Some Finite Abelian Group Theory and Some $q$-Series Identities

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Abstract. The purpose of this article is to compute certain weighted sums over various subposets of the poset of isomorphism classes of finite abelian $\ell$-groups, where $\ell$ is a fixed odd prime. These sums are similar to previous sums computed by Hall and Cohen-Lenstra. The computation expands upon the previous analysis of Cohen-Lenstra while also using the tools of hypergeometric functions and $q$-series. The identities computed in this article, while aesthetically pleasing in their own right, also turn out to be applicable to the construction of a random matrix version of the Cohen-Lenstra-Martinet heuristics.

Keywords: posets, finite abelian groups, $q$-series

1. Background

1.1. Introduction

In 1984, Cohen and Lenstra [2] famously presented a family of conjectures about the structure of ideal class groups of number fields. These conjectures follow from this heuristic:

Heuristic 1.1. For any odd prime $\ell$, a finite abelian $\ell$-group should appear as the $\ell$-Sylow subgroup of the ideal class group of an imaginary quadratic extension of $\mathbb{Q}$ with frequency inversely proportional to the order of its automorphism group.

Their heuristic leads to a probability distribution on the poset of isomorphism classes of finite abelian $\ell$-groups. They computed various averages on this poset using this distribution, which lead directly to their conjectures. Along the way, they proved a variety of identities concerning finite abelian $\ell$-groups, such as the identity of Hall [9], which shows that their heuristic does give a probability distribution:

$$\sum_{\text{finite abelian } \ell \text{-groups } A} \frac{1}{|\text{Aut}A|} = \prod_{i=1}^{\infty} \left( 1 - \ell^{-i} \right)^{-1}.$$
More recently, Delaunay and Delaunay-Jouhet ([3–5]) computed new averages using this distribution. They proceeded to use these averages to study the structure of Tate-Shafarevich groups and Selmer groups of elliptic curves. Moreover, they showed that the averages they computed are consistent with the conjectures of Poonen-Rains ([12]). Like Cohen and Lenstra, they discovered new and interesting identities along the way to compute these averages.

In this paper, we continue the tradition of studying finite abelian group identities to investigate number theoretic phenomena. The simplest identity is Lemma 3.2: for any finite abelian $\ell$-group $A$ of $\ell$-rank $r$,

$$
\sum_{\text{finite abelian } \ell\text{-groups } B \atop \text{rank}_\ell B = r} \frac{|\{C \leq B \mid C \cong A\}|}{|\text{Aut } B|} = \frac{1}{|\text{Aut } A|} \prod_{i=1}^{r} \left(1 - \ell^{-i}\right)^{-1};
$$

but see also Corollary 3.5, Lemma 3.11, and Corollary 3.12. These identities are crucial in constructing a random matrix justification for Malle’s conjectural distribution of relative ideal class groups of number fields in the presence of unexpected roots of unity (see [10, 11]). See [7] for this random matrix construction.

1.2. Notation and Definitions

Let $\ell$ be an odd prime and let $G$ be the poset of isomorphism classes of finite abelian $\ell$-groups, with the relation $[A] \leq [B]$ if and only if there exists an injection $A \hookrightarrow B$. (For notational simplicity, we will conflate finite abelian $\ell$-groups and the equivalence classes containing them.) If $i \in \mathbb{Z}^{\geq 0}$, let

$$
\text{rank}_\ell A := \dim_{\mathbb{F}_\ell} \left( \ell^i A / \ell^i A \right).
$$

We will abbreviate $\text{rank}_\ell A$ by $\text{rank} A$.

**Definition 1.2.** For any $r_1, \ldots, r_i \in \mathbb{Z}^{\geq 0}$, define the following subposet of $G$:

$$
G(r_1, \ldots, r_i) := \{ A \in G \mid \text{rank}_\ell(A) = r_j, \text{ for all } 1 \leq j \leq i \}.
$$

Next, we define a function $S: G \times G \to \mathbb{Z}$ that is related to the Möbius function on $G$. See [6] for a deeper discussion of this relationship.

**Definitions 1.3.** For any $A, B \in G$, let $\text{sub}(A, B)$ be the number of subgroups of $B$ that are isomorphic to $A$. If $A \in G$, an $A$-chain is a finite (possibly empty) linearly ordered subset of $\{ B \in G \mid B > A \}$. Now, given an $A$-chain $\mathcal{C} = \{ A_j \}_{j=1}^{i}$, define

$$
\text{sub(}\mathcal{C}) := (-1)^i \text{sub}(A, A_1) \prod_{j=1}^{i-1} \text{sub}(A_j, A_{j+1}).
$$

(We set $\text{sub}(\mathcal{C}) = 1$ if $\mathcal{C}$ is empty.) Finally, for any $A, B \in G$, let

$$
S(A, B) = \begin{cases} 
0, & \text{if } A \not\leq B, \\
1, & \text{if } A = B, \\
\sum_{\text{max } \mathcal{C} = B} \text{sub(}\mathcal{C}), & \text{if } A < B.
\end{cases}
$$
The following definitions are standard in the study of \(q\)-series.

**Notation 1.4.** For \(q, z \in \mathbb{C}\) with \(|q| < 1\) and any \(i \in \mathbb{Z}_{\geq 0}\), let
\[
(z; q)_i := \prod_{j=0}^{i-1} (1 - q^j z).
\]

To ease notation, set \((q)_i := (q; q)_i\). Next, we define the \(q\)-binomial coefficients: for any \(k, m \in \mathbb{Z}_{\geq 0}\), let
\[
\binom{k}{m}_q := \frac{(q)_k}{(q)_m (q)_{k-m}},
\]
with \((k)_q := 0\) if \(k < m\).

In Sections 3.1 and 3.3, we will set \(q = 1/\ell\), but the results in Section 3.2 hold for any complex number \(q\) of modulus less than 1.

2. Methods

There are two main methods used to prove our results. The first method is a combinatorial analysis of the groups in question, which expands upon the previous analysis of Cohen-Lenstra in [2]. This method is introduced in Section 3.1. The second method is the application of results on hypergeometric series to weighted sums over finite abelian \(\ell\)-groups; Section 3.2 lists these results. Finally, Section 3.3 combines the results and methods from Sections 3.1 and 3.2 to achieve our deepest results.

3. Results and Discussion

3.1. Some Identities from Finite Abelian Group Theory

Before proving the first identity, we will follow [2,9] in making two more definitions, which we will use only in the proof of Lemma 3.2. In this section, we set \(q = 1/\ell\).

**Definition 3.1.** For any \(A \in \mathcal{G}\) and \(i \in \mathbb{Z}_{\geq 0}\), define the \(i\)-weight of \(A\) to be
\[
w_i(A) := \frac{|\text{Surj} (\mathbb{Z}^i, A)|}{|A|^i |\text{Aut} A|},
\]
and for any \(j \in \mathbb{Z}_{\geq 0}\), define
\[
w_i (\ell^j) := \sum_{A \in \mathcal{G} \atop |A| = \ell^j} w_i(A).
\]

**Lemma 3.2.** Suppose that \(A \in \mathcal{G}\) and \(\text{rank} A = r\). Then,
\[
\sum_{B \in \mathcal{G}(r)} \frac{\text{sub}(A, B)}{|\text{Aut} B|} = \frac{1}{(q)_r |\text{Aut} A|}.
\]
Proof. By [2, Proposition 3.1], we know

$$\sum_{B \in \mathcal{G}(r)} \frac{\text{sub}(A, B)}{|\text{Aut} B|} = \sum_{B \in \mathcal{G}(r)} \frac{w_r(B)}{(q)_r} \text{sub}(A, B).$$

Next, we will simplify this summation by making two complementary observations concerning any $B \in \mathcal{G}$. First, if rank $B > r$, then $w_r(B) = 0$ by the definition of the $r$-weight of $G$. Second, if rank $B < r$, then sub$(A, B) = 0$. Thus, our sum becomes

$$\sum_{B \in \mathcal{G}(r)} \frac{\text{sub}(A, B)}{|\text{Aut} B|} = \frac{1}{(q)_r} \sum_{B \in \mathcal{G}} w_r(B) \text{sub}(A, B) = \frac{1}{(q)_r} \sum_{i=0}^{\infty} \sum_{B \in \mathcal{G}, |B| = \ell^i} w_r(B) \text{sub}(A, B).$$

By [2, Proposition 4.1], if $\ell^i \geq |A|$, then

$$\sum_{B \in \mathcal{G}, |B| = \ell^i} w_r(B) \text{sub}(A, B) = w_r\left(\ell^i |A|^{-1}\right) w_r(A).$$

If, on the other hand, $\ell^i < |A|$, then the above sum clearly vanishes. Thus,

$$\sum_{B \in \mathcal{G}(r)} \frac{\text{sub}(A, B)}{|\text{Aut} B|} = \frac{1}{(q)_r} \cdot w_r(A) \cdot \sum_{i=0}^{\infty} w_r(\ell^i).$$

Finally, we apply Proposition 3.1 (again) and [2, Corollary 3.7] to obtain

$$\sum_{B \in \mathcal{G}(r)} \frac{\text{sub}(A, B)}{|\text{Aut} B|} = \frac{1}{(q)_r} \cdot \frac{(q)_r}{|\text{Aut} A|} \cdot \frac{1}{(q)_r},$$

as desired.

Before our next result, we need a few more definitions.

**Definitions 3.3.** For any $A \in \mathcal{G}$ and any $A$-chain $\mathcal{C} = \{A_j\}_{j=1}^i$, define

$$\text{sub}_+(\mathcal{C}) := \text{sub}(A, A_1) \prod_{j=1}^{i-1} \text{sub}(A_j, A_{j+1}).$$

If $A, B \in \mathcal{G}$, let

$$S_+(A, B) := \begin{cases} 0, & \text{if } A \not\preceq B, \\ 1, & \text{if } A = B, \\ \sum_{\text{A-chains } \mathcal{C}, \text{max } \mathcal{C} = B} \text{sub}_+(\mathcal{C}), & \text{if } A < B. \end{cases}$$
Finally, suppose \( A \in \mathcal{G}(r) \) for some \( r \in \mathbb{Z}^{\geq 0} \). Then for any \( i \in \mathbb{Z}^{\geq 0} \), let

\[
\zeta_i(A) := \sum_{\substack{\text{A-chains } \mathcal{C} \in \mathcal{G}(r) \\
\max \mathcal{C} \in \mathcal{G}(r) \\
|\mathcal{C}| = i}} \frac{\text{sub}_+ (\mathcal{C})}{|\text{Aut}(\max \mathcal{C})|}.
\]

We know that \( \zeta_i(A) \) converges by the following lemma.

**Lemma 3.4.** If \( A \in \mathcal{G}(r) \) for some \( r \in \mathbb{Z}^{\geq 0} \), and \( i \in \mathbb{Z}^{\geq 0} \), then

\[
\zeta_i(A) = \frac{((q)_r^{-1} - 1)^i}{|\text{Aut}A|}.
\]

**Proof.** We will induct on \( i \). Note that the case \( i = 0 \) is trivially true and the case \( i = 1 \) is true by Lemma 3.2. Using the induction hypothesis and Lemma 3.2, we see that for any \( i \geq 1 \),

\[
\zeta_i(A) = \sum_{\substack{\text{A-chains } \mathcal{C} \in \mathcal{G}(r) \\
\max \mathcal{C} \in \mathcal{G}(r) \\
|\mathcal{C}| = i-1}} \frac{\text{sub}_+ (\mathcal{C})}{|\text{Aut}(\max \mathcal{C})|} \cdot \sum_{B \in \mathcal{G}(r), B \neq \max \mathcal{C}} \frac{\text{sub}(\max \mathcal{C}, B)}{|\text{Aut}B|}
\]

\[
= \sum_{\substack{\text{A-chains } \mathcal{C} \in \mathcal{G}(r) \\
\max \mathcal{C} \in \mathcal{G}(r) \\
|\mathcal{C}| = i-1}} \frac{\text{sub}_+ (\mathcal{C})}{|\text{Aut}(\max \mathcal{C})|} \cdot \frac{(q)_r^{-1} - 1}{|\text{Aut}(\max \mathcal{C})|}
\]

\[
= \frac{((q)_r^{-1} - 1)^i}{|\text{Aut}A|},
\]

as desired. \(\blacksquare\)

**Corollary 3.5.** If \( A \in \mathcal{G}(r) \) for some \( r \in \mathbb{Z}^{\geq 0} \), then,

\[
\sum_{B \in \mathcal{G}(r)} \frac{S(A, B)}{|\text{Aut}B|} = \frac{(q)_r}{|\text{Aut}A|}.
\]

**Proof.** Since \( \ell \) is odd, we know that \( 0 < (q)_r^{-1} - 1 < 1 \). Moreover,

\[
\sum_{B \in \mathcal{G}(r)} \frac{S(A, B)}{|\text{Aut}B|} = \sum_{i=0}^{\infty} (-1)^i \zeta_i(A).
\]

**Corollary 3.6.** If \( A \in \mathcal{G}(r) \) for some \( r \in \mathbb{Z}^{\geq 0} \), then

\[
\sum_{B \in \mathcal{G}(r)} \frac{S_+(A, B)}{|\text{Aut}B|} = \frac{(q)_r}{(2(q)_r - 1)|\text{Aut}A|}.
\]
3.2. Two $q$-series Identities

We record here two $q$-series identities that we will need in Section 3.3. The first follows from the $q$-binomial theorem, and the second follows from the $q$-Chu-Vandermonde sum. In this section, we allow $q$ to be any complex number of modulus less than 1.

**Corollary 3.7.** Suppose that $k \in \mathbb{Z}_{>0}$. Then,

$$
\sum_{i=0}^{k} (-1)^i \binom{k}{i} q^{i(i+1)/2} a^{-ik} = 0.
$$

**Proof.** We use the following corollary of the $q$-binomial theorem: for any $z \in \mathbb{C}$,

$$
\sum_{i=0}^{k} (-1)^i \binom{k}{i} q^{i(i-1)/2} z^i = (z; q)_k.
$$

This is given as Corollary 10.2.2 (part (c)) in [1] (though, as the authors mention, it is due to Rothe). We substitute $z = q^{-k+1}$ to obtain the result. \[\square\]

**Corollary 3.8.** Suppose that $m, k \in \mathbb{Z}_{\geq 0}$. Then

$$
\sum_{i=0}^{\infty} \left( \begin{array}{c} m \\ i \end{array} \right)_q \binom{k}{i}_q (q)_i q^{(m-i)(k-i)} = 1.
$$

**Proof.** Using the standard notation for hypergeometric functions, a short calculation yields

$$
\sum_{i=0}^{\infty} \left( \begin{array}{c} m \\ i \end{array} \right)_q \binom{k}{i}_q (q)_i q^{(m-i)(k-i)} = q^{mk} \cdot 2\phi_1 \left( q^{-m}, q^{-k}; 0; q, q \right).
$$

The result then follows from the $q$-Chu-Vandermonde sum, which states that

$$
2\phi_1 \left( a, q^{-n}; c, q, q \right) = \frac{(c; q)_n a^n}{(c; q)_n}.
$$

see Formula II.6 from [8, Appendix II]. \[\square\]

3.3. More Identities from Finite Abelian Group Theory

Before continuing, we need one more bit of notation and a lemma that follows from [6]. In this section, we once again set $q = 1/\ell$.

**Notation 3.9.** For any $A \in \mathcal{G}$ and any $i \in \mathbb{Z}_{\geq 0}$, let

$$
A^{\oplus_i} := A \oplus (\mathbb{Z}/\ell) \oplus \cdots \oplus (\mathbb{Z}/\ell).
$$

**Lemma 3.10.** If $s, i \in \mathbb{Z}_{\geq 0}$ and $A \in \mathcal{G}(s, s)$, then

$$
|\text{Aut} A^{\oplus_i}| = q^{-i(2s+i)} (q)_i |\text{Aut} A|.
$$
Proof. This follows from [6, Theorem 2.2].

**Lemma 3.11.** Suppose \( r, s \in \mathbb{Z}^{\geq 0} \) and \( r \geq s \). If \( A \in \mathcal{G}(s, r) \) and \( i \in \{ s, \ldots, r \} \), then

\[
\sum_{B \in \mathcal{G}(r, i)} \frac{\text{sub}(A, B)}{|\text{Aut} B|} = \left( \frac{r-s}{r-i} \right)_q \cdot \frac{q^{i(s)}}{(q)_i |\text{Aut} A|}.
\]

**Proof.** We begin by considering the \( i = r \) case, which we will prove by induction on \( r - s \). (When \( r - s = 0 \), the \( i = r \) case is clearly true by Lemma 3.2.) Now, there exists a unique \( A_0 \in \mathcal{G}(s, s) \) such that \( A = (A_0)_{\oplus (r-s)} \). Next, by Lemma 3.2, Lemma 3.10 (twice), [6, Lemma 2.5], and the induction hypothesis, we see that

\[
\frac{1}{(q)_r |\text{Aut} A|} = \sum_{B \in \mathcal{G}(r)} \frac{\text{sub}(A, B)}{|\text{Aut} B|} = \sum_{j=s}^{r} \sum_{B \in \mathcal{G}(j, j)} \frac{\text{sub}(A, B)}{|\text{Aut} B|}.
\]

Letting \( m = r - s \), we obtain

\[
\frac{1}{(q)_r |\text{Aut} A|} = \frac{1}{|\text{Aut} A|} \cdot \sum_{j=0}^{m-1} \left( \frac{m}{m-j} \right)_q \cdot \frac{q^{i(s)}}{(q)_i |\text{Aut} A|} + \sum_{B \in \mathcal{G}(r, r)} \frac{\text{sub}(A, B)}{|\text{Aut} B|}.
\]

Thus,

\[
\sum_{B \in \mathcal{G}(r, r)} \frac{\text{sub}(A, B)}{|\text{Aut} B|} = \frac{1}{(q)_r |\text{Aut} A|} \cdot \left( 1 - \sum_{j=0}^{m-1} \left( \frac{m}{m-j} \right)_q \cdot \frac{q^{i(s)}}{(q)_r+j-m} \right)
\]

\[
= \frac{1}{(q)_r |\text{Aut} A|} \cdot \left( 1 - \sum_{j=0}^{m-1} \left( \frac{m}{j} \right)_q \cdot \frac{r}{m-j} \cdot (q)_m \cdot q^j \cdot q^{i(s)/(r-j)} \right)
\]
\[
\sum_{B \in G(i, i)} \frac{\text{sub}(A, B)}{|\text{Aut}A|} = \sum_{B \in G(i, i)} \frac{q^{(r-i)(r+i)} \text{sub}\left((A_0) \oplus (i-s), B\right)}{(q)_{r-i} |\text{Aut}B|}
\]
concluding the \( i = r \) case.

For the general case, the same techniques as above show that

\[
\sum_{B \in G(r, i)} \frac{\text{sub}(A, B)}{|\text{Aut}A|} = \sum_{B \in G(i, i)} \frac{q^{(r-i)(r+i)} \text{sub}\left((A_0) \oplus (i-s), B\right)}{(q)_{r-i} |\text{Aut}B|}
\]

completing the proof.

**Corollary 3.12.** Suppose \( r, s \in \mathbb{Z}_{\geq 0} \) and \( r \geq s \). If \( A \in G(r, s) \) and \( i \in \{s, \ldots, r\} \), then

\[
\sum_{B \in G(r, i)} \frac{S(A, B)}{|\text{Aut}B|} = (-1)^{i-s} q^{i(i+1)/2 - j(j+1)/2} \cdot \binom{r-s}{r-i} q^{i-s(i+s)}/q \cdot (q)_s/|\text{Aut}A|.
\]

**Proof.** Throughout this proof, we will rearrange terms of infinite alternating sums. The fact that we can do so follows from Corollary 3.6. To ease notation, let

\[
X_i := \sum_{B \in G(r, i)} \frac{S(A, B)}{|\text{Aut}B|}.
\]

As a first step toward the result, we claim that

\[
X_i = \begin{cases} 
\frac{(q)_i}{|\text{Aut}A|}, & \text{if } i = s, \\
-\sum_{j=s}^{s-1} q^{j(i-j)} \cdot \binom{r-s}{r-i} q^{i-j} \cdot X_j, & \text{if } i > s.
\end{cases}
\]

If \( i = s \), we use Lemma 3.2 to obtain

\[
X_s = \sum_{j=0}^{\infty} \sum_{\text{A-chains } \mathcal{C}} \frac{\text{sub}(\mathcal{C})}{|\text{Aut}(\max \mathcal{C})|} = \frac{1}{|\text{Aut}A|} - \sum_{j=1}^{\infty} \sum_{\text{A-chains } \mathcal{C}} \text{sub}(\mathcal{C}) \cdot \sum_{B \in G(r, s), B \neq \max \mathcal{C}} \frac{\text{sub}(\max \mathcal{C}, B)}{|\text{Aut}B|}
\]
\[
\begin{align*}
&= \frac{1}{|\text{Aut}A|} - \left( \frac{1}{(q)_s} - 1 \right) \sum_{j=1}^{\infty} \sum_{A\text{-chains } \mathcal{C} \in \mathcal{G}(r,s) \atop |\mathcal{C}| = j} \frac{\text{sub}(\mathcal{C})}{|\text{Aut}(\text{max } \mathcal{C})|} \\
&= \frac{1}{|\text{Aut}A|} - \left( \frac{1}{(q)_s} - 1 \right) X_s.
\end{align*}
\]

The result follows by solving for \(X_s\). If, on the other hand, \(i > s\), we can apply Lemma 3.11 to see that

\[
X_i = \sum_{k=1}^{\infty} \sum_{A\text{-chains } \mathcal{C} \in \mathcal{G}(r,i) \atop |\mathcal{C}| = k} \frac{\text{sub}(\mathcal{C})}{|\text{Aut}(\text{max } \mathcal{C})|}
\]

\[
= - \sum_{B \in \mathcal{G}(r,i)} \frac{\text{sub}(A, B)}{|\text{Aut}B|} - \sum_{j=s}^{i-1} \sum_{k=2}^{\infty} \sum_{A\text{-chains } \mathcal{C} \in \mathcal{G}(r,j) \atop |\mathcal{C}| = k-1} \frac{\text{sub}(\mathcal{C})}{|\text{Aut}(\text{max } \mathcal{C})|} \cdot \sum_{B \in \mathcal{G}(r,i)} \frac{\text{sub}(\text{max } \mathcal{C}, B)}{|\text{Aut}B|}
\]

\[
- \sum_{k=2}^{\infty} \sum_{A\text{-chains } \mathcal{C} \in \mathcal{G}(r,i) \atop |\mathcal{C}| = k-1} \frac{\text{sub}(\mathcal{C})}{|\text{Aut}(\text{max } \mathcal{C})|} \cdot \sum_{B \in \mathcal{G}(r,i)} \frac{\text{sub}(\text{max } \mathcal{C}, B)}{|\text{Aut}B|}
\]

\[
= - \left( \frac{r-s}{r-i} \right) q^{i-s} \frac{(q)_i}{(q)_s} \cdot \frac{1}{|\text{Aut}A|} - \sum_{j=s}^{i-1} \left( \frac{r-j}{r-i} \right) \frac{q^{i-j}}{(q)_i} \cdot \sum_{k=2}^{\infty} \sum_{A\text{-chains } \mathcal{C} \in \mathcal{G}(r,k) \atop |\mathcal{C}| = k-1} \frac{\text{sub}(\mathcal{C})}{|\text{Aut}(\text{max } \mathcal{C})|}
\]

\[
- \left( \frac{1}{(q)_i} - 1 \right) \sum_{k=2}^{\infty} \sum_{A\text{-chains } \mathcal{C} \in \mathcal{G}(r,k) \atop |\mathcal{C}| = k-1} \frac{\text{sub}(\mathcal{C})}{|\text{Aut}(\text{max } \mathcal{C})|}
\]

\[
= - \sum_{j=s}^{i-1} \left( \frac{r-j}{r-i} \right) \frac{q^{i-j}}{(q)_i} \cdot X_j - \left( \frac{1}{(q)_i} - 1 \right) X_i.
\]

The claim follows by solving for \(X_i\).

By the claim above, the lemma is true for \(i = s\); we will now prove the lemma by induction on \(i\). Using the induction hypothesis, note that

\[
X_i = - \sum_{j=s}^{i-1} q^{i-j} \cdot \left( \frac{r-j}{r-i} \right) q \cdot X_j
\]

\[
= - \sum_{j=s}^{i-1} \frac{q^{i-j}}{(q)_{r-i}(q)_{i-j}} \cdot \frac{(-1)^{j-s} q^{\frac{j(j+1)}{2}} (q)_{r-s}(q)_s}{q^{\frac{j(j+1)}{2}} (q)_{j-s} |\text{Aut}A|}
\]

\[
= - \frac{(q)_{r-s}(q)_s}{q^{\frac{i(i+1)}{2}} (q)_{r-i} |\text{Aut}A|} \sum_{j=s}^{i-1} (-1)^{j-s} q^{\frac{j(j+1)}{2}} \frac{(q)_{i-j}(q)_{j-s}}{(q)_{i-j}(q)_{j-s}}
\]
\[
\frac{(q)_{r-s}(q)_s}{q^{\frac{s(s+1)}{2}}(q)_{r-i} \cdot \text{Aut} A} \cdot \sum_{j=0}^{i-s-1} (-1)^j q^{i(j-s)+\frac{(i+j)(i+j+1)}{2}}(q)_{i-j-s}(q)_j.
\]

Letting \(k = i - s\), and using Corollary 3.7, we see that

\[
X_i = -q^{ik}(q)_{r-s}(q)_s \cdot \frac{k-1}{(q)_{r-i} \cdot \text{Aut} A} \cdot \sum_{j=0}^{k-1} (-1)^j q^{\frac{i(j+1)-jk}{2}}(q)_{k-j}(q)_j
\]

\[
= -q^{ik}(q)_{r-s}(q)_s \cdot \frac{k-1}{(q)_{r-i} \cdot \text{Aut} A} \cdot \left( -(-1)^k q^{\frac{k(k+1)}{2}k^2} \right).
\]

Substituting for \(k\) gives the result. \(\blacksquare\)

4. Conclusions

As mentioned in Section 1, the identities in this paper are crucial in constructing a random matrix justification for Malle’s conjectural distribution of relative ideal class groups of number fields in the presence of unexpected roots of unity (see [10, 11]).

Moreover, the function \(S: \mathcal{G} \times \mathcal{G} \rightarrow \mathbb{Z}\) from Definition 1.3, in addition to appearing in the aforementioned matrix construction, is an important ingredient in studying distributions and moments of distributions on \(\mathcal{G}\) in general. It is a natural analog of the Möbius function, and is discussed more in [6].

Finally, a clear avenue for future work is an extension of Corollary 3.12 to the posets \(\mathcal{G}(r_1, \ldots, r_i)\) with \(i > 2\).

5. Competing Interests

The author declares that he has no competing interests.

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