Topological substitution for the aperiodic Rauzy fractal tiling

Nicolas Bédaride∗ Arnaud Hilion† Timo Jolivet‡

Abstract

We consider two families of planar self-similar tilings of different nature: the tilings consisting of translated copies of the fractal sets defined by an iterated function system, and the tilings obtained as a geometrical realization of a topological substitution (an object of purely combinatorial nature, defined in [BH13]). We establish a link between the two families in a specific case, by defining an explicit topological substitution and by proving that it generates the same tilings as those associated with the Tribonacci Rauzy fractal.

1 Introduction

1.1 Main result and motivation

Self-similar tilings of the plane are characterized by the existence of a common subdivision rule for each tile, such that the tiling obtained by subdivising each tile is the same as the original one, up to a contraction. These tilings have been introduced by Thurston [Thu89] and they are studied in several fields including dynamical systems and theoretical physics, see [BG13]. A particular class of self-similar tilings arises from substitutions, which are “inflation rules” describing how to replace a geometrical shape by a union of other geometrical shapes (within a finite set of basic shapes). Among these, an important class consists of the planar tilings by the so-called Rauzy fractals associated with some one-dimensional substitutions. These fractals are used to provide geometrical interpretations of substitution dynamical systems. They also provide an interesting class of aperiodic self-similar tilings of the plane, see [PF02, BR10].

The aim of this article is to establish a formal link between two self-similar tilings constructed from two different approaches:

• Using an iterated function system (IFS), that is, specifying the shapes and the positions of the tiles with planar set equations (using contracting linear maps), which define the tiles as unions of smaller copies of other tiles. In particular, an IFS does make use of the Euclidean metric of the plane.
• Using a topological substitution, that is, specifying which tiles are allowed to be neighbors, and how the neighboring relations are transferred when we “inflate” the tiles by substitution to construct the tiling. With this kind of substitution, there is no use in anyway of a the Euclidean metric: the tiles do not have a metric shape (they are just topological discs).

In other words, we tackle the following question:

Given a tiling defined by an IFS, is there a topological substitution which generates an equivalent tiling? If yes, how can we construct it? In other words, when is it possible to describe the geometry of a self-similar tiling (geometrical constraints) by using a purely combinatorial rule (combinatorial constraints) ?
In this article we answer this question for the tilings of the plane by translated copies of the Rauzy fractals associated with the Tribonacci substitution (which are defined by an IFS). We define a particular topological substitution $\sigma$ (Figure 4, p. 10) and we prove that the Tribonacci fractal tiling $T_{\text{frac}}$ and the tiling $T_{\text{top}}$ generated by the topological substitution are equivalent in a strong way. More precisely:

- Associated with the Tribonacci substitution $s : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$, there is a dual substitution $E$ (see Section 4.2) which acts on facets in $\mathbb{R}^3$. Iteration of this dual substitution gives rise to a stepped surface $\Sigma_{\text{step}}$ (a surface which is a union of facets), that is included in the 1-neighbourhood of some (linear) plane $P$ in $\mathbb{R}^3$. Projecting the stepped surface $\Sigma_{\text{step}}$ (and its facets) on $P$ gives rise to a tiling $T_{\text{step}}$ of $P$. It is known [ABI02, BR10] that the tiling $T_{\text{frac}}$ is strongly related to a tiling $T_{\text{step}}$.

- The topological substitution $\sigma$ can be iterated on a tile $C$, giving rises to a 2-dimensional CW-complex $\sigma^\infty(C)$ homeomorphic to a plane, see Section 3.2. However, this complex is not embedded a priori in a plane, even if it turns out that $\sigma^\infty(C)$ can be effectively realized as a tiling $T_{\text{top}}$ of the plane, see Proposition 3.11. To locate a tile $T$ in $\sigma^\infty(C)$ relatively to another one $T'$, we build a vector (an “position”) $\omega_0(T, T') \in \mathbb{Z}^3$: by construction, this vector depends a priori on the choice of a combinatorial path from $T$ to $T'$ in $\sigma^\infty(C)$, and we have to prove that in fact it is independent of the path, see Section 5.1.

- Since it is already explained in the literature how to relate $T_{\text{frac}}$ and $\Sigma_{\text{step}}$, and since we explain how $T_{\text{top}}$ is build from $\sigma^\infty(C)$, the main result of the paper is Theorem 5.16 that states an explicit formula which define a bijection $\Psi$ between tiles in $\sigma^\infty(C)$ and facets in $\Sigma_{\text{step}}$: we reproduce it just below.

**Theorem.** The map $\Psi$ defined, for every tile $T$ of $\sigma^\infty(C)$, by:

$$\Psi(T) = [M^3_3(\omega_0(T, C) + u_{\text{type}(T)}), \theta(\text{type}(T))]^*$$

(1.1)

is a bijection from the set of tiles of $\sigma^\infty(C)$ to the set of facets of $\Sigma_{\text{step}}$.

The notation used to state this theorem will be introduced along the paper. But we want to stress that the fact the formula (1.1) makes use of the position map $\omega_0$ ensures that if two tiles $T$ and $T'$ are close in $\sigma^\infty(C)$, then their images $\Psi(T)$ and $\Psi(T')$ will be close in $\Sigma_{\text{step}}$. In fact, it is easy to convince oneself that something like that should be true by having a look at Figure 1, where three corresponding subsets of the tilings $T_{\text{top}}, T_{\text{frac}}$ and $T_{\text{step}}$ are given.

![Figure 1: The three tilings $T_{\text{top}}, T_{\text{frac}}$ and $T_{\text{step}}$ (from left to right).](image)

On Figure 1 it is also worth to notice that the underlying CW-complexes of $T_{\text{top}}$ and $T_{\text{step}}$ are not the same. Indeed, the valence of a vertex in $T_{\text{top}}$ is either 2 or 3, whereas the valence of a vertex in $T_{\text{step}}$ can be equal to 3, 4, 5 or 6. In that sense, the two tilings $T_{\text{step}}$ and $T_{\text{top}}$ are really different.

We have chosen to present our results on a specific substitution rather than in a general form because it makes presentation clearer and it avoids many “artificial” technicalities. Moreover, we do not know what a general answer to the above question may look like. However, we give some insight about this general question in Section 6.
1.2 Comparison of some different notions of substitutions

The word “substitution” is used in many different ways in the literature. The list below reviews several such notions, going from the most geometrical one (IFS) to the most combinatorial one (topological substitutions). Indeed, as observed by Peyrière [Pey86], having a combinatorial description of substitutive tiling turns out to be very useful in many situations. This list is not exhaustive, it only contains the notions of substitutions that we explicitly use in this article. See [Fra08] for another survey about geometrical substitutions.

One-dimensional symbolic substitutions These substitutions are used to generated infinite one-dimensional words which are studied mostly for their word-theoretical and dynamical properties. An example is the Tribonacci substitution $1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$ defined in Section 4.3. See [PF02] for a classical reference. This is the only notion of the present list which is only symbolic (not geometrical).

Self-affine substitutions (iterated function systems) Also known as substitution Delone sets [LW03], this notion is a particular class of iterated functions systems, where it is required that the geometrical objects defined by the IFS are compact sets which are the closure of their interior, in such a way that tilings can be defined. See Proposition 4.5 for an example of such a definition for the Tribonacci fractal.

Dual (or “generalized”) substitutions These substitutions, introduced in [AI01] can be seen as a discrete version of self-affine substitutions. Instead of defining fractal tilings in a purely geometrical way (like with IFS), these substitutions act on unions of faces of unit cubes located at integer coordinates. We define the associated fractal sets and tilings by iterating the dual substitution and by taking a Hausdorff limit of the (renormalized) unions of unit cube faces. The fact that we deal with unit cube faces allows us to exploit some fine combinatorial and topological properties of the resulting patterns. This provides some powerful tools in the study of substitution dynamics and Rauzy fractal topology. Dual substitutions are usually denoted by $E_1(\sigma)$, where $\sigma$ is a one-dimensional symbolic substitution, See [BR10, ST09] for many references and results, and Definition 4.2 for the particular example studied in this article.

Local substitution rules This notion has been used to tackle combinatorial questions about substitution dynamics [IO93, ABI02, ABS04, BBJS15] and have also been studied in a more general context [Fer07, JK14]. Their aim is to get a “more combinatorial” version of dual substitutions. Instead of computing explicitly the coordinates of the image of each unit cube face (like we do for dual substitutions), we give some local rules (or concatenation rules) for “gluing together” the images of two adjacent faces. The map defined in Figure 10, p. 20 is an example of such a substitution (except that it is defined over topological tiles and not unit cube faces).

Topological substitutions Introduced in [BH13], topological substitutions do not make any use of geometry: the tiles are topological disks (with no Euclidean shape), the boundaries of which have a simplicial structure (made of vertices and edges). It is a notion less geometrically rigid than the previous ones. They act on CW-complexes, and the “gluing rules” are more abstract and combinatorial than local substitution rules. A topological substitution generates a CW-complex homeomorphic to the plane. If this complex can be geometrized as a tiling of the plane, we say that the tiling is a topological substitutive tiling. Topological substitutions allowed for instance to prove that there is no substitutive primitive tiling of the hyperbolic plane, even though an explicit example of a non-primitive topological substitution which generates a tiling of the hyperbolic plane is given in [BH13].
In order to distinguish this notion of substitution used in the present article from the other combinatorial notions discussed in this introduction, we use the term topological substitution instead of combinatorial substitution.

The examples of topological substitutions given in the present article (Figure 4 and Figure 12) are interesting because they provide new examples of topological substitutive tilings, which can be realized as (substitutive) tilings of the plane.

Other related notions There is another notion, elaborated by Fernique and Ollinger [FO10](and developed in details in the specific case of Tribonacci), which lies between local substitution rules and topological substitutions. For these so-called combinatorial substitutions, the Euclidean shape of the tiles is specified, and the matching rules are stated in terms of colors associated with some subintervals on the boundaries of the tiles and their images. We stress that, in that case, the Euclidean geometry is used both to give the shape of the tiles and to specify that two tiles with same shape differ with a translation of the plane.

Purely combinatorial notions of substitutions have already been defined. For instance, Priebe-Frank [Fra03] introduced a very natural notion of (labelled) graph substitutions. In the case of a substitutive tiling, this graph substitution has to be understood as a substitution on the dual graph to the tiling. The main issue with this formalism is that there is no a priori control on the planarity of the graph obtained by iteration of the substitution, and thus in general the limit graph obtained by iteration can not be the dual graph to any tiling of the plane. Topological substitutions of [BH13] remedy this problem.

Topological substitutions have also some worth to be met cousins: the so-called subdivision rules, introduced by Cannon, Floyd and Parry in [CFP01]. The natural context where these subdivision rules have hatched is the one of conformal geometry: on one hand, they can be seen as topological models for postcritically finite rational map of the Riemann sphere [CFP07], on the other hand, they are likely useful to prove Cannon’s conjecture for hyperbolic groups whose Gromov boundary is the 2-dimensional sphere as suggested by the results of [CS98]. Nevertheless, subdivision rules can be also used to produce conformal substitutive tilings of the plane, see [BS97, RS13]. Even if, by iterating both a system of subdivision rules or a substitution, one get a 2-complex homeomorphic to the plane, these processes do differ in their nature: morally, in the case of subdivision rules the 2-complex is obtained as an inverse limit whereas in the case of a substitution it is obtained as a direct limit. It is not clear at all to the authors when, given a 2-complex obtained by one of two processes, one can also recover it using the other process.

1.3 Organization of the article

In Section 2 we quickly review some usual facts about tilings. In Section 3 we recall the general definition of a topological substitution and we define the Tribonacci topological substitution we are interested in. In Section 4 we recall the definition of the Tribonacci dual substitution and its associated IFS Rauzy fractal. The link between the IFS and the topological substitution is finally studied in Section 5. In Section 6 we describe how our results can be extended to some other Rauzy fractal tilings, and we explain that finding a suitable topological substitution has some dynamical implications for the underlying one-dimensional substitution.

Acknowledgements We thank the referee for a very careful reading of the paper and several useful suggestions. This work was supported by the ANR through projects LAM ANR-10-JCJC-0110, QUASICOOL ANR-12-JS02-0011 and FAN ANR-12-IS01-0002.

1Here conformal means that the underlying geometry is the conformal geometry and not the Euclidean one. That is to say that the group $\Gamma$ defining the tiling (see Definition 2.1) is not a subgroup of isometries of $\mathbb{R}^2$, but a subgroup of biholomorphisms of $\mathbb{C}$. 
2 Tilings

2.1 Basic definitions

In this section we recall standard notions on tiling in \( \mathbb{R}^2 \). For the references about this material we refer the reader to [Rob04, Sol07, BHI13, GS87]. We denote by \( \Gamma \) a subgroup of transformations of \( \mathbb{R}^2 \): here, \( \Gamma \) will be the group of translations of \( \mathbb{R}^2 \). We keep \( \Gamma \) in the notation, just to have in mind that some classical tilings need rotations of tiles.

A **tile** is a compact subset of \( \mathbb{R}^2 \) which is the closure of its interior (in most of the basic examples, a tile is homeomorphic to a closed ball). We denote by \( \partial T \) the boundary of a tile, i.e. \( \partial T = T \setminus \hat{T} \). Let \( A \) be a finite set of labels. A **labelled tile** is a pair \((T, a)\) where \( T \) is a tile and \( a \) an element of \( A \). Two labelled tiles \((T, a)\) and \((T', a')\) are **equivalent** if \( a = a' \) and there exists a translation \( g \in \Gamma \) such that \( T' = g T \). An equivalence class of labelled tiles is called a **prototile**: the class of \((T, a)\) is denoted by \([T, a]\), or simply by \([T]\) when the context is sufficiently clear. We will say that \((T, a)\) belongs to the prototile \([T, a]\). In some cases, one does not need the labelling to distinguish different prototiles: for example if we consider a family of prototiles such that the tiles in two different prototiles are not isometric.

**Definition 2.1.** A tiling \( X = (\mathbb{R}^2, \Gamma, P, T) \) of the plane modeled on a set of prototiles \( P \), is a set \( T \) of tiles, each belonging to a prototile in \( P \), such that:

- \( \mathbb{R}^2 = \bigcup_{T \in \Gamma} T \),
- two distinct tiles of \( T \) have disjoint interiors.

A connected finite union of (labelled) tiles is called a (labelled) **patch**. Two finite patches are **equivalent** if they have the same number \( k \) of tiles and these tiles can be indexed \( T_1, \ldots, T_k \) and \( T'_1, \ldots, T'_k \), such that there exists \( g \in \Gamma \) with \( T'_i = g T_i \) for every \( i \in \{1, \ldots, k\} \). Two labelled patches are equivalent if moreover \( T_i, T'_i \) have same labelling for all \( i \in \{1, \ldots, k\} \). An equivalence class of patches is called a **protopatch** and denoted \([P]\) if \( P \) is one of these patches.

The **support** of a patch \( P \), denoted by \( \text{supp}(P) \), is the subset of \( \mathbb{R}^2 \) which consists of points belonging to a tile of \( P \). A **subpatch** of a patch \( P \) is a patch which is a subset of the patch \( P \).

Let \( X = (\mathbb{R}^2, \Gamma, P, T) \) be a tiling, and let \( A \) be a subset of \( \mathbb{R}^2 \). A patch \( P \) **occurs in** \( A \) if there exists \( g \in \Gamma \) such that for any tile \( T \in P \), \( g T \) is a tile of \( T \) which is contained in \( A \):

\[
g T \in T \quad \text{and} \quad \text{supp}(g T) \subseteq A.
\]

We note that any patch in the protopatch \([P]\) defined by \( P \) occurs in \( A \). We say that the protopatch \([P]\) **occurs in** \( A \). The **language** of \( X \), denoted \( \Lambda_X \), is the set of protopatches of \( X \).

When all tiles of \( P \) are euclidean polygons, the tiling is called a **polygonal tiling**.

2.2 Delone set defined by a tiling of the plane

A Delone set in \( \mathbb{R}^2 \) is a set \( D \) of points such that there exists \( r, R > 0 \) such that every euclidean ball of radius \( r \) contains at most one point of \( D \) and every euclidean ball of radius \( R \) contains at least one point of \( D \).

When a tiling of the plane \( X = (\mathbb{R}^2, \Gamma, P, T) \) is modeled on a finite set of prototiles \( P \), there is a standard way (among others) to derive a Delone set \( D \) from a tiling of the plane \( X = (\mathbb{R}^2, \Gamma, P, T) \). We first choose a point in the interior of each prototile. This choice gives us a point \( x_T \) in each tile \( T \) of \( X \). \( D = \{x_T, T \in \Gamma\} \) is a Delone set.

2.3 The 2-complex defined by a tiling of the plane

Let \( X \) be a 2-dimensional CW-complex (see, for instance, [Hat02] for basic facts about CW-complexes). The 0-cells will be called vertices, the 1-cells edges and the 2-cells faces. The
subcomplex of \( X \) which consists of cells of dimension at most \( k \in \{0, 1, 2\} \) is denoted by \( X^k \) (in particular \( X^2 = X \)). We denote by \( |X^k| \) the number of \( k \)-cells in \( X^k \).

Let \( X = (\mathbb{R}^2, \Gamma, \mathcal{P}, \tau) \) be a tiling of the plane. We suppose that the tiles are homeomorphic to a closed disc \( \mathbb{D}^2 \). This tiling defines naturally a 2-complex \( X \) in the following way. The set \( X^0 \) of vertices of \( X \) is the set of points in \( \mathbb{R}^2 \) which belong to (at least) three tiles of \( \mathcal{T} \). Each connected component of the set \( \bigcup_{f \in \mathcal{T}} \partial f \setminus X^0 \) is an open arc. Any closed edge of \( X \) is the closure of one of these arcs.

Such an edge \( e \) is glued to the endpoints \( x, y \in X^0 \) of the arc. The set of faces of \( X \) is the set of tiles of \( X \). We remark that the boundary of a tile is a subcomplex of \( X^1 \) homeomorphic to the circle \( S^1 \): this gives the gluing of the corresponding face on the 1-skeleton.

Let \( Y \) be a 2-dimensional CW-complex homeomorphic to the plane \( \mathbb{R}^2 \). A polygonal tiling \( X \) is the geometric realization of \( Y \) if the 2-complex \( X \) isomorphic (as CW-complex) to \( Y \). In that case, each face of the complex \( Y \) can be naturally labelled by the corresponding prototile of the tiling \( X \).

3 The Tribonacci topological substitution

Before giving the definition of the Tribonacci topological substitution in section 3.2, we first recall some facts about (2-dimensional) topological substitutions in section 3.1. These two sections can be read in parallel: along section 3.1, we illustrate the notions with examples referring to section 3.2.

3.1 Topological substitutions

The mathematical content of this section is essentially contained in [BH13]: we include it here for completeness. The vocabulary we will use in the present setting is often common to the one of tilings: the context is in general sufficient to prevent any ambiguity.

3.1.1 General definition

A topological \( (k \geq 3) \)-gon is a 2-dimensional CW-complex made of one face, \( k \) edges and \( k \) vertices, which is homeomorphic to a closed disc \( \mathbb{D}^2 \), and such that the 1-skeleton is the boundary \( S^1 \) of the closed disc. A topological polygon is a topological \( k \)-gon for some \( k \geq 3 \).

We consider a finite set \( \mathcal{T} = \{T_1, \ldots, T_d\} \) of topological polygons. The elements of \( \mathcal{T} \) are called prototiles, and \( \mathcal{T} \) is called the set of prototiles. If \( T_i \) is a \( n_i \)-gon, we denote by \( E_i = \{e_{1,i}, \ldots, e_{n_i,i}\} \) the set of edges of \( T_i \). In practice, we will need later to consider these \( e_{n_i,i} \) as oriented edges: we first fix an orientation on the boundary of \( T_i \), and equip the \( e_{n_i,i} \) with the induced orientation. We set \( E_i^{-1} = \{e_{1,i}^{-1}, \ldots, e_{n_i,i}^{-1}\} \) and \( E_i^+ = E_i \cup E_i^{-1} \), where \( e^{-1} \) denotes the edge \( e \) equipped with the reverse orientation.

A patch \( P \) modeled on \( \mathcal{T} \) is a 2-dimensional CW-complex homeomorphic to the closed disc \( \mathbb{D}^2 \) such that for each closed face \( f \) of \( P \), there exists a prototile \( T_i \in \mathcal{T} \) and a homeomorphism \( \tau_f : f \to T_i \) which respects the cellular structure. Then \( T_i = \tau_f(f) \) is called the type of \( f \), and denoted by \( \type(f) \). The type of an edge \( e \) of \( T_i \), denoted by \( \type(e) \), is \( \tau(e) \). An edge \( e \) of \( P \) is called a boundary edge if it is contained in the boundary \( S^1 \) of the disc \( \mathbb{D}^2 \cong P \). Such a boundary edge is contained in exactly one closed face of \( P \). An edge \( e \) of \( P \) which is not a boundary edge is called an interior edge. An interior edge is contained in exactly two closed faces of \( P \). In the following definition, and for the rest of this article, the symbol \( \sqcup \) stands for the disjoint union.

Definition 3.1. A topological pre-substitution is a triplet \((\mathcal{T}, \sigma(\mathcal{T}), \sigma)\) where:

1. \( \mathcal{T} = \{T_1, \ldots, T_d\} \) is a set of prototiles,
2. \( \sigma(\mathcal{T}) = \{\sigma(T_1), \ldots, \sigma(T_d)\} \) is a set of patches modeled on \( \mathcal{T} \),
3. \( \sigma : \bigcup_{i \in \{1, \ldots, d\}} T_i \to \bigcup_{i \in \{1, \ldots, d\}} \sigma(T_i) \) is a homeomorphism, which restricts to homeomorphisms \( T_i \to \sigma(T_i) \), such that the image of a vertex of \( T_i \) is a vertex of the boundary of \( \sigma(T_i) \).

**Example 3.2.** In Figure 3, we show one example of a pre-topological substitution defined on 3 prototiles. This is the Tribonacci topological pre-substitution.

**Compatible topological pre-substitution** Let \( T = \{T_1, \ldots, T_d\} \) be the set of prototiles of \( \sigma \), and let \( E_i^\pm = E_i \cup E_i^{-1} \) be the set of oriented edges of \( T_i \) \((i \in \{1, \ldots, d\})\). We denote by \( E^\pm \) the set of all oriented edges: \( E^\pm = \sqcup_i E_i^\pm \).

A pair \((e, e') \in E^\pm \times E^\pm\) is balanced if \( \sigma(e) \) and \( \sigma(e') \) have the same length (= the number of edges in the edge path). The flip is the involution of \( E^\pm \times E^\pm \) defined by \((e, e') \mapsto (e', e)\), and the reversion is the involution of \( E^\pm \times E^\pm \) defined by \((e, e') \mapsto (e^{-1}, e'^{-1})\). The quotient of \( E^\pm \times E^\pm \) obtained by identifying a pair and its image by the flip and also a pair and its image by the reversion is denoted by \( E_2 \). We denote by \([e, e']\) the image of a pair \((e, e') \in E^\pm \times E^\pm \) in \( E_2 \). Since the flip and the reversion preserve balanced pairs, the notion of “being balance” is well defined for elements of \( E_2 \). The subset of \( E_2 \) which consists of balanced elements is called the set of balanced pairs, and denoted by \( \mathcal{B} \). Let \([e, e'] \in \mathcal{B}\) a balanced pair. In other words, \( \sigma(e) \) and \( \sigma(e') \) are paths of edges which have same length say \( p \geq 1 \): \( \sigma(e) = e_1 \ldots e_p, \sigma(e') = e'_1 \ldots e'_p \). Let \( \varepsilon_i = \text{type}(e_i) \) and let \( \varepsilon_i' = \text{type}(e'_i) \): \( \varepsilon_i, \varepsilon_i' \in E^\pm \) for \( i = 1 \ldots p \). Then the \([\varepsilon_i, \varepsilon_i']\) are called the descendants of \([e, e']\).

**Example 3.3.** For the Tribonacci topological pre-substitution, consider for example the edges \( e = B_{45}, e' = C_{56} \). By Figure 3, we have \( \sigma(e) = C_{34}C_{45}C_{56}, \sigma(e') = C_{10}C_{69}C_{98} \). Thus \([e, e']\) is a balanced pair. The descendants of this pair are \([C_{34}, C_{10}], [C_{45}, C_{96}], [C_{56}, C_{98}]\).

Now, we consider a patch \( P \) modeled on \( T \). An interior edge \( e \) of \( P \) defines an element \([\varepsilon, \varepsilon']\) of \( E_2 \). Indeed, let \( f \) and \( f' \) be the two faces adjacent to \( e \) in \( P \). We denote by \( \varepsilon = \tau_f(e) \) the edge of type \( f \) corresponding to \( e \), and by \( \varepsilon' = \tau_{f'}(e) \) the edge of type \( f' \) corresponding to \( e \). The edge \( e \) is said to be balanced if \([\varepsilon, \varepsilon']\) is balanced.

We define, by induction on \( p \in \mathbb{N} \), the notion of a \( p \)-compatible topological pre-substitution \( \sigma \). To any \( p \)-compatible topological pre-substitution \( \sigma \) we associate a new pre-substitution which will be denoted by \( \sigma^p \).

**Definition 3.4.**

a) Any pre-substitution \((T, \sigma(T), \sigma)\) is 1-compatible. We set \( \sigma^1 = \sigma \).

b) A pre-substitution \((T, \sigma(T), \sigma)\) is said to be \( p \)-compatible \((p \geq 2)\) if:

1. \((T, \sigma(T), \sigma)\) is \((p - 1)\)-compatible
2. for all \( i \in \{1, \ldots, d\} \), every interior edge \( e \) of \( \sigma^{p-1}(T_i) \) is balanced.

c) We suppose now \((T, \sigma(T), \sigma)\) is a \( p \)-compatible pre-substitution \((p \geq 2)\). Then we define \( \sigma^p(T_i) \) \((i \in \{1, \ldots, d\}) \) as the patch obtained in the following way:

We consider the collection of patches \( \sigma(\text{type}(f)) \) for each face \( f \) of \( \sigma^{p-1}(T_i) \). Then, if \( f \) and \( f' \) are two faces of \( \sigma^{p-1}(T_i) \) adjacent along some edge \( e \), we glue, edge to edge, \( \sigma(\text{type}(f)) \) and \( \sigma(\text{type}(f')) \) along \( \sigma(\tau_f(e)) \) and \( \sigma(\tau_{f'}(e)) \). This is possible since the \( p \)-compatibility of \( \sigma \) ensures that the edge \( e \) is balanced. The resulting patch \( \sigma^p(T_i) \) is defined by:

\[
\sigma^p(T_i) = \left( \bigcup_{\text{face of } \sigma^{p-1}(T_i)} \sigma(\text{type}(f)) \right) / \sim
\]

where \( \sim \) denotes the gluing. We define \( \sigma^p(T) \) to be the set \( \{\sigma^p(T_1), \ldots, \sigma^p(T_d)\} \).

**Remark 3.5.** Definition 3.4 is recursive. Indeed, conditions b) and c) should be denoted \( b_p \) and \( c_p \) since they do depend on \( p \). Then the definition should be read in the following order: a), \( b_2), (c_2), \ldots, b_p), (c_p), (b_{p+1}), (c_{p+1}), \ldots \).
The map \( \sigma \) induces a natural map on the faces of each \( \sigma^{i-1}(T_i) \) which factorizes to a map \( \sigma_{i,p} : \sigma^{p-1}(T_i) \to \sigma^p(T_i) \) thanks to the \( p \)-compatibility hypothesis:

\[
\begin{array}{ccc}
\bigcup_{f \text{ face of } \sigma^{i-1}(T_i)} \text{type}(f) & \xrightarrow{\sigma} & \bigcup_{f \text{ face of } \sigma^{p-1}(T_i)} \text{type}(f) \\
\sim & & \sim \\
\sigma^{p-1}(T_i) & \xrightarrow{\sigma_{i,p}} & \sigma^p(T_i)
\end{array}
\]

We note that \( \sigma_{i,p} \) is a homeomorphism which sends vertices to vertices. Then we define the map

\( \sigma^p_i : T_i \to \sigma^p(T_i) \)

as the composition: \( \sigma^p_i = \sigma_{i,p} \circ \sigma^{i-1}_i \). This is an homomorphism which sends vertices to vertices. Then \( \sigma^p \) is naturally defined such that the restriction of \( \sigma^p \) on \( T_i \) is \( \sigma^p_i \). We remark that \( \sigma^1 = \sigma \). We have thus obtained a topological pre-substitution \((T, \sigma^p(T), \sigma^p)\). A topological pre-substitution is compatible if it is \( p \)-compatible for every integer \( p \).

**Checking compatibility** In this subsection we give an algorithm which decides whether a pre-substitution is compatible.

Suppose that \( \sigma \) is \( p \)-compatible. We define \( W_p \) as the set of elements \([\varepsilon, \varepsilon'] \in E_2\) such that there exists \( i \in \{1, \ldots, d\} \), as well as two faces \( f \) and \( f' \) in \( \sigma^p(T_i) \) glued along an edge \( e \), such that \( \tau_f(\varepsilon) = \varepsilon \) and \( \tau_{f'}(\varepsilon) = \varepsilon' \). The topological pre-substitution \( \sigma \) is \((p + 1)\)-compatible if and only if \( W_p \) is contained in the set of balanced pairs \( B : W_p \subseteq B \). Then:

- either \( W_p \notin B \): the algorithm stops, telling us that the substitution is not compatible,
- or \( W_p \subseteq B \): then we define \( V_p = V_{p-1} \cup W_p \).

By convention we settle \( V_0 = \emptyset \).

Suppose that \( \sigma \) is compatible. The sequence \((V_p)_{p \in \mathbb{N}}\) is an increasing sequence (for the inclusion) of subsets of the finite set \( E_2 \). Hence there exists some \( p_0 \in \mathbb{N} \) such that \( V_{p_0+1} = V_{p_0} \) (and thus \( V_p = V_{p_0} \) for all \( p \geq p_0 \)). The algorithm stops at step \( p_0 + 1 \) (where \( p_0 \) is the smallest integer such that \( V_{p_0+1} = V_{p_0} \)), telling that \( \sigma \) is compatible.

The heredity graph of edges of \( \sigma \), denoted \( E(\sigma) \), is defined in the following way. The set of vertices of \( E(\sigma) \) is \( V_{p_0} \). There is an oriented edge from vertex \([\varepsilon, \varepsilon']\) to vertex \([\varepsilon, \varepsilon']\) if \([\varepsilon, \varepsilon']\) is a descendant of \([\varepsilon, \varepsilon']\).

**Example 3.6.** For the Tribonacci topological pre-substitution, consider one again the edges \( e = B_{45}, e' = C_{76} \). We have shown in a previous example that \( \sigma(e) = C_{34}C_{45}C_{56}, \sigma(e') = C_{10}C_{89}C_{98} \). Thus \([B_{45}, C_{76}]\) is a vertex of the heredity graph of edges. There are three edges which start from this vertex and go to the vertices defined by the balanced pairs – see Figure 5 and the proof of Lemma 3.9 for more details.

**Core of a topological pre-substitution** Let \( P \) be a patch modeled on \( T = \{T_1, \ldots, T_d\} \). The thick boundary \( B(P) \) of \( P \) is the closed sub-complex of \( P \) consisting of the closed faces which contain at least one vertex of the boundary \( \partial P \) of \( P \). The core \( \text{Core}(P) \) of \( P \) is the closure in \( P \) of the complement of \( B(P) \): in particular, \( \text{Core}(P) \) is a closed subcomplex of \( P \), see Figure 2.

A topological pre-substitution \((T, \sigma(T), \sigma)\) has the core property if there exist \( i \in \{1 \ldots d\} \) and \( k \in \mathbb{N} \) such that the core of \( \sigma^k(T_i) \) is non-empty.

**Example 3.7.** For the Tribonacci topological pre-substitution we show in Figure 6 that the cores of \( \sigma(C) \) and \( \sigma^2(C) \) are empty. But the core of \( \sigma^3(C) \) is not empty.

**Definition 3.8.** A topological substitution is a pre-substitution which is compatible and has the core property.
3.1.2 Topological plane obtained by inflation

Consider a tile $T \in \mathcal{T}$ such that the core of $\sigma(T)$ contains a face of type $T$. Then, we can identify the tile $T$ with a subcomplex of the core of $\sigma(T)$. By induction, $\sigma^k(T)$ is thus identified with a subcomplex of $\sigma^{k+1}(T)$ ($k \in \mathbb{N}$). We define $\sigma^\infty(T)$ as the increasing union:

$$\sigma^\infty(T) = \bigcup_{k=0}^{\infty} \sigma^k(T).$$

By construction, the complex $\sigma^\infty(T)$ is homeomorphic to $\mathbb{R}^2$. (Indeed, denoting $\sigma^k(T)$ by $D_k$, one observes that $\sigma^\infty(T)$ is an increasing union of closed discs $D_k$ with $D_k$ contained in the interior $\text{Int} \ D_{k+1}$ of $D_{k+1}$, and with $D_{k+1} \setminus \text{Int} \ D_k$ homeomorphic to the annulus $S^1 \times I$. This allows to build an homeomorphism between $\sigma^\infty(T)$ and $\mathbb{R}^2$ – see for instance [Hir94, exercise 3 p. 207].) We say that such a complex is obtained by inflation from $\sigma$. Moreover this complex can be labelled by the types of the topological polygons. We notice that $\sigma$ induces an homeomorphism of $\sigma^\infty(T)$.

We denote by $\mathcal{P}_\sigma$ the set of patches in the complex $\sigma^\infty(T)$. We notice that $\sigma$ naturally defines a map $\mathcal{P}_\sigma \to \mathcal{P}_\sigma$, that is still denoted by $\sigma$. To be more precise, given a patch $P \in \mathcal{P}_\sigma$, there is some $k \in \mathbb{N}$ such that $P \subseteq \sigma^k(P) \subset \sigma^\infty(P)$, so that $\sigma(P) \subseteq \sigma^{k+1}(P) \subset \sigma^\infty(P)$: this patch $\sigma(P) \in \mathcal{P}_\sigma$ does not depend on the choice of $k$.

We denote by $\mathcal{T}_\sigma$ the set of tiles in the complex $\sigma^\infty(T)$: $\mathcal{T}_\sigma$ is a subset of $\mathcal{P}_\sigma$. See Figure 3 for examples of such topological complexes generated by topological substitutions.

3.2 The Tribonacci topological substitution

We first define a topological substitution $\sigma$. Then we explain how to derive a tiling of $\mathbb{R}^2$ as a geometrical realization of the patches generated by $\sigma$. The topological substitution is defined on Figure 4 and the first iterations on the polygon C are given in Figure 6.
### 3.2.1 Definition of the topological substitution

We consider the Tribonacci topological pre-substitution $\sigma$ defined on Figure 4. There are three prototiles: two of them, $A$ and $B$, are hexagons, while the third one, $C$, is a decagon. The images of these prototiles (together with the labelling of the vertices and the images of the vertices) are given on Figure 4. In practice, we will denote by $A_i$ the vertex $i$ of $A$, and by $A_i(i+1)$ the edge of $A$ joining $A_i$ and $A_{i+1}$ (and so on for $B$ and $C$).

![Figure 4: The Tribonacci topological substitution.](image)

**Lemma 3.9.** The topological pre-substitution $\sigma$ is a topological substitution.

**Proof.** Using the procedure described at the end of Subsection 3.1.1 (in Paragraph “Checking compatibility”), we first check that $\sigma$ is compatible. We start with the pair of edges that are glued in the images of $A$, $B$ and $C$. In fact, all these gluings occur in $\sigma(C)$: $[A_{45}, C_{54}]$, $[A_{23}, B_{05}]$, $[B_{45}, C_{76}]$, $[A_{43}, C_{36}]$.

We focus now on $[B_{45}, C_{76}]$. The image of the edge $B_{45}$ is the path of edges $C_{34}C_{45}C_{56}$. The image of $C_{76}$ is $C_{10}C_{98}$. Both have length 3, and the gluing gives rise to the pairs of edges: $[C_{34}, C_{10}]$, $[C_{45}, C_{99}]$ and $[C_{56}, C_{98}]$.

Carrying out the other pairs in the same way, and iterating the process, we check that $\sigma$ is compatible. These computations are summed up in the heredity graph of edges, given in Figure 5.

The core property is checked for $\sigma^3(C)$: we see on Figure 6 that Core($\sigma^3(C)$) $\neq \emptyset$. Hence $\sigma$ is a topological substitution.

### 3.2.2 Configurations at the vertices

We denote by $V$ the set of vertices of the prototiles $A$, $B$, $C$. The heredity graph of vertices is an oriented graph denoted by $V(\sigma)$. The set of vertices of $V(\sigma)$ is the set $V$. Let $T, T' \in \{A, B, C\}$, and let $v$ be a vertex of $T$ and $v'$ be a vertex of $T'$: there is an oriented edge in $V(\sigma)$ from $v$ to $v'$ if $\sigma(v)$ is a vertex of type $v'$ of a tile of type $T'$.

The graph $V(\sigma)$ is given in Figure 7. It can be used to control the valences of the vertices in $\sigma^\infty(C)$ thanks to the following property proved in [BH13]. A vertex $v \in V$ is a divided vertex.
if there are at least two oriented edges in \( V(\sigma) \) coming out of \( v \). We denote by \( V_D \) the subset of \( V \) which consists of all divided vertices. The following properties are equivalent, see \( [BH13] \):
• The complex \( \sigma^\infty(C) \) has bounded valence.
• Every infinite oriented path in \( \mathcal{V}(\sigma) \) crosses vertices of \( \mathcal{P}_D \) only finitely many times.
• The oriented cycles of \( \mathcal{V}(\sigma) \) do not cross any vertex of \( \mathcal{P}_D \).

**Lemma 3.10.** The valence of each vertex in \( \sigma^\infty(C) \) is bounded.

**Proof.** We consider Figure [7]. Remark that \( \mathcal{P}_D = \{C_0, C_5\} \). Moreover, neither \( C_0 \) nor \( C_5 \) is crossed by an oriented cycle of \( \mathcal{V}(\sigma) \). Hence the valence of the vertices of \( \sigma^\infty(C) \) is bounded by the previous property. \( \Box \)

Now we introduce another graph, which is called the **configuration graph of vertices** and is denoted by \( \mathcal{C}(\sigma) \). We consider the equivalence relation on \( k \)-tuples of elements of \( V \ (k \in \mathbb{N}) \) generated by:

\[
(x_1, \ldots, x_{k-1}, x_k) \sim (x_k, x_1, \ldots, x_{k-1}) \text{ and } (x_1, x_2, \ldots, x_k) \sim (x_k, \ldots, x_2, x_1).
\]

Let \([x_1, \ldots, x_k]\) denote the equivalence class of \((x_1, \ldots, x_k)\). Let \( K \) be the maximal valence of a vertex in \( \sigma^\infty(C) \). Let \( W \) be the set of equivalence classes of \( k \)-tuples with \( 2 \leq k \leq K \). A vertex \( v \) in the interior of a patch \( \sigma^n(C) \) \((n \geq 1)\) defines an element \([x_1, \ldots, x_k] \in W\) (where \( k \) is the valence of \( v \)). Indeed, the faces adjacent to \( v \) are cyclically ordered, and \( x_i \) is the type of the vertex of the \( i \)-th face which is glued on \( v \).

We define the oriented graph \( \mathcal{C}(\sigma) \) as follows. Let \( W_0 \) be the subset of \( W \) defined by the vertices occurring in the interior of some \( \sigma^n(C) \) for \( n \geq 1 \). An element of \( W_0 \) is called a **vertex configuration**. The set of vertices of \( \mathcal{C}(\sigma) \) is \( W_0 \). For any \( s \in W_0 \), we choose some \( T \in \mathcal{T}, n \geq 1 \) and \( v \) a vertex in the interior of \( \sigma^n(T) \) which defines \( s \). Let \( s' \) the element of \( W_0 \) defined by \( \sigma(v) \). There is an oriented edge in \( \mathcal{C}(\sigma) \) from \( s \) to \( s' \). We notice that this construction does not depend of the choice of \( T \) and \( n \).

In practice, to build the graph \( \mathcal{C}(\sigma) \), we first remark that a vertex \( v \) in the interior of some \( \sigma^n(C) \) \((n \geq 1)\) is either the image of a vertex in the interior of \( \sigma^{n-1}(C) \), or is in the interior of a path of edges which is the image of an interior edge of \( \sigma^{n-1}(C) \). Thus we first make the list of vertex configurations for:

- vertices in the interior of the image of a tile: we get \([C_5, A_4]\) and \([C_6, A_3, B_5]\);
- vertices in the interior of the image of an interior edge. These ones can be derived from the vertices of \( \mathcal{E}(\sigma) \) with a least 2 outing edges. There are 3 such vertices in of \( \mathcal{E}(\sigma) \):
  - \([B_{45}, C_{56}]\) which gives rise to vertex configurations \([C_4, C_0]\) and \([C_5, C_9]\);
  - \([C_{12}, B_{21}]\) and \([C_{12}, C_{70}]\) which gives rise to vertex configurations \([C_4, C_0]\) and \([C_5, C_9]\).

Then we iteratively compute the vertex configurations obtained as the image under \( \sigma^n \) of the vertex configurations in \([C_5, A_4], [C_6, A_3, B_5], [C_4, C_0], [C_5, C_9], C_4, C_0, C_5, C_9])\}. The graph \( \mathcal{C}(\sigma) \) is represented on Figure [8].

### 3.2.3 A geometric realization of \( \sigma^\infty(C) \)

**Proposition 3.11.** The complex \( \sigma^\infty(C) \) can be realized as a tiling of \( \mathbb{R}^2 \). This tiling is denoted by \( \mathcal{T}_{top} \).

**Proof.** We first recall that a 2-dimensional CW-complex with hexagonal faces such that each edge belongs to 2 faces and each vertex belongs to 3 faces is isomorphic to the 2-dimensional CW-complex \( X_{\text{hex}} \) defined by the tiling of \( \mathbb{R}^2 \) by regular Euclidean hexagons.

Let \( C_{\text{hex}} \) the patch made of two hexagonal faces obtained by dividing \( C \) along an edge between vertices \( C_0 \) and \( C_5 \). Given a patch \( P \) made of tiles of types \( A, B \) and \( C \), we build a patch \( P_{\text{hex}} \) made of hexagonal faces by replacing the faces of type \( C \) by \( C_{\text{hex}} \).
We recall that a dual substitution, one gets bigger sets of facets. A priori, there can be some overlaps: in that

4 The Tribonacci dual substitution and its fractal tilings

We claim that $(\sigma^\infty(C))_{\text{hex}} = X_{\text{hex}}$. Indeed, the vertex configurations of $\sigma^\infty(C)$ are given by the vertices of the graph $\mathcal{C}$ (see Figure 8), and for every of them, we check that $P = P_{\text{hex}}$ for the corresponding patch $P$.

Since $(\sigma^\infty(C))_{\text{hex}} = X_{\text{hex}}$, it is straightforward to derive that $\sigma^\infty(C)$ can be geometrically realized as a tiling of $\mathbb{R}^2$, where $A$ and $B$ are realized by regular hexagons, and $C$ by a decagon obtained by gluing two regular hexagons along an edge. \hfill $\square$

**Remark 3.12.** In particular, we can now precise Lemma 3.10: The valence of a vertex in $\sigma^\infty(C)$ is either 2 or 3.

3.3 The pointed topological substitution $\hat{\sigma}$

Let $\mathcal{P}\mathcal{P}$ be the set of pairs $(P, T)$ where $P$ is a patch in $\sigma^\infty(C)$ and $T$ is a tile in $\sigma^\infty(C)$. We stress that $T$ need not lie in $P$. Such a pair $(P, T) \in \mathcal{P}\mathcal{P}$ is a **pointed patch**, and $T$ is the **base tile** of the pointed patch $(P, T)$.

For our purposes, we need to consider a kind of “pointed” version $\hat{\sigma}$ of $\sigma$: this will be a map

$$\hat{\sigma} : \mathcal{P}\mathcal{P} \rightarrow \mathcal{P}\mathcal{P} \quad (P, T) \mapsto (\sigma(P), b(T)).$$

To completely define $\hat{\sigma}$, it remains now to define the map $b$. Let $T$ be a tile of $\sigma^\infty(C)$. Then $\sigma(T)$ is a patch of $\sigma^\infty(C)$. If $\sigma(T)$ is a tile (this is the case when $T$ is of type $A$ or $B$), then we simply set $b(T) = \sigma(T)$. If $T$ is of type $C$, then we define $b(T)$ as the tile of type $C$ in $\sigma(T)$.

**Remark 3.13.** In the example this last choice is a bit arbitrary: in particular, we could have considered other versions of $\hat{\sigma}$ where, for a tile $T$ of type $C$, $b(T)$ would be the tile of type $A$ (or $B$) in $\sigma(T)$. In that case, we would have to modify accordingly the definition of the position map $\omega_0$ of Section 5.1.3.

4 The Tribonacci dual substitution and its fractal tilings

We recall that a **substitution** is a morphism of the free monoid (of rank $d$). There is a general construction introduced in [IO93] and generalized by [AI01] that associates to a substitution a so-called **dual substitution**. To avoid to reproduce the general formalism of [AI01], we focus in sections 4.2 and 4.3 on the **Tribonacci dual substitution** associated to the the Tribonacci substitution $1 \mapsto 12$, $2 \mapsto 13$, $3 \mapsto 1$. In particular, $d = 3$.

Dual substitutions act on facets of $\mathbb{R}^3$ (for the cellular decomposition of $\mathbb{R}^3$ given by $\mathbb{Z}^3$ translated copies of the unit cube): the image of a facet is a set of facets. Hence, when iterating a dual substitution, one gets bigger sets of facets. A priori, there can be some overlaps: in that
case, facets have to be count with some multiplicity, which leads to the notion of multiset of facets.

The data that encode a multiset of facet is given by: the type (an element of the set \{1, 2, 3\}), the position (an element of \(\mathbb{Z}^3\)) and the signed multiplicity (an element of \(\mathbb{Z}\)) of each facet. This setting is formalized in section 4.1, leading to the equivalent notions of weight functions and multisets of facets. We detail the equivalence of the two points of view, so that later in section 5 we will swap from one point of view to the other one according to the context.

### 4.1 Multisets of facets

We denote by \((e_1, e_2, e_3)\) the canonical basis of \(\mathbb{R}^3\). In this article, this basis will be represented as follows in the different figures.

\[
\begin{align*}
  e_3 \\
  e_1 \\
  e_2
\end{align*}
\]

Let \(x \in \mathbb{Z}^3\) and let \(i \in \{1, 2, 3\}\). The **facet** \([x, i]^*\) of vector \(x\) and type \(i\) is a subset of \(\mathbb{R}^3\) defined by:

\[
\begin{align*}
  [x, 1]^* &= \{x + \lambda e_2 + \mu e_3 : \lambda, \mu \in [0, 1]\} = \mathcal{F}_x^1 \\
  [x, 2]^* &= \{x + \lambda e_1 + \mu e_3 : \lambda, \mu \in [0, 1]\} = \mathcal{F}_x^2 \\
  [x, 3]^* &= \{x + \lambda e_1 + \mu e_2 : \lambda, \mu \in [0, 1]\} = \mathcal{F}_x^3.
\end{align*}
\]

On each of the previous pictures, the symbol \(\bullet\) represents the endpoint of the vector \(x\). We set \(\mathcal{F} = \{[x, i]^*, x \in \mathbb{Z}^3, i \in \{1, 2, 3\}\}\).

Let \(W\) be the set of maps from \(\mathcal{F}\) to \(\mathbb{Z}_{\geq 0}\): such a map is called a **weight function**. A weight function \(w \in W\) gives a weight \(w([x, i]^*) \in \mathbb{Z}_{\geq 0}\) to any facet. Equipped with the addition of maps, \(W\) is a monoid.

A **multiset of facets** is a map \(m : \mathbb{Z}^3 \to \mathbb{Z}_{\geq 0}^3\). We denote by \(\mathcal{M}\) the set of multisets of facets. The set \(\mathcal{M}\), equipped with the addition of maps, is a monoid.

Multisets of facets and weight functions are equivalent objects. Indeed, a multiset \(m \in \mathcal{M}\) defines a weight function \(w_m \in W\) by declaring that \(w_m([x, i]^*)\) is the \(i\)th coordinate of \(m(x)\). The map

\[
\begin{align*}
\mathcal{M} &\to W \\
  m &\mapsto w_m
\end{align*}
\]

is an isomorphism of monoids. The inverse of the map is given by

\[
\begin{align*}
W &\to \mathcal{M} \\
  w &\mapsto \left(x \mapsto \left(\underbrace{w([x, 1]^*), w([x, 2]^*), w([x, 3]^*)}_\text{The group } \mathbb{Z}^3 \text{ acts naturally on } \mathcal{M}: \text{ if } v \in \mathbb{Z}^3, m \in \mathcal{M} \text{ then } m + v : x \mapsto m(x - v).\right)\right)
\end{align*}
\]

The **support** \(\text{supp}(w)\) of a weight function \(w\) is the union of facets which have positive weight:

\[
\text{supp}(w) = \bigcup_{w([x, i]^*) > 0} [x, i]^*.
\]

It is a subset of \(\mathbb{R}^3\). The **support** \(\text{supp}(m)\) of a multiset of facets \(m\) is the support of the corresponding weight function:

\[
\text{supp}(m) = \text{supp}(w_m).
\]

Let \(W^0 \subset W\) be the subset of weight functions which take values in \(\{0, 1\}\). We denote by \(\mathcal{M}^0\) the corresponding subset of \(\mathcal{M}\): a multiset of facets \(m\) is in \(\mathcal{M}^0\) if and only if for all \(x \in \mathbb{Z}^3\), the coordinates of \(m(x)\) are in \(\{0, 1\}\).
**Remark 4.1.** We notice that a multiset of facets in \( M^o \) (or a weight function in \( W^o \)) is totally determined by its support.

### 4.2 Dual substitutions

In this Section we quickly review a construction due to Arnoux-Ito [AI01] that associates to a unimodular substitution \( s \) what is called a dual substitution \( E(s) \). For details we refer to [AI01] [BR10]. In particular, this construction can be applied to the Tribonacci substitution to lead the dual substitution \( E \) defined below. The definition of a substitution will be given in Section 4.3.

Consider

\[
M_s = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\]

This matrix has characteristic polynomial \( X^3 - X^2 - X - 1 \). Its dominant eigenvalue \( \beta \) is a Pisot number: \( \beta > 1 \) and its conjugates \( \alpha, \overline{\alpha} \in \mathbb{C} \) are such that \( |\alpha| < 1 \). The euclidean space \( \mathbb{R}^3 \) is hence decomposed as the direct sum of the expanding line (spanned by the left \( \beta \)-eigenvector \( v_\beta \) of \( M_s \)) and the contracting plane \( P \) associated with the complex eigenvalues \( \alpha, \overline{\alpha} \). Let \( \pi_\beta : \mathbb{R}^3 \to P \) be the projection on \( P \) along the line \( \mathbb{R}v_\beta \). We denote by \( h : P \to P \) the action of \( M_s \) on \( P \), which is contracting because \( |\alpha| < 1 \). Remark that \( M_s, h, \pi_\beta \) commute.

**Definition 4.2.** We define

\[
E : \{[x, 1]^* \mapsto M_s^{-1}x + ([0, 1]^* \cup [0, 2]^* \cup [0, 3]^*)
\]

Alternatively \( E \) can be defined using multisets as following:

- The image of \([x, 1]^*\) by \( E \) is the multiset \((\mathbb{Z}^3, m)\) where

  \[
m(y) = \begin{cases} 0 & \text{if } y \neq M_s^{-1}x \\ (1, 1, 1) & \text{if } y = M_s^{-1}x. \end{cases}
\]

- The image of \([x, 2]^*\) by \( E \) is the multiset \((\mathbb{Z}^3, m)\) where

  \[
m(y) = \begin{cases} 0 & \text{if } y \neq M_s^{-1}x \\ (0, 0, 1) & \text{if } y = M_s^{-1}x. \end{cases}
\]

- The image of \([x, 3]^*\) by \( E \) is the multiset \((\mathbb{Z}^3, m)\) where

  \[
m(y) = \begin{cases} 0 & \text{if } y \neq M_s^{-1}x \\ (0, 1, 0) & \text{if } y = M_s^{-1}x. \end{cases}
\]

We extend \( E \) to \( M \) by declaring that the image of a union of faces is the union of the images of these faces (the multiplicities of faces add up). We also note for future application that for all \( x, u \in \mathbb{Z}^3 \), and for all \( i \in \{1, 2, 3\} \), we have

\[
E([x, i]^* + u) = E[x, i]^* + M_s^{-1}u \quad (4.1)
\]

In practice, in order to simplify the notation, we represent \( E \) by the following pictures

\[
\text{where the black dots in the preimages stand for } x, \text{ and the black dots in the images stand for } M_s^{-1}x. \]

We denote the Euclidean scalar product of to vectors \( u, v \in \mathbb{R}^3 \) by \( \langle u, v \rangle \), and we define \( \mathcal{U} \) as the multiset of facets in \( M^o \) whose support is \([0, 1]^* \cup [0, 2]^* \cup [0, 3]^* \) (see Remark 4.1).
Proposition 4.3 ([A101, AB02, ABS04]).

- For every integer $n$, $E^n(U)$ belongs to $M^\circ$, so it can be considered as a subset of $\mathbb{R}^3$ (see Remark [4.7]).
- For every integer $n$, $E^n(U)$ is a subset of $E^{n+1}(U)$. The increasing sequence of $E^n(U)$ converges and we denote

$$\Sigma_{\text{step}} = \lim_{n \to \infty} E^n(U) = \bigcup_{n \in \mathbb{N}} E^n(U).$$

This set $\Sigma_{\text{step}}$ is called the stepped surface.

- Moreover:

$$\Sigma_{\text{step}} = \bigcup_{i \in \{1,2,3\}} \bigcup_{x \in \mathbb{Z}^3} [x,i]^*.\tau$$

- The restriction $\pi_\beta : \Sigma_{\text{step}} \to \mathcal{P}$ of $\pi_\beta$ to $\Sigma_{\text{step}}$ is an homeomorphism.
- This map induces a tiling of the plane $\mathcal{P}$: we denote this tiling by $\mathcal{T}_{\text{step}}$. The set of tiles of $\mathcal{T}_{\text{step}}$ is:

$$\bigcup_{i \in \{1,2,3\}} \{\pi_\beta([0,i]^*) + \pi_\beta(x) : x \in \mathbb{Z}^3, 0 \leq \langle x, v_\beta \rangle < \langle e_i, v_\beta \rangle\}.$$

We say that a vector $x \in \mathbb{Z}^3$ lies in $\Sigma_{\text{step}}$ if there exists $i \in \{1,2,3\}$ such that $[x,i]^*$ is a subset of $\Sigma_{\text{step}}$. Proposition 4.3 implies that the set of vectors lying in $\Sigma_{\text{step}}$ is precisely $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3$, where

$$\mathcal{V}_i = \{x \in \mathbb{Z}^3 : 0 \leq \langle x, v_\beta \rangle < \langle e_i, v_\beta \rangle\}.$$

Remark 4.4. We set $\mathcal{D}_i = \pi_\beta(\mathcal{V}_i)$. Then $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$ is a Delone set in $\mathcal{P}$. The tiling $\mathcal{T}_{\text{step}}$ is obtained by putting in $\mathcal{P}$ a tile of type $i$ (i.e. a translated image of $\pi_\beta([0,i]^*)$) at each vector in $\mathcal{D}_i$.

4.3 Link between the tiling $\mathcal{T}_{\text{step}}$ and the Rauzy fractal

In this section we recall basic facts about Rauzy fractals and substitutions [PF02]. We consider the free monoid $\{1,2,3\}^*$ and the Tribonacci substitution $s : \{1,2,3\}^* \to \{1,2,3\}^*$, which is a morphism defined by

$$s : 1 \mapsto 12 \quad 2 \mapsto 13 \quad 3 \mapsto 1.$$

Denote by $u = 12131\ldots$ the infinite word on the alphabet $\{1,2,3\}$ such that $s(u) = u$. In fact, for all $n \in \mathbb{N}$, $s^n(1)$ is a prefix of $s^{n+1}(1)$, so that $u = \lim_{n \to \infty} s^n(1)$. We denote by $u_i \in \{1,2,3\}$ the $i$-th letter in $w$: $u = u_1 u_2 u_3 \ldots$ with $u_1 = 1$, $u_2 = 2$, $u_3 = 3$.

Let us define $M_s$ the incidence matrix of $s$: its $i$th column vector is equal to $P(s(i))$, where $P$ be the abelianization map from $\{1,2,3\}^*$ to $\mathbb{Z}^3$ defined by $P(w) = (|w|_1, |w|_2, |w|_3)$ and $|w|_i$ stands for the number of occurrences of $i$ in $w$. Consistently with the notation of Section 4.2 we have:

$$M_s = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

In the following proposition, we go on using the notation introduced in Section 4.2. In particular, the map $h : \mathcal{P} \to \mathcal{P}$ is the restriction the action of $M_s$ to contracting eigenplane $\mathcal{P}$.

Proposition 4.5 ([Rau82, CS01, A101]).

- The sets

$$R_i = \{\pi_\beta(P(u_1 \ldots u_{j-1})) : j \in \mathbb{N}, u_j = i\} \text{ for } i \in \{1,2,3\},$$

16
Remark 4.6. Comparing Proposition 4.5 and Proposition 4.3, we see that the positions of the tiles in $\mathcal{T}_{\text{frac}}$ and $\mathcal{T}_{\text{step}}$ are given by the same formula. Indeed, the tiling $\mathcal{T}_{\text{frac}}$ is obtained by putting in $\mathcal{P}$ a tile of type $i$ (i.e. a translated image of $\mathcal{R}_i$) at each vector in $\mathcal{D}_i$, where the sets $\mathcal{D}_i$ are precisely the ones of Remark 4.4. This explicits the strong relation between the two tilings $\mathcal{T}_{\text{frac}}$ and $\mathcal{T}_{\text{step}}$.

5 Link between topological and dual substitutions

5.1 The position map

In this section we define the position map $\omega_0$ from the set $\mathfrak{P}$ of paths of tiles in $\sigma^\infty(C)$ to $\mathbb{Z}^3$. (See Sections 5.1.2 and 5.1.3 below for precise definitions of $\mathfrak{P}$ and $\omega_0$.)

We use the term position map because $\omega_0$ will be used to give a geometric interpretation of the relative positions of two tiles in a common patch. This geometric interpretation is given by a vector in $\mathbb{Z}^3$ (the position of a tile with respect to another one). This will simplify the work done in Section 5.2 where we associate geometric patches of stepped surfaces to abstract topological patches.

5.1.1 Notation

Let $\sigma$ be the Tribonacci topological substitution defined in Section 3.2. We denote by $P$ a patch of $\sigma^\infty(C)$, or possibly $P = \sigma^\infty(C)$.

Definition 5.1. Consider a positive integer $n$. A path of tiles $\gamma$ in $P$ is a sequence $T_0, \ldots, T_n$ of tiles of $P$ such that two consecutive tiles $T_i, T_{i+1}$ are different and share (at least) one common edge for all $i \in \{0, \ldots, n-1\}$. The integer $n+1$ is the length of the path $\gamma = T_0, \ldots, T_n$. The set $\{T_0, \ldots, T_n\} \subset P$ is called the support of $\gamma$ in $P$. When $T_0 = T_n$, $\gamma$ is a loop of tiles. The integer $n$ is the length of the loop $\gamma = T_0, \ldots, T_n$. The path $\gamma = T_0, \ldots, T_n$ and the path $\gamma' = T_0', \ldots, T_m'$ can be concatenated if $T_0' = T_n$. The concatenation of these paths is the path of tiles $\gamma \gamma' = T_0, \ldots, T_n, T_1', \ldots, T_m'$.

Let $\gamma$ be a loop of tiles in $\sigma^\infty(C)$. Among the connected components of the complement of the support of $\gamma$, there is exactly one, denoted by $C_0(\gamma)$, which contains an infinite number of tiles. We denote by $C_0(\gamma)$ the complement of $C_0(\gamma)$: it is a patch, in particular it is homeomorphic to a disc. Alternatively $C_0(\gamma)$ is the smallest subpatch of $\sigma^\infty(C)$ containing the support of $\gamma$. We define the area of $\gamma$ to be the number of tiles in $C_0(\gamma)$:

$$\text{Area}(\gamma) = |C_0(\gamma)|.$$ 

Now we define an equivalence relation on paths of tiles which will define a protopath of tiles: The path $\gamma = T_0, \ldots, T_n$ and the path $\gamma' = T_0', \ldots, T_m'$ are equivalent if
• \( m = n \)
• For every \( i \in \{0, \ldots, n\} \), \( T_i \) and \( T'_i \) have the same prototile type.
• The gluing edges of \( T_i \) and \( T_{i+1} \) have the same type as the gluing edges of \( T'_i \) and \( T'_{i+1} \).

In the same way we define the notion of protoloop. The notions of concatenation, area and length naturally extend to protopaths.

### 5.1.2 Additivity

Let \( P \) be patch in \( \sigma^\infty(C) \). By definition, \( P \) is homeomorphic to a disc and its boundary is homeomorphic to the circle \( S^1 \). Let \( T \) be a tile in \( P \), the **wreath** of \( T \) in \( P \) is the subset of \( P \setminus \{ T \} \) made of tiles that have at least one vertex in common with \( T \). We denote it by \( \text{Wreath}_P(T) \). A **cut tile** of \( P \) is a tile whose wreath in \( P \) is not connected.

Let \( T \) be a cut tile of \( P \). Then \( P \setminus T \) has at least 2 connected components, and each of these components is a patch.

**Lemma 5.2.** Let \( P \) be a finite patch in \( \sigma^\infty(C) \). There exists one tile of \( P \) which is not a cut tile and has one edge in the boundary of \( P \).

**Proof.** Pick a tile \( T_0 \) in \( P \) which has a vertex in the boundary of \( P \). If \( T_0 \) is not a cut tile, we are done. Otherwise, because \( P \) is homeomorphic to a disk, \( P \setminus T_0 \) has at least two connected components: we choose one of them that we denote by \( P_1 \). Pick a tile \( T_1 \) in \( P_1 \) which has a vertex in the boundary of \( P \). If \( T_1 \) is not a cut tile, we are done. Otherwise, \( P \setminus T_1 \) has at least two connected components, and at least one of them is included in \( P_1 \): we choose one of these ones, that we denote by \( P_2 \). If every tile with one edge in the boundary is a cut-tile we obtain an infinite number of nested connected components which is a contradiction with the fact that \( P \) contains a finite number of tiles. \( \square \)

Let \( \mathcal{P} \) be the set of protopaths of tiles in \( \sigma^\infty(C) \). A map \( \omega : \mathcal{P} \rightarrow \mathbb{Z}^3 \) is **additive** if for every \( \gamma, \gamma' \in \mathcal{P} \) that can be concatenated, we have \( \omega(\gamma \gamma') = \omega(\gamma) + \omega(\gamma') \). Hence, an additive map \( \omega : \mathcal{P} \rightarrow \mathbb{Z}^3 \) is uniquely defined by the image of the protopaths of length 2.

**Definition 5.3.** Let \( \mathcal{P}_0 \subseteq \mathcal{P} \) be the subset consisting of protoloops \( \gamma = T_0, \ldots, T_n \) (with \( T_0 = T_n \)) of tiles in \( \sigma^\infty(C) \) such that

- for every \( i \in \{1, \ldots, n-1\} \), \( T_i \in \text{Wreath}_{\mathcal{C}_0(\gamma)}(T_0) \),
- for every \( i \neq j \in \{1, \ldots, n-1\} \), \( T_i \neq T_j \).

We notice that \( \mathcal{P}_0 \) contains all protoloops of length 2.

**Lemma 5.4.** The set \( \mathcal{P}_0 \) is finite.

**Proof.** The valence of every vertex in \( \sigma^\infty(C) \) is bounded (by 3, see Lemma 3.10), we deduce that the cardinality is finite. \( \square \)

It is possible to produce an explicit list of the elements in \( \mathcal{P}_0 \). We detail below all the elements of \( \mathcal{P}_0 \) of length 3.

![Diagram of protoloop elements](image)

**Lemma 5.5.** Let \( \omega : \mathcal{P} \rightarrow \mathbb{Z}^3 \) an additive map such that \( \omega \) vanishes on the elements of \( \mathcal{P}_0 \). Then \( \omega \) vanishes on every protoloop in \( \sigma^\infty(C) \).
Proof. The proof is by induction on the area of the protoloop of tiles $\gamma$ in $\sigma^\infty(C)$. According to Definition 5.1, a loop of tiles has length at least 2, and thus also area at least 2. Moreover, a loop of tiles $\gamma$ with area 2 is the concatenation of a certain number of copies of a same loop of tiles of length 2 $\gamma' = T_0, T_1, T_0$. Since any loop of length 2 is in $\mathcal{P}_0$, we get that $\omega(\gamma') = 0$. By additivity of $\omega$, we derive that $\omega(\gamma) = 0$.

Suppose that $\omega$ vanishes on every loop in $\sigma^\infty(C)$ of area at most $k$. Let $\gamma = T_0, \ldots, T_n$ be a loop of area $k + 1$. By Lemma 5.2 there exists a tile $T$ in $\mathcal{C}_0(\gamma)$ which is not a cut-tile and has one edge in the boundary of $\mathcal{C}_0(\gamma)$. The tile $T$ may occur several times in $\gamma$, and for each occurrence we will successively act as follows.

Let $T_i = T$ be an occurrence of $T$ in $\gamma$.

- If $T_{i-1} = T_{i+1}$, we set $\gamma' = T_0, \ldots, T_i = T_{i+1}, \ldots, T_n$. Then by additivity of $\omega$,
  \[
  \omega(\gamma) = \omega(\gamma') + \omega(T_{i-1}, T_i) + \omega(T_i, T_{i+1}) = \omega(\gamma').
  \]

- If $T_{i-1} \neq T_{i+1}$, see Figure 9. Since $T$ is not a cut-tile of $\mathcal{C}_0(\gamma)$, there exists a path of tiles $T_{i-1}, T_i, \ldots, T_d, T_{i+1}$ in $\text{Wreath}_{\mathcal{C}_0(\gamma)}(T)$ joining $T_{i-1}$ and $T_{i+1}$. Then
  \[
  \gamma'' = T, T_{i-1}, T_i, \ldots, T_d, T_{i+1}, T
  \]

is a loop of tiles, and since $T$ has at least an edge in the boundary of $\mathcal{C}_0(\gamma)$, we see that $\gamma'' \in \mathcal{P}_0$. In particular, $\omega(\gamma'') = 0$. We set

\[
\gamma' = T_0, \ldots, T_{i-1}, T_i, \ldots, T_d, T_{i+1}, \ldots, T_n.
\]

Then by additivity of $\omega$,

\[
\omega(\gamma) = \omega(\gamma') + \omega(\gamma'') = \omega(\gamma').
\]

After proceeding as above for each occurrence of $T$ in $\gamma$, we end up with a loop of tiles $\gamma_0$ such that $\omega(\gamma) = \omega(\gamma_0)$ and $\text{Area}(\gamma_0) \leq k$ (since the support of $\gamma_0$ is included in $\mathcal{C}_0(\gamma) \setminus \{T\}$). We conclude using the induction hypothesis on $\gamma_0$.

\[\square\]

5.1.3 The position map $\omega_0$

Definition 5.6. First we define $\omega_0$ on the set of protopaths of length 2 in $\sigma^\infty(C)$. They form a finite set due to the heredity graph of edges. In Figure 10 we explicitly give this set and define this list and define $\omega_0$ on it. For each protopath $\gamma = (T_0, T_1)$ of Figure 10, $T_0$ is the white tile. Moreover, we set $\omega_0(T_1, T_0) = -\omega_0(T_0, T_1)$.

We are now ready to define the map $\omega_0 : \mathcal{P} \rightarrow \mathbb{Z}^3$. For every protopath $\gamma = T_1, T_2, \ldots, T_n$ of length $n \geq 2$, we set

\[
\omega_0(\gamma) = \sum_{i=1}^{n-1} \omega_0(T_i, T_{i+1}).
\]
Finally, to make sure that $\omega_0$ vanishes on protoloops of $\sigma_\infty(C)$, it remains to check that the map $\omega_0$ vanishes on the elements of $P_0$ (thanks to Lemma 5.5). This a finite process, since Lemma 5.4 ensures that $P_0$ is finite. We detail below an instance of the kind of easy computations that have to be carried on:

$$
\begin{align*}
\omega_0(\begin{pmatrix} A & B \\ C & C \end{pmatrix}) &= \omega_0(\begin{pmatrix} A & C \\ C & B \end{pmatrix}) + \omega_0(\begin{pmatrix} C & B \\ C & C \end{pmatrix}) + \omega_0(\begin{pmatrix} B & C \\ B & C \end{pmatrix}) \\
&= (-1, 0, 2) + (0, 2, -3) + (1, -2, 1) = (0, 0, 0).
\end{align*}
$$

Figure 10: Definition of the map $\omega_0$ over protopaths of length 2. The orientation of the path is indicated using colors: the first tile is white (see Definition 5.6).

According to Lemma 5.5, we thus obtain the following proposition.

**Proposition 5.7.** The map $\omega_0 : P \to \mathbb{Z}^3$ defined previously is additive, and vanishes on each protoloop of tiles.

**Remark 5.8.** Given $T, T'$ two tiles in $\sigma_\infty(C)$, we set

$$
\omega_0(T, T') = \omega_0(\gamma)
$$

where $\gamma$ is any path of tiles joining $T$ to $T'$. Indeed, if $\gamma, \gamma'$ are two such paths, Proposition 5.7 ensures that $\omega_0(\gamma) = \omega_0(\gamma')$ since $\omega_0$ vanishes on the loop of tiles $\gamma \gamma^{-1}$.

The following proposition will be used afterwards.

**Proposition 5.9.** Let $T, T'$ be tiles in $\sigma_\infty(C)$. Then

$$
\omega_0(b(T), b(T')) = M_r^{-1} \omega_0(T, T').
$$

(5.1)

**Proof.** Proposition 5.7 ensures that $\omega_0$ is additive. It is thus sufficient to prove (5.1) for adjacent tiles $T, T'$. Moreover, $\omega_0(T, T')$ only depends on the protopath $(T, T')$ defined by the adjacent tiles $T, T'$. There is only a finite number of protopaths of length 2 to consider: those which are
list on Figure 10. We detail below an instance of the kind of easy computations that have to be carried on. Suppose that \((T, T')\) is the following protopath:

\[
\begin{array}{c}
\begin{array}{c}
\text{By definition of } \omega_0 \text{ (Figure 10) we have } \omega_0(T, T') = (-1, 2, -1). \text{ We compute } \omega_0(b(T), b(T')) \text{ by inspecting the image of } (T, T') \text{ by } \sigma:\n\end{array}
\end{array}
\]

By choosing the path of length three above and by reading Figure 10, we compute
\[
\omega_0(b(T), b(T')) = (1, 1, 0) + (1, 0, -2) = (2, -1, -2),
\]
so the proposition holds in this case because \(\mathbf{M}_s^{-1}(-1, 2, -1) = (2, -1, -2). \square
\]

5.2 From the topological patches to the stepped surface

Let \((P, T)\) a pointed patch formed by a patch \(P\) of \(\sigma^\infty(C)\) and a tile \(T\) of \(\sigma^\infty(C)\). We are going to associate to \((P, T)\) a multiset of facets \(\varphi_0(P, T) \in \mathcal{M}\), see Section 4.1. When \(P\) is equal to the tile \(T\), we simply denote the pointed patch \((P, T)\) by \(T\).

Let \(\mathcal{T}_\sigma\) denote the set of tiles of \(\sigma^\infty(C)\), as in Section 3.1.2. First, we define a map \(\Phi : \mathcal{T}_\sigma \to \mathcal{M}\) so that two tiles of the same type have the same image. This map \(\Phi\) is defined by setting:
\[
\Phi(\text{A}) = \text{E}^3([0, 1]^*), \quad \Phi(\text{B}) = \text{E}^2([0, 1]^*), \quad \Phi(\text{C}) = \text{E}^3([0, 1]^*).
\]

Alternatively:
\[
\Phi(A) = \text{E}^3([0, 3]^*) + e_3 - e_1, \quad \Phi(B) = \text{E}^3([0, 2]^*) + e_2 - e_1, \quad \Phi(C) = \text{E}^3([0, 1]^*).
\]

In the pictures representing multisets, the symbol \(\bullet\) indicates the origin of \(\mathbb{R}^3\). For instance the image of \(A\) is the multiset \(m : \mathbb{Z}^3 \to \mathbb{Z}_{\geq 0}^3\) defined by
\[
m(y) = \begin{cases} 0 & \text{if } y \neq (1, 0, -1) \\ (1, 1, 1) & \text{if } y = (1, 0, -1). \end{cases}
\]

For a patch \(P\) and a tile \(T\), we consider \(T'\) another tile and \(\gamma\) a path of tiles from \(T\) to \(T'\). By Proposition 5.7, the vector \(\omega_0(\gamma)\) only depends on \(T\) and \(T'\), and not on the choice of the path \(\gamma\). Thus we denote it by \(\omega_0(T, T')\).

**Definition 5.10.** Let \(P\) be a patch of \(\sigma^\infty(C)\), and let \(T\) be a tile of \(\sigma^\infty(C)\). The multiset of facets \(\varphi_0(P, T) \in \mathcal{M}\) is defined by:
\[
\varphi_0(P, T) = \sum_{T' \in P} \left(\Phi(T') + \omega_0(T, T')\right).
\]

**Remark 5.11.** By definition, we notice that \(\varphi_0(T, T) = \Phi(T)\) for every tile \(T\) in \(\sigma^\infty(C)\).

The next two lemmas state useful properties of the map \(\varphi_0\). By definition of \(\varphi_0\) and by additivity of \(\omega_0\), we derive immediately the following lemma.
Lemma 5.12. Let $P$ be a patch of $\sigma^\infty(C)$, and let $T$, $T'$ be tiles of $\sigma^\infty(C)$. Then we have
\[ \varphi_0(P,T) = \varphi_0(P,T') + \omega_0(T,T'). \]

Let $P_1$, $P_2$ be patches in $\sigma^\infty(C)$. We denote by $P_1 \cap P_2$ the (possibly empty) patch in $\sigma^\infty(C)$ made of tiles belonging to both $P_1$ and $P_2$: this is the standard definition of “intersection of patches”.

Lemma 5.13. Let $P_1$, $P_2$ be patches of $\sigma^\infty(C)$, and let $T$ be a tile of $\sigma^\infty(C)$.

- If $P_1$ and $P_2$ have no tile in common, then $\varphi_0(P_1 \cup P_2, T) = \varphi_0(P_1, T) + \varphi_0(P_2, T)$.
- In the general situation we have $\varphi_0(P_1 \cup P_2, T) = \varphi_0(P_1, T) + \varphi_0(P_2, T) - \varphi_0(P_1 \cap P_2, T)$.

Proof. The second point is a direct consequence of the first one. By definition, and because $P_1$, $P_2$ have no tile in common, we have
\[
\varphi_0(P_1 \cup P_2, T) = \sum_{T' \in P_1 \cup P_2} \left( \Phi(T') + \omega_0(T,T') \right)
= \sum_{T' \in P_1} \left( \Phi(T') + \omega_0(T,T') \right) + \sum_{T' \in P_2} \left( \Phi(T') + \omega_0(T,T') \right)
= \varphi_0(P_1, T) + \varphi_0(P_2, T).
\]

\[\square\]

5.3 Commutation between $\sigma, E, \varphi_0$

Proposition 5.14. Let $P$ be a simply connected patch of $\sigma^\infty(C)$ and $T$ a tile. We have
\[ \varphi_0 \circ \hat{\sigma}(P,T) = E \circ \varphi_0(P,T). \tag{5.2} \]

In the previous formula, we formally consider that the map $E$ acts on multisets.

Proof. A direct verification shows that for every tile $T$ in $\sigma^\infty(C)$, we have
\[ \varphi_0 \circ \hat{\sigma}(T,T) = E \circ \varphi_0(T,T), \tag{5.3} \]
as detailed on the the following diagrams (where the base tile of a patch is the white tile).

We now establish the relation (5.2) when $P$ is a tile $T'$. Let $T, T'$ be tiles of $\sigma^\infty(C)$. Recall that, by definition of the pointed substitution $\hat{\sigma}$, we have $\hat{\sigma}(T', T) = (\sigma(T'), b(T))$. By Lemma 5.12 and Proposition 5.3, we have
\[
\varphi_0(\sigma(T'), b(T)) = \varphi_0(\sigma(T'), b(T')) + \omega_0(b(T), b(T'))
= \varphi_0(\hat{\sigma}(T', T')) + M^{-1}_x \omega_0(T,T').
\]

Using relation (5.3) and Equation (4.1) we get that:
\[
\varphi_0(\hat{\sigma}(T', T)) = E(\varphi_0(T', T')) + M^{-1}_x \omega_0(T,T')
= E(\varphi_0(T', T') + \omega_0(T,T')).
\]

22
We conclude by using relation (5.4) and the additivity of the map

\[ \varphi_0(\sigma(T'), b(T)) = E(\varphi_0(T', T)). \]  

We now prove the relation (5.2) in full generality. Let \( P \) be a patch in \( \sigma^\infty(C) \) and let \( T \) be a tile of \( \sigma^\infty(C) \). Since \( \sigma(P) = \bigcup_{T'' \in P} \sigma(T'') \), Lemma 5.13 ensures that

\[
\varphi_0(\bar{\sigma}(P, T)) = \varphi_0(\sigma(P), b(T)) = \sum_{T'' \in P} \varphi_0(\sigma(T''), b(T)) = \sum_{T'' \in P} \varphi_0(\bar{\sigma}(T'', T)).
\]

We conclude by using relation (5.4) and the additivity of the map \( E \):

\[
\varphi_0(\bar{\sigma}(P, T)) = \sum_{T'' \in P} E(\varphi_0(T'', T)) = E \left( \sum_{T'' \in P} \varphi_0(T'', T) \right) = E(\varphi_0(P, T)).
\]

\[ \square \]

5.4 From \( \sigma^\infty(C) \) to \( \Sigma_{\text{step}} \)

5.4.1 The map induced by \( \varphi_0 \)

Proposition 5.14 implies that for every integer \( n \),

\[ \varphi_0 \circ \bar{\sigma}^n(C, C) = E^n \circ \varphi_0(C, C) = E^n(\mathcal{U}). \]

In what follows, it is convenient to identify a multiset and its support as explained in Remark 4.1. Since \( E^n(\mathcal{U}) \) converges to the stepped surface \( \Sigma_{\text{step}} = \bigcup_{n \in \mathbb{N}} E^n(\mathcal{U}) \) (see Proposition 4.3) and since \( \sigma^n(C) \) converges to \( \sigma^\infty(C) \), the map \( \varphi_0 \) induces a map, still denoted by \( \varphi_0 \) such that:

- for every tile \( T \) of \( \sigma^\infty(C) \), \( \varphi_0(T, C) \) is a subset of \( \Sigma_{\text{step}} \),
- \( \Sigma_{\text{step}} = \bigcup_{T \text{ tile of } \sigma^\infty(C)} \varphi_0(T, C) \).

**Lemma 5.15.** For every tile \( T \) of \( \sigma^\infty(C) \), there exists a unique facet \( [x, i]^* \) in \( \Sigma_{\text{step}} \) such that

\[ \varphi_0(T, C) = E^3([x, i]^*). \]

**Proof.** First, we recall that Lemma 3 of [AI01] states that the map \( E \) is “injective”: precisely, if \( E([x, i]^*) \) and \( E([x', i']^*) \) have a facet in common, then \( x = x' \) and \( i = i' \). This provides the unicity of the facet \( [x, i]^* \) in the lemma (if it exists).

We know that \( \varphi_0(T, C) = \Phi(T) + \omega_0(T, C) \). By definition of \( \Phi \), there exists an integer \( i \) such that \( \varphi_0(T, C) = E^3([0, i]^*) + u_i + \omega_0(T, C) \)

\[
u_1 = e_3 - e_1 = M_s^{-3}(-2e_1 - e_2), u_2 = e_2 - e_1 = M_s^{-3}(-e_1), u_3 = 0.
\]

That is to say:

\[ \varphi_0(T, C) = E^3([x, i]^*) \quad \text{with} \quad x = M_s^3(\omega_0(T, C) + u_i). \]

We claim that \( [x, i]^* \) lies in \( \Sigma_{\text{step}} \). Indeed since \( \Sigma_{\text{step}} = E^3(\Sigma_{\text{step}}) \), we know that there exists facets of \( \Sigma_{\text{step}} \) which images by \( E^3 \) cover \( \varphi_0(T, C) \). By Lemma 3 of [AI01] again, we conclude that the facet \( [x, i]^* \) is lying in \( \Sigma_{\text{step}} \). \( \square \)
5.4.2 The map $\Psi$

We are now in position to define a bijection $\Psi$ from the set of tiles of $\sigma^\infty(C)$ to the set of facets of $\Sigma_{\text{step}}$:

$$\Psi(T) = [x, i]^*, \quad \text{where} \quad \varphi_0(T) = E^3([x, i]^*).$$

Moreover, since $\Phi(C) = E^3([0, 1]^*), \Phi(B) = E^3([0, 2]^*)$ and $\Phi(A) = E^3([0, 3]^*)$, we see that

- type($T$) = A $\iff$ type($\Psi(T)$) = 3,
- type($T$) = B $\iff$ type($\Psi(T)$) = 2,
- type($T$) = C $\iff$ type($\Psi(T)$) = 1.

We set $\theta(A) = 3, \theta(B) = 2, \theta(C) = 1$. We summarize the previous discussion in the following proposition:

**Theorem 5.16.** The map $\Psi$ defined, for every tile $T$ of $\sigma^\infty(C)$, by:

$$\Psi(T) = [M^3\omega_0(T, C) + u_{\text{type}(T)}, \theta(\text{type}(T))]^*$$

is a bijection from the set of tiles of $\sigma^\infty(C)$ to the set of facets of $\Sigma_{\text{step}}$.

5.5 Link between two tilings $T_{\text{top}}$ and $T_{\text{step}}$

5.5.1 Theorem 5.16 revisited

We recall that, according to Proposition 3.11, $\sigma^\infty(C)$ can be realized as the tiling $T_{\text{top}}$, and according to Proposition 4.3 the tiling $T_{\text{step}}$ is the “image” of the stepped surface $\Sigma_{\text{step}}$ by the projection $\pi_{\beta} : \Sigma_{\text{step}} \rightarrow P$. Hence, Theorem 5.16 explains exactly how the two tilings $T_{\text{top}}$ and $T_{\text{step}}$ are related. In particular:

- The map $\Psi$ sends tiles of the same type to tiles of the same type.
- In $T_{\text{top}}$, the way to locate a tile $T$ with respect to another tile $T'$, via the identification with $\sigma^\infty(C)$, is by using the position $\omega_0(T, T')$. In $T_{\text{step}}$, the corresponding tiles $\Psi(T)$ and $\Psi(T')$ then will differ from the vector $\pi_{\beta}(\omega_0(T, T'))$.

5.5.2 Via the Delone set $D = \cup_{i \in \{1, 2, 3\}} D_i$

We would like to explicit the link between $T_{\text{top}}$ and $T_{\text{step}}$ in the same spirit of what we explain in Remark 4.6 and Remark 4.4. For that, we first notice that, alternatively, $\sigma^\infty(C)$ can be geometrized as follow. According to Lemma 5.15, the set $\{\pi_{\beta}(\varphi_0(T, C)) \mid T \text{ tile of } \sigma^\infty(C)\}$ tiles the plane $P$, and the resulting tiling is a geometric realization of $\sigma^\infty(C)$. The definition of $\Psi$ in Section 5.2 gives us a base point in $\Phi(A), \Phi(B)$ and $\Phi(C)$, and consequently, gives rise to a base point $x_T$ in each $\varphi_0(T, C)$. The set $\{\pi_{\beta}(x_T) \mid T \text{ tile of } \sigma^\infty(C)\}$ is a Delone set in $P$. However, it is not equal to the set $D$ of Remark 4.4, indeed, it is equal to $M^3_{\text{top}} \mathcal{D}$, see the proof of Lemma 5.15.

This leads us to do what follows. For each tile $T$ of $\sigma^\infty(C)$, we consider

$$T_{\text{geo}} = \pi_{\beta}(M^3(\varphi_0(T, C))) \subseteq P.$$ 

The set $\{T_{\text{geo}} \mid T \text{ tile of } \sigma^\infty(C)\}$ tiles the plane $P$. This tiling is again a geometric realization of $\sigma^\infty(C)$, and we denote it by $T'_{\text{geo}}$. By construction, $\pi_{\beta}(M^3x_T)$ lies in $T_{\text{geo}}$, and the set $\{\pi_{\beta}(M^3x_T) \mid T \text{ tile of } \sigma^\infty(C)\}$ is precisely the Delone set $D$. Moreover, Theorem 5.16 ensures that the subset consisting of vectors $\pi_{\beta}(M^3x_T)$ with $\theta(\text{type}(T)) = i \ (i \in \{1, 2, 3\})$ is precisely the set $D_i$ of Remark 4.4.

To sum up: the tiling $T'_{\text{top}}$ is obtained by putting in $P$ a tile of type $i$ (i.e. a translated image of $T_{\text{geo}}$ with $\theta(\text{type}(T)) = i$) at each vector in $D_i$. This explicitly the strong relation between the two tilings $T_{\text{top}}$ and $T_{\text{step}}$, and thus also with $T_{\text{frac}}$ via Remark 4.6.
5.5.3 The map $\Psi$ as a “two-dimensional sliding block code”

It is worth to remark that, by definition of $\Psi$, and because the position map $\omega_0$ is additive, there is an elementary way to rebuild $T_{\text{step}}$ from $T_{\text{top}}$ just by looking at local configurations of tiles. We give in Figure 11 all the information needed for doing that.

![Figure 11: $\Psi$ and $\Psi^{-1}$ as a two-dimensional sliding block code.](image)

In practice, to construct $T_{\text{step}}$ from $T_{\text{top}}$, we can ignore the formula of Theorem 5.16 and simply use the recipe given in Figure 11. We choose a tile $T$ in $T_{\text{top}}$, and put in the plane a tile of $T_{\text{step}}$ of same type as $T$. Then we choose a tile $T'$ adjacent to $T$ and use Figure 11 to place correctly the corresponding tile $\Psi(T')$ relatively to $\Psi(T)$. And we go on, inductively rebuilding $T_{\text{step}}$.

This can also be done to define $\Psi^{-1}$, see Figure 11.

This point of view on $\Psi$ remind us of the so-called sliding block code in classical symbolic dynamics, see for instance [LM95]. In that spirit, $\psi$ could be called a two-dimensional sliding block code.

6 Concluding remarks

Towards more general results The results presented in this article are specifically about the tilings associated with the Tribonacci substitution and its associated Rauzy fractal tiling. We have been able to get such results for some other examples of Pisot substitutions, such as the one shown in Figure 12.

We describe how we derived the topological substitution $\tau$ from the dual substitution $E_1^*(t)$ associated with the symbolic substitution $t : 1 \mapsto 13, 2 \mapsto 1, 3 \mapsto 2$.

1. Start with a single facet $[0, 1]^*$ and compute $E_1^*(t)^k([0, 1]^*)$ with $k$ large enough, in such a way that the patch $E_1^*(t)^k([0, 1]^*)$ contains every possible neighboring couples of facets. This is shown in Figure 13 (left).
2. Compute some more iterates by $E_1^*(t)$ to “inflate” the tiles from single facets to patches of facets (metatiles). This is shown in Figure 13 (center), where each metatile has the same color as its single-facet preimage. We must iterate $E_1^*(t)$ sufficiently many times (3 times in this case), so that every intersection between two tiles is either empty or consists of edges (single points are not allowed).
Figure 12: Definition of the topological substitution \( \tau \) obtained from the dual substitution associated with \( 1 \mapsto 13, 2 \mapsto 1, 3 \mapsto 2 \) (top). Six iterations from the tile \( A \) are shown (bottom).

3. Iterate \( E_1^*(t) \) one more time to “read” how the metatiles should be substituted. This is where we extract the information to define the topological substitution \( \tau \) in two steps:

(a) We define the image of each tile by noticing that the image tiles are either one of the other metatiles, or a union of two metatiles:

\[
\begin{array}{ccc}
1 & 1 & 3 \\
2 & 2 & 3 \\
\end{array} \quad \begin{array}{ccc}
1 & 1 & 3 \\
2 & 2 & 3 \\
\end{array} \quad \begin{array}{ccc}
1 & 1 & 3 \\
2 & 2 & 3 \\
\end{array} \quad \begin{array}{ccc}
1 & 1 & 3 \\
2 & 2 & 3 \\
\end{array} \quad \begin{array}{ccc}
1 & 1 & 3 \\
2 & 2 & 3 \\
\end{array}
\]

(b) We define the boundaries’ images by comparing the common edges between two adjacent metatiles and the common edges between their images in Figure 13 (center and right).

Limits of this approach  Despite the fact that Rauzy fractals tilings have finite local complexity \{ABB+15\}, the method described above is not guaranteed to work in general for an arbitrary dual substitution. The main problem is that in many cases, the topology of the patterns produced by the dual substitution can be complicated (disconnected or not simply connected, for example). This can cause Step 2 above to fail.

These difficulties are linked with some questions about the dynamics of the underlying Pisot substitution. Indeed, it can be proved that the underlying Pisot substitution has pure discrete spectrum if and only if the patterns generated by its associated dual substitution contain arbitrarily large balls \{BR10\}. This property is difficult to check for dual substitutions, but easy to check for topological substitutions (see the core property in Section 3). See \{ABB+15\} for more information about the Pisot conjecture and its different formulations.
Another possible approach to the original question raised in the introduction would be, given an IFS with a topologically complicated attractor, to construct another IFS with a topologically simpler attractor which gives a similar tiling, and then apply the method described above. For example, the tilings associated with the Tribonacci substitution and the “flipped Tribonacci” substitution $1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 1$ are closely related: the tile positions are equal (but the neighbor relations change), even though the topology of the flipped Tribonacci fractal is complicated. However we do not know if this feasible in general, even in the case of Rauzy fractals.

References

[ABB+15] S. Akiyama, M. Barge, V. Berthé, J.-Y. Lee, and A. Siegel. On the Pisot substitution conjecture. In Mathematics of aperiodic order, volume 309 of Progr. Math., pages 33–72. Birkhäuser/Springer, Basel, 2015.

[ABI02] P. Arnoux, V. Berthé, and Sh. Ito. Discrete planes, $\mathbb{Z}^2$-actions, Jacobi-Perron algorithm and substitutions. Ann. Inst. Fourier, 52(2):305–349, 2002.

[ABS04] P. Arnoux, V. Berthé, and A. Siegel. Two-dimensional iterated morphisms and discrete planes. Theoret. Comput. Sci., 319(1-3):145–176, 2004.

[AI01] P. Arnoux and S. Ito. Pisot substitutions and Rauzy fractals. Bull. Belg. Math. Soc. Simon Stevin, 8(2):181–207, 2001.

[BBJS15] V. Berthé, J. Bourdon, T. Jolivet, and A. Siegel. A combinatorial approach to products of pisot substitutions. Ergodic Theory and Dynamical Systems, to appear:1–38, 2015.

[BG13] M. Baake and U. Grimm. Aperiodic order. Vol. 1, volume 149 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2013. A mathematical invitation, With a foreword by Roger Penrose.

[BH13] N. Bédaride and A. Hilion. Geometric realizations of topological substitution. The Quarterly Journal of Mathematics, 64(4):955–979, 2013.

[BR10] V. Berthé and M. Rigo, editors. Combinatorics, automata and number theory, volume 135 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2010.
[BS97] Ph. L. Bowers and K. Stephenson. A “regular” pentagonal tiling of the plane. *Conform. Geom. Dyn.*, 1:58–68 (electronic), 1997.

[CFP01] J. W. Cannon, W. J. Floyd, and W. R. Parry. Finite subdivision rules. *Conform. Geom. Dyn.*, 5:153–196 (electronic), 2001.

[CFP07] J. W. Cannon, W. J. Floyd, and W. R. Parry. Constructing subdivision rules from rational maps. *Conform. Geom. Dyn.*, 11:128–136 (electronic), 2007.

[CS98] J. W. Cannon and E. L. Swenson. Recognizing constant curvature discrete groups in dimension 3. *Trans. Amer. Math. Soc.*, 350(2):809–849, 1998.

[CS01] V. Canterini and A. Siegel. Geometric representation of substitutions of Pisot type. *Trans. Amer. Math. Soc.*, 353(12):5121–5144, 2001.

[Fer07] Th. Fernique. Local rule substitutions and stepped surfaces. *Theoret. Comput. Sci.*, 380(3):317–329, 2007.

[FO10] Th. Fernique and N. Ollinger. Combinatorial substitutions and sofic tilings. In *Journées Automates Cellulaires*, TUCS Lecture Notes, pages 100–110, 2010.

[Fra03] N. Priebe Frank. Detecting combinatorial hierarchy in tilings using derived Voronoi tessellations. *Discrete Comput. Geom.*, 29(3):459–476, 2003.

[Fra08] N. Priebe Frank. A primer of substitution tilings of the Euclidean plane. *Expo. Math.*, 26(4):295–326, 2008.

[GS87] Br. Grünbaum and G. C. Shephard. *Tilings and patterns*. W. H. Freeman and Company, New York, 1987.

[Hat02] Al. Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[Hir94] M. W. Hirsch. *Differential topology*, volume 33 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1976 original.

[IO93] Sh. Ito and Ma. Ohtsuki. Modified Jacobi-Perron algorithm and generating Markov partitions for special hyperbolic toral automorphisms. *Tokyo J. Math.*, 16(2):441–472, 1993.

[JK14] T. Jolivet and J. Kari. Undecidable properties of self-affine sets and multi-tape automata. In *Mathematical foundations of computer science 2014. Part I*, volume 8634 of *Lecture Notes in Comput. Sci.*, pages 352–364. Springer, Heidelberg, 2014.

[LM95] D. Lind and Br. Marcus. *An introduction to symbolic dynamics and coding*. Cambridge University Press, Cambridge, 1995.

[LW03] J. C. Lagarias and Y. Wang. Substitution Delone sets. *Discrete Comput. Geom.*, 29(2):175–209, 2003.

[Pey86] J. Peyrière. Frequency of patterns in certain graphs and in Penrose tilings. *J. Physique, 47*(7, Suppl. Colloq. C3):C3–41–C3–62, 1986. International workshop on aperiodic crystals (Les Houches, 1986).

[PF02] N. Pytheas Fogg. *Substitutions in dynamics, arithmetics and combinatorics*, volume 1794 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002. Edited by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel.
[Rau82] G. Rauzy. Nombres algébriques et substitutions. *Bull. Soc. Math. France*, 110(2):147–178, 1982.

[Rob04] E. Ar. Robinson, Jr. Symbolic dynamics and tilings of $\mathbb{R}^d$. In *Symbolic dynamics and its applications*, volume 60 of *Proc. Sympos. Appl. Math.*, pages 81–119. Amer. Math. Soc., Providence, RI, 2004.

[RS13] M. Ramirez-Solano. Construction of the discrete hull for the combinatorics of a regular pentagonal tiling of the plane. arXiv:1303.5375, 2013.

[Sol97] B. Solomyak. Dynamics of self-similar tilings. *Ergodic Theory Dynam. Systems*, 17(3):695–738, 1997.

[ST09] A. Siegel and J. M. Thuswaldner. Topological properties of Rauzy fractals. *Mém. Soc. Math. Fr. (N.S.*), (118):140, 2009.

[Thu89] W. Thurston. Groups, tilings, and finite state automata. AMS Colloquium lecture notes, 1989. Unpublished manuscript.