Homogeneous Floquet time crystal from weak ergodicity breaking

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Recent works on observation of discrete time-crystalline signatures throw up major puzzles on the necessity of localization for stabilizing such out-of-equilibrium phases. Motivated by these studies, we delve into a clean interacting Floquet system, whose quasi-spectrum conforms to the ergodic Wigner-Dyson distribution, yet with an unexpectedly robust, long-lived time-crystalline dynamics in the absence of disorder or fine-tuning. We relate such behavior to a measure zero set of nonthermal eigenstates with long-range spatial correlations, which coexist with otherwise thermal states at near-infinite temperature and develop a high overlap with a family of translationally invariant, symmetry-broken initial conditions. This resembles the notion of “dynamical scars” that remain robustly localized throughout a thermalizing Floquet spectrum with fractured structure. We dub such a long-lived discrete time crystal formed in partially nonergodic systems, “scarred discrete time crystal” which is distinct by nature from those stabilized by either many-body localization or prethermalization mechanism.

I. INTRODUCTION

Periodically driven (Floquet) quantum systems are of immense recent interest as they can sustain a variety of novel solid state phenomena ranging from Floquet engineering1–3 to extending the theory of localization or Mott insulators to the time domain4–7. They also provide natural platforms for realizing intriguing topological phases, hosting anomalous chiral edge states8–10 or Majorana edge modes11,12, as well as emergent non-equilibrium phases of matter with no static equilibrium counterpart. One of the most significant phases is Floquet discrete time-crystal (DTC)13,14, the so-called “π spin glass” (πSG)15–17, in which a driven system fails to be invariant under the discrete time-translation symmetry of its underlying Hamiltonian.

More broadly, the concept of time crystal has to do with the spontaneous emergence of time-translation symmetry breaking (TTSB) within a time-invariant system. In 2012, Wilczek conceptualized the possibility of continuous TTSB for the ground state of a certain quantum and classical system18,19. However, his original proposition has triggered an intense debate20–22, including subsequent no-go theorems23–25, concerning the existence of time crystals at thermal equilibrium and in the ground states of local time-independent Hamiltonians. On this basis, the search for time crystals shifted toward certain nonequilibrium conditions26, in particular, Floquet systems27–30. The defining diagnostic of a stable DTC phase then reads as non-trivial subharmonic response of certain physical observables, at some multiple of the drive frequency, which is robust to generic perturbations and persists infinitely on approaching the thermodynamic limit.

Nevertheless, such a discernible phase structure is generically nonviable so long as the strong form of “eigenstate thermalization hypothesis” (ETH)27–30 holds in that all Floquet eigenstates look like maximally-entangled featureless states31,32. Thus the key strategy for stabilizing temporal order is to explore possible ways to completely suppress (or at least slow down) the process of Floquet heating toward infinite temperature. This can typically be achieved by either considering (fine-tuned) Bethe-ansatz integrable systems33, or extending the physics of many-body localization (MBL)34,35, driven by spatial disorder, to the Floquet realm36. The latter provides the only known generic mechanism for strong breakdown of ergodicity due to the emergence of a complete set of quasilocal integrals of motion (LIOM) in the so-called “l-bits” formalism.37,38. The fully localized spectrum of Floquet-MBL systems can establish spatio-temporal order even at infinite temperature, giving rise to the concrete example of absolutely stable (space-)time crystals14,16,39–41. However, MBL is not the only game in town. So far, a range of mechanisms have been exploited, both theoretically and experimentally, to realize robust DTC phase (or at least transient DTC signatures) in a broad class of generic clean systems. These mechanisms go from prethermalization42–49 to emergent Floquet integrability in systems with strong interactions50, as well as those relied on protecting “ancillary” symmetry51, e.g., spatial translation52, time-reflection53, or discrete (Abelian) gauge symmetry54.

A rather crisp realization of DTC is also provided by periodically driven mean-field models13,55–61, which can exhibit discrete TTTSB (DTTTSB) even in the presence of quantum chaos. The realization of this kind of DTC, however, is tied to an intrinsically semiclassical few-body phenomenon rather than quantum many-body interactions, whose presence is essential for stabilizing MBL and prethermal DTCs. Such an exotic behavior can be attributed to the phenomenon of mixed classical phase space and its semiclassical correspondence for quantum few-body systems62–65, in which chains of regular “islands” are surrounded by a chaotic “sea”. The periodic jump among separated islands leaves DTC imprint on quantum dynamics when the initial state predominantly falls inside one of these regular regions55–57. The rigidity of mean-filed time crystals then owes to the stability of the mixed phase space under weak integrability-breaking perturbations, which is ensured by the Kolmogorov-Arnold-Moser (KAM) theorem.66 However, away from semiclassical limit for many degrees of freedom, the conditions of the KAM theorem become fragile and one expects the quick disappearance of regular islands, which turns the system into a trivial ETH phase55–57.66

By contrast, here we aim to investigate the formation of robust time-crystalline order, beyond semiclassical limit, in a generically chaotic many-body system as a consequence of weak ergodicity breaking66–74. The weak form of ETH allows for the existence of a measure zero set of ETH-violating eigenstates at finite energy density, which are embedded in a sea
of thermalizing states and now named “quantum many-body scars”\textsuperscript{75,76}. The scar states, then by definition, can evade the prescribed no-go arguments of Ref. 25, which in turn allows for TTTSB-like behavior in quench dynamics of certain kinetically constrained models\textsuperscript{75–80}. However, the perfect scars and the resulting TTTSB typically are limited to rather fine-tuned settings\textsuperscript{72,76–88}, and not expected to be robust under generic perturbations\textsuperscript{89,90}.

It has been recently argued that a more robust types of scars, and hence ergodicity breaking, can arise from “Hilbert-space perturbations”\textsuperscript{89,91–100}, where the Hilbert space fractures into exponentially many finite or even infinite size\textsuperscript{99} Krylov subspaces that remain dynamically disconnected (closed) even after resolving all possible explicit symmetry sectors of the Hamiltonian. The dynamical fracturing leads to an initial-state dependent, effectively localized dynamics that stands beyond the scope of locator-expansion techniques. The most promising candidate in this direction is fractonic Floquet random circuit models\textsuperscript{91,93,94}, in which a subset of robust, localized steady states manifest in the thermalizing Floquet spectrum independent of microscopic details of circuits or driving protocols. These atypical eigenstates, characterized by the subthermal entanglement, are referred to as “dynamical scars”\textsuperscript{93} in analogy to their static counterparts. It is therefore natural to ask whether such partially nonergodic phases can open the door to exploring robust time-crystalline behavior in the presence of many-body quantum chaos?

To answer this question, we begin with a simple nonintegrable Floquet model, as a concrete example of clean true DTCs stabilized by emergent Floquet integrability\textsuperscript{50}. We then show that an infinitesimal deformation of this model is enough to completely destroy signatures of integrability in the Floquet spectrum and induce quantum chaos in the system. However, it still features robust, long-lived DTTSB which does not hinge upon either of fine-tuning or disorder and rather stabilized by long-range correlated dynamical scar states; hence the name “scared discrete time crystal” (SDTC).

The appearance of such an exotic behavior is remarkable in light of the lack of any protecting ancillary symmetry (e.g., time-reflection\textsuperscript{53}) or explicit local constraint (e.g., fracton-like constraints\textsuperscript{91–94,99}) that impedes dynamical scars from mixing with typical thermal states. The rigidity of the SDTC can then be understood through the stability of dynamical scars under generic perturbations. Using confused recurrent neural network (RNN), as a semisupervised machine learning method, we affirmatively confirm the robustness of the SDTC response in the limit of strong interactions, beyond which the system will eventually thermalize (see Fig. 1b). We also explain the formation of the SDTC through the emergence of long-lived local quasi-conservation laws in the form of state-dependent LIOMs. Once the system is properly initialized, the SDTC dynamics cannot strikingly evolve out of the underlying initial sector and approximately preserves the conservation laws in question. Our study thus suggests the existence of a new class of time crystals which is neither localization-driven (via MBL mechanism or gauge invariance) nor symmetry protected, and remarkably would be the case even if the standard prethermal mechanism or mean-field treatment is inapplicable.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{(a) Illustrative representation of the Floquet Hamiltonian (1). (b) Color plot of the level statistics ratio $r_{\text{ave}}$ in the plane of $(V, \Delta)$ at fixed $\lambda = 0.7$ for an open chain with $N = 16$. In almost all regions of the parameter space, except at the exact integrable $V = 0$ line, $r_{\text{ave}}$ flows with system size to random matrix value. Circles indicate the crossover between SDTC and a thermalizing (finite-heating) regime. The results are obtained from the analysis of the dynamics of local imbalance autocorrelators through confused RNN. (c) Upper and bottom panels represent $r_{\text{ave}}$ as a function of $V$ at fixed $\Delta = 1$ and 0, respectively.}
\end{figure}

The paper is structured as follows: in Sec. II we introduce the model and describe its relation with the emergent Floquet integrable model of Ref. 50. Sec. III describes in detail our findings regarding the spectral statistics and structure of Hilbert space from the point of view of the ETH, quantum correlation and entanglement. In Sec. IV we address the persistence, initial-state dependence, and rigidity of the SDTC. Sec. V describes the emergence of local quasi-conservation laws in the SDTC regime. We conclude the paper by briefly summarizing our main results with discussions in Sec. VI.

\section{II. THE MODEL}

We begin by considering a clean one dimensional lattice model of interacting spinless fermions, which undergoes a periodic driving dictated by Floquet unitaries of the form $U_F = \mathcal{P} e^{-i t \rho H_F}$, where $\mathcal{P} = e^{-i t \rho^c H_F}$, $\hbar = 1$ and,

\begin{equation}
H_F = \left(\frac{\pi}{2} - \varepsilon\right) \sum_{i \in \text{odd}} c_{i+1}^\dagger c_i + h.c.,
\end{equation}

\begin{equation}
H_D = V \sum_i (c_i^\dagger c_i + h.c.) + \lambda \sum_{\langle ij \rangle} \hat{n}_i \hat{n}_j + \Delta \sum_i (-1)^i \hat{n}_i,
\end{equation}

\begin{figure}[h]
are the two portions of binary stroboscopic Floquet Hamiltonian during $t_P$ and $t_D$, respectively (see Fig. 1a). Here, $\hat{c}_i$ and $\hat{n}_i = \hat{c}_i^\dagger \hat{c}_i$ respectively represent the annihilation and local occupation operator of fermions at site $i$, $N$ denotes the number of lattice sites at half-filling, and $T = t_P + t_D$ is the drive period. To simplify the notation, the coupling constants ($\varepsilon, \lambda, V, \Delta$) measure in the unit $t_P = t_D = 1$, so $\omega_0 = 2\pi/T = \pi$.

It has been recently shown that this model with $V = 0$, denoted by $U_{F}^{\text{int}} \equiv U_F(\varepsilon, \lambda, \Delta)$, can feature genuine time crystal order protected by emergent Floquet integrability\textsuperscript{50}. Accordingly, we first sketch the dynamics governed by stroboscopic time evolution operator, $U(nT) = (U_{F}^{\text{int}})^n$. For $\varepsilon = 0$, the pumping term $\mathcal{P}$ perfectly exchange particles between even and odd sites regardless of the driving field $H_D$. Hence the local fermion imbalance between even and odd lattice sites of the $i^{\text{th}}$ unit cell, denoted by $\tilde{I}_i = \hat{n}_i^e - \hat{n}_i^o$, changes its sign once per Floquet period. Consequently, measuring the temporal autocorrelation function $\langle \tilde{I}_i(nT)\tilde{I}_i(0) \rangle$ at stroboscopic times leads to $2T$-Rabi oscillations. In the single-particle limit $\lambda = 0$, such a temporal order is unstable to generic imperfection, $\varepsilon \neq 0$. However, in the limit of strong interaction $\lambda/\varepsilon \gg 1$, $H_D$ can act as a collective synchronizer and cause the period to be spontaneously doubled. This coherent dynamics owes to the presence of an "incomplete" set of emergent Floquet-LIOMs, which cause the whole spectrum of $U_{F}^{\text{int}}$ to harbor uncorrelated quasi-energy levels, characterized by "imperfect" Poisson statistics\textsuperscript{50}. Indeed, this emergent integrability is not exact and the distribution of level spacings is close to (and not quite) Poisson as $N \to \infty$.

Despite being in a finite distance from an exact (emergent) integrable manifold, it has been claimed that this model does realize true DTC phase with an exponentially diverging lifetime in system size. Building upon this work, we rule out the formation of such genuine temporal order as it is not absolutely stable (at least) against symmetry-breaking perturbations, e.g., $V$ term in (1), and hence is not generic. As will be shown below, adding an infinitesimal $V$ perturbation, even at $\varepsilon = 0$, substantially modifies the spectral statistics of $U_{F}^{\text{int}}$, making the model generically chaotic (see Figs. 1 and 3). Hence, both signatures of Floquet integrability and the seemingly true DTC order in $U_{F}^{\text{int}}$ are somewhat fine-tuned to a certain manifold at $V = 0$. Nevertheless, we still observe long-lived subharmonic oscillations in the dynamics of the generically chaotic, deformed model $U_F$ (see e.g., Fig. 2a).

Hereunder, we set $\varepsilon = 0$ and drive the imperfect $V$, which leads to beating in $\langle \tilde{I}_i(nT)\tilde{I}_i(0) \rangle$, as the tuning parameter. The interaction strength is also fixed at $\lambda = 0.7$ such that $T\lambda \gg 1$, to avoid possible DTC features emerging in the conventional prethermal regime\textsuperscript{43-45}. Since we are interested in unveiling nonergodic coherent dynamics in a strongly interacting clean model, we need the parameter $V$ to be small enough relative to the interaction strength, but remains in the same order of $\lambda$ such that (i) isolated bands and the resulting nonergodic dynamics due to finite-size effects\textsuperscript{101} are not manifested, (ii) for accessible system sizes, the model locates far away from its near integrable manifold in the vicinity of exact integrable $V = 0$ line. To firm up the absolute stability of the observed time crystallinity, our main focus is on the generic case $V, \Delta \neq 0$, with open boundary condition (OBC), for which the model does not exhibit any explicit microscopic symmetry except the global charge conservation, i.e., a physically natural symmetry not requiring fine-tuning.

### III. TIME TRANSLATION SYMMETRY BREAKING VIA DYNAMICAL SCAR STATES

In the first set of calculations we deliver our findings regarding the level spacing ratio $r_n = \min(\delta_{n+1}/\delta_n, \delta_n/\delta_{n+1})$ where $\delta_n = E_n - E_{n-1}$ is the phase gap and $E_n$ denotes $n^{\text{th}}$ quasi-energy of the Floquet operator. The spectrally averaged $r_n$ over symmetry-resolved Hilbert space sectors ($r_{\text{ave}}$), shown in Fig. 1b, indicates that there exist an apparent phase repulsion in most of the parameter space ($V, \Delta$), explored by $U_F$, as $0.50 \lesssim r_{\text{ave}} \lesssim 0.53$ comes close to the Wigner-Dyson value characteristic of quantum many-body chaos\textsuperscript{31,102}.

Remarkably, an infinitesimal $V$-perturbation is sufficient to generate quantum chaos in the thermodynamic limit. This fact is evinced in Fig. 1c, where the finite-size behavior of $r_{\text{ave}}$ is investigated as a function of $V$ for a fixed value of $\Delta$: At $\Delta = 0$, turning on an infinitesimal value of $V$ immediately breaks exact integrability of $V = 0$ line such that the level spacing exhibits a discernible thermal plateau very close to prediction of the circular orthogonal ensemble (COE) distribution, $r_{\text{COE}} \approx 0.52631$. The near-COE plateaus shorten very slowly in system size and ultimately end up in a completely chaotic regime, where $r_{\text{ave}}$ is enclosed by the random matrix values. The same behaviors also hold for any $\Delta \neq 0$. The only exception is the tiny near integrable region\textsuperscript{104-106} (corresponding to the blue area in Fig. 1b), that appears in the vicinity of both the integrable and inversion symmetric $\Delta = 0$ lines. Within this region, $r_{\text{ave}}$ first fall into a value corresponding to the integrable Poissonian (POI) limit of $r_{\text{POI}} \approx 0.386$,
yet flows towards thermal value with increasing system size (see e.g., $V \lesssim 0.05$ in the upper panel of Fig. 1c). As expected, this near integrable region is strongly narrowed by increasing $N$ (and also $\Delta$), heralding robust thermalization of the model for an arbitrary $V \neq 0$, in the thermodynamic limit.

Despite being generically chaotic, the model (1) can exhibit anomalous DTTSB. To show this, we use the infinite time-evolving block decimation (iTEBD) scheme$^{107}$ which allows to simulate unitary evolution in the infinite volume limit, up to some finite time limited by the maximum bond dimension, $\chi_{\text{max}} \sim 10^{3} - 10^{4}$. We consider a family of short-range correlated initial conditions of the form $|\psi_{\gamma}\rangle = \bigotimes_{i=1}^{N/2} (\cos \gamma \ e_{2i}^{\dagger} + \sin \gamma \ e_{2i+1})|0\rangle$, which are translational invariant up to translations of two lattice spacings. Any nonzero value of $\gamma$ introduces an initial-state imperfection with respect to the perfect charge density wave (CDW), i.e., $|\psi_{0}\rangle = |\ldots 0101 \ldots \rangle$. Figure 2 represents time series of $\langle \hat{Z}(t)\hat{Z}(0)\rangle$, in the strong interaction regime $\lambda/V \gg 1$, which exhibits 2T-periodicity persisting for unusual long times. Additionally, the ballistic spreading of entanglement entropy$^{31,32}$ as well as exponential growth of bond dimension $\chi(t)^{108}$ are both significantly slower than those expected to appear in a common thermalizing phase (e.g., in the opposite extreme limit $\lambda/V \sim 1$), where any temporal feature would be entirely absent.

To provide an initial insight into the nature of such a nonergodic coherent dynamics, we investigate the eigenstate properties of the Floquet unitary (1). For each individual eigenstate $|\Psi_{n}\rangle$, which is trivially a steady state, we evaluate the half-cut entanglement entropy $S_{n} = -\text{Tr} \rho_{n} \log \rho_{n}$, corresponding to the density matrix $\rho_{n} = |\Psi_{n}\rangle/\langle \Psi_{n}|$, and mutual information $I_{n}^{\text{th}}$ between the left- and rightmost $l$ sites. In Fig. 3a, we plot the distribution of $S_{n}$ and $I_{n}^{\text{th}}$ as a function of $V$ for a system of size $N = 18$. Herein lies the essence of weak ergodicity breaking, giving rise to the anomalous DTTSB. Over all considered ranges of $V$, the majority of Floquet eigenstates look like the entropy-maximizing thermal state near the infinite temperature with no long-range order, $I_{n}^{\text{th}} \sim 0$. However, we identify the coexistence of low and high entangled eigenstates over a substantial range of $V \lesssim V_{th} \sim 0.25$, in which the broadening of the entanglement distribution is barely discernible. Such a broadening is in sharp contrast to the usual expectations from the Floquet-ETH, but is analogous to that observed in fracturing phenomenon$^{91,93-96}$. Here, a subset of anomalous nonthermal states, a.k.a. dynamical scars$^{31}$, manifests in the steady-states of Floquet system and can be characterized by their subthermal entanglement.

In particular, for $V \lesssim V_{th}$ there are some number of dynamical scars exhibiting anomalous long-range spatial correlation needed for DTTSB, i.e., $I_{n}^{\text{th}} \sim \log 2$ for 2T-periodicity$^{33}$. The presence of such a special scar subregion of the Hilbert space can underpin spontaneous DTTSB in the thermalizing Floquet spectrum when the system is properly initialized in an experimentally accessible, symmetry broken state. This results in the formation of SDTC dynamics that is distinct from traditional MBL-DTCs, wherein a finite fraction of nonthermal eigenstates (potentially all) can feature stable $\pi$SG order. In the latter case, the $\pi$ spectral pairing structure of entire Floquet spectrum can serve as a practical hallmark of the time crystallinity, which can be quantified using, e.g., $\pi$-translated level spacings$^{10}$. Obviously, this is not the case when detection of the SDTC is concerned.

As is clear from Fig. 3a, the dynamical scar states tend to merge with thermal states around $V_{th}$, indicating precursor to the ergodic behavior and ultimately disappear beyond $V_{pd} \sim 0.4$, where one expects the onset of Floquet thermalization to set in$^{109}$. For the reason that will become clear later, we refer to $V_{th}$ and $V_{pd}$ as thermalization and period-doubling crossover, respectively. It is worth noting that in all mentioned ranges of $V$, the spectral statistics is of Wigner-Dyson type (see, e.g., Fig. 1c), and hence cannot distinguish between completely chaotic regime and those containing a vanishing fraction of dynamical scars.

It is also instructive to look at the expectation value of doublon density $\langle D \rangle = \pi^{2}/12 \sum_{i} n_{i} n_{i+1}$, measured in each individual Floquet eigenstate, $\langle \hat{D}_{n}\rangle$. As clearly seen in Fig. 3b, the main concentration of $\langle \hat{D}_{n}\rangle$ is centered around its infinite temperature value at half-filling, i.e., $\langle \hat{D}_{\infty}\rangle = 1/2^{101}$. Moreover, the distribution of doublon density, $P(\hat{D})$, gradually narrows with increasing system size according to the
end, we evaluate dynamics of the stroboscopic-time staggered total density imbalance, \(\hat{\mathcal{I}}_{\text{tot}} = 2/N \sum_i \hat{I}_i\), evolving from an initially CDW state, \(\mathcal{Z}_{\text{CDW}}(nT) = \langle \psi_0 \rangle \langle -1 \rangle^n \langle \hat{\mathcal{I}}_{\text{tot}}(nT) \hat{\mathcal{I}}_{\text{tot}}(0) \rangle \psi_0 \rangle\), as a measure of the time crystallinity. In order to track the manifestation of DTC order in an initial-state independent manner, it is convenient to consider normalized Hilbert–Schmidt distance, \(\varrho_{\text{HS}}(nT) = \frac{\| \langle -1 \rangle^n \hat{\mathcal{I}}_{\text{tot}}(nT) \hat{\mathcal{I}}_{\text{tot}}(0) \rangle \|_\infty^2}{2 \| \hat{\mathcal{I}}_{\text{tot}} \|_\infty^2} = 1 - \mathcal{Z}_\infty(nT)\), where, \(\mathcal{Z}_\infty(nT) = \frac{1}{\| \hat{\mathcal{I}}_{\text{tot}} \|_\infty^2} \text{Tr} \left( \langle -1 \rangle^n \hat{\mathcal{I}}_{\text{tot}}(nT) \hat{\mathcal{I}}_{\text{tot}}(0) \rangle \right)\), and \(\| \cdots \|_\infty\) denotes the operator norm. Obviously, \(\mathcal{Z}_\infty(nT) \neq 0\) (or equivalently \(\varrho_{\text{HS}}(nT) \neq 1\)) when the Floquet dynamics exhibits DTC order. Autocorrelations of this type have also been used to identify the survival of the MBL and parent \(U(1)\) DTCs at infinite temperature. The results for a typical small value of \(V = 0.1\), shown in Fig. 4a, signify that \(\mathcal{Z}_\infty(nT)\) decays rapidly to zero. The same result also holds for the autocorrelators of local \(\hat{I}_i\) operators. So, any time crystal feature gets lost for this case. By contrast, \(\mathcal{Z}_{\text{CDW}}(nT)\) first drops to a smaller nonvanishing value followed by a long-lived plateau which eventually terminates by some finite-time revivals at the late times. This clearly adds to a strong initial-state dependence in the time crystallinity observed in the presence of many-body chaos.

To give system-size dependence of the crystalline melting time \(\tau_{\text{CDW}}\), followed by Refs. 16 and 50, we calculate the spectral weight for temporal correlation function \(A(\omega_{nm}) = \langle \langle \hat{\Psi}_n \rangle \langle \hat{\Psi}_m \rangle \rangle^2\) where \(\omega_{nm} = E_n - E_m\). This quantity is sharply peaked close to \(\omega_p \sim \pi/T\) (see the inset of Fig. 4a), and the plateaux in \(\mathcal{Z}_{\text{CDW}}\) fall off at times roughly proportional to \(T^{1-\gamma} \sim e^{O(N)}\). As shown in Fig. 4b, for \(V \lesssim V_{\text{th}}\) the melting times experience a strong system-size dependence compared to the \(N\)-independent \(\tau_{\text{approx}}\) extracted from \(\mathcal{Z}_{\infty}\). In this regime, the scaling behavior of \(\tau_{\text{CDW}}\) remains almost independent of \(\Delta\) (see upper panel of Fig. 4c), and hence is relatively insensitive to microscopic (explicit) symmetries of the model. By further increasing \(V\), the system-size dependence becomes weaker and ultimately diminishes for \(V \gtrsim V_{\text{th}}\), consistent with the ETH expectations.

From the first sight, the observed DTC response shows exactly the same diagnostics as those of the parent nonergodic model \(U_{\epsilon,\lambda,\Delta}^{\text{non}}(x,\epsilon,\lambda,\Delta)\), whose level spacing (in the DTC regime) does remain close to Poisson statistics with increasing system size. In particular, the subharmonic oscillations appear to exist for an infinitely long time. However, going to larger system sizes (by taking into account PBC) unveils that the exponential growth of \(\tau_{\text{CDW}}\) persists only up to a time scale \(\tau_v\), associated with a characteristic length \(\ell_{\tau_v,\Delta}\) (see bottom panel of Fig. 4c). For \(N > \ell_{\tau_v,\Delta}\), the melting time gradually stops its initial growth, but still remains exponentially large compared to \(\tau_{\text{approx}}\).

### IV. Persistence, Crossover and Rigidity

The observed anomalous DTTSB leads us to directly examine time-crystalline signature and its fundamentally distinct origin in the presence of quantum many-body chaos.

**Persistence**— In order to settle the dynamical fingerprint of the scar states, we first investigate the persistence of subharmonic response as well as its initial-state dependence. To this

**Figure 4.** (a) The stroboscopic dynamics of time crystal order parameter starting from a prefect initial CDW state at \(V = 0.1\). Inset displays the spectral weight peaked at \(T_{\omega_{nm}} \sim \pi\). In contrast to \(\mathcal{Z}_\infty\) (solid line), obtained from Eq. (4), the decay of \(\mathcal{Z}_{\text{CDW}}\) exhibits a long-lived DTC plateau with an exponentially diverging time scale set by \(T_{\omega_{nm}} \sim a^\gamma\) (dashed lines). (b) The scaling behavior of the crystalline melting times (circles) for various values of \(V\). Triangles denote \(\tau_{\approx}\)'s extracted from \(\mathcal{Z}_\infty\) for \(V = 0.1\). (c) Upper panel: the qualitative independence of the melting times on the potentialionic. Bottom panel demonstrates the existence of a characteristic finite-size length, at which \(\tau_{\text{approx}}\) stops its initial exponential growth.

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We proceed with training a RNN on the stroboscopic time-series of $C_i(nT) \equiv \langle \hat{Z}_i(nT)\hat{Z}_i(0) \rangle$ with $i = 1, \ldots, N/2$, evaluated during the first $n_{\text{max}} = 1000$ periods and sampled at 5 equally spaced points. Thus the input to our networks is of shape $(N/2, n_{\text{max}}/5)$. We choose a single hidden layer network only with 16 long short-term memory (LSTM) units\textsuperscript{114}, for fixed batchsize 100 and 400 epochs. In each epoch all training data lie inside the proposed range of $V_d \in [0, 5]$. The actual training of the network is done by 8000 samples, the learning rate $\alpha = 10^{-5}$, a dropout rate\textsuperscript{115} of 0.2, and minimizing the cross-entropy using Adam optimizer with weight decay ($l_2$ regularization) of 0.01, followed by a final softmax layer of size 2, corresponding to the ergodic and DTC classes we are distinguishing. Figure 5a reveals the W-like RNN performance curve that puts $V_d \equiv V_{th} \approx 0.22$ as true thermalization crossover, which is consistent with the previously estimated value extracted from the structure of entanglement (or mutual information) of Floquet eigenstates, shown in Fig. 3a. Additionally, the position of central peak remains merely intact with system size (Fig. 5b), and is almost independent of the ionic potential (see the phase diagram of Fig. 1b)\textsuperscript{116}. One can further verify this crossover through the Kullback-Leibler divergence (KLD)\textsuperscript{117}

$$D_{\text{ref}}(V) = \sum_\omega F_V(\omega) \log \left( \frac{F_V(\omega)}{F_{\text{ref}}(\omega)} \right),$$

which measures the distance between the normalized Fourier spectrum of $\langle \hat{Z}_{\text{tot}}(nT) \rangle$ at a fixed $V$, denoted by $F_V(\omega)$, and a reference signal corresponding to either a perfect DTC or a completely chaotic response, denoted by $F_{\text{DTC}}$ and $F_{\text{ETH}}$, respectively. At the true critical point $V_c$, one expects $F_{\text{DTC}}(\omega)$ to be equidistant from both $F_{\text{DTC}}$ and $F_{\text{ETH}}$, and thereby $D_{\text{DTC}}(V_c) = D_{\text{ETH}}(V_c)$\textsuperscript{112}. As is clear from Fig. 5, this condition is fulfilled for $V_c \approx V_{th}$, and the critical points extracted in this manner coincide very well with the predictions of the confused RNN.

**Rigidity**— Here we investigate the robustness of the SDTC dynamics in the finite but thermodynamically large system of size $N = 26$. Using numerically exact Krylov space based algorithm, we evaluate the stroboscopic dynamics of the total density imbalance $\langle \hat{Z}_{\text{tot}}(nT) \rangle$, and half-cut entanglement entropy $S_{\text{ent}}(nT)$ as function of $V$. The results depicted in Fig. 6 suggest three distinct dynamical regimes characterized by $V_{th}$ and $V_{pd}$: for small imperfection strength $V \lesssim V_{th}$ (region I in Fig. 6b), $\langle \hat{Z}_{\text{tot}}(nT) \rangle$ displays robust $2T$-oscillations locked at half of the driving frequency $\omega_0/2$. Moreover, the amplitude of the peak, and hence of the oscillations, in the power spectrum $|F_V(\omega_0/2)|$ is apparently large.

In the intermediate regime $V_{th} \lesssim V \lesssim V_{pd}$, at the ergodic side of thermalization crossover (shadow region of Fig. 6b), the broadening of the full width at half maximum (FWHM) still remains negligible, heralding the persistence of period-doubled dynamics. Nonetheless, the system displays a precursor to thermalizing dynamics: $S_{\text{ent}}(nT)$ typifies a logarithmic slow growth leading up to an inevitable thermalization at the late times (see Fig. 6c). Moreover, the time interval over which this logarithmic growth happens does not extend with system size, conveying a bounded, rather than un-

![Figure 5](image-url)

Figure 5. (a) Universal W-like NN performance curves (left axis) in the confused RNN for the model (1) with $N = 16, \Lambda = \Delta = 0.7$ and a fixed set of learning parameters: $l_2 = 0.01, \alpha = 10^{-5}$, dropout 0.2, batchsize of 100 and 400 training epochs. The middle peak points the exact value of the transition at $V_{th} \approx 0.22$ that coincides with the prediction of the KLD calculation in Eq. (5) (right axis). (b) System-size dependence of $V_{th}$ predicted from the machine learning (circles) and KLD analysis (squares). Repeating this procedure for different values of $\Delta$, leads to the phase diagram shown in Fig. 1b.

Our present investigations first suggest that the subharmonic oscillations in the generically chaotic model (1), hosting dynamical scars, should in principle be exponentially but not necessarily infinitely long-lived; the fact that marks the partial persistence of the SDTC dynamics. Second, they rule out the realization of a true crystal in $U_{\text{int}}^V$, as its lifetime does not strictly extend to infinity upon adding generic $V$ perturbation. Moreover, the signature of integrability in the quasi-energy spectrum of $U_{\text{int}}^V$ will be completely lifted by an arbitrary $V \neq 0$ as $N \to \infty$ (see Fig. 1c). Therefore, it cannot generally provide a stable protecting mechanism for realizing true DTC phase in a generic clean system.

**Thermalization crossover**— To specify the boundary between coherent and thermal regimes, from dynamics, we apply method from machine learning based on the “confusion” scheme\textsuperscript{110}, yet with employing recurrent neural networks (RNN) architecture instead of their more common nonrecurrent variants, namely feed-forward networks\textsuperscript{110–112}. The RNNs are designed for processing sequential data with a kind of memory (see e.g., Ref. 113). However, as a supervised method, it requires training on correctly labeled input-output pairs in the extremities of the phase space. Thus it is not directly applicable for the problem where labeling is not known beforehand, specially, from the perspective of finite-size and finite-time data. On the other hand, the heart of the semisupervised confusion algorithm is based on the purposefully mislabeling the input data through proposing dummy critical point $V_d$, and then evaluating the total performance of a trained network with respect to the proposed $V_d$. It is expected that the network performance takes a characteristic universal W-shape as a function of $V_d$, whose middle peak at $V_d$ implies the correct labeling associated with true critical point\textsuperscript{110}, as it would be easiest for the network to classify data for this choice of separation. Tending to do so, the confusion scheme through finding the majority label for the underlying (hidden) structure of dynamics, can be utilized to help RNN in the task of detecting SDTC to Floquet-ETH crossover using a prior unknown labels.
bounded slow heating. Lastly, for \( V \gtrsim V_{pd} \) in region II, FWHM becomes much more pronounced and any temporal feature would entirely disappear. The entanglement dynamics also changes its own behavior from an extremely slow growth in region I, to a fast one in region II where \( S_{ent} \) quickly approaches the maximal Page value within the time scales accessible by our numeric. This feature also verifies that the system size considered here, is thermodynamically large enough to warrant the immunity of our results against finite-size effects.

To firm up the observed period-doubling effect, and its stability, as a direct dynamical manifestation of scar states, we shed light on the structure of Floquet spectrum, when one arranges Floquet eigenstates according to their overlap with CDW state \( |a_0| = |\langle \psi_0 | \Psi_n \rangle| \), together with their mutual information \( F^{11}_n \) and second participation ratio \( PR_n = \sum_{a_n} |\langle a_n | \Psi_n \rangle|^2 \). The results shown in Fig. 7a signify that even deep in the SDTC regime, the dominant eigenstates are short-range correlated, delocalized states with an exponentially small \( F^{11}_n \) and \( PR_n \), which cannot exhibit symmetry breaking. However, they are irrelevant with respect to the initial condition, and hence cannot impede spontaneous DTTSB in a striking sense. Instead, the dynamics is dominated by special outlier states, which are localized on some subsets of thermalizing Floquet spectrum and display nontrivial spatial correlations. Figure 7b demonstrates the scaling of the largest overlap \( |a_0|^2 = \max \{|a_n|^2\} \), corresponding to the eigenstate with the most considerable weight in dynamics, as well as its respective \( F^{11}_0 \) and \( PR_0 \). Within the SDTC regime, we find the values \( |a_0|^2 \sim O(1/2) \), \( F^{11}_0 \sim \log 2 \) and \( PR_0 \sim O(1) \), whose scaling remains fairly constant with system size and only exhibits slow decay upon approaching \( V_{th} \). Beyond this regime, they do appear to be decreasing exponentially with \( N \); the behavior which becomes much more pronounced as system size increases. These results give a clear illustration of the intimate connection between the rigidity of the SDTC dynamics and the robustness of dynamical scars.

A. Coherent thermalizing dynamics: analogues to Floquet supersymmetry

Here we investigate how the presence of dynamical scars affects dynamics of certain observables, irrespective of specific choice of initial condition. We explore the behavior of the autocorrelation function, \( C^{\infty}_{\text{tot}}(nT) \equiv \langle \hat{I}_{\text{tot}}(nT) \hat{I}_{\text{tot}}(0) \rangle_{\infty} \), which takes the same form as Eq. (4) without the factor \( \frac{1}{1 - \mathbf{P}} \). Here our main focus is on the SDTC regime, where the scar states has a tangible effect on the manifestation of DTTSB, once the system evolves from a simple product CDW state. In this regime, the Floquet operator can be restricted to the space spanned by the athermal dynamical scar states, as well as its complement subspace containing otherwise ergodic states, i.e., \( \mathcal{U} = \mathcal{U}_F \mathcal{P}_S + \mathcal{U}_F (1 - \mathcal{P}_S) \), where \( \mathcal{P}_S \) is the projection onto the scarred subspace.

Figure 8 displays a number of representative time-traces for \( C^{\infty}_{\text{tot}}(nT) \) in the extremities of the phase space. For \( V = 0.7 \) where the dynamics is controlled by a set of thermal states, there are no persistent oscillations. However, for \( V = 0.1 \) corresponds to the SDTC regime, \( C^{\infty}_{\text{tot}}(nT) \) exhibits short-time period-two oscillations around its infinite-temperature value. Strikingly, the magnitude of the \( \pi/T \) peak in the power spectrum, \( |F^{\infty}_{\text{tot}}(\omega_0/2)| \), is exponentially decaying with increasing system size (see the inset). Hence, the subharmonic oscillation of \( C^{\infty}_{\text{tot}}(nT) \) can persist only at short times and die.
off exponentially fast in system size, as opposed to a true time crystal. Here, the contribution of $\mathcal{U}_F(1 - P_S)$ tends to drive system towards eventual thermalization, yet with an oscillatory response attributed to the component $\mathcal{U}_F P_S$. The finite-size suppression of oscillations can also be understood through exponential diminution of the number of special scar states with respect to the entire Hilbert space. This nontrivial thermalizing dynamics—with robust $2T$-oscillation of certain observables—is distinct from those prescribed by the conventional Floquet-ETH with no definite frequency.

Such coherent approach to thermal equilibrium emerging in a finite-size, generically chaotic system is reminiscent of that recently observed as a consequence of “Floquet super-symmetry” (FSUSY)\(^{53}\). Similarly, there, $P_S$ can be interpreted as a projector onto a degenerate subspace comprising a measure zero set of nonthermal eigenstates, pinned to 0 and $\pi$ quasi-energy modes, which are protected by the ancillary time-reflection symmetry.\(^{53}\) By contrast, the scarring effect in our generic model is of pure dynamical origin that emerges in the absence of any protecting primary symmetry, and hence does not require such tuning.

**V. EMERGENCE OF QUASI-CONSERVATION LAWS**

We now turn to the explanation of the SDTC dynamics via the emergence of dynamical constraints in the form of long-lived local quasi-conservation laws. To this end, we look at the stroboscopic evolution of participation entropy,

$$S_d(nT) = -\sum_{|i\rangle \in \mathcal{H}_d} |\langle i|\psi(nT)\rangle|^2 \log |\langle i|\psi(nT)\rangle|^2,$$

starting from the CDW state $|\psi_0\rangle$, which measures the spreading of an initial wavefunction over a certain basis in the course of time. Here the computational basis is grouped into the subspaces $\mathcal{H}_d$, each of them has a fixed Hamming distance from $|\psi_0\rangle$; the distance which is defined as a minimum number of particle exchanges required to transform a specific basis into the CDW pattern. Clearly, at the limit of $\lambda/V \to \infty$, e.g., at the exact integrable $V = 0$ line, the system possesses explicit local conservation laws over multiples of two driving periods, i.e., $[\tilde{\mathcal{L}}_i, \mathcal{U}_F^{\pm}] = 0$, and applying $\mathcal{U}_F$ to $|\psi_0\rangle$ displaces the state into its particle-hole counterpart with the maximum distance, $d_{\text{max}} = N/2$. The subsequent action of $\mathcal{U}_F$ will bring it back to itself at $d_{\text{min}} = 0$, closing the cycle at time $2T$. This procedure is carried out perfectly without delocalization in any intervening subspaces such that $S_d(nT) = 0$ for all $d$, even at infinite time.

For a typical finite value of $V$, however, there is no such an exact conservation. However, the behavior of $S_d(nT)$ shown in Fig. 9a, suggests the emergence of long-lived quasi-conservation laws within the SDTC regime: the wavefunction remains almost localized and only partially leaks into the nearby sectors in the vicinity of $d_{\text{min}}$ and $d_{\text{max}}$. Indeed, the SDTC dynamics does not mix different eigenstates in different mutually conserved sectors and approximately preserves the underlying local conservation laws, i.e., $[\tilde{\mathcal{L}}_i, \mathcal{U}_F^{\pm}] \sim 0$. Hence, it infers that if we label $|\psi_0\rangle$ by the set of $\{\mathcal{I}_i\}$, and probe the dynamics stroboscopically in multiples of two driving periods, it cannot strikingly evolve out to a different subspace even in the absence of explicit local conservation laws. This bears some resemblance to the fracturing effect, in which a number of product states (so-called inert\(^{91,92}\) configurations) remain invariant by the dynamics and construct (exactly) localized Krylov subspaces of dimension one, characterized by a set of state-dependent LIOMs. It should be noted that this phenomenon does not necessary require an explicit form of fracton-like constraints, i.e., charge and dipole conservation. Such a constrained dynamics can also asymptotically emerge from the confinement of quasiparticle excitations in the strongly interacting limit of some unconstrained Hamiltonians\(^{96,97}\), specifically those with a similar form to $H_P$ in Eq. (1)\(^{98}\).

Near the ergodic side of the thermalization crossover, e.g., at $V = 0.3$, the wavefunction begins to marginally spread over increasingly distant other sectors, while still preserves its coherent oscillatory behavior during early-to-intermediate times. However, the substantial growth of $S_d(nT)$ eventually happens at the late times, implying the delayed onset of Floquet thermalization. By further increasing $V$ deep in the thermal regime, the rapid expansion of the initial wavefunction indicates the absence of any emergent constraint, which in turn allows the model to thermalize faster.

The same conclusion also holds when one rearranges the computational basis $|i\rangle$, according to the sectors characterized by total doublon density, $D$. Again in the SDTC regime, the wavefunction partially delocalizes about its initial doublon sector, i.e., $D = 0$, and does not explore its entire phase space (see Fig. 9b). Hence, the dynamics starting from $|\psi_0\rangle$ is effectively restricted to approximately preserve the initial doublon number. The emergence of doublon conservation occurs in spite of the fact that the dominant eigenstates of the Floquet operator do not generally exhibit such a conservation law.
and look like the featureless infinite-temperature states (as already mentioned in Fig. 3b). This effect is of pure dynamical origin, through which an initial-state dependent, nonergodic dynamics (though without strict periodicity) would happen in the strongly interacting limit of a generic Floquet system.

VI. DISCUSSION

In summary, we have presented compelling evidence that the quantum many-body dynamics of strongly interacting, chaotic Floquet systems can exhibit anomalous DTTSB protected by weak ergodicity breaking. This breakdown is attributed to the presence of special scar subregion of the Hilbert space with anomalous long-range order, which leads to a robust, long-lasting SDTC dynamics from a family of experimentally accessible initial states. The stability of dynamical scars to generic perturbations of the drive reflects the rigidity of the SDTC response. We utilize the confused RNN to keep track the crossover between SDTC and fully ergodic regimes purely from dynamics. Such machine learning based method can also be highly suitable for classifying partially nonergodic phases, in particular when the exact structure of dynamical phase diagram is not known beforehand.

The chaotic model considered in this work can be seen as a generic deformation of the parent nonergodic model $U^{int}_F$, which contains emergent integrable manifold in a wide range of its parameter space. By deformation of the model towards $U^{int}_F$, the subharmonic response will be enhanced up to a finite time scale $\tau_c$, corresponding to the characteristic length $\ell_{V, \Delta}$, and not necessary being infinitely long-lived. These features are analogous to those of thermalizing (static) Hamiltonians, hosting scar states, that are in proximity to a putative integrable point. It should be pointed out that, arguments along the lines of Ref. 89 suggest a tendency towards thermalization that would arise in $U^{int}_F$, and more generally, in any parent nonergodic model with a finite distance from an exact integrable manifold/point; a natural tendency that may appear at larger time/length scales than are accessible by the simulation methods. This is the reason why in the strongly interacting regime of $U^{int}_F$, $\tau_c$ seems infinite in a finite-sized system.

However, our investigation does not strictly rule out the possibility of the existence of a genuine SDTC phase. One of the most promising directions for future works is finding the signs of such an exotic DTC phase in systems exhibiting strong fracture, which provide a concrete (and more provable) paradigm of partially nonergodic phases. One can examine whether the exactly localized subspaces of the underlying models might be amenable to harbor rSG order persisting for an infinite time. Such a study spells out the minimal ingredients needed for realizing a true clean DTC.

Another outstanding challenge is whether the SDTC phenomenon can be reconciled using the framework of mixed phase space, recently extended to the realm of many-body chaos. While the quantum many-body analogue of the KAM theorem is not available yet, the notion of mixed phase space can still be valid, giving rise to weak ergodicity breaking on general grounds. This standpoint is based on projecting many-body quantum dynamics into effective classical equations of motion through time-dependent variational principle (TDVP) in the restricted MPS manifold; a semiclassical approach that in general stands beyond mean-field description. It may be highly valuable to examine the SDTC dynamics and its stability via the aforementioned TDVP ansätze and characterize possible deformations that increase this stability through the concept of “quantum leakage”. Such a study on the one hand gives an intuition about the notion of dynamical scars, and on the other hand sheds light on the relevant parameter regime, local observables, and initial conditions for which the SDTC behavior may be potentially observed.
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