Continuous-mode effects and photon-photon phase gate performance

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The effects arising from the inherent continuous-mode nature of photonic pulses were poorly understood but significantly influence the performance of quantum devices employing photonic pulse interaction in nonlinear media. Such effects include the entanglement between the continuous wave-vector modes due to pulse interaction as well as the consequence of a finite system bandwidth. We present the first analysis on these effects for interactions between single-photon pulses, demonstrating their impact on the performance of quantum phase gates based on such process. Our study clarifies a realistic picture of this type of quantum devices.

I. INTRODUCTION

Deterministic photon-photon phase gate is a key building block to construct circuits for the scalable all-optical quantum information processing. Cross-phase modulation (XPM) between slow pulses in media under electromagnetically induced transparency (EIT) conditions [1, 2] (or with similar properties) is the main route towards such a gate. Considerable theoretical developments (see, e.g., [3–10]) as well as experimental studies (see, e.g., [11–23]) have been undertaken to explore the EIT-based XPM and alternative approaches aiming at realizing photonic two-qubit gates.

In optics-based quantum computing, a conditional phase of π radians needs to be implemented by photon-photon phase gates. For achieving such a large phase there are the proposals [4, 5, 11] of making pulses co-propagate, so that the generic weak pulse interaction could be compensated by the prolonged interaction time. Another requirement for an ideal phase gate is the uniformity of conditional phase, as given by the mapping |1⟩|1⟩ → e iθ|1⟩|1⟩, i.e., the same phase θ is induced for each mode k of a continuous-mode photon in the state |1⟩ = ∫ −∞ ∞ dk ξ(k)ˆ{a}†(k)|0⟩, where ξ(k) is the pulse profile in wave-vector space. The currently dominant understanding is that a homogeneous phase could be possibly realized by letting one pulse completely go through the other [4, 5, 12, 14].

Since a photon is not a point particle, photon-photon interactions in nonlinear media should be modeled as interacting quantum fields of continuous modes. Using this picture, we show that the above-mentioned notions are generally invalid. The essential effects in realistic single-photon XPM are clarified here for the first time.

II. CONTINUOUS-MODE EFFECTS IN PHOTON-PHOTON INTERACTIONS

We first provide a theoretical framework for the XPM between photons. The interaction between two pulses in a medium of any type of atomic structure realizing EIT can be translated into that between two dark-state polariton fields ˆΨi(z, t) = 1 √ 2π ∫ −∞ ∞ ˆ{a}i(k)e ikz dk (i = 1, 2) with ˆ{a}i(k), ˆ{a}†(k′) = δi,jδ(k − k′) [24]. By neglecting the pulse loss and deformation, as well as the possible self-phase modulation term which has no effect on photon-photon interactions [13, 14], one has the following equations of motion for the slowly varying and transversely well confined polariton fields [2, 13, 15]:

(θi + vi∂2i) ˆΨi(z, t) = −i ˆ{a}i(z, t) ˆΨi(z, t) (1)

where vi are the pulse group velocities. The term ˆ{a}i(z, t) = ∫ dz′ ∆(z − z′) ˆΨ† j(z′, t) ˆΨj(z′, t) could come from a general interaction potential ∆(z − z′). The pulse interaction in the experimentally studied XPM thus far [17, 23], for instance, can be modeled by a contact potential ∆(z − z′) = χ(δ(z − z′)), where χ approximated by a real quantity is the nonlinear rate determined by the specific system parameters. The potential ∆(z − z′) considered here acts instantaneously; see [25] for a study on the non-instantaneous effects. From the field-theoretic viewpoint, Eq. (1) is obtained by the equation of motion iℏ∂t ˆΨi = δH/δ ˆΨ† i for non-relativistic fields, where the Hamiltonian ˆH = ˆK + ˆV consists of the kinetic term ˆK = ∑j=1 ∫ dz ˆΨ† j(z) 1 2 ˆ{V}j ˆΨj(z) and the interaction term ˆV = ˆh ∫ dz ∫ dz′ ˆΨ† j(z) ˆΨ† j′(z′) ∆(z − z′) ˆΨj(z′) ˆΨj′(z) (2)

The pulse interaction would evolve the input state ˆΨ(t)|1⟩|1⟩ = ∫ dz ∫ dz′ ζ(k, k′, t) ˆ{a}†(k) ˆ{b}†(k′)|0⟩, where ˆΨ(t) = ˆTexp(−i ˆh ˆH dt) (T denotes the time-ordering operation and ˆh ≡ 1 is adopted hereafter). It is convenient to use the two-particle function, ψ(z1, z2, t) = ⟨Φi| ˆΨ1(z1, t) ˆΨ2(z2, t)|Φm⟩ = 1 √ 2π ∫ dz dz′ ζ(k, k′, t)e ikz1 e ik′z2 , to study the evolution of the initial state |Φm⟩ = |1⟩|1⟩. To indicate how close a realistic XPM is to the ideal one |1⟩|1⟩ → e iθ|1⟩|1⟩ for phase gates, we use the fidelity F. As figures of merit to characterize a realistic XPM, the fidelity F and conditional phase θ are determined by the overlap [13, 15]:

√F e iθ = ⟨Φ0| ˆΨi⟩ = ∫ dz1 dz2 ψ0(z1, z2, t)ψ(z1, z2, t), (2)

where ψ0(z1, z2, t) is the two-particle function from the field operator equations (∂t + vi∂2i) ˆΨi(z, t) = 0, corresponding to the freely evolved state |Φ0⟩ = e −ikz1 1|Φm⟩. A similar formula in the discrete form is given in [26].
The output state of a realistic XPM, $|\Phi_{out}\rangle = \int dk' dk' d\Omega d\beta (k,k') \hat{\Psi}_1(k') \hat{\Psi}_2(k') |0\rangle$, is generally entangled between the wave-vector modes $k$ and $k'$ of the individual photons. Such field modes entanglement was widely neglected in the previous researches on photon-photon gates, though its effect on XPM was conjectured. It is conceivable that the entanglement would lower the fidelity $F$. Yet, for clarifying the issue, a relation between the amount of entanglement generated in pulse interaction and the corresponding gate operation fidelity has to be found. Here we quantify the field mode entanglement with the linear entropy $S_L = 1 - \text{Tr} \rho_2^2$ ($\rho_2$ are the reduced density matrices of the bipartite state $|\Phi_{out}\rangle$), which takes the following closed form:

$$S_L(t) = 1 - \int dz_1 dz_2 dz_3 dz_4 \left\{ \psi(z_1, z_2, t) \psi^*(z_3, z_4, t) \times \psi(z_3, z_4, t) \psi^*(z_1, z_4, t) \right\};$$

where $\psi(z_1, z_2, t) = \int dk \hat{\Psi}_1(k) e^{ikz_1} dk$ after being imposed a cut-off by the bound of the wave-vector mode $k_s = \Delta \omega_s/(2c)$ becomes

$$\hat{\Psi}_1(z_1) \hat{\Psi}_2(z_2) = \delta_{ij}(k_s/\pi \sin(k_s(z_1 - z_2))) \equiv \delta_{ij} C(z_1 - z_2),$$

where $\sin(z) \equiv \sin(z)/z$. Substituting the formal solution of $\psi(z_1, z_2, t)$ yields the general two-particle function

$$\psi(z_1, z_2, t) = \langle 0 | \hat{\Psi}_1(z_1') \hat{\Psi}_2(z_2') | \Phi_{in}\rangle + \langle 0 | \sum_{n=1}^{\infty} \frac{(i\chi)^n}{n!} \hat{\Psi}_1(z_1') \hat{\Psi}_2(z_2') C(v, t) | \Phi_{in}\rangle$$

where $C(v, t) = \int_0^t dt' dt \chi(v, t) \cdot \chi(v, t') C(v, t - t') C(v, t') C(v, t) C(v, t') C(v, t)$

$$C(z_2' - z_1', t) \equiv C(z_2, t)$$

at the time $t$, where $z_1' = z_1 - vt$ and $v_r = v_1 - v_2$. The second term on the right side of Eq. (5) arises from photon-photon interaction. The norm of the two-particle function $\psi(z_1, z_2, t)$ is not preserved as a consequence of the deviation of the field operator commutator $[\hat{\Psi}_1, \hat{\Psi}_2]$ from the delta function $\delta(z - z')$, so the output two-particle function $\psi(z_1, z_2, t)$ has to be normalized prior to calculating the fidelity and linear entropy using Eqs. (3) and (4). The system bandwidth $\Delta \omega_s$ effectively results in a non-unitary evolution, though the wave-vector modes $k, k'$ of two photons are still continuous after imposing the cut-off $k_s$. This could be understood by the restriction $k_s \leq k, k' \leq k_s$ on the matrix elements $\langle k, k' | \hat{U}(t)| k, k' \rangle$ of the evolution operator $\hat{U}(t)$, causing the loss of the orthonormal relation for the matrix elements $\langle k, k' | \hat{U}(t)| k, k' \rangle$.

III. IMPACT ON PHOTONIC PHASE GATE PERFORMANCE

An important situation we analyze here is the XPM between two co-propagating pulses where $v_r = 0$, the two-particle function in Eq. (5) reduces to (see also Ref. [1])

$$\psi(z_1, z_2, t) = f_1(z_1 - vt) f_2(z_2 - vt) + f_1(z_2 - vt) f_2(z_1 - vt)$$

where $f_1(z) = \langle 0 | \hat{\Psi}_1(z) | \Phi \rangle = \chi \Delta \omega_s t/(2\pi)$, and $v_1 = v_2 = v$. Using the normalized form of this two-particle function, one will obtain from Eq. (6) the following relations to determine the conditional phase and fidelity:

$$\tan \theta = \frac{C_1 \sin \Phi}{1 - C_1 + C_1 \cos \Phi},$$

where $C_1 = \int dz_1 \int dz_2 f_1(Z_1) f_2(Z_2) | \sin[k_0(Z_1 - Z_2)] |^2$, $C_2 = \int dz_1 \int dz_2 f_1(Z_1) f_2(Z_2) | \sin[k_0(Z_1 - Z_2)] |^2$. The dimensionless variables of the integrals are $Z_t = (z_1 - vt)/\sigma$, where $\sigma$ is the pulse size in medium; the system parameter is defined as $k_0 = v/(2c) \Delta \omega_s/\Delta \omega_p$, in proportion to the ratio of the system bandwidth $\Delta \omega_s$ to the pulse bandwidth $\Delta \omega_p$ (the reciprocal of the pulse duration). The factor $v/(2c)$ in the parameter $k_0$ reflects the pulse compression in EIT media.

In Fig. 1 obtained from Eq. (7), one sees two different XPM patterns depending on the values of $C_1$ and $C_2$ at the transitional point $C_1 = 0.5$. This $C_1$ value is the threshold above which the denominator $1 - C_1 + C_1 \cos \Phi$ on the right-hand side of Eq. (7) may become zero or negative with increasing $\Phi$, and it is universal with respect to any pulse shape. As $C_1$ decreases or increases across this point, the gate performance characterized by the achievable $\theta$ values will undergo a transition to the different pattern. For $C_1 < 0.5$, the conditional phase $\theta$ can never assume values greater than $\pi/2$ no matter how large $\Phi$ is.

Without loss of generality, in our discussion below we consider the interaction between two identical pulses in...
the Gaussian profile \( f(z) = \left( \frac{1}{\sqrt{\pi}} \right) \exp\left(-\frac{z^2}{2\sigma^2}\right) \). Then the boundary value \( C_1 = 0.5 \) of the two patterns in Fig. 1 corresponds to the system parameter \( k_0 \approx 2.5 \). As \( k_0 \) increases (decreases) from this value, the coefficient \( C_1 \) will become smaller (larger) into the respective pattern. Eqs. (3) and (8) yield the evolution of the field-mode entanglement and gate operation fidelity with the phase \( \Phi \), which is proportional to the pulse interaction time \( t \). The results are illustrated in Fig. 2 for various \( k_0 \) values.

The expected relation between \( S_L \) and \( F \) manifests in the lower regime of \( C_1 < 0.5 \) in Fig. 1, where a weaker entanglement, at any fixed \( \Phi \) value, is accompanied by a higher fidelity. In this regime the entanglement between the field modes totally vanishes in the limit \( k_0 \to \infty \). In the upper regime of \( C_1 > 0.5 \), the field mode entanglement disappears in the limit \( k_0 \to 0 \). The fidelity value, however, goes down to zero at this point, for any non-zero \( \Phi \) value; c.f. Eq. (5) with a diverging \( C_2 \) at \( k_0 \to 0 \). The non-unitary evolution due to a finite system bandwidth \( \Delta\omega_s \) accounts for this phenomenon. Note that, as the consequence of non-unitary evolution, both factors \( f_1, f_2 \) in the second term of Eq. (6) carry the same variable \( z_2 - vt \), thus deviating from the ideal output two-particle function \( f_1(z_1 - vt)f_2(z_2 - vt)e^{i\phi} \) in the neighborhood of \( k_0 \to 0 \). The XPM between two evenly distributed square pulses shows the same characteristics except for the different relations between the \( C_i \) and \( k_0 \) values, indicating that the effects described above differ, by nature, from the inhomogeneity in pulse interactions.

An interesting observation from Fig. 2 is the feature that, around the transitional point \( k_0 \approx 2.5 \) of the two operation patterns, the plots of \( S_L \), e.g., the thin solid line, reaches the highest value at some \( \Phi < \pi \). Increasing \( \Phi \) beyond this point leads to a continual distortion of the two-photon state without increasing its entanglement. Irrespective of the pulse shapes, the linear entropy plateau at the transitional point is correlated to the fidelity valley like that in Fig. 3, which goes down to \( F = 0 \) at \( k_0 \approx 2.5 \) and \( \Phi = \pi \).

Furthermore, we comment on the notion that pulses moving with matched group velocities make large conditional phase possible [4]. In the previous theoretical studies, the variables \( z_1 \) and \( z_2 \) of the two-particle function in (6) were often mixed up with the pulse-center coordinates, and then the phase \( \Phi \) would be regarded as the conditional phase \( \theta \) for the reason that the overlapped pulse centers with \( z_1 = z_2 = z \) could lead to the ideal output \( f_1(z - vt)f_2(z - vt)e^{i\phi} \) from Eq. (6). In fact, the variable \( \Delta\omega_s(z_1 - z_2)/(2c) \) of the sinc function in (6) assumes any value even if the two pulses co-propagate, because \( z_i \) are the field coordinates over the whole \( z \) axis rather than those of the pulse centers. The conditional phase \( \theta \) should be determined by Eq. (2) giving its relation with the XPM phase \( \Phi \) in Fig. 1. The conditional phase value could reach \( \pi \) in the upper regime of Fig. 1, where the gate operation fidelity is, however, rather low. A high fidelity is possible only in the lowest region in Fig. 1, where the state evolution is close to unitary but the peak of the conditional phase \( \theta \) is vanishing. The vanishing conditional phase in the regime of near unitary evolution also exists in case of interaction between co-propagating single photon and coherent state; see [14].

The trade-off between conditional phase and fidelity in XPM between co-propagating photons is also discussed recently in [20], where a finite mode approximation for pulses is adopted. One problem with the finite mode approximation is its underestimation of XPM intensity—compared with the contribution from the second term in our Eq. (6), the term arising from XPM, the discrete sum

![FIG. 1: (color online) Relation between the conditional phase \( \theta \) and the XPM phase \( \Phi \) for various \( C_1 \) values from the lower to the upper: \( C_1 = 0.20, 0.36, 0.45, 0.50, 0.55, 0.63, 0.78 \) and 0.98. The line \( \theta = \frac{1}{2}\Phi \) corresponding to \( C_1 = 0.5 \) separates the different operation patterns.](image1)

![FIG. 2: (color online) Linear entropy \( S_L \) (left) and fidelity \( F \) (right) plotted vs \( \Phi \) for various \( k_0 \) values: \( k_0 = 0.5 \) (thin dashed line), \( k_0 = 1.0 \) (thick solid line), \( k_0 = 2.5 \) (thin solid line), \( k_0 = 5.0 \) (thick short dashed line), and \( k_0 = 10.0 \) (thick long dashed line). The two pulses are in the identical Gaussian profile.](image2)

![FIG. 3: (color online) Gate operation fidelity \( F \) as a function of the parameters \( k_0 \) and \( \Phi \). The parameter ranges are 0.1 ≤ \( k_0 \) ≤ 8 and 0 ≤ \( \Phi \) ≤ \( \pi \). It is a 3D view for the right plot of Fig. 2.](image3)
The system bandwidth $\Delta t$ described above exists in the limit of an infinite time scale. The two pulses are initially separated by a distance of $d = 10r$, where $r$ is the pulse size in the medium, and run toward each other at a relative velocity $v_r = 2|v_i| = 10^4\sigma$ per unit time (here we adopt a scaled time unit). The pulses completely overlap at $t = 0.001$. The system parameter $k_0 = |v_i|/(2\sigma)\Delta\omega_e\Delta\omega_p$ is 0.001. The XPM phase is defined as $\Phi = \chi\Delta\omega_e t/(\pi v_r c)$.

With the corresponding term in Eq. (22) of [29] (the second term in Eq. (22) of [29]) contributes much less significantly to a coefficient similar to $C_1$ in this paper, limiting its value to less than 0.5. Then only the lower regime in our Fig. 1 can be obtained in the finite mode approach. For two ultrashort pulses the small ratio $v/c$ in the parameter $k_0$ makes the upper regime in Fig. 1 more relevant. The effect of non-unitary evolution of quantum states dominating in this regime is beyond the description by the finite mode approximation.

Next we examine the XPM between pulses colliding head-on [6, 3, 13, 14] via a contact potential $\chi\delta(z-z')$. For two ultrashort pulses satisfying $v_r/(2c)\Delta\omega_e t \ll 1$, there is the approximation $C(v_r t^{-1} - v_t t_k) \approx \Delta\omega_e/(2\pi c)$ in [30], so the infinite sum in Eq. (4) can be approximated by a closed form in this regime. Substituting the normalized two-particle function into (2), we obtain the fidelity evolution in the course of pulse interaction; see the example in Fig. 4. It shows that the fidelity value will decline once a pulse touches the other and stabilize again after they pass through each other. The stronger the interaction (indicated by the $\Phi$ values), the lower the fidelity will become after collision. Against the intuitive notion that an averaged interaction on pulses could generate a uniform conditional phase $\theta$, a realistic XPM between pulses of a non-zero relative velocity can be far away from the ideal process $|1\rangle_1|1\rangle_2 \rightarrow e^{i\theta}|1\rangle_1|1\rangle_2$.

An ideal performance of phase gates based on the XPM described above exists in the limit of an infinite system bandwidth $\Delta\omega_e$. In this limit the two-particle function after two pulses completely going through each other is $f_1(z_1 - vt)f_2(z_2 + vt)e^{i\theta}$, where $\theta = \chi/v_r$ is a fixed value from the contact potential $\chi\delta(z-z')$. This result is also true to the XPM between single photon and coherent state [13]. The time-dependent two-particle functions for such idealized unitary evolution under a general interaction potential $\Delta(z-z')$ take the form $f_1(z_1 - vt)f_2(z_2 - vt)e^{i\Phi(z_1,z_2,t)}$ [13, 13, 13]. The field mode entanglement exhibited by the possibly non-factorisable $\Phi(z_1,z_2,t)$ with respect to its spatial variables is therefore the main factor that determines the gate operation fidelity in a regime of approximately unitary evolution, which could be realized under the condition $\Delta\omega_e \gg \Delta\omega_p$.

IV. CONCLUSION

We have illustrated the effects of continuous field mode entanglement arising from pulse interaction and non-unitary evolution caused by finite system bandwidth, which drastically impair photon-photon phase gate performance. These effects induce more complexity in XPM than what was previously understood. Due to their possible existence in any device working with quantum objects of continuous degrees of freedom, the proper handling of the effects could be a major concern in quantum technology.

Appendix

We provide a brief derivation for the linear entropy formula in Eq. (3). The elements of one reduced density matrix $\rho_1$ for a general bipartite state $\int dk \int dk' \langle k,k'|\psi\rangle^* \langle k'|\psi\rangle|0\rangle$ can be obtained by the following [29] (its discrete form is given in [30]):

$$
\rho_1(k,k',t) = \int dq \zeta(k,q,t)\zeta^*(k',q,t) = \frac{1}{2\pi} \int dz_1 \int dz_2 \int dz_3 \{ \psi(z_1,z_2,t)\psi^*(z_3,z_2,t)e^{-ikz_1}e^{ik'z_3} \}.
$$

The matrix elements of $\rho_1^2$ can thus be obtained by substituting the above into $\rho_1^2(k,k',t) = \int dq \rho_1(k,q,t)\rho_1(q,k',t)$, which leads to a closed form of the linear entropy $S_L(t) = 1 - \int dk \rho_1^2(k,k,t)$.

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[1] H. Schmidt and A. Imamoglu, Opt. Letts. 21, 1936 (1996).

[2] M. Fleischhauer, A. Imamoglu, and J. P. Marangos, Sec.
VI, Rev. Mod. Phys. 77, 633 (2005).
[3] S. E. Harris and L. V. Hau, Phys. Rev. Lett. 82, 4611 (1999).
[4] M. D. Lukin and A. Imamoglu, Phys. Rev. Lett. 84, 1419 (2000).
[5] D. Petrosyan and G. Kurizki, Phys. Rev. A 65, 033833 (2002).
[6] M. Masalas and M. Fleischhauer, Phys. Rev. A 69, 061801(R) (2004).
[7] D. Petrosyan and Y. P. Malakyan, Phys. Rev. A 70, 023822 (2004).
[8] A. Andre, M. Bajcsy, A. S. Zibrov, and M. D. Lukin, Phys. Rev. Lett. 94, 063902 (2005).
[9] I. Friedler, D. Petrosyan, M. Fleischhauer, and G. Kurizki, Phys. Rev. A 72, 043803 (2005).
[10] S. Rebic, D. Vitali, C. Ottaviani, P. Tombesi, M. Artoni, F. Cataliotti, and R. Corbalan Phys. Rev. A 70, 032317 (2004).
[11] Z.-B. Wang, K.-P. Marzlin, and B. C. Sanders, Phys. Rev. Lett. 97, 063901 (2006).
[12] K.-P. Marzlin, Z.-B. Wang, S. A. Moiseev, and B. C. Sanders, J. Opt. Soc. Am. B 27, A36 (2010).
[13] B. He, A. MacRae, Y. Han, A. I. Lvovsky, and C. Simon, Phys. Rev. A 83, 022312 (2011).
[14] E. Shahmoon, G. Kurizki, M. Fleischhauer, and D. Petrosyan, Phys. Rev. A 83, 033806 (2011).
[15] B. He, Q. Lin, and C. Simon, Phys. Rev. A 83, 053826 (2011).
[16] A. Rispe, B. He, and C. Simon, Phys. Rev. Lett. 107, 043601 (2011).
[17] L. V. Hau et al., Nature (London) 397, 594 (1999).
[18] H. Kang and Y. Zhu, Phys. Rev. Lett. 91, 093601 (2003).
[19] Y.-F. Chen, C.-Y. Wang, S.-H. Wang, and I. A. Yu, Phys. Rev. Lett. 96, 043603 (2006).
[20] S. Li, X. Yang, X. Cao, C. Xie, and H. Wang, Phys. Rev. Lett. 101, 073602 (2008).
[21] H.-Y. Lo, P.-C. Su, and Y.-F. Chen, Phys. Rev. A 81, 053829 (2010).
[22] H.-Y. Lo, Y.-C. Chen, P.-C. Su, H.-C. Chen, J.-X. Chen, Y.-C. Chen, I. A. Yu, and Y.-F. Chen, Phys. Rev. A 83, 041804(R) (2011).
[23] B.-W. Shiu, M.-C. Wu, C.-C. Lin, and Y.-C. Chen, Phys. Rev. Lett. 106, 193006 (2011).
[24] Our treatment as well applies to the situations with one of the pulses being a photon at the speed \( c \).
[25] J. H. Shapiro, Phys. Rev. A 73, 062305 (2006).
[26] J. Gea-Banacloche, Phys. Rev. A 81, 043823 (2010).
[27] C. K. Law, I. A. Walmsley, and J. H. Eberly, Phys. Rev. Lett. 84, 5304 (2000).
[28] C. K. Law and J. H. Eberly, Phys. Rev. Lett. 92, 127903 (2004).
[29] M. Srednicki, Phys. Rev. Lett. 71, 666 (1993).
[30] B. He and J. A. Bergou, Phys. Rev. A 78, 062328 (2008).