NONUNIFORM MEAN-SQUARE EXPONENTIAL DICHOTOMIES AND MEAN-SQUARE EXPONENTIAL STABILITY

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ABSTRACT. In this paper, the existence conditions of nonuniform mean-square exponential dichotomy (NMS-ED) for a linear stochastic differential equation (SDE) are established. The difference of the conditions for the existence of a nonuniform dichotomy between an SDE and an ordinary differential equation (ODE) is that the first one needs an additional assumption, nonuniform Lyapunov matrix, to guarantee that the linear SDE can be transformed into a decoupled one, while the second does not. Therefore, the first main novelty of our work is that we establish some preliminary results to tackle the stochasticity. This paper is also concerned with the mean-square exponential stability of nonlinear perturbation of a linear SDE under the condition of nonuniform mean-square exponential contraction (NMS-EC). For this purpose, the concept of second-moment regularity coefficient is introduced. This concept is essential in determining the stability of the perturbed equation, and hence we deduce the lower and upper bounds of this coefficient. Our results imply that the lower and upper bounds of the second-moment regularity coefficient can be expressed solely by the drift term of the linear SDE.

1. Introduction

Mean-square dynamical behavior is one of the important concepts to describe the flows produced by SDEs or random differential equations (RDEs). This is due to the fact that in the case of mean-square setting, the dynamical behavior of SDEs and RDEs are essentially deterministic with the stochasticity built into or hidden in the time-dependent state spaces (under specific conditions, there is no difference between the flows generated by the SDEs and RDEs; in fact, the flow of SDEs is conjugate to the flow of RDEs [26]). Over the years, its many properties and corresponding results have been presented by many researchers. For example, Kloeden and Lorenz [27] provided a definition of mean-square random dynamical systems and studied the existence of pullback attractors. In [20, 32, 53], the concept of mean-square almost automorphy for stochastic process was introduced, the existence, uniqueness and asymptotic stability of mean-square almost automorphic solutions of SDEs were established respectively. Using a stochastic version of theta method, Higham [22] combined analytical and numerical techniques to tackle mean-square asymptotic stability for SDEs. Recently, Zhu and Chu [51] presented the numerical methods for a mean-square exponential dichotomy (MS-ED) of a linear SDE and showed that the MS-ED is equivalent to the numerical results for sufficient

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small step sizes under natural conditions. We also refer to [18, 23, 24, 52] for more related results and techniques about this topic.

The concept of MS-ED is extended from the classical notation of exponential dichotomy, which can be traced back to Perron [40] in 1930s. Since then it has become a very important part of the general theory of dynamical systems, particularly in what concerns the study of stable and unstable invariant manifolds, and therefore has attracted much attention during the last few decades. One can see, for example, [25, 30, 38, 42–44] about evolution equations, [15, 31, 39] about functional differential equations, [13, 14, 29, 45] about skew-product flows, and [18, 49, 50, 52, 53] about random systems or stochastic equations. We also refer to the books [12, 16, 35] for details and further references related to exponential dichotomies.

However, dynamical systems exhibit various different kinds of dichotomic behaviors, and the notion of classical exponential dichotomy cannot contain all possible dichotomic behaviors, as Barreira and Valls mentioned in [7], “the notion of exponential dichotomy demands considerably from the dynamics and it is of considerable interest to look for more general types of hyperbolic behavior”. In these years, many attempts have been made (see, e.g., [36, 37, 41]) to extend the concept of the classical dichotomies. For more recent works we mention in particular the papers [4–10], which, inspired by the fundamental work of nonuniformly hyperbolic trajectory introduced in [2, 3], extend the concept of exponential dichotomy to the nonuniform ones and investigate some related problems. In fact, exponential dichotomy implies nonuniform exponential dichotomy (see e.g., [7–9]). However, the contrary is not true in general. For example, Barreira and Valls [8] showed that the linear equation

\[ u' = (-a - bt \sin t)u, \quad v' = (a + bt \sin t)v \]

with \( a > b > 0 \) admits a nonuniform exponential dichotomy but does not admit a uniform exponential dichotomy.

As our knowledge, the concept of MS-ED was first introduced by Stanzhyts’kyi [47], in which a sufficient condition has been proved to ensure that a linear SDE admits an MS-ED. Based on the definition of MS-ED, Stanzhyts’kyi and Krenevych [48] proved the existence of a quadratic form of the linear SDE. In [52] the robustness of MS-ED for a linear SDE was established, and Stoica [49] studied stochastic cocycles in Hilbert spaces by using MS-ED. Recently, Doan et al. [18] considered the MS-ED spectrum for random dynamical system.

Now we recall the definition of MS-ED. Consider the following linear \( n \)-dimensional Itô stochastic system

\[ du(t) = A(t)u(t)dt + G(t)u(t)d\omega(t), \quad t \in I, \]  

(1.1)

where \( I \) is either the half line \( \mathbb{R}^+ \) or the whole line \( \mathbb{R} \), and \( A(t) = (A_{ij}(t))_{n \times n} \), \( G(t) = (G_{ij}(t))_{n \times n} \) are continuous functions with real entries, which satisfy

\[ \limsup_{t \to +\infty} \log^+ \|A(t)\| = 0, \quad \limsup_{t \to +\infty} \log^+ \|G(t)\| = 0, \]

(1.2)

with \( \log^+ x = \max\{0, \log x\} \). Eq. (1.1) is said to possess a mean-square exponential dichotomy if there exist linear projections \( P(t) : L^2(\Omega, \mathbb{R}^n) \to L^2(\Omega, \mathbb{R}^n) \) such that

\[ \Phi(t) \Phi^{-1}(s) P(s) = P(t) \Phi(t) \Phi^{-1}(s), \quad \forall \ t, s \in I, \]  

(1.3)
and positive constants $K, \alpha$ such that
\[
\mathbb{E}\|\Phi(t)\Phi^{-1}(s)P(s)\|^2 \leq Ke^{-\alpha(t-s)}, \quad \forall \ (t, s) \in I^2_>,
\]
\[
\mathbb{E}\|\Phi(t)\Phi^{-1}(s)Q(s)\|^2 \leq Ke^{-\alpha(s-t)}, \quad \forall \ (t, s) \in I^2_>.
\]
where $\Phi(t)$ is a fundamental matrix solution of (1.1), and $Q(t) = I - P(t)$ is the complementary projection of $P(t)$ for each $t \in I$. $I^2_> := \{(t, s) \in I^2 : t > s\}$ and $I^2_\leq := \{(t, s) \in I^2 : t \leq s\}$ denote the relations of $s$ and $t$ on $I$. The constants $\alpha$ and $K$ are called the exponent and the bound respectively in the case of deterministic systems [21].

This paper, inspired by both the mean-square dynamical properties and the nonuniform behavior, is to study the NMS-ED and its related problems. Eq. (1.1) is said to possess a nonuniform mean-square exponential dichotomy if there exist linear projections $P(t) : L^2(\Omega, \mathbb{R}^n) \rightarrow L^2(\Omega, \mathbb{R}^n)$ such that (1.3) holds, and positive constants $K, \alpha$ and $\beta \in [0, \alpha)$ such that
\[
\mathbb{E}\|\Phi(t)\Phi^{-1}(s)P(s)\|^2 \leq Ke^{-\alpha(t-s)+\beta s}, \quad \forall \ (t, s) \in I^2_>,
\]
\[
\mathbb{E}\|\Phi(t)\Phi^{-1}(s)Q(s)\|^2 \leq Ke^{-\alpha(s-t)+\beta s}, \quad \forall \ (t, s) \in I^2_>.
\]

For the convenience of statement, in the rest of this paper, we call $\alpha$ the exponent, $K$ the bound, and $\varepsilon$ the nonuniform degree. From the point of dichotomic behavior, the standard growth condition (1.4) on $\Phi$ is replaced by a much weaker condition (1.5) so that the main results can be applied to a larger class of equations. The nonuniformity in (1.5) indicates that the bound of the corresponding solution depends on initial time $s$ (while in the uniform case (1.4) this bound must be chosen independently of $s$). Clearly, if one considers $\beta = 0$ in (1.5), we say that (1.1) admits a (uniform) mean-square exponential dichotomy (1.4). That is to say, a mean-square exponential dichotomy is a particular case of the nonuniform ones. On the contrary, the nonuniform part $e^{\beta s}$ in (1.5) cannot be removed in some cases. For example, let $a > b > 0$ be real parameters,
\[
\begin{cases}
  du = (-a - b \sin t)u(t)dt + \sqrt{2b}\cos t \exp(-at + bt \cos t)d\omega(t) \\
  dv = (a + b \sin t)v(t)dt - \sqrt{2b}\cos t \exp(at - bt \cos t)d\omega(t)
\end{cases}
\]

admits an NMS-ED which is not uniform. See Example 6.1 in [54] for details.

The first aim of this paper is: under which conditions the NMS-ED of (1.1) exists? In the process of establishing the existence conditions of nonuniformity, a significant difference between ODEs and SDEs can be observed, that is, for an ODE $x' = A(t)x$, one can assume that $A(t)$ has the block form $A(t) = \begin{pmatrix} A_1(t) & 0 \\ 0 & A_2(t) \end{pmatrix}$.

The blocks $A_1(t), A_2(t)$ correspond, respectively, to stable and unstable components of $A(t)$, under which the system $x' = A(t)x$ can be proved to have a nonuniform exponential dichotomy (see [3] for details). However, this assumption cannot be used directly for SDE (1.1), since it is unreasonable to assume that $A(t)$ and $G(t)$ in system (1.1) can be decoupled into block forms with the same dimensions. To overcome the difficulty caused by the fact that block forms $A(t)$ and $G(t)$ may have different dimensions, a condition called nonuniform Lyapunov matrix is introduced,
under which (1.1) can be transformed into a new system

\[
\begin{align*}
\dot{v}(t) &= B(t)v(t)dt + v(t)d\omega(t) \\
&= \begin{pmatrix}
B_1(t) & 0 \\
0 & B_2(t)
\end{pmatrix} v(t)dt + \begin{pmatrix}
Id_{n_1 \times n_1} & 0_{n_1 \times n_2} \\
0_{n_2 \times n_1} & Id_{n_2 \times n_2}
\end{pmatrix} v(t)d\omega(t).
\end{align*}
\] (1.6)

Thus the drawback in stochasticity can be overcome since the unit matrix can be seen as a block form.

**Theorem 1.1.** Assume that there is a nonuniform Lyapunov matrix \( S(t) \), which transforms (1.1) into the block form (1.6). Then for sufficiently small \( \varepsilon > 0 \), (1.1) admits an NMS-ED with the exponent

\[
\alpha = \max\{-\chi_k + \varepsilon, \chi_k + 1 + \varepsilon\} > 0,
\]

where the notations \( \chi_k, k = 1, \ldots, r \) are the second-moment Lyapunov exponents given in (2.2).

Theorem 1.1 is on the existence of NMS-ED of system (1.1), which is a generalization of nonuniform dichotomy for ODEs. The proof of Theorem 1.1 is presented in Section 3, which is much more delicate than that of previous works for ODEs (see [8]). In fact, a linear SDE which is nonuniformly kinematically similar to (1.1) is constructed by nonuniform Lyapunov matrix, whereby several results are needed before the proof of Theorem 1.1.

Next we consider a nonlinear SDE

\[
\dot{u}(t) = (A(t)u(t) + f(t, u(t)))dt + (G(t)u(t) + h(t, u(t)))d\omega(t), \quad t \in I,
\] (1.7)

which is a perturbation of (1.1). The trivial solution of (1.1) is said to be mean-square exponentially stable (or second-moment exponentially stable) if there exist positive constants \( C, \chi \) such that

\[
E\|x(t)\|^2 \leq C\|x_0\|^2 e^{-\chi(t-t_0)}, \quad \forall t \geq t_0
\]

for all \( x_0 \in \mathbb{R}^n \). It is well-known that mean-square exponential stability is a special case of \( p \)th moment exponential stability. This stability is one of the most effective tools (for example, stability in probability, moment stability and almost sure stability) to describe the stochastic stability (see, e.g., [1, 33, 34] for details), and mean-square exponential stability for SDE can be seen as a natural generalization of the classical concept of exponential stability for ODEs (see e.g., [16]) since the It\( \dot{\text{o}} \) stochastic calculus is a mean-square calculus.

The second aim of this paper is to study the mean-square exponential stability of (1.7) when (1.1) admits an NMS-EC, which is a special case of NMS-ED with \( P(t) = Id \) (see Section 4 for details). Roughly speaking, NMS-EC of (1.1) determines whether or not the trivial solution of the perturbed equation (1.7) is mean-square exponential stability. Example 6.1 in Section 6 indicates that in general the answer is negative. For ODEs, Lyapunov introduced regularity conditions to guarantee exponential stability of the trivial solution of the corresponding perturbed equation (see, e.g., [2, 11]). In order to generalize the Lyapunov stability theorem on the well-established deterministic theory, the notion of regular is stated in the next section. Based on this additional assumption, NMS-EC indeed implies the stability of the trivial solution of (1.7).
**Theorem 1.2.** Assume that Eq. (1.1) admits a nonuniform mean-square exponential contraction, with the second-moment Lyapunov exponent $\chi$ of Eq. (1.1) being regular. Then the trivial solution of the perturbed equation (1.7) is mean-square exponentially stable.

In addition, we draw this conclusion with a weaker hypothesis in the following theorem. Roughly speaking, we obtain the mean-square exponential stability of the perturbed equation (1.7), which does not need the condition that Eq. (1.1) is regular.

**Theorem 1.3.** Assume that Eq. (1.1) admits a nonuniform mean-square exponential contraction with $-q\alpha + \beta < 0$ (see (4.1) and (4.2) in Section 4 for notations and details). Then there exists $\delta > 0$ sufficiently small so that for every initial condition $\xi_0 \in \mathbb{R}^n$ with $\|\xi_0\| \leq \delta$, the solution of Eq. (1.7) starting at $\xi_0$ is mean-square exponential stable which satisfies:

$$E\|u(t)\|^2 \leq \tilde{K}e^{-\alpha t},$$

where $\tilde{K} > 0$ is a constant.

In Section 4, we start by proving this weaker statement. After the proof of Theorem 1.3, the fact that Theorem 1.2 can be obtained directly from Theorem 1.3 is explained in Remark 4.1. In addition, one can find that the second-moment regularity coefficient $\gamma(\chi, \tilde{\chi})$ plays a key role in determining the stability of the perturbed equation (1.7) from the discussion of Remark 4.1. Hence, our aim is to derive the lower and upper bounds of $\gamma(\chi, \tilde{\chi})$ in Section 5.

The paper is organized as follows. The next section introduces some notations and prepares several preliminary results which will be used in later sections. Section 3 proves that (1.1) admits an NMS-ED by using nonuniform Lyapunov matrix $S(t)$. Section 4 devotes to the study of the mean-square exponential stability of (1.1). Section 5 investigates the lower and upper bounds of the second-moment regularity coefficient $\gamma(\chi, \tilde{\chi})$. Finally, an example is given in Section 6, which shows that in general NMS-EC is not enough to guarantee the stability of the perturbed equation of a linear SDE.

### 2. Second-moment Lyapunov exponent

Throughout this paper, we assume that $(\Omega, \mathcal{F}, P)$ is a probability space, $\omega(t) = (\omega_1(t), \ldots, \omega_n(t))^T$ is an $n$-dimensional Brownian motion defined on the space $(\Omega, \mathcal{F}, P)$. $\| \cdot \|$ is used to stand for either the Euclidean vector norm or the matrix norm as appropriate, and $L^2(\Omega, \mathbb{R}^n)$ represents the space of all $\mathbb{R}^n$-valued random variables $x : \Omega \to \mathbb{R}^n$ such that

$$E\|x\|^2 = \int_\Omega \|x\|^2 dP < \infty.$$

For $x \in L^2(\Omega, \mathbb{R}^n)$, let

$$\|x\|_2 = \left( \int_\Omega \|x\|^2 dP \right)^{1/2}.$$

Obviously, $L^2(\Omega, \mathbb{R}^n)$ is a Banach space with the norm $\|x\|_2$. 
Define the second-moment Lyapunov exponent $\chi : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ for a stochastic process $u : \mathbb{R} \to L^2(\Omega, \mathbb{R}^n)$ by the formula

$$\chi(u_0) = \limsup_{t \to +\infty} \frac{1}{t} \log \mathbb{E} \|u(t)\|^2,$$  \hspace{1cm} (2.1)

where $u(t)$ is the solution of (1.1) with the initial point $u(0) = u_0$. The uniqueness of the solution of (1.1) for any given initial value is nicely described in the book by Mao [34, Theorem 2.1, p. 93]. Thus it follows from the abstract theory of Lyapunov exponents (see e.g., [2] for a detailed exposition) that the function $\chi$ takes at most $r \leq n$ distinct values on $\mathbb{R}^n \setminus \{0\}$, say

$$-\infty \leq \chi_1 < \cdots < \chi_k < 0 \leq \chi_{k+1} < \cdots < \chi_r.$$  \hspace{1cm} (2.2)

Let $\Phi(t)$ be a fundamental matrix solution of (1.1). By [34, Theorem 3.2.4], $\Phi(t)$ is invertible with probability 1 in $I$. To introduce the notion of regularity for SDEs, we need the following lemma, which illustrates that the existence of the fundamental matrix solution of the adjoint equation of (1.1).

**Lemma 2.1.** (see [28, Theorem 2.3.1]) Let $\Phi(t)$ be a fundamental matrix solution of (1.1). Then $\Phi^{-1}(t)$ is a fundamental matrix solution of the following stochastic differential equation

$$d\hat{u}(t) = \hat{u}(t)[-A(t) + G^2(t)]dt - \hat{u}(t)G(t)d\omega(t), \hspace{1cm} t \in I.$$  \hspace{1cm} (2.3)

In fact, Lemma 2.1 can be verified by using Itô product rule:

$$d(\Phi\Phi^{-1}) = d\Phi\Phi^{-1} + \Phi d\Phi^{-1} + d\Phi d\Phi^{-1} = d1 = 0.$$

Clearly, $\Phi^{-1}(t)$ is a fundamental matrix solution of the following SDE

$$d\hat{u}(t) = (-A(t) + G^2(t))^T \hat{u}(t)dt - G^T(t)\hat{u}(t)d\omega(t)$$  \hspace{1cm} (2.4)

due to (2.3), where $(-A(t) + G^2(t))^T$ and $G^T(t)$ denote the transpose of $-A(t) + G^2(t)$ and $G(t)$ respectively. For (2.4), consider the associated second-moment Lyapunov exponent $\tilde{\chi} : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ defined by

$$\tilde{\chi}(\hat{u}_0) = \limsup_{t \to +\infty} \frac{1}{t} \log \mathbb{E} \|\hat{u}(t)\|^2,$$  \hspace{1cm} (2.5)

where $\hat{u}(t)$ is the solution of (2.4) with the initial value $\hat{u}(0) = \hat{u}_0$. Again it follows from the abstract theory of Lyapunov exponents that $\tilde{\chi}$ can take at most $s \leq n$ distinct values on $\mathbb{R}^n \setminus \{0\}$, say $-\infty \leq \tilde{\chi}_1 < \cdots < \tilde{\chi}_s$.

Now define the second-moment regularity coefficient of $\chi$ and $\tilde{\chi}$ by

$$\gamma(\chi, \tilde{\chi}) = \min \max \{\chi(u_i) + \tilde{\chi}(\hat{u}_i) : 1 \leq i \leq n\},$$  \hspace{1cm} (2.6)

where the minimum is taken over all bases $u_1, \ldots, u_n$ and $\hat{u}_1, \ldots, \hat{u}_n$ of $\mathbb{R}^n$ such that $\langle u_i, \hat{u}_j \rangle = \delta_{ij}$ for each $i$ and $j$ (here $\delta_{ij}$ is the Kronecker symbol). We say that a basis $(u_1, \ldots, u_n)$ is dual to a basis $(\hat{u}_1, \ldots, \hat{u}_n)$ if $\langle u_i, \hat{u}_j \rangle = \delta_{ij}$ for each $i$ and $j$. The second-moment Lyapunov exponents $\chi$ and $\tilde{\chi}$ are dual, and we write $\chi \sim \tilde{\chi}$ if for any dual bases $(u_1, \ldots, u_n)$ and $(\hat{u}_1, \ldots, \hat{u}_n)$, and every $1 \leq i \leq n$, we have

$$\chi(u_i) + \tilde{\chi}(\hat{u}_i) \geq 0.$$  

In addition, the second-moment Lyapunov exponent $\chi$ is called regular if $\chi \sim \tilde{\chi}$ and $\gamma(\chi, \tilde{\chi}) = 0$. 


Now we illustrate that the exponents $\chi$ associated with (1.1) and $\bar{\chi}$ associated with (2.4) are dual. For this purpose, let $u(t)$ be a solution of (1.1), and $\bar{u}(t)$ be a solution of (2.4). Obviously, $u(t) = \Phi(t)u_0$, and $\bar{u}(t) = \Phi^{-T}(t)\bar{u}_0$. Thus, for every $t \in I$, we have

$$\langle u(t), \bar{u}(t) \rangle = \langle \Phi(t)u_0 \rangle^T(\Phi^{-T}(t)\bar{u}_0) = u_0^T\bar{u}_0 = \langle u_0, \bar{u}_0 \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^n$. Hence

$$\langle u(t), \bar{u}(t) \rangle = \langle u(0), \bar{u}(0) \rangle$$

for any $t \in I$. (1.1) and (2.4) can be called dual due to the fact that (2.7) holds. Now choose dual spaces $(u_1, \ldots, u_n)$ and $(\bar{u}_1, \ldots, \bar{u}_n)$ of $\mathbb{R}^n$. Let $u_i(t)$ be the unique solution of (1.1) with $u_i(0) = u_i$, and $\bar{u}_i(t)$ be the unique solution of (2.4) with $\bar{u}_i(0) = \bar{u}_i$. With the help of Hölder’s inequality, we have

$$\mathbb{E}\|u_i(t)\|^2 \cdot \mathbb{E}\|\bar{u}_i(t)\|^2 \geq 1$$

for every $t \geq 0$, and hence, $\chi(u_i) + \bar{\chi}(\bar{u}_i) \geq 0$ for every $i$. Thus, $\gamma(\chi, \bar{\chi}) \geq 0$ follows immediately from the analysis above.

3. NONUNIFORM MEAN-SQUARE EXPONENTIAL DICHOTOMY

In Section 1 we introduce the notion of NMS-ED for SDEs, which extends the concept of (uniform) MS-ED, and allows us to detect and formulate “random” versions of nonuniform behavior for SDEs. In this section, we will show that (1.1) admits an NMS-ED, if there is a nonuniform Lyapunov matrix $S(t)$, which transforms (1.1) into a new system with block form.

For the convenience of later discussion, we first derive an equivalent definition of the NMS-ED of (1.1).

**Lemma 3.1.** The projector of (1.1) can be chosen as

$$\tilde{P} = \left( \begin{array}{cc} I_{n_1 \times n_1} & 0_{n_1 \times n_2} \\ 0_{n_2 \times n_1} & 0_{n_2 \times n_2} \end{array} \right)$$

with $n_1 = \dim \text{im} \tilde{P}$ and $n_2 = \dim \ker \tilde{P}$ such that $\tilde{P} = \Phi^{-1}(t)P(t)\Phi(t)$ hold for all $t \in I$. Thus the inequalities (1.5) can be rewritten as

$$\mathbb{E}\|\Phi(t)\tilde{P}\Phi^{-1}(s)\|^2 \leq Ke^{-\alpha(t-s)+\beta s}, \quad \forall \ (t, s) \in I_2^2,$$

$$\mathbb{E}\|\Phi(t)\tilde{Q}\Phi^{-1}(s)\|^2 \leq Ke^{-\alpha(s-t)+\beta s}, \quad \forall \ (t, s) \in I_2^2,$$

(3.1)

where $\tilde{Q} = \text{Id} - \tilde{P}$.

**Proof.** Let $P(t) = \Phi(t)\tilde{P}\Phi^{-1}(t)$ for any $t \in I$. Then

$$\mathbb{E}\|\Phi(t)\tilde{P}\Phi^{-1}(s)P(s)\|^2 = \mathbb{E}\|\Phi(t)\tilde{P}\Phi^{-1}(s)\Phi(s)\|^2 = \mathbb{E}\|\Phi(t)\tilde{P}\Phi^{-1}(s)\|^2.$$

Obviously, (3.1) follows immediately from (1.5).

Conversely, it follows from (1.3) that

$$P(t) = P(t)\Phi(t)\Phi^{-1}(s)P(s)\Phi^{-1}(t)$$

$$= \Phi(t)\Phi^{-1}(s)P(s)\Phi(s)\Phi^{-1}(t)$$

for every $t \in I$. Therefore, $\tilde{P}$ is a projector.
for any $t, s \in I$. Then we have
\[ \Phi^{-1}(t)P(t)\Phi(t) = \Phi^{-1}(s)P(s)\Phi(s) \]
for all $t, s \in I$. Denote $\tilde{P} := \Phi^{-1}(t)P(t)\Phi(t)$. Thus ($\text{3.1}$) follows immediately from ($\text{1.5}$) that
\[ E\|\Phi(t)\tilde{P}\Phi^{-1}(s)\|^2 = E\|\Phi(t)\Phi^{-1}(s)P(s)\Phi(s)\Phi^{-1}(s)\|^2 = E\|\Phi(t)\Phi^{-1}(s)P(s)\|^2. \]
Similarly to ($\text{3.2}$), one can prove that
\[ E\|\Phi(t)\tilde{Q}\Phi^{-1}(s)\|^2 = E\|\Phi(t)\Phi^{-1}(s)Q(s)\|^2. \]
In addition,
\[ P(t)\Phi(t)\Phi^{-1}(s) = \Phi(t)\tilde{P}\Phi^{-1}(s) = \Phi(t)\Phi^{-1}(s)P(s), \]
and this completes the proof. 

For $x' = A(t)x$, Barreira and Valls [8] introduced and investigated the nonuniform behavior of $x' = A(t)x$ with the assumption that $A(t)$ has the following block form
\[ A(t) = \begin{pmatrix} A_1(t) & 0 \\ 0 & A_2(t) \end{pmatrix}. \]
But for ($\text{1.1}$), it is unreasonable to assume that $A(t)$ and $G(t)$ in system ($\text{1.1}$) can be decoupled into block forms with the same dimensions. In order to overcome the obstacle caused by the drift term $A(t)$ and the diffusion term $G(t)$ in ($\text{1.1}$), we construct a linear SDE which is kinematically similar to ($\text{1.1}$). For this purpose we establish several auxiliary results.

Consider a linear SDE
\[ dv(t) = B(t)v(t)dt + v(t)d\omega(t) \tag{3.3} \]
with continuous function $B : I \to \mathbb{R}^{n \times n}$. Eq. ($\text{1.1}$) is said to be kinematically similar to Eq. ($\text{3.3}$) if there exists a stochastic process $S(t) = (S_{ij}(t))_{n \times n}$ with
\[ \sup_{t \in I} \|S(t)\|_2 < \infty \quad \text{and} \quad \sup_{t \in I} \|S^{-1}(t)\|_2 < \infty, \]
which satisfies the stochastic differential equation
\[ dS(t) = (A(t)S(t) - S(t)B(t) + S(t) - G(t)S(t))dt + (G(t)S(t) - S(t))d\omega(t). \tag{3.4} \]
The change of variables $u(t) = S(t)v(t)$ then transforms ($\text{1.1}$) into ($\text{3.3}$). This technique is similar to the one used in ODEs (See e.g., [16, p. 38] for a detailed exposition).

**Lemma 3.2.** Let $P : I \to \mathbb{R}^{n \times n}$ be a symmetric projection, and let $\Phi(t)$ be an invertible random matrix for any $t \in I$. The mapping
\[ \tilde{R} : I \to \mathbb{R}^{n \times n}, \quad t \to P\Phi^T(t)\Phi(t)P + (\text{Id} - P)\Phi^T(t)\Phi(t)(\text{Id} - P) \]
is a positive definite, symmetric matrix for every $t \in I$. Moreover, there exists a unique $R(t), t \in I$ with
\[ R^2(t) = \tilde{R}(t), \quad PR(t) = R(t)P. \tag{3.5} \]
In addition, if we put
\[ S : I \to \mathbb{R}^{n \times n}, \quad t \to \Phi(t)R^{-1}(t), \]
then $S(t)$ is an invertible random matrix, which satisfies
\[ S(t)P^{-1}(t) = \Phi(t)P\Phi^{-1}(t) \]
and
\[ \|S(t)\|^2 = \mathbb{E}\|S(t)\|^2 \leq 2, \]
\[ \|S^{-1}(t)\|^2 = \mathbb{E}\|S^{-1}(t)\|^2 \leq \mathbb{E}\|\Phi(t)P\Phi^{-1}(t)\|^2 + \mathbb{E}\|\Phi(t)(Id - P)\Phi^{-1}(t)\|^2. \]

The above lemma is a stochastic version of estimation of kinematical similarity for ODE, which can be proved following the same way as in [16, Lemma 1, p. 39], so we omit the proof. One can also see Lemma A.5 in [46] for details.

In the setting of classical exponential dichotomies, $S^{-1}(t)$ is bounded, which follows from the properties $\|\Phi(t)P\Phi^{-1}(t)\| < +\infty$ and $\|\Phi(t)(Id - P)\Phi^{-1}(t)\| < +\infty$ (see Definition 2.1 in [46] for details). However, in the setting of NMS-ED, $S^{-1}(t)$ can be unbounded on $I$ in the nonuniform mean-square sense due to (3.1), i.e.,
\[ \mathbb{E}\|\Phi(t)P\Phi^{-1}(t)\|^2 \leq Ke^{\beta t} \quad \text{and} \quad \mathbb{E}\|\Phi(t)(Id - P)\Phi^{-1}(t)\|^2 \leq Ke^{\beta t}. \]

Based on this observation, we, unlike the previous work in [17, 46], need to consider the new notion of nonuniform kinematical similarity to overcome the difficulties arising from the lack of boundedness condition.

**Definition 3.1.** Suppose that $S(t)$ is a stochastic process. $S(t)$ is said to be a nonuniform Lyapunov matrix if there exists a constant $M > 0$ such that
\[ \|S(t)\|^2 \leq Me^{\beta t}, \quad \text{and} \quad \|S^{-1}(t)\|^2 \leq Me^{\beta t}, \quad \text{for all} \quad t \in I. \]
(3.6)

(1.1) and (3.3) are said to be nonuniformly kinematically similar if there exists a $\mathbb{R}^{n \times n}$-valued invertible stochastic process $S(t)$ satisfying (3.4).

The following lemma illustrates the construction of (3.4). For the corresponding deterministic version of Lemma 3.3, we refer to [17, Lemma 2.1, p. 158].

**Lemma 3.3.** For a stochastic process $S(t)$, the following statements are equivalent:

1. The systems (1.1) and (3.3) are nonuniformly kinematically similar via $S(t)$ on $I$;
2. Let $\Phi_A(t)$ and $\Phi_B(t)$ denote the fundamental matrix solutions of (1.1) and (3.3) respectively. The identity
   \[ \Phi_A(t)\Phi_A^{-1}(\tau)S(\tau) = S(t)\Phi_B(t)\Phi_B^{-1}(\tau) \]  
   (3.7)
   holds for all $t, \tau \in I$;
3. The stochastic process $S(t)$ solves the SDE (3.4).

**Proof.** First, assume that (1.1) and (3.3) are nonuniformly kinematically similar via $S(t)$ on $I$. Then we obtain from $u(t) = S(t)v(t)$ the relation
\[ \Phi_A(t)\Phi_A^{-1}(\tau)u(\tau) = S(t)\Phi_B(t)\Phi_B^{-1}(\tau)v(\tau). \]
By the arbitrariness of $u(t)$ and the formula $u(\tau) = S(\tau)v(\tau)$, we have
\[ \Phi_A(t)\Phi_A^{-1}(\tau)S(\tau) = S(t)\Phi_B(t)\Phi_B^{-1}(\tau). \]
Second, assume that (3.7) holds for all $t, \tau \in I$. Then we have
\[ \Phi_A(t)\Phi_A^{-1}(0)S(0) = S(t)\Phi_B(t)\Phi_B^{-1}(0). \]
Denote \( \tilde{\Phi}_A(t) = \Phi_A(t)\Phi_A^{-1}(0)S(0) \) and \( \tilde{\Phi}_B(t) = \Phi_B(t)\Phi_B^{-1}(0) \). Hence, the operator \( S(t) \) can be written as:

\[
S(t) = \tilde{\Phi}_A(t)\Phi_B^{-1}(t).
\]

It follows from (1.1), (2.3) and Itô product rule that

\[
dS(t) = d(\tilde{\Phi}_A(t)\Phi_B^{-1}(t))
\]

\[
= d\tilde{\Phi}_A(t)\Phi_B^{-1}(t) + \tilde{\Phi}_A(t)d\Phi_B^{-1}(t) + d\tilde{\Phi}_A(t)d\Phi_B^{-1}(t)
\]

\[
= A(t)S(t)dt + G(t)S(t)d\omega(t) + (S(t) - G(t)S(t))dt + (G(t)S(t) - S(t))d\omega(t),
\]

which means that Statement (3) holds true.

Finally, assuming that \( S(t) \) is a fundamental matrix solution of SDE (3.4), it follows from Itô product rule that

\[
d(S(t)v(t)) = dS(t)v(t) + S(t)dv(t) + dS(t)dv(t)
\]

\[
= (A(t)S(t) - S(t)B(t) + S(t) - G(t)S(t))v(t)dt + (G(t)S(t) - S(t))v(t)d\omega(t)
\]

\[
+ S(t)B(t)v(t)dt + S(t)v(t)d\omega(t) + (G(t)S(t) - S(t))v(t)dt
\]

\[
= A(t)S(t)v(t)dt + G(t)S(t)v(t)d\omega(t)
\]

\[
= A(t)v(t)dt + G(t)x(t)d\omega(t) = du(t).
\]

This completes the proof of the lemma.

\[
\square
\]

**Lemma 3.4.** Assuming that the systems (1.1) and (3.3) are nonuniformly kinematically similar via \( S(t) \) on \( I \), and that the system (3.3) admits an NMS-ED with the form (3.1) and rank(\( \hat{P} \)) = \( k(0 \leq k \leq n) \), then the system (1.1) also admits an NMS-ED with no change in the projector.

**Proof.** Suppose that (1.1) and (3.3) are nonuniformly kinematically similar via \( S(t) \) on \( I \), and (3.6) holds. Namely, let \( \Phi_A(t) \) be the fundamental matrix solution of (1.1), and \( \Phi_A(t) = S(t)\Phi_B(t) \). It follows from the proof of Lemma 3.5 that \( \Phi_B(t) \) is the fundamental matrix solution of (3.3). Hence, for any \( t \in I \),

\[
\mathbb{E}\|\Phi_A(t)\hat{P}\Phi_A^{-1}(t)\|^2 = \mathbb{E}\|S(t)\Phi_B(t)\hat{P}\Phi_B^{-1}(t)\|^2
\]

\[
\leq \|S(t)\|^2 \mathbb{E}\|\Phi_B(t)\hat{P}\Phi_B^{-1}(t)\|^2 \cdot \|S^{-1}(t)\|^2
\]

\[
\leq KM_2e^{\beta t}e^{-\alpha(t-s) + \beta s}e^{\beta s}
\]

\[
= KM_2e^{-(\alpha + \beta)(t-s) + 3\beta s}, \quad \forall (t,s) \in I^2_\varepsilon. \quad (3.8)
\]

Similarly, one can prove that

\[
\mathbb{E}\|\Phi_A(t)\hat{P}\Phi_A^{-1}(t)\|^2 \leq KM_2e^{(\alpha + \beta)(t-s) + 3\beta s}, \quad \forall (t,s) \in I^2_\varepsilon. \quad (3.9)
\]

It follows from (3.8)-(3.9) that (1.1) admits an NMS-ED due to the fact that \( \beta \in [0, \alpha) \), and there is no change in the projector.

\[
\square
\]

**Lemma 3.5.** Assuming that (1.1) admits an NMS-ED of the form (3.1) with invariant projector \( \hat{P} \neq 0, I_d \), the system (1.1) is nonuniformly kinematically similar to a decoupled system (1.6) with

\[
B_1 : \mathbb{R} \to \mathbb{R}^{n_1 \times n_1}, \quad \text{and} \quad B_2 : \mathbb{R} \to \mathbb{R}^{n_2 \times n_2},
\]

where \( n_1 = \dim \ker \hat{P} \), and \( n_2 = \dim \ker \hat{P} \).
Proof. Let \( \Phi_A(t) \) and \( \Phi_B(t) \) be the fundamental matrix solutions of (1.1) and (3.3) respectively. Since (1.1) admits an NMS-ED of the form (3.1) with invariant projector \( \tilde{P} \neq 0, \text{Id} \), by Lemma 3.1, we can choose a fundamental matrix solution \( \Phi_A(t) \) and the projector \( \tilde{P} = \begin{pmatrix} Id_{n_1 \times n_1} & 0 \\ 0 & 0 \end{pmatrix} (n_1 = \dim \text{im} \tilde{P}) \) such that (3.1) holds. For the given fundamental matrix solution \( \Phi_A(t) \), it follows from Lemma 3.2 that there exists an invertible random matrix \( S(t) = \Phi_A(t) \Phi^{-1}_B(t) \) such that

\[
\|S(t)\|_2^2 = \mathbb{E}\|S(t)\|^2 \leq 2,
\]

\[
\|S^{-1}(t)\|_2^2 = \mathbb{E}\|S^{-1}(t)\|^2 \leq \mathbb{E}\|\Phi_A(t)\tilde{P}\Phi^{-1}_A(t)\|^2 + \mathbb{E}\|\Phi_A(t)(\text{Id} - \tilde{P})\Phi^{-1}_A(t)\|^2 \leq 2Ke^{\beta t}.
\]

Let \( M = \max\{2, 2K\} \), and we have

\[
\|S(t)\|_2^2 \leq Me^{\beta t}, \quad \text{and} \quad \|S^{-1}(t)\|_2^2 \leq Me^{\beta t}, \quad \text{for all} \ t \in I,
\]

which implies that \( S(t) \) is a nonuniform Lyapunov matrix. Now we show that \( B(t) \) has the block diagonal form of (1.6). By (3.5), \( \Phi_B(t) \) commutes with matrix \( \tilde{P} \) for every \( t \in I \), i.e.,

\[
\tilde{P}\Phi_B(t) = \Phi_B(t)\tilde{P}.
\]

In addition,

\[
d(\Phi_B(t)\tilde{P}) = B(t)\Phi_B(t)\tilde{P}dt + \Phi_B(t)\tilde{P}d\omega(t)
\]

since \( \Phi_B(t) \) is the fundamental matrix solution of (3.3). By Itô product rule, we have

\[
d(\tilde{P}\Phi_B(t)) = \tilde{P}d\Phi_B(t) = \tilde{P}B(t)\Phi_B(t)dt + \tilde{P}\Phi_B(t)d\omega(t).
\]

Taking the identity (3.10) into (3.11), and comparing with (3.12), we have

\[
\tilde{P}B(t) = B(t)\tilde{P}
\]

for every \( t \in I \). Now we decompose \( B : I \to \mathbb{R}^{n \times n} \) into four functions

\[
B_1 : I \to \mathbb{R}^{n_1 \times n_1}, \quad B_2 : I \to \mathbb{R}^{n_2 \times n_2},
\]

\[
B_3 : I \to \mathbb{R}^{n_1 \times n_2}, \quad B_4 : I \to \mathbb{R}^{n_2 \times n_1},
\]

with

\[
B(t) = \begin{pmatrix} B_1(t) & B_3(t) \\ B_2(t) & B_4(t) \end{pmatrix}.
\]

Identity (3.13) implies that

\[
\begin{pmatrix} B_1(t) & B_3(t) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} B_1(t) & 0 \\ B_4(t) & 0 \end{pmatrix} \quad \text{for} \ t \in I.
\]

So \( B_3(t) \equiv 0 \) and \( B_4(t) \equiv 0 \). Therefore, we get the block diagonal form

\[
B(t) = \begin{pmatrix} B_1(t) & 0 \\ 0 & B_2(t) \end{pmatrix} \quad \text{for} \ t \in I,
\]

and the proof is complete. \( \square \)

Now we can prove Theorem 1.1.
Proof of Theorem 1.1. It suffices to prove that (1.6) admits an NMS-ED due to Lemma 3.4 and Lemma 3.5. From now on we consider (1.6) with initial value $v(0) = v_0 \in \mathbb{R}^n$. Let $\Phi_1(t)$ be a fundamental matrix solution of the equation

$$dx(t) = B_1(t)x(t)dt + x(t)d\omega(t),$$

and denote by $x_1(t), \ldots, x_{n_1}(t)$ the columns of $\Phi_1(t)$. Thus it follows from (2.4) that $\Psi_1(t) := (\Phi_1^{-1}(t))^T$ is a fundamental matrix solution of the equation

$$dy(t) = [-B_1(t) + Id]^T y(t) dt - y(t) d\omega(t).$$

Also let $y_1(t), \ldots, y_{n_1}(t)$ be the columns of $\Psi_1(t)$. Setting

$$a_j = \chi(x_j(0)) \quad \text{and} \quad b_j = \tilde{\chi}(y_j(0))$$

for each $j = 1, \ldots, n_1$, where $\chi$ and $\tilde{\chi}$ are the second-moment Lyapunov exponents defined as in (2.1) and (2.5) respectively, choosing $\varepsilon > 0$ sufficiently small, there is a constant $k_1 > 1$ such that for each $j = 1, \ldots, n_1$ and $t \in I$,

$$E\|x_j(t)\|^2 \leq k_1 e^{(a_j + \varepsilon)t} \quad \text{and} \quad E\|y_j(t)\|^2 \leq k_1 e^{(b_j + \varepsilon)t}. \quad (3.14)$$

For every $i$ and $j$, $(x_i(t), y_j(t)) = \delta_{ij}$ follows directly from the identity $\Psi_1(t)\Phi_1(t) = Id_{n_1 \times n_1}$. In view of (2.6), we can assume

$$\max\{a_j + b_j : j = 1, \ldots, n_1\} = \gamma_1,$$

since the Lyapunov exponents $\chi$ and $\tilde{\chi}$ can only take a finite number of values and the matrix $\Phi_1(t)$ can be chosen repeatedly until we find the minimum value. Hence the elements of the matrix $\Phi_1(t)\Psi_1^T(s) = \Phi_1(t)\Phi_1^{-1}(s)$ are

$$u_{ik}(t, s) = \sum_{j=1}^{n_1} x_{ij}(t)y_{kj}(s) \quad \forall (t, s) \in I_2^2,$$

where $x_{ij}(t)$ is the $i$th coordinate of $x_j(t)$, and $y_{kj}(s)$ is the $k$th coordinate of $y_j(s)$. Therefore,

$$|u_{ik}(t, s)|^2 \leq n_1 \sum_{j=1}^{n_1} |x_{ij}(t)|^2 \cdot |y_{kj}(s)|^2 \leq n_1 \sum_{j=1}^{n_1} ||x_j(t)||^2 \cdot ||y_j(s)||^2.$$

It follows from (3.14) that

$$E|u_{ik}(t, s)|^2 \leq n_1 E\left(\sum_{j=1}^{n_1} ||x_j(t)||^2 \cdot ||y_j(s)||^2\right) \leq n_1 \sum_{j=1}^{n_1} \left(E||x_j(t)||^2 \cdot E||y_j(s)||^2\right) \leq n_1 k_1^2 \sum_{j=1}^{n_1} e^{(a_j + \varepsilon)t + (b_j + \varepsilon)s} \leq n_1 k_1^2 \sum_{j=1}^{n_1} e^{(a_j + \varepsilon)(t-s) + (a_j + b_j + 2\varepsilon)s} \leq n_1 k_1^2 e^{(a_j + \varepsilon)(t-s) + (\gamma_1 + 2\varepsilon)s}, \quad \forall (t, s) \in I_2^2. \quad (3.15)$$
Taking \( v = \sum_{k=1}^{n_1} \alpha_k e_k \) with \( \|v\|^2 = \sum_{k=1}^{n_1} \alpha_k^2 = 1 \), where \( e_1, \ldots, e_{n_1} \) is the canonical basis of \( \mathbb{R}^{n_1} \), we have

\[
\|\Phi_1(t)\Phi_1^{-1}(s)v\|^2 = \left\| \sum_{i=1}^{n_1} \sum_{k=1}^{n_1} \alpha_k u_{ik}(t, s)e_i \right\|^2
\]

\[
= \sum_{i=1}^{n_1} \left( \sum_{k=1}^{n_1} \alpha_k u_{ik}(t, s) \right)^2 \leq \sum_{i=1}^{n_1} \left( \sum_{k=1}^{n_1} \alpha_k^2 \sum_{k=1}^{n_1} u_{ik}(t, s)^2 \right)
\]

\[
= \sum_{i=1}^{n_1} \sum_{k=1}^{n_1} u_{ik}(t, s)^2. \quad (3.16)
\]

Therefore, let \( K_1 = n_1^4 k_2^2 \), take (3.15) into (3.16), and we have

\[
\mathbb{E}\|\Phi_1(t)\Phi_1^{-1}(s)\|^2 \leq \mathbb{E} \left( \sum_{i=1}^{n_1} \sum_{k=1}^{n_1} u_{ik}(t, s)^2 \right)
\]

\[
\leq K_1 e^{(\gamma_2 + 2\varepsilon)(t-s) + (\gamma_1 + 2\varepsilon)s}, \quad \forall (t, s) \in I_2^2. \quad (3.17)
\]

Similarly, consider the matrix \( \Phi_2(t)\Phi_2^{-1}(s) \), where \( \Phi_2(t) \) is a fundamental matrix solution of the equation

\[
dz(t) = B_2(t)z(t)dt + z(t)d\omega(t),
\]

and \( \Psi_2(t) := (\Phi_2^{-1}(t))^T \) is a fundamental matrix solution of the equation

\[
dw(t) = [-B_2(t) + Id]^T w(t)dt - w(t)d\omega(t).
\]

Let now \( z_1(t), \ldots, z_{n_2}(t) \) be the columns of \( \Phi_2(t) \), and \( w_1(t), \ldots, w_{n_2}(t) \) the columns of \( \Psi_2(t) \), and set

\[
\tilde{a}_j = \chi(z_j(0)) \quad \text{and} \quad \tilde{b}_j = \tilde{\chi}(w_j(0))
\]

for each \( j = 1, \ldots, n_2 \), where \( \chi \) and \( \tilde{\chi} \) are the second-moment Lyapunov exponents defined as in (2.14) and (2.5) respectively. Choosing \( \varepsilon > 0 \) sufficiently small, there is a constant \( k_2 > 1 \) such that for each \( j = 1, \ldots, n_2 \) and \( t \in I \),

\[
\mathbb{E}\|z_j(t)\|^2 \leq k_2 e^{(\tilde{a}_j + \varepsilon)t} \quad \text{and} \quad \mathbb{E}\|w_j(t)\|^2 \leq k_2 e^{(\tilde{b}_j + \varepsilon)t}.
\]

For every \( i \) and \( j \), \( \langle z_i(t), w_j(t) \rangle = \delta_{ij} \) follows directly from the identity \( \Psi_2^T(t)\Phi_2(t) = Id_{n_2 \times n_2} \). In view of (2.6), we can assume

\[
\max\{\tilde{a}_j + \tilde{b}_j : j = 1, \ldots, n_1 \} = \gamma_2,
\]

since the Lyapunov exponents \( \chi \) and \( \tilde{\chi} \) can only take a finite number of values and the matrix \( \Phi_2(t) \) can be chosen repeatedly until we find the minimum value. Hence the elements of the matrix \( \Phi_2(t)\Psi_2^T(s) = \Phi_2(t)\Phi_2^{-1}(s) \) are

\[
v_{ik}(t, s) = \sum_{j=1}^{n_2} z_{ij}(t)w_{kj}(s) \quad \forall (t, s) \in I_2^2,
\]

where \( z_{ij}(t) \) is the \( i \)th coordinate of \( z_j(t) \), and \( w_{kj}(s) \) is the \( k \)th coordinate of \( w_j(s) \). Therefore,

\[
|v_{ik}(t, s)|^2 \leq n_2 \sum_{j=1}^{n_2} |z_{ij}(t)|^2 \cdot |w_{kj}(s)|^2 \leq n_2 \sum_{j=1}^{n_2} \| z_j(t) \|^2 \cdot \| w_j(s) \|^2.
\]
Thus for all \((t, s) \in I_2^2\), we have
\[
E|u_k(t, s)|^2 \leq n_2^2k_2^2e^{-(\chi_k + \varepsilon)(s-t) + (\gamma_2 + 2\varepsilon)s}.
\]
Writing \(K_2 = n_2^2k_2^2\), proceeding in a similar manner to that in (3.16)-(3.17), we obtain
\[
E\|\Phi_2(t)\Phi_2^{-1}(s)\|_2^2 \leq K_2e^{-(\chi_k + \varepsilon)(s-t) + (\gamma_2 + 2\varepsilon)s}.
\]
Therefore, we complete the proof of the theorem.

4. Stability of nonuniform mean-square exponential contraction

We consider in this section the problems of mean-square exponential stability under the condition of NMS-EC. Eq. (1.1) is said to admit a \textit{nonuniform mean-square exponential contraction} if for some constants \(K, \alpha > 0\) and \(\beta \in [0, \alpha)\) such that
\[
E\|\Phi(t)\Phi^{-1}(s)\|_2^2 \leq Ke^{-\alpha(t-s) + \beta|s|}, \quad \forall (t, s) \in I_2^2.
\]
Clearly, this statement is a particular case of NMS-ED with projection \(P(t) = Id\) for every \(t \in I\). Throughout this section we assume that \(f, h : \mathbb{R}^+ \times L^2(\Omega, \mathbb{R}^n) \to L^2(\Omega, \mathbb{R}^n)\) in (1.7) are continuous functions such that
\[
f(t, 0) = h(t, 0) = 0, \quad \forall t \geq 0,
\]
and for any \(u, v \in L^2(\Omega, \mathbb{R}^n)\), there exist some constants \(c > 0\) and \(q > 1\) such that
\[
E\|f(t, u) - f(t, v)\|^2 \leq cE\|u - v\|^2(E\|u\|^2 + E\|v\|^2)^q,
\]
for every \(t \geq 0\). Here \(a \lor b\) means the maximum of \(a\) and \(b\). The inequality in (4.2) means that the perturbation in mean-square is small in the neighborhood of zero.

The following is the proof of stability result for (1.7).

\textbf{Proof of Theorem 1.3.} Considering the space
\[
L_c := \{u : t \to L^2(\Omega, \mathbb{R}^n) : u\text{ is continuous and }\|u\|_c \leq r\}
\]
with the norm
\[
\|u\|_c = \sup \left\{ (E\|u(t)\|^2)\frac{1}{2} e^{\frac{2}{3} t} : t \geq 0 \right\},
\]
clearly, \((L_c, \|\cdot\|_c)\) is a Banach spaces. In order to state our result, we need the following lemma.

\textbf{Lemma 4.1.} For any given initial value \(\xi_0 \in \mathbb{R}^n\), the solution of Eq. (1.7) can be expressed as
\[
u(t) = \Phi(t)\Phi^{-1}(s)\xi_0 + \int_s^t \Phi^{-1}(\tau)h(\tau, u(\tau))d\omega(\tau)
\]
\[
+ \int_s^t \Phi^{-1}(\tau)(f(\tau, u(\tau)) - G(\tau)h(\tau, u(\tau)))d\tau.
\]
where \(\Phi(t)\) is the fundamental matrix solution of (1.1) with \(u(s) = \xi_0\).
Proof. Set
\[
\xi(t) = \Phi^{-1}(s)\xi_0 + \int_s^t \Phi^{-1}(\tau)h(\tau, u(\tau))d\omega(\tau)
+ \int_s^t \Phi^{-1}(\tau)(f(\tau, u(\tau)) - G(\tau)h(\tau, u(\tau)))d\tau.
\]
Clearly, \(u(t) = \Phi(t)\xi(t),\) and one can easily verify that \(\xi(t)\) satisfies the differential
\[
d\xi(t) = \Phi^{-1}(t)(f(t, u(t)) - G(t)h(t, u(t)))dt + \Phi^{-1}(t)h(t, u(t))d\omega(t),\]
with \(\xi(0) = \xi_0.\) Since \(\Phi(t)\) is a fundamental matrix solution of \((1.1),\) it follows from Itô product rule that
\[
du(t) = d\Phi(t)\xi(t) + \Phi(t)du(t) + G(t)\Phi(t)\Phi^{-1}(t)h(t, u(t))dt
+ h(t, u(t))d\omega(t) + G(t)h(t, u(t))d\omega(t),
\]
which means that \(u(t) = \Phi(t)\xi(t)\) is a solution of \((1.7).\) In addition, \(u(s) = \xi_0\) is trivial, and this completes the proof of the lemma.

We proceed with the proof of Theorem 1.3. In order to simplify the presentation, write \(f(t, u(t)) = f(t, u(t)) - G(t)h(t, u(t))\) in the following. Squaring both sides of \((4.3),\) and taking expectations, it follows from the elementary inequality
\[
\left\| \sum_{k=1}^m a_k \right\|^2 \leq m \sum_{k=1}^m \|a_k\|^2 \tag{4.4}
\]
that
\[
E\|u(t)\|^2 \leq 3E\|\Phi(t)\Phi^{-1}(0)\xi_0\|^2 + 3E\left\| \int_0^t \Phi(t)\Phi^{-1}(\tau)h(\tau, u(\tau))d\omega(\tau) \right\|^2
+ 3E\left\| \int_0^t \Phi(t)\Phi^{-1}(\tau)f(\tau, u(\tau))d\tau \right\|^2 \tag{4.5}
\]
We define the operator \(\mathcal{T}\) in \((\mathcal{L}_c, \| \cdot \|_c)\) by
\[
(\mathcal{T}u)(t) = \int_0^t \Phi(t)\Phi^{-1}(\tau)h(\tau, u(\tau))d\omega(\tau) + \int_0^t \Phi(t)\Phi^{-1}(\tau)f(\tau, u(\tau))d\tau.
\]
Given \(u_1, u_2 \in \mathcal{L}_c,\) it follows from \((4.5)\) that
\[
E\| (\mathcal{T}u_1)(t) - (\mathcal{T}u_2)(t) \|^2 \leq 3E\left\| \int_0^t \Phi(t)\Phi^{-1}(\tau)[h(\tau, u_1(\tau)) - h(\tau, u_2(\tau))]d\omega(\tau) \right\|^2
+ 3E\left\| \int_0^t \Phi(t)\Phi^{-1}(\tau)[f(\tau, u_1(\tau)) - f(\tau, u_2(\tau))]d\tau \right\|^2 \tag{4.6}
\]
On the other hand, by \((4.2),\) we obtain
\[
E\|f(\tau, u_1(\tau)) - f(\tau, u_2(\tau))\|^2 \leq cE\|u_1(\tau) - u_2(\tau)\|^2(E\|u_1(\tau)\|^2 + E\|u_2(\tau)\|^2)^q
\leq 2^q e r^{2q}(q+1)\alpha\|u_1 - u_2\|_c^2. \tag{4.7}
\]
Similarly, we have
\[ E\|h(\tau, u_1(\tau)) - h(\tau, u_2(\tau))\|^2 \leq 2^q c r^2 q e^{-(q+1)\alpha \tau} \|u_1 - u_2\|^2. \] (4.8)

By (4.1) and (4.8), the first term of right-hand side in (4.6) can be deduced as follows:
\[
E \left\| \int_0^t \Phi(t) \Phi^{-1}(\tau) \left[ h(\tau, u_1(\tau)) - h(\tau, u_2(\tau)) \right] d\omega(\tau) \right\|^2 \\
= \int_0^t E \|\Phi(t)\Phi^{-1}(\tau)\|^2 E\|h(\tau, u_1(\tau)) - h(\tau, u_2(\tau))\|^2 d\tau \\
\leq \frac{2^q c r^2 q \|u_1 - u_2\|^2}{q^2} e^{-\alpha \tau} \int_0^t e^{-q \alpha \tau} d\tau \\
\leq \frac{2^q c r^2 q e^{-\frac{2}{q} t}}{q^2} \|u_1 - u_2\|^2.
\]

As to the second term in (4.6), it follows from (4.1), (4.7), (4.8), \(E\|x\| \leq \sqrt{E\|x\|^2}\), and Cauchy-Schwarz inequality that
\[
E \left\| \int_0^t \Phi(t) \Phi^{-1}(\tau) \left[ f(\tau, u_1(\tau)) - f(\tau, u_2(\tau)) \right] d\tau \right\|^2 \\
= \int_0^t E \|\Phi(t)\Phi^{-1}(\tau)\|^2 \left( \int_0^t \|\Phi^{-1}(\tau)\|^2 \|f(\tau, u_1(\tau)) - f(\tau, u_2(\tau))\|^2 d\tau \right) d\tau \\
\leq \int_0^t E \|\Phi(t)\Phi^{-1}(\tau)\|^2 \int_0^t \|f(\tau, u_1(\tau)) - f(\tau, u_2(\tau))\|^2 d\tau d\tau \\
\leq 2^{1+q} c K (1 + g^2) r^2 q \|u_1 - u_2\|^2 \left( \int_0^t e^{-\frac{2}{q} (t-\tau)} d\tau \right) \left( \int_0^t e^{-\frac{q}{2} (t-\tau)} e^{-\alpha \tau} d\tau \right) \\
\leq \frac{2^{3+q} c K (1 + g^2) r^2 q e^{-\frac{2}{q} t}}{(2q - 1)\alpha^2} \|u_1 - u_2\|^2.
\]

Since \(q > 1\), we can rewrite the inequality (4.6) as
\[ E\|(T u_1)(t) - (T u_2)(t)\|^2 \leq \frac{2^{q+2} c K r^2 q e^{-\frac{2}{q} t}}{\alpha} \left( \frac{1}{q} + \frac{1 + g^2}{(2q - 1)\alpha} \right) \|u_1 - u_2\|^2. \]

We can choose appropriate \(r\) such that
\[ \theta = \sqrt{\frac{2^{q+2} c K r^2 q e^{-\frac{2}{q} t}}{\alpha} \left( \frac{1}{q} + \frac{1 + g^2}{(2q - 1)\alpha} \right)} < \frac{1}{2}. \]

Therefore,
\[ \|T u_1 - T u_2\|_c \leq \theta \|u_1 - u_2\|_c. \] (4.9)

Given \(\|\xi_0\| \leq \delta\), and considering the operator \(\tilde{T}\) in \((L_c, \|\cdot\|_c)\) defined by
\[ (\tilde{T} u)(t) = \xi(t) + (T u)(t) \]
with $\xi(t) = \Phi(t)\Phi^{-1}(0)\xi_0$, it is clear that we have $T u = 0$ for $u = 0$, and it follows from (4.9) that
\[
\|Tu\|_c \leq \theta\|u\|_c.
\]
On the other hand, it follows from (4.1) that
\[
\|\xi(t)\|_c = \sup_{t \geq 0} (\mathbb{E}\|\Phi(t)\Phi^{-1}(0)\xi_0\|^2)^{1/2} e^{\alpha t/2} = \sqrt{K}\delta < \frac{1}{2}r
\]
since $\delta > 0$ is sufficiently small. Therefore,
\[
\|\tilde{T}u\|_c \leq \|\xi(t)\|_c + \|Tu\|_c \leq r,
\]
and this means that $\tilde{T}\mathcal{L}_c \subset \mathcal{L}_c$. In addition, by (4.9), we have
\[
\|\tilde{T}u_1 - \tilde{T}u_2\|_c = \|Tu_1 - Tu_2\|_c \leq \theta\|u_1 - u_2\|_c,
\]
and thus $\tilde{T}$ is a contraction in $(\mathcal{L}_c, \|\cdot\|_c)$. Hence, there exists a unique $u \in \mathcal{L}_c$ such that $\tilde{T}u = u$. By (4.10), we obtain
\[
\|u\|_c \leq \frac{1}{2}r + \theta\|u\|_c,
\]
and thus
\[
\|u\|_c \leq \frac{r}{2(1 - \theta)}.
\]
Therefore the function $u(t)$ satisfies (1.8) with $\tilde{K} = \frac{r^2}{4(1 - \theta^2)} > 0$.

**Remark 4.1.** Let $\chi_{\max}$ denote the maximal value of second-moment Lyapunov exponent of (1.1), and let $\gamma$ denote second-moment regularity coefficient. Using the same techniques as in the proof of Theorem 1.1, it follows easily from (3.17) that $\alpha = -(\chi_{\max} + \varepsilon)$ and $\beta = \gamma + 2\varepsilon$ under the condition of NMS-EC. Since $\varepsilon$ can be chosen arbitrarily small, the assumption $-\gamma + \beta < 0$ in Theorem 1.3 can also be substituted by $q\chi_{\max} + \gamma < 0$.

From the remark above, Theorem 1.2 is an immediate corollary of Theorem 1.3, since regularity means $\gamma = 0$, and clearly, $q\chi_{\max} + \gamma < 0$ implies $\chi_{\max} < 0$. This is a natural condition of NMS-EC.

## 5. Second-moment Regularity Coefficient

Following the discussion of Remark 4.1, exponent $\alpha$ can be estimated by $\chi_{\max}$ in (2.1) in terms of (1.1) and its solutions. Thus it is of special interest to derive the upper and lower bounds of the second-moment regularity coefficient $\gamma(\chi, \tilde{\chi})$, which determines the stability of the perturbed equation. From Section 2 we know that $\gamma(\chi, \tilde{\chi}) \geq 0$. In this section we proceed to derive the more precise lower bound and upper bound of $\gamma(\chi, \tilde{\chi})$, which have the advantage that one does not need to know any explicit information about the solutions of the linear SDE (1.1). More specifically, the lower and upper bounds of $\gamma(\chi, \tilde{\chi})$ can be expressed solely by the drift term $A(t)$, and have nothing to do with the diffusion term $G(t)$. 
5.1. Lower Bound.

**Theorem 5.1.** The second-moment regularity coefficient satisfies

\[ \gamma(\chi, \tilde{\chi}) \geq \frac{2}{n} \left( \limsup_{t \to +\infty} \frac{1}{t} \int_0^t \text{tr} A(\tau) d\tau - \liminf_{t \to +\infty} \frac{1}{t} \int_0^t \text{tr} A(\tau) d\tau \right). \]

**Proof.** Let \( v_1, \ldots, v_n \) be a basis of \( \mathbb{R}^n \), and \( v_i(t) \) be the unique solution of (1.1) such that \( v_i(0) = v_i \). Then \( v_1(t), \ldots, v_n(t) \) are the columns of a fundamental matrix solution of \( \Phi(t) \) of the equation (1.1). Thus it follows from Theorem 3.2.2 in [34] that for every \( t \geq 0 \),

\[ \det \Phi(t) = \exp \left[ \int_0^t \left( \text{tr} A(\tau) - \frac{1}{2} \text{tr} G^2(\tau) \right) d\tau + \int_0^t \text{tr} G(\tau) d\omega(\tau) \right]. \]  

(5.1)

Furthermore,

\[ E| \det \Phi(t)|^2 \leq n \prod_{j=1}^n E|v_j(t)||^2 \]

follows directly from \( |\det \Phi(t)| \leq \prod_{j=1}^n |v_j(t)| \) and the independence of the vectors \( v_1(t), \ldots, v_n(t) \). Thus by using (5.1), log-normal distribution, we have

\[ \sum_{j=1}^n (\chi(v_j) = \limsup_{t \to +\infty} \frac{1}{t} \log \left( \prod_{j=1}^n E|v_j(t)||^2 \right) \]

\[ \geq \limsup_{t \to +\infty} \frac{2}{t} E \left( \exp \left[ \int_0^t \left( \text{tr} A(\tau) - \frac{1}{2} \text{tr} G^2(\tau) \right) d\tau + \int_0^t \text{tr} G(\tau) d\omega(\tau) \right] \right) \]

\[ = \limsup_{t \to +\infty} \frac{2}{t} \int_0^t \text{tr} A(\tau) d\tau. \]

Similarly, let \( w_i(t) \) be the unique solution of (2.4) such that \( w_i(0) = w_i \) for each \( i \), where \( w_1, \ldots, w_n \) is another basis of \( \mathbb{R}^n \). Proceeding in a similar manner, we obtain

\[ -\sum_{j=1}^n (\tilde{\chi}(w_j) \leq -\limsup_{t \to +\infty} \frac{2}{t} \int_0^t \left( \text{tr}\left( -A(\tau) + G^2(\tau) \right) - \frac{1}{2} \text{tr}\left( G(\tau)^2 \right) - \frac{1}{2} \text{tr}\left( G^T(\tau)^2 \right) \right) d\tau \]

\[ = \liminf_{t \to +\infty} \frac{2}{t} \int_0^t \left( \text{tr}(A(\tau) - G^2(\tau)) + \text{tr}(G^2(\tau)) \right) d\tau \]

\[ = \liminf_{t \to +\infty} \frac{2}{t} \int_0^t \text{tr} A(\tau) d\tau. \]

Therefore,

\[ \limsup_{t \to +\infty} \frac{2}{t} \int_0^t \text{tr} A(\tau) d\tau - \liminf_{t \to +\infty} \frac{2}{t} \int_0^t \text{tr} A(\tau) d\tau \]

\[ \leq \sum_{j=1}^n (\chi(v_j) + \tilde{\chi}(w_j)). \]  

(5.2)

Now we require that a basis \( (v_1, \ldots, v_n) \) is dual to a basis \( (w_1, \ldots, w_n) \), and that the minimum in (2.6) is obtained at this pair, i.e.,

\[ \gamma(\chi, \tilde{\chi}) = \max\{\chi(v_i) + \tilde{\chi}(w_i) : 1 \leq i \leq n\}. \]
Hence we have
\[ \sum_{j=1}^{n} (\chi(v_j) + \tilde{\chi}(w_j)) \leq n \max \{ \chi(v_i) + \tilde{\chi}(w_i) : 1 \leq i \leq n \} = n \gamma(\chi, \tilde{\chi}). \]  
(5.3)

Thus the desired result follows immediately from (5.2) and (5.3).

\[ \square \]

5.2. Upper Bound.

For each \( k = 1, \ldots, n \), denote
\[ \alpha_k = \liminf_{t \to +\infty} \frac{1}{t} \int_{0}^{t} a_{kk}(\tau) d\tau, \quad \text{and} \quad \bar{\alpha}_k = \limsup_{t \to +\infty} \frac{1}{t} \int_{0}^{t} a_{kk}(\tau) d\tau, \]
where \( a_{11}(t), \ldots, a_{nn}(t) \) are the diagonal elements of \( A(t) \). In addition, here we assume that \( \bar{\alpha}_k, k = 1, \ldots, n \) are ordered as \( \bar{\alpha}_1 \leq \cdots \leq \bar{\alpha}_n \), since this can be trivially achieved via row permutation of \( A(t) \) and \( G(t) \). Thus we derive that the upper bound for the second-moment regularity coefficient \( \gamma(\chi, \tilde{\chi}) \) can be expressed by these numbers. In the following we use the assumption that \( A(t) \) and \( G(t) \) are upper triangular for every \( t \in I \). In fact, this assumption does not affect the estimation of the upper bound of \( \gamma(\chi, \tilde{\chi}) \) (see Theorem 5.3 and Remark 5.1 after the proof of Theorem 5.2).

**Theorem 5.2.** Assuming that \( A(t) \) and \( G(t) \) are upper triangular, the second-moment regularity coefficient satisfies
\[ \gamma(\chi, \tilde{\chi}) \leq 2 \sum_{k=1}^{k=n} (\bar{\alpha}_k - \alpha_k). \]

**Proof.** Before proving the main result, we first present and prove several lemmas which are useful in the proof of Theorem 5.2. The following two lemmas give the analytic expressions of the solutions of two kind of scalar linear SDEs.

**Lemma 5.1.** (see [34, Lemma 3.2.3]) Let \( a(\cdot), b(\cdot) \) be real-valued Borel measurable bounded functions on \( [t_0, T] \). Let
\[ \tilde{\Phi}(t) = e^{t_0} \int_{0}^{t} (a(\tau) - \frac{1}{2} b^2(\tau)) d\tau + t_0 b(\tau) d\omega(\tau). \]
(5.4)

Then \( x(t) = \tilde{\Phi}(t)x_0 \) is the unique solution to the scalar linear SDE
\[ \begin{aligned}
&dx(t) = a(t)x(t)dt + b(t)x(t)d\omega(t), \\
&x(0) = x_0.
\end{aligned} \]

**Lemma 5.2.** (see [34, p. 98]) Let \( a(\cdot), b(\cdot), c(\cdot) \) and \( d(\cdot) \) be real-valued Borel measurable bounded functions on \( [t_0, T] \). \( \tilde{\Phi}(t) \) is given as in (5.4). Then
\[ x(t) = \tilde{\Phi}(t) \left( x_0 + \int_{t_0}^{t} \tilde{\Phi}^{-1}(\tau)(c(\tau) - b(\tau)d(\tau))d\tau + \int_{t_0}^{t} \tilde{\Phi}^{-1}(\tau)d(\tau)d\omega(\tau) \right) \]
is the unique solution to the scalar linear SDE
\[ \begin{aligned}
&dx(t) = (a(t)x(t) + c(t))dt + (b(t)x(t) + d(t))d\omega(t), \\
&x(0) = x_0.
\end{aligned} \]
Now we denote by $a_{ij}(t)$ and $g_{ij}(t)$ the entries of the matrix $A(t)$ and $G(t)$ respectively for each $i$ and $j$. Denote

$$
\Lambda_i(t) := \int_0^t \left( a_{ii}(\tau) - \frac{1}{2} g_{ii}^2(\tau) \right) d\tau + \int_0^t g_{ii}(\tau) d\omega(\tau),
$$

and define the $n \times n$ matrix function $U(t) = (u_{ij}(t))$ as follows

$$
u_{ij}(t) = \begin{cases} 
0, & \text{if } i > j, \\
e^{\Lambda_i(t)}, & \text{if } i = j, \\
\int_0^t \left( \sum_{k=i+1}^j a_{ik}(\tau)u_{ik}(\tau) - g_{ij}(\tau) \sum_{k=i+1}^j g_{ik}(\tau)u_{ik}(\tau) \right) e^{\Lambda_i(\tau) - \Lambda_j(\tau)} d\tau \\
+ \int_0^t \sum_{k=i+1}^j g_{ik}(\tau)u_{ik}(\tau) e^{\Lambda_i(t) - \Lambda_j(t)} d\omega(\tau), & \text{if } i < j.
\end{cases}
$$

Thus it follows from Lemma 5.1 and Lemma 5.2 that the columns of the matrix function $U(t)$ form a basis for the space of solutions of (1.1). For each $i, j = 1, \ldots, n$, considering

$$\chi(u_{ij}) = \limsup_{t \to +\infty} \frac{1}{t} \log \mathbb{E}|u_{ij}(t)|^2,$$

we have the following result.

**Lemma 5.3.** For every $i, j = 1, \ldots, n$, we have

$$\chi(u_{ij}) \leq 2 \left( \pi_j + \sum_{m=i}^{j-1} (\pi_m - \alpha_m) \right) . \quad (5.6)$$

**Proof.** Firstly, it follows from (5.5), and log-normal distribution that

$$
\chi(u_{ij}) = \limsup_{t \to +\infty} \frac{1}{t} \log \mathbb{E}|u_{ij}(t)|^2 \\
= \limsup_{t \to +\infty} \frac{1}{t} \log \left( e^{2(\int_0^t (a_{ii}(\tau) - \frac{1}{2} g_{ii}^2(\tau)) d\tau + 2 \int_0^t g_{ii}(\tau) d\omega(\tau))} \right) \\
= \limsup_{t \to +\infty} \frac{2}{t} \int_0^t a_{ii}(\tau) d\tau \\
= 2\pi_i.
$$

Now we can apply the backward induction method on $i$. Assuming

$$\chi(u_{kj}) \leq 2\pi_j + 2 \sum_{m=k}^{j-1} (\pi_m - \alpha_m), \quad i + 1 \leq k \leq j \quad (5.7)$$

holds for a given $i < n$, we prove it for $i$, i.e.,

$$\chi(u_{ij}) \leq 2\pi_j + 2 \sum_{m=i}^{j-1} (\pi_m - \alpha_m).$$

Clearly, for each $\varepsilon > 0$, there exists $D > 1$, it is easy to verify from (1.2) and (5.7) that

$$|a_{ik}(t)| \leq De^{\varepsilon}, \quad |g_{ik}(t)| \leq De^{\varepsilon}, \quad \mathbb{E} \left( e^{-\Lambda_i(t)} \right) \leq De^{(-\alpha_i + \varepsilon)t},$$

and

$$\mathbb{E}|u_{kj}(t)|^2 \leq De^{(2\pi_j + 2 \sum_{m=i+1}^{j-1} (\pi_m - \alpha_m) + \varepsilon)t}$$

for every $i, j = 1, \ldots, n$, and $\varepsilon > 0$. This completes the proof.
for \( t \geq 0 \) and \( i + 1 \leq k \leq j \). Therefore, it follows from Itô isometry property, Hölder’s inequality and the elementary inequality (4.4) that

\[
\chi(u_{ij}) \leq \limsup_{t \to +\infty} \frac{1}{t} \log 2\mathbb{E} \left[ \left( \int_0^t \left( \sum_{k=i+1}^j a_{ik}(\tau)u_{ik}(\tau) - g_{ij}(\tau) \sum_{k=i+1}^j g_{ik}(\tau)u_{ik}(\tau) \right) e^{\Lambda_i(t) - \Lambda_i(\tau)} d\tau \right)^2 \right] \\
+ \left( \int_0^t \sum_{k=i+1}^j g_{ik}(\tau)u_{ik}(\tau)e^{\Lambda_i(t) - \Lambda_i(\tau)} d\omega(\tau) \right)^2 \\
\leq \limsup_{t \to +\infty} \frac{1}{t} \log \left[ 8t \int_0^t D^4 \sum_{k=i+1}^j e^{(\tau_j + \sum_{m=i+1}^{j-1} (\tau_m - \alpha_m) - \alpha_i + 3\tau)} \right]^2 dt \\
+ 2 \int_0^t \left( D^3 \sum_{k=i+1}^j e^{(\tau_j + \sum_{m=i+1}^{j-1} (\tau_m - \alpha_m) - \alpha_i + 3\tau)} \right)^2 dt \\
\leq 2\pi_j + \limsup_{t \to +\infty} \frac{1}{t} \log \left[ \left( 8t D^8 n_2 + 2D^6 n_2 \right) e^{(2\tau_j + 2 \sum_{m=i+1}^{j-1} (\tau_m - \alpha_m) - 2\alpha_i + 8\tau)} \right] dt \\
\leq 2\pi_j + 2\pi_j + 2 \sum_{m=i+1}^{j-1} (\tau_m - \alpha_m) - 2\alpha_i + 8\varepsilon \\
= 2 \left( \pi_j + \sum_{m=i}^{j-1} (\tau_m - \alpha_m) \right) + 8\varepsilon.
\]

Note that \( \varepsilon > 0 \) is arbitrary. Thus, (5.6) holds for every \( j \geq i \), and this completes the proof of the lemma. \( \square \)

On the other hand, let \( \tilde{A}(t) := (-A(t) + G^2(t))^T \) and \( \tilde{G}(t) := -G^T(t) \). Thus, it follows from (2.4) that

\[
d\hat{u}(t) = \tilde{A}(t)\hat{u}(t)dt + \tilde{G}(t)\hat{u}(t)d\omega(t) \\
\tag{5.8}
\]

is the adjoint equation of (1.1). Denote the entries of the matrix \( \tilde{A}(t) \) and \( \tilde{G}(t) \) by \( \tilde{a}_{ij}(t) \) and \( \tilde{g}_{ij}(t) \) respectively for each \( i \) and \( j \). Define the \( n \times n \) matrix function \( \tilde{U}(t) = (\tilde{u}_{ij}(t)) \) as follows

\[
\tilde{u}_{ij}(t) = \begin{cases} 
0, & \text{if } i < j, \\
\int_0^t \sum_{k=j+1}^{j-1} \tilde{a}_{ki}(\tau)\tilde{u}_{ki}(\tau) - \tilde{g}_{ij}(\tau) \sum_{k=i+1}^{j-1} \tilde{g}_{ki}(\tau)\tilde{u}_{ki}(\tau) e^{-\Lambda_i(t) + \Lambda_i(\tau)} d\tau \\
+ \int_0^t \sum_{k=i+1}^{j-1} \tilde{g}_{ki}(\tau)\tilde{u}_{ki}(\tau) e^{-\Lambda_i(t) + \Lambda_i(\tau)} d\omega(\tau), & \text{if } i = j, \\
\int_0^t \left( \sum_{k=j+1}^{j-1} \tilde{a}_{ki}(\tau)\tilde{u}_{ki}(\tau) - \tilde{g}_{ij}(\tau) \sum_{k=i+1}^{j-1} \tilde{g}_{ki}(\tau)\tilde{u}_{ki}(\tau) \right) e^{-\Lambda_i(t) + \Lambda_i(\tau)} d\tau \\
+ \int_0^t \sum_{k=i+1}^{j-1} \tilde{g}_{ki}(\tau)\tilde{u}_{ki}(\tau) e^{-\Lambda_i(t) + \Lambda_i(\tau)} d\omega(\tau), & \text{if } i > j.
\end{cases}
\tag{5.9}
\]

Thus it follows from Lemma 5.1 and Lemma 5.2 that the columns of the matrix function \( \tilde{U}(t) \) form a basis for the space of solutions of (5.8). For each \( i, j = 1, \ldots, n \),
considering
\[ \tilde{\chi}(\tilde{u}_{ij}) = \limsup_{t \to +\infty} \frac{1}{t} \log \mathbb{E}\left[|\tilde{u}_{ij}(t)|^2\right], \]
we have the following result.

**Lemma 5.4.** For every \( i, j = 1, \ldots n \), we have
\[ \tilde{\chi}(\tilde{u}_{ij}) \leq 2\left(-\alpha_j + \sum_{m=j+1}^{i} (\overline{\alpha}_m - \underline{\alpha}_m)\right). \quad (5.10) \]

**Proof.** We proceed in a similar way as the proof of Lemma 5.3. Firstly, it follows from (5.5) and log-normal distribution that
\[ \tilde{\chi}(\tilde{u}_{ii}) = \limsup_{t \to +\infty} \frac{1}{t} \log \mathbb{E}\left[|\tilde{u}_{ii}(t)|^2\right] \]
\[ = \limsup_{t \to +\infty} \frac{1}{t} \log \mathbb{E}\left(e^{2\int_{0}^{t}(a_{ii}(\tau) + g_{ii}^2(\tau) - \frac{1}{2} g_{ii}^2(\tau))d\tau - 2 \int_{0}^{t} g_{ii}(\tau)d\omega(\tau)}\right) \]
\[ = \limsup_{t \to +\infty} \frac{-2}{t} \int_{0}^{t} a_{ii}(\tau)d\tau \]
\[ = 2\alpha_i. \]

Now we can apply the induction method on \( i \). Assuming that
\[ \chi(\tilde{u}_{kj}) \leq -2\alpha_j + 2\sum_{m=j+1}^{k} (\overline{\alpha}_m - \underline{\alpha}_m), \quad j \leq k \leq i - 1 \quad (5.11) \]
holds for a given \( i > 1 \), we prove it for \( i \), i.e.,
\[ \tilde{\chi}(\tilde{u}_{ij}) \leq -2\alpha_j + 2\sum_{m=j+1}^{i} (\overline{\alpha}_m - \underline{\alpha}_m). \]

Clearly, for each \( \varepsilon > 0 \), there exists \( D > 1 \), and then one can easily verify from (1.2) and (5.11) that
\[ |\tilde{a}_{ik}(t)| \leq De^{\varepsilon t}, \quad |\tilde{g}_{ik}(t)| \leq De^{\varepsilon t}, \]
\[ \mathbb{E}\left(e^{\Lambda_i(t)}\right) \leq De^{(\overline{\alpha}_i + \varepsilon)t}, \]
and
\[ \mathbb{E}|\tilde{u}_{kj}(t)|^2 \leq De^{(-2\alpha_j + 2\sum_{m=j+1}^{k-1} (\overline{\alpha}_m - \underline{\alpha}_m) + \varepsilon)t} \]
for \( t \geq 0 \) and \( j \leq k \leq i - 1 \). Therefore, it follows from Itô isometry property, Hölder’s inequality and the elementary inequality \((4.4)\) that

\[
\hat{\chi}(\hat{u}_{ij}) \leq \limsup_{t \to +\infty} \frac{1}{t} \log \left[ 4t \int_0^t \mathbb{E} \left( \left( \sum_{k=j}^{i-1} \hat{u}_{ik}(\tau)\hat{u}_{kj}(\tau) \right)^2 + \left( \hat{g}_{ij}(\tau) \sum_{k=j}^{i-1} \hat{g}_{ik}(\tau)\hat{u}_{kj}(\tau) \right)^2 \right) \right] \\
\mathbb{E} \left( e^{-2\Lambda_i(t)+2\Lambda_i(\tau)} \right) d\tau + 2 \int_0^t \mathbb{E} \left[ \sum_{k=j}^{i-1} \hat{g}_{ik}(\tau)\hat{u}_{ik}(\tau) \right] \mathbb{E} \left( e^{-2\Lambda_i(t)+2\Lambda_i(\tau)} \right) d\tau \\
\leq -2\alpha_j + \limsup_{t \to +\infty} \frac{1}{t} \log \left[ \int_0^t \left( 8tD^i \sum_{k=j}^{i-1} e^{(\alpha_j + \sum_{m=j+1}^k (\bar{\alpha}_m - \bar{\alpha}_m) + \bar{\alpha}_k + 3\varepsilon)\tau} \right) d\tau \right] \\
+2 \int_0^t \left( D^i \sum_{k=j}^{i-1} e^{(\alpha_j + \sum_{m=j+1}^k (\bar{\alpha}_m - \bar{\alpha}_m) + \bar{\alpha}_k + 3\varepsilon)\tau} \right) d\tau \\
\leq -2\alpha_j + 2\alpha_j + 2 \sum_{m=j+1}^{i} (\bar{\alpha}_m - \bar{\alpha}_m) + 2\bar{\alpha}_j + 8\varepsilon \\
= 2 \left( -\alpha_j + \sum_{m=j+1}^{i} (\bar{\alpha}_m - \bar{\alpha}_m) \right) + 8\varepsilon.
\]

Note that \( \varepsilon > 0 \) is arbitrary. Thus, \((5.10)\) holds for every \( j \leq i \), and this completes the proof of the lemma.

We now proceed with the proof of Theorem 5.2. It follows from Lemma 5.3 and Lemma 5.4 that

\[
\chi(u_j) = \max \{ \chi(u_{ij}), i = 1, \ldots, n \} \leq 2 \left( \bar{\alpha}_j + \sum_{m=1}^{j-1} (\bar{\alpha}_m - \bar{\alpha}_m) \right),
\]

and

\[
\hat{\chi}(\hat{u}_{ij}) = \max \{ \chi(\hat{u}_{ij}), i = 1, \ldots, n \} \leq 2 \left( -\alpha_j + \sum_{m=j+1}^{n} (\bar{\alpha}_m - \bar{\alpha}_m) \right).
\]

Thus, we have

\[
\chi(u_j) + \hat{\chi}(\hat{u}_{ij}) \leq 2 \sum_{m=1}^{n} (\bar{\alpha}_m - \bar{\alpha}_m) \tag{5.12}
\]

for every \( j = 1, \ldots, n \). Therefore, from the definition of the second-moment regularity coefficient \( \gamma(\chi, \hat{\chi}) \), it suffices to prove that the bases \((u_1, \ldots, u_n)\) and \((\hat{u}_1, \ldots, \hat{u}_n)\) are dual. Clearly, we can let \( \Phi(t) \) and \( \Phi^{-T}(t) \) be fundamental matrix solutions of \((1.1)\) and \((5.8)\) respectively. Note that the columns of the matrix function \( U(t) = (u_{ij}(t)) \) form the basis for the space of solutions of \((1.1)\). Thus we have \( U(t) = \Phi(t)C_1 \) for some constant matrix \( C_1 \). Meanwhile, it is noted that the columns of the matrix function \( \hat{U}(t) = (\hat{u}_{ij}(t)) \) form the basis for the space of
Therefore, solutions of (5.8). Thus we have \( \tilde{U}(t) = \Phi^{-T}(t)C_2 \) for some constant matrix \( C_2 \). Therefore,

\[
\langle u_i(t), \tilde{u}_j(t) \rangle = (U(t)u_i(0))^T (\tilde{U}(t)\tilde{u}_j(0)) \\
= (\Phi(t)C_1u_i(0))^T (\Phi^{-T}(t)C_2\tilde{u}_j(0)) \\
= (C_1u_i(0))^T (C_2\tilde{u}_j(0)) \\
= \langle C_1u_i(0), C_2\tilde{u}_j(0) \rangle
\]

for every \( t \geq 0 \). In addition, it follows from (5.5) and (5.9) that

\[
u_{ij}(0) = \begin{cases} 0, & \text{if } i > j, \\ 1, & \text{if } i = j, \\ 0, & \text{if } i < j, \end{cases} \quad \text{and} \quad \tilde{u}_{ij}(0) = \begin{cases} 0, & \text{if } i < j, \\ 1, & \text{if } i = j, \\ 0, & \text{if } i > j. \end{cases}
\]

Clearly, we have \( \langle u_i(t), \tilde{u}_j(t) \rangle = \langle u_i(0), \tilde{u}_j(0) \rangle = 0 \) for every \( i \neq j \). Furthermore, it follows from (5.5) and (5.9) that

\[
\langle u_i(t), \tilde{u}_i(t) \rangle = e^{A(t)} e^{-\Lambda(t)} = 1.
\]

Thus, it is concluded that \( (u_i(t), \tilde{u}_i(t)) = \delta_{ij} \) for every \( i \) and \( j \). The theorem follows from (5.12) and the definition of the second-moment regularity coefficient \( \gamma(\chi, \tilde{\chi}) \) immediately.

The following result implies that there exist a unitary matrix which can transform (1.1) into a linear SDE with coefficient matrices being upper triangular for every \( t \in I \).

**Theorem 5.3.** There exist a unitary matrix \( S(t) \) such that the change of variable \( x(t) = S^{-1}(t)u(t) \) transforms (1.1) into

\[
dx(t) = B(t)x(t)dt + H(t)x(t)d\omega(t)
\]

with \( B(t) \) and \( H(t) \) being upper triangular for every \( t \in I \).

**Proof.** Assuming that \( U(t) \) is a matrix with the columns \( u_1(t), \ldots, u_n(t) \), where \( u_i(t) \) is the solution of (1.1) satisfying the initial condition \( u_i(0) = u_i \) for \( i = 1, \ldots, n \), and using the Gram-Schmidt orthogonalization procedure to the basis \( u_i(t) \) with \( i = 1, \ldots, n \), we can construct a matrix \( S(t) \) with the columns \( s_1(t), \ldots, s_n(t) \) satisfying \( \langle s_i(t), s_j(t) \rangle = \delta_{ij} \), where \( \delta_{ij} \) is the Kronecker symbol. Obviously, \( S(t) \) is unitary for each \( t \in I \). Moreover, the Gram-Schmidt procedure can be effected in such a way that each function \( s_k(t) \) is a linear combination of functions \( u_1(t), \ldots, u_k(t) \). It follows that the change of variable \( X(t) = S^{-1}(t)U(t) \) is upper triangular for each \( t \in I \), and the columns of \( x_1(t) = S^{-1}(t)u_1(t), \ldots, x_n(t) = S^{-1}(t)u_n(t) \) of the matrix \( X(t) \) form a basis of the space of solutions of (5.13).

Write \( X(t) = (x_1(t), \ldots, x_n(t)) \) as the following

\[
X(t) = \begin{pmatrix}
x_{1,1}(t) & x_{1,2}(t) & x_{1,3}(t) & \cdots & x_{1,n}(t) \\
x_{2,1}(t) & x_{2,2}(t) & x_{2,3}(t) & \cdots & x_{2,n}(t) \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots & x_{n-1,n}(t) \\
& & & \ddots & x_{n,n}(t)
\end{pmatrix},
\]
since $X(t)$ is upper triangular for each $t \in I$. Now we prove that $B(t)$ and $H(t)$ in (5.13) are upper triangular for each $t \in I$. The result follows by induction. Write $B$ and $H$ in block forms:
\[
B = \begin{pmatrix} b_{11} & b_{12} & B_{13} \\ b_{21} & b_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}, \quad \text{and} \quad H = \begin{pmatrix} h_{11} & h_{12} & H_{13} \\ h_{21} & h_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix},
\]
where $B_{13}$, $H_{13}$, $B_{23}$, $H_{23}: I \to \mathbb{R}^{1 \times (n-2)}$, $B_{31}$, $H_{31}$, $B_{32}$, $H_{32}: I \to \mathbb{R}^{(n-2) \times 1}$, $B_{33}$, $H_{33}: I \to \mathbb{R}^{(n-2) \times (n-2)}$ are all continuous and bounded. In order to prove that $b_{21} = h_{21} = 0$ and $B_{31} = H_{31} = 0^{(n-2) \times 1}$, we choose the first column of the matrix $X(t)$, that is
\[
\mathbf{x}_1(t) = (x_{1,1}(t), 0, \ldots, 0)^T_{n-1}
\]
For the second equation of above equality, we have
\[
d0 = b_{21}(t)x_{11}(t)dt + h_{21}(t)x_{11}(t)d\omega(t),
\]
which implies that $b_{21}(t) = h_{21}(t) = 0$ since $x_1$ is a stochastic process and $x_{11} \neq 0$. Moreover, following the same steps as above, we obtain $B_{31} = H_{31} = 0^{(n-2) \times 1}$.

Now we assume that the matrix functions $B$ and $H$ have been progressively upper triangulated in its first $p - 1$ columns so that the transformed coefficient matrices $B$ and $H$ have the forms
\[
B = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ 0 & b_{p,p} & B_{23} \\ 0 & B_{32} & B_{33} \end{pmatrix}, \quad \text{and} \quad H = \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ 0 & h_{p,p} & H_{23} \\ 0 & H_{32} & H_{33} \end{pmatrix},
\]
where $B_{11}$, $H_{11}: I \to \mathbb{R}^{(p-1) \times (p-1)}$ are upper triangular, and $B_{12}$, $H_{12}: I \to \mathbb{R}^{(p-1) \times 1}$, $B_{13}$, $H_{13}: I \to \mathbb{R}^{(p-1) \times (n-p)}$, $B_{23}$, $H_{23}: I \to \mathbb{R}^{1 \times (n-p)}$, $B_{32}$, $H_{32}: I \to \mathbb{R}^{(n-p) \times 1}$, $B_{33}$, $H_{33}: I \to \mathbb{R}^{(n-p) \times (n-p)}$ are all continuous and bounded. Now we prove that $B_{32} = H_{32} = 0^{(n-p) \times 1}$. To obtain this, we choose the $p$th column of the matrix $X(t)$, that is
\[
\mathbf{x}_p(t) = (x_{p,1}(t), \ldots, x_{p,p}(t), 0, \ldots, 0)^T_{n-p}
\]
where we require that the entries $x_{p,j}(t) = 0$, $j = p + 1, \ldots, n$ satisfy
\[
d\begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix} = B_{32}(t)x_{p,p}(t)dt + H_{32}(t)x_{p,p}(t)d\omega(t),
\]
which implies that $B_{32} = H_{32} = 0^{(n-p) \times 1}$ since $x_p$ is a stochastic process and $x_{p,p} \neq 0$, the result follows. \hfill \Box

**Remark 5.1.** The assumption that $A(t)$ and $G(t)$ are upper triangular for every $t \in I$ in Theorem 5.2 does not affect the estimation of the upper bound for the second-moment regularity coefficient $\gamma(\chi, \bar\chi)$. In fact it follows from Theorem 5.3 that there exist a unitary matrix $S(t)$ such that the change of variable $x(t) = S^{-1}(t)u(t)$ transforms (1.1) into (5.13) with $B(t)$ and $H(t)$ being upper triangular for every $t \in I$. Thus one can follow the same idea in Lemma 5.3 to prove that $S(t)$ satisfies the SDE
\[
dS(t) = (A(t)S(t) - S(t)B(t) + S(t)H^2(t) - G(t)S(t)H(t))dt + (G(t)S(t) - S(t)H(t))d\omega(t).
\]
Let $\Phi_{B,H}(t)$ be a fundamental matrix solution of (5.13), and use $\Phi_{A,G}(t) := S(t)\Phi_{B,H}(t)$ to denote the fundamental matrix solution of (1.1). Meanwhile, one can also use the change of variable $\tilde{x}(t) = S^{-1}(t)u(t)$ to transform
\[
d\tilde{u}(t) = \tilde{A}(t)\tilde{u}(t)dt + \tilde{G}(t)\tilde{u}(t)d\omega(t)
\]
into
\[
d\tilde{x}(t) = \tilde{B}(t)\tilde{x}(t)dt + \tilde{H}(t)\tilde{x}(t)d\omega(t)
\]
\[
:= (-A(t) + G^2(t)T)\tilde{u}(t)dt - G^T(t)\tilde{u}(t)d\omega(t) \tag{5.15}
\]
Let $\Phi_{\tilde{B},\tilde{H}}(t)$ be a fundamental matrix solution of (5.16), and use $\Phi_{\tilde{A},\tilde{G}}(t) := \tilde{S}(t)\Phi_{\tilde{B},\tilde{H}}(t)$ to denote the fundamental matrix solution of (5.15). It follows from Lemma 2.1 that
\[
\Phi_{\tilde{A},\tilde{G}}(t) = \Phi_{\tilde{A},\tilde{G}}^T(t), \quad \text{and} \quad \Phi_{\tilde{B},\tilde{H}}(t) = \Phi_{\tilde{B},\tilde{H}}^T(t).
\]
Thus,
\[
\tilde{S}(t) = \Phi_{\tilde{A},\tilde{G}}(t)\Phi_{\tilde{B},\tilde{H}}^{-1}(t) = \Phi_{\tilde{A},\tilde{G}}^T(t)\Phi_{\tilde{B},\tilde{H}}^T(t) = S^{-T}(t).
\]
Let $u(t)$ be a solution of the equation (1.1), and $\tilde{u}(t)$ be a solution of the dual equation (5.15). Obviously, $x(t) = S^{-1}(t)u(t)$ and $\tilde{x}(t) = S^T(t)\tilde{u}(t)$ are solutions of (5.13) and (5.16) respectively. Hence, for every $t \in I$, we have
\[
\langle u(t), \tilde{u}(t) \rangle = (S(t)x(t))^T(S^{-T}(t)\tilde{x}(t)) = x^T(t)\tilde{x}(t) = \langle x(t), \tilde{x}(t) \rangle,
\]
and this means that the change of variables does not affect the inner product. Moreover, the second-moment Lyapunov exponents associated with (5.13) and (5.16) coincide with the second-moment Lyapunov exponents $\chi$ and $\bar{\chi}$ associated with (1.1) and (5.15) respectively since $S(t)$ is unitary for each $t \in I$. This means that the second-moment regularity coefficient of (5.13) and (5.16) is the same as that of (1.1) and (5.15). Thus we can use the assumption that $A(t)$ and $G(t)$ are upper triangular for every $t \in I$ to compute the upper bound for the second-moment regularity coefficient $\gamma(\chi, \bar{\chi})$. 


6. Examples

The following example is on the stability theory of SDE. For the perturbation of a linear SDE, NMS-EC is not enough to guarantee the second-moment exponential stability of its nonlinear perturbation. This example is established by using the ideas of Perron [40, p. 705-706], where the nonuniformity arises from the dependence on the initial time $s$.

Example 6.1. Let

$$0 < b < a < (2e^{-\pi} + 1)b \quad \text{and} \quad 0 < \lambda < \frac{2b}{a-b} - e^\pi.$$ \hspace{1cm} (6.1)

The following linear SDE

$$\begin{cases}
    du_1 = (-a - b(\sin t + \cos t))u_1 dt + \frac{1}{\lambda^{1/2}}u_1 d\omega(t) \\
    dv_1 = (-a + b(\sin t + \cos t))v_1 dt + v_1 d\omega(t)
\end{cases}$$

admits an NMS-EC. However, any nontrivial solution of the following perturbation equation

$$\begin{cases}
    du_2 = (-a - b(\sin t + \cos t))u_2 dt + \frac{1}{\lambda^{1/2}}u_2 d\omega(t) \\
    dv_2 = ((-a + b(\sin t + \cos t))v_2 + u_2^{(\lambda+1)}) dt + v_2 d\omega(t)
\end{cases} \hspace{1cm} (6.2)$$

is not mean-square exponentially stable.

Proof. Let

$$\Phi(t) = \begin{pmatrix} U(t) & 0 \\ 0 & V(t) \end{pmatrix}$$

be a fundamental matrix solution of (6.2). Thus it follows from Lemma 5.1 that $u_1(t) = U(t)U^{-1}(s)u_2(s)$ and $v_1(t) = V(t)V^{-1}(s)v_1(t_0)$ is the unique solution of (6.2) such that

$$U(t)U^{-1}(s) = e^{\int_s^t \left( -a + b(\sin \tau + \cos \tau) + \frac{1}{\lambda^{1/2}} \right) d\tau + \frac{1}{\lambda^{1/2}} \int_s^\tau d\omega(\tau)},$$

and

$$V(t)V^{-1}(s) = e^{\int_s^t \left( -a + b(\sin \tau + \cos \tau) - \frac{1}{\lambda^{1/2}} \right) d\tau + \int_s^\tau d\omega(\tau)},$$

and this implies that

$$\mathbb{E}\|U(t)U^{-1}(s)\|^2 = e^{-2b(t \sin s - s \sin s) + 2a(t-s)} = e^{(-2a+2b)(t-s)} + 2b(\sin s + 1) \leq e^{(-2a+2b)(t-s)+2bs},$$ \hspace{1cm} (6.4)$$

and

$$\mathbb{E}\|V(t)V^{-1}(s)\|^2 = e^{2b(t \sin s - s \sin s) - 2a(t-s)} = e^{(-2a+2b)(t-s) + 2b(\sin s - 1) - 2bs(\sin s - 1)} \leq e^{(-2a+2b)(t-s)+2bs}$$ \hspace{1cm} (6.5)$$

for all $t \geq s$. Furthermore, if $t = e^{2k\pi + \frac{1}{2}\pi}$ and $s = e^{2k\pi + \frac{1}{2}\pi}$ with $k \in \mathbb{N}$, then

$$\mathbb{E}\|U(t)U^{-1}(s)\|^2 = e^{(-2a+2b)(t-s)+2bs}.$$ \hspace{1cm} (6.6)
Similarly, if \( t = e^{2k\pi + \frac{1}{2} \pi} \) and \( s = e^{2k\pi - \frac{1}{2} \pi} \) with \( k \in \mathbb{N} \), then

\[
E\|V(t)V^{-1}(s)\|^2 = e^{(-2a + 2b)(t-s) + 2bs}.
\]

Thus, (6.2) admits an NMS-EC since \(-2a + 2b < 0\). By (6.6) and/or (6.7), the exponential \( e^{2bs} \) in (6.4) and/or (6.5) cannot be removed. This shows that the mean-square exponential contraction is not uniform.

In addition, it follows from Lemma 4.1 that for any initial condition \((u_2(t_0), v_2(t_0))\), the solution of (6.3) is given by

\[
u_2(t) = e^{-b(t \sin \log t - t_0 \sin \log t_0)} - (a + \frac{1}{e^{k\pi}})(t-t_0) + \frac{1}{k\pi} \int_{t_0}^{t} d\omega(\tau) u_2(t_0),
\]

and

\[
v_2(t) = e^{b(t \sin \log t - t_0 \sin \log t_0)} - (a + \frac{1}{e^{k\pi}})(t-t_0) + \int_{t_0}^{t} d\omega(\tau)
\]

\[\times \left( v_2(t_0) + u_2(t_0) \lambda(t_0) \int_{t_0}^{t} e^{-(\lambda + 2)b(\tau \sin \log \tau - t_0 \sin \log t_0)} - \lambda a(\tau - t_0) d\tau \right).
\]

Fix \( 0 < \delta < \frac{\pi}{4} \), and set

\[t'_k = e^{2k\pi - \frac{1}{2} \pi}, \quad t_k = e^{2k\pi - \frac{1}{2} \pi} + \delta
\]

for each \( k \in \mathbb{N} \). Clearly, for every \( \tau \in [t'_k, t_k] \) we have

\[2k\pi - \frac{1}{2} \pi \leq \log \tau \leq 2k\pi - \frac{1}{2} \pi + \delta,
\]

and

\[(2 + \lambda)b \tau \cos \delta \leq -(2 + \lambda)b \tau \sin \log \tau.
\]

This implies that

\[
\int_{t'_k}^{t_k} e^{-(\lambda + 2)b \tau \sin \log \tau - \lambda a \tau} d\tau \geq \int_{t'_k}^{t_k} e^{(\lambda + 2)b \tau \cos \delta - \lambda a \tau} d\tau.
\]

Write \( \rho = (\lambda + 2)b \cos \delta - \lambda a \), thus,

\[
\int_{t'_k}^{t_k} e^{(\lambda + 2)b \tau \cos \delta - \lambda a \tau} d\tau = \int_{t'_k}^{t_k} e^{\rho \tau} d\tau \geq \int_{t'_k}^{t_k} e^{\rho \tau} d\tau = \frac{1 - e^{-\rho \delta}}{\rho} e^{\rho t_k}.
\]

Let \( t'_k = e^{2k\pi + \frac{1}{2} \pi} \). Clearly, \( t'_k = e^{2k\pi + \frac{1}{2} \pi} \). Then for \( k \in \mathbb{N} \) sufficiently large, we obtain

\[e^{bt_k \sin \log t_k} \int_{t'_k}^{t_k} e^{-(\lambda + 2)b \tau \sin \log \tau - \lambda a \tau} \leq e^{bt_k} \int_{t'_k}^{t_k} e^{\rho \tau} d\tau
\]

\[= \frac{1 - e^{-\rho \delta}}{\rho} e^{bt_k + \rho t_k} = \frac{1 - e^{-\rho \delta}}{\rho} e^{(b + \rho \delta - \pi)t'_k}.
\]

On the other hand, we have

\[
E\|v_2(t)\|^2 = e^{-2a(t-t_0) + 2b(t \sin \log t - t_0 \sin \log t_0)}
\]

\[\times \left( v_2(t_0) + u_2(t_0) \lambda(t_0) \int_{t_0}^{t} e^{-(\lambda + 2)b(\tau \sin \log \tau - t_0 \sin \log t_0)} - \lambda a(\tau - t_0) d\tau \right)^2.
\]

Thus it follows from (6.1) that the second-moment Lyapunov exponent of any solution of (6.3) satisfies

\[\chi(v_2) \geq -2a + 2b + 2\rho \delta - \pi = -2a + 2b + 2[(\lambda + 2)b \cos \delta - \lambda a]e^{\delta - \pi} > 0
\]

for any \( \delta \).
if \( u_2(t_0) \neq 0 \). Therefore, the solution \( v_2(t) \) is not mean-square exponentially stable. This completes the construction of the example. 

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