FINITE TYPE APPROXIMATIONS OF GIBBS MEASURES ON SOFIC SUBSHIFTS

J.-R. CHAZOTTES, L. RAMIREZ AND E. UGALDE

Abstract. Consider a Hölder continuous potential \( \phi \) defined on the full shift \( A^\mathbb{N} \), where \( A \) is a finite alphabet. Let \( X \subset A^\mathbb{N} \) be a specified sofic subshift. It is well-known that there is a unique Gibbs measure \( \mu_\phi \) on \( X \) associated to \( \phi \). Besides, there is a natural nested sequence of subshifts of finite type \( (X_m) \) converging to the sofic subshift \( X \). To this sequence we can associate a sequence of Gibbs measures \( (\mu_m^\phi) \). In this paper, we prove that these measures weakly converge at exponential speed to \( \mu_\phi \) (in the classical distance metrizing weak topology). We also establish a strong mixing property (ensuring weak Bernoulicity) of \( \mu_\phi \). Finally, we prove that the measure-theoretic entropy of \( \mu_m^\phi \) converges to the one of \( \mu_\phi \) exponentially fast. We indicate how to extend our results to more general subshifts and potentials. We stress that we use basic algebraic tools (contractive properties of iterated matrices) and symbolic dynamics.

1. Introduction

Existence and uniqueness of equilibrium states/Gibbs measures associated to sufficiently regular potentials is established in the general context of expansive homeomorphisms acting on a compact metric space satisfying specification [1, 7]. This class of systems contains subshifts of finite types (coding Axiom A diffeomorphisms) but more generally all specified subshifts like topologically mixing sofic subshifts (on finite alphabets).

The usual way to prove existence and uniqueness is to construct a sequence of elementary Gibbs measures (which are atomic) and to argue that such a sequence must have an accumulation point in the weak topology. Then one proves that this accumulation point is unique. In the particular case of subshifts of finite types and Hölder continuous potentials, there is a complete theory of Gibbs measures [2].

The point of view adopted here to study Gibbs measures on a specified subshift \( X \) is to approximate it by a nested sequence of subshifts of finite type, \( (X_m) \), in the sense of Hausdorff metric (there is a canonical way to do this). This gives a natural sequences of Gibbs measures (finite-type approximations) which converges weakly to a Gibbs measure whose properties we wish to analyze.

For the sake of definiteness, we assume that the given potential \( \phi \) on \( A^\mathbb{N} \) (\( A \) is a finite alphabet) is Hölder continuous and \( X \subset A^\mathbb{N} \) is a specified sofic subshift. As we shall comment at the end of the paper, we are not restricted to that situation. The two crucial properties on which our method relies are specification and presence of magic words (see definitions below). Sofic subshifts provide a natural class of subshifts with such properties.

Our main result can be phrased as follows: The sequence of finite type approximations \( (\mu_m^\phi) \) defined on \( (X_m) \) weakly converges, as \( X_m \to X \), to a measure \( \mu_\phi \) at an exponential speed. Then this measure must be a Gibbs measure associated to \( \phi \). Moreover, we prove a strong mixing property (implying that \( \mu_\phi \) is Bernoulli). By a classical argument (Bowen), this implies uniqueness. We also prove that the measure-theoretic entropy \( h(\mu_m^\phi) \) converges to \( h(\mu_\phi) \) exponentially fast (as well as the relative entropy \( h(\mu_\phi | \mu_m^\phi) \) to 0). We use and prove the fact that the topological pressure \( P(\phi, X_m) \) converges to \( P(\phi, X) \) exponentially fast.

We use two tools. The first one is algebraic (contraction properties of iteration of primitive matrices with respect to the projective metric): The second one is symbolic dynamics. All our constants have explicit expressions in terms of the ‘data’ of the problem, that is, the cardinality of the alphabet, the supremum norm of the potential, its Hölder constants and the specification length of the subshift \( X \).
We would like to mention a related work to ours due to Gurevich [6]. Therein the author deals with measures of maximal entropy. Informally speaking, he states some sufficient conditions on the way a subshift is approximated in ‘entropy’ by subshifts of finite type in order that the corresponding sequence of measure of maximal entropy have a unique limit. The main tool is graph theory.

The paper is organized as follows. In Section 2 we record basic definitions and notations. Section 3 contains our main results. Section 4 is devoted to some preparatory lemmas that we use for the proof of our main results in Section 5. In Section 6 we indicate some straightforward generalizations of our results as well as examples. We can indeed handle potential with polynomial variations (decaying fast enough). Consequently, the exponential speeds mentioned above become polynomial. We can also deal with more general specified subshifts (for instance, non-sofic but specified $\beta$-shifts).

2. Preliminary notions

2.1. Symbolic dynamics. Let $A$ be a finite alphabet. For all integers $m, n, m \leq n$, in $\mathbb{N}_0 (\mathbb{N}_0 := \mathbb{N} \cup \{0\})$, we denote by $a(m : n)$ the word $a(m)a(m + 1) \cdots a(n - 1)a(n)$ of length $n - m + 1$. The distance
\[
d_A(a, b) := \exp(- \min\{n \geq 0 : a(-n : n) \neq b(-n : n)\})
\]
makes the cartesian product $A^\mathbb{N}$ a compact metric space.

As usual, the shift transformation $T : A^\mathbb{N} \to A^\mathbb{N}$ is the map such that $T(a(i)) = a(i + 1)$.

A subshift is a $T$–invariant compact set $X \subset A^\mathbb{N}$. The subshift $X$ is said to be of finite type, if it is defined by a finite collection of admissible words, which can be taken of the same length for the sake of simplicity (and without loss of generality). So, the subshift of finite type defined by the collection $L \subset A^{n+1}$ of admissible words, is the compact set
\[
A_L := \{ a \in A^\mathbb{N} : a(j : j + n) \in L \ \forall j \in \mathbb{N}_0 \}.
\]

For a given subshift of finite type $X \subset A^\mathbb{N}$, the order of the subshift is the smallest integer $n \in \mathbb{N}$ such that $X$ is defined by a collection of admissible words of length $n$.

A sequence $a \in X$ is periodic of period $p \geq 1$ if $T^p a = a$, and this is its minimal period if in addition $T^k a \neq a$ whenever $0 \leq k < p$. We will denote by $\text{Per}_p(X)$ the collection of all periodic sequences of period $p$ in $X$.

For a general subshift $X \subset A^\mathbb{N}$ and $n \geq 0$, the collection of $X$–admissible words of length $m + 1$ is the set
\[
L_m(X) := \{ a(0 : m) : a \in X \}.
\]

A sofic subshift $X \subset A^\mathbb{N}$ is a continuous $T$–invariant image of a subshift of finite type. More precisely, let $Y \subset A^\mathbb{N}$ be a subshift of finite type, $B$ a finite alphabet, and $\Pi : Y \to B^\mathbb{N}$ a continuous map (with respect to the distances $d_A$ and $d_B$), commuting with $T$. The image $X = \Pi(Y)$, which in general is not of finite type, is a sofic subshift.

A more convenient way to characterize a sofic subshift is as follows. Let $X \subset A^\mathbb{N}$ be a subshift, and let $L^*(X) := \bigcup_{n=0}^{\infty} L_n(X)$ be the language defined by $X$. For each $a \in L^*(X)$ let $f(a) := \{ b \in L^* : ab \in L^* \}$ be the set of followers of $a$, and $p(a) := \{ b \in L^* : ba \in L^* \}$ is the set of predecessors of $a$. The subshift $X$ is sofic if $\{ f(a) : a \in L^* \}$ is a finite collection, in which case $\{ p(a) : a \in L^* \}$ is finite as well [10].

A word $a \in L^*$ is a magic word for $X$ if $b \in p(a)$ and $c \in f(a)$ implies $bac \in L^*$. It is a direct consequence of the finiteness of the collection of followers that every sofic subshift has a magic word (see [10, p. 148]).

For a general subshift $X \subset A^\mathbb{N}$ and a $X$–admissible word $a \in L_m(X)$, the set
\[
[a] := \{ b \in X : b(0 : m) = a \}
\]
is the cylinder of length $m + 1$ determined by $a$.

The subshift $X$ is said to be specified, with specification length $\ell \geq 1$ ($\ell = 0$ means that we have a full shift), if for each pair of $X$–admissible words $a \in \mathcal{L}_m(X)$ and $b \in \mathcal{L}_n(X)$, and $k \geq \ell$, there exists a periodic sequence $c$ of period $m + n + k + 2$, such that $c(0 : m) = a$ and $(T^m + k + 1)c(0 : n) = b$. Specification implies topological mixing and abundance of periodic orbits in the sense that periodic orbits form a dense set in $X$. See [4] for more details on the specification property.

A notational remark: We shall use the symbols $a, b$, etc, both for infinite sequences and finite words for convenience. To avoid any confusion we shall always precise the nature of the $a$’s or $b$’s.

2.2. Gibbs measures. The $\sigma$–field generated by the cylinders of $X \subset A^\mathbb{N}$ coincides with the Borel $\sigma$–field $\mathcal{B}(X)$. The set $\mathcal{M}(X)$ of Borel probability measures in $\mathcal{B}(X)$ is convex and compact in the weak topology. The weak topology can be metrized with the distance (see [16, p. 148])

$$D(\mu, \nu) := \sum_{m=0}^{\infty} 2^{-(m+1)} \left( \sum_{a \in \mathcal{L}_m(X)} |\mu[a] - \nu[a]| \right) .$$

We denote by $\mathcal{M}_T(X)$ the set of $T$–invariant probability measures on $X$.

A function $\phi : A^\mathbb{N} \to \mathbb{R}$ is Hölder continuous if for some $\theta \in [0, 1)$ and $C > 0$, we have $\max \{|\phi(a) - \phi(b)| : a(0 : m) = b(0 : m)\} \leq C \theta^m$ for all $m \geq 0$. The constant $\theta \in [0, 1)$ is the Hölder exponent of $\phi$. As usual, we shall call $\phi$ a potential.

For $\phi : A^\mathbb{N} \to \mathbb{R}$ and $k \in \mathbb{N}_0$ define $S_k \phi : A^\mathbb{N} \to \mathbb{R}$ such that

$$S_k \phi(a) = \sum_{i=0}^{k} \phi \circ T^i(a) .$$

Given a Hölder continuous potential $\phi$ and a subshift $X \subset A^\mathbb{N}$, $\mu \in \mathcal{M}_T(X)$ is a Gibbs measure for the potential $\phi$ if there are constants $C = C(\phi, X) > 0$ and $P(\phi, X) \in \mathbb{R}$ such that

$$C^{-1} \leq \frac{\mu_\phi[a(0 : k)]}{\exp(S_k \phi(a) - (k + 1)P(\phi, X))} \leq C$$

for all $k \in \mathbb{N}_0$.

The constant $P(\phi, X)$ above, is the so called topological pressure of the potential $\phi$. For specified subshifts, it can be defined (see e.g. [1]) by the limit

$$P(\phi, X) := \lim_{k \to \infty} \frac{1}{k+1} \log \left( \sum_{a \in \operatorname{Per}_{k+1}(X)} \exp(S_k \phi(a^*)) \right) ,$$

where $a^* \in X$ is an arbitrary sequence in $[a]$.

For $X$ of finite type and $\phi$ Hölder continuous, there exists a unique Gibbs measure $\mu_\phi$ (a proof of this fact can be found in [2, p. 9 ff.], or [9, ch. 5]). The existence and uniqueness of $\mu_\phi$ for general specified subshifts is a particular instance of the Theorem 2.5 in [7].
3. Main results

Let $X \subset A^\mathbb{N}$ be a specified subshift. The finite type approximation of order $m$, $m \in \mathbb{N}$, to $X$ is the subshift of finite type

\[ X_m := A_{m}(X) = \{ a \in A^\mathbb{N} : a(j : j + m) \in L_m(X), \forall j \in \mathbb{N}_0 \}, \]

determined by the $X$-admissible words of length $m + 1$. It is easy to verify that the sequence of compact sets $\{X_m\}_{m \in \mathbb{N}}$ converges in the Hausdorff metric to $X$ (you can find a definition in [4, p. 111]).

On the finite type approximation $X_m$, the potential $\phi : A^\mathbb{N} \to \mathbb{R}$ defines a unique Gibbs measure $\mu_\phi^m \in \mathcal{M}_T(X_m)$. These measures will be used as finite type approximations of order $m$ of $\mu_\phi \in \mathcal{M}_T(X)$.

For $m, p \in \mathbb{N}$ let $E_{(m,p)} \in \mathcal{M}(X_m)$ be the elementary Gibbs measure with support on $\text{Per}_{p+1}(X_m)$, such that

\[ E_{(m,p)}[b] := \frac{\exp(S_\phi(b))}{\sum_{a \in \text{Per}_{p+1}(X_m)} \exp(S_\phi(a))}, \]

for each $b \in \text{Per}_{p+1}(X_m)$. We will use the fact [8, p. 635] that each Gibbs measure $\mu_\phi$ can be obtained as a weak limit of the sequence of elementary Gibbs measures $E_{(m,p)}$, as $p \to \infty$.

We have the following three main results, whose direct consequence is the constructive proof of existence and uniqueness of Gibbs measures on specified sofic subshifts, associated to Hölder continuous potentials.

**Theorem 3.1 (Speed of convergence of $\mu_\phi^m$).** Let $\phi : A^\mathbb{N} \to \mathbb{R}$ be a Hölder continuous potential, and $X \subset A^\mathbb{N}$ a sofic specified subshift. There exists an invariant measure $\mu^* \in \mathcal{M}_T(X)$, a polynomial $Q_{FT}$ of degree 3, and constants $\theta_{FT} \in (0,1)$, $m^* \in \mathbb{N}$, satisfying

\[ D(\mu_\phi^m, \mu^*) \leq Q_{FT}(m) \theta_{FT}^m, \]

for all $m \geq m^*$.

**Theorem 3.2 ('Gibbs property').** Under the hypotheses of Theorem 3.1, there exists a constant $C_g = C_g(X, \phi) > 0$ such that

\[ C_g^{-1} \leq \frac{\mu^*[a(0:n)]}{\exp(S_n \phi(a) - (n+1)\overline{P}(\phi, X))} \leq C_g \]

for each $n \in \mathbb{N}_0$ and $a \in X$.

**Theorem 3.3 ('Strong mixing').** Under the hypotheses of Theorem 3.1, there exists a polynomial $Q_\mu$ of degree 2, and $\theta_\mu \in (0,1)$ such that, for all $a, b \in \mathcal{L}^*(X)$ there exists $s^* := s^*(a, b)$ satisfying

\[ \left| \frac{\mu^*([a] \cap T^{-s}[b])}{\mu^*[a]} \mu^*[b] - 1 \right| \leq Q_\mu(\sqrt{s}) \theta_\mu^s \]

for all $s \geq s^*$.

Combining the three previous theorems we get the following theorem.

**Theorem 3.4.** Let $\phi : A^\mathbb{N} \to \mathbb{R}$ be a Hölder continuous potential, and $X \subset A^\mathbb{N}$ a specified sofic subshift. The weak limit $\mu_\phi := \lim_{m \to \infty} \mu_\phi^m$ is the unique Gibbs measure associated to the potential $\phi$, i.e. the only $T$-invariant measure on $X$ satisfying (12). Moreover, the finite type approximations $\mu_\phi^m$ converge exponentially fast to $\mu_\phi$ in the sense of (11) and $\mu_\phi$ is mixing in the sense of (13) and Bernoulli.
Proof. Theorems 3.1 and 3.2 ensure the existence of a measure satisfying the inequalities (12) and having exponentially fast converging finite type approximations. To prove uniqueness, we can follow the last part of the proof of Theorem 1.16 in [2]. The mixing property (13) implies weak Bernoullicity, see e.g. [15, p. 169]. The theorem is proved.

□

Remark 3.1. All constants appearing in the above theorems, including the coefficients of the polynomials, have explicit (but somewhat tedious) expressions in terms of the data of the problem, that is, \#A, \|\phi\|, C, \theta (Hölder condition) and \ell (the specification length). These expressions are given in the proofs.

We end this section with the following theorem on speed of convergence of the entropy \( h(\mu^m) \) to \( h(\mu) \), and the relative entropy \( h(\mu|\mu^m) \) to 0.

Theorem 3.5. Under the hypotheses of Theorem 3.1, there exist constants \( C_h > 0, C_P > 0, 0 < \theta_h < 1 \) and \( 0 < \theta_P < 1 \), such that

\[
|h(\mu) - h(\mu^m)| \leq C_h \theta_h^m
\]

and

\[
h(\mu|\mu^m) \leq \frac{C_P}{1 - \theta_P} \theta_P^m.
\]

We refer the reader to [16] for details on entropy of invariant measures. The appendix at the end of the paper contains the necessary informations on entropy and relative entropy regarding our context.

To the best of our knowledge, Theorems 3.1-3.2-3.3 and 3.5 are new. The first three ones imply existence and uniqueness of \( \mu_\phi \). The only known mixing property for this measure is the usual mixing property (which does not assure Bernoullicity). This mixing much less stronger than (13) which implies Bernoullicity.

4. Technical lemmas

In this section we establish some technical lemmas needed to prove theorems of Section 3. We shall use some results coming from the theory of primitive matrices, as well as some elementary facts about weak distance between measures. The Appendix contains these results and some related notions. From now on we assume known those results and notions, as well as the notations established there.

Notations. From now on, an expression of the type \( a = c^{\pm 1} \) stands for the inequalities \( c^{-1} \leq a \leq c \). Similarly \( a = \pm c \) stands for \( -c \leq a \leq c \). By extension, \( a = \exp(\pm b) \) will stand for \( \exp(-b) \leq a \leq \exp(b) \).

Given a Hölder continuous potential \( \phi : A^N \to \mathbb{R} \), for each \( n \in \mathbb{N}_0 \) we define the finite range potential \( \phi^n : A^{n+1} \to \mathbb{R} \) such that

\[
\phi^n(a) = \max\{\phi(b) : b \in [a]\}.
\]

For \( n \geq m \) let \( \mathcal{L}_{(m,n)} := \mathcal{L}_n(X_m) \) be the set of \( X_m \)-admissible words of length \( n + 1 \), which of course contains \( \mathcal{L}_n := \mathcal{L}_n(X) \). Let us define the transfer matrix \( M_{(m,n)} : \mathcal{L}_{(m,n)} \times \mathcal{L}_{(m,n)} \to \mathbb{R}^+ \) such that

\[
M_{(m,n)}(a,b) = \begin{cases} 
\exp(\phi^{n+1}(ab)(n)) & \text{if } a(1 : n) = b(0 : n - 1), \\
0 & \text{otherwise.}
\end{cases}
\]

For a specified subshift \( X \), the matrix \( M_{(m,n)} \) is primitive with primitivity index \( \ell + n + 1 \), and has a unique maximal eigenvalue \( \rho_{(m,n)} := \rho(M_{(m,n)}) \). Associated to \( \rho_{(m,n)} \) there are unique normalized right and left eigenvectors \( v_{(m,n)} := v_{M_{(m,n)}} \) and \( w_{(m,n)} := w_{M_{(m,n)}} \).

The elementary measure \( \mathcal{E}_{(m,p)} \) can be expressed in term of the transfer matrices \( M_{(m,n)} \) as follows.
For $p > m \geq n$, and $a \in \mathcal{L}_{(m,n)}$, we have

\begin{equation}
\mathcal{E}_{(m,p)}[a] = \frac{M_{(m,n)}^{p+1}(a, a)}{\operatorname{Trace}(M_{(m,n)}^{p+1})} \times \exp(\pm 2(p+1)C\theta^{n+1}).
\end{equation}

Now, given $n > m$, for each each $a \in \mathcal{L}_{(m,n)}$ define $R^a$, $L_a : \mathcal{L}_{(m,n)} \to \mathbb{R}^+$ be such that

\begin{equation}
R^a(b) = M_{(m,n)}^{\ell+n+1}(b, a) \quad \text{and} \quad L_a(b) = M_{(m,n)}^{\ell+n+1}(a, b).
\end{equation}

Note that these vectors are positive.

We are able to give a uniform estimate of the values of elementary measures on cylinders, by using Corollary 7.2.

**Lemma 4.1.** Let $X \subset A^\mathbb{N}$ be a specified subshift with specification length $\ell \geq 0$, $\phi : A^\mathbb{N} \to \mathbb{R}$ a Hölder continuous potential with constant $C > 0$ and exponent $\theta \in (0,1)$. There are constants $C_\phi > 0$ and $\theta_\phi \in (0,1)$ such that, for all integers $m, n, p$, such that $m \leq n$, $(n+1)(n+\ell+1) \leq p$ and $a \in \mathcal{L}_{(m,n)}$, we have

\[\mathcal{E}_{(m,p)}[a] = w_{(m,n)}(a) v_{(m,n)}(a) \times \exp(\pm (p+1) C_\phi \theta_\phi^{n+1}).\]

**Proof.** For each $m \leq n$ let $\tau_{(m,n)}$ be the Birkhoff contraction coefficient of $M_{(m,n)}^{n+\ell+1}$. Let $\tilde{M} := M_{(m,n)}^{n+\ell+1}$ and $\Gamma := \Gamma(M)$ is defined in (22) in the Appendix. According to Theorem 7.1, we have

\[\Gamma^{-1} = \min_{a, b, a', b' \in \mathcal{L}_{(m,n)}} \left( \frac{\tilde{M}(a, b) \tilde{M}(a', b')}{\tilde{M}(a, b') \tilde{M}(a', b)} \right)^{-1/2} = \max_{a, b, a', b' \in \mathcal{L}_{(m,n)}} \left( \frac{\tilde{M}(a, b) \tilde{M}(a', b')}{\tilde{M}(a, b') \tilde{M}(a', b)} \right)^{1/2}.
\]

Now, for arbitrary $a, b, a', b' \in \mathcal{L}_{(m,n)}$ we have

\[
\frac{\tilde{M}(a, b) \tilde{M}(a', b')}{\tilde{M}(a, b') \tilde{M}(a', b)} \leq \sum_{c=ab} \exp(S_n + \ell \theta^{n+1}(c)) \sum_{c'=a'b'} \exp(S_n + \ell \theta^{n+1}(c')) \min_{c=ab} \exp(S_n + \ell \theta^{n+1}(c)) \min_{c'=a'b'} \exp(S_n + \ell \theta^{n+1}(c')) 
\]

\[
\leq \left( \frac{\# A}{e^C} \right) \ell \times e^{A\theta}.
\]

Hence

\[
\frac{1}{1 - \tau_{(m,n)}} \leq K_0 := (e^C \# A)^\ell \times e^{A\theta}.
\]

i.e. $\tau_{(m,n)} \leq 1 - K_0^{-1} < 1$.

For each $m \leq n$ let $d_{(m,n)}$ be the projective distance on the simplex of dimension $\# \mathcal{L}_{(m,n)}$, and $F_{(m,n)}$ the transformation defined on the simplex by the transition matrix $M_{(m,n)}$. Note that

\[
d_{(m,n)}(M_{(m,n)} R^a, R^a) = \log \left( \frac{\max_b M_{(m,n)}^{n+\ell+2}(b, a) / M_{(m,n)}^{n+\ell+1}(b, a)}{\min_b M_{(m,n)}^{n+\ell+2}(b, a) / M_{(m,n)}^{n+\ell+1}(b, a)} \right).
\]
We have
\[
\frac{M_{(m,n)}^{n+\ell+2}(b,a)}{M_{(m,n)}^{n+\ell+1}(b,a)} = \sum_{c=bxa \in \mathcal{L}_{(2m+2n+2)}} \exp(S_{n+\ell+1}\phi^{n+1}(c)) \cdot \frac{\exp(S_{n+\ell+1}\phi^{n+1}(c'))}{\exp(S_{n+\ell+1}\phi^{n+1}(c'))}
\]
where \( \|\phi\| := \max\{|\phi(a)| : a \in A^\mathbb{N}\} \). From this we get
\[
\max_{a \in \mathcal{E}_{(m,n)}} d_{(m,n)}(F_{(m,n)}(R^a), R^a) \leq K_1
\]
with
\[
K_1 := 2((\ell + 1)(\log(#A) + C) + \Lambda \theta + ||\phi||).
\]
Finally, with (19), inequalities (18) may be rewritten as
\[
\mathcal{E}_{(m,p)}[a] = \sum_{b \in \mathcal{L}_{(m,n)}} \frac{L_a^b M_{(m,n)}^{p+1-(n+\ell+1)} R^a}{L_b^a M_{(m,n)}^{p+1-(n+\ell+1)} R^b} \times \exp(\pm 2(p + 1)C\theta^{n+1}).
\]
Then, using Corollary 7.2, we have
\[
\mathcal{E}_{(m,p)}[a] = \sum_{b \in \mathcal{L}_{(m,n)}} \frac{w_{(m,n)}^b R^a L_a^b \mathcal{v}_{(m,n)}}{w_{(m,n)}^a R^b L_b^a \mathcal{v}_{(m,n)}} \times \exp\left(\pm 2\left((p + 1)C\theta^{n+1} + K_0K_1(n + \ell + 1)(1 - K_0^{-1})^{p+1}\right)\right)
\]
where \( K_0 \) and \( K_1 \) are given above. On the other hand we have
\[
L_a^b \mathcal{v}_{(m,n)} = \left(M_{(m,n)}^{n+\ell+1} \mathcal{v}_{(m,n)}\right)(a) = \rho_{(m,n)}^{n+\ell+1} \mathcal{v}_{(m,n)}(a),
\]
\[
w_{(m,n)}^b R^a = \left(w_{(m,n)}^{n+\ell+1} M_{(m,n)}^{n+\ell+1}\right)(a) = \rho_{(m,n)}^{n+\ell+1} w_{(m,n)}(a),
\]
and \( w_{(m,n)} \mathcal{v}_{(m,n)} = 1 \). Then, taking into account that \( p + 1 \geq (n + 1)(n + \ell + 1) \) and \( m \leq n \), we obtain
\[
\mathcal{E}_{(m,p)}[a] = w_{(m,n)}(a) \mathcal{v}_{(m,n)}(a) \times \exp\left(\pm (p + 1) C_{\theta^{n+1}}\right)
\]
with \( C_{\theta^{n+1}} := 2(C + K_0K_1) \) and \( \theta := \max(1 - K_0^{-1}, \theta) \). The lemma is proved.

**Lemma 4.2.** Let \( X \subset A^\mathbb{N} \) be a specified sofic subshift, with specification length \( \ell \geq 1 \), and \( \phi : A^\mathbb{N} \to \mathbb{R} \) a Hölder continuous potential with constant \( C > 0 \) and exponent \( \theta \in (0, 1) \). Then there are constants \( m_X \in \mathbb{N}, C_X > 0 \) and \( \theta_X \in (0, 1) \), such that for \( m_X \leq m \leq p \)
\[
1 - (p + 1)C_X \theta_X^m \leq \frac{\mathcal{E}_{(m,p)}[a]}{\mathcal{E}_{(m+1,p)}[a]} \leq 1
\]
for each \( a \in \text{Per}_{p+1}(X_{m+1}) \).
Proof. First note that
\[
\frac{\mathcal{E}_{(m,p)}[a]}{\mathcal{E}_{(m+1,p)}[a]} = \frac{\sum_{b \in \text{Per}_{p+1}(X_{m+1})} e^{S_p(b)}}{\sum_{b \in \text{Per}_{p+1}(X_m)} e^{S_p(b)}} = \\
1 - \frac{\sum_{b \in \text{Per}_{p+1}(X_m \setminus X_{m+1})} e^{S_p(b)}}{\sum_{b \in \text{Per}_{p+1}(X_m)} e^{S_p(b)}} \geq \\
1 - (p+1) \frac{\sum_{b \in \text{Per}_{p+1}(X_m)} \exp(S_p(b))}{\sum_{b \in \text{Per}_{p+1}(X_m)} \exp(S_p(b))}
\]
where \( \partial X_m := \{ a \in X_m : a(0 : m+1) \notin L_{m+1} \} \).

Let \( \partial L_m := L_{(m,m+1)} \setminus L_{m+1} \). Using specification property we obtain
\[
\frac{\sum_{b \in \text{Per}_{p+1}(\partial X_m)} e^{S_p(b)}}{\sum_{b \in \text{Per}_{p+1}(X_m)} e^{S_p(b)}} \leq \frac{\sum_{a \in \partial L_m} e^{S_{m+1}(a^*)}}{\sum_{a \in L_{(m,m+1)}} e^{S_{m+1}(a^*)}} \times \left( \#Ae^{2\|\phi\|} \right)^{\ell} e^{4\Lambda}
\]
for any \( a^* \in [a] \). We will prove that the quotient
\[
\left( \sum_{a \in \partial L_m} \exp(S_{m+1}(a^*)) \right) / \left( \sum_{a \in L_{(m,m+1)}} \exp(S_{m+1}(a^*)) \right)
\]
is exponentially small with \( m \). This is the point at which we use the existence of magic words.

Fix a magic word \( w \in L_k \) with \( k \geq \ell + 1 \). This is always possible since for a magic word \( a \in L^* \), the concatenated word \( ab \) is again magic, for any \( b \in f_X(a) \) (\( f_X(a) \) is the set of followers of \( a \), which contains arbitrary long words). Let \( m \geq 2k(k+\ell) \), so that \( \lfloor (m+1)/(k+\ell +1) \rfloor \geq m/k \) (we will use this condition at the final step of the proof). Note that if \( a \in \partial L_m \), then \( a(i : i+k) \neq w \) for each \( 1 \leq i \leq m-k \). This is because if \( a(i : i+k) = w \) then \( a(0 : i+k) \in L^*(X) \), implying that \( a \in L^*(X) \) which contradicts the hypothesis.

Letting \( q := k + \ell + 1 \) define
\[
\partial L'_m := \{ a \in \partial L_m : a(jq : jq+k) \neq w, \ 0 \leq j \leq \lfloor (m+1)/q \rfloor - 1 \}.
\]
It is clear that \( \partial L'_m \subset \partial L_m \). Define also
\[
e^{w} := \frac{\exp(S_k(b^-))}{\sum_{b \in L_k \setminus \{ w \}} \exp(S_k(b^-))} \times \left( \#A \frac{e^{2\|\phi\|}}{\ell} \right)^{-\ell},
\]
where for each \( b \in L_k \), the sequences \( b^- \), \( b^+ \in [b] \) are such that \( S_k(b^-) = \min_{b^* \in [b]} S_k(b^*) \) and \( S_k(b^+) = \max_{b^* \in [b]} S_k(b^*) \).

Let \( r := \lfloor (m+1)/q \rfloor \). For each \( \omega \in \{0,1\}^r \) define
\[
L^\omega_{(m,m+1)} := \{ a \in L_{(m,m+1)} : a(jq : jq+k) = w \text{ if and only if } \omega(j) = 1 \}.
\]
It is clear that the collection \( \{ L^\omega_{(m,m+1)} : \omega \in \{0,1\}^r \} \) is a partition of \( L_{(m,m+1)} \). Now, it follows from the specification property that for each \( \omega \in \{0,1\}^r \)
\[
\sum_{b \in L^\omega_{(m,m+1)}} \exp(S_{m+1}(b^*)) \geq \left( e^w \right)^{|\omega|} \times \sum_{b \in \partial L^w_m} \exp(S_{m+1}(b^-))
\]
where, as before, \( b^- \in [b] \) minimizes \( S_{m+1}(b^-) \), and \( |\omega| := \sum_{j=0}^{r-1} \omega(i) \). From the previous inequality we readily derive
\[
\sum_{b \in L_{(m,m+1)}} \exp(S_{m+1}(b^*)) \geq (1 + e^w)^{r} \times \sum_{b \in \partial L^w_m} \exp(S_{m+1}(b^-))
\]
Finally, \[
\frac{\sum_{a \in \partial L_m} \exp(S_{m+1} \phi(a^*))}{\sum_{a \in L(m,m+1)} \exp(S_{m+1} \phi(a^*))} \leq \frac{\sum_{a \in \partial L_m} \exp(S_{m+1} \phi(a^*))}{\sum_{a \in L(m,m+1)} \exp(S_{m+1} \phi(a^*))} \leq (1 + \epsilon^m)^{-r}.
\]
Since \( m \geq 2k(k + \ell) \) then \((1 + \epsilon^m)^{-r} \leq (1 + \epsilon^m)^{-m/k} \), and the result follows with \[
C_X := \left( \#A^2 ||\phi|| \right) e^{4A}, \quad \theta_X := (1 + \epsilon^m)^{-1/k}, \quad m_X = 2k(k + \ell).
\]
The lemma is proved.

The following lemma has its own interest.

**Lemma 4.3.** Let \( X \subset A^\mathbb{N} \) be a specified sofic subshift, with specification length \( \ell \). Let \( \phi : A^\mathbb{N} \to \mathbb{R} \) be a Hölder continuous potential with constant \( C > 0 \) and exponent \( \theta \in (0, 1) \). Then there are constants \( m_p \in \mathbb{N}, C_p > 0 \) and \( \theta_p \in (0, 1) \), such that
\[
0 \leq P(\phi, X_m) - P(\phi, X_{m+1}) \leq C_p \theta_p^m
\]
for all \( m \geq m_p \).

**Proof.** Proceeding as in the proof of the previous lemma, we obtain
\[
0 \leq \frac{1}{p+1} \log \left( \frac{\sum_{a \in \text{Per}_{p+1}(X_m)} \exp(S_{p+1} \phi(a))}{\sum_{a \in \text{Per}_{p+1}(X_{m+1})} \exp(S_{p+1} \phi(a))} \right)
\leq \frac{1}{p+1} \log \left( 1 + \frac{(p+1)C_X \theta_X^m}{1 - (p+1)C_X \theta_X^m} \right)
\leq \frac{C_X \theta_X^m}{1 - (p+1)C_X \theta_X^m}
\]
for \( m \geq m_X \).

To make use of the previous inequality, we need to know the speed of convergence of
\[
\frac{1}{p+1} \log \left( \sum_{a \in \text{Per}_{p+1}(X_m)} \exp(S_p \phi(a)) \right) \to P(\phi, X_m).
\]

Some computations like the ones done to prove Lemma 4.1 give we obtain
\[
\sum_{a \in \text{Per}_{p+1}(X_m)} \exp(S_{p+1} \phi(a)) = \text{Trace} \left( \sum_{n=1}^{p+1} \left( M_{(m,n)}^{p+1} \right) \times \exp((p+1)C \theta_n^{n+1}) \right) =
\]
\[
\sum_{a \in \text{Per}_{p+1}(X_m)} \exp(S_{p+1} \phi(a)) = \sum_{b \in \partial L_{(m,n)}} \sum_{v \in \mathcal{V}_{(m,n)}} w_{(m,n)}^b R_b L_{(m,n)}^v \times \rho^{p+1-2(n+\ell+1)} \times \exp((p+1)C \theta_{(m,n)}^{n+1}) \times \exp((p+1)C \theta_{(m,n)}^{n+1})
\]
for each \( m < n \) and \( (n+1)(n+\ell+1) \leq p \). Therefore
\[
\frac{1}{p+1} \log \left( \sum_{a \in \text{Per}_{p+1}(X_m)} \exp(S_p \phi(a)) \right) = \log \rho_{(m,n)} \pm C \theta_{(m,n)}^{n+1}.
\]
Let us now prove that \( \{\rho_{(m,n)}\}_{n > m} \) converges exponentially fast. By definition, the limit has to be equal to \( \exp(P(\phi, X_m)) \).
Let us define $N : \mathcal{L}_{(m,n+1)} \times \mathcal{L}_{(m,n+1)} \rightarrow \mathbb{R}^+$ such that

$$N(a,b) = \begin{cases} \exp(\phi^{n+1}(a)) & \text{if } a(1:n) = b(0:n) \\ 0 & \text{otherwise.} \end{cases}$$

Note that $M_{(m,n)} = N \exp(\pm C\theta^{n+1})$ coordinate-wise and $\rho_{(m,n)} = \exp(\pm C\theta^{n+1})\rho_N$. This can be easily derived from Corollary 7.2, taking into account that $\rho_M = \lim_{n \to \infty} (y^TM^n x)^{1/n}$ for a primitive matrix $M$, and arbitrary positive vectors $x, y$. Let $\nu : \mathcal{L}_{(m,n)} \rightarrow \mathbb{R}$ such that $\nu(a) = \exp(\phi^{n+1}(a)) \times \nu_{(m,n)}(a(1:n+1))$, we have

$$(N\nu)(a) = \exp(\phi^{n+1}(a))(M_{(m,n)}\nu_{(m,n)})(a(1:n+1)) = \exp(\phi^{n+1}(a))\rho_{(m,n)}\nu_{(m,n)}(a(1:n+1)) = \rho_{(m,n)}x(a).$$

Hence, $\nu$ is a positive eigenvector for the matrix $N$, associated to the positive eigenvalue $\rho_{(m,n)}$. Since $N$ primitive, Corollary 7.2 implies that $\rho_N = \rho_{(m,n)}$, therefore $\rho_{(m,n+1)} = \exp(\pm C\theta^{n+1})\rho_{(m,n)}$. From this we obtain,

$$\frac{\rho_{(m,n)}}{\exp(P(\phi,X_m))} = \exp(\pm \Lambda \theta^{n+1}).$$

Since $X_m \supset X_{m+1}$, then $P(\phi,X_m) \geq P(\phi,X_{m+1})$. The previous computations imply on the other hand that

$$P(\phi,X_m) - P(\phi,X_{m+1}) \leq \frac{C_X \theta^m}{1 - ((m + 2)(m + \ell + 2) + 1)C_X \theta^m} + C_X \theta^{m+2} + \Lambda \theta^{m+2},$$

for $m \geq m_X$, by taking $n = m + 1$ and $p = (n + 1)(n + \ell + 1)$. Thus, the lemma follows with

$$\theta_P := \max(\theta, \theta_X, \theta_E),$$

$$C_P := 2C_X + C_E \theta^2 + \Lambda \theta^2$$

and $m_P := \max(m_X, m_0)$, with $m_0$ such that $2C_X((m + 2)(m + \ell + 2) + 1)\theta^m \leq 1$ for all $m \geq m_0$.

\[ \Box \]

5. Proof of the main results

This section is devoted to the proof of Theorems 3.1, 3.2, 3.3 and 3.5.

5.1. Proof of Theorem 3.1. Lemma 4.1 implies that

$$E_{(m,p+1)}[a] = \exp(\pm 2(p + 2)C_E \theta^{-(\ell/2+1)}) \mathcal{E}_{(m,p)}[a],$$

for each $a \in \bigcup_{k=1}^{\sqrt{p}/(\ell/2+1)} \mathcal{L}_{(m,k)}$. Then Lemma 7.1 applies, and we obtain

$$D(E_{(m,p)}, E_{(m,p+1)}) \leq \left( \exp \left( 2C_E (p + 2) \theta^{-(\ell/2+1)} \right) - 1 \right) + 2^{(\ell/2+1)-\sqrt{p}}$$

$$\leq 4C_E (p + 2) \theta^{-(\ell/2+1)} + 2^{(\ell/2+1)-\sqrt{p}}$$

for each $p \geq \max(p_0, (m + 2)(m + \ell + 2))$, with

$$p_0 := \min \left\{ p \in \mathbb{N} : 2C_E (k + 2) \theta^{-(\ell/2+1)} \leq 1 \text{ for all } k \geq p \right\}.$$ 

Since $\sum_{p=0}^{\infty} (p + 2) \theta^{\sqrt{p}} < \infty$, there exists a limit measure $\mu^m := \lim_{p \to \infty} E_{(m,p)}$ belonging to $\mathcal{M}_p(X_m)$. The convergence is such that

$$D(\mu^m, E_{(m,p)}) \leq 4C_E \theta^{-(\ell/2+1)} Q(\sqrt{p})\theta^{\sqrt{p}} + 2^{(\ell/2+3)-\sqrt{p}}(\sqrt{p} + 3)$$
for each $p \geq \max(p_0, (m + 2)(m + \ell + 2))$. Here

$$Q(x) := -\frac{2x(x^2 + 3)}{\log(\theta_E)} + 6\frac{(x^2 + 1)}{\log^2(\theta_E)} - \frac{12x}{\log^3(\theta_E)} + \frac{12}{\log^4(\theta_E)}.$$ 

Let us now prove that the limiting measure $\mu^m$ coincides with the unique Gibbs measure $\mu_\phi \in \mathcal{M}_T(X_m)$. From the specification property we can derive the inequalities

$$\mathcal{E}_{(m,p)}[a] = \exp \left( S_n\phi(a^*) - \log \left( \sum_{b \in \text{Per}_{n+1}(X_m)} \exp(S_n\phi(b)) \right) \right) \times \exp(3\ell\|\phi\| \log(#A) + 5\Lambda)$$

which hold for any $n \leq p$, $a \in \mathcal{L}_{(m,n)}$, and $a^* \in [a]$. On the other hand, the computations performed in the proof of Lemma 4.3 lead us to the inequalities

$$\log \left( \sum_{b \in \text{Per}_{n+1}(X_m)} \exp(S_n\phi(b)) \right) = (n + 1) \left( P(\phi, X_m) \pm (C_E\theta_E^{n+1} + \Lambda\theta^{n+1}) \right),$$

for each $m \leq n$ and $n$ such that $(n + 1)(n + \ell + 1) \leq p$. Since $\mathcal{E}_{(m,p+1)}[a] = \exp(\pm (p + 2)C_E\theta_E^{\ell/2+1})\mathcal{E}_{(m,p)}[a]$, it follows by induction that

$$\mu^m[a] = \mathcal{E}_{(m,p)}[a] \times \exp \left( \pm 2C_E\theta_E^{-(\ell/2+1)}Q(\sqrt{p})\theta^{-\ell/2} \right)$$

for each $a \in \bigcup_{k=1}^{\sqrt{p}-\ell/2+1} \mathcal{L}_{(m,k)}$. Therefore, for each $m \leq n$, $a \in \mathcal{L}_{(m,n)}$, and $a^* \in [a]$, we have

$$\mu^m[a] = \frac{\exp(S_n\phi(a^*) - (n + 1)P(\phi, X_m))}{\exp(S_n\phi(a^*) - (n + 1)P(\phi, X_m))} = \exp(\pm C_{FT}),$$

with

$$C_{FT} := 2C_E\theta_E^{-(\ell/2+1)} \max \left\{ Q(k)\theta_E^k : k \in \mathbb{N} \right\} + 3\ell \|\phi\| \log(#A) + 5\Lambda + \max \left\{ (n + 1) \left( C_E\theta_E^{n+1} + \Lambda\theta^{n+1} \right) : n \in \mathbb{N} \right\}.$$ 

Now, for $a \in \mathcal{L}_n$ with $n \leq m$, we obtain

$$\mu^m[a] = \sum_{b \in \mathcal{L}_{m,n+k} \cap [a]} \mu^m[b] = \exp(\pm C_{FT}) \sum_{b \in \mathcal{L}_{m,n+k} \cap [a]} \exp(S_{n+k}\phi(b^*) - (n + k + 1)P(\phi, X_m)) = \exp \left[ S_n\phi(a^*) - (n + 1)P(\phi, X_m) \pm (C_{FT} + \ell(\log(#A) + \|\phi\|) + \Lambda) \right] \times \sum_{b \in \mathcal{L}_{m,k-1}} e^{S_{k-1}\phi(b^*) - kP(\phi, X_m)} = \exp \left[ S_n\phi(a^*) - (n + 1)P(\phi, X_m) \pm (2C_{FT} + \ell(\log(#A) + \|\phi\|) + \Lambda) \right] \sum_{b \in \mathcal{L}_{m,k-1}} \mu^m[b] = \exp(S_n\phi(a^*) - (n + 1)P(\phi, X_m)) \exp(\pm C_g)$$

by using the specification property, and for $k$ sufficiently large. Here

$$C_g := 2C_{FT} + \ell(\log(#A) + \|\phi\|) + \Lambda.$$ 

In this way we prove that $\mu^m$ satisfies the Gibbs inequality. Theorem 1.16 in [2], establishing the existence and uniqueness of the Gibbs measure $\mu_\phi^m \in \mathcal{M}_T(X_m)$, implies that $\mu^m := \mu_\phi^m$. 
Let \( \hat{m} = \min \{ m \in \mathbb{N} : 4((k + \ell + 1)^2 + 1)C_X\theta_X^k \leq 1 \text{ for all } k \geq m \} \), and \( m_X \) as in Lemma 4.2. From Lemma 7.2, Lemma 4.2, and following the computations in the first part of this proof, we obtain

\[
D(\mathcal{E}(m,(m+\ell+1)^2)), \mathcal{E}(m+1,(m+\ell+2)^2)) \leq D(\mathcal{E}(m,(m+\ell+1)^2)), \mathcal{E}(m+1,(m+\ell+2)^2)) \leq 4((m + \ell + 2)^2 + 1)C_X\theta_X^{m+1} + 2^{-m} + 8((m + \ell + 2)^2 + 1)C_X\theta_X^m,
\]

for all \( m \geq m^* \), with \( m^* = \max(m_X, \hat{m}) \).

Since \( \sum_{m=0}^{\infty} (m + \ell + 2)^2 \max(\theta, \theta_X)^m \) is finite, then \( \mu^* := \lim_{m \to \infty} \mathcal{E}(m,(m+\ell+1)^2) \) is a well defined measure in \( \mathcal{M}_T(X) \). Furthermore, the convergence is such that

\[
D(\mu^*, \mathcal{E}(m,(m+\ell+1)^2)) \leq 2^{-m+1} + 4C_X\mathcal{Q}_E(m)\theta_X^m + 8C_XQ_X(m)\theta_X^{m-1}
\]

with

\[
Q_E(x) := -\frac{(x + \ell + 2)^2 + 1}{\log(\theta_E)} + \frac{2(x + \ell + 2)}{\log^2(\theta_E)} - \frac{2}{\log^3(\theta_E)}
\]

\[
Q_X(x) := -\frac{(x + \ell + 2)^2 + 1}{\log(\theta_X)} + \frac{2(x + \ell + 2)}{\log^2(\theta_X)} - \frac{2}{\log^3(\theta_X)}.
\]

Therefore, for any \( m \geq m^* \), one has

\[
D(\mu^*, \mu_\phi^m) \leq D(\mu^*, \mathcal{E}(m,(m+\ell+1)^2)) + D(\mu_\phi^m, \mathcal{E}(m,(m+\ell+1)^2)) \leq Q_{FT}(m)\theta_X^m
\]

with

\[
Q_{FT}(m) := 4C_X\left(\theta_E^{(\ell/2+1)}Q(m) + \theta_X^{-1}Q_E(m)\right) + 8C_X\theta_X^{-1}Q_X(m) + (m + 3)2^{(\ell/2+3)} + 2
\]

and

\[
\theta_{FT} := \max(\theta_E, \theta_X, 1/2).
\]

\[\Box\]

**Remark 5.1.** In the previous proof, the polynomials \( Q, Q_E, \) and \( Q_X \) were obtained by upper bounding the series \( \sum_{k=m}^{\infty} P(k)\eta^k \), with \( P(x) \) an increasing polynomial, and \( \eta \in (0, 1) \), by the integral \( \eta^{-1}\int_0^\infty P(x)\eta^x\ dx \). Then we used the identity

\[
\int_0^\infty P(x)\eta^x\ dx = \eta^m \times \sum_{k=0}^{\deg(P)} (-1/\log(\eta))^{k+1} P^{(k)}(m)
\]

where \( P^{(k)} \) is the \( k \)th derivative of \( P \).

### 5.2. Proof of Theorem 3.2

In the previous proof we derived the inequalities

\[
\mu_\phi^m[a] = \exp(S_n\phi(a^*) - (n+1)P(\phi, X_m)) = \exp(\pm C_g),
\]

valid for each \( n \in \mathbb{N}, a \in \mathcal{L}_{(m,n)} \) and \( a^* \in [a] \).

On the other hand, Lemma 4.3 ensures that \( P(\phi, X_m) = P(\phi, X) \pm \phi \theta_X^m \), therefore

\[
\mu_\phi^m[a] = \exp(S_n\phi(a^*) - (n+1)P(\phi, X_m)) = \exp(\pm (C_g + (n+1)C_P\theta_X^m)),
\]

valid for each \( n \in \mathbb{N}, a \in \mathcal{L}_{(m,n)} \) and \( a^* \in [a] \). Taking the limit \( m \to \infty \), we obtain the desired result. \( \Box \)
5.3. **Proof of Theorem 3.3.** Proceeding as in the proof of Lemma 4.2, the specification property implies

\[
\mathcal{E}_{(m,(m+\ell+1)^2)}[a] = \sum_{b \in \text{Per}_m(X_{m+1}) \cap [a]} \exp(S_p \phi(b)) \times \sum_{b \in \text{Per}_m(X_{m+1}) \cap [a]} \exp(S_p \phi(b))
\]

for each \( n \leq m \in \mathbb{N} \) and \( a \in \mathcal{L}(m,n) \equiv \mathcal{L}_n \), as long as \( m \geq m^* \). These inequalities can be viewed as extensions to cylinders of the inequalities of Lemma 4.2.

On the other hand, Lemma 4.1 ensures that

\[
\mathcal{E}_{(m,(m+\ell+1)^2)}[a] = \mathcal{E}_{(m,(m+\ell+2)^2)}[a] \times \exp(\pm 2((m+\ell+2)^2+1)C_\varepsilon \theta_{\varepsilon}^{m+1})
\]

These and the previous inequalities imply that \( \mu^*[a] = \mathcal{E}_{(m,(m+\ell+1)^2)}[a] \exp(\pm \gamma_{FT}) \) for each \( m \geq m^* \), \( m \geq n \), and \( a \in \mathcal{L}_n \). Here

\[
\gamma_{FT} := 4\ell(\log(#A) + ||\phi||) + 4\Lambda + \sum_{k=m^*} \left( (k+1)^3 + 1 \right) (4C_\varepsilon \theta_{\varepsilon}^k + 2C_\varepsilon \theta_{\varepsilon}^{k+1})
\]

\[
= 4\ell(\log(#A) + ||\phi||) + 4\Lambda + 4C_\varepsilon Q_X(m^*) \theta_{\varepsilon}^{m^*-1} + 2C_\varepsilon Q_\varepsilon(m^*) \theta_{\varepsilon}^{m^*}.
\]

Because of the previous inequalities,

\[
|\mu^*([a] \cap T^{-s}[b]) - \mu^*[a]| \leq e^{2\gamma_{FT}} |\mathcal{E}_{(m,(m+\ell+1)^2)}([a] \cap T^{-s}[b]) - \mathcal{E}_{(m,(m+\ell+1)^2)}[a] \mathcal{E}_{(m,(m+\ell+1)^2)}[b]|,
\]

for every \( a \in \mathcal{L}_n \), \( b \in \mathcal{L}_n^* \) and \( s \in \mathbb{N} \), as long as \( n + n^* + s \leq m \).

Fix \( a, b \in \mathcal{L}_m \), \( p = p(m) := (m+\ell+1)^2 \), and \( s \leq s^* \) such that \( s + s^* + 4(m+1) = p + 1 \). Following the computations of the proof of Lemma 4.1, we obtain

\[
\mathcal{E}_{(m,p)}([a] \cap T^{-s}[b]) = \sum_{c \in \text{Per}_{p+1}(X_m)} \exp(S_p \phi(c)) \sum_{c \in \text{Per}_{p+1}(X_m)} \exp(S_{p+1} \phi(c))
\]

\[
= L_c^M M_{c} \times L_c^M \times L_{b}^M \times L_{b}^M \times L_{a}^M \times L_{a}^M \times \exp(\pm (3(p+1) \Lambda \theta_{\varepsilon}^{(s/(m+\ell+1))}))
\]

\[
= \mathcal{V}_{(m,m)}(a) \mathcal{V}_{(m,m)}(b) \times \mathcal{W}_{(m,m)}(a) \mathcal{W}_{(m,m)}(b) \times \left( \pm (3(p+1) \Lambda \theta_{\varepsilon}^{(s/(m+\ell+1))}) \right).
\]

Therefore, by using Lemma 4.1 we obtain

\[
\mathcal{E}_{(m,p)}([a] \cap T^{-s}[b]) = \mathcal{E}_{(m,p)}[a] \mathcal{E}_{(m,p)}[b] \times \left( \pm (3(p+1) \Lambda \theta_{\varepsilon}^{(s/(m+\ell+1))}) \right) = \mathcal{E}_{(m,p)}[a] \mathcal{E}_{(m,p)}[b] \times \left( \pm (5((m+\ell+1)^2 + 1) C_\varepsilon \theta_{\varepsilon}^{(s/(m+\ell+1))}) \right),
\]

for each \( a, b \in \mathcal{L}_m \). Because of the additivity of the measure \( \mathcal{E}_{(m,p)} \), these inequalities extend to any \( a, b \in \bigcup_{k=0}^m \mathcal{L}_k \).
Finally, combining the previous inequalities we obtain
\[ e^{4\gamma_P} \left( \exp \left( 5((m + \ell + 1)^2 + 1)C_{\epsilon} \theta_{\epsilon}^{\sqrt{2} / 2} / (m + \ell + 1) \right) - 1 \right) \times \mu^*[a] \mu^*[b] \leq \]
for each \( a, b \in \cup_{k=0}^{m} L_k \). Let
\[ s_0 := \min \{ s \in \mathbb{N} : 5((2\sqrt{k} + \ell + 1)^2 + 1)C_{\epsilon} \theta_{\epsilon}^{\sqrt{2} - (\ell + 5)/4} \leq 1 \text{ for all } k \geq s \} \].
The result follows by taking \( m = m(s) := \lfloor 2\sqrt{s} \rfloor \), so that
\[ \mu^*[a] \cap T^{-s}[b] = \mu^*[a] \mu^*[b] \leq \]
\[ e^{4\gamma_P} \left( \exp \left( 5((2\sqrt{s} + \ell + 1)^2 + 1)C_{\epsilon} \theta_{\epsilon}^{\sqrt{2} - (\ell + 5)/4} \right) - 1 \right) \times \mu^*[a] \mu^*[b] \leq \]
\[ 10C_{\epsilon} e^{4\gamma_P} \theta_{\epsilon}^{-(\ell + 5)/4} ((2\sqrt{s} + \ell + 1)^2 + 1) \theta_{\epsilon}^{\sqrt{2} / 2} \times \mu^*[a] \mu^*[b] \]
for all \( a \in L_n, b \in L_{n'}, \) and \( s > (\max(n, n') + (\ell + 1))^2/4 \). The theorem follows with
\[ \theta_{\mu} := \sqrt{\theta_{\epsilon}} \]
\[ s^*(a, b) := \max(\max(n, n')^2/4, s_0) \]
\[ Q_{\mu}(x) := 10C_{\epsilon} e^{4\gamma_P} \theta_{\epsilon}^{-(\ell + 5)/4} ((2x + \ell + 1)^2 + 1). \]

### 5.4. Proof of Theorem 3.5.
By [2], each measure \( \mu_{\phi}^m \) satisfies the variational principle, as well as the measure \( \mu_{\phi} \) by [1]. This means in particular the following:

\[ P(\phi, X_m) = \int_{X_m} \phi \ d\mu_{\phi}^m + h(\mu_{\phi}^m) \quad \text{and} \quad P(\phi, X) = \int_X \phi \ d\mu_{\phi} + h(\mu_{\phi}). \]

Hence we have
\[ |h(\mu_{\phi}^m) - h(\mu_{\phi})| \leq |P(\phi, X_m) - P(\phi, X)| + \left| \int_X \phi \ d\mu_{\phi} - \int_{X_m} \phi \ d\mu_{\phi}^m \right|. \]
It is obvious from Lemma 4.3 that
\[ 0 < P(\phi, X_m) - P(\phi, X) \leq \frac{C_P}{1 - \theta_P} \theta_P^m. \]

On another hand,
\[ \left| \int_X \phi \ d\mu_{\phi} - \int_{X_m} \phi \ d\mu_{\phi}^m \right| \leq C \theta^m. \]

Statement (14) is thus proved.

Now, applying (24) (see appendix below) and using (20)-(21) we get:
\[ h(\mu_{\phi}^m) = P(\phi, X_m) - \left( \int_X \phi \ d\mu_{\phi} + h(\mu_{\phi}) \right) = P(\phi, X_m) - P(\phi, X) \leq \frac{C_P}{1 - \theta_P} \theta_P^m. \]
This proves (15). The proof of the theorem is now complete.

\[ \square \]
6. Examples, Generalizations and Comments

A natural class of specified sofic subshifts is provided by $\beta$-shifts coding the dynamics of the map on the unit interval $T_\beta : x \mapsto \beta x \mod 1$, where $\beta > 1$ is a real number. For certain $\beta$’s, the corresponding $\beta$-shift is a specified sofic subshift. In [11], the authors construct a sofic coding of hyperbolic automorphisms of the torus. In both cases, the Lebesgue measure on the unit interval or the torus is sent to the measure of maximal entropy on the coding subshift.

In this paper we assumed, for the sake of definiteness, that the potential $\phi$ was Hölder continuous and the subshift $X \subset A^\mathbb{N}$ was a specified sofic subshift. Nevertheless, both assumptions can be weakened. In the proof of Theorem 3.1, and in all other computations, the exponential decay

$$\max\{|\phi(a) - \phi(b)| : a(0 : m) = b(0 : m)\} \leq C\theta^m$$

can be replaced by a polynomial decay

$$\max\{|\phi(a) - \phi(b)| : a(0 : m) = b(0 : m)\} \leq Cm^{-\gamma},$$

as long as $\gamma > 4$. By doing so, the speed of convergence of topological pressure (Lemma 4.3) become polynomial as well. Hence, the speed of convergence in Theorem 3.5 also become polynomial (see the proof).

Regarding the nature of the subshift, the reader can verify that the essential assumptions are specification and presence of magic words. Moreover, the latter assumption is only used in Lemma 4.2. Specified sofic subshifts form a natural class of subshifts having the specification property as well as magic words, but there are huge classes of non–sofic specified subshifts with magic words. Among them, we can mention the class of non–sofic specified $\beta$–shifts (see [13]). One can straightforwardly prove that for each non-sofic specified $\beta$–shifts there exists $k \in \mathbb{N}$ such that $0^k$ is a magic word.

On the other hand, following the examples in [5] we can obtain non–sofic specified subshifts with magic words, as finitary codings of Bernoulli shifts. Take for example the finitary coding $\pi : \{0, 1, 2, 3\}^\mathbb{N} \rightarrow \{0, 1, 2, 3\}^\mathbb{N}$ such that

$$(\pi a)_n = \begin{cases} 0 & \text{if } a(0 : 2k + 1) = 32^k1^k0 \text{ for some } k \in \mathbb{N} \\ a(n) & \text{otherwise} \end{cases}$$

The image subshift $X := \pi(\{0, 1, 2, 3\}^\mathbb{N})$ is not sofic: its description involves a non–regular language. Nevertheless it has the specification property, we may connect any two admissible words by words of the kind $12^k1$, and 3 is magic letter. Any product measure on $\{0, 1, 2, 3\}^\mathbb{N}$ induces a Gibbs measure in $X$, which can be approximated by our method.

Though the class of systems considered here is only a subclass of those covered by Theorem 2.5 in [7], we are able to obtain a speed of convergence (in the weak distance) of finite type approximations to the Gibbs measure on the approximated subshift $X$. We were also able to prove a strong mixing property, implying Bernoullity. Finally, we provide a speed of convergence of the entropy of the finite-type approximations to the entropy of the Gibbs measure on $X$. We also emphasize that all constants appearing in the statements of Section 3 have explicit expressions in terms of the data of the problem. We did not write these explicit formulas in the statements because they are cumbersome. They of course appear in the course of the proofs. It is also worth to notice that we only used classical algebraic tools and symbolic dynamics, except for uniqueness of $\mu_\phi$ for which we used Bowen’s argument.

Further work has to be done in order to generalize our results to more general subshifts. One possible approach requires the a precise control of the convergence of the pressure. A similar approach was already exploited by Gurevich in the proof of the uniqueness of the maximal measure for a class of non–specified subshift [6]. Unfortunately, the systems satisfying the hypotheses of Gurevich’s theorem cannot be explicitly characterized.

Acknowledgment. We thank K. Petersen for providing us reference [6].
References

[1] R. Bowen, Some systems with unique equilibrium states, Math. Systems Theory 8 (1974/75), no. 3, 193–202.
[2] R. Bowen, Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms, Lecture Notes in Mathematics 470, Springer–Verlag, 1975.
[3] J.-R. Chazottes, E. Floriani, R. Lima, Relative entropy and identification of Gibbs measures in dynamical systems, J. Statist. Phys. 90 (1998), no. 3–4, 697–725.
[4] M. Denker, C. Grillenberger, K. Sigmund, Ergodic Theory on Compact Spaces, Lecture Notes in Math. 527, Springer-Verlag, 1976.
[5] M. Denker, Some New Examples of Gibbs Measures, Monat. fur Math. 109 (1990) 49–62.
[6] B. Gurevich, Stationary random sequences of maximal entropy. In Multicomponent random systems, pp. 327–380, Adv. Probab. Related Topics 6 Dekker, New York, 1980.
[7] N. T. A. Haydn and D. Ruelle, Equivalence of Gibbs and Equilibrium States for Homeomorphisms Satisfying Expansiveness and Specification, Commun. Math. Phys. 148 (1992), 155–167.
[8] A. Katok, B. Hasselblatt, Introduction to the modern theory of dynamical systems. Encyclopedia of Mathematics and its Applications 54, Cambridge University Press, Cambridge, 1995.
[9] G. Keller, Equilibrium States in Ergodic Theory, London Mathematical Society Student Texts 42, Cambridge University Press 1998.
[10] B. Kitchens, Symbolic Dynamics, Springer-Verlag, Berlin, 1998.
[11] S. Le Borgne, Un codage sofique des automorphismes hyperboliques du tore, Séminaires de Probabilités de Rennes (1995), 35 pp., Publ. Inst. Rech. Math. Rennes, 1995.
[12] D. Ruelle Statistical Mechanics on compact sets with $\mathbb{Z}^k$ actions satisfying expansiveness and specification, Trans. Amer. Math. Soc. 185 (1973), 237–251.
[13] J. Schmeling, Symbolic dynamics for $\beta$-shifts and self-normal numbers, Ergodic Theory Dynam. Systems 17 (1997), no. 3, 675–694.
[14] E. Seneta, Non-negative Matrices and Markov Chains, Springer Series in Statistics, Springer-Verlag, 1981.
[15] P. Shields, The Ergodic Theory of Discrete Sample Paths, Graduate Studies in Mathematics 13, American Mathematical Society, 1996.
[16] P. Walters, An introduction to Ergodic Theory, Springer Verlag, 1982.

7. Appendix

7.1. Primitive matrices. $M : \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\} \to [0, \infty)$ is said to be primitive if there exists an integer $\ell \geq 1$ such that $M^\ell > 0$. The smallest such integer is the primitivity index of $M$.

For $M$ primitive let

$$\Gamma(M) := \begin{cases} \min_{i,j,k,l} \sqrt{\frac{M(i,j)M(k,l)}{M(i,l)M(k,j)}} & M > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The Birkhoff’s coefficient for $M$ is $\tau(M) := (1 - \Gamma(M))/(1 + \Gamma(M))$.

Consider the function $d : (\mathbb{R}^+)^n \times (\mathbb{R}^+)^n \to \mathbb{R}^+$ such that

$$d(x, y) = \log \left( \frac{\max_{i,j} x(i) / y(j)}{\min_{i,j} x(i) / y(j)} \right).$$

It is the projective distance when restricted to the simplex

$$\Delta_n := \left\{ x : \{1, 2, \ldots, n\} \to (0, 1) : |x|_1 := \sum_{i=1}^n x(i) = 1 \right\}.$$

The Birkhoff’s coefficient gives the contraction rate of the action of $M$ over the vector in $\Delta_n$. 

Theorem 7.1. With $M$, $\Delta_n$ and $d$ be as above, define $F_M : \Delta_n \to \Delta_n$ be such that

$$F_M x := \frac{M x}{|M x|_1}.$$ 

Then $F_M$ is a contraction in $(\Delta_n, d)$ with contraction coefficient $\tau(M)$, i.e.,

$$d(F_M x, F_M y) \leq \tau(M) d(x, y), \quad \forall \ x, y \in \Delta_n.$$ 

A proof of this result can be easily derived from the Theorem 3.12 in [14, p. 108].

The previous result directly implies the Perron-Frobenius Theorem (see [14, ch. 1] for more details): a primitive matrix $M$ has only one maximal eigenvalue $\rho_M > 0$. Associated to it there is a unique right eigenvector $v_M \in \Delta_n$, and a unique left eigenvector $w_M > 0$ such that $w_M^\dagger v_M = 1$.

A rather direct consequence of the previous theorem is the following.

Corollary 7.1. For $M$ primitive with primitivity index $\ell$, let $F := F_M$ and $\tau := \tau(M^\ell)$. Then, for each $x \in \Delta_n$ and $m \in \mathbb{N}$, we have

$$d(F^m x, v_M) \leq \frac{\tau^{\lfloor m/\ell \rfloor} d_M(x)}{1 - \tau}$$

with $d_M(x) := \min \left( \ell d(x, Fx), d(x, F^\ell x) \right)$.

From this we readily deduce the following.

Corollary 7.2. Let $M : \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\} \to \mathbb{R}^+$ be a primitive matrix with primitivity index $\ell$, $F := F_M$, and $\tau := \tau(M^\ell)$. Then, for each $x \in \Delta_n$ and $m \in \mathbb{N}$ we have

$$M^m x = \rho_M^m \left( w_M^\dagger x \right) v_M \exp \left( \pm \frac{\tau^{\lfloor m/\ell \rfloor} d_M(x)}{1 - \tau} \right)$$

with $d_M(x) := \min \left( \ell d(x, Fx), d(x, F^\ell x) \right)$.

Proof. Since $M^m x = |M^m x|_1 F^m x$, then

$$d(F^m x, v_M) = \log \left( \frac{\max_i (M^m x)(i)/v_M(i)}{\min_i (M^m x)(i)/v_M(i)} \right).$$

With

$$C_m(x) := \left( \max_i (M^m x)(i)/v_M(i) \min_i (M^m x)(i)/v_M(i) \right)^{1/2}$$

we have $M^m x = C_m(x) v_M x e^{\pm d(F^m x, v_M)/2}$. Multiplying from the left these inequalities by $w_M^\dagger$ yields $C_m(x) = \rho_M^m (w_M^\dagger x) e^{\pm d(F^m x, v_M)/2}$. Taking into account Corollary 7.1, the desired result follows. □
7.2. Weak distance. In this subsection $X \subset A^\mathbb{N}$ is any subshift. We have the following lemmas.

Lemma 7.1. Let $\nu, \mu \in \mathcal{M}(X)$ be such that $\mu[a] = \nu[a] \exp(\pm \epsilon)$ for each $a \in \mathcal{L}_k(X)$, then $D(\mu, \nu) \leq (\exp(\epsilon) - 1) + 2^{-k}$.

Proof. For $j \leq k$ we have

\[
\sum_{a \in \mathcal{L}_j(X)} |\mu[a] - \nu[a]| \leq \sum_{a \in \mathcal{L}_j(X)} \left( \sum_{b \in \mathcal{L}_k(X): b(0:j) = a} |\mu[b] - \nu[b]| \right) \\
\leq \sum_{a \in \mathcal{L}_j(X)} \left( \sum_{b \in \mathcal{L}_k(X): b(0:j) = a} \mu[b](e^\epsilon - 1) \right) = e^\epsilon - 1.
\]

Hence $D(\mu, \nu) \leq (e^\epsilon - 1) \sum_{j=0}^{k} 2^{-j+1} + \sum_{j=k+1}^{\infty} 2^{-j+1} \left( \sum_{a \in \mathcal{L}_j(X)} |\mu[a] - \nu[a]| \right)$. The result follows taking into account that $\sum_{a \in \mathcal{L}_j(X)} |\mu[a] - \nu[a]| \leq 2$ for all $j \in \mathbb{N}$. \hfill \Box

Lemma 7.2. Let $\nu, \mu \in \mathcal{M}(X)$ be atomic with support $S_\nu := \text{supp}(\nu) \subset S_\mu := \text{supp}(\mu)$. Suppose that $\mu\{x\} \leq \nu\{x\} \leq \mu\{x\} \exp(\epsilon)$ for each $x \in S_\nu$, then $D(\mu, \nu) \leq \exp(\epsilon) - \exp(-\epsilon)$.

Proof. For each $k \in \mathbb{N}$, since $\{|a|: a \in \mathcal{L}_k(X)\}$ is a partition of $X$, we have

\[
\sum_{a \in \mathcal{L}_k(X)} |\mu[a] - \nu[a]| = \sum_{a \in \mathcal{L}_k(X)} (\nu(S_\nu \cap [a]) - \mu(S_\nu \cap [a])) + \sum_{a \in A_m} \mu((S_\mu \setminus S_\nu) \cap [a]) \\
\leq (e^\epsilon - 1)\mu(S_\nu) + \mu(S_\mu \setminus S_\nu) \leq (e^\epsilon - 1) + \mu(S_\mu \setminus S_\nu).
\]

Now, $1 = \nu(S_\nu) \leq \exp(\epsilon)\mu(S_\nu)$, hence $\mu(S_\mu \setminus S_\nu) \leq 1 - \exp(-\epsilon)$ and the result follows. \hfill \Box

7.3. Entropy and relative entropy. Let $\nu$ be a shift-invariant probability measure on a specified subshift $Y \subset A^\mathbb{N}$. The measure-theoretic entropy of $\nu$ is

\[
h(\nu) = -\lim_{n \to \infty} \frac{1}{n+1} \sum_{a \in \mathcal{L}_n(Y)} \nu[a] \log \nu[a].
\]

Since $\nu(A^\mathbb{N} \setminus Y) = 0$, we can replace $\mathcal{L}_n(Y)$ by $A^{n+1}$ by using the usual convention ‘$0 \log 0 = 0’$.

We now turn to relative entropy. We refer the reader to [3] for details. Therein, only subshifts of finite type are considered but the extension to more general subshifts is straightforward. Let $\mu_\psi$ be a Gibbs measure (with Hölder continuous potential $\psi$ defined on $A^\mathbb{N}$) on a specified subshift $Y' \supset Y$. The relative entropy of $\nu$ with respect to $\mu_\psi$ is defined as:

\[
h(\nu|\mu_\psi) = \lim_{n \to \infty} \frac{1}{n+1} \sum_{a \in \mathcal{L}_n(Y')} \nu[a] \log \frac{\nu[a]}{\mu_\psi[a]}.
\]

Notice that the hypothesis $Y \subset Y'$ is crucial to make $\nu[a] \log \frac{\nu[a]}{\mu_\psi[a]}$ well-defined. One can prove that

\[
(24) \quad h(\nu|\mu_\psi) = P(\psi, Y') - \int_Y \psi \, d\nu - h(\nu).
\]

We notice that this result is true whenever $\mu_\psi$ satisfies the ‘Gibbs inequality’ (7), $\psi$ not being necessarily Hölder continuous.