MALLIAVIN CALCULUS FOR INFINITE-DIMENSIONAL SYSTEMS
WITH ADDITIVE NOISE

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ABSTRACT. We consider an infinite-dimensional dynamical system with polynomial nonlinearity and additive noise given by a finite number of Wiener processes. By studying how randomness is spread by the dynamics, we develop in this setting a partial counterpart of Hörmander’s classical theory of Hypoelliptic operators. We study the distributions of finite-dimensional projections of the solutions and give conditions that provide existence and smoothness of densities of these distributions with respect to the Lebesgue measure. We also apply our results to concrete SPDEs such as a Stochastic Reaction Diffusion Equation and the Stochastic 2D Navier–Stokes System.

1. INTRODUCTION

This paper investigates how randomness is spread by an infinite-dimensional nonlinear dynamical system forced by a finite number of independent Wiener processes. The randomness is transferred by the nonlinearity to degrees of freedom other than those where it is initially injected. It would be very interesting to obtain precise information on how the randomness is spread. We will instead show that some transfer happens almost surely. Though we are fundamentally interested in infinite-dimensional systems, we begin with a brief discussion in finite dimensions.

Consider a stochastic differential equation with additive noise:

\[ \begin{aligned}
\delta x &= F_0(x) \, dt + \sum_{k=1}^d F_k \, dW_k(t) \\
x_0 &= x \in \mathbb{R}^m
\end{aligned} \]

where the \( W_k \) are independent standard Brownian Motions, \( F_0 : \mathbb{R}^m \rightarrow \mathbb{R}^m \) is a bounded analytic function and \( F_k : \mathbb{R}^m \rightarrow \mathbb{R}^m \) is a fixed vector for each \( k \in \{1, \ldots, d\} \).

Given a function \( u_0 : \mathbb{R}^m \rightarrow \mathbb{R} \), we can define \( u(x; t) = \mathbb{P}_t u_0(x) = \mathbb{E}_x u_0(x) \).

(Here the notation for the expectation \( \mathbb{E}_x \) reinforces the fact that \( x_0 = x \).) Then \( u(x; t) \) solves the backward-Kolmogorov equation \( \partial_t u = Lu \) with \( u(x; 0) = u_0(x) \) and

\[ L = F_0 \cdot \nabla + \frac{1}{2} \sum_{k=1}^d (F_k \cdot \nabla)^2. \]

If \( \text{span} F \subseteq \mathbb{R}^m \), then the differential operator is uniformly elliptic. In this case, it is classical that \( u(x; t) \) is a smooth function of \( x; t \) and that

*Date: October, 2006.*
\( u(x,t) = R^n \int_{R^n} \tau(x;y)u_0(y)dy, \) where \( \tau(x;y) \) is a smooth, positive function of \( (t;x;y) \).

The function \( \tau(x;y) \) is called the density of \( x_t \) starting from \( x_0 = x \) (see [Bas98]).

The fact that \( \tau \) is smooth and positive is a direct consequence of the randomness spreading through all of the degrees of freedom.

If \( \dim \text{span} F_1; \ldots; F_m < m \), then the preceding conclusions do not necessarily hold. However, if

\[
\text{span } F_i; F_j; F_k ; F_k ; F_i ; F_j ; F_j ; F_0 = R^m ;
\]

then the system is hypoelliptic and the above conclusions again hold (save positivity). Here \( [F_j; F_k] = [F_j; F_k] = [F_j; F_k] = \ldots \) is the Lie bracket (or commutator) of the two vector fields. Since in our setting the \( F_j \) are constant for \( j \neq 1 \), only brackets with \( F_0 \) produce non-zero results. The fact that the system is hypoelliptic follows from Hörmander’s pioneering work. In particular, it falls under a generalization of his “sum of squares” theorem. (The principal part of \( L \) is the sum of squares of vector fields.)

In the 1970’s and 1980’s there was a large body of work to develop a probabilistic understanding of Hörmander’s theorem and related concepts by looking directly at (1) rather than the PDE for \( u_t(x;y) \) and \( u_x(x;y) \). This line of work was initiated by Malliavin and contained substantial contributions from Bismut, Stroock, Kusuoka, Norris and others. The tools developed to address this question go under the heading of Malliavin Calculus (see [Mal78, KS84, Bel87]) which might well be described as the stochastic calculus of variations.

We are interested in developing a version of these results in infinite dimensions (\( m = 1 \)). We wish to understand which of the previous conclusions hold if we assume that some variation on (2) holds with \( m = 1 \), where the SDE in (1) is replaced by a stochastic partial differential equation (SPDE). From the beginning, it is clear that we cannot work directly with the density \( \tau(x;y) \) since in infinite dimensions there is no Lebesgue measure. Ideally, we would like to find a natural replacement for Lebesgue measure in the setting of a given equation. For the moment this escapes us, so we will make statements about the finite-dimensional projection of the spectrum of the Malliavin covariance matrix (see Section 5). One might reasonably ask if we could ever expect some form of Hörmander’s condition to hold when the dimension \( m \) is infinite but the number of Brownian forcing terms is finite. In [EM01, Rom04], it was shown that the finite-dimensional Galerkin truncations of the Navier-Stokes equations satisfy Hörmander’s condition for hypoellipticity independent of the order of the truncation. And thus in some sense, Hörmander’s condition holds, at least formally, for the whole SPDE (\( m = 1 \)).

In [BT05], the authors treat the case when the infinite-dimensional (\( m = 1 \)) evolution generates a fully invertible flow and prove conditions guaranteeing the existence of a density of the finite-dimensional marginals. Because they assume the dynamics generate a flow, their exposition more closely mirrors the finite-dimensional treatment. In particular, they are able to handle diffusion constants which depend on the state of the process. We will see that our treatment will lead to objects not adapted to the Wiener filtration, which makes the general diffusion case more difficult.
Because our PDEs generate only a semi-flow (and not a full flow), we cannot apply directly the same proofs developed using Malliavin Calculus in the finite-dimensional setting or the infinite-dimensional extensions given in [BT05]. However we will see that we can modify the proofs to produce the desired results. D. Ocone [Oco88] first used related ideas in the infinite-dimensional case when the equations were linear in the solution and the noise; and hence, an explicit formula exists for the solution. In [MP06], the 2D Navier-Stokes equations are considered with additive noise. The techniques used there are very close to those used here. However, there the scope is more limited. The calculations are done in coordinates which leads to the restriction that the forcing is diagonal in the same basis. In both cases, as in [Oco88], the time reversed adjoint of the linear flow is used to propagate information backwards in time. This leads to a need for estimates on Wiener polynomials with non-adapted coefficients. In [MP06], only second-order polynomials were considered. Here, by simplifying and streamlining the proofs, we can handle general polynomials of finite order. This allows us to treat PDEs with more general polynomial nonlinearities. Lastly, we observe that if one is only interested in the existence of a density, one can jettison over two-thirds of the paper and all of the technically involved sections.

To further motivate this article, we mention that the type of quantitative estimates on the spectra of the Malliavin covariance matrix obtained in this paper is a critical ingredient in the recent proof of unique ergodicity of the two-dimensional Navier Stokes equations under the type of finite-dimensional forcing considered in this note (see [HM06]). The results of this paper are a major step towards proving similar results for other SPDEs.

In [EH01], the ergodicity of a degenerately forced SPDE was also proven using techniques from Malliavin calculus. In contrast to our setting, there infinitely many directions were forced stochastically. However, the structure of the forcing was such that it caused the asymptotic behavior for the high spatial modes to be close to that of an associated linear equation. The type of analysis used there does not seem to be possible in our setting.

Independently, and contemporaneously to this work M. Wu completed a thesis [Wu06] which carried out the program from [MP06] [HM06] to prove the unique ergodicity of a degenerately forced Boussinesq equation. Since this equation has a quadratic nonlinearity, he was able to use the technical lemmas from [MP06]. However, he also proved a more general technical lemma which can be used to prove the existence, but not smoothness, of finite-dimensional marginal distributions. He used this result to prove the existence of finite-dimensional marginal densities for a degenerately forced cubic reaction-diffusion equation. The technical lemma is similar to Lemma 5.1 and Proposition 5.2 from [MP06], though the proof given is slightly different. In this note, we have used the other, though related, approach from [MP06], since (at least for us) it is more straightforward to use it to obtain the quantitative estimates needed to prove smoothness.

While this paper was in its final stages of completion, the authors became aware of a recent preprint [AKSS] where it is proven that finite-dimensional projections of a randomly forced PDE’s Markov transition kernel are absolutely continuous.
with respect to Lebesgue measure if a certain controllability condition is satisfied. While connections between controllability and the existence of densities are not surprising given what is known for maps and SDEs (see [Kli87, BAL91] for example), the strength of the results in [AKSS] is that they do not require the forcing to be Gaussian. They only need that it satisfies a more general condition of decomposability. However, that approach presently does not provide smoothness of the densities.

Organization: In Section 2 we introduce the abstract setting for the rest of the paper. In Section 3 we give the main results of the paper in a simplified form which is sufficient for the applications we present. Principally, we give results ensuring the existence and smoothness of the finite-dimensional projections of the Markov transition kernels. In Section 4 we specialize the abstract framework and apply it to a scalar reaction-diffusion equation and the two-dimensional Navier–Stokes equation. In Section 5 we give a brief introduction to the ideas from Malliavin calculus we need. In Sections 6 and 7 we respectively state and prove the principal results in their full generality. In Section 8 we give a number of generalizations and refinements tailored to the needs of the arguments in [HM06] which prove the unique ergodicity of the system as already mentioned. The estimates on the spectrum of the Malliavin Covariance matrix in this paper constitute one of the principal ingredients of that work. In Section 9 we give the necessary abstract results on non-adapted polynomials of Wiener processes, one of the main technical tools of the paper. In the remaining two sections, we give a number of auxiliary lemmas needed in the proofs.

Acknowledgments: This work grew from a joint work of JCM with Étienne Pardoux whom he thanks for the many fruitful, interesting and educational discussions which laid the ground work for this work. YB thanks the hospitality of Duke University during the academic year 2004–2005 when the bulk of this work was done. The authors also thank Trevis Litherland, Scott McKinley, and Natesh Pillai for reading and commenting on a preliminary version of this paper. JCM was supported in part by the Sloan Foundation and by an NSF PECASE award DMS04-49910.

2. General Setting

In this section we introduce the framework to define and study solutions of a stochastic evolution equation in a Hilbert space

\[
\begin{align*}
&\dot{u}(t) = L(u)dt + N(u)dt + f(t)dt + \sum_{k=1}^{\infty} g_k \Delta W_k(t); \\
&u(0) = u_0.
\end{align*}
\]

(3)

The three components of the framework are: the space where solutions are to be defined; the deterministic part of the r.h.s. (the drift), namely, the autonomous part given by the vector field $L(u) + N(u)$ and the non-autonomous part $f$; and
We also assume that \( u(4) \) for all \( \in \) introduce the notation polynomial (defined below) with zero linear and constant part. It is convenient to compactly embedded and dense in \( t \) generated by inner products to live. We need two separable Hilbert spaces \( \text{H} \) and \( \text{V} \), with norms \( | \cdot | \) and \( \| \cdot \| \) generated by inner products \( \langle \cdot, \cdot \rangle \) and \( \langle \cdot, \cdot \rangle \) respectively. We assume that \( \text{V} \) is compactly embedded and dense in \( \text{H} \) so that \( \text{H} \) is compactly embedded and dense in \( \text{V}^0 \), the dual of \( \text{V} \). Hence \( \langle \cdot, \cdot \rangle \) also gives the duality pairing between \( \text{V} \) and \( \text{V}^0 \).

We also assume that \( \mathcal{V}j \) \( k \) for any \( v \in \mathcal{V} \).

We shall assume that there is a set \( \mathcal{H}_0 \subseteq \text{H} \) such that with probability one
\[
(4) \quad u(2) \subseteq \mathcal{D} \mathcal{T}; \mathcal{H} \quad \subseteq \mathcal{D} \mathcal{T}; \mathcal{V}^0
\]
for all \( u_0 \in \mathcal{H}_0 \).

The deterministic external force \( \mathcal{F} \) is a bounded \( \mathcal{H} \) -valued function defined on a time interval \( \mathcal{D} \mathcal{T} \). \( \mathcal{F} \) is a linear operator with values in \( \mathcal{V}^0 \) defined on a subspace of \( \mathcal{H} \). The restriction of \( \mathcal{F} \) to \( \mathcal{V} \) is a continuous operator \( \mathcal{V} \to \mathcal{V}^0 \).

The nonlinear vector field \( \mathcal{N} : \mathcal{V} \to \mathcal{H} \) will be assumed to be a continuous polynomial (defined below) with zero linear and constant part. It is convenient to introduce the notation
\[
u^j = (u_1; \ldots; u_j)
\]
Often, for a function \( \mathcal{Q} \) of \( j \) variables, we shall write
\[
\mathcal{Q}(u) = \mathcal{Q}(u^1) = \mathcal{Q}(u_1; \ldots; u_j);
\]

**Definition 2.1.** Given two Banach spaces \( X \) and \( Y \), we say that \( \mathcal{F} : X \to Y \) is a continuous polynomial of positive integer degree \( m \) if
\[
\mathcal{F}(x) = \mathcal{F}(x^m)
\]
for some map \( \mathcal{F} : X^m \to Y \) such that
\[
\mathcal{F}(x_1; \ldots; x_m) = \mathcal{F}_0 + \mathcal{F}_1(x_1) + \mathcal{F}_2(x_1, x_2) + \ldots + \mathcal{F}_m(x_1; \ldots; x_m);
\]
where all functions \( \mathcal{F}_j : X \to Y \) are multilinear (i.e., linear in each variable), symmetric (i.e., invariant under argument permutations), and continuous.

Hence our assumption on \( \mathcal{N} \) states that
\[
\mathcal{N}(u_1; \ldots; u_m) = N_2(u_1; u_2) + \ldots + N_m(u_1; \ldots; u_m);
\]
where all functions \( N_j : \mathcal{V}^j \to \mathcal{H} \) are multilinear, continuous, and symmetric. For notational convenience we will write \( \mathcal{F}(u) = \mathcal{L}(u) + \mathcal{N}(u) \).

Finally, our probability space is \( (\Omega, \mathcal{F}, \mathbb{P}) \), where \( \mathbb{P} = C(\mathcal{D} \mathcal{T} ; \mathbb{R}^d) \), and \( \mathcal{F} \) is the standard Wiener measure on \( \mathcal{D} \mathcal{C} \mathcal{M}(\mathcal{L}) \) equipped with the completion \( \mathcal{F} \) of the Borel -algebra induced by the \( \mathcal{S}p \)-norm. The noise \( \mathcal{W} \) is given by the canonical map \( \mathcal{W}(\cdot) = \cdot, \cdot \in \mathcal{V} \). The \( g_i \) from equation (3) are fixed elements of \( \mathcal{D} \mathcal{C} \mathcal{M}(\mathcal{L}) \) \( \mathcal{V} \).

(Here and in the sequel, we use the notation \( \mathcal{D} \mathcal{C} \mathcal{M}(\mathcal{L}) = \mathcal{F} \mathcal{V} \mathcal{H} : \mathcal{L}(\mathcal{V}) \to \mathcal{H} \) g.) Letting \( e_1; \ldots; e_d \) denote the standard basis in \( \mathbb{R}^d \), we define a linear map \( \mathcal{G} : \mathbb{R}^d \to \mathcal{D} \mathcal{C} \mathcal{M}(\mathcal{L}) \) \( \mathcal{V} \) by \( g_i = \mathcal{G} e_1; \ldots; g_d = \mathcal{G} e_d \).
As usual, the stochastic equation (3) is simply shorthand for the following integral equation:

\[
\frac{d}{dt} u(t; \omega) = u_0 + \int_0^t F(u(s; \omega)) \, ds + \int_0^t f(s) \, ds + GW(t; \omega).
\]

We shall always assume that there exists a stochastic semiflow associated with this equation. More precisely, we assume that there is a family of operators \( t : C([0; t]; R^d) \rightarrow V \), \( t \geq 0 \), such that if \( u(t) = t(W[0; t]) \) for all \( t \in [0; T] \) with probability 1, then \( u \) is a solution of (5) satisfying (4). Here \( W[0; t] \) is the restriction of \( W \) to \([0; t]\). We stress that though the initial data \( u_0 \) may be in \( H \cap V \), the solution is assumed to be in \( C([0; T]; V) \).

### 3. Basic Results

Our main results are the absolute continuity of the distribution of the projection of \( T(W) \) on a finite-dimensional space with respect to the Lebesgue measure on that space, and the smoothness of the density.

We will need some conditions on the linearization of the system (3). Let \( J_{s,t} : H \rightarrow V \) for \( 0 < s < t \) solve the equation in variations:

\[
\frac{\partial}{\partial s} J_{s,t} = (DF)(u(t))J_{s,t}; \quad s < t;
\]

\[
J_{s,s} = \varnothing; \quad 2H.
\]

Here \( (DF)(x)(h) \) is the Fréchet derivative of \( F \) at a point \( x \in V \) applied to a tangent space vector \( h \in V \). Hence, for each \( x \) we have \( (DF)(x) : V \rightarrow V^0 \). Notice that Fréchet derivatives of \( F \) of all orders are well-defined (see Lemma 10.3.)

**Assumption 1.** With probability one, there is a unique solution \( J_{s,t} \) to the equation (6) for every \( 2H \) and \( s \geq 0 \).

\[
J_{s,t} \in C([s,T]; H \setminus L^2([s,T]; V)); \quad \frac{\partial}{\partial s} J_{s,t} \in L^2([s,T]; V^0);
\]

where \( J_{s,t} \) and \( \frac{\partial}{\partial s} J_{s,t} \) are considered as functions of \( t \).

We are also going to consider the time reversed adjoint of \( J_{s,t} \) denoted by \( K_{s,t} : H \rightarrow V; s \geq t \) and defined by the backward equations

\[
\frac{\partial}{\partial s} K_{s,t} = (DF)(u(s))K_{s,t}; \quad s < t;
\]

\[
K_{s,s} = \varnothing; \quad 2H.
\]

Here \( (DF)(y) : V \rightarrow V^0 \) is the adjoint operator for \( (DF)(y) \). (We identify \( V \) and \( V^0 \).)

**Assumption 2.** With probability one, for any \( 0 < t_0 < t \) and \( 2H \), there is a unique solution \( K_{s,t} \) of equation (8) that satisfies

\[
K_{s,t} \in C([t_0; T]; H \setminus L^2([t_0; T]; V)); \quad \frac{\partial}{\partial s} K_{s,t} \in L^2([t_0; T]; V^0);
\]
where $K_{s,t}$ and $\frac{\partial}{\partial s} K_{s,t}$ are considered as functions of $s$.

3.1. Existence of densities. To begin understanding how the randomness spreads through the phase space, we now introduce an increasing collection of sets which characterize some of the directions excited. In Section 6 we will give a more completed, though more complicated, description of the directions excited. However, for many cases, the results of this section are sufficient.

For any positive integer $n$, we introduce the subset $G_n$ of $V$ defined recursively as follows. For $n = 1$, we set $G_1 = \text{span} \{ g_1, \ldots, g_m \}$. For $n > 1$, $G_n$ is defined via $G_n = \text{span} \{ \bigcup_{i=1}^{n-1} G_i \} \cap V \setminus \text{Dom}(L_k) \cup \{ f_1 \}$. For $g \in G_n$, we set

$$E_{s}^{T_0} \left( \int_0^t \int_s^t J(s) ds \right) g_k^{p} u_p(T_0;T;u_0);$$

Finally, we introduce $G_1 = \text{span} \bigcup_{n} G_n$. The following result is a specialization of Theorem 6.2 which is given later.

Assumption 3. For all $u_0 \in H_0$, there is a constant $J(T;u_0)$ such that

$$\sup_{0 \leq s \leq T} E \left( \sum_{k=1}^{d} \left| g_k \right|^2 \right)^2 J(T;u_0).$$

Theorem 3.1. Assume that Assumptions 1, 2, and 3 hold. Suppose that $S \subset H$ is a finite-dimensional linear subspace in $G_1$. Then the distribution of the orthogonal projection $S u(T)$ on $S$ is absolutely continuous with respect to the Lebesgue measure on $S$.

3.2. Smoothness of densities. To prove smoothness of the density obtained in Theorem 3.1 we need stronger assumptions than those made in that theorem. We will replace the assumption of continuity and finiteness of the first derivative in time of $K$ and $u$ with assumptions on the moments of the Lipschitz coefficients in time of related processes.

Assumption 4. In addition to the standing assumptions from section 2 the following conditions hold: For every $u_0 \in H_0$ there exists a fixed $T_0 \geq \{ 0; T \}$ and constants $u_p(T_0;T;u_0)$, for all integers $p \geq 1$, so that

$$E \sup_{T_0 \leq t \leq T} u_p(T_0;T;u_0);$$

$$E \sup_{T_0 \leq t \leq T} kX(t) \kappa^p u_p(T_0;T;u_0);$$

where $X(t) = u(t) GW(t)$. 

where $X(t) = u(t) GW(t)$. 


Assumption 5. In addition to Assumptions 1 and 2 there exists a $T_0$ so that for every $u_0 \in H_0$ and $p \geq 1$ there exists a constant $K_p(T_0;T;u_0)$ with

$$E \sup_{T_0 \leq t \leq T} \mathbb{K}_{u_0} f, v \mathbb{K}_p(T_0;T;u_0);$$

$$E \sup_{T_0 \leq t \leq T} \mathbb{K}_{u_0} f, v \mathbb{K}_p(T_0;T;u_0);$$

where $\mathbb{K}_{u_0}$ denotes the norm of a linear operator mapping $V$ to itself and $\mathbb{K}_{u_0}$ from $V$ into $H$.

Assumption 6. There exists an $2 \in [0,1)$ such that for any $u_0 \in H_0$ and $p \geq 1$ there is a constant $D_p(T;u_0)$ such that

$$E \sup_{t \leq s \leq T} \mathbb{K}_{u_0} f, v \mathbb{K}_p(T;u_0);$$

$$E \sup_{0 \leq s \leq t} \mathbb{K}_{u_0} f, v \mathbb{K}_p(T;u_0);$$

Lastly, we need the following definition which further restricts the class of polynomial nonlinearities we will treat.

Definition 3.2. We define $\text{Poly}_1(V;H)$ as the set of continuous polynomials $Q: V \to H$, with $Q = \sum_{k=1}^{\infty} C_k Q_k$ for some $k$, where the $Q_k$ are homogeneous $i$-linear terms satisfying the following bound for some $C_k$ and all $u \in V$:

$$\|Q(u_1; \ldots; u_k)\|_V \leq C_k \|u_1\|_V \cdots \|u_k\|_V$$

We are now in a position to state precisely our first result on the smoothness of the projections of transition densities.

Theorem 3.3. In the setting of Section 2 assume that Assumptions 4, 5, and 6 hold. Let $S \subset V$ be a finite-dimensional subspace of $G_n$ for some $n$. If $N$ is a continuous polynomial in $\text{Poly}_1(V;H)$ then the density of $S \cap G_n$ with respect to Lebesgue measure exists and is a $C^1$ function on $S$.

4. Applications

We now specialize our setting, restricting ourselves to the case where the linear operator $L$ is dissipative and dominates the nonlinearity. At the end of the section we will fit a reaction-diffusion equation and the 2D Navier-Stokes equation into the setting we now describe.

Let $L$ be a positive, self-adjoint linear operator on a Hilbert space $H$. Additionally, assume that $L$ has compact resolvent. Hence $L$ has a complete orthonormal eigenbasis $e_k: k = 1, 2, \ldots$ with real eigenvalues $0 \leq 2 \in \mathbb{R}$ such that $\lim_{k \to \infty} e_k = 1$. For $s \geq 2$, we define the inner product

$$\langle u, v \rangle_k = \sum_{k=1}^{2^n} \langle h, e_k \rangle \langle e_k, v \rangle$$
and the norm $j_{i}\mathbb{E} = h_{i}u_{i}$. We define the spaces

$$V^2 = f u \in H : j_{i} < 1 \ g$$

and observe that $V^1$ is the dual in $H$ of $V^1$ and that $L$ maps $V^1$ to $V^1$ and $V^2$ to $H$. We assume that $N \geq 2 P \chi_{(V^1; H)}$ and that $f : [0, T] \to V^1$. $V^1$ is uniformly bounded. (See definition [2].)

**Lemma 4.1.** In the setting above, for any $u_0 \in H$ equation (3) has a unique strong solution $u$, generated by a stochastic flow $\varphi : H \to V^2$, satisfying

$$u \in C \left[ 0, T \right] ; H \quad \text{and} \quad \left( 0, T \right] ; V^2$$

In addition, if $2 H$, then there exists a unique solution $J_{st}$ to equation (5), for all $0 < s \leq T$. Furthermore

$$J_{st} \in C \left[ s, T \right] ; H \quad \text{and} \quad \left( s, T \right] ; V^2$$

as a function of $s$.

Lastly, if $2 H$, then there exists a unique solution $K_{st}$ to equation (8), for all $0 < t_0 < s \leq t \leq T$. Furthermore,

$$K_{st} \in C \left( t_0, t \right] ; H \quad \text{and} \quad \left( t_0, t \right] ; V^2$$

as a function of $s$.

**Proof:** Most of the results follow from results about deterministic, time inhomogeneous equations found in [SY02]. As is often done (see for example [Fla94, DPZ96]), we begin by setting $u(t) = X(t) + Gw(t)$. Then $X(t)$ satisfies a standard PDE

$$\frac{\partial}{\partial t} X(t) = F \otimes X(t); t;$$

where the random right hand side is given by $F(x; t) = L(x) + N(x + Gw(t))$. Once it is demonstrated that this equation has a unique solution for every $u_0 \in H$ and almost every $w$, we have constructed a stochastic flow, since all initial conditions can share a single exceptional set in the probability space. Clearly, $(x; t) \quad N (x + Gw(t))$ is almost surely uniformly bounded in $H$ on $f(x; t) : j_{i} + t < C g$, for all $C > 0$. Furthermore, it is a Hölder continuous function of time for all Hölder exponents less than $1/2$. The quoted existence, uniqueness, and regularity for $u$ then follows from Lemma 47.2 from [SY02] applied to the above equation for $X(t)$.

All of the quoted results on the linearization except the fact that the solution is in $L^2 \left( [s, T] ; V^1 \right)$, follow from Theorem 49.1 from [SY02] by arguments similar to those just employed. To see that the solutions are in $L^2 \left( [s, T] ; V^1 \right)$, take the inner product with $v(t) = J_{st}$ to obtain

$$\frac{\partial}{\partial t} (\dot{v}(t)) = \dot{y}(t) \ddot{z} + \dot{h} \; N \; (u(t)) \; v(t) ; v(t) ; t_0$$

(19)

$$\frac{1}{2} \dot{y}(t) \ddot{z} + C (1 + j_{i}(t)^{2^{(n-1)}}) \dot{y}(t) \ddot{z} :$$
Since this implies that
\[
\int_1^T \|v(t)\|_2 dt C j_2 \sup_{t \in [s, T]} v(t) \leq 1;
\]
we are done.

The proofs of the statements for the adjoint linearization are the same as for the linearization after one observes that since \( N \circ Y_1(\mathbb{V}^1; H) \) we have
\[
\sup_{j \geq 1} \|H N_j(u) v_i \|_1 \leq \sup_{j \geq 1} \|H N_j(v_i) \|_1 \|
\]
where the last inequality follows from the Sobolev inequality
\[
\|v(t)\|_2 < 1;
\]
\( u \)

Corollary 4.2. Setting \( V = \mathbb{V}^1, H = H, j' = 1 \), \( j = 1 \), \( j_0 = 0 \) and \( k = 1 \). The standing assumptions of Section 2 and Assumptions 1 and 2 hold in the setting of this section.

4.1. A Reaction-Diffusion Equation. Consider the following reaction-diffusion equation
\[
\begin{align*}
\frac{du}{dt}(x; t) &= u(x; t) + N u(x; t) \, dt + \sum_{k=1}^{X} g_k(x) dW_k(t); \\
\triangledown u(x; 0) &= u_0(x);
\end{align*}
\]
with \( x \in [0, 1] \),
\[
N(u) = \frac{1}{k} u^k,
\]
with \( a_k \in \mathbb{R} \) and \( 0 < a_{2q+1} < 0 \), and under the Dirichlet boundary conditions
\[
u(0) = u(1) = 0 \quad \text{for all} \quad t > 0;
\]
Since in one dimension there exists a constant \( C \) so that for any \( f \in \mathbb{V}^1, f \in \mathbb{V}^1 \) where \( \mathbb{V}^1 \) is the sup-norm, we see that \( N \) is a continuous polynomial from \( \mathbb{V}^1 \) to \( H \) and that
\[
\|D N(u)v\| \leq C \|u\| \|v\|.
\]
The following calculation shows that \( N \circ Y_1(\mathbb{V}^1; H) \). Let \( u \in \mathbb{V}^1 \) and observe that
\[
\int_1^T \|v(t)\|_2 dt C j_2 \sup_{t \in [s, T]} v(t) \leq 1;
\]
\( u \)

where the last inequality follows from the Sobolev inequality \( j \leq C j \).

At the end of the example we will address necessary conditions for the system to be formally Hörmander. For now we address the technical conditions needed to
apply Theorems 3.1 and 3.3. In light of Corollary 4.2 to apply Theorem 3.1 we need to verify Assumption 3. Letting \( v(t) = J_{s\pi}v_0 \), we have

\[
\frac{1}{2} \frac{d}{dt} v(t)^2 = v(t)^2 + \mathcal{A} N(v(t))v(t); v(t) \vert_0
\]

since \( \sup_{a \in \mathbb{R}} N^2(a) = 1 \) for some \( K_1 \). Gronwall’s inequality then implies

\[
\sup_{t \in [0,T]} v(t)^2 + \mathcal{A} e^{2K_1} \gamma_t^T ;
\]

which translates to

\[
\sup_{0 < s < t} \mathcal{A} e^{2K_1} \gamma_s^T ;
\]

for all \( p > 0 \). This ensures that Assumption 3 holds. Having verified all of the assumptions of Theorem 3.1, we have the following result:

**Theorem 4.3.** If \( \mathcal{A} \leq V^2 \), then the conclusions of Theorem 3.1 hold for equation (20).

We now investigate conditions which guarantee that \( G_1 = H \). Let \( I_{0} \) be a finite collection of functions in \( V^2 \). Let \( I \) be the multiplicative algebra generated by \( I_{0} \).

**Lemma 4.4.** If \( I \) is dense in \( V^2 \), then to ensure that \( G_1 = H \) it is sufficient that

\[
ff_1 \quad \mathcal{A} \mathcal{J} I_{0} ; I \quad 2 ; 1 \quad 2qg \quad G_1 = \text{span} f_{g} : j = 1 ; \quad \mathcal{A} g ;
\]

**Remark 4.5.** For \( I \) to be dense in \( V^2 \), it is sufficient, by Stone–Weierstrass, that \( I \) separates points in \( V^2 \) and if \( f(x) = 0 \) for all \( f \in I \) then \( x = 0 \) or 1.

We now turn to proving that the density is smooth. In the sequel, we are going to restrict ourselves to initial data in \( H_0 = H \setminus C(\mathbb{R}; 1) \). It is well known (See [Cer99] Proposition 3.2 or [EH01]) that for all \( p \geq 1 \) and \( u_0 \in H_0 \)

\[
(24) \quad E J_{1}(t)^2 \mathcal{C}_{(p; t)} (1 + J_{1}(t)^2)
\]

and

\[
(25) \quad E \sup_{x \in \mathcal{C}} \sup_{t \in x} J_{1}(t; x)^2 \mathcal{C}_{(p; t)} (1 + J_{1}(t)^2)
\]

4.1.1. **Verification of Assumption** Since for any \( T_0 \geq (0; T) \) and \( t \geq (T_0; T) \)

\[
X(t) = e^{\mathcal{A} t}x_{0} + \mathcal{A} N(u(t))\int_{0}^{t} e^{\mathcal{A} s} N(u(s)) ds + \mathcal{A} G W(t)
\]

we know that

\[
\mathcal{X}(t) = e^{\mathcal{A} t}x_{0} + \mathcal{A} N(u(t))\int_{0}^{t} e^{\mathcal{A} s} N(u(s)) ds + \mathcal{A} G W(t)
\]

\[
C + J_{1}(t)^2 + \sup_{t \in \mathcal{R}} J_{1}(s)^{2q + 1} + \sup_{t \in \mathcal{R}} J_{1}(s)^{2q + 1}
\]
for some positive $a$ and $C$, and all $t \geq 0$. Applying (24), (25) and standard bounds on $\sup_{s \leq r} j^s(t)j^s$ proves the first part of Assumption 4 for any $u_0 \geq 0$.

Similarly for $0 < s < t \leq T$,

$$X(t)X(s) = e^s e^{(t-s)} I u_0^{0} \h \notag$$

$$+ e^{(s-r)} e^{(t-s)} I N(u(r)) + GW(r) \, dr;$$

which implies there exists a constant $C$ depending on $T$ such that

$$J(t)X(s)j^s(t)j^s C \notag$$

$$s \geq 1 + j_10 + \sup_{r \in [s,T]} j_1(r) e^{q+1} + \sup_{r \in [s,T]} j^1(r) (\notag$$

Again combining standard estimates with (24) and (25), we see that the estimate in (13) holds.

4.1.2. Verification of Assumption 5

Again setting $v(t) = j_{s,t} v_0$ for $s \geq 0 \leq T$), we have

$$\notag$$

which when combined with (22) implies that

$$\notag$$

This in turn produces

$$\notag$$

for any $p > 0$ and $u_0 \geq 0$. Since all of the operators on the right-hand side of the governing equation are self-adjoint in this example, the estimates analogous to (22) and (26) hold for $K_{s,T}$.

Using the estimates used to produce (22), it is straightforward to see that there exists a $K(T) > 0$ and $C(T) > 0$ such that for all $r < s \leq T$,

$$\notag$$

which proves the estimate (15).
4.1.3. **Verification of Assumption**

Equation (16) has already been verified above since \( g_k \in V^2 \). To see the second estimate, observe that for \( t, r \in \mathbb{R} \)

\[
J_{st} \triangleq \int_{s}^{t} e^{(t-s)D} N(u_\tau) d\tau \ dr
\]

and hence we have that for \( 0 < s < t < T \),

\[
\mathbb{P} \left( \frac{1}{t-s} \sup_{0 < r < T} \int_{s}^{t} \left[ (t-r) \int_{s}^{r} v \cdot \partial_{x} \tilde{u} \right] \right) \leq C \sup_{0 < s < t < T} \int_{s}^{t} \left( 1 + \sup_{0 < r < T} j_{x} \right) \left( t-s \right)^{p} \ j_{x}^{p}
\]

When Combined with (25), we obtain that for every \( p \geq 1 \) and \( u_0 \in H_0 \)

\[
E \sup_{0 < s < t < T} \int_{s}^{t} \left( t-s \right)^{p} \ j_{x}^{p} \left( t-s \right)^{p} \ j_{x}^{p}
\]

In light of the preceding calculations, we have proven the following result.

**Lemma 4.6.** In the above setting, Assumptions 1, 2, 4, 5, and 6 hold with \( H = H, H_0 = H_0 \) and \( V = V_1 \) and \( N = 2 \mathcal{P} \mathcal{O} \mathcal{Y}_{1}(V_1; H) \). Hence, the conclusions of Theorem 3.2 hold.

4.2. **2D Navier Stokes Equation.** Consider the vorticity formulation of the Navier–Stokes equation in 2D given by:

\[
\begin{align*}
\delta_t w & = w_t + B(K w; w) + f(t) dt + \frac{X^d}{j-1} g_k dW_k(t) \\
\end{align*}
\]

where \( B(u; v) = (u \cdot \nabla)v \) is the usual Navier–Stokes nonlinearity, and \( K \) is the Biot–Savart integral operator which is defined by \( u = K w \) when \( w = \nabla \cdot u \) (see [MB02, MP06] for more details). We denote by \( L^2_0 \) the Hilbert space of square-integrable functions on \( \partial \Omega_2 \frac{\mathbb{R}}{2} \) which are periodic and have spatial mean zero. As before, we form the space \( V^\ast, s \in \mathbb{R} \), from \( H = L^2_0 \) and \( L = \left( \frac{\mathbb{R}}{2} \right) \). We assume that \( f(t) \) is a bounded function in \( V^1 \), \( g_k \in V^2 \).

**Lemma 4.7.** In the above setting, Assumptions 1, 2, 4, 5, and 6 hold with \( H = H, H_0 = H_0 \), and \( V = V_1 \). Additionally, the map \( u \mapsto B(Ku, u) \) \( 2 \mathcal{P} \mathcal{O} \mathcal{Y}_{1}(V_1; H) \). Hence, the conclusions of Theorem 3.2 and 3.3 hold for equation (27).

**Proof.** We begin by proving that \( B(Ku, u) \) \( 2 \mathcal{P} \mathcal{O} \mathcal{Y}_{1}(V_1; H) \). To do so we use the basic facts that \( B(u; v) \) \( C \mathcal{P} \mathcal{J} \mathcal{Q} \mathcal{J} \) and that \( Ku = j \mathcal{B} \) (see for instance [CF88]). Then

\[
B(Ku; u) \subset C \mathcal{P} \mathcal{J} \mathcal{Q} \mathcal{J}
\]

which proves the first result. Assumptions 1 and 2 then follow from Corollary 4.2 or from Proposition 2.1, Proposition 2.2 of [MP06]. The existence of solutions to (27) can also be found in [Fla94]. Assumptions 3, 4, and 5 follow from Corollary
A.2 and Lemma B.1 of [MP06]. The fact that \( w(t) \) \( \in \mathcal{V}^2 \) (see Section 5 for the definition) is also proved in Lemma C.1 of [MP06].

Lastly we we give a fairly weak condition ensuring that the system is formally Hörmander. The following result is a direct consequence of Corollary 4.5 from [HM06].

**Lemma 4.8.** Let \( Z_0 \) be a subset of \( \mathbb{Z}^2 = (0; 0) \) such that the following conditions hold:

i) Integer linear combinations of \( Z_0 \setminus (Z_0) \) generate \( \mathbb{Z}^2 \).

ii) There exist two elements of \( Z_0 \) with non-equal Euclidean norm.

Then \( G_1 = H \) (def \( H \)) if

\[
\cos(k \cdot x); \sin(k \cdot x) : k \in \mathbb{Z} \quad \text{span}_{G_1} \quad \text{def} \quad G_1.
\]

**Remark 4.9.** This result is very similar to one of the principal results in [MP06]. One difference is that we do not require that the set of forcing functions consists of \( \sin \) or \( \cos \) but only that the span of the forcing functions contains the needed collection of \( \sin \) and \( \cos \). For a discussion of what happens when the conditions in Lemma 4.8 fail, see [HM06].

5. **Malliavin Calculus**

Since all of our results use techniques from Malliavin calculus, we give a quick introduction, mainly to fix notation. For a longer introduction see [MP06], for even more background see e.g. [Nua95, Bel87].

First, we define the Malliavin derivative of \( u(T) \) in the direction \( h \in L^2(D; \mathbb{R}^d) \) as

\[
D (u(T)) (h) \equiv H \lim_{\varepsilon \to 0} \frac{\mathbb{E} (u(T; \mathbb{R}^d) + \varepsilon H) - \mathbb{E} (u(T; \mathbb{R}^d))}{\varepsilon}
\]

where \( \mathbb{E} (T; \mathbb{R}^d) = \int_0^T h(s) ds \). It is easy to verify that under Assumption 1 the derivative \( D (u(T)) (h) \) is well-defined for any \( h \in L^2(D; \mathbb{R}^d) \) and that

\[
D (u(T)) (h) = \int_0^T J_{s,T} G h(s) ds.
\]

The Malliavin covariance operator \( M (u(T)) : H \to H \) is defined by

\[
M (u(T)) = \int_0^T J_{s,T} G h(s) ds.
\]

(We shall often write \( M = M (u(T)) \) for brevity). It is clearly nonnegatively definite. Its finite-dimensional projection on the space \( S \) is given by the Malliavin matrix

\[
M_{ij} = M (u(T)) = \int_0^T J_{s,T} G h(s) ds; \quad i, j = 1, \ldots, N
\]

where \( 1, \ldots, N \) is an orthonormal basis in \( S \).
Notice that the definition in (28) involves solving a continuum of linear systems (one for each $s \in [0; T]$). It is more convenient to work with the following representation

\[
\int_{0}^{t} K_{s,t} \, i^2 \, ds
\]

which involves solving only one linear system. This representation follows from the relation $K_{s,t} = \ldots$ and the next lemma:

**Lemma 5.1.** Assume that Assumptions [7] and [2] hold. Then for any $0 \leq s < t \leq T$, the map $\int_{s}^{t} h J_{s,t} \, i$ from $[s;t]$ into $\mathbb{R}$ is constant.

**Proof:** The following essentially recapitulates the proof of Proposition 2.3 from [MP06]. Set $v(t) = J_{s,t}$ and $w(t) = K_{s,t}$. Since $v,w \in L^2([s;t];\mathbb{H})$ and their time derivatives $v', w' \in L^2([s;t];\mathbb{V})$, we may apply integration by parts (see Theorem 2 from [DL88, p.477]):

\[
\int_{s}^{t} \frac{\partial}{\partial r} f(r)(v(r), w(r)) \, dr = \int_{s}^{t} f(r)(v(r), w(r)) \, dr - \int_{s}^{t} f'(r)(v(r), w(r)) \, dr.
\]

for all $r_0 < r_1$.}

**5.1. Higher Malliavin Derivatives.** The existence of a smooth density requires control of higher Malliavin derivatives, which we now introduce. For $n \geq 1$, $s \in [0; T]$ and $h_1, \ldots, h_n \geq 0$, we define

\[
J_{s,t}^{(1)}(\theta) = \int_{s}^{t} \left( \frac{\partial}{\partial h_1} \right) f(t)(\theta) \, dt
\]

where $J_{s,t}^{(1)}(\theta) = \int_{s}^{t} \left( \frac{\partial}{\partial h_1} \right) f(t)(\theta) \, dt$ is the solution of the $n$-th equation in variations defined below.

The first variation of equation (3) is

\[
\frac{\partial}{\partial t} J_{s,t}^{(1)} = \mathcal{D}(u(t)) J_{s,t}^{(1)} ; t > s;
\]

\[
J_{s,t}^{(1)} = \left( \frac{\partial}{\partial t} \right) f(t)(u(t)) \, dt
\]

for all $2 \in \mathbb{V}$. Obviously, $J_{s,t}^{(1)} = J_{s,t}$, where the latter is introduced in (6).

To write down the equations for the higher order variations, we need some additional notation. Suppose we have vectors $(s_1; \ldots; s_n) \in \mathbb{R}^n$ and $\mathcal{I}(\theta) = n \geq 0$ (here $n$ means the number of elements in $\mathcal{I}$). We denote $\mathcal{S}(\theta) = (s_1; \ldots; s_n)$ and $\mathcal{I}(\theta) = (s_1; \ldots; s_n)$. Now for $n \geq 2$, $s_1; \ldots; s_n$, and $\mathcal{S}(\theta) = (s_1; \ldots; s_n) \in \mathbb{V}^n$, the $n$-th equation in variations is given by
Let us fix $\theta \in \mathbb{R}$ and hence
\[ \frac{\partial}{\partial \tau} J_{s_1; \ldots; s_n, \tau} (1; \ldots; n) = D F \left( u (\tau) \right) J_{s_1; \ldots; s_n, \tau} (1; \ldots; n) \]
\[ + G_{s_1; \ldots; s_n, \tau} (u (\tau)) (1; \ldots; n); \quad \tau > -s; \]
\[ J_{s_1; \ldots; s_n, \tau} (1; \ldots; n) = 0; \quad \tau \leq -s; \]
where $-s = s_1; \ldots; s_n$, and for $n \geq 2$.

Proof. \begin{equation}
(31)
\end{equation}
We say that \( \chi \) is the degree of the polynomial \( F \), and the inner sum is taken over all partitions of \( 1; \ldots; s \) into disjoint non-empty sets \( 1; \ldots; i \) (we do not distinguish two partitions obtained from each other by a permutation). The upper limit in the outer sum can be changed to \( m \) since the derivatives of \( F \) of order higher than \( m \) vanish. The lower limit can be changed to \( 2 \) since there are no admissible partitions for \( \tau = 1 \). Since \( N \) has all of the non-linear terms in the equation we can replace \( F \) with \( N \).

Variation of constants for \( (31) \) gives
\begin{equation}
(32)
\end{equation}
for \( n \geq 2 \).

We say that \( u (\tau) \) \( D^1 \) for some Banach space \( Y \) if for all \( n \geq 2 \), and all \( h_1; \ldots; h_n \) \( 2 \textup{R}^d \).

\begin{equation}
(34)
\end{equation}
Lemma 5.2. \textbf{Under assumptions} \( (2) \text{and} (6) \) \textbf{for all} \( n \geq 2 \),
\[ E \sup_{1 \leq \tau} \sup_{x \in I_s} k_{s_1; \ldots; s_n, x} (1; \ldots; n) k^{p} < 1 \quad \text{for all} \quad p > 1; \]
and hence \( u (\tau) \) belongs to \( D^1_u \).

Proof. The fact that \( u (\tau) \) belongs to \( D^1_u \) follows immediately from the first part of the Lemma when combined with \( (30) \). The first claim will follow by induction.

For \( n = 1 \) the statement for \( J_{1} \) follows directly from \( (16) \) in Assumption \( 6 \). Let us fix \( n \geq 2 \) and suppose that the statement holds true for all positive integers less than \( n \). Take any \( 0 < s_1; \ldots; s_n < r < T \) such that \( -s = T_0 \). In the interest of notational compactness we write
\[ \chi \]
\[ \mu \]
}\]
\[ = -2 I_1; \ldots; I_j \]

\[ \chi \] for \( \chi \in \mathbb{R}^n \) X

\[ \mu \]
\[ = -2 I_1; \ldots; I_j \]

\[ \chi \] for \( \chi \in \mathbb{R}^n \) X

\[ \mu \]
\[ = -2 I_1; \ldots; I_j \]
Then by (33), we have
\[
\int_{S} \mathbb{E}[Y] \, d\mathbb{P} = \mathbb{E}[\mathbb{E}[Y | \mathcal{F}_t]] \quad \text{for all } t \geq 0.
\]

By the induction hypothesis, all the moments of the r.h.s. are finite, and we use Hölder's inequality to split the products. The estimates (16), (17), and (12), and the supremum over all of them. Next, taking the expectation of both sides, we use Hölder's inequality to split the products. The estimates (16), (17), and (12), and the induction hypothesis imply that all the moments of the r.h.s. are finite, and we are done.

6. General Results

We now give the proof of the main results of this article. They are generalizations of the results given in Sections 3.1 and 3.2. All of our examples fit into the framework of the previous sections. However, for completeness and to emphasize the connection with the standard finite-dimensional results, we will prove the more general results in this section, which imply the results previously stated.

6.1. Existence of densities. To understand how the randomness spreads through the phase space, we now introduce an increasing collection of sets which characterize the directions excited.

The Lie bracket of two Fréchet-differentiable vector fields \( A; B \) is a new vector field
\[
[A; B](\mathbf{x}) \triangleq \mathcal{D} A(\mathbf{x}) B(\mathbf{x}) - \mathcal{D} B(\mathbf{x}) A(\mathbf{x}) \in \mathbb{V}^0;
\]
defined for all \( \mathbf{x} \in \mathbb{V} \) when it makes sense (i.e., when \( A(\mathbf{x}) B(\mathbf{x}) \) is defined). In the interest of notational brevity, we will write
\[
[A_1; A_2; \ldots; A_n](\mathbf{x}) = [\ldots [A_1; A_2]; \ldots; A_n](\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathbb{V}.
\]

Next, we define the set \( \mathcal{A} \) of admissible vector fields which will play an essential role in the forthcoming iteration scheme. To do so, we fix a time \( t \geq 0 \) and recall the process \( X(t) = u(t) G W(t) \) defined earlier. Notice that \( X(t) \) can also be written as
\[
\int_{0}^{t} F(u(s)) ds + \int_{0}^{t} f(s) ds = u(t) G W(t);
\]
and hence \( X(t) \) is a \( \{0; T\} \)-valued process almost surely.
Definition 6.1. A is the set of all polynomial vector fields \( \mathbb{Q} : \mathbb{V} \to \mathbb{V}^0 \) such that with probability one the following conditions hold:

i) \( \mathbb{Q} \not\in (\mathbb{T}) 2 L^2 [t; \mathbb{T}] \mathbb{V} \),

ii) \( \frac{d}{dt} \mathbb{Q} \not\in (\mathbb{T}) 2 L^2 [t; \mathbb{T}] \mathbb{V}^0 \),

iii) \( \mathbb{F};\mathbb{Q} \) is a continuous polynomial from \( \mathbb{V} \to \mathbb{H} \).

For any \( n \geq 2 \) and any positive integer \( n \), we introduce a set \( H_n \) of smooth vector fields \( \mathbb{Q} : \mathbb{V} \to \mathbb{V}^0 \). For \( n = 1 \), we set \( H_1 = \text{span} \{ g_1, \ldots, g_d \} \). For \( n > 1 \), \( H_n \) is defined recursively from \( H_{n-1} \):

\[
H_n = \text{span} \left( H_{n-1} \right) \bigg[ \mathbb{Q}, h \right]
\]

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iii) \( \mathbb{F};\mathbb{Q} \) is a continuous polynomial from \( \mathbb{V} \to \mathbb{H} \).

For any \( n \geq 2 \) and any positive integer \( n \), we introduce a set \( H_n \) of smooth vector fields \( \mathbb{Q} : \mathbb{V} \to \mathbb{V}^0 \). For \( n = 1 \), we set \( H_1 = \text{span} \{ g_1, \ldots, g_d \} \). For \( n > 1 \), \( H_n \) is defined recursively from \( H_{n-1} \):

\[
H_n = \text{span} \left( H_{n-1} \right) \bigg[ \mathbb{Q}, h \right]
\]

Theorem 6.2. Assume that Assumptions 1 and 2 hold. Suppose that \( S \) is a finite-dimensional linear subspace in \( \mathbb{H} \). If, in addition, \( S \) is a subspace of \( \mathbb{H} \) \( \not\in (\mathbb{T}) \) with probability 1, then the distribution of the orthogonal projection to \( S \) \( \cup (\mathbb{T}) \) on \( S \) is absolutely continuous with respect to the Lebesgue measure on \( S \).

We will see in remark 7.5 that the above theorem holds under a slightly relaxed version of Assumption 3. The following lemma shows that Theorem 3.1 is implied by Theorem 6.2 given above. Its proof will be given after the proof of Theorem 6.2.

Lemma 6.3. Under Assumptions 1 and 2, \( G_n \subset H_n \) for all \( n \).

Proof. We shall proceed by induction. First, notice that \( G_1 = H_1 \) and that all of the elements of \( G_1 \) are constant. Now our induction hypothesis will be that for some \( n \) we have \( G_{n-1} \subset H_{n-1} \) and that all vector fields in \( G_{n-1} \) are constant.

It is sufficient to show that if \( h = N_m \{ g_1, \ldots, g_{k_0} \} \) \( \not\in \mathbb{V} \) for some \( g_2 G_n \not\in \mathbb{V} \setminus \mathbb{Dcm} (\mathbb{L}) \) and \( k_2 \leq 1 \), then \( h \) is constant in \( \mathbb{V} \) (which is trivial), and there is a \( Q \in H_{n-1} \setminus A \) such that \( h = \mathbb{F};\mathbb{Q} \).

To prove the latter, we can choose \( Q = g_m \). Then Lemmas 10.5 and 10.3 imply:

\[
\mathbb{F};\mathbb{Q} \{ g_1, \ldots, g_{k_0} \} = N_m \{ g_1, \ldots, g_{k_0} \} = h.
\]

We shall now check that \( Q \) or, equivalently, \( g \) belongs to \( H_{n-1} \setminus A \). First, notice that \( g \not\in H_{n-1} \) by the induction hypothesis. Next, \( g \not\in A \) since i) \( g \not\in \mathbb{V} \), ii) \( \frac{d}{dt} g = 0 \), and iii) Lemma 10.5 shows that \( \mathbb{F};g = \mathbb{F} (\mathbb{Q}) (g) = F (g \times \rightarrow 1) = Lg + N (g \times \rightarrow 1) \), which is continuous from \( \mathbb{V} \to \mathbb{H} \) by the assumptions on \( N \), since \( L \) is a constant in \( \mathbb{H} \) due to \( g \not\in \mathbb{Dcm} (\mathbb{L}) \).

6.2. Smoothness of densities. We now introduce a second sequence of sets \( H_n \) of vector fields from \( \mathbb{V} \to \mathbb{H} \). The \( H_n \) play the analogous role in our smoothness of density result as the \( H_n \) played in the existence of density result. We begin by...
defining a slightly modified version of the set of admissible vector fields $A$ used in
the last section. Let

$$A \overset{\text{def}}{=} \mathcal{Q} : \mathcal{V} \rightarrow \mathcal{H} : Q \rightarrow 2A \setminus \mathcal{P} \mathcal{O}_{2}(\mathcal{V} ; \mathcal{H}) :$$

(38)

Given a collection of functions $C$ we define the symmetric convex hull, denoted
$\text{SCH}(C)$, by

$$\text{SCH}(C) \overset{\text{def}}{=} X \overset{\text{if}}{=} f_{1} : f_{2} \rightarrow C ; 1 \rightarrow 2 R ; \text{and } \overset{\text{if}}{=} j_{i} j_{1} 1 :$$

For $n = 1$, we set $H_{1} = \text{SCH}(g_{1} ; \ldots ; g_{d}) \rightarrow H_{1}$. For $n > 1$, we construct $H_{n}$
from $H_{n-1}$.

We set

$$H_{n} \overset{\text{def}}{=} \text{SCH}(H_{n-1} \setminus \mathcal{A} \setminus \mathbb{P}_{2} H_{n}|_{\mathcal{Q}_{2} H_{n-1} \setminus \mathcal{A}|_{k_{1}} ; \ldots ; k_{d}} :$$

(39)

Theorem 6.4. Assume that Assumptions 4, 5 and 6 hold. Let $S$ be a deterministic
finite-dimensional subspace of $\mathcal{V}$ such that for some $n$ and some $\theta > 0$

$$p(u_{0} ; T) \overset{\text{def}}{=} \frac{1}{2} \bigg[ \inf_{k} \sup_{h} ; \mathcal{Q}_{2} h(T) \bigg]^{1 \theta} < 1 ;$$

(40)

for all $p \geq 1$. Then the density of $\mathcal{S} \mu(T)$ with respect to Lebesgue measure
(whose existence is guaranteed by Theorem 6.2) is a $C^{1}$ function on $S$.

The next lemma shows that Theorem 6.3 follows from Theorem 6.4.

Lemma 6.5. Recall the definition of $G_{n}$ from (10). If $S \overset{\text{def}}{=} G_{n}$ then the condition in
(40) holds for this $n$. In fact, there exists a subset of constant vector fields $H_{n}^{0}$ such that

$$\inf_{k} \sup_{h} ; \mathcal{Q}_{2} h ; i > 0$$

for some $\theta > 0$.

7. PROOF OF GENERAL RESULTS

7.1. Absolute continuity. Theorem 6.2 will be implied by the following standard
result from Malliavin calculus (see [Nua95, p.86, Section 2.1]; it is straightforward
to check that the definitions of the Malliavin derivative and matrix given in [Nua95]
are equivalent to ours):

Theorem 7.1. Suppose the following conditions are satisfied for a finite-dimensional
random vector $Y$:

i) $E \mathcal{D}(Y) \otimes h f = 1$, for all $h \in L^{2} \mathcal{O}_{2} T) \rightarrow \mathbb{R}^{d}$.

ii) The Malliavin matrix $M(Y)$ is invertible a.s.

Then the law of $Y$ is absolutely continuous with respect to the Lebesgue measure.
Proof of Theorem 6.2. Condition i) of Theorem 7.1 follows from (11).

To verify condition ii), it is sufficient to prove that

\[ P_n \ker M \cap H_1(\mathcal{T}) \circ \circ \circ \circ = 0; \]

where

\[ \ker M = 2V : M \ ; i = 0 \ ; \]

This property is implied by

\[ P_n \ker M \cap H_1(\mathcal{T}) = 1 \]

or, equivalently, by

\[ P_n \ker M \cap H_n(\mathcal{T}) = 1 \]

which in turn follows from

\[ P_n \ker M \cap Q \cap (\mathcal{T}) = 1 \]

and the fact that \( H_1(\mathcal{T}) \) is generated by \( Q \cap (\mathcal{T}) \); \( H_n(\mathcal{T}) \). Relation (42) is a consequence of the following statement which we will prove below.

There is a set \( \mathcal{U} \) with \( P(\mathcal{U}) = 1 \) such that for all \( \mathcal{U} \) in this set \( \mathcal{U} \), all \( \mathcal{U} \in \ker M \), every \( n \in \mathbb{N} \), each \( Q \in H_n \), and all \( s \in [t, T] \), we have that

\[ hQ \cap (s); K_{s, T} i = 0 \]

where \( \mathcal{U} \) was the time fixed at the start of Section 6.1.

This statement will be proved by induction in \( n \). For \( n = 1 \) it follows directly from the representation in (29). The induction step is provided by the next lemma, whose proof will complete the proof of the present result.

Lemma 7.2. There exists a set \( \mathcal{U} \) of probability 1 such that for all \( \mathcal{U} \) in this set \( \mathcal{U} \), the following implication holds true:

Let \( Q \cap V \cap V^0 \) be a polynomial vector field in \( \mathcal{A} \). Then for any \( t_0 \in [t, T] \)

\[ hQ \cap (s); K_{s, T} i = 0; \ s = s_0; T \]

implies that

\[ hF; Q; g_{k_1}; g_{k_2}; \ldots; g_{k_i} \] \( \cap \) \( (s) \); \( K_{s, T} i = 0; \ s = s_0; T \]

for any \( i \) \( 0 \) and \( k_0 \) \( \in \) \( \mathbb{N} \).

Proof. By Theorem 2 from [DL88, p. 477], we can differentiate (44) with respect to \( s \). Equation (8) implies

\[ 0 = \frac{\partial}{\partial s} hQ \cap (s); K_{s, T} i \]

\[ = hD Q \cap (s) F (u(s)); K_{s, T} i \quad hQ \cap (s); D F (u(s)); K_{s, T} i \]

Fix \( s \) and \( X (s) \) and notice that the vector field

\[ R (y) = D Q \cap (s) F (y) \]

\[ D F (y) Q \cap (s) \]
is well-defined and a polynomial from $V \to H$. We also have
\[ \{ F; g_{k_1}; g_{k_2}; \ldots; g_{k_l} \} \mathcal{X}(s) = \{ F; Q; g_{k_1}; g_{k_2}; \ldots; g_{k_l} \} \mathcal{X}(s) : \]
Hence by Lemma 10.6, for $s \in [t_0, T]$ and some $n$
\begin{equation}
0 = h \left[ \mathcal{X}(s) \right] F_{s\tau_T} i
= \sum_{i=0}^{\infty} X_i \left[ h F; Q; g_{k_1}; g_{k_2}; \ldots; g_{k_l} \right] \mathcal{X}(s) F_{s\tau_T} \left[ dW_{k_1} \ldots dW_{k_l} \right] .
\end{equation}
Observe that each of the inner products is a continuous function of time. This follows from the almost sure continuity in $H$ of the two arguments of the inner products. The brackets, by virtue of being in $A$, are continuous from $V \to H$, and $X(s)$ is continuous in $V$ on $[t; T]$ almost surely by assumption. Hence, if $Y(s) = \{ F; Q; g_{k_1}; g_{k_2}; \ldots; g_{k_l} \} \mathcal{X}(s)$, then $Y(t)$ is in $C([t; T]; H)$. By assumption, we know that $K_{s\tau_T}$ is $C([t; T]; H)$ almost surely. For $s < r \leq T$ we have
\[ j_Y(t)K_{s\tau_T} i \mathcal{Y}(t) K_{s\tau_T} j = j_Y(s) K_{s\tau_T} i j + j_Y(t)K_{s\tau_T} j + j_Y(t) K_{r\tau_T} K_{s\tau_T} i j \]
and thus conclude that $h \mathcal{Y}(t) K_{s\tau_T} i$ is continuous in $t$. The proof of the result is now completed using Theorem 9.3.

7.2. Smoothness of the density. Theorem 6.4 will follow from the following classical result from Malliavin Calculus (see for example [Nua95] Corollary 2.1.2) which is a strengthening of Theorem 7.4 which was used to prove the existence of a density:

**Theorem 7.3.** Suppose that $\cup (T)$ is the orthogonal projection onto some finite-dimensional subspace of $Y$ and the following conditions hold:

i) $\cup (T)$ belongs to $D_{1/2}$.

ii) The projected Malliavin matrix $M = M(\cup (T)) = M_{i,j}$ (defined in (28)) satisfies
\[ E \left[ \det M \right] j^p < 1 \quad \text{for all } p > 1. \]

Then the density of $\cup (T)$ with respect to Lebesgue measure on $Y$ exists and is $C^1$-smooth.

We have to check both conditions of this theorem to prove Theorem 6.4. The first condition is implied by Lemma 7.2 and the second one follows from the theorem below. For $n \geq 1$, we define $S_n = \text{span} H_n$. Here $H_1 = [\frac{1}{n} H_n]$.

**Theorem 7.4.** Let $\cup$ be the orthogonal projection onto a finite-dimensional subspace of $S_n$ for some $n$. Fix a number $r > 0$. Let $U = U = \{ 1, 2 \} \cap k \leq k < k'$ and then for any $p > 1$, there is $\text{d} \mathcal{J} = \text{d} \mathcal{J}(p)$ such that
\[ P \left[ \inf_{U} \mathcal{H} \left( \cup (T) \right) \right] \leq \left[ \text{d} \mathcal{J}(p) \right]. \]

if $r = 0$.
Remark 7.5. We notice that Assumption 3 can be relaxed. Specifically, to satisfy the first condition in Theorem 7.4, we only need second moments of the Malliavin derivative of $u(T)$. We only need Assumption 3 to hold with the $j$-norm replaced by a norm dual to a norm, which is finite on $S$. For instance if $S \subseteq V$ then (11) can be replaced by

$$\sup_{k} \sup_{0 \leq s < t} E \|s_t g_k \|_j^j \leq J(T; u_0):$$

7.2.1. The Proof of Theorem 7.2 and Associated Results. The proof of this theorem will use a quantitative version of Lemma 7.2. From this point forward, we fix $T$ to be the maximum of the two $T_0$'s given in Assumptions 4 and 5.

Before stating the result, we need a little notation: For $f : [0; T]$! $\mathbb{R}$ we define

$$\mathcal{A}(f) \overset{\text{def}}{=} \sup_{T} \frac{|f(t)|}{|s|} \text{ and } \mathcal{A}(f) \overset{\text{def}}{=} \sup_{t} \frac{|f(t)|}{|s|};$$

If $f : [0; T]$! $V$, then by $\mathcal{A}(f)$ and $\mathcal{A}(f)$ we mean the same expressions with the absolute values replaced by the indicated norm. When applied to the operator $K_{\mathcal{A}}$, we mean the same expressions where $s$ and $t$ vary over all $s; t \in [0; T]$ with $s < t$. Lastly, we define $\mathcal{A}(f) \overset{\text{def}}{=} \sup_{s; t} |f(t)|$. Lastly, we define $\mathcal{A}(f) \overset{\text{def}}{=} \sup_{s; t} |f(t)|$.

We now give a number of properties of the symmetric convex hull of a set of functions.

Lemma 7.6. Recalling the definition of $A$ from equation (38), let $f_i$; $m$; $k$ be a collection of polynomial vector fields from $V$! $V^0$ with $f_i \geq 2 A$ for all $i$. Let $C = \text{SCH} (f_i)$; $m$; $k$. If $g \geq 2 C$, then $g \geq 2 A$, and for all $x \in V$,

$$\mathcal{L} \mathfrak{P}(g)(x) \leq \sup_{i} \mathcal{L} \mathfrak{P}(f_i)(x);$$

where $\mathcal{L} \mathfrak{P}$ is the local Lipschitz constant defined in (67) and viewed as a function from $H^1 V^0$.

Proof: Let $\text{Ext} (C)$ denote the extreme points of $C$. Clearly, $\text{Ext} (C) = f_i$; $m$; $g$ so it is finite. Being an element of $C$, $g$ is a linear combination of its extreme points. Since this set is finite and each $f_i \geq 2 A$, we see that $g \geq 2 A$. Since $g = \sum_{j \leq m} f_i$ with $j = 1$, we have that

$$\mathcal{L} \mathfrak{P}(g)(x) \overset{\text{def}}{=} \sum_{i} \mathcal{L} \mathfrak{P}(f_i)(x) \leq \sup_{i} \mathcal{L} \mathfrak{P}(f_i)(x);$$

Corollary 7.7. For all $n \leq 1$, $H_n$ and $H_n$ is a collection of uniformly locally Lipschitz functions from $V$! $V^0$, where the $H$ norm is used on the domain. In particular, there is a constant $p(n) > 1$ and $C(n) > 0$ so that

$$\sup_{g \geq H_n} \mathcal{L} \mathfrak{P}(g)(x) \leq C(1 + \|xk\|^p);$$

for all $x \in V$, where $g$ is viewed a polynomial from $H^1 V^0$. 

Proof. Combine Lemma 10.2 with Lemma 7.6

We now give the workhorse lemma which will be used iteratively in the proof of the main result.

**Lemma 7.8.** Recall that $d$ is the number of Wiener processes driving the system. There is a universal, positive number $n_0(d)$ such that for all $n > n_0(d)$ and all $V$ with $k \leq 1$

\[
\sup_{Q} \mathcal{X}(s); K_{sT} \quad \text{i} < n;
\]

\[
\max_{i} \max_{k_1, \ldots, k_l} \sup_{Q} \mathcal{X}(s); K_{sT} \quad \text{i} > \frac{\text{m}}{3};
\]

\[
\mathcal{H}^0 \{ \max_{i} \max_{k_1, \ldots, k_l} \sup_{Q} \mathcal{X}(s); K_{sT} \quad \text{i} \} \quad \text{is} < \frac{\text{m}}{6};
\]

Here $\mathcal{H}^0$ is also universal, depending only on the number $d$; $m$ is the degree of the polynomial $F$.

Furthermore, there are universal, positive constants $K_1(d); K_2(d)$, and $K_3(d)$ such that

\[
\mathbb{P} (\mathcal{H}^0) \quad K_1 < K_2^n ;
\]

for $n > n_0(d)$.

With these results stated we return to the proof of Theorem 7.4 postponing the other proofs to the end of the section.

**Proof of Theorem 7.4.** First observe that the representation (29) implies that

\[
P \left( \inf_{U} \mathcal{H}(u(T)) \right) ; i < n \quad \mathbb{P} \left( \inf_{U} \mathcal{H}(u(T)) \right) ; i < n ;
\]

\[
P \left( \max_{k=1}^{L} \mathcal{H}(u(T)) \right) ; i < n \quad \mathcal{H}(u(T)) ; i < n ;
\]

where $T$ was again the time fixed at the start of Section 7.2.1.

We now need an elementary auxiliary lemma which can be found in [MP06]. We denote by $H \text{ol}(f)$ the Hölder constant of degree $l$ of a function $f$ (see Section 9.2 for a precise definition).

**Lemma 7.9.** [MP06, Lemma 7.6] For any $n > 0$ and $l > 0$, $R \int_{0}^{T} f(s) ds < n$ and $H \text{ol}(f) < c^n$ imply $\int_{0}^{T} f(s) ds < (1 + c)^{n-1}$.

This lemma implies that for a fixed $n > 0$ and any $l = 1; \ldots, d$,

\[
\max_{k=1}^{L} \max_{k_1, \ldots, k_l} \mathcal{H}(u(T)) ; i < n \quad \mathcal{L}(u(T)) ; i < n ;
\]

\[
\max_{k=1}^{L} \mathcal{H}(u(T)) ; i < n \quad \mathcal{L}(u(T)) ; i < n ;
\]
for $i_2 \in (0; 1]$ where $i_1$ is a universal constant independent of everything in the problem. We also have

$$\inf_{\mathbb{R}} \mathbb{P}(\mathcal{E}_i; K_{s^{1/2}}) \approx \inf_{\mathbb{R}} \mathbb{P}(\mathcal{E}_i; K_{s^{1/2}}) \approx \inf_{\mathbb{R}} \mathbb{P}(\mathcal{E}_i; K_{s^{1/2}});$$

where $\mathbb{P}(\mathcal{E}_i; K_{s^{1/2}}) = \sup_{j \neq i} \mathbb{P}(\mathcal{E}_i); j$. Notice that the event in the r.h.s. of (47) does not depend on $i$. Hence if we define $g = \max(1; \sup_{j \neq i} \mathbb{P}(\mathcal{E}_i); j)$, $\mathbb{P}(\mathcal{E}_i; K_{s^{1/2}}) = \sup_{j \neq i} \mathbb{P}(\mathcal{E}_i); j$, and

$$D(\mathbb{R}) = \mathbb{P}(\mathcal{E}_i; K_{s^{1/2}}) = \mathbb{P}(\mathcal{E}_i; K_{s^{1/2}});$$

we have

$$\inf_{\mathbb{R}} \mathbb{P}(\mathcal{E}_i; K_{s^{1/2}}); i < n^{1/3}; \quad \mathbb{P}(\mathcal{E}_i; K_{s^{1/2}}); i < n^{1/3};$$

for all $i_2 \in (0; 1]$. Estimates from section 11 show that $D(\mathbb{R})$ has sufficiently fast decaying probability as $i_1 \to 0$, so we need to obtain a good estimate on the probability of $A_1(\mathcal{E}_i)$. To that end, we define

$$A_1(\mathcal{E}_i) = \begin{cases} A_1(i_1), & i \in \mathbb{Z} \cap \mathbb{R} \cap \mathbb{N}; \\ 0 & \text{otherwise} \end{cases}$$

where

$$A_1(i_1) = \sup_{\mathbb{Q} \subset \mathbb{H}_1} \mathbb{P}(\mathcal{E}_i; K_{s^{1/2}}); i < n^{1/3}; \quad i = 1; 2; \ldots; n > 0;$$

and $(i) = \frac{1}{6} \mathbb{P}(\mathcal{E}_i; K_{s^{1/2}}); i \in \mathbb{Z} \cap \mathbb{R} \cap \mathbb{N}_1$. (In this definition, we set $\mathbb{H}_0 = i$. Notice that this is consistent with the definition of $A_1(\mathcal{E}_i)$ given above.) Next, we define

$$B_1(\mathcal{E}_i) = \begin{cases} B_1(i_1), & i \in \mathbb{Z} \cap \mathbb{R} \cap \mathbb{N}; \\ 0 & \text{otherwise} \end{cases}$$

where

$$B_1(i_1) = A_1(\mathcal{E}_i; K_{s^{1/2}}); i = 1; 2; \ldots;$$

Notice that $A_1 = \mathcal{A}_1 \setminus \mathcal{A}_2 \setminus \mathcal{A}_3 \setminus \mathcal{B}_2$, etc. Integrating this reasoning produces

$$A_1(\mathcal{E}_i) \approx A_1(\mathcal{E}_i) \approx B_1(\mathcal{E}_i); i = 1; 2; \ldots;$$

so that

$$A_1(\mathcal{E}_i) \approx A_1(\mathcal{E}_i) \approx B_1(\mathcal{E}_i); i = 1; 2; \ldots;$$
Now define

\[
C_1(R) \overset{\text{def}}{=} \sup_{Q \in \text{Ext}_1 H} \max_{k} \max_{j_1, \ldots, j_k} \mathbb{E} [ \mathcal{F} ; Q ; g_{j_1} ; \ldots ; g_{j_k} ] (X(t)) ; K ; \mathbb{F} > R
\]

\[
= \max_{Q \in \text{Ext}_1 H} \max_{k} \max_{j_1, \ldots, j_k} \mathbb{E} [ \mathcal{F} ; Q ; g_{j_1} ; \ldots ; g_{j_k} ] (X(t)) ; K ; \mathbb{F} > R
\]

where \( \text{Ext} \) denotes the set of extreme points of a set. The second equality is implied by the fact that a linear function on a convex closed set attains its maximum at an extreme point of the set. Note also that \( \text{Ext}_1 H \) is finite for all \( i \) (this can be proved by induction in \( i \)).

Since Lemma 7.8 implies \( B_1^* \) is finite \( (i) \), we have

\[
A_1^* \overset{\text{def}}{=} \bigwedge_{i=1}^{n} A_1^* \overset{\text{def}}{=} \bigwedge_{i=1}^{n} H \overset{\text{def}}{=} C_1 \end{equation}

We have that

\[
\mathbb{E} [ \mathcal{F} ; Q ; g_{j_1} ; \ldots ; g_{j_k} ] (X(t)) \mathbb{F} > R
\]

Now setting

\[
\mathcal{D}_1(R) \overset{\text{def}}{=} \max_{Q \in \text{Ext}_1 H} \max_{k} \max_{j_1, \ldots, j_k} \mathbb{E} [ \mathcal{F} ; Q ; g_{j_1} ; \ldots ; g_{j_k} ] (X(t)) \mathbb{F} > R
\]

the second inequality in Lemma 11.3 implies that

\[
(49) \quad C_1(R) \overset{\text{def}}{=} \mathcal{D}_1(R=2) \overset{\text{def}}{=} D \overset{\text{def}}{=} P \overset{\text{def}}{=} D \overset{\text{def}}{=} P
\]

(Recall that \( g \), 1). Defining

\[
H \overset{\text{def}}{=} \bigwedge_{i=1}^{n} H \overset{\text{def}}{=} C \overset{\text{def}}{=} D \overset{\text{def}}{=} A
\]

we have that

\[
(50) \quad A_1^* \overset{\text{def}}{=} \bigwedge_{i=1}^{n} A_1^* \overset{\text{def}}{=} \bigwedge_{i=1}^{n} C \overset{\text{def}}{=} \bigwedge_{i=1}^{n} D \overset{\text{def}}{=} \bigwedge_{i=1}^{n} A
\]

Observe that

\[
(51) \quad \mathbb{P} \overset{\text{def}}{=} \bigwedge_{i=1}^{n} \mathbb{P} \overset{\text{def}}{=} \bigwedge_{i=1}^{n} D \overset{\text{def}}{=} \bigwedge_{i=1}^{n} A
\]
We now show that the probability of each of these terms is $o(n^q)$ for any $q > 1$. Applying the Markov inequality and condition (40) yields

$$P(A_{n,q}) = o(n^q);$$

for all $q > 1$ and $n > 0$. For all $n$ sufficiently small, the right hand side is less than $n^{-2}$.

Lemma 7.8 and the finiteness of $\text{Ext}_{r,H_i}$ imply that there are universal constants $K_1, K_2$ and $\epsilon$, depending only on $n$ and the number of Brownian motions, so that

$$P(H_{n,q}) = o(n^q);$$

for all $q > 1$ and $n > 0$. For all $n$ sufficiently small, the right hand side is less than $n^{-2}$. Lemma 7.8 and the finiteness of $\text{Ext}_{r,H_i}$ imply that there are universal constants $K_1, K_2$ and $\epsilon$, depending only on $n$ and the number of Brownian motions, so that

$$P(H_{n,q}) = o(n^q);$$

for all $q > 1$ and $n > 0$. For all $n$ sufficiently small, the right hand side is less than $n^{-2}$.

Lastly from Lemma 11.2 Corollary 7.7 and Assumption 4, we see that for any $q > 1$ there exists $p_1(n) > 1$ and a constant $C(n)$ so that

$$P(C(n) \leq n^q + p_1(n));$$

for any $n > 0$.

Combining these bounds on the probability of the four sets with (51) completes the proof of the lemma.

**Proof of Lemma 7.8** The proof begins the same way as that of Lemma 7.2. Upon reaching (45), we invoke Theorem 9.8 rather than Theorem 9.3.

8. **Refinements and Generalizations**

We now turn to a number of extensions and generalizations of the preceding results. In the first part of the section, we make more explicit the dependence of the estimates on the initial data. Understanding the dependence of the estimates on the initial data is critical to proving results such as unique ergodicity (see [HM06]).

In the second half of this section, we isolate the main arguments of this paper so that they might be better applied to PDEs which do not fit into the precise setting of this text.

8.1. **Dependence on the initial data**

**Theorem 8.1.** In the setting of Section 2 assume that Assumptions 4 and 5 hold. Additionally, assume that there exists a function $\varphi : H \times [0,1]$ such that for any $p \geq 1$ there exist constants $\varphi_p(T,T)$ and $K_p(T,T)$ so that

$$\varphi_p(T,T,u_0) \leq \varphi(T,T,u_0);$$

$$K_p(T,T,u_0) \leq K_p(T,T,u_0);$$

for all $u_0 \in H_0$.

Consider the setting of Theorem 7.4 if either
i) $S$ is a finite-dimensional subset of $G_n$ for some $n < 1$, or

ii) $S$ is a finite-dimensional subset of $V$ so that for some $n < 1$ and for any $p \geq 1$, the condition given in equation (40) holds. Furthermore, for any $p \geq 1$, there exists a positive constant $h_{p}$. For all $p > 0$, there are positive constants $C_{p}$ and $h_{p}$ such that

$$
\inf_{U \in \mathcal{U}} \mathbb{E} (u(T)) < C_{p} (u_{0})^{h_{p}};
$$

for all $u \in (0;0]$ and $u_{0} > 0$. Here, as before, $U = f \in L_{K}$ and $K$ is the projection onto $S$. In the first case $C$ depends on $p_{T};T;S;u_{p}$ and $K_{p}$, and in the second it also depends on $p$. In both cases $u_{0}$ depends only on $S$, and $C$ depends only on $p$ and $S$.

Proof: Looking back at the proof of Theorem 7.4, we need to obtain a bound of the quoted type on the right hand side of (51). In light of the calculations in the proof bounding the size of the various sets, the probabilities of $D$, $\Phi$, and $H$ are all bounded as desired because of the assumptions of Theorem 8.1. The only set left uncontrolled is $A$.

However, all the vector fields in $G_n$ are constant, and hence there is an $u_0$ sufficiently small and depending only on the structure and size of $G_n$ and the $S$ chosen so that, if $u > 2 (0;0]$, then $A$ is empty.

8.2. Generalizations. We now state a few “meta” theorems. The assumptions require extra work to verify but they isolate the main parts of the argument and allow the ideas to be applied to a wider range of PDEs which do not fit exactly into the previous settings. We relax our assumptions on $N$, assuming only that it is a polynomial from $D \circ m (L)$ into $H$. We assume that with probability one

$$
\bigcup_{2} C \bigcup (0;T) \bigcup \mathcal{L}_{\infty} (0;T) \bigcup D \circ m (L);
$$

Lastly we fix a Banach space $(H_{1};j), (H_{1};j)$, with $H_{1} \subset H$, and assume that for each $g_{k}$ and $H_{1}

$$
h_{g_{k}} \in K_{T} \bigcup 2 \bigcup \{ X : T \cup \mathbb{R} \}
$$

with probability one as a function of $t$. We now define a new set of admissible vector fields.

**Definition 8.2.** $\mathcal{E}$ is the set of all polynomial vector fields $Q : V \rightarrow V^{0}$ such that with probability one the following conditions hold:

i) $Q \in (t) \bigcup 2 \bigcup \{ X : T \cup \mathbb{R} \}

$$
\frac{d}{dt} Q \in (t) \bigcup 2 \bigcup \{ X : T \cup \mathbb{R} \}
$$

ii) For all $0 \leq i \leq m$, $k_{j} \in \mathbb{Z}$, and $m$ and $\mathbb{R}$

$$
h_{g_{k}} \in Q \in (t) ; K_{T} \bigcup 2 \bigcup \{ X : T \cup \mathbb{R} \}
$$

is well defined and in $C \bigcup \{ X : T \cup \mathbb{R} \}$ as a function of $t$.
Next, define \( \mathfrak{H} \) exactly as in (37), replacing \( A \) by \( \mathfrak{A} \).

**Theorem 8.3.** Assume that Assumptions [7] and [8] hold. Let \( S \) be a finite-dimensional linear subspace which is a subset of \( \mathfrak{H} \) \( \cap (T) \) with probability one. Then the distribution of the projection of \( X \) \( (T) \) onto \( S \) is absolutely continuous with respect to Lebesgue measure on \( S \).

Turning to smoothness, define \( L^1 \) to be the space of all processes \( f : [0;T] \to \mathbb{R} \) such that

\[
E \sup_{2H^1} h^\#_{i,j} Q \mathfrak{H} g_k \cdots g_k (X (t); K t) i < 1;
\]

for all \( 0 \leq i \leq m \) and \( k \in \{1,2, \ldots, m\} \).

Lastly, define \( \mathfrak{H} \) as in (39), but with \( A \) replaced by \( \mathfrak{H} \).

**Theorem 8.4.** Assume that Assumptions [7] and [8] hold. Let \( S \) be a deterministic finite-dimensional subspace of \( H^1 \) such that for some \( n \) and \( \delta > 0 \),

\[
e_p (u_0;T) = E \inf_{2H^1} \sup_{Q \mathfrak{H} (X (T))} h^\#_{i,j} Q \mathfrak{H} g_k \cdots g_k (X (t); K t) i < 1;
\]

for all \( p \geq 1 \). Here \( \mathfrak{U} = \mathfrak{H} 2 H^1 : j \leq j_1 \leq 1 ; j \leq j_1 \leq g \). If \( s u (T) 2 D^1 \), then the density of \( s u (T) \) with respect to Lebesgue measure (whose existence is guaranteed by Theorem 8.3) is a \( C^1 \)-function on \( S \).

In the spirit of section 8.1, we now give a “meta” theorem which isolates the dependence on the initial data.

**Theorem 8.5.** As above, assume that Assumptions [7] and [8] hold. Let \( S \) be a deterministic finite-dimensional subspace of \( H^1 \) such that, for some \( n \) and \( \delta > 0 \), the bound in (54) holds.

Let \( \#: H^1 \to [0,1] \) be a function such that, for all \( p \geq 1 \), there exists a \( C_p \) such that:

i) For any \( Q \in \mathfrak{H} \),

\[
E \sup_{2H^1} h^\#_{i,j} Q \mathfrak{H} g_k \cdots g_k (X (t); K t) i < 1;
\]

for all \( u_0 \in H^1 \), \( i \leq m \), and \( k \in \{1,2, \ldots, m\} \).

ii) \( e_p (u_0;T) \leq C_p (u_0) \).

Then the conclusion given in (52) holds with \( \mathfrak{U} \) replaced by the \( \mathfrak{U} \) defined in Theorem 8.4 and for constants with the same dependencies as in Theorem 8.1.
9. Non-adapted polynomials of Wiener processes

This section contains the technical estimates which are the heart of the paper. They are the key steps in the proofs in Section 7 which ensure that the randomness moves, with probability one, to all of the degrees of freedom connected to the noise directions through the nonlinearity. The results in section 9.1 are more qualitative and are the basis of the proof of existence of absolutely continuous densities. Section 9.2 contains the more quantitative estimates needed to prove the smoothness of the density and give estimates on the eigenvalues of the Malliavin matrix. That being said, the basic ideas of the two sections are the same. We show that coefficients of a finite Wiener polynomial (see below for more details) are small with high probability if the entire polynomial is small, even if the coefficients are not adapted to the Wiener processes.

The core idea, used in our context, dates back at least to the pioneering work of Malliavin, Bismut, Stroock and others on the probabilistic proof of the existence of smooth densities for hypoelliptic diffusions in finite dimensions. The techniques developed there (see [KS84, Nor86]) used martingale estimates to relate the size of a process to its quadratic variation. Here we cannot make use of such martingale estimates directly since we have non-adapted stochastic processes. The non-adaptedness arose in a natural way because we only have a semiflow and cannot return all estimates to the tangent space at the origin and work with the reduced Malliavin covariance matrix which is adapted. As is often done, we replace an adaptedness assumption with an assumption on the regularity in time of the processes. This section is a generalization of the results in [MP06] which proved similar results for quadratic polynomials of Wiener processes. The proofs here extend these results to polynomials of any order while also simplifying the proofs.

9.1. Qualitative results. Consider a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). For a stochastic process \(X\) defined on \([0; t]\), we define

\[ hX_1; X_2 i_{\mathcal{T}} = \lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} (t_{j+1} - t_j)(X_1(t_j) - X_1(t_{j+1}))(X_2(t_j) - X_2(t_{j+1})) \]

in probability. If this limit exists, where \(T_1 = t_0 < \cdots < t_N = T_2\) for each \(N\) and \(\sup_{j} t_j\) \(t_j \to 0\) as \(N \to 1\). We shall also write \(hX_1 i_2 = hX_1 i_{\mathcal{T}}\) and \(hX_1 i_3 = hX_1 i_{\mathcal{T},X_2}\).

We begin by considering the basic cross quadratic variation between two monomial terms. We emphasize that the processes \(A(s)\) and \(B(s)\) in the following lemma need not be adapted to the filtration generated by the Wiener processes.

**Theorem 9.1.** Let \(\mathcal{W}_1(s) \mathcal{W}_2(s) \cdots \mathcal{W}_d(s)\) be a collection of mutually independent standard one-dimensional Brownian motions on a time interval \(\mathcal{T}\) and let \(A(s) \mathcal{B}(s)\) be two continuous and bounded variation stochastic processes defined
on $I$. Then

$$
\begin{align*}
&\frac{1}{2} n^\lambda \int_0^1 \cdots \int_0^1 \frac{1}{n} \frac{1}{n} \cdots \frac{1}{n} \\
&= \frac{1}{n} A(s) B(s) \frac{1}{n} \frac{1}{n} \cdots \frac{1}{n}.
\end{align*}
$$

Proof. In the proof we write $\mathcal{W}_1(j)$ instead of $\mathcal{W}_1(t_j)$, $A$ $(j)$ instead of $A(t_j)$, $t_k$, and $j$ instead of $j$. We begin by observing that

$$
\begin{align*}
\chi^\lambda_{j} &= \sum_{i=1}^n \chi_{i} (j) \cdots \chi_{i} (j) + A(j) \sum_{i=1}^n \chi_{i} (j) \cdots \chi_{i} (j) + \cdots \\
&= \sum_{i=1}^n \chi_{i} (j) \cdots \chi_{i} (j) + A(j) \sum_{i=1}^n \chi_{i} (j) \cdots \chi_{i} (j) + \cdots \\
&= \sum_{i=1}^n \chi_{i} (j) \cdots \chi_{i} (j) + A(j) \sum_{i=1}^n \chi_{i} (j) \cdots \chi_{i} (j) + \cdots
\end{align*}
$$

where

$$
\begin{align*}
Q_{j}^{(a)} &= \sum_{i=1}^n \chi_{i} (j) \cdots \chi_{i} (j) + A(j) \sum_{i=1}^n \chi_{i} (j) \cdots \chi_{i} (j) + \cdots \\
&= \sum_{i=1}^n \chi_{i} (j) \cdots \chi_{i} (j) + A(j) \sum_{i=1}^n \chi_{i} (j) \cdots \chi_{i} (j) + \cdots
\end{align*}
$$

and

$$
\begin{align*}
Q_{j}^{(b)} &= \sum_{i=1}^n \chi_{i} (j) \cdots \chi_{i} (j) + A(j) \sum_{i=1}^n \chi_{i} (j) \cdots \chi_{i} (j) + \cdots \\
&= \sum_{i=1}^n \chi_{i} (j) \cdots \chi_{i} (j) + A(j) \sum_{i=1}^n \chi_{i} (j) \cdots \chi_{i} (j) + \cdots
\end{align*}
$$

Therefore, the sum in (55) contains the following terms:

$$
\begin{align*}
\chi^\lambda_{j} &= \sum_{i=1}^n \chi_{i} (j) \cdots \chi_{i} (j) + A(j) \sum_{i=1}^n \chi_{i} (j) \cdots \chi_{i} (j) + \cdots \\
&= \sum_{i=1}^n \chi_{i} (j) \cdots \chi_{i} (j) + A(j) \sum_{i=1}^n \chi_{i} (j) \cdots \chi_{i} (j) + \cdots
\end{align*}
$$

Therefore, the sum in (55) contains the following terms:

$$
\begin{align*}
\chi^\lambda_{j} &= \sum_{i=1}^n \chi_{i} (j) \cdots \chi_{i} (j) + A(j) \sum_{i=1}^n \chi_{i} (j) \cdots \chi_{i} (j) + \cdots \\
&= \sum_{i=1}^n \chi_{i} (j) \cdots \chi_{i} (j) + A(j) \sum_{i=1}^n \chi_{i} (j) \cdots \chi_{i} (j) + \cdots
\end{align*}
$$

Therefore, the sum in (55) contains the following terms:

$$
\begin{align*}
\chi^\lambda_{j} &= \sum_{i=1}^n \chi_{i} (j) \cdots \chi_{i} (j) + A(j) \sum_{i=1}^n \chi_{i} (j) \cdots \chi_{i} (j) + \cdots \\
&= \sum_{i=1}^n \chi_{i} (j) \cdots \chi_{i} (j) + A(j) \sum_{i=1}^n \chi_{i} (j) \cdots \chi_{i} (j) + \cdots
\end{align*}
$$

Therefore, the sum in (55) contains the following terms:

$$
\begin{align*}
\chi^\lambda_{j} &= \sum_{i=1}^n \chi_{i} (j) \cdots \chi_{i} (j) + A(j) \sum_{i=1}^n \chi_{i} (j) \cdots \chi_{i} (j) + \cdots \\
&= \sum_{i=1}^n \chi_{i} (j) \cdots \chi_{i} (j) + A(j) \sum_{i=1}^n \chi_{i} (j) \cdots \chi_{i} (j) + \cdots
\end{align*}
$$

The first three sums above converge to zero as $t \to t_j \to 0$, since $A$ and $B$ are of bounded variation and continuous and all $W_i$ are continuous. Lemmas 4.2
and 4.3 from [MP06] imply that the fourth sum above converges to
\[
Z = \sum_{i} A(s) \mathbb{E}(s) \int_{t_{k-1}}^{t_{k}} W_{l_{i} k_{i}}(s) \cdots W_{l_{1} k_{1}}(s) \cdots W_{l_{n} k_{n}}(s) \, ds;
\]
and the theorem is proved.

**Corollary 9.2.** Let \( \mathcal{A} \) be collection of stochastic processes on \( \mathbb{F} \) such that there is a set \( \mathbb{F}_{0} \) with \( \mathbb{P}(\mathbb{F}_{0}) = 1 \), so that for each \( Z \in \mathbb{F}_{0} \) all of the processes in \( \mathcal{A} \) are of bounded variation and continuous.

Then there is a set \( \mathbb{F}_{0} \) with \( \mathbb{P}(\mathbb{F}_{0}) = 1 \), and a sequence of partitions \( t_{N} = t_{0} < \cdots < t_{N}^{N} = t_{N} \) such that for any process \( Z(t) \) of the form
\[
Z = A^{(0)} + \sum_{i} A^{(1)}_{i} W_{i_{1}} + \cdots + \sum_{i} A^{(n)}_{i_{1} \cdots i_{n}} W_{i_{1}} \cdots W_{i_{n}};
\]

with \( A_{i_{1} \cdots i_{n}} \in \mathcal{A} \), one has that the limit
\[
\lim_{N \to \infty} \sum_{j=1}^{2^{N}} t_{j} Z(t_{j})
\]
exists on \( \mathbb{F}_{0} \) and equals \( \mathbb{E}(Z) \).

**Proof.** We notice that the proof of Theorem 9.1 implies that there is a full measure set \( \mathcal{F} \) that is defined in terms of the Wiener processes involved, with the following property: if for all \( Z \in \mathcal{F} \) the realization of a process \( A \) possesses the mentioned regularity properties, then the desired convergence holds. The proof is completed by setting \( \mathbb{F}_{0} = \mathbb{F} \setminus \mathcal{F} \).

We now use the previous results to prove that in the setting of the previous corollary, if \( Z \) is identically zero, then the coefficients \( A_{i_{1} \cdots i_{n}} \) must be identically zero.

**Theorem 9.3.** Let \( \mathcal{A} \) and \( Z \) be as in the above corollary, and let \( \mathbb{F}_{0} \) be the set given in the conclusion of the same corollary. Additionally, assume that, for each \( i_{1} \cdots i_{n} \), the coefficients \( A^{(n)}_{i_{1} \cdots i_{n}} \) are symmetric (i.e. invariant under substitutions on indices \( i_{1} \cdots i_{n} \)).

If \( Z(s) = 0 \) for all \( s \in [0; T] \) with probability one, then all the processes \( A^{(n)}_{i_{1} \cdots i_{n}} \) are identically zero on \([0; T]\) with probability one.
Proof: We proceed by induction. For \( n = 0 \) the statement of the theorem is obvious. Now suppose \( n > 1 \). Then

\[
\begin{align*}
\mathbb{Z} \mathbf{1}_T &= X^n X A_{\mathbf{k}^1} W_{\mathbf{i}^1} \cdots W_{\mathbf{i}^m} X A_{\mathbf{k}^m} W_{\mathbf{k}^m} \cdots W_{\mathbf{k}^m} T \\
Z^T &= X^n X X A_{\mathbf{k}^1} A_{\mathbf{k}^m} X X W_{\mathbf{i}^1} \cdots W_{\mathbf{i}^m} W_{\mathbf{k}^1} \cdots W_{\mathbf{k}^m}
\end{align*}
\]

Since we assumed that \( Z(s) = 0 \) for \( s \leq 0 \), and the integrand is continuous, we conclude that

\[
Z^x(s) = X^n X X A_{\mathbf{k}^1} A_{\mathbf{k}^m} (s) W_{\mathbf{i}^1} (s) \cdots W_{\mathbf{i}^m} (s) = 0
\]

for each \( x = 1, \ldots, d \) and all \( s \leq 0 \). Notice now that due to the symmetry of coefficients \( A \) the process \( Z^x(s) \) satisfies the assumptions of the theorem with \( n \) reduced by one. That \( A_{\mathbf{k}^1} (s) = 0 \) a.s. for \( s > 1 \), follows from the fact that all coefficients of \( Z_i(s) \) are equal to zero a.s. by the induction hypothesis. Since \( Z_0 = 0 \) and \( A^{(0)} = 0 \) for positive \( \), we conclude that \( A^{(0)} = 0 \) as well. The theorem is proved.

9.2. More Quantitative Estimates. Now our aim is to prove a quantitative version of the last theorem. Again we consider a process \( Z(t) \) of the same form as in Corollary 9.2. To do so we introduce a family of Wiener polynomials with constant coefficients which will be used to approximate \( Z \). Namely, for any nonnegative integer \( n \) and collection of coefficients \( \) with

\[
\mathbf{i}_1, \ldots, \mathbf{i}_n = 1, \ldots, d
\]

we define

\[
Z = X^{(0)} + X^{(1)} W_{\mathbf{i}_1} + X^{(2)} W_{\mathbf{i}_2} + \cdots + X^{(n)} W_{\mathbf{i}_n}
\]

We now introduce a collection of typical coefficients, a set of typical Wiener processes, and a collection of atypical \( Z \), which are too small in light of their coefficients not being uniformly small. This last set captures the event which we wish to describe, but for the \( Z \) rather than the \( Z \). We begin with the coefficients \( \), which we do not want to be uniformly too small.
For a real number $\gamma > 0$ and a nonnegative integer $n$ define \( (^n; n) \) to be the set of coefficients \( \binom{n}{i_1, \ldots, i_d} \) such that
\[
\max_{j \leq n} \left\| \sum_{i_1, \ldots, i_d = 0}^{n} i_1^{j_1} \cdots i_d^{j_d} \right\| = 0; \ldots; n; i_1; \ldots; i_d = 1; \ldots; d.
\]

We now define a set of atypical $Z$ with $\gamma > 0$ and divide the segment $[0; T]$ into $m = \lfloor T \gamma^{-1} \rfloor + 1$ segments $I_1 = [0; t_1); I_2 = [t_1; t_2); \ldots; I_m = [t_{m-1}; t_m)$, each one of length less than $\frac{\gamma}{2} \gamma^{-1}$ and greater than $\frac{1}{2} \gamma^{-1}$.

Let
\[
D \left( ^\gamma; I; \left( ^n; n \right) \right) = \inf_{Z \in \left( ^n; n \right)} \sup_{t \in I} \| Z(t) \| < \gamma,
\]
and define
\[
F \left( ^\gamma; \right) = \bigcap_{k=1}^{m} D \left( ^\gamma; I_k; \left( ^n; n \right) \right).
\]

To define the set of typical Wiener trajectories, recall that for any function $f : [0; T] \to \mathbb{R}$ we define its Hölder constant by
\[
\text{Höld} (f) = \sup_{0 < s < r} \frac{|f(s) - f(r)|}{|s - r|};
\]
and
\[
\| f \|_{L^1} = \max_{i_1, \ldots, i_d} \left\| f \right\|_{L^1} < \gamma.
\]

With this definition, we introduce the set of Wiener processes
\[
B (\mathbb{R}) = \bigcap_{i_1, \ldots, i_d = 1}^{n} \left\{ \mathcal{W}_{i_1; \ldots; i_d} < \mathbb{R}; = 1; \ldots; n; i_1; \ldots; i_d = 1; \ldots; d \right\}.
\]

**Remark 9.4.** Notice that the sets $B$ and $F$ are universal in that they do not depend on the processes $A$ in any way other than through the number $n$.

We now are ready to state the quantitative version of Corollary 9.2. We want to conclude that if $Z$ is small it is unlikely that the $A$ processes are not small. The sets $D$ and $E$ below embody the first event and the complement of the second event, respectively:
\[
D \left( ^\gamma \right) = \left\{ \mathcal{F} \mathcal{W}_{i_1; \ldots; i_d} < \gamma \right\};
\]
\[
E \left( ^\gamma \right) = \max_{i_1, \ldots, i_d = 1} \max_{\mathcal{A}_{i_1; \ldots; i_d}} \mathcal{F} \mathcal{W}_{i_1; \ldots; i_d} < \gamma.
\]

To state the result we need to define a localization set which ensures that we can well approximate $Z$ by a $Z$ process with $\gamma > 0$. Defining
\[
C (\mathbb{R}) = \bigcap_{i_1, \ldots, i_d = 1}^{n} \left\{ \mathcal{F} \mathcal{A}_{i_1; \ldots; i_d} < \mathbb{R}; = 1; \ldots; n; i_1; \ldots; i_d = 1; \ldots; d \right\},
\]
we have the desired results.
**Theorem 9.5.** For each $n$ there is $\epsilon_0(n)$ depending only on $n, d$ and $T$ such that

\[(56) \quad D(\mathbb{W}^{n+2}) \setminus E \cap (n) \setminus C(n^{-1}) \supset B \cap (n^{1-5}) \setminus F(\mathbb{W}^{n+1}, 5 \epsilon_0(n^{-1}) \frac{1}{n+T});\]

for all $n < \epsilon_0$:

**Theorem 9.6.** For each $n$ there are positive numbers $\epsilon_1(n); q_1(n); K_1(n); K_2(n)$ depending only on $n, d$ and $T$ such that

\[P \cdot B \cap (n^{1-5}) \setminus F(\mathbb{W}^{n+1}, 5 \epsilon_0(n^{-1}) \frac{1}{n+T}) < K_1 \exp K_2 q_1 g;\]

if $n < \epsilon_1$:

**Remark 9.7.** Theorem 9.6 provides an estimate of the set appearing in the statement of Theorem 9.5. Thus, these two theorems say that if $Z$ is small (the event $D(\mathbb{W}^{n+2})$), then with high probability the coefficients $A$ defining $Z$ are small as well (the event $E(n)$) on the localization set $C(n^{-1})$. Since the $A$ are not necessarily adapted, one aim of Theorem 9.5 is to reduce the problem to the traditional stochastic Itô calculus. Notice also that the events in the r.h.s. of (56) are defined only in terms of the Wiener processes $W$.

We will in fact find not that $Z$ is uniformly small in time, but rather that its integral in time is small. However the following results show how to reduce this case to the previously considered setting.

Consider an arbitrary $R$-valued random variable $g_0$ and define

\[Z(t) = g_0 + Z(s)ds; \quad \overline{D(n)} = \epsilon \sqrt{K_0L}; \quad C(R) = C(R) \setminus E(R);\]

**Theorem 9.8.** For each $n$ there is $\epsilon_0(n)$ depending only on $n, d$ and $T$ such that

\[(57) \quad \overline{D(n)} \setminus E \cap (n^8(n+3) \setminus C(n^8(n+3)) \setminus B \cap (n^{8(n+3)-5}) \setminus F(\mathbb{W}^{n+1}, n^8(n+3) \frac{1}{n+T}));\]

for all $n < \epsilon_0$:

The probability of the r.h.s. is estimated in the following theorem, which is a direct consequence of Theorem 9.6.

**Theorem 9.9.** For each $n$ and numbers $\epsilon_1(n); q_1(n); K_1(n); K_2(n)$ defined in Theorem 9.6

\[P \cdot B \cap (n^8(n+3)-5) \setminus F(\mathbb{W}^{n+1}, n^8(n+3) \frac{1}{n+T})) < K_1 \exp K_2 n^8(n+3) q_1 g;\]

if $n < \epsilon_1$:

**Theorem 9.8** will follow from Theorem 9.5 and the next lemma taken from [MP06]. We will give the proof of Theorem 9.8 before returning to the proof of Theorem 9.5 and Theorem 9.6

**Lemma 9.10.** Let

\[G(t) = G_0 + \int_0^t H(s)ds;\]
where \( G \) and \( H \) are \( R \)-valued functions and \( G_0 \geq 0 \). Suppose \( H \circ l \) (\( H \)) \( c^n \) for some fixed \( n > 0 \) and \( m > 0 \). If \( t \leq \frac{n}{m} \), then \( kH_k \) \( (2 + c)^m \).

**Proof of Theorem 9.8** We begin by considering a generic term \( A_{1_1 \ldots i_k}^{(k)} \) from \( Z \). On \( B^{(m, g, (n + 3))} \setminus \overline{C}^{(m, g, (n + 3))} \), we have that

\[
\begin{align*}
H \circ l_{k-4} (A_{1_1 \ldots i_k}^{(k)}) W_{i_k} : \ldots : W_{i_1} & = L_{k-4} (A_{1_1 \ldots i_k}^{(k)}) k W_{i_k} : \ldots : W_{i_1} k \ L_{k-1} W_{i_1} : \ldots : W_{i_k} k L_{k-1} \\
& + H \circ l_{k-4} (W_{i_1} : \ldots : W_{i_k}) k A_{1_1 \ldots i_k}^{(k)} k L_{k-1}
\end{align*}
\]

Since there are no more than \( c^n \) such terms for each degree between 0 and \( n + 1 \), on \( B^{(m, g, (n + 3))} \setminus \overline{C}^{(m, g, (n + 3))} \) we have

\[
H \circ l_{k-4} (Z) < 2(n + 1) c^n \leq 8^{(n + 3)}.
\]

Then Lemma 9.10 implies

\[
kZk _k \geq (2 + 2(n + 1) c^n)^m
\]

Define

\[
\frac{m}{(2 + 6(n + 3)) - 1 + 2 + (n + 4)(n + 2)} = \left( \frac{n + 3}{n + 4} \right)^m
\]

Then (58) implies that for small \( m \) on \( D^{(m)} \setminus B^{(m, g, (n + 3))} \setminus \overline{C}^{(m, g, (n + 3))} \ D (8^{(n + 2)}) ;

\[
\begin{align*}
& (59) \\
& \overline{D}^{(m)} \setminus B^{(m, g, (n + 3))} \setminus \overline{C}^{(m, g, (n + 3))} \ D (8^{(n + 2)}) ;
\end{align*}
\]

Next,

\[
\begin{align*}
& \overline{D}^{(m)} \setminus B^{(m, g, (n + 3))} \setminus \overline{C}^{(m, g, (n + 3))} \setminus E \ C ( ) \\
& = \overline{D}^{(m)} \setminus B^{(m, g, (n + 3))} \setminus \overline{C}^{(m, g, (n + 3))} \setminus E \ C ( ) \setminus C ( 1 ) \\
& \setminus D (8^{(n + 2)}) \setminus E \ C ( ) \setminus C ( 1 ) \setminus B (1 - 5) \\
& \setminus F (8^{(n + 1)} 5^{(n - 4)} 2^{4 - (n + 1)/2}) ;
\end{align*}
\]

where the identity is implied by \( \overline{C}^{(m, g, (n + 3))} \setminus C ( 1 ) \), the first inclusion is a consequence of (59) and \( B (1 - 5) = B^{(m, g, (n + 3))} \), and the second one from Theorem 9.5. Now (57) is equivalent to (60), and the proof is complete.

We now return to the proofs of the central results of this section.

**Proof of Theorem 9.5** Consider

\[
G (m) = D^{(m \times 2)} \setminus E \ C (m) \setminus C (1) \setminus B (m \times 5) ;
\]
To prove the theorem it is sufficient to show
\[ G (\cdot) \quad F \left( \left( n^{6n+1} \frac{5-4}{n^{2+\frac{3}{n+1}}} n \right) \right); \]

We have
\[ G (\cdot) = \frac{G}{n \cdot k}; \]

where
\[ G_k (\cdot) = D \left( \left( n^{6n+2} \frac{5-4}{n^{2+\frac{3}{n+1}}} n \right) \right) \cap \mathcal{C} (\cdot) \cap \mathcal{B} (\cdot); \]

\[ D (\cdot; I) = \frac{f}{n} \mathcal{J}, \quad (I) < \mathcal{N}; \]

\[ E (\cdot; I) = \max \max \mathcal{J} \mathcal{K} (\cdot) \mathcal{N} \mathcal{K}^\dagger (\cdot); \]

Define
\[ \mathcal{K}_{k} = \mathcal{N} (\cdot; \mathcal{I}) \mathcal{T}_{k}; \]

On \( G_k (\cdot) \)
\[ k \zeta \mathcal{K}_{k} \mathcal{I} (\cdot; \mathcal{I}) = \max \mathcal{J} \mathcal{K}^\dagger (\cdot) \mathcal{N} \mathcal{K}^\dagger (\cdot); \quad k \mathcal{J} \mathcal{K}^\dagger (\cdot; \mathcal{I}) \mathcal{T}_{k} \mathcal{J} \mathcal{I}; \]

for sufficiently small \( \mathcal{N} \), since
\[ 8^{n+2} + (n+1) d_1 \mathcal{K}^\dagger \mathcal{J} \mathcal{K}^\dagger n \mathcal{J} \mathcal{K}^\dagger \mathcal{J} \mathcal{I} \mathcal{T}_{k} \mathcal{J} \mathcal{I} \mathcal{N} \mathcal{K}^\dagger (\cdot; \mathcal{I}) \mathcal{T}_{k} \mathcal{J} \mathcal{I}; \]

for sufficiently small \( \mathcal{N} \), since
\[ 8^{n+2} + 1 + 8^{n+1} 3 = 2 1 = 5 > 8^n + 5 = 4; \]

On the other hand, for \( ! \mathcal{N} \mathcal{K}_{k} \) there exists an \( \mathcal{Q} \mathcal{I} \mathcal{J} \mathcal{I} \mathcal{N} \mathcal{K}^\dagger (\cdot; \mathcal{I}) \mathcal{T}_{k} \mathcal{J} \mathcal{I}; \]

Hence,
\[ G_k (\cdot) \quad D \left( \left( n^{6n+1} \frac{5-4}{n^{2+\frac{3}{n+1}}} n \right) \right) \quad F \left( \left( n^{6n+1} \frac{5-4}{n^{2+\frac{3}{n+1}}} n \right) \right); \]

Proof of Theorem 9.6. We begin by remarking that classical estimates on the supremum
and Hölder continuity of a Wiener process combine to yield
\[ \mathcal{P} \mathcal{N}, \mathcal{C} (\cdot; \mathcal{I}) \mathcal{T}_{k} \mathcal{J} \mathcal{I} \mathcal{N} \mathcal{K}_{k} \mathcal{I} \mathcal{N} \mathcal{K}^\dagger (\cdot; \mathcal{I}) \mathcal{T}_{k} \mathcal{J} \mathcal{I}; \]

for some positive \( \mathcal{K}_{k} \mathcal{I} (\cdot; \mathcal{I}) \mathcal{T}_{k} \mathcal{J} \mathcal{I} \mathcal{N} \mathcal{K}_{k} \mathcal{I} \mathcal{N} \mathcal{K}^\dagger (\cdot; \mathcal{I}) \mathcal{T}_{k} \mathcal{J} \mathcal{I}; \)

Theorem 9.6 is then implied by the identity
\[ \mathcal{B} \mathcal{C} (\cdot; \mathcal{I}) \mathcal{T}_{k} \mathcal{J} \mathcal{I} \mathcal{N} \mathcal{K}_{k} \mathcal{I} \mathcal{N} \mathcal{K}^\dagger (\cdot; \mathcal{I}) \mathcal{T}_{k} \mathcal{J} \mathcal{I}; \]

the estimate from (61), and the following lemma whose proof fills the remainder
of this section.
Lemma 9.11. For every $n$ there are positive numbers $q_3(n)$, $K_5(n)$, $K_6(n)$, and $n_2(n)$ such that for all $k = 1; \ldots; n$,

$$P \left( \left( W_{n+1}^6 \right)^{1 \over 2} \sum_{i=1}^n I_i \right) ; \left( W_{n+1}^{2+{1 \over 2}} \right)^{1 \over 2} \sum_{i=1}^{n_2} I_i \right) \setminus B \left( W_{n+1}^{1+5} \right) K_5 \exp f K_6 \in \mathbb{Q} \ g;$$

for $n < n_2$.

We shall derive this Lemma from the next one.

Lemma 9.12. For every $n$ there are positive numbers $q_3(n)$, $K_7(n)$, $K_8(n)$, $n_3(n)$ with the following property.

Let $\left( \left( I_i \right)_{i=1}^n \right); = 0; \ldots; n; i_1; \ldots; i = 1; \ldots; d$ be a symmetric family of coefficients satisfying

$$\max_{n \geq 0} \sum_{i=1}^n J_{i=1}^{n} \left( W_{n+1}^{2+{1 \over 2}} \right)^{1 \over 2} = 0; \ldots; n; i_1; \ldots; i = 1; \ldots; d \in 1:$$

Define

$$D (n; I; \cdots) = \sup_{t \geq 1} \mathbb{E} (t) \left| j < n \right| :$$

Then

$$P \left( \left( W_{n+1}^6 \right)^{1 \over 2} \sum_{i=1}^n I_i \right) \setminus B \left( W_{n+1}^{1+5} \right) K_7 \exp f K_8 \in \mathbb{Q} \ g;$$

for $n < n_3$.

Proof. We shall prove this lemma by induction in $n$. If $n = 0$, then the statement of the lemma is obvious with the probability in the l.h.s. being equal to 0.

In the induction step we may always assume that

$$\max_{n \geq 0} \sum_{i=1}^n J_{i=1}^{n} \left( W_{n+1}^{2+{1 \over 2}} \right)^{1 \over 2} = 0; \ldots; n; i_1; \ldots; i = 1; \ldots; d \in 1:$$

Since the coefficients are not random, we can use the Itô formula to write down the semimartingale representation of $Z$, namely,

$$Z (t) = V (t) + M (t);$$

where the finite variation part $V$ (which is, in fact, continuously differentiable a.s.) is given by

$$V (t) = \left( 0 \right) + \sum_{i=1}^n \sum_{i_1}^{i_2} W_{i} (t_1) \left( t_2 \right) \cdots \left( t_3 \right)$$

and the martingale part $M$ is given by
For a function $f$ defined on a set $S$ denote

$$\overline{\text{osc}} f = \sup_{s \in S} f(t) : s; t \in S.$$

Since $\sup_{j} j < 2^{n+1}$ implies $\overline{\text{osc}} Z < 2^{n+1}$, the event of interest can be decomposed as

$$D (n^{n+1}; i; t) \setminus B (" 1=5)$$

$$\overline{\text{osc}} V < 2^{n+1}; \sup_{t \in I} M (t) j < 3n^{n+1} \setminus B (" 1=5)$$

$$[ \overline{\text{osc}} V > 2^{n+1} \setminus B (" 1=5) :$$

For small $\nu$ the set $\overline{\text{osc}} V > 2^{n+1} \setminus B (" 1=5)$ in the decomposition above is empty. Indeed, (62) implies that on this event each integral term with coefficient $\overline{\text{osc}}$ in (63) is bounded by $2^{n+1} \setminus B (" 1=5)$ for a positive and sufficiently small $\nu$, and there are only finitely many terms. Now,

$$\overline{\text{osc}} V < 2^{n+1}; \sup_{t \in I} M (t) j < 3n^{n+1} \setminus B (" 1=5)$$

$$\sup_{t \in I} M (t) j < 3n^{n+1} \setminus B (" 1=5)$$

$$\sup_{t \in I} M (t) j < 3n^{n+1}; \text{H} M i > 2^{n+1} 15-8 \setminus B (" 1=5)$$

$$[ \sup_{t \in I} M (t) j < 3n^{n+1}; \text{H} M i > 2^{n+1} 15-8 \setminus B (" 1=5) :$$

Let us denote the sets in the r.h.s. by $D_1$ and $D_2$ respectively. To estimate the probability of the set $D_1$ we need the following lemma (see [Bass,p.209])

**Lemma 9.13.** There exist $c_1; c_2 > 0$ such that if $M_t$ is a continuous martingale, $T$ is a bounded stopping time, and $\nu > 0$, then

$$\mathbb{P} \sup_{t, T} M_{t \wedge T} > \nu c_1 e^{-c_2 \nu} :$$
This result allows to conclude that

\[ P(D_1) \leq C \exp \left( \frac{C_2}{9} n^3 \right)^{\frac{1}{2}}. \]

To estimate \( P(D_2) \), we notice that the proof of Theorem 9.3 and the continuous differentiability of \( V \) imply that

\[ \mathbb{E} \left[ \sum_{r=1}^{\infty} \left( \prod_{i=1}^{r} \mathbb{E} \left[ \int_{0}^{1} \left( \sum_{i=r+1}^{N} \mathbb{E} \left[ \int_{0}^{1} \left( \sum_{i=r+1}^{N} \cdots \right) \right] \right) \right. \right. \right. \]

Therefore,

\[ P(D_2) \leq \max_{r=1}^{\infty} P(D_2) \setminus B(n^{1-5}); \]

where

\[ D_2 = \left\{ \int_{0}^{1} \left( \sum_{i=1}^{r} \mathbb{E} \left[ \prod_{i=1}^{r} \mathbb{E} \left[ \int_{0}^{1} \left( \sum_{i=r+1}^{N} \mathbb{E} \left[ \int_{0}^{1} \left( \sum_{i=r+1}^{N} \cdots \right) \right] \right) \right. \right. \right. \right. \right. \right. \]

There exist \( i_1, i_2, \ldots, i_r \) such that \( j = \frac{1}{n^{2+ \frac{1}{r}}}. \) If \( i \neq 0, \) then choose \( r \) so that the definition of \( D_2 \) contains that \( \frac{1}{n^{2+ \frac{1}{r}}}. \) and define

\[ Z_{\mu} = \left\{ \mathbb{E} \left[ \prod_{i=1}^{r} \mathbb{E} \left[ \int_{0}^{1} \left( \sum_{i=r+1}^{N} \mathbb{E} \left[ \int_{0}^{1} \left( \sum_{i=r+1}^{N} \cdots \right) \right] \right) \right. \right. \right. \]

We want to prove that

\[ D_2 \setminus B(n^{1-5}) = \max_{r=1}^{\infty} \mathbb{E} \left[ \prod_{i=1}^{r} \mathbb{E} \left[ \int_{0}^{1} \left( \sum_{i=r+1}^{N} \mathbb{E} \left[ \int_{0}^{1} \left( \sum_{i=r+1}^{N} \cdots \right) \right] \right) \right. \right. \right. \]

On the set \( B(n^{1-5}) \) the Hölder constant of \( Z_{\mu} \) is bounded by \( n^{c_0} \) \( n^{2+1} \) \( n^{1+1} \). So, if the condition \( \sup_{r=1}^{\infty} \mathbb{E} \left[ \int_{0}^{1} \left( \sum_{i=r+1}^{N} \mathbb{E} \left[ \int_{0}^{1} \left( \sum_{i=r+1}^{N} \cdots \right) \right] \right) \right. \]

for some constant \( c_0 \), since \( 8^n < 1 = 5 + 2 + 1 = (n + 1) + 8^{n+1} \) \( 3 = 8. \) Thus, on \( D_2 \) we have

\[ \max_{r=1}^{\infty} \mathbb{E} \left[ \int_{0}^{1} \left( \sum_{i=r+1}^{N} \mathbb{E} \left[ \int_{0}^{1} \left( \sum_{i=r+1}^{N} \cdots \right) \right] \right) \right. \]

which is impossible for small \( n \). Therefore, our assumption was false and (65) is proved. Now (65) and the induction assumption imply

\[ P(D_2) \setminus B(n^{1-5}) = \max_{r=1}^{\infty} \mathbb{E} \left[ \prod_{i=1}^{r} \mathbb{E} \left[ \int_{0}^{1} \left( \sum_{i=r+1}^{N} \mathbb{E} \left[ \int_{0}^{1} \left( \sum_{i=r+1}^{N} \cdots \right) \right] \right) \right. \right. \]

(65)
Consider now the case where \( j_{0,j}^{(n)} < \frac{n^{2+\frac{1}{n+1}}}{n+1} \) if \( \theta = 0 \); and \( j_{0,j}^{(0)} = n^{2+\frac{1}{n+1}} \). Denote
\[
W = \sup_{j_{0,j}^{(n)}} \sup_{i < j} j_{i} = 1; \ldots; n \quad 1; i_{1}; \ldots; i = 1; \ldots; d:\]
We have
\[
P \left( D \left( n^{n^2+1}; I_{i} \right) \right) \quad P \in n^{2+\frac{1}{n+1}} \quad n d^{n^{2+\frac{1}{n+1}}} W < n^{n^{2+1}} g = P \quad W > \frac{n^{2+\frac{1}{n+1}} n^{n^{2+1}}} n d^{n^{2+\frac{1}{n+1}}} W :
\]
Since \( n^{2+1} > \frac{2}{n+1} \) and \( 2 + \frac{1}{n} > 2 + \frac{1}{n+1} \),
\[
P \left( D \left( n^{n^2+1}; I_{i} \right) \right) \quad K \exp f K_{0} (n)^{6} \quad q g;
\]
for some positive constants \( K \exp f K_{0} (n)^{6} ; q g \). This completes the proof of the lemma.

**Proof of Lemma 9.11** It suffices to show
\[
P \left( D \left( n^{n^2+1}; I_{i} \right) \right) \quad B \left( m^{\frac{1}{n+1}} \right) \quad K_{11} \exp f K_{12} q^{6} q g;
\]
for some positive constants \( K_{11} (n) ; K_{12} (n) ; q g \). where \( (\cdot ; n) \) is the set of vectors \( \left( \frac{n}{i_{0}} ; \ldots ; i_{0} \right) = 0; \ldots; n \); and \( i_{0} ; i_{0} ; \ldots ; i_{0} = 1; \ldots; d \) such that
\[
\max_{j_{0,j}^{(n)}} j_{0,j}^{(n)} = 0; \ldots; n \quad 1; i_{0} ; \ldots ; i_{0} = 1; \ldots; d = m:
\]
For sufficiently small \( \theta > 0 \) there is a set of points \( f (j) ; j = 1; \ldots; \left( \frac{n+1}{d^{6}} \right) \) in \( (n^{n^2+1}; n) \) such that for every \( 2 \left( n^{n^2+1}; n \right) \) there is \( j \) such that \( j_{0,j}^{(n)} j < n^{n^2+\frac{1}{n+1}} \) for all \( \theta \) \( i_{0}; \ldots ; i_{0} \). This implies
\[
\mathcal{E} \left( (j) \right) \quad Z \quad j = \left( n+1 \right) d^{n^{2+\frac{1}{n+1}}} n^{1+5} ;
\]
Choose \( n^{n^2+1} = 3^{2} \); If \( \sup_{j} \mathcal{E} \) \( j < n^{n^2+1} = 5^{4} \), then
\[
\mathcal{E} \left( n^{n^2+1} 3^{2} j \right) \quad n^{n^2+1} = 3^{4} \left( n+1 \right) d^{n^{2+\frac{1}{n+1}}} n^{n^2+1} 3^{4} n^{1+5} < n^{n^2+1} ;
\]
Therefore, Lemma 9.12 implies
\[
P \left( D \left( n^{n^2+1}; I_{i} \right) \right) \quad B \left( m^{n} \right)
\]
\[
\left( \frac{n+1}{d^{6}} + 1 \right) \sup_{x \left( n^{n^2+1}; n \right)} P \left( D \left( n^{n^2+1}; I_{i} \right) \right) \quad B \left( m^{n} \right) \quad K \exp f K_{8} q^{6} q g:
\]
This completes the proof of Lemma 9.11.
10. POLYNOMIAL VECTOR FIELDS. DERIVATIVES AND LIE BRACKETS

We start with a characterization of multilinear continuous operators, which is an obvious generalization of the linear case:

**Lemma 10.1.** Let $X$ and $Y$ be two Banach spaces. Let $Q : X^m \rightarrow Y$ be an $m$-linear operator which is continuous at zero. Then

$$D (x_1, \ldots, x_m) \cdot k = c x_1 \cdots x_m \cdot k;$$

where

$$c = \sup_{x_1, \ldots, x_m} D (x_1, \ldots, x_m) \cdot \frac{z}{k};$$

We define the local Lipschitz constant for a map $Q : X \rightarrow Y$ as

$$Lip (Q) (x) = \lim_{m \rightarrow 0} \sup_{z \in X \atop k \in Y} \frac{D (x) \cdot Q (z) \cdot k}{z \cdot k};$$

**Lemma 10.2.** Let $X$ and $Y$ be two Banach spaces. Suppose $Q : X \rightarrow Y$ is a continuous polynomial vector field of order $m$. Then there is a constant $c$ such that

$$Lip (Q) (x) \leq c (1 + \|x\|^m);$$

**Proof:** This is an easy consequence of Lemma 10.1, since the latter implies a straightforward bound on the local Lipschitz constant for $Q$ in each of the $m$ variables.

The Fréchet derivative of order $i$ of a function $Q : V \rightarrow V^0$ at a point $y$ will be denoted by $D_i (Q) (y) : V^i \rightarrow V^0$. It is an $i$-linear operator and its value at a tangent vector $(1; \ldots; i) \in V^i$ is denoted by $D_i (Q) (y) (1; \ldots; i)$.

**Lemma 10.3.** Let $Q$ be a $j$-linear symmetric function. Then,

$$D_i (Q) (y) (1; \ldots; n) = \begin{cases} 0; & 0 < \frac{i}{j}; \\ \frac{i!}{j!} \frac{i}{j} Q (y (i) (1; \ldots; i); 1; \ldots; j); & i > j; \end{cases}$$

**Proof:** If $i = 1$, the Lemma immediately follows from the chain rule. The general case follows from an iterative application of the statement for $i = 1$.

**Lemma 10.4.** If $Q : V \rightarrow V^0$ is a polynomial vector field of order $m$ such that condition (18) holds true, then for every $i = 2; \ldots; m$ there is a constant $K_i > 0$ such that

$$D_i (Q) (y) (1; \ldots; i) \leq K_i (1 + \|y\|)^m;$$

**Proof:** This is a straightforward consequence of Lemma 10.3

**Lemma 10.5.** Suppose $f_1; \ldots; f_i 2 V$ are constant vector fields and $Q (x) : V \rightarrow V^0$ is a Fréchet differentiable vector field. Then

$$D_i (Q) (x) (f_1; \ldots; f_i) = D (Q ; f_1; f_2; \ldots; f_i)(x);$$
Proof. The lemma is proved by induction:

\[ \mathcal{D}^{-1} Q \cdot \mathcal{X} (f_1; \ldots; f_m) = \mathcal{D}^{-1} Q (f_1; \ldots; f_m) \cdot \mathcal{X} (f_1) \]

Recall that

\[ X (t) = u (0) + \int_0^t F (u (s)) ds + \int_0^t f (s) ds : \]

Lemma 10.6. Let \( Q : V ! V^0 \) be a polynomial vector field. Then

\[ X^n \cdot X \cdot Q u (t) = Q (X (t) + \sum_{i=1}^{k_1} \sum_{i=1}^{k_2} \cdots \sum_{i=1}^{k_m} Q (g_{k_1}; \ldots; g_{k_m}) \cdot \mathcal{W} (k_1; \ldots; k_m) : \]

Proof. Since \( Q \) is polynomial, we have

\[ Q (y) = \sum_{j=0}^{X^n} Q_j (y^j) \]

for some \( n \geq 0 \) where \( Q_j \) is a continuous, symmetric, multilinear vector field for each \( j \). Now

\[ Q (u (t)) = X^n \cdot X \cdot Q_j (X (s) + \sum_{i=0}^{k} g_{k_i} \mathcal{W} (k) \cdot j) \]

Using Lemma [10.3] we have

\[ \sum_{j=1}^{X^n} \frac{j!}{(j-1)!} Q_j (X (s) + \sum_{i} g_{k_i}; \ldots; g_{k_m}) = X^n \cdot \sum_{j=0}^{X^n} Q_j (g_{k_1}; \ldots; g_{k_m}) (X (s)) \]

which completes the proof.

11. Bounds on norms and Lipschitz constants

We define

\[ \mathcal{B} (p) K_{s,p} K_v = \sup_{k \geq 1} \mathcal{B} (p) K_{s,p} K_v \]

and

\[ \mathcal{B} (p) K_{s,p} \mathcal{B} = \sup_{k \geq 1} \mathcal{B} (p) K_{s,p} \mathcal{B} : \]
Lemma 11.1. Under Assumptions \([\mathbb{A}]\) and \([\mathbb{B}]\) for any \(p \geq 1\), there is a universal constant \(C_p\) such that the following bounds hold:

\[
E \sup_{s \leq t \leq T} K_{s:t} \ell_V^0 \leq C_p \left( K_{2p} (1 + u_{2p} m) \right)
\]

\[
E \sup_{s \leq t \leq T} K_{s:t} \ell_V^0 \leq C_p \left( K_{2p} (1 + u_{2p} m) \right) + K_{2p}
\]

Proof. From the equation for \(K_{s:t}\) and the bound on \(F\) (and hence \(DF\)) from \((18)\), we see that for \(2 \leq V\) with \(k \leq 1\)

\[
\sup_{s \leq t \leq T} K_{s:t} \ell_V^0 \geq k_{V^0} \sup_{s \leq t \leq T} F \left( u(t) K_{s:t} \ell_V^0 \right)
\]

\[
\leq C (1 + \sup_{s \leq t \leq T} (u(t) k_m) m) \sup_{s \leq t \leq T} K_{s:t} \ell_V^0
\]

Next, take the supremum over \(q\), then the \(p\)th power, and lastly the expected value. The first inequality of the Lemma follows from the bounds in Assumptions \([\mathbb{A}]\) and \([\mathbb{B}]\) after applying the Cauchy-Schwartz inequality to the right hand side. The second inequality follows from the first one and the assumptions.

Lemma 11.2. Let \(Q : H^1 \rightarrow H^0\) be a continuous polynomial vector field of order \(m\) and let \(f_1 : [0; T]\! \rightarrow H\) for \(i \leq f_1, \; m \leq g\). Then there exists a universal constant \(C_{m} \) such that

\[
\sup_{t \leq T} Q (f_1 (t); \ell_V^0) \leq \sum_{i=1}^{\infty} \left( 1 + \sup_{t \leq T} (f_1) \ell_H^0 \right) \sup_{t \leq T} (f_1)
\]

Proof. The proof is analogous to that of Lemma \([10.2]\).

Lemma 11.3. If \(f, g : [0; T]\! \rightarrow H\) then

\[
\sup_{t \leq T} f (t); g (t) \leq f (t); g (t) + g (t) f (t) i j + g (t) f (s) i j + g (s) f (t) i j
\]

\[
\sup_{t \leq T} f (t) i j + g (t) i j
\]

Proof. The first bound follows from

\[
\sup_{t \leq T} f (t) i j + g (t) i j
\]

We turn to the second bound. Since

\[
\sup_{t \leq T} f (t) i j + g (t) i j
\]

the first inequality of the lemma implies

\[
\sup_{t \leq T} f (t) i j + g (t) i j
\]

and we are done.
REFERENCES

[AKSS] A. Agrachev, S. Kuksin, A. Sarychev, and A. Shirikyan. On finite-dimensional projections of distributions for solutions of randomly forced pde’s. Preprint 2006.

[BAL91] G. Ben Arous and R. Léandre. Décroissance exponentielle du noyau de la chaleur sur la diagonale. I. Probab. Theory Related Fields, 90(2):175–202, 1991.

[Bas98] Richard F. Bass. Diffusions and elliptic operators. Probability and its Applications (New York). Springer-Verlag, New York, 1998.

[Bel87] Denis R. Bell. The Malliavin calculus. Longman Scientific & Technical, Harlow, 1987.

[BT05] Fabrice Baudoin and Josef Teichmann. Hypoellipticity in infinite dimensions and an application in interest rate theory. Ann. Appl. Probab., 15(3):1765–1777, 2005.

[Cer99] Sandra Cerrai. Smoothing properties of transition semigroups relative to SDEs with values in Banach spaces. Probab. Theory Related Fields, 113(1):85–114, 1999.

[CF88] Peter Constantin and Ciprian Foias. Navier-Stokes Equations. University of Chicago Press, Chicago, 1988.

[DL88] Robert Dautray and Jacques-Louis Lions. Analyse mathématique et calcul numérique pour les sciences et les techniques. Masson, Paris, 1988.

[DPZ96] Giuseppe Da Prato and Jerzy Zabczyk. Ergodicity for Infinite Dimensional Systems. Cambridge, 1996.

[EH01] J.-P. Eckmann and M. Hairer. Uniqueness of the invariant measure for a stochastic PDE driven by degenerate noise. Comm. Math. Phys., 219(3):523–565, 2001.

[EM01] Weinan E and Jonathan C. Mattingly. Ergodicity for the Navier-Stokes equation with degenerate random forcing: finite-dimensional approximation. Comm. Pure Appl. Math., 54(11):1386–1402, 2001.

[Fla94] Franco Flandoli. Dissipativity and invariant measures for stochastic Navier-Stokes equations. NoDEA, 1:403–426, 1994.

[HM06] Martin Hairer and Jonathan C. Mattingly. Ergodicity of the degenerate stochastic 2D Navier–Stokes equation. Annals of Mathematics, 164(3), 2006.

[Kli87] Wolfgang Kliemann. Recurrence and invariant measures for degenerate diffusions. The Annals of Probability, 2:690–707, 1987.

[KS84] Shigeo Kusuoka and Daniel Stroock. Applications of the Malliavin calculus. I. In Stochastic analysis (Katata/Kyoto, 1982), pages 271–306. North-Holland, Amsterdam, 1984.

[Mal78] Paul Malliavin. Stochastic calculus of variation and hypoelliptic operators. In Proceedings of the International Symposium on Stochastic Differential Equations (Res. Inst. Math. Sci., Kyoto Univ., Kyoto, 1976), pages 195–263, New York, 1978. Wiley.

[MB02] Andrew J. Majda and Andrea L. Bertozzi. Vorticity and incompressible flow, volume 27 of Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2002.

[MP06] Jonathan C. Mattingly and Étienne Pardoux. Malliavin calculus for the stochastic 2D Navier-Stokes equation. Comm. Pure Appl. Math., 59(12):1742–1790, 2006.

[Nor86] James Norris. Simplified Malliavin calculus. In Séminaire de Probabilités, XX, 1984/85, pages 101–130. Springer, Berlin, 1986.

[Nua95] David Nualart. The Malliavin calculus and related topics. Probability and its Applications. Springer-Verlag, New York, 1995.

[Oco88] Daniel Ocone. Stochastic calculus of variations for stochastic partial differential equations. J. Funct. Anal., 79(2):288–331, 1988.

[Rom04] Marco Romito. Ergodicity of the finite dimensional approximation of the 3D Navier-Stokes equations forced by a degenerate noise. J. Statist. Phys., 114(1-2):155–177, 2004.

[SY02] George R. Sell and Yuncheng You. Dynamics of evolutionary equations, volume 143 of Applied Mathematical Sciences. Springer-Verlag, New York, 2002.

[Wu06] Ming-Yih Wu. Stochastic Boussinesq Equations and the infinite dimensional Malliavin Calculus. PhD thesis, Princeton, 2006.
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