On Double Smoothed Volatility Estimation of Potentially Nonstationary Jump-Diffusion Model

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Abstract

In this paper, we present the double smoothed nonparametric approach for infinitesimal conditional volatility of jump-diffusion model based on high frequency data. Under certain minimal conditions, we obtain the strong consistency and asymptotic normality for the estimator as the time span $T \to \infty$ and the sample interval $\Delta_n \to 0$. The procedure and asymptotic behavior can be applied for both null Harris recurrent and positive Harris recurrent processes.

Keywords: Diffusion models with jumps, infinitesimal conditional moment, consistency and asymptotic normality, variance reduction, nonstationary high frequency financial data.

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JEL classification: C13; C14; C22

1. Introduction

Volatility is a very important variable in the research of financial economics. Portfolio selection, original asset or derivative assets option pricing and risk management depend on the accurate measurement of volatility. In the past few decades, the estimation for volatility has become one of the most active research fields in empirical finance or time series econometrics.

Continuous-time models are widely used in economics and finance, such as interest rate or an asset price, especially the continuous-time diffusion processes, one of the most famous models is the Black and Scholes asset pricing model. Bandi and Phillips [1] proposed a double smoothed approach to the unknown

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coefficients of potentially nonstationary or stationary scalar diffusion models under high frequency data. Recently, an inordinate amount of attention has been focused on the stochastic processes with jumps, which can accommodate the impact of sudden and large shocks to financial markets. Johannes [9] provided the statistical and economic role of jumps in continuous-time interest rate models. In this paper, we are focused on the following time-homogeneous diffusion process with jumps $X = (X_t)_{t \geq 0}$ such that

$$dX_t = \left[ \mu(X_{t-}) - \lambda(X_{t-}) \int_Y c(X_{t-}, y) \Pi(dy) \right] dt + \sigma(X_{t-}) dW_t + dJ_t,$$  \hspace{1em} (1.1)$$

where $J_t$ is a compound Poisson process of the representation

$$dJ_t = \int_Y c(X_{t-}, y) N(dt, dy),$$  \hspace{1em} (1.2)$$

and $N(dt, dy)$ is a time-homogeneous Poisson counting measure with independent increments. In addition, the functions $\mu(\cdot)$ and $\sigma(\cdot)$ are the infinitesimal conditional drift and variation due to the continuous diffusion process, $Y = \mathbb{R} \setminus \{0\}$, $W_t$ is a standard Brownian motion independent of jump process $J_t$, $\lambda(\cdot)$ represents the intensity measure and $c(\cdot, y)$ reflects the conditional magnitude of a jump, where $y$ is a random variable with probability measure $\Pi(dy)$. Furthermore, the coefficients $\mu(\cdot)$, $\sigma(\cdot)$, $\lambda(\cdot)$ and $c(\cdot, y)$ economically represent the time trend referred as expected risk return and the conditional variance of the return for an underlying asset, respectively.

There are many statisticians and economists focused on state-domain nonparametric estimation for volatility functions of jump-diffusion models [11]. Bandi and Nguyen [2] and Johannes [9] considered the kernel weighted version of instantaneous volatility. Hanif [5], Hanif, Wang and Lin [12], Lin and Wang [10], Wang, Zhang and Tang [17], Xu and Phillips [19] and so on improved the properties of the estimators based on outstanding nonparametric approaches. Moreover, Song [15], Chen and Zhang [3] considered the nonparametric volatility estimation of second-order jump-diffusion model under integrated observations based on Wang and Lin [16]. However, these work are only single-smoothing. In this paper, we present the double smoothed nonparametric approach for infinitesimal conditional volatility of jump-diffusion model based on high frequency data. Double smoothed estimator can reduce the asymptotic mean-squared error than the estimator proposed in Bandi and Nguyen [2] for some chosen bandwidths, any level $x$ and any processes. Furthermore, for both null Harris recurrent and positive Harris recurrent processes [14], we obtain
the strong consistency and asymptotic normality for the estimator as the time span $T \to \infty$ and the sample interval $\Delta_n \to 0$. Our result can availably solve the problem proposed in the discussion part of Zhou \cite{20}.

The paper is organized as follows. The large sample properties of the double smoothed volatility estimator are presented in Section 2. Some technical lemmas and detailed proofs for the main theorem are given in Section 3.

2. Technical assumptions and Large sample properties

Model (1.1) can be written in the integral form as

$$X_t + \Delta = X_t + \int_t^{t+\Delta} \mu(X_t) \, dt + \int_t^{t+\Delta} \sigma(X_t-) \, dW_t + \int_t^{t+\Delta} c(X_t-, y) \bar{\nu}(dt, dy),$$

(2.1)

where $\bar{\nu}(dt, dy) := N(dt, dy) - \lambda(X_t-) \Pi(dy) dt$ is a compensated Poisson random measure. We can observe that

$$\int_t^{t+\Delta} \int Y_c(X_t-, y) \bar{\nu}(dt, dy) = \int_t^{t+\Delta} dJ_t - \int_t^{t+\Delta} \lambda(X_t-) \int Y_c(X_t-, y) \Pi(dy) dt$$

(2.2)

represents the conditional variation due to the discontinuous jumps of the process $X_t$.

Due to the Markov properties of model (1.1), we can build the following infinitesimal conditional expectations as those in Bandi and Nguyen \cite{2}:

$$M^2(x) = \lim_{\Delta_n \to 0} E \left[ \frac{(X_{t+\Delta_n} - X_t)^2}{\Delta_n} \bigg| X_t = x \right] = \sigma^2(x) + \lambda(x)E_Y [c^2(x, y)],$$

(2.3)

$$M^k(x) = \lim_{\Delta_n \to 0} E \left[ \frac{(X_{t+\Delta_n} - X_t)^k}{\Delta_n} \bigg| X_t = x \right] = \lambda(x)E_Y [c^k(x, y)],$$

(2.4)

where $k > 2$.

Define

$$t(i\Delta_n)_0 = \inf\{t \geq 0 : |X_t - X_{i\Delta_n}| \leq \varepsilon_n\},$$

$$t(i\Delta_n)_{j+1} = \inf\{t \geq t(i\Delta_n)_j + \Delta_n : |X_t - X_{i\Delta_n}| \leq \varepsilon_n\},$$

$$m_n(i\Delta_n) = \sum_{j=1}^n 1(|X_{i\Delta_n - X_{i\Delta_n}}| \leq \varepsilon_n).$$

For the given $\{X_{i\Delta_n} ; i = 1, 2, \cdots \}$, the double smoothed estimator for volatility $M^2(x)$ based on the infinitesimal conditional expectation (2.3) is defined as

$$M_n^2(x) = \frac{\sum_{i=1}^n K_h_n(X_{i\Delta_n} - x) m_n(i\Delta_n) \Delta_n \sum_{j=0}^{m_n(i\Delta_n)-1} [X_{t(i\Delta_n)j} + \Delta_n - X_{t(i\Delta_n)j}]^2}{\sum_{i=1}^n K_h_n(X_{i\Delta_n} - x)},$$

(2.5)
where \( K_{h_n}(\cdot) := \frac{1}{h_n}K(\frac{\cdot}{h_n}) \) with the kernel function \( K(\cdot) \) and \( h_n \) is a sequence of positive numbers, satisfies \( h_n \to 0 \) as \( n \to \infty \).

The assumptions of this paper are listed below, which confirm the large sample properties of the constructed estimators based on (2.7). In what follows, denote \( \mathcal{D} = (l, u) \) as the admissible range of the process \( X_t \) in model (1.1).

**Assumption 1.** i) For each \( n \in \mathbb{N} \), there exist a constant \( C_1 \) and a function \( \zeta_n : Y \to \mathbb{R}_+ \) with \( \int_Y \zeta_n^2(y)\Pi(dy) < \infty \) such that, for any \( |x| \leq n, |z| \leq n, y \in Y \),

\[
|\mu(x) - \mu(z)| + |\sigma(x) - \sigma(z)| \leq L_n|x - z|,
\]

\[
\lambda(x) \int_Y |c(x, y) - c(z, y)|\Pi(dy) \leq \zeta_n(y)|x - z|.
\]

Moreover, for each \( n \in \mathbb{N} \), there exist \( \zeta_n \) as above and \( C_2 \), such that for all \( x \in \mathbb{R}, y \in Y \),

\[
|\mu(x)| + |\sigma(x)| \leq C_2(1 + |x|), \quad \lambda(x) \int_Y |c(x, y)|\Pi(dy) \leq \zeta_n(y)(1 + |x|).
\]

ii) There exists a constant \( C_3 \) such that

\[
\lambda(x) \int_Y |c(x, y)|^\alpha\Pi(dy) \leq C_3(1 + |x|^\alpha).
\]

for a fixed \( \alpha > 2 \) and \( \forall x \in \mathcal{D} \).

iii) The functions \( \sigma^2(\cdot), \lambda(\cdot) \) and \( c(\cdot, y) \) are at least twice continuously differentiable. \( \lambda(x) \geq 0 \) and \( \sigma^2(x) \geq 0 \) for \( \forall x \in \mathcal{D} \).

**Remark 2.1.** This assumption guarantees the existence and uniqueness of a càdlàg strong solution to stochastic differential equation \( X_t \) in (1.1), see Jacod and Shiryaev [8]. For instance, Bandi and Nguyen [2], Shimizu and Yoshida [14] imposed similar conditions on the coefficients of the underlying stochastic differential equation.

**Assumption 2.** The process \( X = \{X_t\}_{t \geq 0} \) in model (1.1) is Harris recurrent.

**Assumption 3.** The process \( X = \{X_t\}_{t \geq 0} \) in model (1.1) is positive Harris recurrent.

**Remark 2.2.** The Assumption 2 guarantees the existence of a unique invariant measure \( s(x) \), that is, \( s(A) = \int_\mathcal{D} P(X_t^{(x)} \in A)dx \) \( \forall A \in \mathcal{B}(\mathcal{D}) \). The Assumption 3 implies that the process \( X_t \) has a time-invariant probability measure given by \( p(dx) = \frac{s(x)}{s(\mathcal{D})} \). The positive Harris recurrent condition means that the process becomes stationary at any initial level \( x \in \mathcal{D} \). Furthermore, as discussed in Bandi and Phillips [1], the stationary probability measure can increase the asymptotic rate of convergence for underlying estimators.
Assumption 4. The kernel $K(\cdot) : \mathbb{R} \to \mathbb{R}^+$ is a continuously differentiable, bounded and symmetric function satisfying:

$$
\int K(u)du = 1, \quad \int K'(u)du < \infty, \quad K^2_\Gamma := \int K^2(u)u^2du < \infty.
$$

Remark 2.3. In fact, any density function can be considered as a kernel, moreover even unnecessary positive functions can be used. For simplification, we only consider positive and symmetrical kernels used widely. It is well known both empirically and theoretically that the choice of kernel functions is not very important to the kernel estimator, see Gasser and Müller [4].

Assumption 5. $T \to \infty$, $\Delta_n \to 0$, $h_n \to 0$, $\frac{\bar{L}_X(T,x)}{\varepsilon_n^2}(\Delta_n \log(\frac{1}{\Delta_n}))^{\frac{1}{2}} \to 0$, $\varepsilon_n \bar{L}_X(T,x) \to \infty$, $\frac{L_X(T,x)}{\varepsilon_n^2}(\Delta_n \log(\frac{1}{\Delta_n}))^{\frac{1}{2}} \to 0$ as $n \to \infty$.

Remark 2.4. The relationship between $h_n$ and $\Delta_n$ is similar as that in Bandi and Nguyen [2].

We have the following asymptotic results for the double smoothed estimators such as (2.7) based on the assumptions above.

Theorem 2.5. Under Assumptions [1 2 3 4] as $n \to \infty$, we have

(i) $M^n_2(x) \xrightarrow{a.s.} M^2(x)$.

(ii) Furthermore, if $h_n = o(\varepsilon_n)$ and $\varepsilon_n^2 \bar{L}_X(T,x) = O_{a.s.}(1)$, then

$$
\varepsilon_n \bar{L}_X(T,x)(M^n_2(x) - M^2(x) - \varepsilon_n^2 \Gamma M^2) \Rightarrow N \left(0, \frac{1}{2} M^4(x) \right),
$$

where $\Gamma M^2 = \frac{1}{3} \left[ \frac{1}{2} (M^2)''(x) + (M^2)'(x) \frac{\phi'(x)}{\phi(x)} \right]$.

(iii) If $h_n = O(\varepsilon_n)$ with $\frac{h_n}{\varepsilon_n} = \phi$ and $\varepsilon_n^2 \bar{L}_X(T,x) = O_{a.s.}(1)$, then

$$
\varepsilon_n \bar{L}_X(T,x)(M^n_2(x) - M^2(x) - \varepsilon_n^2 \Gamma \phi M^2) \Rightarrow N \left(0, \frac{1}{2} \theta_\phi M^4(x) \right),
$$

where $\Gamma \phi M^2 = (K_2 \phi^2 + \frac{1}{3}) \left[ \frac{1}{2} (M^2)''(x) + (M^2)'(x) \frac{\phi'(x)}{\phi(x)} \right]$ and

$$
\theta_\phi = \frac{1}{2} \int_{-\infty}^{\infty} dz \int_{(z-1)/\phi}^{(z+1)/\phi} da \int_{(z-1)/\phi}^{(z+1)/\phi} da K(a) K(e),
$$

where $\bar{L}_X(T,x)$ is defined as that in lemma [3].

Under Assumption [3] the local time $\bar{L}_X^\oplus(t,a)$ increases consistently with $T$ up to multiplication by a constant as

$$
\frac{\bar{L}_X^\oplus(t,a)}{T} \xrightarrow{a.s.} p(x), \quad \forall x \in \partial,
$$

(2.6)

where $\bar{L}_X^\oplus(t,a)$ is mentioned in equation (3.7) below. Based on the equation (2.6), we can obtain the following corollary.
Corollary 1. [Stationary Case] Under Assumptions\textsuperscript{1} \textsuperscript{2} \textsuperscript{3} \textsuperscript{4} as $n \to \infty$, we have

(i) $\hat{M}_n^2(x) \xrightarrow{P} M^2(x)$.

(ii) Furthermore, \textsuperscript{5} if $h_n = o(\varepsilon_n)$ and $n\Delta_n \varepsilon_n = O_{a.s.}(1)$, then

\[ \sqrt{n\Delta_n \varepsilon_n} (\hat{M}_n^2(x) - M^2(x) - \varepsilon_n^2 * \Gamma_{M^2}) \Rightarrow N \left( 0, \frac{1}{2} \frac{M^4(x)}{p(x)} \right), \]

where $\Gamma_{M^2} = \frac{1}{3} \left[ \frac{1}{2} (M^2)''(x) + (M^2)'(x) \frac{s'(x)}{s(x)} \right]$.

(iii) If $h_n = O(\varepsilon_n)$ with $\frac{h_n}{\varepsilon_n} = \phi$ and $n\Delta_n \varepsilon_n = O_{a.s.}(1)$, then

\[ \sqrt{n\Delta_n \varepsilon_n} (\hat{M}_n^2(x) - M^2(x) - \varepsilon_n^2 * \Gamma_{M^2}) \Rightarrow N \left( 0, \frac{1}{2} \frac{\theta \phi M^4(x)}{p(x)} \right), \]

where $\Gamma_{M^2} = (K_2 \phi^2 + \frac{1}{3}) \left[ \frac{1}{2} (M^2)''(x) + (M^2)'(x) \frac{s'(x)}{s(x)} \right]$ and

$\theta = \frac{1}{\phi} \int_{-\infty}^{+\infty} dz j_{(z+1)/\phi} j_{(z-1)/\phi} j_{(z+1)/\phi} j_{(z-1)/\phi} eK(a) K(e)$.

Remark 2.6. In contrary to the scalar diffusion model without jumps in Bandi and Phillips\textsuperscript{1}, the rate of convergence of the second infinitesimal moment estimator is not the same as the first infinitesimal moment estimator. Apparently, this is due to the presence of discontinuous breaks that have an equal impact on all the functional estimates. As Johannes\textsuperscript{9} pointed out, for the conditional variance of interest rate changes, not only diffusion play a certain role, but also jumps account for more than half at lower interest level rates, almost two-thirds at higher interest level rates, which dominate the conditional volatility of interest rate changes. Thus, it is extremely important to estimate the conditional variance as $M^2(x)$ not only the diffusion part $\sigma^2(x)$, which reflects the fluctuation of the return of the underlying asset. Nonparametric estimation to identify the diffusion coefficient $\sigma^2(x)$, the jump intensity $\lambda(x)$ and the jump sizes $c(x, y)$ for model\textsuperscript{14} is not our objective in this paper and thus it is less of a concern here and left for the future research.

Remark 2.7. There are many statisticians and economists focused on state-domain nonparametric estimation for volatility functions of diffusion models with jumps. Bandi and Nguyen\textsuperscript{2} and Johannes\textsuperscript{9} considered the kernel weighted version of instantaneous volatility combined with the combination of power variation when the price process follows scalar diffusion model with jumps as\textsuperscript{6}. They established the following asymptotic normality for the estimator $\hat{M}_b^2(x)$ of unknown quantity $M^2(x)$ with the Assumptions\textsuperscript{1} \textsuperscript{2} \textsuperscript{3} \textsuperscript{4} and\textsuperscript{5} that is,

\[ \sqrt{h_n} \bar{L}_X(T, x) (\hat{M}_b^2(x) - M^2(x) - \Gamma_{M^2}) \Rightarrow N \left( 0, K_2 M^4(x) \right), \]

where $\hat{M}_b^2(x) = \frac{\sum_{i=1}^{n} K_{h_n}((X_{i\Delta_n} - x)) (X_{i\Delta_n} - X_{i\Delta_n})^2}{\Delta_n \sum_{i=1}^{n} K_{h_n}((X_{i\Delta_n} - x))}$ and
\[ \Gamma_{M^2}^{h_n} = h_n^2 \left[ \frac{1}{2} (M^2)''(x) + (M^2)'(x) \frac{s''(x)}{s(x)} \right]. \]

As mentioned in Bandi and Phillips [1], the estimator \( \hat{M}_{n}^2(x) \) proposed in Bandi and Nguyen [2] is the same as the double smoothed estimator \( M^2(x) \) conducted as (2.7) asymptotically if \( h_n = o(\varepsilon_n) \). Moreover, as discussed in Bandi and Phillips [1], if \( \frac{h_n}{\varepsilon_n} = \phi \), double smoothed estimator can reduce the asymptotic mean-squared error than the estimator \( \hat{M}_{n}^2(x) \) above for some chosen bandwidth \( h_n \), any level \( x \) and any processes.

**Remark 2.8.** It is very important to consider the choice of the bandwidth in nonparametric estimation. Here we will select the optimal bandwidth \( h_n \) based on the mean squared error (MSE) and the asymptotic theory in Theorem 2.5.

The optimal smoothing parameter \( h_n \) for double smoothed estimator of \( M^2(x) \) is given that

\[
 h_{n,\text{opt}} = \phi \left( \frac{1}{L_x(T, x)}, \frac{1}{L_x(T, x)} \right) \sqrt{\frac{1}{2} \theta_p M^4(x)} \left( \frac{1}{2} (M^2)''(x) + (M^2)'(x) \frac{s''(x)}{s(x)} \right) = O_p \left( \frac{1}{L_x(T, x)} \right),
\]

which differs from the continuous case in Bandi and Phillips [1] with \( h_{n,\text{opt}} = O_p \left( \frac{\Delta}{L_x(T, x)} \right) \). Furthermore, one can discuss the optimal bandwidth for double smoothed volatility estimator of jump-diffusion model based on Wang and Zhou [18], which will be under consideration in the future study. If the smoothing parameter \( h_n = O((\bar{L}_x(T, x))^{-1/5}) \), the normal confidence interval for \( M^2(x) \) using double smoothed estimators at the significance level 100(1 - \( \alpha \))% are constructed as follows,

\[
 I_{\mu, \alpha} = \left[ \hat{M}_{n}^2(x) - \varepsilon_n^2 \cdot \hat{\Gamma}^\phi_{M^2} - z_{1-\alpha/2} \cdot \frac{1}{\sqrt{\varepsilon_n \bar{L}_x(T, x)}} \cdot \sqrt{\frac{1}{2} \theta_p M^4(x)}, \right. \]
\[
 \left. \hat{M}_{n}^2(x) - \varepsilon_n^2 \cdot \hat{\Gamma}^\phi_{M^2} + z_{1-\alpha/2} \cdot \frac{1}{\sqrt{\varepsilon_n \bar{L}_x(T, x)}} \cdot \sqrt{\frac{1}{2} \theta_p M^4(x)} \right],
\]

where \( z_{1-\alpha/2} \) is the inverse CDF for the standard normal distribution evaluated at \( 1 - \alpha/2 \). To facilitate statistical inference for \( M^2(x) \) based on Theorem 2.5, we need to conduct consistent estimators for the unknown quantities \( M^4(x) \) in the normal approximation. Based on the infinitesimal moments condition (2.4), the double smoothed estimator for \( M^4(x) \) is conducted as

\[
 \hat{M}_{n}^4(x) = \sum_{i=1}^{n} K_{h_n} (X_i \Delta_n - x) \frac{1}{m_n(i \Delta_n) \Delta_n} \sum_{j=0}^{m_n(i \Delta_n) - 1} \left[ X_{t(i \Delta_n)_j + \Delta_n} - X_{t(i \Delta_n)_j} \right]^4 \frac{1}{\sum_{i=1}^{n} K_{h_n} (X_i \Delta_n - x)},
\]

(2.7)

The consistency and asymptotic normality for \( \hat{M}_{n}^4(x) \) can be done with the similar approach as \( \hat{M}_{n}^2(x) \), which goes beyond the scope here and will be under consideration in the future study.
3. Detailed Proof

In this section, we first present some technical lemmas and the proofs for the main theorems.

3.1. Some Technical Lemmas with Proofs

**Lemma 1.** (Bandi and Nguyen [2]) Let $X$ be a semimartingale with local time $L_X(t, a)_{a \in \mathbb{D}}$ and $f$ be a bounded Borel measurable function, we have

$$
\int_0^tg(X_s-)d[X]^c_s = \int_{-\infty}^\infty L_X(t, a)g(a)da, \quad \text{a.s.}
$$

(3.1)

where $[X]^c_s$ denotes the continuous part of the quadratic variation of $X$.

**Lemma 2.** (Bandi and Nguyen [2]) Let $X$ be a semimartingale satisfying $\sum_{0<s\leq t} |\Delta X_s| < \infty$ a.s. $\forall t$. Then, $\forall (t, a)$ we have

$$
L_X(t, a+) = L_X(t, a) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t 1_{(a, a+\varepsilon]}d[X]^c_s, \quad \text{a.s.}
$$

(3.2)

and

$$
L_X(t, a-) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t 1_{(a-\varepsilon, a]}d[X]^c_s, \quad \text{a.s.}
$$

(3.3)

Also,

$$
\bar{L}_X(t, a) = \frac{L_X(t, a) + L_X(t, a-)}{2} = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t 1_{(|X_s-a|\leq \varepsilon)}d[X]^c_s, \quad \text{a.s.}
$$

(3.4)

**Remark 3.1.** We may employ the following versions of local time in what follows.

$$
\bar{L}_X(t, a+) = \bar{L}_X(t, a) = \frac{1}{\sigma^2(a)} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t 1_{(a, a+\varepsilon]}\sigma^2(X_s)ds, \quad \text{a.s.}
$$

(3.5)

and

$$
\bar{L}_X(t, a-) = \frac{1}{\sigma^2(a)} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t 1_{(a-\varepsilon, a]}\sigma^2(X_s)ds, \quad \text{a.s.}
$$

(3.6)

Also,

$$
\bar{\bar{L}}_X(t, a) = \frac{\bar{L}_X(t, a) + \bar{L}_X(t, a-)}{2} = \frac{1}{\sigma^2(a)} \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t 1_{(|X_s-a|\leq \varepsilon)}\sigma^2(X_s)ds, \quad \text{a.s.}
$$

(3.7)

**Lemma 3.** (Bandi and Nguyen [2]) Under Assumptions [2] - [5] we have

$$
\bar{L}_X(T, x) = \frac{\Delta_n}{h_n} \sum_{i=1}^n K \left( \frac{X_i\Delta_n - x}{h_n} \right) \xrightarrow{a.s.} \bar{\bar{L}}_X(T, x).
$$

(3.8)
3.2. The proof of Theorem 2.5

Proof.

Strong Consistency:

\[
\hat{M}^2_n(x) = \sum_{i=1}^{n} K_{h_n}(X_i \Delta_n - x) \frac{1}{m_n(i \Delta_n) \Delta_n} \sum_{j=0}^{m_n(i \Delta_n)-1} \left( \left[ X_{t(i \Delta_n)} + \Delta_n - X_{t(i \Delta_n)} \right] - M^2(X_i \Delta_n) \right) \\
- \sum_{i=1}^{n} K_{h_n}(X_i \Delta_n - x) M^2(X_i \Delta_n) \\
:= A_{1n} - A_{2n}.
\]

Now, define

\[
\delta_{n,T} = \max_{i \leq n} \sup_{i \Delta_n \leq s \leq (i+1) \Delta_n} |X_s - X_i \Delta_n|.
\]

As is shown in Bandi and Nguyen [2] that

\[
\lim_{n \to \infty} \frac{\delta_{n,T}}{\Delta_n \log(1/\Delta_n)^{1/2}} = C_1 \quad \text{a.s.} \quad (3.9)
\]

for some constant $C_1$, which implies that $\delta_{n,T} = o_{a.s.}(1)$.

As for $A_{1n}$, based on equation (3.9) and the quotient limit theorem for Harris recurrent Markov processes, we can obtain that

\[
A_{1n} = \int_0^T K_{h_n}(X_{s-} - x) M^2(X_{s-}) ds + O_{a.s.} \left( \frac{L_X(T,x)}{h_n} \left( \Delta_n \log(\frac{1}{\Delta_n}) \right)^{1/2} \right)
\]

\[
= \int_0^T K_{h_n}(X_{s-} - x) ds + O_{a.s.} \left( \frac{L_X(T,x)}{h_n} \left( \Delta_n \log(\frac{1}{\Delta_n}) \right)^{1/2} \right)
\]

\[
= M^2(x) s(x) + o_{a.s.}(1) + o_{a.s.}(1)
\]

\[
= M^2(x) = \sigma^2(x) + \lambda(x) E_Y [c^2(x,y)].
\]

For the term $A_{2n}$, it is sufficient to prove that

\[
\frac{1}{m_n(i \Delta_n) \Delta_n} \sum_{j=0}^{m_n(i \Delta_n)-1} \left[ X_{t(i \Delta_n)} + \Delta_n - X_{t(i \Delta_n)} \right] - M^2(X_i \Delta_n) \xrightarrow{a.s.} 0. \quad (3.10)
\]

Using Itô formula to the jump-diffusion setting shown in Protter [12], we can write

\[
(X_{t(i \Delta_n)} + \Delta_n - X_{t(i \Delta_n)})^2
= 2 \int_{t(i \Delta_n)}^{t(i \Delta_n) + \Delta_n} (X_{s-} - X_{t(i \Delta_n)}) \mu(X_{s-}) ds + 2 \int_{t(i \Delta_n)}^{t(i \Delta_n) + \Delta_n} (X_{s-} - X_{t(i \Delta_n)}) \sigma(X_{s-}) dW_s
\]

\[
= 2 \int_{t(i \Delta_n)}^{t(i \Delta_n) + \Delta_n} \mu(X_{s-}) ds + 2 \int_{t(i \Delta_n)}^{t(i \Delta_n) + \Delta_n} \sigma(X_{s-}) dW_s.
\]
of which implies that (3.10) can be divided into five parts as

\[ A \rightarrow 0 \text{ a.s.} \]

By the mean-value theorem, the equation (3.9) and the locally boundedness here we only prove

\[ \int_t^{t+i\Delta_n} \left( X_{s-} - X_t \right) c(X_s, y) \bar{\nu}(ds, dy) + \int_t^{t+i\Delta_n} M^2(X_{s-}) ds \]

\[ + \int_t^{t+i\Delta_n} + \Delta_n \int Y c^2(X_{s-}, y) \bar{\nu}(ds, dy), \]

which implies that (3.10) can be divided into five parts as

\[
\begin{align*}
\frac{1}{m_n(i\Delta_n)\Delta_n} & \sum_{j=0}^{m_n(i\Delta_n)-1} \left[ X_t(i\Delta_n) + \Delta_n \right] - X_t(i\Delta_n) \right]^2 - M^2(X_i\Delta_n) \\
= & \frac{1}{m_n(i\Delta_n)\Delta_n} \sum_{j=0}^{m_n(i\Delta_n)-1} \int_t^{t+i\Delta_n} (M^2(X_{s-}) - M^2(X_t\Delta_n)) ds + \\
+ & \frac{2}{m_n(i\Delta_n)\Delta_n} \sum_{j=0}^{m_n(i\Delta_n)-1} \int_t^{t+i\Delta_n} (X_{s-} - X_t(i\Delta_n)) \mu(X_{s-}) ds + \\
+ & \frac{2}{m_n(i\Delta_n)\Delta_n} \sum_{j=0}^{m_n(i\Delta_n)-1} \int_t^{t+i\Delta_n} (X_{s-} - X_t(i\Delta_n)) \sigma(X_{s-}) dW_s \\
+ & \frac{2}{m_n(i\Delta_n)\Delta_n} \sum_{j=0}^{m_n(i\Delta_n)-1} \int_t^{t+i\Delta_n} (X_{s-} - X_t(i\Delta_n)) \int Y c(X_{s-}, y) \bar{\nu}(ds, dy) \\
+ & \frac{1}{m_n(i\Delta_n)\Delta_n} \sum_{j=0}^{m_n(i\Delta_n)-1} \int_t^{t+i\Delta_n} \int Y c^2(X_{s-}, y) \bar{\nu}(ds, dy) \\
:= & A_{21n} + A_{22n} + A_{23n} + A_{24n} + A_{25n}.
\end{align*}
\]

For instance, we can write \( A_{25n} \) as

\[
\begin{align*}
\frac{1}{m_n(i\Delta_n)\Delta_n} & \sum_{j=0}^{m_n(i\Delta_n)-1} \int_t^{t+i\Delta_n} \int Y c^2(X_{s-}, y) \bar{\nu}(ds, dy) \\
= & \frac{\sum_{j=1}^{n} \{ (X_{j\Delta_n} - X_{i\Delta_n}) \leq \epsilon_n \} \int_{j\Delta_n}^{(j+1)\Delta_n} \int Y c^2(X_{s-}, y) \bar{\nu}(ds, dy)}{\Delta_n \sum_{j=1}^{n} \{ (X_{j\Delta_n} - X_{i\Delta_n}) \leq \epsilon_n \}}.
\end{align*}
\]

\( A_{23n}, A_{24n}, A_{25n} \) are sample averages of martingale difference sequences, which converge to zero a.s.

Due to the locally boundedness of \( \mu(\cdot) \), the term \( A_{22n} \) can be done similarly as \( A_{21n} \), here we only prove \( A_{21n} \xrightarrow{a.s.} 0 \) for simplicity.

By the mean-value theorem, the equation (3.10) and the locally boundedness of \( (M^2)'(\cdot) \), we can obtain

\[
A_{21n} = \frac{1}{m_n(i\Delta_n)\Delta_n} \sum_{j=0}^{m_n(i\Delta_n)-1} \int_t^{t+i\Delta_n} (M^2(X_{s-}) - M^2(X_t\Delta_n)) ds
\]

\[ \int t^{t+i\Delta_n} \left( X_{s-} - X_t \right) c(X_s, y) \bar{\nu}(ds, dy) + \int t^{t+i\Delta_n} \int Y c^2(X_{s-}, y) \bar{\nu}(ds, dy), \]
\[
\begin{align*}
= & \frac{1}{m_n(i\Delta_n)\Delta_n} \sum_{j=0}^{m_n(i\Delta_n) - 1} \int_{t(i\Delta_n)_{j}}^{t((i\Delta_n)_{j} + \Delta_n)} (M^2(X_{s-}) - M^2(X_{j\Delta_n})) \, ds \\
+ & \frac{1}{m_n(i\Delta_n)\Delta_n} \sum_{j=0}^{m_n(i\Delta_n) - 1} \int_{t(i\Delta_n)_{j}}^{t((i\Delta_n)_{j} + \Delta_n)} (M^2(X_{j\Delta_n}) - M^2(X_{i\Delta_n})) \, ds \\
= & \sum_{j=1}^{n} 1\{|X_{j\Delta_n} - X_{i\Delta_n}| \leq \varepsilon_n\} \int_{\Delta_n}^{(j+1)\Delta_n} (M^2(X_{s-}) - M^2(X_{j\Delta_n})) \, ds \\
+ & \sum_{j=1}^{n} 1\{|X_{j\Delta_n} - X_{i\Delta_n}| \leq \varepsilon_n\} \int_{\Delta_n}^{(j+1)\Delta_n} (M^2(X_{j\Delta_n}) - M^2(X_{i\Delta_n})) \, ds \\
= & \sum_{j=1}^{n} 1\{|X_{j\Delta_n} - X_{i\Delta_n}| \leq \varepsilon_n\} \int_{\Delta_n}^{(j+1)\Delta_n} (M^2)(\xi_{n,1}) (X_{s-} - X_{j\Delta_n}) \, ds \\
+ & \sum_{j=1}^{n} 1\{|X_{j\Delta_n} - X_{i\Delta_n}| \leq \varepsilon_n\} \int_{\Delta_n}^{(j+1)\Delta_n} (M^2)(\xi_{n,2}) (X_{j\Delta_n} - X_{i\Delta_n}) \, ds \\
\approx & \text{O}_n, (\Delta_n \log(1/\Delta_n))^{1/2} + C\varepsilon_n \overset{a.s.}{\to} 0, \\
\end{align*}
\]

where \(\xi_{n,1}\) lies between \(X_{s-}\) and \(X_{j\Delta_n}\), \(\xi_{n,2}\) lies between \(X_{i\Delta_n}\) and \(X_{j\Delta_n}\).

We have proved that
\[
\hat{M}_n^2(x) \overset{a.s.}{\to} M^2(x) = \sigma^2(x) + \lambda(x)E_Y[\varepsilon^2(x, y)]. \tag{3.11}
\]

**Asymptotic Normality:**

\[
\begin{align*}
\hat{M}_n^2(x) - M^2(x) \\
= & \Delta_n \sum_{j=1}^{n} K_{h_n}(X_{i\Delta_n} - x) \frac{1}{m_n(i\Delta_n)\Delta_n} \sum_{j=0}^{m_n(i\Delta_n) - 1} \left| X_{t(i\Delta_n)_{j}} + \Delta_n - X_{t(i\Delta_n)_{j}} \right|^2 \\
= & \Delta_n \sum_{j=1}^{n} K_{h_n}(X_{i\Delta_n} - x) \frac{1}{m_n(i\Delta_n)\Delta_n} \sum_{j=0}^{m_n(i\Delta_n) - 1} \left| X_{t(i\Delta_n)_{j}} + \Delta_n - X_{t(i\Delta_n)_{j}} \right|^2 \\
= & \frac{\Delta_n}{\Delta_n} \sum_{j=1}^{n} K_{h_n}(X_{i\Delta_n} - x) \frac{1}{m_n(i\Delta_n)\Delta_n} \sum_{j=0}^{m_n(i\Delta_n) - 1} \left| X_{t(i\Delta_n)_{j}} + \Delta_n - X_{t(i\Delta_n)_{j}} \right|^2 \\
- & \frac{\Delta_n}{\Delta_n} \sum_{j=1}^{n} K_{h_n}(X_{i\Delta_n} - x)M^2(X_{i\Delta_n}) \\
+ & \frac{\Delta_n}{\Delta_n} \sum_{j=1}^{n} K_{h_n}(X_{i\Delta_n} - x)M^2(X_{i\Delta_n}) - \frac{\Delta_n}{\Delta_n} \sum_{i=1}^{n} K_{h_n}(X_{i\Delta_n} - x)M^2(x) \\
= & V + B.
\end{align*}
\]

As for the bias term \(B\), based on equation [3.15] and the quotient limit theorem for Harris recurrent Markov processes, we can obtain that

\[
B = \frac{\Delta_n}{\Delta_n} \sum_{i=1}^{n} K \left( \frac{X_{i\Delta_n} - x}{h_n} \right) \left[ M^2(X_{i\Delta_n}) - M^2(x) \right]
\]
Due to lemma 3, we can obtain that
\[ V = 2 \sum_{h} \Delta h \]

Using the occupation time formula in lemma 1, we can conclude that
\[ \int_{-\infty}^{\infty} K \left( \frac{a - x}{h_n} \right) (M^2(a) - M^2(x)) sa + o_{s.s.}(1) \]  
\[ = \frac{1}{h_n} \int_{-\infty}^{\infty} K \left( \frac{a - x}{h_n} \right) sa + o_{s.s.}(1) \]  
\[ = h_n^2 K_2 \left[ \frac{1}{2} (M^2)'(x) + (M^2)'(x) \frac{\xi(x)}{\xi(x)} \right] + o_{s.s.}(h_n^2), \]

with \( K_2 = \int_{-\infty}^{\infty} u^2 K(u) du \).

For the term \( V \), write \( \tilde{M}^2(X_i \Delta_n) := \frac{1}{m(X_i \Delta_n) \Delta_n} \sum_{j=0}^{m \Delta_n - 1} [X_{i(i \Delta_n) + \Delta_n} - X_{i(i \Delta_n)}]^2 \), we have
\[ V = \frac{\Delta_n}{h_n} \sum_{i=1}^{n} K \left( \frac{X_i \Delta_n - x}{h_n} \right) [\tilde{M}^2(X_i \Delta_n) - M^2(X_i \Delta_n)] \]
\[ = \frac{\Delta_n}{h_n} \sum_{i=1}^{n} K \left( \frac{X_i \Delta_n - x}{h_n} \right) \frac{M^2(X_i \Delta_n) - \xi \Delta_n}{h_n} \sum_{j=1}^{n} \frac{1}{\|X_j \Delta_n - X_i \Delta_n\| \leq \epsilon_n} \]
\[ := V_{1}^{\text{Num}} + V_{2}^{\text{Num}}. \]

Due to lemma 3 we can obtain that
\[ V_{\text{Den}} \xrightarrow{\text{a.s.}} \tilde{L}_{\text{X}}(T, x), \]
so we should deal with the term \( V_{\text{Num}} \) in what follows.
\[ V_{\text{Num}} = \Delta_n \sum_{i=1}^{n} K \left( \frac{X_i \Delta_n - x}{h_n} \right) [\tilde{M}^2(X_i \Delta_n) - M^2(X_i \Delta_n)] \]
\[ = \Delta_n \sum_{i=1}^{n} K \left( \frac{X_i \Delta_n - x}{h_n} \right) \frac{M^2(X_i \Delta_n) - \xi \Delta_n}{h_n} \sum_{j=1}^{n} \frac{1}{\|X_j \Delta_n - X_i \Delta_n\| \leq \epsilon_n} \]
\[ := V_{1}^{\text{Num}} + V_{2}^{\text{Num}}. \]

Using the occupation time formula in lemma 1 we can conclude that
\[ V_{2}^{\text{Num}} = O_{s.s.} \left( \frac{\Delta_n}{\epsilon_n} \right). \]

Using Itô formula to the jump-diffusion setting shown in Protrter [12], we can write
\[ (X_{(j+1) \Delta_n} - X_{j \Delta_n})^2 \]
\[ = 2 \int_{j \Delta_n}^{(j+1) \Delta_n} (X_{s} - X_{j \Delta_n}) \mu(X_{s}) ds + 2 \int_{j \Delta_n}^{(j+1) \Delta_n} (X_{s} - X_{j \Delta_n}) \sigma(X_{s}) dW_{s} \]
\[ + 2 \int_{j+1}^{j+1} (X_s - X_{j\Delta_n}) \int_Y c(X_s, y) \bar{v}(ds, dy) + \int_{j+1}^{j+1} M^2(X_s - ) ds \]

\[ + \int_{j+1}^{j+1} \int_Y c^2(X_s, y) \bar{v}(ds, dy), \]

which implies that the term \( V_1^{Num} \) can be divided into five parts as

\[ V_1^{Num} = V_1^{Num} + V_1^{Num} + V_1^{Num} + V_1^{Num} + V_1^{Num}, \quad \text{(3.14)} \]

where

\[ V_1^{Num} = \frac{\Delta_n}{\overline{h_n}} \sum_{i=1}^{n} K \left( \frac{X_i \Delta_n - x}{\overline{h_n}} \right) \sum_{j=1}^{n-1} 1\{ |X_{j \Delta_n} - X_i \Delta_n| \leq \epsilon_n \} \frac{f^{j+1} \Delta_n (M^2(X_s) - M^2(X_{j \Delta_n})) ds}{\sum_{j=1}^{n} 1\{ |X_{j \Delta_n} - X_i \Delta_n| \leq \epsilon_n \}}, \]

\[ V_1^{Num} = \frac{\Delta_n}{\overline{h_n}} \sum_{i=1}^{n} K \left( \frac{X_i \Delta_n - x}{\overline{h_n}} \right) \sum_{j=1}^{n-1} 1\{ |X_{j \Delta_n} - X_i \Delta_n| \leq \epsilon_n \} \frac{f^{j+1} \Delta_n (X_s - X_{j \Delta_n}) \mu(X_s) ds}{\sum_{j=1}^{n} 1\{ |X_{j \Delta_n} - X_i \Delta_n| \leq \epsilon_n \}}, \]

\[ V_1^{Num} = \frac{\Delta_n}{\overline{h_n}} \sum_{i=1}^{n} K \left( \frac{X_i \Delta_n - x}{\overline{h_n}} \right) \sum_{j=1}^{n-1} 1\{ |X_{j \Delta_n} - X_i \Delta_n| \leq \epsilon_n \} \frac{f^{j+1} \Delta_n (X_s - X_{j \Delta_n}) \sigma(X_s) dW_s}{\sum_{j=1}^{n} 1\{ |X_{j \Delta_n} - X_i \Delta_n| \leq \epsilon_n \}}, \]

\[ V_1^{Num} = \frac{\Delta_n}{\overline{h_n}} \sum_{i=1}^{n} K \left( \frac{X_i \Delta_n - x}{\overline{h_n}} \right) \frac{1}{\overline{h_n}} \sum_{j=1}^{n-1} 1\{ |X_{j \Delta_n} - X_i \Delta_n| \leq \epsilon_n \} \frac{f^{j+1} \Delta_n (X_s - X_{j \Delta_n}) \int_Y c(X_s, y) \bar{v}(ds, dy)}{\sum_{j=1}^{n} 1\{ |X_{j \Delta_n} - X_i \Delta_n| \leq \epsilon_n \}}, \]

\[ V_1^{Num} = \frac{\Delta_n}{\overline{h_n}} \sum_{i=1}^{n} K \left( \frac{X_i \Delta_n - x}{\overline{h_n}} \right) \frac{1}{\overline{h_n}} \sum_{j=1}^{n-1} 1\{ |X_{j \Delta_n} - X_i \Delta_n| \leq \epsilon_n \} \frac{f^{j+1} \Delta_n (X_s - X_{j \Delta_n}) \int_Y c^2(X_s, y) \bar{v}(ds, dy)}{\sum_{j=1}^{n} 1\{ |X_{j \Delta_n} - X_i \Delta_n| \leq \epsilon_n \}}. \]

With the equation (3.12), we can easily get

\[ V_1^{Num} = o_{a.s.} \left( \Delta_n \log \left( \frac{1}{\Delta_n} \right) \right), \]

\[ V_1^{Num} = \left( \Delta_n \log \left( \frac{1}{\Delta_n} \right) \right)^{1/2} O_P \left( V_1^{Num} \right) = o_P \left( V_1^{Num} \right), \]

\[ V_1^{Num} = \left( \Delta_n \log \left( \frac{1}{\Delta_n} \right) \right)^{1/2} O_P \left( V_1^{Num} \right) = o_P \left( V_1^{Num} \right). \]

For the bias effect term \( V_1^{Num} \), we have

\[ \frac{\Delta_n}{\overline{h_n}} \sum_{i=1}^{n} K \left( \frac{X_i \Delta_n - x}{\overline{h_n}} \right) \]

\[ = \frac{\Delta_n}{\overline{h_n}} \sum_{i=1}^{n} K \left( \frac{X_i \Delta_n - x}{\overline{h_n}} \right) \frac{\sum_{j=1}^{n-1} 1\{ |X_{j \Delta_n} - X_i \Delta_n| \leq \epsilon_n \} \frac{f^{j+1} \Delta_n (M^2(X_s) - M^2(X_{j \Delta_n})) ds}{\sum_{j=1}^{n} 1\{ |X_{j \Delta_n} - X_i \Delta_n| \leq \epsilon_n \}}}{\sum_{i=1}^{n} K \left( \frac{X_i \Delta_n - x}{\overline{h_n}} \right)} \]

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Moreover, by use of Taylor expansion, we can formulate

\[
(\frac{M_b a \epsilon_n}{h_n} - 2) = \frac{1}{h_n} \sum_{j=1}^{\infty} \int_0^1 (\frac{M^2 (X_{j\Delta_n} - M^n) (M^2 (X_{j \Delta_n} - M^n)}{\Delta_n} ds \frac{f^{(j+1)}(a \epsilon_n)}{\Delta_n} M^n (X_{j \Delta_n} - M^n) ) ds
\]

\[
= D_{1n} + D_{2n}.
\]

By the mean-value theorem, the equation (3.15) and the locally boundedness of \((M^2)'(\cdot)\), it can be shown that

\[
D_{2n} = O_{a.s.} \left( \left( \frac{1}{\Delta_n} \right)^{1/2} \right)
\]

Moreover,

\[
D_{1n} = \frac{1}{h_n} \int_{-\infty}^{+\infty} K \left( \frac{a - x}{h_n} \right) s(x) da + o_{a.s.}(1)
\]

By use of Taylor expansion, we can formulate

\[
M^2(x + a \epsilon_n) - M^2(x) = (M^2)'(x)a \epsilon_n + \frac{1}{2}(M^2)''(x)(a \epsilon_n)^2 + o(\epsilon_n^2),
\]

\[
M^2(x + ch_n) - M^2(x) = (M^2)'(x)ch_n + \frac{1}{2}(M^2)''(x)(ch_n)^2 + o(h_n^2),
\]

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\[ s(x + a\varepsilon_n) = s(x) + s'(x)a\varepsilon_n + \frac{1}{2}(s)''(x)(a\varepsilon_n)^2 + o(\varepsilon_n^2), \]
\[ s(x + c\varepsilon_n) = s(x) + s'(x)c\varepsilon_n + \frac{1}{2}(s)''(x)(c\varepsilon_n)^2 + o(h_n^2). \]

If \( h_n = o(\varepsilon_n) \), we have
\[
D_{11n} = \frac{\varepsilon_n^2}{3} \left[ \frac{1}{2}(M^2)''(x) + (M^2)'(x) \frac{s'(x)}{s(x)} \right] + o(\varepsilon_n^2). \quad (3.16)
\]

If \( h_n = O(\varepsilon_n) \) with \( \frac{h_n}{\varepsilon_n} = \phi \), then we can get
\[
D_{1n} = \int_{-\infty}^{+\infty} K(c) \left[ \frac{\int_{|x-c\phi| \leq 1} (M^2)'(x)\varepsilon_n + \frac{1}{2}(M^2)''(x)(a\varepsilon_n)^2) s(x+a\varepsilon_n) \varepsilon_n^2}{\int_{|x-c\phi| \leq 1} s(x+a\varepsilon_n) \varepsilon_n^2} \right] s(x + c\varepsilon_n) dc 
\]
\[
= \int_{-\infty}^{+\infty} K(c) s(x + c\varepsilon_n) dc 
\]
\[
= \int_{-\infty}^{+\infty} K(c) \left[ \frac{\int_{|x-c\phi| \leq 1} (M^2)'(x)\varepsilon_n + \frac{1}{2}(M^2)''(x)(c\varepsilon_n)^2) s(x+c\varepsilon_n) \varepsilon_n^2}{\int_{|x-c\phi| \leq 1} s(x+c\varepsilon_n) \varepsilon_n^2} \right] s(x + c\varepsilon_n) dc 
\]
\[
= D_{11n} - D_{12n}.
\]

Set \( g = a - c\phi \), it follows for \( D_{11n} \)
\[
D_{11n} = \int_{-\infty}^{+\infty} K(c) \left[ \frac{\int_{|x-c\phi| \leq 1} (M^2)'(x)(g+c\varepsilon_n)\varepsilon_n + \frac{1}{2}(M^2)''(x)((g+c\varepsilon_n)^2) s(x+(g+c\varepsilon_n)\varepsilon_n) \varepsilon_n^2}{\int_{|x-c\phi| \leq 1} s(x+(g+c\varepsilon_n)\varepsilon_n) \varepsilon_n^2} \right] s(x + c\varepsilon_n) dc 
\]
\[
= \int_{-\infty}^{+\infty} K(c) \left[ \frac{1}{2}(M^2)''(x) + (M^2)'(x) \frac{s'(x)}{s(x)} \right] + o(h_n^2) 
\]
\[
= \int_{-\infty}^{+\infty} K(c) \left[ \frac{1}{2}(M^2)''(x) + (M^2)'(x) \frac{s'(x)}{s(x)} \right] + o(h_n^2) 
\]
\[
= \frac{1}{3} \varepsilon_n^2 \left[ \frac{1}{2}(M^2)''(x) + (M^2)'(x) \frac{s'(x)}{s(x)} \right] + h_n^2 K_2 \left[ \frac{1}{2}(M^2)''(x) + (M^2)'(x) \frac{s'(x)}{s(x)} \right] + o(\varepsilon_n^2 + h_n^2) 
\]
\[
= \varepsilon_n^2 \left( \frac{1}{3} K_2 \phi^2 + \frac{1}{3} \right) \left[ \frac{1}{2}(M^2)''(x) + (M^2)'(x) \frac{s'(x)}{s(x)} \right] + o(\varepsilon_n^2 + h_n^2). 
\]
As for $D_{12n}$, it can be concluded that

$$D_{12n} = h_n^2 K_2 \left[ \frac{1}{2} (M^2)'(x) + \frac{s'(x)}{s(x)} \right] + o_n.s.(h_n^2), \quad (3.17)$$

based on the following similar definite integration

$$\frac{1}{2} \int_{-\infty}^{\infty} 1_{\{|a-c\phi| \leq 1\}} ac \frac{da}{\phi} = \frac{c}{\phi} \int_{1+c\phi}^{1+\phi} ada = c^2.$$

To conclude, when $h_n = o(\varepsilon_n)$, the bias term for $\hat{M}_n^2(x) - M^2(x)$ is

$$V_{11}^{\text{Num}} = D_{11n} - D_{12n} + D_{2n} + B$$

$$= \varepsilon_n^2 \left[ \frac{1}{2} (M^2)'(x) + \frac{s'(x)}{s(x)} \right] + o(\varepsilon_n^2). \quad (3.18)$$

If $h_n = O(\varepsilon_n)$ with $h_n = \varphi$, the total bias term is

$$V_{11}^{\text{Num}} = D_{11n} - D_{12n} + D_{2n} + B$$

$$= \varepsilon_n^2 \left( K_2 \varphi^2 + \frac{1}{3} \right) \left[ \frac{1}{2} (M^2)'(x) + \frac{s'(x)}{s(x)} \right] + o(\varepsilon_n^2). \quad (3.19)$$

For the variance effect term $V_{15}^{\text{Num}}$, we have

$$\sqrt{\varepsilon_n} V_{15}^{\text{Num}} = \sum_{i=1}^{n} K \left( \frac{X_i \Delta_n - x}{h_n} \right) \frac{1}{\varepsilon_n} \sum_{j=1}^{n-1} 1_{\{|X_j \Delta_n - X_i \Delta_n| \leq \varepsilon_n\}} \int_{j+1}^{(j+1)\Delta_n} \int_{Y_{c_2}} c^2(X_{s-}, y) \bar{\nu}(ds, dy)$$

Denote $J_j := \int_{j \Delta_n}^{(j+1)\Delta_n} \int_{Y_{c_2}} c^2(X_{s-}, y) \bar{\nu}(ds, dy)$, which is a martingale difference series. By Gaussian approximation of locally square-integrable martingales (more technical details seen in Lin and Wang [11] and Philipp and Stout [13]), on an extension of the filtered probability space we have

$$\max_{1 \leq j \leq n} \left| J_j - B \int_{j \Delta_n}^{(j+1)\Delta_n} \lambda(X_{s-}) \int_{Y_{c_2}} c^4(X_{s-}, y) \bar{\Pi}(dy) ds \right| = o_n.s.(1). \quad (3.20)$$

Based on the equation (3.20), $\sqrt{\varepsilon_n} V_{15}^{\text{Num}}$ has the same asymptotic distribution with $E_n$.  

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\[
\Delta_n \sum_{i=1}^{n} K \left( \frac{X_i \Delta_n - x}{h_n} \right) \frac{1}{2h_n} \sum_{j=1}^{n-1} 1 \{ |X_j \Delta_n - X_i \Delta_n| \leq \varepsilon_n \} B_{\int_{\Delta_n}^{(j+1)\Delta_n} \lambda(x_{s-}) ds} \int_Y \varepsilon^4(s_{x-}, y) \Pi(dy) ds
\]

which can be embedded in a time-changed Brownian motion with the quadratic variation process under assumption in what follows,

\[
[\varepsilon_n] = \frac{\Delta_n}{h_n} \sum_{i=1}^{n} \sum_{k=1}^{n} K \left( \frac{X_i \Delta_n - x}{h_n} \right) K \left( \frac{X_k \Delta_n - x}{h_n} \right) \times \frac{1}{2h_n} \sum_{j=1}^{n-1} 1 \{ |X_j \Delta_n - X_i \Delta_n| \leq \varepsilon_n \} \int_{\Delta_n}^{(j+1)\Delta_n} \lambda(x_{s-}) ds \int_Y \varepsilon^4(s_{x-}, y) \Pi(dy) ds
\]

To conclude, when \( h_n = o(\varepsilon_n) \), we have

\[
[E_n] \overset{\alpha.s.}{\to} \frac{1}{2} M^4(x) \bar{L}_X(T, x). \tag{3.21}
\]

So with lemma 33 we have

\[
\sqrt{\varepsilon_n} \bar{L}_X(T, x) \frac{V \sum_{i=1}^{N_{\text{num}}} K \left( \frac{X_i \Delta_n - x}{h_n} \right)}{\Delta_n \sum_{i=1}^{n} K \left( \frac{X_i \Delta_n - x}{h_n} \right)} \Rightarrow N \left( 0, \frac{1}{2} M^4(x) \right), \tag{3.22}
\]
which implies with the equation (4.18)
\[ \sqrt{\varepsilon_n} L_X(T, x)(\hat{M}_n^2(x) - M^2(x) - \Gamma_{M^2}) \Rightarrow N \left( 0, \frac{1}{2} M^4(x) \right), \] (3.23)
where \( \Gamma_{M^2} = \frac{\varepsilon_n}{3} \left[ \frac{1}{4}(M')''(x) + (M')'(x) \frac{\phi(x)}{\phi'(x)} \right] \).

If \( h_n = O(\varepsilon_n) \) with \( \frac{h_n}{\varepsilon_n} = \phi \), we can obtain
\[ [E_n] \xrightarrow{a.s.} \frac{1}{2} \theta_{\phi} M^4(x) \hat{L}_X(T, x), \] (3.24)
where
\[ \theta_{\phi} = \int_{-\infty}^{+\infty} da \int_{-\infty}^{+\infty} de K(a) K(e) \frac{1}{2} \int_{-\infty}^{+\infty} dz 1_{\{|z-a\phi|\leq 1\}} 1_{\{|z-e\phi|\leq 1\}} \left( \frac{1}{2} \int_{-\infty}^{+\infty} 1_{\{|z-a\phi|\leq 1\}} dz \right) \]
\[ = \frac{1}{2} \int_{-\infty}^{+\infty} dz \int_{(z-1)/\phi}^{(z+1)/\phi} da \int_{(z-1)/\phi}^{(z+1)/\phi} de K(a) K(e) \]
So with lemma 3 we have
\[ \sqrt{\varepsilon_n} L_X(T, x) \rightarrow_{a.s.} \frac{V_{15}^{Num}}{\Delta_{h_n}} \sum_{i=1}^{n} K \left( \frac{X_{i1} - x}{h_n} \right) \Rightarrow N \left( 0, \frac{1}{2} \theta_{\phi} M^4(x) \right), \] (3.25)
which implies with the equation (4.19)
\[ \sqrt{\varepsilon_n} L_X(T, x)(\hat{M}_n^2(x) - M^2(x) - \Gamma_{M^2}^\phi) \Rightarrow N \left( 0, \frac{1}{2} \theta_{\phi} M^4(x) \right), \] (3.26)
where \( \Gamma_{M^2}^\phi = \varepsilon_n \left( K_2 \phi^2 + \frac{\phi}{3} \right) \left[ \frac{1}{4}(M')''(x) + (M')'(x) \frac{\phi(x)}{\phi'(x)} \right] \).

We have proved the main results in Theorem 2.5 based on (3.23) and (3.24).

\[ \square \]

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