ON COMPLEX-VALUED 2D EIKONALS.
PART FOUR: CONTINUATION PAST A CAUSTIC

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Abstract. Theories of monochromatic high-frequency electromagnetic fields have been designed by Felsen, Kravtsov, Ludwig and others with a view to portraying features that are ignored by geometrical optics. These theories have recourse to eikonals that encode information on both phase and amplitude — in other words, are complex-valued. The following mathematical principle is ultimately behind the scenes: any geometric optical eikonal, which conventional rays engender in some light region, can be consistently continued in the shadow region beyond the relevant caustic, provided an alternative eikonal, endowed with a non-zero imaginary part, comes on stage.

In the present paper we explore such a principle in dimension 2. We investigate a partial differential system that governs the real and the imaginary parts of complex-valued two-dimensional eikonals, and an initial value problem germane to it. In physical terms, the problem in hand amounts to detecting waves that rise beside, but on the dark side of, a given caustic. In mathematical terms, such a problem shows two main peculiarities: on the one hand, degeneracy near the initial curve; on the other hand, ill-posedness in the sense of Hadamard. We benefit from using a number of technical devices: hodograph transforms, artificial viscosity, and a suitable discretization. Approximate differentiation and a parody of the quasi-reversibility method are also involved. We offer an algorithm that restrains instability and produces effective approximate solutions.

1. Introduction

1.1. Geometrical optics fits well a variety of issues, but especially survives as an asymptotic theory of monochromatic high-frequency electromagnetic fields — [BB], [Du], [FKN], [GS], [Jn], [Ke], [KL], [Kl1] and [Kl2], [KO1], [Ln], [MO], [Ru] are selected apropos references. Generalizations have been worked out by Felsen, Kravtsov, Ludwig and their followers — see e.g. [CF1] and [CF2], [EF], [Fe1] and [Fe2], [HF], [Kr1], [Kr2], [Kr3] and [KPA], [LBL], [Ln1] and [Ln2], or consult [BM], [CLOT], [KO2]. One is enough for successfully modeling basic optical processes, such as the propagation of light and the development of caustics. The others embrace geometrical optics and are additionally apt to account for certain optical phenomena — for instance, the rise of evanescent waves past a caustic — that are beyond the reach of geometrical optics. A leitmotif of these is allowing a keynote parameter to adjust itself to a standard equation, and simultaneously take complex values.

1991 Mathematics Subject Classification. Primary, 35F25, 35Q60, 78A05; Secondary, 65D25.
Key words and phrases. Eikonal, caustics, evanescent waves, nonlinear partial differential systems, initial value problems, hodograph transforms, viscosity, quasi-reversibility, approximate differentiation.
The following partial differential equation
\begin{equation}
\left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 = n^2(x, y)
\end{equation}
underlies the mentioned theories in case the spatial dimension is 2. Here \(x\) and \(y\) denote rectangular coordinates in the Euclidean plane; \(n\) is a real-valued, strictly positive function of \(x\) and \(y\); \(w\) is allowed to take both real and complex values. Function \(n\) represents the refractive index of an appropriate (isotropic, non-conducting) two-dimensional medium — its reciprocal stands for velocity of propagation. Function \(w\) is named eikonal according to usage, and relates to the asymptotic behavior of an electromagnetic field as the wave number grows large — the real part of \(w\) accounts for oscillations, the imaginary part of \(w\) accounts for damping. Geometrical optics deals exclusively with real-valued eikonals, complex-valued eikonals distinguish a more advanced context. Throughout the present paper we assume the refractive index is conveniently smooth, and consider sufficiently smooth eikonals.

The following partial differential system
\begin{equation}
\begin{aligned}
u_x^2 + \nu_y^2 - \nu_x^2 - \nu_y^2 &= n^2(x, y) \\
u_xv_x + \nu_yv_y &= 0
\end{aligned}
\end{equation}
governs those complex-valued solutions to (1.1) that obey
\(u = \text{Re} w, \quad v = \text{Im} w\).

Observe the architecture of (1.2): gradients are involved through their orthogonal invariants — lengths and inner product — only. Observe also the following properties, which result from a standard test and easy algebraic manipulations. First, system (1.2) is elliptic-parabolic or degenerate elliptic. Second, a solution array \([u \ v]\) to (1.2) is elliptic if and only if its latter entry \(v\) is free from critical points.

The Bäcklund transformation, which relates \(u\) and \(v\) thus
\begin{equation}
\nabla v = f \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \nabla u,
\end{equation}
\(f^2 = 1 - \frac{n^2}{|\nabla u|^2}\), \(\text{sgn } f = \text{sgn} (u_xv_y - u_yv_x)\),
and implies both
\(|\nabla u| \geq n\)
and
\begin{equation}
\text{div} \left\{ \sqrt{1 - \frac{n^2}{|\nabla u|^2}} \nabla u \right\} = 0,
\end{equation}
is another, decoupled form of (1.2).

System (1.2) discloses two scenarios — the former is tantamount to conventional geometrical optics, the latter opens up new vistas. Either the following equations
\(u_x^2 + u_y^2 = n^2\) and \(v_x = v_y = 0\)
hold, or the following inequalities
\(|\nabla u| > n\) and \(|\nabla v| > 0|\)
and the following equations prevail.

\begin{align}
(1.5) & \quad \nabla u = \frac{1}{f} \left[ \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right] \nabla v, \quad \frac{1}{f^2} = 1 + \frac{n^2}{|\nabla v|^2}, \\
(1.6) & \quad \text{div} \left\{ \frac{1 + \frac{n^2}{|\nabla v|^2}}{\sqrt{1 + \frac{n^2}{|\nabla v|^2}}} \nabla v \right\} = 0,
\end{align}

\begin{align}
(1.7) & \quad \left( |\nabla u|^4 - n^2 u_y^2 \right) u_{xx} + 2n^2 u_x u_y u_{xy} + \left( |\nabla u|^4 - n^2 u_x^2 \right) u_{yy} = n|\nabla u|^2 (\nabla n, \nabla u), \\
(1.8) & \quad \left( |\nabla v|^4 + n^2 v_y^2 \right) v_{xx} - 2n^2 v_x v_y v_{xy} + \left( |\nabla v|^4 + n^2 v_x^2 \right) v_{yy} + n|\nabla v|^2 (\nabla n, \nabla v) = 0.
\end{align}

Equations (1.5) invert the Bäcklund transformation mentioned above, and imply (1.6). (1.7) and (1.8) are quasi-linear partial differential equations of the second order in non-divergence form. The former has a \textit{mixed elliptic-hyperbolic} character: a solution \( u \) is \textit{elliptic} or \textit{hyperbolic} depending on whether the length of \( \nabla u \) exceeds, or is smaller than \( n \). The latter is \textit{elliptic-parabolic} or \textit{degenerate elliptic}: a solution \( v \) such that \( \nabla v \) is free from zeros is \textit{elliptic}, degeneracy occurs at the \textit{critical points} of \( v \).

Any real-valued sufficiently smooth solution to (1.4) satisfies (1.7). A real-valued function is an elliptic solution to (1.7) if and only if it coincides with the former entry of an elliptic solution to (1.2). A real-valued function \( u \), smooth and without critical points, is a hyperbolic solution to (1.7) if and only if two real-valued smooth functions \( \varphi \) and \( \psi \) exist such that

\[ \varphi_x^2 + \varphi_y^2 = n^2, \quad \psi_x^2 + \psi_y^2 = n^2, \quad \varphi_x \psi_y - \varphi_y \psi_x \neq 0, \]

and

\[ u = \frac{1}{2} (\varphi + \psi). \]

Any real-valued, sufficiently smooth solution to (1.6) satisfies (1.8). Any \textit{elliptic} solution to (1.8) satisfies (1.6). A solution to (1.8) \textit{need not} satisfy (1.6): for instance, perfectly smooth solutions to (1.8) exist whose gradient vanishes exclusively in a set of measure 0, and which make the left-hand side of (1.6) a non-zero distribution.

Let \( J \) be endowed with an appropriate domain and obey

\[ J(v) = \iint j \left( \frac{|\nabla v|}{n} \right) n^2 \ dx \ dy \]

for any \( v \) from that domain — here \( j \) is the arc length along a parabola, videlicet

\[ j(\rho) = \frac{\rho}{2} \sqrt{1 + \rho^2} + \frac{1}{2} \log(\rho + \sqrt{1 + \rho^2}) \]

for any real \( \rho \). The following properties hold. (i) \( J \) is convex, coercive and subdifferentiable, but not Fréchet-differentiable. (ii) Any critical point of \( J \), i.e. any function \( v \) such that

\[ \partial J(v) \ni 0, \]

satisfies (1.6) in any open set \( \mathcal{O} \) such that

\[ \mathcal{O} \text{ is essentially contained in } \{(x, y) \in \text{domain of } v : \nabla v(x, y) \neq 0\}. \]
Consequently, a critical point of $J$ solves a free-boundary problem for equation (1.6) — the relevant free boundary is

$$(\text{domain of } v) \cap \partial \{ (x,y) \in \text{domain of } v : \nabla v(x,y) \neq 0 \}$$

and plays the role of a caustic. (ii) Any function $v$ such that $J(v) = \text{minimum}$ satisfies (1.8) in an appropriate viscosity sense. In other words, a minimizer of $J$ solves in a generalized sense a boundary value problem for equation (1.8).

An early treatment of (1.2) traces back to [ER]. Further apropos information is offered in [MT1], [MT2], [MT3], and [MT4], where solutions in closed form, qualitative features, exterior boundary value problems, related free boundaries, variational and viscosity methods are discussed.

### 1.2. Geometrical optics ultimately amounts to manipulating:

(i) the Riemannian metric known as travel time, videlicet

$$n(x,y)\sqrt{dx^2 + dy^2};$$

(ii) the members of appropriate one-parameter families of travel time geodesics — called rays; (iii) the envelopes of rays — called caustics.

Geometric optical eikonals are entirely controlled by rays. They shine in light regions (those spanned by relevant rays), burn out beside caustics (where the ray system breaks down), and shut down in shadow regions (that rays avoid). As a consequence, geometrical optics is unable to account for any optical process that takes place beyond a caustic, on the dark side of it. Essentials of two-dimensional geometrical optics, which are instrumental here, are outlined in Appendix A.

Complex-valued eikonals are more flexible. As the cited work of Felsen, Kravtsov and Ludwig may prompt, complex-valued eikonals look apt to consistently continue geometric optical cognates into shadow regions. Such a continuation is the main theme of the present paper.

Let us pave the way by heuristically considering the case where refractive index $n$ is 1. A classical recipe informs how general geometric optical eikonals can be cooked up: start from a complete integral, derive a one-parameter family of solutions, take the relevant envelope, and shake well. Let $f$ be an arbitrary, but sufficiently smooth, real function. The following pair

$$w = x \cos t + y \sin t + f(t), \quad 0 = -x \sin t + y \cos t + f'(t)$$

causes $w$ and $t$ to enjoy the following properties:

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = 1,$$

the pertinent eikonal equation governing $w$;

$$\cos t \frac{\partial t}{\partial x} + \sin t \frac{\partial t}{\partial y} = 0,$$

a Burgers-type equation governing $t$;

$$w_xt_x + w_yt_y = 0,$$

showing that the gradients of $w$ and $t$ are orthogonal;

$$w_x = \cos t, \quad w_y = \sin t,$$
a Bäcklund transformation further relating $w$ and $t$. Both the rays of $w$ and the level lines of $t$ are the straight-lines where

$$-x \sin t + y \cos t + f'(t) = 0$$

and $t$ equals a constant. Such straight-lines span the light region and envelope the caustic. We have

$$-f' = \text{the support function of the caustic},$$

and

$$x = f''(t) \cos t + f'(t) \sin t, \quad y = f''(t) \sin t - f'(t) \cos t, \quad w = f(t) + f''(t)$$

along the caustic. Therefore the second-order derivatives of $w$ and the gradient of $t$ simultaneously blow up there — in particular, the caustic of $w$ is also the shock-line of $t$.

We claim that both $w$ and $t$ can be continued beyond the caustic, in a subset of the shadow region, if complex values are allowed. Suppose $f$ is analytic — so as it can be continued by a holomorphic function of a complex variable. Let $i = \sqrt{-1}$, the unit imaginary number. Let $u, v, \lambda, \mu$ be real; put the following equations

$$w = u + iv, \quad t = \lambda + i\mu,$$

and equations (1.9) together, but force (1.9) to produce real $x$ and $y$. The following formulas

$$x = \frac{\sin \lambda}{\cosh \mu} \text{Re} f'(t) + \frac{\cos \lambda}{\sinh \mu} \text{Im} f'(t), \quad y = -\frac{\cos \lambda}{\cosh \mu} \text{Re} f'(t) + \frac{\sin \lambda}{\sinh \mu} \text{Im} f'(t),$$

$$u = \text{Re} f'(t) + (x \cos \lambda + y \sin \lambda) \cosh \mu, \quad v = \text{Im} f'(t) + (-x \sin \lambda + y \cos \lambda) \sinh \mu,$$

ensue, then an inspection testifies that the claimed continuation ensues too.

Incidentally, we have also shown that solutions to the non-viscous Burgers equation can be continued past the shock-line by suitable complex-valued solutions to the same equation. Let us mention that complex-valued solutions to viscous and non-viscous Burgers equation are dealt with in [KWYZ], [PS] and [KO]. A more exhaustive analysis is carried out in Appendix B.

1.3. A certain initial value problem for system (1.2) — the one described in items (i) and (ii) below, and in figure 1 — summarizes our purposes.

(i) An initial curve IC is given. The following alternative applies: either IC is specified exactly — no error infects the definition of IC; or else IC is a phantom — some coarse and polluted sampling of IC is gotten. In the former case assume IC is smooth enough. In the latter case recover IC, i.e. fed the available data into an appropriate denoising process, and then elect the consequent output as an operative substitute of IC — ad hoc tools are provided in Appendix C.

Represent IC (either the authentic one, or else its surrogate) by the following equations

$$(1.10) \quad x = \alpha(t), \quad y = \beta(t),$$

and adjust parameter $t$ so as

$$t = \text{a travel time}$$

without any loss of generality.
Assume travel time is an extra metric in action and the relevant geodesic curvature of IC is free from zeroes. In other words, postulate that (1.10) and either of the following equations

\[ \kappa \text{(velocity)}^3 = \text{Geodesic curvature,} \]

\[ \kappa \text{(velocity)}^2 = \text{Euclidean curvature} - \langle \text{unit normal}, \nabla \log n(x, y) \rangle \]

result in \( \kappa \) vanishes nowhere.

(ii) A pair \([u \ v]\) is sought that obeys system (1.2) and fulfills the following conditions. First,

\[ u(x, y) = t, \quad v(x, y) = 0, \]

if \( x, y \) and \( t \) are subject to (1.10). Second, \( u \) and \( v \) are defined in the side of IC that \((\text{sgn} \ \kappa) \times \text{(unit normal to IC)}\)

points to.

Arguments from Appendix A allow us to comment as follows. IC and the mentioned side of it can be viewed as a caustic and a shadow region, respectively. Any geometric optical eikonal, which makes IC a caustic, lives in the illuminated side of IC; the complex-valued eikonal, whose real part is \( u \) and whose imaginary part is \( v \), lives in the opposite, dark side of IC; both equal a travel time along IC. An extension of the geometric optical eikonal in hand ensues. Such an extension does obey the eikonal equation, lives in both the light region and a subset of the shadow region, and takes complex values where shadow prevails. In physical terms, problem (i) and (ii) accepts a caustic in input, then models evanescent waves that rise in the dark side of it.
In the present paper we focus our attention on solutions \( uv \) to the problem (i) and (ii) that meet the following extra requirements:

(iii) they are elliptic;
(iv) their Jacobian determinant obeys 
\[
\text{sgn}(u_xv_y - u_yv_x) = \text{sgn}\ k.
\]
Condition (iii) ensures that the latter entry \( v \) is not constant; as will emerge from subsequent developments, initial conditions plus conditions (iii) and (iv) ensure that the same entry is nonnegative.

The solutions to problem (i)-(iv) develop singularities near IC, as any geometric optical eikonal does in the vicinity of the relevant caustic. We will show that they obey the following expansions
\begin{align*}
(1.12) \quad u(x,y) &= s + o(r), \quad v(x,y) = 2\sqrt{3} |\kappa(s)|^{1/2}|r|^{3/2} + o(r^{3/2})
\end{align*}
as \((x,y)\) belongs to the appropriate side of, and is close enough to IC. Here \( r \) and \( s \) stand for the curvilinear coordinates described in Appendix A — informally, \( r \) is a signed distance from IC, \( s \) is a lifting of a travel time inherent to IC. Note the physical meaning of (1.12): the damping effects, which are encoded in the imaginary part of a complex-valued eikonal, are tuned by the geodesic curvature of the relevant caustic.

2. Framework

2.1. A convenient coordinate system must be called for. We choose to recast (1.2) by reversing the roles of dependent and independent variables — i.e. we think of \( u \) and \( v \) as curvilinear coordinates, and think of \( x \) and \( y \) as functions of \( u \) and \( v \). In other words, we subject (1.2) to the change of variables that is called hodograph transformation at times — see [Ev] and [Zw], for instance.

Let \([uv]\) be any smooth elliptic solution to (1.2), and observe the following.
(i) The level lines of \( u \) and those of \( v \) are free from singular points, and cross at a right angle. Moreover, the Jacobian determinant
\[
(2.1) \quad \Delta = u_xv_y - u_yv_x \quad \text{vanishes nowhere.}
\]

Let 
\[
[u,v] \mapsto [x(u,v), y(u,v)] \quad \text{be a local inverse of} \quad [x,y] \mapsto [u(x,y), v(x,y)],
\]
then observe the following equation
\[
(u_xv_y - u_yv_x)(x_au_v - x_vy_u) = 1
\]
and the propositions (ii)-(iv) below.
(ii) The following partial differential system holds
\begin{align*}
(2.2) \quad 1/E - 1/G &= 1, \quad F = 0.
\end{align*}
Here
\[
E = n^2(x,y)(x_u^2 + y_u^2), \quad F = n^2(x,y)(x_u x_v + y_u y_v), \quad G = n^2(x,y)(x_v^2 + y_v^2)
\]
in other words,
\[
n^2(x,y)((dx)^2 + (dy)^2) = E(du)^2 + 2Fdu
dv + G(dv)^2.
\]
(iii) The following systems and equations hold
\begin{equation}
\frac{\partial}{\partial u} \begin{bmatrix} x \\ y \end{bmatrix} = f \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{\partial}{\partial u} \begin{bmatrix} x \\ y \end{bmatrix},
\end{equation}
\begin{equation}
\frac{\partial}{\partial v} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{f} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial u} \begin{bmatrix} x \\ y \end{bmatrix},
\end{equation}
\begin{equation}
\frac{1}{f^2} = 1 + n^2(x,y)(x_u^2 + y_u^2), \quad f^2 = 1 - n^2(x,y)(x_u^2 + y_u^2).
\end{equation}
(iv) The following equations hold
\begin{equation}
f^2 = 1 - \frac{n^2(x,y)x_u^2}{1 + n^2(x,y)x_u^2}, \quad 2f = 1 - n^2(x,y)\frac{y_u^2}{x_u^2}.
\end{equation}
\begin{equation}
\frac{\partial^2}{\partial u^2} \begin{bmatrix} x \\ y \end{bmatrix} + f^4 \frac{\partial^2}{\partial v^2} \begin{bmatrix} x \\ y \end{bmatrix} = (1 - f^2) \begin{bmatrix} x_u^2 - y_u^2 \\ 2x_u y_u \end{bmatrix} \nabla \log n(x,y)
\end{equation}
in the event that \( n \) is identically 1, these equations read
\begin{equation}
x_{uu} + \left(1 - \frac{x_u^2}{1 + x_u^2}\right)^2 x_{uv} = 0, \quad y_{uu} + \left(1 - \frac{y_u^2}{1 + y_u^2}\right)^2 y_{uv} = 0.
\end{equation}

**Proof of (i).** System (1.2) tells us that \(|\nabla u| > 0,\) and that the gradients of \( u \) and \( v \) are orthogonal. An assumption gives \(|\nabla v| > 0.\)

The following equation
\begin{equation}
(u_x v_y - u_y v_x)^2 = |\nabla u|^2 |\nabla v|^2 - \langle \nabla u, \nabla v \rangle^2
\end{equation}
concludes the proof. \(\square\)

**Proof of (ii).** Since
\begin{equation}
\frac{\partial (u, v)}{\partial (x, y)} \times \frac{\partial (x, y)}{\partial (u, v)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\end{equation}
\begin{equation}
\frac{\partial (u, v)}{\partial (x, y)} \times \left[ \frac{\partial (u, v)}{\partial (x, y)} \right]^T = \begin{bmatrix} |\nabla u|^2 \\ \langle \nabla u, \nabla v \rangle \end{bmatrix},
\end{equation}
\begin{equation}
\frac{\partial (x, y)}{\partial (u, v)} \times \frac{\partial (x, y)}{\partial (u, v)} = \begin{bmatrix} x_u^2 + y_u^2 \\ x_u x_v + y_u y_v \end{bmatrix},
\end{equation}
we have
\begin{equation}
\frac{1}{x_u^2 + y_u^2} = |\nabla u|^2 - \langle \nabla u, \nabla v \rangle^2, \quad \frac{1}{x_v^2 + y_v^2} = |\nabla v|^2 - \langle \nabla u, \nabla v \rangle^2,
\end{equation}
\begin{equation}
x_u x_v + y_u y_v = -\langle \nabla u, \nabla v \rangle (x_u y_v - x_v y_u)^2.
\end{equation}

The conclusion ensues. \(\square\)

**Proof of (iii) and (iv).** The latter equation from (2.2) yields
\begin{equation}
x_u : y_v = y_u : (-x_v),
\end{equation}
hence the following systems result
\[
\begin{bmatrix}
x_u \\
y_u
\end{bmatrix} = f \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} \begin{bmatrix}
x_v \\
y_v
\end{bmatrix},
\begin{bmatrix}
x_v \\
y_v
\end{bmatrix} = \frac{1}{f} \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
x_u \\
y_u
\end{bmatrix}.
\]

Factor \( f \) is easily identified. Both the above systems give
\[
\text{sgn } f = \text{sgn } (x_u y_v - x_v y_u);
\]
the former and system (2.2) imply
\[
\frac{1}{f^2} = 1 + n^2(x, y)(x_v^2 + y_v^2);
\]
the latter and system (2.2) imply
\[
f^2 = 1 - n^2(x, y)(x_u^2 + y_u^2).
\]

Another arrangement reads
\[
\begin{bmatrix}
x_u \\
x_v
\end{bmatrix} = \begin{bmatrix}
0 & f \\
-1/f & 0
\end{bmatrix} \begin{bmatrix}
y_u \\
y_v
\end{bmatrix},
\begin{bmatrix}
y_u \\
y_v
\end{bmatrix} = \begin{bmatrix}
0 & -f \\
1/f & 0
\end{bmatrix} \begin{bmatrix}
x_u \\
x_v
\end{bmatrix}
\]
— two Bäcklund transformations, inverse of one another. The former and system (2.2) imply
\[
f^2 = \frac{1 - n^2(x, y)y_u^2}{1 + n^2(x, y)y_v^2},
\]
the latter and system (2.2) imply
\[
f^2 = \frac{1 - n^2(x, y)x_u^2}{1 + n^2(x, y)x_v^2}.
\]

The integrability conditions, which pertain to the Bäcklund transformations in hand, read
\[
\frac{\partial}{\partial u} \begin{bmatrix}
x_u \\
x_v
\end{bmatrix} = \frac{1}{f} \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix} \frac{\partial}{\partial u} \begin{bmatrix}
x \\
y
\end{bmatrix}
\]
and result in equation (2.7) after algebraic manipulations.

Equations (2.5) and (2.6) are consistent with one another and with the early equations (1.3) and (1.5), as Proposition (ii) and its proof show.

The proof is complete. \( \Box \)

2.2. In view of the discussion above, problem (i)-(iv) stated in Subsection 1.3 amounts to looking for solutions \([x, y]\) to the following partial differential system
\[
\frac{\partial}{\partial v} \begin{bmatrix}
x \\
y
\end{bmatrix} = \frac{1}{f} \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix} \frac{\partial}{\partial u} \begin{bmatrix}
x \\
y
\end{bmatrix},
\]
\[
\text{sgn } f = \text{sgn } \kappa, \quad f^2 = 1 - n^2(x, y)(x_u^2 + y_u^2),
\]
that are defined either in the half-strip
\[
a < u < b, \quad 0 < v < \infty
\]
or in an appropriate bounded subset if it, and satisfy the following initial conditions
\[
x(u, 0) = \alpha(u), \quad y(u, 0) = \beta(u) \quad \text{for } a \leq u \leq b.
\]

We will concentrate on such a problem. A behavior of solutions \([x, y]\) to (2.9) and (2.10) as \( v \) is close to 0 is fixed up in the next section. An algorithm for computing the same solutions is offered in Section 4. Section 5 is devoted to an example.
3. Behavior near the caustic

The state of affairs causes any solution of (2.9) and (2.10) to suffer from singularities near the initial line — indeed,

\[ x_{u}^{2}(u, v) + y_{u}^{2}(u, v) \to \infty \]

as \( a \leq u \leq b \) and \( v \) approaches 0. The proposition below offers more details on the subject, as well as a proof of expansions (1.12).

**Proposition 3.1.** Let \( x \) and \( y \) obey system (2.9) and initial conditions (2.10). Assume \( x(u, v) \) and \( y(u, v) \) depend smoothly upon \( u \) for every nonnegative, sufficiently small \( v \). Then the following asymptotic expansion holds

\[
\begin{bmatrix}
  x(u, v) \\
  y(u, v)
\end{bmatrix} = \begin{bmatrix}
  \alpha(u) \\
  \beta(u)
\end{bmatrix} + \frac{(3v)^{2/3}\text{sgn }\kappa(u)}{2|\kappa(u)|^{1/3}\sqrt{\alpha'(u)^2 + \beta'(u)^2}} \begin{bmatrix}
  -\beta'(u) \\
  \alpha'(u)
\end{bmatrix} + o(v^{2/3})
\]

as \( a \leq u \leq b \) and \( v \) is positive and approaches 0.

**Proof.** A hypothesis made on \( \alpha \) and \( \beta \) in Subsection 1.3, equations (2.9) and initial conditions (2.10) tell us that

\[ f(u, v) \to 0 \]
as \( a \leq u \leq b \) and \( v \) is positive and approaches 0. Therefore,

\[
\lim_{v \to 0} \frac{v^{1/3}}{f(u, v)} = \lim_{v \to 0} \frac{\text{sgn } f(u, v)}{\mathcal{F}(u, v)} \left\{ \text{sgn } f(u, v) \left[ \frac{\partial}{\partial u} f^{3}(u, v) \right]^{1/3} \right\},
\]

by L’Hospital’s rule. Equations (2.9) give successively

\[
\frac{\partial}{\partial v} n^{2}(x, y) = \frac{2n^{2}(x, y)}{f(u, v)} \left\{ \nabla \log n(x, y), \begin{bmatrix}
  -y_{u} \\
  x_{u}
\end{bmatrix} \right\},
\]

\[
\frac{\partial}{\partial v} (x_{u}^{2} + y_{u}^{2}) = -\frac{2}{f(u, v)} (x_{u}y_{uu} - x_{uu}y_{u}),
\]

and

\[
\frac{\partial}{\partial v} f^{3}(u, v) = 3 (x_{u}^{2} + y_{u}^{2})^{3/2} n^{2}(x, y) \left\{ \frac{x_{u}y_{uu} - x_{uu}y_{u}}{(x_{u}^{2} + y_{u}^{2})^{1/2}} - \left\{ \nabla \log n(x, y), (x_{u}^{2} + y_{u}^{2})^{-1/2} \begin{bmatrix}
  -y_{u} \\
  x_{u}
\end{bmatrix} \right\} \right\}
\]

We infer

\[
\lim_{v \to 0} (3v)^{1/3} \left\{ \frac{x_{v}}{y_{v}} \right\} = \frac{\text{sgn } \kappa(u)}{|\kappa(u)|^{1/3}\sqrt{\alpha'(u)^2 + \beta'(u)^2}} \begin{bmatrix}
  -\beta'(u) \\
  \alpha'(u)
\end{bmatrix}
\]

because of the very definition of \( \kappa \).

The conclusion follows. \( \square \)

4. Discrete setting

4.1. Rendering problem (2.9) and (2.10) into an effective discrete form entails coping with singularities of solutions, overflows, and ill-posedness in the sense of Hadamard.

Singularities result from features of both the system and the initial conditions in hand, as already remarked in the previous section. Overflows take place whenever the constraint

\[ n^{2}(x, y)(x_{u}^{2} + y_{u}^{2}) < 1 \]
chances to be violated. Note that the system
\[
\frac{\partial}{\partial v} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{\partial}{\partial u} \begin{bmatrix} x \\ y \end{bmatrix}
\]
— Cauchy-Riemann, a possible linearized version of \(2.9\) — possesses obvious solutions
\[
\begin{bmatrix} x \\ y \end{bmatrix} = \exp(-\sqrt{t} + tv) \begin{bmatrix} \cos(tu) \\ \sin(tu) \end{bmatrix}
\]
which are highly instable, i.e.
\[
\sup \left\{ \left[ \frac{\partial^j x}{\partial u^j} (u, 0) \right]^2 + \left[ \frac{\partial^j y}{\partial u^j} (u, 0) \right]^2 : -\infty < u < \infty \right\} \rightarrow 0
\]
as \( t \uparrow \infty \) and \( j = 1, 2, \ldots \), and
\[
\inf \left\{ x^2(u, v) + y^2(u, v) : -\infty < u < \infty \right\} \rightarrow \infty
\]
as \( t \uparrow \infty \) and \( v \) is positive.

Ill-posedness is a typical drawback of initial value problems for partial differential equations and systems of elliptic type. It was observed by Hadamard \([Ha1]-[Ha2]\) and deeply investigated by John \([Jo1]-[Jo3]\), Lavrentiev jr. \([Lv1]-[Lv3]\), \([LV]\), Miller \([Mi1]-[Mi2]\), Payne \([Pa1]-[Pa4]\) and \([PS1]-[PS2]\), Pucci \([Pc1]-[Pc4]\), and Tikhonov \([Ti1]-[Ti5]\). Classical surveys on the subject have been authored by Lavrentiev jr., \([Lv4]\), [LRS], Payne \([Pa5]\), and Tikhonov \([TA]\). Related information is in \([Br1]-[Br2]\), \([BDV]\), \([BG]\), \([Is]\), \([IVT]\), \([Mr]\), \([Na]\), \([Ri]\) and \([Ta]\). A sample of more recent contributions includes \([Ali]\), \([AI]\), \([AK]\), \([Be]\), \([Br3]\), \([BKP]\), \([BE]\), \([BR]\), \([BV]\), \([Bs]\), \([Ca]\), \([Cr]\), \([CD]\), \([CE]\), \([Co]\), \([Cd]\), \([De]\), \([El1]-[El2]\), \([Ew1]-[Ew2]\), \([HH]\), \([HR]\), \([Hf]\), \([KV]\), \([Kn]\), \([LV]\), \([Li]\), \([Ma1]-[Ma2]\), \([Mo]\), \([Nt1]-[Nt2]\), \([Pa6]-[Pa8]\), \([Ph]\), \([RE]\), \([RHH]\), \([RS]\), \([Rm]\), \([St]\), \([TGSY]\), \([TLY]\), \([VFC]\), \([Wa]\).

As experience suggests, one might attempt to contend with ill-posedness via a priori bounds on solutions and similar devices. We opt not to touch on this issue in the present paper, and focus our attention on constructive aspects instead.

We rely upon: (i) asymptotic expansions, describing how relevant solutions behave near the initial line; (ii) a technique of approximate differentiation, especially designed for working in presence of errors; (iii) an appropriate injection of artificial viscosity, which softens a coefficient and protects against overflows; (iv) an imitation of the quasi-reversibility method.

4.2. Besides data exposed to view, our method takes six parameters in input: \(M, N, \lambda, \mu, \nu, \xi\). The first two are large integers that specify the number of samples in hand — e.g. \(M = N = 100\). The remaining four parameters set the tone of smoothing processes: \(\lambda\) and \(\mu\) relate to approximate differentiation; \(\nu\) stands for viscosity; \(\xi\) relates to quasi-reversibility.

4.3. The following equations and inequality
\[
ustep = (b - a)/(M - 1), \quad vstep > 0,
\]
\[
u_j = a + (j - 1) ustep (j = 1, \ldots, M), \quad v_k = (k - 1) vstep (k = 1, \ldots, N),
\]
will be in force throughout this section. We choose a mesh to consist of the following points
\[
(u_j, v_k) (j = 1, \ldots, M; k = 1, \ldots, N),
\]
and store sample values at mesh points in the following matrices
\[
\begin{bmatrix}
  x(j, k) \mid j = 1, \ldots, M; k = 1, \ldots, N,
  y(j, k) \mid j = 1, \ldots, M; k = 1, \ldots, N
\end{bmatrix}
\]

The columns of these matrices, i.e.
\[x(\cdot, 1), x(\cdot, 2), \ldots, x(\cdot, N), \quad y(\cdot, 1), y(\cdot, 2), \ldots, y(\cdot, N),\]
are recursively generated in the following way.

**First step.**
\[x(j, 1) = \alpha(u_j), \quad y(j, 1) = \beta(u_j) \quad (j = 1, \ldots, M),\]
according to initial conditions (2.10).

**Second step.**
\[
x(j, 2) = x(j, 1) + \frac{\text{sgn}\kappa(u_j)}{2|\kappa(u_j)|^{1/3}} \left(\frac{-\beta'(u_j)}{\sqrt{\alpha'(u_j)^2 + \beta'(u_j)^2}}(3\nu_2)^{2/3}\right),
\]
\[
y(j, 2) = y(j, 1) + \frac{\text{sgn}\kappa(u_j)}{2|\kappa(u_j)|^{1/3}} \left(\frac{\alpha'(u_j)}{\sqrt{\alpha'(u_j)^2 + \beta'(u_j)^2}}(3\nu_2)^{2/3}\right) \quad (j = 1, \ldots, M),
\]
according to expansion (3.1).

**Further steps.** For \(k = 3, \ldots, N\) do actions (i)-(iii) below.

(i) Differentiate.
\[
X = x(\cdot, k - 1), \quad Y = y(\cdot, k - 1),
\]
\[
A = DX, \quad B = DY.
\]

Here
\[
\lambda = \text{a dimensionless positive parameter},
\]
\[
\frac{\mu}{b - a} \simeq 0.0415 + 0.5416 \times M^{-1/2} - 0.6426 \times M^{-1} + 1.3706 \times M^{-3/2},
\]
\[
K(u) = \frac{1}{\pi} \int_0^\infty \frac{\cos(ut)}{1 + t^8} \, dt
\]
for every \(u\), and
\[
D = \frac{1}{\mu} \left[ K' \left( \frac{u_i - u_j}{\mu} \right) \right]_{i,j=1,\ldots,M}^{-1} \left\{ \lambda \text{Id} + \left[ K \left( \frac{u_i - u_j}{\mu} \right) \right]_{i,j=1,\ldots,M} \right\}^{-1}
\]
—a matrix that mimics differentiation with respect to variable \(u\), and is analyzed in Appendix C.

(ii) Enter viscosity.
\[
f(j) = P(\nu, n(X(j), Y(j))\sqrt{A(j)^2 + B(j)^2} \text{sgn}\kappa(u_j) \quad (j = 1, \ldots, M),
\]
\[
U = \begin{bmatrix}
  f(1) & 0 & \cdots & 0 \\
  0 & f(2) & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & f(M)
\end{bmatrix}^{-1}, \quad B = \begin{bmatrix}
  f(1) & 0 & \cdots & 0 \\
  0 & f(2) & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & f(M)
\end{bmatrix}, \quad V = \begin{bmatrix}
  f(1) & 0 & \cdots & 0 \\
  0 & f(2) & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & f(M)
\end{bmatrix}^{-1}
\]

Here
\[
0 < \nu = \text{artificial viscosity} \leq \pi/2,
\]
and $P$ is given by

$$P(\nu,\rho) = \begin{cases} \sqrt{1 - \rho^2} & \text{if } 0 \leq \rho \leq (1 + \sin^2 \nu)^{-1/2}, \\ \frac{1}{\rho + \sqrt{\rho^2 - \cos^2 \nu}} & \text{if } \rho > (1 + \sin^2 \nu)^{-1/2} \end{cases}$$

— observe that

$0 \leq \rho \mapsto P(\nu,\rho)$ is strictly positive and continuously differentiable,

$P(\nu,\rho)$ approaches $\sqrt{1 - \rho^2}$ uniformly as $0 \leq \rho \leq 1$ and $\nu$ approaches 0.

(iii) Enter quasi-reversibility.

$x(\cdot, k) = \varphi(v_k), \ y(\cdot, k) = \psi(v_k)$.

Here $\varphi$ and $\psi$ are the vector-valued mappings that are generated thus:

$$\xi = \text{a dimensionless positive parameter;}$$

$$R = (\text{ustep})^{-2} \begin{bmatrix} 2 & -5 & 4 & -1 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & -1 & 4 & -5 & 2 \end{bmatrix},$$

a caricature of a second-order derivative;

$$\varphi(v_{k-1}) = X, \ \varphi'(v_{k-1}) = U,$$

$$3\xi \|R\varphi(v_k)\|^2 + (\text{ustep}) \int_{v_{k-1}}^{v_k} \|\varphi''(v)\|^2 dv = \text{minimum},$$

$$\psi(v_{k-1}) = Y, \ \psi'(v_{k-1}) = V,$$

$$3\xi \|R\psi(v_k)\|^2 + (\text{ustep}) \int_{v_{k-1}}^{v_k} \|\psi''(v)\|^2 dv = \text{minimum}.$$ As is easy to see,

$$\varphi'''(v) = 0 \ \text{if } v_{k-1} < v < v_k,$$

$$\varphi''(v_k) = 0, \ \varphi'''(v_k) = 3\xi (\text{ustep})^{-2}(R^TR) \varphi(v_k);$$

$$\psi'''(v) = 0 \ \text{if } v_{k-1} < v < v_k,$$

$$\psi''(v_k) = 0, \ \psi'''(v_k) = 3\xi (\text{ustep})^{-2}(R^TR) \psi(v_k).$$

Therefore,

$$x(\cdot, k) = (\text{Id} + \xi \text{ustep}(R^TR))^{-1} (X + \text{ustep}U),$$

$$y(\cdot, k) = (\text{Id} + \xi \text{ustep}(R^TR))^{-1} (Y + \text{ustep}V).$$

— in other words $x(\cdot, k)$ and $y(\cdot, k)$ are mollified versions of $X + \text{ustep}U$ and $Y + \text{ustep}V$, respectively.

Last step. End.
4.4. As a matter of fact, the above process simulates the following partial differential system

\[
\frac{\partial}{\partial v} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{f} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial u} \begin{bmatrix} x \\ y \end{bmatrix} - \xi^4 \frac{\partial^4}{\partial u^4} \begin{bmatrix} x \\ y \end{bmatrix},
\]

\[f = P\left(\nu, n(x, y)\sqrt{x_u^2 + y_u^2}\right) \text{sgn} \kappa(u),\]

where the modified, and extra, terms protect against overflows and instability. The methods based on either artificial viscosity or quasi-reversibility share basic features: they all suggest perturbing the underlying partial differential equation or system in a way or another, in order to palliate obstructions. The quasi-reversibility method was introduced in [LL], and improved in [Mi5], [GZ]; other references are [AT], [ATY], [Bo1]-[Bo2], [DR], [Ex1], [Go], [Hu], [HZ], [KS], [La], [Pa5], [Sh1]-[Sh2], [TT].

5. Example

For simplicity, suppose refractive index \( n \) is 1. Consider the curve — known as Tschirnhausen’s cubic or trisectrix of Catalan [Lw], [Se] — whose parametric equations read

\[x = \frac{1}{2}(1 - 3t^2), \quad y = \frac{t}{2}(3 - t^2), \quad (t = \text{parameter}),\]

and imply

\[
\text{arc length} = \frac{t}{2}(t^2 + 3), \quad t = 2 \sinh\left(\frac{1}{3} \arcsinh(\text{arc length})\right).
\]

---

**Figure 2.** An arc of Tschirnhausen’s cubic: a gross sampling (stars) and a denoised version (circles).
Figure 3. Level lines where either $u = \text{constant}$ or $v = \text{constant}$.

A geometric optical eikonal $w$ making Tschirnhausen’s cubic a caustic is represented by the following equations

$$x = \frac{1}{2}(1 - 3t^2) - \frac{2st}{1 + t^2}, \quad y = \frac{t}{2}(3 - t^2) + \frac{s(1 - t^2)}{1 + t^2},$$

$$w = s + \frac{t}{2}(3 + t^2) \quad (s, t = \text{parameters})$$

in the light region. As arguments from Appendix B show, the same eikonal can be continued in the shadow region via the following equations

$$x = 1 - 2(s^2 + t^2) + s^4 - 2s^2t^2 - 3t^4 \quad \frac{2}{2(1 + s^2 + t^2)}, \quad y = \frac{t}{2}\left[3 - 2(s^2 - t^2) - (s^2 + t^2)^2\right] \quad \frac{2}{2(1 + s^2 + t^2)},$$

$$w = is + t - \frac{2(is + t)}{1 + (is + t)^2}x + \frac{1 - (is + t)^2}{1 + (is + t)^2}y \quad (s, t = \text{real parameters}).$$

The method from Sections 1 to 4 goes in the following way.

(i) Consider the arc of the Tschirnhausen’s cubic where

$$-1.5 \leq t \leq 2.2,$$

and let

$$(\alpha_j, \beta_j) \quad (j = 1, \ldots, 31)$$

be a gross sampling of such an arc — in other words,

$$t_j = -1.5 + 3.7 \frac{j - 1}{30} \quad (j = 1, \ldots, 31),$$

$$\alpha_j = \frac{1}{2}(1 - 3t_j^2) + (5\% \text{ random noise}), \quad \beta_j = \frac{t_j}{2}(3 - t_j^2) + (5\% \text{ random noise}).$$
(ii) Plug gross data into the following denoising process

\[
\lambda = 0.005, \quad \mu = 0.1260,
\]

\[
\sum_{j=1}^{31} \left[ \alpha \left( \frac{j-1}{30} \right) - \alpha_j \right]^2 + \lambda \int_{-\infty}^{\infty} [\mu^7(\alpha'''^2 + \mu^{-1} \alpha^2)] dt = \text{minimum},
\]

\[
\sum_{j=1}^{31} \left[ \beta \left( \frac{j-1}{30} \right) - \beta_j \right]^2 + \lambda \int_{-\infty}^{\infty} [\mu^7(\beta'''^2 + \mu^{-1} \beta^2)] dt = \text{minimum},
\]

and let the path that is represented by the following equations

\[
x = \alpha(t), \quad y = \beta(t), \quad 0 \leq t \leq 1,
\]

surrogate the original Tschirnhausen’s cubic.

(iii) Adjust parametric equations as follows:

\[
\frac{dt}{du} = [\alpha'(t)^2 + \beta'(t)^2]^{-1/2}, \quad t(0) = 0,
\]

\[
x = \alpha(t(u)), \quad y = \beta(t(u)), \quad 0 \leq u \leq \text{Length}.
\]

(iv) Select requisite parameters thus

\[
M = 101, \quad N = 91, \quad v_{\text{step}} = 0.005, \quad \nu = 0.5, \quad \xi = 0.9;
\]

and then set the algorithm from Section 4 to work.

Results are shown in figures 2, 3 and 4, and comfortably agree with those drawn from closed formulas.

**Acknowledgement**

The Authors wish to thank L. Sgheri and G.A. Viano for their valuable help and advice.
Appendix A.

Basic mathematical lineaments of two-dimensional geometrical optics are outlined in the next paragraphs. Selected references on geometrical optics, and on some of its generalizations and applications are [BB], [Du], [GS], [Jn], [Ke], [KL], [Kl1]-[Kl2], [KK], [KO1], [Ln], [MF], [MO], [Ru].

A.1. Terminology. Let \( n \) be a refractive index — i.e. a tractable function of two real variables \( x \) and \( y \), which takes positive values only and is bounded away from zero locally. Any real-valued, suitably smooth solution \( w \) to (1.1) is a geometric optical eikonal (GOE). The domain of \( w \), plus parts of the relevant boundary, is a light region; the complement of it is a shadow region. The trajectories of \( \nabla w \), namely the orbits of the following dynamical system

\[
\begin{vmatrix}
\frac{dx}{w_x(x,y)} & \frac{dy}{w_y(x,y)}
\end{vmatrix} = 0
\]

are called lines of steepest descent — irrespective of whether they are genuine lines or not. A line of steepest descent is a ray if \( w \) is twice continuously differentiable in some neighborhood of it; any line of steepest descent, which is not a ray, is a caustic. (Rays are smooth curves, which have one degree of freedom and travel all over areas without intersecting one another. Caustics are exceptions in a sense: loosely speaking, they can be thought of as envelopes of rays.) The Riemannian arc length, whose element takes the following form

\[
\sqrt{n(x,y)^2 dx^2 + dy^2}
\]

is known as travel time — travel time is an alias of the customary arc length in the case where \( n \equiv 1 \).

A.2. GOEs and travel time. (i) The restriction of any GOE \( w \) to either an appertaining ray or caustic automatically coincides with a properly rescaled travel time \( t \).

(ii) Let nodal line be an alias of locus of zeros. The value of any GOE \( w \) at any point \((x, y)\) equals either the travel time between \((x, y)\) and a nodal line of \( w \) or the negative of such a travel time — provided \((x, y)\) is not a long way off.

Proof of (i). By definition, both rays and caustics of \( w \) obey

\[
\frac{dx}{w_x(x,y)} = \frac{dy}{w_y(x,y)};
\]

the following equation

\[
t = \text{a rescaled travel time}
\]

is an alias of

\[
dt = \pm n(x,y) \sqrt{(dx)^2 + (dy)^2}.
\]

Consequently,

\[
(dw/dt)^2 = n^{-2}(x,y)(w_x^2 + w_y^2)
\]

along any ray or caustic in question. Property (i) follows. □

Property (ii) is a consequence of (i) and Fermat’s principle below.
A.3. Fermat’s principle. The travel time geodesics, i.e. those curves which render
\[ \int n(x, y) \sqrt{dx^2 + dy^2} \]
either a minimum or stationary, are characterized by the following second-order ordinary differential equation
\[ \text{curvature} = \langle \text{unit normal}, \nabla \log n(x, y) \rangle \]
— they have geodesic curvature 0, and are perfect straight lines in the case where \( n \equiv 1 \). The rays of any GOE are geodesics with respect to travel time. The travel time geodesics that are trajectories of some proper vector field — i.e. have one degree of freedom and are free from mutual intersections — are the rays of some GOE.

The foregoing statements rest upon first principles of calculus of variations, differential geometry and ordinary differential equations. Recall that the following formulas apply to any smooth parametric curve:
\[ \text{velocity} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}, \]
\[ \text{unit tangent} = (\text{velocity})^{-1} \frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{unit normal} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{(unit tangent)} \]
\[ (\text{velocity})^{-1} \frac{d}{dt} (\text{unit tangent}) = (\text{curvature}) \times (\text{unit normal}), \]
\[ (\text{velocity})^{-1} \frac{d}{dt} (\text{unit normal}) = -(\text{curvature}) \times (\text{unit tangent}), \]
\[ \text{curvature} = (\text{velocity})^{-3} \begin{vmatrix} \frac{dx}{dt} & \frac{d^2x}{dt^2} \\ \frac{dy}{dt} & \frac{d^2y}{dt^2} \end{vmatrix}; \]
\[ \text{curvature} = (\text{velocity})^{-3} h^{-3} \begin{vmatrix} x(t-h) & y(t-h) & 1 \\ x(t) & y(t) & 1 \\ x(t+h) & y(t+h) & 1 \end{vmatrix} + O(h^2), \]
\[ \text{curvature} = (\text{velocity})^{-3} h^{-3} \left\{ \begin{vmatrix} x(t) & y(t) & 1 \\ x(t+h) & y(t+h) & 1 \\ x(t+2h) & y(t+2h) & 1 \end{vmatrix} - \frac{1}{2} \begin{vmatrix} x(t+h) & y(t+h) & 1 \\ x(t+2h) & y(t+2h) & 1 \\ x(t+3h) & y(t+3h) & 1 \end{vmatrix} \right\} + O(h^2). \]

Recall that the following formula
\[ \text{curvature of the lines of steepest descent} = \text{Jacobian determinant of } |\nabla w|^{-1} \& w \]
applies whenever \( w \) is smooth and has no critical point. Recall also that
\[ n(x, y) \times (\text{Geodesic curvature}) = \text{Euclidean curvature} - \langle \text{unit normal}, \nabla \log n(x, y) \rangle \]
if travel time is an alternative Riemannian metric in force.
A.4. Initial value problems and geometry of their solutions. The condition of taking given values along some given path is qualified initial according to usage. Seeking a GOE, which obeys some initial condition, is an initial value problem. Such a problem has either two different solutions or no solution at all, depending on whether the eikonal equation and the initial condition match or conflict. Generally speaking, a solution \( w \) can be detected in the former case by successively detecting the objects listed below, based upon the arguments provided.

- The values of \( \nabla w \) along the initial curve.

Since the eikonal equation specifies the length of \( \nabla w \) and the initial condition identifies a tangential component of \( \nabla w \), the normal component of \( \nabla w \) along the initial curve comes out in two different modes.

- The rays of \( w \).

An ODE reasoning demonstrates that the travel time geodesics, which live near the initial curve and leave it with the same direction as \( \nabla w \), are the trajectories of a smooth vector field. By Fermat’s principle, these geodesics are the requested rays indeed.

- The values of \( w \) itself on each ray.

Property (i) from Subsection 2 fits the situation well.

A.5. Standard initial value problems. The present item concerns existence, regularity and the number of GOEs that satisfy orthodox initial conditions.

Assume henceforth all ingredients are smooth and let IC stand for initial curve in shorthand. Let

\[
\begin{align*}
 x &= \alpha(t), \quad y = \beta(t), \quad a \leq t \leq b \\
 t &= \text{a travel time.}
\end{align*}
\]

be a parametric representation of IC such that

\[
 t = \text{a travel time.}
\]

Let the initial condition imply

\[
 w(x, y) = \gamma(t)
\]

as \( x, y \) and \( t \) satisfy (A.1), and assume

\[
 |\frac{d\gamma}{dt}(t)| < 1
\]

for \( a \leq t \leq b \). Then exactly two GOEs satisfy the initial condition displayed. Moreover, these eikonals are smooth in a full neighborhood of IC — the relevant light regions surround it completely.

The case where refractive index \( n \) is constant involves explicit formulas, as well as gives evidence to interactions among the eikonal equation, Burgers-type equations and Bäcklund transformations. Assume

\[
 n = 1,
\]

and let (A.1) to (A.3) be in force. Define \( \varphi, \psi, \omega, p, q \) by

\[
\begin{align*}
 \cos \varphi &= \alpha', \quad \sin \varphi = \beta' \\
 \cos \psi &= \gamma', \quad \sin \psi = \pm \sqrt{1 - (\gamma')^2} \\
 \omega &= \varphi + \psi \\
 p &= \cos \omega, \quad q = \sin \omega.
\end{align*}
\]
Since the appurtenant Jacobian matrix equals
\[
\begin{bmatrix}
q(s) & p(s) \\
-p(s) & q(s)
\end{bmatrix}
\begin{bmatrix}
\sin \psi(s) - (w - \gamma(s)) \frac{d\omega(s)}{ds} & 0 \\
0 & 1
\end{bmatrix}
\]
the following pair
\[
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \begin{bmatrix}
\alpha(s) \\
\beta(s)
\end{bmatrix} + (w - \gamma(s)) \begin{bmatrix}
p(s) \\
q(s)
\end{bmatrix}
\]
makes \( s \) and \( w \) implicit functions of \( x \) and \( y \) in a neighborhood of IC. The properties listed below ensue. Function \( s \) obeys the following equations
\[
\begin{align*}
-q(s) (x - \alpha(s)) + p(s) (y - \beta(s)) &= 0, \\
p(s) \frac{\partial s}{\partial x} + q(s) \frac{\partial s}{\partial y} &= 0,
\end{align*}
\]
\[
\left( \frac{\partial s}{\partial x} \right)^2 + \left( \frac{\partial s}{\partial y} \right)^2 = \left[ \sin \psi(s) - (w - \gamma(s)) \frac{d\omega(s)}{ds} \right]^2.
\]
(The first assures that the level lines of \( s \) are straight, the second is a PDE of Burgers type.) Function \( w \) obeys the following eikonal equation
\[
\left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 = 1.
\]
Functions \( s \) and \( w \) are related by the following equations
\[
\nabla w = \begin{bmatrix}
p(s) \\
q(s)
\end{bmatrix},
\]
\[
\begin{bmatrix}
w_{xx} & w_{xy} \\
w_{xy} & w_{yy}
\end{bmatrix} = \frac{d\omega(s)}{ds} \begin{bmatrix}
-w_y & -w_y \\
w_x & w_x
\end{bmatrix} \begin{bmatrix}
s_x \\
s_y
\end{bmatrix}.
\]
(The former is a \textit{Bäcklund transformation}, which pairs solutions to the Burgers and the eikonal equations mentioned above. It assures that \( \nabla s \) and \( \nabla w \) are orthogonal, and the level lines of \( s \) are both lines of steepest descent and isoclines of \( w \).) The Euclidean metric obeys
\[
dx^2 + dy^2 = |\nabla s|^{-2} ds^2 + dw^2.
\]
There holds
\[
s(x, y) = t, \quad w(x, y) = \gamma(t)
\]
as \( x, y \) and \( t \) satisfy (A.1). If \( \omega \) is free from critical points, the line where
\[
\begin{bmatrix}
x \\
y
\end{bmatrix} = \begin{bmatrix}
\alpha(s) \\
\beta(s)
\end{bmatrix} + \sin \psi(s) \frac{d\omega(s)}{ds} \begin{bmatrix}
p(s) \\
q(s)
\end{bmatrix}
\]
is a caustic. (Such a line is an envelope of the level lines of \( s \). Function \( s \) develops shocks along it, the restriction of \( w \) to it equals arc length, and the second-order derivatives of \( w \) blow up there.)
A.6. **Borderline initial value problems and caustics.** The present item is a recipe for producing caustics, which involves coupling the eikonal equation with borderline initial conditions. In the case where refractive index $n$ is identically 1, any smooth convex curve can be viewed as a caustic provided a GOE is detected, whose restriction to the curve in hand equals a relevant arc length.

Let (A.1) specify IC and let (A.2) hold. Let the initial condition imply

$$w(x, y) = t$$

as $x, y$ and $t$ satisfy (A.1). Assume that the appropriate *geodesic curvature* of IC is free from zeros — in other words, let (A.1) and the following equation

$$\kappa \text{ (velocity)}^2 = \text{Euclidean curvature} - \langle \text{unit normal}, \nabla \log n(x, y) \rangle$$

result in

$$\kappa \text{ vanishes nowhere.}$$

Then exactly two GOEs satisfy the present initial condition. Both these eikonals fail to exist on both sides of, and be smooth near IC. They turn IC into a *caustic*, and make the side of it, which

$$(\text{sgn } \kappa) \times \text{(unit normal)}$$

points to, a *shadow* region. Either eikonal in hand obeys

$$w(x, y) = s \pm \frac{2}{3} \sqrt{2} |r| \sqrt{\kappa(s)} + O(r^2), \quad \Delta w(x, y) = \pm \frac{1}{\sqrt{2}} |r| \sqrt{-\frac{1}{2} \kappa(s)} + O(1)$$

at every point $(x, y)$ that belongs to the light region and is close enough to IC — in particular, the two-sheeted surface, made up of the two eikonals in hand, exhibits an *edge of regression* above IC.

Here $r$ and $s$ are the curvilinear coordinates that the following pair

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha(s) \\ \beta(s) \end{bmatrix} + r \left( \alpha'(s)^2 + \beta'(s)^2 \right)^{-\frac{1}{2}} \begin{bmatrix} -\beta'(s) \\ \alpha'(s) \end{bmatrix}$$

relates to rectilinear coordinates $x$ and $y$. Coordinate $r$ is a signed distance from IC, coordinate $s$ makes $(\alpha(s(x, y)), \beta(s(x, y)))$ the orthogonal projection of $(x, y)$ on IC. The former is constant on the parallel lines to IC and obeys the following eikonal equation

$$\left( \frac{\partial r}{\partial x} \right)^2 + \left( \frac{\partial r}{\partial y} \right)^2 = 1;$$

the latter is constant on the normal straight-lines to IC and obeys the following Burgers-type equation

$$\begin{vmatrix} \partial s/\partial x & \partial s/\partial y \\ \alpha'(s) & \beta'(s) \end{vmatrix} = 0.$$

Both are subject to the following Bäcklund transformation

$$\nabla r = \left[ \alpha'(s)^2 + \beta'(s)^2 \right]^{-\frac{1}{2}} \begin{bmatrix} -\beta'(s) \\ \alpha'(s) \end{bmatrix}$$

and exhibit singularities along the evolute of IC.
Here we sketch a method of continuing a two-dimensional geometric optical eikonal past a caustic. Though rigorous, such a method is slightly reminiscent of the so-called theory of complex rays — cf. [CLOT] or [KFA], for instance. It applies in the case where the refractive index equals 1, and can be used for simultaneously continuing solutions to non-viscous Burgers equation beyond shock lines.

B.1. Our method involves analytic continuation from the real-number axis into the complex plane — an ill-posed process in the sense of Hadamard. Let \( h \) be a real or complex-valued function of a real variable, or even a list of samples. An analytic continuation of \( h \) is a holomorphic function of a complex variable, whose domain surrounds the real axis and whose restriction to the real axis fits \( h \) well — in other words, a solution \( H \) of the following initial value problem for Cauchy-Riemann equation

\[
\frac{\partial H}{\partial y} = i \frac{\partial H}{\partial x}, \quad H(\cdot, 0) \simeq h.
\]

If \( h \) is an analytic function, and is not polluted by noise, an analytic continuation \( H \) of \( h \) results from obvious formulas. For example,

\[
H(x, y) = \sum_{k=0}^{\infty} \frac{d^k h(x)}{dx^k} (iy)^k \frac{1}{k!};
\]

or

\[
H(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[i\xi(x+iy)] \hat{h}(\xi) d\xi,
\]

where \( \hat{h} \) denotes Fourier transformation.

If \( h \) collects gross data, an effective analytic continuation \( H \) of \( h \) can be obtained by analytically continuing an appropriate, smoothed and denoised version of \( h \). Consider for instance the case where

\[-\infty < a < b < \infty,\]

\[N = \text{an integer larger than 1},\]

a mesh size is given thus

\[\Delta x = (b-a)/(N-1),\]

nodes are given by

\[x_j = a + (j-1)\Delta x \quad (j = 1, \ldots, N);\]

and \( h \) stands for

\[h_1, \ldots, h_N,\]

a sequence of noisy samples.

An ad hoc analytic continuation \( H \) solves the following least square problem

\[
\Delta x \sum_{j=1}^{N} [H(x_j, 0) - h_j]^2 + \frac{\lambda}{2L} \int_{a-\Delta x/2}^{b+\Delta x/2} dx \int_{-L}^{L} |H(x, y)|^2 dy = \text{minimum}
\]
in a convenient class of holomorphic functions — e.g. the class of trigonometric polynomials of a suitable degree. Here
\[
\frac{\lambda}{2L} \int_{a-\Delta x/2}^{b+\Delta x/2} dx \int_{-L}^{L} |H(x,y)|^2 dy
\]
plays the role of a penalty; \( \lambda \) and \( L \) are regulating parameters — \( \lambda \) is related to noise, \( L \) is related to a priori information.

An explicit expression of \( H \) can be derived via discrete Fourier transforms. Suppose for simplicity that \( N \) is odd, say \( N = 2n + 1 \ (n = 1, 2, \ldots) \);

let \( T = N \Delta x \),

and let \( DFT \) be the discrete Fourier transform that obeys the following equations
\[
DFT_k(h) = \sum_{j=1}^{N} h_j \exp \left( -2\pi ik \frac{x_j}{T} \right) \quad (k = -n, \ldots, n),
\]
\[
h_j = \frac{1}{N} \sum_{k=-n}^{n} DFT_k(h) \exp \left( 2\pi ik \frac{x_j}{T} \right) \quad (j = 1, \ldots, N),
\]
\[
\frac{1}{N} \sum_{k=-n}^{n} |DFT_k(h)|^2 = \sum_{j=1}^{N} |h_j|^2
\]
— cf. [BH], for instance. If
\[
C_0 = 1, \ C_k = \frac{\sinh(4\pi k L/T)}{4\pi k L/T}, \ C_{-k} = C_k \quad (k = 1, \ldots, n),
\]
then
\[
H(x, y) = \sum_{k=-n}^{n} (1 + \lambda C_k)^{-1} DFT_k(h) \exp \left( 2\pi i k \frac{x + iy}{T} \right).
\]
This is a \( T \)-periodic trigonometric polynomial of degree \( n \) that enjoys the following properties
\[
\sum_{j=1}^{N} |H(x_j, 0) - h_j|^2 \leq \left( \frac{\lambda}{\lambda + 1/C_n} \right)^2 \sum_{j=1}^{N} |h_j|^2,
\]
\[
\frac{1}{2L} \int_{a-\Delta x/2}^{b+\Delta x/2} dx \int_{-L}^{L} |H(x,y)|^2 dy \leq \frac{\Delta x}{4\lambda},
\]
and
\[
H(x, y) = \sum_{j=1}^{N} H(x_j, 0) D_N \left( 2\pi \frac{x - x_j + iy}{T} \right)
\]
— here \( D_N \) denotes the Dirichlet, or periodic sinc function obeying
\[
D_N(x) = \frac{\sin(Nx/2)}{N \sin(x/2)}
\]
if \( x/(2\pi) \) is not an integer.
More information on analytic continuation can be found in [A1], [A2], [BV1],
[BK], [CM], [CS1]-[CS2], [DS], [DG1]-[DG2], [FD1], [FD2], [Fr2], [FIT1], [Gu, He],
[Lv5], [LA], [Le], [LRS], [Mi3]-[Mi4], [MV], [Re], [Sa], [Sy], [TA], [Uz], [Ve], [Vu],
[Zh].

B.2. Consider a plane curve \( C \) that either is inherently smooth or results from a
suitable smoothing process of raw data. Assume \( C \) is analytic and its curvature
vanishes nowhere. For simplicity, assume \( C \) is the graph of the following equation
\[ y = f(x), \]
and \( f \) is convex.

Alternative parametric representations of \( C \), which are instrumental throughout,
include
\[
\begin{align*}
(B.1) & \quad x = t, \quad y = f(t), \\
(B.2) & \quad x = g'(t), \quad y = tg'(t) - g(t),
\end{align*}
\]
where parameter \( t \) is the slope of the tangent straight-line. Here \( g \) denotes the
Legendre conjugate of \( f \) — recall from e.g. [Re] that \( f \) and \( g \) are related thus
\[
\begin{align*}
\gamma(t) & = \int \sqrt{\alpha'(s)^2 + \beta'(s)^2} \, ds, \\
\kappa & = \frac{\alpha'\beta'' - \alpha''\beta'}{[(\alpha')^2 + (\beta')^2]^{3/2}}.
\end{align*}
\]
The following pair
\[
(B.4) \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha(s) \\ \beta(s) \end{bmatrix} + \frac{w - \gamma(s)}{\sqrt{\alpha'(s)^2 + \beta'(s)^2}} \begin{bmatrix} \alpha'(s) \\ \beta'(s) \end{bmatrix}
\]
makes \( s \) and \( w \) curvilinear coordinates. As is easy to see, the level lines of \( s \) are
tangent straight-lines to \( C \), the level lines of \( w \) are involutes of \( C \) — orthogonal to
one another.

Equations (B.4) imply
\[ y \leq f(x), \]
and
\[
s(x, y) = t, \quad w(x, y) = \gamma(t),
\]
if \( x, y \) and \( t \) obey (B.3) — \( s \) and \( w \) live below \( C \) and satisfy precise conditions along
\( C \).

We compute
\[
\frac{\partial(x, y)}{\partial(s, w)} = \begin{bmatrix} -\beta'(s) & \alpha'(s) \\ \alpha'(s) & \beta'(s) \end{bmatrix} \begin{bmatrix} \kappa(s)[w - \gamma(s)] & 0 \\ 0 & [(\alpha')^2 + (\beta')^2]^{-1/2} \end{bmatrix}
\]
to draw the following set:

(B.5) \[ \frac{\partial}{\partial x} \alpha(s) + \frac{\partial}{\partial y} \beta(s) = 0, \]

(B.6) \[ (\frac{\partial w}{\partial x})^2 + (\frac{\partial w}{\partial y})^2 = 1, \]

(B.7) \[ \frac{\partial s}{\partial x} \frac{\partial w}{\partial x} + \frac{\partial s}{\partial y} \frac{\partial w}{\partial y} = 0, \]

(B.8) \[ \nabla w = [\alpha'(s)^2 + \beta'(s)^2]^{-1/2} \begin{bmatrix} \alpha'(s) \\ \beta'(s) \end{bmatrix}, \]

(B.9) \[ |\nabla s|^{-2} = [w - \gamma(s)]^2 \kappa(s)^2 [\alpha'(s)^2 + \beta'(s)^2], \]

(B.10) \[ \begin{bmatrix} w_{xx} & w_{xy} \\ w_{yx} & w_{yy} \end{bmatrix} = \frac{1}{\gamma(s) - w} \begin{bmatrix} -w_y \\ w_x \end{bmatrix} (-w_y w_x). \]

Equation (B.5) is a conservation law; it reads

\[ \frac{\partial s}{\partial x} + s \frac{\partial s}{\partial y} = 0, \]

the standard Burgers equation, if (B.2) is in force. Equation (B.6) is the equation of geometrical optics in hand. Equation (B.7) shows that the gradients of \( s \) and \( w \) are orthogonal. Equation (B.8) shows that both \( C \) and the tangent straight-lines to \( C \) are lines of steepest descent of \( w \); it also shows that the straight-lines in question are isolines of \( w \). Equation (B.8) can be viewed as a Bäcklund transformation, which converts any solution to (B.5) into a solution to (B.6). It reads

\[ \nabla w = [1 + f'(s)^2]^{-1/2} \begin{bmatrix} 1 \\ f'(s) \end{bmatrix}, \quad s = g' \left( \frac{w_y}{w_x} \right), \]

or simply

\[ \nabla w = [1 + s^2]^{-1/2} \begin{bmatrix} 1 \\ s \end{bmatrix}, \quad s = \frac{w_y}{w_x}, \]

depending on whether (B.1) or (B.2) is in effect. Equations (B.9) and (B.10) show that both the gradient of \( s \) and the second-order derivatives of \( w \) blow up near \( C \).

We infer that \( s \) is governed by a Burgers-type equation, and develops shocks along \( C \). The following objects — \( w, C, \) the tangent straight-lines to \( C, \) and the region below \( C \) — are a geometric optical eikonal, the relevant caustic, the rays, and the light region, respectively.

We now claim: (i) \( s \) and \( w \) can be continuously extended into the region where \( y > f(x) \),

the dark side of \( C \), if suitable imaginary parts are provided; (ii) the relevant extensions obey equations (B.5) to (B.10).

The points above \( C \) are reached by no tangent straight-line to \( C \), of course. We insist in drawing tangent straight-lines from these points, but allow complex slopes. In other words, we recast (B.5) this way

\[ -\beta'(s)[x - \alpha(s)] + \alpha'(s)[y - \beta(s)] = 0, \]

\[ [w - \gamma(s)][\alpha'(s)^2 + \beta'(s)^2]^{1/2} = \alpha'(s)[x - \alpha(s)] + \beta'(s)[y - \beta(s)]. \]
and force such equations to hold in the situation where
\[ \text{Re}(s) = \lambda, \quad \text{Im}(s) = \mu, \quad \text{Im}(x) = \text{Im}(y) = 0. \]

The following formulas result
\[
x = \frac{\text{Im}[\alpha(s)\beta(s) - \alpha'(s)\beta'(s)]}{\text{Im}[\alpha'(s)\beta'(s)]}, \quad y = \frac{\text{Im}[\beta(s)\alpha'(s) - \beta'(s)\alpha(s)]}{\text{Im}[\beta'(s)\alpha'(s)]},
\]
(B.12)
\[ s = \lambda + i\mu, \]
\[
w = \gamma(s) - \frac{1}{\sqrt{\alpha'(s)^2 + \beta'(s)^2}} \left\{ \frac{\alpha'(s)}{\text{Im}[\alpha'(s)\beta'(s)]} \frac{\text{Im}(\beta(s))}{\text{Im}[\beta'(s)\alpha'(s)]} + \frac{\beta'(s)}{\text{Im}[\alpha'(s)\beta'(s)]} \frac{\text{Im}(\alpha(s))}{\text{Im}[\beta'(s)\alpha'(s)]} \right\},
\]
where \( \alpha, \beta \) and \( \gamma \) stand for the analytic continuations of the original objects. Formulas (B.12) answer the claim. Among other things, they give
\[
\left[ \begin{array}{c} x \\ y \end{array} \right] = \left[ \begin{array}{c} \alpha(\lambda) \\ \beta(\lambda) \end{array} \right] + \frac{dA(\lambda)}{d\lambda} \left[ \begin{array}{c} \alpha'(\lambda) \\ \beta'(\lambda) \end{array} \right] \mu^2 + B(\lambda) \left[ \begin{array}{c} -\beta'(\lambda) \\ \alpha'(\lambda) \end{array} \right] \mu^2 + O(\mu^4),
\]
\[ A = -\frac{1}{3} \log |\kappa| - \frac{1}{2} \log (\alpha'(s)^2 + (\beta')^2), \quad B = \frac{\kappa}{2} \sqrt{\alpha'(s)^2 + (\beta')^2}, \]
as \( \mu \) approaches zero, and
\[
\det \left( \frac{\partial(x, y)}{\partial(\lambda, \mu)} \right) = \frac{\text{Im}[\alpha(s)\beta(s) - \beta(s)\alpha(s)]^2}{\text{Im}[\alpha'(s)\beta'(s)]^3} \left| \alpha''(s)\beta'(s) - \alpha'(s)\beta''(s) \right| \]
consequently
\[
y - f(x) = \frac{1}{2} f''(\alpha(\lambda)) |\text{Im}(\alpha(\lambda + i\mu))|^2 + O(\mu^4)
\]
and
\[
\det \left( \frac{\partial(x, y)}{\partial(\lambda, \mu)} \right) = \mu [\alpha'(\lambda)\beta''(\lambda) - \alpha''(\lambda)\beta'(\lambda) + O(\mu^2)]
\]
as \( \mu \) approaches zero. Thus (B.12) imply (B.11), as well as
\[
\det \left( \frac{\partial(x, y)}{\partial(\lambda, \mu)} \right) \neq 0
\]
if \( \mu \) is different from, and sufficiently close to 0.

B.3. Here is an example. A catenary is the graph of either the following equation
\[ y = \cosh x \]
or the following equations
(B.13)
\[
x = \log(t + \sqrt{1 + t^2}), \quad y = \sqrt{1 + t^2},
\]
where parameter \( t \) coincides with both an arc length and the slope of the tangent straight-line.

Consider solutions to the following equations
\[
\left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 = 1, \quad \frac{\partial s}{\partial x} + s \frac{\partial s}{\partial y} = 0,
\]
which obey the following conditions
\[
w(x, y) = s(x, y) = t
\]
as \( x \) and \( y \) obey (B.13). Function \( w \) and the catenary in question are an eikonal and the relevant caustic, respectively; \( s \) obeys Burgers equation, takes a constant value on each tangent straight-line to the catenary and develops shocks along the catenary.

The light region is the set where
\[
-\infty < x < \infty, \quad y \leq \cosh x;
\]
the shadow region is the set above the catenary, where
\[
-\infty < x < \infty, \quad \cosh x < y.
\]
The following pair
\[
s x - y = s \log(s + \sqrt{1 + s^2}) - \sqrt{1 + s^2},
\]
\[
w = \frac{1}{\sqrt{1 + s^2}} \left[ s x + y - \log(s + \sqrt{1 + s^2}) \right],
\]
makes \( s \) and \( w \) implicit functions of \( x \) and \( y \) in the light region. Eikonal \( w \) and its partner \( s \) can be continued in a subset of the shadow region via the following equations
\[
x = \lambda + \tanh \lambda (\mu \cot \mu - 1), \quad y = \frac{1}{\cosh \lambda} \left( \sinh^2 \lambda \frac{\mu}{\sin \mu} + \mu \sin \mu + \cos \mu \right),
\]
\[
s = \sinh(\lambda + i\mu),
\]
\[
w = \sinh \lambda \frac{\mu}{\sin \mu} + \frac{i}{\cosh \lambda} (\sin \mu - \mu \cos \mu)
\]
— here \( \lambda \) and \( \mu \) are parameters such that
\[
-\infty < \lambda < \infty, \quad 0 \leq \mu \leq \pi/2.
\]
The foregoing equations imply that
\[
y - \cosh x = (\cosh \lambda) \mu^2 \left[ \frac{1}{2} + O(\mu^2) \right],
\]
as \( \mu \) approaches 0. Furthermore,
\[
\det \frac{\partial(x, y)}{\partial(\lambda, \mu)} = \frac{\mu^2 + \tanh^2 \lambda (1 - \mu \cot \mu)^2 (\sinh^2 \lambda + \cos^2 \mu)}{\cosh \lambda \sin \mu},
\]
\[
\frac{\partial w}{\partial x} = \frac{1}{\cosh(\lambda + i\mu)}, \quad \frac{\partial w}{\partial y} = \tanh(\lambda + i\mu)
\]
— in particular, a singularity occurs at the point whose coordinates are
\[
x = 0, \quad y = \pi/2.
\]
Figures 5 and 6 show plots of the imaginary parts of \( w \) and \( s \).

Appendix C.

C.1. Differentiating a real-valued function of one real variable is among the most elementary processes of mathematical and numerical analysis, but is also a significant prototype of those problems that are nowadays qualified ill-posed in the sense of Hadamard. Methods of approximating derivatives of smooth functions under non-exact data have been widely experimented over the years. Here we take the opportunity of sketching one more of such methods. We consider the case where
Figure 5. Eikonal equation: the imaginary part of $w$ beyond a caustic.

Figure 6. Burgers equation: the imaginary part of $s$ beyond a shock-line.

data consist of discrete and noisy samples, nodes are equally spaced, and information is available on both the relevant noise and the underlying smoothness. Our method is inspired by ideas that the theory of statistical learning has recently revived — see e.g. [CS], [EPP], [SZ1]-[SZ4], [Vp1]-[Vp4] — and of course mimics several of its ascendants — see e.g. [ACR], [Al], [AB1]-[AB2], [AH], [Be], [Br], [CJW], [C], [Cl], [Da], [Do], [DI], [EgK], [Fr], [Gr], [JR], [KM], [KW], [Ko], [LP], [LW], [MM], [Mu1]-[Mu2], [MG], [MMZ], [Ol], [Ra1]-[Ra2], [Re], [RR], [Sk], [SL], [Su], [Va], [Wn].
Items in input include:

(i) the end points of a bounded interval — $a$ and $b$;
(ii) the number of both nodes and samples — an integer $N$, larger than 2;
(iii) nodes from $a$ to $b$ — specifically,
$$x_k = a + (k - 1) \frac{b - a}{N - 1} \quad (k = 1, \ldots, N);$$
(iv) noisy samples — a sequence $g_1, g_2, \ldots, g_N$

Goals include recovering some function $f$ and the derivative $f'$ of $f$ based upon
the following information only:

(v) $f$ is smooth;
(vi) $f(x_k)$ is close to $g_k$, for $k = 1, 2, \ldots, N$.

Our recipe segments into the following three steps. First, let $\lambda$ and $\mu$ be positive
parameters, and solve the following variational problem

(C.1) $$\sum_{k=1}^{N} [u(x_k) - g_k]^2 + \lambda \int_{-\infty}^{\infty} \left[ \mu^7 (u''')^2 + \mu^{-1} u^2 \right] dx = \text{minimum},$$

under the following condition

(C.2) $u$ belongs to Sobolev space $W^{4,2}(-\infty, \infty)$.

Second, adjust $\lambda$ and $\mu$ properly. Third, take $u$, $u'$ as approximations of $f$, $f'$.

C.2. Effective formulas read as follows.

Rudiments of the calculus of variations demonstrate that Problem (C.1) \& (C.2) possesses
a unique solution; moreover that such a solution — named $u$ throughout — obeys

(C.3) $$\lambda \left( \mu^7 \frac{d^8 u}{dx^8} + \mu^{-1} u \right) + \sum_{k=1}^{N} [u(x_k) - g_k] \delta(x - x_k) = 0$$

for $-\infty < x < \infty$. The following features are decisive: equation (C.3) is affine;

$$\mu^7 \frac{d^8}{dx^8} + \mu^{-1}$$

is a positive operator in $L^2(-\infty, \infty)$, whose inverse mollifies;

$$\sum_{k=1}^{N} [u(x_k) - g_k] \delta(\cdot - x_k)$$

is a spike train or a shah-function.

Condition (C.2) and equation (C.3) give

(C.4) $$-\lambda u(x) = \sum_{k=1}^{N} [u(x_k) - g_k] K\left( \frac{x - x_k}{\mu} \right).$$
for $-\infty < x < \infty$. Equation (C.4) gives

(C.5) $A \begin{bmatrix} u(x_1) \\ \vdots \\ u(x_N) \end{bmatrix} + \lambda \begin{bmatrix} u(x_1) \\ \vdots \\ u(x_N) \end{bmatrix} = A \begin{bmatrix} g_1 \\ \vdots \\ g_N \end{bmatrix}.$

Figure 7. Plots of $K, K'$ and $K''$.

Here $K$ denotes an appropriate fundamental solution to $d^8/dx^8 + 1$, namely the solution to

$$\frac{d^8 K}{dx^8} + K = \delta(x)$$

that decays at infinity — $K$ is given by

(C.6) \[ \pi K(x) = \int_0^\infty \frac{\cos(x\xi)}{1 + \xi^8} d\xi \]

for $-\infty < x < \infty$, is even and positive definite. Secondly,

$$A = \begin{bmatrix} K(0) & K\left(\frac{x_1 - x_2}{\mu}\right) & \ldots & K\left(\frac{x_1 - x_N}{\mu}\right) \\ K\left(\frac{x_2 - x_1}{\mu}\right) & K(0) & \ldots & K\left(\frac{x_2 - x_N}{\mu}\right) \\ \vdots & \vdots & \ddots & \vdots \\ K\left(\frac{x_N - x_1}{\mu}\right) & K\left(\frac{x_N - x_2}{\mu}\right) & \ldots & K(0) \end{bmatrix}$$

— a symmetric, positive definite Toeplitz matrix.
Let \( \nu = \pi/8 \). Manipulating formula (C.6) gives

\[
K(x) = \frac{1}{8 \sin \nu} - \frac{1}{8 \cos \nu} \frac{x^2}{2} + \frac{1}{8 \cos \nu} \frac{x^4}{24} - \frac{1}{8 \sin \nu} \frac{x^6}{720} + \frac{1}{2 \cdot 5040} + O(x^8)
\]
as \( x \) approaches zero — \( K \) behaves near zero like a spline of order seven. Calculus of residues, or convenient formulas from [PBM, Section 2.5.10], give

\[
4K(x) = e^{-|x| \cos \nu} \cos(|x| \sin \nu - \nu) + e^{-|x| \sin \nu} \sin(|x| \cos \nu + \nu)
\]
as \(-\infty < x < \infty\). Therefore,

\[
\frac{d^n K}{dx^n}(x) = \left( -\frac{\operatorname{sgn} x}{4} \right)^n \left\{ e^{-|x| \cos \nu} \cos(|x| \sin \nu - (n+1)\nu) + e^{-|x| \sin \nu} \sin(|x| \cos \nu + (n+1)\nu - n\pi/2) \right\}
\]
as \( x \neq 0 \) and \( n = 1, 2, 3, \cdots \). Figure 7 shows plots of \( K, K', K'' \).

![Figure 7 showing plots of K, K', K'']

**Figure 7.** Reciprocal condition estimator of \( A \), plotted versus the ratio \((b - a)/(N - 1)\mu\).

Analysis shows the following. The spectrum of \( A \) lies in the open interval \([0, N/(8 \sin \nu)]\). All eigenvalues of \( A \) are close to \( 1/(8 \sin \nu) \), if \( \mu(N - 1)/(b - a) \) is small; otherwise, the largest eigenvalue of \( A \) is close to \( N/(8 \sin \nu) \) and most remaining eigenvalues of \( A \) are close to 0. In particular, \( A \) is invertible anyway; \( A \) is either well-conditioned or ill-conditioned depending on whether \( \mu(N - 1)/(b - a) \) is small or large. Figure 8 shows plots of a reciprocal condition estimator of \( A \) versus \((b - a)/(N - 1)\mu\).

Equation (C.4) implies that \( u \) belongs to the linear span of

\[
K \left( \frac{\cdot - x_1}{\mu} \right), \ldots, K \left( \frac{\cdot - x_N}{\mu} \right)
\]
translations and dilations of $K$. Such items own either a spike-shaped or a well-rounded profile depending on whether $\mu$ is small or large, inasmuch as

$$\frac{\int_{-\infty}^{\infty} |(d/dx)\text{Item}|^2 \, dx}{\int_{-\infty}^{\infty} |\text{Item}|^2 \, dx} = \frac{5}{7} \mu^{-2} \tan \nu, \quad \frac{\int_{-\infty}^{\infty} |(d^2/dx^2)\text{Item}|^2 \, dx}{\int_{-\infty}^{\infty} |\text{Item}|^2 \, dx} = \frac{3}{7} \mu^{-4} \tan \nu.$$  

The same items are definitely linearly independent, although appropriate formulas and analysis show that their Gram matrix is well-conditioned only if $\mu(N-1)/(b-a)$ is small enough.

Equation (C.5) determines $u(x_1), \ldots, u(x_N)$ in terms of data. Note that they solve

$$\sum_{k=1}^{N} [u(x_k) - g_k]^2 + \lambda [u(x_1) \cdots u(x_N)] A^{-1} \begin{bmatrix} u(x_1) \\ \vdots \\ u(x_N) \end{bmatrix} = \text{minimum}$$

— a standard finite-dimensional least-square problem, where $\lambda$ and the inverse of $A$ imitate a regulating parameter à la Tikhonov and a penalty, respectively.

Now we are in a position to draw conclusions. Let $\text{Id}$ be the $N \times N$ unit matrix, and

$$R = (\lambda \text{Id} + A)^{-1}$$

— a resolvent. Let $B$ the vector-valued function such that

$$B(x) = R \begin{bmatrix} K \left( \frac{x - x_1}{\mu} \right) \\ \vdots \\ K \left( \frac{x - x_N}{\mu} \right) \end{bmatrix}$$

for $-\infty < x < \infty$ — an alternative basis in the linear span mentioned above. Define $C$ and $D$ thus

$$C = A R$$

$$D = \mu^{-1} \begin{bmatrix} 0 & K' \left( \frac{x_1 - x_2}{\mu} \right) & \cdots & K' \left( \frac{x_1 - x_N}{\mu} \right) \\ K' \left( \frac{x_2 - x_1}{\mu} \right) & 0 & \cdots & K' \left( \frac{x_2 - x_N}{\mu} \right) \\ \vdots & \vdots & \ddots & \vdots \\ K' \left( \frac{x_N - x_1}{\mu} \right) & K' \left( \frac{x_N - x_2}{\mu} \right) & \cdots & 0 \end{bmatrix} R.$$  

The following equations hold.

$$u(x) = [g_1 \cdots g_N] B(x)$$

for $-\infty < x < \infty$,

$$\begin{bmatrix} u(x_1) \\ \vdots \\ u(x_N) \end{bmatrix} = C \begin{bmatrix} g_1 \\ \vdots \\ g_N \end{bmatrix},$$
\[
\begin{bmatrix}
u'_{1}(x_1) \\
\vdots \\
\vdots \\
u'_{N}(x_N)
\end{bmatrix}
= D
\begin{bmatrix}
g_1 \\
\vdots \\
g_N
\end{bmatrix}.
\]

**Figure 9.** The alternative basis $B$ : plots of $B_1(x), \ldots, B_N(x)$ versus $x$.

**Figure 10.** Matrix $C$ playing the role of a regularizing filter.

As figures 9, 10 and 11 show, $B$ and $C$ mimic a typical Green’s function from two different perspectives, $D$ mimics a derivative of a Green’s function. Observe
incidentally that
\[ \int_{-\infty}^{\infty} \left[ \mu^7 (B''')^T B''' + \mu^{-1} B^T B \right] dx = \text{tr} \left[ A (\lambda \text{Id} + A)^{-2} \right], \quad C = \text{Id} - \lambda (\lambda \text{Id} + A)^{-1}, \]
and that \( B \) and \( C \) solve the following variational problem
\[ \lambda \int_{-\infty}^{\infty} \left[ \mu^7 (B''')^T B''' + \mu^{-1} B^T B \right] dx + \text{tr} \left[ (C^T - \text{Id})(C - \text{Id}) \right] = \text{minimum}, \]
subject to the following conditions
\[ B \in [W^{4,2}(-\infty, \infty)]^N, \quad C = \begin{bmatrix}
B_1(x_1) & B_2(x_1) & \cdots & B_N(x_1) \\
B_1(x_2) & B_2(x_2) & \cdots & B_N(x_2) \\
\vdots & \vdots & \ddots & \vdots \\
B_1(x_N) & B_2(x_N) & \cdots & B_N(x_N)
\end{bmatrix}. \]

C.3. Here we offer directions for adjusting \( \lambda \) and \( \mu \) properly.

Parameter \( \lambda \) — *dimensionless* — discriminates whether solution \( u \) to Problem (C.1) & (C.2) either fits data well (but is simultaneously sensible of noise), or else is little affected by noise (but departs somewhat from data). In loose terms, the following statements hold. First, \( u \) virtually *interpolates* \( g_1, \ldots, g_N \) if \( \lambda \) is close to zero; however, \( u \) is liable to own an *irregular* profile at the same time. Second, \( u \) *quenches smoothly* if \( \lambda \) grows larger and larger. Indeed,
\[ \int_{-\infty}^{\infty} \left[ \mu^7 (B''')^T B''' + \mu^{-1} B^T B \right] dx = \text{tr} \ A^{-1} + O(\lambda), \]
\[ C = \text{Id} + O(\lambda) \]
as $\lambda$ approaches zero;

$$\|d^n B(x)/dx^n\| \leq \lambda^{-1} \frac{\sqrt{N}}{8\sin((n+1)\nu)}$$

as $-\infty < x < \infty$ and $n = 0, 1, \ldots, 6$.

Parameter $\mu$ — making $\mu/(b-a)$ dimensionless — determines how much $u$ is close to, or departs from a spike train. Indeed,

$$B(x) = \mu \frac{8\sin \nu}{1 + \lambda \sin \nu} \left[ \begin{array}{c} \delta(x-x_1) \\ \vdots \\ \delta(x-x_N) \end{array} \right] + O(\mu^3)$$

as $-\infty < x < \infty$ and $\mu$ approaches zero;

$$B(x) = \frac{1}{N + 8\lambda \sin \nu} \left[ \begin{array}{c} 1 \\ \vdots \\ 1 \end{array} \right] + O(\mu^{-2})$$

as $-\infty < x < \infty$ and $\mu$ grows larger and larger.

---

**Figure 12.** Plots of S.R.Q. versus log $\rho$. Here $\rho = (b-a)/(N-1)\mu$.

In the case where suitable information is available a priori, a theorem from Subsection C.4 below suggests which values of $\lambda$ and $\mu$ work properly. Otherwise, parameter $\lambda$ may be determined based upon the discrepancy principle, the cross-validation, the $L$-curve criterion, or other customary devices — see e.g. [Hn] Chapter 7] for details.

Parameter $\mu$ is expediently identified by the following recipe. Let

$$\text{R.Q. of } v = \mu^{14} \frac{\int_{-\infty}^{\infty} [(d/dx)^7 v(x)]^2 dx}{\int_{-\infty}^{\infty} [v(x)]^2 dx}$$
a dimensionless Rayleigh quotient; define a relevant minimum thus
\[ \text{S.R.Q.} = \min \left\{ \text{R.Q. of } v : 0 \neq v \in \text{span of } K \left( \frac{\cdot - x_1}{\mu} \right), \ldots, K \left( \frac{\cdot - x_N}{\mu} \right) \right\} ; \]
then determine \( \mu \) so that
\[ \text{(C.7)} \quad \text{S.R.Q.} = \text{minimum} . \]
Equation (C.7) causes solution \( u \) of Problem (C.1) & (C.2) to retain its most favorable Rayleigh quotient. Equation (C.7) can be approached via equation (C.8) below and tools from linear algebra, although care must be taken of the ill-condition of the involved matrices.

Let
\[ G = \mu^{-1} \times \text{Gram matrix of } K \left( \frac{\cdot - x_1}{\mu} \right), \ldots, K \left( \frac{\cdot - x_N}{\mu} \right) ; \]
\[ H = \mu^{-1} \times \text{Gram matrix of } K^{(14)} \left( \frac{\cdot - x_1}{\mu} \right), \ldots, K^{(14)} \left( \frac{\cdot - x_N}{\mu} \right) ; \]
let \( L \) denote the autocorrelation function of \( K \), videlicet
\[ \pi L(x) = \int_{-\infty}^{\infty} (1 + \xi^8)^{-2} \cos(x\xi) \, d\xi \]
for \(-\infty < x < \infty\). We have
\[ G = \left[ L \left( \frac{x_j - x_k}{\mu} \right) \right]_{j,k=1,\ldots,N} , \quad H = \left[ -L^{(14)} \left( \frac{x_j - x_k}{\mu} \right) \right]_{j,k=1,\ldots,N} ; \]
\[ L(x) = \frac{7}{8} K(x) - \frac{x}{8} K'(x) , \quad L^{(14)}(x) = \frac{7}{8} K^{(6)}(x) + \frac{x}{8} K^{(7)}(x) \]
for \(-\infty < x < \infty\), and
\[ \text{(C.8)} \quad \text{S.R.Q.} = \text{least eigenvalue of } H \text{ with respect to } G \]
— in other words,
\[ \text{S.R.Q.} = \text{least eigenvalue of } G^{-1/2} H G^{-1/2} . \]
Figure 12 shows plots of S.R.Q. versus \((b - a)/(N - 1)\mu\). The following table

| \( N \) | minimum value \((b - a)/(N - 1)\mu\) | \( \mu/(b - a) \) |
|---|---|---|
| 5 | 0.11532075700 | 0.90012773000 | 0.277738360532 |
| 10 | 0.0428250969700 | 0.57886588800 | 0.191946026019 |
| 20 | 0.015529762000 | 0.36110444000 | 0.14575289726 |
| 30 | 0.008487160550 | 0.27110850400 | 0.127191726235 |
| 40 | 0.005502126710 | 0.22017128000 | 0.11645947577 |
| 50 | 0.003920874640 | 0.18711758500 | 0.109065982576 |
| 60 | 0.002967795760 | 0.16352070000 | 0.103651818045 |
| 70 | 0.002342495850 | 0.14755376000 | 0.09431789245 |
| 80 | 0.001906845850 | 0.13182589700 | 0.086468699576 |
| 90 | 0.001349762400 | 0.11146636100 | 0.08180522580 |
lists sample solutions to equation (C.7).

The following formula

\[
\frac{\mu}{b-a} \approx 0.0415 + 0.5416 \times N^{-1/2} - 0.6426 \times N^{-1} + 1.3706 \times N^{-3/2}
\]

gives an effectual estimate of such solutions.

C.4. The following theorem holds.

**Theorem C.1.** Suppose \( u \) solves problem (C.1) \& (C.2); suppose \( f, \varepsilon \) and \( E \) obey

\[
|f(x_k) - g_k| \leq \varepsilon \quad (k = 1, \ldots, N),
\]

\[
\mu^{-4} \int_{-\infty}^{\infty} f(x)^2 dx + \mu^4 \int_{-\infty}^{\infty} |f'''|^2 dx \leq 2(b - a)^{-3} E^2;
\]

let

\[
\delta = \max \left\{ (1 - 1/N)^{-1/2} \frac{\varepsilon}{E}, (N - 1)^{-2} \right\}.
\]

If \( \delta \) approaches zero, and

\[
\frac{\lambda}{N} \left( \frac{\mu}{b-a} \right)^3 \left( \frac{E}{\varepsilon} \right)^{-2}
\]

is bounded and bounded away from zero, then

\[
E^{-1} \max \{|f(x) - u(x)| : a \leq x \leq b\} = O(\delta^{3/4}),
\]

\[
E^{-1}(b - a) \max \{|f'(x) - u'(x)| : a \leq x \leq b\} = O(\delta^{1/4}).
\]

**Proof.** Let

(C.10) \[ v = f - u. \]

Functional \( J \), whose domain is \( W^{4,2}(-\infty, \infty) \) and whose value at any trial function \( \varphi \) obeys

\[
J(\varphi) = \sum_{k=1}^{N} [\varphi(x_k) - g_k]^2 + \lambda \int_{-\infty}^{\infty} \left[ \mu^7 (\varphi'''')^2 + \mu^{-1} \varphi'^2 \right] dx,
\]

attains its minimum value at \( u \). Consequently,

\[
J(f) = J(u) + \text{(a remainder)},
\]

remainder \[ \sum_{k=1}^{N} |v(x_k)|^2 + \lambda \int_{-\infty}^{\infty} \left[ \mu^7 (v''')^2 + \mu^{-1} v'^2 \right] dx. \]

Since

\[
J(f) \leq N \varepsilon^2 + 2\lambda \left( \frac{\mu}{b-a} \right)^3 E^2,
\]

\[
J(u) \geq 0,
\]

\[
\int_{-\infty}^{\infty} \left[ \mu^7 (v''')^2 + \mu^{-1} v'^2 \right] dx \geq 2\mu^3 \int_{-\infty}^{\infty} (v'')^2 dx,
\]

we infer

(C.11) \[ \sum_{k=1}^{N} |v(x_k)|^2 + 2\lambda \mu^3 \int_{-\infty}^{\infty} (v'')^2 dx \leq E^2 \left\{ N \left( \frac{\varepsilon}{E} \right)^2 + 2\lambda \left( \frac{\mu}{b-a} \right)^3 \right\}. \]
Combining inequality (C.11) and Lemma C.2 below results in

\[
(b - a)^{-1} \int_a^b v^2 dx \leq E^2 \left\{ \frac{N}{N - 1} + \frac{1}{\pi^4 (N - 1)^4} \frac{N}{2\lambda} \left( \frac{\mu}{b - a} \right)^{-3} \right\} \left\{ \left( \frac{\varepsilon}{E} \right)^2 + \frac{2\lambda}{N} \left( \frac{\mu}{b - a} \right)^3 \right\} ;
\]

inequality (C.11) also yields

\[
(b - a)^3 \int_a^b (v'')^2 dx \leq E^2 \left\{ 1 + \frac{N}{2\lambda} \left( \frac{\mu}{b - a} \right)^{-3} \left( \frac{\varepsilon}{E} \right)^2 \right\} ;
\]

The last two inequalities and a hypothesis imply

\[
\text{(C.12)} \quad (b - a)^{-1} \int_a^b v^2 dx \leq (\text{Const.}) E^2 \delta^2 ,
\]

\[
\text{(C.13)} \quad (b - a)^3 \int_a^b (v'')^2 dx \leq (\text{Const.}) E^2 .
\]

The conclusions follow from (C.10), (C.12) and (C.13), by virtue of Lemma C.3 below.

![Figure 13. Original function (dotted), and samples polluted by noise (circled).](image)

**Lemma C.2.** Let \(-\infty < a < b < \infty\), and \(N = 2, 3, \ldots\); let

\[
\Delta x = \frac{b - a}{N - 1}, \quad x_k = a + (k - 1) \Delta x \quad (k = 1, \ldots, N).
\]
The following inequality
\[
\left\{ \int_{a}^{b} v^2 \, dx \right\}^{1/2} \leq (\Delta x)^{-1/2} \left\{ \sum_{k=1}^{N} v(x_k)^2 \, dx \right\}^{1/2} + \frac{(\Delta x)^2}{\pi} \left\{ \int_{a}^{b} (v')^2 \, dx \right\}^{1/2}
\]
holds for every \( v \) from \( W^{2,2}(a,b) \).

**Lemma C.3.** Suppose \( v \) is in \( W^{2,2}(a,b) \), and
\[
(b-a)^{-1/2} \left\{ \int_{a}^{b} v^2 \, dx \right\}^{1/2} = ||v||, \quad (b-a)^3 \int_{a}^{b} (v')^2 \, dx = 1.
\]

The following inequalities hold
\[
\max |v| \leq 2^{1/4} 3^{-3/8} ||v||^{3/4} + O(||v||),
\]
\[
(b-a) \max |v'| \leq 2^{1/4} 3^{-3/8} ||v||^{1/4} + O(||v||).
\]

![Figure 14. Original and recovered functions.](image)

The proofs of Lemma C.2 and C.3 are beyond the scope of the present paper, and are omitted.

C.5. Here is an example, demonstrating how the present method works. Let
\[
a = 0, \quad b = 1,
\]
\[
f(x) = \exp \left[ -72 \left( x - \frac{1}{2} \right)^2 \right] \left[ \cos(25x) - 4 \sin(25x) \right],
\]
\[
N = 150,
\]
\[
g_k = f(x_k) \pm (5\% \text{ random noise}) \quad (k = 1, \ldots, N).
\]
Let parameter \( \lambda \) obeys
\[
\lambda = 10^{-6},
\]
let parameter $\mu$ be given by formula (C.9), and let $u$ solve problem (C.1) & (C.2). Figure 13 plots $g_1, g_2, \ldots, g_N$ versus $x_1, x_2, \ldots, x_N$; figure 14 plots $f$ and $u$, and figure 15 plots $f'$ and $u'$.

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