APPLICATIONS OF ARTHUR’S MULTIPLICITY FORMULA TO SIEGEL MODULAR FORMS

HIRAKU ATOBE

Abstract. We give two applications of Arthur’s multiplicity formula to Siegel modular forms. The one is a lifting theorem for vector valued Siegel modular forms, which contains Miyawaki’s conjectures and Ibukiyama’s conjectures. The other is the strong multiplicity one theorem for Siegel modular forms of scalar weights and level one.

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1. Introduction

In the modern number theory, modular forms and automorphic representations are indispensable tools. In particular, they give automorphic $L$-functions, which enjoy Euler products, meromorphic continuations, and functional equations.

We review the classical theory of Siegel modular forms. Let $S_{2k}(\text{SL}_2(\mathbb{Z}))$ (resp. $S_{k,j}(\text{Sp}_2(\mathbb{Z}))$) be the space of elliptic cusp forms of weight $2k$, (resp. the space of Siegel modular cusp forms of degree 2 and of vector weight $\det^k \text{Sym}(j)$). When $f \in S_{2k}(\text{SL}_2(\mathbb{Z}))$ (resp. $f \in S_{k,j}(\text{Sp}_2(\mathbb{Z}))$) is a Hecke...
Ibukiyama’s conjecture of Type I: For any Hecke eigenforms \( f \in S_{2k-4}(\text{SL}_2(\mathbb{Z})) \) and \( g \in S_{k}(\text{SL}_2(\mathbb{Z})) \), there should exist a Hecke eigenform \( F_{f,g} \in S_{k}(\text{Sp}_3(\mathbb{Z})) \) such that
\[
L(s,F_{f,g},\text{std}) = L(s,g,\text{std})L(s+k-2,f)L(s+k-3,f).
\]

Miyawaki’s conjecture of Type II: For any Hecke eigenforms \( f \in S_{2k-2}(\text{SL}_2(\mathbb{Z})) \) and \( g \in S_{k-2}(\text{SL}_2(\mathbb{Z})) \), there should exist a Hecke eigenform \( F_{f,g} \in S_{k}(\text{Sp}_3(\mathbb{Z})) \) such that
\[
L(s,F_{f,g},\text{std}) = L(s,g,\text{std})L(s+k-1,f)L(s+k-2,f).
\]

Ibukiyama’s conjecture of Type I: For any Hecke eigenform \( f \in S_{n+2,2m-3n-2}(\text{Sp}_2(\mathbb{Z})) \) with even \( n,m \) such that \( m > 2n \) and \( n \geq 2 \), there should exist a Hecke eigenform \( F_{f} \in S_{m}(\text{Sp}_{2n}(\mathbb{Z})) \) such that
\[
L(s,F_{f},\text{std}) = \zeta(s) \prod_{i=1}^{n} L \left( s + \frac{n+1}{2} - i, f, \text{spin} \right).
\]

Ibukiyama’s conjecture of Type II: For any Hecke eigenforms \( f \in S_{m-2n+2,2n-2}(\text{Sp}_2(\mathbb{Z})) \) with even \( m \) and \( m > 2n-2 \), there should exist a Hecke eigenform \( F_{f,g} \in S_{m}(\text{Sp}_{2n}(\mathbb{Z})) \) such that
\[
L(s,F_{f,g},\text{std}) = L(s,g,\text{std}) \prod_{i=1}^{2n-2} L(s+m-1-i,f).
\]

In fact, Miyawaki [14] numerically computed the actions of Hecke operators on \( F_{12} \) and \( F_{14} \) which belong to the one dimensional vector spaces \( S_{12}(\text{Sp}_2(\mathbb{Z})) \) and \( S_{14}(\text{Sp}_2(\mathbb{Z})) \), respectively, and predicted the conjectures for them. The general forms of Miyawaki’s conjectures were given by Heim [7]. Ibukiyama [8] also considered his lifting conjectures for the non-cuspidal cases in slightly wider situations.

Nowadays, these conjectures should follow from **Arthur’s multiplicity formula** [3, Theorem 1.5.2]. Let \( \mathbb{A} = \mathbb{A}_\mathbb{Q} \) be the ring of adeles of \( \mathbb{Q} \), and \( \mathbb{A}_\text{fin} \) be its finite part. Arthur’s multiplicity formula decomposes the space \( \mathcal{A}^2(\text{Sp}_n(\mathbb{A})) \) of square-integrable automorphic forms on \( \text{Sp}_n(\mathbb{Q})/\text{Sp}_n(\mathbb{A}) \) into a direct sum of simple \( \text{Sp}_n(\mathbb{A}_\text{fin}) \times (g_\infty,K_\infty) \)-modules using global \( A \)-packets, where \( g_\infty = \text{sp}_n(\mathbb{C}) \) is the complexification of the Lie algebra of \( \text{Sp}_n(\mathbb{R}) \), and
\[
K_\infty = \left\{ \left( \begin{array}{cc} \alpha & \beta \\ -\beta & \alpha \end{array} \right) \in \text{Sp}_n(\mathbb{R}) \middle| t\alpha\alpha + t\beta\beta = 1_n \right\}.
\]
Remark 1.2.

Given by a Siegel modular form is the standard maximal compact subgroup of $\text{Sp}_n(\mathbb{R})$. Using this formula, we may try to decompose the space $S_{\rho_k}(\text{Sp}_n(\mathbb{Z}))$ of Siegel modular forms. However, the automorphic form $\varphi_F \in A^2(\text{Sp}_n(\mathbb{A}))$ given by a Siegel modular form $F \in S_{\rho_k}(\text{Sp}_n(\mathbb{Z}))$ is a “holomorphic cusp form of vector weight $\rho_k$”. This condition implies that the archimedean components of the automorphic representations appearing in the representation generated by $\varphi_F$ are the lowest weight module $L(V_k)$. Therefore, to obtain a decomposition of $S_{\rho_k}(\text{Sp}_n(\mathbb{Z}))$, we need to consider when local $\Lambda$-packets for $\text{Sp}_n(\mathbb{R})$ contain $L(V_k)$. When $k_n > n$, this problem is solved by the works of Adams–Johnson [1] and Arancibia–Mœglin–Renard [2] (see Proposition 3.2). By this observation, we obtain a decomposition of $S_{\rho_k}(\text{Sp}_n(\mathbb{Z}))$ in this case, and we can conclude the following lifting theorem.

**Theorem 1.1** (Lifting Theorem). Let $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ with $k_1 \geq \cdots \geq k_n > n$, and let $g \in S_{\rho_k}(\text{Sp}_n(\mathbb{Z}))$ be a Hecke eigenform.

(A) For positive integers $k$ and $d$, we assume one of the following:
- $k + d - 1 < k_n - n$, $k \equiv d + n \mod 2$; or
- $k - d < k_1 - 1$, $k > d$ and $k \equiv d \mod 2$.

Define $k' = (k_1', \ldots, k_{n+d}) \in \mathbb{Z}^{n+2d}$ so that $k_1' \geq \cdots \geq k_{n+2d}$ and

$$
\{k_1' - 1, k_2' - 2, \ldots, k_{n+2d} - (n + 2d)\} = \{k_1 - 1, k_2 - 2, \ldots, k_n - n\} \cup \{k + d - 1, k + d - 2, \ldots, k - d\}.
$$

Then for any Hecke eigenform $f \in S_{2k}(\text{SL}_2(\mathbb{Z}))$, there exists a Hecke eigenform $F_{f,g} \in S_{\rho_k'}(\text{Sp}_{n+2d}(\mathbb{Z}))$ such that

$$
L(s, F_{f,g}, \text{std}) = L(s, g, \text{std}) \prod_{i=1}^{2d} L(s + k + d - i, f).
$$

(B) For positive integers $k$, $j$ and $d > 0$, we assume all of the following:
- $k \equiv j \equiv 0 \mod 2$;
- $k > 2d + 1$ and $j > 2d - 1$;
- $k_i - i \not\in [\frac{j}{2} - d + 1, \frac{j}{2} + k + d - 2]$ for $i = 1, \ldots, n$.

Define $k' = (k_1', \ldots, k_{n+4d}) \in \mathbb{Z}^{n+4d}$ so that $k_1' \geq \cdots \geq k_{n+4d}$ and

$$
\{k_1' - 1, k_2' - 2, \ldots, k_{n+4d} - (n + 4d)\} = \{k_1 - 1, k_2 - 2, \ldots, k_n - n\} \cup \left\{ \frac{j}{2} + k + d - 2, \frac{j}{2} + k + d - 3, \ldots, \frac{j}{2} + k - d \right\} \cup \left\{ \frac{j}{2} + d, \frac{j}{2} + d - 1, \ldots, \frac{j}{2} - d + 1 \right\}.
$$

Then for any Hecke eigenform $f \in S_{k,j}(\text{Sp}_2(\mathbb{Z}))$, there exists a Hecke eigenform $F_{f,g} \in S_{\rho_k'}(\text{Sp}_{n+4d}(\mathbb{Z}))$ such that

$$
L(s, F_{f,g}, \text{std}) = L(s, g, \text{std}) \prod_{i=1}^{2d} L\left(s + d + \frac{1}{2} - i, f, \text{spin}\right).
$$

Here, when $n = 0$, we interpret $L(s,g,\text{std})$ to be the Riemann zeta function $\zeta(s)$.

**Remark 1.2.**

1. When $n = 0$ and $k > d$, $k \equiv d \mod 2$ in Lifting Theorem (A), the lifting $F_{f,1} \in S_{k+d}(\text{Sp}_{2d}(\mathbb{Z}))$ is the Duke–Imamoglu–Ibukiyama–Ikeda lift of $f \in S_{2k}(\text{SL}_2(\mathbb{Z}))$ [9], which is determined up to a constant multiple.

2. When we set $(n,k,d)$ to be $(1,(k),k-2,1)$ with even $k \geq 12$ (resp. to be $(1,(k-2),k-1,1)$ with even $k \geq 14$) in Lifting Theorem (A), we obtain Miyawaki’s conjecture of Type I (resp. of Type II).
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2. Arthur’s multiplicity formula and supplementary results

In this section, we recall the general theory for Arthur’s multiplicity formula for symplectic groups together with several supplementary results.

(3) When we set \((n,k,k,j,d)\) to be \((0,0,n+2,2m-3n-2,\frac{n}{2})\) with even \(n,m\) such that \(m > 2n\) and \(n \geq 2\) in Lifting Theorem (B), we obtain Ibukiyama’s conjecture of Type I.

(4) Ibukiyama’s conjecture of Type II is not contained in Lifting Theorem. However, using a fact in Remark 3.6 below, one can prove this conjecture.

(5) It is not known how to construct \(F_{f,g}\) in general. In [10], Ikeda suggested a way for the construction of \(F_{f,g}\) for some case in Lifting Theorem (A), which is called the Miyawaki lifting of \(g\). However, the non-vanishing of this lifting is unknown, i.e., Ikeda’s construction might be identically zero.

As another application of Arthur’s multiplicity formula together with several supplementary results, we can get the following theorem:

**Theorem 1.3** (Strong multiplicity one theorem). For \(i = 1,2\), let \(F_i \in S_{k_i}(\text{Sp}_n(\mathbb{Z}))\) be a Hecke eigenform of scalar weight \(k_i\). Suppose that for almost all prime \(p\), the Satake parameter of \(F_1\) at \(p\) is equal to the one of \(F_2\) at \(p\). Assume further that \(\{k_1,k_2\} \neq \{\frac{n}{2},\frac{n}{2}+1\}\) if \(n\) is even. Then there exists a constant \(c \in \mathbb{C}^\times\) such that \(F_2 = cF_1\).

We shall explain an outline of the proof.

**Step 1**: Consider the \(\text{Sp}_n(\mathbb{A}_{\text{fin}}) \times (g_\infty, K_\infty)\)-module \(\pi_{F_i}\) generated by the cusp form \(\varphi_{F_i}\) corresponding to \(F_i\). Since \(F_i\) is level one, we see that \(\pi_{F_i}\) is irreducible (Lemma 3.3).

**Step 2**: By the assumption, \(\pi_{F_1}\) and \(\pi_{F_2}\) are nearly equivalent to each other. This means that \(\pi_{F_1}\) belongs to the same \(A\)-packet as \(\pi_{F_2}\) (Corollary 2.8).

**Step 3**: By the uniqueness of the unramified representation in a \(p\)-adic local \(A\)-packet (Proposition 2.6), and by the uniqueness of the lowest weight module in a real local \(A\)-packet (Proposition 3.3), we see that \(\pi_{F_1}\) and \(\pi_{F_2}\) are isomorphic to each other as \(\text{Sp}_n(\mathbb{A}_{\text{fin}}) \times (g_\infty, K_\infty)\)-modules.

**Step 4**: By the multiplicity one result in local \(A\)-packets (Propositions 2.4, 2.9, 3.3), we see that \(\pi_{F_1}\) and \(\pi_{F_2}\) are equal to each other as subspaces of \(A^2(\text{Sp}_n(\mathbb{A}))\).

**Step 5**: By the uniqueness of the everywhere unramified lowest weight vectors in an automorphic representation, we see that \(\varphi_{F_2}\) is a constant multiple of \(\varphi_{F_1}\). This means that \(F_2 = cF_1\) for some \(c \in \mathbb{C}^\times\).

This paper is organized as follows. In the first part of \([2\] (\[2.1\]–\[2.3\]), we recall the local and global \(A\)-packets and Arthur’s multiplicity formula. In the last part of \([2\] (\[2.4\]–\[2.5\]), we review Mœglin’s supplementary results and Adams–Johnson packets. In section 3, we explain the theory of Siegel modular forms and holomorphic cusp forms. Lifting Theorem (Theorem 1.1) and the strong multiplicity one theorem (Theorem 1.3) are proven in \([3.1\] and \[3.5\] respectively. In Appendix A, we explain the original definition of Arthur’s character formally, and compute it.

**Acknowledgments.** This article is mainly based on author’s talk in the conference “21st Autumn Workshop on Number Theory”. The author is grateful to the organizers Tamotsu Ikeda and Tomokazu Kashio for giving him the opportunity of this talk. The author thanks to Tomoyoshi Ibukiyama and Hidenori Katsurada for telling several lifting conjectures. This work was supported by the Foundation for Research Fellowships of Japan Society for the Promotion of Science for Young Scientists (PD) Grant 29-193.
2.1. Local $A$-parameters and local $A$-packets. First, we recall the notion of local $A$-packets. Let $F$ be a local field of characteristic zero, and $L_F$ be the Weil–Deligne group of $F$, i.e.,

$$L_F = \begin{cases} W_F & \text{if } F \text{ is archimedean,} \\ W_F \times \mathrm{SL}_2(\mathbb{C}) & \text{if } F \text{ is non-archimedean,} \end{cases}$$

where $W_F$ is the Weil group of $F$. The symplectic group of rank $n$ is the split algebraic group over $F$ defined by

$$\mathrm{Sp}_n(R) = \left\{ g \in \mathrm{GL}_{2n}(R) \mid t g \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \right\}$$

for an $F$-algebra $R$, whose Langlands dual group is $\mathrm{SO}_{2n+1}(\mathbb{C})$. A local $A$-parameter for $\mathrm{Sp}_n/F$ is a homomorphism $\psi: L_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_{2n+1}(\mathbb{C})$ such that

1. $\psi|W_F$ is continuous;
2. $\psi(W_F)$ consists of semisimple elements;
3. $\psi(W_F)$ projects onto a relatively compact subset in $\mathrm{SO}_{2n+1}(\mathbb{C})$;
4. $\psi|\mathrm{SL}_2(\mathbb{C})$ is algebraic for each $\mathrm{SL}_2(\mathbb{C}) \subset L_F \times \mathrm{SL}_2(\mathbb{C})$.

Let $\Psi(\mathrm{Sp}_n/F)$ be the set of conjugacy classes of local $A$-parameters for $\mathrm{Sp}_n/F$. For $\psi \in \Psi(\mathrm{Sp}_n/F)$, one can decompose

$$\psi = m_1 \psi_1 + \cdots + m_r \psi_r + (\psi_0 \oplus \psi'_0),$$

where $\psi_1, \ldots, \psi_r$ are distinct irreducible orthogonal representations of $L_F \times \mathrm{SL}_2(\mathbb{C})$ with multiplicities $m_1, \ldots, m_r \geq 1$, and $\psi_0$ is a sum of irreducible representations which are not orthogonal. Define the component group $A_\psi$ of $\psi$ by

$$A_\psi = \bigoplus_{i=1}^r (\mathbb{Z}/2\mathbb{Z}) \alpha_{\psi_i}.$$

Namely, $A_\psi$ is a free $\mathbb{Z}/2\mathbb{Z}$-module of rank $r$, and $\{\alpha_{\psi_1}, \ldots, \alpha_{\psi_r}\}$ is a basis of $A_\psi$ with $\alpha_{\psi_i}$ associated to $\psi_i$. For a subrepresentation

$$\psi' = m'_1 \psi_1 + \cdots + m'_r \psi_r + (\psi'_0 \oplus \psi'_0) \subset \psi$$

with $0 \leq m'_i \leq m_i$ and $\psi'_0 \subset \psi_0$, we set

$$\alpha_{\psi'} = \sum_{i=1}^r m'_i \alpha_{\psi_i} \in A_\psi.$$

The element $\alpha_\psi = \sum_{i=1}^r m_i \alpha_{\psi_i}$ is called the central element in $A_\psi$, and is denoted by $z_\psi$. The Pontryagin dual of $A_\psi$ is denoted by $\hat{A}_\psi$.

Let $\mathrm{Irr}(\mathrm{Sp}_n(F))$ (resp. $\mathrm{Irr}_{\mathrm{unit}}(\mathrm{Sp}_n(F))$) be the set of equivalence classes of irreducible admissible (resp. unitary) representations of $\mathrm{Sp}_n(F)$. The following is the local main theorem of Arthur’s endoscopic classification.

**Theorem 2.1** ([3, Theorem 1.5.1]). For each $\psi \in \Psi(\mathrm{Sp}_n/F)$, there is a finite multiset $\Pi_\psi$ over $\mathrm{Irr}_{\mathrm{unit}}(\mathrm{Sp}_n(F))$ with a map

$$\Pi_\psi : \hat{A}_\psi \to \mathbb{R}, \quad \pi \mapsto \langle \cdot, \pi \rangle_\psi$$

satisfying (twisted and standard) endoscopic character identities and $\langle z_\psi, \pi \rangle_\psi = 1$. Moreover, if $F$ is non-archimedean and $\pi$ is unramified, then $\langle \cdot, \pi \rangle_\psi = 1$.

We call $\Pi_\psi$ the local $A$-packet associated to $\psi$. 
2.2. Global $A$-parameters and global $A$-packets. Next, we define global $A$-packets. Let $F$ be a number field. A (discrete) global $A$-parameter for $\text{Sp}_n/F$ is a symbol
\[ \psi = \tau_1[d_1] \boxplus \cdots \boxplus \tau_r[d_r], \]
where
- $\tau_i$ is an irreducible unitary cuspidal self-dual automorphic representation of $\text{GL}_{m_i}(\mathbb{A})$;
- $d_i$ is a positive integer such that $\sum_{i=1}^r m_i d_i = 2n + 1$;
- if $d_i$ is odd, then $\tau_i$ is orthogonal, i.e., $L(s, \tau_i, \text{Sym}^2)$ has a pole at $s = 1$;
- if $d_i$ is even, then $\tau_i$ is symplectic, i.e., $L(s, \tau_i, \wedge^2)$ has a pole at $s = 1$;
- the central character $\omega_{\tau_i}$ of $\tau_i$ satisfies $\omega_1^{d_i} \cdots \omega_r^{d_r} = 1$;
- if $i \neq j$ and $\tau_i \cong \tau_j$, then $d_i \neq d_j$.

Two global $A$-parameters $\psi = \oplus_{i=1}^r \tau_i[d_i]$ and $\psi' = \oplus_{i=1}^r \tau'_i[d'_i]$ are said to be equivalent if $r = r'$ and there exists a permutation $\sigma \in \mathfrak{S}_r$ such that $d'_i = d_{\sigma(i)}$ and $\tau'_i \cong \tau_{\sigma(i)}$. We denote by $\Psi_2(\text{Sp}_n/F)$ be the set of equivalence classes of discrete global $A$-parameters for $\text{Sp}_n/F$. For $\psi = \oplus_{i=1}^r \tau_i[d_i] \in \Psi_2(\text{Sp}_n/F)$, we define the component group $A_\psi$ of $\psi$ by
\[ A_\psi = \bigoplus_{i=1}^r (\mathbb{Z}/2\mathbb{Z})\alpha_{\tau_i[d_i]}. \]

Namely, $A_\psi$ is a free $\mathbb{Z}/2\mathbb{Z}$-module of rank $r$, and $\{\alpha_{\tau_1[d_1]}, \ldots, \alpha_{\tau_r[d_r]}\}$ is a basis of $A_\psi$ with $\alpha_{\tau_i[d_i]}$ associated to $\tau_i[d_i]$. We call $z_\psi = \alpha_{\tau_1[d_1]} + \cdots + \alpha_{\tau_r[d_r]} \in A_\psi$ the central element.

We define Arthur’s character $\varepsilon_\psi : A_\psi \to \{\pm 1\}$ by
\[ \varepsilon_\psi(\alpha_{\tau_i[d_i]}) = \prod_{j \neq i} \varepsilon(\tau_i \times \tau_j)^{\min\{d_i, d_j\}}, \]
where $\varepsilon(\tau_i \times \tau_j) = \varepsilon(1/2, \tau_i \times \tau_j) \in \{\pm 1\}$ is the central value of the Rankin–Selberg epsilon factor $\varepsilon(s, \tau_i \times \tau_j)$. We note that $\varepsilon_\psi(z_\psi) = \prod_{i=1}^r \varepsilon_\psi(\alpha_{\tau_i[d_i]}) = 1$. An important result of Arthur [3, Theorem 1.5.3] states that $\varepsilon(\tau_1 \times \tau_j) = 1$ if $d_1 \equiv d_j \mod 2$. In particular, if $d_1 = \cdots = d_r = 1$, then $\varepsilon_\psi$ is the trivial character of $A_\psi$.

**Remark 2.2.** This definition of $\varepsilon_\psi$ might seem to be different from Arthur’s original definition. In Appendix A, we formally explain the original definition, and show that it coincides with our definition (Proposition A.1).

Let $\psi = \oplus_{i=1}^r \tau_i[d_i] \in \Psi_2(\text{Sp}_n/F)$ be a global $A$-parameter with $\tau_i$ being an irreducible unitary cuspidal representation of $\text{GL}_{m_i}(\mathbb{A})$. For each place $v$ of $F$, we denote by $\phi_{i,v}$ the $m_i$-dimensional representation of $L_{F_v}$ corresponding to the irreducible representation $\tau_{i,v}$ of $\text{GL}_{m_i}(F_v)$. We define a homomorphism $\psi_v : L_{F_v} \times \text{SL}_2(\mathbb{C}) \to \text{GL}_{2n+1}(\mathbb{C})$ by
\[ \psi_v = (\phi_{1,v} \boxtimes S_{d_1}) \oplus \cdots \oplus (\phi_{r,v} \boxtimes S_{d_r}), \]
where $S_d$ is the unique irreducible algebraic representation of $\text{SL}_2(\mathbb{C})$ of dimension $d$. We call $\psi_v$ the localization of $\psi$ at $v$. By [3, Proposition 1.4.2], the homomorphism $\psi_v$ factors through $\text{SO}_{2n+1}(\mathbb{C}) \hookrightarrow \text{GL}_{2n+1}(\mathbb{C})$. However, the localization $\psi_v$ is not necessarily a local $A$-parameter for $\text{Sp}_n/F_v$ because of the lack of the generalized Ramanujan conjecture. One can define the component group $A_{\psi_v}$ of $\psi_v$ similar to local $A$-parameters. There is a localization map
\[ A_\psi \to A_{\psi_v}, \quad \alpha_{\tau_i[d_i]} \mapsto \alpha_{\phi_{i,v} \boxtimes S_{d_i}}. \]

In particular, we obtain the diagonal map
\[ \Delta : A_\psi \to \prod_v A_{\psi_v}. \]
Let $\psi \in \Psi_2(\text{Sp}_n/F)$ be a global $A$-parameter and $\psi_v$ be the localization of $\psi$ at $v$. We can decompose

$$\psi_v = \psi_1 | \cdot |_{\mathbb{F}_v}^{s_1} \otimes \cdots \otimes \psi_r | \cdot |_{\mathbb{F}_v}^{s_r} \otimes \psi_0 \otimes \psi_V' | \cdot |_{\mathbb{F}_v}^{-s_r} \otimes \cdots \otimes \psi_1' | \cdot |_{\mathbb{F}_v}^{-s_1},$$

where

- $\psi_1$ is an irreducible representation of $L_{\mathbb{F}_v} \times \text{SL}_2(\mathbb{C})$ of dimension $d_i$ such that $\psi_i(W_{\mathbb{F}_v})$ is bounded for $i = 1, \ldots, r$;
- $\psi_0 \in \Psi(\text{Sp}_{n_0}/F_v)$;
- $d_1 + \cdots + d_r + n_0 = n$ and $s_1 \geq \cdots \geq s_r > 0$.

We note that $s_i < 1/2$ by the toward Ramanujan conjecture (see [11] (2.5) Corollary] and [22 Appendix]). We define a representation $\phi_{\psi_i}$ of $L_{\mathbb{F}_v}$ by

$$\phi_{\psi_i}(w) = \psi_i \left( w, \begin{pmatrix} \frac{1}{2} |w|_{\mathbb{F}_v} & 0 \\ 0 & \frac{1}{2} |w|_{\mathbb{F}_v}^{-\frac{1}{2}} \end{pmatrix} \right), \quad w \in L_{\mathbb{F}_v},$$

and we denote by $\tau_{\psi_i}$ the irreducible representation of $\text{GL}_{d_i}(\mathbb{F}_v)$ corresponding to $\phi_{\psi_i}$. Let $\Pi_{\psi_0}$ be the local $A$-packet associated to $\psi_0$, which is a multiset over $\text{Irr}_{\text{unit}}(\text{Sp}_{n_0}(\mathbb{F}_v))$. For $\pi_0 \in \Pi_{\psi_0}$, we put

$$I_{\psi_v}(\pi_0) = \text{Ind}_{P_{\mathbb{F}_v}(\mathbb{F}_v)}^{\text{Sp}_n(\mathbb{F}_v)}(r_{\psi_1} | \cdot |_{\mathbb{F}_v}^{s_1} \otimes \cdots \otimes r_{\psi_r} | \cdot |_{\mathbb{F}_v}^{s_r} \otimes \pi_0),$$

where $P$ is a parabolic subgroup of $\text{Sp}_n$ with Levi subgroup $M_P = \text{GL}_{d_1} \times \cdots \times \text{GL}_{d_r} \times \text{Sp}_{n_0}$. The local $A$-packet $\Pi_{\psi_v}$ associated to the localization $\psi_v$, which is a multiset over $\text{Irr}(\text{Sp}_n(F_v))$, is defined by the disjoint union of the multisets of the Jordan–Hölder series of $I_{\psi_v}(\pi_0)$, i.e.,

$$\Pi_{\psi_v} = \bigsqcup_{\pi_0 \in \Pi_{\psi_0}} \{ \pi \mid \pi \text{ is an irreducible constituent of } I_{\psi_v}(\pi_0) \}.$$

Note that $A_{\psi_v} = A_{\psi_0}$. Define a map

$$\Pi_{\psi_v} \to \widehat{A_{\psi_v}}, \quad \pi \mapsto \langle \cdot, \pi \rangle_{\psi_v}$$

by

$$\langle \cdot, \pi \rangle_{\psi_v} = \langle \cdot, \pi_0 \rangle_{\psi_0}$$

if $\pi$ is an irreducible constituent of $I_{\psi_v}(\pi_0)$.

When $v \mid \infty$, we fix a maximal compact subgroup $K_v$ of $\text{Sp}_n(F_v)$, and set $K_{\infty} = \prod_{v \mid \infty} K_v$. When $v$ is a real place, we choose $K_v$ as

$$K_v = \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \in \text{Sp}_n(F_v) \cong \text{Sp}_n(\mathbb{R}) \mid t\alpha + t\beta = 1_n \right\}.$$

For an irreducible admissible representation $\pi_{\infty} = \otimes_{v \mid \infty} \pi_v$ of $\text{Sp}_n(F \otimes \mathbb{Q} \mathbb{R}) \cong \prod_{v \mid \infty} \text{Sp}_n(F_v)$, we denote the $K_{\infty}$-finite part of $\pi_{\infty}$ by $\pi_{\infty,K_{\infty}} = \otimes_{v \mid \infty} \pi_v, K_v$. This is a simple $(\mathfrak{g}_\infty, K_{\infty})$-module, where $\mathfrak{g}_\infty = \text{Lie}(\text{Sp}_n(F \otimes \mathbb{Q} \mathbb{R})) \otimes \mathbb{R} \mathbb{C}$ is the complexification of the Lie algebra of $\text{Sp}_n(F \otimes \mathbb{Q} \mathbb{R})$.

Now we are ready to define global $A$-packets. For a global $A$-parameter $\psi \in \Psi_2(\text{Sp}_n/F)$, we define the global $A$-packet $\Pi_\psi$ by

$$\Pi_\psi = \left\{ \pi = \left( \bigotimes_{v < \infty}^{'} \pi_v \right) \otimes \left( \bigotimes_{v \mid \infty} \pi_v, K_v \right) \mid \pi_v \in \Pi_{\psi_v}, \langle \cdot, \pi_v \rangle_{\psi_v} = 1 \text{ for almost all } v \right\}.$$
This is a multiset over the set of equivalence classes of simple admissible $\text{Sp}_n(\mathbb{A}_{\text{fin}}) \times (g_\infty, K_\infty)$-modules. By abuse of notation, we simply write $\pi = \otimes'_v \pi_v$ for $\pi = (\otimes'_v g_\infty \pi_v) \otimes (\otimes'_v K_\infty \pi_v, K_v)$. For $\pi = \otimes'_v \pi_v \in \Pi_\psi$, since $\langle \cdot, \pi_v \rangle_{\psi_v} = 1$ for almost all $v$, one can define a character $\langle \cdot, \pi \rangle_{\psi}$ of $\prod_v A_{\psi_v}$ by

$$\langle \cdot, \pi \rangle_{\psi} = \bigotimes_v \langle \cdot, \pi_v \rangle_{\psi_v}.$$

2.3. **Arthur’s multiplicity formula.** We state Arthur’s multiplicity formula. Let $F$ be a number field. For a finite place $v < \infty$, we denote by $\mathfrak{o}_v$ the ring of integers of $F_v$. A smooth function

$$\varphi: \text{Sp}_n(\mathbb{A}) \to \mathbb{C}$$

is a square-integrable automorphic form on $\text{Sp}_n(\mathbb{A})$ if $\varphi$ satisfies the following conditions:

1. $\varphi$ is left $\text{Sp}_n(F)$-invariant;
2. $\varphi$ is right $K$-invariant, where $K = K_{\text{fin}} \times K_\infty$ is the maximal compact subgroup of $\text{Sp}_n(\mathbb{A})$ with $K_{\text{fin}} = \prod_{v < \infty} \text{Sp}_n(\mathfrak{o}_v)$;
3. $\varphi$ is $3$-finite, where $3$ is the center of the universal enveloping algebra $U(g_\infty)$ of $g_\infty$.
4. $\varphi$ is square-integrable, i.e.,

$$\int_{\text{Sp}_n(F) \backslash \text{Sp}_n(\mathbb{A})} |\varphi(g)|^2 dg < \infty.$$  

By [6, §4.3], such a function is of moderate growth. We also note that any cusp form on $\text{Sp}_n(\mathbb{A})$ is square-integrable. The set of square-integrable automorphic forms on $\text{Sp}_n(\mathbb{A})$ is denoted by $A^2(\text{Sp}_n(\mathbb{A}))$. It is an $\text{Sp}_n(\mathbb{A}_{\text{fin}}) \times (g_\infty, K_\infty)$-module.

Recall that for an $A$-parameter $\psi \in \Psi_2(\text{Sp}_n/F)$,

- its component group $A_\psi$ has Arthur’s character $\varepsilon_\psi$;
- there is a diagonal map $\Delta: A_\psi \to \prod_v A_{\psi_v}$;
- we have a character $\langle \cdot, \pi \rangle_{\psi}$ on $\prod_v A_{\psi_v}$ associated to $\pi \in \Pi_\psi$.

Arthur’s multiplicity formula ([3, Theorem 1.5.2]) gives a decomposition of $A^2(\text{Sp}_n(\mathbb{A}))$.

**Theorem 2.3** (Arthur’s multiplicity formula). The discrete spectrum $A^2(\text{Sp}_n(\mathbb{A}))$ decomposes into a direct sum

$$A^2(\text{Sp}_n(\mathbb{A})) \cong \bigoplus_{\psi \in \Psi_2(\text{Sp}_n/F)} \bigoplus_{\pi \in \Pi_\psi} m_{\pi, \psi} \pi$$

as an $\text{Sp}_n(\mathbb{A}_{\text{fin}}) \times (g_\infty, K_\infty)$-module, where the non-negative integer $m_{\pi, \psi}$ is given by

$$m_{\pi, \psi} = \begin{cases} 1 & \text{if } \langle \cdot, \pi \rangle_{\psi} \circ \Delta = \varepsilon_\psi, \\ 0 & \text{otherwise.} \end{cases}$$

We emphasize that Arthur’s multiplicity formula does not tell us whether $A^2(\text{Sp}_n(\mathbb{A}))$ is multiplicity-free or not since the $A$-packets are multisets. In order to investigate $A^2(\text{Sp}_n(\mathbb{A}))$ more precisely, we need to study the structures of local $A$-packets $\Pi_\psi$.

2.4. **Mœglin’s results.** In the successive works [15, 16, 17, 18, 19], Mœglin has defined local $A$-packets herself for $p$-adic fields, and given many important results. Xu [24] proved that Mœglin’s $A$-packets coincide with Arthur’s ones. Hence one can use Mœglin’s results for Arthur’s $A$-packets.

First, we let $F$ be a non-archimedean local field of characteristic zero, with the ring of integers $\mathfrak{o}$. We denote the cardinality of the residual field of $F$ by $q$.

For a local $A$-parameter $\psi \in \Psi(\text{Sp}_n/F)$, we have an $A$-packet $\Pi_\psi$, which is a multiset over $\text{Irr}_{\text{unit}}(\text{Sp}_n(F))$. One of the most important result of Mœglin is as follows:
Theorem 2.4 ([18, 24 Theorem 8.12]). The $A$-packet $\Pi_\psi$ is multiplicity-free, i.e., it is in fact a subset of $\text{Irr}_{\text{unit}}(\text{Sp}_n(\mathbb{F}))$.

Recall that an $A$-parameter is a homomorphism $\psi: W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C}) \to \text{SO}_{2n+1}(\mathbb{C})$. Let $\Delta: W_F \times \text{SL}_2(\mathbb{C}) \to W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C})$ be the diagonal map given by $\Delta(w, a) = (w, a, a)$. For two $A$-parameters $\psi, \psi' \in \Psi(\text{Sp}_n/\mathbb{F})$, their $A$-packets $\Pi_\psi$ and $\Pi_{\psi'}$ can have intersection.

Proposition 2.5 ([17 Corollary 4.2]). If $\Pi_\psi \cap \Pi_{\psi'} \neq \emptyset$, then the diagonal restrictions $\psi \circ \Delta$ and $\psi' \circ \Delta$ are conjugate by $\text{SO}_{2n+1}(\mathbb{C})$.

Recall that an irreducible representation $\pi$ of $\text{Sp}_n(\mathbb{F})$ is nearly equivalent to $\psi, \psi'$, if a nonzero $\text{Sp}_n(\mathbb{F})$-fixed vector. In this case, $\pi$ is uniquely determined by its Satake parameter $c(\pi)$, which is a semisimple conjugacy class of $\text{SO}_{2n+1}(\mathbb{C})$. We say that an $A$-parameter $\psi: L_F \times \text{SL}_2(\mathbb{C}) \to \text{SO}_{2n+1}(\mathbb{C})$ for $\text{Sp}_n/\mathbb{F}$ is unramified if the restriction of $\psi$ to $L_F = W_F \times \text{SL}_2(\mathbb{C})$ factors through the quotient $W_F \times \text{SL}_2(\mathbb{C}) \to W_F/I_F$, where $I_F$ is the inertia subgroup of $W_F$. In this case, one can write

$$\psi = |\cdot|^{s_1} \otimes S_{d_1} \oplus \cdots \oplus |\cdot|^{s_r} \otimes S_{d_r}$$

with $s_i \in \sqrt{-1} \mathbb{R}$.

Proposition 2.6 ([17 Proposition 4.4]). When $\psi$ is unramified and is of the above form, the $A$-packet $\Pi_\psi$ has a unique unramified representation $\pi$. Its Satake parameter $c(\pi)$ is given by

$$c(\pi) = \sum_{i=1}^{r} \left( q^{-s_i + \frac{d_i - 1}{2}} \right) = \sum_{i=1}^{r} \left( q^{-s_i + \frac{d_i - 3}{2}} \right) \cdots q^{-s_i + \frac{d_i - 1}{2}}.$$

By Theorem 2.7, the character $\langle \cdot, \pi \rangle_{\psi}$ of $A_\psi$ is trivial.

Next, we let $\mathbb{F}$ be a number field and $v$ be a finite place of $\mathbb{F}$. Recall that for a global $A$-parameter $\psi \in \Psi_2(\text{Sp}_n/\mathbb{F})$, its localization $\psi_v$ at $v$ might not be a local $A$-parameter. The local $A$-packet $\Pi_{\psi_v}$ is the disjoint union of the Jordan–Hölder series of induced representations $I_{\psi_v}(\pi_0)$, where $\pi_0$ runs over the $A$-packet associated to a local $A$-parameter $\psi_0 \in \Psi(\text{Sp}_{n_0}/\mathbb{F})$.

Proposition 2.7 ([19 Proposition 5.1]). For $\pi_0 \in \Pi_{\psi_v}$, the induced representation $I_{\psi_v}(\pi_0)$ is irreducible. Moreover, if $\pi_0, \pi_0' \in \Pi_{\psi_v}$ are not isomorphic to each other, then $I_{\psi_v}(\pi_0) \not\cong I_{\psi_v}(\pi_0')$. In conclusion, $\Pi_{\psi_v}$ is a (multiplicity-free) subset of $\text{Irr}(\text{Sp}_n(\mathbb{F}))$.

For two irreducible admissible representations $\pi \cong \otimes_v' \pi_v$ and $\pi' \cong \otimes_v' \pi'_v$ of $\text{Sp}_n(\mathbb{A})$, we say that $\pi$ is nearly equivalent to $\pi'$ if $\pi_v \cong \pi'_v$ for almost all $v$.

Corollary 2.8. Let $\psi, \psi' \in \Pi_{\psi_v}$ be two global $A$-parameters, and take $\pi \in \Pi_{\psi}$ and $\pi' \in \Pi_{\psi'}$. Then $\pi$ is nearly equivalent to $\pi'$ if and only if $\psi$ is equivalent to $\psi'$.

Proof. Suppose that $\pi, \pi' \in \Pi_{\psi_v}$. Since $\pi_v$ and $\pi'_v$ are unramified for almost all $v$, by Propositions 2.6 and 2.7 we have $\pi_v \cong \pi'_v$ for almost all $v$.

Conversely, suppose that $\pi$ is nearly equivalent to $\pi'$. Let $v$ be a finite place of $\mathbb{F}$ such that all of $\pi_v, \pi'_v, \psi_v$ and $\psi'_v$ are unramified, and $\pi_v \cong \pi'_v$. Write $\pi_v = I_{\psi_v}(\pi_0)$ and $\pi'_v = I_{\psi_v'}(\pi'_0)$ with $\pi_0 \in \Pi_{\psi_v}$ and $\pi_0' \in \Pi_{\psi_v'}$ for $\psi_0 \in \Psi(\text{Sp}_{n_0}/\mathbb{F})$ and $\psi'_0 \in \Psi(\text{Sp}_{n'_0}/\mathbb{F})$. Comparing the real parts of the exponents of the eigenvalues of the Satake parameters of $\pi$ and $\pi'$, we see that $n_0 = n'_0$ and $\pi_0 \cong \pi'_0$. By Proposition 2.5 we have $\psi_0 \circ \Delta \cong \psi'_0 \circ \Delta$. This implies that $\psi_0 \cong \psi'_0$ since they are both unramified. Comparing the Satake parameters of $\pi$ and $\pi'$ again, we see that $\psi_v \cong \psi'_v$. As in [3, §1.3], global $A$-parameters $\psi$ and $\psi'$ define isobaric sums of representations $\tau_\psi$ and $\tau_{\psi'}$, which are
automorphic representations of $GL_{2n+1}(\mathbb{A})$. By the generalized strong multiplicity one theorem of Jacquet–Shalika [12, Theorem (4.4)], the condition $\psi_v \cong \psi'_v$ for almost all $v$ implies that $\tau_\psi \cong \tau_{\psi'}$. Since $\psi \mapsto \tau_\psi$ is injective, we conclude that $\psi \cong \psi'$.

2.5. Adams–Johnson packets. In this subsection, we consider $\mathbb{F} = \mathbb{R}$. In this case, the theory of local $A$-packets is not satisfactory now. On the other hand, Adams and Johnson [1] constructed $A$-packets for a certain special class of $A$-parameters. These $A$-packets are sets and their elements are given by cohomological induction so that these $A$-packets are relatively easy to understand. Recently, Arancibia, Mœglin and Renard [2] proved that the $A$-packets of Adams–Johnson coincide with Arthur’s ones. Hence one can easily understand Arthur’s $A$-packets for certain spacial parameters.

Recall that the Weil groups of $\mathbb{C}$ and $\mathbb{R}$ are given by

$$W_\mathbb{C} = \mathbb{C}^\times, \quad W_\mathbb{R} = \mathbb{C}^\times \sqcup \mathbb{C}^\times j,$$

respectively, where

$$j^2 = -1, \quad jzj^{-1} = \overline{z} \quad \text{for } z \in \mathbb{C}^\times.$$

There exists a canonical exact sequence

$$1 \longrightarrow W_\mathbb{C} \longrightarrow W_\mathbb{R} \longrightarrow \text{Gal}(\mathbb{C}/\mathbb{R}) \longrightarrow 1.$$

There are exactly two quadratic characters of $W_\mathbb{R}$. One is the trivial representation $1$, and the other is the sign character

$$\text{sgn} : W_\mathbb{R} \rightarrow \{\pm 1\},$$

given by $\text{sgn}(j) = -1$ and $\text{sgn}(z) = 1$ for $z \in \mathbb{C}^\times$.

For each integer $\alpha$, we define a 2-dimensional representation

$$\rho_\alpha : W_\mathbb{R} \rightarrow GL_2(\mathbb{C})$$

by

$$\rho_\alpha(j) = \begin{pmatrix} 0 & (-1)^\alpha \\ 1 & 0 \end{pmatrix}, \quad \rho_\alpha(z) = \begin{pmatrix} \chi_\alpha(z) & 0 \\ 0 & \chi_{-\alpha}(z) \end{pmatrix} \quad \text{for } z \in \mathbb{C}^\times,$$

where the character $\chi_\alpha$ of $\mathbb{C}^\times$ is defined by $\chi_\alpha(z) = \overline{z}^{-\alpha}(\overline{z^2})^{\frac{\alpha}{2}}$. We also write $\chi_\alpha(z) = (z/\overline{z})^{\frac{\alpha}{2}}$.

Then we see that:

- $\rho_\alpha$ is irreducible when $\alpha \neq 0$;
- $\rho_0 \cong 1 \oplus \text{sgn}$;
- $\rho_\alpha \cong \rho_{-\alpha}$;
- $\rho_\alpha$ is orthogonal (resp. symplectic) if $\alpha$ is even (resp. $\alpha$ is odd).

For $\alpha \geq \beta > 0$, the root number $\varepsilon(\rho_\alpha \otimes \rho_\beta)$ is given by

$$\varepsilon(\rho_\alpha \otimes \rho_\beta) = \begin{cases} -1 & \text{if } \alpha \equiv 0, \beta \equiv 1 \mod 2, \\ 1 & \text{otherwise.} \end{cases}$$

Moreover, for $\alpha > 0$ and $\delta \in \{0, 1\}$, we have $\varepsilon(\rho_\alpha \otimes \text{sgn}^\delta) = \sqrt{-1}^{\alpha + 1}$.

For an $A$-parameter $\psi : W_\mathbb{R} \times SL_2(\mathbb{C}) \rightarrow SO_{2n+1}(\mathbb{C})$, define $\psi_d : W_\mathbb{C} \rightarrow SO_{2n+1}(\mathbb{C})$ by

$$\psi_d(z) = \psi(z, \begin{pmatrix} (z/\overline{z})^{\frac{1}{2}} & 0 \\ 0 & (z/\overline{z})^{-\frac{1}{2}} \end{pmatrix}) \quad \text{for } z \in \mathbb{C}^\times.$$

We call an $A$-parameter $\psi \in \Psi(\text{Sp}_n/\mathbb{R})$ Adams–Johnson if $\psi$ is a direct sum of irreducible orthogonal representations of $W_\mathbb{R} \times SL_2(\mathbb{C})$, and $\psi_d$ is multiplicity-free. Let $\Psi_{\text{AJ}}(\text{Sp}_n/\mathbb{R})$ be the subset of
Ψ(\(\text{Sp}_{n}/\mathbb{R}\)) consisting of Adams–Johnson A-parameters. It is easy to see that \(\psi\) is Adams–Johnson if and only if \(\psi\) is of the form

\[
\psi = \left( \bigoplus_{i=1}^{r} \rho_{\alpha_i} \otimes S_{d_i} \right) \oplus \text{sgn}^\delta \otimes S_{d_0},
\]

where

- \(\alpha_i > 0\) and \(d_i > 0\) for \(1 \leq i \leq r\);
- \(2 \sum_{i=1}^{r} d_i + d_0 = 2n + 1\);
- \(\alpha_i + d_i \equiv 1 \mod 2\) for \(1 \leq i \leq r\), and \(d_0 \equiv 1 \mod 2\);
- \(\delta \in \{0, 1\}\) such that \(\delta \equiv \sum_{i=1}^{r} d_i \mod 2\);
- \(\alpha_i - \alpha_{i+1} \geq d_i + d_{i+1}\) for \(1 \leq i < r\), and \(\alpha_r \geq d_r + d_0\).

In this subsection, we fix such \(\psi\).

We denote the standard Cartan involution \(g \mapsto {}^t g^{-1}\) of \(\text{Sp}_n(\mathbb{R})\) by \(\theta\). Let

\[
T(\mathbb{R}) = \left\{ \begin{pmatrix} a_1 & \cdots & b_1 \\ \vdots & \ddots & \vdots \\ -b_1 & \cdots & a_n \end{pmatrix} \in \text{Sp}_n(\mathbb{R}) \mid a_i^2 + b_i^2 = 1 \right\}
\]

be a maximal torus of \(\text{Sp}_n(\mathbb{R})\), which is compact. Fix a (standard) Borel subgroup \(B\) of \(\text{Sp}_n\) containing \(T\) (c.f., see [21 §2.2]). We write \(L(\mathbb{R})\) for the subgroup of \(\text{Sp}_n(\mathbb{R})\) consisting of elements of the form

\[
\begin{pmatrix}
  a_1 \\
  \vdots \\
  a_r \\
  -b_1 \\
  \vdots \\
  -b_r \\
\end{pmatrix}
\begin{pmatrix}
  b_1 \\
  \vdots \\
  b_r \\
  a_1 \\
  \vdots \\
  a_r \\
\end{pmatrix}
\begin{pmatrix}
  A & B \\
  C & D \\
\end{pmatrix}
\]

where \(a_i + \sqrt{-1}b_i \in U(d_i)\) for \(i = 1, \ldots, r\) and

\[
\begin{pmatrix}
  A & B \\
  C & D \\
\end{pmatrix} \in \text{Sp}_{d_0-1}(\mathbb{R}).
\]

Hence

\[
L(\mathbb{R}) \cong U(d_1) \times \cdots \times U(d_r) \times \text{Sp}_{d_0-1}(\mathbb{R}).
\]

Set

\[
\Sigma_\psi = W(L(\mathbb{C}), T(\mathbb{C})) \backslash W(\text{Sp}_n(\mathbb{C}), T(\mathbb{C})) / W(\text{Sp}_n(\mathbb{R}), T(\mathbb{R})).
\]

For a positive integer \(d\), we define \(\mathcal{P}_2(d)\) by the set of pairs of non-negative integers \((p, q)\) such that \(p + q = d\). Then we have a canonical bijection

\[
\Sigma_\psi \cong \prod_{i=1}^{r} \mathcal{P}_2(d_i).
\]
For \( \{(p_i, q_i)\}_i \in \prod_{i=1}^r \mathcal{P}_2(d_i) \), if we set
\[
t(p_i, q_i) = \text{diag}(1, \ldots, 1, \sqrt{-1}, \ldots, \sqrt{-1}) \in \text{GL}_{d_i}(\mathbb{C}),
\]
then a representative of the element in \( \Sigma_\psi \) corresponding to \( \{(p_i, q_i)\}_i \in \prod_{i=1}^r \mathcal{P}_2(d_i) \) is given by
\[
t(\{(p_i, q_i)\}_i) = \begin{pmatrix}
  t(p_1, q_1) & \cdots & \cdots & t(p_r, q_r) \\
  \cdots & 1_{d_0-1} & \cdots & \cdots \\
  t(p_1, q_1)^{-1} & \cdots & t(p_r, q_r)^{-1} & 1_{d_0-1}
\end{pmatrix}.
\]

It is easy to see that \( L_{(p_i, q_i)} = t(\{(p_i, q_i)\}_i) \cdot L \cdot t(\{(p_i, q_i)\}_i)^{-1} \) is defined over \( \mathbb{R} \), and
\[
L_{(p_i, q_i)}(\mathbb{R}) \cong U(p_1, q_1) \times \cdots \times U(p_r, q_r) \times \text{Sp}_{d_0-1}(\mathbb{R}).
\]

Set
\[
\lambda_j = \left( \frac{\alpha_j + d_j - 1}{2}, \frac{\alpha_j + d_j - 3}{2}, \ldots, \frac{\alpha_j - d_j + 1}{2} \right) \in \mathbb{Z}^{d_j}
\]
and
\[
\lambda_\psi = \left( \lambda_1, \ldots, \lambda_r, \frac{d_0-1}{2}, \frac{d_0-3}{2}, \ldots, 1 \right) \in \mathbb{Z}^n.
\]

Let \( \rho = (n, n-1, \ldots, 1) \) be the half sum of the positive roots of \( T \) with respect to \( B \). Then there exists a unitary character \( \chi_{\lambda_\psi} : L(\mathbb{R}) \to \mathbb{C}^* \) such that the restriction to \( T(\mathbb{R}) \) is \( \lambda_\psi - \rho \in \mathbb{Z}^n \cong X^*(T) \).

For \( w \in W(\text{Sp}_n(\mathbb{C}), T(\mathbb{C})) \), we define
\[
\pi_w = A_{w^{-1}}Q_w(w^{-1}\chi_{\lambda_\psi})
\]
to be the derived functor module, where \( Q \) is a \( \theta \)-stable parabolic subgroup of \( \text{Sp}_n \) with Levi subgroup \( L \). It is nonzero and irreducible with infinitesimal character \( \lambda_\psi \). Moreover, \( \pi_w \cong \pi_{w'} \) if and only if the images of \( w \) and \( w' \) in \( \Sigma_\psi \) are equal to each other.

**Theorem 2.9 (\[2\]).** For \( \psi = \oplus_{i=1}^r \rho_{\alpha_i} \otimes S_{d_i} \otimes S_{d_0} \in \Psi_{AJ}(\text{Sp}_n(\mathbb{R})) \) with \( \alpha_i - \alpha_{i+1} \geq d_i + d_{i+1} \) for \( 1 \leq i < r \). Then the A-packet \( \Pi_\psi \) is given by the (multiplicity-free) set
\[
\Pi_\psi = \{ \pi_w \mid w \in \Sigma_\psi \}.
\]

When \( w \in \Sigma_\psi \) corresponds to \( \{(p_i, q_i)\}_i \in \prod_{i=1}^r \mathcal{P}_2(d_i) \), the character \( \langle \cdot, \pi_w \rangle \psi \) is given so that
\[
\langle z_\psi, \pi_w \rangle \psi = 1
\]
and
\[
\langle \alpha_{\rho_{\alpha_i}} \otimes S_{d_i}, \pi_w \rangle \psi = (-1)^{\frac{\rho_{\alpha_i} - d_i}{2}}
\]
for \( i = 1, \ldots, r \), where
\[
\delta_i = \begin{cases} 0 & \text{if } d_i \text{ is even}, \\ (-1)^{\sum_{j=1}^{i-1} d_j} & \text{if } d_i \text{ is odd}. \end{cases}
\]

For example, suppose that \( d_1 = \cdots = d_r = d_0 = 1 \). We note \( r = n \) and \( \delta \equiv n \mod 2 \). If \( w \in \Sigma_\psi \) corresponds to \( \{(p_i, q_i)\}_i \in \mathcal{P}_2(1)^n \), then \( \pi_w \) is the discrete series representation of \( \text{Sp}_n(\mathbb{R}) \) with
Harish-Chandra parameter \((p_1 - q_1)\lambda_1, \ldots, (p_n - q_n)\lambda_n\), where the infinitesimal character of \(\pi_w\) is \(\lambda_\psi = (\lambda_1, \ldots, \lambda_n)\). The character \(\langle \cdot, \pi_w \rangle_\psi\) satisfies that
\[
\langle \alpha_{p_i}, \pi_w \rangle_\psi = (-1)^{i-1}(p_i - q_i)
\]
for \(i = 1, \ldots, n\).

3. Siegel modular forms and holomorphic cusp forms

In this section, we apply Arthur’s multiplicity formula to Siegel modular forms. We prove two theorems in Introduction (11).

3.1. Siegel modular forms. In this subsection, we put \(F = \mathbb{Q}\). Let
\[
\mathcal{H}_n = \{Z \in \text{Mat}_n(\mathbb{C}) \mid ^tZ = Z, \text{Im}(Z) > 0\}
\]
be the Siegel upper half space of degree \(n\). Here, for a real symmetric matrix \(Y\), we write \(Y > 0\) if \(Y\) is positive definite. The symplectic group \(\text{Sp}_n(\mathbb{R})\) acts on \(\mathcal{H}_n\) by
\[
g(Z) = (AZ + B)(CZ + D)^{-1}, \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_n(\mathbb{R}), \ Z \in \mathcal{H}_n.
\]
The canonical automorphy factor \(J(g, Z)\) is defined by
\[
J(g, Z) = CZ + D \in \text{GL}_n(\mathbb{R}), \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_n(\mathbb{R}), \ Z \in \mathcal{H}_n.
\]
For \(k = (k_1, \ldots, k_n) \in \mathbb{Z}^n\) with \(k_1 \geq \cdots \geq k_n\), the irreducible representation of \(U(n)\) with highest weight \((k_1, \ldots, k_n)\) is denoted by \((\rho_k, V_k)\). It is extended to a holomorphic representation of \(\text{GL}_n(\mathbb{R})\). Then \(\rho_k(J(g, Z)) \in \text{GL}(V_k)\) is an automorphy factor. When \(k = (k, k, \ldots, k)\), we have \((\rho_k, V_k) = (\text{det}^k, \mathbb{C})\).

Definition 3.1. Let \(k = (k_1, \ldots, k_n) \in \mathbb{Z}^n\) with \(k_1 \geq \cdots \geq k_n\). A \(V_k\)-valued holomorphic function \(F: \mathcal{H}_n \to V_k\) is a Siegel modular cusp form of vector weight \(\rho_k\) if
\begin{enumerate}
\item \(F(\gamma(Z)) = (\rho_k(J(\gamma, Z))F(Z)\) for \(\gamma \in \text{Sp}_n(\mathbb{Z})\) and \(Z \in \mathcal{H}_n\);
\item \(F\) has a Fourier expansion of the form

\[
F(Z) = \sum_{T \in \text{Sym}_n(\mathbb{Q})} A_F(T)e^{2\pi \sqrt{-1} \text{tr}(TZ)}, \quad A_F(T) \in V_k.
\]
\end{enumerate}

The space of Siegel modular cusp forms of vector weight \(\rho_k\) is denoted by \(S_{\rho_k}(\text{Sp}_n(\mathbb{Z}))\). There is a Hecke theory for \(S_{\rho_k}(\text{Sp}_n(\mathbb{Z}))\). When \(F \in S_{\rho_k}(\text{Sp}_n(\mathbb{Z}))\) is a Hecke eigenform, for each prime \(p\), the Satake parameter
\[
(\beta^\pm_{p,1}, \ldots, \beta^\pm_{p,n}) \in (\mathbb{C}^\times)^n / \mathbb{G}_n \times \{\pm 1\}^n
\]
is associated to \(F\). Then the standard \(L\)-function attached to \(F\) is defined by
\[
L(s, F, \text{std}) = \prod_p \left(1 - p^{-s} \right)^{-1} \prod_{i=1}^n (1 - \beta_{p,i} p^{-s})^{-1} (1 - \beta_{p,i}^{-1} p^{-s})^{-1}
\]
for \(\text{Re}(s) \gg 0\).

Set \(i = \sqrt{-1} \cdot 1_n \in \mathcal{H}_n\). The stabilizer of \(i\) in \(\text{Sp}_n(\mathbb{R})\) is the standard maximal compact subgroup
\[
K_\infty = \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \in \text{Sp}_n(\mathbb{R}) \mid t\alpha + t\beta = 1_n \right\}.
\]
For a Siegel modular cusp form $F \in S_{\rho_k}(\text{Sp}_n(\mathbb{Z}))$, one can define a $V_{\rho_k}$-valued function $\Phi_F: \text{Sp}_n(\mathbb{F}) \backslash \text{Sp}_n(\mathbb{A}) \to V_{\rho_k}$ by

$$
\Phi_F(\gamma g) = \rho_k((J(g, i))^{-1} F(g, i))
$$

for $\gamma \in \text{Sp}_n(\mathbb{Q})$, $g_{\infty} \in \text{Sp}_n(\mathbb{R})$ and $\kappa_{\text{fin}} \in K_{\text{fin}} = \text{Sp}_n(\mathbb{Z})$. By a similar argument to [1], Lemmas 5, 7], $\Phi_F$ satisfies the following:

1. $\Phi_F(\gamma g) = \rho_k(\gamma) \Phi_F(g)$ for $\gamma \in K_{\infty} \subset \text{Sp}_n(\mathbb{R})$, where $K_{\infty}$ is identified with $U(n)$ by

$$
\begin{pmatrix}
\alpha & \beta \\
-\beta & \alpha
\end{pmatrix} \mapsto \alpha + \beta \sqrt{-1},
$$

and the representation $\rho_k$ of $U(n)$ is defined by $\rho_k(a) = \rho_k(\alpha)$ for $a \in U(n)$;

2. $p^{-}\Phi_F = 0$, where

$$
p^{-} = \left\{ \begin{pmatrix} A & \sqrt{-1}B \\
-\sqrt{-1}B & A \end{pmatrix} \bigg| A \in \text{Sym}_2(\mathbb{C}) \right\} \subset g_{\infty};
$$

3. $\Phi_F$ is a cusp form, i.e.,

$$
\int_{N(F) \backslash N(\mathbb{A})} \Phi_F(n g) d g = 0
$$

for $g \in \text{Sp}_n(\mathbb{A})$ and for $N$ being the unipotent radical of any proper $F$-parabolic subgroup $P$ of $\text{Sp}_n$.

Note that $\rho_k$ is isomorphic to the contragredient representation of $\rho_k$, i.e., there exists a non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ on $V_k \times V_k$ such that

$$
\langle \rho_k(a) v, \rho_k(\alpha) v' \rangle = \langle v, v' \rangle
$$

for $v, v' \in V_k$ and $a \in U(n)$. Then for $v \in V_k$, the function

$$
\varphi_{F,v}(g) = \langle v, \Phi_F(g) \rangle \in \mathbb{C}
$$

is a cusp form on $\text{Sp}_n(\mathbb{Q}) \backslash \text{Sp}_n(\mathbb{A})$. Note that for fixed $g \in \text{Sp}_n(\mathbb{A})$, the function $K_{\infty} \ni \kappa_{\infty} \mapsto \varphi_{F,v}(g \kappa_{\infty})$ is a matrix coefficient of $\rho_k$. In particular, the right translations of $\varphi_{F,v}$ under $K_{\infty}$ for $v \in V_k$ form an irreducible representation of $K_{\infty}$ which is isomorphic to $\rho_k$. We call $\varphi_{F,v}$ a holomorphic cusp form of vector weight $\rho_k$.

3.2. Lowest weight modules. Let

$$
\mathfrak{e}_C = \text{Lie}(K_{\infty}) \otimes_{\mathbb{R}} \mathbb{C} = \left\{ \begin{pmatrix} A & B \\
-B & A \end{pmatrix} \in \text{Mat}_{2n}(\mathbb{C}) \bigg| A = -tA, B = tB \right\}
$$

be the complexification of Lie algebra of $K_{\infty}$. Fix $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ with $k_1 \geq \cdots \geq k_n$. We denote the differential of the representation $\rho_k$ of $K_{\infty}$ by $d\rho_k: \mathfrak{e}_C \to \text{End}(V_k)$. Since $[\mathfrak{e}_C, p^-] \subset p^-$, the representation $d\rho_k$ can be extended to a representation of $\mathfrak{e}_C \oplus p^-$ by setting $d\rho_k(p^-) = 0$. Let $\mathcal{U}(g_{\infty})$ (resp. $\mathcal{U}(\mathfrak{e}_C \oplus p^-)$) be the universal enveloping algebra of $g_{\infty}$ (resp. $\mathfrak{e}_C \oplus p^-$). Consider the generalized Verma module

$$
\mathcal{M}(V_k) = \mathcal{U}(g_{\infty}) \otimes_{\mathcal{U}(\mathfrak{e}_C \oplus p^-)} V_k.
$$

It is a $(g_{\infty}, K_{\infty})$-module by setting $\kappa_{\infty} \cdot (1 \otimes v) = 1 \otimes \rho_k(\kappa_{\infty}) v$ for $\kappa_{\infty} \in K_{\infty}$ and $v \in V_k$. It is known that $\mathcal{M}(V_k)$ has a unique irreducible quotient, which we denote by $L(V_k)$. We call the $(g_{\infty}, K_{\infty})$-module $L(V_k)$ the lowest weight module of vector weight $\rho_k$. This module is characterized so that it contains $V_k$ as a $\mathcal{U}(\mathfrak{e}_C \oplus p^-)$-submodule with multiplicity one. The infinitesimal character of $L(V_k)$ is given by

$$
(k_1 - 1, k_2 - 2, \ldots, k_n - n, 0, -(k_n - n), \ldots, -(k_2 - 2), -(k_1 - 1)).
$$

Note that $L(V_k)$ is discrete series if $k_n > n$. 


Let $\psi \in \Psi_2(\text{Sp}_n/\mathbb{R})$. Suppose that

$$\psi_d(z) = \psi\left( z, \begin{pmatrix} (z/z)_{\frac{1}{2}} & 0 \\ 0 & (z/z)^{-\frac{1}{2}} \end{pmatrix} \right) = \begin{pmatrix} z^{\alpha_1}z^{\beta_1} \\ \vdots \\ z^{\alpha_{2n+1}}z^{\beta_{2n+1}} \end{pmatrix}$$

for some $\alpha_i, \beta_i \in \mathbb{C}$ such that $\alpha_i - \beta_i \in \mathbb{Z}$. Here, for $\alpha, \beta \in \mathbb{C}$ with $\alpha - \beta \in \mathbb{Z}$, we write $z^{\alpha - \beta} = z^{\alpha}z^{-\beta}$. In general, any representation $\pi \in \Pi_\psi$ has an infinitesimal character $(\alpha_1, \ldots, \alpha_{2n+1})$. We call $(\alpha_1, \ldots, \alpha_{2n+1})$ the infinitesimal character of $\psi$. In particular, if $\Pi_\psi$ contains a discrete series representation, then $\psi$ is Adams–Johnson (see also [21 Proposition 4.3, Théorème 4.4]).

We determine $A$-packets which contain $L(V_k)$. First, we consider the case where $L(V_k)$ is discrete series, i.e., $k_1 \geq \cdots \geq k_n > n$.

**Proposition 3.2.** Suppose that $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ with $k_1 \geq \cdots \geq k_n > n$. Then for $\psi \in \Psi(\text{Sp}_n/\mathbb{R})$, the $A$-packet $\Pi_\psi$ contains the lowest weight module $L(V_k)$ if and only if $\psi$ is of the form

$$\psi = \left( \bigoplus_{i=1}^t \rho_{\alpha_i} \otimes S_{d_i} \right) \oplus \text{sgn}^n,$$

where

- $\sum_{i=1}^t d_i = n$;
- $\alpha_i + d_i \equiv 1 \mod 2$;
- $\alpha_1 > \cdots > \alpha_t > 0$ and

$$\bigcup_{i=1}^t \left\{ \frac{\alpha_i + d_i - 1}{2}, \frac{\alpha_i + d_i - 3}{2}, \ldots, \frac{\alpha_i - d_i + 1}{2} \right\} = \{ k_1 - 1, k_2 - 2, \ldots, k_n - n \}.$$

In this case, the character $\langle \cdot, L(V_k) \rangle_\psi$ of $A_\psi$ is determined by $\langle z, L(V_k) \rangle_\psi = 1$ and

$$\langle \alpha_{\rho_k}^\circ \otimes \text{sgn}^n, L(V_k) \rangle_\psi = (-1)^{d_0 - \delta_i}$$

for $i = 1, \ldots, t$, where $\delta_i$ is given in Theorem 2.9.

**Proof.** By the above remark, if $L(V_k) \in \Pi_\psi$, then $\psi$ is Adams–Johnson. It is known that the cohomological induction $\pi_w = \mathbb{A}_{w^{-1}Q_w(w^{-1}\chi_\psi)}$ is discrete series if and only if $w^{-1}Lw(\mathbb{R})$ is compact. It happens for some $w \in \Sigma_\psi$ only when $\text{sgn}^n \subset \psi$, i.e., $d_0 = 1$ in the notation in Theorem 2.9.

By comparing the infinitesimal character, we see that if $L(V_k) \in \Pi_\psi$, then $\psi$ is of the form in the proposition.

Suppose that $\psi$ is of the form in the proposition. Then we have $\pi_1 = \mathbb{A}_Q(\chi_\psi) \cong L(V_k)$. The associated character $\langle \cdot, L(V_k) \rangle_\psi$ of $A_\psi$ is computed in Theorem 2.9. \qed

When $k = (k, k, \ldots, k)$, i.e., when $(\rho_k, V_k) = (\det^k, \mathbb{C})$, we write $D^{(n)}_k = L(V_k)$ and call it the lowest weight module of the scalar weight $k$. Some properties of $\psi \in \Psi(\text{Sp}_n/\mathbb{R})$ satisfying $D^{(n)}_k \in \Pi_\psi$ are established by Mœglin–Renard [21] even when $D^{(n)}_k$ is not discrete series.

**Proposition 3.3.** If $0 < k \leq n$, then the multiplicity of $D^{(n)}_k$ in $\Pi_\psi$ is at most one. Moreover, for $k_1, k_2 > 0$ with $k_1 \neq k_2$, if $D^{(n)}_{k_1}$ and $D^{(n)}_{k_2}$ belong to the same $A$-packet $\Pi_\psi$, then $n$ is even and \{k_1, k_2\} = \{\frac{n}{2}, \frac{n}{2} + 1\}.

**Proof.** The first assertion is a part of [21 Théorème 6.1]. If $D^{(n)}_k \in \Pi_\psi$, then the infinitesimal character of $D^{(n)}_k$ is determined by the $A$-parameter $\psi$. If $k > n$, since the infinitesimal character
of $T_n$ is regular, $\psi$ determines $T_n$ uniquely. Hence we may assume that $0 < k_1, k_2 \leq n$. In this case, the last assertion follows from [21, Théorème 6.2].

### 3.3. Automorphic representations generated by Siegel modular forms.

In this subsection, we let $F = \mathbb{Q}$. We prove some lemmas.

**Lemma 3.4.** Let $F \in S_{p_k}(\text{Sp}(\mathbb{Z}))$ be a Hecke eigenform with Satake parameter $c_p(F) \in (\mathbb{C}^\times)^n / \mathbb{S}_n \times \{\pm 1\}^n$ at $p$. Then the cuspidal automorphic representation $\pi_F \subset \hat{A}^2(\text{Sp}(\mathbb{A}))$ generated by the cusp form $\varphi_{F,v}$ for $v \in V_k$ is irreducible. Moreover, its local factors $\pi_{F,p}$ and $\pi_{F,\infty}$ are given as follows:

- $\pi_{F,p}$ is unramified with Satake parameter $c(\pi_{F,p}) = c_p(F)$ for each (finite) prime $p$;
- $\pi_{F,\infty}$ is the lowest weight module $L(V_k)$.

**Proof.** The proof is the same as [5, Theorem 4.3].

Since $\pi_F$ is cuspidal, it is a direct sum of irreducible automorphic representations. Fix an isomorphism $\pi_F \cong \pi_1 \oplus \cdots \oplus \pi_r$ with $\pi_i \cong \otimes'_v \pi_{i,v}$ irreducible. Let $\varphi_{i,v}$ be the image of $\varphi_{F,v} \in \pi_F$ for $v \in V_k$ under the projection $\pi_p \cong \oplus'_{j=1}^{r} \pi_j \to \pi_j$. Since $\pi_F$ is generated by $\{\varphi_{F,v} | v \in V_k\}$, for each $i = 1, \ldots, r$, some $\varphi_{i,v}$ is nonzero. By considering the action of the spherical Hecke algebra $\mathcal{H}(\text{Sp}(\mathbb{Q}_p), \text{Sp}(\mathbb{Z}_p))$ on $\varphi_{i,v}$, we see that $\pi_{i,p}$ is unramified with Satake parameter $c(\pi_{i,p}) = c_p(F)$ for each prime $p$. Moreover, since $\kappa \cdot \varphi_{F,v} = \varphi_{F,K^\infty}$ for $\kappa \in K^\infty$, for each $i = 1, \ldots, r$, we have $\kappa \cdot \varphi_{i,v} = \varphi_{i,K^\infty}$ for $\kappa \in K^\infty$. This means that there is a nonzero $K^\infty$-intertwining operator $V_k \mapsto \pi_{i,\infty} \cong \pi_{i,K^\infty}$. Hence $\pi_{i,\infty} \cong L(V_k)$ since $\pi_{i,\infty}$ is irreducible.

We conclude that $\pi_F$ is isotypic, i.e., $\pi_F \cong \pi \otimes \mathbb{C}'$ for some irreducible automorphic representation $\pi$. However, since $\pi_{F,\infty} \cong \pi_{K^\infty} \otimes \mathbb{C}'$ is generated by $\{\varphi_{F,v} | v \in V_k\}$ as $(g, K^\infty)$-module, it must be irreducible. Hence we have $r = 1$ so that $\pi_F$ is irreducible. \qed

Now we consider the elliptic forms. Let $f \in S_{2k}(\text{SL}_2(\mathbb{Z}))$ be a Hecke eigenform. Consider the irreducible unitary cuspidal automorphic representation $\tau_f$ of $\text{GL}_2(\mathbb{A})$ generated by $\varphi_f$, where $\varphi_f$ is a cusp form on $\text{GL}_2(\mathbb{A})$ defined by

$$\varphi_f(\gamma g_{\infty}) = \det(g_{\infty})^k f(g_{\infty} \langle \sqrt{-1} \rangle) j(g_{\infty}, \sqrt{-1})^{-2k}$$

for $\gamma \in \text{GL}_2(\mathbb{Q})$, $g_{\infty} \in \text{GL}_2(\mathbb{R})$ with $\det(g_{\infty}) > 0$, and $\kappa_{\infty} \in \text{GL}_2(\mathbb{Z})$. Note that $\tau_f$ is symplectic, i.e., $L(s, \tau_f, \Lambda^2)$ has a pole at $s = 1$ since the central character of $\tau_f$ is trivial (see [13, Corollary 7.5]). Moreover, the representation of $W_\mathbb{R}$ corresponding to $\tau_{f,\infty}$ is $\rho_{2k-1}$.

In the rest of this subsection, we consider the Siegel modular forms of degree 2. When $k = (k + j, k)$ with $j \geq 0$, we have $\rho_k = \det^k \text{Sym}(j)$, whose dimension is $j + 1$. In this case, we write $S_{k,j}(\text{Sp}(\mathbb{Z})) = S_{\rho_{k+1},j}(\text{Sp}(\mathbb{Z}))$, and call $f \in S_{k,j}(\text{Sp}(\mathbb{Z}))$ a Siegel modular form of degree 2 and of weight $\det^k \text{Sym}(j)$. Note that $S_{k,j}(\text{Sp}(\mathbb{Z})) = 0$ unless $j \equiv 0 \mod 2$. We assume that $j$ is even in the rest of this subsection. For a Siegel eigenform $f \in S_{k,j}(\text{Sp}(\mathbb{Z}))$, we can define a degree four $L$-function

$$L(s, f, \text{spin}) = \prod_p L_p(s, f, \text{spin})$$

which is called the spinor $L$-function associated to $f$.

We would prove the following lemma.

**Lemma 3.5.** Assume $k \geq 4$ and $j \geq 1$. Let $f \in S_{k,j}(\text{Sp}(\mathbb{Z}))$ be a Hecke eigenform. Then there is an irreducible unitary cuspidal symplectic automorphic representation $\tau_f = \otimes'_v \tau_{f,v}$ of $\text{GL}_4(\mathbb{A})$ such that

$$L(s, \tau_{f,p}) = L_p(s, f, \text{spin})$$

for any prime $p < \infty$, and the $L$-parameter of $\tau_{f,\infty}$ is $\rho_{j+2k-3} \oplus \rho_{j+1}$.\欠け
To prove this lemma, we use Arthur’s multiplicity formula for \( \text{SO}(3, 2) \) via an accidental isomorphism \( \text{PGSp}_2 \cong \text{SO}(3, 2) \), where \( \text{SO}(3, 2) \) is the split special orthogonal group of type \( B_2 \) and of discriminant 1. We omit the detail for Arthur’s multiplicity formula for \( \text{SO}(3, 2) \). See Arthur’s book [3].

For \( v \in V_{(k+j, k)} \), one can define a cusp form \( \varphi_{f,v} \) on \( \text{PGSp}_2(\mathbb{Q}) \backslash \text{PGSp}_2(\mathbb{A}) \) by a similar way to the elliptic modular forms case. We regard \( \varphi_{f,v} \) as a cusp form on \( \text{SO}(3, 2)(\mathbb{Q}) \backslash \text{SO}(3, 2)(\mathbb{A}) \) via an accidental isomorphism \( \text{PGSp}_2 \cong \text{SO}(3, 2) \). By a similar argument to the proof of Lemma 3.4, we see that \( \varphi_{f,v} \) for \( v \in V_{(k+j, k)} \) generates an irreducible cuspidal automorphic representation \( \sigma_f \) of \( \text{SO}(3, 2)(\mathbb{A}) \). One can associate a global \( A \)-parameter \( \psi_f \in \Psi_2(\text{SO}(3, 2)/\mathbb{Q}) \) to \( \sigma_f \). Note that \( \sigma_{f,\infty} \) is a discrete series representation, and its \( L \)-parameter is \( \rho_{j+2k-3} \oplus \rho_{j+1} \). Comparing the infinitesimal character, we see that

\[
(\psi_{f,\infty})_d \cong \chi_{j+2k-3} \oplus \chi_{j+1} \oplus \chi_{-(j+1)} \oplus \chi_{-(j+2k-3)}.
\]

An \( A \)-parameter \( \psi \in \Psi_2(\text{SO}(3, 2)/\mathbb{Q}) \) is one of the following forms:

1. \( \psi = \chi[4] \), where \( \chi \) is a quadratic character of \( \mathbb{A}^\times / \mathbb{Q}^\times \);
2. \( \psi = \chi_1[2] \oplus \chi_2[2] \), where \( \chi_1, \chi_2 \) are quadratic characters of \( \mathbb{A}^\times / \mathbb{Q}^\times \);
3. \( \psi = \tau[2] \), where \( \tau \) is an irreducible cuspidal orthogonal representation of \( \text{GL}_2(\mathbb{A}) \);
4. \( \psi = \chi[2] \oplus \tau[1] \), where \( \chi \) is a quadratic character of \( \mathbb{A}^\times / \mathbb{Q}^\times \) and \( \tau \) is an irreducible cuspidal symplectic representation of \( \text{GL}_3(\mathbb{A}) \);
5. \( \psi = \tau_1[1] \oplus \tau_2[1] \), where \( \tau_1, \tau_2 \) are irreducible cuspidal symplectic representations of \( \text{GL}_2(\mathbb{A}) \);
6. \( \psi = \tau[1] \), where \( \tau \) is an irreducible unitary cuspidal symplectic automorphic representation of \( \text{GL}_4(\mathbb{A}) \).

When \( \psi = \psi_f \), the representations \( \chi \) or \( \tau \) appearing in \( \psi \) should be unramified everywhere. In particular, \( \chi \) must be the trivial character of \( \mathbb{A}^\times / \mathbb{Q}^\times \) since \( \mathbb{A}^\times / \mathbb{Q}^\times \mathbb{Z}^\times \mathbb{R}^\times = \{1\} \). When \( \psi = 1[4] \), the global \( A \)-packet \( \Pi_\psi \) consists of the trivial representation of \( \text{SO}(3, 2)(\mathbb{A}) \). Since \( f \in S_{k,j}(\text{Sp}_2(\mathbb{Z})) \) is not a constant function, \( \psi_f \not\equiv 1[4] \). Hence the case (1) cannot occur. The case (2) is impossible since \( 1[2] \oplus 1[2] \) is not a discrete \( A \)-parameter.

Now consider \( \psi = \tau[2] \) is in the case (3). Then \( (\psi_{\infty})_d \cong \chi_{\alpha+1} \oplus \chi_{\alpha-1} \oplus \chi_{-\alpha+1} \oplus \chi_{-\alpha-1} \) for some \( \alpha \in \mathbb{Z} \) with \( \alpha \geq 0 \). If \( \psi_f = \psi \), we have \( \alpha + 1 = j + 2k - 3 \) and \( \alpha - 1 = j + 1 \). This implies that \( k = 3 \), which contradicts to our assumption that \( k \geq 4 \).

Next, we consider \( \psi = 1[2] \oplus \tau[1] \) as in the case (4). Then \( (\psi_{\infty})_d \cong \chi_\alpha \oplus \chi_1 \oplus \chi_1 \oplus \chi_\alpha \) for some \( \alpha \in \mathbb{Z} \) with \( \alpha \geq 0 \). If \( \psi_f = \psi \), we have \( j = 0 \) an \( \alpha = 2k - 3 \). This contradicts to our assumption that \( j \geq 1 \). Remark that this is compatible that there is no Saito–Kurokawa lifting of vector weight.

Suppose that \( \psi = \tau_1[1] \oplus \tau_2[1] \) as in the case (5) such that \( \psi_{\infty} = \rho_{j+2k-3} \oplus \rho_{j+1} \). Consider the representation \( \sigma = \otimes_i \sigma_i \in \Pi_\psi \) such that \( \sigma_p \) is unramified at any \( p < \infty \), and \( \sigma_{\infty} \cong \sigma_{f,\infty} \). Since \( \sigma_{f,\infty} \) is not generic, the character \( \langle \cdot, \sigma_{\infty} \rangle_{\psi_{\infty}} \) is not trivial. However, the Arthur’s character \( \varepsilon \psi \) must be trivial. This implies that \( \sigma \) is not automorphic by Arthur’s multiplicity formula for \( \text{SO}(3, 2) \). Remark that this is compatible that there is no Yoshida lifting of level one.

In conclusion, if \( k \geq 4 \) and \( j \geq 1 \), the \( A \)-parameter \( \psi_f \in \Psi_2(\text{SO}(3, 2)/\mathbb{Q}) \) is \( \psi_f = \tau_f[4] \) as in the case (6). By the compatibility of Satake parameters, we see that

\[
L(s, \tau_f, p) = L_p(s, f, \text{spin})
\]

for any \( p < \infty \). This completes the proof of Lemma 3.5.

3.4. Proof of Lifting Theorem. Now we prove Lifting Theorem (Theorem 1.1). Let \( \mathbb{F} = \mathbb{Q} \).

For fixed \( k = (k_1, \ldots, k_n) \in \mathbb{Z}^n \) with \( k_1 \geq \cdots \geq k_n > n \), we consider a Hecke eigenform \( g \in S_{k_1}(\text{Sp}_n(\mathbb{Z})) \). Let \( \pi_g = \mathbb{F}_{d=1}^n \tau_{[d]}(d) \in \Psi_2(\text{Sp}_n(\mathbb{Q})) \) be the \( A \)-parameter associated to \( g \), i.e., \( \pi_d \in \Pi_{\psi_d} \), where \( \pi_g \) is the irreducible cuspidal automorphic representation generated by the cusp form \( \varphi_{g,v} \) for
$v \in V_k$. Since $\pi_{g, \text{fin}}$ is unramified everywhere, and $\pi_{g, \infty} \cong L(V_k)$, by Arthur’s multiplicity formula (Theorem 2.3), we have
\[ \varepsilon_{\psi_g}(\alpha_{\tau_i(d_i)}) = \langle \Delta(\alpha_{\tau_i(d_i)}), \pi_g \rangle_{\psi_g} = \langle \alpha_{\phi_i, \infty} \otimes S_{d_i}, L(V_k) \rangle_{\psi_{g, \infty}} \]
for $i = 1, \ldots, r$, where $\phi_i, \infty$ is the representation of $W_\mathbb{R}$ corresponding to $\tau_i, \infty$. Moreover, since $L(V_k)$ is discrete series, by Proposition 3.2, the localization $\psi_{g, \infty}$ at $\infty$ is Adams–Johnson and is of the form
\[ \psi_{g, \infty} = \bigoplus_{l=1}^{t} \rho_{\alpha_l} \otimes S_{d_l} \oplus \text{sgn}^n, \]
where $\alpha_1 > \cdots > \alpha_t > 0$ and
\[ \bigcup_{l=1}^{t} \left\{ \frac{\alpha_l + d_l}{2} - 1, \frac{\alpha_l + d_l}{2}, \ldots, \frac{\alpha_l - d_l + 1}{2} \right\} = \{k_1 - 1, k_2 - 2, \ldots, k_n - n\}. \]
In particular, there is a unique index $i_0$ such that
\[ \phi_{i_0, \infty} \cong \bigoplus_{l=1}^{m_0} \rho_{\alpha_{i_0, l}} \oplus \text{sgn}^n \]
with even $\alpha_{i_0, l}$ (and $d_{i_0} = 1$), and for $i \neq i_0$,
\[ \phi_i, \infty \cong \bigoplus_{l=1}^{m_i} \rho_{\alpha_{i, l}} \]
with $\alpha_{i, l} \neq d_l \mod 2$. Since $\det(\phi_{i, \infty})(-1) = \omega_{\tau_i}(-1) = 1$, where $\omega_{\tau_i}$ is the central character of $\tau_i$, we have $(-1)^{m_{i_0}+n} = 1$ and $(-1)^{m_i d_l} = 1$ for $i \neq i_0$. Moreover, by Proposition 3.2 for $i \neq i_0$, we have $\langle \alpha_{\phi_i, \infty} \otimes S_{d_l}, L(V_k) \rangle_{\psi_{g, \infty}} = (-1)^{\frac{m_i d_l}{2}}$.

Now we prove Lifting Theorem (A).

**Proof of Theorem 7.1 (A).** Let $f \in S_{2k}(\text{SL}_2(\mathbb{Z}))$ be a Hecke eigenform, and $\tau_f$ be the irreducible cuspidal symplectic automorphic representation of $\text{GL}_2(\mathbb{A})$. Take an integer $d > 0$ with $k > d$ such that $k + d - 1 < k_n - n$ or $k - d > k_1 - 1$. This condition implies that $\tau_{f, \infty} \not\cong \tau_{i, \infty}$ for any $1 \leq i \leq r$. We set
\[ \psi_{f, g} = \tau_f[2d] \boxplus \psi_g. \]
One can easily see that $\psi_{f, g} \in \Psi_2(\text{Sp}_{n+2d}(\mathbb{Q}))$. Moreover, since $\psi_{f, g, p}$ is unramified for any (finite) prime $p$, the local $A$-packet $\Pi_{\psi_{f, g, p}}$ contains a unique unramified representation $\pi_{f, g, p}$. In addition, since
\[ \psi_{f, g, \infty} \cong (\rho_{2k-1} \otimes S_{2d}) \oplus \left( \bigoplus_{l=1}^{t} \rho_{\alpha_l} \otimes S_{d_l} \right) \oplus \text{sgn}^{n+2d} \]
is Adams–Johnson, the local $A$-packet $\Pi_{\psi_{f, g, \infty}}$ contains $L(V_{k'})$, where $k' = (k_1', \ldots, k_{n+2d}') \in \mathbb{Z}^{n+2d}$ is given so that $k_1' \geq \cdots \geq k_{n+2d}'$ and
\[ \{k_1' - 1, k_2' - 2, \ldots, k_{n+2d}' - n - 2d\} = \{k_1 - 1, k_2 - 2, \ldots, k_n - n\} \cup \{k + d - 1, k + d - 2, \ldots, k - d\}. \]

Let us consider
\[ \pi_{f, g} = \left( \bigotimes_{p<\infty} \pi_{f, g, p} \right) \otimes L(V_{k'}) \in \Pi_{\psi_{f, g}}. \]
Using Arthur’s multiplicity formula (Theorem 2.3), we study when \( \pi_{f,g} \) is automorphic. It is automorphic if and only if
\[
\varepsilon_{\psi_{f,g}}(\alpha_{\tau_f[2d]}) = \langle \Delta(\alpha_{\tau_f[2d]}), \pi_{f,g} \rangle_{\psi_{f,g}},
\]
and
\[
\varepsilon_{\psi_{f,g}}(\alpha_{\tau_i[d_i]}) = \langle \Delta(\alpha_{\tau_i[d_i]}), \pi_{f,g} \rangle_{\psi_{f,g}}
\]
for \( i = 1, \ldots, r \).

Before checking these conditions, we compute the root number \( \varepsilon(\tau_i \times \tau_f) \). Since \( \tau_i,\text{fin} \) and \( \tau_f,\text{fin} \) are unramified everywhere, \( \varepsilon(\tau_i \times \tau_f) \) is equal to the local root number \( \varepsilon(\phi_{i,\infty} \otimes \rho_{2k-1}) \). For \( i = i_0 \), we have
\[
\varepsilon(\phi_{i_0,\infty} \otimes \rho_{2k-1}) = \left( \prod_{l=1}^{m_{i_0}} \varepsilon(\rho_{\alpha_{l,i_0}} \otimes \rho_{2k-1}) \right) \varepsilon(\text{sgn}^n \otimes \rho_{2k-1}) = \begin{cases} (-1)^{m_{i_0}+k} & \text{if } k + d - 1 < k_n - n, \\ (-1)^k & \text{if } k - d > k_1 - 1. \end{cases}
\]

For \( i \neq i_0 \), we have
\[
\varepsilon(\phi_{i,\infty} \otimes \rho_{2k-1}) = \left( \prod_{l=1}^{m_i} \varepsilon(\rho_{\alpha_{l,i}} \otimes \rho_{2k-1}) \right) = \begin{cases} (-1)^{m_{i,d_i}} & \text{if } k + d - 1 < k_n - n, \\ 1 & \text{if } k - d > k_1 - 1, \end{cases}
\]
which is equal to 1.

Hence we have
\[
\varepsilon_{\psi_{f,g}}(\alpha_{\tau_f[2d]}) = \prod_{i=1}^{r} \varepsilon(\tau_i \times \tau_f)^{\min\{d_i,2d\}} = (-1)^k \times \begin{cases} (-1)^n & \text{if } k + d - 1 < k_n - n, \\ 1 & \text{if } k - d > k_1 - 1. \end{cases}
\]

On the other hand
\[
\langle \Delta(\alpha_{\tau_f[2d]}), \pi_{f,g} \rangle_{\psi_{f,g}} = \langle \alpha_{\rho_{2k-1} \otimes S_{2d}}, L(V_{k'}) \rangle = (-1)^d.
\]

Therefore, the condition \( \varepsilon_{\psi_{f,g}}(\alpha_{\tau_f[2d]}) = \langle \Delta(\alpha_{\tau_f[2d]}), \pi_{f,g} \rangle_{\psi_{f,g}} \) is equivalent that
\[
k \equiv \begin{cases} d + n & \text{mod } 2 & \text{if } k + d - 1 < k_n - n, \\ d & \text{mod } 2 & \text{if } k - d > k_1 - 1. \end{cases}
\]

Next, we check the condition \( \varepsilon_{\psi_{f,g}}(\alpha_{\tau_i[d_i]} \rangle = \langle \Delta(\alpha_{\tau_i[d_i]}), \pi_{f,g} \rangle_{\psi_{f,g}} \) for \( i \neq i_0 \). We have
\[
\langle \Delta(\alpha_{\tau_i[d_i]}), \pi_{f,g} \rangle_{\psi_{f,g}} = \langle \alpha_{\phi_{i,\infty} \bar{S}_{d_i}, L(V_{k'})} \rangle_{\psi_{f,g,\infty}} = \langle \alpha_{\phi_{i,\infty} \bar{S}_{d_i}, L(V_{k'})} \rangle_{\psi_{g,\infty}} = \varepsilon_{g}(\alpha_{\tau_i[d_i]}),
\]
\[
= \varepsilon_{\psi_{f,g}}(\alpha_{\tau_i[d_i]}),
\]
\[
= \varepsilon_{\psi_{f,g}}(\alpha_{\tau_i[d_i]}), \cdot \varepsilon(\tau_i \times \tau_f)^{-\min\{d_i,2d\}} = \varepsilon_{\psi_{f,g}}(\alpha_{\tau_i[d_i]}),
\]
as desired.

Finally, since
\[
\left( \prod_{i=1}^{r} \varepsilon_{\psi_{f,g}}(\alpha_{\tau_i[d_i]} \rangle \cdot \varepsilon_{\psi_{f,g}}(\alpha_{\tau_f[2d]}) = 1
\]
and
\[
\left( \prod_{i=1}^{r} \langle \Delta(\alpha_{\tau_i[d_i]}), \pi_{f,g} \rangle \psi_{f,g} \right) \cdot \langle \Delta(\alpha_{\tau_f[2d]}), \pi_{f,g} \rangle \psi_{f,g} = 1,
\]
when \( k \equiv d + n \mod 2 \) if \( k + d - 1 < k_n - n \) (resp. when \( k \equiv d \mod 2 \) if \( k - d > k_1 - 1 \)), we have
\[
\varepsilon_{\psi_{f,g}}(\alpha_{\tau_i[d_i]}) = \langle \alpha_{\phi_i[\infty]} S_{d_i}, L(V_k) \rangle \psi_{\phi_i[\infty]} = (-1)^{m_{d_i} / 2}
\]
for \( i \neq i_0 \). In particular, if \( \psi_g \) is tempered, then we must have \( m_{d_i} d_i \equiv 0 \mod 4 \) for \( i \neq i_0 \). When \( n = 2 \), i.e., \( g \in S_{pK}(Sp_2(\mathbb{Z})) \), the \( A \)-parameter \( \psi_g \) does not of the form \( \tau[1] \oplus \chi[1] \) with \( \tau \) being an irreducible cuspidal symplectic automorphic representation of \( GL_4(\mathbb{A}) \). Using this fact, one can prove Ibukiya’s conjecture of Type II in Introduction ([?]) by a similar argument to the proof of Lifting Theorem (A).

Next, we prove Lifting Theorem (B).

Proof of Theorem 1.1 (B). Assume that \( k \geq 4 \) and \( j \geq 1 \). Let \( f \in S_{k,j}(Sp_2(\mathbb{Z})) \) be a Hecke eigenform, and \( \tau_f \) be the associated irreducible cuspidal symplectic automorphic representation of \( GL_4(\mathbb{A}) \) by Lemma 3.5. Take an integer \( d > 0 \) such that \( d < \min\{ (k/2) - 1, (j/2) + 1 \} \) and that \( k_i - i \notin [(j/2) - d + 1, (j/2) + k + d - 2] \) for each \( i = 1, \ldots, n \). The last condition implies that \( \tau_{f,i} \notin \tau_i \) for any \( 1 \leq i \leq r \). We set
\[
\psi_{f,g} = \tau_f[2d] \boxplus \psi_g.
\]
One can easily see that \( \psi_{f,g} \in \Psi_2(Spn_{+4d}/\mathbb{Q}) \). Moreover, since \( \psi_{f,g,p} \) is unramified for any (finite) prime \( p \), the local \( A \)-packet \( \Pi_{\psi_{f,g,p}} \) contains a unique unramified representation \( \pi_{f,g,p} \). In addition, since
\[
\psi_{f,g,\infty} \cong (\rho_{j+2k-3} \boxtimes S_{2d}) \oplus (\rho_{j+1} \boxtimes S_{2d}) \oplus \left( \bigoplus_{j=1}^{t} \rho_{\alpha_j} \boxtimes S_{d_j} \right) \oplus \text{sgn}^{n+4d}
\]
is Adams–Johnson, the local $A$-packet $\Pi_{\psi,g,\infty}$ contains $L(V_{k'})$, where $k' = (k'_1, \ldots, k'_{n+4d}) \in \mathbb{Z}^{n+4d}$ is given by $k'_1 \geq \cdots \geq k'_{n+4d}$ and
\[
\{ k'_1 - 1, k'_2 - 2, \ldots, k'_{n+4d} - n - 4d \} = \{ k_1 - 1, k_2 - 2, \ldots, k_n - n \} \cup \left\{ \frac{j}{2} + k + d - 2, \frac{j}{2} + k + d - 3, \ldots, \frac{j}{2} + k - d - 1 \right\} \cup \left\{ \frac{j}{2} + d, \frac{j}{2} + d - 1, \ldots, \frac{j}{2} - d + 1 \right\}.
\]

Let us consider
\[
\pi_{f,g} = \left( \bigotimes_{p < \infty} \pi_{f,g,p} \right) \otimes L(V_{k'}) \in \Pi_{\psi,g}.
\]

Using Arthur’s multiplicity formula (Theorem 2.3), we study when $\pi_{f,g}$ is automorphic. It is automorphic if and only if
\[
\varepsilon_{\psi,g}(\alpha_{\tau_0}[2d]) = \langle \Delta(\alpha_{\tau_0}[2d]), \pi_{f,g} \rangle_{\psi,g},
\]
and
\[
\varepsilon_{\psi,g}(\alpha_{\tau_0}[d_i]) = \langle \Delta(\alpha_{\tau_0}[d_i]), \pi_{f,g} \rangle_{\psi,g}
\]
for $i = 1, \ldots, r$.

Note that $\alpha_{i,\ell} \not\in \left[ j + 1, j + 2k - 3 \right]$ for any $(i, \ell)$ since $k_i - i \not\in \left[ (j/2) - d + 1, (j/2) + k + d - 2 \right]$ for any $i = 1, \ldots, n$. By a similar argument to the proof of Lifting Theorem (A), we have
\[
\varepsilon(\tau_i \times \tau_f) = \begin{cases} (-1)^{j+i} & \text{if } i = i_0, \\ 1 & \text{if } i \neq i_0. \end{cases}
\]

Hence we have $\varepsilon_{\psi,g}(\alpha_{\tau_0}[2d]) = (-1)^{j+i} = (-1)^{k}$ since $j$ is even. On the other hand,
\[
\langle \Delta(\alpha_{\tau_0}[2d]), \pi_{f,g} \rangle_{\psi,g} = \langle \alpha_{\rho_{j+2k-3} \otimes S_{2d}}, L(V_{k'}) \rangle_{\psi,g,\infty} \langle \alpha_{\rho_{j+1} \otimes S_{2d}}, L(V_{k'}) \rangle_{\psi,g,\infty} = 1.
\]

Hence the condition $\varepsilon_{\psi,g}(\alpha_{\tau_0}[2d]) = \langle \Delta(\alpha_{\tau_0}[2d]), \pi_{f,g} \rangle_{\psi,g}$ is equivalent that $k$ is even.

For $i \neq i_0$, since $\varepsilon(\tau_i \times \tau_f) = 1$, we have $\varepsilon_{\psi,g}(\alpha_{\tau_0}[d_i]) = \langle \Delta(\alpha_{\tau_0}[d_i]), \pi_{f,g} \rangle_{\psi,g}$. Therefore, this condition also holds for $i = i_0$ when $k$ is even.

In conclusion, $\pi_{f,g}$ is automorphic if and only if $k \equiv j \equiv 0 \bmod 2$. In this case, since $\pi_{f,g,\infty} \cong L(V_{k'})$ is discrete series, by a result of Wallach [23], we see that $\pi_{f,g}$ is cuspidal. The space $\pi_{f,g}^{K_n,p}$ gives a Hecke eigenform $F_{f,g} \in S_{\rho_{k'}}(\text{Sp}_{n+4d}(\mathbb{Z}))$ such that
\[
L(s, F_{f,g,\text{std}}) = L(s, g, \text{std}) \prod_{i=1}^{2d} L \left( s + d + \frac{1}{2} - i, f, \text{spin} \right).
\]

This completes the proof of Lifting Theorem (B) (Theorem 1.1 (B)).

\[\square\]

### 3.5. Arthur’s multiplicity formula for holomorphic cusp forms

Let $\mathbb{F}$ be a totally real field. For each infinite place $v \mid \infty$, we consider the standard maximal compact group $K_v \subset \text{Sp}_n(\mathbb{F}_v) = \text{Sp}_n(\mathbb{R})$ and the subalgebra $\mathfrak{p}_{\mathbb{F}}^v \subset \text{Lie(\text{Sp}_n(\mathbb{F}_v))} \otimes_{\mathbb{R}} \mathbb{C}$ defined in [33].

**Definition 3.7.** Let $k = (k_v)_{v} \in \prod_{v \mid \infty} \mathbb{Z}_{>0}$. A cusp form $\varphi: \text{Sp}_n(\mathbb{A}) \to \mathbb{C}$ is called holomorphic cusp form of weight $k$ if $k_v \in K_v$ and $g \in \text{Sp}_n(\mathbb{A})$;

1. $\varphi(gk_v) = \det(k_v)^k_v \varphi(g)$ for $k_v \in K_v$ and $g \in \text{Sp}_n(\mathbb{A})$;
2. $\mathfrak{p}_{\mathbb{F}}^v \varphi = 0$.

Here, we identify $K_v$ with $U(n)$, and we denote by $\det$ the determinant of $K_v \cong U(n)$. 
The space of holomorphic cuspidal forms of weight $k$ is denoted by $S_k(Sp_n(\mathbb{A}))$. This is an $Sp_n(\mathbb{A}_{\text{fin}})$-subrepresentation of $A^2(Sp_n(\mathbb{A}))$. Note that $g_{\infty}$ does not act on $S_k(Sp_n(\mathbb{A}))$.

**Example 3.8.** Suppose that $F = \mathbb{Q}$. Then the Siegel modular form $F \in S_k(Sp_n(\mathbb{Z}))$ gives a cuspidal form $\varphi_F \in A^2(Sp_n(\mathbb{A}))$. As explained in [3.4.1], $\varphi_F$ is a holomorphic cuspidal form of weight $k$.

For $k = (k_v) \in \prod_{v \mid \infty} \mathbb{Z}_{>0}$, we set $\Psi_2(Sp_n/F, D_k^{(n)})$ to be the subset of $\Psi_2(Sp_n/F)$ consisting of global $A$-parameters $\psi$ such that $D_k^{(n)} = L(V_{(k_v, \ldots, k_v)}) \subset \Pi_\psi$ for any $v | \infty$. For $\psi \in \Psi_2(Sp_n/F, D_k^{(n)})$, define

$$\Pi_{\psi}^{\text{fin}} = \left\{ \pi_{\text{fin}} = \bigotimes_{v < \infty}^{'} \pi_v \mid \pi_v \in \Pi_\psi, \langle \cdot, \pi_v \rangle_\psi = 1 \text{ for almost all } v \right\}.$$

Now we obtain Arthur's multiplicity formula for holomorphic cusps forms.

**Theorem 3.9.** For $k = (k_v) \in \prod_{v \mid \infty} \mathbb{Z}_{>0}$, we have

$$S_k(Sp_n(\mathbb{A})) \subseteq \bigoplus_{\psi \in \Psi_2(Sp_n/F, D_k^{(n)})} \bigoplus_{\pi_{\text{fin}} \in \Pi_{\psi}^{\text{fin}}} m_{\pi_{\text{fin}}, \psi} \pi_{\text{fin}}$$

as a representation of $Sp_n(\mathbb{A}_{\text{fin}})$, where the non-negative integer $m_{\pi_{\text{fin}}, \psi}$ is given by

$$m_{\pi_{\text{fin}}, \psi} = \begin{cases} 1 & \text{if } \langle \cdot, \pi \rangle_\psi \circ \Delta = \varepsilon_\psi, \\ 0 & \text{otherwise} \end{cases}$$

with setting $\pi = \pi_{\text{fin}} \otimes (\otimes_{v \mid \infty} D_k^{(n)} v)_{k_v}$ for $\pi_{\text{fin}} \in \Pi_{\psi}^{\text{fin}}$. In particular, the space $S_k(Sp_n(\mathbb{A}))$ is multiplicity-free as a representation of $Sp_n(\mathbb{A}_{\text{fin}})$. Moreover, when $k_v > n$ for any $v | \infty$, the inclusion is in fact an equality.

**Proof.** By Arthur’s multiplicity formula (Theorem 2.3) together with the multiplicity one results for $D_k^{(n)}$ in real local $A$-packets (Theorem 2.9, Proposition 3.3) and for the lowest weight vectors in $D_k^{(n)}$, the right hand side of the inclusion is the direct sum of the subspaces of lowest weight vectors in $\pi = \otimes_{v}^{'} \pi_v \subset A^2(Sp_n(\mathbb{A}))$ such that $\pi_v \cong D_k^{(n)}$ for each $v | \infty$. Hence we have the inclusion.

The integer $m_{\pi_{\text{fin}}, \psi} = m_{\pi, \psi}$ is given in Arthur’s multiplicity formula (Theorem 2.3). By the multiplicity one results for $p$-adic local $A$-packets (Theorem 2.4, Proposition 2.7), we see that the global $A$-packet $\Pi_{\psi}^{\text{fin}}$ is a (multiplicity-free) set. Hence the multiplicity of $\pi_{\text{fin}} \in \Pi_{\psi}^{\text{fin}}$ in $S_k(Sp_n(\mathbb{A}))$ is less than or equal to $m_{\pi_{\text{fin}}, \psi}$. In particular, $S_k(Sp_n(\mathbb{A}))$ is multiplicity-free.

When $k_v > n$ for any $v | \infty$, the automorphic representation $\pi = \pi_{\text{fin}} \otimes (\otimes_{v \mid \infty} D_k^{(n)} v)_{k_v} \subset A^2(Sp_n(\mathbb{A}))$ is cuspidal by Wallach [23]. Hence the inclusion is an equality in this case. \hfill \Box

We can now prove the strong multiplicity one theorem (Theorem 1.3). Let $F = \mathbb{Q}$.

**Proof of Theorem 1.3.** We consider the cuspidal automorphic representations $\pi_{F_1}, \pi_{F_2} \subset A^2(Sp_n(\mathbb{A}))$ generated by the cusp forms $\varphi_{F_1}$ and $\varphi_{F_2}$, respectively. By Lemma 3.4 and by the assumption, they are irreducible and are nearly equivalent to each other. Hence they belong to the same $A$-packet, say $\Pi_\psi$, by Corollary 2.8. In particular, for each place $v$, the local factors $\pi_{F_1,v}$ and $\pi_{F_2,v}$ are in $\Pi_\psi$.

When $v = p < \infty$, by Propositions 2.6 and 2.7, the unramified representations $\pi_{F_1,p}$ and $\pi_{F_2,p}$ are determined uniquely by $\psi_p$. In particular, $\pi_{F_1,p} \cong \pi_{F_2,p}$. When $v = \infty$, since $\{k_1, k_2\} \neq \{\frac{3}{2}, \frac{5}{2} + 1\}$, by Proposition 3.3 we have $k_1 = k_2$ so that $\pi_{F_1,\infty} \cong \pi_{F_2,\infty}$. Hence we conclude that $\pi_{F_1} \cong \pi_{F_2}$ as $Sp_n(\mathbb{A}_{\text{fin}}) \times (g_{\infty}, K_{\infty})$-modules.
By the multiplicity one result for $S_{k_1}(\text{Sp}_n(\mathbb{A}))$ in Theorem 3.9 we have $\pi_{F_1} = \pi_{F_2}$ as a subspace of $A^2(\text{Sp}_n(\mathbb{A}))$. Since both $\varphi_{F_1}$ and $\varphi_{F_2}$ are $\text{Sp}_n(\mathbb{Z})$-fixed lowest weight vectors in $\pi_{F_1} = \pi_{F_2}$, there exists a constant $c \in \mathbb{C}^\times$ such that $\varphi_{F_2} = c\varphi_{F_1}$. This implies that $F_2 = cF_1$. □

**Appendix A. The hypothesis Langlands group and Arthur’s character**

In §2.2 for each global $A$-parameter $\psi \in \Psi_2(\text{Sp}_n/\mathbb{F})$, we have defined Arthur’s character $\varepsilon_\psi$ of $A_\psi$. One might seem that our definition differs from Arthur’s original definition. In this appendix, we formally explain the original definition of Arthur’s character using the hypothesis Langlands group, and show that it agrees with our definition.

**A.1. Hypothesis Langlands group.** Let $\mathbb{F}$ be a number field and $W_\mathbb{F}$ be the Weil group of $\mathbb{F}$. We denote the set of irreducible unitary cuspidal automorphic representations of $\text{GL}_m(\mathbb{A})$ by $A_{\text{cusp}}(\text{GL}_m(\mathbb{A}))$. It is hoped that there exists a locally compact group $L_\mathbb{F}$ together with a surjection $L_\mathbb{F} \twoheadrightarrow W_\mathbb{F}$ such that it has a canonical bijection

$$A_{\text{cusp}}(\text{GL}_m(\mathbb{A})) \leftrightarrow \{m\text{-dimensional irreducible unitary representations of } L_\mathbb{F}\}.$$ 

The group $L_\mathbb{F}$ is called the hypothesis Langlands group of $\mathbb{F}$. Moreover, when $\tau \in A_{\text{cusp}}(\text{GL}_m(\mathbb{A}))$ corresponds to $\phi : L_\mathbb{F} \rightarrow \text{GL}_m(\mathbb{C})$, it is expected that $\tau$ is orthogonal (resp. symplectic) if and only if $\phi$ is orthogonal (resp. symplectic).

Using the hypothesis Langlands group $L_\mathbb{F}$, the global Langlands conjecture roughly states that there is a canonical surjection

$$\{\text{irreducible automorphic representations of } \text{Sp}_n(\mathbb{A})\} \twoheadrightarrow \{\psi : L_\mathbb{F} \times \text{SL}_2(\mathbb{C}) \rightarrow \text{SO}_{2n+1}(\mathbb{C})\}.$$ 

Here we consider a semisimple representation $\psi : L_\mathbb{F} \times \text{SL}_2(\mathbb{C}) \rightarrow \text{SO}_{2n+1}(\mathbb{C})$ so that $\psi$ decomposes into a direct sum

$$\psi = (\phi_1 \boxtimes S_{d_1}) \oplus \cdots \oplus (\phi_r \boxtimes S_{d_r})$$

for some irreducible representations $\phi_1, \ldots, \phi_r$ of $L_\mathbb{F}$. Replacing $\phi_i$ as corresponding $\tau_i \in A_{\text{cusp}}(\text{GL}_{m_i}(\mathbb{A}))$, we obtain the notion of global $A$-parameters (see §2.2).

**A.2. Component groups.** Let $\psi = \oplus_{i=1}^r \phi_i \boxtimes S_{d_i} : L_\mathbb{F} \times \text{SL}_2(\mathbb{C}) \rightarrow \text{SO}_{2n+1}(\mathbb{C})$. Set $S_\psi$ and $S_\psi^+$ to be

$$S_\psi = \text{Cent}(\text{Im}(\psi), \text{SO}_{2n+1}(\mathbb{C})), \quad S_\psi^+ = \text{Cent}(\text{Im}(\psi), \text{O}_{2n+1}(\mathbb{C})), $$

respectively. We define the component group $S_\psi$ and $S_\psi^+$ by

$$S_\psi = S_\psi^+/S_\psi^+ \circ (S_\psi^+) \circ ,$$

respectively. We note that $S_\psi^+ = S_\psi \times \{\pm 1_{2n+1}\}$ so that $S_\psi^+ = S_\psi \times \{\pm 1_{2n+1}\}$.

Now we assume that $\psi$ is discrete. This means that $\psi$ is a multiplicity-free sum of irreducible orthogonal representations. Then the image of $\phi_i \boxtimes S_{d_i}$ is contained in an orthogonal group $\text{O}_{m_i d_i}(\mathbb{C})$, and we have

$$S_\psi^+ \cong S_\psi^+ = \{\pm 1_{m_1} d_1\} \times \cdots \times \{\pm 1_{m_r} d_r\} \cong \{\pm 1\}^r.$$ 

By this observation, we obtain the definition of $A_\psi$. There exists a canonical bijection

$$A_\psi^+ \leftrightarrow S_\psi^+, \quad \alpha_{r[d]} \leftrightarrow -1_{m[d]}.$$ 

In particular, the central element $z_\psi \in A_\psi$ corresponds to $-1_{2n+1} \in S_\psi^+$, which is the center of $\text{O}_{2n+1}(\mathbb{C})$. 
A.3. Formal definition of Arthur’s characters. Now we explain the definition of Arthur’s character formally. Let \( \psi = \oplus_{i=1}^r \phi_i \boxtimes S_{d_i} : \mathcal{L}_F \times \text{SL}_2(\mathbb{C}) \to \text{SO}_{2n+1}(\mathbb{C}) \) be discrete. Define a representation

\[
\text{Ad}_\psi : S_\psi \times \mathcal{L}_F \times \text{SL}_2(\mathbb{C}) \to \text{GL}(\mathfrak{s}\mathfrak{o}_{2n+1}(\mathbb{C}))
\]
on the Lie algebra \( \mathfrak{s}\mathfrak{o}_{2n+1}(\mathbb{C}) \) of \( \text{SO}_{2n+1}(\mathbb{C}) \) by setting

\[
\text{Ad}_\psi(s, g, h) = \text{Ad}(s \cdot \psi(g, h))
\]
for \( s \in S_\psi \) and \( (g, h) \in \mathcal{L}_F \times \text{SL}_2(\mathbb{C}) \), where \( \text{Ad} \) is the adjoint representation of \( \text{SO}_{2n+1}(\mathbb{C}) \). Since \( \text{Ad} \) is invariant under Killing form on \( \mathfrak{s}\mathfrak{o}_{2n+1}(\mathbb{C}) \), this representation is orthogonal. We decompose

\[
\text{Ad}_\psi = \bigoplus_{\alpha} (\eta_\alpha \boxtimes \phi_\alpha \boxtimes S_{d_\alpha}),
\]
where \( \eta_\alpha \) and \( \phi_\alpha \) are irreducible representations of \( S_\psi \) and \( \mathcal{L}_F \), respectively. After stating [3, Theorem 1.5.2], Arthur defined a character \( \varepsilon_\psi \) of \( S_\psi = S_\psi = S_\psi \) by

\[
\varepsilon_\psi(s) = \prod_{\alpha} \varepsilon_\alpha(s), \quad s \in S_\psi,
\]
where \( \prod_\alpha \) denotes the product over those indices \( \alpha \) such that \( \phi_\alpha \) is symplectic and \( \varepsilon(1/2, \phi_\alpha) = -1 \).

Here, when \( \phi_\alpha \leftrightarrow \tau_\alpha \in \mathcal{A}_{\text{cusp}}(\text{GL}_{m_\alpha}(\mathbb{A})) \), we mean that \( \varepsilon(1/2, \phi_\alpha) = \varepsilon(1/2, \tau_\alpha) \). We extend \( \varepsilon_\psi \) to \( S_\psi^+ = S_\psi^+ \) by setting \( \varepsilon_\psi(-1_{2n+1}) = 1 \).

**Proposition A.1.** Let \( \psi = \oplus_{i=1}^r \phi_i \boxtimes S_{d_i} : \mathcal{L}_F \times \text{SL}_2(\mathbb{C}) \to \text{SO}_{2n+1}(\mathbb{C}) \) be discrete, where \( \phi_i \) is an irreducible representation of \( \mathcal{L}_F \) corresponding to \( \tau_i \in \mathcal{A}_{\text{cusp}}(\text{GL}_{m_i}(\mathbb{A})) \). Then we have

\[
\varepsilon_\psi(-1_{m_i d_i}) = \prod_{j \neq i} \varepsilon\left(\frac{1}{2}, \tau_i \times \tau_j\right)^{\min\{d_i, d_j\}}.
\]

**Proof.** We decompose \( \text{Ad}_\psi = \oplus_\alpha (\eta_\alpha \boxtimes \phi_\alpha \boxtimes S_{d_\alpha}) \) as above. Note that \( \phi_\alpha \) is symplectic if and only if \( d_\alpha \) is even since \( \eta_\alpha \) is a quadratic character. Hence, for \( s \in S_\psi \),

\[
\varepsilon_\psi(s) = \prod_{\alpha} \varepsilon_\alpha(s) = \prod_{\alpha} \varepsilon\left(\frac{1}{2}, \phi_\alpha\right),
\]
where \( \prod_\alpha \) denotes the product over those indices \( \alpha \) such that \( d_\alpha \) is even and \( \eta_\alpha(s) = -1 \). This means that when we compute \( \varepsilon_\psi(s) \), we only consider the \((-1)\)-eigenspace of \( \text{Ad}(s) \), which is a subrepresentation of \( \text{Ad} \circ \psi = \oplus_\alpha (\phi_\alpha \boxtimes S_{d_\alpha}) \).

Let \( I \) be a subset of \( \{1, \ldots, r\} \) such that the sum \( \sum_{i \in I} m_i d_i \) is even. Set \( s_I = \prod_{i \in I} -1_{m_i d_i} \in S_\psi \). Namely \( s_I \) acts on \( \phi_i \boxtimes S_{d_i} \) by \(-1\) if \( i \in I \), and by \( 1 \) if \( i \notin I \). Note that any element in \( S_\psi \) is of this form.

Now we compute \( \varepsilon_\psi(s_I) \). First note that

\[
\text{Ad} \circ \psi \cong \bigoplus_i \text{Ad}(\phi_i \boxtimes S_{d_i}) \oplus \bigoplus_{i < j} (\phi_i \otimes \phi_j) \boxtimes (S_{d_i} \otimes S_{d_j}).
\]

We see that \( \text{Ad}(s_j) \) preserves each summands, and

- \( \text{Ad}(s_I) \) acts on \( \text{Ad}(\phi_i \boxtimes S_{d_i}) \) by \( 1 \);
- \( \text{Ad}(s_I) \) acts on \( (\phi_i \otimes \phi_j) \boxtimes (S_{d_i} \otimes S_{d_j}) \) by \(-1\) if and only if exactly one of \( i \) and \( j \) belongs to \( I \).
On the other hand,

\[ S_{d_i} \otimes S_{d_j} \cong S_{d_i+d_j-1} \oplus S_{d_i+d_j-3} \oplus \cdots \oplus S_{|d_i-d_j|+1}. \]

In particular, \( S_{d_i} \otimes S_{d_j} \) has \( \min\{d_i, d_j\} \) irreducible summands, which are of the form \( S_{d_\alpha} \) such that \( d_\alpha \equiv d_i + d_j - 1 \mod 2 \). Therefore, by the multiplicativity of \( \varepsilon \)-factors, we have

\[
\varepsilon_{\psi}(s_I) = \prod_{i<j} \varepsilon \left( \frac{1}{2}, \tau_i \times \tau_j \right)^{\min\{d_i, d_j\}},
\]

where \( \prod_{i<j} \) denotes the product over those pairs of indices \( (i, j) \) such that \( i < j, d_i \neq d_j \mod 2 \), and exactly one of \( i \) and \( j \) belongs to \( I \). However, [3, Theorem 1.5.3] states that \( \varepsilon(1/2, \tau_i \otimes \tau_j) = 1 \) if \( d_i \equiv d_j \mod 2 \). Hence one can remove the condition \( d_i \neq d_j \mod 2 \) in the definition of \( \prod_{i<j} \). In particular, when \( I = \{i\} \) (which implies that \( m_i d_i \) is even), we obtain

\[
\varepsilon_{\psi}(-1_{m_i d_i}) = \prod_{j \neq i} \varepsilon \left( \frac{1}{2}, \tau_i \times \tau_j \right)^{\min\{d_i, d_j\}}.
\]

Now suppose that \( m_i d_i \) is odd. Then we can consider \( I = \{1, \ldots, r\} \setminus \{i\} \). Since we define \( \varepsilon_{\psi}(-1_{2n+1}) = 1 \), we have

\[
\varepsilon_{\psi}(-1_{m_i d_i}) = \varepsilon_{\psi}(s_I) = \prod_{j \neq i} \varepsilon \left( \frac{1}{2}, \tau_i \times \tau_j \right)^{\min\{d_i, d_j\}}.
\]

This completes the proof. \( \square \)

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Department of Mathematics, Hokkaido University Kita 10, Nishi 8, Kita-Ku, Sapporo, Hokkaido, 060-0810, Japan

E-mail address: atobe@math.sci.hokudai.ac.jp