Research Article

New S-Type Bounds of M-Eigenvalues for Elasticity Tensors with Applications

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In this paper, based on the extreme eigenvalues of the matrices arisen from the given elasticity tensor, S-type upper bounds for the M-eigenvalues of elasticity tensors are established. Finally, S-type sufficient conditions are introduced for the strong ellipticity of elasticity tensors based on the S-type M-eigenvalue inclusion sets.

1. Introduction

Let $M = \{1, 2, \ldots, m\}$ and $N = \{1, 2, \ldots, n\}$; a real tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{m \times n \times m \times n}$ is called an elasticity tensor, if

$$a_{ijkl} = a_{klij} = a_{i,j,k,l}, \quad i, k \in M, j, l \in N. \quad (1)$$

Consider the following optimization problem with an elasticity tensor $\mathcal{A} = (a_{ijkl})$ [1, 2]:

$$\max f(x, y) = \sum_{i,k=1}^{m} \sum_{j,l=1}^{n} a_{ijkl} x_i y_j x_k y_l,$$

s.t. $x^T x = 1, y^T y = 1$,

$$x \in \mathbb{R}^m, y \in \mathbb{R}^n. \quad (2)$$

Qi et al. introduced the following definition of M-eigenvalues of an elasticity tensor [3, 4].

Definition 1. (see [3, 4]). Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{m \times n \times m \times n}$ be an elasticity tensor, if there exist nonzero vectors, $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$, and a real number $\lambda \in \mathbb{R}$, such that

$$\mathcal{A} x y y = \lambda x,$$

$$\mathcal{A} x y x = \lambda y,$$

$$x^T x = 1, y^T y = 1,$$

where

$$\mathcal{A} x y y = \sum_{k \in M, j,l \in N} a_{ijkl} x_k y_j x_l,$$

$$\mathcal{A} x y x = \sum_{i,k \in M, j \in N} a_{ijkl} x_i y_j x_k. \quad (3)$$

Then, $\lambda$ is called an M-eigenvalue of $\mathcal{A}$, and the nonzero vectors $x$ and $y$ are called the corresponding M-eigenvectors.

Qi in [3, 5, 6] presented some basic studies for tensor computations and approximations. Li et al. [7–10], Bu et al. [11], Che et al. [12], and Zhao et al. [13, 14] worked on analyzing the M-eigenvalues for various elasticity tensors. The authors in [15] proposed a tensor-based FTV model for the three-dimensional image deblurring problem, and some properties for Z-eigenvalues of tensor are given in [16–18]. Let

$$f(x, y) = \sum_{i,k \in M, j,l \in N} a_{ijkl} x_i y_j x_k y_l = \sum_{i,k \in M} x_i y_k x_l y_l B_{ik} y,$$

where $B_{ik} \in \mathbb{R}^{m \times n}$ and $C_{ij} \in \mathbb{R}^{m \times n}$ are symmetric matrices with entries

$$x^T x = 1, y^T y = 1, \quad (4)$$

(5)
Journal of Mathematics

Theorem 1 (see [19]). Let \( \mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{m \times n \times m \times n} \) be an elasticity tensor and \( \lambda \) be an M-eigenvalue of \( \mathcal{A} \). Then,  
\[
\max \{ \delta_1, \delta_2 \} \leq \lambda \leq \min \{ \theta_1, \theta_2 \},
\]
where \( \delta_1 := \min \{ \lambda_{\min}(C_{il}) - g_1(l) \}, \theta_1 := \min \{ \lambda_{\max}(C_{il}) + g_1(l) \}, \delta_2 := \min \{ \lambda_{\min}(B_{il}) - g_2(i) \}, \theta_2 := \min \{ \lambda_{\max}(B_{il}) + g_2(i) \}, \)
\[
(6)
\]

and

\[
g_1(l) = \sum_{j \in N, k \neq l} \rho(C_{jl}), \quad g_2(i) = \sum_{k \in M, k \neq i} \rho(B_{ik}).
\]

Theorem 2 (see [19]). Let \( \mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{m \times n \times m \times n} \) be an elasticity tensor and \( \rho_M(\mathcal{A}) \) be the M-spectral radius of \( \mathcal{A} \). Then,  
\[
\rho_M(\mathcal{A}) \leq \gamma := \min \{ \gamma_1, \gamma_2 \},
\]
where

\[
\gamma_1 = \max_{j, k \in N, i} \frac{1}{2} \left\{ \rho(C_{il}) + \sqrt{\rho^2(C_{il}) + 4 g_1(l) (g_1(j) + \rho(C_{jj}))} \right\},
\]
\[
\gamma_2 = \max_{i, k \in M, k \neq i} \frac{1}{2} \left\{ \rho(B_{il}) + \sqrt{\rho^2(B_{il}) + 4 g_2(i) (g_2(k) + \rho(B_{kk}))} \right\}.
\]

(7)

The following necessary and sufficient condition for strong ellipticity for general anisotropic elastic materials is presented by Han et al. [20].

Theorem 3 (see [20]). Let \( \mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{m \times n \times m \times n} \) be an elasticity tensor. The strong ellipticity condition holds, i.e.,  
\[
f(x, y) = \sum_{i, k=1}^{m} \sum_{j, l=1}^{n} a_{ijkl} x_i x_j y_k y_l > 0,
\]
for all nonzero vectors \( x \in \mathbb{R}^m, y \in \mathbb{R}^n \) if and only if the smallest M-eigenvalue of \( \mathcal{A} \) is positive.

(12)

One application of the lower bound in Theorem 1 is to identify the strong ellipticity condition of an elasticity tensor, and the upper bound in Theorem 2 is given to accelerate convergence of the WQZ-algorithm [19]. In this paper, by breaking \( N \) into disjoint subsets \( S \) and its complement, new S-type upper bounds for the M-spectral radius of an elasticity tensor are given in Section 2. In Section 3, S-type sufficient conditions are also given to identify the strong ellipticity condition of an elasticity tensor.

2. S-Type Upper Bounds

In this section, we give S-type upper bounds for the largest M-eigenvalues of an elasticity tensor, and the relationship between the S-type upper bounds and existed upper bounds is also established. The sets \( S_m, \mathcal{S}_m, S_n, \mathcal{S}_n \) are defined by \( M = S_m \cup \mathcal{S}_m \) and \( S_n \mathcal{n} \mathcal{S}_n = \emptyset \), \( N = S_n \cup \mathcal{S}_n \) and \( S_n \mathcal{n} \mathcal{S}_n = \emptyset \).

Theorem 4. Let \( \mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{m \times n \times m \times n} \) be an elasticity tensor and \( \rho_M(\mathcal{A}) \) be the M-spectral radius of \( \mathcal{A} \). Then,  
\[
\rho_M(\mathcal{A}) \leq \tau := \min \{ \tau_1, \tau_2 \},
\]
where

\[
\tau_1 = \max_{i \in S_m, k \in \mathcal{S}_n} \frac{1}{2} \left\{ g_2^S(i) + g_2^\mathcal{S}(k) + \sqrt{g_2^S(i) - g_2^\mathcal{S}(k) \right\}^2 + 4 g_2^S(i) g_2^\mathcal{S}(k),
\]
\[
\tau_2 = \max_{j \in S_n, l \in \mathcal{S}_m} \frac{1}{2} \left\{ g_1^S(j) + g_1^\mathcal{S}(l) + \sqrt{g_1^S(j) - g_1^\mathcal{S}(l) \right\}^2 + 4 g_1^S(j) g_1^\mathcal{S}(l),
\]

(9)
Proof. Let $\lambda$ be an $M$-eigenvalue of $\mathcal{A}$ with the $M$-eigenvectors $x, y$.

$$g^{S_n}_1(I) = \sum_{j \in S_n} \rho(C_{ji}), \quad g^{S_n}_1(I) = \sum_{j \in S_n} \rho(C_{ji}),$$

$$g^{S_n}_2(I) = \sum_{k \in S_n} \rho(B_{ik}), \quad g^{S_n}_2(I) = \sum_{k \in S_n} \rho(B_{ik}).$$

(14)

Taking modulus in the above equation, we have

$$|\lambda|x_p| \leq \sum_{k \in S_n} |x_k|y^T B_{ik}y + \sum_{k \in S_n} |x_k|\rho|B_{ik}|y.$$ 

$$\leq g^{S_n}_2(p)|x_p| + g^{S_n}_3(p)|x_s|.$$ 

Then,

$$\left( |\lambda| - g^{S_n}_2(p) \right)|x_p| \leq g^{S_n}_3(p)|x_s|.$$ 

If $|\lambda| - g^{S_n}_2(p) > 0$, similarly we can get

$$\left( |\lambda| - g^{S_n}_3(s) \right)|x_s| \leq g^{S_n}_3(p)|x_p|.$$ 

(20)

Multiplying (20) with (21), we have

$$\left( |\lambda| - g^{S_n}_2(p) \right)\left( |\lambda| - g^{S_n}_3(s) \right) \leq g^{S_n}_3(p)g^{S_n}_3(s).$$ 

(21)

Therefore,

$$|\lambda| \leq \frac{1}{2} \left\{ g^{S_n}_3(p) + g^{S_n}_3(s) + \sqrt{\left( g^{S_n}_3(p) - g^{S_n}_3(s) \right)^2 + 4g^{S_n}_3(p)g^{S_n}_3(s)} \right\}.$$ 

(22)

If $|\lambda| - g^{S_n}_2(p) < 0$, then

$$|\lambda| < g^{S_n}_2(p),$$

which means that (23) also holds.

Case II. $|x_s||x_s| = 0$. If $|x_s| = 0$, by inequality (5), then $|\lambda| - g^{S_n}_2(p) \leq 0$; it yields that (7) also holds. If $|x_p| = 0$, by inequality (6), then $|\lambda| - g^{S_n}_2(s) \leq 0$; it yields that (7) also holds.

Let $|y_j| = \max_{i \in S_n} |y_i|$ and $|y_j| = \max_{j \in S_n} |y_j|$, from the $q$-th equation of $\lambda y = \mathcal{A}xy$, we have

$$\lambda y_q = \sum_{i,k \in M} \sum_{j \in S_n} a_{ijk} x_i y_j x_k + \sum_{i,k \in M} \sum_{j \in S_n} a_{ijk} x_i y_j x_k.$$ 

(24)

and similarly, we can get

$$|\lambda| \leq \frac{1}{2} \left\{ g^{S_n}_1(q) + g^{S_n}_1(t) + \sqrt{\left( g^{S_n}_1(q) - g^{S_n}_1(t) \right)^2 + 4g^{S_n}_1(q)g^{S_n}_1(t)} \right\}.$$ 

(25)

We compare the S-type upper bounds in Theorem 4 with the results in [19], which shows that our new S-type upper bounds are always tighter than the results in [19].

**Theorem 5.** Let $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{m \times n \times n}$ be an elasticity tensor. Then,

$$\rho_M(\mathcal{A}) \leq \tau \leq \gamma.$$ 

(26)
Proof. If $\rho_M(\mathcal{A}) \leq \tau$, then

$$\rho_M(\mathcal{A}) \leq \frac{1}{2} \left\{ g_2^S(i) + g_2^S(k) + \sqrt{\left( g_2^S(i) - g_2^S(k) \right)^2 + 4g_2^S(i)g_2^S(k)} \right\}$$

(27)

or

$$\rho_M(\mathcal{A}) \leq \frac{1}{2} \left\{ g_1^S(j) + g_1^S(l) + \sqrt{\left( g_1^S(j) - g_1^S(l) \right)^2 + 4g_1^S(j)g_1^S(l)} \right\}.$$  

(28)

We only proof the following case, and the other case can be proved similarly. If

$$\rho_M(\mathcal{A}) \leq \frac{1}{2} \left\{ g_2^S(i) + g_2^S(s) + \sqrt{\left( g_2^S(i) - g_2^S(s) \right)^2 + 4g_2^S(i)g_2^S(s)} \right\},$$

(29)

from the proof of Theorem 4,

$$\left| \lambda - g_2^S(i) \right| \left| \lambda - g_2^S(s) \right| \leq g_2^S(i)g_2^S(s).$$

(30)

Let $S_m = i$, $S_m = M \setminus i$, then

$$\left( |\lambda| - \rho(B_{ii}) \right) \left( |\lambda| - g_2(s) \right) \leq g_2(i)\rho(B_{ii}).$$

(31)

From inequalities (20) or (21), there is an $i \in M$ with $|\lambda| - \rho(B_{ii}) \leq g_2(i)$; for this index $i$, we have

$$\left( |\lambda| - \rho(B_{ii}) \right) |\lambda| \leq \left( |\lambda| - \rho(B_{ii}) \right) g_2(s) + g_2(i)\rho(B_{ii}),$$

(32)

and therefore, $\rho_M(\mathcal{A}) \leq \gamma$. \hfill \Box

In 2009, the following WQZ-algorithm was presented to compute the largest M-eigenvalue of an elasticity tensor [4].

$$e_{ijkl} = \begin{cases} 1, & \text{if } i = k \text{ and } j = l, \\ 0, & \text{otherwise}. \end{cases}$$

(33)

$$\mathbf{x}_{t+1} = \mathcal{A}\mathbf{y}_t \mathbf{x}_t,$$

$$\mathbf{x}_{t+1} = \frac{\mathbf{x}_{t+1}}{\|\mathbf{x}_{t+1}\|},$$

$$\mathbf{y}_{t+1} = \mathcal{A}\mathbf{x}_{t+1} \mathbf{y}_t,$$

$$\mathbf{y}_{t+1} = \frac{\mathbf{y}_{t+1}}{\|\mathbf{y}_{t+1}\|},$$

$$t = t + 1.$$  

(34)

The following example in [4] is taken to show that the tighter upper bound can accelerate convergence of the WQZ-algorithm.

Example 1. Consider the tensor $\mathcal{A} = (a_{ijkl})$ of Example 4.1 in [4, 21], where

$$\mathcal{A}(i, i, 1, 1) = \begin{bmatrix} -0.9727 & 0.3169 & -0.3437 \\ -0.6332 & -0.7866 & 0.4257 \\ -0.3350 & -0.9896 & -0.4323 \end{bmatrix},$$

$$\mathcal{A}(i, i, 2, 1) = \begin{bmatrix} 0.7387 & 0.6873 & -0.3248 \\ -0.7986 & -0.5988 & -0.9485 \end{bmatrix},$$

$$\mathcal{A}(i, i, 3, 1) = \begin{bmatrix} -0.3350 & -0.9896 & -0.4323 \\ -0.7986 & -0.5988 & -0.9485 \end{bmatrix},$$

$$\mathcal{A}(i, i, 1, 2) = \begin{bmatrix} 0.3169 & 0.6158 & -0.0184 \\ -0.7866 & 0.0160 & 0.0085 \end{bmatrix},$$

$$\mathcal{A}(i, i, 3, 1) = \begin{bmatrix} 0.5853 & 0.5921 & 0.6301 \end{bmatrix}.$$
Consider the elasticity tensor $a$.

In [4], $v$ is taken as follows:

$$\sum_{1 \leq i, j \leq 3, k, l} |A_{ij}| = 23.3503. \quad (38)$$

Let $S_m = S_n = \{1\}$; by Corollary 2 in [22], we have

$$\rho (\mathcal{A}) \leq 11.7253. \quad (39)$$

By Theorem 2, we have

$$\rho (\mathcal{A}) \leq 4.2523. \quad (40)$$

Let $S_n = \{1, 3\}$; by Theorem 4, we have

$$\rho (\mathcal{A}) \leq 4.1528. \quad (41)$$

Example 2. Consider the elasticity tensor $a = (a_{ijkl})$ of CaMg(CO3)2-dolomite [21], whose nonzero entries are

$$a_{1111} = 196.6, a_{3311} = a_{2233} = 83.2, a_{3333} = 110,$$

$$a_{1313} = a_{1313} = 54.7, a_{1212} = a_{2121} = 64.4,$$

$$a_{1213} = -a_{1213} = -a_{1213} = -31.7, a_{1122} = 132.2,$$

$$a_{2321} = a_{2321} = -35.84, a_{3112} = a_{3121} = 44.8,$$

$$a_{3212} = a_{2321} = -a_{3112} = -25.3. \quad (42)$$

In [4], $v$ is taken as follows:

$$\sum_{1 \leq i, j \leq 3, k, l} |A_{ij}| = 23.3503. \quad (43)$$

Let $S_m = S_n = \{1\}$; by Corollary 2 in [22], we have

$$\rho (\mathcal{A}) \leq 491.7400. \quad (44)$$

By Theorem 2, we have

$$\rho (\mathcal{A}) \leq 462.2316. \quad (45)$$

Let $S_m = \{2, 3\}$; by Theorem 4, we have

$$\rho (\mathcal{A}) \leq 211.4729. \quad (46)$$

In Figure 1, we can find that, when taking $v = 211.4729$, the sequence generated in the WQZ-algorithm converges to the largest M-eigenvalue more rapidly than taking $v = 1998.6000$ and $v = 462.2316$.

3. S-Type M-Eigenvalue Inclusion Sets and Strong Ellipticity Conditions

In this section, based on the S-type M-eigenvalue inclusion sets of an elasticity tensor, S-type sufficient conditions for strong ellipticity conditions are given. Let $(sfx^y)_j = \sum_{j=1}^n a_{ijkl}x_iy_k$ and $(sfx^y)_j = \sum_{j=1}^n a_{ijkl}x_iy_k$, we need the following lemma.

Lemma 1 (see [23]). Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{m \times n \times m \times n}$ be an elasticity tensor. Then, the strong ellipticity condition holds if and only if the matrix $\mathcal{A}x^y \in \mathbb{R}^n$ (or $\mathcal{A}x^y \in \mathbb{R}^m$) is positive definite for each nonzero $x \in \mathbb{R}^m$ (or $y \in \mathbb{R}^n$).

Theorem 6. Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{m \times n \times m \times n}$ be an elasticity tensor and $\lambda$ be an M-eigenvalue of $\mathcal{A}$ with the M-eigenvectors $x$, $y$. Then,

$$\lambda \in \Delta_1 (\mathcal{A}) \cap \Delta_2 (\mathcal{A}), \quad (47)$$

where

$$\Delta_1 (\mathcal{A}) = \bigcup_{j \in S_n} \left\{ z \in \mathbb{R} : \left| z - \lambda x^T C_{ij} x \right| \leq \frac{\lambda^5}{\lambda^5} (j) \right\},$$

$$\Delta_2 (\mathcal{A}) = \bigcup_{j \in S_n} \left\{ z \in \mathbb{R} : \left| z - \lambda x^T C_{ij} x \right| \leq \frac{\lambda^5}{\lambda^5} (j) \right\}.$$
\[ \Delta_2(\mathcal{A}) = \bigcup_{i \in S_{m,k}} \bigcup_{i \in S_m} \left\{ z \in \mathbb{R} : \left| z - y^T B_{ii} y \right| - h_{\mathcal{S}_m}^2(i) \right\} \bigcup \left\{ z \in \mathbb{R} : \left| z - y^T B_{kk} y \right| - h_{\mathcal{S}_m}^2(k) \leq g_{\mathcal{S}_m}^2(i) g_{\mathcal{S}_m}^2(k) \right\} \]

\[ h_{\mathcal{S}_m}^1(l) = \sum_{j \in S_{m,j} \neq l} \rho(C_{jl}), \quad h_{\mathcal{S}_m}^1(l) = \sum_{j \in S_{m,j} \neq l} \rho(C_{jl}), \]
\[ h_{\mathcal{S}_m}^2(i) = \sum_{k \in S_{m,k} \neq i} \rho(B_{ik}), \quad h_{\mathcal{S}_m}^2(i) = \sum_{k \in S_{m,k} \neq i} \rho(B_{ik}). \]

**Proof.** Let \( \lambda \) be an M-eigenvalue of \( \mathcal{A} \) with the M-eigenvectors \( x \) and \( y \),

\[ |x_p| = \max_{k \in S_m} |x_k|, \quad |x_i| = \max_{k \in S_m} |x_k|. \]  

(49)

Obviously, at least one of \( |x_p| \) and \( |x_i| \) is nonzero.

**Case I.** If \( |x_p| \neq 0 \), from the \( p \)-th equation of \( \lambda x = \mathcal{A} y x y \), we have

\[ \lambda x_p = \sum_{k=1}^m \sum_{j=1}^n a_{p,j,k} y_j x_k y_l. \]  

(50)

Then, we can get

\[ \lambda x_p - y^T B_{pp} y x_p = \sum_{k \in S_m,k \neq p} x_k y^T B_{pk} y + \sum_{k \in S_m} x_k y^T B_{pk} y. \]  

(51)

Taking modulus in the above equation, we have

\[ |\lambda - y^T B_{pp} y| |x_p| \leq \sum_{k \in S_m,k \neq p} |x_k y^T B_{pk} y| + \sum_{k \in S_m} |x_k y^T B_{pk} y| \]
\[ \leq h_{\mathcal{S}_m}^2(p) |x_p| + g_{\mathcal{S}_m}^2(p) |x_i|. \]  

(52)

Then,
\begin{align}
\left( |\lambda - y^T B_{pp} y| - h_{2m}^s (p) \right) |x_p| \leq g_{2m}^s (p) |x_p|. \tag{53}
\end{align}

If \(|\lambda - y^T B_{pp} y| - h_{2m}^s (p) > 0\), similarly we can get

\begin{align}
\left( |\lambda - y^T B_{pp} y| - h_{2m}^s (p) \right) \left( |\lambda - y^T B_{ss} y| - h_{2m}^s (s) \right) \leq g_{2m}^s (p) g_{2m}^s (s), \tag{55}
\end{align}

so that \(\lambda \in \Delta_2 (\mathcal{A})\). If \(|\lambda - y^T B_{pp} y| - h_{2m}^s (p) \leq 0\); then,

\begin{align}
|\lambda - y^T B_{pp} y| \leq h_{2m}^s (p), \tag{56}
\end{align}

which means that \(\lambda \in \Delta_2 (\mathcal{A})\).

Case II. \(|x_p| |x_s| = 0\). Without loss of generality, let \(|x_s| = 0\), by inequality (8), then \(|\lambda - y^T B_{pp} y| - h_{2m}^s (p) \leq 0\); it yields that \(\lambda \in \Delta_2 (\mathcal{A})\).

\begin{align}
\lambda_{\text{min}} (B_{ii}) > h_{2m}^s (i) & \quad \text{for all } i \in S_m, \tag{57}
\end{align}

\begin{align}
\lambda_{\text{min}} (C_{jj}) > h_{2m}^s (j) & \quad \text{for all } j \in S_n, \tag{58}
\end{align}

then the strong ellipticity condition holds.

\textbf{Proof.} Let \(\lambda\) be an M-eigenvalue of \(\mathcal{A}\) and \(\lambda \leq 0\). From Theorem 6, we obtain \(\lambda \in \Delta (\mathcal{B})\). If \(\lambda \in \Delta_2 (\mathcal{B})\), there are \(i \in S_m\) and \(k \in \overline{S}_n\) such that

\begin{align}
\left( |\lambda - y^T B_{ii} y| - h_{2m}^s (i) \right) \left( |\lambda - y^T B_{kk} y| - h_{2m}^s (k) \right) \geq g_{2m}^s (i) g_{2m}^s (k), \tag{59}
\end{align}

or

\begin{align}
|\lambda - y^T B_{ii} y| \leq h_{2m}^s (i). \tag{60}
\end{align}

Then,

\begin{align}
\begin{align}
|\lambda - y^T B_{ii} y| - h_{2m}^s (i) & \leq \begin{cases} 0, & \text{if } \lambda \leq h_{2m}^s (i) \text{;} \\
|\lambda - y^T B_{kk} y| - h_{2m}^s (k) & \leq \begin{cases} 0, & \text{if } \lambda \leq h_{2m}^s (k) \text{;}
\end{cases}
\end{cases}
\end{align}
\end{align}

\begin{align}
\begin{align}
\lambda_{\text{min}} (B_{ii}) & > h_{2m}^s (i) \quad \text{for all } i \in M, \tag{63}
\end{align}
\end{align}

or

\begin{align}
\lambda_{\text{min}} (C_{jj}) & > h_{2m}^s (j) \quad \text{for all } j \in N, \tag{64}
\end{align}

then the strong ellipticity condition holds.

Based on the above theorems, we introduce the definitions strictly diagonally dominated (M-SDD) and S-type strictly diagonally dominated (M-SSDD) elasticity tensors, which are based on the eigenvalues of matrices of \(B_{ik}\) and \(C_{ij}\).

\textbf{Definition 2.} Let \(\mathcal{A} = (a_{ikj}) \in \mathbb{R}^{m \times n \times m \times n}\) be an elasticity tensor. If

\begin{align}
\lambda_{\text{min}} (B_{ii}) > g_{1} (i), \quad \text{for all } i \in M, \tag{65}
\end{align}

or

\begin{align}
\lambda_{\text{min}} (C_{jj}) > g_{1} (j), \quad \text{for all } j \in N, \tag{66}
\end{align}

then the strong ellipticity condition holds.
\[ \lambda_{\min}(B_{ii}) > g_2(i), \quad \text{for all } i \in M, \]  
(65)

or

\[ \lambda_{\min}(C_{ii}) > g_1(i), \quad \text{for all } i \in N, \]  
(66)

then the elasticity tensor \( \mathcal{A} \) is called strictly diagonally dominated (M-SSDD).

**Definition 3.** Let \( \mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{m \times m \times n \times n} \) be an elasticity tensor. If there exists \( S_m \) or \( S_n \) such that

\[ \lambda_{\min}(B_{ii}) > h_i^S(i) \quad \text{for all } i \in S_m, \]  
(67)

\[ \lambda_{\min}(C_{jj}) > h_j^S(j) \quad \text{for all } j \in S_n, \]  
(68)

then the elasticity tensor \( \mathcal{A} \) is called S-type strictly diagonally dominated (M-SDD).

Next, we give the relationships between the M-SDD elasticity tensor and the M-SSDD elasticity tensor.

**Theorem 9.** Let \( \mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{m \times m \times n \times n} \) be an elasticity tensor. If \( \mathcal{A} \) is an M-SDD elasticity tensor, then \( \mathcal{A} \) is an M-SSDD elasticity tensor.

**Proof.** If \( \mathcal{A} \) is an M-SDD elasticity tensor, we only prove the following case; the other case can be proved similarly. For all \( i \in M, \)

\[ \lambda_{\min}(B_{ii}) > g_2(i), \]  
(69)

Then, for all \( i \in S_m \) and \( k \in \mathcal{S}_m, \)

\[ \lambda_{\min}(B_{ii}) > g_2(i) > h_i^S(i), \]  
(70)

\[ \lambda_{\min}(B_{ii}) - h_i^S(i) > g_2^S(i), \lambda_{\min}(B_{kk}) - h_k^S(k) > g_2^S(k), \]  
(71)

which imply that

\[ \lambda_{\min}(B_{ii}) > h_i^S(i), \]  
(72)

\[ (\lambda_{\min}(B_{ii}) - h_i^S(i))(\lambda_{\min}(B_{kk}) - h_k^S(k)) > g_2^S(i)g_2^S(k), \]  
(73)

and then \( \mathcal{A} \) is an M-SSDD elasticity tensor. \( \square \)

\[ (\lambda_{\min}(B_{11}) - h_2^S(1))(\lambda_{\min}(B_{22}) - h_2^S(2)) = 4.5 > 4 = g_2^S(1)g_2^S(2). \]  
(74)

Then the elasticity tensor \( \mathcal{A} \) is called strictly diagonally dominated (M-SSDD).

**Example 3.** Let \( \mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{2 \times 2 \times 2 \times 2} \) be an elasticity tensor, where

\[ a_{1111} = a_{1212} = 2.5, \ a_{1221} = a_{2222} = 4, \]  
(75)

and other \( a_{ijkl} = 0. \) Obviously, we have

\[ B_{11} = \begin{bmatrix} 2.5 & 1 \\ 1 & 2.5 \end{bmatrix}, \]  
(76)

\[ B_{22} = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}, \]  
(77)

\[ B_{12} = B_{21} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \]  
(78)

\[ C_{11} = C_{22} = \begin{bmatrix} 2.5 & -1 \\ -1 & 4 \end{bmatrix}, \]  
(79)

\[ C_{12} = C_{21} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \]  
(80)

Let \( S_m = \{1\} \), by direct computation, we have

\[ \lambda_{\min}(B_{11}) = 1.5 > 0 = h_2^S(1), \lambda_{\min}(B_{22}) = 3 > 0 = h_2^S(2), \]  
(81)

and

\[ (\lambda_{\min}(B_{11}) - h_2^S(1))(\lambda_{\min}(B_{22}) - h_2^S(2)) = 4.5 > 4 = g_2^S(1)g_2^S(2). \]  
(82)
Then, $\mathcal{A}$ satisfies the sufficient conditions of Theorem 7, and the conditions of Theorem 8 do not hold by $\lambda_{\text{min}}(B_{11}) = 1.5 < 2 = g_2(1)$ and $\lambda_{\text{min}}(C_{11}) = 1.6707 < 2 = g_1(1)$. Therefore, the strong ellipticity condition holds for the elasticity tensor $\mathcal{A}$ by Theorem 7. In fact, the smallest M-eigenvalue of $\mathcal{A}$ is 3.5.

Let $S_0 = S_0 = \{1\}$; by Theorem 11 in [22], we have
\[
(\alpha_1 - r_1^3(\mathcal{A})) (\alpha_2 - r_2^3(\mathcal{A})) = 7.5 < r_1^3(\mathcal{A}) r_2^3(\mathcal{A}) = 20, \tag{76}
\]
where $\alpha_1, r_1^3(\mathcal{A}), \alpha_2, r_2^3(\mathcal{A}), r_1^3(\mathcal{A}), r_2^3(\mathcal{A})$ are defined in [22], which shows that the conditions of Theorem 11 in [22] do not hold.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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**References**

[1] G. Dahl, J. Leinaas, J. Myrheim, and E. Ovrum, “A tensor product matrix approximation problem in quantum physics,” *Linear Algebra and Its Applications*, vol. 420, no. 2-3, pp. 711–725, 2007.

[2] C. Ling, J. Nie, L. Qi, and Y. Ye, “Bi-quadratic optimization over unit spheres and semidefinite programming relaxations,” *SIAM Journal on Optimization*, vol. 20, no. 3, pp. 286–310, 2009.

[3] L. Qi, H.-H. Dai, and D. Han, “Conditions for strong ellipticity and M-eigenvalues,” *Frontiers of Mathematics in China*, vol. 4, no. 2, pp. 349–364, 2009.

[4] Y. Wang, L. Qi, and X. Zhang, “A practical method for computing the largest M-eigenvalue of a fourth-order partially symmetric tensor,” *Numerical Linear Algebra with Applications*, vol. 16, no. 7, pp. 589–601, 2009.

[5] L. Qi, “Eigenvalues of a real supersymmetric tensor,” *Journal of Symbolic Computation*, vol. 40, no. 6, pp. 1302–1324, 2005.

[6] L. Qi, “The best rank-one approximation ratio of a tensor space,” *SIAM Journal on Matrix Analysis and Applications*, vol. 32, no. 2, pp. 430–442, 2011.

[7] C. Li, Y. Li, and X. Kong, “New eigenvalue inclusion sets for tensors,” *Numerical Linear Algebra with Applications*, vol. 21, no. 1, pp. 39–50, 2014.

[8] C. Li, F. Wang, J. Zhao, Y. Zhu, and Y. Li, “Criteria for the positive definiteness of real supersymmetric tensors,” *Journal of Computational and Applied Mathematics*, vol. 255, no. 1, pp. 1–14, 2014.

[9] C. Li and Y. Li, “Double B-tensors and quasi-double B-tensors,” *Linear Algebra and Its Applications*, vol. 466, no. 1, pp. 343–356, 2015.

[10] C. Li, J. Zhou, and Y. Li, “A new Brauer-type eigenvalue localization set for tensors,” *Linear and Multilinear Algebra*, vol. 64, no. 4, pp. 727–736, 2016.

[11] C. Bu, Y. Wei, L. Sun, and J. Zhou, “Braudi-type eigenvalue inclusion sets of tensors,” *Linear Algebra and Its Applications*, vol. 480, no. 1, pp. 168–175, 2015.

[12] H. Che, H. Chen, and Y. Yang, “On the M-eigenvalue estimation of fourthorder partially symmetric tensors,” *Journal of Industrial and Management Optimization*, vol. 16, no. 1, pp. 09–32, 2020.

[13] J. Zhao and C. Li, “Singular value inclusion sets for rectangular tensors,” *Linear and Multilinear Algebra*, vol. 66, no. 7, pp. 1333–1350, 2018.

[14] J. Zhao and C. Sang, “An S-type upper bound for the largest singular value of nonnegative rectangular tensors,” *Open Mathematics*, vol. 14, no. 1, pp. 925–933, 2016.

[15] L. Guo, X. Zhao, X. Gu, Y. Zhao, Y. Zheng, and T. Huang, “Three-dimensional fractional total variation regularized tensor optimized model for image deblurring,” *Applied Mathematics and Computation*, vol. 404, no. 1, Article ID 126224, 2021.

[16] G. Wang, Y. Wang, and Y. Wang, “Some Ostrowski-type bound estimations of spectral radius for weakly irreducible nonnegative tensors,” *Linear and Multilinear Algebra*, vol. 68, no. 9, pp. 1–18, 2020.

[17] G. Wang, G. Zhou, and L. Caccetta, “Z-eigenvalue inclusion theorems for tensors,” *Discrete and Continuous Dynamical Systems - Series B*, vol. 22, no. 1, pp. 1187–1198, 2017.

[18] C. Sang, “A new Brauer-type Z-eigenvalue inclusion set for tensors,” *Numerical Algorithms*, vol. 80, no. 1, pp. 781–794, 2019.

[19] S. Li, Z. Chen, Q. Liu, and L. Lu, “Bounds of M-eigenvalues and strong ellipticity conditions for elasticity tensors,” *Linear and Multilinear Algebra*, pp. 1–14, 2021.

[20] D. Han, H. H. Dai, and L. Qi, “Conditions for strong ellipticity of anisotropic elastic materials,” *Journal of Elasticity*, vol. 97, no. 1, pp. 1–13, 2009.

[21] S. Li, C. Li, and Y. Li, “M-eigenvalue inclusion intervals for a fourth-order partially symmetric tensor,” *Journal of Computational and Applied Mathematics*, vol. 356, no. 1, pp. 391–401, 2019.

[22] J. He, Y. Liu, and G. Xu, “New S-type inclusion theorems for the M-eigenvalues of a 4th-order partially symmetric tensor with applications,” *Applied Mathematics and Computation*, vol. 398, no. 1, Article ID 125992, 2021.

[23] W. Ding, J. Liu, L. Qi, and H. Yan, “Elasticity M-tensors and the strong ellipticity condition,” *Applied Mathematics and Computation*, vol. 373, no. 15, Article ID 24982, 2020.