On the Privacy of Euclidean Distance Preserving Data Perturbation

Chris R. Giannella, Kun Liu, Hillol Kargupta Senior Member, IEEE

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Chris R. Giannella is with the The MITRE Corporation, 300 Sentinel Dr. Suite 600, Annapolis Junction, MD 20701, USA Email: cgiannel@acm.org. Kun Liu is with Yahoo! Labs, 4301 Great America Pkwy, Santa Clara, CA USA, Email: kun@yahoo-inc.com. Hillol Kargupta is with the Department of Computer Science and Electrical Engineering, University of Maryland Baltimore County, Baltimore MD, USA, Email: hillol@cs.umbc.edu. Hillol Kargupta is also affiliated with AGNIK LLC, Columbia MD, USA.
Abstract

We examine Euclidean distance preserving data perturbation as a tool for privacy-preserving data mining. Such perturbations allow many important data mining algorithms, with only minor modification, to be applied to the perturbed data and produce exactly the same results as if applied to the original data, \textit{e.g.} hierarchical clustering and k-means clustering. However, the issue of how well the original data is hidden needs careful study. We take a step in this direction by assuming the role of an attacker armed with two types of prior information regarding the original data. We examine how well the attacker can recover the original data from the perturbed data and prior information. Our results offer insight into the vulnerabilities of Euclidean distance preserving transformations.

Index Terms

Euclidean distance preservation, privacy-preserving data mining, principal component analysis

I. INTRODUCTION

Recent interest in the collection and monitoring of data using data mining technology for the purpose of security and business-related applications has raised serious concerns about privacy issues. For example, mining health-care data for security/fraud issues may require analyzing clinical records and pharmacy transaction data of many individuals over a certain area. However, releasing and gathering such diverse information belonging to different parties may violate privacy laws and eventually be a threat to civil liberties. Privacy-Preserving Data Mining (PPDM) strives to provide a solution to this dilemma. It aims to allow useful data patterns to be extracted without compromising privacy.

Data perturbation represents one common approach in PPDM. Here, the original private dataset \(X\) is perturbed and the resulting dataset \(Y\) is released for analysis. Perturbation approaches typically face a “privacy/accuracy” trade-off. On the one hand, perturbation must not allow the original data records to be adequately recovered. On the other, it must allow “patterns” in the original data to be recovered. In many cases, increased privacy comes at the cost of reduced accuracy and vice versa. For example, Agrawal and Srikant [1] proposed adding randomly generated \textit{i.i.d.} noise to the dataset. They showed how the distribution from which the original data arose can be estimated using only the perturbed data and the distribution of the noise. However, Kargupta \textit{et al.} [2] and Huang \textit{et al.} [3] pointed out how, in many cases, the noise can
be filtered off leaving a reasonably good estimation of the original data (further investigated by Guo et al. [4]). These results point to the fact that unless the variance of the additive noise is sufficiently large, original data records can be recovered unacceptably well. However, this increase in variance reduces the accuracy with which the original data distribution can be estimated. This privacy/accuracy trade-off is not limited to additive noise, some other perturbation techniques suffer from a similar problem, e.g. k-anonymity [5].

Recently, Euclidean distance preserving data perturbation for census model \(^1\) has gained attention ([7]–[13]) because it mitigates the privacy/accuracy trade-off by guaranteeing perfect accuracy. The census model using Euclidean distance preserving data perturbation can be illustrated as follows. An organization has a private, real-valued dataset \(X\) (represented as a matrix where each column is a data record) and wishes to make it publicly available for data analysis while keeping the individual records (columns) private. To accomplish this, \(Y = T(X)\) is released to the public where \(T(\cdot)\) is a function, known only to the data owner, that preserves Euclidean distances between columns. With this nice property, many useful data mining algorithms, with only minor modification, can be applied to \(Y\) and produce exactly the same patterns that would be extracted if the algorithm was applied directly to \(X\). For example, assume single-link, agglomerative hierarchical clustering (using Euclidean distance) is applied directly to \(Y\). The cluster memberships in the resulting dendogram will be identical to those in the dendogram produced if the same algorithm is applied to \(X\).

However, the issue of how well the private data is hidden after Euclidean distance preserving data perturbation needs careful study. Without any prior knowledge, the attacker can do very little (if anything) to accurately recover the private data. However, no prior knowledge seems an unreasonable assumption in many situations. Consideration of prior knowledge based attack techniques against Euclidean distance preserving transformations is an important avenue of study. In this paper, we take a step in this direction by considering two types of prior knowledge and, for each, develop an attack technique by which the attacker can estimate private data in \(X\) from the perturbed data \(Y\).

\(^1\)The census model is widely studied in the field of security control for statistical databases [6].
A. Attacker Prior Knowledge Assumptions

1) **Known input:** The attacker knows some small collection of private data records.

2) **Known sample:** The attacker knows that the private data records arose as independent samples of some \( n \)-dimensional random vector \( \mathcal{V} \) with unknown \( p.d.f. \). And, the attacker has another collection of independent samples from \( 2 \mathcal{V} \). For technical reasons, we make a mild additional assumption that holds in many practical situations [14, pg. 27]: the covariance matrix of \( \mathcal{V} \) has distinct eigenvalues.

It is important to stress that we do not assume that both hold simultaneously. Rather, we consider each assumption separately and develop two separate attacks.

Regarding the known input assumption, as pointed out in [13], this knowledge could be obtained through insider information. For example, consider a dataset where each record corresponds to information about an individual (e.g. medical data, census data). It is reasonable to assume that the individuals know (1) that a record for themselves appears in the dataset, and (2) the attributes of the dataset. As such, each individual knows one private record in the original dataset. A small group of malicious individuals could then combine their insider information to produce a larger known input set. As we will show through experiments, a set of four known inputs is enough to breach the privacy of another unknown input on a 16-dimensional, real dataset.

Regarding the known sample assumptions, as pointed out in [13], this knowledge could be obtained through an insider with access to a competitor’s dataset. For example, consider a pair of competing companies offering a very similar service to the same population (e.g. insurance companies). These companies each store information about individuals from the population in a dataset, one record per individual. It is reasonable to assume that (1) the records from each companies’ dataset are drawn independently from the same underlying distribution, (2) each company collects the same or a heavily overlapping set of attributes (perhaps after some derivation), and (3) if one company releases a perturbed dataset, the other knows the attributes of that dataset. As such, a malicious insider at the other company has a known sample.

\(^2\)These samples are not assumed to have been drawn from the original dataset \( X \), rather, independently from the same distribution that \( X \) was.
B. Results Summary

The first attack technique we develop is called the known input attack, the attacker is assumed to have known input prior knowledge, and proceeds as follows. (1) The attacker links as many of the known private tuples (inputs) to their corresponding columns in $Y$ (outputs). (2) The attacker chooses a Euclidean distance preserving transformation uniformly from the space of such transformations that satisfy these input-output constraints. Based on the links established in step 1, we develop a closed-form expression for the privacy breach probability. Experiments on real and synthetic data indicate that even with a small number of known inputs, the attack can achieve a high privacy breach probability.

The second attack technique we develop is called the known sample attack, the attacker is assumed to have known sample prior knowledge. The attacker uses the relationship between the eigenvectors of the perturbed data and the known sample data to estimate the private data, $X$, from the public perturbed data, $Y$. On real and synthetic data, we empirically study this attack and observe decreasing accuracy in three cases: (1) as the known sample size decreases, (2) as the separation between the eigenvalues of $\Sigma_V$ (the covariance matrix of $V$) decreases, and (3) as certain types of symmetries become more pronounced in the p.d.f. of $V$. The quality decrease in the first two cases is due to the fact that the eigenstates of $V$ are difficult to estimate well. The quality decrease in the third case is due to inherent ambiguity present in the eigenstates of $V$, namely, they are determined up to minor flips of the normalized eigenvectors.

II. RELATED WORK

This section presents a brief overview of the literature on data perturbation for PPDM. There is another class of PPDM techniques using secure multi-party computation (SMC) protocols for implementing common data mining algorithms across distributed datasets. We refer interested readers to [15] for more details.

Additive perturbation: Adding $i.i.d.$ white noise to protect data privacy is one common approach for statistical disclosure control [6]. The perturbed data allows the retrieval of aggregate statistics of the original data (e.g. sample mean and variance) without disclosing values of individual records. Moreover, additive white noise perturbation has received attention in the data mining literature from the perspective (described at the beginning of Section I). Clearly, additive noise does not preserve Euclidean distance perfectly. However, it can be shown that additive noise
preserves the squared Euclidean distance between data tuples on expectation, but, the associated variance is large. We defer the details of this analysis to future work and do not consider additive noise further in this paper.

**Multiplicative perturbation:** Two traditional multiplicative data perturbation schemes were studied in the statistics community [16]. One multiplies each data element by a random number that has a truncated Gaussian distribution with mean one and small variance. The other takes a logarithmic transformation of the data first, adds multivariate Gaussian noise, then takes the exponential function $\exp(.)$ of the noise-added data. These perturbations allow summary statistics (e.g., mean, variance) of the attributes to be estimated, but do not preserve Euclidean distances among records.

To assess the security of traditional multiplicative perturbation together with additive perturbation, Trottini *et al.* [17] proposed a Bayesian intruder model that considers both prior and posterior knowledge of the data. Their overall strategy of attacking the privacy of perturbed data using prior knowledge is the same as ours. However, they particularly focused on linkage privacy breaches, where an intruder tries to identify the identity (of a person) linked to a specific record; while we are primarily interested in data record recovery. Moreover, they did not consider Euclidean distance preserving perturbation as we do.

**Data anonymization:** Samarati and Sweeney [5], [18] developed the *k-anonymity* framework wherein the original data is perturbed so that the information for any individual cannot be distinguished from at least $k$-1 others. Values from the original data are generalized (replaced by a less specific value) to produce the anonymized data. This framework has drawn lots of attention because of its simple privacy definition. A variety of refinements have been proposed, see discussions on k-anonymity in various chapters in [19]. None of these approaches consider Euclidean distance preserving perturbation as we do.

**Data micro-aggregation:** Two multivariate micro-aggregation approaches have been proposed by researchers in the data mining area. The technique presented by Aggarwal and Yu [20] partitions the original data into multiple groups of predefined size. For each group, a certain level of statistical information (e.g., mean and covariance) is maintained. This statistical information is used to create anonymized data that has similar statistical characteristics to the original dataset.

\(^5\)To our knowledge, such observations have not been made before.
Li et al. [21] proposed a kd-tree based perturbation method, which recursively partitions a dataset into subsets which are progressively more homogeneous after each partition. The private data in each subset is then perturbed using the subset average. The relationships between attributes are argued to be preserved reasonably well. Neither of these two approaches preserve Euclidean distance between the original data tuples.

**Data swapping and shuffling:** Data swapping perturbs the dataset by switching a subset of attributes between selected pairs of records so that the individual record entries are unmatched, but the statistics are maintained across the individual fields. A variety of refinements and applications of data swapping have been addressed since its initial appearance. We refer readers to [22] for a thorough treatment. Data shuffling [23] is similar to swapping, but is argued to improve upon many of the shortcomings of swapping for numeric data. However, neither swapping or shuffling preserves Euclidean distance which is the focus of this paper.

**Some other data perturbation techniques:** Evfimievski et al. [24], Rizvi and Haritza [25] considered the use of categorical data perturbation in the context of association rule mining. Their algorithms delete real items and add bogus items to the original records. Association rules present in the original data can be estimated from the perturbed data. Along a related line, Verykios et al. [26] considered perturbation techniques which allow the discovery of some association rules while hiding others considered to be sensitive.

**A. Most Related Work**

In this part, we describe research most related to this paper. The majority of this focuses on Euclidean distance preserving data perturbation.

Oliveira and Zaiane [8], [9], Chen and Liu [7] discussed the use of geometric rotation for clustering and classification. These authors observed that the distance preserving nature of rotation makes it useful in PPDM, but did not analyze its privacy limitations, nor did they consider prior knowledge.

Chen et al. [12] also discussed a known input attack technique. Unlike ours, they considered a combination of distance preserving data perturbation followed by additive noise. And, they assumed a stronger form of known input prior knowledge: the attacker knows a subset of private data records and knows to which perturbed tuples they correspond. Finally, they assume that the number of linearly independent known input data records is no smaller than \( n \) (the dimensionality
of the records). They pointed out that linear regression can be used to re-estimate private data tuples.

Mukherjee et al. [11] considered the use of discrete Fourier transformation (DFT) and discrete cosine transformation (DCT) to perturb the data. Only the high energy DFT/DCT coefficients are used, and the transformed data in the new domain approximately preserves Euclidean distance. The DFT/DCT coefficients were further permuted to enhance the privacy protection level. Note that DFT and DCT are (complex) orthogonal transforms. Hence their perturbation technique can be expressed as left multiplication by a (complex) orthogonal matrix (corresponding to the DFT/DCT followed by a perturbation of the resulting coefficients), then a left multiplication by an identity matrix with some zeros on the diagonal (corresponding to dropping all but the high-energy coefficients). They did not consider attacks based on prior knowledge. As future work, it would be interesting to do so.

Turgay et al. [13] extended some of the results in our conference version of this work [10]. They assume that the similarity matrix of the original data is made public rather than, $Y$, the perturbed data itself. They describe how an attacker, given at least $n + 1$ linearly independent original data tuples and their corresponding entries in the similarity matrix, can recover the private data. Like Chen et al., this differs from our known input attack in two main ways: (i) we do not require prior knowledge beyond the known input tuples; (ii) our attack analysis smoothly encompasses the case where the number of linearly independent known input tuples is greater than $n$ as well as less. Turgay et al. also describe how an attacker, given the underlying probability distribution of the original data, can use PCA to re-estimate the original data. This approach is based on ours in [10], with the following differences. First, they assume that the global distribution of the private data is known, but we only assume a small sample drawn from the same distribution is known. Second, they use a simple (and clever) heuristic to find the best eigenvector mirror directions while we use a complete, enumerative search. While their approach has only linear computational complexity with respect to data dimensionality and our is exponential, their approach will not produce as good an eigenvector matching as ours. It is an interesting direction for future work to explore empirically and analytically how well the results of their heuristic search fare against the results of our complete, enumerative search.

Ting et al. [27] considered left-multiplication by a randomly generated orthogonal matrix. However, they assume the original data tuples are rows rather than columns as we do. As a result,
Euclidean distance between original data tuples is not preserved, but, sample mean and covariance are. If the original data arose as independent samples from multi-variate Gaussian distribution, then the perturbed data allows inferences to be drawn about this underlying distribution just as well as the original data. For all but small or very high-dimensional datasets, their approach is more resistant to prior knowledge attacks than Euclidean distance preserving perturbations. Their perturbation matrix is \( m \times m \) (\( m \) the number of original data tuples), much bigger than Euclidean distance preserving perturbation matrices, \( n \times n \) (\( n \) the number of entries in each original data tuple).

Mukherjee et al. [28] considered additive noise to the most dominate principal components of the dataset along with a modification of k-nearest-neighbor classification on the perturbed data to improve accuracy. Moreover, they nicely extend to additive noise the \( \rho_1 \)-to-\( \rho_2 \) privacy breach measure originally introduced for categorical data in [24]. Their approach, however, does not preserve Euclidean distance, thus is fundamentally different than the perturbation techniques we consider.

Before we briefly describe another two attacks based on independent component analysis (ICA) [29], it is necessary to give a brief ICA overview.

1) ICA Overview: Given an \( n' \)-variate random vector \( V \), one common ICA model posits that this random vector was generated by a linear combination of independent random variables, \( i.e., V = AS \) with \( S \) an \( n \)-variate random vector with independent components. Typically, \( S \) is further assumed to satisfy the following additional assumptions: (i) at most one component is distributed as a Gaussian; (ii) \( n' \geq n \); and (iii) \( A \) has rank \( n \).

One common scenario in practice: there is a set of unobserved samples (the columns of \( n \times q \) matrix \( S \)) that arose from \( S \) which satisfies (i) - (iii) and whose components are independent. But observed is \( n' \times q \) matrix \( V \) whose columns arose as linear combination of the rows of \( S \). The columns of \( V \) can be thought of as samples that arose from a random vector \( V \) which satisfies the above generative model. There are ICA algorithms whose goal is to recover \( S \) and \( A \) from \( V \) up to a row permutation and constant multiple. This ambiguity is inevitable due to the fact that for any diagonal matrix (with all non-zeros on the diagonal) \( D \), and permutation matrix \( P \), if \( A, S \) is a solution, then so is \( (ADP), (P^{-1}D^{-1}S) \).

2) ICA Based Attacks: Liu et al. [30] considered matrix multiplicative data perturbation, \( Y = MX \), where \( M \) is an \( n' \times n \) matrix with each entry generated independently from the same
distribution with mean zero and variance $\sigma^2$. They discussed the application of the above ICA approach to estimate $X$ directly from $Y$: $S = \mathcal{X}$, $V = \mathcal{Y}$, $S = X$, $V = Y$, and $A = M$. They argued the approach to be problematic because the ICA generative model imposes assumptions not likely to hold in many practical situations: the components of $\mathcal{X}$ are independent with at most one such being Gaussian distributed. Moreover, they pointed out that the row permutation and constant multiple ambiguity further hampers accurate recovery of $X$. A similar observation is made later by Chen et al. [12].

Guo and Wu [31] considered matrix multiplicative perturbation assuming only that $M$ is an $n \times n$ matrix (orthogonal or otherwise). They assumed the attacker has known input prior knowledge, i.e. she knows, $\tilde{X}$, a collection of original data columns from $X$. They develop an ICA-based attack technique for estimating the remaining columns in $X$. To avoid the ICA problems described in the previous paragraph, they instead applied ICA separately to $\tilde{X}$ and $Y$ producing representations $(A_{\tilde{X}}, S_{\tilde{X}})$ and $(A_Y, S_Y)$. They argued that these representations are related in a natural way allowing $X$ to be estimated. Their approach is similar in spirit to our known sample attack which related $S$ and $Y$ through representations derived through eigen-analysis.

III. EUCLIDEAN DISTANCE PRESERVING PERTURBATION AND PRIVACY BREACHES

In this section, the definition of $T$, a Euclidean distance preserving data perturbation is provided, as well as the definition of a privacy breach.

A. Notation and Conventions

Throughout this paper, unless otherwise stated, the following notations and conventions are used. “Euclidean distance preserving” and “distance preserving” are used interchangeably. All matrices and vectors discussed are assumed to have real entries (unless otherwise stated). All vectors are assumed to be column vectors and $M'$ denotes the transpose of any matrix $M$. Given a vector $x$, $||x||$ denotes its Euclidean norm. An $m \times n$ matrix $M$ is said to be orthogonal if $M'M = I_n$, the $n \times n$ identity matrix. The set of all $n \times n$, orthogonal matrices is denoted by $O_n$.

\footnote{If $M$ is square, it is orthogonal if and only if $M' = M^{-1}$ [32, pg. 17].}
Given \( n \times p \) and \( n \times q \) matrices \( A \) and \( B \), let \([A|B]\) denote the \( n \times (p + q) \) matrix whose first \( p \) columns are \( A \) and last \( q \) are \( B \). Likewise, given \( p \times n \) and \( q \times n \) matrices \( A \) and \( B \), let 
\[
\begin{bmatrix}
A \\
B
\end{bmatrix}
\]
denote the \((p + q) \times n\) matrix whose first \( p \) rows are \( A \) and last \( q \) are \( B \).

The data owner’s private dataset is represented as an \( n \times m \) matrix \( X \), with each column a record and each row an attribute (each record is assumed to be non-zero). The data owner applies a Euclidean distance preserving perturbation to \( X \) to produce an \( n \times m \) data matrix \( Y \), which is then released to the public or another party for analysis. That \( Y \) was produced from \( X \) by a Euclidean distance preserving data perturbation (but not which one) is also made public.

### B. Euclidean Distance Preserving Perturbation

A function \( H : \mathbb{R}^n \to \mathbb{R}^n \) is Euclidean distance preserving if for all \( x, y \in \mathbb{R}^n \), \(|\|x - y\|\| = |\|H(x) - H(y)\|\|. \) Here \( H \) is also called a rigid motion. It has been shown that any distance preserving function is equivalent to an orthogonal transformation followed by a translation [32, pg. 128]. In other words, \( H \) may be specified by a pair \((M, v) \in O_n \times \mathbb{R}^n\), in that, for all \( x \in \mathbb{R}^n \), \( H(x) = Mx + v \). If \( v = 0 \), \( H \) preserve Euclidean length: \(|\|x\|\| = |\|H(x)\|\|\), as such, it moves \( x \) along the surface of the hyper-sphere with radius \(|\|x\|\|\) and centered at the origin.

We do not assume that the correspondence between the columns of the perturbed dataset \( T(X) = Y \) (denoted \( y_1, \ldots, y_m \)) and the columns of the private dataset \( X \) (denoted \( x_1, \ldots, x_m \)) is known; \( i.e. \) the perturbed version of \( x_i \) is not necessarily \( y_i \). Instead, the columns of \( X \) are transformed using a Euclidean distance preserving function, then are permuted to produce the columns of the perturbed dataset \( Y \). Formally, the perturbed dataset \( Y \), is produced as follows.

The private data owner chooses \((M_T, v_T)\), a secret Euclidean distance preserving function, and \( \pi \), a secret permutation of \( \{1, \ldots, m\} \). Then, for \( 1 \leq i \leq m \), the data owner produces \( y_{\pi(i)} = M_Tx_i + v_T \).

Euclidean distance between the private data tuples is preserved in the perturbed dataset: for all \( 1 \leq i, j \leq m \), \(|\|x_i - x_j\|\| = |\|y_{\pi(i)} - y_{\pi(j)}\|\|\). Moreover, if \( v_T = 0 \), then length of the private data tuples is also preserved: for all \( 1 \leq i \leq m \), \(|\|x_i\|\| = |\|y_{\pi(i)}\|\|\).
C. Privacy Breaches

Based on the assumptions described earlier, the attacker will employ a stochastic attack technique and produce $1 \leq j \leq m$ and non-zero, $\hat{x} \in \mathbb{R}^n$. Here, $\hat{x}$ is an estimate of $x_j$ (with $\hat{j}$ denoting $\pi^{-1}(j)$), the private data tuple that was perturbed to produce $y_j$. Given $\epsilon > 0$, we consider three different privacy breach definitions.

**Definition 3.1:**
1) An $\epsilon$-privacy breach occurs if $||\hat{x} - x_j|| \leq ||x_j||\epsilon$, i.e. if the attacker’s estimate is wrong with Euclidean relative error no more than $\epsilon$.
2) An $\epsilon$-MED-privacy breach (Minimum Entry Difference) occurs if $\min_i \{NAD(x_{j,i}, \hat{x}_i)\} \leq \epsilon$ where $x_{j,i}$ and $\hat{x}_i$ are the $i^{th}$ entries and $NAD(a, \hat{a})$ is the normalized absolute difference: equals $\hat{a}$ if $a = 0$, otherwise, equals $|a - \hat{a}|/|a|$.
3) An $\epsilon$-cos-privacy breach occurs if $1 - \cos(x_j, \hat{x}) \leq \epsilon$ where $\cos(w, \hat{w})$ denotes the cosine distance $\langle w'\hat{w}\rangle/(||w||||\hat{w}||)$.

The relative Euclidean distance breach definition is inappropriate in situations where the accurate recovery of even one entry of a private data tuple is unacceptable to the data owner. The MED breach definition is intended for this situation. Moreover, the relative Euclidean distance breach definition is inappropriate for very high dimensional data (due to the curse of dimensionality) or where accuracy recovery of a private data tuple up to a scaling factor is unacceptable to the data owner. The cos breach definition is intended for these situations.

In the next two sections, we describe and analyze an attack technique for each type of prior knowledge listed in Section I. The main focus of analysis concerns, $\rho(\epsilon)$, the probability that an $\epsilon$-privacy breach occurred. However, we briefly discuss how the analysis can be applied to the probability that an $\epsilon$-MED-privacy breach and $\epsilon$-cos-privacy breach occurred.

IV. KNOWN INPUT ATTACK

For $1 \leq a \leq m - 1$, let $X_a$ denote the first $a$ columns of $X$. The attacker is assumed to know $X_a$ and her attack proceeds in two steps. (1) Infer as many as possible of the input-output

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5Note that $X$ and $Y$ are fixed.
6The attacker does not need to know $\hat{j}$; she is merely producing an estimate of the private data tuple that was perturbed to produce $y_j$.
7Note, $0 \leq \cos(w, \hat{w}) \leq 1$, equaling 1 if and only if $w$ and $\hat{w}$ differ only by a scaling factor.
mappings in \( \pi_a \) (the restriction of \( \pi \) to \( \{1, \ldots, a\} \))\(^8\). (2) Using known inputs along with their inferred outputs, produce \( \hat{x} \).

The bulk of our work involves the development and analysis of an attack technique in the case where the data perturbation is assumed to be orthogonal (does not involve a fixed translation, \( v_T = 0 \)). The majority of this section is dedicated to developing and analyzing an attack in this case. Then, in Appendix I we briefly describe how the attack and analysis can be extended to arbitrary Euclidean distance preserving perturbation (\( v_T \neq 0 \)).

A. Inferring \( \pi_a \)

The attacker may not have enough information to infer \( \pi_a \), so, her goal is to infer \( \pi_I \) (the restriction of \( \pi \) to \( I \subseteq \{1, \ldots, a\} \)), for as large an \( I \) as possible. Next, we describe how the goal can be precisely stated as an algorithmic problem that the attacker can address given her available information.

Given \( I \subseteq \{1, \ldots, a\} \), an assignment on \( I \) is a 1-1 function \( \alpha : I \rightarrow \{1, \ldots, m\} \). An assignment \( \alpha \) on \( I \) is valid if it satisfies both of the following conditions for all \( i, j \in I \), (1) \( ||x_i|| = ||y_{\alpha(i)}|| \) and (2) \( ||x_i - x_j|| = ||y_{\alpha(i)} - y_{\alpha(j)}|| \). There is at least one valid assignment on \( I \), namely \( \pi_I \), but, there may be more. \( I \) is uniquely valid if \( \pi_I \) is the only valid assignment on \( I \). Given uniquely valid \( I \subseteq \{1, \ldots, a\} \), \( I \) is said to be maximal if there does not exist uniquely valid \( J \subseteq \{1, \ldots, a\} \) such that \( |J| > |I| \). It can be shown that there exists only one maximal uniquely valid subset of \( \{1, \ldots, a\} \). Thus, the attacker’s goal is to find the maximal uniquely valid subset of \( \{1, \ldots, a\} \) along with its corresponding assignment.

The following straight-forward algorithm will meet the attackers goal by employing a top-down, level-wise search of the subset space of \( \{1, \ldots, a\} \). The inner for-loop uses an implicit linear ordering to enumerate the size \( \ell \) subsets without repeats and requiring \( O(1) \) space.

\begin{algorithm}
1: For \( \ell = a, \ldots, 1 \), do
2: For all \( I \subseteq \{1, \ldots, a\} \) and \( |I| = \ell \), do
3: If \( I \) is uniquely valid, then output \( I \) along with its corresponding assignment and terminate the algorithm.
4: Otherwise output \( \emptyset \).
\end{algorithm}

\(^8\)That is, find as many as possible perturbed counterparts of \( X_a \) in \( Y \).
Now we develop an algorithm that, given \( I \subseteq \{1, \ldots, a\} \), determines if \( I \) is uniquely valid, and, if so, also computes the corresponding assignment. The idea is search the space of all assignments on \( I \) for valid ones. Once more than one valid assignment is identified, the search is cut-off and the algorithm outputs that \( I \) is not uniquely valid. Otherwise, exactly one valid assignment, \( \pi_I \), will be found. In this case, the algorithm outputs that \( I \) is uniquely valid and returns the corresponding assignment. The algorithm performs a depth-first search employing a simple, but effective, pruning rule to eliminate possible assignment choices at each node in the search tree. The rationale for this pruning rule is described next.

Given \( I_1 \subseteq I \), \( \alpha_1 \) a valid assignment on \( I_1 \), and \( \hat{i} \in (I \setminus I_1) \), let \( C(\alpha_1, \hat{i}) \) denote the set of all \( j \in (\{1, \ldots, m\} \setminus \alpha_1(I_1)) \) which satisfies both of the following conditions: (1) \(|x_i| = |y_j|\), and (2) for all \( i_1 \in I_1 \), \(|x_{i_1} - x_i| = |y_{\alpha_1(i_1)} - y_j|\). \( C(\alpha_1, \hat{i}) \) can be thought of as the set of all valid candidate assignments for \( \hat{i} \) as an extension of valid assignment \( \alpha_1 \). This provides pruning in the search through the assignment space over \( I \) as validated by the following theorem (whose proof is straight-forward).

**Theorem 4.1:** Given \( I_1 \subseteq I \) and \( \hat{i} \in (I \setminus I_1) \), let \( \alpha_1 \) and \( \hat{\alpha}_1 \) be valid assignments on \( I_1 \) and \( (I_1 \cup \{\hat{i}\}) \), respectively. Further assume that \( \hat{\alpha}_1 \) extends \( \alpha_1 \) in the following sense: for all \( \ell \in I_1 \), \( \hat{\alpha}_1(\ell) = \alpha_1(\ell) \). Then, it is the case that \( \hat{\alpha}_1(\hat{i}) \in C(I_1, \hat{i}) \).

Each node in the depth first search tree represents a valid assignment \( \alpha_1 \) over some subset, \( I_1 \), of \( I \). The next step in the search extends the assignment by choosing \( \hat{i} \in (I \setminus I_1) \) and choosing an assignment for \( \hat{i} \) from \( C(I_1, \hat{i}) \).

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**Algorithm IV-A.2** Determining Unique Validity Main

**Inputs:** \( I \subseteq \{1, \ldots, a\} \).

1: Set global variable \( NumValidAssignFound = 0 \).

2: Call Algorithm [IV-A.3] on inputs \( \emptyset \) and \( \alpha_{\emptyset} \) (\( \alpha_{\emptyset} \) denotes the unique valid assignment on \( \emptyset \)).

3: If \( NumValidAssignFound > 1 \), then return “\( I \) IS NOT UNIQUELY VALID”. Else, return “\( I \) IS UNIQUELY VALID WITH ASSIGNMENT” \( \alpha_I \).

**Comment:** The order by which the elements of \( (I \setminus I_1) \) and \( C(I_1, \hat{i}) \) are chosen in iterating through the for loops in Algorithm [IV-A.3] does not affect the correctness of the algorithm. However, it may affect efficiency. For simplicity, the loops order the elements in these sets from smallest to largest index number.
Algorithm IV-A.3 Determining Unique Validity Recursive

**Inputs:** $I_1 \subseteq I$ and $\alpha_1$ a valid assignment on $I_1$.

1. If $I_1 = I$, then
2. $\text{NumValidAssignFound}++$
3. If $\text{NumValidAssignFound} == 1$, then set $\alpha_I$ to $\alpha_1$.
4. End If.
5. Else, do
6. For $\hat{i} \in (I \setminus I_1)$ and as long as $\text{NumValidAssignFound} \leq 1$, do
7. For $j \in C(I_1, \hat{i})$ and as long as $\text{NumValidAssignFound} \leq 1$, do
8. Extend $\alpha_1$ to $\hat{\alpha}_1$ s.t. $\hat{\alpha}_1(\hat{i}) = j$. Let $\hat{I}_1 = I_1 \cup \hat{i}$.
9. Call algorithm IV-A.3 on inputs $\hat{I}_1$ and $\hat{\alpha}_1$.

Algorithm IV-A.1 has worst-case computational complexity $O(m^a)$. While this is no better than a simple brute-force approach, in our experiments, quite reasonable running times are observed because, few original data tuples will have the same length and/or few pairs of original data tuples will have the same Euclidean distance.

**B. Known Input-Output Attack**

Assume, without loss of generality, that the attacker applies Algorithm IV-A.1 and learns $\pi_q$ $(0 \leq q \leq a)$, i.e. \{1, \ldots, q\} is the maximal uniquely valid subset of \{1, \ldots, a\}. Further, to simply notation, we may also assume that $\pi_q(\hat{i}) = \hat{i}$. Let $Y_q$ denote the first $q$ columns of $Y$. As such, the attacker is assumed to know $X_q$ and the fact that $Y_q = M_T X_q$ where $M_T$ is an unknown orthogonal matrix. Based on this, she will apply an attack, called the known input-output attack, to produce $q < j \leq m$, and $\hat{x}$, which is an estimate of $x_j$, the private tuple that was perturbed to produce $y_j$.

The attack is performed in two steps: 1) Using $X_q$ and $Y_q$, the attacker will produce $\hat{M}$, an estimation of $M_T$; 2) Then, for any $q < j \leq m$, the attacker can produce estimate

$$\hat{x} = \hat{M}' y_j.$$ (1)

Let $\rho(x_j, \epsilon)$ denote $Pr(||\hat{x} - x_j|| \leq ||x_j||\epsilon)$, the probability that an $\epsilon$-privacy breach will result from the attacker estimating $x_j$ as $\hat{x}$. We will develop a closed-form expression for $\rho(x_j, \epsilon)$.

---

9This can be achieved by the attacker appropriately reordering the columns of $X_a$ and $Y$.

10If $\hat{M} \approx M_T$, then $\hat{x} \approx M_T' y_j = M_T'(M_T x_j) = x_j$ where $x_j$ was the private tuple perturbed to produce $y_j$. 
This expression will only involve information known to the attacker, therefore, she can choose

$q < j \leq m$ so as to maximize $\rho(x_j, \epsilon)$.

Since $Y_q = M_T X_q$, the attacker knows that $M_T$ must have been drawn from $\mathcal{M}(X_q, Y_q)$, which is the set of all $M \in \mathcal{O}_n$ such that $MX_q = Y_q$. However, since the attacker has no additional information for further narrowing down this space of the possibilities, she will assume each is equally likely to be $M_T$. She will choose $\hat{M}$ uniformly from $\mathcal{M}(X_q, Y_q)$. In most cases, $\mathcal{M}(X_q, Y_q)$ is uncountable. As such, it is not obvious how to choose $\hat{M}$ uniformly from $\mathcal{M}(X_q, Y_q)$ and also not obvious how to compute $\rho(x_j, \epsilon) = Pr(||\hat{x} - x_j|| \leq ||x_j||\epsilon)$. These issues will be discussed in Section IV-D. Before doing so, we discuss some important linear algebra background.

C. Linear Algebra background

Let $Col(X_q)$ denote the column space of $X_q$ and $Col_\perp(X_q)$ denote its orthogonal complement, i.e., \{ $z \in \mathbb{R}^n : z'w = 0, \forall w \in Col(X_q)$ \}. Likewise, let $Col(Y_q)$ denote the column space of $Y_q$ and $Col_\perp(Y_q)$ denote its orthogonal compliment. Let $k$ denote the dimension of $Col(X_q)$. The “Fundamental Theorem of Linear Algebra” [33, pg. 95] implies that the dimension of $Col_\perp(X_q)$ is $n - k$. Since $Y_q = M_T X_q$ and $M_T$ is orthogonal, then it can be shown that $Col(Y_q)$ has dimension $k$. Thus, $Col_\perp(Y_q)$ has dimension $n - k$.

Let $U_k$ and $V_k$ denote $n \times k$ matrices whose columns form an orthonormal basis for $Col(X_q)$ and $Col(Y_q)$, respectively. It can easily be shown that $Col(M_T U_k) = Col(Y_q) = Col(V_k)$. Let $U_{n-k}$ and $V_{n-k}$ denote $n \times (n - k)$ matrices whose columns form an orthonormal basis for $Col_\perp(X_q)$ and $Col_\perp(Y_q)$, respectively. It can easily be shown that $Col(M_T U_{n-k}) = Col_\perp(Y_q) = Col(V_{n-k})$.

D. A Closed-Form Expression for $\rho(x_j, \epsilon)$

Now we return to the issue of how to choose $\hat{M}$ uniformly from $\mathcal{M}(X_q, Y_q)$ and how to compute $\rho(x_j, \epsilon) = Pr(||\hat{x} - x_j|| \leq ||x_j||\epsilon) = Pr(||\hat{M}' M_T x_j - x_j||)$.

To choose $\hat{M}$ uniformly from $\mathcal{M}(X_q, Y_q)$, the basic idea is to utilize standard algorithms for choosing a matrix $P$ uniformly from $\mathcal{O}_{n-k}$, then apply an appropriately designed transformation.

\[11\text{This uniform choice of } \hat{M} \text{ is equivalent to a maximum likelihood estimate of } x_j \text{ for any } q < j \leq m.\]
bijection from $\mathcal{O}_{n-k}$ to $\mathbb{M}(X_q, Y_q)$. The following technical result, proven in Appendix I, provides this transformation.

**Theorem 4.2:** Let $L$ be the mapping $P \in \mathcal{O}_{n-k} \mapsto M_T U_k^t U_k' + V_{n-k} P U_{n-k}'$. Then, $L$ is an affine bijection from $\mathcal{O}_{n-k}$ to $\mathbb{M}(X_q, Y_q)$. And, $L^{-1}$ is the mapping $M \in \mathbb{M}(X_q, Y_q) \mapsto V_{n-k} M U_{n-k}$.

**Algorithm IV-D.1 Uniform Choice From $\mathbb{M}(X_q, Y_q)$**

| Inputs: $U_k$, an $n \times k$ matrix whose columns form an orthonormal basis of $\text{Col}(X_q)$, and $M_T U_k$ ($M_T$ is unknown); $U_{n-k}$ and $V_{n-k}$, $n \times (n-k)$ matrices whose columns form an orthonormal basis of $\text{Col}_L(X_q)$ and $\text{Col}_L(Y_q)$, respectively. |
|---|
| Outputs: $\hat{M}$ a uniformly chosen matrix from $\mathbb{M}(X_q, Y_q)$. |
| 1: Choose $P$ uniformly from $\mathcal{O}_{n-k}$ using algorithm [34]. |
| 2: Set $\hat{M} = L(P)$, i.e., $M_T U_k U_k' + V_{n-k} P U_{n-k}'$. |

Some special cases are interesting to highlight: when $k = n$, $\hat{M}$ is chosen as $M_T$; when $k = n-1$, $\hat{M}$ is one of two choices (one of which equals $M_T$); otherwise, $\hat{M}$ is, in theory, chosen from an uncountable set (containing $M_T$).

Now we develop a closed-form expression for $\rho(x_j, \epsilon)$. The key points are outlined, while a more rigorous justification is provided in Appendix I. First of all, from Algorithm IV-D.1 $\hat{M} = M_T U_k U_k' + V_{n-k} P U_{n-k}'$ where $P$ is chosen uniformly from $\mathcal{O}_{n-k}$. Therefore,

$$\rho(x_j, \epsilon) = Pr(||\hat{M} M_T x_j - x_j|| \leq ||x_j|| \epsilon)$$

$$= Pr(||U_k U_k' x_j + U_{n-k} P' V_{n-k}' M_T x_j - x_j|| \leq ||x_j|| \epsilon).$$

Since $\begin{bmatrix} U_k' \\ U_{n-k}' \end{bmatrix} \in \mathcal{O}_n$, then it can left-multiply each term in the left $||\ldots||$ of the second probability without changing the equality. As a result, the derivation continues

$$\cdots = Pr\left(||\begin{bmatrix} U_k' x_j \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ P' V_{n-k}' M_T x_j \end{bmatrix} - \begin{bmatrix} U_k' x_j \\ U_{n-k}' x_j \end{bmatrix}|| \leq ||x_j|| \epsilon\right)$$

$$= Pr(||P' V_{n-k}' M_T x_j - U_{n-k}' x_j|| \leq ||x_j|| \epsilon).$$

12 That the resulting $\hat{M}$ was chosen uniformly from $\mathbb{M}(X_q, Y_q)$ could be more rigorously justified using left-invariance of probability measures and the Haar probability measure over $\mathcal{O}_{n-k}$. But, such a discussion is not relevant to this paper and is omitted.

13 We define $\mathcal{O}_0$ to contain a single, empty matrix. And, for $P \in \mathcal{O}_0$, we define $V_{n-k} P U_{n-k}'$ to be the $n \times n$ zero matrix.
Since $Col(MT_{n-k}) = Col(V_{n-k})$, then there exists $(n-k) \times (n-k)$ matrix $B$ such that $MT_{n-k}B = V_{n-k}$. It follows that (i) $V'_{n-k} = B'U'_{n-k}M_{T}^{T}$, (ii) $B = U'_{n-k}M_{T}^{T}V_{n-k}$. Thus, $B$ is orthogonal\footnote{$B' = B'U'_{n-k}M_{T}^{T}V_{n-k} = V_{n-k}V_{n-k} = I_{n-k}$.}

Using (i), the derivation continues

\[ \cdots = Pr(||P'B'(U'_{n-k}x_{j}) - (U'_{n-k}x_{j})|| \leq ||x_{j}||\epsilon) \tag{2} \]

\[ = Pr(||P'(U'_{n-k}x_{j}) - (U'_{n-k}x_{j})|| \leq ||x_{j}||\epsilon) \tag{3} \]

where the second equality is due to the fact that $B' \in \mathbb{O}_{n-k}$, and thus $(P'B')$ can be regarded as having been uniformly chosen from $\mathbb{O}_{n-k}$ just like $P'$ (a rigorous proof of the second equality is provided in Appendix I). Putting the whole derivation together,

\[ \rho(x_{j}, \epsilon) = Pr(P \text{ uniformly chosen from } \mathbb{O}_{n-k} \text{ satisfies } ||P'(U'_{n-k}x_{j}) - (U'_{n-k}x_{j})|| \leq ||x_{j}||\epsilon). \tag{4} \]

Let $S_{n-k}(||U'_{n-k}x_{j}||)$ denote the hyper-sphere in $\mathbb{R}^{n-k}$ with radius $||U'_{n-k}x_{j}||$ and centered at the origin. Since $P$ is chosen uniformly from $\mathbb{O}_{n-k}$, then any point on the surface of $S_{n-k}(||U'_{n-k}x_{j}||)$ is equally likely to be $P'(U'_{n-k}x_{j})$. Let $S_{n-k}(U'_{n-k}x_{j}, ||x_{j}||\epsilon)$ denote the “hyper-sphere cap” consisting of all points in $S_{n-k}(||U'_{n-k}x_{j}||)$ with distance from $U'_{n-k}x_{j}$ no greater than $||x_{j}||\epsilon$. Therefore, (4) becomes

\[ \rho(x_{j}, \epsilon) = Pr(\text{a uniformly chosen point on } S_{n-k}(||U'_{n-k}x_{j}||) \text{ is also in } S_{n-k}(U'_{n-k}x_{j}, ||x_{j}||\epsilon)) \]

\[ = \frac{SA(S_{n-k}(U'_{n-k}x_{j}, ||x_{j}||\epsilon))}{SA(S_{n-k}(||U'_{n-k}x_{j}||))} \tag{5} \]

where $SA(.)$ denotes the surface area of a subset of a hyper-sphere\footnote{$S_{1}(||U'_{n-k}x_{j}||)$ consists of two points. We define $\frac{SA(S_{1}(U'_{n-k}x_{j}, ||x_{j}||\epsilon))}{SA(S_{1}(||U'_{n-k}x_{j}||))}$ as 0.5 if $S_{1}(U'_{n-k}x_{j}, ||x_{j}||\epsilon)$ is one point, and as 1 otherwise. Moreover, we define $\frac{SA(S_{1}(U'_{n-k}x_{j}, ||x_{j}||\epsilon))}{SA(S_{1}(||U'_{n-k}x_{j}||))}$ as 1.}

Based on equations (5), we prove, in Appendix I, the following closed form expression, for $\rho(x_{j}, \epsilon)$, where, $\Gamma(.)$ denotes the standard gamma function, $ac\Gamma \,(x)$ denotes $arccos \left( \frac{||x_{j}||\epsilon}{||U'_{n-k}x_{j}||\sqrt{2}} \right)$, and $ac\Gamma \,(x)$ denotes $arccos \left( 1 - \left[ \frac{||x_{j}||\epsilon}{||U'_{n-k}x_{j}||\sqrt{2}} \right]^{2} \right)$.
\[
\rho(x_j, \epsilon) = \begin{cases} 
1 & \text{if } n - k = 0; \\
1 & \text{if } ||x_j|| \geq ||U'_{n-k}x_j||/2 \text{ and } n - k \geq 1; \\
1 & \text{if } ||x_j|| < ||U'_{n-k}x_j||/2 \text{ and } n - k = 1; \\
0.5 & \text{if } ||U'_{n-k}x_j||/\sqrt{2} < ||x_j|| < ||U''_{n-k}x_j||/2 \text{ and } n - k = 2; \\
\left(1/\pi\right)ac_{n-1}(x) & \text{if } ||x_j|| \leq ||U''_{n-k}x_j||/\sqrt{2} \text{ and } n - k = 2; \\
\left(1/\pi\right)ac_{n-1}(y) & \text{if } ||x_j|| \leq ||U''_{n-k}x_j||/\sqrt{2} \text{ and } n - k \geq 3. \\
\end{cases}
\]

\[
\rho(x_j, \epsilon) = \begin{cases} 
1 & \text{if } n - k = 0; \\
1 & \text{if } ||y_j|| \geq ||V'_{n-k}y_j||/2 \text{ and } n - k \geq 1; \\
1 & \text{if } ||y_j|| < ||V'_{n-k}y_j||/2 \text{ and } n - k = 1; \\
0.5 & \text{if } ||V'_{n-k}y_j||/\sqrt{2} < ||y_j|| < ||V''_{n-k}y_j||/2 \text{ and } n - k = 2; \\
\left(1/\pi\right)ac_{n-1}(y) & \text{if } ||y_j|| \leq ||V''_{n-k}y_j||/\sqrt{2} \text{ and } n - k = 2; \\
\left(1/\pi\right)ac_{n-1}(y) & \text{if } ||y_j|| \leq ||V''_{n-k}y_j||/\sqrt{2} \text{ and } n - k \geq 3. \\
\end{cases}
\]

Comment: it can be shown that $||U'_{n-k}x_j||$ is the distance from $x_j$ to its closest point in $Col(X_q)$. Thus, the sensitivity of a tuple to breach is dependent upon its length relative to its distance to the column space of $X_q$.

Recall that the attacker seeks to use the closed-form expressions for $\rho(x_j, \epsilon)$ to decide for which $q < j \leq m$ does $\hat{x} = M'y_j$ produce the best estimation of $x_j$. This is naturally done by choosing $j$ to maximize $\rho(x_j, \epsilon)$. To allow for this, observe that $||x_j|| \epsilon$ and $||U'_{n-k}x_j||$ equal $||y_j|| \epsilon$ and $||V'_{n-k}y_j||$, respectively, which are known to the attacker. Therefore, (6) can be rewritten as follows, where $ac_{||-1}(y)$ denotes $arccos\left(\frac{||y_j||\epsilon}{||U''_{n-k}y_j||\sqrt{2}} - 1\right)$, and $ac_{1-||}(y)$ denotes $arccos\left(1 - \left(\frac{||y_j||\epsilon}{||V''_{n-k}y_j||\sqrt{2}}\right)^2\right)$.

To spell out the attack algorithm, first note that $U_k$, $U_{n-k}$, $V_k$, and $V_{n-k}$ can be computed from $X_q$ and $Y_q$ using standard procedures [33]. Second, $M_T'U_k = Y_qA$ where $A$ is an $q \times k$

\[^{16}M_Tx_j = y_j, \text{ so, } ||x_j|| = ||M_Tx_j|| = ||y_j||. \text{ Moreover, as shown earlier, there exists } B \in \mathbb{O}_{n-k} \text{ such that } V'_{n-k} = B'U'_{n-k}M_T'. \text{ Thus, } ||U'_{n-k}x_j|| = ||B'U'_{n-k}M_T'M_Tx_j|| = ||V'_{n-k}y_j||.\]
matrix that can be computed\textsuperscript{17} from $U_k$ and $X_q$. Third, a recursive procedure for computing (7) is described in Appendix\textsuperscript{1} The precise details of the attack technique can be seen in Algorithm IV-D.2. The $\epsilon$-privacy breach probability $\rho(\epsilon)$ equals $\max_{q < j \leq m} \rho(x_j', \epsilon)$.

Algorithm IV-D.2 Known Input Attack Algorithm

\begin{itemize}
  \item **Inputs:** $Y$, $\epsilon \geq 0$, and $X_q$. The attacker knows $Y_q = M_F X_q$ ($M_F$ is unknown).
  \item **Outputs:** $q < j \leq m$ and $\hat{x} \in \mathbb{R}^n$ the corresponding estimate of $x_j$.
\end{itemize}

\begin{algorithmic}
  \State Compute $U_k, V_k, U_{n-k}, V_{n-k}$, and $M_F U_k$ as described earlier.
  \For {each $q < j \leq m$}
  \State Compute $\rho(x_j', \epsilon)$ using (7) as described in Appendix\textsuperscript{1}
  \EndFor.
  \State Choose the $j$ from the previous loop producing the largest $\rho(x_j, \epsilon)$.
  \State Choose $\hat{M}$ uniformly from $M(X_q, Y_q)$ by applying Algorithm IV-D.1.
  \State Set $\hat{x} \leftarrow \hat{M}' y_j$.
\end{algorithmic}

E. Experiments

The experiments are designed to assess the computational efficiency of the overall known input attack and its effectiveness at breaching privacy. We used two datasets as the input $X$, respectively: 1) a 100,000 tuple synthetic dataset generated from a 100-variate Gaussian distribution\textsuperscript{18}, 2) the Letter Recognition dataset, 20,000 tuples and 16 numeric attributes, from UCI machine learning repository – we removed tuples which were duplicated over the numeric attributes yielding a final dataset of 18,668 tuples. The attacks were implemented in Matlab 7 (R14) and all experiments were carried out on a Thinkpad laptop with 1.83GHz Intel Core 2 CPU, 1.99GB RAM, and WindowsXP system.

The first experiment fixes $X$ and its perturbed version $Y$, but changes the number of known input tuples, $a$. It proceeds by carrying out ten iterations as follows. Select $a$ linearly independent tuples randomly from $X$ (these become the known inputs). Use Algorithm IV-A.1 to compute

\textsuperscript{17}Since $\text{Col}(U_k) = \text{Col}(X_q)$, then by solving $k$ systems of linear equations (one for each column of $U_k$), a $q \times k$ matrix $A$ can be computed such that $X_q A = U_k$.

\textsuperscript{18}The mean vector is specified by independently generating 100 numbers from a univariate Guassian with mean zero and variance one. The covariance matrix is specified by (i) independently generating 100 data tuples each with 100 independently generated entries a from a univariate Guassian with mean zero and variance one, (ii) computing the empirical covariance of this 100 tuple dataset.
I, the maximal uniquely valid assignment. Use steps 2-5 in Algorithm IV-D.2 to compute the \( \rho(\epsilon) \), the \( \epsilon \)-privacy breach probability (a closed-form was given immediately above Algorithm IV-D.2).

To measure the accuracy of the attack, we report the average of \( \rho(\epsilon) \) and \(|I|\) over all iterations. To measure the efficiency, we report the average time taken to compute \( I \) (the rest is ignored as the overall attack computation time is dominated by Algorithm IV-A.1). In Figures 1 and 2, results are shown with \( \epsilon = 0.15 \). In Figure 3, accuracy results are shown with varying \( \epsilon \) and \( a \) fixed at four. In all Figures, the error bars show one standard deviation above and below the average.

**Fig. 1**

**KNOWN INPUT ATTACK ON GAUSSIAN DATA WITH DIFFERENT NUMBER OF KNOWN INPUTS AND \( \epsilon = 0.15 \).**

**Fig. 2**

**KNOWN INPUT ATTACK ON LETTER RECOGNITION DATA WITH DIFFERENT NUMBER OF KNOWN INPUTS AND \( \epsilon = 0.15 \).**

The second experiment fixes the number of known input tuples (and \( \epsilon \) at 0.15) but changes the size of the original data \( X \) in order to assess the computational efficiency of the attack. For the Gaussian data, it uses the first \( k \) tuples as \( X \) where \( k \) takes a value in \{10000, 20000, \ldots, 100000\}. 
Then the attack proceeds by carrying out the following operations ten times. Select \(a = 50\) linearly independent tuples randomly from \(X\) and use Algorithm IV-A.1 to compute the maximal uniquely valid assignment \(I\). The average time taken to compute \(I\) is given in Figure 4 left. For the Letter Recognition data, \(k\) takes a value in \(\{2000, 4000, \ldots, 18000\}\) and the attack randomly select \(a = 10\) linearly independent tuples as the known inputs. The average time taken to find \(I\) is given in Figure 4 right.

Regarding the known input attack accuracy, the linking phase of the attack (Algorithm IV-A.1), exhibits excellent performance. For synthetic data, its performance is perfect in that all known input tuples have their corresponding perturbed tuple inferred (see Figure 1 left). For real data, its performance is nearly perfect – see Figure 2 left. As expected, \(\rho(\epsilon)\) approaches one as \(a\) increases see Figures 1 and 2 right. Interestingly on the synthetic dataset, the transition from \(\rho(\epsilon) = 0 \rightarrow 1\) occurs very sharply around \(a = 60\). Moreover, on the real dataset, \(\rho(\epsilon) = 1\) with \(a\) as small as 4 (and we also observe in Figure 3 that the probability remains fairly high for \(\epsilon\) as small as 0.07).

Regarding computational efficiency, the algorithm appears to require quite reasonable time in all cases observed, e.g. less that 450 seconds on the synthetic dataset with 100 known tuples (see Figure 1 center) and less than 45 seconds on the real dataset with 16 known inputs (see Figure 2 center). With respect to known input set size \((a)\), the average computation time exhibits
a linear (synthetic data) or slower (real data) trend (see Figure 1 center and Figure 2 center). With respect to dataset size (number of private data tuples), the average computation time exhibits a clear linear trend for both synthetic and real data (see Figure 4 left and right). These results demonstrate that, despite the high worst-case computational complexity, the computation times on both real and synthetic data are quite reasonable.

The experimental results support the conclusion that the attack can breach privacy in plausible situations. For example, on the 16-dimensional, 18688 tuple real dataset, the known input attack achieves a privacy breach with probability one using four known inputs and less than 30 seconds of run-time.

F. Analysis Over MED and Cos Privacy Breach Definitions

\( \epsilon \text{-MED-privacy breach:} \) It is shown in Appendix I that
\[
\min_{i=1}^{n} \{ \text{NAD}(x_{j,i} \hat{x}_{i}) \} \leq \frac{||x_{j}-\hat{x}||}{||x_{j}||}.
\]
Hence, if an \( \epsilon \)-privacy breach occurs, then so does an \( \epsilon \)-MED-privacy breach. Therefore, the analysis of the known input attack can be used to lower-bound the probability that an \( \epsilon \)-MED-privacy breach occurred. As a result, our experiments show that on the Letter Recognition data, four known inputs produce an \( \epsilon \)-MED-privacy breach with probability one. Unfortunately, the lower-bound is not tight as examples can be found making the relative Euclidean distance arbitrarily larger than the minimum MED distance.

\( \epsilon \text{-cos-privacy breach:} \) It is shown in Appendix I that
\[
1 - \cos(\hat{x}, x_{j}) = \frac{||\hat{x} - x_{j}||^2}{2||x_{j}||^2}.
\]
Therefore, an \( \epsilon \)-cos-privacy breach occurred if and only if an \( (\sqrt{2\epsilon}) \)-privacy breach occurred. Therefore, the analysis of the known input attack can be easily modified to produce a closed-form expression for the probability that an \( \epsilon \)-cos-privacy breach occurred.

V. Known Sample Attack

In this scenario, we assume that each data column of \( X \) arose as an independent sample from a random vector \( V \) with unknown p.d.f. We also make the following mild technical assumption: the covariance matrix \( \Sigma_{V} \) of \( V \) has all distinct eigenvalues. Furthermore, we assume that the attacker has a collection of \( q \) samples that arose independently from \( V \) – these are denoted as the columns of matrix \( S \). It is important to stress that the columns of \( S \) are not assumed to be samples from the private data \( X \), rather, they are samples drawn from \( V \) independently of \( X \).
Using these assumptions, we will design a Principal Component Analysis (PCA)-based attack technique and analyze its privacy breach probability through experiments.

Attack Intuition: The basic procedure is to estimate $M_T$ and use this estimate to undo the data perturbation applied to $X$. The key idea in estimating $M_T$ is that the principle components of $\Sigma (M_T V + v_T)$ equal the perturbation by $M_T$ of the principal components of $\Sigma_V$ up to a mirror flip about each component. Since these covariance matrices can be estimated from $Y$ and $S$, respectively, then so can the corresponding principal components. The equality above then allows $M_T$ to be estimated up to mirror flips. To choose the right mirror flip, an equality of distributions test is applied using $S$ and $Y$.

A. PCA Preliminaries and a Key Property

Because $\Sigma_V$ is an $n \times n$, symmetric matrix with all distinct eigenvalues, it has $n$ real eigenvalues $\lambda_1 > \ldots > \lambda_n$ and their associated eigenspaces, $\{ z \in \mathbb{R}^n : \Sigma_V z = z \lambda_i \}$, are pair-wise orthogonal with dimension one [33, pg. 295]. As is standard practice, we restrict our attention to only a small number of eigenvectors. Let $Z(V)_i$ denote the set of all vectors $z \in \mathbb{R}^n$ such that $\Sigma_V z = z \lambda_i$ and $||z|| = 1$. We call this the normalized eigenspace of $\lambda_i$. The normalized eigenspaces of $\Sigma_V$ are related in a natural way to those of $\Sigma (M_T V + v_T)$, as shown by the following theorem (proven in Appendix I). This theorem is important as it will provide the foundation for a technique by which the attacker can estimate $M_T$ from $Y$ and $S$.

Theorem 5.1: The eigenvalues of $\Sigma_V$ and $\Sigma (M_T V + v_T)$ are the same. Let $Z(M_T V + v_T)_i$ denote the normalized eigenspace of $\Sigma (M_T V + v_T)$ associated with $\lambda_i$, i.e. the set of vectors $w \in \mathbb{R}^n$ such that $\Sigma (M_T V + v_T) w = w \lambda_i$ and $||w|| = 1$. It follows that $M_T Z(V)_i = Z(M_T V + v_T)_i$, where $M_T Z(V)_i$ equals $\{ M_T z : z \in Z(V)_i \}$.

Because all the eigenspaces of $\Sigma_V$ have dimension one, it can be shown that each normalized eigenspace, $Z(V)_i$, contains only two vectors and these differ only by a factor of $-1$. Thus, letting $z_i$ denote the lexicographically larger vector, $Z(V)_i$, can be written as $\{ z_i, -z_i \}$. Let $Z$ denote the $n \times n$ eigenvector matrix whose $i^{th}$ column is $z_i$. Because the eigenspaces of $\Sigma_V$ are pairwise orthogonal and $||z_i|| = 1$, $Z$ is orthogonal. Similarly, $Z(M_T V + v_T)_i$ can be written as $\{ w_i, -w_i \}$ ($w_i$ is the lexicographically larger among $w_i, -w_i$) and $W$ is the eigenvector matrix $^{19}\Sigma (M_T V + v_T)$ denotes the covariance matrix of random vector $M_T V + v_T$.

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with \(i^{th}\) column \(w_i\) (\(W\) is orthogonal). Note again that columns in both \(Z\) and \(W\) are ordered such that the \(i^{th}\) eigenvector is associated with the \(i^{th}\) eigenvalue. The following result, proven in Appendix I, forms the basis of the attack algorithm.

**Corollary 5.2:** Let \(I_n\) be the space of all \(n \times n\), matrices with each diagonal entry \(\pm 1\) and each off-diagonal entry 0 (\(2^n\) matrices in total). There exists \(D_0 \in I_n\) such that \(M_T = WD_0Z'\).

### B. Known Sample Attack (PCA Attack) Algorithm on Orthogonal Data Perturbation

Like Section IV, we first develop the attack technique in the case where the data perturbation is assumed to be orthogonal (does not involve a fixed translation, \(v_T = 0\)). Then, in Section V-E, we discuss how the attack technique can be extended to arbitrary Euclidean distance preserving perturbation (\(v_T \neq 0\)).

First assume that the attacker knows the covariance matrices \(\Sigma_V\) and \(\Sigma_{M_TV}\) and, thus, computes \(W\), the eigenvector matrix of \(\Sigma_{M_TV}\), and \(Z\), the eigenvector matrix of \(\Sigma_V\). By Corollary 5.2, the attacker can perfectly recover \(M_T\) if she can choose the right \(D\) from \(I_n\). To do so, the attacker utilizes \(S\) and \(Y\), in particular, the fact that these arose as independent samples from \(V\) and \(M_TV\), respectively. For any \(D \in I_n\), if \(D = D_0\), then \(WDZ'S\) and \(Y\) have both arisen as independent samples from \(M_TV\). The attacker will choose \(D \in I_n\) such that \(WDZ'S\) is most likely to have arisen as an independent sample from the same random vector as \(Y\). To make this choice, the attacker can use a multi-variate two-sample hypothesis test for equal distributions [35]. Let \(p(WDZ'S,Y)\) denote the resulting \(p\)-value. The smaller the \(p\)-value, the more convincingly the null hypothesis (that \(WDZ'S\) and \(Y\) have arisen as independent samples from identically distributed random vectors) can be rejected. Therefore, \(D \in I_n\) is chosen to maximize \(p(WDZ'S,Y)\).

Finally, the attacker can eliminate the assumption at the start of the previous paragraph by replacing \(\Sigma_V\) and \(\Sigma_{M_TV}\) with estimates computed from \(S\) and \(Y\). In experiments we use the standard, sample covariance matrices \(\hat{\Sigma}_S\) and \(\hat{\Sigma}_Y\). Algorithm V-B.1 shows the complete PCA-based attack procedure.

Since the two-sample test requires \(O((m+q)^2)\) computation for \(p(.,.)\), the overall computation cost of Algorithm V-B.1 is \(O(2^n(m+q)^2)\).

Take note that the quality of covariance matrix estimation from \(S\) and \(Y\) impacts the effectiveness of the attack. Clearly, poor quality estimation will result in low attack accuracy.
Algorithm V-B.1 PCA-based Attack Algorithm on Orthogonal Data Perturbation

**Inputs:** $S$, an $n \times q$ matrix where each column arose as an independent sample from $V$ – a random vector with unknown $p.d.f.$ whose covariance matrix has all distinct eigenvalues; and such that the columns of $X$ arose as independent samples from $V$, $Y$ where $M_T$ is an unknown, $n \times n$, orthogonal matrix.

**Outputs:** $1 \leq j \leq m$ and $\hat{x} \in \mathbb{R}^n$ the corresponding estimate of $x_j$.

1. Compute sample covariance matrix $\hat{\Sigma}_S$ from $S$ and sample covariance matrix $\hat{\Sigma}_V$ from $Y$.
2. Compute the eigenvector matrix $\hat{Z}$ of $\hat{\Sigma}_S$ and $\hat{W}$ of $\hat{\Sigma}_V$. Each eigenvector has unit length and is sorted in the matrix by the corresponding eigenvalue.
3. Choose $D = \arg\max\{p(\hat{W}D\hat{Z}'S, Y) : D \in I_n\}$, choose $1 \leq j \leq m$ randomly, and set $\hat{x} \rightarrow$ the $j^{th}$ column in $\hat{Z}D\hat{W}'Y$.

With the exception of sample size, we do not consider other sampling factors (departures from independence, noise, outliers, etc.) that effect the quality of covariance matrix estimation. We feel such issues are orthogonal to this work as any technique for covariance matrix estimation can be used in the attack. For simplicity, we stick with the standard, sample covariance matrices.

**C. Experiments – Orthogonal Data Perturbation**

We conduct experiments on both synthetic and real world data to evaluate the performance of PCA-based attack on orthogonal data perturbation. We choose the perturbation matrix uniformly from $\mathcal{O}_n$ and keep it fixed for the same private data. Since the choice of $\pi$ does not affect the experiments, we choose the identity permutation throughout. To approximate the probability of privacy breach, we compute a fraction of the breach out of 100 independent runs. In all figures demonstrated in this section, a solid line is added showing a best polynomial fit to the points. This line is generated with Matlab’s curve fitting toolbox. The attack was implemented in Matlab 6 (R13) and all experiments were carried out on a Dell dual-processor workstation with 3.00GHz and 2.99GHz, Xeon CPUs, 3.00GB RAM, and WindowsXP system.

The synthetic dataset contains 10,000 data points, and it is generated from a multi-variate Gaussian distribution with mean $(10, 10, 10)$ and covariance

$$
\begin{pmatrix}
1 & 1.5 & 0.5 \\
1.5 & 3 & 2.5 \\
0.5 & 2.5 & 75
\end{pmatrix}
$$

The attacker has a sample generated independently from the same distribution. We conduct experiments to examine how sample size affects the quality of the attack. Figure 5 shows that when the relative error bound is fixed, the probability of privacy breach increases as the sample size increases.

For the real world data, we choose the Letter Recognition Database and Adult Database from November 16, 2009 DRAFT.
the UCI machine learning repository.\textsuperscript{20} The Letter Recognition data has 20,000 tuples and 16 numeric features. We choose the first 6 attributes (excluding the class label) for the experiments. Note that unlike the experiments in Section IV-E, here we do not remove duplicates. The Adult data contains 32,561 tuples, and it is extracted from the census bureau database. We select three numeric attributes: age, education-num and hours-per-week, for the experiments. We randomly separate each dataset into two disjoint sets. One set is viewed as the original data, and the other one is the attacker’s sample data. To examine the influence of sample size, we perform the same series of experiments as we do for Gaussian data. Figure 6 gives the results for Letter Recognition data. Figure 7 gives the results for Adult data.

From the above experiments, we have the following observations: (1) the larger the sample size, the better the quality of data recovery and (2) among these three datasets, the PCA-based attack works best for Gaussian data, next Letter Recognition data, and then Adult data. The first observation require no explanations. We will discuss the second one in the next section.

\textsuperscript{20}http://mlearn.ics.uci.edu/MLSummary.html
D. Effectiveness of the Known Sample Attack (PCA Attack) Algorithm on Orthogonal Data Perturbation

The effectiveness of the PCA Attack algorithm can be hampered by the presence of either two of the following properties of the p.d.f., \( f \), of \( \mathcal{V} \). (1) The eigenvalues of \( \Sigma_{\mathcal{V}} \) are nearly identical. (2) For some \( D_i \neq D_0 \in \mathbb{I}_n \), \( f \) is invariant over \( D_i \) in the sense that \( f_{D_i} \) and \( f_{D_0} \) can’t be distinguished, where \( f_{D_0} \) and \( f_{D_i} \) are the p.d.f.s of \( WD_0Z'\mathcal{V} \) and \( WD_iZ'\mathcal{V} \).

First, suppose the eigenvalues of \( \Sigma_{\mathcal{V}} \) are nearly identical. Without loss of generality, we can assume \( \mathcal{V} \) has a diagonal covariance matrix whose diagonal entries (from top-left to bottom-right) are \( d, d - \beta, d - 2\beta, \ldots, d - n\beta \) where \( d - n\beta > 0 \) and \( 0 < \beta < 1 \) is small. In this case, small errors in estimating \( \Sigma_{\mathcal{V}} \) from sample \( S \) can produce a different ordering of the eigenvectors, hence, large errors in the attacker’s recovery. As an extreme case, when \( \mathcal{V} \) is the \( n \)-variate Gaussian with covariance matrix \( I_n \gamma \) for some constant \( \gamma \), all the eigenvalues are the same, and there is only one eigenspace, \( \mathbb{R}^n \). The PCA attack algorithm will fail.

Consider the minimum ratio of any pair of eigenvalues, i.e., \( \min \{ \lambda_i/\lambda_j : \forall i \neq j; i, j = 1, \ldots, n \} \) (we call this the minimum eigen-ratio). We would expect that, the smaller this value, the smaller the attacker’s success probability. To examine this hypothesis, we generate a three-dimensional dataset of tuples sampled independently from a Gaussian with mean \((10, 10, 10)\) and covariance \[
\begin{pmatrix}
0.1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & b
\end{pmatrix}
\]. By changing the value of \( b \) from 2 to 40, we can change the minimum eigen-ratio from 1 to 20. The original data contains 10,000 tuples. We fix the sample ratio to be 2% and relative error bound \( \epsilon = 0.05 \). Figure 8 shows that when all other parameters are fixed, the higher the eigen-ratio, the better the performance of the attack algorithm. This actually explains why, in our previous experiments, PCA attack works best for Gaussian data, then Letter Recognition data, and then Adult data. A simple computation shows that the minimum eigen-ratios of the Gaussian data, Letter Recognition data and Adult data are 19.6003, 1.3109, 1.2734, respectively.

Second, suppose \( f \) is invariant over some \( D_i \neq D_0 \in \mathbb{I}_n \). Then the \( p(WD_0Z'S,Y) \) may not be larger than \( p(WD_iZ'S,Y) \), and the attack algorithm will fail. We would expect that the closer \( f \) is to invariance, the smaller the attacker’s success probability. To examine this hypothesis we need a metric for quantifying the degree to which \( f \) is invariant. Intuitively, the invariance of \( f \) can be quantified as the degree to which \( f_{D_i} \) and \( f_{D_0} \) are distinguishable (minimized over all
minimum eigen-ratio
fraction of runs with a privacy breach ($\epsilon = 0.05$)

Fig. 8
PCA-BASED ATTACK W.R.T. MINIMUM EIGEN-RATIO
($\epsilon = 0.05$ AND SAMPLE RATIO 2%).

Fig. 9
PCA-BASED ATTACK W.R.T. $\alpha$ ($\epsilon = 0.05$ AND SAMPLE RATIO 2%).

$D_i \neq D_0 \in \mathbb{I}_n$). To formalize this definition, we use the symmetric Kullback-Leibler divergence $KL(g||h) + KL(h||g)$ to measure the difference between two continuous distributions $g$ and $h$. This measurement is symmetric and nonnegative, and when it is equal to zero, the distributions can be regarded as indistinguishable. So, we quantify invariance as

$$Inv(f) = \min_{D_i \neq D_0 \in \mathbb{I}_n} \{KL(f_{D_i}||f_{D_0}) + KL(f_{D_i}||f_{D_0})\}, \quad (8)$$

Clearly $Inv(f) \geq 0$ with equality exactly when $f$ is invariant. The behavior of $Inv$ in the general case is quite complicated. However, under certain (fairly strong) assumptions, $Inv(f)$ can be nicely characterized. In Appendix I we provide derivation details of the following result. Let $\mu$ be some fixed element of $\mathbb{R}^n$. Assume that $f$ is an $n$-variate Gaussian distribution with mean vector $\mu_V = \alpha \mu$ for some $\alpha \geq 0$ and invertible covariance matrix $\Sigma_V$. We have,

$$Inv(f) = \alpha^2 \min_{D_i \neq D_0 \in \mathbb{I}_n} (\mu'Z(D_i - D_0)\Lambda_V^{-1}(D_i - D_0)Z'\mu), \quad (9)$$

where $\Lambda_V$ and $Z$ are the eigenvalue and eigenvector matrices of $\Sigma_V$, respectively. Hence, we see that $Inv(f)$ approaches zero quadratically as $\alpha \to 0$.

With this result we can carry out experiments to measure the effect of the degree to which $f$ is invariant on the attacker’s success probability. We generate a dataset by sampling each tuple
independently from a three-dimensional Gaussian with covariance
\[
\begin{pmatrix}
0.1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 40
\end{pmatrix}
\]
and mean vector \( \mu_V = \alpha (1, 1, 1)' \). Note that the minimum eigen-ratio is 20, sufficiently large to isolate the effect of decreasing invariance on attacker’s success probability. We change the value of \( \alpha \) from 0 to 10. The original dataset contains 10,000 tuples. We fix the sample ratio to be 2\%, and relative error bound \( \epsilon = 0.05 \). Figure 9 shows that as the mean approaches zero, the probability of privacy breach drops to zero too; however, as the mean runs away from zero, the probability of privacy breach increases very fast.

E. Known Sample Attack (PCA Attack) Algorithm on General Distance-Preserving Data Perturbation

In the previous subsections, we had considered the case where the data perturbation is assumed to be orthogonal (does not involve a fixed translation, \( v_T = 0 \)). Now we consider how the attack technique can be extended to arbitrary Euclidean distance preserving perturbation (\( v_T \neq 0 \)). The basic idea is very similar to that regarding the known input attack described in Appendix I. Since the same \( v_T \) is added to all tuples in the perturbation of \( X \), then by considering differences, we can transform the situation back to the orthogonal data perturbation case and apply the same attack technique described above. However, since the PCA attack assumes that the tuples in \( X \) arose \textit{independently} from \( V \), then the difference tuples over \( Y \) cannot be computed with respect to a single fixed tuple (the resulting tuples could not be regarded as having arisen independently). Instead, disjoint pairs of tuples from \( Y \) must be used. Further since the tuples in \( S \) are also assumed to have arisen independently from \( V \), then difference tuples must also be formed from \( S \) using disjoint pairs.

Let \( s_1, \ldots, s_q \) denote the sample tuples (columns of \( S \)). We assume that \( q \) and \( m \) (the number of tuples in \( Y \)) are even; if not, we simply discard a randomly chosen tuple from \( Y \) or \( S \) or both. Let \( S^* \) denote the \( n \times (q/2) \) matrix whose \( i^{th} \) column is \( s_i^* = s_i - s_{q/2+i} \). Let \( Y^* \) denote the \( n \times (m/2) \) matrix whose \( i^{th} \) column is \( y_i^* = y_i - y_{m/2+i} \). The \( s^* \) tuples have arisen independently from \( V - W \) where \( W \) is a random vector independent of \( V \) but identically distributed. Moreover, the covariance matrix of \( V - W \), \( \Sigma_{(V-W)} \), has eigenvalues \( 0.5\lambda_1 > 0.5\lambda_2 > \ldots > 0.5\lambda_n \) (all distinct). Finally, the \( y^* \) tuples have arisen independently from \( M_T(V-W) \). Therefore, the PCA attack algorithm can be used with \( S^* \) and \( Y^* \) to produce \( \hat{M} \) an estimation of \( M_T \). Using this
and the sample means, \(\hat{\mu}_{M^TV + v_T}\) and \(\hat{\mu}_V\) (computed from \(Y\) and \(S\)), for any \(1 \leq j \leq m\), data
tuple \(x_j\) is estimated as

\[
\hat{x} = \hat{M}'(y_j - [\hat{\mu}_{M^TV + v_T} - \hat{M}\hat{\mu}_V]).
\] (10)

The intuitive rationale for (10): if \(\hat{M} \approx M_T, \hat{\mu}_{M^TV + v_T} \approx M_T\mu_V + v_T, \) and \(\hat{\mu}_V \approx \mu_V, \) then it follows that \(\hat{x} \approx x_j.\)

F. Effectiveness of the Known Sample Attack (PCA Attack) Algorithm on General Distance-Preserving Data Perturbation

Similar to the discussion in Section V-D, we focus on the \(p.d.f., f^*,\) of \(\mathcal{V} - \mathcal{W},\) and more specifically, (1) the difference in the eigenvalues of \(\Sigma_{\mathcal{V} - \mathcal{W}}\) expressed as the minimum eigen-ratio, and (2) the invariance of \(f^*, \text{Inv}(f^*).\) It can easily be shown that the minimum eigen-ratio of \(\Sigma_{\mathcal{V} - \mathcal{W}}\) is the same as that of \(\Sigma_{\mathcal{V}}.\) Regarding the invariance of \(f^*, if f is multi-variate Gaussian, then the discussion in Section V-D implies that \text{Inv}(f^*) = 0 because the mean of \(f^* \) is 0. Hence, the PCA attack algorithm will likely fail in the case where the original data arose from a multi-variate Gaussian distribution and the data perturbation is distance-preserving but not orthogonal (i.e. \(v_T \neq 0\)).

G. Experiments – General Distance-Preserving Data Perturbation

In this section, we evaluate the performance of the PCA attack on general distance-preserving data perturbation. We produce the translation vector with Matlab’s random number generator. Once generated, the translation is fixed for the same better data for all the experiments. The translation vector is set sufficiently large to distinguish general distance-preserving perturbation from orthogonal transformation.

We first experiment with the same Gaussian data (with a non-zero mean and a high minimum eigen-ratio) we used in Section V-C. As expected, the attack achieves very low frequency of privacy breach regardless of the sample ratio (see Figure 10). Next, we generate data from a \(p.d.f.\) which is a non-symmetric mixture of two Gaussians with \(\mu_1 = (10, 10, 10), \Sigma_1 = \begin{pmatrix} 1 & 1.5 & 0.5 \\ 1.5 & 3 & 2.5 \\ 0.5 & 2.5 & 75 \end{pmatrix}\) and \(\mu_2 = (20, 30, 40), \Sigma_2 = \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 40 \end{pmatrix}.\) The mixture weight for the first Gaussian is 0.2 and the weight for the second is 0.8. Note that this mixture \(p.d.f.\) has high
minimum eigen-ratio thereby reducing the effect of this factor in the experiment. The results are depicted in Figure 11. Here we see that the attack works significantly better. We believe this is due to the asymmetry of the Gaussian mixture allows problems with invariance to be better avoided.

We also conducted experiments with the Adult real dataset and found the frequency of privacy breach to degrade by approximately 10% as compared to the Adult dataset with only an orthogonal perturbation as discussed in Section V-C. We believe this is due to the fact that the underlying generation mechanism for the three attributes of the Adult dataset that we consider is sufficiently close to a multivariate Gaussian to cause attack problems due to invariance. This claim is based on observing a visualization of the dataset.

H. Analysis Over MED and Cos Privacy Breach Definitions

In the case of orthogonal data perturbation or the case of arbitrary Euclidean distance preserving perturbation, the PCA-based attack does not depend upon the definition of privacy breach. Of course, the empirical analysis does depend upon the privacy breach definition. For brevity, we leave to future work the empirical analysis of the known sample attack with respect to other $\epsilon$-MED-privacy-breach or $\epsilon$-cos-privacy-breach.
VI. DISCUSSION: VULNERABILITIES AND A POSSIBLE REMEDY

When considering known input prior knowledge, dimensionality significantly affects the vulnerability of the data to breach. The larger the difference between the number of linearly independent known inputs and the dimensionality, the lower the vulnerability of the data to breach.

When considering known sample prior knowledge, our results point out three factors which affect the vulnerability of the data to breach. 1) Dimensionality: our approach has time complexity exponential in the number of data attributes. Hence, breaching medium to high dimensional data is infeasible. 2) Eigenvalue distinction: the quality of the attacker’s estimate depends upon the size of the separation between the eigenvalues of the underlying covariance matrix. The smaller the separation, the lower the quality of the attacker’s estimate. 3) Underlying p.d.f. symmetry (invariance): the quality of the attacker’s estimate depends upon the symmetries present in the data generation p.d.f. If certain types of symmetries are present (called invariances earlier), the quality of the attack is low. For orthogonal transformations plus translations, the situation is even more difficult for the attacker as these symmetries only need be present after shifting the data to have zero mean.

We conclude the paper by pointing out a potential remedy to the privacy problems described earlier for the known sample attack. The data owner generates \( \tilde{R} \), a \( \ell \times n \) matrix with each entry sampled independently from a distribution with mean zero and variance one and releases \( Y = RX \) where \( R = \ell^{-1/2} \tilde{R} \) (this type of data perturbation for \( \ell \leq n \) was discussed in [30]). It can be shown that matrix \( R \) is orthogonal on expectation and the probability of orthogonality approaches one exponentially fast with \( \ell \). By increasing \( \ell \), the data owner can guarantee that distances are preserved with arbitrarily high probability. Moreover, it can be shown that the randomness introduced by \( R \) kills the covariance in \( Y \) used by the known sample attack. Specifically, given random vector \( V \), it can be shown that, \( \Sigma_{RV} \) (the covariance matrix of \( RV \)) equals \( I_n \gamma \) for some constant \( \gamma \). Therefore, the separation between the eigenvalues is zero, so, as mentioned above, the known sample attack fails.

With respect to the known input attack, the \( RX \) perturbation is potentially vulnerable. We have begun investigating a maximum-likelihood attack technique (see [19] for a summary). Further investigation of \( RX \) perturbation is left to future work.
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A. Known Sample Attack Proofs

**Theorem 5.1** The eigenvalues of $\Sigma_{\mathcal{V}}$ and $\Sigma_{(M_T\mathcal{V}+v_T)}$ are the same. Let $\mathcal{Z}(M_T\mathcal{V}+v_T)$, denote the normalized eigenspace of $\Sigma_{(M_T\mathcal{V}+v_T)}$ associated with $\lambda_i$, i.e. the set of vectors $w \in \mathbb{R}^n$ such that $\Sigma_{(M_T\mathcal{V}+v_T)}w = w\lambda_i$ and $||w|| = 1$. It follows that $M_T\mathcal{Z}(\mathcal{V})_i = \mathcal{Z}(M_T\mathcal{V}+v_T)_i$, where $M_T\mathcal{Z}(\mathcal{V})_i$ equals $\{M_Tz : z \in \mathcal{Z}(\mathcal{V})_i\}$.

**Proof:** First we derive an expression for $\Sigma_{\mathcal{V}}$ in terms of $\Sigma_{(M_T\mathcal{V}+v_T)}$.

\[
\Sigma_{(M_T\mathcal{V}+v_T)} = E[(M_T\mathcal{V}+v_T - E[M_T\mathcal{V}+v_T])(M_T\mathcal{V}+v_T - E[M_T\mathcal{V}+v_T])'] \\
= M_T\Sigma_{\mathcal{V}}M_T'.
\]

Now consider any non-zero $\lambda \in \mathbb{R}^n$, and observe that $det(\Sigma_{\mathcal{V}} - I_n\lambda) = det(M_T\Sigma_{\mathcal{V}}M_T' - I_n\lambda) = det(\Sigma_{(M_T\mathcal{V}+v_T)} - I_n\lambda)$. Therefore $\lambda$ is an eigenvalue of $\Sigma_{\mathcal{V}}$ if and only if $\lambda$ is an eigenvalue of $\Sigma_{(M_T\mathcal{V}+v_T)}$. Finally, consider any non-zero $w \in \mathbb{R}^n$. We have that $[w \in \mathcal{Z}(M_T\mathcal{V}+v_T)_i] \Leftrightarrow [\Sigma_{(M_T\mathcal{V}+v_T)}w = w\lambda_i \text{ and } ||w|| = 1] \Leftrightarrow [M_T\Sigma_{\mathcal{V}}M_T'w = w\lambda_i \text{ and } ||w|| = 1] \Leftrightarrow [\Sigma_{\mathcal{V}}(M_T'w) = (M_T'w)\lambda_i \text{ and } ||M_T'w|| = 1] \Leftrightarrow [M_T'w \in \mathcal{Z}(\mathcal{V})_i] \Leftrightarrow [w \in M_T\mathcal{Z}(\mathcal{V})_i]$.

**Corollary 5.2** Let $\mathbb{I}_n$ be the space of all $n \times n$, matrices with each diagonal entry $\pm 1$ and each off-diagonal entry 0 ($2^n$ matrices in total). There exists $D_0 \in \mathbb{I}_n$ such that $M_T = WD_0Z'$.

**Proof:** Theorem 5.1 implies that for all $1 \leq i \leq n$, $M_Tz_i = w_i$ or $-M_Tz_i = w_i$. Therefore, for some $D_0 \in \mathbb{I}_n$, $M_TZD_0 = W$. Because $D_0^{-1} = D_0$ and $Z$ is orthogonal the desired result follows.

Now we provide the derivation details of (9) under the assumption that $f$ is an $n$ variate Gaussian distribution with mean vector $\mu_{\mathcal{V}} = \alpha\mu$ for some $\alpha \geq 0$ and invertible covariance matrix $\Sigma_{\mathcal{V}}$.

First of all, for $n$-variate Gaussian distributions $g$ and $h$ with the same covariance matrix $\Sigma$ (assumed to be invertible) and mean vectors $\mu_g$ and $\mu_h$, we have
\[ KL(g||h) + KL(h||g) = (\mu_g - \mu_h)\Sigma^{-1}(\mu_g - \mu_h). \quad (11) \]

Second of all, for any \( D \) in \( \mathbb{R}^n \): (1) the covariance matrix of \( f_D \) is \( W\Lambda_W W' \); (2) the mean vector of \( f_D \) is \( WDZ'\mu_V \); and (3) \( f_D \) is multivariate Gaussian. Therefore, if \( f \) is multi-variate Gaussian, then Equations (8) and (11) imply

\[
Inv(f) = \min_{D_i \neq D_0 \in Y_n} (WD_i Z' \mu_V - WD_0 Z' \mu_V)' \Sigma_V^{-1} (WD_i Z' \mu_V - WD_0 Z' \mu_V) \\
= \min_{D_i \neq D_0 \in Y_n} \mu_V'(ZD_i W' - ZD_0 W')(W\Lambda_W W')^{-1} (WD_i Z' - WD_0 Z' ) \mu_V \\
= \min_{D_i \neq D_0 \in Y_n} \mu_V' Z(D_i - D_0) \Lambda_V^{-1} (D_i - D_0) Z' \mu_V \\
= \alpha^2 \min_{D_i \neq D_0 \in Y_n} (\mu' Z(D_i - D_0) \Lambda_V^{-1} (D_i - D_0) Z \mu).
\]

B. Known Input Attack: Proof of Theorem 4.2 and MED/COS Privacy Breach Derivations

**Theorem 4.2** Let \( L \) be the mapping \( P \in \Omega_{n-k} \mapsto M^T U_k U'_k + V_{n-k} P U'_{n-k} \). Then, \( L \) is an affine bijection from \( \Omega_{n-k} \) to \( \mathbb{M}(X_q, Y_q) \). And, \( L^{-1} \) is the mapping \( M \in \mathbb{M}(X_q, Y_q) \mapsto V'_{n-k} M U_{n-k} \).

To prove this theorem we rely upon the following key technical result.

**Lemma 1.1:** Let \( \mathbb{P} \) denote the set \( \{M^T U_k U'_k + V_{n-k} P U'_{n-k} : P \in \Omega_{n-k}\} \). Then \( \mathbb{M}(X_q, Y_q) = \mathbb{P} \).

**Proof:** Let \( \mathbb{M}(U_k, M^T U_k) \) denote the set of all \( M \in \Omega_n \) such that \( MU_k = M^T U_k \). First we show that \( \mathbb{M}(X_q, Y_q) = \mathbb{M}(U_k, M^T U_k) \). Since \( \text{Col}(X_q) = \text{Col}(U_k) \), then there exists \( k \times p \) matrix \( A \) such that \( U_k A = X_q \). Since \( A \) has \( k \) columns, then \( \text{rank}(A) \leq k \). Furthermore, [33, pg. 201] implies that \( k = \text{rank}(U_k A) \leq \min\{k, \text{rank}(A)\} \), thus, \( \text{rank}(A) = k \). Therefore, from [33, pg. 90], \( A \) has a right inverse.

For any \( M \in \Omega_n \), we have

\[
M \in \mathbb{M}(X_q, Y_q) \iff MU_k A = M^T U_k A \\
\iff MU_k = M^T U_k.
\]

The last \( \iff \) follows from the fact that \( A \) has a right inverse. We conclude that \( \mathbb{M}(X_q, Y_q) = \mathbb{M}(U_k, M^T U_k) \). Now we complete the proof by showing that \( \mathbb{M}(U_k, M^T U_k) = \mathbb{P} \).

(1) For any \( M \in \mathbb{P} \), there exists \( P \in \Omega_{n-k} \) such that \( M = \{M^T U_k U'_k + V_{n-k} P U'_{n-k}\} \). We have then
\[ MU_k = M_T U_k U_k' U_k + V_{n-k} P U_{n-k}' U_k \]
\[ = M_T U_k. \]

If we can show that \( M \) is orthogonal, then \( M \in \mathbb{M}(U_k, M_T U_k) \), so, \( P \subseteq \mathbb{M}(U_k, M_T U_k) \), as desired. Let \( U \) denote \([U_k | U_{n-k}]\) (clearly \( U \in \mathbb{O}_n \)). Observe

\[ M' M = U_k U_k' M_T M_T U_k U_k' + U_k U_k' M_T V_{n-k} P U_{n-k}' \]
\[ = U_{n-k} P' V_{n-k}' M_T U_k U_k' + U_{n-k} P' V_{n-k}' M_T U_k U_k' P' U_{n-k}' \]
\[ = U_k U_k' + 0 + 0 + U_{n-k} U_{n-k}' \]
\[ = U U' = I_n. \]

where the first zero in the second equality is due to the fact that \( \text{Col}(M_T U_k) = \text{Col}(Y_q) \), so, \( V_{n-k}' M_T U_k = 0 \).

(2) Now consider \( M \in \mathbb{M}(U_k, M_T U_k) \). It can be shown that \( \text{Col}(V_{n-k}) = \text{Col}(M U_{n-k}) \)\(^{21}\)

Thus, there exists \((n-k) \times (n-k)\) matrix \( P \) with \( V_{n-k} P = M U_{n-k} \). Observe that

\[ P' P = P'(V_{n-k}' V_{n-k}) P \]
\[ = (V_{n-k} P)' (V_{n-k} P) \]
\[ = (M U_{n-k})'(M U_{n-k}) = I_{n-k}. \]

Thus, \( P \in \mathbb{O}_{n-k} \). Moreover,

\[ MU = M[U_k | U_{n-k}] \]
\[ = [M_T U_k | M U_{n-k}] \]
\[ = [M_T U_k | V_{n-k} P]. \]

Thus,

\[ M = [M_T U_k | V_{n-k} P] \begin{bmatrix} U_k' \\ U_{n-k}' \end{bmatrix} \]
\[ = M_T U_k U_k' + V_{n-k} P U_{n-k}'. \]

\(^{21}\)Since \((M U_{n-k})' M U_k = 0\), then \( \text{Col}(M U_{n-k}) = \text{Col}_\perp (M U_k) \). Since \( M U_k = M_T U_k \) and \( \text{Col}(M_T U_k) = \text{Col}(Y_q) \), then it follows that \( \text{Col}_\perp (M U_k) = \text{Col}_\perp (M_T U_k) = \text{Col}_\perp (Y_q) = \text{Col}(V_{n-k}) \).
Therefore, $M \in \mathbb{P}$, so, $M(U_k, M_T U_k) \subseteq \mathbb{P}$, as desired.

Now we prove Theorem 4.2

Proof: Clearly $L$ is an affine map. Moreover, Lemma 1.1 directly implies that $L$ maps $\mathbb{O}_{n-k}$ onto $M(X_q, Y_q)$. To see that $L$ is one-to-one, consider $P_1, P_2 \in \mathbb{O}_{n-k}$ such that $L(P_1) = L(P_2)$. By definition, $M_T U_k U'_k + V_{n-k} P_1 U'_{n-k} = M_T U_k U'_k + V_{n-k} P_2 U'_{n-k}$, thus, $V_{n-k} P_1 U'_{n-k} = V_{n-k} P_2 U'_{n-k}$. Therefore $P_1 = V_{n-k} P_1 U'_{n-k} U_{n-k} = V_{n-k} P_2 U'_{n-k} U_{n-k} = P_2$.

To complete the proof, consider $P \in \mathbb{O}_{n-k}$. We have, $V'_{n-k} L(P) U_{n-k} = V'_{n-k} M_T U_k U'_k U_{n-k} + V'_{n-k} V_{n-k} P U'_{n-k} U_{n-k} = 0 + P$. Moreover, consider $M \in M(X_q, Y_q)$. By Lemma 1.1 there exists $P_M \in \mathbb{O}_{n-k}$ such that $M = M_T U_k U'_k + V_{n-k} P_M U'_{n-k}$. We have $L(V'_{n-k} M U_{n-k}) = L(P_M) = M$. Therefore, the inverse of $L$ is $M \in M(X_q, Y_q) \mapsto V'_{n-k} M U_{n-k}$.

Now we provide the details of the results crucial to establishing the connections between an $\epsilon$-privacy-breach and an $\epsilon$-MED-privacy-breach or $\epsilon$-cos-privacy-breach. First we show that:

$1 - \cos(\hat{x}, x_j) = \frac{||\hat{x} - x_j||^2}{2||x_j||^2}$.

From (1) and the discussion immediately above it, we have $\hat{x} = \hat{M}' y_j = \hat{M}' M_T x_j$, and thus, $||\hat{x}|| = ||x_j||$. It follows that

$$
\frac{||\hat{x} - x_j||^2}{2||x_j||^2} = \frac{2||x_j||^2 - 2\hat{x}' x_j}{2||x_j||^2} = \frac{\hat{x}' x_j}{||x_j||^2} = 1 - \frac{\hat{x}' x_j}{||x_j||}.
$$

Now we show that: $\min_{i=1}^n \{NAD(x_{j,i}, \hat{x}_i)\} \leq \frac{||x_{j} - \hat{x}||}{||x_j||}$. Let $i(\min) = \arg\min_{i=1}^n \{NAD(x_{j,i}, \hat{x}_i)\}$.

Without loss of generality, assume that $\forall 1 \leq i \leq \ell, x_{j,i} \neq 0$ and $\forall \ell + 1 \leq i \leq n, x_{j,i} = 0$. We have:

$$
\frac{||\hat{x} - x_j||^2}{||x_j||^2} = \frac{\sum_{i=1}^{\ell} (x_{j,i} - \hat{x}_i)^2 + \sum_{i=\ell+1}^{n} (\hat{x}_i)^2}{\sum_{i=1}^{\ell} (x_{j,i})^2} = \frac{\sum_{i=1}^{\ell} (x_{j,i})^2 NAD(x_{j,i}, \hat{x}_i)^2 + \sum_{i=\ell+1}^{n} NAD(x_{j,i}, \hat{x}_i)^2}{\sum_{i=1}^{\ell} (x_{j,i})^2}.
$$
\[ \geq \frac{\sum_{i=1}^{\ell} (x_{j,i})^2 NAD(x_{j,i}(\min)) \cdot \hat{x}_i(\min))^2 + \sum_{i=\ell+1}^{n} NAD(x_{j,i}(\min)) \cdot \hat{x}_i(\min))^2}{\sum_{i=1}^{\ell} (x_{j,i})^2} \geq NAD(x_{j,i}(\min), \hat{x}_i(\min))^2. \]

C. Known Input Attack: A Rigorous Development of the Closed-Form Expression for \( \rho(x_j, \epsilon) \)

Up to (2), we had derived the following result (for \( P \) chosen uniformly from \( \mathcal{O}_{n-k} \)):

\[ \rho(x_j, \epsilon) = Pr(||P'B'(U'_{n-k}x_j) - (U'_{n-k}x_j)|| \leq ||x_j|| \epsilon), \] (12)

where \( B \in \mathcal{O}_{n-k} \) and satisfies \( MTU_{n-k}B = V_{n-k} \). Now we provide a rigorous proof of (3), \textit{i.e.} the r.h.s. above equals \( Pr(||P'(U'_{n-k}x_j) - (U'_{n-k}x_j)|| \leq ||x_j|| \epsilon) \). To do so, we need some material from measure theory.

Because \( \mathcal{O}_{n-k} \) is a locally compact topological group [32, pg. 293], it has a Haar probability measure, denoted by \( \mu \), over \( \mathbb{B} \), the Borel algebra on \( \mathcal{O}_{n-k} \). This is commonly regarded as the standard uniform probability measure over \( \mathcal{O}_{n-k} \). Its key property is \textit{left-invariance}: for all \( B \in \mathbb{B} \) and all \( M \in \mathcal{O}_{n-k} \), \( \mu(B) = \mu(MB) \), \textit{i.e.}, shifting \( B \) by a rigid motion does not change its probability assignment.

Let \( \mathcal{O}_{n-k}(U'_{n-k}x_j, ||x_j|| \epsilon) \) denote the set of all \( P \in \mathcal{O}_{n-k} \) such that \( ||P'(U'_{n-k}x_j) - (U'_{n-k}x_j)|| \leq ||x_j|| \epsilon \). Let \( \mathcal{O}^{B'}_{n-k}(U'_{n-k}x_j, ||x_j|| \epsilon) \) denote the set of all \( P \in \mathcal{O}_{n-k} \) such that \( ||P'B'(U'_{n-k}x_j) - (U'_{n-k}x_j)|| \leq ||x_j|| \epsilon \). By definition of \( \mu \) we have,

\[ \mu(\mathcal{O}_{n-k}(U'_{n-k}x_j, ||x_j|| \epsilon)) = Pr(P \text{ uniformly chosen from } \mathcal{O}_{n-k} \text{ lies in } \mathcal{O}_{n-k}(U'_{n-k}x_j, ||x_j|| \epsilon)) \]
\[ = Pr(||P'(U'_{n-k}x_j) - (U'_{n-k}x_j)|| \leq ||x_j|| \epsilon), \]

and,

\[ \mu(\mathcal{O}^{B'}_{n-k}(U'_{n-k}x_j, ||x_j|| \epsilon)) = Pr(P \text{ uniformly chosen from } \mathcal{O}^{B'}_{n-k}(U'_{n-k}x_j, ||x_j|| \epsilon) \leq ||x_j|| \epsilon), \]

Therefore,

\footnote{Since \( \mathcal{O}_{n-k}(U'_{n-k}x_j, ||x_j|| \epsilon) \) and \( \mathcal{O}^{B'}_{n-k}(U'_{n-k}x_j, ||x_j|| \epsilon) \) are topologically closed sets, then they are Borel subsets of \( \mathcal{O}_{n-k} \), therefore, \( \mu \) is defined on each of these.}
whose distance from
theory, it can be shown that
as 1 otherwise. Moreover, we define
≥
c
where the second equality is due to the left-invariance of
expression (6). To simplify exposition, we prove the following result for
Recall that
Since the last equality above was for intuitive purposes only, we will ignore it in completing
(\text{Pr}(\|P' \mathbf{B}'(U'_{n-k}\mathbf{x}_j) - (U'_{n-k}\mathbf{x}_j)\| \leq \|\mathbf{x}_j\|\epsilon)) = \mu(O_{n-k}(U'_{n-k}\mathbf{x}_j, \|\mathbf{x}_j\|\epsilon))
\mu(BO_{n-k}(U'_{n-k}\mathbf{x}_j, \|\mathbf{x}_j\|\epsilon))
\mu(O_{n-k}(U'_{n-k}\mathbf{x}_j, \|\mathbf{x}_j\|\epsilon))
\Pr(||P'(U'_{n-k}\mathbf{x}_j) - (U'_{n-k}\mathbf{x}_j)|| \leq ||\mathbf{x}_j||\epsilon)
(13)

where the second equality is due to the left-invariance of \(\mu\) and the third equality is due to the
fact that \(BO_{n-k}(U'_{n-k}\mathbf{x}_j, ||\mathbf{x}_j||\epsilon)\) can be shown to equal \(O_{n-k}(U'_{n-k}\mathbf{x}_j, ||\mathbf{x}_j||\epsilon)\).

Since the last equality above was for intuitive purposes only, we will ignore it in completing
the derivation of a closed form expression. (12) and (13) imply
\[
\rho(x_j, \epsilon) = \mu(O_{n-k}(U'_{n-k}\mathbf{x}_j, ||\mathbf{x}_j||\epsilon)).
\]

Recall that \(S_{n-k}(||U'_{n-k}\mathbf{x}_j||)\) denotes the hyper-sphere in \(\mathbb{R}^{n-k}\) with radius \(||U'_{n-k}\mathbf{x}_j||\) and centered at the origin and \(S_{n-k}(U'_{n-k}\mathbf{x}_j, ||\mathbf{x}_j||\epsilon)\) denotes the points contained by \(S_{n-k}(||U'_{n-k}\mathbf{x}_j||)\) whose distance from \(U'_{n-k}\mathbf{x}_j\) is no greater than \(||\mathbf{x}_j||\epsilon\). Using basic principles from measure
theory, it can be shown that\(^{23}\)
\[
\mu(O_{n-k}(U'_{n-k}\mathbf{x}_j, ||\mathbf{x}_j||\epsilon)) = \frac{SA(S_{n-k}(U'_{n-k}\mathbf{x}_j, ||\mathbf{x}_j||\epsilon))}{SA(S_{n-k}(||U'_{n-k}\mathbf{x}_j||))}
\]
We have arrived at Equation (5) from Section [IV-D]. Next, we derive the desired closed-form
expression (6). To simplify exposition, we prove the following result for \(m \geq 0, z \in \mathbb{R}^m\), and
\(c \geq 0\) (by plugging in \(m = n - k, z = U'_{n-k}\mathbf{x}_j\), and \(c = ||\mathbf{x}_j||\epsilon\), (6) follows).

\[
\frac{SA(S_m(z, c))}{SA(S_m(\|\mathbf{z}\|))} = \begin{cases} 
1 & \text{if } m = 0; \\
1 & \text{if } c \geq ||\mathbf{z}||2 \text{ and } m \geq 1; \\
0.5 & \text{if } c < ||\mathbf{z}||2 \text{ and } m = 1; \\
1 - (1/\pi)\arccos\left(\left|\frac{c}{\sqrt{\mathbf{z}^T}}\right|^{2} - 1\right) & \text{if } ||\mathbf{z}|| \sqrt{2} < c < ||\mathbf{z}||2 \text{ and } m = 2; \\
1 - (m-1)/(m+1/2) & \text{if } c < ||\mathbf{z}||2 \text{ and } m \geq 3; \\
(1/\pi)\arccos\left(1 - \left|\frac{c}{\sqrt{\mathbf{z}^T}}\right|^{2}\right) & \text{if } ||\mathbf{z}|| \sqrt{2} < c < ||\mathbf{z}||2 \text{ and } m = 2; \\
(1/(m)^{1/2}) \int_{0}^{\infty} \arccos\left(1 - \left|\frac{c}{\sqrt{\mathbf{z}^T}}\right|^{2}\right) \sin^{m-1}(\theta_1) d\theta_1 & \text{if } c \leq ||\mathbf{z}|| \sqrt{2} \text{ and } m = 2; \\
\end{cases}
\]

(14)

\(^{23}\)S\(_1(||U'_1\mathbf{x}_j||)\) consists of two points. Recall that we define \(\frac{SA(S_m(U'_1\mathbf{x}_j, ||\mathbf{x}_j||\epsilon))}{SA(S_m(\|\mathbf{U}'_1\mathbf{x}_j\|))}\) as 0.5 if \(S_1(U'_1\mathbf{x}_j, \|\mathbf{x}_j\|\epsilon)\) is one point, and
as 1 otherwise. Moreover, we define \(\frac{SA(S_m(U'_1\mathbf{x}_j, ||\mathbf{x}_j||\epsilon))}{SA(S_m(\|\mathbf{U}'_1\mathbf{x}_j\|))}\) as 1.

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Before proving (14) we establish:

For \( b \geq 2 \) and \( r > 0 \), \( \text{SA}(S_b(r)) = \frac{br^{b-1} \pi^{b/2}}{\Gamma((b+2)/2)} \).

(15)

Indeed, with \( Vol(.) \) denoting volume, it can be shown that \( \text{SA}(S_b(r)) = \frac{dVol(S_b(r))}{dr} = Vol(S_b(1)) \frac{dr}{dr} \) = \( \frac{b^{b/2} \pi^{b-1}}{\Gamma((b+2)/2)} \). The last equality follows from [36]. Now we return to proving (14).

If \( m = 0 \), then the surface area ratio equals 1 by definition. If \( c \geq ||z||2 \) and \( m \geq 1 \), then the ratio equals 1 since \( S_m(z,c) = S_m(||z||) \). If \( c < ||z||2 \) and \( m = 1 \), then, the ratio equals 0.5 since \( S_1(z,c) = \{z\} \) and \( S_1(||z||) = \{z,-z\} \). For the remainder of the derivation, we assume that \( m \geq 2 \) and, without loss of generality, \( z \) is at the “north pole” of the hyper-sphere \( S_m(||z||) \), i.e. \( z = (1,0,0,\cdots,0) \).

Case \( c \leq ||z||\sqrt{2} \): The set of points on \( S_m(||z||) \) whose distance from \( z \) equals \( c \) is the intersection of \( S_m(||z||) \) with the hyper-plane whose perpendicular to \( z \) is of length \( h \) as seen in Figure 12. Thus, \( S_m(z,c) \) are all those points on \( S_m(||z||) \) not below that hyper-plane.

Sub-case \( m = 2 \): Since \( S_2(||z||) \) is an ordinary circle, then the angle \( \theta \) in Figure 12 determines
the surface area ratio as follows \( \frac{(2\theta/2\pi)SA(S_2(||z||))}{SA(S_2(||z||))} = \theta/\pi \). Moreover, since \( \theta \) is the top angle of an isosceles triangle with sides of length \(||z||\) and base of length \(c\), then \( \sin(\theta/2) = c/(2||z||) \).

The half-angle formula implies that \( \theta = \arccos(1 - [c/(||z||\sqrt{2})]^2) \). Therefore, as desired,

\[
\frac{SA(S_2(z, c))}{SA(S_2(||z||))} = (1/\pi)\arccos(1 - [c/(||z||\sqrt{2})]^2).
\]

**Sub-case** \( m \geq 3 \): Here, computing the surface area ratio is more complicated and requires an appeal to the integral definition of the cap surface area. Consider the intersection of \( S_m(||z||) \) with the hyper-plane whose perpendicular to \( z \) is of length \( 0 \leq h_1 \leq h \) as seen in Figure 12. The surface area of this intersection equals the surface area of \( S_{m-1}(r(h_1)) \). Thus, (15) implies

\[
SA(S_m(z, c)) = \int_{h_1=0}^{h} SA(S_{m-1}(r(h_1))) dh_1
\]

\[
= \left( \frac{(m-1)\pi^{(m-1)/2}}{\Gamma((m+1)/2)} \right) \int_{h_1=0}^{h} (r(h_1))^{m-2} dh_1.
\]

To evaluate the integral, we change coordinates with \( h_1 = ||z||(1 - \cos(\theta_1)) \). So, \( h_1 = 0, h \) implies that \( \theta_1 = 0, \arccos(1 - h/||z||) \). And, \( r(||z||(1 - \cos(\theta_1))) = ||z||\sin(\theta_1) = \frac{dh_1}{d\theta_1} \). Therefore,

\[
\int_{h_1=0}^{h} (r(h_1))^{m-2} dh_1 = \int_{\theta_1=0}^{\arccos(1-h/||z||)} r(||z||(1 - \cos(\theta_1)) = \frac{m-2}{m-1} \int_{\theta_1=0}^{\arccos(1-h/||z||)} ||z||^{m-2}sin^{m-2}(\theta_1) ||z||\sin(\theta_1) d\theta_1
\]

\[
= ||z||^{m-1} \int_{\theta_1=0}^{\arccos(1-h/||z||)} sin^{m-1}(\theta_1) d\theta_1.
\]

Plugging this into the previous equations for \( SA(S_m(z, c)) \) and using (15), we get

\[
\frac{SA(S_m(z, c))}{SA(S_m(||z||))} = \left( \frac{(m-1)\pi^{(m-1)/2}||z||^{m-1}\Gamma((m+2)/2)}{\Gamma((m+1)/2)m||z||^{m-1}m^{m/2}} \right) \int_{\theta_1=0}^{\arccos(1-h/||z||)} sin^{m-1}(\theta_1) d\theta_1
\]

\[
\frac{SA(S_m(z, c))}{SA(S_m(||z||))} = \left( \frac{(m-1)\Gamma((m+2)/2)}{\Gamma((m+1)/2)m\sqrt{\pi}} \right) \int_{\theta_1=0}^{\arccos(1-h/||z||)} sin^{m-1}(\theta_1) d\theta_1.
\]

Since \( h = \frac{c^2}{2||z||} \), then, as desired, we get

\[
\frac{SA(S_m(z, c))}{SA(S_m(||z||))} = \left( \frac{(m-1)\Gamma((m+2)/2)}{\Gamma((m+1)/2)m\sqrt{\pi}} \right) \int_{\theta_1=0}^{\arccos(1-c/||z||\sqrt{2})} sin^{m-1}(\theta_1) d\theta_1.
\]

(17)
Case $\|z\|\sqrt{2} < c < \|z\|2$: As depicted in Figure 13, $S_m(z, c)$ contains the entire northern hemisphere of $S_m(\|z\|)$. Let $S_n(-z, c)$ denote the “south pole” cap defined by $h'$ (and $c'$) in Figure 13 (clearly $c' \leq \|z\|\sqrt{2}$). We have

$$\frac{SA(S_m(z, c))}{SA(S_m(\|z\|))} = 1 - \frac{SA(S_m(-z, c'))}{SA(S_m(\|z\|))}. \quad (18)$$

By replacing “c” with “c’” in (16) and (17) then plugging the resulting expression into (18) we get,

$$\frac{SA(S_m(z, c))}{SA(S_m(\|z\|))} = \begin{cases} 1 - \frac{(1/\pi)\arccos(1 - [c'/\|z\|\sqrt{2}])^2}{m\sqrt{\Gamma((m+1)/2)}} \Gamma(1-\arccos(c'/\|z\|\sqrt{2}))^2 sin^{m-1}(\theta_1) d\theta_1 & \text{if } m = 2; \\ 1 - \frac{(m-1)\Gamma((m+2)/2)}{m\sqrt{\Gamma((m+1)/2)}} \Gamma(1-\arccos(c'/\|z\|\sqrt{2}))^2 sin^{m-1}(\theta_1) d\theta_1 & \text{if } m \geq 3. \end{cases} \quad (19)$$

From Figure 13 it can be seen that $\theta$ is the top angle on an isosceles triangle with sides of length $\|z\|$ and base of length $c'$. So, $sin(\theta/2) = \frac{c'}{2\|z\|}$. The half-angle formula implies $cos(\theta) = 1 - [c'/\|z\|\sqrt{2}]^2$. Similar reasoning shows $cos(\pi - \theta) = 1 - [c'/\|z\|\sqrt{2}]^2$. Since $0 \leq \theta \leq \pi/2$, then $cos(\pi - \theta) = -cos(\theta)$. Thus, $[c'/\|z\|\sqrt{2}]^2 - 1 = 1 - [c'/\|z\|\sqrt{2}]^2$. Plugging $2 - [\frac{c'}{\|z\|\sqrt{2}}]^2$ in for $[\frac{c'}{\|z\|\sqrt{2}}]^2$ in (19) yields the desired results.

**D. Known Input Attack: Computing the Closed-Form Expression for $\rho(x_j, \epsilon)$**

Next we develop recursive procedures for computing (7). This amounts to computing the following two functions: (i) $GR(m) = \Gamma([m + 2]/2)/\Gamma([m + 1]/2)$ for $m \geq 1$; (ii) $SI(z, m) = \int_{\theta_1=0}^{\arccos(z)} sin^{m-1}(\theta_1) d\theta_1$ for $1 \leq z \leq 0$ and $m \geq 1$. Indeed, (7) is equivalent to

$$\rho(x_j, \epsilon) = \begin{cases} 1 & 1 \\ 1 & 0.5 \\ 1 - \frac{(1/\pi)\arccos\left(\frac{|y_j|}{\sqrt{|V'_{n-k}y_j|}}\right)^2}{\sqrt{|V'_{n-k}y_j|}^2} & \text{if } n - k = 0; \\ 1 - \frac{(n-k-1)\Gamma((n-k)/2)}{(n-k)\sqrt{\Gamma((n-k+1)/2)}} SI\left(\frac{|y_j|}{\sqrt{|V'_{n-k}y_j|}}\right)^2 & \text{if } ||y_j|| \epsilon \geq ||V'_{n-k}y_j||^2 \text{ and } n - k \geq 1; \\ 1 - \frac{n-k-1)\Gamma((n-k)/2)}{(n-k)\sqrt{\Gamma((n-k+1)/2)}} SI\left(\frac{|y_j|}{\sqrt{|V'_{n-k}y_j|}}\right)^2 & \text{if } ||y_j|| \epsilon < ||V'_{n-k}y_j||^2 \text{ and } n - k = 1; \\ 1 - \frac{(n-k-1)\Gamma((n-k)/2)}{(n-k)\sqrt{\Gamma((n-k+1)/2)}} SI\left(\frac{|y_j|}{\sqrt{|V'_{n-k}y_j|}}\right)^2 & \text{if } ||V'_{n-k}y_j||^2 < ||y_j|| \epsilon \leq ||V'_{n-k}y_j||^2 \text{ and } n - k = 2; \\ 1 - \frac{n-k-1)\Gamma((n-k)/2)}{(n-k)\sqrt{\Gamma((n-k+1)/2)}} SI\left(\frac{|y_j|}{\sqrt{|V'_{n-k}y_j|}}\right)^2 & \text{if } ||y_j|| \epsilon < ||V'_{n-k}y_j||^2 \text{ and } n - k = 3; \end{cases} \quad (20)$$
To compute $GR(m)$ for $m \geq 1$, we use the following facts: $\Gamma(z + 1) = z\Gamma(z)$ for $z > 0$, $\Gamma(1/2) = \sqrt{\pi}$, and $\Gamma(1) = 1$. Thus, we get a recursive procedure for computing $GR(m)$.

$$GR(m) = \begin{cases} \frac{\sqrt{\pi}}{2} & \text{if } m = 1; \\ \frac{\sqrt{\pi}}{2} & \text{if } m = 2; \\ \left(\frac{m}{m-1}\right) GR(m-2) & \text{if } m \geq 3. \end{cases}$$

(21)

To compute $SI(z, m)$ for $1 \geq z \geq 0$ and $m \geq 1$, we use the following facts. $\sin^{m-2}(\arccos(z)) = [1-z^2]^{(m-2)/2}$ if $m \geq 3$. And, $SI(z, m) = \left[ \int \sin^{m-1}(\theta) \, d\theta \right] (\arccos(z)) - \left[ \int \sin^{m-1}(\theta) \, d\theta \right] (0)$. And,

$$\left[ \int \sin^{m-1}(\theta) \, d\theta \right] (w) = \begin{cases} w & \text{if } m - 1 = 0; \\ -\cos(w) & \text{if } m - 1 = 1; \\ \frac{m-2}{m-1} \left[ \int \sin^{m-3}(\theta) \, d\theta \right] (w) - \frac{\sin^{m-2}(w)}{m-1} & \text{if } m - 1 \geq 2; \end{cases}$$

(22)

Therefore,

$$SI(z, m) = \begin{cases} \arccos(z) & \text{if } m = 1; \\ 1 - z & \text{if } m = 2; \\ \frac{m-2}{m-1} SI(z, m - 2) - \frac{z(1-z^2)^{(m-2)/2}}{m-1} & \text{if } m \geq 3. \end{cases}$$

(23)

E. Known Input Attack on General Distance-Preserving Data Perturbation

Previously, we had considered the case where the data perturbation is assumed to be orthogonal (does not involve a fixed translation, $v_T = 0$). Now we briefly discuss how the attack technique and its analysis can be extended to arbitrary Euclidean distance preserving perturbation ($v_T \neq 0$).

Extending the algorithms for inferring $\pi_a$: Since the length of the private data tuples may not be preserved, then the definition of validity in Section [IV-A] must be changed: $\alpha$ on $I$ is valid if $\forall i, j \in I$, $||x_i - x_j|| = ||y_{\alpha(i)} - y_{\alpha(j)}||$. As well, the definition of $C(\alpha_1, \hat{i})$ (given $I_1 \subseteq I$, $\alpha_1$ a valid assignment on $I_1$, and $\hat{i} \in (I \setminus I_1)$), must change: the set of all $j \in (\{1, \ldots, m\} \setminus \alpha_1(I_1))$ such that for all $i_1 \in I_1$, $||x_{i_1} - x_i|| = ||y_{\alpha_1(i_1)} - y_j||$. With these changes, Algorithms [IV-A.1], [IV-A.2] and [IV-A.3] work correctly as stated.

Extending the known input attack: The basic idea is simple and relies upon the fact that the same $v_T$ is added to all tuples in the perturbation of $X_q$. Fix one tuple, say $x_1$ and $y_1$, and consider the following differences $x_1^- = (x_q - x_1)$, $\ldots$, $x_{q-1}^- = (x_q - x_{q-1})$ and $y_1^- = (y_q - y_1)$,
\ldots, y_{q-1}^{-1} = (y_q - y_{q-1}). \ Let \ X_{q-1}^{-} denote the matrix with columns \ x_1^{-}, \ldots, x_{q-1}^{-} \ and \ Y_{q-1}^{-} \ denote
the matrix with columns \ y_1^{-}, \ldots, y_{q-1}^{-}. \ Observe \ that \ Y_{q-1}^{-} = M_T X_{q-1}^{-}, \ hence, \ the \ attack \ and \ its
analysis \ from \ the \ orthogonal \ data \ perturbation \ case \ can \ be \ applied. \ The \ details \ are \ straightforward \ and \ are \ omitted \ for \ brevity. \ However, \ a \ caveat \ is \ in \ order. \ The \ attack \ depends \ upon \ the
choice \ of \ the \ tuple \ to \ fix. \ Therefore, \ the \ attacker \ examines \ them \ all \ and \ chooses \ the \ highest
privacy \ breach \ probability.