Nodal solutions for the fractional Yamabe problem on Heisenberg groups

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Dedicated to Professor Patrizia Pucci on the occasion of her 65th birthday.

We prove that the fractional Yamabe equation $L_\gamma u = |u|^{((4\gamma)/(Q-2\gamma))}u$ on the Heisenberg group $\mathbb{H}^n$ has $[n+1/2]$ sequences of nodal (sign-changing) weak solutions whose elements have mutually different nodal properties, where $L_\gamma$ denotes the CR fractional sub-Laplacian operator on $\mathbb{H}^n$, $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$, and $\gamma \in \bigcup_{k=1}^n [k, ((kQ)/(Q-1))].$ Our argument is variational, based on a Ding-type conformal pulling-back transformation of the original problem into a problem on the CR sphere $S^{2n+1}$ combined with a suitable Hebey-Vaugon-type compactness result and group-theoretical constructions for special subgroups of the unitary group $U(n+1)$.

Keywords: CR fractional sub-Laplacian; nodal solution; Heisenberg group

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1. Introduction

After the seminal paper of Caffarelli and Silvestre [8], considerable efforts have been made concerning the study of elliptic problems involving the fractional Laplace operator both in Euclidean and non-Euclidean settings. As expected, the Euclidean framework is much more developed; although many results concerning the pure Laplace operator can be transposed to the fractional setting in $\mathbb{R}^n$, there are also subtle differences which require a deep understanding of certain nonlinear phenomena, see for example, Cabré and Sire [6], Caffarelli [7], Caffarelli, Salsa and Silvestre [9], Di Nezza, Palatucci and Valdinoci [10], and references therein.

By exploring analytical and spectral theoretical arguments, important contributions have been obtained recently within the CR setting concerning the fractional Laplace operator with various applications in sub-elliptic PDEs, see Branson, Fontana and Morpurgo [5], Frank and Lieb [14], and Frank, del Mar González, Monticelli and Tan [15]. In particular, in the latter papers, sharp Sobolev and
Moser-Trudinger inequalities are established on the Heisenberg group $\mathbb{H}^n$, $n \geq 1$, the simplest non-trivial CR structure.

In the present paper, we shall consider the fractional Yamabe problem on the Heisenberg group $\mathbb{H}^n$, namely,

$$\begin{align*}
\{ L_\gamma u &= |u|^{(4\gamma)/(Q-2\gamma)} u \quad \text{on } \mathbb{H}^n, \\
u &\in D^{\gamma}(\mathbb{H}^n). \tag{FYH}_\gamma
\end{align*}$$

Hereafter, $Q := Q_n = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$, $\gamma > 0$ is a parameter specified later, $L_\gamma$ denotes the CR fractional sub-Laplacian operator on $\mathbb{H}^n$, and the functional space $D^{\gamma}(\mathbb{H}^n)$ contains real-valued functions from $L^{((2Q)/(Q-2\gamma))}(\mathbb{H}^n)$ whose energy associated with the CR fractional sub-Laplacian operator $L_\gamma$ is finite; see §2.4 for details.

Due to the recent paper of Frank, del Mar González, Monticelli and Tan [15], we know the existence of positive solutions of $(FYH)_\gamma$ for $\gamma \in (0, Q/2)$, having the form

$$u(z, t) = c_0 ((1 + |z|^2)^2 + t^2)^{(2\gamma-Q)/(4)}, \quad (z, t) \in \mathbb{H}^n, \tag{1.1}$$

for some $c_0 > 0$, allowing any left translation and dilation. In the special case $\gamma = 1$, when $L_1 = L$ is the usual sub-Laplacian operator on $\mathbb{H}^n$, the existence and uniqueness (up to left translation and dilation) of positive solutions of the form (1.1) for problem $(FYH)_1$ have been established by Jerison and Lee [18, 19]; see also Garofalo and Vassilev [16] for generic Heisenberg-type groups (e.g. Iwasawa groups).

Our main result guarantees sign-changing solutions for the fractional Yamabe problem $(FYH)_\gamma$ as follows:

**Theorem 1.1.** Let $\gamma \in \bigcup_{k=1}^{n} [k, ((kQ)/(Q-1))]$, where $Q = 2n + 2$. Then problem $(FYH)_\gamma$ admits at least $\lfloor n + 1/2 \rfloor$ sequences of sign-changing weak solutions whose elements have mutually different nodal properties. (Hereafter, $\lfloor r \rfloor$ denotes the integer part of $r \geq 0$.)

Before commenting on theorem 1.1, we recall that similar results are well known in the Euclidean setting; indeed, Bartsch, Schneider and Weth [4] proved the existence of infinitely many sign-changing weak solutions for the polyharmonic problem

$$\begin{align*}
\{ (-\Delta)^m u &= |u|^{((4m)/(N-2m))} u \quad \text{in } \mathbb{R}^N, \\
u &\in D^{m,2}(\mathbb{R}^N). \tag{P}_m
\end{align*}$$

where $N > 2m$, $m \in \mathbb{N}$, and $D^{m,2}(\mathbb{R}^N)$ denotes the usual higher order Sobolev space over $\mathbb{R}^N$. In fact, their proof is based on Ding’s original idea, see [11], who considered the case $m = 1$, by pulling back the variational problem $(P)_m$ to the standard sphere $S^N$ by stereographic projection. In this manner, by exploring certain properties of suitable subgroups of the orthogonal group $O(N+1)$, the authors are able to obtain compactness by exploring a suitable Sobolev embedding result of Hebey and Vaugon [17] which is indispensable in the application of the symmetric mountain pass theorem.

We notice that sign-changing solutions are already guaranteed to the usual CR-Yamabe problem $(FYH)_1$ by Maalaoui and Martino [20], and Maalaoui, Martino
and Tralli [21] by exploring Ding’s approach; their results are direct consequences of theorem 1.1 for $\gamma = 1$.

Coming back to theorem 1.1, we shall mimic Ding’s original idea as well, emphasizing that our CR fractional setting requires a more delicate analysis than either the polyharmonic setting in the Euclidean case (see [4]) or the usual CR framework, that is, when $\gamma = 1$ (see [20,21]). In the sequel, we sketch our strategy. As expected, we first consider the fractional Yamabe problem on the CR sphere $S^{2n+1}$, that is,

\[
\begin{cases}
\mathcal{A}_\gamma U = |U|^{((4\gamma)/(Q-2\gamma))}U & \text{on } S^{2n+1}, \\
U \in H^\gamma(S^{2n+1}),
\end{cases} \tag{FYS}_\gamma
\]

where the intertwining operator $\mathcal{A}_\gamma$ and Sobolev space $H^\gamma(S^{2n+1})$ are introduced in §2.4. By using the Cayley transform between the Heisenberg group $\mathbb{H}^n$ and the CR sphere $S^{2n+1}$, we prove that there is an explicit correspondence between the weak solutions of $(\text{FYH})_\gamma$ and $(\text{FYS})_\gamma$, respectively, see proposition 3.1 (and remark 3.2 for an alternative proof). Being in the critical case, the energy functional associated with problem $(\text{FYS})_\gamma$ does not satisfy the usual Palais-Smale condition due to the lack of compactness of the embedding $H^\gamma(S^{2n+1}) \hookrightarrow L^{2\gamma/(Q-\gamma)}(S^{2n+1})$. In order to regain some compactness, we establish a CR fractional version of the Ding-Hebey-Vaugon compactness result on the CR sphere $S^{2n+1}$, see proposition 3.3. In fact, subgroups of the unitary group $U(n+1)$ having the form $G = U(n_1) \times \cdots \times U(n_k)$ with $n_1 + \cdots + n_k = n + 1$ will imply the compactness of the embedding of $G$-invariant functions of $H^\gamma(S^{2n+1})$ into $L^{(2Q)/(Q-2\gamma)}(S^{2n+1})$. Here, we shall explore the compactness result of Maalaoui and Martino [20] combined with an iterative argument of Aubin [1] and the technical assumption $\gamma \in \bigcup_{k=1}^{n} [k, ((kQ)/(Q-1))];$ some comments on the necessity of the latter assumption are formulated in remark 3.4. Now, having such a compactness, the fountain theorem and the principle of symmetric criticality applied to the energy functional associated with $(\text{FYS})_\gamma$ will guarantee the existence of a whole sequence of $G$-invariant weak solutions for $(\text{FYS})_\gamma$, so for $(\text{FYH})_\gamma$. The number of $[n+1/2]$ sequences of sign-changing weak solutions for $(\text{FYH})_\gamma$ with mutually different nodal properties will follow by careful choices of the subgroups $G = U(n_1) \times \cdots \times U(n_k)$ of the unitary group $U(n+1)$ with $n_1 + \cdots + n_k = n + 1$, see proposition 3.6.

Plan of the paper. In §2, we recall those notions and results that are indispensable to present our argument (e.g. basic facts about Heisenberg groups, the Cayley transform, spherical/zonal harmonics on $S^{2n+1}$, fractional Sobolev spaces on $S^{2n+1}$ and $\mathbb{H}^n$). Section 3 is devoted to the proof of theorem 1.1; in §3.1, we prove the equivalence between the weak solutions of problems $(\text{FYS})_\gamma$ and $(\text{FYH})_\gamma$; in §3.2, we establish the compactness result on the CR fractional setting for $S^{2n+1}$; in §3.3, we treat the group-theoretical aspects of our problem concerning the choice of the subgroups $G = U(n_1) \times \cdots \times U(n_k)$ of the unitary group $U(n+1)$ which is needed to produce $[n+1/2]$ sequences of sign-changing weak solutions for $(\text{FYH})_\gamma$ with different nodal properties. Finally, in §3.4, we assemble the aforementioned pieces in order to conclude the proof of Theorem 1.1.
2. Preliminaries

In order the paper to be self-contained, we recall in this section, some basic notions from [5, 13–15, 23] which are indispensable in the sequel.

2.1. Heisenberg groups

An element in the Heisenberg group $H^n$ is denoted by $(x, y, t)$, where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, $t \in \mathbb{R}$, and we identify the pair $(x, y)$ with $z \in \mathbb{C}^n$ having coordinates $z_j = x_j + iy_j$ for all $j = 1, \ldots, n$. The correspondence with its Lie algebra via the exponential coordinates induces the group law

$$(z, t) \star (z', t') = (z + z', t + t' + 2 \text{Im} \ z \cdot \overline{z'}), \quad \forall (z, t), (z', t') \in \mathbb{C}^n \times \mathbb{R},$$

where $\text{Im}$ denotes the imaginary part of a complex number and $z \cdot \overline{z'} = \sum_{j=1}^n z_j \overline{z_j'}$ is the Hermitian inner product. In these coordinates, the neutral element of $H^n$ is $0_{H^n} = (0, \ldots, 0)$ and the inverse $(z, t)^{-1}$ of the element $(z, t)$ is $(-z, -t)$. Note that $(x, y, t) = (z, t)$ forms a real coordinate system for $H^n$ and the system of vector fields given as differential operators

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad j \in \{1, \ldots, n\}, \quad T = \frac{\partial}{\partial t},$$

forms a basis of the left-invariant vector fields on $H^n$. The vectors $X_j, Y_j, j \in \{1, \ldots, n\}$ form the basis of the horizontal bundle. Let

$$N(z, t) = (|z|^4 + t^2)^{1/4}$$

be the homogeneous gauge norm on $H^n$ and $d_{KC} : H^n \times H^n \to \mathbb{R}$ be the Korányi-Cygan metric given by

$$d_{KC}((z, t), (z', t')) = N((z', t')^{-1} \star (z, t)) = (|z - z'|^4 + (t - t' - 2 \text{Im} \ z \cdot \overline{z'})^2)^{1/4}.$$

The Lebesgue measure of $\mathbb{R}^{2n+1}$ will be the Haar measure on $H^n$ (uniquely defined up to a positive multiplicative constant).

2.2. Cayley transform

Let

$$S^{2n+1} = \{\zeta = (\zeta_1, \ldots, \zeta_{n+1}) \in \mathbb{C}^{n+1} : \langle \zeta, \overline{\zeta} \rangle = 1\}$$

be the unit sphere in $\mathbb{C}^{n+1}$, where $\langle \zeta, \overline{\eta} \rangle = \sum_{j=1}^{n+1} \zeta_j \overline{\eta_j}$. The distance function on $S^{2n+1}$ is given by

$$d_S(\zeta, \eta) = \sqrt{2|1 - \langle \zeta, \overline{\eta} \rangle|}, \quad \zeta, \eta \in S^{2n+1}.$$

The Cayley transform $C : H^n \to S^{2n+1} \setminus \{(0, \ldots, 0, -1)\}$ is defined by

$$C(z, t) = \left(\frac{2z}{1 + |z|^2 + it}, 1 - |z|^2 - it \over 1 + |z|^2 + it, 1 + |z|^2 + it\right),$$

where $\langle \zeta, \overline{\eta} \rangle = \sum_{j=1}^{n+1} \zeta_j \overline{\eta_j}$. The distance function on $S^{2n+1}$ is given by

$$d_S(\zeta, \eta) = \sqrt{2|1 - \langle \zeta, \overline{\eta} \rangle|}, \quad \zeta, \eta \in S^{2n+1}.$$
whose Jacobian determinant is given by

\[ \text{Jac}_C(z, t) = \frac{2^{2n+1}}{((1 + |z|^2)^2 + t^2)^{n+1}}, \quad (z, t) \in \mathbb{H}^n. \]

Accordingly, for any integrable function \( f : S^{2n+1} \to \mathbb{R} \), we have

\[ \int_{S^{2n+1}} f(\eta) d\eta = \int_{\mathbb{H}^n} f(C(z, t)) \text{Jac}_C(z, t) dz dt. \quad (2.1) \]

If \( w = (z, t) \), \( v = (z', t') \) and \( \zeta = C(w), \eta = C(v) \), one has that

\[ d_S(\zeta, \eta) = d_{KC}(w, v) \left( \frac{4}{((1 + |z'|^2)^2 + (t')^2)} \right)^{1/4} \left( \frac{4}{((1 + |z|^2)^2 + t^2)} \right)^{1/4}. \quad (2.2) \]

### 2.3. Spherical/zonal harmonics on \( S^{2n+1} \)

The Hilbert space \( L^2(S^{2n+1}) \), endowed by the inner product

\[ (U, V) = \int_{S^{2n+1}} U \nabla d\eta, \]

can be decomposed into \( U(n + 1) \)-irreducible components as

\[ L^2(S^{2n+1}) = \bigoplus_{j, k \geq 0} \mathcal{H}_{j,k}, \]

where \( \mathcal{H}_{j,k} \) denotes the space of harmonic polynomials \( p(z, \overline{z}) \) on \( \mathbb{C}^{n+1} \) restricted to \( S^{2n+1} \) that are homogeneous of degree \( j \) and \( k \) in the variables \( z \) and \( \overline{z} \), respectively.

We notice that the dimension of \( \mathcal{H}_{j,k} \) is

\[ m_{j,k} = \frac{(j + n - 1)!(k + n - 1)!(j + k + n)}{n!(n - 1)!j!k!}. \]

Moreover, if \( \{Y_{j,k}^l\}_{l=1}^{m_{j,k}} \) is an orthonormal basis of \( \mathcal{H}_{j,k} \), then the zonal harmonics are defined by

\[ \Phi_{j,k}(\zeta, \eta) = \sum_{l=1}^{m_{j,k}} Y_{j,k}^l(\zeta) \overline{Y_{j,k}^l(\eta)}, \quad \zeta, \eta \in S^{2n+1}. \quad (2.3) \]

We recall that \( \Phi_{j,k} \) can be represented as

\[ \Phi_{j,k}(\zeta, \eta) = \frac{(\max\{j, k\} + n - 1)!(j + k + n)}{\omega_{2n+1} n!(\max\{j, k\})!} \langle \zeta, \overline{\eta} \rangle^{j-k} P_{k}^{(n, j-k)}(2\langle \zeta, \overline{\eta} \rangle^2 - 1), \quad (2.4) \]

where \( P_k^{(n, l)} \) denotes the Jacobi polynomials and \( \omega_{2n+1} \) is the usual measure of \( S^{2n+1} \).
2.4. Fractional Sobolev spaces on $S^{2n+1}$ and $\mathbb{H}^n$

The usual sub-Laplacian on $\mathbb{H}^n$ is defined as

$$L = -\frac{1}{4} \sum_{j=1}^{n} (X_j^2 + Y_j^2).$$

If we introduce the differential operators

$$T_j = \frac{\partial}{\partial \eta_j} - \eta_j \sum_{k=1}^{n+1} \eta_k \frac{\partial}{\partial \eta_k}, \quad T_j = \frac{\partial}{\partial \eta_j} - \eta_j \sum_{k=1}^{n+1} \eta_k \frac{\partial}{\partial \eta_k}, \quad j \in \{1, \ldots, n+1\},$$

the conformal sub-Laplacian on $S^{2n+1}$ is given by

$$D = -\frac{1}{2} \sum_{j=1}^{n+1} (T_j T_j + T_j T_j) + \frac{n^2}{4}.$$

Note that for every $Y_{j,k} \in H_{j,k}$, one has

$$DY_{j,k} = \lambda_j \lambda_k Y_{j,k},$$

where $\lambda_j = j + n/2$.

Let $0 < \gamma < Q/2 = n + 1$ be fixed. Given $U \in L^2(S^{2n+1})$, its Fourier representation is

$$U = \sum_{j,k \geq 0} \sum_{l=1}^{m_{j,k}} c_{j,k}^l(U) Y_{j,k}^l$$

with Fourier coefficients $c_{j,k}^l(U) = \int_{S^{2n+1}} U Y_{j,k}^l d\eta$. Accordingly, we may define

$$D^{\gamma/2} U = \sum_{j,k \geq 0} \sum_{l=1}^{m_{j,k}} (\lambda_j \lambda_k)^{\gamma/2} c_{j,k}^l(U) Y_{j,k}^l.$$

The fractional Sobolev space over $S^{2n+1}$ is defined as

$$H^\gamma(S^{2n+1}) = W^{\gamma,2}(S^{2n+1}) = \left\{ U \in L^2(S^{2n+1}) : D^{\gamma/2} U \in L^2(S^{2n+1}) \right\},$$

deed with the inner product and norm

$$(U, V)_{H^\gamma} = \int_{S^{2n+1}} D^{\gamma/2} U \overline{D^{\gamma/2} V} d\eta \quad \text{and} \quad \|U\|_{H^\gamma} = (U, U)_{H^\gamma}^{1/2}$$

$$= \left( \sum_{j,k \geq 0} \sum_{l=1}^{m_{j,k}} (\lambda_j \lambda_k)^{\gamma} |c_{j,k}^l(U)|^2 \right)^{1/2}.$$

The norm $\| \cdot \|_{H^\gamma}$ is equivalent to the norm coming from the inner product

$$(U, V)_\gamma = \sum_{j,k \geq 0} \sum_{l=1}^{m_{j,k}} \lambda_j(\gamma) \lambda_k(\gamma) c_{j,k}^l(U) \overline{c_{j,k}^l(V)},$$
where 
\[ \lambda_j(\gamma) = \frac{\Gamma(((Q+2\gamma)/(4)) + j)}{\Gamma(((Q-2\gamma)/(4)) + j)}, \quad j \in \mathbb{N}_0 = \{0,1,2,\ldots\}; \]

indeed, by asymptotic approximation of the Gamma function \(\Gamma\), one has \(\lambda_j(\gamma) \sim j^\gamma\).

The intertwining operator \(\mathcal{A}_\gamma\) of order \(2\gamma\) on \(S^{2n+1}\) is given by
\[ \text{Jac}_\gamma((Q+2\gamma)/(2Q)) \circ (\mathcal{A}_\gamma U) \circ \tau = \mathcal{A}_\gamma((\text{Jac}_\gamma((Q-2\gamma)/(2Q))U \circ \tau)) \quad \text{for all } \tau \in \text{Aut}(S^{2n+1}), \]
\[ U \in C^\infty(S^{2n+1}), \]
where \(\text{Aut}(S^{2n+1})\) and \(\text{Jac}_\gamma\) denote the group of automorphisms on \(S^{2n+1}\) and the Jacobian of \(\tau \in \text{Aut}(S^{2n+1})\), respectively. In fact, the latter definition can be extended to every \(U \in H^\gamma(S^{2n+1})\). Note that \(\mathcal{A}_\gamma\) may be characterized (up to a constant) by its action on \(\mathcal{H}_{j,k}\) as
\[ \mathcal{A}_\gamma Y_{j,k} = \lambda_j(\gamma) \lambda_k(\gamma) Y_{j,k}, \quad Y_{j,k} \in \mathcal{H}_{j,k}. \quad (2.5) \]

Therefore,
\[ (U,V)_\gamma = \int_{S^{2n+1}} \nabla \mathcal{A}_\gamma U d\eta. \quad (2.6) \]

In particular, \(\lambda_j(1) = \lambda_j\) for every \(j \in \mathbb{N}_0\) and \(\mathcal{A}_1 = \mathcal{D}\). Moreover, according to Frank and Lieb [14], for every real-valued function \(U \in H^\gamma(S^{2n+1})\), one has the sharp fractional Sobolev inequality on the CR sphere \(S^{2n+1}\), that is,
\[ \left( \int_{S^{2n+1}} |U(\eta)|^{((2Q)/(Q-2\gamma))} d\eta \right)^{((Q-2\gamma)/(Q))} \leq C(\gamma, n) \int_{S^{2n+1}} U(\eta) \mathcal{A}_\gamma U(\eta) d\eta, \quad (2.7) \]
where
\[ C(\gamma, n) = \frac{\Gamma((n+1-\gamma)/(2))}{\Gamma((n+1+\gamma)/(2))} \omega_{2n+1}^{-\gamma/n+1}. \]

The CR fractional sub-Laplacian operator on \(\mathbb{H}^n\) is defined by
\[ \mathcal{L}_\gamma = |2T|^\gamma \frac{\Gamma(\mathcal{L}|2T|^{-1} + ((1+\gamma)/(2)))}{\Gamma(\mathcal{L}|2T|^{-1} + 1 - \gamma/(2))}. \]

Direct computation shows that \(\mathcal{L}_1 = \mathcal{L}, \mathcal{L}_2 = \mathcal{L}^2 - |T|^2\). Moreover, the relationship between \(\mathcal{L}_\gamma\) and \(\mathcal{A}_\gamma\) is given by
\[ \mathcal{L}_\gamma((2\text{Jac}_\gamma((Q-2\gamma)/(2Q))(U \circ C)) = (2\text{Jac}_\gamma((Q+2\gamma)/(2Q))\mathcal{A}_\gamma U) \circ C, \]
\[ \forall U \in H^\gamma(S^{2n+1}). \quad (2.8) \]

The fractional Sobolev space over \(\mathbb{H}^n\) is defined by
\[ D^\gamma(\mathbb{H}^n) = \left\{ u \in L^{((2Q)/(Q-2\gamma))}(\mathbb{H}^n) : a_\gamma[u] < +\infty \right\}, \]
where the quadratic form $a_\gamma$ is associated with the operator $\mathcal{L}_\gamma$, that is,

$$a_\gamma[u] = \int_{\mathbb{H}^n} \overline{u} \mathcal{L}_\gamma u \, dz \, dt.$$ 

The form $a_\gamma$ can be equivalently represented by means of spectral decomposition, see [15, p. 126].

3. Proof of Main Theorem

3.1. Equivalent critical problems on $\mathbb{H}^n$ and $S^{2n+1}$.

Let $\gamma \in (0, n+1)$ be fixed. We consider the fractional Yamabe problem on the CR sphere, that is,

$$\begin{cases}
    \mathcal{A}_\gamma U = |U|^{((4\gamma)/(Q-2\gamma))}U & \text{on } S^{2n+1}, \\
    U \in H^\gamma(S^{2n+1}).
\end{cases} \quad (\text{FYS})_\gamma$$

Hereafter, we are considering real-valued functions both in $H^\gamma(S^{2n+1})$ and $D^\gamma(\mathbb{H}^n)$, respectively. The main result of this subsection constitutes the bridge between $(\text{FYS})_\gamma$ and $(\text{FYH})_\gamma$, as follows:

**Proposition 3.1.** Let $0 < \gamma < Q/2 = n + 1$. Then $U \in H^\gamma(S^{2n+1})$ is a weak solution of $(\text{FYS})_\gamma$ if and only if $u = (2\text{Jac}_C)^{(Q-2\gamma)/(2Q)}U \circ C \in D^\gamma(\mathbb{H}^n)$ is a weak solution of $(\text{FYH})_\gamma$.

**Proof.** We first prove the following

**Claim:** Let $U : S^{2n+1} \to \mathbb{R}$ and $u : \mathbb{H}^n \to \mathbb{R}$ be two functions such that $u = \text{Jac}_C^{((Q-2\gamma)/(2Q))}U \circ C$. Then $U \in H^\gamma(S^{2n+1})$ if and only if $u \in D^\gamma(\mathbb{H}^n)$.

Fix $U \in H^\gamma(S^{2n+1})$; we shall prove first that $(z, t) \mapsto u(z, t) = \text{Jac}_C(z, t)^{(Q-2\gamma)/(2Q)}U(C(z, t))$ belongs to $D^\gamma(\mathbb{H}^n)$. By (2.1) one has

$$\int_{\mathbb{H}^n} |u(z, t)|^{((2Q)/(Q-2\gamma))} \, dz \, dt = \int_{\mathbb{H}^n} \text{Jac}_C(z, t)|U(C(z, t))|^{((2Q)/(Q-2\gamma))} \, dz \, dt$$

$$= \int_{S^{2n+1}} |U(\eta)|^{((2Q)/(Q-2\gamma))} \, d\eta. \quad (3.1)$$

Furthermore, by the fractional Sobolev inequality (2.7) and relation (2.5), one has that

$$\left( \int_{S^{2n+1}} |U(\eta)|^{((2Q)/(Q-2\gamma))} \, d\eta \right)^{(Q-2\gamma)/(Q)} \leq C(\gamma, n) \int_{S^{2n+1}} U(\eta)^2 \, d\eta$$

$$= C(\gamma, n) \sum_{j,k \geq 0} \sum_{l=1}^{m_{j,k}} \lambda_j(\gamma) \lambda_k(\gamma) |c_{j,k}^l(U)|^2$$

$$\leq C'(\gamma, n) \|U\|^2_{H^\gamma} < +\infty,$$
where \( C'(\gamma, n) = C_{\gamma} C(\gamma, n) \) and \( C_{\gamma} > 0 \) is such that \( (V, V)_{\gamma} \leq C_{\gamma} \| V \|^2_{H^\gamma} \), for every \( V \in H^\gamma(S^{2n+1}) \); thus \( u \in L^2((2Q)/(Q-2\gamma))(\mathbb{H}^n) \). Moreover, by (2.8) and (2.1) one has

\[
a_{\gamma}[u] = \int_{\mathbb{H}^n} u L_{\gamma} udzdt =
\]

\[
= 2^\alpha' \int_{\mathbb{H}^n} J\text{ac}_C(z, t)((Q-2\gamma)/(2Q))U(C(z, t))\]

\[
L_{\gamma}((2J\text{ac}_C(z, t))((Q-2\gamma)/(2Q))U(C(z, t)))dzdt
\]

\[
= 2^\alpha' \int_{\mathbb{H}^n} J\text{ac}_C(z, t)((Q-2\gamma)/(2Q))U(C(z, t))\]

\[
(2J\text{ac}_C(z, t))((Q+2\gamma)/(2Q))(A_{\gamma} U)(C(z, t))dzdt
\]

\[
= 2^\alpha'' \int_{\mathbb{H}^n} U(C(z, t))(A_{\gamma} U)(C(z, t))J\text{ac}_C(z, t)dzdt
\]

\[
= 2^\alpha'' \int_{S^{2n+1}} U(\eta)A_{\gamma} U(\eta)d\eta
\]

\[
< +\infty,
\]

where \( \alpha' = -((Q-2\gamma)/(2Q)) \) and \( \alpha'' = \alpha' + ((Q+2\gamma)/(2Q)) = ((2\gamma)/(Q)) \). Therefore, \( u \in D^{\gamma}(\mathbb{H}^n) \).

Conversely, let us assume that \( u \in D^{\gamma}(\mathbb{H}^n) \). In particular, we have that \( u \in L^2((2Q)/(Q-2\gamma))(\mathbb{H}^n) \), thus by relation (3.1) it turns out that \( U \in L^2((2Q)/(Q-2\gamma))(S^{2n+1}) \); therefore, \( U \in L^2(S^{2n+1}) \). Furthermore, by (3.2) we also have that

\[
\int_{S^{2n+1}} U(\eta)A_{\gamma} U(\eta)d\eta = 2^{-\alpha''}a_{\gamma}[u] < +\infty,
\]

that is, \( U \in H^\gamma(S^{2n+1}) \), which concludes the proof of Claim.

Let \( U \in H^\gamma(S^{2n+1}) \) be a weak solution of \((\text{FYS})_{\gamma}\); then we have

\[
\int_{S^{2n+1}} A_{\gamma} UV d\eta = \int_{S^{2n+1}} |U|^{(4\gamma)/(Q-2\gamma)}UV d\eta \quad \text{for every } V \in H^\gamma(S^{2n+1}).
\]

(3.3)

Let \( v \in D^{\gamma}(\mathbb{H}^n) \) be arbitrarily fixed and define \( V = (J\text{ac}_C \circ C^{-1})((2\gamma-Q)/(2Q))v \circ C^{-1} \). Since \( v = J\text{ac}_C((Q-2\gamma)/(2Q))V \circ C \), by the Claim we have that \( V \in H^\gamma(S^{2n+1}) \). Accordingly, the function \( V \) can be used as a test-function in (3.3), obtaining

\[
\int_{S^{2n+1}} A_{\gamma} U(J\text{ac}_C \circ C^{-1})((2\gamma-Q)/(2Q))v \circ C^{-1}d\eta
\]

\[
= \int_{S^{2n+1}} |U|^{(4\gamma)/(Q-2\gamma)}U(J\text{ac}_C \circ C^{-1})((2\gamma-Q)/(2Q))v \circ C^{-1}d\eta.
\]
By a change of variables, it follows that
\[
\int_{\mathbb{H}^n} (A \gamma U \circ C)(\text{Jac}_C)^{(2\gamma - Q)/(2Q)} + 1)v dz dt
\]
\[
= \int_{\mathbb{H}^n} |U \circ C|^{((4\gamma)/(Q-2\gamma))}(U \circ C)(\text{Jac}_C)^{(2\gamma - Q)/(2Q)} + 1)v dz dt.
\]

This relation and (2.8) imply that
\[
2^{-((Q+2\gamma)/2Q)} \int_{\mathbb{H}^n} L_\gamma((2\text{Jac}_C)^{(2\gamma - Q)/(2Q)}(U \circ C))v dz dt
\]
\[
= \int_{\mathbb{H}^n} |U \circ C|^{((4\gamma)/(Q-2\gamma))}U \circ C(\text{Jac}_C)^{(2\gamma + Q)/(2Q)}v dz dt.
\]

Since \( u = (2\text{Jac}_C)^{(2\gamma - Q)/(2Q)}U \circ C \), the latter equality is equivalent to
\[
\int_{\mathbb{H}^n} L_\gamma uv dz dt = \int_{\mathbb{H}^n} |u|^{((4\gamma)/(Q-2\gamma))}uv dz dt,
\]
which means precisely that \( u \in D^\gamma(\mathbb{H}^n) \) is a weak solution of \((\text{FYH})_\gamma\). The converse argument works in a similar way. □

**Remark 3.2.** One can provide an alternative proof to proposition 3.1 by exploring the explicit form of the fundamental solution of \( L_\gamma \); a similar approach is due to Bartsch, Schneider and Weth [4] for the polyharmonic operator \((-\Delta)^m \) in \( \mathbb{R}^N \), where \( m \in \mathbb{N} \) and \( N > 2m \). For completeness, we sketch the proof.

We recall that the fundamental solution of \( L_\gamma \) is
\[
L_\gamma^{-1}((z,t), (z', t')) = \frac{c_\gamma}{2} d_{KC}^{2\gamma - Q}((z,t), (z', t')),
\]
where
\[
c_\gamma = \frac{2^{n-\gamma} \Gamma((Q - 2\gamma)/(4))^2}{\pi^{n+1} \Gamma(\gamma)},
\]
see Branson, Fontana and Morpurgo [5, p. 21]. For every \( \psi \in L^{((2Q)/(Q+2\gamma))}(S^{2n+1}) \) we introduce the function
\[
[K_\gamma \psi](\zeta) = c_\gamma \int_{S^{2n+1}} \psi(\eta)|1 - \langle \zeta, \eta \rangle|^{((2\gamma - Q)/(2))} d\eta.
\]
One can prove that \( K_\gamma \psi \in H^\gamma(S^{2n+1}) \) for every \( \psi \in L^{((2Q)/(Q+2\gamma))}(S^{2n+1}) \). Moreover, the Funk-Hecke theorem on the CR sphere \( S^{2n+1} \) gives
\[
[K_\gamma Y_{j,k}](\zeta) = \frac{2^{Q/2-\gamma}}{\lambda_j(\gamma) \lambda_k(\gamma)} Y_{j,k}(\zeta),
\]
see Frank and Lieb [14, corollary 5.3]. Thus, a direct computation yields that
\[
(K_\gamma \psi, V)_\gamma = 2^{Q/2-\gamma} \int_{S^{2n+1}} \psi V d\eta \quad \text{for every} \; V \in H^\gamma(S^{2n+1}).
\]
Nodal solutions for the fractional Yamabe problem on Heisenberg groups 781

Note that if $U \in H^\gamma(S^{2n+1})$ is a weak solution of \((\text{FYS})_\gamma\), the latter relation implies that

$$K_\gamma(|U|^{(4\gamma)/(Q-2\gamma)}U) = 2^{Q/2-\gamma}U \quad \text{on} \ S^{2n+1}. \quad (3.6)$$

Accordingly, by relations (3.6), (3.5), (2.1) and (2.2), it turns out that

$$u(z, t) = (2\text{Jac}_C(z, t))^{(Q-2\gamma)/(2Q)}U(C(z, t))$$

$$= 2^{-Q/2+\gamma}(2\text{Jac}_C(z, t))^{(Q-2\gamma)/(2Q)}K_\gamma(|U(C(z, t))|^{(4\gamma)/(Q-2\gamma)}U(C(z, t)))$$

$$= \frac{c_\gamma}{2} \int_{H^n} d^{2\gamma-Q}_{KC}((z, t), (z', t'))|u(z', t')|^{(4\gamma)/(Q-\gamma)}u(z, t')dz'dt', \quad (z, t) \in H^n.$$  

The latter relation is equivalent to the fact that

$$u(z, t) = \frac{c_\gamma}{2}(|u|^{(4\gamma)/(Q-\gamma)}u) * d^{2\gamma-Q}_{KC}((z, t), \cdot), \quad (z, t) \in H^n, \quad (3.7)$$

where ‘*’ denotes the (noncommutative) convolution operation on the Heisenberg group $H^n$. By (3.4), a similar argument as in Folland [12, theorem 2] gives that $L_\gamma u = |u|^{(4\gamma)/(Q-\gamma)}u$ on $H^n$, which concludes the claim.

3.2. Compactness

According to Frank and Lieb [14], see also (2.7), the embedding $H^\gamma(S^{2n+1}) \hookrightarrow L^{((2Q)/(Q-2\gamma))}(S^{2n+1})$ is continuous, but not compact. This subsection is devoted to regain certain compactness by using suitable group actions on the CR sphere $S^{2n+1}$.

To complete this purpose, let $n_j \in \mathbb{N}$, $j = 1, \ldots, k$, with $n_1 + \cdots + n_k = n + 1$. Associated with these numbers, let

$$G = U(n_1) \times \cdots \times U(n_k) \quad (3.8)$$

be the subgroup of the unitary group $U(n+1) = \{g \in O(2n+2) : gJ = Jg\}$, where

$$J = \begin{bmatrix} 0 & I_{\mathbb{R}^{n+1}} \\ -I_{\mathbb{R}^{n+1}} & 0 \end{bmatrix}.$$  

Let

$$H^\gamma_G(S^{2n+1}) = \{U \in H^\gamma(S^{2n+1}) : g \circ U = U \quad \text{for every} \ g \in G\}$$

be the subspace of $G$-invariant functions of $H^\gamma(S^{2n+1})$, where

$$(g \circ U)(\eta) = U(g^{-1}\eta), \quad \eta \in S^{2n+1}. \quad (3.9)$$

It is clear that $H^\gamma_G(S^{2n+1})$ is an infinite-dimensional closed subspace of $H^\gamma(S^{2n+1})$, whenever $k \geq 2$ in the splitting (3.8).

With the above notations in our mind, a Ding-Hebey-Vaugon-type compactness result reads as follows:

**Proposition 3.3.** Let $\gamma \in \bigcup_{i=1}^n [k, ((kQ)/(Q-1))]$. The embedding $H^\gamma_G(S^{2n+1}) \hookrightarrow L^{((2Q)/(Q-2\gamma))}(S^{2n+1})$ is compact, where $G = U(n_1) \times \cdots \times U(n_k)$ is any choice with $n_j \in \mathbb{N}$, $j = 1, \ldots, k$, and $n_1 + \cdots + n_k = n + 1$. 
Proof. First, when \( G = \text{U}(n+1) \), the space \( H_G^\gamma(S^{2n+1}) \) contains precisely the constant functions defined on \( S^{2n+1} \); in this case, the proof is trivial.

In the general case, we recall by Maalaoui and Martino \([20, \text{lemma 3.3}]\) that the embedding \( W_{G}^{1,2}(S^{2n+1}) = H_{G}^{1}(S^{2n+1}) \hookrightarrow L^q(S^{2n+1}) \) is compact for every \( 1 \leq q < q_1^* \), where \( q_1^* = \left((2(Q - 1))/(Q - 3)\right) \) is the Riemannian critical exponent on the \((Q - 1)\)-dimensional sphere \( S^{2n+1} \).

By our assumption \( \gamma \in \bigcup_{k=1}^{n} \{k,(kQ)/(Q - 1))\} \) we have that \( l := [\gamma] \geq 1 \) and
\[
\gamma\left(1 - \frac{1}{Q}\right) < l \leq \gamma. \tag{3.10}
\]

The iterative argument developed by Aubin \([1, \text{proposition 2.11}]\), applied for \( l \) times, gives that the embedding \( W_{G}^{l,2}(S^{2n+1}) = H_{G}^{l}(S^{2n+1}) \hookrightarrow L^q(S^{2n+1}) \) is compact for every \( 1 \leq q < q_l^* \), where \( q_l^* = \left((2(Q - 1))/(Q - 1 - 2l)\right) \). On one hand, since \( l \leq \gamma \), we have that \( H_{G}^{l}(S^{2n+1}) = W_{G}^{l,2}(S^{2n+1}) \subset W_{G}^{l,2}(S^{2n+1}) \). On the other hand, the left-hand side of (3.10) is equivalent to \( q_l^* > \left((2Q)/(Q - 2\gamma)\right) \). Combining these facts, we have the chain of inclusions
\[
H_{G}^{\gamma}(S^{2n+1}) \subset W_{G}^{l,2}(S^{2n+1}) \hookrightarrow L^{((2Q)/(Q - 2\gamma))}(S^{2n+1}),
\]
where the latter embedding is compact. \( \square \)

Remark 3.4. Our assumption \( \gamma \in \bigcup_{k=1}^{n} \{k,(kQ)/(Q - 1))\} \) is important to guarantee the left-hand side of (3.10), which in turn, implies that \( ((2Q)/(Q - 2\gamma)) \) is within the range \([1, q_l^*)\) where the embedding \( W_{G}^{l,2}(S^{2n+1}) \hookrightarrow L^q(S^{2n+1}) \) is compact, \( q \in [1, q_l^*) \). We are wondering if this assumption can be removed in order to prove the compactness of the above embedding for the whole spectrum \((0, Q/2)\) of the parameter \( \gamma \).

3.3. Special group actions

The goal of this subsection is to describe symmetrically different functions belonging to \( H^\gamma(S^{2n+1}) \) via subgroups of the form \( G = \text{U}(n_1) \times \cdots \times \text{U}(n_k) \) with \( n_1 + \cdots + n_k = n + 1 \). To handle this problem, we explore a Rubik-cube technique, described in a slightly different manner in Balogh and Kristály \([2]\); roughly speaking, \( n + 1 \) corresponds to the number of total sides of the cube, while the sides of the cube are certain blocks in the decomposition subgroup \( G = \text{U}(n_1) \times \cdots \times \text{U}(n_k) \).

To be more precise, let \( n \geq 1 \) and for \( i \in \{1, \ldots, [n + 1/2]\} \), we consider the subgroup of the unitary group \( \text{U}(n + 1) \) as
\[
G_i = \begin{cases} 
\begin{bmatrix} \text{U}\left(\frac{n + 1}{2}\right) & 0 \\
0 & \text{U}\left(\frac{n + 1}{2}\right) 
\end{bmatrix}, & \text{if } n + 1 = 2i, \\
\begin{bmatrix} \text{U}(i) & 0 & 0 \\
0 & \text{U}(n + 1 - 2i) & 0 \\
0 & 0 & \text{U}(i) 
\end{bmatrix}, & \text{if } n + 1 \neq 2i.
\end{cases}
\]
Nodal solutions for the fractional Yamabe problem on Heisenberg groups

It is clear that a particular $G_i$ does not act transitively on the sphere $S^{2n+1}$. However, to recover the transitivity, we shall combine different groups of the type $G_i$; for further use, let $[G_i; G_j]$ be the group generated by $G_i$ and $G_j$.

**Lemma 3.5.** Let $i, j \in \{1, \ldots, [n + 1/2]\}$ with $i \neq j$. Then the group $[G_i; G_j]$ acts transitively on the CR sphere $S^{2n+1}$.

**Proof.** Without loss of generality, we assume that $j > i$. For further use, let $0_k = (0, \ldots, 0) \in \mathbb{C}^k = \mathbb{R}^{2k}$, $k \in \{1, \ldots, n\}$. Let us fix $\eta = (\eta_1, \eta_2, \eta_3) \in S^{2n+1}$ arbitrarily with $\eta_1, \eta_3 \in \mathbb{C}^j$ and $\eta_2 \in \mathbb{C}^{n+1-2j}$; clearly, $\eta_2$ disappears from $\eta$ whenever $2j = n + 1$. Taking into account the fact that $U(j)$ acts transitively on $S^{2j-1}$, there are $g_1^j, g_2^j \in U(j)$ such that if $g_j = g_1^j \times I_{\mathbb{C}^{n+1-2j}} \times g_2^j \in G_j$, then $g_j \eta = (0, 0_j, \eta_1) \eta_2, \eta_3, 0_j, 0_j, 0_j$. Since $j - 1 \geq i$, the transitive action of $U(n + 1 - 2i)$ on $S^{2n+1-4i}$ implies the existence of $g_i^1 \in U(n + 1 - 2i)$ such that $g_i^1(0_j, 0, \eta_1) \eta_2, \eta_3, 0, 0_j, 0_j, 0_j = (1, 0, 0_{n - 2i})$. Therefore, if $g_i = I_{\mathbb{C}^i} \times g_i^1 \times I_{\mathbb{C}^{n+1-2i}} \in G_i$ then $g_i g_j \eta = (0, 1, 0, 0_{n - i}) \in S^{2n+1}$.

By repeating the same procedure for another element $\tilde{\eta} \in S^{2n+1}$, there exists $\tilde{g}_i \in G_i$ and $\tilde{g}_j \in G_j$ such that $\tilde{g}_i \tilde{g}_j \tilde{\eta} = (0, 1, 0, 0_{n - i}) \in S^{2n+1}$. Accordingly,

$$\eta = g_j^{-1} g_i^{-1} \tilde{g}_i \tilde{g}_j \tilde{\eta} = g_j^{-1} \tilde{\eta},$$

where $\tilde{g}_i = g_i^{-1} \tilde{g}_i \in G_i$, which concludes the proof.

For every fixed $i \in \{1, \ldots, [n + 1/2]\}$, we introduce the matrix $A_i$ as

$$A_i = \begin{cases} \begin{bmatrix} 0 & I_{\mathbb{C}^{(n+1)/(2)}} \\ I_{\mathbb{C}^{((n+1)/(2))}} & 0 \end{bmatrix}, & \text{if } n + 1 = 2i, \\
0 & I_{\mathbb{C}^{i}} \\
I_{\mathbb{C}^{i}} & 0 \\
0 & I_{\mathbb{C}^{i}} \\
0 & 0 \end{cases}, & \text{if } n + 1 \neq 2i. 
$$

The following construction is inspired by Bartsch and Willem [3]. Since one has $A_i \in U(n + 1) \setminus G_i$, $A_i^2 = I_{\mathbb{C}^{n+1}}$ and $A_i G_i = G_i A_i$, the group generated by $G_i$ and $A_i$ is $\hat{G}_i = [G_i; A_i] = G_i \cup A_i G_i$, that is,

$$\hat{G}_i = \begin{cases} \begin{bmatrix} U(n + 1/2) & 0 \\ 0 & U(n + 1/2) \end{bmatrix} \cup \begin{bmatrix} 0 & U(n + 1/2) \\ U(n + 1/2) & 0 \end{bmatrix}, & \text{if } n + 1 = 2i, \\
\begin{bmatrix} U(i) & 0 \\ 0 & U(n + 1 - 2i) \end{bmatrix} \cup \begin{bmatrix} 0 & U(i) \\ U(i) & 0 \end{bmatrix} \cup \begin{bmatrix} 0 & U(n + 1 - 2i) \\ 0 & U(n + 1 - 2i) \end{bmatrix} & \text{if } n + 1 \neq 2i. 
\end{cases}$$

In fact, in $\hat{G}_i$, there are only two types of elements: either of the form $g \in G_i$, or $A_i g \in G_i \setminus G_i$ (with $g \in G_i$), respectively.
The action \( \hat{G}_i \odot H^{\gamma}(S^{2n+1}) \mapsto H^{\gamma}(S^{2n+1}) \) of the group \( \hat{G}_i \) on \( H^{\gamma}(S^{2n+1}) \) is defined by

\[
(\hat{g} \odot U)(\eta) = \begin{cases} 
U(g^{-1}\eta), & \text{if } \hat{g} = g \in G_i, \\
-U(g^{-1}A_i^{-1}\eta), & \text{if } \hat{g} = A_i g \in \hat{G}_i \setminus G_i,
\end{cases}
\]

(3.12)

for every \( \hat{g} \in \hat{G}_i \), \( U \in H^{\gamma}(S^{2n+1}) \) and \( \eta \in S^{2n+1} \). We notice that this action is well-defined, continuous and linear. Similarly, as in (3.9), we introduce the space of \( \hat{G}_i \)-invariant functions of \( H^{\gamma}(S^{2n+1}) \) as

\[
H^{\gamma}_{G_i}(S^{2n+1}) = \{ U \in H^{\gamma}(S^{2n+1}) : g \circ U = U \quad \text{for every } g \in G_i \},
\]

where the action \(^\prime \circ \) corresponds to the first relation in (3.12). Furthermore, let

\[
H^{\gamma}_{G_i}(S^{2n+1}) = \{ U \in H^{\gamma}(S^{2n+1}) : \hat{g} \odot U = U \quad \text{for every } \hat{g} \in \hat{G}_i \}
\]

be the space of \( \hat{G}_i \)-invariant functions of \( H^{\gamma}(S^{2n+1}) \).

The following result summarizes the constructions in this subsection.

**Proposition 3.6.** Let \( \gamma \in \bigcup_{k=1}^n \{ k, kQ/Q - 1 \} \), and fix \( i, j \in \{ 1, \ldots, [n+1/2] \} \) such that \( i \neq j \). The following statements hold:

(i) The embedding \( H^{\gamma}_{G_i}(S^{2n+1}) \hookrightarrow L^{((2Q)/(Q-2\gamma))}(S^{2n+1}) \) is compact;

(ii) \( H^{\gamma}_{G_i}(S^{2n+1}) \cap H^{\gamma}_{G_j}(S^{2n+1}) = \{ \text{constant functions on } S^{2n+1} \} \);

(iii) \( H^{\gamma}_{G_i}(S^{2n+1}) \cap H^{\gamma}_{G_j}(S^{2n+1}) = \{ 0 \} \).

**Proof.**

(i) It is clear that \( H^{\gamma}_{G_i}(S^{2n+1}) \subset H^{\gamma}_{G_j}(S^{2n+1}) \). Moreover, by proposition 3.3, we have that the embedding \( H^{\gamma}_{G_i}(S^{2n+1}) \hookrightarrow L^{((2Q)/(Q-2\gamma))}(S^{2n+1}) \) is compact.

(ii) Let us fix \( U \in H^{\gamma}_{G_i}(S^{2n+1}) \cap H^{\gamma}_{G_j}(S^{2n+1}) \). Since \( U \) is both \( G_i \)- and \( G_j \)-invariant, it is also \([G_i; G_j]\)-invariant, that is, \( U(g\eta) = U(\eta) \) for every \( g \in [G_i; G_j] \) and \( \eta \in S^{2n+1} \). According to lemma 3.5, the group \([G_i; G_j]\) acts transitively on the CR sphere \( S^{2n+1} \), that is, the orbit of every element \( \eta \in S^{2n+1} \) by the group \([G_i; G_j]\) is the whole sphere \( S^{2n+1} \). Thus, \( U \) is a constant function.

(iii) Let \( U \in H^{\gamma}_{G_i}(S^{2n+1}) \cap H^{\gamma}_{G_j}(S^{2n+1}) \). On one hand, by (ii), we first have that \( U \) is constant. On the other hand, the second relation from (3.12) implies that \( U(\eta) = -U(A_i\eta) \) for every \( \eta \in S^{2n+1} \). Therefore, we necessarily have that \( U = 0 \).
3.4. Proof of Theorem 1.1.

We associate to problem $({\text{FYS}})_\gamma$ the energy functional $E : H^\gamma(S^{2n+1}) \to \mathbb{R}$ defined by

$$E(U) = \frac{1}{2} \int_{S^{2n+1}} U A_{\gamma} U \, d\eta - \frac{Q - 2\gamma}{2Q} \int_{S^{2n+1}} |U|^{((2Q)/(Q - 2\gamma))} \, d\eta, \quad U \in H^\gamma(S^{2n+1}).$$

Due to (2.7), the functional $E$ is well-defined, belonging to $C^1(H^\gamma(S^{2n+1}), \mathbb{R})$. Moreover, $U \in H^\gamma(S^{2n+1})$ is a critical point of $E$ if and only if $U$ is a weak solution of $({\text{FYS}})_\gamma$.

Let us fix $i \in \{1, \ldots, [n + 1/2]\}$. In order to guarantee critical points for $E$, we first consider the functional $E_i : H^\gamma_{\hat{G}_i}(S^{2n+1}) \to \mathbb{R}$, the restriction of $E$ to the space $H^\gamma_{\hat{G}_i}(S^{2n+1})$. It is clear that $E_i$ is an even functional and it has the mountain pass geometry. Since the embedding $H^\gamma_{\hat{G}_i}(S^{2n+1}) \hookrightarrow L((2Q)/(Q - 2\gamma))(S^{2n+1})$ is compact, see proposition 3.6 (i), we may apply the fountain theorem, see for example, Bartsch and Willem [3, theorem 3.1], guaranteeing a sequence $\{U^k_i\}_{k \in \mathbb{N}} \subset H^\gamma_{\hat{G}_i}(S^{2n+1})$ of critical points for $E_i$ with the additional property that $\|U^k_i\|_{H^\gamma} \to \infty$ as $k \to \infty$.

By using the principle of symmetric criticality of Palais [22], we are going to prove that $\{U^k_i\}_{k \in \mathbb{N}} \subset H^\gamma_{\hat{G}_i}(S^{2n+1})$ are in fact critical points for the original energy functional $E$, thus weak solutions of $({\text{FYS}})_\gamma$. To do this, it suffices to verify that $E$ is a $\hat{G}_i$-invariant functional, that is,

$$E(\hat{g} \odot U) = E(U) \quad \text{for every } \hat{g} \in \hat{G}_i, \; U \in H^\gamma(S^{2n+1}).$$

On one hand, according to relation (2.6), for the quadratic term in $E$, it is enough to prove that $\hat{G}_i$ acts isometrically on $H^\gamma(S^{2n+1})$, that is,

$$(\hat{g} \odot U, \hat{g} \odot U)_\gamma = (U, U)_\gamma \quad \text{for every } \hat{g} \in \hat{G}_i, \; U \in H^\gamma(S^{2n+1}). \quad (3.13)$$

To see this, let us fix $\hat{g} \in \hat{G}_i$ and $U \in H^\gamma(S^{2n+1})$ arbitrarily. We recall that by definition

$$(\hat{g} \odot U, \hat{g} \odot U)_\gamma = \sum_{j, k \geq 0} \lambda_j(\gamma) \lambda_k(\gamma) \sum_{l=1}^{m_{j,k}} |c^l_{j,k}(\hat{g} \odot U)|^2.$$

By using (2.3), one has

$$\sum_{l=1}^{m_{j,k}} |c^l_{j,k}(\hat{g} \odot U)|^2 = \int_{S^{2n+1}} \int_{S^{2n+1}} (\hat{g} \odot U)(\zeta)(\hat{g} \odot U)(\eta) \sum_{l=1}^{m_{j,k}} Y^l_{j,k}(\zeta) \overline{Y^l_{j,k}(\eta)} \, d\zeta \, d\eta$$

$$= \int_{S^{2n+1}} \int_{S^{2n+1}} (\hat{g} \odot U)(\zeta)(\hat{g} \odot U)(\eta) \Phi_{j,k}(\zeta, \eta) \, d\zeta \, d\eta. \quad (3.14)$$
Note that for every \( g \in G_i \subseteq U(n+1) \) and \( \zeta, \eta \in S^{2n+1} \), we have
\[
\langle g\zeta, \overline{g}\eta \rangle = \langle A_i g\zeta, A_i \overline{g}\eta \rangle = \langle \zeta, \eta \rangle;
\]
therefore, by the representation (2.4) of the zonal harmonics we also have that
\[
\Phi_{j,k}(g\zeta, g\eta) = \Phi_{j,k}(A_i g\zeta, A_i g\eta) = \Phi_{j,k}(\zeta, \eta).
\]
Thus, relation (3.12) and suitable changes of variables in (3.14) imply that
\[
\sum_{l=1}^{m_{j,k}} |c_{j,k}^l(g \otimes U)|^2 = \int_{S^{2n+1}} \int_{S^{2n+1}} U(\zeta) U(\eta) \Phi_{j,k}(\zeta, \eta) d\zeta d\eta = \sum_{l=1}^{m_{j,k}} |c_{j,k}^l(U)|^2,
\]
which proves (3.13).

On the other hand, the \( \hat{G}_i \)-invariance of the nonlinear term \( U \mapsto \int_{S^{2n+1}} |U|^{Q/(Q-\gamma)} \) trivially follows by a change of variable, by using the isometric character of the group \( U(n+1) \) on \( S^{2n+1} \).

Accordingly, for every \( i \in \{1, \ldots, [n+1/2]\} \), the functions \( \{U_k\}_{k \in \mathbb{N}} \subseteq H^1_{\hat{G}_i} \) \((S^{2n+1})\) are non-trivial weak solutions of \((\text{FYS})_\gamma\). Due to proposition 3.1, \( u_i = (2\text{Jac}_c)^{(Q-\gamma)/(Q-2\gamma)} U_k \circ \mathcal{C} \in D^\gamma(\mathbb{H}^n) \) are non-trivial weak solutions of the original fractional Yamabe problem \((\text{FYH})_\gamma\); by construction, \( u_i \) are sign-changing functions.

Due to proposition 3.6 (iii), we state that the sequences \( \{U_k\}_{k \in \mathbb{N}} \subseteq H^1_{\hat{G}_i} (S^{2n+1}) \) and \( \{U_j\}_{k \in \mathbb{N}} \subseteq H^1_{\hat{G}_j} (S^{2n+1}) \) with \( i, j \in \{1, \ldots, [n+1/2]\}, \ i \neq j\), cannot be compared from symmetrical point of view. Therefore, the sequences \( \{u_k\} \subseteq D^\gamma(\mathbb{H}^n) \) and \( \{u_k\} \subseteq D^\gamma(\mathbb{H}^n) \) have mutually different nodal properties for every \( i, j \in \{1, \ldots, [n+1/2]\}, \ i \neq j\), which concludes the proof.

**Remark 3.7.** Consider a nonzero solution \( u_i = (2\text{Jac}_c)^{(Q-\gamma)/(Q-2\gamma)} U_k \circ \mathcal{C} \in D^\gamma(\mathbb{H}^n) \) of \((\text{FYH})_\gamma\), with \( \{U_k\}_{k \in \mathbb{N}} \subseteq H^1_{\hat{G}_i} (S^{2n+1}) \setminus \{0\} \). For simplicity, we consider the case \( n+1 = 2i \). Let us introduce the nodal domain of \( U_k \) (or \( u_k \)) as the connected components of \( C_i^k = S^{2n+1} \setminus N_i^k \), where \( N_i^k = \{\eta \in S^{2n+1} : U_k(\eta) = 0\} \).

Since \( U_k \in H^1_{\hat{G}_i} (S^{2n+1}) \), by relation (3.12) it follows that \( U_k(\eta) = U_k(\eta_1, \eta_2) \) with the property that \( U_k(\eta_1, \eta_2) = -U_k(\eta_2, \eta_1) \), \( \eta = (\eta_1, \eta_2) \in S^{2n+1}, \eta_1, \eta_2 \in \mathbb{C}^i \). Accordingly, since \( U_k(\pm \eta_1, \pm \eta_2) = U_k(\pm \eta_1, | \eta_2|) \), \( U_k(\pm \eta_1, | \eta_2|) \) is sign-changing with at least four non-degenerate nodal domains in \( C_i^k \); in two of them the function \( U_k \) is negative, while in the other two it is positive, respectively. When \( n + 1 \neq 2i \), a similar discussion can be performed.

We conclude the paper by the following table providing explicit forms of subgroups of the unitary group \( U(n+1) \) and admissible intervals for the parameter \( \gamma \), depending on the dimension \( n \), where our main theorem applies; we only consider the cases when \( n \in \{1, \ldots, 8\} \):
Nodal solutions for the fractional Yamabe problem on Heisenberg groups

| $n$ | $Q = 2n + 2$ | $G_i$, $i \in \{1, \ldots, [n + 1/2]\}$ | Admissible domains for $\gamma \in (0, Q/2)$ | Number of symmetrically distinct sequences of solution of $(FYH)_\gamma$ |
|-----|-------------|---------------------------------|---------------------------------|---------------------------------|
| 1   | 4           | $G_1 = U(1) \times U(1)$       | $[1, 4/3)$                       | 1                               |
| 2   | 6           | $G_1 = U(1) \times U(1) \times U(1)$ | $[1, 6/5) \cup [2, 12/5)$       | 1                               |
| 3   | 8           | $G_1 = U(1) \times U(2) \times U(1)$ | $[1, 8/7) \cup [2, 16/7) \cup [3, 24/7)$ | 2                               |
| 4   | 10          | $G_1 = U(1) \times U(3) \times U(1)$ | $\bigcup_{k=1}^{5} [k, 10k/9)$  | 2                               |
|     |             | $G_2 = U(2) \times U(1) \times U(2)$ |                                   |                                 |
|     |             | $G_1 = U(1) \times U(4) \times U(1)$ |                                   |                                 |
| 5   | 12          | $G_2 = U(2) \times U(2) \times U(2)$ | $\bigcup_{k=1}^{5} [k, 12k/11)$  | 3                               |
|     |             | $G_3 = U(3) \times U(3)$         |                                   |                                 |
|     |             | $G_1 = U(1) \times U(5) \times U(1)$ |                                   |                                 |
| 6   | 14          | $G_2 = U(2) \times U(3) \times U(2)$ | $\bigcup_{k=1}^{6} [k, 14k/13)$  | 3                               |
|     |             | $G_3 = U(3) \times U(1) \times U(3)$ |                                   |                                 |
|     |             | $G_1 = U(1) \times U(6) \times U(1)$ |                                   |                                 |
| 7   | 16          | $G_2 = U(2) \times U(4) \times U(2)$ | $\bigcup_{k=1}^{7} [k, 16k/15)$  | 4                               |
|     |             | $G_3 = U(3) \times U(2) \times U(3)$ |                                   |                                 |
|     |             | $G_1 = U(4) \times U(4)$         |                                   |                                 |
|     |             | $G_1 = U(1) \times U(7) \times U(1)$ |                                   |                                 |
| 8   | 18          | $G_2 = U(2) \times U(5) \times U(2)$ | $\bigcup_{k=1}^{8} [k, 18k/17)$  | 4                               |
|     |             | $G_3 = U(3) \times U(3) \times U(3)$ |                                   |                                 |
|     |             | $G_4 = U(4) \times U(1) \times U(4)$ |                                   |                                 |

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A. Kristály

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