DIMENSION THEORY OF FLOWS: A SURVEY

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ABSTRACT. This is a survey on recent developments of the dimension theory of flows, with emphasis on hyperbolic flows. In particular, we describe various results of the dimension theory and multifractal analysis of flows, including the dimension of hyperbolic sets, the dimension of invariant measures, the multifractal analysis of equilibrium measures, conditional variational principles, multidimensional spectra, and dimension spectra taking both into account past and future. The dimension theory and the multifractal analysis of dynamical systems grew out exponentially during the last two decades, but for various reasons flows have initially been given less attention than maps. We emphasize that this is not the case anymore and the survey is also an invitation to the theory.

1. Introduction.
1.1. Dimension theory. The dimension theory of dynamical systems is a very active field of research. Its main objective is to measure the complexity from the dimensional point of view of the objects that are invariant under the dynamics. We refer the reader to the books [1, 20] for detailed accounts of substantial parts of the dimension theory of maps.

As a motivation for the development of a dimension theory for flows, we start with a brief discussion of the somewhat simpler theory for maps. Our main aim is to illustrate the type of problems that are considered in dimension theory. The Hausdorff dimension of a set $Z \subset \mathbb{R}^m$ is defined by

$$\dim_H Z = \inf \{ \alpha \in \mathbb{R} : m(Z, \alpha) = 0 \},$$

where

$$m(Z, \alpha) = \liminf_{\varepsilon \to 0} \inf_{U \in \mathcal{U}} (\text{diam } U)^\alpha,$$

with the infimum taken over all finite or countable collections $\mathcal{U}$ of sets of diameter at most $\varepsilon$ such that $\bigcup_{U \in \mathcal{U}} U \supset Z$. Moreover, the lower and upper box dimensions of a set $Z \subset \mathbb{R}^m$ are defined, respectively, by

$$\dim_B Z = \liminf_{\varepsilon \to 0} \frac{\log N(Z, \varepsilon)}{-\log \varepsilon},$$

and

$$\overline{\dim}_B Z = \limsup_{\varepsilon \to 0} \frac{\log N(Z, \varepsilon)}{-\log \varepsilon},$$

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where \( N(Z, \varepsilon) \) is the least number of balls of radius \( \varepsilon \) that are needed to cover \( Z \). One can easily verify that
\[
\dim_H Z \leq \dim_B Z \leq \overline{\dim_B} Z.
\]

We first consider invariant sets of a noninvertible dynamics. Let \( g: M \to M \) be a smooth map on a smooth manifold \( M \). A compact \( g \)-invariant set \( J \subset M \) is said to be a repeller for \( g \) if there exist constants \( c > 0 \) and \( \beta > 1 \) such that
\[
\| d_x g^n v \| \geq c \beta^n \| v \|
\]
for every \( n \in \mathbb{N} \), \( x \in J \) and \( v \in T_x M \). Moreover, the map \( g \) is said to be conformal on \( J \) if \( d_x g \) is a multiple of an isometry for every \( x \in J \) (this happens for example when \( M \) is 1-dimensional). We define a function \( \varphi: J \to \mathbb{R} \) by
\[
\varphi(x) = -\log \| d_x g \|.
\]
The following result expresses the Hausdorff dimension and the lower and upper box dimensions of \( J \) in terms of the topological pressure (see Section 2.2 for the definition).

**Theorem 1.1.** If \( J \) is a repeller for a \( C^{1+\delta} \) map \( g \) that is conformal on \( J \), then
\[
\dim_H J = \dim_B J = \overline{\dim_B} J = s,
\]
where \( s \) is the unique real number such that
\[
P(s\varphi) = 0.
\]

Ruelle [28] showed that \( \dim_H J = s \). The remaining equalities in (1) are due to Falconer [14]. Equation (2) was introduced by Bowen [9] in his study of quasi-circles and establishes an important connection between the thermodynamic formalism and the dimension theory of dynamical systems. Virtually, all known equations used to compute or estimate the dimension of an invariant set are particular cases of this equation.

Now we consider invariant sets of an invertible dynamics. Let \( f: M \to M \) be a diffeomorphism on a smooth manifold \( M \). A compact \( f \)-invariant set \( \Lambda \subset M \) is said to be a hyperbolic set for \( f \) if there exist a splitting
\[
T_x M = E^s \oplus E^u
\]
and constants \( \lambda \in (0, 1) \) and \( c > 0 \) such that for each \( x \in \Lambda \):
1. \( d_x f E^s(x) = E^s(f(x)) \) and \( d_x f E^u(x) = E^u(f(x)) \);
2. \( \| d_x f^n v \| \leq c \lambda^n \| v \| \) for \( v \in E^s(x) \) and \( n \in \mathbb{N} \);
3. \( \| d_x f^{-n} v \| \leq c \lambda^n \| v \| \) for \( v \in E^u(x) \) and \( n \in \mathbb{N} \).

Moreover, a set \( \Lambda \) is said to be locally maximal if it has an open neighborhood \( U \) such that
\[
\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U).
\]
We define functions \( \varphi_s: \Lambda \to \mathbb{R} \) and \( \varphi_u: \Lambda \to \mathbb{R} \) by
\[
\varphi_s(x) = \log \| d_x f E^s(x) \| \quad \text{and} \quad \varphi_u(x) = -\log \| d_x f E^u(x) \|.
\]
The following result is a version of Theorem 1.1 for hyperbolic sets.
Theorem 1.2. If Λ is a locally maximal hyperbolic set for a $C^1$ surface diffeomorphism and $\dim E^s(x) = \dim E^u(x) = 1$ for every $x \in \Lambda$, then
\[ \dim_H \Lambda = \liminf_B \Lambda = \limsup_B \Lambda = t_s + t_u, \] (3)
where $t_s$ and $t_u$ are the unique real numbers such that
\[ P(t_s \varphi_s) = P(t_u \varphi_u) = 0. \]
It follows from work of McCluskey and Manning [18] that
\[ \dim_H \Lambda = t_s + t_u. \]

The remaining equalities in (3) are due to Takens [29] for $C^2$ diffeomorphisms and to Palis and Viana [19] for arbitrary $C^1$ diffeomorphisms. More generally, one can consider higher dimensional manifolds together with an appropriate conformality condition along the stable and unstable directions (that replaces the requirement that both of them have dimension 1). Namely, $f$ is said to be conformal on $\Lambda$ if the maps $d_x f|E^s(x)$ and $d_x f|E^u(x)$ are multiples of isometries for every $x \in \Lambda$.

The study of the dimension of repellers and hyperbolic sets for nonconformal maps is much less developed and in many situations only partial results have been obtained. For example, some results were established for Lebesgue almost all values of some parameter and often only dimension estimates were obtained. On the other hand, some new phenomena occur in the study of the dimension of invariant sets for nonconformal maps. In particular, the Hausdorff and box dimensions of a repeller may not coincide. An example was given by Pollicott and Weiss [24], modifying a construction of Przytycki and Urbański [25] that depends on number-theoretical properties.

1.2. Multifractal analysis. We also want to consider the multifractal analysis of dynamical systems. Roughly speaking, it studies the complexity of the level sets of invariant local quantities obtained from a dynamical system. For example, one can consider Birkhoff averages, Lyapunov exponents, pointwise dimensions or local entropies. The concept of multifractal analysis was suggested by Halsey, Jensen, Kadanoff, Procaccia and Shraiman [15]. The first rigorous approach is due to Collet, Lebowitz and Porzio [13], for a class of measures that are invariant under 1-dimensional Markov maps. Lopes [17] considered the measure of maximal entropy for hyperbolic Julia sets and Rand [26] studied Gibbs measures for a class of repellers.

Considering here for simplicity only the case of maps, we describe briefly the main elements of multifractal analysis. Let $T: X \to X$ be a continuous map of a compact metric space and let $g: X \to \mathbb{R}$ be a continuous function. For each $\alpha \in \mathbb{R}$, consider the level set
\[ K_\alpha = \left\{ x \in X : \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n} g(T^i(x)) = \alpha \right\}. \]

Moreover, let
\[ K = \left\{ x \in X : \liminf_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n} g(T^i(x)) < \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n} g(T^i(x)) \right\}. \]
Clearly,
\[ X = K \cup \bigcup_{\alpha \in \mathbb{R}} K_\alpha. \] (4)
This union is formed by pairwise disjoint $T$-invariant sets. For each $\alpha \in \mathbb{R}$, let
\[ \mathcal{D}(\alpha) = \dim H K_{\alpha}. \]

The function $\mathcal{D}$ is called the \textit{dimension spectrum for the Birkhoff averages of $g$}. For several classes of “hyperbolic” dynamical systems (for example, topological Markov chains, expanding maps and hyperbolic diffeomorphisms) and certain classes of functions (for example, Hölder continuous functions that are not cohomologous to a constant), it was proved that:

1. $\{ \alpha \in \mathbb{R} : K_{\alpha} \neq \emptyset \}$ is an interval;
2. $\mathcal{D}$ is analytic and strictly convex;
3. $K$ is dense and $\dim H K = \dim H X$.

In particular, the decomposition in (4) is often composed of an uncountable number of dense invariant sets of positive Hausdorff dimension. For example, Pesin and Weiss [22, 23] considered repellers and hyperbolic sets for $C^{1+\delta}$ conformal maps.

We refer the reader to the books [1, 20] for details and further references.

1.3. \textbf{The case of flows.} To a large extent, the theory for flows is analogous to the theory for maps, both in what respects to the dimension theory and the multifractal analysis, but there are some differences that will also be described in the following sections.

In this survey we describe various results of the dimension theory and multifractal analysis of flows, including:

1. a formula for the dimension of a hyperbolic set for a conformal flow, as a consequence of the formulas for the dimensions along the stable and unstable directions, with an appropriate version of Theorem 1.2;
2. a description of the dimension of an invariant measure, as a consequence of a formula for its pointwise dimension, and a description of the behavior along an ergodic decomposition;
3. a multifractal analysis of the dimension spectrum for the pointwise dimensions of an equilibrium measure and a multifractal analysis of the entropy spectrum for the Birkhoff averages of a given function;
4. a description of conditional variational principles for various multifractal spectra, including for the local entropies and the Lyapunov exponents, also with a discussion of generic conditions for their analyticity;
5. a multifractal analysis of multidimensional spectra for the general class of continuous flows with an upper semicontinuous metric entropy, as a natural generalization of the former results;
6. a multifractal analysis of the dimension spectrum taking both into account past and future, with formulas for the dimensions of the level sets and of their intersections.

In Section 2 we recall the necessary notions from the theory of hyperbolic flows and from the thermodynamic formalism, including the notion of topological pressure. Finally, in Appendix A we give a description of Markov systems and of the associated suspension flows. These play an important role in many proofs, by essentially transferring some of the problems to a dynamics with discrete time on the base of the suspension.

We refer the reader to the book [2] for a detailed account of many results of the dimension theory and multifractal analysis of flows, including for suspension flows (these flows are only briefly mentioned in the survey).
2. Basic notions.

2.1. Hyperbolic flows. Let $\Phi = (\varphi_t)_{t \in \mathbb{R}}$ be a $C^1$ flow on a smooth manifold $M$. This means that we are given maps $\varphi_t : M \to M$ for $t \in \mathbb{R}$ such that $\varphi_0 = \text{id}$,

$\varphi_t \circ \varphi_s = \varphi_{t+s}$ for $t, s \in \mathbb{R}$

and $(t, x) \mapsto \varphi_t(x)$ is of class $C^1$. A compact $\Phi$-invariant $\Lambda \subset M$ (that is, such that $\varphi_t(\Lambda) = \Lambda$ for $t \in \mathbb{R}$) is called a hyperbolic set for $\Phi$ if there exist a splitting

$$T_\Lambda M = E^s \oplus E^u \oplus E^0$$

and constants $\lambda \in (0, 1)$ and $c > 0$ such that for each $x \in \Lambda$:

1. the vector $(\partial / \partial t)\varphi_t(x)|_{t=0}$ generates $E^0(x)$;
2. $d_x \varphi_t E^s(x) = E^s(\varphi_t(x))$ and $d_x \varphi_t E^u(x) = E^u(\varphi_t(x))$ for $t \in \mathbb{R}$;
3. $\|d_x \varphi_t v\| \leq c\lambda^t \|v\|$ for $v \in E^s(x)$ and $t > 0$;
4. $\|d_x \varphi_{-t} v\| \leq c\lambda^t \|v\|$ for $v \in E^u(x)$ and $t > 0$.

For example, for any geodesic flow on a compact Riemannian manifold with strictly negative sectional curvature the unit tangent bundle is a hyperbolic set. Moreover, any sufficiently small $C^1$ perturbation of a flow with a hyperbolic set also has a hyperbolic set.

Let $\Lambda$ be a hyperbolic set for a flow $\Phi$. For each $x \in \Lambda$ and $\varepsilon > 0$, consider the sets

$$A^s(x) = \{ y \in B(x, \varepsilon) : d(\varphi_t(y), \varphi_t(x)) \to 0 \text{ when } t \to +\infty \}$$

and

$$A^u(x) = \{ y \in B(x, \varepsilon) : d(\varphi_t(y), \varphi_t(x)) \to 0 \text{ when } t \to -\infty \}.$$

Moreover, let

$$V^s(x) \subset A^s(x) \quad \text{and} \quad V^u(x) \subset A^u(x) \quad (5)$$

be the connected components containing $x$. For any sufficiently small $\varepsilon > 0$ these are smooth manifolds with

$$T_x V^s(x) = E^s(x) \quad \text{and} \quad T_x V^u(x) = E^u(x).$$

They are called, respectively, local stable and unstable manifolds (of size $\varepsilon$) at $x$. If $\Lambda$ is locally maximal, that is, if there exists an open neighborhood $U$ of $\Lambda$ such that

$$\Lambda = \bigcap_{t \in \mathbb{R}} \varphi_t(U), \quad (6)$$

then for any sufficiently small $\varepsilon > 0$, there exists $\delta > 0$ such that if $x, y \in \Lambda$ are at a distance $d(x, y) \leq \delta$, then there exists a unique $t = t(x, y) \in [-\varepsilon, \varepsilon]$ for which

$$[x, y] = V^s(\varphi_t(x)) \cap V^u(y)$$

is a single point in $\Lambda$.

2.2. Thermodynamic formalism. Let $\Phi = (\varphi_t)_{t \in \mathbb{R}}$ be a continuous flow on a compact metric space $(X, d)$. Given $x \in X$ and $t, \varepsilon > 0$, consider the set

$$B(x, t, \varepsilon) = \{ y \in X : d(\varphi_{\tau}(y), \varphi_\tau(x)) < \varepsilon \text{ for } \tau \in [0, t] \}.$$

Moreover, let $a : X \to \mathbb{R}$ be a continuous function and write

$$a(x, t, \varepsilon) = \sup \left\{ \int_0^t a(\varphi_{\tau}(y)) \, d\tau : y \in B(x, t, \varepsilon) \right\}. $$
For each $Z \subset X$ and $\alpha \in \mathbb{R}$, let

$$M(Z, a, \alpha, \varepsilon) = \lim_{T \to \infty} \inf \Gamma \sum_{(x, t) \in \Gamma} \exp(a(x, t, \varepsilon) - \alpha t),$$

where the infimum is taken over all finite or countable sets $\Gamma = \{(x_i, t_i) : i \in I\}$ such that $x_i \in X$ and $t_i \geq T$ for $i \in I$ and $\bigcup_{i \in I} B(x_i, t_i, \varepsilon) \supset Z$. The number $P_\Phi\mid Z(a) = \lim_{\varepsilon \to 0} \inf \{\alpha \in \mathbb{R} : M(Z, a, \alpha, \varepsilon) = 0\}$

is called the topological pressure of $a$ on the set $Z$. For simplicity of the notation, we shall also write $P_\Phi(a) = P_\Phi\mid X(a)$. The number $h(\Phi\mid Z) = P_\Phi\mid Z(0)$ is called the topological entropy of $\Phi$ on $Z$.

Now let $M$ be the set of all $\Phi$-invariant probability measures on $X$. We recall that a measure $\mu$ is said to be $\Phi$-invariant if

$$\mu(\varphi_t(A)) = \mu(A)$$

for every measurable set $A \subset X$ and every $t \in \mathbb{R}$. Moreover, a measure $\mu$ is said to be ergodic if for any $\Phi$-invariant set $A \subset X$ (that is, such that $\varphi_t(A) = A$ for $t \in \mathbb{R}$) we have

$$\mu(A) = 0 \text{ or } \mu(X \setminus A) = 0.$$

For each $\mu \in M$, writing

$$h(Z, \varepsilon) = \inf \{\alpha \in \mathbb{R} : M(Z, 0, \alpha, \varepsilon) = 0\},$$

the limit

$$h_\mu(\Phi) = \lim_{\varepsilon \to 0} \inf \{h(Z, \varepsilon) : \mu(Z) = 1\}$$

exists. When $\mu$ is ergodic, the number $h_\mu(\Phi)$ coincides with the entropy of $\Phi$ with respect to $\mu$, that is, the entropy of the time-1 map $\varphi_1$ with respect to $\mu$.

**Theorem 2.1.** If $\Phi$ is a continuous flow on a compact metric space $X$, then

$$P_\Phi(a) = \sup \left\{ h_\mu(\Phi) + \int_X a \, d\mu : \mu \in M \right\}$$

for any continuous function $a : X \to \mathbb{R}$.

A measure $\mu \in M$ is said to be an equilibrium measure for the function $a$ if the supremum in (7) is attained at this measure, that is,

$$P_\Phi(a) = h_\mu(\Phi) + \int_X a \, d\mu.$$

We denote by $C(X)$ the space of all continuous functions $a : X \to \mathbb{R}$ equipped with the supremum norm and by $D(X) \subset C(X)$ the set of all continuous functions with a unique equilibrium measure. Given a finite set $K \subset C(X)$, we denote by span $K \subset C(X)$ the linear space generated by the functions in $K$.

**Theorem 2.2.** If $\Phi$ is a continuous flow on a compact metric space $X$ such that the map $\mu \mapsto h_\mu(\Phi)$ is upper semicontinuous, then:

1. each $a \in C(X)$ has equilibrium measures and $D(X)$ is dense in $C(X)$;
2. given \( a, b \in C(X) \), the map \( \mathbb{R} \ni t \mapsto P_\Phi(a + tb) \) is differentiable at \( t = 0 \) if and only if \( a \in D(X) \), in which case the unique equilibrium measure \( \mu_a \) for the function \( a \) is ergodic and satisfies

\[
\frac{d}{dt} P_\Phi(a + tb)|_{t=0} = \int_X b d\mu_a;
\]

3. if \( \text{span}\{a, b\} \subset D(X) \), then the function \( t \mapsto P_\Phi(a + tb) \) is of class \( C^1 \).

We note that if \( \Phi \) is an expansive flow, then \( \mu \mapsto h_\mu(\Phi) \) is upper semicontinuous (see [30]). For example, if \( \Lambda \) is a hyperbolic set for \( \Phi \), then \( \Phi|\Lambda \) is expansive.

For flows with a hyperbolic set, Theorem 2.2 can be strengthened as follows.

We say that \( \Phi|\Lambda \) is topologically mixing if for any nonempty open sets \( U \) and \( V \) intersecting \( \Lambda \) there exists \( s \in \mathbb{R} \) such that

\[
\varphi_t(U) \cap V \cap \Lambda \neq \emptyset \quad \text{for} \quad t > s.
\]

Moreover, a function \( a: \Lambda \to \mathbb{R} \) is said to be \( \Phi \)-cohomologous to a function \( b: \Lambda \to \mathbb{R} \) if there exists a bounded measurable function \( q: \Lambda \to \mathbb{R} \) such that

\[
a(x) - b(x) = \lim_{t \to 0} \frac{q(\varphi_t(x)) - q(x)}{t}.
\]

**Theorem 2.3.** If \( \Phi \) is a \( C^1 \) flow with a locally maximal hyperbolic set \( \Lambda \) such that \( \Phi|\Lambda \) is topologically mixing, then:

1. the map \( \mu \mapsto h_\mu(\Phi) \) is upper semicontinuous;
2. each Hölder continuous function \( a: \Lambda \to \mathbb{R} \) has a unique equilibrium measure;
3. given Hölder continuous functions \( a, b: \Lambda \to \mathbb{R} \), the map \( t \mapsto P_\Phi(a + tb) \) is analytic and

\[
\frac{d^2}{dt^2} P_\Phi(a + tb) \geq 0 \quad \text{for} \quad t \in \mathbb{R},
\]

with equality if and only if \( b \) is \( \Phi \)-cohomologous to a constant.

3. Dimension of hyperbolic sets. In this section we describe a formula for the dimension of a locally maximal hyperbolic set for a conformal flow. The dimension is expressed in terms of the topological pressure. Let \( \Lambda \) be a hyperbolic set for a \( C^1 \) flow \( \Phi \). The restriction \( \Phi|\Lambda \) is said to be conformal if the maps

\[
d_x \varphi_t|E^s(x): E^s(x) \to E^s(\varphi_t(x)) \quad \text{and} \quad d_x \varphi_t|E^u(x): E^u(x) \to E^u(\varphi_t(x))
\]

are multiples of isometries for \( x \in \Lambda \) and \( t \in \mathbb{R} \).

Consider the families of local stable and unstable manifolds \( V^s(x) \) and \( V^u(x) \) for \( x \in \Lambda \) (see (5)). The following result of Pesin and Sadovskaya [21] expresses the dimensions of the sets \( V^s(x) \cap \Lambda \) and \( V^u(x) \cap \Lambda \) in terms of the topological pressure. We define functions \( \zeta_s, \zeta_u: \Lambda \to \mathbb{R} \) by

\[
\zeta_s(x) = \frac{\partial}{\partial t} \log \|d_x \varphi_t|E^s(x)\| \bigg|_{t=0} = \lim_{t \to 0} \frac{1}{t} \log \|d_x \varphi_t|E^s(x)\|
\]

and

\[
\zeta_u(x) = \frac{\partial}{\partial t} \log \|d_x \varphi_t|E^u(x)\| \bigg|_{t=0} = \lim_{t \to 0} \frac{1}{t} \log \|d_x \varphi_t|E^u(x)\|.
\]

Since the flow is of class \( C^1 \), these functions are well defined.
Theorem 3.1. Let $\Phi$ be a $C^{1+\delta}$ flow with a locally maximal hyperbolic set $\Lambda$ such that $\Phi|\Lambda$ is conformal and topologically mixing. Then
\[ \dim_H(V^s(x) \cap \Lambda) = \dim_B(V^s(x) \cap \Lambda) = \dim_B(V^s(x) \cap \Lambda) = t_s \]
and
\[ \dim_H(V^u(x) \cap \Lambda) = \dim_B(V^u(x) \cap \Lambda) = \dim_B(V^u(x) \cap \Lambda) = t_u \]  
(8)
for every $x \in \Lambda$, where $t_s$ and $t_u$ are the unique real numbers such that
\[ P_{\Phi|\Lambda}(t_s \zeta_s) = P_{\Phi|\Lambda}(-t_u \zeta_u) = 0. \]  
(9)

We note that the dimensions of $V^s(x) \cap \Lambda$ and $V^u(x) \cap \Lambda$ are independent of $x$. The proof uses a Markov system, which is an appropriate flow version of a Markov partition (see Appendix A for a detailed description), to construct special covers of finite multiplicity (Moran covers) that as in the case of maps allow one to compute the dimensions along the stable and unstable manifolds using the topological pressure. More generally, if $\Phi|\Lambda$ is not conformal but the map $d_x \varphi_t|E^u(x): E^u(x) \to E^u(\varphi_t(x))$ is a multiple of an isometry for every $x \in \Lambda$ and $t \in \mathbb{R}$, then the identities in (8) still hold. A similar observation holds for the dimensions along the stable manifolds.

In view of the conformality, the dimension is obtained by adding the dimensions along the stable and unstable manifolds, plus the dimension along the flow.

Theorem 3.2 ([21]). If $\Phi$ is a $C^{1+\delta}$ flow with a locally maximal hyperbolic set $\Lambda$ such that $\Phi|\Lambda$ is conformal and topologically mixing, then
\[ \dim_H \Lambda = \dim_B \Lambda = \dim_B \Lambda = t_s + t_u + 1. \]  
(10)

We explain briefly the idea of the proof. Consider a small disk $D$ transverse to the flow and define
\[ \{x,y\} = \pi([x,y]) \]  
(11)
on $(\Lambda \cap D) \times (\Lambda \cap D)$, with $[x,y]$ as in (6), where $\pi$ is the projection onto $D$ along the flow. The map $(x,y) \mapsto \{x,y\}$ is Hölder continuous and has a Hölder continuous inverse. But since $\Phi$ is conformal on $\Lambda$, it follows from results of Hasselblatt [16] that the distributions
\[ x \mapsto E^s(x) \oplus E^0(x) \quad \text{and} \quad x \mapsto E^u(x) \oplus E^0(x) \]
are Lipschitz. This implies that the map in (11) is in fact Lipschitz and has a Lipschitz inverse. Therefore,
\[ \dim_H(\Lambda \cap D) = \dim_H(\Lambda \cap D, \Lambda \cap D) = t_s + t_u, \]
with analogous identities for the lower and upper box dimensions. Identity (10) follows now readily from adding the direction of the flow.

4. Dimension of invariant measures. In this section we describe various ways of expressing the dimension of an invariant measure on a hyperbolic set. Let $\Lambda$ be a hyperbolic set for a $C^1$ flow $\Phi$ and let $\mu$ be a $\Phi$-invariant probability measure on $\Lambda$. By Birkhoff’s ergodic theorem, the limits
\[ \lambda_s(x) = \lim_{t \to \infty} \frac{1}{t} \log \|d_x \varphi_t|E^s(x)\| \quad \text{and} \quad \lambda_u(x) = \lim_{t \to \infty} \frac{1}{t} \log \|d_x \varphi_t|E^u(x)\| \]
exist for $\mu$-almost every $x \in \Lambda$ (for a conformal flow these are, respectively, the negative and positive values of the Lyapunov exponent). Moreover, by the flow version of the Brin–Katok formula (see [11]), we have
\[
h_\mu(x) = \lim_{\varepsilon \to 0} \lim_{t \to \infty} -\frac{1}{t} \log \mu(B(x, t, \varepsilon))
\]
for $\mu$-almost every $x \in \Lambda$, where
\[
B(x, t, \varepsilon) = \{ y \in M : d(\varphi_\tau(y), \varphi_\tau(x)) < \varepsilon \text{ for } \tau \in [0, t] \}. \tag{12}
\]
The number $h_\mu(x)$ is called the local entropy of $\mu$ at $x$. The function $x \mapsto h_\mu(x)$ is $\mu$-integrable, everywhere and satisfies
\[
h_\mu(\Phi) = \int_\Lambda h_\mu(x) \, d\mu(x). \tag{13}
\]

The following result of Barreira and Wolf [7] gives an explicit formula for the pointwise dimensions of $\mu$ in terms of the Lyapunov exponents and the local entropies. The lower and upper pointwise dimensions of $\mu$ at a point $x \in \Lambda$ are defined, respectively, by
\[
d_\mu(x) = \liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} \quad \text{and} \quad \overline{d}_\mu(x) = \limsup_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}.
\]
The pointwise dimensions give information about the local nature of an invariant measure from the point of view of dimension, in a similar manner as the local entropies give information about the local nature of the entropy of an invariant measure (see (13)). We refer the reader to the book [1] for a detailed discussion.

**Theorem 4.1.** Let $\Phi$ be a $C^{1+\delta}$ flow with a locally maximal hyperbolic set $\Lambda$ such that $\Phi|\Lambda$ is conformal and let $\mu$ be a $\Phi$-invariant probability measure on $\Lambda$. For $\mu$-almost every $x \in \Lambda$, we have
\[
d_\mu(x) = \overline{d}_\mu(x) = h_\mu(x) \left( \frac{1}{\lambda_u(x)} - \frac{1}{\lambda_s(x)} \right) + 1. \tag{14}
\]

Pesin and Sadovskaya [21] established identity (14) for the class of equilibrium measures of a Hölder continuous function. We note that these measures are ergodic and have a local product structure, while Theorem 4.1 considers arbitrary measures.

The Hausdorff dimension and the lower and upper box dimensions of $\mu$ are defined, respectively, by
\[
\dim_H \mu = \inf \{ \dim_H Z : \mu(\Lambda \setminus Z) = 0 \},
\]
\[
\dim_B \mu = \liminf_{\delta \to 0} \{ \dim_B Z : \mu(Z) \geq \mu(\Lambda) - \delta \},
\]
\[
\overline{\dim}_B \mu = \liminf_{\delta \to 0} \{ \overline{\dim}_B Z : \mu(Z) \geq \mu(\Lambda) - \delta \}.
\]
One can easily verify that
\[
\dim_H \mu \leq \dim_B \mu \leq \overline{\dim}_B \mu.
\]
The following result is a consequence of Theorem 4.1.

**Theorem 4.2** ([7]). If $\Phi$ is a $C^{1+\delta}$ flow with a locally maximal hyperbolic set $\Lambda$ such that $\Phi|\Lambda$ is conformal and $\mu$ is an ergodic $\Phi$-invariant probability measure on $\Lambda$, then
\[
\dim_H \mu = \dim_B \mu = \overline{\dim}_B \mu = h_\mu(\Phi) \left( \frac{1}{\int_\Lambda \zeta_u \, d\mu} - \frac{1}{\int_\Lambda \zeta_s \, d\mu} \right) + 1. \tag{15}
\]
It was shown by Young [31] that if there exists a constant \( d \geq 0 \) such that 
\[
d_{\mu}(x) = d
\]
for \( \mu \)-almost every \( x \in \Lambda \), then
\[
\dim_{H\mu} = \dim_{B\mu} = \dim_{B\mu} = d.
\]
Hence, identity (15) follows from the fact that the quantities \( h_{\mu}(x) \), \( \lambda_u(x) \) and \( \lambda_s(x) \) are invariant \( \mu \)-almost everywhere. Indeed, since they are invariant, when \( \mu \) is ergodic we have
\[
h_{\mu}(x) = h_{\mu}(\Phi),
\]
\[
\lambda_u(x) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \zeta_u(\varphi_\tau(x)) \, d\mu(x) = \int_\Lambda \zeta_u \, d\mu,
\]
\[
\lambda_s(x) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \zeta_s(\varphi_\tau(x)) \, d\mu(x) = \int_\Lambda \zeta_s \, d\mu
\]
for \( \mu \)-almost every \( x \in \Lambda \).

Now we describe the behavior of the Hausdorff dimension of an invariant measure with respect to an ergodic decomposition. Let \( \mathcal{M} \) be the set of all \( \Phi \)-invariant probability measures on \( \Lambda \) endowed with the weak* topology. Moreover, let \( \mathcal{M}_E \subset \mathcal{M} \) be the subset of all ergodic measures. Given \( \mu \in \mathcal{M} \), a probability measure \( \tau \) on \( \mathcal{M} \) (that is, on the Borel \( \sigma \)-algebra generated by the weak* topology) is said to be an ergodic decomposition of \( \mu \) if \( \tau(\mathcal{M}_E) = 1 \) and
\[
\int_\Lambda \varphi \, d\mu = \int_\mathcal{M} \left( \int_\Lambda \varphi \, d\nu \right) \, d\tau(\nu)
\]
for any continuous function \( \varphi : \Lambda \to \mathbb{R} \).

**Theorem 4.3** ([7]). Let \( \Phi \) be a \( C^{1+\delta} \) flow with a locally maximal hyperbolic set \( \Lambda \) such that \( \Phi|\Lambda \) is conformal and let \( \mu \) be a \( \Phi \)-invariant probability measure on \( \Lambda \). For each ergodic decomposition \( \tau \) of \( \mu \), we have
\[
\dim_{H\mu} = \text{ess sup}\{ \dim_{H\nu} : \nu \in \mathcal{M}_E \},
\]
with the essential supremum taken with respect to \( \tau \).

5. **Multifractal analysis.** In this section we describe the multifractal analysis of spectra that are defined in terms of invariant local quantities of a flow. We first consider the dimension spectrum for the pointwise dimensions of an invariant measure. Let \( \Lambda \) be a hyperbolic set for a \( C^1 \) flow \( \Phi \) and let \( \mu \) be a \( \Phi \)-invariant probability measure on \( \Lambda \). For each \( \alpha \in \mathbb{R} \), consider the set
\[
K_\alpha = \left\{ x \in \Lambda : \lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} = \alpha \right\}.
\]
The dimension spectrum for the pointwise dimensions of \( \mu \) is defined by
\[
D(\alpha) = \dim_{H} K_\alpha.
\]
Moreover, given a continuous function \( g : \Lambda \to \mathbb{R} \), for each \( q \in \mathbb{R} \) let \( T_s(q) \) and \( T_u(q) \) be the unique real numbers such that
\[
P_{\Phi|\Lambda}(T_s(q) \zeta_s + qg) = P_{\Phi|\Lambda}(-T_u(q) \zeta_u + qg) = 0.
\]
We write
\[
T(q) = T_s(q) + T_u(q) - q + 1.
\]
The following result of Pesin and Sadovskaya [21] describes the dimension spectrum for the pointwise dimensions of a class of equilibrium measures.

**Theorem 5.1.** Let $\Phi$ be a $C^{1+\delta}$ flow with a locally maximal hyperbolic set $\Lambda$ such that $\Phi|\Lambda$ is conformal and topologically mixing and let $\mu$ be an equilibrium measure for a Hölder continuous function $g$ with $P_{\Phi|\Lambda}(g) = 0$. Then:

1. for $\mu$-almost every $x \in \Lambda$, we have
   $$\lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} = h_{\mu}(\Phi) \left( \frac{1}{\int_{\Lambda} \zeta_u \, d\mu} - \frac{1}{\int_{\Lambda} \zeta_s \, d\mu} \right) + 1;$$
2. if $\alpha = -T'$, then
   $$D(\alpha(q)) = T(q) + q\alpha(q) \quad \text{for} \quad q \in \mathbb{R};$$
3. if $\dim_H \mu \neq \dim_H \Lambda$, then the functions $D$ and $T$ are analytic and strictly convex.

Now we consider the entropy spectrum for the Birkhoff averages of a continuous function $g : \Lambda \to \mathbb{R}$. For each $\alpha \in \mathbb{R}$, consider the set $$K_\alpha = \left\{ x \in \Lambda : \lim_{t \to \infty} \frac{1}{t} \int_0^t g(\varphi_{\tau}(x)) \, d\tau = \alpha \right\}.$$ The entropy spectrum for the Birkhoff averages of $g$ is defined by $$E(\alpha) = h(\Phi|K_\alpha).$$

For each $q \in \mathbb{R}$, let $\nu_q$ be the equilibrium measure for $qg$ and write $T(q) = P_{\Phi}(qg)$. The following result of Barreira and Saussol [5] describes the entropy spectrum.

**Theorem 5.2.** Let $\Phi$ be a $C^{1+\delta}$ flow with a locally maximal hyperbolic set $\Lambda$ such that $\Phi|\Lambda$ is topologically mixing and let $g : \Lambda \to \mathbb{R}$ be a Hölder continuous function with $P_{\Phi}(g) = 0$. Then:

1. the domain of $E$ is a closed interval in $[0, +\infty)$ coinciding with the range of the function $\alpha = -T'$ and for each $q \in \mathbb{R}$ we have
   $$E(\alpha(q)) = T(q) + q\alpha(q) = h_{\nu_q}(\Phi);$$
2. if $g$ is not $\Phi$-cohomologous to a constant on $\Lambda$, then the functions $E$ and $T$ are analytic and strictly convex.

Burns and Gelfert [12] considered the geodesic flow for a compact rank 1 surface and obtained some generalizations of Theorems 5.1 and 5.2 in this setting.

The irregular set for the Birkhoff averages of $g$ is defined by $$\mathcal{B}(g) = \left\{ x \in \Lambda : \liminf_{t \to \infty} \frac{1}{t} \int_0^t g(\varphi_{\tau}(x)) \, d\tau < \limsup_{t \to \infty} \frac{1}{t} \int_0^t g(\varphi_{\tau}(x)) \, d\tau \right\}.$$ By Birkhoff’s ergodic theorem, this set has zero measure with respect to any invariant probability measure on $\Lambda$. On the other hand, the following result gives a necessary and sufficient condition so that the irregular set has full topological entropy $h(\Phi|\Lambda)$.

**Theorem 5.3 ([5]).** Let $\Phi$ be a $C^{1+\delta}$ flow with a locally maximal hyperbolic set $\Lambda$ such that $\Phi|\Lambda$ is topologically mixing and let $g : \Lambda \to \mathbb{R}$ be a Hölder continuous function. Then the following properties are equivalent:

1. $g$ is not $\Phi$-cohomologous to a constant on $\Lambda$;
2. $h(\Phi|\mathcal{B}(g)) = h(\Phi|\Lambda)$. 
In fact, most Hölder continuous functions are not \( \Phi \)-cohomologous to a constant. Given \( \gamma \in (0,1) \), let \( C^\gamma(\Lambda) \) be the set of all Hölder continuous functions on \( \Lambda \) with Hölder exponent \( \gamma \), endowed with the norm

\[
\| \varphi \|_\gamma = \sup \{|\varphi(x)| : x \in \Lambda\} + \sup \left\{ \frac{|\varphi(x) - \varphi(y)|}{d(x,y)^\gamma} : x, y \in \Lambda \text{ and } x \neq y \right\},
\]

where \( d \) is the distance on \( M \). We recall that \( \Phi|\Lambda \) is said to be topologically transitive if for any nonempty open sets \( U \) and \( V \) intersecting \( \Lambda \) there exists \( t \in \mathbb{R} \) such that \( \varphi_t(U) \cap V \cap \Lambda \neq \emptyset \).

**Theorem 5.4 ([5]).** Let \( \Phi \) be a \( C^1 \) flow with a locally maximal hyperbolic set \( \Lambda \) such that \( \Phi|\Lambda \) is topologically transitive. For each \( \gamma \in (0,1) \), the set of all functions in \( C^\gamma(\Lambda) \) that are not \( \Phi \)-cohomologous to a constant is open and dense.

6. **Conditional variational principles.** In this section we present an alternative description of the entropy spectrum for the Birkhoff averages of a continuous function in terms of a conditional variational principle. In fact, we consider the general case of ratios of Birkhoff averages. Let \( \Lambda \) be a hyperbolic set for a \( C^1 \) flow \( \Phi \). Given continuous function \( a, b: \Lambda \to \mathbb{R} \) with \( b > 0 \) and \( \alpha \in \mathbb{R} \), consider the set

\[
K_\alpha = \left\{ x \in \Lambda : \lim_{t \to \infty} \frac{\int_0^t a(\varphi_\tau(x)) \, d\tau}{\int_0^t b(\varphi_\tau(x)) \, d\tau} = \alpha \right\}.
\]

The **entropy spectrum** for the pair of functions \((a, b)\) is defined by

\[
\mathcal{F}(\alpha) = h(\Phi|K_\alpha).
\]

Moreover, let

\[
\underline{\alpha} = \inf \left\{ \frac{\int_\Lambda a \, d\mu}{\int_\Lambda b \, d\mu} : \mu \in \mathcal{M} \right\} \quad \text{and} \quad \overline{\alpha} = \sup \left\{ \frac{\int_\Lambda a \, d\mu}{\int_\Lambda b \, d\mu} : \mu \in \mathcal{M} \right\},
\]

where \( \mathcal{M} \) is the set of all \( \Phi \)-invariant probability measures on \( \Lambda \). The following result of Barreira and Saussol [6] provides a conditional variational principle for the spectrum \( \mathcal{F} \).

**Theorem 6.1.** Let \( \Phi \) be a \( C^1 \) flow with a locally maximal hyperbolic set \( \Lambda \) such that \( \Phi|\Lambda \) is topologically mixing and let \( a, b: \Lambda \to \mathbb{R} \) be Hölder continuous functions with \( b > 0 \). Then:

1. if \( \alpha \notin [\underline{\alpha}, \overline{\alpha}] \), then \( K_\alpha = \emptyset \);
2. if \( \alpha \in (\underline{\alpha}, \overline{\alpha}) \), then \( K_\alpha \neq \emptyset \),

\[
\mathcal{F}(\alpha) = \max \left\{ h_\mu(\Phi) : \frac{\int_\Lambda a \, d\mu}{\int_\Lambda b \, d\mu} = \alpha \text{ and } \mu \in \mathcal{M} \right\}
\]

\[
= \min \{ P_\Phi(qa - qab) : q \in \mathbb{R} \}.
\]

The following result shows that the spectrum \( \mathcal{F} \) is analytic when \( a \) is not cohomologous to a multiple of \( b \).

**Theorem 6.2 ([6]).** Let \( \Phi \) be a \( C^1 \) flow with a locally maximal hyperbolic set \( \Lambda \) such that \( \Phi|\Lambda \) is topologically mixing and let \( a, b: \Lambda \to \mathbb{R} \) be Hölder continuous functions with \( b > 0 \). Then:

1. if \( a \) is \( \Phi \)-cohomologous to a multiple of \( b \), then \( \alpha = \overline{\alpha} \);
2. if \( a \) is not \( \Phi \)-cohomologous to a multiple of \( b \), then \( \alpha < \overline{\alpha} \) and the function \( \mathcal{F} \) is analytic on \((\underline{\alpha}, \overline{\alpha})\).
When $b = 1$ the statement in Theorem 6.2 was first established in [5].

Again, most Hölder continuous functions satisfy the second alternative in Theorem 6.2. Let $C^1_+(\Lambda)$ be the set of all positive functions in $C^\gamma(\Lambda)$.

**Theorem 6.3 ([6]).** Let $\Phi$ be a $C^1$ flow with a locally maximal hyperbolic set $\Lambda$ such that $\Phi|\Lambda$ is topologically transitive. For each $\gamma \in (0, 1)$, the set of all pairs of functions $(a, b) \in C^\gamma(\Lambda) \times C^\gamma(\Lambda)$ such that $a$ is not $\Phi$-cohomologous to a multiple of $b$ is open and dense.

In the remainder of this section we consider briefly multifractal spectra for the local entropies and for the Lyapunov exponents. These spectra are particular cases of those considered in Theorems 6.1 and 6.2.

Let $\mu$ be a $\Phi$-invariant probability measure on $\Lambda$. For each $x \in \Lambda$, the lower and upper $\mu$-local entropies of $\Phi$ at $x$ are defined, respectively, by

$$h^-_\mu(x) = \lim_{\varepsilon \to 0} \liminf_{t \to \infty} -\frac{1}{t} \log \mu(B(x, t, \varepsilon))$$

and

$$\overline{h}_\mu(x) = \lim_{\varepsilon \to 0} \limsup_{t \to \infty} -\frac{1}{t} \log \mu(B(x, t, \varepsilon)),$$

with $B(x, t, \varepsilon)$ as in (12). When $h^-_\mu(x) = \overline{h}_\mu(x)$, the common value is denoted by $h_\mu(x)$ and is called the $\mu$-local entropy of $\Phi$ at $x$. By the Shannon–McMillan–Breiman theorem, the $\mu$-local entropy of $\Phi$ is defined $\mu$-almost everywhere. Moreover, if $\mu$ is ergodic, then $h_\mu(x) = h(\Phi)$ for $\mu$-almost every $x \in \Lambda$. The entropy spectrum for the local entropies of $\mu$ is defined by

$$\mathcal{H}(\alpha) = h(\Phi|K^h_\alpha),$$

where

$$K^h_\alpha = \{x \in \Lambda : h^-_\mu(x) = \overline{h}_\mu(x) = \alpha\}.$$

When $\Lambda$ is a locally maximal hyperbolic set for $\Phi$, for any sufficiently small $\varepsilon > 0$ we have

$$K^h_\alpha = \{x \in \Lambda : \lim_{t \to \infty} \frac{1}{t} \log \mu(B(x, t, \varepsilon)) = \alpha\}$$

and there exists a unique invariant measure $m$ with $h_m(\Phi) = h(\Phi)$. We write

$$\alpha^h = \inf \left\{ -\int_{\Lambda} a \, d\nu : \nu \in M \right\} \quad \text{and} \quad \overline{\alpha}^h = \sup \left\{ -\int_{\Lambda} a \, d\nu : \nu \in M \right\}.$$

**Theorem 6.4 ([6]).** Let $\Phi$ be a $C^1$ flow with a locally maximal hyperbolic set $\Lambda$ such that $\Phi|\Lambda$ is topologically mixing and let $\mu$ be an equilibrium measure for a Hölder continuous function $\alpha : \Lambda \to \mathbb{R}$ with $P_\Phi(\alpha) = 0$. Then:

1. if $\alpha \in (\alpha^h, \overline{\alpha}^h)$, then $K^\mu_\alpha \neq \emptyset$ and

$$\mathcal{H}(\alpha) = \max \left\{ h_\nu(\Phi) : -\int_{\Lambda} a \, d\nu = \alpha \text{ and } \nu \in M \right\}$$

$$= \min \{ P_\Phi(q\alpha) + q\alpha : q \in \mathbb{R} \};$$

2. if $\mu \neq m$, then $\underline{\alpha}^h < \overline{\alpha}^h$ and the function $\mathcal{H}$ is analytic on $(\alpha^h, \overline{\alpha}^h)$.

Now we consider the multifractal spectrum for the Lyapunov exponents. The stable and unstable entropy spectra for the Lyapunov exponents are defined, respectively, by

$$\mathcal{L}_s(\alpha) = h(\Phi|K^s_\alpha) \quad \text{and} \quad \mathcal{L}_u(\alpha) = h(\Phi|K^u_\alpha),$$
where

\[ K_s^{\alpha} = \left\{ x \in \Lambda : \lim_{t \to \infty} \frac{1}{t} \log \| d_x \varphi_t | E_s^u(x) \| = \alpha \right\} \]

and

\[ K_u^{\alpha} = \left\{ x \in \Lambda : \lim_{t \to \infty} \frac{1}{t} \log \| d_x \varphi_t | E_u^s(x) \| = \alpha \right\}. \]

We write

\[ \alpha_s = \inf \left\{ \int_{\Lambda} \zeta_s \, d\mu : \mu \in \mathcal{M} \right\} \]

and

\[ \alpha_u = \sup \left\{ \int_{\Lambda} \zeta_s \, d\mu : \mu \in \mathcal{M} \right\}. \]

**Theorem 6.5** ([6]). Let \( \Phi \) be a \( C^{1+\delta} \) flow with a locally maximal hyperbolic set \( \Lambda \) such that \( \Phi|\Lambda \) is conformal and topologically mixing. Then:

1. if \( \alpha \in (\alpha_s, \alpha_u) \), then \( K_s^{\alpha} \neq \emptyset \) and

\[ \mathcal{L}_s(\alpha) = \max \left\{ h_{\mu}(\Phi) : \int_{\Lambda} \zeta_s \, d\mu = \alpha \text{ and } \mu \in \mathcal{M} \right\} \]

\[ = \min \{ P_{\Phi}(q\zeta_s) - q\alpha : q \in \mathbb{R} \}; \]

2. if \( \zeta_s \) is not \( \Phi \)-cohomologous to zero, then \( \alpha_s < \alpha_u \) and the function \( \mathcal{L}_s \) is analytic on \( (\alpha_s, \alpha_u) \).

One can formulate a corresponding result for the spectrum \( \mathcal{L}_u \).

**7. Multidimensional spectra.** In this section we describe a multidimensional version of the entropy spectrum for the ratios of Birkhoff averages. We consider the general case when the entropy map is upper semicontinuous. Let \( \Phi \) be a continuous flow on a compact metric space \( X \). Consider functions \( A, B \in C(X)^d \) for some \( d \in \mathbb{N} \), with components

\[ A = (a_1, \ldots, a_d) \quad \text{and} \quad B = (b_1, \ldots, b_d), \]

such that \( b_i > 0 \) for \( i = 1, \ldots, d \). For each \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d \), let

\[ K_\alpha = \left\{ x \in X : \lim_{t \to \infty} \frac{1}{t} \int_0^t a_i(\varphi_\tau(x)) \, d\tau = \alpha_i \text{ for } i = 1, \ldots, d \right\}. \]

The *entropy spectrum* for the pair \( (A, B) \) is defined by

\[ \mathcal{F}(\alpha) = h(\Phi|K_\alpha). \]

Moreover, consider the function \( \mathcal{P} : \mathcal{M} \to \mathbb{R}^d \) defined by

\[ \mathcal{P}(\mu) = \left( \frac{\int_X a_1 \, d\mu}{\int_X b_1 \, d\mu}, \ldots, \frac{\int_X a_d \, d\mu}{\int_X b_d \, d\mu} \right), \]

where \( \mathcal{M} \) is the set of all \( \Phi \)-invariant probability measures on \( X \). For each \( \alpha = (\alpha_1, \ldots, \alpha_d) \) and \( \beta = (\beta_1, \ldots, \beta_d) \) in \( \mathbb{R}^d \), we write

\[ \alpha \ast \beta = (\alpha_1 \beta_1, \ldots, \alpha_d \beta_d) \quad \text{and} \quad \langle \alpha, \beta \rangle = \sum_{i=1}^{d} \alpha_i \beta_i. \]

The following result of Barreira and Doutor [3] provides a partial variational principle for the spectrum \( \mathcal{F} \).
Theorem 7.1. Let \( \Phi \) be a continuous flow on a compact metric space \( X \) such that the map \( \mu \mapsto h_\mu(\Phi) \) is upper semicontinuous and consider functions \( (A,B) \in C(X)^d \times C(X)^d \) such that

\[
\text{span}\{a_1, b_1, \ldots, a_d, b_d\} \subset D(X).
\]

If \( \alpha \in \mathbb{P}(\mathcal{M}) \), then:

1. \( K_\alpha \neq \emptyset \) and

\[
\mathcal{F}(\alpha) = \max\{h_\mu(\Phi) : \mu \in \mathcal{M} \text{ and } \mathbb{P}(\mu) = \alpha\}
\]

\[
= \min\{P_\theta((q,A - \alpha \ast B)) : q \in \mathbb{R}^d\};
\]

2. there exists an ergodic measure \( \mu_\alpha \in \mathcal{M} \) such that

\[
\mathbb{P}(\mu_\alpha) = \alpha, \quad \mu_\alpha(K_\alpha) = 1 \quad \text{and} \quad h_{\mu_\alpha}(\Phi) = \mathcal{F}_\alpha(\alpha).
\]

For a topologically mixing flow on a locally maximal hyperbolic set, the statement in Theorem 7.1 was first established in [6] (see Theorem 6.1).

The following result gives conditions for the analyticity of the spectrum.

Theorem 7.2 ([3]). Let \( \Phi \) be a \( C^1 \) flow with a locally maximal hyperbolic set \( \Lambda \) such that \( \Phi|\Lambda \) is topologically mixing. If the functions \( a_i, b_i : \Lambda \to \mathbb{R} \), for \( i = 1, \ldots, d \), are Hölder continuous, then \( \mathcal{F} \) is analytic on \( \mathbb{P}(\mathcal{M}) \).

8. Dimension spectra. Finally, in this section we consider Birkhoff averages both into the future and into the past and we describe the Hausdorff dimension of the corresponding level sets. Let \( \Lambda \) be a hyperbolic set for a \( C^1 \) flow \( \Phi \). Given \( d \in \mathbb{N} \), let

\[
F = C^\gamma(\Lambda)^d \times C^\gamma(\Lambda)^d.
\]

Moreover, given functions \( (A^\pm, B^\pm) \in F \), write

\[
A^+ = (a_1^+, \ldots, a_d^+), \quad B^+ = (b_1^+, \ldots, b_d^+)
\]

and

\[
A^- = (a_1^-, \ldots, a_d^-), \quad B^- = (b_1^-, \ldots, b_d^-).
\]

We shall always assume that all components of \( A^- \) and \( B^- \) are positive. For each \( \alpha = (a_1, \ldots, a_d) \) and \( \beta = (\beta_1, \ldots, \beta_d) \) in \( \mathbb{R}^d \), let

\[
K^+_\alpha = \left\{x \in \Lambda : \lim_{t \to +\infty} \frac{1}{t} \int_0^t a_i^+(\varphi_t(x)) \, d\tau = \alpha_i \text{ for } i = 1, \ldots, d\right\}
\]

and

\[
K^-_\beta = \left\{x \in \Lambda : \lim_{t \to -\infty} \frac{1}{t} \int_0^t a_i^-(\varphi_t(x)) \, d\tau = \beta_i \text{ for } i = 1, \ldots, d\right\}.
\]

Barreira and Doutor [4] described the dimensions of these sets as follows.

Theorem 8.1. Let \( \Phi \) be a \( C^{1+\delta} \) flow with a locally maximal hyperbolic set \( \Lambda \) such that \( \Phi|\Lambda \) is conformal and topologically mixing and let \( (A^\pm, B^\pm) \in F \). For each \( \alpha, \beta \in \mathbb{R}^d \), \( x^+ \in K^+_\alpha \) and \( x^- \in K^-_\beta \), we have

\[
\dim_H K^+_\alpha = \dim_H (K^+_\alpha \cap V^u(x^+)) + t_s + 1
\]

and

\[
\dim_H K^-_\beta = \dim_H (K^-_\beta \cap V^s(x^-)) + t_u + 1,
\]

with \( t_s \) and \( t_u \) as in (9).
The dimension spectrum for the functions in (16) and (17) is defined by
\[ D(\alpha, \beta) = \dim_H(K_\alpha^+ \cap K_\beta^-). \]
Moreover, consider the functions \( P^+, P^- : M \to \mathbb{R}^d \) defined by
\[
P^+(\mu) = \left( \frac{\int_\Lambda a_1^+ d\mu}{\int_\Lambda b_1^+ d\mu}, \ldots, \frac{\int_\Lambda a_d^+ d\mu}{\int_\Lambda b_d^+ d\mu} \right)
\]
and
\[
P^-(\mu) = \left( \frac{\int_\Lambda a_1^- d\mu}{\int_\Lambda b_1^- d\mu}, \ldots, \frac{\int_\Lambda a_d^- d\mu}{\int_\Lambda b_d^- d\mu} \right),
\]
where \( M \) is the set of all \( \Phi \)-invariant probability measures on \( \Lambda \). The following result is a conditional variational principle for the spectrum \( D \).

**Theorem 8.2** ([4]). Let \( \Phi \) be a \( C^{1+\delta} \) flow with a locally maximal hyperbolic set \( \Lambda \) such that \( \Phi|\Lambda \) is conformal and topologically mixing and let \( (A^\pm, B^\pm) \in \mathcal{F} \). Then:
1. if \( \alpha \in \text{int} P^+(M) \) and \( \beta \in \text{int} P^-(M) \), then
   \[
   D(\alpha, \beta) = \dim_H K_\alpha^+ + \dim_H K_\beta^- - \dim_H \Lambda
   \]
   
   \[
   = \max \left\{ \frac{h_\mu(\Phi)}{\int_\Lambda \xi_\mu d\mu} : \mu \in M \text{ and } P^+(\mu) = \alpha \right\}
   \]
   \[
   + \max \left\{ \frac{h_\mu(\Phi)}{\int_\Lambda \xi_\mu d\mu} : \mu \in M \text{ and } P^-(\mu) = \beta \right\} + 1;
   \]
2. the function \( D \) is analytic on \( \text{int} P^+(M) \times \text{int} P^-(M) \).

**Appendix A. Markov systems.** Let \( \Lambda \) be a locally maximal hyperbolic set for a \( C^1 \) flow \( \Phi \) on \( M \). Consider an open disk \( D \subset M \) of dimension \( \dim M - 1 \) that is transverse to the flow. Moreover, given \( x \in D \), let \( U(x) \) be an open neighborhood of \( x \) that is diffeomorphic to the product \( D \times (-\varepsilon, \varepsilon) \) and define a map \( \pi_D : U(x) \to D \) by
\[
\pi_D(\varphi_t(y)) = y.
\]
A closed set \( R \subset \Lambda \cap D \) is said to be a rectangle if \( R = \overline{\text{int} R} \) (with the interior computed with respect to the induced topology on \( \Lambda \cap D \)) and \( \pi_D([x, y]) \in R \) whenever \( x, y \in R \).

Now we consider rectangles \( R_1, \ldots, R_k \subset \Lambda \) and \( \varepsilon > 0 \) such that:
1. \( R_i \cap R_j = \partial R_i \cap \partial R_j \) for \( i \neq j \);
2. \( \Lambda = \bigcup_{t \in [0, \varepsilon]} \varphi_t(Z) \), where \( Z = \bigcup_{i=1}^k R_i \);
3. for each \( i \neq j \) either
   \[
   \varphi_t(R_i) \cap R_j = \emptyset \quad \text{for} \quad t \in [0, \varepsilon]
   \]
   or
   \[
   \varphi_t(R_j) \cap R_i = \emptyset \quad \text{for} \quad t \in [0, \varepsilon].
   \]
We define a transfer function \( \tau : \Lambda \to [0, +\infty) \) by
\[
\tau(x) = \min \{ t > 0 : \varphi_t(x) \in Z \}
\]
and a transfer map \( T : \Lambda \to Z \) by
\[
T(x) = \varphi_{\tau(x)}(x).
\]
The collection of rectangles $R_1, \ldots, R_k$ is said to be a Markov system for $\Phi$ on $\Lambda$ if

$$T(\text{int}(V^s(x) \cap R_i)) \subset \text{int}(V^s(T(x)) \cap R_j)$$

and

$$T^{-1}(\text{int}(V^u(T(x)) \cap R_j)) \subset \text{int}(V^u(x) \cap R_i)$$

for $x \in \text{int}(T(R_i) \cap \text{int} R_j)$. It was shown by Bowen [8] and Ratner [27] that there exist Markov systems of arbitrary small diameter.

Now we consider the $k \times k$ matrix $A$ with entries

$$a_{ij} = \begin{cases} 1 & \text{if } \text{int}(T(R_i) \cap \text{int} R_j) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, we consider the set

$$\Sigma_A = \{\cdots i_{-1}i_0i_1 \cdots : a_{i_{n+1}i_n} = 1 \text{ for } n \in \mathbb{Z}\} \subset \{1, \ldots, k\}^\mathbb{Z}$$

and the topological Markov chain $\sigma: \Sigma_A \to \Sigma_A$ defined by $\sigma(\cdots i_0 \cdots) = (\cdots j_0 \cdots)$, where $j_n = i_{n+1}$ for each $n \in \mathbb{Z}$. We define a coding map $\pi: \Sigma_A \to \bigcup_{i=1}^k \text{int} R_i$ for (a part of) the hyperbolic set by

$$\pi(\cdots i_0 \cdots) = \bigcap_{j \in \mathbb{Z}} (T|Z)^{-j}(\text{int} R_i).$$

One can easily verify that

$$\pi \circ \sigma = T \circ \pi.$$  (18)

By (18), the restriction of the flow to the set $\Lambda$ is a factor of a suspension flow over a topological Markov chain. More precisely, consider the space

$$Z = \{(x,s) \in X \times \mathbb{R} : 0 \leq s \leq \tau(x)\}$$

and let $Y$ be the set obtained from $Z$ by identifying the points $(x, \tau(x))$ and $(T(x), 0)$ for each $x \in X$. One can introduce in a natural way a topology on $Y$ obtained from the product topology on $Z \subset X \times \mathbb{R}$, with respect to which $Y$ is a compact topological space. This topology is induced by a certain distance introduced by Bowen and Walters [10]. The suspension flow over $T$ with height function $\tau$ is the flow $\Psi = (\psi_t)_{t \in \mathbb{R}}$ on $Y$ composed of the maps $\psi_t: Y \to Y$ defined by

$$\psi_t(x, s) = (x, s + t).$$

We note that any suspension flow is indeed a flow.

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