Luminosity, redshift and gas abundance in general relativistic radiation hydrodynamics

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Abstract

Quasi-stationary flows of gas accreting onto a compact center are analyzed in the framework of general-relativistic radiation hydrodynamics, under assumptions of spherical symmetry and thin gas approximation. Numerical investigation shows that luminosity, redshift and gas abundance are correlated. The gas can constitute up to one third of the total mass of brightest low-redshift sources, but its abundance goes down to 1/30 for sources with luminosities close to the Eddington limit.
We investigate a steady gas accretion onto a compact core in the framework of general relativity. The main goal of this letter is to show that bright sources — with the luminosity approaching the Eddington limit — must contain a significant fraction of gas. Our model assumes spherical symmetry, polytropic equation of state and thin gas approximation in the transport equation [1]

We use comoving coordinates $t, r, 0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi$: time, coordinate radius and two angle variables, respectively. The metric is

$$ds^2 = -N^2 dt^2 + \hat{a} dr^2 + R^2 d\Omega^2$$

where $R$ denotes the area radius. The radial velocity of gas is given by $U = \frac{1}{N} \frac{dR}{dt}$.

The energy-momentum tensor reads $T_{\mu\nu} = T_{\mu\nu}^B + T_{\mu\nu}^E$, where the baryonic part is given by $T_{\mu\nu}^B = (\rho + p) U_{\mu} U_{\nu} + pg_{\mu\nu}$ with the time-like and normalized four-velocity $U_{\mu}, U_{\mu} U^{\mu} = -1$. The radiation part has only four nonzero components: $T_{00}^E \equiv -\rho^E = -T_{r r}^E$ and $T_{r 0}^E = T_{0 r}^E$. A comoving observer would measure local mass densities, material $\rho = T_{\mu\nu}^B U^{\mu}$ and radiation $\rho^E$, respectively. The baryonic current is defined as $j^0 \equiv \rho_0 U^\mu$, where $\rho_0$ is the baryonic mass density. Define $n_{\mu}$ as the unit normal to a centered (coordinate) sphere lying in the hypersurface $t = \text{const}$ and $k$ as the related mean curvature scalar, $k = \frac{R}{2} \nabla_{\mu} n^\mu = \frac{1}{\sqrt{a}} \partial_t R$. The comoving radiation flux density reads $j = U_{\mu} n^\nu NT_{\nu}^{\mu E} / \sqrt{a} = NT_{r 0}^{0 E} / \sqrt{a}$. The baryonic matter satisfies the polytropic equation of state $p = K \rho_0^\gamma$ (with constants $K$ and $\Gamma$). The internal energy $h$ and the rest and baryonic mass densities are related by $\rho = \rho_0 + h$, where $h = p/(\Gamma - 1)$.

The equation

$$\nabla_{\mu} j^{\mu} = 0$$

expresses the conservation of baryonic matter.

There are four conservation equations resulting (due to the contracted Bianchi identities) from the Einstein equations, namely $\nabla_{\nu} T_{\nu}^{\mu B} = -\nabla_{\nu} T_{\nu}^{\mu E} = F_{\nu}$ (here $\nu = 0, r$). The quantity $F_{\nu}$ is the radiation force density and it describes interaction between baryons and radiation. The present formulation of general-relativistic radiation hydrodynamics agrees with that of Park [3], Miller and Rezzola [4] and (on a Schwarzschildian background) Thorne et. al [5].

One can solve formally the Einstein constraint equations $G_{\mu 0} = 8\pi T_{\mu 0}$, arriving at ([2],[6])

$$k = \sqrt{1 - \frac{2m(R)}{R} + U^2}.$$  

Above $m(R)$ is the quasilocal mass given by

$$m(R) = m - 4\pi \int_R^{R_0} dr r^2 \left( \rho + \rho^E + \frac{U j}{k} \right).$$
The integration in (4) extends from $R$ to the outer boundary $R_\infty$. A ball of gas is comprised between a hard core of a radius $R_0$ and sphere $S_\infty$ of a radius $R_\infty$. Its external boundary is connected to the Schwarzschild vacuum spacetime by a transient zone of a negligible (due to special transitory data) mass. Thence the asymptotic mass $M$ is approximately equal to $m(R_\infty)$. Similar picture emerges in the recent construction of quasistars [7].

In an alternative, polar gauge foliation, one has a new time $t_S(t, r)$ with $\partial_t S = \partial_t - NU \partial_R$. The expression $4\pi NkR^2 \left( j \left( 1 + \left( \frac{U}{k} \right)^2 \right) + 2U\rho^E / k \right)$ represents the radiation flux measured by an observer located at $R$ in coordinates $(t_S, R)$. One can show that

$$\partial_R m(R) = -4\pi \left( NkR^2 \left( j \left( 1 + \left( \frac{U}{k} \right)^2 \right) + 2\rho^E / k \right) + NU^2 (\rho + p) \right) R_\infty. \quad (5)$$

The mass contained in the annulus $(R, R_\infty)$ changes if the fluxes on the right hand side, one directed outward and the other inward, do not cancel.

The local baryonic flux will be denoted as $\dot{M} = -4\pi UR^2 \rho_0$ and its boundary value reads $\dot{M}_\infty$. The accretion process is said to be quasistationary if all relevant observables measured at $R$ are approximately constant during time intervals much smaller than the runaway instability time scale $T = M/\dot{M}_\infty$. Analytically, we assume that $\partial_t X = (\partial_t - NU \partial_R)X = 0$ for $X = \rho_0, \rho, j, U$…

The above assumptions imply, in the thin gas approximation [1], that $F_0 = 0$ and the radiation force density has only one nonzero component $F_r = \kappa k N \rho_0 j$. The only direct interaction between baryons and radiation is through elastic Thomson scattering. $\kappa$ is a material constant, depending in particular on the Thomson cross section $\sigma$, $\kappa = \sigma / (4\pi m_p c)$. $c$ is the speed of light and $m_p$ is the proton mass.

The full system of equations in a form suitable for numerics has been derived elsewhere [11]. It consists of:

i) The total energy conservation

$$\dot{M} N \frac{\Gamma - 1}{\Gamma - 1 - a^2} + 2\dot{M} N \rho^E \rho_0 = 4\pi R^2 j N k \left( 1 + \frac{U^2}{k^2} \right) + C; \quad (6)$$

The constant $C$ is the asymptotic energy flux flowing through the sphere of a radius $R_\infty$ (see (5)).

ii) The local radiation energy conservation (below $a = \sqrt{dp/d\rho}$ is the speed of sound)

$$\left( 1 - \frac{2m(R)}{R} \right) N \frac{d}{dR} \left( R^2 \rho^E \right) = -\kappa k N j \rho_0 + 2N \left( U \rho^E - k j \right) \frac{dU}{dR} + 2k \left( jU - \kappa \rho^E \right) \frac{dN}{dR} + 8\pi NRk \left( j^2 - j \rho^E \frac{U}{k^2} \right). \quad (7)$$

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iii) The relativistic Euler equation

\[ \frac{d}{dR} \ln a^2 = -\frac{\Gamma - 1 - a^2}{a^2 - \frac{U^2}{k^2}} \times \]

\[ \left[ \frac{1}{k^2 R} \left( \frac{m(R)}{R} - 2U^2 + 4\pi R^2 \left( \rho + p + \frac{jU}{k} \right) \right) \right] - \]

\[ \kappa j \left( 1 - \frac{a^2}{\Gamma - 1} \right) \]. \hspace{1cm} (8)

iv) The baryonic mass conservation

\[ \frac{dU}{dR} = -\frac{U}{\Gamma - 1 - a^2} \frac{d}{dR} \ln a^2 - \frac{2U}{R} + \frac{4\pi R j}{k}. \] \hspace{1cm} (9)

v) The equation for the lapse

\[ \frac{dN}{dR} = N \left( \kappa j \frac{\Gamma - 1 - a^2}{\Gamma - 1} + \frac{d}{dR} \ln \left( \Gamma - 1 - a^2 \right) \right). \] \hspace{1cm} (10)

Equations (6)–10) constitute, with \( k \) and \( m(R) \) given by (3) and (4), the complete model used in numerical calculations. The asymptotic data are such that \( a_{\infty}^2 \gg \frac{M}{R_{\infty}} \gg \frac{U_{\infty}^2}{R_{\infty}} \), which guarantees the fulfillment of the Jeans criterion for the stability (see a discussion in [8] and studies of stability of accreting flows in newtonian hydrodynamics [9]), suggesting in turn the stability of solutions. One can put \( j_{\infty} = \rho_{\infty} E \). The total luminosity is well approximated by \( L_0 = 4\pi R_{\infty}^2 j_{\infty} \) and it should be related to the accretion rate by the formula

\[ L_0 = \alpha \dot{M}_{\infty} \equiv \left( 1 - \frac{N(R_0)}{k(R_0)} \sqrt{1 - \frac{2m(R_0)}{R_0}} \right) \dot{M}_{\infty}. \] \hspace{1cm} (11)

The last formula is justified by two arguments. i) In the nonrelativistic limit one gets \( \alpha = |\phi(R_0)| \), where \( \phi \) is the newtonian potential. Eq. (11) states now that the binding energy is transformed into radiation with the implicit assumption that the heat capacity of the core is negligible. ii) The condition of stationarity implies the existence of the approximate time-like Killing vector and it appears that \( \alpha \) gives the standard measure of the gravitational redshift. If stationary observers detect \( \omega_0 \) at \( R_0 \) and \( \omega \) at infinity, and \( 1/\omega \ll 2M \) (the geometric optics condition — see [10] for a discussion) then \( \omega = (1 - \alpha) \omega_0 \). Thus \( \alpha \) can be regarded as a proper binding energy and again one arrives at formula (11).
Let us remark that from above definitions and the equation \ref{eq:8} one infers \( L_0 \leq 4\pi M/\kappa \), for accretion solutions; it is notable here that the limiting luminosity involves the total mass \( M \) instead of the mass \( m(R_0) \) of the central core.

The triple of independent boundary data can consist of \( \alpha, L_0 \) and \( a^2_\infty \). These quantities can be determined from observations of highest redshift, total luminosity and asymptotic temperature, respectively. Then one chooses \( j_\infty = \rho^E_\infty = L_0 / (4\pi R^2_\infty) \), and the mass accretion rate \( \dot{M} = L_0 / \alpha \). These data specify supersonic flows up to, possibly, a bifurcation \([11]\). In the case of subsonic flows another boundary condition is needed, for instance the asymptotic baryonic mass density \( \rho_\infty \).

Eqs. \ref{eq:7}-\ref{eq:10} are in the evolution form. Numerical calculation starts from the outer boundary \( R_\infty \), taking into account Eq. \ref{eq:6}, and evolves inwards until the equality \( \alpha = 1 - \frac{N(R)}{k(R)} \sqrt{1 - \frac{2m(R)}{R}} \) is met at some \( R \), denoted as \( R_0 \) and being regarded as the radius of the compact core. The numerical integration employs the 8th order Runge-Kutta method \([12]\). Choosing \( \rho_0_\infty \) at random one either obtains no solution at all or a subsonic solution. Using the bisection method and automating the search process, one can obtain a boundary of the solution set (on the plane \( L_0 - \rho_0_\infty \)), which (interestingly enough) appears to bifurcate from a brightest flow. This boundary will be called later on as the bifurcation curve.

We choose \( M_0 / M = 5.95496 \times 10^{-7} \), where \( M_0 \) is the Solar mass. In the standard gravitational units \( G = c = 1 \) and in the scaling \( M = 1 \) one gets \( \kappa = 2.1326762 \times 10^{21} (M_0 / M) \), that is \( \kappa = 1.27 \times 10^{15} \). The size of the system is \( R_\infty = 10^6 \). The speed of sound is given in successive runs by \( a^2_\infty = 4 \times 10^{-3}, 4 \times 10^{-4}, 4 \times 10^{-5} \). The Eddington luminosity reads \( L_E = 4\pi M/\kappa = 9.9847 \times 10^{-15} \).

Figures \ref{fig:1}-\ref{fig:3} show accreting solutions on the luminosity-(mass of the central core) diagram for \( \alpha = 25 \times 10^{-4}, 0.5, 0.9 \), respectively. Each figure depicts solution sets for three different values of the asymptotic speed of sound \( a^2_\infty \). For small \( \alpha \) there can exist two accreting solutions possessing sonic points, with asymptotic densities \( \rho_{0_\infty 1} \) and \( \rho_{0_\infty 2} \). Subsonic flows then exist for each \( \rho_{0_\infty} \in (\rho_{0_\infty 1}, \rho_{0_\infty 2}) \). Subsonic solutions are not specified uniquely for given boundary data but the length of the interval of allowed values of the asymptotic baryonic density \( \rho_{0_\infty} \) becomes shorter with the increase of \( L_0 \).

For larger \( L_0 \) and/or \( \alpha \) the bifurcation curve can consist either of subsonic or supersonic flows and its interior consists of subsonic solutions \([11]\). The brightest system coincides, as before, with the bifurcation point and it is unique. The luminosity of the bifurcation point increases with the decrease of the asymptotic speed of sound and it goes up with the increase of \( \alpha \). Its gas abundance
FIG. 1: Small binding energy, $\alpha = 0.0025$. Three asymptotic values of the speed of sound, $a_\infty^2 = 0.004, 0.0004, 0.00004$. Two bifurcation branches encompass the set of subsonic flows. The abscissa shows the luminosity and the ordinate shows the mass of the compact core.

depends both on luminosity and redshift.

The gas abundance for $\alpha = 25 \times 10^{-4}$ decreases from almost $1/3$ at $a_\infty^2 = 0.004$ to $1/5$ at $a_\infty^2 = 0.00004$, as shown on Fig. 1. This value of $\alpha$ implies $2M(R_0)/R_0 \approx 0.005$. Interestingly, the abundance $1/3$ can be shown analytically to characterize those general relativistic polytropic flows without radiation that maximize the accretion rate \[13\] and low luminosity newtonian sources \[8\]. The case of $\alpha = 0.5$ corresponds to a very compact central object with $2M(R_0)/R_0 \approx 0.75$, close to the Buchdahl limit \[14\]. Fig. 2 demonstrates that gas contribution equals about 0.16 for $a_\infty^2 = 0.004$ and goes down to 0.04 for $a_\infty^2 = 0.00004$. When $\alpha = 0.9$, then $2M(R_0)/R_0 \approx 0.99$ at the surface of the compact central object, well beyond the Buchdahl limit. Only exotic matter violating standard energy conditions can be responsible for such compact bodies \[15\]. We find from Fig. 3 that gas contribution to the mass changes from $1/8$ ($a_\infty^2 = 0.004$) to $1/30$ ($a_\infty^2 = 0.00004$).
It is clear that a reformulation of the problem would allow one to estimate the mass of an isolated system, assuming that the mass of the central core is known. This can be possibly applied to Thorne-Żytkow stars.

One can show that in models with test fluids the gas density is bounded from below. In particular, in a Shakura model and for general relativistic systems with low luminosity and redshift, the bound is provided by a supersonic flow [11]. The full general-relativistic analysis reveals a new qualitative effect. Namely in steady (sub-or supersonic) accretion solutions the gas abundance is bounded both from below and from above by bounds that depend on the redshift and luminosity. This is a clear demonstration of the importance of backreaction in accretion processes.

The triple of observables $\alpha, L_0, a^2_{\infty}$ does not specify accretion completely, but for high luminosities the remaining freedom (in choosing the asymptotic baryonic mass density) is severely restricted and the brightest flow is unique. It is interesting to note that the concept of Eddington luminosity still applies – in the light of our data – in the general relativistic case. In all stud-
FIG. 3: High binding energy, $\alpha = 0.9$, $a^2 = 0.004$, 0.0004, 0.00004. The axes are as in Fig.1.

ied examples we have $L_0 < L_E$; this is supported also by an analytic argument, discussed above. Now $L_E = 4\pi M/\kappa$; it is the global mass rather than the mass of the compact core, that enters the expression for the Eddington luminosity.

Numerical data show that gas can be abundant in quasi-stationary accreting systems. Brightest systems can possess even 33% of gas for small redshifts and still more than 10% of gas for $\alpha = 0.9$. It is an open problem whether this conclusion is true in nonspherical steady flows.

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[1] D. Mihalas and B. W. Mihalas, Foundation of Radiation Hydrodynamics, Oxford University Press New York Oxford 1984.

[2] M. Iriondo, E. Malec and N. O’ Murchadha, Phys. Rev. D54, 4792(1996).

[3] M.-G. Park & G. S. Miller ApJ, 371, 708(1991).

[4] L. Rezzolla & J. C. Miller, Class. and Quantum Grav., 11, 1815(1994).

[5] K. S. Thorne, R. A. Flammang & A. N. Żytkow, MNRAS, 194, 475(1981).

[6] E. Malec, Phys. Rev. D60, 104043(1999).

[7] M. C. Begelman, E. M. Rossi and Ph. J. Armitage, Quasistars: Accreting black holes inside massive envelopes, arXiv:0711.4078v2 [astro-ph].

[8] J. Karkowski, E. Malec and K. Roszkowski, Astronomy and Astrophysics, 479, 161(2008).

[9] P. Mach, Acta Phys. Pol. B38, 3935(2007); B. Kinasiewicz, P. Mach and E. Malec, Int. J. of Geometric Meth. in Mod. Phys. 4, 197(2007).

[10] J. Karkowski and E. Malec, Classical and Quantum Gravity 20, 85-92, (2003).

[11] J. Karkowski, E. Malec, K. Roszkowski and Z. Świerczyński, Supersonic and subsonic flows in general relativistic radiation hydrodynamics, submitted for publication 2008.

[12] E. Hairer, S.P. Norsett and G. Wanner, Solving Ordinary Differential Equations I. Nonstiff Problems. 2nd Edition, Springer Series in Computational Mathematics, Springer-Verlag, 1993.

[13] J. Karkowski, B. Kinasiewicz, P. Mach, E. Malec and Z. Świerczyński, Phy. Rev. D73, 021503(R)(2006).

[14] A. Buchdahl, Phys. Rev. 116, 1027(1957).

[15] Mazur, P. O. and Mottola, E., Proc. Nat. Acad. Sci., 101, 9545(2004).

[16] Thorne, K. S. and Zytkowski, A. N., ApJ, 212, 832(1977).