Landau equation for self-gravitating classical and quantum particles: application to dark matter

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Abstract We develop the kinetic theory of classical and quantum particles (fermions and bosons) in gravitational interaction. The kinetic theory of quantum particles may have applications in the context of dark matter. For simplicity, we consider an infinite and spatially homogeneous system (or make a local approximation) and neglect collective effects. This leads to the quantum Landau equation derived heuristically in (Physica A 332: 89, 2004). We establish its main properties: conservation laws, $H$-theorem, equilibrium state, relaxation time, quantum diffusion and friction coefficients, quantum Rosenbluth potentials, self-consistent evolution, (thermal) bath approximation, quantum Fokker–Planck equation, quantum King model. For bosonic particles, the Landau equation can describe the process of Bose–Einstein condensation. We discuss the relation of our study with the works of Levkov et al. (Phys Rev Lett 121: 151301, 2018) and Bar-Or et al. (Astrophys J 871: 28, 2019) on fuzzy dark matter halos and the formation of Bose stars and solitons.

1 Introduction

The kinetic theory of self-gravitating systems was pioneered by Chandrasekhar [1] following previous works by Jeans [2–4], Schwarzschild [5], Ambarzumian [6] and Spitzer [7]. He evaluated the relaxation time of a stellar system due to gravitational encounters and found that it scales as $t_R \sim (N / \ln N) t_D$, where $N$ is the number of stars in the system and $t_D$ is the dynamical time. For stellar systems which contain a large number of stars, like spiral or elliptical galaxies, the relaxation time is larger than the age of the universe by many orders of magnitude ($N \sim 10^{12}$, $t_D \sim 10^8$ yrs, age $\sim 10^{10}$ yrs $\sim 100 t_D$, $t_R \sim 10^{20}$ yrs for our Galaxy). As a result, galaxies are essentially collisionless. They are described by the Vlasov–Poisson equations [8,9]. Their apparent regularity is due to a process of violent collisionless relaxation, operating on a few dynamical times, first identified by King [10] and Hénon [11] and formalized by Lynden-Bell [12]. For stellar systems which contain a moderate number of stars, like globular clusters or galactic nuclei, the relaxation time is comparable to their age ($N \sim 10^5$, $t_D \sim 10^5$ yrs, age $\sim 10^{10}$ yrs $\sim 10^5 t_D$, $t_R \sim 10^{10}$ yrs for globular clusters). Therefore, globular clusters are “collisional” and undergo a slow (secular) relaxation.

1 Close gravitational encounters may be important in galactic nuclei.

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evolution due to gravitational encounters. In the simplest description, which is based on a local approximation (equivalent to assuming that the system is infinite and homogeneous) and neglects collective effects, their evolution is described by the gravitational Landau equation. The Landau equation can be obtained from the Boltzmann equation in a limit of weak deflections. An equivalent equation was obtained independently by Chandrasekhar and generalized by Rosenbluth et al. from the Fokker–Planck equation of Brownian theory. The kinetic theory of self-gravitating systems was initially used to study the evaporation of globular clusters due to star encounters. Recently, this kinetic theory has been generalized in order to describe fully spatially inhomogeneous systems and take collective effects into account. This leads to the inhomogeneous Lenard–Balescu equation written with angle-action variables which is essentially exact at the order $1/N$. The Landau (or Lenard–Balescu) equation conserves the mass and the energy, increases the Boltzmann entropy ($H$-theorem), and relaxes toward the Boltzmann distribution function (DF). In the thermal bath approximation, where the field particles are at statistical equilibrium, it takes the form of a Kramers equation with an anisotropic velocity-dependent diffusion tensor. From this kinetic equation, one can derive a truncated model called the King model.

In addition to baryonic matter (stars), the universe contains a large amount of dark matter (DM). The nature of DM is still unknown and remains one of the greatest mysteries of modern cosmology. DM may be made of quantum particles, either fermions (like sterile neutrinos) or bosons (like axions). The collisional evolution of a gas of self-gravitating fermions is described by the fermionic Landau equation which involves a term of the form $f(1 - f)$ taking into account the Pauli exclusion principle. This equation conserves the mass and the energy, increases the Fermi–Dirac entropy ($H$-theorem), and relaxes towards the Fermi–Dirac DF. In the thermal bath approximation, where the field particles are at statistical equilibrium, it takes the form of a fermionic Kramers equation with an anisotropic velocity-dependent diffusion tensor. From this kinetic equation, one can derive a truncated model called the fermionic King model. However, the relaxation time towards the Fermi–Dirac DF of quantum statistical mechanics due to gravitational encounters is extremely long because the number of particles in a DM halo is astoundingly large (typically $N \sim 10^{75}$ for fermions of mass $m \sim 100 \text{eV}/c^2$). In addition, we will see that, in the case of fermions, quantum effects (Pauli’s blocking) have the tendency to increase the relaxation time even more, especially at low temperatures. Therefore, fermionic DM halos are essentially collisionless. They are described in good approximation by the classical Vlasov–Poisson equations. More fundamentally, one should use the Wigner-Poisson equations for a multistate system. However, in the case of fermions, the quantum potential arising from the Heisenberg uncertainty principle can usually be neglected (Thomas-Fermi approximation) leading to the classical Vlasov equation.

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2 The Chandrasekhar equation appears in a less neat (less symmetric) form than the Landau equation. The connection between these two equations is discussed in [21] and in Sect. 3 of the present paper.

3 See the introduction of [29–31] (resp. [32,33]) for a short history and an exhaustive list of references on fermionic (resp. bosonic) DM. See also the reviews [34–40] on bosonic DM.

4 More fundamentally, one should use the Wigner-Poisson equations for a multistate system. However, in the case of fermions, the quantum potential arising from the Heisenberg uncertainty principle can usually be neglected (Thomas-Fermi approximation) leading to the classical Vlasov equation.
relevant for self-gravitating fermions [30,31]. As a result, self-gravitating fermions should violently relax towards an out-of-equilibrium quasistationary state which is similar to the Fermi–Dirac DF in quantum mechanics. This process takes place on a few dynamical times $t_{D}$. In addition, the kinetic equation for the coarse-grained DF is similar to the fermionic Vlasov–Landau equation [41–44]. Accordingly, in fermionic DM halos, the Fermi–Dirac distribution and the fermionic Landau equation may be justified by the theory of violent relaxation of Lynden-Bell, not by the usual statistical mechanics and kinetic theory of quantum particles. Since violent collisionless relaxation occurs on a much shorter timescale than collisional relaxation, the Lynden-Bell theory may solve the timescale problem of fermionic DM halos, i.e., it may explain why fermionic DM halos are described by the Fermi–Dirac DF although they are collisionless [30,31]. Fermionic DM halos described by the Lynden-Bell (Fermi–Dirac-like) DF generically have a “core-halo” structure [45]. They are made of a quantum core (fermion ball) which corresponds to the ground state of the self-gravitating Fermi gas at $T = 0$ surrounded by an isothermal atmosphere (in the sense of Lynden-Bell) with an effective temperature $T_{\text{eff}}$. The quantum core (fermion ball) is stabilized against gravitational collapse by the pressure arising from the exclusion principle (Lynden-Bell and Pauli) so that the density does not diverge at $r = 0$. This may solve the cusp-core problem of the classical CDM model. On the other hand, an isothermal halo where the density decreases as $r^{-2}$ at large distances yields flat rotation curves in agreement with the observations.

Let us now turn to the case of self-gravitating bosons. At sufficiently high temperatures (above the condensation temperature $T_c$), the collisional evolution of uncondensed self-gravitating bosons is described by the bosonic Landau equation [41] which involves a term of the form $f (1 + f)$ taking into account Bose enhancement. This equation conserves the mass and the energy, increases the Bose–Einstein entropy ($H$-theorem), and relaxes toward the Bose–Einstein DF. In the thermal bath approximation, where the field particles are at statistical equilibrium, it takes the form of a bosonic Kramers equation with an anisotropic velocity-dependent diffusion tensor. From this kinetic equation one can derive a truncated model called the bosonic King model [41]. As we shall see, the relaxation time of uncondensed bosons above $T_c$ is very large and it only slightly decreases due to Bose enhancement when $T \to T_c$. Therefore, self-gravitating bosons above $T_c$ are essentially collisionless. They are described in good approximation by the classical Vlasov–Poisson equations. In this regime, they can experience a process of violent relaxation leading to an out-of-equilibrium quasistationary state on a few dynamical times. Degeneracy in the sense of Lynden-Bell can be neglected for bosons so the quasistationary state is similar to the Boltzmann distribution. In addition, the kinetic equation for the coarse-grained DF is similar to the classical Vlasov–Landau equation.

We can also use the bosonic Landau equation to study the process of Bose–Einstein condensation below $T_c$. Let us assume that, at $t = 0$, the bosons are uncondensed but that $T < T_c$. In that case, we expect that, after some evolution, the DF develops a Dirac peak at $v = 0$ so that $f (v, t) = f_{\text{gas}} (v, t) + M_c (t) \delta (v)$, where $f_{\text{gas}} (v, t)$ is the DF of the uncondensed

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5 The fact that this kinetic equation relaxes toward the Lynden-Bell DF on a few dynamical times is interpreted by Kadomtsev and Pogutse [43] in terms of “collisions” between macroparticles with a large effective mass.

6 Numerical simulations of classical collisionless CDM lead to a universal density profile called the Navarro–Frenk–White (NFW) profile [46]. This profile presents an $r^{-1}$ central cusp which is not consistent with the observations that rather favor flat cores [47].

7 In the case of fuzzy dark matter (FDM), with bosons of mass $m \sim 10^{-22} \text{eV}/c^2$, we are not in this situation (see “Appendix A”). We are rather in the situation, considered below, where $T \ll T_c$.

8 More fundamentally, one should use the Wigner-Poisson equations but, when $T > T_c$, quantum effects are negligible, or weak, during the collisionless regime.
bosons (gas) and $M_c(t)$ is the mass of the condensed bosons. The mass $M_c(t)$ of the Bose–Einstein condensate (BEC) increases with time until a statistical equilibrium state is reached. This equilibrium state contains a fraction $(T / T_c)^{3/2}$ of uncondensed bosons with vanishing chemical potential, the rest being in the form of condensed bosons. For nongravitational systems, the temporal formation of a BEC (Dirac peak) when $T < T_c$ has been analytically studied by Sopik et al. [48] in the canonical ensemble (fixed $T$) from the bosonic Kramers equation with a constant isotropic diffusion coefficient. They considered the exact Bose–Einstein model and a simplified model obtained by making the approximation $f(1 + f) \simeq f^2$ to simplify the equations. They obtained a self-similar solution in the precondensation regime. During this regime, the central DF increases and becomes infinite in a finite time $t_{\text{coll}}$ when $T < T_c$ and in an infinite time when $T = T_c$. At that moment, the chemical potential vanishes. The condensate (Dirac peak) appears in a postcollapse regime and its mass grows initially as $M_0(t) \propto t - t_{\text{coll}}$ before saturating exponentially rapidly. The statistical equilibrium state is reached for $t \to +\infty$. More recently, Levkov et al. [50] have considered the mechanism of Bose–Einstein condensation and the formation of Bose stars in virialized DM halos and miniclusters by gravitational interactions (see also [51–54]). They started from an incoherent initial configuration of bosons described by a random classical field $\psi(r, t)$ satisfying the Schrödinger–Poisson equations and assumed that the initial velocity DF is Maxwellian with $T_0 \ll T_c$. They derived a bosonic Landau equation in the microcanonical ensemble (fixed $E$) for the DF $f(v, t)$ in which Bose stimulation is accounted for by a quadratic term $f^2$ (their kinetic equation coincides with the bosonic Landau equation derived in [41] when we make the approximation $f(1 + f) \simeq f^2$). They estimated the kinetic relaxation (or condensation) time $t_{\text{gr}}$ due to gravitational interactions and compared it with direct numerical simulations. They showed that collisional relaxation is enhanced by the collective interaction of large fluctuations of the boson gas at large distances. Starting from a spatially homogeneous distribution, they observed the phenomenon of Bose–Einstein condensation and the formation of an isolated Bose star. Once the Bose star nucleates, a small $\delta$ peak with mass $M_\delta$ appears in the DF. With time the peak grows in height with an initial rate $M_\delta(t) \propto (t - t_{\text{gr}})^{1/2}$ as it acquires particles from the surroundings (the scaling changes from $t^{1/2}$ to $t^{1/8}$ at later times [51,54]). The Dirac peak $M_\delta(t)\delta$ in the velocity or energy distribution gives rise to a Bose star in configuration space (recall that the wave function $\psi(r, t)$ and the density $\rho = |\psi|^2$) are related to the Fourier transform of the velocity distribution $f(v, t)$ while the smooth distribution $f_{\text{gas}}$ describes a sort of homogeneous halo surrounding the Bose star and feeding it.

In the study of Levkov et al. [50], it is assumed that the bosons are initially uncondensed, that they are spatially homogeneously distributed, and that their distribution is dynamically stable (they also consider the formation of Bose stars in virialized miniclusters resulting from Jeans instability). The bosons initially have an out-of-equilibrium Maxwellian DF with a temperature (or velocity dispersion) $T_0 \ll T_c$ so that Bose–Einstein condensation takes place. The study of Levkov et al. [50] therefore describes the slow (secular) evolution of a stable homogeneous system of bosons, and the process of condensation, caused by gravitational interactions. Even if the “collisional” relaxation is considerably accelerated by Bose stimulation, the Bose star still forms on a relatively long timescale, much longer

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9 Interestingly, there is a striking similarity between the Bose–Einstein condensation in the canonical ensemble and the gravitational collapse of a gas of classical self-gravitating Brownian particles [49].

10 In the precondensation phase, the central DF grows more and more and diverges at $t_{\text{gr}}$ at which time the Dirac appears. The mass of the Dirac peak grows in the postcondensation regime. This is very similar to the results of Sopik et al. [48] except that the exponents characterizing the initial growth of $M_0(t)$ and $M_\delta(t)$ are different because the models describe different statistical ensembles (canonical versus microcanonical).

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than the dynamical time $t_{gr} \gg t_D$. Other authors [55–63] considered a different situation where the bosons are in an initial distribution that is spatially inhomogeneous, unsteady, or dynamically unstable. In that case, a violent collisionless relaxation leading to a BEC occurs on a short dynamical timescale before a slower collisional relaxation, corresponding to the one described by Levkov et al. [50], takes place. This is the situation that we now review.

Let us consider the dynamical evolution of an inhomogeneous gas of bosons in gravitational interaction in relation to the formation of DM halos. Self-gravitating bosons are basically described by the Schrödinger–Poisson equations which govern the evolution of the wavefunction $\psi(r, t)$. These equations are equivalent to the Wigner-Poisson equations which govern the evolution of the Wigner DF $f_W(r, v, t)$. Let us assume that we start from an initial condition that is an unsteady or an unstable state of the Schrödinger–Poisson or Wigner-Poisson equations. In a first regime, taking place on a free fall time, the Schrödinger–Poisson equations experience a process of gravitational cooling (like in the case of boson stars [64–66]) which is similar to the violent relaxation of collisionless stellar systems [12]. The self-gravitating boson gas undergoes gravitational collapse, displays damped oscillations, and finally settles down on a quasistationary (virialized) state by radiating part of the scalar field. This is a purely mean field process which takes place on a few dynamical times. The process of gravitational cooling and violent relaxation leads to DM halos with a “core-halo” structure. The quantum core (soliton) corresponds to the ground state of the Schrödinger–Poisson equations. It results from the balance between the gravitational attraction and the quantum pressure arising from the Heisenberg uncertainty principle or from the repulsive self-interaction of the bosons. Quantum mechanics is important at “small” scales of the order of the de Broglie length ($\lambda_{dB} = \hbar / (m_b \sigma_b) \sim 1 \text{ kpc}$) determining the size of the soliton. It stabilizes the system against gravitational collapse and may solve the cusp-core problem of classical CDM. The quantum core (ground state) is surrounded by a halo of scalar radiation resulting from the quantum interferences of excited states. This extended halo is made of uncondensed bosons. It is the counterpart of the halo observed in numerical simulations of classical collisionless CDM. The smooth density profile of this halo (on a scale larger than de Broglie length) is similar to the NFW profile [46]. It is also similar to the isothermal profile predicted by the statistical theory of Lynden-Bell [12]. It is important to note that this isothermal halo is an out-of-equilibrium structure with an effective temperature $T_{eff} \ll T_c$.

11 If we consider self-interacting bosons with a scattering length $a_s$, the Schrödinger equation has to be replaced by the Gross–Pitaevskii equation [32]. Basically, one has to interpret the wavefunction in the Schrödinger and Gross–Pitaevskii equations as an operator (Heisenberg representation). However, when the occupation number is large $N = n_b \lambda_{dB}^3 \gg 1$, which is the case for FDM (see “Appendix A”), the coherent state dynamics of the bosons can be approximated by a classical field.

12 The Wigner equation is the counterpart of the Klimontovich equation for classical self-gravitating particles but it takes into account properties specific to bosons (Heisenberg uncertainty principle and Bose stimulation). In the collisional (secular) regime, the smoothed-out Wigner DF satisfies the bosonic Landau equation [50]. In this regime, we can neglect collisions and make a mean field approximation. In that case, the Wigner equation for the smoothed-out Wigner DF is the counterpart of the Vlasov equation for classical collisionless self-gravitating systems but it takes into account properties specific to bosons (Heisenberg uncertainty principle) [59].

13 In a cosmological setting, if one starts from an infinite homogeneous distribution of bosons, the gas is expected to undergo Jeans instability and form clumps (miniclusters). This corresponds to the linear regime of structure formation. These clumps grow and, in the nonlinear regime of structure formation, undergo free fall, gravitational cooling and violent relaxation leading to the DM halos that we observe today. During their cosmic evolution, the halos merge and grow in size to form larger halos. This corresponds to the hierarchical scenario of bottom-up cosmic structure formation.

14 A repulsive self-interaction ($a_s > 0$) stabilizes the quantum core. By contrast, an attractive self-interaction destabilizes the quantum core above a maximum mass $M_{max} = 1.01 \hbar / \sqrt{Gm_b|a_s|}$ first identified in [32].
an isothermal halo where the density decreases as $r^{-2}$ at large distances yields flat rotation curves in agreement with the observations.\footnote{The isothermal nature of the halo (Maxwell-Boltzmann distribution) seems to be confirmed by the numerical results of [52,59,60]. The halo cannot be exactly isothermal otherwise it would have an infinite mass [67]. In reality, the density in the halo decreases at large distances as $r^{-3}$, similarly to the NFW [46] and Burkert [47] profiles, instead of $r^{-2}$ corresponding to the isothermal sphere [67]. This extra-confinement may be due to incomplete relaxation, tidal effects, and stochastic perturbations as discussed in [68]. The $r^{-2}$ isothermal profile is established in an intermediate region between the core and the external halo.} Therefore, in the FDM model, the quantum core (soliton) solves the cusp problem of the CDM model and the approximately isothermal halo accounts for the flat rotation curves of the galaxies. In a companion paper [69], we have proposed a heuristic parametrization of the process of gravitational cooling by using an approach similar to the kinetic theory [41–44,70] developed in connection to the Lynden-Bell theory of violent relaxation [12].\footnote{In that case, the relaxation of the coarse-grained Wigner DF $\tilde{f}_W(r, v, t)$ is due to the fluctuations of the violently changing gravitational potential while the system is unsteady.} In the case of bosons, degeneracy in the sense of Lynden-Bell can be neglected. As a result, the kinetic equation for the coarse-grained Wigner DF $\tilde{f}_W(r, v, t)$ has the form of a Wigner-Landau equation with a Wigner advection term (taking into account the Heisenberg uncertainty principle) responsible for the quantum core (soliton) and an effective classical Landau “collision” term (taking into account the strong fluctuations of the gravitational potential) responsible for the isothermal halo. This core-halo structure has been clearly evidenced in numerical simulations of the Schrödinger–Poisson equations [55–63]. The core mass–halo mass relation $M_c(M_v)$ has been obtained numerically in Ref. [56] and explained in Refs. [68,71] from an effective maximum entropy principle in the sense of Lynden-Bell (see also [51,56,58,60,72] for other justifications). For noninteracting bosons, the core mass scales as $M_c \propto M_v^{1/3}$.\footnote{These quasiparticles are fundamentally different from the macroparticles introduced by Kadomtsev and Pogutse [43] in the context of the violent relaxation of collisionless stellar systems.}

On a longer timescale, the bosons in the halo undergo a collisional evolution. As shown by Schive et al. [55,56], quantum wave interferences produce time-dependent small-scale density granules of the size $\lambda_{DB} = h/(m_b\sigma_b) \sim 1$ kpc of the solitonic core. These granules have been clearly evidenced in numerical simulations of the Schrödinger–Poisson equations [55–63]. They have been interpreted by Hui et al. [73] as quasiparticles with an effective mass $m_{\text{eff}} \sim \rho_b\lambda_{DB}^3 \sim 10^7 M_\odot \gg m_b$ that depends on the local halo density $\rho_b$ and velocity distribution.\footnote{These quasiparticles are fundamentally different from the macroparticles introduced by Kadomtsev and Pogutse [43] in the context of the violent relaxation of collisionless stellar systems.} These authors argued that the fluctuating gravitational force in a FDM halo of mean density $\rho_b$ is similar to that of a classical $N$-body system composed of such quasiparticles. In CDM halos, the mass $m_b$ of the particles is so small that the relaxation time is much larger than the age of the universe. Therefore, CDM halos are collisionless. In FDM halos, the effective mass $m_{\text{eff}}$ of the quasiparticles is large enough that relaxation can be important. Relaxation between quasiparticles can lead to the formation and/or growth of a central BEC (soliton). The granules can also cause the diffusion (heating) of light test particles such as stars (with $m \ll m_{\text{eff}}$) on a secular time. This can heat and expand the central regions of a stellar system embedded in the halo. On the other hand, heavy test particles such as black holes or globular clusters (with $m \gg m_{\text{eff}}$) experience dynamical friction (cooling). The variation of the effective mass across the system can stall the inspiral of the massive object toward the center of the galaxy at the distance where the effective mass of the quasiparticles $m_{\text{eff}}(r)$ becomes equal to the mass $m$ of the test particles.

The evolution of test particles in a halo of FDM due to “collisions” with quasiparticles has been analyzed by Bar-Or et al. [74]. Starting from the Schrödinger–Poisson equations,
they developed a kinetic theory and quantitatively showed that FDM halos indeed behave as classical self-gravitating systems (similar to globular clusters) made of quasiparticles with an effective mass $m_{\text{eff}}$. Because of Bose enhancement, the quasiparticles are much heavier than the bosons ($m_{\text{eff}} \gg m_b$). This strongly accelerates the collisional relaxation with respect to the one that would be caused by particles of mass $m_b$. They identified a heating time (coinciding with the condensation time of [50]) strongly reduced by the effective mass and a cooling time independent of the effective mass. A similar description has been developed by Marsh and Niemeyer [75] and El-Zant et al. [76]. We will see that their results can be recovered from the bosonic Landau equation of Ref. [41] by considering classical test particles in collision with bosonic field particles in an out-of-equilibrium quasistationary state (resulting from the violent relaxation process) for which we can make the approximation $f(1 + f) \simeq f^2$ and assume that the DF of the field particles $f$ is Maxwellian with a temperature $T \ll T_c$.

The granules (quasiparticles) also provoke the secular evolution of the halo itself. This evolution can be described by the self-consistent bosonic Landau equation [41]. On a secular timescale, the bosons of the halo condense (because $T \ll T_c$) and “feed” the central soliton. As a result, the mass of the soliton increases while the halo is slowly depleted. This is the spatially “inhomogeneous” version of the “homogeneous” situation studied by Levkov et al. [50]. Note that this mechanism of condensation, which operates on a secular timescale, is physically different from the process of gravitational cooling and violent relaxation mentioned above, which operates on a dynamical timescale. In the “homogeneous” approach of Levkov et al. [50] (where gravitational cooling and violent relaxation are absent because the initial state is virialized), the formation of the soliton is exclusively due to the condensation caused by gravitational interactions (collisions). In the “inhomogeneous” approach of [55–63] (where the initial state is dynamically unstable), the soliton and the halo form during the phase of free fall through a process of violent collisionless relaxation. However, on a longer (secular) timescale, the soliton continues to grow like in [50], being fed by the halo. If the bosons have an attractive self-interaction (see footnote 15), the soliton may grow overcritical ($M_s > M_{\text{max}}$ [32]) and collapse [77] or explode into relativistic axions (bosenova) and emit radio-photons via parametric resonance [78]. The distinction between the phase of violent relaxation (formation of the soliton) and the phase of slow collisional relaxation (growth of the soliton) is illustrated in [51]. The collapse of the soliton and the formation of multi boson stars in miniclusters by a process of fragmentation when the bosons have an attractive self-interaction and $M_s > M_{\text{max}}$ is shown in [54].

For completeness, we would like to mention another possible scenario of evolution [68]. If the bosons have a strongly repulsive self-interaction, the collisional evolution of the halo may be due to self-interaction instead of gravitational encounters (these comments also apply to fermions). Such an evolution can be described by the ordinary Boltzmann kinetic equation.\footnote{If the bosons are self-interacting, they may experience a relaxation due to direct “collisions” with a cross section $\sigma = \pi a_b^2$ rather than a relaxation due to gravitational scattering with a Rutherford cross section $\sigma_{gr} \sim \pi G^2 m_b^2 \ln \Lambda/\sigma^4$. As discussed in footnote 52 of [68], a relevant situation corresponds to bosons with a mass $m = 1.10 \times 10^{-3}$ eV/c$^2$ and scattering length $a_s = 4.41 \times 10^{-6}$ fm. For QCD axions and ultralight axions, Levkov et al. [50] show that the collisional relaxation due to the self-coupling is slower than the collisional relaxation due to two-body gravitational encounters. However, for bosons with $m = 1.10 \times 10^{-3}$ eV/c$^2$ and $a_s = 4.41 \times 10^{-6}$ fm, we find the opposite. Self-interaction wins over gravity by $\tau_{gr}/\tau_{self} \sim \sigma_{self}/\sigma_{gr} \sim \sigma_{self}^2/(G^2 m^2 \ln \Lambda) \sim 10^{77}$. In that case, the self-interaction relaxation time $\tau_{self} \sim 1/(\ln m a_b^2) \sim 10^{11}$ yrs is comparable to the age of the universe (it is shorter in the core of the galaxy where the density is higher). In the numerical applications, we have taken $\sigma \sim 100$ km/s and $\rho_b \sim 4 \times 10^{-3} M_\odot$/pc$^3$ (see “Appendix A”).} This corresponds to self-interacting dark matter (SIDM) [79]. Close collisions can establish an isothermal distribution, especially in the center of the halo. We are then led to the scenario of evolution discussed by [68,80] according to which the DM halo is isothermal and, because of...
evaporation, slowly evolves along the series of equilibria towards states of higher and higher central density. At some point (when the central density reaches a critical value corresponding to the minimum energy in the series of equilibria), it becomes thermodynamically unstable and undergoes a gravothermal catastrophe on a secular timescale. Core collapse can be followed by a dynamical instability of general relativistic origin leading to the formation of a supermassive black hole. This scenario only works for sufficiently large DM halos such that the gravothermal catastrophe can take place (see [68] for details). This may explain why supermassive black holes are observed only at the center of large galaxies.

The goal of the present paper is to establish the general kinetic equations that describe the collisional evolution of self-gravitating systems of quantum particles (fermion and bosons). We remain at a general level and treat all possible situations, even those that do not correspond to situations of most astrophysical interest (for example, we consider the case of bosons at $T > T_c$ in spite of the fact that the situation relevant to FDM is $T \ll T_c$). The paper is organized as follows. In Sect. 2, we recall the heuristic derivation of the quantum Landau equation presented in [41] and extend it to a multispecies system of particles. In Sect. 3, we review the kinetic theory of classical self-gravitating particles. In Sect. 4, we extend these results to the case of quantum particles. In Sect. 5, we consider the case of classical particles in collision with quantum particles. In Sect. 6, we show at a general level that the diffusion caused by quantum particles may be interpreted in terms of quasiparticles with an effective mass $m_{\text{eff}}$ and recover, in the case of FDM, the results of [74]. In Sect. 7, we consider the evolution of the system as a whole governed by the self-consistent Landau equation. We discuss the possibility, in the case of bosons, to describe the process of Bose–Einstein condensation. In “Appendix A”, we discuss the validity of the approximations made in the context of FDM. In “Appendix B”, we establish the main properties of the multi-species quantum Landau equation. In “Appendix C”, we discuss the simplification of the bosonic Landau equation in the limit of large occupation numbers. In “Appendix D”, we study the rate of change of the energy of classical particles due to collisions with other classical particles. In “Appendix E”, we extend these results to the case of classical particles experiencing collisions with quasiparticles in FDM halos. In “Appendix F”, we derive a self-similar solution of the Spitzer-Schwarzschild equation describing the diffusion of classical self-gravitating systems when $m \ll m_p$.

2 Multispecies classical and quantum Landau equation

2.1 Classical Landau equation

When considering the “collisional” evolution of an isolated system of classical particles, it is natural to start from the Boltzmann equation [14]

$$\frac{\partial f}{\partial t} = \int d\mathbf{v}_1 d\Omega \ w(\mathbf{v}, \mathbf{v}_1; \mathbf{v}', \mathbf{v}_1') \{ f(\mathbf{v}', t) f(\mathbf{v}_1', t) - f(\mathbf{v}, t) f(\mathbf{v}_1, t) \},$$

(1)

where $d\Omega$ is the element of solid angle and $w(\mathbf{v}, \mathbf{v}_1; \mathbf{v}', \mathbf{v}_1')$ is the density probability of a collision transforming the velocities $\mathbf{v}, \mathbf{v}_1$ into $\mathbf{v}', \mathbf{v}_1'$ or the converse (see, e.g., [81] for details). This transition rate depends only on the nature of the two-body particle interaction. According to the principle of detailed balance, the transition rate satisfies the symmetry property $w(\mathbf{v}, \mathbf{v}_1; \mathbf{v}', \mathbf{v}_1') = w(\mathbf{v}', \mathbf{v}_1'; \mathbf{v}, \mathbf{v}_1)$ expressing the fact that the probability densities of the direct and inverse collisions are equal. The above description assumes that the system is a dilute (low density) gas so that it can be described by a one-particle DF. We also assume
that the system is spatially homogeneous so that the DF depends only on the velocity and time, i.e., \( f = f(v, t) \).

If we now consider weakly coupled gases of particles in interaction, like charges in a plasma or stars in a stellar system, we can make an additional approximation. In the case of Coulombian or Newtonian potentials, the interactions are very weak compared to the mean kinetic energy of the particles and this implies that the deviation resulting from a collision is small (on the average) compared to the initial velocity of the particles. In that case, we can assume that each encounter provokes a weak deflection of the particle trajectory. This is the approach followed by Landau [13] in the case of Coulombian plasmas. His results also apply to gravitational plasmas and we shall write them directly in that context. Considering the weak deflection limit of the Boltzmann equation, Landau obtained a kinetic equation of the form [13]

\[
\frac{\partial f}{\partial t} = 2\pi G^2 m \ln \frac{\Lambda}{\lambda} \int d\mathbf{v}' K_{ij} \left( \frac{f'}{\partial v'_j} - f \frac{\partial f'}{\partial v'_j} \right)
\]

with

\[
K_{ij} = \frac{u^2 \delta_{ij} - u_i u_j}{u^3},
\]

where \( u = v' - v \) is the relative velocity between the particles engaged in a collision. We have noted \( f \) for \( f(v, t) \) and \( f' \) for \( f(v', t) \). The Landau equation can be viewed as a Fokker–Planck equation involving a diffusion term and a friction term. There is a well-known difficulty with the Landau approach in the sense that the Landau equation yields a logarithmically diverging quantity \( \ln \frac{\Lambda}{\lambda} = \int db/b \), where \( b \) is the impact parameter (\( \ln \Lambda \) is called the Coulomb logarithm). Since the divergence is weak (logarithmic), the procedure is marginally valid. The divergence can be regularized by introducing cutoffs. The small scale cutoff is the Landau length \( \lambda_L \sim Gm/\sigma^2 \sim GM^2/k_B T \) (where \( \sigma \sim (k_B T/m)^{1/2} \) denotes the typical root mean square velocity of the particles and \( T \) the temperature) corresponding to the distance of the closest approach at which a collision produces a deflection at 90°. This cutoff eliminates very close encounters which produce large deflections. The large scale cutoff is the Jeans length \( \lambda_J \sim \sigma t_D \sim \sqrt{G\rho} \sim (k_B T/G \rho m)^{1/2} \) (where \( t_D \sim 1/\sqrt{G\rho} \) is the dynamical time and \( \rho = nm \) is the mass density) which is the presumable analogue of the Debye length in plasma physics. This yields

\[
\ln \Lambda \sim \ln \left( \frac{\lambda_J}{\lambda_L} \right) \sim \ln \left( n\lambda_J^3 \right) \sim \ln N,
\]

where \( N \) is the number of particles in the system.

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20 In a Coulombian plasma, the divergence at large scales can be avoided by taking into account collective effects (Debye shielding). This leads to the Lenard–Balescu equation [82, 83]. In a gravitational plasma (stellar system), the divergence at large scales can be avoided by taking into account spatial inhomogeneity. This leads to the inhomogeneous Landau equation [21]. One can also take into account collective effects (anti-shielding) leading to the inhomogeneous Lenard–Balescu equation [26, 27]. These equations can be obtained from systematic methods, the BBGKY hierarchy (based on the Liouville equation) or the quasilinear theory (based on the Klimontovich equation), by considering an expansion of the exact equations of motion in power of \( 1/N \) (see the introduction of [21, 84] for a short history on kinetic theories). In this paper, for simplicity, we will neglect collective effects and assume that the system is spatially homogeneous.

21 According to the virial theorem, we have \( \sigma^2 \sim GM/R \), where \( M \) is the total mass of the stellar system and \( R \) is its radius. Therefore, \( \lambda_J \sim R \) showing that the Jeans length is of the order of the system’s size.
The Landau equation can be generalized to a multi-species system as
\[
\frac{\partial f_a}{\partial t} = 2\pi G^2 \sum_b \ln \Lambda_{ab} \frac{\partial}{\partial v_i} \int d\mathbf{v}' K_{ij} \left( m_b f_b' \frac{\partial f_a}{\partial v_j} - m_a f_a \frac{\partial f_b'}{\partial v_j} \right),
\]
where \( f_a(\mathbf{v}, t) \) is the DF of particles of species “a” with mass \( m_a \), \( f_b(\mathbf{v}, t) \) is the DF of particles of species “b” with mass \( m_b \) and \( \ln \Lambda_{ab} \) is the Coulomb logarithm constructed with the Landau length \( \lambda_L = G(m + m_b)/V_{ab}^2 \) (where \( V_{ab} \) is a typical relative velocity).

The Landau equation conserves the mass \( M_a \) of each species and the total energy \( E \) of the system. It also satisfies an \( H \)-theorem for the Boltzmann entropy (see “Appendix B”). As a result, it relaxes towards the Boltzmann distribution
\[
f^\text{eq}_a(\mathbf{v}) = \rho_a \left( \frac{\beta m_a}{2\pi} \right)^{3/2} e^{-\beta m_a \mathbf{v}^2},
\]
which maximizes the Boltzmann entropy \( S \) at fixed masses \( M_a \) and energy \( E \) (see “Appendix B”). Here \( \rho_a = \int f_a d\mathbf{v} \) is the mass density of species \( a \) and \( \beta = 1/k_B T \) is the inverse temperature. We note that the temperature \( T \) is the same for all the species of particles (this is the Lagrange multiplier associated with the conservation of the total energy). This leads to the theorem of equipartition of energy:
\[
\frac{1}{2} m_a \langle \mathbf{v}^2 \rangle_a = \frac{3}{2} k_B T.
\]

Equipartition of energy usually implies that heavy particles sink at the center of the system while light particles wander around.

**Remark** We can obtain an estimate of the relaxation time as follows.\(^{22}\) Writing \( v \sim \sigma \) and \( f \sim \rho/\sigma^3 \) and considering the scaling of the different terms appearing in the classical Landau equation (2) we obtain
\[
t_R \sim \sigma^3 G^2 \rho m \ln \Lambda.
\]
If we extend this formula to inhomogeneous systems by making a local approximation (see, e.g., [21]), we find that the relaxation time is inversely proportional to the local density \( \rho(\mathbf{r}) \) [assuming that \( \sigma \sim (k_B T/m)^{1/2} \) is uniform throughout the system]. Therefore, the relaxation time is shorter in regions of high density (core) and longer in regions of low densities (halo). Introducing the dynamical time
\[
t_D \sim \frac{\lambda_J}{\sigma} \sim \frac{1}{(G \rho)^{1/2}},
\]
we can rewrite Eq. (8) under the form [1,67]
\[
t_R \sim \frac{n \lambda_J^3}{\ln \Lambda} t_D \sim \frac{N}{\ln N} t_D.
\]
This shows that the relaxation time measured in terms of the dynamical time scales like \( N \) (up to a logarithmic correction). This is connected to the fact that the Landau equation can be obtained as the first deviation from the Vlasov equation (or collisionless Boltzmann equation) in an expansion of the equations of the BBGKY hierarchy in powers of \( 1/N \ll 1 \) (see, e.g., [21]).

\(^{22}\) We consider a single species system for simplicity. More accurate expressions of the relaxation time for single and multispecies systems are given in Sect. 3.7.
2.2 Generalized Landau equation

In Ref. [41], we have introduced a generalized Landau equation associated with a formalism of generalized thermodynamics. We started from the generalized Boltzmann equation [85]

$$\frac{\partial f}{\partial t} = \int d\mathbf{v}_1 d\Omega \, w(\mathbf{v}, \mathbf{v}_1; \mathbf{v}', \mathbf{v}_1') \{a(f')b(f)a(f_1)b(f_1') - a(f)b(f')a(f_1)b(f_1')\}, \quad (11)$$

where $f = f(\mathbf{v}, t)$, $f_1 = f(\mathbf{v}_1, t)$, $f' = f(\mathbf{v}', t)$, and $f_1' = f(\mathbf{v}_1', t)$. This equation can be obtained from a kinetical interaction principle (KIP) which allows the probabilities of transition to depend on the occupation numbers (concentration) of the starting and arrival sites. The factor $a(f)$ is an arbitrary function of the particle population of the starting site and the factor $b(f')$ is an arbitrary function of the arrival site particle population. Usually, the probability of transition is proportional to the density of the starting site and independent of the density of the arrival site so that $a(f) = f$ and $b(f') = 1$. This leads to the ordinary Boltzmann equation (1). However, we can consider a more general dependence on the occupancy in the starting and arrival sites. This can account for microscopic constraints leading to exclusion (close-packing effects, steric hindrance, Pauli exclusion principle for fermions, Lynden-Bell exclusion principle for collisionless self-gravitating systems...) or inclusion (Bose enhancement) effects that can inhibit or stimulate the particle transition $\mathbf{v} \rightarrow \mathbf{v}'$. These constraints may be of classical or quantum origin. The factors $a(f)$ and $b(f')$ take these constraints into account. For example, if we take $a(f) = f$ and $b(f') = 1 \mp f'/\eta_0$ we recover the fermionic and bosonic Boltzmann equations

$$\frac{\partial f}{\partial t} = \int d\mathbf{v}_1 d\Omega \, w(\mathbf{v}, \mathbf{v}_1; \mathbf{v}', \mathbf{v}_1')(f'(1 \mp f/\eta_0)f_1'(1 \mp f_1/\eta_0) - f(1 \mp f'/\eta_0)f_1(1 \mp f_1'/\eta_0))$$

(12)

introduced by Nordheim [86] and Uehling and Uhlenbeck [87]. For more general functions, this generalization encompasses the case of quantum particles (fermions and bosons) with exclusion or inclusion principles.

Considering the gravitational interaction and making a weak deflection approximation, we obtained in Ref. [41] the generalized Landau equation

$$\frac{\partial f}{\partial t} = A \frac{\partial}{\partial v_i} \int d\mathbf{v}_1 K_{ij} \left\{a_1 b_1 (ba' - b'a) \frac{\partial f}{\partial v_j} - ab (b_1 a'_1 - b'_1 a_1) \frac{\partial f_1}{\partial v_{1j}} \right\}, \quad (13)$$

where $a = a(f), b = b(f), a_1 = a(f_1), b_1 = b(f_1)$ and $'$ denotes the derivative with respect to $f$, e.g., $a' = a'(f)$. On the other hand, $A = 2\pi G^2 m \ln \Lambda$. If we define the functions $g$ and $h$ by

$$g(f) = a(f)b(f) \quad \text{and} \quad h(f) = b(f)a'(f) - b'(f)a(f),$$

(14)

the generalized Landau equation (13) can be rewritten as [41]

$$\frac{\partial f}{\partial t} = A \frac{\partial}{\partial v_i} \int d\mathbf{v}' K_{ij} \left\{g(f') h(f) \frac{\partial f}{\partial v_j} - g(f) h(f') \frac{\partial f'}{\partial v'_j} \right\},$$

(15)

where $f = f(\mathbf{v}, t)$ and $f' = f(\mathbf{v}', t)$. It is associated with a generalized entropy of the form

$$S = - \int C(f) \, d\mathbf{v},$$

(16)
where $C(f)$ is a convex function determined by $h$ and $g$ according to

$$C''(f) = \frac{h(f)}{g(f)}.$$  \hfill (17)

One can show [41] that the generalized Landau equation (15) conserves the mass and the energy and satisfies an $H$-theorem for the generalized entropy (16). As a result, it relaxes towards the distribution

$$f_{eq}(v) = (C')^{-1} \left( \alpha - \beta m v^2/2 \right),$$  \hfill (18)

which maximizes the generalized entropy at fixed mass and energy. It is obtained by writing the variational problem as

$$\delta S - \beta \delta E + \left( \alpha/m \right) \delta M = 0$$

where $\beta$ (inverse temperature) and $\alpha$ (chemical potential) are Lagrange multipliers (see Ref. [41] for a more detailed discussion).

2.3 Quantum Landau equation

We can use the formalism of [41] to obtain a kinetic equation for quantum particles in gravitational interaction in a semiclassical approximation. To that aim, we can proceed as follows.

It is natural to assume that the transition probability is proportional to the density of the starting site so that $a(f) = f$. In that case, the generalized Landau equation (13) reduces to

$$\frac{\partial f}{\partial t} = A \frac{\partial}{\partial v_i} \int d v_1 K_{ij} \left\{ f_1 b_1 (b - b' f) \frac{\partial f}{\partial v_j} - f b (b_1 - b'_1 f_1) \frac{\partial f_1}{\partial v_{1j}} \right\}. $$  \hfill (19)

We note that the coefficients of diffusion and friction are not independent since they both depend on $b(f)$. Choosing $b(f) = 1$, i.e., a probability of transition which does not depend on the population of the arrival site, leads to the ordinary Landau equation (2). If we now assume that the transition probability is blocked (inhibited) if the concentration of the arrival site is $\eta_0$, it is natural to take $b(f) = 1 - f/\eta_0$. This corresponds to the case of fermions in quantum mechanics where

$$\eta_0 = g \frac{m^4}{\hbar^3}$$  \hfill (20)

is the maximum value of the DF fixed by the Pauli exclusion principle ($h = 2\pi\hbar$ is the Planck constant and $g = 2s + 1$ is the spin multiplicity of the quantum states). A similar exclusion principle also arises in the theory of violent relaxation developed by Lynden-Bell [12] for collisionless self-gravitating systems described by the Vlasov–Poisson equations (in that case, $\eta_0$ is related to the initial value of the DF). Inversely, if we assume that the transition probability is stimulated if the arrival site is occupied, it is natural to take $b(f) = 1 + f/\eta_0$. This corresponds to the case of bosons in quantum mechanics which experience Bose enhancement. In that case, $\eta_0$ is given by Eq. (20) with $g = 1$ for spin-zero bosons. Therefore, we shall take

$$b(f) = 1 \mp \frac{f}{\eta_0},$$  \hfill (21)

where the upper sign corresponds to fermions and the lower sign corresponds to bosons. In that case, we find that

$$b - b' f = 1,$$  \hfill (22)
implying that the diffusion coefficient in Eq. (19) does not explicitly depend on \( f(v, t) \).\(^{23}\)

Conversely, if we impose that the diffusion coefficient does not explicitly depend on \( f(v, t) \), we obtain the differential equation (22) which can be integrated into

\[
b(f) = 1 - \kappa \frac{f}{\eta_0}.
\]

Interestingly, this condition selects the case of fermions (\( \kappa = 1 \)) and bosons (\( \kappa = -1 \)), and also the case of intermediate quantum statistics (\( \kappa \) arbitrary). Regrouping the previous results, we obtain the quantum Landau equation \(^{41}\)

\[
\frac{\partial f}{\partial t} = 2\pi G^2 m \ln \frac{\Lambda}{\Lambda_1} \int d\nu K_{ij} \left\{ f' \left( 1 - \kappa \frac{f'}{\eta_0} \right) \frac{\partial f}{\partial \nu_j} - f \left( 1 - \kappa \frac{f}{\eta_0} \right) \frac{\partial f'}{\partial \nu_j} \right\}.
\]

For quantum particles, the Coulomb logarithm \( \ln \frac{\Lambda}{\Lambda_1} \) is constructed with the de Broglie length \( \lambda_{dB} = \frac{\hbar}{m \sigma} \) instead of the Landau length \( \lambda_L \).\(^{24}\) It can be generalized to several species as

\[
\frac{\partial f_a}{\partial t} = 2\pi G^2 \sum_b \Lambda_{ab} \frac{\partial}{\partial \nu_i} \int d\nu' K_{ij} \left\{ m_b f_b' \left( 1 - \kappa_b \frac{f_b'}{\eta_b} \right) \frac{\partial f_a}{\partial \nu_j} - m_a f_a \left( 1 - \kappa_a \frac{f_a}{\eta_a} \right) \frac{\partial f_b'}{\partial \nu_j} \right\},
\]

where

\[
\eta_a = \frac{m_a^4}{\hbar^3} \quad \text{and} \quad \eta_b = \frac{m_b^4}{\hbar^3}.
\]

The quantum Landau equation conserves the total mass \( M_a \) of each species and the total energy \( E \) of the system. It also satisfies an \( H \)-theorem for the Fermi–Dirac or Bose–Einstein entropy (see “Appendix B”). As a result, it relaxes towards the Fermi–Dirac or Bose–Einstein distribution\(^{25}\)

\[
f_{eq}^a(v) = \frac{\eta_a}{\lambda_a e^{\beta m_a \frac{v^2}{2}} + \kappa_a},
\]

which maximizes the Fermi–Dirac or Bose–Einstein entropy \( S \) at fixed masses \( M_a \) and energy \( E \) (see “Appendix B”). The temperature \( T \) is the same for all the species. The inverse fugacity \( \lambda_a \) is determined by the mass density \( \rho_a = \int f_a dv \) as detailed in “Appendix B7”.

**Remark** We can obtain an estimate of the relaxation time as follows.\(^{26}\) Writing \( v \sim \sigma \) and \( f \sim \rho/\sigma^3 \) and considering the scaling of the different terms appearing in the quantum Landau equation (24), we obtain

\[
t_R \sim \frac{\sigma^3}{G^2 \rho m (1 - \kappa \chi) \ln \Lambda}
\]

with

\[
\chi \equiv \frac{f}{\eta_0} \sim \frac{\rho \hbar^3}{\sigma^3 m^4}.
\]

\(^{23}\) It is, however, a functional of \( f \) since it is expressed as an integral of \( f(v_1, t) \) over \( v_1 \).

\(^{24}\) The quantum Coulomb logarithm has been discussed in detail by Bar-Or et al. \([74]\) in the context of FDM so we shall not re-discuss it here; we leave it under the generic form \( \ln \frac{\Lambda}{\Lambda_1} \) both in the classical and quantum cases.

\(^{25}\) In the case of bosons, the Bose–Einstein distribution is valid only for \( T > T_c \) (see “Appendix B9”).

\(^{26}\) We consider a single species system for simplicity. More accurate expressions of the relaxation time for single and multispecies systems (including the case of FDM) are given in Sect. 6.
Expression (28) differs from the classical relaxation time from Eq. (8) by the fact that the mass \( m \) of the particles is replaced by an effective mass \( m_{\text{eff}} \equiv m(1 - \kappa \chi) \), corresponding to the mass of quasiparticles. In the case of fermions (\( \kappa = 1 \)), we have \( m_{\text{eff}} < m \) so that quantum mechanics (Pauli’s blocking) has the effect of increasing the relaxation time (\( t_{R}^{\text{Fermi}} > t_{R}^{\text{class}} \)). In the case of bosons (\( \kappa = -1 \)), we have \( m_{\text{eff}} > m \) so that quantum mechanics (Bose enhancement) has the effect of reducing the relaxation time (\( t_{R}^{\text{Bose}} < t_{R}^{\text{class}} \)). The classical limit is recovered when \( \chi \ll 1 \) and \( m_{\text{eff}} \sim m \). On the other hand, for bosons (\( \kappa = -1 \)), there are situations where \( \chi \gg 1 \) (see “Appendix A”). In that case, we can make the approximation

\[
\frac{\partial f}{\partial t} \approx \frac{2\pi}{G m^3} \ln \Lambda \frac{\partial}{\partial v_i} \int d'v K_{ij} \left( f^{2} \frac{\partial f}{\partial v_j} - f^{2} \frac{\partial f'}{\partial v'_j} \right).
\]

The stationary solution of this equation is the Rayleigh–Jeans DF (C3) which is associated with the log-entropy (C2) [88]. The corresponding relaxation time is

\[
t_{R} \sim \frac{\sigma^3}{G^2 \rho m \chi \ln \Lambda} \sim \frac{\sigma^3}{G^2 \rho m_{\text{eff}} \ln \Lambda} \sim \frac{\sigma^6 m^3}{G^2 \rho^2 h^3 \ln \Lambda}.
\]

The effective mass of the particles is given by

\[
m_{\text{eff}} \sim m \chi \sim \frac{\rho h^3}{\sigma^3 m^3} \sim \rho \lambda_{DB}^3,
\]

where \( \lambda_{DB} = h/(m \sigma) \) is the de Broglie length. The effective mass \( m_{\text{eff}} \) corresponds to the mass contained within the de Broglie sphere. The effective number of particles is \( N_{\text{eff}} = M/m_{\text{eff}} \) and the relaxation time can be written as

\[
t_{R} \sim \frac{\rho R^3}{m_{\text{eff}} \ln \Lambda} t_{D} \sim \left( \frac{R}{\lambda_{DB}} \right)^3 \frac{1}{\ln \Lambda} t_{D} \sim \frac{N_{\text{eff}}}{\ln N_{\text{eff}}} t_{D},
\]

where \( t_{D} \sim R/\sigma \sim (G \rho)^{-1/2} \) is the dynamical time. It can be shown that \( t_{R} \) represents the condensation time for the self-consistent evolution of a system of bosons in gravitational interaction below the critical temperature \( T_c \) [50] or the heating time for a system of classical particles in “collisions” with quasiparticles of mass \( m_{\text{eff}} \) arising from wave interferences in FDM halos [74]. Furthermore, the kinetic theory is valid for \( R \gg \lambda_{DB} \), i.e., \( N_{\text{eff}} \gg 1 \). Numerical applications to justify the previous approximations and give characteristic values of the relaxation time for FDM are made in “Appendix A”.

3 Classical kinetic theory

In this section, we review and complete the kinetic theory of classical particles in gravitational interaction such as stars in globular clusters. As discussed previously, we use an idealization in which the system is assumed to be spatially homogeneous and we ignore collective effects.

3.1 Classical Landau equation

We consider a two-species system consisting of test particles of mass \( m \) and DF \( f(v, t) \) experiencing “collisions” with field (background) particles of mass \( m_b \) and DF \( f_b(v, t) \). The DF of the field particles is assumed to be given. It may be independent of time (fixed) or may evolve with time according to another equation that is not specified here (it could be the
Landau equation of species $b$ in the self-consistent treatment of collisions described above or, possibly, another equation of evolution). If we neglect the “collisions” between the test particles (when $b \neq a$) and only consider the “collisions” between the test particles and the field particles, the classical Landau equation writes

$$\frac{\partial f}{\partial t} = 2\pi G^2 \ln \Lambda \frac{\partial}{\partial v_i} \int d\nu' K_{ij} \left( m_b f'_b \frac{\partial f}{\partial v_j} - m f \frac{\partial f'_b}{\partial v'_j} \right).$$

(34)

If we assume that $f_b(\nu)$ is fixed (independent of time), then Eq. (34) is just a partial differential equation of the Fokker–Planck type (see below). This corresponds to the “bath” approach in which the test particles interact with field particles that have a prescribed DF. In particular, the “thermal bath” approach corresponds to the situation where the field particles are at statistical equilibrium with the Maxwell-Boltzmann DF (this assumes that they are not disturbed by collisions with the test particles). On the other hand, by taking $m_b = m$ and $f_b = f$, Eq. (34) describes a single species system of particles of mass $m$ and DF $f(\nu, t)$ evolving in time in a self-consistent manner. In that case, we recover the classical Landau equation (2) which is an integrodifferential equation.

**Remark** As emphasized in previous works (see, e.g., [89]), the “bath” approximation transforms an integrodifferential equation (Landau) where the diffusion and friction coefficients are functionals of $f$ into a differential equation (Fokker–Planck) where the diffusion and friction coefficients are independent of $f$.

### 3.2 Classical Fokker–Planck equation

The classical Landau equation (34) can be written in the form of a Fokker–Planck equation

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v_i} \left( D_{ij} \frac{\partial f}{\partial v_j} - f F^\text{pol}_i \right)$$

(35)

involving a diffusion tensor\(^{27}\)

$$D_{ij} = 2\pi G^2 m_b \ln \Lambda \int d\nu' K_{ij} f'_b$$

(36)

and a friction by polarization\(^{28}\)

$$F^\text{pol}_i = 2\pi G^2 m \ln \Lambda \int d\nu' K_{ij} \frac{\partial f'_b}{\partial v'_j}.$$  

(37)

The usual form of the Fokker–Planck equation [90] is

$$\frac{\partial f}{\partial t} = \frac{\partial^2}{\partial v_i \partial v_j} \left( D_{ij} f \right) - \frac{\partial}{\partial v_i} \left( f F^\text{friction}_i \right),$$

(38)

where the first two moments of the velocity increment $\Delta \nu$ are

$$D_{ij} = \frac{\langle \Delta v_i \Delta v_j \rangle}{2\Delta t} \quad \text{and} \quad F^\text{friction}_i = \frac{\langle \Delta v_i \rangle}{\Delta t}.$$  

(39)

\(^{27}\) The diffusion arises from the fluctuations of the field particles. This is why it is proportional to the mass $m_b$ of the field particles.

\(^{28}\) This terminology was introduced in [91] in order to distinguish the friction force $F^\text{pol}_i$ that appears naturally in the Landau equation from the “true” friction $F^\text{friction}_i$ that appears in the Fokker–Planck equation (see below). As explained in [91], the friction by polarization arises from the retroaction of the field particles to the perturbation caused by the test particles. This is why it is proportional to the mass $m$ of the test particles.
Equation (39) defines the true (total) friction force $F_{\text{friction}}$. The relation between the friction by polarization and the true friction is

$$F_{\text{friction}} = F_{\text{pol}}^i + \frac{\partial D_{ij}}{\partial v_j}. \quad (40)$$

For classical particles, we have

$$\frac{\partial D_{ij}}{\partial v_j} = 2\pi G^2 m_b \ln \Lambda \int d\nu' \frac{\partial K_{ij}}{\partial v_j'} f_b'. \quad (41)$$

Therefore, we obtain

$$F_{\text{friction}} = \frac{m + m_b}{m} F_{\text{pol}}. \quad (42)$$

When $m_b = m$ we find that $F_{\text{friction}} = 2 F_{\text{pol}}$. In that case, the true friction and the friction by polarization differ by a factor 2. This factor 2, or more generally the ratio $(m + m_b)/m$, appeared at several occasions in the literature and was not always clearly understood.

### 3.3 Rosenbluth potentials

Recalling Eq. (3), the diffusion tensor (36) is given explicitly by

$$D_{ij} = 2\pi G^2 m_b \ln \Lambda \int d\nu' \frac{u^2 \delta_{ij} - u_i u_j}{u^3} f_b'. \quad (43)$$

On the other hand, integrating Eq. (37) by parts and using the identity

$$\frac{\partial K_{ij}}{\partial v_j'} = 2 \frac{u_i}{u^3}, \quad (44)$$

we can rewrite the friction by polarization as

$$F_{\text{pol}}^i = -4\pi G^2 m \ln \Lambda \int d\nu' \frac{u_i}{u^3} f_b'. \quad (45)$$

Then, using Eq. (42), the true friction is

$$F_{\text{friction}}^i = -4\pi G^2 (m + m_b) \ln \Lambda \int d\nu' \frac{u_i}{u^3} f_b'. \quad (46)$$

Now, using the identities

$$K_{ij} = \frac{u^2 \delta_{ij} - u_i u_j}{u^3} = \frac{\partial^2 u}{\partial v_i \partial v_j} \quad \text{and} \quad \frac{\partial K_{ij}}{\partial v_j} = -2 \frac{u_i}{u^3} = 2 \frac{\partial}{\partial v_i} \left( \frac{1}{u} \right). \quad (47)$$
the diffusion tensor and the friction can be rewritten as

\[ D_{ij} = 2\pi G^2 m_b \ln \Lambda \frac{\partial^2 \chi}{\partial v_i \partial v_j}(v), \]  
(48)

\[ F_{\text{pol}} = 4\pi G^2 m \ln \Lambda \frac{\partial \lambda}{\partial v}(v), \]  
(49)

\[ \frac{\partial D_{ij}}{\partial v_j} = 4\pi G^2 m_b \ln \Lambda \frac{\partial \lambda}{\partial v}(v), \]  
(50)

\[ F_{\text{friction}} = 4\pi G^2 (m + m_b) \ln \Lambda \frac{\partial \lambda}{\partial v}(v) \]  
(51)

with

\[ \chi(v) = \int f'_b |v - v'| d'v' \quad \text{and} \quad \lambda(v) = \int \frac{f'_b}{|v - v'|} d'v'. \]  
(52)

The functions \( \chi(v) \) and \( \lambda(v) \) are the so-called Rosenbluth potentials [18]. They are the solutions of the differential equations

\[ \Delta_v \chi = 2\lambda, \quad \Delta_v \lambda = -4\pi f_b, \]  
(53)

where we have used \( \Delta |v - v'| = 2/|v - v'| \) and \( \Delta (1/|v - v'|) = -4\pi \delta(v - v') \).

**Remark** The expressions of the diffusion tensor [Eq. (43)] and true friction [Eq. (46)] can be directly obtained from the two-body encounters theory developed by Chandrasekhar [1, 15, 20] and Rosenbluth et al. [18]. In these approaches, contrary to the Landau approach, the Coulomb logarithm does not diverge at short distances since strong collisions occurring at small impact parameters are taken into account exactly through the Rutherford cross section of gravitational scattering. These coefficients of diffusion and friction can then be substituted into the general form of the Fokker–Planck equation [Eq. (38)] to obtain a self-consistent kinetic equation. This equation is equivalent to the Landau equation but it appears in a less symmetric form which is not as elegant as the Landau equation (see [21] for a comparison between the Chandrasekhar and Landau approaches).

### 3.4 Isotropic bath

When \( f_b(v) \) is isotropic, i.e., \( f_b = f_b(v) \), the Rosenbluth potentials can be simplified and the coefficients of diffusion and friction can be calculated explicitly (see, e.g., [18, 67]). The diffusion tensor takes the form

\[ D_{ij} = \left( D_{||} - \frac{1}{2} D_{\perp} \right) \frac{v_i v_j}{v^2} + \frac{1}{2} D_{\perp} \delta_{ij}, \]  
(54)

where \( D_{||} \) and \( D_{\perp} \) are the diffusion coefficients in the directions parallel and perpendicular to the velocity \( v \) of the test particle. They are given by

\[ D_{||} = \frac{16\pi^2}{3} G^2 m_b \ln \Lambda \frac{1}{v} \left[ \int_0^v \frac{v_1^4}{v^4} f_b(v_1) dv_1 + v \int_v^{+\infty} v_1 f_b(v_1) dv_1 \right], \]  
(55)

\[ D_{\perp} = \frac{16\pi^2}{3} G^2 m_b \ln \Lambda \frac{1}{v} \left[ \int_0^v \left( v_1^2 - \frac{v_1^4}{v^2} \right) f_b(v_1) dv_1 + 2v \int_v^{+\infty} v_1 f_b(v_1) dv_1 \right]. \]  
(56)
On the other hand, the friction force is given by

$$F_{\text{pol}} = -16\pi^2 G^2 m \ln \Lambda \frac{v}{v^3} \int_0^v v_1^2 f_b(v_1) \, dv_1,$$

(57)

$$\frac{\partial D_{ij}}{\partial v_j} = -16\pi^2 G^2 m_b \ln \Lambda \frac{v}{v^3} \int_0^v v_1^2 f_b(v_1) \, dv_1,$$

(58)

$$F_{\text{friction}} = -16\pi^2 G^2 (m + m_b) \ln \Lambda \frac{v}{v^3} \int_0^v v_1^2 f_b(v_1) \, dv_1.$$

(59)

Equation (59) is the celebrated Chandrasekhar formula of dynamical friction [15]. We note that it depends only on the DF of the field particles with a velocity $v_1$ smaller than $v$. This is because the Rosenbluth potential $\lambda(\mathbf{r})$ given by Eq. (52) is similar to the gravitational potential produced by a distribution of matter with density $\rho(\mathbf{r})$ (if we identify $\mathbf{v}$ with $\mathbf{r}$ and $f_b(\mathbf{v})$ with $\rho(\mathbf{r})$). Consequently, the total friction force given by Eq. (51) is similar to the gravitational force. If the DF of the field particles is isotropic (corresponding to a spherically symmetric distribution of matter in the gravitational analogy), we immediately obtain Eq. (59) which is the counterpart of Newton’s law $a = -GM(r)r/r^3$ with $M(r) = \int_0^r \rho(r')4\pi r'^2 \, dr$.

### 3.5 Thermal bath: Einstein relation

We now assume that the field particles are in a statistical equilibrium state described by the Maxwellian DF:

$$f_b(\mathbf{v}) = \rho_b \left( \frac{\beta m_b}{2\pi} \right)^{3/2} e^{-\beta m_b v^2/2}.$$

(60)

This corresponds to the so-called “thermal bath”. This DF may be written as

$$f_b(\mathbf{v}) = \frac{\rho_b}{(2\pi \sigma_b^2)^{3/2}} e^{-v^2/2\sigma_b^2},$$

(61)

where we have introduced the velocity dispersion of the field particles in one direction (see “Appendix B8”)

$$\sigma_b^2 = \frac{k_B T}{m_b} = \frac{1}{\beta m_b} \frac{\langle v^2 \rangle}{3}.$$

(62)

Substituting the identity

$$\frac{\partial f_b}{\partial \mathbf{v}} = -f_b \beta m_b \mathbf{v}$$

(63)

into Eq. (37), we obtain

$$F_{i}^{\text{pol}} = -2\pi G^2 \beta m m_b \ln \Lambda \int d\mathbf{v}' K_{ij} f_b' v_j'$$

$$= -2\pi G^2 \beta m m_b \ln \Lambda \int d\mathbf{v}' K_{ij} f_b'(u_j + v_j)$$

$$= -2\pi G^2 \beta m m_b v_j \ln \Lambda \int d\mathbf{v}' K_{ij} f_b'',$$

(64)

where we have used $K_{ij} u_j = 0$ according to Eq. (3) to get the third line. Comparing this expression with Eq. (36) we obtain

$$F_{i}^{\text{pol}} = -\beta m D_{ij} v_j.$$
Using the identity \( D_{ij} v_j = D_{\parallel} v_i \) [see Eq. (54)], we can also write

\[
\mathbf{F}_{\text{pol}} = -\beta m D_{\parallel} \mathbf{v}.
\]  

(66)

We see that the friction by polarization is proportional and opposite to the velocity of the test particle. Furthermore, the friction coefficient is given by the Einstein relation

\[
\xi = \beta m D_{\parallel},
\]  

(67)

which expresses the fluctuation-dissipation theorem. We note that the Einstein relation is valid for the friction by polarization, not for the true friction.\(^{29}\) Substituting Eq. (65) into Eq. (35), we obtain the Kramers–Chandrasekhar equation\(^{30}\)

\[
\frac{\partial f}{\partial t} = \frac{\partial f}{\partial v_i} [D_{ij} \left( \frac{\partial f}{\partial v_j} + \beta m f v_j \right)].
\]  

(70)

This equation relaxes towards the Maxwellian DF:

\[
f_{\text{eq}}(\mathbf{v}) = \rho \left( \frac{\beta m}{2\pi} \right)^{3/2} e^{-\frac{\beta m v^2}{2}},
\]  

(71)

involving the mass \( m \) of the test particles. This DF may be written as

\[
f_{\text{eq}}(\mathbf{v}) = \rho \left( \frac{2\pi \sigma_{\text{eq}}^2}{3} \right)^{3/2} e^{-\frac{v^2}{2\sigma_{\text{eq}}^2}},
\]  

(72)

where we have introduced the equilibrium velocity dispersion of the test particles in one direction

\[
\sigma_{\text{eq}}^2 = \frac{k_B T}{m} = \frac{1}{\beta m} = \frac{\langle v^2 \rangle_{\text{eq}}}{3}.
\]  

(73)

\(^{29}\) This is because \( D_{ij} \) depends on the velocity. We do not have this subtlety for ordinary Brownian motion where the diffusion coefficient is constant. In his seminal paper on dynamical friction, Chandrasekhar \([15]\) calculates the true friction and obtains the following relation between the friction force and the diffusion coefficient

\[
\mathbf{F}_{\text{friction}} = -\beta(m + m_p)D_{\parallel} \mathbf{v} \quad \text{or} \quad \frac{\Delta v_i}{\Delta t} = -\beta(m + m_p) \frac{\langle \Delta v_i \Delta v_j \rangle}{2 \Delta t} v_j
\]  

(68)

that he calls the Einstein relation. The r.h.s. differs from Eq. (66) in the ratio \((m + m_p)/m\) (see the comment at the end of Sect. 3.2). However, Eq. (68) is valid only for classical particles while Eq. (66) remains valid for quantum particles (see below) and, more generally, for an arbitrary form of entropy \([41]\). In addition, Eq. (68) is valid only when collective effects are neglected while Eq. (66) remains valid when they are taken into account (see, e.g., \([89]\)). Therefore, Eq. (66) involving the friction by polarization appears to be more fundamental than Eq. (68) involving the true friction.

\(^{30}\) The ordinary Kramers \([92]\) equation, first derived by Klein \([93]\), is a Fokker–Planck equation describing the evolution of the DF \( f(\mathbf{r}, \mathbf{v}, t) \) of a spatially inhomogeneous system of Brownian particles submitted to an external force \( \mathbf{F} \). It involves an advection term in phase space and a constant isotropic diffusion tensor \( D_{ij} = D \delta_{ij} \). It writes

\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m} \cdot \frac{\partial f}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \left[ D \left( \frac{\partial f}{\partial \mathbf{v}} + \beta m f \mathbf{v} \right) \right].
\]  

(69)

A drift–diffusion equation in velocity space similar to Eq. (69) but without the advection term was first derived by Lord Rayleigh \([94]\) to describe the dynamics of massive particles bombarded by numerous small projectiles. This paper can be seen as a precursor of the theory of Brownian motion that is usually considered to start with the seminal work of Einstein \([95]\) (see \([96]\) for additional comments about the connection between the paper of Lord Rayleigh \([94]\) and Brownian theory). The Fokker–Planck equation (69) with a velocity-dependent diffusion coefficient was studied by Chandrasekhar \([15–17]\) in relation to the evaporation of globular clusters.
At statistical equilibrium, the test particles and the field particles have the same temperature \( T \) (equipartition of energy) but not the same typical velocities since, according to Eq. (7), we have

\[
\frac{1}{2} m \langle v^2 \rangle_{\text{eq}} = \frac{1}{2} m_b \langle v_b^2 \rangle = \frac{3}{2} k_B T \quad \Rightarrow \quad \sigma_{\text{eq}} = \left( \frac{m_b}{m} \right)^{1/2} \sigma_b.
\]  

(74)

If \( m > m_b \) (resp. \( m < m_b \)) the typical velocity of the test particles is smaller (resp. larger) than the typical velocity of the field particles. On the other hand, the DF of the test particles at statistical equilibrium is related to the DF of the field particles (thermal bath) by

\[
f_{\text{eq}}(v) \propto f_b(v)^{m/m_b}.
\]  

(75)

3.6 Diffusion and friction terms in the thermal bath approximation

For a thermal bath [see Eq. (60)], the diffusion coefficients of the test particles obtained from Eqs. (55) and (56) are given by

\[
D_\parallel = 4 \pi G^2 m_b \ln \Lambda_\rho_b G(x) \frac{1}{v},
\]  

(76)

\[
D_\perp = 4 \pi G^2 m_b \ln \Lambda_\rho_b \left[ \text{erf}(x) - G(x) \right] \frac{1}{v},
\]  

(77)

where \( x \) and \( G(x) \) are defined below. On the other hand, the friction force [obtained from Eqs. (57)–(59)] is given by

\[
F_{\text{friction}} = -4 \pi G^2 \beta (m + m_b) m_b \ln \Lambda_\rho_b G(x) \frac{v}{v}.
\]  

(80)

We can check from these expressions that the Einstein relation (67) is satisfied. In the foregoing equations, we have defined

\[
x = \sqrt{\frac{\beta m_b}{2}} v = \frac{v}{\sqrt{2} \sigma_b}
\]  

(81)

and

\[
G(x) = \frac{2}{\sqrt{\pi} x^2} \int_0^x t^2 e^{-t^2} dt = \frac{1}{2x^2} \left[ \text{erf}(x) - \frac{2x}{\sqrt{\pi}} e^{-x^2} \right],
\]  

(82)

where

\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt
\]  

(83)
is the error function. We have \( G(x) \sim 2x/(3\sqrt{\pi}) \) for \( x \to 0 \) and \( G(x) \sim 1/(2x^2) \) for \( x \to +\infty \). We can then write the diffusion tensor as

\[
D_{ij}(v) = \left( \frac{2}{\pi} \right)^{1/2} \frac{G^2 m_b \ln \Lambda \rho_b \sqrt{\beta m_b} G_{ij}(x)}{\sigma_b},
\]

(84)

where

\[
G_{ij} = \left( G_\parallel - \frac{1}{2} G_\perp \right) \frac{x_i x_j}{x^2} + \frac{1}{2} G_\perp \delta_{ij}
\]

(85)

with

\[
G_\parallel = \frac{2\pi^{3/2}}{x} G(x), \quad G_\perp = \frac{2\pi^{3/2}}{x} [\text{erf}(x) - G(x)].
\]

(86)

We have the asymptotic behaviors

\[
G_\parallel(0) = \frac{4\pi}{3}, \quad G_\perp(0) = \frac{8\pi}{3},
\]

(87)

\[
G_\parallel(x) \sim +\infty \frac{\pi^{3/2}}{x^3}, \quad G_\perp(x) \sim +\infty \frac{2\pi^{3/2}}{x}.
\]

(88)

The diffusion coefficients \( D_\parallel(v) \) and \( D_\perp(v) \) are plotted in Fig. 1 of [21]. We note in particular that \( D_\parallel(v) \) decreases as \( v^{-3} \) for \( v \to +\infty \).

### 3.7 Dimensionless Kramers–Chandrasekhar equation

It is convenient to write the Kramers–Chandrasekhar equation (70) under a dimensionless form [21,97,98]. Making the change of variables from Eq. (81) and using Eq. (84), we obtain

\[
\frac{\partial f}{\partial t} = \frac{1}{t_H} \frac{\partial}{\partial x_i} \left[ G_{ij}(x) \left( \frac{\partial f}{\partial x_j} + \frac{2 m}{m_b} f x_j \right) \right],
\]

(89)

where we have introduced the relaxation time (heating time) [21,97,98]

\[
t_H = \frac{\sqrt{2\pi}}{G^2 \rho_b m_b (\beta m_b)^{3/2} \ln \Lambda}
\]

\[
= \frac{\sqrt{2\pi} \sigma_b^3}{G^2 \rho_b m_b \ln \Lambda}.
\]

(90)

The heating time depends only on the mass of the field particles. When the field stars have a larger mass than the test stars (\( m_b > m \)) the heating time \( t_H \) is reduced as compared to the case where all the stars have the same mass (\( m_b = m \)). On the other hand, when \( m \ll m_b \), the friction term is negligible and we just have a diffusive heating (see Sect. 3.8).

We can also write

\[
\frac{\partial f}{\partial t} = \frac{1}{t_C} \frac{\partial}{\partial x_i} \left[ G_{ij}(x) \left( \frac{m_b}{m} \frac{\partial f}{\partial x_j} + 2 f x_j \right) \right],
\]

(91)

We can estimate the heating time by writing \( (\langle \Delta v \rangle^2) \sim D t_H \sim \sigma_b^2 \). This corresponds to the typical time needed by the test particles to acquire the velocity of the field particles through diffusion. According to Eq. (84), the diffusion coefficient scales as \( D \sim G^2 \rho_b m_b \ln \Lambda / \sigma_b \). This gives \( t_H \sim \sigma_b^2 / (G^2 \rho_b m_b \ln \Lambda) \) in agreement with Eq. (90).

---

\( ^{31} \) We can estimate the heating time by writing \( (\langle \Delta v \rangle^2) \sim D t_H \sim \sigma_b^2 \). This corresponds to the typical time needed by the test particles to acquire the velocity of the field particles through diffusion. According to Eq. (84), the diffusion coefficient scales as \( D \sim G^2 \rho_b m_b \ln \Lambda / \sigma_b \). This gives \( t_H \sim \sigma_b^2 / (G^2 \rho_b m_b \ln \Lambda) \) in agreement with Eq. (90).
where we have introduced the relaxation time (cooling time) \[21,97,98\]32

\[ t_C = \frac{\sqrt{2\pi}}{G^2 \rho_b m (\beta m_b)^{3/2} \ln \Lambda} = \frac{\sqrt{2\pi} \sigma_b^3}{G^2 \rho_b m \ln \Lambda}. \tag{92} \]

The cooling time depends only on the mass of the test particles. It has the same expression as when all the stars have the same mass \((m_b = m)\). When \(m \gg m_b\), the diffusion term is negligible and we just have a frictional cooling (see Sect. 3.9).

We note that

\[ \frac{t_C}{t_H} = \frac{m_b}{m}. \tag{93} \]

The dimensionless Kramers–Chandrasekhar equation can therefore be written as

\[ \frac{\partial f}{\partial t} = \frac{\partial}{\partial x_i} \left[ G_{ij}(x) \left( \frac{1}{t_H} \frac{\partial f}{\partial x_j} + \frac{2}{t_C} f x_j \right) \right]. \tag{94} \]

When \(f(x, t)\) is isotropic, i.e., \(f = f(x, t)\), using \(G_{ij} x_j = G_{\parallel}(x) x_i\), the foregoing equations take the form

\[ \frac{\partial f}{\partial t} = \frac{1}{t_H} \frac{1}{x^2} \frac{\partial}{\partial x} \left[ x^2 G_{\parallel}(x) \left( \frac{\partial f}{\partial x} + \frac{2}{m} f x \right) \right], \tag{95} \]

\[ \frac{\partial f}{\partial t} = \frac{1}{t_C} \frac{1}{x^2} \frac{\partial}{\partial x} \left[ x^2 G_{\parallel}(x) \left( \frac{m_b}{m} \frac{\partial f}{\partial x} + 2 f x \right) \right], \tag{96} \]

\[ \frac{\partial f}{\partial t} = \frac{1}{x^2} \frac{\partial}{\partial x} \left[ x^2 G_{\parallel}(x) \left( \frac{1}{t_H} \frac{\partial f}{\partial x} + \frac{2}{t_C} f x \right) \right]. \tag{97} \]

### 3.8 Diffusion equation \((m \ll m_b)\)

When the test particles are much lighter than the field particles \((m \ll m_b)\) we can neglect the force by polarization:33

\[ F_{\text{pol}} = 0. \tag{98} \]

In that case, the stars experience just a process of diffusion (heating). This is the situation considered by Spitzer and Schwarzschild \[97\]. The Fokker–Planck equation (35) reduces to a pure diffusion equation

\[ \frac{\partial f}{\partial t} = \frac{\partial}{\partial v_i} \left( D_{ij} \frac{\partial f}{\partial v_j} \right), \tag{99} \]

\footnote{We can estimate the cooling time by writing \(\langle (\Delta v)^2 \rangle \sim D t_C \sim \sigma_{\text{eq}}^2\). This corresponds to the typical time needed by the test particles to acquire their equilibrium velocity. Using \(D \sim G^2 \rho_b m \ln \Lambda/\sigma_b\) and \(m \sigma_{\text{eq}}^2 = m_b \sigma_b^2\), we obtain \(t_C \sim \sigma_b^3/(G^2 \rho_b m \ln \Lambda)\) in agreement with Eq. (92). Alternatively, we can estimate the cooling time by writing \(t_C \sim \xi^{-1}\). This is the friction time. Using the Einstein relation \(\xi \sim D \beta m\) we get \(t_C \sim \sigma_b^3/(G^2 \rho_b m \ln \Lambda)\) as before.}

\footnote{This amounts to making \(m \to 0\) in Eq. (37). When \(m \ll m_b\) the test particles do not significantly perturb the field particles (see footnote 28). As a result, there is no retroaction from the field particles, hence no friction by polarization. Note, however, that the true friction is nonzero [see Eq. (100)].}
where the diffusion coefficient is given by Eq. (36). The mass \( m \) of the test particles does not appear in this equation. According to Eq. (40), the total friction force is given by

\[
F_{i}^{\text{friction}} = \frac{\partial D_{i j}}{\partial v_{j}}.
\]

(100)

Using Eq. (39), this relation can be rewritten as

\[
\frac{\langle \Delta v_{i} \rangle}{\Delta t} = \frac{\partial}{\partial v_{j}} \left( \frac{\langle \Delta v_{i} \Delta v_{j} \rangle}{2 \Delta t} \right).
\]

(101)

This relation is sometimes referred to as the fluctuation-dissipation theorem but this terminology is misleading because the friction by polarization (which is at the origin of the Einstein relation (66)) is zero in the present case (see footnote 29).

If we assume that the field stars have the Maxwellian distribution (60) and that the DF of the test stars is isotropic, we find, using the normalized variables introduced in Sect. 3.7, that the diffusion equation (99) takes the form

\[
\frac{\partial f}{\partial t} = \frac{1}{t_H} \frac{1}{x^2} \frac{\partial}{\partial x} \left( x^2 G_{\parallel}(x) \frac{\partial f}{\partial x} \right).
\]

(102)

This is the Spitzer-Schwarzschild diffusion equation [97]. We obtain in “Appendix F” an analytical self-similar solution of this equation and compare this solution with the numerical results of Spitzer and Schwarzschild [97].

3.9 Deterministic equation \((m \gg m_b)\)

When the test particles are much heavier than the field particles \((m \gg m_b)\) we can neglect the diffusion:

\[
D_{i j} = 0.
\]

(103)

In that case, the test particles experience just a process of friction (cooling). The Fokker–Planck equation (35) reduces to the purely frictional equation

\[
\frac{\partial f}{\partial t} = \frac{\partial}{\partial v_{i}} \left( -f F_{i}^{\text{pol}} \right),
\]

(104)

where the friction force is given by Eq. (37). The mass \( m_b \) of the field particles does not appear in this equation. According to Eq. (40), we have

\[
F_{i}^{\text{friction}} = F_{i}^{\text{pol}}.
\]

(105)

Eq. (104) is equivalent to the deterministic equation of motion

\[
\frac{d\mathbf{v}}{dt} = \mathbf{F}_{\text{pol}},
\]

(106)

where \( \mathbf{v} \) is the velocity of the test particle. For an isotropic bath, \( \mathbf{F}_{\text{pol}} \) is given by Eq. (57) and for a thermal bath it takes the form of Eq. (66).

Remark  Dynamical friction is responsible for the decay of globular clusters and black hole orbits in a galaxy (this corresponds to the so-called sinking satellite problem). As they orbit through a galaxy they are subject to dynamical friction and spiral toward the galaxy center [67,99].

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34 This amounts to making \( m_b \rightarrow 0 \) in Eq. (36). When \( m_b \ll m \) the field particles do not produce significant fluctuations to induce a diffusion of the test particles (see footnote 27).
4 Quantum kinetic theory

In this section, we develop the kinetic theory of quantum particles in gravitational interaction such as fermions (massive neutrinos) or bosons (axions) in DM halos. Again, we use an idealization in which the system is assumed to be spatially homogeneous and we ignore collective effects.

4.1 Quantum Landau equation

Under the same assumptions as those made in Sect. 3.1, but considering now quantum particles instead of classical particles, the quantum Landau equation writes

\[
\frac{\partial f}{\partial t} = 2\pi G^2 \ln \Lambda \frac{\partial}{\partial v_i} \int d\mathbf{v}' K_{ij} \left\{ m_b f'_b \left(1 - \kappa_b \frac{f'_b}{\eta_b} \right) \frac{\partial f}{\partial v_j} - m f \left(1 - \kappa \frac{f}{\eta_0} \right) \frac{\partial f'_b}{\partial v'_j} \right\}
\]

(107)

with \( \kappa = +1 \) for fermions and \( \kappa = -1 \) for bosons (and \( \kappa = 0 \) for classical particles). As in Sect. 2.3, we have defined

\[
\eta_0 = g \frac{m^4}{h^3}, \quad \eta_b = g \frac{m_b^4}{h^3}.
\]

(108)

4.2 Quantum Fokker–Planck equation

The quantum Landau equation (107) can be written in the form of a quantum Fokker–Planck equation

\[
\frac{\partial f}{\partial t} = \frac{\partial}{\partial v_i} \left[ D_{ij} \frac{\partial f}{\partial v_j} - f \left(1 - \kappa \frac{f}{\eta_0} \right) F_{\text{pol}}^i \right]
\]

(109)

with a diffusion tensor

\[
D_{ij} = 2\pi G^2 m_b \ln \Lambda \int d\mathbf{v}' K_{ij} f'_b \left(1 - \kappa_b \frac{f'_b}{\eta_b} \right)
\]

(110)

and a friction by polarization

\[
F_{\text{pol}}^i = 2\pi G^2 m \ln \Lambda \int d\mathbf{v}' K_{ij} \frac{\partial f'_b}{\partial v'_j}.
\]

(111)

We note that the force by polarization has the same form as in the classical case. The quantum factor \( f'_b \left(1 - \kappa_b \frac{f'_b}{\eta_b} \right) \) occurs only in the diffusion coefficient. The usual form of the quantum Fokker–Planck equation [85] is

\[
\frac{\partial f}{\partial t} = \frac{\partial}{\partial v_i} \left[ D_{ij} \frac{\partial f}{\partial v_j} - f \left(1 - \kappa \frac{f}{\eta_0} \right) \left( F_{\text{friction}}^i - \frac{\partial D_{ij}}{\partial v_j} \right) \right],
\]

(112)

where the first two moments of the velocity increment \( \Delta \mathbf{v} \) are given by Eq. (39). Equation (39) defines the true (complete) friction force \( F_{\text{friction}} \). The relation between the friction by polarization and the true friction is

\[
F_{\text{friction}}^i = F_{\text{pol}}^i + \frac{\partial D_{ij}}{\partial v_j}.
\]

(113)
as in the classical case. We can also write the quantum Fokker–Planck equation as
\[
\frac{\partial f}{\partial t} = \frac{\partial}{\partial v_i} \left[ \frac{\partial}{\partial v_j} (D_{ij} f) - f \frac{\partial D_{ij}}{\partial v_j} - f \left( 1 - \frac{f}{\eta_0} \right) F_i^{\mathrm{pol}} \right].
\] (114)

4.3 Quantum Rosenbluth potentials

Using the identities from Eq. (47), the diffusion tensor (110) may be rewritten as
\[
D_{ij} = 2\pi G^2 m_b \ln \Lambda \frac{\partial^2 \chi}{\partial v_i \partial v_j} (v)
\] (115)
with
\[
\chi(v) = \int f_b \left( 1 - \kappa_b f_b \right) |v - v'| d v'.
\] (116)

Similarly, the friction by polarization (111) may be rewritten as (using an integration by parts)
\[
F_{\mathrm{pol}} = 4\pi G^2 m \ln \Lambda \frac{\partial \lambda}{\partial v} (v)
\] (117)
with
\[
\lambda(v) = \int \frac{f_b'}{|v - v'|} d v'.
\] (118)

Using Eq. (47), we also have
\[
\frac{\partial D_{ij}}{\partial v_j} = 4\pi G^2 m_b \ln \Lambda \frac{\partial \sigma}{\partial v_i} (v)
\] (119)
with
\[
\sigma(v) = \int \frac{f_b \left( 1 - \kappa_b f_b \right)}{|v - v'|} d v'.
\] (120)

The functions \( \chi(v), \lambda(v) \) and \( \sigma(v) \) are the quantum Rosenbluth potentials introduced in [41]. They are the solutions of the differential equations
\[
\Delta_v \lambda = -4\pi f_b, \quad \Delta_v \sigma = -4\pi f_b \left( 1 - \kappa_b \frac{f_b}{\eta_b} \right), \quad \Delta_v \chi = 2\sigma.
\] (121)

4.4 Isotropic bath

When \( f_b(v) \) is isotropic, i.e., \( f_b = f_b(v) \), the quantum Rosenbluth potentials can be simplified [41]. In that case, the diffusion tensor is of the form of Eq. (54) with
\[
D_\parallel = \frac{16\pi^2}{3} G^2 m_b \ln \Lambda \frac{1}{v} \left[ \int_0^v \frac{v_1^4}{v^2} f_b(v_1) \left( 1 - \kappa_b \frac{f_b(v_1)}{\eta_b} \right) d v_1 
+ v \int_v^{+\infty} v_1 f_b(v_1) \left( 1 - \kappa_b \frac{f_b(v_1)}{\eta_b} \right) d v_1 \right].
\] (122)

In this paper, they are introduced for an arbitrary form of entropy including the Fermi–Dirac and Bose–Einstein entropies.
\[ D_\perp = \frac{16\pi^2}{3} G^2 m_b \ln \Lambda \frac{1}{v} \left[ \int_0^v \left( 3v_1^2 - \frac{v_1^4}{v^2} \right) f_b(v_1) \left( 1 - \kappa_b \frac{f_b(v_1)}{\eta_b} \right) dv_1 \right. \\
\left. + 2v \int_v^{+\infty} v_1 f_b(v_1) \left( 1 - \kappa_b \frac{f_b(v_1)}{\eta_b} \right) dv_1 \right]. \]  

(123)

On the other hand, the friction force is given by

\[ F_{\text{pol}} = -16\pi^2 G^2 m_b \ln \Lambda \frac{v}{v_3} \int_0^v v_1^2 f_b(v_1) dv_1, \]  

(124)

\[ \frac{\partial D_{ij}}{\partial v_j} = -16\pi^2 G^2 m_b \ln \Lambda \frac{v}{v_3} \int_0^v v_1^2 f_b(v_1) \left( 1 - \kappa_b \frac{f_b(v_1)}{\eta_b} \right) dv_1, \]  

(125)

\[ F_{\text{friction}} = -16\pi^2 G^2 \ln \Lambda \frac{v}{v_3} \left[ \int_0^v v_1^2 f_b(v_1) dv_1 + m_b \int_0^v v_1^2 f_b(v_1) \left( 1 - \kappa_b \frac{f_b(v_1)}{\eta_b} \right) dv_1 \right]. \]  

(126)

4.5 Thermal bath: Einstein relation

We now assume that the field particles are in a statistical equilibrium state described by the Fermi–Dirac or Bose–Einstein DF:

\[ f_b(v) = \frac{\eta_b}{\lambda_b e^{\beta mb v^2/2} + \kappa_b}. \]  

(127)

This corresponds to the so-called “thermal bath”.\(^{36}\) The inverse fugacity \(\lambda_b\) is related to the density \(\rho_b\) by the relation from Eq. (B31). Using the relation

\[ f_b \left( 1 - \kappa_b \frac{f_b}{\eta_b} \right) = \frac{\eta_b \lambda_b e^{\beta mb v^2/2}}{(\lambda_b e^{\beta mb v^2/2} + \kappa_b)^2}, \]  

(128)

we find that

\[ \frac{\partial f_b}{\partial v} = -f_b \left( 1 - \kappa_b \frac{f_b}{\eta_b} \right) \beta m_b v. \]  

(129)

Substituting this identity into Eq. (111), we get

\[ F_{\text{pol}}^{\text{pol}} = -2\pi G^2 \beta mm_b \ln \Lambda \int dv' K_{ij} f_b' \left( 1 - \kappa_b \frac{f_b'}{\eta_b} \right) v'_j \\
= -2\pi G^2 \beta mm_b \ln \Lambda \int dv' K_{ij} f_b' \left( 1 - \kappa_b \frac{f_b'}{\eta_b} \right) (u_j + v_j) \\
= -2\pi G^2 \beta mm_b v_j \ln \Lambda \int dv' K_{ij} f_b' \left( 1 - \kappa_b \frac{f_b'}{\eta_b} \right), \]  

(130)

where we have used \(K_{ij} u_j = 0\) according to Eq. (3) to get the third line. Comparing this expression with Eq. (110), we obtain

\[ F_{\text{pol}} = -\beta m D_{ij} v_j. \]  

(131)

Using the identity \(D_{ij} v_j = D_{\parallel} v_i\) [see Eq. (54)], we can also write

\[ F_{\text{pol}} = -\beta m D_{\parallel} v. \]  

(132)

\(^{36}\) For bosons, the DF (127) is valid for \(T > T_c\), where \(T_c\) is constructed with \(m_b\) and \(\rho_b\) (see “Appendix B9”).
We see that the friction by polarization is proportional and opposite to the velocity of the test particle. Furthermore, the friction coefficient is given by the Einstein relation

\[ \xi = \beta m D_\parallel, \]  

(133)
as in the classical case [see Eq. (67)].\textsuperscript{37} As noted previously, the Einstein relation is valid for the friction by polarization, not for the true friction which cannot be easily related to the diffusion coefficient in the case of quantum particles. Substituting Eq. (131) into Eq. (109), we obtain the quantum Kramers–Chandrasekhar equation [41]\textsuperscript{38}

\[ \frac{\partial f}{\partial t} = \frac{\partial}{\partial v_i} \left[ D_{ij} \left( \frac{\partial f}{\partial v_j} + \beta m f \left( 1 - \frac{v_j}{\eta_0} \right) v_j \right) \right]. \]  

(134)

This equation relaxes towards the Fermi–Dirac or Bose–Einstein DF:

\[ f_{eq}(v) = \eta_0 \lambda e^{\beta m v^2 / 2} + \kappa. \]  

(135)

At statistical equilibrium, the test particles and the field particles have the same temperature.

Remark In the case of bosons (\( \kappa = -1 \)), the Kramers–Chandrasekhar equation (134) relaxes towards the Bose–Einstein DF from Eq. (135) only if \( T > T_c \), where \( T_c \) is constructed with \( m \) and \( \rho \) (see “Appendix B9”). When \( T < T_c \), this equation displays a process of Bose–Einstein condensation leading to the formation of a Dirac peak at \( v = 0 \). In that case, the DF can be written as \( f(v, t) = f_{\text{gas}}(v, t) + M_c(t) \delta(v) \) where \( f_{\text{gas}}(v, t) \) is the DF of the uncondensed bosons (gas) and \( M_c(t) = M - M_{\text{gas}}(t) = M - \int_{0}^{+\infty} f_{\text{gas}}(v, t) 4\pi v^2 dv \) is the mass of the condensed bosons. This mass increases with time until a statistical equilibrium state, comprising both uncondensed and condensed bosons, is reached [see Eq. (B50)]. Since the bosonic Kramers–Chandrasekhar equation assumes the existence of a thermal bath, it corresponds to a canonical description of Bose–Einstein condensation. The formation of a BEC has been studied by Sopik et al. \cite{48} by solving the bosonic Kramers equation in the case \( D_{ij} = D \delta_{ij} \) with \( D \) constant.

### 4.6 Diffusion and friction terms in the thermal bath approximation

For a thermal bath [see Eq. (127)], the diffusion coefficients of the test particles obtained from Eqs. (122) and (123) are given by

\[ D_\parallel = \frac{16\pi^2}{3} G^2 m_b \ln \Lambda \left( \frac{v^3}{L_4 + v^3 M_1} \right), \]  

(136)

\[ D_\perp = \frac{16\pi^2}{3} G^2 m_b \ln \Lambda \left( \frac{1}{v} \left( 3L_2 - \frac{1}{v^2} L_4 + 2v M_1 \right) \right), \]  

(137)

\textsuperscript{37} It is shown in [41] that relation (132) is valid for an arbitrary form of entropy.

\textsuperscript{38} This is a particular case of nonlinear Fokker–Planck equations [100]. It satisfies an H-theorem in the canonical ensemble for a generalized free energy of the form \( F = E - TS \) where \( S \) is the generalized entropy (16) associated with the Landau equation from which it is issued (here the Fermi–Dirac ou Bose–Einstein entropy).
Performing the change of variables

\[ x = \sqrt{\frac{\beta m_b}{2}} v. \]  

To obtain the second equalities in Eqs. (138)–(140), we have used Eq. (128) and performed straightforward integrations by parts. Combining these expressions, we get

\[
D_\parallel = 16\pi^2 G^2 \ln \Lambda \frac{1}{v^2} \int_0^v \frac{v^2}{\lambda_b e^{\lambda m_b v^2/2} + \kappa_b} \, dv_1, \\
D_\perp = 16\pi^2 G^2 \ln \Lambda \frac{1}{v} \left( \int_0^v \frac{1}{\lambda_b e^{\lambda m_b v^2/2} + \kappa_b} \, dv_1 - \frac{1}{v^2} \int_0^v \frac{v^2}{\lambda_b e^{\lambda m_b v^2/2} + \kappa_b} \, dv_1 \right).
\]

Performing the change of variables \( x_1 = \beta m_b v_1^2/2 \) and using Eq. (B31), we finally obtain

\[
D_\parallel = 2\pi G^2 m_b \rho_b \ln \Lambda \frac{1}{x} \frac{1}{I_{1/2}(\lambda_b, x)} \frac{1}{v}, \\
D_\perp = 2\pi G^2 m_b \rho_b \ln \Lambda \left[ \frac{I_{1/2}(\lambda_b)}{I_{1/2}(\lambda_b)} - \frac{I_{1-2}(\lambda_b, x) - I_{1-2}(\lambda_b, x)}{x^2} \right] \frac{1}{v},
\]

where we have defined

\[
I_n(t, x) = \frac{1}{I_n(t)} \int_0^x \frac{y^n}{t e^y + \kappa} \, dy
\]

and

\[
D_{ij} = 2\pi G^2 \beta m b^2 \rho_b \ln \Lambda \left[ \frac{I_{1-2}(\lambda_b)}{I_{1/2}(\lambda_b)} - \frac{I_{1-2}(\lambda_b, x)}{x^2} \right] \frac{1}{v},
\]

Here, \( I_n(t, x) \) denote the incomplete Fermi and Bose integrals [41]. The complete Fermi and Bose integrals \( I_n(t) \) are defined in “Appendix B7”. Using Eqs. (124)–(126) and (132), we similarly obtain

\[
F_{\text{pol}} = -2\pi G^2 \beta m m_b \rho_b \ln \Lambda \frac{1}{x^2} \frac{1}{I_{1/2}(\lambda_b, x)} \frac{v}{x},
\]

\[
\frac{\partial D_{ij}}{\partial v_j} = 2\pi G^2 \beta m b^2 \rho_b \ln \Lambda \left[ \frac{1}{x^2} \frac{1}{I_{1/2}(\lambda_b)} \frac{v}{x} \left( \frac{x}{\lambda_b e^{x^2/2} + \kappa_b} - \frac{1}{2} I_{1-2}(\lambda_b, x) / I_{1/2}(\lambda_b) \right) \right],
\]

\[
F_{\text{friction}} = F_{\text{pol}} + \frac{\partial D_{ij}}{\partial v_j}.
\]
In the classical limit where $\lambda_b \to +\infty$ (see “Appendix B8”), using the identities [41]

$$I_n(t) \sim \frac{1}{t} \Gamma(n + 1) \quad (t \to +\infty),$$  

$$I_n(\infty, x) = \frac{2}{\Gamma(n + 1)} \int_0^x y^{2n + 1} e^{-y^2} dy,$$  

$$I_{1/2}(\infty, x) = 2x^2 G(x), \quad I_{-1/2}(\infty, x) = \text{erf}(x),$$  

we recover the results from Sect. 3.6.

For completely degenerate fermions described by the DF (see “Appendix B10”)

$$f_b(v_1) = \eta_b H(v_1 - v_F),$$  

where $v_F$ is the Fermi velocity, we see from Eq. (110) that $D_{ij} \to 0$. This is because the fermionic field particles are in the ground state at $T = 0$ so there is no fluctuation. Therefore, the diffusion tensor vanishes. The test particles undergo only a friction force $F_{\text{friction}} = F_{\text{pol}}$ given by Eq. (124). We are in the same situation as in the sinking satellite problem of Sect. 4.7. Using Eq. (153), we find that

$$F_{\text{pol}} = -\frac{16}{3} \pi^2 G^2 m \ln \eta_b v \quad (v < v_F),$$  

$$F_{\text{pol}} = -\frac{16}{3} \pi^2 G^2 m \ln \eta_b v_F^3 v \quad (v > v_F).$$  

### 4.7 Fermionic and bosonic King models

We can use the kinetic theory to derive a truncated DF accounting for an escape of particles above a limit energy $\varepsilon_m$. Let us consider a possibly inhomogeneous system of quantum particles in gravitational interaction. Making a local approximation, the evolution of the DF is governed by the quantum Vlasov–Landau equation [39]

$$\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial r} - \nabla \Phi \cdot \frac{\partial f}{\partial v} = 2\pi G^2 \ln \Lambda \frac{\partial}{\partial v_i} \int dv' K_{ij} \left[ m_b f'_b \left( 1 - \frac{f'_b}{\eta_b} \right) \frac{\partial f}{\partial v_j} - mf \left( 1 - \frac{f}{\eta_b} \right) \frac{\partial f}{\partial v_j} \right],$$  

where $\Phi(r, t)$ is the gravitational potential and we have noted $f = f(r, v, t)$ and $f'_b = f_b(r, v', t)$. Making a thermal bath approximation, we obtain the quantum Vlasov–Kramers equation [40]

$$\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial r} - \nabla \Phi \cdot \frac{\partial f}{\partial v} = \frac{\partial}{\partial v} \left[ \frac{K}{v^3} \left( \frac{\partial f}{\partial v} + \beta mf \left( 1 - \frac{f}{\eta_0} \right) v \right) \right].$$  

To simplify the equation, we have assumed that the velocity DF of the field particles is isotropic and replaced the diffusion coefficient $D_{\parallel}(v)$ by its asymptotic expression $D_{\parallel} \sim$.

---

39 The classical Vlasov–Landau equation ($\kappa = 0$) relying on a local approximation was used by Ipser [101] to study spatially inhomogeneous stellar systems. This approximation was criticized by Kandrup [102] who derived a more complicated kinetic equation. We refer to the introduction of [21] for a review of different approaches attempting to take into account spatial inhomogeneity in the kinetic theory of self-gravitating systems going beyond the local approximation. These efforts finally led to the inhomogeneous Lenard–Balescu equation [26,27] which is the exact kinetic equation of self-gravitating systems at the order $O(1/N)$.

40 When self-consistently coupled to gravity, we obtain the Kramers–Poisson equation. In the strong friction limit $\xi \to +\infty$ (which may be relevant for certain self-gravitating systems), we obtain the Smoluchowski–Poisson equation that has been extensively studied in relation to a model of self-gravitating Brownian particles [49].
$K/v^3$ valid for high velocities $v \rightarrow +\infty$.\textsuperscript{41} We now assume that high energy particles are removed by a tidal field. In the case of globular clusters or DM halos, this field may be due to the gravitational attraction of a nearby galaxy. We then look for a stationary solution of Eq. (157) of the form $f = f(\epsilon)$ satisfying the boundary condition $f(\epsilon_m) = 0$. Here, $\epsilon \equiv m[v^2/2 + \Phi(r)]$ denotes the energy of a particle and $\epsilon_m$ is the escape energy above which $f = 0$. A DF of the form $f = f(\epsilon)$ cancels the advection term (l.h.s. of Eq. (157)) according to the Jeans theorem.\textsuperscript{67} Then, using the identity $(\partial/\partial v)(v/v^3) = 0$ for $v \neq 0$, we get

$$\frac{d}{d\epsilon} \left[ \frac{df}{d\epsilon} + \beta f \left( 1 - \kappa \frac{f}{\eta_0} \right) \right] = 0$$

or, equivalently,

$$\frac{df}{d\epsilon} + \beta f \left( 1 - \kappa \frac{f}{\eta_0} \right) = -J,$$  

(159)

where $J$ is a constant of integration representing the diffusion current in energy space. If we set $J = 0$, we recover the Fermi–Dirac and Bose–Einstein DFs of statistical equilibrium [see Eq. (27)] with $\epsilon$ instead of $mv^2/2$. However, for spatially inhomogeneous self-gravitating systems, these DFs have an infinite mass [67]. This is why it is important to take into account the escape of high energy particles. If $J \neq 0$, Eq. (159) accounts for an escape of particles at a constant rate $J$. The system is not truly static since it looses gradually particles but we can consider that it passes by a succession of quasistationary states that are the solutions of Eq. (159). This is a Riccati equation that can be solved analytically by usual means. Repeating the calculations of [42], that are valid both for fermions and bosons, we find in good approximation that\textsuperscript{42}

$$f = \eta_0 e^{-\beta \epsilon} - e^{-\beta \epsilon_m} \frac{\lambda + \kappa e^{-\beta \epsilon}}{\lambda \eta_0},$$

(160)

In the classical limit ($\kappa = 0$), we recover the King model [24]. Therefore, the DF from Eq. (160) may be called the quantum (fermionic or bosonic) King model. This DF could describe DM halos made of massive neutrinos or axions limited in extension by tidal forces [30,31]. On a secular timescale, we should take into account the slow change of the constants $\lambda$ and $\beta$ in Eq. (160) due to evaporation in a sort of adiabatic approximation.

\section{5 Classical particles in collision with quantum particles}

In this section, we consider the case of classical test particles in “collision” with quantum field particles. For example, the classical particles may be stars, globular clusters or black holes moving in a DM halo made of fermions or bosons (see Sect. 6.4). As before, we use an idealization in which the system is assumed to be spatially homogeneous and we ignore collective effects.

\textsuperscript{41} Recall that the diffusion coefficient of quantum particles at large $v$ behaves like the diffusion coefficient of classical particles studied in Sect. 3.6.

\textsuperscript{42} This expression is valid for $J/\beta \eta_0 \ll 1$. The general case will be treated in a specific paper [103] (see also [42]). For bosons, there is a maximum current $J_{\text{max}} = \beta \eta_0/4$ corresponding to $f = (\eta_0/2)\beta(\epsilon_m - \epsilon)/(2 - \beta(\epsilon_m - \epsilon))$ with $\beta \epsilon_m < 2$.  

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Assuming that the test particles are classical ($\kappa = 0$) and that the field particles are quantum ($\kappa_b = \pm 1$), we obtain the Landau equation

$$\frac{\partial f}{\partial t} = 2\pi G^2 \ln \Lambda \frac{\partial}{\partial v_i} \int d\nu' K_{ij} \left\{ m_b f'_b \left( 1 - \kappa_b \frac{f'_b}{\eta_b} \right) \frac{\partial f}{\partial v_j} - m f \frac{\partial f'_b}{\partial v'_j} \right\}. \quad (161)$$

It can be written in the form of a Fokker–Planck equation

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v_i} \left( D_{ij} \frac{\partial f}{\partial v_j} - f F_{i}^{\text{pol}} \right) \quad (162)$$

with a diffusion tensor

$$D_{ij} = 2\pi G^2 m_b \ln \Lambda \int d\nu' K_{ij} f'_b \left( 1 - \kappa_b \frac{f'_b}{\eta_b} \right) \quad (163)$$

and a friction by polarization

$$F_{i}^{\text{pol}} = 2\pi G^2 m \ln \Lambda \int d\nu' K_{ij} \frac{\partial f'_b}{\partial v'_j} \quad (164)$$

As before, the diffusion coefficient is affected by quantum mechanics while the friction is not (it has the same expression as for classical particles). The usual form of the Fokker–Planck equation is

$$\frac{\partial f}{\partial t} = \frac{\partial^2}{\partial v_i \partial v_j} \left( D_{ij} f \right) - \frac{\partial}{\partial v_i} \left( f F_{i}^{\text{friction}} \right), \quad (165)$$

with the total friction

$$F_{i}^{\text{friction}} = F_{i}^{\text{pol}} + \frac{\partial D_{ij}}{\partial v_j}. \quad (166)$$

In the thermal bath approximation, where the field particles have the Fermi–Dirac or Bose–Einstein DF (127), the friction by polarization is given by (see Sect. 4.5)

$$F_{i}^{\text{friction}} = -\beta m D_{ij} \nu, \quad (167)$$

and we get the Kramers–Chandrasekhar equation

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v_i} \left[ D_{ij} \left( \frac{\partial f}{\partial v_j} + \beta m f \nu_j \right) \right], \quad (168)$$

where the diffusion tensor is given by Eq. (54) with Eqs. (143) and (144). This equation relaxes towards the Maxwell–Boltzmann distribution (71). At statistical equilibrium, the (classical) test particles and the (quantum) field particles have the same temperature.

### 6 Effective mass of quasiparticles

In this section, we consider the case of classical or quantum particles in “collision” with quantum particles and introduce the notion of quasiparticles with effective mass $m_{\text{eff}}$. 
6.1 General expression

We redefine the DF of the field particles through the relation

$$F_{\text{eff}}(v) = \frac{\int f_b \, dv}{\int f_b \left(1 - \kappa_b \frac{f_b}{\eta_b}\right) \, dv} \, f_b(v) \left(1 - \kappa_b \frac{f_b(v)}{\eta_b}\right).$$  \hspace{1cm} (169)

The effective DF $F_{\text{eff}}(v)$ is normalized such that

$$\int F_{\text{eff}}(v) \, dv = \int f_b(v) \, dv = \rho_b.$$

The diffusion tensor from Eq. (110) or (163) can then be rewritten as

$$D_{ij} = 2\pi G^2 m_{\text{eff}} \ln \Lambda \int d'v' K_{ij} F'_{\text{eff}},$$  \hspace{1cm} (170)

where we have introduced the effective mass

$$m_{\text{eff}} = \frac{\int f_b \left(1 - \kappa_b \frac{f_b}{\eta_b}\right) \, dv}{\int f_b \, dv}. \hspace{1cm} (171)$$

In this manner, we see that the diffusion coefficient of the test particles due to collisions with quantum particles of mass $m_b$ and DF $f_b$ is the same as the diffusion coefficient (36) due to collisions with classical particles of effective mass $m_{\text{eff}}$ and effective DF $F_{\text{eff}}$. We note that $m_{\text{eff}} < m_b$ for fermions and $m_{\text{eff}} > m_b$ for bosons. We stress that this interpretation in terms of quasiparticles of effective mass $m_{\text{eff}}$ is valid only for the diffusion coefficient, not for the friction by polarization [see Eq. (111) or (164)] which keeps the same expression as for classical particles.

6.2 Thermal bath

If the quantum particles are at statistical equilibrium, using Eq. (128), we get

$$\int f_b \left(1 - \kappa_b \frac{f_b}{\eta_b}\right) \, dv = \int_0^{+\infty} \frac{\eta_b \lambda_b \beta m_b v^2/2}{(\lambda_b \beta m_b v^2/2 + \kappa_b)^2} \cdot 4\pi v^2 \, dv = \frac{4\pi}{\beta m_b} \int_0^{+\infty} f_b \, dv = \frac{1}{\beta m_b} \int \frac{f_b \, dv}{v^2},$$  \hspace{1cm} (172)

where we have used an integration by parts to obtain the second equality. This relation can also be obtained by using Eq. (129), implying

$$v \cdot \frac{\partial f_b}{\partial v} = -f_b \left(1 - \kappa_b \frac{f_b}{\eta_b}\right) \beta m_b v^2.$$  \hspace{1cm} (173)

Then,

$$\int f_b \left(1 - \kappa_b \frac{f_b}{\eta_b}\right) \, dv = -\frac{1}{\beta m_b} \int \frac{1}{v^2} v \cdot \frac{\partial f_b}{\partial v} \, dv = -\frac{1}{\beta m_b} \int f'_b(v) \frac{v}{u} \, dv = \frac{1}{\beta m_b} \int \frac{f_b \, dv}{v^2},$$  \hspace{1cm} (174)

where the last equality is obtained by integrating by parts. As a result, the effective mass of the field particles may be rewritten as

$$m_{\text{eff}} = \frac{1}{\beta} \int \frac{f_b}{v^2} \, dv = \frac{1}{\beta} \left(\frac{1}{v^2}\right).$$  \hspace{1cm} (175)
Fig. 1  Reduction factor for fermions as a function of the temperature: $m_{\text{eff}}/m_b$ (solid line); $D_F^F/D_C^F$ (dashed line); $D_F^C/D_C^C$ (dashed-dotted line) for $x = 1$. The relaxation time is increased with respect to the classical case.

More specifically, since

$$\int f_b \left( 1 - \kappa_b f_b \right) \, dv = \frac{4\pi \eta_b}{\beta m_b} \int_0^\infty \frac{dv}{\lambda_b e^{\beta m_b v^2/2} + \kappa_b} = \frac{4\pi \eta_b}{\sqrt{2}\beta m_b^{3/2}} I_{-1/2}(\lambda_b),$$

we obtain

$$m_{\text{eff}} = \frac{1}{2} m_b \frac{I_{-1/2}(\lambda_b)}{I_{1/2}(\lambda_b)}. \quad (177)$$

In the case of fermions, using the results of “Appendix B10”, we find that

$$\frac{T}{T_F} = \left( \frac{2}{3} \right)^{2/3} \frac{1}{I_{1/2}(\lambda_b)^{2/3}}, \quad \frac{m_{\text{eff}}}{m_b} = \frac{1}{2} \frac{I_{-1/2}(\lambda_b)}{I_{1/2}(\lambda_b)}. \quad (178)$$

The first relation of Eq. (178) links the temperature to the inverse fugacity ($T_F$ is the Fermi temperature). The second relation of Eq. (178) links the effective mass to the inverse fugacity. Eliminating the inverse fugacity between these two relations, we obtain the effective mass as a function of the temperature. This relation is plotted in Fig. 1. Because of Pauli’s blocking, the effective mass $m_{\text{eff}}$ of the quasiparticles is smaller than the mass $m_b$ of the fermions. Furthermore, the effective mass decreases as the temperature decreases. For $\lambda_b \to +\infty$ (classical limit), using Eq. (B43), we find $T/T_F \sim (16/9\pi)^{1/3} \lambda_b^{2/3} \to +\infty$ and $m_{\text{eff}}/m_b \simeq 1 - 1/(2\sqrt{2}\lambda_b) \to 1$ yielding

$$m_{\text{eff}} \approx m_b - \frac{1}{3} \left( \frac{T_F}{T} \right)^{3/2} \quad (T \to +\infty). \quad (179)$$

For $\lambda_b \to 0$ (completely degenerate limit), using Eq. (B63), we find $T/T_F \sim 1/(- \ln \lambda_b) \to 0$ and $m_{\text{eff}}/m_b \sim (3/2)/(- \ln \lambda_b) \to 0$ yielding

$$m_{\text{eff}} \approx \frac{3}{2} \frac{T}{T_F} \quad (T \to 0). \quad (180)$$
Fig. 2 Amplification factor for bosons as a function of the temperature: \( m_{\text{eff}}/m_b \) (solid line); \( D_B^B / D_C^C \) (dashed line); \( D_B^B / D_C^\perp \) (dashed-dotted line) for \( x = 1 \). The relaxation time is reduced with respect to the classical case.

In the case of bosons, using the results of “Appendix B9”, we find that

\[
\frac{T}{T_c} = \left[ \frac{I_{1/2}(1)}{I_{1/2}(1/b)} \right]^{2/3}, \quad \frac{m_{\text{eff}}}{m_b} = \frac{1}{2} \frac{I_{-1/2}(1/b)}{I_{1/2}(1/b)}. \tag{181}
\]

The first relation of Eq. (181) links the temperature to the inverse fugacity (\( T_c \) is the condensation temperature). The second relation of Eq. (181) links the effective mass to the inverse fugacity. Eliminating the inverse fugacity between these two relations, we obtain the effective mass as a function of the temperature. This relation is plotted in Fig. 2. Because of Bose enhancement, the effective mass \( m_{\text{eff}} \) of the quasiparticles is larger than the mass \( m_b \) of the bosons. Furthermore, the effective mass increases as the temperature decreases. For \( \lambda_b \to +\infty \) (classical limit), using Eq. (B43), we find \( T/T_c \sim \zeta(3/2)^{2/3} \lambda_b^{-2/3} \to +\infty \) and \( m_{\text{eff}}/m_b \sim 1 + 1/(2\sqrt{2}\lambda_b) \to 1 \) yielding

\[
\frac{m_{\text{eff}}}{m_b} \simeq 1 + \zeta \left( \frac{3}{2} \right)^{3/2} \left( \frac{1}{2} \right)^{3/2} \left( \frac{T_c}{T} \right)^{3/2} + \cdots \quad (T \to +\infty). \tag{182}
\]

For \( \lambda_b \to 1 \) (limit of condensation), using Eqs. (B52) and (B53), we get \( T/T_c \simeq 1 + [4\sqrt{\pi}/3\zeta(3/2)]\sqrt{\ln \lambda_b} \) and \( m_{\text{eff}}/m_b \sim [\sqrt{\pi}/\zeta(3/2)]\sqrt{\ln \lambda_b} \), yielding

\[
\frac{m_{\text{eff}}}{m_b} \sim \frac{4\pi}{3\zeta(3/2)^2} \left( \frac{T}{T_c} - 1 \right)^{-1} \quad (T \to T_c^+). \tag{183}
\]

The previous calculations show that \( m_{\text{eff}} \leq m_b \) for fermions and \( m_{\text{eff}} \geq m_b \) for bosons. Using the heuristic arguments of Sect. 2.3 this implies that the relaxation time due to collision with fermions (resp. bosons) is larger (resp. smaller) than the relaxation time due to collision with classical particles of the same mass. We also note that the effective mass of bosons diverges when \( T \to T_c \). This could suggest that the relaxation time diverges at \( T_c \). Actually, we must be careful that the diffusion coefficient (170) does not necessarily diverge at \( T_c \) because it involves an integral over \( F_{\text{eff}} \) that may tend to zero. To determine the relaxation...
time, it is therefore better to come back to the explicit expressions (143) and (144) of the diffusion coefficient. The relaxation time can then be estimated by

$$t_R \sim \frac{\sigma^2}{D},$$

(184)

where $\sigma$ is the typical value of the velocity and $D$ is the typical value of the diffusion coefficient (see footnotes 31 and 32). The relaxation time is inversely proportional to the diffusion coefficient. To measure the effect of quantum mechanics, instead of considering the ratio $m_{\text{eff}}/m_b$ between the effective mass and the bare mass, we can consider the ratio between the quantum diffusion coefficients (143) and (144) and the classical diffusion coefficients (76) and (77). We find

$$\frac{D_Q}{D_C} = \frac{I_{1/2}(\lambda_b, x)}{2x^2G(x)},$$

(185)

and

$$\frac{D_Q}{D_C} = \frac{1}{2} \text{erf}(x) - G(x) \left[ \frac{I_{-1/2}(\lambda_b)}{I_{1/2}(\lambda_b)} - \frac{I_{1/2}(\lambda_b, x)}{x^2} \right].$$

(186)

In the classical limit $\lambda_b \to +\infty$ we recover $D_Q/D_C = 1$. In the case of fermions (resp. bosons), one can show that $D_F/D_C \leq 1$ (resp. $D_B/D_C \geq 1$) which confirms that the relaxation time for fermions (resp. bosons) is larger (resp. smaller) than for classical particles of the same mass. On the other hand, in the case of bosons, we see that $D_Q$ does not diverge at $T_c$ (contrary to $m_{\text{eff}}$) while $D_Q$ diverges at $T_c$ (similarly to $m_{\text{eff}}$). To get more quantitative results, we can consider particular values of $x$. For $x \to 0$, we get

$$\frac{D_Q}{D_C} \sim \frac{\sqrt{\pi}}{2} \frac{1}{I_{1/2}(\lambda_b)} \frac{1}{\lambda_b + \kappa}.$$

(187)

For $x \to +\infty$, we get

$$\frac{D_Q}{D_C} \to 1,$$

(188)

and

$$\frac{D_Q}{D_C} \sim \frac{1}{2} \frac{I_{-1/2}(\lambda_b)}{I_{1/2}(\lambda_b)} = \frac{m_{\text{eff}}}{m_b}.$$

(189)

For $x = 1$, the factors $D_Q/D_C$ and $D_Q/D_C$ are plotted as a function of $T/T_Q$ in Figs. 1 and 2 and compared with $m_{\text{eff}}/m_b$. In the case of fermions, we see on Fig. 1 that the reduction factors $D_F/D_C$ and $D_F/D_C$ both behave similarly to $m_{\text{eff}}/m_b$. This implies that the increase of the relaxation time due to Pauli’s blocking is relatively important for $T \ll T_F$. In the case of bosons, we see on Fig. 2 that the amplification factor $D_B/D_C$ behaves similarly to $m_{\text{eff}}/m_b$ (in particular, it diverges at $T_c$) while the amplification factor $D_B/D_C$ remains approximately constant (it tends to 1.72 at $T_c$). If we consider isotropic systems, this implies that the reduction of the relaxation time due to Bose enhancement is not very important for $T \geq T_c$. If we consider anisotropic systems, the divergence of the diffusion coefficient $D_B$ at $T_c$ implies that system becomes rapidly isotropic when $T \to T_c$ (the corresponding relaxation time tends to zero).
6.3 A nice relation

Using the formalism of [41], the results of the previous sections can be generalized to an arbitrary entropy of the form (16). According to Eq. (18), the DF of the field particles is

$$f_b(v) = \left( C' \right)^{-1} \left( \alpha_b - \beta m_b \frac{v^2}{2} \right).$$  \hspace{1cm} (190)

The effective DF and the effective mass can be written in the general case as

$$F_{\text{eff}}(v) = \frac{\int f_b \, dv}{\int f_b \, dv} = \frac{1}{\int f_b \, dv}$$  \hspace{1cm} (191)

and

$$m_{\text{eff}} = m_b \frac{\int f_b \, dv}{\int f_b \, dv} = \frac{1}{\beta} \frac{\int f_b \, dv}{\int f_b \, dv} = \frac{1}{\beta} \frac{\int f_b \, dv}{\int f_b \, dv} = \frac{1}{\beta} \left( \frac{1}{v^2} \right).$$  \hspace{1cm} (192)

To obtain the second and subsequent equalities, we have used the identity

$$C'(f_b) = \alpha_b - \beta m_b \frac{v^2}{2} \Rightarrow \frac{\partial f_b}{\partial v} = - \beta m_b \frac{v^2}{C''(f_b)},$$  \hspace{1cm} (193)

which follows from Eq. (190) and performed integrations par parts.

On the other hand, introducing the density \( \rho_b = \int f_b \, dv \) and the pressure \( P_b = \frac{1}{3} \int f_b v^2 \, dv \), where \( f_b \) is given by Eq. (190), and eliminating \( \alpha_b \) between these two expressions, we obtain the equation of state \( P_b(\rho_b) \) of the field particles. The squared speed of sound \( c_s^2 = P_b'(\rho_b) \) is then given by (see also Eq. (50) of [104])

$$c_s^2 = \frac{1}{3} \int F'(x) v^2 \, dv = \frac{1}{3} \int v^2 \cdot \frac{\partial f_b}{\partial v} \, dv = \frac{1}{\beta} \frac{\int f_b \, dv}{\int f_b \, dv} = \frac{1}{\beta} \left( \frac{1}{v^2} \right).$$  \hspace{1cm} (194)

where \( F(x) = (C')^{-1}(-x) \) with \( x = \beta m_b \frac{v^2}{2} - \alpha_b \). To obtain the second and subsequent equalities, we have used the identity

$$f_b(v) = F \left( \beta m_b \frac{v^2}{2} - \alpha_b \right) \Rightarrow \frac{\partial f_b}{\partial v} = F'(x) \beta m_b v,$$  \hspace{1cm} (195)

which follows from Eq. (190) and performed integrations par parts. Comparing Eqs. (193) and (195), we obtain \( F'(x) = -1/C''(f_b) \). Finally, comparing Eqs. (192) and (194), we obtain the nice relation

$$m_{\text{eff}} = \frac{1}{\beta c_s^2},$$  \hspace{1cm} (196)

which is valid when the field particles are at statistical equilibrium.

6.4 The case of FDM

In this section we focus on the case of bosons \( \kappa_b = -1 \). We consider the situation studied by Bar-Or et al. [74] in the context of FDM (see the introduction and “Appendix A”), namely the collisional evolution of test particles (like stars, globular clusters, black holes...) moving...
in a FDM halo. We assume that the bosons that compose the halo are described by an out-of-equilibrium DF with an effective temperature $T_{\text{eff}} \ll T_c$ resulting from a process of violent collisionless relaxation. As shown by Hui et al. [73], quantum interferences produce a form of scalar radiation made of quasiparticles (granules) of effective mass $m_{\text{eff}} \gg m_b$ and approximately Maxwellian DF. Everything happens as if the test particles were in collision with these quasiparticles. To simplify the expression of the effective mass $m_{\text{eff}}$ of these quasiparticles, we make two approximations:

(i) We assume that the amplification due to Bose stimulation is large ($f_b \gg \eta_b$) so that

$$f_b \left(1 + \frac{f_b}{\eta_b}\right) \simeq \frac{f_b^2}{\eta_b},$$

(197)

With this approximation, the effective DF of the bosons and their effective mass defined by Eqs. (169) and (171) become

$$F_{\text{eff}}(v) = \frac{\int f_b dv}{\int f_b^2 dv} f_b(v)^2$$

(198)

and

$$m_{\text{eff}} = \frac{m_b}{\eta_b} \frac{\int f_b^2 dv}{\int f_b dv}.$$  

(199)

We note that the effective mass of the quasiparticles is much higher than the mass of the bosons ($m_{\text{eff}} \gg m_b$). Therefore, Bose stimulation significantly accelerates the collisional relaxation.

(ii) We assume that $f_b$ is Maxwellian

$$f_b(v) = \frac{\rho_b}{(2\pi \sigma_b^2)^{3/2}} e^{-\frac{v^2}{2\sigma_b^2}}$$

(200)

with a velocity dispersion $\sigma_b^2$. Substituting Eq. (200) into Eq. (198), we find that the effective DF of the bosons is a Maxwellian with velocity dispersion $\sigma_b^2/2$, i.e.,

$$F_{\text{eff}}(v) = \frac{\rho_b}{(\pi \sigma_b^2)^{3/2}} e^{-\frac{v^2}{4\sigma_b^2}}.$$  

(201)

On the other hand, their effective mass from Eq. (199) is

$$m_{\text{eff}} = \frac{m_b}{\eta_b} \frac{\rho_b}{(2\pi \sigma_b^2)^{3/2}} \frac{\int_0^{+\infty} e^{-v^2/2\sigma_b^2} v^2 dv}{\int_0^{+\infty} e^{-v^2/2\sigma_b^2} dv}$$

$$= \frac{m_b}{\eta_b} \frac{\rho_b}{(4\pi \sigma_b^2)^{3/2}}.$$  

(202)

Recalling Eq. (108), and taking $g = 1$, we obtain

$$m_{\text{eff}} = \frac{\pi^{3/2} h^3 \rho_b}{m_b^3 \sigma_b^3}.$$  

(203)

This can be written as

$$m_{\text{eff}} = \frac{1}{8\pi^{3/2}} \rho_b \lambda_{\text{dB}}^3,$$

(204)

where $\lambda_{\text{dB}} = h/(m_b \sigma_b)$ is the de Broglie length.
According to Eqs. (170), (201) and (202), the diffusion tensor is the same as the one due to classical particles with effective mass \( m_{\text{eff}} \) and Maxwellian DF with a velocity dispersion \( \sigma_{\text{eff}} = \sigma_b / \sqrt{2} \). Using Eqs. (76) and (77), we therefore obtain

\[
D_\parallel = 4\pi G^2 m_{\text{eff}} \ln \Lambda \rho_b G(\chi_{\text{eff}}) \frac{1}{v}, \\
D_\perp = 4\pi G^2 m_{\text{eff}} \ln \Lambda \rho_b \left[ \text{erf}(\chi_{\text{eff}}) - G(\chi_{\text{eff}}) \right] \frac{1}{v},
\]

with

\[
\chi_{\text{eff}} = \frac{v}{\sigma_b}.
\]

The diffusion tensor can be written as

\[
D_{ij} = \frac{2}{\sqrt{\pi}} G^2 m_{\text{eff}} \ln \Lambda \rho_b \frac{1}{\sigma_b} G_{ij}(\chi_{\text{eff}}),
\]

where \( G_{ij} \) is defined by Eq. (85). According to Eq. (79), we have

\[
\frac{\partial D_{ij}}{\partial v_j} = -\frac{8\pi G^2 m_{\text{eff}} \ln \Lambda \rho_b}{\sigma^2} G(\chi_{\text{eff}}) \frac{v}{v}.
\]

On the other hand, according to Eqs. (164) and (200), the force by polarization is the same as the one due to classical particles with mass \( m_b \) and Maxwellian DF with a velocity dispersion \( \sigma_b \). Using Eq. (78), we therefore obtain

\[
F_{\text{pol}} = -\frac{4\pi G^2 m \ln \Lambda \rho_b}{\sigma_b^2} G(x) \frac{v}{v},
\]

with

\[
\chi = \frac{v}{\sqrt{2\sigma_b}}.
\]

We note that \( \chi_{\text{eff}} = \sqrt{2} \chi \). According to Eqs. (166), (209) and (210) the total friction force is

\[
F_{\text{friction}} = -\frac{4\pi G^2 \ln \Lambda \rho_b}{\sigma_b^2} [m G(x) + 2m_{\text{eff}} G(\chi_{\text{eff}})] \frac{v}{v}.
\]

We note that it is affected by Bose stimulation through the term \( \partial D_{ij} / \partial v_j \). The friction by polarization can be written as

\[
F_{\text{pol}} = -\frac{1}{\sigma_b^2} \frac{m_{\text{eff}}}{m} G(\chi_{\text{eff}}) \frac{v}{v}.
\]

It differs from the Einstein relation (167) obtained by assuming that the field particles (bosons) are at statistical equilibrium with the Bose–Einstein DF at \( T > T_c \). This is because we are in a completely different situation where the bosons are in an out-of-equilibrium state with \( T_{\text{eff}} \ll T_c \). Finally, assuming that the test particles are classical and using \( D_{ij} v_j = D_\parallel v_i \) according to Eq. (54), the Fokker–Planck equation (162) can be written as

\[
\frac{\partial f}{\partial t} = \frac{\partial}{\partial v_i} \left[ D_{ij} \left( \frac{\partial f}{\partial v_j} + \frac{1}{\sigma_b^2} \frac{m}{m_{\text{eff}}} G(\chi_{\text{eff}}) f v_j \right) \right].
\]
It relaxes towards an equilibrium DF determined by the differential equation

\[ \frac{\partial f}{\partial v} + \frac{1}{\sigma_b^2} \frac{m}{m_{\text{eff}}} \frac{G(x)}{G(x_{\text{eff}})} f v = 0. \] (215)

Its solution is

\[ f_{\text{eq}}(v) = A e^{-\frac{1}{2} \frac{m}{m_{\text{eff}}} \int_0^v \frac{G(x)}{G(x_{\text{eff}})} v \, dv}. \] (216)

We note that the equilibrium state of the test (classical) particles is non-Maxwellian (see Fig. 1 of [74]) contrary to the case where the bosons are at statistical equilibrium [see Eq. (71)]. Again, this is because we are considering a completely different situation where the bosons are out-of-equilibrium.43

The Fokker–Planck equation (214) can be written under a dimensionless form as

\[ \frac{\partial f}{\partial t} = \frac{1}{t_H} \frac{\partial}{\partial x_i} \left[ G_{ij}(x_{\text{eff}}) \left( \frac{\partial f}{\partial x_j} + 2 \frac{m}{m_{\text{eff}}} \frac{G(x)}{G(x_{\text{eff}})} f x_j \right) \right], \] (217)

where we have introduced the heating time [74]

\[ t_H = \frac{\sqrt{\pi} \sigma_b^3}{G^2 \rho_b m_{\text{eff}} \ln \Lambda} = \frac{m_b^3 \sigma_b^6}{\pi G^2 \rho_b^2 h^3 \ln \Lambda}. \] (218)

Alternatively, it can be written as

\[ \frac{\partial f}{\partial t} = \frac{1}{t_C} \frac{\partial}{\partial x_i} \left[ G_{ij}(x_{\text{eff}}) \left( \frac{m_{\text{eff}}}{m} \frac{\partial f}{\partial x_j} + 2 \frac{G(x)}{G(x_{\text{eff}})} f x_j \right) \right], \] (219)

where we have introduced the cooling time [74]

\[ t_C = \frac{\sqrt{\pi} \sigma_b^3}{G^2 \rho_b m \ln \Lambda}. \] (220)

We can also write

\[ \frac{\partial f}{\partial t} = \frac{1}{t_H} \frac{\partial}{\partial x_i} \left[ G_{ij}(x_{\text{eff}}) \left( \frac{1}{t_H} \frac{\partial f}{\partial x_j} + \frac{1}{t_C} \frac{2}{t_C} \frac{G(x)}{G(x_{\text{eff}})} f x_j \right) \right], \] (221)

with

\[ \frac{t_C}{t_H} = \frac{m_{\text{eff}}}{m}. \] (222)

Apart from a numerical factor, the cooling time has the same expression as in the case of classical field particles of mass \( m_b \) and velocity dispersion \( \sigma_b \). By contrast, the heating time is very different as it involves the effective mass of the quasiparticles \( m_{\text{eff}} \) instead of the boson mass \( m_b \). Since \( m_{\text{eff}} \gg m_b \), the heating time is considerably reduced.

43 Using the asymptotic behaviors of \( G(x) \) (see Sect. 3.6), we find that \( f_{\text{eq}} \propto \exp[-\frac{1}{2} \frac{m}{m_{\text{eff}}} m_{\text{eff}} v^2] \) for \( v \rightarrow 0 \) and \( f_{\text{eq}} \propto \exp[-\frac{1}{2} \frac{2m}{m_{\text{eff}}} m_{\text{eff}} v^2] \) for \( v \rightarrow +\infty \). The first expression is also valid for all \( v \) when \( m \gg m_{\text{eff}} \) and the second when \( m \ll m_{\text{eff}} \). Therefore, the DF of the test particles is Maxwellian in these limits. Furthermore, the velocity dispersion of the test particles varies from \( (m_{\text{eq}})^2 G^2 / \sigma_b^2 = m_{\text{eff}} / (2m) \) when \( m \ll m_{\text{eff}} \) to \( (m_{\text{eq}})^2 G^2 / \sigma_b^2 = \sqrt{2m_{\text{eff}} / m} \) when \( m \gg m_{\text{eff}} \). These results are consistent with Figs. 1 and 2 of [74] (note that our expression of \( f_{\text{eq}} \) is slightly simpler than their Eq. (90)). They can also be recovered from Eq. (E15).
Remark Bar-Or et al. [74] obtained these results by a very different method based on the Schrödinger–Poisson equations. It is interesting to recover their results directly from the formalism of Ref. [41] based on the bosonic Landau equation (24).

7 Self-consistent Landau equation in the single species case

In this section we consider the self-consistent (integrodifferential) Landau equation for a single species system. We make a change of variables to write it in energy space.

7.1 Quantum Landau equation in energy space

The self-consistent quantum Landau equation can be written as

\[
\frac{\partial f}{\partial t} = 2\pi G^2 m \ln \Lambda \frac{\partial}{\partial v_i} \int dv' K_{ij} \left\{ f'(1 - \kappa \frac{f'}{\eta_0}) \frac{\partial f}{\partial v_j} - f \left(1 - \kappa \frac{f}{\eta_0}\right) \frac{\partial f'}{\partial v_j'} \right\}
\] (223)

or, equivalently, as

\[
\frac{\partial f}{\partial t} = \frac{\partial}{\partial v_i} \left[ D_{ij} \frac{\partial f}{\partial v_j} - f \left(1 - \kappa \frac{f}{\eta_0}\right) F_{pol}^i \right],
\] (224)

where \(D_{ij}\) and \(F_{pol}\) are defined by Eqs. (110) and (111) with \(m_b = m\) and \(f_b = f\). If the DF is isotropic, i.e., \(f = f(v, t)\), the quantum Landau equation reduces to the form

\[
\frac{\partial f}{\partial t} = \frac{1}{v^2} \frac{\partial}{\partial v} \left[ v^2 \left( D_{ij} \frac{\partial f}{\partial v_j} - f \left(1 - \kappa \frac{f}{\eta_0}\right) F_{pol}^i \right) \right],
\] (225)

where \(D_{ij}\) and \(F_{pol}^i\) are given by Eqs. (122) and (124). In that case, the quantum Landau equation can be written explicitly as

\[
\frac{\partial f}{\partial t} = 16\pi^2 G^2 m \ln \Lambda \frac{1}{v^2} \frac{\partial}{\partial v} \left[ A \frac{\partial f}{\partial v} + Bf \left(1 - \kappa \frac{f}{\eta_0}\right) \right]
\] (226)

with

\[
A(v, t) = \frac{1}{3v} \int_0^v v_1 f(v_1, t) \left(1 - \kappa \frac{f(v_1, t)}{\eta_0}\right) d v_1 + \frac{v^2}{3} \int_v^{+\infty} v_1 f(v_1, t) \left(1 - \kappa \frac{f(v_1, t)}{\eta_0}\right) d v_1
\] (227)

and

\[
B(v, t) = \int_0^v f(v_1, t) v_1^2 d v_1.
\] (228)

This is a self-consistent (integrodifferential) equation for \(f(v, t)\). In the case of bosons \((\kappa = -1)\), this equation should display the process of Bose–Einstein condensation when \(E < E_c\), where \(E_c = (3/2)M\sigma^2\) is the critical energy corresponding to \(T = T_c\) (see “Appendix B9”). In that case, the DF can be written as \(f(v, t) = f_{\text{gas}}(v, t) + M_c(t)\delta(v)/(4\pi v^2)\) where \(f_{\text{gas}}(v, t)\) is the DF of the uncondensed bosons (gas) and \(M_c(t) = M - M_{\text{gas}}(t)\) is the mass of the condensed bosons. The mass of the Dirac peak (condensate) increases with time until a statistical equilibrium state comprising both uncondensed and condensed bosons is reached [see Eq. (B50)]. Since the bosonic Landau
equation conserves the energy, it corresponds to a microcanonical description of Bose–Einstein condensation. This can be contrasted from the canonical situation studied by Sopik et al. [48] (see the Remark at the end of Sect. 4.5).

It is convenient to rewrite the quantum Landau equation (226) in terms of the individual energy $\epsilon = (1/2)mv^2$ instead of the velocity $v$ by using the relation $F(\epsilon, t) d\epsilon = 4\pi v^2 f(v, t) dv$ yielding

$$F(\epsilon, t) = \frac{4\pi}{m} v f(v, t).$$

(229)

With this change of variables, we obtain

$$\frac{\partial F}{\partial t} = 16\pi^2 G^2 m^3 \ln \Lambda \frac{\partial}{\partial \epsilon} \left[ A \frac{\partial F}{\partial \epsilon} - \frac{AF}{2\epsilon} + \frac{BF}{\sqrt{2m\epsilon}} \left( 1 - \kappa \frac{m^{3/2}F}{4\pi \eta_0 \sqrt{2\epsilon}} \right) \right]$$

(230)

with

$$A(\epsilon, t) = \frac{1}{6\pi \sqrt{2m\epsilon}} \int_0^\epsilon \epsilon_1 F(\epsilon_1, t) \left( 1 - \kappa \frac{m^{3/2}F(\epsilon_1, t)}{4\pi \eta_0 \sqrt{2\epsilon_1}} \right) d\epsilon_1$$

$$+ \frac{\epsilon}{6\pi} \int_\epsilon^{+\infty} \frac{1}{\sqrt{2m\epsilon_1}} F(\epsilon_1, t) \left( 1 - \kappa \frac{m^{3/2}F(\epsilon_1, t)}{4\pi \eta_0 \sqrt{2\epsilon_1}} \right) d\epsilon_1$$

(231)

and

$$B(\epsilon, t) = \frac{1}{4\pi} \int_0^\epsilon F(\epsilon_1, t) d\epsilon_1.$$  

(232)

If we consider the case of bosons ($\kappa = -1$) with $E \ll E_c$, we can make the approximation

$$f \left( 1 + \frac{f}{\eta_0} \right) \simeq \frac{f^2}{\eta_0}.$$  

(233)

In that case, the bosonic Landau equation (230) reduces to

$$\frac{\partial F}{\partial t} = 16\pi^2 G^2 m^3 \ln \Lambda \frac{\partial}{\partial \epsilon} \left[ A \frac{\partial F}{\partial \epsilon} + \left( \frac{BmF}{4\pi \eta_0} - A \right) \frac{F}{2\epsilon} \right]$$

(234)

with

$$A(\epsilon, t) = \frac{m}{48\pi^2 \eta_0} \left[ \frac{1}{\sqrt{\epsilon}} \int_0^\epsilon \sqrt{\epsilon_1} F(\epsilon_1, t)^2 d\epsilon_1 + \epsilon \int_\epsilon^{+\infty} \frac{1}{\epsilon_1} F(\epsilon_1, t)^2 d\epsilon_1 \right]$$

(235)

and

$$B(\epsilon, t) = \frac{1}{4\pi} \int_0^\epsilon F(\epsilon_1, t) d\epsilon_1.$$  

(236)

When $E < E_c$, the mass of condensed bosons is $M_{BEC}(t) = M - M_{\text{gas}}(t)$ with $M_{\text{gas}}(t) = \int_0^{+\infty} F(\epsilon, t) d\epsilon$. Eqs. (234)–(236) are equivalent to those derived by Levkov et al. [50] from the Schrödinger–Poisson equations. They used them to study the process of Bose–Einstein condensation in the context of DM, resulting in the formation of a Bose star. Using scaling arguments, they showed that when $E \ll E_c$ the relaxation (condensation) time is given by Eq. (31).

---

44 We can make a similar simplification in Eqs. (226)–(228).
Remark It is interesting to calculate the initial current of collisions

\[(J_i)_0 = - \left[ D_{ij} \frac{\partial f}{\partial v_j} - f \left( 1 - \kappa \frac{f}{\eta_0} \right) F_i^{\text{pol}} \right]_{t=0} \quad (237)\]

in the case of fermions and bosons. Assuming that the initial DF is Maxwellian (interpreted as an out-of-equilibrium DF)

\[f_0(v) = \frac{\rho}{(2\pi\sigma^2)^{3/2}} e^{-\frac{v^2}{2\sigma^2}}, \quad (238)\]

and noting that the Maxwellian DF is the stationary solution of the classical Landau equation \((\kappa = 0)\), we conclude that the initial current of quantum particles involves only terms proportional to \(\kappa\). Therefore, it takes the form

\[(J_i)_0 = - \left[ D_{ij}^{(\kappa)} \frac{\partial f}{\partial v_j} + \kappa \frac{f^2}{\eta_0} F_i^{\text{pol}} \right]_{t=0}, \quad (239)\]

where \(F_i^{\text{pol}}\) is given by Eq. (111) without the subscript \(b\) and [see Eq. (110)]

\[D_{ij}^{(\kappa)} = -2\pi \kappa G^2 m \ln \Lambda \int d\nu' K_{ij} \frac{f'^2}{\eta_0}. \quad (240)\]

We can therefore use the results of Sect. 6.4 both for bosons and fermions. We get

\[(J_i)_0 = - \left[ -D_{ij}^{(\kappa)} f \frac{\nu_i}{\sigma^2} + \kappa \frac{f^2}{\eta_0} F_i^{\text{pol}} \right]_{t=0}, \quad (241)\]

where \(D_{ij}^{(\kappa)}\) and \(F_i^{\text{pol}}\) are given by Eqs. (205) and (210) without the subscript \(b\), and we have used Eq. (238) to calculate the derivative. Introducing the variable \(x\) defined by Eq. (211) and writing \(J_0 = J_0 v / v\), the normalized initial current of collisions is given by

\[J_0 \frac{J^*}{J^*} = e^{-x^2} \left[ G(\sqrt{2} x) + \kappa 2\sqrt{2} e^{-x^2} G(x) \right] \quad (242)\]

with

\[J^* = \frac{\sqrt{\pi} G^2 \rho^3 \hbar^3 \ln \Lambda}{g m^3 \sigma^6}. \quad (243)\]

Using Eq. (82), it can be written explicitly as

\[J_0 \frac{J^*}{J^*} = \frac{1}{4x^2} \left[ e^{-x^2} \text{erf}(\sqrt{2} x) + 4\sqrt{2} x e^{-2x^2} \text{erf}(x) - 2\sqrt{2} x (1 + 4\kappa) x e^{-3x^2} \right]. \quad (244)\]

It is represented in Fig. 3. In the case of fermions, we see that the collisions tend to reduce the value of the DF for small \(v\) (where \(f\) is large) since \(J_0 / J^* \sim 2(2/\pi)^{1/2} x > 0\). This is a dynamical consequence of Pauli’s blocking. Inversely, in the case of bosons, collisions tend to increase the value of the DF for small \(v\) since \(J_0 / J^* \sim -(2/3)(2/\pi)^{1/2} x < 0\). This is a dynamical consequence of Bose stimulation. Finally, we note that, in this particular situation, the typical timescale of evolution is given by Eq. (31) [see Eq. (243)] both for fermions and bosons (without having to assume \(f / \eta_0 \gg 1\)). This timescale should not be interpreted, however, as a time of relaxation towards thermal equilibrium.
Fig. 3 Normalized initial current of collisions as a function of the normalized velocity \( x = v / \sqrt{2} \sigma \). The collisions reduce the DF at small \( v \) in the case of fermions \((J_0 > 0)\) and increase the DF at small \( v \) in the case of bosons \((J_0 < 0)\).

7.2 Classical Landau equation in energy space

The classical Landau equation (2) is recovered from Eq. (223) in the dilute limit \( f \ll \eta_0 \) where we can make the approximation

\[
f \left( 1 - \kappa \frac{f}{\eta_0} \right) \simeq f. \tag{245}\]

If the DF is isotropic, i.e., \( f = f(v, t) \), the Landau equation can be written explicitly as

\[
\frac{\partial f}{\partial t} = 16\pi^2 G^2 m \Lambda \frac{1}{v^2} \frac{\partial}{\partial v} \left( A \frac{\partial f}{\partial v} + B f \right) \tag{246}\]

with

\[
A(v, t) = \frac{1}{3v} \int_0^v v_1^4 f(v_1, t) \, dv_1 + \frac{v^2}{3} \int_v^{+\infty} v_1 f(v_1, t) \, dv_1 \tag{247}\]

and

\[
B(v, t) = \int_0^v f(v_1, t) v_1^2 \, dv_1. \tag{248}\]

In terms of the energy, i.e., for \( F = F(\epsilon, t) \), it takes the form

\[
\frac{\partial F}{\partial t} = 16\pi^2 G^2 m^3 \ln \Lambda \frac{\partial}{\partial \epsilon} \left( A \frac{\partial F}{\partial \epsilon} - \frac{AF}{2\epsilon} + \frac{BF}{\sqrt{2m\epsilon}} \right) \tag{249}\]

with

\[
A(\epsilon, t) = \frac{1}{6\pi \sqrt{2m\epsilon}} \int_0^\epsilon \epsilon_1 F(\epsilon_1, t) \, d\epsilon_1 + \frac{\epsilon}{6\pi} \int_\epsilon^{+\infty} \frac{1}{\sqrt{2m\epsilon_1}} F(\epsilon_1, t) \, d\epsilon_1 \tag{250}\]

45 This equation is implicit in the papers of Chandrasekhar [15] and Rosenbluth et al. [18]. It was first explicitly written by MacDonald et al. [105] and King [106].
and
\[ B(\epsilon, t) = \frac{1}{4\pi} \int_0^\epsilon F(\epsilon_1, t) \, d\epsilon_1. \] (251)

7.3 Kramers–Chandrasekhar equation in energy space

We now assume that the field particles are at statistical equilibrium and make a thermal bath approximation. This transforms a self-consistent integrodifferential equation (Landau) into a differential equation (Fokker–Planck). Using the fact that the functions \( A \) and \( B \) are defined by
\[ D_{\parallel} = 16\pi^2 G^2 m \ln \Lambda \frac{1}{v^2} A, \quad F_{\text{pol}} = -16\pi^2 G^2 m \ln \Lambda \frac{1}{v^2} B, \] (252)
the Einstein relation from Eq. (132) can be rewritten as
\[ B = A \beta m v. \] (253)

In the thermal bath approximation, using Eq. (253), the quantum Landau equation (226) is transformed into the quantum Kramers–Chandrasekhar equation
\[ \frac{\partial f}{\partial t} = \frac{1}{v^2} \frac{\partial}{\partial v} \left\{ v^2 D_{\parallel} \left[ \frac{\partial f}{\partial v} + \beta m f \left( 1 - \kappa \frac{f}{\eta_0} \right) v \right] \right\}, \] (254)
where \( D_{\parallel} \) is given by Eq. (143). For classical particles (\( \kappa = 0 \)), Eq. (254) reduces to the Kramers–Chandrasekhar
\[ \frac{\partial f}{\partial t} = \frac{1}{v^2} \frac{\partial}{\partial v} \left[ v^2 D_{\parallel} \left( \frac{\partial f}{\partial v} + \beta m f v \right) \right], \] (255)
where \( D_{\parallel} \) is given by Eq. (76). For bosons (\( \kappa = -1 \)) with \( E \ll E_c \), making the approximation from Eq. (233), we obtain
\[ \frac{\partial f}{\partial t} = \frac{1}{v^2} \frac{\partial}{\partial v} \left[ v^2 D_{\parallel} \left( \frac{\partial f}{\partial v} + \beta \frac{m}{\eta_0} f^2 v \right) \right], \] (256)
where \( D_{\parallel} \) is given by Eq. (C21).

On the other hand, the quantum Landau equation in energy space (230) is transformed into the quantum Kramers–Chandrasekhar equation in energy space
\[ \frac{\partial F}{\partial t} = 2m \frac{\partial}{\partial \epsilon} \left\{ \epsilon D_{\parallel} \left[ \frac{\partial F}{\partial \epsilon} - \frac{F}{2\epsilon} + \beta F \left( 1 - \kappa \frac{m^3/2 F}{4\pi \eta_0 \sqrt{2} \epsilon} \right) \right] \right\}, \] (257)
where \( D_{\parallel} \) is given by Eq. (143) with the change of variables \( \epsilon = mv^2/2 \). For classical particles (\( \kappa = 0 \)), Eq. (257) reduces to
\[ \frac{\partial F}{\partial t} = 2m \frac{\partial}{\partial \epsilon} \left\{ \epsilon D_{\parallel} \left( \frac{\partial F}{\partial \epsilon} - \frac{F}{2\epsilon} + \beta F \right) \right\}, \] (258)
where \( D_{\parallel} \) is given by Eq. (76) with the change of variables \( \epsilon = mv^2/2 \). For bosons (\( \kappa = -1 \)) with \( E \ll E_c \), making the approximation from Eq. (233), we obtain
\[ \frac{\partial F}{\partial t} = 2m \frac{\partial}{\partial \epsilon} \left\{ \epsilon D_{\parallel} \left[ \frac{\partial F}{\partial \epsilon} - \frac{F}{2\epsilon} + \frac{m^3/2 \beta F^2}{4\pi \eta_0 \sqrt{2} \epsilon} \right] \right\}, \] (259)
where \( D_{\parallel} \) is given by Eq. (C21) with the change of variables \( \epsilon = mv^2/2 \).
Remark If we make the change of variables \(\epsilon = m v^2/2\) but work in terms of \(f(\epsilon, t)\) rather than in terms of \(F(\epsilon, t)\), we get

\[
\frac{\partial f}{\partial t} = 2m \frac{1}{\sqrt{\epsilon}} \frac{\partial}{\partial \epsilon} \left\{ \epsilon^{3/2} D_\parallel \left[ \frac{\partial f}{\partial \epsilon} + \beta f \left( 1 - \kappa \frac{f}{\eta_0} \right) \right] \right\},
\]

where \(D_\parallel\) is given by Eq. (143) with the change of variables \(\epsilon = m v^2/2\). For classical particles \((\kappa = 0)\), Eq. (260) reduces to

\[
\frac{\partial f}{\partial t} = 2m \frac{1}{\sqrt{\epsilon}} \frac{\partial}{\partial \epsilon} \left[ \epsilon^{3/2} D_\parallel \left( \frac{\partial f}{\partial \epsilon} + \beta f \right) \right],
\]

where \(D_\parallel\) is given by Eq. (76) with the change of variables \(\epsilon = m v^2/2\). Since \(D_\parallel \propto v^{-3}\) when \(v \to +\infty\) (see Sect. 4.7), we see that \(\epsilon^{3/2} D_\parallel \to \text{cst}\) when \(\epsilon \to +\infty\). With this approximation, the stationary solution of Eq. (260), corresponding to a constant current \(J\), returns the quantum King model of Sect. 4.7.46 We note that Eq. (260) is an exact equation for the time-dependent evolution of \(f(\epsilon, t)\). However, it is restricted to spatially homogeneous self-gravitating systems.

8 Conclusion

In this paper, we have developed the kinetic theory of self-gravitating fermions and bosons. Although we have worked at a general level, our study can have applications in the context of fermionic or bosonic DM. For simplicity, we have considered an infinite homogeneous system and we have neglected collective effects. We have studied the self-consistent evolution of the system “as a whole” and the relaxation of test particles experiencing collisions with field particles that are either at statistical equilibrium (thermal bath) or in an out-of-equilibrium quasistationary state. We have shown that the quantum nature of the particles affects the relaxation time with respect to the case of classical particles (assuming for comparison that they have the same mass and the same DF). The diffusion of the test particles is different depending on whether the field particles are classical or quantum. In this respect, quantum particles of mass \(m_b\) are equivalent to classical quasiparticles of mass \(m_{\text{eff}}\).47 The heating time due to collisions with quantum particles is larger than the heating time due to collisions with classical particle if the quantum particles are fermions \((m_{\text{eff}} < m_b)\) and smaller if they are bosons \((m_{\text{eff}} > m_b)\). The friction by polarization experienced by a test particle has the same expression for classical and quantum field particles (except for a small change in the Coulomb logarithm). As a result, the cooling time is not affected by the quantum nature of the particles. This confirms the results obtained by Bar-Or et al. [74] in the context of FDM. For FDM, the regime of most interest is when \(T \ll T_c\). In that case, \(m_{\text{eff}} \gg m_b\) and the heating time is strongly reduced with respect to CDM, becoming smaller than the age of the universe. Therefore, collisional effects affect the evolution of test particles (stars, globular clusters, black holes... ) traveling through FDM halos [74]. They also induce the secular evolution of the halo and the slow growth of the mass of the soliton which is fed by the halo [50]. A detailed description of the interaction between the soliton and the halo should be considered in future works.

46 For classical particles, the stationary solution of Eq. (261) or, equivalently, Eq. (255) leads to the King model [28] without being required to make the approximation \(\epsilon^{3/2} D_\parallel \to \text{cst}\) (see [24] and Appendix D of [25]).

47 In the case of bosons, these quasiparticles [73] correspond to incoherent granular density fluctuations produced by wave interference [55, 56]. In the case of fermions, they are more difficult to interpret physically.
We have shown that the results of Levkov et al. [50] and Bar-Or et al. [74] could be understood in a unified manner from the quantum Landau equation (24) introduced in [41]. In the case of bosons, this equation is more general than the one considered by Levkov et al. [50] because we do not necessarily make the approximation $f(1 + f/\eta_0) \approx f^2/\eta_0$. As a result, the bosonic Landau equation (24) relaxes towards the Bose–Einstein DF when $E > E_c$ and displays the phenomenon of Bose–Einstein condensation when $E < E_c$. In the regime $E \ll E_c$, relevant to FDM, we can make the approximation $f(1 + f/\eta_0) \approx f^2/\eta_0$ and we recover the results of Levkov et al. [50]. On the other hand, if we make a bath approximation, we can derive from the quantum Landau equation (24) a nonlinear Fokker–Planck equation which generalizes the classical Kramers–Chandrasekhar equation. If the field particles are at statistical equilibrium (thermal bath), the diffusion and friction coefficients satisfy an Einstein relation and the Fokker–Planck equation relaxes towards the Fermi–Dirac or Bose–Einstein DF. If the field particles are not at statistical equilibrium, the Fokker–Planck equation relaxes towards a different DF. In the case of FDM, if we assume that the bosons have an out-of-equilibrium Maxwellian DF (resulting from a process of collisionless violent relaxation) and make the approximation $f(1 + f/\eta_0) \approx f^2/\eta_0$, we recover the results of Bar-Or et al. [74] that were obtained in a different manner. Although we have focused in this paper on the kinetic theory of quantum particles because of its possible applications to fermionic and bosonic DM, it is shown in [41] that the results of kinetic theory can be extended to other situations where the system is described by a generalized entropy (e.g., the Tsallis entropy). On the other hand, it would be interesting to generalize the quantum kinetic theory to inhomogeneous systems and take collective effects into account in order to extend the classical inhomogeneous Lenard–Balescu equation [26,27] to the quantum realm.

Note added: While this paper was in course of finalization, I came across the very interesting paper of Bar-Or et al. [107] who, independently, derived similar results. In particular, they derived the bosonic Landau equation (24) from a heuristic approach (equivalent to the one presented in [41]) and also directly from the Schrödinger–Poisson or Wigner-Poisson equations by using a quasilinear theory. They used this equation to directly obtain the coefficients of diffusion and friction as done in the present paper (and previously in Ref. [41] at a general level). The scope of the two papers is relatively different so they complete each other. A draft of the present paper was transmitted to J.B. Fouvry on 21 October 2020 and I thank him for his kind correspondence.

Appendix A: Application of the kinetic theory to FDM: validity of the approximations

The kinetic theory developed in this paper can be applied in the context of fermionic and bosonic DM. In this Appendix, we focus on the case of bosonic DM. It has been proposed that DM may be made of bosons like ultralight axions with a mass of the order of $m_b \sim 10^{-22}$ eV/c$^2$ (see, e.g., [73]). For such a small mass, the wave nature of the bosons manifests itself at the scale of DM halos. This has been called FDM [108]. As recalled in the Introduction, two situations have been considered in the literature.

In the first situation, studied by Levkov et al. [50], it is assumed that the gas of bosons is spatially homogeneous and dynamically stable (virialized). In that case, it can only evolve under the effect of gravitational interactions (“collisions”). This is a slow process taking place on a secular timescale. The evolution of the DF $f(v, t)$ is governed by the self-consistent bosonic Landau equation (230). In most applications of astrophysical interest, we are far below the condensation temperature so we can make the approximation $f(1 + f/\eta_0) \approx f^2/\eta_0$ leading to the simplified self-consistent bosonic Landau equation (234). This equation...
describes the condensation of the boson gas. As shown by Levkov et al. [50], this condensation process leads to the formation of a Bose star associated with the emergence of a Dirac peak in the velocity DF (i.e., in $k$-space) surrounded by a homogeneous (or weakly inhomogeneous) halo of uncondensed bosons that feeds it.

In the second situation, studied by [55–63], the system is initially unsteady or dynamically unstable so it rapidly becomes spatially inhomogeneous. In that case, it experiences two successive kinds of relaxation. In a first regime, it undergoes a process of gravitational cooling [64–66] and violent collisionless relaxation [12] on a few dynamical times, arising from the strong fluctuations of the gravitational potential accompanying the formation of the DM halo. This leads to a DM halo with a core-halo structure made of a quantum core (soliton) and an approximately isothermal halo (see the Introduction for more details). The halo has a granular structure arising from wave interferences [55,56]. It is made of quasiparticles of large mass $m_{\text{eff}} \gg m_b$ due to Bose stimulation [73]. These quasiparticles induce a slow collisional relaxation of the DM halo on a secular timescale like in the case of globular clusters. The initial DF of the halo can be approximated by the Maxwell-Boltzmann distribution with a velocity dispersion $\sigma_b$ corresponding to an effective temperature $T_{\text{eff}} \ll T_c$. This is an out-of-equilibrium DF arising from the process of violent relaxation. In a sense, this isothermal DF can be justified by Lynden-Bell’s statistical theory. Treating the halo as approximately homogeneous (i.e., making a local approximation), its secular evolution is governed by the self-consistent bosonic Landau equation (230), or its simplified form (234), as discussed above. The DF of the halo is expected to slowly change with time due to gravitational interactions. Since $T_{\text{eff}} \ll T_c$, it also undergoes a process of condensation feeding the solitonic core. This corresponds to the problem studied by Levkov et al. [50] except that the system (core + halo) is fundamentally inhomogeneous in the present case. On the other hand, if we consider the evolution of classical test particles in the halo, like stars, globular clusters or black holes, their evolution is governed by a classical Fokker–Planck equation that can be obtained from the Landau equation (161) by making a bath approximation, i.e., by replacing the DF of the bosonic field particles by their (out-of-equilibrium) Maxwell-Boltzmann DF. Since the amplification factor is large we can also make the approximation $f (1 + f / \eta_0) \simeq f^2 / \eta_0$. This is the situation considered by Bar-Or et al. [74] who studied different applications of the kinetic theory to DM halos.

Let us now make orders of magnitude estimates in order to justify the approximations that are relevant for FDM. The DF of the halo [see Eq. (200)] can be estimated by

$$f_b \sim \frac{\rho_b}{\sigma_b^3},$$

(A1)

where $\rho_b$ is the mass density of the halo and $\sigma_b$ is the velocity dispersion of the bosons that compose it. Recalling that

$$\eta_b = \frac{m_b^4}{h^3},$$

(A2)

the amplification factor due to Bose stimulation is given by

$$\chi \equiv \frac{f_b}{\eta_b} = \frac{\rho_b}{\eta_b \sigma_b^3} = \frac{\rho_b h^3}{\sigma_b^3 m_b^4}.$$  

(A3)

48 This is a state of dynamical equilibrium with respect to a collisionless evolution (virialized state or quasistationary state) but not a state of thermodynamical equilibrium with respect to a collisional relaxation.
Assuming that the halo is approximately isothermal with an effective temperature $T_{\text{eff}}$, we have

$$\sigma_b^2 \sim \frac{k_B T_{\text{eff}}}{m_b}.$$  
(A4)

On the other hand, recalling the expression of the condensation temperature $T_c$ from Eq. (B45), we can write

$$\sigma_c^2 \sim \frac{k_B T_c}{m_b} \sim \left(\frac{\rho_b}{\eta_b}\right)^{2/3}.$$  
(A5)

Therefore, the amplification factor can be written in terms of the effective temperature as

$$\chi \sim \left(\frac{T_c}{T_{\text{eff}}}\right)^{3/2} \sim \left(\frac{\sigma_c}{\sigma_b}\right)^3.$$  
(A6)

The condition that the amplification due to Bose stimulation is important can be expressed as

$$\chi \gg 1 \iff f_b \gg \eta_b \iff T_{\text{eff}} \ll T_c.$$  
(A7)

We will see that this condition is fulfilled in FDM halos. We can also obtain a more explicit expression of $\chi$ as follows. The velocity dispersion of the bosons in the halo can be estimated from the virial theorem yielding

$$\sigma_b^2 \sim \frac{GM_h}{r_h},$$  
(A8)

where $M_h$ is the halo mass and $r_h$ is the halo radius. They are connected by the relation

$$M_h \sim \Sigma_0 r_h^2,$$  
(A9)

where $\Sigma_0 = 141 M_\odot/\text{pc}^2$ is the universal surface density of the DM halos (see, e.g., [68] for a more detailed discussion). Therefore, $\sigma_b^2 \sim G \Sigma_0 M_h$. Substituting this relation into Eq. (A3) and using $\rho_b \sim M_h/r_h^3 \sim \Sigma_0^{3/2}/M_h^{1/2}$, we get

$$\chi \sim \frac{\Sigma_0^{3/4} h^3}{G^{3/2} m_b^4 M_h^{5/4}}.$$  
(A10)

This equation determines the amplification factor $\chi$ in a halo of mass $M_h$. When $\chi \gg 1$, the halo acts as if it were composed of quasiparticles with effective mass (see Sect. 6)

$$m_{\text{eff}} \sim \chi m_b \sim \frac{\rho_b h^3}{\sigma_b^3 m_b^3} \sim \rho_b \lambda_{\text{dB}}^3,$$  
(A11)

where $\lambda_{\text{dB}} = h/(m_b \sigma_b)$ is the de Broglie length. We see that $\chi$ is equal to the occupation number $N = n_b \lambda_{\text{dB}}^3$ (number of bosons in the de Broglie sphere) so that $m_{\text{eff}} \sim N m_b$. For $\chi \gg 1$, the effective mass $m_{\text{eff}}$ of the quasiparticles is much larger than the mass $m_b$ of the bosons. This is a striking effect of Bose stimulation. The number of quasiparticles in the DM

49 If we apply the condition $\chi \gg 1$ to the “minimum halo” of mass $(M_h)_{\text{min}} \sim (h^4 \Sigma_0/G^2 m_b^4)^{1/3} \sim 10^8 M_\odot$ [71], we get the condition $m_b \ll (h^4 \Sigma_0/G^2)^{1/7} \sim 10^{10} \text{GeV}/c^2$. Using $\Sigma_0 \sim c \sqrt{\Lambda}/G$ [71], this condition can be rewritten as $m_b \ll (h^4 c \sqrt{\Lambda}/G^3)^{1/7}$, where $\Lambda = 1.00 \times 10^{-35} \text{ s}^{-2}$ is the cosmological constant.
halo is \( N_{\text{eff}} = N/m_{\text{eff}} \) and the relaxation time induced by these quasiparticles can be written as (see Sect. 2.3)

\[
t_R \sim \frac{N_{\text{eff}}}{\ln N_{\text{eff}}} t_D, \tag{A12}
\]

where \( t_D \) is the dynamical time

\[
t_D \sim \frac{R}{\sigma} \sim \frac{1}{(G\rho)^{1/2}}. \tag{A13}
\]

Because of Bose stimulation, the effective number of particles and the collisional relaxation time of FDM halos are strongly reduced as compared to classical CDM halos.

**Numerical application:** For a FDM halo of mass \( M_h \sim 10^{11} M_\odot \), radius \( r_h \sim 30 \text{ kpc} \), density \( \rho_b \sim 4 \times 10^{-3} M_\odot/\text{pc}^3 \) and velocity dispersion \( \sigma_b \sim 100 \text{ km/s} \) (medium spiral), and for a boson mass \( m_b \sim 10^{-22} \text{ eV}/c^2 \), we get \( T_{\text{eff}} \sim 10^{-25} \text{ K} \) and \( T_c \sim 10^{38} \text{ K} \).\(^{50}\) This yields \( \chi \sim (T_c/T_{\text{eff}})^{3/2} \sim 10^{94} \gg 1 \). Therefore, the amplification factor due to Bose stimulation is gigantic. The description of FDM halos in terms of quasiparticles is fully justified, as well as the approximation \( f(1 + f/\eta_b) \simeq f^2/\eta_b \). The de Broglie length is \( \lambda_{\text{dB}} \sim 1 \text{ kpc} \), corresponding to the typical size of the soliton.\(^{51}\) The occupation number is \( N = n_b\lambda_{\text{dB}}^3 = 10^{94} \gg 1 \) justifying to treat the wavefunction as a classical field. The effective mass of the quasiparticles is \( m_{\text{eff}} \sim Nm_b \sim 10^6 M_\odot \). Because of Bose stimulation, the quasiparticles are much heavier than the bosons. The number of bosons in a FDM halo is \( N \sim 10^{99} \) but the number of quasiparticles is only \( N_{\text{eff}} \sim 10^5 \), which is comparable to the number of stars in a globular cluster. The dynamical time is \( t_D \sim 10^8 \text{ yrs} \) and the relaxation time is \( t_R \sim 10^{12} \text{ yrs} \), which is comparable to the age of the Universe (the relaxation time due to particles of mass \( m_b \sim 10^{-22} \text{ eV}/c^2 \) would be \( t_R \sim 10^{105} \text{ yrs} \)). Therefore, “collisions” of quasiparticles in FDM halos are relevant on a secular timescale of the order of the age of the universe, similarly to “collisions” of stars in globular clusters (recall that our numerical application is indicative since the effective mass \( m_{\text{eff}} \) depends on the position and is larger in regions of high density implying a shorter relaxation time in the core of the galaxy). Finally, as a comparison, we note that \( \chi \sim 0.1 \) for fermions of mass \( m \sim 100 \text{ eV}/c^2 \), so that quantum effects – Pauli’s exclusion principle – are important in the core of the galaxy (where the density is higher than the average estimate made above).

**Appendix B: Conservation laws and \( H \)-theorem**

In this Appendix, we establish general properties of the multi-species quantum Landau equation (25).

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\(^{50}\) The very small temperature \( T_{\text{eff}} \sim 10^{-25} \text{ K} \) of the bosons is clearly unphysical confirming that \( T_{\text{eff}} \) is an effective temperature. On the other hand, since \( T_{\text{eff}} \ll T_c \), the isothermal halo is an out-of-equilibrium structure otherwise it would be condensed (or have a non-Maxwellian DF corresponding to the Bose–Einstein DF below \( T_c \)). As we have previously indicated, the isothermal DF of the halo is established by a process of violent collisionless relaxation \([12]\), not by a collisional process. FDM is a striking example where the out-of-equilibrium isothermal Lynden-Bell DF is very different from the thermal equilibrium state of the system. For classical particles, the Lynden-Bell DF and the Maxwell-Boltzmann DF are similar (in the nondegenerate limit) even though they have a very different interpretation and a very different domain of validity.

\(^{51}\) The mass-radius relation of the soliton is \( M_c R_c \sim h^2/\text{Gm}_b^2 \) (see, e.g., \([32]\)). Using the velocity dispersion tracing relation \( \sigma_b^2 \sim GM_c/R_c \sim GM_b/R_h \) \([71]\), we get \( R_c \sim h/m_b\sigma_b = \lambda_{\text{dB}} \).
B1: Conservation of the mass of each species

Since $\partial f_a / \partial t$ is the divergence of a current, the mass of each species

$$M_a = \int f_a \, d\mathbf{r} \, d\mathbf{v}$$

is conserved.

B2: Conservation of the total impulse

The quantum Landau equation (25) conserves the total impulse

$$\mathbf{P} = \sum_a \int f_a \mathbf{v} \, d\mathbf{r} \, d\mathbf{v}. \quad \text{(B2)}$$

For each component we have

$$\dot{P}_k = \sum_a \int d\mathbf{r} d\mathbf{v} v_k \frac{\partial f_a}{\partial t}, \quad \text{(B3)}$$

hence

$$\dot{P}_k = 2\pi G^2 \ln \Lambda \sum_{a, b} \int d\mathbf{r} d\mathbf{v} v_k \frac{\partial}{\partial v_j} \int d\mathbf{v'} K_{ij} \left\{ m_b f'_b \left(1 - \kappa_b \frac{f'_b}{\eta_b} \right) \frac{\partial f_a}{\partial v_j} - m_a f_a \left(1 - \kappa_a \frac{f_a}{\eta_a} \right) \frac{\partial f'_b}{\partial v'_j} \right\}$$

$$= -2\pi G^2 \ln \Lambda \sum_{a, b} \int d\mathbf{r} d\mathbf{v} d\mathbf{v'} K_{kj} \left\{ m_b f'_b \left(1 - \kappa_b \frac{f'_b}{\eta_b} \right) \frac{\partial f_a}{\partial v_j} - m_a f_a \left(1 - \kappa_a \frac{f_a}{\eta_a} \right) \frac{\partial f'_b}{\partial v'_j} \right\}$$

$$= 2\pi G^2 \ln \Lambda \sum_{a, b} \int d\mathbf{r} d\mathbf{v} d\mathbf{v'} K_{kj} \left\{ m_b f'_b \left(1 - \kappa_b \frac{f'_b}{\eta_b} \right) \frac{\partial f_a}{\partial v_j} - m_a f_a \left(1 - \kappa_a \frac{f_a}{\eta_a} \right) \frac{\partial f'_b}{\partial v'_j} \right\} = 0. \quad \text{(B4)}$$

The third line is obtained by integrating by parts and the conservation law results from the antisymmetry of the collision current with respect to the interchange $(a, v) \leftrightarrow (b, v')$.

B3: Conservation of the total energy

The quantum Landau equation (25) conserves the total energy

$$E = \sum_a \int f_a \frac{v^2}{2} \, d\mathbf{r} \, d\mathbf{v}. \quad \text{(B5)}$$

We have

$$\dot{E} = \sum_a \int d\mathbf{r} d\mathbf{v} \frac{v^2}{2} \frac{\partial f_a}{\partial t}, \quad \text{(B6)}$$

There is no potential (gravitational) energy since we have assumed that the system is infinite and homogeneous.
hence

\[ \dot{E} = 2\pi G^2 \ln \Lambda \sum_{a,b} \int d\mathbf{r} d\mathbf{v} \frac{v^2}{2} \frac{\partial}{\partial v_i} \int d\mathbf{v}' K_{ij} \left\{ m_b f'_b \left(1 - \kappa_b f'_b \eta_b \right) \frac{\partial f_a}{\partial v_j} - m_a f_a \left(1 - \kappa_a f_a \eta_a \right) \frac{\partial f'_b}{\partial v'_j} \right\} \]

\[ = -2\pi G^2 \ln \Lambda \sum_{a,b} \int d\mathbf{r} d\mathbf{v} d\mathbf{v}' v_i K_{ij} \left\{ m_b f'_b \left(1 - \kappa_b f'_b \eta_b \right) \frac{\partial f_a}{\partial v_j} - m_a f_a \left(1 - \kappa_a f_a \eta_a \right) \frac{\partial f'_b}{\partial v'_j} \right\} \]

\[ = 2\pi G^2 \ln \Lambda \sum_{a,b} \int d\mathbf{r} d\mathbf{v} d\mathbf{v}' v_i K_{ij} \left\{ m_b f'_b \left(1 - \kappa_b f'_b \eta_b \right) \frac{\partial f_a}{\partial v_j} - m_a f_a \left(1 - \kappa_a f_a \eta_a \right) \frac{\partial f'_b}{\partial v'_j} \right\} \]

\[ = \pi G^2 \ln \Lambda \sum_{a,b} \int d\mathbf{r} d\mathbf{v} d\mathbf{v}' v_i K_{ij} \left\{ m_b f'_b \left(1 - \kappa_b f'_b \eta_b \right) \frac{\partial f_a}{\partial v_j} - m_a f_a \left(1 - \kappa_a f_a \eta_a \right) \frac{\partial f'_b}{\partial v'_j} \right\} = 0. \]  

(B7)

The third line is obtained by integrating by parts and the conservation law results from the interchange \((a, \mathbf{v}) \leftrightarrow (b, \mathbf{v}')\) and the identity \(K_{ij} u_j = 0\).

B4: The Fermi–Dirac or Bose–Einstein distribution is a stationary solution

The Fermi–Dirac or Bose–Einstein distribution

\[ f_a^{\text{eq}} = \frac{\eta_a}{\kappa_a e^{\beta m_a (\mathbf{v} - \mathbf{U})^2/2} + \kappa_a}, \]  

(B8)

where \(U\) and \(\beta\) are the same for all the species, is a steady state of the quantum Landau equation (25). When the DF is given by Eq. (B8) we can easily establish that

\[ \frac{\partial f_a}{\partial \mathbf{v}} = -f_a \left(1 - \kappa_a \frac{f_a}{\eta_a} \right) \beta m_a (\mathbf{v} - \mathbf{U}). \]  

(B9)

As a result

\[ \left\{ m_b f'_b \left(1 - \kappa_b \frac{f'_b}{\eta_b} \right) \frac{\partial f_a}{\partial v_j} - m_a f_a \left(1 - \kappa_a \frac{f_a}{\eta_a} \right) \frac{\partial f'_b}{\partial v'_j} \right\} \]

\[ = -\beta m_a f_a \left(1 - \kappa_a \frac{f_a}{\eta_a} \right) m_b f'_b \left(1 - \kappa_b \frac{f'_b}{\eta_b} \right) (v_j - v'_j). \]  

(B10)

Substituting this identity into Eq. (25) and using \(K_{ij} u_j = 0\), we find that the DF is stationary: \(\partial_t f_a = 0\).

B5: \(H\)-theorem

The quantum Landau equation (25) satisfies an \(H\)-theorem for the Fermi–Dirac or Bose–Einstein entropy

\[ S = -k_B \sum_a \frac{g m_a^3}{h^3} \int \left\{ \frac{f_a}{\eta_a} \ln \frac{f_a}{\eta_a} + \kappa_a \left(1 - \kappa_a \frac{f_a}{\eta_a} \right) \ln \left(1 - \kappa_a \frac{f_a}{\eta_a} \right) \right\} d\mathbf{r} d\mathbf{v}. \]  

(B12)

\[ \text{In the classical limit } f_a/\eta_a \ll 1, \text{ we recover the Boltzmann entropy} \]

\[ S = -k_B \sum_a \int \left( \frac{f_a}{m_a} \ln \frac{f_a}{\eta_a} - \frac{f_a}{m_a} \right) d\mathbf{r} d\mathbf{v}. \]  

(B11)
We have
\[ \dot{S} = -k_B \sum_a \int d\mathbf{r} d\mathbf{v} \ln \left( \frac{f_a/\eta_a}{1 - \kappa_a f_a/\eta_a} \right) \frac{1}{m_a} \frac{\partial f_a}{\partial t}, \] (B13)

hence
\[
\dot{S} = -2\pi k_B G^2 \ln \Lambda \sum_{a,b} \int d\mathbf{r} d\mathbf{v} \ln \left( \frac{f_a/\eta_a}{1 - \kappa_a f_a/\eta_a} \right) \frac{1}{m_a} \frac{\partial f_a}{\partial v_i} \int d^{\prime}v_i K_{ij}
\times \left\{ m_b f_{b}^{\prime} \left( 1 - \kappa_b \frac{f_{b}^{\prime}}{\eta_b} \right) \frac{\partial f_a}{\partial v_j} - m_a f_a \left( 1 - \kappa_a \frac{f_a}{\eta_a} \right) \frac{\partial f_{b}^{\prime}}{\partial v_i} \right\}
\]
\[
= 2\pi k_B G^2 \ln \Lambda \sum_{a,b} \int d\mathbf{r} d\mathbf{v} \ln \left( \frac{f_a/\eta_a}{1 - \kappa_a f_a/\eta_a} \right) \frac{1}{m_a} \frac{\partial f_a}{\partial v_i} K_{ij}
\times \left\{ m_b f_{b}^{\prime} \left( 1 - \kappa_b \frac{f_{b}^{\prime}}{\eta_b} \right) \frac{\partial f_a}{\partial v_j} - m_a f_a \left( 1 - \kappa_a \frac{f_a}{\eta_a} \right) \frac{\partial f_{b}^{\prime}}{\partial v_i} \right\}
\]
\[
= 2\pi k_B G^2 \ln \Lambda \sum_{a,b} \int d\mathbf{r} d\mathbf{v} \ln \left( \frac{f_a/\eta_a}{1 - \kappa_a f_a/\eta_a} \right) \frac{1}{m_a} \frac{\partial f_a}{\partial v_i} K_{ij}
\times \left\{ m_b f_{b}^{\prime} \left( 1 - \kappa_b \frac{f_{b}^{\prime}}{\eta_b} \right) \frac{\partial f_a}{\partial v_j} - m_a f_a \left( 1 - \kappa_a \frac{f_a}{\eta_a} \right) \frac{\partial f_{b}^{\prime}}{\partial v_i} \right\}
\]
\[
= \pi k_B G^2 \ln \Lambda \sum_{a,b} \int d\mathbf{r} d\mathbf{v} \ln \left( \frac{f_a/\eta_a}{1 - \kappa_a f_a/\eta_a} \right) \frac{1}{m_a} \frac{\partial f_a}{\partial v_i} K_{ij} X_i K_{ij} X_j, \] (B14)

where
\[
X_i \equiv m_b f_{b}^{\prime} \left( 1 - \kappa_b \frac{f_{b}^{\prime}}{\eta_b} \right) \frac{\partial f_a}{\partial v_i} - m_a f_a \left( 1 - \kappa_a \frac{f_a}{\eta_a} \right) \frac{\partial f_{b}^{\prime}}{\partial v_i}. \] (B15)

The third line is obtained by integrating by parts and the last line results from the interchange \((a, \mathbf{v}) \leftrightarrow (b, \mathbf{v}^\prime)\). Since
\[
X_i K_{ij} X_j = \frac{u^2 X^2 - (\mathbf{X} \cdot \mathbf{u})^2}{u^3} \geq 0, \] (B16)

we conclude that \( \dot{S} \geq 0 \) with equality if, and only if, \( \mathbf{X} \) is parallel to \( \mathbf{u} = \mathbf{v}^\prime - \mathbf{v} \). Therefore, the entropy increases (H-theorem). On the other hand, we show in the next subsection that \( S \) is bounded from above so that \( S \leq S_{\text{max}} \), where \( S_{\text{max}} \) corresponds to the value of the entropy calculated with the Fermi–Dirac or Bose–Einstein distribution \( f_a^{\text{eq}} \) (maximum entropy state). Finally, we show below that \( \dot{S} = 0 \) if, and only if, \( f_a \) is the Fermi–Dirac or Bose–Einstein distribution \( f_a^{\text{eq}} \). As a result, we can conclude that \( f_a(\mathbf{v}, t) \) tends to the Fermi–Dirac or Bose–Einstein distribution \( f_a^{\text{eq}} \) when \( t \to +\infty \).\(^{54}\)

Proof The condition \( \dot{S} = 0 \) is equivalent to the condition that \( \mathbf{X} \) is parallel to \( \mathbf{v}^\prime - \mathbf{v} \). Using
\[
\frac{\partial}{\partial \mathbf{v}} \ln \left( \frac{f/\eta}{1 - \kappa f/\eta} \right) = \frac{1}{f(1 - \kappa f/\eta)} \frac{\partial f}{\partial \mathbf{v}}, \] (B17)

\(^{54}\) In the case of bosons, this is true only for \( E \geq E_c \). For \( E < E_c \), the total DF involves a Dirac-\( \delta \) which accounts for the process of Bose–Einstein condensation (see “Appendix B9”).
this condition can be rewritten as
\[
\frac{1}{m_a} \frac{\partial}{\partial v} \ln \left( \frac{f_a/\eta_a}{1 - \kappa_a f_a/\eta_a} \right) - \frac{1}{m_b} \frac{\partial}{\partial v'} \ln \left( \frac{f_b'/\eta_b}{1 - \kappa_b f_b'/\eta_b} \right) = -A_{ab}(v, v')(v' - v). \quad (B18)
\]
From the symmetry of the left hand side of Eq. (B18), it can be shown [82] that \( A_{ab}(v, v') \) must be a constant \( A \). This implies that
\[
\frac{1}{m_a} \frac{\partial}{\partial v} \ln \left( \frac{f_a/\eta_a}{1 - \kappa_a f_a/\eta_a} \right) = -A v + B, \quad (B19)
\]
where \( B \) is another constant. Eq. (B19) may be rewritten as
\[
\frac{\partial}{\partial v} \ln \left( \frac{f_a/\eta_a}{1 - \kappa_a f_a/\eta_a} \right) = -\beta m_a (v - U) \quad (B20)
\]
with \( \beta = A \) and \( U = B/A \). These constants are the same for all the species. After integration, Eq. (B20) leads to the Fermi–Dirac or Bose–Einstein distribution (B8).

B6: Maximum entropy state

From very general considerations, we expect that the statistical equilibrium state in the microcanonical ensemble is the “most probable” state, i.e., the state that maximizes the entropy at fixed total energy, total impulse and fixed mass of each species. We thus have to solve the maximization problem

\[
\max \{ S \mid E, P, M_a \text{ fixed} \}. \quad (B21)
\]
An extremum of entropy at fixed total energy, total impulse and fixed mass of each species is determined by the variational problem (first variations)

\[
\frac{\delta S}{k_B} - \beta \delta E + \beta U \cdot \delta P + \sum_a \frac{\alpha_a}{m_a} \delta M_a = 0, \quad (B22)
\]
where \( \beta, U \) and \( \alpha_a = \mu_a/k_B T \) are Lagrange multipliers taking into account the constraints of fixed \( E, P \) and \( M_a \). They have the interpretation of an inverse temperature, a velocity of translation and a chemical potential divided by \( k_B T \). Using

\[
\delta S = -k_B \sum_a \int d\mathbf{r} d\mathbf{v} \ln \left( \frac{f_a/\eta_a}{1 - \kappa_a f_a/\eta_a} \right) \frac{1}{m_a} \delta f_a, \quad (B23)
\]
\[
\delta E = \sum_a \int d\mathbf{r} d\mathbf{v} \frac{v^2}{2} \delta f_a, \quad (B24)
\]
\[
\delta P = \sum_a \int d\mathbf{r} d\mathbf{v} v \delta f_a, \quad (B25)
\]
\[
\delta M_a = \int d\mathbf{r} d\mathbf{v} \delta f_a, \quad (B26)
\]
we obtain

\[
\ln \left( \frac{f_a/\eta_a}{1 - \kappa_a f_a/\eta_a} \right) = -\beta m_a \frac{v^2}{2} + \beta m_a U \cdot v + \alpha_a, \quad (B27)
\]
yielding the Fermi–Dirac or Bose–Einstein distribution

\[
f_a^{eq}(v) = \frac{\eta_a}{\lambda_a e^{\beta m_a(v-U)^2/2} + \kappa_a}, \quad (B28)
\]
where \( \lambda_a = e^{-\alpha_a - \beta m_a U^2/2} \) in the inverse fugacity. The second variations of entropy write

\[
\delta^2 S = -\frac{k_B}{2} \sum_a g m_a^3 \int \frac{\left( \frac{\delta f_a}{\eta_a} \right)^2}{\eta_a (1 - \kappa_a \frac{f_a}{\eta_a})} d\rho d\nu,
\]

and they are clearly negative. Therefore, the Fermi–Dirac or Bose–Einstein distribution is the unique maximum of \( S \) at fixed \( E, P \) and \( M_a \). Hence it is a global maximum.

**Remark** In the case of bosons, as in the previous section (see footnote 54), we have implicitly assumed that \( E > E_c \) otherwise a Dirac peak occurs in the DF in relation to the process of Bose–Einstein condensation.

**B7: Equation of state**

Let us consider a single species system of fermions or bosons at statistical equilibrium and take \( U = 0 \) for simplicity. In that case, the Fermi–Dirac or Bose–Einstein distribution writes

\[
f(v) = \frac{\eta_0}{\lambda e^{\beta m v^2/2} + \kappa}, \quad \text{with} \quad \eta_0 = g \frac{m^4}{h^3}.
\]

We recall that \( \lambda = e^{-\alpha} \) with \( \alpha = \beta \mu = \mu/k_B T \). The density \( \rho = \int f d\nu \) and the pressure \( P = \frac{1}{3} \int f v^2 d\nu \) are given by

\[
\rho = \frac{4\pi \sqrt{2} \eta_0}{(\beta m)^{3/2}} I_{1/2}(\lambda), \quad P = \frac{8\pi \sqrt{2} \eta_0}{3(\beta m)^{5/2}} I_{3/2}(\lambda),
\]

where \( I_n(t) \) are the Fermi or Bose integrals

\[
I_n(t) = \int_0^{+\infty} \frac{x^n}{t e^x + \kappa} dx,
\]

with \( \kappa = +1 \) for fermions and \( \kappa = -1 \) for bosons. For fermions, they are defined for \( t \geq 0 \). For bosons, they are defined for \( t \geq 1 \) when \( n > 0 \) and for \( t > 1 \) when \( -1 < n \leq 0 \). We have the identity

\[
I_n'(t) = -\frac{n}{t} I_{n-1}(t).
\]

The inverse fugacity \( \lambda \) is determined by the mass density \( \rho \) through the first relation of Eq. (B31). On the other hand, the two relations from Eq. (B31) determine the quantum equation of state \( P = P(\rho) \). Finally, the velocity dispersion in one direction is

\[
\sigma^2 = \frac{\langle v^2 \rangle}{3} = \frac{1}{3\rho} \int f v^2 d\nu = \frac{P}{\rho} = \frac{2}{3} \frac{k_B T}{m} \frac{I_{3/2}(\lambda)}{I_{1/2}(\lambda)}
\]

and the squared speed of sound is

\[
c_s^2 = P'(\rho) = \frac{2}{\beta m} \frac{I_{1/2}(\lambda)}{I_{-1/2}(\lambda)},
\]

where we have used Eq. (B33). The Jeans wavenumber for an infinite homogeneous self-gravitating gas of fermions or bosons (neglecting the quantum potential)\(^{55}\) is [104]

\[
k_J = \frac{\sqrt{4\pi G \rho}}{c_s}.
\]

\(^{55}\) See [32] for more general results taking into account the quantum potential.
The Jeans length \( \lambda_J = 2\pi/k_J \sim c_s t_D \) corresponds to the typical distance traveled by a sound wave at velocity \( c_s \) during the dynamical time \( t_D \sim (G\rho)^{-1/2} \).

**B8: Classical limit**

In the classical limit, the statistical equilibrium state is determined by the Boltzmann distribution

\[
f(v) = \frac{\eta_0}{\lambda} e^{-\beta m v^2/2}.
\]

The density and the pressure are given by

\[
\rho = \left(\frac{2\pi}{\beta m}\right)^{3/2} \frac{\eta_0}{\lambda}, \quad P = \frac{1}{2\pi} \left(\frac{2\pi}{\beta m}\right)^{5/2} \frac{\eta_0}{\lambda},
\]

leading to the classical (linear) equation of state

\[
P = \frac{\rho k_B T}{m},
\]

and to the velocity dispersion

\[
\sigma^2 = \frac{\langle v^2 \rangle}{3} = \frac{P}{\rho} = \frac{k_B T}{m}.
\]

The squared speed of sound is \( c_s^2 = k_B T/m \). The classical limit corresponds to \( T \to +\infty \) (implying \( \lambda \to +\infty \) according to Eq. (B38), \( \alpha \to -\infty \) and \( \mu \to -\infty \)). The transition between the classical regime and the quantum regime corresponds to \( \lambda \sim 1 \), \( f \sim \eta_0 \) (i.e. \( \chi \sim 1 \)) and \( T \sim T_Q \), where the quantum temperature \( T_Q \) is approximately determined by

\[
\frac{k_B T_Q}{m} \sim \left(\frac{\rho}{\eta_0}\right)^{2/3} \quad \text{i.e.} \quad T_Q \sim \frac{k_B^2 n^{2/3}}{m k_B}.
\]

Physically, the classical regime is valid when \( T \gg T_Q, \lambda \gg 1 \) and \( f \ll \eta_0 \) (i.e. \( \chi \ll 1 \)).

**Remark** The classical results can be recovered from the general formulae of “Appendix B7” by using the identity [41]

\[
I_n(t) \sim \frac{1}{t} \Gamma(n + 1), \quad (t \to +\infty).
\]

More precisely, for \( t \gg 1 \), we have the expansion

\[
I_n(t) = \Gamma(n + 1) \sum_{k=1}^{\infty} (-\kappa)^{k-1} \left(\frac{1}{t}\right)^k \frac{1}{k^{n+1}}.
\]

**B9: Bose–Einstein condensation**

The Bose–Einstein DF is given by

\[
f(v) = \frac{\eta_0}{\lambda e^{\beta m v^2/2} - 1} \quad (T \geq T_c)
\]

with \( \eta_0 = m^4/h^3 \) and \( \lambda = e^{-\alpha} = e^{-\beta \mu} \). The value of the DF for \( v = 0 \) is \( f(0) = \eta_0/(\lambda - 1) \) so we need to impose \( \lambda \geq 1 \) (i.e., \( \alpha \leq 0 \) and \( \mu \leq 0 \)) to have \( f \) positive for all \( v \).
condensation temperature $T_c$ corresponds to $\lambda = 1$ (i.e., $\alpha = \mu = 0$). Using Eq. (B31), we obtain

$$\frac{k_B T_c}{m} = \frac{1}{2\pi \xi (3/2)^{2/3}} \left( \frac{\rho}{\eta_0} \right)^{2/3} \quad \text{i.e.,} \quad T_c = \frac{2\pi \hbar^2 n_{1/3}}{mk_B \xi (3/2)^{2/3}},$$

(B45)

where we have used $I_{1/2}(1) = \frac{\sqrt{\pi}}{2} \zeta(3/2)$ with $\zeta(3/2) = 2.612 \ldots$. At the condensation point,

$$f(v) = \frac{\eta_0}{e^{\beta m \frac{v^2}{2}} - 1} \quad (T = T_c).$$

(B46)

For $v \to 0$, we get

$$f(v) \sim \frac{2\eta_0}{\beta m} v^{-2} \quad (T = T_c),$$

(B47)

which is similar to the Rayleigh–Jeans spectrum of radiation (see “Appendix C”). On the other hand, the pressure and the velocity dispersion at the condensation point are

$$P_c = \frac{(2\pi)^{3/2} \eta_0}{(\beta m) \xi (5/2)}, \quad \sigma_c^2 = \frac{P_c}{\rho} = \frac{\zeta(5/2) k_B T_c}{\zeta(3/2)} m,$$

(B48)

where we have used $I_{3/2}(1) = \frac{3\sqrt{\pi}}{4} \zeta(5/2)$ with $\zeta(5/2) = 1.341 \ldots$. The energy of condensation is $E_c = (3/2) M \sigma_c^2$. We note that $\sigma_c^2$ is of the order of $k_B T_c/m$. This is because, at the condensation point, $P_c \sim P_{\text{class}}$. For $T \to T_c^+$, using Eqs. (B52) and (B53), we have

$$T/T_c \simeq 1 + \frac{4\sqrt{\pi}}{3 \zeta(3/2)} \sqrt{\ln \lambda} \quad \text{(see Eq. (181))}$$

and $c_s^2 \sim \left( \frac{\zeta(3/2)}{\sqrt{\pi} \beta c m} \right) \sqrt{\ln \lambda} \quad \text{(see Eq. (B35))},$ leading to

$$c_s^2 \sim \frac{3 \zeta(3/2)^2}{4\pi} \frac{k_B T_c}{m} \left( \frac{T}{T_c} - 1 \right).$$

(B49)

The squared speed of sound vanishes at the critical temperature (this is also the case for the Jeans length from Eq. (B36)). When $T > T_c$, the bosons are uncondensed and their DF is given by Eq. (B44). When $T < T_c$, a fraction of the bosons is condensed in the state $v = 0$. The total DF is

$$f(v) = \frac{\eta_0}{e^{\beta m \frac{v^2}{2}} - 1} + \rho \left[ 1 - \left( \frac{T}{T_c} \right)^{3/2} \right] \delta(v) \quad (T \leq T_c).$$

(B50)

The first term corresponds to uncondensed bosons with $\lambda = 1$ and the second term corresponds to condensed bosons (the proportion of condensed bosons is obtained from the normalization condition $\rho = \int f \, dv$). At $T = 0$, all the bosons are condensed: $f = \rho \delta(v)$. We note that the velocity dispersion (or pressure) vanishes at $T = 0$.

The completely condensed limit corresponds to $T = 0$ and $\lambda = 1$ (implying $\alpha = \mu = 0$). Physically, the system is completely condensed when $T \ll T_c$, $\lambda = 1$ and $f \gg \eta_0$ (i.e., $\chi \gg 1$). The classical limit corresponds to $T \gg T_c$, $\lambda \gg 1$ and $f \ll \eta_0$ (i.e., $\chi \ll 1$). The transition between the completely condensed limit and the classical limit corresponds to $T \sim T_c$, $\lambda \sim 1$ and $f \sim \eta_0$ (i.e., $\chi \sim 1$).

Remark The de Broglie (thermal) wavelength of a particle is $\lambda_{\text{dB}} = \sqrt{2\pi \hbar^2/mk_B T}$. Particles become correlated with each other when their wavelengths overlap, that is, when the thermal wavelength is greater than the mean inter-particle distance $l = n^{-1/3}$. The condition $\lambda_{\text{dB}} > l$.
can be written \( n_{\text{dB}}^3 > 1 \) which yields the inequality \( T < T_c \) with \( T_c \approx 2 \pi \hbar^2 n^{2/3} / m k_B \) corresponding, up to a factor of order unity, to the condensation temperature from Eq. (B45).

Close to the condensation temperature, i.e., for \( t \to 1^+ \), we have the Robinson expansion

\[
I_n(t) = \Gamma(n + 1) \left[ \Gamma(-n)(\ln t)^n + \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} \xi(n + 1 - k)(\ln t)^k \right],
\]

which is valid for \( n < 0 \) and for non-integer \( n > 0 \). In particular,

\[
I_{-1/2}(t) \sim \frac{\Gamma(1/2)^2}{\sqrt{\ln t}},
\]

\[
I_{1/2}(t) = \Gamma(3/2)[\zeta(3/2) + \Gamma(-1/2)\sqrt{\ln t} + \cdots],
\]

\[
I_{3/2}(t) = \Gamma(5/2)[\zeta(5/2) + \Gamma(-3/2)(\ln t)^{3/2} + \cdots],
\]

with \( \Gamma(1/2) = \sqrt{\pi}, \Gamma(3/2) = \sqrt{\pi}/2, \Gamma(-1/2) = -2\sqrt{\pi}, \zeta(3/2) = 2.612 \ldots, \Gamma(5/2) = 3\sqrt{\pi}/4, \Gamma(-3/2) = 4\sqrt{\pi}/3 \) and \( \zeta(5/2) = 1.341 \ldots \)

B10: Completely degenerate fermions

The Fermi–Dirac DF is given by

\[
f(v) = \frac{\eta_0}{\lambda e^{\beta v_F^2 / \tau} + 1},
\]

with \( \eta_0 = g m^4 / h^3 \) and \( \lambda = e^{-\alpha} = e^{-\beta \mu} \). In the completely degenerate limit, the Fermi–Dirac DF reduces to the step function

\[
f(v) = \eta_0 H(v - v_F) \quad (T = 0),
\]

where \( v_F = (2\mu / m)^{1/2} \) is the Fermi velocity (the Fermi energy \( \epsilon_F = mv_F^2 / 2 = \mu \) is equal to the chemical potential) and \( H \) is the Heaviside function defined such that \( H(x) = 1 \) if \( x < 0 \) and \( H(x) = 0 \) if \( x > 0 \). The density and the pressure are given by

\[
\rho = \frac{4\pi}{3} \eta_0 v_F^3, \quad P = \frac{4\pi}{15} \eta_0 v_F^5.
\]

This leads to the polytropic equation of state

\[
P = \frac{1}{5} \left( \frac{3}{4\pi \eta_0} \right)^{2/3} \rho^{5/3},
\]

to the velocity dispersion

\[
\sigma^2 = \frac{\langle v^2 \rangle}{3} = \frac{P}{\rho} = \frac{1}{5} \left( \frac{3}{4\pi \eta_0} \right)^{2/3} \rho^{2/3} = \frac{v_F^2}{5},
\]

and to the squared speed of sound

\[
\epsilon_s^2 = P'(\rho) = \frac{1}{3} \left( \frac{3}{4\pi \eta_0} \right)^{2/3} \rho^{2/3} = \frac{v_F^2}{3}.
\]

We note that the pressure, the velocity dispersion and the speed of sound do not vanish at \( T = 0 \) as a consequence of the Pauli exclusion principle (this is also the case for the Jeans length from Eq. (B36)).
If we define the Fermi temperature \( T_F \) by the relation
\[
\frac{k_B T_F}{m} = \frac{1}{2} \left( \frac{3 \rho}{4 \pi \eta_0} \right)^{2/3} \quad \text{i.e.} \quad T_F = \frac{1}{2} \left( \frac{6 \pi^2}{g} \right)^{2/3} \frac{\hbar^2 n^{2/3}}{mk_B}.
\] (B61)

Physically, the Fermi temperature determines the transition between the completely degenerate limit and the nondegenerate limit:
\[
P_{\text{deg}} \sim P_{\text{class}} \Rightarrow \frac{1}{5} \left( \frac{3}{4 \pi \eta_0} \right)^{2/3} \rho^{5/3} \sim \rho \frac{k_B T}{m} \Rightarrow T \sim T_F.
\] (B62)

The completely degenerate limit corresponds to \( T = 0 \) and \( \mu > 0 \) finite (implying \( \alpha \rightarrow +\infty \) and \( \lambda \rightarrow 0 \)). Physically, it is valid when \( T \ll T_F, \lambda \ll 1 \) and \( f \sim \eta_0 \) (i.e., \( \chi \sim 1 \)). The nondegenerate (classical) limit corresponds to \( T \gg T_F, \lambda \gg 1 \) and \( f \ll \eta_0 \) (i.e., \( \chi \ll 1 \)). The transition between the completely degenerate limit and the nondegenerate limit corresponds to \( T \sim T_F, \lambda \sim 1 \) and \( f \sim \eta_0 \) (i.e., \( \chi \sim 1 \)).

Remark The completely degenerate limit can be recovered from the general formulae of “Appendix B7” by using the identity [41]
\[
I_n(t) \sim \frac{(-\ln t)^{n+1}}{n+1} (t \rightarrow 0).
\] (B63)

More precisely, for \( t \ll 1 \), we have the Sommerfeld expansion
\[
I_n(t) = \frac{(-\ln t)^{n+1}}{n+1} \left[ 1 + \sum_{k=1}^{+\infty} 2C_{2k-1}^n(n+1)(-\ln t)^{-2k} \Gamma(2k)\zeta(2k) \left( 1 - \frac{1}{2^{2k-1}} \right) \right].
\] (B64)

Appendix C: Kinetic theory of bosons in the approximation \( f/\eta \gg 1 \)

In this Appendix, we consider the case of bosons in the limit of large occupation numbers \( f/\eta \gg 1 \) (see “Appendix A”) where we can make the approximation
\[
f \left( 1 + \frac{f}{\eta} \right) \simeq \frac{f^2}{\eta}.
\] (C1)

C1: Log-entropy and Rayleigh–Jeans distribution

When \( f/\eta \gg 1 \), the Bose–Einstein entropy from Eq. (B12) reduces to
\[
S = k_B \sum_a \frac{g m_a^3}{\hbar^3} \int \ln \left( \frac{f_a}{\eta_a} \right) \, d\mathbf{r}d\mathbf{v}.
\] (C2)

In the context of generalized thermodynamics, it corresponds to a generalized entropy of the form \( S \sim \int \ln f \, d\mathbf{r}d\mathbf{v} \) that we called the log-entropy [88]. Extremizing this entropy at fixed energy and fixed mass of each species, we obtain the DF:
\[
f_a(v) = \frac{\eta_a}{\beta m_a \frac{v^2}{2} - \alpha_a}.
\] (C3)

This DF is positive provided that \( \alpha_a \leq 0 \). In that case, it reduces to the Cauchy or Lorentz distribution [88]. In the present context, we will call it the Rayleigh–Jeans distribution because
of its similarity with the Rayleigh–Jeans spectrum of radiation for a vanishing chemical potential \( \alpha = 0 \).\(^{56}\) The DF from Eq. (C3) is the limit form of the Bose–Einstein DF from Eq. (B44) when \( v \to 0 \) and \( \alpha_a \to 0^- \). Therefore, it becomes exact for small velocities close to the condensation temperature \( T_c \). In that case, we have \( f_a/\eta_a \gg 1 \).\(^{57}\) It has to be noted that the Rayleigh–Jeans DF decreases as \( v^{-2} \) for \( v \to +\infty \) so it is not normalizable. Therefore, the density \( \rho_b \) is not defined, except if we introduce a cutoff at \( V_{\text{max}} \).\(^{48}\) In that case, for a given density \( \rho_a \), the chemical potential is related to the temperature by the formula

\[
\rho_a = 4\pi \eta_a |\alpha_a|^{1/2} \left( \frac{2}{\beta m_a} \right)^{3/2} \left[ \frac{\beta m_a}{2|\alpha_a|} V_{\text{max}} - \tan^{-1} \left( \frac{\beta m_a}{2|\alpha_a|} V_{\text{max}} \right) \right]. \tag{C4}
\]

The condensation temperature \( T_c \), corresponding to \( \alpha_a = 0 \), is given by

\[
k_B T_c = \frac{m_a \rho_a}{8\pi \eta_a V_{\text{max}}}. \tag{C5}
\]

At that point, the DF from Eq. (C3) reduces to

\[
f_a(v) = \frac{2\eta_a}{\beta m_a v^2}. \tag{C6}
\]

On the other hand, Eq. (C4) can be rewritten as

\[
\frac{T_c}{T} = 1 - \frac{\tan^{-1}(X)}{X}, \quad \text{with } X = \sqrt{\frac{\beta m_a}{2|\alpha_a|} V_{\text{max}}}. \tag{C7}
\]

C2: Bosonic Landau equation

With the approximation from Eq. (C1), the bosonic Landau equation (107) becomes

\[
\frac{\partial f}{\partial t} = 2\pi G^2 \ln \Lambda \frac{\partial}{\partial v_i} \int d\nu' K_{ij} \left( \frac{m_b}{\eta_b} f_b^{\nu'2} \frac{\partial f}{\partial v_j} - \frac{m}{\eta_0} f^{\nu 2} \frac{\partial f}{\partial v'_j} \right). \tag{C8}
\]

In the single-species case, we obtain the self-consistent kinetic equation

\[
\frac{\partial f}{\partial t} = 2\pi G^2 \frac{m}{\eta_0} \ln \Lambda \frac{\partial}{\partial v_i} \int d\nu' K_{ij} \left( f'^{\nu'2} \frac{\partial f}{\partial v_j} - f^{\nu 2} \frac{\partial f'}{\partial v'_j} \right), \tag{C9}
\]

which can be used as an approximation of Eq. (223) to study the process of Bose–Einstein condensation when \( E < E_c \) (see Sect. 7.1).

C3: Bosonic Fokker–Planck equation

The bosonic Landau equation (C8) can be written under the form of a bosonic Fokker–Planck equation

\[
\frac{\partial f}{\partial t} = \frac{\partial}{\partial v_i} \left( D_{ij} \frac{\partial f}{\partial v_j} - f^{\nu 2} \frac{F_{ij}^{\text{pol}}}{\eta_0} \right) \tag{C10}
\]

\(^{56}\) Note that the Rayleigh–Jeans spectrum of radiation corresponds to ultrarelativistic bosons (photons) while we consider here nonrelativistic bosons.

\(^{57}\) If we consider the core of the Bose–Einstein DF, we have \( f(0) = \eta_0/(\lambda - 1) \). Therefore, the limit \( f(0)/\eta = 1/(\lambda - 1) \gg 1 \) is valid for \( \lambda \to 1 \) hence for \( T \to T_c \).
with a diffusion tensor
\[ D_{ij} = 2\pi G^2 m_b \ln \Lambda \int d\mathbf{v}' K_{ij} \frac{f_{b}^{v'}}{\eta_b} \]  
(C11)
and a friction by polarization
\[ F_{i}^{\text{pol}} = 2\pi G^2 m \ln \Lambda \int d\mathbf{v}' K_{ij} \frac{\partial f_{b}^{v'}}{\partial v'_j}. \]  
(C12)

The relation between the friction by polarization and the true friction is still given by Eq. (113). The results from Secs. 4.3 and 4.4 remain valid with the approximation from Eq. (C1).

C4: Thermal bath

We now assume that the field particles are in a statistical equilibrium state described by the Rayleigh–Jeans DF
\[ f_b(v) = \frac{\eta_b}{\beta m_b v^2 + |\alpha_b|}. \]  
(C13)

We can check that Eq. (131) remains valid so the bosonic Kramers–Chandrasekhar equation takes the form
\[ \frac{\partial f}{\partial t} = \frac{\partial}{\partial v_i} \left[ D_{ij} \left( \frac{\partial f}{\partial v_j} + \beta \frac{m}{\eta_0} f^2 v_j \right) \right]. \]  
(C14)

This equation relaxes towards the Rayleigh–Jeans DF
\[ f_{\text{eq}}(v) = \frac{\eta_0}{\beta m \frac{v^2}{2} + |\alpha|}. \]  
(C15)

At statistical equilibrium, the test particles and the field particles have the same temperature.

C5: Diffusion and friction terms in the thermal bath approximation

For a thermal bath [see Eq. (C13)], the diffusion coefficients of the test particles obtained from Eqs. (136)–(140) with the approximation from Eq. (C1) are given by
\[ D_{\parallel} = \frac{16\pi^2}{3} G^2 m_b \ln \Lambda \frac{1}{v^3} (L_4 + v^3 M_1) \]  
(C16)
and
\[ D_{\perp} = \frac{16\pi^2}{3} G^2 m_b \ln \Lambda \frac{1}{v} \left( 3L_2 - \frac{1}{v^2} L_4 + 2v M_1 \right) \]  
(C17)
with
\[ M_1 = \frac{1}{\eta_b} \int_{v_1}^{\infty} v_1^2 f_b(v_1)^2 dv_1 = \frac{\eta_b}{\beta m_b |\alpha_b| + \beta m_b \frac{v^2}{2}}, \]  
(C18)
\[ L_2 = \frac{1}{\eta_b} \int_0^v v^2 f_b(v)^2 dv_1 = -\frac{\eta_b}{\beta m_b |\alpha_b| + \beta m_b \frac{v^2}{2}} + \frac{\eta_b}{\beta m_b} \left( \frac{2}{\beta m_b |\alpha_b|} \right)^{1/2} \tan^{-1} \left[ \left( \frac{\beta m_b}{2|\alpha_b|} \right)^{1/2} v \right], \]  
(C19)
\[ L_4 = \frac{1}{\eta_b} \int_0^v v^4 f_b(v)^2 dv_1 = -\frac{\eta_b}{\beta m_b |\alpha_b| + \beta m_b \frac{v^2}{2}} + \frac{6\eta_b}{(\beta m_b)^2} \left( v - \left( \frac{2|\alpha_b|}{\beta m_b} \right)^{1/2} \tan^{-1} \left[ \left( \frac{\beta m_b}{2|\alpha_b|} \right)^{1/2} \right] \right). \]  
(C20)
This leads to the explicit expressions

\[
D_\parallel = 32\pi^2 G^2 m_b \ln \Lambda \frac{1}{v^3} \frac{\eta_b}{(\beta m_b)^2} \left\{ v - \left( \frac{2|\alpha_b|}{\beta m_b} \right)^{1/2} \tan^{-1} \left( \frac{\beta m_b}{2|\alpha_b|} \right)^{1/2} v \right\}
\]

(C21)

and

\[
D_\perp = 16\pi^2 G^2 \ln \Lambda \frac{\eta_b}{v^3} \left\{ \frac{2|\alpha_b|}{\beta m_b} \right\}^{1/2} \tan^{-1} \left( \frac{\beta m_b}{2|\alpha_b|} \right)^{1/2} v \right\}
\]

\[- \frac{2}{\beta m_b v^2} \left\{ v - \left( \frac{2|\alpha_b|}{\beta m_b} \right)^{1/2} \tan^{-1} \left( \frac{\beta m_b}{2|\alpha_b|} \right)^{1/2} v \right\} \}
\]

(C22)

Despite the fact that the density \( \rho_b \) is not defined (the Rayleigh–Jeans DF is not normalizable), the diffusion coefficients are perfectly well-defined. We note that \( D_\parallel \) decreases as \( v^{-2} \), instead of \( v^{-3} \) in Eq. (143), when \( v \to +\infty \). Using Eqs. (124)–(126) and (132), we also obtain

\[
F_{\text{pol}} = -32\pi^2 G^2 m_b \ln \Lambda \frac{1}{v^3} \frac{\eta_b}{(\beta m_b)^2} \left\{ v - \left( \frac{2|\alpha_b|}{\beta m_b} \right)^{1/2} \tan^{-1} \left( \frac{\beta m_b}{2|\alpha_b|} \right)^{1/2} v \right\} v.
\]

(C23)

\[
\frac{\partial D_{ij}}{\partial v_j} = -16\pi^2 G^2 m_b \ln \Lambda \frac{v_i}{v^3} \frac{\eta_b}{\beta m_b} \left\{ \frac{v_i}{|\alpha_b| + \beta m_b v^2} + \left( \frac{2}{\beta m_b |\alpha_b|} \right)^{1/2} \tan^{-1} \left( \frac{\beta m_b}{2|\alpha_b|} \right)^{1/2} v \right\}.
\]

(C24)

\[
F_{\text{friction}} = F_{\text{pol}} + \frac{\partial D_{ij}}{\partial v_j}.
\]

(C25)

At the critical temperature \( T_c \), corresponding to \( \alpha_b \to 0 \), the diffusion coefficient in the direction parallel to the velocity of the test particle converges and achieves the expression

\[
D_\parallel = 32\pi^2 G^2 m_b \ln \Lambda \frac{1}{v^2} \frac{\eta_b}{(\beta_c m_b)^2}.
\]

(C26)

while the diffusion coefficient in the direction perpendicular to the velocity of the test particle diverges like

\[
D_\perp \sim 8\pi^3 G^2 \ln \Lambda \frac{\eta_b}{\beta_c v} \left( \frac{2}{\beta_c m_b |\alpha_b|} \right)^{1/2}.
\]

(C27)

Similarly, the friction by polarization converges at \( T_c \) while the total friction diverges (this is also the case when the bosons are described by the exact Bose–Einstein DF). One can also define an effective mass of the particles. From Eqs. (175) and (C13), we obtain

\[
m_{\text{eff}} = \frac{2\pi^2 \eta_b}{\beta^{3/2} \rho_b} \left( \frac{2}{m_b |\alpha_b|} \right)^{1/2}.
\]

(C28)

We can then express \( D_\perp \) close to \( T_c \) under the form

\[
D_\perp \sim 4\pi G^2 \ln \Lambda \frac{\rho_b}{v} m_{\text{eff}}.
\]

(C29)

Using Eq. (C7), we see that the effective mass diverges like

\[
m_{\text{eff}} \sim \frac{\pi^2}{4} \frac{1}{\beta_c V_{\text{max}}^2} \left( \frac{T}{T_c} - 1 \right)^{-1}
\]

(C30)

\[58\] Note that the integral in the numerator of Eq. (175) converges while the integral in the denominator diverges in the absence of a cutoff \( V_{\text{max}} \).
when $T \to T_c$.

**Remark** The Rayleigh–Jeans (Lorentz or Cauchy) DF is a rare case, with the Maxwellian, where the diffusion and friction coefficients can be calculated analytically (for a thermal bath).

### Appendix D: Rate of change of the energy of classical particles

In this Appendix we compute the rate of change of the energy of a classical system of particles of mass $m$ (test particles) due to collisions with classical particles of mass $m_b$ (field particles).\(^{59}\) We use this result to estimate the relaxation time of the system towards statistical equilibrium (thermalization). We also study the evolution of the velocity dispersion of the test particles due to heating and cooling (see Appendix D of [109] for more general results).

The (kinetic) energy of the test particles is

$$E = \int f \frac{v^2}{2} dv. \quad (D1)$$

Using the Fokker–Planck equation (35) and performing integrations by parts, we find that its rate of change is

$$\dot{E} = \int \frac{\partial f}{\partial t} \frac{v^2}{2} dv = \int \frac{\partial}{\partial v_i} \left( D_{ij} \frac{\partial f}{\partial v_j} - f F_{i}^{\text{pol}} \right) \frac{v^2}{2} dv = -\int \left( D_{ij} \frac{\partial f}{\partial v_j} - f F_{i}^{\text{pol}} \right) v_i dv$$

$$= -\int D_{ij} v_i \frac{\partial f}{\partial v_j} dv + \int f (F_{\text{pol}} \cdot v) dv. \quad (D2)$$

The mean change in energy arises from the competition between diffusion (heating) and dynamical friction (cooling). We assume that, initially, the test particles have a Maxwellian DF

$$f_0(v) = \rho \left( \frac{\beta_0 m}{2\pi} \right)^{3/2} e^{-\beta_0 m v^2/2} \quad (D3)$$

with a temperature $T_0$. Using

$$\frac{\partial f_0}{\partial v} = -\beta_0 m f_0 v, \quad (D4)$$

we find that the initial rate of change of the energy of the test particles is given by

$$(\dot{E})_0 = \beta_0 m \int D_{ij} v_i v_j f_0 dv + \int f_0 (F_{\text{pol}} \cdot v) dv. \quad (D5)$$

\(^{59}\) We note that the collisions of the test particles between themselves do not change their energy. We also note that the total energy (test + field particles) is conserved if the two populations evolve self-consistently (see “Appendix B”).
If the DF of the field particles is isotropic, using the identity \( D_{ij} v_i v_j = D \parallel v^2 \) [see Eq. (54)], we obtain
\[
(\dot{E})_0 = \beta_0 m \int D \parallel v^2 f_0 \, dv + \int f_0 (F_{\text{pol}} \cdot \mathbf{v}) \, dv. \tag{D6}
\]

If the field particles are at statistical equilibrium with a temperature \( T \) (thermal bath) then, using Eq. (66), we get
\[
F_{\text{pol}} \cdot \mathbf{v} = -\beta m D \parallel v^2. \tag{D7}
\]

In that case, the initial rate of change of the energy of the test particles can be written as
\[
(\dot{E})_0 = (\beta_0 - \beta) m \int D \parallel v^2 f_0 \, dv. \tag{D8}
\]

From this formula, we see that \((\dot{E})_0 < 0\) if \( T_0 > T \) and \((\dot{E})_0 > 0\) if \( T_0 < T \). In the first case, the test particles have to lose energy in order to acquire the bath temperature \( T \). In the second case they have to gain energy. Using Eqs. (76) and (D3), we have
\[
\int f_0 D \parallel v^2 \, dv = \rho \left( \frac{\beta_0 m}{2\pi} \right)^{3/2} 4\pi G^2 \rho_b \frac{4\pi}{\beta m_b} I \left( \frac{m T}{m_b T_0} \right), \tag{D9}
\]

where \( I(a) \) is the integral
\[
I(a) = \int_0^{+\infty} e^{-ax^2} G(x) x^3 \, dx. \tag{D10}
\]

Recalling the definition of \( G(x) \) from Eq. (82) and integrating by part, we get
\[
I(a) = \frac{1}{a \sqrt{\pi}} \int_0^{+\infty} e^{-(1+a)x^2} x^2 \, dx. \tag{D11}
\]

With the change of variables \( y = \sqrt{1+a} \, x \), this integral can be rewritten as
\[
I(a) = \frac{1}{a \sqrt{\pi} (1+a)^{3/2}} \int_0^{+\infty} e^{-y^2} y^2 \, dy. \tag{D12}
\]

Using the identity
\[
\int_0^{+\infty} e^{-y^2} y^2 \, dy = \frac{\sqrt{\pi}}{4}, \tag{D13}
\]

we finally obtain
\[
I(a) = \frac{1}{4a(1+a)^{3/2}}. \tag{D14}
\]

Collecting the foregoing results, we find that
\[
\int f_0 D \parallel v^2 \, dv = 4\sqrt{2\pi} \rho_b G^2 \ln \Lambda \rho B T \frac{k_B T_0}{m} \frac{1}{\left( \frac{k_B T_0}{m} + \frac{k_B T}{m_b} \right)^{3/2}}. \tag{D15}
\]

Therefore, the initial rate of change of the energy of the test particles is
\[
(\dot{E})_0 = 4\sqrt{2\pi} \rho_b G^2 \ln \Lambda \rho B \frac{T - T_0}{\left( \frac{k_B T_0}{m} + \frac{k_B T}{m_b} \right)^{3/2}}. \tag{D16}
\]

\[60\] We can obtain this equation directly from the Kramers–Chandrasekhar equation (70).
This equation is exact at $t = 0$ under the preceding assumptions. Now, assuming that the DF of the test particles can be approximated at any time by a Maxwellian distribution with temperature $T_t(t)$, and using the relation

$$E(t) = \frac{3}{2} \rho \frac{k_B T_t(t)}{m}, \quad (D17)$$

we can extend Eq. (D16) to any time and write

$$\frac{d T_t}{dt} = \frac{8}{3} \sqrt{\frac{2\pi \rho b G^2 m}{\Lambda}} \ln \frac{T - T_t}{(\frac{k_B T_t}{m} + \frac{k_B T}{m_b})^{3/2}}. \quad (D18)$$

This corresponds to the result of Spitzer [7, 98, 110] (see also Landau [13] and Eq. (2.377) of Chandrasekhar [1]) obtained in a different manner. This equation shows that the temperature of the test particles relaxes towards the temperature of the bath ($T_t = T$). For short times, we have

$$T_t(t) = T_0 + \frac{8}{3} \sqrt{\frac{2\pi \rho b G^2 m}{\Lambda}} \ln \frac{T - T_0}{(\frac{k_B T_0}{m} + \frac{k_B T}{m_b})^{3/2}} t + \cdots \quad (D19)$$

For late times, we have $T_t \simeq T$ (thermalization), and we can write Eq. (D18) as

$$\frac{d T_t}{dt} = \frac{T - T_t}{t_{\text{relax}}}, \quad (D20)$$

where, following Spitzer [110, 111], we have introduced the relaxation (or equipartition) time

$$t_{\text{relax}} = \frac{3}{8\sqrt{2\pi \rho b G^2 m}\ln \Lambda} \left(1 + \frac{m_{\text{b}}}{m}\right)^{3/2}. \quad (D21)$$

The solution of Eq. (D20) is

$$T_t(t) = T + (T_0 - T)e^{-t/t_{\text{relax}}}. \quad (D22)$$

Close to equilibrium, the temperature of the test particles relaxes exponentially rapidly towards the temperature of the bath on a time scale $t_{\text{relax}}$. If we introduce the velocity dispersion (in one direction) of the test and field particles defined by

$$\sigma_t^2 = \frac{k_B T_t}{m} = \frac{1}{\beta_t m} = \frac{1}{3\rho} \int f v^2 d\mathbf{v} \quad \text{and} \quad \sigma_b^2 = \frac{k_B T}{m_b} = \frac{1}{\beta_b m_b} = \frac{1}{3\rho_b} \int f_b v^2 d\mathbf{v}, \quad (D23)$$

we can rewrite Eq. (D18) as

$$\frac{d \sigma_t^2}{dt} = \frac{8}{3} \sqrt{2\pi \rho b G^2 \ln \Lambda} \frac{m_{\text{b}} \sigma_b^2}{(\sigma_t^2 + \sigma_b^2)^{3/2}}. \quad (D24)$$

At equilibrium, we have equipartition of energy:

$$T_t = T \quad \text{and} \quad m \sigma_t^2 = m_b \sigma_b^2. \quad (D25)$$

---

61 The DF which is the solution of the Fokker–Planck equation (70) is not exactly Maxwellian (see “Appendix F”) but we may expect that a Maxwellian ansatz provides a reasonable approximation of the exact DF to evaluate $\dot{E}$. 

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On the other hand, the relaxation time from Eq. (D21) can be written as

\[
 t_{\text{relax}} = \frac{3\sigma_b^3}{8\sqrt{2\pi \rho_b G^2 m \ln \Lambda}} \left(1 + \frac{m_b}{m}\right)^{3/2}.
\]  

(D26)

For a single species system \((m_b = m)\) it reduces to\(^{62}\)

\[
 t_{\text{relax}} = \frac{3\sigma^3}{4\sqrt{\pi \rho G^2 m \ln \Lambda}}.
\]  

(D27)

We stress that Eqs. (D20), (D21) and (D26) are valid only close to the equilibrium state so they may not accurately characterize the whole relaxation process. To obtain a better description of the relaxation process we have to come back to Eq. (D18) or, equivalently Eq. (D24), as discussed in the following sections.

D1: General case

In the general case, Eq. (D24) can be written as

\[
 \frac{d\sigma_i^2}{dt} = \frac{1}{t_{\text{cool}} - \sigma_i^2 t_{\text{cool}}} \left( \frac{\sigma_i^2}{t_{\text{heat}}} - \frac{\sigma_i^2}{t_{\text{cool}}} \right) \frac{1}{1 + \frac{\sigma_i^2}{\sigma_b^2}}^{3/2},
\]  

(D28)

where, following Spitzer \(^{63}\), we have introduced the heating time

\[
 t_{\text{heat}} = \frac{3\sigma_b^3}{8\sqrt{2\pi \rho_b G^2 m_b \ln \Lambda}}
\]  

(D29)

and the cooling time

\[
 t_{\text{cool}} = \frac{3\sigma_b^3}{8\sqrt{2\pi \rho_b G^2 m \ln \Lambda}}.
\]  

(D30)

They satisfy the relations

\[
 \frac{t_{\text{cool}}}{t_{\text{heat}}} = \frac{m_b}{m} \quad \text{and} \quad \frac{t_{\text{relax}}}{t_{\text{cool}}} = \left(1 + \frac{m_b}{m}\right)^{3/2}.
\]  

(D31)

We can also write Eq. (D28) as

\[
 \frac{d\sigma_i^2}{dt} = \frac{1}{t_{\text{cool}}} \left( \frac{m_b}{m} \sigma_i^2 - \sigma_i^2 \right) \frac{1}{1 + \frac{\sigma_i^2}{\sigma_b^2}}^{3/2}.
\]  

(D32)

\(^{62}\) This result may look paradoxical at first sight since it is based on the rate of change of energy [see Eq. (D16)] while the energy of a single species system of particles is conserved. Actually, it gives the relaxation time of a test particle (or an ensemble of noninteracting test particles) of mass \(m\) in collision with field particles of the same mass \(m\). The test particles are initially out-of-equilibrium while the field particles are at statistical equilibrium (this is how they can be distinguished). It also provides the typical relaxation time of the system “as a whole” evolving according to the self-consistent Landau equation (2). In that case, it can be obtained qualitatively from scaling arguments (see Sect. 2.1).

\(^{63}\) These characteristic times are implicit in the work of Spitzer [7] (we note that this paper, which includes the effects of diffusion and dissipation was written before the seminal paper of Chandrasekhar [15] on dynamical friction). They have also been introduced by Bar-Or et al. [74] in the case of FDM (see Sect. 6.4 and “Appendix E”).
Setting \( x = \frac{\sigma_t^2}{\sigma_b^2} \) and \( a = m_b / m \), the solution of this equation can be written as [7]

\[
\int \frac{(1 + x)^{3/2}}{x-a} \, dx = -\frac{t}{t_{\text{cool}}},
\]  
(D33)

where the integral is explicitly given by

\[
\int \frac{(1 + x)^{3/2}}{x-a} \, dx = \frac{2}{3} \sqrt{1 + x} (4 + 3a + x) - 2(1 + a)^{3/2} \tanh^{-1} \left( \sqrt{\frac{1 + x}{1 + a}} \right) + C.
\]  
(D34)

Let us consider particular limits of this equation.

(i) When \( \sigma_t \gg \sigma_b \) \( (x \gg 1) \), Eq. (D33) reduces to

\[
\int \frac{x^{3/2}}{x-a} \, dx = -\frac{t}{t_{\text{cool}}},
\]  
(D35)

and the integral is explicitly given by

\[
\int \frac{x^{3/2}}{x-a} \, dx = 2a \sqrt{x} + \frac{2}{3} x^{3/2} - 2a^{3/2} \tanh^{-1} \left( \sqrt{\frac{x}{a}} \right) + C.
\]  
(D36)

We recall that, at equilibrium, \( (\sigma_t^2/\sigma_b^2)_{\text{eq}} = m_b/m \). Therefore, if we want the condition \( \sigma_t \gg \sigma_b \) to be valid at any time, we have to assume that \( m_b \gg m \). In that case, \( t_{\text{cool}} \simeq (m/m_b)^{3/2}t_{\text{relax}} \gg t_{\text{heat}} \).

(ii) When \( \sigma_t \ll \sigma_b \) \( (x \ll 1) \), Eq. (D32) reduces to

\[
\frac{d\sigma_t^2}{dt} = \frac{1}{t_{\text{cool}}} \left( \frac{m_b}{m} \sigma_b^2 - \sigma_t^2 \right),
\]  
(D37)

and its solution is [7]

\[
\sigma_t^2(t) = \frac{m_b}{m} \sigma_b^2 + \left[ \sigma_t^2(0) - \frac{m_b}{m} \sigma_b^2 \right] e^{-t/t_{\text{cool}}},
\]  
(D38)

At equilibrium, \( (\sigma_t^2/\sigma_b^2)_{\text{eq}} = m_b/m \). Therefore, if we want the condition \( \sigma_t \ll \sigma_b \) to be valid at any time, we have to assume that \( m_b \ll m \). In that case, \( t_{\text{cool}} \simeq t_{\text{relax}} \ll t_{\text{heat}} \).

(iii) For a single species system \( (m_b = m) \), Eq. (D32) reduces to (see footnote 62)

\[
\frac{d\sigma_t^2}{dt} = \frac{1}{t_{\text{cool}}} \left( \sigma_b^2 - \sigma_t^2 \right) \left( 1 + \frac{\sigma_t^2}{\sigma_b^2} \right)^{3/2}
\]  
(D39)

with \( t_{\text{cool}} = t_{\text{heat}} = t_{\text{relax}}/2^{3/2} \). At equilibrium \( \sigma_t = \sigma_b \). Setting \( x = \sigma_t^2/\sigma_b^2 \), the solution of this equation can be written as

\[
\int \frac{(1 + x)^{3/2}}{x-1} \, dx = -\frac{t}{t_{\text{cool}}},
\]  
(D40)

where the integral is explicitly given by

\[
\int \frac{(1 + x)^{3/2}}{x-1} \, dx = \frac{2}{3} \sqrt{1 + x} (7 + x) - 4 \sqrt{2} \tanh^{-1} \left( \sqrt{\frac{x}{2}} \right) + C.
\]  
(D41)
When $\sigma_t \gg \sigma_b$ we get

$$\frac{\sigma_t^2}{\sigma_b^2} = \left[ \frac{\sigma_t(0)^3}{\sigma_b^3} - \frac{3}{2} \frac{t}{t_{\text{cool}}} \right]^{2/3}. \quad (D42)$$

When $\sigma_t \ll \sigma_b$ we obtain

$$\frac{\sigma_t^2}{\sigma_b^2} = 1 + \left[ \frac{\sigma_t(0)^2}{\sigma_b^2} - 1 \right] e^{-t/t_{\text{cool}}}. \quad (D43)$$

These two approximate expressions are only valid for sufficiently short times since $\sigma_t = \sigma_b$ at equilibrium.

D2: $m_b \gg m$: diffusion only (heating)

When $m_b \gg m$, we have $t_{\text{cool}} \gg t_{\text{heat}}$. This is the situation studied by Spitzer and Schwarzschild [97] (see also “Appendix F”). In that case, the friction can be neglected and the evolution of the test particles is purely diffusive (on the timescale $t_{\text{heat}}$). Eqs. (D24) and (D28) reduce to

$$\frac{d\sigma_t^2}{dt} = \frac{8}{3} \sqrt{\frac{2}{\pi}} \rho_b G^2 m_b \ln \Lambda \frac{\sigma_b^2}{(\sigma_t^2 + \sigma_b^2)^{3/2}} \quad (D44)$$

and

$$\frac{d\sigma_t^2}{dt} = \frac{\sigma_b^2}{t_{\text{heat}}} \frac{1}{\left( 1 + \frac{\sigma_t^2}{\sigma_b^2} \right)^{3/2}}. \quad (D45)$$

The solution of Eq. (D45) is [7,97,112]

$$\frac{\sigma_t^2(t)}{\sigma_b^2} = \left[ \left( 1 + \frac{\sigma_t^2(0)}{\sigma_b^2} \right)^{5/2} + \frac{5}{2} \frac{t}{t_{\text{heat}}} \right]^{2/5} - 1. \quad (D46)$$

For $t \to +\infty$ (formally), we get $\sigma_t^2(t)/\sigma_b^2 \sim (5t/2t_{\text{heat}})^{2/5}$. If we take $\sigma_t(0) = \sigma_b$ we find that $\sigma_t/\sigma_b = 1.15$ at $t = t_{\text{heat}}$. On the other hand, $\sigma_t/\sigma_b = 2$ at $t = 20.1 t_{\text{heat}}$.

D3: $m \gg m_b$: friction only (cooling)

When $m \gg m_b$, we have $t_{\text{heat}} \gg t_{\text{cool}}$. In that case, the diffusion can be neglected and the test particles just feel the effect of the friction (on the timescale $t_{\text{cool}}$). Eqs. (D24) and (D28) reduce to

$$\frac{d\sigma_t^2}{dt} = -\frac{8}{3} \sqrt{\frac{2}{\pi}} \rho_b G^2 m_b \ln \Lambda \frac{\sigma_t^2}{(\sigma_t^2 + \sigma_b^2)^{3/2}} \quad (D47)$$

and

$$\frac{d\sigma_t^2}{dt} = -\frac{\sigma_t^2}{t_{\text{cool}}} \frac{1}{\left( 1 + \frac{\sigma_t^2}{\sigma_b^2} \right)^{3/2}}. \quad (D48)$$
Setting \( x = \sigma_t^2 / \sigma_b^2 \), the solution of Eq. (D48) can be written as
\[
\int (1 + x)^{3/2} \frac{dx}{x} = -\frac{t}{t_{\text{cool}}},
\] (D49)
where the integral is explicitly given by
\[
\int (1 + x)^{3/2} \frac{dx}{x} = \frac{2}{3} \sqrt{1 + x} (4 + x) + \ln \left( \frac{1 - \sqrt{1 + x}}{1 + \sqrt{1 + x}} \right) + C.
\] (D50)
For \( t \to +\infty \) (formally), we get \( \sigma_t^2(t) \propto e^{-t / t_{\text{cool}}} \).

### Appendix E: Rate of change of the energy of classical particles in FDM halos

We consider classical particles of mass \( m \) in collision with bosons of mass \( m_b \) – or quasi-particles of effective mass \( m_{\text{eff}} \) – in FDM halos (see Sect. 6.4). The Fokker–Planck equation (162) has the same form as Eq. (35) so the results (D1)–(D6) of Appendix D remain valid.

The initial rate of change of the energy of the test particles is
\[
(\dot{E})_0 = \beta_0 m \int D_{||} v^2 f_0 \, dv + \int f_0 (F_{\text{pol}} \cdot \mathbf{v}) \, dv.
\] (E1)
It is the sum of two terms, heating and cooling. To evaluate these terms, we make the same approximations as in Sect. 6.4. We assume that the DF of the field particles (bosons) is approximately Maxwellian with a velocity dispersion \( \sigma_b^2 \ll \sigma_c^2 \) [see Eq. (200)] and we make the approximation \( f_b(1 + f_b / \eta_b) \simeq f_b^2 / \eta_b \) [see Eq. (197)] in the diffusion tensor from Eq. (163). When the corresponding expressions of \( D_{||} \) and \( F_{\text{pol}} \) are substituted into Eq. (E1), we recover the results of Bar-Or et al. [74] obtained in a slightly different manner.

#### E1: Heating

We first consider the rate of energy due to heating:
\[
(\dot{E})_0^{\text{heat}} = \beta_0 m \int D_{||} v^2 f_0 \, dv.
\] (E2)
We have to compute the diffusion coefficient \( D_{||} \) of the test particles due to the collisions with the bosons. As explained in Sect. 6.4, the diffusion coefficient is the same as the one created by classical particles of effective mass \( m_{\text{eff}} \) and Maxwellian DF with a velocity dispersion \( \sigma_{\text{eff}}^2 / 2 \) [74]. Replacing \( m_b \) by \( m_{\text{eff}} \) and \( \sigma_b^2 \) by \( \sigma_{\text{eff}}^2 / 2 \) in Eq. (D44), we obtain
\[
\frac{d\sigma_t^2}{dt} = \frac{4}{3} \sqrt{2\pi} G^2 m_{\text{eff}} \ln \Lambda \rho_b \sigma_b^2 \frac{1}{(\sigma_t^2 + \frac{1}{2} \sigma_b^2)^{3/2}}.
\] (E3)
This equation can be rewritten as
\[
\frac{d\sigma_t^2}{dt} = \frac{\sigma_b^2}{t_{\text{heat}}} \frac{1}{(1 + \frac{2\sigma_t^2}{\sigma_b^2})^{3/2}},
\] (E4)
where we have introduced the heating time [74]
\[
t_{\text{heat}} = \frac{3\sigma_b^3}{16 \sqrt{\pi} G^2 \rho_b m_{\text{eff}} \ln \Lambda}.
\] (E5)
This expression differs from the classical heating time \( (D29) \) by the fact that \( m_b \) is replaced by \( m_{\text{eff}} \). Since \( m_{\text{eff}} \gg m_b \), the heating time due to collisions with bosons of mass \( m_b \) — or quasiparticles of effective mass \( m_{\text{eff}} \) — is much shorter than the heating time due to collisions with classical particles of mass \( m_b \) as in the CDM model. This is due to Bose enhancement.

Using Eq. (203), we can rewrite Eq. (E5) as

\[
t_{\text{heat}} = \frac{3m_b^3\sigma_b^6}{16\pi^2G^2\rho_b^3h^3\ln \Lambda}. \tag{E6}
\]

The solution of Eq. (E4) is

\[
\frac{\sigma_t^2(t)}{\sigma_b^2} = \frac{1}{2} \left[ \left( 1 + 2\frac{\sigma_t^2(0)}{\sigma_b^2} \right)^{5/2} + 5\frac{t}{t_{\text{heat}}} \right]^{-2/5} - \frac{1}{2}, \tag{E7}
\]

which may be compared with Eq. (D46). For \( t \to +\infty \) (formally), we get \( \sigma_t^2(t)/\sigma_b^2 \sim \frac{1}{2}(5t/t_{\text{heat}})^{2/5} \). If we take \( \sigma_t(0) = \sigma_b \) we find that \( \sigma_t/\sigma_b = 1.08 \) at \( t = t_{\text{heat}} \). On the other hand, \( \sigma_t/\sigma_b = 2 \) at \( t = 45.5t_{\text{heat}} \).

E2: Cooling

We now consider the rate of energy due to cooling:

\[
(\dot{E})_0^{\text{cool}} = \int f_0(F_{\text{pol}} \cdot \mathbf{v}) \, d\mathbf{v}. \tag{E8}
\]

We have to compute the friction by polarization \( F_{\text{pol}} \) experienced by the test particles due to the collisions with the bosons. As explained in Sect. 6.4, the friction by polarization is the same as the one created by classical particles of mass \( m_b \) and Maxwellian DF with a velocity dispersion \( \sigma_b^2 \). Therefore, Eq. (D47) remains unchanged and we obtain

\[
\frac{d\sigma_t^2}{dt} = -\frac{8}{3} \frac{\sqrt{2}\pi G^2 m}{\ln \Lambda \rho_b \sigma_t^2} \frac{1}{(\sigma_t^2 + \sigma_b^2)^{3/2}}. \tag{E9}
\]

This equation can be rewritten as

\[
\frac{d\sigma_t^2}{dt} = -\frac{\sigma_t^2}{t_{\text{cool}}} \frac{1}{\left( 1 + \frac{\sigma_t^2}{\sigma_b^2} \right)^{3/2}}, \tag{E10}
\]

where we have introduced the cooling time \([74]\)

\[
t_{\text{cool}} = \frac{3\sigma_b^3}{8\sqrt{2}\pi G^2 \rho_b m \ln \Lambda}. \tag{E11}
\]

It corresponds to the classical cooling time \( (D30) \). Therefore, the cooling time due to collisions with bosons of mass \( m_b \) is the same as the cooling time due to collisions with classical particles of mass \( m_b \) as in the CDM model (it is independent of the mass of the field particles) except for a change in the Coulomb logarithm \([74]\). The solution of Eq. (E10) is given by Eqs. (D49) and (D50).
E3: General case

If we account simultaneously for the processes of heating and cooling, and combine Eqs. (E4) and (E10), we obtain the following equation

$$
\frac{d\sigma_i^2}{dt} = \frac{\sigma_b^2}{t_{\text{heat}}} \cdot \frac{1}{\left(1 + \frac{2\sigma_i^2}{\sigma_b^2}\right)^{3/2}} - \frac{\sigma_i^2}{t_{\text{cool}}} \cdot \frac{1}{\left(1 + \frac{\sigma_i^2}{\sigma_b^2}\right)^{3/2}}.
$$

(E12)

According to Eqs. (E5) and (E11), we have

$$
t_{\text{cool}} = t_{\text{heat}} \sqrt{\frac{2m_{\text{eff}}}{m}}.
$$

(E13)

The diffusion (heating) dominates the friction (cooling) when $m_{\text{eff}} \gg m$. The friction (cooling) dominates the diffusion (heating) when $m \gg m_{\text{eff}}$.

Setting $x = \sigma_i^2/\sigma_b^2$ and $a = m_{\text{eff}}/m$, the formal solution of Eq. (E12) is

$$
\int \frac{dx}{x (1+x)^{3/2}} - \frac{\sqrt{2a}}{(1+2x)^{3/2}} = \frac{t}{t_{\text{cool}}}.
$$

(E14)

When $x \gg 1$ the solution is given by Eqs. (D35) and (D36) with the substitution $m_b \rightarrow m_{\text{eff}}/2$. When $x \ll 1$ the solution is given by Eq. (D38) with the substitution $m_b \rightarrow \sqrt{2m_{\text{eff}}}$. According to Eq. (E12), the normalized velocity dispersion of the test particles relaxes towards the equilibrium value $x_e = (\sigma_i^2)_{\text{eq}}/\sigma_b^2$ determined by the equation

$$
x_e (1 + 2x_e)^{3/2} = \sqrt{2m_{\text{eff}}/m}.
$$

(E15)

When $m \ll m_{\text{eff}}$, we get $x_e \sim m_{\text{eff}}/2m$. When $m \gg m_{\text{eff}}$, we get $x_e \sim \sqrt{2m_{\text{eff}}/m}$. When $m_{\text{eff}} = m$, we obtain $x_e = 0.812$. We recall that these results are approximate since they assume that the DF of the test particles is Maxwellian which is usually not the case [see Eq. (216)]. This is, however, the case when $m \ll m_{\text{eff}}$ and when $m \gg m_{\text{eff}}$, so we recover the results of footnote 43 (Fig. 4).

Appendix F: Self-similar solution of the Spitzer–Schwarzschild equation

For $x \rightarrow +\infty$, using Eq. (88), the Spitzer-Schwarzschild equation (102) reduces to

$$
\frac{\partial f}{\partial t} = \frac{\pi^{3/2}}{t_H} \cdot \frac{1}{x^2} \frac{\partial}{\partial x} \left( \frac{1}{x} \frac{\partial f}{\partial x} \right).
$$

(F1)

Measuring the time in units of $t_H/\pi^{3/2}$ and measuring the DF in units of $\rho/(2^{3/2}\sigma_b^3)$, we obtain

$$
\frac{\partial f}{\partial t} = \frac{1}{x^2} \frac{\partial}{\partial x} \left( \frac{1}{x} \frac{\partial f}{\partial x} \right)
$$

(F2)

with the normalization condition $\int f \, dx = 1$. On the other hand, the velocity dispersion of the test particles [see Eq. (D23)] is given by

$$
\frac{\sigma_i^2}{\sigma_b^2} = \frac{2}{3} \int f x^2 \, dx.
$$

(F3)
The diffusion equation (F2), which was not explicitly written in [97], admits a self-similar solution of the form

\[ f(x, t) = t^{-\alpha} F\left(\frac{x}{t^{\beta}}\right). \]  

\[ \text{(F4)} \]

The scaling \( x \sim t^{1/5} \) deduced from Eq. (F2) yields \( \beta = 1/5 \). On the other hand, the normalization condition \( \int f \, dx = 1 \) implies \( t^{-\alpha} t^{3\beta} \sim 1 \), hence \( \alpha = 3\beta = 3/5 \). Therefore, we can rewrite Eq. (F4) as

\[ f(x, t) = t^{-3/5} F\left(\frac{x}{t^{1/5}}\right), \]  

\[ \text{(F5)} \]

with \( \int_0^{+\infty} F(X) 4\pi X^2 \, dX = 1 \), where \( X = x/t^{1/5} \). Substituting Eq. (F5) into Eq. (F2), we find that the invariant profile \( F(X) \) is determined by the differential equation

\[ \frac{1}{X^2} \frac{d}{dX} \left( \frac{1}{X} \frac{dF}{dX} \right) + \frac{1}{5} X^4 \frac{dF}{dX} + \frac{3}{5} F = 0 \]  

\[ \text{(F6)} \]

or, equivalently,

\[ \frac{d^2F}{dX^2} + \left( \frac{1}{5} X^4 - \frac{1}{X} \right) \frac{dF}{dX} + \frac{3}{5} X^3 F = 0. \]  

\[ \text{(F7)} \]

If we make the change of variables

\[ F(X) = e^{-X^5/25} V(X), \]  

\[ \text{(F8)} \]

we find that \( V(X) \) satisfies

\[ \frac{d^2V}{dX^2} - \left( \frac{1}{5} X^4 + \frac{1}{X} \right) \frac{dV}{dX} = 0. \]  

\[ \text{(F9)} \]

\[ ^{64} \text{Coming back to the Spitzer-Schwarzschild equation (102), the self-similar solution (F4) is valid for } t \to +\infty, \text{ when the typical value of } x \text{ is large (see the scaling below), so that the approximation from Eq. (F1) is justified.} \]
This is a first order differential equation for \( V'(X) \) whose solution is

\[
V'(X) = AXe^{X^5/25}.
\]  
(F10)

Therefore, the general solution of Eq. (F7) is

\[
F(X) = Ae^{-X^5/25} \int_0^X we^{u^5/25} \, dw + Be^{-X^5/25},
\]  
(F11)

where \( A \) and \( B \) are integration constants. Defining the function

\[
\Phi(s, x) = \int_0^x t^{s-1} e^t \, dt,
\]  
(F12)

we can rewrite Eq. (F11) as

\[
F(X) = Ae^{-X^5/25} \Phi \left( \frac{2}{5}, \frac{X^5}{25} \right) + Be^{-X^5/25}.
\]  
(F13)

The physical solution corresponds to \( A = 0 \) leading to

\[
F(X) = Be^{-X^5/25}.
\]  
(F14)

The constant \( B \) is determined by the normalization condition \( \int_0^{+\infty} F(X)4\pi X^2 \, dX = 1 \)
yielding

\[
B = \frac{1}{4\pi^{5/3} \Gamma(3/5)} = 0.0387298 \ldots.
\]  
(F15)

According to Eqs. (F3) and (F5), the velocity dispersion of the test particles is given by

\[
\frac{\sigma_t^2}{\sigma_b^2} \sim \frac{2}{3} t^{2/5} \int_0^{+\infty} F(X)4\pi X^4 \, dX.
\]  
(F16)

With the results from Eqs. (F14) and (F15), we obtain

\[
\frac{\sigma_t^2}{\sigma_b^2} \sim \frac{2 \times 5^{4/5}}{3\Gamma(3/5)} t^{2/5} = 1.62231 \ldots t^{2/5}.
\]  
(F17)

This asymptotic result, valid for \( t \to +\infty \), is in very good agreement with formula (18) of Spitzer and Schwarzschild [97] obtained by solving the diffusion equation (102) numerically. We also note that the velocity distribution (F14) is non-Maxwellian.

If we come back to the original variables, we get

\[
\frac{\sigma_t^2}{\sigma_b^2} \sim 3.22421 \ldots \left( \frac{t}{t_H} \right)^{2/5}.
\]  
(F18)

Now, comparing Eqs. (90) and (D29), we find \( t_H/t_{\text{heat}} = 16\pi/3 \), yielding

\[
\frac{\sigma_t^2}{\sigma_b^2} \sim 1.04415 \ldots \left( \frac{t}{t_{\text{heat}}} \right)^{2/5}.
\]  
(F19)

This exact result may be compared with the approximate formula [see Eq. (D46)]:

\[
\left( \frac{\sigma_t^2}{\sigma_b^2} \right)_{\text{app}} \sim \left( \frac{5}{2} \frac{t}{t_{\text{heat}}} \right)^{2/5} \sim 1.4427 \ldots \left( \frac{t}{t_{\text{heat}}} \right)^{2/5},
\]  
(F20)
based on the assumption that the velocity distribution of the test particles is always Maxwellian (which is not true for large times as we have seen above). This Maxwellian assumption provides the correct exponent $2/5$ but the prefactor is slightly in error.

Remark These results are also valid for the diffusion of light particles with mass $m \ll m_{\text{eff}}$ in FDM halos (see Sect. 6.4) provided that we replace $t_H$ in Eq. (F1) by $2\sqrt{2}t_H$, where $t_H$ is now given by Eq. (218).

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