Space-time from Symmetry: The Moyal Plane from the Poincaré-Hopf Algebra

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Abstract

We show how to get a non-commutative product for functions on space-time starting from the deformation of the coproduct of the Poincaré group using the Drinfel’d twist. Thus it is easy to see that the commutative algebra of functions on space-time (\(\mathbb{R}^4\)) can be identified as the set of functions on the Poincaré group invariant under the right action of the Lorentz group provided we use the standard coproduct for the Poincaré group. We obtain our results for the noncommutative Moyal plane by generalizing this result to the case of the twisted coproduct. This extension is not trivial and involves cohomological features.

As is known, spacetime algebra fixes the coproduct on the diffeomorphism group of the manifold. We now see that the influence is reciprocal: they are strongly tied.

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I. INTRODUCTION

During the last fifteen years, excellent physical arguments have emerged suggesting that space-time at Planck scale is noncommutative. In particular it has been argued [1] that the coexistence of Einstein’s theory of relativity and basic quantum mechanics, namely Heisenberg’s uncertainty principle, makes the quantum nature of spacetime important at energy scales close to Planck scale. A simple model which reflects this noncommutativity is the Moyal plane $\mathcal{A}_\theta(\mathbb{R}^N)$. It is defined by the $\ast$-product

$$f_1 \ast f_2(x) = f_1(x) e^{i\theta_{\alpha\beta} \partial_\alpha \otimes \partial_\beta} f_2(x)$$

(1)

on functions $f_i$ on $\mathbb{R}^N$. The latter implies the commutation relation

$$[\hat{x}_\mu,\hat{x}_\nu] = i\theta_{\mu\nu}$$

(2)

where $\theta_{\mu\nu} = -\theta_{\nu\mu}$ are constants and $\hat{x}_\mu$ on the coordinate functions:

$$\hat{x}_\mu(x) = x_\mu.$$

(3)

Now the noncommutative multiplication rule (1) can be written using the so called Drinfel’d twist $F_\theta$ [2]:

$$f_1 \ast f_2 = m_\theta(f_1 \otimes f_2) = m_0 \circ F_\theta(f_1 \otimes f_2)$$

(4)

$$F_\theta := \exp \left( \frac{i}{2} \theta_{\mu\nu} \partial^\mu \otimes \partial^\nu \right)$$

(5)

where $m_0$ is the commutative, point-wise multiplication and $m_\theta$ is the noncommutative one given by the noncommutativity in the coordinates (2).

Also it was thought for a long time that the relation (2) spoils Poincaré invariance completely since under the naive Lorentz transformations of $\hat{x}_\mu$ and $\hat{x}_\nu$, the L.H.S. of (2), transforms in a non-trivial way whereas the R.H.S. being a constant does not change. Later, the Poincaré symmetry was restored (at least partially [3, 4, 5, 6]) by changing the action of the Poincaré group to the so-called twisted action [7]. It involves a deformation of the coproduct $\Delta_0$ of the Poincaré-Hopf algebra to a coproduct $\Delta_\theta$ for compatibility with the product (4) (for a discussion on Hopf algebras
The deformation is as follows:

\[ \Delta_0 \rightarrow \Delta_\theta = F_\theta^{-1} \Delta_0 F_\theta, \quad (6) \]

\[ F_\theta = \exp \left( -\frac{i}{2} \theta_{\mu\nu} P^\mu \otimes P^\nu \right), \quad (7) \]

\[ P^\mu = \text{Translations generators} \quad (8) \]

(Note that \( F_\theta \) is the realization of \( F_\theta \) on functions on \( \mathbb{R}^N \).)

In this paper, we will show that from the deformation (6) of the Hopf algebra, it is possible to get a deformation on the algebra of functions on the Poincaré group and then obtain the Moyal algebra \( A_\theta(\mathbb{R}^N) \) therefrom using non-trivial considerations. This is done by constructing the dual Hopf algebra to the Poincaré-Hopf algebra and using the fact that \( \mathbb{R}^N \) can be identified with the Poincaré group quotiented by the Lorentz group. For the standard coproduct \( (\theta_{\mu\nu} = 0) \), the procedure leads to the commutative algebra of functions on spacetime. The dual Hopf algebra and its relevant properties are recalled in section 2. Subsequent sections discuss the construction of spacetime algebra therefrom.

II. HOPF ALGEBRA DUALITY

We will briefly recall now how to construct the dual of a Hopf algebra, for details see [9, 10]. Given a Hopf algebra \((H, \Delta, \mu, \eta, \epsilon, S)\), where \( \Delta \) is the co-product, \( \mu \) is the multiplication map, and \( \eta, \epsilon \) and \( S \) are respectively the unit, co-unit and the antipode, we want to construct another Hopf algebra \((H^*, \Delta^*, \mu^*, \eta^*, \epsilon^*, S^*)\) which will be called the dual of \( H \). Since we do not want to deeply go into mathematical formality, we will just focus on how to get \( \Delta^* \) and \( \mu^* \) from \( H \) assuming that once we find these structures, the unit, co-unit and the antipode can also be found.

Now \( H^* \) is the dual of \( H \). We want to identify \( H^* \) with (linear) functions \( \mathcal{F}(H) \) on \( H \). For this, we need a pairing of functions \( f \in \mathcal{F}(H) \) with elements of \( H \). For this, we use the natural choice:

\[ \forall f \in \mathcal{F}(H), \ h \in H \quad \langle f, h \rangle \equiv f(h) \in \mathbb{C} \quad . \quad (9) \]

With this pairing, we identify \( H^* \) with \( \mathcal{F}(H) \). Then we can define \( \mu^* \) as follows:

\[ \mu^* : \forall f_1, f_2 \in H^*, h \in H \quad \langle \mu^*(f_1 \otimes f_2), h \rangle := \langle f_1 \otimes f_2, \Delta(h) \rangle \ \text{or} \ \langle (f_1 \cdot^* f_2)(h), h \rangle := (f_1 \cdot f_2)(\Delta(h)) \quad (10) \]
where we have indicated $\mu^*(f_1 \otimes f_2)$ as $f_1 \cdot^* f_2$.

In the same way, we can construct the coproduct on $H^*$:

$$\Delta^*: \forall h_1, h_2 \in H, f \in H^* \quad (\Delta^*(f), h_1 \otimes h_2) := \langle f, \mu(h_1 \otimes h_2) \rangle \text{ or } \Delta(f)(h_1 \otimes h_2) := f(h_1 \cdot h_2) \quad (11)$$

where in this case we have indicated $\mu(h_1 \otimes h_2)$ as $h_1 \cdot h_2$. From the relations above, it is clear that the co-structure of $H$ will give rise to the multiplication in $H^*$ and vice-versa. In particular if we deform, in the sense of the Hopf algebra deformation theory [9, 10], the co-product of the first one, it will translate into a deformation of the multiplication rule for $H^*$. This already suggests that somehow starting from the co-deformation of the Poincaré-Hopf algebra, namely the Poincaré group algebra (see below) using a deformed co-product, we can get a deformation of the multiplication of the dual, that is of functions on the Poincaré group, which can then be translated into a deformation of the algebra of functions on spacetime.

Once the pairing is given, there is also an intrinsic and natural way of lifting the left and right actions of the group on itself,

$$\rho_L(\tilde{h}) \triangleright h := \tilde{h} \cdot h \quad (12)$$

$$\rho_R(\tilde{h}) \triangleright h := h \cdot \tilde{h}^{-1} \quad (13)$$

to the dual $H^*$. It goes as follows:

$$f_L \equiv \rho^*(\tilde{h}) \triangleright_L f : \quad \langle f_L, h \rangle := \langle f, \rho_L(\tilde{h}^{-1}) \triangleright h \rangle = f(\tilde{h}^{-1} \cdot h) \quad (14)$$

$$f_R \equiv \rho^*(\tilde{h}) \triangleright_R f : \quad \langle f_R, h \rangle := \langle f, \rho_R(\tilde{h}^{-1}) \triangleright h \rangle = f(h \cdot \tilde{h}) \quad (15)$$

The action on $(H \otimes H)^*$ is lifted, similarly, using the co-product $\Delta$.

These observations will be very useful for us in the following.

As a final remark, we emphasise that to identify $(H \otimes H)^*$ with say $\mathcal{F}(H) \otimes \mathcal{F}(H)$, we need a pairing of the latter with $H \otimes H$. In other words, we need an identification of $H^* \otimes H^*$ with $(H \otimes H)^*$ by introducing a pairing of the former with $H \otimes H$. A suitable pairing has been assumed in (11). An example of this ambiguity is already present in the possibility of taking the “flipped pairing”, $\langle f_1 \otimes f_2, g_1 \otimes g_2 \rangle = f_1(g_2)f_2(g_1)$. The significance of this remark will emerge below.
III. THE GROUP ALGEBRA AND ITS DUAL

To get a feeling of the construction above as well as to introduce the two Hopf algebras we will be working with, we want now to construct the dual of a specific Hopf algebra, namely the group algebra $\mathbb{C}G$ of a group $G$, regarded as a Hopf algebra, in terms of functions on $\mathbb{C}G$.

For a given group $G$, its group algebra is the vector space over complex numbers obtained from any linear combination of the elements of the group upon which we define a multiplication and a co-multiplication rule. The product between two elements $g_1 = \sum_k \lambda_k \cdot g_k$ and $g_2 = \sum_l \theta_l \cdot g_l$ of $\mathbb{C}G$ where $\lambda_k, \theta_l \in \mathbb{C}$ and $g_k, g_l \in G$ is inherited from group multiplication:

$$g_1 \cdot \mathbb{C}g_2 := \sum_{k,l} (\lambda_k \theta_l) g_k \cdot g_l \quad (16)$$

where $\cdot G$ is the group multiplication of $G$. The canonical co-product $\Delta$ is defined as follows:

$$\forall g \in G, \Delta(g) = g \otimes g \quad (17)$$

Using linearity we can extend $\Delta$ to the whole group algebra.

We want now to find its dual, namely the Hopf algebra of functions upon the group $G$. We identify functions $f$ on $G$ with elements of $\mathbb{C}G^*$ using the natural pairing

$$\langle f, g \rangle = f(g), \quad g \in G \quad (18)$$

where $f(g)$ is just the value of the function on the point $g$. Again we can extend the pairing of $f$ to all elements of $\mathbb{C}G$ using linearity.

It is now time to use (10-11) to get the structures induced by the ones on $\mathbb{C}G$.

First let us identify $\mathbb{C}G^* \otimes \mathbb{C}G^*$ with $(\mathbb{C}G \otimes \mathbb{C}G)^*$ by assuming the pairing

$$\langle f_1 \otimes f_2 , g \otimes g \rangle = f_1(g)f_2(g), \quad f_i \in \mathbb{C}G^*, \quad g \in G. \quad (19)$$

This pairing is then as usual extended to all of $\mathbb{C}G \otimes \mathbb{C}G$ using linearity.

The multiplication rule for functions on $G$ is, using (17):

$$\langle \mu^*(f_1 \otimes f_2) , g \rangle = \langle f_1 \otimes f_2 , g \rangle \text{ or } (f_1 \cdot^* f_2)(g) = f_1(g)f_2(g) \quad (20)$$

which is simply the point-wise, commutative, product. The co-product induced is instead less trivial:

$$\langle \Delta^*(f) , g_1 \otimes g_2 \rangle = \langle f, g_1 \cdot_G g_2 \rangle \text{ or } \Delta^*(f)(g_1 \otimes g_2) = f(g_1 \cdot_G g_2) \quad (21)$$
We thus find that the Hopf algebra dual to $C^*$ is nothing but the commutative algebra of functions on the group $G$, with the co-product above.

We emphasise that to arrive at (20), we have chosen a specific pairing between $f_1 \otimes f_2$ and $g \otimes g$. We can instead choose another pairing $\langle f_1 \otimes f_2, g \otimes g \rangle = \langle f_1 \otimes f_2, K(g \otimes g) \rangle$ where $K$ is an invertible linear operator chosen from $C^* \otimes C^*$ without setting $K = 1 \otimes 1$. This would have different consequences.

The actions given by (14-15) in this case give the right and left regular representations of the group $G$ on the functions. If we choose the Poincaré group $P$ for $G$, the algebra of functions on spacetime can be obtained as the coset of the dual of $C^P$ with respect to the right action of the Lorentz group. This can be understood since spacetime is topologically $\mathbb{R}^4$ and so is the coset $P / \mathbb{L}^+$. We notice that in this case the construction can be consistently carried through since the point-wise product of right-invariant functions under the Lorentz group is still right invariant. This will not be the case when we consider the dual of the deformed Poincaré group algebra, this problem has been already investigated, look at [11] and references therein\(^1\). We will now show a possible way of circumventing this issue using the above duality construction.

**IV. MOYAL FROM POINCARÉ**

In this section we show how to get Moyal spacetime applying the above considerations to the deformation $C^P_\theta$ of the Poincaré group algebra. $C^P_\theta$ is a particular Hopf algebra deformation of $C^P \equiv C^P_0$ in which only the co-product has been deformed (see (6)). Since the co-product in $C^P_\theta$ is no longer co-commutative, we expect the multiplication induced on the algebra of functions by duality to be noncommutative like the Moyal one in fact.

As we said at the end of the previous section, the problem in this case is that the preceding coset operation cannot be carried through in an obvious manner. In order to be able to consider right-invariant functions on $C^P_\theta$, we have to modify the pairing of $f_1 \otimes f_2$ with $C^P_\theta \otimes C^P_\theta$ where

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\(^1\) In these approaches the quantum spacetime algebra is provided by the translation sector of the dual of the deformed Hopf-Poincaré. In this case we end up with different commutation relation among spacetime coordinates in which an “anomalous” term dependent upon Lorentz group parameter appears.
are functions on $C\mathcal{P}_\theta$ with the natural pairing (18) from which we start the construction. This modification of the pairing, gotten by just asking for compatibility with invariance under the right action of the Lorentz group, turns out to be exactly what we need to obtain the multiplication rule (4) on the algebra of function on $\mathbb{R}^4$. As mentioned above we will present the calculation in the example of the Moyal twist, $F_\theta = \exp\left(-\frac{i}{2} P^\mu \theta_{\mu\nu} \otimes P^\nu\right)$, that is enough to see the procedure for say the Wick-Voros twist.

$C\mathcal{P}_\theta$ can be thought of as generated by elements $U(a, \Lambda)$, where

$$U(\Lambda)U(a) = U(\Lambda a)U(\Lambda).$$

Using (22), we can write all the elements of $C\mathcal{P}_\theta$ with the translation on the left and Lorentz transformation on the right. This is hereafter assumed.

Note also that

$$(b \cdot P)U(a) = -i \frac{d}{d\lambda} U(a + \lambda b)\bigg|_{\lambda=0}$$

where $P_\mu$ is the generator of translations.

It is convenient now to find a basis for the dual. Following Dirac, any element in $C\mathcal{P}_\theta^*$ can be written in terms of the $\delta$-distributions $\langle a', \Lambda' \rangle$ where

$$\langle a', \Lambda' | U(a)U(\Lambda) \rangle = \delta^4(a' - a)\delta(\Lambda' - \Lambda)$$

Fourier transforming in $a'$ we can thus regard $C\mathcal{P}_\theta^*$ as spanned by

$$\langle e_p \otimes \alpha | : \langle e_p \otimes \alpha | U(a)U(\Lambda) \rangle = e_p(a)\alpha(\Lambda)$$

where $e_p(a) = e^{ip \cdot a}$ and $\alpha(\Lambda) \in \mathbb{C}$. Note that:

$$\langle e_p \otimes \alpha | U(\Lambda)U(a) \rangle = \langle e_p \otimes \alpha | U(\Lambda a)U(\Lambda) \rangle = e^{ip \cdot \Lambda a}\alpha(\Lambda).$$

We are now ready to compute explicitly the deformed product $\mu^*$ induced on $C\mathcal{P}_\theta^*$. Recalling (6), (7) and (10) and with $f_1 = e_p \otimes \alpha$ and $f_2 = e_q \otimes \beta$,

$$\langle \mu^*_\theta(f_1 \otimes f_2) | U(a)U(\Lambda) \rangle = \langle f_1 \otimes f_2 | \Delta_\theta(U(a)U(\Lambda)) \rangle = \langle f_1 \otimes f_2 | F_\theta^{-1}\left[U(a)U(\Lambda) \otimes U(\Lambda a)U(\Lambda)\right]F_\theta\rangle$$

$$= \langle e_p \otimes e_q \beta | e^{\frac{i}{2}(\Lambda P - \frac{1}{2}(\Lambda P)\wedge(\Lambda P))} \left[U(a)U(\Lambda) \otimes U(a)U(\Lambda)\right]\rangle$$

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where $P \land P = P^\mu \theta_{\mu\nu} \otimes P^\nu$. To evaluate the expression above, we need to know how to evaluate expressions like

$$\langle e_p | (b \cdot P) U(a) \rangle$$  \hspace{1cm} (28)

From (23), this is given by

$$-i \frac{d}{d\lambda} \langle e_p | U(a + \lambda b) \rangle \bigg|_{\lambda=0} = b \cdot p \, e^{ip \cdot a}$$  \hspace{1cm} (29)

where $P$ has become $p$. Then pairing $f_1 \otimes f_2$ with $C_\theta \otimes C_\theta$,

$$\langle \mu_\theta ^* (f_1 \otimes f_2) | U(a) U(\Lambda) \rangle := (f_1 \ast f_2) \left( U(a) U(\Lambda) \right) = e^{-\frac{i}{2} P^L \land P^L - \frac{i}{2} P^R \land P^R} e_{p+q}(a(\Lambda) \beta (\Lambda)).$$  \hspace{1cm} (30)

where in the evaluation, following [7], we have used what above has been called “flipped pairing”. This pairing is assumed from now on for convenience since otherwise finally we will end up with $A_{-\theta}(\mathbb{R}^N)$.

Using the right and left action on $C_\theta^* \otimes C_\theta^*$ described above (14-15), this can be written as

$$\mu_\theta \circ (e_p \alpha \otimes e_q \beta) = \mu_0 \left[ e^{-\frac{i}{2} P^L \land P^L - \frac{i}{2} P^R \land P^R} e_{p+q}(a(\Lambda) \beta (\Lambda)) \right].$$  \hspace{1cm} (31)

Before taking the coset with respect the right action of the Lorentz group, we need to ensure that it can be taken consistently. This is the case if the product of two right-invariant functions is still right-invariant. From the explicit expression (30), we can already see that the product acquires a dependence on $\Lambda$ spoiling this condition. More formally, in the basis we have chosen, a right invariant function on $C_\theta^*$ is one in which the “Lorentz part” $\alpha$ is the constant function. Now the Lorentz group does not act on the right trivially on the product of two such functions. Thus from (15):

$$\rho_R^\theta \left( U(a') U(\Lambda') \right) \triangleright (f_1 \ast f_2) \left( U(a) U(\Lambda) \right) = \langle e_p \alpha \otimes e_q \beta | \Delta_\theta \left( U(a + \Lambda a') U(\Lambda \Lambda') \right) \rangle$$

$$= e^{-\frac{i}{2} p \land q e^{\frac{1}{2} (\Lambda \Lambda')} \land (\Lambda' q)} e_{p+q}(a + \Lambda a') \alpha (\Lambda \lambda') \beta (\Lambda \lambda')$$  \hspace{1cm} (32)

where we used the compatibility of the co-product with the product multiplication in $C_\theta$, $\Delta_\theta(g) \cdot \Delta_\theta(g') = \Delta_\theta(g \cdot g')$ and (22). The equation above shows that there is a non-trivial dependence on the Lorentz group coming from twisting the co-product. That prevents us from taking the coset and still retaining an algebra structure. Thus from the product (30) obtained from the pairing (18), we cannot proceed with the coset and obtain an algebra of functions on the spacetime.
V. MOYAL FROM A MODIFIED PAIRING

As we already emphasised above, in the identification of $f_1 \otimes f_2 \ (f_i \in \mathbb{CP}_\theta^*)$ with an element of $(\mathbb{CP}_\theta \otimes \mathbb{CP}_\theta)^*$ we have a certain freedom. Thus we can carry out the whole construction modifying how $\mathbb{CP}_\theta^* \otimes \mathbb{CP}_\theta^*$ pairs with $\mathbb{CP}_\theta \otimes \mathbb{CP}_\theta$. If $\sigma$ be an invertible element of $\mathbb{CP}_\theta \otimes \mathbb{CP}_\theta$, then a possible pairing is

$$\langle f_1 \otimes f_2 | g_1 \otimes g_2 \rangle_{\sigma R} = (f_2 \otimes f_1) \left( (g_1 \otimes g_2) \circ \sigma \right) = (f_2 \otimes f_1) \left( \sigma^R \circ (g_1 \otimes g_2) \right) \quad (33)$$

where composition $\circ$ should be understood in terms of products and actions on the Hopf algebra. In the previous construction we implicitly assumed the “trivial” pairing (with $\sigma = 1 \otimes 1$) of $f_1 \otimes f_2$ with elements of $\mathbb{CP}_\theta \otimes \mathbb{CP}_\theta$.

The modification (33) does not effect the induced co-product $\Delta^*_\theta$ whereas the new pairing defines a new multiplication map $\tilde{\mu}^*_\theta$:

$$\langle \tilde{\mu}^*_\theta (f_1 \otimes f_2) | h \rangle_{\sigma R} = \langle f_1 \otimes f_2 | \Delta_\theta(h) \rangle_{\sigma R} = (f_2 \otimes f_1) \left( \sigma^R \circ \Delta_\theta(h) \right). \quad (34)$$

A natural question to pose is under what conditions $\sigma$ gives rise to an associative product. We will show in the final section that associativity of $\tilde{\mu}^*_\theta$ puts cohomological constraints on $\sigma$ in the sense of the Hopf algebra deformation theory [10].

With a particular choice of the map $\sigma$, we can ensure that the product (30) is compatible with the right invariance under the action of the Lorentz group. Thus if we assume that

$$\sigma \equiv \sigma^R = e^{\frac{i}{2} P^R L^R} \quad (35)$$

and use (34) and the expression (31) for $\mu^*_\theta$, we get an expression for a new multiplication map:

$$\tilde{\mu}^*_\theta (e_p \alpha \otimes e_q \beta) \left( U(a) U(\Lambda) \right) = \left[ \mu_0 \left( e^{-\frac{i}{2} P^L L^L} e_p \alpha \otimes e_q \beta \right) \right] \left( U(a) U(\Lambda) \right) = \left[ \mu_0 \mathcal{F}_\theta (e_p \alpha \otimes e_q \beta) \right] \left( U(a) U(\Lambda) \right) \quad (36)$$

where $\mathcal{F}_\theta$ acts only on $e_p$ and $e_q$.

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2 We are not considering here compatibility of the two structures $(\Delta^*_\theta, \tilde{\mu}^*_\theta)$. This will be briefly discussed below.
Acting on the right, we can check that the choice of pairing made above allows us to take cosets. Thus now

\[ \rho_R^*\left(U(a')U(\Lambda')\right) \triangleright \tilde{\nu}_\theta^*(f_1 \otimes f_2)\left(U(a)U(\Lambda)\right) = e^{-\frac{i}{2}\nu^q \epsilon_{p+q}(a + \Lambda a')\alpha(\Lambda\Lambda')\beta(\Lambda\Lambda')} \]  

(37)

Assuming that \(\alpha\) and \(\beta\) are invariant under the right action of the Lorentz group (that is that they are constant functions on the Lorentz group) and putting \(a' = 0\), we get from above that also the product is right-invariant under Lorentz transformations. This is what we required.

After the coset \(\mathbb{C} \mathcal{P}_\theta^*/\mathcal{L}_+^1\) has been taken, we thus obtain a deformation of the algebra of functions on spacetime with product \(\tilde{\nu}_\theta^*\). A comparison of (36) with (4) shows that this deformation of the algebra of functions is exactly the one given by Moyal twist \(m_\theta = m_0 \circ \mathcal{F}_\theta\), just as we claimed.

VI. RIGIDITY OF SPACETIME ALGEBRA

The above construction only involves a modification of the pairing on the right. Asking for the multiplication map in (34) to preserve the invariance of functions under the right-action of the Lorentz group does not however give rise to constraints on possible modifications of the pairing on the left. Thus the following choice would still be compatible with the above construction:

\[ \langle f_1 \otimes f_2 | g_1 \otimes g_2 \rangle_{\delta \triangleright \sigma_R} := (f_2 \otimes f_1)\left((\delta^L \circ \sigma^R) \circ (g_1 \otimes g_2)\right) \equiv (f_2 \otimes f_1)\left(\delta \circ (g_1 \otimes g_2) \circ \sigma\right) \]  

(38)

where \(\delta\) is an invertible element of \(H \otimes H\). This coupling would define a new product, namely:

\[ \tilde{\nu}^*_\theta (f_1 \otimes f_2)(g) := (f_2 \otimes f_1)(\delta \circ \Delta_\theta(g) \circ \sigma) \]  

(39)

We will show that this freedom in changing the pairing from the left can be ruled out by requiring that there is a left action of the Hopf-Poincaré group on the algebra of functions we get from the coset operation. Thus we require that

\[ \rho^*(\tilde{h}) \triangleright_L \left(\tilde{\nu}_\theta^*(f_1 \otimes f_2)\right)(h) = \tilde{\nu}_\theta^*\left(\rho_{H^* \otimes H^*}^*(\tilde{h}) \triangleright_L (f_1 \otimes f_2)\right)(h) \]  

(40)

where \(\rho_{H^* \otimes H^*}^*(\tilde{h}) = \left(\rho_{H^* \otimes H^*}^* \circ \rho_{H^*}^*\right)\Delta(\tilde{h})\) and \(\rho_{H^*}^*\) is defined by (14). We can now compute both sides:

\[ \text{L.H.S.} = \tilde{\nu}_\theta^*(f_1 \otimes f_2)(\tilde{h}^{-1} \cdot h) = (f_2 \otimes f_1)\left(\delta \circ \Delta(\tilde{h}^{-1}) \circ \Delta(h) \circ \sigma\right) \]  

(41)

where compatibility between \(\Delta_\theta\) and group composition law has been used.
On the R.H.S. we get:

\[ \text{R.H.S.} = \left( \rho_{\tilde{H}^* \otimes H^*}(\tilde{h}) \triangleright_L (f_2 \otimes f_1) \right) (\delta \circ \Delta(h) \circ \sigma) = (f_2 \otimes f_1)(\Delta(\tilde{h}^{-1}) \circ \delta \circ \Delta(h) \circ \sigma) \]  

(42)

Since (41) and (42) have to be equal for all elements \( \tilde{h} \in \mathbb{C} \mathcal{P} \), the unique choice of \( \delta \) is the trivial one.

VII. REMARKS

As already indicated above, we now discuss the conditions on the product (34), induced on \( \mathbb{C} \mathcal{P}^* \) by the coupling (33), to be associative, that is:

\[ \tilde{\mu}_\theta(f_1 \otimes \tilde{\mu}_\theta(f_2 \otimes f_3))(h) = \tilde{\mu}_\theta(\tilde{\mu}_\theta(f_1 \otimes f_2) \otimes f_3)(h) \]  

(43)

with \( f_i \in \mathbb{C} \mathcal{P}^*_\theta \) and \( h \in \mathbb{C} \mathcal{P}_\theta \). Using (34) we can compute both sides getting:

\[ \text{L.H.S.} = \left( \tilde{\mu}_\theta(f_2 \otimes f_3)(f_1) \right) (\sigma \circ \Delta_\theta(h)) = (f_3 \otimes f_2 \otimes f_1)(\left( \mathbbm{1} \otimes \sigma \right)(\left( \mathbbm{1} \otimes \Delta_\theta \right)(\sigma \circ \Delta_\theta(h))) \]  

(44)

\[ \text{R.H.S.} = \left( f_3 \otimes \tilde{\mu}_\theta(f_1 \otimes f_2) \right)(\sigma \circ \Delta_\theta(h)) = (f_3 \otimes f_2 \otimes f_1)(\left( \sigma \otimes \mathbbm{1} \right)(\Delta_\theta \otimes \mathbbm{1})(\sigma \circ \Delta_\theta(h))) \]  

(45)

Using compatibility of the coproduct with the multiplication map, \( \Delta_\theta(h_1 \cdot h_2) = \Delta_\theta(h_1) \cdot \Delta_\theta(h_2) \), and co-associativity of the deformed coproduct, \( (\mathbbm{1} \otimes \Delta_\theta)(\Delta_\theta(h)) = (\Delta_\theta \otimes \mathbbm{1})(\Delta_\theta(h)) \) we find that the associativity of the product \( \tilde{\mu}_\theta^* \) translates into

\[ (\mathbbm{1} \otimes \sigma)(\mathbbm{1} \otimes \Delta_\theta)(\sigma) = (\sigma \otimes \mathbbm{1})(\Delta_\theta \otimes \mathbbm{1})(\sigma) \]  

(46)

In the Hopf algebra cohomology, the condition above corresponds exactly to \( \sigma \) being a 2-cocycle \[10\]. We have therefore shown, as we claimed, that the associativity of the product induced by the new pairing (33) restricts in a cohomological way the possible choices of the map \( \sigma \). In the case under discussion it is not necessary to check that the choice made in (35) fulfills the condition given above since the induced product on spacetime is Moyal which is known to be associative.

Another important point we can comment on here is the following. If we relax the condition of the multiplication is compatible with the left action of the Lorentz group, the whole family of multiplication maps (39) is allowed. This allows us to get different deformations \( \mathcal{A}_\theta(\mathbb{R}^4) \) of the algebras of functions on the spacetime choosing different \( \delta \).
Now deformations of $\mathcal{F}(\mathbb{R}^4)$ are classified by deformation quantization cohomology (see [12]) which is Hochshild cohomology. Also it can be shown [13] that maps $\delta$ in the same Hopf-algebra cohomology class (see [10]) induce isomorphic deformations. We thus get an action of Hopf algebra cohomology on deformation quantization cohomology. This action merits further study.

We want also to stress how the freedom in defining a map from $(H \otimes H)^*$ to $H^* \otimes H^*$, once the pairing $\langle \cdot , \cdot \rangle$ is given, is the only one available to characterize $H^*$ as a Hopf algebra. Both the product and the coproduct structures are then in fact canonically defined. The compatibility between the two, which characterizes completely $H^*$ as a Hopf algebra, can be also written in term of the pairings, although this is a very strong constraint\(^3\). It can be shown, in fact, that the only product $\tilde{\mu}_\sigma$ compatible with $\Delta_\delta$ (the coproduct induced on $\mathcal{F}(\mathbb{C}P_\theta)$ by the canonical pairing (18)) is the one given by the trivial choice $\sigma = 1 \otimes 1$. However we are not interested in $\mathcal{F}(\mathbb{C}P_\theta)$ as a Hopf algebra. We are interested in it as an algebra and in recovering the Moyal plane therefrom. We are hence allowed to choose a non-trivial $\sigma$.

Finally we point out that the above construction can be generalized to any group $\mathcal{G}$ which is the semi-direct product of a subgroup $\mathcal{G}$ and translations.

VIII. CONCLUSIONS

The construction explained in this paper shows how the deformation of spacetime algebra is tightly connected with the deformation of the coproduct of its associated Hopf algebra (such as the Poincaré-Hopf algebra). So far the latter was primarily seen as a consequence of the former one. We have now shown how the connection goes in the other direction as well.

The deformations of Hopf algebras are classified by the Hopf algebra cohomology [9, 10]. The deformation of spacetime algebra is classified by Hochshild cohomology [12]. The requirement that a deformed Hopf algebra acts on a deformed spacetime algebra when such an action exists in the undeformed case clearly puts constraints on the possible deformations of both algebras and consequently ties the two cohomologies. It is of considerable interest to properly understand these

\(^3\) We gratefully acknowledge Alberto Ibort for having pointed this out.
connections. We have taken the first step in this direction by showing the two-way connection between the Moyal deformation of spacetime and the Drinfel’d twist of the Hopf-Poincaré algebra. We plan to more fully study these mutual dependences in later works.

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