Stability with respect to domain of the low Mach number limit of compressible viscous fluids

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Abstract
We study the asymptotic limit of solutions to the barotropic Navier-Stokes system, when the Mach number is proportional to a small parameter \( \varepsilon \to 0 \) and the fluid is confined to an exterior spatial domain \( \Omega_\varepsilon \) that may vary with \( \varepsilon \). As \( \varepsilon \to 0 \), it is shown that the fluid density becomes constant while the velocity converges to a solenoidal vector field satisfying the incompressible Navier-Stokes equations on a limit domain. The velocities approach the limit strongly (a.a.) on any compact set, uniformly with respect to a certain class of domains. The proof is based on spectral analysis of the associated wave propagator (Neumann Laplacian) governing the motion of acoustic waves.

Key words: Incompressible limit, domain dependence, Navier-Stokes system

1 Introduction

There is a vast number of mathematical models of incompressible fluids that can be identified as a singular limit of more complex systems describing the motion of compressible

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and/or heat conducting fluids, see the review papers by Gallagher [18], Masmoudi [32], [35], or the monograph [15] and the references therein. In many cases the resulting system is driven by an exterior force related to gravitation of rigid objects outside the fluid domain whereas, at the same time, the fluid occupies the whole physical space $R^3$, see the study of the Oberbeck-Boussinesq approximation by Brandolese and Schonbek [3]. Such a situation, if physically relevant, can be viewed as a singular limit, where the primitive system is posed on a family of exterior domains $\Omega_\varepsilon$, with $R^3 \setminus \Omega_\varepsilon$ being the rigid body(ies) acting on the fluid by their gravitation, and $\Omega_\varepsilon \to R^3$ as $\varepsilon \to 0$ in a certain sense. Our goal in the present study is to develop a method for studying the incompressible limits, with the Mach number $Ma = \varepsilon \to 0$, on a family of exterior domains $\Omega_\varepsilon \subset R^3$ varying with $\varepsilon > 0$. In particular, we identify a class of domains giving rise to uniform convergence of the fluid velocities, independent of the specific shape of their boundaries. To this end, we adapt the technique introduced in [14], based on spectral theory for the corresponding acoustic wave propagator - the Neumann Laplacian on $\Omega_\varepsilon$. The adaptation leans on delicate estimates of the associated Helmholtz projections based on the results by Farwig, Kozono, and Sohr in [12] and [13].

For the sake of simplicity, we focus only on the mechanical aspects of the fluid motion, ignoring completely the effect of temperature changes. Accordingly, we consider the compressible Navier-Stokes system in Eulerian reference coordinates:

\[
\begin{align*}
\partial_t \varrho + \text{div}_x (\varrho u) &= 0, \\
\partial_t (\varrho u) + \text{div}_x (\varrho u \otimes u) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho) &= \text{div}_x S(\nabla_x u), \\
S(\nabla_x u) &= \mu \left( \nabla_x u + \nabla^t_x u - \frac{2}{3} \text{div}_x u I \right), \quad \mu > 0,
\end{align*}
\]

where the unknown functions are the mass density $\varrho = \varrho(t,x)$ and the vector velocity $u = u(t,x)$, with $t \in (0,T)$, and $x \in \Omega_\varepsilon$. The symbol $p = p(\varrho)$ denotes the pressure - a given function of the density - whereas $S$ stands for the viscous stress tensor. The small parameter $\varepsilon$ represents the Mach number tending to zero in the asymptotic limit.

The system (1.1-1.3) is supplemented by the (acoustically hard) complete slip boundary conditions

\[
\begin{align*}
u \cdot n|_{\partial \Omega_\varepsilon} &= 0, \\
[S \cdot n] \times n|_{\partial \Omega_\varepsilon} &= 0,
\end{align*}
\]

where the $n$ denotes the (outer) normal vector to $\partial \Omega_\varepsilon$. Moreover, as the fluid occupies an exterior domain, the behavior of $\varrho, u$ at infinity must be specified:

\[
\varrho \to \varrho > 0, \quad u \to 0 \text{ as } |x| \to \infty,
\]
where $\overline{\rho}$ is a constant.

In the singular limit $\varepsilon \to 0$, the fluid is driven to incompressibility as the speed of sound tends to infinity. The incompressible limit for system (1.1 - 1.3) as well as related problems have been studied by many authors, see Alazard [1], Klainerman and Majda [24], Lions and Masmoudi [30], or Schochet [40], to name only a few. Similarly to Lions and Masmoudi [30], [31], our approach is based on the concept of weak solutions to the Navier-Stokes system, where the presence of viscosity plays a crucial role. As is well known (cf. [30]), convergence of solutions of the system (1.1 - 1.3) to the incompressible limit may be disturbed by the presence of rapidly oscillating acoustic waves, here represented by the gradient part of the velocity field. Since the physical domains $\Omega_\varepsilon$ are unbounded, however, we expect that acoustic waves disperse leaving very fast any bounded part of the physical space as was observed by Alazard [1], Bresch and Metivier [4], Isozaki [21], and [17]. The main novelty of this paper is the fact that the physical domains $\Omega_\varepsilon$ are allowed to change their shape together with the Mach number. In particular, we show that the rate of convergence is uniform within a certain class of domains specified in Section 2.2 below. An interesting aspect of the problem is the boundary behavior of the limit velocity field $U$. As shown in [8], the slip boundary conditions (1.4) may give rise to the more standard no-slip condition

$$U|_{\partial \Omega} = 0,$$

or to the kind of friction-driven boundary conditions identified in [17].

In order to see the principal difficulties involved, we rewrite the Navier-Stokes system in the form of Lighthill’s acoustic analogy [27], [28]:

$$\varepsilon \partial_t r + \text{div}_x V = 0 \text{ in } (0, T) \times \Omega_\varepsilon,$$  

(1.7)

$$\varepsilon \partial_t V + p'(\rho) \nabla_x r = \varepsilon \text{div}_x L \text{ in } (0, T) \times \Omega_\varepsilon,$$  

(1.8)

supplemented with the boundary condition

$$V \cdot n|_{\partial \Omega_\varepsilon} = 0,$$  

(1.9)

where

$$r \equiv \frac{\rho - \overline{\rho}}{\varepsilon}, \quad V \equiv \rho u,$$  

(1.10)

and $L$ is the so-called Lighthill’s tensor,

$$L \equiv S - \rho u \otimes u - \frac{1}{\varepsilon^2} \left( p(\rho) - p'(\overline{\rho})(\rho - \overline{\rho}) - p(\overline{\rho}) \right) I.$$

(1.11)

Applying, formally, the Helmholtz projection to (1.8) we obtain the wave (acoustic) equation

$$\varepsilon \partial_t r + \Delta \Phi = 0,$$  

(1.12)
\[ ε \partial_t Φ + p'(\bar{ρ})r = ε \Delta_{ε,N}^{-1} \text{div}_x \text{div}_x L, \quad (1.13) \]

\[ \nabla_x Φ \cdot \mathbf{n}|_{∂Ω_ε} = 0, \quad (1.14) \]

for the acoustic potential \( Φ = \Delta_{ε,N}^{-1} \text{div}_x V \), where the symbol \( \Delta_{ε,N} \) denotes the standard Laplace operator supplemented with homogeneous Neumann boundary conditions on \( ∂Ω_ε \).

Accordingly, our main task consists in:

- estimating the forcing term \( \Delta_{ε,N}^{-1} \text{div}_x \text{div}_x L \) as well as the initial data in terms of a suitable power of the operator \( \Delta_{ε,N} \), cf. \([17]\);
- evaluating local rate of decay of solutions to the wave equation \((1.12), (1.13)\).

In general, bounds of \( \Delta_{ε,N}^{-1} \text{div}_x \text{div}_x L \) in terms of \( \Delta_{ε,N} \) depend on the shape of \( ∂Ω_ε \), where the latter must be at least of class \( C^{1,1} \) to recover the standard \( W^{2,p} \)-theory, not available on less smooth, say, Lipschitz domains, see Grisvard \([19]\). On the other hand, our method is applicable to families \( \{Ω_ε\}_{ε>0} \), whose smoothness parameters blow up for \( ε \to 0 \). In particular, they may approach a less regular domain in the asymptotic limit and/or their boundaries may oscillate similarly to \([8]\), see Section 2.2. As a result, the forcing term involving Lighthill’s tensor become unbounded for \( ε \to 0 \), and this defect must be compensated by uniform dispersive estimates of order \( \sqrt{ε} \) which we achieve in the spirit of a result due to Kato \([23]\).

It is interesting to note that similar results on bounded domains, supplemented with the no-slip boundary condition \((1.6)\), where convergence of the velocities is enforced by a viscous boundary layer (see Desjardins et al. \([11]\)) seem very sensitive and much less stable with respect to domain perturbations, in particular they completely fail on balls.

The organization of the paper is as follows. In Section 2, we recall some known facts concerning the compressible Navier-Stokes system \((1.1 - 1.3)\), including the available existence theory, introduce the principal hypotheses concerning the admissible class of domains, and state our main result. Section 3 reviews the standard uniform bounds on solutions to \((1.1 - 1.3)\) in \((0,T) \times Ω_ε\) independent of the singular parameter \( ε \to 0 \). In Section 4, we introduce the acoustic equation and identify the terms in Lighthill’s tensor. In particular, we deduce estimates on Lighthill’s tensor in terms of \( ε \), based on the \( W^{2,p} \)-theory for the Neumann Laplacian \( \Delta_{ε,N} \) defined on \( Ω_ε \). Section 5 deals with the spectral theory of the operator \(-Δ_{ε,N}\) in \( L^2(Ω_ε) \). We introduce the associated spectral measures and show local decay of acoustic waves of rate \( \sqrt{ε} \). Finally, the incompressible limit is performed in Section 6. In addition, we shortly discuss the limit boundary conditions in the spirit of \([7]\).
2 Preliminaries and main result

Throughout the paper, the pressure $p$ is a continuously differentiable function of the density such that

$$p \in C[0, \infty) \cap C^1(0, \infty), \quad p'(\varrho) > 0 \text{ for all } \varrho > 0, \quad \lim_{\varrho \to \infty} \frac{p'(\varrho)}{\varrho^{\gamma-1}} = p_\infty > 0, \quad (2.1)$$

for a certain $\gamma > 3/2$.

Multiplying the momentum equation $(1.2)$ by $u$ and integrating by parts leads to the energy inequality:

$$\frac{d}{dt} \int_{\Omega_\epsilon} E_\epsilon (\varrho, u)(\tau, \cdot) \, dx + \int_{\Omega_\epsilon} S(\nabla_x u) : \nabla_x u \, dx \leq 0, \quad \tau \in (0, T), \quad (2.2)$$

where we have set

$$E_\epsilon (\varrho, u) := \frac{1}{2} \varrho |u|^2 + \frac{1}{\epsilon^2} (P(\varrho) - P'(\overline{\varrho})(\varrho - \overline{\varrho}) - P(\overline{\varrho})), \quad (2.3)$$

As $P''(\varrho) = p'(\varrho)/\varrho > 0$, the function $P$ is strictly convex and $P(\varrho) - P'(\overline{\varrho})(\varrho - \overline{\varrho}) - P(\overline{\varrho}) \approx c(\varrho - \overline{\varrho})^2$ provided $\varrho \approx \overline{\varrho}$. Consequently, with initial data of the form

$$\varrho(0, \cdot) = \varrho_{0, \epsilon} = \overline{\varrho} + \epsilon r_{0, \epsilon}, \quad u(0, \cdot) = u_{0, \epsilon}, \quad (2.4)$$

with

$$\|r_{0, \epsilon}\|_{L^2(\Omega_\epsilon)} + \|r_{0, \epsilon}\|_{L^\infty(\Omega_\epsilon)} + \|u_{0, \epsilon}\|_{L^2(\Omega_\epsilon; \mathbb{R}^3)} \leq c, \quad (2.5)$$

we get the total (mechanical) energy associated to the initial data

$$\int_{\Omega_\epsilon} E_\epsilon (\varrho_{0, \epsilon}, u_{0, \epsilon}) \, dx = \int_{\Omega_\epsilon} \left( \frac{1}{2} \varrho_{0, \epsilon} |u_{0, \epsilon}|^2 + \frac{1}{\epsilon^2} (P(\varrho_{0, \epsilon}) - P'(\overline{\varrho})(\varrho_{0, \epsilon} - \overline{\varrho}) - P(\overline{\varrho})) \right) \, dx,$$

bounded uniformly for $\epsilon \to 0$.

2.1 Weak solutions

We say that a pair of functions $\varrho$, $u$ represents a weak solution to the Navier-Stokes system $(1.1 - 1.3)$, with the boundary conditions $(1.4)$, $(1.5)$, and the initial data $(2.4)$ if:
\[ \varrho \geq 0, (\varrho - \bar{\varrho}) \in (L^2 + L^\infty)(\Omega_\varepsilon), \ u \in L^2(0,T; W^{1,2}(\Omega_\varepsilon; \mathbb{R}^3)) \text{ such that } \mathbf{u} \cdot \mathbf{n}|_{\partial \Omega_\varepsilon} = 0; \]

- the equation of continuity (1.1) is satisfied in the sense of renormalized solutions:
  \[ \int_0^T \int_{\Omega_\varepsilon} \left[ (\varrho + b(\varrho)) \partial_t \varphi + (\varrho + b(\varrho)) \mathbf{u} \cdot \nabla_x \varphi 
    + (b(\varrho) - b'(\varrho) \varrho) \text{div}_x \mathbf{u} \varphi \right] d\mathbf{x} \, dt = 
    - \int_{\Omega_\varepsilon} (\varrho_{0,\varepsilon} + b(\varrho_{0,\varepsilon})) \varphi(0,\cdot) d\mathbf{x}, \]
  for any test function \( \varphi \in C^\infty((0,T) \times \overline{\Omega_\varepsilon}) \), and any \( b \in C^\infty[0,\infty), b' \in C^\infty_c[0,\infty) \);

- the momentum equation (1.2) holds in the sense of the integral identity
  \[ \int_0^T \int_{\Omega_\varepsilon} \left[ \varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + \frac{1}{\varepsilon^2} p(\varrho)\text{div}_x \varphi \right] d\mathbf{x} \, dt = 
    \int_0^T \int_{\Omega_\varepsilon} S(\nabla_x \mathbf{u}) : \nabla_x \varphi d\mathbf{x} \, dt 
    - \int_{\Omega_\varepsilon} \varrho_{0,\varepsilon} \mathbf{u}_{0,\varepsilon} \cdot \varphi(0,\cdot) d\mathbf{x}, \]
  for any test function \( \varphi \in C^\infty((0,T) \times \overline{\Omega_\varepsilon}; \mathbb{R}^3), \nabla_x \varphi \cdot \mathbf{n}|_{\partial \Omega_\varepsilon} = 0; \)

- the energy inequality
  \[ - \int_0^T E_\varepsilon(\varrho, \mathbf{u}) \partial_t \psi \, dt + \int_0^T \psi \int_{\Omega_\varepsilon} S(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \, d\mathbf{x} \, dt \leq \int_{\Omega_\varepsilon} E_\varepsilon(\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}) \psi(0) \, d\mathbf{x}, \]
  holds for any \( \psi \in C^\infty_c[0,T), \psi \geq 0. \)

### 2.2 Admissible domains

Motivated by [8], we introduce a class of admissible domains that allows for “oscillating” boundaries. As first observed by Casado-Diaz, Fernandez-Cara, and Simon [9], such a family of domains may give rise to the \textit{no-slip} boundary condition (1.6) for the limit velocity field. A general description of all possible limit boundary conditions was obtained in [7].

Introducing a cone
\[ C(x, \omega, \delta, \xi) = \{ y \in \mathbb{R}^3 \mid 0 < |y - x| \leq \delta, \ (y - x) \cdot \xi > \cos(\omega)|y - x| \}, \]
with vertex at \( x \), aperture \( 2\omega < \pi \), height \( \delta \), and orientation given by a unit vector \( \xi \), we say that \( \Omega_\varepsilon \) satisfies the uniform \( \delta \)--cone condition if for any \( x_0 \in \partial \Omega_\varepsilon \), there exists a unit vector \( \xi_{x_0} \in \mathbb{R}^3 \) such that

\[
C(x, \omega, \delta, \xi_{x_0}) \subset \Omega_\varepsilon \text{ whenever } x \in \Omega_\varepsilon, \ |x - x_0| < \delta,
\]

see Henrot and Pierre [20, Definition 2.4.1].

Assume we are given a family of domains \( \{\Omega_\varepsilon\}_{\varepsilon>0} \) complying with the following hypotheses:

- \( \Omega_\varepsilon \subset \mathbb{R}^3 \) is an exterior domain with \( C^2 \)--boundary for each fixed \( \varepsilon > 0 \);
- there is a \( d > 0 \) such that \( \mathbb{R}^3 \setminus \Omega_\varepsilon \subset B_d \equiv \{ x \in \mathbb{R}^3 \mid |x| < d \} \) for all \( \varepsilon > 0 \);
- \( \Omega_\varepsilon \) satisfy the uniform \( \delta \)--cone condition with \( \delta > 0 \) (and \( \omega \) independent of \( \varepsilon \));
- for each \( x_0 \in \partial \Omega_\varepsilon \), there are two (open) balls \( B_r[x_i] \equiv \{ x ; |x - x_i| < r \} \subset \Omega_\varepsilon \), \( B_r[x_e] \subset \mathbb{R}^3 \setminus \Omega_\varepsilon \) of radius \( r > c_b \varepsilon^\beta \) such that

\[
B_r[x_i] \cap B_r[x_e] = x_0,
\]

with \( c_b > 0, \beta > 0 \) independent of \( \varepsilon \).

The above hypotheses give rise to the following properties enjoyed by the family \( \{\Omega_\varepsilon\}_{\varepsilon>0} \):

- **Uniform extension property (see Jones [22]).**
  
  There exists an extension operator \( E_\varepsilon \),
  \[
  E_\varepsilon : W^{1,p}(\Omega_\varepsilon) \to W^{1,p}(\mathbb{R}^3), \quad E_\varepsilon[v]|_{\Omega_\varepsilon} = v, \quad \|E_\varepsilon[v]\|_{W^{1,p}(\mathbb{R}^3)} \leq c\|v\|_{W^{1,p}(\Omega_\varepsilon)},
  \]
  where the constant \( c \) is independent of \( \varepsilon \to 0 \).

- **Uniform Korn’s inequality (see [6, Proposition 4.1]).**
  
  Let \( v \in W^{1,2}(\Omega_\varepsilon \cap B; \mathbb{R}^3) \), and \( M \subset \Omega_\varepsilon \cap B \) such that \( |M| > m > 0 \), where \( B \) is a bounded ball. Then
  \[
  \|v\|_{W^{1,2}(\Omega_\varepsilon \cap B; \mathbb{R}^3)}^2 \leq c(m) \left( \left\| \nabla_x v + \nabla_x' v - \frac{2}{3} \text{div}_x v \right\|^2_{L^2(\Omega_\varepsilon \cap B; \mathbb{R}^{3 \times 3})} + \int_M |v|^2 \, dx \right),
  \]
  with \( c(m) \) independent of \( \varepsilon \to 0 \).
• Compactness (see Henrot and Pierre [20, Theorem 2.4.10]).

There exists an exterior domain $\Omega$, satisfying the uniform $\delta$–cone condition, and a suitable subsequence of $\varepsilon'$s (not relabeled) such that

$$|\Omega_{\varepsilon} \setminus \Omega| + |\Omega \setminus \Omega_{\varepsilon}| \to 0 \text{ as } \varepsilon \to 0. \quad (2.12)$$

For each $x_0 \in \partial\Omega$, there is $x_{\varepsilon,0} \in \partial\Omega_{\varepsilon}$ such that $x_{\varepsilon,0} \to x_0$, in particular,

$$R^3 \setminus \Omega \subset B_d. \quad (2.13)$$

For any compact $K \subset \Omega$, there exists $\varepsilon(K)$ such that

$$K \subset \Omega_{\varepsilon} \text{ for all } \varepsilon < \varepsilon(K). \quad (2.14)$$

Property (2.12) is important when studying stability of the spectral properties of the Neumann Laplacian $\Delta_{\varepsilon,N}$ with respect to $\varepsilon$, see Arrieta and Krejčířík [2]. Note that the limit domain need not be of class $C^2$ but merely Lipschitz, see Henrot and Pierre [20, Theorem 2.4.7].

### 2.3 Main result

Before stating our main result, we introduce the limit problem - the *incompressible* Navier-Stokes system - satisfied by the limit velocity field $U$:

$$\text{div}_x U = 0, \quad (2.15)$$

$$\bar{\rho} (\partial_t U + \text{div}_x(U \otimes U)) + \nabla_x \Pi = \mu \Delta U, \quad (2.16)$$

in $(0,T) \times \Omega$, with the condition at infinity

$$|U| \to 0 \text{ as } |x| \to \infty, \quad (2.17)$$

and the initial condition

$$U(0, \cdot) = U_0. \quad (2.18)$$

In the *weak* formulation, the decay condition (2.17) is replaced by a single stipulation $U \in L^2(0,T; W^{1,2}(\Omega, R^3))$, the incompressibility constraint (2.15) is satisfied a.a. in $(0,T) \times \Omega$, while the momentum equation (2.16), together with (2.18), are replaced by a family of integral identities;

$$\int_0^T \int_{\Omega} (\bar{\rho}U \cdot \partial_t \varphi + \bar{\rho}U \otimes U : \nabla_x \varphi) \, dx \, dt \quad (2.19)$$
\[
= \mu \int_0^T \int_{\Omega} \nabla_x U : \nabla_x \varphi \, dx \, dt - \int_{\Omega} \overline{U}_0 \cdot \varphi(0, \cdot) \, dx,
\]
for any test function \( \varphi \in C^\infty_c([0, T) \times \Omega; \mathbb{R}^3) \) satisfying \( \text{div}_x \varphi = 0 \).

Note that we have deliberately omitted to specify any boundary conditions on \( \partial \Omega \).

In the low Mach number limit, we can easily show that the impermeability condition \( u_\varepsilon \cdot n_{|\partial \Omega} = 0 \) gives rise to the same property \( U \cdot n_{|\partial \Omega} = 0 \) for the limit velocity field whereas the boundary behavior of the tangential component of \( U \) may be quite complex depending sensitively on the asymptotic shape of the boundaries \( \partial \Omega_\varepsilon \), cf. [7]. Sufficient conditions for \( U \) to satisfy the no-slip condition

\[
U_{|\partial \Omega} = 0
\]
will be discussed in Section 6.2.

Our main result reads as follows.

**Theorem 2.1** Suppose that a family of domains \( \{\Omega_\varepsilon\}_{\varepsilon > 0} \) belongs to the class specified in Section 2.2 with

\[
0 < \beta < \frac{1}{4},
\]
where \( \varepsilon^{\beta} \) is the radius of the balls in (2.9). Let \( \{\rho_\varepsilon, u_\varepsilon\}_{\varepsilon > 0} \) be a family weak solutions to the compressible Navier-Stokes system (1.1 - 1.5) in \( (0, T) \times \Omega_\varepsilon \), supplemented with the initial conditions (2.4), where

\[
r_{0,\varepsilon} = \frac{\rho_{0,\varepsilon} - \overline{\rho}}{\varepsilon} \to r_0 \text{ weakly in } L^2(\mathbb{R}^3), \quad \|r_{0,\varepsilon}\|_{L^\infty(\mathbb{R}^3)} \leq c, \quad \overline{\rho} > 0,
\]

\[
u_{0,\varepsilon} \to u_0 \text{ weakly in } L^2(\mathbb{R}^3, \mathbb{R}^3),
\]
and the pressure satisfies (2.7) with \( \gamma > 3/2 \).

Then

\[
\text{ess sup}_{t \in (0, T)} \|\rho_\varepsilon(t, \cdot) - \overline{\rho}\|_{(L^2 + L^q)(\Omega_\varepsilon)} \to 0 \text{ as } \varepsilon \to 0 \text{ for } 1 \leq q < \min\{\gamma, 2\},
\]

\[
\|u_\varepsilon\|_{L^2(0, T; W^{1,2}(\Omega_\varepsilon; \mathbb{R}^3))} \leq c,
\]
and, at least for a suitable subsequence,

\[
u_\varepsilon \to U \text{ in } L^2((0, T) \times K; \mathbb{R}^3) \text{ for any compact } K \subset \Omega_\varepsilon,
\]
where

\[
U \in L^2(0, T; W^{1,2}(\Omega; \mathbb{R}^3)), \quad U \cdot n_{|\partial \Omega} = 0,
\]
is a weak solution of the incompressible Navier-Stokes system (2.15 - 2.17) in \((0,T) \times \Omega\), emanating from the initial data

\[ U_0 = H[u_0], \]

where \(\Omega\) is the limit domain identified through (2.12 - 2.14) and \(H\) denotes the standard Helmholtz projection in \(\Omega\).

The rest of the paper is devoted to the proof of Theorem 2.1. Note that existence of solution to the compressible Navier-Stokes system, in the framework of weak solutions, was first established in the seminal work of Lions [29] for \(\gamma \geq 9/5\), and the result then extended in [16] to the “technically” optimal range \(\gamma > 3/2\). A detailed discussion of various choices of boundary conditions, including the case of exterior domains, may be also found in the monograph by Novotný and Straškraba [37]. The problem of limit boundary conditions, here discussed in Section 6, was studied by Casado-Díaz, Fernández-Cara, and Simon [9] in the periodic setting, and later extended in [7], [8]. Finally, we remark there is an analogue of Theorem 2.1 for a family of bounded domains obtained by completely different methods (see [6]), based on analysis of a viscous boundary layer similar to Desjardins et al. [11].

3 Uniform bounds

All uniform bounds presented below may be viewed as a direct consequence of the energy inequality (2.8). To begin, similarly to [15], we introduce the essential and residual part of a function \(h_\varepsilon\) as

\[ [h_\varepsilon]_{\text{ess}} = \chi(\rho_\varepsilon)h_\varepsilon, \quad [h_\varepsilon]_{\text{res}} = h - h_{\text{ess}}, \]

where \(\chi \in C_\infty(0, \infty), \ 0 \leq \chi \leq 1, \ \chi \equiv 1\) in an open neighborhood of \(\overline{\Omega}\).

3.1 Energy bounds

Since the initial data satisfy (2.4), (2.5), the initial energy \(E(\rho_0, u_0)\) in (2.8) remains bounded uniformly for \(\varepsilon \to 0\), where we have used hypothesis (2.1). Consequently, we deduce the following list of estimates:

\[ \text{ess sup}_{t \in (0,T)} \int_{\Omega_\varepsilon} \rho_\varepsilon |u_\varepsilon|^2 \, dx \leq c, \quad (3.1) \]
ess sup \( t \in (0, T) \) \( \int_{\Omega} \left[ \frac{\partial \varepsilon - \bar{\varepsilon}}{\varepsilon} \right]_{\text{ess}}^2 \, dx \leq c \), ess sup \( t \in (0, T) \) \( \int_{\Omega} [\partial \varepsilon]_{\text{res}} \, dx \leq \varepsilon^2 c \), ess sup \( t \in (0, T) \) \( \int_{\Omega} 1_{\text{res}} \, dx \leq \varepsilon^2 c \),

(3.2)

and

\[
\int_0^T \int_{\Omega} \left\| \nabla_x u_\varepsilon + \nabla_x^2 u_\varepsilon - \frac{2}{3} \text{div}_x u_\varepsilon \right\|^2 \, dx \, dt \leq c,
\]

(3.3)

where the constants are independent of \( \varepsilon \to 0 \).

Moreover, relation (3.2) immediately yields

\[
\text{ess sup } t \in (0, T) \int_{\Omega} \left[ \frac{\partial \varepsilon - \bar{\varepsilon}}{\varepsilon} \right]_{\text{res}} \leq c \varepsilon^{2-q} \quad \text{for any } 1 \leq q \leq \min\{\gamma, 2\},
\]

(3.4)

which, together with (3.2) gives rise to (2.24).

Finally, since the family of domains \( \{\Omega_\varepsilon\}_{\varepsilon > 0} \) admits the uniform Korn’s inequality (2.11), we can combine (3.2), (3.3) to conclude that

\[
\|u_\varepsilon\|_{L^2(0, T; W^{1,2}(\Omega_\varepsilon; \mathbb{R}^3))} \leq c,
\]

(3.5)

uniformly for \( \varepsilon \to 0 \).

### 3.2 Convergence

As the family \( \{\Omega_\varepsilon\}_{\varepsilon > 0} \) possesses the uniform extension property (2.10), we may assume that

\[
u_\varepsilon \to U \text{ weakly in } L^2(0, T; W^{1,2}(\mathbb{R}^3; \mathbb{R}^3)).
\]

(3.6)

Moreover, by virtue of the uniform bounds (3.2), (3.5), we can perform the limit in (2.6) to obtain

\[
\text{div}_x U = 0 \text{ a.a. in } (0, T) \times \Omega,
\]

and, similarly, one can pass to the limit in the weak formulation of momentum equation (2.7) to deduce that

\[
\int_0^T \int_{\Omega} (\bar{\varrho} U \cdot \partial t \varphi + \bar{\varrho} \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi) \, dx \, dt
\]

\[
\quad = \mu \int_0^T \int_{\Omega} \nabla_x U : \nabla_x \varphi \, dx \, dt - \int_{\Omega} \bar{\varrho} U_0 \cdot \varphi(0, \cdot) \, dx,
\]

for any test function \( \varphi \in C_0^\infty([0, T) \times \Omega; \mathbb{R}^3) \) satisfying \( \text{div}_x \varphi = 0 \), where the symbol \( \bar{\varrho} \mathbf{u} \otimes \mathbf{u} \) denotes a weak limit of \( \{\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon\}_{\varepsilon > 0} \). Accordingly, in order to finish the proof of Theorem 2.1 we have to show

\[
\bar{\varrho} \mathbf{u} \otimes \mathbf{u} = \bar{\varrho} U \otimes U,
\]
or, equivalently, the strong convergence of the velocities claimed in (2.23). This will be our goal in the remaining part of the paper. We remark that, by virtue of (2.12 - 2.14), it is easy to check that the the limit velocity field satisfies the impermeability condition

\[ U \cdot n|_{\partial \Omega} = 0, \]

in a weak sense.

4 The acoustic equation

In this section, we introduce a weak formulation of the acoustic equation (1.12 - 1.14) and discuss its basic properties.

4.1 Weak formulation

With

\[ r_\varepsilon \equiv \frac{\rho_\varepsilon - \overline{\rho}}{\varepsilon}, \quad V_\varepsilon \equiv \rho_\varepsilon u_\varepsilon, \]

the Navier-Stokes system (2.6), (2.7) can be written in the form:

\[
\int_0^T \int_{\Omega_\varepsilon} (\varepsilon r_\varepsilon \partial_t \varphi + V_\varepsilon \cdot \nabla_x \varphi) \, dx \, dt = - \int_{\Omega_\varepsilon} \varepsilon r_{0,\varepsilon} \varphi(0, \cdot) \, dx, \tag{4.1}
\]

for any \( \varphi \in C^\infty_c([0,T) \times \overline{\Omega}) \),

\[
\int_0^T \int_{\Omega_\varepsilon} (\varepsilon V_\varepsilon \cdot \partial_t \varphi + p'(\overline{\rho})r_\varepsilon \varphi) \, dx \, dt = - \int_{\Omega_\varepsilon} \varepsilon q_{0,\varepsilon} u_{0,\varepsilon} \cdot \varphi(0, \cdot) \, dx \tag{4.2}
\]

\[
+ \int_0^T \int_{\Omega_\varepsilon} \left( \varepsilon S(\nabla_x u_\varepsilon) : \nabla_x \varphi - \varepsilon \overline{\rho} \varepsilon u_\varepsilon \otimes u_\varepsilon : \nabla_x \varphi \right) \, dx \, dt
\]

\[
- \int_0^T \int_{\Omega_\varepsilon} \varepsilon \left( \frac{1}{\varepsilon^2} (p(\varepsilon) - p'\overline{\rho})(\varepsilon - \overline{\rho}) - p(\overline{\rho}) \right) \, dx \, dt,
\]

for any \( \varphi \in C^\infty_c([0,T) \times \overline{\Omega_\varepsilon}; R^3), \varphi \cdot n|_{\partial \Omega_\varepsilon} = 0. \)

Now, thanks to the slip boundary condition (1.4), observe that \( \nabla_x \Delta_{\varepsilon,N}^{-1}[\varphi] \), where \( \Delta_{\varepsilon,N} \) is the Neumann Laplacian in \( \Omega_\varepsilon \), is an admissible test function in (4.2). Consequently, we obtain

\[
\int_0^T \int_{\Omega_\varepsilon} (\varepsilon F_\varepsilon \partial_t \varphi - p'(\overline{\rho})r_\varepsilon \varphi) \, dx \, dt = - \int_{\Omega_\varepsilon} \varepsilon q_{0,\varepsilon} u_{0,\varepsilon} \cdot \nabla_x \Delta_{\varepsilon,N}^{-1}[\varphi(0, \cdot)] \, dx, \tag{4.3}
\]
for all $\varphi \in C^\infty_c([0,T] \times \overline{\Omega})$, where $\Phi_\varepsilon$ is the acoustic potential, meaning,

$$
\mathbf{V}_\varepsilon = \mathbf{H}_\varepsilon \mathbf{V}_\varepsilon + \nabla_x \Phi_\varepsilon,
$$

where $\mathbf{H}_\varepsilon$ denotes the standard Helmholtz projection in $\Omega_\varepsilon$. Note that for $\varphi \in C^\infty_c([0,T] \times \Omega) \cap C^\infty_c(\Omega_\varepsilon$) and $\varepsilon > 0$ fixed, the test function $\nabla_x \Delta_\varepsilon^{-1}[\varphi]$ is continuously differentiable in $\Omega_\varepsilon$ and belongs to the space $C^\infty_c([0,T]; W^{1,2}(\Omega_\varepsilon; \mathbb{R}^3))$.

The Helmholtz projection $\mathbf{H}_\varepsilon[v]$ of a function $v \in L^p(\Omega_\varepsilon; \mathbb{R}^3)$ is defined as

$$
\mathbf{H}_\varepsilon[v] = v - \nabla \Phi,
$$

where $\Phi \in D^{1,p}(\Omega_\varepsilon)$ is the unique solution of the problem

$$
\int_{\Omega_\varepsilon} \nabla_x \Phi \cdot \nabla_x \varphi \, dx = \int_{\Omega_\varepsilon} v \cdot \nabla_x \varphi \, dx \quad \text{for all } \varphi \in C^\infty_c(\overline{\Omega_\varepsilon}),
$$

in other words, at least formally,

$$
\Delta \Phi = \text{div}_x v \quad \text{in } \Omega_\varepsilon, \quad \nabla_x \Phi \cdot \mathbf{n}|_{\partial \Omega_\varepsilon} = v \cdot \mathbf{n}|_{\partial \Omega_\varepsilon}, \quad |\Phi| \to 0 \quad \text{for } |x| \to \infty.
$$

Here, the symbol $D^{1,p}(\Omega_\varepsilon)$ denotes the homogeneous Sobolev space - a completion of $C^\infty_c(\overline{\Omega_\varepsilon})$ with respect to the norm $\|\nabla_x \Phi\|_{L^p(\Omega_\varepsilon; \mathbb{R}^3)}$. We have Sobolev’s inequality

$$
\|\Phi\|_{L^q(\Omega_\varepsilon)} \leq c(p) \|\nabla_x \Phi\|_{L^p(\Omega_\varepsilon; \mathbb{R}^3)}, \quad q = \frac{3p}{3-p} \quad \text{for any } 1 \leq p < 3, \quad (4.4)
$$

for any $\Phi \in D^{1,p}(\Omega_\varepsilon)$, where, since $\{\Omega_\varepsilon\}_{\varepsilon>0}$ admits the uniform extension property $\text{(2.10)}$, the constant $c(p)$ is independent of $\varepsilon$.

Finally, the equation $\text{(1.1)}$ reads

$$
\int_0^T \int_{\Omega_\varepsilon} \left( \varepsilon \partial_t \varphi + \nabla_x \Phi_\varepsilon \cdot \nabla_x \varphi \right) \, dx \, dt = - \int_{\Omega_\varepsilon} \varepsilon r_\varepsilon \varphi(0, \cdot) \, dx \quad \text{(4.5)}
$$

for any $\varphi \in C^\infty\left([0,T] \times \overline{\Omega}\right)$. Equations $(4.3), (4.5)$ represent a weak formulation of the acoustic equation $(1.12), (1.13)$, with the Neumann boundary condition $(1.14)$ implicitly included through the class of test functions in $(1.5)$. 

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4.2 Uniform bounds, part I

It follows from the uniform bounds established in (3.1), (3.2), (3.4) that

\[ r_\varepsilon = [r_\varepsilon]_{\text{ess}} + [r_\varepsilon]_{\text{res}}, \]

satisfies

\[ \text{ess sup}_{t \in (0, T)} \| [r_\varepsilon]_{\text{ess}} \|_{L^2(\Omega_\varepsilon)} \leq c, \]  
(4.6)

and

\[ \text{ess sup}_{t \in (0, T)} \| [r_\varepsilon]_{\text{res}} \|_{L^q(\Omega_\varepsilon)} \leq \varepsilon \frac{2^q}{q} c \text{ for any } 1 \leq q < \min\{\gamma, 2\}. \]  
(4.7)

Similarly,

\[ V_\varepsilon = [V_\varepsilon]_{\text{ess}} + [V_\varepsilon]_{\text{res}}, \]

where, in accordance with (3.1), (3.2),

\[ \text{ess sup}_{t \in (0, T)} \| [V_\varepsilon]_{\text{ess}} \|_{L^2(\Omega_\varepsilon; \mathbb{R}^3)} = \text{ess sup}_{t \in (0, T)} \| \sqrt{\varrho_\varepsilon} u_\varepsilon \|_{L^2(\Omega_\varepsilon; \mathbb{R}^3)} \leq c, \]  
(4.8)

and

\[ \text{ess sup}_{t \in (0, T)} \| [V_\varepsilon]_{\text{res}} \|_{L^q(\Omega_\varepsilon; \mathbb{R}^3)} = \text{ess sup}_{t \in (0, T)} \| \sqrt{\varrho_\varepsilon} u_\varepsilon \|_{L^q(\Omega_\varepsilon; \mathbb{R}^3)} \leq c \varepsilon^{1/\gamma}, \quad q = \frac{2\gamma}{\gamma + 1}. \]  
(4.9)

It remains to find suitable bounds on the forcing terms in acoustic equation (4.3 - 4.5).

To this end, we need the elliptic estimates for the Neumann Laplacian \( \Delta_{\varepsilon, N} \) discussed in the next section.

4.3 Elliptic estimates and Helmholtz decomposition in \( \Omega_\varepsilon \)

In order to control the forcing terms as well as the initial data in the acoustic equation, we need bounds on \( \nabla^2 v \) in terms of \( \Delta_{\varepsilon, N}[v] \). As the curvature, represented by the radius of the balls in (2.9), is not uniformly bounded, the \( W^{2, p} \)–elliptic bounds may “blow-up” for \( \varepsilon \to 0 \).

4.3.1 \( W^{2, p} \)–bounds

In order to obtain \( W^{2, p} \)–bounds, we consider the rescaled family of domains

\[ \tilde{\Omega}_\varepsilon \equiv \frac{1}{\varepsilon^\beta} \Omega_\varepsilon, \]  
(4.10)
where the exponent $\beta > 0$ is the same as in (2.9). Accordingly, the rescaled domains $\hat{\Omega}_\varepsilon$ are of uniform $C^2$-class, in particular, the standard elliptic theory yields

$$\|\nabla^2_x v\|_{L^p(\hat{\Omega}_\varepsilon; \mathbb{R}^3)} \leq c(p) (\|\Delta_x v\|_{L^p(\hat{\Omega}_\varepsilon)} + \|v\|_{L^p(\hat{\Omega}_\varepsilon)})$$

for any $v \in C^\infty_c(\hat{\Omega}_\varepsilon)$ satisfying $\nabla_x v \cdot n|_{\partial \hat{\Omega}_\varepsilon} = 0$. It is important to notice that, by virtue of the hypotheses introduced in Section 2.2, the constant $c(p)$ depends only on the rescaled radius $c_b$ of the balls appearing in (2.9).

Consequently, returning to the original domains $\Omega_\varepsilon$ we may infer that

$$\|\nabla^2_x v\|_{L^p(\Omega_\varepsilon; \mathbb{R}^3)} \leq \varepsilon^{-\beta \left(\frac{3}{2} - \frac{3}{p}\right)} c(p) \|v\|_{L^p(\Omega_\varepsilon)}$$

for any $v \in C^\infty_c(\Omega_\varepsilon)$ such that $\nabla_x v \cdot n|_{\partial \Omega_\varepsilon} = 0$, with $c(p)$ independent of $\varepsilon \to 0$.

### 4.3.2 Helmholtz decomposition

Consider the family of rescaled domains $\hat{\Omega}_\varepsilon$ introduced in (4.10), with the associated Helmholtz projections $\hat{\mathbb{H}}_\varepsilon$. By virtue of the result by Farwig, Kozono, and Sohr [12], we have

$$\|\hat{\mathbb{H}}_\varepsilon[v]\|_{(L^p \cap L^2)(\hat{\Omega}_\varepsilon; \mathbb{R}^3)} \leq c(p) \|v\|_{(L^p \cap L^2)(\hat{\Omega}_\varepsilon; \mathbb{R}^3)}$$

for any $2 \leq p < \infty$, (4.13)

where, similarly to the preceding part, the constant $c(p)$ depends only on $c_b$. Going back to the original domain $\Omega_\varepsilon$ we therefore obtain

$$\|\mathbb{H}_\varepsilon[v]\|_{(L^p \cap L^2)(\Omega_\varepsilon; \mathbb{R}^3)} \leq \varepsilon^{-\beta \left(\frac{3}{2} - \frac{3}{p}\right)} c(p) \|v\|_{(L^p \cap L^2)(\Omega_\varepsilon; \mathbb{R}^3)}$$

for any $2 \leq p < \infty$, (4.14)

uniformly for $\varepsilon \to 0$.

Similarly, by means of a duality argument,

$$\|\mathbb{H}_\varepsilon[v]\|_{(L^p + L^2)(\Omega_\varepsilon; \mathbb{R}^3)} \leq \varepsilon^{-\beta \left(\frac{3}{p} - \frac{3}{2}\right)} c(p) \|v\|_{(L^p + L^2)(\Omega_\varepsilon; \mathbb{R}^3)}$$

for any $1 < p < 2$. (4.15)

The estimates (4.14), (4.15) arise also in problems of homogenization and need not be optimal, cf. Masmoudi [33], [34]. On the other hand, as the limit domain $\Omega$ may be only Lipschitz, it is not surprising that the $L^p$-bounds in (4.14), (4.15) blow up for $\varepsilon \to 0$.

### 4.4 Uniform bounds, part II

With the bounds established in the previous section at hand, we are able to control the forcing terms in the acoustic equation (4.3). To begin, relation (4.12) implies that

$$\left| \int_{\Omega_\varepsilon} \mathcal{S}(\nabla_x u_\varepsilon) : \nabla^2_x \Delta^{-1}_{\varepsilon,N}[\varphi] \, dx \right|$$
\[ \leq c \| S(\nabla_x u_\varepsilon) \|_{L^2(\Omega_\varepsilon; R^{3\times3})}\left( \| \varphi \|_{L^2(\Omega_\varepsilon)} + \frac{1}{\varepsilon^{2\beta}} \| (-\Delta_{\varepsilon,N})^{-1}[\varphi] \|_{L^2(\Omega_\varepsilon)} \right), \]

therefore, by means of the Riesz representation theorem,

\[ \int_0^T \int_{\Omega_\varepsilon} S(\nabla_x u_\varepsilon) : \nabla_x^2 \Delta_{\varepsilon,N}^{-1}[\varphi] \, dx \, dt = \frac{1}{\varepsilon^{2\beta}} \int_0^T \int_{\Omega_\varepsilon} \left( F_\varepsilon^1 \varphi + F_\varepsilon^2 (-\Delta_{\varepsilon,N})^{-1}[\varphi] \right) \, dx \, dt, \tag{4.16} \]

where

\[ \| F_\varepsilon^i \|_{L^2((0,T)\times\Omega_\varepsilon)} \leq c, \quad i = 1, 2, \text{ uniformly for } \varepsilon \to 0. \tag{4.17} \]

Similarly, we can write

\[ \int_{\Omega_\varepsilon} \partial_\varepsilon u_\varepsilon \otimes u_\varepsilon : \nabla_x^2 \Delta_{\varepsilon,N}^{-1}[\varphi] \, dx \]

\[ = \int_{\Omega_\varepsilon} [\partial_\varepsilon]_{\text{ess}} u_\varepsilon \otimes u_\varepsilon : \nabla_x^2 \Delta_{\varepsilon,N}^{-1}[\varphi] \, dx + \int_{\Omega_\varepsilon} [\sqrt{\partial_\varepsilon}]_{\text{res}} \sqrt{\partial_\varepsilon} u_\varepsilon \otimes u_\varepsilon : \nabla_x^2 \Delta_{\varepsilon,N}^{-1}[\varphi] \, dx, \]

where, by virtue of (4.12),

\[ \left| \int_{\Omega_\varepsilon} [\partial_\varepsilon]_{\text{ess}} u_\varepsilon \otimes u_\varepsilon : \nabla_x^2 \Delta_{\varepsilon,N}^{-1}[\varphi] \, dx \right| \]

\[ \leq \|[\partial_\varepsilon]_{\text{ess}} u_\varepsilon \|_{L^2(\Omega_\varepsilon; R^3)} \| u_\varepsilon \|_{L^6(\Omega_\varepsilon; R^3)} \| \nabla_x^2 \Delta_{\varepsilon,N}^{-1}[\varphi] \|_{L^3(\Omega_\varepsilon; R^{3\times3})} \]

\[ \leq c \|[\partial_\varepsilon]_{\text{ess}} u_\varepsilon \|_{L^2(\Omega_\varepsilon; R^3)} \| u_\varepsilon \|_{L^6(\Omega_\varepsilon; R^3)} \left( \| \varphi \|_{L^3(\Omega_\varepsilon)} + \frac{1}{\varepsilon^{2\beta}} \| (-\Delta_{\varepsilon,N})^{-1}[\varphi] \|_{L^3(\Omega_\varepsilon)} \right), \]

and, by interpolation, the uniform extension property, and Sobolev’s inequality,

\[ \| \varphi \|_{L^3(\Omega_\varepsilon)} \leq c_1 \left( \| \varphi \|_{L^2(\Omega_\varepsilon)} + \| \varphi \|_{L^6(\Omega_\varepsilon)} \right) \]

\[ \leq c_2 \left( \| \varphi \|_{L^2(\Omega_\varepsilon)} + \| \nabla_x \varphi \|_{L^2(\Omega_\varepsilon)} \right) = c_2 \left( \| \varphi \|_{L^2(\Omega_\varepsilon)} + \| (-\Delta_{\varepsilon,N})^{1/2}[\varphi] \|_{L^2(\Omega_\varepsilon)} \right), \]

while, by the same token,

\[ \| (-\Delta_{\varepsilon,N})^{-1}[\varphi] \|_{L^3(\Omega_\varepsilon)} \leq c_1 \left( \| (-\Delta_{\varepsilon,N})^{-1}[\varphi] \|_{L^2(\Omega_\varepsilon)} + \| (-\Delta_{\varepsilon,N})^{-1/2}[\varphi] \|_{L^2(\Omega_\varepsilon)} \right). \]

Consequently, there exist functions \( F_\varepsilon^i, \ i = 3, \ldots, 6, \)

\[ \| F_\varepsilon^i \|_{L^2((0,T)\times\Omega_\varepsilon)} \leq c, \quad i = 3, \ldots, 6 \text{ uniformly for } \varepsilon \to 0, \tag{4.18} \]

such that

\[ \int_0^T \int_{\Omega_\varepsilon} [\partial_\varepsilon]_{\text{ess}} u_\varepsilon \otimes u_\varepsilon : \nabla_x^2 \Delta_{\varepsilon,N}^{-1}[\varphi] \, dx \, dt \tag{4.19} \]
\[
- \frac{1}{\epsilon^{2\beta}} \int_0^T \int_{\Omega_\epsilon} \left( F_\epsilon^3 \varphi + F_\epsilon^4 (-\Delta_{\epsilon,N})^{-1/2}[\varphi] + F_\epsilon^5 (-\Delta_{\epsilon,N})^{1/2}[\varphi] + F_\epsilon^6 (-\Delta_{\epsilon,N})^{-1}[\varphi] \right) \, dx \, dt.
\]

Furthermore, in accordance with (4.9),
\[
\left| \int_{\Omega_\epsilon} \theta_\epsilon \otimes u_\epsilon \cdot \nabla_x^2 \Delta_{\epsilon,N}^{-1}[\varphi] \, dx \right| \leq \epsilon^{1/\gamma} c \| u \|_{L^6(\Omega_\epsilon ; R^3)} \| \nabla_x^2 \Delta_{\epsilon,N}^{-1}[\varphi] \|_{L^r(\Omega_\epsilon ; R^{3 \times 3})},
\]
where, by virtue of (4.12),
\[
\| \nabla_x^2 \Delta_{\epsilon,N}^{-1}[\varphi] \|_{L^r(\Omega_\epsilon ; R^{3 \times 3})} \leq c \left( \frac{\| \varphi \|_{L^2(\Omega_\epsilon)} + \frac{1}{\epsilon^{2\beta}} \| (-\Delta_{\epsilon,N})^{-1}[\varphi] \|_{L^2(\Omega_\epsilon)} \right),
\]
with
\[
\frac{\gamma + \frac{1}{2}}{2\gamma} + \frac{1}{6} + \frac{1}{r} = 1.
\]

In what follows, we suppose \( \gamma < 2, r \geq 3 \) as, otherwise, the estimates are the same as in (4.19). Thus, applying once more (4.12), we obtain
\[
\| \varphi \|_{L^r(\Omega_\epsilon)} \leq c \left( \frac{1}{\epsilon^{2\beta}} \| (-\Delta_{\epsilon,N})^{-1}[\varphi] \|_{L^2(\Omega_\epsilon)} \right),
\]
and, similarly,
\[
\| (-\Delta_{\epsilon,N})^{-1}[\varphi] \|_{L^r(\Omega_\epsilon)} \leq c \left( \frac{1}{\epsilon^{2\beta}} \| (-\Delta_{\epsilon,N})^{-1}[\varphi] \|_{L^2(\Omega_\epsilon)} \right).
\]
Consequently, we get the same result as (4.19) provided \( \max \{ \frac{1}{\gamma}, \frac{1}{2} \} \geq 2\beta \), in particular if \( \beta < 1/4 \).

Thus, we may infer that
\[
\int_0^T \int_{\Omega_\epsilon} \theta_\epsilon \otimes u_\epsilon \cdot \nabla_x^2 \Delta_{\epsilon,N}^{-1}[\varphi] \, dx \, dt = \frac{1}{\epsilon^{2\beta}} \int_0^T \int_{\Omega_\epsilon} \left( F_\epsilon^3 \varphi + F_\epsilon^4 (-\Delta_{\epsilon,N})^{-1/2}[\varphi] + F_\epsilon^5 (-\Delta_{\epsilon,N})^{1/2}[\varphi]
\quad + F_\epsilon^6 (-\Delta_{\epsilon,N})^{-1}[\varphi] + F_\epsilon^7 (-\Delta_{\epsilon,N})^{-1}[\varphi] \right) \, dx \, dt,
\]
where \( F_\epsilon^i \) satisfy (4.18). Note that the same symbol \( F_\epsilon^i \) may stand for different functions than above.

Finally,
\[
\left| \int_{\Omega_\epsilon} \left( \frac{1}{2} \left( p(\theta_\epsilon) - p'(\overline{p})(\theta_\epsilon - \overline{p}) - p(\overline{p}) \right) \varphi \right) \, dx \right|
\]
\[
\leq \left\| \frac{1}{\epsilon^2} \left( p(\varrho_\epsilon) - p'(\varrho_\epsilon)(\varrho_\epsilon - \varrho) - \varrho \right) \right\|_{L^1(\Omega_\epsilon)} \| \varphi \|_{L^\infty(\Omega_\epsilon)},
\]
where, in accordance with the uniform extension property,
\[
\| \varphi \|_{L^\infty(\Omega_\epsilon)} \leq c \left( \| \nabla_x \varphi \|_{L^6(\Omega_\epsilon; R^3)} + \| \varphi \|_{L^6(\Omega_\epsilon)} \right),
\]
and, by virtue of (4.12),
\[
\| \nabla^2_x \varphi \|_{L^2(\Omega_\epsilon; R^{3\times3})} \leq c \left( \left\| (-\Delta_{\epsilon,N})^{1/2} \varphi \right\|_{L^2(\Omega_\epsilon)} + \frac{1}{\epsilon^{2\beta}} \| \varphi \|_{L^2(\Omega_\epsilon)} \right);
\]
\[
\| \nabla_x \varphi \|_{L^2(\Omega_\epsilon; R^3)} = \left\| (-\Delta_{\epsilon,N})^{1/2} \varphi \right\|_{L^2(\Omega_\epsilon)}.
\]

Seeing that, by virtue of the uniform bounds established (3.2), (3.4),
\[
\text{ess sup}_{t \in (0,T)} \left\| \frac{1}{\epsilon^2} \left( p(\varrho_\epsilon) - p'(\varrho_\epsilon)(\varrho_\epsilon - \varrho) - \varrho \right) \right\|_{L^1(\Omega_\epsilon)} \leq c,
\]
we conclude that
\[
\int_0^T \int_{\Omega_\epsilon} \left( \frac{1}{\epsilon^2} \left( p(\varrho_\epsilon) - p'(\varrho_\epsilon)(\varrho_\epsilon - \varrho) - \varrho \right) \varphi \right) \, dx \, dt \quad (4.21)
\]
\[
= \frac{1}{\epsilon^{2\beta}} \int_0^T \int_{\Omega_\epsilon} \left( F^8_{\epsilon} \varphi + F^9_{\epsilon} (-\Delta_{\epsilon,N})^{1/2} \varphi + F^{10}_{\epsilon} (-\Delta_{\epsilon,N}) \varphi \right) \, dx \, dt,
\]
where
\[
\| F^i_{\epsilon} \|_{L^2((0,T) \times \Omega_\epsilon)} \leq c, \quad i = 8, 9, 10 \text{ uniformly for } \epsilon \to 0. \quad (4.22)
\]

### 4.5 The acoustic equation revisited

Using relations (4.8), (4.9), together with (4.15), we can write the acoustic potential $\Phi_{\epsilon}$ in the form
\[
\Phi_{\epsilon} = \Phi^1_{\epsilon} + \Phi^2_{\epsilon},
\]
where
\[
\text{ess sup}_{t \in (0,T)} \| \Phi^1_{\epsilon} \|_{L^{1,2}(\Omega_\epsilon; R^3)} \leq c, \quad (4.23)
\]
\[
\text{ess sup}_{t \in (0,T)} \| \Phi^2_{\epsilon} \|_{L^2(\Omega_\epsilon; R^3)} \to 0 \text{ as } \epsilon \to 0. \quad (4.24)
\]
In view of the uniform bounds obtained in the previous section, the acoustic equation (4.3), (4.5) can be written in the concise form

$$
\int_0^T \int_{\Omega_\varepsilon} \left( \varepsilon \partial_t \varphi + \nabla_x \Phi_\varepsilon \cdot \nabla_x \varphi \right) \, dx \, dt = - \int_{\Omega_\varepsilon} \varepsilon \rho_{0,\varepsilon} \varphi(0, \cdot) \, dx
$$

(4.25)

for any $\varphi \in C^\infty_c([0,T) \times \Omega), and

$$
\int_0^T \int_{\Omega_\varepsilon} \left( \varepsilon \Phi_\varepsilon \partial_t \varphi - \rho'(\varepsilon) r_{\varepsilon} \varphi \right) \, dx \, dt = - \int_{\Omega_\varepsilon} \varepsilon \Phi_{0,\varepsilon} \varphi(0, \cdot) \, dx
$$

(4.26)

$$
+ \varepsilon^{1-2\beta} \int_0^T \int_{\Omega} \left( G_1^1 \varphi + G_2^2 (-\Delta_{\varepsilon,N})^{-1/2} [\varphi] + G_3^3 (-\Delta_{\varepsilon,N})^{1/2} [\varphi] 
+ G_4^4 (-\Delta_{\varepsilon,N}) [\varphi] + G_5^5 (-\Delta_{\varepsilon,N})^{-1} [\varphi] \right) \, dx \, dt,
$$

for all $\varphi \in C^\infty_c([0,T) \times \Omega), \nabla_x \varphi \cdot n_{\partial \Omega_\varepsilon} = 0$, where

$$
\| r_{0,\varepsilon} \|_{L^2(\Omega_\varepsilon)} + \| (-\Delta_{\varepsilon,N})^{-1/2} [\Phi_{0,\varepsilon}] \|_{L^2(\Omega_\varepsilon)} \leq c,
$$

(4.27)

$$
\| G_i^i \|_{L^2(\Omega_\varepsilon)} \leq c, \quad i = 1, \ldots, 5,
$$

(4.28)

uniformly for $\varepsilon \to 0$.

5 Spectral analysis of Neumann Laplacian on varying domains

As observed, the Neumann Laplacian $\Delta_{\varepsilon,N}$ plays a crucial role in the analysis of acoustic waves. We recall that $-\Delta_{\varepsilon,N}$ may be viewed as a non-negative self-adjoint operator on the space $L^2(\Omega_\varepsilon)$, with

$$
\mathcal{D}(-\Delta_{\varepsilon,N}) = \left\{ w \in W^{1,2}(\Omega_\varepsilon) \mid \int_{\Omega_\varepsilon} \nabla_x w \cdot \nabla_x \varphi \, dx = \int_{\Omega_\varepsilon} g \varphi \, dx \right\}
$$

for all $\varphi \in C^\infty_c(\Omega_\varepsilon)$ and a certain $g \in L^2(\Omega_\varepsilon)$, $-\Delta_{\varepsilon,N} w = g$.

Since the boundaries $\partial \Omega_\varepsilon$ are regular, the standard elliptic theory yields

$$
\mathcal{D}(-\Delta_{\varepsilon,N}) = \left\{ w \in W^{2,2}(\Omega_\varepsilon) \mid \nabla_x w \cdot n|_{\partial \Omega_\varepsilon} = 0 \right\}.
$$

We denote by $\{P_{\varepsilon,\lambda}\}_{\lambda \geq 0}$ the associated family of spectral projections. The following analysis is a slight modification of [14 Section 2], similar problems were studied by Rauch and Taylor [38].
5.1 Spectral measures

Our goal is to obtain dispersive estimates, and, in particular, local decay of acoustic waves, with a rate independent of the scaling parameter $\varepsilon$. To this end, we introduce the spectral measure $\mu_{\varepsilon, \varphi}$ associated to a function $\varphi \in L^2(\Omega_\varepsilon)$ through Stone’s formula (see Reed and Simon [39, Theorem VII.13])

$$
\mu_{\varepsilon, \varphi}(a,b) = \lim_{\delta \to 0+} \lim_{\eta \to 0+} \int_{a+\delta}^{b-\delta} \left\langle \left( \frac{1}{-\Delta_{\varepsilon,N} - \lambda + i\eta} - \frac{1}{-\Delta_{\varepsilon,N} - \lambda - i\eta} \right) [\varphi]; \varphi \right\rangle_{\Omega_\varepsilon} d\lambda,
$$

where the symbol $\langle \cdot; \cdot \rangle_{\Omega_\varepsilon}$ denotes the standard (complex) scalar product in $L^2(\Omega_\varepsilon)$.

Now, the crucial observation is that we may perform the limit $\eta \to 0+$ in (5.1) as soon as $\varphi \in C^\infty_c(\Omega_\varepsilon)$ since the operators $-\Delta_{\varepsilon,N}$ satisfy the limiting absorption principle, see Leis [26], Vainberg [41, Chapter VIII]:

**Limiting absorption principle (LAP)**

The operators $(1 + |x|^2)^{-s/2} \circ (-\Delta_{\varepsilon,N} - \lambda \pm i\eta)^{-1} \circ (1 + |x|^2)^{-s/2}$ are bounded on $L^2(\Omega_\varepsilon)$ for any $s > 1$ uniformly for $\lambda$ belonging to compact subsets of $(0, \infty)$ and $\eta > 0$.

Consequently, the spectral measure $\mu_{\varepsilon, \varphi}$, $\varphi \in C^\infty_c(\Omega_\varepsilon)$, is absolutely continuous with respect to the Lebesgue measure on $(0, \infty)$ and Stone’s formula (5.1) reduces to

$$
\mu_{\varepsilon, \varphi}(a,b) = \int_a^b \left\langle \left( w^-_{\varepsilon,\lambda} - w^+_{\varepsilon,\lambda} \right); \varphi \right\rangle_{\Omega_\varepsilon} d\lambda, \quad 0 < a < b,
$$

where $w_{\varepsilon,\lambda}^\pm$ are solutions of the Neumann problem

$$
\Delta w_{\varepsilon,\lambda}^\pm + \lambda w_{\varepsilon,\lambda}^\pm = \varphi \text{ in } \Omega_\varepsilon, \quad \nabla_x w_{\varepsilon,\lambda}^\pm \cdot n|_{\partial \Omega_\varepsilon} = 0,
$$

uniquely determined by Sommerfeld’s radiation condition

$$
\lim_{r \to \infty} r \left( \partial_r \pm i\sqrt{\lambda} \right) w_{\varepsilon,\lambda}^\pm = 0, \quad r \equiv |x|,
$$

see Vainberg [41, Chapter VIII].

Our goal is a uniform bound on the norm of the functions $w_{\varepsilon,\lambda}^\pm$ independent of the scaling parameter $\varepsilon$. To this end, we fix $R > 2d$ so that $\partial \Omega_\varepsilon \subset B_{R/2}$ for all $\varepsilon$. Following [14] we recall the explicit formula for the exterior Dirichlet problem

$$
\Delta v_{\varepsilon,\lambda}^\pm + \lambda v_{\varepsilon,\lambda}^\pm = 0 \text{ in } R^3 \setminus B_R, \quad v_{\varepsilon,\lambda}^\pm|_{\partial B_R} = \tilde{v}_{\varepsilon,\lambda}^\pm,
$$

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supplemented with the radiation condition (5.4), namely
\[ v^{\pm}_{\varepsilon,\lambda}(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_l^m Y_l^m(\theta, \phi) \frac{h_l^{(1)}(\pm \sqrt{\lambda} r)}{h_l^{(1)}(\pm \sqrt{\lambda} R)} \] for all \( x \in R^3 \setminus B_R \), \hspace{1cm} (5.5)

where \((r, \theta, \phi)\) are polar coordinates, \(Y_l^m\) are spherical harmonics of order \(l\), \(h_l^{(1)}\) are spherical Bessel functions, and
\[ \tilde{v}^{\pm}_{\varepsilon,\lambda}(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_l^m Y_l^m(\theta, \phi) \] for \( |x| = R \), see Chandler-Wilde and Monk [10], Nédélec [36].

Assume that \( \varphi \in C^\infty_c(\Omega_\varepsilon) \) in (5.3) is such that
\[ \text{supp}[\varphi] \subset B_R. \]

We claim that the functions \( w^{\pm}_{\varepsilon,\lambda} \) solving (5.3), (5.4) admit a uniform bound
\[ \| w^{\pm}_{\varepsilon,\lambda} \|_{L^2(B_3R \cap \Omega_\varepsilon)} \leq c \| \varphi \|_{L^2(\Omega_\varepsilon)}, \hspace{1cm} (5.6) \]
with \( c \) independent of \( \varepsilon \to 0 \), provided \( \lambda \) belongs to a compact subinterval of \((0, \infty)\). In order to see (5.6), we argue by contradiction assuming the existence of sequences
\[ \varphi_\varepsilon \in C^\infty_c(B_R \cap \Omega_\varepsilon), \hspace{1cm} \| \varphi_\varepsilon \|_{L^2(B_3R \cap \Omega_\varepsilon)} = 1, \hspace{1cm} \lambda_\varepsilon \to \lambda \in (0, \infty), \]
such that the corresponding (unique) solutions \( w^{\pm}_{\varepsilon,\lambda_\varepsilon} \) of (5.3), (5.4) satisfy
\[ \| w^{\pm}_{\varepsilon,\lambda_\varepsilon} \|_{L^2(B_3R \cap \Omega_\varepsilon)} \to \infty \text{ for } \varepsilon \to 0. \]

Setting
\[ v^{\pm}_{\varepsilon,\lambda_\varepsilon} = \frac{w^{\pm}_{\varepsilon,\lambda_\varepsilon}}{\| w^{\pm}_{\varepsilon,\lambda_\varepsilon} \|_{L^2(B_3R \cap \Omega_\varepsilon)}}, \]
we check that \( v^{\pm}_{\varepsilon,\lambda_\varepsilon} \) is the unique solution of the problem
\[ \Delta v^{\pm}_{\varepsilon,\lambda} + \lambda_\varepsilon v^{\pm}_{\varepsilon,\lambda} = \frac{\varphi_\varepsilon}{\| w^{\pm}_{\varepsilon,\lambda_\varepsilon} \|_{L^2(B_3R \cap \Omega_\varepsilon)}} \text{ in } \Omega_\varepsilon, \hspace{1cm} \nabla v^{\pm}_{\varepsilon,\lambda} \cdot \mathbf{n}|_{\partial \Omega_\varepsilon} = 0, \]
satisfying
\[ \lim_{r \to \infty} r \left( \partial_r \pm i \sqrt{\lambda_\varepsilon} \right) v^{\pm}_{\varepsilon,\lambda_\varepsilon} = 0, \]

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and
\[ \|v_{\varepsilon, \lambda}^\pm\|_{L^2(B_{3R} \cap \Omega_\varepsilon)} = 1. \] (5.7)

Since
\[ \Delta v_{\varepsilon, \lambda}^\pm = -\lambda v_{\varepsilon, \lambda}^\pm \] in \( B_{3R} \setminus B_R \),
the standard elliptic estimates yield
\[ v_{\varepsilon, \lambda}^\pm \to v_\lambda^\pm \] in \( C^m(K) \) for any \( m \geq 0 \) and any compact \( K \subset B_{3R} \setminus \overline{B}_R \),
in particular,
\[ v_{\varepsilon, \lambda}^\pm \to v_\lambda^\pm \] in \( C^m(\partial B_{2R}) \) for any \( m \geq 0 \),
which, together with formula (5.5) yields
\[ v_{\varepsilon, \lambda}^\pm \to v_\lambda^\pm \] in \( C^m(\Omega_\varepsilon \cap B_{3R}) \) for any \( m \geq 0 \),
where \( v_\lambda^\pm \) satisfy
\[ \Delta v_\lambda^\pm + \lambda v_\lambda^\pm = 0 \] in \( \Omega_\varepsilon \cap B_{3R} \),
\[ \nabla_x v_\lambda^\pm \cdot n|_{\partial \Omega} = 0. \] (5.8)

Finally, we claim that \( v_\lambda^\pm \) satisfies
\[ \Delta v_\lambda^\pm + \lambda v_\lambda^\pm = 0 \] in \( \Omega \cap B_{3R} \),
\[ \nabla_x v_\lambda^\pm \cdot n|_{\partial \Omega} = 0. \] (5.9)

In order to see (5.9) observe that
\[ \int_{\Omega_\varepsilon} \left( \nabla_x v_{\varepsilon, \lambda}^\pm \cdot \nabla_x \psi - \lambda v_{\varepsilon, \lambda}^\pm \psi + \frac{\omega_{\varepsilon, \lambda}^\pm}{\|w_{\varepsilon, \lambda}^\pm\|_{L^2(B_{3R} \cap \Omega_\varepsilon)}} \right) d\varepsilon = 0 \] (5.10)
for any \( \psi \in C^\infty_c(\overline{\Omega} \cap B_{3R}) \). Since \( v_{\varepsilon, \lambda}^\pm \) are bounded by (5.7), we conclude that
\[ \|v_{\varepsilon, \lambda}^\pm\|_{W^{1,2}(B_{3R} \cap \Omega_\varepsilon)} \leq c, \] (5.11)
uniformly for \( \varepsilon \to 0 \).

As \( \Omega_\varepsilon \) possess the uniform extension property we may assume that (5.11) holds in \( R^3 \),
and, by virtue of (2.12), we may pass to the limit in (5.10) for each fixed \( \psi \) to obtain the desired conclusion (5.9). Moreover, by the same token, relation (5.7) implies that
\[ \|v_\lambda^\pm\|_{L^2(\Omega)} = 1. \] (5.12)
However, relations (5.8), (5.9) imply \( v_\lambda^\pm \equiv 0 \) in contrast with (5.12). Thus we have shown (5.6).
Summing up the previous discussion we may infer that
\[ 0 \leq \langle (w_{\varepsilon,\lambda}^+ - w_{\varepsilon,\lambda}^-); \varphi \rangle_{\Omega_{\varepsilon}} \leq c(a, b, \varphi) \text{ for any } 0 < a < b, \varphi \in C^\infty_c(\Omega_{\varepsilon}), \tag{5.13} \]
uniformly for \(\varepsilon \to 0\). Moreover, repeating the arguments of the proof of (5.6), we conclude that
\[ \lim_{\varepsilon \to 0} \langle (w_{\varepsilon,\lambda}^+ - w_{\varepsilon,\lambda}^-); \varphi \rangle_{\Omega_{\varepsilon}} = \langle (w_{\lambda}^+ - w_{\lambda}^-); \varphi \rangle_{\Omega} \text{ for any } \lambda > 0, \varphi \in C^\infty_c(\Omega), \tag{5.14} \]
where
\[ \Delta w_{\lambda}^\pm = \lambda w_{\lambda}^\pm \text{ in } \Omega, \quad \nabla_x w_{\lambda}^\pm \cdot n|_{\partial \Omega} = 0, \]
\[ \lim r \to \infty r \left( \partial_r \pm i\sqrt{\lambda} \right) w_{\lambda}^\pm = 0, \quad r \equiv |x|. \]

Seeing that
\[ \|\nabla_x \varphi\|_{L^2(\Omega_{\varepsilon}; R^3)} = \|\sqrt{-\Delta_{\varepsilon,N}}[\varphi]\|_{L^2(\Omega_{\varepsilon}; R^3)}, \tag{5.15} \]
and
\[ \langle G(-\Delta_{\varepsilon,N})[\varphi]; \varphi \rangle_{\Omega_{\varepsilon}} = \int_0^\infty G(\lambda) \, d\mu_{\varepsilon,\varphi} \]
for any \(\varphi \in C^\infty_c(\Omega_{\varepsilon})\), we deduce from (5.13), (5.14) that
\[ \|G(-\Delta_{\varepsilon,N})[\varphi]\|_{W^{1,2}(\Omega_{\varepsilon})} \leq c(G, \varphi), \tag{5.16} \]
and
\[ 1_{\Omega_{\varepsilon}}G(-\Delta_{\varepsilon,N})[\varphi] \to 1_{\Omega}G(-\Delta_{N})[\varphi] \text{ in } L^2(R^3), \tag{5.17} \]
for any \(G \in C^\infty_c(0, \infty), \varphi \in C^\infty_c(\Omega)\).

### 5.2 Uniform decay for \(|x| \to \infty\)

Our goal in this section is to establish the following decay estimate:

**Lemma 5.1** For \(G \in C^\infty_c(0, \infty), \varphi \in C^\infty_c(\Omega), \text{ supp}[\varphi] \subset B_R\), we have
\[ \int_{|x| \geq R} |x|^{2s} |G(\sqrt{-\Delta_{\varepsilon,N}})[\varphi]|^2 \, dx \leq c(G, s, R)\|\varphi\|^2_{L^2(\Omega)} \text{ for any } s \geq 0, \]
uniformly for \(\varepsilon \to 0\).
Proof: Extending $G$ as an even function on $R$ we have

$$G(\sqrt{-\Delta_{\delta,N}})[\varphi] = \frac{1}{2} \int_{-\infty}^{\infty} \hat{G}(t) \left( \exp(i\sqrt{-\Delta_{\delta,N}}t) + \exp(-i\sqrt{-\Delta_{\delta,N}}t) \right) [\varphi] \, dt,$$

where $\hat{G}$ stands for the Fourier transform of $G$.

Denoting

$$\|v\|_{s,R}^2 = \int_{|x|>R} |v|^2 |x|^{2s} \, dx,$$

we get

$$\left\| G(\sqrt{-\Delta_{\delta,N}})[\varphi] \right\|_{s,R} \leq \frac{1}{2} \int_{-\infty}^{\infty} |\hat{G}(t)| \left\| \left( \exp(i\sqrt{-\Delta_{\delta,N}}t) + \exp(-i\sqrt{-\Delta_{\delta,N}}t) \right) [\varphi] \right\|_{s,R} \, dt,$$

where

$$\left\| \left( \exp(i\sqrt{-\Delta_{\delta,N}}t) + \exp(-i\sqrt{-\Delta_{\delta,N}}t) \right) [\varphi] \right\|_{s,R}^2 = \int_{\Omega_{\delta}} \text{sgn}^+(|x|-R)|x|^{2s} \left| \left( \exp(i\sqrt{-\Delta_{\delta,N}}t) + \exp(-i\sqrt{-\Delta_{\delta,N}}t) \right) [\varphi] \right|^2 \, dx.$$

On the other hand, the wave operator

$$\left( \exp(i\sqrt{-\Delta_{\delta,N}}t) + \exp(-i\sqrt{-\Delta_{\delta,N}}t) \right) = 2 \cos(\sqrt{-\Delta_{\delta,N}}t)$$

admits a finite speed of propagation 1, specifically,

$$\text{supp} \left( \exp(i\sqrt{-\Delta_{\delta,N}}t) + \exp(-i\sqrt{-\Delta_{\delta,N}}t) \right) [\varphi] \subset B_{R+|t|};$$

whence

$$\int_{\Omega_{\delta}} \text{sgn}^+(|x|-R)|x|^{2s} \left| \left( \exp(i\sqrt{-\Delta_{\delta,N}}t) + \exp(-i\sqrt{-\Delta_{\delta,N}}t) \right) [\varphi] \right|^2 \, dx$$

$$\leq (|t|+R)^{2s} \int_{\Omega_{\delta}} \left| \left( \exp(i\sqrt{-\Delta_{\delta,N}}t) + \exp(-i\sqrt{-\Delta_{\delta,N}}t) \right) [\varphi] \right|^2 \, dx$$

$$= (|t|+R)^{2s} \|\varphi\|_{L^2(\Omega_{\delta})}^2.$$

As $G \in C_c^\infty(0, \infty)$, we have $(|t|+R)^s \hat{G} \in L^1(R)$ for any $s$, which completes the proof.

Q.E.D.
5.3 Functional calculus - decay estimates

Our ultimate goal in this section is the following result.

**Lemma 5.2** We have

$$\int_0^T \left| \left\langle \exp \left( i \sqrt{-\Delta_{\varepsilon,N}} t \varepsilon \right) \left[ \Psi, G(-\Delta_{\varepsilon,N})[\varphi] \right] \right\rangle_{\Omega_{\varepsilon}} \right|^2 dt \leq \varepsilon c(\varphi, G) \| \Psi \|^2_{L^2(\Omega_{\varepsilon})}$$

for any $\varphi \in C_c^\infty(\Omega)$, $\Psi \in L^2(\Omega_{\varepsilon})$, and any $G \in C_c^\infty(0, \infty)$.

**Proof:** We adapt the arguments of [14]. By virtue spectral theorem (see Reed and Simon [39, Chapter VIII]) we have

$$\left\langle \exp \left( i \sqrt{-\Delta_{\varepsilon,N}} t \varepsilon \right) \left[ \Psi, G(-\Delta_{\varepsilon,N})[\varphi] \right] \right\rangle_{\Omega_{\varepsilon}} = \int_0^\infty \exp \left( \frac{i \lambda t}{\varepsilon} \right) G(\lambda) \tilde{\Psi}_\varepsilon(\lambda) \, d\mu_{\varepsilon,\varphi}(\lambda),$$

where $\mu_{\varepsilon,\varphi}$ is the spectral measure associated to the function $\varphi$, and

$$\tilde{\Psi}_\varepsilon \in L^2(0, \infty; d\mu_{\varepsilon,\varphi}), \quad \| \tilde{\Psi}_\varepsilon \|_{L^2_{\mu_{\varepsilon,\varphi}}} \leq \| \Psi \|_{L^2(\Omega_{\varepsilon})}.$$

Following Last [25] we deduce

$$\int_0^T \left| \left\langle \exp \left( i \sqrt{-\Delta_{\varepsilon,N}} t \varepsilon \right) \left[ \Psi, G(-\Delta_{\varepsilon,N})[\varphi] \right] \right\rangle_{\Omega_{\varepsilon}} \right|^2 dt$$

$$\leq eT \sqrt{\pi} \int_0^\infty \int_0^\infty \tilde{\Psi}_\varepsilon(x) \tilde{\Psi}_\varepsilon(y) \exp \left( -\frac{T^2 |\sqrt{x} - \sqrt{y}|^2}{4\varepsilon^2} \right) G(x)G(y) \, d\mu_{\varepsilon,\varphi}(x) \, d\mu_{\varepsilon,\varphi}(y);$$

therefore, by Cauchy-Schwartz inequality,

$$\int_0^T \left| \left\langle \exp \left( i \sqrt{-\Delta_{\varepsilon,N}} t \varepsilon \right) \left[ \Psi, G(-\Delta_{\varepsilon,N})[\varphi] \right] \right\rangle_{\Omega_{\varepsilon}} \right|^2 dt$$

$$\leq eT \sqrt{\pi} \int_0^\infty |\Psi_\varepsilon(x)|^2 \left( \int_0^\infty \exp \left( -\frac{|\sqrt{x} - \sqrt{y}|^2 T^2}{4\varepsilon^2} \right) d\mu_{\varepsilon,\varphi}(y) \right) G^2(x) \, d\mu_{\varepsilon,\varphi}(x).$$

Furthermore,

$$\int_0^\infty \exp \left( -\frac{|\sqrt{x} - \sqrt{y}|^2 T^2}{4\varepsilon^2} \right) d\mu_{\varepsilon,\varphi}(y)$$

(5.18)

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\[
\begin{align*}
&= \sum_{n=0}^{\infty} \int_{\varepsilon n \leq |\sqrt{y} - \sqrt{x}| < \varepsilon(n+1)} \exp \left( -\frac{|\sqrt{x} - \sqrt{y}|^2 T^2}{\varepsilon^2} \right) \, d\mu_{\varepsilon,\phi}(y) \\
&\leq \sup_{n \geq 0} \int_{\varepsilon n \leq |\sqrt{y} - \sqrt{x}| < \varepsilon(n+1)} 1 \, d\mu_{\varepsilon,\phi}(y) \sum_{n=0}^{\infty} \exp \left( -\frac{y^2 T^2}{4} \right).
\end{align*}
\]

Since only the points \(x\) belonging to \(\text{supp}[G]\) are relevant in (5.18), the length of the intervals

\[I_n(x) = \{y \in [0, \infty) \mid \varepsilon n \leq |\sqrt{y} - \sqrt{x}| < \varepsilon(n+1)\}\]

never exceeds \(\varepsilon\),

\[|I_n(x)| \leq c_G \varepsilon.\]

Thus we conclude combining (5.18), with the uniform bounds on the spectral measures (see (5.2)) established in (5.13).

Q.E.D.

6 Dispersive estimates, local decay of acoustic waves

Returning to the acoustic equation (4.25), (4.26) we can use the dispersive estimate established in Lemma 5.2 to show that

\[\Big\{ t \mapsto \int_{\Omega_{\varepsilon}} \Phi_{\varepsilon} G(-\Delta_{\varepsilon,N})[\phi] \, dx \Big\} \to 0 \text{ in } L^2(0, T) \quad (6.1)\]

for any fixed \(G \in C_\infty(0, \infty), \phi \in C_\infty^0(\Omega)\). Indeed, by means of Duhamel’s formula, we have

\[
\Phi_{\varepsilon}(t, \cdot) = \frac{1}{2} \exp \left( i \sqrt{-\Delta_{\varepsilon,N}} \frac{t}{\varepsilon} \right) \left[ \Phi_{0,\varepsilon} + \frac{1}{\sqrt{-\Delta_{\varepsilon,N}}} [r_{0,\varepsilon}] \right] + \frac{1}{2} \exp \left( -i \sqrt{-\Delta_{\varepsilon,N}} \frac{t}{\varepsilon} \right) \left[ \Phi_{0,\varepsilon} - \frac{1}{\sqrt{-\Delta_{\varepsilon,N}}} [r_{0,\varepsilon}] \right] + \varepsilon^{-2\beta} \frac{1}{2} \int_0^t \left( \exp \left( i \sqrt{-\Delta_{\varepsilon,N}} \frac{t-s}{\varepsilon} \right) + \exp \left( -i \sqrt{-\Delta_{\varepsilon,N}} \frac{t-s}{\varepsilon} \right) \right) [H_{\varepsilon}(s)] \, ds,
\]

with

\[H_{\varepsilon} = G_{\varepsilon}^1 + (-\Delta_{\varepsilon,N})^{-1/2}[G_{\varepsilon}^2] + (-\Delta_{\varepsilon,N})^{1/2}[G_{\varepsilon}^3] + (-\Delta_{\varepsilon,N})[G_{\varepsilon}^4] + (-\Delta_{\varepsilon,N})^{-1}[G_{\varepsilon}^5],\]

(see (4.26)), where we have assumed, for the sake of simplicity, that \(p'(\overline{p}) = 1\). Consequently, the desired conclusion (6.1) follows from Lemma 5.2 as soon as \(\beta < 1/4\), see [14] for details.
6.1 Compactness in time of the momenta

In view of estimate (3.5), the desired strong convergence claimed in (2.23) follows as soon as we show that

\[ \left\{ t \mapsto \int_{\Omega} V_\varepsilon(\cdot, t) \cdot w \, dx \right\} \to \left\{ t \mapsto \overline{\vartheta} \int_{\Omega} U(\cdot, t) \cdot w \, dx \right\} \text{ in } L^2(0, T) \text{ for } w \in C_c^\infty(\Omega; \mathbb{R}^3). \tag{6.2} \]

Indeed relation (6.2) with (3.6) imply that

\[ \int_T^0 \int_K \eta_\varepsilon^2 |u_\varepsilon|^2 \, dx \, dt \to \overline{\vartheta} \int_T^0 \int_K |U|^2 \, dx \, dt \text{ for any compact } K \subset \Omega, \]

yielding (2.23).

In order to show (6.2), we use Helmholtz decomposition to obtain

\[ \int_{\Omega_\varepsilon} V_\varepsilon \cdot w \, dx = \int_{\Omega_\varepsilon} H_\varepsilon[u_\varepsilon] \cdot w \, dx - \int_{\Omega_\varepsilon} \Phi_\varepsilon \text{div}_x w \, dx \]

\[ = \int_{\Omega_\varepsilon} \varrho_\varepsilon u_\varepsilon \cdot H_\varepsilon[w] \, dx - \int_{\Omega_\varepsilon} \Phi_\varepsilon \text{div}_x w \, dx \]

\[ = \int_{\Omega_\varepsilon} \varrho_\varepsilon u_\varepsilon \cdot H[w] \, dx + \int_{\Omega_\varepsilon} \varrho_\varepsilon u_\varepsilon \cdot (H_\varepsilon[w] - H[w]) \, dx - \int_{\Omega_\varepsilon} \Phi_\varepsilon \text{div}_x w \, dx, \]

where, in accordance with (4.2) and the standard Aubin-Lions argument,

\[ \left\{ t \mapsto \int_{\Omega_\varepsilon} \varrho_\varepsilon u_\varepsilon \cdot H[w] \, dx \right\} \to \left\{ t \mapsto \overline{\vartheta} \int_{\Omega} U \cdot w \, dx \right\} \text{ in } L^2(0, T). \tag{6.3} \]

Here, we have extended \( H[w] \) and \( H_\varepsilon[w] \) by zero outside \( \Omega \) and \( \Omega_\varepsilon \), respectively. Furthermore, we write

\[ \int_{\Omega_\varepsilon} \varrho_\varepsilon u_\varepsilon \cdot (H_\varepsilon[w] - H[w]) \, dx \]

\[ = \int_{\Omega_\varepsilon} (\varrho_\varepsilon - \overline{\vartheta}) u_\varepsilon \cdot (H_\varepsilon[w] - H[w]) \, dx + \overline{\vartheta} \int_{\Omega_\varepsilon} u_\varepsilon \cdot (H_\varepsilon[w] - H[w]) \, dx. \]

In view of estimates (4.6), (4.7), and boundedness of \( H_\varepsilon, H \) in \( L^p \) (see (4.14), (4.15)), we get

\[ \left\{ t \mapsto \int_{\Omega_\varepsilon} (\varrho_\varepsilon - \overline{\vartheta}) u_\varepsilon \cdot (H_\varepsilon[w] - H[w]) \, dx \right\} \to 0 \text{ in } L^2(0, T). \tag{6.4} \]

Moreover, as

\[ H_\varepsilon[w] \to H[w] \text{ weakly in } L^2(\Omega; \mathbb{R}^3), \]
we get, by virtue of (2.22),

\[ \left\{ t \mapsto \int_{\Omega} \mathbf{u}_\varepsilon \cdot (\mathbf{H}_\varepsilon[w] - \mathbf{H}[w]) \, dx \right\} \to 0 \text{ in } L^2(0, T). \]  

(6.5)

Finally,

\[ \int_{\Omega} \Phi_\varepsilon \text{div}_x w \, dx = \int_{\Omega} \Phi_\varepsilon G(-\Delta_\varepsilon,N) [\text{div}_x w] \, dx + \int_{\Omega} \Phi_\varepsilon (1 - G(-\Delta_\varepsilon,N)) [\text{div}_x w] \, dx, \]

where, as stated in (6.1),

\[ \left\{ t \mapsto \int_{\Omega} \Phi_\varepsilon G(-\Delta_\varepsilon,N) [\text{div}_x w] \, dx \right\} \to 0 \text{ in } L^2(0, T). \]  

(6.6)

Writing \( \Phi_\varepsilon = \Phi^1_\varepsilon + \Phi^2_\varepsilon \) as in (4.23), (4.24), we have

\[ \left\{ t \mapsto \int_{\Omega} \Phi^2_\varepsilon (1 - G(-\Delta_\varepsilon,N)) [\text{div}_x w] \, dx \right\} \to 0 \text{ in } L^2(0, T), \]  

(6.7)

while, in agreement with (5.17) and Lemma 5.1,

\[ \left\{ t \mapsto \int_{\Omega} \Phi^1_\varepsilon (1 - G(-\Delta_\varepsilon,N)) [\text{div}_x w] \, dx \right\} \]  

\[ \to \left\{ t \mapsto \int_{\Omega} \Phi^1 (1 - G(-\Delta_N)) [\text{div}_x w] \, dx \right\} \text{ in } L^2(0, T), \]  

(6.8)

where the resulting expression is small as soon as \( G \approx 1_{[0, \infty)} \). Indeed

\[ \int_{\Omega} \Phi^1 (1 - G(-\Delta_N)) [\text{div}_x w] \, dx = \int_{\Omega} (-\Delta_N)^{1/2} \Phi^1 \frac{1}{(-\Delta_N)^{1/2}} (1 - G(-\Delta_N)) [\text{div}_x w] \, dx, \]

where

\[ \| (-\Delta_N)^{1/2} [\Phi^1] \|_{L^2(\Omega)} = \| \nabla_x \Phi^1 \|_{L^2(\Omega)}, \]

while

\[ \left\| \frac{1}{(-\Delta_N)^{1/2}} [\text{div}_x w] \right\|_{L^2(\Omega)} = \left\| (-\Delta_N)^{1/2} (-\Delta_N)^{-1} [\text{div}_x w] \right\|_{L^2(\Omega)} \]

\[ = \left\| \nabla_x (-\Delta_N)^{-1} [\text{div}_x w] \right\|_{L^2(\Omega)} \leq \| w \|_{L^2(\Omega; R^3)}. \]

Relations (6.3 - 6.8) imply (6.2), in particular, we have shown (2.23). The proof of Theorem 2.1 is now complete.
6.2 Boundary behavior of the limit velocity field $U$

In the previous analysis, we left open the problem of the boundary conditions satisfied by the limit velocity field $U$. To this end, we revoke the results of [8]. Suppose that, after suitable translation and rotation of the coordinate system, a part $\Gamma$ of the boundary of $\Omega$ can be described by a graph of a function $b \in W^{1,\infty}(U)$, $U \subset \mathbb{R}^2$,

$$\Gamma = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in U, \ x_3 = b(x_1, x_2)\},$$

while $\Gamma_\varepsilon = \partial \Omega_\varepsilon \cap U \times \mathbb{R}$ are represented as

$$\Gamma_\varepsilon = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in U, \ x_3 = b_\varepsilon(x_1, x_2)\},$$

where $\{b_\varepsilon\}_{\varepsilon > 0}$ is a bounded sequence in $W^{1,\infty}(U)$, $b_\varepsilon \to b$ in $C(\overline{U})$. Similarly to [8], we assume that the boundaries $\Gamma_\varepsilon$ are oscillating for $\varepsilon \to 0$. More specifically, introducing a Young measure $\mathcal{R}[y]$, $y \in U$, associated to the gradients $\{\nabla_y b_\varepsilon\}_{\varepsilon > 0}$, we suppose that

$$\text{supp}[\mathcal{R}[y]] \text{ contains two independent vectors in } \mathbb{R}^2 \text{ for a.a. } y \in U. \quad (6.9)$$

As shown in [8], condition (6.9) implies

$$U|_{\Gamma} = 0,$$

see also Březina [5] for refined results in this direction.

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