On Algebraic Functions

Nikos Bagis
Stenimahou 5 Edessa
Pella 58200, Greece
bagkis@hotmail.com

Abstract

In this note we consider functions with Moebius-periodic rational coefficients. These functions under some conditions take algebraic values and can be recovered by theta functions and the Dedekind eta function. Special cases are the elliptic singular moduli, the Rogers-Ramanujan continued fraction, Eisenstein series and functions associated with Jacobi symbol coefficients.

Keywords: Theta functions; Algebraic functions; Special functions; Periodicity;

1 The elementary algebraic functions

The elliptic singular moduli \( k_r \) is the solution \( x \) of the equation

\[
\frac{2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; -x^2 \right)}{2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; x^2 \right)} = \sqrt{r} \tag{1}
\]

where

\[
2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; x^2 \right) = \sum_{n=0}^{\infty} \frac{\left( \frac{1}{2} \right)^2}{(n!)^2} x^{2n} = \frac{2}{\pi} K(x) = \int_{0}^{\pi/2} \frac{d\phi}{\sqrt{1 - x^2 \sin^2(\phi)}} \tag{2}
\]

The 5th degree modular equation which connects \( k_{25r} \) and \( k_r \) is (see [13]):

\[
k_r k_{25r} + k'_r k'_{25r} + 2^{5/3} (k_r k_{25r} k'_r k'_{25r})^{1/3} = 1 \tag{3}
\]

The problem of solving (3) and finding \( k_{25r} \) reduces to solve the depressed equation after named by Hermite (see [3]):

\[
u^6 - v^6 + 5u^2v^2(u^2 - v^2) + 4uv(1 - u^4v^4) = 0 \tag{4}
\]

where \( u = k_r^{1/4} \) and \( v = k_{25r}^{1/4} \).

The function \( k_r \) is also connected to theta functions by the relations

\[
k_r = \frac{\theta_2(q)^2}{\theta_3(q)^3}, \text{ where } \theta_2(q) = \theta_2 = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} \text{ and } \theta_3(q) = \theta_3 = \sum_{n=-\infty}^{\infty} q^{n^2} \tag{5}
\]
\( q = e^{-\pi \sqrt{r}} \).

Hence one can understand that we may call the solution of the depressed equation with

\[ k_{25r} = \frac{\theta_2^2(q^5)}{\theta_3^2(q^5)} \]  

(6)

But this is not satisfactory.

For example in the case of \( \pi \) formulas of Ramanujan, one has to know from the exact value of \( k_r \) the exact value of \( k_{25r} \) in radicals. Here we must mention the concept that when \( r \) is positive rational then the value of \( k_r \) is algebraic number.

Another example is the Rogers-Ramanujan continued fraction (RR CF) which is

\[ R(q) = \frac{q^{1/5}}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \ldots}}}} \]  

(7)

(see [4],[5],[6],[8],[9],[10],[13],[14],[16],[21]), the value of which depends form the depressed equation.

By finding the value of (RRCF) we can find the value of \( j \)-invariant from Klein’s equation (see [19],[8] and Wolfram pages 'Rogers Ramanujan Continued fraction'):

\[ j_r = -\frac{(R^{20} - 228R^{15} + 494R^{10} + 228R^5 + 1)^3}{R^5(R^{10} + 11R^5 - 1)^5}, \text{ where } R = R(q^2) \]  

(8)

One can also prove that Klein’s equation (8) is equivalent to depressed equation (4).

Using the 5th degree modular equation of Ramanujan

\[ R(q^{1/5})^5 = R(q) \frac{1 - 2R(q) + 4R(q)^2 - 3R(q)^3 + R(q)^4}{1 + 3R(q) + 4R(q)^2 + 2R(q)^3 + R(q)^4} \]  

(9)

we can find form (8) the value of \( j_r/25 \) and hence from the relation

\[ j_r = \frac{256(k_r^2 + k_r'^4)^3}{(k_r k_r')^4}. \]  

(10)

\( k_r/25 \). This is done in [7]. Knowing \( k_r \) and \( k_{r/25} \), we have evaluated \( k_{25r} \) and give relations of the form

\[ k_{25r} = \Phi(k_r, k_{r/25}) \text{ and } k_{25r} = \Phi_n(k_r, k_{r/25}), n \in \mathbb{N} \]  

(11)

Hence when we know the value of \( R(q) \) in radicals we can find \( k_r \) and \( k_{25r} \) in radicals and the opposite.

Also in [3] and Wikipedia 'Bring Radical' one can see how the depressed equation can be used for the extraction of the solution of quintic equation

\[ ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0 \]  

(12)
The above equation can be solved exactly with theta functions and in some cases in radicals. The same holds and with the sextic equation (see [8])

\[ \frac{b^2}{20a} + bY + aY^2 = cY^{5/3} \]  

(13)

which have solution

\[ Y = Y_r = \frac{b}{250a} \left( R(q^2)^{-5} - 11 - R(q^2)^{3/5} \right), \quad q = e^{-\pi\sqrt{r}}, \quad r > 0 \]  

(14)

and \( r \) is related with the constants by the relation \( j_r = 250 \frac{a^2}{a^2}, \) in order to generate the solution.

The Ramanujan-Dedekind eta function is defined as

\[ \eta(\tau) = \prod_{n=1}^{\infty} (1 - q^n), \quad q = e^{\pi\tau}, \quad \tau = \sqrt{-r} \]  

(15)

The \( j \)-invariant can evaluated in terms of Ramanujan-Dedekind eta function as

\[ j_r = \left[ \left( q^{-1/24} \frac{\eta(\tau)}{\eta(2\tau)} \right)^{16} + 16 \left( q^{1/24} \frac{\eta(2\tau)}{\eta(\tau)} \right)^8 \right]^3 \]  

(16)

and hold the following evaluation (see [11],[22])

\[ \eta(\tau)^8 = \frac{2^{8/3}}{\pi^4} q^{-1/3} (k_r)^{2/3} (k'_r)^{8/3} K(k_r) K(k'_r) \]  

(17)

In this article we examine the general case of algebraic functions. Numerical evaluations of such functions of Main Theorem bellow are given with the Program Mathematica and the routine 'Recognize'. By this method these exact values coming from the middle of nowhere, however they still remaining conjectures.

It is great challenge to find the polynomials and modular equations of these Moebius-coefficients functions and united them with a general theory. Many various scientists are working for special functions (mentioned above) such the singular moduli (\( j \)-invariant) and the related to them Hilbert polynomials, as also for (RRCF) and theta functions and other similar functions.

In [7] is presented a way to evaluate the fifth singular moduli and the Rogers-Ramanujan continued fraction with the function \( w_r = \sqrt{k_rk'_{25r}}. \) This function can replace the classical singular moduli in the case of Rogers-Ramanujan continued fraction and Klein’s invariant. Since scientists cover the gaps of the ”elementary” algebraic functions a general theory will needed. In [5] and [11] attempts are made. Hence we concern to construct a theory of such functions and characterize them.

We begin by giving a definition and a conjecture which will help us for the proof of the main theorem.
Definition 1.
Let $a, p$ be positive rational numbers with $a < p$ and $q = e^{-\pi \sqrt{r}}$, $r > 0$. We call "agile" the quantity

$$\[a, p; q\] := \prod_{n=0}^{\infty} (1 - q^{pn+a})(1 - q^{pn+p-a})$$  \hspace{1cm} (18)

The "agiles" have the following very interesting conjecture-property.

Conjecture.
If $q = e^{-\pi \sqrt{r}}$, $r$ is positive rational and $a, b$ are positive rationals then

$$[a, p; q]^* := q^{p/12 - a/2 + a^2/(2p)}[a, p; q] = \text{Algebraic Number}$$  \hspace{1cm} (19)

Assuming the above unproved property we will show the following

Main Theorem.
Let $f$ be a function analytic in $(-1,1)$. Set by Moebius theorem bellow $X(n)$ to be

$$X(n) = \frac{1}{n} \sum_{d|n} \frac{f(d)(0)}{\Gamma(d)} \mu \left( \frac{n}{d}\right)$$  \hspace{1cm} (20)

If $X(n)$ is $T$ periodic sequence taking rational values and catoptric in the every period-interval i.e. if for every $n \in \mathbb{N}$ is $a_k = X(k + nT) = X(k)$ with $a_T = 0$ and we have $a_1 = a_{T-1}, a_2 = a_{T-2}, \ldots, a(T-1)/2 = a(T+1)/2$, then exist rational number $A$, such that

$$q^A e^{-f(q)} = \text{Algebraic Number},$$  \hspace{1cm} (21)

where $q = e^{-\pi \sqrt{r}}$, $r$ positive rational.

The number $A$ is given from

$$A = \sum_{j=1}^{T+1} \left( \frac{j}{2} + \frac{j^2}{2T} + \frac{T}{12} \right) X(j)$$  \hspace{1cm} (22)

For to prove the Main Theorem we will use the next known (see [2]) Moebius inversion Theorem

Theorem. (Moebius inversion Theorem)
If $f(n)$ and $g(n)$ are arbitrary arithmetic functions, then

$$\sum_{d|n} f(d) = g(n) \Leftrightarrow f(n) = \sum_{d|n} g(d) \mu \left( \frac{n}{d}\right)$$  \hspace{1cm} (23)
The Moebius Mu function $\mu$ is defined as $\mu(n) = 0$ if $n$ is not square free, and $(-1)^r$ if $n$ have $r$ distinct primes.

Hence some values are $\mu(1) = 1$, $\mu(3) = -1$, $\mu(15) = 1$, $\mu(12) = 0$, etc.

For to prove the Main Theorem we will use the next

**Lemma 1.**

If $|x| < 1$

$$\log \left( \prod_{n=1}^{\infty} (1 - x^n)^{X(n)} \right) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \sum_{d|n} X(d)d$$

(24)

**Proof.**

It is $|x| < 1$, hence

$$\log \left( \prod_{n=1}^{\infty} (1 - x^n)^{X(n)} \right) = \sum_{n=1}^{\infty} X(n) \log (1 - x^n) =$$

$$= -\sum_{n=1}^{\infty} X(n) \sum_{m=1}^{\infty} \frac{x^{mn}}{m} = -\sum_{n,m=1}^{\infty} \frac{x^{nm}}{nm} X(m)m = -\sum_{n=1}^{\infty} \frac{x^n}{n} \sum_{d|n} X(d)d.$$

**Proof of Main Theorem.**

From Taylor expansion theorem we have

$$e^{-f(x)} = \exp \left( -\sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \right)$$

From the Moebius inversion theorem exists $X(n)$ such that

$$X(n) = \frac{1}{n} \sum_{d|n} \frac{f^{(d)}(0)}{\Gamma(d)} \mu \left( \frac{n}{d} \right)$$

or

$$\frac{f^{(n)}(0)}{n!} = \frac{1}{n} \sum_{d|n} X(d)d$$

Hence from Lemma 1

$$e^{-f(x)} = \exp \left( -\sum_{n=1}^{\infty} \frac{x^n}{n} \sum_{d|n} X(d)d \right) = \prod_{n=1}^{\infty} (1 - x^n)^{X(n)}$$

and consequently because of the periodicity and the catoptric property of $X(n)$, we get

$$e^{-f(x)} = \prod_{j=1}^{\left[ \frac{T-1}{x} \right]} [j, T; x]^{X(j)}$$

(25)
which is a finite product of "agiles" and from the Conjecture exist $A$ rational such that (21) hold, provided that $x = q^{-\pi\sqrt{r}}$ and $r$ positive rational.

**Examples.**
1) For $X(n) = \left(\frac{\pi}{n}\right)$, where $G = 2^m g_1^{m_1} g_2^{m_2} \cdots g_s^{m_s}$, $m, m_1, \ldots, m_s$ not negative integers, $m \neq 1$ and $g_1 < g_2 < \ldots < g_s$ primes of the form $1 \mod 4$, then exist $A$ rational such that

$$q^C \prod_{n=1}^{\infty} (1 - q^n)^{\left(\frac{\pi}{n}\right)} = q^{\frac{G-1}{2}} \prod_{j=1}^{[j, G, q]} X(j) = \text{Algebraic}$$

when $q = e^{-\pi\sqrt{r}}$, $r$ positive rational.

A special case is $G = 5$ which gives the Rogers-Ramanujan continued fraction. More precisely

$$q^{1/5} \prod_{n=1}^{\infty} (1 - q^n)^{\left(\frac{\pi}{5}\right)} = R(q) = q^{1/5} \frac{1}{1 + \frac{q}{1 + \frac{q^2}{1 + \ldots}}}$$

2) If $X = \{1, 1, 0, 1, 1, 0, \ldots\}$, then $T = 3$ and $A = -\frac{1}{12}$ we get that if $q = e^{-\pi}$, then

$$q^{-1/12} e^{-f(q)} = \sqrt[12]{81 \left(885 + 511\sqrt{3} - 3\sqrt{174033} + 100478\sqrt{3}\right)}$$

3) If $X = \{1, 1, 1, 0, 1, 1, 1, 0, 1, 1, 0, \ldots\}$, then $T = 5$ and $A = -\frac{1}{6}$ we get that

i) If $q = e^{-\pi\sqrt{2}}$, then $q^{-1/6} e^{-f(q)}$, is root of

$$3125 + 250v^6 - 20v^{10} + v^{12} = 0$$

We can solve the above equation observing that is of the form (13)

$$3125 + 250Y_r^6 + Y_r^{12} = j_r^{1/3} Y_r^{10} \quad \text{(eq)}$$

where $j_r$ is the $j$-invariant, hence

$$q^{-1/6} e^{-f(q)} = \sqrt[6]{Y_{1/2}}$$

then see [21]:

$$R(e^{-2\pi\sqrt{2}}) = \frac{\sqrt{5(g + 1) + 2g\sqrt{5} - \sqrt{5}g - 1}}{2}$$

where

$$(g^3 - g^2)/(g + 1) = (\sqrt{5} + 1)/2$$
One can use the duplication formula of RRCF (see [16]) to find \( R(e^{-\pi\sqrt{2}}) \) in radicals and hence the value of \( Y_{1/2} \) in radicals.

ii) If \( q = e^{-2\pi} \), then

\[
q^{-1/6}e^{-f(q)} = \sqrt[6]{Y_1} = \sqrt{\frac{5}{2} + \frac{5\sqrt{5}}{2}}
\]

...etc

If \( q = e^{-\pi\sqrt{r}} \)

\[
q^{-1/6}e^{-f(q)} = \sqrt[6]{Y_{r/4}}
\]

**Note.** The Rogers-Ramanujan continued fraction is \( X = \{1, -1, -1, 1, 0, ...\} \) and evaluations can be given.

2 The representation of \( e^f(q) \)

We will show in Theorem 1 bellow the representation of a function \( f \) in terms of known functions.

For \(|q| < 1\) the Jacobi theta functions are

\[
\vartheta(a, b; q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{an^2+bn}
\]  

One can evaluate the agiles by the theta function

\[
M(c, q) := \sum_{n=0}^{\infty} c^n q^{n(n+1)/2} = \frac{1 - cq - c(q - q^2) - cq^3 - c(q^4 - q^2)}{1 + 1 + 1 + 1 + 1 + ...}
\]  

then from [11] a way to express the agiles is

\[
[a, \rho; q] = \frac{M(-q^{-a}, q^p) - q^a M(-q^a, q^p)}{\eta(\rho)}
\]  

however we shall use the general theta functions evaluation see relation (32) bellow for more concentrated forms. The reader can change from one form to the other.

A first result in the agiles also given in [11] was the evaluation of the duplication formula

\[
\frac{[a, \rho; q^2]^*}{[a, \rho; q]^*} = \tau^* (a, \rho; q)
\]  

**Proposition 1.**

If \( a, b \) are positive reals and \( n \) integer, then

\[
\tau^* (a, \rho; q) = \tau^* (np \pm a, \rho; q)
\]  

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**Theorem 1.**
For every $f$ analytic in $(-1,1)$, which
$$X(n) = \frac{1}{n} \sum_{d | n} \frac{f^{(d)}(0)}{\Gamma(d)} \mu \left( \frac{n}{d} \right)$$
is periodic-symmetric with period $T$ (Möbius periodic) and real-valued, then hold
$$e^{-f(q)} = \eta(T) \sum_{j=1}^{\lfloor \frac{T-1}{2} \rfloor} X(j) \prod_{j=1}^{\lfloor \frac{T-1}{2} \rfloor} \vartheta \left( \frac{T}{2}, \frac{T-2j}{2}, q \right) X(j)$$
for every $|q| < 1$. In case that $X(j)$ are rational, then $q^A e^{-f(q)}$ is algebraic.

**Proof.**
Use the expansion found in [11]:
$$[a, p; q] = \frac{1}{\eta(pr)} \sum_{n=-\infty}^{\infty} (-1)^n q^{pn^2/2 + (p-2a)n/2} = \frac{1}{\eta(pr)} \vartheta \left( \frac{p}{2}, \frac{p-2a}{2}, q \right)$$
along with relation (25).

**Theorem 2.**
If $X(n)$ is real $T$-periodic and catoptric, then
$$\sum_{n=1}^{\infty} \frac{nX(n)q^n}{1-q^n} = -q \frac{d}{dq} \log \left( \eta(T) \sum_{j=1}^{\lfloor \frac{T-1}{2} \rfloor} X(j) \prod_{j=1}^{\lfloor \frac{T-1}{2} \rfloor} \vartheta \left( \frac{T}{2}, \frac{T-2j}{2}, q \right) X(j) \right)$$

**Proof.**
If
$$X(n) = \frac{1}{n} \sum_{d | n} \frac{f^{(d)}(0)}{\Gamma(d)} \mu(n/d)$$
then it holds
$$\int \frac{1}{q} \sum_{n=1}^{\infty} \frac{nX(n)q^n}{1-q^n} dq = f(q)$$
From Theorem 1 we get the result.

**Remark.**
If $R(a, b, p; q) = q^{C[a,p,q]} [b,p,q]$ denotes a Ramanujan quantity (see [5]), then we have the next closed form evaluation, with theta functions
$$\frac{R'(X_p, q)}{R(X_p, q)} = C + \frac{d}{dq} \log \left( \prod_{j=1}^{\lfloor \frac{p-1}{2} \rfloor} \vartheta \left( \frac{p}{2}, \frac{p-2j}{2}, q \right) X_p(j) \right)$$
by using (33).

**Example.**

4) The Jacobi symbol \( \left( \frac{a}{b} \right) \) is 5 periodic and symmetric, hence

\[
\sum_{n=1}^{\infty} \left( \frac{n}{5} \right) \frac{\eta^n}{1 - q^n} = -q \frac{d}{dq} \log \left( \frac{\vartheta(5/2, 3/2; q)}{\vartheta(5/2, 1/2; q)} \right) = -q \frac{d}{dq} \log (q^{-1/5} R(q)).
\]  

(35)

From Example 3 and the above one can prove that

\[
\sum_{n=1}^{\infty} \frac{\eta^n}{1 - q^n} = 5 \sum_{n=1}^{\infty} \frac{\eta^n}{1 - q^n} = -q \frac{d}{dq} \log \left( \frac{\vartheta(5/2, 3/2; q)}{\vartheta(5/2, 1/2; q)} \right)
\]  

(36)

and

\[
\frac{1}{6} + \sum_{n=1}^{\infty} \frac{\eta^n}{1 - q^n} = \sum_{n=1}^{\infty} \frac{\eta^n}{1 - q^n} = -q \frac{d}{dq} \log (Y(\sqrt{q}))
\]  

(37)

In view of [5] relation (92) and the expansion of \( L_1(q) \) in the same paper, we get

\[
-\frac{q}{6} \frac{d}{dq} \log \left( Y \left( q^{1/2} \right) \right) = -\frac{q^{1/2}}{12} Y' \left( q^{1/2} \right) = -\frac{1}{6} \frac{K[r]^2}{\pi^2 \sqrt{r}} + a(r) \frac{K[r]^2}{\pi^2 \sqrt{r}} + 5K[25r]^2 - \frac{a(25r)K[25r]^2}{\pi^2 \sqrt{r}}
\]  

(38)

where \( a(r) \) is the elliptic alpha function (see [17]) and \( K[r] = K(k_r) \) is the complete elliptic integral of the first kind at singular values. Hence for certain \( r \) we can find special values of \( Y' \left( e^{-\pi \sqrt{r}} \right) \).

From (36) and (37) we get the following evaluation for the theta function

\[
\theta = \frac{\vartheta(1, 5/3; q) \vartheta(3, 5/3; q)^6}{q^2 \eta(10/3)^{12}} = (R(q^2)^{-5} - 11 - R(q^2)^5)
\]  

(39)

and in view of [9] we get the following, similar to inverse elliptic nome theorem

**Theorem 3.**

\[
-\frac{1}{5} \int_{+\infty}^{\theta} \frac{dt}{t^{1/6} \sqrt{125 + 22t + t^2}} = \frac{1}{5 \sqrt{5}} B(k_4r, 1/6, 2/3)
\]  

(40)

and

\[
\theta = H(k_4r)
\]  

(41)

where

\[
-\frac{1}{5} \int_{+\infty}^{G(x)} \frac{dt}{t^{1/6} \sqrt{125 + 22t + t^2}} = x \quad \text{and} \quad H(x) = G \left( B \left( \frac{1}{5}, \frac{2}{3} \right) \right)
\]  

(42)
also
\[ \frac{d}{dr} B(k_r^2, 1/6, 2/3) = -\frac{\pi}{2} \sqrt{r} q^{1/6} \eta(\tau)^4 \]
and \( \theta^{1/6} \) is root of (eq).

For example with \( r = 1/5 \), then \( \theta = 5\sqrt{5} \) and
\[ k_{4/5} = \frac{2 - \sqrt{2 - 4\sqrt{2 + 5}}}{2 + \sqrt{2 - 4\sqrt{2 + 5}}} \]
\[ 5\sqrt{5} = H \left( \frac{2 - \sqrt{2 - 4\sqrt{2 + 5}}}{2 + \sqrt{2 - 4\sqrt{2 + 5}}} \right). \]

In general if \( X(n) = \left( \frac{n}{m} \right) \) have period \( p \) and \( G_0 = 2^{m_0} p_1^{m_1} p_2^{m_2} \ldots p_s^{m_s} \), with \( p_j \)-primes of the form \( 1 (mod 4) \), \( m_s, s, j = 0, 1, 2, \ldots \) and \( m_0 \neq 1 \), then
\[ \sum_{n=1}^{\infty} \left( \frac{n}{G_0} \right) \frac{nq^n}{1 - q^n} = -q \frac{d}{dq} \log \left( \prod_{j=1}^{n} \eta \left( \frac{G_0 - 2j}{2} ; q \right) \right) \]
(44)

Also

**Conjecture 2.**

If \( g \) is perfect square and \( p_1, p_2, \ldots, p_\lambda \) are all the prime factors of \( g \), then
\[ \prod_{n=1}^{\infty} (1 - q^n)^{\left( \frac{g}{2n} \right)} = \eta(\tau) \prod_{i=1}^{\lambda} \eta(p_i \tau)^{-1} \prod_{i < j} \eta(p_i p_j \tau)^{-1} \prod_{i < j < k} \eta(p_i p_j p_k \tau)^{-1} \ldots \]
(45)

and
\[ \prod_{n=1}^{\infty} (1 - q^n)^{-\left( \frac{g}{2n} \right)} \frac{d}{dq} \prod_{n=1}^{\infty} (1 - q^n)^{\left( \frac{g}{2n} \right)} = \sum_{n=1}^{\infty} \left( \frac{n}{g} \right) \frac{nq^n}{1 - q^n} = \]
\[ = -q \left[ \sum_{j=1}^{\infty} \frac{g-2j}{2} \prod_{j=1}^{\infty} \eta \left( \frac{g-2j}{2} ; q \right) \right] = \]
\[ = -q^{-1} \left[ L(q) - \sum_{i=1}^{\lambda} p_i L(q^{p_i}) + \sum_{i<j} p_i p_j L(q^{p_i p_j}) - \ldots \right] \]
where \( \eta(n \tau) \) and \( L(q^n) \) can evaluated explicity from (17) and
\[ L(q) = \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n} = 1 - \frac{3\pi}{2\sqrt{r}} (1 - 2k_r^2) K(k_r) + 6 \left( \frac{\alpha(r)}{\sqrt{r}} - k_r^2 \right) K(k_r)^2 \]
(46)
Another interesting case is when \( A_n = \sum_{d|n} a_d \mu(n/d) \) and \( a_n \) is \( T \)-periodic, then

\[
\sum_{n=1}^{\infty} \frac{A_n q^n}{1 - q^n} = \sum_{n=1}^{\infty} A_n q^n = \sum_{n=1}^{\infty} a_n q^n = (1 - q^T) \sum_{n=1}^{T} A_d q^n \tag{47}
\]

hence the series

\[
\sum_{n=1}^{\infty} \frac{A_n q^n}{1 - q^n}
\]

is polynomial of \( q \).

**Example.**

If \( a_n = \sqrt{2} \cos(\pi n/4) \), then \( T = 8 \) and

\[
\sqrt{2} \sum_{n=1}^{\infty} \sum_{d|n} \cos(\pi d/4) \mu(n/d) \frac{x^n}{1 - x^n} = \frac{x - x^3 - \sqrt{2} x^4 - x^5 + x^7 + \sqrt{2} x^8}{1 - x^8}
\]

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