Generalized equation of relativistic quantum mechanics

A. A. Ketsaris
19-1-83, ul. Krasniy Kazanetz, Moscow 111395, Russian Federation
(February 13, 2017)

We develop a new concept of quantum mechanics which is based on a generalized space-time and on an action vector space similar to it. Both spaces are provided by algebraic properties. This allows to calculate the Dirac matrixes and to derive quantum mechanics equations from structure equations of the specified algebras. A new interpretation of the wave function is given as differential of the action vector. A generalization of the Dirac equation for 8-component wave function is derived. It is interpreted as the equation for two leptons of the same generation. A procedure of the approximate description of free leptons is formulated. The generalized equation of quantum mechanics is reduced to the Dirac, Pauli and Schrödinger equations by the sequential use of this procedure. We explain the existence of three lepton generations.

Our special purpose is to derive the relativistic quantum mechanics equations. It is assumed that such a derivation will lead to the understanding wave function through properties of generalized space-time and generalized action and will also introduce a new quality to the equations themselves.

1. The space-time, \( X \), is generalized up to \( \bar{X} \), space of all contravariant tensors over \( X \). The vector space \( X \) supplemented by the vector multiplication rule is algebra. The space \( \bar{X} \) is defined as a generalized space-time. It is assumed that the generalized space-time is a space of elementary particles. The Clifford algebra, \( \mathbb{C} \), is further selected from \( \bar{X} \). Its space is considered as a space of leptons.

2. Apart from of the generalized space-time \( \bar{X} \), a generalized conjugate space-time, \( \bar{X}_c \), is introduced as a set of all covariant tensors over \( X \). \( \bar{X}_c \) is also algebra. The conjugate space-time is identified with a space of elementary antiparticles. The space of the conjugate Clifford algebra \( \bar{\mathbb{C}} \) selected from \( \bar{X}_c \) is considered as a space of antileptons. For the algebra \( \bar{\mathbb{C}} \), as we show, the approximate regular representation of basis vectors is given by the Pauli and Dirac matrices.

3. The action is considered as the vector quantity. The action vectors form an algebra, \( S \), similar to \( X \). Moreover a space of conjugate action vectors, \( \bar{S} \), is introduced as similar to \( \bar{X}_c \). The spaces \( S \) and \( \bar{S} \) are related to elementary particles and antiparticles. The Clifford algebras \( S \mathbb{C} \) and \( \bar{S} \mathbb{C} \), selected from algebras \( S \) and \( \bar{S} \), are related with leptons and antileptons. They are similar to the Clifford algebras \( \mathbb{C} \) and \( \bar{\mathbb{C}} \), respectively.

4. The partial derivation of the multiplication rule in the algebras \( X \) and \( \bar{X} \), \( S \) and \( \bar{S} \) produce specific differential relations called structure equations. The structure equations for the Clifford algebras \( S \mathbb{C} \) and \( \bar{S} \mathbb{C} \) are reduced to the generalized equations of relativistic quantum mechanics for leptons and antileptons, respectively. The relativistic quantum mechanics equations are reduced to the Dirac equations with the simplified assumptions.
II. GENERALIZATION OF SPACE-TIME

A. Space-time

Consider the space-time $X$ as a vector space over the reals, $\mathbb{R}$. A vector $x \in X$ can be expressed through basis vectors

$$x = e_i x^i,$$

where $i = 1, 2, 3, 4$; $e_1$, $e_2$, $e_3$ are the basis vectors of geometric space, $e_4$ is the basis vector of time, and $x^i \in \mathbb{R}$ are vector coordinates.

Let us introduce an operation inverse to multiplication of vector by number. For each vector $a \in X$, there is a linear transformation of vector $x \in X$ onto $X$, which is called the scalar product of vectors $a$ and $x$ and is denoted by

$$\langle a, x \rangle \in \mathbb{R}.$$

The vectors $a$ and $x$ are orthogonal to each other if

$$\langle a, x \rangle = 0.$$

The scalar product of vector $x$ by itself defines its length square

$$\langle x, x \rangle = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2.$$

The metric tensor in $X$ is given by the scalar product of basis vectors:

$$g_{ik} \equiv \langle e_i, e_k \rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We omit hereafter zero matrix elements for convenience. The inverse metric tensor $g^{ik}$ is defined by condition

$$g^{ik} g_{kl} = \delta^i_l.$$

Let us introduce basis vectors $E^i \in X$ for which

$$\langle E^i, x \rangle = x^i.$$

The basis vectors $E^i$ are called conjugate with respect to basis vectors $e_i$. For conjugate basis vectors

$$\langle E^i, e_k \rangle = \delta^i_k.$$

The conjugate basis vectors $E^i$ are connected to the basis vectors $e_i$ by a relation:

$$E^i = e_k g^{ik}.$$

The conjugate vector

$$\tilde{x} = x_i E^i$$

can be put into correspondence with the vector $x = e_i x^i$. Here $x_i = \delta_{ik} x^k$ are coordinates of conjugate vector. The scalar product of vector $x$ by conjugate vector $\tilde{x}$ is

$$\langle x, \tilde{x} \rangle = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2.$$

A set of conjugate vectors forms the vector space over the reals which will be called a conjugate space-time and will be denoted by $\tilde{X}$.

B. Generalized space-time

Let us generalize the space-time $X$ up to the universal algebra of contravariant tensors, $\mathbb{X}$. For this purpose we introduce a tensor multiplication $\otimes$ for vectors $x_1 \in X$ and consider, in addition to vectors $x_1$, pairs $x_1 \otimes x_2$, triples $x_1 \otimes x_2 \otimes x_3$ and, in the common case, a set of $\otimes$-products of $p$ vectors, $x_1 \otimes x_2 \otimes \ldots \otimes x_p$, where $x_2, \ldots, x_p$ belong also to $X$. A space of vectors

$$x_1 \otimes x_2 \otimes \ldots \otimes x_p,$$

will be denoted by $X^p$. A vector $x \in X^p$ can be expressed using basis vectors

$$x = e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_p},$$

where

$$e_{i_1} \otimes e_{i_2} = e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_p}.$$

Let us introduce a vector space

$$\mathbb{X} = X^0 + X^1 + \ldots + X^p + \ldots,$$

where $X^0 = \mathbb{R}$, $X^1 = X$. The space $\mathbb{X}$ with the addition and multiplication of vectors is the universal algebra of contravariant tensors or the universal contravariant algebra. The vector $x \in X$ can be expressed through basis vectors

$$x = e_0 x^0 + e_{i_1} x^{i_1} + e_{i_2} x^{i_2} + \ldots + e_{i_m} x^{i_m} + \ldots,$$

Here the unit of reals $\mathbb{R}$ is designated by $e_0$. The vector space $\mathbb{X}$ will be called a generalized space-time.

For basis vectors of $\mathbb{X}$-space, the multiplication is given by

$$e_{i_{m+n}} \otimes e_{i_{m+n}} = e_{i_{m+n}} \otimes e_{i_{m+n}},$$

or more commonly,

$$e_{i_{m+n}} \otimes e_{i_{m+n}} = e_{i_{m+n}} \otimes e_{i_{m+n}} \cdot \delta^{i_{m+n}}_{i_{m+n}} = e_{i_{m+n}} \otimes e_{i_{m+n}} \cdot \delta^{i_{m+n}}_{i_{m+n}}.$$

An upper generalized index

$$I_m = i_1 i_2 \ldots i_m$$

and a lower generalized index

$$I_m = i_m \ldots i_m$$

are introduced to make this expression more compact. We can rewrite

$$e_{I_m} = e_{i_{m+n}} \cdot \delta^{i_{m+n}}_{i_{m+n}} = e_{i_{m+n}} \cdot \delta^{i_{m+n}}_{i_{m+n}}.$$

Then the rule of multiplication of basis vectors in $\mathbb{X}$-algebra takes the form

$$e_{I_m} \otimes e_{K_n} = e_{I_m+n} \cdot \delta^{I_m+n}_{I_m}.$$
Here the Kronecker deltas $\delta^{L+m+n}_{l+m,K}$ can be considered as structure constants of $X$-algebra.

Let us introduce an upper collective index $I$ running values

$$0, i_1, (i_1i_2), \ldots, (i_1i_2 \ldots i_n), \ldots$$

and a lower collective index $I$ running values

$$0, i_1, (i_2i_1), \ldots, (i_n \ldots i_2i_1), \ldots$$

Then the vector $x \in X$ can be rewritten in a compact form:

$$x = e_I \cdot x^I.$$  

**C. Subspace of generalized space-time**

In this section we study how subspace and subalgebra are selected from the generalized space-time.

In spite of the fact that the $X$-space is infinite-dimensional, we introduce formally its dimensionality, $N$. Let $q$ coordinates be expressed through $p = N - q$ the other coordinates for any vector $x \in X$:

$$x^{I(q)} = A^{I(q)}_{K(p)} \cdot x^{K(p)}.$$  

Here the collective index $I(q)$ runs index values of dependent coordinates, the collective index $K(p)$ runs index values of independent coordinates, and $A^{I(q)}_{K(p)}$ are connecting constants. Then vector

$$x = e_K \cdot x^K = e_{K(p)} \cdot x^{K(p)} + e_{I(q)} \cdot x^{I(q)} = \left(e_{K(p)} + e_{I(q)} \cdot A^{I(q)}_{K(p)} \right) x^{K(p)}$$

belongs to the subspace of $X$ with dimensionality $p$. This subspace will be denoted by $D$. Basis vectors of $D$ are

$$e_{K(p)} = e_{K(p)} + e_{I(q)} \cdot A^{I(q)}_{K(p)} = e_{I(N)} \cdot A^{I(N)}_{K(p)},$$

where we have introduced the notation

$$A^{I(N)}_{K(p)} = \begin{cases} 
\delta^{I(p)}_{K(p)}, \text{ for } I(N) = I(p); \\
\delta^{I(q)}_{K(p)}, \text{ for } I(N) = I(q). 
\end{cases}$$

Hereafter we use the large latin letters for indices of basis vectors and of coordinates in $D$-subspace as well as in $X$-space (for example, $\varepsilon_K$ instead of $\varepsilon_K(n_1)$). Thus it is meant that the indices in $D$-subspace accept only $p$ values.

We shall now find the condition such that the subspace of $X$ is also subalgebra of $X$. If $D$ is algebra with finite dimensionality then the multiplication rule of basis vectors $\varepsilon_K$ can be written as

$$\varepsilon_I \circ \varepsilon_K = \varepsilon_{L} \cdot C_{IK}^L.$$  

Here $C_{IK}^L$ are the structure constants or the parastrophic matrices of algebra $D$, the symbol "$\circ$" designates multiplication unlike $\otimes$-multiplication used for $X$-algebra. We establish connection between the multiplication rules in $D$ and in $X$. We rewrite the relation (2) as

$$\varepsilon_K = e_{K(n)} \cdot A^K_{K},$$

the summation is over all basis vectors of the space $X$. From it and from (2) we derive

$$\varepsilon_I \circ \varepsilon_K = e_{m} \cdot e_{n} \cdot A^{I(m)} \cdot A^{K(n)} = e_{m+n} \cdot \delta^{L+m+n}_{m+n} A^{I(m)} \cdot A^{K(n)} = e_{m+n} \cdot A^{L+m+n} \cdot C_{IK}^L.$$  

Hereof we obtain the relation between the structure constants of subalgebra $D$ and the relation constants:

$$\delta^{L+m+n}_{m+n} A^{I(m)} \cdot A^{K(n)} = A^{L+n+m} \cdot C_{IK}^L.$$  

This relation represents the condition when the subspace of generalized space-time is the subalgebra of $X$.

In a specific case the multiplication (2) determines the scalar product of basis vectors

$$\langle \varepsilon_I, \varepsilon_K \rangle = \varepsilon_{0} C_{IK}^0$$

and the metric tensor

$$g_{IK} = C_{IK}^0.$$  

Note also that

$$C_{0I}^L = \delta^L_I, \quad C_{00}^L = \delta^L_K.$$  

$D$ is algebra with division if for each vector $x \in D$ except a zero-vector, there is the inverse vector $x^{-1}$ that satisfies the relation

$$x \circ x^{-1} = \varepsilon_0.$$  

Or in the coordinate form

$$g_{IK} \cdot x^I (x^{-1})^K = 1.$$  

**D. Regular representation of subalgebra of generalized space-time**

The universal contravariant algebra $X$ is associative. Therefore its subalgebra $D$ is also associative. From the associativity of a subalgebra $D$

$$(\varepsilon_N \circ \varepsilon_I) \circ \varepsilon_K = \varepsilon_N \circ (\varepsilon_I \circ \varepsilon_K)$$

its regular representation follows. Using (3) we obtain
Herefrom

$$C_{LM}^{N1} = C_{ML}^{N1}.$$  

Comparing this expression with (3) we see that it is possible to put parastrophic matrices $C_{LM}^{N1}$ into a correspondence with the basis vectors $\varepsilon_{I}$. This correspondence is called the regular (joined) representation of the $\mathbb{D}$-algebra and is denoted as

$$\varepsilon_{I} \sim C_{LM}^{N1}.$$  

The number $I$ of parastrophic matrix is the index of basis vector that can be represented by this matrix.

### E. Generalized space-time of leptons

Consider the Clifford algebra as subalgebra of universal contravariant algebra $X$. As we shall see later, this algebra holds a central position in lepton physics. We shall define the Clifford algebra in two steps.

At first we define a contracted algebra, $\mathbb{R}$, as subalgebra of $X$ through the following conditions on coordinates of vectors:

$$x^{i_1 i_2 \ldots i_p (k_1 k_1)}_{p+1} i_{q+1} \ldots i_{q+1} (m m)_{q+1} \ldots i_n = x^{i_1 i_2 \ldots i_p} \prod_{l=1}^{m} g^{k_l k_l}.$$  

Here the $k$-indices in brackets are equal to each other, $m$ is the number of pairs of such indices; any two neighbouring $i$-indices are not equal to each other. Thus according to (3) the vector space of algebra $\mathbb{R}$ is built on the basis vectors

$$\varepsilon_{i_n \ldots i_2 i_1} = \varepsilon_{in \ldots i_2 i_1} + \sum_{m} \varepsilon_{i_n \ldots i_2 i_1} (k m k m) i_{q+1} \ldots i_{q+1} \prod_{l=1}^{m} g^{k_l k_l},$$

non-containing identical neighbouring indices. In particular,

$$\varepsilon_{i_2 i_1} = \varepsilon_{i_1} \varepsilon_{i_2} = \begin{cases} \varepsilon_{0} g_{i_1 i_2}, & \text{for } i_1 = i_2; \\ \varepsilon_{i_1} \varepsilon_{i_2} = \varepsilon_{11}, & \text{for } i_1 \neq i_2. \end{cases}$$  

Thus $\varepsilon_{i i} = C^{0}_{ii} \varepsilon_{0}$ is the scalar product $\langle \varepsilon_{i}, \varepsilon_{i} \rangle = (\varepsilon_{i})^2$, and $C_{ii}^{0}$ is the metric tensor $g_{ii}$. For example, if the $X$-space is one-dimensional and $(\varepsilon_{1})^2 = 1$ then the $\mathbb{R}$-space is constructed on the two basis vectors:

$$\varepsilon_{0} = \varepsilon_{0} + \varepsilon_{11} + \varepsilon_{1111} + \ldots + \varepsilon_{111111} + \ldots,$$

$$\varepsilon_{1} = \varepsilon_{1} + \varepsilon_{111} + \varepsilon_{111111} + \ldots + \varepsilon_{11111111} + \ldots,$$

Now we define the Clifford algebra $\mathbb{C}$ as subalgebra of $\mathbb{R}$ by the following system of linear equations:

$$x^\sigma(i_1 i_2 \ldots i_n) = -x^{i_1 i_2 \ldots i_n},$$

where $\sigma$ is the permutation of any two neighbouring distinct indices. For example,

$$x^{34142} = -x^{34142} = x^{31442} = x^{312} g^{44} = -x^{312}.$$  

Thus the Clifford algebra is built on basis vectors

$$\varepsilon_{0} = e_{0},$$
$$\varepsilon_{1} = e_{1},$$
$$\varepsilon_{121} = (e_{11} \otimes e_{11} - e_{12} \otimes e_{11}),$$
$$\ldots,$$

$$\varepsilon_{i_p \ldots i_2 i_1} = \sum_{\sigma} \varepsilon_{11111111} \sigma(\varepsilon_{11} \otimes e_{12} \otimes \ldots \otimes e_{i_p}),$$

non-containing indices with identical values. These basis vectors obey the multiplication rule

$$\varepsilon_{i_n \ldots i_2 i_1} \varepsilon_{i_p \ldots i_2 i_1} = \varepsilon_{i_n \ldots i_2 i_1} \varepsilon_{i_p \ldots i_2 i_1},$$

where $i$ enumerates distinct indices and $k$ enumerates conterminous indices of comultipliers; $\sigma_{i_p k_l}$ is the permutation of index $i_p$ with index $k_l$ in the second comultiplier, $\sigma_{k_l i_p+1}$ is the permutation of index $k_l$ with index $i_p+1$ in the first comultiplier. For basis vectors with distinct indices

$$\varepsilon_{i_n \ldots i_2 i_1} \varepsilon_{i_p \ldots i_2 i_1} = \varepsilon_{i_n \ldots i_2 i_1} \varepsilon_{i_p \ldots i_2 i_1}.$$  

The space with basis vectors $\varepsilon_{i_n \ldots i_2 i_1}$ will be denoted by $\mathbb{C}^{p}$. If the dimensionality of the initial space $X$ is denoted by $n$, the dimensionality of $\mathbb{C}^{p}$ is equal to number of combinations of $n$ things $p$ at a time

$$\mathbb{C}^{p} = \frac{n(n-1) \ldots (n-p+1)}{p!}.$$  

Therefore

$$\dim \mathbb{C}^{p} = \dim \mathbb{C}^{n-p}, \quad \dim \mathbb{C}^{n} = 1, \quad \dim \mathbb{C}^{n-1} = n.$$  

The space of the Clifford algebra $\mathbb{C}$ is a sum of spaces:

$$\mathbb{C} = \mathbb{C}^{0} + \mathbb{C}^{1} + \ldots + \mathbb{C}^{n},$$

where $\mathbb{C}^{0} = \mathbb{I} \mathbb{K}$, $\mathbb{C}^{1} = X$. The dimensionality of the Clifford algebra

$$N = \mathbb{C}^{0} + \mathbb{C}^{1} + \ldots + \mathbb{C}^{n} = (1 + 1)^n = 2^n.$$  

From here on the space $\mathbb{C}$ will be identified with a generalized space-time of leptons.

### F. Regular representation of Clifford algebra

We consider further parastrophic matrices that represent the basis vectors of the Clifford algebra $\mathbb{C}$.  

4
1. Product of Clifford algebras

First let us discuss the representation of product of Clifford algebras.

Hereinafter, when it is necessary to stress the dimensionality of Clifford algebra we shall use the notation $C_n$, where $n$ is the dimensionality of $X$.

The Clifford algebra $C_n$ can be written as the product $C_m \times C_{n-m}$. The representation of basis vectors of $C_n$ in the subalgebra $C_m$ over the field of hypernumbers forming algebra $C_{n-m}$ corresponds to this product. Consider such a representation for vector $x = \varepsilon_K \cdot x^K$. The basis vectors $\varepsilon_K$ can be written as

$$\varepsilon_K = \varepsilon_{k_2} \circ \varepsilon_{k_1} = \varepsilon_{d_1} C_{d_1 k_1 k_2} \cdot$$

where $\varepsilon_{k_1}$ are basis vectors of subalgebra $C_m$, $\varepsilon_{k_2}$ are basis vectors of subalgebra $C_{n-m}$, and $C_{d_1 k_1 k_2}$ are the parastrophic matrices of $C_{n-m}$ in $C_m$ over field of hypernumbers $C_{n-m}$. We assume here that these parastrophic matrices can be expressed through basis hypernumbers, $\varepsilon_{k_2}$, of field $C_{n-m}$ as

$$C_{d_1 k_1 k_2} = \delta_{d_1} \varepsilon_{k_2} \cdot$$

Then

$$\varepsilon_{k_2} \circ \varepsilon_{k_1} = \varepsilon_{k_1} \varepsilon_{k_2} \cdot$$ (6)

Thus the representation of vectors of algebra $C_n$ in the subalgebra $C_m$ over field of hypernumbers $C_{n-m}$ has the form

$$x = \varepsilon_K \cdot x^K = \varepsilon_{k_2} \circ \varepsilon_{k_1} x_{k_2 k_1} = \varepsilon_{k_1} \left( \varepsilon_{k_2} x_{k_2 k_1} \right) \cdot$$

This representation will be called complex or quaternion, when hypernumbers $\varepsilon_{k_2}$ are complex numbers or quaternions, respectively. The complex and quaternion representations considered below are convenient by compactness.

2. Classification of Clifford algebras

We shall now elaborate a classification of Clifford algebras by means of signatures of basis vectors.

Let us assign the basis vectors $\varepsilon_0$ and $\varepsilon_i$ to forming, and the remaining basis vectors to produced bearing in mind that these latter vectors are formed from $\varepsilon_0$ and $\varepsilon_i$ by $\circ$-multiplication.

The square of produced vector is expressed through the squares of forming vectors. For example,

$$\varepsilon_{i_2 i_1} \circ \varepsilon_{i_2 i_1} = - (\varepsilon_{i_2})^2 (\varepsilon_{i_1})^2$$

$$\varepsilon_{i_3 i_2 i_1} \circ \varepsilon_{i_3 i_2 i_1} = - (\varepsilon_{i_3})^2 (\varepsilon_{i_2})^2 (\varepsilon_{i_1})^2$$

$$\varepsilon_{i_4 i_3 i_2 i_1} \circ \varepsilon_{i_4 i_3 i_2 i_1} = + (\varepsilon_{i_4})^2 (\varepsilon_{i_3})^2 (\varepsilon_{i_2})^2 (\varepsilon_{i_1})^2 \cdot$$

But as $(\varepsilon_i)^2$ is equal either to $+\varepsilon_0$ or to $-\varepsilon_0$ the Clifford algebras may be classified by the signature of forming vector squares. Consider such a classification for several cases of the dimensionality $n$ of space $X$.

(a) $n = 0$, $N = 1$, $\varepsilon_A = \{\varepsilon_0\}$. The signature of square of $\varepsilon_0$ is

$$(+) \cdot$$

(b) $n = 1$, $N = 2$, $\varepsilon_A = \{\varepsilon_0, \varepsilon_1\}$. The two variants of the signatures of forming vector squares are possible:

$$(+, +), (+, -) \cdot$$

The last case is the algebra of complex numbers.

(c) $n = 2$, $N = 4$, $\varepsilon_A = \{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_21\}$. The three alternative sets of the signatures are possible:

$$(+, +, +, -)$$

$$(+, +, -)$$

$$(+, -) \cdot$$

In the last case the Clifford algebra is called the quaternion algebra.

(d) $n = 3$, $N = 8$, $\varepsilon_A = \{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_21, \varepsilon_13, \varepsilon_32, \varepsilon_123\}$. The possible variants of the signatures are

$$(+, +, +, +, +, +, +, +)$$

$$(+, +, +, +, +, +, - +, +)$$

$$(+, +, +, +, +, +, +, -)$$

$$(+, +, +, +, +, +, +, +) \cdot$$

The first case corresponds to the Clifford algebra constructed on the geometric space.

(e) $n = 4$, $N = 16$, $\varepsilon_A = \{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_21, \varepsilon_13, \varepsilon_32, \varepsilon_14, \varepsilon_24, \varepsilon_34,$

$$(\varepsilon_12, \varepsilon_124, \varepsilon_134, \varepsilon_224, \varepsilon_1234) \cdot$$

The possible variants of the signatures are

$$(+, +, +, +, +, +, +, +)$$

$$(+, +, +, +, +, +, +, +)$$

$$(+, +, +, +, +, +, +, +) \cdot$$

The second case corresponds to the Clifford algebra constructed on the space-time.

We now formulate the rule of calculation of parastrophic matrices for the Clifford algebra. This rule follows from the regular representation of basis vectors:

$$\varepsilon_i \sim C_{L K} = C_I(i) \cdot$$

Here $I$ is the parastrophic matrix number, $K$ is the column number, $L$ is the row number. In order to calculate element $C_{L K} I$, we proceed as follows: we evaluate the product $\varepsilon_K \circ \varepsilon_I$ by using the multiplication rule (the basis vector $\varepsilon_i$ is the right comultiplier); the index $L$ of the resulting basis vector $\varepsilon_L$ will be the row number for the desired element $C_{L K} I$, and the coefficient at the specified basis vector will determine its value.

Let us consider different representations of basis vectors of the Clifford algebra $\mathbb{C}$ when the initial space $X$ is: 1) the geometric space, 2) the space-time.
3. Clifford algebra on geometric space

The algebra $\mathbb{C}_3$ with signature $(+, +++, -, -, -)$ corresponds to this case. Let us choose the special order of indices:

$$(32, 13, 21, 0, 1, 2, 3, 123).$$

Such a choice is justified by that the parastrophic matrices of the conjugate Clifford algebra are represented by Pauli and Dirac matrices at this order of indices (see Section II B). Using the above rule one can obtain parastrophic matrices $C_{KI}^I$ (see Appendix A1).

The vector $x \in \mathbb{C}_3$ can be expressed as

$$x = \varepsilon_{13} \circ (\varepsilon_{21} x^{32} + \varepsilon_0 x^{13}) + \varepsilon_0 \circ (\varepsilon_{21} x^{21} + \varepsilon_0 x^0) + \varepsilon_2 \circ (\varepsilon_{21} x^1 + \varepsilon_0 x^2) + \varepsilon_{123} \circ (\varepsilon_{21} x^3 + \varepsilon_0 x^{123}).$$

This decomposition corresponds to the representation of algebra $\mathbb{C}_3$ as product $\mathbb{C}_2 \times \mathbb{C}_1$ and is complex. The basis vectors of $\mathbb{C}_2$ are $\varepsilon_{13}, \varepsilon_0, \varepsilon_2, \varepsilon_{123}$, the basis vectors of $\mathbb{C}_1$ are $\varepsilon_{21}, \varepsilon_0$. The direction 21, defining algebra $\mathbb{C}_1$, will be called basic. We put the basis vector $\varepsilon_{21}$ into correspondence with the imaginary unit $i$ bearing in mind that $(\varepsilon_{21})^2 = -1$. The complex representation of the basis vectors $\varepsilon_4$ corresponding to (8) is given by $4 \times 4$ matrices (see Appendix A1), where the basis units $I, i, a, b$ replace the following blocks:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It is significant that in the complex representation the matrix

$$C_{121}^{i1} = i \delta^{i1}_{121}$$

corresponds to the basis vector $\varepsilon_{21}$. As it will be shown in a forthcoming paper [1], this basic vector is closely connected with an consideration of interaction between the lepton and the electromagnetic field.

Note that $\varepsilon_{21}$ is the selected direction in the above representation. However the directions $\varepsilon_{13}$ and $\varepsilon_{32}$ are equivalent to the direction $\varepsilon_{21}$ from an algebraic point of view and can also be taken as basic. In order to distinguish these cases from the previous one, we denote imaginary unit by $j$ when $\varepsilon_{13}$ is taken as basic direction, and by $k$ when $\varepsilon_{32}$ is taken as basic direction. In these cases, the regular representation matrices differ from the matrices presented in Appendix A1 by replacement of the imaginary unit $i$ either on $j$, or on $k$.

On the other hand, the vector $x \in \mathbb{C}_3$ can be expressed in the form of quaternion representation

$$x = (\varepsilon_{32} x^{32} + \varepsilon_{13} x^{13} + \varepsilon_{21} x^{21} + \varepsilon_0 x^0) \circ \varepsilon_0 + (\varepsilon_{32} x^1 + \varepsilon_{13} x^2 + \varepsilon_{21} x^3 + \varepsilon_0 x^{123}) \circ \varepsilon_{123}.$$  

It corresponds to the representation of algebra $\mathbb{C}_4$ as product $\mathbb{C}_2 \times \mathbb{C}_2$. The basis vectors of $\mathbb{C}_1$ are $\varepsilon_0, \varepsilon_{123}$; the basis vectors of $\mathbb{C}_2$ are $\varepsilon_{32}, \varepsilon_{13}, \varepsilon_{21}, \varepsilon_0$. The quaternion representation of the basis vectors $\varepsilon_A$ corresponding to (8) is given by $2 \times 2$ matrices presented also in Appendix A1.

4. Clifford algebra on space-time

The Clifford algebra $\mathbb{C}_4$ with the signature

$$(+, +++, - , - - - + +, + + +, - )$$

corresponds to this case. Choose the special order of indices:

$$(32, 13, 21, 0, 42, 14, 1324, 34, 1, 2, 3, 123, 134, 234, 4, 124).$$

Now one can calculate the parastrophic matrices $C_{KI}^I$ by using the rule formulated above. They are presented in Appendix A2.

The complex representation of the basis vectors $\varepsilon_I$, corresponding to the decomposition:

$$x = \varepsilon_{13} \circ (\varepsilon_{21} x^{32} + \varepsilon_0 x^{13}) + \varepsilon_0 \circ (\varepsilon_{21} x^{21} + \varepsilon_0 x^0) + \varepsilon_2 \circ (\varepsilon_{21} x^1 + \varepsilon_0 x^2) + \varepsilon_{123} \circ (\varepsilon_{21} x^3 + \varepsilon_0 x^{123})$$

is given by $8 \times 8$ matrices where the above blocks are replaced by the basis units $I, i, a, b$ (see Appendix A2). The quaternion representation of the vectors $\varepsilon_I$, corresponding to decomposition:

$$x = (\varepsilon_{32} x^{32} + \varepsilon_{13} x^{13} + \varepsilon_{21} x^{21} + \varepsilon_0 x^0) \circ \varepsilon_0 + (\varepsilon_{32} x^1 + \varepsilon_{13} x^2 + \varepsilon_{21} x^3 + \varepsilon_0 x^{123}) \circ \varepsilon_{123}$$

is given by $4 \times 4$ matrices presented also in Appendix A3.

G. Generalized conjugate space-time. Conjugate Clifford algebra.

To describe the space structure of antiparticles we introduce a generalized conjugate space-time. For this purpose we generalize the conjugate space-time $\bar{X}$ introduced in Section II A up to universal algebra of covariant tensors $\bar{X}$ in line with Section II B. The vector $\bar{x} \in \bar{X}$ can be expressed through basis vectors

$$\bar{x} = x_0 E^0 + x_i E^{i1} + x_{i1} E^{i12} + \ldots$$

Here $E^0$ is the unit of reals $\mathbb{K}$. Note that the $\otimes$-multiplication in $\bar{X}$ differs from multiplication in $X$ by an order of multipliers.
The convolution of linear spaces $X$ and $\tilde{X}$ is generalized up to convolution of linear spaces $X$ and $\tilde{X}$. In particular the basis vectors $e_{k_n}...e_{k_1}$ and $E_{1}^{i_2...i_n}$ can be chosen so that

$$\langle E_{1}^{i_2...i_n}, e_{k_n}...e_{k_1} \rangle = \delta_{k_n}^{i_n}...\delta_{k_1}^{i_1}.$$

Let $\tilde{\mathbb{D}}$ be subalgebra of $\tilde{X}$. Then the rule of multiplication of the basis vectors is written as

$$E^I \circ E^K = C^{IK}E_L \cdot E^L,$$

where $C^{IK}E_L$ are the structure constants of the conjugate subalgebra $\tilde{\mathbb{D}}$. In a specific case, the multiplication $E^I \circ E^K$ defines the convolution

$$\langle E^I, E^K \rangle = C^{IK}_{0}E^0$$

and the inverse metric tensor $g^{I0} = C^{IK}_{0}$. Note also that $C^{0K}_{I} = C^{K0}_{I} = \delta_{I}^{K}$. From the condition of associativity of multiplication

$$E^I \circ (E^K \circ E^N) = (E^I \circ E^K) \circ E^N,$$

it follows that

$$C^{KN}E_L \cdot C^{IL}M = C^{IK}E_L \cdot C^{LN}M.$$  

This expression demonstrates the possibility of regular representation of basis vectors $E^I$.

$$E^I \sim C^{IL}M.$$  

The relation between the structure constants of algebra $\tilde{\mathbb{D}}$ and of algebra $\mathbb{D}$ is given by the operation of conjugation:

$$C^{RQ}_{LP} = g^{R1} \cdot g^{QK} \cdot C^{KL}_{1} \cdot g_{LP}.$$  

A conjugate Clifford algebra $\tilde{\mathbb{C}}$ can be defined as subalgebra of universal covariant algebra $\tilde{X}$ by analogy with the Clifford algebra $\mathbb{C}$. From here on the space $\tilde{\mathbb{C}}$ will be identified with a generalized space-time of antileptons. The basis vectors of $\tilde{\mathbb{C}}$ will be denoted by $\tilde{E}_{i}^{i_2...i_n}$. In particular, for the basis vectors with distinct indices

$$\tilde{E}_{i_1}^{i_2...i_n} \circ \tilde{E}_{i_2}^{i_3...i_n} = \tilde{E}_{i_1}^{i_2...i_n}.$$  

H. Regular representation of conjugate Clifford algebra. Pauli and Dirac matrices

From the regular representation of basis vectors

$$\tilde{E}^I \sim C^{IK}E_L \equiv C^{IL}_{K},$$

the calculation rule of parastrophic matrices for the conjugate Clifford algebra follows. Here $I$ is the parastrophic matrix number, $L$ is the column number, $K$ is the row number. In order to calculate element $C^{IK}L$, we proceed as follows: we evaluate the product $\tilde{E}^I \circ \tilde{E}^K$ by using the multiplication rule of the type (3) (the basis vector $\tilde{E}^I$ is the left comultiplier); the index $L$ of the resulting basis vector $\tilde{E}_L$ will be the column number for the desired element $C^{IK}L$, and the coefficient at the specified basis vector will determine its value. The resulting matrix must be multiplied by $(\tilde{E}^I)^2$.

Consider different representations of basis vectors of the conjugate Clifford algebra $\tilde{\mathbb{C}}$ when the initial space $\tilde{X}$ is: 1) the geometric space ($\tilde{X}_3 \equiv X_3$), 2) the conjugate space-time.

1. Conjugate Clifford algebra on geometric space

The algebra $\tilde{\mathbb{C}}_3$ with the signature $(+, +, +, −, −, −)$ corresponds to this case. Choose the index order applied previously: (32, 13, 21, 0, 1, 2, 3, 123). The parastrophic matrices $C^{K}_{L}I$ can be calculated as from the above rule so and from the relation (9). In conjugate Clifford algebra $\tilde{\mathbb{C}}_3$ the metric tensor is

$$g^{LP} \approx \begin{bmatrix}
1 & 3 & 32 & 21 & 0 & 1 & 2 & 3 & 123 \\
3 & 1 & 1 & 21 & 0 & 1 & 2 & 3 & 123 \\
32 & 21 & 0 & 1 & 2 & 3 & 123 & 1 & 1 & 21 \\
21 & 0 & 1 & 2 & 3 & 123 & 1 & 1 & 21 & 32 \\
0 & 1 & 2 & 3 & 123 & 1 & 1 & 21 & 32 & 21 \\
1 & 2 & 3 & 123 & 1 & 1 & 21 & 32 & 21 & 0 \\
2 & 3 & 123 & 1 & 1 & 21 & 32 & 21 & 0 & 1 \\
3 & 123 & 1 & 1 & 21 & 32 & 21 & 0 & 1 & 2 \\
32 & 123 & 1 & 1 & 21 & 32 & 21 & 0 & 1 & 2 \\
21 & 0 & 1 & 2 & 3 & 123 & 1 & 1 & 21 & 32 & 21 & 0 & 1 & 2 & 3 & 123
\end{bmatrix}.$$  

If we substitute this tensor and the parastrophic matrices of Clifford algebra $\mathbb{C}_3$ (see Appendix A3) in (9), we find the regular representation matrices of basis vectors for conjugate Clifford algebra $\tilde{\mathbb{C}}_3$.

The eight parastrophic matrices $C^{K}_{L}I$ of algebra $\tilde{\mathbb{C}}_3$ are written in the real, complex, and quaternion representations in Appendix A3. The complex representation is based on the decomposition

$$\tilde{x} = E^{13} \circ (x_{32} E^{21} + x_{13} E^{0}) + E^{0} \circ (x_{21} E^{21} + x_{0} E^{0}) + E^{2} \circ (x_{1} E^{21} + x_{2} E^{0}) + E^{123} \circ (x_{3} E^{21} + x_{123} E^{0}).$$

The quaternion representation is based on the decomposition

$$\tilde{x} = (x_{32} E^{32} + x_{13} E^{13} + x_{21} E^{21} + x_{0} E^{0}) \circ E^{0} + (x_{1} E^{32} + x_{2} E^{13} + x_{3} E^{21} + x_{123} E^{0}) \circ E^{123}.$$  

In quaternion representation the parastrophic matrices contain the Pauli matrices $\sigma_1$, $\sigma_2$, $\sigma_3$ as basis blocks.

Let us introduce matrices $\gamma_0$, $\gamma_1$, $\gamma_2$, $\gamma_3$, $\gamma_{21}$, $\gamma_{13}$, $\gamma_{32}$ and $\gamma_{123}$ through the relations

$$E^{0} = I \gamma_0, \quad E^{1} = i \gamma_1, \quad E^{2} = i \gamma_2, \quad E^{3} = i \gamma_3, \quad E^{21} = \gamma_1 \gamma_2 = \gamma_{12}, \quad E^{13} = \gamma_3 \gamma_1 = \gamma_{31}, \quad E^{32} = \gamma_2 \gamma_3 = \gamma_{23}, \quad E^{123} = (-i) \gamma_1 \gamma_2 \gamma_3 = (-i) \gamma_{123}.$$
These matrices
\[
\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma_{123} = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]
\[
\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
\]
\[
\gamma_{12} = i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \gamma_{31} = i \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma_{23} = i \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]
form the incomplete set of Dirac matrices describing only spatial basis vectors.

2. Conjugate Clifford algebra on conjugate space-time

The Clifford algebra \( \tilde{C}_4 \) with signature
\[
(-, +, +, +, -, -, +, +, -, +, +, -)
\]
corresponds to this case. The parastrophic matrices \( C_{IL_\tilde{p}} \) are calculated for the index order applied previously
\[
(32, 13, 21, 0, 42, 14, 1324, 34, 1, 2, 3, 123, 134, 234, 4, 124).
\]
The regular representation of basis vectors for conjugate Clifford algebra \( \tilde{C}_4 \) can also be obtained from relation (1) by using the metric tensor
\[
\tilde{g}^{LP} \sim \begin{array}{cccc|cccc}
32 & 13 & 21 & 0 & 42 & 1324 & 34 & 1 \\
13 & -1 & 1 & 0 & 1 & -1 & 1 & 0 \\
21 & 1 & -1 & 1 & 0 & 1 & -1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
42 & 1324 & 34 & 1 & 123 & 134 & 234 & 4 \\
1324 & -1 & 1 & 0 & 1 & -1 & 1 & 0 \\
34 & 1 & -1 & 1 & 0 & 1 & -1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
123 & 134 & 234 & 4 & 124 & 4 & 124 & 123 & 134 & 234 & 4 & 124
\end{array}
\]

16 parastrophic matrices of algebra \( \tilde{C}_4 \) are written in the real, complex, and quaternion representations in Appendix B. The complex representation of the basis vectors is based on decomposition of vector \( \tilde{x} \in \tilde{C}_4 \):
\[
\tilde{x} = \gamma_{13} \circ (x_{32} E_{21} + x_{13} E_0) + E_0 \circ (x_{21} E_{21} + x_{0} E_0) + E_0 \circ (x_{12} E_{21} + x_{14} E_0) + E_{34} \circ (x_{1324} E_{21} + x_{34} E_0) + E_2 \circ (x_{1} E_{21} + x_{2} E_0) + E_{123} \circ (x_{3} E_{21} + x_{123} E_0) + E_{234} \circ (x_{134} E_{21} + x_{234} E_0) + E_{124} \circ (x_{4} E_{21} + x_{124} E_0).
\]
It is given by \( 8 \times 8 \) matrices where blocks are replaced by the basis units 1 and \( i \).

The quaternion representation of vectors \( E^L \) corresponds to decomposition:
\[
\tilde{x} = (x_{32} E_{32} + x_{13} E_{13} + x_{21} E_{21} + x_{0} E_0) \circ E^0 + (x_{42} E_{42} + x_{14} E_{14} + x_{1324} E_{21} + x_{34} E_0) \circ E^{34} + (x_{1} E_{1} + x_{2} E_{2} + x_{3} E_{3} + x_{123} E_0) \circ E^{123} + (x_{134} E_{134} + x_{234} E_{234} + x_{124} E_0) \circ E^{124}.
\]
It is given by \( 4 \times 4 \) matrices.

I. Approximate representation of basis vectors

In this Section consider the regular representation of basis vectors of algebra \( C_n \) in its subalgebra \( C_{n-k} \) for \( k < n \). Such a representation will be called approximate. Further it will be used to obtain the Dirac and Pauli matrices.

For example, consider the approximate representation of basis vectors of algebra \( \mathbb{C}_3 \) in its subalgebra \( \mathbb{C}_{3-1} \).

Let us separate the basis vectors \( E^L \) of algebra \( \mathbb{C}_3 \) into two groups \( E^{I_1} \) and \( E^{I_2} \) with the same number of vectors so that the vectors \( E^{I_1} \) make up algebra. Because of symmetries of the Clifford algebra, the relations (3) take the form
\[
E^{I_1} \circ E^{K_1} = C^{I_1} C^{K_1} E^{L_1}, \quad (10)
\]
\[
E^{I_2} \circ E^{K_1} = C^{I_2} C^{K_1} E^{L_2}, \quad (11)
\]
\[
E^{I_1} \circ E^{K_2} = C^{I_1} C^{K_2} E^{L_2},
\]
\[
E^{I_2} \circ E^{K_2} = C^{I_2} C^{K_2} E^{L_1}.
\]

We assume that, approximately by calculating the representation matrices of basis vectors of \( \mathbb{C}_3 \) in \( \mathbb{C}_{3-1} \), the basis vectors \( E^{L_2} \) can be replaced by \( E^{L_1} \) through the relation:
\[
E^{L_2} = P^{L_2} E^{L_1},
\]
where \( P^{L_2} \) is the correspondence matrix. Then the relation (11) takes the form
\[
E^{I_2} \circ E^{K_1} = C^{I_2} C^{K_1} P^{L_2} E^{L_1}.
\]
From (10) and (12) the representation matrices of basis vectors of the algebra \( \mathbb{C}_3 \) in its subalgebra \( \mathbb{C}_{3-1} \) can be calculated. Note that the basis vectors of \( \mathbb{C}_{3-1} \) are exactly represented but the other basis vectors are approximately represented.

Similarly it is possible to consider the approximate representation of basis vectors of algebra \( C_n \) in its subalgebra \( C_{n-k} \), where \( k < n \).

1. First approximate representation

Consider the approximate representation of basis vectors of algebra \( \mathbb{C}_4 \) in its subalgebra \( \mathbb{C}_3 \) constructed on the
basis vectors $E^{32}, E^{13}, E^{21}, E^0, E^1, E^2, E^3, E^{123}$. For this purpose, we assume that by calculating the parastrophic matrices by (12) the basis vectors with indices

$42, 14, 1324, 34, 134, 324, 124$

are replaced by the basis vectors with indices

$32, 13, 21, 0, 1, 2, 3, 123$

respectively. Then the dimensionality of matrices of algebra $\bar{C}_4$ is reduced by half and equal to $8 \times 8$ for the real representation, $4 \times 4$ for the complex representation, and $2 \times 2$ for the quaternion representation. For example,

The matrices $\gamma$ make the full set of Dirac matrices. Thus the Dirac matrices correspond to the approximate representation of basis vectors of the conjugate Clifford algebra $\bar{C}_4$ in the conjugate algebra $\bar{C}_3$. Such a representation will be called the first approximate one and will be denoted by

$\bar{R}_1 : \bar{C}_4 \rightarrow \bar{C}_3 \{E^{32}, E^{13}, E^{21}, E^0, E^1, E^2, E^3, E^{123}\}$.

The approximate representation

$R_1 : C_4 \rightarrow C_3 \{E_{32}, E_{13}, E_{21}, E_0, E_1, E_2, E_3, E_{123}\}$

can be considered by an analogous way. As a result, we obtain

As a result, we obtain

The matrices of spatial vectors for the conjugate space-time in the approximate representation coincide with the exact matrices for the three-dimensional case.

Let us introduce matrices $\gamma_K$ in the correspondence with expressions:

$E^0 = \gamma_0, \ E^1 = i \gamma_1, \ E^2 = i \gamma_2, \ E^3 = i \gamma_3, \ E^4 = i \gamma_4,$

$E^{21} = \gamma_1 \gamma_2 = \gamma_{12}, \ E^{13} = \gamma_3 \gamma_1 = \gamma_{31}, \ E^{32} = \gamma_2 \gamma_3 = \gamma_{23}, \ E^{41} = \gamma_4 \gamma_1 = \gamma_{41},$

$E^{123} = (-i) \gamma_1 \gamma_2 \gamma_3 = (-i) \gamma_{123}, \ E^{124} = (-i) \gamma_1 \gamma_2 \gamma_4 = (-i) \gamma_{124},$

$E^{234} = (-i) \gamma_2 \gamma_3 \gamma_4 = (-i) \gamma_{234}, \ E^{314} = (-i) \gamma_3 \gamma_1 \gamma_4 = (-i) \gamma_{314}, \ E^{1324} = \gamma_1 \gamma_3 \gamma_2 \gamma_4 = \gamma_{1324}.$

The matrices $\gamma$ make the full set of Dirac matrices. Thus the Dirac matrices correspond to the approximate representation of basis vectors of the conjugate Clifford algebra $\bar{C}_4$ in the conjugate algebra $\bar{C}_3$. Such a representation will be called the first approximate one and will be denoted by

$\bar{R}_1 : \bar{C}_4 \rightarrow \bar{C}_3 \{E^{32}, E^{13}, E^{21}, E^0, E^1, E^2, E^3, E^{123}\}$.

The approximate representation

$R_1 : C_4 \rightarrow C_3 \{E_{32}, E_{13}, E_{21}, E_0, E_1, E_2, E_3, E_{123}\}$

can be considered by an analogous way. As a result, we obtain

$E^0 \sim I, \ E^{123} \sim i I,$

$E^1 \sim 1, \ E^2 \sim i -\sigma_2,$

$E^3 \sim i \sigma_1, \ E^4 \sim i \sigma_2,$

$E^{21} \sim i \sigma_3, \ E^{13} \sim 1 \sigma_3,$

$E^{32} \sim i \sigma_1, \ E^{14} \sim 1 \sigma_1,$

$E^{42} \sim i \sigma_2, \ E^{34} \sim 1 \sigma_2,$

$E^{123} \sim 1 \sigma_1, \ E^{124} \sim i \sigma_1,$

$E^{234} \sim 1 \sigma_2, \ E^{314} \sim i \sigma_2.$

$E^0 \sim I, \ E^{1324} \sim i I,$

$E^1 \sim 1, \ E^2 \sim i -\sigma_2,$

$E^3 \sim i \sigma_1, \ E^4 \sim i \sigma_2,$

$E^{21} \sim i \sigma_3, \ E^{13} \sim 1 \sigma_3,$

$E^{32} \sim i \sigma_1, \ E^{14} \sim 1 \sigma_1,$

$E^{42} \sim i \sigma_2, \ E^{34} \sim 1 \sigma_2,$

$E^{123} \sim 1 \sigma_1, \ E^{124} \sim i \sigma_1,$

$E^{234} \sim 1 \sigma_2, \ E^{314} \sim i \sigma_2.$

2. Second approximate representation

Consider the representation

$\bar{R}_2 : \bar{C}_4 \rightarrow \bar{C}_2 \{E^{32}, E^{13}, E^{21}, E^0\}$

which will be called the second approximate representation. For this purpose let’s assume that, by calculating the parastrophic matrices by (12), the basis vectors with indices $(42, 14, 1324, 34), (134, 234, 4, 124)$, and $(1, 2, 3, 123)$ are replaced by the basis vectors with indices $(32, 13, 21, 0)$, respectively. Then the dimensionality of matrices of basis vectors of $\bar{C}_4$ is reduced by half with respect to the first approximate representation and is equal to $4 \times 4$ for the real representation, $2 \times 2$ for the complex representation, and $1 \times 1$ for the quaternion representation.

For the regular representation of basis vectors of the conjugate space-time in the algebra $\bar{C}_2$ we have
As a result, the basis vectors of algebra $\mathbb{C}_4$ are represented as
\[
\begin{align*}
\varepsilon_0 &= I \mathbb{I}, & \varepsilon_{21} &= i \mathbb{I}, & \varepsilon_{34} &= i \mathbb{I}, \\
\varepsilon_2 &= a I, & \varepsilon_{13} &= b I, & \varepsilon_{123} &= I \mathbb{I}, \\
\varepsilon_{23} &= b I, & \varepsilon_{32} &= i I, & \varepsilon_{124} &= I \mathbb{I}, \\
\varepsilon_{34} &= i I, & \varepsilon_{14} &= a I, & \varepsilon_{134} &= i I, \\
\varepsilon_{4} &= i \mathbb{I}, & \varepsilon_{42} &= b I, & \varepsilon_{234} &= b I, \\
\varepsilon_{1324} &= i \mathbb{I}.
\end{align*}
\]

3. Third approximate representation

Consider the representation
\[
\tilde{R}_3 : \tilde{\mathbb{C}}_4 \rightarrow \tilde{\mathbb{C}}_1 \{\mathcal{E}^{21}, \mathcal{E}^{0}\},
\]
which will be called the third approximate representation. For this purpose let us assume that the basis vectors with indices $(42, 14)$, $(1324, 34)$, $(134, 234)$, $(4, 124)$, $(1, 2)$, $(3, 123)$, $(32, 13)$ are replaced by the basis vectors with indices $(21, 0)$, respectively. Then the dimensionality of matrices of basis vectors of $\mathbb{C}_4$ is reduced by half with respect to the second approximate representation and is equal to $2 \times 2$ for the real representation, $1 \times 1$ for the complex representation. As a result, the basis vectors of algebra $\tilde{\mathbb{C}}_4$ are represented as
\[
\begin{align*}
\mathcal{E}^0 &= I, & \mathcal{E}^1 &= -i, & \mathcal{E}^2 &= i, & \mathcal{E}^3 &= i, \\
\mathcal{E}^4 &= i, & \mathcal{E}_{12} &= 1, & \mathcal{E}_{13} &= -i, & \mathcal{E}_{14} &= i, \\
\mathcal{E}_{23} &= i, & \mathcal{E}_{24} &= 1, & \mathcal{E}_{34} &= -i, & \mathcal{E}_{1324} &= i.
\end{align*}
\]

In the approximate representation
\[
R_3 : \mathbb{C}_4 \rightarrow \mathbb{C}_1 \{\varepsilon_{21}, \varepsilon_0\}
\]
the basis vectors of algebra $\mathbb{C}_4$ are written in the form
\[
\begin{align*}
\varepsilon_0 &= I, & \varepsilon_1 &= a, & \varepsilon_2 &= b, & \varepsilon_3 &= i, \\
\varepsilon_4 &= i, & \varepsilon_{21} &= i, & \varepsilon_{13} &= b, & \varepsilon_{32} &= a, \\
\varepsilon_{14} &= b, & \varepsilon_{42} &= a, & \varepsilon_{34} &= I, & \varepsilon_{123} &= I, \\
\varepsilon_{124} &= I, & \varepsilon_{134} &= a, & \varepsilon_{234} &= b, & \varepsilon_{1324} &= i.
\end{align*}
\]

J. Derivation of vectors of generalized space-time.

Structure equations

The existence of the structure equations is the important feature of derivation of algebras. It is connected to derivation of the multiplication rule for vectors. We shall consider the structure equations for the algebra $\mathbb{D}$ being the subalgebra of contravariant universal algebra $\mathbb{X}$.

Consider vectors $x, x_1, x_2 \in \mathbb{D}$ connected by the multiplication rule:
\[
x = x_1 \circ x_2.
\]
From (3) the coordinate form of the multiplication rule follows
\[ x^K = C^K_{LI} (x^I)_L. \]

Use the inverse vector \( x^{-1} \) for which the condition (1) is fulfilled. We obtain for the inverse vector:
\[ x^{-1} = (x_1 \circ x_2)^{-1} = (x_2)^{-1} \circ (x_1)^{-1}. \] (14)

Let us introduce a differential operator \( \delta \) acting on the right expression. Consider the differential \( \delta x \). We shall distinguish differentials \( \delta_1, \delta_2, \ldots \) by index, the differential of vector \( x \) by variation of vector \( x_p \) will be denoted by \( \delta_p x \). From (3) follows
\[ \delta x_1 = \delta_1 x \circ (x_2)^{-1}, \quad \delta x_2 = (x_1)^{-1} \circ \delta_2 x. \] (15)

Let us introduce the second differential \( \delta_2 \delta_1 x \). From (3) the second differential is written as
\[ \delta_2 \delta_1 x = \delta_1 x \circ \delta_2 x. \]

Using (15) and (14) we get
\[ \delta_2 \delta_1 x = \delta_1 x \circ (x)^{-1} \circ \delta_2 x. \] (16)

In the neighbourhood of the algebra unit, i.e., when \( x = (x)^{-1} = \varepsilon_0 \), the relation (14) takes the form
\[ \delta_2 \delta_1 x = \delta_1 x \circ \delta_2 x. \] (17)

This relation is the structure equation of algebra \( \mathbb{D} \) in the vector form. If we substitute in (17) differentials expressed through basis vectors \( \delta x = \varepsilon I_{\delta} x(I) \) and use the multiplication rule for basis vectors (3), we obtain the structure equations in the coordinate form
\[ \delta_2 \delta_1 x^L = C^L_{JI} \cdot \delta_2 x^J \circ \delta_1 x^K. \]

The previous considerations are readily generalized to the differential of \( n \)-th order \( \delta_n \delta_{n-1} \ldots \delta_2 \delta_1 x \). The common structure equation has the form
\[ \delta_n \delta_{n-1} \ldots \delta_2 \delta_1 x = \delta_1 x \circ \delta_2 x \circ \ldots \circ \delta_{n-1} x \circ \delta_n x. \]

III. RELATIVISTIC QUANTUM MECHANICS

EQUATIONS AND LEPTONS

A. Generalized action vector. The quantization equations in differentials

In this section we shall generalize the notion of action. We suppose that the action is vector instead of scalar. A space of action vectors will be denoted by \( \mathbb{S}X \). We also assume that the space \( \mathbb{S}X \) is similar to the space \( \mathbb{X} \) bearing in mind that the basis vectors of \( \mathbb{X} \) can be accepted as the basis vectors in the space \( \mathbb{S}X \). Thus the action vector \( S \in \mathbb{S}X \) can be written as \( S = e_K \cdot S^K \). We endow the coordinates \( S^K \) with the dimensionality \([\text{erg} \times s] \) in contrast to dimensionless coordinates of vector of \( \mathbb{X} \). Then the scalar component of this vector, \( e_0 \cdot S^0 \), is the action in a classical sense.

The set \( \mathbb{S}X \) is algebra as well as \( \mathbb{X} \) with the same multiplication rule for basis vectors. The multiplication rule for vectors in algebra \( \mathbb{S}X \) can be written in form
\[ S = -\frac{1}{S^0} S_1 \circ S_2, \]
where \( S, S_1, S_2 \in \mathbb{S}X, S^0 \) is a constant with the dimensionality of action used for the agreement of dimensionalities of the right and left sides of equation.

For algebra \( \mathbb{S}X \) as well as for algebra \( \mathbb{X} \), there are structure equations which can be written as follows
\[ \delta_2 \delta_1 S = -\frac{1}{S^0} \delta_1 S \circ \delta_2 S. \] (18)

These equations are similar to the equations (17). Or in the coordinate form
\[ \delta_2 \delta_1 S^L = -\frac{1}{S^0} C^L_{IJR} \cdot \delta_2 S^R \cdot \delta_1 S^L. \] (19)

We shall consider the vector \( S \) as a function of vector \( x \in \mathbb{X} \): \( S = S(x) \).

In the equations (18) and (19), let us introduce the notation
\[ \psi = \delta_1 S \]
and the notation \( d \) for the differential \( \delta_2 \). In the new notations the structure equation takes the form
\[ d\psi = -\frac{1}{S^0} \psi \circ dS. \]
The vector \( \psi \) will be identified with a wave function. The structure equations in the wave function coordinates \( \psi^L = \delta_1 S^L \):
\[ d\psi^L = -\frac{1}{S^0} C^L_{IJR} \cdot dS^R \cdot \psi^L \]
will be called quantization equations in differentials.

Express the differential \( dS \) as
\[ dS = \partial_M S \cdot dx^M \]
and introduce generalized impulses as
\[ p_M = -\partial_M S = -e_R \cdot \partial_M S^R = e_R \cdot p^R_M. \]

Here
\[ p^R_M = -\partial_M S^R \]
are the coordinates of generalized impulses. From the quantization equations in differentials, it follows that
\[ \partial_M \psi^L(x) = \frac{1}{S^0} C^L_{IJR} \cdot p^R_M \cdot \psi^L. \] (20)

These relations will be called quantum postulates.
B. Relativistic quantum mechanics equations

Consider the quantum postulates (20). We multiply these relations by the structure constant $C^{MK}_{I}$:

$$C^{MK}_{I} \cdot \partial_{m} \psi^{l}(x) = \frac{1}{S_{0}} C^{MK}_{I} \cdot C^{LR}_{I} \cdot p^{R}_{M} \cdot \psi^{L}.$$  \hspace{1cm} (21)

These equations will be called relativistic quantum mechanics equations in Dirac's form.

Further we switch from the spaces $X$ and $SX$ to them subspaces $C_{4}$ and $SC_{4}$. Thus the parastrophic matrices for the Clifford algebras $C_{4}$ and $SC_{4}$ will be used as $C^{LR}_{I}$ and $C^{MK}_{I}$. In this case the components of wave function, $\psi^{l}(x)$, are sixteen real functions. We set $S_{0} = \hbar$, the Plank constant. Let also the wave function $\psi(x)$ depend only on coordinates of the usual space-time $X$. Moreover we assume that the generalized impulse $p^{R}_{M}$ has only two components

\begin{align*}
p^{0}_{0} &= -\partial_{0} S^{0} = \frac{m c}{2}, \\
p^{34}_{0} &= -\partial_{0} S^{34} = \frac{m c}{2}.
\end{align*}

Then we obtain

$$C^{MK}_{I} \partial_{m} \psi^{l} = \frac{m c}{2 \hbar} (\delta^{K}_{L} + C^{K}_{L34}) \psi^{L}.\,$$

As we shall see later, these relations can be considered as the relativistic quantum mechanics equations for free leptons.

Rewrite these equations for the representation of algebra $C_{4}$ in the subalgebra $C_{n}$ ($n < 4$) over the field of hypernumbers $C_{4-n}$ in the vector form:

$$C^{MK}_{I} \cdot \partial_{m} \psi^{l} \cdot \varepsilon_{K} = \frac{m c}{2 \hbar} (\delta^{K}_{L} + C^{K}_{L34}) \psi^{L} \cdot \varepsilon_{K}.$$  \hspace{1cm} (22)

In this expression we represent the basis vector $\varepsilon_{K}$ as the product:

$$\varepsilon_{K} = \varepsilon_{k_{2}} \circ \varepsilon_{k_{1}},$$

where $\varepsilon_{k_{1}}$ are the basis vectors of subalgebra $C_{n}$, and $\varepsilon_{k_{2}}$ are the basis vectors of subalgebra $C_{4-n}$. We next pass from the basis vectors $\varepsilon_{k_{2}}$ to the basis numbers $\xi_{k_{2}}$ in the correspondence with (1) and to the parastrophic matrices $C^{mk_{1}k_{2}}_{I}$, expressed through these numbers. As a result, we obtain

$$C^{mk_{1}k_{2}}_{I} \partial_{m} \Psi^{l} = \frac{m c}{2 \hbar} (\delta^{k_{1}k_{2}} + C^{k_{1}k_{2}}_{I34}) \Psi^{l},\,$$

where $\Psi^{l} = \xi_{k_{2}} \psi^{l}_{k_{2}}$.

In the complex representation

\begin{align*}
\xi_{i2} &= \{ I, i \}, \\
\varepsilon_{ni} &= \{ \varepsilon_{13}, \varepsilon_{0}, \varepsilon_{34}, \varepsilon_{14}, \varepsilon_{2}, \varepsilon_{123}, \varepsilon_{234}, \varepsilon_{124} \}
\end{align*}

the components of wave function are complex: \begin{align*}
\psi^{13} &= i \psi^{32} + \psi^{13}, \\
\psi^{14} &= i \psi^{42} + \psi^{14}, \\
\psi^{2} &= i \psi^{1} + \psi^{2}, \\
\psi^{234} &= i \psi^{134} + \psi^{234},
\end{align*} \hspace{1cm} (23)

In the quaternion representation

\begin{align*}
\xi_{i2} &= \{ 1, i, 1, 1 \}, \\
\varepsilon_{ni} &= \{ \varepsilon_{0}, \varepsilon_{34}, \varepsilon_{123}, \varepsilon_{124} \}
\end{align*}

the components of wave function are quaternion:

\begin{align*}
\Psi^{0} &= a I \psi^{32} + b I \psi^{13} + i \psi^{21} + \psi^{0}, \\
\Psi^{34} &= a I \psi^{42} + b I \psi^{14} + i \psi^{1324} + \psi^{34}, \\
\Psi^{123} &= a I \psi^{1} + b I \psi^{2} + i \psi^{3} + \psi^{123}, \\
\Psi^{124} &= a I \psi^{134} + b I \psi^{234} + i \psi^{4} + \psi^{124}.
\end{align*} \hspace{1cm} (24)

Let us write the quantum mechanics equations for the quaternion components:

\begin{align*}
\begin{pmatrix}
i & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{pmatrix} & \begin{pmatrix}
\partial_{0} \\
\partial_{1} \\
\partial_{2} \\
\partial_{3}
\end{pmatrix}
= \begin{pmatrix}
\varepsilon_{0} \\
\varepsilon_{34} \\
\varepsilon_{123} \\
\varepsilon_{124}
\end{pmatrix} \\
= \frac{m c}{2 \hbar} \begin{pmatrix}
\psi^{0} \\
\psi^{34} \\
\psi^{123} \\
\psi^{124}
\end{pmatrix}&
\end{align*}

Or

\begin{align*}
i \partial_{0} \psi^{124} - i \sigma^{a} \partial_{a} \psi^{123} &= \frac{m c}{2 \hbar} (\psi^{0} + \psi^{34}), \\
i \partial_{0} \psi^{123} - i \sigma^{a} \partial_{a} \psi^{124} &= \frac{m c}{2 \hbar} (\psi^{0} + \psi^{34}), \\
i \partial_{0} \psi^{34} + i \sigma^{a} \partial_{a} \psi^{0} &= \frac{m c}{2 \hbar} (\psi^{123} + \psi^{124}), \\
i \partial_{0} \psi^{0} + i \sigma^{a} \partial_{a} \psi^{34} &= \frac{m c}{2 \hbar} (\psi^{123} + \psi^{124}).
\end{align*} \hspace{1cm} (25)

We transform these equations as follows. Add the first equation with the second one and the third one with the fourth one:

\begin{align*}
i \partial_{0} \varphi_{2} - i \sigma^{a} \partial_{a} \varphi_{2} &= \frac{m c}{2 \hbar} \varphi_{1}, \\
i \partial_{0} \varphi_{1} + i \sigma^{a} \partial_{a} \varphi_{1} &= \frac{m c}{2 \hbar} \varphi_{2},
\end{align*} \hspace{1cm} (26)

where $\varphi_{1} = \psi^{0} + \psi^{34}$ and $\varphi_{2} = \psi^{123} + \psi^{124}$. Then we subtract the third equation from the fourth one, and the first one from the second one:

\begin{align*}
i \partial_{0} \chi_{2} - i \sigma^{a} \partial_{a} \chi_{2} &= 0, \\
i \partial_{0} \chi_{1} + i \sigma^{a} \partial_{a} \chi_{1} &= 0,
\end{align*}

where $\chi_{1} = \psi^{0} - \psi^{34}$ and $\chi_{2} = \psi^{123} - \psi^{124}$. Thus the system of four equations can be transformed to two independent systems of two equations.
C. Special Cases

1. Dirac theory

The generalization considered can be reduced to the Dirac theory in the following way.

1. The wave function is represented as the vector of the subalgebra $\mathbb{SC}_3$ constructed on the basis vectors $\varepsilon_{32}, \varepsilon_{13}, \varepsilon_{21}, \varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_{123}$.

For this the components of wave function

$$
\Psi^{34}(\psi^{42}, \psi^{14}, \psi^{1324}, \psi^{34}) = \Psi^{124}(\psi^{134}, \psi^{234}, \psi^4, \psi^{124}) = 0,
$$

$$
\chi_1 = \varphi_1 = \Psi^0(\psi^{32}, \psi^{13}, \psi^{21}, \psi^0),
$$

$$
\chi_2 = \varphi_2 = \Psi^{123}(\psi^1, \psi^3, \psi^2, \psi^{123}).
$$

2. The first approximate representation $\widetilde{R}_1$ is used for the basis vectors of $\mathbb{C}_4$ (see Section II).

Then the equation system (22) is reduced to the system equivalent the Dirac equations.

2. Pauli theory

To obtain the Pauli theory as a special case it is necessary 1) to consider the wave function as the vector of the subalgebra $\mathbb{SC}_1$, constructed on the basis vectors $\varepsilon_{32}, \varepsilon_{13}, \varepsilon_{21}, \varepsilon_0$;

2) to use the second approximate representation $\widetilde{R}_2$ for the basis vectors of $\mathbb{C}_4$. Then the basis vectors of the conjugate space-time will have the form

$$
\mathcal{E} = (-i)\sigma^a (a = 1, 2, 3), \quad \mathcal{E}^4 = i \mathbb{1}.
$$

The equation (22) can be rewritten as follows

$$
(\mathcal{E}^4 \partial_4 + \mathcal{E}^a \partial_a) \Psi^{123} = \frac{mc}{\hbar} \Psi^0,
$$

$$
(\mathcal{E}^4 \partial_4 - \mathcal{E}^a \partial_a) \Psi^{123} = \frac{mc}{\hbar} \Psi^{123},
$$

where $\Psi^0$ and $\Psi^{123}$ are the quaternion functions defined by (23). Eliminating the quaternion $\Psi^{123}$ from this system we obtain the equation

$$
(\mathcal{E}^4 \partial_4 + \mathcal{E}^a \partial_a) \circ (\mathcal{E}^4 \partial_4 - \mathcal{E}^a \partial_a) \Psi^0 = \frac{m^2c^2}{\hbar^2} \Psi^0,
$$

which is reduced to the Klein-Gordon equation with respect to the quaternion $\Psi^0$.

(\mathcal{E}^4 \circ \mathcal{E}^4 \partial_4^2 - \mathcal{E}^a \circ \mathcal{E}^b \partial_a \partial_b) \Psi^0 = \frac{m^2c^2}{\hbar^2} \Psi^0.

This equation is equivalent to the Klein-Gordon equation for two complex functions

$$
\Psi^0 = \begin{bmatrix} i\psi^{32} + \psi^{13} \\
 i\psi^{21} + \psi^0 \end{bmatrix}.
$$

In the non-relativistic approximation, it is reduced to the Pauli equation.

3. Schrödinger theory

To obtain the Schrödinger theory as a special case it is necessary 1) to consider the wave function as the vector of the subalgebra $\mathbb{SC}_1$, constructed on the basis vectors $\varepsilon_{21}, \varepsilon_0$;

2) to use the third approximate representation $\widetilde{R}_3$ for the basis vectors of $\mathbb{C}_4$. Then the basis vectors of the conjugate space-time will have the form

$$
\mathcal{E}^1 = -i, \quad \mathcal{E}^2 = 1, \quad \mathcal{E}^3 = -i, \quad \mathcal{E}^4 = i.
$$

The equation (22) can be rewritten as follows

$$
\mathcal{E}^4 \partial_4 \psi^2 + \mathcal{E}^1 \partial_1 \psi^{123} - \mathcal{E}^2 \partial_2 \psi^{123} - \mathcal{E}^3 \partial_3 \psi^2 = \frac{mc}{\hbar} \psi^{13},
$$

$$
\mathcal{E}^4 \partial_4 \psi^{123} + \mathcal{E}^1 \partial_1 \psi^2 + \mathcal{E}^2 \partial_2 \psi^2 + \mathcal{E}^3 \partial_3 \psi^{123} = \frac{mc}{\hbar} \psi^0,
$$

$$
\mathcal{E}^4 \partial_4 \psi^{13} - \mathcal{E}^1 \partial_1 \psi^0 + \mathcal{E}^2 \partial_2 \psi^0 + \mathcal{E}^3 \partial_3 \psi^{13} = \frac{mc}{\hbar} \psi^2,
$$

$$
\mathcal{E}^4 \partial_4 \psi^0 - \mathcal{E}^1 \partial_1 \psi^{13} - \mathcal{E}^2 \partial_2 \psi^{13} - \mathcal{E}^3 \partial_3 \psi^0 = \frac{mc}{\hbar} \psi^{123},
$$

where $\psi^0, \psi^{123}, \psi^2, \psi^{13}$ are complex functions defined by (23). If we eliminate from the second equation of this system the complex functions $\psi^{123}$ and $\psi^2$ by means of the third and fourth equations, we obtain the equation with respect to $\psi^0$

$$
(\mathcal{E}^4 \circ \mathcal{E}^4 \partial_4^2 - \mathcal{E}^1 \circ \mathcal{E}^1 \partial_1^2 + \mathcal{E}^2 \circ \mathcal{E}^2 \partial_2^2 - \mathcal{E}^3 \circ \mathcal{E}^3 \partial_3^2) \psi^0 = \frac{m^2c^2}{\hbar^2} \psi^0,
$$

which is equivalent to the Klein-Gordon equation and, in a non-relativistic approximation, is reduced to the Schrödinger equations.

D. Symmetries of wave function components and leptons

In this section we give the interpretation of the components $\Psi^0 + \Psi^{34}, \Psi^{123} + \Psi^{124}, \Psi^{123} - \Psi^{124}, \Psi^0 - \Psi^{34}$ of wave function as the wave functions of different particles. Our interpretation is based on the following reasons:
1. The relativistic quantum mechanics equations obtained can be presented as two systems of equations, each of them applies to the two-component wave function.

2. The independence of the specified two systems of equations from each other allows to refer these systems to different particles. One of these particles is massive but the other is massless.

3. With the passage from the generalized equations to the Dirac equations, when the component $\Psi^0$ passes in a left component of the Dirac wave function, and the component $\Psi^{123}$ passes in a right component of the Dirac wave function.

The specified circumstances allow to present facts as follows. The relativistic quantum mechanics equations concern two particles whose wave functions have two components. These particles are leptons of the same generation, i.e. electron and its neutrino $\{e, \nu_e\}$, muon and its neutrino $\{\mu, \nu_\mu\}$, tau-lepton and its neutrino $\{\tau, \nu_\tau\}$. In our case the neutrino is considered as two-component particle. However the left neutrino is only observed. This fact can be explained by that interactions involving the right neutrino is significantly weaker than those involving the left neutrino. In our following paper it will be shown that such a difference between the left and right neutrinos exists.

In order to answer the question of how wave functions (and quantum mechanics equations) are distinguished for neutrinos exists. Shown that such a difference between the left and right neutrinos is significantly weaker than those involving the left neutrino. In our following paper it will be shown that such a difference between the left and right neutrinos exists.

Thus we set the following correspondence between the components of wave function and leptons. The components of wave function $\psi^A$, where $A = 32, 13, 21, 0, 42, 14, 1324, 34, 1, 2, 3, 123, 134, 234, 4, 124$, are separated into four groups:

- The left component of electron
  \[ e_L = \Psi^0(\psi^{12}, \psi^{13}, \psi^{21}, \psi^0) + \Psi^{123}(\psi^{12}, \psi^{13}, \psi^{21}, \psi^{123}) \]

- The right component of electron
  \[ e_R = \Psi^{123}(\psi^1, \psi^2, \psi^3, \psi^{123}) + \Psi^{124}(\psi^{134}, \psi^{234}, \psi^4, \psi^{124}) \]

- The left component of $\epsilon$-neutrino
  \[ \nu_{eL} = \Psi^{123}(\psi^{21}, \psi^1, \psi^{123}) - \Psi^{124}(\psi^{134}, \psi^{234}, \psi^4, \psi^{124}) \]

- The right component of $\epsilon$-neutrino
  \[ \nu_{eR} = \Psi^0(\psi^{32}, \psi^{13}, \psi^{21}, \psi^0) - \Psi^{34}(\psi^{42}, \psi^{14}, \psi^{1324}, \psi^{34}) \]

The components of wave function for leptons of the second and third generations differ from the above ones by the cyclic permutation of spatial indices:

- for muon and $\mu$-neutrino $3 \rightarrow 2, 2 \rightarrow 1, 1 \rightarrow 3$;
- for $\tau$-lepton and $\tau$-neutrino $3 \rightarrow 1, 2 \rightarrow 3, 1 \rightarrow 2$.

The relativistic quantum mechanics equations, for example, for leptons of the second generation will have the form:

\[
\begin{align*}
    j & \partial_4 \mu_L - j \sigma^a \partial_a \mu_L = \frac{m_\mu c}{\hbar} \mu_L, \\
    j & \partial_4 \mu_R + j \sigma^a \partial_a \mu_R = \frac{m_\mu c}{\hbar} \mu_R, \\
    j & \partial_4 \nu_\mu_R - j \sigma^a \partial_a \nu_\mu_R = 0, \\
    j & \partial_4 \nu_\mu_L + j \sigma^a \partial_a \nu_\mu_L = 0, \\
    p^2 = p_0^2 &= \frac{m_\mu c}{2}.
\end{align*}
\]

The difference of masses of electron, muon, and $\tau$-lepton testifies probably to an anisotropy of directions $\varepsilon_{21}, \varepsilon_{13}, \varepsilon_{32}$ in the generalized space-time $C_4$ and the action space $SC_4$.

We give the following definitions. The space of Clifford algebra $C_4$ serves to describe lepton motion and thus will be called a space of leptons. Respectively, the action space $SC_4$ will be called an action space of leptons.

### E. Conjugate action vector. Quantization equations in differentials for antileptons

To derive the quantum mechanics equations for antiparticles it is necessary to pass from the space $X$ to the conjugate space $\tilde{X}$ which will be called a space of antiparticles in this connection. In addition, it is necessary to pass from the action space $\tilde{S}X$ to the conjugate space $\tilde{S}\tilde{X}$.

We suppose that space $\tilde{S}X$ is similarly to space $\tilde{X}$. That is, vector $\tilde{S} \in \tilde{S}X$ can be written as $\tilde{S} = S_K \cdot E^K$. For algebra $\tilde{S}\tilde{X}$, the structure equation takes place

\[ \delta_2 \delta_1 \tilde{S} = - \frac{1}{S_0} \delta_2 \tilde{S} \circ \delta_1 \tilde{S}. \]

Or in the coordinate form:

\[ \delta_2 \delta_1 S_j = - \frac{1}{S_0} \delta_2 S_P \cdot \delta_1 S_L \cdot C^{PL}_j. \] (27)

We also suppose that the vector $\tilde{S}$ is a function of vector $\tilde{x} \in \tilde{X}$: $\tilde{S} = \tilde{S}(\tilde{x})$.

Let us introduce the notations $\psi = \delta_1 S$ and $d = \delta_2$ in the equations (27). The function $\psi(\tilde{x})$ will be called
a wave function of antileptons. The structure equations are written for the components of wave function as
\[ d\psi_L = -\frac{1}{S_0} dS_P \cdot \psi_L \cdot C^{PL}_L. \]

They will be called quantization equations in differentials for antiparticles.

Write the differential \( d\tilde{S} \) as follows
\[ d\tilde{S} = \partial^M \tilde{S}(\tilde{x}) \, dx_M \]
and introduce generalized conjugate impulses as
\[ p^M = -\partial^M \tilde{S}(\tilde{x}) = -\partial^M \tilde{S}(\tilde{x}) \, E^P = \tilde{p}^M \cdot E^N. \]

From the quantization equations in differentials, the relations follow
\[ \partial^M \psi_L(\tilde{x}) = \frac{1}{S_0} \tilde{p}^M \cdot \psi_L \cdot C^{PL}_L, \tag{28} \]
which will be called quantum postulates for antiparticles.

F. Relativistic quantum mechanics equations for antileptons

Consider the quantum postulates for antiparticles \(^{(28)}\). We multiply these relations by the structure constant \( C^{KM}_L \):
\[ \partial^M \psi_L(\tilde{x}) \cdot C^{KM}_L = \frac{1}{S_0} \tilde{p}^M \cdot C^{PL}_L \cdot C^{KM}_L \cdot \psi_L. \]

To consider the quantum mechanics equations for antileptons it is necessary to pass from the conjugate action space \( \bar{S} \mathcal{X} \) to its subspace \( \bar{C}_4 \), which will be named a space of antileptons in this connection. The matrices of basis vectors \( \bar{E}^i \) of this space are presented in Appendix \( \bar{E} \).

We set again \( S_0 = \hbar \). Let also the wave function \( \psi_i(\tilde{x}) \) depend only on coordinates of the conjugate space-time \( \tilde{X} \). Moreover we assume that the generalized conjugate impulse \( \tilde{p}^M \) has only two components
\[ \tilde{p}^{1324}_{1324} = \partial^{1324} S_{1324}(\tilde{x}) = \frac{mc}{2}, \]
\[ \tilde{p}^{123}_1 = \partial^0 S_{123}(\tilde{x}) = \frac{mc}{2}. \]

Then we obtain
\[ C^{KM}_L \partial^M \psi_L = \frac{mc}{2\hbar} \left( C^{1324L}_I \cdot C^{K}_{1324} + C^{123L}_K \right) \psi_L, \]

These equations will be named relativistic quantum mechanics equations for antileptons.

Consider these equations for the representation of algebra \( \bar{C}_4 \) in the subalgebra \( \bar{C}_n \) \((n < 4)\) over the field of hypernumbers \( \bar{C}_{4-n} \) in the vector form:
\[ C^{I}_{KM} \partial^m \psi_L = \frac{mc}{2\hbar} \left( C^{1324L}_I \cdot C^{K}_{1324} + C^{123L}_K \right) \psi_L. \]

In this expression we represent \( \bar{E}^K \) as the product:
\[ \bar{E}^K = \bar{E}^{k_1} \circ \bar{E}^{k_2} = \xi_{k_2} \xi_{k_1}, \]
where \( \xi_{k_1} \) are the basis vectors of subalgebra \( \bar{C}_n \), \( \xi_{k_2} \) are the basis vectors of subalgebra \( \bar{C}_{4-n} \), and \( \xi_{k_2} \) are the basis numbers. As a result, we obtain
\[ C^{i_{k_1}i_{k_2}} \partial^m \psi_i = \frac{mc}{2\hbar} \left( C^{1324L}_{i_{k_1}} \cdot C^{L}_{i_{k_1}i_{k_2}} + C^{123L}_{k_1} \right) \psi_i, \]

where \( \psi_i \) and \( \xi_{i_{k_2}} \xi_{k_2} \), and the parastratop matrices \( C^{i_{k_1}i_{k_2}} \) are expressed through the basis numbers.

In the complex representation
\[ \xi^0 = \{I, i\}, \]
\[ \xi^1 = \{E^{13}, E^{10}, E^{14}, E^{34}, E^2, E^{123}, E^{234}, E^{124}\} \]
the components of wave function are complex:
\[ \psi_{13} = i\psi_{32} + \psi_{13}, \quad \psi_0 = i\psi_{21} + \psi_0, \]
\[ \psi_{14} = i\psi_{42} + \psi_{14}, \quad \psi_{34} = i\psi_{324} + \psi_{34}, \]
\[ \psi_2 = i\psi_1 + \psi_2, \quad \psi_{123} = i\psi_3 + \psi_{123}, \]
\[ \psi_{234} = i\psi_{134} + \psi_{234}, \quad \psi_{124} = i\psi_4 + \psi_{124}. \]

In the quaternion representation
\[ \xi^1 = \{I, \sigma_1, i\sigma_2, i\sigma_3\}, \]
\[ \xi^0 = \{E^{13}, E^{34}, E^{123}, E^{124}\} \]
the components of wave function are quaternion:
\[ \psi_0 = i\sigma_1 \psi_{32} + i\sigma_2 \psi_{13} + i\sigma_3 \psi_{21} + \psi_0, \]
\[ \psi_{34} = i\sigma_1 \psi_{42} + i\sigma_2 \psi_{14} + i\sigma_3 \psi_{324} + \psi_{34}, \]
\[ \psi_{123} = i\sigma_1 \psi_1 + i\sigma_2 \psi_2 + i\sigma_3 \psi_3 + \psi_{123}, \]
\[ \psi_{124} = i\sigma_1 \psi_{134} + i\sigma_2 \psi_{234} + i\sigma_3 \psi_4 + \psi_{124}. \]

The quantum mechanics equations in relation to the quaternion components can be written as follows
\[ i \partial^4 \psi_{123} + (a I \partial^1 + b I \partial^2 + i \partial^3) \psi_{123} = \frac{mc}{2\hbar} \varphi_1, \]
\[ -i \partial^4 \psi_{123} + (a I \partial^1 + b I \partial^2 - i \partial^3) \psi_{124} = \frac{mc}{2\hbar} \varphi_2, \]
\[ -i \partial^4 \psi_{34} - (a I \partial^1 + b I \partial^2 + i \partial^3) \psi_0 = \frac{mc}{2\hbar} \varphi_1, \]
\[ i \partial^4 \psi_0 - (a I \partial^1 + b I \partial^2 - i \partial^3) \psi_{34} = \frac{mc}{2\hbar} \varphi_2, \]
where \( \varphi_1 = \psi_{123} - \psi_0, \) and \( \varphi_2 = \psi_{124} - \psi_{34}. \) This equation system is readily reduced to the two systems from two equations:
\[ i \partial^4 \varphi_2 + (a I \partial^1 + b I \partial^2 + i \partial^3) \varphi_1 = \frac{mc}{h} \varphi_1, \]
\[ -i \partial^4 \varphi_1 + (a I \partial^1 + b I \partial^2 - i \partial^3) \varphi_2 = \frac{mc}{h} \varphi_2, \]
and

\[ i \partial^4 \chi_2 + (a I \partial I + b I \partial^2 + i \partial^3) \chi_1 = 0, \]
\[ -i \partial^4 \chi_1 + (a I \partial I + b I \partial^2 - i \partial^3) \chi_2 = 0, \]

where \( \chi_1 = \Psi_{123} + \Psi_0 \) and \( \chi_2 = \Psi_{124} + \Psi_{34} \). Therefore these equations concern two antiparticles whose wave functions have two components. The one of the antiparticles is massive, and the other is massless. We consider that the equations obtained describe antileptons of the same generation.

Repeating the reasons of Section III D one can set correspondence between the components of wave function and the antileptons of different generations.

G. Relativistic quantum mechanics equations for arbitrary action vector

The relativistic quantum mechanics equations obtained were derived from the structure equations (18) for the action algebra \( \mathcal{SX} \). However these equations are a special case of the common structure equations similar to the equations (19) for the algebra \( \mathcal{X} \):

\[ \delta_2 \delta_1 S = -\delta_1 S \circ S^{-1} \circ \delta_2 S. \]

Using the coordinate form of action vector we obtain

\[ e_I \cdot \delta_2 \delta_1 S^J = -(e_L \circ e_Q \circ e_R) \delta_2 S^R (S^{-1})^Q \cdot \delta_1 S^L. \]

From the expression for the product of basis vectors

\[ e_L \circ e_Q \circ e_R = e_1 \cdot C^I_{LP} \cdot C^P_{QR}, \]

it follows that

\[ \delta_2 \delta_1 S^J = -\delta_2 S^R (S^{-1})^Q \cdot \delta_1 S^L (C^P_{QR} \cdot C^I_{LP}). \]

If we introduce the wave function \( \psi \) as \( \delta_1 S \), we obtain the quantization equations in differentials

\[ \delta_2 \psi^J = -(C^I_{LP} \cdot C^P_{QR}) \delta_2 S^R (S^{-1})^Q \cdot \psi^L. \]

From them the quantum postulates follow

\[ \partial_M \psi^J = (C^I_{LP} \cdot C^P_{QR}) p^R_M (S^{-1})^Q \cdot \psi_L. \]

If we multiply these relations by the structure constants \( C^{MK}_I \) of the conjugate algebra, we get

\[ C^{MK}_I \cdot \partial_M \psi^J = C^{MK}_I (C^I_{LP} \cdot C^P_{QR}) p^R_M (S^{-1})^Q \cdot \psi^L. \]

(29)

These relations are the relativistic quantum mechanics equations for the arbitrary action vector. In the case when the relativistic quantum mechanics equations are considered in the neighbourhood of action vector

\[ S^Q = S^0 = \hbar, \]

from (24) we obtain the equations (21).

IV. CONCLUSIONS

We summarize the more important results found in the previous Sections.

1. The action for elementary particles should be considered as vector in the space of contravariant tensors of all ranks \( \mathcal{SX} \). The action for elementary antiparticles should be considered as vector in the space of covariant tensors of all ranks \( \mathcal{SX} \). The specified sets \( \mathcal{SX} \) and \( \mathcal{SX} \), supplied by tensor multiplication, are algebras. For leptons, these algebras are reduced to the Clifford algebras \( \mathcal{SC} \) and \( \tilde{\mathcal{SC}} \).

2. The wave function is identified with the partial differential of action vector.

3. The quantum equations are the structure equations typical for the action vector algebra. Thus, the quantization effect is the consequence of the algebraic structure of the action vector set.

4. The Clifford algebra \( \tilde{\mathcal{SC}} \) generalizes the Dirac algebra. For the Clifford algebra the wave function has four quaternion components. In the Dirac approximation two of them transform to the left component of the Dirac wave function, and the other components transform to the right one. It is one of reasons why the above four components are interpreted as the components of wave functions of leptons of the same generation.

5. Any of the basis vectors \( \varepsilon_{21}, \varepsilon_{13}, \varepsilon_{32} \) can be taken as the basic direction for the complex representation of wave functions of leptons of the same generation. Thus these basis vectors should be put into the correspondence with three generations of leptons. This assumption allows to obtain the quantum equations for leptons of three generations.

6. The transformation from the Clifford algebra to the conjugate Clifford algebra is nonequivalent to the transformation to complex conjugate vectors and matrices. For this reason, the quantum mechanics equations for antileptons in Dirac’s approximation differ from the appropriate equations of the Dirac theory.

7. Both the known quantum mechanics equations and those derived here can be applied only for the action vector near to the Plank constant. In a common case it is necessary to use the equations (24).

ACKNOWLEDGMENTS

The author is very grateful to E. A. Cherkashina for assistance in preparing manuscript.
APPENDIX A: REGULAR REPRESENTATION OF CLIFFORD ALGEBRAS

In this Appendix we give the Clifford algebra parastrophic matrices which realize the regular representation of basis vectors:

$$\varepsilon_1 \sim C_{KL}^{L} \equiv C_i^j_k.$$ 

1. Clifford algebra $C_3$

For the Clifford algebra constructed on the geometric space, the basis vectors are represented by the following matrices.

\[
\begin{align*}
\varepsilon_0 & \sim 
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix} = I &\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix} = I \\
\varepsilon_1 & \sim 
\begin{bmatrix}
1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix} = a &\begin{bmatrix}
1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix} = a \\
\varepsilon_2 & \sim 
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix} = b &\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix} = b \\
\varepsilon_3 & \sim 
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix} = i &\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix} = i \\
\varepsilon_{21} & \sim 
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix} = i &\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix} = i
\end{align*}
\]

By the transformation of matrices $C_{KL}^{L}$ the special notations are used for matrix blocks. At first, the blocks $2 \times 2$ were denoted as follows

\[
\begin{align*}
1 & = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} & a & = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & b & = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & i & = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{align*}
\]

The algebra of numbers $\{1, a, b, i\}$ is represented by the multiplication rules:

\[
a^2 = b^2 = 1, \quad i^2 = -1, \quad ab = -ba = i, \quad a i = i a = b, \quad b i = i b = -a.
\]

Thereupon the blocks $2 \times 2$ were denoted as follows

\[
\begin{align*}
1 & = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} & I & = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{align*}
\]

2. Clifford algebra $C_4$

For the Clifford algebra constructed on the space-time, the basis vectors are represented by the following matrices.
| 13 | 0 | 14 | 34 | 2 | 123 | 234 | 124 |
|----|---|----|----|---|-----|-----|-----|
| 32 | 13 | 21 | 42 | 0 | 14   | 1324| 1   |
| 14 | 1  | 2  | 34 | 1 | 2    | 1   | -1  |
| 34 | 1  | 23 | 24 | 1 | 123  | 2   | 1   |
| 234| 1  | 124| 4  | 1 | 1    | 4   | 1   |

\[ \varepsilon_0 \sim \]

\[ \varepsilon_1 \sim \]

\[ \varepsilon_2 \sim \]

\[ \varepsilon_3 \sim \]
\[
\begin{array}{cccc}
32 & 13 & 0 & 42 \\
13 & 21 & 0 & 14 \\
0 & 14 & -1 & 1324 \\
42 & 1324 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccc}
32 & 13 & 0 & 42 \\
13 & 21 & 0 & 14 \\
0 & 14 & -1 & 1324 \\
42 & 1324 & 1 & 1 \\
\end{array}
\]
APPENDIX B: REGULAR REPRESENTATION OF CONJUGATED CLIFFORD ALGEBRAS

In this Appendix we give the conjugated Clifford algebra parastrophic matrices which realize the regular representation of basis vectors:

$$\mathcal{E}^I \sim C^{IK} \mathcal{L} \equiv C^{I(K)} \mathcal{L}.$$
1. Clifford algebra $\tilde{\mathbb{C}}_3$

For the conjugated Clifford algebra constructed on the geometric space, the basis vectors are represented by the following matrices.

$$E^0 \sim \begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1
\end{pmatrix} = I$$

$$E^1 \sim \begin{pmatrix}
1 & -1 \\
-1 & 1 \\
1 & -1 \\
1 & 1
\end{pmatrix} = i\begin{pmatrix}
\sigma_1 \\
\sigma_1 \\
\sigma_2 \\
\sigma_2
\end{pmatrix}$$

$$E^2 \sim \begin{pmatrix}
1 & 1 \\
1 & -1 \\
-1 & 1 \\
1 & -1
\end{pmatrix} = i\begin{pmatrix}
\sigma_1 \\
\sigma_1 \\
\sigma_2 \\
\sigma_2
\end{pmatrix}$$

$$E^3 \sim \begin{pmatrix}
1 & -1 \\
1 & 1 \\
-1 & 1 \\
-1 & -1
\end{pmatrix} = i\begin{pmatrix}
\sigma_1 \\
\sigma_1 \\
\sigma_2 \\
\sigma_2
\end{pmatrix}$$

By the transformation of matrices $C^{IKL}$ the special notations are used for matrix blocks. At first, blocks $2 \times 2$ were denoted as follows

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$  

This is tantamount to changing from real matrices $2 \times 2$ to complex numbers. Thereupon the complex matrices $2 \times 2$ were denoted as follows

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

The matrices $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices, with the difference that the opposite sign matrix was denoted as $\sigma_3$ for reasons of symmetry.

2. Clifford algebra $\tilde{\mathbb{C}}_4$

For the conjugated Clifford algebra constructed on the conjugated space-time, the basis vectors are represented by the following matrices.
\[ E_0 \sim \begin{array}{cccccccc}
32 & 13 & 0 & 14 & 34 & 2 & 123 & 234 \\
32 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
31 & & & & & & & \\
30 & & & & & & & \\
29 & & & & & & & \\
28 & & & & & & & \\
27 & & & & & & & \\
26 & & & & & & & \\
25 & & & & & & & \\
24 & & & & & & & \\
23 & & & & & & & \\
22 & & & & & & & \\
21 & & & & & & & \\
20 & & & & & & & \\
19 & & & & & & & \\
18 & & & & & & & \\
17 & & & & & & & \\
16 & & & & & & & \\
15 & & & & & & & \\
14 & & & & & & & \\
13 & & & & & & & \\
12 & & & & & & & \\
11 & & & & & & & \\
10 & & & & & & & \\
9 & & & & & & & \\
8 & & & & & & & \\
7 & & & & & & & \\
6 & & & & & & & \\
5 & & & & & & & \\
4 & & & & & & & \\
3 & & & & & & & \\
2 & & & & & & & \\
1 & & & & & & & \\
0 & & & & & & & \\
\end{array} \]

\[ E_0 \sim = 1 \begin{array}{cccccccc}
13 & 0 & 14 & 34 & 2 & 123 & 234 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & & & & & & & \\
14 & & & & & & & \\
34 & & & & & & & \\
123 & & & & & & & \\
234 & & & & & & & \\
124 & & & & & & & \\
\end{array} \]

\[ E_1 \sim \begin{array}{cccccccc}
32 & 13 & 0 & 14 & 34 & 2 & 123 & 234 \\
32 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
31 & & & & & & & & \\
30 & & & & & & & & \\
29 & & & & & & & & \\
28 & & & & & & & & \\
27 & & & & & & & & \\
26 & & & & & & & & \\
25 & & & & & & & & \\
24 & & & & & & & & \\
23 & & & & & & & & \\
22 & & & & & & & & \\
21 & & & & & & & & \\
20 & & & & & & & & \\
19 & & & & & & & & \\
18 & & & & & & & & \\
17 & & & & & & & & \\
16 & & & & & & & & \\
15 & & & & & & & & \\
14 & & & & & & & & \\
13 & & & & & & & & \\
12 & & & & & & & & \\
11 & & & & & & & & \\
10 & & & & & & & & \\
9 & & & & & & & & \\
8 & & & & & & & & \\
7 & & & & & & & & \\
6 & & & & & & & & \\
5 & & & & & & & & \\
4 & & & & & & & & \\
3 & & & & & & & & \\
2 & & & & & & & & \\
1 & & & & & & & & \\
0 & & & & & & & & \\
\end{array} \]

\[ E_1 \sim = i \begin{array}{cccccccc}
13 & 0 & 14 & 34 & 2 & 123 & 234 \\
13 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & & & & & & & & \\
14 & & & & & & & & \\
34 & & & & & & & & \\
123 & & & & & & & & \\
234 & & & & & & & & \\
124 & & & & & & & & \\
\end{array} \]

\[ E_2 \sim \begin{array}{cccccccc}
32 & 13 & 0 & 14 & 34 & 2 & 123 & 234 \\
32 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
31 & & & & & & & & \\
30 & & & & & & & & \\
29 & & & & & & & & \\
28 & & & & & & & & \\
27 & & & & & & & & \\
26 & & & & & & & & \\
25 & & & & & & & & \\
24 & & & & & & & & \\
23 & & & & & & & & \\
22 & & & & & & & & \\
21 & & & & & & & & \\
20 & & & & & & & & \\
19 & & & & & & & & \\
18 & & & & & & & & \\
17 & & & & & & & & \\
16 & & & & & & & & \\
15 & & & & & & & & \\
14 & & & & & & & & \\
13 & & & & & & & & \\
12 & & & & & & & & \\
11 & & & & & & & & \\
10 & & & & & & & & \\
9 & & & & & & & & \\
8 & & & & & & & & \\
7 & & & & & & & & \\
6 & & & & & & & & \\
5 & & & & & & & & \\
4 & & & & & & & & \\
3 & & & & & & & & \\
2 & & & & & & & & \\
1 & & & & & & & & \\
0 & & & & & & & & \\
\end{array} \]

\[ E_2 \sim = -i \begin{array}{cccccccc}
13 & 0 & 14 & 34 & 2 & 123 & 234 \\
13 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & & & & & & & & \\
14 & & & & & & & & \\
34 & & & & & & & & \\
123 & & & & & & & & \\
234 & & & & & & & & \\
124 & & & & & & & & \\
\end{array} \]

\[ E_3 \sim \begin{array}{cccccccc}
32 & 13 & 0 & 14 & 34 & 2 & 123 & 234 \\
32 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
31 & & & & & & & & \\
30 & & & & & & & & \\
29 & & & & & & & & \\
28 & & & & & & & & \\
27 & & & & & & & & \\
26 & & & & & & & & \\
25 & & & & & & & & \\
24 & & & & & & & & \\
23 & & & & & & & & \\
22 & & & & & & & & \\
21 & & & & & & & & \\
20 & & & & & & & & \\
19 & & & & & & & & \\
18 & & & & & & & & \\
17 & & & & & & & & \\
16 & & & & & & & & \\
15 & & & & & & & & \\
14 & & & & & & & & \\
13 & & & & & & & & \\
12 & & & & & & & & \\
11 & & & & & & & & \\
10 & & & & & & & & \\
9 & & & & & & & & \\
8 & & & & & & & & \\
7 & & & & & & & & \\
6 & & & & & & & & \\
5 & & & & & & & & \\
4 & & & & & & & & \\
3 & & & & & & & & \\
2 & & & & & & & & \\
1 & & & & & & & & \\
0 & & & & & & & & \\
\end{array} \]

\[ E_3 \sim = -i \begin{array}{cccccccc}
13 & 0 & 14 & 34 & 2 & 123 & 234 \\
13 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & & & & & & & & \\
14 & & & & & & & & \\
34 & & & & & & & & \\
123 & & & & & & & & \\
234 & & & & & & & & \\
124 & & & & & & & & \\
\end{array} \]

\[ \boxed{-i} \]

\[ \boxed{-1} \]

\[ \boxed{1} \]

\[ \boxed{\sigma^{-1}} \]

\[ \boxed{-\sigma^{3}} \]

\[ \boxed{i} \]

\[ \boxed{-i} \]
\[ E^4 \sim (-1) \]

\[ E^{13} \sim (-1) \]

\[ E^{21} \sim (-1) \]

\[ E^{32} \sim (-1) \]
\[
\begin{array}{c}
\mathcal{E}^{14} \sim \\
\begin{array}{cccccccc}
32 & 13 & 0 & 14 & 34 & 2 & 1 & 1324 \\
32 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
14 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\
34 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\
1324 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\
\end{array}
\end{array}
\]

\[
\mathcal{E}^{34} \sim \\
\begin{array}{cccccccc}
32 & 13 & 0 & 14 & 34 & 2 & 1 & 1324 \\
32 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
14 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\
34 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\
1324 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\mathcal{E}^{42} \sim \\
\begin{array}{cccccccc}
32 & 13 & 0 & 14 & 34 & 2 & 1 & 1324 \\
32 & -1 & -1 & 1 & 1 & 1 & 1 & 1 \\
14 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\
34 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\
1324 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\mathcal{E}^{123} \sim (-1) \\
\begin{array}{cccccccc}
32 & 13 & 0 & 14 & 34 & 2 & 1 & 1324 \\
32 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\
14 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\
34 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\
1324 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\mathcal{E}^{123} \sim (-1) \\
\begin{array}{cccccccc}
32 & 13 & 0 & 14 & 34 & 2 & 1 & 1324 \\
32 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\
14 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\
34 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\
1324 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
= 1 \\
\begin{array}{cccc}
1 & 1 \\
1 & 1 \\
-1 & -1 \\
1 & 1 \\
\end{array}
\]

\[
\sigma^1 \sigma^1 \\
\sigma^1 \sigma^1 \\
-\sigma^1 -\sigma^1 \\
\sigma^1 \sigma^1 \\
\]

\[
= 1 \\
\begin{array}{cccc}
1 & 1 \\
1 & 1 \\
-1 & -1 \\
1 & 1 \\
\end{array}
\]

\[
\sigma^3 \sigma^3 \\
\sigma^3 \sigma^3 \\
-\sigma^3 -\sigma^3 \\
\sigma^3 \sigma^3 \\
\]

\[
= 1 \\
\begin{array}{cccc}
1 & 1 \\
1 & 1 \\
-1 & -1 \\
1 & 1 \\
\end{array}
\]

\[
\lll \\
\lll \\
\lll \\
\lll \\
\]

\[
= 1 \\
\begin{array}{cccc}
1 & 1 \\
1 & 1 \\
-1 & -1 \\
1 & 1 \\
\end{array}
\]
\[ E^{124} \sim \]

\[
\begin{array}{ccccccc}
32 & 13 & 0 & 14 & 34 & 2 & 123 & 34 \\
32 & -1 & 1 & -1 \\
13 & 1 & 1 \\
0 & -1 & 1 \\
14 & -1 & 1 \\
34 & 1 & 1 \\
2 & 1 \\
3 & 1 \\
123 & 1 \\
34 & -1 \\
4 & 1 \\
124 & 1 \\
\end{array}
= I
\]

\[ E^{234} \sim \]

\[
\begin{array}{ccccccc}
32 & 13 & 0 & 14 & 34 & 2 & 123 & 34 \\
32 & -1 & 1 & -1 \\
13 & 1 & 1 \\
0 & -1 & 1 \\
14 & -1 & 1 \\
34 & 1 & 1 \\
2 & 1 \\
3 & 1 \\
123 & 1 \\
34 & -1 \\
4 & 1 \\
124 & 1 \\
\end{array}
= I
\]

\[ E^{134} \sim \]

\[
\begin{array}{ccccccc}
32 & 13 & 0 & 14 & 34 & 2 & 123 & 34 \\
32 & -1 & 1 & -1 \\
13 & 1 & 1 \\
0 & -1 & 1 \\
14 & -1 & 1 \\
34 & 1 & 1 \\
2 & 1 \\
3 & 1 \\
123 & 1 \\
34 & -1 \\
4 & 1 \\
124 & 1 \\
\end{array}
= (-1) I
\]

\[ E^{1324} \sim (-1) \]

\[
\begin{array}{ccccccc}
32 & 13 & 0 & 14 & 34 & 2 & 123 & 34 \\
32 & -1 & 1 & -1 \\
13 & 1 & 1 \\
0 & -1 & 1 \\
14 & -1 & 1 \\
34 & 1 & 1 \\
2 & 1 \\
3 & 1 \\
123 & 1 \\
34 & -1 \\
4 & 1 \\
124 & 1 \\
\end{array}
= i
\]

[1] A. A. Ketsaris, *Generalized relativistic quantum mechanics equation in gauge field*, in preparation.