Correlations and superfluidity of a one-dimensional Bose gas in a quasiperiodic potential

Alberto Cetoli and Emil Lundh
Department of Physics, Umeå University, SE-90187 Umeå, Sweden

We consider the correlations and superfluid properties of a Bose gas in an external potential. Using a Bogoliubov scheme, we obtain expressions for the correlation function and the superfluid density in an arbitrary external potential. These expressions are applied to a one-dimensional system at zero temperature subject to a quasiperiodic modulation. The critical parameters for the Bose glass transition are obtained using two different criteria and the results are compared. The Lifshits glass is seen to be the limiting case for vanishing interactions.

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I. INTRODUCTION

Research on the problem of a Bose gas in a disordered potential gained momentum with the realization of well-controlled disordered potentials for ultracold bosonic atoms, which inspired a surge in theoretical and experimental activity [1–7], and the subsequent observation of Anderson localization by two groups [8, 9]. The quest is now on to better understand the effect of disorder on an interacting quantum system.

The phase diagram of an interacting Bose gas subject to a disordered potential was outlined in, e.g., Refs. [2–7, 10, 11]. While the exact picture of the phase diagram depends on the employed potential, the qualitative behavior appears to be the same for all kinds of disorder realizations. In absence of disorder, the gas is in a superfluid phase, which in a two- or three-dimensional Bose system at zero temperature is identical to a Bose-Einstein condensate (BEC), and in one dimension is a quasicondensate with suppressed density fluctuations and algebraically decaying correlations. Upon raising the disordered potential, the system can enter a new quantum phase: the Bose glass state, also called a fragmented Bose-Einstein condensate. The Bose glass state lacks long-range phase coherence, having exponentially decaying correlations and zero superfluid density, but it is compressible. In the weakly interacting limit, it has been conjectured that the Bose glass goes over into a Lifshits (or Anderson) glass [4, 6], where the density profile is that of a superposition of the lowest-lying single-particle states, which are exponentially localized. Finally, a gas of noninteracting bosons will have all the particles occupying the lowest of the single-particle states.

Experimentally, two basic types of disordered potential for ultracold atoms have been realized, both by optical means: speckle and quasiperiodic. The first type of disorder uses laser speckle, the two-dimensional diffraction pattern of a laser beam passing through a roughened plate. Speckle potentials for ultracold atoms were first used in Ref. [12]. A quasiperiodic potential in one dimension is created simply by superposing two standing waves with different wavelength; if the wavelengths differ by an irrational factor – the most popular being the golden ra-
properties is a key to understanding the behavior of the Bose gas; the superfluid properties is another. Both of these can be calculated in the Bogoliubov scheme, where the Gross-Pitaevskii equation gives the (quasi-) condensate density and superfluid velocity and the Bogoliubov equations yield the excitation spectrum and corrections to the density. The Bose glass transition in a Gaussian correlated potential was explored recently for strong interparticle potentials by Fontanesi et al. in Ref. [2]. The Bogoliubov scheme is usually derived starting from the assumption that the system is Bose-Einstein condensed and there exists a condensate wavefunction \( \Phi(r) \), defined either as the expectation value of the bosonic field operator or as the wavefunction for the single-particle mode that is occupied by a macroscopic fraction of the particles. Clearly, this assumption would appear to preclude the description of both quasicondensate and Bose glass, but, as we shall see, it turns out that the Bogoliubov scheme can in fact describe these states as well. The explanation is that BEC is convenient, but not a necessary requirement, for deriving the Bogoliubov equations; it suffices to make the weaker assumption that the quantum fluctuations in the density and gradient of the phase are small. Several papers have presented different sketches of the derivation of Bogoliubov theory for a quasicondensate [20–23]. In this article we put up what we believe to be a complete, simple and consistent derivation of the Bogoliubov equations in a Bose gas using a minimum of assumptions, in order to be able to analyze the quasicondensate - Bose glass transition in one dimension. Using the same formalism we derive an expression for the superfluid density in order to better understand the transition.

In this work we consider the experimentally relevant case of a 1D Bose gas in a quasiperiodic potential and examine a wide parameter regime, allowing for a discussion of the Bose glass transition and the conjectured Lifshits glass. It is important to stress that the Bogoliubov approximation has to be used with some care in one-dimensional systems. This problem has been treated in detail for a uniform gas by Lieb and Lininger [26], where it is shown that the Bogoliubov perturbation theory agrees with the exact answer for \( g/n < \hbar^2/2m \). Moreover, our numerical scheme is consistent only if the fluctuation in the phase and in the density are small, and this clearly cannot be the case upon increasing indefinitely the interaction within the particles. For this reason, due to the intrinsic limit of the Bogoliubov approach, we expect our results to be reliable only for weak interaction and high density. Within these limits, the application of the Bogoliubov recipe has been proven to be theoretically sound, and it is known to give the same correlation function of the Luttinger liquid theory, as shown in Ref. [27]. We wish to compare the different predictions for the phase transition obtained from analyzing the behavior of the correlation function and of the superfluid density.

As we shall see, our analysis suggests the existence of a glassy phase for small interparticle interaction. The plot of this phase appears as a series of peaks with little overlap. According to the language used in Ref. [3], this phase is called “fragmented BEC”, or Bose glass; in the same reference, the Anderson glass appears when the overlap between the peaks is negligible. We maintain these definitions through our article.

This paper is organized as follows. In Section II we present a derivation of Bogoliubov theory and in particular an expression for a correlation function. In Sec. III an expression for the superfluid density is derived. Sec. IV presents numerical results for the superfluid-Bose glass phase transition in a 1D Bose gas. Finally, in Sec. V we summarize and conclude.

### II. CORRELATION FUNCTION

Inspired by the works of Ho and Ma [22], and Xia and Silbey [25], we use a path integral formalism, considering a system of bosons described by the Euclidean action

\[
S[\psi^*, \psi] = \int d\tau d\mathbf{r} \psi^*(\mathbf{r}, \tau) \left( -\hbar \partial_\tau + \frac{\hbar^2}{2m} \nabla^2 - U(\mathbf{r}) + \mu - \frac{g}{2} |\psi(\mathbf{r}, \tau)|^2 \right) \psi(\mathbf{r}, \tau). \tag{1}
\]

where \( \psi \) is the scalar boson field. Assuming that the quantum fluctuations of density and gradient of the phase are small we expand \( \psi \) as

\[
\psi(\mathbf{r}, \tau) = e^{i \theta(\mathbf{r}, \tau)} \sqrt{n_0(\mathbf{r})} + \delta n(\mathbf{r}, \tau)
\]

\[
\approx e^{i \theta(\mathbf{r}, \tau)} \sqrt{n_0(\mathbf{r})} \left( 1 + \frac{1}{2} \frac{\delta n(\mathbf{r}, \tau)}{n_0(\mathbf{r})} - \frac{1}{8} \frac{\delta n^2(\mathbf{r}, \tau)}{n_0(\mathbf{r})^2} \right)
\]

\[
= \psi_0(\mathbf{r}, \tau) + \delta \psi(\mathbf{r}, \tau). \tag{2}
\]

The equation of motion for \( \psi_0 \) is found by means of the variational principle \( \delta S/\delta n_0^* = 0 \), and the result is the Gross-Pitaevskii equation for \( n_0 \) [28]

\[
- \frac{\hbar^2}{2m} \nabla^2 \sqrt{n_0(\mathbf{r})} + U(\mathbf{r}) \sqrt{n_0(\mathbf{r})} + g n_0(\mathbf{r})^2 = \mu \sqrt{n_0(\mathbf{r})}. \tag{3}
\]
This amounts to treating \( n_0(\mathbf{r}) \) as a scalar field representing the (quasi-) condensate density, and \( \delta n, \theta \) as perturbations. In a three-dimensional system, and in two dimensions at zero temperature, the condensate wavefunction \( \Phi \) would in the absence of currents be equal to the square root of the classical density, \( \sqrt{n_0} \). In principle one could describe a state with a current by assuming a slowly varying phase \( \theta_0 \) that is also treated classically, but we refrain from doing so for convenience.

Ignoring terms of order higher than \( \theta^2 \) and \( \delta n^2 \) the action becomes

\[
S = S_0 + S_1 + S_2.
\]  

Here, \( S_0 \) contains only the classical density \( n_0 \) and is minimized by the equation of motion for \( n_0 \), \( S_1 \) must vanish for \( n_0 \) to be a stationary solution, while \( S_2 \) is

\[
S_2 = \frac{1}{2} \int d\tau d\mathbf{r} \left( \frac{\delta n(\mathbf{r},\tau)}{\sqrt{2n_0(\mathbf{r})}} \right) S \left( \frac{\delta n(\mathbf{r},\tau)}{\sqrt{2n_0(\mathbf{r})}} \right),
\]

with

\[
S = \left( \begin{array}{ccc} \frac{-\hbar^2}{2m} \nabla^2 + U + 3g n_0 - \mu & -\hbar \partial_r & \frac{-\hbar^2}{2m} \nabla^2 + U + g n_0 - \mu \\ \hbar \partial_r & -\hbar \partial_r & \end{array} \right)
\]  

\[
= -\hbar \partial_r \sigma_1 + \left( \begin{array}{ccc} H_3 & 0 \\ 0 & H_1 \end{array} \right),
\]

where we have implicitly defined the scalar operators \( H_3 \) and \( H_1 \). In order to find the correlators between \( \delta n \) and \( \theta \) we need to find the function \( G \) that inverts \( S \),

\[
\left[ -\hbar \partial_r \sigma_1 + \left( \begin{array}{ccc} H_3 & 0 \\ 0 & H_1 \end{array} \right) \right] G(\mathbf{r},\tau,\mathbf{r}',\tau') = \delta(\mathbf{r} - \mathbf{r}').
\]

Of course, what we have derived so far is mathematically identical to the well-known Bogoliubov theory, as we now show. Introducing the transformation \( T \) as

\[
T = \frac{1}{\sqrt{2}} \left( \begin{array}{ccc} 1 & 1 \\ 1 & -1 \end{array} \right),
\]

we obtain

\[
T \mathcal{L} T^{-1} = \left( \begin{array}{ccc} 0 & H_3 \\ H_3 & 0 \end{array} \right),
\]

with

\[
\mathcal{L} = \left( \begin{array}{ccc} \frac{-\hbar^2}{2m} \nabla^2 + U + 2g n_0 - \mu & n_0 g \\ n_0 g & \frac{\hbar^2}{2m} \nabla^2 - U - 2g n_0 + \mu \end{array} \right).
\]

The diagonalization of \( \mathcal{L} \) leads to the Bogoliubov equations for the Bogoliubov amplitudes \( u_j(\mathbf{r}) \) and \( v_j(\mathbf{r}) \),

\[
\mathcal{L} \left( \begin{array}{c} u_j \\ v_j \end{array} \right) = E_j \left( \begin{array}{c} u_j \\ v_j \end{array} \right).
\]

Moreover, the Green function for the action \( S = i\hbar \partial_\theta + \mathcal{L} \) is already known \[29\], and equal to

\[
G(\mathbf{r},\mathbf{r}',\omega) = -\sum_{j \neq 0} \frac{\hbar}{\hbar \omega - E_j} \left( \begin{array}{ccc} u_j & \left( u_j^* \right)^\dagger \\ v_j & \left( v_j^* \right)^\dagger \end{array} \right).
\]

Defining \( \chi_j \) as

\[
\chi_j = \left( \begin{array}{c} \chi_j^1 \\ \chi_j^2 \end{array} \right) = T \left( \begin{array}{c} u_j \\ v_j \end{array} \right) = \left( \frac{\delta n_j}{\sqrt{2n_0}} \frac{\sqrt{2n_0}}{i \sqrt{2n_0} \theta_j} \right),
\]

\[
\sum_{j \neq 0} \frac{\hbar}{\hbar \omega - E_j} \left( \begin{array}{c} u_j \\ v_j \end{array} \right) = -\sum_{j \neq 0} \frac{\hbar}{\hbar \omega - E_j} \left( \begin{array}{c} u_j \\ v_j \end{array} \right).
\]

where \( \chi_j^1 \) and \( \chi_j^2 \) are the components of the matrix \( \chi_j \).
Furthermore let us define
\[ \tilde{\chi}_j = T \left( \frac{v_j^*}{u_j^*} \right), \] (14)
we can apply the transformation \( T \) to Eq. (11) and eventually find
\[ G(r, \tau, r', \tau') = \sum_{\omega_n} \frac{e^{i \omega_n \eta}}{\beta} G(r, r, \omega_n) \]

\[ = -\sum_{j \neq 0} \sum_{\omega_n} \frac{e^{i \omega_n \eta}}{\beta} \left[ \frac{1}{i \hbar \omega_n - E_j} \chi_j \chi_j^\dagger - \frac{1}{i \hbar \omega_n + E_j} \tilde{\chi}_j \tilde{\chi}_j^\dagger \right] \]

\[ = \sum_{j \neq 0} \chi_j \chi_j^\dagger N(E_j) + \tilde{\chi}_j \tilde{\chi}_j^\dagger (N(E_j) + 1) \]

\[ = \left( \sum_{j \neq 0} \frac{1}{2 \sqrt{n_0 n_0'}} \left( \delta n \delta n' - \sqrt{n_0} \frac{n_0'}{n_0} \delta n i \theta \right) - \sqrt{n_0} \frac{n_0'}{n_0} \delta n' \right), \] (15)

with \( \omega_n \) being the Matsubara frequencies, \( \eta \) a positive infinitesimal, and \( N(E_j) = 1/(e^{\beta E_j} - 1) \) is the Bose-Einstein distribution function. For convenience we have defined the notation

\[ \theta = \theta(r, \tau) \]
\[ \theta' = \theta(r', \tau') \]
\[ \delta n = \delta n(r, \tau) \]
\[ \delta n' = \delta n(r', \tau'). \] (16)

Furthermore let us define
\[ \Delta \theta = \theta' - \theta. \] (17)

Using Eq. (2) the one-body correlation function becomes
\[ \langle \psi^* (r) \psi(r') \rangle = \left( \sqrt{n_0 + \delta n} e^{-i \theta} e^{i \theta'} \sqrt{n_0' + \delta n'} \right) \]

\[ = \left( \sqrt{n_0 + \delta n} e^{i \Delta \theta} \sqrt{n_0 + \delta n'} \right) \]

\[ = \sqrt{n_0 n_0'} (e^{i \Delta \theta} + \frac{1}{2} \frac{\delta n}{n_0} e^{i \Delta \theta} + \frac{1}{2} e^{i \Delta \theta} \frac{\delta n'}{n_0'}) \]

\[ - \frac{1}{8} \left( \frac{\delta n}{n_0} \right)^2 e^{i \Delta \theta} - \frac{1}{8} e^{i \Delta \theta} \left( \frac{\delta n'}{n_0'} \right)^2 - \frac{1}{4} \frac{\delta n}{n_0} e^{i \Delta \theta} \frac{\delta n'}{n_0'} \right). \] (18)

This expression can be evaluated using Wick’s theorem. To lowest order in \( \delta n \) and \( \theta \) one finds \[ 24 \]
\[ \langle e^{i \Delta \theta} \rangle = e^{-\frac{1}{2} \langle (\Delta \theta)^2 \rangle} \]
\[ \langle \frac{\delta n}{n_0} e^{i \theta} \rangle = e^{-\frac{1}{2} \langle (\Delta \theta)^2 \rangle} \langle \frac{\delta n}{n_0} \frac{i \Delta \theta}{n_0} \rangle \]
\[ \langle \left( \frac{\delta n}{n_0} \right)^2 e^{i \Delta \theta} \rangle \approx e^{-\frac{1}{2} \langle (\Delta \theta)^2 \rangle} \left( \frac{\delta n}{n_0} \right)^2 \]
\[ \langle \frac{\delta n}{n_0} e^{i \Delta \theta} \frac{\delta n'}{n_0'} \rangle \approx e^{-\frac{1}{2} \langle (\Delta \theta)^2 \rangle} \langle \frac{\delta n}{n_0} \frac{\delta n'}{n_0'} \rangle, \] (19)

so that, eventually, to second order \[ 22 \]
\[ \langle \psi^* (r) \psi(r') \rangle = \sqrt{n_0 n_0'} e^{-\frac{1}{2} \langle (\Delta \theta)^2 \rangle} \left[ 1 + \frac{1}{2} \left( \frac{\delta n}{n_0} \frac{i \Delta \theta}{n_0} \right) + \left( \frac{\delta n'}{n_0'} \right) \right] \]

\[ + \frac{1}{4} \left( \frac{\delta n}{n_0} \frac{\delta n'}{n_0'} \right) \]

\[ - \frac{1}{8} \left( \left( \frac{\delta n}{n_0} \right)^2 \right) + \left( \frac{\delta n'}{n_0'} \right)^2 \right] \]. (20)
Since both \( \delta n \) and \( \theta \) are small, the expression between square brackets can be thought as a first order expansion of an exponential. Using Eqs. (13) and (14) the following expression is obtained,

\[
\ln g_1 (r, r') = \ln \psi^*(r) \psi(r') - \ln \sqrt{n n'} \\
= -\frac{1}{2} \sum_{j \neq 0} \left\{ \left| \frac{v_j}{\sqrt{n}} - \frac{v_j'}{\sqrt{n'}} \right|^2 + N_j \left[ \frac{u_j}{\sqrt{n}} - \frac{u_j'}{\sqrt{n'}} \right]^2 + \left| \frac{v_j}{\sqrt{n}} - \frac{v_j'}{\sqrt{n'}} \right|^2 \right\} \\
+ i \frac{1}{2} \sum_{j \neq 0} \left[ I(r, r') + N_j I(r, r') \right],
\]

where we used the shorthand \( N_j = N(E_j) \). The quantity \( i I(r, r') \) is purely imaginary, and equal to

\[
i I(r, r') = \left( \frac{v_j'}{\sqrt{n'}} - \frac{v_j}{\sqrt{n}} \right) + \left( \frac{u_j'}{\sqrt{n'}} - \frac{u_j}{\sqrt{n}} \right) \\
+ \left( \frac{v_j'}{\sqrt{n'}} - \frac{v_j}{\sqrt{n}} \right) + \left( \frac{u_j'}{\sqrt{n'}} - \frac{u_j}{\sqrt{n}} \right).
\]

If the excitation energies and the condensate wavefunction are real, then the Bogoliubov amplitudes \( u_j \) and \( v_j \) can also be chosen as real. The imaginary part in Eq. (22) therefore vanishes, and the resulting expression coincides with that found by Mora and Castin using a different formalism \(24\). In particular, for the uniform case the expressions for the Bogoliubov functions are known, leading to

\[
\ln g_1 = -\frac{1}{4} \sum_{k \neq 0} \left( 1 - \cos \left( \frac{k}{2} |r - r'| \right) \right)^2 \left[ |v_k|^2 + N_k \left| |u_k|^2 + |v_k|^2 \right] \right).
\]

In one dimension, Eq. (21) is an expression for the correlation in a Bose system which is ultraviolet and infrared convergent. Even at \( T = 0 \) the sum has a finite value without assuming a cutoff or a modification in the interparticle potential.

### III. SUPERFLUID DENSITY

According to the two-fluid model \(30, 31\), the mass density of a quantum fluid can be divided into a superfluid part \( \rho_s (r) \) and a normal one \( \rho_n (r) \), the total mass current being \( J(r) = \rho_s (r) v_s (r) + \rho_n (r) v_n (r) \). While the superfluid density is in general different from the condensate density, the superfluid velocity is the condensate velocity. In particular, upon imposing a phase twist on the condensate wavefunction, the superfluid part will be proportional to the additional kinetic energy.

It is in this sense that the superfluid density can be defined as a response to a twist of the order parameter \(32, 33\), by means of rewriting Eq. (2) as

\[
\psi(r) = \exp^{i \theta(r)} \sqrt{\bar{n}_0} \left( \exp^{i k_0 \cdot r} + \frac{1}{2} \frac{\delta n}{n_0} - \frac{1}{8} \frac{\delta n^2}{n_0^2} \right),
\]

with \( k_0 = \Theta \hat{e}_0 / L \), \( L \) being the length of the system in the direction of the unit vector \( \hat{e}_0 \), and \( \Theta \) a small twist angle. For convenience, let us take the order parameter normalized to unity; the superfluid density - in the direction of \( \hat{e}_0 \) - is then defined by the thermodynamic limit of

\[
\rho_s = \frac{2 L^2 m^2 N}{\hbar^2 \Theta^2 V} \left[ F^\Theta (\mu, T) - F^0 (\mu, T) \right]
\]

\[
= \frac{2 m^2 N}{\hbar^2 k_0^2 V} \left[ F^\Theta (\mu, T) - F^0 (\mu, T) \right].
\]

The substitution \(24\) results in the twisted action

\[
S^\Theta = S + \int dr dt \frac{\hbar^2}{2 m} \left[ k_0^2 n_0 (r, \tau) - 2 i k_0 \delta n (r, \tau) \nabla \theta (r, \tau) \right]
\]

\[
= S + k_0 \int d\tau V (\tau),
\]

(26)
where in the second line we ignored higher order terms in the fluctuating fields $\delta n$ and $\theta$. If the system is large enough ($L \gg 1$), then $V(\tau)$ can be seen as a perturbation; moreover, let us assume that the symmetry of the problem imposes that the odd terms in $k_0$ vanish in $F^\Theta$, since the cases with $k_0$ and $-k_0$ lead to the same physical situations. The new free energy can be computed using the linked cluster theorem (see, for example [34])

$$F^\Theta = F^0 + \sum_{n=1}^{\infty} \frac{k_0^2}{n! \beta \hbar^n} \int d\tau_1 ... d\tau^n (V(\tau_1) ... V(\tau^n))_{\text{connected}}$$

$$= F^0 + \frac{\hbar^2}{2m} k_0^2 - \frac{\hbar^4}{2m^2} k_0^2 \frac{1}{\beta} \int d\tau d\tau' \left[ \langle \delta n \delta n' \rangle \langle \nabla \theta \nabla \theta' \rangle + \langle \delta n \nabla \theta \rangle \langle \nabla \theta \delta n' \rangle \right] + O(k_0^4)$$

$$= F^0 + \frac{\hbar^2}{2m} k_0^2 V N \rho_s + O(k_0^4). \quad (27)$$

Here, the notation $\langle \ldots \rangle_{\text{connected}}$ stands, as usual, for a diagrammatic expansion where only connected diagrams are retained. The superfluid density can be found in terms of the Green functions $\tilde{G}$, defining

$$G(r_1, r_2, r_3, r_4, \tau, \tau') = G_{11}(r_1, \tau, r_3, \tau') G_{00}(r_2, \tau, r_4, \tau')$$

$$+ G_{10}(r_1, \tau, r_4, \tau') G_{01}(r_2, \tau, r_3, \tau'), \quad (28)$$

so that, by partial integration of Eq. (27),

$$\rho_s = \frac{N}{V} - \lim_{r_1 \to r_2 \to r_3 \to r_4} \frac{\hbar^2}{m \beta \hbar} \frac{N}{V} \int d\tau d\tau' \nabla_1 \nabla_2 \int d\tau d\tau' \tilde{G}(r, \tau', r_1, \tau, \tau')$$

$$- \lim_{r_1 \to r_2 \to r_3 \to r_4} \frac{\hbar^2}{m \beta \hbar} \frac{N}{V} \int d\tau d\tau' \left( \frac{\nabla \sqrt{n_0(r')}}{\sqrt{n_0(r')}} \right) \nabla_1 \int d\tau d\tau' \tilde{G}(r, \tau', r_1, \tau, \tau')$$

$$- \lim_{r_2 \to r_1 \to r_3 \to r_4} \frac{\hbar^2}{m \beta \hbar} \frac{N}{V} \int d\tau d\tau' \left( \frac{\nabla \sqrt{n_0(r')}}{\sqrt{n_0(r')}} \right) \nabla_2 \int d\tau d\tau' \tilde{G}(r, \tau', r_2, \tau, \tau')$$

$$- \frac{\hbar^2}{m \beta \hbar} \frac{N}{V} \int d\tau d\tau' \left( \frac{\nabla \sqrt{n_0(r')}}{\sqrt{n_0(r')}} \right) \left( \frac{\nabla \sqrt{n_0(r'')}}{\sqrt{n_0(r'')}} \right) \int d\tau d\tau' \tilde{G}(r, \tau', r, \tau, \tau'). \quad (29)$$

The evaluation of the averaged operators is done by summing over the Matsubara frequencies in expressions like

$$\frac{1}{\beta} \int_{0}^{\beta} d\tau \int_{0}^{\beta} d\tau' G_{\alpha\beta}(r_1, \tau, r_2, \tau') G_{\gamma\delta}(r_3, \tau, r_4, \tau')$$

$$= \frac{1}{\beta^2} \int d\tau d\tau' \sum_{m,n} \exp(i (\omega_n + \omega_m) (\tau - \tau')) G_{\alpha\beta}(r_1, r_2, \omega_n) G_{\gamma\delta}(r_3, r_4, \omega_m)$$

$$= \frac{1}{\beta^2} \sum_n G_{\alpha\beta}(r_1, r_2, \omega_n) G_{\gamma\delta}(r_3, r_4, -\omega_n)$$

$$= \sum_{i \neq j} \left[ \frac{N(E_i) + N(E_j)}{E_i + E_j} \chi_i^\alpha(r_1) \chi_i^\beta(r_2) \chi_j^\gamma(r_3) \chi_j^\delta(r_4) \right.$$

$$- \frac{N(E_i) - N(E_j)}{E_i - E_j} \chi_i^\alpha(r_1) \chi_i^\beta(r_2) \chi_j^\gamma(r_3) \tilde{\chi}^\delta_j(r_4)$$

$$- \frac{N(E_i) - N(E_j)}{E_i - E_j} \chi_i^\alpha(r_1) \chi_j^\beta(r_2) \chi_j^\gamma(r_3) \chi_j^\delta(r_4)$$

$$+ \frac{N(E_i) + N(E_j) + 1}{E_i + E_j} \chi_i^\alpha(r_1) \tilde{\chi}_i^\beta(r_2) \chi_j^\gamma(r_3) \tilde{\chi}_j^\delta(r_4) \right]$$

$$+ \sum_i \frac{2}{E_i} \left[ \chi_i^\alpha(r_1) \chi_i^\beta(r_2) \chi_i^\gamma(r_3) \chi_i^\delta(r_4) + \chi_i^\alpha(r_1) \chi_i^\beta(r_2) \chi_i^\gamma(r_3) \tilde{\chi}_i^\delta(r_4) \right.$$

$$+ \beta \sum_i N(E_i) (N(E_i) + 1) \left[ \chi_i^\alpha(r_1) \tilde{\chi}_i^\beta(r_2) \chi_i^\gamma(r_3) \chi_i^\delta(r_4) + \chi_i^\alpha(r_1) \chi_i^\beta(r_2) \chi_i^\gamma(r_3) \tilde{\chi}_i^\delta(r_4) \right], \quad (30)$$

with $\chi_i$ and $\tilde{\chi}_i$ defined in Eq. (13) and (14). In particular, for the uniform case, all the terms proportional to $\nabla \sqrt{n_0}$ vanish; moreover, since the expressions for the Bogoliubov functions are known analytically, it is possible to see that
only the last term of (30) survives, each term of the sum between square parenthesis giving a contribution of \( k^2 / 2 \). In one dimension one obtains

\[
\rho_s = \frac{N}{V} - \frac{\hbar^2}{mV} \sum_{k \neq 0} k^2 \frac{\partial N(E_k)}{\partial E_k},
\]

which is the well known Landau result [31].

IV. BOSE GLASS TRANSITION

We now apply our expressions for the correlation function and the superfluid density to a system at \( T = 0 \). The external potential is quasiperiodic, obtained as a sum of two potentials whose periods are incommensurate with each other,

\[
U(x) = V_1 \cos \left( \frac{2 \pi}{d} x \right) + V_2 \cos \left( \frac{2 \pi}{\lambda d} x \right).
\]

For our specific realization we have chosen \( \lambda \) to be the golden ratio, approximated as a fraction of two consecutive Fibonacci numbers. In the following we shall consider two systems with length \( L = 377d \) and \( L = 89d \); the value of \( \lambda \) is, respectively, \( \lambda = 377/233 \) and \( \lambda = 89/55 \).

For \( V_2 = 0 \), upon increasing the value of \( V_1 \), the system ceases to be superfluid and enters a Mott insulating phase [10, 35, 36]. As stated in the introduction, this phase cannot be seen using the Bogoliubov ansatz we employ for the excitations, and our method should not be applied when the physics of the system is affected by the presence of the Mott phase. However, when the second lattice is turned on a new quantum phase, the Bose glass, becomes possible. According to Ref. [10], when \( V_2 \) is greater than the interaction energy, the Mott phase disappears from the phase diagram of the gas, and the only insulating phase is the glass phase. We believe that in this situation of strong disorder our approach can be quantitatively correct, and for definiteness we have chosen the weights of the two potentials to be equal (\( V_1 = V_2 = V \)).

Taking \( d \) as the unit of length, the Gross-Pitaevskii equation can be rewritten as

\[
\left[ -\frac{1}{2} \frac{d^2}{dx^2} + g N |\Phi(x)|^2 + U(x) \right] \Phi(x) = \mu \Phi(x),
\]

where the energy is measured in units of \( E_d = \hbar^2 / m d^2 \), and the norm of the quasicondensate wavefunction \( \Phi \) is set to 1. For each value of the parameters \( V \) and \( g \), we have found the ground state of Eq. (33) using an imaginary time evolution with the split-operator method, along with periodic boundary conditions. Using the ground-state wavefunction the Bogoliubov excitations are found by means of a direct diagonalization of the Bogoliubov equations [11]. In all our calculations the Laplacian term is represented using the Fourier transform, while the derivative in the expression for superfluid density [29] is approximated using a finite difference expression.

The Bogoliubov analysis is expected to be relevant for values of the interparticle interaction \( g_n \ll 1 \). This means that our work is realistic for

\[
g_n \ll n^2,
\]

i.e., the approximation we are using starts to be meaningful for systems that contain a few particles per site.

We first solve for the ground state in a system with \( L = 377d \), using a numerical grid of 3016 points. Choosing a mean density \( n = N/L = 4 \), we show in Fig. 1 for the two cases \( V = 5E_d \) and \( V = 9E_d \). Upon decreasing \( g \), the condensate density develops dips that become more and more pronounced. Indeed, in some regions the condensate seems to be broken up into several pieces that hardly overlap. As we shall see, this has consequences for the behavior of the correlation function.

Using the same mean density \( n = N/L = 4 \), Fig. 2 plots the exponent of the correlation function

\[
\ln g_1(0, x) = \ln \langle \psi(0) \psi(x) \rangle - 1/2 \ln n_0(0)n_0(x).
\]

FIG. 1: (Color online) Density profiles for some values of \( g_n \) for \( V = 5E_d \) (top panel) and \( V = 9E_d \) (bottom panel); \( g_n \) is expressed in units of \( E_d = \hbar^2 / m d^2 \) and \( x \) in units of \( d \). Upon decreasing the interaction strength the condensate breaks up in sets of spikes with little overlap.
While for higher $gn$ the values of $\ln g_1$ follow a logarithmic behavior (insets of Fig. 2), leading to a power law decay in the correlation function, for lower $gn$ the function shows a linear fall, therefore giving an exponential decay. Since our system is finite the transition to a different decay behavior is gradual, but as a quantitative measure we record the point where a linear fit for $\ln g_1$ (Eq. (35)) gives a smaller error than a logarithmic fit, i.e., at the left side of the filled line the correlation function decays exponentially. The dashed line indicates the point at which the smallest excitation energies cannot be resolved numerically. The inset shows the sum of the residuals of the fits (in arbitrary units) as a function of $gn$, for $V = 9E_d$: the triangles refer to a linear regression, the diamonds to a logarithmic interpolation. $V$ and $gn$ are expressed in units of $E_d = h^2/m d^2$.

The interpretation of the phase diagram in Fig. 3 is the following: at the left side of the line there is an exponentially decaying correlation function, along with a vanishing superfluidity; we believe that at this point our Bogoliubov scheme detects the Bose glass phase, as described previously in Sec. 1. The Bose glass phase is seen to disappear completely below a finite value for the disorder strength $V$ (approximately $V = 3$), in contrast to the findings of Ref. [7]. This is the main difference with the results of Fontanesi et al... Besides finite size...
function $\Phi$, inverse participation ratio of the quasicondensate wavefunction, our work agrees with this conclusion.

Within the limit of the Bogoliubov approximation, Fig. (5) presents the behavior of the system for a finer resolution in the values of $g n$ and $V$ than Fig. (3). As we can see, the plots do not show any distinct transition. The glassy phase does not appear to be related to a radical change in the density profile. Moreover, as stated in Sec. I, a new phase, the Lifshits glass, has been conjectured to exist for weak interparticle potential, where the peaks in the density profile do not overlap. We argue that such a phase should have a distinctive signal in the quantity $P$. However, the plots of the inverse participation ratio do not show only a smooth decrease. We conclude that - within our approximation - the Lifshits glass does not seem to exist as a phase in its own right, but only as the limit of vanishing interaction.

V. CONCLUSIONS

Using a path integral formalism we have obtained – within the Bogoliubov approximation – expressions for the correlation function and the superfluid density in a Bose gas for an arbitrary external potential, at zero or finite temperature. Applying these expressions to a one-dimensional system at $T = 0$, a quasi-periodic external potential was seen to cause a transition to a phase where the correlation decays exponentially and the superfluid density vanishes. We believe this is the Bose glass phase. A comparison with the experiment of Fallani et al. [1] is not straightforward, because they employed a quasiperiodic potential with different relative weights [i.e., $V_1 \neq V_2$ in Eq. (32), and the proximity to the Mott insulating phase in the experiment makes the Bogoliubov approxi-
imation questionable. However, they spotted signs of a Bose glass transition when the peak height of the external potential varies between $2V \approx 16 E_d$ and $2V \approx 18 E_d$, with $g n \approx 1 E_d$, which is where the transition takes place in our study for the bigger system ($L = 377 d$).

We have not found a perfect match between the results for the superfluid density and the correlation behavior for the location of the Bose glass transition. We would like to point out that the formula for the superfluid density [29] assumes that the velocity $v_s$ of the condensate does not depend on the position, and in this sense it is an approximation even within the Bogoliubov approach. For this reason we plan to improve the expression for $\rho_s$ in the future.

Finally, we notice that our expression for the correlation function and the superfluid density are valid also at a finite temperature, and they can be used to study the Bose glass transition for $T \neq 0$. This relevant issue will be considered in a future work.

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