Electroweak renormalization based on gauge-invariant vacuum expectation values of non-linear Higgs representations: 2. extended Higgs sectors

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ABSTRACT: A recently proposed scheme for a gauge-invariant treatment of tadpole corrections in spontaneously broken gauge theories—called Gauge-Invariant Vacuum expectation value Scheme (GIVS)—is applied to a singlet Higgs extension of the Standard Model and to the Two-Higgs Doublet Model. In contrast to previously used tadpole schemes, the GIVS unifies the gauge-invariance property with perturbative stability. For the Standard Model this was demonstrated for the conversion between on-shell and \( \overline{\text{MS}} \) renormalized masses, where the GIVS leads to very moderate, gauge-independent electroweak corrections. In models with extended scalar sectors, issues with tadpole renormalization exist if Higgs mixing angles are renormalized with \( \overline{\text{MS}} \) conditions, which is the major subject of this article. In detail, we first formulate non-linear representations of the extended scalar sectors, which is an interesting subject in its own right. Then we formulate the GIVS which employs these non-linear representations in the calculation of the tadpole renormalization constants, while actual higher-order calculations in the GIVS proceed in linear representations as usual. Finally, for the considered models we discuss the next-to-leading-order (electroweak and QCD) corrections to the decay processes \( h/H \to WW/ZZ \to 4 \) fermions of the CP-even neutral Higgs bosons \( h \) and \( H \) using \( \overline{\text{MS}} \)-renormalized Higgs mixing angles with the GIVS and previously used tadpole treatments.
1 Introduction

Current particle physics can be, very globally, characterized by two facts: On the one hand, the Standard Model (SM) of particle physics describes almost all collider data very well; on the other hand, the SM is not able to explain some well established phenomena like Dark Matter and the matter–antimatter asymmetry in the Universe, and it is not yet clear which is the right generalization of the SM to accommodate neutrino masses. Rather independent of theoretical expectations or speculations, the Higgs sector of the SM either is modified or extended in more comprehensive models or at least provides a portal to some new sector. For this reason, experimental analyses of the Higgs boson and its properties as well as the search for further Higgs bosons remain a cornerstone in the LHC physics programme and beyond.
On the theory side, this programme implies an immense effort for delivering predictions of adequate precision both in the Standard Model (SM) of particle physics and its most prominent extensions (see, e.g., Refs. [1–6]). In this context, the calculation of QCD and electroweak (EW) radiative corrections plays a central role, which in turn involves the issue of renormalization and the choice of appropriate input parameters (see, e.g., the review [7] for details and original references). The choice of appropriate renormalization and input-parameter schemes is particularly important and delicate in SM extensions. New-physics parameters are often defined via $\overline{\text{MS}}$ renormalization conditions. On the one hand, this choice is made for simplicity reason; on the other hand, it is also a natural choice if the considered model can be treated as an effective low-energy model the parameters of which are determined via the matching to the complete model at a high-energy scale and the evolution of the parameters down to the low-energy scale via renormalization group equations, typically given in the $\overline{\text{MS}}$ scheme. However, choosing $\overline{\text{MS}}$ renormalization conditions leads to issues in EW corrections to mass parameters and mixing angles. These issues are related to the precise higher-order definition of vacuum expectation values (vevs) of Higgs fields, a subject that in turn is connected to some scheme choice for treating the tadpole diagrams, which have only one external Higgs-boson leg. In the SM, this problem shows up as large EW corrections or gauge dependences in the conversion between $\overline{\text{MS}}$-renormalized and on-shell (OS)-renormalized masses. Recall that predictions entirely based on OS renormalization do not depend on the tadpole treatment. In SM extensions, similar issues appear in the renormalization of Higgs mixing angles and severely limit the use of $\overline{\text{MS}}$ conditions, as discussed for singlet Higgs extensions and Two-Higgs Doublet Models (THDMs) in the literature in detail [8–12].

Perturbative calculations become significantly more transparent if tadpole contributions are avoided by an appropriate introduction of tadpole counterterms $\delta L_{\delta t} = \delta t \, h$ in the Lagrangian for each Higgs field $\phi(x) = v + h(x)$ that might acquire a non-vanishing vev $v$. Adjusting $\delta t$ in such a way that the vev of $h$ vanishes, $\langle 0 | h(x) | 0 \rangle = 0$, redistributes the explicit tadpole terms from loop diagrams to counterterms of other couplings as implicit tadpole contributions. In the last decades mostly two different tadpoles schemes for introducing $\delta L_{\delta t}$ have been used. One possibility, called Parameter Renormalized Tadpole Scheme (PRTS) in the following, is to include $\delta t$ in the parameter renormalization transformation which expresses bare parameters in terms of renormalized parameters, as, e.g., done in Refs. [13, 14]. The renormalized vev parameter $v$ is chosen as the value of $\phi$ in the minimum of the renormalized (corrected) effective Higgs potential, which absorbs potentially large corrections to the vev into renormalized parameters. These corrections are induced by tadpole loops which are gauge dependent in the usual loop calculation machinery. Unfortunately this in general results in a gauge-dependent parametrization of predicted observables if $\overline{\text{MS}}$ masses or $\overline{\text{MS}}$ Higgs mixing angles are used (see discussions in Refs. [8, 9, 12]); for OS-renormalized mass or mixing parameters such gauge dependences do not arise.

Alternatively to the PRTS, a tadpole counterterm can be introduced by shifting the field $h(x)$ according to $h(x) \to h(x) + \Delta v$ in the bare Lagrangian with a properly adjusted constant $\Delta v$ [15]. In the following, we refer to this scheme as Fleischer–Jegerlehner Tadpole
Scheme (FJTS)\(^1\). Note that the FJTS produces the same result as if including all explicit tadpole diagrams wherever they appear in a calculation, because the field shift implements an unobservable reparametrization of the path integral over \( h(x) \) employed in the quantization. The benefit of this procedure is that no gauge dependences between bare parameters are introduced, resulting in a gauge-independent parametrization of observables in terms of renormalized parameters. The downside of this approach is the appearance of potentially large EW corrections in predictions with \( \overline{\text{MS}} \) mass parameters, resulting from the fact that the original Higgs field \( \phi(x) \) is not expanded around the “true” (corrected) minimum of the effective Higgs potential, as e.g. discussed for \( \overline{\text{MS}} \) masses in the SM in Refs. [17–19]. More aspects of the technical and conceptual differences in the PRTS and FJTS treatments are for instance discussed in Refs. [7–9, 20, 21].

In Ref. [21], we have proposed a hybrid scheme called Gauge-Invariant Vacuum expectation value Scheme (GIVS) unifying the good features of the PRTS and FJTS.\(^2\) In the GIVS Higgs fields are expanded about the “true” (corrected) minimum of the effective Higgs potential, so that no additional corrections arise from correcting the expansion point. Gauge dependences in tadpole contributions are completely avoided by using non-linear representations of Higgs fields [22, 23]. In these non-linear representations, tadpole renormalization constants are gauge independent, because CP-even neutral components of Higgs multiplets and, thus, Higgs vacuum expectation values are gauge invariant. Moreover, Higgs potentials are completely free of Goldstone-boson fields. If a higher-order calculation is performed in this non-linear Higgs representation, in the GIVS no further effort in the tadpole renormalization is needed, i.e. there the GIVS proceeds as the PRTS of a linear representation.

Since, however, actual higher-order calculations for decay or scattering processes are more conveniently performed using linear Higgs representations, in the latter a second type of tadpole renormalization constant is introduced by field shifts as in the FJTS to guarantee a full compensation of explicit tadpole diagrams, rendering the GIVS a hybrid scheme. The two types of tadpole counterterms can be directly read from PRTS and FJTS Feynman rules, as e.g. given in App. A of Ref. [7] for the SM. As a first application of the GIVS, in Ref. [21] we have evaluated the differences between OS- and \( \overline{\text{MS}} \)-renormalized masses of SM particles, demonstrating the perturbative stability of the mass conversion in the GIVS, similar to the PRTS, but without gauge dependences.

In this paper, we formulate the GIVS for a Higgs Singlet Extension of the SM (SESM) inspired by Refs. [24–26] and the THDM [27, 28] and investigate the perturbative stability of radiative corrections to the phenomenologically important decays of CP-even Higgs bosons into four-fermion final states, \( h/H \to WW/ZZ \to 4f \), when the Higgs mixing angles are renormalized with \( \overline{\text{MS}} \) conditions. Our study completes the discussions of next-to-

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\(^1\)This scheme is equivalent to the \( \beta_t \) scheme of Ref. [16].

\(^2\)After completing this work, we became aware of Ref. [20] where the impact of gauge-fixing on tadpole renormalization is discussed in detail. In their remark 3 of Sect. D the authors of Ref. [20] suggest a tadpole scheme that seems to be equivalent to the GIVS in the SM at the one-loop level. As in the GIVS, the gauge-dependent part of the tadpole function \( T_h \) is treated as in the FJTS, and the remaining contribution according to the PRTS prescription. The gauge-dependent part of the tadpole function is, however, determined by the difference of \( T_h \) in \( R_\xi \) gauge to \( T_h \) in the Landau gauge (\( \xi = 0 \)), in which \( R_\xi \) gauge-fixing does not break global gauge symmetry.
leading-order (NLO) predictions for these decays in the SESM and THDM which started in Refs. [11] and [10, 29], respectively, with $\overline{\text{MS}}$ renormalization schemes for the Higgs mixing angles and were extended to OS and symmetry-inspired renormalization schemes in Ref. [12]. To this end, we have extended the Monte Carlo program Prophecy4f 3.0 [30–32] by adding $\overline{\text{MS}}$ schemes based on the GIVS tadpole treatment for the Higgs mixing angles in the SESM and THDM. As a byproduct of this work, we had to construct an appropriate non-linear parametrization of the Higgs sector of the THDM, which is more involved than in the SM and the SESM and a subject that might be of interest in its own right.

The article is organized as follows: In Section 2 we describe the non-linear Higgs representations of the SESM and THDM in detail. The formulation of the GIVS for those models, the issue of tadpoles in the $\overline{\text{MS}}$ renormalization of the Higgs mixing angle, and the phenomenological applications to the four-body Higgs decays are presented in Section 3. Our conclusions are given in Section 4, and the appendix provides some further information on the calculational setup for the evaluation of the Higgs-boson decays in the SESM and THDM.

2 Linear and non-linear Higgs representations

In this section, we introduce the linear and non-linear Higgs representations for the SESM and the THDM. All parameters and fields are considered as “bare” in this section, i.e. the renormalization transformation for introducing renormalized quantities and renormalization constants, including the choice of the tadpole scheme, will be the next step after this section. For the definition of field-theoretical quantities we consistently follow the notation and conventions of Ref. [7], and the transition to the non-linear Higgs representation uses Refs. [22, 23, 33, 34] as guideline.

2.1 Higgs singlet extension of the Standard Model

2.1.1 Kinetic Higgs Lagrangian

Higgs Singlet Extensions of the SM have been introduced in the literature in different variants, see, e.g., Refs. [24–26]. Here, we consider a simple variant with a complex Higgs doublet $\Phi$ and charge conjugate $\Phi^c$ with the same quantum numbers as in the SM,

$$\Phi = \begin{pmatrix} \phi^+ \\ \frac{1}{\sqrt{2}} (v_2 + \eta_2 + i\chi) \end{pmatrix}, \quad \Phi^c = \begin{pmatrix} \frac{1}{\sqrt{2}} (v_2 - \eta_2 - i\chi) \\ -\phi^- \end{pmatrix}, \quad (2.1)$$

with the complex would-be Goldstone-boson fields $\phi^+$, $\phi^- = (\phi^+)^*$, the real would-be Goldstone-boson field $\chi$, the physical Higgs field $\eta_2$, and the constant $v_2$ parametrizing the vev of the Higgs doublet. Apart from the doublet $\Phi$, we introduce a real Higgs singlet field

$$\sigma = v_1 + \eta_1 \quad (2.2)$$
with vev \( v_1 \), as also considered in Refs. [11, 12, 35–37]. Our conventions closely follow Refs. [11, 12, 37]. As in the treatment of the SM in Ref. [21], we switch to the \((2 \times 2)\) matrix notation for the Higgs doublet

\[
\Phi \equiv (\Phi^c, \Phi) = \frac{1}{\sqrt{2}} (v_2 + \eta_2) \mathbb{1} + 2i \phi, \quad \phi \equiv \frac{\phi_j \sigma_j}{2} = \frac{\vec{\phi} \cdot \vec{\sigma}}{2},
\]

where \( \mathbb{1} \) is the \(2 \times 2\) unit matrix, \( \sigma_j \) \((j = 1, 2, 3)\) denote the Pauli matrices, and \( \phi_j \) are the three real would-be Goldstone-boson degrees of freedom in a more generic notation. As in Ref. [21], we use the summation convention over the Goldstone index \( j \), a vector-like notation \( \vec{\phi} = (\phi_1, \phi_2, \phi_3)^T \), \( \vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)^T \), etc., and boldface characters like \( \phi \) for matrix structures. The new field components \( \phi_j \) can be identified according to

\[
\phi^\pm = (\phi_2 \pm i\phi_1)/\sqrt{2}, \quad \chi = -\phi_3.
\]

Since the additional Higgs singlet is invariant under SU(2)_w \times U(1)_Y transformations by definition, only the parametrization of \( \Phi \) changes in the transition to the non-linear representation,

\[
\sigma = v_1 + h_1, \quad \Phi = \frac{1}{\sqrt{2}} (v_2 + h_2) U(\zeta),
\]

with the physical Higgs fields \( h_1 \equiv \eta_1 \) and \( h_2 \), and the same unitary Goldstone-boson matrix \( U(\zeta) \) as in the SM [21],

\[
U(\zeta) = \exp \{ 2i \zeta / v \}, \quad \zeta \equiv \frac{\zeta_j \sigma_j}{2},
\]

with the real would-be Goldstone-boson components \( \vec{\zeta} = (\zeta_1, \zeta_2, \zeta_3)^T \) and \( v = v_2 \). The relation between the fields in the linear and non-linear transformation can be obtained in full analogy to the SM [21, 23],

\[
\eta_2 = c_\zeta (v + h_2) - v = h_2 - \frac{\zeta^2}{2v} \left( 1 + \frac{h_2}{v} \right) + \mathcal{O}(\zeta^4),
\]

\[
\vec{\phi} = \frac{s_\zeta}{\zeta} (v + h_2) \vec{\zeta} = \left( 1 + \frac{h_2}{v} \right) \vec{\zeta} + \mathcal{O}(\zeta^3),
\]

where

\[
s_\zeta \equiv \sin \left( \frac{\zeta}{v} \right), \quad c_\zeta \equiv \cos \left( \frac{\zeta}{v} \right), \quad \zeta \equiv |\vec{\zeta}| = (\vec{\zeta}^2)^{1/2}.
\]

The gauge transformation of \( \Phi \) in the matrix representation is given by [23]

\[
\Phi \rightarrow S(\theta) \Phi S_Y(\theta_Y)
\]

with the unitary transformation matrices

\[
S(\theta) = \exp \{ ig_2 \theta \}, \quad S_Y(\theta_Y) = \exp \left\{ \frac{i}{2} g_1 \theta_Y \sigma_3 \right\}, \quad \theta \equiv \frac{\theta_j \sigma_j}{2},
\]

where \( g_2 \) and \( g_1 \) are the weak and electromagnetic coupling constants, respectively.
depending on the group parameters $\theta = \vec{\theta} \cdot \vec{\sigma}/2$ and $\theta_Y$ for the SU(2)$_w$ and the U(1)$_Y$ transformations, respectively. Taking into account that $\sigma$ is gauge invariant, the Higgs and would-be Goldstone fields of the non-linear representation transform according to
\begin{equation}
\begin{align*}
h_1 &\to h_1, \quad h_2 \to h_2, \\
U(\zeta) &\to S(\theta) U(\zeta) S(\theta_Y),
\end{align*}
\tag{2.11}
\end{equation}
i.e. Higgs fields $h_1$ and $h_2$ are gauge invariant, while the would-be Goldstone fields $\zeta_j$ change under gauge transformations.

The kinetic terms of the Higgs Lagrangian just adds a term for the field $h_1$ to the SM contribution,
\begin{equation}
L_{H,\text{kin}} = \frac{1}{2}(\partial \sigma)^2 + \frac{1}{2} \text{tr} \left[(D_\mu \Phi)^\dagger (D^\mu \Phi)\right],
\tag{2.12}
\end{equation}
where
\begin{equation}
D^\mu \Phi = \partial^\mu \Phi - ig_2 W^\mu \Phi - ig_1 B^\mu \sigma_3/2, \quad W^\mu = \frac{W_j^\mu}{2} \sigma_j,
\tag{2.13}
\end{equation}
is the covariant derivative in matrix notation, with the matrix-valued SU(2)$_w$ gauge field $W^\mu$ and the U(1)$_Y$ gauge field $B^\mu$ and respective gauge couplings $g_2$ and $g_1$. Inserting the field parametrizations (2.5), this leads to
\begin{equation}
L_{H,\text{kin}} = \frac{1}{2}(\partial h_1)^2 + \frac{1}{2}(\partial h_2)^2 + \frac{g_2^2}{8} (v_2 + h_2)^2 \tilde{C}_\mu^{(u)} \cdot \tilde{C}_\mu^{(u),\mu},
\tag{2.14}
\end{equation}
with $\tilde{C}_\mu^{(u)}$ as given for the SM,
\begin{equation}
C_\mu^{(u)} = W_\mu^{(u)} + \frac{g_1}{g_2} B^\mu \sigma_3/2 = \frac{\tilde{C}_\mu^{(u)} \cdot \vec{\sigma}}{2},
\tag{2.15}
\end{equation}
\begin{equation}
W_\mu^{(u)} = U(\zeta)^\dagger W_\mu U(\zeta) + \frac{i}{g_2} U(\zeta)^\dagger \partial_\mu U(\zeta).
\tag{2.16}
\end{equation}
More explicit expressions for $C_\mu^{(u)}$ and $\tilde{C}_\mu^{(u)} \cdot \tilde{C}_\mu^{(u),\mu}$ can be found in Sect. 2.1 of Ref. [21]. The gauge-fixing part of the Lagrangian in the non-linear representation, which is the same as in the SM, can be found there as well.

### 2.1.2 Higgs potential and tadpoles

In the considered variant of the SESM, the interactions among the Higgs fields are ruled by the following Higgs potential,
\begin{equation}
V = -\mu_2^2 \Phi^\dagger \Phi - \mu_1^2 \sigma^2 + \frac{\lambda_2}{4} (\Phi^\dagger \Phi)^2 + \lambda_1 \sigma^4 + \lambda_{12} \Phi^\dagger \Phi \sigma^2
\tag{2.17}
\end{equation}
with real coupling constants $\mu_1^2$, $\mu_2^2$, $\lambda_1$, $\lambda_2$, $\lambda_{12}$. The potential $V$ has the most general form that supports renormalizability and a $\mathbb{Z}_2$ symmetry under $\sigma \to -\sigma$. As in the SM, $V$ involves would-be Goldstone-boson fields in the linear Higgs representation, but not in the non-linear representation, where it is given by
\begin{equation}
V = -\frac{\mu_2^2}{2} \text{tr}[\Phi^\dagger \Phi] - \mu_1^2 \sigma^2 + \frac{\lambda_2}{16} (\text{tr}[\Phi^\dagger \Phi])^2 + \lambda_1 \sigma^4 + \frac{\lambda_{12}}{2} \text{tr}[\Phi^\dagger \Phi] \sigma^2
\end{equation}
\[ = -\frac{\mu_0^2}{2}(v_2 + h_2)^2 - \mu_0^2(v_1 + h_1)^2 + \frac{\lambda_0}{16}(v_2 + h_2)^4 + \lambda_1(v_1 + h_1)^4 \]
\[ + \frac{\lambda_{12}}{2}(v_1 + h_1)^2(v_2 + h_2)^2. \] (2.18)

Both in the linear and non-linear representations, the original Higgs-boson fields \((\eta_1, \eta_2)\) and \((h_1, h_2)\), respectively, can be rotated into a field basis of \(h\) and \(H\), which correspond to mass eigenstates,

\[
\begin{pmatrix}
\eta_1 \\
\eta_2
\end{pmatrix} = R(\alpha) \begin{pmatrix} H \\ h \end{pmatrix}, \quad \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = R(\alpha) \begin{pmatrix} H \\ h \end{pmatrix}_{\text{nl}}, \quad R(\alpha) = \begin{pmatrix} c_\alpha & -s_\alpha \\ s_\alpha & c_\alpha \end{pmatrix},
\] (2.19)

where \(\alpha\) is a real-valued mixing angle which is determined by the parameters of the Higgs potential.\(^3\) In the following, we will often use the shorthands \(s_\alpha \equiv \sin \alpha\), \(c_\alpha \equiv \cos \alpha\), \(t_\alpha \equiv \tan \alpha\) and analogously for other angles. The fact that the field bases \((h, H)\) in the two representations are not the same is indicated by the subscript “nl” in Eq. (2.19) indicating the non-linear representation. By convention, \(H\) corresponds to the heavier and \(h\) to the lighter Higgs state.\(^4\) Since \(\eta_\alpha = h_n + O(\xi^2) (n = 1, 2)\) holds up to terms of higher powers in the would-be Goldstone fields \(\xi_j\), the mixing angle \(\alpha\) is the same in the two representations (at least in lowest perturbative order).

The tadpole constants for the non-linear and linear Higgs representations are easily calculated to

\[
\Gamma_{h_n}^{h_n} = T_{nl}^{h_n} = \frac{1}{16\pi^2 v_2} \left\{ \lambda_n^h v_2^2 A_0(M_n^2) + \lambda^n H v_2^2 A_0(M_H^2) - 4\delta_{n2} \sum_f N_f^2 M_f^2 A_0(m_f^2) \right\}
\]
\[ + \delta_{n2} M_2 \left[ 3A_0(M_2^2) - 2M_2^2 \right] + 2\delta_{n2} M_W^2 \left[ 3A_0(M_W^2) - 2M_W^2 \right] \] (2.20)

and

\[
\Gamma_{\eta_n}^{\eta_n} = T_{nl}^{\eta_n} + \lambda_{nl}^G v_2^2 \Delta \nu_\xi,
\] (2.21)

respectively, where the gauge-dependent constant \(\Delta \nu_\xi\) is defined as in the SM \([21]\),

\[
\Delta \nu_\xi = \frac{1}{16\pi^2 v_2} \left\{ \frac{1}{2} A_0(\xi_Z M_Z^2) + A_0(\xi_W M_W^2) \right\},
\] (2.22)

with \(\xi_Z\) and \(\xi_W\) representing the arbitrary \(R_\xi\) gauge parameters of the \(Z\)- and \(W\)-boson fields. As mentioned before, in the SESM we have \(v = v_2\). In Eq. (2.20) \(\delta_{n2}\) represents the usual Kronecker \(\delta\), and the scalar coupling factors \(\lambda_n^h\), \(\lambda_n^H\), and \(\lambda_n^G\) with \(n = 1, 2\) are

\[
\lambda_1^h = \frac{12s_\alpha^2 v_1 \lambda_1}{v_2}, \quad \lambda_2^h = \frac{3c_\alpha^2 \lambda_2}{4} - \frac{(2c_\alpha s_\alpha v_1}{v_2} - s_\alpha^2 \lambda_{12}^h,
\]

\[^3\]To avoid confusion between the angle \(\alpha\) and the electromagnetic coupling, we denote the fine-structure constant \(\alpha_{em}\) in the following.

\[^4\]In Ref. \([12]\), the two physical Higgs bosons were called \(H_{1/2}\) which corresponds to \(H \equiv H_1\) and \(h \equiv H_2\) compared to our conventions here.
\[
\lambda_1^H = \frac{12 c_a^2 v_1 \lambda_1}{v_2} + \left( \frac{s_a^2 v_1}{v_2} + 2 c_a s_a \right) \lambda_1, \quad \lambda_2^H = \frac{3 s_a^2 \lambda_2}{4} + \left( \frac{2 c_a s_a v_1}{v_2} + c_a^2 \right) \lambda_1, \\
\lambda_1^G = \frac{2 v_1 \lambda_1}{v_2}, \quad \lambda_2^G = \lambda_2^G/2.
\]

In Eqs. (2.20) and (2.22), we made use of the scalar one-point one-loop integral in \( D = 4 - 2\epsilon \) dimensions,

\[
A_0(m^2) = \frac{(2\pi\mu)^{4-D}}{4\pi^2} \int \frac{d^D q}{q^2 - m^2 + i0} = m^2 \left[ \Delta + \ln \left( \frac{\mu^2}{m^2} \right) + 1 \right] + \mathcal{O}(\epsilon)
\]

with the arbitrary reference mass \( \mu \) and the standard UV divergence

\[
\Delta = \frac{2}{4-D} + \ln 4\pi - \gamma_E,
\]

in which \( \gamma_E \) denotes the Euler–Mascheroni constant.

Obviously, the tadpole constants \( T_{nl}^H \) of the non-linear representation are gauge independent, while the \( T_{nl}^G \) of the linear representation involve gauge-dependent contributions proportional to \( \Delta v^\xi \). The gauge independence of \( T_{nl}^H \) holds to all perturbative orders by virtue of the Nielsen identities [38–40] as discussed in Sect. 2.2 of Ref. [21] for the SM.

### 2.2 Two-Higgs-Doublet Model

#### 2.2.1 Kinetic Higgs Lagrangian

Two-Higgs Doublet Models (THDMs) [27, 28] contain two complex Higgs doublets \( \Phi_n (n = 1, 2) \), both with weak hypercharge \( Y_w. \Phi_n = +1 \). As also done in previous literature on the renormalization of the THDM [8–10, 12, 37, 41–45], we only consider CP-conserving Higgs potentials with additional discrete symmetries that allow only for one type of Yukawa coupling per Higgs doublet and fermion type to avoid flavour-changing neutral currents at tree level. In the two-component notation, the doublets \( \Phi_n \) and their charge conjugates \( \Phi_n^c = i\sigma_2 \Phi_n^\dagger \) are parametrized according to

\[
\Phi_n = \left( \begin{array}{c}
\frac{1}{\sqrt{2}} (v_n + \eta_n + i\chi_n) \\
\frac{1}{\sqrt{2}} (v_n + \eta_n - i\chi_n)
\end{array} \right), \quad \Phi_n^c = \left( \begin{array}{c}
\frac{1}{\sqrt{2}} (v_n + \eta_n + i\chi_n) \\
-\phi_n^\dagger
\end{array} \right),
\]

where \( \eta_n \) are the CP-even neutral Higgs fields that mix to the two physical CP-even neutral Higgs bosons h and H and from which the corresponding vevs \( v_n \) have been split off. The remaining component fields \( \phi_n^\dagger, \phi_n^\dagger = (\phi_n^\dagger)^* \), and \( \chi_n \) are superpositions of would-be Goldstone-bosons fields and the fields of the physical Higgs bosons, as described further below. Analogously to Eq. (2.3) in the SESM, in the linear parametrization the two Higgs doublets of the THDM can be written as \( 2 \times 2 \) matrices,

\[
\Phi_n \equiv (\Phi_n^c, \Phi_n) = \frac{1}{\sqrt{2}} [(v_n + \eta_n) \mathbb{1} + 2i\phi_n^\dagger], \quad \phi_n \equiv \frac{\phi_{nj}\sigma_j}{2}
\]

with the same fields \( \eta_n \) as in Eq. (2.26), but the components \( \phi_n^\dagger \) and \( \chi_n \) are absorbed into the two field matrices \( \phi_n \) with the real component fields \( \phi_{nj} (n = 1, 2; j = 1, 2, 3) \). In analogy to the SM, the two sets of fields are related according to

\[
\phi_n^\pm = (\phi_n^a \pm i\phi_n^b)/\sqrt{2}, \quad \chi_n = -\phi_n^3.
\]
In the non-linear representation, we have to take care of the fact that after splitting off the Goldstone-boson matrix $U(\zeta)$ from $\Phi_n$ with $U(\zeta)$ given in Eq. (2.6), apart from the physical CP-even neutral fields $h_n$ some dependence on three further real physical Higgs fields $\rho_j$ ($j = 1, 2, 3$) remains, which correspond to one CP-odd neutral and two charged Higgs bosons. We, thus, can make the following ansatz for the non-linearly parametrized Higgs matrix fields,

$$\Phi_n = U(\zeta) \Phi^{(u)}_n, \quad \Phi^{(u)}_n = \frac{1}{\sqrt{2}} [(v_n + h_n) \mathbb{1} + i c_{nj} \sigma_j \rho_j], \quad (2.29)$$

where the coefficients $c_{nj}$ ($n = 1, 2; j = 1, 2, 3$) are real constants, which can be determined upon requiring canonical field normalization. To this end, it is sufficient to take the terms in $\partial_{\mu} \Phi_n$ that are linear in all scalar fields and to evaluate the (bilinear) terms in the kinetic Lagrangian of the Higgs fields for free propagation. These kinetic terms are canonically normalized and do not mix the free scalar fields $\zeta_j$, $h_n$, and $\rho_j$ if we set

$$c_{1j} = -\frac{v_0}{v} \equiv -\sin \beta, \quad c_{2j} = \frac{v_1}{v} \equiv \cos \beta, \quad v \equiv \sqrt{v_1^2 + v_2^2}. \quad (2.30)$$

Since the constants $c_{nj}$ do not depend on $j$, we can write

$$\Phi^{(u)}_n = \frac{1}{\sqrt{2}} [(v_n + h_n) \mathbb{1} + 2i c_n \rho], \quad \rho \equiv \frac{p_j \sigma_j}{2}, \quad c_n \equiv c_{nj}. \quad (2.31)$$

The gauge-invariant trace of $\Phi_m^\dagger \Phi_n$ reads

$$\text{tr}[\Phi_m^\dagger \Phi_n] = \text{tr}[(\Phi_m^{(u)})^\dagger \Phi^{(u)}_n] = (v_m + h_m)(v_n + h_n) + c_m c_n \rho^2. \quad (2.32)$$

The relations between the fields of the two representations are given by

$$\eta_n = c_\zeta (v_n + h_n) - v_n - s_\zeta c_n \frac{\bar{\rho} \cdot \zeta}{\zeta}, \quad \tilde{\phi}_n = c_\zeta c_n \bar{\rho} + s_\zeta \left[(v_n + h_n) \frac{\zeta}{\zeta} + c_n \frac{\bar{\rho} \times \zeta}{\zeta}\right], \quad (2.33)$$

where $\times$ denotes the usual cross product of 3-dimensional vectors.

Analogously to Eq. (2.9), the Higgs doublets transform as

$$\Phi_n \rightarrow S(\theta) \Phi_n S_Y(\theta_Y) \quad (2.34)$$

under $\text{SU}(2)_w \times \text{U}(1)_Y$ transformations with $S$ and $S_Y$ given in Eq. (2.10). Applying these transformations to Eq. (2.29), we can determine the gauge transformation of $\Phi_n^{(u)}$ by isolating the transformation of $U(\zeta)$ as described in (2.11),

$$\Phi_n \rightarrow \left[S(\theta) \Phi_n \Phi(\theta_Y)\right] \left[S_Y(-\theta_Y) \Phi^{(u)}_n S_Y(\theta_Y)\right]. \quad (2.35)$$

The resulting gauge transformation rules of the fields $h_n$, $\rho_j$, and $\zeta_j$ are given by

$$h_n \rightarrow h_n, \quad (2.36)$$

$$\sigma_j \rho_j \rightarrow S_Y(-\theta_Y) \sigma_j \rho_j S_Y(\theta_Y) = \left(\sum_{j=1,2} \sigma_j \rho_j\right) S_Y(2\theta_Y) + \sigma_3 \rho_3$$
\[ \rho^\pm = \frac{1}{\sqrt{2}}(\rho_2 \pm i\rho_1) \rightarrow \exp\{\mp ig_1\theta_Y\} \rho^\pm, \quad \rho_3 \rightarrow \rho_3. \quad (2.37) \]

That means that the fields \( h_n \) and \( \rho_3 \) are completely gauge invariant while \( \rho_1 \) and \( \rho_2 \) are only invariant under the SU(2)\(_w\) gauge transformations. The fields \( \rho_1, \rho_2 \) mix into each other under the U(1)\(_Y\) hypercharges transformations, but \( \rho^\pm \) correspond to eigenstates of weak hypercharge. As SU(2)\(_w\) singlets, the fields \( \rho^\pm \), thus, have electric charge \( Q_{\rho^\pm} = Y_{\rho^\pm}/2 = \pm 1 \) according to the Gell-Mann–Nishijima relation. The vev parameters \( v_n \) again are associated with the gauge-invariant fields \( h_n \). The Goldstone-boson fields \( \zeta_j \) transform under SU(2)\(_w\) × U(1)\(_Y\) exactly as in the SM.

Using the matrix representation (2.27) for \( \Phi_n \), the Higgs kinetic terms are given by

\[
\mathcal{L}_{\text{H,kin}} = \frac{1}{2} \text{tr} \left[ (D_\mu \Phi_n)^\dagger (D^\mu \Phi_n) \right],
\]

\[
= \frac{1}{2} (\partial h_n)^2 + \frac{1}{2} (\partial \tilde{\rho})^2 + \frac{g_2^2}{2} \left[ (v_n + h_n)^2 + \tilde{\rho}^2 \right] \tilde{C}_\mu^{(u)\cdot} \tilde{C}^{(u)\cdot}_\mu + \frac{g_2}{2} \tilde{C}_\mu^{(u)\cdot} \left[ c_n(\partial^\mu h_n)\tilde{\rho} - c_n h_n(\partial^\mu \tilde{\rho}) + (\tilde{\rho} \times \partial^\mu \tilde{\rho}) \right] + \frac{g_1^2}{2} B^2 (\rho_1^2 + \rho_2^2) + g_1 B^\mu \left[ (\partial_\mu \rho_1)(\rho_2 - (\partial_\mu \rho_2)\rho_1) \right] + \frac{g_1 g_2}{2} B_\mu \left[ c_n h_n \left( \rho_1 C_2^{(u)\cdot} - \rho_2 C_1^{(u)\cdot} \right) + \rho_3 \tilde{\rho} \cdot \tilde{C}^{(u)\cdot} - \tilde{\rho}^2 C_3^{(u)\cdot} \right] \quad (2.38) \]

where a summation over \( n = 1, 2 \) is implicitly understood and \( C_\mu^{(u)\cdot} \) is again as defined in Eq. (2.15). Up to quadratic order in \( \zeta_j \), the Higgs kinetic Lagrangian reads

\[
\mathcal{L}_{\text{H,kin}} = \frac{1}{2} (\partial h_n)^2 + \frac{1}{2} (\partial \tilde{\rho})^2 + \frac{1}{2v^2} \left[ (v_n + h_n)^2 + \tilde{\rho}^2 \right] \left\{ (\partial_\mu \zeta) \cdot (\partial^\mu \zeta) + \frac{g_2^2 v^2}{4} \tilde{C}_\mu \cdot \tilde{C}^\mu + g_1 g_2 B_\mu \left[ -W_\mu^\alpha \zeta^2 + (\bar{W}^\mu \cdot \zeta) \zeta_3 \right] - g_2 v \tilde{C}_\mu \cdot (\bar{W}^\mu \times \zeta) - g_2 v \tilde{C}_\mu \cdot (\bar{W}^\mu \times \zeta) \right. \\
+ \frac{g_2^2}{2} \left( \tilde{C}_\mu - \frac{2}{g_2 v} \partial_\mu \tilde{\rho} \cdot \tilde{C}_\mu - \frac{2}{g_2 v^2} (\tilde{W}_\mu \times \zeta) - g_2 v \tilde{C}_\mu \cdot (\bar{W}^\mu \times \zeta) \right) \\
\left. \cdot \left[ c_n(\partial^\mu h_n)\tilde{\rho} - c_n h_n(\partial^\mu \tilde{\rho}) + (\tilde{\rho} \times \partial^\mu \tilde{\rho}) + g_1 B^\mu \rho_3 \tilde{\rho} \right] + \frac{g_1^2}{2} B^2 (\rho_1^2 + \rho_2^2) + g_1 B^\mu \left[ (\partial_\mu \rho_1)(\rho_2 - (\partial_\mu \rho_2)\rho_1) \right] + \frac{g_1 g_2}{2} B_\mu \left[ c_n h_n \left( \rho_1 C_2^{(u)\cdot} - \rho_2 C_1^{(u)\cdot} \right) - \frac{2}{g_2 v} (\rho_1\partial^\mu \zeta_2 + \rho_2 \partial^\mu \zeta_1) \right. \\
\left. - \frac{2}{v} \left[ (\tilde{\rho} \cdot \zeta) W_3^\mu - (\tilde{\rho} \cdot \bar{W}^\mu) \zeta_3 \right] - \frac{2}{g_2 v^2} \left[ (\tilde{\rho} \cdot \partial^\mu \zeta) \zeta_3 - (\tilde{\rho} \cdot \zeta) \partial^\mu \zeta_3 \right] \right. \\
+ \frac{2}{v^2} \left[ (\bar{W}^\mu \cdot \zeta) (\rho_1 \zeta_2 - \rho_2 \zeta_1) - \zeta^2 (\rho_1 W_2^\mu - \rho_2 W_1^\mu) \right] \\
- \tilde{\rho}^2 \left[ C_3^{(u)\cdot} - \frac{2}{g_2 v} \partial^\mu \zeta_3 - \frac{2}{v} (W_1^\mu \zeta_2 - W_2^\mu \zeta_1) - \frac{2}{g_2 v^2} (\zeta_1 \partial^\mu \zeta_2 - \zeta_2 \partial^\mu \zeta_1) \right. \\
\left. - \frac{2}{v^2} \left[ \zeta^2 W_3^\mu - (\bar{W}^\mu \cdot \zeta) \zeta_3 \right] \right) + \mathcal{O}(\zeta^3). \quad (2.39) \]
The gauge-fixing part of the Lagrangian in the non-linear representation is the same as in the SM, see Sect. 2.1 of Ref. [21].

2.2.2 Higgs potential and tadpoles

In order to rule out flavour-changing neutral currents at tree level, we assume the \( Z_2 \) symmetry \( \Phi_1 \to -\Phi_1 \) and \( \Phi_2 \to \Phi_2 \) that is only softly broken by the \( m_{12}^2 \) term in the Higgs potential, which in terms of bare parameters and fields reads

\[
V = m_{11}^2 \Phi_1 \Phi_1 + m_{22}^2 \Phi_2 \Phi_2 - m_{12}^2 (\Phi_1 \Phi_2 + \Phi_2 \Phi_1)
+ \frac{\lambda_1}{2} (\Phi_1^\dagger \Phi_1)^2 + \frac{\lambda_2}{2} (\Phi_2^\dagger \Phi_2)^2 + \lambda_3 (\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2)
+ \lambda_4 (\Phi_1^\dagger \Phi_2)(\Phi_2^\dagger \Phi_1) + \frac{\lambda_5}{2} \left[ (\Phi_1^\dagger \Phi_1)^2 + (\Phi_2^\dagger \Phi_2)^2 \right].
\]

(2.40)

Moreover, we assume all couplings in \( V \) to be real in order to conserve CP. In the non-linear Higgs representation this translates to

\[
V = \frac{m_{11}^2}{2} \text{tr}[\Phi_1^\dagger \Phi_1] + \frac{m_{22}^2}{2} \text{tr}[\Phi_2^\dagger \Phi_2] - m_{12}^2 \text{tr}[\Phi_1^\dagger \Phi_2]
+ \frac{\lambda_1}{8} (\text{tr}[\Phi_1^\dagger \Phi_1])^2 + \frac{\lambda_2}{8} (\text{tr}[\Phi_2^\dagger \Phi_2])^2 + \frac{\lambda_3}{4} \text{tr}[\Phi_1^\dagger \Phi_1] \text{tr}[\Phi_2^\dagger \Phi_2]
+ \lambda_4 \text{tr}[\Phi_1^\dagger \Phi_2 \Omega_+] \text{tr}[\Phi_1^\dagger \Phi_2 \Omega_-] + \frac{\lambda_5}{2} \left[ (\text{tr}[\Phi_1^\dagger \Phi_2 \Omega_+])^2 + (\text{tr}[\Phi_1^\dagger \Phi_2 \Omega_-])^2 \right]
\]

(2.41)

with the two-dimensional projection operators \( \Omega_\pm = \frac{1}{2}(1 \pm \sigma_3) \), which select the original Higgs doublet \( \Phi \) or its charge conjugate from the matrix field \( \Phi \). Obviously, the unitary Goldstone-boson matrix \( U(\zeta) \) again drops out in \( V \) in this representation.

In the linear Higgs representation, the CP-even Higgs fields \( \eta_n \) of the doublets \( \Phi_n \) are related to the fields \( H, h \) corresponding to mass eigenstates of the CP-even neutral Higgs bosons by a rotation with an angle \( \alpha \) as given in Eq. (2.19). Similarly, there is an analogous rotation about an angle \( \beta \) between the CP-odd neutral fields \( \chi_n \) to the CP-odd neutral Higgs field \( A_0 \) and the neutral would-be Goldstone boson field \( G_0 \). Similarly, in the charged sector the charged fields \( \phi_n^\pm \) and the charged Higgs fields \( H^\pm \) are rotated into the charged would-be Goldstone boson fields \( G^\pm \);

\[
\begin{pmatrix} -\phi_{13} \\ -\phi_{23} \end{pmatrix} = R(\beta) \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \quad \begin{pmatrix} \phi_1^\pm \\ \phi_2^\pm \end{pmatrix} = R(\beta) \begin{pmatrix} G_0 \\ A_0 \end{pmatrix}.
\]

(2.42)

The matrix \( R(\beta) \) is defined as \( R(\alpha) \) in Eq. (2.19), with \( \alpha \) replaced by \( \beta \). In lowest order, the angle \( \beta \) is given by the ratio of the vev parameters \( v_2 \) and \( v_1 \), as already indicated in Eq. (2.30), i.e. we have

\[
t_\beta \equiv \tan \beta = \frac{v_2}{v_1}.
\]

(2.43)

The parameters \( \beta \), which is usually taken as independent input parameter, and \( v = \sqrt{v_1^2 + v_2^2} \), which is directly related to the W-boson mass by \( M_W = g_2 v/2 \), then fix the vevs \( v_1 \), \( v_2 \) in terms of \( M_W \), \( g_2 \), and \( \beta \).
Owing to the relations $\eta_n = h_n + \ldots$ up to higher powers in the fields, the fields $h_n$ of the non-linear Higgs representation are transformed into the fields $h$ and $H$ of the CP-even Higgs bosons by a rotation with the same angle $\alpha$ as for the $\eta_n$ in the linear representation (at least in lowest perturbative order), i.e. Eq. (2.19) holds in the THDM as well. The rotation of the CP-odd field components $\phi_{n,j}$ to fields corresponding to mass eigenstates and would-be Goldstone-boson fields is already contained in the factorization of the Goldstone matrix $U(\zeta)$ from the matrix fields $\Phi_n$ in the non-linear representation. The relations $\phi_{n,j} = c_n \rho_j + v_n \zeta_j / v + \ldots$, which are valid up to higher powers in the fields, just provide this rotation,

$$
\left(\begin{array}{c}
\phi_{1j} \\
\phi_{2j}
\end{array}\right) = R(\beta) \left(\begin{array}{c}
\zeta_j \\
\rho_j
\end{array}\right) + \ldots 
$$ (2.44)

We can, thus, identify

$$
\left(\begin{array}{c}
-\zeta_3 \\
-\rho_3
\end{array}\right) = \left(\begin{array}{c}
G_0 \\
A_0
\end{array}\right)_{nl}, \quad \left(\begin{array}{c}
\zeta^+ \\
\rho^+
\end{array}\right) = \left(\begin{array}{c}
H^+ \\
\lambda^+_{nl}
\end{array}\right)
$$ (2.45)

with the same angle $\beta$ as in the linear representation (at least in lowest order).

Finally, we again calculate and compare the tadpole contributions. They are given by

$$
\Gamma_{nl}^{h_n} = T_{nl}^{h_n} = \frac{1}{16\pi^2 v} \left\{ \lambda_n^h v^2 A_0(M_H^2) + \lambda_n^H v^2 A_0(M_H^2) + \lambda_n^A v^2 A_0(M_A^2) + \lambda_n^{H \pm} v^2 A_0(M_H^{\pm})
\right.
$$

$$
- 4 \sum_f \xi_n^f N_f^2 m_f^2 A_0(m_f^2) + \frac{v_n M_Z^2}{v} \left[ 3 A_0(M_Z^2) - 2 M_W^2 \right]
$$

$$
\left. + \frac{2 v_n M_W^2}{v} \left[ 3 A_0(M_W^2) - 2 M_W^2 \right] \right\}
$$ (2.46)

and

$$
\Gamma^{h_n} = T^{h_n} = T_{nl}^{h_n} + \lambda_n^G v^2 \Delta v_{\xi}
$$ (2.47)

for the non-linear and the linear representation, respectively, where $\Delta v_{\xi}$ given in Eq. (2.22) and $v^2 = v_1^2 + v_2^2$ in the THDM. The parameters $\xi_n^f$ are the coupling strengths to the fermions relative to the SM value of $m_f / v$. The factors $\xi_n^f$ depend on the type of the THDM model and are related to the parameters $\xi_n^H$ and $\xi_n^h$ given in Tab. 2 of Ref. [10] by

$$
\left(\begin{array}{c}
\xi_n^f \\
\xi_n^h
\end{array}\right) = R(\alpha) \left(\begin{array}{c}
\xi_n^H \\
\xi_n^h
\end{array}\right).
$$ (2.48)

The trilinear coupling factors $\lambda_n^h$, $\lambda_n^H$, $\lambda_n^A$, and $\lambda_n^{H \pm}$ with $n = 1, 2$ are

$$
\lambda_1^h = \frac{3}{2} s^2_\alpha c_\beta \lambda_1 + \frac{c_\alpha}{2} [c_\beta c_\alpha - 2 s_\beta s_\alpha] \lambda_{345}, \quad \lambda_2^h = \frac{3}{2} c_\alpha^2 s_\beta \lambda_2 + \frac{s_\alpha}{2} [s_\beta s_\alpha - 2 c_\beta c_\alpha] \lambda_{345},
$$

$$
\lambda_1^H = \frac{3}{2} c^2_\alpha c_\beta \lambda_1 + \frac{s_\alpha}{2} [c_\beta s_\alpha + 2 s_\beta c_\alpha] \lambda_{345}, \quad \lambda_2^H = \frac{3}{2} s^2_\alpha s_\beta \lambda_2 + \frac{c_\alpha}{2} [s_\beta c_\alpha + 2 c_\beta s_\alpha] \lambda_{345},
$$

$$
\lambda_1^A = \frac{c_\beta}{2} \left[ v_\beta^2 \lambda_1 + c^2_\beta \lambda_{345} - 2 \lambda_5 \right], \quad \lambda_2^A = \frac{c_\beta}{2} \left[ c^2_\beta \lambda_2 + v_\beta^2 \lambda_{345} - 2 \lambda_5 \right],
$$
\[
\begin{align*}
\lambda_1^{H^\pm} &= c_\beta \left[ s_\beta^2 (\lambda_1 - \lambda_{345}) + \lambda_3 \right], \\
\lambda_2^{H^\pm} &= s_\beta \left[ c_\beta^2 (\lambda_2 - \lambda_{345}) + \lambda_3 \right], \\
\lambda_1^T &= c_\beta \left[ c_\beta^2 \lambda_1 + s_\beta^2 \lambda_{345} \right], \\
\lambda_2^T &= s_\beta \left[ s_\beta^2 \lambda_2 + c_\beta^2 \lambda_{345} \right],
\end{align*}
\]  

(2.49)

where the shorthand \(\lambda_{345} = \lambda_3 + \lambda_4 + \lambda_5\) was used with \(\lambda_i, i = 1, \ldots, 5\), denoting the quartic couplings from the Higgs potential. The tadpole constants \(T_{nl}^{H}\) are gauge independent, a fact that even holds to all perturbative orders by virtue of the Nielsen identities [38–40] (see discussion in Sect. 2.2 of Ref. [21] for the SM).

3 Gauge-invariant vacuum expectation value renormalization

3.1 Higgs singlet extension of the Standard Model

3.1.1 Schemes for tadpole and vacuum expectation value renormalization

To prepare the renormalization of the scalar sector of the SESM, we split the bare scalar fields \(\sigma_B, \Phi_B\) into bare field excitations \(\eta_{1,B}, \eta_{2,B}\), etc., and bare constants \(v_{1,0}\) and \(v_{2,0}\),

\[
\sigma_B = v_{1,0} + \eta_{1,B}, \quad \Phi_B = \left( \frac{1}{\sqrt{2}} (v_{2,0} + \eta_{2,B} + i\chi_B) \right).
\]  

(3.1)

The bases of bare Higgs-boson fields \((\eta_{1,B}, \eta_{2,B})\) and \((h_B, H_B)\) are related to each other by rotations about the bare mixing angle \(\alpha_0\), as defined in Eq. (2.19). As in the SM case described in Ref. [21], we need not consider renormalization in the non-linear Higgs representation, because only the tadpole constants \(T_{nl}^{h}\) and \(T_{nl}^{H}\) will be required from this representation.

Since the gauge structure of the SESM is the same as in the SM, and since only \(\Phi\), but not \(\sigma\), interacts with the EW gauge bosons, the W-boson and Z-boson masses \(M_W\) and \(M_Z\) are determined by \(v_2\) and the gauge couplings.

Different renormalization schemes for the SESM were proposed in Refs. [11, 12, 35–37], where the most delicate parameter to be renormalized is the mixing angle \(\alpha\). Ref. [12] contains a comprehensive overview of different ways to renormalize \(\alpha\) and discusses the renormalization scheme dependence of some Higgs-boson production and decay processes obtained with \(\alpha\) renormalized with OS, MS, or symmetry-inspired schemes.

Before briefly recapitulating the application of the FJTS and PRTS to the SESM and formulating the GIVS for the SESM, we first summarize the common features of these schemes. Inserting the field decomposition (3.1) into the bare Lagrangian \(\mathcal{L}\), produces a term \(t_{\eta_1,0}\eta_1 + t_{\eta_2,0}\eta_2\) in the Lagrangian \(\mathcal{L}\) with

\[
\begin{align*}
t_{\eta_1,0} &= v_{1,0} \left( 2\mu_{1,0}^2 - 4\lambda_{1,0}v_{1,0}^2 - \lambda_{12,0}v_{2,0}^2 \right), \\
t_{\eta_2,0} &= v_{2,0} \left( \mu_{2,0}^2 - \frac{1}{4}\lambda_{2,0}v_{2,0}^2 - \lambda_{12,0}v_{1,0}^2 \right)
\end{align*}
\]  

(3.2)

at the one-loop level. Since the fields \(\eta_1\) and \(\eta_2\) correspond to components of Higgs fields that develop vevs, the corresponding one-point vertex functions \(\Gamma_{\eta_1}\) and \(\Gamma_{\eta_2}\) only vanish in higher orders after imposing appropriate renormalization conditions. To this end, a
contribution $\delta t_{\eta_1 \eta_1} + \delta t_{\eta_2 \eta_2}$ in the counterterm Lagrangian $\delta L$ is required with appropriate tadpole renormalization constants $\delta t_{\eta_n}$, and we demand for the renormalized one-point functions $\Gamma_{R}^{\eta_n}$

$$\Gamma_{R}^{\eta_n} = T^{\eta_n} + \delta t_{\eta_n} = 0 \Rightarrow \delta t_{\eta_n} = -T^{\eta_n}, \quad n = 1, 2. \quad (3.3)$$

The tadpole renormalization constants $\delta t_{\eta_n}$ are generated in different ways by different choices of the bare tadpole constants $t_{\eta_n,0}$, possibly accompanied by further field redefinitions of the bare Higgs fields $\eta_n$. These choices define different tadpole schemes. It is convenient to rotate the two vertex functions and the corresponding tadpole renormalization constants $\delta t_{\eta_k}$ into the field basis $H$, $h$ of mass eigenstates,

$$\left( \begin{array}{c} T_{\eta_1} \\ T_{\eta_2} \end{array} \right) = R(\alpha) \left( \begin{array}{c} T^H \\ T^h \end{array} \right), \quad \left( \begin{array}{c} \delta t_{\eta_1} \\ \delta t_{\eta_2} \end{array} \right) = R(\alpha) \left( \begin{array}{c} \delta t_H \\ \delta t_h \end{array} \right). \quad (3.4)$$

In this field basis, the tadpole renormalization condition reads

$$\Gamma_{R}^{H} = T^{H} + \delta t_{H} = 0 \Rightarrow \delta t_{H} = -T^{H},$$

$$\Gamma_{R}^{h} = T^{h} + \delta t_{h} = 0 \Rightarrow \delta t_{h} = -T^{h}. \quad (3.5)$$

**FJTS:**

In complete analogy to the SM case, in the FJTS the bare tadpole constants are set to zero by definition, $t_{\eta_n,0} = 0$, determining the vev parameters according to

$$v_{1,0} = \sqrt{\frac{\lambda_{2,0} \mu_{1,0}^2 - 2\lambda_{12,0} \mu_{2,0}^2}{2(\lambda_{1,0} \lambda_{2,0} - \lambda_{12,0}^2)}}, \quad v_{2,0} = \sqrt{\frac{2(2\lambda_{1,0} \mu_{2,0}^2 - \lambda_{12,0} \mu_{1,0}^2)}{\lambda_{1,0} \lambda_{2,0} - \lambda_{12,0}}}. \quad (3.6)$$

The tadpole counterterms $\delta t_{\eta_1 \eta_1} + \delta t_{\eta_2 \eta_2}$ in the counterterm Lagrangian $\delta L$ are generated from field shifts

$$\eta_n \rightarrow \eta_n + \Delta v_{n}^{\text{FJTS}}, \quad n = 1, 2, \quad (3.7)$$

in the bare Lagrangian, or equivalently

$$h_B \rightarrow h_B + \Delta v_{h}^{\text{FJTS}}, \quad H_B \rightarrow H_B + \Delta v_{H}^{\text{FJTS}} \quad (3.8)$$

in the field basis corresponding to mass eigenstates. The constants $\Delta v_{1,2}^{\text{FJTS}}$ and $\Delta v_{h/H}^{\text{FJTS}}$ are related to each other like the corresponding fields,

$$\left( \begin{array}{c} \Delta v_{1}^{\text{FJTS}} \\ \Delta v_{2}^{\text{FJTS}} \end{array} \right) = R(\alpha) \left( \begin{array}{c} \Delta v_{1}^{\text{FJTS}} \\ \Delta v_{2}^{\text{FJTS}} \end{array} \right). \quad (3.9)$$

The field shifts lead to the terms $-\Delta v_{h} M_{h}^2 h - \Delta v_{H} M_{H}^2 H$ linear in the fields $h$, $H$ in the Lagrangian, so that the identification with the tadpole counterterms fixes the constants $\Delta v_{h/H}^{\text{FJTS}}$ to

$$\Delta v_{H}^{\text{FJTS}} = -\frac{\delta t_{H}}{M_{H}} = \frac{T^H}{M_{H}^2}, \quad \Delta v_{h}^{\text{FJTS}} = -\frac{\delta t_{h}}{M_{h}} = \frac{T^h}{M_{h}^2}. \quad (3.10)$$
We finally note that $\Delta v_{1,FJTS}$ is gauge independent because of the gauge invariance of $\langle \sigma_B \rangle$, but $\Delta v_{2,FJTS}$ is gauge dependent.

In the FJTS, the parameters $v_{n,0}$ correspond to the location of the minimum of the tree-level Higgs potential. To define the fields $\eta_{n,B}$, $h_B$, $H_B$ as excitations about the minimum of the effective Higgs potential, the shifts $\Delta v_{1,2,FJTS}$ and $\Delta v_{h/H}$ had to be introduced, which are a source of potentially large corrections in the FJTS if $\overline{\text{MS}}$ parameters are used.

**PRTS:**

In the PRTS, the identification of fields as excitations about the minimum of the effective Higgs potential is achieved without further field redefinition, i.e. with an appropriate definition of the bare parameters $t_{n,0}$ and $v_{n,0}$. The condition that the renormalized vev minimizes the effective Higgs potential implies that the sum of bare and loop-induced tadpole contributions vanish,

$$0 = t_{n,0} + T^{n_0}.$$  \hfill (3.11)

Splitting the bare tadpole parameters $t_{n,0} = t_{n} + \delta t^{PRTS}_{\eta_n}$ into a renormalized parts $t_{n}$ and a tadpole renormalization constants $\delta t^{PRTS}_{\eta_n}$ and using Eq. (3.3), this leads to

$$0 = t_{n} + \delta t^{PRTS}_{\eta_n} + T^{n_0} = t_{n},$$  \hfill (3.12)

i.e. the renormalized parts $t_{n}$ vanish. In turn, this means that the bare tadpole parameters directly provide the tadpole renormalization constants, $t_{n,0} = \delta t^{PRTS}_{\eta_n}$, which are given by

$$\delta t^{PRTS}_{\eta_1} = v_{1,0} \left( 2 \mu_1^2 - 4 \lambda_1 v_{1,0}^2 - \lambda_{12} v_{2,0}^2 \right),$$

$$\delta t^{PRTS}_{\eta_2} = v_{2,0} \left( \mu_2^2 - \frac{1}{4} \lambda_{22} v_{2,0}^2 - \lambda_{12} v_{1,0}^2 \right).$$  \hfill (3.13)

The bare vev parameters $v_{n,0} = v_{n} + \delta v_{n}$ are split into renormalized parts $v_{n}$ and renormalized constants $\delta v_{n}$ accordingly, where the renormalized parameters $v_{n}$ are directly related to measured quantities. As in the SM, $v_2$ is directly related to the W-boson mass and the gauge coupling $g_2$, while $v_1$ is related to the masses $M_1$, $M_H$ of the light and heavy Higgs bosons $h$ and $H$, the renormalized mixing angle $\alpha$, and to one of the scalar couplings $\lambda_k$, for which we choose $\lambda_{12}$ as in Ref. [11]. These relations are demanded both for bare and renormalized quantities,

$$v_{1,0} = \frac{s_{200} (M_H^2 - M_h^2)}{4 v_{2,0} \lambda_{12}}, \quad v_{2,0} = \frac{2 M_{W,0}}{g_2,0} = \frac{2 M_{W,0} s_{w,0}}{e_0},$$  \hfill (3.14)

$$v_1 = \frac{s_{200} (M_H^2 - M_h^2)}{4 v_2 \lambda_{12}}, \quad v_2 = \frac{2 M_W}{g_2} = \frac{2 M_{W} s_{w}}{e},$$  \hfill (3.15)

where $e$ is the elementary charge and $s_w$ the sinus of the weak mixing angle $\theta_w$, which is fixed by the ratio of the $W$-/$Z$-boson masses, $c_w = \cos \theta_w = M_W/M_Z$. The relation of $v_{n}$ to measured quantities which serve as input for the model then fixes the renormalization constants $\delta v_{n}$ via the renormalization conditions for the input quantities.
Having fixed the renormalization of the vev parameters $v_n$ and the tadpoles, the renormalization of the Higgs sector can be completed by splitting the bare versions of the original parameters of the Higgs potential, $\mu_{1,0}^2$, $\mu_{2,0}^2$, $\lambda_{1,0}$, $\lambda_{2,0}$, $\lambda_{12,0}$, into renormalized parameters and renormalization constants as usual. As done in Ref. [11], we take the parameters $M_h$, $M_H$, $\alpha$, $\lambda_{12}$, and $v_2 = 2M_W s_W/e$ as input to parametrize the Higgs sector, so that the corresponding renormalization constants are directly fixed by renormalization conditions. The corresponding bare parameters are tied to the bare original Higgs parameters via the diagonalization of the bare Higgs mass matrix. In total, the five relations providing the link between the two sets of bare parameters, $\mu_{1,0}^2$, $\mu_{2,0}^2$, $\lambda_{1,0}$, $\lambda_{2,0}$, $\lambda_{12,0}$ and $M_h$, $M_H$, $\alpha_0$, $\lambda_{12,0}$, $v_2$, are given by Eq. (3.13), the first relation in Eq. (3.14), and the following two relations,

$$\lambda_{1,0} = \frac{1}{8\delta t_{1,0}^PRTS} \left( c_{\alpha,0}^2 M_{H,0}^2 + M_{h,0}^2 s_{\alpha,0}^2 + \frac{\delta t_{PRTS}}{v_{1,0}} \right),$$

$$\lambda_{2,0} = \frac{2}{v_{2,0}} \left( c_{\alpha,0}^2 M_{h,0}^2 + M_{H,0}^2 s_{\alpha,0}^2 + \frac{\delta t_{PRTS}}{v_{2,0}} \right).$$

The corresponding relations between renormalized parameters are obtained by replacing the bare parameters by renormalized ones and by setting the tadpole constants $\delta t_{PRTS}$ to zero; these relations are given in Eq. (2.15) of Ref. [11]. The bare and renormalized relations, finally fix the renormalization constants $\delta \mu_{1}^2$, $\delta \mu_{2}^2$, $\delta \lambda_{1}$, $\delta \lambda_{2}$, $\delta \lambda_{12}$ in terms of $\delta M_{h}^2$, $\delta M_{H}^2$, $\delta \alpha$, $\delta \lambda_{12}$, $\delta v_2$ as given in Eq. (3.7) of Ref. [11]. From the results given there, we can deduce that the tadpole contributions to all counterterms can be obtained from the bare Lagrangian without tadpole terms by the substitutions

$$\mu_{1,0}^2 \to \mu_{1,0}^2 + \frac{3\delta t_{PRTS}^{\eta_1}}{4v_{1}}, \quad \mu_{2,0}^2 \to \mu_{2,0}^2 + \frac{3\delta t_{PRTS}^{\eta_2}}{2v_{2}}, \quad \lambda_{1,0} \to \lambda_{1,0} + \frac{\delta t_{PRTS}^{\eta_1}}{8v_{1}^2}, \quad \lambda_{2,0} \to \lambda_{2,0} + \frac{\delta t_{PRTS}^{\eta_2}}{v_{2}^2}, \quad \lambda_{12,0} \to \lambda_{12,0}.$$

In complete analogy to the SM, the PRTS induces gauge-dependent relations between the bare input parameters and the bare original parameters of the Higgs sector, leading to a gauge-dependent relation between predictions for observables and input parameters if $\overline{\text{MS}}$-renormalized masses or an $\overline{\text{MS}}$-renormalized Higgs mixing angle $\alpha$ are used.

**GIVS:**

As in the SM, the GIVS is fully equivalent to the formulation of the PRTS in the non-linear Higgs representation. In some slight abuse of notation, we keep the letters $H$, $h$ for the Higgs fields corresponding to mass eigenstates also in the non-linear Higgs representation. This should not cause confusion, since the fields $H$, $h$ of the non-linear representation are only appearing as labels for vertex functions $\Gamma_{nl}$ and tadpoles $T_{nl}$, where the non-linear representation is explicitly indicated. The PRTS tadpole renormalization constants of the non-linear representation are determined by the unrenormalized Higgs one-point functions,

$$\delta t_{H,nl}^{PRTS} = -T_{nl}^H, \quad \delta t_{h,nl}^{PRTS} = -T_{nl}^h.$$

(3.18)
Transferred to the linear representation, we keep the PRTS part of the tadpole renormalization constants, defining

\[ \delta t_{H,1}^{\text{GIVS}} = \delta t_{H,1}^{\text{PRTS}} = -T_{nl}^H, \quad \delta t_{h,1}^{\text{GIVS}} = \delta t_{h,1}^{\text{PRTS}} = -T_{nl}^h, \]  

and supplement them with FJTS parts,

\[ \delta t_{H,2}^{\text{GIVS}} = -M_H^2 \Delta v_H^{\text{GIVS}} = T_{nl}^H - T^H = -s_\alpha M_H^2 \Delta v_\xi, \]
\[ \delta t_{h,2}^{\text{GIVS}} = -M_h^2 \Delta v_h^{\text{GIVS}} = T_{nl}^h - T^h = -c_\alpha M_h^2 \Delta v_\xi, \]  

(3.19)

where we have used Eq. (2.21) with Eq. (2.23) for the differences \( T_{nl}^H - T^H \) and \( T_{nl}^h - T^h \). Specifically, we get

\[ \Delta v_H^{\text{GIVS}} = s_\alpha \Delta v_\xi, \quad \Delta v_h^{\text{GIVS}} = c_\alpha \Delta v_\xi. \]  

(3.20)

In total, we obtain the tadpole renormalization constants from the sum of the two respective parts in order to cancel all explicit tadpole diagrams,

\[ \delta t_{H}^{\text{GIVS}} = \delta t_{H,1}^{\text{GIVS}} + \delta t_{H,2}^{\text{GIVS}} = -T^H, \quad \delta t_{h}^{\text{GIVS}} = \delta t_{h,1}^{\text{GIVS}} + \delta t_{h,2}^{\text{GIVS}} = -T^h. \]  

(3.21)

In analogy to the SM, the tadpole contributions \( \delta t_{H,1}^{\text{GIVS}} \) and \( \delta t_{h,1}^{\text{GIVS}} \) enter relations between the bare parameters of the Higgs sector, causing no problems with gauge dependences, since these terms are gauge independent. On the other hand, the gauge-dependent parts \( \delta t_{H,2}^{\text{GIVS}} \) and \( \delta t_{h,2}^{\text{GIVS}} \) have no effects on predictions for observables, because they only enter the calculation via field shifts.

The generation of the two types of tadpole counterterms can be deduced from the generation of the PRTS and FJTS tadpole counterterms exactly as in the SM [21], resulting in the following rules to be applied to the bare Lagrangian,

\[ \mu_{1,0}^2 \rightarrow \mu_{1,0}^2 + \frac{3\delta t_{\eta_1,1}^{\text{GIVS}}}{4v_1^2}, \quad \mu_{2,0}^2 \rightarrow \mu_{2,0}^2 + \frac{3\delta t_{\eta_2,1}^{\text{GIVS}}}{2v_2^2}, \]
\[ \lambda_{1,0} \rightarrow \lambda_{1,0} + \frac{\delta t_{\eta_1,1}^{\text{GIVS}}}{8v_1^4}, \quad \lambda_{2,0} \rightarrow \lambda_{2,0} + \frac{2\delta t_{\eta_2,1}^{\text{GIVS}}}{v_2^4}, \quad \lambda_{12,0} \rightarrow \lambda_{12,0}, \]
\[ \eta_{1,B} \rightarrow \eta_{1,B} + \Delta v_1^{\text{GIVS}}, \quad \eta_{2,B} \rightarrow \eta_{2,B} + \Delta v_2^{\text{GIVS}}, \]  

(3.22)

where the renormalization constants of the field basis \( \eta_1, \eta_2 \) are related to the ones of the \( H, h \) basis as follows,

\[ \begin{pmatrix} \delta t_{\eta_1,1}^{\text{GIVS}} \\ \delta t_{\eta_2,1}^{\text{GIVS}} \end{pmatrix} = R(\alpha) \begin{pmatrix} \delta t_{H,1}^{\text{GIVS}} \\ \delta t_{h,1}^{\text{GIVS}} \end{pmatrix}, \]
\[ \begin{pmatrix} \Delta v_1^{\text{GIVS}} \\ \Delta v_2^{\text{GIVS}} \end{pmatrix} = R(\alpha) \begin{pmatrix} \Delta v_H^{\text{GIVS}} \\ \Delta v_h^{\text{GIVS}} \end{pmatrix} = -R(\alpha) \begin{pmatrix} \delta t_{H,2}^{\text{GIVS}}/M_H^2 \\ \delta t_{h,2}^{\text{GIVS}}/M_h^2 \end{pmatrix}. \]  

(3.23)
3.1.2 Mixing-angle renormalization

Finally, we address the \( \overline{\text{MS}} \) renormalization variants for the mixing angle \( \alpha \) with the different tadpole treatments if \( \alpha \) is used as one of the basic input parameters of the SESM. For the FJTS and PRTS we translate the results given in Refs. [11, 12, 37] into the conventions and notation used in this paper. In detail, we trade the original parameters \( \mu^2_1, \mu^2_2, \lambda_1, \lambda_2 \), and \( \lambda_{12} \) of the Higgs potential \( V \) given in Eq. (2.17) for the parameters \( v_2 \) (fixed by the unit charge \( e \) and the masses \( M_W \) and \( M_Z \)), \( M_h, M_H, s_\alpha \), and one of the quartic scalar couplings, which is either taken as \( \lambda_{12} \) as in Ref. [11] or as \( \lambda_1 \) as in Ref. [12]. Since the \( \overline{\text{MS}} \) renormalization of \( \lambda_{12} \) or \( \lambda_1 \) is independent of the tadpole scheme, we do not discuss the renormalization of those parameters below and refer to Refs. [11, 12] for details.

In the \( \overline{\text{MS}} \) scheme, the renormalization constant \( \delta \alpha \) for the angle \( \alpha \) can be determined according to

\[
\delta \alpha_{\overline{\text{MS}}} = \frac{1}{4} \left( \delta Z_{Hh} - \delta Z_{hH} \right)_{\text{UV}},
\]

where \( \delta Z_{ij} \) are the OS field renormalization constants for the \( H, h \) fields defined by

\[
\begin{pmatrix} H_B \\ h_B \end{pmatrix} = \begin{pmatrix} 1 + \frac{1}{2} \delta Z_{HH} & \frac{1}{2} \delta Z_{hH} \\ \frac{1}{2} \delta Z_{hH} & 1 + \frac{1}{2} \delta Z_{hh} \end{pmatrix} \begin{pmatrix} H \\ h \end{pmatrix},
\]

and the subscript “UV” indicates that in dimensional regularization only the UV-divergent parts proportional to \( \Delta \) as given in (2.25) are taken into account. Recall that the reference mass scale \( \mu \) takes over the role of the renormalization scale in \( \overline{\text{MS}} \) schemes. The constants \( \delta Z_{ij} \) are determined by the Higgs-boson self-energies \( \Sigma^{ij}(p^2) \) with momentum transfer \( p \) evaluated at the on-shell points \( p^2 = M_H^2 \) or \( p^2 = M_h^2 \). In the following, we decompose \( \Sigma^{ij}(p^2) = \Sigma_{1PI}^{ij}(p^2) + \Sigma_{\text{tad}}^{ij} \) into the contribution \( \Sigma_{1PI}^{ij}(p^2) \) induced by one-particle-irreducible (1PI) diagrams and the contribution \( \Sigma_{\text{tad}}^{ij} \) of all (momentum-independent) explicit tadpole diagrams and tadpole counterterms. In detail, the diagonal quantities \( \delta Z_{ii} \) involve only derivatives w.r.t. \( p^2 \) and thus do not receive tadpole contributions, but the mixing constants \( \delta Z_{Hh}, \delta Z_{hH} \) involve tadpole contributions\(^5\), which are the source for the different \( \overline{\text{MS}} \) results for \( \delta \alpha \) in the various tadpole schemes. The tadpole contributions to \( \delta \alpha_{\overline{\text{MS}}} \) are given by

\[
\delta \alpha_{\overline{\text{MS}}, \text{tad}} = \left. \frac{\Sigma_{\text{tad}}^{Hh}}{M_H^2 - M_h^2} \right|_{\text{UV}},
\]

where inserting all tadpole counterterms contributing to the \( hH \) mixing self-energy in the various tadpole schemes into Eq. (3.28), we find

\[
\delta \alpha_{\text{FJTS}} = \left. \frac{e(C_{hhH} \Delta v_{hH}^{\text{FJTS}} + C_{hhH} \Delta v_{hH}^{\text{FJTS}})}{M_H^2 - M_h^2} \right|_{\text{UV}},
\]

\(^5\)Following the conventions of Refs. [7, 12, 21], our unrenormalized one-loop self-energy functions \( \Sigma^{ij} \) include one-particle-irreducible diagrams as well as explicit tadpole loops and tadpole counterterm contributions, as pictorially indicated in Eq. (141) of Ref. [7].
\[ \delta \alpha_{\text{MS,tad}}^{\text{PRTS}} = 0, \quad (3.30) \]

\[ \delta \alpha_{\text{MS,tad}}^{\text{GIVS}} = \frac{e (c_a C_{hhH} + s_a C_{hHH}) \Delta \nu_c}{M_H^2 - M_h^2} \bigg|_{\text{UV}}, \quad (3.31) \]

with the shorthands

\[ C_{hhH} = \frac{s_a}{e} \left( 2M_h^2 + M_H^2 \right) \left( \frac{2v_2 \lambda_{12}}{M_H^2 - M_h^2} - \frac{c_a^2}{v_2} \right), \]

\[ C_{hHH} = \frac{c_a}{e} \left( M_h^2 + 2M_H^2 \right) \left( \frac{2v_2 \lambda_{12}}{M_H^2 - M_h^2} - \frac{s_a^2}{v_2} \right) \]

(3.32)

for the scalar self-couplings \( e(C_{hhH} h^2 H + C_{hHH} hH^2)/2 \) in the Lagrangian.

As already explained in Refs. [11, 12, 37], changing the tadpole treatment in the \( \overline{\text{MS}} \) renormalization of the mixing angle \( \alpha \) changes the physical meaning of \( \alpha \), similar to a change in its renormalization condition. This change leads to a finite shift in the numerical value of the renormalized parameter \( \alpha_{\text{TS}}^{\overline{\text{MS}}} \) in the tadpole scheme TS when TS is changed. This shift, which is related to the finite parts of tadpole contributions, was calculated in Ref. [37] by considering NLO corrections to specific vertices. In Ref. [11] arguments for a universal relation between \( \alpha_{\text{TS}}^{\overline{\text{MS}}} \) and \( \alpha_{\text{FJTS}}^{\overline{\text{MS}}} \) were given, based on the observation that the FJTS is equivalent to a procedure in which no tadpole renormalization is performed at all.

In the following we derive the general relation between the input parameters of the SESM when changing the tadpole scheme, again basing the arguments on the equivalence of the FJTS and performing no tadpole renormalization.

While the bare original Higgs parameters \( \mu_{1,0}^2, \mu_{2,0}^2, \lambda_{1,0}, \lambda_{2,0}, \lambda_{12,0} \) are the same in the FJTS and PRTS, this is in general not the case between the derived bare parameters \( M_{h,0}, M_{H,0}, \alpha_0, \lambda_{12,0} \) (or \( \lambda_{1,0} \)), and \( v_{2,0} \) corresponding to the input parameters. This is due to the fact that tadpole terms \( \delta t_{\text{PRTS}} \) appear in the relations linking the two sets of bare parameters in the PRTS, which do not appear in the FJTS. As pointed out in the previous section, these PRTS tadpole terms can be introduced into the relations without tadpole terms, which are valid in the FJTS, via the substitutions (3.17). Applying this consideration to the bare mixing angle \( \alpha_0 \), we get the one-loop relation

\[ \alpha_0^{\text{FJTS}} = \alpha_0 \bigg|_{(3.17), p_0 \rightarrow p_0^{\text{PRTS}}}, \]

(3.33)

which means that the substitution (3.17) is applied to \( \alpha_0 \) expressed in terms of the bare parameters \( \{p_0\} = \{\mu_{1,0}^2, \mu_{2,0}^2, \lambda_{1,0}, \lambda_{2,0}, \lambda_{12,0}\} \) in the absence of tadpole terms. Analogous substitutions hold for the other bare parameters such as the Higgs masses \( M_{h,0} \) and \( M_{H,0} \).

For the mixing angle \( \alpha_0 \) the explicit result of the substitution is

\[ \alpha_0^{\text{PRTS}} = \alpha_0^{\text{FJTS}} + \frac{e}{M_H^2 - M_h^2} \left( C_{hhH} \frac{\delta t_{h}^{\text{FJTS}}}{M_h^2} + C_{hHH} \frac{\delta t_{H}^{\text{FJTS}}}{M_H^2} \right), \]

(3.34)

the derivation of which is somewhat cumbersome but straightforward. The relation between the two renormalized parameters \( \alpha_{\text{PRTS}}^{\text{TS}} \) and \( \alpha_{\text{FJTS}}^{\text{TS}} \) simply follows from the renormalization transformation

\[ \alpha_0^{\text{TS}} = \alpha_0^{\text{TS}} + \delta \alpha_0^{\text{TS}} \]

(3.35)
with TS being the PRTS or FJTS. The shift between the $\overline{\text{MS}}$-renormalized parameters in the PRTS and FJTS, thus, reads
\[
\alpha_{\text{PRTS}}^{\overline{\text{MS}}} - \alpha_{\text{FJTS}}^{\overline{\text{MS}}} = (\alpha_{0}^{\text{PRTS}} - \alpha_{0}^{\text{FJTS}}) - (\delta\alpha_{\text{PRTS}}^{\overline{\text{MS}}} - \delta\alpha_{\text{FJTS}}^{\overline{\text{MS}}})
\] (3.36)
and can be explicitly calculated using Eq. (3.34) and the difference between the corresponding renormalization constants $\delta\alpha_{\text{TS}}^{\overline{\text{MS}}}$. As explained above, the constants $\delta\alpha_{\text{TS}}^{\overline{\text{MS}}}$ receive contributions from the 1PI self-energy part $\Sigma_{\text{1PI}}^{Hh}$ and from tadpole loops and tadpole renormalization constants. Since the self-energy contributions do not depend on the tadpole scheme, the difference $\delta\alpha_{\text{PRTS}}^{\overline{\text{MS}}} - \delta\alpha_{\text{FJTS}}^{\overline{\text{MS}}}$ in Eq. (3.36) can be calculated from the tadpole contributions given in Eqs. (3.30) and (3.29) alone. Combining all those parts, we finally get
\[
\alpha_{\text{PRTS}}^{\overline{\text{MS}}} - \alpha_{\text{FJTS}}^{\overline{\text{MS}}} = -\frac{e}{M_{H}^{2} - M_{h}^{2}} \left( C_{hhH} \frac{T_{h}}{M_{h}^{2}} + C_{hHH} \frac{T^{H}}{M_{H}^{2}} \right) \bigg|_{\text{finite}},
\] (3.37)
where the subscript “finite” indicates that the UV-divergent part proportional to the parameter $\Delta$ defined in Eq. (2.25) is omitted.

The conversion between the FJTS and the GIVS proceeds along the same lines. In Eqs. (3.34)–(3.36) we merely have to change the label PRTS to GIVS on all parameters and to replace the tadpole renormalization constants $\delta t_{\text{PRTS}}^{X}$ by $\delta t_{\text{GIVS}}^{X}$, because only part 1 of $\delta t_{\text{GIVS}}^{X}$ enters the relations between bare parameters in the GIVS. This finally results in
\[
\alpha_{\text{GIVS}}^{\overline{\text{MS}}} - \alpha_{\text{FJTS}}^{\overline{\text{MS}}} = -\frac{e}{M_{H}^{2} - M_{h}^{2}} \left( C_{hhH} \frac{T_{h}}{M_{h}^{2}} + C_{hHH} \frac{T^{H}}{M_{H}^{2}} \right) \bigg|_{\text{finite}},
\] (3.38)
which is identical to the conversion (3.37) between the PRTS and the FJTS except that the tadpole contributions are calculated in the non-linear Higgs representation, rendering the relation between $\alpha_{\text{GIVS}}^{\overline{\text{MS}}}$ and $\alpha_{\text{FJTS}}^{\overline{\text{MS}}}$ gauge independent.

### 3.1.3 NLO decay widths for h/H → 4f in the SESM

In Ref. [11], the renormalization of the SESM is described in detail for an $\overline{\text{MS}}$-renormalized Higgs mixing angle $\alpha$, both with the PRTS and the FJTS for treating tadpoles. In Ref. [12] other renormalization schemes based on OS conditions or prescriptions inspired by symmetries are formulated in addition. In those articles, explicit NLO predictions for the important class of Higgs decay processes $h/H \rightarrow WW/ZZ \rightarrow 4f$ are discussed, in particular the renormalization scale and renormalization scheme dependences of these decay widths for some selected SESM scenarios. The underlying calculations are the basis for the implementation of these predictions in the public Monte Carlo program PROPHECY4F 3.0 [30–32].

Here we continue the discussion of Refs. [11, 12] by adding results for the scheme with an $\overline{\text{MS}}$-renormalized mixing angle $\alpha$ with GIVS tadpole treatment. To this end, we have implemented this new scheme into PROPHECY4F as new option.\textsuperscript{6} The numerical input for

\textsuperscript{6}An updated public version of PROPHECY4F will appear on https://prophecy4f.hepforge.org/ soon; meanwhile a non-public version of the program may be obtained from the authors on request.
The scenarios called BHM200±, BHM400, BHM600 as well as the details of the calculational setup can be found in Refs. [11, 12]. We just recall that the mass of the lighter Higgs boson is set to $M_H = 125.1$ GeV, the mass of the heavier is indicated in the name of the scenario (i.e. $M_H = 200$ GeV for BHM200±, etc.), and the mixing angle $\alpha$ is chosen as $\pm 0.29, 0.26$, and 0.22 in BHM200±, BHM400, and BHM600, respectively, so that $\alpha$ is of the order of magnitude that is maximally allowed by LHC Higgs analyses. Results of the scheme conversion of the input values for the MS parameters $\alpha$ and $\lambda_{12}$ are given in App. A.1. As in Refs. [11, 12], the PRTS is evaluated in 't Hooft–Feynman gauge ($\xi_a = 1$).

Tables 1 and 2 summarize the NLO predictions for the partial decay widths of $h \rightarrow 4f$ and $H \rightarrow 4f$, respectively, based on the various $\overline{\text{MS}}$ schemes using the OS scheme as reference scheme in which the input parameters are defined. The values for the widths in the OS and $\overline{\text{MS}}$ schemes with PRTS or FJTS tadpole treatment are taken from Tabs. 4

| Ren. scheme | tadpoles | \( \text{BHM200}^+ \) | \( \text{BHM200}^- \) | \( \text{BHM400} \) | \( \text{BHM600} \) |
|-------------|----------|------------------|------------------|------------------|---------------------|
|             |          | LO               | NLO              | LO               | NLO                 |
| OS          |          | 0.84034(3)       | 0.90553(6)       | 0.84034(3)       | 0.90552(6)          |
| MS          | FJTS     | 0.82292(3)+.3%   | 0.90550(7)+.7%   | 0.82614(3)+.6%   | 0.90558(7)+.0%      |
|             |          | $\Delta_{\text{OS}} = +0.07\%$ | $\Delta_{\text{OS}} = -0.20\%$ | $\Delta_{\text{OS}} = +0.09\%$ | $\Delta_{\text{OS}} = -0.00\%$ |
| MS          | PRTS     | 0.83361(3)+.4%   | 0.90539(6)+.5%   | 0.83261(3)+.3%   | 0.90546(7)+.5%      |
|             |          | $\Delta_{\text{OS}} = +0.80\%$ | $\Delta_{\text{OS}} = -0.02\%$ | $\Delta_{\text{OS}} = -0.92\%$ | $\Delta_{\text{OS}} = -0.01\%$ |
| MS          | GIVS     | 0.83440(3)+.3%   | 0.90540(6)+.5%   | 0.83363(3)+.4%   | 0.90554(6)+.6%      |
|             |          | $\Delta_{\text{OS}} = -0.71\%$ | $\Delta_{\text{OS}} = +0.01\%$ | $\Delta_{\text{OS}} = -0.08\%$ | $\Delta_{\text{OS}} = -0.01\%$ |

**Table 1:** LO and NLO decay widths $\Gamma^{h \rightarrow 4f}[\text{MeV}]$ of the light SESM Higgs boson $h$ for various SESM scenarios in different renormalization schemes, with the OS scheme as input scheme (and full conversion of the input parameters into the other schemes). The scale variation (given in percent as sub- and superscripts) corresponds to the scales $\mu = \mu_0/2$ and $\mu = 2\mu_0$ with central scale $\mu_0 = M_H$. The quantity $\Delta_{\text{OS}} = \Gamma_{\overline{\text{MS}}} / \Gamma_{\text{OS}} - 1$ shows the relative difference between the $\overline{\text{MS}}$ predictions and the OS value; its spread illustrates the scheme dependence of the prediction at LO and NLO.
and 5 of Ref. [12], i.e. the $\overline{MS}$ results with GIVS tadpole treatment extend the results of Ref. [12]. For completeness, we mention that those results are based on the set of independent parameters containing a running $\lambda_1$, not $\lambda_{12}$, but the scenarios are still defined via the values for $\lambda_{12}$. Interchanging the roles of $\lambda_1$ and $\lambda_{12}$ as running parameters has only a marginal effect on the presented numbers.

Since the input values of the mixing angle $\alpha$ and the parameter $\lambda_{12}$ are properly converted from the OS scheme to the other $\overline{MS}$ schemes, ideal predictions would not show any residual dependence on the renormalization scale $\mu$ and on the renormalization scheme.\footnote{In the OS scheme, there is a tiny scale dependence in the NLO predictions resulting from the $\overline{MS}$ definition of the scalar coupling $\lambda_1$, which formally enters beyond NLO.} An estimate for the scheme dependence is given by the spread of the quantity $\Delta_{OS} = \Gamma_{\overline{MS}}/\Gamma_{OS} - 1$, which shows the relative difference between the $\overline{MS}$ predictions and the OS value for the integrated widths. The results show a significant reduction of the scheme dependence in the transition from LO to NLO, with a residual scheme dependence mostly of the naively expected size of $< 0.1\%$ for $h \to 4f$ and of $\lesssim 0.5\%$ for $H \to 4f$.

Similar comments apply to the residual renormalization scale dependence of the predictions in the $\overline{MS}$ schemes, which is, however, generally larger than the uncertainties due

| Ren. scheme | tadpoles | $\text{BHM200}^+$ | | \text{BHM200}^- | |
|-------------|-----------|-----------------|----------------|-----------------|
|             |           | LO | NLO | LO | NLO |
| OS          |           | 109.430(4) | 119.84(8)$^{+0.9\%}_{-0.6\%}$ | 109.430(4) | 119.812(8)$^{+0.6\%}_{-0.0\%}$ |
| $\overline{MS}$ | FJTS | 134.126(4)$^{+30.1\%}_{-23.2\%}$ | $\Delta_{OS} = +22.6\%$ | 129.570(4)$^{+5.6\%}_{-6.0\%}$ | $\Delta_{OS} = +18.4\%$ |
| $\overline{MS}$ | PRTS | 118.976(4)$^{+39.9\%}_{-43.6\%}$ | $\Delta_{OS} = +8.7\%$ | 120.307(4)$^{+32.4\%}_{-44.6\%}$ | $\Delta_{OS} = +10.0\%$ |
| $\overline{MS}$ | GIVS | 117.847(4)$^{+32.5\%}_{-44.7\%}$ | $\Delta_{OS} = +7.7\%$ | 118.947(4)$^{+34.1\%}_{-46.0\%}$ | $\Delta_{OS} = +8.7\%$ |

| Ren. scheme | tadpoles | $\text{BHM400}$ | $\text{BHM600}$ |
|-------------|-----------|-----------------|-----------------|
|             |           | LO | NLO | LO | NLO |
| OS          |           | 1533.42(4) | 1643.86(8)$^{+0.0\%}_{+0.0\%}$ | 4295.9(1) | 4532.4(2)$^{+0.0\%}_{+0.0\%}$ |
| $\overline{MS}$ | FJTS | 1582.44(4)$^{+27.6\%}_{-21.7\%}$ | $\Delta_{OS} = +3.2\%$ | 4007.1(1)$^{+32.5\%}_{-24.8\%}$ | $\Delta_{OS} = -6.7\%$ |
| $\overline{MS}$ | PRTS | 1617.26(4)$^{+9.2\%}_{+6.3\%}$ | $\Delta_{OS} = +5.5\%$ | 4530.1(1)$^{+2.3\%}_{+2.1\%}$ | $\Delta_{OS} = +5.5\%$ |
| $\overline{MS}$ | GIVS | 1609.86(4)$^{+6.6\%}_{+6.7\%}$ | $\Delta_{OS} = +5.0\%$ | 4511.4(1)$^{+2.6\%}_{+2.4\%}$ | $\Delta_{OS} = +5.0\%$ |

Table 2: As in Table 1, but for the decay width $\Gamma^{H \to 4f}[\text{MeV}]$ of the heavy SESM Higgs boson H.
to scheme dependence. The seemingly higher stability of the FJTS w.r.t. scale variations in the BHM200$^-$ scenario is accidental, as can be seen from the more detailed discussion of scale variations in Ref. [11]. Systematic differences between the $\overline{\text{MS}}$ predictions based on the various tadpole schemes are only visible for the decay width of the heavy H boson with high masses $M_H = 400$ GeV and $600$ GeV. The larger scale dependence of the FJTS numbers reflect the onset of the potentially larger corrections in this scheme, which were already mentioned in the introduction. Such enhanced corrections in the FJTS are also expected from the different decoupling properties of a heavy H boson, depending on the tadpole scheme, as worked out in Ref. [46].$^8$ While decoupling is predicted in the $\overline{\text{MS}}$ scheme with PRTS tadpole treatment and in the OS scheme, in the FJTS scheme the NLO corrections do not go over into the SM prediction, but receive contributions enhanced by terms $\propto \ln(M_H^2/\mu^2)$. The overall picture of the shown results, however, demonstrates perturbative stability of all schemes for the considered scenarios, with the residual scale dependence indicating a realistic estimate of missing higher-order corrections. We, finally, emphasize that the new GIVS results are very close to the earlier PRTS results from Refs. [11, 12], as already expected from the analytical results for the tadpole corrections.

3.2 Two-Higgs-Doublet Model

3.2.1 Schemes for tadpole and vacuum expectation value renormalization

Several different renormalization procedures for the THDM have been proposed in the literature [8–10, 12, 37, 41–44], where the most important differences concern the renormalization conditions imposed on the two mixing angles $\alpha$ and $\beta$. As in the SESM considered above, Ref. [12] compares the strengths and weaknesses of the different types of renormalizations, such as renormalization schemes for $\alpha$ and $\beta$ based on OS, $\overline{\text{MS}}$, or symmetry-inspired conditions, including in particular the $\overline{\text{MS}}$ renormalization variants with tadpoles treated in the FJTS or PRTS (see also Refs. [10, 37]).

The bare vev parameters $v_{n,0}$, which quantify the vevs of the bare doublets

$$\Phi_{n,B} = \left(\frac{1}{\sqrt{2}}(v_{n,0} + \eta_{n,B} + i\chi_{n,B})\right), \quad n = 1, 2, \quad (3.39)$$

will again be specified in the context of tadpole renormalization. Inserting this field decomposition into the bare Lagrangian $\mathcal{L}$, produces the tadpole contribution $t_{\eta_{1,0}}\eta_1 + t_{\eta_{2,0}}\eta_2$ in the Lagrangian $\mathcal{L}$ with

$$t_{\eta_{1,0}} = -\frac{v_{1,0}}{2} (2m_{11,0}^2 + \lambda_{1,0}v_{1,0}^2) + \frac{v_{1,0}}{2} (2m_{12,0}^2 - \lambda_{345,0}v_{1,0}v_{2,0}),$$
$$t_{\eta_{2,0}} = -\frac{v_{2,0}}{2} (2m_{22,0}^2 + \lambda_{2,0}v_{2,0}^2) + \frac{v_{1,0}}{2} (2m_{12,0}^2 - \lambda_{345,0}v_{1,0}v_{2,0}). \quad (3.40)$$

$^8$To be precise, the heavy field $H$ was integrated out in Ref. [46] in the non-linear representation of the Higgs doublet, so that the PRTS of Ref. [46] is identical to the GIVS proposed in this paper. However, in that publication only potential non-decoupling effects were considered, i.e. terms in the effective Lagrangian that do not vanish for $M_H \rightarrow \infty$ (with $s_\alpha$ scaling like $1/M_H$), and in this order of the $1/M_H$ expansion there is no difference between the PRTS and the GIVS in the SESM anyhow.
In all versions for tadpole renormalization considered here, tadpole counterterms $\delta t_{n_1} + \delta t_{n_2}$ will be generated in the counterterm Lagrangian $\delta \mathcal{L}$ with tadpole renormalization constants $\delta t_{n_0}$ which cancel the explicit tadpole loop contributions $T_{n_0}$ as expressed in Eq. (3.3), or in its equivalent form (3.5) in the field basis $H, h$ with the transition given in Eq. (3.4). As described for the SESM in Section 3.1 in detail, the PRTS, FJTS, and GIVS generate the tadpole renormalization constants $\delta t_{n_0}$ in different ways out of the bare tadpole constants $t_{n_0,0}$ and possible field redefinitions. The procedure in the THDM follows the same pattern as in the SESM, so that we spell out only the most salient steps in the following.

**FJTS:**
In the FJTS, the bare tadpole constants $t_{n_0,0}$ are set to zero in Eq. (3.40). The tadpole counterterms $\delta t_{n_1} + \delta t_{n_2}$ are generated by the field shifts given in Eq. (3.7), or equivalently by the shifts (3.8) in the field basis of $h, H$. The constants $\Delta v_{\text{FJTS}}$ and $\Delta v_{h/H}$ are again related to each other as given in Eq. (3.9) and related to the tadpole renormalization constants $\delta t_{h/H}$ and the loop-induced tadpole constants $T_{h/H}$ as given in Eq. (3.10).

**PRTS:**
In the PRTS, the bare tadpole constants $t_{n_0,0} = t_{n_0} + \delta t_{n_0}^{\text{PRTS}}$ are again split into the renormalized tadpole constants $t_{n_0}$, which are demanded to vanish, and the tadpole renormalization constants $\delta t_{n_0}$, which by virtue of Eq. (3.40) are given by

$$\delta t_{n_1}^{\text{PRTS}} = -\frac{v_{1,0}^2}{2} \left(2m_{11,0}^2 + \lambda_{1,0} v_{1,0}^2\right) + \frac{v_{2,0}^2}{2} \left(2m_{12,0}^2 - \lambda_{345,0} v_{1,0} v_{2,0}\right) = -T_{n_1},$$

$$\delta t_{n_2}^{\text{PRTS}} = -\frac{v_{2,0}^2}{2} \left(2m_{22,0}^2 + \lambda_{2,0} v_{2,0}^2\right) + \frac{v_{1,0}^2}{2} \left(2m_{12,0}^2 - \lambda_{345,0} v_{1,0} v_{2,0}\right) = -T_{n_2}. \quad (3.41)$$

Again, all tadpole terms translate into the $h, H$ field basis by the rotation with the angle $\alpha$ as specified in Eq. (3.4).

The bare vev parameters $v_{n,0} = v_n + \delta v_n$ are decomposed into renormalized parameters $v_n$ and renormalization constants as usual. The vev parameters $v_n$ are translated into the parameters $v$ and $\beta$ which are closely related to input parameters,

$$v_{1,0}^2 + v_{2,0}^2 = v_0^2 = \frac{4M_W^2}{g_2^2} = \frac{4M_W^2}{g_2^2} \frac{s_w^2}{c_w^2}, \quad \tan \beta_0 = t_{\beta,0} = \frac{v_{2,0}}{v_{1,0}}, \quad (3.42)$$

$$v_1^2 + v_2^2 = v^2 = \frac{4M_W^2}{g_2^2} = \frac{4M_W^2}{g_2^2} \frac{s_w^2}{c_w^2}, \quad \tan \beta = t_{\beta} = \frac{v_2}{v_1}. \quad (3.43)$$

The renormalization constants $\delta v_n$ are, thus, tied to the renormalization constants $\delta M_W^2$, $\delta s_w$, $\delta Z$, and $\delta \beta$, which are directly fixed by renormalization conditions of the respective parameters.

Equations (3.41) and (3.42) represent four independent relations between the eight bare original parameters of the Higgs potential $m_{11,0}^2, m_{22,0}^2, m_{12,0}^2, \lambda_{1,0}, \ldots, \lambda_{5,0}$, and the parameters that serve as phenomenological input of the THDM. For the latter, we follow Refs. [10, 12, 29] and take the parameters $v_0 = 2M_{W,0}/g_2, \alpha_0, \beta_0, M_{h,0}, M_{H,0}, M_{A,0}$,
\(M_{H^\pm,0}\), and \(\lambda_{5,0}\). The four missing relations follow from the diagonalization of the mass matrices in the Higgs sector and can, e.g., be written as

\[
M_{h,0}^2 = \frac{2c_{\alpha-\beta,0}}{s_{\beta,0}} m_{12,0}^2 + \frac{v_0^2}{2} \left( 2\lambda_{1,0} c_{\beta,0}^2 s_{\alpha,0}^2 + 2\lambda_{2,0} c_{\alpha,0}^2 s_{\beta,0}^2 - s_{2\alpha,0} s_{2\beta,0} \lambda_{345,0} \right) \\
+ t_{\eta_1,0} s_{\alpha,0}^2 / v_0 c_{\beta,0} + t_{\eta_2,0} c_{\alpha,0}^2 / v_0 s_{\beta,0},
\]

\[
M_{H,0}^2 = \frac{2s_{\alpha-\beta,0}}{s_{\beta,0}} m_{12,0}^2 + \frac{v_0^2}{2} \left( 2\lambda_{1,0} s_{\beta,0}^2 c_{\alpha,0}^2 + 2\lambda_{2,0} s_{\alpha,0}^2 c_{\beta,0}^2 + s_{2\alpha,0} s_{2\beta,0} \lambda_{345,0} \right) \\
+ t_{\eta_1,0} c_{\alpha,0}^2 / v_0 c_{\beta,0} + t_{\eta_2,0} s_{\alpha,0}^2 / v_0 s_{\beta,0},
\]

\[
M_{A,0}^2 = \frac{2m_{12,0}^2}{s_{\beta,0}} - \lambda_{5,0} v_0^2 + t_{\eta_1,0} c_{\beta,0}^2 / v_0 c_{\beta,0} + t_{\eta_2,0} c_{\beta,0}^2 / v_0 s_{\beta,0},
\]

\[
M_{H^\pm,0}^2 = \frac{2m_{12,0}^2}{s_{\beta,0}} - \frac{v_0^2}{2} \left( \lambda_{4,0} + \lambda_{5,0} \right) + t_{\eta_1,0} s_{\beta,0} / v_0 c_{\beta,0} + t_{\eta_2,0} c_{\beta,0} / v_0 s_{\beta,0},
\]

(3.44)

which can be directly read from Eqs. (2.16a/b), (2.17a), and (2.18a) of Ref. [10]. These eight bare relations carry over to eight renormalized relations upon replacing the bare quantities by the respective renormalized ones and setting all tadpole terms to zero. The bare and renormalized relations then determine the renormalization constants \(\delta m_{11}^2\), \(\delta m_{22}^2\), \(\delta m_{12}^2\), \(\delta \lambda_1\), \ldots, \(\delta \lambda_5\) of the original parameters in terms of the renormalization constants \(\delta v\), \(\delta \alpha\), \(\delta \beta\), \(\delta M_h\), \(\delta M_H\), \(\delta M_A\), \(\delta M_{H^\pm}\), and \(\delta \lambda_5\). From the tadpole renormalization constants contained in these relations, we can, for instance, read off the substitutions

\[
m_{11,0}^2 \to m_{11,0}^2 - \frac{\left(9 + 4c_{2\beta} - c_{4\beta}\right) \delta t_{t_1}^{\text{PRTS}}}{8c_{\beta} v} + \frac{c_{2\beta} s_{\beta} \delta t_{t_2}^{\text{PRTS}}}{v},
\]

\[
m_{22,0}^2 \to m_{22,0}^2 + \frac{c_{\beta} s_{\beta} \delta t_{t_1}^{\text{PRTS}}}{v} - \frac{\left(9 - 4c_{2\beta} - c_{4\beta}\right) \delta t_{t_2}^{\text{PRTS}}}{8s_{\beta} v},
\]

\[
m_{12,0}^2 \to m_{12,0}^2 + \frac{s_{\beta} \delta t_{t_1}^{\text{PRTS}}}{v} + \frac{c_{\beta} \delta t_{t_2}^{\text{PRTS}}}{v},
\]

\[
\lambda_{1,0} \to \lambda_{1,0} + \frac{\left(3 - c_{2\beta}\right) \delta t_{t_1}^{\text{PRTS}}}{2c_{\beta} v^3} - \frac{s_{\beta} \delta t_{t_2}^{\text{PRTS}}}{v^3},
\]

\[
\lambda_{2,0} \to \lambda_{2,0} - \frac{c_{\beta} \delta t_{t_1}^{\text{PRTS}}}{v^3} + \frac{\left(3 + c_{2\beta}\right) \delta t_{t_2}^{\text{PRTS}}}{2s_{\beta} v^3},
\]

\[
\lambda_{3,0} \to \lambda_{3,0} + \frac{s_{\beta} \delta t_{t_1}^{\text{PRTS}}}{c_{\beta} v^3} + \frac{c_{\beta} \delta t_{t_2}^{\text{PRTS}}}{s_{\beta} v^3}, \quad \lambda_{4,0} \to \lambda_{4,0}, \quad \lambda_{5,0} \to \lambda_{5,0},
\]

(3.45)

which generate the PRTS tadpole counterterms out of the bare Lagrangian with vanishing tadpole terms.

Again, owing to the gauge dependences in the relations among bare input parameters and the bare original parameters of the Higgs sector, the PRTS leads to a gauge-dependent relation between predictions for observables and input parameters if \(\overline{\text{MS}}\)-renormalized masses or \(\overline{\text{MS}}\)-renormalized Higgs mixing angles \(\alpha, \beta\) are used.
Formally, the construction of the GIVS tadpole renormalization constants for the fields $H, h$ follows exactly the same pattern as in the SESM, described in the previous section, i.e. Eqs. (3.18)–(3.20) and (3.22) literally apply in the THDM as well. Again, the gauge-independent constants $\delta t_{GIVS}^{H,I}$ and $\delta t_{GIVS}^{h}$ enter relations between the bare parameters of the Higgs potential, the gauge-dependent parts $\delta t_{H,2}^{GIVS}$ and $\delta t_{h,2}^{GIVS}$ have no effects on predictions for observables, and the full tadpole constants $\delta t_{GIVS}^{H}$ and $\delta t_{GIVS}^{h}$ exactly cancel all explicit tadpole diagrams by construction. Since the rotation (2.19) between the $\eta_1, \eta_2$ and $H, h$ field bases is formally the same in the SESM and THDM, the translation (3.24) of the tadpole constants into the $\eta_1, \eta_2$ basis holds in the THDM without change. In the determination of the renormalization constants of the mixing angles $\alpha$ and $\beta$ below, the explicit results for $\delta t_{H,2}^{GIVS}$ and $\delta t_{h,2}^{GIVS}$ will be very useful,

\[
\begin{align*}
\Delta v^\text{GIVS}_H &= -\frac{\delta t_{H,2}^{GIVS}}{M_H^2} = \frac{T^H - T_{nl}^H}{M_H^2} = c_{\beta - \alpha} \Delta v_\xi, \\
\Delta v^\text{GIVS}_h &= -\frac{\delta t_{h,2}^{GIVS}}{M_h^2} = \frac{T^h - T_{nl}^h}{M_h^2} = s_{\beta - \alpha} \Delta v_\xi.
\end{align*}
\]

To derive these relations, the tree-level relations between the Higgs masses $M_H, M_h$ and the original parameters of the THDM Higgs potential have been used.

What of course changes in the THDM w.r.t. SESM, is the generation of tadpole counterterms from the bare Lagrangian because of the completely different form of the Higgs potential. The generation of the counterterms connected with $\delta t_{nl}^{GIVS}$ follow from the ones for $\delta t_{nl}^{PRTS}$, as given in Eq. (3.45), while the $\delta t_{nl}^{GIVS}$ again originate from field shifts,

\[
\begin{align*}
m^2_{11,0} \to m^2_{11,0} - \frac{(9 + 4c_2\beta - c_4\beta)\delta t^{GIVS}_{nl,1}}{8c_3\beta^2} + \frac{c_2^2\beta^2\delta t^{GIVS}_{nl,1}}{v}, \\
m^2_{22,0} \to m^2_{22,0} + \frac{c_3^2\beta^2\delta t^{GIVS}_{nl,3}}{v} - \frac{(9 - 4c_2\beta - c_4\beta)\delta t^{GIVS}_{nl,3}}{8s_2\beta^2}, \\
m^2_{12,0} \to m^2_{12,0} + \frac{s_2^2\beta^2\delta t^{GIVS}_{nl,1}}{v} + \frac{c_2^2\beta^2\delta t^{GIVS}_{nl,3}}{v}, \\
\lambda_{1,0} \to \lambda_{1,0} + \frac{(3 - c_2\beta)\delta t^{GIVS}_{nl,1}}{2c_3\beta^2} - \frac{s_2\beta^2\delta t^{GIVS}_{nl,3}}{v^3}, \\
\lambda_{2,0} \to \lambda_{2,0} - \frac{c_2^2\beta^2\delta t^{GIVS}_{nl,1}}{v^3} + \frac{(3 + c_2\beta)\delta t^{GIVS}_{nl,3}}{2s_2\beta^2}, \\
\lambda_{3,0} \to \lambda_{3,0} + \frac{s_2^2\delta t^{GIVS}_{nl,1}}{c_3^2\beta^2} - \frac{c_2^2\delta t^{GIVS}_{nl,1}}{s_2\beta^2}, \\
\eta_{1,B} \to \eta_{1,B} + \Delta \xi^{GIVS}, \\
\eta_{2,B} \to \eta_{2,B} + \Delta \xi^{GIVS}.
\end{align*}
\]

### 3.2.2 Mixing-angle renormalization

The $\overline{MS}$ renormalization constant $\delta \alpha_{\overline{MS}} = \alpha_0 - \alpha$ of $\alpha$ can be determined as in Eq. (3.25) from the OS field renormalization constants $\delta Z_{ij}$ of the $h/H$ system exactly as in the
SES M. As a result, Eq. (3.28) holds in the THDM as well. Inserting all tadpole loop and counterterm contributions into the $hH$ self-energy in Eq. (3.28), we get

$$\delta\Omega^{\text{FJTS}}_{\text{MS, tad}} = \frac{e (C_{hhH} \Delta v_H^{\text{FJTS}} + C_{hhH} \Delta v_h^{\text{FJTS}})}{M_H^2 - M_h^2} |_{\text{UV}},$$

$$\delta\Omega^{\text{PRTS}}_{\text{MS, tad}} = 0,$$

$$\delta\Omega^{\text{GIVS}}_{\text{MS, tad}} = \frac{e (s_{\beta - \alpha} C_{hhH} + c_{\beta - \alpha} C_{hHH}) \Delta v_H}{M_H^2 - M_h^2} |_{\text{UV}},$$

with the shorthands

$$C_{hhH} = \frac{c_{\beta - \alpha}}{\epsilon s_{2\beta}} \left[ (3s_{2\alpha} - s_{2\beta}) \left( M_A^2 + v^2 \lambda_3 \right) - s_{2\alpha} (2M_h^2 + M_H^2) \right],$$

$$C_{hHH} = \frac{s_{\beta - \alpha}}{\epsilon s_{2\beta}} \left[ -(3s_{2\alpha} + s_{2\beta}) \left( M_A^2 + v^2 \lambda_3 \right) + s_{2\alpha} (2M_h^2 + 2M_H^2) \right]$$

for the scalar self-couplings $e (C_{hhH} h^2 H + C_{hHH} hH^2)/2$ in the Lagrangian.

The $\overline{\text{MS}}$ renormalization constant $\delta \beta_{\overline{\text{MS}}} = \beta_0 - \beta$ of $\beta$ can be obtained according to

$$\delta \beta_{\overline{\text{MS}}} = \frac{1}{4} \left( \delta Z_{G_0 A_0} - \delta Z_{A_0 G_0} \right) |_{\text{UV}},$$

from the field renormalization constants $\delta Z_{ij}$ of the $G_0/A_0$ system defined by

$$\begin{pmatrix} G_{0,B} \\ A_{0,B} \end{pmatrix} = \begin{pmatrix} 1 + \frac{1}{2} \delta Z_{G_0 G_0} & \frac{1}{2} \delta Z_{G_0 A_0} \\ \frac{1}{2} \delta Z_{A_0 G_0} & 1 + \frac{1}{2} \delta Z_{A_0 A_0} \end{pmatrix} \begin{pmatrix} G_0 \\ A_0 \end{pmatrix}. $$

For field renormalization constants $\delta Z_{ij}$ relevant for external physical $A_0$ bosons, OS renormalization conditions for momentum transfer $p^2 = M_{A_0}^2$ can be chosen as usual; for the Goldstone field $G_0$ OS conditions may be taken for any virtuality $p^2 = M^2$ in order to absorb UV divergences (without changing the final result). Taking $M = 0$, we, thus, can define

$$\delta Z_{A_0 G_0} = \frac{2 \Sigma_{A_0 G_0} (0)}{M_{A_0}^2}, \quad \delta Z_{G_0 A_0} = -\frac{2 \text{Re} \{ \Sigma_{A_0 G_0} (M_{A_0}^2) \}}{M_{A_0}^2}.$$  

The tadpole contributions to $\delta \beta_{\overline{\text{MS}}}$ are then given by

$$\delta \beta_{\overline{\text{MS}}, \text{tad}} = - \frac{\Sigma_{\text{tad}}^{A_0 G_0}}{M_{A_0}^2} |_{\text{UV}},$$

where $\Sigma_{\text{tad}}^{A_0 G_0}$ is the sum of all (momentum-independent) explicit tadpole diagrams and tadpole counterterms to the self-energy $\Sigma_{A_0 G_0} (p^2)$. Inserting all tadpole counterterms contributing to the $A_0 G_0$ mixing self-energy in the various tadpole schemes into Eq. (3.55), we obtain

$$\delta \beta_{\overline{\text{MS}}, \text{tad}}^{\text{FJTS}} = \left[ c_{\beta - \alpha} \frac{\Delta v_H^{\text{FJTS}}}{v} \left( \frac{M_h^2}{M_{A_0}^2} - 1 \right) - s_{\beta - \alpha} \frac{\Delta v_H^{\text{FJTS}}}{v} \left( \frac{M_H^2}{M_{A_0}^2} - 1 \right) \right] |_{\text{UV}},$$

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\[ \delta \beta^\text{PRTS}_{\text{MS}, \text{tad}} = \left( -c_\alpha - a \frac{\delta \alpha^{\text{PRTS}}}{v M_0^2} + s_\beta - a \frac{\delta \beta^{\text{PRTS}}}{v M_0^2} \right) \bigg|_{\text{UV}}, \]
\[ \delta \beta^\text{GIVS}_{\text{MS}, \text{tad}} = \left( -c_\alpha - a \frac{\delta \alpha^{\text{GIVS}}}{v M_0^2} + s_\beta - a \frac{\delta \beta^{\text{GIVS}}}{v M_0^2} \right) \bigg|_{\text{UV}} + \left( -c_\alpha - a \frac{\Delta \alpha^{\text{GIVS}}}{v} + s_\beta - a \frac{\Delta \beta^{\text{GIVS}}}{v} \right) \bigg|_{\text{UV}}, \]

where Eq. (3.46) was used in the last equation. The agreement of \( \delta \beta^\text{MS, tad} \) in the PRTS and GIVS, which we have established in the class of \( R_g \) gauges, is particularly interesting. It means that the \( \overline{\text{MS}} \) renormalizations of \( \beta \) in the PRTS and GIVS in \( R_g \) gauges are the same, at least at the one-loop level. Thus, all results obtained with \( \overline{\text{MS}} \)-renormalized \( \beta \) in the PRTS in \( R_g \) gauges can be reinterpreted as results obtained in the GIVS, which is gauge independent. It would be interesting to investigate in how far this statement holds beyond the one-loop level as well, but this task is beyond the scope of this paper. According to the arguments given in App. D of Ref. [9], where it is shown that \( \delta \beta^\text{PRTS}_{\text{MS}} \) in fact depends on the gauge in general, the equivalence of \( \beta^\text{PRTS}_{\text{MS}} \) and \( \beta^\text{GIVS}_{\text{MS}} \) should, however, not hold outside the class of \( R_g \) gauges.

Analogously to the situation in the SESM discussed in Section 3.1.2, and as discussed in Refs. [9, 10, 12, 37], changing the tadpole treatment in the \( \overline{\text{MS}} \) renormalization of the mixing angles \( \alpha \) and \( \beta \) changes the physical meaning of \( \alpha \) and \( \beta \), similar to a change in their renormalization conditions. The corresponding finite shifts in the numerical values of the renormalized parameters \( \alpha^{\text{TS}}_{\overline{\text{MS}}} \) and \( \beta^{\text{TS}}_{\overline{\text{MS}}} \) when the tadpole scheme TS is changed from the PRTS to the FJTS were calculated in Refs. [9, 10, 37] by considering NLO corrections to specific vertices. In Ref. [10] arguments for a universal relation between \( (\alpha_{\overline{\text{MS}}}^{\text{PRTS}}, \beta_{\overline{\text{MS}}}^{\text{PRTS}}) \) and \( (\alpha_{\overline{\text{MS}}}^{\text{FJTS}}, \beta_{\overline{\text{MS}}}^{\text{FJTS}}) \) were given, based on the observation that the FJTS is equivalent to a procedure in which no tadpole renormalization is performed at all. In the following we proceed as in Section 3.1.2 for the SESM and derive the general relation between the input parameters of the THDM when changing the tadpole scheme, again basing the arguments on the equivalence of the FJTS and performing no tadpole renormalization. As in the SESM, the tadpole renormalization constants entering the relations between the bare input parameters can be obtained via the substitutions (3.45), which generate the PRTS tadpole counterterms in the bare Lagrangian without tadpole terms as follows,

\[ \alpha_0^{\text{FJTS}} = \alpha_0 \bigg|_{(3.45), p_0 \to p_0^{\text{PRTS}}}, \quad \beta_0^{\text{FJTS}} = \beta_0 \bigg|_{(3.45), p_0 \to p_0^{\text{PRTS}}}, \]

which means that the substitution (3.45) is applied to \( \alpha_0 \) and \( \beta_0 \) expressed in terms of the bare parameters \( \{p_0\} = \{m_{1,0}^2, m_{2,0}^2, m_{12,0}^2, \lambda_{1,0}, \ldots, \lambda_{5,0}\} \) in the absence of tadpole terms. Note that there are no shifts in the OS-renormalized parameters for \( v, M_h, M_H, M_A, \) and \( M_{H^\pm} \). Owing to the last relation of Eq. (3.45), there is no shift in the \( \overline{\text{MS}} \)-renormalized \( \lambda_5 \) either. For \( \alpha \) and \( \beta \), however, Eq. (3.45) implies the one-loop relations

\[ \alpha_0^{\text{PRTS}} = \alpha_0^{\text{FJTS}} + \frac{e}{M_H^2 - M_h^2} \left( C_{hhH} \frac{\delta \alpha^{\text{FJTS}}}{M_h^2} + C_{hHH} \frac{\delta \beta^{\text{FJTS}}}{M_H^2} \right), \]
\[ \beta_0^{\text{PRTS}} = \beta_0^{\text{FJTS}} + \frac{1}{v} \left( -c_\beta - a \frac{\delta \alpha^{\text{FJTS}}}{M_h^2} + s_\beta - a \frac{\delta \beta^{\text{FJTS}}}{M_H^2} \right), \]
the derivation of which is again somewhat cumbersome but straightforward. The relations between the two renormalized mixing angles in the PRTS and FJTS follows from the renormalization transformations
\[ \alpha^\text{TS}_0 = \alpha^\text{TS} + \delta\alpha^\text{TS}, \quad \beta^\text{TS}_0 = \beta^\text{TS} + \delta\beta^\text{TS} \] (3.62)

with TS being the PRTS or FJTS. The resulting shifts between the \( \overline{\text{MS}} \)-renormalized renormalized parameters in the two tadpole schemes read
\[ \alpha^\text{PRTS}_{\overline{\text{MS}}} - \alpha^\text{FJTS}_{\overline{\text{MS}}} = (\alpha^\text{PRTS}_0 - \alpha^\text{FJTS}_0) - (\delta\alpha^\text{PRTS}_{\overline{\text{MS}}} - \delta\alpha^\text{FJTS}_{\overline{\text{MS}}}) \] (3.63)
\[ \beta^\text{PRTS}_{\overline{\text{MS}}} - \beta^\text{FJTS}_{\overline{\text{MS}}} = (\beta^\text{PRTS}_0 - \beta^\text{FJTS}_0) - (\delta\beta^\text{PRTS}_{\overline{\text{MS}}} - \delta\beta^\text{FJTS}_{\overline{\text{MS}}}) \] (3.64)

The differences between the bare mixing angles can be read from Eqs. (3.60) and (3.61), and the differences between the corresponding renormalization constants \( \delta\alpha^\text{TS}_{\overline{\text{MS}}} \) and \( \delta\beta^\text{TS}_{\overline{\text{MS}}} \) again receive only contributions from the corresponding tadpole contributions, which can be read from Eqs. (3.48), (3.49), (3.56), and (3.57). Combining all those parts, we get
\[ \alpha^\text{PRTS}_{\overline{\text{MS}}} - \alpha^\text{FJTS}_{\overline{\text{MS}}} = -\frac{e}{M_H^2 - M_h^2} \left( C_{hhH} \frac{T^h}{M_h^2} + C_{hhH} \frac{T^H}{M_H^2} \right) \bigg|_{\text{finite}}, \] (3.65)
\[ \beta^\text{PRTS}_{\overline{\text{MS}}} - \beta^\text{FJTS}_{\overline{\text{MS}}} = \frac{1}{v} \left( c_{\beta-a} \frac{T^h}{M_h^2} - s_{\beta-a} \frac{T^H}{M_H^2} \right) \bigg|_{\text{finite}}. \] (3.66)

As already explained in Section 3.1.2 for the SESM in more detail, the conversion between the FJTS and the GIVS proceeds analogously, with the GIVS constants \( \delta t^\text{GIVS}_X \) (\( X = \eta_1, \eta_2, H, h \)) playing the roles of \( \delta t^\text{PRTS}_X \) in the PRTS. The final results for the conversion are
\[ \alpha^\text{GIVS}_{\overline{\text{MS}}} - \alpha^\text{FJTS}_{\overline{\text{MS}}} = -\frac{e}{M_H^2 - M_h^2} \left( C_{hhH} \frac{T^h}{M_h^2} + C_{hhH} \frac{T^H}{M_H^2} \right) \bigg|_{\text{finite}}, \] (3.67)
\[ \beta^\text{GIVS}_{\overline{\text{MS}}} - \beta^\text{FJTS}_{\overline{\text{MS}}} = \frac{1}{v} \left( c_{\beta-a} \frac{T^h}{M_h^2} - s_{\beta-a} \frac{T^H}{M_H^2} \right) \bigg|_{\text{finite}}, \] (3.68)

which have the same form as the conversion between PRTS and FJTS with the only difference that the tadpole contributions are calculated in the non-linear Higgs representation. Owing to the gauge independence of the tadpole constants in the non-linear representation, the relation between \( (\alpha^\text{GIVS}_{\overline{\text{MS}}}, \beta^\text{GIVS}_{\overline{\text{MS}}}) \) and \( (\alpha^\text{FJTS}_{\overline{\text{MS}}}, \beta^\text{FJTS}_{\overline{\text{MS}}}) \) is gauge independent. Note that Eqs. (3.66) and (3.68) together with Eq. (3.46) imply the notable relation
\[ \beta^\text{GIVS}_{\overline{\text{MS}}} = \beta^\text{PRTS}_{\overline{\text{MS}}}, \] (3.69)
i.e. the GIVS and PRTS \( \overline{\text{MS}} \) renormalizations of \( \beta \) coincide in the class of \( R_\xi \) gauges (at least at the one-loop level).

The equality (3.69) can be used to put some phenomenological predictions based on \( \overline{\text{MS}} \)-renormalized angles \( \beta \) in the PRTS on more solid theoretical ground. If the \( R_\xi \) gauge is employed in those predictions, as most frequently done, the PRTS results can be reinterpreted as GIVS results, which are gauge independent, at least as long as no other sources
of gauge dependences exist (as, e.g., from the renormalization of $\alpha$ in the THDM). This arguments, for instance, holds for a scheme proposed and used in Refs. [30, 31] for the THDM, as discussed in the next section in more detail. Note that the argument is also applicable to many higher-order calculations within the Minimal Supersymmetric extension of the SM (MSSM) which contains a THDM of Type II as Higgs sector. The corresponding parameter $\tan \beta$ is very often renormalized in the $\overline{\text{MS}}$ scheme with PRTS tadpole treatment, and the $R_\xi$ gauge is most often used in higher-order calculations, see for example Refs. [47–50], where this renormalization condition is applied in Higgs-mass-spectrum calculations or other higher-order calculations in the MSSM. In those applications the $\overline{\text{MS}}$ PRTS $\tan \beta$ renormalization is found to be a convenient scheme leading to perturbatively well-behaved results. The NLO results obtained in this way can, thus, be interpreted as gauge-independent results obtained within the GIVS. We even expect that this statement carries over to predictions beyond the NLO level, but the investigation of this conjecture is beyond the scope of this paper.

3.2.3 NLO decay widths for $h \to 4f$ in the THDM

In this section we extend the discussion of NLO predictions for the Higgs decay processes $h \to WW/ZZ \to 4f$ in the THDM started in Refs. [10, 12, 29] with the Monte Carlo program Prophecy4f 3.0 [30–32]. To this end, we have implemented the GIVS tadpole treatment as new option for $\overline{\text{MS}}$ renormalization schemes of the THDM in Prophecy4f.

In detail, in Refs. [10, 29] two types of $\overline{\text{MS}}$ renormalization schemes for the mixing angles $\alpha$ and $\beta$ were considered, both with PRTS and FJTS tadpole treatments: one scheme with $\overline{\text{MS}}$-renormalized angles $\alpha$, $\beta$, and another scheme in which the coupling constant $\lambda_3$ was taken as independent variable instead of the angle $\alpha$. According to Eq. (3.69) of the previous section, there is no difference between the PRTS (in the $R_\xi$ gauge) and the GIVS tadpole treatment in the renormalization of $\beta$, so that the $\overline{\text{MS}}$ scheme with input $(\lambda_3, \beta)$, which was called “$\overline{\text{MS}}(\lambda_3)$” in Refs. [10, 29], is the same in the PRTS and GIVS tadpole variants. This, in particular, means that the $\overline{\text{MS}}(\lambda_3)$ results of Refs. [10, 29] are fully gauge independent when interpreted as results obtained in the GIVS for the tadpoles. Ref. [32] extended the discussion of Refs. [10, 29] by including renormalization schemes based on OS or symmetry-inspired renormalization conditions for $\alpha$ and $\beta$.

The numerical input for the scenarios called A1, A2, B1, B2 as well as the details of the calculational setup can be found in Refs. [10, 12, 29]. The mass of the lighter Higgs boson is set to $M_h = 125 \text{ GeV}$ in all considered scenarios, and the mixing angles are varied in their empirically allowed range with central values in the vicinity of $|c_{\beta-\alpha}| = |\cos(\beta - \alpha)| = 0.1$. The scenarios A1 and A2 are “low-mass scenarios” with $M_H = 300 \text{ GeV}$ and $M_{A_0} = M_{H^\pm} = 460 \text{ GeV}$; B1 and B2 are “high-mass scenarios” with $M_H = 600 \text{ GeV}$ and $M_{A_0} = M_{H^\pm} = 690 \text{ GeV}$. Results of the scheme conversion of the input values for the $\overline{\text{MS}}$ parameters $c_{\alpha\beta} = c_{\beta-\alpha}$ and $\beta$, are given in App. A.2. The following results for $h \to 4f$ are given for a Type I THDM, but they hardly change when going over to Types II, “lepton-specific”, or “flipped”, as shown in Refs. [10, 29].

\footnote{A1 is called Aa in Refs. [10, 29].}
| Ren. scheme | tadpoles  | A1 LO ± 0.1% | NLO ± 0.1% | A1 LO ± 0.1% | NLO ± 0.1% |
|-------------|-----------|-------------|-------------|-------------|-------------|
| **OS12 (α, β)** | 0.89832(3) | 0.96194(7) | 0.87110(3) | 0.92947(7) |
| **MS (α, β)** | FJTS 0.89996(3) | 0.96283(7) | 0.88508(3) | 0.93604(7) |
| **MS (α, β)** | PRTS 0.89035(3) | 0.96103(7) | 0.86131(3) | 0.92784(7) |
| **MS (α, β)** | GIVS 0.89082(3) | 0.96106(7) | 0.86249(3) | 0.92808(7) |
| **MS (λ, β)** | FJTS 0.89246(3) | 0.96108(7) | 0.85590(3) | 0.92723(7) |
| **MS (λ, β)** | PRTS/GIVS 0.89156(3) | 0.96111(7) | 0.85841(3) | 0.92729(7) |

| Ren. scheme | tadpoles  | B1 LO ± 0.5% | NLO ± 0.5% | B1 LO ± 0.5% | NLO ± 0.5% |
|-------------|-----------|-------------|-------------|-------------|-------------|
| **OS12 (α, β)** | 0.88698(3) | 0.94070(8) | 0.82573(3) | 0.87192(6) |
| **MS (α, β)** | FJTS 0.90406(3) | 0.96069(7) | 0.90654(3) | 0.96981(7) |
| **MS (α, β)** | PRTS 0.88608(3) | 0.94049(8) | 0.87943(3) | 0.92421(7) |
| **MS (α, β)** | GIVS 0.88661(3) | 0.94082(8) | 0.87990(3) | 0.92469(7) |
| **MS (λ, β)** | FJTS 0.75976(3) | 1.0889(3) | 0.90615(3) | 0.97290(7) |
| **MS (λ, β)** | PRTS/GIVS 0.88417(3) | 0.93934(8) | 0.87475(3) | 0.91958(7) |

**Table 3**: LO and NLO decay widths $\Gamma^{h\rightarrow4f}$ [MeV] of the light CP-even Higgs boson $h$ of the THDM for various scenarios in different renormalization schemes, with the OS12 scheme as input scheme (and full conversion of the input parameters into the other schemes). The scale variation (given in percent) corresponds to the scales $\mu = \mu_0/2$ and $\mu = 2\mu_0$ with central scale $\mu_0 = (M_{H_2} + M_{H_1} + M_{A_0} + 2M_{H^+})/5$. The quantity $\Delta_{OS12} = \Gamma_{\overline{MS}}/\Gamma_{OS12} - 1$ shows the relative difference between the $\overline{MS}$ predictions and the OS12 value; its spread illustrates the scheme dependence of the prediction at LO and NLO.

Table 3 shows NLO predictions for the partial decay width of $h \rightarrow 4f$ for the considered scenarios based on the various $\overline{MS}$ schemes using the OS-type scheme OS12 as reference scheme in which the input parameters are defined. In the reference scheme OS12, which
was suggested in Ref. [12], both $\alpha$ and $\beta$ are renormalized with OS conditions imposed on appropriate ratios of Higgs-boson decay amplitudes into neutrinos. Although involving some process dependence, results obtained in the OS12 scheme exhibit many desirable features such as gauge independence and excellent perturbative stability even in exceptional parameter regions (mass degeneracies, large Higgs masses, extreme mixing angles). The OS12 scheme is, thus, an appropriate choice as reference scheme. The $\overline{\text{MS}}$ results with GIVS tadpole treatment extend Tab. 7 of Ref. [12], where results for further renormalization schemes can be found. Since the input values of the mixing angles $\alpha$, $\beta$, and the parameter $\lambda_5$ are properly converted from the OS12 scheme to the other $\overline{\text{MS}}$ schemes, ideal predictions would not show any residual dependence on the renormalization scale $\mu$ and on the choice of renormalization scheme. Recall that the small $\mu$ dependence of the OS12 predictions only originates from the dependence of the genuine NLO correction on $\lambda_5$, while $\alpha$ and $\beta$ are fixed. The comprehensive comparison of predictions based on several other schemes carried out in Ref. [12], moreover, shows that the OS12 predictions are perturbatively perfectly stable with very small corrections. The comparison of $\overline{\text{MS}}$ numbers to the OS12 numbers, thus, nicely quantify the scheme dependence of the LO and NLO results.

Compared to the discussion of the renormalization scale and scheme dependence of the corresponding results in the SESM in Section 3.1.3, the situation for the THDM results is much more diverse, and the results for the low-mass and high-mass scenarios A1, A2 and B1, B2, respectively, differ considerably. In the low-mass scenarios A1, A2 the $\overline{\text{MS}}$ predictions based on PRTS or GIVS show a nice reduction of their scale and scheme uncertainty in the transition from LO to NLO, as already discussed in Refs. [10, 12, 29] for the PRTS variant. As also pointed out there, the FJTS variants already show issues with perturbative stability in A1, A2; nevertheless the FJTS $\overline{\text{MS}}$ predictions look still consistent with a reasonable, albeit significant, scale uncertainty reflecting the overall theoretical uncertainty realistically.

In the high-mass scenario B1, the $\overline{\text{MS}}(\alpha, \beta)$ schemes behave properly for all tadpole schemes. For B1, the $\overline{\text{MS}}(\lambda_3, \beta)$ scheme produces still reasonable results for the PRTS/GIVS tadpole variants, though without significant reduction of the scale uncertainty. However, the scheme $\overline{\text{MS}}(\lambda_3, \beta)$ FJTS totally fails, as can already be conjectured from the corrections observed in the parameter conversion shown in App. A.2. The situation for high-mass scenario B2 is even more extreme. The failure of the FJTS $\overline{\text{MS}}$ schemes can already be anticipated from the corrections to the parameter conversions, see again App. A.2. The PRTS and GIVS $\overline{\text{MS}}$ variants behave better, but do not show uncertainty reductions in the transition from LO to NLO. The question marks in Table 3 indicate that the scale variation could not be evaluated consistently, since the conversion effects pushed the parameter $s_\alpha$ out of its bounds $|s_\alpha| \leq 1$ for the extreme scales.

In summary, we confirm the expectation that the new (gauge-independent) GIVS tadpole variant produces results in the $\overline{\text{MS}}$-type schemes that are very close to the corresponding (gauge-dependent) PRTS results; for the $\overline{\text{MS}}(\lambda_3, \beta)$ scheme the variants are even identical. The superiority of the PRTS variant over the FJTS variant in view of perturbative stability, which was discussed in Refs. [10, 12, 29] in greater detail, thus, nicely carries over to the GIVS, which rescues gauge independence in addition.
It should, however, also be mentioned that none of the $\overline{\text{MS}}$ schemes with any tadpole treatment is able to produce fully satisfactory results for the four-body decays of the heavier CP-even neutral Higgs boson $H$ of the THDM. Predictions for the decays $H \rightarrow 4f$ typically receive larger corrections than for $h \rightarrow 4f$, because the leading-order decay widths of the former are reduced by small values of $c_{\beta-\alpha}^2$ w.r.t. the latter which are proportional to $s_{\beta-\alpha}^2 \lesssim 1$. The potentially larger corrections to $H \rightarrow 4f$ render the corresponding NLO predictions more prone to perturbative instabilities. The GIVS results for the NLO decay widths are again similar to the PRTS results, which are included in the detailed study of NLO predictions for $H \rightarrow WW/ZZ \rightarrow 4f$ based on various types of renormalization schemes presented in Ref. [12]. For $H \rightarrow 4f$, predictions, thus, should be based on the OS-type or symmetry-inspired schemes proposed in Ref. [12].

4 Conclusions

Applying $\overline{\text{MS}}$ renormalization conditions to mass parameters or mixing angles leads to subtle issues in the precise definition of vev parameters if EW corrections are included in predictions. Technically, these issues concern the treatment of tadpoles in the course of renormalization. In the past, mostly two tadpole schemes have been in use, called PRTS and FJTS in this paper, which, however, both have benefits and drawbacks. The PRTS is based on the idea to renormalize parameters in such a way that a field is expanded about the minimum of the effective Higgs potential. For Higgs fields developing vevs this means that no potentially large corrections to the vevs enter predictions, but gauge-dependent tadpole loop contributions enter the relations between bare parameters of the theory (at least in the usually adopted Higgs representations), leading to gauge-dependent parametrizations of observables. The alternative scheme, called FJTS, avoids gauge dependences by just redistributing tadpole contributions in predictions via field shifts without altering parameter relations, but on cost of an expansion about a field configuration that corresponds to the vacuum only in lowest order. By experience this procedure is prone to artificially large corrections which jeopardize the stability of perturbation theory.

To improve on this unsatisfactory situation, in a preceding paper [21] we have proposed a hybrid scheme for treating tadpole contributions, dubbed Gauge-Invariant Vacuum expectation value Scheme (GIVS), which unifies the benefits and avoids the shortcomings of the PRTS and FJTS. Perturbative stability is achieved by an expansion about the true vacuum field configuration, and gauge dependences are avoided by employing non-linear Higgs representations, in which CP-even Higgs fields and their vevs are gauge-invariant quantities. As an application in the SM, we had shown in Ref. [21] that the conversion of OS-renormalized to $\overline{\text{MS}}$-renormalized masses involves only very moderate EW corrections, in contrast to the situation in the FJTS. In this paper, we have applied the GIVS to the $\overline{\text{MS}}$ renormalization of Higgs mixing angles in two scalar extensions of the SM: a Higgs singlet extension (called SESM), and the Two-Higgs-Doublet model (THDM), the renormalization of which has been extensively discussed in the literature in recent years.

To apply the GIVS to the SESM and THDM, we have first formulated the Higgs sectors in a non-linear way. While the SESM requires only minor modifications beyond the SM
formulation, the generalization to the THDM is quite non-trivial and interesting in its own right. We have described the transition from linearly to non-linearly realized Higgs doublets in detail and have presented the (non-polynomial) kinetic Lagrangians of the THDM Higgs field in a way that renders the generation of all Feynman rules (which involve arbitrarily many Goldstone legs) quite simple. While kinetic Lagrangians become more complicated in non-linear Higgs representations, the Higgs potentials become simpler, because all would-be Goldstone fields drop out there by construction.

To illustrate the perturbative stability of the results based on \( \overline{\text{MS}} \) renormalization with the GIVS tadpole treatment, we have discussed the NLO predictions for the important four-body decays of CP-even neutral Higgs bosons, \( h/H \to WW/ZZ \to 4 \) fermions based on \( \overline{\text{MS}} \)-renormalized Higgs mixing angles. In all cases, we have found that the (gauge-independent) GIVS results are very close to the corresponding (gauge-dependent) PRTS results, thus, sharing the superiority over the FJTS results w.r.t. perturbative stability. The new GIVS variants of the \( \overline{\text{MS}} \) schemes will be made available in a forthcoming public version of the Monte Carlo program PROPHETY4F for the considered four-body Higgs decays.

Finally, in the THDM we have made the observation that the GIVS variant for an \( \overline{\text{MS}} \)-renormalized parameter \( \tan \beta = v_2/v_1 \) produces exactly the same one-loop renormalization constants for \( \tan \beta \) as the PRTS variant in the class of \( R_\xi \) gauges, i.e. that effectively the \( \overline{\text{MS}} \) PRTS renormalization for \( \tan \beta \) does not suffer from gauge dependences after reinterpretation as GIVS results. This observation is particularly interesting in the context of EW renormalization of supersymmetric models, in which \( \tan \beta \) is often \( \overline{\text{MS}} \)-renormalized with PRTS-like tadpole prescriptions. We conjecture that this interesting fact generalizes to higher orders as well and, thus, can put higher-order predictions in supersymmetric models obtained with the PRTS in \( R_\xi \) gauges on a gauge-invariant basis.

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Appendix

A Parameter conversion tables

The full set of input parameters of the various SESM and THDM scenarios, as used in Sections 3.1.3 and 3.2.3, can be found in Refs. [11] and [10, 29], respectively. Here we list results for the conversion of those non-standard input parameters that are subject to different renormalization schemes for the SESM and THDM. The input parameters are defined in the on-shell input schemes OS and OS12\(^{10}\) in the SESM and THDM, respectively.

\(^{10}\) Although NLO results in the OS and OS12 schemes are completely independent of the tadpole treatment, there is some slight dependence on the tadpole scheme (beyond NLO accuracy) used in these schemes.
Table 4: Conversion of parameters from the OS scheme as input scheme to the different \(\overline{\text{MS}}\) renormalization schemes at the central scale \(\mu_0 = M_h\) performed in the PRTS, using the bare parameters for \(\tan \alpha\) and \(\lambda_{12}\) in the matching.

| Ren. scheme | tadpoles | BHM200⁺ | BHM200⁻ | BHM400 | BHM600 |
|-------------|----------|---------|---------|--------|--------|
| OS          |          | 0.29    | 0.07    | −0.29  | −0.07  |
| \(\overline{\text{MS}}\) FJTS | 0.321    | 0.077   | −0.316  | −0.076 | 0.264  | 0.173  | 0.212  | 0.222  |
| \(\overline{\text{MS}}\) PRTS | 0.302    | 0.073   | −0.304  | −0.073 | 0.267  | 0.175  | 0.226  | 0.236  |
| \(\overline{\text{MS}}\) GIVS | 0.301    | 0.073   | −0.302  | −0.073 | 0.266  | 0.174  | 0.225  | 0.236  |

Table 4 shows the results for the (full) conversion of parameters from the OS renormalization scheme used as input scheme to \(\overline{\text{MS}}\) schemes for the considered benchmark scenarios. This table completes Tab. 14 of Ref. [12], where the input conversion to other schemes can be found. We recall that the included renormalization schemes employ \(\lambda_1\) (not \(\lambda_{12}\)) as independent running \(\overline{\text{MS}}\)-renormalized parameter, although the scenarios are fixed by the values of \(\lambda_{12}\).

The moderate corrections in the input parameter conversions reflect the perturbative stability of the \(\overline{\text{MS}}\) schemes with all the considered tadpole treatments in all considered scenarios. Note also that the corrections are almost identical for the PRTS and GIVS variants, as expected from the small differences in the tadpole terms of the renormalization constant \(\delta \alpha\) of the mixing angle \(\alpha\).

A.2 THDM scenarios

Table 5 shows the corresponding conversion in the THDM from the OS12 renormalization scheme used as input scheme to the different \(\overline{\text{MS}}\) renormalization schemes. This table completes Tab. 15 of Ref. [12], where the input conversion to other schemes can be found. As compared to the SESM, the conversion effects for the considered THDM scenarios are more pronounced. The conversion to \(\overline{\text{MS}}\) (FJTS) becomes perturbatively unstable (and fails completely) for scenarios B1 and B2. On the other hand, the PRTS and GIVS tadpole treatments lead to moderate corrections, reflecting perturbative stability. As expected, the results of the PRTS and GIVS tadpole variants almost agree for the \(\overline{\text{MS}}\) \((\alpha, \beta)\) scheme; in the \(\overline{\text{MS}}\) \((\lambda_3, \beta)\) scheme these two tadpole variants produce identical results as explained in Section 3.2.1.

if input parameter conversions between schemes are done in a self-consistent way i.e. not truncated at the NLO level (see Ref. [7] for more details). For completeness, we mention that we apply the PRTS variant in the OS and OS12 schemes.
Table 5: Conversion of parameters from the OS12 scheme as input scheme to the different $\overline{\text{MS}}$ renormalization schemes at the central scale $\mu_0 = (M_h + M_H + M_{A_0} + 2M_{H^\pm})/5$ performed in the PRTS, using the bare parameters for $\tan \alpha$ and $\tan \beta$ in the matching.

| Ren. scheme | tadpoles | $A_1$ | $A_2$ | $B_1$ | $B_2$ |
|-------------|----------|-------|-------|-------|-------|
| OS12$(\alpha, \beta)$ |  | $c_{\alpha \beta}$ | $t_\beta$ | $c_{\alpha \beta}$ | $t_\beta$ |
| $\overline{\text{MS}} (\alpha, \beta)$ | FJTS | 0.090 | 1.93 | 0.157 | 1.91 |
| $\overline{\text{MS}} (\alpha, \beta)$ | PRTS | 0.137 | 1.90 | 0.225 | 1.91 |
| $\overline{\text{MS}} (\lambda_3, \beta)$ | GIVS | 0.135 | 1.90 | 0.222 | 1.91 |
| $\overline{\text{MS}} (\lambda_3, \beta)$ | FJTS | 0.128 | 1.92 | 0.238 | 1.88 |
| $\overline{\text{MS}} (\lambda_3, \beta)$ | PRTS/GIVS | 0.132 | 1.90 | 0.232 | 1.91 |

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