$\mathcal{PT}$ Symmetric Aubry-Andre Model

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(Dated: February 13, 2014)

$\mathcal{PT}$ Symmetric Aubry-Andre Model describes an array of $N$ coupled optical waveguides with position dependent gain and loss. We show that the reality of the spectrum depends sensitively on the degree of disorder for small number of lattice sites. We obtain the Hofstadter Butterfly spectrum and discuss the existence of the phase transition from extended to localized states. We show that rapidly changing periodical gain/loss materials almost conserves the total intensity.

PACS numbers: 11.30.Er, 42.82.Et, 03.65.-w

I. INTRODUCTION

Recent experimental realization of $\mathcal{PT}$ symmetric optical systems with balanced gain and loss has attracted a lot of attention \cite{1,2}. The $\mathcal{PT}$ symmetric optical systems lead to interesting results such as unconventional beam refraction and power oscillation \cite{4,5}, nonreciprocal Bloch oscillations \cite{6}, unidirectional invisibility \cite{7}, an additional type of Fano resonance \cite{8}, and chaos \cite{9}. In the $\mathcal{PT}$ symmetric optical systems, the net gain or loss of particles vanishes due to the balanced gain and loss mechanism. These systems are described by non-Hermitian Hamiltonian with real energy eigenvalues provided that non-Hermitian degree is below than a critical number, $\gamma_{\mathcal{PT}}$. If it is beyond the critical number, spontaneous $\mathcal{PT}$ symmetry breaking occurs. It implies the eigenfunctions of the Hamiltonian are no longer simultaneous eigenfunction of $\mathcal{PT}$ operator and consequently the energy spectrum becomes either partially or completely complex. The critical number of non-Hermitian degree is shown to be different for planar and circular array configurations \cite{10}, and it can be increased if impurities and tunneling energy are made position-dependent in an extended lattice \cite{11}. However, $\gamma_{\mathcal{PT}}$ decreases with increasing the lattice sites \cite{12,13}, hence the $\mathcal{PT}$ symmetric phase is fragile. An important consequence of $\mathcal{PT}$ symmetric optical systems is the power oscillations. It was shown that the beam power in a one dimensional tight binding chain doesn’t depend on the microscopic details such as disorder and periodicity \cite{14}. The probability-preserving time evolution in terms of the Dirac inner product for $\mathcal{PT}$ symmetric tight-binding ring was considered \cite{15}. It is interesting to note that the $\mathcal{PT}$ operator coincides with time evolution operator at some certain times that allows perfect state transfer in the $\mathcal{PT}$ symmetric optical lattice with position dependent tunneling energy \cite{16}. The equivalent Hermitian Hamiltonian for a tight-binding chain can also be constructed to understand the non-Hermitian system \cite{17}. In this paper, we investigate disordered array of $\mathcal{PT}$ symmetric tight binding chain \cite{18,19}. It is well known that disorder in quantum mechanical systems induces localization. Here, we show that localization occurs if some certain conditions are satisfied. Our system is described by the $\mathcal{PT}$ symmetric extension of the Aubry Andre model \cite{20}. The energy spectrum associated with Hermitian Aubry-Andre model at certain strength of parameters has fractal structure, which is known as the Hofstadter butterfly spectrum \cite{21,22}. We also investigate the Hofstadter butterfly spectrum in the presence of non-Hermitian impurities.

II. MODEL

Consider an array of $N$ coupled optical waveguides with position dependent gain and loss and constant tunneling amplitude $J$ through which light is transferred from site to site. We adopt open boundary conditions. The beam propagation in the tight-binding structure can be described by a set of equations for the electric field amplitudes $c_n$,

$$i \frac{dc_n}{dz} = -J(c_{n+1} + c_{n-1}) + i \gamma_n c_n , \quad (1)$$

where $n = 1, 2, ..., N$ is the waveguide number and the position dependent non-Hermitian degree $\gamma_n$ describes the strength of gain/loss material that is assumed to be balanced, i.e., $\sum_{n=1}^{N} \gamma_n = 0$. The field amplitude transforms as $c_n \rightarrow c_{N-n+1}$ under parity transformation and the complex number transforms as $i \rightarrow -i$ under anti-linear time reversal transformation. Thus the global $\mathcal{PT}$ symmetry is lost unless a precise relation between $\gamma_n$ holds. To model disorder, $\gamma_N$ can be chosen randomly with zero mean \cite{13}. In this case, the system would be no longer $\mathcal{PT}$ symmetric and the corresponding energy eigenvalues are not real. Bendix et al. studied a disordered system by considering a pair of $N$ coupled dimers with impurities $(\gamma_n, -\gamma_n)$ \cite{14}. They noted that the system is not $\mathcal{PT}$ symmetric as a whole (global symmetry), but it possesses a local $\mathcal{PT}$ symmetry that admits real spectrum. Here, we consider a disordered system with global $\mathcal{PT}$ symmetry and study localization, which is well known to occur in a disordered Hermitian lattice. Consider the following gain/loss parameter

$$i \gamma_n = V \cos (2\pi \beta n + \phi_N) + i \gamma_0 \sin (2\pi \beta n + \phi_N) , \quad (2)$$
where $V$ and $\gamma_0$ are constants, $\beta$ determines the degree of the disorder and $\phi_N$ is the constant phase difference which depends on the total number of sites. We require that gain and loss are balanced, so we demand $\phi_N = -\pi \beta (N + 1) + \phi_0$, where the constant $\phi_0$ is an integer multiple of $\pi$. Without loss of generality, we take $\phi_0 = 0$. We emphasize that the system is $\mathcal{PT}$ symmetric globally. The Equ. (1) with (2) can be called the $\mathcal{PT}$ symmetric Aubry-Andre model [25], which can now be engineered experimentally [1–3]. The most interesting result of the Hermitian Aubry-Andre model is that the states at the center of the lattice is localized (Anderson localization) for irrational values of $\beta$ when $V > 2$. Apparently, the non-Hermitian character of the Aubry-Andre equation (1) could change the physics of this system dramatically. Note also that Aubry-Andre model coincides with the Harper model when $V = 2$ and $\gamma_0 = 0$ and the energy spectrum as a function of $\beta$ is known as Hofstadter butterfly spectrum, which is an example of fractal structure that appears in physics [26, 27]. Here, we study the Hofstadter butterfly spectrum and localization effect for the $\mathcal{PT}$ symmetric Aubry-Andre model.

It is sufficient to analyze the region $0 < \beta < 1$ since the system repeats itself in equal intervals of $\beta$. Furthermore, the energy spectrum is symmetric with respect to $\beta = 0.5$ axis and the spectrum does not depend on the sign of $\gamma_0$. As a special case, if $\beta$ is either 0 or 1, then the gain/loss terms vanish. If $\beta = 1/2$ and $N$ is even, the system has gain and loss with amplitudes $\mp i\gamma_0$ at alternating lattice sites. The gain/loss materials change periodically if $\beta$ is a rational number and quasi-periodically if $\beta$ is an irrational number. In the latter case, the gain/loss impurities are disordered. Note that $\beta$ can be given with a finite number of digits in a real experiment. To increase the incommensurability of $\beta = p/q$ ($p, q$ are two coprime positive integers), one can choose sufficiently large $p$ and $q$. Then the system becomes strongly disordered.

We look for stationary solutions of the equation (1). Suppose first that $V = 0$. In the absence of gain and loss, the system has the well known energy spectrum of width $4J$: $E = -2J \cos n \pi/N$. In the presence of gain and loss, the real part of the energy eigenvalues, $\mathcal{R}\{E\}$, are still contained in $[-2J, 2J]$ for any $N$. More precisely, the energy width is a decreasing function $|\gamma_0|$. The distribution of $\mathcal{R}\{E\}$ crucially depends on the strength of disorder through the value of $\beta$. It consists of a finite number of bands when $\beta$ is rational. In this case, $\mathcal{R}\{E\}$ is the union of bands and the length of the gap between any two bands depends on $q$ ($\beta = p/q$). On the other hand, the fractal structure appears and the spectrum is a Cantor set when $\beta$ is irrational (for the mathematicians, this property is known as the 10 Martinis conjecture [28] in the Hermitian limit). Such a fractal structure can be seen in the Fig. (1), where we plot the $\mathcal{PT}$ symmetric Hofstadter butterfly spectrum at $\gamma_0 = 2$, $V = 0$ (a) and at $V = 2$, $\gamma_0 = 0.1$ (b). An important difference between $V = 0$ and $V \neq 0$ cases is that the symmetry with respect to zero energy axis is lost for the latter case. However, the real part of the energy eigenvalues is symmetric with respect to $\beta = 0.5$ axis for any $V$. Note also that the width of $\mathcal{R}\{E\}$ increases with $V$ and it takes its maximum value when $\gamma_0 = 0$. We show the nice fractal picture for the real part of the spectrum. As can be seen below, the $\mathcal{PT}$ symmetry breaking point is very small for large $N$ and thus the corresponding energy spectrum is not real. However, there exists some special values of $\beta$ for the Fig. (1) with entirely real spectrum. For example, the spectrum is real when $\beta = 1/5$. To gain more insight on the role of disorder, let us study how the real and imaginary parts of the energy change with $\gamma_0$ for weakly and strongly disordered system. The Fig. (2) plots $\mathcal{R}\{E\}$ as a function $\gamma_0$ for a weak $\beta = 1/3$ and strong disorders $\beta = 11/30$ when $N = 30$. As can be seen from the figures, the degree of disorder in the lattice has a dramatic effect for large values of $\gamma_0$. The real part of energy shrink to zero (they become degenerate and the energy width becomes zero) for very large values of $\gamma_0$ if the system is periodic while this is not the case if it is quasi-periodical. However, the corresponding imaginary part of energy eigenvalues are different from zero for such a large number

\[ \begin{align*}
\text{(a) } N=50, J=1, V=0 \text{ and } \gamma_0=2 \\
\text{(b) } N=50, J=1, V=2 \text{ and } \gamma_0=0.1
\end{align*} \]

FIG. 1: The $\mathcal{PT}$ symmetric Hofstadter butterfly spectrum for the real part of the energy eigenvalues at $N = 50$ and $\gamma_0 = 2$ (a), $\gamma_0 = 0.1$ (b). The graph has a line of reflection at $\beta = 1/2$. 

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of $\gamma_0$. If the impurity strength, $\gamma_0$, exceeds a critical point, $\gamma_{PT}$, $PT$ symmetry is spontaneously broken and thus the energy eigenvectors are not simultaneous eigenvectors of the Hamilton and $PT$ operators. In this case, the energy eigenvalues become partially or entirely complex. An important consequence for our system is that strong disorder increases the critical point $\gamma_{PT}$ considerably. The Fig. 3 plots the imaginary part of the energy eigenvalues for various values of $N$ at $\beta = 0.6$ and the inverse of the golden ratio $\beta = (\sqrt{5} - 1)/2 \approx 0.618$, which is the common choice in the study of the Aubry-Andre model. We numerically find that due to the disorder, $\gamma_{PT}$ increases by a factor of nearly 1.6 when $N = 25$. However, the number of lattice sites $N$ has dominant effect on $\gamma_{PT}$ and thus increasing the degree of the disorder has slightly changes $\gamma_{PT}$ for large $N$. The critical point decreases with increasing $N$ and approaches zero when $N$ is large. The $PT$ symmetric phase is said to be fragile since $\gamma_{PT}$ is zero as $N \to \infty$.

It is well known that the Hermitian Aubry-Andre model, $\gamma_0 = 0$, displays a phase transition from extended to exponentially localized states (It is also known as metal insulator transition [23]). Of particular importance is the self-dual point $V/J = 2$ where the localization transition occurs [23]. Let us now study whether localization takes place for the $PT$ symmetric Aubry-Andre model. Suppose first $V = 0$. We take the inverse of the golden ratio $\beta = \frac{\sqrt{5} - 1}{2}$, $J = 1$ and $N = 49$ with the initial condition $|c_n(z = 0)|^2 = \delta_{n,25}$. We find numerically that initially localized wave packet delocalizes in time when $\gamma_0 = 2$. We repeat numerical solution for large values of $\gamma_0$, but exponentially localized states do not emerge for the $PT$ symmetric Aubry-Andre model. This result is interesting that disorder does not induce localization for the $PT$ symmetric Aubry Andre model contrary to Hermitian one. This is because of the fragile nature of the $PT$ symmetric phase. Before metal insulator phase transition takes place, the system enters broken $PT$ symmetric phase and the corresponding intensity grows exponentially, where the intensity is given by $I = \sum_{n=1}^{N} |c_n|^2$ and satisfies

$$\frac{dI}{dt} = 2 \sum_{n=1}^{N} \Re \{\gamma_n \} |c_n|^2.$$  \hspace{1cm} (3)

The intensity grows exponentially in the broken $PT$ symmetric case while it oscillates when the energy spectrum is entirely real.

Suppose now $V \neq 0$. We find numerically the time evolution of the single site excitation. It is well known that the metal-insulator transition occurs at $V = 2$ and $\gamma_0 = 0$ and the wave packet is localized around the single site when $V > 2$. If $V < 2$, the probability $|c_{25}|^2$ goes to zero rapidly with $z$. The presence of gain/loss change the dynamics significantly. Although the probability $|c_{25}|^2$ doesn’t rapidly go to zero when $\gamma_0 \neq 0$, this can not be considered localization in the rigorous sense. This is because the introduction of gain/loss to the system does not conserve the total intensity and the generated particles enter the system not only $n = 25$-th lattice site but also the other lattice sites. Thus the occupation on the waveguide away from $n = 25$ is not negligible. For large $z$, the particles are generated even at the edges of the system.

A question arises. Does the phase transition from extended to exponentially localized state occurs if we some-
how find a way to make the total intensity bounded for large values of $\gamma_0$? To answer this question, consider $z$-dependent periodic impurity strength [30–35]

$$i\gamma_n = V \cos(2\pi \beta_n + \phi_N) + i\gamma_0 \cos(2\pi \omega z) \sin(2\pi \beta_n + \phi_N), \quad (4)$$

where $\omega$ is a constant. Note that the corresponding Hamiltonian is still $\mathcal{PT}$ invariant. The gain and loss are also locally balanced after one period.

The intensity oscillates in time when $\omega = 0$ if $\gamma_0 < \gamma_{PT}$. The oscillation is not in general periodic. Introducing periodically changing impurity, $\omega \neq 0$, makes the oscillation periodical with $z$. Increasing $\omega$ decreases the period of the intensity. We assert that the intensity is in principle conserved in the limit $\omega \to \infty$ since impurities do not have enough time to transfer intensity to the system. So, we expect that rapidly changing impurities practically conserves the intensity. To check this argument, we solve the equation (1) numerically. We find that the intensity is almost constant when $\omega = 3$ as can be seen from the Fig. (4). The disorder has nothing to do with the intensity conservation and the intensity is almost conserved for any values of $\beta$.

To predict localization, let us define the variance of the probability distribution as $\sigma(z) = \sqrt{\sum_n (n - \bar{n})^2 |c_n|^2 / P}$, where $\bar{n} = \sum_n n|c_n|^2 / P$ is the average site $z$-dependent occupation [33]. We plot the variance in the Fig. (4). The linearly increasing $\sigma(z)$ with respect to $z$ implies that the wave packet delocalizes (spreads ballistically with $z$).

On the contrary, oscillating $\sigma(z)$ shows us that the wave packet is localized. We find that the onset of localization appears for $\mathcal{PT}$ symmetric Aubry-Andre model provided that $\sqrt{V/J} > 2$ and the system is disordered. We emphasize that localization doesn’t take place if the system is ordered, i.e. $\beta$ is a rational number. In the localization regime, the occupation at $n = 25$-th well oscillates periodically with $z$ and is almost constant for large values of $V$. We note that the underlying mechanism of localization studied here is essentially the same of Anderson localization.

To summarize, we have studied $\mathcal{PT}$ symmetric tight binding optical lattice with disordered impurities. We have considered the complex extension Aubry-Andre model. We have plotted complex Hofstatder butterfly spectrum and shown that the reality of the spectrum depends sensitively on the impurity strength and $\beta$. We have shown that the critical point $\gamma_{PT}$ increases with the increasing degree of disorder. We have demonstrated that the transition from extended to localized states does not occur for the system described by $\mathcal{PT}$ symmetric Aubry Andre model. The metal insulator transition occurs if the impurities changes periodically with $z$ at each site. We have also shown that rapidly changing periodical impurities conserves the total intensity.
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