Weak approximation and Manin group of R-equivalences
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Abstract
We extend one exact sequence of Colliot-Thélène and Sansuc for tori over number fields to one for arbitrary connected groups.

Introduction. Let $G$ be a connected linear algebraic group defined over a number field $k$. Denote by

$$\text{A}(G) = \prod_v G(k_v)/\text{Cl}(G(k))$$

the obstruction to weak approximation in $G$,

$$\text{III}(G) = \text{Ker} \left( H^1(k, G) \rightarrow \prod_v H^1(k_v, G) \right)$$

the Tate - Shafarevich group of $G$ where $H^1(.,.)$ denotes Galois cohomology and $k_v$ denotes the completion of the field $k$ at a valuation $v$. We denote also by $G(k)/R$ the Manin group of $R$-equivalences of $G(k)$, $RG(k)$ the subgroup of $G(k)$ consisting of all elements of $G(k)$ which are $R$-equivalent to 1 in $G(k)$ (see [M]). One defines similar groups in the case of local fields $k_v$ which are

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completions of $k$.
If $G = T$ is a $k$-torus, then Voskresenski [V1] showed that there is an exact sequence connecting arithmetic, geometric and cohomological invariants of $T$

$$1 \to A(T) \to H^1(k, Pic \bar{X})* \to \text{III}(T) \to 1,$$

(1)

where $X$ is a smooth compactification of $T$ over $k$, $ar{X} = X \times_k \bar{k}$ and $()^*$ means taking the character group with values in $\mathbb{Q}/\mathbb{Z}$. Later on, Sansuc [Sa] showed that this sequence also holds for arbitrary connected $k$-group $G$ over a number field $k$ without factors of type $E_8$, thus also for any connected $k$-group by combining this with the result on the Hasse principle for $E_8$ by Chernousov. In their fundamental paper [CTS] Colliot-Thélène and Sansuc have established, among the others, the following exact sequence which connects various important arithmetic and geometric invariants of algebraic tori defined over a number field $k$. Namely we have

$$1 \to \text{III}(S) \to T(k)/R \xrightarrow{\beta} \prod_v T(k_v)/R \to A(T) \to 1,$$

(2)

an exact sequence where $S$ is the Neron - Severi torus of $T$ which comes from a flasque resolution of $T$.

This exact sequence gives us the information about the kernel of $\beta$: it is the group $\text{III}(S)$, and the cokernel of $\beta$: it is the group $A(T)$. It is a natural question to ask if one can extend (2) to the case of connected reductive groups over global fields. Denote by $\text{III}RG$ the "local-global" group of $R$-equivalent classes modulo $RG(k)$ which are trivial locally everywhere, i.e.,

$$\text{III}RG = \cap_v RG(k_v)/RG(k).$$

Then (2) can be written in the following form

$$1 \to \text{III}RT \to T(k)/R \xrightarrow{\beta} \prod_v T(k_v)/R \to A(T) \to 1.$$  

(3)

We show in this note that (3) also holds for any connected reductive $k$-group $G$. Namely we have

**Theorem.** For any connected group $G$ defined over a global field $k$, assumed reductive if $\text{char} k > 0$, there is an exact sequence of groups

$$1 \to \text{III}RG \to G(k)/R \xrightarrow{\beta} \prod_v G(k_v)/R \to A(G) \to 1.$$
We give some preliminary results in Section 1 and we prove the main result in Section 2. In Section 3 we give a cohomological interpretation of the group $IIIRG$. I would like to thank J.-L. Colliot-Thélène for the interest in the paper.

1 Some auxiliary results

We need the following auxiliary results in the arithmetic of algebraic groups.

1.1. **Theorem.** Any semisimple simply connected group defined over a global field $k$ satisfies weak approximation.

This result is due to Kneser and Harder. We refer to [Sa] for historical remarks and references.

The following lemma of Langlands allows one to reduce many problems in reductive groups to the case of a reductive group with simply connected semisimple part.

1.2. **Lemma.** [L] Let $G$ be a connected reductive group defined over a field $k$. Then there exists a connected reductive group $H$, an induced torus $Z$ and a central extension

$$1 \to Z \to H \to G \to 1,$$

all defined over $k$.

In literature, such central extension of $G$ is called a $z$-extension of $G$.

1.3. **Theorem.** Let $G$ be a connected reductive group defined over a local or global field $k$. Then the Manin group $G(k)/R$ is finite.

In [G2] Gille proved this theorem in the case of number field, but the proof can be modified to give other cases as well (see [T1,T2]).

1.4. **Proposition.** Let $G$ be a unirational group over a field $k$ and $S$ a
finite set of discrete valuations of $k$. Then via the diagonal embedding, the
closure $\text{Cl}(G(k))$ of $G(k)$ in $\prod_{v \in S} G(k_v)$ in the product topology contains an
open subgroup of the latter.

This result is implicitly due to Kneser, who used it in the study of strong
approximation. The proof follows from the Implicit Function Theorem.

1.5. Theorem. [CTS] Let $T$ be a torus defined over a field $k$. Then
there exists a flasque resolution of $T$, i.e., an exact sequence of $k$-tori
\[ 1 \to S \to N \to T \to 1, \]
where $N$ is an induced torus and $S$ is a flasque torus. Moreover $T(k)/R \simeq H^1(k, S)$.

2 Main theorem and the proof

First we need the following

2.1. Lemma. Let $G$ be a connected reductive and $H$ a $z$-extension of $G,$
all defined over a field $k$. Assume that either $H(k)/R$ or $G(k)/R$ is finite.
Then there are canonical isomorphism of groups and bijection of factor sets,
respectively
\[ G(k)/R \simeq H(k)/R, \quad A(G, S) \simeq A(H, S), \]
where $A(G, S)$ denotes the factor $\prod_{v \in S} G(k_v)/\text{Cl}(G(k))$ with $S$ a finite set of
discrete valuations of $k$ and $\text{Cl}(G(k))$ the closure of $G(k)$ via diagonal em-
bedding in the product topology of $\prod_{v \in S} G(k_v)$.

Proof. Let $1 \to Z \to H \xrightarrow{\pi} G \to 1$ be the corresponding $z$-extension. Since $Z$ is an induced torus, it has trivial cohomology so $\pi$ is surjective
while restricted to $H(K)$ for any field extension $K$ of $k$. It is clear that $\pi(RH(k)) \subset RG(k)$ hence $\pi$ defines a surjective homomorphism of groups
\[ \pi': H(k)/R \to G(k)/R. \]

It is well-known that $H \simeq Z \times G$ (birationally equivalent) and we know by
[CTS] that this induces a bijection
\[ H(k)/R \leftrightarrow G(k)/R. \]
Since $H(k)/R$ is finite, $\pi'$ is an isomorphism of groups. To prove the second bijection, it suffices to prove the following equality

$$\pi^{-1}(Cl(G(k))) = Cl(H(k)).$$

One inclusion is obvious. Let $h \in \pi^{-1}(Cl(G(k)))$. Then $\pi(h) = \lim_n g_n$, with $g_n \in G(k)$. There are $h_n \in H(k)$ such that $g_n = \pi(h_n)$, so $\lim_n \pi(h_n^{-1}h) = 1$. Let $U_n$ be a system of open neighbourhoods of 1 in $G_S = \prod_{v \in S} G(k_v)$ such that $U_n \subset U_{n-1}$, $\cap_n U_n = \{1\}$ and $\pi(h^{-1}_n h) \in U_n$. Thus for all $n$ we have

$$\pi^{-1}(1) = \pi^{-1}(\cap_n U_n) = \cap_n \pi^{-1}(U_n) = \prod_{v \in S} Z(k_v),$$

and

$$h^{-1}_n h \in \pi^{-1}(U_n) \quad (4)$$

It is well-known by a result of Grothendieck - Rosenlicht that connected reductive groups are unirational over the field of definition. Thus by Proposition 1.4, $Cl(H(k))$ is an open subgroup of $H_S$, hence $Cl(H(k)) \prod_{v \in S} Z(k_v)$ is an open neighbourhood of $\prod_{v \in S} Z(k_v)$ in $\prod_{v \in S} H(k_v)$. Therefore for $n$ large we have

$$\pi^{-1}(U_n) \subset Cl(H(k)) \prod_{v \in S} Z(k_v) \subset Cl(H(k)),$$

since $Z$ has weak approximation property. Thus for $n$ large we derive from (4)

$$h \in h_n \pi^{-1}(U_n) \subset h_n Cl(H(k)) = Cl(H(k))$$

and the lemma follows. ■

2.2. Lemma. Let $G$ be a connected reductive group defined over a field $k$, $H$ a $z$-extension of $G$. For a finite set $S$ of valuations of $k$, let $RG_S = \prod_{v \in S} RG(k_v)$.

a) If $RG_S \subset Cl(G(k))$ then $RH_S \subset Cl(H(k))$.

b) Conversely, assume that $H(k_v)/RH(k_v)$ is finite for $v \in S$ and $RH_S \subset Cl(H(k))$. Then $RG_S \subset Cl(G(k))$.

Proof. a) We have $\pi(RH_S) \subset RG_S$ where we denote by the same symbol $\pi$ the homomorphism $H_S \to G_S$ induced from $\pi : H \to G$. Hence

$$RH_S \subset \pi^{-1}(RG_S) \subset \pi^{-1}(Cl(G(k))) = Cl(H(k))$$

and...
by Lemma 2.1.
b) By Lemma 2.1 we have $\pi(RH_S) = RG_S$, thus

$$RG_S \subset \pi(Cl(H(k))) = Cl(G(k)).$$

The lemma is proved.

The following was first mentioned in [V2] in the case the finite set of valuations consists of only one element.

2.3. **Lemma.** Let $T$ be a torus defined over a field $k$, $S$ a finite set of valuations of $k$. Then $RT_S \subset Cl(T(k))$.

*Proof.* Consider the following flasque resolution of $T$

$$1 \rightarrow S \rightarrow N \xrightarrow{p} T \rightarrow 1.$$

Since this is also a flasque resolution of $T$ over $k_v$, from the proof of Theorem 1.5 (in [CTS]) it follows that

$$p(N(k)) = RT(k), \ p(N(k_v)) = RT(k_v)$$

for any $v \in S$ and since $N$ has weak approximation over $k$, it follows that

$$\prod_{v \in S} RT(k_v) = p(\prod_{v \in S} N(k_v)) = p(Cl(N(k))) \subset Cl(T(k)).$$

The lemma is proved.

For arbitrary $k$-group it is not known if the Lemma 2.3 is true in general, but in the important case of connected reductive groups over global fields we have the following result, which answers a (implicit) question of Voskresenski [V2] in this case.

2.4. **Theorem.** Let $k$ be a global field and $G$ a connected reductive group over $k$. For any finite set $S$ of valuations of $k$ we have

$$RG_S \subset Cl(G(k)).$$

*Proof.* By 2.3 we may assume that $G$ is not a torus. By 1.3 and 2.2 we may assume that the semisimple part $G'$ of $G$ is simply connected. By enlarging
S we may also assume that S contains all archimedean valuations of k. Let \( G = G'T \), where T is a central torus. We have the following exact sequence of \( k \)-groups

\[
1 \to G' \to G \xrightarrow{p} T' \to 1,
\]

where \( T' \) is a factor of T. Consider the following commutative diagram, where the rows are exact

\[
\begin{array}{ccc}
G(k) & \to & T'(k) \\
\downarrow & \delta & \downarrow \\
G(S) & \to & T'(S)
\end{array}
\]

\[
\delta \gamma
\]

\[
G(S) \to T'(S) \to \prod_{v \in S} H^1(k_v, G'),
\]

where \( G(S) = \prod_{v \in S} G(k_v) \), \( T'(S) = \prod_{v \in S} T'(k_v) \). Since \( G' \) is simply connected, we know that \( \gamma \) is bijective. Let \( g \in RG_S \). Then \( p(g) \in RT_S \subset Cl(T'(k)) \) by 2.3. Therefore there is a sequence \((t_n), t_n \in T'(k)\) such that

\[
p(g) = \lim_n t_n,
\]

hence

\[
\delta(p(g)) = \delta(\lim_n t_n) = 1.
\]

Since the product \( \prod_{v \in S} H^1(k_v, G') \) is finite, for large \( n \) we have

\[
\delta(t_n) = 1,
\]

i.e., for large \( n \), there are \( g_n \in G(k) \) such that \( t_n = p(g_n) \). Therefore

\[
\lim_n p(g_n^{-1}g) = 1.
\]

By the same argument we used in the proof of 2.1, and by using weak approximation property in simply connected groups (see 1.1) we see that for large \( n \)

\[
g_n^{-1}g \in G'(S)Cl(G(k)) \subset Cl(G(k)),
\]

i.e., \( g \in Cl(G(k)) \). □

To deduce the relation between weak approximation and R-equivalence we need the following well-known
2.5. Lemma. With the same notation as in 2.1, $RG_S$ is an open subgroup of $\prod_{v \in S} G(k_v)$.

Proof. It follows from the fact that $G$ is unirational and that affine spaces have only trivial $R$-equivalent classes. \hfill \blacksquare

Now we have

2.6. Theorem. Let $k$ be a global field and $G$ a connected $k$-group, which is reductive if char.$k > 0$. For any finite set $S$ of valuations of $k$ we have a bijection

$$A(S, G) \leftrightarrow \text{Coker } (G(k)/R \rightarrow \prod_{v \in S} G(k_v)/R).$$

In particular, the obstruction $A(S, G)$ has a natural group structure inherited from that of $\text{Coker } (G(k)/R \rightarrow \prod_{v \in S} G(k_v)/R)$.

Proof. By 2.4 we have

$$RG_S \subset Cl(G(k)),$$

and by 2.5

$$Cl(G(k)) = RG_S G(k).$$

Thus

$$\text{Coker } (G(k)/R \rightarrow \prod_{v \in S} G(k_v)/R) = \prod_{v \in S} G(k_v)/Cl(G(k)) = A(S, G).$$

The theorem is proved. \hfill \blacksquare

It is well-known (see [CTS], [Sa]) that for almost all $v$, $G(k_v)/R = 1$ and there is a finite set $S_0$ of valuations of $k$ such that $G$ has weak approximation over $k$ with respect to any finite set of valuations of $k$ outside $S_0$. Thus we have

2.7. Theorem. Let the notation be as in 2.6 and the introduction. Then we have the following exact sequence of groups

$$1 \rightarrow \Pi IRG \rightarrow G(k)/R \rightarrow \prod_{v \in S} G(k_v)/R \rightarrow A(G) \rightarrow 1. \quad (5)$$
3 A cohomological interpretation of III$RG$.

As we have seen, in the case $G = T$ is a torus, the group III$RG$ is the Tate - Shafarevich group of the Neron - Severi torus $S$ of $T$. We would like to find a similar interpretation for the case of reductive groups over number fields. From [Sa] we know that for any connected reductive group $G$ over a field $k$, there is a number $n$, induced $k$-tori $P, Q$ and a central $k$-isogeny

$$1 \to F \to \tilde{G} \times P \to G^n \times Q \to 1,$$

where $\tilde{G}$ is a semisimple simply connected $k$-group (namely the simply connected covering of $G^n$, $G'$ being the semisimple part of $G$. From the proof in the case of tori and the case of reductive groups, we see that for $n$ smallest possible, the groups $P, Q$ are determined uniquely. The corresponding group $G \times P$ is called canonical special covering of $G$ and $F$ the canonical special fundamental group of $G$. In the case $G$ is semisimple, these notions are just the usual notions of simply connected covering and fundamental group.

First we consider the case $G = T$ is a torus. There is a canonical special covering of $T$

$$1 \to F \to P \to T^n \times Q \to 1,$$

where $P, Q$ are induced tori, $F$ a finite group.

For any extension $K$ of $k$ we have the following exact sequence of groups

$$P(K) \xrightarrow{p} (T^n \times Q)(K) \xrightarrow{\delta} H^1(K, F) \to 0,$$

and similar sequence when we take the group of $R$-equivalences

$$1 \to T^n(K)/R \to H^1(K, F)/R \to 0,$$

where in the case of $H^1(K, F)$, we consider the $R$-equivalence induced from $(T^n \times Q)(K)$, (see [G1]). Therefore

$$T^n(K)/R = H^1(K, F)/R,$$

and we have

$$\text{III}RT^n = \text{III}F/R := \ker (H^1(k, F)/R \to \prod_v H^1(k_v, F)/R).$$
Thus from (5) we deduce the following exact sequence for tori

$$1 \to \Pi F/R \to T^n(k)/R \to \prod_v T^n(k_v)/R \to A(T^n) \to 1.$$  \hfill (6)

Next we discuss the analog of (6) in the case of connected reductive groups. Let $G$ be a connected reductive $k$-group with canonical special fundamental group $F$. There is an embedding

$$H^1(k, F) \to H^1(k(t), F),$$

where $k(t)$ is the rational function field in the variable $t$. There is an equivalence relation $R$ on $H^1(k, F)$, defined as follows:

for $x, y \in H^1(k, F)$, $x \sim_R y$ iff there exists $z(t) \in H^1(k(t), F)$ such that there are specializations $t \to 0, t \to 1$ with $z(0) = x, z(1) = y$.

We denote by $H^1(k, F)/R$ the set of $R$-equivalence classes. Let $\delta : (G^n \times Q)(k) \to H^1(k, F)$ be the connecting map and we denote by $\alpha$ the composite map

$$G^n(k)/R \to \text{Im} (\delta)/R \to H^1(k, F)/R.$$

The second map is induced from the embedding $\text{Im} (\delta) \to H^1(k, F)$. We have

3.1. Theorem. Let $G$ be a connected reductive group defined over a global field $k$ and other notation be as above.

a) If $k$ is a global field of characteristic $> 0$ and $G$ contains no anisotropic factors of type $2A_n$, then the following sequence is exact

$$1 \to \Pi F/R \to G^n(k)/R \to \prod_v G^n(k_v)/R \to A(G^n) \to 1.$$  \hfill (7)

b) In general, we have the following exact sequence of groups

$$1 \to \alpha^{-1}(\Pi F/R) \to G^n(k)/R \to \prod_v G^n(k_v)/R \to A(G^n) \to 1.$$

Proof. a) By a well-known theorem of Kneser [K] (in characteristic 0) and Bruhat - Tits [BT] (in characteristic $> 0$), $H^1(k_v, \tilde{G}) = 0$ where $\tilde{G}$ is a semisimple simply connected group and $k_v$ is a non-archimedean local field. Also, if $k$ is a global field of characteristic $> 0$, $H^1(k, \tilde{G}) = 0$ for simply connected semisimple group $\tilde{G}$ by [Ha]. It is well-known that $\tilde{G}(k)$ is projectively
simple, hence also $\tilde{G}(k)/R = 1$ for simply connected $k$-group $\tilde{G}$ containing no anisotropic factors of type $^2A_n$ (see [PR] and [Ti2]). Thus the exact sequence (7) follows as in the case of tori.

$b)$ In the general case, for any non-archimedean valuation $v$ we still have $G^n(k_v)/R \cong H^1(k_v, F)/R$. It is known and easy to prove that $H(R)/R = 1$ for any connected $R$-group. Hence

$$\text{Ker } (G^n(k)/R \to \prod_v G^n(k_v)/R) = \text{Ker } (G^n(k)/R \to \prod_v H^1(k_v, F)/R)$$

$$= \alpha^{-1}(\text{III } F/R).$$

The theorem is proved. ■

Conjecturally, $\alpha$ is an isomorphism and $\tilde{G}(k)/R = 1$ for any simply connected semisimple group $\tilde{G}$ over a global field $k$. Moreover, the term $\text{III } F/R$ might be expressed as a function of $S := \text{Pic}(V(G))^*$, where $k$ is assumed a number field and $V(G)$ is a smooth compactification of $G$.

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