Stochastic Monotone Submodular Maximization with Queries

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Abstract

We study a stochastic variant of monotone submodular maximization problem as follows. We are given a monotone submodular function as an objective function and a feasible domain defined on a finite set, and our goal is to find a feasible solution that maximizes the objective function. A special part of the problem is that each element in the finite set has a random hidden state, active or inactive, only the active elements contribute to the objective value, and we can conduct a query to an element to reveal its hidden state. The goal is to obtain a feasible solution having a large objective value by conducting a small number of queries. This is the first attempt to consider nonlinear objective functions in such a stochastic model.

We prove that the problem admits a good query strategy if the feasible domain has a uniform exchange property. This result generalizes Blum et al.’s result on the unweighted matching problem and Behnezhad and Reyhani’s result on the weighted matching problem in both objective function and feasible domain.

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1 Introduction

1.1 Background and Motivation The stochastic combinatorial optimization with queries is the following type of problems: Let $E$ be a finite set, $f : 2^E \to \mathbb{R}_{\geq 0}$ be an objective function, $D \subseteq 2^E$ be a feasible domain, and $p \in (0, 1)$ be a probability parameter. At the beginning, nature selects a random subset $A \subseteq E$ such that $\Pr(e \in A) = p$ for all $e \in E$ independently. An element $e \in E$ is active if $e \in A$ and inactive otherwise. We do not know whether $e$ is active or not in advance, but by conducting a query to $e$, we can obtain this information. Let $Q \subseteq E$ be the set of query targets. We say that $Q$ has an approximation factor of $c \in \mathbb{R}$ if

$$\max_{X \in D} f(X \cap A \cap Q) \geq c \max_{Z \in D} f(Z \cap A)$$

holds (with high probability or in expectation). The goal of the problem is to design a query strategy that conducts a small number of queries having a large approximation factor. We evaluate not only the number of queries but also the degree of adaptivity of a query strategy. The degree of adaptivity is the number of rounds of the query strategy, where it may conduct multiple queries in each round. Smaller degree of adaptivity is preferred because it corresponds to the number of adaptive decision makings for queries; in particular, a query strategy of the degree of adaptivity $1$ is completely non-adaptive.

The above problem generalizes the stochastic matching problem of Blum et al. [4], in which the objective function is the cardinality function, and the feasible domain is the set of matchings in a given graph. They showed that, for any $\epsilon > 0$, there is a query strategy that conducts $1/p^{O(1/\epsilon)}$ queries per vertex (more precisely, it has the degree of adaptivity of $1/p^{O(1/\epsilon)}$, and in each round it conducts a query to a matching), and gives $1 - \epsilon$ approximate solution in expectation. Assadi et al. [1] considered the same problem and improved the number of queries to $O(1/p\epsilon)$ and the guarantee of the approximation factor to with high probability. Behnezhad and Reyhani [3] considered the stochastic weighted matching whose objective function is $f(X) = \sum_{e \in X} w_e$ for $w : E \to \mathbb{R}_{\geq 0}$, and obtained the same guarantee as that of Blum et al. [4] for the unweighted problem. There are several attempts on non-adaptive strategies; see Assadi et al. [1] and Behnezhad et al. [2].

Beyond the matching problems, the authors [9] considered in the following research question:

**Problem 1.1.** What class of problems admits efficient query strategy?

To answer this question, the authors [9] considered a general problem, stochastic packing integer programming problem, in which the objective function is $f(X) = \sum_{e \in X} c_e$ with $c_e = O(1)$ and the feasible domain is $D = \{X \subseteq E : \sum_{e \in X} a_{i,e} \leq b_i \ (\forall i \in [n])\}$, where all the coefficients are nonnegative integers, and derived a sufficient condition of having a good query strategy, which is described in terms of the dual problem. This result typically gives a query strategy that conducts $O(\text{poly}(1/p, 1/\epsilon) \log n)$ queries per constraint to obtain a $(1 - \epsilon)\alpha$ approximate solution with high probability, where $\alpha$ is the integrality gap of the problem.

In this study, we further explore this research question (Problem 1.1). Specifically, we consider stochastic monotone submodular maximization with queries problem, which is a stochastic combinatorial optimization with queries problem whose objective function is a monotone submodular function.

1.2 Our Contribution There are two approaches in the literature of the stochastic combinatorial optimization with queries: the first one is the local-search based framework, and the other is the duality-based approach. In our submodular objective case, we employ the local-search based framework since there is no existing duality theory for submodular maximization problems.

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[1] All the results in this study can be generalized for the activation model of $\Pr(e \in A) \geq p$. 
We show that an “exchange property” gives a sufficient condition for the existence of an efficient query strategy.

1.2.1 First Attempt with Large Degree of Adaptivity If we ignore the degree of adaptivity, we can simply construct a query strategy via a local search. To be precise, we here consider a $k$-exchange system.

A hereditary set system $\mathcal{I} \subseteq 2^E$ is a $k$-exchange system if for every $X, Y \in \mathcal{I}$, there exists a collection of subsets $\{T_y\}_{y \in Y \setminus X} \subseteq 2^X \setminus Y$ such that (1) each $T_y$ has the cardinality at most $k$, (2) each $x \in X \setminus Y$ appears at most $k$ times in the collection, and (3) for every subset $S \subseteq Y \setminus X$, we have $X \cup S \setminus \bigcup_{y \in S} T_y \in \mathcal{I}$.

We first introduce a local search algorithm for a $k$-exchange system. At 0-th step, we set $X_0 = \emptyset$. For each step $t$, we find $e \in E \setminus X_t$ and $T \subseteq X_t$ with $|T| \leq k$ such that $X_t \cup \{e\} \setminus T$ is in $\mathcal{D}$ and has the largest objective value. Then, we update the solution by $X_{t+1} \leftarrow X_t \cup \{e\} \setminus T$.

We can see that this local-search algorithm has an approximation factor of $(1 - \epsilon)/(k + 1)$ as follows. Let $Y$ be the optimal solution. Then, by the exchange property between $X_t$ and $Y$, there exists a set family $\{T_e\}_{e \in Y \setminus X}$ that satisfies the above three conditions. This set family satisfies the following inequality (we omit the proof; see Lemma 3.5 for a generalization):

$$\sum_{e \in Y \setminus X_t} (f(X_t \cup \{e\} \setminus T_e) - f(X_t)) \geq f(Y) - (k + 1)f(X_t). \tag{1.2}$$

Thus, by taking the maximum summand, and by the definition of $X_{t+1}$, we obtain $f(X_{t+1}) - f(X_t) \geq (1/n)(f(Y) - (k + 1)f(X_t))$, where $n$ is the maximum cardinality of the solution. From this inequality, after $N$ iterations, we obtain the following (we also omit the proof; see Lemma 3.5):

$$f(Y) - (k + 1)f(X_N) \leq \left(1 - \frac{1}{n}\right)^N f(Y). \tag{1.3}$$

Thus, by choosing $N = n \log(1/\epsilon)$, we obtain a solution with an approximation factor of $(1 - \epsilon)/(k + 1)$.

Now we convert this local search algorithm as a query strategy. In each step of the local search, we conduct a query to the selected $e$. If $e$ is active, then we perform the exchange; otherwise, we skip $e$ and continue to the next element. This query strategy gives the same solution to the local search algorithm applied to the omniscient problem. Hence, it has an approximation factor of $(1 - \epsilon)/(k + 1)$. Moreover, it conducts linearly many queries in the solution size, i.e., $O(n \log(1/\epsilon)/p)$ queries, with high probability.

1.2.2 Our Contribution: Uniform Exchange Map The only one issue of the above strategy is that it has a large degree of adaptivity because it conducts one query per each round. To reduce the degree of adaptivity, we have to conduct multiple queries simultaneously. In terms of the local search, it corresponds to perform multiple augmentation simultaneously. This lead us to a new structural property of set systems.

Again, we consider a $k$-exchange system. In the above analysis of the local search algorithm, we exchanged a current solution $X$ by a single element $e$ as $X \setminus \{e\} \setminus T_e$. However, in reality, the property (3) allows us to exchange any subset $S \subseteq Y$ simultaneously as $X \cup S \setminus \bigcup_{e \in S} T_e$. This property indicates the following strategy: conduct a query to all $e \in Y \setminus X$ and observe the set of active elements $R$ in $Y \setminus X$. Then, exchange all $R$ simultaneously as $X \cup R \setminus \bigcup_{e \in R} T_e$. We can prove that this strategy has a provable approximation factor with a small degree of adaptivity.

We generalize the above strategy to a general set system as follows. For two feasible sets, $X, Y \in \mathcal{D}$, an exchange map between $X$ and $Y$ is a pair of (possibly random) functions $S_{X,Y}: 2^{Y \setminus X} \rightarrow 2^{Y \setminus X}$ and $T_{X,Y}: 2^{Y \setminus X} \rightarrow 2^{X \setminus Y}$ such that for any $R \subseteq Y \setminus X$, we have (1) $S_{X,Y}(R) \subseteq R$ and (2)
Algorithm 1 Query Strategy

for $t = 1, 2, \ldots, N$ do
    Compute the optimal solution $Y_t$ to the optimistic problem.
    Query all $e \in Y_t$
end for

Output the optimal solution $X_N$ to the pessimistic problem.

$X \cup S_{X,Y}(R) \setminus T_{X,Y}(R) \in \mathcal{D}$. An exchange map $(S,T)$ is $(\alpha,\beta)$-uniform if $\Pr(y \in S_{X,Y}(R)) \geq \alpha$ for all $y \in Y \setminus X$ and $\Pr(x \in T_{X,Y}(R)) \leq \beta$ for all $x \in X \setminus Y$, where $\Pr$ is the probability over $R \subseteq Y \setminus X$ where $\Pr(y \in R) = p$ independently randomly. Note that $\alpha, \beta$ depend on $p$. We say that $\mathcal{D}$ admits an $(\alpha,\beta)$-uniform exchange map if for all $X,Y \in \mathcal{D}$ there is an $(\alpha,\beta)$-uniform exchange map. We refer the ratio $\alpha/\beta$ as the uniformity of the exchange map.

A typical example of a uniform exchange map comes from a $k$-exchange system. From the collection of subsets $\{T_e\}_{e \in Y \setminus X}$ in the property of the $k$-exchange system, we obtain the following exchange map:

$$S_{X,Y}(R) = R, \quad (1.4)$$
$$T_{X,Y}(R) = \bigcup_{y \in R} T_y. \quad (1.5)$$

By the property (3) of the $k$-exchange system, this forms an exchange map. Moreover, we can see that $\Pr(y \in S_{X,Y}(R)) = p$ and $\Pr(x \in T_{X,Y}(R)) \leq kp$. Therefore, it admits $(p,pk)$-uniform exchange map.

Next, we propose a query strategy, which is shown in Algorithm 1. As same as in [9], we introduce two problems. A pessimistic problem is the problem in which all the non-queried elements are supposed to be inactive, and an optimistic problem is the problem in which all the non-queried elements are supposed to be active. Our algorithm iteratively solves the optimistic problem, and conducts queries to the solution. After sufficient iterations, it computes the optimal solution to the pessimistic problem. Note that this type algorithm has been employed in all the existing studies of stochastic combinatorial optimization with queries.

Our main lemmas are presented below; one is for linear objective case, and the other is for submodular objective case; these are proved in a similar way. These lemma says that the algorithm gives a good solution if $\mathcal{D}$ admits a uniform exchange map, where the approximation factor depends on the uniformity of the exchange map.

**Lemma 1.1. (Linear Objective Case)** Suppose that $f : 2^E \rightarrow \mathbb{R}_+$ be a monotone linear function and $\mathcal{D}$ admits an $(\alpha,\beta)$-uniform exchange map. Then, for any $\epsilon, \delta \in (0,1)$, after $N = 16\log(1/\delta)/\alpha \max\{\alpha, \beta\} \epsilon$ iterations, the output of Algorithm 1 gives a $(1-\epsilon)\alpha/\max\{\alpha, \beta\}$ approximate solution with probability at least $1-\delta$.

**Lemma 1.2. (Submodular Objective Case)** Suppose that $f : 2^E \rightarrow \mathbb{R}_+$ be a monotone submodular function and $\mathcal{D}$ admits an $(\alpha,\beta)$-uniform exchange map. Then, for any $\epsilon, \delta \in (0,1)$, after $N = 16\log(1/\delta)/\alpha(\alpha+\beta) \epsilon$ iterations, the output of Algorithm 1 gives a $(1-\epsilon)\alpha/(\alpha+\beta)$ approximate solution with probability at least $1-\delta$.

It should be emphasized that the uniform exchange map is only used in the analysis (i.e., the algorithm does not explicitly use it). Thus, if $\mathcal{D}$ admits multiple uniform exchange maps, the performance of the algorithm is the maximum of them.

The proof of the above lemmas are not so complicated (see Section 3). The actual contribution is introducing the concept of uniform exchange map to separate the probabilistic argument required to
| Constraint                  | Degree of Adaptivity | Approximation Factor |
|-----------------------------|----------------------|---------------------|
| $k$-Exchange System         | $\log(1/\delta)/p^{O(k/\epsilon)}$ | $(1-\epsilon)/k$   |
| $k$-Intersection System    | $O(k \log(1/\delta)/p^2 \epsilon)$ | $(1-\epsilon)/(k+1)$ |
| Knapsack Constraint         | $O(\log(1/\delta)/p^2 \epsilon)$ | $(1-\epsilon)/6$   |

Table 1: Our query strategies for monotone submodular maximization problem. The approximation guarantee is with probability at least $1-\delta$.

We prove the existence of uniform exchange maps for $k$-exchange system (Lemma 4.1), $k$-intersection system (Lemma 4.5), and knapsack constraint (Lemmas 4.8 and 4.9). These implies the query strategies summarized in Table 1.

We compare our result and the existing results. On the unweighted matching problems, which is a special case where the objective function is the cardinality function and the constraint is a 2-exchange system, our result outperforms Blum et al. [4]'s result since our approximation guarantee is “with high probability,” whereas theirs is “in expectation.” Also, ours is outperformed by Assadi et al. [11]'s result since ours requires exponentially larger number of queries than theirs. On the weighted matching problem, our result outperforms Beznezhad and Beyhani [3]'s result since our approximation guarantee is “with high probability,” whereas theirs is “in expectation.” On more general problems, our new result basically outperforms the authors’ previous result [9] since ours requires a constant number of queries, whereas the previous result requires a logarithmic number of queries. Also, our approach can be applied to non packing-type constraint (e.g., matroid bases), whereas the previous result can only be applied to packing-type problems.

2 Preliminaries

2.1 Submodular Functions

A function $f: 2^E \to \mathbb{R}$ is normalized if $f(\emptyset) = 0$. $f$ is monotone if

$$f(X) \leq f(Y)$$

for all $X, Y \subseteq E$ with $X \subseteq Y$. $f$ is submodular if

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)$$

for all $X, Y \subseteq E$, or equivalently, it satisfies the diminishing return property

$$f(X \cup \{e\}) - f(X) \geq f(Y \cup \{e\}) - f(Y)$$

for all $X \subseteq Y$ and $e \in E \setminus Y$.

2.2 Probabilistic Inequality

**Lemma 2.1. (Markov inequality)** Let $Z$ be nonnegative random variable. Then,

$$\Pr(Z \geq a) \leq \frac{\mathbb{E}[Z]}{a},$$

where $\mathbb{E}$ denotes the expectation.
Lemma 2.2. (Reverse Markov Inequality) Let $Z$ be a nonnegative random such that $E[Z] \geq a$ and $Z \leq b$ for some $a, b \in \mathbb{R}_{\geq 0}$. Then,

$$\Pr(Z \geq (1/2)E[Z]) \geq \frac{a}{2b}.$$  

(2.5)

Proof. We use the Markov inequality to $b - Z$, which is a nonnegative random variable:

$$\Pr(Z < (1/2)E[Z]) = \Pr(b - Z > (1/2)E[Z]) \leq \frac{b - E[Z]}{b - (1/2)E[Z]} = 1 - \frac{(1/2)E[Z]}{b - (1/2)E[Z]}.$$  

(2.6)

Thus,

$$\Pr(Z \geq (1/2)E[Z]) \geq \frac{(1/2)E[Z]}{b - (1/2)E[Z]} \geq \frac{E[Z]}{2b} \geq \frac{a}{2b}.$$  

(2.7)

Lemma 2.3. (Tail Inequality for Binomial Distribution [5]) For any $N \in \mathbb{Z}_{\geq 0}$ and $q \in (0, 1)$, a random variable $X$ that follows a binomial distribution $\text{Binomial}(4N/q, q)$ satisfies

$$\Pr(X \leq N) \leq \exp(-N).$$  

(2.8)

3 Proof of Main Lemmas

The proofs of the main lemmas are similar on both linear and submodular case. For simplicity, we first give a proof for linear objective case. Then, we give a proof for submodular objective case.

3.1 Linear Objective Function

We first evaluate the expected gain of the random uniform exchange as follows.

Lemma 3.1. Let $X, Y \in \mathcal{D}$ and $(S_{X,Y}, T_{X,Y})$ be an $(\alpha, \beta)$-uniform exchange map between $X$ and $Y$. Then, for any monotone linear function $f: 2^E \to \mathbb{R}_{\geq 0}$, we have

$$E[f(X \cup S_{X,Y}(R) \setminus T_{X,Y}(R)) - f(X)] \geq \alpha f(Y) - \max\{\alpha, \beta\} f(X).$$  

(3.1)

Proof. We have

$$E[f(X \cup S_{X,Y}(R) \setminus T_{X,Y}(R)) - f(X)] = E[f(S_{X,Y}(R)) - f(T_{X,Y}(R))] \geq \alpha f(Y \setminus X) - \beta f(X \setminus Y) \geq \alpha f(Y) - \max\{\alpha, \beta\} f(X).$$  

(3.3)

(3.4)

Using this lemma, we obtain $\alpha/\beta$ approximate solution with high probability.

Lemma 3.2. Let $X \in \mathcal{D}$, $Y \in \text{argmax}_{Y' \in \mathcal{D}} f(Y')$, and $(S_{X,Y}, T_{X,Y})$ be an $(\alpha, \beta)$-uniform exchange map between $X$ and $Y$. Suppose that $(1 - \epsilon)\alpha f(Y) \geq \beta f(X)$ for some $\epsilon \in (0, 1)$. Then, for any monotone linear function $f: 2^E \to \mathbb{R}_{\geq 0}$, we have

$$\Pr\left[f(X \cup S_{X,Y}(R) \setminus T_{X,Y}(R)) - f(X) \geq \frac{1}{2} (\alpha f(Y) - \max\{\alpha, \beta\} f(X)) \right] \geq \frac{\alpha \epsilon}{2}.$$  

(3.5)
Proof. Let \( \Delta = f(X \cup S_{X,Y}(R) \setminus T_{X,Y}(R)) - f(X) \) be a random variable in \( R \) and let \( \bar{\Delta} = f(Y) - f(X) \). Then, by Lemma 3.1 and the assumption of the lemma, we have \( \mathbb{E}[\Delta] \geq \epsilon \alpha f(Y) \). Also, since \( Y \) is an optimal solution, we have \( \Delta \leq \bar{\Delta} \). Therefore, by the reverse Markov inequality (Lemma 2.2), we have
\[
\Pr(\Delta \geq (1/2)\mathbb{E}[\Delta]) \geq \frac{\epsilon \alpha f(Y)}{2f(Y) - 2f(X)} \geq \frac{\epsilon \alpha}{2}. \tag{3.6}
\]
By expanding \( \mathbb{E}[\Delta] \) using Lemma 3.1, we obtain the lemma. \( \square \)

Proof. (Proof of Lemma 1.1). Let \( \mathcal{F}_t \) be the filtration about the known active elements at \( t \)-th step. Let \( X_t, Y_t, X^* \) be optimal solutions to the pessimistic, optimistic, and omniscient problems, respectively. Note that all of these quantities (including \( X^* \)) are random variables depending on the activation of elements. Let \( \Delta_t = \alpha f(X^*) - \max\{\alpha, \beta\} f(X_t) \). Then, by Lemma 3.2 under the filtration \( \mathcal{F}_t \),
\[
\Delta_{t+1} - \Delta_t = -\max\{\alpha, \beta\}(f(X_{t+1}) - f(X_t))
\leq -\frac{\max\{\alpha, \beta\}}{2}(\alpha f(Y) - \max\{\alpha, \beta\} f(X))
= -\frac{\max\{\alpha, \beta\}}{2} \Delta_t \tag{3.7}
\]
if \((1 - \epsilon)\alpha f(X^*) \geq \beta f(X_t)\) holds and with probability at least \( \alpha \epsilon/2 \). Since \((1 - \epsilon)\alpha f(X^*) - \beta f(X_t) \geq \Delta_t - \epsilon \Delta_0 \), we have
\[
\Delta_{t+1} \leq \begin{cases} 
\left(1 - \frac{\max\{\alpha, \beta\}}{2}\right) \Delta_t, & \text{if } \Delta_t \geq \epsilon \Delta_0 \text{ and with probability at least } \frac{\alpha \epsilon}{2}, \\
\Delta_t, & \text{otherwise}.
\end{cases} \tag{3.9}
\]
For any filtration until \( N \)-th steps, the first event occurs at most \( 2 \log(1/\epsilon)/\max\{\alpha, \beta\} \) times; otherwise \( \Delta_N \leq \epsilon \Delta_0 \) holds. Thus, by Lemma 2.3 if \( N \geq 16 \log(1/\delta)/\alpha \max\{\alpha, \beta\} \epsilon \), the probability of having such filtration is at most \( \delta \). \( \square \)

### 3.2 Submodular Objective Function

The proof for the submodular case is basically the same. We use the following lemmas as an alternative to the linearity of the objective function.

**Lemma 3.3.** (Probabilistic version of \[8\] Lemma 1.1) Let \( S \subseteq E \) be a random variable such that \( \Pr(e \in S) \geq \alpha \) for all \( e \in E \). Then, for any normalized monotone submodular function \( f : 2^E \rightarrow \mathbb{R}_{\geq 0} \), we have
\[
\mathbb{E}[f(S)] \geq \alpha f(E). \tag{3.11}
\]

**Proof.** Without loss of generality, we assume that \( E = [m] \) for some positive integer \( m \), where \( [m] = \{1, \ldots, m\} \). Then, for any monotone submodular function \( f : 2^E \rightarrow \mathbb{R}_{\geq 0} \), we have
\[
f(S) = \sum_i (f(S \cap [i]) - f(S \cap [i-1])) \tag{3.12}
\geq \sum_i (f((S \cap [i]) \cup [i-1]) - f([i-1])) \tag{3.13}
= \sum_i 1[i \in S] (f([i]) - f([i-1])), \tag{3.14}
\]

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where \(1[i \in S]\) is the indicator of the event \(i \in S\). By taking the expectation over \(S\) and using the monotonicity of the function, i.e., \(f([i]) - f([i - 1]) \geq 0\), we have
\[
\mathbb{E}[f(S)] \geq \alpha \sum_i (f([i]) - f([i - 1])) = \alpha f([m]), \tag{3.15}
\]
which concludes the proof.

\[\square\]

**Lemma 3.4. (Probabilistic version of [8, Lemma 1.2])** Let \(T \subseteq E\) be a random variable such that \(\Pr(e \in T) \leq \beta\) for all \(e \in E\). Then, for any normalized monotone submodular function \(f : 2^E \to \mathbb{R}_{\geq 0}\), we have
\[
\mathbb{E}[f(E) - f(E \setminus T)] \leq \beta f(E). \tag{3.16}
\]

**Proof.** Without loss of generality, we assume that \(E = [m]\) for some positive integer \(m\). Then, for any monotone submodular function \(f : 2^E \to \mathbb{R}_{\geq 0}\) we have
\[
f(E) - f(E \setminus T) = \sum_i (f(E \setminus (T \cap [i - 1])) - f(E \setminus (T \cap [i]))) \tag{3.17}
\]
\[
\leq \sum_i (f(E \setminus [i - 1]) - f(E \setminus [i - 1] \setminus (T \cap \{i\}))) \tag{3.18}
\]
\[
= \sum_i 1[i \in T] (f(E \setminus [i - 1]) - f(E \setminus [i])). \tag{3.19}
\]

where \(1[i \in T]\) is the indicator of the event \(i \in T\). By taking the expectation over \(T\), and using the monotonicity of the function, i.e., \(f(E \setminus [i - 1]) \geq f(E \setminus [i])\), we obtain
\[
\mathbb{E}[f(E) - f(E \setminus T)] \leq \beta \sum_i (f(E \setminus [i - 1]) - f(E \setminus [i])) = \beta f(m), \tag{3.20}
\]
which concludes the proof. \[\square\]

Using these lemmas, we obtain the following lemma as a submodular version of Lemma 3.1:

**Lemma 3.5. (Submodular version of Lemma 3.1)** Let \(X, Y \in \mathcal{D}\) and \((S_{X,Y}, T_{X,Y})\) be \((\alpha, \beta)\)-uniform exchange map between \(X\) and \(Y\). Then, for any normalized monotone submodular function \(f : 2^E \to \mathbb{R}_{\geq 0}\), we have
\[
\mathbb{E}[f(X \cup S_{X,Y}(R) \setminus T_{X,Y}(R)) - f(X)] \geq \alpha f(X \cup Y) - (\alpha + \beta)f(X). \tag{3.21}
\]

**Proof.** By the submodularity, for any \(R\), we have
\[
f(X \cup S_{X,Y}(R) \setminus T_{X,Y}(R)) - f(X) \geq f(X \cup S_{X,Y}(R)) - f(X) + f(X \setminus T_{X,Y}(R)) - f(X). \tag{3.22}
\]

Then, we take the expectation over \(R\). (1) is lower-bounded by \(\alpha f(X \cup Y) - \alpha f(X)\) by Lemma 3.3 applied to the function \(2^Y \setminus X \ni S \mapsto f(X \cup S) - f(X)\), and (2) is lower-bounded by \(-\beta f(X)\) by Lemma 3.3 applied to the function \(2^X \ni S \mapsto f(S)\). Thus, we obtain (3.21). \[\square\]

Then, the remaining part of the proof of Lemma 1.2 is the same as that of Lemma 1.1.

**Proof.** (Proof of Lemma 1.2). The proof is the same as proof of Lemma 1.1 where we use Lemma 3.5 instead of Lemma 3.1. \[\square\]
4 Examples of Uniform Exchange Maps

In this section, we present several examples of uniform exchange maps.

4.1 Exchange System As we see in Introduction, a \( k \)-exchange system admits a \( (p,kp) \)-uniform exchange map. Here, we prove the existence of an exchange map with a better uniformity.

**Lemma 4.1.** For any \( h \in \mathbb{Z}_{\geq 1} \), a \( k \)-exchange system admits \( (p^h/3h^2,p^h(k-1+1/h)/3h^2) \)-uniform exchange map.

**Corollary 4.1.** For any \( \epsilon > 0 \), Algorithm 7 with \( N = \log(1/\delta)/p^{O(k/\epsilon)} \) iterations gives an approximation factor of \( (1 - \epsilon)/k \) with probability at least \( 1 - \delta \).

This result generalizes Blum et al. [4]'s and Behnezhad and Reyhani [3]'s results on stochastic unweighted and weighted matching problems since the set of matchings forms a 2-exchange system. Our proof also generalizes these proofs using the technique in [7] for a local search on \( k \)-exchange systems. We use the following lemma.

**Lemma 4.2.** (Theorem 5) Let \( G \) be an undirected graph whose maximum degree is at most \( k \in \mathbb{Z} \) with \( k \geq 2 \). Then, for every \( h \in \mathbb{Z} \) with \( h \geq 1 \) there exists a multiset \( \mathcal{P}(G,k,h) \) of simple paths in \( G \) and a labeling \( \ell: V \times \mathcal{P}(G,k,h) \to \{0\} \cup [h] \) such that

1. For every \( P \in \mathcal{P}(G,k,h) \), the labeling \( \ell \) of the nodes of \( P \) is consecutive and increasing with labels from \([h]\). Vertices not in \( P \) receive label 0.

2. For every \( P \in \mathcal{P}(G,k,h) \) and \( v \in P \), if \( \deg_G(v) = j \) and \( \ell(v,P) \notin \{1,h\} \), then at least two of the neighbors of \( v \) are in \( P \).

3. For every \( v \in V \) and label \( i \in [h] \), there are \( n(k,h) = k(k-1)^{h-2} \) paths \( P \in \mathcal{P}(G,k,h) \) for which \( \ell(v,P) = i \).

**Proof.** (Proof of Lemma 4.1) We fix \( X,Y \in \mathcal{I} \) and construct an exchange map as follows. Let \( \{T_y\}_{y \in Y \setminus X} \) be the subsets in the definition of the \( k \)-exchange system. Let \( \mathcal{G} = (V,E) \) be a bipartite graph where \( V = (X \setminus Y) \cup (Y \setminus X) \) and \( E = \{(x,y) \in (X \setminus Y) \times (Y \setminus X) : x \in T_y\} \). By the definition of a \( k \)-exchange system, \( \mathcal{G} \) has the degree at most \( k \). Hence, by Lemma 4.2 we obtain paths \( \{P_1, \ldots, P_M\} = \mathcal{P}(G,k,2h) \). Let \( S_i = P_i \cap (X \setminus Y) \) and \( T_i = \bigcup_{x \in S_i} T_x \) for each \( i \in [M] \). Then, we have \( |S_i| \leq h \) and \( |T_i| \leq 1 + h(k-1) \).

We consider the intersection graph of \( \{S_i\}_{i \in [M]} \): the vertices are \( \{S_i\}_{i \in [M]} \) and there is an edge \( (S_i, S_j) \) if \( S_i \cap S_j \neq \emptyset \). By the third property of the multiset \( \mathcal{P} \), this graph has the degree at most \( 2h^2n(k,2h) \). Therefore, it has \( 3h^2n(k,2h) \) coloring; we choose any such coloring.

Now we define \( S_{X,Y} \) and \( T_{X,Y} \). We first draw a color class uniformly randomly. Then, we select each \( S_i \) if all \( y \in S_i \) is active and with probability \( p^{-|S_i|} \). Here, the additional probability makes the selection probability uniform. We define \( S_{X,Y} \) and \( T_{X,Y} \) by

\[
S_{X,Y}(R) = \bigcup_{i : S_i \text{ is selected}} S_i, \quad (4.1)
\]
\[
T_{X,Y}(R) = \bigcup_{i : S_i \text{ is selected}} T_i. \quad (4.2)
\]

We check these form \( (p^h/3h^2,p^h(k-1+1/h)/3h^2) \)-uniform exchange map. By the construction, for any \( R \subseteq Y \setminus X, S_{X,Y}(R) \subseteq R \) holds. Also, by the definition of the \( k \)-exchange system, \( X \cup S_{X,Y}(R) \setminus T_{X,Y}(R) \)
The event “$x \in S_{X,Y}(R)$” occurs when the selected color class contains $S_i$ with $x \in S_i$ and $S_i$ is selected. This probability is exactly $p^h = p^h/3h^2$. The event “$y \in T_{X,Y}(R)$” occurs when the selected color class contains $S_i$ with $y \in T_i$ and $S_i$ is selected. This probability is at most $(p^h/3h^2) \times (k - 1 + 1/h)$.

**4.2 Matroid and Matroid Intersection** A family of subsets $\mathcal{I} \subseteq 2^E$ is an independent set family of a matroid if (1) $\emptyset \in \mathcal{I}$, (2) $Y \in \mathcal{I}$ implies $X \in \mathcal{I}$ for all $X \subseteq Y$, and (3) if $X, Y \in \mathcal{I}$ and $|X| < |Y|$ then there exists $e \in Y \setminus X$ such that $X \cup \{e\} \in \mathcal{I}$. An element $e \in \mathcal{I}$ is called an independent set. A maximal element $B \in \mathcal{I}$ is called a base. The set $\mathcal{B}$ of all the bases are called the base family of the matroid.

The both independence family and the base family satisfies the following property called the generalized Rota-exchange property.

**Theorem 4.1.** (Generalized Rota-Exchange Property (Base), Lemma 2.6) Let $X, Y \in \mathcal{B}$. For any subsets $A_1, \ldots, A_n \subseteq Y \setminus X$ that cover each $y \in Y \setminus X$ exactly $q$ times, there exists subsets $B_1, \ldots, B_n \subseteq X \setminus Y$ such that $X \cup A_i \setminus B_i \in \mathcal{B}$ for all $i \in [n]$ and each $x \in X \setminus Y$ is covered exactly $q$ times.

**Theorem 4.2.** (Generalized Rota-Exchange Property (Independent Set), Lemma 2.7) Let $X, Y \in \mathcal{I}$. For any subsets $A_1, \ldots, A_n \subseteq Y \setminus X$ that cover each $y \in Y \setminus X$ at most $q$ times, there exists subsets $B_1, \ldots, B_n \subseteq X \setminus Y$ such that $X \cup A_i \setminus B_i \in \mathcal{I}$ for all $i \in [n]$ and each $x \in X \setminus Y$ is covered at most $q$ times.

These theorems immediately give the uniform exchange map as follows.

**Lemma 4.3.** An independent set family and a base family of a matroid admit $(p, p)$-uniform exchange maps.

**Proof.** We apply the generalized Rota-exchange property to the family $2^{Y \setminus X}$ to obtain a family of subsets $\{B_R\}_{R \subseteq Y \setminus X}$. Then, we define $S_{X,Y}(R) = R$ and $T_{X,Y}(R) = B_R$. Then, the generalized Rota-exchange property guarantees that $\Pr(y \in S_{X,Y}(R)) = p$ ( = $\alpha$) for all $y \in Y \setminus X$ and $\Pr(x \in T_{X,Y}(R)) \leq p$ ( = $\beta$) for all $x \in X \setminus Y$.

We then consider the intersection of $k$ matroids. The exchange map has the following composition property.

**Lemma 4.4.** (Composition of Uniform Exchange Map) Let $\mathcal{D}_1, \ldots, \mathcal{D}_n \subseteq 2^E$ be set families each of which admit $(\alpha, \beta_i)$-uniform exchange maps $(S_{X,Y}, T_{i,X,Y})$ with a common $S$-map. Then, $\mathcal{D}_1 \cap \cdots \cap \mathcal{D}_n$ has an $(\alpha, \beta_1 + \cdots + \beta_n)$-uniform exchange map.

**Proof.** We can define $T_{X,Y}(R) = T_{1,X,Y}(R) \cup \cdots \cup T_{n,X,Y}(R)$.

Using this lemma, we immediately obtain the following result.

**Lemma 4.5.** A $k$-intersection system admits a $(p, pk)$-uniform exchange map.

**Corollary 4.2.** For any $\epsilon > 0$, Algorithm 7 with $N = O(k \log(1/\delta)/p^2 \epsilon)$ iterations gives an approximation factor of $(1 - \epsilon)/(k + 1)$ with probability at least $1 - \delta$.

**Remark.** We tried to obtain an exchange map with better uniformity using the local search technique in [8], but it has not succeeded.
4.3 **Knapsack Constraint** We refer as a knapsack constraint to a family \(\mathcal{D} = \{X \subseteq E : \sum_{x \in X} c_x \leq 1\}\) for some positive numbers \(c_e \in \mathbb{R}_{>0} (e \in E)\).

In general, a knapsack constraint may not have a uniform exchange map with a small uniformity because adding one “heavy” item may require to remove almost all the items. To handle this situation, we handle heavy items and light items separately; then combine these results. Note that a similar technique can be found in the literature of prophet inequality \([6]\).

We first generalize our definition as follows. For a positive integer \(\gamma\), we define \(\gamma\mathcal{D} := \{X_1 \cup \cdots \cup X_\gamma : X_1, \ldots, X_\gamma \in \mathcal{D}\}\). For two feasible sets \(X, Y \in \mathcal{D}\), a \(\gamma\)-related exchange map is a pair of functions \((S_{X,Y}, T_{X,Y})\) such that for any \(R \subseteq Y \setminus X\), we have (1) \(S_{X,Y}(R) \subseteq R\) and (2) \(X \cup S_{X,Y}(R) \setminus T_{X,Y}(R) \in \gamma\mathcal{D}\). The \((\alpha, \beta)\)-uniformity is defined similarly to the exchange map. A feasible domain \(\mathcal{D}\) admits \(\gamma\)-related \((\alpha, \beta)\)-exchange map if for any \(X, Y \in \mathcal{D}\), there exists \(\gamma\)-related \((\alpha, \beta)\)-exchange map.

**Lemma 4.6.** (Generalization of Lemma [1,2]) Suppose that \(\mathcal{D}\) admits \(\gamma\)-related \((\alpha, \beta)\)-uniform exchange map. For any \(\epsilon, \delta \in (0, 1)\), after \(N = 16\log(1/\delta)/\alpha(\alpha + \beta)\epsilon\) iterations, the output of Algorithm 7 gives an \((1 - \epsilon)\alpha/\gamma(\alpha + \beta)\)-approximate solution with probability at least \(1 - \delta\).

**Proof.** The proof is almost the same as that of Lemma [1,2] where we use the \(\gamma\)-related condition to guarantee the optimal value at \((t + 1)\)-th step is at least \(\gamma\) times the exchanged solution.

**Special Case 1: No Heavy Items** We say that an item \(e \in E\) is heavy item if \(c_e > 1/3\). We first consider the case that there is no heavy item.

**Lemma 4.7.** Suppose that there is no item with size greater than 1/3. Then, \(\mathcal{D}\) admits a \(2\)-related \((p, p)\)-uniform exchange map.

**Proof.** Without loss of generality, we consider \(X \cap Y = \emptyset\). We set \(S_{X,Y}(R) = R\), which gives \(\Pr(y \in S_{X,Y}(R)) = p\). We define \(T_{X,Y}(R)\) as follows. We write a unit length circle and pack each element \(x \in X\) as an arc of length \(c_x\). Then, we select an arc of length \(\sum_{r \in R} c_r\) uniformly at random. We remove the items intersecting the arc with probability of the intersecting length. Due to the randomness of the arc, it gives \(\Pr(x \in T_{X,Y}(R)) = p\). Since the boundary of the arc overlaps at most two subsets, the exchanged solution has the capacity at most \(5/3\), which is decomposed to two sets of capacity 1 (: place the items into the interval of length \(5/3\) and partition by length 1). Thus, this procedure gives \(2\)-related \((p, p)\)-uniform exchange map.

This lemma immediately gives the following result.

**Lemma 4.8.** For a knapsack problem without items of capacity greater than 1/3, there is a query strategy that conducts \(O(\log(1/\delta)/p^2\epsilon)\) queries and has an approximation factor of \((1 - \epsilon)/4\) with probability at least \(1 - \delta\).

**Special Case 2: Only Heavy Items** If all the items have size at least 1/3, the cardinality of a solution is at most 2. In general, if the cardinality of a solution is at most \(k\), we obtain a query strategy with a good approximation guarantee as follows.

**Lemma 4.9.** If all \(X \in \mathcal{D}\) has cardinality at most \(k\), there exists a query strategy that conducts \(O(\log(1/\delta)/p^2\epsilon)\) and has an approximation factor of \((1 - \epsilon)p/(1 - (1 - p)^k) \geq (1 - \epsilon)/k\) with probability at least \(1 - \delta\).
Proof. We construct a uniform exchange map as follows. We define $S_{X,Y}(R) = R$. Then $\Pr(y \in S_{X,Y}(R)) = p$. We define $T_{X,Y}(R)$ by

$$T_{X,Y}(R) = \begin{cases} 
\emptyset, & R = \emptyset, \\
X, & \text{otherwise}.
\end{cases}$$

(4.3)

Then, $\Pr(x \in T_{X,Y}(R)) = 1 - (1 - p)^k$. Using this map, we can see that

$$f(X \cup S_{X,Y}(R) \setminus T_{X,Y}(R)) - f(X) = \begin{cases} 
0, & R = \emptyset, \\
f(R) - f(X), & \text{otherwise}.
\end{cases}$$

(4.4)

Thus, by taking the expectation, we obtain

$$\mathbb{E}[f(X \cup S_{X,Y}(R) \setminus T_{X,Y}(R))] = \mathbb{E}[f(R)] - (1 - (1 - p)^k)f(X)$$

$$\geq pf(Y) - (1 - (1 - p)^k)f(X).$$

(4.5)

(4.6)

Using this inequality instead of (3.1), we obtain the desired result. □

General Case  By combining the query strategies in Lemmas 4.8 and 4.9 we obtain the following theorem.

**Theorem 4.3.** The knapsack problem admits a query strategy that conducts $O(\log(1/\delta)/p^2\epsilon)$ queries and having approximation factor of $(1 - \epsilon)/6$ with probability at least $1 - \delta$.

Proof. We run the query strategies of two special cases simultaneously. We evaluate the performance of this strategy.

Let $X^*$ be the optimal solution, $X^{*(l)}$ be the optimal solution that consists of items of size at most 1/3, and $X^{*(h)}$ be the optimal solution that consists of items of size greater than 1/3, respectively, to the omniscient problem. Also, let $X_N$ be the optimal solution, $X^{(l)}_N$ be the subset of $X_N$ that consists of items of size at most 1/3, and $X^{(h)}_N$ be the subset of $X_N$ that consists of items of size greater than 1/3, respectively, to the pessimistic problem. Then, we have

$$f(X_N \cap A \cap Q) \geq f(X^{(l)}_N \cap A \cap Q) \geq \frac{1 - \epsilon}{4} f(X^{*(l)} \cap A),$$

$$f(X_N \cap A \cap Q) \geq f(X^{(h)}_N \cap A \cap Q) \geq \frac{1 - \epsilon}{2} f(X^{*(h)} \cap A),$$

(4.7)

(4.8)

simultaneously with probability at least $1 - 2\delta$. Therefore, we obtain

$$f(X_N \cap A \cap Q) \geq \frac{1 - \epsilon}{6} \left(f(X^{*(l)} \cap A) + f(X^{*(h)} \cap A)\right)$$

$$\geq \frac{1 - \epsilon}{6} f(X^* \cap A).$$

(4.9)

(4.10)

□

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