About the Chasm Separating the Goals of Hilbert’s Consistency Program From the Second Incompleteness Theorem

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Abstract

We have published several articles about generalizations and boundary-case exceptions to the Second Incompleteness Theorem during the last 25 years, The current paper will review some of our prior results and also introduce an “enriched” refinement of semantic tableaux deduction. While there is no question that the Second Incompleteness Theorem is a strong result, the current article will emphasize its boundary-case exceptions are significant because they can own a simultaneous knowledge about their own consistency, together with an understanding of the $\Pi_1$ implications of Peano Arithmetic.

Keywords and Phrases: Gödel’s Second Incompleteness Theorem, Consistency, Hilbert’s Second Open Question, Hilbert-styled Deduction.

Mathematics Subject Classification: 03B52; 03F25; 03F45; 03H13

Comment: The bibliography section of this article contains citations to all Willard’s major prior papers about logic.

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1 Introduction

The existence of a deep chasm between the goals of Hilbert’s consistency program and the implications of Gödel’s Second Incompleteness Theorem was immediately apparent when Gödel announced his famous millennial discovery [17].

Interestingly, neither Gödel (in 1931) nor Hilbert (during the remainder of his life) dismissed the existence of possible compromise solutions, whereby a fragment of the goals of the Consistency Program could remain intact. For instance, Hilbert never withdrew his statement * for justifying his Constancy Program in [25]:

* “Let us admit that the situation in which we presently find ourselves with respect to paradoxes is in the long run intolerable. Just think: in mathematics, this paragon of reliability and truth, the very notions and inferences, as everyone learns, teaches, and uses them, lead to absurdities. And where else would reliability and truth be found if even mathematical thinking fails?”

Indeed, the motto of Hilbert’s Consistency Program (“Wir müssen wissen—Wir werden wissen”) was engraved onto Hilbert’s tombstone.

Also, Gödel was cautious (at least in the early 1930’s) not to speculate about whether all facets of Hilbert’s Consistency program would come to a termination. He thus inserted the following cautious caveat into his famous 1931 millennial paper [17]:

** “It must be expressly noted that Theorem XI” (e.g. the Second Incompleteness Theorem) “represents no contradiction of the formalistic standpoint of Hilbert. For this standpoint presupposes only the existence of a consistency proof by finite means, and there might conceivably be finite proofs which cannot be stated in P or in ... ”

Several biographies of Gödel [8,19,64] have noted that Gödel’s intention (prior to 1930) was to establish Hilbert’s proposed objectives, before he formalized his famous result that led in the opposite direction. Yourgrau’s biography of Gödel [64], indeed surprisingly, records how von Neumann, found it necessary during the early 1930’s to “argue against Gödel himself” about the definitive termination of Hilbert’s consistency program, which “for several years” after [17]’s publication, Gödel “was cautious not to prejudge”. It is known that Gödel had hinted that the Second Incompleteness Theorem was more significant during a 1933 Vienna lecture [18]. He did not, however, completely embrace

1 English translation: “We must know: We will know.”
the Second Incompleteness until after learning about Turing’s work [44] during the mid-
1930’s.

Most of the research that has followed G¨ odel’s historic 1931 seminal discovery has focused on studying generalizations of the Second Incompleteness Theorem, rather then exploring its possible boundary-case exceptions. Many of these generalizations of the Second Incompleteness Theorem [2, 4, 7, 10, 14, 21, 22, 29, 32, 33, 34, 36, 37, 41, 42, 45, 46, 51, 52, 54, 56] are quite beautiful. The author of this paper is especially impressed by a generalization of the Second Incompleteness Effect, arrived at by the combined work of Pudlák and Solovay (abetted by the research of Nelson and Wilkie-Paris) [31, 36, 41, 46]. These results, which also have been more recently discussed in [7, 20, 42, 51], have noted the Second Incompleteness Theorem does not require the presence of the Principle of Induction to apply to most formalisms that use a Hilbert-style form of deduction. (The next chapter will offer a detailed summary of this important generalization of the Second Incompleteness Theorem in its Remark 2.5.)

Our research, during the last 25 years has had a different focus, exploring Boundary-Case exceptions to the Second Incompleteness Theorem more intensively than its generalizations. It would be natural for many readers to ask why such exceptions should also be studied, so intensively?

The reason is that while generalizations of the Second Incompleteness Theorem are very pure form a mathematical standpoint, it must not be forgotten that Mankind’s survival in the future will require developing formalisms that own enough ingenuity to solve a variety of pressing ecological problems, such as Global Warming, in a satisfactory manner.

More specifically, the solution of mundane problems, threatening human survival, do not require use of a formalism producing short proofs of the existence of large integers, whose binary encodings employ more digits than the number of atoms in the universe. It is, however, vital for mature logical formalisms to appreciate their own consistency, in at least a fragmentary sense, when they reason about the implications of their own reasoning. (Otherwise, a Thinking Being, whether computerized or human, would not be able to explain to itself fully why it is of foundational importance to study its own thinking process, as a fundamental problem-solving facility.)
In particular, there is no doubt that a branch of mathematics, that makes it difficult to manufacture *abbreviated* proofs of the existence of numbers as large as a google-plex (e.g. $2^{2^{100}}$), does fall short of the Utopian ideals that the intellectual community wishes for Mathematics. We will argue, however, that the striking engineering needs that confront modern Mankind requires the evolution of alternate formalisms, however *theoretically weak*, for an adequate result to be obtained for many more pressing issues.

In other words, we will contend that Hilbert and Gödel were essentially correct when their statements * and ** suggested that a nihilistic approach, which ignores the engineering-style capacities of weaker formalisms, that own a fragmented conception of their own consistency, has serious short-comings. This is because the dangers of Global Warming and other imminent threats that endanger 21-st century Mankind are too serious for logicians to entertain using anything less than a formalism, which possesses at least a fragmentary conception of its own consistency.

Also, we will discuss an addition to our prior research about self-justifying logics, called *Deductive Enrichment*, which should convince skeptical readers that our formalisms do indeed have practical value. Especially within a special context where modern computers can perform arithmetic operations with more than a billion times the speed of a human being, we will argue our boundary-case exceptions to the prior century’s Second Incompleteness Theorem have noteworthy pragmatic significance.

As the reader examines this paper, it should be kept in mind that all our self-justifying axiom systems (since 1993) contain an ability to prove analogs of all the $\Pi_1$ theorems of Peano Arithmetic under a slightly revised language (such as $L^*$ formalism). This fact is non-trivial because an axiom system that recognizes its own consistency will contain little pragmatic significance, if it does not maintain an ability to prove all the quite central $\Pi_1$ theorems of Peano Arithmetic.

What will make our formalisms tempting in the current article is the new notion of “Deductive Enrichment”. It will allow a formalism to maintain a simultaneous knowledge about its own consistency together with a recognition about the truth behind the $\Pi_1$ theorems of Peano Arithmetic.

In particular, we do not dispute that our formalism will fall short of the Utopian ideals for mathematics when it is unable to produce a *brief* proof for the existence of
large numbers, such as a google-plex (e.g. $2^{2^{100}}$). We will, however, claim our formalism contains some pragmatic value when it can own a simultaneous knowledge about its own consistency and the truth behind the $\Pi_1$ theorems of Peano Arithmetic.

2 General Notation and Literature Survey

Let us call an ordered pair $(\alpha, d)$ a “Generalized Arithmetic” when its first and second components are defined as follows:

1. The **Axiom Basis** “$\alpha$” of a Generalized Arithmetic will be defined as the set of proper axioms it employs.

2. The second component “$d$” of a Generalized Arithmetic will be defined as the combination of its formal rules of inference and the logical axioms “$L_d$” it employs. This second component, “$d$” of a Generalized Arithmetic will be called its Deductive Apparatus.

**Example 2.1** This notation allows us to conveniently separate the logical axioms $L_d$, associated with $(\alpha, d)$, from its “basis axioms” $\alpha$. It also allows one to compare the various deductive apparatus techniques that have appeared in the literature. For instance, the $d_E$ apparatus, introduced in §2.4 of Enderton’s textbook [9], had used only modus ponens as a rule of inference, combined with a complicated 4-part schema of logical axioms. This differs from the $d_M$, $d_H$ and $d_F$ approaches of Mendelson [30], Hájek-Pudlák [22] and Fitting [11]. The former two textbooks employ a simpler set of logical axioms than $d_E$, but they require two rules of inference (modus ponens and generalization). The $d_F$ apparatus, appearing in Fitting’s textbook [11], as well as its predecessor due to Smullyan [40], actually employ no logical axioms. Instead, Fitting and Smullyan rely upon a “tableaux style” method for generating a consequently larger number of rules of inference.

**Definition 2.2** Let $\alpha$ once again denote an axiom basis, and $d$ designate a deduction apparatus. Then the ordered pair $(\alpha, d)$ will be called a **Self Justifying configuration** when:
i one of \((\alpha,d)\)'s theorems (or possibly one of \(\alpha\)'s axioms) do state that the deduction
method \(d\), applied to the basis system \(\alpha\), produces a consistent set of theorems, and

ii the axiom system \(\alpha\) is in fact consistent.

Example 2.3 Using Definition 2.2's notation, our prior research in [49, 51, 55, 56, 59, 61] developed arithmetics \((\alpha,d)\) that were “Self Justifying”. It also proved the Second Incompleteness Theorem implies specific limits beyond which self-justifying formalisms cannot transgress. For any \((\alpha,d)\), it is almost trivial to construct a system \(\alpha^d \supseteq \alpha\) that satisfies the Part-i condition (in an isolated context where the Part-ii condition is not also satisfied). For instance, \(\alpha^d\) could consist of all of \(\alpha\)'s axioms plus an added “SelfRef(\(\alpha,d\))” sentence, defined as stating:

\[ \oplus \text{ There is no proof (using } d\text{'s deduction method) of } 0 = 1 \text{ from the union of the axiom system } \alpha \text{ with this sentence “SelfRef}(\alpha,d) “ (looking at itself). \]

Kleene discussed in [28] how to encode rough analogs of the above “I Am Consistent” axiom statement. Each of Kleene, Rogers and Jeroslow [28, 39, 27], however, emphasized that \(\alpha^d\) may be inconsistent (e.g. violating Part-ii of self-justification’s definition), despite SelfRef(\(\alpha,d\))’s formalized assertion. This is because if the pair \((\alpha,d)\) is too strong then a quite conventional Gödel-style diagonalization argument can be applied to the axiom basis of \(\alpha^d = \alpha + \text{SelfRef}(\alpha,d)\), where the added presence of the statement SelfRef(\(\alpha,d\)) will cause this extended version of \(\alpha\), ironically, to become automatically inconsistent. Thus, the encoding of “SelfRef(\(\alpha,d\))” is relatively easy, via an application of the Fixed Point Theorem, but this sentence is often, ironically, entirely useless!

Definition 2.4 Let \(\text{Add}(x,y,z)\) and \(\text{Mult}(x,y,z)\) denote two 3-way predicate symbols specifying that \(x+y = z\) and \(x*y = z\) (under \(\Pi_1\) styled-encodings for the associative, commutative, identity and distributive principles using these two 3-way predicate symbols). Let \(\alpha\) denote what the first paragraph of this section had called an “axiom basis”. We will then say that \(\alpha\) recognizes successor, addition and multiplication as Total Functions iff it can prove (1) - (3) as theorems.
∀x ∃z  Add(x, 1, z)  \hspace{1cm} (1)
∀x ∀y ∃z  Add(x, y, z)  \hspace{1cm} (2)
∀x ∀y ∃z  Mult(x, y, z)  \hspace{1cm} (3)

Also, an axiom basis \( \alpha \) will be called **Type-M** iff it includes \( \{1\} \) - \( \{3\} \) as theorems, **Type-A** if it includes only \( \{1\} \) and \( \{2\} \) as theorems, and **Type-S** if it contains only \( \{1\} \) as a theorem. Also, \( \alpha \) is called **Type-NS** iff it can prove none of these theorems.

**Remark 2.5** The separation of basis axiom systems into the four categories of Type-NS, Type-S, Type-A and Type-M systems enables us to nicely summarize the prior literature about generalizations and boundary-case exceptions for the Second Incompleteness Theorem. This is because:

a. The combined research of Pudlák, Solovay, Nelson and Wilkie-Paris \[31, 36, 41, 46\], as is formalized by Theorem ++ , implies no natural Type-S generalized arithmetic \((\alpha, d)\) can recognize its own consistency when \( d \) is one of Example 2.1’s three examples of Hilbert-style deduction operators of \( d_E \), \( d_H \) or \( d_M \). In particular, it establishes the following result:

\[
\text{++ (Solovay’s modification \[41\] of Pudlák’s formalism using some of Nelson and Wilkie-Paris \[31, 46\]’s methods)}: \text{ Let } (\alpha, d) \text{ denote a generalized arithmetic supporting the } (1)’s \text{ Type-S statement and assuring the successor operation will satisfy both } x’ \neq 0 \text{ and } x’ = y’ \Leftrightarrow x = y. \text{ Then } (\alpha, d) \text{ cannot verify its own consistency whenever simultaneously } d \text{ is a Hilbert-style deductive apparatus and } \alpha \text{ treats addition and multiplication as 3-way relations, satisfying their usual associative, commutative distributive and identity axioms.}
\]

Essentially, Solovay \[41\] privately communicated to us in 1994 an analog of theorem ++. Many authors have noted Solovay has been reluctant to publish his nice privately communicated results on many occasions \[7, 22, 31, 33, 36, 46\]. Thus, approximate analogs of ++ were explored subsequently by Buss-Ignjatovic, Hájek and Švejdar in \[7, 20, 42\], as well as in Appendix A of our paper \[51\]. Also, Pudlák’s initial 1985 article \[36\] captured the majority of ++’s essence, chronologically before Solovay’s observations. Also, Friedman did related work in \[14\].
b. Part of what makes ++ interesting is that explored two methods for generalized arithmetics to confirm their own consistency, whose natural hybridizations are precluded by ++. Specifically, these results involve using Example 2.3’s self-referencing “I am consistent” axiom (from its statement ⊕). They will enable several Type-NS basis systems to verify their own consistency under a Hilbert-style deductive apparatus, or alternatively allow a Type-A system to corroborate its own self-consistency under a more restricted semantic tableaux style deductive apparatus. Also, Willard observed how one could refine ++ with Adamowicz-Zbierski’s methodology to show Type-M systems cannot recognize their own tableaux-style consistency.

3 General Perspective

This section will explain how some seemingly minor hair-thin Boundary-Case exceptions to the Second Incompleteness Theorem can be transformed into major chasms when one contemplates the facts that 21-st century computers can perform arithmetic computations with more than a billion times the speed of the human mind (with a similar accompanying increase in the lengths of the logical sentences being manipulated). This distinction will raise questions about whether 21-st century engineering projects will ultimately be forced to encounter questions about the Second Incompleteness Theorem, which were ignored when the scientific world had first learned about Gödel’s work during the early 1930’s.

3.1 Linguistic Notation

Our language for formalizing exceptions to the Second Incompleteness theorem will be called $L^*$. It will include the symbols $C_0$, $C_1$ and $C_2$ for representing the integers of 0, 1 and 2. The language $L^*$ will discuss the properties of non-negative integers greater than 2, in terms of these three starting integers, as will be explained in this section.

The predicate symbols used by our language $L^*$ will be the equality and less-than-or-equal predicates, denoted as “=” and “≤”. Sometimes, we will informally also use the symbols ≥, < and >.

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2 The Example provided three examples of Hilbert-style deduction operators, called $d_E$, $d_H$, and $d_M$. It explained how these deductive operators differ from a tableaux-style deductive apparatus by containing a modus ponens rule.
Define $F(a_1, a_2, \ldots a_j)$ to be a **NON-GROWTH FUNCTION** iff for all values of $a_1, a_2, \ldots a_j$, the function $F$ satisfies $F(a_1, a_2, \ldots a_j) \leq \text{Maximum}(a_1, a_2, \ldots a_j)$. Our axiom systems will employ a set of eight non-growth functions, called the **GROUNDING FUNCTIONS**. They will include:

1. Integer Subtraction where $x - y$ is defined to equal zero when $x < y$,
2. Integer Division where $\frac{x}{y} = x$ when $y = 0$, and it otherwise equals $\lfloor \frac{x}{y} \rfloor$.
3. $\text{Predecessor}(x) = \text{Max}(x - 1, 0)$,
4. $\text{Maximum}(x, y)$,
5. $\text{Logarithm}(x) = \lceil \log_2(x + 1) \rceil$,
6. $\text{Root}(x, y) = \lfloor x^{1/y} \rfloor$ when $y \geq 1$, and $\text{Root}(x, 0) = x$.
7. $\text{Count}(x, j)$ designating the number of “1” bits among $x$’s rightmost $j$ bits.
8. $\text{Bit}(x, i)$ designating the value of the integer $x$’s $i$–th rightmost bit. (Note that $\text{Bit}(x, i) = \text{Count}(x, i) - \text{Count}(x, i - 1)$.)

In addition to the above non-growth functions, our language $L^*$ will employ two growth functions. They will correspond to Integer-Addition and Double$(x) = x + x$. We will use the term **U-Grounding Function** to refer to a function that is one of our eight Grounding Functions or the operations of Addition and Doubling.

**Comment 3.1** We do not technically need both the operations of Addition and Doubling in our U-Grounding language $L^*$. However, it much is easier to encode large integers when we have access to both these function symbols. For example, any integer $N > 1$ can be encoded by a term of length $O(\log(N))$, using only the constant symbol for “1”, when both the addition and doubling function symbols are present. For instance, below is our binary-like encoding for the number eleven.

$$1 + \text{Double}(1 + \text{Double}(\text{Double}(1)))$$  \hspace{1cm} (4)

Henceforth, the symbol $\overline{N}$ will denote such a binary-like encoding for the integer $N$. (In the degenerate case where $N = 0$, $\overline{0}$ will simply be defined as being the constant symbol “$C_0$”, that represents zero’s value.)
Definition 3.2 We will follow mostly conventional logic notation when discussing the U-Grounding functions. Thus, a term is defined to be a constant symbol, a variable symbol or a function symbol (followed by some input arguments, which are similarly defined terms). If \( t \) is a term then the quantifiers in \( \forall v \leq t \ \Psi(v) \) and \( \exists v \leq t \ \Psi(v) \) will be called bounded quantifiers. These two wffs will be semantically equivalent to the respective formulae of \( \forall v \ ( v \leq t \Rightarrow \Psi(v) ) \) and \( \exists v \ ( v \leq t \land \Psi(v) ) \). A formula \( \Phi \) will be called \( \Delta^*_0 \) iff all its quantifiers are bounded. Thus a \( \Delta^*_0 \) formula is defined to be a wff that is any combination of atomic formulae (using our ten U-Grounding functions and the equals and \( \leq \) predicates) combined by bounded quantifiers and the boolean operations of AND,OR, NOT and IMPLIES in an arbitrary manner.

Definition 3.3 For any integer \( i \geq 0 \), this paragraph will define the notions of a \( \Pi^*_i \) and \( \Sigma^*_i \) formula. Their definition has three parts, and it is given below.

1. Every \( \Delta^*_0 \) formula is defined to be also both a \( \Pi^*_0 \) and \( \Sigma^*_0 \) formula.

2. A formula is called \( \Pi^*_{i+1} \) iff for some \( \Sigma^*_i \) formula \( \Phi(v_1, v_2, ... v_n) \), it can be written in the form \( \forall v_1 \forall v_2 \ldots \forall v_n \Phi(v_1, v_2, ... v_n) \). (Since this rule also applies when the number of quantifiers \( n \) equals zero, it follows that every \( \Sigma^*_i \) formula is automatically by default also \( \Pi^*_{i+1} \).

3. Similarly, a formula is called \( \Sigma^*_{i+1} \) iff for some \( \Pi^*_i \) formula \( \Phi(v_1, v_2, ... v_n) \), it can be written in the form \( \exists v_1 \exists v_2 \ldots \exists v_n \Phi(v_1, v_2, ... v_n) \). (Since this rule also applies when the number of quantifiers \( n \) equals zero, it follows that every \( \Pi^*_i \) formula is automatically by default also \( \Sigma^*_{i+1} \).

Example 3.4 Lines (5) are (6) are examples of \( \Pi^*_2 \) sentences, and Lines (7) are (8) are examples of \( \Pi^*_1 \) sentences. Note that some \( \Pi^*_2 \) sentences can be proven to be logically equivalent to \( \Pi^*_1 \) sentences. Thus for example, the sentences in Lines (5) and (8) are logically equivalent to each other.

\[
\begin{align*}
\forall x \ \forall y \ \exists z \quad z - x &= y \\
\forall x \ \forall y \ \exists z \quad x > 0 \Rightarrow \frac{z}{x} &= y \\
\forall x \ \forall y \quad x + y &= y + x
\end{align*}
\]
\[ \forall x \forall y \exists z \leq x + y \quad z - x = y \]  \hfill (8)

Note that while Line (5) can be transformed into a logically equivalent \( \Pi_1^* \) sentence (encoded by Line (8)), there is no analogous \( \Pi_1^* \) sentence equivalent to Line (6). (This is because Addition but not Multiplication belongs to our set of U-Grounding functions.)

**Remark 3.5** Throughout all our papers, the symbols \( \text{Add}(x, y, z) \) and \( \text{Mult}(x, y, z) \) denote two \( \Delta_0^* \) formulae that are satisfied precisely when the respective conditions of \( x + y = z \) and \( x \cdot y = z \) are true. It turns out that we can define both these formulae using only the non-growth functions of integer subtraction and division. Such definitions of \( \text{Add}(x, y, z) \) and \( \text{Mult}(x, y, z) \) are provided by Lines (9) and (10) below:

\[
\begin{align*}
  z - x &= y \quad \land \quad z \geq y \\
  [ (x = 0 \lor y = 0) \Rightarrow z = 0 ] \quad \land \quad [ (x \neq 0 \land y \neq 0) \Rightarrow \left( \frac{z}{x} = y \quad \land \quad \frac{z - 1}{x} < y \right) ]
\end{align*}
\]  \hfill (9)

In this context, an axiom system \( \alpha \) will be said to recognize Addition “as a total function” iff it can prove

\[ \forall x \forall y \exists z \quad \text{Add}(x, y, z) \]  \hfill (11)

Likewise, we will say an axiom system \( \alpha \) can recognize Multiplication “as a total function” iff it can prove

\[ \forall x \forall y \exists z \quad \text{Mult}(x, y, z) \]  \hfill (12)

Also, we will say an axiom system \( \alpha \) can recognize Successor (i.e. the operation of “plus one”) as a total function iff it can prove

\[ \forall x \exists z \quad \text{Add}(x, 1, z) \]  \hfill (13)

Some axiom systems \( \alpha \) are unable to prove Multiplication is a total function, but they can prove every true \( \Delta_0^* \) sentence. Other axiom systems are unable to recognize any of Addition, Multiplication or Successor as total functions, but they can still prove every true \( \Delta_0^* \) sentence. It will turn out these facts will be central to understanding the generality and limitations of Gödel’s Second Incompleteness Theorem.
Definition 3.6 A sentence \( \Phi \) will be said to be written in **Prenex* Normal Form** iff for some \( i \geq 0 \), it can be written as a \( \Pi^*_i \) or a \( \Sigma^*_i \) sentence. (It can be easily established that a predicate logic, using the language \( L^* \), can show that every sentence \( \Phi \) can be mapped onto a Prenex* sentence \( \Phi^* \) such that \( \Phi \iff \Phi^* \). Thus without any loss in generality, we may assume that all the proper axioms within a basis system \( \alpha \) can be encoded in a Prenex* normalized form.)

### 3.2 Enriched Forms of Tableaux Deduction

We will first employ our preceding language \( L^* \) to review the definition of “Semantic Tableaux” deduction in this section. We will then define two minor variations of this construct, called the Rank-Zero and Rank-Zero-Plus enriched versions of Tableaux deduction. The addition of these Rank-Zero constructs to the tableaux formalism will, likely, first appear to the a minor wrinkle to most readers. It will turn out, however, that the factor-billion difference between the speeds of a human mind and of a digital computer will make our added “Rank-Zero” refinement useful.

Our definition of a semantic tableaux proof will be very similar to its counterparts in Fitting’s and Smullyan’s textbooks [11, 40]. Define a \( \Phi \)-**Focused Candidate Tree** for the axiom system \( \alpha \) to be a tree structure whose root corresponds to the sentence \( \neg \Phi \) and whose all its other nodes are either formal axioms of \( \alpha \) or deductions from higher nodes of the tree. Let the notation “\( \mathcal{A} \implies \mathcal{B} \)” indicate that \( \mathcal{B} \) is a valid deduction when \( \mathcal{A} \) is an ancestor of \( \mathcal{B} \). In this notation, the deduction rules allowed in a candidate tree are:

1. \( \Upsilon \wedge \Gamma \implies \Upsilon \) and \( \Upsilon \wedge \Gamma \implies \Gamma \). 
2. \( \neg \neg \Upsilon \implies \Upsilon \). Other Tableaux rules for the “\( \neg \)” symbol are: \( \neg (\Upsilon \vee \Gamma) \implies \neg \Upsilon \wedge \neg \Gamma \), \( \neg (\Upsilon \implies \Gamma) \implies \Upsilon \wedge \neg \Gamma \), \( \neg (\Upsilon \wedge \Gamma) \implies \neg \Upsilon \vee \neg \Gamma \), \( \neg \exists v \, \Upsilon(v) \implies \forall v \neg \Upsilon(v) \), and \( \neg \forall v \, \Upsilon(v) \implies \exists v \neg \Upsilon(v) \).
3. A pair of sibling nodes \( \Upsilon \) and \( \Gamma \) is allowed when their ancestor is \( \Upsilon \vee \Gamma \).
4. A pair of sibling nodes \( \neg \Upsilon \) and \( \Gamma \) is allowed when their ancestor is \( \Upsilon \implies \Gamma \).
5. \( \exists v \, \Upsilon(v) \implies \Upsilon(u) \) where \( u \) is a newly introduced “Parameter Symbol”.
6. Our variation of Rule 5 for **bounded existential quantifiers** of the form “\( \exists v \leq s \)” is the identity: \( \exists v \leq s \, \Upsilon(v) \implies u \leq s \wedge \Upsilon(u) \).
7. \( \forall v \, \Upsilon(v) \implies \Upsilon(t) \) where \( t \) denotes any U-Grounded term. These terms are defined to be parameter symbols, constant symbols, or U-Grounding functions with recursively defined inputs.

8. Our variation of Rule 7 for \textit{bounded universal quantifiers} of the form “\( \forall v \leq s \) ” is the identity: \( \forall v \leq s \, \Upsilon(v) \implies t \leq s \implies \Upsilon(t) \).

Define a particular leaf-to-root branch in a candidate tree \( T \) to be \textbf{Closed} iff it contains both some sentence \( \Upsilon \) and its negation \( \neg \Upsilon \). A \textbf{Semantic Tableaux} proof of \( \Phi \) will then be defined to be a candidate tree, \textit{all of whose root-to-leaf branches are closed}, such that the tree’s root stores the sentence \( \neg \Phi \) and where all its other nodes are either axioms of \( \alpha \) or deductions from higher nodes.

**Definition 3.7** Let \( Z \) denote an arbitrary set of sentences in our language \( L^* \). Recall that a node in semantic tableaux proof from an axiom system \( \alpha \) is allowed to include any axiom of \( \alpha \) as one of its stored sentences. Such a proof will be called a \textbf{Z-Enriched} proof if it may also include any version of (14)’s formalization of the “Law of the Excluded Middle” as a permissible logical axiom when \( \Psi \in Z \).

\[
\Psi \lor \neg \Psi
\]  
(14)

It is well known that semantic tableaux proofs satisfy Gödel’s Completeness Theorem [11, 40]. This implies that the set of theorems that are proven from an axiom system \( \alpha \) via a conventional (unenriched) version of semantic tableaux deduction is the same as the set of theorems proven from a Z-enriched version of this deductive mechanism. Our main result in this section will show, however, that such proofs can have their efficiency often exponentially improved via such enrichments, when Line (14)’s schema is treated as a set of logical axioms (rather than as a collection of derived theorems).

**Definition 3.8** Let \( \alpha \) be an arbitrary set of proper axioms and \( D \) denote a deduction method. An arbitrary theorem \( \Phi \) of \( \alpha \) will be said to satisfy a \textbf{Ψ-Based Linear Constrained Cut Rule} iff \( \Phi \)’s shortest proof from \( \alpha \) (via \( D \)’s apparatus) is guaranteed to be no longer than proportional to the sum of the lengths of the proofs of \( \Psi \) and \( \Psi \Rightarrow \Phi \) from \( \alpha \).
Example 3.9  All Hilbert-style deduction methods (including Example 2.1’s \( d_E, d_H, \) and \( d_M \) Hilbert-style methodologies) employ Linear Constrained Cut Rules for any arbitrary input sentence \( \Psi \) (on account of the presence of their modus ponens rules). This is the intuitive reason that Theorem ++’s generalization of the Second Incompleteness Theorem (from Remark 2.5) applies to them. We will soon see that a similar generalization of the Second Incompleteness Theorem applies to modifications of semantic tableaux deduction (where Line (14)’s invocation of the Law of the Excluded Middle is available as a logical axiom for every input sentence \( \Psi \)).

Lemma 3.10  Let \( D_\Psi \) denote an “enriched” deduction method, identical to semantic tableaux deduction, except that Line (14)’s version of the Law of the Excluded Middle is available as a logical axiom under \( D_\Psi \). Then for an arbitrary theorem \( \Phi \), a \( \Psi \)-Based Linear Constrained Cut Rule will be satisfied by the deduction method \( D_\Psi \).

A Brief Sketched Justification: The germane proof \( p \) of \( \Phi \) will follow the usual semantic tableaux format by having its root store the sentence \( \neg \Phi \). The child of this root will then consist of Line (14)’s version of the Law of the Excluded Middle. Also, the two children of this node will consist of the sentences of \( \neg \Psi \) and \( \Psi \). We omit the details, but it is easy to then verify that:

a. one may insert a subtree below \( \neg \Psi \) that is no longer than linearly proportional to the length of \( \Psi \)’s proof.

b. one may insert a subtree below \( \Psi \) that is no longer than linearly proportional to the length of the proof of \( \Psi \Rightarrow \Phi \).

These constraints imply that the \( \Psi \)-Based Linear Constrained Cut Rule will be satisfied. \( \square \).

Definition 3.11  There will be several types of “Enrichments” of the semantic tableaux deduction method that we will examine in the context of Definitions 3.7 and 3.8. These will include:

1. **Infinitely Enriched** formalisms that allow Line (14)’s variation of the “Law of the Excluded Middle” to become a logical axiom, for any sentence \( \Psi \) from \( L^* \)’s language.
2. **Rank-k Enriched formalisms** that allow Line (14)’s variation of the “Law of the Excluded Middle” to be a logical axiom when $\Psi$ is any $\Pi^*_k$ or $\Sigma^*_k$ sentence.

3. **Rank-Zero Enriched formalisms** that allow Line (14)’s variation of the “Law of the Excluded Middle” to be a logical axiom when $\Psi$ is any $\Delta^*_0$ sentence.

4. **Rank-Zero-Plus Enriched formalisms** that are a slightly stronger version of the Rank-Zero formalism that take (15) as a logical axiom for any $\Delta^*_0$ formula $\psi(x)$.

$$\forall x \, \psi(x) \lor \neg \psi(x)$$

(15)

**Remark 3.12** Let $\alpha$ denote any axiom basis that includes the ten U-Grounding symbols. Then if $D$ denotes the semantic tableaux deductive methodology and if all of $\alpha$’s axioms hold true in the standard model, it will follow that $55$’s $IS_D(\alpha)$ formalism will be a self-justifying system which proves all $\alpha$’s $\Pi^*_1$ theorems and additionally recognizes its own semantic tableaux consistency. It turns out this result will also generalize when $D$ corresponds to either Item 3’s Rank-Zero enriched form of semantic tableaux deduction or Item 4’s Rank-Zero-Plus enriched form. (E.g. these two systems will be also capable of recognizing their own consistency under their enriched deduction methods.)

In contrast, one may easily apply Lemma 3.10 to show that the invariant $++$ (appearing in Remark 2.5) will generalize to establish that Type-S axiom systems cannot verify their own consistency under infinite enrichments of the semantic tableaux. Indeed, the methods from 54 imply the Second Incompleteness Theorem also generalizes for the case of Rank-2 or higher enriched formalisms under Type-A arithmetics. Thus, there is only a meaningful open question about the application of Remark 3.12’s paradigm to Rank-1 Enriched systems.

An attached appendix will review 55’s definition of the $IS_D(\alpha)$ axiom system, for the benefit of those readers who have not read 55. (It will amplify upon the claims made in the previous two paragraphs.) Our recommendation is that the reader postpone examining this appendix until after the main sections of the current paper are finished. This is because Definition 3.11’s notion of “Deductive Enrichment” is quite subtle, and the next two sections shall need to first consider it in more detail.

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This is because infinite enrichments of tableaux make its deductive procedure resemble Hilbert deduction because such enrichments uniformly obey Definition 3.8’s Linear Constrained Cut Rule.
4 The Significance of Deductive Enrichment

A general rule of thumb in Proof Theory is that an axiom system is typically extended in order to expand the class of theorems it can prove. Since semantic tableaux deduction satisfies Gödel’s Completeness Theorem, the Definition 3.11’s four variations of deductive enrichment will, however, not change the theorems they can be derive from an initial base axiom system. Instead, their function will be to improve the overall proof efficiency. (This is because an invoked version of the Law of the Excluded Middle will shorten proof lengths when it is treated as a logical axiom, rather than as a derived theorem.)

This issue was perhaps not so central in the early 20-th century when Gödel announced his initial 2-part Incompleteness paradigm. At that time, the only available medium of thought was the Human Brain, which performed arithmetic computations at approximately a billion times a slower speed than that of the typical 21st century household computer. Also, the potential lengths of logical sentences during the 1930’s was much shorter than many potentially lengthy present-day computer-generated sentences.

Within the context of the longer expressions that computers can physically produce, the task of separating true from false $\Delta^0_3$ sentences will likely become increasingly daunting (assuming $P \neq NP$ ), even when this task is technically decidable. Hence a Rank-Zero Enrichment of a tableaux deductive formalism is a useful instrument, with the improved efficiency of its Rank-Zero Linear Constrained Cut Rule. Moreover, the results from our earlier paper [55] do trivially imply that our Rank-Zero and Rank-Zero-Plus extensions of Self-Justifying formalism are guaranteed to be consistent. (This is because an application of either the Rank-Zero or Rank-Zero-Plus versions of the Law of the Excluded Middle can be formally encoded within a $\Pi^1_1$ format, and the attached Appendix explains our results from [55] imply that the addition of any logically valid $\Pi^1_1$ sentences are compatible with $IS_D(\alpha)$ retaining its internal consistency.) In contrast, [54] showed the same is not true for Rank-2 and higher enrichments of tableaux deduction thus causing these formalisms to lose their self-justification property.

Remark 4.1 Many readers will be initially disappointed that Rank-2 and other higher enrichments levels will be of infeasible under self-justifying semantic tableaux deductive systems. This will mean that if $\Psi$ denotes (16)’s declaration that multiplication is a total function then neither can it be assumed to be true by our self-justifying formalisms,
nor can the theorem $\Psi \lor \neg\Psi$ be promoted into becoming a logical axiom (under the self-justifying methods of IS$_D(\alpha)$ without producing an inconsistency).

$$\forall x \forall y \exists z \ Mult(x, y, z)$$

Nevertheless, there is a method whereby our Rank-Zero Enriched formalisms can partially formalize Line (16)’s meaning. Thus, let $\Psi_k$ denote the $\Delta^*_0$ formula (shown below) indicating that its localized version of multiplication is a total function among input integers less than $2^k$.

$$\forall x \leq \text{Double}^k(2) \ \forall y \leq \text{Double}^k(2) \ \exists z \leq \text{Double}^{2k}(2) \ Mult(x, y, z) \ (17)$$

Then it turns out that our Rank-Zero enriched versions of tableau deduction can prove $\Psi_k$ as a theorem, as well as treat $\Psi_k \lor \neg\Psi_k$ as a logical axiom. We will call $\Psi_k$ and the logical axiom $\Psi_k \lor \neg\Psi_k$ the K-Localizations of the sentences $\Psi$ and $\Psi \lor \neg\Psi$.

In many pragmatic applications, one does not technically need $\Pi^*_j$ and $\Sigma^*_j$ theorems $\Phi$ (with $j \geq 1$): Instead, it suffices to prove a K-Localized theorem $\Phi_k$, that employs analogs of Line (17)’s three specified bounded quantifiers, for some sufficiently large fixed constant $k$. In particular, a transition of higher sentences into $\Delta^*_0$ formulae is especially pragmatic for 21st century computers, whose speed and allowed byte-lengths can exceed by factors of many millions their counterparts produced by human mind.

Within such a context, the self-justifying capacities of even a Rank-Zero Enriched form of Semantic tableaux deduction are much more tempting during the 21st century than they were at the time of Gödel 1931 discovery (when computers were unavailable). It is mainly for this reason that we suspect the modern world should not fully dismiss the capacities of self-justifying logic formalisms.

Moreover, we suspect that the futuristic civilization within our solar system, including that on the planet Earth, may have no choice but to rely upon Self-Justifying computerized logic systems. This is because many scientists (including the late physicist Stephen Hawking) have speculated that if current trends continue, then Global Warming will cause the planet Earth to become too hot for mammals to survive on it, within one or two centuries. In such a context, where computers will not need the Earth’s cooler temperatures and/or Oxygen to survive (e.g. see footnote 4), Stephen Hawking [23, 24] has predicted...

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4In particular, solar powered computers, physically residing outside the planet Earth, will need neither
that computers may become the main form of thinking agent during what will hopefully be only a temporary period of Global Warming.

Such computers will need, presumably, to rely upon some form of Self-Justifying cognitive process to organize the motivations of their thought processes. In particular, humans seem to have relied upon some type of instinctive appreciation of the coherence of their thought processes, as a prerequisite for motivating their cognitive process. Our suspicion is that computers will need to imitate this self-reinforcing process.

Our conjecture is, thus, that the continuation of human civilization, within our Solar System, may require computers taking temporary control of the larger part of its destiny. More specifically, we suspect that a computer-and-robot technology shall be needed to reverse Global Warming and enable a saved sample of frozen mammal embryos to subsequently populate the planet Earth, again.

In particular, some readers may shutter at the thought that planet Earth could become temporarily uninhabitable during a perhaps thousand-year era of Global Over-Heating. This difficulty, however, may actually amount to only be a temporary phenomena, if computers and robots can restore Earth into a more hospitable environment after a period of several thousand years (and also frozen human embryos are saved).

More precisely in a context where life has existed on Earth for approximately 4 billion years, our perspective is that the danger posed by Global Warming would be tragic but temporary, if robots and computers can reverse Global Warming after a period of a few thousand years. In contrast, the implications of Global Warming would be much more severe, if either it cannot be reversed or no frozen embryos are saved before Global Warming occurs. It is for this reason that we suggest it is imperative that a fleet of self-justifying computers, along with accompanying robots and saved frozen embryos, be assembled as at least a partial response to a potentially very serious Global Warming crisis. (See footnote 5 for some clarifications about the nature and limits of this proposal.)

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5 It is possible that if computer and robotics technologies do advance quickly enough than a tragic Global Over-Heating can be completely avoided. Moreover, Steven Pinker discusses in [35] some technologies that could possibly eschew global warming. Our point is, however, that it would be wise to also investigate the potential application of self-justifying formalisms because their use may be necessary in the future for Earth's mammals to survive and prosper. The Corollary A.3, in Appendix A, thus illustrates a potential use of self-justifying computerized systems that possibly could be urgent.
5 Further Remarks

The author of this article has published several articles about generalizations and boundary-case exceptions for Gödel’s Second Incompleteness Theorem [49–62], including six papers that have appeared in the JSL and APAL. Unfortunately, the author has experienced both a stroke and heart attack during the summer of 2016.

These events did not prevent the author from continuing his teaching during 2017 and 2018. They did, however, cause a change in the specific goals of our research. Thus, the current article was intended, mostly, to encourage others to join in this research project. It has, thus, focused on explaining why this topic warrants further investigation.

This is subtle because our IS\textsubscript{D}(α) axiom system (formally defined in the Appendix) has three disadvantages when \( D \) denotes semantic tableaux deduction. These drawbacks are that:

1. IS\textsubscript{D}(α) is an unorthodox “Type-A” axiom system, which recognizes addition but not also multiplication as a total function.

2. IS\textsubscript{D}(α) has employed a semantic tableaux deductive apparatus, that is much weaker than a more conventional Hilbert-styled deductive apparatus.

3. IS\textsubscript{D}(α) is able to recognize its consistency only by employing a version of Example 2.3’s self-referencing “I am consistent” axiom.

Our reply to Item 1 is that while IS\textsubscript{D}(α) is unconventional, it is it not quite as weak as it may appear. This is partly because IS\textsubscript{D}(α) treats multiplication as a 3-way predicate Mult\((x, y, z)\), formalized by Line \([10]\), rather than as a total function. Moreover, IS\textsubscript{D}(α) can be easily arranged to prove all the \( \Pi_1 \) theorems of Peano Arithmetic (except for minor changes in notation) when its input axiom system \( \alpha \) is made to correspond to the trivial extension of Peano Arithmetic that includes our ten U-Grounding functions symbols. While these adjustments may not be ideally Utopian, they are sufficient to reply to the main difficulties raised by Item 1.

Our reply to Item 3 is also easy because the goal of IS\textsubscript{D}(α) IS NOT TO prove its own consistency under a skinny definition of a proof. It is rather to find an axiom system that is comfortable with a built-in internal assumption about its own consistency. IS\textsubscript{D}(α) can
do this with its physically built-in “I am consistent” axiom. However, most conventional logics are simply incompatible with their counterparts of this axiom.

Many readers will probably be especially leery of Item 2 because conventional Hilbert-style deductive proofs satisfy Definition 3.8’s Linear Constrained Cut Rule, while Semantic Tableaux Deduction does not obey this property. In particular, our ISD(α) formalisms are required to treat the Law of the Excluded Middle as a set of derived theorems (rather than as a formally stronger built-in schema of logical axioms).

We can, fortunately, partially reply to this daunting challenge because Definition 3.11 indicated that there were four methods for enriching semantic tableaux deductive machineries. Two of these four methods (i.e. the Rank-Zero and Rank-Zero-Plus enrichments) were formally compatible with self-justifying extensions of the semantic tableaux deductive machinery existing. Thus at least the Rank-Zero and Rank-Zero-Plus versions of the Law of the Excluded Middle may be incorporated into self-justifying semantic tableaux formalisms.

Naturally, it would be better if Remark 3.12 indicated enrichment methodologies of Rank-2 and higher were also compatible with self justification. Unfortunately, however, excessive enrichments of the semantic tableaux deductive systems preclude self-justifying systems from existing (in a manner roughly analogous to how excessive enrichments of Uranium Ore can cause nuclear reactors to become dangerously unstable). Thus, we are currently confined to study Rank-Zero enrichments of semantic tableaux deduction, in a context where Rank-2 enrichments are infeasible, and Rank-1 enrichments are a remaining open question. (See [54] for a discussion of Rank-2 enrichments.)

While we do not wish to ignore the fact that only that only Rank-Zero and Rank-Zero-Plus enrichments of semantic tableaux deduction are known to be compatible with self justification, it should be remarked that even such Rank-Zero enrichments are interesting. This is because one can philosophically argue that a logical sentence loses its purely sensuous quality when it employs unbounded quantifiers. Thus even $\Pi_1$ and $\Sigma_1$ sentences lie slightly above the “touch-and-feel level” on account of their use of unbounded quantifiers. In other words, our available ability to muster self-justifying Rank-Zero and Rank-Zero-Plus enrichments of semantic tableaux deduction is significant because this level of enriching is broad enough to include the crucial “touch-and-feel” sentences of the language $L^*$. 
The preceding point is important because it allows us to summarize both the strengths and weaknesses of Gödel’s 1931 Second Incompleteness Theorem. Thus, the traditional literature has been certainly correct in viewing Gödel’s discovery as a seminal result, when our boundary-case exceptions to it have persisted at only the Rank-Zero and Rank-Zero-Plus enrichment levels. On the other hand, the existence of such enrichments show that Gödel and Hilbert were partially correct when their statements ∗ and ∗∗ foresaw that some types of exceptions to the Second Incompleteness Effect would persist (e.g. see §). Moreover, the study of how to efficiently process ∆0 sentences is important both because of their “touch-and-feel” property and because it will be challenging to process these sentences efficiently, assuming that \( P \neq NP \).

Moreover, we remind the reader that Stephen Hawking and other scientists have expressed concern that Global Warming could possibly lead to, at least, a temporary era where digital computers replace human brains as the primary economically efficient mechanism for generating thoughts [23, 24]. In a context where computers can generate thoughts more than a billion times faster than the human mind, the preceding chapter suggested that a self-justifying formalism, using only Rank-Zero and Rank-Zero-Plus enrichments of semantic tableaux, could help computers function more efficiently.

In particular, our fervent hope is that humans will continue to be important and central in the future. It is likely that computerized simulations of the human thought processes will be also important, even if Global Warming occurs as, hopefully, a very temporary phenomena.

The writing of this article has been a quite painful task because this paper conveys a message that is an awkward mixture of insight, hope and humbling realization. Thus our fundamental insight is that some type of borderline exception to the Second Incompleteness Theorem will persist, whereby logical formalisms can retain at least some type of partial internalized appreciation of their own consistency. Our accompanying hope is that computers can use this knowledge to help life on Earth survive Global Warming and other futuristic challenges. And finally this paper’s humbling but partially optimistic conclusion is that a prosperous life can continue on planet Earth, albeit seemingly only if (?) humans share with computers a joint control over our future destiny.

**Added Remark:** This paper has been written as an “extended abstract”, rather than
as a fully detailed exposition, primarily because many of our other papers have discussed related topics in much greater detail. The writing style of an informal extended abstract also seemed preferable because our chief goal has been to stimulate a larger audience of researchers to join us in a project, centered around futuristic self-justifying logic systems.

6 About the Author

In addition to publishing several papers about symbolic logic, Dan Willard has been also active in several other areas of academic research. This other activity has included:

1. Co-Authoring with Robert Trivers [43, 63] a hypothesis that Darwinian evolution exercises partial control over the sex determination of offspring. Google Scholar has recently recorded 3,586 citations to article [43], and it was also discussed twice in the New York Times on February 17, 1981 and January 21, 2017 under the respective titles of “Species Survival Linked to Lopsided Sex Ratios” and “Does Breast Milk Have A Sex Bias”.

2. Co-Authoring with Michael Fredman the articles [12, 13] about Sorting and Searching. These papers were the chronologically first among six items mentioned in the Mathematics and Computer Science Section of the 1991 Annual Report of the National Science Foundation.

3. Developing the Y-Fast Trie data structure [47], the Down Pointer Method [48] and many other results germane to Computational Geometry.

The momentum from the preceding articles enabled Dan Willard to persuade the University of Albany to make available sufficient time to undertake detailed research into symbolic logic, as did appear in the subsequent papers [49]-[62].

Acknowledgment: I thank Seth Chaiken for several comments about how to improve the style of presentation.
Appendix

Our article [55] had introduced both the definitions of the IS$_D$(α) axiom system and of a formalism owning a “Level(J) Appreciation” of its own consistency. This appendix will review the definitions of these two concepts, as well as explain how they are related to enriched versions of semantic tableaux deduction. This appendix should be read only after Sections 1-5 are completed.

During our discussion, $L^*$ will denote the language defined in §3.1 and β will denote a basis axiom system whose deductive apparatus is denoted as D. In this context, the basis system β will be said to own:

1. a **Level(n) Appreciation** of its own consistency under D’s deductive apparatus iff β can prove that there exists no $\Pi^*_n$ sentence $\Upsilon$ such that $(\beta, D)$ supports simultaneous proofs of both $\Upsilon$ and $\neg \Upsilon$.

2. a **Level(0-) Appreciation** of its own consistency under D’s deductive apparatus iff it can prove there exists no proof of $0=1$ from the $(\beta, D)$ formalism.

All these definitions of consistency, from Level(0-) up to Level(n), are equivalent to each other under strong enough models of Arithmetic. However, many weak axiom systems do not possess the mathematical strength to recognize their equivalence.

In particular, an axiom system β owning a Level(1) appreciation of its own consistency is much stronger than such a system possessing a Level(0-) appreciation of its own consistency. This is because Level(1) systems can use a proof of a $\Pi^*_1$ theorem $\Upsilon$ to gain the practical knowledge that no proof of $\neg \Upsilon$ exists.

In a context where α is essentially any axiom system that employs $L^*$ ’s language and where D denotes any deductive apparatus, our axiom system IS$_D$(α) in [55] was designed to achieve two specific goals. These were:

1. To prove all true $\Delta^*_0$ sentences, as well as to prove all the $\Pi^*_1$ theorems that are implied by the axiom basis α. In particular, the formal definition of IS$_D$(α) system, in §3 of [55], accomplished the first task trivially via its “Group-Zero” and “Group-1 schema”. It performed the second task because its “Group-2 scheme”
employed Line (18)’s particular generic structure for each $\Pi_1^*$ sentence $\Phi$ (e.g. see Footnote $^6$).

$$\forall p \{ \text{Prf}_\beta(\lceil \Phi \rceil , p) \rightarrow \Phi \} \quad (18)$$

2. To “formally recognize” its own Level-1 consistency under $D$’s deductive apparatus. This was accomplished by having IS$_D(\alpha)$’s “Group-3” axiom formalize a $\Pi_1^*$ sentence that amounts to (19)’s statement. (The symbol “Pair($x, y$)” in Line (19) is a $\Delta_0^*$ formula indicating that $x$ is the Gödel number of a $\Pi_1$ sentence and that $y$ represents $x$’s “mechanically formalized negation”. Also, Prf$_{\text{IS}_D(\alpha)}(a, b)$ in (19) denotes a $\Delta_0^*$ formula indicating that $b$ is a proof, using the deduction method $D$ , of the theorem $a$ from the axiom system IS$_D(\alpha)$.) The nice aspect of (19) is that IS$_D(\alpha)$ can unambiguously interpret the meaning of its three $\Delta_0^*$ formulae because its Group-Zero and Group-1 schema allow it to correctly decipher Line (19)’s statement.

$$\forall x \forall y \forall p \forall q \neg [ \text{Pair}(x, y) \land \text{Prf}_{\text{IS}_D(\alpha)}(x, p) \land \text{Prf}_{\text{IS}_D(\alpha)}(y, q) ] \quad (19)$$

**Remark A.1** It is unnecessary to provide a formal description of the axiom system IS$_D(\alpha)$ here because §3 of [55] already explained how it (and especially Line (19) ) can be exactly encoded. There are two clarifications relevant to IS$_D(\alpha)$’s definition that should, however, be mentioned.

a. Our IS$_D(\alpha)$ axiom system is a well defined entity for any deductive apparatus $D$ and for any axiom basis system $\alpha$ (that have recursively enumerable formal structures). Since Line (19) is a self-referencing sentence, one needs some meticulous care, however, to assure that the Fixed Point Theorem can encode a $\Pi_1^*$ sentence that is equivalent to Line (19)’s statement. (We do not discuss this topic here because it was discussed in adequate detail in [55]).

b. Although the IS$_D(\alpha)$ axiom system is well-defined entity for all inputs $D$ and $\alpha$, this fact does not also guarantee that IS$_D(\alpha)$ is consistent. Indeed, the Second Incompleteness Theorem implies IS$_D(\alpha)$ is inconsistent for most inputs $D$ and $\alpha$.

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$^6$The symbol $\lceil \Phi \rceil$ in (18) denotes $\Phi$’s Gödel number, and the symbol $\text{Prf}_\beta(\lceil \Phi \rceil , p)$ will designate a $\Delta_0^*$ stating that $p$ is a proof of $\Phi$.

$^7$In particular, if $x$ is the Gödel number of a $\Pi_1$ sentence $\Phi$ then its “mechanically formalized negation” $y$ is the Gödel number for “$\neg \Phi$”.

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23
This issue was previously visited by us in Example 2.3. It had emphasized that “I am consistent” axioms, similar to Line (19), can be easily encoded (via the Fixed Point theorem). However, these axiom sentences are typically useless, on account of the inconsistencies that they usually produce.

Our main result in [55] is related to Item (b)’s ironical paradigm. In particular, [55] established that Remark A.1.b’s paradigm can be evaded when \( D \) corresponds to semantic tableaux deduction. Thus, [53 55] established that the following partial boundary case exception to the Second Incompleteness Theorem does arise:

**Theorem A.2.** [53 55] Let \( D_S \) denote the semantic tableaux deductive apparatus, and \( \alpha \) denote any axiom system all of whose \( \Pi^*_1 \) theorems are true sentences under the standard model using the language \( L^* \). Then \( IS_{D_S}(\alpha) \) will prove all \( \alpha \)'s \( \Pi^*_1 \) theorems, and it will also be consistent.

We remind the reader that Theorem A.2 is significant because the Second Incompleteness Theorems implies that most systems, formally verifying their own consistency, actually fail to be consistent. Theorem A.2 is germane to the current article because it implies the following corollary:

**Corollary A.3** Let \( D_E \) denote the Rank-Zero-Plus enrichment of the semantic tableaux apparatus. Also, let \( \alpha \) again denote an axiom system all of whose \( \Pi^*_1 \) theorems are true sentences under the standard model. Then \( IS_{D_E}(\alpha) \) will also be consistent.

Corollary A.3 is an easy consequence of Theorem A.2. We will now present a brief sketch of its proof.

**Proof Outline:** Let \( \alpha^* \) denote the extension of the basis system \( \alpha \) that includes one instance of axiom (20) for every \( \Delta^*_0 \) formula \( \Psi(x) \).

\[
\forall x \quad \Psi(x) \lor \lnot \Psi(x)
\]

(20)

It is apparent that all the theorems of \( \alpha^* \) hold true under the standard model because (20) holds true under the Standard Model, as do also all the particular \( \Pi^*_1 \) theorems of \( \alpha \). Hence, Theorem A.2 implies that \( IS_{D_S}(\alpha^*) \) is consistent. This implies that \( IS_{D_E}(\alpha) \) must also be consistent (since the \( IS_{D_S}(\alpha^*) \) and \( IS_{D_E}(\alpha) \) formalisms are essentially identical to each other, except for minor changes in notation). □
Remark A.4. The simplicity of Corollary A.3’s proof may tempt one to partially overlook its significance. This corollary is significant because one of the main themes of our article has been that a deductive apparatus does not capture the core intentions of most logics, unless it contains some form of Definition 3.8’s Linear Constrained Cut Rule. The significance of Corollary A.3 is that it shows that the Rank-Zero-Plus variant of Definition 3.8’s linear cut rule is actually formally supported by the self-justifying IS_{DE}(\alpha) axiom system.

Remark A.5. We again remind the reader that the formalism IS_{DS}(\alpha) and IS_{DE}(\alpha) in Propositions A.2 and A.3 are capable of proving all Peano Arithmetic’s \Pi_1^* theorems when \alpha designates the trivial extension of Peano Arithmetic which includes the ten U-Grounding functions symbols. (These formalisms can thus appreciate the fundamental \Pi_1 significance of traditional arithmetic.)

References

[1] Z. Adamowicz, “Herbrand consistency and bounded arithmetic”, Fundamenta Mathematicae 171 (2002) pp. 279-292.
[2] Z. Adamowicz and P. Zbierski, “On Herbrand consistency in weak theories”, Archives for Mathematical Logic 40(2001) 399-413
[3] L. D. Beklemishev, “Induction rules, reflection principles and provably recursive functions”, Annals of Pure and Applied Logic 85 (1997) pp. 193-242.
[4] A. Bezboruah and J. C. Shepherdson, “Gödel’s second incompleteness theorem for Q”, Journal of Symbolic Logic 41 (1976) pp. 503-512
[5] S. R. Buss, Bounded Arithmetic, (Ph D Thesis) Proof Theory Notes #3, Bibliopolis 1986.
[6] S. R. Buss, Informal private conversations during a lunch in Vienna at the 5-th meeting of the Kurt Gödel Colloquium, 1997.
[7] S. R. Buss and A. Ignjatovic, “Unprovability of consistency statements in fragments of bounded arithmetic”, Annals of Pure and Applied Logic 74 (1995) pp. 221-244.
[8] J. W. Dawson, Logical Dilemmas: The life and work of Kurt Gödel, AKPeters, 1997
[9] H. B Enderton, Mathematical Introduction Logic, Academic Press, 2011
[10] S. Feferman, “Arithmetization of mathematics in a general setting”, Fundamenta Mathematicae 49 (1960) pp. 35-92.
[11] M. Fitting, First Order Logic and Automated Theorem Proving, Springer-Verlag, 1996.
[12] M. L. Fredman and D. E. Willard, “Surpassing the information theoretic barrier with fusion trees”, The Journal of Computer and Systems Sciences 47 (1993), pp. 424-433.
[13] M. L. Fredman and D. E. Willard, “Transdichotomous algorithms for minimum spanning trees and shortest paths”, The Journal of Computer and Systems Sciences 48 (1994), pp. 533-551.

[14] H. M. Friedman, “On the consistency, completeness and correctness problems”, Ohio State Tech Report, 1979.

[15] H. M. Friedman, “Translatability and relative consistency”, Ohio State Tech Report, 1979.

[16] H. M. Friedman, “Gödel’s blessing and Gödel’s curse”. (This is “Lecture 4” within a 5-part Ohio State YouTube lecture, dated March 14, 2014, whose last few minutes is devoted to summarizing the mysteries left unsolved by the Second Incompleteness Theorem.)

[17] K. Gödel, “Über formal unentscheidbare sätze der principia mathematica und verwandte systeme I”, Monatshefte für Mathematik und Physik 37 (1931) pp. 349-360.

[18] K. Gödel, “The present situation in the foundations of mathematics”, in Collected Works Volume III: Unpublished Essays and Lectures, 2004, pp. 45–53, edited by S. Feferman, Oxford University Press.

[19] R. Goldstein, Incompleteness: The Proof and Paradox of Kurt Gödel, Norton Press, 2005.

[20] P. Hájek, “Mathematical fuzzy logic and natural numbers”, Fundamenta Mathematicae 81 (2007), pp. 155-163.

[21] P. Hájek, “Towards metamathematics of weak arithmetics over fuzzy logic”, Logic Journal of the IPL 19 (2011), pp. 467-475.

[22] P. Hájek and P. Pudlák, Metamathematics of First Order Arithmetic, Springer Verlag 1991.

[23] Stephen Hawking interviewed by Dana Dovey, Newsweek December 26 2017, available at www.newsweek.com under the title of “Stephen Hawking’s six wildest predictions from 2017 - from a robot apocalypse to the demise of earth”

[24] Stephen Hawking interviewed by Racheal McNemy on November 1, 2017, available at https://www.cambridg-news.co.uk under the title of “Stephen Hawking says he fears artificial intelligence will replace humans”

[25] D. Hilbert, “Über das unendliche”, Mathematische Annalen 95 (1926) pp. 161-191.

[26] D. Hilbert and P. Bernays, Grundlagen der Mathematik, Springer 1939.

[27] R. Jeroslow, “Consistency statements in mathematics”, Fundamenta Math 72 (1971) pp.17-40.

[28] S. C. Kleene, “On the notation of ordinal numbers”, Journal of Symbolic Logic 3 (1938), 150-156.

[29] G. Kreisel and G. Takeuti, “Formally self-referential propositions for cut-free classical analysis”, Dissertationes Mathematicae 118 (1974).

[30] E. Mendelson, Introduction to Mathematical Logic, CRC Press, 2010.

[31] E. Nelson, Predicative Arithmetic, Mathematics Notes, Princeton University Press, 1986.
[32] R. Parikh, “Existence and feasibility in arithmetic”, *Journal of Symbolic Logic* 36 (1971), pp. 494-508

[33] J. B. Paris and C. Dimitracopoulos, “A note on undefinability of cuts”, *Journal Symbolic Logic* 48 (1983) 564-569

[34] C. Parsons, “On $n$–quantifier elimination”, *Journal of Symbolic Logic* 37 (1972), pp. 466-482.

[35] S. Pinker, *Enlightenment Now the case for Reason, Science, Humanism and Progress*, Viking, New York, 2018.

[36] P. Pudlák, “Cuts, consistency statements and interpretations”, *Journal of Symbolic Logic* 50 (1985), pp. 423-442

[37] P. Pudlák, “On the lengths of proofs of consistency”, in *Collegium Logicum: 1996 Annals of the Kurt Gödel Society* (Volume 2), Springer-Wien-NewYork, pp 65-86.

[38] P. Pudlák, Private emailed communications, 2001.

[39] H. A. Rogers, *Theory of Recursive Functions and Effective Compatibility*, McGrawHill 1967.

[40] R. Smullyan, *First Order Logic*, Springer-Verlag, 1968.

[41] R. M. Solovay, Private telephone conversation in 1994 describing Solovay’s generalization of one of Pudlák’s theorems [36], using some methods of Nelson and Wilkie-Paris [31, 46]. (The Appendix A of [51] offers a 4-page summary of this conversation.)

[42] V. Švejdar, ”An interpretation of Robinson arithmetic in its Grzegorczjk’s weaker variant” *Fundamenta Informaticae* 81 (2007) pp. 347-354.

[43] R. L. Trivers and D. E. Willard, “Natural selection of parental ability to vary sex ratio of offspring”, *Science* (New Series, Volume 179), January 5, 1973, pp. 90-92,

[44] A. Turing, “On the computable numbers with an application to the entscheidungsproblem”, *Proceedings of London Mathematics Socieity*, 1936, pp. 230-265.

[45] A. Visser, “Faith and falsity”, *Annals of Pure and Applied Logic* 131 (2005) pp. 103-131.

[46] A. J. Wilkie and J. B. Paris, “On the scheme of induction for bounded arithmetic”, *Annals of Pure and Applied Logic* (35) 1987, 261-302.

[47] D. Willard, “Log-Logarithmic Worst-Case Range Queries are Possible in Space O(N),” *Information Processing Letters* 17 (1983), pp. 81-84.

[48] D. Willard, “New Data Structures for Orthogonal Range Queries,” *SIAM Journal on Computing*, 14(1985), pp. 232-253.

[49] D. Willard, “Self-verifying axiom systems”, *Computational Logic and Proof Theory: The Third Kurt Gödel Colloquium* (1993), Springer-Verlag LNCS#713, 325-336.

[50] D. Willard, “The Tangibility Reflection Principle”, *Fifth Kurt Gödel Colloquium* (1997), Springer-Verlag LNCS#1289, pp. 319–334.
[51] D. Willard, “Self-verifying systems, the incompleteness theorem and the tangibility reflection principle”, in Journal of Symbolic Logic 66 (2001) pp. 536-596.

[52] D. Willard, “How to extend the semantic tableaux and cut-free versions of the second incompleteness theorem almost to robinson’s arithmetic Q”, Journal of Symbolic Logic 67 (2002) pp. 465–496.

[53] D. Willard, “Some New Exceptions for the Semantic Tableaux Version of the Second Incompleteness Theorem”, Proceedings of the Tableaux 2002 Conference SpringerVerlag LNAI#2381, pp. 281–297.

[54] D. Willard, “A version of the second incompleteness theorem for axiom systems that recognize addition but not multiplication as a total function”, First Order Logic Revisited, Logos Verlag, pp. 337–368.

[55] D. Willard, “An exploration of the partial respects in which an axiom system recognizing solely addition as a total function can verify its own consistency”, Journal of Symbolic Logic 70 (2005) pp. 1171-1209.

[56] D. Willard, “A generalization of the second incompleteness theorem and some exceptions to it”. Annals of Pure and Applied Logic 141 (2006) pp. 472-496.

[57] D. Willard, “On the available partial respects in which an axiomatization for real valued arithmetic can recognize its consistency”, Journal of Symbolic Logic 71 (2006) pp. 1189-1199.

[58] D. Willard, “Passive induction and a solution to a Paris-Wilkie question”, Annals of Pure and Applied Logic 146(2007) pp. 124-149.

[59] D. Willard, “Some specially formulated axiomizations for $\Sigma_0$ manage to evade the herbrandized version of the second incompleteness theorem”, Information and Computation 207 (2009), pp. 1078-1093.

[60] D. Willard, “A detailed examination of methods for unifying, simplifying and extending several results about self-justifying logics”, Cornell Library Archives 2011 (a helpful summary of Willard’s research), http://arxiv.org/abs/1108.6330

[61] D. Willard, “On the broader epistemological significance of self-justifying axiom systems”, Proceedings of 21st Wollic Conference, Springer Verlag LNCS 8652 (2014), pp. 221-236.

[62] D. Willard, “On how the introducing of a new $\theta$ function symbol into arithmetic’s formalism is germane to devising axiom systems that can appreciate fragments of their own Hilbert consistency”, http://arxiv.org/abs/1612.08071 Cornell Library Archives 2016.

[63] D. Willard, “Implications of the Trivers-Willard Sex Ratio Hypothesis for Avian Species and Poultry Production, And a Summary of the Historic Context of this Research” http://arxiv.org/abs/1707.00039v2 Cornell Library Archives 2017.

[64] P. Yourgrau, A World Without Time: The Forgotten Legacy of Gödel and Einstein, Basic Books, 2005. (See page 58 for the passages we have quoted.)