CLASSIFICATION OF CALABI HYPERSURFACES WITH PARALLEL FUBINI-PICK FORM

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Abstract: In this paper, we present the classification of 2 and 3-dimensional Calabi hypersurfaces with parallel Fubini-Pick form with respect to the Levi-Civita connection of the Calabi metric.

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1. Introduction

In equiaffine differential geometry, the problem of classifying locally strong convex affine hypersurfaces with parallel Fubini-Pick form (also called cubic form) has been studied intensively, from the earlier beginning paper by Bokan-Nomizu-Simon [1], and then [5], [6], [7], to complete classification of Hu-Li-Vrancken [8]. Here we recall results about the classification of locally strong convex affine hypersurfaces with parallel Fubini-Pick form $\nabla A = 0$ with respect to the Levi-Civita connection of the affine Berwald-Blaschke metric. The condition $\nabla A = 0$ implies that $M$ is an affine hypersphere with constant affine scalar curvature. Thus Theorem 1 of Li and Penn [14] (see also Theorem 3.7 in [12]) can be restated as follows:

Theorem 1.1. [14] Let $x : M \rightarrow A^3$ be a locally strongly convex affine surface with $\nabla A = 0$. Then, up to an affine transformation, $x(M)$ lies on the surface $x_1x_2x_3 = 1$ or strongly convex quadric.

The classification of 3-dimensional affine hypersurfaces with parallel Fubini-Pick form, due to Dillen and Vrancken [5].

Theorem 1.2. [5] Let $x : M \rightarrow A^4$ be a locally strongly convex affine hypersurface with $\nabla A = 0$. Then, up to an affine transformation, either $x(M)$ is an open part of a locally strongly convex quadric or $x(M)$ is an open part of one of the following two hypersurfaces:

(i) $x_1x_2x_3x_4 = 1$;
(ii) $(x_1^2 - x_2^2 - x_3^2)x_4^2 = 1$.

In [8], Hu-Li-Vrancken introduced some typical examples and gave the complete classification of locally strongly convex affine hypersurfaces of $\tilde{R}^{m+1}$ with parallel cubic form with respect to the Levi-Civita connection of the affine Berwald-Blaschke metric.

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In centroaffine differential geometry, Cheng-Hu-Moruz [4] obtained a complete classification of locally strong convex centroaffine hypersurfaces with parallel cubic form. On the other hand, Liu and Wang [16] gave the classification of the centroaffine surfaces with parallel traceless cubic form relative to the Levi-Civita connection. In [3], Cheng and Hu established a general inequality for locally strongly convex centroaffine hypersurfaces in $\mathbb{R}^{n+1}$ involving the norm of the covariant differentiation of both the difference tensor and the Tchebychev vector field. Applying the classification result of [4], Cheng-Hu [3] completely classified locally strongly convex centroaffine hypersurfaces with parallel traceless difference tensor.

A centroaffine hypersurface is said to be Canonical if its Blaschke metric is flat and its Fubini-Pick form is parallel with respect to its Blaschke metric. In [15], Li and Wang classified the Canonical centroaffine hypersurfaces in $\mathbb{R}^{n+1}$. In this paper, we first classify the canonical Calabi hypersurfaces in Calabi geometry. As a corollary, we classify Calabi surfaces with parallel Fubini-Pick form with respect to the Levi-Civita connection of the Calabi metric.

**Theorem 1.3.** Let $f$ be a smooth strictly convex function on a domain $\Omega \subset \mathbb{R}^n$. If its graph $M = \{(x, f(x)) \mid x \in \Omega\}$ has a flat Calabi metric and parallel cubic form. Then $M$ is Calabi affine equivalent to an open part of the following hypersurfaces:

(i) elliptic paraboloid; or

(ii) the hypersurfaces $Q(c_1, \ldots, c_r; n)$, $1 \leq r \leq n$.

The definitions of Calabi affine equivalent and hypersurfaces $Q(c_1, \ldots, c_r; n)$ will be given in Section 2 (see Definition 2.1 and Example 2.1, respectively).

**Corollary 1.4.** Let $f$ be a smooth strictly convex function on a domain $\Omega \subset \mathbb{R}^2$. If its graph $M = \{(x, f(x)) \mid x \in \Omega\}$ has parallel cubic form, $M$ is Calabi affine equivalent to an open part of the following surfaces:

(i) elliptic paraboloid; or

(ii) the surfaces $Q(c_1, \ldots, c_r; 2)$, $1 \leq r \leq 2$.

Motivated by above classification results in equiaffine differential geometry and centroaffine differential geometry, we present the classification of 3-dimensional Calabi hypersurfaces with parallel Fubini-Pick form with respect to the Levi-Civita connection of the Calabi metric in Section 4 and Section 5.

**Theorem 1.5.** Let $f$ be a smooth strictly convex function on a domain $\Omega \subset \mathbb{R}^3$. If its graph $M = \{(x, f(x)) \mid x \in \Omega\}$ has parallel cubic form. Then $M$ is Calabi affine equivalent to an open part of one of the following three types of hypersurfaces:

(i) elliptic paraboloid; or

(ii) the hypersurfaces $Q(c_1, \ldots, c_r; 3)$, $1 \leq r \leq 3$; or

(iii) the hypersurface

$$x_4 = -\frac{1}{2c^2} \ln(x_1^2 - (x_2^2 + x_3^2)),$$

where the constant $-2c^2$ is the scalar curvature of $M$.

**Remark:** (1) In [20], Xu-Li proved that
Theorem 1.6. [20] Let $M^n (n \geq 3)$ be a Calabi complete Tchebychev affine Kähler hypersurface with nonnegative Ricci curvature. Then it must be Calabi affine equivalent to either an elliptic paraboloid or one of the hypersurfaces $Q(c_1, ..., c_r; n)$.

Case (iii) of Theorem 1.5 shows that there exists a class of Calabi complete Tchebychev affine Kähler hypersurfaces with negative Ricci curvature. Thus the restriction on Ricci curvature in Theorem 1.6 is essential.

(2) The Euler-Lagrange equation of the volume variation with respect to the Calabi metric can be written as the following fourth order PDE (see [17] or [13])

$$\Delta \ln \det \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right) = 0,$$

(1.1)

where $\Delta$ is the Laplacian of the Calabi metric $G = \sum \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j$. Graph hypersurfaces $M^n$ defined by solutions of (1.1) are called affine extremal hypersurfaces. Case (iii) of Theorem 1.5 shows that there exists a class of new Euclidean complete and Calabi complete affine extremal hypersurfaces.

2. Preliminaries

2.1. Calabi geometry. In this section, we shall show some basic facts for the Calabi geometry, see [2] or [19]. Let $f$ be a strictly convex $C^\infty$-function on a domain $\Omega \subset \mathbb{R}^n$. Consider the graph hypersurface

$$M^n := \{(x_i, f(x_i)) \mid x_{n+1} = f(x_1, \ldots, x_n), (x_1, \ldots, x_n) \in \Omega\}.\quad (2.1)$$

This Calabi geometry can also be essentially realised as a very special relative affine geometry of the graph hypersurface (2.1) by choosing the so-called Calabi affine normalization with

$$Y = (0, 0, \ldots, 1)^t \in \mathbb{R}^{n+1}$$

being the fixed relative affine normal vector field which we call the Calabi affine normal.

For the position vector $x = (x_1, \ldots, x_n, f(x_1, \ldots, x_n))$ we have the decomposition

$$x_{ij} = c_{ij}^k x_k \frac{\partial}{\partial x_k} + f_{ij} Y,$$

(2.2)

with respect to the bundle decomposition $\mathbb{R}^{n+1} = x_\ast TM^n \oplus \mathbb{R} \cdot Y$, where the induced affine connection $c_{ij}^k \equiv 0$. It follows that the relative affine metric is nothing but the Calabi metric

$$G = \sum \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j.$$

The Levi-Civita connection with respect to the metric $G$ has the Christoffel symbols

$$\Gamma^k_{ij} = \frac{1}{2} \sum f^{kl} f_{ijl},$$

where and hereafter

$$f_{ijk} = \frac{\partial^2 f}{\partial x_i \partial x_j \partial x_k}, \quad (f^{kl}) = (f_{ij})^{-1}.\quad (f_{ij})^{-1}$$
Then we can rewrite the Gauss structure equation as follows:

\[ x_{ij} = \sum A^k_{ij}x_k \frac{\partial}{\partial x_k} + G_{ij}Y. \]  

(2.3)

The Fubini-Pick tensor (also called cubic form) \( A_{ijk} \) and the Weingarten tensor satisfy

\[ A_{ijk} = A^l_{ij}G_{kl} = -\frac{1}{2}f_{ijk}, \quad B_{ij} = 0, \]  

(2.4)

which means \( A_{ijk} \) are symmetric in all indexes. Classically, the tangent vector field

\[ T := \frac{1}{n} \sum G^k G^{ij} A_{ijk} \frac{\partial}{\partial x_l} = -\frac{1}{2n} \sum f^{kl} f^{ij} f_{ijk} \frac{\partial}{\partial x_l}, \]  

(2.5)

is called the Tchebychev vector field of the hypersurface \( M^n \), and the invariant function

\[ J := \frac{1}{n(n-1)} \sum G^{il} G^{jp} G^{kq} A_{ijk} A_{lpq} = \frac{1}{4n(n-1)} \sum f^{il} f^{jp} f^{kq} f_{ijk} f_{lpq} \]  

(2.6)

is named as the relative Pick invariant of \( M^n \). As in the report \[10\], \( M^n \) is called a Tchebychev affine Kähler hypersurface, if the Tchebychev vector field \( T \) is parallel with respect to the Calabi metric \( G \).

The Gauss integrability conditions and the Codazzi equations read

\[ R_{ijkl} = \sum f^{mh}(A_{jkm}A_{hil} - A_{ikm}A_{hjl}), \]  

(2.7)

\[ A_{ijk,l} = A_{iji,k}. \]  

(2.8)

From (2.7) we get the Ricci tensor

\[ R_{ik} = \sum f^{jl} f^{mh}(A_{jkm}A_{hil} - A_{ikm}A_{hjl}). \]  

(2.9)

Thus, the scalar curvature is given by

\[ R = n(n-1)J - n^2|T|^2. \]  

(2.10)

Using Ricci identity, (2.7) and (2.8), we have two useful formulas.

**Lemma 2.1.** For a Calabi hypersurface, the following formulas hold

\[ \frac{1}{2}\Delta |T|^2 = \sum T^2_{ij} + \sum T_i T_j + \sum R_{ij} T_i T_j, \]  

(2.11)

\[ \frac{n(n-1)}{2}\Delta J = \sum (A_{ijk,l})^2 + \sum A_{ijk} A_{llij,jk} + \sum (R_{ijkl})^2 + \sum R_{ij} A_{lpq} A_{jqp}. \]  

(2.12)

Let \( A(n+1) \) be the group of \( (n+1) \)-dimensional affine transformations on \( R^{n+1} \). Then \( A(n+1) = GL(n+1) \ltimes R^{n+1} \), the semi-direct product of the general linear group \( GL(n+1) \) and the group \( R^{n+1} \) of all the parallel transports on \( R^{n+1} \). Define

\[ SA(n+1) = \{ \phi = (M, b) \in A(n+1) = GL(n+1) \ltimes R^{n+1}; \ M(Y) = Y \}. \]  

(2.13)
where $Y = (0, \ldots, 0, 1)^t$ is the Calabi affine normal. Then the subgroup $SA$ consists of all the transformations $\phi$ of the following type:

$$X := (X^t, X^{n+1})^t \equiv (X^1, \ldots, X^n, X^{n+1})^t$$

$$\mapsto \phi(X) := \begin{pmatrix} a^i_j & 0 \\ a^{n+1}_j & 1 \end{pmatrix} X + b, \quad \forall X \in \mathbb{R}^{n+1}, \quad (2.14)$$

for some $(a^i_j) \in A(n)$, constants $a^{n+1}_j (j = 1, \ldots, n)$ and some constant vector $b \in \mathbb{R}^{n+1}$. Clearly, the Calabi metric $G$ is invariant under the action of $SA(n + 1)$ on the graph hypersurfaces or, equivalently, under the induced action of $SA(n + 1)$ on the strictly convex functions, which is naturally defined to be the composition of the following maps:

$$f \mapsto (x_i, f(x_i)) \mapsto (\tilde{x}_i, \tilde{f}(\tilde{x}_i)) \equiv \phi(x_i, f(x_i)) \mapsto \phi(f) := \tilde{f}, \quad \forall \phi \in SA(n + 1).$$

**Definition 2.1.** [20] Two graph hypersurfaces $(x_i, f(x_i))$ and $(\tilde{x}_i, \tilde{f}(\tilde{x}_i))$, defined respectively in domains $\Omega, \tilde{\Omega} \subset \mathbb{R}^n$, are called Calabi-affine equivalent if they differ only by an affine transformation $\phi \in SA(n + 1)$.

Accordingly, we have

**Definition 2.2.** [20] Two smooth functions $f$ and $\tilde{f}$ respectively defined on domains $\Omega, \tilde{\Omega} \subset \mathbb{R}^n$ are called affine equivalent (related with an affine transformation $\varphi \in A(n)$) if there exist some constants $a_1^{n+1}, \ldots, a^n_{n+1}, b^{n+1} \in \mathbb{R}$ such that $\varphi(\Omega) = \tilde{\Omega}$ and

$$\tilde{f}(\varphi(x_j)) = f(x_j) + \sum a^i_j x_j + b^{n+1} \quad \text{for all } (x_j) \in \Omega. \quad (2.15)$$

Clearly, the above two definitions are equivalent to each other.

2.2. **Canonical Calabi hypersurfaces.** A Calabi hypersurface is called canonical if its Fubini-Pick form is parallel with respect to the Levi-Civita connection of the Calabi metric and its Calabi metric $G$ is flat. In [20] Xu and Li introduced a large class of new canonical Calabi hypersurfaces, which are denoted by $Q(c_1, \ldots, c_r; n)$. It turns out that these new examples are all Euclidean complete and Calabi complete.

**Example 2.1.** [20] Given the dimension $n$ and let $1 \leq r \leq n$. For any positive numbers $c_1, \ldots, c_r$, define

$$\Omega_{c_1, \ldots, c_r; n} = \{(x_1, \ldots, x_n); \ x_1 > 0, \ldots, x_r > 0\}$$

and consider the following smooth functions

$$f(x_1, \ldots, x_n) \equiv Q(c_1, c_2, \ldots, c_r; n)(x_1, \ldots, x_n)$$

$$:= - \sum_{i=1}^r c_i \ln x_i + \sum_{j=r+1}^n \frac{1}{2} x_j^2, \quad (x_1, \ldots, x_n) \in \Omega_{c_1, \ldots, c_r; n}. \quad (2.16)$$
3. Proof of the Theorem 1.3 and Corollary 1.4

Lemma 3.1. Let \( \{A^i\}_{1 \leq i \leq n} \) be real symmetric matrices satisfying \( A^iA^j = A^jA^i \), \( \forall 1 \leq i, j \leq n \). Then there exists an orthogonal matrix such that matrices \( \{A^i\}_{1 \leq i \leq n} \) can be simultaneously diagonalized.

Proof. As we know the conclusion obviously holds for the case of \( n = 2 \). Now we assume that it holds for \( n = k \). Namely, there is an orthogonal matrix \( P \) such that \( A^1, \cdots, A^k \) can be simultaneously diagonalized:

\[
PA^iP^{-1} = diag(\lambda^i_1E_{n^1_i}, \cdots, \lambda^i_kE_{n^k_i}), \quad 1 \leq i \leq k;
\]

where \( \lambda^i_1, \cdots, \lambda^i_k \) are different eigenvalues for any fixed \( i \).

In the next we will prove the conclusion still holds for the case of \( n = k + 1 \). Since \( A^iA^{k+1} = A^{k+1}A^i, \forall 1 \leq i \leq k \), we obtain

\[
(PA^iP^{-1})(PA^{k+1}P^{-1}) = (PA^{k+1}P^{-1})(PA^iP^{-1}).
\]

Denote

\[
B^{k+1} := PA^{k+1}P^{-1},
\]

where \( B^{k+1} \) is a real symmetric matrix. From (3.1) and (3.2) we have

\[
\lambda^i_pB^k_{pq}B^{k+1}_{q} = \lambda^i_qB^{k+1}_{q}, \quad \forall p, q, \quad 1 \leq i \leq k.
\]

Fix an arbitrary index \( i \in \{1, \cdots, k\} \). If \( \lambda^i_p \neq \lambda^i_q \) for some indices \( p \neq q \) then, by (3.3), it must hold that \( B^k_{pq} = 0 \). Therefore, for any pair of indices \( p \neq q \), if \( B^k_{pq} \neq 0 \), then it holds that \( \lambda^i_p = \lambda^i_q \) for each \( i = 1, \cdots, k \). Thus we get

\[
B^{k+1} = diag(B^{k+1}_{n^1_{k+1}}, \cdots, B^{k+1}_{n^k_{k+1}}),
\]

where \( B^{k+1}_{n^j_{k+1}}(1 \leq j \leq r) \) are real symmetric matrices of order \( n^j_{k+1} \), and, for any fixed \( 1 \leq j \leq r \) and \( 1 \leq i \leq k \), the \( n^1_{k+1} + \cdots + n^j_{k+1} + 1 \) th to \( n^1_{k+1} + \cdots + n^j_{k+1} \) th eigenvalues of \( A^i \) are equal. Thus there are a set of orthogonal matrices \( R_{n^j_{k+1}}, 1 \leq j \leq r \) such that \( R_{n^j_{k+1}}B^{k+1}_{n^j_{k+1}}R^{-1}_{n^j_{k+1}} \) are diagonal matrices. Let

\[
R = diag(R_{n^1_{k+1}}, \cdots, R_{n^k_{k+1}}).
\]

Then the real symmetric matrices \( A^1, \cdots, A^{k+1} \) can be simultaneously diagonalized by orthogonal matrix \( RP \). \( \square \)

Proof of the Theorem 1.3. The canonical Calabi hypersurface means

\[
\nabla A = 0 \quad \text{and} \quad R_{ijkl} = 0.
\]

Hence \( M^n \) locally is a Euclidean space. We choose local coordinates \( \{u^1, \cdots, u^n\} \) such that the Calabi metric is given by \( G = \sum (du^i)^2 \), and \( A_{ijk} = \text{const} \) in this coordinates. We consider the following two subcases:

Case 1. \( A_{ijk} = 0, \forall i, j, k \). Obviously, in this case, \( M^n \) is an open part of elliptic paraboloid.
Case 2. Otherwise. Let \( p \in M^n \) be a fixed point with coordinates \((0, \cdots, 0)\). Choose the local orthonormal frame field \( e_i = \frac{\partial}{\partial u^i} \), and \( e = (0, \cdots, 0, 1) \) on \( M^n \). Let \( \{\omega^i\} \) be the dual frame field of \( \{e_i\} \). Denote

\[
A^{(k)} := A^{e_k} = \sum A^{(k)}_{ij} du^i du^j, \\
A^{(k)}_{ij} := A(e_i, e_j, e_k) \equiv A_{ijk}.
\]

By the Gauss integrability conditions \([7, 7]\) and the flatness of the metric \( G \), we have

\[
\sum A_{iml} A_{jmk} - \sum A_{imk} A_{jml} = 0, \tag{3.4}
\]

which means the following matrix equalities:

\[
(A^{(k)}_{ij})(A^{(l)}_{ij}) = (A^{(l)}_{ij})(A^{(k)}_{ij}), \quad \forall 1 \leq k, l \leq n.
\]

By Lemma 3.1, we get that matrices \((A^{(k)}_{ij})\) can be simultaneously diagonalized. There exists an orthogonal constant matrix \( C = (c_{ij}) \), for any fixed \( 1 \leq k \leq n \), such that

\[
(A^{(k)}_{ij}) = C (A^{(k)}_{ij}) C^{-1} = diag(\lambda_1^k, \lambda_2^k, \cdots, \lambda_n^k).
\]

Here \( A^{(k)}_{ij} = A(\bar{e}_i, \bar{e}_j, e_k) \) and \( \bar{e}_i = \sum c_{ij} e_j, 1 \leq i \leq n \), then

\[
\bar{A}_{ijk} := A(\bar{e}_i, \bar{e}_j, e_k) = \sum c_{kl} A(\bar{e}_i, \bar{e}_j, e_l) = \sum c_{kl} \lambda_i^l \delta_{ij}.
\]

Since the matrices \((\bar{A}_{ijk})\) are symmetric in all indexes, we get:

\[
\bar{A}_{ijk} = \begin{cases} 
\bar{A}_{iii}, \quad &1 \leq i = j = k \leq n, \\
0, \quad &\text{otherwise}.
\end{cases} \tag{3.5}
\]

From \( dx = \omega^i e_i = \bar{\omega}^i \bar{e}_i \), we can get \( \bar{\omega}^i = \sum c^{ij} \omega^j = \sum \bar{c}^{ij} du^j \), where \((\bar{c}^{ij})\) denotes the inverse matrix of \((c_{ij})\). Let \( \bar{u}^i = \sum c_{ij} u^j, 1 \leq i \leq n \), then \((\bar{u}^1, \cdots, \bar{u}^n)\) are new Euclidean coordinates of \( M^n \), such that \( \frac{\partial}{\partial \bar{u}^i} = \bar{e}_i, 1 \leq i \leq n \). Under these new coordinates, the tensor \( \bar{A} \) is expressed as \((3.5)\), thus we have:

\[
\begin{aligned}
\left\{ 
\begin{array}{l}
\bar{d} \bar{e}_i = \sum \bar{\omega}^j \bar{e}_j + d\bar{u}^i e_i, \quad &1 \leq i \leq n, \\
\bar{d}x = \sum d\bar{u}^i \bar{e}_i.
\end{array}
\right.
\tag{3.6}
\end{aligned}
\]

Since the Calabi metric is flat and \( \frac{\partial}{\partial \bar{u}^i} \) are orthonormal, we obtain \( \bar{\omega}^j = \bar{A}_{ijk} d\bar{u}^k \). Assume that \( x \in M^n \) is an arbitrary point with coordinates \((v^1, \cdots, v^n)\). We draw a curve connecting \( p \) and \( x \)

\[
\bar{u}^i(t) = v^i t, \quad &0 \leq t \leq 1.
\]

Along this curve the equations \((3.6)\) become

\[
\begin{aligned}
\left\{ 
\begin{array}{l}
\frac{d\bar{u}^i}{dt} = \bar{A}_{iii} v^i \bar{e}_i + v^i e_i, \quad &1 \leq i \leq n, \\
\frac{dx}{dt} = \sum v^i \bar{e}_i.
\end{array}
\right.
\tag{3.7}
\end{aligned}
\]

Consider the ordinary differential equation

\[
\frac{du}{dt} = au + b.
\]
It is easy to find out its solution
\[ u(t) = \begin{cases} (u(0) + \frac{b}{a})e^{at} - \frac{b}{a}, & a \neq 0, \\ u(0) + bt, & a = 0. \end{cases} \]

We may assume that \( \bar{e}_i(0) = (0, \ldots, 1, \ldots, 0), 1 \leq i \leq n \), where 1 is on \( i \)-th entry and \( \bar{A}_{iii} \geq 0 \) at point \( p \). By an arrangement, we can get
\[
\begin{align*}
\bar{A}_{iii} > 0, & \quad 1 \leq i \leq r; \\
\bar{A}_{jjj} = 0, & \quad r + 1 \leq j \leq n,
\end{align*}
\]
where \( 1 \leq r \leq n \) and \( r = n \) means that \( \bar{A}_{iii} > 0 \) for all \( 1 \leq i \leq n \).

Without loss of generality, we assume \( v^i > 0, 1 \leq i \leq n \). Solve equations (3.7), we obtain:
\[
\begin{align*}
\bar{e}_i = \exp(\bar{A}_{iii}v^i)\bar{e}_i(0) + \frac{1}{\bar{A}_{iii}} \exp(\bar{A}_{iii}v^i) - \frac{1}{\bar{A}_{iii}} e, & \quad 1 \leq i \leq r; \\
\bar{e}_j = \bar{e}_j(0) + v^j t e, & \quad r + 1 \leq j \leq n; \\
(x(t)) = x(0) + \int_0^t v^i \bar{e}_i(s) ds.
\end{align*}
\]
Thus
\[
x(t) = x(0) + \sum_{i=1}^r \frac{1}{\bar{A}_{iii}} [\exp(\bar{A}_{iii}v^i) - 1] \bar{e}_i(0) + \sum_{i=1}^r \frac{1}{\bar{A}_{iii}^2} [\exp(\bar{A}_{iii}v^i) - 1] e \\
- \sum_{i=1}^r \frac{1}{\bar{A}_{iii}} v^i t e + \sum_{j=r+1}^n v^j t \bar{e}_j(0) + \sum_{j=r+1}^n \frac{1}{2} (v^j)^2 t^2 e.
\]

Evaluate (3.10) at \( t = 1 \) we have:
\[
x_i = x_i(0) + \frac{1}{\bar{A}_{iii}} [\exp(\bar{A}_{iii}v^i) - 1], & \quad 1 \leq i \leq r; \\
x_j = x_j(0) + v^j, & \quad r + 1 \leq j \leq n; \\
x_{n+1} = x_{n+1}(0) + \sum_{i=1}^r \left( \frac{1}{\bar{A}_{iii}^2} [\exp(\bar{A}_{iii}v^i) - 1] - \frac{1}{\bar{A}_{iii}} v^i \right) + \sum_{j=r+1}^n \frac{1}{2} (v^j)^2.
\]

Inserting \( x_i \) and \( x_j \) into \( x_{n+1} \), we find
\[
x_{n+1} = \sum_{i=1}^r \frac{1}{\bar{A}_{iii}} x_i - \sum_{i=1}^r \frac{1}{\bar{A}_{iii}^2} \ln(\bar{A}_{iii}x_i + 1) + \frac{1}{2} \sum_{j=r+1}^n (x_j)^2.
\]

It is easy to find that \( x_{n+1} \) is affine equivalent to
\[
x_{n+1} = -\sum_{i=1}^r \frac{1}{\bar{A}_{iii}^2} \ln x_i + \sum_{j=r+1}^n \frac{1}{2} (x_j)^2.
\]

This completes the proof of theorem 1.3. \( \square \)

**Proof of Corollary 1.4**

By \( \nabla A = 0 \), the definition of the Tchebychev vector field \( T \) and the Pick invariant \( J \), we can get:
\[
\nabla T = 0 \quad \text{and} \quad J = \text{const}.
\]
It follows that $|T| = \text{const.}$

**Case 1.** $|T| = 0$. It means that
\[
\det(f_{ij}) = \text{const} > 0,
\]
and the Ricci formula
\[
R_{ij} = A_{iml}A_{jml} - A_{ijm}A_{ml} = A_{iml}A_{jml}.
\]
Hence, by (3.14), (2.12) and $\nabla A = 0$, we have
\[
\frac{n(n - 1)}{2} \Delta J = \sum (R_{ij})^2 + \sum (R_{ijkl})^2.
\]
It follows that $R_{ijkl} = 0$. Then, by (2.10), we obtain the relative Pick invariant
\[
n(n - 1)J = R + n^2|T|^2 = 0.
\]
Thus $f$ is a strictly convex quadratic function.

**Case 2.** $|T| = \text{const} > 0$. In this case, we can choose an orthonormal frame field
\[
\{\tilde{e}_1, \tilde{e}_2\}
\]
on $M^2$ with $\tilde{e}_1 = \frac{T}{|T|}$, where $\nabla \tilde{e}_1 = 0$, since $\nabla T = 0$. From the definition of the Riemannian curvature tensor, we get
\[
R_{ijkl} = 0.
\]
Thus, by Theorem 1.3, we complete the proof of the corollary 1.4.

4. **The classification of 3-dimension case**

4.1. **Elementary discussions in terms of a typical basis.** Now, we fix a point $p \in M^n$. For subsequent purpose, we will review the well known construction of a typical orthonormal basis for $T_p M^n$, which was introduced by Ejiri and has been widely applied, and proved to be very useful for various situations, see e.g., [7], [18] and [3]. The idea is to construct from the $(1,2)$ tensor $A$ a self adjoin operator at a point; then one extends the eigenbasis to a local field. Let $p \in M^n$ and $U_p M^n = \{v \in T_p M^n \mid G(v,v) = 1\}$. Since $M^n$ is locally strong convex, $U_p M^n$ is compact. We define a function $F$ on $U_p M^n$ by $F(v) = A(v,v,v)$. Then there is an element $e_1 \in U_p M^n$ at which the function $F(v)$ attains an absolute maximum, denoted by $\mu_1$. Then we have the following lemma. For its proof, we refer the reader to [7] or [12].

**Lemma 4.1.** There exists an orthonormal basis $\{e_1, \cdots, e_n\}$ of $T_p M^n$ such that the following hold:
(i) $A(e_i, e_i, e_j) = \mu_i \delta_{ij}$, for $i = 1, \cdots, n$.
(ii) $\mu_1 \geq 2 \mu_i$, for $i \geq 2$. If $\mu_1 = 2 \mu_i$, then $A(e_i, e_i, e_i) = 0$.

Consider the function
\[
F(v) = A(v,v,v) \quad \text{on } U_p M^n.
\]
Let $e_1 \in U_p M^n$ be a vector at which $F(v)$ attains an absolute maximum $A_{111}(\geq 0)$. From Lemma 4.1, we can further choose $e_2, \cdots, e_n$ such that $\{e_1, \cdots, e_n\}$ form an
orthonormal basis of $T_pM^n$, which possesses the following properties:

\[ G(e_i, e_j) = \delta_{ij}, \quad A_{1ij} = \mu_i \delta_{ij}, \quad 1 \leq i, j \leq n; \]

\[ \mu_1 \geq 2 \mu_i \text{ and if } \mu_1 = 2 \mu_i, \text{ then } A(e_i, e_i, e_i) = 0 \text{ for } i \geq 2. \]

Using $\nabla A = 0$ and the Ricci identity, for $i \geq 2$, we have

\[ 0 = A_{1i,1i} - A_{1i,ii} = 2A_{p1i}R_{p1i} + A_{11p}R_{p1i} = \mu_i(\mu_1 - 2 \mu_i)(\mu_i - \mu_1). \] (4.1)

Therefore we have the following lemma.

**Lemma 4.2.** Let $M^n$ be a Calabi hypersurface with parallel Fubini-Pick form. Then, for every point $p \in M^n$, there exists an orthonormal basis $\{e_j\}_{1 \leq j \leq n}$ of $T_pM^n$ (if necessary, we rearrange the order), satisfying $A(e_1, e_j) = \mu_j e_j$, and there exists a number $i$, $0 \leq i \leq n$, such that

\[ \mu_2 = \mu_3 = \cdots = \mu_i = \frac{1}{2} \mu_1; \quad \mu_{i+1} = \cdots = \mu_n = 0. \]

Therefore, for a strictly convex Calabi hypersurface with parallel Fubini-Pick form, we have to deal with $(n + 1)$ cases as follows:

**Case $\mathfrak{C}_0$.** $\mu_1 = 0$.

**Case $\mathfrak{C}_1$.** $\mu_1 > 0; \mu_2 = \mu_3 = \cdots = \mu_n = 0$.

**Case $\mathfrak{C}_i$.** $\mu_2 = \mu_3 = \cdots = \mu_i = \frac{1}{2} \mu_1 > 0; \quad \mu_{i+1} = \cdots = \mu_n = 0$ for $2 \leq i \leq n - 1$.

**Case $\mathfrak{C}_n$.** $\mu_2 = \mu_3 = \cdots = \mu_n = \frac{1}{2} \mu_1 > 0$.

When working at the point $p \in M^n$, we will always assume that an orthonormal basis is chosen such that Lemma 4.1 is satisfied.

4.2. **The settlement of the Cases $\mathfrak{C}_0$ and $\mathfrak{C}_n$.** Firstly, about the Case $\mathfrak{C}_0$, we have the following lemma.

**Lemma 4.3.** If the Case $\mathfrak{C}_0$ occurs, then $M^n$ is an open part of elliptic paraboloid.

**Proof.** If $\mu_1 = 0$, then

\[ A(v, v, v) = 0 \text{ for any } v \in U_pM^n. \] (4.2)

Put $v = \frac{1}{\sqrt{2}}(e_i + e_j) \in U_pM^n$ in (4.2), then

\[ 0 = A(e_i, e_i, e_j) + A(e_i, e_j, e_j). \]

On the other hand, put $v = \frac{1}{\sqrt{2}}(e_i - e_j) \in U_pM^n$ in (4.2), then

\[ 0 = -A(e_i, e_i, e_j) + A(e_i, e_j, e_j). \]

Thus we have

\[ A(e_i, e_i, e_j) = 0, \quad \forall i, j. \]

From $0 = \frac{1}{2} A(e_i + e_k, e_i + e_k, e_j)$, we have

\[ A(e_i, e_j, e_k) = 0, \quad \forall i, j, k. \]
Therefore $J \equiv 0$, and $M^n$ is an open part of elliptic paraboloid.

Secondly, we have the following important observation:

**Lemma 4.4.** The Case $\mathfrak{C}_n$ does not occur.

**Proof.** Assume that this case does occur. For any $i \geq 2$, $\mu_i = \frac{1}{2} \mu_1 > 0$, then $A(e_1, v, v) = \frac{1}{2} \mu_1$ and $A(v, v, v) = 0$ for any $v \in \{e_1^+\} \cap U_p M^n$. From the proof of Lemma 4.3, we see that

$$A(e_i, e_j, e_k) = 0, \quad 2 \leq i, j, k \leq n.$$  

Then, for any unit vector $v \in \{e_1^+\} \cap U_p M^n$, we have

$$A(e_1, e_1) = \mu_1 e_1, \quad A(e_1, v) = \frac{1}{2} \mu_1 v, \quad A(v, v) = \frac{1}{2} \mu_1 e_1. \quad (4.3)$$

By $\nabla A = 0$, we know that the curvature operator of Levi-Civita $R$ and Fubini-Pick tensor $A$ satisfy

$$R(e_1, v)A(v, v) = 2A(R(e_1, v)v, v). \quad (4.4)$$

By (4.4), (4.3) and (2.7), we get $\mu_1 = 0$. This contradiction completes the proof of Lemma 4.4. \hfill \Box

In the following we only consider 3-dimensional Calabi hypersurfaces with parallel Fubini-Pick form. Therefore, we only need to deal with the Case $\mathfrak{C}_1$ and Case $\mathfrak{C}_2$. In sequel of this paper, we are going to discuss these cases separately.

### 4.3. The settlement of the case $\mathfrak{C}_1$.

**Lemma 4.5.** If the Case $\mathfrak{C}_1$ occurs, then $M^3$ is Calabi affine equivalent to an open part of the hypersurfaces $Q(c_1, \cdots, c_r; 3), \ 1 \leq r \leq 3$.

**Proof.** Denote

$$A^k_{ij} := A(e_i, e_j, e_k) \equiv A_{ijk},$$

and put

$$a := A_{222}, \quad b := A_{233}, \quad c := A_{333}, \quad d := A_{223}.$$

By $\mu_2 = \mu_3$, we can further choose $e_2$ as a unit vector for which the function $F$, restricted to $\{e_1^+\} \cap U_p M^3$, attains its maximum $A_{222} \geq 0$. It follows that $A_{223} = 0$, and $A_{222} \geq 2A_{233}$. Thus we get

$$\begin{pmatrix} A^1_{ij} \\ \end{pmatrix} = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} A^2_{ij} \\ \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}, \quad \begin{pmatrix} A^3_{ij} \\ \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & c \end{pmatrix}. \quad (4.5)$$

By a direct calculation, we have

$$R_{22} = R_{33} = b(b - a),$$

$$R_{11} = R_{12} = R_{13} = R_{23} = 0. \quad (4.6)$$

By (2.11) and (2.12), it yields

$$0 = R_{22}(T_2)^2 + R_{33}(T_3)^2 = \frac{1}{9} b(b - a)[(a + b)^2 + c^2], \quad (4.7)$$
\[
0 = \sum (R_{ijkl})^2 + R_{22} \sum (A_{2pq})^2 + R_{33} \sum (A_{3pq})^2 \\
= \sum (R_{ijkl})^2 + b(b - a)(a^2 + 3b^2 + c^2). \tag{4.8}
\]

If \(b > 0\), it contradicts to (4.7). If \(b < 0\), it also contradicts to (4.8). Therefore \(b = 0\). By (4.8) we get \(R_{ijkl}(p) = 0\). Since the arbitrary of point \(p\), we have the Calabi metric is flat. Combining \(\nabla A = 0\) and Theorem 1.3, one can get the following classification results:

1. if \(a = 0\), \(c = 0\), then \(M^3\) is Calabi affine equivalent to an open part of the hypersurface \(Q(c_1; 3)\);

2. if \(a \neq 0\), \(c = 0\), then \(M^3\) is Calabi affine equivalent to an open part of the hypersurface \(Q(c_1, c_2; 3)\);

3. if \(a \neq 0\), \(c \neq 0\), then \(M^3\) is Calabi affine equivalent to an open part of the hypersurface \(Q(c_1, c_2, c_3; 3)\).

\[\square\]

5. CLASSIFICATION OF CASE \(\mathcal{C}_2\)

By \(\mu_1 = 2\mu_2 > 0\), we know \(A_{222} = 0\). Thus we have

\[(A^1_{ij}) = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, (A^2_{ij}) = \begin{pmatrix} 0 & \mu_2 & 0 \\ \mu_2 & 0 & d \\ 0 & d & b \end{pmatrix}, (A^3_{ij}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & d & b \\ 0 & b & c \end{pmatrix}. \tag{5.1}\]

By (2.9), we obtain

\[
R_{11} = -\mu_2^2, \quad R_{22} = -\mu_2^2 + b^2 + d^2 - cd, \quad R_{33} = b^2 + d^2 - cd, \\
R_{12} = -\mu_2 b, \quad R_{13} = \mu_2 d, \quad R_{23} = 0.
\]

Using \(\nabla A = 0\) and the Ricci identity, we have

\[
0 = A_{223,13} - A_{223,31} = 2A_{p23}R_{p213} + A_{222}R_{p313} = 2b^2\mu_2. \tag{5.2}
\]

\[
0 = A_{222,12} - A_{222,21} = 3A_{22p}R_{p212} = 3\mu_2(d^2 - \mu_2^2). \tag{5.3}
\]

\[
0 = A_{123,23} - A_{123,32} = A_{p23}R_{p123} + A_{1p3}R_{p223} + A_{12p}R_{p323} \tag{5.4}
\]

\[
= \mu_2(2b^2 + 2d^2 - cd).
\]

By (5.2), (5.3) and (5.4), we obtain

\[
b = 0, \quad d^2 = \mu_2^2 \neq 0, \quad c = 2d.
\]

Thus the Pick invariant and the scalar curvature are

\[
J = \frac{7}{3}\mu_2^2, \quad R = -4\mu_2^2. \tag{5.5}
\]

Now put tangent vectors

\[
\tilde{e}_1 := \frac{\sqrt{2}}{2}(e_1 + e_3), \quad \tilde{e}_3 := \frac{\sqrt{2}}{2}(-e_1 + e_3), \tag{5.6}
\]

\[
\tilde{e}_2 := \frac{\sqrt{2}}{2}(e_1 - e_3).
\]

\[
\tilde{e}_1, \tilde{e}_2, \tilde{e}_3
\]

are linearly independent and \(\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}\) is a basis of \(TM_0\). By the hypothesis of Theorem 1.3, we get the following results:

(1) if \(\mu_1 > 0\), \(\mu_2 > 0\), then \(M^3\) is Calabi affine equivalent to an open part of the hypersurface \(Q(c_1; 3)\);
then \( \{\tilde{e}_1, e_2, \tilde{e}_3\} \) forms an orthonormal basis of \( T_p M^3 \), with respect to which, the Fubini-Pick tensor \( A \) takes the following form:

\[
A(\tilde{e}_1, \tilde{e}_1) = \sqrt{2} \mu_2 \tilde{e}_1; \quad A(\tilde{e}_1, e_2) = \sqrt{2} \mu_2 e_2; \quad A(\tilde{e}_1, \tilde{e}_3) = \sqrt{2} \mu_2 \tilde{e}_3, \tag{5.7}
\]

and

\[
A(e_2, e_2) = \sqrt{2} \mu_2 \tilde{e}_1; \quad A(e_2, \tilde{e}_3) = 0; \quad A(\tilde{e}_3, \tilde{e}_3) = \sqrt{2} \mu_2 \tilde{e}_1.
\]

By parallel translation along geodesics (with respect to the Levi-Civita connection \( \nabla \)) through \( p \) to a normal neighborhood around \( p \), we can extend \( \{\tilde{e}_1, e_2, \tilde{e}_3\} \) to obtain a local orthonormal basis \( \{E_1, E_2, E_3\} \) on a neighborhood of \( p \) such that

\[
A(E_1, E_1) = \sqrt{2} \mu_2 E_1; \quad A(E_1, E_2) = \sqrt{2} \mu_2 E_2; \quad A(E_1, E_3) = \sqrt{2} \mu_2 E_3 \tag{5.8}
\]

holds at every point in a normal neighborhood. Denote by \( \omega^i_i \) the connection form with respect to the orthonormal frame \( \{E_i\} \). By \( \nabla A = 0 \),

\[
A_{11i,j} \omega^j_i = dA_{11i} - 2A_{j1i} \omega^j_i - A_{11j} \omega^j_i,
\]

and choose \( i = 3 \), we have

\[
\omega^3_3 = 0. \tag{5.9}
\]

Similarly, by

\[
A_{22i,j} \omega^j_i = dA_{22i} - 2A_{j2i} \omega^j_i - A_{22j} \omega^j_i,
\]

and choose \( i = 2 \), we have

\[
\omega^2_2 = 0. \tag{5.10}
\]

Then (5.9) and (5.10) show that \( E_1 \) is a parallel vector field with respect to the Levi-Civita connection. Thus, by (5.5), we have

\[
R_{2323} = \frac{1}{2} R = -2 \mu_2^2 = \text{const.} \tag{5.11}
\]

By the above these equalities we have the following lemma:

**Lemma 5.1.** We have

(i) \( \nabla E_i = 0 \);

(ii) \( \langle \nabla_{E_i} E_j, E_1 \rangle = 0 \), for any \( i, j = 2, 3 \).

This lemma tell us that the distribution by \( D_1 := \{RE_1\} \) and \( D_2 := \text{span}\{E_2, E_3\} \) are totally geodesic. Therefore it follows from the de Rham decomposition theorem [9, pp.187] that as a Riemannian manifold, \( (M^3, G) \) is locally isometric to a Riemannian product \( R \times H^2(-2 \mu_2^2) \), where \( H^2(-2 \mu_2^2) \) is the hyperbolic plane of constant negative curvature \(-2 \mu_2^2 \), and after identification, the local vector field \( E_1 \) is tangent to \( R \) and \( D_2 \) is tangent to \( H^2(-2 \mu_2^2) \).

Denote by \( x = (x_1, x_2, x_3, x_4)^t \) the position vector of \( M^3 \) in \( A^4 \). Using the standard parametrization of the hypersphere model of \( H^2(-2 \mu_2^2) \), we see that there exists local coordinates \( (y_1, y_2, y_3) \) on \( M^3 \), such that the metric is given by

\[
G = (dy_1)^2 + (dy_2)^2 + \sinh^2(\sqrt{2} \mu_2 y_2)(dy_3)^2, \tag{5.12}
\]
and \( E_1 = \frac{\partial x}{\partial y_1} \), and \( \frac{\partial x}{\partial y_2} \) \((\sinh(\sqrt{2\mu_2}y_2))^{-1}\frac{\partial x}{\partial y_2}\), form a G-orthonormal basis. We may assume that \( E_2 = \frac{\partial x}{\partial y_2} \) and \((\sinh(\sqrt{2\mu_2}y_2))E_3 = \frac{\partial x}{\partial y_3}\). Then a straightforward computation shows that

\[
\nabla \frac{\partial x}{\partial y_2} = 0, \quad (5.13)
\]

\[
\nabla \frac{\partial x}{\partial y_3} = \nabla \frac{\partial x}{\partial y_2} = \sqrt{2\mu_2} \coth(\sqrt{2\mu_2}y_2) \frac{\partial x}{\partial y_3}, \quad (5.14)
\]

\[
\nabla \frac{\partial x}{\partial y_2} = -\sqrt{2\mu_2} \sinh(\sqrt{2\mu_2}y_2) \cosh(\sqrt{2\mu_2}y_2) \frac{\partial x}{\partial y_2}. \quad (5.15)
\]

Using the definition of \( A \), we get the following system of differential equations, where, in order to simplify the equations, we have put \( c = \sqrt{2\mu_2} \) and \( Y = (0, 0, 0, 1)^t \).

\[
\frac{\partial^2 x}{\partial y_1 \partial y_1} = c \frac{\partial x}{\partial y_1} + Y, \quad (5.16)
\]

\[
\frac{\partial^2 x}{\partial y_1 \partial y_2} = c \frac{\partial x}{\partial y_2}, \quad (5.17)
\]

\[
\frac{\partial^2 x}{\partial y_1 \partial y_3} = c \frac{\partial x}{\partial y_3}, \quad (5.18)
\]

\[
\frac{\partial^2 x}{\partial y_2 \partial y_2} = c \frac{\partial x}{\partial y_1} + Y, \quad (5.19)
\]

\[
\frac{\partial^2 x}{\partial y_2 \partial y_3} = c \coth(cy_2) \frac{\partial x}{\partial y_3}, \quad (5.20)
\]

\[
\frac{\partial^2 x}{\partial y_3 \partial y_3} = c \sinh^2(cy_2) \frac{\partial x}{\partial y_1} - c \sinh(cy_2) \cosh(cy_2) \frac{\partial x}{\partial y_2} + \sinh^2(cy_2)Y. \quad (5.21)
\]

To solve the above equations, first we solve its corresponding system of homogeneous equations.

\[
\frac{\partial^2 x}{\partial y_1 \partial y_1} = c \frac{\partial x}{\partial y_1}, \quad (5.22)
\]

\[
\frac{\partial^2 x}{\partial y_1 \partial y_2} = c \frac{\partial x}{\partial y_2}, \quad (5.23)
\]

\[
\frac{\partial^2 x}{\partial y_1 \partial y_3} = c \frac{\partial x}{\partial y_3}, \quad (5.24)
\]

\[
\frac{\partial^2 x}{\partial y_2 \partial y_2} = c \frac{\partial x}{\partial y_1}, \quad (5.25)
\]

\[
\frac{\partial^2 x}{\partial y_2 \partial y_3} = c \coth(cy_2) \frac{\partial x}{\partial y_3}, \quad (5.26)
\]

\[
\frac{\partial^2 x}{\partial y_3 \partial y_3} = c \sinh^2(cy_2) \frac{\partial x}{\partial y_1} - c \sinh(cy_2) \cosh(cy_2) \frac{\partial x}{\partial y_2}. \quad (5.27)
\]
From (5.22), we know that there exist vector valued functions $P_1(y_2, y_3)$ and $P_2(y_2, y_3)$ such that
\[ x = P_1(y_2, y_3)e^{cy_2} + P_2(y_2, y_3). \] (5.28)
From (5.23) and (5.24) it then follows that the vector function $P_2$ is independent of $y_2$ and $y_3$. Hence there exists a constant vector $A_1$ such that $P_2(y_2, y_3) = A_1$. Next, it follows from (5.25) that $P_1(y_2, y_3)$ satisfies that the following differential equation:
\[ \frac{\partial^2 P_1}{\partial y_2 \partial y_2} = c^2 P_1. \] (5.29)
Hence we can write
\[ P_1(y_2, y_3) = Q_1(y_3) \cosh(cy_2) + Q_2(y_3) \sinh(cy_2). \] (5.30)
From (5.26), we then deduce that there exists a constant vector $A_2$ such that $Q_1(y_3) = A_2$. The last formula (5.27) implies there exist constant vectors $A_3$ and $A_4$ such that
\[ Q_2(y_3) = A_3 \cos(cy_3) + A_4 \sin(cy_3). \] (5.31)
Therefore the general solution of system (5.22-5.27) are
\[ x = e^{cy_2}(A_2 \cosh(cy_2) + [A_3 \cos(cy_3) + A_4 \sin(cy_3)] \sinh(cy_2)) + A_1, \] (5.32)
where $A_i$ are constant vectors. On the other hand, we know that
\[ \bar{x} = (0, 0, 0, -\frac{y_1}{c})^t \]
is a special solution of equations (5.16-5.21). Therefore the general solutions of equations (5.16-5.21) are
\[ x = e^{cy_2} \{A_2 \cosh(cy_2) + [A_3 \cos(cy_3) + A_4 \sin(cy_3)] \sinh(cy_2)\} + A_1 + \bar{x}. \] (5.33)

Since $M^3$ is nondegenerate, $x - A_1$ lies linearly full in $A^4$. Hence $A_2, A_3, A_4$ and $(0, 0, 0, 1)$ are linearly independent vectors. Thus there exists an affine transformation $\phi \in SA(4)$ such that
\[ A_1 = (0, 0, 0, 0)^t, A_2 = (1, 0, 0, 0)^t, A_3 = (0, 1, 0, 0)^t, A_4 = (0, 0, 1, 0)^t. \]
Then the position vector
\[ x = (\cosh(cy_2)e^{cy_2}, \cos(cy_3) \sinh(cy_2)e^{cy_2}, \sin(cy_3) \sinh(cy_2)e^{cy_2}, -\frac{y_1}{c})^t. \] (5.34)
It follows that, up to an affine transformation $\phi \in SA(4)$, $M^3$ locally lies on the graph hypersurface of function
\[ x_4 = -\frac{1}{2c^2} \ln(x_1^2 - (x_2^2 + x_3^2)). \] (5.35)
Thus we finally arrive at the following lemma.

**Lemma 5.2.** If the Case $\mathcal{C}_2$ occurs, then $M^3$ is Calabi affine equivalent to an open part of the hypersurface
\[ x_4 = -\frac{1}{2c^2} \ln(x_1^2 - (x_2^2 + x_3^2)), \]
where the constant $-2c^2$ is the scalar curvature of $M^3$. 
Combining Lemma 4.3, Lemma 4.4, Lemma 4.5 and Lemma 5.2, we complete the proof of Theorem 1.5.

\[ \square \]

References

[1] N. Bokan, K. Nomizu, U. Simon: Affine hypersurfaces with parallel cubic forms. Tôhoku Math. J. 42(1990), 101-108.

[2] E. Calabi: Improper affine hyperspheres of convex type and a generalization of a theorem of Jörgens. Michigan J. Math. 5(1958), 105-126.

[3] X. Cheng, Z. Hu: An optimal inequality on locally strongly convex centroaffine hypersurfaces. J. Geom. Anal. 28(2018), 643-655.

[4] X. Cheng, Z. Hu, M. Morus: Classification of the locally strongly convex centroaffine hypersurfaces with parallel cubic form. Results Math. 72(2017), 419-469.

[5] F. Dillen, L. Vrancken: 3-dimensional affine hypersurfaces in $R^4$ with parallel cubic form. Nagoya Math. J. 124(1991), 41-53.

[6] F. Dillen, L. Vrancken, L. Yaprak: Affine hypersurfaces with parallel cubic form. Nagoya Math. J. 135(1994), 153-164.

[7] Z. Hu, H. Li, U. Simon, L. Vrancken: On locally strongly convex affine hypersurfaces with parallel cubic form. Part I. Diff. Geom. Appl. 27(2009), 188-205.

[8] Z. Hu, H. Li, L. Vrancken: Locally strongly convex affine hypersurfaces with parallel cubic form. J. Diff. Geom. 87(2011), 239-307.

[9] S. Kobayashi, K. Nomizu: Foundations of Differential Geometry, vol. I, Interscience Publishers, New York, 1963.

[10] A.-M. Li: Affine Kähler manifolds. Report on international congress in Banach Center of Poland, 2005.

[11] A.-M. Li, H. Li, U. Simon: Centroaffine Bernstein problems. Diff. Geom. Appl. 20(2004), no. 3, 331-356.

[12] A.-M. Li, U. Simon, G. Zhao, Z. Hu: Global Affine Differential Geometry of Hypersurfaces. Second revised and extended edition. De Gruyter Expositions in Mathematics, 11. De Gruyter, Berlin, 2015.

[13] A.-M. Li, R.W. Xu, U. Simon, F. Jia: Affine Bernstein Problems and Monge-Ampère Equations. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2010.

[14] A.-M. Li, G. Penn: Uniqueness theorems in affine differential geometry. Part II. Results Math. 13(1988), 308-317.

[15] A.-M. Li, C.P. Wang: Canonical centroaffine hypersurfaces in $R^n+1$. Affine differential geometry (Oberwolfach, 1991). Results Math. 20(1991), no. 3-4, 660-681.

[16] H.L. Liu, C.P. Wang: Centroaffine surfaces with parallel traceless cubic form. Bull. Belg. Math. Soc. 4(1997), 493-499.

[17] H. Li: Variational problems and PDEs in affine differential geometry. PDEs, submanifolds and affine differential geometry, 9-41, Banach Center Publ., 69, Polish Acad. Sci. Inst. Math., Warsaw, 2005.

[18] H. Li, L. Vrancken: A basic inequality and new characterization of Whitney spheres in a complex space form. Israel J. Math. 146 (2005), 223-242.

[19] A.V. Pogorelov: The Minkowski Multidimensional Problem. John Wiley & Sons, New York - London - Toronto, 1978.

[20] R.W. Xu, X.X. Li: On the complete solutions to the Tchebychev-Affine-Kähler equation and its geometric significance. Preprint in 2019.