P_{\alpha, \mathcal{F}}(\mathcal{E}_v, v \in V) \cdot \text{rk}_{\mathcal{F}}(\mathcal{E}_v, v \in V)).
\]

The number

\[
\mu(\mathcal{E}_v, \alpha; \tau)
\]

is defined by the formula in Remark 3.4.2(ii).

Now, we call a \(V\)-split sheaf \((\mathcal{E}_v, v \in V; \tau)\) with a decoration of type \((a, b, c, m)\) (semi)stable (or more precisely \((\sigma, \eta, \delta)-(\text{semi})\text{stable}\), if for every weighted filtration \((\mathcal{E}_v, \alpha)\) of \((\mathcal{E}_v, v \in V)\),

\[
M_{\alpha, \mathcal{F}}(\mathcal{E}_v, \alpha) + \delta \cdot \mu(\mathcal{E}_v, \alpha; \tau)(\geq 0).
\]

The first main result

We define the functors

\[
M^{(a, b, c, m)} : \text{Sch}_C \rightarrow \text{Sets}
\]

\[
S \mapsto \left\{ \text{Equivalence classes of families of (semi)stable \(V\)-split sheaves of type} \; P \; \text{with a decoration of type} \; (a, b, c, m) \right\}.
\]

**Theorem 3.5.1.** (i) There exist a projective scheme \(M := M^{(a, b, c, m)}(\mathcal{E}_v, \alpha; \tau)\) and a natural transformation \(\theta : M^{(a, b, c, m)} \rightarrow h_M\), such that for any other scheme \(M'\) and any other natural transformation \(\theta' : M^{(a, b, c, m)} \rightarrow h_{M'}\), there is a unique morphism \(\xi : M \rightarrow M'\) with \(\theta' = h_{\xi} \circ \theta\). (ii) The space \(M\) contains an open subscheme \(M^0\) which is a coarse moduli scheme for the functor \(M^{(a, b, c, m)}\).

3.6 Applications to quiver problems

Let \(Q = (V, A, t, h)\) be a quiver with vertices \(V = \{v_1, \ldots, v_t\}\), arrows \(A = \{a_1, \ldots, a_u\}\), the tail map \(t : A \rightarrow V\), and the head map \(h : A \rightarrow V\). We assume that no multiple arrows occur. Fix a tuple of coherent sheaves \(G = (G_a, a \in A)\). An (augmented) representation of \(Q\) of type \((P, \mathcal{G})\) is a tuple \((\mathcal{E}_v, v \in V; f_a, a \in A; \epsilon)\), consisting of

- a \(V\)-split sheaf \((\mathcal{E}_v, v \in V)\) of type \(P\),
- homomorphisms \(f_a : G_a \otimes \mathcal{E}_{t(a)} \rightarrow \mathcal{E}_{h(a)}, a \in A\),
- a complex number \(\epsilon\),

such that either \(\epsilon \neq 0\) or one of the \(f_a, a \in A\), is non-trivial. For simplicity, we will often drop the term ‘augmented’ in the following. Two representations \((\mathcal{E}_v, v \in V; f_a, a \in A; \epsilon)\) and \((\mathcal{E}_v', v \in V; f'_a, a \in A; \epsilon')\) are called equivalent, if there are isomorphisms \(\psi_v : \mathcal{E}_v \rightarrow \mathcal{E}_v', v \in V, z \in \mathbb{C}^*, \) such that

\[
z \cdot (\psi_{h(a)} \circ f_a \circ (id_{G_a} \otimes \psi_{t(a)})^{-1}) = f'_a, \quad a \in A, \quad \text{and} \quad z \cdot \epsilon = \epsilon'.
\]
A family of representations of $Q$ of type $(P, \mathcal{C})$ parametrized by $S$ is a tuple $\langle \mathcal{E}_{S,v}, v \in V: f_{S,a}, a \in A; \mathcal{L}_S, \mathcal{E}_S \rangle$ which consists of
- $S$-flat families $\mathcal{E}_{S,v}$ on $S \times X$ of torsion free sheaves on $X$ with Hilbert polynomial $P_v$, $v \in V$;
- a line bundle $\mathcal{L}_S$ on $S$;
- a section $\mathcal{E}_S \in H^0(S, \mathcal{L}_S)$;
- homomorphisms $f_{S,a}: \pi_X^* \mathcal{G}_a \otimes \mathcal{E}_{S,(a)} \rightarrow \mathcal{E}_{S,(a)} \otimes \mathcal{L}_S$.

An equivalence of the families $\mathcal{E}_{S,v}^1, v \in V; f_{S,a}^1, a \in A; \mathcal{L}_S^1, \mathcal{E}_S^1$ and $\mathcal{E}_{S,v}^2, v \in V; f_{S,a}^2, a \in A; \mathcal{L}_S^2, \mathcal{E}_S^2$ consists of an isomorphism $z_S: \mathcal{L}_S^1 \rightarrow \mathcal{L}_S^2$ and isomorphisms $\psi_v: \mathcal{E}_{S,v}^1 \rightarrow \mathcal{E}_{S,v}^2, v \in V$, such that

$$f_{S,a}^2 = (\psi_{h(a)} \otimes \pi_X^*(z_S)) \circ f_{S,a}^1 \circ (\text{id}_{\pi_X^* \mathcal{G}_a} \otimes \psi_{f(a)})^{-1}, a \in A, z_S \circ \mathcal{E}_S^1 = \mathcal{E}_S^2.$$

Associated decorations

Fix the parameters $\sigma = (\sigma_v \in \mathbb{Z}_{\geq 0}, v \in V)$. There are $m \geq 0$ and $b > 0$, such that we have surjections $\nu_v: \mathcal{O}_X(-m) \otimes \mathcal{E}_{S,v} \rightarrow \mathcal{G}_a$ for all $a \in A$, and an embedding $\nu_0: \mathcal{O}_X \rightarrow \mathcal{O}_X(m)$. Set $M := \bigoplus_{v \in V} \mathcal{C}^v \mathcal{E}_{S,v}$, and $(\mathcal{L}_S^m)_{\mathcal{E}_S^m}$ as $\text{Hom}(\mathcal{L}_S, \mathcal{E}_S^m)$.

Let $(\mathcal{E}_{S,v}, v \in V; f_{S,a}, a \in A; \mathcal{L}_S, \mathcal{E}_S)$ be a family of representations on $S \times X$ and $\iota: U \subset S \times X$ the maximal open subset over which all the $\mathcal{E}_{S,v}, v \in V$, are locally free. Then, the restrictions of the $f_{S,a}^{\mathcal{E}_{S,v}(\mathcal{E}_{S,v})}$ to $U$ together with the pullback of $\mathcal{E}_S$ to $U$ may be interpreted as a homomorphism

$$\tau': \text{Hom}
\left(\bigoplus_{v \in V} \mathcal{E}_{S,v}^\otimes \otimes \bigoplus_{v \in V} \mathcal{E}_{S,v}^\otimes \right) \oplus \mathcal{O}_U \rightarrow \iota^*(\pi_X^* \mathcal{L}_S \otimes \pi_X^* \mathcal{O}_X(m)).$$

The splitting of $Z$ yields a natural projection

$$\text{det}(\mathcal{E}_{S,v}) \otimes (-1) \mathcal{O}_U \rightarrow \text{Hom}
\left(\bigoplus_{v \in V} \mathcal{E}_{S,v}^\otimes \otimes \bigoplus_{v \in V} \mathcal{E}_{S,v}^\otimes \right) \oplus \mathcal{O}_U,$$

so that we get

$$\tau'' : \left(\bigoplus_{v \in V} \mathcal{E}_{S,v}^\otimes \otimes \bigoplus_{v \in V} \mathcal{E}_{S,v}^\otimes \right) \oplus \mathcal{O}_U \rightarrow \iota^*(\text{det}(\mathcal{E}_{S,v}) \otimes \pi_X^* \mathcal{L}_S \otimes \pi_X^* \mathcal{O}_X(m)).$$

We finally define

$$\tau_S : \left(\bigoplus_{v \in V} \mathcal{E}_{S,v}^\otimes \otimes \bigoplus_{v \in V} \mathcal{E}_{S,v}^\otimes \right) \oplus \mathcal{O}_U \rightarrow \iota_* \left(\left(\bigoplus_{v \in V} \mathcal{E}_{S,v}^\otimes \otimes \bigoplus_{v \in V} \mathcal{E}_{S,v}^\otimes \right) \oplus \mathcal{O}_U \right).$$

Prop. A.1.1

$$\text{det}(\mathcal{E}_{S,v}) \otimes \pi_X^* \mathcal{L}_S \otimes \pi_X^* \mathcal{O}_X(m).$$
We call \((\mathcal{E}_{S,v}, v \in V; \tau_S)\) the associated family of \(V\)-split sheaves with a decoration of type \((s, b + 1, 1, m), s := \sum_{v \in V} \sigma_v \cdot r_v\). When \(S\) is just a point, we simply speak of the associated decoration \(\tau\).

**The semistability condition**

Fix the same data as before. We call a representation \((\mathcal{E}_v, v \in V; f_a, a \in A; \varepsilon)\) \((\sigma, \eta, \delta)\)-(semi)stable, if the \(V\)-split sheaf \((\mathcal{E}_v, v \in V; \tau)\) with the associated decoration is \((\sigma, \eta, \delta)\)-(semi)stable.

**The second main result**

Define the functors
\[
\mathbb{R}(Q)^{(\sigma, \eta, \delta)-ss}_{P, G} : \text{Sch}_C \rightarrow \text{Sets}
\]
\[
S \mapsto \left\{ \text{Equivalence classes of families of } (\sigma, \eta, \delta)\text{-} (\text{semi})\text{stable representations of } Q \text{ of type } (P, G) \right\}.
\]

**Theorem 3.6.1.** (i) There exist a projective scheme \(R := \mathbb{R}(Q)^{(\sigma, \eta, \delta)-ss}_{P, G}\) and a natural transformation \(\theta : \mathbb{R}(Q)^{(\sigma, \eta, \delta)-ss}_{P, G} \rightarrow h_R\), such that for any other scheme \(R'\) and any other natural transformation \(\theta' : \mathbb{R}(Q)^{(\sigma, \eta, \delta)-ss}_{P, G} \rightarrow h_{R'}\), there exists a unique morphism \(\zeta : R \rightarrow R'\) with \(\theta' = h_\zeta \circ \theta\). (ii) The scheme \(R\) contains an open subscheme \(R_s\) which is a coarse moduli scheme for the functor \(\mathbb{R}(Q)^{(\sigma, \eta, \delta)-s}_{P, G}\).

**3.7 Behaviour for large \(\delta\)**

Intuitively, one would like to have a semistability concept for representations of quivers which poses conditions on subrepresentations only. However, as illustrated in [24] for the example of the quiver consisting of one vertex and an arrow, connecting the vertex to itself, this property cannot be expected for general \(\delta\). However, as in the case of the aforementioned quiver, for large \(\delta\), the semistability concept will stabilize to one which is a condition on subrepresentations only. Another nice feature is that, for large \(\delta\), one has a generalized Hitchin map. Our first result is the following:

**Theorem 3.7.1.** Fix the data \(\sigma, \eta, P,\) and \(G\). Let \(\delta\) be a positive polynomial of degree exactly \(\dim X - 1\). Then, there exists a natural number \(n_\infty\), such that for all \(n \geq n_\infty\), the following two conditions on a representation \((\mathcal{E}_v, v \in V; f_a, a \in A; \varepsilon)\) of \(Q\) of type \((P, G)\) are equivalent

1. The representation \((\mathcal{E}_v, v \in V; f_a, a \in A; \varepsilon)\) is \((\sigma, \eta/n, n \cdot \delta)\)-(semi)stable, \(\eta/n := (\eta_v/n, v \in V)\).

2. (a) For any non-trivial, proper subrepresentation \((\mathcal{F}_v, v \in V)\) (i.e., \(V\)-split subsheaf, such that \(f_a(G_a \otimes \mathcal{F}_v) \subset F_{h(a)}\) for all \(a \in A\)) one has
\[
\frac{P_{\sigma, \eta}((\mathcal{F}_v, v \in V))}{\rk_a((\mathcal{F}_v, v \in V))} \leq \frac{P_{\sigma, \eta}((\mathcal{E}_v, v \in V))}{\rk_a((\mathcal{E}_v, v \in V))},
\]
and
(b) either \( \varepsilon \neq 0 \), or the restriction of \((E_v, v \in V; f_a, a \in A; \varepsilon)\) to a general point \( x \in X \) in the open subset where all the \( E_v, v \in V \), are locally free is semistable.

We call a representation \((E_v, v \in V; f_a, a \in A; \varepsilon)\) of \( Q \) of type \((P, \mathcal{G}) (\sigma, \chi)\)-(semi)stable, if it satisfies Condition 2 in Theorem 3.7.1 with \((\leq)\). Note that, for representations of the form \((E_v, v \in V; f_a, a \in A; \varepsilon = 1)\), this is exactly the ‘Gieseker-analogue’ of the semistability definition given by Álvarez-Cónsul and García-Prada [1]. Observe that for \((\sigma, \chi)\)-(semi)stable representations, one has the concepts of a Jordan–Hölder filtration, the associated graded object, and S-equivalence. Therefore, one can also speak of \((\sigma, \chi)\)-polystable representations.

**Invariants of quivers and the generalized Hitchin map**

Recall that we may find \( b > 0 \) and any point \( x \) in the open subset \( \text{Rep}^r o (Q_b) \) for all \( r \) as well as an embedding \( v_0: \mathcal{O}_X(-m)^{\oplus b} \to \mathcal{G}_a \) for all \( a \in A \) as well as an embedding \( v_0: \mathcal{O}_X \to \mathcal{O}_X(m) \). Therefore, we now look at the quiver \( Q^b := (V, A^b) := A \times \{1, \ldots, b\}, t, h \) where the tail and head maps are given by the projection onto the first factor followed by the tail and head map of \( Q \). In other words, any arrow in \( Q \) is replaced by \( b \) copies of the same arrow. We choose \( \underline{r} = (r_v, v \in V) \) as the dimension vector. The variety of representations with this dimension vector is thus

\[
\text{Rep}_{\underline{r}}(Q^b) := \bigoplus_{a \in A^b} \text{Hom}(\mathbb{C}^{r_{t(a)}}, \mathbb{C}^{r_{h(a)}}).
\]

The variety \( \text{Rep}_{\underline{r}}(Q^b) \) comes with an action of the group \( X_{v \in V} \text{GL}(r_v, \mathbb{C}) \). By the work of LeBruyn and Procesi [13], one knows explicit generators for the ring of \((X_{v \in V} \text{GL}(r_v, \mathbb{C}))\)-invariant regular functions on \( \text{Rep}_{\underline{r}}(Q^b) \). To state the result, let \( \sigma = (a_1, \ldots, a_l) \) be an oriented cycle, i.e., a sequence of arrows \( a_1, \ldots, a_l \), such that \( h(a_i) = t(a_{i-1}) \), \( i = 2, \ldots, l \), and \( h(a_1) = t(a_l) =: v(\sigma) \). We call \( l \) the length of the cycle. For any such cycle and any point \( x \in \text{Rep}_{\underline{r}}(Q^b) \), we get an endomorphism \( f_{x,\sigma} : \mathbb{C}^{r_{v(\sigma)}} \to \mathbb{C}^{r_{v(\sigma)}} \). We then define

\[
t_0 : \text{Rep}_{\underline{r}}(Q^b) \to \mathbb{C}, \quad x \mapsto \text{Trace}(f_{x,\sigma}).
\]

The function \( t_0 \) is obviously invariant under the \((X_{v \in V} \text{GL}(r_v, \mathbb{C}))\)-action. The result of [13] states that the invariants of the form \( t_0, \sigma \) an oriented cycle, generate the ring of invariants \( \mathcal{R}_{\underline{r}}(Q^b) := \mathbb{C}[\text{Rep}_{\underline{r}}(Q^b)]^{X_{v \in V} \text{GL}(r_v, \mathbb{C})} \). Moreover, one may restrict to oriented cycles of length at most \( (\sum_{v \in V} r_v)^2 + 1 \). We also look at the affine variety

\[
\text{Rep}_{\underline{r}}(Q^b) := \text{Rep}_{\underline{r}}(Q^b) \oplus \mathbb{C}.
\]

This is also a \((X_{v \in V} \text{GL}(r_v, \mathbb{C}))\)-variety, the action on \( \mathbb{C} \) being trivial. Denote by \( t_0 \) the projection onto the second factor. This is a \((X_{v \in V} \text{GL}(r_v, \mathbb{C}))\)-invariant function, and the above result implies

\[
\mathcal{R}_{\underline{r}}(Q^b) := \mathbb{C}[\text{Rep}_{\underline{r}}(Q^b)]^{X_{v \in V} \text{GL}(r_v, \mathbb{C})} = \mathbb{C} \left[ t_0; t_0, \sigma \text{ an oriented cycle of length } \leq \left( \sum_{v \in V} r_v \right)^2 + 1 \right].
\]
Next, assign degree one to \( t_0 \) and degree length of \( o \) to \( t_0 \), an oriented cycle. Then, \( R^e_\mathbb{C}(Q^b) \) is a graded ring, and \( \text{Proj}(R^e_\mathbb{C}(Q^b)) \) identifies with the \((\mathbb{C}^* \times \mathbb{X}_{v \in V} \text{GL}_v, \mathbb{C})\)-quotient of \( \text{Rep}_e(Q^b) \) where \( \mathbb{C}^* \) acts by scalar multiplication. We may choose a degree \( d \) such that the subring \( R^{(d)} \subset R^e_\mathbb{C}(Q^b) \) of elements the degree of which is a multiple of \( d \) is generated by elements of degree \( d \), say, \( i_0, \ldots, i_q \) (see [18], § III.8, Lemma). This yields an embedding

\[
\text{Proj}(R^e_\mathbb{C}(Q^b)) \hookrightarrow \mathbb{P}_q.
\]

Now, we return to the setting of representations of \( Q \) of type \((P, G)\) where we fix \( b, m, v_a, a \in A \), and \( v_0 \) as before. Set \( H(Q, P, G) := H^0(O_X(d \cdot m)) \oplus \epsilon \).

Let \( S \) be a scheme, and \((E_{S,v}, v \in V; f_{S,a}, a \in A; L_S, \epsilon)\) a family of \((\sigma, \chi)\)-semistable representations of \( Q \) of type \((P, G)\) parametrized by \( S \). Denote by \( i_0 : U \subset C \times X \) the maximal open subset where all the \( E_{S,v}, v \in V \), are locally free. To the invariant \( t_0 \) corresponds the homomorphism

\[
t_0'' : \mathcal{O}_{S \times X} \xrightarrow{\pi^*_S \mathcal{E}_{S,v} \otimes \pi^*_X v_0} \pi^*_S \mathcal{L}_S \otimes \pi^*_X \mathcal{O}_X(m).
\]

Furthermore, using the quotients \( v_a, a \in A \), for any oriented cycle \( o \) of length \( l \), we get a homomorphism

\[
f'_o : E_{S,v(o)} \rightarrow E_{S,v(o)} \otimes (\pi^*_S \mathcal{L}_S \otimes \pi^*_X \mathcal{O}_X(m))^{\otimes l}.
\]

If we restrict \( f'_o \) to \( U \) and take traces, we obtain a section

\[
t'_o : \mathcal{O}_U \rightarrow t'(\pi^*_S \mathcal{L}_S \otimes \pi^*_X \mathcal{O}_X(m))^{\otimes l}.
\]

By Proposition A.1.1, this extends to

\[
t'_o : \mathcal{O}_{S \times X} \rightarrow (\pi^*_S \mathcal{L}_S \otimes \pi^*_X \mathcal{O}_X(m))^{\otimes l}.
\]

Therefore, any invariant \( i_j, j = 0, \ldots, q \), provides a section

\[
i_j := i_j(E_{S,v}, v \in V; f_{S,a}, a \in A; L_S, \epsilon) : \mathcal{O}_{S \times X} \rightarrow (\pi^*_S \mathcal{L}_S \otimes \pi^*_X \mathcal{O}_X(m))^{\otimes d}, \quad j = 0, \ldots, q.
\]

Condition 2(b) now grants that one of the homomorphisms \( i_j, j = 0, \ldots, q \), will be non-zero. Hence, we get a morphism

\[
\text{Hit}(E_{S,v}, v \in V; f_{S,a}, a \in A; L_S, \epsilon) : S \rightarrow \mathbb{H} := \mathbb{H}(Q, P, G)
\]

with \( \text{Hit}^{\ast} \mathcal{O}_{\mathbb{H}}(1) = \mathcal{L}_S^{\otimes d} \).

**The third main result**

This time, we look at the functors

\[
\text{R}(Q)_{(\sigma, \chi)-(\text{semi})stable} : \text{Sch}_C \rightarrow \text{Sets}
\]

\[
S \mapsto \left\{ \begin{array}{l}
\text{Equivalence classes of families of } (\sigma, \chi) \\
\text{(semi)stable representations of } Q \text{ of type } (P, G)
\end{array} \right\}.
\]
Theorem 3.7.2. (i) There exist a projective scheme $\mathcal{R} := \mathcal{R}(\mathcal{Q})_{\mathcal{P}/\mathcal{G}}^{(\sigma, \chi) - \text{ss}}$ and a natural transformation $\theta: \mathcal{R}(\mathcal{Q})_{\mathcal{P}/\mathcal{G}}^{(\sigma, \chi) - \text{ss}} \to h_\mathcal{R}$, such that for any other scheme $\mathcal{R}'$ and any other natural transformation $\theta': \mathcal{R}(\mathcal{Q})_{\mathcal{P}/\mathcal{G}}^{(\sigma, \chi) - \text{ss}} \to h_\mathcal{R}'$, there exists a unique morphism $\zeta: \mathcal{R} \to \mathcal{R}'$ with $\theta' = h_\mathcal{R}' \circ \theta$. (ii) The scheme $\mathcal{R}$ contains an open subscheme $\mathcal{R}'$ which is in bijection to the set of $S$-equivalence classes of $(\sigma, \chi)$-semistable representations of $\mathcal{Q}$ of type $(\mathcal{P}, \mathcal{Q})$, or, equivalently, to the set of isomorphism classes of $(\sigma, \chi)$-polystable representations of $\mathcal{Q}$ of type $(\mathcal{P}, \mathcal{Q})$. (iii) The closed points of $\mathcal{R}$ are in bijection to the set of $S$-equivalence classes of $(\sigma, \chi)$-semistable representations of $\mathcal{Q}$ of type $(\mathcal{P}, \mathcal{Q})$, or, equivalently, to the set of isomorphism classes of $(\sigma, \chi)$-polystable representations of $\mathcal{Q}$ of type $(\mathcal{P}, \mathcal{Q})$. (iv) There is a generalized Hitchin morphism

$$\text{Hit}(\mathcal{Q}, \mathcal{P}, \mathcal{Q}): \mathcal{R} \to \mathbb{H}(\mathcal{Q}, \mathcal{P}, \mathcal{Q}).$$

Note that the theorem in the introduction now follows by taking $\mathcal{D}$ and $\mathbb{D}$ as the open subscheme $\mathfrak{e} = 1$ in $\mathcal{R}$ and $\mathbb{H}$, respectively (cf. Remark 3.4.2).

4. The proofs

4.1 Proof of Theorem 3.5.1

In order to prove Theorem 3.5.1, one can copy almost word by word the proofs in [24], §2.3.6, or [8]. The only point which has to be given special attention and which is indeed rather tricky is the correct choice of a linearization on the parameter space. We will, therefore, construct the parameter space, give the linearization of the respective group action, and show in a sample computation that it is the correct one.

The parameter space

We denote by $\mathfrak{A}_v$, $v \in V$, the union of those components of Pic($X$) which contain line bundles of the form $\text{det}(\mathcal{E}_v)$ where $(\mathcal{E}_v, v \in V; \tau)$ is a semistable $V$-split sheaf of type $\mathcal{P}$ with a decoration of type $(a, b, c, m)$. We also set $\mathfrak{A} := X_{v \in V} \mathfrak{A}_v$. By the usual boundedness arguments, we can find an $l_0$, such that for all $l \geq l_0$, all semistable $V$-split sheaves $(\mathcal{E}_v, v \in V; \tau)$ of type $\mathcal{P}$ with a decoration of type $(a, b, c, m)$, all $v \in V$, all $[\mathcal{L}] \in \mathfrak{A}_v$, and all $\mathcal{N} = \bigotimes_{v \in V} \mathcal{L}_v^{\otimes r_v}$ with $[\mathcal{L}_v] \in \mathfrak{A}_v$, $v \in V$,

- $\mathcal{E}_v(l)$ is globally generated and $H^i(\mathcal{E}_v(l)) = 0$ for all $i > 0$;
- $\mathcal{L}(r_v \cdot l)$ is globally generated and $H^i(\mathcal{L}(r_v \cdot l)) = 0$ for all $i > 0$;
- $\mathcal{N}^{\otimes c}(a \cdot l)$ is globally generated and $H^i(\mathcal{N}^{\otimes c}(a \cdot l)) = 0$ for all $i > 0$.

We fix such an $l$, and set $p_v := P_v(l)$, $v \in V$, and

$$p := \sum_{v \in V} p_v \cdot p_v.$$

Moreover, we choose vector spaces $W_v$ of dimension $p_v$ and let $\Omega_v^0$ be the quasi-projective quot scheme parametrizing quotients $q: W_v \otimes O_X(-l) \to \mathcal{F}$ with $\mathcal{F}$ a torsion free coherent $O_X$-module with Hilbert polynomial $P_v$ and $H^0(q(l))$ an isomorphism, $v \in V$. Let $\mathcal{E}_v$ be the universal quotient on $\Omega_v^0 \times X$, $v \in V$, and

$$\mathcal{E}_{\text{total}} := \bigoplus_{v \in V} \pi_{\Omega_v^0}^* \mathcal{E}_v^{\otimes r_v}$$

be the resulting sheaf on $(X_{v \in V} \Omega_v^0) \times X$. Define $M := \bigoplus_{v \in V} W_v^{\otimes r_v}$ and

$$\mathfrak{P} := \mathbb{P}(((M^{\otimes a})^{\otimes b})^\vee \otimes \pi_{X_{v \in V} \Omega_v^0}^* \text{det}(\mathcal{E}_{\text{total}}^{\otimes c} \otimes \pi_{X}^* O_X(a \cdot l))).$$
This is a projective bundle over \( X_{v \in V} \Omega^0_v \), and the parameter space \( \mathcal{M} \) is constructed in the usual way as a closed subscheme of \( \mathcal{P} \). In particular, it is projective over \( X_{v \in V} \Omega^0_v \). Furthermore, \( \mathcal{M} \) comes with an action of the group \( (X_{v \in V} \text{GL}(W_v))/\mathbb{C}^* \), \( \mathbb{C}^* \) being diagonally embedded. We define

\[
\tilde{G} := \left( \bigotimes_{v \in V} \text{GL}(W_v) \right) \cap \text{SL}(M)
\]

\[
= \left\{ (h_1, \ldots, h_t) \in \bigotimes_{v \in V} \text{GL}(W_v) \mid \det (h_1)^{\sigma_{v_1}} \cdots \det (h_t)^{\sigma_{v_t}} = 1 \right\}.
\]

The group \( \tilde{G} \) maps with finite kernel onto \( (X_{v \in V} \text{GL}(W_v))/\mathbb{C}^* \), whence we may restrict our attention to the action of \( \tilde{G} \).

The linearization of the above group action will be induced via a Gieseker morphism to some other scheme. For this, we fix Poincaré line bundles \( P_v \) over \( A_v \times X \), \( v \in V \), and set

\[
\mathcal{G}_v := \mathbb{P} \left( \left( r_v \bigotimes_{W_v} \chi \right) \otimes (P_v \otimes \pi^* X \mathcal{O}_X (r_v \cdot l)) \right).
\]

Choosing \( P_v \) appropriately, we may assume that \( \mathcal{O}_{\mathcal{G}_v}(1) \) is very ample for all \( v \in V \). On \( A \times X \), we get the line bundle

\[
\mathcal{P} := \bigotimes_{v \in V} \pi^*_{\mathcal{A}_v \times X} \mathcal{P}^{\otimes \sigma_v}.
\]

Then, we define

\[
\mathcal{P}' := \mathbb{P} \left( (M^{\otimes a})^{\otimes b} \otimes \pi^*_{\mathcal{A}_v \times X} \mathcal{P}^{\otimes \sigma_v} \right)
\]

as a projective bundle over \( \mathcal{A} \). Again, \( \mathcal{O}_{\mathcal{P}'}(1) \) can be assumed to be ample. We now have a \( \tilde{G} \)-equivariant and injective morphism

\[
\Gamma: \mathcal{M} \longrightarrow \mathcal{P}' \times \bigotimes_{v \in V} \mathcal{G}_v.
\]

For a given \( \beta \in \mathbb{Z}_{>0} \) and \( \kappa_v \in \mathbb{Z}_{>0} \), \( v \in V \), there is a natural linearization of the \( \tilde{G} \)-action on \( \mathcal{P}' \times X_{v \in V} \mathcal{G}_v \) in the ample line bundle \( \mathcal{O}(\beta; \kappa_v, v \in V) \). This may be altered by any character of \( X_{v \in V} \text{GL}(W_v) \). Let \( d := \delta(l), x_v := -\chi_v(l)/d, \] \( x'_v := r_v x_v/p_v, v \in V, \]

\[
\rho := \frac{p - a \cdot d}{r \cdot d}, \quad \varepsilon_v := \frac{x_v}{\rho} = \sigma_v + \frac{r \cdot \chi_v(l)}{p - a \cdot d}, \quad v \in V,
\]

and

\[
x''_v := \varepsilon_v \cdot \frac{r_v}{p_v} - r_v, \quad v \in V.
\]

**Remark 4.1.1.** To be very precise, the quantities \( \varepsilon \) and \( \varepsilon_v, v \in V \), are functions in \( l \). Since \( p = P(l) \) is a positive polynomial of degree \( X \) and both \( \delta \) and \( \chi \) are polynomials of degree at most \( \text{dim } X - 1 \), it is clear that \( \varepsilon \) and \( \varepsilon_v, v \in V \), will be positive for \( l \gg 0 \), i.e., the line bundle in which the action is linearized is really ample.
Now, we choose $\beta \in \mathbb{Z}_{>0}$ and $\kappa_v \in \mathbb{Z}_{>0}$ such that

$$\frac{\kappa_v}{\beta} = \varepsilon \cdot \varepsilon_v, \quad v \in V.$$  

We modify the linearization of the $\tilde{G}$-action on $X_v \in V$ in $\mathcal{O}(\kappa_v, v \in V)$ by a character, such that $C^\ast \cdot \text{id}_{W_{v_1}} \times \cdots \times C^\ast \cdot \text{id}_{W_{v_l}}$ acts via $a \mapsto \Pi v \in V z_v^{\rho_v}e_v$ with 

$$e_v := \beta \cdot (x_1' + x_2''), \quad v \in V.$$  

Note that this character is just the restriction of the character $(m_1, \ldots, m_l) \mapsto \det(m_1)^{e_1} \cdots \det(m_l)^{e_l}$ of $X_v \in V$ to the center $Z$. We work with the resulting linearization of the $\tilde{G}$-action on $P' \times X_v \in V$ in $\mathcal{O}(\beta, \kappa_v, v \in V)$.

A sample computation

In order to illustrate that our choice of the linearization is accurate, we go through part of the calculations which are analogous to those in §2.3 of [24]. More precisely, we show the following: Let $m = (\xi_v : W_v \otimes \mathcal{O}_X(-l) \rightarrow \mathcal{E}_v, v \in V; \tau)$ be a point in the parameter space $\mathfrak{M}$, such that $\Gamma(m)$ is (semi)stable with respect to the chosen linearization in $\mathcal{O}(\beta; \kappa_v, v \in V)$, then $(\mathcal{E}_v, v \in V; \tau)$ is a (semi)stable $V$-split sheaf with a decoration of type $(a, b, c, m)$. First, as in [24], one verifies that the (semi)stability condition has to be checked only for those weighted filtrations $(\varepsilon_j, \alpha)$ which satisfy 

$$\mathcal{E}_j(l) \text{ is globally generated and } H^i(\mathcal{E}_j(l)) = 0,$$

$$i > 0, \quad j = 1, \ldots, s, \quad v \in V.$$

For weighted filtrations of that type, we have to prove that 

$$M_{\mathfrak{Z}}(\mathcal{E}, \alpha)(l) + \delta(l) \cdot \mu(\mathcal{E}, \alpha; \tau)(\geq 0).$$

(3)

Define $\gamma = (\gamma_1, \ldots, \gamma_{s+1})$ by the conditions 

$$\frac{\gamma_j + 1 - \gamma_j}{p} = \alpha_j, \quad j = 1, \ldots, s,$$

and, setting $\mathcal{E}_j^{\text{total}} := \bigoplus v \in V \mathcal{E}_v^{\mathcal{E}_j^{\text{total}}}, j = 1, \ldots, s + 1,$ 

$$\sum_{j=1}^{s+1} \gamma_j \cdot (h^0(\mathcal{E}_j^{\text{total}}(l)) - h^0(\mathcal{E}_j^{\text{total}}(l))) = 0.$$

Then, we obtain a weighted filtration $(\mathcal{E}_j, \gamma)$ and, thus, weighted filtrations $(\mathcal{E}_v^v, \gamma_v^v)$ of $\mathcal{E}_v, v \in V$. Next, we choose bases $w_v^v = (w_{v_1}^v, \ldots, w_{v_{s+1}}^v)$ of $W_v$ with 

$$\langle w_1^v, \ldots, w_{s+1}^v \rangle = H^0(\mathcal{E}_j^v(l)), \quad j = 1, \ldots, s_v, \quad v \in V.$$
and set
\[ \tilde{\gamma}^v := (\gamma_1^v, \ldots, \gamma_s^v, \ldots, \gamma_{s_v+1}^v, \ldots, \gamma_{s_v+1}^v). \]

This yields the one-parameter subgroup
\[ \lambda := (\lambda(w^v_1, \tilde{\gamma}^v_1), \ldots, \lambda(w^v_t, \tilde{\gamma}^v_t)) \]
of \( \tilde{G} \). Now, with \( \Gamma(m) = ([m'_1]; [m_v], v \in V) \),
\[ \frac{\mu(\lambda, [m_v])}{\beta} = \mu(\lambda, [m']) \]
\[ + \varepsilon \cdot \left( \sum_{v \in V} \varepsilon_v \cdot \mu(\lambda, [m_v]) - \sum_{v \in V} \sigma_v \left( \frac{r_v}{p_v} - \frac{r}{p} \right) \cdot \left( \sum_{j=1}^{s_v+1} \gamma_j^v (h^0(\hat{E}_j^v(l)) - h^0(\hat{E}_{j-1}^v(l))) \right) \right) =: A \]
\[ + \sum_{v \in V} \left( x'_v \cdot \sum_{j=1}^{s_v+1} \gamma_j^v (h^0(\hat{E}_j^v(l)) - h^0(\hat{E}_{j-1}^v(l))) \right) =: B \]
Observe
\[ \sum_{j=1}^{s_v+1} \gamma_j^v (h^0(\hat{E}_j^v(l)) - h^0(\hat{E}_{j-1}^v(l))) \]
\[ = \gamma_{s_v+1}^v \cdot p_v - \sum_{j=1}^{s_v} (\gamma_{j+1}^v - \gamma_j^v) \cdot h^0(\hat{E}_j^v(l)) \]
\[ = \gamma_{s+1}^v \cdot p_v - \sum_{j=1}^{s_v} (\gamma_{j+1}^v - \gamma_j^v) \cdot P(\hat{E}_j^v)(l). \]
As \( \sum_{v \in V} x'_v \cdot p_v = 0 \), we find
\[ B = - \sum_{v \in V} \left( x'_v \cdot \sum_{j=1}^{s_v} (\gamma_{j+1}^v - \gamma_j^v) \cdot P(\hat{E}_j^v)(l) \right). \]
Next,
\[ \mu(\lambda, [m_v]) = \sum_{j=1}^{s_v} \frac{\gamma_{j+1}^v - \gamma_j^v}{p_v} \cdot (p_v \cdot \text{rk} \hat{E}_j^v - h^0(\hat{E}_j^v(l)) \cdot r_v) \]
\[ = \sum_{j=1}^{s_v} \frac{\gamma_{j+1}^v - \gamma_j^v}{p_v} \cdot (p_v \cdot \text{rk} \hat{E}_j^v - P(\hat{E}_j^v)(l) \cdot r_v). \]
Thus,
\[
\epsilon \cdot \epsilon_v \cdot \mu(\lambda, [m_v]) - x'_v \cdot \sum_{j=1}^{s_v} (\gamma^v_{j+1} - \gamma^v_j) \cdot h^0(\mathcal{E}^v_j(l))
\]
\[
= \sum_{j=1}^{s_v} \frac{\gamma^v_{j+1} - \gamma^v_j}{p_v} \cdot \left(\epsilon \cdot \epsilon_v \cdot (p_v \cdot \text{rk} \mathcal{E}^v_j(l)) - P(\mathcal{E}^v_j(l) \cdot r_v) - x'_v \cdot p_v \cdot P(\mathcal{E}^v_j(l))\right)
\]
\[
= \sum_{j=1}^{s_v} \frac{\gamma^v_{j+1} - \gamma^v_j}{p_v} \cdot \left(\epsilon \cdot \epsilon_v \cdot (p_v \cdot \text{rk} \mathcal{E}^v_j(l)) - P(\mathcal{E}^v_j(l) \cdot r_v) - x_v \cdot r_v \cdot P(\mathcal{E}^v_j(l))\right)
\]
\[
= \epsilon \cdot \sigma_v \cdot \sum_{j=1}^{s_v} \left(\frac{\gamma^v_{j+1} - \gamma^v_j}{p_v} \cdot (p_v \cdot \text{rk} \mathcal{E}^v_j(l) \cdot r_v)\right)
\]
\[
- \sum_{j=1}^{s_v} (x_v \cdot (\gamma^v_{j+1} - \gamma^v_j) \cdot \text{rk} \mathcal{E}^v_j).
\]

For a given vertex \(v_0 \in V\) and a given index \(j_0 \in \{1, \ldots, s_v\}\), let \(s(v_0, j_0) \leq s'(v_0, j_0) \in \{1, \ldots, s\}\) be the minimal and the maximal index among those indices \(j\) with \(\mathcal{E}^v_{j} = \mathcal{E}^v_{j_0}\). Then, by definition,
\[
p \cdot \sum_{j=s(v_0, j_0)}^{s'(v_0, j_0)} \alpha_j = \gamma^v_{j_0+1} - \gamma^v_{j_0}.
\]

Hence,
\[
\sum_{v \in V} \left(\sum_{j=1}^{s_v} (x_v \cdot (\gamma^v_{j+1} - \gamma^v_j) \cdot \text{rk} \mathcal{E}^v_j)\right) = p \cdot \sum_{j=1}^{s} \alpha_j \left(\sum_{v \in V} (x_v \cdot \text{rk} \mathcal{E}^v_j)\right).
\]

Using Proposition 3.3.5, we discover that \(\epsilon \cdot A + B\) equals
\[
\epsilon \cdot \sum_{j=1}^{s} \alpha_j (p \cdot \text{rk} \mathcal{E}^{\text{total}}_j - P(\mathcal{E}^{\text{total}}_j(l)) \cdot r) - p \cdot \sum_{j=1}^{s} \alpha_j \left(\sum_{v \in V} (x_v \cdot \text{rk} \mathcal{E}^{\text{total}}_j)\right)
\]
\[
= \sum_{j=1}^{s} \alpha_j \cdot \left(\frac{p^2 \cdot \text{rk} \mathcal{E}^{\text{total}}_j}{r \cdot d} - \frac{p \cdot a \cdot \text{rk} \mathcal{E}^{\text{total}}_j}{r} - \frac{p \cdot P(\mathcal{E}^{\text{total}}_j(l))}{d} + a \cdot P(\mathcal{E}^{\text{total}}_j(l))\right)
\]
\[
- p \cdot \sum_{j=1}^{s} \alpha_j \left(\sum_{v \in V} (x_v \cdot \text{rk} \mathcal{E}^{\text{total}}_j)\right).
\]

In order to conclude, we have to compute \(\mu(\lambda, [m'])\). Under the identification of \(M\) with the space \(H^0(\mathcal{E}^{\text{total}}(l))\), we define
\[
\text{gr}_j(M) = H^0((\mathcal{E}^{\text{total}}_j / \mathcal{E}^{\text{total}}_{j-1})(l)), \quad j = 1, \ldots, s + 1.
\]
The basis $m$ of $M$ induced by the bases $w^v$ for $W_v$, $v \in V$, yields a natural isomorphism

$$M \cong \bigoplus_{j=1}^{s+1} \text{gr}_j(M).$$

For an index tuple $\iota \in J^a := \{1, \ldots, s + 1\}^a$, we define $M_{\iota} := \text{gr}_{\iota_1}(M) \otimes \cdots \otimes \text{gr}_{\iota_a}(M)$, and for $k \in \{1, \ldots, b\}$, we let $M_k^{\iota}$ be $M_{\iota}$ embedded into the $k$-th copy of $M^{\otimes \alpha}$ in $M^{\otimes \alpha \otimes b}$. If we denote $P(E^{\text{total}}_j(l)) = h_0^{\lambda}(E^{\text{total}}_j(l))$ by $m_j$, $j = 1, \ldots, s$, then

$$\lambda = \sum_{j=1}^{s} \alpha_j \lambda(m_j)$$

as a one-parameter subgroup of $\text{SL}(M)$. Therefore,

$$\mu(\lambda, [m']) = -\min \left\{ \sum_{j=1}^{s} \alpha_j (a \cdot m_j - v_j(\iota) \cdot p) \mid k \in \{1, \ldots, b\}, \iota \in J^a: M_k^{\iota} \not\subset \ker(m') \right\}.$$

Here,

$$v_j(\iota) = \# \{i \leq j \mid \iota_i = (i_1, \ldots, i_a), i = 1, \ldots, a\}.$$

Let $i_0 \in J^a$ be an index which realizes the precise value of $\mu(\lambda, [m'])$. Then, altogether, we find

$$\sum_{j=1}^{s} \alpha_j \left( \frac{p^2 \cdot \text{rk} E^{\text{total}}_j}{r \cdot d} - \frac{p \cdot a \cdot \text{rk} E^{\text{total}}_j}{r} - \frac{p \cdot P(E^{\text{total}}_j(l))}{d} \right)$$

as the value for $\mu(\lambda, \Gamma(m))$. We multiply this by $r \cdot d / p$ and get

$$\sum_{j=1}^{s} \alpha_j \cdot (p \cdot \text{rk} E^{\text{total}}_j - r \cdot P(E^{\text{total}}_j(l))) + \frac{d}{r} \left( \sum_{j=1}^{s} \alpha_j \cdot (v_j(l_0) \cdot r - a \cdot \text{rk} E^{\text{total}}_j) \right)$$

as in [24], one verifies that

$$\mu(E_{\bullet}, \alpha; \tau) = \sum_{j=1}^{s} \alpha_j \cdot (v_j(l_0) \cdot r - a \cdot \text{rk} E^{\text{total}}_j),$$

so, by Remark 3.4.3, $\mu(\lambda, \Gamma(m)) \geq 0$ implies inequality (3).

4.2 Proof of Theorem 3.6.1

We use the same set up and the same notation as in §3.6 on the associated decorations and, with respect to the corresponding parameters, at the beginning of §4.1. This time, we set $N := \bigoplus_{v \in V} W_v$. The space
\[ \mathfrak{P}'' := \mathbb{P} \left( N^\vee \otimes \pi(X_{\in V} \Omega^0_v)_* \left( \bigoplus_{v \in V} \pi^*_X \mathcal{E}_v \otimes \pi^*_X \mathcal{O}_X (l + m) \right) \right)^{\oplus b} \]

is a projective bundle over \( X_{v \in V} \Omega^0_v \). Denote by \( \mathcal{E}_{\mathfrak{P}'', v} \) the pullback of \( \mathcal{E}_v \) to \( \mathfrak{P}'' \times X \). On \( \mathfrak{P}'' \times X \), there are the tautological homomorphisms

\[ \phi'': \ N \otimes \mathcal{O}_{\mathfrak{P}'' \times X} \longrightarrow \left( \bigoplus_{v \in V} \mathcal{E}_{\mathfrak{P}'', v} \otimes \pi^*_X \mathcal{O}_X (l + m) \right)^{\oplus b} \otimes \pi^*_X \mathcal{O}_{\mathfrak{P}''} (1) \]

and

\[ \varepsilon'': \ \mathcal{O}_{\mathfrak{P}'' \times X} \longrightarrow \pi^*_X \mathcal{O}_{\mathfrak{P}''} (1) \otimes \pi^*_X \mathcal{O}_X (m). \]

First, we define \( \mathcal{R}' \) as the closed subscheme where \( \phi'' \) factorizes over the quotient \( \bigoplus_{v \in V} \mathcal{E}_{\mathfrak{P}'', v} \otimes \pi^*_X \mathcal{O}_X (l) \). Then, \( \phi''_{|\mathfrak{R}' \times X} \) may be considered as a collection of homomorphisms

\[ f'_{v, v'} : \ \mathcal{E}_{\mathfrak{R}', v} \longrightarrow \mathcal{E}_{\mathfrak{R}', v'} \otimes \pi^*_X \mathcal{O}_X (m)^{\oplus b} \otimes \pi^*_X \mathcal{O}_{\mathfrak{R}'} (1), \quad v, v' \in V. \]

Moreover, we have

\[ \varepsilon' : \ \mathcal{O}_{\mathfrak{R}' \times X} \longrightarrow \pi^*_X \mathcal{O}_{\mathfrak{R}'} (1) \otimes \pi^*_X \mathcal{O}_X (m). \]

Now, we can define \( \mathfrak{R} \) as a closed subscheme of \( \mathfrak{R}' \) by the following conditions:

- The restriction of \( f'_{v, v'} \) to \( \mathfrak{R} \times X \) is trivial, if \( (v, v') \notin A \);
- If \( (v, v') = a \in A \), then the corresponding homomorphism

\[ f''_{v, v'} : \ \pi^*_X \mathcal{O}_X (-m)^{\oplus b} \otimes \mathcal{E}_{\mathfrak{R}'} \longrightarrow \mathcal{E}_{\mathfrak{R}'} \otimes \pi^*_X \mathcal{O}_{\mathfrak{R}'} (1) \]

vanishes on

\[ \pi^*_X \ker(\mathcal{O}_X (-m))^{\oplus b} \longrightarrow \mathcal{G}_a \otimes \mathcal{E}_{\mathfrak{R}', v}. \]

Note that this is a closed condition, by Proposition A.2.1.

- The restriction of

\[ \mathcal{O}_{\mathfrak{R}' \times X} \longrightarrow \pi^*_X \mathcal{O}_X (m) \otimes \pi^*_X \mathcal{O}_{\mathfrak{R}'} (1) \longrightarrow \pi^*_X (\mathcal{O}_X (m)/\mathcal{O}_X) \otimes \pi^*_X \mathcal{O}_{\mathfrak{R}'} (1) \]

to \( \mathfrak{R}' \times X \) is trivial, too.

The space \( \mathfrak{R} \) is the correct parameter space and parametrizes a universal family \((\mathcal{E}_{\mathfrak{R}', v}, v \in V; f_{\mathfrak{R}, a}, a \in A; \mathcal{L}_{\mathfrak{R}}, \varepsilon_{\mathfrak{R}})\). It comes with an action of \( X_{v \in V} \text{GL}(W_v) \), and the universal family is linearized with respect to that group action. The parameter space is also projective over \( X_{v \in V} \Omega^0_v \).
The associated family of $V$-split sheaves $(\mathcal{E}_{\mathcal{R}, v}, v \in V; \mathcal{L}_{\mathcal{R}}, \tau_{\mathcal{R}})$ of type $P$ with a decoration of type $(s, b + 1, 1, m)$ defines a $(X_{v \in V} \text{GL}(W_v))$-equivariant morphism

$$I : \mathcal{R} \longrightarrow \mathcal{M}$$

over the base scheme $X_{v \in V} \mathcal{Q}^0_v$. Since $\mathcal{R}$ is proper over $X_{v \in V} \mathcal{Q}^0_v$, the morphism $I$ is automatically proper ([9], ch. II, Cor. 4.8(e)). It is also injective. To see this, let $r \in \mathcal{R}$ be a point which corresponds to the representation $(\mathcal{E}_v, v \in V; f_a, a \in A; \varepsilon)$ of $Q$ of type $(P, \mathcal{G})$. For any $a \in A$, the surjection $\mathcal{O}_X(-m)^{\oplus b} \otimes W_t(a) \otimes \mathcal{O}_X(-l) \longrightarrow \mathcal{G}_a \otimes \mathcal{E}_{t(a)}$ yields an injective homomorphism

$$\text{Hom} \left( \mathcal{G}_a \otimes \mathcal{E}_{t(a)}, \mathcal{E}_{h(a)} \right) \subset \text{Hom} \left( \mathcal{O}_X(-m)^{\oplus b} \otimes W_t(a) \otimes \mathcal{O}_X(-l), \mathcal{E}_{h(a)} \right)$$

$$= H^0 \left( W_t(a)^{\oplus b} \otimes \mathcal{E}_{h(a)}(l + m) \right).$$

Since $\mathcal{E}_{h(a)}$ is torsion free, the restriction map

$$H^0 \left( X, W_t(a)^{\oplus b} \otimes \mathcal{E}_{h(a)}(l + m) \right) \longrightarrow H^0 \left( U, W_t(a)^{\oplus b} \otimes \mathcal{E}_{h(a)}(l + m) \right)_{\mathcal{M}}$$

is injective for any open subset $U = X \setminus Z$ with $\text{codim}_X(Z) \geq 2$. If $\mathcal{U} \subset \mathcal{R} \times X$ is the maximal open subset where all the $\mathcal{E}_{t(a)}$, $v \in V$ are locally free, then $U := \mathcal{U} \cap (\{ r \} \times X)$ is the maximal open subset where all the $\mathcal{E}_v$, $v \in V$, are locally free ([10], Lemma 2.1.7). In particular, the complement of $U$ in $X$ has codimension at least two. Since $\tau_{\mathcal{R}[g] \times X}$ determines all the $f_a, a \in A$, and $\varepsilon$ over $U$, we are done. Because $I$ is injective and proper and, thus finite, Theorem 3.6.1 follows immediately from Theorem 3.5.1 and its proof. □

4.3 Proof of Theorem 3.7.1

The proof of Theorem 3.7.1 is basically a formal adaptation of the corresponding result for Hitchin pairs. If, in the following, a representation $(\mathcal{E}_v, v \in V; f_a, a \in A; \varepsilon)$ is given, $\tau$ will always stand for the associated decoration. We first observe the following.

**Lemma 4.3.1.** Suppose we are given $\sigma$, $\eta$, and $\delta$, as well as a $(\sigma, \eta, \delta)$-semistable representation $(\mathcal{E}_v, v \in V; f_a, a \in A)$ of type $(P, \mathcal{G})$. Then, it satisfies Condition 2(a) of Theorem 3.7.1.

**Proof.** Let $(\mathcal{F}_v, v \in V)$ be a non-trivial, proper subrepresentation of $(\mathcal{E}_v, v \in V; f_a, a \in A)$. Set

$$\mathcal{E}_\bullet : \ 0 \subsetneq (\mathcal{F}_v, v \in V) \subsetneq (\mathcal{E}_v, v \in V).$$

Then, one verifies $\mu(\mathcal{E}_\bullet, (1); \tau) \leq 0$, from which the assertion follows. □

**PROPOSITION 4.3.2**

Fix $\sigma$ and $\chi$. Then, the set of torsion free sheaves occurring in representations of type $(P, \mathcal{G})$ which satisfy Condition 2(a) of Theorem 3.7.1 is bounded.

**Proof.** We fix surjections $\nu_a : \mathcal{O}_X(-m)^{\oplus b} \longrightarrow \mathcal{G}_a, a \in A$. We may now adapt Nitsure’s argument ([20], Proposition 3.2). Let

$$0 = (\mathcal{F}_0^v, v \in V) \subsetneq (\mathcal{F}_1^v, v \in V) \subsetneq \cdots \subsetneq (\mathcal{F}_s^v, v \in V) \subsetneq (\mathcal{F}_{s+1}^v, v \in V)$$

$$= (\mathcal{E}_v, v \in V)$$
be the Harder–Narasimhan filtration of \((E_v, v \in V)\) defined with respect to the parameters \(\sigma\) and \(\chi\). It will suffice to bound \(\mu_{\sigma, \chi}(F^v, v \in V)\). Define

\[
D := \deg \mathcal{O}_X(m) + \max_{v \in V} \left\{ \frac{\chi_v}{\sigma_v} \right\} - \min_{v \in V} \left\{ \frac{\chi_v}{\sigma_v} \right\}.
\]

We claim that

\[
\mu_{\sigma, \chi}(F^v, v \in V) \leq \min \left\{ \mu_{\sigma, \chi}(E_v, v \in V), \mu_{\sigma, \chi}(E_v, v \in V) + \frac{((\sum_{v \in V} r_v) - 1)^2}{\sum_{v \in V} r_v} \cdot D \right\}.
\]

We can view the collection \(f_a, a \in A\), together with the zero homomorphisms \(E^v_1 \rightarrow E^v_2 \otimes \mathcal{O}_X(m)^{\oplus b}\) for \((v_1, v_2) \notin A\) as a homomorphism

\[
f : \bigoplus_{v \in V} E_v \rightarrow \left( \bigoplus_{v \in V} E_v \right) \otimes \mathcal{O}_X(m)^{\oplus b}.
\]

For any \(V\)-split subsheaf \((F^v, v \in V)\), the condition of being a subrepresentation is, thus, equivalent to \(f(\bigoplus_{v \in V} F^v) \subseteq (\bigoplus_{v \in V} F^v) \otimes \mathcal{O}_X(m)^{\oplus b}\). We simply say that \(\bigoplus_{v \in V} F^v\) is \(f\)-invariant. If the condition

\[
\mu_{\sigma, \chi}(F^v, v \in V) \leq \mu_{\sigma, \chi}(E^v, v \in V)
\]

is violated, then, by definition, none of the sheaves \(F^v_j := \bigoplus_{v \in V} F^v_j\) can be \(f\)-invariant, \(j = 1, \ldots, s\), i.e., the homomorphisms \(\phi_j : F^v_{j+1} \rightarrow (F^v_{j+1} / F^v_j) \otimes \mathcal{O}_X(m)^{\oplus b}\) are non-trivial, \(E^v_{\{j\}} := \bigoplus_{v \in V} E^v_j, j = 1, \ldots, s\). For any \(j = 1, \ldots, s\), there exist \(\iota \leq j - 1\) and \(\kappa \geq j + 1\), such that \(\phi_j\) induces a non-trivial homomorphism \(\overline{\phi}_j : F^v_{\iota + 1} / F^v_{\iota} \rightarrow (F^v_{\kappa} / F^v_{\kappa - 1}) \otimes \mathcal{O}_X(m)^{\oplus b}\). Now, Propositions 3.3.3 and 3.3.4 imply

\[
\mu_{\sigma, \chi}(F^v_j / F^v_{j-1}, v \in V) \leq \mu_{\sigma, \chi}(F^v_{j+1} / F^v_{j}, v \in V)
\]

\[
\leq \mu_{\sigma, \chi}(F^v_{\kappa} / F^v_{\kappa - 1}, v \in V) + D
\]

\[
\leq \mu_{\sigma, \chi}(F^v_{j+1} / F^v_{j}, v \in V) + D.
\]

Thus,

\[
\mu_{\sigma, \chi}(F^v_1, v \in V) \leq \mu_{\sigma, \chi}(E^v_1 / F^v_s, v \in V) + s \cdot D
\]

\[
\leq \mu_{\sigma, \chi}(E^v_1 / F^v_s, v \in V) + \left( \sum_{v \in V} r_v - 1 \right) \cdot D.
\]

Finally, one finds that

\[
\mu_{\sigma, \chi}(E^v_1 / F^v_s, v \in V) \leq \frac{\deg_{\sigma, \chi}(E^v_1, v \in V) - \mu_{\sigma, \chi}(F^v_1, v \in V)}{(\sum_{v \in V} r_v) - 1}
\]

from which the assertion follows. \(\square\)
We first show that (1) implies (2). We have already checked that 2(a) holds. Before we check Condition 2(b), we review the linear algebra setting. The space \( \bigoplus_{a \in A} \operatorname{Hom}(C^{\ell(a)}, C^{rh(a)}) \oplus b \) can be \((X_{v \in V} \operatorname{GL}(C^{r_v}))\)-equivariantly embedded into \( E := \operatorname{End}(M)^{\oplus b} \), \( M := \bigoplus_{v \in V} C^{\sigma_v r_v} \).

Let \([f_a : C^{\ell(a)} \to (C^{rh(a)})^{\oplus b}, a \in A] \) in \( \mathbb{P}( \bigoplus_{a \in A} \operatorname{Hom}(C^{\ell(a)}, C^{rh(a)})^{\oplus b} ) \) be an element which is unstable with respect to the \( \tilde{G} \)-action. Let \([f] \in \mathbb{P}(E)\) be the associated element which is equally unstable. As explained before, a one-parameter subgroup \( \lambda : \mathbb{C}^* \to \tilde{G} \) yields a weighted flag \((M, \alpha)\) with

\[
M : \ 0 \subset U_1 \subset \cdots \subset U_s \subset C \sum_{v \in V} \sigma_v r_v.
\]

Here, \( U_j = \bigoplus_{v \in V} U_{j-1} \sigma_v \) for suitable subspaces \( U_j \subset C^{r_v}, \ j = 1, \ldots, s \). Then, \( \mu(\lambda, \ [f] ) < 0 \) will occur if and only if

\[
f(U_j) \subset U_{j-1} \oplus b,
\]

i.e.,

\[
f \left( \bigoplus_{v \in V} U_j^{v, \oplus \sigma_v} \right) \subset \left( \bigoplus_{v \in V} U_{j-1}^{v, \oplus \sigma_v} \right)^{\oplus b}, \quad j = 1, \ldots, s + 1,
\]

or equivalently

\[
U_j^{v, \oplus \sigma_v} \subset \ker \left( M \xrightarrow{f} \bigoplus_{v \in V} (C^{\sigma_v r_v} / U_{j-1}^{v, \oplus \sigma_v})^{\oplus b} \right), \quad \text{for all } v \in V.
\]

Conversely, we may define

\[
Y_1^v := \ker(C^{r_v} \xrightarrow{\operatorname{diag}} C^{\sigma_v r_v} \xrightarrow{f} M^{\oplus b}), \quad v \in V,
\]

and

\[
Y_j^v := \ker \left( C^{r_v} \xrightarrow{\operatorname{diag}} C^{\sigma_v r_v} \xrightarrow{f} \bigoplus_{v \in V} (C^{\sigma_v r_v} / Y_{j-1}^{v, \oplus \sigma_v})^{\oplus b} \right), \quad v \in V.
\]

By our previous observations, this process will stop after at most \( s \) steps, i.e., we get a flag

\[
M' : \ 0 \subset \bigoplus_{v \in V} Y_1^{v, \oplus \sigma_v} \subset \cdots \subset \bigoplus_{v \in V} Y_s^{v, \oplus \sigma_v} \subset C \sum_{v \in V} \sigma_v r_v,
\]

and \((M', (1, \ldots, 1))\) comes from a suitable one-parameter subgroup \( \lambda : \mathbb{C}^* \to \tilde{G} \) with \( \mu(\lambda, \ [f] ) = - \sum_{v \in V} \sigma_v r_v \).

The latter construction can be extended to the setting of sheaves, i.e., given a representation \((E_v, v \in V; f_a, a \in A; \epsilon = 0)\) as in (1) for which Condition 2(b) fails, we define

\[
\mathcal{F}_1^v := \ker \left( E_v \xrightarrow{\operatorname{diag}} E_v^{\oplus \sigma_v} \xrightarrow{f} E_v^{\oplus \sigma_v} \otimes \mathcal{O}_X(m)^{\oplus b} \right), \quad v \in V,
\]
and

\[ \mathcal{F}_j^v := \ker \left( \mathcal{E}_v \xrightarrow{\text{diag}} \mathcal{E}_v^\oplus \mathbf{\sigma}_v \xrightarrow{f} \bigoplus_{v \in V} (\mathcal{E}_v^\oplus \mathbf{\sigma}_v / \mathcal{F}_{j-1}^v) \otimes \mathcal{O}_X(m)^\oplus b \right), \]

\[ v \in V, j > 1. \]

Then, we find the weighted filtration \((\mathcal{E}_\bullet, (1, \ldots, 1))\) with

\[ \mathcal{E}_\bullet: 0 \subseteq (\mathcal{F}_1^v, v \in V) \subseteq \cdots \subseteq (\mathcal{F}_j^v, v \in V) \subseteq (\mathcal{E}_v, v \in V) \]

and

\[ \mu(\mathcal{E}_\bullet, (1, \ldots, 1); \tau) = -\sum_{v \in V} \sigma_v \cdot r_v. \]

By the boundedness result 4.3.1, it is clear that the sheaves of the form \(\mathcal{F}_j^v\) as just-defined live in bounded families, too. In particular, there is a constant \(C > 0\), such that

\[ \deg_{\mathcal{E}_v}(\mathcal{E}_v, v \in V) \cdot \rk_{\mathcal{E}_v}(\mathcal{F}_j^v, v \in V) \]

\[ - \deg_{\mathcal{E}_v}(\mathcal{F}_j^v, v \in V) \cdot \rk_{\mathcal{E}_v}(\mathcal{E}_v, v \in V) < C \]

for any filtration as above. But then, with \(\bar{\delta} > 0\), the coefficient of \(x^{\dim X - 1}\) in \(\delta\), the condition of \((\sigma, \eta/n, n \cdot \delta)\)-semistability requires

\[ 0 \leq M_{\mathcal{E}_v}(\mathcal{E}_\bullet, (1, \ldots, 1)) + n \cdot \delta \cdot \mu(\mathcal{E}_\bullet, (1, \ldots, 1); \tau) \]

\[ \leq \left( \left( \sum_{v \in V} r_v \right) - 1 \right) \cdot C - n \cdot \bar{\delta} \cdot \sum_{v \in V} (\sigma_v \cdot r_v) \cdot x^{\dim X - 1}, \]

but for large \(n\), this is impossible.

The converse is an easy adaptation of the argument given in ([24], Example 3.6), and is left as an exercise to the reader. \(\square\)

4.4 Proof of Theorem 3.7.2

The points (i) and (ii) are just a reformulation of Theorem 3.6.1. Point (iii) is proved by standard arguments and will be omitted here. Finally, the constructions carried out in Section 3.7 show that the universal family on the parameter space \(\mathcal{R}\) defines a morphism \(\mathcal{R} \rightarrow \mathbb{H}(\mathcal{Q}, \mathcal{P}, \mathcal{G})\). This morphism is invariant under the \(\mathcal{G}\)-action and, thus, descends to the moduli space \(\mathcal{R}(\mathcal{Q})_{\mathcal{G}^\text{ss}}^{(\sigma, \chi)}\). \(\square\)

Appendix A: Two auxiliary results

A.1 Restrictions of families of locally free sheaves to open subsets

Let \(X\) be a smooth projective manifold and \(S\) a noetherian scheme. Let \(i: U \subset S \times X\) be an open subset, such that

\[ \text{codim}(X \setminus (U \cap \{s\} \times X), X) \geq 2, \quad \text{for all } s \in S. \]
PROPOSITION A.1.1

In the above situation, the natural homomorphism $\mathcal{O}_{S \times X} \longrightarrow t_\ast \mathcal{O}_U$ is an isomorphism. In particular, for any locally free sheaf $\mathcal{V}$ on $S \times X$, we have

$$\mathcal{V} = t_{\ast t}^\ast \mathcal{V}.$$ 

Proof. We refer to ([16], p. 111f).

A.2 Zero loci of sheaf homomorphisms

The following result may be found in ([8], Lemma 3.1).

PROPOSITION A.2.1

Let $S$ be a noetherian scheme, $\mathcal{A}_S^1$ and $\mathcal{A}_S^2$ coherent sheaves on $S \times X$, and $\phi_S: \mathcal{A}_S^1 \longrightarrow \mathcal{A}_S^2$ a homomorphism. Assume that $\mathcal{A}_S^2$ is $S$-flat. Then, there is a closed subscheme $\mathcal{Y} \subset S$ the closed points of which are those $s \in S$ for which $\phi_S|_{\{s\} \times X} \equiv 0$. More precisely, it has the property that any morphism $f: T \longrightarrow S$ factors through $\mathcal{Y}$, if and only if $(f \times \text{id}_X)\ast \phi_S$ is the zero homomorphism.

Appendix B: A concluding remark

It was pointed out to me by Balázs Szendrői that the case in which $X$ is a point is formally not covered by our formalism. It would be formally included, if one allowed $\delta$ to have degree $\dim(X)$ (or higher). Then, for $X$ a point, $\delta = 1$, $\eta_v, v \in V$, with $\sum_{v \in V} \eta_v r_v = 0$, and a representation $(f_a, a \in A) \in \bigoplus_{a \in A} \text{Hom}(\mathbb{C}^\delta a, \mathbb{C}^\delta a)$, the condition of (semi)stability would read

$$\sum_{v \in V} \sigma_v \dim(W_v) - \sum_{v \in V} \eta_v \dim(W_v) = 1 - \sum_{v \in V} \eta_v \dim(W_v) \sum_{v \in V} \sigma_v \dim(W_v) \leq \sum_{v \in V} \eta_v r_v - \sum_{v \in V} \eta_v r_v \sum_{v \in V} \sigma_v r_v = 1,$$

i.e.,

$$-\sum_{v \in V} \eta_v \dim(W_v) \leq 0$$

for any non-trivial subrepresentation $(W_v, v \in V)$. This is precisely King’s definition with respect to the character

$$(U_v, v \in V; f_a, a \in A) \longmapsto -\sum_{v \in V} \eta_v \cdot \dim(U_v).$$

Note that for $\dim(X) \geq 0$, a positive polynomial $\delta$ of degree $\dim(X)$, and $\eta_v, v \in V$, with $\sum_{v \in V} \eta_v r_v = 0$, the condition of (semi)stability can be restated as follows:

- For any non-trivial subrepresentation $(F_v, v \in V)$ one has

$$-\sum_{v \in V} \eta_v \cdot \text{rk}(F_v) \leq 0,$$
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and,

• if ‘=’ occurs, then

\[
\frac{\sum_{v \in V} \sigma_v P(F_v)}{\sum_{v \in V} \sigma_v \text{rk} (F_v)} \leq \frac{\sum_{v \in V} \sigma_v P(E_v)}{\sum_{v \in V} \sigma_v \text{rk} (E_v)}.
\]

Let us call representations which satisfy this condition asymptotically (semi)stable. The moduli spaces for asymptotically (semi)stable objects might be obtained as follows:

• Fix the data \(\sigma_v, \eta_v, v \in V\).

• There is a positive polynomial \(\delta_0\) (depending on the Hilbert polynomials \(P_v, \sigma_v\), and \(\eta_v, v \in V\)) of degree \(\dim(X) - 1\), such that, for any \(\delta > \delta_0\), a representation \((E_v, v \in V; f_a, a \in A)\) will be (semi)stable with respect to the parameters \(\sigma_v, \eta_v, v \in V\), and \(\delta\), if and only if it is asymptotically (semi)stable.

The techniques to prove this should be adapted from my recent paper ‘Global boundedness for decorated sheaves’ Int. Math. Res. Not. 68 (2004) 3637–71.

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