BEILINSON'S HODGE CONJECTURE FOR $K_1$ REVISITED

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Abstract. Let $U/\mathbb{C}$ be a smooth quasiprojective variety and $\text{CH}^r(U,1)$ a special instance of Bloch's higher Chow groups ([Blo]). Jannsen was the first to show that the cycle class map $\text{cl}_{r,1} : \text{CH}^r(U,1) \otimes \mathbb{Q} \to \text{hom}_{\text{MHS}}(\mathbb{Q}(0), H^{2r-1}(U,\mathbb{Q}(r)))$ is not in general surjective, contradicting an earlier conjecture of Beilinson. In this paper we give a refinement of Jannsen's counterexample, and further show that the aforementioned cycle class map becomes surjective at the generic point.

§0. Introduction

Let $U/\mathbb{C}$ be a smooth quasiprojective variety, $\text{CH}^r(U,m) := \text{CH}^r(U) \otimes \mathbb{Q}$, where $\text{CH}^r(U,m)$ is Bloch’s higher Chow group, and

$$\text{cl}_{r,m}^U : \text{CH}^r(U,m) \to \Gamma(H^{2r-m}(U,\mathbb{Q}(r))) := \text{hom}_{\text{MHS}}(\mathbb{Q}(0), H^{2r-m}(U,\mathbb{Q}(r))),$$

the Betti cycle class map. If $m = 0$, then the Hodge conjecture (classical form) implies that $\text{cl}_{r,0}^U$ is surjective. Beilinson ([Be]) once conjectured that $\text{cl}_{r,m}^U$ is always surjective. It was Jannsen ([J3]) who was the first to find a counterexample, in the case $m = 1$, where the complex numbers $\mathbb{C}$ are used in an essential way. In contrast to this, one expects the surjectivity of $\text{cl}_{r,m}^U$ in the case where $U$ is obtained via base extension from a variety defined over a number field. It turns out that Jannsen’s counterexample is indeed very special, being the complement of a closed subscheme of codimension $r$ in a projective algebraic manifold $X$. In contrast to this, we show rather easily that the corresponding limit map

$$\text{cl}_{r,1} : \text{CH}^r(\text{Spec}(\mathbb{C}(X)),1) \to \Gamma(H^{2r-1}(\mathbb{C}(X),\mathbb{Q}(r))),$$

is onto, provided that a certain reasonable conjectural type statement in [J1] holds, and unconditionally in the case $r = \dim X$. We review this in Example 2.7 below. Broadly speaking, besides providing a closer examination of this cycle class map, we want to also consider the relative situation as well. Consider a proper morphism $\rho : \mathcal{X} \to S$ of smooth complex quasiprojective varieties. Let $\eta$ be the generic point of $S$.

Question 0.1. Is the induced map

$$\text{cl}_{r,m}^\eta : \text{CH}^r(\mathcal{X}_\eta,m;\mathbb{Q}) \to \Gamma(H^{2r-m}(\mathcal{X}_\eta,\mathbb{Q}(r))),$$

surjective? [Here $H^{2r-m}(\mathcal{X}_\eta,\mathbb{Q}(r)) = \lim_{U \subset S} H^{2r-m}(\rho^{-1}(U)(\mathbb{C}),\mathbb{Q}(r))$.]

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Note that 0.1 in the case \( m = 0 \) is equivalent to the classical Hodge conjecture, using a standard localization argument ([J3]). Based on our limited results in §2 for the case \( m = 1 \), we feel that the answer to this question is yes. The main reason for considering the relative situation is as follows. As a formal consequence of M. Saito’s theory of mixed Hodge modules (see [As], [KL]), there is an exact sequence:

\[
0 \to \text{Ext}^1_{\text{PMHS}}(\mathbb{Q}(0), H^{2r-\nu-m}(\eta, R^2r-\nu-\rho \ast \mathbb{Q}(r))) \to \left\{ \begin{array}{l}
\text{Germs of} \\
\text{higher order} \\
\text{normal functions} \\
\text{at } \eta
\end{array} \right. \\
\to \text{hom}_{\text{MHS}}(\mathbb{Q}(0), H^{2r-\nu-m}(\eta, R^2r-\nu-\rho \ast \mathbb{Q}(r))) \to 0
\]

Here PMHS stands for the category of graded polarizable MHS (mixed Hodge structures). The key point is, is there lurking a generalized Poincaré existence theorem for higher normal functions? (Namely, are these normal functions cycle induced?) An affirmative answer to 0.1, would imply a generalized Poincaré existence theorem, and conversely such an existence theorem should lead to an affirmative answer to 0.1.

In §2 we discuss a relative version of the cycle class map on “the generic fiber”, as well as examples pointing to an affirmative answer to 0.1 in the case \( m = 1 \). In §3 we study the relative situation more carefully, by looking at those open subsets of a complete base variety \( \mathfrak{S} \), obtained by deleting codimension \( p \) subvarieties from \( \mathfrak{S} \). The cokernel of a resulting limiting cycle class map admits a rather explicit description in terms of two filtrations on a given Chow group. The main results of §3 are summarized in Corollary 3.6. For the remainder of the paper, we restrict to the case \( X = S \), with \( \rho = \text{identity} \), and where \( X := \mathcal{X} \) is a projective algebraic manifold. We then describe the aforementioned cokernel in terms of the complexity of the Chow ring \( \text{CH}^*(X; \mathbb{Q}) \). This is accomplished with the aid of our main technical result in Theorem 4.2, with subsequence corollary consequences given in §5. Finally in §6 we present some key examples.

§1. Notation

The following notation will be used throughout the paper. Let \( X \) be a smooth complex projective variety of dimension \( d \).

(i) \( \Gamma(\cdot) = \text{hom}_{\text{MHS}}(\mathbb{Q}(0), \cdot) \)

(ii) \( J(\cdot) = \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), \cdot) \)

(iii) \( J^r(X) = \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^{2r-1}(X, \mathbb{Q}(r))) \)

(iv) \( \text{CH}^r(X; \mathbb{Q}) = \text{CH}_{d-r}(X; \mathbb{Q}) \) is the Chow group of cycles of codimension \( r \) (dim \( d-r \)) in \( X \) modulo rational equivalence, tensored with \( \mathbb{Q} \), \( \text{CH}^r_{\text{alg}}(X; \mathbb{Q}) \subset \text{CH}^r_{\text{hom}}(X; \mathbb{Q}) \subset \text{CH}^r(X; \mathbb{Q}) \) are the subgroups of cycles algebraically equivalent to zero (resp. nullhomologous), and \( \text{Griff}^r(X; \mathbb{Q}) = \text{CH}_{\text{hom}}^r(X; \mathbb{Q})/\text{CH}_{\text{alg}}^r(X; \mathbb{Q}) \) is the Griffiths group (\( \otimes \mathbb{Q} \)). More generally, one has the higher Chow groups \( \text{CH}^r(\mathcal{X}, m) \) due to Bloch, and defined in [Blo].

(v) \( AJ : \text{CH}^r_{\text{hom}}(X; \mathbb{Q}) \to J^r(X) \) the Abel-Jacobi map

(vi) \( \text{CH}^r(X; \mathbb{Q})_{AJ} = \ker(AJ) \)

(vii) \( \text{HC} = \text{Hodge conjecture (classical form)} \)

(viii) \( \text{GHC} = \text{(Grothendieck amended) general Hodge conjecture ([G], [L]).} \)
\section*{\textsc{Beilinson-Hodge Conjecture}}

The cycle map on the generic fiber

The setting is the following diagram

\begin{equation}
\begin{array}{ccc}
\mathcal{X} & \hookrightarrow & \overline{X} \\
\rho & \downarrow & \overline{\rho} \\
S & \hookrightarrow & \overline{S}
\end{array}
\end{equation}

where $\overline{X}$ and $\overline{S}$ are nonsingular complex projective varieties, $\overline{\rho}$ is a dominating flat morphism, $D \subset S$ a divisor, $\mathcal{Y} := \overline{\pi}^{-1}(D)$, $S := \overline{S} \setminus D$, $X := \overline{X} \setminus \mathcal{Y}$ and $\rho := \overline{\pi}|_X$. For notational simplicity, we will identify $H^{2r-1}(X, \mathbb{Q}(r))$ with its image in $H^{2r-1}(\overline{X}, \mathbb{Q}(r))$. There is a short exact sequence

\begin{equation}
0 \to \frac{H^{2r-1}(\overline{X}, \mathbb{Q}(r))}{H^{2r-1}(\mathcal{X}, \mathbb{Q}(r))} \to H^{2r-1}(\mathcal{X}, \mathbb{Q}(r)) \to H^{2r}(\overline{X}, \mathbb{Q}(r))^\circ \to 0,
\end{equation}

and corresponding diagram:

\begin{equation}
\begin{array}{ccc}
\text{CH}^r(\mathcal{X}, 1; \mathbb{Q}) & \to & \text{CH}^r(\overline{X}; \mathbb{Q})^\circ \\
\text{cl}^r_{\mathcal{X}} & \downarrow \beta_Y & \downarrow A_{\mathcal{X} \overline{\mathcal{X}}} \\
\Gamma(H^{2r-1}(\mathcal{X}, \mathbb{Q}(r))) & \to & \Gamma(H^{2r}(\overline{X}, \mathbb{Q}(r))^\circ) \\
\end{array}
\end{equation}

where $A_{\mathcal{X} \overline{\mathcal{X}}}$ is a corresponding reduced Abel-Jacobi map. Let us assume that $\beta_Y$ is surjective (such is the case if the Hodge conjecture holds.) If we apply the snake lemma, we arrive at

\begin{equation}
\text{ker}(\text{cl}^r_{\mathcal{X}}) \simeq \frac{\ker[A_{\mathcal{X} \overline{X}} \cdot \text{Im}(\alpha_Y) : \text{Im}(\beta_Y) \to J\left(\frac{H^{2r-1}(\overline{X}, \mathbb{Q}(r))}{H^{2r}(\overline{X}, \mathbb{Q}(r))^\circ}\right)]}{\alpha_Y(\text{ker}(\beta_Y))}.
\end{equation}

Now take the limit over all $D \subset S$ to arrive at an induced cycle map:

\begin{equation}
\text{cl}^r_{\eta, 1} : \text{CH}^r(\mathcal{X}_\eta, 1; \mathbb{Q}) \to \Gamma(H^{2r-1}(\mathcal{X}_\eta, \mathbb{Q}(r))).
\end{equation}

where $\eta$ is the generic point of $S$. We arrive at:

\begin{equation}
\frac{\Gamma(H^{2r-1}(\mathcal{X}_\eta, \mathbb{Q}(r)))}{\text{cl}^r_{\eta, 1}(\text{CH}^r(\mathcal{X}_\eta, 1; \mathbb{Q}))} \simeq \frac{\ker[\mathcal{K} \cdot J\left(\frac{H^{2r-1}(\overline{X}, \mathbb{Q}(r))}{H^{2r}(\overline{X}, \mathbb{Q}(r))^\circ}\right)]}{\mathcal{N} \cdot \text{CH}^r(\mathcal{X}; \mathbb{Q})},
\end{equation}

where $\mathcal{K} := \ker[\text{CH}_{\text{hom}}(\overline{X}; \mathbb{Q}) \to \text{CH}^r(\mathcal{X}_\eta; \mathbb{Q})]$, and $\mathcal{N} \cdot \text{CH}^r(\overline{X}) \subset \text{CH}^r_{\text{hom}}(\overline{X}; \mathbb{Q})$ is the subgroup generated by cycles which are homologous to zero on some codimension $q$ subscheme of $\overline{X}$ obtained from a (pure) codimension $q$ subscheme of $\overline{S}$ via $\overline{\pi}^{-1}$, and where
Here we give some evidence that the RHS (hence LHS) of (2.6) is zero. Suppose that

\[ \Gamma(H^{2r-1}(\mathcal{X}, \mathbb{Q}(r))) \text{ becomes:} \]

\[ \frac{\operatorname{cl}_{r,1}(\operatorname{CH}^r(\mathcal{X}, 1; \mathbb{Q}))}{N_1^X \operatorname{CH}^r(\mathcal{X}; \mathbb{Q})} \simeq \frac{N_1 \operatorname{CH}^r(\mathcal{X}; \mathbb{Q}) + \ker [\mathcal{X} \to \mathcal{X}] J(H^{2r-1}(\mathcal{X}, \mathbb{Q}(r)))]}{N_1^X \operatorname{CH}^r(\mathcal{X}; \mathbb{Q})} \]

Example 2.7. Suppose that \( \mathcal{X} = \mathcal{S} \) with \( \mathcal{p} \) the identity. In this case (2.6) becomes:

\[ \frac{\Gamma(H^{2r-1}(\mathcal{C}(\mathcal{X}), \mathbb{Q}(r)))}{\operatorname{cl}_{r,1}(\operatorname{CH}^r(\operatorname{Spec}(\mathcal{C}(\mathcal{X})), 1; \mathbb{Q}))} \simeq \frac{N_1 \operatorname{CH}^r(\mathcal{X}; \mathbb{Q})}{N_1^X \operatorname{CH}^r(\mathcal{X}; \mathbb{Q})} \]

where \( N_1 \operatorname{CH}^r(\mathcal{X}; \mathbb{Q}) \) is the subgroup of cycles, that are homologous to zero on codimension 1 subschemes of \( \mathcal{X} \). According to Jannsen ([J1], p. 227), there is reason to believe that the right hand side of 2.7 should be conjecturally zero. In particular, since \( \operatorname{Spec}(\mathcal{C}(\mathcal{X})) \) is a point, this implies that \( \Gamma(H^{2r-1}(\mathcal{C}(\mathcal{X}), \mathbb{Q}(r))) = 0 \) for \( r > 1 \). The reader can easily check that

\[ \operatorname{cl}_{r,1}(\operatorname{CH}^r(\operatorname{Spec}(\mathcal{C}(\mathcal{X})), 1; \mathbb{Q})) = \Gamma(H^{2r-1}(\mathcal{C}(\mathcal{X}), \mathbb{Q}(r))) \]

holds unconditionally in the case \( r = \dim \mathcal{X} \).

Example 2.8. Here we give some evidence that the RHS (hence LHS) of (2.6) is zero. Suppose \( \mathcal{X} = X \times \mathcal{S} \), and let us assume the condition

\[ \operatorname{CH}^r(\mathcal{X}; \mathbb{Q}) = \bigoplus_{\ell=0}^{r} \operatorname{CH}^{r-\ell}(X; \mathbb{Q}) \otimes \operatorname{CH}^{\ell}(\mathcal{S}; \mathbb{Q}). \]

An example situation is when \( \mathcal{S} \) is a flag variety, such as a projective space; however conjecturally speaking, this condition is expected to hold for a much broader class of examples (see [CL]). Thus

\[ \xi \in \operatorname{CH}^r(\mathcal{X}; \mathbb{Q}) \Rightarrow \xi = \sum_{\ell=0}^{r} \xi_{\ell} \in \bigoplus_{\ell=0}^{r} \operatorname{CH}^{r-\ell}(X; \mathbb{Q}) \otimes \operatorname{CH}^{\ell}(\mathcal{S}; \mathbb{Q}). \]

Fix an \( \ell \) and write

\[ \xi_{\ell} = \sum_{i=1, j=1}^{N_1, M_1} \gamma_i^\ell \otimes \beta_j^\ell. \]

We can assume that \( \{[\gamma_i^\ell] \ldots, [\gamma_{N_1}^\ell]\} \) is a basis for \( \mathbb{Q}[\gamma_i^\ell] + \cdots + \mathbb{Q}[\gamma_{N_1}^\ell] \subset H^{2r-2\ell}(X, \mathbb{Q}) \), and that \( \{[\beta_i^\ell] \ldots, [\beta_{M_1}^\ell]\} \) is a basis for \( \mathbb{Q}[\beta_i^\ell] + \cdots + \mathbb{Q}[\beta_{M_1}^\ell] \subset H^{2\ell} (\mathcal{S}, \mathbb{Q}) \). Therefore we can write

\[ \xi_{\ell} = \sum_{i=1, j=1}^{N_1, M_1} \gamma_i^\ell \otimes \beta_j^\ell + \sum_{i=1, j=1}^{N_1, M_1} \gamma_i^\ell \otimes (\beta_j^\ell)' + \sum_{i=1, j=1}^{N_1, M_1} (\gamma_i^\ell)' \otimes \beta_j^\ell + \xi_{\ell}, \]
where $\xi'_\ell \in \text{CH}^{r-\ell}(X; \mathbb{Q}) \otimes \text{CH}^r_\text{hom}(\overline{S}; \mathbb{Q})$, $(\gamma'_i) \in \text{CH}^{r-\ell}(X; \mathbb{Q})$, $(\beta'_j) \in \text{CH}^r_\text{hom}(\overline{S}; \mathbb{Q})$. It is obvious that $\xi \in \text{CH}^r_\text{hom}(\overline{X}; \mathbb{Q}) \Rightarrow \sum_{i=1,j=1}^{N_{r,1},M_{r,1}} \gamma_i \otimes \beta_j = 0$ for each $\ell$. Furthermore, $AJ(\xi') = 0$, where $\xi' = \sum_{\ell=0}^r \xi'_\ell$. This is easily seen from the description

$$\text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^{2r-2\ell-1}(X, \mathbb{Q}) \otimes H^{2\ell}(\overline{S}, \mathbb{Q}))(r) = \bigoplus_{\ell=0}^r \left[ \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^{2r-2\ell}(X, \mathbb{Q}) \otimes H^{2\ell-1}(\overline{S}, \mathbb{Q}))(r) \right].$$

Note that for $\ell \geq 1$, $\sum_{i=1,j=1}^{N_{r,1},M_{r,1}} (\gamma'_i) \otimes \beta'_j + \xi'_\ell \in \mathbb{N}_{\overline{S}}^{1} \text{CH}^r(\overline{X}; \mathbb{Q})$. Recall that $\mathcal{K} = \text{ker}[\text{CH}^r_\text{hom}(\overline{X}; \mathbb{Q}) \to \text{CH}^r(\mathcal{X}_n; \mathbb{Q})]$. Then:

$$\mathcal{K} \subset \bigoplus_{\ell \geq 1} \text{CH}^{r-\ell}(X; \mathbb{Q}) \otimes \text{CH}^r(\overline{S}; \mathbb{Q}),$$

and

$$\xi \in \mathcal{K} \cap \text{CH}^r(\overline{X}; \mathbb{Q})_{AJ} \Rightarrow AJ_{\overline{S}}((\beta'_j)) = 0.$$ 

Thus if we assume that $\text{ker}(AJ_{\overline{S}}) \subset \mathbb{N}_{\overline{S}}^{1} \text{CH}^r(\overline{X}; \mathbb{Q})$, then we arrive at $\xi \in \mathbb{N}_{\overline{S}}^{1} \text{CH}^r(\overline{X}; \mathbb{Q})$.

Conjecture 2.9. The map $cl^r_{\overline{S}}$ in (2.4.1) is surjective.

§3. A (RELATIVE) FILTRATION ON CHOW GROUPS

We recall the setting in (2.1) and the definitions of $\mathcal{K}$, $\mathbb{N}_{\overline{S}}^{1} \text{CH}^r(\overline{X}; \mathbb{Q})$, $\mathbb{N}_{\overline{S}}^{2} H^{2r-1}(\overline{X}, \mathbb{Q}(r))$ in (2.5) above, and introduce the following descending filtration $\{ L^p_{X/\overline{S}} \text{CH}^r(\overline{X}; \mathbb{Q}) \}_{p \geq 0}$:

$$L^p_{X/\overline{S}} \text{CH}^r(\overline{X}; \mathbb{Q}) = \begin{cases} 
\text{ker}[AJ^p|_{\mathcal{K}_p} : \mathcal{K}_p \to J \left( \frac{H^{2r-1}(\overline{X}, \mathbb{Q}(r))}{N^p H^{2r-1}(\overline{S}, \mathbb{Q}(r))} \right) ] & \text{for } 0 \leq p \leq r \\
0 & \text{for } p \geq r + 1
\end{cases}$$

where $\mathcal{K}_p = \text{ker}[\text{CH}^r_\text{hom}(\overline{X}; \mathbb{Q}) \to \lim_{D \in Z^r(\overline{S})} \text{CH}^r(\overline{X} \setminus \overline{1}(D); \mathbb{Q})], Z^r(\overline{S})$ is the free abelian group generated by cycles of codimension $p$ subvarieties of $\overline{S}$.

Example 3.2. Suppose that $X := \overline{X} = \overline{S}$ where $\overline{p}$ is the identity. In this case put $\{ L^p \text{CH}^r(X; \mathbb{Q}) \}_{p \geq 0} = \{ L^p_{X/\overline{S}} \text{CH}^r(\overline{X}; \mathbb{Q}) \}_{p \geq 0}$. We then have:

$$L^p \text{CH}^r(X; \mathbb{Q}) = \begin{cases} 
\text{ker}[AJ^p : \text{CH}^r_\text{hom}(X; \mathbb{Q}) \to J \left( \frac{H^{2r-1}(X, \mathbb{Q}(r))}{N^p H^{2r-1}(X, \mathbb{Q}(r))} \right) ] & \text{for } 0 \leq p \leq r \\
0 & \text{for } p \geq r + 1
\end{cases}$$

where $AJ^p : \text{CH}^r_\text{hom}(X; \mathbb{Q}) \to J \left( \frac{H^{2r-1}(X, \mathbb{Q}(r))}{N^p H^{2r-1}(X, \mathbb{Q}(r))} \right)$ is the reduced Abel-Jacobi map. Note that $N^r H^{2r-1}(X; \mathbb{Q}) = 0$ and hence $AJ^r = AJ$ with:

$$L^r \text{CH}^r(X; \mathbb{Q}) = \text{ker}[AJ : \text{CH}^r_\text{hom}(X; \mathbb{Q}) \to J^r(X)] = \text{CH}^r(X; \mathbb{Q})_{AJ}$$

The coniveau filtration $\{ N^p \text{CH}^r(X; \mathbb{Q}) \}_{p \geq 0}$ on Chow groups introduced above appears in [J1]. Recall its $p$–th level $N^p \text{CH}^r(X; \mathbb{Q})$ is defined to be the subgroup generated by cycles which are homologous to zero on some $p$–codimensional possibly reducible subvariety of $X$.

We have the following relations:
Proposition 3.3. For each $p$, we have
\[ N^p_S \text{CH}^r(\overline{X}; \mathbb{Q}) \subseteq L^p_{\overline{X}/S} \text{CH}^r(\overline{X}; \mathbb{Q}) \subseteq \text{CH}^r_{\text{hom}}(\overline{X}; \mathbb{Q}). \]

Proof. The first inclusion is a consequence of functoriality of the Abel-Jacobi map, and the second inclusion is by definition. \(\square\)

For $r \geq 1$ we consider
\[ \text{cl}_{r,1}(\mathcal{Y}) : \text{CH}^r(\overline{X}\setminus \mathcal{Y}, 1; \mathbb{Q}) \to \Gamma(\mathbb{H}^{2r-1}(\overline{X}\setminus \mathcal{Y}, \mathbb{Q}(r))) \]
for a closed subscheme $\mathcal{Y}$ of $\overline{X}$. Set
\[ cl^p_{r,1} := \lim_{D \in Z^r(S)} \text{cl}_{r,1}(\langle \mathbb{p}^{-1}(D) \rangle) : \]
\[ \lim_{D \in Z^r(S)} \text{CH}^r(\overline{X}\setminus \mathbb{p}^{-1}(D), 1; \mathbb{Q}) \to \lim_{D \in Z^r(S)} \Gamma(\mathbb{H}^{2r-1}(\overline{X}\setminus \mathbb{p}^{-1}(D), \mathbb{Q}(r))). \]

Proposition 3.4. Assume the HC. Then,
\[ \text{coker}(cl^p_{r,1}) \cong \frac{L^p_{\overline{X}/S} \text{CH}^r(\overline{X}; \mathbb{Q})}{N^p_S \text{CH}^r(\overline{X}; \mathbb{Q})}, \quad \text{for } 0 \leq p \leq r. \]
Further, one does not need the HC assumption if $p \geq \min\{\dim \overline{X} - 3, r - 1\}$.

Proof. The proof proceeds exactly the same way as in §2 in obtaining the expression in (2.5), with $D$ now a closed subscheme of $S$ of (pure) codimension $p$, and $\mathcal{Y} = \langle \mathbb{p}^{-1}(D) \rangle$ being a closed subscheme of $\overline{X}$ of codimension $p$, and where $N^p_S$ in (2.5) is replaced by $N^p_S$. As in §2, the surjectivity of $\beta_Y$ in diagram (2.3) is only guaranteed if the homological version of the HC ([J3]) holds for $\mathcal{Y}$. For $p \geq \dim \overline{X} - 3$, that is clearly the case since $\dim \mathcal{Y} \leq 3$. A similar story holds for $p \geq r - 1$, by the Lefschetz (1, 1) theorem. \(\square\)

Proposition 3.5. Assume the HC. For any $p$ such that $0 \leq p \leq r$, we have
\[ L^p_{\overline{X}/S} \text{CH}^r(\overline{X}; \mathbb{Q}) = N^p_S \text{CH}^r(\overline{X}; \mathbb{Q}) + \text{CH}^r(\overline{X}; \mathbb{Q})_{\text{AJ}} \bigcap K_p. \]
Further, one does not need the HC assumption when both conditions hold: $p \geq \min\{\dim \overline{X} - 3, r - 1\}$ and $N^p_S \mathbb{H}^{2r-1}(\overline{X}, \mathbb{Q}(r)) = 0$.

Proof. Let $\xi \in L^p_{\overline{X}/S} \text{CH}^r(\overline{X}; \mathbb{Q})$. Then by definition, there is $\mathcal{Y} = \langle \mathbb{p}^{-1}(D) \rangle \in Z^r(\overline{X})$ such that if $j : \mathcal{Y} \hookrightarrow \overline{X}$ is the inclusion, then $\xi \in \text{Im}(\alpha_{\mathcal{Y}})$ represents an element in the numerator of the right hand side of the following expression derived similarly to that in (2.4):
\[ \text{coker}(\text{cl}_{r,1}(\mathcal{Y})) \simeq \frac{\ker \left[ AJ_{\text{Im}(\alpha_{\mathcal{Y}})} : \text{Im}(\alpha_{\mathcal{Y}}) \to J\left( \mathbb{H}^{2r-1}(\overline{X}\setminus \mathcal{Y}, \mathbb{Q}(r)) \right) \right]}{\alpha_{\mathcal{Y}}(\ker(\beta_{\mathcal{Y}}))}. \]
By the HC, the inclusion $j, \mathbb{H}^{2r-1}(\overline{X}, \mathbb{Q}(r)) \hookrightarrow \mathbb{H}^{2r-1}(\overline{X}, \mathbb{Q}(r))$ has a cycle induced left inverse
\[ [w] : \mathbb{H}^{2r-1}(\overline{X}, \mathbb{Q}(r)) \to j_*, \mathbb{H}^{2r-1}(\overline{X}, \mathbb{Q}(r)) \subseteq \mathbb{H}^{2r-1}(\overline{X}, \mathbb{Q}(r)), \]
where \(|w| \subset \mathcal{Y} \times \mathcal{Y}\). Using functoriality of the Abel-Jacobi map and the fact that \(\text{Im}(\alpha_Y) \subset \mathcal{K}_p\), we have \(\xi - \alpha_Y(w, \xi) \in \ker(AJ) \cap \mathcal{K}_p\), and \(w, \xi) \in \ker(\beta_Y)\). Thus
\[
\xi \in \ker(AJ) \cap \mathcal{K}_p + \alpha_Y(\ker(\beta_Y)) \subset \ker(AJ) \cap \mathcal{K}_p + N^p_S S^r CH^r(\mathcal{Y}; \mathbb{Q}).
\]

Hence this shows that
\[
L^p_{X/S} CH^r(\mathcal{Y}; \mathbb{Q}) \subset \ker(AJ) \cap \mathcal{K}_p + N^p_S S^r CH^r(\mathcal{Y}; \mathbb{Q}).
\]

Also by using Proposition 3.3, we have
\[
L^p_{X/S} CH^r(\mathcal{Y}; \mathbb{Q}) \subset \ker(AJ) \cap \mathcal{K}_p + N^p_S S^r CH^r(\mathcal{Y}; \mathbb{Q}) \subset L^p_{X/S} CH^r(\mathcal{Y}; \mathbb{Q}), \quad \text{for } 1 \leq p \leq r.
\]

\[\square\]

**Corollary 3.6.** Assume the HC. Then for any \(p\) such that \(1 \leq p \leq r\),
\[
\text{coker}(\text{cl}_{p, r, 1}^i) \cong \frac{L^p_{X/S} CH^r(\mathcal{Y}; \mathbb{Q})}{N^p_S S^r CH^r(\mathcal{Y}; \mathbb{Q})} \cong \frac{CH^r(\mathcal{Y}; \mathbb{Q})_{AJ} \cap \mathcal{K}_p}{CH^r(\mathcal{Y}; \mathbb{Q})_{AJ} \cap \ker \mathcal{K}_p \cap N^p_S S^r CH^r(\mathcal{Y}; \mathbb{Q})}.
\]

Further, the HC assumption is not needed if both conditions hold: \(p \geq \min\{\dim \mathcal{X} - 3, r - 1\}\) and \(N^p_S S^{2r-1}(\mathcal{X}, \mathbb{Q}(r)) = 0\).

For the remainder of this section, and indeed for the rest of this paper, we are going to restrict to the special situation in Example 3.2 with \(X := \mathcal{X} = \mathcal{S}\) and where \(\mathcal{P}\) is the identity.

**Remarks.** (i) Note that Corollary 3.6 implies that
\[
\text{coker}(\text{cl}_{p, r, 1}^i) \cong CH^r(X; \mathbb{Q})_{AJ}.
\]

Since the right hand side need not be zero, we recover the counterexample in [J3].

(ii) Assuming the HC, together with \(N^r-1 CH^r(X; \mathbb{Q}) = CH^r_{\text{alg}}(X; \mathbb{Q}) ([J1] (Lemma 5.7)), and \(J(N^r-1 H^{2r-1}(X, \mathbb{Q}(r))) = J^r_q(X) := AJ(CH^r_{\text{alg}}(X; \mathbb{Q}))\) (see [L]), the reader can check the existence of the commutative diagram:

\[
\begin{array}{ccccccccc}
0 & 0 & 0 & & 0 & & 0 & & 0 \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & CH^r_{\text{alg}}(X; \mathbb{Q})_{AJ} & \rightarrow & CH^r(X; \mathbb{Q})_{AJ} & \rightarrow & \frac{L^{r-1} CH^r(X; \mathbb{Q})}{N^{r-1} CH^r(X_{\text{alg}})} & \rightarrow & 0 \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
(3.8) 0 & \rightarrow & CH^r_{\text{alg}}(X; \mathbb{Q}) & \rightarrow & CH^r_{\text{hom}}(X; \mathbb{Q}) & \rightarrow & \text{Griff}^r(X; \mathbb{Q}) & \rightarrow & 0. \\
\downarrow AJ_a^{-1} & \rightarrow & AJ & \rightarrow & AJ & \rightarrow & AJ & \rightarrow & 0. \\
\downarrow & & AJ_a^{-1} & \rightarrow & AJ & \rightarrow & AJ & \rightarrow & 0 \\
0 & \rightarrow & J^r_a(X) & \rightarrow & J^r(X) & \rightarrow & J\left(\frac{H^{2r-1}(X, \mathbb{Q}(r))}{N^{r-1} H^{2r-1}(X_{\text{alg}}(r))}\right) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & & & & & & & 0
\end{array}
\]

where \(AJ_a^{-1} = AJ\big|_{CH^r_{\text{alg}}(X; \mathbb{Q})}, CH^r_{\text{alg}}(X; \mathbb{Q})_{AJ} = \ker(AJ_a^{-1}).\)
§4. LEVEL OF CHOW GROUPS

We recall the following notion introduced in [L], that measures the complexity of Chow groups.

**Definition 4.1.** We define

\[
\text{Level}(H^r(X)) = \max \{ |p - q| \mid H^{p,q}(X) \neq 0, p, q \geq 0 \},
\]

\[
\text{Level}(\text{CH}^r(X;\mathbb{Q})) = \min \left\{ \ell \geq 0 \left| \begin{array}{c}
\text{CH}^r(X;Y;\mathbb{Q}) = 0 \\
\text{over all closed } Y \subset X \\
\text{with codim}_X Y = r - \ell
\end{array} \right. \right\},
\]

\[
\text{Level}(\text{CH}^r(X;\mathbb{Q})) = \max \{ \text{Level}(\text{CH}^r(X;\mathbb{Q})) \mid r \geq 0 \}.
\]

The notion of a conjectured descending Bloch-Beilinson (B-B) filtration \( \{ F^\nu \text{CH}^r(X;\mathbb{Q}) \}_{\nu=0}^\infty \) is widely introduced in the literature. We refer the reader to [J1] for the definition, as well as the fact that if it exists, then it is unique. Among the many properties of this filtration, one has

\[ F^0 \text{CH}^r(X;\mathbb{Q}) = \text{CH}^r(X;\mathbb{Q}), \quad F^1 \text{CH}^r(X;\mathbb{Q}) = \text{CH}_{\text{hom}}^r(X;\mathbb{Q}), \quad F^2 \text{CH}^r(X;\mathbb{Q}) \subset \ker(AJ), \]

and that the induced action of a correspondence on \( \text{Gr}_{\nu}^p \) depends only on the cohomology class of that correspondence. Various candidate B-B filtrations have been introduced in the literature, such as by Griffiths/Green, J. P. Murre, M. Saito/M. Asakura, S. Saito, W. Raskind, J. D. Lewis, U. Jannsen, et al. Under the aforementioned uniqueness property, we will make the blanket assumption that all such candidates define the same filtration.

**Theorem 4.2.** Assume GHC and existence of the B-B filtration. Then for \( \ell \geq 1 \),

\[
\text{Level}(\text{CH}^r(X;\mathbb{Q})) \leq \ell \quad \Rightarrow \quad \text{CH}_{\text{hom}}^r(X;\mathbb{Q}) \subset N^{r-\ell} \text{CH}^r(X;\mathbb{Q})
\]

and hence \( \text{CH}_{\text{hom}}^r(X;\mathbb{Q}) = L^p \text{CH}^r(X;\mathbb{Q}) = N^p \text{CH}^r(X;\mathbb{Q}) \) for any \( p \) satisfying \( 1 \leq p \leq r-\ell \).

**Proof.** First note that Level\((H^r(X)) = \text{Level}(\text{CH}^r(X;\mathbb{Q})) \leq \ell \) ([L], Corollary 15.64), hence \( N^p H^{2r}(X, Q(r)) = H^{2r}(X, Q(r)) \) for any \( p \leq (2r - \ell)/2 \). By definition of level and by the GHC, there is a closed \( Y \subset X \) possibly reducible of codimension \( r - \ell \) such that both of \( \text{CH}^r_Y(X;Q) \to \text{CH}^r(X;Q) \) and \( H^{2r}_Y(X, Q(r)) \to H^{2r}(X, Q(r)) = N^{r-\ell} H^{2r}(X, Q(r)) \) are surjective. Let \( \tilde{Y} \to Y \) be a desingularization of \( Y \) and \( \sigma : \tilde{Y} \to Y \to X \) be the composition of desingularization followed by an inclusion. Then we have the following diagram:

\[
\begin{array}{ccc}
\text{CH}^r(\tilde{Y};\mathbb{Q}) & \xrightarrow{\sigma_*} & \text{CH}^r(X;\mathbb{Q}) \\
\downarrow \text{cl}_{\tilde{Y}} & & \downarrow \text{cl}_X \\
H^{2r}(\tilde{Y}, Q(\ell)) & \xrightarrow{[\sigma]_*} & H^{2r}(X, Q(r)) = N^{r-\ell} H^{2r}(X, Q(r))
\end{array}
\]

Note that \([\sigma]_*\) is surjective by Deligne’s mixed Hodge theory ([D] Proposition 8.2.8). Then by the HC, there is a cycle induced map

\[
[\Gamma]_* : H^{2r}(X, Q(r)) \to H^{2l}(\tilde{Y}, Q(\ell))
\]

such that \([\sigma]_* \circ [\Gamma]_* = \text{id} \) on \( H^{2r}(X, Q(r)) \). We now assume our given B-B filtration \( F^* \text{CH}^r(-;\mathbb{Q}) \), let \( \xi_1 \in F^p \text{CH}^r(X;\mathbb{Q}) \), and set \( \xi_0 := \Gamma_*(\xi_1) \in \text{CH}^r(\tilde{Y};\mathbb{Q}) \). Then \( \xi_0 \in \text{CH}^r(\tilde{Y};\mathbb{Q}) \) and it is in the image of the restriction map \( \text{CH}^r(X;\mathbb{Q}) \to \text{CH}^r(\tilde{Y};\mathbb{Q}) \). Therefore, \( \xi_0 \in L^p \text{CH}^r(X;\mathbb{Q}) \) for some \( p \leq (2r - \ell)/2 \) and hence \( \xi_0 \in N^p \text{CH}^r(X;\mathbb{Q}) \) for any \( p \leq (2r - \ell)/2 \). Therefore, \( \text{Level}(\text{CH}^r(X;\mathbb{Q})) \leq \ell \).
Claim 1: $q_X^p(\xi_1 - \sigma_*(\xi_0)) = 0$:

Note that on graded level $Gr_q^p$, the action of correspondences depends only on the cohomology class of such correspondences, so

$$q_X^p(\xi_1 - \sigma_*(\xi_0)) = q_X^p(\xi_1) - q_X^p(\sigma_*(\xi_1)) = q_X^p(\xi_1) - \sigma_* q_X^p(\Gamma_*(\xi_1))$$

$$= q_X^p(\xi_1) - \sigma_*(\Gamma_* q_X^p(\xi_1)) = q_X^p(\xi_1) - (\sigma I \ast [\Gamma_*])q_X^p(\xi_1) = 0.$$

Claim 2: For any $\xi \in CH^{\text{hom}}_r(X; \mathbb{Q}) = F^1CH(X; \mathbb{Q})$, there is $\xi_Y \in F^2CH(X; \mathbb{Q})$ and $\xi_2 \in F^{\ell+1}CH(X; \mathbb{Q})$ such that $\xi_1 = \sigma_*(\xi_Y) + \xi_2$:

Let $\xi \in CH^{\text{hom}}_r(X; \mathbb{Q}) = F^1CH(X; \mathbb{Q})$. Set $\xi_1 = \Gamma_*(\xi) \in F^1CH(X; \mathbb{Q})$. Then by Claim 1,

$$\xi_1 := \xi - \sigma_*(\xi_2) \in \ker(q_X^1) = F^2CH(X; \mathbb{Q})$$

Set $\xi_2 = \Gamma_*(\xi_1) \in F^2CH(X; \mathbb{Q})$. By applying Claim 1 to $\xi_2 \in F^2CH(X; \mathbb{Q})$, we get

$$\xi_2 := \xi_2 - \sigma_*(\xi_2) \in \ker(q_X^2) = F^3CH(X; \mathbb{Q})$$

By repeating this process $\ell$ times, we arrive at

$$\xi = \sigma_*(\xi_1) + \sigma_*(\xi_2) + \cdots + \sigma_*(\xi_\ell) + \xi_\ell,$$

where $\xi_i \in F^iCH(X; \mathbb{Q})$ and $\xi_\ell \in F^{\ell+1}CH(X; \mathbb{Q})$. Set $\xi_Y = \sum_{i=1}^{\ell} \xi_i \in F^1CH(X; \mathbb{Q})$.

Claim 3: $F^{\ell+1}CH(X; \mathbb{Q}) = 0$:

Note that

$$Gr_q^pCH(X; \mathbb{Q}) = \Delta_X(2d - 2r + p, 2r - p)CH(X; \mathbb{Q}),$$

where $d = \dim X$ and $\Delta_X \in CH^4(X \times X)$ is the diagonal class. Then,

$$[\Delta_X(2d - 2r + p, 2r - p)] \in H^{2d-2r+p}(X, \mathbb{Q}) \otimes H^{2r-p}(X, \mathbb{Q}),$$

and by the GHC together with Level$(H^*(X)) \leq \ell$ we can assume that $[\Delta_X(2d - 2r + p, 2r - p)]$ is supported on some $W_{1,p} \times W_{p,2} \subset X \times X$ with codim$(W_{1,p}, X) \geq (2d - 2r + p - \ell)/2$ and codim$(W_{p,2}, X) \geq (2r - p - \ell)/2$, where one can easily check that $\Delta_X(2d - 2r + p, 2r - p) = 0$ on $CH^r(X; \mathbb{Q})$ for $p \geq \ell + 1$. Hence $F^{\ell+1}CH(X; \mathbb{Q}) = 0$.

From Claims 1, 2 and 3, we get, for any $\xi \in CH^{\text{hom}}_r(X; \mathbb{Q})$,

$$\xi = \sigma_*(\sum_{i=1}^{\ell} \xi_i) \in \sigma_*(F^1CH(X; \mathbb{Q}))$$

since $F^\bullet CH(X; \mathbb{Q})$ is a descending filtration.
Claim 4: $\sigma_*(F^1\operatorname{CH}^\ell(\tilde{Y}; \mathbb{Q})) \subset N^{r-\ell} \operatorname{CH}^r(X; \mathbb{Q})$.

This follows easily from the fact that $F^1\operatorname{CH}^\ell(\tilde{Y}; \mathbb{Q}) = \operatorname{CH}_{\hom}^\ell(\tilde{Y}; \mathbb{Q})$.

By the above claims, we have now completed the proof of the lemma. □

Recall that $N^{r-1} \operatorname{CH}^r(X; \mathbb{Q}) = \operatorname{CH}^r_{\alg}(X; \mathbb{Q})$. We deduce:

**Corollary 4.3.** Assume the GHC and the existence of the B-B filtration. Then

$$\operatorname{Level}((\operatorname{CH}^*)^r(X; \mathbb{Q})) \leq 1 \Rightarrow \operatorname{Griff}^r(X; \mathbb{Q}) = 0.$$  

**Remark 4.4.** The converse of Corollary 4.3 is false, as can be seen by taking $X$ to be any surface of positive geometric genus.

§5. Summary Consequences

**Proposition 5.1.** Assume the HC and a given $p$. Then the following statements are equivalent:

(i) $\operatorname{cl}_{r,1}^p$ is surjective,

(ii) $\operatorname{CH}^r(X; \mathbb{Q})_{\AJ} \subset N^p \operatorname{CH}^r(X; \mathbb{Q})$,

(iii) $\operatorname{cl}_{r,1}^\nu$ is surjective for all $1 \leq \nu \leq p$.

**Proof.** Use Corollary 3.6. □

**Corollary 5.2.** Assume the GHC and existence of the B-B filtration. Then for $\ell \geq 1$, if $\operatorname{Level}((\operatorname{CH}^*)^r(X; \mathbb{Q})) \leq \ell$, then $\operatorname{cl}_{r,1}^p$ is surjective for all $p$ satisfying $1 \leq p \leq r - \ell$.

**Proof.** In order to show the surjectivity of $\operatorname{cl}_{r,1}^p$ for $1 \leq p \leq r - \ell$, it is enough by Proposition 5.1 to show that $\operatorname{cl}_{r,1}^{r-\ell}$ is surjective, equivalently

$$\operatorname{coker}(\operatorname{cl}_{r,1}^{r-\ell}) \cong \frac{L^{r-\ell} \operatorname{CH}^r(X; \mathbb{Q})}{N^{r-\ell} \operatorname{CH}^r(X; \mathbb{Q})} = 0$$

First note that $\operatorname{Level}(H^*(X)) \leq \ell$ implies $N^{r-\ell} H^{2r-1}(X, \mathbb{Q}(r)) = H^{2r-1}(X, \mathbb{Q}(r))$ and in turn

$$L^{r-\ell} \operatorname{CH}^r(X; \mathbb{Q}) = \ker \left[ \operatorname{CH}_{\hom}^r(X; \mathbb{Q}) \xrightarrow{J} \left( \frac{H^{2r-1}(X, \mathbb{Q})}{N^{r-\ell} H^{2r-1}(X, \mathbb{Q}(r))} \right) \right] = \operatorname{CH}_{\hom}^r(X; \mathbb{Q}) \subset N^{r-\ell} \operatorname{CH}^r(X; \mathbb{Q}),$$

where the last inclusion follows from Theorem 4.2. Hence we are done. □

In case when $\ell = 1$, Corollary 5.2 gives the surjectivity of $\operatorname{cl}_{r,1}^p$ up to $p = r - 1$. However we can extend this to the surjectivity of $\operatorname{cl}_{r,1}^p$ by the result of S. Saito $F^2 \operatorname{CH}_{\alg}^r(X; \mathbb{Q}) = \operatorname{CH}^r(X; \mathbb{Q})_{\AJ} \cap \operatorname{CH}_{\alg}^r(X; \mathbb{Q})$ ([S] Corollary 3.7). In fact, we have following:

**Corollary 5.3.** Assume the GHC and existence of the B-B filtration. Then the following statements are equivalent:

(i) $\operatorname{Level}((\operatorname{CH}^*)^r(X; \mathbb{Q})) \leq 1$,

(ii) $\operatorname{cl}_{r,1}^p$ is surjective for any $r$, $p$, 


(iii) \( \text{CH}^*_{\text{alg}}(X; \mathbb{Q}) \cong J_a(X) \),

(iv) \( \text{CH}^*_{\text{alg}}(X; \mathbb{Q}) \) is representable.

**Proof.** (ii) \( \Rightarrow \) (iii) : Suppose \( \text{cl}^p_{\nu, 1} \) is surjective for all \( p, r \). Then, in particular, \( \text{cl}^r_{\nu, 1} \) is surjective for any \( r \) and hence \( \text{CH}^r(X; \mathbb{Q})_{AJ} = 0 \) for any \( r \) (Example 3.2). This implies that \( AJ_{a}^{r-1} : \text{CH}^r_{\text{alg}}(X; \mathbb{Q}) \xrightarrow{\cong} J_a^{r}(X) \) is an isomorphism for any \( r \). Thus \( \text{CH}^*_{\text{alg}}(X; \mathbb{Q}) \cong J_a^{r}(X; \mathbb{Q}) \).

(iii) \( \Rightarrow \) (iv) : Obvious.

(iv) \( \Rightarrow \) (i) : [L] (Corollary 15.42).

(i) \( \Rightarrow \) (ii) : Suppose \( \text{Level}(\text{CH}^r(X; \mathbb{Q})) \leq 1 \). We refer to diagram (3.8). Then by Corollary 4.3,

\[
\frac{L^{r-1}\text{CH}^r(X; \mathbb{Q})}{N^{r-1}\text{CH}^r(X; \mathbb{Q})} = 0, \quad \text{hence } \text{CH}^r_{\text{alg}}(X; \mathbb{Q})_{AJ} = \text{CH}^r(X; \mathbb{Q})_{AJ}.
\]

Now statement (ii) follows from Claim 3 in the proof of Theorem 4.2, since in this case \( \text{Level}(\text{CH}^r(X; \mathbb{Q})) \leq 1 \Rightarrow F^2\text{CH}^r(X; \mathbb{Q}) = 0 \), together with the aforementioned result due to S. Saito, viz., \( \text{CH}^r_{\text{alg}}(X; \mathbb{Q})_{AJ} = F^2\text{CH}^r_{\text{alg}}(X; \mathbb{Q}) \). □

§6. SOME KEY EXAMPLES

Our first example, which illustrates the nontriviality of \( \text{coker}(\text{cl}^r_{\nu, 1}) \) for \( 2 \leq \nu \leq r \), is based on a rewording of a theorem in Nori’s paper [N]. In order to reword Nori’s theorem, we have to make the following assumptions:

1. GHC

2. The conjecture in Jannsen’s paper ([J1]) about Nori’s filtration. Specifically

\[ A_t \text{CH}^r(X; \mathbb{Q}) = N^r \text{CH}^r(X; \mathbb{Q}), \quad \nu = r - \ell - 1, \]

where \( A_t \text{CH}^r(X; \mathbb{Q}) \) is defined in [N]. Nori’s theorem translated in our language is now the following:

**Theorem 6.0.** ([N]) Let \( W \) be a smooth complex projective variety and \( X \subset W \) a sufficiently general complete intersection of sufficiently high multidegree. Let \( \xi_W \in \text{CH}^r(W; \mathbb{Q}) \) and put \( \xi := |\xi_W|_X \). Assume \( 2r - \nu - 1 < d := \dim X \). If \( \xi \in N^\nu \text{CH}^r(X; \mathbb{Q}) \), then:

1. \( [\xi_W] = 0 \in H^{2r}(W; \mathbb{Q}) \),

2. \( A_{1-t}^{\nu}([\xi_W]) = 0 \in \text{Ext}^1_{\text{MHS}}(Q(0), \frac{H^{2r-1}(W, \mathbb{Q}(r))}{N^\nu H^{2r-1}(W, \mathbb{Q}(r))}) \).

**Example 6.1.** The following example is essentially due to Nori ([N]). Let \( W = Q_{2r} \subset \mathbb{P}^{2r+1} \) be a smooth quadric. Let \( \xi_W \in \text{CH}^r(W; \mathbb{Q}) \) be given such that \( \text{Prim}^{2r}(W; \mathbb{Q}) = Q[\xi_W] \). Let \( X \subset W \) be a general complete intersection of sufficiently high multidegree of dimension \( d \leq 2r - 1 \). Note that \( H^{2r-1}(W, \mathbb{Q}) = 0 \), and we also want to arrange for \( H^{2r-1}(X, \mathbb{Q}) = 0 \). To ensure that \( H^{2r-1}(X, \mathbb{Q}) = 0 \), we must have \( d \neq 2r - 1 \). In particular, in the relation \( 2r - \nu - 1 < d < 2r - 1 \), we must necessarily have \( 2 \leq \nu \leq r \). Put \( \xi = |\xi_W|_X \in \text{CH}^r_{\text{hom}}(X; \mathbb{Q}) \).

Then by Nori’s theorem with the range of \( \nu \),

\[ \xi \neq 0 \in \frac{L^\nu \text{CH}^r(X; \mathbb{Q})}{N^\nu \text{CH}^r(X; \mathbb{Q})}. \]
Thus \( \text{coker}(cl_{r,1}^\nu) \neq 0 \) for \( 2 \leq \nu \leq r \). Complementary results appear in our next example.

Example 6.2. Suppose \( X \subset \mathbb{P}^N \) be a smooth complete intersection of dimension \( d \). Then by using the Lefschetz theorem, for \( \nu \geq 1 \), \( \text{Gr}^{p}\text{CH}^r(X; \mathbb{Q}) = 0 \) if \( 2r - \nu \neq d \) (S. Saito). Let \( F^*\text{CH}^r(X; \mathbb{Q}) \) be a the (unique) conjectured B-B filtration. It is known that

\[
F^2\text{CH}^r(X; \mathbb{Q}) \subset \text{CH}^r(X; \mathbb{Q})_{AJ} \subset F^1\text{CH}^r(X; \mathbb{Q}),
\]

and let us assume the following conjectural statement of Jannsen [J1]:

\[
F^{\nu+1}\text{CH}^r(X; \mathbb{Q}) \subset N^\nu\text{CH}^r(X; \mathbb{Q}).
\]

For a given \( p \), if we impose the condition that \( \text{Gr}^{p}\text{CH}^r(X; \mathbb{Q}) = 0 \) for all \( 1 \leq \nu \leq p \), (i.e. \( F^1\text{CH}^r(X; \mathbb{Q}) = \cdots = F^p\text{CH}^r(X; \mathbb{Q}) \)), then this implies

\[
\text{CH}^r(X; \mathbb{Q})_{AJ} \subset F^1\text{CH}^r(X; \mathbb{Q}) = F^{p+1}\text{CH}^r(X; \mathbb{Q}) \subset N^p\text{CH}^r(X; \mathbb{Q}),
\]

hence by Proposition 5.1, \( cl_{r,1}^\nu \) is surjective for all \( 1 \leq \nu \leq p \). Note that the requirement \( 2r - \nu \neq d \) for all \( 1 \leq \nu \leq p \) is equivalent to \( 2r - d \notin \{1, \ldots, p\} \). Hence we have either \( 2r - d \leq 0 \) or \( 2r - d \geq p + 1 \).

1. If \( 2r - d \leq 0 \), then for any \( \nu \geq 1 \), we have

\[
2r - \nu \leq d - \nu < d.
\]

Hence for any \( \nu \geq 1 \), \( cl_{r,1}^\nu \) is surjective.

2. If \( 2r - d \geq p + 1 \), then:

\[
\text{Gr}^{p}\text{CH}^r(X; \mathbb{Q}) = 0, \quad \text{for } 1 \leq \nu \leq 2r - d - 1,
\]

and hence \( cl_{r,1}^\nu \) is surjective for \( 1 \leq \nu \leq 2r - d - 1 \).

Remark 6.3. We want to make it clear that in the previous example, we have either \( d > 2r-1 \) or \( d \leq 2r - \nu - 1 \). This is in total contrast to Example 6.1, where \( 2r - \nu - 1 < d < 2r - 1 \), and where in this case \( cl_{r,1}^\nu \) is not surjective for \( 2 \leq \nu \leq r \).

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