Effects of time delay in no-knowledge quantum feedback control

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Abstract. The no-knowledge quantum feedback, introduced in Phys. Rev. Lett., 113, 020407 (2014), is a measurement-based feedback protocol for decoherence suppression in a quantum system coupled to noisy environment. By continuously measuring the environmental noise, without directly gathering any information about the system, the decoherence effect can be suppressed by feeding back quantum controls proportional to the measured signal. In the original work, the feedback control was assumed instantaneous, leading to perfect cancellation of noise backaction on the quantum system. However, the instantaneous feedback is difficult to achieve in practice, and close-loop feedback protocols are always associated with finite delayed time. Therefore, in this work, we consider the effects of the delay between the time at which the measurement signal is acquired and the time that such signal is fed back to the system. We investigate the problem with an example of a two-level system (qubit) coupled to a Markovian reservoir, via a Hermitian coupling operator, where a homodyne detection is used to measure the environmental noise. We numerically simulate quantum stochastic trajectories of the qubit and analyse their averaged dynamics. We find that the feedback control with time delay can either enhance or reduce the decoherence effects, depending on whether the delayed time is in-phase or out-of-phase with the unitary dynamics of the qubit system.

1. Introduction

Emerging quantum technologies such as quantum computing, quantum metrology, quantum communication and quantum information processing have shown great potential and promises [1–6]. The possibility to realize these technologies rely heavily on the ability to keep individual quantum systems highly coherence. Since quantum systems are susceptible to decoherence from their surrounding [7,8], it is the most important task to search for techniques that can minimize decoherence effect while maintaining necessary controls and high fidelity readout. There have been many proposed techniques on suppressing decoherence for different systems [9–12].

In a close quantum system, the evolution of its quantum state can be completely characterized by a unitary operator $\hat{U}(t) = e^{it\hat{H}/\hbar}$, which depends on the nature of the Hamiltonian operator $\hat{H}$. However, in an open quantum system, where the system also interacts with its environment, there may be energy dissipation to the environment [13,14] and other decoherence effects, turning a quantum system into a classical one [8,15,16]. Apart from the unitary and decoherence effects, there is also a measurement backaction, which refers to the effect from measuring the quantum system, or from information gathering via detection devices. The measurement process can be
continuous in time, which will not collapse the state instantaneously to observable eigenstates, but gradually change the state in a stochastic sense [17, 18].

In this work, we are interested in decoherence suppression process with continuous quantum measurement. One approach to suppress decoherence is to perform a protocol similar to noise cancellation. It involves measuring noises affecting the systems and then feedback controlling the systems to counteract the decoherence effects from noises. The process is preferably done continuously in time, and therefore one can use the quantum trajectory theory [19, 20] to describe the dynamics of quantum systems under continuous measurement and control. We consider a specific case of a qubit being continuously measured with a homodyne measurement, with a coupling operator $\hat{L}$, a homodyne phase $\theta$, and an efficiency $\eta = 1$. We can write the unnormalized Stratonovich stochastic differential equation for a quantum trajectory as [21]:

$$
\partial_t \rho_t = \mathcal{L} \rho_t + \mathcal{A}[\hat{L} e^{i\theta}][\hat{n} y_0(t)] - \frac{1}{2} \mathcal{A}^2[\hat{L} e^{i\theta}] \rho_t,
$$

where $\rho_t$ is the unnormalized density operator and it can be normalized as $\rho_t = \rho_t / \text{Tr}[\rho_t]$. We note that the superoperators $\mathcal{L}$, $\mathcal{D}$ and $\mathcal{A}$ are defined as $\mathcal{L} \rho_t = -i [\hat{H}, \rho_t] + \mathcal{D}[\hat{L} \rho_t]$, $\mathcal{D}[\hat{Z} \rho_t] = \hat{Z} \rho_t \hat{Z}^\dagger - \frac{1}{2} (\hat{Z}^\dagger \hat{Z} \rho_t + \rho_t \hat{Z}^\dagger \hat{Z})$, $\mathcal{A}[\hat{Z} \rho_t] = \hat{Z} \rho_t \hat{Z}^\dagger$, and $\mathcal{A}^2[\hat{Z} \rho_t] = \hat{Z} [\mathcal{A}[\hat{Z} \rho_t]]$. In equation (1), the first term is the well-known Lindblad term, which consists of the unitary evolution and decoherence. The second term is a stochastic term due to the measurement backaction. The last term is the Stratonovich correction term [22]. The measurement result $y_0(t) = \langle \hat{L} e^{i\theta} + \hat{L} e^{-i\theta} \rangle + \xi(t)$ comprises two parts: the part containing the system’s information and the part containing only the stochastic noise $\xi(t)$.

We follow the no-knowledge quantum feedback protocol introduced in [21], which occurs when $\hat{L}$ is Hermitian and setting $\theta = \pi/2$ in equation (1). The no-knowledge measurement does not extract any information about the system, thus only acquire information about the noise signal, i.e., $y_{\pi/2}(t) = \xi(t)$. Upon using this setting, the stochastic master equation then becomes

$$
\partial_t \rho_t = -\frac{i}{\hbar} [\hat{H} - \hat{L} y_{\pi/2}(t), \rho_t].
$$

From equation (2), it is clear that the decoherence can be suppressed and controlled by simultaneously feeding back the measured noise signal in the form of $\hat{H} \rightarrow \hat{H}_{\text{eff}} := \hat{H} + \hat{L} y_{\pi/2}(t)$. By substituting this form of $\hat{H}$, the dynamics becomes purely the unitary evolution $\partial_t \rho_t = -\frac{i}{\hbar} [\hat{H}_{\text{eff}}, \rho_t]$. However, this instantaneous feedback is impractical. In practice, there is always time delay due to measurement and signal processing, which leads us to investigating the effects of time delay in the no-knowledge quantum feedback control.

2. Numerical simulation using discrete-time operation

We will investigate the delay for the no-knowledge quantum feedback protocol using discrete-time operation numerical simulation. The discrete-time operation refers to discretizing the time to equal steps of size $dt$. The quantum dynamics is obtained by applying the usual quantum operations, e.g., unitary, decoherence, or measurement operations, to quantum states, resulting in system’s dynamics that converge to continuous-time limit when the step size $dt \rightarrow 0$. The reason we choose to work in discrete time is that we can manipulate the delayed time simply as integers of the time step.

We first rewrite equation (2) as a composition of two different operators; namely, the unitary operator $\hat{U} = e^{-i\hat{H} dt}$ and the measurement operator $\hat{M} = e^{i\hat{L} y_0 dt}$. Here, the Planck’s constant $\hbar$ has been set to one. In the next infinitesimal time step $dt$, an unnormalized state can be expressed as $\hat{\rho}(t + dt) = \hat{U} \hat{M} \hat{\rho}(t) \hat{M}^\dagger \hat{U}^\dagger$. As noted earlier, this discrete-time representation is...
equivalent to equation (2) when we expand the exponential operators in Taylor’s series and keep only terms of an order $O(dt^2)$. This approximation is valid when $dt$ is sufficiently small, and the contribution of higher-order terms can be ignored. Moreover, the ordering of $\hat{M}$ and $\hat{U}$ is also irrelevant for such an infinitesimal scale of $dt$, because any errors terms with a commutator $[\hat{U}, \hat{M}]$ are of an order $O((dt)^2)$ [23, 24].

This discrete-time operation method can be useful in numerical simulation, especially for feedback with time delay. This is because we can easily add feedback operator at anytime during the evolution. We denote a feedback operator $\hat{F}_t$, where the subscript $t$ refers to the time when the feedback is applied to the system. With the time delay, the feedback operator is a function of the actual measurement signal $y_\theta$, not at the same time $t$, but at earlier time $t - \tau$. That is, the feedback process is delayed by a duration $\tau$. Following the no-knowledge protocol, the feedback operation can be written as an inverse of the measurement operation, i.e., $\hat{F}_t = \hat{M}_t^\dagger = e^{-iL\rho(t-\tau)dt}$. The evolution of the quantum state when the feedback is turned on is then given by $\hat{\rho}_F(t + dt) = \hat{F}_t \hat{U} \hat{\rho}(t) \hat{M}_t \hat{U}^\dagger \hat{F}_t^\dagger$.

We can summarize our numerical simulation procedure as the following steps: (i) First, discretize the total time interval to $N$ steps of the size $dt$. (ii) Set up an initial state $\rho(0)$. (iii) For $t < \tau$ (no feedback), compute unnormalized state dynamics using $\hat{\rho}(t + dt) = \hat{M}_t \hat{\rho}(t) \hat{M}_t^\dagger \hat{U}^\dagger$, then normalize the state using $\rho(t + dt) = \hat{\rho}(t + dt)/\text{Tr}[\hat{\rho}(t + dt)]$. (iv) For $t \geq \tau$ (with feedback), compute unnormalized state dynamics using $\hat{\rho}_F(t + dt) = \hat{F}_t \hat{U} \hat{\rho}(t) \hat{M}_t \hat{U}^\dagger \hat{F}_t^\dagger$, and normalize the state as before. Then, we repeat the steps (i) through (iv) for a large number of realizations (5000 realizations), and then average them to determine mean quantum state dynamics.

3. Results and discussion

We are interested in a case where a qubit is coupled to a Markovian environment via a Hermitian coupling operator $\hat{L} = \sigma_z$. The qubit’s Hamiltonian for its unitary dynamics is $H = \Omega \sigma_z$. We have implemented the measurement and the feedback protocols as described in the previous section. We have simulated qubit’s trajectories using Python programming, generating random white noises $\xi(t)$ with a Gaussian probability distribution of mean zero and a standard deviation $\sqrt{dt}$. The numerical parameters are set as the following: (i) Time step $dt = 10^{-5}$; (ii) Homodyne phase $\theta = \pi/2$; (iii) Qubit’s initial state $\rho(0) = \frac{1}{2} (\mathbb{1} + \frac{1}{\sqrt{2}} (\sigma_x + \sigma_y))$. In order to investigate the effect of feedback with time delay, we consider three cases of a qubit dynamics: the case with only measurement and feedback operations, the case with added commuting unitary dynamics, and the case with added non-commuting unitary dynamics.

3.1. Measurement and feedback

In the first case, which is the simplest case, we neglect the dynamics from the qubit’s unitary evolution and only consider the measurement backaction and the feedback. The measurement backaction is described by $\hat{\rho}(t + dt) = \hat{M} \hat{\rho}(t) \hat{M}^\dagger$, which gives,

$$\hat{\rho}(t + dt) = \rho(t) - ig \xi dt [\hat{L}, \rho] + dt \mathcal{D}[\hat{L}] \rho \tag{3}$$

where its ensemble average (denoted by $\mathbb{E}[\rho]$) can be computed from

$$\mathbb{E}[\rho(t + dt)] = \rho(t) + dt \mathcal{D}[\hat{L}] \rho(t) = e^{\mathcal{L}dt} \rho(t). \tag{4}$$

When the feedback is implemented, we expect to see that the feedback operator cancels the measurement backaction from the earlier times. Because of the commutative property between the measurement and feedback operators in this case, that is $[\hat{M}, \hat{F}] = 0$, the feedback can cancel the measurement backaction perfectly. Hence, we can obtain an analytic solution for the
Figure 1. Numerical simulation of qubit trajectories showing the mean trajectories of the state with different values of delay time (labeled in increments of $dt$). (1a): the final amplitude is stabilized when feedback is turned on. (1b): the qubit’s state follows the Lindblad trajectory (labeled as “no fb”) before $t = \tau$, but the feedback restores the amplitude of the Rabi oscillation.

The averaged trajectory as $\rho(t) = \begin{cases} e^{Lt}, & t < \tau \\ e^{L\tau}, & t \geq \tau \end{cases}$ which describes the dynamics that follows the usual Lindblad decay until the feedback starts. Since the feedback completely cancels the noise effects, the qubit’s dynamics is stabilized at constant values after the feedback control is on. We show in Figure 1a the stabilization effects for different delayed times.

3.2. Commuting measurement, feedback and unitary operators

In the second case, we add unitary dynamics for the qubit $\hat{U} = e^{-i\hat{H}dt}$ such that $[\hat{H}, \hat{L}] = 0$, that is the unitary dynamics commutes with the measurement backaction. In this case, we choose $\hat{H} = \hat{L} = \hat{\sigma}_z$. Therefore, the feedback operation is able to cancel the effect of noise measurement similar to the previous case. However, the dynamics after backaction cancellation is not constant, but is rather the pure unitary dynamics.

In Figure 1b, the dynamics of the qubit follows the Lindblad decay before the feedback (the pink line with a label “no fb”) and then it follows the Rabi oscillation after the feedback is on. However, the restoration of the Rabi oscillation is not perfect. This is because the state right before the feedback started is no longer a pure state, as compared to the actual initial state. The effect of the Lindblad decay reduces the purity of the qubit’s state, yielding a mixed state.

3.3. Non-commuting measurement, feedback and unitary operators

In the third case, we choose the Hamiltonian of the unitary dynamics $\hat{H} = \hat{\sigma}_x$ that does not commute with the measurement $\hat{L} = \hat{\sigma}_z$. The delayed feedback appears to cause an interesting effect on the qubit dynamics. The simulation results shown in Figure 2a exhibit not only the amplitude damping, but also dephasing on the Rabi oscillation. When the time delay is large ($\tau > 300dt$ in Figure 2b), we can see that the averaged dynamics strangely deviates from the Lindblad dynamics. We speculate that the strange behavior is a result of feeding back control to the system when it is out of phase. We therefore consider choosing the delayed time to be an integer of the Rabi period $T$. A period of the qubit Rabi oscillation is given by the Rabi frequency in the Hamiltonian term as $T = 2\pi/\Omega$. Hence, we set $\Omega$ to be $2\pi$, therefore, $T = 1$. The time-delayed feedback provides an almost-perfect restoring results when $\tau$ is an integer multiplied by $1000dt$, which is exactly the period of the Rabi oscillation. When $\tau$ is a half
Figure 2. Quantum state dynamics for non-commuting Hamiltonian and feedback operators. (2a): the effects of time delay are displayed as damping and dephasing, shown for all three coordinates of the qubit Bloch sphere. (2b): the averaged trajectories when the feedback with time delay is applied. The destructive effects of the delayed feedback can be seen in the cases of \( \tau = 300dt, 500dt, \) and \( 700dt \).

Figure 3. Plots of qubit dynamics as a function of time, with the delayed feedback. The label “Lindblad” refers to the solution of the Lindblad equation \( \dot{\rho} = \mathcal{L}\rho(t) \equiv -i[H, \rho(t)] + \mathcal{D}[\hat{L}]\rho(t) \). (3a) shows averaged dynamics with \( \tau \) being an integer multiple of the Rabi period \( T = 2\pi/\Omega \). (3b) shows averaged dynamics with \( \tau \) being a half-integer multiple of the Rabi period \( T \).

To further explore, we simulate qubit trajectories when the delayed is in-phase (i.e. \( \tau = nT \)) and out-of-phase (i.e. \( \tau = \frac{1}{2}nt \)) for \( n = 1, 2, 3, 4, 5 \). We find that when the time delay is in-phase (see figure 3a), the feedback leads to a constructive effect (stabilization effect) where it reduces the amplitude damping rate of the system. However, the stabilization effect is not as good as in the commuting cases. On the other hand, for the out-of-phase time delay (see figure 3b), the destructive effects from feedback control is obvious, where it accelerates the decoherence rate.
4. Conclusion
We have investigated how the time delayed feedback affects the decoherence suppression offered by the no-knowledge quantum feedback. We consider a qubit evolution measured by a homodyne measurement and analyze the qubit’s averaged trajectories for different types of dynamics. We have shown that, if the qubit’s unitary dynamics commutes with the measurement operator (as well as the feedback operator), then the timing of the feedback delay is not important and the feedback can suppress the decoherence and stabilize the final quantum state. The purity of the stabilized state only depends on how fast the feedback is applied. The faster the feedback, the better the state purity. However, for the case when the unitary operator does not commute with the measurement one, the time delay can have detrimental effects on the averaged dynamics. The delayed feedback causes not only the amplitude damping, but also the Rabi oscillation dephasing. We have shown that the stabilization effect can still be achieved if the delayed feedback is in sync with the unitary Rabi oscillation. That is the delay time should be an integer multiple of the Rabi period in order to get an effective decoherence cancellation similar to the commuting case. Otherwise, the decoherence effect and the amplitude damping can be worse than the Lindblad dynamics, especially if feedback delayed times are at half-integer multiples of the Rabi period.

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