Finite Temperature Magnetism in Fractional Quantum Hall Systems: Composite Fermion Hartree-Fock and Beyond

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Using the Hamiltonian formulation of Composite Fermions developed recently, the temperature dependence of the spin polarization is computed for the translationally invariant fractional quantum Hall states at \( \nu = 1/3 \) and \( \nu = 2/5 \) in two steps. In the first step, the effect of particle-hole excitations on the spin polarization is computed in a Composite Fermion Hartree-Fock approximation. The computed magnetization for \( \nu = 1/3 \) lies above the experimental results for intermediate temperatures indicating the importance of long wavelength spin fluctuations which are not correctly treated in Hartree-Fock. In the second step, spin fluctuations beyond Hartree-Fock are included for \( \nu = 1/3 \) by mapping the problem on to the coarse-grained continuum quantum ferromagnet.

The parameters of the effective continuum quantum ferromagnet description are extracted from the preceding Hartree-Fock analysis. After the inclusion of spin fluctuations in a large-\( N \) approach, the results for the finite-temperature spin polarization are in quite good agreement with the experiments.

I. INTRODUCTION

The fractional quantum Hall (FQH) effect has introduced us to new, highly correlated, incompressible states of electrons in high magnetic fields. A unified understanding of all fractions \( \nu = p/(2sp + 1) \) was achieved by the Composite Fermion picture of Jain, in which the electrons are dressed by 2\( s \) units of statistical flux to form Composite Fermions (CFs). At a mean field level, the CFs see a reduced field \( B^* = B/(2sp + 1) \), in which they fill \( p \) CF-Landau levels (CF-LLs), and exhibit the integer quantum Hall effect.

Due to the small \( g \) factor of electrons in GaAs, spins may not be fully polarized in FQH states. Transitions between singlet, partially polarized, and finally fully polarized states (based on gap measurements) have been observed for a number of fillings \( \nu \) which can be understood in terms of CF's with a spin \( \nu \). The transitions happen when an unoccupied CF-LL of one spin crosses the occupied CF-LL of the opposite spin.

While these low temperature measurements are in satisfactory agreement with the ground states predicted in the CF picture, in order to understand the temperature dependence of the polarization \( P(T) \) one has to consider all excited states as well. Detailed measurements of \( P(T) \) for the \( \nu = 1/3 \) state have recently appeared in the literature. It is well-known that the \( \nu = 1/3 \) state is spontaneously polarized at \( T = 0 \), even when the Zeeman coupling \( E_Z \) is zero. In this it is analogous to the \( \nu = 1 \) state which has been extensively studied theoretically and experimentally. There are, however, significant differences between the two cases at finite \( T \).

In a recent paper, MacDonald and Palacios identified a key qualitative feature that makes \( \nu = 1/3 \) very different from \( \nu = 1 \). In the \( \nu = 1 \) case the particle-hole excitations are very high in energy compared to \( E_Z \), and are frozen out at all low temperatures of interest. Consequently, the \( T \) dependence of \( P \) comes mainly from spin wave excitations and their interactions. This is the reason why long-wavelength effective theories such as the continuum quantum ferromagnet approach are successful. However, for \( \nu = 1/3 \), particle-hole excitations are on the same scale of energy as \( E_Z \), and cannot be ignored at any \( T \). MacDonald and Palacios use a simplified model to illustrate this feature, but the model is not sufficiently detailed to enable a calculation of \( P(T) \) for a realistic sample, and is difficult to extend to non-Laughlin fractions.

The goal of this paper is to describe a general analytical method for approximately computing \( P(T) \) for an arbitrary principal fraction for realistic samples. To this end, an approximate hamiltonian formalism in which Composite Fermion variables explicitly appear will be used. This hamiltonian approach is based on the field theoretic idea of attaching flux to electrons by using a Chern-Simons field. The CF-Hamiltonian approach has many features suited to the computation of the physical properties of fractional Hall systems. The CFs see the effective field, and fill CF-Landau levels (CF-LLs). In the principal fractions \( \nu = p/(2p + 1) \) with which we will be concerned, the effective field is \( B^* = B/(2p + 1) \), and the CFs fill \( p \) CF-LLs. The energies of the CF-LLs are controlled entirely by interactions, which is a correct feature of the physics of electrons in the lowest Landau level. Note that this is not a theory in which Composite Fermions are free. On the contrary, the theory is fully interacting, with both the kinetic energies and the residual interactions of the CFs being controlled by the electron-electron interaction. Finally, the nonperturbative charge and dipole moment of the excitations appear explicitly in the theory. The fact that all these nontrivial features are built into the theory raises the expectation that very simple approximate treatments of this Hamiltonian theory (such as Hartree-Fock) will suffice to produce reason-
HAMILTONIAN FORMALISM

The Hamiltonian approach is particularly suited to the computation of finite-temperature properties in fractional Hall systems. In the Composite Fermion Hartree-Fock (CFHF) approximation, one self-consistently finds the single-particle energies and occupations of the various CF-LLs at any finite temperature. Since the energies of the states are controlled entirely by interactions and occupations, these energies will be temperature-dependent. This is a familiar feature in other interacting many-body systems, such as the BCS superconductor, where the single-particle gap is a collective effect, and depends on temperature. Thus the Hamiltonian theory gives us valuable information on the evolution of the collective state as a function of temperature, which is then used in the mapping to the effective theory. Once the occupations of the single particle CF-LLs have been determined, the polarization is computed simply as the difference of the total \( \uparrow \)-spin and \( \downarrow \)-spin occupations. This yields the CFHF prediction for the temperature dependence of the spin magnetization. This procedure and the results for \( \nu = 1/3 \) and \( \nu = 2/5 \) are described in Section III. A brief description of this work has appeared previously\[11\].

It turns out that for the spontaneously polarized \( \nu = 1/3 \) state the CFHF prediction is higher than the experimental values for all temperatures, with the discrepancy being considerable for intermediate temperatures, as shown in Figure 1. To put it in the proper context, the agreement between CFHF and experiment is considerably better than for \( \nu = 1 \), where HF does a very poor job of predicting the spin polarization\[12\].

However, the discrepancy is nonetheless there, and is presumably the result of spin fluctuations which are not treated correctly in the CFHF approximation. One can imagine integrating out the fermions to obtain an effective theory that has low energy spin degrees of freedom. One is then led to map the low energy physics on to the continuum quantum ferromagnet (CQFM). The CQFM has two free parameters, the magnetization per unit volume \( M_0 \), and the spin stiffness \( \rho_s \). In the traditional CQFM theory these are temperature-independent parameters. However, since the theory is fermionic at the microscopic level, with the fermionic energy levels and occupations being temperature dependent, one should expect \( M_0 \) and \( \rho_s \) to acquire a \( T \) dependence in the effective theory. It turns out that these parameters can be easily extracted from the CFHF treatment of Section III. Armed with this information, we proceed to include spin fluctuations in a large-N approach as described by Read and Sachdev\[13\]. The results are in quite good agreement with the experimental data over the whole range of temperature, as shown in Figure 1 for the 10W sample of Khandelwal et al\[11\]. The same comparison is shown for the data of Melinte et al\[12\] in Figure 2.

The details of the figures will be explained in Sections III and IV. The mapping to the effective theory and the subsequent calculations are described in Section IV. We end with some conclusions, caveats, and open questions.

II. HAMILTONIAN FORMALISM

Since detailed descriptions of the Hamiltonian theory of CFs have appeared elsewhere\[14\], we will restrict ourselves to a summary of the essential features of this formalism.

The Composite Fermion picture was originally used by Jain\[15\] to generate electronic wave functions with good correlations. These wave functions have excellent over-
lap with the exact wave functions for finite systems, and encode all the right physics. In order to compute dynamical response functions, it is desirable to have an operator or field theoretic formulation of CFs. The fundamental property that CFs carry statistical flux can be implemented by a Chern-Simons(CS) transformation, which performs flux attachment via the CS gauge field to obtain a field theoretic description with either bosons or fermions. These theories have provided us with a link between the microscopic formulation of the problem and experiment, both for incompressible and compressible states.

Recently R. Shankar and the present author developed a hamiltonian CS theory for the fractional quantum Hall states. Inspired by the work of Bohm and Pines on the 3D electron gas, the Hilbert space was enlarged to introduce $n$ high-energy magnetoplasmons degrees of freedom, ($n$ also being the number of electrons) at the same time imposing an equal number of constraints on physical states. However, the fermions still had the bare mass (recall that in the lowest Landau level, the electrons should lose all memory of the bare mass, and acquire an effective mass controlled by interactions), and the frequency of the magnetoplasmons was incorrect. Hence a final canonical transformation was employed to decouple the fermions from the oscillators in the infrared limit.

The final fermions are called the Composite Fermions for the following reasons. Firstly, the final fermions have no dispersion in the absence of interactions and acquire an effective mass dependent on interactions alone. Next, the final canonical transformation assigns to each fermion the magnetic moment $e/2m$ as mandated by very general arguments. The central result of the formalism is the formula for the electronic charge density, which takes the following form, separable into high- and low-energy pieces, at small $q$:

$$\rho_e(q) = \frac{q}{\sqrt{8\pi}} \left( \frac{2p}{2p+1} (A(q) + A^\dagger(-q)) + \sum_j e^{-iqx_j} \frac{2}{2p+1} - il^2 \sum_j (q \times \Pi_j) e^{-iqx_j} \right)$$

where $A, A^\dagger$ refer to the annihilation and creation operators of the magnetoplasmon oscillators, $l = 1/\sqrt{eB}$ is the magnetic length, and $\Pi_j = \vec{P}_j + e\vec{A}^\ast(r_j)$ is the velocity operator of the CFs. The oscillator piece saturates Kolm's theorem. The rest, to be called $\bar{\rho}$, is obtained by adding to the canonically transformed electron charge density a particular multiple of the constraint (in the physical subspace, one can add any multiple of the constraint without physical consequences, but we wish to work in the full space). It has some very useful properties in the full space:

- Note that $\bar{\rho}$ is a sum of a monopole with charge $e^* = e/(2p+1)$, which is the charge associated with the CF, and a dipole piece which alone survives at $\nu = 1/2$ and has the value proposed by Read (A number of recent constructions have emphasized this dipolar aspect).

- We also find that as $q \to 0$ all transition matrix elements of $\bar{\rho}$ from the HF ground state vanish at least as $q^2$.

The final property is an essential property of physical charge density matrix elements from incompressible liquid ground states in the LLL. It is easy to see that if one intends to use the Hartree-Fock approximation ignoring constraints, these properties of $\bar{\rho}$ are essential. They make it plausible that $\bar{\rho}$ does not suffer vertex corrections.

The Hamiltonian of the low-energy sector (dropping the magnetic moment term) is

$$H = \frac{1}{2} \int \frac{d^2q}{(2\pi)^2} v(q)\bar{\rho}(-q)\bar{\rho}(q).$$

where $v(q)$ is the electron-electron interaction. Real samples have a finite thickness $\Lambda$ of the same order as $l$, so that the Coulomb interaction is cutoff at large wavevector. We will model this interaction by

$$v(q) = \frac{2\pi e^2}{q} \exp\frac{-\lambda q}{l}$$

Finally, the constraint will be approximately implemented by cutting off the number of CF-LLs to maintain the correct number of electronic states. For $\nu = p/(2p+1)$ this means keeping $2p+1$ CF-LLs. More details of the formalism can be found in refs. [19,29].

### III. THE COMPOSITE FERMION HARTREE-FOCK APPROXIMATION

The CFHF approximation has been applied to the above Hamiltonian, and reasonable success has been obtained in computing various physical quantities in the gapped fractions, including a very recent calculation of the temperature-dependent polarization $P(T)$ for the compressible half-filled LLL.

Before one employs the CFHF approximation, one needs to express the Hamiltonian as an operator acting on a set of states. Since the CFs see the effective field $B^\ast$, it is natural to represent the Hilbert space as Slater determinants of single-particle CF-Landau level states in the effective field. The wave functions of these states in the Landau gauge are

$$\phi_{n,X}(\vec{r}) = \frac{1}{\sqrt{L}} e^{iXy/(l^*)^2} \phi_n((x - X)/l^*)$$

Here $l^* = l\sqrt{2p+1}$ is the effective magnetic length, $L$ is the linear dimension of the system, $X = 2\pi(l^*)^2 j/L$ is the
where the HF single-particle energy is expressed in this basis (after the spin labels have been included) as
\[ \hat{\rho}(q) = \sum_{\sigma,X[n_n]} \rho_{n_1n_2}(q)d_{\sigma,n_1,X-n_1\sigma}d_{\sigma,n_2,X+n_2\sigma} \]  
(5)

where \( \sigma = \uparrow, \downarrow \) is the spin index, and \( d_{\sigma,n,X} \) destroys a CF in the single-particle state labelled by \( \sigma, n, X \), and \( \rho_{n_1n_2}(q) \) is a matrix element given by
\[ \rho_{n_1n_2}(q) = \frac{(-1)^{n_2-n_1}}{2^{p+1}} \sqrt{\frac{\pi}{2}} \exp(\theta_q) \left( \frac{q^\sigma}{\sqrt{x}} \right)^{n_2-n_1} \times e^{-y/2}(n_2-n_1)_2 + 2L_{n_2-n_1} - (n_1+1)L_{n_2+n_1} \]  
(6)

where \( n_2 \) (\( n_1 \)) is the lesser (greater) of \( n_1, n_2 \), \( \theta_q \) is the angle of the vector \( q \) measured from the \( x \)-axis, \( y = (q^\sigma)^2/2 \) is the argument of the Laguerre polynomials \( L_n \).

The Hamiltonian is now a four-fermion operator. It can easily be shown that the single-particle states defined above form a good HF basis. In the CFHF approximation, one reduces the four-fermion Hamiltonian to a two-fermion operator by taking averages according to the rules
\[ \langle GS|d^\dagger_{\sigma}\sigma^\dagger_{n,X}GS \rangle = \delta_{\sigma\nu} N_F(\nu) \]  
(7)
\[ \langle GS|d_{\sigma}^\dagger d_{\nu}GS \rangle = \delta_{\sigma\nu}(1 - N_F(\nu)) \]  
(8)

where \( \nu \) is a shorthand for all the state labels. This results in the HF Hamiltonian
\[ H_{HF} = \sum_{\sigma,n,X} \epsilon(\sigma,n)d_{\sigma,n,X}^\dagger d_{\sigma,n,X} \]  
(9)

where the HF single-particle energy is
\[ \epsilon(\sigma,n) = \pm \frac{E^\sigma}{\hbar^2} + \frac{1}{2} \int \frac{d^2q}{(2\pi)^2} V(q) \sum_m (1 - 2N_F(\sigma,m))|\rho_{n,m}(q)|^2 \]  
(10)

in which the Zeeman energy has been added \( (E_Z = \gamma^\sigma\mu_B B) \).

Finding the energies at \( T = 0 \) is quite simple, since the occupations of the CF-LLs can only be 0 or 1. For example, for the \( \nu = 1/3 \) state, \( N_F(\uparrow,0) = 1 \) and all other occupations are zero, and for the \( \nu = 2/3 \) singlet state \( N_F(\uparrow,0) = N_F(\downarrow,0) = 1 \), and all other occupations are zero. For nonzero \( T \) one has to carry out a self-consistent procedure. First one chooses trial values for the energies (say the \( T = 0 \) values) and a trial value for chemical potential \( \mu \) (say halfway between the lowest unoccupied and highest occupied CF-LL). Then one assigns the occupations of the single-particle levels according to Fermi occupation function
\[ N_F(\sigma,n) = \frac{1}{1 + \exp(\epsilon(\sigma,n) - \mu)/T} \]  
(11)

with the trial value of \( \mu \). Then one recomputes the HF energies using equation (11). Note that the structure of degenerate CF-LLs remains intact, and only the energies and occupations change. Since the filling has to remain fixed, the chemical potential will change when the energies change. One then recomputes the chemical potential as the root of the equation
\[ \sum_n (N_F(\uparrow,n) + N_F(\downarrow,n)) = p \]  
(12)

for the principal fraction \( p/(2p+1) \) and iterates the whole process until self-consistency is achieved. Finally, the spin polarization is given by
\[ P = \sum_n (N_F(\uparrow,n) - N_F(\downarrow,n))/p \]  
(13)

From the above procedure it is clear that only single-particle excitations have been taken into account in obtaining \( P(T) \) so far, and spin fluctuations have been explicitly ignored. If it so happens that for the system under consideration the effects of spin fluctuations are small, then this approximation should be accurate, otherwise not.

Let us proceed to compare the CFHF results to experiments. We first consider the 10W sample of Kandelwal \textit{et al.} The sample parameters are \( B_{z} = 9.61T \), and \( B_{\perp} = 12T \). This implies that the Coulomb energy scale is \( E_C = e^2/\varepsilon l \approx 160K \) and the Zeeman energy is \( E_Z = 0.0175E_C \). We will use a value of \( \lambda = 1.5l \) for the thickness parameter for illustrative purposes throughout this paper. This value ought to be in the physical regime for most samples, and also approximately agrees with that extracted from an analysis of the compressible states. It should be emphasized that the CFHF analysis, and the mapping to the effective spin theory that follows, can be performed for any potential \( v(q) \).

Figure 1 (in the Introduction) shows the HF prediction from our theory for \( \lambda = 1.5l \) compared to the experimental data. The agreement is good at very low \((T \leq 1K)\) and very high \((T \geq 8K)\) temperatures. However, at intermediate \( T \), there is a big discrepancy between the CFHF prediction and the data. This indicates that effects not treated correctly in HF, notably long wavelength thermal spin fluctuations, are important in this intermediate regime of \( T \). Nonetheless, to put the result in context, one should note that the CFHF prediction agrees much better with the data than the corresponding HF prediction for \( \nu = 1 \) (see, for example, ref [15] for a comparison of the different predictions for \( \nu = 1 \)). This is because at \( \nu = 1 \) particle-hole excitations are completely unimportant at all temperatures of interest, while thermal spin fluctuations dominate. Since long wavelength spin fluctuations are treated very poorly in HF the agreement is bad. However, at \( \nu = 1/3 \), particle-hole excitations play a major role in reducing \( P(T) \) for \( T > 5K \). In the next section we will see how to incorporate spin fluctuations into our calculation, resulting in much better agreement with the data.
Once again a value of $s_{P=1} \approx 1$. In any reasonable theory one expects to find that $P = 1$ for $T \ll E_Z$, and expects to see this saturated value of $P$ up to about $T = 0.5E_Z$ or so.

Based on these considerations two values of the saturation Knight shift $K_{s_{P=1}}$, $21kH_z$ and $19kH_z$ have been used to fit the data here, both of which lie within the error bars of the low $T$ data. One possibility that can explain this spread is that spin-reversed quasiparticles are present in the ground state due to disorder, which can bring down the “saturated” value of the Knight shift. The $21kH_z$ value was used by Melinte et al in a phenomenological tanh($\Delta/4k_BT$) fit to obtain $\Delta = 1.7E_Z$. However, one must note that the fit for $K_{s_{P=1}} = 19kH_z$ seems slightly better, since then the experimental saturation region is about $0.5E_Z$. The agreement between theory and experiment for this value of $K_{s_{P=1}}$ are also better than for $K_{s_{P=1}} = 21kH_z$. The $19kH_z$ value will be used in the mapping to the effective theory in the next section. Overall the agreement is slightly worse than for the Khandelwal et al data, but leads to the same conclusion: It is quite important to treat thermal spin fluctuations correctly at intermediate temperatures for $\nu = 1/3$.

Before we proceed to approximately incorporate spin fluctuations into the theoretical prediction, let us address the following interesting question: Why is HF so good in this case relative to the case of $\nu = 1$? To answer this question let us turn to the spin wave dispersions. The spin wave is a collective spin-flip excitation, and at wave vector $\vec{q}$ corresponds to a plane wave state in which a majority spin quasihole and a minority spin quasiparticle are at a separation of $q(l^*)^2$.

The $q \rightarrow 0$ limit is required to be $E_Z$ by Larmor’s theorem, while in the $q \rightarrow \infty$ limit the particle and hole become infinitely separated, so that the energy of the excitation is the spin-reversed particle-hole gap $\Delta_{SR}$. The dispersion of these excitations can be computed for $\nu = 1/3$ in the manner described in ref.[29], and is shown in Figure 4 for $\lambda = 1.5l$ for $E_Z = 0.0175E_C$ and $T = 0$. Figure 4 explicitly illustrates the feature that the spin-flip particle-hole excitations are at the same energy scale as $E_Z$.

How does this evolve as temperature increases? Recall that all the energy splittings in the CF-Hamiltonian formalism come from interactions, and as the occupations of the states change with $T$ so do their HF energies. As $T$ increases the occupations of the minority spin levels increase while that of the lowest majority spin level decreases. This means that the exchange splitting between the minority and majority spin levels decreases, as can be seen from equation (10). It is clear that as $T$ becomes very large the occupations of all the levels should tend to become the same, and therefore $\Delta_{SR}$ should tend towards $E_Z$. Since the spin wave dispersion has to be $E_Z$ for very small $q$, and $\Delta_{SR}$ for very large $q$, this implies that the spin wave dispersion becomes increasingly flat at $T$ increases. We will estimate the spin stiffness in the next section and corroborate this conclusion. Figure 5 shows this behavior of $\Delta_{SR}$ explicitly for the same parameters as in Figure 4.
be proposed to explain intriguing observations by Kukushkin et al. of a state with half the maximal polarization for $\nu = 2/5$, which is not allowed as a translationally invariant CF state at $T = 0$.

Let us compare our results to the only other method that can compute $P(T)$ for arbitrary fractions, which is exact diagonalization (keeping all the excited states) and subsequent calculation of thermodynamic quantities. Due to computational limitations, this method is restricted to fairly small systems. For example, the largest system studied by Chakraborty and Pietilainen for $\nu = 1/3$ has 5 electrons, and for $\nu = 2/5$ has 4 electrons. For $\nu = 1/3$ the exact diagonalization result lies above our predictions (and the experiment) for $T > 4K$. This discrepancy might be the result of finite thickness and/or finite size corrections. However, at low $T$ the exact diagonalization result follows the data more closely than our HF approximation (in all the above comparisons the $g = 0.5$ line in Figure 2 of ref[48] has been used and compared to the 10W sample of Khandelwal et al.)

This sample has the closest parameters to those used in ref[48]). For $\nu = 2/5$, the CFHF results reproduce the nonmonotonicity of $P(T)$ for those values of $E_z$ where the singlet state is the ground state, and the peaks in $P(T)$ occur at roughly the same $T$ in the CFHF and the exact diagonalization results. However, the same overall pattern holds for $\nu = 2/5$, namely, the results of Chakraborty and Pietilainen are below the CFHF prediction for low $T$, but are higher for $T > 0.02E_C$, where they once again see a $1/T$ tail with a large coefficient. It would be interesting to explore the finite size systematics to see if the large $T$ tail is suppressed for larger sizes.

IV. SPIN FLUCTUATIONS: MAPPING TO THE CONTINUUM QUANTUM FERROMAGNET

As can be seen in Figures 1 and 2, spin fluctuations are quite important for $\nu = 1/3$ at intermediate temperatures. For $\nu = 1$, the coarse-grained effective theory of the continuum quantum ferromagnet (CQFM) coupled with the large-$N$ approximation, first applied to this problem by Read and Sachdev, has been quite successful in explaining the temperature dependence of the polarization. In this section a method is presented to map the $\nu = 1/3$ problem to the CQFM.

In the CQFM description one starts with the action

$$S = \int d^4x \int_0^{1/T} d\tau (iM_0 \vec{A}(\vec{n}) \cdot \nabla_x \vec{n} + \rho_s (\nabla_x \vec{n})^2 - M_0 \vec{H} \cdot \vec{n} + \cdots)$$

where $\vec{n}$ is a local vector of unit length pointing in the direction of the magnetization, $\vec{A}(\vec{n})$ is the field that implements the Berry’s phase needed to obtain the correct quantum commutation relations between the spin components, and $\vec{H} = g^* \mu_B \vec{B}$ is the Zeeman field ($|\vec{H}| = E_z$). The two crucial parameters which enter the action are $M_0$, the magnetization density, and $\rho_s$, the spin stiffness. There are other omitted terms in the action, which
produce at most a logarithmic correction to the magnetization in two dimensions\(^4\). There are various ways one can proceed at this point\(^4\), but the most convenient one for our purposes involves mapping the spin problem to a problem with Schwinger bosons\(^4\), and making the large-\(N\) approximation\(^4\). There are two common ways of mapping the spin problem to Schwinger bosons: The \(SU(N)\) approach and the \(O(N)\) approach\(^4\), which give slightly different answers.

In the following, we will restrict the theoretical results to the leading large-\(N\) approximation, where the magnetization is given as \(M(T) = M_0 \Phi_M(r, h)\), where \(r = \rho_s / T\) and \(h = E_Z / T\) are scaling variables, and \(\Phi_M\) is a scaling function. In this approximation, the results of Read and Sachdev\(^4\) for the magnetization in the \(SU(N)\) approach are

\[
\Phi_M(r, h) = \frac{\log(q_1 - e^{-h/2}) - \log(q_1 - e^{h/2})}{8 \pi r} \tag{15}
\]

and \(q_1 > 1\) is a root of the equation

\[
(q_1 - e^{-h/2})(q_1 - e^{h/2}) = q_1^2 e^{-8 \pi r} \tag{16}
\]

The corresponding results in the leading \(O(N)\) approximation\(^4\) are

\[
\Phi_M(r, h) = \frac{\log(q_2 - e^{-h}) - \log(q_2 - e^{h})}{4 \pi r} \tag{17}
\]

where \(q_2 > 1\) is the solution of

\[
(q_2 - e^{-h})(q_2 - 1)(q_2 - e^{h}) = q_2^1 e^{-4 \pi r} \tag{18}
\]

The principal assumption underlying the CQFM description is that long-wavelength ferromagnetic spin fluctuations are the only low energy modes that affect the spin polarization in the temperature range of interest. In the regime where this assumption holds for the \(\nu = 1/3\) state, it should be possible to map the CF-Hamiltonian theory to the CQFM. Conceptually, one can think of “integrating out” the fermions and leaving behind an effective theory of the spin fluctuations. Operationally, one needs to find the values of \(M_0\) and \(\rho_s\) corresponding to the \(\nu = 1/3\) state. An additional complication arises here: Since the underlying fermionic theory responds to temperature by self-consistently modifying occupations and energies, one should expect to obtain temperature dependent values \(M_0(T)\) and \(\rho_s(T)\). We will extract these values from the CFHF Hamiltonian results.

First consider \(M_0(T)\). We already have a value for the CFHF magnetization \(M_{HF}(T)\). Let us first write down the correct relation between \(M_0\) and \(M_{HF}\) and then justify it.

\[
M_{HF}(T) = M_0(T) \Phi_M(r = 0, h = \Delta_{SR}/T) \tag{19}
\]

In the CFHF theory the particles and holes are treated as independent, or noninteracting, with a gap equal to \(\Delta_{SR}\) (to be more precise, this is the lowest energy spin-flip excitation). To put it differently, the energy gap to create this spin-flip excitation is always \(\Delta_{SR}\), no matter what the distance between the particle and the corresponding hole. This is as though the collective mode dispersion were completely flat, \(\omega(q) = \Delta_{SR}\). The CQFM description that corresponds most closely to the CFHF is the one that has the same spin-flip excitation spectrum, namely one with no spin stiffness, that is, \(r = 0\), and an effective Zeeman field \(E_{Zeff} = \Delta_{SR}\). The CQFM prediction for the magnetization of such a theory is the right hand side of equation (15), which has to be equated to the CFHF prediction, hence the above formula, equation (19).

Knowing that, for \(SU(N)\),

\[
\Phi_M(0, h) = \tanh(h/2) \tag{20}
\]

while, for \(O(N)\),

\[
\Phi_M(0, h) = \frac{\sinh(h)}{1/2 + \cosh(h)} \tag{21}
\]

one extracts \(M_0(T)\) from equation (19), using the value of \(M_{HF}(T)\) computed in Section III\(^4\).

Next consider the spin stiffness \(\rho_s(T)\). At a given temperature \(T\) the self-consistent occupations \(N_{F,GS}(\sigma, n)\) and energies \(\epsilon(\sigma, n)\) in the ground state are computed using the procedure described in Section II. Now one creates a twisted spin state by defining

\[
d_{\uparrow, n, X} = \cos(qX/2)d_{\alpha, n, X} + \sin(qX/2)d_{\beta, n, X}
\]

\[
d_{\downarrow, n, X} = \cos(qX/2)d_{\alpha, n, X} + \sin(qX/2)d_{\beta, n, X} \tag{22}
\]

where \(\alpha, \beta\) define local directions of up and down. In the twisted spin state the occupations of the local up and down spins remain the same as in the ground state, that is

\[
< d_{\alpha, n, X}^\dagger d_{\alpha, n', X} > = \delta_{nn'} \delta X \delta \epsilon_{N_{F,GS}(\uparrow)}
\]

\[
< d_{\beta, n, X}^\dagger d_{\beta, n', X} > = \delta_{nn'} \delta X \delta \epsilon_{N_{F,GS}(\downarrow)}
\]

\[
< d_{\alpha, n, X}^\dagger d_{\beta, n', X} = 0 \tag{23}
\]

This leads to the following expectation values for the actual (global) \(\uparrow\) and \(\downarrow\) spin directions.

\[
< d_{\uparrow, n, X}^\dagger d_{\uparrow, n', X} > = \delta_{nn'} \delta X \times \cos^2(qX/2)N_{F,GS}(\uparrow, n) + \sin^2(qX/2)N_{F,GS}(\downarrow, n)
\]

\[
< d_{\downarrow, n, X}^\dagger d_{\downarrow, n', X} > = \delta_{nn'} \delta X \times \sin^2(qX/2)N_{F,GS}(\uparrow, n) + \cos^2(qX/2)N_{F,GS}(\downarrow, n)
\]

\[
< d_{\uparrow, n, X}^\dagger d_{\downarrow, n', X} > = \delta_{nn'} \delta X \times \sin(qX/2) \cos(qX/2)(N_{F,GS}(\uparrow, n) - N_{F,GS}(\downarrow, n))
\]

\[
< d_{\downarrow, n, X}^\dagger d_{\uparrow, n', X} > = < d_{\uparrow, n, X}^\dagger d_{\downarrow, n', X} > \tag{24}
\]

and corresponds to a state where the unit vector \(\vec{n}\) pointing in the direction of the local magnetization has components \(\vec{n} = (\sin(qX), 0, \cos(qX))\). Using the expectation values from equation (24), one can compute the HF energy of the twisted ground state, and thence the excess energy to order \(q^2\). Comparing to the energy cost of a
twist in the CQFM, which is \( (\rho_s/2)L^2\eta^2 \), one finds the spin stiffness
\[
\rho_s = \frac{1}{16\pi^2} \int \frac{d^2 s}{(2\pi)^2} v(s) \sum_{n_1,n_2} |\rho_{n_1,n_2}(s)|^2 \times 
(N_F(\uparrow,n_1) - N_F(\downarrow,n_1))(N_F(\uparrow,n_2) - N_F(\downarrow,n_2))
\]
where \( L^2 \) is the area of the system, and \( \rho_{n_1,n_2} \) is the density matrix element of equation (24). One caveat should be mentioned here: The above should be regarded as an estimate for the twist rather than a rigorous calculation (even in HF), since ideally one should compute the free energy cost of a twist, rather than just the internal energy cost, as we have done. The free energy cost would be computed by carrying out a self-consistent HF at finite \( T \) in the presence of a twist. However, a fully relaxed HF calculation of an inhomogeneous state at \( T \neq 0 \) is computationally prohibitive. In the following we use the value given by equation (24).

![Figure 7](image_url)

**FIG. 7.** The parameters \( M_0(T) \) (normalized to 1 at \( T = 0 \)) for SU(N) (dashed line) and O(N) (dot-dashed line), and \( \rho_s(T \times 400 \text{ in units of } \varepsilon_C) \) extracted from the CFHF for \( \varepsilon_L = 0.0186\varepsilon_C \) and \( \lambda = 1.5l \). While \( M_0 \) shows no dramatic behavior, \( \rho_s \) decreases precipitously beyond 3K.

Figure 7 shows the results for \( M_0(T) \) and \( \rho_s(T) \) for the parameters corresponding to the Melinte et al M242 sample [1]. As can be seen, \( \rho_s \) in particular has a dramatic \( T \) dependence, and essentially vanishes for \( T > 6K \). This corroborates the earlier observation that the spin-reversed gap collapses to \( \varepsilon_L \) at around the same temperature, as seen in Figure 4.

Having extracted the parameters \( M_0(T) \) and \( \rho_s(T) \), it is now easy to calculate the effects of spin fluctuations in the leading large-\( N \) approximation from equations (15,17). Figure 1 (in the Introduction) presents the results for the Khandelwal et al data [1]. As can be seen, the agreement between the theory and experiment is now quite good, indicating that the proper treatment of long-wavelength thermal spin fluctuations has remedied most of the defects of the CFHF prediction. Turning to the M242 sample of Melinte et al [12], a similar dramatic improvement in the agreement between theory and experiment is seen in Figure 2 (in the Introduction). There are still some discrepancies in both the figures, but they never amount to more than 10—15%.

One might wonder how important it is to keep the temperature dependence of the parameters \( M_0 \) and \( \rho_s \). Figure 8 shows the prediction of the large-\( N \) approach using the values of \( M_0 \) and \( \rho_s \) computed at \( T = 0 \) for the parameters corresponding to the Khandelwal et al data [1]. This fit ignores finite-temperature single particle excitations. It is clear that while this prediction is somewhat better than the CFHF result for lower temperatures, it is worse beyond 6K. Also, this prediction is uniformly worse than the one including both CFHF and spin fluctuation effects.

![Figure 8](image_url)

**FIG. 8.** The theoretical prediction for the Khandelwal et al data [1] based on a zero temperature fit to the parameters of the CQFM. The thickness parameter is assumed to be \( \lambda = 1.5l \). Circles represent the experimental data [1].

Clearly, the reason is that this fit does not incorporate the finite-temperature fermionic single-particle effects that renormalize the spin stiffness substantially downwards beyond about 3K, as seen in Figure 6. It must be concluded that the temperature dependence of the parameters \( M_0 \) and \( \rho_s \) is crucial, and that both single-particle and thermal spin wave effects must be treated correctly if one is to accurately predict the spin polarization.

V. CONCLUSIONS, CAVEATS, AND OPEN QUESTIONS

In summary, an approximate analytical method for computing the temperature dependence of the polarization for an arbitrary fractional quantum Hall state has been presented, and illustrated for \( \nu = 1/3 \) and 2/5. It consists of two steps. The first step is a finite-temperature Composite Fermion Hartree-Fock approximation on the CF Hamiltonian. This already incorporates important effects on the polarization resulting from particle-hole excitations, and is sufficient to produce all the correct qualitative features, such as the nonmonotonicity of the polarization for 2/5. For the spontaneously polarized 1/3 state, however, there is a sub-
stantial discrepancy between this prediction and experiments at intermediate temperatures due to the incorrect treatment of spin fluctuations. The second step, for the $\nu = 1/3$ state, is to map the long-wavelength low-energy physics onto the continuum quantum ferromagnet [14]. The parameters that enter the effective theory are extracted from the CFHF approximation. Finally, the approximate solution to the continuum quantum ferromagnet in the large-$N$ approach [14] is used to produce a prediction for the spin polarization which incorporates the effects of both single particle and spin wave excitations. Note that the reduction of the spin polarization due to single-particle and spin wave effects is not additive in the two effects: the parameters $M_0$ and $\rho_s$ extracted from the CFHF approximation encode information about single-particle excitations in a very nonlinear way. The resulting prediction is in quite good agreement with experimental data, as seen in Figures 1 and 2. The comparison between theory and experiment was made using the model potential of equation (8) for illustrative purposes, with a thickness parameter $\lambda = 1.5\lambda_c$, which ought to be in the physical regime for most samples. However, the above two-step procedure of first carrying out the CFHF, and then mapping the problem on to the CQFM, can be executed for any potential $\nu(q)$.

A number of caveats must be noted at this point. In order for the mapping to the CQFM to make sense, spin wave excitations must be the only low energy spin-flip excitations. However, as $T$ increases, other spin-flip excitations also become relevant (such as a transition from the $\uparrow$-spin $n = 0$ CF-LL to the $\downarrow$-spin $n = 1$ CF-LL). Another complicating issue is that while the CQFM assumes the spin wave dispersion to be quadratic at all $q$, in reality it turns over and asymptotes at the spin-reversed gap $\Delta_{SR}$. Both these considerations suggest that the mapping is to be trusted only in the regime where $T \leq \Delta_{SR}(T)$. One can verify from Figure 5 that this corresponds to $T \leq 6K_F$ or so, which fortunately includes the most interesting region of the data. It is somewhat puzzling that the prediction from the mapping works well to twice this temperature. Finally, $1/N$ corrections can be expected to the leading large-$N$ prediction, which reduce the magnetization slightly beyond the leading large-$N$ result (see, for example, Timm et al. [14]). These corrections would improve the agreement between the theory and the data at low $T$.

An important open problem is the development of a formalism in which one can systematically integrate out fermions, leaving behind a theory of the low-energy excitations, perhaps along the lines of Ref. [15]. Such excitations need not be restricted to spin wave excitations, but may perhaps include other spin-flip modes or even density modes, depending on the temperature and the system. Such a theory is necessary to address the effects of thermal spin fluctuations on the temperature dependent spin polarization of fractional Hall states which are not fully polarized in the absence of a Zeeman coupling. The spin-flip modes of such systems do not have the simple quadratic CQFM form even at small $q$ (for example, the spin-flip collective mode of the fully polarized 2/5 state starts with a negative quadratic term). A further interesting application would be to the interplay of spin and density fluctuations in the compressible fractional Hall states [14].

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51 Note that neither of the above formulas agrees with the result for itinerant noninteracting electrons at ν = 1, namely, \( \phi_M = \tanh(h/4) \). However, since we will be using the large-\( N \) forms with nonzero \( r \) later, we have to use the above forms for consistency.