ON THE INTERIOR REGULARITY CRITERIA OF THE 3-D NAVIER-STOKES EQUATIONS INVOLVING TWO VELOCITY COMPONENTS

WENDONG WANG∗
School of Mathematical Sciences, Dalian University of Technology
Dalian 116024, China

LIQUN ZHANG† AND ZHIFEI ZHANG♯
†School of Mathematical Sciences, UCAS and Institute of Mathematics
AMSS, Hua Loo-Keng Key Laboratory of Mathematics
Chinese Academy of Sciences
Beijing 100190, China

♯School of Mathematical Sciences, Peking University
Beijing 100871, China

(Communicated by Yanyan Li)

ABSTRACT. We present some interior regularity criteria of the 3-D Navier-Stokes equations involving two components of the velocity. These results in particular imply that if the solution is singular at one point, then at least two components of the velocity have to blow up at the same point.

1. Introduction. In this paper, we study the incompressible Navier-Stokes equations
\[
\begin{aligned}
\partial_t u - \Delta u + u \cdot \nabla u + \nabla \pi &= 0, \\
\text{div} u &= 0,
\end{aligned}
\]
(1)
where \((u(x,t), \pi(x,t))\) denote the velocity and the pressure of the fluid respectively.

In a seminal paper [11], Leray proved the global existence of weak solution with finite energy. In two spatial dimensions, Leray weak solution is unique and regular. In three spatial dimensions, the regularity and uniqueness of weak solution is an outstanding open problem in the mathematical fluid mechanics. It was known that if the weak solution \(u\) of (1) satisfies so called Ladyzhenskaya-Prodi-Serrin(LPS) type condition
\[
u \in L^q(0,T; L^p(\mathbb{R}^3)) \text{ with } \frac{2}{q} + \frac{3}{p} \leq 1, \quad p \geq 3,
\]
then it is regular in \(\mathbb{R}^3 \times (0,T)\), see [18, 6, 19, 5], where the regularity in the class \(L^\infty(0,T; L^3(\mathbb{R}^3))\) was proved by Escauriaza, Seregin and Šverák [5].
Concerning the partial regularity of weak solution satisfying the local energy inequality, it was started in a series of papers by Scheffer [14, 15, 16], and later Caffarelli, Kohn and Nirenberg [1] showed that one dimensional Hausdorff measure of the possible singular set is zero. The proof relies on the following small energy regularity result: there exists some \( \varepsilon_0 > 0 \) so that if \( u \) is a suitable weak solution of the Navier-Stokes equations and satisfies
\[
\sup_{R > 0} \frac{1}{R} \int_{Q_R(z)} |\nabla u|^2 \, dx \, dt \leq \varepsilon_0,
\]
then \( u \) is regular at the point \( z \) (i.e., \( u \) is bounded in a \( Q_r(z) \) for some \( r > 0 \)). Here and in what follows \( z = (x, t) \), \( Q_R(z) = (-R^2 + t, t) \times B_R(x) \) and \( B_R(x) \) is a ball of radius \( r \) centered at \( x \). One could check Lin [12] and Ladyzhenskaya-Seregin [10] for the simplified proof and improvements. More generalizations could be found from Tian-Xin [20], Seregin [17], Gustafson-Kang-Tsai [7], Vasseur [21], Kukavica [8] and the references therein.

Recently, there are many interesting works devoted to the LPS type criterions involving the partial components of the velocity, see Cao-Titi [2, 3], Chemin-Zhang [4], Kukavica-Ziane [9], Pokorny-Zhou [23] and references therein. Especially, they proved one velocity component implies the regularity if it satisfies a more stronger condition than the LPS condition, i.e.
\[
u_3 \in L^p_t(0,T; L^p_x(R^3)), \quad \frac{2}{q} + \frac{3}{q} = \frac{3}{4} + \frac{1}{2q}, \quad q > \frac{10}{3},
\]
see [2] or [23].

We wander whether one velocity component satisfying the LPS condition is sufficient for ensuring the regularity of (1)?

Motivated by this problem, we began to consider interior regularity criteria depending the components of the velocity. Different with previous energy estimates in the whole space-time domain, we consider the local domain and using some refined analysis of local scaling invariant quantities. In [22], The authors considered the interior regularity criteria involving the partial components of the velocity. Let
\[
G(u, p; r) = r^{1 - \frac{3}{q} - \frac{3}{2} \frac{p}{q}} \|u\|_{L^p_t L^q_x(Q_r)}.
\]
It was proved in [22] that if \((u, \pi)\) is a suitable weak solution of (1) in \( Q_1 \) and satisfies
\[
\sup_{0 < r < 1} G(u_3, p; q; r) < M \quad \text{for some } M > 0,
\]
and
\[
\limsup_{r \to 0} G(u_h, p; q; r) = 0,
\]
where \( u_h = (u_1, u_2) \) and \( 1 \leq \frac{3}{p} + \frac{2}{q} < 2, \ 1 < q \leq \infty \), then \((0, 0)\) is a regular point.

The goal of this paper is to get rid of the extra condition (2). Making full use of the structure of nonlinear term and \( \text{div}u = 0 \), we obtain the following interior regularity criteria involving two components of the velocity.

**Theorem 1.1.** Let \((u, \pi)\) be a suitable weak solution of (1) in \( R^3 \times (-1, 0) \). If \( u \) satisfies one of the following three conditions:
1. \( u_h \in L^p_t L^q_x(Q_{r_0}), \ 3 \frac{p}{q} = 1, 2 < q < \infty \);
2. \( \nabla u_h \in L^p_t L^q_x(Q_{r_0}), \ 3 \frac{p}{q} = 2, 2 < q < \infty \);
3. \( \nabla u_h \in L^q_t L^p_x(Q_{r_0}) \) and \( \limsup_{r \to 0} \sup_G(u_h, p, q; r) = 0 \), \( \frac{3}{p} + \frac{2}{q} = 2, 1 < q \leq 2 \); for some \( r_0 \in (0, 1) \), then \( u \) is regular at \((0, 0)\).

**Remark 1.** The key step is to obtain a class of suitable local energy inequality in order to prove Theorem 1.1, since we only have the local smallness of the horizontal velocity. First of all, the nonlinear term \( u \cdot \nabla \phi(|u|^2) \) from Definition 2.1 is easily dealt with due to \( \text{div} u = 0 \), which can rewrite as the form of \( u \cdot F(x, t) \). Secondly, the difficulty comes from the nonlocal pressure term \( u \cdot \nabla \phi \pi \), and we have the following observation. The pressure \( \pi \) satisfies

\[
\pi = \frac{1}{4\pi} \int_{R^3} \frac{1}{|x-y|} \sum_{i,j=1}^{3} \partial_i \partial_j (u_i u_j) dy = \pi_1 + \pi_2,
\]

where

\[
\pi_1 = \frac{1}{4\pi} \int_{R^3} \frac{1}{|x-y|} \sum_{i+j \leq 0} \partial_i \partial_j (u_i u_j) dy, \quad \pi_2 = \frac{1}{4\pi} \int_{R^3} \frac{1}{|x-y|} \partial_3 \partial_3 (u_3 u_3) dy.
\]

Since there is a \( u_h \) term in \( \pi_1 \), and we compute the term \( \pi_2 \) by using \( \nabla \cdot u = 0 \)

\[
\int_{Q_{r_0}} \phi u_3 \partial_3 \pi_2 dx dt = \int_{Q_{r_0}} \phi u_3 \partial_3 \left[ \frac{1}{2\pi} \int_{R^3} \frac{1}{|x-y|} \partial_3 \nabla_h \cdot (-u_3 u_h) dy \right] dx dt + \int_{Q_{r_0}} \phi u_3 \left[ \frac{1}{2\pi} \int_{R^3} \frac{1}{|x-y|} \partial_3 \partial_3 (u_3 u_3) d y \right] dx dt \\
\hat{=} \int_{Q_{r_0}} \phi u_3 \partial_3 \pi_3 dx dt + \int_{Q_{r_0}} \phi u_3 \partial_3 \pi_4 dx dt,
\]

and more details see Lemma 3.2. Finally, the estimates of the pressure terms \( \pi_1, \pi_3, \pi_4 \) by Calderon-Zygmund theory and iterations can be found in Lemma 3.3 and 3.4.

The range of \((p, q)\) can be extended if we impose a similar condition on the velocity in a cylinder domain. The proof relies on a new pressure decomposition formula.

**Theorem 1.2.** Let \((u, \pi)\) be a suitable weak solution of (1) in \( R^3 \times (-1, 0) \). If \( u \) satisfies

\[
\limsup_{r \to 0} r^{1-\frac{3}{p} - \frac{2}{q}} \left( \int_{-r^2}^0 \left( \int_{|x| \leq r, x_3 \in R} |u_h(x, t)|^p dx dx_3 \right)^{\frac{q}{2}} dt \right)^{\frac{1}{2}} = 0,
\]

where \((p, q)\) satisfies

\[
1 \leq \frac{3}{p} + \frac{2}{q} \leq 2, \quad \frac{3}{2} \leq p \leq \infty, \quad (p, q) \neq (\infty, 1),
\]

then \( u \) is regular at \((0, 0)\).

**Remark 2.** An interesting consequence of Theorem 1.1 and Theorem 1.2 is that if the solution is singular at one point, then at least two components of the velocity have to blow up at the same point.

**Remark 3.** In Theorem 1.2, we relax the request of the indexes \((p, q)\), which is weaker than the LPS condition in Theorem 1.1. To achieve the goal, we exploit the
2D fundamental solution and prove a weak local energy inequality:
\[
A(u; r) + E(u; r) 
\leq C \left( \frac{1}{r^3} \right) A(u; \rho) + C \left( \frac{1}{r} \right)^2 G(u_h, p, q; \rho) \left( G(u, 2p', 2q'; \rho) + \hat{H}(\pi, p', q'; \rho) \right),
\]
which has the decay of one order \((\frac{1}{r^3})\) and the main obstacle is the pressure term \(\hat{H}(\pi, p', q'; \rho)\). Our new ingredient is a gradient estimate for harmonic function, which is new even for harmonic function to our knowledge. Let \(f\) be a harmonic function in a cubic \(D_1 \subset \mathbb{R}^3\). Let
\[
P_3 f(x_h) = \frac{1}{2} \int_{-1}^1 f(x_1, x_2, x_3) dx_3,
\]
Then it holds that
\[
\sup_{x \in B_1} |\nabla f_3| \leq C \int_{B_1} |f(x) - P_3 f(x_h)| dx
\]
Using this, we can obtain the required pressure estimate (see Lemma 4.4), which yields an effective iterative scheme. On the other hand, it seems that it’s difficult to obtain the similar interior criterion depending on one velocity component from the local energy inequality, since will get the following local inequality without decay before the term \(A(u; \rho)\):
\[
A(u; r) + E(u; r) 
\leq CA(u; \rho) + C \left( \frac{1}{r} \right)^2 G(u_h, p, q; \rho) \left( G(u, 2p', 2q'; \rho) + \hat{H}(\pi, p', q'; \rho) \right),
\]
if we only use the 1D fundamental solution.

The rest of the paper is organized as follows. In section 2, we introduce some notations and some known interior regularity criteria. In section 3, we prove Theorem 1.1 by some technical lemmas; especially, we exploit a new local energy inequality by the pressure decomposition. Section 4 is devoted to the proof of Theorem 1.1 by proving a new gradient estimate for harmonic function.

2. Suitable weak solution and \(\varepsilon\)-regularity criterion. Let us first introduce the definition of suitable weak solution.

**Definition 2.1.** Let \(\Omega \subset \mathbb{R}^3\) and \(T > 0\). We say that \((u, \pi)\) is a suitable weak solution of (1) in \(\Omega_T = \Omega \times (-T, 0)\) if
1. \(u \in L^\infty(-T, 0; L^2(\Omega)) \cap L^2(-T, 0; H^1(\Omega))\) and \(\pi \in L^2(\Omega_T)\);
2. the (NS) equation is satisfied in the sense of distribution;
3. the local energy inequality: for any nonnegative \(\phi \in C^\infty_c(\mathbb{R}^3 \times \mathbb{R})\) vanishing in a neighborhood of the parabolic boundary of \(\Omega_T\),
\[
\int_{\Omega} |u(x,t)|^2 \phi dx + 2 \int_{-T}^t \int_{\Omega} |\nabla u|^2 \phi dx ds 
\leq \int_{-T}^t \int_{\Omega} u^2 (\partial_t \phi + \Delta \phi) + u \cdot \nabla \phi (|u|^2 + 2\pi) dx ds
\]
for any \(t \in [-T, 0]\).
Let \((u, \pi)\) be a solution of (1) and introduce the following scaling
\[
\begin{align*}
   u^\lambda(x, t) &= \lambda u(\lambda x, \lambda^2 t), \\
   \pi^\lambda(x, t) &= \lambda^2 \pi(\lambda x, \lambda^2 t),
\end{align*}
\] (3)
for any \(\lambda > 0\), then the family \((u^\lambda, \pi^\lambda)\) is also a solution of (1). Let us introduce some invariant quantities under the scaling (3):
\[
\begin{align*}
   A(u, r, z_0) &= \sup_{-r^2+t_0 \leq t \leq t_0} r^{-1} \int_{B_r(x_0)} |u(y, t)|^2 dy, \\
   E(u, r, z_0) &= r^{-1} \int_{Q_r(z_0)} |\nabla u(y, s)|^2 dy ds.
\end{align*}
\]
We also introduce
\[
\begin{align*}
   G(f, p, q; r, z_0) &= r^{1 - \frac{3}{2} - \frac{2}{q}} \|f\|_{L^p_t L^q_x(Q_r(z_0))}, \\
   H(f, p, q; r, z_0) &= r^{2 - \frac{3}{2} - \frac{2}{q}} \|f\|_{L^p_t L^q_x(Q_r(z_0))}, \\
   \widetilde{G}(f, p, q; r, z_0) &= r^{1 - \frac{3}{2} - \frac{2}{q}} \|(f)_{B_r(x_0)}\|_{L^p_t L^q_x(Q_r(z_0))}, \\
   \widetilde{H}(f, p, q; r, z_0) &= r^{2 - \frac{3}{2} - \frac{2}{q}} \|(f)_{B_r(x_0)}\|_{L^p_t L^q_x(Q_r(z_0))},
\end{align*}
\]
where the mixed space-time norm \(\| \cdot \|_{L^p_t L^q_x(Q_r(z_0))}\) is defined by
\[
\|f\|_{L^p_t L^q_x(Q_r(z_0))} \doteq \left( \int_{t_0}^{t_0} \left( \int_{B_r(x_0)} |f(x, t)|^p dx \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}},
\]
and \((f)_{B_r(x_0)}\) is the average of \(f\) in the ball \(B_r(x_0)\). These scaling invariant quantities will play an important role in the interior regularity theory.

For the simplicity, we denote \(Q_r(0)\) by \(Q_r\) and \(B_r(0)\) by \(B_r\), and we will use the following notations:
\[
A(u, r, (0, 0)) = A(u, r), \quad E(u, r, (0, 0)) = E(u, r).
\]

Here and in what follows, we define a solution \(u\) to be regular at \(z_0 = (x_0, t_0)\) if \(u \in L^\infty(Q_r(z_0))\) for some \(r > 0\). We recall the following \(\varepsilon\)-regularity result.

**Proposition 1.** [7] Let \((u, \pi)\) be a suitable weak solution of (1) in \(Q_1(z_0)\) and \(w = \nabla \times u\). There exists \(\varepsilon_1 > 0\) such that if one of the following two conditions holds,
\[
\begin{align*}
   1. & \quad r^{1 - \frac{3}{2} - \frac{2}{q}} \|u\|_{L^p_t L^q_x(Q_r(z_0))} \leq \varepsilon_1 \text{ for any } 0 < r < \frac{1}{2}, \text{ where } 1 \leq \frac{3}{p} + \frac{2}{q} \leq 2; \\
   2. & \quad r^{2 - \frac{3}{2} - \frac{2}{q}} \|w\|_{L^p_t L^q_x(Q_r(z_0))} \leq \varepsilon_1 \text{ for any } 0 < r < \frac{1}{2}, \text{ where } 2 \leq \frac{3}{p} + \frac{2}{q} \leq 3 \text{ and } (p, q) \neq (1, \infty);
\end{align*}
\]
then \(u\) is regular at \(z_0\).

**3. Proof of Theorem 1.1.** Throughout this section, we assume that \((u, \pi)\) is a suitable weak solution of (1) in \(R^3 \times (-1, 0)\) and \(u \in L^\infty(-1, 0; L^2(R^3)) \cap L^2(-1, 0; H^1(R^3))\).

**3.1. Proof of Case 1.** In this subsection, we assume that \(\nabla h = (\partial_1, \partial_2)\) and \(u_h = (u_1, u_2) \in L^p_t L^q_x(Q_{r_0})\) for some \(r_0 \in (0, 1)\), where \(\frac{3}{p} + \frac{2}{q} = 1, 2 < q < \infty\). We denote by \((p', q')\) the conjugate index of \((p, q)\).

**Lemma 3.1.** It holds that for any \(r \in (0, 1)\),
\[
\begin{align*}
   1. & \quad \text{if } \frac{3}{p} + \frac{2}{q} = \frac{3}{2}, 2 \leq l \leq 6, \text{ we have} \\
   & \quad \|u\|_{L^l_t L^q_x(Q_r)} \leq C(\|u\|_{L^p_t L^q_x(Q_r)} + \|
abla u\|_{L^p_t L^q_x(Q_r)});
\end{align*}
\]
2. If $\frac{3}{4} + \frac{1}{\pi} = 4.1 \leq l \leq \frac{3}{2}$, we have
\[
\|u\|_{L^1_t L^l_x(Q_r)} \leq C\left(\|u\|_{L^\infty_t L^2_x(Q_r)} + \|\nabla u\|_{L^2_t L^2_x(Q_r)}\right)^2.
\]
Here $C$ is a constant independent of $r$.

Proof. By scaling invariance, it suffices to consider the case of $r = 1$. By Hölder inequality and Sobolev interpolation inequality (for example, see [1]), we get
\[
\|u\|_{L^1_t L^l_x(Q_1)} \leq C\left(\|u\|_{L^2_t L^2_x(Q_1)}^{\frac{3}{2}} + \|\nabla u\|_{L^2_t L^2_x(Q_1)}^{\frac{1}{2}}\right) \leq C\left(\|u\|_{L^\infty_t L^2_x(Q_1)} + \|\nabla u\|_{L^2_t L^2_x(Q_1)}\right).
\]
This gives the first inequality. The proof of the second inequality is similar. \hfill \Box

In the following, we derive the local energy inequality. We denote
\[
G_1(f, p, q, r) = r^{3\frac{1}{\pi} - \frac{3}{2}}\|f\|_{L^2_t L^2_x(Q_r)}.
\]

Lemma 3.2. Let $0 < 4r < \rho < r_0$ and $1 \leq p, q \leq \infty$. Then we have
\[
A(u, r) + E(u, r) \leq C\left(\frac{L}{p}\right)^2 A(u, \rho) + C\left(\frac{L}{p}\right)^2 G_1(u_h, p, q; \rho)\left(A(u, \rho) + E(u, \rho) \right) + C\left(\frac{L}{p}\right)^2 G_1(u, \rho)\left(A(u, \rho) + E(u, \rho) \right)
\]
where the constant $C$ is independent of $r, \rho, \pi_1, \pi_3$ and $\partial_3 \pi_4$ is given by
\[
\pi_1 = \frac{1}{4\pi} \int_{R^3} \frac{1}{|x-y|} \sum_{i+j=6} \partial_i \partial_j (u_i u_j) dy, \quad \pi_3 = \frac{1}{2\pi} \int_{R^3} \frac{1}{|x-y|} \partial_3 \nabla_h \cdot (-u_3 u_h) dy,
\]
\[
\partial_3 \pi_4 = \frac{1}{2\pi} \int_{R^3} \frac{1}{|x-y|} \partial_3 \partial_3 (u_h \cdot \nabla_h u_3) dy.
\]

Proof. Let $\zeta$ be a cutoff function, which vanishes outside of $Q_\rho$ and equals 1 in $Q_{\frac{3}{2}}$, and satisfies
\[
|\nabla \zeta| \leq C\rho^{-1}, \quad |\partial_3 \zeta| + |\Delta \zeta| \leq C\rho^{-2}.
\]
Define the backward heat kernel as
\[
\Gamma(x, t) = \frac{1}{4\pi(r^2-t)^{3/2}} e^{-\frac{|x|^2}{4(r^2-t)}}.
\]
Let $\phi = \Gamma \zeta$. Due to the local energy inequality and noting that $(\partial_t + \Delta)\Gamma = 0$, we obtain
\[
\sup_t \int_{B_\rho} |u|^2 \phi dx + \int_{Q_\rho} |\nabla u|^2 \phi dx dt \leq \int_{Q_\rho} \left(|u|^2 (\Delta \phi + \partial_t \phi) - \phi u \cdot \nabla (|u|^2 + 2\pi - 2\pi B_\rho)\right) dx dt
\]
\[
\leq \int_{Q_\rho} \left(|u|^2 (\Gamma \Delta \zeta + \Gamma \partial_t \zeta + 2\nabla \Gamma \cdot \nabla \zeta) - \phi u \cdot \nabla (|u|^2) - \phi u \cdot \nabla (2\pi - 2\pi B_\rho)\right) dx dt.
\]
It is easy to verify the following facts:
\[
\Gamma(x, t) \geq C^{-1} r^{-3} \quad \text{in } Q_r;
\]
\[
\phi \leq C r^{-3}, \quad |\nabla \phi| \leq |\nabla \Gamma| |\zeta| + |\Gamma| |\nabla \zeta| \leq C t^{-4};
\]
\[
|\Gamma \Delta \zeta| + |\Gamma \partial_t \zeta| + |2\nabla \Gamma \cdot \nabla \zeta| \leq C t^{-5}.
\]
Let
\[ I = \int_{Q_r} \phi u \cdot \nabla(|u|^2) \, dx \, dt = I_1 + I_2, \]
where
\[ I_1 = \int_{Q_r} \phi u_h \cdot \nabla_h(|u|^2) \, dx \, dt, \quad I_2 = \int_{Q_r} \phi u_3 \cdot \nabla_3(|u|^2) \, dx \, dt. \]

By Hölder inequality and Lemma 3.1, we have
\[ |I_1| \leq Cr^{-3}\|u_h\|_{L^p_{t}L^q_{x}(Q_r)}\|\nabla(|u|^2)\|_{L^{q'}_{t}L^{p'}_{x}(Q_r)} \leq Cr^{-2}\left(\frac{p}{r}\right)G(u_h, p, q; \rho)(A(u, \rho) + E(u, \rho)), \]

and using the facts that \( \nabla \cdot u = 0 \) and \( \frac{3}{2p'} + \frac{2}{2q'} = 2 \), we get by integrating by parts and Hölder inequality that
\[ |I_2| \leq \left| \int_{Q_r} \phi u_3 \partial_3(|u|^2) \, dx \, dt - 2 \int_{Q_r} \phi u_3 \nabla_h \cdot u_3 \, dx \, dt \right| \]
\[ \leq \left| \int_{Q_r} (\phi \partial_3 u_3 + u_3 \partial_3 \phi)(|u|^2) \, dx \, dt - 2 \int_{Q_r} u_3 \cdot \nabla_h (\phi u_3^2) \, dx \, dt \right| \]
\[ \leq Cr^{-2}\left(\frac{p}{r}\right)G(u_h, p, q; \rho)(A(u, \rho) + E(u, \rho)) \]
\[ \leq Cr^{-2}\left(\frac{p}{r}\right)^2 G(u_h, p, q; \rho)(A(u, \rho) + E(u, \rho)). \]

This gives that
\[ |I| \leq Cr^{-2}\left(\frac{p}{r}\right)^2 G(u_h, p, q; \rho)(A(u, \rho) + E(u, \rho)). \]

The main trouble comes from the term including the pressure. Let
\[ II = \int_{Q_r} \phi u \cdot \nabla(\pi - (\pi)_{B_r}) \, dx \, dt = II_1 + II_2, \]
where
\[ II_1 = \int_{Q_r} \phi u_h \cdot \nabla_h (\pi - (\pi)_{B_r}) \, dx \, dt, \quad II_2 = \int_{Q_r} \phi u_3 \partial_3 (\pi - (\pi)_{B_r}) \, dx \, dt. \]

We get by Hölder inequality that
\[ |II_1| \leq Cr^{-2}\left(\frac{p}{r}\right)G(u_h, p, q; \rho)G_1(\nabla_h \pi, p', q'; \rho). \]

To deal with \( II_2 \), recall that the pressure \( \pi \) satisfies
\[ -\Delta \pi = \partial_i \partial_j (u_i u_j), \]

hence,
\[ \pi = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x - y|} \sum_{i,j=1}^{3} \partial_i \partial_j (u_i u_j) \, dy \doteq \pi_1 + \pi_2, \]

where
\[ \pi_1 = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x - y|} \sum_{i+j<6} \partial_i \partial_j (u_i u_j) \, dy, \quad \pi_2 = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x - y|} \partial_3 \partial_3 (u_3 u_3) \, dy. \]
We get by using $\nabla \cdot u = 0$ that
\[
\int_{Q_\rho} \phi u_3 \partial_3 \pi_3^2 dx dt \\
= \int_{Q_\rho} \phi u_3 \partial_3 \left[ \frac{1}{4\pi} \int_{R^3} \frac{1}{|x-y|} \partial_3 \partial_3 (u_3 u_3) dy \right] dx dt \\
= \int_{Q_\rho} \phi u_3 \partial_3 \left[ \frac{1}{2\pi} \int_{R^3} \frac{1}{|x-y|} \partial_3 (u_h \cdot \nabla_h u_3 - \nabla_h \cdot (u_3 u_h)) dy \right] dx dt \\
= \int_{Q_\rho} \phi u_3 \partial_3 \left[ \frac{1}{2\pi} \int_{R^3} \frac{1}{|x-y|} \partial_3 \nabla_h \cdot (-u_3 u_h) dy \right] dx dt \\
+ \int_{Q_\rho} \phi u_3 \partial_3 \left[ \frac{1}{2\pi} \int_{R^3} \frac{1}{|x-y|} \partial_3 \partial_3 (u_h \cdot \nabla_h u_3) dy \right] dx dt \\
= \int_{Q_\rho} \phi u_3 \partial_3 \pi_3^2 dx dt + \int_{Q_\rho} \phi u_3 \partial_3 \pi_4 dx dt.
\]
Consequently, we obtain
\[
|II_2| \leq |\int_{Q_\rho} \phi u_3 \partial_3 (\pi_1 + \pi_3) dx dt| + |\int_{Q_\rho} \phi u_3 \partial_3 \pi_4 dx dt| \\
\leq C \|\partial_3 (\phi u_3)\|_{L^2(Q_\rho)} (\|\pi_1 - (\pi_1)_{B_\rho}\|_{L^2(Q_\rho)} + \|\pi_3 - (\pi_3)_{B_\rho}\|_{L^2(Q_\rho)}) \\
+ Cr^{-3} \|u_3\|_{L^1(Q_\rho)} \|\partial_3 \pi_4\|_{L_{t'}^{m'}(Q_\rho)},
\]
where $(m', n')$ is the conjugate index of $(m, n)$ satisfying
\[
\frac{1}{m'} = \frac{1}{p} + \frac{1}{2}, \quad \frac{1}{n'} = \frac{1}{q} + \frac{1}{2}.
\]

hence, $m = \frac{2p}{p-2}$, $n = \frac{2q}{q-2}$. Thus,
\[
|II_2| \leq Cr^{-2} \left( \frac{p}{r} \right)^2 (A(u, \rho) + E(u, \rho)) \left( \tilde{H}(\pi_1, 2, 2; \rho) + \tilde{H}(\pi_3, 2, 2; \rho) \right) \\
+ Cr^{-2} \left( \frac{p}{r} \right) G(u, \frac{2p}{p-2}, \frac{2q}{q-2}; \rho) G_1(\partial_3 \pi_4, \frac{2p}{p+2}, \frac{2q}{q+2}; \rho).
\]
Noting that $\frac{3}{p' + 2} + \frac{2}{q' + 2} = \frac{3}{2}$, we get by Lemma 3.1 that
\[
G(u, \frac{2p}{p-2}, \frac{2q}{q-2}; \rho) \leq C (A(u, \rho) + E(u, \rho)) \frac{1}{2},
\]

hence,
\[
|II_2| \leq Cr^{-2} \left( \frac{p}{r} \right)^2 (A(u, \rho) + E(u, \rho)) \left( \tilde{H}(\pi_1, 2, 2; \rho) + \tilde{H}(\pi_3, 2, 2; \rho) \right) \\
+ Cr^{-2} \left( \frac{p}{r} \right) (A(u, \rho) + E(u, \rho)) \frac{1}{2} G_1(\partial_3 \pi_4, \frac{2p}{p+2}, \frac{2q}{q+2}; \rho).
\]
Now the lemma follows by summing up the estimates of $I, II_1$ and $II_2$.

The following lemma is devoted to the estimates of the pressure.
Lemma 3.3. Let $\pi_1, \pi_3, \partial_t \pi_4$ be as in Lemma 3.2. Then it holds that for $0 < 8r < \rho < r_0$,

$$\tilde{H}(\pi_1, 2; 2; r) \leq C(\frac{\rho}{r})^2 G(u_h, p, q; \rho)(A(u, \rho) + E(u, \rho))^{\frac{3}{2}} + C(\frac{r}{\rho})^2 \tilde{H}(\pi_1, 1, 2; \rho),$$

$$\tilde{H}(\pi_3, 2; 2; r) \leq C(\frac{\rho}{r})^2 G(u_h, p, q; \rho)(A(u, \rho) + E(u, \rho))^{\frac{3}{2}} + C(\frac{r}{\rho})^2 \tilde{H}(\pi_3, 1, 2; \rho),$$

$$G_1(\nabla_h \pi, p', q'; r) \leq C(\frac{\rho}{r}) (A(u, \rho) + E(u, \rho)) + C(\frac{r}{\rho})^3 1 G_1(\nabla_h \pi, 1, q'; \rho),$$

$$G_1(\partial_3 \pi_4, 2p + 2q + 2; r) \leq C(\frac{\rho}{r}) G(u_h, p, q; \rho) E(u, \rho)^{\frac{3}{2}} + C(\frac{r}{\rho})^{\frac{3p+q}{2p}} G_1(\partial_3 \pi_4, 1, 2q + 2; r),$$

where $C$ is a constant independent of $r, \rho$.

Proof. Let $\zeta$ be a cut-off function, which equals 1 in $Q_\frac{r}{2}$ and vanishes outside of $Q_\rho$.

We decompose $\pi_1$ into $\tilde{\pi}_1 + \tilde{\pi}_2$ with

$$\tilde{\pi}_1 = \frac{1}{2\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} \sum_{i+j<6} \partial_i \partial_j (u, u_j \zeta^2).$$

By Calderon-Zygmund inequality, we have

$$\int_{B_{\rho}} |\tilde{\pi}_1|^2 dx \leq C \int_{B_{\rho}} (|u_h| u)|^2 dx.$$

Since $\tilde{\pi}_2$ is harmonic in $Q_{\frac{r}{2}}$, we have

$$\int_{B_{\rho}} |\tilde{\pi}_2 - (\tilde{\pi}_2)_{B_{\rho/4}}|^2 \leq C \rho^3 \sup_{B_{\rho/4}} |\nabla \tilde{\pi}_2|^2 \leq C(\frac{r}{\rho})^5 \rho^{-3} \left( \int_{B_{\frac{r}{2}}} |\tilde{\pi}_2 - (\tilde{\pi}_2)_{B_{\frac{r}{2}}}|^2 dx \right)^2.$$

Then we get by Lemma 3.1 that

$$\tilde{H}(\pi_1, 2; 2; r) \leq Cr^{-\frac{3}{2}} |u_h| u|_{L^2 L^{\frac{6}{2}}(Q_\rho)} + C(\frac{r}{\rho})^2 \tilde{H}(\pi_1, 1, 2; \rho)$$

$$\leq C r^{-\frac{3}{2}} G(u_h, p, q; \rho) \|u\|_{L^2 L^{\frac{6}{2}}(Q_\rho)} + C(\frac{r}{\rho})^2 \tilde{H}(\pi_1, 1, 2; \rho)$$

$$\leq C(\frac{\rho}{r})^2 G(u_h, p, q; \rho)(A(u, \rho) + E(u, \rho))^{\frac{3}{2}} + C(\frac{r}{\rho})^2 \tilde{H}(\pi_1, 1, 2; \rho).$$

The first equality of the lemma is proved. The proof of the second inequality is almost the same. Let us turn to the proof of the third inequality. Recall that $\pi$ satisfies

$$-\Delta \pi = \partial_t \partial_j (u_i u_j).$$

We decompose $\nabla_h \pi$ into $\tilde{\pi}_1 + \tilde{\pi}_2$ with

$$\tilde{\pi}_1 = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} \partial_i \partial_j (\nabla_h (u_i u_j) \zeta^2) dx.$$

By Calderon-Zygmund inequality, we have

$$\int_{B_{\rho}} |\tilde{\pi}_1|^2 dx \leq C \int_{B_{\rho}} (|u| \|\nabla_h u\|)|^2 dx.$$
Since $\nabla \tilde{h} \tilde{\pi}_2$ is harmonic in $Q_{\tilde{z}}$, we have
\[
\int_{B_r} |\tilde{\pi}_2|^p \, dx \leq C r^3 \sup_{B_{r/4}} |\tilde{\pi}_2|^{p'}
\leq C (\frac{L}{\rho})^3 \rho^{-3p' + 3} \left( \int_{B_{\tilde{z}}} |\tilde{\pi}_2| \, dx \right)^{p'}.
\]
Noting that $\frac{3}{p'} + \frac{2}{q'} = 4$, we infer from Lemma 3.1 that
\[
G_1(\nabla \pi, p', q'; r) \leq C \left( \frac{L}{\rho} \right)^3 \rho^{-3p' + 3} G_1(\pi_2, q'; r) \leq C (\frac{L}{\rho})^3 \rho^{-3p' + 3} \left( \int_{B_{\tilde{z}}} |\tilde{\pi}_2| \, dx \right)^{p'}.
\]
The third inequality is proved. The proof of the fourth inequality is similar.

**Lemma 3.4.** Let $\pi_1, \pi_3, \partial_3 \pi_4$ be as in Lemma 3.2. It holds that for any $r_0 \in (0, 1)$,
\[
\bar{H}(\pi_1, 1, 2; r_0) + \bar{H}(\pi_3, 1, 2; r_0) \leq C,
\]
\[
G_1(\nabla \pi, 1, q'; r_0) + G_1(\partial_3 \pi_4, 1, \frac{2q}{q + 2}; r_0) \leq C,
\]
where the constant $C$ depends on $r_0$ and $\|u\|_{L^\infty((-1,0;L^2(\mathbb{R}^3)) \cap L^2((-1,0;H^1(\mathbb{R}^3)))}$.

**Proof.** As in the proof of Lemma 3.1, we have
\[
\|u\|_{L^2_1 L^1_2((-1,0) \times \mathbb{R}^3)} \leq C \|u\|_{L^\infty_1 L^2_2 \cap L^2_1 H^1_2((-1,0) \times \mathbb{R}^3)},
\]
where $\frac{3}{7} + \frac{2}{s} = \frac{9}{7}$, $2 \leq l \leq 6$, and
\[
\|\nabla u\|_{L^1_1 L^1_2((-1,0) \times \mathbb{R}^3)} \leq C \|u\|_{L^\infty_1 L^2_2 \cap L^2_1 H^1_2((-1,0) \times \mathbb{R}^3)},
\]
where $\frac{3}{7} + \frac{2}{s} = \frac{9}{7}$, $1 \leq l \leq \frac{9}{7}$. By Calderon-Zygmund inequality, it follows that
\[
\| (\pi_1, \pi_3) \|_{L^2_1 L^2_2((-1,0) \times \mathbb{R}^3)} \leq C \|u\|_{L^2_1 L^1_2((-1,0) \times \mathbb{R}^3)},
\]
where $\frac{3}{7} + \frac{2}{s} = \frac{9}{7}$, $2 \leq l \leq 6$, and
\[
\| (\nabla \pi, \partial_3 \pi_4) \|_{L^1_1 L^1_2((-1,0) \times \mathbb{R}^3)} \leq C \|u\|_{L^2_1 L^1_2((-1,0) \times \mathbb{R}^3)},
\]
where $\frac{3}{7} + \frac{2}{s} = \frac{9}{7}$, $1 \leq l \leq \frac{9}{7}$. The lemma follows by taking suitable $(s, l)$ and Hölder inequality.

Now we are in position to prove Case 1 in Theorem 1.1. Given any $\varepsilon > 0$, there exists $\rho \in (0, r_0) \text{ so that}$
\[
G(u_h, p, q; \rho) \leq \varepsilon.
\]
Take $r$ so that $0 < 8r < \rho < r_0$. It follows from Lemma 3.2 that
\[
A(u, r) + E(u, r) \leq C \left( \frac{L}{\rho} \right)^2 A(u, \rho) + C_0 \left( \frac{L}{r} \right)^2 \varepsilon (A(u, \rho) + E(u, \rho) + G_1(\nabla \pi, p', q'; \rho))
+ C \left( \frac{L}{r} \right)^2 (A(u, \rho) + E(u, \rho)) \lesssim \left( \bar{H}(\pi_1, 2, 2; \rho) + \bar{H}(\pi_3, 2, 2; \rho) \right)^2
\]
Then it follows from Lemma 3.3 that iteration argument ensures that there exists $r$ small enough so that

$$
\text{We first choose } \theta \text{ such that} \\
\delta > 0 \text{ will be determined later. Let} \\
F(r) = A(u; r) + E(u; r) + \frac{1}{2}G_1(\nabla h, p', q'; r) \\
+ C\left(\left(\frac{r}{\rho}\right)^2 + (\varepsilon + \delta + \sqrt{\varepsilon})\left(\frac{\rho}{r}\right)^4 + \sqrt{\delta}\right)F(\rho) \\
+ C\left(\sqrt{\varepsilon}\left(\frac{\rho}{r}\right) + \left(\frac{r}{\rho}\right)^{2-1}\right)F(\rho) \\
+ C\left(\delta^{-\frac{2}{3}}\left(\frac{\rho}{r}\right)^{2-1} + \left(\frac{r}{\rho}\right)^4 \right)F(\rho).
$$

Take $r = \theta \rho$ with $0 < \theta < \frac{1}{8}$. The above inequality yields that

$$
F(\theta \rho) \leq C(\theta^2 + \sqrt{\delta} + (\varepsilon + \delta + \sqrt{\varepsilon})\theta^{-4} + \sqrt{\varepsilon}\theta^{-1} + \theta^{\frac{2}{3}}\theta^{-1} + \delta^{-\frac{2}{3}}\theta^{-2}\varepsilon^2 + \theta^{1+\frac{\varepsilon}{2}})F(\rho).
$$

We first choose $\theta$ small enough, then choose $\delta$ small, and finally choose $\varepsilon$ small enough so that

$$
F(\theta \rho) \leq \frac{1}{2}F(\rho).
$$

On the other hand, Lemma 3.3 and Lemma 3.4 imply that

$$
F(r_0) \leq C
$$

with $C$ depending on $r_0$ and $\|u\|_{L^\infty(-1; 0; L^q(-1; 0; H^1(\Omega)))}$. Then a standard iteration argument ensures that there exists $r_1 > 0$ such that

$$
F(r) \leq \varepsilon_1 \quad \text{for any} \quad 0 < r < r_1 < r_0,
$$

which implies Case 1 of Theorem 1.1 by Proposition 1.

3.2. Proof of Case 2 and Case 3. Let us claim that Case 2 and Case 3 in Theorem 1.1 can be deduced from the following theorem.

**Theorem 3.5.** Let $(u, \pi)$ be a suitable weak solution of (1) in $\mathbb{R}^3 \times (-1, 0)$. If it satisfies

$$
\lim_{r \to 0} \left( H(\nabla u_h, p, q; r) + G(u_h, p, q; r) \right) = 0, \quad \frac{3}{p} + \frac{2}{q} = 2, \quad 1 < q < \infty \quad (5)
$$

then $u$ is regular at $(0, 0)$.

Indeed, the assumptions in Case 3 obviously imply (5). Let us verify (5) in Case 2. In such case, $2 < q < \infty$ and $\frac{2}{3} < p < 3$. By Poincâre inequality, we have

$$
G(u_h - (u_h)_B, p, q; r) \leq CH(\nabla u_h, p, q; r)
$$
for any $0 < r < r_0$. Since $\frac{3}{p} + \frac{2}{q} = 2$, we have
\[
G(u_h, p, q; r) \leq G(u_h - (u_h)_{B_r}, p, q; r) + G((u_h)_{B_r}, p, q; r) \\
\leq C(\frac{r}{\rho})^3G(u_h - (u_h)_{B_r}, p, q; \rho) + C(\frac{r}{\rho})^2G((u_h)_{B_r}, p, q; \rho) \\
\leq C(\frac{r}{\rho})H(\nabla u_h, p, q; \rho) + C(\frac{r}{\rho})^\frac{3}{2}G((u_h)_{B_r}, p, q; \rho).
\]
Note that $p < 3$ and $\limsup_{r \to 0} H(\nabla u_h, p, q; r) = 0$. Then by a standard iteration, there holds
\[
\limsup_{r \to 0} G(u_h, p, q; r) = 0,
\]
which implies (5).
In what follows, we assume that $\frac{3}{p} + \frac{2}{q} = 2, 1 < q < \infty$. We denote by $(p', q')$ the conjugate index of $(p, q)$. To prove Theorem 3.5, we need the following local energy inequality.

**Lemma 3.6.** Let $0 < 4r < \rho < r_0$. It holds that
\[
A(u, r) + E(u, r) \\
\leq C(\frac{r}{\rho})^2A(u, \rho) + C(\frac{r}{\rho})G_1(\partial_3 \pi, \frac{2p}{p+1}, \frac{2q}{q+1}; \rho)(A(u, \rho) + E(u, \rho))^\frac{1}{2} \\
+ C((\frac{r}{\rho})H(\nabla u_h, p, q; \rho) + (\frac{r}{\rho})^2G((u_h)_{B_r}, p, q; \rho))(A(u, \rho) + E(u, \rho) + \tilde{H}(\pi, p', q'; \rho))
\]
where the constant $C$ is independent of $r, \rho$.

**Proof.** Since the proof is very similar to Lemma 3.2, we only present a sketch. Using the same test function $\phi$ in the proof of Lemma 3.2, we have
\[
\sup_t \int_{B_r} |u|^2 \phi dx + \int_{Q_r} |\nabla u|^2 \phi dx dt \\
\leq \int_{Q_r} (|u|^2(\Delta \phi + \partial_3 \phi) - \phi u \cdot \nabla(|u|^2 + 2\pi - 2(\pi)_{B_r}))dx dt \\
\leq \int_{Q_r} (|u|^2(\Gamma \Delta \zeta + \Gamma \partial_3 \zeta + 2\nabla \Gamma \cdot \nabla \zeta) - \phi u \cdot \nabla(|u|^2) - \phi u \cdot \nabla(2\pi - 2(\pi)_{B_r}))dx dt.
\]
Let
\[
I = \int_{Q_r} \phi u \cdot \nabla(|u|^2)dx dt \doteq I_1 + I_2,
\]
where
\[
I_1 = \int_{Q_r} \phi u_h \cdot \nabla_h(|u|^2)dx dt, \quad I_2 = \int_{Q_r} \phi u_3 \cdot \partial_3(|u|^2)dx dt.
\]
By Hölder inequality and $\nabla \cdot u = 0$, we have
\[
|I_1| \leq \int_{Q_r} \phi \nabla_h \cdot u_h(|u|^2)dx dt + \int_{Q_r} (u_h \cdot \nabla_h \phi)(|u|^2)dx dt \\
\leq C r^{-2}(\frac{\rho}{r})H(\nabla u_h, p, q; \rho) + (\frac{\rho}{r})^2G(u_h, p, q; \rho))G(u, 2p', 2q'; \rho)^2,
\]
and noting that $\partial_3|u|^2 \leq |\nabla u_h||u|$, we get

$$|I_2| \leq Cr^{-2}(\frac{\rho}{r})H(\nabla u_h, p, q; \rho)G(u, 2p', 2q'; \rho)^2,$$

which along with Lemma 3.1 imply that

$$|I| \leq Cr^{-2}(\frac{\rho}{r})H(\nabla u_h, p, q; \rho) + \left(\frac{\rho}{r}\right)^2G(u_h, p, q; \rho))(A(u, \rho) + E(u, \rho)).$$

Let

$$II = \int_{Q_r} \phi u \cdot \nabla (\pi - (\pi)_{B_\rho})dxdt = II_1 + II_2,$$

where

$$II_1 = \int_{Q_r} \phi u_h \cdot \nabla_h (\pi - (\pi)_{B_\rho})dxdt, \quad II_2 = \int_{Q_r} \phi u_3 \partial_3 (\pi - (\pi)_{B_\rho})dxdt.$$

We have by Hölder inequality and Lemma 3.1 that

$$|II_1| \leq \left| \int_{Q_r} \phi \nabla_h \cdot u_h((\pi - (\pi)_{B_\rho})dxdt \right| + \left| \int_{Q_r} (u_h \cdot \nabla_h \phi)(\pi - (\pi)_{B_\rho})dxdt \right|$$

$$\leq Cr^{-2}(\frac{\rho}{r})H(\nabla u_h, p, q; \rho) + \left(\frac{\rho}{r}\right)^2G(u_h, p, q; \rho))(\pi, p', q'; \rho),$$

$$|II_2| \leq Cr^{-2}(\frac{\rho}{r})G_1(\partial_3\pi, \frac{2p}{p+1}, \frac{2q}{q+1}; \rho)G(u, 2p', 2q'; \rho)$$

$$\leq Cr^{-2}(\frac{\rho}{r})G_1(\partial_3\pi, \frac{2p}{p+1}, \frac{2q}{q+1}; \rho)(A(u, \rho) + E(u, \rho))^{\frac{1}{2}}.$$

The lemma follows by summing up the estimates of $I, II_1$ and $II_2$. \hfill \Box

The proof of the following lemma is similar to Lemma 3.3. So, we omit the details.

**Lemma 3.7.** It holds that for any $0 < 8r < \rho < r_0$,

$$\tilde{H}(\pi, p', q'; r) \leq C\left(\frac{\rho}{r}\right)\tilde{G}(u, 2p', 2q'; \rho)^2 + C\left(\frac{r}{\rho}\right)^{\frac{\delta}{2}}\tilde{H}(\pi, 1, q'; \rho),$$

$$G_1(\partial_3\pi, \frac{2p}{p+1}, \frac{2q}{q+1}; r) \leq C\left(\frac{\rho}{r}\right)^{\frac{1}{2}}\tilde{G}(u, 2p', 2q'; \rho)H(\nabla u_h, p, q; \rho)$$

$$+ C\left(\frac{r}{\rho}\right)^{1+\frac{\delta}{2}}G_1(\partial_3\pi, 1, \frac{2q}{q+1}; \rho),$$

where the constant $C$ is independent of $r, \rho$.

Now let us turn to prove Theorem 3.5.

**Proof.** By the assumption, given any $\varepsilon > 0$, there exists $\rho \in (0, r_0)$ so that

$$H(\nabla u_h, p, q; \rho) + G(u_h, p, q; \rho) \leq \varepsilon.$$

Take $r > 0$ so that $0 < 8r < \rho < r_0$. It follows from Lemma 3.6 that

$$A(u, r) + E(u, r) \leq C\left(\frac{r}{\rho}\right)^2A(u, \rho) + C\delta^{-1}G_1(\partial_3\pi, \frac{2p}{p+1}, \frac{2q}{q+1}; \rho)^2$$

$$+ C\left(\frac{\rho}{r}\right)^{\frac{1}{2}}(\varepsilon + \delta)(A(u, \rho) + E(u, \rho)) + \varepsilon\tilde{H}(\pi, p', q'; \rho),$$

where $\delta > 0$ will be determined later. Let

$$F(r) = A(u, b; r) + E(u, b; r) + \varepsilon^{1/2}\tilde{H}(\pi, p', q'; r) + \delta^{-\frac{1}{2}}G_1(\partial_3\pi, \frac{2p}{p+1}, \frac{2q}{q+1}; r)^2.$$
Then it follows from Lemma 3.7 that
\[
F(r) \leq C \left( \frac{L}{\rho} \right)^2 + \sqrt{\delta + (\varepsilon + \delta + \sqrt{\varepsilon}) (\frac{L}{\rho})^2} F(\rho)
\]
\[+ C \left( \sqrt{\varepsilon (\frac{L}{\rho})} + \frac{L}{\rho} \right) F(\rho) + C \left( \frac{\sqrt{\delta}}{\rho} \right) \varepsilon^2 + C \left( \frac{\sqrt{\delta}}{\rho} \right)^2 F(\rho).
\]

Take \( r = \theta \rho \) with \( 0 < \theta < \frac{1}{5} \). The above inequality yields that
\[
F(\theta \rho) \leq C (\theta^2 + \sqrt{\delta + (\varepsilon + \delta + \sqrt{\varepsilon}) \theta^{-2}} + \sqrt{\varepsilon} \theta^{-1} + \theta^{\frac{3}{2}} + \delta^{-\frac{3}{2}} \theta^{-1} \varepsilon^2 + \theta^{2+\frac{3}{2}}) F(\rho).
\]

We first choose \( \theta \) small enough, then choose \( \delta \) small, finally choose \( \varepsilon \) small enough so that
\[
F(\theta \rho) \leq \frac{1}{2} F(\rho).
\]

On the other hand, it is easy to see that
\[
F(r_0) \leq C
\]
with \( C \) depending on \( r_0 \) and \( \|u\|_{L^\infty(-1,0;L^2(R^3))} \). Then a standard iteration argument ensures that there exists \( r_1 > 0 \) such that
\[
F(r) \leq \varepsilon_1 \quad \text{for all} \quad 0 < r < r_1 < r_0,
\]
which implies Theorem 3.5 by Proposition 1. \( \square \)

4. **Proof of Theorem 1.2.** Throughout this section, we assume that \((u, \pi)\) be a suitable weak solution of (1) in \( R^3 \times (-1, 0) \). Let us first introduce some notations.

Let \( B_R^2 = \{(x_1, x_2); |(x_1, x_2)| \leq R \} \), \( B_r^2 = B_R^2 \times R = \{(x_1, x_2, x_3); (x_1, x_2) \in B_R^2, x_3 \in R \} \), and \( Q_r^\ast = B_r^2 \times (-r^2, 0) \). Moreover, \( Q_r^\ast(z_0) = (-r^2 + t_0, t_0) \times B_r^2(x_0) \), \( B_r^2(x_0) = B_2^2(x_0) \times R \) and \( B_r^2 \) is a ball of radius \( r \) centered at the horizontal part of \( x_0 \). For the simplicity, we denote \( Q_r^\ast(0) \) by \( Q_r^\ast \) and \( B_r^2(0) \) by \( B_r^2 \). As in Section 2, we will still use the notations like \( A(u, r) \), \( E(u, r) \), \( G(f, p, q; r) \), \( H(f, p, q; r) \), \( \tilde{G}(f, p, q; r) \), \( \tilde{H}(\pi, p, q; r) \) etc. The differences are that here the integral domain is replaced by \( Q_r^\ast \) or \( B_r^2 \), and the mean value in \( G, \tilde{H} \) is taken only on \( B_r^2 \). We denote by \((p', q')\) the conjugate index of \((p, q)\).

**Lemma 4.1.** Let \( 0 < 4r < \rho < r_0 \) and \( 1 \leq p, q \leq \infty \). We have
\[
A(u; r) + E(u; r)
\]
\[\leq C \left( \frac{L}{\rho} \right) A(u; \rho) + C \left( \frac{L^2}{\rho} \right) G(u_h, p, q; \rho) \left( G(u, 2p', 2q'; \rho) \right)^2 + \tilde{H}(\pi, p', q'; \rho),
\]
where \( C \) is a constant independent of \( r, \rho \).

**Proof.** Let \( \zeta \) be a cutoff function, which vanishes outside of \( Q_\rho \) and equals 1 in \( Q_2 \), and satisfies
\[
|\nabla \zeta| \leq C_0 \rho^{-1}, \quad |\partial_t \zeta| + |\Delta \zeta| \leq C_0 \rho^{-2}.
\]

Define the backward heat kernel as
\[
\Gamma(x, t) = \frac{1}{4\pi(r^2 - t)} e^{-\frac{|x|^2}{4(r^2 - t)}},
\]
Taking the test function \( \phi = \Gamma \zeta \) in the local energy inequality, and noting \((\partial_t + \Delta_h)\Gamma = 0\), where \( \Delta_h = \partial^2_{x_1} + \partial^2_{x_2}, \) we obtain
from which and Hölder inequality, it follows that it is easy to verify that

$$\sup_t \int_{B_r^*} |u|^2 \phi dx + \int_{Q_r^*} |\nabla u|^2 \phi dx dt$$

$$\leq \int_{Q_r^*} \left[ |u|^2 (\nabla \phi + \partial_t \phi) + u \cdot \nabla \phi (|u|^2 + 2\pi - 2(\pi)_{B_r^2}) \right] dx dt$$

$$\leq \int_{Q_r^*} \left[ |u|^2 (\nabla \Delta \zeta + \nabla \partial_t \zeta + 2 \nabla \cdot \nabla \zeta) + |\nabla \phi||u^3| (|u|^2 + 2\pi - (\pi)_{B_r^2}) \right] dx dt.$$  

It is easy to verify that

$$\Gamma(x, t) \geq C_0^{-1} r^{-2} \quad \text{in} \quad Q_r^*,$$

$$|\nabla \phi| \leq |\nabla \Gamma| \zeta + \Gamma |\nabla \zeta| \leq C_0 r^{-3},$$

$$|\Gamma \Delta \zeta| + |\partial_t \zeta| + 2 |\nabla \cdot \nabla \zeta| \leq C_0 r^{-4},$$

from which and Hölder inequality, it follows that

$$A(u, r) + E(u, r)$$

$$\leq C \left( \frac{\Gamma}{\rho} \right) A(u, p) + C \left( \frac{\rho}{\Gamma} \right)^2 \rho^{-2} \int_{Q_r^*} (|u_h||u|^2 + |u_h||\pi - (\pi)_{B_r^2}) dx dt$$

$$\leq C \left( \frac{\Gamma}{\rho} \right) A(u, p) + C \left( \frac{\rho}{\Gamma} \right)^2 G(u_h, p, q; \rho) \left( G(u, 2p', 2q'; \rho)^2 + \tilde{H}(\pi, p', q'; \rho) \right).$$

This completes the proof of the lemma.

In the sequel, we assume that \((p, q)\) satisfies

$$\frac{3}{p} + \frac{2}{q} = 2, \quad \frac{3}{2} \leq p < \infty.$$

**Lemma 4.2.** For any \(0 < r < r_0\), we have

$$G(u, 2p', 2q'; r)^2 \leq C \left( E(u, 2r) + A(u, 2r) \right),$$

where \(C\) is a constant independent of \(r\).

**Proof.** Recall a well-known Sobolev’s interpolation inequality (for example, see [1]):

$$\int_{R^3} |f|^{\ell} \leq C \left( \int_{R^3} |\nabla f|^2 dx \right)^a \left( \int_{R^3} |f|^2 dx \right)^{\frac{\ell-a}{2}},$$

(6)

where \(2 \leq \ell \leq 6\) and \(a = \frac{3}{4}(\ell-2)\). Applying (6) with \(\ell = 2p'\) (Note that \(2p' \leq 6\) since \(p \geq \frac{3}{2}\)) and a suitable localization, we get

$$G(u, 2p', 2q'; r)^2 = r^\frac{3}{p} + \frac{2}{q} \frac{3}{2} ||u||_{L^{2p', 2q'}(Q_r^*)}^2$$

$$\leq C r^{-1} \left\{ \int_{-r - r} \left[ \int_{B_{2r}} |\nabla u|^2 \right]^\frac{a'}{a'} \left( \int_{B_{2r}} |u|^2 \right)^{\frac{p'}{2} - a} + r^{2-a} \left( \int_{B_{2r}} |u|^2 \right)^{\frac{p'}{2}} dt \right\}^\frac{1}{a'}$$

$$\leq C r^{-1} \left\{ \int_{-r} \left( \int_{B_{2r}} |\nabla u|^2 \right)^{\frac{a'}{p'}} \left( \int_{B_{2r}} |u|^2 \right)^{q'(1-\frac{a'}{p'})} dt \right\}^\frac{1}{a'} + r^{2-a} \frac{2a}{p'} + 2 \left( \sup_t \int_{B_{2r}} |u|^2 \right)^\frac{1}{q'},$$

then the lemma follows by noting that \(\frac{aq'}{p} = 1\) and \(-\frac{2a}{p'} + \frac{2}{q'} = -\frac{3}{p} - \frac{2}{q} + 2 = 0.\)

In the following, we will introduce a new pressure decomposition formula in a cylinder domain based on the following properties of harmonic function, which is new even for harmonic function to our knowledge.
Lemma 4.3. Let $f$ be a harmonic function in a cubic $D_1 \subset \mathbb{R}^3$. Let

$$P_3 f(x) = \frac{1}{2} \int_{-1}^{1} f(x_1, x_2, x_3) dx_3,$$

$$P_3 f(x_3) = \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} f(x_1, x_2, x_3) dx_h.$$

Then it holds that

$$\sup_{x \in B_{1/2}} |\nabla_3 f| \leq C \int_{B_1} |f(x) - P_3 f(x_h)| dx,$$

$$\sup_{x \in B_{1/2}} |\nabla_3 f| \leq C \int_{B_1} |f(x) - P_3 f(x)| dx.$$

Proof. For $|h| \leq \frac{1}{5}$, let

$$f^h(x) = f^h(x_1, x_2, x_3) = f(x_1, x_2, x_3 + h),$$

and $g(x) = f(x) - P_3 f(x_1, x_2)$. It is easy to see that

$$f(x) - f^h(x) = g(x) - g^h(x), \quad x \in B_{1/2}.$$

Since $f$ is a harmonic function in $D_1$, we have

$$\Delta (f(x) - f^h(x)) = \Delta (g(x) - g^h(x)) = 0, \quad x \in B_{1/2}.$$

The gradient estimate of harmonic function yields that

$$\sup_{B_{1/2}} |\partial_3 f - \partial_3 f^h| \leq C \sup_{B_{1/2}} |f - f^h| \leq C \sup_{B_{1/2}} |g - g^h|,$$

$$\sup_{B_{1/2}} |g - g^h| \leq C \int_{B_{1/2}} |g - g^h| dx \leq C \int_{B_1} |f - P_3 f| dx.$$

This proves that for any $|h| \leq \frac{1}{5}$,

$$\sup_{B_{1/2}} |\partial_3 f - \partial_3 f^h| \leq C \sup_{B_{1/2}} |f - f^h| \leq C \int_{B_1} |f - P_3 f| dx. \quad (7)$$

The second inequality of (7) implies by Mean value theorem that given $x \in B_{1/2}$, there exists $h = h(x)$ with $|h| \leq \frac{1}{5}$ so that

$$|\partial_3 f(x_1, x_2, x_3 + h)| \leq C \int_{B_1} |f - P_3 f| dx,$$

which along with (7) gives the first inequality of the lemma. The proof of the second inequality of the lemma is similar.

Let

$$\tilde{H}(\pi, p', q'; r) = r^{2 - \frac{2}{p'} - \frac{2}{q'}} \left( \int_{r^2} \left( \int_{B_r^2} |\pi - P_{h,r} \pi(x_3)|^{p'} dx_h dx_3 \right)^{\frac{q'}{2}} \frac{dr}{r^2} \right)^{1/q'},$$

where

$$P_{h,r} \pi(x_3) = \frac{1}{|B_r^2|} \int_{B_r^2} \pi(x_h, x_3) dx_h.$$
Lemma 4.4. For any $0 < 8r < \rho < r_0$, it holds that
\[
\hat{H}(\pi, p', q'; r) \leq C \left( \frac{p}{r} \right) \hat{G}(u, 2p', 2q'; \rho)^2 + C \left( \frac{r}{\rho} \right)^2 \hat{H}(\pi, p', q'; \rho),
\]
where $C$ is a constant independent of $r, \rho$.

Proof. Recall that the pressure $\pi$ satisfies
\[
\triangle \pi = -\partial_i \partial_j (u_i u_j).
\]
Let $\pi = \pi_1 + \pi_2$ where $\pi_1$ is defined by
\[
\triangle \pi_1 = -\partial_i \partial_j (u_i u_j \chi(x_h)),
\]
here $\chi(x_h)$ is a smooth function with $\chi(x_h) = 1$ for $|x_h| \leq \frac{\rho}{2}$ and $\chi(x_h) = 0$ for $|x_h| \geq \rho$. So, $\pi_2$ is harmonic in $B^*_\frac{\rho}{2}$.

Due to $p' > 1$, by Calderon-Zygmund inequality we have
\[
\hat{H}(\pi_1, p', q'; \rho) \leq C \hat{G}(u, 2p', 2q'; \rho)^2. \tag{8}
\]
We denote $B^*_{\rho, k} = B^*_{\rho} \times (-k\rho, k\rho)$. Since $\pi_2$ is harmonic in $B^*_\frac{\rho}{2}$, we have
\[
\int_{B^*_{\rho, k}} |\pi_2 - (\pi_2)_{B^*_{\rho/2}}|^2 dx \leq C r^{2+p'} \sum_{j=1}^{2k} \sup_{B_{\frac{\rho}{2}}(z_j)} |\nabla h \pi_2|^2,
\]
where $z_j \in \{x; x_h = 0, x_3 = 3r - kr, 0 \leq k \leq 2k\}$. We infer from Lemma 4.3 that
\[
\sup_{B_{\frac{\rho}{2}}(z_j)} |\nabla h \pi_2|^2 \leq C \rho^{-3-p'} \int_{B^*_{\rho}} |\pi_2 - P_{h, \frac{\rho}{2}} \pi_2|^2 dx.
\]
Note that the ball $B^*_{\rho}(z_j)$ intersects each other at most $C^2 \rho$ times. We infer that
\[
\int_{B^*_{\rho, k}} |\pi_2 - (\pi_2)_{B^*_{\rho}}|^2 dx \leq C \left( \frac{r}{\rho} \right)^{2+p'} \int_{B^*_{\rho}} |\pi_2 - P_{h, \frac{\rho}{2}} \pi_2|^2 dx,
\]
and letting $k \to \infty$, we get
\[
\hat{H}(\pi_2, p', q'; r) \leq C \left( \frac{r}{\rho} \right)^{3-\frac{1}{p}+\frac{2}{q}} \hat{H}(\pi_2, p', q'; \rho).
\]
which along with (8) gives
\[
\hat{H}(\pi, p', q'; r) \leq \hat{H}(\pi_1, p', q'; r) + \hat{H}(\pi_2, p', q'; r) \leq C \left( \frac{r}{\rho} \right)^{2-\frac{1}{p}+\frac{2}{q}} \hat{G}(u, b, 2p', 2q'; \rho)^2 + C \left( \frac{r}{\rho} \right)^{3-\frac{1}{p}+\frac{2}{q}} \hat{H}(\pi, p', q'; \rho).
\]
The proof is finished. \qed

Now we are in position to prove Theorem 1.2.

Proof. Let
\[
F(r) = A(u, r) + E(u, r) + \frac{1}{2} \hat{H}(\pi, p', q'; r).
\]
Take $(r, \rho, \kappa)$ so that $0 < 8r < \rho$ and $8\rho < \kappa < r_0$ and
\[
G(u_h, p, q; \rho) \leq \varepsilon.
\]
It suffices to consider the case
\[
\frac{3}{p} + \frac{2}{q} = 2, \quad \frac{3}{2} \leq p \leq \infty, \quad (p, q) \neq (\infty, 1),
\]
since the other cases can be reduced to such case by Hölder inequality. We know from Lemma 4.2 and Lemma 4.4 that
\[ \tilde{H}(\pi, p', q'; \rho) \leq C\left(\frac{\kappa}{\rho}\right)\tilde{G}(u, b, 2p', 2q'; \kappa)^2 + C\left(\frac{\rho}{\kappa}\right)^{2\sigma} \tilde{H}(\pi, p', q'; \kappa) \]
\[ \leq C\left(\frac{\kappa}{\rho}\right)F(\kappa) + C\left(\frac{\rho}{\kappa}\right)^{2\sigma} \tilde{H}(\pi, p', q'; \kappa), \]
which along with Lemma 4.1 and Lemma 4.4 gives
\[
F(r) \\
\leq C\left(\frac{\kappa}{\rho}\right)A(u, \rho) + C\left(\frac{\rho}{\kappa}\right)^2 (\varepsilon G(b, 2p', 2q'; \rho)^2 + \varepsilon \tilde{H}(\pi, p', q'; \rho)) \\
+ \varepsilon^{\frac{1}{2}} \tilde{H}(\pi, p', q'; r) \\
\leq C\left(\frac{\kappa}{\rho}\right)F(\rho) + C\varepsilon\left(\frac{\rho}{\kappa}\right)^2 F(\rho) + C\left(\frac{\rho}{\kappa}\right)^2 \varepsilon \tilde{H}(\pi, p', q'; \rho) + \varepsilon^{\frac{1}{2}} \tilde{H}(\pi, p', q'; r) \\
\leq C\left(\frac{\kappa}{\rho}\right)F(\rho) + C\varepsilon\left(\frac{\rho}{\kappa}\right)^2 F(\rho) + C\left(\frac{\rho}{\kappa}\right)^2 \varepsilon \left(\frac{\rho}{\kappa}\right)^{2\sigma} \tilde{H}(\pi, p', q'; \kappa) \\
+ C\varepsilon^{\frac{1}{2}} \left(\frac{\rho}{\kappa}\right)^{2\sigma} \tilde{H}(\pi, p', q'; \kappa) \\
\leq C\left(\frac{\kappa}{\rho}\right) + \left(\frac{\rho}{\kappa}\right)^2 \varepsilon \left(\frac{\kappa}{\rho}\right)^{2\sigma} \tilde{H}(\pi, p', q'; \kappa) + C\varepsilon^{\frac{1}{2}} \left(\frac{\rho}{\kappa}\right)^{2\sigma} \tilde{H}(\pi, p', q'; \kappa).
\]
Take \( r = \theta^2 \rho \), \( \rho = \theta \kappa \) with \( 0 < \theta < \frac{1}{2} \). The above inequality yields that
\[
F(r) \leq C\left(\theta + \varepsilon \theta^{-5} + \varepsilon^{\frac{1}{2}} \theta^{-4+2\sigma} + \varepsilon^{\frac{1}{2}} \theta^{-3} + \theta^{2\sigma}\right) F(\kappa).
\]
Choose \( \theta \) small enough, and then choose \( \varepsilon \) small enough so that
\[
C\left(\theta + \varepsilon \theta^{-5} + \varepsilon^{\frac{1}{2}} \theta^{-4+2\sigma} + \varepsilon^{\frac{1}{2}} \theta^{-3} + \theta^{2\sigma}\right) \leq \frac{1}{2}.
\]
This gives the following iterative inequality
\[
F(\theta^2 \kappa) \leq \frac{1}{2} F(\kappa).
\]
On the other hand, it is easy to see that
\[
F(R) \leq C
\]
with \( C \) depending on \( R \) and \( \|u\|_{L^\infty((-1,0);L^2(R^3)) \cap L^2((-1,0);H^1(R^3))}. \) Indeed, since \( \pi \) satisfies
\[
-\Delta \pi = \partial_i \partial_j (u_i u_j),
\]
by Calderon-Zygmund inequality and interpolation inequality, we get
\[
\|\pi\|_{L^p((-T,0);L^p(R^3))} \leq C_0 \|u\|^2_{L^2((-T,0);L^{2\sigma}(R^3))} \\
\leq C_0 \|u\|^2_{L^\infty((-T,0);L^2(R^3)) \cap L^2((-T,0);H^1(R^3))}.
\]
Then a standard iteration argument ensures that there exists \( r_1 > 0 \) such that
\[
F(r) \leq \varepsilon \quad \text{for all} \quad 0 < r < r_1,
\]
which implies Theorem 1.2 by Proposition 1.
REFERENCES

[1] L. Caffarelli, R. Kohn and L. Nirenberg, Partial regularity of suitable weak solutions of the Navier-Stokes equations, *Comm. Pure Appl. Math.*, **35** (1982), 771–831.

[2] C. Cao and E. S. Titi, Regularity criteria for the three dimensional Navier-Stokes equations, *Indiana Univ. Math. J.*, **57** (2008), 2643–2661.

[3] C. Cao and E. S. Titi, Global regularity criterion of the 3D Navier-Stokes equations involving one entry of the velocity gradient tensor, *Arch. Ration. Mech. Anal.*, **202** (2011), 919–932.

[4] J. Y. Chemin and P. Zhang, On the critical one component regularity for 3-D Navier-Stokes system, *Ann. Sci. Éc. Norm. Supér.*, **49** (2016), 131–167.

[5] L. Escauriaza, G. A. Seregin and V. Šverák, $L^{3,\infty}$ solutions to the Navier-Stokes equations and backward uniqueness, *Russ. Math. Surveys*, **58** (2003), 211–250.

[6] Y. Giga, Solutions for semilinear parabolic equations in $L^p$ and regularity of weak solutions of the Navier-Stokes system, *J. Differential Equations*, **62** (1986), 186–212.

[7] S. Gustafson, K. Kang and T.-P. Tsai, Interior regularity criteria for suitable weak solutions of the Navier-Stokes equations, *Comm. Math. Phys.*, **273** (2007), 161–176.

[8] I. Kukavica, On partial regularity for the Navier-Stokes equations, *Discrete Contin. Dyn. Syst.*, **21** (2008), 717–728.

[9] I. Kukavica and M. Ziane, One component regularity for the Navier-Stokes equation, *Nonlinearity*, **19** (2006), 453–469.

[10] O. A. Ladyzhenskaya and G. A. Seregin, On partial regularity of suitable weak solutions to the three-dimensional Navier-Stokes equations, *J. Math. Fluid Mech.*, **1** (1999), 356–387.

[11] J. Leray, Sur le mouvement d’un liquides visqueux emplissant l’espace, *Acta Math.*, **63** (1934), 193–248.

[12] F. H. Lin, A new proof of the Caffarelli-Kohn-Nirenberg theorem, *Comm. Pure Appl. Math.*, **51** (1998), 241–257.

[13] J. Nečas, M. Říček and V. Šverák, On Leray’s self-similar solutions of the Navier-Stokes equations, *Acta Math.*, **176** (1996), 283–294.

[14] V. Scheffer, Partial regularity of solutions to the Navier-Stokes equations, *Pacific J. Math.*, **66** (1976), 533–562.

[15] V. Scheffer, Hausdorff measure and the Navier-Stokes equations, *Commun. Math. Phys.*, **55** (1977), 97–112.

[16] V. Scheffer, The Navier-Stokes equations on a bounded domain, *Commun. Math. Phys.*, **73** (1980), 1–42.

[17] G. A. Seregin, Estimate of suitable solutions to the Navier-Stokes equations in critical Morrey spaces, *Journal of Mathematical Sciences*, **143** (2007), 2961–2968.

[18] J. Serrin, The initial value problem for the Navier-stokes equations, in *Nonlinear problems(R. E. Langer Ed.)*, *Univ. of Wisconsin Press*, Madison, (1963), 69–98.

[19] M. Struwe, On partial regularity results for the Navier-Stokes equations, *Comm. Pure Appl. Math.*, **41** (1988), 437–458.

[20] G. Tian and Z. Xin, Gradient estimation on Navier-Stokes equations, *Comm. Anal. Geom.*, **7** (1999), 221–257.

[21] A. Vasseur, A new proof of partial regularity of solutions to Navier-Stokes equations, *Nonlinear Differential Equations Appl.*, **14** (2007), 753–785.

[22] W. Wang and Z. Zhang, On the interior regularity criterion and the number of singular points to the Navier-Stokes equations, *J. Anal. Math.*, **123** (2014), 139–170.

[23] Y. Zhou and M. Pokorný, On the regularity of the solutions of the Navier-Stokes equations via one velocity component, *Nonlinearity*, **23** (2010), 1097–1107.

Received July 2017; revised November 2017.

*E-mail address: wendong@dlut.edu.cn*

*E-mail address: lqzhang@math.ac.cn*

*E-mail address: zfzhang@math.pku.edu.cn*