Asymptotic theory for differentially private generalized $\beta$-models with parameters increasing*

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Modelling edge weights play a crucial role in the analysis of network data, which reveals the extent of relationships among individuals. Due to the diversity of weight information, sharing these data has become a complicated challenge in a privacy-preserving way. In this paper, we consider the case of the non-denosing process to achieve the trade-off between privacy and weight information in the generalized $\beta$-model. Under the edge differential privacy with a discrete Laplace mechanism, the Z-estimators from estimating equations for the model parameters are shown to be consistent and asymptotically normally distributed. The simulations and a real data example are given to further support the theoretical results.

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1. INTRODUCTION

With the rapid development of computer and network technology, the analysis of network data has aroused widespread concerns in various fields. Unfortunately, collecting, storing, analyzing and sharing these data is challenging, due to the privacy of individuals (e.g., financial transactions). Besides, more privacy protection may reduce the validity of data [Duncan et al. (2004)]. Many approaches have been proposed to guarantee the trade-off between individual privacy and the utility of published data, which focus on data encryption, identity authentication, data perturbation [Samarati and Sweeney (1998); Fung et al. (2007); Machanavajjhala et al. (2006); Ghinita et al. (2008); Li et al. (2007); Aggarwal and Yu (2007)]. Dwork (2006a) proposed a rigorous notion of privacy named $\varepsilon$-differential privacy to control strong worst-case privacy risks. More formally, adding or removing a single record in the dataset does not have a serious effect on the outcome of any analysis. Starting from Dwork (2006a), various types of data and queries were widely applied by researchers under differential privacy constraints [Holohan et al. (2017); McSherry and Talwar (2007); Wasserman and Zhou (2010)].

Random graphs are powerful statistical tools in the study of network data. These graph models are based on degree sequences $d$, which are used in modelling the realistic networks. In the undirected case, the $\beta$-model is well-known for the binary network, renamed by Chatterjee et al. (2011). Many scholars have focused on the studies of the $\beta$-model [Jackson (2008); Lauritzen (2008); Blitzstein and Diaconis (2011)]. Chatterjee et al. (2011) proved the existence and consistency of the maximum likelihood estimator (MLE) of the $\beta$-model as the number of parameters goes to infinity. Yan and Xu (2013) further derived its asymptotic normality. On the other hand, edge weights reveal the strength of relationships among individuals, which are critical for understanding many phenomena. For example, in friendship networks, we can assign close friends with a higher weight and acquaintances or normal friends with a lower weight, which are also referred to as the strong tie and weak tie reported by Granovetter (1993). To this point, Hillar et al. (2013) studied the maximum entropy distributions on weighted graphs with the $\beta$-model as the special case and proved the consistency of the MLE under the assumption that all parameters are bounded by a constant; Yan et al. (2015) proved the asymptotic normality of the MLE.

In the privacy analysis of network data, the raw data is published via pre-processing so that the confidential and sensitive information is captured as less as possible. One of the popular approaches is to add some noises $\varepsilon$ into the degree sequence $d$. For example, Hay et al. (2009) applied the Laplace noise-addition mechanism to release the degree partition of a graph, and designed to reduce the error with the $\ell_2$-norm between the true and released degree partitions. However, the process of adding noises is often ignored when summary statistics are published in a privacy-preserving way. As a result, the estimated parameters may not be consistent, even not exist [e.g., Hay et al. (2009)]. Duchi et al. (2018) illustrated that the estimator operated on private data has a larger error than the non-private estimator. Based on privatized data, estimating summary statistics and estimating parameters of models are totally different problems.

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To this point, Karwa et al. (2016) paid attention to the noise addition process through the denoised method to achieve valid inference and obtained the consistency and asymptotic normality of a differential privacy estimator in the β-model.

In this paper, we adopt the non-denoised method to establish the asymptotic properties of the Z-estimator of the parameter in the generalized β-model with finite discrete weighted edges under the discrete Laplace mechanism, which is different from the work of Karwa et al. (2016). Moreover, Karwa et al. (2016) only considered binary edges. In some scenarios, edge weights play important roles in the complex social interactions among individuals than binary networks [e.g., Farine (2014)]. Furthermore, edge weights may further increase the risk of privacy disclosure, due to the diversity of weight-related information. For instance, weighted social networks often provide a more realistic representation of the social networks [e.g., Farine (2014)].

In the context of network data, there are two main variants of differential privacy: edge differential privacy (EDP) [Nissim et al. (2007)] and node differential privacy (NDP) [Hay et al. (2009), Kasiviswanathan et al. (2013)], which are based on the different definitions of graph neighbors. Specifically, EDP guarantees that released databases do not reveal the addition or removal of a special edge, while NDP hides the addition or removal of a node (along with its edges) in a graph \( G \). In this paper, we refer to EDP, where two graphs \( G \) and \( G' \) are said to be neighbors if they differ in exactly one edge.

**Definition 2.1** (Edge differential privacy). Let \( \varepsilon \geq 0 \) be a privacy parameter. A randomized mechanism (or a family of conditional probability distributions) \( Q(\cdot|G) \) is \( \varepsilon \)-edge differentially private if

\[
\sup_{G,G' \in \mathcal{G}, \delta(G,G')=1} \sup_{S \in \mathcal{S}} \frac{Q(S|G)}{Q(S|G')} \leq e^\varepsilon,
\]

where \( \mathcal{G} \) is the set of all undirected graphs of interest on \( n \) nodes, \( \delta(G,G') \) is the number of edges on which \( G \) and \( G' \) differ, \( S \) is the set of all possible outputs (or the support of \( Q \)).

The above definition of EDP is based on ratios of probabilities. Generally, the data curator chooses an appropriate privacy parameter \( \varepsilon \) to achieve the trade-off between privacy and validity. As the value of \( \varepsilon \) is extremely small, more privacy is protected. Under EDP constraints, changing one record in the dataset cannot affect seriously on the distribution of the output. For example, a hospital can release some medical information about their patients to the public, while simultaneously ensuring very high levels of privacy in the case of EDP. This is because EDP offers a guarantee no matter whether or not the patient participates in the study, the probability of a possible output is almost the same. As a result, an attacker can not find whether a single individual is in the original database or not. As we know, the effective implementation of \( \varepsilon \)-differential privacy is associated with the magnitude of additional random noise.
To this end, Dwork et al. (2006b) introduced the notion of global sensitivity, which is referred to as the maximum $\ell_1$-norm among various dataset pairs $(G, G')$.

**Definition 2.2** (Global sensitivity). Let $f : G \rightarrow \mathbb{R}^k$. The global sensitivity of $f$ is defined as

$$\Delta_G(f) = \max_{G \rightarrow G'} \| f(G) - f(G') \|_1,$$

where $\| \cdot \|_1$ is the $\ell_1$-norm for vector.

Although there are many mechanisms for releasing the output of any function $f$ under differential privacy, the Laplace mechanism is the most common one. Karwa et al. (2016) presented a discrete Laplace mechanism to achieve edge differential privacy, which is given below.

Let $f : G \rightarrow \mathbb{Z}^k$, and let $Z_1, \ldots, Z_k$ be independent and identically distributed discrete Laplace random variables with probability mass function defined by

$$P(Z = z) = \frac{1 - \lambda}{1 + \lambda} |z|, z \in \mathbb{Z}, \lambda \in (0, 1).$$

Then the algorithm which outputs $f(G) + (Z_1, \ldots, Z_k)$ with inputs $G$ is $\varepsilon$-edge differentially private, where $\varepsilon = -\Delta_G(f) \log \lambda$.

Based on the definition of differential privacy, Dwork et al. (2006b) found that any function of a differentially private mechanism is also differentially private, as follow: Let $f$ be an output of an $\varepsilon$-differentially private mechanism and $g$ be any function. Then $g(f(G))$ is also $\varepsilon$-differentially private. This result indicates that any post-processing done on the noisy degree sequence obtained as an output of a differentially private mechanism is also differentially private.

More generally, we may consider the skew discrete Laplace mechanism. When the positive noises and negative noises arising with different probability law, the skew discrete Laplace distribution [Kozubowski and Inusah (2006)] as a discretization of non-symmetric Laplace distribution could be used. The skew Laplace distribution is useful in applications to communications, engineering, and finance and economics, see Kotz et al. (2012) and references therein. For more information on the skew discrete Laplace mechanism, see the supplementary material for details.

### 2.3 Estimation

Let $G_n$ be a simple undirected graph including $n$ nodes. Let $a_{ij}$ be the weight of edge $(i, j)$. Let $A = (a_{ij})$ be the adjacency matrix of $G_n$. Note that $G_n$ has no self-loops, $a_{ii} = 0$. Define $d_j = \sum_{j \neq i} a_{ij}$ and $d = (d_1, \ldots, d_n)^T$ as the degree sequence of $G_n$. The density or probability mass function on $G_n$ with respect to some canonical measure $\nu$ has the exponential-family random graph models with the degree sequence as sufficient statistic, i.e.,

$$p(G_n; \alpha) = \exp(\alpha^T d - Z(\alpha)),$$

where $Z(\alpha)$ is the normalizing constant, $\alpha = (\alpha_1, \cdots, \alpha_n)^T$ is a vector parameter.

We assume that the edge weights $\{a_{ij}\}$ are independently multinomial random variables with the probability mass function:

$$P(a_{ij} = a) = \frac{e^{\alpha(a_i + a_j)}}{\sum_{k=0}^{q-1} e^{k(a_i + a_j)}}, \quad a = 0, 1, \ldots, q - 1.$$

where $q \geq 2$ is a fixed number of the class. Thus the likelihood of $\alpha$ is

$$L(\alpha) \propto \prod_{j \neq a} \prod_{a=0}^{q-1} [P(a_{ij} = a)]^{1(a_{ij} = a)},$$

which gives the log-likelihood of $\alpha$,

$$\log L(\alpha) \propto \sum_{j \neq a} \sum_{a=0}^{q-1} \left(1(a_{ij} = a) \cdot \left(\alpha(a_i + a_j) - \log \left(\sum_{k=0}^{q-1} e^{k(a_i + a_j)}\right)\right)\right).$$

So the log-partition function in (1) is given by $Z(\alpha) = \sum_{j \neq i} \log \sum_{a=0}^{q-1} e^{k(a_i + a_j)}$. This model is a direct generalization of the $\beta$-model, which only considers the dichotomous edges.

Moreover, the first order condition for the log-likelihood function w.r.t. $\alpha_i$ are

$$\frac{\partial \log L(\alpha)}{\partial \alpha_i} = \sum_{j \neq i} a_{ij} - \sum_{j=1, j \neq i}^{n} \sum_{a=0}^{q-1} \frac{ae^{\alpha(a_i + a_j)}}{\sum_{k=0}^{q-1} e^{k(a_i + a_j)}}, i = 1, \ldots, n.$$

Let $e_i$ be a noise independently drawn from discrete Laplace distribution with parameter $\lambda_n$. We output $\tilde{d}_i := d_i + e_i$ using the discrete Laplace mechanism. However, the degree $d_i = \sum_{j \neq i} a_{ij}$ of vertex $i$ is not attainable, since the observed degree contains unknown noise in private data set. We resort to the moment equations which are given by the following system of functions:

$$F_i(\alpha) = \tilde{d}_i - E(\tilde{d}_i) = \tilde{d}_i - E(d_i), \quad i = 1, \ldots, n,$$

$$F(\alpha) = (F_1(\alpha), \ldots, F_n(\alpha))^T.$$

Under this case, since adding or removing an edge can change the degree of at most two nodes, by 1 each, the global sensitivity for the degree sequence $d$ is 2. Therefore, we have the privacy parameter

$$\varepsilon_n := -\Delta_G(f) \log \lambda_n = -2 \log \lambda_n.$$

So, $\lambda_n = \exp(-\varepsilon_n / 2)$.

We use $\hat{\alpha}$ to denote the Z-estimator of $\alpha$ satisfying $F(\hat{\alpha}) = 0$. Since the noises $e_i$’s ($i = 1, 2, \cdots, n$) are independently drawn from symmetric discrete Laplace distribution.

**Asymptotic theory for differentially private generalized $\beta$-models with parameters increasing**

with parameter \( \lambda_n \), \( E(\varepsilon_i) = 0 \). Note that \( d_i \) is a sum of edge weights \( a_{ij} \)'s \( (j = 1, \ldots, n, j \neq i) \). So we have

\[
E(\hat{d}_i) = E(d_i + \varepsilon_i) = E(d_i) = \sum_{j=1, j \neq i}^{n} E(a_{ij})
\]

\[
= \sum_{j=1, j \neq i}^{n} \sum_{k=0}^{q-1} kP(a_{ij} = k) = \sum_{j=1, j \neq i}^{n} \sum_{k=0}^{q-1} \sum_{i,j}^{q-1} k e^{k(a_{ij} + \alpha_j)}.
\]

Therefore the moment-based estimating equations with noisy degree sequences are

\[
(2) \quad \hat{d}_i = \sum_{j=1, j \neq i}^{n} \sum_{k=0}^{q-1} \frac{a_{ij} e^{k(\hat{\alpha}_i + \hat{\alpha}_j)}}{\sum_{j=1, j \neq i}^{n} \sum_{k=0}^{q-1} e^{k(\hat{\alpha}_i + \hat{\alpha}_j)}}, \quad i = 1, \ldots, n.
\]

### 2.4 Consistency and asymptotic normality

In this section, we obtain that the Z-estimator of the parameter involving noisy degree sequence is asymptotically consistent and normally distributed.

Given \( m, M > 0 \), we say an \( n \times n \) matrix \( V_n = (v_{ij}) \) belongs to the matrix class \( \mathcal{L}_n(m, M) \) if \( V_n \) is a symmetric nonnegative matrix satisfying

\[
v_{ii} = \sum_{j=1, j \neq i}^{n} v_{ij}; \quad M \geq v_{ij} = v_{ji} \geq m, \quad 0 \neq j \neq i.
\]

Generally, the inverse of \( V_n, V_n^{-1} \), does not have a closed form. Yan and Xu (2013) proposed a simple matrix \( \hat{S} \) by \( \hat{S}_{ij} \) to approximate \( V_n^{-1} \), where \( \hat{S}_{ij} = \frac{\delta_{ij}}{v_{ii} - v_{ij}} \), \( \delta_{ij} \) is the Kronecker delta function, and \( v_{ii} = \sum_{j=1, j \neq i}^{n} (1 - \delta_{ij})v_{ij} = \sum_{j=1, j \neq i}^{n} v_{ii} \).

Similar to Yan et al. (2016b), let the parameter vector \( \alpha = (\alpha_1, \ldots, \alpha_n)^T \) belong to the symmetric parameter space

\[
D = \{ \alpha \in \mathbb{R}^n : -Q_n \leq \alpha_i + \alpha_j \leq Q_n, 1 \leq i < j \leq n \},
\]

where \( \{Q_n\} \) is the sequence of upper bound of the parameters. Let \( F'(\alpha) \) be the Jacobian matrix of \( F(\alpha) \) at \( \alpha \), then for \( i, j = 1, \ldots, n \),

\[
\partial F_i = \sum_{j=1, j \neq i}^{n} \sum_{k=0}^{q-1} \frac{a_{ij} e^{k(a_{ij} + \alpha_j)}}{\sum_{j=1, j \neq i}^{n} \sum_{k=0}^{q-1} e^{k(a_{ij} + \alpha_j)}}
\]

\[
\partial F_i = \sum_{j=1, j \neq i}^{n} \sum_{k=0}^{q-1} \frac{(k-l) e^{k(a_{ij} + \alpha_j)}}{\sum_{j=1, j \neq i}^{n} \sum_{k=0}^{q-1} e^{k(a_{ij} + \alpha_j)}}
\]

Here we need assume \( V_n = F'(\alpha) \in \mathcal{L}_n(m, M) \), i.e., \( v_{ij} = \partial F_i / \partial \alpha_i \), and \( v_{ij} = \partial F_i / \partial \alpha_j \), where

\[
m = (2(1 + e^{Q_n}))^{-1} \quad \text{and} \quad M = \frac{q^2}{2}.
\]

First, we give the convergence rate of the \( \epsilon_n \) error which directly shows the consistency of the parameters under some mild conditions.

**Theorem 2.1.** Consider the discrete Laplace mechanism with \( \lambda_n = \exp(-\epsilon_n/2) \) and assume that \( \alpha \in D \) and \( e^{Q_n} = o((n/\log n)^{1/2}) \), where \( D = \{ \alpha \in \mathbb{R}^n : -Q_n \leq \alpha_i + \alpha_j \leq Q_n, 1 \leq i < j \leq n \} \). If \( \epsilon_n \geq c \sqrt{\log n} \) (denoted as \( \epsilon_n = \Omega(\sqrt{\log n}) \)), where \( c \geq 4 \) is a constant, then as \( n \) goes to infinity, \( \hat{\alpha} \) exists and satisfies

\[
\|\hat{\alpha} - \alpha\| = O_p(\epsilon^{3Q_n} \sqrt{\log n} / n) = o_p(1).
\]

**Remark 2.1.** In Theorem 2.1, we use Newton’s method to obtain the existence and consistency of \( \hat{\alpha} \). This indicates that the Z-estimator of the parameter \( \alpha \) involving a noisy sequence is accurate under the non-denoised process. If \( Q_n \) is bounded and thus \( \alpha \) is a sparse vector, this convergence matches the minimax optimal upper bound \( \|\hat{\beta} - \beta\|_\infty = O_p(\sqrt{\log n}) \) for the Lasso estimator in the linear model with \( p = n - 1 \)-dimensional true parameter vector \( \beta^* \) and the sample size \( n \), see Lounici (2008).

Second, we get the asymptotic normality of the estimator in the restricted parameter space under the slower rate condition for \( e^{Q_n} \) compared with the rate for \( e^{Q_n} \) in Theorem 2.1, as follow.

**Theorem 2.2.** Consider the discrete Laplace mechanism with \( \lambda_n = \exp(-\epsilon_n/2) \) and assume that the conditions in Theorem 2.1 hold. If we assign a smaller \( e^{Q_n} = o((n^{1/18} / (\log n)^{1/9})) \), then as \( n \) goes to infinity, for any fixed \( r \geq 1 \), the vector

\[
(v_{11}^{1/2}(\alpha_1 - \alpha_i), \ldots, v_{rr}^{1/2}(\alpha_r - \alpha_r)) \xrightarrow{d} N_r(0, I_r)
\]

where \( I_r \) is a \( r \times r \) identity matrix.

**Remark 2.2.** By Theorem 2.2, for any fixed \( i \), the convergence rate of \( \hat{\alpha}_i \) is \( 1/v_{ii}^{1/2} \), when \( \epsilon_n = \tilde{O}(\sqrt{\log n}) \). Since \( (n - 1)(2(1 + e^{Q_n}))^{-1} \leq v_{ii} \leq (n - 1)q^{2} / 2 \), the convergence rate is between \( O(n^{-1/2} e^{Q_n}/2) \) and \( O(n^{-1/2}) \), which is the same as the non-privacy estimator [Yan et al. (2016b)].

The proofs of Theorems 2.1 and 2.2 are postponed in the Appendix section. After deriving the theoretical results, numerical studies are carried out in the next section to verify the asymptotic properties of the Z-estimate. Theorem 2.2 can also be used to construct a confidence interval for the parameters. For instance, an approximate \( 1 - \alpha \) confidence interval for \( \alpha_i - \alpha_j \) is \( \hat{\alpha}_i - \hat{\alpha}_j \pm Z_{1-\alpha/2}(1/v_{ii} + 1/v_{jj})^{1/2} \), where \( Z_{1-\alpha/2} \) is the \( 1 - \alpha \)-quantile of the standard normal distribution, \( \hat{\varepsilon}_{ii} \) and \( \hat{\varepsilon}_{jj} \) are the Z-estimates of \( \varepsilon_{ii} \) and \( \varepsilon_{jj} \) by replacing all \( \alpha_i \) with their \( \hat{\alpha}_i \)'s.
3. NUMERICAL STUDIES

3.1 Simulations

We first consider simulations under a discrete weight $q = 3$. In this case, we evaluate asymptotic properties by simulating finite sample data in finite networks. We consider the changes of $n, \varepsilon$ and $L$. Based on Yan and Leng (2015) and Yan et al. (2016a), the setting of the true parameter vector $\alpha^*$ takes a linear form. Specifically, we set $\alpha_{ij}^* = (n - i + 1)L/n$, for $i = 1, \ldots, n$. We discuss three distinct values for $L$, $L = 0, \log(\log n), (\log n)^{1/2}$, respectively. We simulate three distinct values for $\varepsilon$: one is fixed ($\varepsilon = 2$) and the other two values tend to zero with $n$, i.e., $\varepsilon = \log(n)/n^{1/4}, \log(n)/n^{1/2}$. Here we discuss three values for $n$, $n = 100, 200$ and 500. Each simulation is repeated 10,000 times.

By Theorem 2.2, $\tilde{\xi}_{ij} = [\tilde{\alpha}_i - \bar{\alpha}_j - (\alpha_{ij}^* - \alpha_{i}^*)]/(1/\bar{v}_{ii} + 1/\bar{v}_{jj})^{1/2}$ converges to the standard normal distribution, where $\bar{v}_{ii}$ is the estimator of $v_{ii}$ by replacing $\alpha_{ij}^*$ with $\hat{\alpha}_{ij}$. Hence, we apply the quantile-quantile (QQ) plot to demonstrate the asymptotic normality of $\tilde{\xi}_{ij}$. Three special pairs $(1, 2), (n/2, n/2 + 1)$ and $(n - 1, n)$ for $(i, j)$ are presented in Figure 1. Further, we list the coverage probability of the 95% confidence interval, the length of the confidence interval, and the frequency that the estimate does not exist.

For $\varepsilon = 2, \log(n)/n^{1/4}$, the QQ-plots under $n = 100, 200$ and 500 are similar. Thus, we here only show the QQ-plots for $\tilde{\xi}_{ij}$ under the case of $\varepsilon = 2$ and $n = 100$ in Figure 1 to save space. In Figure 1, the horizontal and vertical axes are the theoretical and empirical quantiles, respectively, and the red lines correspond to the reference lines $y = x$. From Figure 1, we see that for fixed pair $(i, j) = (1, 2)$, the empirical quantiles coincide well with the ones of the standard normality for noisy estimates (i.e., $\tilde{\xi}_{ij}$) expect for $L = (\log n)^{1/2}$. When $L = (\log n)^{1/2}$, notable deviations exist for pair $(1, 2)$ in Figure 1. For other pairs $(n/2, n/2 + 1)$ and $(n - 1, n)$, the approximation of asymptotic normality is good when $L = 0, \log(\log n), (\log n)^{1/2}$. While $\varepsilon = \log(n)/n^{1/2}$, the approximation of asymptotic normality respecting to $\tilde{\xi}_{ij}$ is bad, see Figure 1 of the supplementary material.

The coverage probability of the 95% confidence interval for $\alpha_{ij}^* - \alpha_{ij}$, the length of the confidence interval, and the frequency that the estimate does not exist are summarized in Table 1. The length of the confidence interval is related to $L$ and $n$. That is, the length increases as $L$ increases, or the length decreases as $n$ increases. Under the case of $\varepsilon = 2, \log(n)/n^{1/4}$, the coverage frequencies of pair $(1, 2)$ are higher than the nominal level 95% expect for $L = 0$; for other pairs $(n/2, n/2 + 1)$ and $(n - 1, n)$, the coverage frequencies are all close to the nominal level 95% for all $L$, where the ones are the closest at $n = 500$. For $\varepsilon = \log(n)/n^{1/2}$, the coverage frequencies are lower than the nominal level 95% for all $L$. This indicates that as $\varepsilon$ reduces to a specific value (e.g., $\log(n)/n^{1/2}$), notable deviations exist between the coverage frequencies and the nominal level 95%, especially the probabilities of the non-existent estimates are very high when $L = (\log n)^{1/2}$.

Second, we compare the simulation results between with the denoising process [Karwa et al. (2016)] and without the denoising process in the case of $q = 2$. Here the settings of $\alpha^*, \alpha^*, L$ and $\varepsilon$ are the same as those in the first simulation. Here, we only consider $n = 100, 200$ without 500. Each simulation is repeated 10,000 times.

According to the results in Karwa et al. (2016), $\xi_{ij} = [\bar{\alpha}_i - \bar{\alpha}_j - (\alpha_{ij}^* - \alpha_{i}^*)]/(1/\bar{v}_{ii} + 1/\bar{v}_{jj})^{1/2}$ converges to the standard normal distributions, where $\bar{\alpha}_i$ is the estimate of $\alpha_i$ with the denoising process and $\bar{v}_{ii}$ is the estimate of $v_{ii}$ by replacing $\alpha_{ij}$ with $\hat{\alpha}_{ij}$. We apply the quantile-quantile (QQ) plot and record the coverage probability of the 95% confidence interval, the length of confidence interval, and the frequency that the estimate does not exist, to compare the performance of $\tilde{\xi}_{ij}$ and $\xi_{ij}$. The QQ-plots are shown in Figure 2 and numerical comparison results are given in Table 2. In Figure 2, the QQ-plots for both $\tilde{\xi}_{ij}$ denoted by the red color and $\xi_{ij}$ denoted by the blue color are very close and coincide well with the ones of the standard normality when $\varepsilon = 2, \log(n)/n^{1/4}$ and $L \leq \log(\log n)$. (We only show the QQ-plots of $\varepsilon = 2$ and $n = 100$ in Figure 2 to save space and the other cases are similar.) This indicates that the parameter estimates are nearly the same with and without the
denoising process. However, when $\varepsilon = \log(n)/n^{1/2}$, the approximation of asymptotic normality of both $\xi_{ij}$ and $\xi_{ij}$ is not good, see Figure 2 of the supplementary material.

In Table 2, Type “A” and “B” represent the estimates without and with the denoised process, respectively. From this table, we can see that the difference between both estimates is very small. Similar to the analysis of Table 1, the length of confidence interval increases as $L$ increases and decreases as $n$ increases. Under the case of $\varepsilon = 2$, $\log(n)/n^{1/4}$, the coverage frequencies of all pairs are all close to the nominal level 95% when $L = 0$, $\log(n)$; for $L = (\log(n))^{1/2}$, both the non-denoised and denoised estimates often failed to exist for $n = 100$, while $n = 200$ the non-existent frequencies of estimates are lower. For $\varepsilon = \log(n)/n^{1/2}$, the coverage frequencies for both non-denoised and denoised estimates exist a great gap compared with the nominal level 95% for all $L$, and the probabilities of the non-existent estimates also increase as $L$ increases.

### 3.2 Real data example

We use the affiliation network dataset in Sundaresan et al. (2007) as a data example. As discussed in Haratym (2017), it remains an interesting issue that the animals should have some sort of privacy rights. In some ways, society has already begun moving in that direction. This network dataset is based on a study of a community of 28 Grevy’s zebras. Sundaresan et al. (2007) showed that Grevy’s zebra individuals are more selective in their choices of associates, tending to form bonds with others in the same reproductive state. In the dataset, Grevy’s zebras are labelled from 1 to 28, and 111 edges with finite weight $q = 3$. The edge weight of 0 denotes

| $n$ | $(i,j)$ | $L = 0$ | $L = \log(\log n)$ | $L = (\log(n))^{1/2}$ |
|-----|---------|----------|---------------------|----------------------|
| 100 | (1, 2)  | 94.63(0.35(0) | 96.75(0.81(0.41) | 99.75(1.13(31.79) |
|     | (50, 51)| 94.80(0.35(0) | 94.79(0.55(0.41) | 95.07(0.73(31.79) |
|     | (99, 100)| 94.90(0.35(0) | 94.04(0.41(0.41) | 94.55(0.46(31.79) |
| 200 | (1, 2)  | 94.40(0.25(0) | 97.68(0.62(0) | 99.86(0.85(6.42) |
|     | (50, 51)| 94.51(0.25(0) | 94.54(0.41(0) | 95.74(0.54(6.42) |
|     | (99, 100)| 94.44(0.25(0) | 94.45(0.30(0) | 94.92(0.33(6.42) |
| 500 | (1, 2)  | 95.17(0.16(0) | 98.62(0.42(0) | 99.98(0.57(0.03) |
|     | (50, 51)| 94.79(0.16(0) | 94.67(0.27(0) | 96.81(0.36(0.03) |
|     | (99, 100)| 95.05(0.16(0) | 94.91(0.19(0) | 95.20(0.21(0.03) |

| $n$ | $(i,j)$ | $L = 0$ | $L = \log(\log n)$ | $L = (\log(n))^{1/2}$ |
|-----|---------|----------|---------------------|----------------------|
| 100 | (1, 2)  | 94.41(0.35(0) | 95.80(0.81(1.42) | 99.48(1.12(51.83) |
|     | (50, 51)| 94.40(0.35(0) | 94.07(0.56(1.42) | 93.96(0.73(51.83) |
|     | (99, 100)| 94.39(0.35(0) | 93.66(0.41(1.42) | 94.12(0.46(51.83) |
| 200 | (1, 2)  | 94.40(0.25(0) | 96.90(0.62(0.03) | 99.70(0.84(17.92) |
|     | (50, 51)| 94.34(0.25(0) | 94.10(0.41(0.03) | 95.13(0.54(17.92) |
|     | (99, 100)| 94.24(0.25(0) | 94.09(0.30(0.03) | 94.42(0.33(17.92) |
| 500 | (1, 2)  | 95.03(0.16(0) | 98.30(0.42(0) | 99.94(0.57(0.73) |
|     | (50, 51)| 94.60(0.16(0) | 94.53(0.27(0) | 96.40(0.36(0.73) |
|     | (99, 100)| 94.92(0.16(0) | 94.82(0.19(0) | 95.00(0.21(0.73) |

Sundaresan et al. (2007) as a data example. As discussed in Haratym (2017), it remains an interesting issue that the animals should have some sort of privacy rights. In some ways, society has already begun moving in that direction. This network dataset is based on a study of a community of 28 Grevy’s zebras. Sundaresan et al. (2007) showed that Grevy’s zebra individuals are more selective in their choices of associates, tending to form bonds with others in the same reproductive state. In the dataset, Grevy’s zebras are labelled from 1 to 28, and 111 edges with finite weight $q = 3$. The edge weight of 0 denotes
that a pair of zebras never appeared during the study, the edge weight of 1 denotes that a pair of zebras appeared together at least once, while the edge weight of 2 indicates a statistically significant tendency of pairs to appear together.

On the other hand, the estimate \( \hat{\alpha} \) does not exist when the degree of a vertex is zero. Hence we removed the vertex 8 whose degree is zero before analysis. The network with the left 27 vertices is shown in Figure 3. We chose the privacy parameter \( \epsilon \) as 1. Figure 4 reports the scatter plots of noisy degree sequence \( \bar{d} \) vs the estimates \( \hat{\alpha} \) for the 27 Grevy’s zebra dataset. From Figure 4, the value of \( \hat{\alpha} \) increases as the number of \( \bar{d} \) increases. Furthermore, the estimates can reveal a trend in these zebras' choices of associates. The larger the estimates \( \hat{\alpha} \), the more zebras have associates or the higher the frequency of pairs to appear together. As shown in Figure 4, the number of zebras’ associates and the frequency of pairs to appear together are more and higher under the case that \( \hat{\alpha} \) is around zero. Table 3 reports the estimates, the 95% confidence interval, the corresponding standard errors and the noisy degree sequence. In Table 3, the larger estimates correspond to the larger noisy degrees. The largest degree is 23 for vertex 4, which also has the largest estimate 0.447. On the other hand, the vertex for 22 with the smallest estimate -2.260, has degree 2 in Table 3.

4. SUMMARY AND FUTURE STUDY

In this paper, we have established the uniform consistency and asymptotic normality of the Z-estimator in the generalized \( \beta \)-model involving noisy degree sequence \( \bar{d} = d + e \), where \( d \) is the sufficient statistic and \( e \) are some noises from discrete Laplace distribution. By using the Newton-Kantorovich theorem, we try to ignore adding noisy process, and obtain the existence and consistency of the Z-estimator.
Table 2. Estimated coverage probabilities ($\times$100\%) of $\alpha_i^* - \alpha_j^*$ for pair (i, j) as well as the length of confidence intervals (in square brackets), and the probabilities ($\times$100\%) that the estimate does not exist (in parentheses). Type “A” denotes the estimate with the non-denoised process and “B” denotes the estimate with the denoised process.

| n   | (i, j) | Type | $L = 0$ | $L = \log(\log(n))$ | $L = (\log(n))^{1/4}$ |
|-----|-------|------|--------|----------------------|-----------------------|
| 100 | (1,2) | A    | 93.62(0.57)(0) | 93.38(1.01)(1.25) | 97.38(1.46)(43.88) |
|     |       | B    | 93.79(0.57)(0) | 93.76(1.01)(1.30) | 97.35(1.47)(44.17) |
|     | (50,51) | A    | 93.44(0.57)(0) | 93.57(0.76)(1.25) | 93.16(0.94)(43.88) |
|     |       | B    | 93.62(0.57)(0) | 93.48(0.76)(1.30) | 93.19(0.94)(44.17) |
|     | (99,100) | A    | 93.42(0.57)(0) | 93.63(0.63)(1.25) | 93.16(0.68)(43.88) |
|     |       | B    | 93.82(0.57)(0) | 92.90(0.63)(1.30) | 93.43(0.68)(44.17) |
| 200 | (1,2) | A    | 94.55(0.40)(0) | 93.80(0.75)(0.03) | 96.60(1.11)(9.94) |
|     |       | B    | 94.85(0.40)(0) | 94.04(0.75)(0.03) | 96.75(1.11)(10.55) |
|     | (100,101) | A    | 95.09(0.40)(0) | 94.28(0.55)(0.03) | 93.93(0.68)(9.94) |
|     |       | B    | 94.77(0.40)(0) | 94.58(0.55)(0.03) | 93.87(0.68)(10.55) |
|     | (199,200) | A    | 94.89(0.40)(0) | 94.20(0.45)(0.03) | 93.75(0.48)(9.94) |
|     |       | B    | 94.28(0.40)(0) | 94.23(0.45)(0.03) | 93.56(0.48)(10.55) |
|     |       |       | $\epsilon = \log(n)/n^{1/4}$ |                  |                      |
| 100 | (1,2) | A    | 92.62(0.58)(0) | 91.31(1.02)(4.46) | 96.04(1.46)(65.58) |
|     |       | B    | 92.74(0.58)(0) | 91.74(1.02)(5.14) | 95.99(1.45)(66.34) |
|     | (50,51) | A    | 92.56(0.58)(0) | 91.88(0.76)(4.46) | 91.11(0.95)(65.58) |
|     |       | B    | 92.67(0.58)(0) | 92.00(0.76)(5.14) | 91.21(0.95)(66.34) |
|     | (99,100) | A    | 92.70(0.58)(0) | 92.58(0.63)(4.46) | 91.81(0.68)(65.58) |
|     |       | B    | 92.78(0.58)(0) | 91.79(0.64)(5.14) | 91.23(0.68)(66.34) |
| 200 | (1,2) | A    | 94.14(0.40)(0) | 92.03(0.76)(0.19) | 95.34(1.12)(26.08) |
|     |       | B    | 94.31(0.40)(0) | 92.69(0.76)(0.21) | 95.00(1.12)(26.44) |
|     | (100,101) | A    | 94.72(0.40)(0) | 93.40(0.55)(0.19) | 92.48(0.68)(26.08) |
|     |       | B    | 94.21(0.40)(0) | 93.46(0.55)(0.21) | 92.62(0.68)(26.44) |
|     | (199,200) | A    | 94.46(0.40)(0) | 93.51(0.45)(0.19) | 93.06(0.48)(26.08) |
|     |       | B    | 93.92(0.40)(0) | 93.46(0.45)(0.21) | 92.71(0.48)(26.44) |
|     |       |       | $\epsilon = \log(n)/n^{1/4}$ |                  |                      |
| 100 | (1,2) | A    | 79.34(0.58)(0.24) | 72.51(1.05)(88.03) | 100.00(1.44)(99.97) |
|     |       | B    | 78.94(0.58)(0.24) | 75.06(1.05)(87.29) | 66.67(1.27)(99.97) |
|     | (50,51) | A    | 78.51(0.58)(0.24) | 73.68(0.81)(88.03) | 100.00(0.87)(99.97) |
|     |       | B    | 79.20(0.58)(0.24) | 72.46(0.80)(87.29) | 100.00(0.82)(99.97) |
|     | (99,100) | A    | 78.81(0.58)(0.24) | 75.86(0.65)(88.03) | 100.00(0.70)(99.97) |
|     |       | B    | 78.73(0.58)(0.24) | 75.30(0.66)(87.29) | 100.00(0.67)(99.97) |
| 200 | (1,2) | A    | 82.64(0.41)(0) | 69.26(0.80)(70.75) | 100.00(1.23)(99.96) |
|     |       | B    | 82.51(0.41)(0) | 70.91(0.79)(71.16) | 100.00(0.83)(99.96) |
|     | (50,51) | A    | 83.03(0.41)(0) | 74.50(0.57)(70.75) | 50.00(0.94)(99.96) |
|     |       | B    | 82.74(0.41)(0) | 75.24(0.57)(71.16) | 50.00(0.64)(99.96) |
|     | (99,100) | A    | 82.66(0.41)(0) | 77.88(0.45)(70.75) | 100.00(0.50)(99.96) |
|     |       | B    | 82.54(0.41)(0) | 79.96(0.45)(71.16) | 50.00(0.47)(99.96) |
Lemma A.1. Consider the discrete Laplace mechanism with \( \lambda_n = \exp(-\epsilon_n/2) \), if \( \epsilon_n = \Omega(\sqrt{\log n}) \) and \( e^\Delta_n = o((n/ \log n)^{\frac{1}{2}}) \), then as \( n \) goes to infinity, for any fixed \( r \geq 1 \),

\[
\left( \frac{\bar{d}_1 - E(d_1)}{\sqrt{v_{11}}}, \ldots, \frac{\bar{d}_r - E(d_r)}{\sqrt{v_{rr}}} \right) \xrightarrow{d} N_r(0, I_r).
\]

Lemma A.1 indicates that the components of \( (\bar{d}_1 - E(d_1), \ldots, \bar{d}_r - E(d_r)) \) are asymptotically independent and normally distributed with variances \( v_{11}, \ldots, v_{rr} \), respectively.

Lemma A.2 (Karwa et al. (2016), Proposition E). Let \( e_1, \ldots, e_n \) be i.i.d random variables drawn from discrete Laplace distribution with probability mass function defined by

\[
P(e_i = c) = \frac{1 - \lambda}{1 + \lambda} \lambda^{|c|}, e \in Z, \lambda \in (0, 1).
\]

Then we have \( E(e_i) = 0 \) and \( \text{Var}(e_i) = \frac{2\lambda}{(1 - \lambda)^2} \). Moreover,

\[
P(|e_i| > c) = \frac{2\lambda^{c+1}}{1 + \lambda}, \quad E[|e_i|] = \frac{2\lambda}{1 - \lambda^2};
\]

\[
P(\max_i |e_i| > c) = 1 - \left( 1 - \frac{2\lambda^{c+1}}{1 + \lambda} \right)^n.
\]

Proof of Lemma A.1. By \( \bar{d}_i = d_i + e_i \), we can analysis the asymptotic normality of Lemma A.1 in two parts, i.e., \( (d_i - E(d_i))/v_{ii}^{1/2} \) and \( (e_i - E(e_i))/v_{ii}^{1/2} \). On the one hand, Yan et al. (2015) have verified the result of the first part by Liapounov’s central limit theorem [Chung (2001)]. On the other hand, we can easily obtain the stochastic order of the second part by Chebychev inequality.

Let \( \bar{d}_i = \bar{d}_i + e_i, \quad i = 1, \ldots, r \), then

\[
\left( \frac{\bar{d}_1 - E(d_1)}{\sqrt{v_{11}}}, \ldots, \frac{\bar{d}_r - E(d_r)}{\sqrt{v_{rr}}} \right) \xrightarrow{d} N_r(0, I_r).
\]
Now, we only discuss the property of $\frac{e_i}{\sqrt{v_i}}$. In fact, by Chebyshev inequality and Lemma A.2, for any constant $a > 0$, as $n$ goes to infinity, we have

$$P\left(\frac{e_i}{\sqrt{v_i}} \geq a\right) = P\left(|e_i| \geq a\sqrt{v_i}\right) \leq \frac{\Var(e_i)}{a^2v_i} \leq \frac{2(1+e^{Q_n})}{a^2(n-1)} \times \frac{2e^{-\frac{2}{h}}}{(1-e^{-\frac{2}{h}})^2} \leq O\left(\frac{e^{Q_n}}{n}\right) \rightarrow 0,$$

by noticing that $e^{-\frac{2}{h}} < 1$ for all $n$.

\begin{proof}

**Step 2:** We apply the Newton-Kantorovich theorem \cite{Gragg and Tapia (1974)} to obtain the existence and consistence of the estimator satisfying the equation (2).

For a subset $C \subset \mathbb{R}^n$, let $C^0$ and $\overline{C}$ denote the interior and closure of $C$ in $\mathbb{R}^n$, respectively. Let $\Omega(x, r)$ denote the open ball $\{y : \|y - x\| < r\}$, and $\overline{\Omega(x, r)}$ be its closure.

**Proposition A.1** (Gragg and Tapia (1974)). Let $F(x) = (F_1(x), \ldots, F_n(x))^T$ be a function vector on $x \in \mathbb{R}^n$. Assume that the Jacobian matrix $F'(x)$ is Lipschitz continuous on an open convex set $D$ with the Lipschitz constant $\kappa$. Given $x_0 \in D$, assume that $[F'(x_0)]^{-1}$ exists,

$$\|F'(x_0)^{-1}\|_{\infty} \leq R,$$

$$\|F'(x_0)^{-1}F'(x_0)\|_{\infty} \leq \delta, h = 2\kappa \delta \leq 1,$$

$$\Omega(x_0, t^*) \subset D^0, t^* := \frac{2}{h}(1 - \sqrt{1 - h}) \delta = \frac{2}{1 + \sqrt{1 - h}} \delta \leq 2\delta,$$

where $N$ and $\delta$ are positive constants that may depend on $x_0$ and the dimension $n$ of $x_0$. Then the Newton iteration $x_{k+1} = x_k - [F'(x_k)]^{-1}F(x_k)$ exists and $x_k \in \Omega(x_0, t^*) \subset D^0$ for all $k \geq 0, x = \lim x_k$ exists, $x \in \Omega(x_0, t^*) \subset D$ and $F(x) = 0$.

Besides, we also need the following four lemmas to prove Theorem 2.1. Specifically, Lemmas A.3-A.5 are served for Newton-Kantorovich theorem, and Lemma A.6 is based on Hoeffding’s inequality and Lemma A.2.

**Lemma A.3** (Yan and Xu (2013)). If $V_n \in \mathcal{L}_n(m, M)$, and $n$ is large enough, then

$$\|V_n^{-1} - S_n\| \leq \frac{cM^2}{m^3(n-1)^2},$$

where $(S_n)_{ij} := \frac{\delta_{ij}1}{v_i}, v_i := \sum_{i=1}^n v_{ii}, c$ is a constant that not depends on $M, m$ and $n$.

To establish the form $\|F'(\alpha)^{-1}F(\alpha)\|_{\infty} \leq \delta$ in Theorem 2.1, we first use a simple matrix $S_n = (s_{ij})$ to approximate $V_n^{-1}$. The upper bound of the approximation error is given below.

**Lemma A.4.** Assume that $\alpha \in D$, where $D = \{\alpha \in \mathbb{R}^n : -Q_n \leq \alpha_i + \alpha_j \leq Q_n, 1 \leq i < j \leq n\}$. If $n$ is large enough, then

$$\|V_n^{-1} - S_n\| \leq \frac{c_1(1 + e^{Q_n})^3}{(n-1)^2},$$

where $V_n := F'(\alpha), (S_n)_{ij} := \frac{\delta_{ij}1}{v_i}, c_1$ is a constant that not depends on $e^{Q_n}$ and $n$.

\begin{proof}

By (3), $v_i = \sum_{i=1}^n v_{ii} \geq n(n-1)m$ and Lemma A.3, we can easily obtain

$$\|V_n^{-1} - S_n\| \leq \|V_n^{-1} - \tilde{S}_n\| + \|\tilde{S}_n - S_n\|$$

$$\leq \frac{cM^2}{m^3(n-1)^2} + \frac{1}{n(n-1)m}$$

$$\leq \frac{c + n^2}{M^2} = \frac{M^2}{m^3(n-1)^2} = \frac{c_1(1 + e^{Q_n})^3}{(n-1)^2},$$

where $c_1$ is a constant that not depends on $e^{Q_n}$ and $n$.

To confirm the value of $\kappa$ in Newton-Kantorovich theorem, we use triangle inequality and Lemma A.4 to obtain the upper bound of $V_n^{-1}$, as follow.

**Lemma A.5.** Assume that $\alpha \in D$, where $D = \{\alpha \in \mathbb{R}^n : -Q_n \leq \alpha_i + \alpha_j \leq Q_n, 1 \leq i < j \leq n\}$. If $n$ is large enough, then

$$\|V_n^{-1}\|_{\infty} \leq \frac{c_2(1 + e^{Q_n})^3}{n-1},$$

where $V_n := F'(\alpha), (S_n)_{ij} := \frac{\delta_{ij}1}{v_i}, c_2$ is a constant that not depends on $e^{Q_n}$ and $n$.

\begin{proof}

By (3) and Lemma A.4, we obtain

$$\|V_n^{-1}\|_{\infty} \leq \|V_n^{-1} - S_n\|_{\infty} + \|S_n\|_{\infty}$$

$$\leq \frac{c_1(1 + e^{Q_n})^3}{(n-1)^2} + \frac{2(1 + e^{Q_n})}{n-1}$$

$$\leq \frac{c_2(1 + e^{Q_n})^3}{n-1},$$

where $c_2$ is a constant that not depends on $e^{Q_n}$ and $n$.

The following lemma guarantees that the upper bound of $\|d - E(d)\|_{\infty}$ is the magnitude of $(n \log n)^{1/2}$.

**Lemma A.6.** Let $\kappa_n = 2(q-1)\sqrt{(n-1) \log(n-1)}$. Then, with probability approaching one as $n \to \infty$,

$$\max_{1 \leq i \leq n} |d_i - E(d_i)| \leq 2(q-1)\sqrt{(n-1) \log(n-1)}.$$
Proof. Let \( \kappa_n = 2(q-1)\sqrt{(n-1)\log(n-1)} \), then
\[
P(\max_i | d_i - E(d_i) | \geq \kappa_n)
\leq P(\max_i | d_i - E(d_i) | \geq \frac{\kappa_n}{2}) + P(\max_i | e_i | \geq \frac{\kappa_n}{2}).
\]
(7)

Here, the inequality (7) is divided into two parts and is given respective discussions. For the first part, we have
\[
P(\max_i | d_i - E(d_i) | \geq \frac{\kappa_n}{2}) \leq \sum_i P( | d_i - E(d_i) | \geq \frac{\kappa_n}{2}).
\]

By Hoeffding’s inequality, it implies
\[
P( | d_i - E(d_i) | \geq \frac{\kappa_n}{2}) \leq 2 \exp \left( - \frac{2(\frac{\kappa_n}{2})^2}{(n-1)(q-1)^2} \right)
\leq 2 \exp \left( - \frac{2(q-1)^2(n-1)\log(n-1)}{(n-1)(q-1)^2} \right)
= \frac{2}{(n-1)^2}.
\]

For the second part, by definition of maximum and \( \lambda_n = \exp(-\epsilon_n/2) \), we obtain
\[
P(\max_i | e_i | \geq \frac{\kappa_n}{2}) = 1 - \prod_{i=1}^{n} P( | e_i | \leq \frac{\kappa_n}{2})
= 1 - \left( 1 - \frac{2\lambda_n(\frac{\kappa_n}{2})+1}{1 + \lambda_n} \right)^n
= 1 - \left( 1 - \frac{2e^{-\epsilon_n}(\frac{\kappa_n}{2})+1/2}{1 + e^{-\epsilon_n/2}} \right)^n.
\]

Therefore, by (8) and (9), with probability approaching one as \( n \to \infty \), we have
\[
\max_i | d_i - E(d_i) | \leq 2(q-1)\sqrt{(n-1)\log(n-1)}.
\]

Proof of Theorem 2.1. In the Newton's iterative step, putting the initial value \( \alpha^0 := \alpha \). Let \( V_n = F'(\alpha) \in \mathcal{L}_\alpha(m,M) \) and \( W_n = V_n^{-1}S_n \). Let \( F(\alpha) = d - E(d) \), by Lemma A.4 and (6), we get
\[
||[F'(\alpha)]^{-1}F(\alpha)|| \leq n||W_n|||F(\alpha)||_{\infty} + \max_i |\frac{F_i(\alpha)}{v_{ii}}|
\leq (n||W_n|| + \frac{1}{v_{ii}})||F(\alpha)||_{\infty}
\leq \left( c_1n(1 + e^{Q_{\alpha}})^3 + 2(1 + e^{Q_{\alpha}}) \right)||F(\alpha)||_{\infty}
\leq c_3(1 + e^{Q_{\alpha}})^3 \sqrt{\frac{\log(n-1)}{n-1}},
\]
where \( c_3 \) is a constant.

Combining Lemma A.5, we can set \( \delta = c_3(1 + e^{Q_{\alpha}})^3 \sqrt{\log(n-1)} \) and \( \mathcal{R} = c_3(1 + e^{Q_{\alpha}})^3 \) in Newton-Kantorovich theorem.

Next, we indicate that the Jacobian matrix \( F'(\alpha) \) is Lipschitz continuous with \( \kappa = 4(q-1)^3(n-1) \). Here, our method is similar to Yan et al. (2016b).

Let \( g_{ij}(\alpha) = \left( \frac{\partial^2 F_i}{\partial \alpha_j \partial \alpha_j}, \frac{\partial^2 F_i}{\partial \alpha_j \partial \alpha_l} \right)^T \).

By some computations, we have
\[
\frac{\partial^2 F_i}{\partial \alpha_j^2} = \sum_{j,k,l} \frac{k(k-l)^2(2a)\epsilon^{(k+l+\alpha)}(\alpha_i + \alpha_j)}{2(\sum_{a=0}^{k-1} \epsilon^{(k+\alpha)}(\alpha_j+\alpha_j))}.
\]

As \( \sum_{k,l,a} \epsilon^{(k+l+\alpha)}(\alpha_i + \alpha_j) \leq (n-1)^{-1} \epsilon^{(k+\alpha)}(\alpha_j+\alpha_j) \), we have
\[
\frac{\partial^2 F_i}{\partial \alpha_j^2} \leq (q-1)^3(n-1), \quad \frac{\partial^2 F_i}{\partial \alpha_j \partial \alpha_l} \leq (q-1)^3.
\]

Thus \( ||g_{ij}(\alpha)||_1 \leq 2(n-1)(q-1)^3 \).

If \( i \neq j \) and \( k \neq i,j \),
\[
\frac{\partial^2 F_i}{\partial \alpha_i \partial \alpha_j} = 0.
\]

Then we get \( ||g_{ij}(\alpha)||_1 \leq 2(q-1)^3, i \neq j \). Therefore, by the mean-value for vector-valued functions (Lang (1993)),

**Asymptotic theory for differentially private generalized β-models with parameters increasing**
for a vector \( \mathbf{v} \),

\[
\max \left\{ \sum_{j=1}^{n} \left| \frac{\partial F_i}{\partial x_j}(x) - \frac{\partial F_i}{\partial x_j}(y) \right| v_j \right\}
\]

\[
\leq \|v\|_{\infty} \max \left\{ \sum_{j=1}^{n} \left| \frac{\partial F_i}{\partial x_j}(x) - \frac{\partial F_i}{\partial x_j}(y) \right| \right\}
\]

\[
= \|v\|_{\infty} \max \left\{ \sum_{j=1}^{n} \left| \int_{0}^{1} g_{ij}(tx + (1-t)y)(x-y)dt \right| \right\}
\]

\[
\leq 4(q-1)^3(n-1)\|v\|_{\infty}\|x-y\|_{\infty}.
\]

Thus, all conditions in the Newton-Kantorovich theorem are satisfied. Since the inequality (6) holds with probability approaching one, then (4) is fulfilled.

**A.2 Proof of Theorem 2.2**

The aim of proving Theorem 2.2 is to establish the following equation

\[
(\mathbf{a} - \mathbf{c})_i = [S_n(\mathbf{d} - E(\mathbf{d}))]_i + o_p(n^{-1/2}).
\]

This will follow directly from \([W_n(\mathbf{d} - E(\mathbf{d}))]_i = o_p(n^{-1/2}), \) \(n \to \infty\), and by Theorem 2.1. To this end, we introduce the following lemma.

**Lemma A.7.** Assume that \( \alpha \in D \), where \( D = \{\alpha \in \mathbb{R}^n : -Q_n \leq \alpha_i + \alpha_j \leq Q_n, \text{ for } 1 \leq i < j \leq n \} \). Let \( \epsilon_n = \Omega(\sqrt{\log n}) \), \( e^{Q_n} = o((n/\log n)^{1/2}) \), and \( U_n = \text{cov}[W_n(\mathbf{d} - E(\mathbf{d}))] \). Then

\[
[W_n(\mathbf{d} - E(\mathbf{d}))]_i = o_p(n^{-1/2}).
\]

**Proof.** Let \( \nabla_n = \text{cov}(\mathbf{d} - E(\mathbf{d})) \), \( V_n = \text{cov}(\mathbf{d} - E(\mathbf{d})) \) and \( E_n = \text{cov}(\mathbf{e}) \). For \( 1 \leq i \leq n \), the random variables \( d_i, e_i \) are mutually independent, then

\[
\text{cov}(d_i - E(d_i), d_j - E(d_j)) = \text{cov}(d_i - E(d_i), d_j + e_j - E(d_j))
\]

\[
+ \text{cov}(e_i, d_j) - \text{cov}(e_i, d_j - E(d_j)) + \text{cov}(e_i - E(e_i), e_j)
\]

Thus the elements of the matrix \( \nabla_n \) are denoted by \( \bar{v}_{ij} = v_{ij}, \bar{v}_{ii} = v_{ii} + \text{var}(e_i), 1 \leq i \neq j \leq n \). Let \( U_n = \text{cov}[W_n(\mathbf{d} - E(\mathbf{d}))] \). Then

\[
U_n = W_n \nabla_n W_n^T = W_n(V_n + E_n)W_n^T = W_nV_nW_n^T + W_nE_nW_n^T.
\]

On the one hand, \( W_nV_nW_n^T = (V_n^{-1} - S_n) - S_n(I_n - V_nS_n) \), where \( n \times n \) matrix \( I_n \) is an identity matrix.

By (3), we obtain

\[
(11)
\]

\[
|\{S_n(I_n - V_nS_n)\}_{ij}| = |(\delta_{ij} - 1)v_{ij}| \leq \frac{2q^2(1 + e^{Q_n})^2}{(n-1)^2}.
\]

By Lemma A.4 and (11), we have

\[
\|W_nV_nW_n^T\| \leq \frac{c_1(1 + e^{Q_n})^3}{(n-1)^2} + \frac{2q^2(1 + e^{Q_n})^2}{(n-1)^2}
\]

\[
\leq O \left( \frac{e^{3Q_n}}{n^3} \right).
\]

On the other hand,

\[
\|W_nE_nW_n^T\| \leq \sum_{k=1}^{n} w_{ik} e_k w_{kj} \]

\[
\leq \max_k |e_k| \sum_{k=1}^{n} w_{ik} \|w_{kj}\|
\]

\[
\leq n \max_k |e_k| \|W_n\|^2
\]

\[
\leq \frac{2ne^{-2}}{(1 - e^{-2})^2} \times \frac{c_1^2(1 + e^{Q_n})^6}{(n-1)^4}
\]

\[
\leq O \left( \frac{e^{6Q_n}}{n^3} \right).
\]
Hence, $\|U_n\| \leq O\left(\frac{e^{3Q_n}}{n^2}\right)$. Furthermore, by Chebyshev inequality, for any constant $\alpha > 0$, we get

$$P\left(\frac{[W_n(d - E(d))]}{\alpha n^{-1/2}} \geq \alpha\right) = P\left(\frac{[W_n(d - E(d))]}{\alpha n^{-1/2}} \geq an^{-1/2}\right) \leq \frac{n[\text{cov}(W_n(d - E(d)))]}{a^2} \leq O\left(\frac{e^{3Q_n}}{n}\right).$$

Then while $e^{Q_n} = o(n(\log n)^{1/2})$,

$$P\left(\frac{[W_n(d - E(d))]}{n^{-1/2}} \geq \alpha\right) \rightarrow 0,$

as $n \rightarrow \infty$. Therefore,

$$[W_n(d - E(d))] = o_p(n^{-1/2}).$$

**Proof of Theorem 2.2.** Let $\hat{r}_{ij} = \hat{\alpha}_i + \hat{\alpha}_j - \alpha_i - \alpha_j$. Under the conditions in Theorem 2.1, we have from the consistency property

$$\max_{i \neq j} \hat{r}_{ij} = O_p(\sqrt{\log n}).$$

Let $u(t) = \sum_{k=0}^{n-1} a_k e^{-k^2 t}$. For $i = 1, \ldots, n$, by the Taylor's expansion, we get

$$\hat{d}_i - E(d_i) = \sum_{j \neq i} [u(\hat{\alpha}_i + \hat{\alpha}_j) - u(\alpha_i + \alpha_j)]$$

$$= \sum_{j \neq i} [u'(\alpha_i + \alpha_j)((\hat{\alpha}_i + \hat{\alpha}_j) - (\alpha_i + \alpha_j)) + h_i],$$

where $h_i = (1/2) \sum_{j \neq i} u''(\hat{r}_{ij})[(\hat{\alpha}_i + \hat{\alpha}_j) - (\alpha_i + \alpha_j)]^2$ and $\hat{r}_{ij} = t_{ij}(\alpha_i + \alpha_j) + (1 - t_{ij})(\hat{\alpha}_i + \hat{\alpha}_j), t_{ij} \in (0, 1)$. Writing the above expressions into a matrix, we have

$$\hat{d} - E(d) = V_n(\hat{\alpha} - \alpha) + h,$$

thus

$$\hat{\alpha} - \alpha = V_n^{-1}(\hat{d} - E(d)) + V_n^{-1}h,$$

where $h = (h_1, \ldots, h_n)^T$.

By (10), we know $|h_i| \leq \frac{1}{2}(n-1)(q-1)^2\hat{r}_{ij}^2$. Therefore,

$$\|V_n^{-1}h_i\| = |(S_n h_i)| + \|W_n h_i\| \leq \max_i |h_i| + \|W_n\| \sum_i |h_i| \leq O\left(\frac{e^{3Q_n} \log n}{n}\right).$$

If $e^{Q_n} = o\left(\frac{n^{1/18}}{(\log n)^{1/6}}\right)$, then $V_n^{-1}h_i = o(n^{-1/2}).$

By Theorem 2.1 and Lemma A.7, for $i = 1, \ldots, r$, we have

$$(\hat{\alpha} - \alpha)_i = [S_i(d - E(d))]_i + o_p(n^{-1/2})$$

$$= \frac{\hat{d}_i - E(d_i)}{v_{ii}} + o_p(n^{-1/2}).$$

Hence, Theorem 2.2 follows directly from Lemma A.1. Finally, we conclude the proof by multiplying $\sqrt{v_{ii}}$ to left and right of the last display. $\square$

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**SUPPLEMENTARY MATERIAL**

Supplement to “Asymptotic Theory for Differentially Private Generalized β-models with Parameters Increasing.” The supplementary material contains a brief introduction of skew discrete Laplace mechanism in Subsection 2.2, and the QQ plots of parameter estimates with $\epsilon = \log(n)/n^{1/2}$ in Subsection 3.1 (http://intlpress.com/site/pub/files/supp/sii/2020/0013/0003/SII-2020-0013-0003-s003.pdf).

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**REFERENCES**

Aggarwal C. C., Yu P. S., 2007. On privacy-preservation of text and sparse binary data with sketches. In Proceedings of the 2007 SIAM International Conference on Data Mining, 57–67. MR3523923

Blitzstein J., Diaconis P., 2011. A sequential importance sampling algorithm for generating random graphs with prescribed degrees. Internet Math. 6, 489–522. MR2860836

Chatterjee S., Diaconis P., Sly A., 2011. Random graphs with a given degree sequence. Ann. Appl. Probab. 21, 1400–1435. MR2857452

Chung K. L., 2001. A course in probability theory, 3rd. Academic press. MR1758626

Duncan G. T., Keller-McNulty S. A., Stokes S. L., 2004. Disclosure risk vs. data utility: the R-U confidentiality map. Technical Report Number 121 of National Institute of Statistical Sciences. MR2061932

Duchi J. C., Jordan M. I., Wainwright M. J., 2018. Minimax optimal procedures for locally private estimation. J. Am. Stat. Assoc. 113(521), 182–201. MR3803452

Dwork C., 2006a. Differential privacy. In Proceedings of the 33rd international colloquium on Automata, Languages and Programming, 1–12. MR2307219

Dwork C., McSherry F., Nissim K., Smith A., 2006b. Calibrating noise to sensitivity in private data analysis. In TCC, 265–284. MR2241676

Farine D. R., 2014. Measuring phenotypic assortment in animal social networks: Weighted associations are more robust than binary edges. Anim. Behav. 89, 141–153.

Fung B. C. M., Wang K., Yu P. S., 2007. Anonymizing classification data for privacy preservation. IEEE Transactions on Knowledge and Data Engineering (TKDE) 19(5May), 711–725.
Grigg W. B., Tapia R. A., 1974. Optimal error bounds for the Newton-Kantorovich theorem. *SIAM J. Numer. Anal.* **11**, 10–13. MR0344594

Granovetter M. S., 1993. The strength of weak ties. *Am. J. Sociol.* 1360–1380.

Haratym E., 2017. Animals’ right to privacy. *World Scientific News* **85**, 13–77.

Hay M., Li C., Miklau G., Jensen D., 2009. Accurate estimation of the degree distribution of private networks. In *ICDM’09. Ninth IEEE International Conference on Data Mining*. Springer, Berlin, Heidelberg. 169–178.

Hillar C., Wibisono A., 2013. Maximum entropy distributions on graphs. Available at http://arxiv.org/abs/1301.3321.

Holohan N., Leith D. J., Mason O., 2017. Extreme points of the local differential privacy polytope. *Linear. Algebra. Appl.* **534**, 78–96. MR3697057

Inouash S., Kozubowski T. J., 2006. A discrete analogue of the Laplace distribution. *J. Stat. Plan. Infrer.* **136**(3), 1090–1102. MR2184990

Jackson M. O., 2008. *Social and economic networks*. Princeton University Press, Princeton. MR2435744

Karwa V., Slavkovic A. B., 2016. Inferencing using noisy degrees: differentially private \( \beta \)-model and synthetic graphs. *Ann. Stat.* **44**, 87–112. MR3449763

Kashivwanathian S. P., Nissim K., Raskhodnikova S., Smith A., 2013. Analyzing Graphs with Node Differential Privacy. In *Theory of Cryptography*, A. Sahai (eds), Lecture Notes in Computer Science, vol 7785. Springer, Berlin, Heidelberg.

Kotz S., Kozubowski T., Podgórski K., 2012. *The Laplace distribution and generalizations: a revisit with applications to communications, economics, engineering, and finance*. Springer. MR1935481

Kozubowski T. J., Inouash S., 2006. A skew Laplace distribution on intervals. Ann. I. Stat. Math. **58**(3), 555–571. MR2327893

Lang S., 1993. *Real and Functional Analysis*. Springer. MR1216137

Lauritzen S. L., 2008. Exchangeable Rasch matrices. Rendiconti di Matematica, Series VII. 28, 83–95. MR2463441

Li N., Li T., Venkatasubramanian S., 2007. t-closeness: Privacy beyond \( k \)-anonymity and 1-diversity. In *Proceedings of the 23rd International Conference on Data Engineering*, 106–115.

Li Y., Shen H., Lang C., Dong H., 2016. Practical anonymity models on protecting private weighted graphs. *Neurocomputing* **218**, 359–370.

Lounici, K., 2008. Sup-norm convergence rate and sign concentration property of Lasso and Dantzig estimators. *Electron. J. Stat.* **2**, 90–102. MR2380087

Machanavajjhala A., Gehrke J., Kifer D., Venkatasubramania M., 2006. \( \ell_\infty \)-diversity: Privacy beyond kappa-anonymity. In *Proceedings of the 22nd International Conference on Data Engineering*, 24.

McSherry F., Talwar K., 2007. Mechanism design via differential privacy. In *Proceedings of the 39th Annual Symposium on Foundations of Computer Science*. IEEE. 94–103.

Nissim K., Raskhodnikova S., Smith A., 2007. Smooth sensitivity and sampling in private data analysis. In *Proceedings of the thirty-ninth annual ACM Symposium on Theory of Computing*. ACM. 75–84. MR2402430

Van der Vaart A. W., 1998. *Asymptotic statistics* (Vol. 3). Cambridge university press. MR1652247

Woodruff D. P., 2014. Graph-Querying with Differential Privacy. In *NIPS 2014*.

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