Certain Semisymmetry Curvature Conditions on Paracontact Metric \((k, \mu)\)-Manifolds

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Abstract
The object of the present paper is to characterize paracontact metric \((k, \mu)\)-manifolds satisfying some semisymmetry curvature conditions.

Keywords: Paracontact metric \((k, \mu)\)-manifolds; Weyl semisymmetric manifolds; Projective semisymmetric manifolds; \(\phi\)-Weyl semisymmetry; \(h\)-Weyl semisymmetry.

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1. Introduction

In modern contact geometry, the study of nullity distribution on paracontact geometry is one among the most interesting topics. Paracontact metric structures have been introduced in [3], as a natural odd-dimensional counterpart to para-Hermitian structures, like contact metric structures correspond to the Hermitian ones. Paracontact metric manifolds have been studied by many authors in the recent years. A systematic study of paracontact metric manifolds was carried out by Zamkovoy [11].

An important class among paracontact metric manifolds is that of the \((k, \mu)\)-manifolds, which satisfy the nullity condition

\[
R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY),
\]

for all \(X, Y\) vector fields on \(M\), where \(k\) and \(\mu\) are constants and \(h = \frac{1}{2}L_\xi\phi\) [1]. This class includes the para-Sasakian manifolds [3, 11], the paracontact metric manifolds satisfying \(R(X, Y)\xi = 0\) for all \(X, Y\) vector fields on \(M\) [12].

Among the geometric properties of manifolds symmetry is an important one. Local point of view it was introduced by Shirokov [5] as a Riemannian manifold with covariant constant curvature tensor \(R\), that is, with \(\nabla R = 0\), where \(\nabla\) is the Levi-Civita connection. An extensive theory of symmetric Riemannian manifolds was introduced by Cartan [2]. A manifold is called semisymmetric if the curvature tensor \(R\) satisfies \(R(X, Y) \cdot R = 0\), where \(R(X, Y)\) is considered to be a derivation of the tensor algebra at each point of the manifold for the tangent vectors \(X, Y\). Semisymmetric manifolds were locally classified by Szabó [7]. Also in [10] Yildiz and De studied \(h\)-Weyl semisymmetric, \(\phi\)-Weyl semisymmetric, \(h\)-projectively semisymmetric and \(\phi\)-projectively semisymmetric non-Sasakian \((k, \mu)\)-contact metric manifolds. Recently Mandal and De studied certain curvature conditions on paracontact \((k, \mu)\)-spaces [4].

The projective curvature tensor is an important tensor from the differential geometric point of view. Let \(M\) be a \((2n+1)\)-dimensional semi-Riemannian manifold with metric \(g\). The Ricci operator \(Q\) of \((M, g)\) is defined by \(g(QX, Y) = S(X, Y)\), where \(S\) denotes the Ricci tensor of type \((0, 2)\) on \(M\). If there exists a one-to-one correspondence between each coordinate neighbourhood of \(M\) and a domain in Euclidean space such that any geodesic of the semi-Riemannian manifold corresponds to a straight line in the Euclidean space, then \(M\) is said to be locally projectively flat. For \(n \geq 1\), \(M\) is locally projectively flat if and only if the well known projective curvature tensor \(P\) vanishes.

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Here \( P \) is defined by [6]

\[
P(X, Y)Z = R(X, Y)Z - \frac{1}{2n} \{ S(Y, Z)X - S(X, Z)Y \},
\]

(1.2)

for all \( X, Y, Z \) vector fields on \( M \), where \( R \) is the curvature tensor and \( S \) is the Ricci tensor of \( M \).

In fact, \( M \) is projectively flat if and only if it is of constant curvature [8]. Thus the projective curvature tensor is the measure of the failure of a semi-Riemannian manifold to be of constant curvature.

In semi-Riemannian geometry one of the important curvature properties is conformal flatness. The \textit{Weyl conformal curvature tensor} is a measure of the curvature of spacetime and differs from the semi-Riemannian curvature tensor. It is the traceless component of the Riemannian tensor which has the same symmetries as the Riemannian tensor. The \textit{Weyl conformal curvature tensor} is defined by

\[
C(X, Y)Z = R(X, Y)Z - \frac{1}{2n} \{ S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY \} + \frac{r}{2n(2n - 1)} \{ g(Y, Z)X - g(X, Z)Y \},
\]

(1.3)

for all \( X, Y, Z \) vector fields on \( M \), where \( r = tr(S) \) is scalar curvature [9].

A paracontact metric \((k, \mu)\)-manifold is said to be an \textit{Einstein} manifold if the Ricci tensor satisfies \( S = ag \), where \( a \) a smooth function.

The outline of the article goes as follows: After introduction in section 2, we recall basic facts and some basic results of paracontact metric manifolds with characteristic vector field \( \xi \) belonging to the \((k, \mu)\)-nullity distribution. In section 3, we characterize paracontact metric \((k, \mu)\)-manifolds satisfying some semisymmetry curvature conditions. We prove that Weyl semisymmetric and projective semisymmetric paracontact metric \((k, \mu)\)-manifolds are Einstein manifolds and \( h \)-Weyl semisymmetric and \( \phi \)-Weyl semisymmetric paracontact metric \((k, \mu)\)-manifolds are \( \eta \)-Einstein manifolds provided \( k \neq -1 \).

### 2. Preliminaries

An \((2n + 1)\)-dimensional manifold \( M \) is said to have an \textit{almost paracontact structure} if it admits a \((1, 1)\)-tensor field \( \phi \), a vector field \( \xi \) and a 1-form \( \eta \) satisfying the following conditions ([3], [11]):

(i) \( \eta(\xi) = 1 \), \( \phi^2 = I - \eta \otimes \xi \),

(ii) the tensor field \( \phi \) induces an almost paracomplex structure on each fibre of \( D = \ker(\eta) \), i.e., the \( \pm 1 \)-eigendistributions, \( D^\pm = D_\phi(\pm 1) \) of \( \phi \) have equal dimension \( n \).

Thus from the definition it follows that \( \phi \xi = 0 \), \( \eta \circ \phi = 0 \) and the endomorphism \( \phi \) has rank \( 2n \). The Nijenhius torsion tensor field \([\phi, \phi] \) is given by

\[
[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y].
\]

When the tensor field \( N_\phi = [\phi, \phi] - 2d\eta \otimes \xi = 0 \), the almost paracontact manifold is said to be normal. If an almost paracontact manifold admits a pseudo-Riemannian metric \( g \) such that

\[
g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y),
\]

(2.1)

for all \( X, Y \) vector fields on \( M \), we say that \((M, \phi, \xi, \eta, g)\) is an \textit{almost paracontact metric manifold}. Notice that such a pseudo-Riemannian metric is necessarily of signature \((n + 1, n)\). For an almost paracontact metric manifold, there always exists an orthogonal basis \( \{ X_1, \ldots, X_n, Y_1, \ldots, Y_n, \xi \} \), such that \( g(X_i, X_j) = \delta_{ij}, g(Y_i, Y_j) = -\delta_{ij}, g(X_i, Y_j) = 0, g(\xi, X_i) = g(\xi, Y_j) = 0, \) and \( Y_i = \phi X_i, \) for any \( i, j \in \{ 1, \ldots, n \} \). Such basis is called a \( \phi \)-basis.

We define the \textit{fundamental form} of the almost paracontact metric manifold by \( \theta(X, Y) = g(X, \phi Y) \). If \( d\eta(X, Y) = g(X, \phi Y) \), then \( M \) is said to be \textit{paracontact metric manifold}. In a paracontact metric manifold one defines a symmetric, trace-free operator \( h = \frac{1}{2} L_\xi \phi \), where \( L_\xi \phi \), denotes the Lie derivative. It is known [11] that \( h \) anti-commutes with \( \phi \) and satisfies \( h\xi = 0, \) \( \text{tr} h = \text{tr} h\phi = 0 \) and

\[
\nabla\xi = -\phi + \phi h,
\]

where \( \nabla \) is the Levi-Civita connection of the pseudo-Riemannian manifold \((M, g)\).
Moreover $h = 0$ if and only if $\xi$ is Killing vector field. In this case $M$ is said to be a $K$-paracontact manifold. A normal paracontact metric manifold is called a para-Sasakian manifold satisfies holds on $M$ equivalent to $k$ and only if $Q$ operator Lemma 2.1. 

Let $k$, or an Einstein manifold if and only if $\eta = 2(1-n) + n(2(1-\mu))$, or an Einstein manifold if and only if $k = 0 = \mu$ and $n = 1$ (in this case the manifold is Ricci-flat).

For $k < -1$, $(M, g)$ is an $\eta$-Einstein manifold if and only if $\mu = 2(1-n)$, or an Einstein manifold if and only if $k = 1-n^2$ and $\mu = 2(1-n)$.

**3. Main results**

In this section we study some semisymmetry curvature conditions on paracontact metric $(k, \mu)$-manifolds. Firstly we give the following:

**Definition 3.1.** A semi-Riemannian manifold $(M^{2n+1}, g)$, $n > 1$, is said to be Weyl semisymmetric if

$$R(U, X) \cdot C = 0,$$

holds on $M$ for all $U, X$ vector fields on $M$.

Let $M$ be a Weyl semisymmetric paracontact metric $(k, \mu)$-manifold with $k \neq -1$. Then above equation is equivalent to

$$\langle R(U, X) \cdot C \rangle(W, Y)Z = 0,$$

for any $U, X, W, Y, Z$ vector fields on $M$. Thus we have

$$R(U, X)C(W, Y)Z - C(R(U, X)W, Y)Z - C(W, R(U, X)Y)Z - C(W, Y)R(U, X)Z = 0.\tag{3.2}$$

Substituting $U = W = \xi$ in (3.2) yields

$$R(\xi, X)C(\xi, Y)Z - C(R(\xi, X)\xi, Y)Z - C(\xi, R(\xi, X)Y)Z - C(\xi, Y)R(\xi, X)Z = 0,\tag{3.3}$$

where

$$C(\xi, Y)Z = (\frac{r - 2nk}{2n(2n - 1)})(g(Y, Z)\xi - \eta(Z)\xi) - \frac{1}{2n - 1} (S(Y, Z)\xi - \eta(Z)QY).\tag{3.4}$$
With the help of (3.3) and (3.4), we get
\[ kS(X, Y) + \mu S(hX, Y) - 2nk^2g(X, Y) - 2nk\mu g(hX, Y) = 0. \]  
(3.5)

Putting \( Y = hY \) in (3.5) and using (2.3), we obtain
\[ \mu(k + 1)S(X, Y) + kS(hX, Y) - 2nk^2g(hX, Y) - 2nk\mu(k + 1)g(X, Y) = 0. \]  
(3.6)

Now suppose \( k \neq -1 \) and \( \mu \neq 0 \). Multiplying (3.5) by \( k \) and (3.6) by \( \mu \), we have
\[ k^2S(X, Y) + \mu kS(hX, Y) - 2nk^3g(X, Y) - 2nk^2\mu g(hX, Y) = 0, \]  
(3.7)

and
\[ \mu^2(k + 1)S(X, Y) + \mu kS(hX, Y) - 2nk^2\mu g(hX, Y) - 2nk(k + 1)\mu^2 g(X, Y) = 0, \]  
(3.8)

respectively. Subtracting (3.8) from (3.7), we get
\[ \{k^2 - \mu^2(k + 1)\}\{S(X, Y) - 2nk\mu g(X, Y)\} = 0. \]  
(3.9)

If \( k \neq -1 \), then \( k^2 - \mu^2(k + 1) \neq 0 \). Therefore from (3.9) it follows that \( S(X, Y) = 2nk\mu g(X, Y) \), which implies that the manifold \( M \) is an Einstein manifold. Thus we have the following:

**Theorem 3.1.** If \( M \) is a \((2n + 1)\)-dimensional Weyl semisymmetric paracontact metric \((k, \mu)\)-manifold with \( k \neq -1 \) then the manifold \( M \) is an Einstein manifold.

**Definition 3.2.** A semi-Riemannian manifold \((M^{2n+1}, g), n > 1\), is said to be projective semisymmetric if
\[ R(U, X) \cdot P = 0, \]
holds on \( M \) for all \( U, X \) vector fields on \( M \).

Let \( M \) be a projective semisymmetric paracontact metric \((k, \mu)\)-manifold with \( k \neq -1 \). Then above equation is equivalent to
\[ (R(U, X) \cdot P)(W, Y)Z = 0, \]  
(3.10)

for any \( U, X, W, Y, Z \) vector fields on \( M \). Thus we have
\[ R(U, X)P(W, Y)Z - P(R(U, X)W, Y)Z - P(W, R(U, X)Y)Z - P(W, Y)R(U, X)Z = 0. \]  
(3.11)

Substituting \( U = W = \xi \) in (3.11) yields
\[ R(\xi, X)P(\xi, Y)Z - P(R(\xi, X)\xi, Y)Z - P(\xi, R(\xi, X)\xi)Z - P(\xi, Y)R(\xi, X)Z = 0, \]  
(3.12)

where
\[ P(\xi, Y)Z = kg(Y, Z)\xi + \mu(g(hY, Z)\xi - \eta(Z)hY) - \frac{1}{2n} S(Y, Z)\xi. \]  
(3.13)

With help of (3.13) and (3.12), we get
\[ \mu\{\eta(Z)g(R(\xi, X)hY, \xi) + g(R(\xi, X)Y, hZ) + g(R(\xi, X)Z, hY)\} + \frac{1}{2n}\{S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z)\} = 0, \]
which implies that
\[ \mu\{kg(hX, Z)\eta(Y) + \mu g(hX, hZ)\eta(Y)\} \]  
(3.14)
Putting $Z = \xi$ in (3.14), we have
\[
2kS(X, Y) + \mu S(hX, Y) - 2nk^2g(X, Y) - 2n\mu g(hX, Y) = 0. \tag{3.15}
\]

Putting $X = hX$ in (3.15) and using $h^2 = (k + 1)\phi^2$, we obtain
\[
\mu(k + 1)S(X, Y) + kS(hX, Y) - 2nk^2g(hX, Y) - 2nk(k + 1)\mu g(X, Y) = 0. \tag{3.16}
\]

Multiplying (3.15) by $k$ and (3.16) by $\mu$, we have
\[
k^2S(X, Y) + k\mu S(hX, Y) - 2nk^2g(X, Y) - 2nk\mu g(hX, Y) = 0, \tag{3.17}
\]

and
\[
\mu^2(k + 1)S(X, Y) + \mu kS(hX, Y) - 2nk^2\mu g(hX, Y) - 2nk(k + 1)\mu^2 g(X, Y) = 0. \tag{3.18}
\]

respectively. Subtracting (3.18) from (3.17), we get
\[
\{k^2 - \mu^2(k + 1)\}S(X, Y) - 2nkg(X, Y) = 0. \tag{3.19}
\]

If $k \neq -1$ then $k^2 - \mu^2(k + 1) \neq 0$. Therefore from (3.19) it follows that $S(X, Y) = nkg(X, Y)$. Thus the manifold $M$ is an Einstein manifold. Hence we have the following:

**Theorem 3.2.** If $M$ is a $(2n + 1)$-dimensional projective semisymmetric paracontact metric $(k, \mu)$-manifold with $k \neq -1$ then the manifold $M$ is an Einstein manifold.

**Definition 3.3.** A semi-Riemannian manifold $(M^{2n+1}, g), n > 1$, is said to be $h$-Weyl semisymmetric if
\[
C(X, Y) \cdot h = 0, \tag{3.20}
\]
holds on $M$.

Now let $M$ be a $h$-Weyl semisymmetric paracontact metric $(k, \mu)$-manifold with $k \neq -1$. Then equation (3.20) is equivalent to
\[
C(X, Y)hZ - hC(X, Y)Z = 0,
\]
for any $X, Y, Z$ vector fields on $M$. Firstly, we get
\[
R(X, Y)hZ - hR(X, Y)Z = \mu(k + 1)\{g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X
+ k\{g(hY, Z)\eta(X)\xi - g(hX, Z)\eta(Y)\xi
+ \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX
+ g(\phi Y, Z)\phi hX - g(\phi X, Z)\phi hY
+ \{\mu + k\}\{g(\phi h X, Z)\phi Y - g(\phi h Y, Z)\phi X
+ 2\mu g(\phi Y, X)\phi h Z. \tag{3.21}
\]

Then we can write
\[
C(X, Y)hZ - hC(X, Y)Z = R(X, Y)hZ - hR(X, Y)Z
- \frac{1}{2n - 1}\{S(Y, hZ)X - S(X, hZ)Y + g(Y, hZ)QX
- g(X, hZ)QY - S(Y, hZ)hX + S(X, hZ)hY
- g(Y, hZ)hQX + g(X, hZ)hQY
+ \frac{r}{2n(2n - 1)}\{g(Y, hZ)X - g(X, hZ)Y
- g(Y, hZ)hX + g(X, hZ)hY\} = 0. \tag{3.22}
\]
Thus from (3.25) and (3.26), we get

\[ \mu(k + 1)\{g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\} + k\{g(hY, Z)\eta(X)\xi - g(hX, Z)\eta(Y)\xi + \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX\} + g(\phi Y, Z)\phi h X - g(\phi X, Z)\phi h Y\] 
\[+ (\mu + k)\{g(\phi h X, Z)\phi Y - g(\phi h Y, Z)\phi X\} + 2\mu g(\phi X, Y)\phi h Z\] 
\[- \frac{1}{2n - 1}\{S(hY, hZ)X - S(hX, hZ)Y + g(hY, hZ)QX\]
\[-g(hY, hZ)QY - S(hY, hZ)hX + S(hX, hZ)hY\]
\[-g(Y, hZ)hQX + g(X, hZ)hQY\] 
\[+ \frac{r}{2n(2n - 1)}\{g(Y, hZ)X - g(X, hZ)Y - g(hY, hZ)hX + g(X, hZ)hY\} = 0.\] 

(3.23)

Putting \(Y = hY\) in (3.23), we have

\[\mu(k + 1)\{g(hY, Z)\eta(X)\xi + \eta(X)\eta(Z)hY\} + k\{g(h^2Y, Z)\eta(X)\xi + \eta(X)\eta(Z)h^2Y\} + g(\phi h Y, Z)\phi h X - g(\phi h X, Z)\phi h^2 Y\] 
\[+ (\mu + k)\{g(\phi h X, Z)\phi h Y - g(\phi h^2 Y, Z)\phi h X\} + 2\mu g(\phi X, Y)\phi h^2 Z\] 
\[- \frac{1}{2n - 1}\{S(hY, hZ)X - S(hX, hZ)hY + g(hY, hZ)QX\]
\[g(hY, hZ)QY - S(hY, hZ)hX + g(hY, hZ)hQX + g(X, hZ)hQhY\] 
\[+ \frac{r}{2n(2n - 1)}\{g(hY, hZ)X - g(X, hZ)hY\} = 0.\] 

(3.24)

Multiplying with \(\xi\) in (3.24), we obtain

\[(k + 1)\eta(X)\mu g(hY, Z) + k\{g(Y, Z) - \eta(Y)\eta(Z)\} - \frac{1}{2n - 1}\{S(Y, Z) - 2nk\eta(Y)\eta(Z)\}
\[+ \frac{r}{2n(2n - 1)}\{g(Y, Z) - \eta(Y)\eta(Z)\}] = 0,\]
i.e.,

\[\mu g(hY, Z) + (k + \frac{r}{2n(2n - 1)} + 2nk)\{g(Y, Z) - \eta(Y)\eta(Z)\} - \frac{1}{2n - 1}\{S(Y, Z) - 2nk\eta(Y)\eta(Z)\} = 0.\] 

(3.25)

Now from (2.6), we have

\[g(hY, Z) = \frac{1}{2(n - 1) + \mu}S(Y, Z) - \frac{2(1 - n) + n\mu}{2(n - 1) + \mu}g(Y, Z) - \frac{(2(n - 1) + n(2k - \mu))}{2(n - 1) + \mu}\eta(Y)\eta(Z).\] 

(3.26)

Thus from (3.25) and (3.26), we get

\[\frac{\mu}{2(n - 1) + \mu} - \frac{1}{2n - 1})S(Y, Z)\]
\[-\frac{\mu(2(1 - n) + n\mu)}{2(n - 1) + \mu} - k - \frac{r}{2n(2n - 1)} - 2nk)g(Y, Z)\]
\[-\frac{\mu(2(n - 1) + n(2k - \mu))}{2(n - 1) + \mu} + k + \frac{r}{2n(2n - 1)} + 2nk - \frac{2nk}{2n - 1})\eta(Y)\eta(Z) = 0,\]
which turns to

$$S(Y, Z) = \frac{\lambda_2}{\lambda_1} g(Y, Z) + \frac{\lambda_3}{\lambda_1} \eta(Y) \eta(Z),$$

where

$$\lambda_1 = \frac{\mu}{2(n-1) + \mu} - \frac{1}{2n-1},$$

$$\lambda_2 = \frac{\mu(2(1-n) + n\mu)}{2(n-1) + \mu} - k - \frac{r}{2n(2n-1)} - 2nk,$$

$$\lambda_3 = \frac{\mu(2(n-1) + n(2k - \mu))}{2(n-1) + \mu} + k + \frac{r}{2n(2n-1)} + 2nk - \frac{2nk}{2n-1}.$$

Thus the manifold $M$ is an $\eta$-Einstein manifold. Hence we state the following:

**Theorem 3.3.** If $M$ is a $(2n+1)$-dimensional $h$-Weyl semisymmetric paracontact metric $(k, \mu)$-manifold with $k \neq -1$ then $M$ is an $\eta$-Einstein manifold.

**Definition 3.4.** A semi-Riemannian manifold $(M^{2n+1}, g), n > 1$, is said to be $\phi$-Weyl semisymmetric if

$$C(X, Y) \cdot \phi = 0,$$

holds on $M$.

Let $M$ be a $\phi$-Weyl semisymmetric paracontact metric $(k, \mu)$-manifold with $k \neq -1$. Then above equation is equivalent to

$$C(X, Y)\phi Z - \phi C(X, Y)Z = 0.$$

for any $X, Y, Z$ vector fields on $M$. Firstly we get

$$R(X, Y)\phi Z - \phi R(X, Y)Z = g(X, \phi Z)Y - g(Y, \phi Z)X + g(Y, Z)\phi X$$

$$-g(X, Z)\phi Y - g(X, \phi Z)hY + g(Y, \phi Z)hX$$

$$+g(hY, \phi Z)X - g(hX, \phi Z)Y - g(Y, Z)\phi hX$$

$$+g(X, Z)\phi hY - g(hY, Z)\phi X + g(hX, Z)\phi Y$$

$$+\frac{-1 - \mu}{k+1} \{g(hY, \phi Z)hX - g(hX, \phi Z)hY - g(hY, Z)\phi hX$$

$$+g(hX, Z)\phi hY\} - \frac{-k + \mu}{k+1} \{g(hX, \phi Z)\phi hY - g(hY, \phi Z)\phi hX$$

$$-g(\phi hY, Z)hX + g(\phi hX, Z)hY\}$$

$$+(k + 1) \{g(\phi X, Z)\eta(Y)\xi - g(\phi Y, Z)\eta(X)\xi$$

$$+\eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi X\}$$

$$+(\mu - 1) \{g(\phi hX, Z)\eta(Y)\xi - g(\phi hY, Z)\eta(X)\xi$$

$$+\eta(X)\eta(Z)\phi hY - \eta(Y)\eta(Z)\phi hX\}.\quad (3.27)$$
Then we have

\[
C(X, Y)\phi Z - \phi C(X, Y)Z = g(X, \phi Z)Y - g(Y, \phi Z)X + g(Y, Z)\phi X \\
- g(X, Z)\phi Y - g(X, \phi Z)hY + g(Y, \phi Z)hX \\
+ g(hY, \phi Z)X - g(hX, \phi Z)Y - g(Y, Z)\phi hX \\
+ g(X, Z)\phi hY - g(hY, Z)\phi X + g(hX, Z)\phi Y \\
+ \frac{-1 - \frac{n}{k + 1}}{\frac{2n}{2n - 1}} \{ g(hY, \phi Z)hX - g(hX, \phi Z)hY - g(hY, Z)\phi hX \\
+ g(hX, Z)\phi hY \} - \frac{-k + \frac{4}{k + 1}}{\frac{2n}{2n - 1}} \{ g(hX, \phi Z)\phi hY - g(hY, \phi Z)\phi hX \\
- g(\phi hY, Z)hX + g(\phi hX, Z)hY \} + (k + 1) \{ g(\phi X, Z)\eta(Y)\xi - g(\phi Y, Z)\eta(X)\xi \\
+ \eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi hX \} + (\mu - 1) \{ g(\phi hX, Z)\eta(Y)\xi - g(\phi hY, Z)\eta(X)\xi \\
+ \eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi hX \} - \frac{1}{2n - 1} \{ S(Y, \phi Z)X - S(X, \phi Z)Y + g(Y, \phi Z)QX \\
- g(X, \phi Z)QY - S(Y, \phi Z)\phi X + S(X, \phi Z)\phi Y \\
- g(Y, \phi Z)QX + g(X, \phi Z)QY \} + \frac{r}{2n(2n - 1)} \{ g(Y, \phi Z)X - g(X, \phi Z)Y \\
- g(Y, \phi Z)\phi X + g(X, \phi Z)\phi Y \} = 0. \tag{3.28}
\]

Putting \( X = \phi X \) and multiplying with \( W \) in (3.28), we obtain

\[
g(\phi X, \phi Z)g(Y, W) - g(Y, \phi Z)g(\phi X, W) - g(Y, Z)g(\phi X, \phi W) \\
- g(\phi X, Z)g(\phi Y, W) - g(\phi X, \phi Z)g(hY, W) + g(Y, \phi Z)g(h\phi X, W) \\
+ g(hY, \phi Z)g(\phi X, W) - g(h\phi X, \phi Z)g(Y, W) - g(Y, Z)g(\phi h\phi X, W) \\
+ g(\phi X, Z)g(\phi hY, W) + g(hY, Z)g(\phi X, \phi W) + g(h\phi X, Z)g(\phi Y, W) \\
+ \frac{-1 - \frac{n}{k + 1}}{\frac{2n}{2n - 1}} \{ g(hY, \phi Z)g(\phi h\phi X, W) - g(\phi h\phi X, \phi Z)g(hY, W) \\
- g(hY, Z)g(\phi h\phi X, W) + g(\phi h\phi X, Z)g(hY, W) \} \\
- (k + 1) \{ g(\phi X, \phi Z)\eta(Y)\eta(W) - \eta(Y)\eta(Z)g(\phi X, \phi W) \} \\
- (\mu - 1) \{ g(\phi h\phi X, \phi Z)\eta(Y)\eta(W) - \eta(Y)\eta(Z)g(\phi h\phi X, \phi W) \} - \frac{1}{2n - 1} \{ S(Y, \phi Z)g(\phi X, W) - S(\phi X, \phi Z)g(\phi Y, W) + g(Y, \phi Z)S(\phi X, W) \\
- g(\phi X, \phi Z)S(Y, W) - S(Y, \phi Z)g(\phi^2 X, W) + S(\phi X, \phi Z)g(\phi Y, W) \\
+ g(Y, \phi Z)S(\phi X, \phi W) - g(\phi X, \phi Z)S(Y, \phi W) \} + \frac{r}{2n(2n - 1)} \{ g(Y, \phi Z)g(\phi X, W) - g(\phi X, \phi Z)g(Y, W) \\
+ g(Y, \phi Z)g(\phi X, \phi W) + g(\phi X, \phi Z)g(\phi Y, W) \} = 0. \tag{3.29}
\]

Putting \( Y = W = \xi \) in (3.29), we get

\[
(-k + \frac{2nk}{2n - 1} - \frac{r}{2n(2n - 1)})g(\phi X, \phi Z) + \mu g(\phi h\phi X, \phi Z) + \frac{1}{2n - 1}S(\phi X, \phi Z) = 0. \tag{3.30}
\]
Using (2.1) and (2.6) in (3.30), we have
\[
(k - \frac{2nk}{2n-1} + \frac{r}{2n(2n-1)})\{g(X, Z) - \eta(X)\eta(Z)\}
\]
\[
-S(X, Z) + 2n\eta(X)\eta(Z)
\]
\[
-(\frac{\mu(2n+1) + 4(n-1)}{2n-1})\{\frac{1}{2(n-1)+\mu}S(X, Z) - \frac{2(1-n)+n\mu}{2(n-1)+\mu}g(X, Z)
\]
\[
-(\frac{2(n-1) + n(2k-\mu)}{2(n-1)+\mu}\eta(X)\eta(Z)) = 0,
\]
i.e.,
\[
[4(n-1) + \frac{2(n-1)(2-\mu) + (2-2n+n\mu)}{2(n-1)+\mu}]S(X, Z)
\]
\[
= [\frac{2nk(2n-1) - 4n^2k + r}{2n} - \frac{2(n-1)(2-\mu) + (2-2n+n\mu)}{2(n-1)+\mu}]g(X, Z)
\]
\[
-\frac{2nk(2n-1) - 4n^2k + r - 4n^2k}{2n} + \frac{2(n-1)(2-\mu) + (2-2n+2n-\mu)}{2(n-1)+\mu}\eta(X)\eta(Z).
\]
Hence we have
\[
S(X, Z) = \frac{\lambda_1'}{\lambda_1}g(X, Z) + \frac{\lambda_2'}{\lambda_1}\eta(X)\eta(Z),
\]
where
\[
\lambda_1' = 4(n-1) + \mu - \frac{2(n-1)(2-\mu) + (2-2n+n\mu)}{2(n-1)+\mu},
\]
\[
\lambda_2' = \frac{2nk(2n-1) - 4n^2k + r}{2n} - \frac{2(n-1)(2-\mu) + (2-2n+n\mu)}{2(n-1)+\mu},
\]
\[
\lambda_3' = \frac{2nk(2n-1) - 4n^2k + r - 4n^2k}{2n} + \frac{2(n-1)(2-\mu) + (2-2n+2n-\mu)}{2(n-1)+\mu}.
\]
Thus the manifold \(M\) is an \(\eta\)-Einstein manifold. Hence we can state the following:

**Theorem 3.4.** If \(M\) is a \((2n+1)\)-dimensional \(\phi\)-Weyl semisymmetric paracontact metric \((k, \mu)\)-manifold with \(k \neq -1\) then \(M\) is an \(\eta\)-Einstein manifold.

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