Bispectral analysis: comparison of two windowing functions

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Abstract. Amongst all the normalized forms of bispectrum, the bicoherence is shown to be a very useful diagnostic tool in experimental studies of nonlinear wave interactions in plasma, as it measures the fraction of wave power due to the quadratic wave coupling in a self-excited fluctuation spectrum [1, 2]. In order to avoid spectral leakage, the application of a windowing function is needed during the bicoherence computation. Spectral leakage from statistically dependent components are of crucial importance in the discrimination between coupled and uncoupled modes, as they will introduce in the bicoherence spectrum phase-coupled modes which in reality do not exist. Therefore, the windowing function plays a key role in the bicoherence estimation. In this paper, two windowing methods are compared: the multiplication of the initial signal by the Hanning function and the subtraction of the straight line which links the two extremities of the signal. The influence of these two windowing methods on both the power spectrum and the bicoherence spectrum is showed. Although both methods give precise results, the Hanning function appears to be the more suitable window.

1. Introduction
A time series of any fluctuating physical quantity (including fluctuating plasma density and/or potential) may be, to a first approximation, regarded as a superposition of statistically uncorrelated waves and can be described, in part, by its power spectrum which shows the frequency distribution of power of the fluctuations. The power spectrum is of limited value when various spectral components interact with one another due to some nonlinear or parametric process (nonlinear wave-wave interaction in plasma, for example). In such a case, higher order spectral techniques are necessary to accurately and completely characterize the fluctuating signal, since the nonlinearities result in new spectral components being formed which are phase coherent [1]. The detection of such phase coherence may be carried out with the aid of the bicoherence, the normalized form of the bispectrum (third-order spectrum). This is the reason of the actual spreading interest on the bicoherence spectrum in the Fusion research field. In the next Section (II) we are going to briefly present definitions and computational procedures regarding the bicoherence spectrum. In Section III, we show the influence of two windowing functions in both the Fourier and bicoherence spectrum. Lastly, in Section IV we sum up the main results.

2. Bispectrum and bicoherence
The bispectrum is the generalization of the power spectrum (second order cumulant spectrum) to the third order. The (discrete) power spectrum \( P(k) \), with discrete integer frequency \( k \) can be written either as the discrete Fourier transform (DFT) of the second order cumulant \( c_2(\tau_j) \), where \( \tau_j \) is a discrete lag, or as the expectation of the product of two DFTs \( \hat{X}(\tau) \) whose sum frequency is zero [2]:
where $X^*(k) = X(-k)$ denotes the complex conjugate of $X(k)$. In the same way the bispectrum $B(k,l)$, with discrete integer frequencies $k$ and $l$, can be written either as the Double-DFT of the third order cumulant $c_3(\tau_1,\tau_2)$, or as the expectation of the product of three DFTs whose sum frequency is zero:

$$B(k, l) = D^2DFT\left(c_3(\tau_1, \tau_2)\right) = E\left[X(k)X(l)X^*(k + l)\right]$$

To clarify how the bispectrum can be used to discriminate between nonlinear coupled waves and spontaneously excited waves we are going to make a simple example. We suppose an experimental situation where only three waves at $\omega_1$, $\omega_2$ and $\omega_m$ satisfy the resonance conditions to have nonlinear wave-wave interaction ($\omega_m = \omega_1 + \omega_2$ and $k_3 = k_1 + k_2$) [3]. Then the waves have to satisfy the following relation:

$$X(m) = A_{kl}X(k)X(l)$$

where $m = k + l$ and $A_{kl}$ is the coupling coefficient. The primary waves at $\omega_1$ and $\omega_2$ interact and generate a third wave at their sum (or difference) frequency $\omega_m = \omega_1 + \omega_2$. By writing $X(k) = |X(k)|\exp(i\phi(k))$, it is clear that the phases must satisfy the condition $\phi(m) = \phi(k) + \phi(l)$ for equation (3) and that the phase of the bispectrum at the frequencies $k$ and $l$ is $\phi_B(k,l) = \phi(k) + \phi(l) - \phi(m) = 0$ for equation (2) [2]. Therefore, when three spectral components are nonlinearly coupled to each other, the bispectrum phase will not be random at all, although phases of each wave are randomly changing for each realization. Consequently, when a statistical averaging denoted by the expectation operator in (2) is carried out, the bispectrum will take a value different from zero. On the other hand, if the three waves present at $\omega_1$, $\omega_2$ and $\omega_3$ are spontaneously excited independent waves, each wave may be characterized by statistically independent random phases. When the statistical averaging is carried out, the bispectrum will vanish due to the random phase mixing effect [1].

In order to analyze nonlinear wave couplings, the bicoherence is the most appropriate tool. It is defined as:

$$b^2(k, l) = \frac{|\theta(k,l)|^2}{E[|X(k)X(l)|^2]E[|X(m)|^2]}$$

which is a normalized bispectrum. The bicoherence is often preferred to other normalized bispectrum as it represents the fraction of power at $\omega_m$ due to three-wave coupling and is bounded between 0 and 1 [1].

2.1. Numerical computation of the bicoherence

The bicoherence of deterministic signals can be estimated from a single realization without averaging only if the noise of the measurement is zero [4]. As this situation is rarely, if ever, encountered in practice, some form of averaging is required to obtain estimates with statistical stability. Averaging can be carried out in one of two ways:

- Multiple realisations of the process can be obtained, and estimates averaged over these realisations (ensemble averaging).
- Only one realisation is used and it is divided up and averaged accordingly.

In the second case (which is the more frequent in practice) we can compute correctly the bicoherence spectrum only if the statistical properties obtained from a single record are the same as those of an ensemble average (ergodic measures) [2].

The computational procedures to compute the bicoherence are as follow.

1) Form $M$ sets of data records of length $N$. They will be denoted as $x^{(i)}(l)$, with $i = 1, \ldots, M$ and $l = 1, \ldots, N$ where $x(l) = x(l\Delta t)$. We assume that each data series $x^{(i)}(l)$ is band-limited and that the sampling interval $\Delta t$ is sufficiently small so that the Nyquist frequency $f_N = 1/(2\Delta t)$ is
larger than any spectral component present in \( x^{(i)}(l) \) (if not an antialiasing filter must be applied [5]). Moreover, we assume that the record length \( T = N \Delta t \) is large enough to yield sufficient frequency resolution (the elementary bandwidth \( \Delta f = 1/T \) must be smaller than the frequency gap between two neighbour spectral components \( [f_i - f_{i+1}] \)).

2) Subtract the mean value \( m = 1/N \sum_{l=1}^{N} x^{(i)}(l) \) from each record \( x^{(i)}(l) = x^{(i)}(l) - m \).

3) Apply an appropriate window to each record \( x^{(i)}(l) = x^{(i)}(l) w(l) \) to reduce spectral leakage [6].

4) Compute the DFT \( X^{(i)}(k) \) of each data series:

\[
X^{(i)}(k) = \frac{1}{N} \sum_{l=1}^{N} x^{(i)}(l) \exp \left( -\frac{j2\pi kl}{N} \right) \quad \text{with} \quad k = 1, \ldots, N/2 \quad \text{and} \quad i = 1, \ldots, M. \quad (5)
\]

5) Estimate the bicoherence by:

\[
b^2(k,l) = \frac{\left| \frac{1}{M} \sum_{l=1}^{M} x^{(i)}(k) x^{(j)}(l) x^{(i)}(k+l) \right|^2}{\left( \frac{1}{M} \sum_{l=1}^{M} x^{(i)}(k) x^{(i)}(l) \right) \left( \frac{1}{M} \sum_{l=1}^{M} x^{(i)}(k+l) \right)^2} \quad (6)
\]

It is not necessary to compute the bicoherence spectrum over the entire \((k,l)\)-plane due to Nyquist frequency limit and due to symmetry relations fulfilled by the bispectrum. It is possible to show that all the bicoherence spectrum is contained in the inner triangular region defined by \( 0 \leq l \leq a/2 \) and \( l \leq k \leq a - l \) where \( a = f_N / \Delta f \) [7].

3. Influence of two windowing functions on the bicoherence spectrum

Usually the measured signal is composed of a large number of spectral modes. Therefore, when a nonlinear interaction between two modes (say at \( f_k \) and \( f_l \)) occurs, usually the fraction of power at \( f_{k+l} \) due to this nonlinear interaction is not high, since other spontaneously excited modes are present at that frequency. So the bicoherence will usually take small values.

In general, spectral leakage from statistically independent or random phase components will have a similar effect on the bicoherence spectrum as the addition of white random noise [8]. It will lower the fraction of power that is phase-coupled. Likewise, spectral leakage from statistically dependent components will introduce phase-coupled modes which in reality do not exist. If we add these effects to the background noise of the measurement, which also lowers the fraction of power that is phase-coupled, it appears clear that it would be difficult to use the bicoherence spectrum to discriminate between coupled and uncoupled modes. So the windowing function plays a key role in the bicoherence estimation.

Two windowing methods will be analyzed: The Hanning window and the subtraction of the straight line which links the two extremities of the data series (straight line window).

3.1 Fourier spectra

To study the influence of these two window functions on the Fourier spectrum we use a test signal of record length \( T = N \Delta t = 2\pi \), \( x(l) = \sin(5l\Delta t/4) \) with \( l = 1, \ldots, N \), which has \( x(1) \neq x(N) \). We already know that a sinus signal with \( x(1) = x(N) \) has modes at \( k = \pm 1 \), so we are looking for the leakage around these modes. The Hanning window consists in the multiplication of the sampled signal by the Hanning function:

\[
w_H(l) = \frac{1}{2} \left[ 1 - \cos \left( \frac{2\pi l}{N} \right) \right] = \sin^2 \left( \frac{\pi l}{N} \right) \quad \text{with} \quad l = 1, \ldots, N \quad (7)
\]

The straight line window consists in the subtraction of the straight line which links the two extremities of the sampled signal:

\[
w_S(l) = \left[ \frac{x(N) - x(1)}{N} \right] l + x(1) \quad \text{with} \quad l = 1, \ldots, N \quad (8)
\]

to sampled signal itself.
When these windows are applied to the initial signal, a new windowed signal is obtained with \( x(1) = x(N) \), as we can see from figure 1. Then the DFTs of the windowed signals are computed (\( N = 1024 \)).

![Figure 1](image1.png)

**Figure 1.** Application of the Hanning window (left) and of the straight line window (right) to the signal.

As we can see from figure 2 the spectral leakage has a faster decay when the Hanning windowing is applied. Also, we note that the application of the straight line window introduces an additive spectral leakage at \( k = 0 \) since the mean value of the windowed signal is different from zero. This additive leakage will be removed when, during the bicoherence computation, the subtraction of the mean value of each record is carried out (in this case the windowing operation will be applied before subtracting the mean value).

![Figure 2](image2.png)

**Figure 2.** Power spectrum of the test signal in logarithmic scale when the Hanning window (left) or the straight line window (right) are applied.

3.2 Bicoherence spectra

To study the influence of the two windowing functions on the bicoherence spectrum we generated \( M = 150 \) records of such test signal:

\[
x(t) = \cos(\omega_b t + \theta_b) + \cos(\omega_c t + \theta_c) + \frac{1}{2} \cos(\omega_d t + \theta_d) + \cos(\omega_b t + \theta_b) \cos(\omega_c t + \theta_c) + \eta(t)
\]

where \( f_b/f_N = 0.157, f_c/f_N = 0.268, f_d = f_b + f_c, \eta(t) \) is a small amplitude Gaussian noise (−10 dB) and \( \theta_b, \theta_c \) and \( \theta_d \) are chosen from a set of random numbers which are uniformly distributed over (−\( \pi \), \( \pi \)). Each test signal is composed of \( N = 128 \) data points. The product term in the test signal will
generate the sum and difference frequency waves, which have a phase given by the sum or the
difference of \( \theta_b \) and \( \theta_c \). Thus half of the power at \( f_d = f_b + f_c \) is due to the product interaction of waves
at \( f_b \) and \( f_c \) and the other half is independent from the product interaction. The power at \( f_a = f_c - f_b \) will
be entirely due to the product interaction of waves at \( f_b \) and \( f_c \) except for a small amount of noise
power.

\[ b^2(b,c) = 0.49 \text{ which implies that only half of the power at } f_d \text{ is due to the coupling of} \]
the waves at \( f_b \) and \( f_c \). Furthermore the computed bicoherence at the difference interaction was \( b^2(-b,c) = b^2(a,c) = 0.97 \) which shows that the power at \( f_a \) is entirely due to the interaction of waves at \( f_b \) and \( f_c \).

These peaked values are almost the same whether we use the Hanning window or the straight line
window. The main difference between the two spectra is due to the effect of spectral leakage when the
DFTs are computed. Since the Hanning window reduces markedly the spectral leakage (as we saw
from figure 2), the related bicoherence spectrum has a better peak resolution. On the contrary, the
straight line window does not reduce sufficiently spectral leakage and so the related bicoherence
spectrum has a worst peak resolution, showing peaks with a conic shape which reflects the DFT
spectral leakage. This characteristic will affect negatively the bispectral analysis when the signal is
composed of many different modes. In this case spectral leakages will make it impossible to clearly
distinguish between spontaneously excited waves and nonlinearly coupled waves. Therefore, the
Hanning window is found to be more suitable for bispectral analysis than the straight line one.

Figure 3. Power spectrum in logarithmic scale of the test signal

Figure 3 shows the power spectrum of one record of the test signal. We can clearly see the principal
four modes of the signal but we cannot understand whether there are some coupled modes or not (due
to the lack of phase information). On the other hand, the bicoherence spectrum allows us to have a
clearer picture of wave coupling. Figures 4 and 5 show that the computed bicoherence for the sum
interaction was \( b^2(b,c) = 0.49 \) which implies that only half of the power at \( f_d \) is due to the coupling of
the waves at \( f_b \) and \( f_c \). Furthermore the computed bicoherence at the difference interaction was \( b^2(-b,c) = b^2(a,c) = 0.97 \) which shows that the power at \( f_a \) is entirely due to the interaction of waves at \( f_b \) and \( f_c \).

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**Figure 4.** 2-D view of the bicoherence spectrum when the Hanning window (left) or the straight line window (right) is applied to a signal which involves the quadratic coupling of two waves.

![Bicoherence spectrum](image)

**Figure 5.** 3-D view of the bicoherence spectrum when the Hanning window (left) or the straight line window (right) is applied to a signal which involves the quadratic coupling of two waves.

**4. Conclusion**

To compute the bicoherence spectrum, a window function must be applied to the data series in order to avoid spectral leakage. In general, spectral leakage from statistically independent or random phase components will lower the fraction of power that is phase-coupled [8]. On the other hand, spectral leakage from statistically dependent components will introduce phase-coupled modes which in reality do not exist. If we add these effects to the background noise of the measurement, which also lowers the fraction of phase-coupled power, it is clear that it would be difficult to use the bicoherence spectrum to discriminate between coupled and uncoupled modes. So the windowing function plays a key role in the bicoherence estimation. A comparison between the Hanning window and the so-called straight line window was developed. The analysis of Fourier spectra shows that the application of the Hanning window reduces spectral leakage more than the straight line window. This characteristic is found to have a marked impact on the bicoherence spectrum. When the Hanning window is applied on the test signal, the bicoherence spectrum shows a good peak resolution. On the other hand, when the test signal is windowed by the straight line, the peaks in the bicoherence spectrum have a conic shape, reflecting the spectral leakages in the DFT calculation. Therefore, the straight line window is found not to be suitable for the bicoherence computation.

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