We develop a Lie algebraic approach to systematically calculate the evolution operator of the generalized two-dimensional quadratic Hamiltonian with time-dependent coefficients. Although the development of the Lie algebraic approach presented here is mainly motivated by the two-dimensional quadratic Hamiltonian, it may be applied to investigate the evolution operators of any Hamiltonian having a dynamical algebra with a large number of elements. We illustrate the method by finding the propagator and the Heisenberg picture position and momentum operators for a two-dimensional charge subject to uniform and constant electro-magnetic fields.
I. INTRODUCTION

In many applications as radio-frequency ion traps [1–8], quantum optics [9–12], cosmology [13, 14], quantum field theory [15], quantum dissipation [16–22], magneto transport in lateral heterostructures [23–26] and even gravitational waves [27] the time evolution of particles in quadratic potentials is frequently examined. The one-dimensional, generalized time-dependent quadratic Hamiltonian is given by

$$\hat{H} = a_1(t) + a_2(t)\hat{x} + a_3(t)\hat{\hat{p}} + a_4(t)\hat{x}^2 + a_5(t)\hat{\hat{p}}^2 + a_6(t) (\hat{x}\hat{p} + \hat{p}\hat{x}),$$

(1)

where $\hat{x}$ and $\hat{p}$ are the usual position and momentum operators following the standard commutation relation $[\hat{x}, \hat{p}] = i\hbar$. Aside from the simple harmonic oscillator, a large number of interesting systems arise from this Hamiltonian as the linear potential [28, 29], the driven harmonic oscillator [30, 31], Kanai-Caldirano Hamiltonians [16–21], and time dependent harmonic oscillators i.e. an oscillator with time-varying frequency [21, 32, 33]. The time evolution generated by the most general version of (1) has been studied by means of Lewis and Riesenfeld [34] invariants [35] and through linear invariants [1].

Combining two one-dimensional generalized quadratic Hamiltonians along the $x$ and $y$ coordinates and adding cross terms for the position and momentum operators one arrives to the most general form of the two-dimensional quadratic Hamiltonian

$$\hat{H} = a_1(t) + a_2(t)\hat{x} + a_3(t)\hat{y} + a_4(t)\hat{\hat{p}}_x + a_5(t)\hat{\hat{p}}_y + a_6(t)\hat{x}^2 + a_7(t)\hat{y}^2 + a_8(t)\hat{x}\hat{y} + a_9(t)\hat{p}_x^2$$

$$\qquad + a_{10}(t)\hat{p}_y^2 + a_{11}(t)\hat{p}_x\hat{p}_y + a_{12}(t) (\hat{\hat{p}}_x\hat{p}_x + \hat{p}_x\hat{x}) + a_{13}(t) (\hat{\hat{p}}_y\hat{p}_y + \hat{p}_y\hat{y})$$

$$\qquad + a_{14}(t)\hat{x}\hat{p}_y + a_{15}(t)\hat{y}\hat{p}_x,$$

(2)

where newly the position and momentum operators along the $x$ and $y$ axes follow the standard commutation relations $[\hat{x}, \hat{y}] = [\hat{p}_x, \hat{p}_y] = 0$ and $[\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = i\hbar$. Hamiltonians as the one of a charged particle subject to variable electromagnetic fields, two coupled one-dimensional oscillators or the two dimensional harmonic oscillator stem from this Hamiltonian. In particular, the Hamiltonian of a one-dimensional generalized harmonic oscillator arises from (2).

Some special cases emerging from these Hamiltonians have been studied by diverse mathematical methods. For instance the time dependent linear potential has been treated through the Lewis and Riesenfeld [34] invariant theory [28, 29, 36], Feynman’s path integrals [37–41], time-space transformation methods [42] and others [43, 44]. The quantum oscillator with time-dependent mass and frequency has been dealt through the group-theoretical approach[45], unitary transformations [4, the Lewis and Riesenfeld invariant theory [46–48].

Even though the most general form of the one-dimensional quadratic Hamiltonian (1) has been treated through the Lewis Riesenfeld theory [21, 35, 49] and linear invariants [1], the two-dimensional quadratic Hamiltonian (twoDQH) has only been studied for a limited number of special cases. One of these corresponds to a charged particle subject to a constant uniform magnetic field and a quadratic potential [50] whose propagator was calculated by means of the path integral method. The isotropic harmonic oscillator in the presence of a time dependent magnetic field was investigated through the unitary transformation approach[22, 51]. The Lewis and Riesenfeld invariant theory[34, 52, 53] and quadratic invariants [54] were applied to the study of a charged particle subject to time-varying electromagnetic fields [25].

Therefore, even though a wide variety of systems stemming from the Hamiltonian in Eq. (2) have been studied by diverse methods, the evolution of the two-dimensional generalized quadratic Hamiltonian’s most general case has not been treated by any method to the extent of our knowledge.

The aim of this paper is therefore to develop a systematic method based on the Lie algebraic approach mainly with the purpose of obtaining the evolution operator of the two-dimensional, generalized time-dependent quadratic Hamiltonian presented in (1). Although most of this paper is devoted to the Lie algebra of (2), that consists of 15 generators, the presentation on the Lie algebraic approach is general enough to be applied to systems whose Hamiltonians can be expanded in terms of an arbitrarily large number of generators.

This paper is organized as follows. In Section II we develop the Lie algebraic approach for a rather general Hamiltonian consisting of the linear combination of an arbitrary number of generators. The Lie algebraic approach is applied to the generalized two-dimensional Harmonic oscillator in Section III. The general method is illustrated through the example of a two-dimensional charged particle subject to an in-plane electric field and a perpendicular magnetic field in Section IV. In Section V we summarise and give general conclusions.

II. THE LIE ALGEBRAIC APPROACH

The Lie algebraic approach relies on the existence of a set of $n$ operators $\{\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n\}$ that form a closed Lie algebra $\mathcal{L}_n$. This means that the commutator of any two elements of $\mathcal{L}_n$ should be expressible as a linear combination
of its own elements

$$[\hat{h}_i, \hat{h}_j] = i\hbar \sum_{k=1}^{n} c_{i,j,k} \hat{h}_k,$$

(3)

where $c_{i,j,k}$ are named structure constants of the algebra and contain all of the information concerning the unitary group.

The Hamiltonian $\hat{H}$ of a given system is said to have a dynamical algebra if it can be expressed as the linear combination of the elements of $\mathcal{L}_n$

$$\hat{H} = \sum_{k=1}^{n} a_k(t) \hat{h}_k = \hat{h}^\top a,$$

(4)

where for the sake of simplicity we have defined the vectors

$$\hat{h}^\top = (\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n),$$

(5)

$$a^\top = (a_1, a_2, \ldots, a_n).$$

(6)

The coefficients $a_i$ may in general be functions of time. The key element behind the Lie algebraic approach is that the general form of the evolution operator of such type of Hamiltonian can be expressed as [55–59]

$$\hat{U} = U^\dagger = \exp \left[ -i \hbar \sum_{i=1}^{n} \gamma_k(t) \hat{h}_k \right] = \prod_{k=1}^{n} \hat{U}_k = \prod_{k=1}^{n} \exp \left[ -i \hbar \alpha_k(t) \hat{h}_k \right]$$

(7)

where $U$ is an auxiliary unitary operator and

$$\hat{U}_k = \exp \left[ i \alpha_k(t) \hat{h}_k \right], \quad k = 1, \ldots n,$$

(8)

are the elements of unitary group generated by $\hat{h}_k$ with transformation parameters $\alpha_k$. As a direct consequence of the algebra closure, any transformation $\hat{U}_i$ acting on any generator $\hat{h}_j$ yields the linear combination of the same generators

$$\hat{U}_i \hat{h}_j \hat{U}_i^\dagger = \sum_{k=1}^{n} (\mathcal{M}_i)_{j,k} \hat{h}_k, \quad i, j, k = 1, \ldots n.$$

(9)

This expression can also be conveniently expressed as

$$\hat{U}_i (\alpha_i) \hat{h} \hat{U}_i^\dagger (\alpha_i) = \mathcal{M}_i (\alpha_i) \hat{h}.$$

(10)

These represent the transformation rules of $\mathcal{L}_n$ that are completely determined by the $\mathcal{M}_i$ matrices.

Additionally, referring to (A3), we see that all of the transformations given above acting on the energy operator yield

$$\hat{U}_i \hat{p}_i \hat{U}_i^\dagger = \hat{p}_i + \alpha_i(t) \hat{h}_i = \hat{p}_i + \hat{h}^\top \mathcal{I}_i \hat{\alpha}, \quad i = 1, \ldots n.$$

(11)

where $(\mathcal{I}_i)_{jk} = \delta_{i,j} \delta_{i,k},$

$$\hat{\alpha}^\top = (\alpha_1, \alpha_2, \ldots, \alpha_n)$$

(12)

and the overdot denotes the time derivative thus

$$\dot{\hat{\alpha}}^\top = \left( \frac{d\alpha_1}{dt}, \frac{d\alpha_2}{dt}, \ldots, \frac{d\alpha_n}{dt} \right).$$

(13)

Let us now proceed to finding the evolution operator. The transformation parameters $\alpha_i$ are in general time-dependent functions yet to be found. Once these functions are known, the evolution operator is completely determined as can be seen from Eq. (7). However, calculating them is not an easy task. To do so, let us first consider Schrödinger equation

$$\hat{H} |\psi(t)\rangle = \hat{p}_i |\psi(t)\rangle,$$

(14)
where \( \hat{p}_t = i\hbar \partial / \partial t \) is the energy operator. It is convenient to introduce the Floquet operator[60]

\[
\hat{\mathcal{H}} = \hat{H} - \hat{p}_t,
\]

since it allows to express Schrödinger equation in the compact form

\[
\hat{\mathcal{H}} |\psi (t)\rangle = 0.
\]

Let us now assume that a set of unitary transformation parameters \( \alpha_1, \alpha_2, \ldots, \alpha_n \), exists such that if \( \hat{U} = \hat{U}_n \cdots \hat{U}_2 \hat{U}_1 \) is applied to the Shrödinger equation (16), the Floquet operator is reduced to the energy operator \( \hat{p}_t \), namely

\[
\hat{U}\hat{\mathcal{H}}\hat{U}^\dagger |\psi (t)\rangle = -\hat{p}_t \left[ \hat{U} |\psi (t)\rangle \right] = 0.
\]

Reminding that \( \hat{p}_t \) is \( \hbar \) times a time derivative, it is clear that \( \hat{U} |\psi (t)\rangle \) must be a time-independent ket, say

\[
|\psi (t)\rangle = \hat{U}^\dagger |\psi (0)\rangle,
\]

or equivalently

\[
|\psi (t)\rangle = \hat{U}^\dagger |\psi (0)\rangle.
\]

According to the considerations above this equation states that

\[
\hat{U} = \hat{U}^\dagger = \hat{U}_n^\dagger \cdots \hat{U}_2^\dagger \hat{U}_1^\dagger,
\]

is in fact the time evolution operator (7). Hence Eq. (17) gives us a prescription for finding the evolution operator: if the transformation \( \hat{U} \) reduces the Floquet operator \( \hat{\mathcal{H}} \) to the energy operator \( \hat{p}_t \) then \( \hat{U} = \hat{U}^\dagger \) is the evolution operator.

At this point it is clear that in order to calculate any operator in the Heisenberg picture we can successively apply \( \hat{U}_1 \) to \( \hat{U}_n \). Thus, the Heisenberg picture operator of \( \hat{H} \) is given by

\[
\hat{H}_H = \hat{U}_n \cdots \hat{U}_2 \hat{U}_1 \hat{H} \hat{U}_1^\dagger \hat{U}_2^\dagger \cdots \hat{U}_n^\dagger = \mathcal{M}_1 \mathcal{M}_2 \cdots \mathcal{M}_n \hat{H}.
\]

This expression is easily evaluated by using the transformation rules (10). Moreover, if \( A(\hat{H}) \) is an analytic function of the generators \( \hat{H}^\dagger = (\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n) \) the Heisenberg picture of \( A \) is given by

\[
A_H(\hat{H}) = \hat{U}_n \cdots \hat{U}_2 \hat{U}_1 A(\hat{H}) \hat{U}_1^\dagger \hat{U}_2^\dagger \cdots \hat{U}_n^\dagger = A(\hat{H}_H) = A(\hat{h}_H, \hat{h}_{H2}, \ldots, \hat{h}_{Hn}).
\]

Even though (20) gives the general form of the evolution operator, we have not yet established the relation between the transformation parameters \( \boldsymbol{\alpha} \) and the Hamiltonian coefficients \( \mathbf{a} \) that insure that condition (17) is met. In order to accomplish this, we use Eqs. (9) and (11) and infer that the general structure of the transformed Floquet operator must be

\[
\hat{U} \hat{\mathcal{H}} \hat{U}^\dagger = \sum_{i=1}^{n} u_i (\mathbf{a}, \boldsymbol{\alpha}, \hat{\boldsymbol{\alpha}}) \hat{h}_i - \hat{p}_t = \hat{H}^\dagger \mathbf{u} - \hat{p}_t.
\]

Furthermore, according to Eqs. (10) and (11), \( \mathbf{u} \) must be a linear function of \( \hat{\boldsymbol{\alpha}} \) of the form

\[
\mathbf{u} (\mathbf{a}, \boldsymbol{\alpha}, \hat{\boldsymbol{\alpha}}) = \mathbf{w} (\mathbf{a}, \boldsymbol{\alpha}) + \nu (\boldsymbol{\alpha}) \hat{\boldsymbol{\alpha}},
\]

where

\[
\mathbf{w} (\mathbf{a}, \boldsymbol{\alpha}) = \mathcal{M}_1^\dagger (\alpha_1) \cdots \mathcal{M}_2^\dagger (\alpha_2) \mathcal{M}_3^\dagger (\alpha_3) \mathcal{M}_4^\dagger (\alpha_4) \mathcal{M}_5^\dagger (\alpha_5) \mathcal{M}_6^\dagger (\alpha_6) \mathcal{M}_7^\dagger (\alpha_7) \cdots + \mathcal{M}_n^\dagger (\alpha_n) \mathcal{I}_{n-1} + \mathcal{I}_n.
\]

In order for (23) to reduce to the energy operator \( \hat{p}_t \) the \( u_i \) coefficients must vanish giving rise to the following system of \( n \) ordinary coupled differential equations

\[
\mathbf{u} (\mathbf{a}, \boldsymbol{\alpha}, \hat{\boldsymbol{\alpha}}) = 0.
\]
These equations provide the means to establish the explicit form of the transformation parameters that fulfill condition (17). Although in principle (26) would suffice to determine the transformation parameters \( \alpha \), algebras formed by a large number of operators yield very complex system of ordinary differential equations hindering their solution. Notwithstanding it is possible to simplify the \( u_i \) coefficients even further into the linear combination

\[
\mathbf{u} (\mathbf{a}, \alpha, \dot{\alpha}) = \nu (\alpha) \mathbf{E} (\mathbf{a}, \alpha, \dot{\alpha}),
\]

where the elements of the vector \( \mathbf{E}^T = (E_1, E_2, \ldots, E_n) \) are more simple differential equations of the form

\[
\mathbf{E} (\mathbf{a}, \alpha, \dot{\alpha}) = \nu (\alpha) \mathbf{E} (\mathbf{a}, \alpha, \dot{\alpha}) = 0.
\]

Even though from the above expressions it is evident that \( \mu (\mathbf{a}, \alpha) = \nu^{-1} (\alpha) \mathbf{E} (\mathbf{a}, \alpha) \), \( \mathbf{E} (\mathbf{a}, \alpha, \dot{\alpha}) \) is not essential to know the explicit form of the coefficients \( u_i \) as functions of \( \mathbf{a}, \alpha \) and \( \dot{\alpha} \); it suffices to work them out from (23) by successively applying the transformation rules (10) to the Floquet operator. Thereby, from Eqs. (24) and (27)

\[
\nu_{i,j} (\mathbf{a}, \alpha) = \frac{\partial u_i}{\partial \alpha_j} = \frac{\partial u_i}{\partial \mathbf{E}_j}.
\]

and since \( u \) might be expressed either as a linear combination of \( \dot{\alpha} \) or \( \mathbf{E} \) the equations of the form (28) may be obtained from

\[
\mathbf{E} (\mathbf{a}, \alpha, \dot{\alpha}) = \nu^{-1} (\alpha) \mathbf{u} (\mathbf{a}, \alpha, \dot{\alpha}) = 0,
\]

provided that \( \text{det} \nu \neq 0 \). In order to know the evolution operator i.e. the transformation parameters’ explicit form one must find the solution to the system of ordinary differential equations (30).

To summarise we can reduce the method into five steps: a) The Floquet operator is transformed by applying the whole set of unitary transformations generated by \( \mathcal{L}_n \). b) Identify the \( u_i \) coefficients from the transformed Floquet operator. c) Derive the \( \nu \) matrix through Eq. (29). d) Obtain the simplified set of ordinary equations by using Eq. (30). e) Solve the set of ordinary differential equations for the \( \alpha \) parameters. f) The evolution operator is finally obtained by plugging this solution into the general form of the evolution operator in Eq. (7).

Finally, the Green function may be obtained by splitting the evolution operator’s matrix element into the the ones concerning each of the \( n \) unitary transformations as

\[
G(x, y, t; x', y', 0) = \left\langle x, y \right| \hat{U}^\dagger (t) \left| x', y' \right\rangle = \int dx_1 dy_1 \int dx_2 dy_2 \cdots \int dx_n dy_{n-1} \int dx_{n-1} dy_{n-1} \left\langle x_1, y_1 \right| \hat{U}_1^\dagger (t) \left| x_2, y_2 \right\rangle \cdots \left\langle x_n, y_n \right| \hat{U}_n^\dagger (t) \left| x', y' \right\rangle.
\]

The matrix elements of \( \hat{U}_1^\dagger, \hat{U}_2^\dagger, \ldots, \hat{U}_n^\dagger \) are readily calculated by using the transformation rules.

### III. GENERALIZED TWO-DIMENSIONAL QUADRATIC HAMILTONIANS

In this section we develop the Lie algebraic approach to obtain the evolution operator of the general two-dimensional quadratic Hamiltonian. To motivate the discussion let us take the Hamiltonian of a two-dimensional charged particle in perpendicular magnetic field and in-plane electric fields as the starting point. This Hamiltonian is given by

\[
\hat{H} = \frac{m}{2} (\hat{p}_x^2 + \hat{p}_y^2) + \frac{e^2 B^2}{8m} (\hat{x}^2 + \hat{y}^2) + \frac{eB}{2m} (\hat{x}\hat{p}_y - \hat{y}\hat{p}_x) + eE_x \hat{x} + eE_y \hat{y},
\]

where \( m \) and \( q = -e \) are the particle’s mass and charge, \( B \), \( E_x \) and \( E_y \) are the perpendicular magnetic and in-plane electric field components. The scalar and vector potentials are expressed in the symmetric gauge as \( \phi = -E_x (\hat{x}) - E_y (\hat{y}) \), \( A_x = -B\hat{y}/2 \) and \( A_y = B\hat{x}/2 \). The position and momentum operators \( \hat{x}, \hat{y}, \hat{p}_x \) and \( \hat{p}_y \) fulfill the usual commutation relations \( [\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = i\hbar \) and \( [\hat{x}, \hat{y}] = [\hat{p}_x, \hat{p}_y] = [\hat{x}, \hat{p}_y] = [\hat{y}, \hat{p}_x] = 0 \).

In principle this Hamiltonian is expressed as a linear combination of \( \hat{x}, \hat{y}, \hat{x}^2, \hat{y}^2, \hat{p}_x^2, \hat{p}_y^2, \hat{x}\hat{p}_y \) and \( \hat{y}\hat{p}_x \) with time dependent coefficients. However, these eight operators alone do not form a closed Lie algebra under commutation given that, for example, the commutators \( [\hat{x}, \hat{p}_x] = i\hbar \mathbf{1}, [\hat{x}, \hat{p}_x^2] = 2i\hbar \hat{p}_x \) and \( [\hat{x}^2, \hat{p}_x^2] = 2i\hbar (\hat{x}\hat{p}_y + \hat{y}\hat{p}_x) \) yield operators...
outside the original set, namely \( \hat{1}, \hat{\rho}_x \) and \( \hat{x}, \hat{p}_x + \hat{\rho}_x \). Then there follows that, in order to close the algebra, the set must be extended to

\[
\begin{align*}
\hat{h}_1 &= \hat{1}, & \quad \hat{h}_2 &= \hat{x}, & \quad \hat{h}_3 &= \hat{y}, & \quad \hat{h}_4 &= \hat{\rho}_x, & \quad \hat{h}_5 &= \hat{\rho}_y, \\
\hat{h}_6 &= \hat{x}^2, & \hat{h}_7 &= \hat{y}^2, & \hat{h}_8 &= \hat{x} \hat{y}, & \hat{h}_9 &= \hat{\rho}_x^2, & \hat{h}_{10} &= \hat{\rho}_y^2, \\
\hat{h}_{11} &= \hat{\rho}_x \hat{\rho}_y, & \hat{h}_{12} &= \hat{x} \hat{\rho}_x + \hat{\rho}_x \hat{x}, & \hat{h}_{13} &= \hat{y} \hat{\rho}_y + \hat{\rho}_y \hat{y}, & \hat{h}_{14} &= \hat{x} \hat{\rho}_y, & \hat{h}_{15} &= \hat{y} \hat{\rho}_x.
\end{align*}
\]

Indeed, the commutation relations for these operators yield a closed algebra summarised in Table I. We henceforth call this algebra \( \mathcal{L}_{15} \). These commutation relations can be comprehended as a particular realization of (3) where the structure constants are related to the coefficients found in Table I. Also in this Table, the particular generator ordering in Eq. (33) reveals the two following different sub-algebras (enclosed in squares) \( \{ \hat{h}_1, \hat{h}_2, \hat{h}_3, \hat{h}_4, \hat{h}_5 \} \) and \( \{ \hat{h}_1, \hat{h}_2, \hat{h}_3, \hat{h}_4, \hat{h}_5, \hat{h}_6, \hat{h}_7, \hat{h}_8 \} \). Not as evident as the previous ones, one can find even more sub-algebras in \( \mathcal{L}_{15} \) that correspond to relevant physical problems. For example \( \{ \hat{h}_1, \hat{h}_2, \hat{h}_4 \} \) and \( \{ \hat{h}_1, \hat{h}_3, \hat{h}_5 \} \) are also sub-algebras of \( \mathcal{L}_{15} \).

In particular \( \{ \hat{h}_6, \hat{h}_9, \hat{h}_{12} \} \) or \( \{ \hat{h}_7, \hat{h}_{10}, \hat{h}_{13} \} \) form the \( SU(1,1) \) Lie algebra that has been used to study Kanai-Caldirola Hamiltonians through the Lie algebraic approach [27, 58]. The sub-algebras \( \{ \hat{h}_1, \hat{h}_2, \hat{h}_4, \hat{h}_6, \hat{h}_9, \hat{h}_{12} \} \) or \( \{ \hat{h}_1, \hat{h}_3, \hat{h}_5, \hat{h}_7, \hat{h}_{10}, \hat{h}_{13} \} \) correspond to the generalised one-dimensional harmonic oscillator[1, 35] along the \( x \) and \( y \) axis respectively.

One can easily express Hamiltonian (32) as a linear combination of the \( \mathcal{L}_{15} \) elements

\[
\hat{H} = a_1 \hat{h}_1 + a_2 \hat{h}_2 + a_3 \hat{h}_3 + a_4 \hat{h}_4 + a_5 \hat{h}_5 + a_6 \hat{h}_6 + a_7 \hat{h}_7 + a_8 \hat{h}_8 + a_9 \hat{h}_9 + a_{10} \hat{h}_{10} + a_{11} \hat{h}_{11} + a_{12} \hat{h}_{12} + a_{13} \hat{h}_{13} + a_{14} \hat{h}_{14} + a_{15} \hat{h}_{15},
\]

where \( a_2 = cE_x \), \( a_3 = cE_y \), \( a_6 = a_7 = eB^2/8m \), \( a_9 = a_{10} = 1/2m \), \( a_{14} = -a_{15} = eB/2m \) and \( a_1 = a_4 = a_5 = a_8 = a_{11} = a_{12} = a_{13} = 0 \). In the most general case, when the coefficients \( a_1 \) to \( a_{15} \) are non-vanishing functions of time, we call (34) the generalized two-dimensional quadratic Hamiltonian. Many Hamiltonians of physical significance arise from (34) for example, a single electron in an elliptically shaped quantum dot with quadratic confining potential, an electron subject to variable electromagnetic field or two-dimensional quadratic Kanai-Caldirola Hamiltonians among others.

Now we can start to build the evolution operator by identifying the elements of the unitary group generated by \( \mathcal{L}_{15} \). The transformations produced by the 15 generators of \( \mathcal{L}_{15} \) are given by

\[
\begin{align*}
\hat{U}_1 &= \exp \left( i\alpha_1 \hat{x} / \hbar \right), & \hat{U}_2 &= \exp \left( i\alpha_2 \hat{x} / \hbar \right), & \hat{U}_3 &= \exp \left( i\alpha_3 \hat{y} / \hbar \right), \\
\hat{U}_4 &= \exp \left( i\alpha_4 \hat{\rho}_x / \hbar \right), & \hat{U}_5 &= \exp \left( i\alpha_5 \hat{\rho}_x / \hbar \right), & \hat{U}_6 &= \exp \left( i\alpha_6 \hat{\rho}_x / \hbar \right), \\
\hat{U}_7 &= \exp \left( i\alpha_7 \hat{y}^2 / \hbar \right), & \hat{U}_8 &= \exp \left( i\alpha_8 \hat{y} / \hbar \right), & \hat{U}_9 &= \exp \left( i\alpha_9 \hat{\rho}_x^2 / \hbar \right), \\
\hat{U}_{10} &= \exp \left( i\alpha_{10} \hat{y}^2 / \hbar \right), & \hat{U}_{11} &= \exp \left( i\alpha_{11} \hat{\rho}_x \hat{\rho}_y / \hbar \right), & \hat{U}_{12} &= \exp \left[ i\alpha_{12} \left( \hat{x} \hat{\rho}_x + \hat{\rho}_x \hat{x} \right) / \hbar \right], \\
\hat{U}_{13} &= \exp \left[ i\alpha_{13} \left( \hat{y} \hat{\rho}_y + \hat{\rho}_y \hat{y} \right) / \hbar \right], & \hat{U}_{14} &= \exp \left[ i\alpha_{14} \hat{\rho}_y / \hbar \right], & \hat{U}_{15} &= \exp \left( i\alpha_{15} \hat{\rho}_x / \hbar \right).
\end{align*}
\]

The first five transformations shift the energy, position and momentum operators by the time-dependent functions \( \alpha_1 \) to \( \alpha_5 \). We prove later on that these parameters are related with the classical action \( S \) position \( x \), \( y \), and momentum \( -\hat{p}_x, -\hat{p}_y \). Whereas \( \hat{U}_6 \) and \( \hat{U}_7 \) shift the momentum operator by \( -\alpha_6 \hat{x} \) and \( -\alpha_7 \hat{y} \), \( \hat{U}_8 \) and \( \hat{U}_{10} \) shift the position operator by \( -\alpha_9 \hat{\rho}_x \) and \( \alpha_{10} \hat{\rho}_y \). The dilatations \( \hat{U}_{12} \) and \( \hat{U}_{13} \) preserve the commutation relations between the transformed position and momentum operators by expanding the position operators \( \hat{x} \) and \( \hat{y} \) by the factors \( \exp (2\alpha_{12}) \) and \( \exp (2\alpha_{13}) \), respectively, while contracting the momentum operators by the inverse factors \( \exp (-2\alpha_{12}) \) and \( \exp (-2\alpha_{13}) \), respectively. In principle it is possible to compute the 240 transformation rules summarized in Tables II and III, however in order to obtain them all, only a few are needed. Eq. (11) yields the first 15 transformation rules. Eq. (33) allows to derive the transformation rules for all the remaining transformations except the dilatations \( \hat{U}_{12} \) and \( \hat{U}_{13} \). For example

\[
\hat{U}_9 \hat{x} \hat{U}_9^\dagger = \hat{x} + \hat{U}_9 [\hat{x}, \hat{\rho}_x] = \hat{x} + \hat{U}_9 \frac{\partial \hat{U}_9^\dagger}{\partial \hat{\rho}_x} = \hat{x} + 2\alpha_9 \hat{\rho}_x.
\]
hence, integrating we get

\[
\frac{\partial}{\partial \alpha_{12}} \hat{U}_{12} \hat{\mathcal{U}}_{12}^\dagger = \frac{i}{\hbar} \hat{U}_{12} \left[ \hat{x}, \hat{x} \hat{p}_x + \hat{p}_x \hat{x} \right] \hat{U}_{12}^\dagger = 2 \hat{U}_{12} \hat{\mathcal{U}}_{12}^\dagger.
\]

(37)

On the other hand, the action of the dilatations is better calculated by taking the derivative with respect to the transformation parameter. Then, for the dilatation \( \hat{U}_{12} \) we have

\[
\hat{U}_{12} \hat{\mathcal{U}}_{12}^\dagger = \exp(2\alpha_{12})\hat{x},
\]

(38)
The upshot of the method presented in Section II is that by calculating the matrix transformation rules are presented for reference in the supplemental material at [URL will be inserted by AIP].

Let us now reduce the Floquet operator of the generalized two-dimensional quadratic Hamiltonian in Eq. (34) by means of these unitary transformations. We thus calculate the transformed Floquet operator by applying the unitary transformation $\hat{U} = \hat{U}_{15} \hat{U}_{14} \hat{U}_{13} \hat{U}_{12} \hat{U}_{11} \hat{U}_{9} \hat{U}_{8} \hat{U}_{7} \hat{U}_{6} \hat{U}_{5} \hat{U}_{4} \hat{U}_{3} \hat{U}_{2} \hat{U}_{1}$ stepwisely. Proceeding in this way through the 15 transformations we obtain

$$\hat{U}\hat{H}\hat{U}^\dagger = \hat{U} \left( \hat{H} - \hat{p}_t \right) \hat{U}^\dagger = u_1 h_1 + u_2 h_2 + u_3 h_3 + u_4 h_4 + u_5 h_5 + u_6 h_6 + u_7 h_7$$

$$+ u_8 h_8 + u_9 h_9 + u_{10} h_{10} + u_{11} h_{11} + u_{12} h_{12} + u_{13} h_{13} + u_{14} h_{14} + u_{15} h_{15} - \hat{p}_t,$$

(39)

where the explicit form of the $u$ coefficients as functions of $a$, $\alpha$ and $\dot{\alpha}$ is to involved to be presented here (see supplemental material at [URL will be inserted by AIP] for the explicit form of the $u$ coefficients). However, the upshot of the method presented in Section II is that by calculating the matrix $\nu$ through Eq. (29) and using (27) it
is possible to express the $u$ coefficients as compact functions of $E$ and $\alpha$ in the following form

\begin{align}
  u_1 &= E_1 + \alpha_4 E_2 + \alpha_5 E_3, \\
  u_2 &= e^{2\alpha_1} E_2 + e^{2\alpha_1} \alpha_1 E_3 - (e^{2\alpha_1} \alpha_6 + e^{2\alpha_1} \alpha_8) E_4 - (e^{2\alpha_1} \alpha_8 + e^{2\alpha_1} \alpha_7) E_5, \\
  u_3 &= e^{2\alpha_1} \alpha_1 E_2 + e^{2\alpha_1} \alpha_5 E_2 - (e^{2\alpha_1} \alpha_8 \alpha_{15} + 2e^{2\alpha_1} \alpha_7 \alpha_{15} + 1) E_5 \\
  &- [2e^{2\alpha_1} \alpha_6 \alpha_{15} + e^{2\alpha_1} \alpha_8 \alpha_{15} + 1] E_4, \\
  u_4 &= \left[2e^{2\alpha_1} \alpha_9 \alpha_{14} \alpha_{15} + 1 - e^{-2\alpha_1} \alpha_{11} \alpha_{15} \right] E_2 \\
  &+ \left[ e^{-2\alpha_1} \alpha_{11} \alpha_{14} \alpha_{15} + 1 - 2e^{-2\alpha_1} \alpha_{10} \alpha_{15} \right] E_3 \\
  &+ \left[ 2e^{-2\alpha_1} \alpha_6 \alpha_{10} + \alpha_6 \alpha_{11} \alpha_{15} - e^{-2\alpha_1} \left(4\alpha_6 \alpha_9 + \alpha_8 \alpha_{11} - 1\right) \alpha_{14} \alpha_{15} + 1 \right] E_4 \\
  &+ \left[ e^{-2\alpha_1} \left(4\alpha_7 \alpha_{10} + \alpha_8 \alpha_{11} - 1\right) \alpha_{15} - 2e^{-2\alpha_1} \left(\alpha_8 \alpha_9 + \alpha_7 \alpha_{11}\right) \alpha_{14} \alpha_{15} + 1 \right] E_5, \\
  u_5 &= \left(e^{-2\alpha_1} \alpha_{11} - 2e^{-2\alpha_1} \alpha_9 \alpha_{14} \right) E_2 + \left(e^{-2\alpha_1} \alpha_{10} - e^{-2\alpha_1} \alpha_{11} \alpha_{14}\right) E_3 \\
  &+ \left[ e^{-2\alpha_1} \left(4\alpha_6 \alpha_9 + \alpha_8 \alpha_{11} - 1\right) \alpha_{15} - 2e^{-2\alpha_1} \left(\alpha_8 \alpha_9 + \alpha_7 \alpha_{11}\right) \alpha_{14} \alpha_{15} + 1 \right] E_4 \\
  &+ \left[ e^{-2\alpha_1} \left(4\alpha_7 \alpha_{10} + \alpha_8 \alpha_{11} - 1\right) \alpha_{15} - 2e^{-2\alpha_1} \left(\alpha_8 \alpha_9 + \alpha_7 \alpha_{11}\right) \alpha_{14} \alpha_{15} + 1 \right] E_5, \\
  u_6 &= e^{4\alpha_1} \alpha_7^2 E_6 + e^{4\alpha_1} \alpha_1 \alpha_4 E_7 + e^{2(\alpha_1+\alpha_3)} \alpha_1 \alpha_4 E_8, \\
  u_7 &= e^{4\alpha_1} \alpha_5 E_6 + e^{4\alpha_1} \alpha_3 \alpha_{15} + 1 \alpha_7 E_7 + e^{2(\alpha_1+\alpha_3)} \alpha_{14} \alpha_{15} + 1 \alpha_7 E_8, \\
  u_8 &= 2e^{4\alpha_1} \alpha_5 E_6 + 2e^{4\alpha_1} \alpha_3 \alpha_{14} \alpha_{15} + 1 \alpha_7 E_7 + e^{2(\alpha_1+\alpha_3)} \left(2\alpha_{14} \alpha_{15} + 1\right) \alpha_7 E_8, \\
  u_9 &= e^{-4(\alpha_1+\alpha_3)} \left[ e^{2\alpha_1} \alpha_{11} \alpha_{15} + 1 - 2e^{2\alpha_1} \alpha_9 \alpha_{14} \alpha_{15} + 1 \right]^2 E_6 \\
  &+ e^{-4(\alpha_1+\alpha_3)} \left[ e^{2\alpha_1} \alpha_{11} \alpha_{14} \alpha_{15} + 1 - 2e^{2\alpha_1} \alpha_9 \alpha_{10} \alpha_{15} \right]^2 E_7 \\
  &+ e^{-4(\alpha_1+\alpha_3)} \left[ 2e^{2\alpha_1} \alpha_9 \left(\alpha_{14} \alpha_{15} + 1 - e^{2\alpha_1} \alpha_{12} \alpha_{11} \alpha_{15} \right) \alpha_{14} \alpha_{15} + 1 \right] \alpha_7 E_8 \\
  &+ e^{-4(\alpha_1+\alpha_3)} \left(2e^{2\alpha_1} \alpha_9 \alpha_{14} \alpha_{15} + 1 - 2e^{2\alpha_1} \alpha_9 \alpha_{10} \alpha_{15} \right) \alpha_7 E_9 \\
  &+ e^{-4(\alpha_1+\alpha_3)} \left(e^{-2(\alpha_1+\alpha_3)} \alpha_{14} \alpha_{15} + 1\right) \alpha_7 E_{10} \\
  &+ e^{-4(\alpha_1+\alpha_3)} \left(e^{-2(\alpha_1+\alpha_3)} \alpha_{14} \alpha_{15} + 1\right) \alpha_7 E_{11}, \\
  u_{10} &= e^{-4(\alpha_1+\alpha_3)} \left(e^{-2(\alpha_1+\alpha_3)} \alpha_{11} - 2e^{2\alpha_1} \alpha_9 \alpha_{14} \alpha_{15} \right)^2 E_6 + e^{-4(\alpha_1+\alpha_3)} \left(e^{2\alpha_1} \alpha_{11} \alpha_{14} - 2e^{2\alpha_1} \alpha_{10} \alpha_{15} \right)^2 E_7 \\
  &+ e^{-4(\alpha_1+\alpha_3)} \left(2e^{2\alpha_1} \alpha_9 \alpha_{14} \alpha_{15} + 1 - 2e^{2\alpha_1} \alpha_{11} \alpha_{14} \alpha_{15} + 1\right) \alpha_7 E_8 \\
  &+ e^{-4(\alpha_1+\alpha_3)} \left(e^{-2(\alpha_1+\alpha_3)} \alpha_{14} \alpha_{15} + 1\right) \alpha_7 E_{10} \\
  &+ e^{-4(\alpha_1+\alpha_3)} \left(e^{-2(\alpha_1+\alpha_3)} \alpha_{14} \alpha_{15} + 1\right) \alpha_7 E_{11}, \\
  u_{11} &= -\left[ e^{-4(\alpha_1+\alpha_3)} \alpha_{11} - 2e^{2\alpha_1} \alpha_9 \alpha_{14} \alpha_{15} \right] \alpha_7 E_6 \\
  &- 2e^{-4(\alpha_1+\alpha_3)} \left(e^{2\alpha_1} \alpha_{11} \alpha_{14} \alpha_{15} + 1 - 2e^{2\alpha_1} \alpha_{10} \alpha_{15} \right) \alpha_7 E_7 \\
  &+ e^{-4(\alpha_1+\alpha_3)} \left(2e^{2\alpha_1} \alpha_9 \alpha_{11} \alpha_{14} - 4e^{-4(\alpha_1+\alpha_3)} \alpha_{11} \alpha_{14} \alpha_{15} + 1\right) \alpha_7 E_{10} \\
  &+ e^{-4(\alpha_1+\alpha_3)} \left(2e^{2\alpha_1} \alpha_9 \alpha_{11} \alpha_{14} - 4e^{-4(\alpha_1+\alpha_3)} \alpha_{11} \alpha_{14} \alpha_{15} + 1\right) \alpha_7 E_{11}. 
\end{align}
\[ u_{12} = \left[ 2a_9 (a_{14}a_{15} + 1) - e^{2a_{12} - 2a_{13}a_{11}a_{15}} \right] E_6 + a_{14} \left[ e^{2a_{13} - 2a_{12}a_{11} (a_{14}a_{15} + 1) - 2a_{10}a_{15}} \right] E_7 \\
+ \left[ \frac{a_{11}}{2} - e^{2a_{12} - 2a_{13}a_{10}a_{15}} + e^{2a_{13} - 2a_{12}a_{9}a_{14} (a_{14}a_{15} + 1)} \right] E_8 + (a_{14}a_{15} + 1) E_{12} \\
+ \frac{1}{2} a_{15} E_{14} - a_{14}a_{15} E_{13}, \]  

\[ u_{13} = \left( e^{2a_{12} - 2a_{13}a_{11} - 2a_9a_{14}} \right) a_{15} E_6 + e^{-2a_{12}} \left( e^{2a_{12}a_{10} - e^{2a_{13}a_{11}a_{14}}} \right) (a_{14}a_{15} + 1) E_7 \\
+ \left[ \frac{a_{11}}{2} + e^{2a_{12} - 2a_{13}a_{10}a_{15}} - e^{2a_{13} - 2a_{12}a_{9}a_{14} (a_{14}a_{15} + 1)} \right] E_8 - a_{14}a_{15} E_{12} \\
+ (a_{14}a_{15} + 1) E_{13} + \frac{a_{11}}{2} E_{14}, \]  

\[ u_{14} = \left( e^{2a_{12} - 2a_{13}a_{11} - 4a_9a_{14}} \right) E_6 + \left( 4a_{10}a_{14} - 2e^{2a_{13} - 2a_{12}a_{11}a_{14}} \right) E_7 \\
+ e^{-2(a_{12}+a_{13})} \left( 2e^{a_{12}a_{10} - 2e^{a_{13}a_{10}}a_{14}^2} \right) E_8 - 2a_{14} E_{12} + 2a_{14} E_{13} + E_{14}, \]  

\[ u_{15} = 2 \left[ 2a_9 (a_{14}a_{15} + 1) - e^{2a_{12} - 2a_{13}a_{11}a_{15}} \right] a_{15} E_6 \\
+ 2 (a_{14}a_{15} + 1) \left[ e^{2a_{12} - 2a_{11}a_{14} (a_{14}a_{15} + 1) - 2a_{10}a_{15}} \right] E_7 \\
+ 2e^{-2(a_{12}+a_{13})} \left[ e^{4a_{13}a_9 (a_{14}a_{15} + 1)^2} - e^{4a_{12}a_{10}a_{15}^2} \right] E_8 + 2 (a_{14}a_{15} + 1) a_{15} E_{12} \\
- 2 (a_{14}a_{15} + 1) a_{15} E_{13} - a_{15}^2 E_{14} + E_{15}. \]  

The \( \nu \) matrix can be obtained from the previous equations by using the right-hand side of Eq. (29). For the particular transformation ordering used here det \( \nu = 1 \), thus upon calculating the inverse of \( \nu \), the explicit form of the equations \( \mathcal{E} \) if readily obtained by using Eq. (30)

\( E_1 = a_9 a_2^2 - a_4 a_2 + a_{11} a_3 a_2 + a_{10} a_3^2 - a_6 a_4^2 - a_7 a_5^2 - a_5 a_3 - a_8 a_4 a_5 + a_1 - \dot{a}_1, \)

\( E_2 = -2a_{12} a_2 - a_{14} a_3 + 2a_9 a_4 + a_8 a_5 + a_2 - \dot{a}_2, \)

\( E_3 = -a_{15} a_2 - 2a_{13} a_3 + a_8 a_4 + 2a_7 a_5 + a_3 - \dot{a}_3, \)

\( E_4 = -2a_9 a_2 - a_{11} a_3 + 2a_{12} a_4 + a_{15} a_5 + a_4 - \dot{a}_4, \)

\( E_5 = -a_{11} a_2 - 2a_{10} a_3 + a_{14} a_4 + 2a_{13} a_5 + a_5 - \dot{a}_5, \)

\( E_6 = 4a_9 a_2^2 - 4a_{12} a_6 + 2a_{11} a_8 a_6 + a_{10} a_8^2 - a_{14} a_8 + a_6 - \dot{a}_6, \)

\( E_7 = 4a_{10} a_2^2 - 4a_{13} a_7 + 2a_{11} a_8 a_7 + a_9 a_8^2 - a_{15} a_8 + a_7 - \dot{a}_7, \)

\( E_8 = -2a_{14} a_7 - 2a_{15} a_6 - 2a_{12} a_6 - 2a_{13} a_8 + 4a_9 a_6 a_8 + 4a_{10} a_7 a_8 \\
+ a_{11} (a_8^2 + 4a_6 a_7) + a_8 - \dot{a}_8, \)

\( E_9 = 4a_{12} a_9 + a_{13} a_{11} - 2a_9 (a_8 a_9 + a_7 a_{11}) + a_9 (1 - 8a_6 a_9 - 2a_8 a_{11}) - \dot{a}_9, \)

\( E_{10} = 4a_{13} a_{10} + a_{14} a_{11} - 2a_9 (a_8 a_{10} + a_6 a_{11}) + a_{10} (1 - 8a_7 a_{10} - 2a_8 a_{11}) - \dot{a}_{10}, \)

\( E_{11} = 2a_{14} a_9 + 2a_{15} a_{10} + 2a_{12} a_{11} + 2a_{13} a_{11} \\
- a_9 [4a_8 a_{10} + 4a_6 a_{11}] - a_{10} [4a_9 a_8 + 4a_7 a_{11}] \\
+ a_{11} [1 - 4a_6 a_9 - 4a_7 a_{10} - 2a_8 a_{11}] - \dot{a}_{11}, \)

\( E_{12} = \frac{1}{2} e^{2a_{13} - 2a_{12} a_{15} a_{14}} + a_{11} \left( \frac{a_8}{2} + e^{2a_{13} - 2a_{12} a_7 a_{14}} \right) \\
- a_9 (2a_6 + e^{2a_{13} - 2a_{12} a_8 a_{14}}) + a_{12} - \dot{a}_{12}, \)

\( E_{13} = -a_{10} a_7 - \frac{1}{2} e^{2a_{13} - 2a_{12} a_{15} a_{14}} + e^{2a_{13} - 2a_{12} a_9 a_8 a_{14}} \\
+ a_{11} \left( e^{2a_{13} - 2a_{12} a_7 a_{14}} - \frac{a_8}{2} \right) + a_{13} - \dot{a}_{13}, \)

\( E_{14} = e^{2a_{13} - 2a_{12} a_{15} a_{14}} - 2e^{2a_{13} - 2a_{12} a_9 a_8 a_{14}} + e^{2a_{13} - 2a_{12} a_7 a_{14}} - 2e^{2a_{12} - 2a_{13} a_{10} a_{8}} \\
- 2e^{-2(a_{12} + a_{13}) a_{11}} (e^{4a_{13} a_8 a_{14}} + e^{4a_{12} a_6}) - \dot{a}_{14}, \)

\( E_{15} = e^{2a_{13} - 2a_{12} a_{15} - 2e^{2a_{13} - 2a_{12} a_1 a_{11} a_{7}} - 2e^{2a_{13} - 2a_{12} a_9 a_8} - \dot{a}_{15}. \)
parameters vanish at \( t = 0 \)

\[ \alpha_i(0) = 0, \quad i = 1, 2, \ldots, 15, \quad (70) \]

in order to ensure that the evolution operator equals the identity at \( t = 0 \), namely \( \hat{U}(0) = \hat{1} \) and therefore \( \hat{U}_1(0) = \hat{U}_2(0) = \ldots \hat{U}_{15}(0) = \hat{1} \).

The solution to the system of ordinary differential equations (55)-(69), together with the initial conditions (70), yields the explicit form of the set of transformation parameters as functions of time.

The first five differential equations (55)-(59) may be solved independently for the transformation parameters \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) and \( \alpha_5 \). This is a direct consequence of the fact that \( \hat{h}_1, \hat{h}_2, \hat{h}_3, \hat{h}_4 \) and \( \hat{h}_5 \) form a sub-algebra of \( \mathcal{L}_{15} \) as can be verified in Table I. There is a close analogy between these five equations and the equations of motion of the classical version of (32). Replacing \( \xi_1, \xi_2 \) and \( \xi_3 \) from Eqs. (55), (56) and (57) into \( u_1 \) in Eq. (40) we get

\[ u_1 = a_9 \alpha_2^2 - a_4 \alpha_2 + a_{11} \alpha_3 \alpha_2 - 2a_{12} \alpha_4 \alpha_2 - a_{15} \alpha_2^2 + a_{10} \alpha_5^2 + a_6 \alpha_3^2 + a_7 \alpha_5^2 - a_5 \alpha_3 \\
+ a_2 \alpha_4 - a_{14} \alpha_3 \alpha_4 + a_3 \alpha_5 - 2a_{13} \alpha_2 \alpha_5 + a_8 \alpha_4 \alpha_5 + a_1 - a_4 \alpha_2 - a_5 \alpha_3 - \dot{\alpha}_1, \quad (71) \]

Since \( u_1 \) must vanish in order to reduce the Floquet operator, we may identify \( \alpha_1 \) with the classical action \( S = \alpha_1 \) and therefore, from the above equation the classical Lagrangian \( L = u_1 + \dot{\alpha}_1 = \mathcal{S} \) is given by

\[ L = a_9 \alpha_2^2 - a_4 \alpha_2 + a_{11} \alpha_3 \alpha_2 - 2a_{12} \alpha_4 \alpha_2 - a_{15} \alpha_2^2 + a_{10} \alpha_5^2 + a_6 \alpha_3^2 + a_7 \alpha_5^2 - a_5 \alpha_3 \\
+ a_2 \alpha_4 - a_{14} \alpha_3 \alpha_4 + a_3 \alpha_5 - 2a_{13} \alpha_2 \alpha_5 + a_8 \alpha_4 \alpha_5 + a_1 - a_4 \alpha_2 - a_5 \alpha_3. \quad (72) \]

This analogy goes even further. Indeed the Euler equations arising from this Lagrangian yield

\begin{align}
\frac{d}{dt} \frac{\partial L}{\partial \alpha_2} - \frac{\partial L}{\partial \alpha_2} &= -2a_9 \alpha_2 - a_{11} \alpha_3 + 2a_{12} \alpha_4 + a_{15} \alpha_5 + a_4 - \dot{\alpha}_4 = 0 = \mathcal{E}_4, \quad (73) \\
\frac{d}{dt} \frac{\partial L}{\partial \alpha_3} - \frac{\partial L}{\partial \alpha_3} &= -a_{11} \alpha_2 - 2a_{10} \alpha_3 + a_{14} \alpha_4 + 2a_{13} \alpha_5 + a_5 - \dot{\alpha}_5 = \mathcal{E}_5, \quad (74) \\
\frac{d}{dt} \frac{\partial L}{\partial \alpha_4} - \frac{\partial L}{\partial \alpha_4} &= 2a_{12} \alpha_2 + a_{14} \alpha_3 - 2a_6 \alpha_4 - a_8 \alpha_5 - a_2 + \dot{\alpha}_2 = -\mathcal{E}_2 = 0, \quad (75) \\
\frac{d}{dt} \frac{\partial L}{\partial \alpha_5} - \frac{\partial L}{\partial \alpha_5} &= a_{15} \alpha_2 + 2a_{13} \alpha_3 - a_8 \alpha_4 - 2a_7 \alpha_5 - a_3 + \dot{\alpha}_3 = -\mathcal{E}_3 = 0, \quad (76) \end{align}

which precisely correspond to Eqs (56)-(59). In these equations it is clear that there is a correspondence between the transformation parameters and the classical position and momentum. In particular, \( \alpha_2, \alpha_3, \alpha_4 \) and \( \alpha_5 \) may be identified with the classical position and momentum variables \( -p_x, -p_y, x \) and \( y \) respectively.

The remaining transformation parameters \( \alpha_6 \) to \( \alpha_{15} \) are obtained from the solution of the system of ordinary differential equations (60)-(69) and the initial conditions (70).

With all the transformation parameters at hand we can write the evolution operator as

\[ \hat{U} = \hat{U}^\dagger = \exp \left( -\frac{i}{\hbar} \alpha_1 \hat{h}_1 \right) \exp \left( -\frac{i}{\hbar} \alpha_2 \hat{h}_2 \right) \ldots \exp \left( -\frac{i}{\hbar} \alpha_{15} \hat{h}_{15} \right). \quad (77) \]

This equation together with the transformation rules in Tables II and III allow us to compute the evolution of any operator belonging to the \( \mathcal{L}_{15} \) algebra through Eqs. (21) and (22). For example, the Heisenberg picture position and momentum operators are obtained from Eq. (21) by acting \( \hat{U} \) on the Schrödinger picture position and momentum
operators and following the transformation rules. For the most general case we have

\begin{align}
\hat{x}_H &= \hat{U} \hat{x} \hat{U}^\dagger = \alpha_4 \hat{h}_1 + e^{2\alpha_{12}} \hat{h}_2 + e^{2\alpha_{12}} \alpha_{15} \hat{h}_3 \\
&\quad + 2e^{2\alpha_{12}} \alpha_9 (\alpha_{14} \alpha_{15} + 1) - e^{-2\alpha_{13}} \alpha_{11} \alpha_{15} \hat{h}_4 + \left( e^{-2\alpha_{13}} \alpha_{11} - 2e^{-2\alpha_{12}} \alpha_9 \alpha_{14} \alpha_{15} \right) \hat{h}_5 \\
&= \alpha_4 + e^{2\alpha_{12}} \hat{x} + e^{2\alpha_{12}} \alpha_{15} \hat{y} + 2e^{-2\alpha_{12}} \alpha_9 (\alpha_{14} \alpha_{15} + 1) - e^{-2\alpha_{13}} \alpha_{11} \alpha_{15} \hat{p}_x \\
&\quad + \left( e^{-2\alpha_{13}} \alpha_{11} - 2e^{-2\alpha_{12}} \alpha_9 \alpha_{14} \alpha_{15} \right) \hat{p}_y,
\end{align}

\begin{align}
\hat{y}_H &= \hat{U} \hat{y} \hat{U}^\dagger = \alpha_5 \hat{h}_1 + e^{2\alpha_{13}} \alpha_{14} \hat{h}_2 + e^{2\alpha_{13}} (\alpha_{14} \alpha_{15} + 1) \hat{h}_3 \\
&\quad + e^{-2\alpha_{12}} \alpha_{11} (\alpha_{14} \alpha_{15} + 1) - e^{-2\alpha_{13}} \alpha_{10} \alpha_{15} \hat{h}_4 + \left( 2e^{-2\alpha_{13}} \alpha_{10} - e^{-2\alpha_{12}} \alpha_{11} \alpha_{14} \alpha_{15} \right) \hat{h}_5 \\
&= \alpha_5 + e^{2\alpha_{13}} \alpha_{14} \hat{x} + e^{2\alpha_{13}} (\alpha_{14} \alpha_{15} + 1) \hat{y} + 2e^{-2\alpha_{12}} \alpha_9 (\alpha_{14} \alpha_{15} + 1) - e^{-2\alpha_{13}} \alpha_{10} \alpha_{15} \hat{p}_x \\
&\quad + \left( 2e^{-2\alpha_{13}} \alpha_{10} \alpha_{11} - e^{-2\alpha_{12}} \alpha_{11} \alpha_{14} \alpha_{15} \right) \hat{p}_y,
\end{align}

\begin{align}
\hat{p}_x H &= \hat{U} \hat{p}_x \hat{U}^\dagger = \hat{U} \hat{h}_3 \hat{U}^\dagger = -\alpha_2 \hat{h}_1 - (2e^{2\alpha_{12}} \alpha_6 + e^{2\alpha_{13}} \alpha_8 \alpha_{14}) \hat{h}_2 \\
&\quad - \left[ 2e^{2\alpha_{12}} \alpha_6 \alpha_{15} + e^{2\alpha_{13}} \alpha_8 (\alpha_{14} \alpha_{15} + 1) \right] \hat{h}_3 \\
&\quad + 2e^{-2\alpha_{13}} (\alpha_8 \alpha_{10} + \alpha_6 \alpha_{11}) \alpha_{15} - e^{-2\alpha_{12}} \left( 4\alpha_6 \alpha_9 + \alpha_8 \alpha_{11} - 1 \right) \alpha_{14} \alpha_{15} \hat{h}_4 \\
&\quad + \left[ 2e^{2\alpha_{12}} (4\alpha_6 \alpha_9 + \alpha_8 \alpha_{11} - 1) \alpha_{14} - 2e^{-2\alpha_{13}} (\alpha_8 \alpha_{10} + \alpha_6 \alpha_{11}) \right] \hat{h}_5 \\
&= -\alpha_2 + (2e^{2\alpha_{12}} \alpha_6 + e^{2\alpha_{13}} \alpha_8 \alpha_{14}) \hat{x} - \left[ 2e^{2\alpha_{12}} \alpha_6 \alpha_{15} + e^{2\alpha_{13}} \alpha_8 \alpha_{14} \alpha_{15} \right] \hat{y} \\
&\quad + \left[ 2e^{-2\alpha_{13}} (\alpha_8 \alpha_{10} + \alpha_6 \alpha_{11}) \alpha_{15} - e^{-2\alpha_{12}} \left( 4\alpha_6 \alpha_9 + \alpha_8 \alpha_{11} - 1 \right) \alpha_{14} \alpha_{15} \right] \hat{p}_x \\
&\quad + \left[ 2e^{-2\alpha_{12}} (4\alpha_6 \alpha_9 + \alpha_8 \alpha_{11} - 1) \alpha_{14} - 2e^{-2\alpha_{13}} (\alpha_8 \alpha_{10} + \alpha_6 \alpha_{11}) \right] \hat{p}_y,
\end{align}

\begin{align}
\hat{p}_y H &= \hat{U} \hat{p}_y \hat{U}^\dagger = \hat{U} \hat{h}_3 \hat{U}^\dagger = -\alpha_3 \hat{h}_1 - (2e^{2\alpha_{12}} \alpha_8 + e^{2\alpha_{13}} \alpha_7 \alpha_{14}) \hat{h}_2 \\
&\quad - \left[ 2e^{2\alpha_{12}} \alpha_8 \alpha_{15} + e^{2\alpha_{13}} \alpha_7 (\alpha_{14} \alpha_{15} + 1) \right] \hat{h}_3 \\
&\quad + e^{-2\alpha_{13}} (4\alpha_7 \alpha_{10} + \alpha_8 \alpha_{11} - 1) \alpha_{15} - 2e^{-2\alpha_{12}} (4\alpha_7 \alpha_{10} \alpha_9 + \alpha_7 \alpha_{11}) \alpha_{14} \alpha_{15} \hat{h}_4 \\
&\quad + \left[ 2e^{2\alpha_{12}} (4\alpha_7 \alpha_{10} \alpha_9 + \alpha_7 \alpha_{11} - 1) \alpha_{14} - 2e^{-2\alpha_{13}} (4\alpha_7 \alpha_{10} \alpha_8 \alpha_{11} - 1) \right] \hat{h}_5 \\
&= -\alpha_3 + (2e^{2\alpha_{12}} \alpha_8 + e^{2\alpha_{13}} \alpha_7 \alpha_{14}) \hat{x} - \left[ 2e^{2\alpha_{12}} \alpha_8 \alpha_{15} + e^{2\alpha_{13}} \alpha_7 \alpha_{14} \alpha_{15} \right] \hat{y} \\
&\quad + \left[ 2e^{-2\alpha_{13}} (4\alpha_7 \alpha_{10} + \alpha_8 \alpha_{11} - 1) \alpha_{15} - 2e^{-2\alpha_{12}} (4\alpha_7 \alpha_{10} \alpha_9 + \alpha_7 \alpha_{11}) \alpha_{14} \alpha_{15} \right] \hat{p}_x \\
&\quad + \left[ 2e^{-2\alpha_{12}} (4\alpha_7 \alpha_{10} \alpha_9 + \alpha_7 \alpha_{11} - 1) \alpha_{14} - 2e^{2\alpha_{13}} (4\alpha_7 \alpha_{10} + \alpha_8 \alpha_{11} - 1) \right] \hat{p}_y.
\end{align}

The propagator associated to the evolution operator (77) can be readily evaluated by separating the individual propagators corresponding to each of the 15 unitary transformations

\begin{align}
G(x, y; t; x', y', 0) &= \langle x, y | \hat{U}(t) | x', y' \rangle \\
&= \int dx_1 dy_1 \int dx_2 dy_2 \ldots \int dx_{n-1} dy_{n-1} \langle x, y | \hat{U}_1(t) | x_1, y_1 \rangle \langle x_1, y_1 | \hat{U}_2(t) | x_2, y_2 \rangle \\
&\quad \times \ldots \langle x_{n-2}, y_{n-2} | \hat{U}_{n-1}(t) | x_{n-1}, y_{n-1} \rangle \langle x_{n-1}, y_{n-1} | \hat{U}_n(t) | x', y' \rangle.
\end{align}
The propagators for the 15 unitary transformations are given by

\[
\begin{align*}
\langle x, y | \hat{U}_1^\dagger (t) | x_1, y_1 \rangle &= \exp \left( -\frac{i}{\hbar} \alpha_1 \right) \delta (x - x_1) \delta (y - y_1), \\
\langle x_1, y_1 | \hat{U}_2^\dagger (t) | x_2, y_2 \rangle &= \exp \left( -\frac{i}{\hbar} \alpha_2 x_2 \right) \delta (x_1 - x_2) \delta (y_1 - y_2), \\
\langle x_2, y_2 | \hat{U}_3^\dagger (t) | x_3, y_3 \rangle &= \exp \left( -\frac{i}{\hbar} \alpha_3 y_3 \right) \delta (x_2 - x_3) \delta (y_2 - y_3), \\
\langle x_3, y_3 | \hat{U}_4^\dagger (t) | x_4, y_4 \rangle &= \delta (x_3 - x_4 - \alpha_4) \delta (y_3 - y_4), \\
\langle x_4, y_4 | \hat{U}_5^\dagger (t) | x_5, y_5 \rangle &= \delta (x_4 - x_5) \delta (y_4 - y_5 - \alpha_5), \\
\langle x_5, y_5 | \hat{U}_6^\dagger (t) | x_6, y_6 \rangle &= \exp \left( -\frac{i}{\hbar} \alpha_6 x_6 \right) \delta (x_5 - x_6) \delta (y_5 - y_6), \\
\langle x_6, y_6 | \hat{U}_7^\dagger (t) | x_7, y_7 \rangle &= \exp \left( -\frac{i}{\hbar} \alpha_7 y_7 \right) \delta (x_6 - x_7) \delta (y_6 - y_7), \\
\langle x_7, y_7 | \hat{U}_8^\dagger (t) | x_8, y_8 \rangle &= \exp \left( -\frac{i}{\hbar} \alpha_8 x_8 y_8 \right) \delta (x_7 - x_8) \delta (y_7 - y_8), \\
\langle x_8, y_8 | \hat{U}_9^\dagger (t) | x_9, y_9 \rangle &= \exp \left( -\frac{i}{\hbar} \alpha_9 x_9 \right) \delta (y_8 - y_9), \\
\langle x_9, y_9 | \hat{U}_{10}^\dagger (t) | x_{10}, y_{10} \rangle &= \delta (x_9 - x_{10}) \exp \left( -\frac{i}{\hbar} \alpha_{10} \delta (y_9 - y_{10}) \right), \\
\langle x_{10}, y_{10} | \hat{U}_{11}^\dagger (t) | x_{11}, y_{11} \rangle &= \exp \left( -\frac{i}{\hbar} \alpha_1 x_{11} \right) \left[ \delta (y_9 - y_{10}) \right], \\
\langle x_{11}, y_{11} | \hat{U}_{12}^\dagger (t) | x_{12}, y_{12} \rangle &= \exp (-\alpha_{12} \delta (e^{-2\alpha_{12}} x_{11} - x_{12}) \delta (y_{11} - y_{12}), \\
\langle x_{12}, y_{12} | \hat{U}_{13}^\dagger (t) | x_{13}, y_{13} \rangle &= \exp (-\alpha_{13} \delta (x_{12} - x_{13}) \delta (e^{-2\alpha_{13}} y_{12} - y_{13}), \\
\langle x_{13}, y_{13} | \hat{U}_{14}^\dagger (t) | x_{14}, y_{14} \rangle &= \delta (x_{13} - x_{14}) \delta (y_{13} - y_{14} - \alpha_{13} x_{14}), \\
\langle x_{14}, y_{14} | \hat{U}_{15}^\dagger (t) | x', y' \rangle &= \delta (x_{14} - x' - \alpha_{15} y') \delta (y_{14} - y').
\end{align*}
\]

After substituting (83)-(97) in to the general expression for the propagator (82) and integrating, the Green function takes the final form

\[
G(x, y, t; x', y', 0) = \frac{(1 + i)^2 \eta}{4\pi \hbar \alpha_11} \exp (\alpha_{12} + \alpha_{13})
\times \exp \left\{ -\frac{i}{\hbar} \left[ \left( \frac{4\alpha_9 \alpha_6 - 1}{4\alpha_9} \right) (x - \alpha_4)^2 + \alpha_7 (y - \alpha_5)^2 + \alpha_8 (x - \alpha_4) (y - \alpha_5) + \frac{f}{\alpha_{11}} (y - \alpha_5) + \alpha_3 y + \alpha_2 x + \frac{\alpha_{10}}{\alpha_{11}} f + \alpha_1 \right] \right\}
\times \exp \left\{ -\frac{i}{\hbar} \alpha_9 \eta^2 \left[ \frac{x - \alpha_4}{2\alpha_9} - \frac{y - \alpha_5}{\alpha_{11}} + \frac{1}{\alpha_{11}} \left( g - \frac{2\alpha_{10}}{\alpha_{11}} f \right) \right]^2 \right\},
\]

where

\[
\begin{align*}
f &= \exp (2\alpha_{12}) \left( x' + \alpha_{15} y' \right), \\
g &= \exp (2\alpha_{13}) \left( y' + \alpha_{14} x' + \alpha_{14} \alpha_{15} y' \right), \\
\eta^2 &= \frac{\alpha_{11}^2}{\alpha_{11}^2 - 4\alpha_9 \alpha_{10}}.
\end{align*}
\]
IV. TWO-DIMENSIONAL CHARGED PARTICLE SUBJECT TO AN IN-PLANE ELECTRIC FIELD AND A PERPENDICULAR MAGNETIC FIELD

In order to illustrate the use of the Lie algebraic approach let us study the dynamics of a two-dimensional charged particle subject to an in-plane time dependent electric field and a perpendicular magnetic field given by the Hamiltonian in Eq. (32). In this case $a_2 = eE_x, a_3 = eE_y, a_6 = a_7 = m\omega_c^2/8, a_9 = a_{10} = 1/2m, a_{14} = -a_{15} = \omega_c/2$ and $a_1 = a_4 = a_5 = a_8 = a_{11} = a_{12} = a_{13} = 0$ where $\omega_c = eB/m$ is the cyclotron frequency. For the sake of simplicity we consider the case where $E_x, E_y$ and $B$ are constant although the more general case where these quantities are time-dependent can, in principle, be dealt with [61]. Substituting the previous parameters into the system of ordinary differential equations given by (55)-(69) we obtain the explicit form of the $\alpha$ parameters. As stated above, the generators corresponding to the first five parameters form a closed sub-algebra of $\mathcal{L}_{15}$ therefore the first five differential equations (55)-(59) may be solved independently from the rest of the system. The solution to the first five differential equations is

$$
\begin{align*}
\alpha_1 &= \frac{e^2}{2m\omega_c^2} \left( E_x^2 + E_y^2 \right) \left( \sin \omega_c t - \omega_c t \cos \omega_c t \right), \\
\alpha_2 &= -\frac{eE_y}{2\omega_c} + e \left( \frac{E_x}{2\omega_c} \right) \left( E_x \sin \omega_c t + E_y \cos \omega_c t \right), \\
\alpha_3 &= \frac{eE_x}{2\omega_c} + e \left( \frac{E_y}{2\omega_c} \right) \left( E_y \sin \omega_c t - E_x \cos \omega_c t \right), \\
\alpha_4 &= -\frac{eE_x}{m\omega_c} + \frac{e}{m\omega_c} \left( E_y \sin \omega_c t - E_x \cos \omega_c t \right), \\
\alpha_5 &= -\frac{eE_y}{m\omega_c} + \frac{e}{m\omega_c} \left( E_x \sin \omega_c t + E_y \cos \omega_c t \right),
\end{align*}
$$

where it can be easily verified that these functions correspond to the classical solution for the position and momentum of a charged particle moving in constant and uniform electromagnetic fields. The generators $\hat{h}_6, \hat{h}_7$ and $\hat{h}_8$ also form a sub-algebra of $\mathcal{L}_{15}$ and therefore yield three differential equations that may be solved for $\alpha_6, \alpha_7, \alpha_8$, apart from the remaining differential equations. By recasting Eqs. (60)-(62) in terms of $\alpha_6 - \alpha_7, \alpha_6 + \alpha_7$ and $\alpha_8$ and reminding that the initial conditions are $\alpha_6(0) = \alpha_7(0) = \alpha_8(0) = 0$ the solution for these three parameters is readily obtained as

$$
\alpha_6 = \alpha_7 = \frac{m\omega_c}{4} \tan \frac{\omega_c t}{2}, \quad \alpha_8 = 0.
$$

The following three generators $\hat{h}_9, \hat{h}_{10}$ and $\hat{h}_{11}$ do not form a closed algebra and therefore the corresponding differential equations have dependencies in parameters others than $\alpha_9, \alpha_{10}$ and $\alpha_{11}$. However, having obtained $\alpha_6, \alpha_7$ and $\alpha_8$ and by rewriting Eqs. (63)-(65) in terms of $\alpha_9 - \alpha_{10}, \alpha_9 + \alpha_{10}$ and $\alpha_{11}$ we find

$$
\alpha_9 = \alpha_{10} = \frac{1}{m\omega_c} \cos \frac{\omega_c t}{2} \sin \frac{\omega_c t}{2}, \quad \alpha_{11} = 0,
$$

where we have used $\alpha_9(0) = \alpha_{10}(0) = \alpha_{11}(0) = 0$. After substituting the results for $\alpha_6, \alpha_7, \alpha_8$ in the differential equations (66)-(68) and rewriting for $\alpha_{13} - \alpha_{12}, \alpha_{13} + \alpha_{12}$ and $\alpha_{14}$ the three differential equations yield the following Riccati differential equation

$$
\dot{\beta} - \beta^2 - \left( \frac{\omega_c}{2} \right)^2 = 0,
$$

where $\beta = \dot{\alpha}_{13} - \dot{\alpha}_{12}$. Solving this equation and using the initial conditions $\alpha_{12}(0) = \alpha_{13}(0) = \alpha_{14}(0) = 0$ the solution for $\alpha_{12}, \alpha_{13}$ and $\alpha_{14}$ is readily found

$$
\begin{align*}
\alpha_{12} &= \ln \left( \cos \frac{\omega_c t}{2} \right), \quad \alpha_{13} = 0, \quad \alpha_{14} = \cos \frac{\omega_c t}{2} \sin \frac{\omega_c t}{2},
\end{align*}
$$

The parameter $\alpha_{15}$ is calculated by direct integration of Eq. (69) giving

$$
\alpha_{15} = -\tan \frac{\omega_c t}{2}.
$$
Upon replacing the explicit form of the parameters $\alpha_6-\alpha_{15}$ in Eqs. (103), (104), (106) and (107) in the general form for the Heisenberg picture position and momentum operators (78)-(81) we get

$$
\hat{x}_H = \alpha_1 + \cos \frac{\omega_c t}{2} \left( \cos \frac{\omega_c t}{2} \hat{x} - \sin \frac{\omega_c t}{2} \hat{y} \right) \\
\hat{y}_H = \alpha_5 + \cos \frac{\omega_c t}{2} \left( \sin \frac{\omega_c t}{2} \hat{x} + \cos \frac{\omega_c t}{2} \hat{y} \right) \\
\hat{p}_x H = -\alpha_2 - \frac{m\omega_c}{2} \sin \frac{\omega_c t}{2} \left( \cos \frac{\omega_c t}{2} \hat{x} - \sin \frac{\omega_c t}{2} \hat{y} \right) \\
\hat{p}_y H = -\alpha_3 - \frac{m\omega_c}{2} \sin \frac{\omega_c t}{2} \left( \sin \frac{\omega_c t}{2} \hat{x} + \cos \frac{\omega_c t}{2} \hat{y} \right)
$$

Finally, replacing the explicit form of the $\alpha_6-\alpha_7$ parameters, the general form of the propagator (82) yields the expression

$$
G(x, y, t; x', y', 0) = \frac{m\omega_c}{4\pi \hbar \sin \frac{\omega_c t}{2}} \exp \left[ -\frac{i}{\hbar} \left( \alpha_1 + \alpha_2 x + \alpha_3 y \right) \right] \\
\times \exp \left[ \frac{im\omega_c}{4\hbar \sin \frac{\omega_c t}{2}} \left\{ \left( x - \alpha_4 \right)^2 + \left( y - \alpha_5 \right)^2 + \left( x' - \alpha_4 \right)^2 + \left( y' - \alpha_5 \right)^2 \right\} \cos \frac{\omega_c t}{2} \\
- 2 \cos \frac{\omega_c t}{2} x' (x - \alpha_4) - 2 \sin \frac{\omega_c t}{2} y' (y - \alpha_5) \\
+ 2 \sin \frac{\omega_c t}{2} y' (x - \alpha_4) - 2 \cos \frac{\omega_c t}{2} y' (y - \alpha_5) \right\} \right].
$$

Comparable results for the propagator and the Heisenberg picture position and momentum operator have been obtained via the path integral method [50] or time-dependent perturbation approach of the Fock-Darwin Hamiltonian [62].

V. CONCLUSIONS

We have developed a systematic method based on the Lie algebraic approach to obtain the evolution operator and its corresponding propagator for the generalized two-dimensional quadratic Hamiltonian. This method relies on the possibility of expressing the Hamiltonian as a linear combination of elements that form Lie algebra with coefficients that in general are time-dependent functions. In this case the evolution operator is a member of the unitary group generated by these elements, and therefore is expressible in terms of the elements of the same algebra and the corresponding time-dependent transformation parameters.

Finding the explicit time-dependence of the the transformation parameters determines completely the evolution operator. Therefore we have exploited the properties of Hamiltonians having a dynamical algebra to find analytical expressions for the ordinary differential equations that govern the dynamics of the transformation parameters.

Even though the method presented here is mainly intended to obtain the evolution operator for the generalized two-dimensional quadratic Hamiltonian, the results from Section II are general enough that may be applied to any Hamiltonian having a dynamical algebra with a large number of elements.

To illustrate the method we have presented the example of a two-dimensional charged particle in uniform electromagnetic fields. The obtained propagator and Heisenberg picture position and momentum operators are consistent
with the ones calculated with the path integral method [50] and time-dependent perturbation theory of the Fock-Darwin Hamiltonian [62].

The rather general form of the two-dimensional quadratic Hamiltonian allows this method to tackle a wide variety of significant physical situations such as two-dimensional single electrons trapped in asymmetric quantum dots with parabolic confinement, or charged particle subject to time-varying uniform electro-magnetic fields among others.

VI. ACKNOWLEDGMENTS

The authors would like to thank the “Departamento de Ciencias Básicas UAM-A” for the financial support. J. C. Sandoval-Santana and V. Ibarra-Sierra would like to acknowledge the support received from “Becas de Posgrado UAM”.

Appendix A: Usefull relations

Since it is widely used to calculate most of the transformation rules, we enunciate the next commutation relation. If the commutor

$$[\hat{A}, \hat{B}] = \hat{C},$$

(A1)

commutes with the operators $\hat{A}$ and $\hat{B}$, i.e.

$$[\hat{A}, \hat{C}] = [\hat{B}, \hat{C}] = 0.$$  

(A2)

then it follows that

$$[\hat{A}, F(\hat{B})] = [\hat{A}, \hat{B}] \frac{\partial F(\hat{B})}{\partial \hat{B}},$$

(A3)

provided that $F$ is an analytical function.

[1] G. Harari, Y. Ben-Aryeh, and A. Mann, Phys. Rev. A 84, 062104 (2011).
[2] L. Yan, M. Feng, and K. Wang, Phys. Rev. A 89, 035401 (2014).
[3] T. S. H"aberle and M. Freyberger, Phys. Rev. A 89, 052332 (2014).
[4] L. S. Brown, Phys. Rev. Lett. 66, 527 (1991).
[5] J. I. Cirac, L. J. Garay, R. Blatt, A. S. Parksins, and P. Zoller, Phys. Rev. A 49, 421 (1994).
[6] D. Leibfried, R. Blatt, C. Monroe, and D. Wineland, Rev. Mod. Phys. 75, 281 (2003).
[7] S. Mavadia, G. Stutter, J. F. Goodwin, D. R. Crick, R. C. Thompson, and D. M. Segal, Phys. Rev. A 89, 032502 (2014).
[8] K. Abe and T. Hasegawa, Phys. Rev. A 81, 033402 (2010).
[9] B. Baseia, S. S. Mizrahi, and M. H. Y. Moussa, Phys. Rev. A 46, 5885 (1992).
[10] A. B. Nassar, Journal of Optics B: Quantum and Semiclassical Optics 4, S226 (2002).
[11] S. K. Singh and S. Mandal, Optics Communications 283, 4685 (2010).
[12] S. Mandal, Physics Letters A 321, 308 (2004).
[13] A. L. Matacz, Phys. Rev. D 49, 788 (1994).
[14] I. Pedrosa, C. Furtado, and A. Rosas, Physics Letters B 651, 384 (2007).
[15] D. G. Vergel and E. J. Villaseor, Annals of Physics 324, 1360 (2009).
[16] P. Caldirola, Nuovo Cimento 18, 393 (1941).
[17] E. Kanai, Prog. Theor. Phys. 3, 440 (1948).
[18] H. Bateman, Phys. Rev. 38, 815 (1931).
[19] C.-I. Um, K.-H. Yeon, and T. F. George, Physics Reports 362, 63 (2002).
[20] J. M. Manoyan, Journal of Physics A: Mathematical and General 19, 3013 (1986).
[21] K. H. Yeon, C. I. Um, and T. F. George, Phys. Rev. A 68, 052108 (2003).
[22] V. Ibarra-Sierra, A. Anzaldo-Meneses, J. Cardoso, H. H.-S. na, A. Kunold, and J. Roa-Neri, Annals of Physics 335, 86 (2013).
[23] Q. Shi, M. Khodas, A. Levchenko, and M. A. Zudov, Phys. Rev. B 88, 245409 (2013).
[24] X. L. Lei and S. Y. Liu, Journal of Applied Physics 115, 233711 (2014).
[25] J. Iarrea, Physica B: Condensed Matter 436, 10 (2014).
[26] A. Kunold and M. Torres, Physica B: Condensed Matter 425, 78 (2013).
[27] Y. Ben-Aryeh, Journal of Physics A: Mathematical and Theoretical 42, 055307 (2009).
[28] I. Guedes, Phys. Rev. A 63, 034102 (2001).
[29] H. Bekkar, F. Benamira, and M. Maamache, Phys. Rev. A 68, 016101 (2003).
[30] I. Urdaneta, L. Sandoval, and A. Palma, Journal of Physics A: Mathematical and Theoretical 43, 385204 (2010).
[31] A. Palma, M. Villa, and L. Sandoval, International Journal of Quantum Chemistry 112, 2441 (2012).
[32] R. V. Bunyi, F. Colombo, I. Sabadini, and D. C. Struppa, Journal of Mathematical Physics 55, 113511 (2014).
[33] Z. Yang, Journal of Mathematical Physics 56, 032102 (2015), http://dx.doi.org/10.1063/1.4914337.
[34] H. R. Lewis and W. B. Riesenfeld, Journal of Mathematical Physics 10, 1458 (1969).
[35] A. L. de Lima, A. Rosas, and I. Pedrosa, Annals of Physics 323, 2253 (2008).
[36] Y. Saadi, J. R. Choi, and K. H. Yeon, Journal of the Korean Physical Society 56, 1063 (2010).
[37] K. Hira, European Journal of Physics 34, 777 (2013).
[38] D. C. Khandekar and S. V. Lawande, Journal of Mathematical Physics 20, 1870 (1978).
[39] R. P. Feynman, Rev. Mod. Phys. 20, 367 (1948).
[40] R. P. Feynman, Phys. Rev. 80, 440 (1950).
[41] E. Merzbarcher, “Quantum mechanics,” (John Wiley & Sons, Inc., USA, 1998) Chap. 15, 3rd ed.
[42] C.-Y. Long, S.-J. Qin, Z.-H. Yang, and G.-J. Guo, International Journal of Theoretical Physics 48, 981 (2009).
[43] M. Feng, Phys. Rev. A 64, 034101 (2001).
[44] P.-G. Luan and C.-S. Tang, Phys. Rev. A 71, 014101 (2005).
[45] G. Profilo and G. Soliana, Phys. Rev. A 44, 2057 (1991).
[46] I. A. Pedrosa, Phys. Rev. A 55, 3219 (1997).
[47] V. V. Dodonov and V. I. Man’ko, Phys. Rev. A 20, 550 (1979).
[48] K.-H. Yeon, S.-S. Kim, Y.-M. Moon, S.-K. Hong, C.-I. Um, and T. F. George, Journal of Physics A: Mathematical and General 34, 7719 (2001).
[49] K.-H. Yeon, D.-H. Kim, C.-I. Um, T. F. George, and L. N. Pandey, Phys. Rev. A 55, 4023 (1997).
[50] B. K. Cheng, Journal of Physics A: Mathematical and General 17, 819 (1984).
[51] M.-L. Liang and F.-L. Zhang, Physica Scripta 73, 677 (2006).
[52] C. A. S. Ferreira, P. T. S. Alencar, and J. M. F. Bassalo, Phys. Rev. A 66, 024103 (2002).
[53] M. Maamache, A. Bounames, and N. Ferkous, Phys. Rev. A 73, 016101 (2006).
[54] M. S. Abdalla and P. G. L. Leach, Journal of Mathematical Physics 52, 083504 (2011).
[55] W. Magnus, Communications on Pure and Applied Mathematics 7, 649 (1954).
[56] J. Wei and E. Norman, Journal of Mathematical Physics 4, 575 (1963).
[57] Y. Alhassid and R. D. Levine, Phys. Rev. A 18, 89 (1978).
[58] C. M. Cheng and P. C. W. Fung, Journal of Physics A: Mathematical and General 21, 4115 (1988).
[59] F. Boldt, J. D. Nulton, B. Andresen, P. Salamon, and K. H. Hoffmann, Phys. Rev. A 87, 022116 (2013).
[60] H. Breuer and F. Petruccione, “The theory of open quantum systems,” (Oxford University Press, New York, 2006) Chap. 8.
[61] V. Ibarra-Sierra, J. Sandoval-Santana, J. Cardoso, and A. Kunold, Annals of Physics 362, 83 (2015).
[62] J. E. Santos, N. M. R. Peres, and J. a. M. B. L. dos Santos, Phys. Rev. A 80, 053401 (2009).