LIOUVILLE TYPE THEOREMS FOR THE $p$-HARMONIC
FUNCTIONS ON CERTAIN MANIFOLDS

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Abstract. We show that the Dirichlet problem at infinity is unsolvable for the $p$-Laplace equation for any nonconstant continuous boundary data, for certain range of $p > n$, on an $n$-dimensional Cartan-Hadamard manifold constructed from a complete noncompact shrinking gradient Ricci soliton. Using the steady gradient Ricci soliton, we find an incomplete Riemannian metric on $\mathbb{R}^2$ with positive Gauss curvature such that every positive $p$-harmonic function must be constant for $p \geq 4$.

1. Introduction

In this article, we study two questions about the $p$-Laplace equation on Riemannian manifolds. The first one is the solvability of the Dirichlet problem at infinity on a negatively curved complete noncompact manifold, and the second one is the Liouville property for positive solutions on $\mathbb{R}^2$ equipped with an incomplete metric with positive Gauss curvature. In both cases, the $n$-dimensional manifold $M$ under consideration is equipped with a Riemannian metric $e^{\frac{M}{p-n}}g$ where $(M, g, f)$ is a complete gradient Ricci soliton which is shrinking for the first case and steady for the second case.

On a Riemannian manifold, for a constant $p > 1$, a function $v$ in $W^{1,p}_{\text{loc}} \cap L^\infty_{\text{loc}}$ is $p$-harmonic if it is a weak solution to the $p$-Laplace equation
\begin{equation}
\text{div} \left( |\nabla v|^{p-2} \nabla v \right) = 0.
\end{equation}
It is known that $p$-harmonic functions are in $C^{1,\alpha}$ (22) and the reference therein).

The behaviour of harmonic, more generally $p$-harmonic, functions depends on the sign of the curvature of the manifold in an essential way. We will discuss negatively curved and non-negatively curved manifolds separately.

A Cartan-Hadamard manifold is a complete simply connected Riemannian manifold with nonpositive sectional curvature everywhere. It is well-known that a Cartan-Hadamard manifold $M$ can be compactified by attaching a sphere $M(\infty)$ at the infinity. In the cone topology, the compactification is homeomorphic to a closed Euclidean $n$-ball [9]. The Dirichlet problem at infinity for $p$-harmonic functions is to solve the $p$-Laplace equation:
\begin{equation}
\text{div} \left( |\nabla v|^{p-2} \nabla v \right) = 0
\end{equation}

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on $M$ such that $v$ agrees with a given continuous function $\varphi$ on $M(\infty)$. For $p = 2$, the Dirichlet problem at infinity for harmonic functions is solvable if there are suitable lower and upper bounds for the sectional curvature (Andersen [2], Andersen-Schoen [3], Choi [8], Hsu [14], Sullivan [20]). Ancona [1] constructed an example showing the Dirichlet problem is unsolvable if only a negative constant upper bound is imposed. For $p \in (1, \infty)$, the Dirichlet problem at infinity is solvable under similar curvature assumptions like those in the case $p = 2$, in particular, it is solvable if the sectional curvature is bounded by

$$- r^{2\alpha - 4 - \epsilon} \leq K \leq - \frac{\alpha(\alpha - 1)}{r^2}$$

near $M(\infty)$ where $\epsilon > 0, \alpha > 1$ where $r$ is the distance to a fixed point, for $p \in (1, 1 + (n - 1)\alpha)$ (Holopainen [12], Holopainen-Vähäsniemi [13], Pansu [19]).

Our first result is to show the unsolvability of the Dirichlet problem at infinity on certain Cartan-Hadamard manifolds constructed from shrinking gradient Ricci solitons, for certain range of $p > n$, which include the shrinking Gaussian soliton $(\mathbb{R}^n, dx^2, \frac{|x|^2}{4})$ for every $p > n$. It is interesting to observe that the sectional curvature of the complete negatively curved metric $e^{\frac{|x|^2}{4(2-p-n)}} dx^2$ is not bounded above by $- \frac{\alpha(\alpha - 1)}{r^2}$, for any constant $\alpha > 1$, at certain sections for sufficiently large $r$ (see Remark 2.1). This indicates the upper bound in (1.3) is sharp in some sense for the solvability of the Dirichlet problem at infinity.

**Theorem 1.1.** Suppose $(M, g, f)$ is a simply connected $n$-dimensional complete noncompact shrinking gradient Ricci soliton whose the sectional curvatures is bounded above by a constant $K_0$ with $0 < K_0 < \frac{1}{2(n-1)}$. Then the Dirichlet problem at infinity for the $p$-Laplace equation on $(M, e^{\frac{2f}{p-n}}g)$ is unsolvable for any nonconstant continuous boundary value $\varphi$ and $n < p < \frac{1}{K_0} + 2 - n$.

The proof replies on a Liouville type property (Proposition 2.1) for positive solutions to the $p$-Laplace equation on $(M, e^{\frac{2f}{p-n}}g)$ for every $p > 1$, where Cao-Zhou’s estimates on $f$ and on the volume growth for gradient shrinking Ricci solitons [5] are crucial as they imply that $e^{-f}$ is integrable on $(M, g)$. The advantage for considering the range $p > n$ is that, under the conformal change of metric, it yields a complete metric $\tilde{g}$ and it guarantees the negativity of the curvature of $\tilde{g}$ under the curvature assumption $K \leq K_0$, while one does not have such flexibility for $p = 2$.

However, the integration argument in the proof of Proposition 2.1 is no longer valid for steady gradient Ricci solitons due to different behaviour of $f$ (typically $f$ tends to $-\infty$ along a sequence of points $x_k$ that go to infinity [17], [26]). Alternatively, a powerful way to prove Liouville type theorems for positive harmonic functions on complete manifolds with non-negative Ricci curvature is via Yau’s gradient estimate [24]. The $p$-harmonic version of Yau’s estimate is established by Wang-Zhang [25] (see [21] for a sharp form of the estimate). For a positive $p$-harmonic function $u$ in the conformally changed metric $\tilde{g} = e^{-\frac{2f}{p-n}}g$, we will first derive a maximum principle for $|\nabla \log u|$ for steady (or shrinking) gradient Ricci solitons, via a Bochner type formula. However, the required assumption on
Ricci curvature for the gradient estimates cannot hold globally for steady gradient Ricci solitons if \( \dim M > 2 \) because it would imply the scalar curvature of \( g \) possesses a positive constant lower bound but this is impossible as shown in \([17]\) and \([26]\). In dimension 2, we can combine the maximum principle (Proposition 3.3) and the gradient estimate to prove a Liouville type result on the 2-plane with a positively curved incomplete metric.

**Theorem 1.2.** Let \((\mathbb{R}^2, g, f)\) be Hamilton’s cigar soliton. Then there does not exist any nonconstant positive \( p \)-harmonic function on \((\mathbb{R}^2, \tilde{g})\) for \( p \geq 4 \).

Harmonic functions on the complete gradient Ricci solitons have been studied by Munteanu-Sesum \([17]\) and Munteanu-Wang \([18]\) with applications to the geometry and topology of the solitons; Moser \([16]\) observed an interesting connection between the inverse mean curvature flow formulated as level sets in \( \mathbb{R}^n \) and 1-harmonic functions; Kotschwar-Ni \([15]\) generalize this to Riemannian ambient manifolds.

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2. The Dirichlet Problem at Infinity

In this section, the triple \((M, g, f)\) is assumed to be a complete noncompact shrinking gradient Ricci soliton. We first establish the following Liouville property for positive \( p \)-harmonic functions for \( p > 1 \) without additional curvature assumption.

An \( n \)-dimensional Riemannian manifold \((M, g)\) is gradient Ricci soliton if

\[
\text{Ric} + \nabla \nabla f + \varepsilon g = 0
\]

for some smooth function \( f \) and \( \varepsilon = -\frac{1}{2}, 0, \frac{1}{2} \). Corresponding to the three values of \( \varepsilon \), the gradient Ricci soliton \((M, g, f)\) is shrinking, steady, or expanding \((7), (11)\).

**Proposition 2.1.** Let \((M, g, f)\) be a complete noncompact gradient shrinking Ricci soliton. Then there is no nonconstant positive \( p \)-harmonic function on \((M, e^{-\frac{2f}{n-p}} g)\) for \( p > 1 \).

**Proof.** Since \( u \) is a \( p \)-harmonic function on \((M, \tilde{g})\) where \( \tilde{g} = e^{-\frac{2f}{n-p}} g \), it holds

\[
\text{div}_{\tilde{g}} \left( |\tilde{\nabla} w|^{p-2} \tilde{\nabla} w \right) = |\tilde{\nabla} w|^p
\]

where \( w = -(p-1) \log u \). For any smooth cut-off function \( \phi \in C_0^\infty(M) \), in the complete metric \( g \), we require

\[
\begin{align*}
\phi &= 1, & \text{on } B_{x_0}(\rho, g) \\
\phi &= 0, & \text{on } M \setminus B_{x_0}(2\rho, g) \\
0 &\leq \phi \leq 1, & \text{on } M \\
|\nabla \phi|^2 &\leq C/\rho^2, & \text{on } M.
\end{align*}
\]
Here $B_{x_0}(r, g)$ stands for the geodesic ball centred at $x_0$ with radius $r$ in the metric $g$ in $M$. Multiplying $\phi^2$ to (2.2) then integrating and applying the Stokes’ theorem, we have

$$
\int_M |\tilde{\nabla} w|_g^p \phi^2 \, d\mu_{\tilde{g}} = -2 \int_M \phi |\tilde{\nabla} w|_g^{p-2} \tilde{\nabla} w \tilde{\nabla} \phi \, d\mu_{\tilde{g}}
$$

$$
\leq 2 \left( \int_M \phi^2 |\tilde{\nabla} w|_g^p \, d\mu_{\tilde{g}} \right)^{\frac{p-1}{p}} \left( \int_M \phi^2 |\tilde{\nabla} \phi|_g^p \, d\mu_{\tilde{g}} \right)^{\frac{1}{p}}
$$

by the Cauchy-Schwarz inequality $(p > 1)$. Therefore, we have

$$
\int_M \phi^2 |\tilde{\nabla} w|_g^p \, d\mu_{\tilde{g}} \leq 2^p \int_M \phi^2 |\tilde{\nabla} \phi|_g^p \, d\mu_{\tilde{g}}.
$$

Converting back to the metric $g$, we are led to

$$
(2.3) \quad \int_M \phi^2 |\nabla w|^p e^{-f} \, d\mu_g \leq 2^p \int_M \phi^2 |\nabla \phi|^p e^{-f} \, d\mu_g.
$$

By Theorem 1.1 in [5], the potential function $f$ for a shrinking gradient Ricci soliton satisfies the pointwise estimate

$$
(2.4) \quad \frac{1}{4}(r(x) - c)^2 \leq f(x) \leq \frac{1}{4}(r(x) + c)^2
$$

for $x \in M \setminus B_{x_0}(1, g)$, where $r(x)$ is the distance from $x$ to a fixed point $x_0$ in $M$ and $c$ is a positive constant.

Therefore, by (2.3) and (2.4)

$$
\int_{B(x_0, \rho)} |\nabla w|^p e^{-\frac{1}{4}(r+c)^2} \, d\mu_g \leq \int_M \phi^2 |\nabla w|^p e^{-f} \, d\mu_g
$$

$$
\leq \frac{2^p C e^{-\frac{1}{4}(\rho-c)^2}}{\rho^p} \int_{B_{x_0}(2\rho, g) \setminus B_{x_0}(\rho, g)} \, d\mu_g
$$

$$
\leq \frac{2^p C e^{-\frac{1}{4}(\rho-c)^2}}{\rho^p} \rho^n
$$

where the last inequality follows from the volume growth estimate (Theorem 1.2 in [5]) on shrinking gradient Ricci solitons:

$$
\text{Vol} \left( B_{x_0}(\rho, g) \right) \leq C \rho^n
$$

for sufficiently large $\rho$ and $C$ stands for some uniform constant. Now letting $\rho \to \infty$, we conclude $|\nabla w| \equiv 0$ on $M$, so $u$ is a constant. \qed

Next, we show that $(M, \tilde{g})$ can be turned into a negatively curved manifold under suitable assumptions on $p$ and the sectional curvature $K$ of $(M, g)$:

**Proposition 2.2.** Let $(M, g, f)$ be a simply connected $n$-dimensional complete noncompact shrinking gradient Ricci soliton whose sectional curvature is bounded above by a constant $K_0$ with $0 < K_0 < \frac{1}{2(n-1)}$. Then $(M, e^{-\frac{2f}{n-p}} g)$ is a Cartan-Hadamard manifold for $n < p \leq \frac{1}{K_0} + 2 - n$. 

Proof. When $p > n$, the metric $\tilde{g} = e^{-\frac{2f}{n-p}} g$ is complete since

$$\frac{2f(x)}{n-p} = \frac{2f(x)}{p-n} \geq \frac{(r-c)^2}{2(p-n)}$$

by [5] and that $g$ is complete.

We use the conventions in [7] for curvatures. The Riemann curvature tensor is written as

$$R \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \frac{\partial}{\partial x^k} = R^l_{ijkl} \frac{\partial}{\partial x^l}$$

$$R_{ijkl} = \left\langle R \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right), \frac{\partial}{\partial x^k} \right\rangle$$

and if $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ is orthonormal at $x_0 \in M$, the sectional curvature of the plane $P_{ij}$ spanned by $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}$ at $x_0$ is

$$K(P_{ij}) = R_{ijji}$$

and the Ricci curvature at $x_0$ is

$$R_{jk} = \sum_{i=1}^{n} R^i_{ijk}.$$

Under the conformal change of metric $\tilde{g} = e^{\frac{2f}{n-p}} g$, the sectional curvature at $x_0$ changes as (cf. p.27 in [7]):

$$(2.5) \quad \tilde{K} (P_{ij}) = \frac{\tilde{g}(\tilde{R}_{ijij}, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})}{\tilde{g}_{ii} \tilde{g}_{jj} - \tilde{g}_{ij}^2}$$

$$= e^{\frac{4f}{n-p}} \tilde{R}_{ijji}$$

$$= e^{\frac{4f}{n-p}} \cdot e^{\frac{2f}{p-n}} \left( R_{ijij} - \frac{f_{ii} + f_{jj}}{p-n} - \frac{\nabla f^2 - f_i^2 - f_j^2}{(p-n)^2} \right)$$

$$= e^{\frac{2f}{n-p}} \left( K(P_{ij}) - \frac{f_{ii} + f_{jj}}{p-n} - \frac{\nabla f^2 - f_i^2 - f_j^2}{(p-n)^2} \right).$$

On the gradient shrinking Ricci soliton, we therefore have

$$\tilde{K}(P_{ij}) \leq e^{\frac{2f}{n-p}} \left( K(P_{ij}) + \frac{R_{ii} + R_{jj} - 1}{p-n} \right)$$

by using the defining equation for shrinking gradient Ricci solitons and dropping the last term above that is nonpositive for $i \neq j$. 
From the assumption on $K_0$ and $p > n$, it follows
\[
K(P_{ij}) + \frac{R_{ii} + R_{jj} - 1}{p - n} = K(P_{ij}) + \frac{\sum_{s \neq i} K(P_{is}) + \sum_{s \neq j} K(P_{sj}) - 1}{p - n} \leq \left(1 + \frac{2(n - 1)}{p - n}\right) K_0 - \frac{1}{p - n} (p + n - 2) K_0 - 1.
\]
Therefore the sectional curvature $\tilde{K}$ of $(M, e^{\frac{2t}{p-n}} g)$ is nonpositive since $p + n - 2 \leq \frac{1}{K_0}$. \(\Box\)

**Proof of Theorem 1.1.** Suppose there is a solution $u$ to the Dirichlet problem at infinity and $u = \varphi$ on $M(\infty)$ for some nonconstant function $\varphi \in C^0(M(\infty))$. Then $u$ is continuous on $M \cup M(\infty)$ hence it is bounded. Then $u - \inf_M u + 1$ is a positive solution to the $p$-Laplace equation on $(M, \tilde{g})$, therefore it must be constant from Proposition 2.1. In turn, $u$ is constant on $M$ and $\varphi$ must be constant on $M(\infty)$. The contradiction concludes the proof.

When $\mathbb{R}^n$ is viewed as a shrinking gradient Ricci soliton with $f(x) = \frac{|x|^2}{4}$, we can take $K_0 = 0$ and we have

**Corollary 2.3.** The Dirichlet problem at infinity for the $p$-Laplace equation is unsolvable on $(\mathbb{R}^n, e^{\frac{|x|^2}{4(p-n)}} dx^2)$ for every $p > n$.

**Remark 2.1.** The sectional curvature of $\tilde{g} = e^{\frac{|x|^2}{4(p-n)}} dx^2$ can be computed from (2.5)
\[
\tilde{K}(P_{ij})(x) = - e^{-\frac{|x|^2}{2(p-n)}} \left( \frac{1}{p - n} + \frac{|x|^2 - (x^i)^2 - (x^j)^2}{4(p-n)^2} \right)
\]
where $P_{ij}(x)$ is the plane spanned by $\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \}$ at $x \in \mathbb{R}^n$. The Riemannian distance from $x$ to the origin is
\[
r(x) = \int_0^{\frac{|x|}{\sqrt{4(p-n)}}} e^{\frac{s^2}{4(p-n)}} ds.
\]
If we take $x = (0, ..., 0, x^i, 0, ..., 0)$, then $|x|^2 - (x^i)^2 - (x^j)^2 = 0$ and
\[
\lim_{|x| \to \infty} -\tilde{K}(P_{ij}(x)) r^2(x) = \lim_{|x| \to \infty} \left( \int_0^{\frac{|x|}{\sqrt{4(p-n)}}} e^{\frac{s^2}{4(p-n)}} ds \right)^2 = \frac{1}{p - n} \left( \lim_{|x| \to \infty} \frac{2(p - n)}{|x|} \right)^2 = 0
\]
by l’Hôpital’s rule. This in particular shows that there does not exist constant $\alpha > 1$ which makes true
\[
K(x) \leq -\frac{\alpha(\alpha - 1)}{r^2(x)}
\]
for all sections at $x$ for large $r(x)$. 


3. **A Liouville theorem on $\mathbb{R}^2$ with an incomplete metric with positive curvature**

In this section, we consider the $p$-Laplace equation weighted by a smooth function $f$ on a manifold $(M, g)$, which is equivalent to the $p$-Laplace equation on $(M, e^{-\frac{2f}{n-p}}g)$ and derive a Bochner formula for its solutions. Specialized to the shrinking or steady gradient Ricci solitons, the Bochner formula yields a maximum principle, and this is applied to Hamilton’s cigar soliton.

3.1. **A Bochner type formula for the weighted $p$-Laplace equation.** Let $g$ be a Riemannian metric on an $n$-dimensional manifold $M$, $f$ is a smooth real valued function on $M$. Consider the following equation

$$(3.1) \quad \text{div} \left( |\nabla u|^{p-2} \nabla u \right) - |\nabla u|^{p-2} \langle \nabla f, \nabla u \rangle = 0$$

on $M$. Equation (3.1) has a variational structure, in fact, it is the Euler-Lagrange equation of the following weighted $p$-energy functional

$$E_{p,f}(u) = \int_M |\nabla u|^p e^{-f} d\mu_g.$$ 

We call (3.1) the $f$-weighted $p$-Laplacian equation on $(M, g)$.

**Proposition 3.1.** Under a conformal change $\tilde{g} = e^{-\frac{2f}{n-p}}g$, $u$ is a solution to (3.1) on $(M, g)$ if and only if $u$ is a solution to the $p$-Laplace equation (1.2) on $(M, \tilde{g})$.

**Proof.** We write $\nabla$ for $\nabla_g$ and $\tilde{\nabla}$ for $\nabla_{\tilde{g}}$. For any $\varphi \in C^\infty_0(M)$,

$$\int_M \langle \tilde{\nabla} \varphi, |\nabla u|^{p-2} \tilde{\nabla} u \rangle_{\tilde{g}} d\mu_{\tilde{g}} = \int_M |\tilde{\nabla} u|^{p-2} \langle \tilde{\nabla} \varphi, \tilde{\nabla} u \rangle_{\tilde{g}} d\mu_{\tilde{g}}$$

$$= \int_M \left( e^{\frac{2f}{n-p}} |\nabla u|^{p-2} \right) e^{-\frac{2f}{n-p}} \langle \nabla \varphi, \nabla u \rangle_g e^{-\frac{2f}{n-p}} d\mu_g$$

$$= \int_M \langle \nabla \varphi, |\nabla u|^{p-2} \nabla u \rangle_g e^{-f} d\mu_g.$$ 

This shows that any weak solution to (3.1) on $(M, g)$ is also a weak solution to (1.2) on $(M, \tilde{g})$ and vice versa. $\square$

Suppose $u(x, t)$ is a positive solution of (3.1). Define

$$w = -(p-1) \log u \quad \quad h = |\nabla w|^2.$$ 

We consider the symmetric $n \times n$ matrix

$$A = \text{id} + (p-2) \frac{\nabla w \otimes \nabla w}{h}.$$ 

Note that $A$ is well defined whenever $h > 0$ and is positive definite for $p > 1$. Arising from the linearized operator of the nonlinear $p$-harmonic equations, this matrix was first introduced in [16] and was used in [15] and [25] to study positive $p$-harmonic functions.
For the $f$-weighted $p$-Laplace equation (3.1), the linearized operator is
\[ \mathcal{L}(\psi) = \text{div} \left( h^{\frac{p}{2}-1} A(\nabla \psi) \right) - h^{\frac{p}{2}-1} \langle \nabla f, A(\nabla \psi) \rangle - ph^{\frac{p}{2}-1} \langle \nabla w, \nabla \psi \rangle \]
for smooth functions $\psi$ on $M$ and the following Bochner type formula holds.

**Proposition 3.2.** Let $u$ be a positive smooth solution to (3.1) in an open subset $U$ in $M$ and assume $h > 0$ on $U$. Then
\[ \text{div} \left( h^{\frac{p}{2}-1} A(\nabla u) \right) - h^{\frac{p}{2}-1} \langle \nabla f, A(\nabla u) \rangle - ph^{\frac{p}{2}-1} \langle \nabla w, \nabla u \rangle = \left( \frac{p}{2} - 1 \right) |\nabla h|^2 h^{\frac{p}{2}-2} + 2h^{\frac{p}{2}-1} \left( |\nabla \nabla w|^2 + \text{Ric}(\nabla w, \nabla w) + \nabla \nabla f(\nabla w, \nabla w) \right). \]

**Proof.** Using (3.1), we first observe
\[ \text{div} \left( h^{\frac{p}{2}-1} A(\nabla h) \right) - h^{\frac{p}{2}-1} \langle \nabla f, A(\nabla h) \rangle - ph^{\frac{p}{2}-1} \langle \nabla w, \nabla h \rangle = \left( \frac{p}{2} - 1 \right) |\nabla h|^2 h^{\frac{p}{2}-2} + 2h^{\frac{p}{2}-1} \left( |\nabla \nabla w|^2 + \text{Ric}(\nabla w, \nabla w) + \nabla \nabla f(\nabla w, \nabla w) \right). \]

Then we calculate directly
\[ \left( \frac{p}{2} - 1 \right) |\nabla h|^2 h^{\frac{p}{2}-2} + h^{\frac{p}{2}-1} \Delta h + \left( \frac{p}{2} - 2 \right) (p - 2) h^{\frac{p}{2}-3} \langle \nabla w, \nabla h \rangle^2 + (p - 2) h^{\frac{p}{2}-2} \langle \nabla w, \nabla h \rangle \Delta w + (p - 2) h^{\frac{p}{2}-2} \langle \nabla \langle \nabla w, \nabla h \rangle, \nabla w \rangle. \]

Using the standard Bochner type formula for $h = |\nabla w|^2$:
\[ \Delta h = 2|\nabla \nabla w|^2 + 2\text{Ric}(\nabla w, \nabla w) + 2\langle \nabla \Delta w, \nabla w \rangle \]
we have
\[ \left( \frac{p}{2} - 1 \right) h^{\frac{p}{2}-2} |\nabla h|^2 + 2h^{\frac{p}{2}-1} \left( |\nabla \nabla w|^2 + \text{Ric}(\nabla w, \nabla w) + \langle \nabla \Delta w, \nabla w \rangle \right) + \left( \frac{p}{2} - 2 \right) (p - 2) h^{\frac{p}{2}-3} \langle \nabla w, \nabla h \rangle^2 + (p - 2) h^{\frac{p}{2}-2} \langle \nabla w, \nabla h \rangle \Delta w + (p - 2) h^{\frac{p}{2}-2} \langle \nabla \langle \nabla w, \nabla h \rangle, \nabla w \rangle. \]

Rewrite (3.3) by using $h = |\nabla w|^2$ as
\[ h^{\frac{p}{2}-1} \Delta w + \left( \frac{p}{2} - 1 \right) h^{\frac{p}{2}-2} \langle \nabla h, \nabla w \rangle - h^{\frac{p}{2}} = h^{\frac{p}{2}-1} \langle \nabla f, \nabla w \rangle. \]

Taking the gradient of both sides of (3.5) and then taking the product with $\nabla w$, we are led to
Then \( \nabla f \) is a row vector and the prime denotes its transpose. Moreover, \( (3.9) \) yields the desired result.

**3.2. A maximum principle.** When the triple \( (M,g,f) \) is either shrinking or steady, Proposition 3.2 can be used to prove the following maximum principle.

**Proposition 3.3.** Let \( u \) be a positive smooth solution to \( (3.1) \) in a bounded connected open subset \( U \) in \( M \) with smooth boundary \( \partial U \), \( p > 1 \). Suppose \( (M,g,f) \) is a shrinking or steady gradient Ricci soliton. Then \( \frac{\nabla u}{u} \) attains its maximum on \( \partial U \).

**Proof.** Let \( h = (p - 1)^2 \frac{\nabla u}{u}^2 \). Assume \( \max_{\partial U} h > \max_{\partial U} h \). Then, there exists \( x_0 \in U \) such that \( h(x_0) = \max_{\partial U} h > 0 \). Since \( u \in C^{1,\alpha} \) and \( u > 0 \), \( h \) is continuous. Let

\[
V = \{ x \in U : h(x) = h(x_0) \}.
\]

By the continuity of \( h \), \( V \) is a closed subset of \( U \) and \( V \) does not intersect \( \partial U \). There exists a point \( x_1 \in V \) such that the geodesic ball \( B_{x_1}(r,g) \subset U \) is not contained in \( V \) for any \( 0 < r < r_0 \) for some \( r_0 \), i.e. \( x_1 \) is a boundary point of \( V \). By the continuity of \( h \)

\[
\frac{\nabla u}{u} = \left( \frac{2}{p - 1}\right) \left( \frac{2}{p - 2}\right) h^{\frac{3}{2}} \langle \nabla f, \nabla w \rangle + \left( \frac{2}{p - 1}\right) h^{\frac{1}{2}} \langle \nabla w, \nabla f \rangle \nabla w - \frac{2}{p} h^{\frac{1}{2}} \langle \nabla h, \nabla f \rangle \nabla w.
\]
again, there is geodesic ball $B_{x_1}(r_1, g)$ in $U$ on which $h$ is positive. Observe
\[
\text{RHS of (3.2)} = \frac{p - 2}{2} |\nabla h|^2 h^{\frac{n}{2} - 2} + 2h^{\frac{n}{2} - 1}|\nabla \nabla w|^2 + 2h^{\frac{n}{2} - 1} (\text{Ric} + \nabla \nabla f)(\nabla w, \nabla w)
\geq 2h^{\frac{n}{2} - 1} (\text{Ric} + \nabla \nabla f)(\nabla w, \nabla w)
= \begin{cases} 
2h^{\frac{n}{2} - 1}|\nabla w|^2 \geq 0, & \text{if } (M, g, f) \text{ is a shrinking soliton} \\
0, & \text{if } (M, g, f) \text{ is a steady soliton}
\end{cases}
\]
where for the first inequality, we argue as
\[
4h|\nabla \nabla w|^2 + (p - 2)|\nabla h|^2 \geq 4|\nabla w|^2 |\nabla \nabla w| |\nabla \nabla w| - |\nabla |\nabla w||^2 
= 4|\nabla w|^2 (|\nabla \nabla w|^2 - |\nabla |\nabla w||^2)
\geq 0
\]
by Itô’s inequality and $p \geq 1$. Then it follows that the linear differential operator $\mathcal{L}$ satisfies $\mathcal{L}(h) \geq 0$ on $U$. Next, since $A$ is positive definite and symmetric on $B_{x_1}(r_1, g)$, so is $h^{\frac{n}{2} - 1}A$; therefore, $\mathcal{L}$ is uniformly elliptic on $B_{x_1}(r_1, g)$. By Hopf’s strong maximum principle (cf. Theorem 3.5 in [10]), $h$ must be a constant on $B_{x_1}(r_1, g)$ since it attains its maximum at the interior point $x_1$. However, this contradicts the maximality of $V$ as $B_{x_1}(r_1, g)$ contains points not in $V$.

3.3. Gradient estimates. Let us first recall a gradient estimate in [25]:

**Theorem 3.4. (Wang-Zhang)** Let $(M^n, g)$ be a complete Riemannian manifold with $\text{Ric} \geq -(n - 1)\kappa$ for some positive constant $\kappa$. Assume that $v$ is a positive $p$-harmonic function on the geodesic ball $B_{x_0}(R, g) \subset M$. Then
\[
\frac{|\nabla v|}{v} \leq C(p, n) \left( \frac{1}{R} + \sqrt{\kappa} \right)
\]
on $B_{x_0}(\frac{R}{2}, g)$ for some constant $C(p, n)$.

We prove the following gradient estimate for the $f$-weighted $p$-Laplacian equation.

**Proposition 3.5.** Let $(M^n, g, f)$ be a complete gradient Ricci soliton with
\[
\text{Ric} \geq -(n - 1)\kappa e^{\frac{-2L}{n-p}}g - \frac{2\varepsilon g}{n-p} - \frac{S g}{n-p} - (df \otimes df - |\nabla f|^2 g) \frac{n - 2}{(n - p)^2}
\]
where $S$ is the scalar curvature of $(M, g)$. Assume that $u$ is a positive solution of equation (3.1). Then there exists a constant $C(p, n)$ such that
\[
\frac{|\nabla u(x)|}{u(x)} \leq C(p, n) \left( \frac{1}{R} + \sqrt{\kappa} \right) e^{-\frac{f(x)}{n-p}}
\]
for $x \in B_{x_0}(\frac{R}{2}, e^{\frac{-2L}{n-p}} g)$. 
Proof. For a smooth function $f$, let $\nabla f$ be the gradient, $\Delta f$ the Laplacian and $\nabla \nabla f$ the Hessian with respect to $g$. For the conformal change of metrics $\tilde{g} = e^{-\frac{2f}{n-p}}g$, the Ricci tensors of $\tilde{g}$ and $g$ are related by (see [3], page 59):

$$\tilde{\text{Ric}} = \text{Ric} - (n-2) \left( -\frac{\nabla \nabla f}{n-p} - \frac{df \otimes df}{(n-p)^2} \right) + \left( -\frac{\Delta f}{n-p} - \frac{n-2}{(n-p)^2} |\nabla f|^2 \right) g.$$

From the gradient Ricci soliton equation (2.1), the scalar curvature $S$ of $M$ satisfying the following two equations (see [4]):

$$S + \Delta f - n\varepsilon = 0,$$
$$S + |\nabla f|^2 + \varepsilon f = 0.$$

Putting (2.1) and (3.12) into (3.11), we have

$$\tilde{\text{Ric}} = \text{Ric} + (n-2) \left( -\frac{Ric - \varepsilon g}{n-p} + \frac{df \otimes df}{(n-p)^2} \right) + \left( \frac{S + n\varepsilon}{n-p} - \frac{n-2}{(n-p)^2} |\nabla f|^2 \right) g$$

$$= \frac{2 - p}{n-p} Ric + \frac{2\varepsilon g}{n-p} + \frac{S g}{n-p} + (df \otimes df - |\nabla f|^2 g) \frac{n-2}{(n-p)^2} g.$$n

Therefore, the curvature assumption in Theorem 3.5 implies

$$\tilde{\text{Ric}} \geq -(n-1)\kappa.$$

By Proposition 3.1, we know that $u$ is also a positive solution to (1.2) for the metric $\tilde{g}$, hence by Theorem 3.4 we have

$$\frac{|\nabla u|_{\tilde{g}}}{u} \leq C(p, n) \left( \frac{1}{R} + \sqrt{\kappa} \right)$$

on $B_{x_0}(\frac{R}{2}, \tilde{g})$. This is equivalent to

$$\frac{|\nabla u(x)|}{u(x)} \leq C(p, n) \left( \frac{1}{R} + \sqrt{\kappa} \right) e^{-\frac{f(x)}{n-p}}$$

for $x \in B_{x_0}(\frac{R}{2}, \tilde{g})$. \qed

3.4. A Liouville type theorem for $p$-Laplace equation in dimension 2. For a steady gradient Ricci soliton, the condition (3.10) on the Ricci curvature in Proposition 3.5 cannot hold globally when $n \geq 3$ because it would imply, by taking trace, that the scalar curvature is bounded below by a positive constant but this is impossible. However, the condition (3.10) is satisfied when $n = 2$ for $p \geq 4$ or $1 < p < 2$ because

$$\text{Ric} = \frac{1}{2} S g \geq \frac{1}{p-2} S g$$

since $S \geq 0$ for any steady gradient Ricci soliton [6] and $\kappa = 0$.

Note that Hamilton’s cigar soliton is the unique 2-dimensional complete noncompact steady gradient Ricci soliton. The cigar soliton is $\mathbb{R}^2$ equipped with the complete metric (cf. [7]):

$$g = \frac{dx^2 + dy^2}{1 + x^2 + y^2}$$
and the potential function
\[ f(x, y) = -\log(1 + x^2 + y^2). \]

The conformally altered metric is
\[ \tilde{g} = e^{-2 \log(1 + x^2 + y^2)} g = (1 + x^2 + y^2)^p (dx^2 + dy^2). \]

In particular, \( \tilde{g} \) is complete if \( 1 < p < 2 \) and incomplete if \( p > 2 \). However, to use the gradient estimate in proving Liouville type result, we will need \( p \geq 4 \). It is straightforward to compute the Gauss curvature of \( \tilde{g} \):
\[
\tilde{K} = -\frac{1}{2} (1 + r^2) \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \partial_r \right) \log(1 + r^2)^{-\frac{p}{p-2}}
= \frac{2p}{p-2} (1 + r^2)^{-\frac{p}{p-2}}
= \frac{2p}{p-2} (1 + r^2)^{-\frac{4}{p-2}}
\]
which is positive and tends to 0 as \( r \to \infty \) if \( p > 4 \). When \( p = 4 \), the incomplete metric \((1 + x^2 + y^2)^{-2} (dx^2 + dy^2)\) has constant curvature \( \tilde{K} = 4 \).

**Theorem 3.6.** Let \((\mathbb{R}^2, g, f)\) be Hamilton’s cigar soliton. Then there does not exist any nonconstant positive \( p \)-harmonic function on \((\mathbb{R}^2, \tilde{g})\) for \( p \geq 4 \).

**Proof.** Let \( u \) be a positive solution to (3.1). For any point \( x_0 \in M \), the maximum principle (Corollary 3.3) asserts
\[
\frac{\nabla u(x_0)}{u(x_0)} \leq \max_{x \in \partial B_0(R, g)} \frac{\nabla u(x)}{u(x)} = \frac{\nabla u(x_R)}{u(x_R)}
\]
for some \( x_R \in \partial B_0(R, g) \) where \( x_0 \in B_0(R, g) \) and \( r(x_0, 0) < R \). From the discussion above, when \( n = 2 \) and \( p \geq 4 \), the Ricci curvature condition (3.10) in Proposition 3.5 is satisfied. The diameter of \((\mathbb{R}^2, \tilde{g})\) is
\[
2R_0 = 2 \int_0^\infty \frac{dr}{(1 + r^2)^{\frac{p}{2}(p-2)}} < \infty.
\]
It is clear that \( r(x_R, 0) \to \infty \) if and only if \( r(x_R, 0) \to R_0 \), where \( \tilde{r} \) denotes the distance function for the metric \( \tilde{g} \). Let
\[
r_R = \int_0^\infty \frac{dr}{(1 + r^2)^{\frac{p}{2}(p-2)}}.
\]
It follows from Proposition 3.5, applied on the ball $B_{x_R}(r, \tilde{g})$, that

$$\left| \nabla \frac{u(x_R)}{u(x_R)} \right| \leq C(n, p) \left( \frac{r_x}{2} \right)^{-1} e^{-\frac{r}{p-2} \log(1+|x_R|^2)}$$

$$= 2C(n, p) \left( \int_{R}^{\infty} \frac{dr}{1 + r^2} (1 + R^2)^{\frac{p-2}{2}} \right)^{-1}$$

$$\leq 2C(n, p) \left( (1 + R^2)^{\frac{p-2}{2}} \int_{R}^{\infty} \frac{dr}{r^p} \right)^{-1}$$

$$= 2C(n, p) \left( \frac{p-2}{2} (1 + R^2)^{\frac{2}{p-2} R^{-\frac{2}{p-2}}} \right)^{-1}.$$

Since $p > 2$, letting $R \to 0$ we conclude $|\nabla u(x_0)| = 0$, hence $u$ is constant as $x_0$ is arbitrary. \qed

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