Factorization formulas for tree amplitudes

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Abstract We present a coordinate space version of the factorization formula for the connected tree part of the chronological products. We consider a general framework, and then we apply it for the QCD case.

1 Introduction

The most natural way to arrive at the Bogoliubov axioms of perturbative quantum field theory (pQFT) is by analogy with non-relativistic quantum mechanics [1–3]. So we start from Bogoliubov axioms [4,5] as presented in [6]; for every set of monomials $A_1(x_1), \ldots, A_n(x_n)$ in some jet variables (associated to some classical field theory) one associates the operator-valued distributions $T^{A_1 \ldots A_n}(x_1, \ldots, x_n)$ called chronological products; it will be convenient to use another notation: $T(A_1(x_1), \ldots, A_n(x_n))$.

The Bogoliubov axioms, presented in Sect. 3, express essentially some properties of the scattering matrix understood as a formal perturbation series with the “coefficients” the chronological products: (1) (skew)symmetry properties in the entries $A_1(x_1), \ldots, A_n(x_n)$; (2) Poincaré invariance; (3) causality; (4) unitarity; (5) the “initial condition” which says that $T(A(x))$ is a Wick monomial. So we need some basic notions on free fields and Wick monomials which will be presented in Sect. 2 also following [6]. One can supplement these axioms by requiring (6) power counting; (7) Wick expansion property. It is a highly non-trivial problem to find solutions for the Bogoliubov axioms, even in the simplest case of a real scalar field.

There are, at least to our knowledge, three rigorous ways to do that; for completeness, we remind them following [7]: (a) Hepp axioms [2]; (b) Polchinski flow equations [8,9]; (c) the causal approach due to Epstein and Glaser [1,5] which we prefer. It is a recursive procedure for the basic objects $T(A_1(x_1), \ldots, A_n(x_n))$ and reduces the induction procedure to a distribution splitting of some distributions with causal support. In an equivalent way, one can reduce the induction procedure to the process of extension of distributions [10]. An equivalent point of view uses retarded products [11] instead of chronological products. For gauge models, one has to deal with non-physical fields (the so-called ghost fields) and impose

To the memory of Günter Scharf.

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a supplementary axiom (8) namely gauge invariance, which guarantees that the physical states are left invariant by the chronological products.

In this paper, we will prove some factorization properties of connected tree contributions of the chronological products. Such sort of factorization formulas have appeared a long time ago [12–14] in momentum space. For a pedagogical exposition, see, for instance, [15] and [16]. We give a purely combinatorial proof of some formulas of such a type in coordinate space. The framework is quite general: it works for any trilinear interaction Lagrangian. In Sect. 5, we will consider the particular case of QCD and try to obtain the formulas of [12] stripping the Feynman amplitudes of the color factors. We present a quite general method, and we use only well defined mathematical objects: the chronological products. A similar analysis appears in [17].

2 Wick products

We follow the formalism from [6]. We consider a classical field theory on the Minkowski space $\mathcal{M} \simeq \mathbb{R}^4$ (with variables $x^\mu, \mu = 0, \ldots, 3$ and the metric $\eta$ with $\text{diag}(\eta) = (1, -1, -1, -1)$ described by the Grassmann manifold $\Xi_0$ with variables $\xi_a, a \in \mathcal{A}$ (here $\mathcal{A}$ is some index set) and the associated jet extension $J^r(\mathcal{M}, \Xi_0)$, $r \geq 1$ with variables $x^\mu, \xi_{a;\mu_1,\ldots,\mu_n}, n = 0, \ldots, r$; we denote generically by $\xi_p, p \in P$ the variables corresponding to classical fields and their formal derivatives and by $\Xi_r$ the linear space generated by them. The variables from $\Xi_r$ generate the algebra $\text{Alg}(\Xi_r)$ of polynomials.

To illustrate this, let us consider a real scalar field in Minkowski space $\mathcal{M}$. The first jet-bundle extension is

$$J^1(\mathcal{M}, \mathbb{R}) \simeq \mathcal{M} \times \mathbb{R} \times \mathbb{R}^4$$

with coordinates $(x^\mu, \phi, \phi_\mu), \mu = 0, \ldots, 3$.

If $\varphi : \mathcal{M} \to \mathbb{R}$ is a smooth function we can associate a new smooth function $j^1 \varphi : \mathcal{M} \to J^1(\mathcal{M}, \mathbb{R})$ according to

$$j^1 \varphi(x) = (x^\mu, \varphi(x), \partial_\mu \varphi(x)).$$

For higher-order jet-bundle extensions, we have to add new real variables $\phi_{[\mu_1,\ldots,\mu_r]}$ considered completely symmetric in the indexes. For more complicated fields, one needs to add supplementary indexes to the field, i.e., $\phi \to \phi_a$ and similarly for the derivatives. The index $a$ carries some finite dimensional representation of $\text{SL}(2, \mathbb{C})$ (Poincaré invariance) and maybe a representation of other symmetry groups. In classical field theory, the jet-bundle extensions $j^r \varphi(x)$ do verify Euler–Lagrange equations. To write them, we need the formal derivatives defined by

$$d_v \phi_{[\mu_1,\ldots,\mu_r]} \equiv \phi_{[v,\mu_1,\ldots,\mu_r]}.$$  \hspace{1cm} (2.1)

We suppose that in the algebra $\text{Alg}(\Xi_r)$ generated by the variables $\xi_p$ there is a natural conjugation $A \to A^\dagger$. If $A$ is some monomial in these variables, there is a canonical way to associate with $A$ a Wick monomial: we associate with every classical field $\xi_a, a \in \mathcal{A}$ a quantum free field denoted by $\xi^\text{quant}_a(x), a \in \mathcal{A}$ and determined by the 2-point function

$$\langle \Omega, \xi^\text{quant}_a(x), \xi^\text{quant}_b(y)\rangle = -i D^{(+)}(\xi_a(x), \xi_b(y)) \times 1.$$  \hspace{1cm} (2.2)

Here

$$D_{ab}(x - y) \equiv D(\xi_a(x), \xi_b(y))$$  \hspace{1cm} (2.3)

is the causal Pauli–Jordan distribution associated with the two fields; it is (up to some numerical factors) a polynomial in the derivatives applied to the Pauli–Jordan distribution. We
understand by $D_{ab}(x)$ the positive and negative parts of $D_{ab}(x)$. The $n$-point functions for $n \geq 3$ are obtained assuming that the truncated Wightman functions are null: see [18], relations (8.74) and (8.75) and proposition 8.8 from there. The definition of these truncated Wightman functions involves the Fermi parities $|\xi_p|$ of the fields $\xi_p$, $p \in P$.

Afterward, we define

$$\xi_a^{\text{quant}}_{\mu_1, \mu_2, \ldots, \mu_n}(x) \equiv \partial_{\mu_1} \ldots \partial_{\mu_n} N^\text{quant}_a(x), a \in \mathcal{A}$$

which amounts to

$$D(\xi_a^{\text{quant}}_{\mu_1, \mu_2, \ldots, \mu_n}(x), \xi_{b,v_1, \ldots, v_n}(y)) = (-1)^n i \partial_{\mu_1} \ldots \partial_{\mu_m} \partial_{v_1} \ldots \partial_{v_n} D_{ab}(x - y) \times 1. \quad (2.4)$$

More sophisticated ways to define the free fields involve the GNS construction.

The free quantum fields are generating a Fock space $F$ in the sense of the Borchers algebra: formally it is generated by states of the form $\xi_1^{\text{quant}}(x_1) \ldots \xi_n^{\text{quant}}(x_n) \Omega$ where $\Omega$ the vacuum state. The scalar product in this Fock space is constructed using the $n$-point distributions, and we denote by $F_0 \subseteq F$ the algebraic Fock space.

One can prove that the quantum fields are free, i.e., they verify some free field equation; in particular, every field must verify Klein Gordon equation for some mass $m$

$$(\Box + m^2) \xi_a^{\text{quant}}(x) = 0 \quad (2.5)$$

and it follows that in momentum space they must have the support on the hyperboloid of mass $m$. This means that they can be split in two parts $\xi_a^{\text{quant}(\pm)}$ with support on the upper (resp. lower) hyperboloid of mass $m$. We convene that $\xi_a^{\text{quant}(+)}$ resp. $\xi_a^{\text{quant}(-)}$ correspond to the creation (resp. annihilation) part of the quantum field. The expressions $\xi_a^{\text{quant}(+)}$ resp. $\xi_a^{\text{quant}(-)}$ for a generic $\xi_p$, $p \in P$ are obtained in a natural way, applying partial derivatives.

For a general discussion of this method of constructing free fields, see Ref. [18]—especially prop. 8.8. The Wick monomials are leaving invariant the algebraic Fock space. The definition for the Wick monomials is contained in the following proposition.

Proposition 2.1 The operator-valued distributions $N(\xi_{q_1}(x_1), \ldots, \xi_{q_n}(x_n))$ are uniquely defined by:

$$N(\xi_{q_1}(x_1), \ldots, \xi_{q_n}(x_n))\Omega = \xi_1^{(+)}(x_1) \ldots \xi_n^{(+)}(x_n)\Omega \quad (2.6)$$

$$[\xi_p(y), N(\xi_{q_1}(x_1), \ldots, \xi_{q_n}(x_n))] = -i \sum_{m=1}^n \prod_{l < m} (-1)^{|\xi_p||\xi_{q_l}|} D_{p q_m}(y - x_m) N(\xi_{q_1}(x_1), \ldots, \hat{m}, \ldots, \xi_{q_n}(x_n)) \quad (2.7)$$

$$N(\emptyset) = 1. \quad (2.8)$$

The expression $N(\xi_{q_1}(x_1), \ldots, \xi_{q_n}(x_n))$ is (graded) symmetrical in the arguments.

The expression $N(\xi_{q_1}(x_1), \ldots, \xi_{q_n}(x_n))$ are called Wick monomials. There is an alternative definition based on the splitting of the fields into the creation and annihilation part for which we refer to [6].

It is a non-trivial result of Wightman and Gårding [19] that in $N(\xi_{q_1}(x_1), \ldots, \xi_{q_n}(x_n))$ one can collapse all variables into a single one and still get a well-defined expression:

Proposition 2.2 The expressions

$$W_{q_1, \ldots, q_n}(x) \equiv N(\xi_{q_1}(x), \ldots, \xi_{q_n}(x)) \quad (2.9)$$
are well defined. They verify:

\[
W_{q_1,\ldots,q_n}(x) = \frac{\langle \xi^{(+)}(x) \rangle}{\Omega_1} \cdots \frac{\langle \xi^{(+)}(x) \rangle}{\Omega_1}
\]  

(2.10)

\[
[\xi^{(e)}_p(y), W_{q_1,\ldots,q_n}(x)] = -i \sum_{m=1}^{n} \prod_{l<m} (-1)^{|\xi^{(e)}_p||\xi^{(e)}_q|} \|p_{q_m}^{(e)}(y-x_m) W_{q_1,\ldots,q_n}(x)\]

(2.11)

\[
W(\emptyset) = I.
\]  

(2.12)

We call expressions of the type \(W_{q_1,\ldots,q_n}(x)\) Wick monomials. By

\[
|W| = \sum_{l=1}^{n} |\xi^{(e)}_l|
\]  

(2.13)

we mean the Fermi number of \(W\). We define the derivative

\[
\frac{\partial}{\partial \xi^p} W_{q_1,\ldots,q_n}(x) \equiv \sum_{s=1}^{n} \prod_{l<s} (-1)^{|\xi^{(e)}_p||\xi^{(e)}_q|} \|p_{q_s}^{(e)} W_{q_1,\ldots,q_s,\ldots,q_n}(x)\]

(2.14)

and we have a generalization of the preceding proposition.

**Proposition 2.3** Let \(W_j = W_{q_1^{(j)},\ldots,q_n^{(j)}}(x), j = 1,\ldots,n\) be Wick monomials. Then, the expression \(N(W_1(x_1),\ldots,W_n(x_n))\) is well defined through

\[
N(W_1(x_1),\ldots,W_n(x_n)) \Omega = \prod_{j=1}^{n} \prod_{l=1}^{r_j} \langle \xi^{(+)}_{q_l^{(j)}}(x_j) \rangle \Omega
\]  

(2.15)

\[
[\xi^{(e)}_p(y), N(W_1(x_1),\ldots,W_n(x_n))] = -i \sum_{m=1}^{n} \prod_{l<m} (-1)^{|\xi^{(e)}_p||\xi^{(e)}_q|} \|p_{q_m}^{(e)}(y-x_m) N(W_1(x_1),\ldots,\frac{\partial}{\partial \xi^{(e)}_q} W_m(x_m),\ldots,W_n(x_n))
\]  

(2.16)

\[
N(W_1(x_1),\ldots,W_n(x_n), 1) = N(W_1(x_1),\ldots,W_n(x_n))
\]  

(2.17)

\[
N(W(x)) = W(x).
\]  

(2.18)

The expression \(N(W_1(x_1),\ldots,W_n(x_n))\) is symmetric (in the Grassmann sense) in the entries \(W_1(x_1),\ldots,W_n(x_n)\).

One can prove that

\[
[N(A(x)), N(B(y))] = 0, \quad (x-y)^2 < 0
\]  

(2.19)

where by \([,\cdot]\) we mean the graded commutator. This is the most simple case of causal support property. Now we are ready for the most general setting. We define for any monomial \(A \in \text{Alg}(\Xi_r)\) the derivation

\[
\xi \cdot A \equiv (-1)^{|\xi||A|} \frac{\partial}{\partial \xi} A
\]  

(2.20)

for all \(\xi \in \Xi_r\). Here, \(|A|\) is the Fermi parity of \(A\) and we consider the left derivative in the Grassmann sense. So, for the moment, the product \(\cdot\) is defined as an map \(\Xi_r \times \text{Alg}(\Xi_r) \to \text{Alg}(\Xi_r)\). An expression \(E(A_1(x_1),\ldots,A_n(x_n))\) is called of Wick type iff verifies:
\[
[\xi_p(y), E(A_1(x_1), \ldots, A_n(x_n))] = -i \sum_{m=1}^{n} \prod_{l \leq m} (-1)^{|\xi_p||A_l|}
\sum_q D_{pq}(y - x_m) \ E(A_1(x_1), \ldots, \xi_q \cdot A_m(x_m), \ldots, A_n(x_n))
\]  
(2.21)

\[
E(A_1(x_1), \ldots, A_n(x_n), 1) = E(A_1(x_1), \ldots, A_n(x_n))
\]  
(2.22)

\[
E(1) = 1.
\]  
(2.23)

The expression \(N(W_1(x_1), \ldots, W_n(x_n))\) from proposition 2.3 is of Wick type. Then, we easily have:

**Proposition 2.4** If \(E(A_1(x_1), \ldots, A_k(x_k))\) and \(F(A_{k+1}(x_1), \ldots, A_n(x_n))\) are expressions of Wick type, then \(E(A_1(x_1), \ldots, A_k(x_k)) \ F(A_{k+1}(x_1), \ldots, A_n(x_n))\) is also an expression of Wick type.

Now we formulate Wick theorem. First, we extend the product (2.20) to more factors through iteration:

\[
(\xi \eta) \cdot A = \xi \cdot (\eta \cdot A), \quad \xi, \eta \in \mathbb{E}_0
\]  
(2.24)

and \(A\) an arbitrary monomial. So now \(\cdot\) is a map \(\text{Alg}(\mathbb{E}_r) \times \text{Alg}(\mathbb{E}_r) \rightarrow \text{Alg}(\mathbb{E}_r)\) In particular, it makes sense to consider expressions of the type \(B \cdot A\) where \(A\) and \(B\) are both monomials. One gets something non-null if \(B\) is a submonomial of \(A\). One easily derives that

\[
A \cdot A = C(A)1
\]  
(2.25)

where \(C(A)\) is a numerical factor. Then, we have:

**Theorem 2.5** (Wick) Let \(E(A_1(x_1), \ldots, A_n(x_n))\) be an expression of Wick type. The following formula is true:

\[
E(A_1(x_1), \ldots, A_n(x_n)) = \sum_{B \in \mathbb{E}} \epsilon(B_1, \ldots, B_n; A_1, \ldots, A_n)
\langle \Omega, E(B_1(x_1), \ldots, B_n(x_n))\Omega \rangle \ N(B_1 \cdot A_1(x_1), \ldots, B_n \cdot A_n(x_n))
\]  
(2.26)

where \(B_j\) are distinct Wick submonomials of \(A_j\) and

\[
\epsilon(B_1, \ldots, B_n; A_1, \ldots, A_n) = (-1)^s \prod_{l=1}^{n} C(B_l)^{-1}
\]  
(2.27)

with

\[
s \equiv \sum_{l=1}^{n} |B_l| \left( \sum_{p=l+1}^{n} (|A_p| + |B_p|) \right) = \sum_{p=2}^{n} (|A_p| + |B_p|) \left( \sum_{l=1}^{p-1} |B_l| \right).
\]  
(2.28)

In the same way, we prove:

**Theorem 2.6** The following formula is true:

\[
N(\xi_p(y), A_1(x_1), \ldots, A_n(x_n)) = \xi_p(y) \cdot N(A_1(x_1), \ldots, A_n(x_n))
\]

\[+i \sum_{m=1}^{n} \prod_{l < m} (-1)^{|\xi_p||A_l|} \sum_q D_{pq}^{(+)}(y - x_m) \ N(A_1(x_1), \ldots, \xi_q \cdot A_m(x_m), \ldots, A_n(x_n))
\]  
(2.29)
3 Bogoliubov axioms

Suppose the monomials $A_1, \ldots, A_n \in \text{Alg}(\Xi)$ are self-adjoint: $A_j^\dagger = A_j$, $\forall j = 1, \ldots, n$ and of Fermi number $f_i$.

The chronological products

$$T(A_1(x_1), \ldots, A_n(x_n)) \equiv T^{A_1, \ldots, A_n}(x_1, \ldots, x_n) \quad n = 1, 2, \ldots$$

are some distribution-valued operators leaving invariant the algebraic Fock space and verifying the following set of axioms:

- **Skew-symmetry** in all arguments:

  $$T(\ldots, A_i(x_i), A_{i+1}(x_{i+1}), \ldots) = (-1)^{f_i f_{i+1}} T(\ldots, A_{i+1}(x_{i+1}), A_i(x_i), \ldots) \quad (3.1)$$

- **Poincaré invariance**: we have a natural action of the Poincaré group in the space of Wick monomials and we impose that for all $g \in \text{inSL}(2, \mathbb{C})$ we have:

  $$U_g T(A_1(x_1), \ldots, A_n(x_n)) U_g^{-1} = T(g \cdot A_1(x_1), \ldots, g \cdot A_n(x_n)) \quad (3.2)$$

where in the right hand side we have the natural action of the Poincaré group on $\Xi$.

Sometimes it is possible to supplement this axiom by other invariance properties: space and/or time inversion, charge conjugation invariance, global symmetry invariance with respect to some internal symmetry group, supersymmetry, etc.

- **Causality**: if $y \cap (x + \bar{V}^+) = \emptyset$ then we denote this relation by $x \succeq y$. Suppose that we have $x_i \succeq x_j$, $\forall i \leq k$, $j \geq k + 1$. then we have the factorization property:

  $$T(A_1(x_1), \ldots, A_n(x_n)) = T(A_1(x_1), \ldots, A_k(x_k)) \ T(A_{k+1}(x_{k+1}), \ldots, A_n(x_n)); \quad (3.3)$$

- **Unitarity**: We define the *anti-chronological products* using a convenient notation introduced by Epstein-Glaser, adapted to the Grassmann context. If $X = \{j_1, \ldots, j_s\} \subset N \equiv \{1, \ldots, n\}$ is an ordered subset, we define

  $$T(X) \equiv T(A_{j_1}(x_{j_1}), \ldots, A_{j_s}(x_{j_s})). \quad (3.4)$$

Let us consider some Grassmann variables $\theta_j$, of parity $f_j$, $j = 1, \ldots, n$ and let us define

$$\theta_X \equiv \theta_{j_1} \cdots \theta_{j_s}. \quad (3.5)$$

Now let $(X_1, \ldots, X_r)$ be a partition of $N = \{1, \ldots, n\}$ where $X_1, \ldots, X_r$ are ordered sets. Then, we define the (Koszul) sign $\epsilon(X_1, \ldots, X_r)$ through the relation

$$\theta_1 \cdots \theta_n = \epsilon(X_1, \ldots, X_r) \ \theta_{X_1} \cdots \theta_{X_r} \quad (3.6)$$

and the antichronological products are defined according to

$$(-1)^n \tilde{T}(N) \equiv \sum_{r=1}^{n} (-1)^r \sum_{I_1, \ldots, I_r \in \text{Part}(N)} \epsilon(X_1, \ldots, X_r) \ T(X_1) \ldots T(X_r) \quad (3.7)$$

Then, the unitarity axiom is:

$$\tilde{T}(N) = T(N)^\dagger. \quad (3.8)$$
• The “initial condition”:

\[ T(A(x)) = N(A(x)). \]  

(3.9)

• **Power counting**: We can also include in the induction hypothesis a limitation on the order of singularity of the vacuum averages of the chronological products associated with arbitrary Wick monomials \( A_1, \ldots, A_n \); explicitly:

\[
\omega(\langle \Omega, T^{A_1, \ldots, A_n}(X)\Omega \rangle) \leq \sum_{l=1}^{n} \omega(A_l) - 4(n - 1)
\]  

(3.10)

where by \( \omega(d) \) we mean the order of singularity of the (numerical) distribution \( d \) and by \( \omega(A) \) we mean the canonical dimension of the Wick monomial \( W \).

• **Wick expansion property**: In analogy to (2.21), we require

\[
\left[ \xi_p(y), T(A_1(x_1), \ldots, A_n(x_n)) \right] = -i \sum_{m=1}^{n} \prod_{l \leq m} (-1)^{\xi_p[|A_l|]} \sum_{q} D_{pq}(y - x_m) \ T(A_1(x_1), \ldots, \xi_q \cdot A_m(x_m), \ldots, A_n(x_n))
\]  

(3.11)

Up to now, we have defined the chronological products only for self-adjoint Wick monomials \( W_1, \ldots, W_n \), but we can extend the definition for Wick polynomials by linearity.

The construction of Epstein–Glaser is based on a recursive procedure [5]. We can derive from these axioms the following result [20].

**Theorem 3.1** One can fix the causal products such that the following formula is true

\[
T(\xi_p(y), A_1(x_1), \ldots, A_n(x_n))
\]

\[
= -i \sum_{m=1}^{n} \prod_{l \leq m} (-1)^{\xi_p[|A_l|]} \sum_{q} D_{pq}^F(y - x_m) \ T(A_1(x_1), \ldots, \xi_q \cdot A_m(x_m), \ldots, A_n(x_n)) + \xi^{(+)}_p(y) T(A_1(x_1), \ldots, A_n(x_n)) + \prod_{l \leq n} (-1)^{\xi_p[|_{fi}]} T(A_1(x_1), \ldots, A_n(x_n)) \xi^{(-)}_p(y)
\]

(3.12)

where \( D_{pq}^F \) is a Feynman propagator associated with the causal distribution \( D_{pq} \).

Some times (3.12)—or variants of it—is called the **equation of motion** axiom [21].

**4 Tree contributions**

For simplicity, we assume that the jet space is generated by scalars variables \( \xi_a, a \in \mathcal{A} \).

If \( A \) is a monomial in the jet variables, then we denote by \( \rho(A) \) the number of factors. Then, we have:

**Proposition 4.1** Let \( A_1, \ldots, A_n \) such that

\[
\sum_{j=1}^{n} \rho(A_j) = 2n - 2.
\]

(4.1)

Then, the chronological product \( T^{\text{conn}}(A_1(x_1), \ldots, A_n(x_n)) \) is a c-number.
Proof We use the topological relations valid for the connected part of the chronological products:

\[
L = I - n + 1
\]

\[
\sum_{j=1}^{n} \rho(A_j) = 2I + B
\]

(4.2)

where \(L\) is the number of loops, \(I\) is the number of internal lines, and \(B\) the number of external lines. If we eliminate \(I\) and use the relation from the statement, we end up with

\[
2L + B = 0
\]

so we have \(L = 0\), \(B = 0\). In particular, from \(B = 0\) it follows that we do not have external lines, so \(T_{\text{conn}}(A_1(x_1), \ldots, A_n(x_n))\) is a \(c\)-number.

Corollary 4.2 In the preceding conditions, suppose that we have \(p_1\) entries \(A\) with \(\rho(A) = 1\), \(p_2\) entries with \(\rho(A) = 2\) and \(p_3\) entries with \(\rho(A) = 3\). Then, we have

\[
p_1 = p_3 + 2
\]

\[
p_2 = n - 2 - 2p_3
\]

(4.3)

so in particular we must have \(n \geq 2(p_3 + 1)\).

Proposition 4.3 Let \(A_2, \ldots, A_{n-1}\) with \(\rho(A_j) = 2\), \(\forall j\). Then, the following formula is valid

\[
T_{\text{conn}}(\xi_{a_1}(x_1), A_2(x_2), \ldots, A_{n-1}(x_{n-1}), \xi_{a_n}(x_n))
\]

\[
= (-i)^{n-1}(n-2)! S[D^F(\xi_{a_1}(x_1), \xi_{a_2}(x_2))
\]

\[
\prod_{j=2}^{n-2} D^F(\xi_{a_j} \cdot A_j(x_j), \xi_{a_{j+1}}(x_{j+1})) D^F(\xi_{a_{n-1}} \cdot A_{n-1}(x_{n-1}), \xi_{a_n}(x_n))\] \(n \geq 3\)

(4.4)

where \(S\) symmetrizes in \(A_2(x_2), \ldots, A_{n-1}(x_{n-1})\).

Proof By induction on \(n\). For \(n = 3\), this follows immediately from the field equation property; then the induction goes rather easily using again the field equation property. We remark that because the monomials \(A\) are bilinear, then the formal derivative \(\xi \cdot A\) is a linear expression in the fields, so the expression \(D^F(\xi \cdot A(x), \eta(y))\) makes sense.

From the preceding proposition, we have

Corollary 4.4 The preceding formula can be written in the following form

\[
T_{\text{conn}}(\xi_{a_1}(x_1), A_2(x_2), \ldots, A_{n-1}(x_{n-1}), \xi_{a_n}(x_n))
\]

\[
= \sum_{p=2}^{n-1} C_{n-3}^{p-2} S[T_{\text{conn}}(\xi_{a_1}(x_1), A_2(x_2), \ldots, A_{p-1}(x_{p-1}), \xi_{a_p}(x_p))
\]

\[
T_{\text{conn}}(\xi_{a_p} \cdot A_p(x_p), A_{p+1}(x_{p+1}), \ldots, A_{n-1}(x_{n-1}), \xi_{a_n}(x_n))\] \(n \geq 3\)

(4.5)

We have our first form of a factorization formula. However, we would want to get rid of the combinatorial factors.

The first step is:
Corollary 4.5  Suppose that

\[ A_p = \frac{1}{2} g^{(p)}_{bc} \xi_b \xi_c, \quad g^{(p)}_{bc} = b \leftrightarrow c. \]

Then, the preceding formula can be written in the following form

\[ T^{\text{conn}}(\xi_{a_1}(x_1), A_2(x_2), \ldots, A_{n-1}(x_{n-1}), \xi_{a_n}(x_n)) \]

\[ = \frac{1}{n-2} \sum_{p=2}^{n-1} g^{(p)}_{bc} C_{n-3}^{p-2} S(p)[T^{\text{conn}}(\xi_{a_1}(x_1), A_2(x_2), \ldots, A_{p-1}(x_{p-1}), \xi_b(x_p)) \]

\[ T^{\text{conn}}(\xi_c(x_p), A_{p+1}(x_{p+1}), \ldots, A_{n-1}(x_{n-1}), \xi_{a_n}(x_n))] \quad (n \geq 3) \]

where \( S(p) \) symmetrizes in \( A_2(x_2), \ldots, A_{p-1}(x_{p-1}), A_{p+1}(x_{p+1}), \ldots, A_{n-1}(x_{n-1})). \)

Proof  We use \( S = \frac{1}{n-2} \sum_{p=2}^{n-1} S(p) \) and \( \xi_b \cdot A_p = g^{(p)}_{bc} \xi_c. \)

Next we have:

Proposition 4.6  Let \( A_1, \ldots, A_n \) with \( \rho(A_j) = 2, \forall j. \) Then

\[ T^{\text{conn}}(\xi_{a_1}(z_1), \xi_{a_2}(z_2), A_1(x_1), \ldots, A_n(x_n)) \]

\[ = \frac{1}{n} \sum_{p=1}^{n} g^{(p)}_{bc} \sum_{I_1, I_2 \in \text{Part}(N_p)} T^{\text{conn}}(\xi_{a_1}(z_1), \xi_b(x_p), I_1) T^{\text{conn}}(\xi_{a_2}(z_2), \xi_c(x_p), I_2) \]

(4.7)

Here \( N_p \equiv \{1, \ldots, n\} - \{p\} \) and for \( I = \{i_1, \ldots, i_l\}, \ p \notin I \) we denote

\[ T^{\text{conn}}(\xi_a(z), \xi_b(x_p), I) \equiv T^{\text{conn}}(\xi_a(z), \xi_b(x_p), A_{i_1}(x_{i_1}), \ldots, A_{i_l}(x_{i_l})). \]  

(4.8)

Equivalently

\[ T^{\text{conn}}(\xi_{a_1}(z_1), \xi_{a_2}(z_2), A_1(x_1), \ldots, A_n(x_n)) \]

\[ = \frac{1}{n} \sum_{I_0, I_1, I_2 \in \text{Part}(N), |I_0| = 1} g^0_{bc} T^{\text{conn}}(\xi_{a_1}(z_1), \xi_b(x_{I_0}), I_1) T^{\text{conn}}(\xi_{a_2}(z_2), \xi_c(x_{I_0}), I_2) \]

(4.9)

where if \( I_0 = \{p\} \) we denote \( g^0_{bc} \equiv g^{(p)}_{bc} \) and \( x_{I_0} \equiv x_p. \) From the last formula the symmetry property at \( \xi_{a_1}(z_1) \leftrightarrow \xi_{a_2}(z_2) \) is obvious, as it is the symmetry in \( A_1(x_1), \ldots, A_n(x_n). \)

Proof  We consider the last formula from the statement. The symmetry at \( \xi_{a_1}(z_1) \leftrightarrow \xi_{a_2}(z_2) \) follows if we use the redefinitions \( I_1 \leftrightarrow I_2, \ b \leftrightarrow c. \) For the symmetry at \( A_j(x_j) \leftrightarrow A_k(x_k) \) (with \( j, k \) fixed), we split the sum in three contributions: a) \( j, k \in I_1 \) (or \( j, k \in I_2 \); b) \( j \in I_1, k \in I_2; \ c) p = j \) of \( p = k. \) The first two contributions are obviously invariant to \( A_j(x_j) \leftrightarrow A_k(x_k) \) and the two contributions from case c) are mapped one into each other.

Now we prove that the last formula from the statement coincides with the formula from the preceding proposition. Because of the symmetry property in \( A_1(x_1), \ldots, A_n(x_n) \) just proved we can introduce the symmetrization operator \( S \) without changing anything:

\[ T^{\text{conn}}(\xi_{a_1}(z_1), \xi_{a_2}(z_2), A_1(x_1), \ldots, A_n(x_n)) \]

\[ = \frac{1}{n} \sum_{I_0, I_1, I_2 \in \text{Part}(N), |I_0| = 1} g^0_{bc} S[T^{\text{conn}}(\xi_{a_1}(z_1), \xi_b(x_{I_0}), I_1) T^{\text{conn}}(\xi_{a_2}(z_2), \xi_c(x_{I_0}), I_2)] \]
Now we consider all contributions with \(|I_1| = p - 1\). There are \(C_{n-1}^{p-1}\) such contributions and all are equal because of the presence of the symmetrization operator \(S\). So, we can take \(I_1 = \{1, \ldots, p - 1\}, \ I_2 = \{p + 1, \ldots, n\}\) and \(I_0 = \{p\}\). The formula from the preceding proposition follows. \(\square\)

Next we have

**Proposition 4.7** The \(n\) contributions from the preceding proposition are equal. It follows that we have

\[
T^{\text{conn}}(\xi_{a_1}(z_1), \xi_{a_2}(z_2), A_1(x_1), \ldots, A_n(x_n)) = g_{bc}^{(1)} \sum_{I_1, I_2 \in \text{Part}([2, \ldots, n])} T^{\text{conn}}(\xi_{a_1}(z_1), \xi_{b}(x_1), I_1) T^{\text{conn}}(\xi_{a_2}(z_2), \xi_{c}(x_1), I_2)
\]

(4.10)

**Proof** By induction. For \(n = 2\), the equality follows easily. We suppose that we have the equality for \(2, \ldots, n - 1\) \((n \geq 3)\) and consider the case \(n\). It is sufficient to prove the equality of the first two contributions. In the expression

\[
E_1 \equiv g_{bc}^{(1)} \sum_{I_1, I_2 \in \text{Part}([2, \ldots, n])} T^{\text{conn}}(\xi_{a_1}(z_1), \xi_{b}(x_1), I_1) T^{\text{conn}}(\xi_{a_2}(z_2), \xi_{c}(x_1), I_2)
\]

we have two types of terms: a) \(2 \in I_1\), b) \(2 \in I_2\); we apply in both contributions the induction hypothesis. We use the same idea in the contribution

\[
E_2 \equiv g_{bc}^{(2)} \sum_{I_1, I_2 \in \text{Part}([1, 3, \ldots, n])} T^{\text{conn}}(\xi_{a_1}(z_1), \xi_{b}(x_2), I_1) T^{\text{conn}}(\xi_{a_2}(z_2), \xi_{c}(x_2), I_2)
\]

and we obtain \(E_1 = E_2\). \(\square\)

Next we introduce in the game a Wick monomial with three factors.

**Proposition 4.8** Let \(A_1, \ldots, A_n\) with \(\rho(A_j) = 2\), \(\forall j\) and

\[
B = \frac{1}{3!} g_{b_1b_2b_3} \xi_{b_1} \xi_{b_2} \xi_{b_3}
\]

(4.11)

with \(g_{b_1b_2b_3}\) completely symmetric. Then,

\[
T^{\text{conn}}(\xi_{a_1}(z_1), \xi_{a_2}(z_2), \xi_{a_3}(z_3), A_1(x_1), \ldots, A_n(x_n), B(y)) = g_{b_1b_2b_3} \sum_{I_1, I_2, I_3 \in \text{Part}([1, \ldots, n])} T^{\text{conn}}(\xi_{b_1}(z_1), \xi_{b}(y), I_1) T^{\text{conn}}(\xi_{a_2}(z_2), \xi_{b_2}(y), I_2) T^{\text{conn}}(\xi_{a_3}(z_3), \xi_{b_3}(y), I_3).
\]

(4.12)

The symmetry in \(\xi_{a_j}(z_j)\) and in \(A_1(x_1), \ldots, A_n(x_n)\) is manifest.

**Proof** By induction on \(n\). For \(n = 0\), we use the equation of motion axiom and obtain the formula. We consider the formula valid for \(1, \ldots, n - 1\), and we have for \(n\) with the equation

\(\square\)
of motion axiom:

\[ T^{\text{conn}}(\xi_{a_1}(z_1), \xi_{a_2}(z_2), \xi_{a_3}(z_3), A_1(x_1), \ldots, A_n(x_n), B(y)) \]

\[ = -i \sum_{j=1}^{n} D^F(\xi_{a_1}(z_1), \xi_{d}(x_j)) \]

\[ + T^{\text{conn}}(\xi_{a_2}(z_2), \xi_{a_3}(z_3), A_1(x_1), \ldots, \xi_{d} \cdot A_j(x_j), \ldots, A_n(x_n), B(y)) \]

\[ - i \sum_{j=1}^{n} D^F(\xi_{a_1}(z_1), \xi_{d}(y)) \]

\[ + T^{\text{conn}}(\xi_{a_2}(z_2), \xi_{a_3}(z_3), A_1(x_1), \ldots, A_n(x_n), \xi_{d} \cdot B(y)) \]  

(4.13)

In the first contribution, we apply the induction hypothesis, and in the second contribution we use the preceding proposition.

Next we have:

**Proposition 4.9** In the preceding conditions, we have

\[ T^{\text{conn}}(\xi_{a_1}(z_1), \xi_{a_2}(z_2), \xi_{a_3}(z_3), A_1(x_1), \ldots, A_n(x_n), B(y)) = 3 g^{(1)}_{bc} \]

\[ \sum_{I_1, I_2 \in \text{Part}([2, \ldots, n])} \mathcal{S}[T^{\text{conn}}(\xi_{a_1}(z_1), \xi_{b}(x_1), I_1) T^{\text{conn}}(\xi_{a_2}(z_2), \xi_{a_3}(z_3), \xi_{c}(x_1), I_2, B(y))] \]  

(4.14)

where \( \mathcal{S} \) symmetrizes in \( \xi_{a_j}(z_j) \), \( j = 1, 2, 3 \). We have similar formulas with \( g^{(1)}_{bc} \rightarrow g^{(j)}_{bc} \) (\( \forall j \)).

**Proof** In the right hand side of the formula from the statement, we consider three contributions corresponding to \( 1 \in I_1, 1 \in I_2, 1 \in I_3 \) and use the preceding proposition. Summing the three contributions we get the left hand side of the formula from the statement.

Finally, we have the main result.

**Theorem 4.10** In the preceding conditions, the following formula is true for \( m \geq 1 \)

\[ T^{\text{conn}}(\xi_{a_1}(z_1), \ldots, \xi_{a_{m+2}}(z_{m+2}), A_1(x_1), \ldots, A_n(x_n), B(y_1), \ldots, B(y_m)) \]

\[ = \frac{1}{3!} g^{(1)}_{bcd} \sum_{X_1, X_2, X_3 \in \text{Part}([\xi_{a_1}, \ldots, \xi_{a_{m+2}}])} T^{\text{conn}}(Z_1, \xi_{b}(y_1), X_1, Y_1) T^{\text{conn}}(Z_2, \xi_{c}(y_1), X_2, Y_2) T^{\text{conn}}(Z_3, \xi_{d}(y_1), X_3, Y_3) \]  

(4.15)

where the sum \( \sum \) runs over the following partitions:

\[ Z_1, Z_2, Z_3 \in \text{Part}([\xi_{a_1}, \ldots, \xi_{a_{m+2}}]) \]

\[ X_1, X_2, X_3 \in \text{Part}([A_1, \ldots, A_n]) \]

\[ Y_1, Y_2, Y_3 \in \text{Part}([B_2, \ldots, B_m]) \]

constrained by

\[ \text{card}(Z_j) = \text{card}(Y_j) + 2, \ j = 1, 2, 3. \]  

(4.16)

We also have for \( n \geq 1 \)

\[ T^{\text{conn}}(\xi_{a_1}(z_1), \ldots, \xi_{a_{m+2}}(z_{m+2}), A_1(x_1), \ldots, A_n(x_n), B(y_1), \ldots, B(y_m)) \]

\[ = \frac{1}{2!} g^{(1)}_{ef} \sum_{X_1, X_2 \in \text{Part}([A_1, \ldots, A_n])} T^{\text{conn}}(Z_1, \xi_{e}(x_1), X_1, Y_1) T^{\text{conn}}(Z_2, \xi_{f}(x_1), X_2, Y_2) \]  

(4.17)
where the sum $\sum$ runs over the following partitions:

\[ Z_1, Z_2 \in \text{Part}([\xi_{a_1}, \ldots, \xi_{a_{m+2}}]) \]
\[ X_1, X_2 \in \text{Part}([A_2, \ldots, A_n]) \]
\[ Y_1, Y_2 \in \text{Part}([B_1, \ldots, B_m]) \]

constrained by

\[ \text{card}(Z_j) = \text{card}(Y_j) + 1, \ j = 1, 2. \]  \hfill (4.18)

We denote formula (4.15) by $(B_{m,n})$ and formula (4.17) by $(A_{m,n})$.

**Proof** We use a double induction over $m$ and $n$.

(i) For $m = 0$, we have only $(A_{0,n})$ which is Proposition 4.7. For $m = 1$, we have $(B_{1,n})$ which is Proposition 4.8 and $(A_{1,n})$ which is Proposition 4.9.

(ii) First we prove that $(B_{m,n}) \implies (A_{m,n})$ using the same argument as in Proposition 4.9.

(iii) The induction hypothesis is

\[ (B_{1,n}), \ldots, (B_{m-1,n}), \ (m \geq 2) \]

and we prove $(B_{m,n})$. We prove this last formula by induction over $n$.

Then, we start the induction proving $(B_{m,0})$.

(iv) Using the equation of motion axiom, we have

\[ T^{\text{conn}}(\xi_{a_1}(z_1), \ldots, \xi_{a_{m+2}}(z_{m+2}), B_1(y_1), \ldots, B_m(y_m)) \]

\[ = -i \sum_{j=1}^{m} D^F(\xi_{a_1}(z_1), \xi_b(y_j)) \]

\[ T^{\text{conn}}(\xi_{a_1}(z_2), \ldots, \xi_{a_{m+2}}(z_{m+2}), B_1(y_1), \ldots, \xi_b \cdot B_j(y_j), \ldots, B_m(y_m)) \]

\hfill (4.19)

and we have two contributions corresponding to $j = 1$ and $j > 1$. In the first contribution, we apply the induction hypothesis $(A_{m-1,1})$, and in the second contribution we apply $(B_{m-1,1})$. After some rearrangements of the terms, we get $(B_{m,0})$.

(v) Now we suppose that

\[ (B_{m,0}), \ldots, (B_{m,n-1}) \]

are valid and we prove $(B_{m,n})$. As before we have

\[ T^{\text{conn}}(\xi_{a_1}(z_1), \ldots, \xi_{a_{m+2}}(z_{m+2}), A_1(x_1), \ldots, A_n(x_n), B_1(y_1), \ldots, B_m(y_m)) \]

\[ = -i \sum_{j=1}^{m} D^F(\xi_{a_1}(z_1), \xi_b(y_j)) \]

\[ T^{\text{conn}}(\xi_{a_2}(z_2), \ldots, \xi_{a_{m+2}}(z_{m+2}), A_1(x_1), \ldots, A_n(x_n), B_1(y_1), \]

\[ \ldots, \xi_b \cdot B_j(y_j), \ldots, B_m(y_m)) \]

\[ -i \sum_{j=1}^{n} D^F(\xi_{a_1}(z_1), \xi_b(x_j)) \]

\[ T^{\text{conn}}(\xi_{a_2}(z_2), \ldots, \xi_{a_{m+2}}(z_{m+2}), A_1(x_1), \ldots, \xi_b \cdot A_j(x_j), \]

\[ \ldots, A_n(x_n), B_1(y_1), \ldots, B_m(y_m)) \]  \hfill (4.20)
The first contribution is analyzed as at (iv): for the term corresponding to \( j = 1 \) we use \((A_{m-1,n+1}, B_{m-1,n+1})\) and for the contribution corresponding to \( j > 1 \) we use \((B_{m,n-1})\).

Finally, for the second contribution from the right hand side of (4.20) we use the induction hypothesis \((B_{m,n-1})\). If we cleverly combine the various contributions, we obtain \((B_{m,n})\).

Corollary 4.11 (i) Let \( A_1, \ldots, A_n \) trilinear monomials. Then, we have for the connected tree contributions \( T_{\text{conn}}^{(0)} \) the following formula for \( n \geq 4 \)

\[
T_{\text{conn}}^{(0)} (A_1(x_1), \ldots, A_n(x_n)) = \frac{1}{3!} g_{bcd}^{(1)} \sum : T_{\text{conn}}^{(0)} (I_1, \xi_b(x_1)) T_{\text{conn}}^{(0)} (I_2, \xi_c(x_1)) T_{\text{conn}}^{(0)} (I_3, \xi_d(x_1)) : \tag{4.21}
\]

where the sum runs over the partitions of the set \( A_2, \ldots, A_n \).

(ii) Let \( A_1 \) be a bilinear Wick monomial and \( A_2, \ldots, A_n \) trilinear monomials. Then, we have for \( n \geq 3 \)

\[
T_{\text{conn}}^{(0)} (A_1(x_1), \ldots, A_n(x_n)) = \frac{1}{2!} g_{ef}^{(1)} \sum : T_{\text{conn}}^{(0)} (I_1, \xi_e(x_1)) T_{\text{conn}}^{(0)} (I_2, \xi_f(x_1)) : \tag{4.22}
\]

where the sum runs over the partitions of the set \( A_2, \ldots, A_n \).

Proof We combine Wick theorem 2.26 and the preceding theorem.

Now we are ready to study the so-called Berends–Giele currents [12]. We observe from (i) of the preceding corollary that the connected part of the tree contributions to the chronological products \( T_{\text{conn}}^{(0)} (B_1(y_1), \ldots, B_n(y_n)) \) can be expressed in terms of chronological products of the type \( T_{\text{conn}}^{(0)} (\xi_a(z), B_1(y_1), \ldots, B_m(y_m)), m < n \). These are the Berends–Giele currents in coordinate space. There is a special entry linear in the fields which corresponds to the special line of [12]: The field \( \xi_a(z) \) is taken to be off-shell in this reference. First, we give the recursion formula for such currents which is a coordinates version of the recursion formula of [12].

Proposition 4.12 Suppose that

\[
B_j = \frac{1}{3} g_{bcd}^{(j)} \xi_b \xi_c \xi_d. \tag{4.23}
\]

with \( g_{bcd}^{(j)} \) completely symmetric. Then, the following recursion formulas are valid for \( n \geq 3 \)

\[
T_{\text{conn}}^{(0)} (\xi_a_0(z_0), B_1(y_1), \ldots, B_n(y_n)) = -i \frac{\sum_{j=1}^{m} g_{bcd}^{(j)} D^F (\xi_a_0(z_0), \xi_b(y_j)) \sum_{l_1, l_2} : T_{\text{conn}}^{(0)} (\xi_c(y_j), I_1) T_{\text{conn}}^{(0)} (\xi_d(y_j), I_2) : }{2} \tag{4.24}
\]

where the sum \( \sum_{l_1, l_2} \) goes over the partitions of \( \{B_1(y_1), \ldots, B_n(y_n)\} \).
Proof First, we use the equation of motion formula:
\[
T^{\text{conn}}_{(0)}(\xi_{a_0}(z_0), B_1(y_1), \ldots, B_n(y_n))
= -i \sum_{j=1}^{m} D^F(\xi_{a_0}(z_0), \xi_{b}(y_j)) T^{\text{conn}}_{(0)}(B_1(y_1), \ldots, \xi_{b} \cdot B_j(y_j), \ldots, B_n(y_n))
\]
(4.25)
and then we use for the chronological product from the right hand side part (ii) of the preceding Corollary with \(A_1 \rightarrow \xi_{b} \cdot B_j\) and \(A_2, \ldots, A_n \rightarrow B_1, \ldots, \hat{j}, \ldots, B_n\). \qed

We should compare the preceding formula with formula (52) of [13], more precisely the first line and notice a similar structure.

We now go to momentum space. A typical experiment is described by the matrix element
\[
\langle \tilde{\xi}_{a}^{(+)}(p_1) \cdot \cdots \tilde{\xi}_{a}^{(+)}(p_n) \Omega, T^{\text{conn}}_{(0)}(\xi_{a}(z), B_1(y_1), \ldots, B_m(y_m)) \Omega \rangle
\]
which creates \(n\) particles from the vacuum. Here, \(\tilde{\xi}_{a}(p)\) are the Fourier transforms of the fields and \(\tilde{\xi}_{a}^{(+)}(p)\) are the positive and negative parts.

We have

Proposition 4.13 The preceding expression can be nonzero only for \(n = m + 1\).

Proof From Wick theorem and the first Proposition 4.1. Indeed, from Wick theorem we have
\[
T^{\text{conn}}_{(0)}(\xi_{a}(z), B_1(y_1), \ldots, B_m(y_m))
\sim \sum \langle \Omega, T^{\text{conn}}_{(0)}(\xi_{a}(z), B_1(y_1), \ldots, B_m(y_m)) \Omega : B'_1(y_1) \cdot \cdots \cdot B''_m(y_m) : \rangle
\]
where \(B = B'B''\) is a factorization of \(B\) in Wick submonomials. Suppose that in the set of Wick monomials \(B_1'(y_1), \ldots, B_m'(y_m)\) we have \(m_j\) elements with \(\rho = j, j = 1, 2, 3\), i.e., with 1, 2, 3 factor fields. Then, in \(B'_1(y_1), \ldots, B'_m(y_m)\) : we have \(2m_1 + m_2\) factors so the matrix element is nonzero only iff \(n = 2m_1 + m_2\). Using Proposition 4.1, we get the equality from the statement. \qed

In momentum space, we want to obtain recursion formulas for the expressions
\[
T^{B_1, \ldots, B_m}_{a_0; a_1, \ldots, a_n}(p_0; p_1, \ldots, p_n) = \frac{1}{(2\pi)^d} \int dy_1 \cdots dy_mdz \ e^{-ip_0 \cdot z_0} \langle \tilde{\xi}_{a_1}^{(+)}(p_1) \cdots \tilde{\xi}_{a_n}^{(+)}(p_n) \Omega, T^{\text{conn}}_{(0)}(\xi_{a}(z), B_1(y_1), \ldots, B_m(y_m)) \Omega \rangle
\]
(4.27)
and we need to consider only the case \(n = m + 1\) as we have proved above.

We will give a recursion formula for these objects. First, we need a technical lemma.

Lemma 4.14 Let
\[
A = \prod_{j=1}^{n} \xi_{a_j}^{(-)}(z_j), \quad B_k = \prod_{j \in P_k} \xi_{b_j}^{(+)}(y_j), \quad (k = 1, 2)
\]
(4.28)
with \(P_1 \cup P_2 = \{1, \ldots, n\}, P_1 \cap P_2 = \emptyset, P_1 \neq \emptyset, P_2 \neq \emptyset\). Then, we have
\[
\langle \Omega, AB_1B_2 \Omega \rangle = \sum \langle \Omega, A_1B_1 \Omega \rangle \langle \Omega, A_2B_2 \Omega \rangle
\]
(4.29)
where the sum goes over all factorizations \(A = A_1A_2\) into submonomials such that \(\rho(A_j) = \rho(B_j), j = 1, 2\).
Proof We use induction over \( n \). For \( n = 2 \), the result is easy to obtain. We suppose that we have proved the result for \( 2, \ldots, n \) and consider the case \( n + 1 \). We take \( A = A_0 \xi_{\alpha_{n+1}}^(-(z_{n+1})} \) where \( A_0 = \prod_{j=1}^{n} \xi_{a_j}^(-(z_j)} \) and compute

\[
\langle \Omega, A_0 \xi_{\alpha_{n+1}}^(-(z_{n+1})} B_1 B_2 \Omega \rangle
\]

\[
= \langle \Omega, A_0 [\xi_{\alpha_{n+1}}^(-(z_{n+1})}, B_1 ]B_2 \Omega \rangle + \langle \Omega, A_0 B_1 [\xi_{\alpha_{n+1}}^(-(z_{n+1})}, B_2 ]\Omega \rangle
\]

The commutators are sums of expressions of the type \( \prod_j \xi_j^{(y_j)} \) with one less factor \( \xi \) and some numerical coefficients. So, we can apply the induction hypothesis and obtain

\[
\langle \Omega, A B_1 B_2 \Omega \rangle
\]

\[
= \sum \langle \Omega, A_1 [\xi_{\alpha_{n+1}}^-(z_{n+1})}, B_1 ] \rangle \langle \Omega, A_2 B_2 \Omega \rangle
\]

\[
+ \sum \langle \Omega, A_1 B_1 \Omega \rangle \langle \Omega, A_2 [\xi_{\alpha_{n+1}}^-(z_{n+1})}, B_2 ]\Omega \rangle
\]

where the first sum is restricted to \( \rho(A_1) = \rho(B_1) - 1 \), \( \rho(A_2) = \rho(B_2) \) and the second sum to \( \rho(A_1) = \rho(B_1) \), \( \rho(A_2) = \rho(B_2) - 1 \). We can rewrite the preceding formula as

\[
\langle \Omega, A B_1 B_2 \Omega \rangle
\]

\[
= \sum \langle \Omega, A_1 \xi_{\alpha_{n+1}}^-(z_{n+1})}, B_1 \Omega \rangle \langle \Omega, A_2 B_2 \Omega \rangle
\]

\[
+ \sum \langle \Omega, A_1 B_1 \Omega \rangle \langle \Omega, A_2 \xi_{\alpha_{n+1}}^-(z_{n+1})}, B_2 \Omega \rangle \]

and observe that the factorizations \( A = A_1 A_2 \) into submonomials such that \( \rho(A_j) = \rho(B_j) \). \( j = 1, 2 \) are of two types \( A = (A_1 \xi_{\alpha_{n+1}}^-(z_{n+1})) A_2 \) such that \( \rho(A_1) = \rho(B_1) - 1 \), \( \rho(A_2) = \rho(B_2) \) and \( A = A_1 (A_2 \xi_{\alpha_{n+1}}^-(z_{n+1})) \) with \( \rho(A_1) = \rho(B_1) \), \( \rho(A_2) = \rho(B_2) - 1 \). They correspond to the two sums above. \( \square \)

Using this lemma, we obtain from Proposition 4.12:

**Proposition 4.15** Suppose that

\[
B_j = \frac{1}{3} g_{bcd, b}^j \xi_{b} \xi_{c} \xi_{d},
\]

(4.30)

with \( g_{bcd, b}^j \) completely symmetric. Then, the following recursion formulas are valid for \( n \geq 4 \)

\[
\langle \xi_{a_1}^{+(p_1)} \cdots \xi_{a_m}^{+(p_m)} \Omega, T_{(0)}^{\text{con}}(\xi_{a_0}(z_0), B_1(y_1), \ldots, B_m(y_m))\Omega \rangle
\]

\[- \frac{i}{2} \sum_{j=1}^{m} g_{bcd, b}^j D_F(\xi_{a_0}(z_0), \xi_{b}(y_j)) \sum_{l_1, l_2} \sum_{P_1, P_2} \]

\[
\langle \Omega, \prod_{j \in P_1} \xi_{a_j}^{-(p_j)} T_{(0)}^{\text{con}}(\xi_{c}(y_j), I_1) \Omega \rangle
\]

\[
\langle \Omega, \prod_{j \in P_2} \xi_{a_j}^{-(p_j)} T_{(0)}^{\text{con}}(\xi_{d}(y_j), I_2) \Omega \rangle
\]

(4.31)

where the sum \( \sum_{l_1, l_2} \) goes over the partitions of \( B_1(y_1), \ldots, j, \ldots, B_m(y_m) \) and the sum \( \sum_{P_1, P_2} \) over partitions of \( \{1, \ldots, n\} \) such that \( \text{card}(P_j) = \text{card}(I_j) + 1 \), \( j = 1, 2 \).

Finally, we have
Theorem 4.16 The expressions (4.27) are of the form
\begin{equation}
T_{a_0; a_1, \ldots, a_n}^{B_1, \ldots, B_m} (p_0; p_1, \ldots, p_n) = \delta \left( \sum_{j=0}^{n} p_j \right) A_{a_0; a_1, \ldots, a_n}^{B_1, \ldots, B_m} (p_1, \ldots, p_n)
\end{equation}
and the following recursion relations are true
\begin{equation}
A_{a_0; a_1, \ldots, a_n}^{B_1, \ldots, B_m} (p_0; p_1, \ldots, p_n) = - \frac{i}{2} (2\pi)^6 \sum_{j=1}^{m} g^{(j)} \hat{D}^{F}_{a_0 b} (p_0) \sum_{I_1, I_2} A_{c, A_1}^{I_1} (P_1) A_{d, A_2}^{I_2} (P_2)
\end{equation}
where the sums are as above and we have denoted for simplicity \( p_0 \equiv - \sum_{k=1}^{n} p_k \) and \( A_1 = \{ a_j \}_{j \in P_1}, \ P_1 = \{ p_j \}_{j \in P_1}, \ A_2 = \{ a_j \}_{j \in P_2}, \ P_2 = \{ p_j \}_{j \in P_2} \).

Proof We integrate over \( y_1, \ldots, y_n \) the formula from the preceding proposition. The integral over \( y_k \neq y_j \) corresponding to \( B_k \in I_1 \) and \( B_k \in I_2 \), respectively, are producing, essentially, the two factors \( A \) from the right hand side. The integral over \( y_j \) and \( z_0 \) are producing the overall \( \delta \) factor multiplied by the propagator \( \hat{D}^{F}_{a_0 b} (p_0) \). \( \square \)

5 Yang–Mills fields

First, we can generalize the preceding formalism to the case when some of the scalar fields are odd Grassmann variables. One simply insert everywhere the Koszul sign. The next generalization is to arbitrary vector and spinorial fields. If we consider, for instance, the Yang–Mills interaction Lagrangian corresponding to pure QCD [6], then the jet variables \( \xi_a, a \in \Xi \) are \( (v^\mu_A, u_A, \tilde{u}_A) \), \( A = 1, \ldots, r \) where \( v^\mu_A \) are Grassmann even and \( u_A, \tilde{u}_A \) are Grassmann odd variables.

The interaction Lagrangian is determined by gauge invariance. Namely, we define the gauge charge operator by
\begin{equation}
d_Q v^\mu_A = i d^\mu u_A, \quad d_Q u_A = 0, \quad d_Q \tilde{u}_A = -i d_\mu v^\mu_A, \ A = 1, \ldots, r
\end{equation}
where \( d^\mu \) is the formal derivative. The gauge charge operator squares to zero:
\begin{equation}
d_Q^2 \simeq 0
\end{equation}
where by \( \simeq \) we mean, modulo the equation of motion. Now we can define the interaction Lagrangian by the relative cohomology relation:
\begin{equation}
d_Q T (x) \simeq \text{total divergence}.
\end{equation}
If we eliminate the corresponding coboundaries, then a trilinear Lorentz covariant expression is uniquely given by
\begin{equation}
T = f_{ABC} \left( \frac{1}{2} v_A^{\mu} v_B^\nu F^{\nu\mu}_C + u_A v_B^\mu d_\mu \tilde{u}_C \right)
\end{equation}
where
\begin{equation}
F^{\mu\nu}_A \equiv d^\mu v^\nu_A - d^\nu v^\mu_A, \quad \forall a = 1, \ldots, r
\end{equation}
and \( f_{ABC} \) are real and completely anti-symmetric. (This is the trilinear part of the usual QCD interaction Lagrangian from classical field theory.)

Then, we define the associated Fock space by the nonzero 2-point distributions as

\[
\langle \Omega, v^\mu_A(x_1)v^\nu_B(x_2)\rangle = i \eta^{\mu\nu} \delta^+_A D_0^+(x_1 - x_2),
\]

\[
\langle \Omega, u_A(x_1)\bar{u}_B(x_2)\rangle = -i \delta^+_A D_0^+(x_1 - x_2),
\]

\[
\langle \Omega, \bar{u}_A(x_1)u_B(x_2)\rangle = i \delta^+_A D_0^+(x_1 - x_2).
\]

(5.6)

and construct the associated Wick monomials. Then, the expression (5.4) gives a Wick polynomial \( T^{\text{quant}} \) formally the same, but: (a) the jet variables must be replaced by the associated quantum fields; (b) the formal derivative \( d^\mu \) goes in the true derivative in the coordinate space; (c) Wick ordering should be done to obtain well-defined operators. We also have an associated gauge charge operator in the Fock space given by

\[
\left[ Q, v^\mu_A \right] = i \partial^\mu u_A,
\]

\[\{ Q, u_A \} = 0, \quad \{ Q, \bar{u}_A \} = -i \partial^\mu v^\mu_A \]

\( Q\Omega = 0. \)

(5.7)

Then, it can be proved that \( Q^2 = 0 \) and

\[
\left[ Q, T^{\text{quant}}(x) \right] = \text{total divergence}
\]

(5.8)

where the equations of motion are automatically used because the quantum fields are on-shell. From now on, we abandon the super-script \( \text{quant} \) because it will be obvious from the context if we refer to the classical expression (5.4) or to its quantum counterpart.

Next, we notice that we can write (5.4) in the form (4.11):

\[
T = \frac{1}{3!} g_{pqr} \xi_p \xi_q \xi_r
\]

(5.9)

with \( \xi_p \) jet variables and

\[
g_{pqr} \equiv g(\xi_p, \xi_q, \xi_r)
\]

(5.10)

having Grassmann permutation symmetries. For the QCD Lagrangian from above, we have the nonzero entries:

\[
g(v^\mu_A, v^\nu_B, F^\rho_C) = \frac{1}{2} (\eta^{\mu\rho} \eta^{\nu\sigma} - \eta^{\nu\rho} \eta^{\mu\sigma}) f_{ABC}
\]

\[
g(u_A, v^\mu_B, \bar{u}_C; v) = \delta^\mu_0 f_{ABC}
\]

(5.11)

and the rest of the nonzero expressions \( g_{pqr} \) obtained by permutations, taking into account the Grassmann parity sign.

To construct the tree contribution of the chronological products in the second order, one follows the causal prescription. First one determines the causal commutator. By direct computations, one gets:

\[
[T(x), T(y)]^{\text{tree}} = D_0(x - y)W(x, y) + \partial_\mu D_0(x - y)W^\mu(x, y)
\]

\[
+ \partial_\mu \partial_\nu D_0(x - y)W^{\mu\nu}(x, y)
\]

(5.12)

(with \( W, W^\mu, W^{\mu\nu} \) Wick polynomials) which comes from the generic expression

\[
[T(x), T(y)]^{\text{tree}} = \epsilon D(\xi_p(x), \xi_q(y)) : \xi_p \cdot T(x) \xi_q \cdot T(y):
\]

(5.13)

if one considers various values of the jet variables \( \xi_a, \xi_b = (v^\mu_A, u_A, \bar{u}_A), A = 1, \ldots, r \) and their formal derivatives. Here, \( \epsilon \) is a Koszul sign and we also use the derivative \( \xi \cdot W \) from
formula \( (2.20) \). In the same way, we can determine other causal commutators \( [T^\mu(x), T(y)] \), etc. In particular, if we use \( v_{C\mu:v} \cdot T = f_{ABC} v_{A\mu} v_{B\nu} \), we have the contribution

\[
[T(x), T(y)]^{\text{free,vvvv}} = D(v_{A\mu:v}(x), v_{B\nu;\sigma}(y)) f_{CAB} f_{CDE} :v^\mu_A(x)v^\nu_B(x)v^\rho_D(y)v^\sigma_E(y) : \tag{5.14}
\]

Then, we must perform a causal splitting of the causal commutator. This amounts to causally split

\[
D(\xi_p(x), \xi_q(y)) = D^{\text{adv}}(\xi_p(x), \xi_q(y)) - D^{\text{ret}}(\xi_p(x), \xi_q(y)) \tag{5.15}
\]

and make the corresponding substitution 

\[
D^{\text{adv}}(\xi_p(x), \xi_q(y)) \rightarrow D^F(\xi_p(x), \xi_q(y)) \text{ where:}
\]

\[
D^F = D^{\text{ret}} + D^{(+)} = D^{\text{adv}} - D^{(-)} . \tag{5.16}
\]

We obtain for the chronological products:

\[
T(T(x), T(y))^{\text{free}} = \varepsilon D^F(\xi_p(x), \xi_q(y)) : \xi_p \cdot T(x) \xi_q \cdot T(y) : \tag{5.17}
\]

and in particular

\[
T(T(x), T(y))^{\text{free,vvvv}} = D^F(v_{A\mu:v}(x), v_{B\nu;\sigma}(y)) f_{CAB} f_{CDE} :v^\mu_A(x)v^\nu_B(x)v^\rho_D(y)v^\sigma_E(y) : . \tag{5.18}
\]

The central splitting is not unique. The so-called central splitting amounts to the following choice:

\[
D^{\text{adv(ret)}, \text{central}}(\xi_{a_1;\mu_1 \ldots \mu_m}(x), \xi_{b_1;\nu_1 \ldots \nu_n}(y)) = (-1)^n \partial_{\mu_1} \cdots \partial_{\mu_m} \partial_{\nu_1} \cdots \partial_{\nu_n} D^{\text{adv(ret)}}(\xi_{a}(x), \xi_{b}(y)) \tag{5.19}
\]

by analogy with \( (2.4) \).

It amounts to the substitutions \( D_0 \rightarrow D_0^{\text{adv(ret)}}, \partial_\mu D_0 \rightarrow \partial_\mu D_0^{\text{adv(ret)}}, \partial_\mu \partial_\nu D_0 \rightarrow \partial_\mu \partial_\nu D_0^{\text{adv(ret)}} \) in the formula \( (5.12) \).

Then, it follows that to obtain the (tree) chronological product \( T(T(x), T(y))^{\text{free,central}} \), one must make in \( (5.18) \) the substitution \( D^F \rightarrow D^F, \text{central} \) where

\[
D^F, \text{central}(\xi_{a_1;\mu_1 \ldots \mu_m}(x), \xi_{b_1;\nu_1 \ldots \nu_n}(y)) = (-1)^n \partial_{\mu_1} \cdots \partial_{\mu_m} \partial_{\nu_1} \cdots \partial_{\nu_n} D^F(\xi_{a}(x), \xi_{b}(y)) \tag{5.20}
\]

or more explicitly \( D_0 \rightarrow D_0^F, \partial_\mu D_0 \rightarrow \partial_\mu D_0^F, \partial_\mu \partial_\nu D_0 \rightarrow \partial_\mu \partial_\nu D_0^F \) in the formula \( (5.12) \).

In particular, from \( (5.14) \) we have

\[
T(T(x), T(y))^{\text{tree,vvvv,central}} = \partial_\mu \partial_\nu D_0^F(x - y) f_{CAB} f_{CDE} :v^\mu_A(x)v^\nu_B(x)v^\rho_D(y)v^\sigma_E(y) : . \tag{5.21}
\]

In this way the causal products are verifying Bogoliubov axioms, but the gauge invariance condition

\[
d_Q T(T(x), T(y)) = i \frac{\partial}{\partial x^\mu} T(T^\mu(x), T(y)) + i \frac{\partial}{\partial y^\mu} T(T(x), T^\mu(y)) \tag{5.22}
\]

is broken. One can save gauge invariance if we impose that the constants \( f_{ABC} \) are verifying Jacobi identity and add a finite renormalization of the type \cite{22}

\[
N(T(x), T(y)) = \delta(x - y) N(x), \quad N = \frac{i}{2} f_{ABE} f_{CDE} v^\mu_A v^\nu_B v^\rho_C v^\sigma_D . \tag{5.23}
\]
to the expression $T(T(x), T(y))^{\text{tree, central}}$. The Wick polynomial $N$ is the quadri-linear part of the usual QCD interaction Lagrangian from classical field theory, and it is the justification for the quadri-linear vertexes from [12] (see more precisely [13], the second line of formula (52)).

We notice that we can get rid of the finite renormalization if one redefines:

$$T(v_{A\mu;\nu}(x), v_{B\rho;\sigma}(y)) = DF(v_{A\mu;\nu}(x), v_{B\rho;\sigma}(y))$$

$$= -i \delta_{AB} \eta_{\mu\rho} \delta_{\nu\sigma} D^F(x - y) + c \eta_{\mu\rho} \eta_{\nu\sigma} \delta_{AB} \delta(x - y)$$

(5.24)

with a clever choice $c = \frac{i}{T}$. The first term in the right hand side follows from the causal splitting of formula (2.4). The second term is possible because it does not change the order of singularity and the covariance properties of the chronological product. Indeed, if we substitute (5.24) in (5.18), the $\delta$ contribution produces exactly the finite renormalization (5.23). So, with this clever trick, we do not need the quadri-linear vertexes of [12].

Now, we try to make the connection with the momentum space formulas, i.e., we consider expressions of the type (4.27). In the case of gauge models, we should pay attention to the choice of the physical external states of the type $\xi(p,1,\ldots,\epsilon(p_n,1,\ldots)$.

It can be proved [23] that the physical states of the Yang–Mills setting are of the form

$$\epsilon(p_1, \pm) \cdot \tilde{v}_{A_1}(p_1) \cdots \epsilon(p_n, \pm) \cdot \tilde{v}_{A_n}(p_n)$$

where the expressions $\epsilon(p_j, \pm)$ are selecting the two polarizations of a gluon state. (It means that we can ignore all tree graphs with ghost lines.) It follows that the basic Feynman amplitudes are of the form

$$A(A_0, \epsilon_0, p_0; A_1, \epsilon_1, p_1; \ldots; A_n, \epsilon_n, p_n)$$

(5.25)

where $\epsilon_j = \pm$. We note that we have taken in (4.27) $B_1 = \cdots = B_n = T$ where $T$ is the interaction Lagrangian given by (5.4).

The idea from [12] is to separate the color dependence; mathematically it means to find an orthogonal basis in the space of tensors $i^{(K)}_{A_0 \ldots A_n}$ and write

$$A(A_0, \epsilon_0, p_0; A_1, \epsilon_1, p_1; \ldots; A_n, \epsilon_n, p_n) = \sum_{K} i^{(K)}_{A_0 \ldots A_n} A_K(\epsilon_0, p_0; \epsilon_1, p_1; \ldots; \epsilon_n, p_n).$$

(5.26)

If such a writing would be possible, we would use it in theorem 4.16 and determine a recursion relation for the reduced amplitude of the type $A_K(\epsilon_0, p_0; \epsilon_1, p_1; \ldots; \epsilon_n, p_n)$. However, to find such a basis is not elementary. We give some details. By $A \rightarrow F_A$, we mean the adjoint representations of the Lie algebra associated with $f_{ABC}$

$$(F_A)_{BC} = - f_{ABC}.$$ 

(5.27)

We will assume that the Lie algebra associated with the structure constants $f_{ABC}$ is semi-simple, so the Killing–Cartan form $g_{AB} = f_{ACD} f_{BDC}$ can be chosen $g_{AB} = \delta_{AB}$. It follows that the $r \times r$ matrices $F_A$, $A = 1, \ldots, r$ are linear independent. In this case, we can prove the following “completeness” relation:

**Lemma 5.1** Suppose that $X, Y$ are in the linear span of $F_A$, $A = 1, \ldots, r$. Then, the following formula is true

$$f_{ABC} Tr(F_B X) Tr(F_C Y) = Tr(F_A[X, Y])$$

(5.28)

where we sum over the dummy indexes.
We introduce in $V \equiv \text{Span}(F_A)_{A=1,\ldots,r}$ the scalar product

$$(X,Y) = Tr(XY).$$

In particular

$$(F_A, F_B) = \delta_{AB}.$$  

(5.30)

Because $X, Y \in V$ by hypothesis, we have the writings

$$X = x_B F_B, \quad Y = y_B F_B$$

(5.31)

from where

$$[X,Y] = f_{ABC} x_B y_C F_A.$$ 

(5.32)

If we substitute in the relation from the statement, we obtain an identity. \hfill \Box

Next we define the iterated commutators \cite{15} $C(A_1, \ldots, A_n)$ through the recursion relations:

$$C(\emptyset) = 1, \quad C(A_1) = F_{A_1}$$

$$C(A_1, \ldots, A_{n-1}) = [C(A_1, \ldots, A_{n-2}), F_{A_{n-1}}].$$

(5.33)

and note that they belong to $V$.

We now prove the following

\textbf{Proposition 5.2} The amplitudes of color factors (5.25) are of the form

$$Tr(C(A_1, \ldots, A_n) F_{A_0}).$$

Proof The first non-trivial case is $n = 2$ (and $m = 1$); the expression (4.26) is proportional to $g_{a_0 a_1 a_2}$, so for the particular case of QCD we get the factor $f_{A_0 A_1 A_2} \sim Tr((C(A_1, A_2) F_{A_0})$.

If we suppose that the assertion from the statement is true for $2, \ldots, n - 1$, we have for $n$ from (4.33) that the color factor is of the form

$$f_{BCD} Tr(C(A_1, \ldots, A_k) F_B) Tr(C(A_{k+1}, \ldots, A_n) F_C).$$

If we apply the preceding lemma, we obtain the color factor

$$Tr([C(A_1, \ldots, A_k), C(A_{k+1}, \ldots, A_n)] F_A).$$

But the commutator of two iterated commutators is a sum of iterated commutators, so we obtain the result from the statement. \hfill \Box

So it follows that the Feynman amplitudes are sums of the type

$$A = \sum Tr(C(A_{\sigma(1)}, \ldots, A_{\sigma(n)}) F_{A_0}) A_{\sigma}.$$  

(5.34)

If we make explicit the commutators, we arrive at the formula from \cite{24}:

$$A_{A_0; A_1 \ldots A_n} = \sum Tr(F_{A_{\sigma(0)}} \ldots F_{A_{\sigma(n)}}) A_{\sigma}$$  

(5.35)

However, the expressions $Tr(F_{A_0} \ldots F_{A_n})$ above are not linear independent. They verify many identities presented in \cite{24} and \cite{25}. If one takes into account all these identities, one can find, at least for the Lie algebra $su(N)$, an orthogonal basis, up to terms of order $1/N^2$—see \cite{26} formula (3.3).

Next we have
Proposition 5.3 The following formula is valid:

\[(F_{A_2} \ldots F_{A_{n-1}})_{A_1 A_n} = \text{Tr}(C(A_1, \ldots, A_{n-2}) [F_{A_{n-1}}, F_{A_n}]) = \text{Tr}(C(A_1, \ldots, A_{n-2}, A_n) F_{A_n})\] (5.36)

Proof By induction. For \(n = 3\), the assertion is immediate. If the formula from the statement is true, then we have

\[
(F_{A_2} \ldots F_{A_n})_{A_1 A_{n+1}} = (F_{A_2} \ldots F_{A_{n-1}})_{A_1 B} (F_{A_n})_{B A_{n+1}}
= \text{Tr}(C(A_1, \ldots, A_{n-2}) [F_{A_{n-1}}, F_B]) f_{B A_n A_{n+1}}
= \text{Tr}(C(A_1, \ldots, A_{n-2}) [F_{A_{n-1}}, [F_{A_n}, F_{A_{n+1}}]])
= \text{Tr}(C(A_1, \ldots, A_{n-2}) F_{A_{n-1}} [F_{A_n}, F_{A_{n+1}}])
- \text{Tr}(C(A_1, \ldots, A_{n-2}) [F_{A_n}, F_{A_{n+1}}] F_{A_{n-1}}).
\]

We have used the induction hypothesis in the second equality. Now, in the very last term from above we use the cyclic property of the trace and immediately obtain the formula from the statement for \(n \to n + 1\) and this finishes the induction. \(\Box\)

If we use this formula, we get color factors of the form \((F_{A_2} \ldots F_{A_{n-1}})_{A_1 A_n}\); in [27] (see also [15]) the linear independence for this basis was proved for \(A_1, A_n\) fixed; however, orthogonality is lacking. From the result of [27], we can obtain the result of [24]. For an orthogonal basis, see [28].

6 Conclusions

We have proved that the factorization formula is very simple in the coordinate space; it is based on Proposition 4.12 and Corollary 4.11. To go in the momentum space requires some computations, and the result is given in Theorem 4.16.

One can easily particularize the formulas for the Yang–Mills case. However, to strip the amplitudes of the color factors one needs an linear independent and orthogonal basis in the space of tensors associated with the Lie algebra with some special tensor properties and this is not an elementary problem.

We stress in the end the fact that our factorization formula is similar, but not identical to the factorization formula of [12]. In our opinion, it remains an interesting problem to derive the factorization formula of [12] using only well-defined mathematical objects, namely the chronological products.

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