A Simple Approximation for a Hard Routing Problem

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Abstract We consider a routing problem which plays an important role in several applications, primarily in communication network planning and VLSI layout design. The original underlying graph algorithmic task is called Disjoint Connecting Paths problem. In most applications one can encounter its capacitated generalization, which is known as the Unsplittable Flow problem. These algorithmic tasks are very hard in general, but various efficient (polynomial-time) approximate solutions are known. Nevertheless, the approximations tend to be rather complicated, often rendering them impractical in large, complex networks. Our goal is to present a solution that provides a simple, efficient algorithm for the unsplittable flow problem in large directed graphs. The simplicity is achieved by sacrificing a small part of the solution space. This also represents a novel paradigm of approximation: rather than giving up finding an exact solution, we restrict the solution space to its most important subset and exclude those that are marginal in some sense. Then we find the exact optimum efficiently within the subset. Specifically, the sacrificed parts (i.e., the marginal instances) only contain scenarios where some edges are very close to saturation. Therefore, the excluded part is not significant, since the excluded almost saturated solutions are typically undesired in practical applications, anyway.

Keywords: disjoint paths problem, unsplitting flow, randomized rounding, network design.
1 Introduction

The Disjoint Connecting Paths problem is the following decision task:

**Input:** a set of node pairs \((s_1, t_1), \ldots, (s_k, t_k)\) in a graph.

**Task:** Find edge disjoint paths \(P_1, \ldots, P_k\), such that \(P_i\) connects \(s_i\) with \(t_i\) for each \(i\).

This is one of the classical NP-complete problems that appears already at the sources of NP-completeness theory, among the original problems of Karp [5]. It remains NP-complete both for directed and undirected graphs, as well as for the edge disjoint and vertex disjoint paths version. The corresponding natural optimization problem, when we are looking for the maximum number of terminator pairs that can be connected by disjoint paths is NP-hard.

There is also a capacitated version of the Disjoint Paths Problem, also known as the Unsplitting Flow problem. In this task a flow demand value is given for each origin-destination pair \((s_i, t_i)\), as well as a capacity value is known for each edge. The requirement is to find a system of paths, connecting the respective source-destination pairs, such that the capacity constraint of each edge is obeyed, i.e., the sum of the flows of paths that traverse the edge cannot be more than the capacity of the edge. The name Unsplitting Flow expresses the requirement that between each source-destination pair the flow must follow a single route, it cannot split. Note that here the disjointness of the paths themselves is not required a priori, but can be enforced by the capacity constraints. The Unsplitting Flow problem is important in communication network design and routing applications.

In this paper, after reviewing some existing results, we show that the Unsplitting Flow problem, which is NP-complete, becomes efficiently solvable by a relatively simple algorithm if we impose a mild and practically well justifiable restriction on the instance.

2 Previous Results

Considerable work was done on the Disjoint Paths Problem, since its first appearance as an NP-complete problem in [5] in 1972.

One direction of research deals with finding the “heart” of the difficulty: which are the simplest restricted cases that still remain NP-complete? (Or NP-hard if the optimization version is considered, where we look for the maximum number of connecting paths, allowing that possibly not all source-destination pairs will be connected). Kramer and van Leeuwen [8] proves, motivated by VLSI layout design, that the problem remains NP-complete even for graphs as regular as a two dimensional mesh. If we restrict ourselves to undirected
planar graphs with each vertex having degree at most three, the problem also remains **NP**-complete, as proven by Middendorf and Pfeiffer [9]. The optimization version remains **NP**-hard for trees with parallel edges, although there the decision problem is already solvable in polynomial time [4].

The restriction that we only allow paths which connect each source node with a dedicated target is essential. If this is relaxed and we are satisfied with edge disjoint paths that connect each source \( s_i \) with *some* of destinations \( t_j \) but not necessarily with \( t_i \), then the problem becomes solvable with classical network flow techniques. Thus, the prescribed matching of sources and destinations causes a dramatic change in the problem complexity. Interestingly, it becomes already **NP**-complete if we require that just *one* of the sources is connected to a dedicated destination, the rest is relaxed as above (Faragó [3]).

Another group of results produces polynomial time algorithmic solutions for finding the paths, possibly using randomization, in *special classes* of graphs. For example, Middendorf and Pfeiffer [9] proves the following. Let us represent the terminator pairs by *demand edges*. These are additional edges that connect a source with its destination. If this extended graph is embeddable in the plane such that the demand edges lie in a bounded number of faces of the original graph, then the problem is solvable in polynomial time. (The faces are the planar regions bordered by the curves that represent the edges in the planar embedding, i.e., in drawing the graph in the plane). Thus, this special case requires that, beyond the planarity of the extended graph, the terminators are concentrated in a constant number of regions (independent of the graph size), rather than spreading over the graph.

A deep theoretical result, due to Robertson and Seymour [11], is that for general graphs the problem can be solved in polynomial time if the number \( k \) of paths to be found is **constant** (i.e., cannot grow with the size of the graph). Broder, Frieze, Suen and Upfal [2] consider the case of *random graphs* and provide a randomized algorithm that, under some technical conditions, finds a solution with high probability in time \( O(nm^2) \) for a graph of \( n \) vertices and \( m \) edges.

Another line of research aims at finding approximations to the optimization version. An algorithm is said to be an \( f(n) \)-approximation if it can connect a subset of the terminator pairs by disjoint paths such that this subset is at most \( f(n) \) times smaller than the optimum in a graph of \( n \) vertices. For example, in this terminology a 2-approximation algorithm always reaches at least the half of the optimum, or an \( O(\log n) \)-approximation reaches at least a \( c/\log n \) fraction of the optimum, for \( n > n_0 \) with some constants \( c, n_0 \).

Various approximations have been presented in the literature. For example, Garg, Vazirani and Yannakakis [4] provide a 2-approximation for trees with parallel edges. Aumann and Rabani [1] gives an \( O(\log n) \)-approximation for the 2-dimensional mesh. Kleinberg and
Tardos [6] present an \(O(\log n)\)-approximation for a larger subclass of planar graphs, they call “nearly Eulerian, uniformly high-diameter planar graphs” (the rather technical definition is omitted here). For the general case an approximation factor of \(\min\{\sqrt{m}, m/\text{opt}\} = O(\sqrt{m})\) is known to be achievable (Srinivasan [12]), where \(m\) is the number of edges and \(\text{opt}\) is the optimum, i.e., the maximum number of disjoint connecting paths between the source-destination pairs. Similar bounds apply for the Unsplitting Flow problem, as well. Bounds have been also found in terms of special (less trivial) graph parameters. For example, Kolman and Scheideler [7] proves that an efficient \(O(F)\) approximation exists, where \(F\) is the so called flow number of the graph. Although the flow number can be computed in polynomial time [7], it is an indirect characterization of the graph.

3 A Simple Practical Approximation

The various above referenced solutions are rather complicated, which is certainly not helpful for practical applications, in particular in large, complex networks. Our approach for providing a simple solution to the unsplitting flow problem based on the following idea. We “cut down” a small part of the solution space by slightly reducing the edge capacities. In other words, we exclude solutions that are close to saturating some edge, as explained below.

Let \(V_i\) be the given flow demand of the \(i^{th}\) connecting path. We normalize these demands such that \(V_i \leq 1\) for every \(i\). Let \(C_j\) be the capacity of edge \(j\). The graph is assumed directed and the edges are numbered from 1 through \(m\). Recall that a feasible solution of the problem is a set of connecting (directed) paths that satisfy the edge capacity constraints, that is, on each edge \(j\) the sum of the \(V_i\) values of those paths that traverse the edge does not exceed \(C_j\). As mentioned earlier, deciding whether a feasible solution exist at all is a difficult (\(NP\)-complete) problem.

On the other hand, not all feasible solutions are equally good from the practical viewpoint. For example, if a route system in a network saturates or nearly saturates some links, then it is not preferable because it is close to being overloaded. For this reason, let us assign a parameter \(0 < \rho_j < 1\) to each edge \(j\), such that \(\rho_j\) will act as a “safety margin” for the edge. More precisely, let us call a feasible solution a safe solution with parameters \(\rho_j\), \(j = 1, \ldots, m\), where \(m\) is the number of edges, if it uses at most \(\tilde{C}_j = \rho_j C_j\) capacity on edge \(j\).

Now, the interesting thing is that if we restrict ourselves to only those cases when a safe solution exists, then the hard algorithmic problem becomes solvable by a relatively simple randomized algorithm. With very high probability the algorithm finds a solution in polynomial time, whenever there exists a safe solution.
The price is that we exclude those cases when a feasible solution still possibly exists, but there is no safe solution. This means, in these cases all feasible solutions are undesirable, in the sense that they make some edges nearly saturated. In these marginal cases the algorithm may find no solution at all. This approach constitutes a new avenue to approximation, in the sense that instead of giving up finding an exact solution, we rather restrict the search space to a (slightly) smaller one. When, however, the algorithm finds any solution, then it is an exact (not just approximate) solution.

Now let us choose the safety margin $\rho_j$ for a graph of $m$ edges as

$$\rho_j = 1 - (e - 1) \sqrt{\frac{\ln 2m}{C_j}} \approx 1 - 1.71 \sqrt{\frac{\ln 2m}{C_j}}$$

where $\ln$ denotes the natural logarithm $\log_e$. Note that $\rho_j$ tends to 1 with growing $C_j$, even if the graph also grows, but $C_j$ grows faster than the logarithm of the graph size, which is very reasonable (note that doubling the number of edges will increase the natural logarithm by less than 1). For example, if in a graph each edge capacity is 1000 units, measured in relative units, such that the maximum path flow demand is 1, and the graph has 200 edges, then $\rho \approx 0.97$.

Now we outline how the algorithm works. To make it even closer to practical applications, we also assume that cost factors are assigned to the edges and we are looking for a feasible solution with small cost, where the cost incurred on an edge is proportional with the demand routed through it.

**Algorithm**

**Step 1. Initialization**
Compute the $\tilde{C}_j = \rho_j C_j$ values with $\rho_j$ set according to (1).

**Step 2. Flow relaxation**
Solve the continuous minimum cost multicommodity flow relaxation of the problem, using the $\tilde{C}_j$ capacities. This can be done by standard linear programming. In case the flow problem has no solution then declare “no safe solution exists” and STOP.

**Step 3. Randomized Rounding via Random Walk**
For each source-destination pair $u_i, v_j$ find a path via the following randomized rounding

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1 Note that although this phase finds a flow of the required value between each source-destination pair, it does not yet provide the required unsplittable flow, since the found flow typically branches arbitrarily into small parts rather than going on one path, this is why it is called a relaxation of the problem.
procedure. Start at the source and take the next node such that it is drawn randomly among the successor neighbors of the source, with probabilities proportional to the \(i^{th}\) commodity flow values on the edges from \(u_i\) to the successor neighbors in the directed graph. Continue this in a similar way: at each node choose the next one among its successor neighbors randomly, with probabilities that are proportional to the \(i^{th}\) commodity flow values. Finally, upon arrival at \(v_i\), we store the found \((u_i, v_i)\) path.

**Step 4: Feasibility Check and Repetition**

Having found a system of paths in the previous steps, check whether it is a feasible solution. If so, then STOP, else repeat from **Step 2**.

If after repeating \(r\) times (\(r\) is a fixed parameter) none of the runs are successful then declare “No solution is found” and STOP.

It is clear from the above informal description that the algorithm has practically feasible complexity, since the most complex part of it is solving a multicommodity flow problem that can be done by linear programming. It is repeated \(r\) times where \(r\) is a parameter, chosen by us. The main property of the algorithm is shown in the following theorem.

**Theorem 1** If a safe solution exists, the algorithm finds a feasible solution with probability at least \(1 - 2^{-r}\).

**Proof.** Since a safe solution is also a feasible solution of the multicommodity flow relaxation, therefore, if there is no flow solution in **Step 2**, then no safe solution can exist either.

**Step 3** transforms the flow solution into paths. To see that they are indeed paths, observe that looping cannot occur in the randomized branching procedure, because if a circle arises on the way, that would mean a circle with all positive flow values for a given commodity, which could be canceled from the flow of that commodity, thus contradicting to the minimum cost property of the flow. Furthermore, since looping cannot occur, we must reach the destination via the procedure in at most \(n\) steps, where \(n\) is the number of nodes.

Now a key observation is that if we build the paths with the described randomization between the \(i^{th}\) source-destination pair, then the expected value of the load that is put on any given edge by these paths will be exactly the value of the \(i^{th}\) commodity flow on the link. This follows from the fact that the branching probabilities are flow-proportional.

From the above we know that the total expected load of an edge, arising from the randomly chosen paths, is equal to the total flow value on the edge. What we have to bound is the deviation of the actual load from this expected value. Let \(F_j\) be the flow (=expected
load) on edge $j$. This arises in the randomized procedure as

$$F_j = E\left( \sum_i V_i X_i \right),$$

where $X_i$ is a random variable that takes the value 1 if the $i^{th}$ path contributes to the edge load, otherwise it is 0. The construction implies that these random variables are independent.

Now consider the random variable

$$\Psi_j = \sum_i V_i X_i.$$

We have $E(\Psi_j) = F_j$. The probability that $\Psi_j$ deviates from its expected value by more than a factor of $\delta$ can be bounded by the tail inequality found in [10]:

$$\Pr(\Psi_j > (1 + \delta)F_j) < \left( \frac{e^{\delta}}{(1 + \delta)^{(1+\delta)}} \right)^{F_j}.$$

It can be calculated from this [10] that if we want to guarantee that the bound does not exceed a given value $\epsilon > 0$, then it is sufficient to satisfy

$$\delta \leq (e - 1) \sqrt{\frac{\ln(1/\epsilon)}{F_j}}.$$

Let us choose $\epsilon = 1/(2m)$. Then we have

$$\Pr(\Psi_j > (1 + (e - 1) \sqrt{(\ln 2m)/F_j})F_j) < \frac{1}{2m}.$$

Since the bound that we do not want to exceed is the edge capacity $C_j$, therefore, if

$$C_j \geq \left( 1 + (e - 1) \sqrt{(\ln 2m)/F_j} \right)F_j$$

is satisfied, then we have

$$\Pr(\Psi_j > C_j) < \frac{1}{2m}.$$
If this holds for all edges, that yields

$$\Pr(\exists j : \Psi_j > C_j) \leq \sum_{j=1}^{m} \Pr(\Psi_j > C_j)$$

$$< \frac{m}{2m}$$

$$= \frac{1}{2}.$$

Thus, the probability that the found path system is not feasible is less than 1/2. Repeating the procedure $r$ times with independent randomness, the probability that none of the trials provide a feasible solution is bounded by $1/2^r$, that is, the failure probability becomes very small, already for moderate values of $r$.

Finally, expressing $F_j$ from (2) we obtain

$$F_j \leq C_j \left(1 - (e - 1) \sqrt{\frac{\ln 2m}{C_j}}\right) = \rho_j C_j,$$

which shows that the safety margin is correctly chosen, thus completing the proof.

\[\diamondsuit\]

4 Conclusion

We have presented a simple, efficient solution for the NP-complete Unsplittable Flow problem in directed graphs. The simplicity and efficiency is achieved by sacrificing a small part of the solution space. The sacrificed part only contains scenarios where some edges are very close to saturation. Therefore, the loss is not significant, since the almost saturated solutions are typically undesired in practical applications, such as network design.

The approach constitutes a new avenue to approximation, in the sense that instead of giving up finding an exact solution, we rather restrict the search space to a (slightly) smaller one. When, however, the algorithm finds any solution, which happens with high probability, then it is an exact (not just approximate) solution. In this paper we only laid down the theoretical foundations of the approach, numerical validation is planned for a later paper.
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