CONVERGENCE OF AN ITERATIVE SCHEME FOR THE MONGE-AMPE`RE EIGENVALUE PROBLEM WITH GENERAL INITIAL DATA

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Abstract. In this note, we revisit an iterative scheme, due to Abedin and Kitagawa (Inverse Iteration for the Monge-Amp`ere Eigenvalue Problem, arXiv:2001.01291v2, to appear in Proc. Amer. Math. Soc.), to solve the Monge-Amp`ere eigenvalue problem on a general bounded convex domain. Using a nonlinear integration by parts, we show that the scheme converges for all convex initial data having finite and nonzero Rayleigh quotient to a nonzero Monge-Amp`ere eigenfunction.

1. Introduction and statement of the main result

In this note, we revisit an iterative scheme, due to Abedin and Kitagawa in their recent paper [1], to solve the Monge-Amp`ere eigenvalue problem on a bounded open convex domain Ω in \( \mathbb{R}^n \) \((n \geq 2)\):

\[
\begin{aligned}
\left\{ 
\begin{array}{ll}
\det D^2 w = \lambda |w|^n & \text{in } \Omega, \\
w = 0 & \text{on } \partial \Omega.
\end{array}
\right.
\end{aligned}
\]  

Before recalling relevant results, it is convenient to introduce some notation. Let

\[ K = \{ w \in C(\overline{\Omega}) : w \text{ is convex, nonzero in } \Omega, w = 0 \text{ on } \partial \Omega \}. \]

When \( u \) is merely a convex function on \( \Omega \), by an abuse of notation, we use \( \det D^2 u \, dx \) to denote the Monge-Amp`ere measure associated with \( u \); see Section 2.

For a convex function \( u \) on \( \Omega \), we define its Rayleigh quotient by

\[
R(u) = \frac{\int_{\Omega} |u| \det D^2 u \, dx}{\int_{\Omega} |u|^{n+1} \, dx}.
\]

Implicit in the definition (1.2) is the requirement that \( \|u\|_{L^{n+1}(\Omega)} < \infty \).

For general bounded convex domains \( \Omega \subset \mathbb{R}^n \), the existence, uniqueness and variational characterization of the Monge-Amp`ere eigenvalue, and uniqueness of convex Monge-Amp`ere eigenfunctions were obtained in [11]. They are the singular counterparts of those established by Lions [13] and Tso [15] in the smooth, uniformly convex setting. For the purpose of this note, we recall here part of [11, Theorem 1.1].

**Theorem 1.1.** ([11]) Let \( \Omega \) be a bounded open convex domain in \( \mathbb{R}^n \). Define \( \lambda = \lambda[\Omega] \) by

\[
\lambda[\Omega] = \inf_{w \in K} R(w).
\]

Then, the following facts hold:

(i) (Existence) The infimum in (1.3) is achieved by a nonzero convex solution \( w \in C^{0,\beta}(\overline{\Omega}) \cap C^\infty(\Omega) \) for all \( \beta \in (0,1) \) to the eigenvalue problem (1.1). The constant \( \lambda[\Omega] \) is called the Monge-Amp`ere eigenvalue of \( \Omega \) and \( w \) is called a Monge-Amp`ere eigenfunction of \( \Omega \).

(ii) (Uniqueness) If the pair \( (\Lambda, \tilde{w}) \) satisfies \( \det D^2 \tilde{w} = \Lambda |\tilde{w}|^n \) in \( \Omega \) where \( \Lambda > 0 \) is a positive constant and \( \tilde{w} \in K \), then \( \Lambda = \lambda[\Omega] \) and \( \tilde{w} = mw \) for some positive constant \( m \).

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In [1], Abedin and Kitagawa introduce an iterative scheme
\begin{equation}
\begin{cases}
\det D^2 u_{k+1} = R(u_k)|u_k|^n & \text{in } \Omega, \\
u_{k+1} = 0 & \text{on } \partial \Omega
\end{cases}
\end{equation}
(1.4)
to solve the Monge-Ampère eigenvalue problem (1.1). Here $u_{k+1}$ is a convex Aleksandrov solution of (1.4). We refer to Theorem 2.3 for the existence of $u_{k+1}$ and to Definition 2.2 for the notion of Aleksandrov solutions to the Monge-Ampère equation.

An interesting feature of the iterative scheme (1.4) is that the sequence \( \{u_k\}_{k=0}^{\infty} \) is obtained by repeatedly inverting the Monge-Ampère operator with Dirichlet boundary condition. One notes that similar inverse iteration methods have been considered for the p-Laplace equation [2 3 10]. Abedin and Kitagawa establish the first inverse iteration result for the eigenvalue problem of a fully nonlinear degenerate elliptic equation for a large class of initial data. Their main convergence result states as follows.

**Theorem 1.2.** ([1, Theorem 1.4]) Let $\Omega \subset \mathbb{R}^n$ be a bounded and convex domain. Let $u_0 \in C(\overline{\Omega})$ be a function satisfying:
(i) $u_0$ is convex and $u_0 \leq 0$ on $\partial \Omega$; (ii) $R(u_0) < \infty$; (iii) $\det D^2 u_0 \geq 1$ in $\Omega$.

For $k \geq 0$, define the sequence $u_k \in \mathbb{K}$ to be the solutions of the Dirichlet problem (1.4). Then \( \{u_k\} \) converges uniformly on $\overline{\Omega}$ to a non-zero Monge-Ampère eigenfunction $u_\infty$ of $\Omega$. Furthermore, $\lim_{k \to \infty} R(u_k) = \lambda(\Omega)$.

The conditions (i) and (iii) in Theorem 1.2 were used in [1] to show that, in the iterative scheme (1.4), the sequence \( \{u_k\} \) satisfies $u_k \leq \hat{w}$ for all $k \geq 0$ where $\hat{w}$ is a Monge-Ampère eigenfunction of $\Omega$ with $\|\hat{w}\|_{L^\infty(\Omega)} = (\lambda(\Omega))^{-1/n}$. This implies the lower bound $\|u_k\|_{L^\infty(\Omega)} \geq (\lambda(\Omega))^{-1/n}$ which guarantees the nontriviality of the limit $u_\infty$ of $u_k$. Here we call a function $w$ trivial if $w \equiv 0$ in $\Omega$.

**Remark 1.3.** Clearly (i) and (iii) in Theorem 1.2 imply that $R(u_0) > 0$. Without (i) and (iii) in Theorem 1.2, $R(u_0)$ can be 0 and thus the scheme (1.4) gives $u_k \equiv 0$ for all $k \geq 1$. For example, if $u_0$ is a nonzero affine function, then $R(u_0) = 0$. Thus, to get the nontriviality of the limit $u_\infty$ of $u_k$, if it exists, we need to require that $R(u_0)$ be nonzero.

In this note, we remove the restrictions (i) and (iii) in Theorem 1.2. We show that the iterative scheme (1.4) converges for all convex initial data having finite and nonzero Rayleigh quotient to a nonzero Monge-Ampère eigenfunction of $\Omega$. Thus our result covers all possible convex functions on $\Omega$ as initial data for the scheme (1.4).

**Theorem 1.4.** Let $\Omega \subset \mathbb{R}^n$ \( (n \geq 2) \) be a bounded and convex domain. Let $u_0 \in C(\Omega)$ be a nonzero convex function on $\Omega$ with $0 < R(u_0) < \infty$. For $k \geq 0$, define the sequence $u_k \in \mathbb{K}$ to be the solutions of the Dirichlet problem (1.4). Then \( \{u_k\} \) converges uniformly on $\overline{\Omega}$ to a non-zero Monge-Ampère eigenfunction $u_\infty$ of $\Omega$. Furthermore, for all $k \geq 3$ and any fixed nonzero Monge-Ampère eigenfunction $w$, we have

$$
\frac{[R(u_k)]^{1/n}}{(\lambda(\Omega))^{1/n}} \leq \frac{\int_{\Omega} |u_{k+1} - u_k| |w|^n \, dx}{\int_{\Omega} |u_3|^n |w|^n \, dx} \leq C(u_0, \Omega, n) \int_{\Omega} |u_\infty - u_k| \, dx.
$$

The nontriviality of our limit under the nonzero finiteness of the Rayleigh quotient $R(u_0)$ is due to an eventual regularity of the scheme (Proposition 3.1) and an important monotonicity formula during the scheme (Lemma 3.2). The proof of this monotonicity formula is based on a nonlinear integration by parts, established in [11], which was designed to prove uniqueness results for the Monge-Ampère equations and systems of Monge-Ampère equations [11 12].

**Remark 1.5.** Theorem 1.4 also bounds the convergence rate of $R(u_k)$ to the Monge-Ampère eigenvalue $\lambda(\Omega)$ in terms of the convergence rate of $u_k$ to the nonzero Monge-Ampère eigenfunction $u_\infty$. 
Compared to the inverse iteration methods for the $p$-Laplace equation in [2, 3, 10], this type of estimate seems to be new.

The rest of this note is organized as follows. In Section 2, we recall basic facts on the Monge-Ampère equation and prove a reverse Aleksandrov estimate in Proposition 2.5. In Section 3, we show the eventual smoothness and a new monotonicity formula for the iterative scheme (1.4). The proof of Theorem 1.4 will be given in Section 4. In Section 5, we make some remarks on the energy characterization of the Monge-Ampère eigenfunctions.

## 2. The Monge-Ampère equation and a reverse Aleksandrov estimate

Here, we recall some basic facts on the Monge-Ampère equation on open convex domain $\Omega$ of $\mathbb{R}^n$; see the books by Figalli [7] and Gutiérrez [8] for more details. We will establish a reverse Aleksandrov estimate in Proposition 2.5 that could be of independent interest.

For a convex function $u : \Omega \to \mathbb{R}$, we define the subdifferential of $u$ at $x \in \Omega$ by
\[ \partial u(x) := \{ p \in \mathbb{R}^n : u(y) \geq u(x) + p \cdot (y - x) \quad \forall y \in \Omega \}. \]

Below is a precise definition of the Monge-Ampère measure of a convex function $u : \Omega \to \mathbb{R}$; see also [7, Definition 2.1] and [8, Theorem 1.1.13].

**Definition 2.1** (Monge-Ampère measure). Let $u : \Omega \to \mathbb{R}$ be a convex function. The Monge-Ampère measure, $Mu$, associated with the convex function $u$ is defined by
\[ Mu(E) = |\partial u(E)| \quad \text{where} \quad \partial u(E) = \bigcup_{x \in E} \partial u(x), \quad \text{for each Borel set} \quad E \subset \Omega. \]

If $u \in C^2(\Omega)$, then $Mu = \det D^2u(x) \, dx$ in $\Omega$.

**Definition 2.2** (Aleksandrov solutions). Given an open convex set $\Omega$ and a Borel measure $\mu$ on $\Omega$, we call a convex function $u : \Omega \to \mathbb{R}$ an **Aleksandrov solution** to the Monge-Ampère equation
\[ \det D^2u = \mu, \]
if $\mu = Mu$ as Borel measures. When $\mu = f \, dx$ we will say for simplicity that $u$ solves
\[ \det D^2u = f. \]

In this note, we use $\det D^2u$ to denote the Monge-Ampère measure $Mu$ for a general convex function $u$. Thus, for all Borel set $E \subset \Omega$,
\[ \int_E |u| \det D^2u \, dx = \int_E |u| Mu. \]

Now, we recall the basic existence and uniqueness result for solutions to the Dirichlet problem with zero boundary data for the Monge-Ampère equation; see [7, Theorem 2.13], [8, Theorem 1.6.2], and [9, Theorem 1].

**Theorem 2.3** (The Dirichlet problem). Let $\Omega$ be a bounded open convex domain in $\mathbb{R}^n$, and let $\mu$ be a nonnegative Borel measure in $\Omega$. Then there exists a unique convex function $u \in C(\overline{\Omega})$ that is an Aleksandrov solution of
\[ \begin{cases} \det D^2u = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega. \end{cases} \]

For later reference, we state the celebrated Aleksandrov’s maximum principle for the Monge-Ampère equation; see [7, Theorem 2.8] and [8, Theorem 1.4.2].
Theorem 2.4 (Aleksandrov’s maximum principle). Let $\Omega \subset \mathbb{R}^n$ be an open, bounded and convex domain. Let $u \in C(\Omega)$ be a convex function. If $u = 0$ on $\partial \Omega$, then

$$|u(x)|^n \leq C(n)(\text{diam}\Omega)^{n-1}\text{dist}(x, \partial \Omega) \int_{\Omega} \det D^2u \, dx \quad \text{for all } x \in \Omega.$$  

We have the following proposition which will play a crucial role in the proof of Theorem 1.4.

Proposition 2.5 (Reverse Aleksandrov estimate). Let $\Omega$ be a bounded open convex domain in $\mathbb{R}^n$. Let $\lambda[\Omega]$ be the Monge-Ampère eigenvalue of $\Omega$ and let $w$ be a nonzero Monge-Ampère eigenfunction of $\Omega$. Assume that $u \in C^5(\Omega) \cap C(\bar{\Omega})$ is a strictly convex function in $\Omega$ with $u = 0$ on $\partial \Omega$ and satisfies

$$\int_{\Omega} (\det D^2u)^{1/n}|w|^{n-1} \, dx < \infty.$$  

Then

$$(2.1) \quad \int_{\Omega} (\lambda[\Omega])^{1/n}|u||w|^{n} \, dx \geq \int_{\Omega} (\det D^2u)^{1/n}|w|^n \, dx.$$  

Remark 2.6. Compared to Theorem 2.4, the function $u$ appears on the dominating side in (2.1) in Proposition 2.5. For this reason, (2.1) can be viewed as a sort of reverse Aleksandrov estimate. Moreover, this estimate is sharp. When $u$ is a Monge-Ampère eigenfunction of $\Omega$, (2.1) is an equality.

To prove Proposition 2.5 we recall the following nonlinear integration by parts established in [11] Proposition 1.7.

Proposition 2.7 (Nonlinear integration by parts). Let $\Omega$ be a bounded open convex domain in $\mathbb{R}^n$. Let $u, v \in C(\Omega) \cap C^5(\Omega)$ be strictly convex functions in $\Omega$ with $u = v = 0$ on $\partial \Omega$. If

$$\int_{\Omega} (\det D^2u)^{1/n}(\det D^2v)^{-\frac{n-1}{n}} \, dx < \infty, \quad \text{and} \quad \int_{\Omega} \det D^2v \, dx < \infty,$$

then

$$\int_{\Omega} |u| \det D^2v \, dx \geq \int_{\Omega} |v|(\det D^2u)^{\frac{1}{n}}(\det D^2v)^{-\frac{n-1}{n}} \, dx.$$  

Proof of Proposition 2.5. We apply Proposition 2.7 to $u$ and $v = w$. Then, using $\det D^2w = \lambda[\Omega]|w|^n$, we get

$$\int_{\Omega} |u|\lambda[\Omega]|w|^n = \int_{\Omega} |u| \det D^2w \, dx \geq \int_{\Omega} |w|(\det D^2u)^{1/n}(\det D^2w)^{-\frac{n-1}{n}} \, dx$$

$$= \int_{\Omega} (\lambda[\Omega])^{\frac{n-1}{n}}(\det D^2u)^{1/n}|w|^n \, dx.$$  

Dividing the first and last expressions in the above estimates by $(\lambda[\Omega])^{\frac{n-1}{n}}$, we obtain (2.1). □

3. Eventual Smoothness and a New Monotonicity Formula for the Iterative Scheme

In this section, we show the eventual smoothness and a new monotonicity formula for the iterative scheme (1.4). They are stated in Proposition 3.1 and Lemma 3.2.

We have the following eventual smoothness of solutions to the iterative scheme (1.4). The proof is a modification of the proof of Proposition 2.8 in [11].

Proposition 3.1. Let $\Omega$ be an open, bounded convex set in $\mathbb{R}^n$ with nonempty interior. Let $u_0 \in C(\Omega)$ be a nonzero convex function on $\Omega$ with $0 < R(u_0) < \infty$. For $k \geq 0$, define the sequence $u_k \in K$ to be the solutions of the Dirichlet problem (1.4). Then, $u_1 \in C^{0,\frac{1}{n}}(\Omega)$, and $u_{k+1}$ is strictly convex in $\Omega$ and $u_{k+1} \in C^{2k,\frac{1}{n}}(\Omega)$ for all $k \geq 1$.  

We recall that a convex function \( u \) on an open bounded convex domain \( \Omega \) is said to be strictly convex in \( \Omega \), if for any \( x \in \Omega \) and \( p \in \partial u(x) \),
\[
 u(z) > u(x) + p \cdot (z - x) \quad \text{for all} \quad z \in \Omega \setminus \{x\}.
\]

**Proof of Proposition 3.1.** First, using the Aleksandrov’s maximum principle in Theorem 2.4, we note that each \( u_{k+1} \) is uniformly bounded, that is \( M_{k+1} = \|u_{k+1}\|_{L^\infty(\Omega)} < \infty \). The regularity \( C^{0,\frac{1}{n}}(\Omega) \) of \( u_1 \) is a consequence of the Aleksandrov maximum principle. Since \( u_0 \neq 0 \), and \( R(u_0) > 0 \), we have \( u_1 \neq 0 \). The convexity of \( u_1 \) shows that \( u_1 < 0 \) in \( \Omega \).

We show by induction the following:

**Claim.** \( u_{k+1} \) is strictly convex in \( \Omega \) and \( u_{k+1} \in C^{2k,\frac{1}{n}}(\Omega) \) for all \( k \geq 1 \).

We start with the base case \( k = 1 \). For each \( \varepsilon \in (0, M_2) \), let \( \Omega' := \Omega(\varepsilon) = \{x \in \Omega : u_2(x) \leq -\varepsilon\} \). Since \( u_2 \in C(\Omega) \) is convex, the set \( \Omega(\varepsilon) \) is convex with nonempty interior. Note that, since \( |u_1| > 0 \) in \( \Omega \) and \( u_1 \in C^{0,\frac{1}{n}}(\Omega) \), by continuity, \( |u_1| \geq m(n, \varepsilon) > 0 \) in \( \Omega' \). Since \( \lambda[\Omega]|m''(n, \varepsilon) \leq \det D^2 u_2 = R(u_1)|u_1|^n \leq R(u_1)M_1^n \) in \( \Omega' \) and \( u_2 = -\varepsilon \) on \( \partial \Omega' \),

the function \( u_2 \) is strictly convex in \( \Omega' \) by the localization theorem of Caffarelli [4] (see also [7, Theorem 4.10] and [8, Corollary 5.2.2]). Moreover, \( u_1 \in C^{0,\frac{1}{n}}(\Omega') \). Now, using Caffarelli’s \( C^{2,\alpha} \) estimates [5], we have \( u_2 \in C^{2,\frac{1}{n}}(\Omega') \). Since \( \varepsilon \in (0, M_2) \) is arbitrary, we conclude \( u_2 \in C^{2,\frac{1}{n}}(\Omega) \) and \( u_2 \) is strictly convex in \( \Omega \).

Suppose the claim holds up to \( k - 1 \) where \( k \geq 2 \). We show it also holds for \( k \). For each \( \varepsilon_k \in (0, M_{k+1}) \), let \( \Omega(\varepsilon_k) = \{x \in \Omega : u_{k+1}(x) \leq -\varepsilon_k\} \). Since \( u_{k+1} \in C(\Omega) \) is convex, the set \( \Omega(\varepsilon_k) \) is convex with nonempty interior. Let us denote \( \Omega_k' = \Omega(\varepsilon_k) \) for brevity. Note that, by continuity, \( |u_k| \geq m(n, k, \varepsilon) > 0 \) in \( \Omega_k' \).

Since \( \lambda[\Omega]|m''(n, k, \varepsilon) \leq \det D^2 u_{k+1} = R(u_k)|u_k|^n \leq R(u_k)M_k^n \) in \( \Omega_k' \) and \( u_{k+1} = -\varepsilon_k \) on \( \partial \Omega_k' \), the function \( u_{k+1} \) is strictly convex in \( \Omega_k' \) by the localization theorem of Caffarelli. By the induction hypothesis, \( u_k \in C^{2(k-1),\frac{1}{n}}(\Omega_k') \). In the interior of \( \Omega_k' \), the equation \( \det D^2 u_{k+1} = R(u_k)|u_k|^n \) now becomes uniformly elliptic with \( C^{2(k-1),\frac{1}{n}} \) right hand side. Therefore we have \( u_{k+1} \in C^{2k,\frac{1}{n}}(\Omega_k') \).

Since \( \varepsilon_k \in (0, M_{k+1}) \) is arbitrary, we conclude \( u_{k+1} \in C^{2k,\frac{1}{n}}(\Omega) \) and \( u_{k+1} \) is strictly convex in \( \Omega \).

Our key observation is the following monotonicity result for the iterative scheme (1.4). Note that
\[
 R(u_k) \geq \lambda[\Omega] \quad \text{for all} \quad k \geq 1.
\]

**Lemma 3.2** (Monotonicity formula for the iterative scheme). Let \( \Omega \) be a bounded open convex domain in \( \mathbb{R}^n \). Let \( u_0 \in C(\Omega) \) be a nonzero convex function on \( \Omega \) with \( 0 < R(u_0) < \infty \). Let \( w \) be a nonzero Monge-Ampère eigenfunction of \( \Omega \). Consider the iterative scheme (1.4). If \( k \geq 3 \), then
\[
 \int_\Omega |u_{k+1}|w^n\ dx \geq \int_\Omega |u_k|w^n\ dx + \frac{R(u_k)^{1/n} - (\lambda[\Omega])^{1/n}}{(\lambda[\Omega])^{1/n}} \int_\Omega |u_k|w^n\ dx.
\]

**Proof of Lemma 3.2.** By Proposition 3.1 we have \( u_{k+1} \in C^{0,\frac{1}{n}}(\Omega) \) for all \( k \geq 3 \). We apply Proposition 2.5 to \( u_{k+1} \) and recall
\[
 \det D^2 u_{k+1} = R(u_k)|u_k|^n,
\]
to get
\[
\int_{\Omega} |u_{k+1}|w^n \, dx \geq \frac{1}{(\lambda[\Omega])^{1/n}} \int_{\Omega} (\det D^2 u_{k+1})^{1/n} |w|^{n} \, dx
\]
\[
= \frac{|R(u_k)|^{1/n}}{(\lambda[\Omega])^{1/n}} \int_{\Omega} |u_k|w^n \, dx
\]
\[
= \int_{\Omega} |u_k|w^n \, dx + \frac{[R(u_k)]^{1/n} - (\lambda[\Omega])^{1/n}}{(\lambda[\Omega])^{1/n}} \int_{\Omega} |u_k|w^n \, dx.
\]

The monotonicity property is thus proved.

We recall the following monotonicity property in [1] Lemma 3.1.

**Lemma 3.3.** ([1] Lemma 3.1) Let $\Omega$ be an open, bounded convex set in $\mathbb{R}^n$ with nonempty interior. Let $u_0 \in C(\Omega)$ be a nonzero convex function on $\Omega$ with $0 < R(u_0) < \infty$. Consider the iterative scheme (1.4). Then for all $k \geq 0$, we have
\[
R(u_{k+1})||u_{k+1}|n^2_{L^{n+1}(\Omega)} \leq R(u_k)||u_k|n^2_{L^{n+1}(\Omega)}.
\]

Lemma 3.3 was stated and proved in [1] for $u_0$ satisfying (i), (ii) and (iii) in Theorem 1.2. However, the proof in [1] only used the assumptions $0 < R(u_0) < \infty$ and $u_0$ is convex. We include here the short proof of Lemma 3.3 for reader’s convenience.

**Proof of Lemma 3.3.** The proof follows by multiplying both sides of the first equation of (1.4) by $|u_{k+1}|$, integrating over $\Omega$ and then using the H"{o}lder inequality:
\[
R(u_{k+1})||u_{k+1}|n^2_{L^{n+1}(\Omega)} = \int_{\Omega} |u_{k+1}| \det D^2 u_{k+1} \, dx
\]
\[
= R(u_k) \int_{\Omega} |u_k| |u_{k+1}| \leq R(u_k)||u_k|n^2_{L^{n+1}(\Omega)}||u_{k+1}|n^2_{L^{n+1}(\Omega)}.
\]

Using $u_{k+1} \neq 0$ for all $k \geq 0$, we obtain the claimed monotonicity property. \hfill \Box

4. CONVERGENCE OF THE ITERATIVE SCHEME

In this section, we prove Theorem 1.4.

Some of our arguments in Step 2 of the proof of Theorem 1.4 are similar to those in the proof of Theorem 1.2 in [1]. However, since we can obtain the convergence of $R(u_k)$ to $\lambda[\Omega]$ from Lemma 3.2 we can avoid using the continuity property of the energy $\int_{\Omega} |u_k| \det D^2 u_k \, dx$ for a converging sequence of convex functions $u_k$ with an upper bound on the density of the Monge-Ampère measure $\det D^2 u_k$ (see [1] Lemma 2.9). Moreover, the monotone property in Lemma 3.2 also allows us to quickly conclude that the whole sequence $u_k$ converges to the same limit.

**Proof of Theorem 1.4.** We fix a nonzero Monge-Ampère eigenfunction $w$. The assumptions on $u_0$ imply that $u_k \neq 0$ for all $k \geq 0$. The proof is split into several steps.

**Step 1:** The whole sequence $R(u_k)$ converges to $\lambda[\Omega]$.

Using the monotonicity property established in Lemma 3.2 we find that if $k \geq 3$, then
\[
(4.1) \quad \|u_k\|_{L^\infty(\Omega)} \geq \frac{\int_{\Omega} |u_k|w^n \, dx}{\int_{\Omega} |w|n \, dx} \geq \frac{\int_{\Omega} |u_3|w^n \, dx}{\int_{\Omega} |w|n \, dx} \geq c(n, \Omega, u_0) > 0.
\]

For each $k \geq 1$, using $R(u_k) \geq \lambda[\Omega]$, we obtain from Lemma 3.3 that
\[
(4.2) \quad \|u_k\|_{L^{n+1}(\Omega)}^n \leq \frac{R(u_0)}{R(u_k)} \|u_0\|_{L^{n+1}(\Omega)}^n \leq \frac{R(u_0)}{\lambda[\Omega]} \|u_0\|_{L^{n+1}(\Omega)}^n < \infty.
\]
This implies that the increasing sequence $\int_{\Omega} |u_k||w|^n \, dx$ is bounded from above and thus converges to a limit $L$

$$\lim_{k \to \infty} \int_{\Omega} |u_k||w|^n = L \in (0, \infty)$$

where we used (1.1) to get $L > 0$.

Now, for $k \geq 3$, taking into account the full monotonicity property in Lemma 5.2, we get

$$[R(u_k)]^{1/n} - (\lambda[\Omega])^{1/n} \leq \frac{(\lambda[\Omega])^{1/n} \int_{\Omega} (|u_{k+1}| - |u_k|)|w|^n \, dx}{\int_{\Omega} |u_k||w|^n \, dx}$$

$$\leq \frac{(\lambda[\Omega])^{1/n} \int_{\Omega} (|u_{k+1}| - |u_k|)|w|^n \, dx}{\int_{\Omega} |u_k||w|^n \, dx}.$$ (4.4)

Letting $k \to \infty$ in (4.4) and recalling (4.3), we conclude that the whole sequence $R(u_k)$ converges to $\lambda[\Omega]$:

$$\lim_{k \to \infty} R(u_k) = \lambda[\Omega].$$ (4.5)

**Step 2: Convergence of $u_k$ to a nontrivial Monge-Ampère eigenfunction $u_\infty$ of $\Omega$.**

Next, applying the Aleksandrov estimate in Theorem 2.4 to $u_{k+1}$ where $k \geq 0$, and then using the Hőlder inequality together with (4.5), we find

$$\|u_{k+1}\|_{L^n(\Omega)}^n \leq C(n, \Omega) \int_{\Omega} \det D^2 u_{k+1} \, dx = C(n, \Omega) R(u_k) \int_{\Omega} |u_k|^n \, dx \leq C(n, \Omega) R(u_k) \|u_k\|_{L^{n+1}(\Omega)}^{1/(n+1)} \leq C(n, \Omega, u_0).$$

Hence, we obtain the uniform $L^\infty$ bound

$$\|u_k\|_{L^\infty(\Omega)} \leq C(n, \Omega, u_0) < \infty.$$ From the Aleksandrov estimate, we have the uniform $C^{0, 1/(n)}(\Omega)$ bound for $u_k$ when $k \geq 1$:

$$\|u_k\|_{C^{0, 1/(n)}(\Omega)} \leq C(n, \Omega, u_0).$$

Therefore, up to extracting a subsequence, we have the following uniform convergence

$$u_{k_j} \to u_\infty \neq 0$$

for a convex function $u_\infty \in C(\Omega)$ with $u_\infty = 0$ on $\partial \Omega$ while we also have the uniform convergence

$$w_{k+j} \to w_\infty \neq 0$$

for a convex function $w_\infty \in C(\Omega)$ with $w_\infty = 0$ on $\partial \Omega$.

Thus, letting $j \to \infty$ in

$$\det D^2 u_{k_j+1} = R(u_{k_j}) |u_{k_j}|^n,$$

using (4.5) and the weak convergence of the Monge-Ampère measure (see [21 Corollary 2.12] and [8, Lemma 5.3.1]), we get

$$\det D^2 w_\infty = \lambda[\Omega]|u_\infty|^n.$$ (4.6)

Letting $j \to \infty$ in the following monotonicity property (see Lemma 3.3)

$$R(u_{k+j+1}) \|u_{k+j+1}\|_{L^{n+1}(\Omega)} \leq R(u_{k+j}) \|u_{k+j+1}\|_{L^{n+1}(\Omega)} \leq R(u_{k+j}) \|u_{k+j+1}\|_{L^{n+1}(\Omega)},$$

and recalling (4.5), we find that

$$\|w_\infty\|_{L^{n+1}(\Omega)} = \|u_\infty\|_{L^{n+1}(\Omega)}.$$
However, from (4.6), we have

\[ R(w_\infty)\|w_\infty\|_{L^{n+1}(\Omega)}^{n+1} = \int_\Omega |w_\infty| \det D^2 w_\infty \, dx = \lambda[\Omega] \int_\Omega |u_\infty|^n |w_\infty| \, dx \]

\[ \leq \lambda[\Omega] \|u_\infty\|_{L^{n+1}(\Omega)}^n \|w_\infty\|_{L^{n+1}(\Omega)} \]

\[ = \lambda[\Omega] \|w_\infty\|_{L^{n+1}(\Omega)}^{n+1}. \]

Since \( R(w_\infty) \geq \lambda[\Omega] \), we must have \( R(w_\infty) = \lambda[\Omega] \), and the inequality above must be an equality, but this gives \( u_\infty = cw_\infty \) for some constant \( c > 0 \). Thus, from (4.6), we have

\[ \det D^2 w_\infty = c^n \lambda[\Omega] |w_\infty|^n. \]

It follows from the uniqueness part of Theorem 1.1 that \( c = 1 \) and \( w_\infty = u_\infty \) is a Monge-Ampère eigenfunction of \( \Omega \). Passing to the limit in Lemma 3.2 we have

\[ \int_\Omega |u_\infty|^n |w|^n = \lim_{k \to \infty} \int_\Omega |u_k|^n |w|^n = L. \]

With this property and the uniqueness up to positive multiplicative constants of the Monge-Ampère eigenfunctions of \( \Omega \), we conclude that the limit \( u_\infty \) does not depend on the subsequence \( u_{k_j} \). This shows that the whole sequence \( u_k \) converges to a nonzero Monge-Ampère eigenfunction \( u_\infty \) of \( \Omega \).

**Step 3: Convergence estimate for** \([R(u_k)]^{1/n} - (\lambda[\Omega])^{1/n}\).

Let \( k \geq 3 \). By (4.4), and the fact that \( \int_\Omega |u_k|^n |w|^n \, dx \) increases to \( L = \int_\Omega |u_\infty|^n |w|^n \, dx \), we have the estimates

\[ [R(u_k)]^{1/n} - (\lambda[\Omega])^{1/n} \leq \frac{\int_\Omega (|u_{k+1}| - |u_k|) |w|^n \, dx}{\int_\Omega |w|^n \, dx} \]

\[ \leq \frac{\int_\Omega (|u_\infty| - |u_k|) |w|^n \, dx}{\int_\Omega |w|^n \, dx} \]

\[ \leq C(n, \Omega, u_0) \int_\Omega |u_\infty - u_k| \, dx. \]

The last statement of the theorem follows. \( \square \)

5. **Energy characterization of Monge-Ampère eigenfunctions**

In this section, we make some remarks on the energy characterization of the Monge-Ampère eigenfunctions motivated from the proof of Theorem 1.4 in Section 4.

Observe that the Monge-Ampère measure of each \( u_{k_j+1} \) has density \( R(u_{k_j}) |u_{k_j}|^n \) which is uniformly bounded from above by a positive constant independent of \( k \). Thus, using the continuity property of the energies \( \int_\Omega |u_{k_j+1}| \det D^2 u_{k_j+1} \, dx \) (see 11 Lemma 2.9), we get

\[ \lim_{j \to \infty} \int_\Omega |u_{k_j+1}| \det D^2 u_{k_j+1} \, dx = \int_\Omega |u_\infty| \det D^2 u_\infty \, dx \]

so, by (4.5)

\[ R(w_\infty) = \lim_{j \to \infty} R(u_{k_j+1}) = \lambda[\Omega]. \]

We would like to show that \( w_\infty \) is a Monge-Ampère eigenfunction of \( \Omega \). In the proof of Theorem 1.4 we used the monotonicity property of the scheme (1.4) given by Lemma 3.3. Finding a direct proof from (5.1) leads us to the following question:

**Question 5.1.** Assume that \( u \in K \) satisfies \( R(u) = \lambda[\Omega] \). Is \( u \) a Monge-Ampère eigenfunction?
It is well known that for a connected, open and bounded domain in $\Omega \subset \mathbb{R}^n$, if $v \in W^{1,2}_0(\Omega) \setminus \{0\}$ satisfies
\[
\int_\Omega |Dv|^2 \, dx = \lambda_1 \int_\Omega |v|^2 \, dx
\]
where $\lambda_1$ is the first eigenvalue of the Laplace operator with zero boundary condition in $\Omega$ then $v$ is in fact a first eigenfunction of the Laplace operator on $\Omega$.

An affirmative answer to Question 5.1 will provide a nonlinear analogue of the above result. We partially answer Question 5.1 in the affirmative under the hypotheses of Lemma 5.2 so we can also use this lemma to conclude that $u$ is a Monge-Ampère eigenfunction of $\Omega$.

**Lemma 5.2.** Let $\Omega$ be a bounded open convex domain in $\mathbb{R}^n$. Let $\lambda[\Omega]$ be the Monge-Ampère eigenvalue of $\Omega$. Assume that $u \in C^3(\Omega)$ with $D^2 u > 0$ in $\Omega$ and $R(u) = \lambda[\Omega]$. Then $u$ is a Monge-Ampère eigenfunction of $\Omega$.

Using Proposition 3.1 and (4.6), we find that the function $w_\infty$ in the proof of Theorem 1.4 satisfies the hypotheses of Lemma 5.2 so we can also use this lemma to conclude that $w_\infty$ is a Monge-Ampère eigenfunction of $\Omega$, thus bypassing the arguments after (4.6) in Step 2 there.

**Proof of Lemma 5.2.** Suppose that $u \in C^3(\Omega)$, with $D^2 u > 0$ in $\Omega$ and $R(u) = \lambda[\Omega]$. Then $u$ is uniform convex in each compact subset of $\Omega$, that is, if $E \subset \Omega$ is compact then $D^2 u \geq c(E, u) > 0$ in $E$.

By multiplying a positive constant to $u$, we can assume that $\|u\|_{L^{n+1}(\Omega)} = 1$. Let $v \in C^\infty_c(\Omega)$. Then, using the uniform convexity of $u$ in each compact subset of $\Omega$, we have $u + tv \in C^\infty_c(\Omega)$ for $|t|$ small. Let
\[
f(t) = R(u + tv) = \frac{\int_\Omega (-u - tv) \det D^2(u + tv) \, dx}{\int_\Omega (-u - tv)^{n+1} \, dx}.
\]
Then, using $\|u\|_{L^{n+1}(\Omega)} = 1$ and $R(u) = \lambda[\Omega]$, we can compute
\[
f'(0) = \int_\Omega (-v) \det D^2 u \, dx + \frac{d}{dt} \int_\Omega (-u) \, dx \bigg|_{t=0} \det D^2(u + tv) \, dx - (n + 1)\lambda[\Omega] \int_\Omega |u|^n(-v) \, dx.
\]
Let $(U^{ij})$ be the cofactor matrix of the Hessian $D^2 u = (u_{ij}) \equiv \frac{\partial^2 u}{\partial x_i \partial x_j}$. Then
\[
\frac{d}{dt} \bigg|_{t=0} \det D^2(u + tv) = U^{ij}v_{ij}.
\]
Integrating by parts twice, using $u \in C^3(\Omega)$, and the fact that each row and each column of $(U^{ij})$ is divergence free, we get
\[
\int_\Omega (-u) \frac{d}{dt} \bigg|_{t=0} \det D^2(u + tv) \, dx = \int_\Omega (-u)U^{ij}v_{ij} \, dx = \int_\Omega (-v)U^{ij}u_{ij} \, dx = n \int_\Omega (-v) \det D^2 u \, dx.
\]
Therefore,
\[
f'(0) = (n + 1) \left[ \int_\Omega (-v) \det D^2 u - \lambda[\Omega] \int_\Omega |u|^n(-v) \, dx \right].
\]
Since $f$ has a minimum value at 0, we have $f'(0) = 0$. As a consequence, we find that
\[
\int_\Omega (-v) \det D^2 u - \lambda[\Omega] \int_\Omega |u|^n(-v) \, dx = 0
\]
for all $v \in C^\infty_c(\Omega)$. It follows that $\det D^2 u = \lambda[\Omega]|u|^n$ in $\Omega$ and $u$ is a Monge-Ampère eigenfunction of $\Omega$. \hfill \Box
It would be interesting to remove the assumption of strict convexity in Lemma 5.2. The difficulty here is that we could not use the variations $u + tv$ for all $v \in C^\infty_c(\Omega)$ and all $t$ small. Without the strict convexity of $u$, we can only use the variations of the form $u + tv$ where $v \in \mathbb{K}$ and $t \geq 0$. Related to this issue, we have the following result for smooth and uniformly convex domains.

**Lemma 5.3.** Assume that $\Omega$ is a bounded, open, and uniformly convex domain in $\mathbb{R}^n$ with $\partial \Omega \in C^3$. Let $\lambda[\Omega]$ be the Monge-Ampère eigenvalue of $\Omega$. Suppose that $u \in \mathbb{K} \cap C^3(\Omega) \cap C^{1,1}(\overline{\Omega})$ and $R(u) = \lambda[\Omega]$. Then, $u$ is a Monge-Ampère eigenfunction of $\Omega$.

The proof of Lemma 5.3 relies on an interesting result of Lions [14] on the characterization of the dual cone of the cone of convex functions using second derivatives of positive symmetric matrices. In fact, Question 5.1 fits into the framework of Calculus of variations with a convexity constraint. The assumption $u \in C^{1,1}(\overline{\Omega})$ in Lemma 5.3 is motivated by the classical result of Lions [13] which says that if $\Omega$ is a bounded open, and uniformly convex domain in $\mathbb{R}^n$ with $\partial \Omega \in C^3$, then the Monge-Ampère eigenfunctions of $\Omega$ are $C^{1,1}(\overline{\Omega})$.

**Proof of Lemma 5.3.** Suppose that $u \in \mathbb{K} \cap C^3(\Omega) \cap C^{1,1}(\overline{\Omega})$, and $R(u) = \lambda[\Omega]$. By multiplying a positive constant to $u$, we can assume that $\|u\|_{L^{n+1}(\Omega)} = 1$. Let $v \in \mathbb{K} \cap C^2(\overline{\Omega})$. Then, $u + tv \in \mathbb{K}$ for all $t \geq 0$. Let

$$f(t) = \frac{\int_\Omega (u - tv) \det D^2(u + tv) \, dx}{\int_\Omega (u - tv)^{n+1} \, dx}.$$  

As in the proof of Lemma 5.2, we can compute

$$f'(0) = \int_\Omega (-v) \det D^2 u \, dx + \int_\Omega (-u) U^{ij} v_{ij} \, dx - (n + 1) \lambda[\Omega] \int_\Omega |u|^n (-v) \, dx,$$

where $(U^{ij})$ is the cofactor matrix of the Hessian $D^2 u = (u_{ij}) = \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right)$. Integrating by parts twice, using $u \in C^3(\Omega) \cap C^{1,1}(\overline{\Omega})$, we get

$$\int_\Omega (-u) U^{ij} v_{ij} \, dx = \int_\Omega (-v) U^{ij} u_{ij} \, dx = n \int_\Omega (-v) \det D^2 u \, dx.$$

Therefore,

$$f'(0) = (n + 1) \int_\Omega \left[ \lambda[\Omega] |u|^n - \det D^2 u \right] v \, dx.$$

Since $f$ has a minimum value at 0 on $[0, \infty)$, we have $f'(0) \geq 0$. As a consequence, we find that

$$\int_\Omega \left[ \lambda[\Omega] |u|^n - \det D^2 u \right] v \, dx \geq 0$$  

for all $v \in C^2(\overline{\Omega}) \cap \mathbb{K}$. Using an approximation argument and recalling that $u \in \mathbb{K} \cap C^{1,1}(\overline{\Omega})$, we find that the inequality (5.2) also holds for all $v \in \mathbb{K}$.

Therefore, the bounded function $\lambda[\Omega] |u|^n - \det D^2 u$ belongs to the dual cone of the cone of convex functions $\mathbb{K}$. By a result of Lions [14, p. 1389] (see also Carlier [6, Theorem 2]), there exist bounded Radon measures $(\mu_{ij})_{1 \leq i, j \leq n}$ on $\Omega$ such that $\mu_{ij} = \mu_{ji}$, $(\mu_{ij}) \geq 0$ in the sense of symmetric matrices, and

$$\lambda[\Omega] |u|^n - \det D^2 u = \frac{\partial^2 \mu_{ij}}{\partial x_i \partial x_j}$$

in the sense of distributions, that is, for all $v \in C^2(\overline{\Omega})$ with $v = 0$ on $\partial \Omega

$$\int_\Omega \left( \lambda[\Omega] |u|^n - \det D^2 u \right) v \, dx = \int_\Omega v_{ij} d\mu_{ij}.$$  

We can say more about the measures $(\mu_{ij})$. From the proof of Theorem 2 in [6], there exist $C^\infty_c(\Omega)$ functions $\psi_{ij}^{(k)}$ $(1 \leq i, j \leq n)$ such that for each $k = 1, 2, \cdots$, the matrix $(\psi_{ij}^{(k)})$ is symmetric and
nonnegative definite, and the weak limit of \((\psi^{(k)}_{ij})\), as \(k \to \infty\), is \((\mu_{ij})\). In particular, for any compact set \(K \subset \Omega\) with nonempty interior \(\overset{\circ}{K}\) and any nonnegative definite matrix \((a^{ij})\) with entries \(a^{ij} \in C(K)\), we have

\[
\lim_{k \to \infty} \int_K a^{ij} \psi^{(k)}_{ij} \, dx = \int_K a^{ij} d\mu_{ij}.
\]

Using the boundedness of \((\mu_{ij}(\Omega))\) and approximations, we find that (5.3) also holds for \(v \in C^{1,1}(\overline{\Omega}) \cap C^2(\Omega)\) with \(v = 0\) on \(\partial \Omega\).

In particular, for \(v = u\), using \(R(u) = \lambda[\Omega]\), we obtain

\[
\int_{\Omega} u_{ij} d\mu_{ij} = \int_{\Omega} (\lambda[\Omega]|u|^n - \det D^2 u) \, u \, dx = -\int_{\Omega} (\lambda[\Omega]|u|^n - \det D^2 u) \, |u| \, dx = 0.
\]

Let

\[
E = \{x \in \Omega : D^2 u > 0\}.
\]

We show that

\[
\lambda[\Omega]|u|^n - \det D^2 u = 0 \quad \text{on} \ E.
\]

Indeed, let \(K\) be any compact subset of \(E\) with nonempty interior \(\overset{\circ}{K}\). Then, \(D^2 u \in C^2(K)\) and there is a positive constant \(c_K > 0\) such that \(D^2 u - c_K I_n \geq 0\) on \(K\) where \(I_n\) denotes the identity \(n \times n\) matrix. Therefore, for each \(k\), using the fact that the matrix \((\psi^{(k)}_{ij})\) is symmetric and nonnegative definite, we have

\[
\int_{\overset{\circ}{K}} u_{ij} \psi^{(k)}_{ij} \, dx \geq c_K \int_{\overset{\circ}{K}} \text{Trace}(\psi^{(k)}_{ij}) \, dx \geq 0.
\]

From (5.5), we deduce that

\[
0 = \int_{\Omega} u_{ij} d\mu_{ij} \geq \int_{\overset{\circ}{K}} u_{ij} d\mu_{ij}.
\]

Now, letting \(k \to \infty\) in (5.7) and using (5.4), we obtain

\[
0 \geq \int_{K} u_{ij} d\mu_{ij} \geq c_K \int_{K} d \text{Trace}(\mu_{ij}) \geq 0.
\]

Hence,

\[
\text{Trace}(\mu_{ij})(K) = 0.
\]

Since this holds for all compact subsets \(K\) of \(E\), it follows that

\[
\text{Trace}(\mu_{ij})(E) = 0.
\]

Now, let \(v \in C^\infty_c(E)\). Then, there is a positive constant \(C_v\) such that \(-C_v I_n \leq D^2 v \leq C_v I_n\) in \(E\). Using (5.8) and (5.4), we obtain

\[
\left| \int_{E} v_{ij} d\mu_{ij} \right| \leq C_v \int_{E} d \text{Trace}(\mu_{ij}) = C_v \text{Trace}(\mu_{ij})(E) = 0.
\]

Recalling (5.3), we find that

\[
\int_{E} (\lambda[\Omega]|u|^n - \det D^2 u) \, v \, dx = 0 \quad \text{for all} \ v \in C^\infty_c(E)
\]

from which we obtain (5.6) as claimed.

From (5.6), we find that the last equation of (5.5) reduces to

\[
\int_{\Omega \setminus E} (\lambda[\Omega]|u|^n - \det D^2 u) \, |u| \, dx = 0.
\]
Since $|u| > 0$ and $\det D^2u = 0$ in $\Omega \setminus E$, we find that the Lebesgue measure of $\Omega \setminus E$ is 0. Thus, from (5.6), we in fact have

$$\lambda[\Omega]|u|^n - \det D^2u = 0 \text{ on } \Omega.$$

Hence $u$ is a Monge-Ampère eigenfunction of $\Omega$. \hfill \Box

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