Elliptic integral evaluation of a Bessel moment by contour integration of a lattice Green function

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Abstract

A proof is found for the elliptic integral evaluation of the Bessel moment

$$M := \int_0^\infty t I_0^2(t) K_0^2(t) K_0(2t) \, dt = \frac{1}{12} K(\sin(\pi/12)) K(\cos(\pi/12)) = \frac{\Gamma^6\left(\frac{1}{3}\right)}{64\pi^2 2^{2/3}}$$

resulting from an angular average of a 2-loop 4-point massive Feynman diagram, with one internal mass doubled. This evaluation follows from contour integration of the Green function for a hexagonal lattice, thereby relating $M$ to a linear combination of two more tractable moments, one given by the Green function for a diamond lattice and both evaluated by using W.N. Bailey’s reduction of an Appell double series to a product of elliptic integrals. Cubic and sesquiplicate modular transformations of an elliptic integral from the equal-mass Dalitz plot are proven and used extensively. Derivations are given of the sum rules

$$\int_0^\infty \left( I_0(at) K_0(at) - \frac{2}{\pi} K_0(4at) K_0(t) \right) K_0(t) \, dt = 0$$

with $a > 0$, proven by analytic continuation of an identity from Bailey’s work, and

$$\int_0^\infty t I_0(at) \left( I_0^3(at) K_0(8t) - \frac{1}{4\pi^2} I_0(t) K_0^3(t) \right) \, dt = 0$$

with $2 \geq a \geq 0$, proven by showing that a Feynman diagram in two spacetime dimensions generates the enumeration of staircase polygons in four dimensions.

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1 Introduction

In a recent joint work with David Bailey, Jonathan Borwein and Larry Glasser [2] we conjectured, on the basis of numerical computation, that the moment

\[ M := \int_0^\infty t I_0^2(t)K_0^2(t)K_0(2t) \, dt \]  

has the evaluation

\[ M = \frac{1}{12} K_3 K'_3 = \frac{\sqrt{3}}{12} K_3^2 = \frac{\Gamma^6(\frac{1}{3})}{64\pi^{2/3}} \]  

where \( I_0 \) and \( K_0 \) are Bessel functions and the complete elliptic integral \( K_3 = K(\sin(\pi/12)) \) is evaluated at the third singular value [7], where the complementary elliptic integral \( K'_3 = K(\cos(\pi/12)) \) gives \( K'_3/K_3 = \sqrt{3} \) and hence a nome [1, 17.3.7] \( q = \exp(-\pi \sqrt{3}) \).

The moment \( M \) is obtained by an angular average, in two-dimensional Euclidean momentum space, of a 2-loop 4-point Feynman diagram obtained by cutting two of the 5 lines in a 4-loop vacuum diagram. The arguments of the Bessel functions show that the external particles have the same mass as two of the three internal particles, while the third internal mass is doubled. In [2, Sect. 5.10], a proof was found for Laporta’s conjectural evaluation [15] of the equal-mass moment \( \int_0^\infty t I_0^2(t)K_0^2(t) \, dt \) at the 15th singular value with the much smaller nome \( q = \exp(-\pi \sqrt{15}) \approx 5.2 \times 10^{-6} \). Notwithstanding the substantial progress made in [2] since my talk Reciprocal PSLQ and the Tiny Nome of Bologna at Bielefeld in June 2007, we were unable to prove that doubling one of the internal masses leads to the nome \( q = \exp(-\pi \sqrt{3}) \), though it was possible to prove that this nome results from doubling one of the external masses.

Alternative forms of conjecture [2] were given in [2]. These correspond to the proven evaluations

\[ M = \frac{1}{4} \int_0^\infty \int_0^\infty \frac{dx \, dy}{\sqrt{(1 + x^2)(1 + y^2)(1 + (x + y)^2)(1 + (x - y)^2)}} \]  

and

\[ \int_0^1 K \left( \frac{z \sqrt{5 - 4z}}{2 - z} \right) \, dz = \frac{1}{4} \int_0^1 y K \left( \frac{2 + y \sqrt{1-y}}{2-y} \right) \, dy \]  

It was also proven that each of the integrals

\[ \int_0^\infty \int_0^\infty \frac{dx \, dy}{\sqrt{(1 + x^2)(1 + y^2)(1 + (x + y)^2)}} = \frac{2}{3} K_3 K'_3 \]  

\[ \int_0^\infty \int_0^\infty \frac{dx \, dy}{\sqrt{(1 + x^2)(1 + y^2)(1 + (x - y)^2)}} = \frac{4}{3} K_3 K'_3 \]  

\[ \int_0^1 K(y) \, dy / \sqrt{(1 - y^2)(1 + 3y^2)} = \frac{1}{2} K_3 K'_3 \]  

yields the third singular value. Yet no combination of these 6 interesting formulae appeared to offer a direct route to proving the outstanding conjecture that \( M = \frac{1}{12} K_3 K'_3 \).

1Some equal-mass moments with only 4 Bessel functions evaluate [3, 11, 14] in terms of \( \zeta(2) \) and \( \zeta(3) \).

2See [http://www.physik.uni-bielefeld.de/igs/schools/ZiF2007/Broadhurst.pdf](http://www.physik.uni-bielefeld.de/igs/schools/ZiF2007/Broadhurst.pdf).
I have now found a proof by a route that is very far from direct, namely by using the vanishing of a suitable contour integral of the Green function [13, Eq. 4.7]

\[ \bar{D}(z) := \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \frac{d\theta_1 d\theta_2}{1 - z^2(3 + 2 \cos \theta_1 + 4 \cos \theta_1 \cos 3\theta_2)} \]  

for a two-dimensional hexagonal (or “honeycomb”) lattice. Then, after many intermediate transformations, I am able to show that \( M \) is the difference of an even and odd Bessel moment, each of which are now proven to yield the third singular value.

In Section 2, I use contour integration to derive the vanishing of

\[ \int_1^1 \sigma_1(x) dx + \int_1^\infty \sigma_2(x) dx - \int_0^\infty \bar{D}(ix) dx = 0 \]  

where \( \sigma_1 \) and \( \sigma_2 \) are the reciprocals of arithmetic-geometric means (AGMs).

In Section 3, I evaluate the integral of \( \bar{D} \) on the imaginary axis as an even moment of 3 Bessel functions. By analytic continuation of results obtained by Wilfrid Norman Bailey [4, 5], I then obtain the third singular value for this term in the contour integral (10). Moreover, I prove the remarkable result that for arbitrary real positive \( a \)

\[ K_0(a, t) := I_0(at)K_0(at) - \frac{2}{\pi} K_0(4at)K_0(t) \]  

is orthogonal to \( K_0(t) \), giving a continuous infinity of sum rules of the form

\[ \int_0^\infty K_0(a, t)K_0(t) dt = 0. \]

In Section 4, I derive a cubic modular transformation that relates the integral of \( \sigma_2 \) to the integral of \( \bar{D} \) between the origin and the branchpoint at \( z = \frac{1}{3} \). I also give a simple proof of a result recorded in [7, Eq. 4.6.14] and used in [2, Sect. 5.10] to derive a modular identity from a transformation in [8] of the equal mass Dalitz plot.

In Section 5, I derive a sesquiplicate modular transformation of the form \( q \to q^{3/2} \).

In Section 6, I combine these two modular transformations to relate the integral of \( \sigma_2 \) to an odd moment of 5 Bessel functions that yields the third singular value via its relation to a Green function on a three-dimensional diamond lattice.

In Section 7, I prove that \( M = \frac{1}{12} K_3 K_3' \).

In Section 8, I derive further identities, by combining cubic and sesquiplicate modular transformations.

### 2 Spectral function and contour integral

The Taylor series [2, Eq. 184]

\[ \bar{D}(z) = \sum_{k=0}^{\infty} a_k z^{2k} \]  

is valid for \( |z| < \frac{1}{3} \) and has integer coefficients:\[3\]

\[ a_k = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} \]  

\[ \text{See } \text{http://www.research.att.com/~njas/sequences/A002893} \].
enumerating closed walks with $2k$ steps on a hexagonal lattice. For $\frac{1}{3} > y > -\frac{1}{3}$, there is an exponentially fast method for computing [13, Eq. 4.10]

$$\tilde{D}(y) = \frac{1}{\text{AGM}(\sqrt{(1-3y)(1+y)^3}, \sqrt{(1+3y)(1-y)^3})}$$  \hspace{1cm} (15)

as the reciprocal of an arithmetic-geometric mean.

I now give the analytic continuation of (15) in the complex $z$-place with cuts on the real axis running from $z = -\infty$ to $z = -\frac{1}{3}$ and from $z = \frac{1}{3}$ to $z = +\infty$. To construct $\tilde{D}$, I use the dispersion relation [6]

$$\tilde{D}(z) = \frac{2}{\pi} \int_{\frac{1}{3}}^{1} \frac{\sigma_1(x)x}{x^2 - z^2} dx + \frac{2}{\pi} \int_{1}^{\infty} \frac{\sigma_2(x)x}{x^2 - z^2} dx$$  \hspace{1cm} (16)

for which it suffices to know the spectral function [6, Chap. 6] $\sigma(x) = \Im \tilde{D}(x + i\varepsilon)$, which is the imaginary part on the top lip of the right-hand cut, with real $x > \frac{1}{3}$ and infinitesimal real $\varepsilon > 0$. Denoting $\sigma(x) = \sigma_1(x)$, for $\frac{1}{3} < x < 1$, and $\sigma(x) = \sigma_2(x)$, for $1 < x$, I obtain

$$\sigma_1(x) = \frac{1}{\text{AGM}(\sqrt{16x^3}, \sqrt{(1+3x)(1-x)^3})}$$  \hspace{1cm} (17)

$$\sigma_2(x) = \frac{1}{\text{AGM}(\sqrt{(3x-1)(x+1)^3}, \sqrt{(3x+1)(x-1)^3})}$$  \hspace{1cm} (18)

from analytic continuation of (15).

Thus moments of these reciprocal AGMs yield the Taylor coefficients

$$\frac{2}{\pi} \int_{\frac{1}{3}}^{1} \frac{\sigma_1(x)x}{x^{2k+1}} dx + \frac{2}{\pi} \int_{1}^{\infty} \frac{\sigma_2(x)x}{x^{2k+1}} dx = \sum_{j=0}^{k} \binom{k}{j} \left(\frac{2}{j}\right)$$  \hspace{1cm} (19)

as was confirmed by Pari-GP, which yielded 200 good decimal digits for each of the first 100 Taylor coefficients in 73 CPU-seconds.

To prove that $M = \frac{1}{12} K_3 K_3'$ it will be sufficient to take the imaginary part of the contour integral

$$\oint_C \tilde{D}(z) dz = \int_0^{\frac{1}{3}} \tilde{D}(y) dy + \int_{\frac{1}{3}}^{1} \tilde{D}(x + i\varepsilon) dx - i \int_0^{\infty} \tilde{D}(ix) dx = 0$$  \hspace{1cm} (20)

where $C$ is a counterclockwise contour that encloses the quadrant in which the real and imaginary parts of $z$ are positive. Cauchy’s theorem ensures the vanishing of (20), since the quarter circle at infinity gives no contribution. Indeed, large $x$ behaviour

$$\sqrt{3} \sigma_2(x) = \frac{1}{x^2} + O\left(\frac{1}{x^4}\right)$$  \hspace{1cm} (21)

shows that $\pi \sqrt{3} \tilde{D}(z) = \log(-z^2)/z^2 + O(1/z^2)$. More precisely, I obtained from [2, Sect. 5.7] the large $z$ behaviour

$$\pi \sqrt{3} \tilde{D}(z) = -\frac{\log(-9z^2)}{z^2} + O\left(\frac{\log(-9z^2)}{z^4}\right).$$  \hspace{1cm} (22)

Taking the imaginary part of (20), I obtain the sum rule (11) and commence the contour clockwise, from the origin, starting on the imaginary axis.
3 An even Bessel moment on the imaginary axis

From [2, Eq. 23] I obtain the odd moments

$$\int_0^\infty t^{2k+1} I_0(t) K_0^2(t) \, dt = \frac{\pi}{3\sqrt{3}} \left( \frac{2k!}{3^k} \right)^2 a_k$$

and hence for \( \frac{1}{3} > x > -\frac{1}{3} \) the representation

$$\tilde{D}(x) = \frac{3\sqrt{3}}{\pi} \int_0^\infty t I_0(3xt) I_0(t) K_0^2(t) \, dt$$

using the expansion [1, 9.6.12]

$$I_0(z) = \sum_{k=0}^{\infty} \left( \frac{z^k}{2^k k!} \right)^2.$$  

The analytic continuation to the imaginary axis is

$$\tilde{D}(ix) = \frac{3\sqrt{3}}{\pi} E(3x)$$

with the Bessel moment

$$E(x) = \int_0^\infty t J_0(t) I_0(t) K_0^2(t) \, dt$$

obtained from [24] and [26] by the analytic continuation \( J_0(xt) = I_0(ixt) \). Its evaluation

$$E(x) = \frac{\pi}{\text{AGM}(\Re h, |h|)}, \quad \text{with} \quad h = \sqrt{(1-ix)(3+ix)^3},$$

is obtained by substituting \( y = ix/3 \) in [15] and by using \( \text{AGM}(h, \overline{h}) = \text{AGM}(\Re h, |h|) \), where \( \overline{h} \) is the complex conjugate of \( h \). I remark that [28] is a computationally efficient rewriting of [2, Eq. 138], since the former requires only a complex square root, while the latter used a less frugal, but equivalent, combination of trigonometric and inverse trigonometric functions.

Using [27], I obtain

$$\int_0^\infty E(x) \, dx = \int_0^\infty I_0(t) K_0^2(t) \, dt$$

as an even moment of Bessel functions, by interchanging the order of integrations over \( x \) and \( t \) and using the evaluation [11, 11.4.17] \( \int_0^\infty t J_0(at) \, dx = 1 \).

Thanks to [11, 5], I was able to obtain the remarkable identity

$$\int_0^\infty I_0(at) K_0(at) K_0(t) \, dt = \frac{2}{\pi} \int_0^\infty K_0(4at) K_0^2(t) \, dt$$

for all real \( a > 0 \). I proved this by using [5, Eq. 3.3] in the case \( 0 < a \leq 1/2 \), obtaining the left-hand side of (30) as a product of complete elliptic integrals that is identical to the product that I had earlier obtained by the delicate limiting process that led to [2, Eq. 36], for the evaluation of the more demanding right-hand side. Then, for \( a \geq 1/2 \), the analytic continuation given in [2, Eqs. 34,35] provides the evaluation

$$4a \int_0^\infty I_0(at) K_0(at) K_0(t) \, dt = K^2(\sin \alpha) + K^2(\cos \alpha), \quad \alpha = \frac{1}{2} \arcsin\left(\frac{1}{2a}\right)$$
and hence the third singular value in the neat evaluation

\[ \int_0^\infty E(x) \, dx = \int_0^\infty I_0(t) K_0^2(t) \, dt = K_3^2 \]

(32)

obtained by setting \( a = 1 \) and hence \( \alpha = \pi/12 \) in (31).

Thus I obtain

\[ \int_1^3 \sigma_1(x) \, dx + \int_1^\infty \sigma_2(x) \, dx - \frac{1}{\pi} K_3 K'_3 = 0 \]

(33)

from (10), by using (26) and the relation \( K'_3 = \sqrt{3} K_3 \).

Thus I obtain

\[ \int_1^3 \sigma_1(x) \, dx + \int_1^\infty \sigma_2(x) \, dx - \frac{1}{\pi} K_3 K'_3 = 0 \]

(33)

from (10), by using (26) and the relation \( K'_3 = \sqrt{3} K_3 \).

## 4 A cubic modular transformation

Proceeding clockwise, I pass along the quarter circle at infinity with impunity, thanks to (22), and arrive at the top lip of the cut, with \( z = x + i \varepsilon \) and large \( x \), where the spectral function \( \sigma_2(x) \) is needed. Here I shall substitute \( 3y = 1/x \) in the identity

\[ \frac{1}{\text{AGM}(\sqrt{(1-3y)(1+y)^3}; \sqrt{(1+3y)(1-y)^3})} = \frac{1}{\text{AGM}(\sqrt{(1-y)(1+3y)^3}; \sqrt{(1+y)(1-3y)^3})} \]

(34)

which is valid for \( \frac{1}{3} > y > -\frac{1}{3} \) and was proven using the EllipticK and HeunG functions of Maple to show that each side of (34) satisfies the same second-order differential as HeunG(9, 3; 1, 1, 1, 1; 9y^2). Since each side has the Taylor expansion \( 1 + 3y^2 + \mathcal{O}(y^4) \), each evaluates to this HeunG function, for \( \frac{1}{3} > y > -\frac{1}{3} \).

I remark that (34) is a cubic modular transformation. Setting

\[ k = \sqrt{\frac{16y^3}{(1+3y)(1-y)^3}}, \quad k' = \sqrt{1-k^2}, \quad l = \sqrt{\frac{16y}{(1-y)(1+3y)^3}}, \quad l' = \sqrt{1-l^2}, \]

(35)

one trivially obtains

\[ \sqrt{k'l'} + \sqrt{kl} = \frac{(1+y)(1-3y) + 4y}{(1-y)(1+3y)} = 1 \]

(36)

for \( \frac{1}{3} > y > -\frac{1}{3} \). This proves [7] that the modular transformation is cubic, i.e. that the nome [17, 3.7] associated with \( k \) is the cube of the nome associated with \( l \). Denoting the latter by \( q = \exp(-\pi K(l')/K(l)) \), one may rewrite (34) as

\[ \frac{\theta_2^2(q^3)}{\sqrt{16y^3}} = \frac{\theta_2^2(q)}{\sqrt{16y}} \]

(37)

with \( \theta_2(q) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} \). Then \( \theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2} \) determines \( l^2 = \theta_2^4(q)/\theta_3^4(q) \). Hence I have proven that the cubic multiplier \( y = \theta_2^2(q^3)/\theta_2^2(q) \) is a root of the quartic polynomial \( (1-y)(1+3y)^3 - 16y\theta_3^4(q)/\theta_2(q) \). Thus I recover a result of Joubert and Cayley that was recorded in [7, Eq. 4.6.14] and used in the proof of the modular identity in [2, Eq. 157], obtained from an analysis of the equal mass Dalitz plot in [8].
More importantly, for present considerations, one may rewrite (34) as
\[
\tilde{D}(y) = \sqrt{3}x^2 \sigma(x), \quad \text{for} \quad 3xy = 1,
\]
and hence obtain
\[
\int_0^{\frac{1}{3}} \tilde{D}(y) \, dy = \frac{1}{\sqrt{3}} \int_1^\infty \sigma(x) \, dx
\]
which transforms (33) to
\[
\int_0^{\frac{1}{3}} \sigma_1(x) \, dx + \sqrt{3} \int_0^{\frac{1}{3}} \tilde{D}(y) \, dy - \frac{1}{\pi} K_3 K_3' = 0.
\]

5 A sesquiplicate modular transformation

The integral over \( \tilde{D} \) in (40) may be transformed to yield a Bessel moment by using the transformation
\[
2\tilde{D}(y) = \sqrt{3}(1 - x^2) \sigma_1(x), \quad \text{for} \quad 9(1 - x^2)(1 - y^2) = 8,
\]
which is valid for \( 0 < x < 1 \). It maps the region \( 0 < x < \frac{1}{3} \) to \( \frac{1}{3} > y > 0 \) and the region \( \frac{1}{3} < x < 1 \) to the positive imaginary \( y \)-axis. To prove (41), I used the representations
\[
\tilde{D}(y) = \text{HeunG}(9, 3; 1, 1, 1, 1; 9y^2), \quad \frac{4\sigma_1(x)}{3\sqrt{3}} = \text{HeunG}(-8, -2; 1, 1, 1, 1; 1 - 9x^2),
\]
with the latter proven in the same manner as was used to proved the former, in Section 4. Applying the penultimate identity of Maple’s FunctionAdvisor(identities,HeunG) to \( \sigma_1(x) \), one obtains (41), provided that one avoids the cut with \( 9y^2 \geq 1 \). This is a modular transformation of the form \( q \to q^{3/2} \), with the nomes “in sesquiplicate proportion” (as Newton’s translator Motte said of the corresponding power law relation in Kepler’s third law). It may be obtained by combining an ascending cubic modular transformation with a descending quadratic modular transformation. Thanks to the explicit form of Heun’s differential equation [16] I was able to find a more direct proof, by using (42).

Applying (41) in the region \( 0 < y < \frac{1}{3} \), I obtain
\[
\frac{\pi \sqrt{3}}{2} \int_0^{\frac{1}{3}} \tilde{D}(y) \, dy = \int_0^{\frac{1}{3}} \frac{D(x)}{\sqrt{(1 - x^2)(1 - 9x^2)}} \, dx
\]
where the square root comes from the Jacobian
\[
\left| \frac{dy}{dx} \right| = \frac{8x}{3(1 - x^2)} \frac{1}{\sqrt{(1 - x^2)(1 - 9x^2)}}
\]
of the transformation in (41) and
\[
D(x) := 2\pi x \sigma_1(x) = \frac{4xK\left(\sqrt{(1-3x)(1+x)^3}/(1+3x)(1-x)^3\right)}{\sqrt{(1+3x)(1-x)^3}}
\]
is the function defined in [2, Eq. 64]. I remark that \( D \) provides an evaluation of the moment [2, Eq. 149]
\[
\int_0^\infty t I_0^2(t)K_0(t)K_0(ct) \, dt = \frac{1}{6c} D\left(\frac{1}{c}\right)
\]
for $c > 1$. The physical significance of $D$ derives from the dispersion relation [2 Sect. 4.2] for an odd moment with 4 Bessel functions,

$$S(w^2) := \int_0^\infty t J_0(wt)K_0^3(t) \, dt = \int_0^{\frac{1}{3}} \frac{D(x) \, dx}{1 + w^2 x^2},$$  \hspace{1cm} (47)$$

which is the sunrise diagram [11, 12] with 3 unit internal masses and Euclidean external momentum with norm $w^2$, in two-dimensional spacetime. On the cut with $-w^2 = c^2 > 9$, the elliptic integral in $D(1/c)$ appears in the imaginary part of $S$ from integration over the Dalitz plot [8] for the decay of a particle of mass $c > 3$ into 3 particles of unit mass.

Then the 5-Bessel moment

$$T(u^2, v^2) := \int_0^\infty t J_0(ut)J_0(vt)K_0^3(t) \, dt = \frac{1}{\pi} \int_0^\pi S(u^2 + 2uv \cos \theta + v^2)$$  \hspace{1cm} (48)$$
is obtained as an angular average [2 Sect. 5.3] of a diagram in which the external momentum is shared by a pair of particles. Exchanging the order of integration in (47) and (48), I obtain

$$T(-a^2, -1) = \int_0^\infty t I_0(at)I_0(t)K_0^3(t) \, dt = \int_0^{\frac{1}{3}} \frac{D(x) \, dx}{\sqrt{(1 - (ax - x)^2)(1 - (ax + x)^2)}}$$  \hspace{1cm} (50)$$

for $2 \geq a \geq 0$

6 Bessel moments from the diamond lattice

In [2 Eq. 55], the Bessel moments

$$\int_0^\infty t^{2k+1}I_0(t)K_0^3(t) \, dt = \frac{\pi^2}{16} \left(\frac{k!}{4^k}\right)^2 b_k$$  \hspace{1cm} (51)$$

were evaluated in terms of the integers

$$b_k = \sum_{j=0}^{k} \left(\begin{array}{c} k \\ j \end{array}\right)^2 \left(\begin{array}{c} 2k - 2j \\ k - j \end{array}\right) \left(\begin{array}{c} 2j \\ j \end{array}\right)$$  \hspace{1cm} (52)$$

that enumerate closed walks on a diamond lattice. They give the Taylor coefficients of

$$\int_0^\infty t I_0(at)I_0(t)K_0^3(t) \, dt = \frac{\pi^2}{16} \sum_{k=0}^\infty b_k \left(\frac{a}{8}\right)^{2k}$$  \hspace{1cm} (53)$$
as may seen by expanding $I_0(at)$ and using (51). It follows that this Bessel moment is given by the evaluation [13 Eq. 5.4] of the diamond lattice Green function,

$$\int_0^\infty t I_0(at)I_0(t)K_0^3(t) \, dt = \frac{1}{4} K(k_-) K(k_+),$$  \hspace{1cm} (54)$$

\[\text{See } \text{http://www.research.att.com/~njas/sequences/A002895.}\]
where the arguments of the complete elliptic integrals are determined by

\[ k_*^2 = \frac{1}{2} \pm \frac{a^2}{8} \sqrt{1 - \frac{a^2}{16} - \left(\frac{1}{2} - \frac{a^2}{16}\right) \sqrt{1 - \frac{a^2}{4}}} \]  

(55)

by setting \( z = a/2 \) in [13, Eq. 5.5]. Once again, a cubic modular transformation is involved, since \( \sqrt{k'_+ k'_-} + \sqrt{k_- k'_+} = 1 \), with \( k'_+ = \sqrt{1 - k_*^2} \).

Then setting \( a = 2 \) in (50) and (54), I obtain an evaluation of the integrals (43)

\[ \pi \sqrt{3} \int \frac{1}{1+3} \sigma_1(x) dx + \frac{1}{2\pi} K_3 K'_3 - \frac{1}{\pi} K_3 K'_3 = 0 \]  

(57)

which shows that the first term also has an evaluation at the third singular value.

I remark that the diamond lattice integers (52) also enumerate staircase polygons [9, 10] in four dimensions, for which the generating function [9, Eq. 6a] is an odd 5-Bessel moment containing \( I_0^4 \). Thus I derive from (54) the remarkable sum rule

\[ \int_0^\infty t I_0(at) \left( t^3 I_0(8t) - \frac{1}{4\pi^2} I_0(t) K_0^3(t) \right) dt = 0 \]  

(58)

for \( 2 \geq a \geq 0 \).

7 Proof of the conjecture

Now I set \( y = iw/3 \) in the sesquiplicate modular transformation (11) and obtain an integral of \( \tilde{D} \) on the imaginary axis. The result is

\[ \pi \int \sigma_1(x) dx = \int_0^\infty \frac{E(w)w}{(w^2 + 1)(w^2 + 9)} dw \]  

(59)

with a square root from the inverse of the Jacobian (11) and

\[ E(w) = \int_0^\infty t J_0(wt) I_0(t) K_0^2(t) dt \]  

(60)

coming from the analytic continuation (26).

Then I observe that

\[ \int_0^\infty s J_0(ws)J_0(vs)K_0(cs) ds = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \frac{d\theta}{w^2 + 2wv \cos \theta + v^2 + c^2} \]  

(61)

\[ = \frac{1}{\sqrt{(w - v)^2 + c^2)((w + v)^2 + c^2)} \]  

(62)
since the Bessel moment corresponds to an angular average \[2\,\text{Sect. 5.3}\] of a tree diagram in two-dimensional Euclidean momentum space. Using the distribution
\[
\int_0^\infty w J_0(ws) J_0(wt) \, dw = 2\delta(s^2 - t^2)
\]
(63)
I evaluate the 5-Bessel moment
\[
\int_0^\infty t J_0(vt) K_0(ct) I_0(t) K_2^2(t) \, dt = \int_0^\infty \frac{E(w)w}{\sqrt{((w - v)^2 + c^2)((w + v)^2 + c^2)}} \, dw
\]
(64)
as a folding of (60) with (61).

Then I set \(v = i\) and \(c = 2\) and obtain
\[
\frac{\pi}{6} \int_\frac{1}{3}^1 \sigma_1(x) \, dx = \int_0^\infty t I_0^2(t) K_0^2(t) K_0(2t) \, dt := M
\]
(65)
from (59). Hence the contour integral (57) gives
\[
\frac{6}{\pi} M + \frac{1}{2\pi} K_3 K'_3 - \frac{1}{\pi} K_3 K'_3 = 0
\]
(66)
which completes the proof that \(M = \frac{1}{12} K_3 K'_3\).

The reader may consider (as does the author) that this is a rather indirect proof, since it involves the difference between an even and odd Bessel moment, obtained via delicate contour integration and several modular transformations. However, a great deal of unrewarded effort had previously been expended on searching for a more direct proof.

8 Further identities

Now I consider the real part on the cut, remarking that
\[
\rho(x) = \frac{1}{\text{AGM}(\sqrt{(3x - 1)^4(x + 1)}, \sqrt{16x})}
\]
(67)
is real for \(x > \frac{1}{3}\) and satisfies the same differential equation as \(\tilde{D}\), \(\sigma_1\) and \(\sigma_2\). Thus a multiple of \(\rho\) should give the real part of \(\tilde{D}\) on each of the two portions of the cut, with \(\frac{1}{3} < x < 1\) and \(1 < x\). However, these multiples may be distinct, since the two portions are separated by a singular point of the differential equation, at \(x = 1\). Indeed the multiples are not the same; the Green function on the lip of its cut is given by
\[
\tilde{D}(x \pm i\varepsilon) = \begin{cases} 
\rho(x) \pm i\sigma_1(x) & \text{for } \frac{1}{3} < x < 1, \\
-2\rho(x) \pm i\sigma_2(x) & \text{for } 1 < x,
\end{cases}
\]
(68)
where the unit multiple, on the first portion of the cut, results from analytic continuation of (15), while the factor of \(-2\), for the second portion, is entirely determined by the asymptotic behaviour (22).

Thus I derive the vanishing of
\[
\Re \oint_C \tilde{D}(z) \, dz = \int_0^\frac{1}{3} \tilde{D}(y) \, dy + \int_\frac{1}{3}^1 \rho(x) \, dx - 2 \int_1^\infty \rho(x) \, dx = 0
\]
(69)
and use the cubic modular transformation complementary to \(38\),

\[
3\sqrt[3]{3}x^2\rho(x) = \sigma_1(y), \quad \text{for} \quad 3xy = 1,
\]

(70)
to write the contour integral as

\[
\int_0^{\frac{i}{\sqrt{3}}} \bar{D}(y) \, dy + \frac{1}{\sqrt{3}} \int_0^{1} \sigma_1(y) \, dy - \frac{2}{\sqrt{3}} \int_0^{\frac{i}{\sqrt{3}}} \sigma_1(y) \, dy = 0.
\]

(71)
The first term was evaluated in \([56]\) and the second in \([57]\). Hence I prove the identity

\[
K_3K_3' = \int_0^{\frac{i}{\sqrt{3}}} \frac{D(y)}{y} \, dy
\]

(72)

obtained by using the definition \(D(y) := 2\pi y\sigma_1(y)\) in the third term.

There are now four proven evaluations at the third singular value from integrals of \(D\), \(\bar{D}\) and \(E\), namely

\[
K_3^2 = 2\pi \int_0^{\frac{i}{\sqrt{3}}} \bar{D}(y) \, dy = \int_0^{\infty} E(x) \, dx
\]

(73)

\[
K_3K_3' = \int_0^{\frac{i}{\sqrt{3}}} \frac{D(y)}{y} \, dy = \int_0^{1} \frac{D(y)}{y} \, dy
\]

(74)

obtained from two Bessel moments and from the vanishing of the real and imaginary parts of a contour integral.

There are now three proven evaluations of odd Bessel moments at the third singular value, namely

\[
K_3K_3' = 4 \int_0^{\infty} t I_0(2t) I_0(t) K_0^3(t) \, dt
\]

(75)

\[
= 4\pi^2 \int_0^{\infty} t I_0^4(t) K_0(4t) \, dt
\]

(76)

\[
= 12 \int_0^{\infty} t I_0^2(t) K_0^3(t) K_0(2t) \, dt := 12M
\]

(77)

with the first proven in \([2]\), the second in \([9]\) and the third by the contour integration \((66)\).

There are now four proven evaluations of even Bessel moments at the third singular value, namely

\[
K_3^2 = \int_0^{\infty} I_0(t) K_0^2(t) \, dt = \frac{2}{\pi} \int_0^{\infty} K_0^2(t) K(4t) \, dt,
\]

(78)

\[
K_3K_3' = 4 \int_0^{\infty} I_0(t) K_0(t) K_0(4t) \, dt
\]

(79)

of which only the last was proven in \([2]\). The first is proven in \([32]\), the second results from sum rule \((12)\) at \(a = 1\) and the third from setting \(a = \frac{1}{4}\) in \((12)\) and then rescaling \(t\) by a factor of 4.

Moreover, by using the Taylor expansion of \(\bar{D}\) in \((73)\) and the Clausen product formula \((7)\) for \(K_3^2\), I obtain a novel relation

\[
\sum_{k=0}^{\infty} \frac{a_k}{(2k+1)3^{2k+1}} = \frac{\pi}{8} \sum_{k=0}^{\infty} \frac{(2k)^3}{2^{3k}}
\]

(80)
between a sum over the integers \(a_k\), which enumerate closed walks on a two-dimensional hexagonal lattice, and a sum over the integers \((2k)_k^{3}\), which enumerate closed walks on a three-dimensional body centred cubic lattice \([13]\).

Relations between integrals of products of AGMs and odd moments of 6 Bessel functions may be obtained from the vanishing of the contour integral \(\oint_C \tilde{D}(z) z \, dz\). From its imaginary part, I obtain the superconvergence relation

\[
\Im \oint_C \tilde{D}^2(z) z \, dz = \int_0^1 \rho(x) \sigma_1(x) \, dx - 2 \int_1^\infty \rho(x) \sigma_2(x) \, dx = 0 \quad (81)
\]

and hence, by cubic modular transformation of both \(\sigma_2\) and \(\rho\), the identity

\[
\frac{1}{4\sqrt{3\pi}} \int_{\frac{1}{3}}^1 \frac{D(y)D\left(\frac{1}{3y}\right)}{y} \, dy = \int_0^\frac{1}{3} D(y) \tilde{D}(y) \, dy = \int_0^\infty t I_0^3(t) K_0^3(t) \, dt \quad (82)
\]

where the Bessel moment was derived from the integral of \(D \tilde{D}\) in \([2, \text{Eq. 223}]\).

For the real part of the contour integral, I use the sesquiplicate transformation \([41]\) to prove that

\[
I_1 := \int_0^\infty E^2(w) w \, dw = \int_0^\frac{1}{3} \frac{D^2(x)}{18x} \, dx := I_2 \quad (83)
\]

\[
I_3 := \pi^2 \int_0^\frac{1}{3} \tilde{D}^2(y) \, dy = \int_0^\frac{1}{3} \frac{D^2(x)}{6x} \, dx := I_4 \quad (84)
\]

and make cubic transformations of \(\sigma_2\) and \(\rho\) to obtain

\[
\pi^2 \Re \oint_C \tilde{D}^2(z) z \, dz = 3(I_1 - I_2) + 2(I_4 - I_3) = 0 \quad (85)
\]

with no further relation obtained by contour integration.

Then I use the distribution \([63]\) to prove that

\[
I := \int_0^\infty t I_0^2(t) K_0^4(t) \, dt = \int_0^\infty E^2(w) w \, dw \quad (86)
\]

by folding two copies of \([63]\). Next I remark that the appearance of \(D\) as a moment, in \([16]\), and also as a spectral function, in \([17]\), proves that \(I = I_4\), as was remarked in \([2, \text{Sect. 6.1}]\). Thus I obtain

\[
I_1 = I_2 = I_3 = I_4 = I := \int_0^\infty t I_0^2(t) K_0^4(t) \, dt \quad (87)
\]

with each integral in \([83, 84]\) yielding the same Bessel moment.

Finally, I show how to compute \(\tilde{D}(z)\) throughout the quadrant \(z = x + iy\) with \(x > 0\) and \(y > 0\), using the separatrices

\[
y_1(x) = \frac{x + 1}{\sqrt{1 + \frac{1}{4x}}}, \quad y_2(x) = \frac{x - 1}{\sqrt{1 - \frac{1}{4x}}} \quad (88)
\]


to distinguish the three cases

\[
\tilde{D}(z) = \begin{cases} 
2\rho(z) + \frac{1}{4}i\sigma_2(z) & \text{if } x > 0 \text{ and } y_1(x) < y \\
-2\rho(z) + i\sigma_2(z) & \text{if } x > 1 \text{ and } y_2(x) > y > 0 \\
i\sigma_2(z) & \text{otherwise}
\end{cases} \quad (89)
\]

obtained by comparing results of numerical integration in \([16]\) with the easier evaluations of complex AGMs in \([18]\) and \([67]\) computed as convergents of the defining iteration

\[
\text{AGM}(a, b) = \text{AGM}\left(\frac{a + b}{2}, \sqrt{ab}\right).
\]
9 Conclusion

In 1936, Wilfrid Norman Bailey proved an identity [5, Eq. 3.3] that leads, via the analysis in [2, 8, 9, 13], to remarkable connections between Feynman diagrams, integrals of the reciprocals of arithmetic-geometric means, lattice Green functions and the enumeration of staircase polygons. The cubic modular transformation (3 4) and the sesquiplicate modular transformation (4 1) provide wonderful relations between the integrals and generating functions for these four allied structures.

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References

[1] M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions, NBS (now NIST), 1965.

[2] David H. Bailey, Jonathan M. Borwein, David Broadhurst and M.L. Glasser, “Elliptic integral evaluations of Bessel moments,” preprint, 6 January 2008, http://arxiv.org/PS_cache/arxiv/pdf/0801/0801.0891v1.pdf.

[3] David H. Bailey, Jonathan M. Borwein and Richard E. Crandall, “Integrals of the Ising class,” J. Physics A: Mathematical and General, 39 (2006), 12271–12302, http://crd.lbl.gov/~dhbailey/dhbpapers/ising.pdf.

[4] W. N. Bailey, “Some infinite integrals involving Bessel functions,” Proc. London Math. Soc., 40 (1936), 37–48.

[5] W. N. Bailey, “Some infinite integrals involving Bessel functions (II),” J. London Math. Soc., 11 (1936), 16–20.

[6] Gabriel Barton, Introduction to dispersion techniques in field theory, Lecture Notes and Supplements in Physics, W.A. Benjamin, New York and Amsterdam, 1965.

[7] Jonathan M. Borwein and Peter B. Borwein, Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity, CMS Monographs and Advanced books in Mathematics, John Wiley, Hoboken, NJ, 1987.

[8] A.I. Davydychev and R. Delbourgo, “Explicitly symmetrical treatment of three body phase space,” J. Phys., A37 (2004), 4871–4886, http://arxiv.org/PS_cache/hep-th/pdf/0311/0311075v1.pdf.

[9] M.L. Glasser and E. Montaldi, “Staircase polygons and recurrent lattice walks,” Phys. Rev., E47 (1993), 2339–2342.

[10] A.J. Guttmann and T. Prellberg, “Staircase polygons, elliptic integrals, Heun functions and lattice Green functions,” Phys. Rev., E47 (1993), 2233–2236.

5Lecture recorded at http://durpgap.googlepages.com/qftw.
[11] S. Groote, “Lectures on configuration space methods for sunrise type diagrams,” International Research Workshop on Calculations for Modern and Future Colliders (CALC 2003), http://arxiv.org/PS_cache/hep-ph/pdf/0307/0307290v1.pdf.

[12] S. Groote, J.G. Korner and A.A. Pivovarov, “On the evaluation of a certain class of Feynman diagrams in x-space: Sunrise-type topologies at any loop order,” Annals Phys., 322 (2007), 2374–2445, http://arxiv.org/PS_cache/hep-ph/pdf/0506/0506286v1.pdf.

[13] G.S. Joyce, “On the cubic lattice Green functions,” Philosophical Transactions of the Royal Society of London, Mathematical and Physical Sciences, 445 (1994), 463–477.

[14] Stéphane Ouvry, “Random Aharonov-Bohm vortices and some exactly solvable families of integrals,” Journal of Statistical Mechanics: Theory and Experiment, 1 (2005), P09004.

[15] S. Laporta, “High precision epsilon expansions of massive four loop vacuum bubbles,” Phys. Lett., B549 (2002), 115–122, http://arxiv.org/PS_cache/hep-ph/pdf/0210/0210336v1.pdf.

[16] A. Ronveaux (Ed.), Heun’s Differential Equations, The Clarendon Press, 1995.