An alternate model for protective measurements of two-level systems.

Anirban Das†
Indian Institute of Technology, Kharagpur, India

N.D. Hari Dass‡
Institute of Mathematical Sciences, Chennai, India

In this article we propose an alternate model for the so-called protective measurements, more appropriately adiabatic measurements of a spin $\frac{1}{2}$ system where the apparatus is also a quantum system with a finite dimensional Hilbert space. This circumvents several technical as well as conceptual issues that arise when dealing with an infinite dimensional Hilbert space as in the analysis of conventional Stern-Gerlach experiment. Here also it is demonstrated that the response of the detector is continuous and it directly measures expectation values without altering the state of the system when the unknown original state is a nondegenerate eigenstate of the system Hamiltonian, in the limit of ideal adiabatic conditions. We have also computed the corrections arising out of the inevitable departures from ideal adiabaticity i.e the time of measurement being large but finite. To overcome the conceptual difficulties with a quantum apparatus, we have simulated a classical apparatus as a large assembly of spin-$1/2$ systems. We end this article with a conclusion and a discussion of some future issues.

I. INTRODUCTION

Interpretation of measurements in quantum theory and the issue of reality of wave functions have occupied a central position in physics. The conventional wisdom is that the property of linear superposition of physical states necessitates an ensemble interpretation of physical states (wave functions) and the outcome of a measurement on a single quantum state is random and consequently no reality, at least in the same sense of the word as was associated in pre-quantum era, can be ascribed to a wave function. For generic quantum states this is indeed the accepted view point and has been beautifully supported even by recent developments like the no-cloning theorem etc. But Aharonov and Vaidman [1], suitably supported even by recent developments like the no-cloning theorem etc. But Aharonov and Vaidman [1], and subsequently Aharonov, Anandan and Vaidman [2] showed that adiabatic measurements on a restricted class of unknown single quantum states can preserve the original state in the limit of infinitely long measurement times and at the same time give as output the expectation value of an arbitrary observable. Since the original state is unaltered, it can be used in further measurements of this type, and with the right number of independent measurements, the state can be fully determined without affecting it. As shown by these authors the scheme works only when the original unknown state is restricted to be a nondegenerate eigenstate of the system Hamiltonian. The main caveat is that in every realisable set up the measurement time can be made quite large but is never infinite. This leads to a correction which is harmless for ensembles but critical for a single quantum state. As shown in [3] this correction precludes ascribing reality to single wavefunctions of even this restricted class. However, the present work may point to a way out of this difficulty.

In their original analysis [1,2] what played the role of apparatus was a quantum mechanical system whose Hilbert space dimensionality is infinite. This leads to a variety of technical issues which have been addressed in detail in [3]. In this paper we have found an alternate model for adiabatic (protective) measurements where we have replaced the ‘apparatus’ also by a spin-$1/2$ system. To avoid various conceptual pitfalls of a ‘quantum apparatus’ we have subsequently considered an extension whereby a large number of spin-$1/2$ particles is made to simulate a ‘classical apparatus.’

II. CONVENTIONAL MEASUREMENT

To set the stage for a discussion of the main contents of this paper, let us first discuss the idea of a conventional (quantum)measurement.

The current viewpoint, developed by the founding fathers of Quantum Mechanics, is that the measurement of an observable corresponding to an operator $A$ of a system in the state $\psi$ will have as its outcome only the eigenvalues of $A$. The choice of the eigenvalue is completely random and the system after measurement will ‘collapse’ into the eigenfunction which corresponds to the eigenvalue which is the outcome. As a consequence, the original state is irretrievably altered. Even though the individual outcomes are random, the frequency with which the eigenvalue $\lambda_i$ occurs is given by $|\langle \psi | \chi_i \rangle|^2$ where $|\chi_i \rangle$ is the corresponding eigenstate.

Now we discuss how conventional or impulsive measurement is realised when the apparatus is described by an infinite dimensional Hilbert space. Let $Q$ be an operator, corresponding to the observable of the system we wish to measure, and let it interact with the appropriate apparatus through an interaction,
\[ H_I = \hbar g(t)Q_A \otimes Q_S \]  

where \( Q_A \) is an observable of the apparatus, and \( g(t) \) is the strength of the interaction normalized such that \( \int_0^\tau g(t) = 1 \). The interaction is nonzero only in the short interval \([0, \tau]\). Let the system be initially in state \(|\nu\rangle\) which is not necessarily an eigen state of \( Q_S \), and the apparatus be in state \(|\phi(r_0)\rangle\) which is a wave packet of eigen state of the operator \( R_A \) conjugate to \( Q_A \), centered at the eigen value \( r_0 \). The interaction \( H_I \) is of short duration, and assumed to be so strong that the effect of the free Hamiltonians of the apparatus and the system can be neglected. More explicitly, the total Hamiltonian is given by

\[ H(t) = I_S \otimes H_A + H_S \otimes I_A + \hbar g(t)Q_A \otimes Q_S \]  

If \(|t = 0\rangle\) is the state vector of the combined apparatus-system just before the measurement process begins, the state vector after \( T \) is given by,

\[ |t = T\rangle = T e^{\frac{\hbar}{2} \int_0^T H(r) d\tau |t = 0\rangle} \]  

where \( T \) is the time ordering operator. Consequently, after time \( \tau \) the state of the system is

\[ |t = \tau\rangle = e^{-iQ_A \otimes Q_S |\nu\rangle \langle \phi(r_0)|} \]  

If we expand \(|\nu\rangle\) in the eigenstates of \( Q_S \), \(|s_i\rangle\), we get,

\[ |\psi(\tau)\rangle = \sum_i e^{-iQ_A s_i c_i |s_i\rangle \langle \phi(r_0)|} \]  

where \( s_i \) are the eigen values of \( Q_S \) and \( c_i \) are the expansion coefficients. The exponential term shifts the center of the wave packet by \( s_i \):

\[ |\psi(\tau)\rangle = \sum_i c_i |s_i\rangle \langle \phi(r_0 + s_i)| \]  

To see this, first consider a \( r\)-representation of \(|\phi(r_0)|\):

\[ |\phi(r_0)\rangle = \int dr f(r-r_0)|R = r\rangle \]  

\[ = \int dq dr f(r-r_0)\langle q(r)|q\rangle \]  

On using \( \langle q(r)| = e^{-iqr} \)

\[ e^{-iQ_A s_i} = \int dq \, dr f(r-r_0)\langle q(r)\rangle e^{-iqs_i}|a\rangle \]  

\[ = dq \, dr f(r-r_0)\langle q(r)\rangle e^{-iqs_i}|a\rangle \]  

\[ = |\phi(r_0 + s_i)| \]  

In the last step we have shifted the variable of integration to \( r + s_i \) and used the \( r\)-representation eqn(8) again.

The state given in eqn(7) is an entangled state, where the position of the wave packet gets correlated with the eigen state \(|s_i\rangle\). Detecting the center of the wave packet at \( r_0 + s_i \) will throw the system into the eigenstate \(|s_i\rangle\). Thus we see that impulsive measurement leads to complete destruction of the wave function and once a measurement is made no further measurement can be made. Also the result of the outcome has to be statistically determined. Such a measurement is realized through the Stern-Gerlach experiment where the apparatus is really the position of the silver atom and hence is a continuous variable and the Hilbert space is unbounded.

Now we propose an apparatus to measure spin for a two-state system \((\text{spin} 1/2)\) like the Stern-Gerlach apparatus but unlike the Stern-Gerlach apparatus where the position is unbounded and is an observable operating on an infinite dimensional Hilbert space, the Hilbert space of our apparatus will be taken to be finite dimensional.

The main advantage in using a finite dimensional Hilbert space is that our operators are also bounded so perturbations can be applied safely because the interaction Hamiltonian will never be arbitrarily large. Adiabatically measuring spin this way is a new concept which can be realized experimentally with trapped atoms. The considerations of this paper can be potentially significant for reading out problem in quantum computation.

We consider a measurement on a two-state system \( S \) (\( \text{spin} 1/2 \)) in which a quantum two-state system acts as a detector in the precise sense that information about the state of the measured system is carried by the state of the quantum detector. The Hilbert space \( \mathcal{H}_S \) of the system is spanned by the orthonormal states \(|\uparrow\rangle\) and \(|\downarrow\rangle\) while the states \(|d_i\rangle\) span the detector \( \mathcal{H}_D \) of the detector. The detector is initially in \(|d_i\rangle\) state and clicks only when the spin of the system is \(|\uparrow\rangle\) (\([5–8]\)). In other words, the measurement interaction can be represented as

\[ |\uparrow\rangle|d_i\rangle \quad U_M |\uparrow\rangle |\uparrow\rangle|d_i\rangle \]  

\[ |\downarrow\rangle|d_i\rangle \quad U_M |\downarrow\rangle |\downarrow\rangle|d_i\rangle \]  

We try to figure out the most general unitary transformation \( U_M \) that will lead to such an interaction:

\[ U_M |\uparrow\rangle |d_i\rangle = |\uparrow\rangle |d_i\rangle e^{i\theta} \]  

\[ U_M |\downarrow\rangle |d_i\rangle = |\downarrow\rangle |d_i\rangle e^{i\phi} \]  

The explicit form of \( U_M \) is

\[ U = \begin{pmatrix} 0 & e^{i\theta} & 0 & 0 \\ B_1 e^{i\theta_1} & 0 & B_3 e^{i\phi_3} & 0 \\ \pm B_3 e^{i\phi_3} & 0 & \pm B_1 e^{i\phi_3} & 0 \\ 0 & 0 & 0 & e^{i\phi} \end{pmatrix} \]  

where \( \theta, \phi, \theta_1, \theta_3, \phi_1, \phi_3 \) are arbitrary phases and can have any values and \( B_1^2 + B_3^2 = 1 \) (Due to the fact that \( U \) is
unitary). Thus there can be infinitely many such unitary transformations possible and hence infinitely many interaction Hamiltonians. We try to find a unitary transformation of the type in eqn(12) which gives the simplest possible interaction Hamiltonian. The structure of the most general interaction Hamiltonian is given by

$$H_I = \hbar g(t) \sum_{i,j} c_{ij} Q_A^i Q_S^j$$

with $g(t)$ normalized according to $\int_0^T g(t) = 1$. For the impulsive case the interaction takes place during a short time interval $[0, \tau]$. $Q_A$ are observables corresponding to the apparatus and $Q_S$ observables corresponding to the system. The choice $\theta = 0$, $B_1 = 1$, $B_3 = 0$, $\phi = 0$, $\theta_1 = 0$, $\theta_3 = 0$, $\phi_1 = 0$, $\phi_3 = 0$ and taking only the positive signs one indeed gets a simple form for $U_M$ which also gives a simple form of $H_I$(not always obvious):

$$U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

leading to,

$$\ln U = \begin{pmatrix} \frac{i \pi}{2} & \frac{i \pi}{2} & 0 & 0 \\ \frac{-i \pi}{2} & \frac{i \pi}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The Hamiltonian is given by,

$$H_I = i \hbar g(t) \ln U.$$  

From now onwards we adopt the convention $\hbar = 1$. The Hamiltonian that leads to (13) is,

$$H_I = -\pi g(t)(P_{z,+}^S \otimes P_{x,-}^A)$$

where $P_{x,-}$ is the projection operator for the apparatus spin down along $x$ direction and $P_{z,+}$ is the projection operator for system spin up along $z$ direction. More explicitly

$$P_{z,+}^S = \frac{1 + \sigma_z}{2}, \quad P_{x,-}^A = \frac{1 - \sigma_x}{2}$$

Thus $Q_A = P_{z,+}^A$, $Q_S = -\pi P_{z,+}^S$ in this case.

### III. ALTERNATE MODEL FOR PROTECTIVE MEASUREMENTS

Aharonov and Vaidman [1], and subsequently Aharonov, Anandan and Vaidman [2] proposed a radically different approach to quantum measurements where there is no collapse of the wavefunction after measurement to one of the eigenstates of the measured observable and remarkably the original state is preserved even after the measurement. However this scheme has certain limitations; first of all the scheme applies only when the original state of the system is a non-degenerate eigenstate of its Hamiltonian. More importantly their claim is valid only in the limit of ideal adiabaticity i.e in the limit of the measurement time being infinite. They thought that their type of measurement would enable ascribing reality to the wavefunction of a single unknown quantum state, albeit of a restricted class, departing from the conventional ensemble interpretation of quantum mechanics. This premise rested on the property of their measurements which maintained the original unknown state while giving full information about it. However, as shown by Qureshi and Hari Dass [3] even the very small correction term arising out of the departure from ideal adiabaticity can spoil this property in the case of measurements on a single quantum state. Question of ascribing reality to the wavefunction arises only in that case. In the case of ensembles, this tiny correction is irrelevant, but so is the question of the reality of wavefunctions. Nevertheless this scheme is attractive in a practical sense since a protective measurement on an ensemble of states (with the restrictions mentioned above) maintains the ensemble in its original state to a very high degree [4].

First we briefly recapitulate the idea of protective measurements. In contrast to the case of impulsive measurements discussed earlier, the interaction of the system with the apparatus now is weak and adiabatic. Hence one cannot neglect the free Hamiltonians. As mentioned before, the system is assumed to be in a non-degenerate eigenstate of its Hamiltonian. Let the Hamiltonian of the combined system be

$$H(t) = H_A \otimes I_S + I_A \otimes H_S + g(t)Q_A \otimes Q_S$$

Where $H_A$ and $H_S$ are the Hamiltonians of the apparatus and the system. The coupling $g(t)$ acts for a long time $T$ and goes to zero smoothly before and after the interaction. It is normalized again by $\int_0^T g(t) = 1$. Therefore $g(t) \approx \frac{1}{T}$ is small and constant for the most part. Corrections due to $g(t)$ deviating from a constant have been carefully analysed in [4] and shown to be negligible. If $|t = 0\rangle$ is the state vector of the combined apparatus-system just before the measurement process begins, the state vector after $T$ is given by,

$$|t = T\rangle = T e^{-i \int_0^T H(\tau) d\tau} |t = 0\rangle$$

where $T$ is the time ordering operator. We divide the interval $[0, T]$ into $N$ equal intervals $\Delta T$, so that $\Delta T = \frac{T}{N}$, and because the full Hamiltonian commutes with itself at different times during $[0, T]$ (since we have taken $g(t) = \frac{1}{T}$), we can write eqn (19) as

$$|t = T\rangle = \{ e^{-i \Delta T (H_A + H_S + \frac{1}{T} + Q_A Q_S) N} |t = 0\rangle \}$$

Let us now examine the case when $Q_A$ commutes with $H_A$. Let $|n\rangle$ be the simultaneous eigenstates of $Q_A$ and
$H_A$ such that $Q_A|a_i⟩ = a_i|a_i⟩$ and $H_A|a_i⟩ = E_i^a|a_i⟩$. The $|a_i⟩$ form an orthonormal basis in the apparatus Hilbert space $\mathcal{H}_A$. Let $|\chi_j⟩$ be any orthonormal basis spanning the system Hilbert space $\mathcal{H}_S$. Let the exact eigenstates of the instantaneous Hamiltonian of eqn(18) be expanded as

$$|\psi_\mu⟩ = \sum_{ij} c_{ij}^\mu |\chi_j⟩|a_i⟩$$

(21)

satisfying

$$\langle H_A \otimes I_S + I_A \otimes H_S + \frac{1}{T}Q_A \otimes Q_S |\psi_\mu⟩ = \lambda_\mu |\psi_\mu⟩$$

(22)

Substituting the expansion eqn(21) in eqn(22) one gets

$$\sum_{kj} c_{kj}^\mu \{ E_i^a + H_S + \frac{a_i}{T}Q_S \} |\chi_j⟩|a_k⟩ = \lambda_\mu \sum_{pq} c_{pq}^\mu |\chi_p⟩|a_p⟩$$

(23)

Taking the inner product of both sides of this equation with $|a_i⟩$ one gets

$$\langle E_i^a + H_S + \frac{a_i}{T}Q_S \rangle \sum_{pj} c_{pj}^\mu |\chi_j⟩ = \lambda_\mu \sum_{pj} c_{pj}^\mu |\chi_j⟩$$

(24)

On introducing $|\tilde{\mu}_i⟩ = \sum_{pj} c_{pj}^\mu |\chi_j⟩$, this can be recast as

$$\langle E_i^a + H_S + \frac{a_i}{T}Q_S \rangle |\tilde{\mu}_i⟩ = \lambda_\mu |\tilde{\mu}_i⟩$$

(25)

So the exact eigenstates can be written in a factorized form $|a_i⟩|\tilde{\mu}_i⟩$ where $|\tilde{\mu}_i⟩$ are system states which depend on the eigenvalue of $Q_A$, i.e., they are the eigenstates of $\frac{1}{T}a_iQ_S + H_S$. In the extreme adiabatic limit $T \to \infty$, the states $|\tilde{\mu}_i⟩$ approach $|\mu⟩$ which are eigenstates of $H_S$ with eigenvalue $\mu$. Now let us assume the initial state of the system and apparatus to be a direct product of a non-degenerate eigenstate $|\nu⟩$ of $H_S$ and a wave-packet state of the apparatus $|\phi(r_0)⟩$ that is centred around $r = r_0$:

$$|t = 0⟩ = |\nu⟩|\phi(r_0)⟩$$

(26)

Introducing the complete set of exact eigenstates in the above equation, the wave function at time $T$ can now be written as,

$$|t = T⟩ = \sum_{i,\mu} e^{-iE_i^a T} \langle \tilde{\mu}_i |\nu⟩ \langle a_i |\phi(r_0)⟩ |a_i⟩|\tilde{\mu}_i⟩$$

(27)

The exact instantaneous eigenvalues $E(a_i, \mu)$ can be written as

$$E(a_i, \mu) = E_i^a + \frac{\langle \tilde{\mu}_i |Q_S |\nu⟩ a_i}{T} + \langle \tilde{\mu}_i |H_S |\tilde{\mu}_i⟩$$

(28)

On using first order perturbation theory,

$$|\tilde{\mu}_i⟩ = |\mu⟩ + \frac{1}{T} \sum_{\chi \neq \mu} \frac{|\chi⟩ a_i⟨Q_S |\chi⟩ y_\mu}{E_\mu - E_\chi} + O(\frac{1}{T^2})$$

(29)

Using equation (29), and the fact that $|\mu⟩$ and $|\chi⟩$ are orthogonal we get, to first order in $\frac{1}{T}$

$$E(a_i, \mu) = E_i^a + \mu + \frac{a_i}{T} (\mu |Q_S |\mu⟩ + O(\frac{1}{T^2})$$

(30)

To obtain the leading order result, we need to keep $O(\frac{1}{T})$ terms in $E(a_i, \mu)$ but only the leading order terms for $|\tilde{\mu}_i⟩$ (this is because $E(a_i, \mu)$ is multiplied by $T$ in the exponent). Summing over $\mu$ in eqn(27) (as a result of which only the term with $\mu = \nu$ survives) we get,

$$|t = T⟩ \approx \sum_i e^{-iE_i T} |\nu⟩ e^{-iE_i T - i(Q_S)\nu} |a_i⟩⟨a_i|\phi(r_0)⟩$$

(31)

On using the identity

$$\sum_i e^{-iE_i T - i(Q_S)\nu} |a_i⟩⟨a_i| = e^{-iH_A T - i(Q_S)\nu} Q_A$$

(32)

we get

$$|t = T⟩ \approx e^{-iE_i T} |\nu⟩ e^{-iH_A T - i(Q_S)\nu} |\phi(r_0)⟩$$

(33)

In the case of systems governed by a Heisenberg algebra for which displacement operators exist, we can further simplify this to

$$|t = T⟩ \approx e^{-iE_i T} e^{-iH_A T |\nu⟩ |\phi(r_0)⟩}$$

(34)

where we have made use of eqn(9) in the last step. This is the central result of [1,2] and one sees that in the extreme adiabatic limit the apparatus measures the expectation value of a system observable directly and that too without disturbing the state of the system!

A. The alternate model

Now we consider the case when $H_A$ represents a finite dimensional Hilbert space. To be specific, we take it to be a spin-1/2 system. Because of the finite dimensionality of the Hilbert space, the algebra of observables can not be of the Heisenberg type and consequently there are no displacement operators. However, the relevant algebra now is the Lie algebra and we have rotation operators instead.

For $H_A$ we choose a rotationally invariant operator which in the present case means it is a constant. Here too we have an improvement over the original situation of [2]. Let this constant be $E_a^a$ so that $E_a^a$ is independent of $i$. For the system Hamiltonian $H_S$ we take the Hamiltonian of the spin-1/2 system in the presence of an unknown (to the person making the measurements) magnetic field $\vec{B} = B_0\vec{\sigma}$. Thus $H_S = -B_0\vec{\sigma} \cdot \vec{n}$. Let the eigenstates of $H_S$ be $|\pm⟩$ which are obviously nondegenerate. Furthermore, $\vec{\sigma} |\pm⟩ = \pm |\pm⟩$. Following the discussions of sec.II we choose the interaction Hamiltonian
to be \( H_I = - \frac{i}{\hbar} (P_{\frac{S}{^2}x} + P_{\frac{A}{^2}x}) \). For this case \( Q_A \) and \( H_A \) commute.

Let the initial unknown state of the system be \( |\uparrow\rangle \) and let its representation in terms of the basis vectors of the system used in eqn(10) be

\[
|\uparrow\rangle = \alpha |\uparrow\rangle + \beta |\downarrow\rangle
\]

which is an eigen function of \( H_S \) and let the detector be initially in the state \( |d_i\rangle \).

Putting together all of these details in eqn (31) we get, for our model,

\[
|t = T\rangle \approx e^{iB_0T - \frac{\pi}{T} T} e^{i\pi (P_{\frac{S}{^2}x} + P_{\frac{A}{^2}x}) |\nu\rangle} |d_i\rangle
\]

Now \( \langle P_{\frac{S}{^2}x} \rangle \nu = |\alpha|^2 \) and \( P_{\frac{A}{^2}x} = \frac{1}{2} - S^z_x \). Hence the state after the measurement is,

\[
|t = T\rangle \approx e^{iB_0T - \frac{\pi}{T} (\nu)} e^{-iS^z_x |\alpha|^2} |d_i\rangle
\]

Thus we see that in our version of the protective measurements, the apparatus state instead of being displaced by an amount proportional to the expectation value of the system observable being measured is actually rotated around the x-axis by an angle that is proportional to the same expectation value. Again, the original wavefunction is completely preserved in the limit of ideal adiabatic conditions. As the original state is preserved even after the measurement as in the treatments of [1,2] it can be used to determine all the expectation values \( \langle S^z_x \rangle, \langle S^y_x \rangle, \langle S^z_x \rangle \) thereby fully determining the unknown initial state.

IV. CORRECTIONS DUE TO DEPARTURE FROM IDEAL ADIABATICITY

So far the treatments were based on ideal adiabatic conditions and neglected the \( \frac{1}{T} \) correction terms arising out of the measurement time \( T \) being finite even though arbitrarily large in actual experiments. There are essentially two distinct types of \( \frac{1}{T} \) corrections that arise. First arises out of \( O(\frac{1}{T}) \) corrections to eqn(30) contributing to corrections to phases. The second arises out of keeping the \( O(\frac{1}{T^2}) \) corrections to \( |\bar{\mu}\rangle \) in eqn(29). We give below the ingredients for calculating both types of corrections:

\[
|\bar{\mu}\rangle \langle \bar{\mu} |\nu\rangle \approx \delta_{\mu\nu} |\mu\rangle + \frac{1}{T} \sum_{k\neq \mu} |\mu\rangle |d_{\beta} \rangle \langle d_{\beta} | E_{\beta} - E_{\nu}
\]

\[
|Q_s\rangle \langle Q_s |\bar{\mu}\rangle \approx |Q_s\rangle + \frac{2a_i}{T} \sum_{k \neq \mu} \frac{\langle Q_s \rangle_{\mu k} (Q_s)_{k \mu}}{E_{\mu} - E_{k}}
\]

\[
|H_s\rangle \langle H_s |\bar{\mu}\rangle \approx \langle H_s \rangle \mu + \frac{a_i^2}{T^2} \sum_{k \neq \mu} \frac{\langle H_s \rangle_{\mu k} (Q_s)_{k \mu}}{E_{\mu} - E_{k}}
\]

Using eqns (??) it is easy to show that the corrections to \( E(\alpha_i, \mu) \) are:

\[
E(\alpha_i, \mu) \approx E^0 + \langle H_s \rangle \mu + \frac{a_i}{T} \langle Q_s \rangle_{\mu} + \frac{a_i^2}{T^2} \sum_{k \neq \mu} \frac{\langle H_s \rangle_{\mu k} (Q_s)_{k \mu}}{(E_{\mu} - E_{k})^2}
\]

As discussed at length in [3] the \( O(\frac{1}{T^2}) \) corrections to the post-measurement state are the ones that come in the way of a truly protective measurements on unknown single quantum states. As far as measurements on ensemble of states are concerned none of these corrections are serious. Again the \( O(\frac{1}{T}) \) corrections to phases do not alter the leading order conclusions about the preservation of the original state, so we shall neglect such corrections to phases.

With \( |\uparrow\rangle = \alpha |\uparrow\rangle + \beta |\downarrow\rangle \) as the initial state \( |\nu\rangle \) of the system, the state \( |\chi\rangle \) orthogonal to it is \( |\downarrow\rangle = \beta^* |\uparrow\rangle - \alpha^* |\downarrow\rangle \). Then we have: \( E_{\nu} = -B_0 \), \( E_{\chi} = B_0 \), \( \langle Q_s \rangle_{\nu} = -\pi |\alpha|^2 \), \( \langle Q_s \rangle_{\chi} = -\pi |\beta|^2 \) and \( \langle Q_s \rangle_{\chi\nu} = -\pi |\alpha\beta|^2 \). Consequently With \( |\nu\rangle = |\uparrow\rangle \) we separate the contributions from \( \mu = +, - \) in eqn(27):

\[
e^{-i(E^0 - B_0)T + \pi |\alpha|^2 |\uparrow\rangle + \frac{a_i |\alpha\beta|^2}{2B_0 T} |\bar{\mu}\rangle \langle \bar{\mu} |\nu\rangle}
\]

Using eqn(32) along with the definition \( \eta = \frac{2B_0 T}{E^0 - B_0} \) we can combine these into the final result:

\[
|t = T\rangle = e^{iB_0 T + \pi |\alpha|^2 P_{\frac{A}{^2}x} - |\uparrow\rangle} |d_i\rangle
\]

\[
\langle \eta^* e^{-iB_0 T - \pi |\alpha|^2 |\uparrow\rangle} \rangle = e^{iB_0 T + \pi |\alpha|^2 P_{\frac{A}{^2}x} - |\uparrow\rangle} |d_i\rangle
\]

Using the properties of the projection operator \( P_{\frac{A}{^2}x} \) we can simplify this to

\[
|t = T\rangle = e^{iB_0 T + \pi |\alpha|^2 |\uparrow\rangle} + \langle \eta^* e^{-iB_0 T + \pi |\alpha|^2 |\uparrow\rangle} \rangle
\]

We thus see that correcting the final expression to the order of \( \frac{1}{T} \) there exists a finite probability of \( O(\frac{1}{T^2}) \) that the system is transferred to another orthogonal, non-degenerate state of its Hamiltonian thus leading to a complete loss of the original state. Hence we cannot carry out protective measurement on a single state safely because there exists a low but finite probability of the system loosing its original state after measurement. Hence to safely carry it out we must use an ensemble of state rather than a single state so that we may identify which outcome corresponds to the case where the original state is being preserved. But then the whole issue of the reality of wavefunctions becomes irrelevant. However, as pointed out in [4], the protective measurements are still useful because they maintain the purity of the original ensemble to a very high degree (of the order of \( 1 - \frac{1}{T^2} \)).
V. CLASSICAL APPARATUS IN TERMS OF QUANTUM SPINS

Till now we have used a quantum mechanical system with Hilbert space $\mathcal{H}_A$ to describe the ‘apparatus’. This is of course unsatisfactory as it leads to unsurmountable technical and philosophical problems. In this section we make a proposal as to how one may overcome this difficulty. The idea is to simulate a classical detector out of several quantum detectors. A large collection of microscopic quantum systems is expected to behave like a macroscopic system and hence simulate a classical detector. Specifically, fluctuations in certain “macroscopic” observables will be small. We consider a system which has $N$ quantum detectors described by the tensor product Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2 \ldots \otimes \mathcal{H}_N$ initially in the state $|d_{11}⟩|d_{21}⟩\ldots|d_{N1}⟩$. It is easy to generalise the considerations of our paper till now to this enlarged situation.

$$Q_A = -\pi \sum_{l=1}^{N} (I \otimes I \otimes \ldots \otimes P^{(l)}_{x,z} \otimes \ldots I \otimes I) \quad (46)$$

Putting $P^{(l)}_{x,z} = \frac{(I-\sigma^{(l)}_x)}{2}$, we get

$$Q_A = -\pi I_A^2 + \pi \sum_{l=1}^{N} I \otimes I \otimes \ldots S^{(l)}_x \otimes \ldots I \otimes I \quad (47)$$

When this detector is used to detect the system state $|\uparrow⟩$ using the interaction Hamiltonian $H_I = \frac{1}{\hbar}(P^S_{z,+} \otimes Q_A)$, we get, in the extreme adiabatic limit,

$$|t = T⟩ \approx e^{iB_0T + \frac{\pi|\alpha|^2}{2}} \Pi_{n=1}^N e^{\pi \sigma_x^N} \exp(-i\pi|\alpha|^2)S_{x,n}|d_{n1}⟩ \quad (48)$$

Thus we see that each individual quantum spin making up the apparatus is rotated about the x-axis by the same magnitude. The relevant macroscopic observable for the apparatus is the total spin $\vec{S} = \sum_{i=1}^N \vec{s}_i$. The fluctuation in the mean of the total spin is proportional to $\frac{1}{\sqrt{N}}$. Thus increasing $N$ reduces the fluctuations and make the detector becomes essentially classical.

VI. CONCLUSION AND FUTURE ISSUES

In this article we have considered an alternate realisation of the idea of protective measurements first discussed by [1,2]. The apparatus has been replaced by a finite dimensional Hilbert space system. Apart from overcoming the difficulties arising out of unbounded operators (discussed in detail in [3]), the algebra of observables changes from the Heisenberg type to the Lie algebra of rotation operators. Instead of the apparatus being displaced by an amount proportional to the expectation value of measured observables as in the original treatments [1,2], in our case the apparatus spin is rotated by an amount proportional to the respective expectation values. Unlike in the original treatment the apparatus corresponding to the case where protective measurement fails because of corrections due to finite but large times of observations is predetermined. The consequences of this for protective measurements will be dealt in future publications. We have also tried to overcome the conceptual issues of working with a quantum detector by considering an ensemble of detectors. We believe this is a potentially interesting way of addressing the thorny issues of Quantum Measurements. This can also open up interesting ways for readouts from Quantum Computers. This will also be explored in a future publication.

VII. ACKNOWLEDGEMENT

AD would like to thank Institute of Mathematical Sciences for granting the Summer Research fellowship for carrying out this research. AD would also like to thank his family and friends for their encouragemnt and support.

[1] Y. Aharonov and L. Vaidman, Phys. Lett. A 178 38(1993)
[2] Y. Aharonov, J. Anandan and L. Vaidman, Phys. Rev. A 47 4616(1993)
[3] N.D. Hari Dass and T.Qureshi, Phys.Rev.A, 59:2590(1999)
[4] N.D. Hari Dass quant-ph, 9908085 v1(28 Aug 1999)
[5] H.D. Zeh, Foundations of Physics 1:69 (1970)
[6] E. Wigner, Am. J. Phys 31: 615(1963)
[7] M.O. Scully, B.G. Englert and J. Schwinger Phys. Rev. A 401775(1989)
[8] W.H.Zurek, Los Alamos Science, 27(2002)