FLUCTUATIONS FOR ZEROS OF GAUSSIAN TAYLOR SERIES

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ABSTRACT. We study fluctuations in the number of zeros of random analytic functions given by a Taylor series whose coefficients are independent complex Gaussians. When the functions are entire, we find sharp bounds for the asymptotic growth rate of the variance of the number of zeros in large disks centered at the origin. To obtain a result that holds under no assumptions on the variance of the Taylor coefficients we employ the Wiman-Valiron theory. We demonstrate the sharpness of our bounds by studying well-behaved covariance kernels, which we call admissible (after Hayman).

1. Introduction

Some of the earliest works concerning random analytic functions are the ones of Littlewood and Offord [20, 19], who showed that the structure of the zero set of these functions is very regular. More recently, these classical results were sharpened in the papers [15, 22]. In this paper we consider the typical size of fluctuations in the number of zeros of random analytic functions whose coefficients are independent complex Gaussians. This is the most well-studied and best understood model (see the book [14] and ICM notes [24]).

Given a sequence \(\{a_n\}_{n \geq 0}\) of non-negative numbers, we consider random Taylor series

\[
f(z) = \sum_{n \geq 0} \xi_n a_n z^n,
\]

where \(\xi_n\) are independent and identically distributed standard complex Gaussians. We only consider transcendental analytic functions, that is, sequences \(\{a_n\}\) which contain infinitely many non-zero terms. Denote by \(\mathcal{Z}_f = f^{-1}\{0\}\) the zero set of \(f\); its properties are determined by the covariance kernel

\[
K(z, w) = E\left[f(z) \overline{f(w)}\right] = G(z \overline{w}), \quad \text{where} \quad G(z) := \sum_{n \geq 0} a_n^2 z^n.
\]

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We will call $G$ the covariance function of $f$; denote by $R_G$ the radius of convergence of $G$ around the origin. We consider both $R_G < \infty$ and $R_G = \infty$, and in the former case, without loss of generality, we may assume $R_G = 1$. Then, it is not difficult to check that the radius of convergence of $f$ is almost surely $R_G$ in both cases (see [14, Lemma 2.2.3]). When $R_G = \infty$ we call $f$ a Gaussian entire function.

Let $n_f(r)$ be the number of zeros of the function $f$ inside the disk $\{|z| \leq r\}$, where $r < R_G$. We are interested in the asymptotic statistical properties of the random variable $n_f(r)$, as $r \to R_G$. In order to study this asymptotics, it will be convenient to define the following functions

$$a(r) = a_G(r) := r \left(\log G(r)\right)', \quad b(r) = b_G(r) := ra'(r),$$

borrowing the notation used in [12]. Since the Taylor coefficients of $G$ are non-negative, the function $r \mapsto \log G(e^r)$ is convex, hence $a(r)$ is increasing, and $b(r)$ is non-negative for all $r < R_G$.

The Edelman-Kostlan formula [14, p. 25] (see also Appendix A) states that for any Gaussian analytic function $f$ of the form (1.1) we have

$$\mathbb{E}[n_f(r)] = a\left(r^2\right), \quad \text{for all } r < R_G.$$

However, the expected value provides little information about the distribution of the random variable. Here we will be interested in the asymptotic growth rate of the variance $\text{Var}(n_f(r))$ in terms of the functions $a$ and $b$, in general under no additional assumptions on the nature of the coefficients $a_n$.

In order to present the results we will need the following notation. We say that a set $E \subset \mathbb{R}^+$ is of finite logarithmic measure if

$$\int_E \frac{dt}{t} < \infty.$$

If $g_1, g_2 : \mathbb{R}^+ \to \mathbb{R}^+$ are non-negative functions, we write $g_1 \lesssim_L g_2$ if there is a constant $C > 0$, and a set $E \subset \mathbb{R}^+$ of finite logarithmic measure, such that $g_1 \leq Cg_2$ in $\mathbb{R}^+ \setminus E$. Finally, we write $g_1 \asymp_L g_2$ if $g_1 \lesssim_L g_2$ and $g_1 \gtrsim_L g_2$ both hold.

**Theorem 1.1.** For any Gaussian entire function $f$ with a transcendental covariance function $G$, and any $\varepsilon > 0$

$$\text{Var}(n_f(r)) \gtrsim_L \frac{b^2\left(r^2\right)}{a\left(r^2\right)^{2+\varepsilon}}.$$

In addition, if $b$ is a non-decreasing function, then

$$\text{Var}(n_f(r)) \gtrsim_L b\left(r^2\right).$$

**Remark 1.1.** With some more work the factor $a\left(r^2\right)^\varepsilon$ in Theorem 1.1 can be replaced by a power of $\log a\left(r^2\right)$, we will not pursue this here.
Remark 1.2. By [29, Lemma 1] it follows that for every $\varepsilon > 0$ we have that $b(r) \lesssim a(r)^{1+\varepsilon}$.

It turns out that the upper bound for the variance may be considerably larger asymptotically. The following result holds without any restrictions on $G$.

**Theorem 1.2.** Let $f$ be a Gaussian analytic function with covariance function $G$, then for every $r < R_G$

$$\text{Var}(n_f(r)) \leq b(r^2).$$

**Remark 1.3.** If $G(z)$ is of the form $a_n^2 z^n + a_m^2 z^m$, then one can check that $\text{Var}(n_f(r)) = b_G(r^2)$. This implies that for any non-decreasing and unbounded function $\beta : \mathbb{R}^+ \to \mathbb{R}^+$, there is a covariance function $G_\beta$, so that if $f$ is the Gaussian entire function whose covariance function is $G_\beta$, there is a sequence $r_n \to \infty$ so that

$$\text{Var}(n_f(r_n)) = (1 + o(1)) \beta(r_n) = (1 + o(1)) b_{G_\beta}(r_n^2), \quad n \to \infty,$$

and in particular

$$\limsup_{r \to \infty} \frac{\text{Var}(n_f(r))}{b_{G_\beta}(r^2)} = 1.$$ 

By Theorem 1.2 and Remark 1.2 we get the following conclusion.

**Corollary 1.3.** For any Gaussian entire function $f$ with a transcendental covariance function $G$, and every $\varepsilon > 0$,

$$\text{Var}(n_f(r)) \lesssim \mathbb{E}[n_f(r)]^{1+\varepsilon}.$$ 

1.1. **Well-behaved covariance functions.** If the covariance function $G$ of $f$ is sufficiently well-behaved, such as $e^z$, $e^{e^z}$, and the Mitag-Leffler functions

$$G(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)},$$

then it is possible to find the asymptotics of the variance. A notable example is the Gaussian Entire Function (GEF), with $G(z) = e^z$, whose zero set is invariant with respect to the isometries of the complex plane (see the book [14, Chapter 2.3]). Forrester and Honner [8] found the precise asymptotic growth of the variance for the GEF (see also [23]). In order to extend this result, we define two classes of admissible covariance functions, which in particular include all the previous examples. For the precise definitions see Sections 4.1 and 6.2. More examples, including Gaussian analytic functions with an admissible covariance function in the unit disk are described in Section 7.

**Theorem 1.4.** Let $f$ be a Gaussian analytic function with a type I admissible covariance function $G$. Then

$$\text{Var}(n_f(r)) = (1 + o(1)) \frac{\zeta(\frac{3}{2})}{4\sqrt{\pi}} \sqrt{b(r^2)}, \quad r \to R_G^-,$$

where $\zeta(u)$ is the Riemann zeta function.
Remark 1.4. This indicates that the lower bound in Theorem 1.1 is sharp.

Suppose $G$ is a sufficiently regular covariance function (see Section (6.2) for the precise requirements). In the next theorem we construct a Gaussian entire function $\tilde{f}$ with covariance function $\tilde{G}$, so that the variance of the number of zeros of $\tilde{f}$ is large outside a small exceptional set of values of $r$. The statement of the theorem requires the following definitions.

**Definition 1.5.** We will say that two covariance kernels $G$ and $\tilde{G}$ are similar if

$$a_{\tilde{G}}(r) \asymp_{L} a_{G}(r) \quad \text{and} \quad b_{\tilde{G}}(r) \asymp_{L} b_{G}(r).$$

**Definition 1.6.** Let $G(z) = \sum_{n=0}^{\infty} c_{n}z^{n}$ be an analytic function. A function $\tilde{G}$ is a Taylor series restriction of $G$, if $\tilde{G}(z) = \sum_{n=0}^{\infty} \delta_{n}c_{n}z^{n}$ with $\delta_{n} \in \{0, 1\}$ for all $n \in \mathbb{N}$.

**Theorem 1.7.** Let $G$ be a type II admissible function. There exists $\tilde{G}$ which is a Taylor series restriction of and similar to $G$, so that

$$\text{Var} \left( n_{\tilde{f}}(r) \right) \asymp_{L} b_{\tilde{G}}(r^{2}).$$

Remark 1.5. The theorem shows that the upper bound in Theorem 1.2 is sharp (up to a constant) for certain transcendental entire functions outside a set of finite logarithmic measure (cf. Remark 1.3).

**Remark 1.6.** By the example in Section 7.2 (which is type II admissible) it follows that in general $\varepsilon$ cannot be removed in Corollary 1.3.

1.2. **Background and related results.** Following earlier work by Edelman and Kostlan [6], and Offord [27], some fundamental properties of zeros of Gaussian analytic functions (GAFs) were developed by Sodin [33] (see also [16, Chapter 13]). Sodin and Tsirelson [34] found the asymptotics of the variance and proved a central limit theorem for smooth linear statistics for planar, spherical, and hyperbolic GAFs. More general results about linear statistics were obtain by Nazarov and Sodin [23, 25].

For the family of hyperbolic GAFs, whose zero sets are invariant with respect to the isometries of the unit disk, Buckley [4] found the asymptotics of the variance of the number of zeros (see also Section 7.4). Buckley and Sodin [5] studied fluctuations in the increment of the argument along curves for the planar GAF (GEF). Feldheim [7] derived bounds for the growth of the variance of zeros for GAFs which are invariant with respect to shifts. Ghosh and Peres [9] showed that the fast decay rate of the variance of smooth linear statistics of the GEF implies the rigidity of the zero set. Their technique was recently used by the authors in [17] to construct examples of “completely rigid” Gaussian entire functions.

Considerable amount of research is devoted to the study of zero sets of random algebraic and trigonometric polynomials. Maslova [21], Granville and Wigman [11], Azaïs, Dalmao, and León [2], and Nguyen and Vu [26] (by no means an exhaustive list) proved central limit theorems for real zeros of random polynomials. Bally, Caramellino, and Poly [3] studied the
dependence of the variance of the number of zeros on the distribution of the coefficients. Very recently, following the earlier work [32], Shiffman [30] found an asymptotic expansion for the variance of smooth statistics of random zeros on complex manifolds.

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## 2. Definitions and Preliminaries

Given an analytic function \( G(z) = \sum_{n \geq 0} a_n^2 z^n \), we denote its radius of convergence by \( R_G \), and assume from here on that \( R_G \in \{1, \infty\} \). We always assume \( a_n \) are non-negative, and contain infinitely many non-zero terms (i.e. \( G \) is transcendental). We recall the following notation

\[
a(z) = a_G(z) = z \frac{G'(z)}{G(z)}, \quad b(z) = b_G(z) = za_G'(z).
\]

We use little-\( o \) and big-\( O \) notation in the standard way. Given two functions \( g_1, g_2 : \mathbb{R} \to \mathbb{R}^+ \), we write \( g_1 \lesssim g_2 \) if \( g_1 = O(g_2) \), possibly on a subset of \( \mathbb{R} \) (depending on the context). We also write \( g_1 \sim g_2 \) when \( g_1(x) = (1 + o(1))g_2(x) \) as \( x \to \infty \). Recall that \( g_1 \lesssim_L g_2 \) when there exists a set \( \mathcal{N} \subset \mathbb{R}^+ \) and a constant \( C > 0 \) so that \( g_1(x) \leq Cg_2(x) \) for all \( x \in \mathcal{N} \), and \( \mathbb{R}^+ \setminus \mathcal{N} \) is a set of finite logarithmic measure. Let \( I \subset \mathbb{R}^+ \) be an open interval, we denote the fact that \( h : I \to \mathbb{R}^+ \) is a non-decreasing and unbounded function on \( I \) by writing \( h \uparrow \infty \).

### 2.1. A formula for the variance

Let \( G(z) = \sum_{n=0} a_n^2 z^n \) be the covariance function of a Gaussian analytic function \( f \), with radius of convergence \( R_G \in \{1, \infty\} \). For the rest of the paper it will be convenient to put \( e^t = r^2 \), and use the exponential change of variables

\[
H(t) = G(e^t) = \sum_{n \geq 0} a_n^2 e^{nt},
\]

and also define

\[
t_G := \log R_G, \quad A(t) := (\log H(t))' = a(e^t), \quad B(t) := A'(t) = b(e^t).
\]

Notice that \( A(z), B(z) \) are meromorphic functions which are given by

\[
A(z) = \frac{H'(z)}{H(z)}, \quad B(z) = \frac{H''(z)}{H(z)} - \left( \frac{H'(z)}{H(z)} \right)^2 = H^{-2}(z) \left( \sum_{n \leq m} (n - m)^2 a_n^2 a_m^2 e^{(m+n)z} \right).
\]

We will repeatedly use the following formula for the variance of \( n_f(r) \):

\[
\text{Var}(n_f(r)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|A(t + i\theta) - A(t)|^2}{\exp \left( 2 \int_0^\theta \text{Im}[A(t + i\varphi)] \, d\varphi \right) - 1} \, d\theta,
\]
for its proof see Claim A.2 in Appendix A (cf. [16, p. 195]). We will also use the following equivalent form

\begin{equation}
\text{Var} \left( n_f (r) \right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\left| H(t) H'(t + i\theta) - H(t + i\theta) H'(t) \right|^2}{H^2(t) (H^2(t) - |H^2(t + i\theta)|)} \, d\theta.
\end{equation}

2.2. Local admissibility. In order to bound the integral in (2.1) from below, we will introduce the following definition, which is motivated by a result from the Wiman-Valiron theory about the value distribution of entire functions, more precisely the asymptotics of such functions near their points of maximum modulus (see [13, Theorem 10]).

**Definition 2.1.** An analytic function $H$ is called local $\delta$-admissible on a set $T \subset (-\infty, t_G)$ if there is a function $\delta (t) : [-\infty, t_G) \to (0, \pi)$ so that for any $\varepsilon > 0$ there exists an $\eta > 0$, such that for $t \in T \cap (t_0 (\varepsilon), t_G)$ and $|\tau| \leq \eta \delta (t)$

\begin{equation}
\log \frac{H(t + \tau)}{H(t)} = \tau A(t) + \frac{1}{2} \tau^2 B(t) + h_t (\tau), \quad \text{where } |h_t (\tau)| \leq \varepsilon |\tau|^2 B(t).
\end{equation}

**Remark 2.1.** It is implicitly assumed that $t + \delta (t) < t_G$ for all $t < t_G$.

We will show in Section 3.2 that if $B \uparrow \infty$, then $H$ is local $\delta$-admissible outside a set of finite logarithmic measure, with $\delta (t) = \frac{1}{\sqrt{B(t)}}$. With a (smaller) appropriate choice of $\delta$ this statement is also true without making any assumptions on $B$, for the details see Section 3.2.3.

2.3. Lower bound for the variance assuming local admissibility. Let $f$ be a Gaussian analytic function with covariance function $G(z) = \sum_{n=0}^{\infty} a_n^2 z^n$, and radius of convergence $R_G \in \{1, \infty\}$. Recall our notation

\[ H(t) = G(e^t) = \sum_{n \geq 0} a_n^2 e^{nt}, \]

and

\[ t_G = \log R_G, \quad A(t) = a(e^t), \quad B(t) = b(e^t) = A'(t). \]

In the next lemma we use Cauchy’s integral formula to obtain estimates for $A, B$ when $H$ is a local $\delta$-admissible function.

**Lemma 2.2.** Let $H$ be a local $\delta-$admissible function on $T$ with $t_G \in \{0, \infty\}$. For any $\varepsilon > 0$, there exists $\eta > 0$ such that for any $t \in T \cap (t_0 (\varepsilon), t_G)$, and $|\theta| \leq \frac{\varepsilon}{2} \delta (t)$ we have

\[ (1 - \varepsilon) B(t) |\theta| \leq |A(t + i\theta) - A(t)| \leq (1 + \varepsilon) B(t) |\theta|, \]

and

\[ \frac{1 - \varepsilon}{2} \theta^2 B(t) \leq \int_{0}^{\theta} \text{Im} [A(t + i\varphi)] \, d\varphi \leq \frac{1 + \varepsilon}{2} \theta^2 B(t). \]
Proof. Given $\varepsilon > 0$, choose $\eta > 0$ as in the definition of a local $\delta$–admissible function, and let $0 < |\tau| \leq \frac{\eta}{2}\delta(t)$. Differentiating (2.3) with respect to $\tau$ we obtain

\begin{equation}
(2.4) \quad A(t + \tau) = \frac{H'(t + \tau)}{H(t + \tau)} = A(t) + \tau B(t) + h'_t(\tau),
\end{equation}

with

\[ h'_t(\tau) = \frac{1}{2\pi i} \int_{\Gamma} \frac{h_t(z)}{(z - \tau)^2} dz, \quad \text{where } \Gamma = \{ z : |z - \tau| = |\tau| \}. \]

By local-admissibility we have

\[ |h'_t(\tau)| \leq \frac{1}{|\tau|} \cdot \varepsilon |\tau|^2 B(t) = \varepsilon |\tau| B(t). \]

It follows from (2.4) that for $t \in T \cap (t_0(\varepsilon), t_G)$ and $|\theta| \leq \frac{\eta}{2}\delta(t)$ we have

\[ (1 - \varepsilon) B(t) |\theta| \leq |A(t + i\theta) - A(t)| \leq (1 + \varepsilon) B(t) |\theta|. \]

Since $A(t) \in \mathbb{R}$ we also have there

\[ \int_0^\theta \text{Im} [A(t + i\varphi)] d\varphi \leq \int_0^\theta \text{Im} [A(t) + i\varphi B(t) (1 + \varepsilon)] d\varphi = \frac{(1 + \varepsilon)}{2} \theta^2 B(t), \]

and similarly for the lower bound. $\square$

For $J \subset T$ we define the following integral

\begin{equation}
(2.5) \quad I(H, t, J) := \frac{1}{2\pi} \int_J \frac{|A(t + i\theta) - A(t)|^2}{\exp \left( 2 \int_0^\theta \text{Im} [A(t + i\varphi)] d\varphi \right) - 1} d\theta.
\end{equation}

**Corollary 2.3.** Let $H$ be a local $\delta$–admissible function on $T$ with $t_G \in \{0, \infty\}$, then for $t \in T$ we have

\[ \text{Var}(n_f(r)) = I(H, t, T) \gtrsim \min \left\{ \delta(t) B(t), \sqrt{B(t)} \right\}. \]

**Proof.** Applying Lemma 2.2 with $\varepsilon = \frac{1}{2}$, there exists an $\eta > 0$ so that for $t$ sufficiently large and $|\theta| \leq \frac{\eta}{2}\delta(t)$,

\[ \frac{|A(t + i\theta) - A(t)|^2}{\exp \left( 2 \int_0^\theta \text{Im} [A(t + i\varphi)] d\varphi \right) - 1} \geq \frac{\frac{1}{4} B^2(t) \theta^2}{\exp \left( \frac{1}{2} \theta^2 B(t) \right) - 1}. \]
Put $\Delta (t) := \min \left\{ \frac{\eta}{2} \delta (t), \frac{1}{\sqrt{B(t)}} \right\}$, by the inequality $e^x - 1 \leq 4x$ which is valid for $x \in [0, 2]$, we find

$$\mathcal{I} (H, t, \mathbb{T}) \geq \mathcal{I} (H, t, [-\Delta (t), \Delta (t)]) \geq \frac{1}{4} B^2 (t) \int_{-\Delta(t)}^{\Delta(t)} \frac{\theta^2}{6B(t)\theta^2} \, d\theta \geq \frac{1}{24} B (t) \Delta (t).$$

\[
\square
\]

3. Lower bound for the variance

In this section we prove Theorem 1.1. First we assume that $b$ is non-decreasing. Below the letters $\lambda, t, \theta, y$ denote real quantities, and $\tau$ is a complex number. It will be convenient to put $e^t = r^2$.

3.1. Normal values of $t$ and the set $\mathcal{X}$. We will now define a set $\mathcal{X} \subset \mathbb{R}^+$ whose complement is of finite Lebesgue measure, where the function $B$ increases slowly. Since $B$ is unbounded we may choose a sequence $t_\ell \uparrow \infty$ so that

$$B (t_\ell) = \ell^6, \quad \ell \geq 1.$$  

We then define a sequence of intervals $\{T_\ell\}_{\ell=1}^\infty$ by

$$T_\ell = [t_\ell, t_{\ell+1}], \quad |T_\ell| = t_{\ell+1} - t_\ell.$$  

**Definition 3.1.** The interval $T_\ell$ is long if

$$|T_\ell| \geq \frac{8}{\ell^2},$$  

otherwise it is short. For a long interval $T_\ell$ we define its interior by

$$\hat{T}_\ell = \left[t_\ell + \frac{2}{\ell^2}, t_{\ell+1} - \frac{2}{\ell^2}\right].$$  

**Remark 3.1.** Notice that long intervals have a non-trivial interior.

**Definition 3.2.** The set $\mathcal{X}$ of normal values of $t$ is given by

$$\mathcal{X} := \bigcup_{T_\ell \text{ long}} \hat{T}_\ell.$$  

**Remark 3.2.** Notice that $\sum_{T_\ell \text{ short}} |T_\ell| < \infty$ and also $\sum_{T_\ell \text{ long}} |T_\ell \setminus \hat{T}_\ell| < \infty$. Therefore, the Lebesgue measure of the set $\mathbb{R}^+ \setminus \mathcal{X}$ is finite.

**Remark 3.3.** Throughout the proof we may need to take the value of $\ell$ to be sufficiently large, thus we may drop finitely many intervals $\hat{T}_\ell$ from $\mathcal{X}$ without explicitly stating it.
3.2. Proof of local admissibility for non-decreasing $B$. Let $T_\ell$ be a long interval with $\ell \geq 4$. Since $B$ is non-decreasing, we have for all $t \in T_\ell$

$$B(t_\ell) \leq B(t) \leq B(t_{\ell+1}) = (\ell + 1)^6 \leq 4\ell^6 = 4B(t_\ell).$$

By the Lagrange formula for the remainder in the Taylor approximation for \(\log \frac{H(y+\lambda)\,H(y)}{H(y)}\) near \(\lambda = 0\), we have

$$\log \frac{H(y+\lambda)\,H(y)}{H(y)} = \lambda A(y) + \frac{\lambda^2}{2}B(c),$$

where \(|c - y| \leq |\lambda|\). If \(y + \lambda, y \in T_\ell\), then \(c \in T_\ell\), and we deduce that

$$\left|\log \frac{H(y+\lambda)\,H(y)}{H(y)} - \lambda A(y)\right| \leq \frac{\lambda^2}{2}B(c) \leq 2\lambda^2B(t_\ell).$$

3.2.1. An adaptation of Rosenbloom’s method. Recall that

$$H(z) = \sum_{n=0}^{\infty} a_n^2 e^{nz},$$

where \(a_n\) are non-negative. Following Rosenbloom [29] we define for \(t \in \mathbb{R}\), the random variable \(X_t \in \mathbb{N}\) as follows:

$$\mathbb{P}[X_t = k] = \frac{a_k^2 e^{kt}}{H(t)}, \quad k \in \mathbb{N}.$$  

Then

$$\mathbb{E}[X_t] = \frac{1}{H(t)} \sum_{k=0}^{\infty} ka_k^2 e^{kt} = \frac{H'(t)}{H(t)} = A(t),$$

and moreover

$$\text{Var}(X_t) = \mathbb{E}[X_t^2] - (\mathbb{E}[X_t])^2 = \frac{1}{H(t)} \sum_{k=0}^{\infty} k^2 a_k^2 e^{kt} - A^2(t)$$

$$= \frac{H''(t)}{H(t)} - \left(\frac{H'(t)}{H(t)}\right)^2 = B(t).$$

In order to approximate \(H\) by an appropriate (exponential) polynomial, we first prove the following lemma.

Lemma 3.3. For \(t \in T_\ell\), and \(|\tau - t| < \frac{1}{\sqrt{B(t)}}\), we have

$$|E(\tau)| := \left|\sum_{|k-A(t)| > s\sqrt{B(t)}} a_k^2 e^{kt}\right| \leq 2H(\text{Re}[\tau]) \exp\left(-\frac{1}{8}(s - 4)^2\right),$$

for all \(4 < s < B^{1/6}(t)\).
Proof. Since $|a_k^2 e^{k\tau}| = a_k^2 e^{k \text{Re}[\tau]}$, by the triangle inequality we may assume $\tau$ is real. Notice that

$$
\sum_{|k-A(t)| > s} a_k^2 e^{k\tau} = H(\tau) \mathbb{P} \left[ |X_\tau - A(t)| > s\sqrt{B(t)} \right],
$$

and also

$$
\mathbb{E} \left[ e^{\lambda X_\tau} \right] = \frac{1}{H(\tau)} \sum_{k=0}^\infty e^{\lambda k} a_k^2 e^{k\tau} = \frac{H(\tau + \lambda)}{H(\tau)}.
$$

Fix $\lambda > 0$ so that $\tau + \lambda \in T_\ell$. By Markov’s inequality

$$
\mathbb{P} \left[ X_\tau > A(t) + s\sqrt{B(t)} \right] \leq \frac{\mathbb{E} \left[ e^{\lambda X_\tau} \right]}{\exp \left( \lambda \left( A(t) + s\sqrt{B(t)} \right) \right)}
$$

$$
= \exp \left( \log \frac{H(\tau + \lambda)}{H(\tau)} - \lambda \left( A(t) + s\sqrt{B(t)} \right) \right).
$$

By (3.3) and (3.2) we have

$$
\log \frac{H(\tau + \lambda)}{H(\tau)} - \lambda A(t) = \log \frac{H(\tau + \lambda)}{H(\tau)} - \lambda A(\tau) + \lambda [A(\tau) - A(t)]
$$

$$
\leq \lambda [A(\tau) - A(t)] + 2\lambda^2 B(t_\ell)
$$

$$
\leq 4\lambda |\tau - t| B(t_\ell) + 2\lambda^2 B(t_\ell) \leq 4\lambda \sqrt{B(t)} + 2\lambda^2 B(t).
$$

Therefore, by taking $\lambda = \frac{s-4}{4\sqrt{B(t)}}$, we get

$$
\log \mathbb{P} \left[ X_\tau > A(t) + s\sqrt{B(t)} \right] \leq 4\lambda \sqrt{B(t)} + 2\lambda^2 B(t) - \lambda s\sqrt{B(t)}
$$

$$
\leq -\frac{1}{8} (s - 4)^2.
$$

Since $s < B^{1/6}(t)$ we obtain by (3.2)

$$
\tau + \lambda < t_{\ell+1} - \frac{2}{\ell^2} + \frac{1}{4\sqrt{B(t)}} + \frac{1}{4B(t)^{1/3}} \leq t_{\ell+1} - \frac{2}{\ell^2} + \frac{1}{\sqrt{B(t_\ell)}} + \frac{1}{4B(t_\ell)^{1/3}}
$$

$$
= t_{\ell+1} - \frac{2}{\ell^2} + \frac{1}{\ell^3} + \frac{1}{4\ell^2} < t_{\ell+1},
$$

in the same way we see that $\tau + \lambda > t_\ell$, so that $\tau + \lambda \in T_\ell$ as required. The bound for

$$
\mathbb{P} \left[ X_t < A(t) + s\sqrt{B(t)} \right]
$$

is obtained similarly. \hfill \Box

Definition 3.4. We say that an exponential polynomial with real exponents is of width at most $\omega$, if it is of the form

$$
\sum_{k=0}^m c_k e^{\alpha_k z}, \quad \text{with } \max_k |\alpha_k| \leq \omega.
$$
The following lemma adapts [13, Lemma 8] to exponential polynomials.

**Lemma 3.5.** Let \( P(z) \) be an exponential polynomial of width at most \( \omega \), and non-negative coefficients. We have for any \( t \in \mathbb{R} \), and \( |\tau - t| \leq \frac{1}{5\omega} \) that

\[
\frac{3}{4} P(t) \leq |P(\tau)| \leq \frac{5}{4} P(t).
\]

**Proof.** Suppose \( P(z) = \sum_{k=0}^{m} c_k e^{\alpha_k z} \) with \( \max_k |\alpha_k| \leq \omega \), and \( c_k \geq 0 \). Then

\[
|P'(\tau)| = \left| \sum_{k=0}^{m} c_k \alpha_k e^{\alpha_k \tau} \right| = \left| \sum_{k=0}^{m} c_k e^{\alpha_k \tau} \alpha_k e^{\alpha_k (\tau - t)} \right| \leq \max_k |\alpha_k e^{\alpha_k (\text{Re}[\tau] - t)}| \cdot P(t)
\]

\[
\leq \max_k |\alpha_k| e^{|\alpha_k||\tau - t|} \cdot P(t) \leq \omega e^{e^{\omega|\tau - t|}} P(t).
\]

Now,

\[
|P(\tau) - P(t)| = \left| \int_{t}^{\tau} P'(s) \, ds \right| \leq \omega e^{e^{\omega|\tau - t|}} P(t) \cdot |\tau - t| \leq \frac{1}{5} e^{1/5} P(t) \leq \frac{1}{4} P(t).
\]

\[\square\]

3.2.2. **Proof of second part of Theorem 1.1.** The proof of Theorem 1.1 in the case \( b \uparrow \infty \) follows by combining the next proposition with Corollary 2.3. We will use the Schwarz integral formula (see [1, Chapter 4, Section 6.3]), if \( g \) is an analytic function in the closed unit disk, then

\[
g(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \text{Re}[g(\zeta)] \frac{d\zeta}{\zeta} + i\text{Im}[g(0)],
\]

for all \( |z| < 1 \), where \( \mathbb{T} \) is the unit circle.

**Proposition 3.6** (cf. [13, Theorem 10]). If \( B(t) \uparrow \infty \), then \( H \) is local \( \delta \)-admissible on \( \mathcal{X} \), with \( \delta(t) = \frac{1}{\sqrt{B(t)}} \), that is, for any \( \varepsilon > 0 \) there exists \( \eta > 0 \), such that for \( t \in \mathcal{X} \cap (t_0(\varepsilon), t_G) \) and \( |w| \leq \frac{\varepsilon}{\sqrt{B(t)}} \) we have

\[
\log \frac{H(t + w)}{H(t)} = wA(t) + \frac{1}{2} w^2 B(t) + h_t(w), \quad \text{where } |h_t(w)| \leq \varepsilon |w|^2 B(t).
\]

**Proof.** Let \( \varepsilon > 0 \) be given, and \( \eta > 0 \) depending on \( \varepsilon \) to be chosen later, also let \( t \in \mathcal{X} \). Put

\[
Q(\tau) := \sum_{|k - A(t)| \leq s \sqrt{B(t)}} a_k^2 e^{k\tau} = H(\tau) - E(\tau),
\]

then by Lemma 3.3 for \( |\tau - t| < \frac{1}{\sqrt{B(t)}} \) we have \( |E(\tau)| \leq 2H(\text{Re}[\tau]) \exp \left(-\frac{1}{8}(s - 4)^2\right) \)

for all \( 4 < |s| < B^{1/6}(t) \). We also write \( P(\tau) = e^{A(t)\tau} Q(\tau) \) so that \( P \) is an exponential
polynomial with non-negative coefficients of width at most \( s \sqrt{B(t)} \). Choosing \( s = 9 \) so that 
\[
2 \exp \left( -\frac{1}{8} (s - 4)^2 \right) < \frac{1}{4},
\]
we have

\[
|E(\tau)| < \frac{1}{4} H(\text{Re}[\tau]),
\]

and in particular

\[
\frac{3}{4} H(\text{Re}[\tau]) < Q(\text{Re}[\tau]) = H(\text{Re}[\tau]) - E(\text{Re}[\tau]) < \frac{5}{4} H(\text{Re}[\tau]).
\]

By Lemma 3.5 and the previous inequality for \( \text{Re}[\tau] = t \), we have for \( |\tau - t| \leq \frac{1}{45 \sqrt{B(t)}} \)

\[
\left( \frac{3}{4} \right)^2 H(t) \leq \frac{3}{4} Q(t) \leq |Q(\tau)| \leq \frac{5}{4} Q(t) \leq \frac{5}{4} H(t).
\]

From (3.5), (3.6), and (3.7), we get

\[
|E(\tau)| \leq \frac{1}{4} \cdot \frac{4}{3} Q(\text{Re}[\tau]) \leq \frac{1}{3} \cdot \frac{5}{4} H(t).
\]

Thus, we have

\[
|H(\tau)| \leq |Q(\tau)| + |E(\tau)| \leq \left[ \frac{5}{4} + \frac{1}{3} \cdot \frac{5}{4} \right] H(t) \leq 2 H(t),
\]

and

\[
|H(\tau)| \geq |Q(\tau)| - |E(\tau)| \geq \left[ \left( \frac{3}{4} \right)^2 - \frac{1}{3} \cdot \frac{5}{4} \right] H(t) \geq \frac{1}{8} H(t).
\]

We found that

\[
- \log 8 \leq \log \frac{|H(\tau)|}{H(t)} \leq \log 2.
\]

Now let us define the analytic function

\[
\phi(w) = \log H(t + w) - \log H(t), \quad \phi(w) = \sum_{n=1}^{\infty} \phi_n w^n, \quad |w| \leq \frac{1}{45 \sqrt{B(t)}} =: \lambda_0.
\]

By (3.8) we have that \( |\text{Re}[\phi(w)]| \leq \log 8 \), and therefore by (3.4) for \( |w| \leq \frac{1}{2} \lambda_0 \) we get (since \( \phi(0) = 0 \))

\[
|\phi(w)| \leq \frac{1 + \frac{1}{2}}{1 - \frac{1}{2}} \cdot \log 8 < 7.
\]
By Cauchy’s estimates $|\phi_n| \leq 7 \left(\frac{2}{\lambda_0}\right)^n$, and therefore, for $|w| \leq \frac{1}{4}\lambda_0$

$$\left| \phi (w) - wA(t) - \frac{1}{2}w^2B(t) \right| = \left| \sum_{n=3}^{\infty} \phi_n w^n \right| \leq 7 \sum_{n=3}^{\infty} \left(\frac{2w}{\lambda_0}\right)^n \leq 7 \cdot 8 \left(\frac{w}{\lambda_0}\right)^3 \sum_{n=0}^{\infty} \frac{1}{2^n} = 112 \left(\frac{w}{\lambda_0}\right)^3.$$ 

Thus in order to obtain the result it remains to choose $\eta = \frac{\varepsilon}{112 \cdot 45}$. □

3.2.3. Proof of first part of Theorem 1.1. Fix an entire function $G(z) = \sum_{n \geq 0} a_n^2 z^n$, with non-negative coefficients $a_n$, and recall that $a(r) = rG'(r)/G(r)$, $b(r) = ra'(r)$.

In order to get a lower bound for the variance without any assumptions on the function $b(r)$ we will use some results about $G$ obtained by the Wiman-Valiron method (see [13]). We recall some of the terminology regarding entire functions: $\mu(r) = \max_n \{a_n^2 r^n\}$ is called the maximal term of $G$, and $N(r) = \max \{n : a_n^2 r^n = \mu(r)\}$ the central index.

One of the main results of the Wiman-Valiron method is that there is a set $\mathcal{N} \subset \mathbb{R}^+$, such that $G(z)$ has desirable properties if $|z| \in \mathcal{N}$, and that $\mathbb{R}^+ \setminus \mathcal{N}$ has finite logarithmic measure (see [13, Sections 2 and 3]). We fix a parameter $\gamma \in (0, \frac{1}{2})$; the set $\mathcal{N}$ will depend on $G$ and $\gamma$. By [13, Theorem 2] for $r \in \mathcal{N}$ we have for all $n \in \mathbb{N}$

$$a_n^2 r^n \leq \mu(r) \exp\left(-\frac{k^2}{(|k| + N(r))^{1+\gamma}}\right),$$

where $k = n - N(r)$, hence the summands $a_n^2 |z|^n$ of the series $G(z)$ with $|z| = r$, corresponding to the “window” of indices

$$\{n \in \mathbb{N} : |n - N(r)| \leq K(r)\} \quad \text{with} \quad K(r) := N(r)^{\frac{1}{1+\gamma}},$$

are the largest ones. In particular, this implies that $N(r)$ and $a(r)$ are asymptotically comparable, see Claim 3.7 below. Notice that by applying the change of variables $x \mapsto \sqrt{x}$, the set $\{r > 0 : r^2 \notin \mathcal{N}\}$ is of finite logarithmic measure as well.

It is known that for any $\gamma > 0$ we have

$$b(r) \leq a(r)^{1+\gamma}$$

outside a set of finite logarithmic measure (see [29, Lemma 1]). Thus, we may and will assume that (3.10) is satisfied for all $r \in \mathcal{N}$.

Claim 3.7. For $r \in \mathcal{N}$ we have

$$a(r) = N(r) + O(K(r)), \quad \text{as} \quad r \to \infty.$$
Proof. Since \( a_n^2 \geq 0 \) we have that \( \max_{|z|=r} |G(z)| = G(r) \) and hence, by [13, Theorem 12] with \( q = 1, f = G, k = K(r) \) we have
\[
\frac{r}{N(r)} G'(r) = G(r) + O \left( \frac{K(r)}{N(r)} \right) G(r).
\]
and therefore
\[
a(r) = N(r) + O(K(r)).
\]
\[\square\]

Fixing \( r \in \mathcal{N} \) we can approximate the function \( G \) near \( |z| = r \) by a polynomial of degree about \( K(r) \) (cf. Lemma 3.3). This allows us to obtain rather precise Taylor expansion asymptotics for \( \log G(e^\tau) \) near \( \tau = 2 \log r \).

The following lemma is a special case of [13, Lemma 2].

Lemma 3.8. Suppose \( e^t = r^2 \in \mathcal{N} \) and \( K, N \) are as above, then for \( |\tau - t| \leq \frac{2}{K(e^t)} \)
\[
\left| \sum_{|k-N(e^t)|>K(e^t)} a_k^2 e^{kr} \right| \leq \frac{1}{4} H(\text{Re} [\tau]).
\]

Combining the lemma above with Claim 3.7, we find that we can replace \( N \) by \( A \).

Corollary 3.9. There exists a constant \( s > 0 \), so that for \( e^t = r^2 \in \mathcal{N} \) and \( |\tau - t| \leq \frac{2}{K(e^t)} \),
\[
\left| \sum_{|k-A(t)|>sK(e^t)} a_k^2 e^{kr} \right| \leq \frac{1}{4} H(\text{Re} [\tau]).
\]

Repeating the proof of Proposition 3.6 using the previous corollary instead of Lemma 3.3 we obtain the following result.

Proposition 3.10. Any entire function \( G \) with non-negative coefficients is local \( \delta \)-admissible on a set \( \mathcal{N} = \mathcal{N}_G \) with \( \delta(t) = \frac{B(t)}{K^3(e^t)} \). Here \( \mathbb{R}^+ \setminus \mathcal{N} \) is a set of finite logarithmic measure.

We are now ready to prove the first part of Theorem 1.1. We recall that \( f \) is a Gaussian entire function with covariance kernel \( G \).

Proof of Theorem 1.1. Choose \( \gamma \) sufficiently small so that \( \frac{3}{2} (1 + \gamma) + \gamma \leq \frac{3}{2} + \varepsilon \). By (3.10), (3.9), and Claim 3.7, we have
\[
\frac{B^2(t)}{K^3(e^t)} \leq \frac{2B^2(t)}{A(t)^{2(1+\gamma)}} \leq 2\sqrt{B(t)} \quad \text{as } t \to \infty, \quad e^t \in \mathcal{N},
\]
and therefore by Corollary 2.3, with \( e^t = r^2 \), \( H(z) = G(e^z) \), and the previous proposition
\[
\text{Var} (n_f(r)) = \mathcal{I}(H, t, \mathbb{T}) \geq \min \left\{ \delta(t) B(t), \sqrt{B(t)} \right\} \gtrsim \frac{B^2(t)}{K^{3+\gamma}(e^t)} \gtrsim \frac{b^2(r^2)}{a^{2(1+\gamma)+\gamma}}\Big( r^2 \Big).
\]
4. Asymptotics of the variance - proof of Theorem 1.4

Let \( f \) be a Gaussian analytic function with covariance function \( G \), recall \( H(t) = G(e^t) \), and that
\[
\log R_G = t_G \in \{0, \infty\}, \quad A(t) = (\log H(t))' = a(e^t), \quad B(t) = A'(t) = b(e^t).
\]
In this section we find the asymptotic growth of \( \text{Var}(n_f(r)) \) as \( r \to R^- \) when the covariance kernel \( G \) is type I admissible.

4.1. Type I admissible covariance functions. To find precise asymptotics for the integral (2.1) we make certain assumptions on the function \( H \), motivated by the Hayman [12] admissibility condition.

**Definition 4.1.** We call \( G \) type I admissible if the function \( H \) has the following properties:

1. \( B(t) \to \infty \) as \( t \to t_G^- \).
2. \( A(t) = O(B^2(t)) \) as \( t \to t_G^- \).
3. There is a constant \( C_G > 2 \) such that
   \[
   \log \left( \frac{H(t + i\theta)}{H(t)} \right) = i\theta A(t) - \frac{1}{2} \theta^2 B(t) (1 + o(1)), \quad \text{as } t \to t_G^-,
   \]
   uniformly for all \( |\theta| \leq \delta(t) := \sqrt{C_G \frac{\log B(t)}{B(t)}} \).
4. \( |H(t + i\theta)| = O\left( \frac{H(t)}{B^2(t)} \right) \) and \( |H'(t + i\theta)| = O\left( \frac{H'(t)}{B^2(t)} \right) \) as \( t \to t_G^- \), uniformly in \( |\theta| \in [\delta(t), \pi] \).

**Remark 4.1.** By the proof of [12, Lemma 4] it follows that an admissible function is local \( \delta \)-admissible on \( \mathbb{R}^+ \) with \( \delta(t) = \frac{c}{\sqrt{B(t)}} \) with some constant \( c > 0 \).

**Remark 4.2.** Since \( B = A' = (\log H)' \) it follows that Assumption 2 puts a restriction on the minimal growth rate of the function \( H \).

**Remark 4.3.** The choice of the constant \( C_G \) is not important, we choose it in this way so that Assumptions 3 and 4 will agree for \( \theta = \delta(t) \).

4.2. Asymptotics of the variance. We put \( r^2 = e^t \), and split the integral in (2.1) into two parts:
\[
\text{Var}(n_f(r)) = \mathcal{I}(H, t, \mathbb{T}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \frac{A(t + i\theta) - A(t)^2}{2 \int_0^\theta \text{Im}[A(t + i\varphi)] d\varphi} \right) d\theta
\]
\[= \mathcal{I}(H, t, [-\delta(t), \delta(t)]) + \mathcal{I}(H, t, \mathbb{T} \setminus [-\delta(t), \delta(t)])
\]
\[=: J_1(r) + J_2(r),
\]
where we used the definition of $\mathcal{T}$ in (2.5).

4.2.1. Evaluating $J_1$. For $|\theta| \leq \delta(t)$, Assumption 3 implies

$$|A(t + i\theta) - A(t)| = B(t) |\theta| (1 + o(1)), \quad t \to t_G^-,$$

uniformly in $|\theta| \leq \delta(t)$. Since $A(t) \in \mathbb{R}$

$$\int_0^\theta \text{Im} [A(t + i\varphi)] d\varphi = \int_0^\theta \text{Im} [A(t) + i\varphi B(t) (1 + o(1))] d\varphi$$

$$\sim -\frac{1}{2} \theta^2 B(t), \quad t \to t_G^-,$$

also uniformly in $|\theta| \leq \delta(t)$. Making the change of variables

$$u = \sqrt{-2 \int_0^\theta \text{Im} [A(t + i\varphi)] d\varphi} \sim \theta \sqrt{B(t)}$$

yields by (4.1) and (4.2)

$$J_1(r) \sim \frac{\sqrt{B(t)}}{2\pi} \int_{|u| \leq \delta(t)\sqrt{B(t)(1+o(1))}} \frac{u^2}{\exp (u^2) - 1} du.$$

By Assumption 1 and the definition of $\delta$, we have that $\delta(t) \sqrt{B(t)} \to \infty$, as $t \to t_G^-$. Thus

$$\int_{|u| \leq \delta(t)\sqrt{B(t)(1+o(1))}} \frac{u^2}{\exp (u^2) - 1} du \sim \int_{\mathbb{R}} \frac{u^2}{\exp (u^2) - 1} du = \frac{\sqrt{\pi}}{2} \zeta \left( \frac{3}{2} \right), \quad \text{as } t \to t_G^-,$$

where $\zeta$ is the Riemann zeta function. This gives the main term in the asymptotic behavior of the variance

$$J_1(r) \sim \frac{\zeta \left( \frac{3}{2} \right)}{4\sqrt{\pi}} \sqrt{B(t)}, \quad t \to t_G^-.$$

4.2.2. Bounding $J_2$. Again we write $r^2 = e^t$. The admissibility assumptions allow us to control the size of $G$ also in the range $\delta(t) \leq |\theta| \leq \pi$. Note the identity

$$\frac{|A(t + i\theta) - A(t)|^2}{\exp \left( 2 \int_0^\theta \text{Im} [A(t + i\varphi)] d\varphi \right) - 1} = \frac{|H(t) H'(t + i\theta) - H(t + i\theta) H'(t)|^2}{H^2(t) (H^2(t) - |H^2(t + i\theta)|)}.$$
By Assumptions 1 and 4, we have that $H^2(t) - |H^2(t + i\theta)| \sim H^2(t)$. Therefore, again by Assumption 4 we get
\[
\frac{|A(t + i\theta) - A(t)|^2}{\exp \left(2 \int_0^\theta \text{Im} \left[A(t + i\varphi)\right] d\varphi\right) - 1} \leq (2 + o(1)) \left[\frac{|H(t) H'(t + i\theta)|^2}{H^4(t)} + \frac{|H(t + i\theta) H'(t)|^2}{H^4(t)}\right]
\]
\[
= (2 + o(1)) \left[\frac{|H' (t + i\theta)|^2}{H^2(t)} + \frac{|H (t + i\theta)|^2}{H^2(t)} A^2 (t)\right]
\]
\[
= O \left(\frac{A^2 (t)}{B^4(t)}\right).
\]
Finally, by Assumption 2
\[
J_2 (r) = O \left(\frac{A^2 (t)}{B^4(t)}\right) = O (1), \quad t \to t^-_G,
\]
and combining this with (4.3) we conclude that
\[
\text{Var} \left(n_f (r)\right) = J_1 (r) + J_2 (r) \sim \frac{\zeta \left(\frac{3}{2}\right)}{4\sqrt{\pi}} \sqrt{b (r^2)}, \quad r \to R^-_G,
\]
thus proving Theorem 1.4.

\[\square\]

5. Upper bound for the variance - proof of Theorem 1.2

The upper bound for the variance is derived from the following algebraic identity.

Claim 5.1. For $a_1, a_2, a_3 \in \mathbb{R}$ and $b_1, b_2, b_3 \in \mathbb{C}$ the following holds
\[
|a_1 b_3 - \overline{b_1} b_2|^2 = (a_1 a_3 - |b_2|^2) (a_1 a_2 - |b_1|^2) - a_1 \cdot \det \begin{pmatrix} a_1 & \overline{b_1} & b_2 \\ b_1 & a_2 & \overline{b_3} \\ b_2 & b_3 & a_3 \end{pmatrix}.
\]

Proof. One can directly show this equality holds, but here we will give a geometric reasoning which holds when the matrix in the expression above is positive definite (which is the case we use). In that case we may take vectors $(v_1, v_2, v_3)$ so that their Gram matrix satisfies
\[
\begin{pmatrix} a_1 & \overline{b_1} & b_2 \\ b_1 & a_2 & \overline{b_3} \\ b_2 & b_3 & a_3 \end{pmatrix} = \begin{pmatrix} \langle v_1, v_1 \rangle & \langle v_2, v_1 \rangle & \langle v_3, v_1 \rangle \\ \langle v_1, v_2 \rangle & \langle v_2, v_2 \rangle & \langle v_3, v_2 \rangle \\ \langle v_1, v_3 \rangle & \langle v_2, v_3 \rangle & \langle v_3, v_3 \rangle \end{pmatrix}
\]
and we denote the corresponding Gram determinant by $\text{Gram} (v_1, v_2, v_3)$. On one side we have
\[
\text{dist}^2 (v_3, \text{span} \{v_1, v_2\}) = \frac{\text{Gram} (v_1, v_2, v_3)}{\text{Gram} (v_1, v_2)} = \frac{\text{Gram} (v_1, v_2, v_3)}{a_1 a_2 - |b_1|^2}.
\]
On the other hand, using the Gram-Schmidt process to find orthogonal vectors \((w_1, w_2, w_3)\) we get

\[
\begin{align*}
    w_1 &= v_1, \\
    w_2 &= v_2 - \frac{b_1}{a_1} v_1, \\
    w_3 &= v_3 - \frac{b_1}{a_1} v_1 - \frac{b_2}{a_2} v_2.
\end{align*}
\]

Taking into account (5.1), we find that

\[
\begin{align*}
    w_1 &= v_1, \quad w_2 = v_2 - \frac{b_1}{a_1} v_1, \quad \langle w_2, w_2 \rangle = a_2 - \frac{|b_1|^2}{a_1} \\
    \langle v_3, w_2 \rangle &= \langle v_3, v_2 \rangle - \frac{b_1}{a_1} \langle v_3, v_1 \rangle = \frac{b_1 b_2}{a_2} - \frac{b_1 b_3}{a_1} \frac{b_2}{a_2}.
\end{align*}
\]

Finally

\[
\begin{align*}
    \text{dist}^2 (v_3, \text{span} \{v_1, v_2\}) &= \langle v_3, v_3 \rangle - \frac{|\langle v_3, w_1 \rangle|^2}{\langle w_1, w_1 \rangle} - \frac{|\langle v_3, w_2 \rangle|^2}{\langle w_2, w_2 \rangle} \\
    &= a_3 - \frac{|b_2|^2}{a_1} - \frac{|a_1 b_3 - b_1 b_2|^2}{a_1 (a_1 a_2 - |b_1|^2)},
\end{align*}
\]

and we get the required identity by multiplying the two expressions for \(\text{dist}^2 (v_3, \text{span} \{v_1, v_2\})\) by \(a_1 (a_1 a_2 - |b_1|^2)\). By continuity the identity extends to positive semi-definite matrices. \(\square\)

Put

\[
g_j (\theta) = \sum_{n \in \mathbb{Z}} n^j c_n e^{i n \theta}, \quad j \in \{0, 1, 2\},
\]

and write \(g\) for \(g_0\).

**Claim 5.2.** Assume that \(c_n \in \mathbb{R}\) for all \(n \in \mathbb{Z}\) and that \(\sum_{n \in \mathbb{Z}} n^2 c_n^2 < \infty\), then

\[
|g (0) g_1 (\theta) - g_1 (0) g (\theta)|^2 \leq (g (0) g_2 (0) - g_1^2 (0)) (g^2 (0) - |g (\theta)|^2).
\]

**Proof.** By Claim 5.1, and using \(g_j (\theta) = g_1 (-\theta)\), we have

\[
|g (0) g_1 (\theta) - g_1 (0) g (\theta)|^2 = (g (0) g_2 (0) - g_1^2 (0)) (g^2 (0) - |g (\theta)|^2)
\]

\[
- g (0) \cdot \det \begin{pmatrix} g (0) & g (\theta) & g_1 (0) \\ g (-\theta) & g (0) & g_1 (-\theta) \\ g_1 (0) & g_1 (\theta) & g_2 (0) \end{pmatrix}.
\]
Let $h_j(\theta) = \sum_{n \in \mathbb{Z}} \xi_n n^i c_n e^{in\theta}$ for $j \in \{0, 1\}$, put $V = (h(0), h(-\theta), h_1(0))$ (again with $h = h_0$), then we have

$$(E[V_j V_k])_{j,k=1}^3 = \begin{pmatrix} g(0) & g(\theta) & g_1(0) \\ g(-\theta) & g(0) & g_1(-\theta) \\ g_1(0) & g_1(\theta) & g_2(0) \end{pmatrix}.$$ 

Thus the matrix above is a covariance matrix, hence positive semi-definite and its determinant is non-negative. □

5.1. Proof of Theorem 1.2. Now let $f$ be a Gaussian analytic function with covariance kernel $G$, whose radius of convergence is $R_G$. For the proof again it will be more convenient to use the exponential parameterization

$$H(z) := G(e^z) = \sum_{n=0}^{\infty} a_n^2 e^{nz},$$

so that

$$t_G = \log R_G, \quad A(z) = \frac{H'(z)}{H(z)} = (\log H(z))', \quad B(z) = A'(z) = (\log H(z))''.$$

We have (see Claim A.2 in Appendix A)

$$\text{Var}(n_f(r)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|H(t) H'(t + i\theta) - H(t + i\theta) H'(t)|^2}{H^2(t) (H^2(t) - |H^2(t + i\theta)|)} d\theta,$$

where $e' = r^2$.

The following claim will finish the proof of Theorem 1.2.

Corollary 5.3. For all $t < t_G$ and $\theta \in [-\pi, \pi]$

$$\frac{|H(t) H'(t + i\theta) - H(t + i\theta) H'(t)|^2}{H^2(t) (H^2(t) - |H^2(t + i\theta)|)} \leq B(t).$$

Proof. The result follows immediately from Claim 5.2, by taking $c_n^2 = a_n^2 e^{nt}$ for $n \in \mathbb{N}$, and $c_n = 0$ otherwise, so that

$$g(\theta) = \sum_{n=0}^{\infty} a_n^2 e^{nt} e^{in\theta} = H(t + i\theta), \quad g_1(\theta) = H'(t + i\theta), \quad g_2(\theta) = H''(t + i\theta),$$

and

$$\frac{g(0) g_2(0) - g_1^2(0)}{g^2(0)} = \frac{H(t) H''(t) - (H'(t))^2}{H^2(t)} = (\log H(t))'' = B(t).$$

Notice that for $t < t_G$ we have that $\sum_{n \in \mathbb{Z}} n^2 c_n^2 = \sum_{n \geq 0} n^2 a_n^2 e^{nt} < \infty$ by the definition of $t_G$. □
6. Gaussian Entire Functions with Large Variance - Proof of Theorem 1.7

In this section, we will prove Theorem 1.7. For an entire function \( G \) we recall that

\[
a_{\tilde{G}}(r) = r \left( \log G(r) \right)', \quad b_{\tilde{G}}(r) = ra_{\tilde{G}}'(r),
\]

where in this section, we will sometimes add the subscript \( \tilde{G} \) in order to distinguish between functions \( a, b \) associated with different entire functions \( G \). Given a type II admissible covariance function \( G \) (see Section 6.2), we will construct a Gaussian entire function \( \tilde{f} \) with a transcendental covariance function \( \tilde{G} \), which is similar to \( G \), that is

\[
a_{\tilde{G}}(r) \approx_L a_G(r), \quad \text{and} \quad b_{\tilde{G}}(r) \approx_L b_G(r),
\]

moreover, \( \tilde{G} \) is a restriction of \( G \) (see Definition 1.6). We then prove

\[
(6.1) \quad \text{Var} \left( n_f(r) \right) \approx_L b_{\tilde{G}} \left( r^2 \right),
\]

thus showing that the bound in Theorem 1.2 is sharp up to a constant (and an exceptional set of values of \( r \)).

6.1. Estimates for Sums of Gaussians. In this section we fix the parameters \( A, B, s > 0 \) and \( p \) a positive integer.

Claim 6.1. Suppose \( s \geq 1 \), and \( p \leq \sqrt{B} \). For \( j \in \{0, 1, 2\} \), we have

\[
\frac{1}{\sqrt{e}} (A - s \sqrt{B})^j \leq \sum_{|kp-A| \leq s \sqrt{B}} (kp)^j \exp \left(-\frac{(kp-A)^2}{2B}\right) \leq e^2 \left[ \frac{\sqrt{2\pi B}}{p} \left( A^j + B \cdot 1_{\{j=2\}} \right) + 1 \right],
\]

where the sum runs over integers \( k \), and

\[
1_{\{j=2\}} = \begin{cases} 0, & j \neq 2; \\ 1, & j = 2. \end{cases}
\]

Proof. Since \( p \leq \sqrt{B} \) the sum is non-empty and the lower bound follows. In the other direction, put \( \phi(x) = \exp \left(-\frac{(x-A)^2}{2B}\right) \). If \( (k-1)p \geq A \), then we have

\[
(kp)^j \phi(kp) \leq \frac{1}{p} \left( \frac{k}{k-1} \right)^j \int_{(k-1)p}^{kp} x^j \phi(x) \, dx \\
\leq \frac{1}{p} \exp \left( \frac{j}{k-1} \right) \int_{(k-1)p}^{kp} x^j \phi(x) \, dx \\
\leq \frac{e^2}{p} \int_{(k-1)p}^{kp} x^j \phi(x) \, dx,
\]

where in the last inequality we used $k \geq 2$. Thus,

$$\sum_{A+p \leq kp \leq s\sqrt{B}} (kp)^j \phi(kp) \leq \frac{e^2}{p} \int_{A-s\sqrt{B}}^{A+s\sqrt{B}} x^j \phi(x) \, dx.$$ 

Using a similar argument for $(k-1)p \leq A$ we get (adding the integral from $-p$ to $p$ twice)

$$\sum_{|kp-A| \leq s\sqrt{B}} (kp)^j \phi(kp) \leq \frac{e^2}{p} \int_{A-s\sqrt{B}}^{A+s\sqrt{B}} x^j \phi(x) \, dx + e^2$$

$$= \sum_{k_1 \leq k \leq k_2} \psi(k_1, k_2).$$

**Claim 6.2.** Suppose $s \geq 1$, $p \leq \sqrt{B}$. We have,

$$\frac{p^2}{e} \leq \sum_{|k_1p - A| \leq s\sqrt{B}} (k_1p - k_2p)^2 \exp \left( -\frac{(k_1p - A)^2}{2B} - \frac{(k_2p - A)^2}{2B} \right) \leq \frac{24e^4\pi B^2}{p^2},$$

where the double sum runs over integers $k_1, k_2$.

**Proof.** The lower bound follows since $p \leq \sqrt{B}$, so that the double sum is not empty. Put $\psi(x, y) = (x - y)^2 \phi(x) \phi(y)$ so that

$$\sum_{|k_1p - A| \leq s\sqrt{B}} (pk_1 - pk_2)^2 \exp \left( -\frac{(k_1p - A)^2}{2B} - \frac{(k_2p - A)^2}{2B} \right)$$

$$= \sum_{|k_1p - A| \leq s\sqrt{B}} \psi(k_1p, k_2p).$$
Writing \( n_1 = k_1 + k_2 \) and \( n_2 = k_1 - k_2 \), we find that

\[
\sum_{|k_1 p - A| \leq s\sqrt{B}} \psi(k_1 p, k_2 p) \leq \sum_{|k_2 p - A| \leq s\sqrt{B}} \sum_{|n_2 p| \leq 2s\sqrt{B}} \psi\left(\frac{(n_2 - n_1) p}{2}, \frac{(n_2 + n_1) p}{2}\right)
\]

where we used the following identity in the last equality

\[
\psi\left(\frac{x - y}{2}, \frac{x + y}{2}\right) = y^2 \exp\left(-\frac{y^2}{4B}\right) \exp\left(-\frac{(x - 2A)^2}{4B}\right).
\]

Bounding both factors in (6.2) using the previous claim, we conclude that

\[
\sum_{|k_1 p - A| \leq s\sqrt{B}} \sum_{|k_2 p - A| \leq s\sqrt{B}} \psi(k_1 p, k_2 p) \leq e^4 \left(\frac{\sqrt{2\pi \cdot 2B}}{p} \cdot 2B + 1\right) \left(\frac{\sqrt{2\pi \cdot 2B}}{p} + 1\right)
\]

\[
\leq e^4 \left(\frac{\sqrt{4\pi B}}{p}\right)^2 \cdot (2B + 1) \cdot 2
\]

\[
\leq \frac{24e^4 \pi B^2}{p^2}.
\]

6.2. **Type II admissible functions and their Taylor coefficients.** As before, we associate with \( G \) the functions

\[
H(t) = G(e^t), \quad A(t) = \frac{H'(t)}{H(t)}, \quad B(t) = A'(t) = \frac{H''(t)}{H(t)} - \left(\frac{H'(t)}{H(t)}\right)^2.
\]

In this section we will use a stronger version of the Hayman admissibility condition, which allows us to improve the error term in [12, Theorem I] (see the lemma below).

**Definition 6.3.** We say that an entire function \( G \) is **type II admissible** if the function \( H \) has the following properties:

1. \( B \uparrow \infty \)
2. \( A(t) = O(B^2(t)) \) as \( t \to \infty \).
There exist constants $C_G > 2$ and $\varepsilon > 0$ such that
\[
\log \frac{H(t + i\theta)}{H(t)} = i\theta A(t) - \frac{1}{2} \theta^2 B(t) + \Delta(t, \theta), \quad \forall \ |\theta| \leq \delta(t),
\]
where
\[
\delta(t) := \sqrt{C_G \log B(t)} /
\]
\[
|\Delta(t, \theta)| \leq B^{3/2 - \varepsilon}(t) |\theta|^3.
\]

We again put $r^2 = e^t$.

Remark 6.1. Throughout this section, in order make the expressions shorter, we may suppress the dependence of $H, A, B,$ and other parameters on $t$ inside the proofs.

Lemma 6.4 (cf. [12, Theorem 1]). Suppose that $G(z) = \sum_{n=0}^{\infty} a_n^2 z^n$ is type II admissible, then there exists an $\varepsilon \in (0, 1)$ such that for all $t$ sufficiently large,
\[
a_n^2 e^{nt} = \frac{H(t)}{\sqrt{2\pi B(t)}} \left[ \exp \left( -\left( \frac{n - A(t)}{2B(t)} \right)^2 \right) + O \left( \frac{1}{B^\varepsilon(t)} \right) \right], \quad \text{as } t \to \infty,
\]
uniformly in $n$.

Proof. Put $\delta = \delta(t)$. By Cauchy’s formula
\[
a_n^2 e^{nt} = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(t + i\theta) e^{-in\theta} \, d\theta,
\]
which we write as $a_n^2 e^{nt} = I_1 + I_2$, where
\[
I_1 = \frac{1}{2\pi} \int_{|\theta| \leq \delta} H(t + i\theta) e^{-in\theta} \, d\theta, \quad I_2 = \frac{1}{2\pi} \int_{\delta < |\theta| \leq \pi} H(t + i\theta) e^{-in\theta} \, d\theta.
\]
By Assumption 4, we have, uniformly in $n$,
\[
I_2 = O \left( \frac{H(t)}{B(t)} \right).
\]
By Assumption we have

\[ I_1 = \frac{H}{2\pi} \int_{|\theta| \leq \delta} \exp \left( -i\theta (n - A) - \frac{1}{2} \theta^2 B + \Delta (\theta) \right) \, d\theta \]

\[ = \frac{H}{2\pi} \int_{|\theta| \leq \delta} \left(1 + O (\Delta (\theta))\right) \exp \left( -i\theta (n - A) - \frac{1}{2} \theta^2 B \right) \, d\theta + O \left( B^{3/2 - \varepsilon} \delta^4 \right) \]

\[ = \frac{H}{2\pi} \int_{|\theta| \leq \delta} \exp \left( -i\theta (n - A) - \frac{1}{2} \theta^2 B \right) \, d\theta + O \left( \frac{(\log B)^2}{B^{1/2 + \varepsilon}} \right). \]

In order to find the asymptotic behavior of \( I_1 \), we make the change of variables

\[ y = \theta \sqrt{\frac{B}{2}}, \quad \alpha := \frac{n - A}{\sqrt{B/2}}, \]

and obtain (uniformly in \( n \))

\[ I_1 = \frac{H}{\pi \sqrt{2B}} \int_{|y| \leq \delta \sqrt{\frac{B}{2}}} \exp \left( -y^2 - i\alpha y \right) \, dy + O \left( \frac{1}{B^{\frac{1}{2} (1 + \varepsilon)}} \right) \]

\[ = \frac{H}{\pi \sqrt{2B}} \left[ \int_{\mathbb{R}} - \int_{|y| \geq \delta \sqrt{\frac{B}{2}}} \right] \exp \left( -y^2 - i\alpha y \right) \, dy + O \left( \frac{1}{B^{\frac{1}{2} (1 + \varepsilon)}} \right) \]

\[ = \frac{H}{\sqrt{2\pi B}} \exp \left( -\frac{1}{4} \alpha^2 \right) + O \left( \exp \left( - \left( \delta \sqrt{\frac{B}{2}} \right)^2 \right) \right) + O \left( \frac{1}{B^{\frac{1}{2} (1 + \varepsilon)}} \right) \]

\[ = \frac{H}{\sqrt{2\pi B}} \exp \left( -\frac{1}{4} \alpha^2 \right) + O \left( \frac{1}{B} \right) + O \left( \frac{1}{B^{\frac{1}{2} (1 + \varepsilon)}} \right) \]

\[ = \frac{H}{\sqrt{2\pi B}} \exp \left( -\frac{(n - A)^2}{2B} \right) + O \left( \frac{1}{B^{\frac{1}{2} (1 + \varepsilon)}} \right). \]

\[ \square \]

6.3. Construction of the covariance function \( \tilde{G} \). Recall that in Section 3.1 we defined a sequence of intervals \( \{T_{\ell}\}_{\ell=1}^{\infty} \) so that

\[ T_{\ell} = [t_{\ell}, t_{\ell+1}], \quad |T_{\ell}| = t_{\ell+1} - t_{\ell}, \quad B (t_{\ell}) = \ell^6, \quad \ell \geq 1. \]

Moreover, the interval \( T_{\ell} \) is long if \( |T_{\ell}| \geq \frac{8}{\ell^2} \), and in that case its interior is given by

\[ \tilde{T}_{\ell} = \left[ t_{\ell} + \frac{2}{\ell^2}, t_{\ell+1} - \frac{2}{\ell^2} \right]. \]
The set $\mathcal{X}$ of normal values is given by
$$
\mathcal{X} := \bigcup_{T_\ell \text{ long}} \hat{T}_\ell.
$$

We also remind that the Lebesgue measure of the set $\mathbb{R}^+ \setminus \mathcal{X}$ is finite.

We now fix the sequences $p_\ell = \ell^3 = \sqrt{8} (t_\ell)$, and $s_\ell = c_1 \sqrt{\log \ell}$ (see Proposition 6.5) and define the following sets
$$
\mathcal{I}_\ell := [A (t_\ell), A (t_{\ell+1})) \cap \mathbb{N}, \quad \mathcal{I}_\ell^p := \{ n \in \mathcal{I}_\ell : p_\ell | n \},
$$
and the corresponding exponential polynomials
$$
(6.3) \quad P_\ell (t) = \sum_{n \in \mathcal{I}_\ell^p} a_n^2 e^{nt}.
$$

The function $\tilde{G}$ is constructed as follows
$$
(6.4) \quad \tilde{G} (e^t) = \tilde{H} (t) = \sum_{\ell=0}^{\infty} P_\ell (t) =: \sum_{n=0}^{\infty} \delta_n a_n^2 e^{nt}.
$$

For $t \in \hat{T}_\ell$ the main indices $n$ corresponding to $t$ are included in the following window
$$
\mathcal{I}_\ell^p (t) := \{ n \in \mathcal{I}_\ell^p : |n - A (t)| \leq s_\ell \sqrt{B (t)} \}.
$$

Finally, we put
$$
R_{\ell}^{[j]} (t) = \sum_{n \in \mathcal{I}_\ell^p (t)} n^j a_n^2 e^{nt}, \quad R_{\ell}^{[j]} (t) = \sum_{n \in \mathcal{I}_\ell^p (t)} (n - A (t))^j a_n^2 e^{nt},
$$
and
$$
Q_{\ell}^{(j)} (\tau, t) = \sum_{|n-A(t)| > s_\ell \sqrt{B(t)}} n^j a_n^2 e^{nt}, \quad Q_{\ell}^{[j]} (\tau, t) = \sum_{|n-A(t)| > s_\ell \sqrt{B(t)}} (n-A(t))^j a_n^2 e^{nt}.
$$

**Proposition 6.5.** For $t \in \mathcal{X}$ and $|\tau - t| \leq \frac{1}{2 \sqrt{B(t)}}$, we have
$$
\max_{j \in \{0,1,2\}} \left\{ \left| Q_{\ell}^{(j)} (\tau) \right|, \left| Q_{\ell}^{[j]} (\tau, t) \right| \right\} \leq \frac{H \left( \Re \left[ \tau \right] \right)}{B^3 (t)}.
$$

**Remark 6.2.** The choice of the exponent 3 is arbitrary, it can be replaced by any large positive number.

**Proof.** Fix an interval $\hat{T}_\ell \subset \mathcal{X}$. By Lemma 3.3 we will have
$$
|Q_{\ell} (\tau)| \leq \sum_{|k-A(t)| > s \sqrt{B(t)}} a_k^2 e^{k \Re [\tau]} \leq 2H (\Re [\tau]) \exp \left( -\frac{1}{8} (s - 4)^2 \right),
$$
as long as \(4 < s < B^{1/6} (t_{\ell})\). Notice that in order to guarantee that
\[
2 \exp \left( -\frac{1}{8} (s - 4)^2 \right) \leq \frac{1}{\ell_{48}} = \frac{1}{B^8 (t_{\ell})} \leq \frac{1}{B^8 (t)}.
\]
it is sufficient to take \(s = 4 + 2\sqrt{2 \log 2 + 96 \log (t)}\). Thus, we may choose any \(c_1 > 8\sqrt{6}\) in the definition of \(s_{\ell}\), and obtain
\[
|Q_{\ell} (\tau)| \leq \frac{H (\text{Re} [\tau])}{B^8 (t)}.
\]

For \(\tau \in \mathbb{C}\) which satisfies \(|\tau - t| < \frac{1}{2 \sqrt{B (t)}}\), consider the contour \(\Gamma = \left\{ z : |z - \tau| = \frac{1}{2 \sqrt{B (t)}} \right\}\). Using Cauchy’s integral formula, we get
\[
\left| Q_{\ell}^{(j)} (\tau) \right| = \left| \frac{1}{2\pi i} \int_{\Gamma} \frac{Q_{\ell} (z)}{(z - \tau)^{j+1}} \, dz \right| \lesssim \frac{H (\text{Re} [\tau])}{[B (t)]^{8-\frac{4}{j}}}.
\]

In addition, since the the coefficients \(a^2_n\) are non-negative, we also notice that when \(|\text{Re} [\tau] - t| \leq \frac{1}{2 \sqrt{B (t)}}\), and \(j \in \{0, 1, 2\}\),
\[
(6.5) \quad \left| Q_{\ell}^{(j)} (\tau) \right| \leq \left| Q_{\ell}^{(j)} (\text{Re} [\tau]) \right| \leq \frac{H (\text{Re} [\tau])}{B^7 (t)}.
\]

To bound \(\left| Q_{\ell}^{[j]} (\tau, t) \right|\), notice that \(\overline{Q}_{\ell}^{[0]} (\tau, t) = Q_{\ell} (\tau, t)\),
\[
\overline{Q}_{\ell}^{[1]} (\tau, t) = Q_{\ell}' (\tau, t) - A (t) \cdot Q_{\ell} (\tau, t),
\]
\[
\overline{Q}_{\ell}^{[2]} (\tau, t) = Q_{\ell}'' (\tau, t) - 2A (t) \cdot Q_{\ell}' (\tau, t) + A (t)^2 \cdot Q_{\ell} (\tau, t),
\]
and use Assumption 2. \(\square\)

**Proposition 6.6.** There are constants \(c, C > 0\) such that for \(t \in \mathcal{X}\), and \(j \in \{0, 1, 2\}\), we have
\[
c \frac{H (t) (A (t))^j}{\sqrt{B (t)}} \leq R_{\ell}^{(j)} (t) \leq C \frac{H (t) (A (t))^j}{\sqrt{B (t)}},
\]
and
\[
\overline{R}_{\ell}^{[j]} (t) \leq C \frac{H (t) (s_{\ell} \sqrt{B (t)})^j}{\sqrt{B (t)}}.
\]
Proof. By Lemma 6.4, Claim 6.1, and (3.10) we obtain

\[ R_{t \ell}(j) = \sum_{|kp-A|<s_t \sqrt{B}} (kp)^j a_{kp}^2 e^{kp t} \]

\[ = \frac{H}{\sqrt{2 \pi B}} \sum_{|kp-A|<s_t \sqrt{B}} (kp)^j \left[ \exp \left( -\frac{(kp-A)^2}{2B} \right) + O \left( \frac{1}{B^c} \right) \right] \]

\[ \lesssim \frac{H}{\sqrt{B}} \left[ \frac{\sqrt{B}}{p_\ell} \left( A^j + B \cdot 1_{(j=2)} \right) + 1 + \frac{s_t \sqrt{B}}{p_\ell} \cdot \frac{(A + s_t \sqrt{B})^j}{B^c} \right] \]

\[ \lesssim \frac{H}{p_\ell} \cdot A^j \left[ 1 + s_t \cdot \frac{1}{B^c} \right] \lesssim \frac{H(t)}{\sqrt{B(t)}} \cdot A(t)^j \left[ 1 + s_t \cdot \frac{1}{B^c(t)} \right]. \]

The lower bound is obtained in a similar way, using the lower bound in Claim 6.1. To bound \( \tilde{R}_{t \ell}(j) \), we simply use

\[ \tilde{R}_{t \ell}(j) \leq \sum_{n \in \mathbb{Z}^p(t)} |n - A(t)|^j a_n^2 e^{nt} \leq \left( s_t \sqrt{B} \right)^j \tilde{R}_{t \ell}(t) \lesssim \frac{H(t) \left( s_t \sqrt{B} \right)^j}{\sqrt{B(t)}}. \]

Now we are ready to prove that \( G \) and \( \tilde{G} \) are similar.

**Lemma 6.7.** We have

\[ a_{\tilde{H}}(r) \asymp_L a(r), \quad b_{\tilde{H}}(r) \asymp_L b(r). \]

**Proof.** Let \( c, C > 0 \) denote constants. Fix \( t \in \mathcal{X} \), it is sufficient to show that

\[ cA(t) \leq A_{\tilde{H}}(t) \leq CA(t), \quad cB(t) \leq B_{\tilde{H}}(t) \leq CB(t). \]

It follows from the identities

\[ A_{\tilde{H}}(t) = \frac{\tilde{H}'(t)}{\tilde{H}(t)}, \quad B_{\tilde{H}}(t) = A_{\tilde{H}}'(t) = \frac{\tilde{H}''(t)}{\tilde{H}(t)} - \left( \frac{\tilde{H}'(t)}{\tilde{H}(t)} \right)^2, \]

and the definition of \( \tilde{H} \) in (6.4), that

\[ A_{\tilde{H}}(t) = \sum_{n=0}^\infty n \delta_n a_n^2 e^{nt} \frac{\tilde{H}(t)}{\tilde{H}(t)}, \quad B_{\tilde{H}}(t) = \sum_{n,m=0}^\infty (n-m)^2 \delta_n \delta_m a_n^2 a_m^2 e^{(n+m)t} \frac{\tilde{H}(t)^2}{2 \left( \tilde{H}(t) \right)^2}. \]

Notice that for \( j \in \{0, 1, 2\} \) we have

\[ |\tilde{H}^{(j)}(t) - R_{t \ell}^{(j)}(t)| \leq Q_{t \ell}^{(j)}(t, t). \]
Therefore, by Propositions 6.5 and 6.6 we have
\[ A_t(H) = \frac{R_t(t) + (H'(t) - R_t(t))}{R_t(t) + (H(t) - R_t(t))} \lesssim \frac{C H(t) A(t)}{\sqrt{B(t)}} + \frac{H(t)}{B^3(t)} \lesssim A(t), \]
and similarly for the lower bound. Put
\[ \Sigma_1 := \sum_{n,m \in I^p(t)} (n - m)^2 a_n^2 a_m^2 e^{(n+m)t}, \]
and notice that clearly
\[ \Sigma_1 \leq 2B_t(H) (H(t))^2. \]
In addition,
\[ 2B_t(H) (H(t))^2 = \sum_{n,m=0}^{\infty} (n - m)^2 \delta_n \delta_m a_n^2 a_m^2 e^{(n+m)t} \leq \sum_{n,m \in I^p(t)} (n - m)^2 a_n^2 a_m^2 e^{(n+m)t} \]
\[ + 2 \cdot \sum_{n \in I^p(t), |m-A(t)|>s_{\ell} \sqrt{B(t)}} (n - m)^2 a_n^2 a_m^2 e^{(n+m)t} \]
\[ + \sum_{|n-A(t)|>s_{\ell} \sqrt{B(t)}} \sum_{|m-A(t)|>s_{\ell} \sqrt{B(t)}} (n - m)^2 a_n^2 a_m^2 e^{(n+m)t} \]
\[ =: \Sigma_1 + \Sigma_2 + \Sigma_3. \]
By Claim 6.8 below, we have \( c H^2(t) \leq \Sigma_1 \leq C H^2(t) \), so that again by (6.6) and Propositions 6.5 and 6.6 we have
\[ B(t) H^2(t) \lesssim \Sigma_1 \lesssim B(t) H^2(t). \]
Writing
\[ (n - m)^2 \leq \frac{1}{2} ((n - A)^2 + (m - A)^2), \]
we get, using Proposition 6.5,
\[ \Sigma_3 \leq \left( \sum_{|n-A(t)|>s_{\ell} \sqrt{B(t)}} \right) \left( \sum_{|m-A(t)|>s_{\ell} \sqrt{B(t)}} \right) a_n^2 a_m^2 e^{nt} = O \left( H^2(t) B(t) \right) \]
\[ \leq \frac{H^2(t)}{B^6(t)} \lesssim \frac{H^2(t)}{B^5(t)}. \]
In addition, combining Propositions 6.5 and 6.6, we have

\[
\Sigma_2 \leq \left( \sum_{n \in I^p(t)} (n - A)^2 a_n^2 e^{nt} \right) \left( \sum_{|m-A(t)| > s \sqrt{B(t)}} a_m^2 e^{m t} \right) + \left( \sum_{n \in I^p(t)} a_n^2 e^{nt} \right) \left( \sum_{|m-A(t)| > s \sqrt{B(t)}} (m - A)^2 a_m^2 e^{m t} \right)
\]

\[
= \overline{R}_\ell^{[2]}(t) \overline{Q}_\ell^{[0]}(t, t) + \overline{R}_\ell^{[0]}(t) \overline{Q}_\ell^{[2]}(t, t)
\]

\[
\lesssim \frac{H(t) \cdot s B(t) \cdot H(t)}{B(t)^3} + \frac{H(t) \cdot H(t)}{B(t)^3} \leq \frac{H^2(t)}{B^2(t)} \lesssim B(t) \cdot \overline{B}(t).
\]

Thus,

\[
B(t) \lesssim B_{\overline{B}}(t) \lesssim B(t).
\]

**Claim 6.8.** There are constants \(c, C > 0\), so that for \(t \in X\), we have

\[
cH^2(t) \leq \Sigma_1 \leq CH^2(t),
\]

where

\[
\Sigma_1 := \sum_{n, m \in I^p(t)} (n - m)^2 a_n^2 a_m^2 e^{(n+m)t}.
\]

**Proof.** By Lemma 6.4 we have

\[
\Sigma_1 = \frac{H^2}{2\pi B} \sum_{n, m \in I^p(t)} (n - m)^2 \left( e^{-\frac{(n-A)^2}{2B}} + O \left( \frac{1}{B^c} \right) \right) \left( e^{-\frac{(m-A)^2}{2B}} + O \left( \frac{1}{B^c} \right) \right),
\]

and therefore

\[
|\Sigma_1 - S_1| \leq S_2,
\]

where

\[
S_1 := \frac{H^2}{B} \sum_{n, m \in I^p(t)} (n - m)^2 \exp \left( -\frac{(n-A)^2}{2B} - \frac{(m-A)^2}{2B} \right),
\]

\[
S_2 := \frac{H^2}{B^{1+s}} \sum_{n, m \in I^p(t)} (n - m)^2.
\]
By Claim 6.2, we have
\[ S_1 \lesssim \frac{H^2}{B} \cdot B = H^2. \]
For \( n, m \in \mathcal{I}_t^p (t) \), \( |n - m| \leq 2s_t \sqrt{B} \), thus we have
\[ S_2 \lesssim \frac{H^2}{B^e} \sum_{n,m \in \mathcal{I}_t^p (t)} s^2_t \lesssim \frac{H^2}{B^e} \cdot s^4_t = o (H^2). \]

6.4. Proof of Theorem 1.7. Let
\[ \tilde{f}(z) = \sum_{\ell=0}^{\infty} \sum_{n \in \mathcal{I}_p^\ell} \xi_n a_n z^n \]
be a Gaussian entire function with covariance function \( \tilde{G} \). In order to prove that \( \text{Var} \ (n \ (r)) \gtrsim p^2 \gtrsim B \ (t) \) it is enough to show that \( n \ (r) = kp_\ell \) for two different values of \( k \), with probability at least \( c > 0 \) each. By Rouché’s theorem, it is enough to show that the term \( \xi_{kp_\ell} a_{kp_\ell} z^{kp_\ell} \) dominates all other terms.

**Proposition 6.9.** There is a constant \( c > 0 \), so that for every \( t \in \mathcal{X} \) the probability of the event \( \{ n \tilde{f} (r) = kp_\ell \} \) is at least \( c \) for two different values \( k \in \mathbb{N} \).

**Proof.** Put
\[ m_1 = \max \{ m \in \mathcal{I}_t^p : m < A \}, \quad m_2 = \min \{ m \in \mathcal{I}_t^p : m > A \}, \]
and notice that
\[ \frac{1}{2} p_\ell \leq \max \{ |m_1 - A|, |m_2 - A| \} \leq p_\ell, \quad p_\ell \leq |m_1 - m_2| \leq 2p_\ell. \]
Define the events
\[ E_j = \{ |\xi_{m_j} a_{m_j} z^{m_j} | > |f (z) - \xi_{m_j} a_{m_j} z^{m_j} | \}, \quad j \in \{ 1, 2 \}. \]
We will prove \( \mathbb{P} [ E_1 ] > c \), the proof for \( E_2 \) is similar. The result will then follow by Rouché’s theorem. We have the crude bound:
\[ \mathbb{E} | f (z) - \xi_{m_j} a_{m_j} z^{m_j} | \leq \sum_{n \neq m_1} \mathbb{E} |\xi_n| \delta_n a_n r^n = \frac{\sqrt{\pi}}{2} [S_1 + S_2], \]
where
\[ S_1 = \sum_{n \neq m_1, n \in \mathcal{I}_t^p} a_n r^n, \quad S_2 = \sum_{|n - m| > s_t \sqrt{B}} \delta_n a_n r^n. \]
By Lemma 6.4, we have, uniformly in $n$,

$$a_n^2 e^{nt} = \frac{H}{\sqrt{2\pi B}} \left[ \exp \left( -\frac{(n-A)^2}{2B} \right) + O \left( \frac{1}{B^\varepsilon} \right) \right].$$

Since $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$, we get by Claim 6.1,

$$S_1 \leq \sum_{n \in I_p} a_n e^{\frac{1}{2}tn} \leq \frac{H}{B^{1/4}} \sum_{n \in I_p} \exp \left( -\frac{(n-A)^2}{4B} \right) + \frac{\sqrt{H}}{B^{1/4}} \cdot \frac{s_t \sqrt{B}}{p_t} \cdot O \left( \frac{1}{B^\varepsilon} \right).$$

Now let $\eta = \frac{1}{A} \leq \frac{1}{2\sqrt{B}}$, where the inequality holds (for $t$ sufficiently large) by (3.10). Writing,

$$r = e^{\frac{1}{2}tn}, \quad \tau = t + \eta, \quad r^n = e^{\frac{1}{2}tn} = e^{\frac{1}{2}(\tau-\eta)n} = e^{\frac{1}{2}\tau n} e^{-\frac{1}{2}\eta n},$$

and using the Cauchy-Schwarz inequality, Proposition 6.5, and (3.3), we have

$$S_2^2 \leq \sum_{|n-A| > s_t \sqrt{B}} a_n^2 e^{\tau n} \cdot \sum_{|n-A| > s_t \sqrt{B}} e^{-\eta n} \leq Q_t^{(0)}(\tau, t) \cdot \sum_{n=0}^{\infty} e^{-\eta n} \leq \frac{H (\Re [\tau])}{B^3} \cdot \frac{2}{\eta} \leq H(t) \exp \left( \eta \cdot A + C\eta^2 B \right) \cdot \frac{A}{B^3} \approx H \cdot \frac{A}{B^3},$$

where we again used (3.10) in the last inequality. Therefore, by Assumption 2 we get

$$S_2 \lesssim \frac{\sqrt{H(t)}}{B^{1/4} (t)}.$$

We conclude by Markov's inequality that for $C > 0$ sufficiently large, we have

$$P \left[ |f(z) - \xi_m a_m z_m^2| > \frac{\sqrt{\pi}}{2} |S_1 + S_2| \right] \leq P \left[ |f(z) - \xi_m a_m z_m^2| > \frac{C \sqrt{H(t)}}{B^{1/4} (t)} \right] < \frac{1}{2}.$$

Finally, again by Lemma 6.4

$$(a_m r_m^2)^2 \gtrsim \frac{H(t)}{\sqrt{B(t)}},$$

and thus with probability at least $c > 0$ we have that $|\xi_m a_m z_m| > |f(z) - \xi_m a_m z_m^2|$. \qed

### 7. Examples of Admissible Covariance Functions

Here are some explicit examples for covariance functions which are type I and type II admissible.
7.1. **The Mittag-Leffler function.** We consider the Gaussian entire function $f$ whose covariance function $G$ is given by the Mittag-Leffler function

$$G(z) = G_{\alpha}(z) = \sum_{n \geq 0} \frac{z^n}{\Gamma(1 + \alpha^{-1}n)},$$

where $\alpha > 0$ is a parameter. Notice that $G_1(z) = e^z$ and $G_{\frac{1}{2}}(z) = \cosh \sqrt{z}$. The asymptotic behavior of $G$ and $G'$ is well-known, see for example [10, Section 3.5.3]. In particular, as $|z| \to \infty$, and uniformly in $\arg z$,

$$G(z) = \begin{cases} \alpha e^{\frac{z}{\alpha}} + O(1), & |\arg z| \leq \frac{\pi}{2\alpha}; \\ O(1), & \text{otherwise,} \end{cases}$$

and

$$zG'(z) = \begin{cases} \alpha^2 e^{\frac{z}{\alpha}} + O(1), & |\arg z| \leq \frac{\pi}{2\alpha}; \\ O(1), & \text{otherwise.} \end{cases}$$

**Remark 7.1.** Notice that for $\alpha \in (0, \frac{1}{2}]$ the complement of $\{|\arg z| \leq \frac{\pi}{2\alpha}\}$ is empty.

From this asymptotic description, one easily verifies that $G$ is type I and type II admissible. Since

$$a(r) = \frac{rG'(r)}{G(r)} \sim \alpha r^\alpha, \quad b(r) = ra'(r) \sim \alpha^2 r^\alpha, \quad r \to \infty,$$

we have

$$\mathbb{E}[n_f(r)] = a(r^2) \sim \alpha r^{2\alpha}, \quad r \to \infty.$$

By Theorem 1.4,

$$\text{Var}(n_f(r)) \sim \frac{\zeta\left(\frac{3}{2}\right)}{4\sqrt{\pi}} \sqrt{b(r^2)} \sim \frac{\zeta\left(\frac{3}{2}\right)}{4\sqrt{\pi}} \cdot \alpha r^\alpha, \quad r \to \infty.$$

7.2. **The double exponent.** Here we consider the Gaussian entire function $f$ with covariance function

$$G(z) = e^{ez}.$$

The function $G$ is type I and type II admissible, and has an infinite order of growth, with

$$a(r) = re^r, \quad b(r) = r(r+1)e^r.$$

Thus,

$$\mathbb{E}[n_f(r)] = r^2e^{r^2},$$

and by Theorem 1.4

$$\text{Var}(n_f(r)) \sim \frac{\zeta\left(\frac{3}{2}\right)}{4\sqrt{\pi}} r^2e^{\frac{1}{2}r^2}, \quad r \to \infty.$$
7.3. The Lindelöf functions. For \( \alpha > 0 \), we consider the Gaussian entire function \( f \) with covariance function
\[
G(z) = G_\alpha(z) = \sum_{n \geq 0} \frac{z^n}{\log^\alpha (n + e)}.
\]
The function \( G \) has infinite order of growth, and it follows from [18, Example 1.4.1], that it is type I and type II admissible with,
\[
\log G(r) \sim \frac{\alpha}{e} r^{-\frac{1}{\alpha}} \exp \left( \frac{r^{\frac{1}{\alpha}}}{\alpha} \right), \quad r \to \infty,
\]
and
\[
a(r) \sim \frac{1}{e} \exp \left( r^{\frac{1}{\alpha}} \right), \quad b(r) \sim \frac{r^{\frac{1}{\alpha}}}{\alpha e} \exp \left( r^{\frac{1}{\alpha}} \right), \quad r \to \infty.
\]
In this case,
\[
\mathbb{E} [n_f(r)] \sim e^{-1} r^{2/\alpha}, \quad r \to \infty,
\]
\[
\operatorname{Var} (n_f(r)) \sim \frac{\zeta \left( \frac{3}{2} \right)}{4\sqrt{\pi e \alpha}} r^{1/\alpha} e^{r^{2/\alpha}}, \quad r \to \infty.
\]

7.4. An example with radius of convergence 1. For \( \alpha > 0 \), we consider the Gaussian analytic function \( f \), where now the covariance function is given by
\[
G(z) = \exp \left( \frac{1}{(1 - z^2)^\alpha} \right).
\]
One can check that this function is type I admissible with \( R_G = 1 \) and \( C_G > 2 \) sufficiently large depending on \( \alpha \). Since
\[
a(r) = \frac{\alpha r}{(1 - r)^{\alpha + 1}}, \quad b(r) = \frac{\alpha r}{(1 - r)^{\alpha + 1}} + \frac{\alpha (\alpha + 1) r^2}{(1 - r)^{\alpha + 2}}.
\]
We have
\[
\mathbb{E} [n_f(r)] = \frac{\alpha r^2}{(1 - r^2)^{\alpha + 1}}, \quad r < 1,
\]
and Theorem 1.4 yields
\[
\operatorname{Var} (n_f(r)) \sim \frac{\zeta \left( \frac{3}{2} \right)}{4\sqrt{\pi e \alpha}} \frac{\sqrt{\alpha (\alpha + 1)}}{(1 - r^2)^{\frac{1}{2} \alpha + 1}}, \quad r \to 1^{-}.
\]

Remark 7.2. For functions \( G \) of slower growth, the above asymptotics no longer holds. Buckley [4] found the asymptotic of the variance for the following special choice
\[
G(z) = \frac{1}{(1 - z)^L}, \quad \text{with } L > 0,
\]
which corresponds to Gaussian analytic functions whose zero sets are invariant with respect to the isometries of the hyperbolic disk (see [14, Chapter 2.3]). Earlier Peres and Virág [28]
computed the variance in the case $L = 1$, where the zero set forms a *determinantal* point process.

**Appendix A. Formulas for the expectation and variance**

For the convenience of the reader, here we give proofs for the formulas of the expected value and variance of the number of zeros in a disk from [16, p. 195]. Let $\Pi \subset \mathbb{C}$ be a compact subset of the plane, and denote by $n_f (\Pi)$ the number of zeros of the Gaussian analytic function $f$ in $\Pi$ (we assume that $\Pi$ is contained inside the domain of convergence of $f$). We denote by $K_f$ the covariance kernel of $f$, and by $J_f$ the normalized covariance kernel, given by

$$J_f (z, w) = \frac{K_f (z, w)}{\sqrt{K_f (z, z) K_f (w, w)}}.$$

We first recall the Edelman-Kostlan formula [14, p. 25], which states

$$E [n_f (\Pi)] = \frac{1}{4\pi} \int_{\Pi} \Delta z \log K_f (z, z) \, dm (z),$$

where $m$ is the Lebesgue measure on $\mathbb{C}$ and $\Delta z = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}$ is the usual Laplace operator. In addition, we have (see [31, Thm. 3.1] or [23, Lemma 2.3])

$$\text{Var} (n_f (\Pi)) = \frac{1}{16\pi^2} \int_{\Pi \times \Pi} \Delta z \Delta w \text{Li}_2 (|J_f (z, w)|^2) \, dm (z) \, dm (w),$$

where

$$\text{Li}_2 (x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

is the dilogarithm function.

We now derive more explicit formulas when $\Pi = r \mathbb{D} := \{|w| \leq r\}$. Recall that $n (r) := n_f (r \mathbb{D})$, $K_f (z, w) = G (z \bar{w})$, $H (t) = G (e^t)$, $A (t) = H' (t)$, $B (t) = A' (t)$, and that we put $e^t = r^2$.

**Claim A.1.** We have

$$E [n (r)] = a (r^2) = A (t).$$

**Proof.** Writing the Laplace operator in polar coordinates and differentiating the covariance kernel, we get

$$\Delta \log K_f (z, z) = \frac{1}{r} \frac{\partial}{\partial r} \left[ \frac{r}{\partial r} \right] \log G (r^2).$$

Now, by (A.1) we have,

$$E [n (r)] = \frac{1}{2} \int_{0}^{r} \frac{\partial}{\partial s} \left[ s \frac{\partial}{\partial s} \right] \log G (s^2) \, ds = \frac{1}{2} \left[ s \frac{\partial}{\partial s} \right] \log G (s^2) \bigg|_{s=r} = \frac{r^2 G' (r^2)}{G (r^2)} = a (r^2) = A (t).$$
Claim A.2 (cf. [4, Lemma 5]). We have

\[
\text{Var} (n_f (r)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\left| G (r^2) G' (r^2 e^{i\theta}) r^2 e^{i\theta} - G (r^2 e^{i\theta}) G' (r^2) r^2 \right|^2}{G^2 (r^2) (G^2 (r^2) - |G^2 (r^2 e^{i\theta})|)} \, d\theta
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|H (t) H' (t + i\theta) - H (t + i\theta) H' (t)|^2}{H^2 (t) (H^2 (t) - |H^2 (t + i\theta)|)} \, d\theta
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|A (t + i\theta) - A (t)|^2}{\exp \left( 2 \cdot \text{Im} \left[ J_0^\theta A (t + i\varphi) d\varphi \right] \right) - 1} \, d\theta.
\]

Proof. Applying Stokes’ Theorem to (A.2) we get

(A.3) \[
\text{Var} (n(r)) = -\frac{1}{4\pi^2} \oint \oint \frac{\partial}{\partial z} \frac{\partial}{\partial w} \text{Li}_2 \left( |J_f (z, w)|^2 \right) \, dz \, dw.
\]

Recall that

\[
\frac{d}{d\zeta} \text{Li}_2 (\zeta) = \frac{1}{\zeta} \log \frac{1}{1 - \zeta},
\]

and therefore

\[
\frac{\partial}{\partial w} \text{Li}_2 \left( |J_f (z, w)|^2 \right) = \frac{\partial}{\partial z} \text{Li}_2 \left( \frac{K_f (z, w) K_f (w, z)}{K_f (z, z) K_f (w, w)} \right) = \frac{\partial}{\partial w} \text{Li}_2 \left( \frac{G (z \overline{w}) G (\overline{z} w)}{G (z \overline{z}) G (w \overline{w})} \right)
\]

\[
= \log \left( 1 - \frac{G (z \overline{w}) G (\overline{z} w)}{G (z \overline{z}) G (w \overline{w})} \right) \left( \frac{w \overline{G} (z \overline{w}) G' (w \overline{w}) - z \overline{G} (w \overline{w}) G' (z \overline{w})}{G (z \overline{z}) G (w \overline{w}) - G (z \overline{w}) G (\overline{z} w)} \right),
\]

hence, after some simplifications

\[
\frac{\partial}{\partial z} \frac{\partial}{\partial w} \text{Li}_2 \left( |J_f (z, w)|^2 \right) = \frac{(w \overline{G} (z \overline{w}) G' (w \overline{w}) - z \overline{G} (w \overline{w}) G' (z \overline{w}))}{G (z \overline{z}) G (w \overline{w}) - G (z \overline{w}) G (\overline{z} w)}.\]

Using the parametrization \( z = re^{i\theta_1}, \ w = re^{i\theta_2} \) in (A.3) (notice the contour \( \partial \ (r \mathbb{D}) \) is oriented clockwise) and after some additional simplifications, we get,

\[
\text{Var} (n_f (r)) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{\left| G (r^2) G' (r^2 e^{i(\theta_1 - \theta_2)}) r^2 e^{i(\theta_1 - \theta_2)} - G (r^2 e^{i(\theta_1 - \theta_2)}) G' (r^2) r^2 \right|^2}{G^2 (r^2)^2 - |G (r^2 e^{i(\theta_1 - \theta_2)})|^2} \, d\theta_1 d\theta_2.
\]

Making a change of variables \( \theta = \theta_1 - \theta_2 \) and integrating out the other variable, we get

\[
\text{Var} (n_f (r)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\left| G (r^2) G' (r^2 e^{i\theta}) r^2 e^{i\theta} - G (r^2 e^{i\theta}) G' (r^2) r^2 \right|^2}{G^2 (r^2) (G^2 (r^2) - |G^2 (r^2 e^{i\theta})|)} \, d\theta.
\]
Now using $r^2 = e^t$, we find that

$$\left| \frac{G(r^2)G'(r^2 e^{i\theta}) r^2 e^{i\theta} - G(r^2 e^{i\theta}) G'(r^2) r^2}{G^2(r^2) (G^2(r^2) - |G^2(r^2 e^{i\theta})|)} \right|^2 = \frac{|H(t) H'(t + i\theta) - H(t + i\theta) H'(t)|^2}{H^2(t) (H^2(t) - |H^2(t + i\theta)|)}$$

$$= \frac{|A(t + i\theta) - A(t)|^2}{\exp \left( -2 \cdot \Re \left[ i \int_0^\theta A(t + i\varphi) d\varphi \right] \right) - 1},$$

and since $\Im(z) = -\Re(iz)$ we get the required result. □

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