In this paper, we define the notions \( q \)-birational morphism and \( q \)-birational divisor and develop the theory about them. We state and prove versions of Kodaira-type vanishing theorem and Zariski decomposition theorem for \( q \)-birational divisors.

1. Introduction

Throughout the paper, \( q \geq 2 \) is a positive integer, \( k \) is an algebraically closed field of characteristic 0 and all varieties are quasi-projective, separated, finite type, reduced and irreducible schemes over \( k \). Any divisor is assumed to be Cartier unless otherwise stated.

For any normal projective varieties \( X, X' \) and any proper birational morphism \( f : X' \to X \), the following property is well-known:

\[
f_* \mathcal{O}_{X'} = \mathcal{O}_X.
\]

Moreover, by extending the property, one can prove a vanishing of the higher direct image of structure sheaf as the following:

**Proposition 1.1** (See Proposition 3.2). Let \( X \) be any normal variety with \((RS_q)\). Then the following are equivalent:

(a) \( X \) is \((S_{q+1})\).

(b) For any proper birational morphism \( f : X' \to X \) with \( X' \) smooth, \( R^if_*\mathcal{O}_{X'} = 0 \) for \( 1 \leq i < q \).

For the definition of \((R_q)\), \((RS_q)\) and \((S_{q+1})\), see Definition 2.3. Note that the idea of the proof of Proposition 1.1 goes back to [Kov99], [Ale08], [Fuj17], [Kol11] and [AH12].

Under the same conditions above, assume that the exceptional locus of \( f \) has codimension 1 and let \( F \) be any effective \( f \)-exceptional divisor on \( X' \). Then we have the following property:

\[
f_* \mathcal{O}_{X'}(F) = \mathcal{O}_X.
\]

Considering Proposition 1.1, we can ask what condition on \( f \) is sufficient for the following vanishing to hold:

\[
R^if_* \mathcal{O}_{X'}(F) = 0 \text{ for } 1 \leq i < q.
\]

To answer the question, we define the following notion of \( q \)-birationality for a birational morphism.

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**Definition 1.2** (See Definition 3.1). Let $X, X'$ be any normal varieties and $f : X' \to X$ be any proper birational morphism.

(a) The center of $f$ is the reduced closed subscheme $C$ of $X$ which is the image of the exceptional locus along $f$.

(b) We say $f$ is a $q$-birational morphism if the exceptional locus has codimension 1 and the center of $f$ has codimension $\geq q + 1$.

Then we can prove the following theorems.

**Theorem 1.3** (See Theorem 3.4). Let $X$ be a variety with $(R_q, (S_{q+1})$ and $f : X' \to X$ be any $q$-birational morphism from a smooth variety $X'$. Then $R^i f_* \mathcal{O}_{X'}(E) = 0$ for any $1 \leq i < q$ and any effective $f$-exceptional divisor $E$.

**Theorem 1.4** (See Theorem 3.7). Let $X$ be a Cohen-Macaulay variety with $(R_q)$ and $f : X' \to X$ be any proper birational morphism from a smooth variety $X'$. Suppose that $R^i f_* \mathcal{O}_{X'}(E) = 0$ for any $1 \leq i < q$ and any effective $f$-exceptional divisor $E$. Then $f$ is $q$-birational.

As a corollary of Ambro’s vanishing theorem (see Theorem 2.3 in [Amb03]) and Theorem 1.3, we have a generalization of Theorem 3.4 in [Ale08] and Theorem 7.1.4 in [Fuj17].

**Corollary 1.5** (See Corollary 3.10). Let $X$ be any normal variety and $\Delta$ be any effective $\mathbb{Q}$-Weil divisor such that $(X, \Delta)$ is log canonical. Suppose $D$ is a reduced Cartier divisor on $X$ such that $(X, \Delta + \varepsilon D)$ is log canonical for some $0 < \varepsilon \ll 1$. If $X$ is $(R_q)$ and $(S_{q+1})$, then $D$ is $(S_{q+1})$.

Note that a special case of Theorem 1.3 is proved in [ST08]. Now, we define the following notion of (weak) $q$-birationality for a divisor.

**Definition 1.6** (See Definition 4.1). Let $X$ be any smooth projective variety and $D$ be any divisor on $X$. We say $D$ is weakly $q$-birational if there are smooth projective varieties $X'$ and $X_0$, a diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & X_0 \\
\pi \downarrow & & \downarrow \\
X & & 
\end{array}
\]

an ample divisor $H$ on $X_0$ and a positive integer $p$ such that

(a) $f'$ is $q$-birational and,

(b) $\pi^*(pD) - (f')^*H$ is a (not necessarily effective) $f'$-exceptional divisor on $X'$.

If $\pi^*(pD) - (f')^*H$ in (b) is effective, we say $D$ is $q$-birational. If $D$ is weakly $q$-birational, we say the minimum of $p$ is the index of $D$ and denote it by $p_D$.

Using Theorem 1.3, we can prove a Kodaira-type vanishing theorem for $q$-birational divisors as follows:
Theorem 1.7 (See Theorem 4.4). Let $X$ be any smooth projective variety and $D$ be any $q$-birational divisor on $X$. Then

$$H^i(X, \mathcal{O}_X(K_X + p_D D)) = 0 \quad \text{for } 1 \leq i < q.$$ 

Moreover, any $q$-birational divisor admits a form of Zariski decomposition.

Theorem 1.8 (See Theorem 4.5). Let $X$ be any smooth projective variety, $D$ be any $q$-birational divisor on $X$ and $p_D D = P + N$ be the weak decomposition of $D$ for the index $p_D$ of $D$. Then

$$H^i(X, \mathcal{O}_X(mp_D D)) = H^i(X, \mathcal{O}_X(mp + mN)) = H^i(X, \mathcal{O}_X(mP))$$ 

for any $m \geq 0$ and $0 \leq i < q$.

One can consider the Kawamata-Viehweg vanishing theorem as a generalization of the Kodaira vanishing theorem. Moreover, divisorial Zariski decomposition exists for any pseudo-effective divisor. Hence, along the same line, we will discuss how to generalize Theorem 1.7 and Theorem 1.8. We also pose questions concerning such generalizations, which remain open.

The rest of the paper is organized as follows. We begin Section 2 by defining some notations and stating some basic theorems. Section 3 is devoted to defining $q$-birational morphism and proving Theorem 1.3 and Theorem 1.4. In Section 4, we define $q$-birational divisors, prove Theorem 1.7 and Theorem 1.8 and discuss how to generalize those theorems.

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2. Preliminaries

In the section, we collect properties and theorems used in the next sections.

From this section and to the end of the paper, for any variety $X$ and any (not necessarily closed) point $x \in X$, we write $\dim x$ for the dimension of the closure of $x$. In addition to this, set $\operatorname{codim} X x := \dim X - \dim x$ and $X_x := \text{Spec} \mathcal{O}_{X,x}$. Moreover, we assume $2 \leq q < \dim X$.

For a proper morphism $f : X' \to X$ between two varieties, set $X'_x := X' \times_X X_x$, $\mathcal{O}_{X',x}$ for the structure sheaf of $X'_x$ and $\omega_{X',x}$ for the dualizing sheaf of $X'_x$.

Now, we recall a well-known result.

Theorem 2.1 (See [KM98], Theorem 5.10). Let $X, X'$ be any smooth projective varieties and $f : X' \to X$ be any proper birational morphism. Suppose that $\mathcal{E}$ is a vector bundle on $X$. Then we have

$$H^i(X, \mathcal{E}) = H^i(X', f^* \mathcal{E}) \quad \text{for } i \geq 0.$$ 

The following theorem is called the relative Kawamata-Viehweg vanishing theorem.
Theorem 2.2 (See [KMM87], Theorem 1-2-3). Let $X'$ be any smooth projective variety, $X$ be any projective variety, and $f : X' \to X$ be any proper birational morphism. Suppose that $D$ is a $f$-nef divisor on $X'$. Then, $R^if_*\mathcal{O}_{X'}(K_{X'} + D) = 0$ for $i \geq 1$.

Let us introduce the following notions.

Definition 2.3. Let $X$ be any normal variety.

(a) We call $X (R_q)$ if the singular locus of $X$ has codimension $\geq q + 1$.

(b) We call $X (RS_q)$ if $\text{codim} \text{Supp} (\bigcup_{i \geq 1} R^if_*\mathcal{O}_{X'}) \geq q + 1$ for any resolution $f : X' \to X$.

(c) We call $X (S_{q+1})$ if $H^i_x(X_x, \mathcal{O}_{X,x}) = 0$ for any $x \in X$ and any $i < \min\{q + 1, \text{codim}_X x\}$.

We need the theory of derived categories to prove Corollary 2.5. Hence, it is worth stating the Grothendieck duality.

Theorem 2.4 (See 0AU3, (4) in [Stacks]). Let $X, X'$ be any noetherian schemes and $f : X' \to X$ be any proper birational morphism. For any $K \in D(X')$,

$$R\text{Hom}_{\mathcal{O}_X}(Rf_*K, \omega^*_X) \cong Rf_*R\text{Hom}_{\mathcal{O}_{X'}}(K, \omega^*_X')$$

in $D(X)$, where $D(X)$ denotes the bounded derived category of coherent sheaves on $X$ and $\omega^*_X$ denotes the normalized dualizing complex on $X$.

Let us prove the following corollary of the Kawamata-Viehweg vanishing theorem and the Grothendieck duality.

Corollary 2.5 (See [Fuj17], Lemma 7.1.2). Let $X, X'$ be any varieties and $X'$ be smooth. For any proper birational morphism $f : X' \to X$ and any point $x \in X$ with $\text{codim}_X x \geq c$, $H^i_{f^{-1}(x)}(X'_x, \mathcal{O}_{X'_x}) = 0$ for $i < c$.

Proof. Let $E$ be the injective hull of the residue field of $\mathcal{O}_{X,x}$ as an $\mathcal{O}_{X,x}$-module. Then we have

$$R\Gamma_{f^{-1}(x)}(X'_x, \mathcal{O}_{X'_x}) = R\Gamma_x(X_x, Rf_*\mathcal{O}_{X'_x})$$

$$= \text{Hom}_{\mathcal{O}_{X,x}}(R\text{Hom}_{\mathcal{O}_x}(Rf_*\mathcal{O}_{X'_x}, \omega^*_{X,x}), E),$$

where we used the Leray spectral sequence for the first equality and the local duality for the second equality (for the statement of the local duality, see Lemma 0AAK in [Stacks]). Moreover,

$$R\text{Hom}_{\mathcal{O}_x}(Rf_*\mathcal{O}_{X'_x}, \omega^*_{X,x}) = Rf_*R\text{Hom}_{\mathcal{O}_{X'_x}}(\mathcal{O}_{X'_x}, \omega^*_{X'_x}, \text{codim}_X x])$$

$$= Rf_*\omega^*_{X'_x, \text{codim}_X x}] = \omega^*_{X'_x, \text{codim}_X x},$$

where we used the Grothendieck duality on the first equality and the Kawamata-Viehweg vanishing theorem for the third equality. Hence we have the assertion. \qed
For any normal projective variety \( X \) and a big and finitely generated divisor \( D \) on \( X \), we may consider the following birational contraction

\[
X \longrightarrow X_0 := \text{Proj} \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mD))
\]

and we say that the map is the \textit{birational contraction induced by} \( D \).

The following lemma is for an equivalent notion of bigness and finite generation. The following lemma explains why we define Definition 1.6 in such a way.

**Lemma 2.6.** Let \( X \) be any normal projective variety and \( D \) be any effective divisor on \( X \). Then the following are equivalent:

(a) \( D \) is big and finitely generated.

(b) There are proper birational morphisms \( f', \pi \), a diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & X_0 \\
\downarrow{\pi} \quad & & \quad \downarrow{\pi} \\
X & \xrightarrow{f} & X_0,
\end{array}
\]

an ample divisor \( H \) on \( X_0 \) and a positive integer \( p \) such that \( X', X_0 \) are normal projective varieties and \( \pi^*(pD) - (f')^*H \) is an effective \( f' \)-exceptional divisor on \( X' \).

**Proof.** For \((a) \implies (b)\), let \( f : X \longrightarrow X_0 \) be the contraction induced by \( D \). Then we have an ample divisor \( H \) on \( X_0 \) and a positive integer \( p \) such that \( pD = f^*H + F \) holds for some effective \( f \)-exceptional divisor \( F \) (see Lemma 1.6 in [HK00]). By the resolution of indeterminacies, we have a normal projective variety \( X' \) and the following diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & X_0 \\
\downarrow{\pi} \quad & & \quad \downarrow{\pi} \\
X & \xrightarrow{f} & X_0,
\end{array}
\]

From this, \( \pi^*(pD) - (f')^*H = \pi^*F \) is an effective \( f' \)-exceptional divisor on \( X' \).

For \((b) \implies (a)\), set \( F' = \pi^*(pD) - (f')^*H \). Then we have \( \pi^*(pD) = (f')^*H + F' \) and hence \( D \) is big. For the proof of finite generation, we have

\[
H^0(X_0, \mathcal{O}_{X_0}(mH)) = H^0(X', \mathcal{O}_{X'}(m(f')^*H)) = H^0(X', \mathcal{O}_{X'}(m(f')^*H + mF')) = H^0(X', \mathcal{O}_{X'}(m\pi^*(pD))) = H^0(X, \mathcal{O}_X(mpD)).
\]

Hence the proof is complete. \( \Box \)

The following result is devoted to defining the notion of \textit{weak decomposition}, which is needed in formulating Theorem 1.8.
Lemma 2.7. Let $X$ be any normal projective variety. Suppose that there are two birational contractions $f : X \longrightarrow X_0, f' : X \longrightarrow X'_0$, two ample divisors $H, H'$ on $X_0, X'_0$, effective $f, f'$-exceptional divisors $F, F'$ respectively and $f^* H + F = (f')^* H' + F'$. Then $X_0 \cong X'_0$, $F = F'$ and $H = H'$.

Proof. By Lemma 1.7 in [HK00], the map $f \circ g^{-1} : X'_0 \longrightarrow X_0$ is a proper birational morphism and $F = F'$. If it is not an isomorphism, then $H' = (f \circ g^{-1})^* H$ is not ample, leading to a contradiction. Thus $f \circ g^{-1}$ is an isomorphism and $H = H'$. □

Definition 2.8. Let $X$ be any normal projective variety, $D$ be any big and finitely generated divisor on $X$ and $f : X \longrightarrow X_0$ be the birational contraction induced by $D$. Suppose $H$ is an ample divisor, $F$ is an effective $f$-exceptional divisor and $p$ is a positive integer such that $pD = f^* H + F$. The decomposition $pD = f^* H + F$ is called the weak decomposition of $pD$.

Remark 2.9. Under the same conditions of Definition 2.8, note that the weak decomposition of $pD$ is unique. In the rest of the paper, we write the weak decomposition of $pD$ as $pD = P + N$, where $P := f^* H$ and $N := F$.

3. $q$-BIRATIONAL MORPHISMS

This section aims to define the notion of $q$-birational morphisms and to prove Theorem 1.3 and Theorem 1.4. Let us define the following notions.

Definition 3.1. Let $X, X'$ be any normal varieties and $f : X' \rightarrow X$ be any proper birational morphism.

(a) The center of $f$ is the reduced closed subscheme $C$ of $X$ which is the image of the exceptional locus along $f$.

(b) We say $f$ is a $q$-birational morphism if the exceptional locus has codimension 1 and the center of $f$ has codimension $\geq q + 1$.

Let us prove the following birational criterion of $(S_{q+1})$.

Proposition 3.2. Let $X$ be any normal variety with $(RS_q)$. Then the following are equivalent:

(a) $X$ is $(S_{q+1})$.

(b) For any proper birational morphism $f : X' \rightarrow X$ with $X'$ smooth, $R^i f_* \mathcal{O}_{X'} = 0$ for $1 \leq i < q$.

Proof. For (a) $\Rightarrow$ (b), choose any resolution $f : X' \rightarrow X$ of $X$ which is an isomorphism outside the singular locus of $X$. For any point $x \in X$, there is a spectral sequence

$$E^s_{2} = H^s_x(X, (R^i f_* \mathcal{O}_{X'})) \Rightarrow H^{s+i}_{f^{-1}(x)}(X', \mathcal{O}_{X'}).$$

Suppose that $x \in C$ is a point of the center $C$ of $f$.

Let us assume that $2 \leq n \leq q$ and $R^i f_* \mathcal{O}_{X'} = 0$ for $1 \leq i < n - 1$. It suffices to show that $x$ is not an associated point of $R^{n-1} f_* \mathcal{O}_{X'}$. Note that we may assume...
codim\(_X\) \(x \geq q + 1\), because \((R^n f_* O_{X'})_x = 0\) for any point \(x \in C\) with codim\(_X\) \(x \leq q\) and any \(i \geq 1\).

Consider the spectral sequence (3.1). For any \(0 \leq i \leq n\), \(E_2^{00} = 0\) since \(X\) is \((S_{q+1})\). Using the our assumption leads to \(E_2^{(n-j)j} = 0\) for \(1 \leq j < n - 1\). Moreover, \(E^{n-1} = 0\) by Corollary 2.5. Hence, we have

\[
E_2^{0(n-1)} = E_3^{0(n-1)} = \cdots = E_\infty^{0(n-1)} = E^{n-1} = 0.
\]

Therefore \(H_2^0(X_x, (R^{n-1}f_* O_{X'})_x) = 0\) and hence \(R^{n-1}f_* O_{X'}\) does not have associated point as \(x\). Thus, we have the assertion.

For \((b) \implies (a)\), consider the spectral sequence (3.1). What we have to prove is for any point \(x \in X\), \(H_2^0(X_x, O_{X,x}) = 0\) for \(i < \min\{q + 1, \text{codim}_X x\}\). Indeed, by \(R^i f_* O_{X'} = 0\) for \(1 \leq i < q\), \(H_2^0(X_x, (R^i f_* O_{X'})_x) = 0\) for \(0 \leq t < q\). Thus, by the spectral sequence argument, we obtain

\[
E_2^0 = E_3^0 = \cdots = E_\infty^0 \subseteq H_{f^{-1}(x)}^1(X'_x, O_{X',x})
\]

for any \(i \leq q\). Using Corollary 2.5, we have the assertion. \(\square\)

Remark 3.3. Let us note that the above proof is similar to the proof of Lemma 5.12 in [KM98]. Furthermore, for \(q = 2\), Proposition 7.1.7 in [Fuj17] induces our proposition. It is worth noting that Proposition 3.2 is similar to Lemma 4.1 in [AH12]. In [AH12], the lemma is for \((C_{q+1})\) because the proof used sheaf cohomology, not local cohomology. Our theorem is for \((S_{q+1})\) because we used local cohomology as in [Fuj17].

Let us prove the main results of this section, Theorem 1.3 and Theorem 1.4.

Theorem 3.4. Let \(X\) be a variety with \((R_q), (S_{q+1})\) and \(f : X' \to X\) be any \(q\)-birational morphism from a smooth variety \(X'\). Then \(R^i f_* O_{X'}(E) = 0\) for any \(1 \leq i < q\) and any effective \(f\)-exceptional divisor \(E\).

Proof. The main idea of the proof of Theorem 3.4 goes back to the proof of Lemma 3.3 in [Kov99]. Let us note that \(R^i f_* O_{X'} = 0\) for \(1 \leq i < q\) by Proposition 3.2.

For any closed point \(x \in X\), set \(F := f^{-1}(x)\) and let \(J\) be the ideal sheaf of \(F\) in \(X'\). Then the theorem of formal functions gives

\[
(R^i f_* O_{X'}(E))^\wedge_x = \lim_{\leftarrow} H^i(X', O_{X'}(E) \otimes O_{X'/J^n})
\]

and by the Serre duality, \(H^i(X', O_{X'}(E) \otimes O_{X'/J^n})\) is the dual of

\[
\Ext_{X'}^{\dim X - i}(O_{X'/J^n}, \omega_{X'} \otimes O_{X'}(-E)) = \Ext_{X'}^{\dim X - i}(O_{X'}(E) \otimes O_{X'/J^n}, \omega_{X'}).\]

By the definition of local cohomology, the direct limit

\[
\lim_{\rightarrow} \Ext_{X'}^{\dim X - i}(O_{X'}(E) \otimes O_{X'/J^n}, \omega_{X'})
\]

coincides with \(H_F^{\dim X - i}(X', \omega_{X'} \otimes O_{X'}(-E))\). Therefore, we have

\[
(R^i f_* O_{X'}(E))^\wedge_x \text{ is dual to } H_F^{\dim X - i}(X', \omega_{X'} \otimes O_{X'}(-E)).
\]
By the Leray spectral sequence
\[ E_2^{st} = H^s_x(X, R^t f_* (\omega_{X'} \otimes O_{X'}(-E))) \longrightarrow H^{s+t}_F(X', \omega_{X'} \otimes O_{X'}(-E)), \]
and the relative Kawamata-Viehweg vanishing theorem, we have
\[ H^i_F(X', \omega_{X'} \otimes O_{X'}(-E)) = 0 \]
for any vector bundle on \( X \). Hence \( \dim X \leq \dim X' - 1 \).

By considering the injection \( \omega_{X'} \otimes O_{X'}(-E) \rightarrow \omega_{X'} \), we obtain an injection \( f_* (\omega_{X'} \otimes O_{X'}(-E)) \rightarrow f_* \omega_{X'} \) and an exact sequence
\[ 0 \rightarrow f_* (\omega_{X'} \otimes O_{X'}(-E)) \rightarrow f_* \omega_{X'} \rightarrow Q \rightarrow 0 \]
for some coherent sheaf \( Q \) on \( X \). Note that the dimension of the support of \( Q \) is \( \leq \dim X - q - 1 \). Hence the long exact sequence of local cohomology implies that
\[ H^i_F(X, f_* (\omega_{X'} \otimes O_{X'}(-E))) = H^i_X(X, f_* \omega_{X'}) = 0 \]
for \( 1 \leq i < q \). Note that \( H^i_F(X, Q) = 0 \) for \( j \leq q \) which follows from dimension count. Hence \( R^i f_* O_{X'}(E) = 0 \) for any closed point \( x \in X \) and \( R^i f_* O_{X'}(E) = 0 \) for \( 1 \leq i < q \).

Remark 3.5. [ST08] proved our theorem for the Cohen-Macaulay case. Indeed, under the settings of Theorem 3.4, let \( C \) be the center of \( f \) and \( a \) be an ideal sheaf on \( X \) which is contained in the ideal sheaf of \( C \). Suppose that \( f \) is a log resolution of \( (X, a) \) and \( E \) is the \( f \)-exceptional divisor on \( X' \) with \( aO_{X'} = O_{X'}(-E) \). Then \( (X, a) \) has rational singularities outside of \( C \) and we have \( R^i f_* O_X(E) = 0 \) by Lemma 4.8 in [ST08].

Lemma 3.6. Let \( X \) be a normal variety, \( f : X' \rightarrow X \) be \( q \)-birational morphism and \( E \) be any vector bundle on \( X' \). Then there is an effective \( f \)-exceptional divisor \( E \) on \( X' \) such that the cokernel of the following map
\[ f_* (E \otimes O_{X'}(-E)) \rightarrow f_* E \]
has the support of codimension \( q + 1 \).

Proof. Let \( Q \) be the cokernel of the map \( E \otimes O_{X'}(-E) \rightarrow E \), \( C \) be the center of \( f \) and \( E' := f^{-1}(C) \). If we use the relative Serre vanishing theorem, we have that for some positive integer \( m \), there is an exact sequence
\[ 0 \rightarrow f_* (E \otimes O_{X'}(-mE')) \rightarrow f_* E \rightarrow f_* Q \rightarrow 0, \]
since \( R^1 f_* (E \otimes O_{X'}(-mE')) = 0 \) holds. Note that \( E' \) is anti \( f \)-ample.

By the theorem of formal functions, we have
\[ f_* E^\wedge = \lim_{n} f_* (E \otimes O_n E'), \]
where \( E^\wedge \) is the formal sheaf associated to \( E \).
where \( nE' \) is the \( n \)th infinitesimal neighborhood of \( E' \subseteq X' \), \( nC \) is the \( n \)th infinitesimal neighborhood of \( C \subseteq X \) and

\[
f_*\mathcal{E}^\wedge := \lim_{n} (f_*\mathcal{E} \otimes \mathcal{O}_{nC}).
\]

Let \( x \in C \) be a generic point of \( C \). Then the natural map \( f_*\mathcal{E}_x \to f_*\mathcal{E}_x^\wedge \) is an injection and hence \( f_*\mathcal{E}_x^\wedge \neq 0 \). Thus there is a positive integer \( n' \) such that \( (f_*\mathcal{E} \otimes \mathcal{O}_{n'C})_x \neq 0 \). Therefore, \( (f_*\mathcal{E} \otimes \mathcal{O}_{n'C})_{x'} \neq 0 \) for any closed point \( x' \in \{x\} \). Set \( E := \max\{m, n'\}E' \).

**Theorem 3.7.** Let \( X \) be a Cohen-Macaulay variety with \( (R_q) \) and \( f : X' \to X \) be any proper birational morphism from a smooth variety \( X' \). Suppose that \( R^i f_*\mathcal{O}_{X'}(E) = 0 \) for any \( 1 \leq i < q \) and any effective \( f \)-exceptional divisor \( E \). Then \( f \) is \( q \)-birational.

**Proof.** Let \( c \geq 2 \) be the codimension of the center of \( f \). For any effective \( f \)-exceptional divisor \( E \), we may consider the injection

\[
\varphi_0 : \omega_{X'} \otimes \mathcal{O}_{X'}(-E) \to \omega_{X'}.
\]

Denote by \( Q \) the cokernel. Note that the support of \( f_*Q \) has codimension \( c \) for some \( E \) by Lemma 3.6. Let us consider the map \( \varphi_1 : f_*\omega_{X'} \to \omega_X \) in Proposition 5.77 in [KM98], the composition

\[
\varphi : f_* (\omega_{X'} \otimes \mathcal{O}_{X'}(-E)) \xrightarrow{f_*\varphi_0} f_*\omega_{X'} \xrightarrow{\varphi} \omega_X,
\]

and the cokernel \( Q' \) of \( \varphi \). Since \( \text{Supp} f_*Q \subseteq \text{Supp} Q' \) holds and the support of the cokernel of \( \varphi_1 \) has codimension \( \geq c \), the support of \( Q' \) has codimension \( c \) for such \( E \).

From the injection, we have an exact sequence

\[
0 \to f_* (\omega_{X'} \otimes \mathcal{O}_{X'}(-E)) \xrightarrow{\varphi} \omega_X \to Q' \to 0. \tag{3.4}
\]

Take local cohomology to (3.4) as in the proof of Theorem 3.4. Then we have an exact sequence

\[
H_x^{\dim X-c}(X, \omega_X) \to H_x^{\dim X-c}(X, Q) \to H_x^{\dim X-c+1}(X, f_* (\omega_{X'} \otimes \mathcal{O}_{X'}(-E))) \to H_x^{\dim X-c+1}(X, \omega_X)
\]

for any closed point \( x \in X \). Since \( X \) is Cohen-Macaulay (note that if \( X \) is Cohen-Macaulay, then \( \omega_X \) is also a Cohen-Macaulay sheaf on \( X \) by Lemma 0AWS in [Stacks]), we obtain

\[
H_x^{\dim X-c}(X, Q) \cong H_x^{\dim X-c+1}(X, f_* (\omega_{X'} \otimes \mathcal{O}_{X'}(-E)))
\]

for any closed point \( x \in X \).

By Theorem 3.5.7 in [BH98], there is a closed point \( x \in X \) such that

\[
H_x^{\dim X-c+1}(X, f_* (\omega_{X'} \otimes \mathcal{O}_{X'}(-E))) \neq 0.
\]

For such \( x \), using the Kawamata-Viehweg vanishing theorem and the Leray spectral sequence as in the proof of Theorem 3.4, we have

\[
H_F^{\dim X-c+1}(X, \omega_X \otimes \mathcal{O}_{X'}(-E)) \neq 0,
\]

where \( F := f^{-1}(x) \). Hence, \( (R^{c+1}f_*\mathcal{O}_{X'}(E))_F \neq 0 \) by (3.2) and the theorem follows. \( \square \)
Theorem 3.8. Let $X, X'$ be any projective varieties, $X$ be $(R_q)$ and Cohen-Macaulay, $X'$ be smooth and $f : X' \to X$ be any proper birational morphism. Then $f$ is a $q$-birational morphism if and only if for any effective $f$-exceptional divisor $E$ on $X'$, $R^i f_* \mathcal{O}_{X'}(E) = 0$ for $1 \leq i < q$.

We might expect that Theorem 3.8 is true for any variety $X$ with $(R_q)$ and $(S_{q+1})$. However, we cannot prove or disprove it.

We may write Theorem 3.8 as an absolute cohomology vanishing.

Corollary 3.9. Let $X, X'$ be any normal projective varieties and $f : X' \to X$ be any proper birational morphism. Suppose that $X'$ is smooth and $X$ is $(R_q)$ and $(S_{q+1})$.

(a) Suppose $f$ is $q$-birational. Then for any Cartier divisor $D$ on $X$ and effective $f$-exceptional divisor $F$,

\[(3.5) \quad H^i(X', \mathcal{O}_{X'}(f^* D + F)) = H^i(X, \mathcal{O}_X(D)) = H^i(X', \mathcal{O}_{X'}(f^* D))\]

for $1 \leq i < q$.

(b) Suppose $X$ is Cohen-Macaulay, $f$ has divisorial exceptional locus and (3.5) holds for $1 \leq i < q$. Then $f$ is $q$-birational.

Proof. For (a), the second equality follows from Proposition 3.2. Hence it suffices to prove the first equality only.

If $f$ is $q$-birational, by the Leray spectral sequence

\[E_2^{ij} = H^i(X, R^j f_* \mathcal{O}_{X'}(f^* D + F)) \implies H^{i+j}(X', \mathcal{O}_{X'}(f^* D + F))\]

and Theorem 3.4, the assertion is proved.

For (b), we follow the idea of the proof of Theorem 1-2-3 in [KMM87]. Let $m$ be any nonnegative integer and $H$ be any ample divisor on $X$. Consider the Leray spectral sequence

\[E_2^{ij} = H^j(X', R^i f_* (\mathcal{O}_{X'}(mf^* H + F))) \implies H^{i+j}(X', \mathcal{O}_{X'}(mf^* H + F)).\]

For $1 \leq j < q$, if $m \gg 0$, then

\[H^j(X', \mathcal{O}_{X'}(mf^* H + F)) = H^j(X, \mathcal{O}_X(mH)) = 0\]

by our assumption and the Serre vanishing theorem. By the same way, if $i \geq 1$, then $E_2^{ij} = 0$ for any $j \geq 0$ and $m \gg 0$. Hence,

\[(3.6) \quad H^0(X, R^t f_* \mathcal{O}_{X'}(F) \otimes \mathcal{O}_X(mH)) = E_2^{0j} = E_\infty^{0j} = 0 \quad \text{for } 1 \leq j < q \text{ and } m \gg 0.\]

Let $1 \leq t < q$. If $R^t f_* \mathcal{O}_{X'}(F) \neq 0$, then for $m \gg 0$, $R^t f_* \mathcal{O}_{X'}(F) \otimes \mathcal{O}_X(mH)$ is a globally generated sheaf on $X$ and hence

\[H^0(X, R^t f_* \mathcal{O}_{X'}(F) \otimes \mathcal{O}_X(mH)) \neq 0.\]

It contradicts (3.6). Therefore $R^t f_* \mathcal{O}_{X'}(F) = 0$ and by Theorem 3.8, we obtain the assertion. \qed
We end this chapter with the following consequence of Theorem 3.4. Note that the proof is akin to the proof of Theorem 3.4 in [Ale08] and the proof of Proposition 7.1.4 in [Fuj17].

**Corollary 3.10.** Let $X$ be any normal variety and $\Delta$ be any effective $\mathbb{Q}$-Weil divisor such that $(X, \Delta)$ is log canonical. Suppose $D$ is a reduced Cartier divisor on $X$ such that $(X, \Delta + \varepsilon D)$ is log canonical for some $0 < \varepsilon \ll 1$. If $X$ is $(R_q)$ and $(S_{q+1})$, then $D$ is $(S_{q+1})$.

**Proof.** We may follow the idea of the proof of Theorem 3 in [Kol11]. Note that the closure of a point of $D$ is not an lc center by the monotonicity property of log discrepancy (see [KM98], Lemma 2.27). Let $x \in D$ be any point. We may assume that $|\Delta| = 0$. Indeed, if $\Delta = \sum d_i D_i$, then consider $\sum (\frac{d_i}{\varepsilon}) (D_{1i} + D_{2i})$, where $D_{1i}, D_{2i}$ are general members of the linear system $|D_i|$.

Consider the following exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0.$$ 

By the local cohomology exact sequence, it suffices to show $H^i_x(X, \mathcal{O}_X(-D)) = 0$ for any $i < \min\{q + 2, \text{codim}_X x\}$.

Let $f : X' \to X$ be a log resolution of $(X, \Delta)$. Then there is a divisor $\Delta_{X'}$ on $X'$ such that $K_{X'} + \Delta_{X'} = f^*(K_X + \Delta)$ and the coefficients of $\Delta_{X'}$ are $\leq 1$. Hence, we have

$$(3.7) \quad K_{X'} + f_*^{-1}\Delta \sim_{\mathbb{Q}} F - F' - F'' + f^*(K_X + \Delta),$$

where $F$ is an effective $f$-exceptional divisor, $F'$ is a simple normal crossing divisor with $[F'] = 0$ and $F''$ is a reduced $f$-exceptional divisor. We may write (3.7) as follows:

$$F \sim_{\mathbb{Q}} K_{X'} + F'' + (f_*^{-1}\Delta + F') = f^*(K_X + \Delta).$$

If we use Ambro’s vanishing theorem (see Theorem 2.3 in [Amb03]), we have $x$ is not an associated point of $R^if_*\mathcal{O}_{X'}(F)$. Moreover, since $f$ is $q$-birational, $R^if_*\mathcal{O}_{X'}(F) = 0$ for $1 \leq i < q$ holds by Theorem 3.4.

Consider the Leray spectral sequence

$$E^2 = H^i_x(X, R^if_*\mathcal{O}_{X'}(F - f^*D)) \implies H^{i+t}_{f^{-1}(x)}(X', \mathcal{O}_{X'}(F - f^*D)).$$

Since $x$ is not an associated point of $R^if_*\mathcal{O}_{X'}(F)$, by the projection formula,

$$H^0_x(X, R^if_*\mathcal{O}_{X'}(F - f^*D)) = 0.$$ 

Moreover, $H^i_x(X, R^if_*\mathcal{O}_{X'}(F - f^*D)) = 0$ for $0 \leq i < q$. Hence, by inspecting the spectral sequence, $E^2 = E^0 \subseteq E^i$ for $0 \leq i < \min\{q + 2, \text{codim}_X x\}$ and

$$(3.8) \quad H^i_x(X, \mathcal{O}_X(-D)) \subseteq H^{i+t}_{f^{-1}(x)}(X', \mathcal{O}_{X'}(F - f^*D)) \text{ for } i < \min\{q + 2, \text{codim}_X x\}.$$ 

By the Kawamata-Viehweg vanishing theorem $R^if_*\mathcal{O}_{X'}(K_{X'} - F + f^*D) = 0$ for $i \geq 1$. Thus, (3.2) gives us that $H^{i+1}_{f^{-1}(x)}(X', \mathcal{O}_{X'}(F - f^*D)) = 0$ for $i < \text{codim}_X x$. Therefore by (3.8), we have the assertion. \qed
4. q-BIRATIONAL DIVISORS

In this section we introduce the notion of q-birational divisors and prove Theorem 1.7 and Theorem 1.8. Moreover, we discuss the questions about the generalizations of those theorems.

**Definition 4.1.** Let $X$ be any smooth projective variety and $D$ be any divisor on $X$. We say $D$ is *weakly q-birational* if there are smooth projective varieties $X'$ and $X_0$, a diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & X_0, \\
\pi & \downarrow & \\
X & \xleftarrow{\pi} & 
\end{array}
\]

an ample Cartier divisor $H$ on $X_0$ and a positive integer $p$ such that

(a) $f'$ is q-birational and,

(b) $\pi^* (pD) - (f')^* H$ is a (not necessarily effective) $f'$-exceptional divisor on $X'$.

If $\pi^* (pD) - (f')^* H$ in (b) is effective, we say $D$ is *q-birational*. If $D$ is weakly q-birational, we say the minimum of $p$ is the *index* of $D$ and denote it by $p_D$.

**Remark 4.2.** By putting $X' = X$, $X_0 = X$ and $\pi, f' = \text{id}$, any ample divisor is q-birational for any $q \geq 2$. In Definition 4.1, if $D$ is q-birational, such ample divisor $H$ is unique by Lemma 2.7.

**Remark 4.3.** The notion of q-birationality is stronger than that of bigness and finite generation. If $X$ is a smooth projective variety and $D$ is an effective divisor on $X$, $D$ is q-birational if and only if it is big and finitely generated, $X'$, $X_0$ are smooth, and $f'$ is q-birational if we use the notions in Lemma 2.6.

The main purposes of introducing q-birationality are to state and prove the following Kodaira-type vanishing theorem and Zariski decomposition theorem.

**Theorem 4.4.** Let $X$ be any smooth projective variety and $D$ be any q-birational divisor on $X$. Then

$$H^i(X, \mathcal{O}_X(K_X + p_D D)) = 0 \text{ for } 1 \leq i < q.$$ 

**Proof.** Let

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & X_0, \\
\pi & \downarrow & \\
X & \xleftarrow{\pi} & 
\end{array}
\]

be the diagram in the definition of q-birationality and $H$ be the ample divisor on $X_0$ such that $p_D \pi^* D = (f')^* H + F_0$ for some effective $f'$-exceptional divisor $F_0$ on $X'$.
By Serre duality and Theorem 2.1,
\[ H^i(X, \mathcal{O}_X(K_X + p_D D)) = H^\dim X - i(X, \mathcal{O}_X(-p_D D)) \]
\[ = H^\dim X - i(X', \mathcal{O}_{X'}(-p_D \pi^* D)) \]
\[ = H^i(X', \mathcal{O}_{X'}(K_{X'} + p_D \pi^* D)). \]

Since \( f' \) is \( q \)-birational and Corollary 3.9 holds, we have
\[ H^i(X', \mathcal{O}_{X'}(K_{X'} + p_D \pi^* D)) = H^i(X', \mathcal{O}_{X'}(K_{X'} + p_D (f')^* H)). \]

Note that we used the fact that \( K_{X'} = (f')^* K_X + F \) for some effective \( f' \)-exceptional divisor \( F \) on \( X' \). Now, we can apply the Kawamata-Viehweg vanishing theorem to prove the assertion. \( \square \)

**Theorem 4.5.** Let \( X \) be any smooth projective variety, \( D \) be any \( q \)-birational divisor on \( X \) and \( p_D D = P + N \) be the weak decomposition of \( p_D D \) for the index \( p_D \) of \( D \). Then
\[ H^i(X, \mathcal{O}_X(mp_D D)) = H^i(X, \mathcal{O}_X(mP + mN)) = H^i(X, \mathcal{O}_X(mP)) \]
for any \( m \geq 0 \) and \( 0 \leq i < q \).

**Proof.** Let

\[
\begin{array}{c}
\pi \\
\downarrow \\
X' \\
\downarrow \\
X_0 \\
\end{array}
\quad \begin{array}{c}
f' \\
\downarrow \\
\end{array}
\]

be the diagram in the definition of \( q \)-birationality. By Theorem 2.1,
\[ H^i(X, \mathcal{O}_X(mp_P + mN)) = H^i(X', \mathcal{O}_{X'}(m\pi^* P + m\pi^* N)). \]

We see that there is an ample divisor \( H \) on \( X_0 \) such that \( \pi^* P = (f')^* H \) and \( \pi^* (mN) \) is an effective \( f' \)-exceptional divisor on \( X' \). Hence Corollary 3.9 gives
\[ H^i(X', \mathcal{O}_{X'}(m\pi^* P + m\pi^* N)) = H^i(X', \mathcal{O}_{X'}(m(f')^* H)), \]
if we use \( q \)-birationality of \( f' \). Therefore, by Theorem 2.1, we have the assertion. \( \square \)

The rest of the paper discusses generalizations of Theorem 4.4 and Theorem 4.5. To do this, let us prove the numerical invariance of weak \( q \)-birationality.

**Proposition 4.6.** Let \( X \) be any smooth projective variety and \( D \) be any weakly \( q \)-birational divisor on \( X \). Then for some \( m \geq 1 \), \( mD \) is also weakly \( q \)-birational. Conversely, for any divisor \( D' \) on \( X \), if \( mD' \) is weakly \( q \)-birational for some \( m \geq 1 \), then \( D' \) is weakly \( q \)-birational.

**Proof.** For the first assertion, under the same setting of Definition 4.1, \( \pi^*(pmD) - f'^*(mH) \) is an \( f' \)-exceptional divisor on \( X' \). The second assertion follows from the definition immediately. \( \square \)

The following shows that weak \( q \)-birationality is preserved by numerical equivalence.
Theorem 4.7. Let $X$ be any smooth projective variety, $D$ be any weakly $q$-birational divisor on $X$ and $D'$ be any divisor on $X$. If $D \equiv D'$, then $D'$ is weakly $q$-birational.

Proof. Let $\pi : X' \to X$ be the diagram in the definition and $H$ be an ample $\mathbb{Q}$-divisor on $X_0$ such that $\pi^*D - (f')^*H$ is $f$-exceptional. By Lemma 2.40 in [Laz04], there is a very ample divisor $N$ such that $N + P$ is globally generated for any numerically trivial divisor $P$ on $X$.

It suffices to show that $\pi^*D - (f')^*H'$ is $f$-exceptional for some ample $\mathbb{Q}$-divisor $H'$ on $X_0$. If we set $P := D' - D$, then $D' = D + P$.

By

$$\pi^*(mD') - (f')^*(mH + (f')_*\pi^*(N + mP) - (f')_*\pi^*N)$$

$$\sim_{\mathbb{Q}} (\pi^*(mD) - (f')^*(mH)) + (\pi^*(N + mP) - (f')^*(f')_*\pi^*(N + mP))$$

$$- (\pi^*N - (f')^*f'_*\pi^*N)$$

and the fact that $mH + (f')_*\pi^*(N + mP) - (f')_*\pi^*N$ is ample for $m \gg 0$, we have $mD'$ is $q$-birational for $m \gg 0$. Therefore $D'$ is $q$-birational. \qed

Remark 4.8. This proof does not work for $q$-birational divisors because we do not know whether

$$(\pi^*(N + mP) - (f')^*(f')_*\pi^*(N + mP)) - (\pi^*N - (f')^*f'_*\pi^*N)$$

is effective or not. We can not expect numerical invariance for $q$-birational divisors because finite generation is not preserved by numerical equivalence in general.

Remark 4.9. We may define (weak) $q$-birationality with weaker condition that $X_0$ in Definition 4.1 is required to be $(R_4)$ and $(S_{q+1})$ in which $X_0$ is not necessarily smooth. If we use such weaker definition of (weak) $q$-birationality instead of the definition in Definition 4.1, we can prove the analogous statements of Theorem 4.4 and Theorem 4.5. However, we do not know whether the analog of Theorem 4.7 is true. The main issue is unless $X_0$ is $\mathbb{Q}$-factorial, the notion $(f')^*(f')_*\pi^*N$ does not make sense because $(f')_*\pi^*N$ is not necessarily $\mathbb{Q}$-Cartier.

Theorem 4.7 gives a new class of divisors related to nefness.

Definition 4.10. Let $X$ be any smooth projective variety and $N^1(X)$ be the Néron-Severi group of $X$ as a finitely generated $\mathbb{Z}$-module.

(a) We call $D \in N^1(X)$ weakly $q$-birational if an inverse of $D$ along $\text{Pic}(X) \to N^1(X)$ is weakly $q$-birational.

(b) For $D \in N^1(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, we call $D$ weakly $q$-birational if there is a positive integer $m$ such that $mD$ is a weakly $q$-birational integral divisor on $X$. 


(c) For $D \in N^1(X) \otimes \mathbb{R}$, we call $D$ q-nef if it is pseudo-effective and an element of the closure of the set of weakly q-birational $\mathbb{Q}$-divisors on $X$.

(d) For $D \in \text{Pic}(X)$, we call $D$ q-nef if the image of $D$ along $\text{Pic}(X) \to N^1(X)$ is q-nef.

Now, we can formulate the following two questions, which are generalizations of Theorem 4.4 and Theorem 4.5. For the definition of divisorial Zariski decomposition, see Definition 3.1.12 in [Nak04].

**Question 4.11.** Let $X$ be any smooth projective variety and $D$ be any big and q-nef $\mathbb{Q}$-divisor on $X$. Does the following vanishing

$$H^i(X, \mathcal{O}_X(K_X + [D])) = 0$$

hold for $1 \leq i < q$?

**Question 4.12.** Let $X$ be any smooth projective variety, $D$ be any big integral divisor on $X$ and $D = P + N$ be the divisorial Zariski decomposition of $D$. If $D$ is q-nef, does the map $\mathcal{O}_X([mP]) \to \mathcal{O}_X(mD)$ induce isomorphisms

$$H^i(X, \mathcal{O}_X([mP])) \cong H^i(X, \mathcal{O}_X(mD))$$

for all $m \geq 0$ and $0 \leq i < q$?

**Remark 4.13.** For $q = \dim X$, Question 4.11 and Question 4.12 have affirmative answers, because any q-nef divisors are nef. Note that for any big and nef divisor, the Kawamata-Viehweg vanishing theorem holds and a nef divisor does not have the negative part in its Zariski decomposition.

**REFERENCES**

[Ale08] V. Alexeev: Limits of stable pairs, Pure Appl. Math. Q. 4 (2008), no. 3, 767–783.

[Amb03] F. Ambro: Quasi-log varieties, Proc. Steklov Inst. Math. 240 (2003), no. 1, 214–233.

[AH12] V. Alexeev and C. Hacon: Non-rational centers of log canonical singularities, J. Algebra. 369 (2012), 1–15.

[BH98] W. Bruns and H. Herzog: Cohen-Macaulay rings, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1998.

[Fuj17] O. Fujino: Foundations of the minimal model program, MSJ Memoirs, vol. 35, Mathematical Society of Japan, Tokyo, 2017.

[HK00] Y. Hu and S. Keel: Mori dream spaces and GIT, Michigan Math. J. 48 (2000), 331–348.

[KMM87] Y. Kawamata, K. Matsuda and K. Matsuki: Introduction to the minimal model problem, Adv. Stud. Pure Math. 10 (1987), 283–360.

[Koll11] J. Kollár: A local version of the Kawamata-Viehweg vanishing theorem, Pure Appl. Math. Q. 7 (2011), no. 4, 1477–1494.

[KM98] J. Kollár and S. Mori: Birational geometry of algebraic varieties, Cambridge Tracts in Mathematics, vol. 134. Cambridge University Press, Cambridge, 1998.

[Kov99] S. Kovács: Rational, log canonical, Du Bois singularities: on the conjectures of Kollár and Steenbrink, Compositio Math. 118 (1999), no. 2, 123–133.
[Laz04] R. Lazarsfeld: *Positivity in algebraic geometry I: Classical setting: line bundles and linear series*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 48, Springer, Berlin, 2004.

[Nak04] N. Nakayama: *Zariski-decomposition and abundance*, Mathematical Society of Japan, vol. 14, Tokyo, 2004.

[ST08] K. Schwede and S. Takagi: *Rational singularities associated to pairs*, Special volume in honor of Melvin Hochster, Michigan Math. J. 57 (2008), 625–658.

[Stacks] A. J. de Jong et al.: *The Stacks Project*, Available at http://stacks.math.columbia.edu

DEPARTMENT OF MATHEMATICS, YONSEI UNIVERSITY, 50 YONSEI-RO, SEODAEMUN-GU, SEOUL 03722, KOREA

Email address: whatisthat@yonsei.ac.kr