The lift of type IIA supergravity with D6 sources: M-theory with torsion

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Abstract

This paper is concerned with an extension of the well known Kaluza-Klein mechanism. As the standard ansatz for Kaluza-Klein reduction implies the existence of a gauge potential associated with the KK field strength, it follows immediately that this field strength satisfies its Bianchi identity. Hence, the standard KK formalism breaks down in the presence of a violated Bianchi identity. This occurs for example in the context of D6 sources.

We will investigate and partially solve this problem in the context of the type IIA/M-theory duality. Our discussion is motivated by the construction of gauge/string duals with backreacting flavor branes using D6-branes, which appear in M-theory as KK-monopoles.

We are able to derive source-modified equations of motion for the eleven-dimensional theory, and are subsequently able to obtain the source-modified type IIA equations by direct dimensional reduction.

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1 Introduction

In the context of M-theory, the relations between type IIA string theory and eleven-dimensional supergravity are by now standard textbook material (see for example [1, 2, 3, 4]). The M2-brane gives rise to the D2 and the fundamental string, the M5 to the D4 and NS5 branes. The D0 and D6-branes on the other hand have a slightly different origin. Not being related to any brane-like object in eleven dimensions, they are results of the Kaluza-Klein (KK) reduction relating the two theories; the former being a particle-like, localized gravitational excitation on the KK-circle, the latter a peculiar fibration of said circle over the ten-dimensional base, known as a Kaluza-Klein monopole (a good review is given by [5]). In this paper, we are concerned with a small gap in this formalism that becomes apparent when one tries to consider the M-theory lift of smeared D6-branes.

The problem can be quickly explained. The bosonic sector of eleven-dimensional supergravity contains only the graviton $\hat{g}_{MN}$ and a four-form field $\hat{F}_{(4)}$. Upon KK reduction, $\hat{F}_{(4)}$ gives rise to the Kalb-Ramond three-form field $H_{(3)}$ as well as the Ramond-Ramond four-form $F_{(4)}$. From $\hat{g}_{MN}$ one obtains the ten-dimensional metric $g_{\mu\nu}$, the dilaton $\Phi$, and a one-form gauge potential $A_{(1)}$, with an associated field strength $F_{(2)} = dA_{(1)}$. If we assume the KK-circle to be parameterized by $z$, the standard KK-ansatz relating the two geometries is:

$$ds_\text{M}^2 = e^{-\Phi} ds_{\text{IIA}}^2 + e^{\Phi} (A_{(1)} + dz)^2$$

$$\hat{F}_{(4)} = F_{(4)} + H_{(3)} \wedge dz$$

Where the distinction is necessary, hats and tildes denote eleven-dimensional quantities. Capital letters describe eleven-dimensional indices. The M-theory circle will be parameterized by either $z$, $\psi_+$, or $\psi$. 

2 Flavored $\mathcal{N} = 1$ string duals from D6-branes

2.1 The eleven-dimensional dual without sources

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3.1.1 The $G_2$-structure

3.1.2 The SUSY variations

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A D6-brane embeddings

B Kaluza-Klein reduction of supergravity variations with torsion
Given any solution to the equations of motion of type IIA supergravity, one can use (1.1) to lift to eleven dimensions and vice versa. However, as $A_{(1)}$ plays the role of a gauge potential, it is actually $F_{(2)} = dA_{(1)}$ that contains the physically relevant degrees of freedom. Thus given a set $\{g_{\mu\nu}, \Phi, F_{(2)}, H_{(3)}, F_{(4)}\}$ one first has to find a gauge potential prior to lifting. Now assume that for some reason $dF_{(2)} \neq 0$. Clearly $A_{(1)}$ cannot exist and we are unable to find a gauge potential. Therefore we cannot use (1.1) to perform the lift. This is the apparent gap in the standard formalism we alluded to earlier.

The problem is not a purely formal one. D6-branes couple magnetically to $A_{(1)}$. As we will explain shortly, the inclusion of D6 sources violates the Bianchi identity $dF_{(2)} = 0$ at the position of the sources. While this is not a problem for localized sources – as a matter of fact it is the reason why the KK-monopole is a gravitational instanton – one encounters the problem at hand once one distributes the branes continuously and thus violates the Bianchi identity on an open subset of space-time.

As an aside it is worthwhile to point out that the relation between D6-branes and the RR two-form is much the same as that between magnetic monopoles and the $F_{EM}$ in standard electro-magnetism. The inclusion of magnetic sources restores the symmetry of the Maxwell equations. Schematically

$$d \ast F_{EM} = \ast j_M$$

Thus, the Bianchi identity is violated by the magnetic current $j_M$. In the context of quantum field theories one speaks of monopole condensation. (See e.g. [6])

In this paper, we will not resolve the issue in full generality, but will focus on the inclusion of D6 sources in type IIA backgrounds of the form

$$M_{10} = \mathbb{R}^{1,3} \times M_6$$

without three of four-form flux, that preserve four supercharges. More precisely, we will be interested in the construction of string duals to 3 + 1-dimensional $SU(N_c)$ gauge theories with $\mathcal{N} = 1$ supersymmetry and $N_f$ flavors using D6-branes. Let us briefly expose some general points concerning gauge/string dualities.

Since its discovery, the AdS/CFT correspondence ([7], [8]) has been used to study a variety of problems ranging from black hole entropy to the physics of condensed matter systems. In the line of work dedicated to the study of increasingly realistic gauge theories with physics similar to the one of QCD, recent years have seen a continuous extension of the duality to gauge theories with matter fields charged under the fundamental representation of the gauge group [9]. On the string theory side, the addition of fundamental matter corresponds to the inclusion of branes extending along both the Minkowski directions associated with the gauge theory as well as a non-compact cycle transverse to them. As long as one remains in the probe limit, $N_f \ll N_c$, it is sufficient to consider these flavor branes as probes in the background space-time. If one wants to go beyond this probe approximation ([10] [11] [12] [13]), one needs to include the backreaction of the flavor branes onto space-time. I.e. one needs to consider the combined action

$$S = S_{IIA} + S_{Branes}$$
where $S_{\text{Branes}} = \sum_{N_f} (S_{\text{DBI}} + S_{\text{WZ}})$ consists of the standard brane action for every single flavor brane. For other examples, see [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26].

Note that apart from [9], all the previously cited duals consider the addition of flavors to string duals with reduced supersymmetry. One of the main techniques used to reduce the amount of supercharges preserved by the geometry consists in wrapping the $N_c$ color branes on compact cycles in the geometry [27, 28, 29, 30, 31]. E.g. one wraps D4-branes on an $S^1$ or D5s on an $S^2$ to study 3 + 1-dimensional gauge theories. In general, the preservation of some supersymmetry requires the gauge theory living on the brane to be topologically twisted – see [28].

This paper was originally born out of the interest to study the addition of flavor branes to type IIA backgrounds dual to $\mathcal{N} = 1$, $SU(N_c)$ super Yang-Mills. Before flavoring, the geometry is that of $N_c$ D6-branes wrapping a three-cycle in the deformed conifold\footnote{To be precise, we will be dealing with conifolds deformed by the presence of branes or $F(2)$ flux. They do carry $SU(3)$-structure but are not of $SU(3)$-holonomy. Therefore, they are not Calabi-Yau and strictly speaking we should not refer to them as (deformed/resolved) conifolds. For the lack of a better term however, we shall refer to the internal six-dimensional manifolds in this paper by that name though, as their topology is the same as that of their Calabi-Yau cousins.}. In the limit $N_c g_{YM}^2 \gg 1$, the backreaction of the color branes causes the system to undergo a geometric transition. The system is now best described in terms of the resolved conifold with the branes having been replaced by $N_c$ units of two-form flux over a two-cycle. This was originally studied in [32, 33] and the geometric transition is based on the work of [34, 35]; an attempt at generalizing the duality to include finite-temperature duals was made in [36]. The resulting ten-dimensional background consists of metric, dilaton and RR two-form $(g_{\mu\nu}, \Phi, F(2))$. Referring back to (1.1) one sees that it lifts to pure geometry in M-theory, as both $H(3)$ and $F(4)$ are set to zero. It is for this reason that it is particularly simple and interesting to study these geometries and dualities from the perspective of eleven-dimensional supergravity. Here, the equations of motion and supergravity variations simplify to

$$\hat{R}_{MN} = 0$$
$$\delta_\epsilon \hat{\psi}_M = \partial_M \hat{\epsilon} + \frac{1}{4} \hat{\omega}_{MAB} \hat{\Gamma}^{AB} \hat{\epsilon} \quad \text{(1.5)}$$

The eleven-dimensional geometry is of the form

$$\mathcal{M}_{11} = \mathbb{R}^{1,3} \times \mathcal{M}_7 \quad \text{(1.6)}$$

As the seven-dimensional manifold $\mathcal{M}_7$ preserves 1/8-SUSY and is Ricci flat, it is a manifold of $G_2$-holonomy. The concept of M-theory compactifications on such manifolds \footnote{For the lack of a better term however, we shall refer to the internal six-dimensional manifolds in this paper by that name though, as their topology is the same as that of their Calabi-Yau cousins.} is pretty much the same as that of the old heterotic string models on Calabi-Yau three-folds used in classic string phenomenology. Mathematically this is reflected by the presence of a three-form $\phi_{G_2}$ that is closed and co-closed

$$d\phi_{G_2} = 0 \quad d(*_7 \phi_{G_2}) = 0 \quad \text{(1.7)}$$

where $*_7$ denotes the seven-dimensional Hodge dual on the internal space.

From the point of view of type IIA string theory, the flavoring procedure is reasonably straightforward. As was shown in [38, 39, 40] and then applied
to gauge/string duality in [41], the brane action can be written as an integral over the ten-dimensional space-time instead of as a sum over integrals over the seven-dimensional world volume,

\[ S_{\text{Branes}} = -T_6 \int_{M_{10}} \left( e^{-\Phi} \phi_{D6} - A_{(7)} \right) \wedge \Xi_{(3)} \]  

(1.8)

where \( \phi_{D6} \) is the so-called calibration form and \( \Xi_{(3)} \) takes the role of a source density for the D6-branes. The presence of \( S_{\text{Branes}} \) in the modified action (1.4) gives source term contributions to the equations of motion. Most prominent among these is the appearance of a magnetic source term for the RR two-form,

\[ dF_{(2)} = -(2\kappa_{10}^2 T_6) \Xi_{(3)} \]  

(1.9)

that violates the standard Bianchi identity. In type IIA one accommodates for this simply by adding a flavor contribution to the RR form,

\[ F_{(2)} = dA_{(1)} + (2\kappa_{10}^2 T_6)B_{(2)} \]  

(1.10)

with \( B_{(2)} \rightarrow 0 \) as \( N_f \rightarrow 0 \). (Note that \( B_{(2)} \) is not to be confused with the Kalb-Ramond two-form potential \( H_{(3)} = dB_{(2)} \) that will not appear in this paper.) On a technical side, one anticipates that the flavor branes will deform the \( N_f = 0 \) geometry, and begins therefore by studying deformations of the original background prior to flavoring. Subsequently one searches for solutions of the fully backreacted problem. Intuitively one would consider a localized stack of \( N_f \) flavor branes from which it follows that \( \Xi_{(3)} \) should contain delta functions localizing the sources on the internal cycles. This is undesirable as localized sources break the isometries of the background and do therefore break global symmetries of the full dual theory (including the KK-modes); the resulting differential equations are also very hard to solve. Therefore, one distributes the flavor branes continuously over their transverse cycles. In the process, the flavor symmetry breaks as \( U(N_f) \rightarrow U(1) \). The procedure is known as smearing and the form \( \Xi_{(3)} \) occasionally referred to as the smearing form. As it was shown in [41], the choice of smearing form is not arbitrary as supersymmetry and the modified Bianchi identities require it to satisfy

\[ d *_{10} d(e^{-\Phi} \phi_{D6}) = -(2\kappa_{10}^2 T_6)\Xi_{(3)} \]  

(1.11)

It is a priori not obvious how to accomodate the violation of the Bianchi identity (1.9) in M-theory. However, as the sources will not only modify the Bianchi identity, yet also the dilaton and Einstein equations, it is reasonable to expect that the eleven-dimensional geometry will not be Ricci flat. Instead, the Einstein equations should be supplemented by the presence of a source term,

\[ \hat{R}_{MN} - \frac{1}{2} \hat{g}_{MN} \hat{R} = \hat{T}_{MN} \]  

(1.12)

From the loss of Ricci flatness it follows that the manifold can no longer be of \( G_2 \)-holonomy; as it preserves the same amount of supersymmetry however it is fair to expect it to carry a \( G_2 \)-structure. Therefore, there is still a three-form \( \hat{\phi}_{G_2} \) that now fails to be (co)closed. One can anticipate that the failure of the
manifold to be of $G_2$-holonomy is parameterized by $N_f$ and thus ultimately by the $B_{(2)}$ contribution to $F_{(2)}$, i.e.

$$d\hat{\phi}_{G_2} \sim (F - dA) \quad (1.13)$$

These expressions, relating forms of different degrees, are to be understood in such a way that the left hand side vanishes when the right hand side does, and vice versa. Now for a manifold carrying a $G$-structure, its failure to be of $G$-holonomy is measured by its intrinsic torsion. Therefore, we expect the flavors in eleven dimensions to appear in the form of intrinsic torsion. A detailed study of the relation between the eleven and ten-dimensional supersymmetry variations will prompt us to consider eleven-dimensional backgrounds with torsion $\tau$, where the torsion is related to $F - dA = B$.

Finally we will see that an uplift of our ten-dimensional equations of motion is given by the relation

$$R^{(\tau)}_{MN} + \frac{1}{2} R_{KLRN}^{(\tau)}(\hat{\phi})_{M}^{KLR} = 0 \quad (1.14)$$

which is the solution to our initial problem. $R^{(\tau)}$ is the eleven-dimensional Riemann (Ricci) tensor with torsion – we have discarded the use of hats to avoid an overly cluttered notation. As one can always rewrite the Riemann tensor as a combination of a torsion free Riemann tensor with additional terms depending on the torsion, it is possible to recast the above equation in the form of (1.12) with the energy-momentum tensor depending only on the torsion.

At first glance, equation (1.14) appears like a modification of M-theory and violates all intuition as eleven-dimensional supergravity is unique. However, one must not forget that we never assumed to solve the problem in its full generality. As a matter of fact, (1.14) has to be taken with several pinches of salt - which might not be a surprise, as the inclusion of source terms in theories of gravity is always a rather difficult business. First of all, (1.14) assumes the background to be of topology $M_{11} = \mathbb{R}^{1,3} \times M_7$, with the internal manifold carrying a $G_2$-structure. Furthermore this means that we are not dealing with maximal eleven-dimensional supergravity, but with a situation with reduced supersymmetry – $1/8$ BPS – in which case the theory is no longer unique. Still, as we will see, equation (1.14) manages what the standard KK-ansatz (1.1) does not. It gives the correct source-modified equations of motion in type IIA.

The structure of this paper is as follows. In section 2 we will begin with a review of the unflavored geometries in ten and eleven dimensions and then continue by studying the flavoring problem from the perspective of type IIA. Following this, we will turn to the issue of the M-theory lift in section 3. The paper is amended by several appendices on brane embeddings, spinor conventions and KK reduction. For illustrative and motivational purposes we will be using a specific case of a M-theory $G_2$-holonomy manifold and its type IIA reduction in section 2. However, the results of section 3 on the M-theory lift of smeared D6-branes do not depend on this example or the type IIA reduction chosen. They only depend on the presence of a $G_2$-structure, four-dimensional Minkowski space and the absence of M-theory fluxes.

3For intrinsic torsion in the context of string theory see [12].
Note that (1.14) is not the only result presented here. As we are studying the flavoring problem in type IIA in order to find an answer to the issue of the M-theory lift, this paper makes also considerable progress towards the construction of a dual to four-dimensional, $\mathcal{N} = 1$ $SU(N_c)$ super Yang-Mills with backreacting flavors. For the specific ansatz of section 2 we are able to derive a set of very generic first-order equations – (2.32) and (2.36) – that have to be satisfied by smeared D6 sources in this geometry. We proceed to derive an analytic one-parameter family of solutions in section 2.3. While the fluxes in this solution satisfy the flux quantization necessary for a string dual, the geometry is that of a cone over $S^2 \times S^3$ with a singularity at the origin. So we expect the interpretation of this solution as a suitable dual to be difficult. The presentation of the flavoring problem is supplemented by a discussion of D6-brane embeddings for the geometries at hand in appendix A.

2 Flavored $\mathcal{N} = 1$ string duals from D6-branes

In this section, we will review the source-free string duals in their ten and eleven-dimensional formulations. Subsequently we will be turning to the issue of adding sources to the type IIA background. Let us once more emphasize that the particular choices of eleven-dimensional geometry (and its dimensional reduction) are of no direct consequence for our results concerning the M-theory lift of smeared D6-branes. The concrete geometry presented here is chosen due to its relevance to the flavoring problem in type IIA.

2.1 The eleven-dimensional dual without sources

Building on the work of Brandhuber [43] (see also [44, 45]) we consider the purely gravitational M-theory background given by the elfbein

$$
\tilde{e}^\mu = dx^{\mu} \quad \tilde{e}^\rho = E(\rho) d\rho \\
\tilde{e}^{1,2} = A(\rho) \sigma_{1,2} \quad \tilde{e}^{3,4} = C(\rho) [\Sigma_{1,2} - f(\rho) \sigma_{1,2}] \\
\tilde{e}^5 = B(\rho) \sigma_3 \quad \tilde{e}^6 = D(\rho) [\Sigma_3 - g(\rho) \sigma_3]$$

(2.1)

$\sigma_i, \Sigma_i$ are left-invariant Maurer-Cartan forms which we chose to be

$$
\sigma_1 = \cos \psi d\theta + \sin \psi \sin \theta d\phi \quad \Sigma_1 = \cos \tilde{\psi} d\tilde{\theta} + \sin \tilde{\psi} \sin \tilde{\theta} d\tilde{\phi} \\
\sigma_2 = -\sin \psi d\theta + \cos \psi \sin \theta d\phi \quad \Sigma_2 = -\sin \tilde{\psi} d\tilde{\theta} + \cos \tilde{\psi} \sin \tilde{\theta} d\tilde{\phi} \\
\sigma_3 = d\psi + \cos \theta d\phi \quad \Sigma_3 = d\tilde{\psi} + \cos \tilde{\theta}
$$

(2.2)

The solutions we are interested in are 1/8-BPS; therefore one can impose the following constraints onto the SUSY spinor $\tilde{\epsilon}$:

$$
\Gamma^{1234} \tilde{\epsilon} = \tilde{\epsilon} \quad \Gamma^{1356} \tilde{\epsilon} = -\tilde{\epsilon} \quad \Gamma^{0126} \tilde{\epsilon} = -\tilde{\epsilon}
$$

(2.3)

As a direct consequence we can calculate the following spinor bilinear, which turns out to be the $G_2$-structure form

$$
\tilde{G}_2 = (\tilde{e} \Gamma_{A_0 A_1 A_2} \tilde{\epsilon}) \tilde{\epsilon}^{A_0 A_1 A_2} = \tilde{\epsilon}^{13} + \tilde{\epsilon}^{024} + \tilde{\epsilon}^{056} + \tilde{\epsilon}^{146} + \tilde{\epsilon}^{345} - \tilde{\epsilon}^{125} - \tilde{\epsilon}^{236}
$$

(2.4)
In the absence of four-form flux the preservation of four supercharges is equivalent to the manifold being of $G_2$-holonomy. A necessary and sufficient condition is the closure and co-closure of the $G_2$-structure form. By imposing $d\tilde{\phi}_{G_2} = 0$ and $d(\tau_GG_2) = 0$ we obtain the BPS equations

\[
A' = \frac{E[BD(g - f^2) + ACf(1 - g)]}{2AB}, \quad B' = \frac{ECf(1 - g)}{A}, \quad C' = \frac{E[ABD - C^3f(1 - g)]}{2ABC}, \quad f = \frac{BC}{2AD}, \quad g = 1 - 2f^2.
\]

The same BPS system follows from demanding that $\delta\tilde{\phi}_M = 0$. The best known solution to (2.5) is the Bryant-Salamon metric [46]. With

\[
A^2 = B^2 = \frac{\rho^2}{12}, \quad C^2 = D^2 = \frac{\rho^2}{9}(1 - \frac{\rho_0^3}{\rho^3}), \quad E^2 = (1 - \frac{\rho_0^3}{\rho^3})^{-1}, \quad f = g = \frac{1}{2}.
\]

the metric takes the form

\[
ds^2 = dx_{1,3}^2 + (1 - \frac{\rho_0^3}{\rho^3})^{-1}d\rho^2 + \frac{\rho^2}{12}\sigma^2 + \frac{\rho^2}{9}(1 - \frac{\rho_0^3}{\rho^3})(\Sigma - \frac{1}{2}\sigma)^2.
\]

The seven-dimensional $G_2$ cone actually turns out to be the cotangent bundle $T^*S^3$. The geometry is that of a cone over $S^3 \times S^3$, with each sphere being parameterized by a set of Maurer-Cartan forms. At $\rho = \rho_0$, the minimum of the radial parameter, one of the spheres ($\Sigma$) collapses, while the other ($\sigma$) remains of finite size. M-theory dynamics on this type of manifold were discussed in [37]. Fluctuations in $\rho_0$ and the gauge potential $A_3$ can be combined into a complex parameter. However, as these fluctuations turn out to be non-normalizable, they do not parameterize a moduli space of vacua, yet rather a moduli space of theories.

There are three $U(1)$ isometries in (2.1) given by $\partial_\rho$, $\partial_\phi$ and $\partial_\phi + \partial_\sigma$, and there are therefore three different dimensional reductions to type IIA. In each case one obtains a conifold geometry with flux, with the conifold singularity being resolved by a deformation or resolution. I.e. there is a cone over $S^3 \times S^3$ and one of the spheres vanishes at at the minimal radius while the other remains of finite size. Furthermore, if we choose to reduce along an isometry embedded in the vanishing sphere, we need to recall that the vanishing of the M-theory circle indicates the presence of D6-branes. Thus the reduction along $\partial_\phi$ yields a deformed conifold with a D6-brane at $\rho = \rho_0$ extending along the Minkowski directions and wrapping the non-vanishing $S^3$. If one mods out the $U(1)$ by $\mathbb{Z}_{N_c}$ before reducing, the corresponding geometry is that of $N_c$ branes. The other two reductions include non-singular $U(1)$'s, so we end up with resolved conifolds. As the M-theory circle is non-singular, there is no D6-brane. There is $F_2$ flux though on the finite-size two-sphere. The different geometries are related by a flop transition between the resolved conifolds and the conifold transition between the deformed and the resolved ones.

In the context of gauge/string duality, the deformed conifold corresponds to the weak ‘t Hooft coupling regime, while the resolved one is to be considered for large ‘t Hooft coupling. Thus the latter provides the appropriate supergravity dual. M-theory realizes the conifold dualities via the aforementioned moduli space of solutions. See [32, 33, 37].
Scherk-Schwarz gauge In what follows we will study the reduction along \( \partial_\psi + \partial_\tilde{\psi} \). In the context of the flavoring problem of section 2.2 one expects the system to be best described by one of the resolved conifold geometries with additional flavor branes. Therefore, out of the three isometries discussed \( \partial_\phi \) and \( \partial_\psi + \partial_\tilde{\psi} \) are the obvious choices. We selected the latter as it leads to simpler equations in type IIA. The choice made here does affect the flavoring problem, yet not our results on the M-theory lift. As we are interested in the reduction of tangent-space quantities, we need to transform the elfbein to Scherk-Schwarz gauge

\[
\hat{e}^A_M = \begin{pmatrix} e^{-\frac{1}{2} \Phi} e^\mu A_\mu \\ 0 \\ e^{\frac{1}{2} \Phi} \end{pmatrix} \quad \hat{E}^A_M = \begin{pmatrix} e^{\frac{1}{2} \Phi} E_\mu \\ 0 \\ -e^{-\frac{1}{2} \Phi} \end{pmatrix}
\]

To obtain the gauge (2.8) from (2.1), we perform the following gauge transformation:

\[
\Lambda = \Lambda^{(3)} \Lambda^{(2)} \Lambda^{(1)}
\]

with the individual transformations \( \Lambda^{(1)}, \Lambda^{(2)}, \Lambda^{(3)} \) being

\[
\Lambda^{(1)} = \begin{pmatrix} I_{9 \times 9} \\ \frac{\cos \alpha}{\sin \alpha} - \frac{\sin \alpha}{\cos \alpha} \end{pmatrix}
\]

\[
\Lambda^{(2)} = \begin{pmatrix} I_{5 \times 5} \\ \frac{\cos \varphi_+}{\sin \varphi_+} - \frac{\sin \varphi_+}{\cos \varphi_+} \end{pmatrix}
\]

\[
\Lambda^{(3)} = \begin{pmatrix} I_{6 \times 6} \\ \frac{\cos \alpha}{0} - \frac{\sin \alpha}{0} \end{pmatrix}
\]

and all other entries zero. Here we defined

\[
\cos \alpha(\rho) = \frac{D(1-g)}{\sqrt{B^2 + (1-g)^2 D^2}} \\
\sin \alpha(\rho) = \frac{B}{\sqrt{B^2 + (1-g)^2 D^2}}
\]

In principle one needs only \( \Lambda^{(1)} \) and \( \Lambda^{(2)} \) to obtain Scherk-Schwarz gauge; yet without \( \Lambda^{(3)} \) the new projections satisfied by the SUSY spinor would be linear combinations of the old ones (2.3) with coefficients \( \cos \alpha, \sin \alpha \). As it is, the form of the SUSY projections remains invariant under \( \Lambda \). I.e.

\[
\hat{\Gamma}^{1234} \hat{\epsilon} = \hat{\epsilon} \quad \hat{\Gamma}^{1356} \hat{\epsilon} = -\hat{\epsilon} \quad \hat{\Gamma}^{0126} \hat{\epsilon} = -\hat{\epsilon}
\]

Thus the \( G_2 \)-structure (2.3) remains formally the same, with the vielbeins \( \check{e}^A \) now replaced by \( \hat{e}^A \). A disadvantage of the reducible gauge is that the new vielbein is rather complicated.
Dimensional reduction and type IIA string theory

The resulting type IIA vielbein is given by

\[ e^\mu = e^{13} dx^\mu \] (2.12a)

\[ e^\rho = e^{13} E d\rho \] (2.12b)

\[ e^1 = e^{13} A (\cos \frac{\psi}{2} d\theta + \sin \theta \sin \frac{\psi}{2} d\phi) \] (2.12c)

\[ e^2 = e^{13} A \cos \alpha (\cos \frac{\psi}{2} \sin \theta d\phi - \sin \frac{\psi}{2} d\theta) \]
\[ + e^{13} C \sin \alpha \left[ \cos \frac{\psi}{2} (\sin \theta d\phi - f \sin \theta d\phi) + \sin \frac{\psi}{2} (d\theta + f d\theta) \right] \] (2.12d)

\[ e^3 = e^{13} C \left[ \cos \frac{\psi}{2} (d\theta - f d\phi) - \sin \frac{\psi}{2} (f \sin \theta d\phi + \sin \theta d\phi) \right] \] (2.12e)

\[ e^4 = -e^{13} A \sin \alpha (\cos \frac{\psi}{2} \sin \theta d\phi - \sin \frac{\psi}{2} d\theta) \]
\[ + e^{13} C \cos \alpha \left[ \cos \frac{\psi}{2} (\sin \theta d\phi - f \sin \theta d\phi) + \sin \frac{\psi}{2} (d\theta + f d\theta) \right] \] (2.12f)

\[ e^5 = e^{13} D \sin \alpha (\cos \theta d\phi - \cos \theta d\phi + d\psi) \] (2.12g)

While the dilaton and gauge potential are

\[ e^{13} = \frac{B}{2 \sin \alpha} = \frac{D(1 - g)}{2 \cos \alpha} \]

\[ A_{(1)} = \cos \theta d\phi + \cos \tilde{\theta} d\phi + \frac{B^2 - D^2(1 - g^2)}{B^2 + (1 - g)^2 D^2} (\cos \theta d\phi - \cos \tilde{\theta} d\phi + d\psi) \]
\[ = \cos \theta d\phi + \cos \tilde{\theta} d\phi + (\sin^2 \alpha - \frac{1 + g}{1 - g} (\cos^2 \alpha) (\cos \theta d\phi - \cos \tilde{\theta} d\phi + d\psi) \] (2.13)

Using \( \hat{\Gamma}^{10} = \Gamma^{11} \), the reduction of the SUSY projections takes a more pleasing form:

\[ \Gamma^{1234} \epsilon = \epsilon \quad \Gamma^{135} \Gamma^{11} \epsilon = -\epsilon \quad \Gamma^{12} \Gamma^{11} \epsilon = -\epsilon \] (2.14)

This allows us to calculate the generalized calibration form for D6-branes in this background.

\[ \phi_{D6} = (\epsilon G_{a0...a6} \epsilon) e^{a0...a6} = e^{125} x^1 x^2 x^3 \wedge (\epsilon^{125} - \epsilon^{345} - \epsilon^{24} - \epsilon^{213}) \] (2.15)

Note that the internal three-form part of this is up to some overall dilaton factor identical to that part of the \( G_2 \)-structure independent of \( e^6 \).

\( G \)-structures

In terms of \( G \)-structures the situation in type IIA is the following. Because we preserve four supercharges, we expect space-time to carry an \( SU(3) \)-structure. As it was shown in [47], it can be directly derived from the \( G_2 \)-structure of the KK-lift. Centerpiece of that reduction are the relations

\[ J = (\hat{\phi}_{G_2})_{a0...a6} e^{ab} \]
\[ \Psi = (\hat{\phi}_{G_2})_{abc} e^{abc} \] (2.16)
For the six-dimensional internal manifold, $J$ defines an almost complex structure, with respect to which we can define from $\Psi$ a $(3,0)$-form $\Omega$ as

$$\Omega = \Psi - i \ast_6 \Psi$$  \hspace{1cm} (2.17)

These satisfy the equations

$$J \wedge \Omega = 0$$

$$J \wedge J \wedge J = \frac{3i}{4} \Omega \wedge \bar{\Omega}$$  \hspace{1cm} (2.18)

In the case at hand we have

$$J = e^{\rho_5} + e^{14} - e^{23}$$

$$\Psi = e^{\rho_{13}} + e^{\rho_{24}} + e^{345} - e^{125}$$  \hspace{1cm} (2.19)

which gives

$$\Omega = \Psi - i \ast_6 \Psi = (e^\phi + ie^5) \wedge (e^1 + ie^4) \wedge (e^3 + ie^2)$$  \hspace{1cm} (2.20)

Thinking about lifting from ten to eleven dimensions, we can invert equations (2.16) to express the eleven-dimensional $G_2$-structure in terms of the ten-dimensional quantities:

$$\hat{\phi}_{G_2} = e^{-\Phi} \Psi + e^{-\frac{4}{3} \Phi} J \wedge \bar{e}^6$$

$$*_{7} \hat{\phi}_{G_2} = -\frac{1}{2} e^{-\frac{4}{3} \Phi} J \wedge J + e^{-\Phi} (\ast_6 \Psi) \wedge e^6$$  \hspace{1cm} (2.21)

As previously stated, Ricci flatness, preservation of four supercharges and absence of four-form flux in eleven dimensions guarantee the $G_2$-holonomy of the internal manifold. This translates in the closure and co-closure of $\hat{\phi}_{G_2}$. As the fibration of the M-theory circle over the ten-dimensional base is non-trivial, one obtains non-vanishing two-form flux upon reduction to type IIA. Hence the internal six-dimensional manifold does not have $SU(3)$-holonomy due to its intrinsic torsion. This means that the forms $J$ and $\Omega$ are not both closed. The relation they will obey can be derived from the closure and co-closure of $\hat{\phi}_{G_2}$ thanks to (2.21)

$$d \hat{\phi}_{G_2} = d(e^{-\Phi} \Psi) + dJ \wedge (A_{(1)} + d\psi^+) + J \wedge dA_{(1)} = 0$$

$$d *_{7} \hat{\phi}_{G_2} = -\frac{1}{2} d(e^{-4\Phi/3} J \wedge J) + d(e^{-\Phi/3} \ast_6 \Psi) \wedge (A_{(1)} + d\psi^+)$$

$$- e^{-\Phi/3} (\ast_6 \Psi) \wedge dA_{(1)} = 0$$  \hspace{1cm} (2.22)

We know that none of the type IIA quantities depends on $\psi^+$. Hence, the contribution to the previous equations coming from $d\psi^+$ must cancel by itself. It gives

$$0 = dJ$$

$$0 = d(e^{-\Phi/3} \ast_6 \Psi)$$

$$0 = d(e^{-\Phi} \Psi) + J \wedge dA_{(1)}$$

$$0 = -\frac{1}{2} d(e^{-4\Phi/3} J \wedge J) - e^{-\Phi/3} (\ast_6 \Psi) \wedge dA_{(1)}$$  \hspace{1cm} (2.23)
These equations can be rephrased (following [47] for example) as
\[ \begin{align*}
\text{d} J &= 0 \\
\text{d} \Phi &= \frac{3}{4} e^{\Phi} \text{d} A_{(1)} \wedge (\ast_6 \Psi) \\
J \ast \text{d} A_{(1)} &= 0
\end{align*} \]  \hspace{1cm} (2.24)
where
\[ G_{(p)} \ast H_{(p+q)} = \frac{1}{p!} G^{\mu_1 \ldots \mu_p} H_{\mu_1 \ldots \mu_p \mu_{p+1} \ldots \mu_{p+q}} \text{d} x^{\mu_{p+1}} \wedge \ldots \wedge \text{d} x^{\mu_{p+q}} \]  \hspace{1cm} (2.25)

We described in this section the construction of a type IIA background from the reduction of eleven-dimensional supergravity. We also derived the equations imposed on the structure by supersymmetry. Now we turn to the problem of adding backreacting flavors in this ten-dimensional context.

### 2.2 Smeared sources in type IIA supergravity

#### 2.2.1 The source-modified first-order system

Applying the method developed in [41], we are now addressing the problem of flavoring the type IIA background obtained in the previous section. It means that we are looking for a solution to the following action, describing the back-reaction of smeared D6-brane sources in a type IIA background:
\[ S = S_{\text{IIA}} - T_6 \int \left( e^{-\Phi} \phi_{D6} - A_{(7)} \right) \wedge \Xi_{(3)} \]  \hspace{1cm} (2.26)

where \( S_{\text{IIA}} \) is the type IIA supergravity action, \( \phi_{D6} \) is the calibration form corresponding to supersymmetric D6-branes, \( A_{(7)} \) is the seven-form potential and \( \Xi_{(3)} \) is the smearing form, representing the distribution of sources. The sources in (2.26) modify the standard type IIA equations of motion and Bianchi identities to
\[ \begin{align*}
\text{d} F_{(2)} &= -(2\kappa_{10}^2 T_6) \Xi_{(3)} \\
0 &= \ast_{10} F_{(2)} \\
0 &= \frac{1}{\sqrt{-g}} \partial_\kappa (\sqrt{-g} \kappa^\lambda e^{-2\Phi} \partial_\lambda \Phi) - \frac{3}{8} F^2 - \frac{3}{4} e^{-\Phi} \Xi_{(6)} (\ast_{10} \phi_{D6}) \\
R_{\mu \nu} &= -2 \nabla_\mu \nabla_\nu \Phi + \frac{e^{2\Phi}}{2} (F_{\mu \kappa} F_{\nu}^\kappa - \frac{1}{4} g_{\mu \nu} F_{(2)}^2) \\
\hspace{2cm} &+ \frac{e^{\Phi}}{4} \left( (\ast_{10} \phi_{D6})_\mu \kappa^\lambda \Xi_{(6) \kappa \lambda} - g_{\mu \nu} \Xi_{(6)} (\ast_{10} \phi_{D6}) \right)
\end{align*} \]  \hspace{1cm} (2.27)

Fortunately, the flavoring procedure does not require us to explicitly solve the complete second-order system. Due to the standard integrability arguments ([15, 19]), it is sufficient to satisfy the Bianchi identities along with the first-order BPS equations\(^{4}\). However, in section [3] we will show how to derive the second-order system directly from M-theory.

\(^{4}\text{Technically there are further mild assumptions to satisfy. I.e. one needs the (0, \mu) components of the Einstein equation to vanish explicitly.}\)
The metric ansatz is given by the vielbein (2.12) and the dilaton is assumed to depend only on the radial coordinate \( \rho \). The calibration associated with \( \kappa \)-symmetric D6-branes is given by (2.15) which is

\[
\phi_{D6} = e^{\phi x^1 x^2 x^3} \land \Psi
\]  

Supersymmetry requires the two-form flux to obey the generalized calibration condition

\[
*_{10} d(e^{-\Phi} \phi_{D6}) = F_{(2)}
\]  

This tells us that the most general ansatz for \( F_{(2)} \) is

\[
F_{(2)} = e^{-4\Phi/3} (F_{\rho 5}(\rho)e^{\phi} + F_{12}(\rho)e^{12} + F_{14}(\rho)e^{14} + F_{23}(\rho)e^{23} + F_{34}(\rho)e^{34})
\]  

The conditions given by supersymmetry on this \( SU(3) \) geometry with intrinsic torsion are still given by (see end of section 2.1)

\[
dJ = 0
\]

\[
d\Phi = \frac{3}{4} e^{\Phi} F_{(2)} \land (*_{6} \Psi)
\]

\[
J \land F_{(2)} = 0
\]  

As mentioned before, the modified equations of motion relate the smearing form to the two-form flux.

\[
dF_{(2)} = -2\kappa_{10}^{2} T_{6} \Xi
\]  

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This equation, combined with (2.30) and (2.32), tells us that the most general ansatz for \( \Xi \) is
\[
\Xi = e^{-5\Phi/3} \left( \Xi_1(\rho)e^{\rho^{14}} + \Xi_2(\rho)(e^{\rho^{23}} + e^{\rho^{14}}) + \Xi_3(\rho)e^{\rho^{12}} + \Xi_4(\rho)(e^{135} + e^{245}) \right)
\]
with
\[
\Xi_3 = -\Xi_1 - 2\Xi_2/\tan \alpha \\
\Xi_4 = \frac{F_{34}}{2\kappa_1^2 T_6 D \sin^2 \alpha}
\]
and the additional conditions
\[
F'_{23} = E \left( -\frac{F_{34}}{D \tan \alpha} - \frac{DF_{23} \cos^2 \alpha + DF_{34} \cos \alpha \sin \alpha}{A^2} - \frac{D^2 \cos \alpha \sin \alpha}{A^2 C^2} + \frac{2F_{23}^2}{\tan \alpha} + \frac{DF_{23} \cos(2\alpha)}{C^2 \sin^2 \alpha} + \frac{D^2 \cos \alpha}{C^4 \sin \alpha} + 3F_{34} F_{23} - 2\kappa_1^2 T_6 \Xi_3 \right)
\]
\[
F'_{34} = E \left( \frac{F_{34}}{D \tan \alpha} - \frac{DF_{34} \cos(2\alpha)}{2A^2} + \frac{DF_{34} \cos(2\alpha)}{2C^2 \sin^2 \alpha} + \frac{F_{34} F_{23}}{\tan \alpha} + \frac{2F_{23}^2}{\tan \alpha} - 2\kappa_1^2 T_6 \Xi_1 \right)
\]
One can verify explicitly that any solution to equations (2.32) and (2.36) automatically verifies the source-modified equations of motion (2.27).

As we want to interpret the two-form flux \( F_{(2)} \) as created by brane sources, we need the flux to be quantized, obeying
\[
\int_S S F_{(2)} = 2\pi N_c.
\]
\( S \) is a suitable two-cycle surrounding the branes in the transverse, three-dimensional space. This adds constraints on \( \Xi \) and \( F_{(2)} \):
\[
\Xi_1 = \Xi_2 \tan \alpha \\
F_{23} = -\frac{A^2 D + C^4 F_{34} \sin^2 \alpha + C^2 (2N_c e^{\Phi} \sin \alpha \tan \alpha + D \sin^2 \alpha + A^2 F_{34})}{(A^2 C^2 + C^4 \sin^2 \alpha) \tan \alpha}
\]
that are compatible with the equation (2.36).

### 2.3 Finding a solution

In this section, we present an analytic solution to the previous system of first-order equations. We will notice that this solution corresponds to the addition of sources in the singular conifold. First, we can directly solve one of the equations in (2.32):
\[
D = e^\Phi e^{5\Phi/3} \frac{N_c C^2 \sin \alpha \tan \alpha}{A^2}
\]
Let us now specialize to the case \( \Xi_2 = 0 \). We see that this reduces the freedom of the smearing form to
\[
\Xi_{(3)} = \frac{e^{-5\Phi/3} F_{34}}{2\kappa_1^2 T_6 D \sin^2 \alpha} (e^{135} + e^{245})
\]
The branes smeared with this particular form would correspond to branes extended in the radial direction \( \rho \) in a trivial way. For a discussion of \( \kappa \)-symmetric brane embeddings in this geometry, see appendix A. This simplification enables
us to solve the equation for the last unknown component of the two-form flux $F_{(2)}$:

$$F_{34} = e^{\frac{2}{3} \Phi} \frac{N_f \sin \alpha}{AC}$$  \hspace{1cm} (2.40)

where $N_f$ is a constant of integration related to the number of flavors in the dual field theory. We now suppose that the two-form flux is independent of the radial coordinate $\rho$, a property verified in other examples of string duals. This imposes that

$$A^2 = C^2 \sin^2 \alpha$$  \hspace{1cm} (2.41)

Finally, we assume $f$ to be constant. A look at the original metric (2.12) tells us that $f$ parameterizes the fibration between the two spheres – this becomes rather more obvious in (2.1). Thus if $f$ is independent of $\rho$, the fibration does not change if we flow along the radial direction. Then we can solve the full BPS system analytically, and we find:

$$D^2 = e^{\frac{2}{3} \Phi} N_c$$

$$A^2 = e^{\frac{2}{3} \Phi} \frac{4N_c^2 (1 - f^2)^2}{3f^2}$$

$$C^2 = e^{\frac{2}{3} \Phi} \frac{4N_c^2 (1 - f^2)}{3f^2}$$

$$E^2 = \frac{16N_c^2 (1 - f^2)^2}{f^2} [(e^{\frac{2}{3} \Phi})']^2$$

$$\cos \alpha = f$$

$$N_f = \pm \frac{N_c (4f^2 - 1)}{3f}$$

where $0 < f < 1$. The two-form flux is

$$F_{(2)} = - N_c \left( \sin \theta d\theta \wedge d\phi + \sin \tilde{\theta} d\tilde{\theta} \wedge d\tilde{\phi} \right) + N_f \sin \psi \left( d\theta \wedge d\tilde{\theta} + \sin \theta \sin \tilde{\theta} d\phi \wedge d\tilde{\phi} \right) + N_f \cos \psi \left( \sin \tilde{\theta} d\tilde{\theta} \wedge d\tilde{\phi} + \sin \theta d\theta \wedge d\phi \right)$$  \hspace{1cm} (2.43)

At this point we notice that we can write the metric explicitly as a cone upon redefinition of the radial coordinate. We take

$$r = \frac{4N_c (1 - f^2)}{f} e^{2\Phi/3}$$  \hspace{1cm} (2.44)

then $dr^2 = E^2 d\rho^2$ and the metric is

$$ds^2_{IIB} = e^{2\Phi/3} \left( dx_{1,3}^2 + dr^2 + r^2 d\Omega^2_{int} \right)$$  \hspace{1cm} (2.45)

where

$$d\Omega^2_{int} = \frac{1}{12} (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{1}{12(1 - f^2)} [(\omega_1 - f d\theta)^2 + (\omega_2 - f \sin \theta d\phi)^2] + \frac{1}{16(1 - f^2)} (\omega_3 - \cos \theta d\phi)^2$$  \hspace{1cm} (2.46)
We can first notice that taking the limit \( N_f \to 0 \) for this solution gives the singular conifold. It indeed corresponds to taking \( f \to \frac{1}{2} \), giving

\[
\text{ds}^2_{N_f \to 0} = \frac{r}{6N_c} \left( \text{d}x_{1,3}^2 + \text{d}r^2 + \frac{r^2}{12} (\text{d} \theta^2 + \sin^2 \theta \text{d} \phi^2) \right) \\
+ \frac{r^2}{9} \left[ (\omega_1 - \frac{1}{2} \text{d} \theta)^2 + (\omega_2 - \frac{1}{2} \sin \theta \text{d} \phi)^2 \right] + \frac{r^2}{12} (\omega_3 - \cos \theta \text{d} \phi)^2
\]

Secondly, we have quantization of the two-form color flux, which is necessary for the gauge/string duality. The interpretation of the additional flavor terms to the flux is not clear. A look at the solution and appendix A prompts us to suspect that the interpretation of the sources as being due to flavor branes is more straightforward if one reduces along \( \partial \phi \). It should be interesting to consider the solution at hand in the context of conifold transitions though. Of course, this is just one solution of the BPS equations of this particular dimensional reduction. Other solutions might also present interesting properties. In any way, we leave the study and interpretation of flavored solutions to future work, and turn in the following back to the problem of the M-theory lift.

3 Back to M-theory

Having studied the flavoring problem of D6-branes in the background (2.12) in the previous section, we have sufficient intuition to turn back towards the more general case of smeared D6 sources in M-theory. The discussion here is fairly generic and requires only the presence of the various \( G \)-structures as well as the overall topology \( \mathbb{R}^{1,3} \times \mathcal{M} \).

3.1 Lifting the SUSY variations

3.1.1 The \( G_2 \)-structure

Our considerations in the introduction about the loss of Ricci flatness prompted us to consider the appearance of intrinsic torsion. So we will begin our attempt at finding a candidate M-theory lift with magnetic \( A_{(1)} \) sources by studying the ten and eleven-dimensional \( G \)-structures. Originally we were dealing with a \( G_2 \)-holonomy manifold in eleven dimensions. Then we reduced it to an \( SU(3) \)-structure in ten dimensions, following the equations (2.11). After this we flavored the theory, which changed the structure equations in ten dimensions (2.23) by replacing \( dA_{(1)} \) by \( F_{(2)} \). However, after adding sources in type IIA supergravity, we have \( F_{(2)} \neq dA_{(1)} \). So, if we try to lift back to eleven dimensions, we start from

\[
\begin{align*}
0 &= \text{d}J \\
0 &= \text{d}(e^{-\Phi/3} *_6 \Psi) \\
0 &= \text{d}(e^{-\Phi} \Psi) + J \wedge F_{(2)} \\
0 &= -\frac{1}{2} \text{d}(e^{-\Phi/3} J \wedge J) - e^{-\Phi/3}(*_6 \Psi) \wedge F_{(2)}
\end{align*}
\]
When we then look at the $G_2$-structure we find, combining (2.22) with the above,

\[
\begin{align*}
\mathrm{d}\hat{\phi}_{G_2} &= -J \wedge (F(2) - \mathrm{d}A(1)) \\
\mathrm{d} \ast_7 \hat{\phi}_{G_2} &= e^{-\Phi/3} (*_7 \Phi) \wedge (F(2) - \mathrm{d}A(1))
\end{align*}
\]

(3.2)

So sources in type IIA supergravity translate in eleven dimensions to the loss of $G_2$-holonomy and the appearance of torsion proportional to $F(2) - \mathrm{d}A(1) = B(2)$.  

3.1.2 The SUSY variations

The previous section gave a first confirmation of our suspicion that geometric torsion should allow us to accommodate for the sources in M-theory. This suggests that all geometric quantities such as covariant derivatives and curvature tensors should be replaced by their torsion-modified relatives. Simplest among these is the covariant derivative, which makes an explicit appearance in the eleven-dimensional supergravity variation $\delta \hat{\psi}_M = D_M \hat{\psi}$, which yields the IIA supergravity variations upon KK-reduction. In appendix B we therefore study how this equation and its Kaluza-Klein reduction change upon inclusion of a torsion tensor

\[
\delta \hat{\psi}_M = D_M \hat{\psi} + \frac{1}{4} \hat{\omega}_{MAB} \Gamma^{AB} \hat{\psi} + \frac{1}{4} \hat{\tau}_{MAB} \Gamma^{AB} \hat{\psi} = D^{(\tau)}_M \hat{\psi}
\]

(3.3)

The result is given in (B.18) and we proceed by investigating what constraints we have to impose on $\hat{\tau}_{MAB}$ in order for the lower-dimensional variations to include magnetic sources.

Now from the form of the dilatino variation (Einstein frame),

\[
\delta \lambda = \frac{3}{16} \frac{1}{\sqrt{2}} e^{\Phi/4} (\mathrm{d}A_{bc} + 2e^{-\Phi/4} \hat{\tau}_{bc}) \Gamma^{bc} \hat{\psi} + \frac{\sqrt{2}}{4} (\partial_b \Phi + \frac{3}{2} e^{-\Phi/4} \hat{\tau}_{zb}) \Gamma^b \Gamma^{11} \hat{\psi}
\]

(3.4)

it follows that we have to demand $\hat{\tau}_{zac} = 0$ and $\hat{\tau}_{zbc} = T_b \kappa_{zbc}^2 B_{bc}$, as (3.4)

then takes the form

\[
\delta \lambda = \frac{3}{16} \frac{1}{\sqrt{2}} e^{\Phi/4} F_{bc} \Gamma^{bc} \hat{\psi} + \frac{\sqrt{2}}{4} \partial_b \Phi \Gamma^b \Gamma^{11} \hat{\psi}
\]

(3.5)

with the two-form now no longer closed, $F = \mathrm{d}A + T_b \kappa_{zbc}^2 B_{bc}$.

Substituting $\hat{\tau}_{zac}$ and $\hat{\tau}_{zbc}$ into the gravitino variation of (B.18) we see that if we impose

\[
\hat{\tau}_{zac} = 0 \quad \hat{\tau}_{zbc} = \frac{e^{\Phi/4}}{2} B_{bc} \quad \hat{\tau}_{abc} = \frac{e^{\Phi/2}}{2} A_{\mu} B_{bc} \quad \hat{\tau}_{abz} = -\frac{e^{\Phi/2}}{2} B_{ab}
\]

(3.6)

the gravitino variation turns also to the desired form

\[
\delta \epsilon \psi_\mu = \partial_\mu \epsilon + \frac{1}{4} \omega_{abc} \Gamma^{abc} \epsilon + \frac{1}{64} e^{\Phi/2} A_{\mu} F_{cd} (\eta_{abc} \Gamma^{cd} - 14 \delta_\mu \Gamma^{cd}) \Gamma^{11} \epsilon
\]

(3.7)

\[\text{Of course, once we include the torsion and proceed from } D_M \hat{\psi} \text{ to } D^{(\tau)}_M \hat{\psi}, \text{ it is not certain whether this defines a SUSY variation of a supergravity theory. What we do know for certain however – and will show in the following – is that the naive dimensional reduction of the usual eleven-dimensional SUSY variation does not yield the correct type IIA one and that } \delta \epsilon \psi_\mu \text{ gives a first order differential on the spinor that does reduce to the correct equations. With this in mind, we write } \delta \epsilon \psi_M = D^{(\tau)}_M \hat{\psi}.\]

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Equations (3.5) and (3.7) are important results. If one performs a KK-reduction of the original supergravity variation without torsion, \( \delta \hat{\psi}_M = D_M \hat{\epsilon} \), one obtains supergravity variations including \( dA_1 \), yet not \( B_2 = F_2 - dA_1 \). By adding the torsion term to the eleven-dimensional supergravity variation, we are able to directly derive the ten-dimensional variations with \( F_2 \) instead of \( dA_1 \). Looking back at (3.3) it is fair to say that the spin connection \( \hat{\omega}_{MAB} \) contains \( dA_1 \), while the torsion carries the \( B_2 \) term necessary to complete \( F_2 \). The right-hand side of (3.3) is constituted of two parts. The first two terms are the ones coming from the lift of the IIA part, and are exactly the terms present in eleven-dimensional supergravity. The last term, which is the only one involving the torsion, corresponds to the lift of the contribution of the sources to the ten-dimensional supergravity variations. Thus, it seems that, mimicking what happens in ten dimensions, we are in presence of the usual eleven-dimensional supergravity plus some sources.

Using the torsion-modified covariant derivative for spinors (3.3) we can also define such an operator \( \nabla^{(\tau)} \) for tensors. The relevant connection coefficients \( \Gamma \) are

\[
\Gamma^K_{LM} = \{ K^L_M \} + K^K_{LM} \]

\[
K_{AMB} = \tilde{\tau}_{MAB}
\]

where \( \{ K^L_M \} \) is the Levi-Civita connection. With the help of \( \nabla^{(\tau)} \), we can rewrite equations (3.2) as

\[
\nabla^{(\tau)}_M \hat{\phi}_{G_2} = 0
\]

\[
\nabla^{(\tau)}_M (\ast_7 \hat{\phi}_{G_2}) = 0
\]

One should remember that the original BPS equations could be written geometrically as \( \nabla_M \hat{\phi}_{G_2} = 0 \) and \( \nabla_M \ast_7 \hat{\phi}_{G_2} = 0 \) yet that these ceased to be valid once we include the sources in ten dimensions – as we discussed in section 3.1.1. Equations (3.9) show however that these geometric BPS equations remain formally invariant once we include torsion.

### 3.2 The equations of motion

We shall finally turn to the search for equations of motion in M-theory that reduce to the source-modified second-order equations in type IIA as given in equation (2.27). To find these equations, we actually reverse the integrability argument that allowed us to consider the first instead of the second-order equations in sections 2.2.1 and 2.3.

To get an idea of what we are about to do, let us briefly digress to the simple case without any flavors or sources. The Bianchi identities are the usual ones, the equation of motion is simple Ricci flatness, \( R_{MN} = 0 \), and the \( G_2 \)-structure is closed and co-closed. Thus the latter satisfies \( \nabla_M \hat{\phi}_{G_2} = 0 \). Taking the commutator

\[
0 = [\nabla_K, \nabla_L] \hat{\phi}_{G_2 MNP}
\]

\[
= -\hat{R}^S_{MKL} \hat{\phi}_{G_2 SNP} - \hat{R}^S_{NKL} \hat{\phi}_{G_2 MSP} - \hat{R}^S_{PKL} \hat{\phi}_{G_2 MNS}
\]

(3.10)
Upon contraction of (3.10) with \( \hat{\phi}_{G2} \), we find

\[
0 = 2 \hat{R}_{KL} + \hat{R}_{MNP}(\ast 7 \hat{\phi}_{G2})_{K}^{MNP}
\]

(3.11)

In the absence of torsion, \( \hat{R}_{MNP}(\ast 7 \hat{\phi}_{G2})_{K}^{MNP} = 0 \), due to the well-known symmetries satisfied by the Riemann tensor,

\[
\hat{R}_{K[LMN]} = 0
\]

\[
\hat{R}_{KLMN} = \hat{R}_{MNKL} = -\hat{R}_{MNLK}
\]

(3.12)

Therefore, our space-time is Ricci flat and the equations of motion are satisfied.

After this brief digression, we return to the original problem. Our aim is to find a suitable equation of motion in M-theory, that reduces to (2.27) upon dimensional reduction. For consistency this equation of motion needs to reduce to simple Ricci flatness in the limit where the type IIA source density \( \Xi \) – equivalently the torsion \( \hat{\tau} \) in M-theory – vanishes. In opposite to our considerations in the previous paragraph, the \( G_2 \)-structure does no longer satisfy \( \nabla_{M} \hat{\phi}_{G2} = 0 \), but instead satisfies \( \nabla_{M} \hat{\phi}_{G2} = 0 \). So we can once more consider the commutator of covariant derivatives. The identities of footnote 6 used to derive (3.11) still hold, yet (3.12) do not, and we arrive at the main result of this paper, the M-theory lift of the source-modified equations of motion

\[
0 = 2 \hat{R}_{(\tau)}^{(\tau)} + \hat{R}_{(\tau)MNP}(\ast 7 \hat{\phi}_{G2})_{K}^{MNP}
\]

(3.13)

where \( \hat{R}_{(\tau)} \) is the Riemann (Ricci) tensor in the presence of torsion.

As we have pointed out before, the BPS equations in their geometric form – \( \nabla_{M} \hat{\phi}_{G2} = 0 \) – are equivalent to those obtained from the SUSY spinor \( \hat{\epsilon} \), \( D_{M} \hat{\epsilon} = 0 \). Therefore we could have derived (3.13) also using (3.3). A commutator of covariant derivatives acting on the SUSY spinor yields

\[
0 = \hat{R}_{CDML}^{(\tau)} \hat{\Gamma}_{CD}^{M} \hat{\epsilon}
\]

(3.14)

We then contract with \( \bar{\hat{\epsilon}} \hat{\Gamma}_{K}^{M} \) and make use of the identity

\[
\Gamma^{ABCD} = \Gamma^{ABCD} + \eta^{AB} \Gamma^{CD} - \eta^{CA} \Gamma^{DB} + \eta^{DA} \Gamma^{BC}
\]

\[
- \eta^{AC} \Gamma^{BD} - \eta^{BD} \Gamma^{AC} + \eta^{AB} \eta^{CD} - \eta^{AD} \eta^{BC}
\]

(3.15)

it follows that

\[
0 = 2(\bar{\hat{\epsilon}} \hat{\epsilon}) \hat{R}_{KL}^{(\tau)} + (\bar{\hat{\epsilon}} \hat{\Gamma}_{K}^{MNP} \hat{\epsilon}) \hat{R}_{MNP} + \mathcal{O}(\bar{\hat{\epsilon}} \hat{\epsilon})
\]

(3.16)

The assumptions made about the SUSY spinor \( \hat{\epsilon} \) imply that there is a \( G_2 \) structure that can be expressed as

\[
(\ast \hat{\phi}_{G2})_{KLMP} = (\hat{\epsilon} \hat{\Gamma}_{ABC} \hat{\epsilon})^{ABC}
\]

(3.17)

As one can verify by direct calculation using (3.3), the \( G_2 \)-structure satisfies

\[
\hat{\phi}_{G2}^{klmn} \hat{\phi}_{G2}^{k} = 6 \delta_{l}^{k}
\]

\[
\hat{\phi}_{G2}^{k} \hat{\phi}_{G2}^{lmp} = (\ast 7 \hat{\phi}_{G2})_{mn}^{kl} + \delta_{l}^{k} \delta_{m}^{p} - \delta_{m}^{l} \delta_{n}^{p}
\]

\( k, l, m, n, p \) denote indices of the seven-dimensional internal manifold.
They also imply that all terms of the form $\hat{\Gamma}^{AB}\hat{\epsilon}$ vanish. Hence (3.13) follows from (3.16).

The equations of motion (3.13) can be rewritten in a more typical and enlightening fashion using the Einstein tensor

$$\hat{R}_{KL} - \frac{1}{2}\hat{g}_{KL}\hat{R} = \hat{T}_{KL}$$

(3.18)

where $\hat{T}_{KL}$ is the energy-momentum tensor of the sources. It can be written in terms of the torsion as

$$\hat{T}_{KL} = \nabla_L K^M_{MK} - \nabla_M K^M_{LK} + K^M_{LP} K^P_{MK} - K^M_{MP} K^P_{LK}$$

$$+ \frac{1}{2}(\nabla_L K_{MPN} - \nabla_P K_{MLN} + K_{MLQ} K^Q_{PN} - K_{MPQ} K^Q_{LN})(\ast_7 \hat{\phi}_{G_2})_{K^{MNP}}$$

$$+ \frac{1}{2}\hat{g}_{KL}(\nabla_M K^M_{Q} Q - \nabla_Q K^M_{M} Q + K^M_{MP} K^P_{Q} Q - K^M_{QP} K^P_{M} Q)$$

$$+ \frac{1}{2}\hat{g}_{KL}(\nabla_P K_{MQN} + K_{MPR} K^R_{QN})(\ast_7 \hat{\phi}_{G_2})_{Q^{MNP}}$$

(3.19)

where $K_{MNP}$ is the contorsion tensor (see (3.8)). From (3.18), we can see that the Einstein equation we are proposing contains two terms: on the left-hand side, one has the Einstein tensor one would get from varying the eleven-dimensional supergravity action with no four-form flux; on the right-hand side, one has an energy-momentum tensor that vanishes when the torsion is set to zero. When the torsion vanishes, so does $\hat{T}$ and one recovers the M-theory Einstein equation.

Writing the equation in this form makes very clear the fact that the lift of type IIA supergravity with sources is eleven-dimensional supergravity supplemented by some sources. Unfortunately, we were not able to find an action that would be responsible for this energy-momentum tensor. To summarize, we claim that having sources in ten dimensions corresponds to having an energy-momentum tensor in eleven dimensions, of the form presented above.

To verify our claim, we will now perform the explicit dimensional reduction of (3.13), and show that we recover all the equations of motion of type IIA with sources. The calculations are – as so often in supergravity – straightforward yet tedious. We found [50] quite helpful, yet not essential. The reader not interested in mathematical details might want to skip ahead to the end of this section, where we summarize our findings. Notice that in the following, despite the fact that we dropped the superscript $(\tau)$ for simplicity of notation, all hatted Riemann and Ricci tensors are considered in the presence of torsion.

Let us start with the $zz$-component of (3.11). We find

$$\hat{R}_{zz} = -\frac{2}{3} e^{4\Phi} \frac{1}{\sqrt{-g}} \partial_{\mu}(\sqrt{-g} e^{-2\Phi} \partial^\mu \Phi) + \frac{1}{4} e^{4\Phi} F^2$$

$$+ \frac{1}{2} \hat{g}_{KL}(\nabla_P K_{MQN} + K_{MPR} K^R_{QN})(\ast_7 \hat{\phi}_{G_2})_{Q^{MNP}}$$

(3.20)

from which it follows that

$$0 = 2\hat{R}_{zz} + (\ast_7 \hat{\phi}_{G_2})_{zz}$$

$$= -\frac{4}{3} e^{4\Phi} \frac{1}{\sqrt{-g}} \partial_{\mu}(\sqrt{-g} e^{-2\Phi} \partial^\mu \Phi) + \frac{1}{2} e^{4\Phi} F^2 + \frac{1}{4} e^{3\Phi} (\ast_6 \Psi)_{\omega} dB$$

(3.21)

Let us start with the $zz$-component of (3.11). We find

$$\hat{R}_{zz} = -\frac{2}{3} e^{4\Phi} \frac{1}{\sqrt{-g}} \partial_{\mu}(\sqrt{-g} e^{-2\Phi} \partial^\mu \Phi) + \frac{1}{4} e^{4\Phi} F^2$$

$$+ \frac{1}{2} \hat{g}_{KL}(\nabla_P K_{MQN} + K_{MPR} K^R_{QN})(\ast_7 \hat{\phi}_{G_2})_{Q^{MNP}}$$

(3.20)

from which it follows that

$$0 = 2\hat{R}_{zz} + (\ast_7 \hat{\phi}_{G_2})_{zz}$$

$$= -\frac{4}{3} e^{4\Phi} \frac{1}{\sqrt{-g}} \partial_{\mu}(\sqrt{-g} e^{-2\Phi} \partial^\mu \Phi) + \frac{1}{2} e^{4\Phi} F^2 + \frac{1}{4} e^{3\Phi} (\ast_6 \Psi)_{\omega} dB$$

(3.21)
Here we used that \( *_6 \Psi = - *_{10} \phi_6 \) and \( dB = dF = - \Xi \). And we notice that we find the source-modified ten-dimensional equation of motion for the dilaton as in (2.27).

Now we investigate the \( \mu z \)-component of (3.11). We find

\[
\hat{R}_{\mu z} = - \frac{1}{2} e^{2\Phi} \partial \nu F_{\nu \mu} + A_{\mu} \hat{R}_{zz}
\]

\[
(*_7 \hat{\phi}_{G2})_{\mu}^{SPK} \hat{R}_{SPKz} = - \frac{1}{6} \epsilon_{\mu \nu \rho \sigma} (*_7 \hat{\phi}_{G2})^{abcd}(dB)_{bcd} + A_{\mu} (e^{3\Phi}(*_6 \Psi)_\nu dB)
\]

(3.22)

Now we have

\[
\frac{1}{6} \epsilon_{\mu \nu \rho \sigma} (*_7 \hat{\phi}_{G2})^{abcd}(dB)_{bcd} = \frac{1}{6} (*_6 J)_{\mu \nu \rho \sigma}(dA - F)_{\nu \rho \sigma} \partial \Phi
\]

\[
= \frac{1}{12} \sqrt{-g} (6) \epsilon^{\alpha \beta \mu \nu \rho \sigma} J_{\alpha \beta} (dA - F)_{\nu \rho \sigma} \partial \Phi
\]

\[
= \frac{1}{12} \sqrt{-g} (6) (dx^{\mu} \wedge *_6 J)_{\nu \rho \sigma} (dA - F)_{\nu \rho \sigma} \partial \Phi
\]

\[
= \frac{1}{12} \sqrt{-g} (6) (dx^{\mu} \wedge d(J \wedge B)) = 0
\]

(3.23)

This agrees with the \( \mu z \)-component. Let us finally look at the \( \mu \nu \) component. One might suspect this to be identical to the \( \mu z \)-component. Due to the presence of torsion however, the Ricci tensor is no longer symmetric and one has to check this independently.

Interestingly, the Kaluza-Klein reductions of \( \mu z \) and \( z \nu \) are already different in the torsion-free case. Here the two differ by \( F - dA \) however, which vanishes in source and torsion-free geometries.
Then from previous computation we know that

\[ R_{\mu\nu} = R_{\mu\nu} + 2\nabla_{\mu} \partial_{\nu} \Phi - \frac{e^{2\Phi}}{2} (F_{\mu\rho} (dA)^{\rho}_{\nu} - \frac{1}{4} g_{\mu\nu} F^2) - \frac{1}{2} A_{\nu} \nabla^{\nu} F_{\mu\nu} + A_{\mu} \hat{R}_{\nu\nu} - \frac{e^{-2\Phi}}{2} g_{\mu\nu} \hat{R}_{\mu\nu} \]

(3.27)

and

\[ (\ast \hat{\Phi}_{G2})_{\mu}^{SPK} \hat{R}_{\nu} = A_{\mu} [(\ast \hat{\Phi}_{G2})_{\mu}^{SPK} \hat{R}_{\nu}] + \frac{4}{3} e^{\Phi} (\ast \Phi)_{\mu} e_{cd} B_{\nu cd} \partial_{\nu} \Phi \]

\[ - \frac{1}{6} A_{\nu} e^{2\Phi} e_{abcd} (d \nu)_{abcd} - e^{\Phi} (\ast \Phi)_{\mu} \nabla_{\nu} B_{\nu cd} \]

\[ + \frac{1}{2} e^{2\Phi} e_{abcd} B_{\nu cd} F_{\nu cd} - \frac{1}{2} e^{\Phi} (\ast \Phi)_{\mu} \partial_{\nu} B_{\nu cd} \]

(3.28)

Let us first notice that

\[ (\ast \Phi)_{\mu} (d \nu)_{vd} + \frac{1}{2} (\ast \Phi)_{\mu} (d \nu)_{vd} = \frac{1}{2} (\ast \Phi)_{\mu} (d \nu)_{vd} \]

(3.29)

Then from previous computation we know that

\[ e_{abcd} (d \nu)_{abcd} = 0 \]

(3.30)

Here are formulae that are going to be useful in the following calculations:

\[ (\ast \Phi)_{ab} (\ast \Phi)_{cd} = \eta_{ac} \eta_{bd} - \eta_{ad} \eta_{bc} + J_{ac} J_{db} + J_{ad} J_{bc} \]

\[ \frac{1}{2} (J \wedge J)_{abcd} = J_{ab} J_{cd} + J_{ac} J_{db} + J_{ad} J_{bc} \]

(3.31)

and once again

\[ \partial_{\nu} \Phi = \frac{3}{4} e^{\Phi} (F_{\nu} (\ast \Phi))_{a} = \frac{3}{4} e^{\Phi} F_{\nu a} \]

(3.32)

So

\[ (\ast \hat{\Phi}_{G2})_{\mu} (d \nu)_{abcd} F_{\nu cd} = \frac{1}{2} (J \wedge J)_{abcd} F_{\nu cd} \]

\[ = F_{\nu cd} (J^{ab} J^{cd} + J^{ac} J^{db} + J^{ad} J^{bc}) \]

\[ = 2 J^{ab} F_{\nu cd} \]

(3.33)

because supersymmetry dictates that \( F_{\nu J} = 0 \). And

\[ (\ast \Phi)_{ab} \partial_{c} \Phi = - \frac{3}{4} e^{\Phi} F_{\nu} (\ast \Phi)_{a b c} \]

(3.34)

So if we now put everything together, we get

\[ (\ast \hat{\Phi}_{G2})_{\mu}^{SPK} \hat{R}_{\nu} = A_{\mu} [(\ast \hat{\Phi}_{G2})_{\mu}^{SPK} \hat{R}_{\nu}] + e^{2\Phi} e_{abcd} J_{ab} F_{\nu cd} \]

\[ - \frac{1}{2} e^{\Phi} (\ast \Phi)_{\mu} (d \nu)_{abcd} - e^{2\Phi} B_{\nu cd} (F_{\nu} d + \Phi J_{f j g}) \]

\[ = A_{\mu} [(\ast \hat{\Phi}_{G2})_{\mu}^{SPK} \hat{R}_{\nu}] - \frac{1}{2} e^{\Phi} (\ast \Phi)_{\mu} (d \nu)_{abcd} \]

\[ - e^{2\Phi} F_{\nu} d B_{\nu cd} \]

(3.35)
So looking finally at the whole picture

\[ 0 = 2 \hat{R}_{\mu\nu} + \left( *_{7} \hat{\phi}_{G_{2}} \right)_{\mu} S^{P K} \hat{R}_{P K \nu} \]

\[ = 2 R_{\mu\nu} + 4 \nabla_{\mu} \partial_{\nu} \Phi - e^{2\phi} (F_{\mu\rho} (dA)_{\nu} \rho \rho - \frac{1}{4} g_{\mu\nu} F_{\rho}^{2}) - A_{\mu} \nabla_{\nu} F_{\rho \rho} + e^{2\phi} F_{\mu \rho} d B_{\nu \rho} \]

\[ - e^{-2\phi} g_{\mu\nu} \hat{R}_{zz} + A_{\mu} \left[ \left( *_{7} \hat{\phi}_{G_{2}} \right)_{z} S^{P K} \hat{R}_{P K \nu} \right] - \frac{1}{2} e^{\phi} \left( *_{6} \Psi \right)_{\mu} e_{\nu \rho \sigma} + A_{\mu} 2 \hat{R}_{z\nu} \]

\[ = 2 \hat{R}_{\mu\nu} + 4 \nabla_{\mu} \partial_{\nu} \Phi - e^{2\phi} (F_{\mu\rho} (dA + B)_{\nu} \rho \rho - \frac{1}{4} g_{\mu\nu} F_{\rho}^{2}) - A_{\mu} \nabla_{\nu} F_{\rho \rho} \]

\[ + A_{\mu} [2 \hat{R}_{\nu} z + \left( *_{7} \hat{\phi}_{G_{2}} \right)_{z} S^{P K} \hat{R}_{P K \nu}] - e^{-2\phi} g_{\mu\nu} \hat{R}_{zz} + \frac{1}{2} e^{3\phi} \left( *_{6} \Psi \right)_{\mu} dB \]

\[ - \frac{1}{2} e^{\phi} \left( *_{10} \phi D_{6} \right)_{\mu} e_{\rho \sigma} \Xi_{\rho \sigma} - \frac{1}{2} e^{-2\phi} g_{\mu\nu} e^{3\phi} \left( *_{6} \Psi \right)_{\mu} dB \]

(3.36)

which gives

\[ 0 = 2 \hat{R}_{\mu\nu} + 4 \nabla_{\mu} \partial_{\nu} \Phi - e^{2\phi} (F_{\mu\rho} F_{\nu} \rho \rho - \frac{1}{4} g_{\mu\nu} F_{\rho}^{2}) \]

\[ - \frac{1}{2} e^{\phi} \left( *_{10} \phi D_{6} \right)_{\mu} e_{\rho \sigma} \Xi_{\rho \sigma} - \frac{1}{2} e^{-2\phi} g_{\mu\nu} e^{3\phi} \left( *_{6} \Psi \right)_{\mu} dB \]

\[ + A_{\nu} \nabla_{\rho} F_{\mu \rho} + A_{\mu} [2 \hat{R}_{z\nu} + \left( *_{7} \hat{\phi}_{G_{2}} \right)_{z} S^{P K} \hat{R}_{P K \nu}] \]

\[ - \frac{1}{2} e^{-2\phi} g_{\mu\nu} \hat{R}_{zz} + \frac{1}{2} e^{3\phi} \left( *_{6} \Psi \right)_{\mu} dB \]

(3.37)

where we recognize the first two lines of this equation as being the Einstein equation of type IIA supergravity with sources and the rest vanishes thanks to other components of (3.11). This completes the reduction of eleven-dimensional Einstein equations to the type IIA supergravity equations of motion.

To summarize, in this section we showed that the equation of motion of eleven-dimensional supergravity with torsion (3.13), which is given to us by integrability, reduces to the source-modified type IIA supergravity equations of motion (2.27). It thus shows that adding torsion to eleven-dimensional supergravity reduces to adding smeared D6 sources in type IIA supergravity.

4 Conclusions, future work

In this paper we have been interested in two related issues: the addition of D6-branes as smeared sources to a type IIA background, and the lifting of such a system to eleven-dimensional supergravity. We considered these in the context of 1/8 BPS solutions of the form $R^{1,3} \times M$, a fact represented by the presence of a $G_{2}$ or $SU(3)$-structure.

Concerning the problem of the M-theory lift, we showed that ordinary eleven-dimensional supergravity cannot accommodate for the presence of the additional sources and argued that a possible solution might lie in the inclusion of geometric torsion. While our argument was founded on the observed loss of Ricci flatness in the higher-dimensional theory, we were able to show by explicit calculation that the supersymmetry variations take the required form upon addition of torsion. Moreover, the torsion must take the form (3.6), related to the distribution $\Xi_{(3)}$ of the sources in the reduced theory. Subsequently we derived a set of second order equations that could be the equations of motion of some eleven-dimensional
supergravity with torsion, and proved that they reduce to the type IIA equations of motion with smeared D6-branes. As we pointed out, this work is not in contradiction with the uniqueness of supergravity in eleven dimensions, because we are only considering a theory that preserves four supercharges. We did not of course show that there is a well defined theory in eleven dimensions that is supersymmetric and has the field content of both eleven dimensional supergravity as well as of the additional torsion. One should not forget however, that we are not studying the uplift of $S_{\text{IIA}}$, which is well known, but of

$$S = S_{\text{IIA}} + S_{\text{D6-sources}}$$  \hspace{1cm} (4.1)

The problem was first addressed in [52] whose authors found a seven-dimensional gauged sigma model action that reduces to the DBI-term of the D6 brane. They were unable to find a suitable uplift of the Wess-Zumino term however. While this paper does not solve the problem in the sense of [52], it does succeed in lifting the ten-dimensional equations of motion to pure eleven-dimensional geometry. The question whether the results are just an accidental rewriting of type IIA dynamics in higher-dimensional notation or do actually point to a higher dimensional supersymmetric theory that includes torsion is left for further work.

While there is a long history of the uses of torsion in the context of string theory, the torsion used in papers such as [51] and [42] is related to the presence of fluxes, not of sources. Therefore the addition of further torsion is a rather unorthodox concept. So it is necessary to wonder if we would not have been able to solve the problem at hand with simpler methods. As mentioned before, our argument was based on the loss of Ricci flatness in eleven dimensions. One might guess that it is possible to use the four-form in M-theory, $\hat{F}^{(4)}$, to obtain a suitable energy-momentum tensor to supplement the Einstein equations. This however leads to four and three-form flux in type IIA, in contradiction with our results of section 2. Another possibility would be to use the KK-monopole action of [52]. There the authors constructed a gauged sigma model action (A.1) that is the dimensional uplift of the DBI term of a D6-brane. Using this, one could try to lift the action (1.4) to M-theory. Yet considered in connection with the standard Kaluza-Klein mechanism, (1.4) is an action in terms of $dA^{(1)}$, not $F^{(2)}$. So even if one were able to lift the brane contribution to (1.4), the supergravity part would still be lacking the source contribution. Still, it might be interesting to try to match the sigma model action [52] with the inclusion of torsion.

As we mentioned before, our calculations here depend on several assumptions. Most notable among these are the presence of the $G_2$-structure and the Minkowski directions in the metric. Relaxing both of these would be a very interesting avenue to follow in the future.

One should also be able to extend the considerations of this paper to the case of KK theories with non-abelian field strengths. While this is not directly related to the issue of smeared D6-branes, one should be facing the same difficulties we did; after all the existence of a gauge potential is again implied by the KK-ansatz.

A further extension and application of the results of this paper lies outside of string theory. A close look at the considerations made in section 3 shows that we hardly make any use of string or M-theory. The setup is merely that of a $U(1)$
Kaluza-Klein theory in $d$ and $d + 1$ dimensions with monopole condensation in the lower dimensional theory. Hence the results of this paper may be reexpressed as follows: a monopole condensate in a $d$-dimensional Kaluza-Klein theory might be described as torsion in $d + 1$ dimensions.

The other problem studied in this paper is the construction of a gravity dual to $\mathcal{N} = 1$, $SU(N_c)$ super Yang-Mills with flavors. We addressed this in section 2. Here we found a system of first-order BPS equations that describes the addition of D6 sources to the type IIA background (2.12). At the end of section 2 we presented a family of exact solutions. The detailed study of these, especially concerning the physics of their gauge theory dual, has not been made and could be of interest as future work, as well as finding other solutions.

Discussing the addition of flavors to (2.12), we ignored the complicated issue of conifold transitions. Recall that depending on the value of $N_c g_s^2$ the unflavored system was best described by either D6-branes on the deformed conifold ($N_c g_s^2 \ll 1$) or pure two-form flux on the two-sphere in the resolved conifold ($N_c g_s^2 \gg 1$). It is a priori not clear that this is still the case upon addition of flavors – a problem that could be studied using topological string theory as in [34], type IIA string theory or M-theory ([32]).

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A D6-brane embeddings

We will now discuss D6-brane embeddings in the three type IIA reductions of the Bryant-Salamon metric (2.7). In principle one would have to study each of the three reductions independently, but as we will show now it is actually possible to search for these embeddings directly in M-theory. Strictly speaking we will do nothing but rewriting the calibration condition of type IIA string theory in eleven-dimensional notation. However this turns out to be quite useful, as the M-theory expressions are more compact and less convoluted than their lower-dimensional counterparts.

The starting point is the gauged sigma model action of [52]. Here, the authors constructed an action that is the eleven-dimensional uplift of the DBI action of a D6-brane. In other words, it can be thought of as the world-volume action of a Kaluza-Klein monopole. Let the M-theory circle be described by the Killing vector $K = \partial_z$. Then

$$S_{KK7} = -T_{KK7} \int d^2 \xi K^2 \sqrt{- \det \partial_i X^M \partial_j X^N \Pi_{MN}}$$

$$\Pi_{MN} = g_{MN} - K^{-1} K_M K_N$$

(A.1)

The action is that of a gauged sigma model. $\Pi_M^N$ projects eleven to ten-dimensional vectors. One verifies by explicit calculation that (A.1) reduces to
the DBI action of a D6-brane. We want to use $\Pi_{MN}$ to describe calibrated cycles of D6-branes in type IIA using M-theory notation. Recall that a D6-brane embedding $X^\mu(\xi)$ is supersymmetric if it satisfies the calibration condition

$$X^*\phi_{D6} = \sqrt{-g_{ind.}}$$  \hspace{1cm} (A.2)

Here $(g_{ind.})_{ij} = \partial_i X^\mu \partial_j X^{\nu} g_{\mu\nu}$ is the induced metric and $\phi_{D6}$ the calibration form \[^\text{(2.15)}\]. Defining $\hat{e}_{IIA}^A M = \Pi_{MN} \hat{E}^N A$ we have, using \[^\text{(2.8)}\], $\hat{e}_{IIA}^a = e^{-\Phi/3} e^a$. We can now define the M-theory lift of the type IIA calibration form as

$$\phi_{KK7} = (\hat{e}_{IIA})^{x_0 x_1 x_2 x_3} \wedge [(\hat{e}_{IIA})^{125} - (\hat{e}_{IIA})^{345} - (\hat{e}_{IIA})^{24} - (\hat{e}_{IIA})^{13}]$$  \hspace{1cm} (A.3)

In an abuse of notation, we have labeled this the calibration form of a KK-monopole. Also $\sqrt{-\Pi} = e^{-\Phi/3} \sqrt{-g_{ind.}}$, and we arrive at a lifted form of the calibration condition \[^\text{(A.2)}\],

$$X^*\phi_{KK7} = \sqrt{-\Pi}$$  \hspace{1cm} (A.4)

We will now use \[^\text{(A.4)}\] to study D6-brane embeddings. Recall that there are three $U(1)$ isometries, with three different dimensional reductions

$$\partial_{\psi+\bar{\psi}} \subset \sigma \times \Sigma \hspace{0.5cm} \text{Resolved conifold}$$
$$\partial_{\phi} \subset \sigma \hspace{0.5cm} \text{Resolved conifold}$$
$$\partial_{\bar{\phi}} \subset \Sigma \hspace{0.5cm} \text{Deformed conifold}$$

**Color embeddings** Color embeddings are those which wrap only a compact cycle. In the case at hand they do not extend along the radial direction at all. If we specify to the deformed conifold, that is, we choose the isometry $K = \partial_{\bar{\phi}}$, we find an embedding parameterized by

$$x^\mu = \begin{array}{cccccc}
\rho & \theta & \phi & \psi & \bar{\theta} & \bar{\phi} & \bar{\psi} \\
\rho_0 & \circ & \circ & \circ & . & K & .
\end{array}$$

(A.6)

The embedding exists only at $\rho = \rho_0$ as

$$X^*\phi_{KK7} = -\frac{2\rho^3 + \rho_0^3}{72\sqrt{3}} \sin \theta \sqrt{-\Pi} = -\frac{4\rho^3 + \rho_0^3}{72\sqrt{3}} \sin \theta$$

(A.7)

So we recover the color brane embedding of the string dual we started with. Note that this cycle is calibrated in M-theory though. I.e. it is a minimum volume cycle of the eleven-dimensional geometry.

For the resolved conifold associated with $K = \partial_\phi$ one might try an embedding as

$$x^\mu = \begin{array}{cccccc}
\rho & \theta & \phi & \psi & \bar{\phi} & \bar{\psi} & \bar{\phi} \\
\rho_0 & . & K & . & \circ & \circ & \circ
\end{array}$$

(A.8)

However, the cycle in question vanishes at $\rho = \rho_0$, as one would expect.

---

\[^8\] The notation for these embedding diagrams is as follows: $a -$ signals a non-compact direction along which the brane extends, $a \circ$ a wrapped compact one. $K$ denotes the M-theory circle associated with the Killing vector $K$, $\circ$ finally stands for localized directions.
Massless flavor embeddings  Massless flavor branes extend fully along the radial direction $\rho$. Therefore they only need to wrap a two-cycle in the internal geometry and one can make the following guess

$$x^\mu \rho \theta \phi \psi \tilde{\theta} \tilde{\phi} \tilde{\psi} \quad (A.9)$$

Note that this embedding works for both the $\partial \phi$ and the $\partial \tilde{\phi}$ isometries.

For the deformed conifold, i.e. reduction along $\partial \tilde{\phi}$, we obtain the relation

$$X^* \phi_{KK7} = \frac{\rho^2}{6\sqrt{3}} \sin^2 \tilde{\theta}$$

$$\sqrt{-\Pi} = \frac{\rho^2}{6\sqrt{3}} \sin \tilde{\theta}$$

(A.10)

demanding $\tilde{\theta} = \frac{\pi}{2}$. The resolved conifold associated with $\partial \phi$ gives

$$X^* \phi_{KK7} = \frac{\rho^2}{6\sqrt{3}} \sin^2 \theta$$

$$\sqrt{-\Pi} = \frac{\rho^2}{6\sqrt{3}} \sin \theta$$

(A.11)

demanding $\theta = \frac{\pi}{2}$; whereas for the $\partial \phi + \partial \tilde{\phi}$ reduction both $X^* \phi$ and $\sqrt{-\Pi}$ vanish. Interestingly, in M-theory the cycle $(x^\mu, \rho, \psi, \tilde{\psi})$ is calibrated in the traditional sense; that is, it is a minimal volume cycle.

Massive flavor embeddings  Naturally one would like to relax the constraints on $\theta$ and $\tilde{\theta}$ respectively from the above paragraph. A good guess to do so lies in assuming a relation between $\rho$ and $\theta$ (or $\tilde{\theta}$).

In the case of the $\partial \tilde{\phi}$ reduction, we assume

$$x^\mu = \rho(\tilde{\theta}) \rho \theta \phi \psi \tilde{\theta} \tilde{\phi} \tilde{\psi} \quad (A.12)$$

Then

$$X^* \phi_{KK7} = \frac{\rho^3 - \rho_0^3}{18\sqrt{3}} \cos \tilde{\theta} + 3\rho^2 \rho' \sin \tilde{\theta} \sin \hat{\theta}$$

$$\sqrt{-\Pi} = \frac{\sqrt{(\rho^3 - \rho_0^3)^2 + 9\rho^4 \rho'^2}}{18\sqrt{3}} \sin \hat{\theta}$$

(A.13)

Demanding the two expressions to agree, it follows that

$$\rho'(\tilde{\theta}) = \frac{\rho^3 - \rho_0^3}{3\rho'} \tan \hat{\theta}$$

$$\rho(\tilde{\theta}) = \left(\rho_0^3 + e^{3C_1 \sec \hat{\theta}}\right)^{1/3}$$

(A.14)

with $C_1$ being a constant of integration, associated with the mass of the flavors, as we will show now. $\sec \hat{\theta} \in [1, \infty)$, so the brane reaches down to $(\rho_0^3 + e^{3C_1})^{1/3}$. Thus the massless limit is given by $C_1 \rightarrow -\infty$. In order to compare this embedding with the massless one of the previous paragraph, we have to solve the
embedding equation for $\tilde{\theta}$ before taking this limit – as we expect the brane to be localized in $\tilde{\theta}$, so the mapping $\tilde{\theta} \mapsto \rho$ is ill defined. The result is

$$\tilde{\theta} = \arccos \frac{e^{3C_1}}{\rho^3 - \rho_0^3}$$

(A.15)

So in the limit $C_1 \to -\infty$, the brane sits once more at $\tilde{\theta} = \frac{\pi}{2}$, which is also the position of the brane for $\rho \gg \rho_0$.

For the $\partial_\rho$ reduction, one needs to swap $\theta$ for $\tilde{\theta}$. Then,

$$x^\mu = \rho(\theta) \circ \phi \circ \psi \circ \tilde{\theta} \circ \phi \circ \tilde{\psi}$$

(A.16)

The calibration condition is given by

$$X^* \phi_{KK7} = \frac{(8\rho^6 - 7\rho^3\rho_0^3 - \rho_0^6) \cos \theta + 6\rho^2(4\rho^3 - \rho_0^3)\rho' \sin \theta}{36\sqrt{3}(4\rho^3 - \rho_0^3)} \sin \theta$$

$$\sqrt{-\Pi} = \frac{\sqrt{4\rho^6 - 6\rho^3\rho_0^3 + \rho_0^6 + 36\rho^2(\rho')^2}}{36\sqrt{3}} \sin \theta$$

(A.17)

leading to a differential equation for $\rho$ that is considerably harder to solve than the previous one. One can study it numerically, obtaining results similar to those of the previous embedding. As to analytic results, setting $\rho_0 \to 0$, leads to simplifications allowing for

$$\rho(\theta) = C_1 (\sec \theta)^{1/3}$$

(A.18)

which is identical to (A.15) in the same limit.

**B Kaluza-Klein reduction of supergravity variations with torsion**

We review the dimensional reduction of the SUSY variations – with an additional torsion term – from eleven to ten-dimensional supergravity. Conceptually we follow [3], our conventions are slightly different though. We assume a space-time with topology $M_{10} \times S^1$ and label the eleventh coordinate as $z$. Naturally all fields will be independent of $z$. Further assuming the eleven-dimensional background to be purely gravitational, we only need to consider the variation of the gravitino,

$$\delta_\epsilon \hat{\psi}_M = \partial_M \hat{\epsilon} + \frac{1}{4} \hat{\omega}_{MAB} \Gamma^{AB} \hat{\epsilon} + \frac{1}{4} \hat{\tau}_{MAB} \hat{\Gamma}^{AB} \hat{\epsilon}$$

(B.1)

which we have modified by the presence of the torsion term $\hat{\tau}$. As in section 2.1 we take the vielbein to be in Scherk-Schwarz gauge (2.8).

We shall perform the reduction of (B.1) step by step and our first aim shall be the reduction of the spin connection

$$\hat{\omega}_{ABC} = \frac{1}{2} \left( \hat{\Omega}_{CAB} - \hat{\Omega}_{BAC} - \hat{\Omega}_{ABC} \right)$$

(B.2)
with the objects of anholomorphicity defined as

\[ \hat{\Omega}_{ABC} = (\partial_M e^K_N - \partial_N e^K_M) \hat{\eta}_{KA} \hat{E}^N_B \hat{E}^M_C \]  

(B.3)

Then

\[ \hat{\omega}_{zbc} = + \frac{e^4}{3} \Phi^2 (dA)_{bc} \hat{\omega}_{abc} = \frac{e}{3} \Phi \omega_{abc} \]

(B.4)

\[ \hat{\omega}_{abz} = - \frac{e^4}{3} \Phi^2 (dA)_{ab} \hat{\omega}_{zaz} = \frac{2}{3} e^1 \Phi \partial_a \Phi \]

(B.5)

Note that we use \( dA_{\mu\nu} \) instead of \( F_{\mu\nu} \) as we are anticipating the inclusion of sources such that \( F \) is no longer exact.

Turning to the gravitino, one could make an ansatz

\[ \hat{\psi}_M = (e^{m\Phi} \psi_\mu, e^{n\Phi} \lambda) \]

(B.5)

and

\[ \hat{\epsilon} = e^{\Phi} \epsilon \]

(B.6)

with \( l, m, n \in \mathbb{C} \). Yet, as we will see, we will need to consider linear combinations such as \( \hat{\psi}_\mu = e^{m\Phi} \psi_\mu + e^{n\Phi} \Gamma_\mu \lambda + e^{p\Phi} \Gamma_\mu \Gamma^{11} \lambda \).

We begin with the covariant derivative of the SUSY spinor, looking first at the vector components:

\[ e^{-i\Phi} \hat{D}_\mu \hat{\epsilon} = (i \partial_\mu \Phi \epsilon + \partial_\mu \epsilon) + e^a \left[ \frac{1}{4} \omega_{abc} + \frac{1}{12} (\eta_{ab} \partial_c \Phi - \eta_{ac} \partial_b \Phi) \right] \Gamma^{bc} \epsilon 
- \frac{1}{4} e^a \partial_a dA_{ab} \Gamma^{11} \epsilon + \left( \frac{1}{8} e^2 \Phi A_{abc} \Gamma^{11} + \frac{1}{3} e^\Phi A_{\mu} \partial_b \Phi \Gamma^{bc} \Gamma^{11} \right) \epsilon 
+ \frac{1}{4} \hat{\tau}_{abc} \Gamma^{bc} \epsilon + \frac{1}{2} \hat{\tau}_{bac} \Gamma^{11} \epsilon \]

(B.7)

The scalar component satisfies

\[ e^{-i\Phi} \hat{D}_z \hat{\epsilon} = \frac{e^2 \Phi}{8} dA_{abc} \Gamma^{bc} \epsilon + \frac{e^\Phi}{3} \partial_b \Phi \Gamma^{11} \epsilon + \frac{1}{4} \hat{\tau}_{a} \Gamma^{bc} \epsilon + \frac{1}{2} \hat{\tau}_{zbc} \Gamma^{11} \epsilon \]

(B.8)

Equations (B.7) and (B.8) hold in string frame. To convert to Einstein frame we need to recall that the gamma matrices are defined in tangent space, from which it follows that only the curved space gamma matrices are affected by Weyl transformations. For a generic Weyl transformation, we have

\[
\begin{align*}
&e^\alpha_\mu \mapsto e^{\delta \Phi} e^\alpha_\mu \\
&E^a_\mu \mapsto e^{-\delta \Phi} E^a_\mu \\
&\partial_a \mapsto e^{-\delta \Phi} \partial_a \\
&\Gamma^a \mapsto \Gamma^a \\
&\hat{\tau}_{abc} \mapsto \hat{\tau}_{abc}
\end{align*}
\]

(9.8)

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&\partial_a \mapsto e^{-\delta \Phi} \partial_a \\
&\Gamma^a \mapsto \Gamma^a \\
&\hat{\tau}_{abc} \mapsto \hat{\tau}_{abc}
\end{align*}
\]

(9.8)
So that with \( (e^S)_{\mu} = e^{\frac{1}{2}\Phi}(e^E)_{\mu} \), \( \delta = \frac{1}{2} \),

\[
e^{-i\Phi}\hat{D}_\mu \hat{\epsilon} = (\partial_\mu \Phi + \delta_\mu \epsilon) + c_\mu \left[ \frac{1}{4} \omega_{abc} + \frac{1}{48} (\eta_{ab} \partial_c \Phi - \eta_{ac} \partial_b \Phi) \right] \Gamma^{abc} \epsilon
\]
\[
- \frac{1}{4} e^{\frac{1}{2}\Phi} e_\mu^a dA_{abc} \Gamma^{\mu} \Gamma^{11} \epsilon + \left( \frac{1}{8} e^{\frac{1}{2}\Phi} A_\mu dA_{abc} \Gamma^{abc} + \frac{1}{3} e^{\frac{1}{2}\Phi} A_\mu \partial_b \Phi \Gamma^{\mu} \Gamma^{11} \right) \epsilon
\]
\[
+ \frac{1}{4} \eta_{abc} \Gamma^{abc} \epsilon + \frac{1}{2} \eta_{abc} \Gamma^{11} \epsilon
\]
\[e^{-i\Phi}\hat{D}_z \hat{\epsilon} = \frac{e^{\frac{1}{2}\Phi}}{8} dA_{zbc} \Gamma^{zbc} \epsilon + e^{\frac{1}{2}\Phi} \partial_b \Phi \Gamma^{11} \epsilon + \frac{1}{4} \tau_{zbc} \Gamma^{11} \epsilon + \frac{1}{2} \tau_{zbc} \Gamma^{11} \epsilon
\]  
(B.10)

One needs to compare \((B.7)\) and \((B.8)\) or \((B.10)\) repectively to the SUSY variations of the ansatz \([B.5]\).

\[
\hat{D}_\mu \hat{\epsilon} = \delta_\mu \psi_\mu = e^{m\Phi} (m \delta_\mu \Phi \psi_\mu + \delta_\mu \psi_\mu) = e^{m\Phi} \delta_\mu \psi_\mu
\]

\[
\hat{D}_z \hat{\epsilon} = \delta_\mu \psi_z = e^{m\Phi} (n \delta_\mu \Phi \lambda + \delta_\lambda \lambda) = e^{m\Phi} \delta_\lambda \lambda
\]  
(B.11)

The last equalities follow from the fact that we assume the spinor fields to vanish. However the resulting variations will explicitly depend on the gauge-potential \(A\). We therefore replace the original ansatz \([B.5]\) with

\[
\psi_\mu = \hat{\psi}_\mu = x_2 e^a_\mu \eta_{ab} \Gamma^{11} \hat{\psi}_z = x_3 A_\mu \hat{\psi}_z
\]

\[
\lambda = x_1 \hat{\psi}_z
\]

\[
\hat{\epsilon} = e^{i\Phi} \epsilon
\]  
(B.12)

which amounts to a field redefinition in ten dimensions. If one was to work properly, one had to perform the dimensional reduction of the action as well in order to make sure that the fermion terms have the proper normalizations. The SUSY variations of \((B.12)\) are

\[
\delta_\mu \psi_\mu = \delta_\mu \hat{\psi}_\mu - x_2 e^a_\mu \eta_{ab} \Gamma^{11} \delta_\mu \hat{\psi}_z = x_3 A_\mu \delta_\mu \hat{\psi}_z
\]

\[
= e^{-i\Phi} \hat{D}_\mu \hat{\epsilon} - x_2 e^a_\mu \eta_{ab} \Gamma^{11} (x_3 A_\mu e^{-i\Phi} \hat{D}_z \hat{\epsilon}) + x_3 A_\mu e^{-i\Phi} \hat{D}_z \hat{\epsilon}
\]  
(B.13)

Note that the variations of the bosonic fields all vanish, as we have set the fermions explicitly to zero. Our aim is to compare \((B.13)\) with the IIA Einstein frame SUSY variations as taken from \([33]\)

\[
\delta \lambda = \sqrt{2} \frac{1}{4} \partial_\mu \Phi \Gamma^{11} \epsilon + \frac{3}{16} \frac{1}{\sqrt{2}} e^{\frac{1}{2}\Phi} dA_{\mu_1 \mu_2} \Gamma^{\mu_1 \mu_2} \epsilon
\]  
(B.14a)

\[
\delta \psi_\mu = D_\mu \epsilon + \frac{1}{64} \frac{1}{\sqrt{2}} e^{\frac{1}{2}\Phi} dA_{\mu_1 \mu_2} (\Gamma^{\mu_1 \mu_2} - 14 \delta^{\mu_1}_\mu \Gamma^{\mu_2}) \Gamma^{11} \epsilon
\]  
(B.14b)
Before evaluating (B.13), we calculate

\[ x_2 e^\alpha_{\mu} \eta_{ab} \Gamma^{b c d} \epsilon \]

\[ = x_2 \frac{1}{8} e^{\frac{3}{2} \Phi} e^\alpha_{\mu} \eta_{ab} A_{c d} (\Gamma^{b c d} + 2 \eta^{b c} \Gamma^{d}) \Gamma^{11} \epsilon \]

\[ - x_2 \frac{1}{3} e^{\frac{3}{2} \Phi} e^\alpha_{\mu} \partial_\alpha \Phi - x_2 \frac{1}{6} e^{\frac{3}{2} \Phi} e^\alpha_{\mu} (\eta_{ab} \partial_\alpha \Phi - \eta_{ba} \partial_\beta \Phi) \Gamma^{b c} \epsilon \]

\[ - \frac{1}{2} x_2 e^\alpha_{\mu} \eta_{ab} \Phi_{\alpha \beta} \epsilon + \frac{1}{4} x_2 ^e_{\mu} \Phi_{ab} \Gamma^{b c d} + 2 \delta^e_{a} \Gamma^{d} \Gamma^{11} \epsilon \]

Putting things together, we use equations (B.10) and (B.13)

\[ \delta \epsilon \psi_\mu = (\partial_\mu \Phi + \partial_\mu \epsilon) + e^a_{\mu} \left[ \frac{1}{4} \omega_{abc} + \frac{1}{48} (\eta_{ab} \partial_\epsilon - \eta_{bc} \partial_\epsilon) \right] \Gamma^{b c} \epsilon \]

\[ - \frac{1}{4} e^{\frac{3}{2} \Phi} e^a_{\mu} A_{d a} \Gamma^{11} \epsilon \]

\[ - x_2 \frac{1}{3} e^{\frac{3}{2} \Phi} e^a_{\mu} \eta_{ab} A_{c d} (\Gamma^{b c d} + 2 \eta^{b c} \Gamma^{d}) \Gamma^{11} \epsilon \]

\[ + \frac{1}{2} e^{\frac{3}{2} \Phi} A_{d b} \Gamma^{11} \epsilon \]

\[ x_3 \left( \frac{1}{3} e^{\frac{3}{2} \Phi} e^a_{\mu} A_{d a} \Phi \Gamma^{11} \epsilon \right) \]

\[ + \frac{1}{4} \Phi_{abcd} \Gamma^{11} \epsilon \]

\[ + \frac{1}{2} \Phi_{abcd} \Gamma^{11} \epsilon \]

\[ x_3 A_{d b} \frac{1}{2} \Phi \Gamma^{11} \epsilon \]

\[ \delta \epsilon \lambda = \frac{1}{3} e^{\frac{3}{2} \Phi} d A_{b c} \epsilon + x_1 \frac{1}{3} e^{\frac{3}{2} \Phi} \partial_b \Phi \Gamma^{11} \epsilon + x_1 (\frac{1}{4} \Phi_{b c d} \Gamma^{11} \epsilon + \frac{1}{2} \Phi_{d b} \Gamma^{11} \epsilon) \]

(B.16)

Investigating this and comparing with (B.14) one sets \( l = \frac{1}{2x} \) and

\[ x_1 = \frac{3 \sqrt{2}}{4} e^{-\frac{3}{2} \Phi} \]

\[ x_2 = -\frac{1}{8} e^{-\frac{3}{2} \Phi} \]

\[ x_3 = 1 \]

(B.17)

to obtain the standard type IIA SUSY variations garnished with some additional

\[^9\] The following is used here:

\[ \Gamma^{a} \Gamma^{b} = \Gamma^{a b} + \eta^{a b} \]

\[ \Gamma^{a} \Gamma^{b} \Gamma^{c} = \Gamma^{a b c} + \eta^{a b} \Gamma^{c} - \eta^{a c} \Gamma^{b} + \eta^{b c} \Gamma^{a} \]
torsion terms:
\[
\delta_t \psi = \delta_t \psi + e^a \frac{1}{4} \epsilon a^{abc} \Gamma^b \epsilon + \frac{1}{64} \epsilon a^a \epsilon b \Gamma^a_{b c d} (\eta_{a b} \Gamma^d \Gamma^c - 14 \delta_a \Gamma^c \Gamma^d) \Gamma^{11} \epsilon \\
+ \frac{\epsilon}{4} \epsilon a^{abc} \Gamma^b \epsilon + \frac{1}{2} \hat{\tau}_{\mu b z} \Gamma^{b c} \epsilon a^a \eta_{a b} \hat{\tau}_{z a z} \Gamma^{b c} \epsilon \\
- \frac{1}{16} e^{\Phi} \epsilon a^{a b c} \eta_{a b} \hat{\tau}_{z a z} \epsilon - \frac{1}{16} e^{-\Phi} \epsilon a^{a b c} \eta_{a b} \hat{\tau}_{z a z} \Gamma^{b c} \epsilon \\
+ \frac{1}{32} e^{-\Phi} \epsilon a^{a b c} \eta_{a b} \hat{\tau}_{z a z} \Gamma^{b c} \epsilon a^a \eta_{a b} \hat{\tau}_{z a z} \Gamma^{b c} \epsilon \\
- A_\mu \left( \frac{1}{4} \hat{\tau}_{z b z} \Gamma^{b c} + \frac{1}{2} \hat{\tau}_{z b z} \Gamma^{b c} \Gamma^{11} \right) \epsilon
\]
\[
\delta_t \lambda = \frac{3}{16} \frac{1}{\sqrt{2}} e^{\frac{3}{4} \Phi} (d A_{b c} + 2 e^{-\frac{3}{4} \Phi} \hat{\tau}_{z b c}) \Gamma^{b c} \epsilon + \frac{\sqrt{2}}{4} (\frac{1}{2} e^{-\frac{3}{4} \Phi} \hat{\tau}_{z b c} \Gamma^{b c} \Gamma^{11} \epsilon \\
(B.18)
\]

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