ON REGULARITY OF STOCHASTIC CONVOLUTIONS OF
FUNCTIONAL LINEAR DIFFERENTIAL EQUATIONS
WITH MEMORY

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(Communicated by María J. Garrido-Atienza)

Abstract. In this work, we consider the regularity property of stochastic convolutions for a class of abstract linear stochastic retarded functional differential equations with unbounded operator coefficients. We first establish some useful estimates on fundamental solutions which are time delay versions of those on \(C_0\)-semigroups. To this end, we develop a scheme of constructing the resolvent operators for the integrodifferential equations of Volterra type since the equation under investigation is of this type in each subinterval describing the segment of its solution. Then we apply these estimates to stochastic convolutions of our equations to obtain the desired regularity property.

1. Introduction. We begin with an example of stochastic delay heat equations without exterior energy source to motivate our work. Let \(h_i : \mathbb{R} \to \mathbb{R}, i = 0, 1\), be two monotonically differentiable functions which satisfy the following conditions (see, e.g., Coleman and Gurtin \cite{1} and Nunziato \cite{6})

\[
x \cdot h_i(x) \geq \gamma_{i,1}|x|^p + \alpha_{i,1}, \quad \forall x \in \mathbb{R},
\]

\[
|h_i(x)| \leq \gamma_{i,2}|x|^{p-1} + \alpha_{i,2}, \quad \forall x \in \mathbb{R},
\]

where \(p \geq 2\), \(\gamma_{i,j} > 0\) and \(\alpha_{i,j} \in \mathbb{R}\) for \(i = 0, 1\) and \(j = 1, 2\). The non-Fourier heat conduction model with delay in the conductor \((0, \pi) \subset \mathbb{R}\) starts from the following constitutive equation

\[
q(t, x) = -h_0(\partial y(t, x)/\partial x) - h_1(y(t-r, x)), \quad t \geq 0, \quad x \in (0, \pi),
\]

and the energy conservative equation without exterior energy sources

\[
d_t p(t, x) + \frac{\partial q(t, x)}{\partial x} dt = 0, \quad t \geq 0, \quad x \in (0, \pi),
\]

where \(y\) denotes the temperature, \(q\) is the heat flux and \(p\) is the internal energy which can be taken, in most situations, as the form: \(p(t, x) = \kappa y(t, x), \kappa > 0\).

In practice, the assumption of zero exterior energy source is artificial, and a more realistic model is that the null exterior energy source is perturbed by a noise process,

\textsuperscript{2010 Mathematics Subject Classification.} Primary: 60H15, 60G15, 60H05.

\textit{Key words and phrases.} Regularity property, fundamental solution, stochastic convolution.

The author is grateful to the Tianjin Thousand Talents Plan for its financial support.
for example, a Gaussian white noise $b(x)\dot{w}(t, x)$, $b \in L^2(0, \pi)$. In other words, we replace (2) by the equation
\[
dt p(t, x) + \frac{\partial g(t, x)}{\partial x} dt = b(x) dw(t, x), \quad t \geq 0, \quad x \in (0, \pi).
\] (3)

Then, by substituting (3) into (1), we obtain, for simplicity, letting $\kappa = 1$, the following equation
\[
\frac{dy(t, x)}{dx} = \frac{\partial h_0(dy(t, x)/dx)}{dx} dt + \frac{\partial h_1(y(t-r, x))}{dx} dt + b(x) dw(t, x), \quad t \geq 0, \\
y(0, \cdot) = \phi_0(\cdot) \in L^2(0, \pi), \quad y(\theta, \cdot) = \phi_1(\theta, \cdot) \in W^{1,p}(0, \pi), \quad \theta \in [-r, 0], \\
y(t, 0) = y(t, \pi) = 0, \quad t \in (0, \infty).
\] (4)

Let $\Delta = \frac{d^2}{dx^2}$, $H = L^2(0, \pi)$, $V = H_0^1(0, \pi)$, $U = W^{1,p}(0, \pi)$ and $Bu = bu, u \in H$, $W(t) = w(t, \cdot)$, $g_0(u) = \frac{dh_0(du(x)/dx)}{dx}$, $g_1(u) = \frac{dh_1(u(x))}{dx}$ for any $u \in W^{1,p}(0, \pi)$.

We have thus a stochastic differential equation with delay in $H$,
\[
\begin{align*}
dy(t) &= g_0(y(t)) dt + g_1(y(t-r)) dt + BdW(t), \quad t \geq 0, \\
y(0) &= \phi_0, \quad y_0 = \phi_1,
\end{align*}
\] (5)

where $U \subset H \subset U^*$ and $g_i, i = 0, 1$, is a continuous monotone operator from $U$ to $U^*$ such that
\[
\begin{align*}
\langle u, g_i(u) \rangle_{U, U^*} &\geq \gamma_{i,1} ||u||_U^p + \alpha_{i,1}, \quad \forall u \in U, \\
||g_i(u)||_{U^*} &\leq \gamma_{i,2} ||u||_U + \alpha_{i,2}, \quad \forall u \in U,
\end{align*}
\]
where $p \geq 2$, $\gamma_{i,j} > 0$ and $\alpha_{i,j} \in \mathbb{R}$ for $i = 0, 1$ and $j = 2$. In particular, if $h_0(x) = x$, $h_1(x) = \gamma x$, $\gamma > 0$, $p = 2$, then $g_0(u) = \Delta u$, $g_1(u) = \gamma (\cdot - \Delta)^{1/2} u$, $V = U$ and the equation (5) reduces to
\[
\begin{align*}
dy(t) &= \Delta y(t) dt + \gamma (\cdot - \Delta)^{1/2} y(t-r) dt + BdW(t), \quad t \geq 0, \\
y(0) &= \phi_0, \quad y_0 = \phi_1.
\end{align*}
\] (6)

The aim of this work is to investigate the regularity property of such stochastic systems as (6).

The organization of this work is as follows. In Section 2, we first introduce the deterministic linear retarded functional differential equation associated in our formulation of stochastic systems. We review the useful variation of constants formula for the equation under consideration by means of its fundamental solution. Also, we state some estimates about fundamental solutions which will play an important role in the subsequent investigation. By employing the main results, we establish in Section 3 the desired regularity property of stochastic convolutions. In Sections 4 and 5, we present the detailed proofs of the main theorem, i.e., Theorem 2.1, by following J. Prüss's method of constructing the resolvent operators for the integrodifferential equations of Volterra type.

2. Fundamental solution. We are concerned with the following linear retarded functional differential equation in a Banach space $X$,
\[
\begin{align*}
dy(t) &= Ay(t) dt + A_1 y(t-r) dt + \int_{-r}^{0} a(\theta) A_2 y(t+\theta) d\theta + f(t), \quad t \geq 0, \\
y(0) &= \phi_0, \quad y(\theta) = \phi_1(\theta), \quad \theta \in [-r, 0], \quad \phi = (\phi_0, \phi_1),
\end{align*}
\] (7)
where $r > 0$ is some constant incurring the system delay, $a(\cdot) \in L^2([-r, 0], \mathbb{R})$ and 
$\phi = (\phi_0, \phi_1)$ is an appropriate initial datum. Here $A : \mathcal{D}(A) \subset X \to X$ is the 
infinite generator of an analytic semigroup $e^{tA}$, $t \geq 0$, and $A_1$, $A_2$ are two 
closed linear operators with domains $\mathcal{D}(A_i) \supset \mathcal{D}(A)$, $i = 1, 2$, and $f$ is a continuous 
function with values in $X$. For simplicity, we assume in this work that the $C_0$- 
semigroup $e^{tA}$ is negative type, i.e., there exist constants $M \geq 1$ and $\mu > 0$ such that 
\[ \|e^{tA}\| \leq Me^{-\mu t}, \quad \|Ae^{tA}\| \leq M/t \quad \text{for all} \quad t > 0, \] 
and for $\gamma \in (0, 1)$, there exists a constant $M_\gamma > 0$ such that 
\[ \|(-A)^\gamma e^{tA}\| \leq M_\gamma/t^\gamma \quad \text{for all} \quad t > 0, \] 
where $(-A)^\gamma$ is the standard fractional power of operator $A$.

Equations of the type (7) were investigated by Di Blasio, Kunisch and Sinestrari [3], Sinestrari [9, 10] and the fundamental solution to (7) was introduced by Jeong, Nakagiri and Tanabe [5]. In particular, it is known that the fundamental solution $G(\cdot) : \mathbb{R} \to \mathcal{L}(X)$ to (7) is an operator-valued function which is strongly continuous 
in $X$ and satisfies 
\[ \frac{d}{dt}G(t) = AG(t) + A_1G(t-r) + \int_{-r}^{0} a(\theta)A_2G(t+\theta)d\theta, \] 
\[ G(0) = I, \quad G(t) = O, \quad t \in (-\infty, 0), \] 
where $O$ is the null operator in $X$. According to the well-known Duhamel’s principle, the problem (10) is transformed to the integral equation 
\[ G(t) = \begin{cases} 
\begin{align*}
& e^{tA} + \int_{0}^{t} e^{(t-s)A}A_1G(s-r)ds + \int_{0}^{t} \int_{-r}^{0} a(\theta)e^{(t-s)A}A_2G(s+\theta)d\theta ds, \\
& \quad t \geq 0,
& 0, \quad t < 0.
\end{align*}
\end{cases} \] 
The fundamental solution $G$ enables us to solve the initial value problem for the 
equation (7). In fact, it may be shown that under some reasonable conditions on $f$ and initial datum $\phi = (\phi_0, \phi_1)$, the unique mild solution $y$ of (7) is represented as 
\[ y(t) = G(t)\phi_0 + \int_{-r}^{0} U_t(\theta)\phi_1(\theta)d\theta + \int_{0}^{t} G(t-s)f(s)ds, \quad t \geq 0, \] 
where 
\[ U_t(\theta) = G(t-\theta-r)A_1 + \int_{-r}^{\theta} G(t-s+\tau)a(\tau)A_2d\tau, \quad \theta \in [-r, 0], \] 
with the initial condition $y(0) = \phi_0$ and $y(\theta) = \phi_1(\theta)$, $\theta \in [-r, 0)$. This is a time 
delay version of the usual variation of constants formula without memory.

In order to apply (12) to such equations as (6) to consider their regularity property 
of solutions, we need establish some inequalities in association with $G(\cdot)$, which 
are the main results of this work. To this end, we shall formulate the following 
condition:

(H) $a(\cdot) \in L^\infty([-r, 0], \mathbb{R})$ and 
\[ \mathcal{D}((-A)^\mu) \subset \mathcal{D}(A_1), \quad \mathcal{D}((-A)^\nu) \subset \mathcal{D}(A_2) \] 
for some $0 < \mu, \nu < 1$. 

Theorem 2.1. Assume that condition (H) holds. Then
(a) for any $\gamma \in [\nu, 1)$, it is true that
\[
\|(-A)^\gamma G(t)\| \leq \frac{C_{n,\gamma}}{(t-nr)^{\gamma}} \quad \text{for all} \quad t \in (nr, (n+1)r),
\] (14)
\[
\left\|\int_s^t (-A)^\gamma G(u)du\right\| \leq C_{n,\gamma} \quad \text{for all} \quad nr \leq s < t \leq (n+1)r,
\] (15)
where $C_{n,\gamma} > 0$, $n \in \mathbb{N} := \{0, 1, \cdots \}$, are constants depending on $n$ and $\gamma$.
(b) for any $\gamma \in [\nu, 1)$ and $0 < \beta < 1 - \gamma$, it is true that
\[
\|(-A)^\gamma (G(t) - G(s))\| \leq C_{n,\beta,\gamma} \frac{(t-s)^{\beta}}{(s-\nu)^{\beta+\gamma}} \quad \text{for all} \quad nr < s < t < (n+1)r,
\] (16)
where $C_{n,\beta,\gamma} > 0$, $n \in \mathbb{N}$, are constants depending on $n$, $\beta$ and $\gamma$.

3. Stochastic convolution. Let $\{\Omega, \mathcal{F}, \mathbb{P}\}$ be a probability space equipped with some filtration $\{\mathcal{F}_t\}_{t \geq 0}$. Let $K$ be a separable Hilbert space and $\{W_Q(t), t \geq 0\}$ denote a $Q$-Wiener process with respect to $\{\mathcal{F}_t\}_{t \geq 0}$ in $K$, defined on $\{\Omega, \mathcal{F}, \mathbb{P}\}$, with covariance operator $Q$, i.e.,
\[
\mathbb{E}\langle W_Q(t), x \rangle_K (W_Q(s), y)_{K} = (t \wedge s)\langle Qx, y \rangle_K \quad \text{for all} \quad x, y \in K,
\]
where $Q$ is a positive, self-adjoint and trace class operator on $K$. We frequently call $W_Q(t)$, $t \geq 0$, a $K$-valued $Q$-Wiener process with respect to $\{\mathcal{F}_t\}_{t \geq 0}$ if the trace $Tr(Q) < \infty$. We introduce a subspace $K_Q = \mathcal{R}(Q^{1/2}) \subset K$, the range of $Q^{1/2}$, which is a Hilbert space endowed with the inner product
\[
\langle u, v \rangle_{K_Q} = \langle Q^{-1/2}u, Q^{-1/2}v \rangle_K \quad \text{for any} \quad u, v \in K_Q.
\]
Let $H$ be a separable Hilbert space and $\mathcal{L}_2(K_Q, H)$ denote the space of all Hilbert-Schmidt operators from $K_Q$ into $H$. Then $\mathcal{L}_2(K_Q, H)$ turns out to be a separable Hilbert space, equipped with the norm
\[
\|\Psi\|_{\mathcal{L}_2(K_Q, H)}^2 = Tr[\Psi Q^{1/2}(\Psi Q^{1/2})^*] \quad \text{for any} \quad \Psi \in \mathcal{L}_2(K_Q, H).
\]
For arbitrarily given $T \geq 0$, let $J(t, \omega)$, $t \in [0, T]$, be an $\mathcal{L}_2(K_Q, H)$-valued process, and we define the following norm for arbitrary $t \in [0, T]$,
\[
|J|_t := \left\{ \mathbb{E} \int_0^t Tr\left[J(s, \omega)Q^{1/2}(J(s, \omega)Q^{1/2})^*\right] ds \right\}^{\frac{1}{2}}.
\] (17)
In particular, we denote by $\mathcal{U}^2([0, T]; \mathcal{L}_2(K_Q, H))$ the space of all $\mathcal{L}_2(K_Q, H)$-valued measurable processes $J$, adapted to the filtration $\{\mathcal{F}_t\}_{t \leq T}$, satisfying $|J|_T < \infty$.

Suppose that $W(\cdot)$ is a $Q$-Wiener process in $K$ such that $Qe_j = \lambda_j e_j$, $\lambda_j > 0$, $j \geq 1$, where $\{e_j\}$ is a complete orthonormal basis in $K$, then it is immediate that
\[
W(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} w_j(t)e_j, \quad t \geq 0,
\]
where $\{w_j(\cdot)\}$ is a group of independent real Wiener processes. The stochastic integral
\[
\int_0^t J(s)dW(s) \in H, \quad t \geq 0,
\]
may be defined for all $J \in \mathcal{U}^2([0, T] \times \Omega; \mathcal{L}_2(K_Q, H))$. 

The reader is referred to Da Prato and Zabczyk [2] for more details on this topic.

We are concerned about the following linear stochastic retarded functional differential equation on $H$,

$$
\begin{cases}
 dy(t) = A_1 y(t - r)dt + A_2 y(t + \theta) d\theta + B dW(t), & t \geq 0, \\
y(0) = \phi_0, & y(\theta) = \phi_1(\theta), \theta \in [-r, 0], \phi = (\phi_0, \phi_1),
\end{cases}
$$

where $A_1, A_2$ are given as in Section 2, $B \in \mathcal{L}_2(K_H, H)$ and $\phi = (\phi_0, \phi_1)$ is an appropriate initial datum. It is well known that the unique mild solution $y$ of (19) is represented as

$$
y(t) = G(t)\phi_0 + \int_{-r}^{0} U_t(\theta)\phi_1(\theta) d\theta + \int_{0}^{t} G(t - s)BdW(s), \quad t \geq 0,
$$

where

$$
U_t(\theta) = G(t - \theta - r)A_1 + \int_{-r}^{\theta} G(t - s + \tau) a(\tau) A_2 d\tau, \quad \theta \in [-r, 0],
$$

with the initial condition $y(0) = \phi_0$ and $y(\theta) = \phi_1(\theta), \theta \in [-r, 0)$. In particular, if $\phi = (0, 0)$, then the unique mild solution (20) is the so-called stochastic convolution process

$$
y(t) := W_G(t) = \int_{0}^{t} G(t - s)BdW(s), \quad t \geq 0.
$$

For any $T > 0$, let $C^\beta([0, T]; H)$ denote the usual Banach space of all Hölder continuous functions on $H$ with order $\beta \in (0, 1)$. The following lemma is referred to Tanabe [11].

**Lemma 3.1.** Suppose that $a(\cdot)$ in (19) is Hölder continuous with order $\rho \in (0, 1)$, then operator $G(t)$ is strongly continuous in $H$ on each $[nr, (n+1)r], n = 0, 1, \cdots$. Moreover, the following estimates hold:

$$
\|G(t) - G(s)\| \leq C_{n, \kappa} \left( \frac{t - s}{s - nr} \right)^\kappa \quad \text{for all} \quad nr < s < t \leq (n + 1)r,
$$

where $\kappa \in (0, \rho)$ and $C_{n, \kappa} > 0$ are some constants depending on $n, \kappa$.

**Theorem 3.2.** Suppose that $a(\cdot)$ in (19) is Hölder continuous with order $\rho \in (0, 1)$. Let $T > 0$. Assume that $\text{Tr}(Q) < \infty$ and $B \in \mathcal{L}(K, H)$, the space of all bounded, linear operators from $K$ into $H$, then the trajectories of $W_G$ are in $C^\beta([0, T]; H)$ where

$$
\beta = \begin{cases}
\frac{1}{2} & \text{if } \rho > \frac{1}{2}, \\
\rho & \text{if } \rho \leq \frac{1}{2}.
\end{cases}
$$
Proof. It suffices to show this theorem for any $0 < s < t < T$ with $t - s < r$. To this end, it is easy to have that

$$
E\|W_G(t) - W_G(s)\|^2_H = \sum_{j=1}^{\infty} \lambda_j \int_s^t \|G(t - u)Be_j\|^2_H du + \sum_{j=1}^{\infty} \lambda_j \int_0^s \|(G(t - u) - G(s - u))Be_j\|^2_H du
$$

$$
=: I_1 + I_2.
$$

(23)

Since $G(\cdot)$ is strongly continuous on $\mathbb{R}$, it is easy to see, by the well-known Principle of Uniform Boundedness, that $G(\cdot)$ is norm bounded on $[0, T]$, and there exists a real number $C(T) > 0$ such that

$$
I_1 = \sum_{j=1}^{\infty} \lambda_j \int_s^t \|G(t - u)Be_j\|^2_H du
$$

$$
\leq C(T)\|B\|^2 \sum_{j=1}^{\infty} \lambda_j (t - s) = C(T)\|B\|^2 Tr(Q)(t - s).
$$

(24)

On the other hand, suppose that $s \in (Nr, (N + 1)r)$ for some $N \in \mathbb{N}$ and $t - s < r$. Then, for the item $I_2$ we have

$$
I_2 = \sum_{j=1}^{\infty} \lambda_j \int_0^s \|[(G(t - u) - G(s - u))Be_j]\|^2_H du
$$

$$
\leq \|B\|^2 Tr(Q) \sum_{k=0}^{N} \int_{kr}^{(k+1)r} \|G(t - u) - G(u)\|^2 du
$$

$$
\leq \|B\|^2 Tr(Q) \sum_{k=0}^{N} \int_{kr}^{(k+1)r} \|G(t - u) - G(u)\|^2 du
$$

$$
+ \|B\|^2 Tr(Q) \sum_{k=0}^{N} \int_{(k+1)r - (t - s)}^{(k+1)r} \|G(t - u) - G(u)\|^2 du.
$$

By using Lemma 3.1 for those values

$$
\beta < \begin{cases} 1/2 & \text{if } \rho > 1/2, \\ \rho & \text{if } \rho \leq 1/2, \end{cases}
$$

one can further obtain

$$
I_2 \leq \|B\|^2 Tr(Q) \sum_{k=0}^{N} \int_{kr}^{(k+1)r - (t - s)} C_{k,\beta}(t - s)^{2\beta}(u - kr)^{-2\beta} du
$$

$$
+ \|B\|^2 Tr(Q) \sum_{k=0}^{N} \int_{(k+1)r - (t - s)}^{(k+1)r} C_{k,\beta}(t - s)^{2\beta} \left[u - ((k + 1)r - (t - s))\right]^{-2\beta} du
$$

$$
\leq \|B\|^2 Tr(Q) \left(\frac{t - s)^{2\beta}}{1 - 2\beta} \sum_{k=1}^{N} C_{k,\beta}(r - (t - s))^{1 - 2\beta} + \sum_{k=1}^{N} C_{k,\beta}(t - s)^{-1 - 2\beta}\right)
$$

$$
\leq \|B\|^2 Tr(Q) \left[\frac{(t - s)^{2\beta}r^{1 - 2\beta}}{1 - 2\beta} \sum_{k=1}^{N} C_{k,\beta} + \frac{t - s}{1 - 2\beta} \sum_{k=1}^{N} C'_{k,\beta}\right]
$$
Since \( m \) is arbitrary, the trajectories of \( W_G \) are in \( C^\beta([0,T];H) \). The proof is complete.  \( \square \)

**Theorem 3.3.** Suppose that condition \((H)\) holds. Let \( T > 0 \) and \( \text{Tr}(Q) < \infty \). For \( \nu \leq \gamma < 1 \) and \( 0 < \beta < \frac{1}{2} - \gamma \), the trajectories of \( W_G \) are in \( C^\beta([0,T],D((-A)^\gamma)) \).

**Proof.** Once again, we intend to use the Kolmogorov test. Let \( 0 \leq s < t \leq T \) with \( s, t \in (Nr,(N+1)r) \) for some \( N > 0 \). Then, by definition, it follows easily that

\[
\mathbb{E}\|(-A)^\gamma W_G(t) - (-A)^\gamma W_G(s)\|_H^2
\]

\[
= \sum_{k=1}^{\infty} \lambda_k \int_s^t \|(-A)^\gamma G(t-u)e_k\|_H^2 du + \sum_{k=1}^{\infty} \lambda_k \int_0^s \|(-A)^\gamma [G(t-u) - G(s-u)]e_k\|_H^2 du
\]

\[= I_1 + I_2. \tag{27} \]

Now we estimate \( I_1 \) and \( I_2 \) separately. First, by using Theorem 2.1 we have for \( s, t \in (Nr,(N+1)r) \) with \( s < t \) that

\[
I_1 \leq \text{Tr}(Q) \int_0^{t-s} \|(-A)^\gamma G(v)\|_H^2 dv
\]

\[\leq \text{Tr}(Q) \int_0^{t-s} \frac{C_{2,\gamma}^2}{v^{2\gamma}} dv = \frac{\text{Tr}(Q)C_{2,\gamma}^2}{1 - 2\gamma} (t-s)^{1-2\gamma}, \quad C_{0,\gamma} > 0. \tag{28} \]

Since \( 0 < \beta < \frac{1}{2} - \gamma < 1 - \gamma \), it follows that \( 1 - 2\beta - 2\gamma > 0 \), and we thus employ Theorem 2.1 to obtain

\[
I_2 = \sum_{k=1}^{\infty} \lambda_k \int_0^s \|(-A)^\gamma [G(t-s+v) - G(v)]e_k\|_H^2 dv
\]

\[= \sum_{k=1}^{\infty} \lambda_k \sum_{j=0}^{N-1} \int_{j^r}^{(j+1)^r} \|(-A)^\gamma [G(t-s+v) - G(v)]e_k\|_H^2 dv \]
\[ + \sum_{k=1}^{\infty} \lambda_k \int_{N_T} (-A)^{\gamma} [G(t - s + v) - G(v)] e_k \|e_k\|^2_H dv \leq \text{Tr}(Q) \left( \sum_{j=0}^{N-1} \int_{jT}^{(j+1)T} C_{j,\beta,\gamma}^2 (t - s)^{2\beta} (v - jT)^{2\beta + 2\gamma} dv + \int_{N_T}^{\infty} C_{j,\beta,\gamma}^2 (t - s)^{2\beta} (v - N_T)^{2\beta + 2\gamma} dv \right) \]

where \([T]\) denotes the biggest integer less than or equal to \(T\). Hence, by substituting (28), (29) into (27) and using the Kolmogorov test, we then obtain the desired result. \(\square\)

4. Proof of Theorem 2.1 (a). We begin with establishing some useful lemmas.

Lemma 4.1. Let \(\gamma \in (0, 1)\). For any \(0 < s < t < \infty\), there exists a constant \(M_{\gamma} > 0\) such that

\[ \|(-A)^{\gamma} e^{tA} - (-A)^{\gamma} e^{sA}\| \leq M_{\gamma} \left( \frac{1}{s^\gamma} - \frac{1}{t^\gamma} \right), \]

\[ \|(-A)^{\gamma} Ae^{tA} - (-A)^{\gamma} Ae^{sA}\| \leq M_{\gamma} \left( \frac{1}{s^{\gamma+1}} - \frac{1}{t^{\gamma+1}} \right). \]

Proof. It is well-known that for any \(\gamma \in (0, 1)\) and \(k = 1, 2\), there exists a constant \(M_{\gamma,k} > 0\) such that

\[ \|(-A)^{\gamma} A^k e^{tA}\| \leq M_{\gamma,k}/t^{k+\gamma}, \quad t > 0. \]

Therefore, by using this estimate and the following equalities

\[ (-A)^{\gamma} e^{tA} - (-A)^{\gamma} e^{sA} = \int_s^t (-A)^{\gamma} Ae^{uA} du, \]
\[ (-A)^{\gamma} Ae^{tA} - (-A)^{\gamma} Ae^{sA} = \int_s^t (-A)^{\gamma} A^2 e^{uA} du, \]

one can easily obtain the desired results. The proof is complete now. \(\square\)

Corollary 1. Let \(\gamma \in (0, 1)\) and \(\beta > 0\). For any \(0 < s < t < \infty\), there exists a constant \(C_{\beta,\gamma} > 0\) such that

\[ \|(-A)^{\gamma} e^{tA} - (-A)^{\gamma} e^{sA}\| \leq C_{\beta,\gamma} (t - s)^\beta \cdot s^{-\beta-\gamma}. \]

Proof. First note that for any real numbers \(a \geq b \geq 0\) and \(0 < \delta \leq 1\), we have the following inequality

\[ a^\delta - b^\delta \leq (a - b)^\delta. \]

For \(0 < \beta \leq \gamma\) and any \(0 < s < t < \infty\), we have by using (33) that

\[ \frac{1}{s^\gamma} - \frac{1}{t^\gamma} = \frac{1}{s^\gamma} \frac{t^\gamma - s^\gamma}{t^\gamma} \leq \frac{1}{s^\gamma} \left( \frac{t^\gamma - s^\gamma}{t^\gamma} \right)^{\beta/\gamma} \leq (t - s)^\beta \cdot s^{-\beta-\gamma}. \]

On the other hand, for \(\beta > \gamma\) and \(0 < s < t < \infty\), we have

\[ \frac{1}{s^\gamma} - \frac{1}{t^\gamma} = \frac{t^\beta - s^\beta - \gamma}{s^\gamma \cdot t^\beta} \leq \frac{t^\beta - s^\beta - \gamma}{s^\gamma \cdot s^\beta} = (t - s)^\beta \cdot s^{-\beta-\gamma}. \]

Now we have by virtue of Lemma 4.1 the desired inequality. \(\square\)
To proceed further, let us denote by \( |·|_\infty \) the essential least upper bound norm in \( L^\infty([-r,0],[\mathbb{R}]) \), i.e.,
\[
|a|_\infty := \text{ess sup}_{\theta \in [-r,0]} |a(\theta)| \quad \text{for any} \quad a \in L^\infty([-r,0],[\mathbb{R}]).
\] (35)

For \( \gamma \in (0,1) \), we define
\[
\Gamma(t) = \int_0^t (-A)^\gamma e^{(t-s)A} a(-s) ds, \quad t \in [0,r].
\]

Proposition 1. The mapping \( \Gamma(\cdot) : [0,r] \to \mathcal{L}(X) \), the space of all bounded linear operators on \( X \), is uniformly bounded and for any \( 0 < \beta < 1 - \gamma \), there exists a number \( C_{\beta,\gamma} > 0 \) such that
\[
\|\Gamma(t) - \Gamma(s)\| \leq C_{\beta,\gamma}(t-s)^\beta, \quad 0 \leq s < t \leq r,
\] (36)
i.e., \( \Gamma \) is Hölder continuous on \( [0,r] \) with order \( \beta \in (0,1-\gamma) \).

Proof. First, it is easy to see by virtue of (9) that
\[
\|\Gamma(t)\| \leq |a|_\infty \int_0^t \|(-A)^\gamma e^{(t-s)A}\| \|ds \leq |a|_\infty \int_0^t \frac{M_\gamma}{(t-s)^\gamma} ds = \frac{|a|_\infty M_\gamma}{1-\gamma} t^{1-\gamma},
\]
which immediately implies
\[
\|\Gamma\|_\infty := \sup_{0 \leq t \leq r} \|\Gamma(t)\| \leq \frac{|a|_\infty M_\gamma r^{1-\gamma}}{1-\gamma} < \infty.
\]

Thus, \( \Gamma \) is uniformly bounded in \([0,r]\). To show the relation (36), we have for \( 0 \leq s < t \leq r \) that
\[
\|\Gamma(t) - \Gamma(s)\|
= \|\int_s^t (-A)^\gamma e^{(t-u)A} a(-u) du - \int_s^t (-A)^\gamma e^{(s-u)A} a(-u) du\|
\leq \|\int_s^t (-A)^\gamma e^{(t-u)A} a(-u) du\|
+ \|\int_s^t \left[ (-A)^\gamma e^{(t-u)A} a(-u) - (-A)^\gamma e^{(s-u)A} a(-u) \right] du\|
=: I_1 + I_2.
\] (37)

By using the relation (9), we easily obtain the inequality
\[
I_1 \leq |a|_\infty \int_s^t \|(-A)^\gamma e^{(t-u)A}\| du
\leq |a|_\infty \int_s^t \frac{M_\gamma}{(t-u)\gamma} du = \frac{|a|_\infty M_\gamma}{1-\gamma} (t-s)^{1-\gamma}.
\] (38)

On the other hand, we can apply Corollary 1 to (37) to obtain
\[
I_2 \leq |a|_\infty \int_0^s \|(-A)^\gamma e^{(t-u)A} - (-A)^\gamma e^{(s-u)A}\| du
\leq |a|_\infty M_\gamma (t-s)^\beta \int_0^s (s-u)^{-\gamma-\beta} du
\leq \frac{|a|_\infty M_\gamma (t-s)^\beta}{1-\gamma-\beta} r^{1-\gamma-\beta}.
\] (39)
Hence, substituting (38) and (39) into (37), we immediately obtain the desired result.

To show Theorem 2.1 (a), we intend to develop an induction scheme. We first consider the case that \( n = 0 \) and set
\[
V(t) = (-A)^\gamma (G(t) - e^{tA}), \quad t \in [0, r].
\]
Then, the integral equation to be satisfied by \( V(t) \) is
\[
V(t) = V_0(t) + \int_0^t \Gamma(t - s)A_2(-A)^{-\gamma}V(s)ds, \quad t \in [0, r],
\]
where
\[
V_0(t) = \int_0^t \Gamma(t - s)A_2 e^{sA}ds = \int_0^t (\Gamma(t - s) - \Gamma(t))A_2 e^{sA}ds + \Gamma(t)A_2 A^{-1}(e^{tA} - I), \quad t \in [0, r].
\]

Since \( \mathcal{D}(A) \subset \mathcal{D}(A_i) \), we have \( A_i A^{-1} \in \mathcal{L}(X), \ i = 1, 2 \). Then for any \( 0 < \beta < 1 - \gamma \), it follows by virtue of Proposition 1 and (8) that
\[
\|V_0(t)\| \leq \int_0^t \|\Gamma(t - s) - \Gamma(t)\|A_2 A^{-1}A_2 e^{sA}\|ds + \|\Gamma(t)\|\|A_2 A^{-1}\|\|e^{tA}\| + 1
\]
\[\leq \int_0^t C_{\beta, \gamma} M\|A_2 A^{-1}\|s^{\beta - 1} ds + \|\Gamma\|\|A_2 A^{-1}\|(M + 1)
\]
\[\leq C_{\beta, \gamma} M\|A_2 A^{-1}\|\beta r^\beta + \|\Gamma\|\|A_2 A^{-1}\|(M + 1), \quad t \in [0, r].
\]
Hence, \( V_0(\cdot) \) is uniformly bounded on \([0, r]\). Since \( \mathcal{D}((-A)^\nu) \subset \mathcal{D}(A_2) \) and \( \gamma \geq \nu \) by assumption, we have
\[
\|A_2(-A)^{-\gamma}\| \leq \|A_2(-A)^{-\nu}\| \cdot \|(-A)^{-(\gamma-\nu)}\| < \infty.
\]
Hence, by virtue of the well-known Gronwall lemma and (40) that
\[
v_0 := \sup_{0 \leq t \leq r} \|V(t)\| < \infty.
\]
Note that
\[
(-A)^\gamma G(t) = (-A)^\gamma e^{tA} + V(t), \quad t \in [0, r].
\]
Hence, for \( t \in (0, r] \) we have
\[
\|(-A)^\gamma G(t)\| \leq \|(-A)^\gamma e^{tA}\| + \|V(t)\|
\]
\[\leq M_\gamma/t^\gamma + v_0 \leq (M_\gamma + v_0 r^\gamma)/t^\gamma, \quad M_\gamma > 0,
\]
which is (14) with \( n = 0 \). In a similar manner, for any \( 0 < s < t \leq r \), we have from (9) and (42) that
\[
\left\| \int_s^t (-A)^\gamma G(u)du \right\| \leq \int_s^t \|(-A)^\gamma e^{uA}\|du + \int_s^t \|V(u)\|du
\]
\[\leq \int_s^t M_\gamma u^\gamma du + v_0(t - s) \leq \frac{M_\gamma \cdot r^{1-\gamma}}{1-\gamma} + rv_0 < \infty,
\]
which is (15) with \( n = 0 \).
Now suppose the fundamental solution $G(\cdot)$ satisfies all the estimates in Theorem 2.1 on the intervals $[0, r], \cdots, [(n-1)r, nr]$. Then in the interval $[nr, (n+1)r]$, the integral equation to be satisfied by

$$V(t) = (-A)^\gamma (G(t) - \int_{nr}^t e^{(t-s)A}A_1G(s-r)ds), \quad t \in [nr, (n+1)r],$$

is

$$V(t) = V_0(t) + \int_{nr}^t \Gamma(t-u)A_2(-A)^{-\gamma}V(u)du,$$

where

$$V_0(t) = (-A)^\gamma e^{tA} + (-A)^\gamma \int_{r}^{nr} e^{(t-s)A}A_1G(s-r)ds$$

$$+ \int_0^{(t-r)} e^{(t-r-u)A}\Gamma(r)A_2G(u)du$$

$$+ \int_{t-r}^{nr} \Gamma(t-u)A_2G(u)du + \int_{nr}^t \Gamma(t-u)A_2 \int_{nr}^u e^{(u-s)A}A_1G(s-r)dsdu$$

$$=: I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t).$$

Now we estimate each term on the right hand side of (43). First of all, it is immediate that

$$\| I_1(t) \| \leq \frac{M_\gamma}{t^\gamma} \leq \frac{M_\gamma}{(nr)^\gamma}, \quad t \in (nr, (n+1)r],$$

and

$$I_2(t) = \sum_{j=1}^{n-1} (-A)^\gamma \int_{jr}^{(j+1)r} e^{(t-s)A}A_1G(s-r)ds, \quad t \in (nr, (n+1)r].$$

By virtue of condition (H), for any $\mu \vee \gamma \leq \tau < 1$,

$$\mathcal{D}((-A)^\tau) \subset \mathcal{D}(A_i) \quad \text{and} \quad A_i(-A)^{-\tau} \in \mathcal{L}(X), \quad i = 1, 2.$$

For arbitrary $t \in (nr, (n+1)r], j = 0, 1, \cdots, n-2$, and $\gamma \vee \mu + \delta < 1$ with $\delta > 0$ sufficiently small, it is immediate to see, by the induction assumption and Corollary 4.1, that

$$\left\| (-A)^\gamma \int_{jr}^{(j+1)r} e^{(t-s)A}A_1G(s-r)ds \right\|$$

$$\leq \int_{jr}^{(j+1)r} \left\| (-A)^\gamma e^{(t-s)A} - (-A)^\gamma e^{(t-jr)A} \right\| \cdot \left\| A_1(-A)^{-(\gamma \vee \mu + \delta)} \right\| ds$$

$$+ \left\| (-A)^\gamma e^{(t-jr)A} \right\| \cdot \left\| A_1(-A)^{-(\gamma \vee \mu + \delta)} \right\| \cdot \left\| \int_{jr}^{(j+1)r} (-A)^{\gamma \vee \mu + \delta}G(s-r)ds \right\|$$

$$\leq C_{\gamma, \mu, \delta} \int_{jr}^{(j+1)r} \frac{(s-jr)^\gamma}{(t-s)^{2\gamma}} \cdot \frac{C_j'}{(s-jr)^{\gamma \vee \mu + \delta}} ds + M_{\gamma, \mu, \delta} C_j' \cdot \frac{1}{(t-jr)^\gamma}$$

$$\leq C_{\gamma, \mu, \delta} C_j' \int_{jr}^{(j+1)r} \frac{(s-jr)^{\gamma \vee \mu - \delta}}{(s-jr)^{\gamma \vee \mu + \delta - \gamma}} ds + M_{\gamma, \mu, \delta} C_j' \cdot \frac{1}{(t-jr)^\gamma}$$
Hence, (48) follows immediately from (49). Now we re-write (44), (45) and (46), we have for

\[ C = \min_{\gamma < 1 - \gamma, \delta > 0} \left( \frac{1}{\gamma} \right) \]

where

\[ C_{\beta, \gamma, \delta}, M_{\delta, \gamma}, C_{n-1} > 0 \]

and \( B(\cdot, \cdot) \) is the standard Beta function. Combining (44), (45) and (46), we have for \( t \in (nr, (n+1)r] \) that

\[ \|I_2(t)\| \leq \sum_{j=1}^{n-2} \left( C_{\beta, \gamma, \delta} C_{n-1} \frac{t^{1-\gamma}}{1 - (\gamma + \mu + \delta - \gamma)} + \frac{M_{\beta, \gamma, \delta} C_{n-1}^\prime}{r^\gamma} \right) \]

To estimate \( I_3(t) \), we first note that

\[ \|e^{tA} - e^{sA}\| \leq C_{\alpha} (t - s)^{\alpha} \cdot s^{-\alpha}, \quad C_{\alpha} > 0, \]

for any \( 0 < s < t < \infty \) and \( \alpha \in (0, 1] \). Indeed, by virtue of (8), we have for any \( 0 < s < t \) that

\[ \|e^{tA} - e^{sA}\| \leq M \int_s^t \frac{1}{u} du \]

\[ = M \log(t/s) = M \log \left( 1 + \frac{t - s}{s} \right). \]

It is elementary to see that

\[ \log(1 + a) \leq a^{\alpha} / \alpha \quad \text{for any} \quad a > 0, \quad 0 < \alpha \leq 1. \]

Hence, (48) follows immediately from (49). Now we re-write \( I_3(t) \) as

\[ \int_0^{t-r} e^{(t-u-r)A} \Gamma(r) A_2 G(u) du \]

\[ = \sum_{j=0}^{n-2} \int_{jr}^{(j+1)r} e^{(t-s)A} \Gamma(r) A_2 G(u) du + \int_{(n-1)r}^{t-r} e^{(t-u-r)A} \Gamma(r) A_2 G(u) du. \]
For $j = 0, 1, \ldots, n - 2$, we have by the induction assumption and (48) that for sufficiently small $\delta > 0$,
\[
\left\| \int_{j r}^{(j+1) r} e^{(t-u-r)A} \Gamma(r) A_2 G(u) du \right\| \\
\leq \int_{j r}^{(j+1) r} \left\| e^{(t-u-r)A} - e^{(t-jr-r)A} \right\| \cdot \left\| \Gamma(r) \right\| \cdot \left\| A_2(-A)^{\gamma,\delta} \right\| \cdot \left\| (-A)^{1-\delta} G(u) \right\| du \\
+ \left\| e^{(t-jr-r)A} \right\| \cdot \left\| \Gamma(r) \right\| \cdot \left\| A_2(-A)^{\gamma,\delta} \right\| \cdot \left\| (-A)^{1-\delta} G(u) \right\| du \\
\leq \int_{j r}^{(j+1) r} \frac{\gamma}{r} \left\| \Gamma(r) \right\| \cdot \left\| A_2(-A)^{\gamma,\delta} \right\| \cdot \left\| (-A)^{1-\delta} G(u) \right\| du \\
\leq C_{\gamma,\delta} B(1-\gamma, \gamma + \delta) + M_\delta C_n' \cdot C_{\gamma,\delta} > 0, C_n' > 0, M_\delta > 0.
\]
(52)

In a similar manner, we have for $t \in (nr, (n+1)r]$ and sufficiently small $\delta > 0$ that
\[
\left\| \int_{(n-1)r}^{tr} e^{(t-u-r)A} \Gamma(r) A_2 G(u) du \right\| \\
\leq \int_{(n-1)r}^{tr} \left\| e^{(t-u-r)A} - e^{(t-(n-1)r)A} \right\| \cdot \left\| \Gamma(r) \right\| \cdot \left\| A_2(-A)^{\gamma,\delta} \right\| \cdot \left\| (-A)^{1-\delta} G(u) \right\| du \\
+ \left\| e^{(t-(n-1)r)A} \right\| \cdot \left\| \Gamma(r) \right\| \cdot \left\| A_2(-A)^{\gamma,\delta} \right\| \cdot \left\| (-A)^{1-\delta} G(u) \right\| du \\
\leq C_{\gamma,\delta} \int_{(n-1)r}^{tr} \frac{\gamma}{r} \left\| \Gamma(r) \right\| \cdot \left\| A_2(-A)^{\gamma,\delta} \right\| \cdot \left\| (-A)^{1-\delta} G(u) \right\| du \\
= C_{\gamma,\delta} B(1-\gamma, \gamma + \delta) + M_\delta C_n' \cdot C_{\gamma,\delta} > 0, C_n' > 0, M_\delta > 0.
\]
(53)

By virtue of (51), (52) and (53), we thus obtain for $t \in (nr, (n+1)r]$ that
\[
\left\| \int_{0}^{tr} e^{(t-u-r)A} \Gamma(r) A_2 G(u) du \right\| \leq (M_\delta + C_{\gamma,\delta} B(1-\gamma, \gamma + \delta)) \left( \sum_{j=0}^{n} C_{j,\delta}^r + 1 \right).
\]
(54)

Now we intend to estimate $I_4(t)$. To this end, we have by virtue of (H), (36) and induction assumption that for sufficiently small $\delta > 0$,
\[
\left\| \int_{t-r}^{nt} \Gamma(t-u) A_2 G(u) du \right\| \\
\leq \int_{t-r}^{nt} \left\| \Gamma(t-u) - \Gamma(r) \right\| \cdot \left\| A_2(-A)^{\gamma,\delta} \right\| \cdot \left\| (-A)^{1-\delta} G(u) \right\| du \\
+ \left\| \Gamma(r) \right\| \cdot \left\| A_2(-A)^{\gamma,\delta} \right\| \cdot \left\| (-A)^{1-\delta} G(u) \right\| du \\
\leq \int_{t-r}^{nt} \frac{\gamma}{r} \left\| \Gamma(r) \right\| \cdot \left\| A_2(-A)^{\gamma,\delta} \right\| \cdot \left\| (-A)^{1-\delta} G(u) \right\| du \\
\leq \frac{C_{\beta,\gamma,\delta} C_n^{r+1} \beta}{\beta + \delta} + \left\| \Gamma(r) \right\| \cdot C_n'
\]
where $0 < \beta < 1 - \gamma$ and $C_{\beta,\gamma,\delta}, C_n' > 0$. 

where \(0 < \tilde{\beta} < 1 - \nu\). Hence, we have for \(t \in (nr, (n+1)r]\) that

\[
\|I_5(t)\| \leq \left(\int_{nr}^t \|\Gamma(t-u)\| \cdot \|A_2(-A)^{-\nu}\| \cdot \|(-A)^\nu \int_{nr}^u e^{(u-s)A}A_1G(s-r)ds\|\right) du \\
\leq \left(\int_{nr}^t \|A_2(-A)^{-\nu}\| \left(\int_{nr}^u \frac{C_{\beta,\nu,\delta}^\nu C_{\beta,n}^\nu (s-nr)^{\tilde{\beta}+\delta-1}}{(s-nr)^{\nu+\tilde{\beta}}} ds du + \int_{nr}^t \frac{C_{\beta,n}^\nu M_\nu}{(u-nr)^\nu} du\right)
\leq \frac{r^{1-\nu} \|A_2(-A)^{-\nu}\|}{1-\nu-\tilde{\beta}} \cdot \frac{C_{\beta,\nu,\delta}^\nu C_{\beta,n}^\nu}{\tilde{\beta} + \delta} + \frac{C_{\beta,n}^\nu C_{\gamma,n}^\nu M_\nu}{1-\nu}.
\]

By combining (47), (54) with (56) and noticing (41), we obtain that \(V_0(t)\) and \(V(t)\) both are uniformly bounded in \([nr, (n+1)r]\).

Finally, for \(nr < t \leq (n+1)r\), we have by (H) and induction assumption that for sufficiently small \(\delta > 0\),

\[
\|(-A)^\gamma \int_{nr}^t e^{(t-s)A}A_1G(s-r)ds\| \\
\leq \|\int_{nr}^t e^{(t-s)A}A_1(-A)^{(1-\delta)}(-A)^{1-\delta}G(s-r)ds\| \\
\leq \|\int_{nr}^t e^{(t-s)A}A_1(-A)^{(1-\delta)}(-A)^{1-\delta}G(s-r)ds\| \\
\leq C_{\beta,\gamma,n}^\nu \|A_1(-A)^{-(1-\delta)}\| \int_{nr}^t (t-s)^{1-\delta^{1-\gamma-1} \cdot (s-nr)^{\beta^{1-\delta^{1+\delta}}} ds \\
+ \frac{C_{\beta,n}^\nu \|A_1(-A)^{-(1-\delta)}\| M_{\gamma,\delta}}{(t-nr)^\gamma} \\
\leq \left\{C_{\beta,\gamma,n}^\nu \cdot B(1-\beta-\gamma, \beta + \delta) + C_{\gamma,n}^\nu \|A_1(-A)^{-(1-\delta)}\| M_{\gamma,\delta}\right\}/(t-nr)^\gamma,
\]

where \(C_{\beta,\gamma,n}^\nu, C_{\beta,\gamma,n}^\nu, C_{\gamma,n}^\nu, M_{\gamma,\delta} > 0, 0 < \beta < 1 - \gamma\) and \(B(\cdot, \cdot)\) is the Beta function.

Now the inequality (14) for \(t \in (nr, (n+1)r]\) follows from (57) and the equality

\[
(-A)^\gamma G(t) = (-A)^\gamma \int_{nr}^t e^{(t-s)A}A_1G(s-r)ds + V(t).
\]
On the other hand, by using (57), we have that
\[
\left\| \int_{nr}^t (A)^\gamma G(u) du \right\| \leq \int_{nr}^t \left\| (A)^\gamma \int_{nr}^u e^{(u-s)A} A_1 G(s-r) ds \right\| du + \int_{nr}^t \|V(u)\| du \\
\leq C_{\gamma,n}^{\gamma} \left( \frac{t^{1-\gamma}}{1-\gamma} + 1 \right), \quad C_{\gamma,n}^{\gamma} > 0,
\]
is uniformly bounded in \([nr,(n+1)r]\), a fact which implies (15). The proof is complete now.

5. Proof of Theorem 2.1 (b). We still want to develop an induction scheme here. First, consider \(V(t)\) and \(V_0(t)\), \(t \in [0,r]\), defined in (40).

**Lemma 5.1.** For any \(0 < s < t < r\) and \(0 < \beta < 1 - \gamma\), there exists a constant \(M_{\beta,\gamma} > 0\) such that
\[
\|V(t) - V(s)\| \leq M_{\beta,\gamma} (t-s)^\beta s^{-\beta}. \tag{59}
\]

**Proof.** We notice, by definition, for \(0 < s < t < r\) that
\[
V_0(t) - V_0(s) = \int_s^t \Gamma(t-u) A_2 e^{uA} du + \int_0^s (\Gamma(t-u) A_2 e^{uA} - \Gamma(s-u) A_2 e^{uA}) du \\
=: I_1(s,t) + I_2(s,t). \tag{60}
\]
For \(0 < s < t < r\), it is easy to see from (50) that
\[
\|I_1(s,t)\| \leq \int_s^t \|\Gamma(t-u)\| \cdot \|A_2 A^{-1}\| \cdot \|A e^{uA}\| du \\
\leq M \|\Gamma\|_{\infty} \cdot \|A_2 A^{-1}\| \cdot \ln \left( 1 + \frac{t-s}{s} \right) \leq M \|\Gamma\|_{\infty} \cdot \|A_2 A^{-1}\| \frac{t-s}{\beta} s^{-\beta}, \tag{61}
\]
where \(M > 0\) and by Proposition 1, we have for \(0 < s < t < r\) that
\[
\|I_2(s,t)\| \leq \int_0^s \|\Gamma(t-u) - \Gamma(s-u)\| \cdot \|A_2 (-A)^{-\nu}\| \cdot \|(-A)^\nu e^{uA}\| du \\
\leq C_{\beta,\gamma} \int_0^s \|t-s\|^\beta \|A_2 (-A)^{-\nu}\| \cdot \frac{M_{\nu}}{u^{\nu}} du \\
\leq C_{\beta,\gamma} M_{\nu} \frac{\|A_2 (-A)^{-\nu}\|}{1-\nu} r^{1-\nu+\beta} (t-s)^\beta s^{-\beta}, \quad C_{\beta,\gamma} > 0, \quad M_{\nu} > 0. \tag{62}
\]
Hence, by combining (60), (61) and (62), we get for \(0 < s < t < r\) that
\[
\|V_0(t) - V_0(s)\| \leq M_{\beta,\gamma} (t-s)^\beta s^{-\beta}, \quad M_{\beta,\gamma} > 0,
\]
from which the desired result follows by (41) and the well-known Gronwall inequality. \(\square\)

By virtue of Corollary 1 and (59), we obtain the estimates in Theorem 2.1 for \(0 < s < t < r\), \(0 < \beta < 1 - \gamma\):
\[
\|(-A)^\gamma (G(t) - G(s))\| \leq \|(-A)^\gamma e^{tA} - (-A)^\gamma e^{sA}\| + \|V(t) - V(s)\| \\
\leq M_{\beta,\gamma} [(t-s)^\beta \cdot s^{-\beta+\gamma} + (t-s)^\beta s^{-\beta}] \\
\leq (M_{\beta,\gamma} + M_{\beta,\gamma} r^\gamma) (t-s)^\beta \cdot s^{-\beta-\gamma}, \quad M_{\beta,\gamma} > 0.
\]
Now suppose that $G(t)$ satisfies those estimates in Theorem 2.1 on the intervals $[0, r], \cdots , [(n - 1)r, nr]$. Then in the interval $[nr, (n + 1)r]$, the integral equation to be satisfied by

$$V(t) = (A)^{\gamma} \left( G(t) - \int_{nr}^{t} e^{(t-s)A} A_1 G(s-r) ds \right), \quad t \in (nr, (n + 1)r], \quad (63)$$

is

$$V(t) = V_0(t) + \int_{nr}^{t} \Gamma(t-u) A_2 (A)^{-\gamma} V(u) du,$$

where $V_0(t)$ is given as in (43). We first show the Hölder continuity of $V_0(t)$ and $V(t)$. Let $nr < s < t < (n + 1)r$. By virtue of (43), we have

$$V_0(t) - V_0(s) = (A)^{\gamma} \left( e^{sA} - e^{rA} \right)$$

$$+ (A)^{\gamma} \left( \int_{nr}^{r} e^{(t-u)A} A_1 G(u-r) du - \int_{nr}^{nr} e^{(s-u)A} A_1 G(u-r) du \right)$$

$$+ \left( \int_{0}^{r-t} e^{(t-u)A} \Gamma(r) A_2 G(u) du - \int_{0}^{s-r} e^{(s-u)A} \Gamma(r) A_2 G(u) du \right)$$

$$+ \left( \int_{nr}^{t} \Gamma(t-u) A_2 \int_{nr}^{u} e^{(u-v)A} A_1 G(v-r) dv du \right.$$

$$- \int_{nr}^{s} \Gamma(s-u) A_2 \int_{nr}^{u} e^{(u-v)A} A_1 G(v-r) dv du \left.) \right).$$

$$= : I_1(t,s) + I_2(t,s) + I_3(t,s) + I_4(t,s) + I_5(t,s).$$

First, by virtue of Corollary 4.1 we have

$$\|I_1(t,s)\| \leq M_\gamma (t-s)^{\beta} (s-nr)^{-\delta}, \quad M_\gamma > 0.$$  \hspace{1cm} (65)

Now let us consider the item $I_2(t,s)$. For sufficiently small $\delta > 0$, there exists, by virtue of (46) and (48), a value $c_{n, \beta, \gamma, \delta} > 0$ such that

$$\|I_2(t,s)\| \leq \|e^{(t-nr)A} - e^{(s-nr)A}\| \cdot \sum_{k=2}^{n} \left\| (A)^{\gamma} \int_{(k-1)r}^{kr} e^{(n-r-u)A} A_1 G(u) du \right\|$$

$$\leq c_{n, \beta, \gamma, \delta} (t-s)^{\beta} (s-nr)^{-\delta}. \quad (66)$$

As for the third term on the right side of (64), we have

$$I_3(t,s) = \int_{s-r}^{t-r} e^{(t-u)A} \Gamma(r) A_2 G(u) du$$

$$+ \int_{(n-1)r}^{s-r} \left( e^{(t-u)A} - e^{(s-r-u)A} A_1 G(s-r) du + e^{(s-nr)A} \right) \Gamma(r) A_2 G(u) du$$

$$+ \left( e^{(t-nr)A} - e^{(s-nr)A} \right) \Gamma(r) \int_{(n-1)r}^{s-r} A_2 G(u) du.$$
By virtue of (9) and induction assumption, we have for sufficiently small \( \delta > 0 \)
\begin{equation}
J_1(t, s) + J_2(t, s) + J_3(t, s) + J_4(t, s).
\end{equation}

On the other hand, by virtue of (48) we have for \((n - 1)r < u < s - r\) and sufficiently small \(\delta > 0\) that
\begin{align*}
\|e^{(t-r)A} - e^{(s-r)A} - e^{(t-nr)A} + e^{(s-nr)A}\| \\
\leq \|e^{(t-r-u)A} - e^{(s-r-u)A}\| + \|e^{(t-nr)A} - e^{(s-nr)A}\|
\leq c_\beta,\gamma,\delta (t-s)^{\beta+\delta} (s-r-u)^{\beta+\delta} + c_\alpha,\gamma,\delta (u-(n-1)r)^{\beta+\delta}
\leq 2c_\beta,\gamma,\delta (t-s)^{\beta+\delta} (s-r-u)^{\beta+\delta},
\end{align*}
where \(c_\beta,\gamma,\delta > 0\). Combining these two inequalities, it follows that
\begin{equation}
\|e^{(t-r-u)A} - e^{(s-r-u)A} - e^{(t-nr)A} + e^{(s-nr)A}\| \\
\leq C_\beta,\delta (t-s)^{\beta} (s-r-u)^{-\beta+\delta} (u-(n-1)r)^{\delta},
\end{equation}
for \(0 < \beta < 1 - \gamma\) and sufficiently small \(\delta > 0\). With the aid of (71), we have for sufficiently small \(\delta > 0\) that
\begin{align*}
\|A_2(-A)^{-r}\| \cdot \|\Gamma(r)\| \cdot \frac{C_{n-1,\delta}}{(u-(n-1)r)^{\nu}} du
\end{align*}
\begin{align*}
&= C_{\beta, \delta}' \|\Gamma(r)\| (t - s)^{\beta} \int_{(n-1)r}^{s-r} (s - r - u)^{-(\beta + \delta)} (u - (n-1)r)^{\delta - \nu} du \\
&= c_{\beta, \gamma, \delta, n}' B(1 - \beta - \delta, 1 + \delta - \nu) (t - s)^{\beta} (s - nr)^{-\beta}, \quad c_{\beta, \gamma, \delta, n}' > 0.
\end{align*}

In a similar manner, one can have by virtue of (49) and (50) that
\begin{align*}
\|J_3(t, s)\| \\
&\leq \|e^{(t-nr)A} - e^{(s-nr)A}\| \cdot \|\Gamma(r)\| \cdot \|A_2(-A)^{-\nu}\| \cdot \left\| \int_{(n-1)r}^{s-r} (-A)^{-\nu} G(u) du \right\|
\end{align*}
\begin{align*}
&\leq c_{\beta, \nu, n}' \|\Gamma(r)\| \left( \frac{t - s}{s - nr} \right)^{\beta}, \quad c_{\beta, \nu, n}' > 0,
\end{align*}
and
\begin{align*}
\|J_4(t, s)\| \\
&\leq \|e^{(t-nr)A} - e^{(s-nr)A}\| \cdot \|\Gamma(r)\| \cdot \|A_2(-A)^{-\nu}\| \left( \sum_{j=0}^{n-2} \int_{jr}^{(j+1)r} \|(-A)^{-\nu} G(u)\| du \right)
\end{align*}
\begin{align*}
&\leq c_{\beta, \nu, n}' \left( \frac{t - s}{s - nr} \right)^{\beta}, \quad c_{\beta, \nu, n}' > 0.
\end{align*}

Combining (67)-(74), we conclude that for some \(M_{\beta, \gamma, \nu}' > 0\),
\begin{align*}
\left\| \int_{t-r}^{t-r} e^{(t-r-u)A} \Gamma(r) A_2 G(u) du - \int_{t-r}^{t-r} e^{(s-r-u)A} \Gamma(r) A_2 G(u) du \right\|
\leq M_{\beta, \gamma, \nu}' (t - s)^{\beta} (s - nr)^{-\beta}.
\end{align*}

For the item \(I_4(t, s)\), let \((n - 1)r < s - r < t - r < u < nr\) and by Proposition 1, (50), (68) and induction assumption, we may obtain for sufficiently small \(\delta > 0\) that
\begin{align*}
&\left\| \int_{t-r}^{t-r} \Gamma(t - u) A_2 G(u) du - \int_{s-r}^{t-r} \Gamma(s - u) A_2 G(u) du \right\|
\end{align*}
\begin{align*}
&= \left\| \int_{t-r}^{t-r} (\Gamma(t - u) - \Gamma(s - u)) (A_2(-A)^{-\nu})(-A)^{-\nu} G(u) du \\
&- \int_{s-r}^{t-r} \Gamma(s - u) (A_2(-A)^{-1-\delta})(-A)^{-1-\delta} G(u) du \right\|
\end{align*}
\begin{align*}
&\leq C_{n, \beta, \gamma, \nu} \int_{t-r}^{t-r} (t - s)^{\beta} \cdot \frac{1}{(u - (n-1)r)^{\delta}} du \\
&+ \|\Gamma\| \infty C_{n-1, \nu, \delta} \int_{s-r}^{t-r} \frac{1}{(u - (n-1)r)^{1-\delta}} du
\end{align*}
\begin{align*}
&\leq C_{n, \beta, \gamma, \nu} (t - s)^{\beta} \int_{t-r}^{t-r} \frac{1}{(u - (n-1)r)^{\delta}} du \\
&+ \|\Gamma\| \infty C_{n-1, \nu, \delta} (t - s)^{\beta} \int_{s-r}^{t-r} \frac{1}{(u - (n-1)r)^{1-\delta}} du
\end{align*}
\begin{align*}
&\leq C_{n, \beta, \gamma, \nu} (t - s)^{\beta} 1 - \nu + C_{n-1, \gamma, \delta} B(1 - \beta, \delta)(t - s)^{\beta} (t - nr)^{-\beta + \delta}
\end{align*}
\begin{align*}
&\leq C_{n, \beta, \gamma, \delta} (t - s)^{\beta} (s - nr)^{-\beta}, \quad C_{n, \beta, \gamma}'' > 0.
\end{align*}
In a similar way, we can have
\[
\left\| \int_{nr}^{t} \Gamma(t-u)A_{2} \int_{nr}^{u} e^{(u-v)A} A_{1} G(v-r)dvdu \right. \\
- \left. \int_{nr}^{s} \Gamma(s-u)A_{2} \int_{nr}^{u} e^{(u-v)A} A_{1} G(v-r)dvdu \right. \\
\leq C_{n,\beta,\gamma}^{m} (t-s)^{\beta} (s-nr)^{-\beta}, \quad C_{n,\beta,\gamma}^{m} > 0.
\]

Combining (64)-(76), we conclude that
\[
\|V_{0}(t) - V_{0}(s)\| \leq M_{n,\beta,\gamma}'(t-s)^{\beta} (s-nr)^{-\beta}, \quad M_{n,\beta,\gamma}' > 0,
\]
and further we have
\[
\|V(t) - V(s)\| \leq M_{n,\beta,\gamma}''(t-s)^{\beta} (s-nr)^{-\beta}, \quad M_{n,\beta,\gamma}'' > 0,
\]
for \(nr < s < t < (n+1)r\).

On the other hand, we have by assumption that
\[
(\mathcal{A})^{\gamma} \int_{nr}^{t} e^{(t-u)A} A_{1} G(u-r)du - (\mathcal{A})^{\gamma} \int_{nr}^{s} e^{(s-u)A} A_{1} G(u-r)du \\
= \int_{s}^{t} (\mathcal{A})^{\gamma} e^{(t-u)A} (A_{1} (\mathcal{A})^{-(1-\delta)}) (\mathcal{A})^{1-\delta} G(u-r)du \\
+ \int_{nr}^{s} (\mathcal{A})^{\gamma} (e^{(t-u)A} - e^{(s-u)A}) (A_{1} (\mathcal{A})^{-(1-\delta)}) (\mathcal{A})^{1-\delta} G(u-r)du \\
=: K_{1}(t,s) + K_{2}(t,s),
\]
for any \(nr < s < t < (n+1)r\) and sufficiently small \(\delta > 0\). Hence, we have by the induction hypothesis and Theorem 2.1 that
\[
\|K_{1}(t,s)\| \leq \int_{s}^{t} \| (\mathcal{A})^{\gamma} e^{(t-u)A} \| \cdot \| A_{1} (\mathcal{A})^{-(1-\delta)} \| \cdot \| (\mathcal{A})^{1-\delta} G(u-r) \| du \\
\leq M (t-s)^{\beta} \int_{nr}^{t} (t-u)^{-\beta-\gamma} \| A_{1} (\mathcal{A})^{-(1-\delta)} \| \cdot \frac{C_{n,\nu}}{(u-nr)^{1-\delta}} du \\
\leq MC_{n,\nu} \| A_{1} (\mathcal{A})^{-(1-\delta)} \| \cdot B(1-\beta-\gamma, \delta)(t-nr)^{-\beta-\gamma-1+\delta}(t-s)^{\beta} \\
\leq M_{n,\beta,\nu,\gamma,\delta}'(t-s)^{\beta} \cdot (s-nr)^{-\beta-\gamma}, \quad M_{n,\beta,\nu,\gamma,\delta}' > 0.
\]

In a similar manner, we have
\[
\|K_{2}(t,s)\| \leq \int_{nr}^{s} M \| A_{1} (\mathcal{A})^{-(1-\delta)} \| \cdot (t-s)^{\beta} \cdot (s-nr)^{-\beta-\gamma}(u-nr)^{-(1-\delta)} du \\
\leq \frac{MC_{n,\nu} \| A_{1} (\mathcal{A})^{-(1-\delta)} \| \cdot (t-s)^{\beta}}{\delta} \cdot (s-nr)^{-\beta-\gamma}(s-nr)^{\delta} \\
\leq M_{n,\delta,\beta,\gamma}''(t-s)^{\beta}(s-nr)^{-\beta-\gamma}, \quad M_{n,\delta,\beta,\gamma}'' > 0.
\]

With the aid of (78)-(80), we thus obtain
\[
\left\| (\mathcal{A})^{\gamma} \int_{nr}^{t} e^{(t-u)A} A_{1} G(u-r)du - (\mathcal{A})^{\gamma} \int_{nr}^{s} e^{(s-u)A} A_{1} G(u-r)du \right. \\
\leq C_{n,\beta,\gamma} \cdot (t-s)^{\beta} (s-nr)^{-\beta-\gamma}
\]
\]
\[
(81)
\]
for some $C_{n,\bar{z},\gamma} > 0$. Now combining (63), (77) and (81), we finally obtain (16). The proof is thus complete.

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Received for publication April 2019.

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