Embedding of hyperbolic spaces in the product of trees

Sergei Buyalo* & Viktor Schroeder†

Abstract

We show that for each \( n \geq 2 \) there is a quasi-isometric embedding of the hyperbolic space \( \mathbb{H}^n \) in the product \( T^n = T \times \cdots \times T \) of \( n \) copies of a (simplicial) metric tree \( T \). On the other hand, we prove that there is no quasi-isometric embedding \( \mathbb{H}^2 \to T \times \mathbb{R}^m \) for any metric tree \( T \) and any \( m \geq 0 \).

1 Introduction

Recall that a map \( f : X \to Y \) between metric spaces is called a large scale uniform embedding if

\[
\varphi_1(|x - x'|) \leq |f(x) - f(x')| \leq \varphi_2(|x - x'|)
\]

for some functions \( \varphi_1, \varphi_2 : [0, \infty) \to [0, \infty) \) tending to infinity and all \( x, x' \in X \). The map \( f \) is called quasi-isometric, if one can take linear functions as \( \varphi_1, \varphi_2, \varphi_i(t) = l_it + m_i, i = 1, 2 \).

We denote by \( \mathbb{H}^n \) the real hyperbolic space of dimension \( n \) and of curvature \(-1\).

Theorem 1.1. For each \( n \geq 2 \) there is a quasi-isometric embedding

\[
f : \mathbb{H}^n \to T^n = T \times \cdots \times T,
\]

where \( T \) is a homogeneous simplicial metric tree, whose edges all have length 1.

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Remark 1.2. Every vertex of the tree $T$ from Theorem 1.1 has infinite (countable) valence, i.e., it is adjacent to infinitely many edges. In particular, $T$ is not locally compact. For $n = 2$ there is a better result [DS], saying that the hyperbolic plane $H^2$ can be quasi-isometrically embedded in the product of two locally compact metric trees.

Remark 1.3. There is a general embedding result [Dr] according to which every metric space of bounded geometry, whose asymptotic dimension $\leq n$, admits a large scale uniform embedding into the product of $n + 1$ locally compact metric trees. The hyperbolic space $H^n$ has bounded geometry and its asymptotic dimension equals $n$. Thus our Theorem 1.1 is stronger than the Dranishnikov’s result applied to $H^n$ w.r.t. the number of trees needed for an embedding and the quality of embeddings: we construct quasi-isometric embeddings. On the other hand, it is weaker w.r.t. finiteness properties of the target trees.

One can ask whether it is possible to embed $H^n$ quasi-isometrically in the product of less than $n$ metric trees. To make this question nontrivial, one should stabilize the product by an additional factor which has arbitrarily large dimension and small growth rate, e.g., by $\mathbb{R}^m$. It easily follows from results of our previous paper [BS1] that there is no quasi-isometric embedding $H^n \to X$, $X = T_1 \times \cdots \times T_p \times \mathbb{R}^m$, for any $p \leq n - 2$ and $m \geq 0$. For the projection $X \to T_1 \times \cdots \times T_p$ defines a subexponential foliation of $X$ of rank $p = \dim(T_1 \times \cdots \times T_p)$, therefore, the subexponential corank of $X$ is $\leq p$, and by the main result of [BS1], the existence of $H^n \to X$ implies $n - 1 \leq p$. In fact, Theorem 1.1 is optimal w.r.t. the number of trees in the product stabilized by $\mathbb{R}^m$. We prove here that this is true for $n = 2$. The general case is considered in a forthcoming paper [BS2].

Theorem 1.4. There is no quasi-isometric embedding $H^2 \to T \times \mathbb{R}^m$ for any metric tree $T$ and $m \geq 0$.

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2 Proof of Theorem 1.1

2.1 Idea of the embedding

We describe the idea of the embedding for the case $n = 2$. We write $H^2$ in horospherical coordinates $H^2 = \mathbb{R} \times \mathbb{R}$ such that the sets $\{t\} \times \mathbb{R}$ are
horocycles. Consider the integer horocycle \( h_i = \{ i \} \times \mathbb{R} \) with intrinsic metric isometric to the real line. The canonical projection \( \pi : h_i \to h_{i-1} \) is a homothety. We choose an integer \( p \geq 5 \) and assume (after scaling the metric of \( H^2 \) suitable) that homothety factor of \( \pi \) is \( 1/p \).

Consider on each horocycle \( h_i \) in a periodic way intervals \( Q_{ij} \), \( j \in \mathbb{Z} \), all of length \( (1 - \frac{2}{p}) < 1 \) such that the gap between two neighboring intervals is \( 2/p < 1/2 \).

**Figure 1: \( p = 5 \)**

It is not difficult to arrange these intervals in a way that

(i) the projection \( \pi(Q_{ij}) \subset h_{i-1} \) on an interval \( Q_{ij} \) is either contained completely in some interval \( Q_{i-1,j'} \) or completely in a gap;

(ii) for every \( Q_{ij} \) there exists a \( k > 0 \) such that \( \pi^k(Q_{ij}) \) is contained in some interval \( Q_{i-k,j'} \subset h_{i-k} \).

The intervals now define a tree \( T \): the vertices are the intervals \( Q_{ij} \). The vertex \( Q_{ij} \) is connected by an edge with \( Q_{i-k,j'} \), where \( k = k(i,j) \) and \( Q_{i-k,j'} \) is the smallest integer and the interval according to (ii). The map \( f : H^2 \to T \) is defined by associating to a point \( x \) an interval \( Q_{ij} \) with minimal distance to \( x \); \( f \) is Lipschitz on a large scale due to (i).

Next we define a second tree \( T' \) in the same way using now intervals \( Q'_{ij} \) such that for every \( i, \cup_{j \in \mathbb{Z}} (Q_{ij} \cup Q'_{ij}) = h_i \), i.e., the intervals \( Q'_{ij}, j \in \mathbb{Z} \), cover the gap of the intervals \( Q_{ij} \). Finally, we will show that \( (f, f') : H^2 \to T \times T' \) is quasi-isometric.

For convenience of notations, we shift the dimension by 1, and construct a quasi-isometric embedding \( H^{n+1} \to T^{n+1} \) assuming that \( n \geq 1 \).

### 2.2 Construction of the target tree \( T \)

To construct the target tree \( T \), we consider the unit cube \( I^n \subset \mathbb{R}^n \), where \( I = [0, 1] \subset \mathbb{R} \). We fix \( a \in (0, 1) \) such that \( p := \frac{2}{1-a} \) is an integer, \( p \in \mathbb{N} \), and moreover \( \frac{1}{p-1} + \frac{1}{p} < \frac{1}{n+1} \). In particular, \( p > 2(n+1) \).

Consider the subsegment \( J \subset I \) of the length \( a \) centered at the middle of \( I \), i.e., at \( 1/2 \). Now, the middle subcube \( A = J^n \subset I^n \) will play the role of a template for the vertices of \( T \). The condition \( \frac{1}{p-1} + \frac{1}{p} < \frac{1}{n+1} \) will be used while constructing \( n + 1 \) appropriate copies of \( T \), see sect. 2.3.
2.2.1 Definition of the vertices of $T$

Using the action of the integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ by shifts on $\mathbb{R}^n$ we define the set

$$Q_0 = \bigcup_{\gamma \in \mathbb{Z}^n} \gamma A.$$  

Note that $Q_0$ is a disconnected subset of $\mathbb{R}^n$, every connected component $\gamma A$, $\gamma \in \mathbb{Z}^n$, of which is a cube of diameter $a\sqrt{n}$. We call the connected components of the set $Q_0$ the vertices of $Q_0$, and we shall identify them with the vertices of the level 0 of the tree $T$.

To define the vertices of the level 1, we apply the following procedure. We subdivide the segment $I$ into $p$ equal subsegments of the length $1/p$, so that $p-2$ of them cover the middle subsegment $J$, and the remaining two cover its complement in $I$. This subdivision induces the subdivision of the cube $I^n$ into $p^n$ congruent and parallel subcubes. There is a natural labeling of these subcubes $I^n_l$, $l \in L$, by the set $L := \{1, \ldots, p\}^n$, and for every $l \in L$ the canonical homothety $h_l : I^n \to I^n_l$ with the coefficient $\lambda = 1/p$ maps the middle subcube $A \subset I^n$ onto the subcube $A_l = h_l(A) \subset I^n_l$. Now, we define

$$Q_1 = \bigcup_{\gamma \in \mathbb{Z}^n} \bigcup_{l \in L} \gamma A_l.$$  

Note that $Q_1$ is a disconnected subset of $\mathbb{R}^n$, every connected component $\gamma A_l$, $\gamma \in \mathbb{Z}^n$, $l \in L$, of which is a cube of diameter $\lambda a\sqrt{n}$. We call the connected components of $Q_1$ the vertices of $Q_1$.

Any vertex of $Q_1$ is either separated from every vertex of $Q_0$ by the distance at least $\lambda^2$, or it lies inside of some vertex of $Q_0$ being separated from its boundary by the distance at least $\lambda^2$. This property is of key importance in what follows and it is called the separation property of $Q_0 \cup Q_1$. We shall identify the vertices of $Q_1$ with the vertex set of level 1 of the tree $T$.

To define the vertex set of any level $k \geq 1$, we apply repeatedly the described procedure. Namely, consider the set $L$ as an alphabet, and let $W_k$ be the set of words of length $k$ in the alphabet $L$, i.e., each $w \in W_k$ is a sequence of $k$ letters from $L$. In particular, $W_0 = \emptyset$ and $W_1 = L$. For each $w \in W_k$, $w = l_1 \ldots l_k$, we define the homothety $h_w : I^n \to I^n$ as the composition $h_w = h_{l_1} \circ \cdots \circ h_{l_k}$, $h_\emptyset = \text{id}$. This is a homothety with the coefficient $\lambda^k$. We let $A_w = h_w(A)$ be a subcube in $I^n$, and note that $A_w \subset A_{w'}$, where $w' \in W_{k-1}$, if and only if $w' \subset w$ is the initial subword and every coordinate of the last letter $l_k \in \{1, \ldots, p\}^n$ is different from 1 and $p$. Now, we define

$$Q_k = \bigcup_{\gamma \in \mathbb{Z}^n} \bigcup_{w \in W_k} \gamma A_w.$$
Figure 2: some small cubes of the next level are hidden behind the large black cubes; here $p = 7$

Again, for each $k \geq 1$, $Q_k$ is a disconnected set of $\mathbb{R}^n$, every connected component $\gamma A_w$, $\gamma \in \mathbb{Z}^n$, $w \in W_k$, of which is a cube of diameter $\lambda^k a \sqrt{n}$. We call the connected components of $Q_k$ the vertices of $Q_k$.

The separation property of the set $Q^+ = \bigcup_{k \geq 0} Q_k$ is the following. For each $0 \leq k' < k$ any vertex of $Q_k$ either is separated from every vertex of $Q_{k'}$ by the distance at least $\lambda^{k+1}$ or it lies inside of some vertex of $Q_{k'}$ being separated from its boundary by the distance at least $\lambda^{k+1}$. This immediately follows from self-similarity of our construction. More precisely, for some vertices $\gamma A_w \subset Q_k$, $\gamma' A_{w'} \subset Q_{k'}$, we have $\gamma A_w \subset \gamma' A_{w'}$ if and only if $\gamma = \gamma'$ and $w'$ is an initial subword of $w$, and the first letter from $w \setminus w'$, $l_{k+1}$, has all coordinates different from 1 and $p$. The vertices of $Q^+$ we shall identify with the vertices of all levels $\geq 0$ of the tree $T$.

To define the vertex set $Q_{-k}$ for $k > 0$ we take the vector $\eta = \frac{1}{p} \theta \in \mathbb{R}^n$, where $\theta := \{1, \ldots, 1\} \in \mathbb{R}^n$, and consider the homothety $H : \mathbb{R}^n \to \mathbb{R}^n$,

$$H(x) = p(x - \eta), \quad x \in \mathbb{R}^n.$$ 

Then we put $Q_{-k} := H^k(Q_0)$ for each $k > 0$. Again, $Q_{-k}$ is a disconnected subset of $\mathbb{R}^n$, every connected component of which is a cube of diameter $\lambda^{-k} a \sqrt{n}$.

Note that $\eta_0 = \frac{1}{p-1} \theta$ is the unique fixed point for $H$, $H(\eta_0) = \eta_0$. Furthermore, since $p \geq 3$, we have $\frac{1}{p} < \frac{1}{p-1} < 1 - \frac{1}{p}$, hence, $\eta_0$ is an interior point.
of the cube $A$. Then for $\delta := \text{dist}(\eta_0, \partial A) > 0$ we have

$$\text{dist}(\eta_0, \partial H^k(A)) = p^k \delta \to \infty$$

as $k \to \infty$.

Therefore, given a vertex $v$ of $Q_{k'}$, $k' \in \mathbb{Z}$, the vertex $H^k(A)$ of $Q_{-k}$ cover $v$ for all sufficiently large $k$. This property will provide connectedness of the tree $T$. Furthermore, the set $Q = \bigcup_{k \in \mathbb{Z}} Q_k$ has the separation property exactly as it is stated above for any $k$, $k' \in \mathbb{Z}$ with $k' < k$ (for more detail see the proof of Proposition 2.2 below).

### 2.2.2 Definition of the edges of $T$

Let $V_k$ be the set of the vertices of $Q_k$, $k \in \mathbb{Z}$. We define the vertex set $V$ of the tree $T$ as the union $V = \bigcup_{k \in \mathbb{Z}} V_k$. Two vertices $v \in V_k$, $v' \in V_{k'}$ are connected by an edge in $T$ if and only if $k \neq k'$, say $k' < k$, $v \subset v'$ (considered as cubes in $\mathbb{R}^n$), and $k'$ is minimal with this property. This defines a graph $T$ with the vertex set $V$ and the edge set $E$.

**Lemma 2.1.** The graph $T$ is a tree, i.e., it is connected and has no circuit.

*Proof.* $T$ has no circuit because every vertex $v \in V_k$ is connected with at most one vertex from $V_{k'}$ for every $k$, $k' \in \mathbb{Z}$, $k' < k$, since the vertices of $Q_{k'}$ are separated subsets of $\mathbb{R}^n$.

By the property of the homothety $H : \mathbb{R}^n \to \mathbb{R}^n$ indicated above, given two vertices $v' \in V_{k'}$, $v'' \in V_{k''}$ the vertex $v = H^k(A)$ of $V_{-k}$ cover $v'$, $v''$ for all sufficiently large $k$; $v' \subset v'$, $v'' \subset v$. By the separation property of $Q$, if $v' \subset u$ for some vertex $u \in V_m$, $m > k$, then $m < k'$ and $u \subset v$. Therefore, the vertices $v'$, $v''$ are connected by paths in $T$ with the vertex $v$. Hence, $T$ is connected. \[\square\]

### 2.3 Construction of colored copies of $T$

Here we describe $n + 1$ colored copies of $T$, $T_c$, $c \in C$, for which the target space $X$ for the embedding $H^{n+1} \to X$ will be $X = \bigcup_{c \in C} T_c$. As the set of the colors we take the cyclic group $C = \mathbb{Z}/(n+1)\mathbb{Z}$ of order $n+1$, and color the tree $T$ by its unit, $T = T_0$. For each $c \in C$ the vertex set $V_c$ of $T_c$ will be the union $V_c = \bigcup_{k \in \mathbb{Z}} V_{c,k}$, where every $V_{c,k}$ is the set of cubes in $\mathbb{R}^n$.

**Proposition 2.2.** For each $c \in C$ there is a copy $T_c$ of the tree $T$ with the vertex set $V_c = \bigcup_{k \in \mathbb{Z}} V_{c,k}$ such that the set $V_c$ of cubes in $\mathbb{R}^n$ satisfies the separation property, and for every $k \in \mathbb{Z}$ the union $\bigcup_{c \in C} V_{c,k}$ covers $\mathbb{R}^n$. 

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The proof is based on Lemma 2.3 below. Identifying the opposite \((n-1)\)-faces of the cube \(I^n\), we obtain \(n\)-torus \(P^n = I^n/\sim\), and we consider the set \(A = J^n \subset I^n\) as a subset of \(P^n\). Recall that \(p > 2(n+1)\), see sect. 2.2.

**Lemma 2.3.** Consider the diagonal action of the group \(C\) on \(P^n\), \((c, x) \mapsto x + cv \mod \mathbb{Z}^n\), where \(\nu = \frac{1}{n+1} \theta\). Then the shifts of \(A\) cover \(P^n\), \(\bigcup_{c \in C} c(A) = P^n\).

**Proof.** We put \(Z = P^n \setminus A\). Since \(c(Z) = P^n \setminus c(A)\) for every \(c \in C\), it suffices to show that
\[
G := \bigcap_{c \in C} c(Z) = \emptyset.
\]
Assume that it is not the case. Then \(G\) contains the \(C\)-orbit of some \(x \in Z\), since \(c(G) = G\). The point \(x = (x_1, \ldots, x_n)\) has a coordinate which lies in
\[
I_0 = [0, 1/p) \cup (1 - 1/p, 1].
\]
Without loss of generality, we may assume that \(x_1 \in I_0\). Since \(c(x) \in Z\), \(x\) has a coordinate lying in
\[
I_n = \left( \frac{n}{n+1} - \frac{1}{p}, \frac{n}{n+1} + \frac{1}{p} \right),
\]
and we may assume that \(x_n \in I_n\), because \(I_0 \cap I_n = \emptyset\) due to the condition \(p > 2(n+1)\). Similarly, since \(c'(x) \in Z\), we find that
\[
x_{n-i} \in I_{n-i} = \left( \frac{n-i}{n+1} - \frac{1}{p}, \frac{n-i}{n+1} + \frac{1}{p} \right)
\]
for all \(i = 0, 1, \ldots, n-1\), in particular, \(x_1 \in I_1\). This is a contradiction, because \(I_0 \cap I_1 = \emptyset\). \(\square\)

We fix a universal covering \(\pi : \mathbb{R}^n \to P^n\), and for \(c \in C\) we put \(Q_{c,0} = \pi^{-1}(c(A))\). Clearly, \(Q_{c,0} = Q_0 + cv\), where \(\nu = \frac{1}{n+1} \theta \in \mathbb{R}^n\) and \(C\) is identified with the set \(\{0, 1, \ldots, n\}\). The cubes of \(Q_{c,0}\) form the vertex set \(V_{c,0}\) of the 0-level of \(T_c\). It follows from Lemma 2.3 that \(\bigcup_{c \in C} V_{c,0} = \mathbb{R}^n\).

Given a letter \(l \in L = \{1, \ldots, p\}\), we assume that the homothety \(h_l : I^n \to I^n_l\) (see sect. 2.2.1) is canonically extended to the homothety \(\mathbb{R}^n \to \mathbb{R}^n\) for which we use the same notation.

**Lemma 2.4.** For every letters \(l, l' \in L\) and each color \(c \in C\), the sets \(h_l(Q_{c,0})\) and \(h_{l'}(Q_{c,0})\) coincide, \(h_l(Q_{c,0}) = h_{l'}(Q_{c,0})\).
Proof. The cube \( c(A) \) is obtained from \( A \) by a shift, \( c(A) = A + cv \). Thus the cubes \( h_t(c(A)) \) and \( h_v(c(A)) \) are obtained from \( A_t = h_t(A) \) and \( A_v = h_v(A) \) respectively by one and the same shift of \( \mathbb{R}^n \). The homothety \( h_v \) can be obtained by composing the homothety \( h_t \) with the shift of \( \mathbb{R}^n \), which moves the cube \( I_t^p \) to the cube \( I_v^p \). Since the shifts of \( \mathbb{R}^n \) commute, the last one moves \( h_t(c(A)) \) to \( h_v(c(A)) \). This easily implies the claim. \( \square \)

Given \( k \geq 0, c \in C \), we define \( Q_{c,k} := h_w(Q_{c,0}) \) for some word \( w \in W_k \). By Lemma 2.4, this is independent of \( w \). The cubes of \( Q_{c,k} \) form the vertex set \( V_{c,k} \) of the \( k \)-level of \( T_c \). It follows that \( \bigcup_{c \in C} V_{c,k} = \mathbb{R}^n \) for each \( k \geq 0 \).

Furthermore, for each color \( c \in C \), the set \( Q^+_c = \bigcup_{k \geq 0} Q_{c,k} \) has the separation property because every \( Q_{c,k}, k \geq 0 \) can be obtained from the cube \( c(A) \) by the self-similarity maps \( \{ h_w : w \in W_k \} \) and then applying the action of \( \mathbb{Z}^n \).

Given a color \( c \in C \), we define the set \( Q_{c,k} \) for negative levels as

\[
Q_{c,k} := H^{-k}(Q_{c,0}), \quad k < 0,
\]

where the homothety \( H : \mathbb{R}^n \to \mathbb{R}^n \) is defined at the end of sect. 2.2.1, \( H(x) = p(x - \eta) \). The cubes of \( Q_{c,k} \) form the vertex set \( V_{c,k} \) of the \( k \)-level of \( T_c \). It follows that \( \bigcup_{c \in C} V_{c,k} = \mathbb{R}^n \) for each \( k < 0 \).

Now, we define the vertex set of \( T_c \) as \( V_c = \bigcup_{k \in \mathbb{Z}} V_{c,k} \). The edges of \( T_c \) are defined by the same condition as for the tree \( T = T_0 \). To prove that \( T_c \) is connected, we need

**Lemma 2.5.** The fixed point of \( H \), \( \eta_0 = \frac{1}{p - 1} \theta \), lies in the interior of the cube \( c(A) \) (taken mod \( \mathbb{Z}^n \)) for every \( c \in C \).

**Proof.** The point \( \eta_0(c) = \eta_0 + (n + 1 - c) \nu \) lies in the interior of the cube \( A \) mod \( \mathbb{Z}^n \) for every color \( c \in C \), because \( \frac{1}{p} < \frac{1}{p - 1} + \frac{n + 1 - c}{n + 1} < 1 - \frac{1}{p} \) due to the condition \( \frac{1}{p - 1} + \frac{1}{p} < \frac{1}{n + 1} \), see sect. 2.2. Therefore, \( \eta_0 = \eta_0(c) + cv \) mod \( \mathbb{Z}^n \) lies in the interior of the cube \( c(A) = A + cv \) mod \( \mathbb{Z}^n \). \( \square \)

It follows from Lemma 2.5 that \( \text{dist}(\eta_0, \partial H^{-k}(c(A))) \to \infty \) as \( k \to -\infty \). Therefore, given a vertex \( v \) of \( Q_{c,k'} \), \( k' \in \mathbb{Z} \), the vertex \( H^{-k}(c(A)) \) of \( Q_{c,k} \) cover \( v \) for all \( k < 0 \) with sufficiently large \( |k| \). The same argument as in the proof of Lemma 2.1 shows that \( T_c \) is a tree for every color \( c \in C \).

**Proof of Proposition 2.2.** To complete the proof of Proposition 2.2 it remains to show that the set \( V_c \) of cubes in \( \mathbb{R}^n \) satisfies the separation property for every color \( c \in C \). That is for each \( k, k' \in \mathbb{Z}, k' < k \), any vertex of \( V_{c,k} \) either is separated from every vertex of \( V_{c,k'} \) by the distance at least \( \lambda^{k+1} \) or
it lies inside of some vertex of $V_{c,k'}$ being separated from its boundary by the distance at least $\lambda^{k+1}$, $\lambda = 1/p$.

This property is already proved for the case $k' \geq 0$. Assume that $k' < 0$. Then, by definition, $V_{c,k'} = H^{-k'}(V_{c,0})$. Note that $H^{-1} = h_l : \mathbb{R}^n \to \mathbb{R}^n$ for the letter $l = (1, \ldots, 1) \in L$. Since $h_w(V_{c,0}) = V_{c,k}$ for each word $w \in W_k$ by Lemma 2.4, we have $V_{c,k} = H^{-k}(V_{c,0})$ for all $k \geq 0$ and, hence, for all $k \in \mathbb{Z}$. It follows

$$H^{-k'}(V_{c,k}) = H^{-k-k'}(V_{c,0}) = V_{c,k-k'},$$

and the general case follows from the case $k' \geq 0$. \hfill $\square$

2.4 Definition of the embedding $f : H^{n+1}_p \to \prod_{c \in C} T_c$

It is convenient to rescale the metric of $H^{n+1}_p$ as follows. The space $\mathbb{R} \times \mathbb{R}^n$ with the warped product metric $ds^2 = dt^2 + e^{2\sigma t}d\rho_n^2$, where $d\rho_n^2 = dx_1^2 + \ldots + dx_n^2$ is the canonical Euclidean metric on $\mathbb{R}^n$, and $\sigma = \ln p$, has the constant curvature $K \equiv -\sigma^2$. In other words, $\mathbb{R} \times \mathbb{R}^n$ with the metric $ds^2$ is the hyperbolic space $H^{n+1}_p$ rescaled by the factor $1/\sigma$, $H^{n+1}_p$ for brevity. Then the shift $pr : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$, $pr(t, x) = (t - 1, x)$, is a homothety of $ds^2$, restricted to any horosphere $t \times \mathbb{R}^n$, with the coefficient $\lambda = 1/p$.

Now, for every color $c \in C$ and every $k \in \mathbb{Z}$ we consider the copy $(k, Q_{c,k}) \subset \mathbb{R} \times \mathbb{R}^n$ of the set $Q_{c,k}$, see sect. 2.3. Recall that $Q_{c,k} \subset \mathbb{R}^n$ consists of cubes with diameter $\lambda^k a \sqrt{n}$. It follows that the diameter of the cubes of $(k, Q_{c,k})$ is $a \sqrt{n}$ (w.r.t. the horospherical metric of $(k, \mathbb{R}^n)$ induced by $ds^2$), i.e., it is one and the same for all levels $k \in \mathbb{Z}$ and all colors $c \in C$.

For every $c \in C$, we define a (discontinuous) map $f_c : H^{n+1}_p \to T_c$ assigning to $z \in H^{n+1}_p$ the vertex $f_c(z) \in V_c$ represented by a cube from $\cup_k (k, Q_{c,k})$ closest to $z$. This defines the required map $f : H^{n+1}_p \to \prod_{c \in C} T_c$.

The following fact allows to simplify the proof of the Lipschitz property of $f$.

**Lemma 2.6.** Let $a, b, c$ be the side lengths of a triangle in a metric space such that $c \geq a$. Then $a + b \leq 3c$.

**Proof.** If $b < 2a$ then $a + b < 3a \leq 3c$. Assume that $b \geq 2a$. Then $2c \geq 2(b - a) \geq b$. Therefore, $3c \geq a + b$ as well. \hfill $\square$

**Proposition 2.7.** The map $f_c : H^{n+1}_p \to T_c$ is large scale Lipschitz for every $c \in C$. 9
Proof. We have to show that

$$\text{dist}(f_c(z), f_c(z')) \leq \Lambda \text{dist}(z, z') + \alpha$$

for some $\Lambda \geq 1$, $\alpha \geq 0$ and all $z, z' \in H_{p+1}$. Fix $z, z' \in H_{p+1}$, and put $v = f_c(z), v' = f_c(z')$. Then $v \in V_{c,k}, v' \in V_{c,k'}$ for some $k, k' \in \mathbb{Z}$. W.L.G. we can assume that $z, z'$ are the centers of the cubes $v \subset (k, Q_{c,k}), v' \subset (k', Q_{c,k'})$, respectively, and that $k' \geq k$. Taking the point $z'' \in (k', \mathbb{R}^n)$ which projects to $z \in (k, \mathbb{R}^n)$, we note that $\text{dist}(f_c(z), f_c(z'')) \leq k' - k$ because the levels of the end point of any edge in $T_c$ differ at least by 1, and $k' - k = \text{dist}(z, z'')$. Using this and Lemma 2.6 we can assume W.L.G. that $k' = k$, i.e., the points $z, z'$ belong to the horosphere $(k, \mathbb{R}^n) \subset H_{p+1}$, and the cubes $v, v'$ have one and the same level $k$.

It follows from the definition of the edges that no shortest path in $T_c$ has an interior vertex with locally maximal level. Thus the shortest path in $T_c$ between $v$ and $v'$ has a unique vertex $v_0$ of a lowest level $k_0$, $k \geq k_0$.

Let $v_1 \in v_0 v, v'_1 \in v_0 v'$ be the vertices adjacent to $v_0$. Note that $v_1$, $v'_1 \subset v_0$ considered as cubes in $\mathbb{R}^n$. Then by the separation property, the cubes $v_1, v'_1$ either are disjoint or one of them is contained in the other. However, the last case is excluded because otherwise we would have a path in $T_c$ between $v_1$ and $v'_1$ missing the vertex $v_0$, and hence the initial path $v_0 v \cup v_0 v'$ would not be the shortest one.

Assuming that $v_1 \subset (k_1, Q_{c,k_1}), v'_1 \subset (k'_1, Q_{c,k'_1})$, where $k \geq k_1 \geq k'_1 > k_0$, we obtain that the Euclidean distance between the cubes $v_1, v'_1 \subset \mathbb{R}^n$ is at least $\lambda^{k_1+1}$. Therefore, the horospherical distance between $z, z'$ w.r.t. the horosphere $(k, \mathbb{R}^n)$ is $\geq e^{\sigma k} \lambda^{k+1} \geq \text{const}(p) p^{k - k_1}$, where $\text{const}(p) > 0$ depends only on $p$. Then for the distance in $H_{p+1}$ we have

$$\text{dist}(z, z') \geq \text{const}_1(p)(k - k_1) - \text{const}_2(p).$$

First, we consider the case that the next vertex $v'_2$ of the path $v_0 v'$ following $v'_1$ has the level $k'_2 \geq k_1$. Then for the distances in the tree $T_c$ we have

$$\text{dist}(v_0, v) = 1 + \text{dist}(v_1, v) \leq 1 + k - k_1$$

and

$$\text{dist}(v_0, v') = 2 + \text{dist}(v'_2, v') \leq 2 + k - k'_2 \leq 2 + k - k_1.$$ 

Therefore,

$$\text{dist}(v, v') \leq 3 + 2(k - k_1) \leq \Lambda \text{dist}(z, z') + \alpha$$

for some constants $\Lambda > 0$, $\alpha \geq 0$ depending only on $p$.

Assume now that $k'_2 < k_1$. Then we have $v' \subset v'_2 \subset v'_1$ for the vertices $v'$, $v'_2, v'_1$ considered as the cubes in $\mathbb{R}^n$. Thus the point $z'$ projected down to the
horosphere \((k'_1, \mathbb{R}^n)\) lies in the interior of the cube \(v'_1\) being separated from its boundary by the Euclidean distance \(\geq \lambda^{k'_2 + 1}\). It follows that the horospherical distance between \(z, z'\) is \(\geq \lambda^{k'_2 + 1}e^{\sigma k} = p^{k - k'_2 - 1}\), and consequently
\[
\text{dist}(z, z') \geq \text{const}_1(p)(k - k'_2) - \text{const}_2(p).
\]

On the other hand,
\[
\text{dist}(v_0, v) \leq 1 + k - k_1 \leq 1 + k - k'_2
\]
and
\[
\text{dist}(v_0, v') \leq 2 + k - k'_2.
\]
Therefore,
\[
\text{dist}(v, v') \leq 3 + 2(k - k'_2) \leq \Lambda \text{dist}(z, z') + \alpha
\]
for some constants \(\Lambda > 0, \alpha \geq 0\) depending only on \(p\).

The following Proposition completes the proof of Theorem 1.1.

**Proposition 2.8.** The map \(f : \mathbb{H}^{n+1}_p \to \prod_{c \in C} T_c\) is quasi-isometric.

**Proof.** By Proposition 2.7, it remains to show that
\[
\text{dist}(z, z') \leq \Lambda \text{dist}(f(z), f(z')) + \alpha
\]
for some constants \(\Lambda \geq 1, \alpha \geq 0\), and all \(z, z' \in \mathbb{H}^{n+1}_p\).

We can assume that \(z = (k, x), z' = (k', x')\) for some \(k, k' \in \mathbb{Z}, k' \geq k\), where \(x, x' \in \mathbb{R}^n\). First, consider the case \(x = x'\). Then the geodesic segment \(zz' \in \mathbb{H}^{n+1}_p\) intersects \(k' - k + 1\) horospheres \((t, \mathbb{R}^n)\) at the points \((k, x), (k + 1, x), \ldots, (k', x)\). Since \(\cup_{c \in C} V_{c,k} = \mathbb{R}^n\), at least \((k' - k + 1)/|C|\) of these points belong to cubes with one and the same color \(c \in C\). All of those cubes contain the cube \(f_c(z') \in V_{c,k'}\). Hence, for the distance in \(T_c\) we have \(\text{dist}(f_c(z), f_c(z')) \geq (k' - k + 1)/|C| - 1\) by the separation property. Therefore,
\[
\text{dist}(f(z), f(z')) \geq \frac{1}{|C|}(k' - k + 1) - 1 \geq \frac{1}{n + 1} \text{dist}(z, z') - 1.
\]

In general case, we consider the points \(\overline{z}, \overline{z}'\) which are projections of \(z, z'\) respectively to a horosphere \((k_0, \mathbb{R}^n)\) with largest level \(k_0\), for which the horospherical distance between \(\overline{z}, \overline{z}'\) is at most \(p\sqrt{n}\). Then this distance is \(> \sqrt{n}\), thus \(f_c(\overline{z}) \neq f_c(\overline{z}')\) for every color \(c \in C\). Since the geodesics in every tree \(T_c\) have no interior point with locally maximal level, it follows that
\[
\text{dist}(f_c(z), f_c(z')) \geq \text{dist}(f_c(z), f_c(\overline{z})) + \text{dist}(f_c(\overline{z}), f_c(\overline{z}')) + \text{dist}(f_c(\overline{z}'), f_c(z')).
\]
Applying the first case, we obtain
\[
\text{dist}(f(z), f(z')) \geq \frac{1}{(n+1)} \max\{\text{dist}(z, \bar{z}), \text{dist}(z', \bar{z}')\} - 1 \\
\geq \text{const}_1(n) \text{dist}(z, z') - \text{const}_2(n, p)
\]
for some positive constants depending only on \(p\) and/or \(n\).

\[\square\]

### 3 Proof of Theorem 1.4

Actually, we prove a stronger result.

**Theorem 3.1.** Given \(l \geq 1\), \(m \geq 0\), \(n \in \mathbb{N}\), there is \(r_0 = r_0(l, m, n) > 0\) such that no ball \(B_r \subset H^2\) of radius \(r \geq r_0\) can be \((l, m)\)-quasi-isometrically embedded in \(T \times \mathbb{R}^n\) for any tree \(T\).

Clearly, Theorem 1.4 follows from Theorem 3.1. In turn, Theorem 3.1 is a corollary of the following more general result about arcs with bounded winding.

Let \(X\) be a CAT\((-1)\) space of bounded geometry. The last means that there are \(\rho_X > 0\) and \(M_X : (0, \infty) \rightarrow (0, \infty)\) such that every ball \(B_r \subset X\) of radius \(r > 0\) contains at most \(M_X(r)\) points which are \(\rho_X\)-separated. We use notation \(|x - x'|\) for the distance in \(X\) between \(x, x' \in X\), and \(\text{diam} A\) for the diameter of \(A \subset X\) in \(X\). We fix an origin \(o \in X\), and denote by \(S_r\) the metric sphere in \(X\) of radius \(r > 0\) centered at \(o\). If \(x, x'\) are different from \(o\), we let \(\angle_o(x, x')\) be the angle at \(o\) of the comparison triangle \(\bar{a}xx' \subset H^2\). If \(A\) misses \(o\), we put \(\angle_o(A) = \sup\{\angle_o(x, x') : x, x' \in A\}\), the angle diameter of \(A\).

Let \(A \subset S_r, r > 0,\) be an arc. Any \(a, a' \in A\) define the subarc \(A(a, a') \subset A\) with the end points \(a, a'\). Given \(\delta, \sigma \in (0, 1)\), one says that the arc \(A\) has a \(\delta\)-bounded winding at the scale \(\sigma\), if for every subarc \(A(a, a') \subset A\) with \(\angle_o(A(a, a')) \geq \sigma \angle_o(A)\) we have \(|a - a'| \geq \delta \text{diam} A\) (this property is useful in the theory of quasi-conformal mappings).

From now on, we assume that some constants \(l \geq 1\), \(m \geq 0\) and an integer \(n \geq 0\) are fixed, and saying about a quasi-isometric map \(f : A \rightarrow Y\) we mean that the map \(f\) is \((l, m)\)-quasi-isometric, i.e.,
\[
\frac{1}{l}|a - a'| - m \leq \text{dist}(f(a), f(a')) \leq l|a - a'| + m
\]
for all \(a, a' \in A\). The constants \(\sigma_0, r_0, N_0\), which will be introduced below, depend, in particular, on \(l, m, n, \rho_X\) and \(M_X\). We do not reflect this dependence in notations for brevity.
Theorem 3.2. For every $\delta \in (0, 1/4]$, $\varepsilon > 0$ there are $\sigma_0 = \sigma_0(\delta) \in (0, 1)$, $r_0 = r_0(\delta, \varepsilon) \geq 1$, such that the following holds true. Let $A \subset X$ be an arc with

1. $A \subset S_r$ for some $r \geq r_0$;
2. $\angle_\sigma(A) \geq \varepsilon$;
3. $A$ has a $\delta$-bounded winding at the scale $\sigma_0$.

Then there is no quasi-isometric map $f : A \to T \times \mathbb{R}^n$ for any metric tree $T$.

Any nondegenerate arc $A \subset \partial B_r \subset H^2$ subtending the angle $\leq \pi$ has 1-bounded winding at any scale $\sigma \in (0, 1)$ (for $H^n$ with $n \geq 3$ this is certainly not true). Hence, Theorem 3.1 follows from Theorem 3.2.

Remark 3.3. The factor $\mathbb{R}^n$ in Theorem 3.2 can be replaced by any geodesic space $Y$, which satisfies the following condition. There is a function $N : (0, 1) \to \mathbb{N}$ such that for every $\rho \in (0, 1)$ every ball $B_R \subset Y$ with sufficiently large radius $R$ contains at most $N(\rho)$ points which are $\rho R$-separated (the constants $r_0, \sigma_0$ then depend also on $N$).

Remark 3.4. Using Theorem 3.2 one can show that there is no quasi-isometric map $f : X \to T \times \mathbb{R}^n$, where $X$ is a CAT($-1$)-space with bounded geometry such that $\dim \partial_\infty X = 1$. That is, the hyperbolic rank (see [Gr], [BS1]) of the product $T \times \mathbb{R}^n$ is zero for any tree $T$ and any $n \geq 0$.

Remark 3.5. Every quasi-isometric map $f : A \to T \times \mathbb{R}^n$ as above can be easily modified to a continuous one. So, W.L.G. we shall prove only that there is no continuous quasi-isometric map $f : A \to T \times \mathbb{R}^n$.

Briefly, the proof proceeds as follows. Assuming that the assertion is not true, we find, for sufficiently large $r$, an arc $A_r \subset S_r$ with bounded winding at some scale, and a continuous quasi-isometric map $f_r : A_r \to T \times \mathbb{R}^n$. Let $g_r : A_r \to T$ be the composition of $f_r$ with the projection $T \times \mathbb{R}^n \to T$ onto the first factor. We study preimages $g_r^{-1}(C)$ of geodesic segments $C$ in the subtree $D_r = g_r(A_r) \subset T$. Since $f_r$ maps $g_r^{-1}(C)$ quasi-isometrically in $C \times \mathbb{R}^n \subset \mathbb{R}^{n+1}$, the preimage $g_r^{-1}(C)$ is small in the sense that it contains a bounded amount of sufficiently separated points. Then it follows from hyperbolicity of the space $X$ that the complement $D_r \setminus C$ contains large subtrees, i.e., subtrees for which preimage has a large subarc in $A_r$. Since $A_r$ has a bounded winding, the end points of such an arc are sufficiently separated in $X$. The key point of the proof (Lemma 3.8) is that the number of large subtrees is sufficiently large for an appropriately chosen segment $C \subset D_r$, and hence there are sufficiently many separated points in $A_r$ mapped by $f_r$ in $C \times \mathbb{R}^n$ to obtain a contradiction with properties of $C \times \mathbb{R}^n$. 

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Assume that for $r \geq 1$ there is an arc $A_r \subset S_r$ and a continuous quasi-isometric map $f_r : A_r \to T \times \mathbb{R}^n$. Then $D_r = g_r(A_r) \subset T$ is a connected compact subset and therefore it is a subtree.

**Lemma 3.6.** Fix $\delta \in (0, 1)$. There is $N_0 = N_0(\delta) \in \mathbb{N}$, such that for every segment $C \subset D_r$ the arc $A_r$ contains at most $N_0$ points from $g_r^{-1}(C)$, which are pairwise separated by the distance $\geq \delta \rho$ in $X$, where

$$\rho = \max\left\{\frac{\rho_X}{\delta}, \frac{2l}{\delta}, \text{diam } g_r^{-1}(C)\right\}.$$

**Proof.** We put $\delta' = \frac{\delta}{4l^2}$, $r(l, m, \delta) = 2lm/\delta$ and note that

$$r(l, m, \delta) \geq ml$$

since $l \geq 1$ and $\delta < 1$. We define $N_0$ as the maximum of two numbers $N'_0$ and $N''_0$, where $N'_0$ is the maximal number of $\delta'$-separated points in the ball of radius 1 in $\mathbb{R}^{n+1}$, $N'_0 = N'_0(l, n, \delta)$, and $N''_0 = \max\{M(\rho_X/\delta), M_X(2lm/\delta)\}$, $N''_0 = N''_0(l, m, X, \delta)$. Therefore,

$$N_0 = N_0(l, m, n, X, \delta) = N_0(\delta)$$

according to our agreement.

If $\text{diam } g_r^{-1}(C) \leq \rho$, then the claim follows from the definition of $N''_0$. Thus we assume that $\rho = \text{diam } g_r^{-1}(C) > r(l, m, \delta)$. Consider a segment $C \subset D_r$. Let $E \subset g_r^{-1}(C)$ be a maximal $\delta\rho$-separated subset. Since $\text{diam } E \leq \rho$, the set $f_r(E)$ lies in a ball of radius $\leq l\rho + m \leq \rho' = 2l\rho$ in $C \times \mathbb{R}^n \subset \mathbb{R}^{n+1}$. Furthermore, $f_r(E)$ is $(\frac{\delta'}{4} - m)$-separated. Since $\frac{\delta'}{4} - m \geq \frac{\delta'}{4l} = \delta'\rho'$, we obtain $|E| \leq N_0$.

Every segment $C \subset D_r$ separates the tree $D_r \subset T$ into a collection $T(C)$ of closed subtrees in $D_r$: every subtree $P \subset D_r$ from $T(C)$ is the closure in $D_r$ of some connected component of the complement $D_r \setminus C$.

Let $P$ be a closed subtree in $D_r$. Every connected component of the preimage $g_r^{-1}(P) \subset A_r$ is an arc, may be degenerate. For $\sigma \in (0, 1)$, the subtree $P$ is called $\sigma$-large w.r.t. $D_r$, if $g_r^{-1}(P)$ contains a connected component with the angle diameter $\geq \sigma \cdot \angle_0(A_r)$ (every such a component is called large). If it is clear, which tree $D_r$ is considered, then we speak about $\sigma$-large trees.

**Lemma 3.7.** Let $\delta \in (0, 1/4]$, $N_0 = N_0(\delta)$ be the constant from Lemma 3.6, $\sigma \in (0, \frac{1}{2(N_0+1)}]$. Then for every $\varepsilon > 0$ there is $r(\delta, \varepsilon) > 0$, such that for all $r \geq r(\delta, \varepsilon)$ we have the following. If the arc $A_r \subset S_r$ has the angle diameter $\angle_0(A_r) \geq \varepsilon$, then for every segment $C \subset D_r$, the collection of trees $T(C)$, into which the segment separates the tree $D_r$, contains at least one $\sigma$-large tree.
Proof. We fix $\varepsilon > 0$ and take $r(\delta, \varepsilon) > \max\{\frac{\rho x}{\delta}, \frac{2\ell}{\delta}\}$ such that for all $r \geq r(\delta, \varepsilon)$ the following holds true: if $|x - x'| \leq r$ for $x, x' \in S_r$, then the angle distance $\angle_o(x, x') \leq \frac{r}{2\delta}$. Such an $r(\delta, \varepsilon)$ exists since $X$ is a CAT(-1)-space. Assume that $r \geq r(\delta, \varepsilon)$ and $\angle_o(A_r) \geq \varepsilon$. Then $\rho = \text{diam } A_r > \max\{\frac{\rho x}{\delta}, \frac{2\ell}{\delta}\}$.

We choose a maximal $\delta r$-separated subset $E \subset \rho^{-1}(C)$. The closed balls $B_{\delta r}(x) \subset X, x \in E$, cover the preimage $\rho^{-1}(C)$, and the number of them $|E| \leq N_0$ according to Lemma 3.6, since

$$\text{diam } \rho^{-1}(C) \leq \rho.$$ 

Assuming that an orientation of the arc $A_r$ is fixed, for every point $x \in E$ we take the point $x^+ \in A_r$ of the first coming in the ball $B_{\delta r}(x)$ and the point $x^- \in A_r$ of the last coming out from the ball $B_{\delta r}(x)$. We have $|x^- - x^+| \leq 2\delta r \leq r$ by the choice of $\delta \leq 1/4$ and because $\rho \leq 2r$. Then the angle distance between these points satisfies $\angle_o(x^-, x^+) \leq \frac{r}{2\delta}$ by the choice of $r$.

It is clear that

$$\rho^{-1}(C) \subset \bigcup_{x \in E} A(x^-, x^+).$$

The complement $A_r \setminus \cup_{x \in E} A(x^-, x^+)$ consists of open intervals, whose number $\leq |E| + 1 \leq N_0 + 1$. It suffices to prove that at least one of those intervals has the angle diameter $\geq \sigma \cdot \angle_o(A_r)$. Assume that it is not the case, and every interval has the angle diameter $< \sigma \cdot \angle_o(A_r)$. Then

$$\angle_o(A_r) < (N_0 + 1) \cdot \sigma \cdot \angle_o(A_r) + \sum_{x \in E} \angle_o(x^-, x^+) \leq \frac{1}{2}(\angle_o(A_r) + \varepsilon).$$

However, this contradicts the condition $\angle_o(A_r) \geq \varepsilon$. \hfill \ensuremath{\Box}

Lemma 3.8. Let $\delta \in (0, 1/4]$, and let $N_0 = N_0(\delta)$ be the constant from Lemma 3.6, $\sigma \in (0, \frac{1}{2(N_0 + 1)})$, $\varepsilon > 0$. Then for every $N \in \mathbb{N}$ and every $r \geq r(\delta, \sigma^N \varepsilon)$ we have: if an arc $A_r \subset S_r$ has the angle diameter $\angle_o(A_r) \geq \varepsilon$, then there exists a segment $C \subset D_r$ such that the tree collection $\mathcal{T}(C)$, into which the segment separates the tree $D_r$, contains at least $N \sigma^N$-large trees.

Proof. The segment $C$ will be constructed inductively as the union of an increasing (in one direction) sequence of subsegments $C_0 \subset C_1 \subset \ldots$. We take as $C_0$ some extreme vertex of the tree $D_r$. Then the collection $\mathcal{T}(C_0)$ consists of one tree $P_1 = D_r$ which is, of course, $\sigma$-large. As $C_1$ we take the edge of the tree $P_1$ adjacent to $C_0$.

Assume that we have already constructed segments $C_0 \subset C_1 \subset \ldots \subset C_{k-1} \subset D_r$ and $\sigma$-large trees $P_1 \in \mathcal{T}(C_0), \ldots, P_{k-1} \in \mathcal{T}(C_{k-2})$ such that $P_i$ is a unique $\sigma$-large tree in $\mathcal{T}(C_{i-1})$ for all $i = 1, \ldots, k - 1$, and $P_{k-1} \cap C_{k-1}$
is the edge of the tree $P_{k-1}$ adjacent to the segment $C_{k-2}$. One can assume that

$$r(\delta, \sigma^N \varepsilon) \geq r(\delta, \sigma^{N-1} \varepsilon) \geq \cdots \geq r(\delta, \varepsilon).$$

Then by Lemma 3.7, the collection $\mathcal{T}(C_{k-1})$ contains at least one $\sigma$-large tree, and by the assumption of uniqueness, all such trees of that collection have one and the same common point with the segment $C_{k-1}$, which is its end (different from $C_0$ if $k \geq 2$). If the collection $\mathcal{T}(C_{k-1})$ also contains a unique $\sigma$-large tree $P_k$, then we take as $C_k$ the union of the segment $C_{k-1}$ and the edge of the tree $P_k$ adjacent to that segment. By construction, the union $C_k$ is a segment.

We assert that for some $k \geq 1$ the collection $\mathcal{T}(C_k)$ contains at least two $\sigma$-large trees. Indeed, the tree $D_r \subset T$ is a compact subset being the continuous image of the compact set $A_r$, $D_r = g_r(A_r)$. Thus it has only finitely many edges. If one assumes that the assertion is not true, then the procedure described above gives after a finite number of steps a segment $C \subset D_r$, that connects some extreme vertices of the tree $D_r$, such that the collection $\mathcal{T}(C)$ contains no $\sigma$-large trees. This contradicts Lemma 3.7.

Let $k \geq 1$ be the least integer for which the collection $\mathcal{T}(C_k)$ contains at least two $\sigma$-large trees. We denote by $Q_1$ one of them, and for another, $Q$, we choose a large connected component $A_1 \subset A_r$ of its preimage $g_r^{-1}(Q)$ and denote by $D_1 = g_r(A_1)$ the subtree $D_1 \subset Q$. Then $\angle_o(A_1) \geq \sigma \cdot \angle(A_r) \geq \sigma \varepsilon$.

At least one of the ends of the arc $A_1$ is an interior point in the arc $A_r$ (otherwise $A_1 = A_r$ and $D_1 = D_r$, which is impossible). Thus the tree $D_1 \subset Q$ has a common point with the segment $C_k$ (which is an extreme point for both). The condition

$$r \geq r(\delta, \sigma^N \varepsilon) \geq \cdots \geq r(\delta, \sigma \varepsilon)$$

allows to apply to $D_1$ the previous arguments and continue the construction of the segment $C$, starting with the end of the segment $C_k$, different from $C_0$. Every $\sigma$-large subtree $P \subset D_1$ w.r.t. $D_1$ is a $\sigma^2$-large w.r.t. $D_r$ since for a large connected component $A'$ of the set $g_r^{-1}(P) \subset A_1 \subset A_r$ its angle diameter $\angle_o(A') \geq \sigma \cdot \angle_o(A_1) \geq \sigma^2 \cdot \angle_o(A_r)$.

Since

$$r \geq r(\delta, \sigma^N \varepsilon),$$

the condition of Lemma 3.7 is satisfied at least for $N$ such steps, and we obtain a segment $C \subset D_r$ and different trees $Q_1, \ldots, Q_N \in \mathcal{T}(C)$, where the tree $Q_i$ is $\sigma^i$-large and hence $\sigma^N$-large w.r.t. $D_r$, $i = 1, \ldots, N$. \hfill $\square$

**Lemma 3.9.** Let $\delta, \sigma \in (0, 1)$, $r > 0$. Assume that an arc $A_r \subset S_r$ has a $\delta$-bounded winding at the scale $\sigma$, and $\text{diam} \ A_r \geq \max\{\frac{2\pi}{\sigma}, \frac{2\pi}{\delta}\}$. Then
for every segment $C \subset D_r = g_r(A_r)$ the collection $\mathcal{T}(C)$ contains at most $N_0 + 2$ subtrees which are $\sigma$-large, where $N_0 = N_0(\delta)$ is the constant from Lemma 3.6.

Proof. We fix a segment $C \subset D_r$. Let $A' \subset A_r$ be a connected component of the set $g_r^{-1}(P)$ for one of the trees $P \in \mathcal{T}(C)$. Note that $g_r(a) \in C$ for an end $a$ of the arc $A'$, which is an interior point in $A_r$. This follows from continuity of the map $g_r$. If the tree $P$ is large and $A'$ is a large component from its preimage, $\angle_o(A') \geq \sigma \angle_o(A_r)$, then the ends $a, a'$ of the arc $A'$ are separated in $X$ by the distance $|a - a'| \geq \delta \rho$, where $\rho = \text{diam } A_r$, according the property of a bounded winding of the arc $A_r$.

Assume that trees $P_1, P_2, P_3 \subset D_r$ are large and there exist connected components $A_1 \subset g_r^{-1}(P_1), A_2 \subset g_r^{-1}(P_2), A_3 \subset g_r^{-1}(P_3)$, such that the arc $A_2$ separates the arcs $A_1$ and $A_3$ on the arc $A_r$. Then the distance in $X$ between each end of the arc $A_1$ and each end of the arc $A_3$ is at least $\delta \rho$, because the pairs of the points above are the ends of subarcs in $A_r$ containing $A_2$ and hence having the angle diameter $\geq \angle_o(A_2) \geq \sigma \angle_o(A_r)$.

Therefore, any maximal collection $\mathcal{T}_0$ of large trees from the collection $\mathcal{T}(C)$, such that for every tree there is a large interior component of the preimage in $A_r$, and each two of them are separated by a large component of the preimage of some large tree (not necessarily from the collection $\mathcal{T}_0$), gives $2k$ points from $g_r^{-1}(C)$, which are pairwise $\delta \rho$-separated in $X$, where $k$ is the number of the trees of the collection $\mathcal{T}_0$. Since $\rho \geq \max\{\frac{\rho \delta}{\delta}, \frac{2m}{\delta}, \text{diam } g_r^{-1}(C)\}$, we have $2k \leq N_0$ according Lemma 3.6. On the other hand, it is clear that the number of the large trees in $\mathcal{T}(C)$ is at most $2k + 2$ (the additional factor 2 takes into account large trees which may not have large interior component in the preimage) and thus that number is $\leq N_0 + 2$. \qed

Proof of Theorem 3.2. Let $N_0 = N_0(\delta)$ be the constant from Lemma 3.6. We put

$$\sigma_0 = (2(N_0 + 1))^{-/(N_0+3)}, \quad r_0 = r(\delta, \sigma_0 \varepsilon),$$

where $r(\delta, \varepsilon)$ is the constant from Lemma 3.7. Then $\sigma_0 = \sigma_0(\delta)$ and $r_0 = r_0(\delta, \varepsilon)$.

Assume now that $r \geq r_0$ and an arc $A_r \subset S_r$ with the angle diameter $\angle_o(A_r) \geq \varepsilon$ is quasi-isometrically (and continuously) mapped into $T \times \mathbb{R}^n$. We let $\sigma = \frac{1}{2(N_0+1)}$. Then for $N = N_0 + 3$ we have $r \geq r(\delta, \sigma^N \varepsilon)$, i.e., for the arc $A_r$ the condition of Lemma 3.8 is satisfied, and $\sigma^N = \sigma_0$. According to that Lemma, there is a segment $C \subset D_r = g_r(A_r)$ such that the collection of trees $\mathcal{T}(C)$ contains at least $N$ trees which are $\sigma_0$-large.
Assume further that the arc $A_r$ has a $\delta$-bounded winding at the scale $\sigma_0$. By Lemma 3.9, the collection $T(C)$ contains at most $N_0 + 2$ trees which are $\sigma_0$-large. This is a contradiction with what we get above, since $N_0 + 2 < N$. Therefore, the arc $A_r$ cannot be quasi-isometrically mapped into $T \times \mathbb{R}^n$. 

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Sergei Buyalo, 
St. Petersburg Dept. of Steklov Math. Institute RAS, Fontanka 27, 191023 St. Petersburg, Russia 
buyalo@pdmi.ras.ru

Viktor Schroeder, 
Institut für Mathematik, Universität Zürich, Winterthurer Strasse 190, CH-8057 Zürich, Switzerland 
vschroed@math.unizh.ch