Poisson and symplectic functions in Lie algebroid theory

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For Murray Gerstenhaber and Jim Stasheff, in admiration and respect

Summary. Emphasizing the role of Gerstenhaber algebras and of higher derived brackets in the theory of Lie algebroids, we show that the several Lie algebroid brackets which have been introduced in the recent literature can all be defined in terms of Poisson and pre-symplectic functions in the sense of Roytenberg and Terashima. We prove that in this very general framework there exists a one-to-one correspondence between non-degenerate Poisson functions and symplectic functions. We also determine the differential associated to a Lie algebroid structure obtained by twisting a structure with background by both a Lie bialgebra action and a Poisson bivector.

Introduction

Towards 1958, Ehresmann [15] introduced the idea of differentiable categories, of which the differentiable groupoids, now called Lie groupoids, are an example, and he developed this theory further in the 1960’s [16]. At the end of the decade, Pradines introduced the corresponding infinitesimal objects which he called Lie algebroids [46]. The theory of Lie algebroids, which has since been developed by many authors, and in particular by Mackenzie [41] [43], encompasses both differential geometry – because the tangent bundle of a smooth manifold is the prototypical Lie algebroid –, and Lie algebra theory – because the Lie algebras are Lie algebroids whose base manifold is a singleton –, while other examples of Lie algebroids occur in the theory of foliations (see, e.g., [45]) and in Poisson geometry [10] [43]. The corresponding, purely algebraic concept, called pseudo-Lie algebras [42] or Lie-Rinehart algebras [20], among many other names, dates back to Jacobson [22], as has been observed in [30].

While the structure of what is now called a Gerstenhaber algebra appeared in the work of Murray Gerstenhaber on the Hochschild cohomology of associative algebras [17], it became clear in the work of Koszul [34] and of many
other authors \[31, 58\] that Gerstenhaber algebras play an essential role in the theory of Lie algebroids. Whenever a vector bundle has the structure of a Lie algebroid, the linear space of sections of its exterior algebra bundle is a Gerstenhaber algebra, the prototypical example of which is the linear space of fields of multivectors equipped with the Schouten-Nijenhuis bracket on any smooth manifold. The close relationship between Poisson geometry and Lie algebroid theory appears clearly in the concept of a Lie bialgebroid defined by Mackenzie and Xu \[14\] as the infinitesimal object of a Poisson groupoid, and characterized in terms of derivations in \[22\]. For any Poisson manifold \(M\) with tangent bundle \(TM\), the pair \((TM, T^*M)\) is a Lie bialgebroid, while the Lie bialgebroids over a point are Drinfeld’s Lie bialgebras of Poisson-Lie group theory \[13\].

When passing from the case of Lie bialgebras to that of the Lie-quasi bialgebras \[14\], or their dual version, the quasi-Lie bialgebras, or the more general case of proto-bialgebras \[24\] \(\text{higher structures}\), in the sense of Jim Stasheff \[52\], appear. The associated algebra is not a Gerstenhaber algebra but only a Gerstenhaber algebra up to homotopy, but with all \(n\)-ary brackets beyond the third vanishing (see \[21\] \[4\] \[5\]). The analogous theory generalizing Lie algebroids was developed by Roytenberg \[47\] and, more recently, by Terashima \[55\]. Their articles form the basis of the present exposition.

The concept of twisting for proto-bialgebroids was defined by Roytenberg \[47\] as a generalization of the twisting of proto-bialgebras introduced in \[24\], itself a generalization of the twisting of Lie bialgebras defined by Drinfeld in the theory of the semi-classical limit of the quasi-Hopf algebras \[14\], while the concept of Poisson function, which was already implicit in \[47\], has now been formally introduced by Terashima in \[55\], with interesting applications which we review and develop here. Poisson functions generalize both Poisson structures on manifolds and triangular \(r\)-matrices on Lie algebras, and, more generally, Poisson structures on Lie algebroids as well as their twisted versions (see \[37\] \[47\] \[55\]).

The cohomological approach to Lie algebroid theory arose from the viewpoint developed for Lie bialgebras by Lecomte and Roger \[35\], itself based on

\[1\] In \[24\] \[6\], Lie-quasi bialgebras were called Jacobian quasi-bialgebras, and quasi-Lie bialgebras were called co-Jacobian quasi-bialgebras. We also point out that, in the translation of Drinfeld’s original paper \[14\], the term “quasi-Lie bialgebra” is used for what we call Lie-quasi bialgebra. Proto-bialgebras were introduced in \[24\] where they were called proto-Lie-bialgebras, to distinguish them from the associative version of this notion.

\[2\] There are some changes in the notations. In particular the notations \(\phi\) and \(\psi\) used by Roytenberg in \[47\] are exchanged in order to return to the conventions of \[24\] \[6\] \[28\].
the even Poisson bracket introduced by Kostant and Sternberg in \[33\]. In \[24\], we extended this approach to the Lie-quasi bialgebras defined by Drinfeld \[14\], and we introduced the dual objects and the more general notion of proto-Lie bialgebra, encompassing both the Lie-quasi bialgebras and their duals. In \[47\] Roytenberg extended the cohomological approach to Lie bialgebra theory to the “oid” case by combining the supermanifold approach due to Vaintrob \[56\] and T. Voronov (see \[57\] citing earlier publications) with the results of \[24\].

The preprint that Terashima communicated to me in 2006 \[55\] goes further along the same lines and provides a beautiful unification of results in both recent \[7\], \[9\] and not so recent papers \[39\], showing that they are special cases of a general construction of Lie algebroid structures obtained by twisting certain basic structures.

The main features of this paper are the following. Our first Section deals with the general definition of a structure on a vector bundle, \( V \). The basic tool for the study of the properties of “structures” is the big bracket, denoted by \( \{ \cdot, \cdot \} \), the bigraded even Poisson bracket which is the canonical Poisson bracket on the cotangent bundle of the supermanifold \( \Pi V \), i.e., \( V \) with reversed parity on the fibers, which, on vector-valued forms or 1-form-valued multivectors, coincides with the Nijenhuis-Richardson bracket up to sign. The “structures” are cubic functions on this cotangent bundle whose Poisson square vanishes. Vector bundles equipped with a “structure” generalize the Lie, Lie-quasi and quasi-Lie bialgebroids, in particular the Lie bialgebras.

In Section 2 we introduce the dual notions of twisting by a bivector and twisting by a 2-form, and we define the Poisson functions and the presymplectic functions with respect to a given structure. Such bivectors (resp., 2-forms) give rise by twisting to quasi-Lie (resp., Lie-quasi) bialgebroids. We show that the twist of Lie-quasi bialgebras in the sense of Drinfeld \[14\] and the twisted Poisson structures on manifolds, introduced by Klímačik and Strobl in \[23\] (under the name WZW-Poisson structures) and studied by Ševera and Weinstein in \[50\] (where they are called Poisson structures with background), are both particular cases of the general notion of a twisted structure.

In Section 3 we prove that the graphs of Poisson functions and of presymplectic functions are Dirac sub-bundles of the Courant algebroid \( V \oplus V^* \), which is the “double” of \( V \).

The aim of Section 4 is to prove Theorem 4.2 which states that non-degenerate Poisson functions are in one-to-one correspondence with symplectic functions, a generalization of the well-known fact that a non-degenerate bivector on a manifold defines a Poisson structure if and only if its inverse is a closed 2-form. We believe that this theorem had not yet been proved in so general a form.

\[^3\]Even Poisson brackets had already appeared in the context of the quantization of systems with constraints in the work of Batalin, Fradkin and Vilkovisky. See \[51\] and references therein.
In Section 5 we study the case where a Poisson function involves both a Poisson structure on a manifold $M$ in the ordinary sense and a Lie algebra action on this manifold. In the general case, with non trivial Lie-quasi bialgebra actions and background 3-forms on the manifold, we determine explicit expressions for the bracket and the differential thus defined. In fact, the twisting of a structure on a vector bundle $V$ by a Poisson function gives rise to a Lie algebroid structure on the dual vector bundle $V^*$ and, dually, to a differential on the sections of $\wedge^\bullet V$, the exterior algebra bundle of $V$. In particular cases, we recover the brackets on vector bundles of the form $T^*M \times g$ which were associated to Poisson actions of Poisson-Lie groups on Poisson manifolds by Lu in [39] and, more generally, to quasi-Poisson $G$-manifolds in the sense of [2] by Bursztyn and Crainic in [7], and to quasi-Poisson $G$-spaces in the sense of [1] by Bursztyn, Crainic and ˇSevera in [9]. This approach gives an immediate proof that these brackets satisfy the Jacobi identity and are indeed Lie algebroid brackets. The formulas for the differential in the general case are, to the best of our knowledge, new.

1 Definition of structures

1.1 Towards a unification

It was already clear in the theory of Lie bialgebras that the “big bracket” was the appropriate tool for their study. Roytenberg extended the definition and the use of the big bracket to the case of Lie algebroids [47], and Terashima’s article [55] proves additional results, by suitably twisting certain basic structures.

1.2 The big bracket

Consider the bigraded supermanifold $X = T^*\Pi V$, where $V$ is a vector bundle over a manifold $M$, and where $\Pi$ denotes the change of parity of the fibers. Then $X$ is canonically equipped with an even Poisson bracket [33], the Poisson structure on $X$ actually being symplectic. This Poisson bracket, called the big bracket, is here denoted by $\{ \ , \}$. The algebra $F$ of smooth functions on $X$ is bigraded in the following way. If $(x^i, \xi^a)$ are local coordinates on $\Pi V$ ($i = 1, \ldots, \dim M, \ a = 1, \ldots, \text{rank} V$), we denote by $(x^i, \xi^a, p_i, \theta_a)$ the corresponding local coordinates on $T^*\Pi V$, and we assign them the bidegrees $(0, 0), (0, 1), (1, 1)$ and $(1, 0)$, respectively. An element of $F$ of bidegree $(k, \ell)$, with $k \geq 0$ and $\ell \geq 0$, is said to be of shifted bidegree $(p, q)$ when $p = k - 1$ and $q = \ell - 1$ ($p \geq -1$ and $q \geq -1$), whence the table

| $x^i$ | $\xi^a$ | $p_i$ | $\theta_a$ | bidegree |
|-------|---------|-------|-----------|----------|
| $(0, 0)$ | $(0, 1)$ | $(1, 1)$ | $(1, 0)$ | $(-1, -1)$ | $(-1, 0)$ | $(0, 0)$ | $(0, -1)$ | shifted bidegree |
The total degree (resp., total shifted degree) will be called, for short, the degree (resp., shifted degree). The big bracket is of shifted bidegree \((0, 0)\), and it satisfies
\[
\{x^i, p_j\} = \delta^i_j = -\{p_j, x^i\}, \quad \{\xi^a, \theta_b\} = \delta^a_b = \{\theta_b, \xi^a\}.
\]

### 1.3 Definition of structures

As in [56] [47] [57] (also see [28]) we consider functions on \(X\) that define bialgebroid structures or generalizations thereof on \((V, V^*)\). See [24] [6] [47] for proofs of the statements in this section.

**Definition 1.1.** A structure on \(V\) is a homological function on \(X\) of degree 3, i.e., an element \(S \in F\) of shifted degree 1 such that \(\{S, S\} = 0\).

Let
\[
S = \phi + \gamma + \mu + \psi
\]
(1)
in the notations of [24] and [6]. Then,
\* \(\phi\), of shifted bidegree \((2, -1)\), is a 3-form on \(V^*\),
\[
\phi = \frac{1}{6} \phi^{abc} \theta_a \theta_b \theta_c,
\]
\* \(\gamma\), of shifted bidegree \((1, 0)\), defines an anchor, \(a^*: V^* \to TM\), and a bracket on \(V^*\),
\[
\gamma = (a^*)^b p_i \theta_b + \frac{1}{2} \gamma^b \theta_b \theta_c \xi^a,
\]
\* \(\mu\), of shifted bidegree \((0, 1)\), defines an anchor, \(a_*: V \to TM\), and a bracket on \(V\),
\[
\mu = (a_*)_b \xi^b + \frac{1}{2} \mu^b \theta_a \xi^b \xi^c,
\]
\* \(\psi\), of shifted bidegree \((-1, 2)\), is a 3-form on \(V\),
\[
\psi = \frac{1}{6} \psi_{abc} \xi^a \xi^b \xi^c.
\]

Then \(S\) is a structure if and only if
\[
\begin{align*}
\frac{1}{2} \{\mu, \mu\} + \{\gamma, \psi\} &= 0, \\
\{\mu, \gamma\} + \{\phi, \psi\} &= 0, \\
\frac{1}{2} \{\gamma, \gamma\} + \{\mu, \phi\} &= 0, \\
\{\mu, \psi\} &= 0, \\
\{\gamma, \phi\} &= 0.
\end{align*}
\]

By definition, when \(S\) is a structure on \(V\), the pair \((V, V^*)\) is a proto-bialgebroid. The anchor and bracket of \(V\) and of \(V^*\) are the following derived brackets [26] [27] [28] [47] [57]:
anchor of $V$, $a_*(X) \cdot f = \{\{X, \mu\}, f\}$,

bracket of $V$, $\mu(X, Y) = \{\{X, \mu\}, Y\}$,

anchor of $V^*$, $a^*(\alpha) \cdot f = \{\{\alpha, \gamma\}, f\}$,

bracket of $V^*$, $\gamma(\alpha, \beta) = \{\{\alpha, \gamma\}, \beta\}$,

for $f \in C^\infty(M)$, $X$ and $Y \in \Gamma(V)$, $\alpha$ and $\beta \in \Gamma(V^*)$. The quasi-Gerstenhaber brackets on $\Gamma(\bigwedge^\bullet V)$, where $\bigwedge^\bullet V$ is the exterior algebra of $V$, and on $\Gamma(\bigwedge^\bullet V^*)$, are expressed by the same formulas. They are denoted by $[,]_\mu$ and $[,]_\gamma$, respectively.

The Lie-quasi bialgebroids, quasi-Lie bialgebroids and Lie bialgebroids are defined as follows:

• $(V, V^*)$ is a Lie-quasi bialgebroid if and only if $S = \phi + \gamma + \mu$, i.e., if $\psi = 0$. Then $V$ is a Lie algebroid, $\Gamma(\bigwedge^\bullet V)$ is a Gerstenhaber algebra, while $\Gamma(\bigwedge^\bullet V^*)$ is a quasi-Gerstenhaber algebra.

• $(V, V^*)$ is a quasi-Lie bialgebroid if and only if $S = \gamma + \mu + \psi$, i.e., if $\phi = 0$. Then $V^*$ is a Lie algebroid, $\Gamma(\bigwedge^\bullet V^*)$ is a Gerstenhaber algebra, while $\Gamma(\bigwedge^\bullet V)$ is a quasi-Gerstenhaber algebra.

• $(V, V^*)$ is a Lie bialgebroid if and only if $S = \gamma + \mu$, i.e., if $\phi = \psi = 0$. Then both $V$ and $V^*$ are Lie algebroids, and both $\Gamma(\bigwedge^\bullet V)$ and $\Gamma(\bigwedge^\bullet V^*)$ are Gerstenhaber algebras.

The quasi-Gerstenhaber algebras (see [17] [21] [4] [5]) are the simplest higher structures beyond the Gerstenhaber algebras themselves; they correspond to the case where all $n$-ary brackets, $\ell_n$, vanish for $n \geq 4$.

On the Poisson manifold $T^{*}IV$, we can consider the Hamiltonian vector field with Hamiltonian $S \in F$, which we denote by $d_S = \{S, \cdot\}$. Because $\{S, S\} = 0$, $d_S$ is a differential on the space of smooth functions on $T^{*}IV$, i.e., a derivation of $F$ of degree 1 and of square zero.

**Example 1.** When $V = TM$ and $S = \mu = p_i \xi^i$, then $\mu(X, Y)$ is the Lie bracket of vector fields $X$ and $Y$, the corresponding Gerstenhaber bracket on $\Gamma(\bigwedge^\bullet TM)$ is the Schouten–Nijenhuis bracket of multivector fields, and the restriction of $d_S = d_\mu$ to the differential forms on $M$ is the de Rham differential.

**Example 2.** When $M$ is a point, then $V = g$ is a vector space and a structure $S = \mu + \gamma$ on $V$ is a Lie bialgebra structure on $(g, g^*)$, also denoted by $S_g + S_{g'}$, in Section 5, while $d_S = d_\mu + d_\gamma$ is the Chevalley–Eilenberg cohomology operator of the double of the Lie bialgebra. More generally, on $V = g$, a structure $S = \mu + \gamma + \phi$, where $\phi \in \bigwedge^3 V$, is a Lie-quasi bialgebra structure on $(g, g^*)$. 
2 Twisting

We consider a structure $S$ on the vector bundle $V$ that defines a proto-bialgebroid structure on $(V, V^*)$, and we shall now study the twisting, $e^{-\sigma}S$, of $S$ by a function $\sigma$ of shifted bidegree $(1, -1)$ or $(-1, 1)$.

2.1 Twisting by Poisson or pre-symplectic functions

Let $\sigma \in \mathcal{F}$ be a function of shifted bidegree $(1, -1)$ or $(-1, 1)$. Since the right adjoint action, $\text{ad}_{\sigma} = \{., \sigma\}$, of an element $\sigma$ of shifted degree 0 is a derivation of degree 0 of $(\mathcal{F}, \{., \})$, and since, for any $a \in \mathcal{F}$, the series $a + \{a, \sigma\} + \frac{1}{2}\{\{a, \sigma\}, \sigma\} + \frac{1}{6}\{\{\{a, \sigma\}, \sigma\}, \sigma\} + \ldots$ terminates for reasons of bidegrees, the exponential of $\text{ad}_{\sigma}$ is well-defined and is an automorphism of $(\mathcal{F}, \{., \})$, which, in an abuse of notation, we shall denote by $e^\sigma$. It follows that, for any structure $S$, and for any $\sigma$ of shifted degree 0, $\{e^\sigma S, e^\sigma S\} = e^\sigma\{S, S\} = 0$, and therefore $e^{-\sigma}S$ is also a structure.

**Definition 2.1.** When $\sigma$ is a function of shifted bidegree $(1, -1)$ or $(-1, 1)$, the structure $e^{-\sigma}S$ is called the twisting of $S$ by $\sigma$.

A function of shifted bidegree $(1, -1)$ is a bivector $\sigma$ on $V$, expressed in local coordinates as 
$$\sigma = \frac{1}{2}\sigma^{ab}\theta_a\theta_b,$$
while a function of shifted bidegree $(-1, 1)$ is a 2-form $\tau$ on $V$, expressed in local coordinates as
$$\tau = \frac{1}{2}\tau_{ab}\xi^a\xi^b.$$

We list the explicit formulas \[47\] for the homogeneous components of twisted structures.

- For $\sigma$ of shifted bidegree $(1, -1)$, let $e^{-\sigma}S = \phi_\sigma + \gamma_\sigma + \mu_\sigma + \psi_\sigma$ be the decomposition (1) of $e^{-\sigma}S$ as a sum of terms of homogeneous bidegrees. Then,
$$\begin{cases}
\phi_\sigma = \phi - \{\gamma, \sigma\} + \frac{1}{2}\{\mu, \sigma\}, \\
\gamma_\sigma = \gamma - \{\phi, \sigma\} + \frac{1}{2}\{\psi, \sigma\}, \\
\mu_\sigma = \mu - \{\gamma, \sigma\}, \\
\psi_\sigma = \psi .
\end{cases}$$
(2)

- For $\tau$ of shifted bidegree $(-1, 1)$, let $e^{-\tau}S = \phi_\tau + \gamma_\tau + \mu_\tau + \psi_\tau$ be the decomposition (1) of $e^{-\tau}S$ as a sum of terms of homogeneous bidegrees. Then,
$$\begin{cases}
\phi_\tau = \phi , \\
\gamma_\tau = \gamma - \{\phi, \tau\} , \\
\mu_\tau = \mu - \{\gamma, \tau\} + \frac{1}{2}\{\phi, \tau\}, \\
\psi_\tau = \psi - \{\phi, \tau\} + \frac{1}{2}\{\gamma, \tau\} - \frac{1}{3}\{\{\phi, \tau\}, \tau\} .
\end{cases}$$
(3)
**Definition 2.2.** Let $S$ be a structure on $V$.

(i) A function $\sigma$ of shifted bidegree $(1,-1)$ such that $\phi_\sigma = 0$ is called a Poisson function with respect to $S$.

(ii) A function $\tau$ of shifted bidegree $(-1,1)$ such that $\psi_\tau = 0$ is called a pre-symplectic function with respect to $S$.

In view of these definitions, we immediately obtain

**Proposition 2.3.** Let $S$ be a structure on $V$ and let $\sigma$ (resp., $\tau$) be a function of shifted bidegree $(1,-1)$ (resp., $(-1,1)$).

(i) If $\sigma$ is a Poisson function, the twisted structure $e^{-\sigma}S$ is a quasi-Lie bialgebroid structure.

(ii) If $\tau$ is a pre-symplectic function, the twisted structure $e^{-\tau}S$ is a Lie-quasi bialgebroid structure.

### 2.2 Twisting by Poisson functions

It follows from the formula for $\phi_\sigma$ in (2) that a section $\sigma$ of $\bigwedge^2 V$ is a Poisson function with respect to a structure $S = \phi + \gamma + \mu + \psi$ if and only if

$$\phi - \{\gamma, \sigma\} + \frac{1}{2}\{\{\mu, \sigma\},\sigma\} - \frac{1}{6}\{\{\psi, \sigma\},\sigma\} = 0. \quad (4)$$

Equation (4) is called a generalized twisted Maurer-Cartan equation, or simply a Maurer-Cartan equation.

For any bivector $\sigma$, we set $\sigma^\# = i_\alpha \sigma$, for $\alpha \in \Gamma(V^*)$, where $i$ denotes the interior product. Whenever $\sigma$ is a Poisson function with respect to $S = \phi + \gamma + \mu + \psi$, the term of shifted bidegree $(1,0)$ in $e^{-\sigma}S$,

$$\gamma_\sigma = \gamma - \{\mu, \sigma\} + \frac{1}{2}\{\{\psi, \sigma\},\sigma\},$$

defines an anchor $a^\# + a_\# \circ \sigma^\#$ and a Lie bracket on $\Gamma(V^*)$, as well as a Gerstenhaber bracket on $\Gamma(\bigwedge V^*)$, which we denote by $[, ]_{\gamma_\sigma}$, and a differential $d_{\gamma_\sigma} = \{\gamma_\sigma, \cdot\}$ on $\Gamma(\bigwedge V^*)$. There is also a bracket, $[, ]_{\mu_\sigma}$, on $\Gamma(\bigwedge V^*)$ defined by the term of shifted bidegree $(0,1)$, $\mu_\sigma = \mu - \{\psi, \sigma\}$, and a derivation of degree 1, $d_{\mu_\sigma} = \{\mu_\sigma, \cdot\}$, of $\Gamma(\bigwedge V^*)$. Then $\frac{1}{2}\{\mu_\sigma, \mu_\sigma\} + \{\gamma_\sigma, \psi\} = 0$, so that $\psi$ measures the defect in the Jacobi identity for $[, ]_{\mu_\sigma}$, and $(d_{\mu_\sigma})^2 \psi = [\psi,]_{\gamma_\sigma}$.

It appears that the twisting of Lie bialgebras in the sense of Drinfeld [14], as well as its generalizations to proto-bialgebras [24] and to proto-bialgebroids [27], and the twisting of Poisson structures in the sense of Severa and Weinstein [50], and its generalizations to structures on Lie algebroids [47], [29], all fit into this general framework, although the meaning of the word “twisting” is not quite the same in both instances. In the first instance, one twists a given structure, in the sense of Definition [14] on a Lie algebra $\mathfrak{g}$ by an element $\sigma \in \bigwedge^2 \mathfrak{g}$ (often denoted by $t$ or $f$), called the “twist” [14] [1]. For any twist, a Lie-quasi bialgebra is twisted into a Lie-quasi bialgebra. In the
second case, it would be more appropriate to speak of “Poisson structures with background”: the given structure on the vector bundle \( V \) is of the form \( \mu + \psi \), where \( \psi \) is a \( d_\mu \)-closed 3-form, and equation (4) which reduces to the twisted Poisson condition (6) below is the condition for \( \sigma \in \Gamma(\wedge^2 V) \) to twist \( \mu + \psi \) into a quasi-Lie bialgebroid structure.

(i) **Twist in the sense of Drinfeld.** In the case of a twist of a Lie-quasi bialgebra, one twists a structure \( S = \phi + \gamma + \mu + 0 \) on a Lie algebra \( g \) by an arbitrary \( \sigma \in \wedge^2 g \) into

\[
e^{-\sigma} S = \left( \phi - \{ \gamma, \sigma \} + \frac{1}{2} \{ \mu, \sigma \}, \sigma \right) + (\gamma - \{ \mu, \sigma \}) + \mu + 0,
\]

and one obtains a “twisted Lie-quasi bialgebra”. The resulting object is a Lie bialgebra, with \( \mu_\sigma = \mu \) and \( \gamma_\sigma = \gamma - \{ \mu, \sigma \} \), if and only if \( \sigma \) is a Poisson function, i.e., satisfies the condition

\[
\frac{1}{2} [\sigma, \sigma]_\mu + d_\gamma \sigma - \phi = 0.
\]

If one twists a Lie bialgebra (\( \psi = \phi = 0 \)), this condition reduces to the usual **Maurer-Cartan equation**,

\[
\frac{1}{2} [\sigma, \sigma]_\mu + d_\gamma \sigma = 0.
\]  

(5)

If one twists a trivial Lie bialgebra (\( \psi = \phi = \gamma = 0 \)), the Maurer-Cartan equation reduces to \( [\sigma, \sigma]_\mu = 0 \), i.e., to the classical Yang-Baxter equation. In fact, for \( \sigma = r \in \wedge^2 g \),

\[
-\frac{1}{2} [r, r]_g = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13} + r_{23}],
\]

and the **classical Yang-Baxter equation** (CYBE) on a Lie algebra \( g \) is the condition \( [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13} + r_{23}] = 0 \), for \( r \in \wedge^2 g \).

When \( S = \mu \), the necessary and sufficient condition for \( \mu + \gamma_\sigma \) to be a Lie bialgebra structure on \( (g, g^*) \) is \( \{ \mu, \{ \mu, \mu \}, \sigma \} = 0 \), the **generalized classical Yang-Baxter equation**, which states that \( [\sigma, \sigma]_\mu \) is \( \text{ad}^\mu \)-invariant.

In the same way, a Lie-quasi bialgebroid can be twisted by a bivector, and a Lie bialgebroid is twisted into a Lie bialgebroid if and only if the bivector satisfies the Maurer-Cartan equation (5) (see [37] [47] [28]).

(ii) **Twisted Poisson structures.** If \( S \) is a structure on a vector bundle \( V \) such that \( \gamma = 0 \) and \( \phi = 0 \), then \( \{ \mu, \mu \} = 0 \), i.e., \( V \) is a Lie algebroid, and \( \psi \) is a \( d_\mu \)-closed section of \( \wedge^3 V^* \). In this case, one twists \( S = 0 + 0 + \mu + \psi \) into

\[
e^{-\sigma} S = \left( \frac{1}{2} \{ \mu, \sigma \}, \sigma \right) - \frac{1}{6} \{ \{ \psi, \sigma \}, \sigma \}, \sigma \right) + \left( -\{ \mu, \sigma \} + \frac{1}{2} \{ \{ \psi, \sigma \}, \sigma \} \right) + (\mu - \{ \psi, \sigma \}) + \psi.
\]
Thus, \( \sigma \) is a Poisson function if and only if

\[
\{ \{ \mu, \sigma \}, \sigma \} - \frac{1}{3} \{ \{ \psi, \sigma \}, \sigma \} = 0,
\]

which is the condition

\[
\frac{1}{2} [\sigma, \sigma]_\mu = (\wedge^3 \sigma^2) \psi,
\]

i.e., \((\sigma, \psi)\) is a twisted Poisson structure on the Lie algebroid \( V \). When \( \sigma \) satisfies the twisted Poisson condition (6), the resulting object is a quasi-Lie bialgebroid. In particular, \(-\{ \mu, \sigma \} + \frac{1}{2} \{ \{ \psi, \sigma \}, \sigma \}\) is a Lie algebroid bracket on \( V^* \).

If, in addition, \( \psi = 0 \), then \( \sigma \) is a Poisson function if and only if

\[
\{ \{ \mu, \sigma \}, \sigma \} = 0,
\]

which is the condition

\[
[\sigma, \sigma]_\mu = 0,
\]

i.e., \( \sigma \) is a Poisson structure in the usual sense, a section of \( \wedge^2 V^* \) with Schouten–Nijenhuis square zero. The Poisson case is also called the triangular case by extension of the terminology used in the theory of Lie bialgebras.

The twisted differential. In the Poisson case \((\gamma = 0 \text{ and } \psi = 0)\), the anchor of \( V^* \) is \( \alpha_\star \circ \sigma^2 \), and the bracket on \( \Gamma(\wedge^1 V^*) \) is \( \gamma_{\sigma} = \{ \sigma, \mu \} \), the Koszul bracket\(^4\). The corresponding differential on \( \Gamma(\wedge^1 V) \) is the Lichnerowicz–Poisson differential [36], \( d_\sigma = \{ \{ \sigma, \mu \}, . \} = [\sigma, .]_\mu \), while the differential on \( \Gamma(\wedge^1 V^*) \) is the Lie algebroid cohomology operator \( d_\mu = \{ \mu, . \} \). The pair \((V, V^*)\) is a Lie algebroid.

In the twisted Poisson case, \( \gamma_\sigma = -\{ \mu, \sigma \} + \frac{1}{2} \{ \{ \psi, \sigma \}, \sigma \} \) restricts to the Lie algebroid bracket on sections of \( V^* \) defined by Ševera and Weinstein [50], and the corresponding differential on \( \Gamma(\wedge^1 V) \) is the twisted Poisson differential, \( d_\sigma + i_{\psi(2)} \), where \( \psi(2) = \frac{1}{2} \{ \{ \psi, \sigma \}, \sigma \} = (\wedge^2 \sigma^2) \psi \), while the derivation \( \{ \mu, . \} \) is the derivation \( d_\mu + i_{\psi(1)} \), where \( \psi(1) = \{ \psi, \sigma \} = \sigma^2 \psi \) (see [50] [47] [29]). The pair \((V, V^*)\) is then a quasi-Lie bialgebroid.

2.3 Twisting by pre-symplectic functions

It follows from formula (3) that a section \( \tau \) of \( \wedge^2 V^* \) is a pre-symplectic function with respect to a structure \( S = \phi + \gamma + \mu + \psi \) if and only if

\(^4\)The Koszul bracket [34] restricts to the bracket of sections of \( \Gamma(V^*) \) generalizing the well-known bracket of 1-forms on a Poisson manifold. The bracket of 1-forms on symplectic manifolds was introduced in the book of Abraham and Marsden (1967). For Poisson manifolds, it was discovered independently in the 1980’s by several authors – Gelfand and Dorfman, Fuchssteiner, Magri and Morosi, Daleskii –, and Weinstein [19] has shown that it is a Lie algebroid bracket.
Equation (7) is dual to (4) and it is also called a generalized twisted Maurer-Cartan equation or again simply a Maurer-Cartan equation. Pre-symplectic functions generalize pre-symplectic structures on manifolds as well as their twisted versions.

If \( \gamma = \phi = 0 \), then \( \{ \mu, \mu \} = 0 \), i.e., \( V \) is a Lie algebroid, and \( \psi \) is a \( d_\mu \)-closed section of \( \wedge^3 V^* \). In this case, \( \tau \) is pre-symplectic if and only if the pair \( (\tau, \psi) \) satisfies the twisted pre-symplectic condition,

\[
\psi - \{ \mu, \tau \} = 0 ,
\]

which is the condition, \( d_\mu \tau = \psi \), i.e., \( (\tau, \psi) \) is a twisted pre-symplectic structure on the Lie algebroid \( V \). (See [50] and see [49] for an example of a twisted symplectic structure arising in the theory of the lattices of Neumann oscillators.)

If, in particular, \( \gamma = \phi = \psi = 0 \), then \( \{ \mu, \mu \} = 0 \) and \( V \) is a Lie algebroid. In this case, \( \tau \) is pre-symplectic if and only if \( \tau \) satisfies the pre-symplectic condition,

\[
\{ \mu, \tau \} = 0 ,
\]

which is the condition, \( d_\mu \tau = 0 \), i.e., \( \tau \) is a \( d_\mu \)-closed section of \( \wedge^2 V^* \), the pre-symplectic case.

3 The graphs of Poisson and of pre-symplectic functions

3.1 Courant algebroids, the Courant algebroid \( V \oplus V^* \)

A Loday algebra (called Leibniz algebra by Loday [38]) is equipped with a bracket (in general non skew-symmetric) satisfying the Jacobi identity in the form \([u, [v, w]] = [[u, v], w] + [v, [u, w]]\). We give the definition of Courant algebroids in [28] which is equivalent to the original definition of Courant and Weinstein [12,11].

A Courant algebroid is a vector bundle \( E \to M \), equipped with a vector bundle morphism, \( a_E : E \to TM \), called the anchor, a fiber-wise non-degenerate symmetric bilinear form \((\cdot, \cdot)\), and a bracket, \([\cdot, \cdot] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)\), called the Dorfman-Courant bracket, such that

- \( \Gamma(E) \) is a Loday algebra,
- for all \( x, u, v \in \Gamma(E) \),
  \[
a_E(x) \cdot (u, v) = (x, [u, v] + [v, u]) = ([x, u], v) + (u, [x, v]) .
\]

A sub-bundle, \( F \subset E \), is called a Dirac sub-bundle if

- \( F \) is maximally isotropic,
- \( \Gamma(F) \) is closed under the bracket.
When $S$ is a structure on $V$, the vector bundle $E = V \oplus V^*$ with the canonical scalar product,

$$(u, v) = \{u, v\},$$

and bracket

$$[u, v]_S = \{\{u, S\}, v\},$$

(8)

for $u, v \in \Gamma(V \oplus V^*)$, is a Courant algebroid [47] [57] [28], called the double of $V$.

**Lemma 3.1.** Let $S$ be a structure on $V$.

(i) The function $\sigma \in \Gamma(\wedge^2 V)$ is a Poisson function with respect to $S$ if and only if $V^*$ is a Dirac sub-bundle of $(V \oplus V^*, [\cdot, \cdot]_{e^{-\sigma}S})$.

(ii) The function $\tau \in \Gamma(\wedge^2 V^*)$ is a pre-symplectic function with respect to $S$ if and only if $V$ is a Dirac sub-bundle of $(V \oplus V^*, [\cdot, \cdot]_{e^{-\tau}S})$.

**Proof.** Part (i) (resp., (ii)) follows from the computation of the bi degrees of the homogeneous terms in $[u, v]_{e^{-\sigma}S}$ (resp., $[u, v]_{e^{-\tau}S}$) for $u, v \in \Gamma(V)$ (resp., $u, v \in \Gamma(V^*)$).

□

### 3.2 Graphs as Dirac structures

Theorem 3.2 below generalizes the characterization of the graphs of Poisson, quasi-Poisson and pre-symplectic structures in [37] and [50], and that of twisted pre-symplectic structures in [3] and [7]. The statement of this theorem can be found in Remark 4.2 in [47] (cf. also Prop. 5 in [9]), and the proof given here is also due to Roytenberg [48]. Both theorems in this section have been proved by Terashima [55].

**Theorem 3.2.** Let $S$ be a structure on $V$.

(i) A section $\sigma$ of $\wedge^2 V$ is a Poisson function with respect to $S$ if and only if its graph in the Courant algebroid $(V \oplus V^*, [\cdot, \cdot]_S)$ is a Dirac sub-bundle.

(ii) A section $\tau$ of $\wedge^2 V^*$ is a pre-symplectic function with respect to $S$ if and only if its graph in the Courant algebroid $(V \oplus V^*, [\cdot, \cdot]_S)$ is a Dirac sub-bundle.

**Proof.** We need only prove (ii), since the proof of (i) is entirely similar. We shall denote by $\tau^h$ the vector bundle morphism from $V$ to $V^*$ induced by $\tau \in \Gamma(\wedge^2 V^*)$, such that $\tau^h X = -i_X \tau$, for $X \in V$, as well as the associated map on sections of $V$. By the graph of $\tau$, we mean the graph of $\tau^h$. Since $\tau^h(X) = \{X, \tau\}$, for all $X \in \Gamma(V)$, and since, for reasons of bidegree, $e^{\tau} X = X + \{X, \tau\}$, it follows that

$$\text{Graph}(\tau) = e^{\tau} V. \quad (9)$$

Since $e^{\tau}$ is an automorphism of $(\mathcal{F}, \{\cdot, \cdot\})$, it is an isomorphism from $(V \oplus V^*, [\cdot, \cdot]_{e^{-\tau}S})$ to $(V \oplus V^*, [\cdot, \cdot]_S)$. Thus $e^{\tau} V$ is a Dirac sub-bundle of $(V \oplus V^*, [\cdot, \cdot]_S)$ if and only if $V$ is a Dirac sub-bundle of $(V \oplus V^*, [\cdot, \cdot]_{e^{-\tau}S})$. Thus (ii) follows from (9) and Lemma 3.1 (ii). □
Theorem 3.3. Let $S = \phi + \gamma + \mu + \psi$ be a structure on $V$.

(i) Let $\sigma$ be a Poisson function with respect to $S$. The projection $\text{Graph}(\sigma) \to \Gamma(V^*)$ is a morphism of Lie algebroids when $\text{Graph}(\sigma)$ is equipped with the Lie bracket induced from the Dorfman-Courant bracket $[,]_S$ and $\Gamma(V^*)$ is equipped with the Lie bracket $\gamma_\sigma = \gamma - \{\mu, \sigma\} + \frac{1}{2}\{\psi, \sigma\}, \sigma$. 

(ii) Let $\tau$ be a pre-symplectic function with respect to $S$. The projection $\text{Graph}(\tau) \to \Gamma(V)$ is a morphism of Lie algebroids when $\text{Graph}(\tau)$ is equipped with the Lie bracket $\mu_\tau = \mu - \{\gamma, \tau\} + \frac{1}{2}\{\phi, \tau\}, \tau$.

Proof. We need only prove (ii), since the proof of (i) is entirely similar. For any $\tau \in \Gamma(\wedge^2 V^*)$, $X$ and $Y \in \Gamma(V)$, \[ [e^\tau X, e^\tau Y]_S = e^\tau [X, Y]_{e^\tau S} \] If $\tau$ is a pre-symplectic function with respect to $S$, then \[ [e^\tau X, e^\tau Y]_S = e^\tau [X, Y]_{\mu_\tau} = [X, Y]_{\mu_\tau} + \{[X, Y]_{\mu_\tau}, \tau\} \] whose $V$-component is $[X, Y]_{\mu_\tau}$. \qed

4 Symplectic functions

Let us now assume that $\sigma \in \Gamma(\wedge^2 V)$ is non-degenerate, i.e., the map $\sigma^4 : V^* \to V$ defined by $\sigma^4 \alpha = i_\alpha \sigma$, for $\alpha \in \Gamma(V^*)$, is invertible. Set $\tau_\sigma = (\sigma^4)^{-1}$, and let $\tau \in \Gamma(\wedge^2 V^*)$ be such that $\tau_\sigma X = -i_X \tau$, for $X \in \Gamma(V)$. We say that $\tau \in \Gamma(\wedge^2 V^*)$ and $\sigma \in \Gamma(\wedge^2 V)$ are inverses of one another. A non-degenerate pre-symplectic function is called symplectic.

4.1 “Non-degenerate Poisson” is equivalent to “symplectic”

Many classical results are corollaries of the general theorem which we state and prove in this section. Recall that $\xi^a \theta_b = -\theta_b \xi^a$, $\{\xi^a, \theta_b\} = \delta^a_b = \{\theta_b, \xi^a\}$ and, for $u, v, w \in \mathcal{F}$,
\[
\{u, vw\} = \{u, v\}w + (\text{degree} u)|v|w \{u, w\}, \\
\{uv, w\} = u\{v, w\} + (\text{degree} v)|u|w \{u, w\},
\]
where $|u|$ is the degree of $u$, and
\[
\{\{u, v\}, w\} = \{\{u, v\}, w\} + (\text{shifted degree} v)|u||w| \{v, \{u, w\}\}, \\
\{\{u, v\}, w\} = \{\{u, v\}, w\} + (\text{shifted degree} w)|u||v| \{\{u, v\}, w\},
\]
where $|u|$ is the shifted degree of $u$. The proof of the theorem depends on the following lemma.

Lemma 4.1. Assume that $\sigma \in \Gamma(\wedge^2 V)$ is non-degenerate and that its inverse is $\tau$. Then

(i) $\{\sigma, \tau\} = -\{\tau, \sigma\} = \text{Id}_V$.

(ii) If $S$ is of shifted bidegree $(p, q)$, then
\[
\{\{\sigma, \tau\}, S\} = (q - p)S .
\]
Proof. This lemma is proved by straightforward computations, using the equality $\text{Id}_V = \xi^a \theta_a$. □

**Theorem 4.2.** Let $S$ be a structure on $V$. Let $\sigma \in \Gamma(\wedge^2 V)$ be a non-degenerate bivector with inverse $\tau \in \Gamma(\wedge^2 V^*)$. Then $\sigma$ is a Poisson function with respect to $S$ if and only if $-\tau$ is a symplectic function with respect to $S$.

**Proof.** Lemma 4(ii) applied in the cases $(p,q) = (2,-1), (1,0), (0,1)$ and $(-1,2)$, and repeated applications of the Jacobi identity yield the following computations. Let $\mu$ be of shifted bidegree $(0,1)$. From

$$\{\{\mu, \tau\}, \sigma\} = \{\mu, \{\tau, \sigma\}\} + \{\{\mu, \sigma\}, \tau\} = \mu + \{\{\mu, \sigma\}, \tau\} ,$$

we obtain

$$\{\{\mu, \tau\}, \sigma\} = \{\mu, \sigma\} + \{\{\mu, \sigma\}, \tau\}$$

$$= \{\mu, \sigma\} + \{\{\mu, \sigma\}, \{\tau, \sigma\}\} + \{\{\mu, \sigma\}, \tau\} = \{\{\mu, \sigma\}, \sigma\} ,$$

Whence

$$\{\{\{\mu, \tau\}, \sigma\}, \sigma\} = \{\{\{\mu, \sigma\}, \tau\}, \sigma\} = \{\{\mu, \sigma\}, \{\tau, \sigma\}\}$$

$$= -3 \{\{\mu, \sigma\}, \sigma\} .$$

Similarly, if $\gamma$ is of shifted bidegree $(1,0)$,

$$\{\{\{\gamma, \tau\}, \sigma\}, \sigma\} = 12 \{\gamma, \sigma\} .$$

If $\phi$ is of shifted bidegree $(2,-1)$,

$$\{\{\{\phi, \tau\}, \sigma\}, \sigma\} = -36 \phi .$$

Let $S = \phi + \gamma + \mu + \psi$. The term of shifted bidegree $(-1,2)$ in $e^{-\tau}S$ is

$$\psi_\tau = \psi - \{\mu, \tau\} + \frac{1}{2} \{\{\phi, \tau\}, \tau\} - \frac{1}{6} \{\{\gamma, \tau\}, \tau\} ,$$

and the term of shifted bidegree $(2,-1)$ in $e^{-\sigma}S$ is

$$\phi_\sigma = \phi - \{\gamma, \sigma\} + \frac{1}{2} \{\{\mu, \sigma\}, \sigma\} - \frac{1}{6} \{\{\psi, \sigma\}, \sigma\} .$$

The preceding equalities and analogous results for other iterated brackets, reversing the roles of $\sigma$ and $\tau$, yield the equalities:

$$\{\{\psi_\tau, \sigma\}, \sigma\} = 6 \phi_{-\sigma}$$

and

$$\{\{\phi_\sigma, \tau\}, \tau\} = 6 \psi_{-\tau} .$$

Therefore $\psi_\tau = 0$ implies $\phi_{-\sigma} = 0$, and conversely. □
The method of proof used above in the general case can be applied to give one-line proofs of some well-known results.

- For the case of non-degenerate Poisson structures, the proof reduces to \(\{\{\{\mu, \tau\}, \sigma\}, \sigma\} = 0\) implies that \(\{\{\{\mu, \tau\}, \sigma\}, \sigma\} = 0\), and a similar argument applies to the converse. This simple argument proves the classical result: non-degenerate closed 2-forms are in one-to-one correspondence with non-degenerate Poisson bivectors.

- For the case of non-degenerate twisted Poisson structures (see Section 2.2 (ii)), the proof reduces to \(\{\mu, \tau\} = -\psi\) implies that \(\{\{\{\mu, \tau\}, \sigma\}, \sigma\} = -\{\{\psi, \sigma\}, \sigma\}, \sigma\}\), which implies that \(\{\{\mu, \sigma\}, \sigma\} = \frac{1}{\tau}\{\{\psi, \sigma\}, \sigma\}, \sigma\}\), and a similar argument for the converse. Thus \(d_\mu \tau = -\psi\) implies \(\frac{1}{2}\{\sigma, \sigma\}_\mu = (\wedge^3 \sigma^2)\psi\) and conversely. This constitutes a direct proof of the following corollary of Theorem 4.2 (see [50] [3] [32]).

**Corollary 4.3.** (i) A non-degenerate bivector on a Lie algebroid defines a twisted Poisson structure if and only if its inverse is a twisted symplectic 2-form.

(ii) The leaves of a twisted Poisson manifold are twisted symplectic manifolds.

It follows from this corollary that, in the case of Lie algebras, considered to be Lie algebroids over a point, a non-degenerate \(r \in \wedge^2 g\) is a solution of the twisted classical Yang-Baxter equation, generalizing the classical Yang-Baxter equation (see Section 2.2),

\[
\frac{1}{2} [r, r]_g = (\wedge^3 r^2) \psi ,
\]

where \(\psi\) is a \(d_g\)-closed 3-form on the Lie algebra \(g\), if and only if its inverse is a non-degenerate 2-form \(\tau\) satisfying the twisted closure condition, \(d_g \tau = -\psi\). Here \(d_g\) is the Chevalley-Eilenberg cohomology operator of \(g\) and the bracket, \([, , ]_g\), is the algebraic Schouten bracket on \(\wedge^\bullet g\).

Recall that a Lie algebra is called quasi-Frobenius if it possesses a non-degenerate 2-cocycle. Thus, we recover in particular the well-known correspondence [54] [18] [19] between non-degenerate triangular \(r\)-matrices, i.e., skew-symmetric solutions of the classical Yang-Baxter equation, and quasi-Frobenius structures.

**Corollary 4.4.** A non-degenerate bivector in \(\wedge^2 g\) is a solution of the classical Yang-Baxter equation if and only if its inverse defines a quasi-Frobenius structure on \(g\).

### 4.2 Regular twisted Poisson structures

We summarize a result from [52] which can now be considered to be a corollary of Theorem 4.2. Let \(A\) be a vector bundle with a bivector \(\pi \in \Gamma(\wedge^2 A)\) such that \(\pi^t\) is of constant rank. Let \(B\) be the image of \(\pi^t\). Then \(B\) is a Lie sub-algebroid of \(A\) and, because \(\pi\) is skew-symmetric, \(\pi^t\) defines an isomorphism,
\( \pi_B^*: B^* \to B \), where \( B^* = A^*/\ker \pi^i \) is the dual of \( B \). Then the inverse of \( \pi_B^* \)
defines a non-degenerate 2-form on \( B \), \( \omega_B \in \Gamma(\wedge^2B^*) \), by \((\pi_B^*)^{-1}X = -i_X\omega_B\), for \( X \in \Gamma(B) \).

Assume that the vector bundle, \( A \), is in fact a Lie algebroid. Let \( \psi \) be a \( d_A \)-closed 3-form on \( A \), and let \( \psi_B \) denote the pull-back of \( \psi \) under the canonical injection \( \iota_B : B \hookrightarrow A \). Then

**Proposition 4.5.** Under the preceding assumptions, \((A, \pi, \psi)\) is a Lie algebroid with a regular twisted Poisson structure if and only if \((B, \omega_B, \psi_B)\) is a Lie algebroid with a twisted symplectic structure, i.e., if and only if \( d_B\omega = -\psi_B \).

This proposition constitutes a linearization of the twisted Poisson condition, and can be applied in particular to the case of Lie algebras [32].

## 5 Another type of Poisson function: Lie algebra actions on manifolds

In this section, we consider the twisting of various structures involving the action of a Lie algebra on a manifold.

### 5.1 Structures on \( TM \times g^* \)

Let \( g \) be a Lie algebra, and let \( M \) be a manifold. We consider the vector bundle \( V = TM \times g^* \) over \( M \) which is, by definition, \( TM \oplus (M \times g^*) \to M \).

We introduce local coordinates on \( T^*\Pi V \), \((x^i, \xi^i, e_A, p_i, \theta_i, \epsilon^A)\), where \( i = 1, \ldots, \dim M \), and \( A = 1, \ldots, \dim g \), with the following bidegrees,

\[
\begin{array}{cccc}
x^i & \xi^i & e_A & p_i \\
(0,0) & (0,1) & (1,1) & (1,0)
\end{array}
\begin{array}{c}
\theta_i \\
(1,0)
\end{array}
\begin{array}{c}
\epsilon^A \\
(-1,-1)
\end{array}
\]

satisfying

\[
\{x^i, p_j\} = \delta^i_j, \quad \{\xi^i, \theta_j\} = \delta^i_j, \quad \{e_A, \epsilon^B\} = \delta^B_A.
\]

Let

\[
S_g = \frac{1}{2} C^{CD}_{AB} e^A e^B e_D
\]

be the function on \( T^*\Pi V \) of shifted bidegree \((1,0)\) defining the Lie bracket of \( g \), and let

\[
S_M = p_i \xi^i
\]

be the function on \( T^*\Pi V \) of shifted bidegree \((0,1)\) which defines the Schouten–Nijenhuis bracket of multivectors on \( M \). Then

\[
[u, v]_g = \{u, S_g\}, v, \quad (11)
\]
for all \(u, v \in \mathfrak{g}\), and
\[
[X, Y]_M = \{\{X, S_M\}, Y\}, \tag{12}
\]
for all \(X, Y \in \Gamma(TM)\). It is easy to show that \(S_\mathfrak{g} + S_M\) is a structure on \(V\).

More generally, consider the following functions on \(T^*\Pi V\) of shifted bidegree \((-1, 2)\), a 3-form \(\Psi_M\) on \(M\),
\[
\Psi_M = \frac{1}{6} \Psi_{ijk} \xi^i \xi^j \xi^k,
\]
and a 3-form \(\Psi_\mathfrak{g}\) on \(\mathfrak{g}^*\),
\[
\Psi_\mathfrak{g} = \frac{1}{6} \Psi^{ABC} e_A e_B e_C.
\]
Then \(S_\mathfrak{g} + S_M + (\Psi_\mathfrak{g} + \Psi_M)\) is a structure on \(V\) if and only if
\begin{itemize}
  \item \(\{S_M, \Psi_M\} = 0\), i.e., \(\Psi_M\) is a closed 3-form on \(M\), and
  \item \(\{S_\mathfrak{g}, \Psi_\mathfrak{g}\} = 0\), i.e., \(\Psi_\mathfrak{g}\) is a 0-cocycle on \(\mathfrak{g}\) with values in \(\wedge^3 \mathfrak{g}\).
\end{itemize}

More generally still, we can, in addition, introduce a function on \(T^*\Pi V\) of shifted bidegree \((0, 1)\) which defines a bracket on \(\mathfrak{g}^*\),
\[
S_\mathfrak{g}^* = \frac{1}{2} \Gamma^{AB} e_A e_B e_C.
\]
Then \(S = S_\mathfrak{g} + (S_\mathfrak{g}^* + S_M) + (\Psi_\mathfrak{g} + \Psi_M)\), a sum of terms of shifted bidegrees \((1, 0)\), \((0, 1)\) and \((-1, 2)\), respectively, is a structure on \(V\) if and only if
\begin{itemize}
  \item \(\{S_M, \Psi_M\} = 0\), i.e., \(\Psi_M\) is a closed 3-form on \(M\), and
  \item \(\{S_\mathfrak{g} + S_\mathfrak{g}^* + \Psi_\mathfrak{g}, S_\mathfrak{g} + S_\mathfrak{g}^* + \Psi_\mathfrak{g}\} = 0\), the condition that \((\mathfrak{g}, \mathfrak{g}^*)\) be a Lie-quasi bialgebra.
\end{itemize}

Let us assume that these conditions are satisfied. By what function can we twist the structure \(S_\mathfrak{g} + (S_\mathfrak{g}^* + S_M) + (\Psi_\mathfrak{g} + \Psi_M)\)? We can twist it by any function of shifted bidegree \((1, -1)\). Therefore we can choose
\[
\rho = \rho_i A^i e^A \theta_i,
\]
and twist \(S\) by \(\rho\), and/or we can twist \(S\) by the bivector
\[
\pi = \frac{1}{2} \pi^{ij} \theta_i \theta_j.
\]
We shall now prove, following Terashima [55], that twisting by \(\rho + \pi\) provides a natural and unified way of determining the Lie algebroid structures discovered by Lu [38] and by Bursztyn, Crainic and Ševera [7, 9]. This method yields an immediate proof of the fact that these are indeed Lie algebroid structures.

5.2 Twisting by a Lie algebra action

Let us first determine the meaning of the condition that \(\rho\) be a Poisson function with respect to \(S = S_\mathfrak{g} + S_M\). We remark that \(\rho\), considered either as a function on \(T^*\Pi V\) or as a map from \(\mathfrak{g}\) to \(\Gamma(TM)\), satisfies, for all \(u \in \mathfrak{g}\),
\[ \{ \rho, u \} = \rho(u) . \]

Computing the terms of shifted bidegrees \((2, -1) \), \((1, 0) \) and \((0, 1) \) of the twisted structure, \( e^{-\rho}S \), we obtain

\[
e^{-\rho}(S_g + S_M) = \left( -\{S_g, \rho \} + \frac{1}{2}\{\{S_M, \rho \}, \rho \} \right) + (S_g - \{S_M, \rho \}) + S_M .
\]

Therefore \( \rho \) is a Poisson function with respect to \( S = S_g + S_M \) if and only if

\[
-\{S_g, \rho \} + \frac{1}{2}\{\{S_M, \rho \}, \rho \} = 0 . \tag{13}
\]

**Lemma 5.1.** The function \( \rho \) is a Poisson function with respect to \( S_g + S_M \) if and only if it is a Lie algebra action of \( g \) on \( M \).

*Proof.* The proof of the fact that relation (13) is equivalent to

\[
\rho([u, v]_g) = [\rho(u), \rho(v)]_M
\]

for all \( u, v \in g \), depends on formulas (11) and (12), the Jacobi identity and the vanishing of all brackets of the form \( \{e_A, \theta_i \} \) and \( \{e_A, \theta_i \} \), whence

\[
\rho([u, v]_g) = \{\{S_g, \rho \}, u \}, v \}
\]

and

\[
[\rho(u), \rho(v)]_M = \frac{1}{2}\{\{S_M, \rho \}, \rho \}, v \} . \quad \square
\]

**5.3 Introducing additional twisting by a bivector**

Let us now twist \( S = S_g + (S_g^* + S_M) + (\Psi_g + \Psi_M) \) by

\[
\sigma = \pi + \rho .
\]

We first observe that the brackets \( \{ \pi, \rho \}, \{S_g, \pi \}, \{S_g^*, \pi \}, \{\{S_g^*, \rho \}, \pi \} \) and \( \{\Psi_g, \pi \} \) vanish. Computing the term of shifted bidegree \((2, -1) \) in \( e^{-(\pi+\rho)}S \), we see that \( \pi + \rho \) is a Poisson function with respect to \( S \) if and only if

\[
-\{S_g, \rho \} + \frac{1}{2}\{\{S_g^*, \rho \}, \rho \} + \frac{1}{2}\{\{S_M, \pi + \rho \}, \pi + \rho \}
- \frac{1}{2}\{\{\Psi_g + \Psi_M, \pi + \rho \}, \pi + \rho \} = 0 .
\]

The computation of the several terms in this generalized twisted Maurer-Cartan equation yields
Proposition 5.2. The function $\pi + \rho$ is a Poisson function with respect to $S = S_g + (S_g^* + S_M) + (\Psi_g + \Psi_M)$ if and only if the following four conditions are satisfied:

(A) $\{\{\Psi_M, \rho\}, \rho\} = 0$ ,

(B) $-\{S_g, \rho\} + \frac{1}{2}\{\{S_M, \rho\}, \rho\} - \frac{1}{2}\{\{\Psi_M, \rho\}, \pi\} = 0$ ,

(C) $\{\{S_M, \pi\}, \rho\} + \frac{1}{2}\{\{S_g^*, \rho\}, \rho\} - \frac{1}{2}\{\{\Psi_M, \rho\}, \pi\} = 0$ ,

(D) $\{\{S_M, \pi\}, \pi\} - \frac{1}{3}\{\{\Psi_g, \rho\}, \rho\} - \frac{1}{3}\{\{\Psi_M, \pi\}, \pi\} = 0$ .

Condition (A) is the relation $i_{\rho(u)}(\gamma(u)) + (\wedge^3 \rho)\Psi_M = 0$, for all $u, v, w \in g$, which means that $\Psi_M$ is in the kernel of $\wedge^3 \rho$, where $\rho^*$ is the dual of $\rho$.

Condition (B) is the relation

$$\rho([u, v]_g) - [\rho(u), \rho(v)]_M = \pi^\sharp(i_{\rho(u)}(\gamma(u))) ,$$

for all $u, v \in g$. This is proved by the same computations as in Lemma 5.1.

Thus (B) expresses the fact that $\rho$ is a twisted action of $g$ on $M$.

Condition (C) is the relation

$$\{\{S_M, \pi\}, \rho\} + \frac{1}{2}\{\{S_g^*, \rho\}, \rho\} - \frac{1}{2}\{\{\Psi_M, \pi\}, \pi\} = 0$$

for all $u \in g$, where $\gamma : g \to \wedge^2 g$ is $S_g^*$ viewed as a cobracket on $g$. In fact,

$$\{\{S_M, \pi\}, \rho\} = \{\{\rho, u\}, S_M\}, \pi\} = \{\{\rho, u\}, \pi\} = \mathcal{L}_{\rho(u)}\pi ,$$

while

$$\frac{1}{2}\{\{S_g^*, \rho\}, \rho\}, u\} = (\wedge^2 \rho)(\gamma(u)) ,$$

and

$$\frac{1}{2}\{\{\Psi_M, \rho\}, \pi\}, u\} = (\wedge^2 \pi^\sharp)(i_{\rho(u)}\Psi_M) .$$

Condition (D) is the relation

$$\frac{1}{2}\{\pi, \pi\} = (\wedge^3 \rho)(\Psi_g) + (\wedge^3 \pi^\sharp)(\Psi_M) .$$

5.4 Particular cases

In the light of Proposition 5.2 and formulas (14), (15) and (16), we can interpret several important particular cases of Poisson functions of the type $\pi + \rho$.

- Case $\rho = 0$, already studied in section 2.2 Conditions (A), (B) and (C) are identically satisfied and (D) is the condition that $M$ be a twisted Poisson
If \( \rho = 0 \) and \( \Psi_M = 0 \), then (D) is the condition that \( M \) be a Poisson manifold.

- **Case** \( \Psi_M = 0 \). While condition (A) is identically satisfied, conditions (B), (C) and (D) express the fact that \( M \) is a quasi-Poisson \( g \)-space, the version of the quasi-Poisson \( G \)-spaces in the sense of [1] in which only an infinitesimal Lie algebra action is assumed. When the Lie group \( G \) is connected and simply connected, conditions (B), (C) and (D) imply that \( M \) is a quasi-Poisson \( G \)-space, and conversely.

- **Case** \( \Psi_M = 0 \) and \( S_{g^*} = 0 \). Conditions (B), (C) and (D) are (B) \( M \) is a \( g \)-manifold, (C) \( \pi \) is a \( g \)-invariant bivector, (D) \( \frac{1}{2} [\pi, \pi]_M = (\wedge^3 \rho)(\Psi_g) \).

  If \( \Psi_g \) is the Cartan 3-vector of the Lie algebra \( g \) of a connected and simply connected Lie group with a bi-invariant scalar product, conditions (B), (C) and (D) express the fact that \( M \) is a quasi-Poisson \( g \)-manifold, the version of the quasi-Poisson \( G \)-manifolds in the sense of [2] in which only an infinitesimal Lie algebra action is assumed. When the Lie group \( G \) is connected and simply connected, conditions (B), (C) and (D) imply that \( M \) is a quasi-Poisson \( G \)-manifold, and conversely.

- **Case** \( \Psi_M = 0 \) and \( \Psi_g = 0 \). In this case, \( (g, g^*) \) is a Lie bialgebra. Condition (D) expresses the fact that \( \pi \) is a Poisson bivector, and equations (14) and (15) show that conditions (B) and (C) express the fact that \( \rho \) is an infinitesimal Poisson action of the Lie bialgebra \( (g, g^*) \) on the Poisson manifold \( M \) in the sense of Lu and Weinstein [40] [39] (which can also be called a Lie bialgebra action), corresponding to a Poisson action of the connected and simply connected Poisson-Lie group with Lie algebra \( g \).

**Remark** The method described here for the characterization of Poisson and quasi-Poisson structures can be used to recover conditions defining Poisson-Nijenhuis [31] and Poisson-quasi-Nijenhuis [53] structures.

### 5.5 The Lie algebroid structure of \( V^* = T^* M \times g \)

Whenever \( \sigma \) is a Poisson function with respect to a structure \( S \) on \( V \), with \( e^{-\sigma} S \), \((V,V^*)\) becomes a quasi-Lie bialgebroid. Therefore when \( \sigma = \pi + \rho \) is a Poisson function with respect to the structure \( S = S_g + (S_{g^*} + S_M) + (\Psi_g + \Psi_M) \) on \( V = TM \times g^* \), there is a Lie algebroid structure on \( V^* = T^* M \times g \), with anchor \( \pi^* + \rho \) and Lie bracket

\[
\gamma_{\sigma} = S_g - \{ S_{g^*} + S_M, \pi^* + \rho \} + \frac{1}{2} \{ \{ \Psi_g + \Psi_M, \pi^* + \rho \}, \pi^* + \rho \} ,
\]

and \( \{ \gamma_{\sigma}, \cdot \} \) is a differential on \( \Gamma(\wedge^*(TM \times g^*)) \). Dually, there is a bracket \( \mu_{\sigma} \) on \( TM \times g^* \), but the Jacobi identity is not satisfied in general and the derivation \( \{ \mu_{\sigma}, \cdot \} \) on \( \Gamma(\wedge^*(T^* M \times g)) \) does not square to zero in general, since \((V,V^*)\) is only a quasi-Lie bialgebroid. From formula (17) and Proposition 5.2 we obtain:
Theorem 5.3. When conditions (A)-(D) are satisfied, $T^*M \times \mathfrak{g}$ is a Lie algebroid with anchor $\pi + \rho$ and Lie bracket
\[
\gamma_\sigma = S_\rho - \{S_{\rho^*}, \rho\} - \{S_M, \pi\} - \{S_M, \rho\} + \frac{1}{2} \{\Psi_\rho, \rho\} + \frac{1}{2} \{\Psi_M, \pi\} + \{\Psi_M, \rho\} + \frac{1}{2} \{\Psi_M, \rho\}, \rho\} .
\]

We shall now show that the preceding general formula yields the brackets of [39], [7] and [9] as particular cases.

Case $\rho = 0$. Formula (18) reduces to $\gamma_\sigma = S_\rho - \{S_{\rho^*}, \rho\} - \{S_M, \pi\} - \{S_M, \rho\} + \frac{1}{2} \{\Psi_\rho, \rho\}$. The Lie algebroid structure of $V^* = T^*M \times \mathfrak{g}$ is the direct sum of the point-wise Lie bracket of sections of $M \times \mathfrak{g} \to M$ and the Lie algebroid bracket of Ševera and Weinstein [50] on $\Gamma(T^*M)$ for the twisted Poisson manifold $(M, \pi, \Psi_M)$.

Case $\Psi_M = 0$. Formula (18) reduces to
\[
\gamma_\sigma = S_\rho - \{S_{\rho^*}, \rho\} - \{S_M, \pi\} - \{S_M, \rho\} + \frac{1}{2} \{\Psi_\rho, \rho\} .
\]

For $u, v \in \Gamma(M \times \mathfrak{g})$ and $\alpha, \beta \in \Gamma(T^*M)$, we obtain the following expressions entering in the brackets of sections of $T^*M \times \mathfrak{g}$.

\[
\begin{align*}
\{\{u, S_\rho\} - \{S_M, \rho\}, \chi\} &= [u, \chi]_\mathfrak{g} + \mathcal{L}_{\rho(u)} \chi - \mathcal{L}_{\rho(v)} u, \\
\{\{\alpha, S_{\rho^*}\}, \chi\} &= -i_{\rho^*(\alpha)} \{S_{\rho^*}, \chi\} = \text{ad}^*_{\rho^*(\alpha)} u, \\
\{\{\alpha, S_M\}, \pi\} &= \mathcal{L}_{\pi^*(\alpha)} u, \\
\{\{\alpha, S_M\}, \rho\} &= \mathcal{L}_{\rho(u)} \alpha, \\
\{\{\alpha, S_M\}, \beta\} &= -[\alpha, \beta]_\pi, \\
\frac{1}{2} \{\{\Psi_\rho, \rho\}, \beta\} &= i_{(\wedge^2 \rho^*) (\alpha \wedge \beta)} \Psi_\mathfrak{g},
\end{align*}
\]

where $\mathcal{L}$ denotes the Lie derivation of vector-valued functions and of forms by vectors, and $\text{ad}^*$ is defined by means of the bracket of $\mathfrak{g}^*$. The bracket defined by $\gamma_\sigma$ is therefore

\[
\begin{align*}
[u, v] &= [u, v]_\mathfrak{g} + \mathcal{L}_{\rho(u)} v - \mathcal{L}_{\rho(v)} u, \\
[\alpha, u] &= \mathcal{L}_{\pi^*(\alpha)} u - \mathcal{L}_{\rho(u)} \alpha - \text{ad}^*_{\rho^*(\alpha)} u, \\
[\alpha, \beta] &= [\alpha, \beta]_\pi + i_{(\wedge^2 \rho^*) (\alpha \wedge \beta)} \Psi_\mathfrak{g}.
\end{align*}
\]

The bracket $[u, v]$ is the transformation Lie algebroid bracket [41] on $M \times \mathfrak{g} \to M$. Summarizing this discussion, we obtain

Proposition 5.4. If $\Psi_M = 0$, then $M$ is a quasi-Poisson $\mathfrak{g}$-space in the sense of [1] and the Lie algebroid bracket of $T^*M \times \mathfrak{g}$ is the bracket of Bursztyn, Crainic and Ševera [9]. In particular, if $\Psi_M = 0$ and $S_\rho = 0$, then $M$ is a quasi-Poisson $\mathfrak{g}$-manifold in the sense of [2], and the Lie algebroid bracket of $T^*M \times \mathfrak{g}$ is the bracket of Bursztyn and Crainic [7].
Case $\Psi_M = \Psi_g = 0$. Formula (18) reduces to

$$\gamma_\sigma = S_g - \{S_g\cdot \rho\} - \{S_M, \pi\} - \{S_M, \rho\}.$$  

Introducing the notations of Lu [39], the bracket of Bursztyn, Crainic and Ševera reduces to the following expressions, for $\alpha, \beta \in \Gamma(T^*M)$, and constant sections $u, v$ of $M \times g$,

$$\begin{cases} 
[u, v] = [u, v]_g \\
[\alpha, u] = D_\alpha u - D_u \alpha \\
[\alpha, \beta] = [\alpha, \beta]_{\pi}
\end{cases}$$  

Proposition 5.5. If $\Psi_M = 0$ and $\Psi_g = 0$, then $M$ is a manifold with a Lie bialgebra action and the Lie algebroid bracket of $T^*M \times g$ is the bracket of Lu [39], defining a matched pair of Lie algebroids.

5.6 The twisted differential

Let us determine the differential $d_{\gamma_\sigma} = \{\gamma_\sigma, \cdot\}$ on $\Gamma(\Lambda^*(TM \times g^*))$, where $\gamma_\sigma$ is defined by (18). The particular case of the quasi-Poisson $g$-spaces was recently treated in [8].

We first prove that the image of a section $X \otimes \eta$ of $\Lambda^kTM \otimes \Lambda^\ell g^*$ is a section of $\sum_{-1 \leq j \leq 2} \Lambda^{k+j}TM \otimes \Lambda^{\ell-j+1}g^*$. We shall write $\Gamma(g^*)$ for $\Gamma(M \times g^* \to M)$. In fact, for $X \in \Gamma(\Lambda^kTM)$,

$$\begin{cases} 
\{\{S_M, \pi\}, X\} \text{ and } \{\{\Psi_M, \pi\}, \pi\}, X\} \in \Gamma(\Lambda^{k+1}TM), \\
\{\{S_M, \rho\}, X\} \text{ and } \{\{\Psi_M, \pi\}, \pi\}, X\} \in \Gamma(\Lambda^kTM \otimes g^*), \\
\{\{\Psi_M, \rho\}, \pi\}, X\} \in \Gamma(\Lambda^{\ell-1}TM \otimes \Lambda^2g^*),
\end{cases}$$

and for $\eta \in \Gamma(\Lambda^\ell g^*)$,

$$\begin{cases} 
\{S_g, \eta\} \text{ and } \{\{S_M, \rho\}, \eta\} \in \Gamma(\Lambda^{\ell+1}g^*), \\
\{\{S_g, \rho\}, \eta\} \in \text{ and } \{\{S_M, \pi\}, \eta\} \in \Gamma(TM \otimes \Lambda^\ell g^*), \\
\{\{\Psi_g, \rho\}, \rho\}, \eta\} \in \Gamma(\Lambda^2TM \otimes \Lambda^{\ell-1}g^*),
\end{cases}$$

while all other brackets vanish.

Each derivation is determined by its values on the elements of degree 0 and 1. If $f \in C^\infty(M)$,

$$(d_{\gamma_\sigma} f)(\alpha + u) = (\pi^\sharp(\alpha) + \rho(u)) \cdot f,$$  

for $\alpha \in \Gamma(T^*M)$ and $u \in \Gamma(g)$. If $X \in \Gamma(TM)$, $d_{\gamma_\sigma}(X)$ is the sum of the following terms,
For and let Theorem 5.6. The sum of the following terms, $\pi = \rho \in \Gamma(\wedge^2 TM)$, $\{\{\Psi_M, \pi\}, \pi\}, X = [\pi, X]_M + (\wedge^2 \pi^2)(i_X \Psi_M)$, $\in \Gamma(\wedge^2 TM)$, $\{-\{S_M, \pi\}, X\} + \{\{\Psi_M, \pi\}, \pi\}, X = [\rho(\cdot), X]_M + (\pi^* \wedge \rho)(i_X \Psi_M)$, $\in \Gamma(TM \otimes g^\ast)$, $\frac{1}{2}\{\{\Psi_M, \rho\}, \rho\}, X = (\wedge^2 \rho)(i_X \Psi_M) \in \Gamma(\wedge^2 g^*)$, where $[\rho(\cdot), X]_M : u \in g \mapsto [\rho(u), X]_M \in \Gamma(TM)$. For $\eta \in \Gamma(g^*)$, $d_{\eta \ast}(\eta)$ is the sum of the following terms,

$\{B, \eta\} - \{\{S_M, \rho\}, \eta\} = d_B \eta \ast \ll L_{\rho(\cdot)} \eta \ast \gg \in \Gamma(\wedge^2 g^*)$,
$\{-\{S_{g^*}, \rho\}, \eta\} - \{\{S_M, \pi\}, \eta\} = \rho(\text{ad}_{\pi}(\cdot)) + L_{\pi(\cdot)} \eta \in \Gamma(TM \otimes g^*)$,
$\frac{1}{2}\{\{\Psi_M, \rho\}, \rho\}, \eta = -(\wedge^2 \rho)(i_{\Psi_M}) \in \Gamma(\wedge^2 TM)$.

where $\ll L_{\rho(\cdot)} \eta \ast \gg : (u, v) \in \wedge^2 g \mapsto (L_{\rho(u)} \eta) \ast (L_{\rho(v)} \eta, u) \in C^\infty(M)$, $\rho(\text{ad}_{\pi}(\cdot)) : u \in g \mapsto \rho(\text{ad}_{\pi}(u)) \in \Gamma(TM)$, and $L_{\pi(\cdot)} \eta : \alpha \in \Gamma(T^*M) \mapsto L_{\pi(\alpha)} \eta \in \Gamma(g^*)$. The derivation $d_{\eta \ast}$ is then extended to all sections of $\wedge^2 (TM \times g^*)$ by the graded Leibniz rule. We have thus obtained the following

**Theorem 5.6.** Let $\sigma = \pi + \rho$ be a Poisson function with respect to the structure $S = S_g + (S_{g^*} \ast S_M) + (\Psi_g \ast \Psi_M)$.
(i) For $\gamma_\sigma$ defined by (18), $d_{\gamma_\sigma} = \{\gamma_\sigma, \cdot\}$ is a differential on $\Gamma(\wedge^2 (TM \times g^*))$.
(ii) $d_{\gamma_\sigma} = \sum_{-1 \leq j \leq 2} d_{(j, 1 - j)}$, where $\quad \quad d_{(j, 1 - j)} : \Gamma(\wedge^k TM \otimes \wedge^\ell g^*) \rightarrow \Gamma(\wedge^{k+j} TM \otimes \wedge^{\ell+1-j} g^*)$.

and $\quad d_{(-1, 2)} = \frac{1}{2}\{\{\Psi_M, \rho\}, \rho\}, \}$, $\quad d_{(0, 1)} = \{-\{S_M, \rho\} + \{\Psi_M, \pi\}, \rho\} + S_{g^*}, \}$, $\quad d_{(1, 0)} = \{-\{S_M, \pi\} + \frac{1}{2}\{\Psi_M, \pi\}, \pi\} - \{S_{g^*}, \rho\}, \}$, $\quad d_{(2, -1)} = \frac{1}{2}\{\{\Psi_g, \rho\}, \rho\}, \}$.

(iii) For $f \in C^\infty(M)$ and $\eta \in \Gamma(M \times g^* \rightarrow M)$, $d_{\gamma_\sigma}(f)$ and $d_{\gamma_\sigma}(\eta)$ are determined by Equations (19) and (20) while, for $X \in \Gamma(TM)$,

$d_{\gamma_\sigma}(X) = [\pi, X]_M + [\rho(\cdot), X]_M + (\wedge^2 \pi^2 + \pi^* \wedge \rho + \wedge^2 \rho)(i_X \Psi_M)$.

These formulas simplify in each of the particular cases listed in Section 6. In the case of the quasi-Poisson $g$-spaces, $d_{\gamma_\sigma}(X) = [\pi, X]_M + L_{\rho(\cdot)} \cdot X$. From this formula and from (19), it follows that the restriction of $d_{\gamma_\sigma}$ to the space of $g$-invariant multivectors on $M$ is the differential of the quasi-Poisson cohomology introduced in [2]. This fact was observed in [55].

**Remark** Throughout this Section, the tangent bundle $TM$ can be replaced by an arbitrary Lie algebroid over $M$, provided that the de Rham differential is replaced by the differential associated with the Lie algebroid in order to yield more general results.
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