VARIATIONAL PROBLEMS ASSOCIATED WITH A SYSTEM OF NONLINEAR SCHRÖDINGER EQUATIONS WITH THREE WAVE INTERACTION

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Abstract. In this paper we study several $L^2$-constrained variational problems associated with a three component system of nonlinear Schrödinger equations with three wave interaction. We consider the existence and the orbital stability of minimizers for these variational problems. We also investigate an asymptotic expansion of the minimal energy and the asymptotic behavior of a minimizer for the variational problem when the attractive effect of three wave interaction is sufficiently large.

1. Introduction and main results. In this paper, we investigate several variational problems associated with the following system of nonlinear Schrödinger equations with three wave interaction

\[
\begin{align*}
  i\partial_t u_1 + \Delta u_1 - V_1(x)u_1 + |u_1|^{p-1}u_1 &= -\alpha u_3 \bar{u}_2, \\
  i\partial_t u_2 + \Delta u_2 - V_2(x)u_2 + |u_2|^{p-1}u_2 &= -\alpha u_3 \bar{u}_1, \\
  i\partial_t u_3 + \Delta u_3 - V_3(x)u_3 + |u_3|^{p-1}u_3 &= -\alpha u_1 \bar{u}_2,
\end{align*}
\]

for $(t,x) \in \mathbb{R} \times \mathbb{R}^N$, where $N = 1, 2, 3$, $1 < p < 1 + 4/N$, $\alpha > 0$, $u_1$, $u_2$ and $u_3$ are complex valued functions of $(t,x) \in \mathbb{R} \times \mathbb{R}^N$. Here, $\bar{u}_j$ ($j = 1, 2$) is the complex conjugate of $u_j$, respectively. Throughout this paper, we assume that the potential $V_j$ ($j = 1, 2, 3$) satisfies the following conditions (V1)–(V3), respectively:

- For the potential $V$,
  - (V1) $V \in L^\infty(\mathbb{R}^N; \mathbb{R})$;
  - (V2) $V(x) \leq \lim_{|y| \to \infty} V(y) = 0$ for almost every $x \in \mathbb{R}^N$;
  - (V3) $V(-x_1, x') = V(x_1, x')$ for almost every $x_1 \in \mathbb{R}$ and $x' \in \mathbb{R}^{N-1}$, $V(s,x') \leq V(t,x')$ for almost every $s, t \in \mathbb{R}$ with $0 \leq s < t$ and $x' \in \mathbb{R}^{N-1}$.

When $V_j \equiv 0$ for all $j = 1, 2, 3$, the system (1.1) was introduced by M. Colin, T. Colin, M. Ohta in [12] as a simplified model of a quasilinear Zakharov system related to the Raman amplification in a plasma considered in [9, 10].

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First, under the same assumptions on $N, p, \alpha$ and potentials, we consider the following variational problem for $\gamma, \mu, s \geq 0$:

\[
I(\gamma, \mu, s) := \inf \{ E(\bar{u}) \mid \bar{u} \in H^1(\mathbb{R}^N; \mathbb{C}^3), \|u_1\|_2^2 = \gamma, \|u_2\|_2^2 = \mu, \|u_3\|_2^2 = s \},
\]

\[
E(\bar{u}) := \frac{1}{2} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |\nabla u_j|^2 \, dx + \frac{1}{2} \sum_{j=1}^{3} \int_{\mathbb{R}^N} V_j(x)|u_j|^2 \, dx
\]

\[
- \frac{1}{p+1} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |u_j|^{p+1} \, dx - \alpha \text{Re} \int_{\mathbb{R}^N} u_1 u_2 \pi_3 \, dx,
\]

where $\bar{u} := (u_1, u_2, u_3) \in H^1(\mathbb{R}^N; \mathbb{C}^3)$. We denote by $I^\infty, E^\infty$ instead of $I, E$ if $V_j \equiv 0$ for all $j = 1, 2, 3$.

We define a minimizing sequence for $I(\gamma, \mu, s)$ to be a sequence $\{\tilde{u}_n\}_{n=1}^\infty$ in $H^1(\mathbb{R}^N; \mathbb{C}^3)$ such that $\|u_{1,n}\|_2^2 \to \gamma, \|u_{2,n}\|_2^2 \to \mu, \|u_{3,n}\|_2^2 \to s$ and $E(\tilde{u}_n) \to I(\gamma, \mu, s)$ as $n \to \infty$. We also define a minimizer for $I(\gamma, \mu, s)$ to be an element $\bar{u}$ in $H^1(\mathbb{R}^N; \mathbb{C}^3)$ such that $\|u_1\|_2^2 = \gamma, \|u_2\|_2^2 = \mu, \|u_3\|_2^2 = s$ and $E(\bar{u}) = I(\gamma, \mu, s)$.

In $N = 1, 1 < p < 1 + 4/N$ and $\alpha > 0$, Ardila [2] had proved the compactness of a minimizing sequence for $I^\infty(\gamma, \mu, s)$.

We generalize the result of Ardila to the higher dimensional case and to the model with potentials as follows.

**Theorem 1.1.** Suppose $\gamma, \mu, s > 0, N = 1, 2, 3, 1 < p < 1 + 4/N, \alpha > 0$ and $V_j(x)$ ($j = 1, 2, 3$) satisfies the conditions (V1)-(V3), respectively. Then, any minimizing sequence $\{\tilde{u}_n\}_{n=1}^\infty$ for $I(\gamma, \mu, s)$ is relatively compact in $H^1(\mathbb{R}^N; \mathbb{C}^3)$ up to translations. That is, there exist $\{y_n\}_{n=1}^\infty \subset \mathbb{R}^N$ and $\bar{u} \in H^1(\mathbb{R}^N; \mathbb{C}^3)$ such that $\{\tilde{u}_n(\cdot + y_n)\}_{n=1}^\infty$ has a subsequence converging strongly in $H^1(\mathbb{R}^N; \mathbb{C}^3)$ to $\bar{u}$. Moreover, $\bar{u}$ is a minimizer for $I(\gamma, \mu, s)$.

**Remark 1.** In Theorem 1.1, if there exists $j \in \{1, 2, 3\}$ such that $V_j \equiv 0$, then we can take $y_n = 0$ for all $n \in \mathbb{N}$. Whether Theorem 1.1 holds even without the condition (V3) or not is an open problem.

**Remark 2.** In the condition (V2), the condition $\lim_{|y| \to \infty} V(y) = 0$ can be relaxed as $\lim_{|y| \to \infty} V(y) = V_\infty$ for some constant $V_\infty \in \mathbb{R}$ without loss of generality, since we deals with the $L^2$-constrained variational problem. However, the condition $V(x) \leq \lim_{|y| \to \infty} V(y) = V_\infty$ (for almost every $x \in \mathbb{R}^N$) is almost necessary even for the scalar case (see e.g. [19, 20]). Although the condition (V3) is not necessary for the scalar case or some systems (see e.g. [4], [16]), we need the condition (V3) to overcome the difficulty to establish the strict subadditivity (see Proposition 3) for our system with three wave interaction by using the coupled rearrangement technique. We note that Bhattarai [4] and Ikoma-Miyamoto [16] studied $L^2$-constrained variational problems with local and nonlocal nonlinearities based on new arguments to establish the strict subadditivity. However, since Bhattarai’s method strongly depends on the property that the energy functional has the quadratic or superquadratic nonlinearities for each component and Ikoma-Miyamoto’s method depends on the exponential decay estimates for solutions, their methods do not work to our system with three wave interaction.

We also note that our strategy of the proof of Theorem 1.1 is different from the one in Ardila [2] even in the case $N = 1$. (see Bhattarai [3] for related results to other three component system.)
The key ingredients of the proof of Theorem 1.1 is the coupled rearrangement technique developed by Shibata [24] in the study of other systems. Gou-Jianjean [15] also used the coupled rearrangement technique for the $L^2$-constrained variational problem to the system arising from the Bose-Einstein condensation phenomena. For other studies on $L^2$-constrained variational problems for nonlinear Schrödinger systems arising in Bose-Einstein condensates with partial confinement or trapping potentials, see [14], [22] and the references therein.

Next, we study the stability of solutions to the system (1.1). We need to consider the following another variational problem:

$$J(\gamma, \mu) := \inf \{ E(\bar{u}) \mid \bar{u} \in H^1(\mathbb{R}^N; \mathbb{C}^3), \; Q_1(\bar{u}) = \gamma, \; Q_2(\bar{u}) = \mu \},$$

where $Q_1(\bar{u}) := \| u_1 \|_2^2 + \| u_2 \|_2^2$, $Q_2(\bar{u}) := \| u_1 \|_2^2 + \| u_3 \|_2^2$. We represent the set of minimizers for $J(\gamma, \mu)$ by $\mathcal{M}_{\gamma, \mu}$. That is,

$$\mathcal{M}_{\gamma, \mu} := \{ \bar{u} \in H^1(\mathbb{R}^N; \mathbb{C}^3) \mid Q_1(\bar{u}) = \gamma, \; Q_2(\bar{u}) = \mu, \; E(\bar{u}) = J(\gamma, \mu) \}.$$

Before stating our result, we note that the system (1.1) has the following conservation laws.

**Proposition 1.** Let $N = 1, 2, 3$, $1 < p < 1 + 4/N$, $\alpha > 0$ and we assume the conditions (V1)–(V3) for $V_j(x)$, $j = 1, 2, 3$. For all $\bar{u}_0 \in H^1(\mathbb{R}^N; \mathbb{C}^3)$, there exists a unique global solution $\bar{u} \in C([0, \infty); H^1(\mathbb{R}^N; \mathbb{C}^3))$ to system (1.1) satisfying $\bar{u}(0) = \bar{u}_0$. Furthermore, the solution $\bar{u}(t)$ satisfies the conservation laws

$$E(\bar{u}(t)) = E(\bar{u}_0), \quad Q_1(\bar{u}(t)) = Q_1(\bar{u}_0), \quad Q_2(\bar{u}(t)) = Q_2(\bar{u}_0)$$

for all $t \in [0, \infty)$.

Since the nonlinearity is $L^2$-subcritical, Proposition 1 can be proved by using the standard techniques (see e.g. Cazenave [8]). The important feature of the system (1.1) is the conservation of the quantities $E(\bar{u})$, $Q_1(\bar{u})$ and $Q_2(\bar{u})$.

Now, as an application of Theorem 1.1, we have the following stability for $\mathcal{M}_{\gamma, \mu}$.

**Theorem 1.2.** Let $N = 1, 2, 3$, $1 < p < 1 + 4/N$, $\alpha > 0$ and $\gamma, \mu > 0$. Assume the conditions (V1)–(V3) for $V_j(x)$, $j = 1, 2, 3$. Then the set $\mathcal{M}_{\gamma, \mu}$ is non-empty and is stable in the following sense: For all $\delta > 0$, there exists $\varepsilon > 0$ such that, if $\bar{u}_0 \in H^1(\mathbb{R}^N; \mathbb{C}^3)$ satisfies

$$\inf_{\bar{\varphi} \in \mathcal{M}_{\gamma, \mu}} \| \bar{u}_0 - \bar{\varphi} \|_{H^1(\mathbb{R}^N; \mathbb{C}^3)} < \delta,$$

then the solution $\bar{u}(t)$ of the system (1.1) with the initial data $\bar{u}(0) = \bar{u}_0$ satisfies

$$\inf_{\bar{\varphi} \in \mathcal{M}_{\gamma, \mu}} \| \bar{u}(t) - \bar{\varphi} \|_{H^1(\mathbb{R}^N; \mathbb{C}^3)} < \varepsilon \quad \text{for all } t \geq 0.$$

**Remark 3.** For the minimizer $\phi(x) = (\phi_1(x), \phi_2(x), \phi_3(x)) \in \mathcal{M}_{\gamma, \mu}$, there exist constants $\omega_j \in \mathbb{R}$ ($j = 1, 2$) such that

$$\begin{cases}
\Delta \phi_1 - 2\omega_1 \phi_1 - V_1(x) \phi_1 + |\phi_1|^{p-1} \phi_1 = -\alpha \phi_3 \bar{\phi}_2, \\
\Delta \phi_2 - 2\omega_2 \phi_2 - V_2(x) \phi_2 + |\phi_2|^{p-1} \phi_2 = -\alpha \phi_3 \bar{\phi}_1, \\
\Delta \phi_3 - 2(\omega_1 + \omega_2) \phi_3 - V_3(x) \phi_3 + |\phi_3|^{p-1} \phi_3 = -\alpha \phi_1 \phi_2.
\end{cases}$$
Thus, $\vec{u} := (u_1, u_2, u_3) = (e^{2i\omega_1 t}\phi_1(x), e^{2i\omega_2 t}\phi_2(x), e^{2i(\omega_1+\omega_2) t}\phi_3(x))$ is a standing wave of
\[
\begin{cases}
  i\partial_t u_1 + \Delta u_1 - V_1(x)u_1 + |u_1|^{p-1}u_1 = -\alpha u_3 \vec{w}_2, \\
  i\partial_t u_2 + \Delta u_2 - V_2(x)u_2 + |u_2|^{p-1}u_2 = -\alpha u_3 \vec{w}_1, \\
  i\partial_t u_3 + \Delta u_3 - V_3(x)u_3 + |u_3|^{p-1}u_3 = -\alpha u_1 u_2.
\end{cases}
\]
(1.1)

Theorem 1.2 is also the generalization of the result of Ardila [2] to the higher dimensional case and to the model with potentials.

Finally, we state an asymptotic behavior of minimizers for $I(\gamma, \mu, s)$ as $\alpha \to \infty$. To do this, we introduce the following new variational problem (cf. [21]):

\[
\Sigma_0(\gamma, \mu, s) := \inf \{ E^0(\vec{u}) | \vec{u} \in H^1(\mathbb{R}^N; \mathbb{C}^3), \| u_1 \|_2^2 = \gamma, \| u_2 \|_2^2 = \mu, \| u_3 \|_2^2 = s \},
\]

\[
E^0(\vec{u}) := \frac{1}{2} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |\nabla u_j|^2 \, dx - Re \int_{\mathbb{R}^N} u_1 u_2 \vec{w}_3 \, dx.
\]

To emphasize $\alpha$–dependence, we denote by $I_{\alpha}, E_{\alpha}$ instead of $I, E$. Then, we have the following result.

**Theorem 1.3.** Let $N \leq 3, 1 < p < 1 + 4/N, \alpha > 0$ and $\gamma, \mu, s > 0$. Assume the conditions (V1)–(V3) for $V_j(x), j = 1, 2, 3$. Then we have the following results:

(i) $I_{\alpha}(\gamma, \mu, s) = \Sigma_0(\gamma, \mu, s)\alpha^{4/(4-N)} + o(\alpha^{4/(4-N)})$ (as $\alpha \to \infty$).

(ii) Assume, furthermore, $N \leq 2$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a positive sequence with $\alpha_n \to \infty$ (as $n \to \infty$) and let $\vec{u}_n$ be a minimizer for $I_{\alpha_n}(\gamma, \mu, s)$. We set

\[
\vec{v}_n(x) := \alpha_n^{-N/(4-N)} \vec{u}_n(\alpha_n^{-2/(4-N)} x).
\]

Then, taking a subsequence, there exist $\{y_n\}_{n=1}^{\infty} \subset \mathbb{R}^N$ and a minimizer $\vec{v}$ for $\Sigma_0(\gamma, \mu, s)$ such that

\[
\| \vec{v}_n(x + y_n) - \vec{v} \|_{L^1} \to 0 \quad \text{(as $n \to \infty$)}.
\]

**Remark 4.** Whether the existence of a minimizer for $\Sigma_0(\gamma, \mu, s)$ and the compactness in the statement (ii) in Theorem 1.3 hold even in the case $N = 3$ or not is an open problem. We explain the reason later in Remark 6 and Proposition 5 why we need the restriction $N \leq 2$ to establish the compactness of the minimizing sequence for $\Sigma_0(\gamma, \mu, s)$.

**Remark 5.** When $|\alpha|$ is sufficiently large, without the constraint of $L^2$ norm, Pomponio had proved the existence of a vector ground state, namely $u_j \neq 0$ for every $j = 1, 2, 3$, of the following system

\[
\begin{cases}
  -\Delta u_1 + V_1(x) u_1 - |u_1|^{p-1} u_1 = \alpha u_3 u_2, \\
  -\Delta u_2 + V_2(x) u_2 - |u_2|^{p-1} u_2 = \alpha u_3 u_1, \\
  -\Delta u_3 + V_3(x) u_3 - |u_3|^{p-1} u_3 = \alpha u_1 u_2.
\end{cases}
\]

Here $u_j$ is a real valued function on $\mathbb{R}^N, \alpha \in \mathbb{R}, N = 1, 2, 3, 2 < p < (N+2)/(N-2)$.

For the potential functions $V_j (j = 1, 2, 3)$, Pomponio assumed only the following conditions (i) and (ii).

(i) for all $j = 1, 2, 3, V_j : \mathbb{R}^N \to \mathbb{R}$ is measurable and there exists $C_j > 0$ such that $V_j(x) \geq C_j$ for almost every $x \in \mathbb{R}^N$.

(ii) for all $j = 1, 2, 3, V_j(x) \leq V_{j,\infty} := \lim_{|y| \to \infty} V_j(y)$ for almost every $x \in \mathbb{R}^N$ and there exists $j = 1, 2, 3$ such that $V_j \neq V_{j,\infty}$. 
This paper is organized as follows. In section 2, we collect the basic estimates and the properties of the coupled rearrangements. In section 3, we show the subadditivity and the strict subadditivity of the energy $I^\infty$ and $I$. In section 4, we give the proof of Theorem 1.1. In section 5, we show that $\mathcal{M}_{\gamma,\mu}$ is non-empty (see Lemma 5.3) and give the proof of Theorem 1.2. In section 6, we first show the key monotonicity property of the energy $\Sigma_0$ for each component and give the proof of the compactness of the minimizing sequence for the energy $\Sigma_0$ for $N \leq 2$. We also establish the asymptotic expansion of the energy $I_\alpha(\gamma, \mu, s)$ as $\alpha \to \infty$ and complete the proof of Theorem 1.3. In the Appendix, we give the proof of the technical lemma (Lemma 2.6).

**Notation.** We denote by $H^1(\mathbb{R}^N)$ the Sobolev space of all complex-valued functions $H^1(\mathbb{R}^N; \mathbb{C})$, its norm will be denoted by $\| \cdot \|_{H^1}$. The space $L^q(\Omega)$ ($1 \leq q \leq \infty$) is the space of all complex-valued functions $L^q(\Omega; \mathbb{C})$, its norm will be denoted by $\| \cdot \|_{L^q(\Omega)}$. In particular, $\Omega = \mathbb{R}^N$, we denote by $\| \cdot \|_q$ instead of $\| \cdot \|_{L^q(\mathbb{R}^N)}$. We also denote by $H^1(\mathbb{R}^N; \mathbb{C}^3)$ the product space $H^1(\mathbb{R}^N; \mathbb{C}^3) = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.

2. Preliminaries.

2.1. Key inequalities. The following Gagliardo-Nirenberg’s inequality is a basic inequality (see [1]).

**Lemma 2.1** (Gagliardo–Nirenberg’s inequality). Let $N \in \mathbb{N}$ and $2 < q \leq 2 + 4/N$. Then, there exists $C(q, N) > 0$ such that

$$\|u\|_q^q \leq C(q, N)\|\nabla u\|_2^{N(q-2)/2}\|u\|_2^{q-N(q-2)/2} \text{ for all } u \in H^1(\mathbb{R}^N).$$

The following inequality is also important in our argument.

**Lemma 2.2.** Let $R > 0$, $N \in \mathbb{N}$ and $2 < q \leq 2 + 4/N$. Then, there exists a positive constant $C = C(R, q, N) > 0$ such that

$$\|u\|_q^q \leq C\left(\sup_{y \in \mathbb{R}^N} \int_{|y-x| < R} |u|^2 \, dx\right)^{(q-2)/2}\|u\|_{H^1}^2 \text{ for all } u \in H^1(\mathbb{R}^N).$$

**Proof.** For the case $q = 2 + 4/N$, we have the desired estimate by [8, Lemma 1.7.7]. Then the desired estimate for the case $2 < q < 2 + 4/N$ follows from Hölder’s inequality. 

2.2. Basic estimates. We prove the following estimate for $E$ to show that $I(\gamma, \mu, s) > -\infty$ and minimizing sequence for $I(\gamma, \mu, s)$ is bounded in $H^1(\mathbb{R}^N; \mathbb{C}^3)$ for $\gamma, \mu, s > 0$.

**Lemma 2.3.** Suppose that $1 \leq N \leq 3$, $1 < p < 1 + 4/N$ and the potential $V_j$ ($j = 1, 2, 3$) satisfies the condition $(V1)$. Then there exist positive constants $C(p, N), C(\alpha, N), \beta(p, N), \beta(N) > 0$ such that

$$E(\bar{u}) + \frac{1}{2} \sum_{j=1}^{3} |V_{j,\min}|\|u_j\|_2^2 + C(p, N)\sum_{j=1}^{3} \|u_j\|_2^{\beta(p,N)} + C(\alpha, N)\sum_{j=1}^{3} \|u_j\|_2^{\beta(N)}$$

$$\geq \frac{1}{4} \sum_{j=1}^{3} \|\nabla u_j\|_2^2 \text{ for all } \bar{u} \in H^1(\mathbb{R}^N; \mathbb{C}^3),$$

where $V_{j,\min} := \text{essinf}_{\mathbb{R}^N} V_j$. 
Proof. We have
\[ \sum_{j=1}^{3} \int_{\mathbb{R}^N} V_j(x)|u_j|^2 \, dx \geq \sum_{j=1}^{3} V_{j,\text{min}} \|u_j\|_2^2. \]  \tag{2.1} 
Since \( N(p - 1)/2 < 2 \) for \( p < 1 + 4/N \), from Gagliardo-Niremberg's inequality and Young's inequality, there exist \( C(p, N) > 0 \) and \( \beta(p, N) > 0 \) such that
\[ \|u_j\|_{p+1}^p \leq \frac{1}{8} \|\nabla u_j\|_2^2 + C(p, N)\|u_j\|_2^{\beta(p, N)}, \quad j = 1, 2, 3. \]  \tag{2.2} 
From Hölder's inequality and Young's inequality,
\[ \alpha \text{Re} \int_{\mathbb{R}^N} u_1 u_2 \nabla_3 \, dx \leq \alpha \int_{\mathbb{R}^N} |u_1||u_2||u_3| \, dx \leq \alpha \|u_1\|_3 \|u_2\|_3 \|u_3\|_3 \leq \frac{\alpha}{3} \sum_{j=1}^{3} \|u_j\|_3^3. \]  \tag{2.3} 
Since \( 1 \leq N \leq 3 \), from (2.2) with \( p = 2 \), there exist \( C(\alpha, N) > 0 \) and \( \beta(N) > 0 \) such that
\[ \frac{\alpha}{3} \sum_{j=1}^{3} \|u_j\|_3^3 \leq \frac{1}{8} \sum_{j=1}^{3} \|\nabla u_j\|_2^2 + C(\alpha, N) \sum_{j=1}^{3} \|u_j\|_2^{\beta(N)}. \]  \tag{2.4} 
From (2.3)–(2.4), we have
\[ \left| \alpha \text{Re} \int_{\mathbb{R}^N} u_1 u_2 \nabla_3 \, dx \right| \leq \frac{1}{8} \sum_{j=1}^{3} \|\nabla u_j\|_2^2 + C(\alpha, N) \sum_{j=1}^{3} \|u_j\|_2^{\beta(N)}. \]  \tag{2.5} 
From (2.1), (2.2), (2.5), we have
\[ E(\vec{u}) \geq \frac{1}{4} \sum_{j=1}^{3} \|\nabla u_j\|_2^2 + \frac{1}{2} \sum_{j=1}^{3} V_{j,\text{min}} \|u_j\|_2^2 - C(p, N) \sum_{j=1}^{3} \|u_j\|_2^{\beta(p, N)} - C(\alpha, N) \sum_{j=1}^{3} \|u_j\|_2^{\beta(N)}. \] 

Lemma 2.4. Suppose that \( \gamma, \mu, s > 0 \), \( 1 \leq N \leq 3 \), \( 1 < p < 1 + 4/N \), and the potential \( V_j \) (\( j = 1, 2, 3 \)) satisfies the condition (V1). Then, we have \( I(\gamma, \mu, s) > -\infty \). Furthermore, if \( \{\vec{u}_n\}_{n=1}^\infty \) is a minimizing sequence for \( I(\gamma, \mu, s) \), then \( \{u_{j,n}\}_{n=1}^\infty \) (\( j = 1, 2, 3 \)) is bounded in \( H^1(\mathbb{R}^N) \).

Proof. Let \( \vec{u} \in H^1(\mathbb{R}^N; \mathbb{C}^3) \) such that \( \|u_1\|_2^2 = \gamma, \|u_2\|_2^2 = \mu, \|u_3\|_2^2 = s \). From Lemma 2.3,
\[ E(\vec{u}) \geq \frac{1}{4} \sum_{j=1}^{3} \|\nabla u_j\|_2^2 + \frac{1}{2} \sum_{j=1}^{3} V_{j,\text{min}} \|u_j\|_2^2 - C(p, N) \sum_{j=1}^{3} \|u_j\|_2^{\beta(p, N)} - C(\alpha, N) \sum_{j=1}^{3} \|u_j\|_2^{\beta(N)}. \] 

Therefore, we have \( I(\gamma, \mu, s) > -\infty \).

Let \( \{\vec{u}_n\}_{n=1}^\infty \) be a minimizing sequence for \( I(\gamma, \mu, s) \). From the definition of \( I(\gamma, \mu, s) \), \( \{E(\vec{u}_n)\}_{n=1}^\infty \) and \( \{\|u_{j,n}\|_2^2\}_{n=1}^\infty \) (\( j = 1, 2, 3 \)) are bounded sequences. From Lemma 2.3, \( \|\nabla u_{j,n}\|_2^2 \) is also a bounded sequence. Hence, \( \{\|u_{j,n}\|_{H^1}\} \) (\( j = 1, 2, 3 \)) is a bounded sequence. \( \square \)
We prepare the following lemma on the local Lipschitz continuity of $E$ on $H^1(\mathbb{R}^N; \mathbb{C}^3)$ to prove the continuity of $I$.

**Lemma 2.5.** Suppose that $R > 0$ and the potential $V_j$ ($j = 1, 2, 3$) satisfies the condition (V1). In addition, we set

$$
\Sigma \leq R := \left\{ \vec{u} \in H^1(\mathbb{R}^N; \mathbb{C}^3) \left| \sum_{j=1}^3 \|u_j\|^2_{H^1} \leq R^2 \right. \right\}.
$$

Then there exists $C = C(R) > 0$ such that

$$
|E(\vec{u}) - E(\vec{v})| \leq C \sum_{j=1}^3 \|u_j - v_j\|_{H^1} \quad \text{for all } \vec{u}, \vec{v} \in \Sigma \leq R.
$$

Since we can prove Lemma 2.5 by the standard argument, we omit the details.

We use the following continuity of $I$ on $[0, \infty) \times [0, \infty) \times [0, \infty)$ in Theorem 1.1 and Lemma 5.2 and so on.

**Lemma 2.6.** $I$ is continuous on $[0, \infty) \times [0, \infty) \times [0, \infty)$.

We prove this lemma in the appendix by using Lemma 2.5.

We note the following lemma on the weak continuity of the potential energy. For the proof of this Lemma, see e.g. Lieb-Loss [18, Theorem 11.4].

**Lemma 2.7.** Suppose that $\{u_n\}_{n=1}^\infty$ is bounded in $H^1(\mathbb{R}^N)$ and $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$ and assume $V \in L^\infty(\mathbb{R}^N; \mathbb{R})$ and $V(x) \to 0$ as $|x| \to \infty$. Then it follows that

$$
\int_{\mathbb{R}^N} V(x)|u_n|^2 \, dx = \int_{\mathbb{R}^N} V(x)|u|^2 \, dx + o(1) \quad \text{as } n \to \infty.
$$

**Lemma 2.8.** Let $u_1, u_2, u_3$ be measurable functions defined on $\mathbb{R}^N$ such that $u_j \geq 0$ almost everywhere in $\mathbb{R}^N$ ($j = 1, 2, 3$) and

$$
\lim_{|x| \to \infty} u_j(x) = 0 \quad (j = 1, 2, 3).
$$

Then we have

$$
\int_{\mathbb{R}^N} u_1 u_2 u_3 \, dx \leq \int_{\mathbb{R}^N} u_1^* u_2^* u_3^* \, dx,
$$

where $u^*$ is the Steiner rearrangement of $u$ with respect to the hyperplane $x_1 = 0$.

We refer [17] for the definition of the Steiner rearrangement.

**Proof.** Fix almost every $x' \in \mathbb{R}^{N-1}$. Set $f_j(x_1) := u_j(x_1, x')$. Note that $g^* = |g|^*$.

By Theorem 1 of [7], we have

$$
\int_{\mathbb{R}} f_1 f_2 f_3 \, dx_1 \leq \int_{\mathbb{R}} f_1^* f_2^* f_3^* \, dx_1,
$$

that is,

$$
\int_{\mathbb{R}} u_1 u_2 u_3 \, dx_1 \leq \int_{\mathbb{R}} u_1^* u_2^* u_3^* \, dx_1.
$$

Integrating both sides over $\mathbb{R}^{N-1}$ with respect to $x'$, we obtain the conclusion. □
We prepare the following lemma on the Steiner rearrangement inequality for the potential energy to show that the rearrangement of a minimizer is also a minimizer for \( I(\gamma, \mu, s) \).

**Lemma 2.9.** Suppose that the potential \( V \) satisfies the conditions \((V1)-(V3)\) and \( u \in H^1(\mathbb{R}^N) \) satisfies

\[
\lim_{|x| \to \infty} u(x) = 0.
\]

Then we have

\[
\int_{\mathbb{R}^N} V(x)(u^*)^2 \, dx \leq \int_{\mathbb{R}^N} V(x)|u|^2 \, dx \quad (j = 1, 2, 3).
\]

This lemma follows from Lemma 2.8 and \((-V)^* = -V\).

### 2.3. Coupled rearrangement.

In this subsection, we collect the definition and useful properties of the coupled rearrangement developed by Shibata [24] for reader’s convenience. For \( x \in \mathbb{R}^N \), we denote by \( x = (x_1, x') \) \( (x_1 \in \mathbb{R}, x' \in \mathbb{R}^{N-1}) \) and also we denote by \( \mathcal{L}^1 \) the 1-dimensional Lebesgue measure.

First, we state the definition of the coupled rearrangement developed by Shibata [24].

**Definition 2.10** (Shibata [24]). Let \( u, v \) be measurable functions defined on \( \mathbb{R}^N \) such that

\[
\lim_{|x| \to \infty} u(x) = \lim_{|x| \to \infty} v(x) = 0.
\]

Then the coupled rearrangement \( u \ast v \) is defined by

\[
(u \ast v)(x_1, x') := \int_0^\infty \chi_{\{|u(x',>t)\ast|v(x')|>t\}}(x_1) \, dt, \quad x_1 \in \mathbb{R}, \ x' \in \mathbb{R}^{N-1}
\]

for any measurable subsets \( A, B \subset \mathbb{R} \), where \( A \ast B \) is defined as follows:

\[
A \ast B := \frac{-(\mathcal{L}^1(A) + \mathcal{L}^1(B))/2, (\mathcal{L}^1(A) + \mathcal{L}^1(B))/2)}{2}.
\]

We state the following Proposition 2, Corollary 1 and Lemma 2.12 to prove the strict subadditivity for \( I \).

**Proposition 2.** Let \( f_j, g_j \) be measurable functions defined on \( \mathbb{R} \) such that \( f_j, g_j \geq 0 \) almost everywhere on \( \mathbb{R} \) \((j = 1, 2, 3)\) and

\[
\lim_{|x| \to \infty} f_j(x) = \lim_{|x| \to \infty} g_j(x) = 0 \quad (j = 1, 2, 3).
\]

Then it follows that

\[
\int_{\mathbb{R}} f_1 f_2 f_3 \, dx + \int_{\mathbb{R}} g_1 g_2 g_3 \, dx \leq \int_{\mathbb{R}} (f_1 \ast g_1)(f_2 \ast g_2)(f_3 \ast g_3) \, dx.
\]

**Proof.** Since we can show by using an argument similar to that in Lemma 5.2 in [24], we omit the details. \( \square \)

**Corollary 1.** Let \( u_j, v_j \) \( (j = 1, 2, 3) \) be measurable functions defined on \( \mathbb{R}^N \) such that \( u_j, v_j \geq 0 \) almost everywhere in \( \mathbb{R}^N \) \((j = 1, 2, 3)\) and

\[
\lim_{|x| \to \infty} u_j(x) = \lim_{|x| \to \infty} v_j(x) = 0 \quad (j = 1, 2, 3).
\]

Then, it follows that

\[
\int_{\mathbb{R}^N} u_1 u_2 u_3 \, dx + \int_{\mathbb{R}^N} v_1 v_2 v_3 \, dx \leq \int_{\mathbb{R}^N} (u_1 \ast v_1)(u_2 \ast v_2)(u_3 \ast v_3) \, dx.
\]
Thus it holds that

\[ f \]

From the assumptions in this Lemma, we have

\[ u \]

Then, we have

\[ v \]

Proof. Fix almost every \( x' \in \mathbb{R}^{N-1} \). Set \( f_j(x_1) := u_j(x_1, x') \) and \( g_j(x_1) := v_j(x_1, x') \) \((j = 1, 2, 3)\). Since \( f_j \) and \( g_j \) satisfy all the assumptions of Proposition 2, it follows that

\[
\int_{\mathbb{R}} f_1 f_2 f_3 \, dx_1 + \int_{\mathbb{R}} g_1 g_2 g_3 \, dx_1 \leq \int_{\mathbb{R}} (f_1 \ast g_1)(f_2 \ast g_2)(f_3 \ast g_3) \, dx_1,
\]

that is,

\[
\int_{\mathbb{R}} u_1 u_2 u_3 \, dx_1 + \int_{\mathbb{R}} v_1 v_2 v_3 \, dx_1 \leq \int_{\mathbb{R}} (u_1 \ast v_1)(u_2 \ast v_2)(u_3 \ast v_3) \, dx_1.
\]

Integrating both sides over \( \mathbb{R}^{N-1} \), we obtain the conclusion.

Here, we note the following simple observation.

**Lemma 2.11.** Let \( f : \mathbb{R} \to \mathbb{C} \) be a measurable function such that

\[
\lim_{|x| \to \infty} f(x) = 0,
\]

\[
f(x) \geq 0 \quad \text{for almost every } x \in \mathbb{R},
\]

\[
f(-x) = f(x) \quad \text{for almost every } x \in \mathbb{R},
\]

\[
f(x) \geq f(y) \quad \text{for almost every } x, y \in \mathbb{R} \text{ with } 0 \leq x < y.
\]

Then it follows that \( f \ast 0 = f \) almost everywhere in \( \mathbb{R} \). Here, \( f \ast 0 \) is the coupled rearrangement of \( f \) and 0. See Definition 2.10.

**Proof.** From the definition of the coupled rearrangement and the Steiner rearrangement, we have

\[
(f \ast 0)(x) = \int_{0}^{\infty} \chi_{\{f \ast 0 > t\}}(x) \, dt = \int_{0}^{\infty} \chi_{\{f > t\} \ast 0}(x) \, dt
\]

\[
= \int_{0}^{\infty} \chi_{\{f > t\}}(x) \, dt = f^*(x).
\]

From the assumptions in this Lemma, we have \( f^* = f \) almost everywhere in \( \mathbb{R} \). Thus it holds that

\[
(f \ast 0)(x) = f(x) \quad \text{for almost every } x \in \mathbb{R}.
\]

We prove the coupled rearrangement inequality for the potential energy.

**Lemma 2.12.** Assume the potential \( V \) satisfies the conditions \((V1)\)–\((V3)\) and let \( u_2, u_3, v_2, v_3 \in H^1(\mathbb{R}^N) \) such that \( u_2, u_3, v_2, v_3 \geq 0 \) almost everywhere in \( \mathbb{R}^N \) and

\[
\lim_{|x| \to \infty} u_j(x) = \lim_{|x| \to \infty} v_j(x) = 0 \quad (j = 2, 3).
\]

Then, we have

\[
\int_{\mathbb{R}^N} V(x)(u_2 \ast v_2)(u_3 \ast v_3) \, dx \leq \int_{\mathbb{R}^N} V(x)u_2u_3 \, dx.
\]

**Proof.** Fix \( x' \in \mathbb{R}^{N-1} \). Set \( f_1(x_1) := (-V)(x_1, x') \), \( f_1 \) satisfies all assumptions in Lemma 2.11. From Lemma 2.11, we have \( f_1 \ast 0 = f_1 \) almost everywhere in \( \mathbb{R} \). Set \( f_k(x_1) := u_k(x_1, x') \), \( g_k(x_1) := v_k(x_1, x') \) \((k = 2, 3)\), \( f_1, f_2, f_3, g_2 \) and \( g_3 \) satisfy all assumptions in Proposition 2. From Proposition 2, it follows that
\[ \int_{\mathbb{R}} f_1(x_1)f_2(x_1)f_3(x_1) \, dx_1 \leq \int_{\mathbb{R}} (f_1 \ast 0)(x_1)(f_2 \ast g_2)(x_1)(f_3 \ast g_3)(x_1) \, dx_1 \]
\[ = \int_{\mathbb{R}} f_1(x_1)(f_2 \ast g_2)(x_1)(f_3 \ast g_3)(x_1) \, dx_1. \]

That is, for almost every \( x' \in \mathbb{R}^{N-1} \),
\[ \int_{\mathbb{R}} (-V)(x_1, x')u_2(x_1, x')u_3(x_1, x') \, dx_1 \]
\[ \leq \int_{\mathbb{R}} (-V)(x_1, x')(u_2 \ast v_2)(x_1, x')(u_3 \ast v_3)(x_1, x') \, dx_1. \]

Integrating both sides over \( \mathbb{R}^{N-1} \) with respect to \( x' \), it holds that
\[ \int_{\mathbb{R}^N} (-V)(x)u_2(x)u_3(x) \, dx \leq \int_{\mathbb{R}^N} (-V)(x)(u_2 \ast v_2)(x)(u_3 \ast v_3)(x) \, dx. \]

Thus we obtain the conclusion. \( \square \)

2.4. **Scalar variational problems** \( S_\infty \) and \( S_V \). We consider the following variational problems \( S_\infty(\gamma) \) and \( S_V(\gamma) \) corresponding to the scalar equation for \( \gamma \geq 0 \):
\[
S_\infty(\gamma) := \inf \{ J_\infty(u) \mid u \in H^1(\mathbb{R}^N), \| u \|_2^2 = \gamma \},
\]
\[
S_V(\gamma) := \inf \{ J_V(u) \mid u \in H^1(\mathbb{R}^N), \| u \|_2^2 = \gamma \},
\]
where
\[
J_\infty(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} \, dx,
\]
\[
J_V(u) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2) \, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} \, dx.
\]

\( N \in \mathbb{N} \) and \( 1 < p < 1 + 4/N \) and the potential \( V \) satisfies the conditions (V1)–(V2). Now, we remark that if \( V \equiv 0 \), \( S_V(\gamma) \) corresponds to \( S_\infty(\gamma) \). It is well known that \( S_\infty(\gamma) < 0 \) for \( \gamma > 0 \). Since \( V(x) \leq 0 \) for almost every \( x \in \mathbb{R}^N \), \( S_V(\gamma) \leq S_\infty(\gamma) \) for \( \gamma \geq 0 \). If \( V \not\equiv 0 \), then it is also well-known that \( S_V(\gamma) < S_\infty(\gamma) \) holds for \( \gamma > 0 \).

The following lemma is used to describe the relationship between \( I \) and \( S_V(\gamma) \).

**Lemma 2.13.** Suppose that \( \gamma > 0 \) and the potential \( V \) satisfies the condition (V1)–(V2). Then there exists a minimizer \( \phi \) for \( S_V(\gamma) \) such that \( \phi > 0 \) almost everywhere in \( \mathbb{R}^N \).

**Proof.** For the proof, see [19, 20]. \( \square \)

3. **Subadditivity and strict subadditivity for** \( I \). To prove the subadditivity for \( I \), we need the following lemma on the construction of a minimizing sequence for \( I(\gamma, \mu, s) \).

**Lemma 3.1.** Let \( \gamma, \mu, s \geq 0 \) and the potential \( V_j \) \( (j = 1, 2, 3) \) satisfies the condition (V1). Then there exists a minimizing sequence \( \{ \bar{u}_n \}_{n=1}^{\infty} \) for \( I(\gamma, \mu, s) \) such that
\[
\text{supp } u_{j,n} \text{ is compact } (j = 1, 2, 3),
\]
\[
u_{j,n} \geq 0 \text{ almost everywhere in } \mathbb{R}^N (j = 1, 2, 3),
\]
\[
\| u_{1,n} \|_2^2 = \gamma, \quad \| u_{2,n} \|_2^2 = \mu, \quad \| u_{3,n} \|_2^2 = s.
\]
From (3.4) and (3.6), it holds that
\[ \gamma > 0, \text{ and } \|v_{2,n}\|_2^2 = \mu, \quad \|v_{3,n}\|_2^2 = s. \]

Since \( v_{j,n} \in H^1(\mathbb{R}^N) \) and \( C_c^\infty(\mathbb{R}^N) \) is dense in \( H^1(\mathbb{R}^N) \), it follows that for all \( j = 1, 2, 3 \) and for all \( n \in \mathbb{N} \), there exists \( w_{j,n} \in C_c^\infty(\mathbb{R}^N) \) such that
\[ \|w_{j,n} - v_{j,n}\|_{H^1} < 1/n. \]

Since \( \{v_{j,n}\}_{n=1}^\infty \) is bounded in \( H^1(\mathbb{R}^N) \) from Lemma 2.4, \( \{w_{j,n}\}_{n=1}^\infty \) is also bounded in \( H^1(\mathbb{R}^N) \) from (3.1). Therefore there exists \( M > 0 \) such that
\[ \sum_{j=1}^3 (\|v_{j,n}\|_{H^1}^2 + \|w_{j,n}\|_{H^1}^2) \leq M^2 \quad \text{for all } n \in \mathbb{N}. \] (3.2)

From this and Lemma 2.5, there exists \( C(M) > 0 \) such that
\[ |E(\vec{w}_n) - E(\vec{v}_n)| \leq C(M) \sum_{j=1}^3 \|w_{j,n} - v_{j,n}\|_{H^1} \quad \text{for all } n \in \mathbb{N}. \] (3.3)

From (3.1) and (3.3), we have
\[ |E(\vec{w}_n) - E(\vec{v}_n)| \leq \frac{3C(M)}{n} \quad \text{for all } n \in \mathbb{N}. \]

Note that \( E(\vec{v}_n) \to I(\gamma, \mu, s) \), it follows that
\[ E(\vec{w}_n) \to I(\gamma, \mu, s). \] (3.4)

Now, set
\[
z_{1,n} := \begin{cases} \frac{\sqrt{\gamma}}{\|w_{1,n}\|_2} w_{1,n} & (\gamma > 0), \\ 0 & (\gamma = 0) \end{cases}, \quad z_{2,n} := \begin{cases} \frac{\sqrt{\mu}}{\|w_{2,n}\|_2} w_{2,n} & (\mu > 0), \\ 0 & (\mu = 0) \end{cases},
\]
\[
z_{3,n} := \begin{cases} \frac{\sqrt{s}}{\|w_{3,n}\|_2} w_{3,n} & (s > 0), \\ 0 & (s = 0) \end{cases},
\]

it holds that
\[ \|z_{1,n}\|_2^2 = \gamma, \quad \|z_{2,n}\|_2^2 = \mu, \quad \|z_{3,n}\|_2^2 = s. \] (3.5)

If the case \( \gamma > 0 \), it follows that \( \sqrt{\gamma/\|w_{1,n}\|_2} \to 1 \) and \( \{w_{1,n}\}_{n=1}^\infty \) is bounded in \( H^1(\mathbb{R}^N) \) from (3.1). From this, it follows that \( \|z_{1,n} - w_{1,n}\|_{H^1} \to 0 \) as \( n \to \infty \).

If the case \( \gamma = 0 \), note that \( v_{1,n} = z_{1,n} = 0 \) for all \( n \in \mathbb{N} \), from (3.1), we have \( \|w_{1,n}\|_{H^1} \to 0 \) as \( n \to \infty \). From this, it follows that \( \|z_{1,n} - w_{1,n}\|_{H^1} \to 0 \) as \( n \to \infty \).

Thus in all cases, we have
\[ \|z_{j,n} - w_{j,n}\|_{H^1} \to 0 \quad \text{as } n \to \infty, \quad j = 1, 2, 3. \]

Also from the same argument as before we have
\[ |E(\vec{z}_n) - E(\vec{w}_n)| \to 0 \quad \text{as } n \to \infty. \] (3.6)

From (3.4) and (3.6), it holds that
\[ E(\vec{z}_n) \to I(\gamma, \mu, s). \]
Moreover set \( u_{j,n} := |z_{j,n}| \) \((j = 1, 2, 3)\), from [18, Theorem 6.17], it follows that

\[
\int_{\mathbb{R}^N} |\nabla z_{j,n}|^2 \, dx \leq \int_{\mathbb{R}^N} |\nabla z_{j,n}|^2 \, dx.
\]

Note that (3.5), we have

\[
I(\gamma, \mu, s) \leq E(\bar{u}_n) \leq E(\bar{z}_n) \rightarrow I(\gamma, \mu, s) \quad (n \rightarrow \infty).
\]

Therefore \( \{\bar{u}_n\}_{n=1}^\infty \) is a minimizing sequence for \( I(\gamma, \mu, s) \) and has the desired properties.

Now we prove the subadditivity for \( I \).

Lemma 3.2. Let \( \gamma, \mu, s > 0 \) and assume the potentials \( V_j \) \((j = 1, 2, 3)\) satisfy the conditions \((V1)\) and \( V_j \leq 0 \) almost everywhere in \( \mathbb{R}^N \). Then for all \( \gamma' \in [0, \gamma] \), \( \mu' \in [0, \mu] \) and \( s' \in [0, s] \), it follows that

\[
I(\gamma, \mu, s) \leq I(\gamma', \mu', s') + I^\infty(\gamma - \gamma', \mu - \mu', s - s'),
\]

where we denote by \( I^\infty \) instead of \( I \) if \( V_j \equiv 0 \) for all \( j = 1, 2, 3 \).

Proof. From Lemma 3.1, there exist minimizing sequences \( \{\tilde{u}_n\}_{n=1}^\infty \) for \( I(\gamma', \mu', s') \) and \( \{\tilde{v}_n\}_{n=1}^\infty \) for \( I^\infty(\gamma - \gamma', \mu - \mu', s - s') \) such that

- \( \text{supp } u_{j,n} \) is compact \((j = 1, 2, 3)\),
- \( u_{j,n} \geq 0 \) almost everywhere in \( \mathbb{R}^N \) \((j = 1, 2, 3)\),
- \( \|u_{1,n}\|^2 = \gamma' \), \( \|u_{2,n}\|^2 = \mu' \), \( \|u_{3,n}\|^2 = s' \),
- \( \text{supp } v_{j,n} \) is compact \((j = 1, 2, 3)\),
- \( v_{j,n} \geq 0 \) almost everywhere in \( \mathbb{R}^N \) \((j = 1, 2, 3)\),
- \( \|v_{1,n}\|^2 = \gamma - \gamma' \), \( \|v_{2,n}\|^2 = \mu - \mu' \), \( \|v_{3,n}\|^2 = s - s' \).

Since \( u_{j,n} \) and \( v_{j,n} \) have compact supports, there exists \( \{x_n\}_{n=1}^\infty \subset \mathbb{R}^N \) such that

\[
\text{supp } u_{j,n} \cap \text{supp } v_{j,n}(· + x_n) = \emptyset \quad \text{for all } n \in \mathbb{N}.
\]

Set \( w_{j,n}(x) := u_{j,n}(x) + v_{j,n}(x + x_n) \). Note that

\[
\|w_{1,n}\|^2 = \gamma, \quad \|w_{2,n}\|^2 = \mu, \quad \|w_{3,n}\|^2 = s,
\]

we have

\[
E(\bar{w}_n) \geq I(\gamma, \mu, s).
\]

Moreover it holds that

\[
\int_{\mathbb{R}^N} |\nabla w_{j,n}(x)|^2 \, dx = \int_{\mathbb{R}^N} |\nabla u_{j,n}(x)|^2 \, dx + \int_{\mathbb{R}^N} |\nabla v_{j,n}(x + x_n)|^2 \, dx
\]

\[
= \int_{\mathbb{R}^N} |\nabla u_{j,n}(x)|^2 \, dx + \int_{\mathbb{R}^N} |\nabla v_{j,n}(x)|^2 \, dx,
\]

\[
\int_{\mathbb{R}^N} V_j(x) |w_{j,n}(x)|^2 \, dx = \int_{\mathbb{R}^N} V_j(x) |u_{j,n}(x)|^2 \, dx + \int_{\mathbb{R}^N} V_j(x) |v_{j,n}(x + x_n)|^2 \, dx
\]

\[
\leq \int_{\mathbb{R}^N} V_j(x) |u_{j,n}(x)|^2 \, dx,
\]

\[
\int_{\mathbb{R}^N} |w_{j,n}(x)|^{p+1} \, dx = \int_{\mathbb{R}^N} |u_{j,n}(x)|^{p+1} \, dx + \int_{\mathbb{R}^N} |v_{j,n}(x + x_n)|^{p+1} \, dx
\]

\[
= \int_{\mathbb{R}^N} |u_{j,n}(x)|^{p+1} \, dx + \int_{\mathbb{R}^N} |v_{j,n}(x)|^{p+1} \, dx,
\]
Lemma 3.3. Let $\gamma, \mu, s \geq 0$ and assume the conditions \textit{(V1)}–\textit{(V3)} for $V_j(x)$, $j = 1, 2, 3$. Suppose $\bar{u}$ is a minimizer for $I(\gamma, \mu, s)$. Then $\bar{u}^*$ is also a minimizer for $I(\gamma, \mu, s)$ and the following properties hold: for all $j \in \{1, 2, 3\}$,

(i) $u_j^* \in H^1(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$.

(ii) $\lim_{|x| \to \infty} u_j^*(x) = 0$.

(iii) $u_j^* \geq 0$ almost everywhere in $\mathbb{R}^N$.

(iv) $u_j^*(-x_1, x') = u_j^*(x_1, x')$ for almost every $x_1 \in \mathbb{R}$ and for almost every $x' \in \mathbb{R}^{N-1}$.

(v) $u_j^*(s, x') \geq u_j^*(t, x')$ for almost every $s, t \in \mathbb{R}$ with $0 \leq s < t$ and for almost every $x' \in \mathbb{R}^{N-1}$.

(vi) if $\gamma > 0$, then $u_j^* > 0$ almost everywhere in $\mathbb{R}^N$.

if $\mu > 0$, then $u_j^* > 0$ almost everywhere in $\mathbb{R}^N$.

if $s > 0$, then $u_j^* > 0$ almost everywhere in $\mathbb{R}^N$.

Proof. From the basic properties of the Steiner rearrangement, Lemma 2.8 and Lemma 2.9, $\bar{u}^*$ is also a minimizer for $I(\gamma, \mu, s)$. By using the elliptic regularity theory, it follows that (i),(ii). From the definition of the Steiner rearrangement, it holds that (iii),(iv),(v). From the Euler-Lagrange equation and the weak Harnack’s inequality (see e.g. [13, Theorem 8.18]), it follows that (vi).

Now we prove the strict subadditivity for $I$ under the assumption that decomposed minimization problems have a minimizer respectively.

Proposition 3 (Strict subadditivity for $I$). Assume the conditions \textit{(V1)}–\textit{(V3)} for $V_j(x)$, $j = 1, 2, 3$. Let $a_1, a_2, a_3, b_1, b_2, b_3 \geq 0$ and we assume $I(a_1, a_2, a_3)$ and $I^\infty(b_1, b_2, b_3)$ have a minimizer respectively. If $a_1, b_1 > 0$ or $a_2, b_2 > 0$ or $a_3, b_3 > 0$, then we have

$$I(a_1 + b_1, a_2 + b_2, a_3 + b_3) < I(a_1, a_2, a_3) + I^\infty(b_1, b_2, b_3),$$

where we denote by $I^\infty$ instead of $I$ if $V_j \equiv 0$ for all $j = 1, 2, 3$.

Proof. Suppose that $\bar{u}$ and $\bar{v}$ are minimizers for $I(a_1, a_2, a_3)$ and $I^\infty(b_1, b_2, b_3)$ respectively. We consider only the case $a_1 > 0$ and $b_1 > 0$. Since $a_1 > 0$ and $b_1 > 0$,
from Lemma 3.3, $u^*$ and $\bar{v}^*$ are minimizers for $I(a_1, a_2, a_3)$ and $I^\infty(b_1, b_2, b_3)$ respectively and for all $j \in \{1, 2, 3\}$,

\[ u^*_1 > 0 \quad \text{almost everywhere in } \mathbb{R}^N, \quad u^*_2, u^*_3 \geq 0 \quad \text{for almost every } x \in \mathbb{R}^N, \]  

(3.7)

\[ u^*_j \in H^1(\mathbb{R}^N) \cap C^1(\mathbb{R}^N), \quad \lim_{|x| \to \infty} u^*_j(x) = 0, \]  

(3.8)

\[ u^*_j(-x_1, x') = u^*_j(x_1, x') \quad \text{for almost every } x_1 \in \mathbb{R} \] 

and

\[ \text{for almost every } x' \in \mathbb{R}^{N-1}, \]  

(3.9)

\[ u^*_j(s, x') \geq u^*_j(t, x') \quad \text{for almost every } s, t \in \mathbb{R} \text{ with } 0 \leq s < t \] 

(3.10)

and

\[ v^*_1 > 0 \quad \text{almost everywhere in } \mathbb{R}^N, \quad v^*_2, v^*_3 \geq 0 \quad \text{almost everywhere in } \mathbb{R}^N, \]  

(3.11)

\[ v^*_j \in H^1(\mathbb{R}^N) \cap C^1(\mathbb{R}^N), \quad \lim_{|x| \to \infty} v^*_j(x) = 0, \]  

(3.12)

\[ v^*_j(-x_1, x') = v^*_j(x_1, x') \quad \text{for almost every } x_1 \in \mathbb{R} \] 

and

\[ \text{for almost every } x' \in \mathbb{R}^{N-1}, \]  

(3.13)

\[ v^*_j(s, x') \geq v^*_j(t, x') \quad \text{almost every } s, t \in \mathbb{R} \text{ with } 0 \leq s < t \] 

(3.14)

From (3.7)–(3.14), Lemma 2.12, Corollary 1 and the results in [24], we have

\[
\int_{\mathbb{R}^N} |\nabla (u^*_1 \ast v^*_1)|^2 \, dx < \int_{\mathbb{R}^N} |\nabla u^*_1|^2 \, dx + \int_{\mathbb{R}^N} |\nabla v^*_1|^2 \, dx,
\]  

(3.15)

\[
\int_{\mathbb{R}^N} |\nabla (u^*_j \ast v^*_j)|^2 \, dx \leq \int_{\mathbb{R}^N} |\nabla u^*_j|^2 \, dx + \int_{\mathbb{R}^N} |\nabla v^*_j|^2 \, dx \quad (j = 2, 3),
\]  

(3.16)

\[
\int_{\mathbb{R}^N} (-V_j(x))(u^*_j)^2 \, dx \leq \int_{\mathbb{R}^N} (\nabla V_j(x))(u^*_j \ast v^*_j)^2 \, dx,
\]  

(3.17)

\[
\int_{\mathbb{R}^N} u^*_j u^*_2 u^*_3 \, dx + \int_{\mathbb{R}^N} v^*_1 v^*_2 v^*_3 \, dx \leq \int_{\mathbb{R}^N} (u^*_1 \ast v^*_1)(u^*_2 \ast v^*_2)(u^*_3 \ast v^*_3) \, dx,
\]  

(3.18)

\[
\int_{\mathbb{R}^N} |u^*_j \ast v^*_j|^q \, dx = \int_{\mathbb{R}^N} |u^*_j|^q \, dx + \int_{\mathbb{R}^N} |v^*_j|^q \, dx \quad (q = 2, p + 1).
\]  

(3.19)

From (3.15)–(3.19), it holds that

\[
I(a_1 + b_1, a_2 + b_2, a_3 + b_3) \leq E(u^*_1 \ast v^*_1, u^*_2 \ast v^*_2, u^*_3 \ast v^*_3) \] 

\[
< E(u^*_1, u^*_2, u^*_3) + E^\infty(v^*_1, v^*_2, v^*_3)
\] 

\[
= I(a_1, a_2, a_3) + I^\infty(b_1, b_2, b_3).
\]

\[\square\]

4. Proof of Theorem 1.1. In this section, we divided the proof of Theorem 1.1 into two cases, the case without potentials and with potentials, respectively. Before going to the proof of Theorem 1.1, we describe the relationship between $I$ and $S_{V_j}$.

Lemma 4.1. Suppose that $\gamma, \mu, s \geq 0$ and the potentials $V_j (j = 1, 2, 3)$ satisfy the condition (V1). Then the following holds:

(i) if $\gamma, \mu, s > 0$, then $I(\gamma, \mu, s) < S_{V_1}(\gamma) + S_{V_2}(\mu) + S_{V_3}(s)$.

(ii) if $\gamma = 0$ or $\mu = 0$ or $s = 0$, $I(\gamma, \mu, s) = S_{V_1}(\gamma) + S_{V_2}(\mu) + S_{V_3}(s)$. 



(iii) $I(0, 0, 0) = 0$.
If $V_j(x) \leq 0$ for almost every $x \in \mathbb{R}^N$, then $S_{V_1}(\gamma), S_{V_2}(\mu), S_{V_3}(s) \leq 0$ for $\gamma, \mu, s \geq 0$. It follows that $I(\gamma, \mu, s) \leq 0$. In particular, if $\gamma > 0$ or $\mu > 0$ or $s > 0$, then $I(\gamma, \mu, s) < 0$.

Proof. (i) : From Lemma 2.13, there exist $\phi_1, \phi_2$ and $\phi_3$ such that $\phi_1, \phi_2$ and $\phi_3$ are minimizers for $S_{V_1}(\gamma), S_{V_2}(\mu)$ and $S_{V_3}(s)$ respectively and $\phi_j > 0$ almost everywhere in $\mathbb{R}^N$. Therefore

$$I(\gamma, \mu, s) \leq E(\bar{\phi})$$

$$= J_{V_1}(\phi_1) + J_{V_2}(\phi_2) + J_{V_3}(\phi_3) - \alpha \int_{\mathbb{R}^N} \phi_1 \phi_2 \phi_3 \, dx$$

$$< J_{V_1}(\phi_1) + J_{V_2}(\phi_2) + J_{V_3}(\phi_3)$$

$$= S_{V_1}(\gamma) + S_{V_2}(\mu) + S_{V_3}(s).$$

(ii) and (iii) follow immediately from the definition of $I$ and $S_{V_j}$.

4.1. Proof of Theorem 1.1 for the case without potentials. First, we show the following lemma which guarantees that a minimizing sequence for $I^\infty(\gamma, \mu, s)$ is non-vanishing. We also apply this lemma for different set of parameters $\gamma, \mu, s$ in our argument throughout this subsection.

Lemma 4.2. Let $\bar{\gamma}, \bar{\mu}, \bar{s} \geq 0$ and $\{\bar{u}_n\}_{n=1}^\infty \subset H^1(\mathbb{R}^N; \mathbb{C}^3)$ be a minimizing sequence for $I^\infty(\bar{\gamma}, \bar{\mu}, \bar{s})$. Then, the following holds: If $\bar{\gamma} > 0$, taking a subsequence, there exist $\varepsilon_1 > 0$ and $R_1 > 0$ such that

$$\sup_{y \in \mathbb{R}^N} \int_{|x-y|< R_1} |u_{1,n}|^2 \, dx \geq \varepsilon_1 \text{ for all } n \in \mathbb{N}.$$

If $\bar{\mu} > 0$ or $\bar{s} > 0$, the same statement holds for $\{u_{2,n}\}_{n=1}^\infty$ or $\{u_{3,n}\}_{n=1}^\infty$, respectively.

Proof. We consider two cases in the followings:

Case (a) $\bar{\mu}, \bar{s} > 0$.
Case (b) $\bar{\mu} = 0$ or $\bar{s} = 0$.

Case (a) : Suppose that for all $R > 0$,

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{|x-y|< R} |u_{1,n}|^2 \, dx = 0.$$

From Lemma 2.2,

$$\|u_{1,n}\|_3^3 \leq C \left( \sup_{y \in \mathbb{R}^N} \int_{|x-y|< R} |u_{1,n}|^2 \, dx \right)^{1/2} \|u_{1,n}\|_{H^1}^2.$$

Combining this and $\{u_{1,n}\}_{n=1}^\infty$ is bounded in $H^1(\mathbb{R}^N)$, we have

$$\|u_{1,n}\|_3 \to 0 \quad (n \to \infty)$$

Therefore it follows that

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} u_{1,n} u_{2,n} \overline{u}_{3,n} \, dx = 0.$$
there exist $C > S$

Therefore \( \{ \\}

Thus we have

Proof of Theorem 1.1 for the case without potentials.

From the continuity of $S_\infty$ on $[0, \infty)$ (we can prove this fact as in Lemma 2.6), it holds that

\[
I^\infty(\tilde{\gamma}, \tilde{\mu}, \tilde{s}) = \lim_{n \to \infty} E^\infty(\tilde{u}_n) = \lim_{n \to \infty} (J^\infty(u_{1,n}) + J^\infty(u_{2,n}) + J^\infty(u_{3,n})) \\
\geq \lim_{n \to \infty} \left( S_\infty(\|u_{1,n}\|^2_2) + S_\infty(\|u_{2,n}\|^2_2) + S_\infty(\|u_{3,n}\|^2_2) \right) \\
= S_\infty(\tilde{\gamma}) + S_\infty(\tilde{\mu}) + S_\infty(\tilde{s}).
\]

This is a contradiction with the inequality

\[ I^\infty(\tilde{\gamma}, \tilde{\mu}, \tilde{s}) < S_\infty(\tilde{\gamma}) + S_\infty(\tilde{\mu}) + S_\infty(\tilde{s}) \]

in Lemma 4.1 (i). Thus we obtain the conclusion.

Case (b): Since $\tilde{\mu} = 0$ or $\tilde{s} = 0$, from Lemma 4.1, we have

\[
I^\infty(\tilde{\gamma}, \tilde{\mu}, \tilde{s}) = S_\infty(\tilde{\gamma}) + S_\infty(\tilde{\mu}) + S_\infty(\tilde{s})
\]

Since \( \{ \tilde{u}_n \} \) is a minimizing sequence for $I^\infty(\tilde{\gamma}, \tilde{\mu}, \tilde{s})$, from Lemma 2.4, \( \{ u_{j,n} \}_{n=1}^\infty \) \((j = 1, 2, 3)\) is bounded in $H^1(\mathbb{R}^N)$. Moreover, Since $\|u_{2,n}\|_2 \to 0$ or $\|u_{3,n}\|_2 \to 0$ \((n \to \infty)\), from Gagliardo-Nirenberg’s inequality, it holds that

\[
\lim_{n \to \infty} \alpha \text{Re} \int_{\mathbb{R}^N} u_{1,n} u_{2,n} \overline{u}_{3,n} \, dx = 0 \quad \text{as } n \to \infty.
\]

Note the continuity of $S_\infty$ on $[0, \infty)$, it follows that

\[
S_\infty(\tilde{\gamma}) + S_\infty(\tilde{\mu}) + S_\infty(\tilde{s}) = I^\infty(\tilde{\gamma}, \tilde{\mu}, \tilde{s}) \\
= E^\infty(\tilde{u}_n) + o(1) \\
= J^\infty(u_{1,n}) + J^\infty(u_{2,n}) + J^\infty(u_{3,n}) + o(1) \\
\geq S_\infty(\tilde{\gamma}) + S_\infty(\tilde{\mu}) + S_\infty(\tilde{s}) + o(1).
\]

Thus we have

\[
\lim_{n \to \infty} J^\infty(u_{1,n}) = S_\infty(\tilde{\gamma}).
\]

Therefore \( \{ u_{1,n} \}_{n=1}^\infty \) is a minimizing sequence for $S_\infty(\tilde{\gamma})$. From Lemma 2.2 and the basic properties of the minimizing sequence for $S_\infty(\tilde{\gamma})$ (see [8, Proposition 8.3.6]), there exist $C > 0$, $R > 0$ and $n_0 \in \mathbb{N}$ such that

\[
\sup_{y \in \mathbb{R}^N} \int_{|x-y| < R} |u_{1,n}|^2 \, dx \geq C \quad \text{for all } n \geq n_0.
\]

Now we prove Theorem 1.1 for the case without potentials.

Proof of Theorem 1.1 for the case without potentials. Let \( \{ \tilde{u}_n \}_{n=1}^\infty \subset H^1(\mathbb{R}^N; \mathbb{C}^3) \) be a minimizing sequence for $I^\infty(\gamma, \mu, s)$. The proof proceeds in five steps:

(Step 1.) First, we prove that taking a subsequence, there exist \( \{ x_n \}_{n=1}^\infty \subset \mathbb{R}^N \) and $\tilde{u} \in H^1(\mathbb{R}^N; \mathbb{C}^3)$ such that

\[
u_j \to u_j \text{ weakly in } H^1(\mathbb{R}^N) \quad (j = 1, 2, 3, \ n \to \infty),
\]

\[u_1 \neq 0.
\]

Since $\gamma > 0$, by Lemma 4.2, taking a subsequence, there exist $\varepsilon_1 > 0$ and $R_1 > 0$ such that

\[
\sup_{y \in \mathbb{R}^N} \int_{|x-y| < R_1} |u_{1,n}|^2 \, dx \geq \varepsilon_1 \quad \text{for all } n \in \mathbb{N}.
\]
Therefore, there exists \( \{x_n\}_{n=1}^{\infty} \subset \mathbb{R}^N \) such that 
\[
\int_{|x-x_n|<R_1} |u_{1,n}|^2 \, dx > \frac{\varepsilon_1}{2} \quad \text{for all } n \in \mathbb{N}.
\]
On the other hand, since \( \{\tilde{u}_n\}_{n=1}^{\infty} \) is bounded in \( H^1(\mathbb{R}^N; \mathbb{C}^3) \) from Lemma 2.4, taking a subsequence, there exists \( \tilde{u} \in H^1(\mathbb{R}^N; \mathbb{C}^3) \) such that 
\[
\tilde{u}_{j,n}(x + x_n) \rightharpoonup u_j \quad \text{weakly in } H^1(\mathbb{R}^N) \quad (j = 1, 2, 3).
\]
Now, we remark that 
\[
\int_{|x|<R_1} |u_{1,n}(x + x_n)|^2 \, dx = \int_{|x-x_n|<R_1} |u_{1,n}|^2 \, dx > \frac{\varepsilon_1}{2} \quad \text{for all } n \in \mathbb{N}.
\]
Since the embedding \( H^1(\{|x|<R_1\}) \subset L^2(\{|x|<R_1\}) \) is compact, we have 
\[
\int_{|x|<R_1} |u_1|^2 \, dx \geq \frac{\varepsilon_1}{2} > 0,
\]
that is, we have \( u_1 \neq 0 \).

**Claim 1.** Taking a subsequence, for \( q = 2 \) and \( q = p + 1 \), we have 
\[
\int_{\mathbb{R}^N} |u_{j,n}|^q \, dx = \int_{\mathbb{R}^N} |u_j|^q \, dx + \int_{\mathbb{R}^N} |v_{j,n}|^q \, dx + o(1) \quad (as \ n \to \infty).
\]
This statement follows immediately from Brezis-Lieb Lemma (see e.g. [6]).

**Claim 2.**
\[
\int_{\mathbb{R}^N} (|\nabla u_{j,n}|^2 - |\nabla u_j|^2 - |\nabla v_{j,n}|^2) \, dx = o(1) \quad (as \ n \to \infty).
\]
Indeed, note that 
\[
\int_{\mathbb{R}^N} (|\nabla u_{j,n}|^2 - |\nabla u_j|^2 - |\nabla v_{j,n}|^2) \, dx = 2\Re \int_{\mathbb{R}^N} \nabla u_j \cdot \nabla v_{j,n} \, dx,
\]
where \( \nabla u \cdot \nabla v := \sum_{k=1}^{N} \frac{\partial u}{\partial x_k} \frac{\partial v}{\partial x_k} \). Since \( v_{j,n} \to 0 \) weakly in \( H^1(\mathbb{R}^N) \), it follows 
\[
\int_{\mathbb{R}^N} \nabla u_j \cdot \nabla v_{j,n} \, dx = o(1) \quad (as \ n \to \infty).
\]
Hence we get Claim 2.

**Claim 3.**
\[
\int_{\mathbb{R}^N} u_{1,n} u_{2,n} \overline{v}_{3,n} \, dx = \int_{\mathbb{R}^N} u_{1} u_{2} \overline{v}_3 \, dx + \int_{\mathbb{R}^N} u_{1,n} v_{2,n} \overline{v}_{3,n} \, dx + o(1) \quad (as \ n \to \infty).
\]
Actually, since 
\[
\int_{\mathbb{R}^N} u_{1,n} u_{2,n} \overline{v}_{3,n} \, dx = \int_{\mathbb{R}^N} (u_1 + v_{1,n})(u_2 + v_{2,n})(\overline{v}_3 + \overline{v}_{3,n}) \, dx
\]
\[
= \int_{\mathbb{R}^N} u_1 u_2 \overline{v}_3 \, dx + \int_{\mathbb{R}^N} u_{1} u_{2} \overline{v}_3 \, dx + \int_{\mathbb{R}^N} u_{1,n} v_{2,n} \overline{v}_{3,n} \, dx + \int_{\mathbb{R}^N} u_{1,n} v_{2,n} \overline{v}_{3,n} \, dx
\]
Since we can prove these easily by using $v_{j,n} \to 0$ ($j = 1, 2, 3$) in $L^3_{\text{loc}}(\mathbb{R}^N)$, we omit the details.

From Claim 1–Claim 3, it follows that
\[ E^\infty(\vec{u}_n) = E^\infty(\vec{u}) + E^\infty(\vec{v}_n) + o(1) \quad \text{as } n \to \infty. \quad (4.3) \]

From Claim 1 we have also
\[ \|u_{j,n}\|^2_2 = \|u_j\|^2_2 + \|v_{j,n}\|^2_2 + o(1) \quad \text{as } n \to \infty. \]

Letting $n \to \infty$ in (4.3), from Lemma 2.6 and Lemma 3.2, we have
\[
I^\infty(\gamma, \mu, s) = E^\infty(\vec{u}) + \lim_{n \to \infty} E^\infty(\vec{v}_n)
\geq I^\infty(\gamma', \mu', s') + \lim_{n \to \infty} I^\infty(\|v_{1,n}\|^2_2, \|v_{2,n}\|^2_2, \|v_{3,n}\|^2_2)
= I^\infty(\gamma', \mu', s') + I^\infty(\gamma - \gamma', \mu - \mu', s - s')
\geq I^\infty(\gamma, \mu, s).
\]

Therefore,
\[
E^\infty(\vec{u}) = I^\infty(\gamma', \mu', s'),
I^\infty(\gamma, \mu, s) = I^\infty(\gamma', \mu', s') + I^\infty(\gamma - \gamma', \mu - \mu', s - s'). \quad (4.4)
\]

That is, $\vec{u}$ is a minimizer for $I^\infty(\gamma', \mu', s')$.

(Step 3.) We prove $\gamma' = \gamma$. Suppose by contradiction that $\gamma' < \gamma$. By the same argument as in (Step 1), there exist a minimizing sequence $\{\vec{v}_n\}_{n=1}^\infty$ for $I^\infty(\gamma - \gamma', \mu - \mu', s - s')$ and a function $\vec{v} \in H^1(\mathbb{R}^N, \mathbb{C}^3)$ with $v_1 \neq 0$ such that
\[ v_{j,n} \rightharpoonup v_j \text{ weakly in } H^1(\mathbb{R}^N) \quad (j = 1, 2, 3). \]

Set $\gamma'' := \|v_1\|^2_2(> 0)$, $\mu'' := \|v_2\|^2_2$, $s'' := \|v_3\|^2_2$. Using the same argument as in (Step 2), we have
\[
I^\infty(\gamma - \gamma', \mu - \mu', s - s') = I^\infty(\gamma'', \mu'', s'') +
+ I^\infty(\gamma - \gamma', \mu - \mu', s - s' - s''), \quad (4.5)
E^\infty(\vec{v}) = I^\infty(\gamma'', \mu'', s'').
\]

Now, $\vec{u}$ and $\vec{v}$ are minimizers for $I^\infty(\gamma', \mu', s')$ and $I^\infty(\gamma'', \mu'', s'')$ respectively. Furthermore since $\gamma' > 0$ and $\gamma'' > 0$, from Proposition 3, it follows that
\[
I^\infty(\gamma' + \gamma'', \mu' + \mu'', s' + s'') < I^\infty(\gamma', \mu', s') + I^\infty(\gamma'', \mu'', s''). \quad (4.6)
\]
Therefore, combining (4.4),(4.5),(4.6) and using Lemma 3.2, we arrive at
\[
I^\infty(\gamma, \mu, s) = I^\infty(\gamma', \mu', s') + I^\infty(\gamma - \gamma', \mu - \mu', s - s')
\]
\[
= I^\infty(\gamma', \mu', s') + I^\infty(\gamma'', \mu'', s'') + I^\infty(\gamma - \gamma' - \gamma'', \mu - \mu' - \mu'', s - s' - s'')
\]
\[
> I^\infty(\gamma' + \gamma'', \mu' + \mu'', s' + s'') + I^\infty(\gamma - \gamma' - \gamma'', \mu - \mu' - \mu'', s - s' - s'')
\]
\[
\geq I^\infty(\gamma, \mu, s).
\]

This is a contradiction. Hence, it follows that \(\gamma' = \gamma\).

(Step 4.) We prove \(\mu' = \mu\) and \(s' = s\). First, we show \(\mu' > 0\). We assume that \(\mu' = 0\), then by \(\gamma' = \gamma\) and the subadditivity for \(S_\infty\) (see Lions [19, 20] for more details), it follows that
\[
I^\infty(\gamma, \mu, s) = I^\infty(\gamma, 0, s') + I^\infty(0, \mu, s - s')
\]
\[
= S_\infty(\gamma) + S_\infty(s') + S_\infty(s - s')
\]
\[
\geq S_\infty(\gamma) + S_\infty(\mu) + S_\infty(s).
\]

This contradicts Lemma 4.1 (i). Thus, we have \(\mu' > 0\). We can show \(\mu' = \mu\) as in (Step 3). In a similar way, we can also prove \(s' = s\).

(Step 5.) We prove that \(\lim_{n \to \infty} ||\bar{u}_n(\cdot + x_n) - \bar{u}||_{H^1} = 0\) and \(\bar{u}\) is minimizer for \(I^\infty(\gamma, \mu, s)\). From (Step 1)–(Step 4), we have proved
\[
u_j, n \to u_j \text{ weakly in } H^1(\mathbb{R}^N) \quad (j = 1, 2, 3),
\]
\[
||u_1||^2_2 = \gamma, \quad ||u_2||^2_2 = \mu, \quad ||u_3||^2_2 = s.
\]

Moreover, note that \(||u_{1,n}||^2_2 \to \gamma, \quad ||u_{2,n}||^2_2 \to \mu\) and \(||u_{3,n}||^2_2 \to s\) \((n \to \infty)\), we have
\[
||u_{j,n}(\cdot + x_n) - u_j||_2 \to 0 \quad (n \to \infty).
\]

Here, we set \(\phi_{j,n} := u_{j,n}(\cdot + x_n) - u_j \quad (j = 1, 2, 3)\). By \(\{\phi_{j,n}\}_{n=1}^\infty\) is bounded in \(H^1(\mathbb{R}^N)\) and Gagliardo–Nirenberg’s inequality, it holds that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |\phi_{j,n}|^{p+1} \, dx = 0, \quad (4.8)
\]
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \phi_{1,n} \phi_{2,n} \phi_{3,n} \, dx = 0. \quad (4.9)
\]

Indeed, by Gagliardo–Nirenberg’s inequality, for each \(q = p + 1\) and \(3\), there exists a constant \(C(q, N) > 0\) respectively such that
\[
||\phi_{j,n}||^q_q^2 \leq C(q, N)||\nabla \phi_{j,n}||^N(q-2)/2^2 ||\phi_{j,n}||^{q-N(q-2)/2}.
\]

Note that \(\{\phi_{j,n}\}_{n=1}^\infty\) is bounded in \(H^1(\mathbb{R}^N)\) and (4.7), it follows that
\[
||\phi_{j,n}||^{p+1}_{p+1} \to 0, \quad ||\phi_{j,n}||^3_3 \to 0. \quad (4.10)
\]

From this, we have (4.8) and (4.9). By (Step 2), since it follows that
\[
E^\infty(\bar{u}_n) = E^\infty(\bar{u}) + E^\infty(\bar{\phi}_n) + o(1) \quad (n \to \infty).
\]

Letting \(n \to \infty\), we have
\[
I^\infty(\gamma, \mu, s) = E^\infty(\bar{u}) + \lim_{n \to \infty} E^\infty(\bar{\phi}_n)
\]
\[
= E^\infty(\bar{u}) + \lim_{n \to \infty} \frac{1}{2} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |\nabla \phi_{j,n}|^2 \, dx
\]
\[
\geq E^\infty(\bar{u}) \geq I^\infty(\gamma, \mu, s).
\]
Hence,
\[ E^\infty(\vec{u}) = I^\infty(\gamma, \mu, s), \]
\[ \lim_{n \to \infty} \frac{1}{2} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |\nabla \phi_{j,n}|^2 \, dx = 0. \]

From this and (4.7), the conclusion follows. \qed

4.2. Proof of Theorem 1.1 for the case with potentials. First, we note the following.

**Lemma 4.3.** Let \( \gamma, \mu, s \geq 0 \). Then \( I^\infty(\gamma, \mu, s) \) has a minimizer.

**Proof.** If \( \gamma, \mu, s > 0 \), this statement follows immediately from Theorem 1.1 for the case without potentials. On the other hand, if \( \gamma = 0 \) or \( \mu = 0 \) or \( s = 0 \), from Lemma 4.1, we have \( I^\infty(\gamma, \mu, s) = S^\infty_\gamma + S^\infty_\mu + S^\infty_s \). If \( \gamma > 0 \), from the compactness of the minimizing sequence for \( S^\infty_\gamma \), \( S^\infty_\gamma \) has a minimizer. If \( \gamma = 0 \), \( S^\infty_0 \) has a minimizer 0. Similarly, \( S^\infty_\mu \) and \( S^\infty_s \) have minimizers respectively. Therefore, \( I^\infty(\gamma, \mu, s) \) has a minimizer. \( \square \)

The next lemma plays an important role in proof of the strict inequality about relationship between \( I \) and \( I^\infty \).

**Lemma 4.4.** Let \( \gamma, \mu, s \geq 0 \). Then there exists a minimizer \( \vec{\phi} \) for \( I^\infty(\gamma, \mu, s) \) satisfying the following conditions:

(i) if \( \gamma > 0 \), then \( \phi_1 > 0 \) almost everywhere in \( \mathbb{R}^N \).

(ii) if \( \mu > 0 \), then \( \phi_2 > 0 \) almost everywhere in \( \mathbb{R}^N \).

(iii) if \( s > 0 \), then \( \phi_3 > 0 \) almost everywhere in \( \mathbb{R}^N \).

Where we denote by \( I^\infty \) instead of \( I \) if \( V \equiv 0 \) for all \( j = 1, 2, 3 \).

**Proof.** By Lemma 4.3, \( I^\infty(\gamma, \mu, s) \) has a minimizer. Thus we can prove this Lemma by the same argument as in the proof of Lemma 3.3. \( \square \)

We describe the relationship between \( I \) and \( I^\infty \). This lemma guarantees that minimizing sequence for \( I(\gamma, \mu, s) \) with potentials does not vanish.

**Lemma 4.5.** Let \( \gamma, \mu, s > 0 \). Then it follows that \( I(\gamma, \mu, s) < I^\infty(\gamma, \mu, s) \).

**Proof.** From Lemma 4.4, there exists a minimizer \( \vec{\phi} = (\phi_1, \phi_2, \phi_3) \) for \( I^\infty(\gamma, \mu, s) \) such that \( \phi_j > 0 \) almost everywhere in \( \mathbb{R}^N \). Since \( V_1 \not\equiv 0 \) or \( V_2 \not\equiv 0 \) or \( V_3 \not\equiv 0 \), it follows that

\[ \sum_{j=1}^{3} \int_{\mathbb{R}^N} V_j(x) \phi_j^2 \, dx < 0. \]

Thus we have

\[ I(\gamma, \mu, s) \leq E(\vec{\phi}) < E^\infty(\vec{\phi}) = I^\infty(\gamma, \mu, s). \]

Now we prove Theorem 1.1 for the case with potentials.

**Proof of Theorem 1.1 for the case with potentials.** Let \( \{\vec{u}_n\}_{n=1}^{\infty} \subset H^1(\mathbb{R}^N; \mathbb{C}^3) \) be a minimizing sequence for \( I(\gamma, \mu, s) \). Since \( \{\vec{u}_n\}_{n=1}^{\infty} \) is bounded in \( H^1(\mathbb{R}^N; \mathbb{C}^3) \), there exists \( \vec{u} \in H^1(\mathbb{R}^N; \mathbb{C}^3) \) such that

\[ u_{j,n} \rightharpoonup u_j \text{ weakly in } H^1(\mathbb{R}^N) \quad (j = 1, 2, 3, \quad n \to \infty). \]
(Step 1.) Set \( \gamma := \|u_1\|^2_2, \mu := \|u_2\|^2_2, s' := \|u_3\|^2_2 \). We prove that \( \vec{u} \) is a minimizer for \( I(\gamma', \mu', s') \) and taking a subsequence, the followings hold:

\[
\|u_{j,n}\|^2_2 = \|u_j\|^2_2 + \|u_{j,n} - u_j\|^2_2 + o(1) \quad (j = 1, 2, 3, \text{ as } n \to \infty),
\]

\[
E(\vec{u}_n) = E(\vec{u}) + E^\infty(\vec{u}_n - \vec{u}) + o(1) \quad (\text{as } n \to \infty),
\]

\[
I(\gamma, \mu, s) = I(\gamma', \mu', s') + I^\infty(\gamma - \gamma', \mu - \mu', s - s'),
\]

\( \gamma' > 0 \text{ or } \mu' > 0 \text{ or } s' > 0 \).

From the same argument as in the Step 2 of the proof of Theorem 1.1 for the case without potentials, taking a subsequence, we have

\[
\int_{\mathbb{R}^N} |u_{j,n}|^2 \, dx = \int_{\mathbb{R}^N} |u_j|^2 \, dx + \int_{\mathbb{R}^N} |u_{j,n} - u_j|^2 \, dx + o(1) \quad (\text{as } n \to \infty), \tag{4.11}
\]

\[
E^\infty(\vec{u}_n) = E^\infty(\vec{u}) + E^\infty(\vec{u}_n - \vec{u}) + o(1) \quad (\text{as } n \to \infty). \tag{4.12}
\]

Moreover, from Lemma 2.7, it follows that

\[
\int_{\mathbb{R}^N} V_j(x)|u_{j,n}|^2 \, dx = \int_{\mathbb{R}^N} V_j(x)|u_j|^2 \, dx + o(1) \quad (\text{as } n \to \infty, \, j = 1, 2, 3). \tag{4.13}
\]

From (4.12),(4.13), we have

\[
E(\vec{u}_n) = E(\vec{u}) + E^\infty(\vec{u}_n - \vec{u}) + o(1) \quad (\text{as } n \to \infty). \tag{4.14}
\]

By the same argument as in the proof of Theorem 1.1 for the case without potentials, we can prove that

\[
E(\vec{u}) = I(\gamma', \mu', s'),
\]

\[
I(\gamma, \mu, s) = I(\gamma', \mu', s') + I^\infty(\gamma - \gamma', \mu - \mu', s - s'). \tag{4.15}
\]

That is, \( \vec{u} \) is a minimizer for \( I(\gamma', \mu', s') \). Suppose \( \gamma' = \mu' = s' = 0 \). From \( I(0, 0, 0) = 0 \) and (4.15),

\[
I(\gamma, \mu, s) = I^\infty(\gamma, \mu, s).
\]

This contradicts Lemma 4.5. Therefore we have \( \gamma' > 0 \text{ or } \mu' > 0 \text{ or } s' > 0 \).

(Step 2.) From (Step 1), \( \gamma' > 0 \text{ or } \mu' > 0 \text{ or } s' > 0 \). We consider only the case \( \gamma' > 0 \). We prove \( \gamma' = \gamma \). Suppose \( \gamma' < \gamma \). From (Step 1), \( \vec{u} \) is a minimizer for \( I(\gamma', \mu', s') \) and from Lemma 4.3, \( I^\infty(\gamma - \gamma', \mu - \mu', s - s') \) has a minimizer. Since \( \gamma' > 0, \gamma - \gamma' > 0 \), from Proposition 3, we have

\[
I(\gamma, \mu, s) < I(\gamma', \mu', s') + I^\infty(\gamma - \gamma', \mu - \mu', s - s').
\]

This contradicts (4.15).

(Conclusion.) We can prove \( \mu' = \mu \) and \( s' = s \) by the same argument as in the Step 4 of the proof of Theorem 1.1 for the case without potentials. Then we can prove that \( \lim_{n \to \infty} \|\vec{u}_n - \vec{u}\|_{H^1} = 0 \) and \( \vec{u} \) is a minimizer for \( I(\gamma, \mu, s) \) by the same argument as in the Step 5 of the proof of Theorem 1.1 for the case without potentials. \( \square \)

5. Proof of Theorem 1.2. The following lemma is the finiteness of \( J(\gamma, \mu) \) and the boundedness of a minimizing sequence for \( J(\gamma, \mu) \).
Lemma 5.1. Let $\gamma, \mu \geq 0$. Then $J(\gamma, \mu) > -\infty$. Also, for any minimizing sequence $\{\tilde{u}_n\}_{n=1}^{\infty}$ for $J(\gamma, \mu)$, $\{\tilde{u}_n\}_{n=1}^{\infty}$ is bounded in $H^1(\mathbb{R}^N; \mathbb{C}^2)$. Here, $\{\tilde{u}_n\}_{n=1}^{\infty}$ is said to be a minimizing sequence for $J(\gamma, \mu)$ if $\{\tilde{u}_n\}_{n=1}^{\infty}$ satisfies the following conditions:

$$Q_1(\tilde{u}_n) \to \gamma, \quad Q_2(\tilde{u}_n) \to \mu,$$

$$E(\tilde{u}_n) \to J(\gamma, \mu) \quad (\text{as } n \to \infty).$$

Proof. We can prove this lemma by the same argument as in the proof of Lemma 2.4. \qed

The following lemma is the relationship between $I$ and $J$. Any minimizing sequence for $J(\gamma, \mu)$ becomes a minimizing sequence for $I(\gamma - a, \mu - a, a)$ for some $a > 0$.

Lemma 5.2. Let $\gamma, \mu > 0$ and $\{\tilde{u}_n\}_{n=1}^{\infty}$ any minimizing sequence for $J(\gamma, \mu)$. Then, taking a subsequence of $\{\tilde{u}_n\}_{n=1}^{\infty}$ (we still denote by $\tilde{u}_n$), there exists $0 < a \leq \min\{\gamma, \mu\}$ (in particular, if $\gamma \neq \mu$, then $a < \min\{\gamma, \mu\}$) such that

$$\|u_{1,n}\|_2^2 \to \gamma - a, \quad \|u_{2,n}\|_2^2 \to \mu - a, \quad \|u_{3,n}\|_2^2 \to a,$$

$$E(\tilde{u}_n) \to I(\gamma - a, \mu - a, a) \quad (\text{as } n \to \infty).$$

That is, $\{\tilde{u}_n\}_{n=1}^{\infty}$ is a minimizing sequence for $I(\gamma - a, \mu - a, a)$. Particularly, we have $J(\gamma, \mu) = I(\gamma - a, \mu - a, a)$.

Proof. Since $\|u_{1,n}\|_2^2 + \|u_{3,n}\|_2^2 \to \gamma$ and $\|u_{2,n}\|_2^2 + \|u_{3,n}\|_2^2 \to \mu$, $\{\|u_{3,n}\|_2^2\}_{n=1}^{\infty}$ is bounded. Therefore, taking a subsequence, there exists $a \in [0, \min\{\gamma, \mu\}]$ such that $\|u_{3,n}\|_2^2 \to a$ (as $n \to \infty$).

Therefore, it follows that

$$\|u_{1,n}\|_2^2 \to \gamma - a, \quad \|u_{2,n}\|_2^2 \to \mu - a.$$  

Note that $I$ is continuous on $[0, \infty) \times [0, \infty) \times [0, \infty)$ (Lemma 2.6), we have

$$J(\gamma, \mu) = \lim_{n \to \infty} E(\tilde{u}_n) \geq \lim_{n \to \infty} I(\|u_{1,n}\|_2^2, \|u_{2,n}\|_2^2, \|u_{3,n}\|_2^2) = I(\gamma - a, \mu - a, a).$$  

(5.1)

On the other hand, it follows that

$$J(\gamma, \mu) \leq I(\gamma - a, \mu - a, a).$$  

(5.2)

From (5.1) and (5.2), we have

$$J(\gamma, \mu) = I(\gamma - a, \mu - a, a).$$

Thus we also obtain that

$$\lim_{n \to \infty} E(\tilde{u}_n) = I(\gamma - a, \mu - a, a).$$

Claim 1. $a > 0$.

Suppose $a = 0$, we have $\|u_{1,n}\|_2^2 \to \gamma$, $\|u_{2,n}\|_2^2 \to \mu$ and $\|u_{3,n}\|_2^2 \to 0$ (as $n \to \infty$). From (5.1), it follows that

$$J(\gamma, \mu) = I(\gamma, \mu, 0) = S_{V_1}(\gamma) + S_{V_2}(\mu).$$
From Lemma 2.13, there exist minimizers \( \phi_1, \phi_2 \) for \( S_{V_1}(\gamma) \) and \( S_{V_2}(\mu) \) such that \( \phi_1, \phi_2 > 0 \). Thus we have

\[
J(\gamma, \mu) \leq E(\phi_1, \phi_2, 0) = J_{V_1}(\phi_1) + J_{V_2}(\phi_2) = S_{V_1}(\gamma) + S_{V_2}(\mu).
\]

From the above, it holds that

\[
J(\gamma, \mu) = S_{V_1}(\gamma) + S_{V_2}(\mu) = E(\phi_1, \phi_2, 0).
\]

Thus (\( \phi_1, \phi_2, 0 \)) is a minimizer for \( J(\gamma, \mu) \). From the Euler-Lagrange equation, for all \( \psi \in H^1(\mathbb{R}^N) \),

\[
a \text{Re} \int_{\mathbb{R}^N} \phi_1 \phi_2 \overline{\psi} = 0.
\]

Thus we have

\[
\phi_1 \phi_2 = 0 \quad \text{almost everywhere in } \mathbb{R}^N.
\]

This contradicts \( \phi_1, \phi_2 > 0 \) almost everywhere in \( \mathbb{R}^N \). Thus we have \( a > 0 \).

**Claim 2.** \( \gamma \neq \mu \implies a < \min \{\gamma, \mu\} \).

We can prove this claim by the same way as in Claim 1.

The next lemma is the compactness of a minimizing sequence for \( J(\gamma, \mu) \).

**Lemma 5.3.** Let \( \gamma, \mu > 0 \) and \( \{\tilde{u}_n\}_{n=1}^\infty \) a minimizing sequence for \( J(\gamma, \mu) \). Then, by Lemma 5.2, taking a subsequence, there exists \( 0 < a \leq \min \{\gamma, \mu\} \) (in particular, if \( \gamma \neq \mu \), then \( a < \min \{\gamma, \mu\} \) such that

\[
\|u_{1,n}\|^2 \to \gamma - a, \quad \|u_{2,n}\|^2 \to \mu - a, \quad \|u_{3,n}\|^2 \to a,
\]

\[
E(\tilde{u}_n) \to I(\gamma - a, \mu - a, a) \quad (n \to \infty).
\]

That is, \( \{\tilde{u}_n\}_{n=1}^\infty \) is a minimizing sequence for \( I(\gamma - a, \mu - a, a) \). Then, the following cases occur:

**Case (i)** \( a < \min \{\gamma, \mu\} \) and \( V_j \equiv 0 \) for all \( j = 1, 2, 3 \);

Taking a subsequence, there exist \( \{y_n\}_{n=1}^\infty \subset \mathbb{R}^N \) and \( \tilde{u} \in \mathcal{M}_{\gamma, \mu} \) such that

\[
\lim_{n \to \infty} \|\tilde{u}_n(\cdot + y_n) - \tilde{u}\|_{H^1} = 0.
\]

Moreover, we have \( \tilde{u}(\cdot - y_n) \in \mathcal{M}_{\gamma, \mu} \).

**Case (ii)** \( a < \min \{\gamma, \mu\} \) and \( V_1 \neq 0 \) or \( V_2 \neq 0 \) or \( V_3 \neq 0 \);

Taking a subsequence, there exists \( \tilde{u} \in \mathcal{M}_{\gamma, \mu} \) such that

\[
\lim_{n \to \infty} \|\tilde{u}_n - \tilde{u}\|_{H^1} = 0.
\]

**Case (iii)** \( \gamma = \mu, \ a = \gamma \) and \( V_3 \equiv 0 \);

Taking a subsequence, there exist \( \{y_n\}_{n=1}^\infty \subset \mathbb{R}^N \) and \( u_3 \in H^1(\mathbb{R}^N) \)

\[
\lim_{n \to \infty} \|\tilde{u}_n(\cdot + y_n) - (0, 0, u_3)\|_{H^1} = 0.
\]

Moreover, \( (0, 0, u_3), (0, u_3), (u_3, - y_n) \) \( \in \mathcal{M}_{\gamma, \mu} \).

**Case (iv)** \( \gamma = \mu, \ a = \gamma \) and \( V_3 \neq 0 \);

Taking a subsequence, there exists \( u_3 \in H^1(\mathbb{R}^N) \) such that

\[
\lim_{n \to \infty} \|\tilde{u}_n - (0, 0, u_3)\|_{H^1} = 0.
\]

Moreover, \( (0, 0, u_3) \in \mathcal{M}_{\gamma, \gamma} \).
Proof. Case (i) Since \( V_j \equiv 0 \) \((j = 1, 2, 3)\), note that \( E(\bar{u}) = E^\infty(\bar{u}) \), \( I(\gamma - a, \mu - a, a) = I^\infty(\gamma - a, \mu - a, a) \). Since \( \gamma - a, \mu - a, a > 0 \) and \( \{\bar{u}_n\}_{n=1}^{\infty} \) is a minimizing sequence for \( I^\infty(\gamma - a, \mu - a, a) \), from Theorem 1.1 for the case without potentials, taking a subsequence, there exist \( \{y_n\}_{n=1}^{\infty} \subset \mathbb{R}^N \) and \( \bar{u} \in H^1(\mathbb{R}^N; C^3) \) such that

\[
\bar{u} \text{ is a minimizer for } I^\infty(\gamma - a, \mu - a, a),
\]

\[
\|u_{j,n}(\cdot + y_n) - u_j\|_{H^1} \to 0 \ (n \to \infty, \ j = 1, 2, 3).
\]

From Lemma 5.2, \( J(\gamma, \mu) = I^\infty(\gamma - a, \mu - a, a) \). Therefore, \( \bar{u} \) is a minimizer for \( J(\gamma, \mu) \). Furthermore, since \( E^\infty \) is translation invariant, \( \bar{u}(\cdot - y_n) \) is also a minimizer for \( J(\gamma, \mu) \).

Case (ii) Since \( \gamma - a, \mu - a, a > 0 \), \( \{\bar{u}_n\}_{n=1}^{\infty} \) is a minimizing sequence for \( I(\gamma - a, \mu - a, a) \) and there exists \( j = 1, 2, 3 \) such that \( V_j \neq 0 \), from Theorem 1.1 for the case with potentials, taking a subsequence, there exists \( \bar{u} \in H^1(\mathbb{R}^N; C^3) \) such that

\[
\bar{u} \text{ is a minimizer for } I(\gamma - a, \mu - a, a),
\]

\[
\|u_{j,n} - u_j\|_{H^1} \to 0 \ (n \to \infty, \ j = 1, 2, 3).
\]

From Lemma 5.2, \( J(\gamma, \mu) = I(\gamma - a, \mu - a, a) \). Therefore, \( \bar{u} \) is a minimizer for \( J(\gamma, \mu) \).

Case (iii) Note that \( \|u_{1,n}\|_{H^1}^2 \to 0 \), \( \|u_{2,n}\|_{H^1}^2 \to 0 \), \( \|u_{3,n}\|_{H^1}^2 \to \gamma \ (n \to \infty) \), since Gagliardo–Nirenberg’s inequality and \( \{u_{j,n}\}_{n=1}^{\infty} \ (j = 1, 2, 3) \) is bounded in \( H^1(\mathbb{R}^N) \), we have

\[
\int_{\mathbb{R}^N} V_1(x)|u_{1,n}|^2 dx \to 0, \quad \int_{\mathbb{R}^N} V_2(x)|u_{2,n}|^2 dx \to 0 dx,
\]

\[
\|u_{1,n}\|_{p+1} \to 0, \quad \|u_{2,n}\|_{p+1} \to 0,
\]

\[
\int_{\mathbb{R}^N} u_{1,n} u_{2,n} u_{3,n} dx \to 0 \ (n \to \infty).
\]

Claim 1. \( \|u_{1,n}\|_{H^1} \to 0 \), \( \|u_{2,n}\|_{H^1} \to 0 \ (n \to \infty) \).

Indeed, since \( V_3 \equiv 0 \), note that \( J_{\gamma}(u_{3,n}) = J_\infty(u_{3,n}) \), \( I(0, 0, \gamma) = I(0, 0, \gamma) \), (5.3)–(5.5) and the continuity of \( S_\infty \) on \([0, \infty)\), it follows that

\[
S_\infty(\gamma) = I(0, 0, \gamma)
\]

\[
= \lim_{n \to \infty} E(\bar{u}_n)
\]

\[
= \lim_{n \to \infty} \left\{ \frac{1}{2} \sum_{j=1}^{2} \|\nabla u_{j,n}\|_2^2 + J_\infty(u_{3,n}) \right\}
\]

\[
\geq \limsup_{n \to \infty} \frac{1}{2} \sum_{j=1}^{2} \|\nabla u_{j,n}\|_2^2 + S_\infty(\gamma).
\]

Therefore, it holds that

\[
\lim_{n \to \infty} \frac{1}{2} \sum_{j=1}^{2} \|\nabla u_{j,n}\|_2^2 = 0.
\]

Combining this and \( \|u_{1,n}\|_{H^1}^2 \to 0 \), \( \|u_{2,n}\|_{H^1}^2 \to 0 \ (n \to \infty) \), we have

\[
\|u_{1,n}\|_{H^1} \to 0, \quad \|u_{2,n}\|_{H^1} \to 0 \ (n \to \infty).
\]
From the above, we have $J_{V_1}(u_{1,n}) \to 0$, $J_{V_2}(u_{2,n}) \to 0$ ($n \to \infty$). Thus, it follows that

$$S_\infty(\gamma) = I(0,0,\gamma) = \lim_{n \to \infty} J_\infty(u_{3,n}).$$

Therefore, $\{u_{3,n}\}_{n=1}^\infty$ is a minimizing sequence for $S_\infty(\gamma)$. From the compactness of the minimizing sequence for $S_\infty(\gamma)$, there exist $\{y_n\}_{n=1}^\infty \subset \mathbb{R}^N$ and $u_3 \in H^1(\mathbb{R}^N)$ such that

- $u_3$ is a minimizer for $S_\infty(\gamma)$,
- $\lim_{n \to \infty} \|u_{3,n}(\cdot + y_n) - u_3\|_{H^1} = 0$.

Combining this and (5.6), we have

$$\lim_{n \to \infty} \|\vec{u}_n(\cdot + y_n) - (0,0,u_3)\|_{H^1} = 0.$$

Next, we prove $(0,0,u_3) \in \mathcal{M}_{\gamma,\gamma}$. From Lemma 5.2, it follows that

$$J(\gamma,\gamma) = I(0,0,\gamma).$$

Note $V_3 \equiv 0$, we have

$$J_\infty(u_3) = E(0,0,u_3), \quad S_\infty(\gamma) = I(0,0,\gamma)$$

Furthermore, since $u_3$ is a minimizer for $S_\infty(\gamma)$, it holds that

$$\|u_3\|^2_2 = \gamma, \quad J_\infty(u_3) = S_\infty(\gamma).$$

From (5.7)–(5.10), we have

$$Q_1(0,0,u_3) = \|u_3\|^2_2 = \gamma, \quad Q_2(0,0,u_3) = \|u_3\|^2_2 = \gamma,$$

$$E(0,0,u_3) = J(\gamma,\gamma).$$

Therefore, $(0,0,u_3) \in \mathcal{M}_{\gamma,\gamma}$. Note that $V_3 \equiv 0$, since $E$ is translation invariant for third component, it follows also that $(0,0,u_3(\cdot - y_n)) \in \mathcal{M}_{\gamma,\gamma}$.

**Case (iv)** In a similar way to Case (iii), we have

$$\|u_{1,n}\|_{H^1} \to 0, \quad \|u_{2,n}\|_{H^1} \to 0 \quad (n \to \infty), \quad \{u_{3,n}\}_{n=1}^\infty \text{ is a minimizing sequence for } S_{V_1}(\gamma).$$

Hence from the compactness of the minimizing sequence for $S_{V_1}(\gamma)$, there exists $u_3 \in H^1(\mathbb{R}^N)$ such that

- $u_3$ is a minimizer for $S_{V_1}(\gamma)$,
- $\lim_{n \to \infty} \|u_{3,n} - u_3\|_{H^1} = 0$.

Combining these and (5.11), we have

$$\lim_{n \to \infty} \|\vec{u}_n - (0,0,u_3)\|_{H^1} = 0.$$

In a similar way to Case (iii), we can prove $(0,0,u_3) \in \mathcal{M}_{\gamma,\gamma}$. 

Now we prove Theorem 1.2 about the stability for the set of minimizers for $J(\gamma,\mu)$. 

Proof of Theorem 1.2. Assume that $M_{\gamma, \mu}$ is unstable. Then there exists $\varepsilon > 0$ such that for all $n \in \mathbb{N}$, there exists $\bar{u}_{n,0} \in H^1(\mathbb{R}^N; \mathbb{C}^N)$ such that

$$
\inf_{\varphi \in M_{\gamma, \mu}} \| \bar{u}_{n,0} - \varphi \|_{H^1} < \frac{1}{n}
$$

and there exists $t_n \in [0, \infty)$ such that

$$
\inf_{\varphi \in M_{\gamma, \mu}} \| \bar{u}_n(t_n) - \varphi \|_{H^1} \geq \varepsilon,
$$

where $\bar{u}_n$ denotes the solution of the Cauchy problem (1.1) with the initial data $\bar{u}_n(0) = \bar{u}_{n,0}$. From (5.12), there exists $\varphi_n \in M_{\gamma, \mu}$ such that

$$
\| \bar{u}_{n,0} - \varphi_n \|_{H^1} < \frac{1}{n}
$$

for all $n \in \mathbb{N}$. Thus we have

$$
\| \bar{u}_{n,0} - \varphi_n \|_{H^1} \to 0 \quad (n \to \infty).
$$

Furthermore, since

$$
\| u_{j,n,0} \|_2 - \| \varphi_{j,n} \|_2 \leq \| u_{j,n,0} - \varphi_{j,n} \|_2 \leq \| \bar{u}_{n,0} - \varphi_n \|_{H^1} \to 0 \quad (n \to \infty),
$$

it follows that

$$
Q_1(\bar{u}_{n,0}) - Q_1(\varphi_n) \to 0, \quad Q_2(\bar{u}_{n,0}) - Q_2(\varphi_n) \to 0 \quad (n \to \infty),
$$

where $\bar{u}_{n,0} = (u_{1,n,0}, u_{2,n,0}, u_{3,n,0})$, $\varphi_n = (\varphi_{1,n}, \varphi_{2,n}, \varphi_{3,n})$. Since $\{\varphi_n\}_{n=1}^{\infty}$ is a minimizing sequence for $J(\gamma, \mu)$, from Lemma 5.1, $\{\varphi_{j,n}\}_{n=1}^{\infty}$ is bounded in $H^1(\mathbb{R}^N)$. From (5.14), $\{u_{j,n,0}\}_{n=1}^{\infty}$ is also bounded in $H^1(\mathbb{R}^N)$. Thus, there exists $M > 0$ such that

$$
\sum_{j=1}^{3} (\| u_{j,n,0} \|_{H^1}^2 + \| \varphi_{j,n} \|_{H^1}^2) \leq M^2.
$$

Therefore from Lemma 2.5, we have there exists $C(M) > 0$ such that

$$
|E(\bar{u}_{n,0}) - E(\varphi_n)| \leq C(M) \sum_{j=1}^{3} \| u_{j,n,0} - \varphi_{j,n} \|_{H^1} \to 0 \quad (n \to \infty).
$$

Since $\varphi_n \in M_{\gamma, \mu}$, it holds that

$$
Q_1(\varphi_n) = \gamma, \quad Q_2(\varphi_n) = \mu, \quad E(\varphi_n) = J(\gamma, \mu).
$$

From (5.15)–(5.18), we have

$$
Q_1(\bar{u}_{n,0}) \to \gamma, \quad Q_2(\bar{u}_{n,0}) \to \mu, \quad E(\bar{u}_{n,0}) \to J(\gamma, \mu) \quad (n \to \infty).
$$

From Proposition 1, it follows that

$$
Q_1(\bar{u}_{n}(t_n)) = Q_1(\bar{u}_{n,0}), \quad Q_2(\bar{u}_{n}(t_n)) = Q_2(\bar{u}_{n,0}), \quad E(\bar{u}_{n}(t_n)) = E(\bar{u}_{n,0}) \quad \text{for all } n \in \mathbb{N}.
$$

From (5.19)–(5.22), we have

$$
Q_1(\bar{u}_{n}(t_n)) \to \gamma, \quad Q_2(\bar{u}_{n}(t_n)) \to \mu, \quad E(\bar{u}_{n}(t_n)) \to J(\gamma, \mu) \quad (n \to \infty).
$$

Therefore $\{\bar{u}_n(t_n)\}$ is a minimizing sequence for $J(\gamma, \mu)$. 

Here, from Lemma 5.3, taking a subsequence, there exist \( \{y_n\}_{n=1}^{\infty} \subset \mathbb{R}^N \) and \( \bar{u} \in \mathcal{M}_{\gamma,\mu} \) such that
\[
\|\bar{u}_n(t_n, \cdot) - \bar{u}\|_{H^1} \to 0 \quad (n \to \infty).
\] (5.25)
Note that if the cases (ii) and (iv) in Lemma 5.3 occur, then \( y_n = 0 \) and if the cases (iii) and (iv) in Lemma 5.3 occur, then \( u_1 = u_2 = 0 \). Thus from Lemma 5.3, \( \bar{u}(\cdot - y_n) \in \mathcal{M}_{\gamma,\mu} \). From (5.25) and \( \bar{u}(\cdot - y_n) \in \mathcal{M}_{\gamma,\mu} \), we have
\[
\inf_{\bar{\varphi} \in \mathcal{M}_{\gamma,\mu}} \|\bar{u}_n(t_n) - \bar{\varphi}\|_{H^1} \leq \|\bar{u}_n(t_n) - \bar{u}(\cdot - y_n)\|_{H^1} \to 0 \quad (n \to \infty).
\]
This contradicts (5.13). Therefore \( \mathcal{M}_{\gamma,\mu} \) is stable. \( \square \)

6. Proof of Theorem 1.3. In this section, unless otherwise mentioned, we assume that \( N \leq 3 \).

6.1. Compactness of the minimizing sequence for \( \Sigma_0(\gamma, \mu, s) \). We prepare the estimates of \( E_0 \) from below.

Lemma 6.1. There exist \( C(N), \beta(N) > 0 \) such that
\[
E_0(\bar{u}) + C(N) \sum_{j=1}^{3} \|u_j\|^2_{L^2} \geq \frac{1}{4} \sum_{j=1}^{3} \|\nabla u_j\|^2_{L^2} \quad \text{for all } \bar{u} \in H^1(\mathbb{R}^N; \mathbb{C}^3).
\]

Proof. We can prove this lemma by the same argument as in Lemma 2.3. \( \square \)

We note the finiteness of \( \Sigma_0(\gamma, \mu, s) \) and the boundedness of minimizing sequence for \( \Sigma_0(\gamma, \mu, s) \).

Lemma 6.2. Let \( \gamma, \mu, s > 0 \). Then it follows that \( \Sigma_0(\gamma, \mu, s) > -\infty \). Moreover, Let \( \{\bar{u}_n\}_{n=1}^{\infty} \) be a minimizing sequence for \( \Sigma_0(\gamma, \mu, s) \), \( \{u_j,n\}_{n=1}^{\infty} \) \( (j = 1, 2, 3) \) is bounded in \( H^1(\mathbb{R}^N) \).

Proof. By using Lemma 6.1, we can prove this lemma by the same argument as in Lemma 2.4. \( \square \)

We describe the properties of \( \Sigma_0(\gamma, \mu, s) \).

Lemma 6.3. Let \( \gamma, \mu, s \geq 0 \). Then the following properties hold:
(i) if \( \gamma, \mu, s > 0 \), then \( \Sigma_0(\gamma, \mu, s) < 0 \).
(ii) if \( \gamma = 0 \) or \( \mu = 0 \) or \( s = 0 \), then \( \Sigma_0(\gamma, \mu, s) = 0 \).
(iii) if \( \gamma = 0 \) or \( \mu = 0 \) or \( s = 0 \) and \( \gamma \neq 0 \) or \( \mu \neq 0 \) or \( s \neq 0 \), then \( \Sigma_0(\gamma, \mu, s) \) has no minimizers.

Proof. We can prove (i) as in Lemma 2.1 in [2] and (ii) as in Lemma 4.1.
(iii) : We prove only for the case \( \gamma > 0 \) and \( \mu = 0 \). Suppose that there exists a minimizer \( \bar{u} \) for \( \Sigma_0(\gamma, \mu, s) \). Since \( \mu = 0 \), we have \( u_2 = 0 \). From (ii), it follows that \( \Sigma_0(\gamma, 0, s) = 0 \). Since \( \bar{u} \) is a minimizer for \( \Sigma_0(\gamma, 0, s) \), we have
\[
0 = \Sigma_0(\gamma, 0, s) = E_0(\bar{u}) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_1|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_3|^2 \, dx.
\]
Thus we have \( |\nabla u_j| = 0 \) \( (j = 1, 3) \). From \( u_j \in L^2(\mathbb{R}^N) \) \( (j = 1, 3) \), it holds that \( u_j = 0 \) \( (j = 1, 3) \). This contradicts \( \|u_1\|^2_{L^2} = \gamma > 0 \). Therefore there is no minimizers for \( \Sigma_0(\gamma, \mu, s) \). \( \square \)

The next lemma is used to guarantee that all minimizing sequence for \( \Sigma_0(\gamma, \mu, s) \) does not vanish.
Lemma 6.4. Let $\gamma, \mu, s > 0$ and let $\{\bar{u}_n\}_{n=1}^\infty \subset H^1(\mathbb{R}^N; \mathbb{C}^3)$ be a minimizing sequence for $\Sigma_0(\gamma, \mu, s)$. Then taking a subsequence, for all $j = 1, 2, 3$, there exist $\varepsilon_j > 0$ and $R_j > 0$ such that
\[
\sup_{y \in \mathbb{R}^N} \int_{|x-y|<R_j} |u_{j,n}|^2 \, dx \geq \varepsilon_j \quad \text{for all } n \in \mathbb{N}.
\]
\[\text{Proof.} \quad \text{We can prove this lemma as in the proof of Lemma 4.2.} \]

We prepare the following lemma to prove that $\Sigma_0$ is strict decreasing.

Lemma 6.5. Let $\gamma, \mu, s > 0$ and let $\{\bar{u}_n\}_{n=1}^\infty \subset H^1(\mathbb{R}^N; \mathbb{C}^3)$ be a minimizing sequence for $\Sigma_0(\gamma, \mu, s)$. Then taking a subsequence, for all $j \in \{1, 2, 3\}$, there exists $\delta_j > 0$ such that
\[
\int_{\mathbb{R}^N} |\nabla u_{j,n}|^2 \, dx \geq \delta_j, \quad \text{for all } n \in \mathbb{N}.
\]
\[\text{Proof.} \quad \text{Suppose that there exists } j = 1, 2, 3 \text{ such that}
\]
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_{j,n}|^2 \, dx = 0.
\]
\[\text{From Gagliardo–Nirenberg’s inequality and } \{u_{j,n}\}_{n=1}^\infty \text{ is bounded in } H^1(\mathbb{R}^N), \text{ we obtain that}
\]
\[
\left| \text{Re} \int_{\mathbb{R}^N} u_{1,n}u_{2,n}u_{3,n} \, dx \right| \to 0 \quad \text{as } n \to \infty.
\]

The rest of the proof is the same argument as in Lemma 4.2.

Next, we prove the strict subadditivity for $\Sigma_0$. First we prove the strict subadditivity for $\Sigma_0$ for the restricted case.

Proposition 4 (Strict subadditivity for $\Sigma_0$ for the restricted case). Let $a_1, a_2, a_3, b_1, b_2, b_3 > 0$ and we assume $\Sigma_0(a_1, a_2, a_3)$ and $\Sigma_0(b_1, b_2, b_3)$ have a minimizer respectively. Then, it follows that
\[
\Sigma_0(a_1 + b_1, a_2 + b_2, a_3 + b_3) < \Sigma_0(a_1, a_2, a_3) + \Sigma_0(b_1, b_2, b_3).
\]
\[\text{Proof.} \quad \text{In a similar way to Proposition 3, we can prove this proposition by using the coupled rearrangement techniques.} \]

Remark 6. By Lemma 6.3 (iii), if $L^2$ mass constraints is 0 for some component and non-zero for some component, then $\Sigma_0$ has no minimizer. Therefore we can’t use the argument in Proposition 4 to the proof of the strict subadditivity for $\Sigma_0$ in the case that the $L^2$ mass component is 0 for some component and non-zero for some component.

To complete the proof of the strict subadditivity for $\Sigma_0$, we prove the following lemma. We use the condition $N \leq 2$ here.

Proposition 5. Let $N \leq 2$ and $a_1, a_2, a_3 > 0$. Then, we have
\[
a_1 < b_1 \implies \Sigma_0(a_1, a_2, a_3) > \Sigma_0(b_1, a_2, a_3), \tag{6.2}
\]
\[
a_2 < b_2 \implies \Sigma_0(a_1, a_2, a_3) > \Sigma_0(a_1, b_2, a_3), \tag{6.3}
\]
\[
a_3 < b_3 \implies \Sigma_0(a_1, a_2, a_3) > \Sigma_0(a_1, a_2, b_3). \tag{6.4}
\]
Proof. We prove only (6.2). Let \( \{ f_n \}_{n=1}^{\infty} \) be a minimizing sequence for \( \Sigma_0(\alpha_1, \alpha_2, \alpha_3) \) such that \( f_{j,n} \geq 0 \) and \( \| f_{j,n} \|_2^2 = \alpha_j \) \( (j = 1, 2, 3) \). Also, we define

\[
g_{j,n}(x) := \begin{cases}
\frac{b_j}{a_j} \theta^{N/2} f_{1,n}(\theta x) & (j = 1) \\
\theta^{N/2} f_{j,n}(\theta x) & (j = 2, 3),
\end{cases}
\]

where \( \theta = (\alpha_1/b_1)^{1/(4-N)} < 1 \). Then

\[
\int_{\mathbb{R}^N} |g_{j,n}|^2 \, dx = \begin{cases}
b_1 & (j = 1) \\
a_j & (j = 2, 3),
\end{cases}
\]

\[
\int_{\mathbb{R}^N} |\nabla g_{j,n}|^2 \, dx = \begin{cases}
\theta^{N-2} \int_{\mathbb{R}^N} |\nabla f_{1,n}|^2 \, dx & (j = 1) \\
\theta^2 \int_{\mathbb{R}^N} |\nabla f_{j,n}|^2 \, dx & (j = 2, 3),
\end{cases}
\]

\[
\int_{\mathbb{R}^N} g_{1,n} g_{2,n} g_{3,n} \, dx = \theta^{N-2} \int_{\mathbb{R}^N} f_{1,n} f_{2,n} f_{3,n} \, dx.
\]

From the above, we have

\[
E^0(\tilde{g}_n) = \frac{1}{2} \theta^{N-2} \int_{\mathbb{R}^N} |\nabla f_{1,n}|^2 \, dx + \frac{1}{2} \theta^2 \sum_{j=2}^{3} \int_{\mathbb{R}^N} |\nabla f_{j,n}|^2 \, dx
\]

\[
- \theta^{N-2} \int_{\mathbb{R}^N} f_{1,n} f_{2,n} f_{3,n} \, dx
\]

\[
= \theta^{N-2} E^0(\tilde{f}_n) + \frac{1}{2} (\theta^2 - \theta^{N-2}) \sum_{j=2}^{3} \int_{\mathbb{R}^N} |\nabla f_{j,n}|^2 \, dx.
\]

Here, note that \( \theta^2 - \theta^{N-2} < 0 \) since \( N \leq 2 \) and \( \theta < 1 \). Moreover from Lemma 6.5, taking a subsequence, there exists \( C \) > 0 such that

\[
\sum_{j=2}^{3} \int_{\mathbb{R}^N} |\nabla f_{j,n}|^2 \, dx \geq C \quad \text{for all } n \in \mathbb{N}.
\]

Therefore, we obtain

\[
E^0(\tilde{g}_n) \leq \theta^{N-2} E^0(\tilde{f}_n) + \frac{1}{2} (\theta^2 - \theta^{N-2}) C.
\]

Also we note that

\[
\Sigma_0(\alpha_1, \alpha_2, \alpha_3) \leq E^0(\tilde{g}_n),
\]

let \( n \to \infty \), it follows that

\[
\Sigma_0(\alpha_1, \alpha_2, \alpha_3) \leq \theta^{N-2} \Sigma_0(\alpha_1, \alpha_2, \alpha_3) - \frac{1}{2} (\theta^{N-2} - \theta^2) C < \theta^{N-2} \Sigma_0(\alpha_1, \alpha_2, \alpha_3).
\]

Note that \( \theta^{N-2} \geq 1 \) since \( \theta < 1 \) and \( N \leq 2 \). Since \( \Sigma_0(\alpha_1, \alpha_2, \alpha_3) < 0 \), we have

\[
\Sigma_0(\alpha_1, \alpha_2, \alpha_3) < \Sigma_0(\alpha_1, \alpha_2, \alpha_3).
\]

We describe the continuity of \( \Sigma_0 \) on \([0, \infty) \times [0, \infty) \times [0, \infty) \).

**Lemma 6.6.** \( \Sigma_0 \) is continuous on \([0, \infty) \times [0, \infty) \times [0, \infty) \).

**Proof.** We can prove this Lemma by the same argument as in Lemma 2.6. \( \square \)
Lemma 6.7. Let $\gamma, \mu, s > 0$. Then for all $\gamma' \in [0, \gamma]$, $\mu' \in [0, \mu]$ and $s' \in [0, s]$,

$$\Sigma_0(\gamma, \mu, s) \leq \Sigma_0(\gamma', \mu', s') + \Sigma_0(\gamma - \gamma', \mu - \mu', s - s').$$

Now we prove the compactness of all minimizing sequence for $\Sigma_0(\gamma, \mu, s)$. Here, we assume that $N \leq 2$.

**Proposition 6.** Let $N \leq 2$, $\gamma, \mu, s > 0$ and $\{\bar{u}_n\}_{n=1}^\infty \subset H^1(\mathbb{R}^N; \mathbb{C}^3)$ a minimizing sequence for $\Sigma_0(\gamma, \mu, s)$. Then, taking a subsequence, there exist $\{y_n\}_{n=1}^\infty \subset \mathbb{R}^N$ and $\bar{u} \in H^1(\mathbb{R}^N; \mathbb{C}^3)$ such that

$$\|u_{j,n}(\cdot + y_n) - u_j\|_{H^1} \to 0 \quad (j = 1, 2, 3, \ n \to \infty).$$

Furthermore, $\bar{u}$ is a minimizer for $\Sigma_0(\gamma, \mu, s)$.

**Proof.** Let $\{\bar{u}_n\}_{n=1}^\infty \subset H^1(\mathbb{R}^N; \mathbb{C}^3)$ be a minimizing sequence for $\Sigma_0(\gamma, \mu, s)$. The proof proceeds in six steps: especially, we use the key Proposition 5 in Step 5.

**Step 1.** By the same argument as in the proof of Theorem 1.1 for the case without potentials, we can prove that taking a subsequence, there exist $\{x_n\}_{n=1}^\infty \subset \mathbb{R}^N$ and $\bar{u} \in H^1(\mathbb{R}^N; \mathbb{C}^3)$ such that

$$u_{j,n}(\cdot + x_n) \rightharpoonup u_j \text{ weakly in } H^1(\mathbb{R}^N) \quad (j = 1, 2, 3, \ n \to \infty),$$

and $u_1 \neq 0$.

**Step 2.** Set $\gamma' := \|u_1\|_2^2 > 0$, $\mu' := \|u_2\|_2^2$, $s' := \|u_3\|_2^2$ and $v_{j,n} := u_{j,n}(\cdot + x_n) - u_j$ $(j = 1, 2, 3)$. By the same argument as in the proof of Theorem 1.1 for the case without potentials, we can prove that $\bar{u}$ is a minimizer for $\Sigma_0(\gamma', \mu', s')$ and taking a subsequence,

$$\|u_{j,n}\|_2^2 = \|u_j\|_2^2 + \|u_{j,n}(\cdot + x_n) - u_j\|_2^2 + o(1) \quad (j = 1, 2, 3, \ n \to \infty),$$

$$E^0(\bar{u}_n) = E^0(\bar{u}) + E^0(\bar{u}_n(\cdot + x_n) - \bar{u}) + o(1) \quad (n \to \infty),$$

$$\Sigma_0(\gamma, s) = \Sigma_0(\gamma', \mu', s') + \Sigma_0(\gamma - \gamma', \mu - \mu', s - s').$$

**Step 3.** We prove $\mu' > 0$ and $s' > 0$. Suppose $\mu' = 0$, since $\gamma' > 0$, from Lemma 6.3 (iii), $\Sigma_0(\gamma', 0, s')$ has no minimizers. However, from (Step 2), $\bar{u}$ is a minimizer for $\Sigma_0(\gamma', 0, s')$. This is a contradiction. Thus, $\mu' > 0$. We can prove similarly $s' > 0$.

**Step 4.** We show $\gamma' = \gamma$ or $\mu' = \mu$ or $s' = s$. Suppose $\gamma' < \gamma$, $\mu' < \mu$ and $s' < s$. By Step 3, we know $\gamma' > 0$, $\mu' > 0$ and $s' > 0$. By using Proposition 4, by the same argument as in the proof of Theorem 1.1, this contradicts (6.5). Thus it follows that $\gamma' = \gamma$ or $\mu' = \mu$ or $s' = s$.

**Step 5.** We prove $\gamma' = \gamma$, $\mu' = \mu$, $s' = s$. By (Step 4), $\gamma' = \gamma$ or $\mu' = \mu$ or $s' = s$. We consider only the case of $\gamma' = \gamma$. From (6.5), we have

$$\Sigma_0(\gamma, s) = \Sigma_0(\gamma, \mu', s').$$

Next, we prove $\mu' = \mu$. Suppose $\mu' < \mu$, from (6.3) and (6.4) in Proposition 5, we have

$$\Sigma_0(\gamma, \mu', s') > \Sigma_0(\gamma, s) \geq \Sigma_0(\gamma, \mu, s).$$
But this contradicts $\Sigma_0(\gamma, \mu, s) = \Sigma_0(\gamma, \mu', s')$. Therefore we conclude $\mu' = \mu$. Similarly, we can prove $s' = s$.

(Step 6.) By the same argument as in the proof of Theorem 1.1, we can prove
\[ \lim_{n \to \infty} \| \tilde{u}_n(x_n) - \tilde{u}\|_{H^1} = 0 \] and $\tilde{u}$ is a minimizer for $\Sigma_0(\gamma, \mu, s)$.

6.2. Proof of Theorem 1.3. In this subsection, we establish the asymptotic behavior of the energy as $\alpha \to \infty$ and complete the proof of Theorem 1.3.

Next we consider asymptotic behavior of $I_\alpha(\gamma, \mu, s)$ as $\alpha \to \infty$. To that end, we prepare the following lemma.

Lemma 6.8 (Upper bound of $I_\alpha(\gamma, \mu, s)$). Let $\gamma, \mu, s > 0$. Then,
\[ I_\alpha(\gamma, \mu, s) \leq \Sigma_0(\gamma, \mu, s) \alpha^{4/(4-N)} \text{ for all } \alpha > 0. \]

Proof. From the definition of $\Sigma_0(\gamma, \mu, s)$, for any $\varepsilon > 0$, there exists $\tilde{v}_0 \in H^1(\mathbb{R}^N; \mathbb{C}^3)$ such that
\[ \| v_{1,0} \|_2^2 = \gamma, \quad \| v_{2,0} \|_2^2 = \mu, \quad \| v_{3,0} \|_2^2 = s, \quad E^0(\tilde{v}_0) < \Sigma_0(\gamma, \mu, s) + \varepsilon. \]

Set $u_{j,0}(x) := \alpha^{N/(4-N)}v_{j,0}(\alpha^{2/(4-N)}x)$ $(j = 1, 2, 3)$. Then it holds that
\[ \int_{\mathbb{R}^N} |u_{j,0}|^2 \, dx = \int_{\mathbb{R}^N} |v_{j,0}|^2 \, dx = \begin{cases} \gamma & (j = 1) \\ \mu & (j = 2) \\ s & (j = 3) \end{cases}. \]

Moreover, we have
\[ \int_{\mathbb{R}^N} |
abla u_{j,0}|^2 \, dx = \alpha^{4/(4-N)} \int_{\mathbb{R}^N} |\nabla v_{j,0}|^2 \, dx, \]
\[ \int_{\mathbb{R}^N} V_j(x)|u_{j,0}|^2 \, dx \leq 0, \]
\[ \int_{\mathbb{R}^N} |u_{j,0}|^{p+1} \, dx \geq 0, \]
\[ \alpha \Re \int_{\mathbb{R}^N} u_{1,0}u_{2,0}u_{3,0} \, dx = \alpha^{4/(4-N)} \Re \int_{\mathbb{R}^N} v_{1,0}v_{2,0}v_{3,0} \, dx. \]

Therefore,
\[ I_\alpha(\gamma, \mu, s) \leq E_\alpha(\tilde{u}_0) \]
\[ \leq \alpha^{4/(4-N)} \left( \frac{1}{2} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |\nabla v_{j,0}|^2 \, dx - \Re \int_{\mathbb{R}^N} v_{1,0}v_{2,0}v_{3,0} \, dx \right) \]
\[ = \alpha^{4/(4-N)} E^0(\tilde{v}_0) \]
\[ \leq \alpha^{4/(4-N)} (\Sigma_0(\gamma, \mu, s) + \varepsilon) \text{ for all } \varepsilon > 0 \text{ and } \alpha > 0. \]

Let $\varepsilon \downarrow 0$, we get the conclusion.

Next, we prepare that the lower bound of $I_\alpha(\gamma, \mu, s)$.

Lemma 6.9 (Lower bound of $I_\alpha(\gamma, \mu, s)$). Let $\gamma, \mu, s > 0$. Then it follows that
\[ I_\alpha(\gamma, \mu, s) \geq \Sigma_0(\gamma, \mu, s) \alpha^{4/(4-N)} + o(\alpha^{4/(4-N)}) \text{ as } \alpha \to \infty. \]
Proof. By the definition of \( I_\alpha(\gamma, \mu, s) \), for all \( 0 < \varepsilon < 1 \), there exists \( \vec{u} = \vec{u}_{\varepsilon, \alpha} \in H^1(\mathbb{R}^N; \mathbb{C}^3) \) such that
\[
\| u_1 \|_2^2 = \gamma, \quad \| u_2 \|_2^2 = \mu, \quad \| u_3 \|_2^2 = s, \quad E_\alpha(\vec{u}) < I_\alpha(\gamma, \mu, s) + \varepsilon. \tag{6.7}
\]
Now, we define \( \vec{v} \) so that \( \vec{u}(x) = \alpha^{N/(4-N)} \vec{v}(\alpha^{2/(4-N)} x) \). Then we have
\[
\int_{\mathbb{R}^N} |v_j|^2 \, dx = \int_{\mathbb{R}^N} |u_j|^2 \, dx = \begin{cases} 
\gamma & (j = 1) \\
\mu & (j = 2) \\
s & (j = 3)
\end{cases}.
\]
Therefore, it follows that
\[
\Sigma_0(\gamma, \mu, s) \leq E^0(\vec{v}). \tag{6.8}
\]
Furthermore, we have
\[
\int_{\mathbb{R}^N} |\nabla u_j|^2 \, dx = \alpha^{4/(4-N)} \int_{\mathbb{R}^N} |\nabla v_j|^2 \, dx, \tag{6.9}
\]
\[
\int_{\mathbb{R}^N} V_j(x)|u_j|^2 \, dx \geq \begin{cases} 
V_{j, \min} \gamma & (j = 1) \\
V_{j, \min} \mu & (j = 2) \\
V_{j, \min} s & (j = 3)
\end{cases}, \tag{6.10}
\]
\[
\int_{\mathbb{R}^N} |u_j|^{p+1} \, dx = \alpha^{(p-1)N/(4-N)} \int_{\mathbb{R}^N} |v_j|^{p+1} \, dx, \tag{6.11}
\]
\[
\alpha \text{Re} \int_{\mathbb{R}^N} u_1 u_2 \overline{v}_3 \, dx = \alpha^{4/(4-N)} \text{Re} \int_{\mathbb{R}^N} v_1 v_2 \overline{v}_3 \, dx, \tag{6.12}
\]
where \( V_{j, \min} := \text{essinf}_{x \in \mathbb{R}^N} V_j(x) \). From (6.7), we have
\[
I_\alpha(\gamma, \mu, s) + \varepsilon > E_\alpha(\vec{u}) = \alpha^{4/(4-N)} E^0(\vec{v}) + \frac{1}{2} \sum_{j=1}^3 \int_{\mathbb{R}^N} V_j(x)|u_j|^2 \, dx \\
- \frac{\alpha^{(p-1)N/(4-N)}}{p+1} \sum_{j=1}^3 \int_{\mathbb{R}^N} |v_j|^{p+1} \, dx \\
= \alpha^{4/(4-N)} \left( \frac{1}{2} \sum_{j=1}^3 \int_{\mathbb{R}^N} |\nabla v_j|^2 \, dx + \frac{1}{2\alpha^{4/(4-N)}} \sum_{j=1}^3 \int_{\mathbb{R}^N} V_j(x)|u_j|^2 \, dx \\
- \frac{\alpha^{(p-1)N-4)/(4-N)}}{p+1} \sum_{j=1}^3 \int_{\mathbb{R}^N} |v_j|^{p+1} \, dx - \text{Re} \int_{\mathbb{R}^N} v_1 v_2 \overline{v}_3 \, dx \right). \tag{6.13}
\]
Now, we prove the following claim.

**Claim.** There exists \( C > 0 \) such that for all \( 0 < \varepsilon < 1 \), \( \alpha \geq 1 \), \( \sum_{j=1}^3 \| v_j \|_{H^1}^2 \leq C \).

Indeed, if \( \alpha \geq 1 \) and \( 0 < \varepsilon < 1 \), from (6.13) and Lemma 6.8, we have
\[
\Sigma_0(\gamma, \mu, s) + 1 \\
\geq \Sigma_0(\gamma, \mu, s) + \frac{\varepsilon}{\alpha^{4/(4-N)}} \\
\geq \frac{1}{2} \sum_{j=1}^3 \int_{\mathbb{R}^N} |\nabla v_j|^2 \, dx + \frac{1}{2} \sum_{j=1}^3 V_{j, \min} \int_{\mathbb{R}^N} |v_j|^2 \, dx
\]
Since 0 < ε < 1 is arbitrary, letting ε ↓ 0, we get a conclusion. \( \square \)

Now we prove that the asymptotic behavior of \( I_\alpha(\gamma, \mu, s) \) as \( \alpha \to \infty \).

**Proof of Theorem 1.3.** From Lemma 6.8 and 6.9, we have

\[
\alpha^{4/(4-N)}(\Sigma_0(\gamma, \mu, s) + o(1)) \leq I_\alpha(\gamma, \mu, s) \leq \Sigma_0(\gamma, \mu, s)\alpha^{4/(4-N)} \quad (\text{as} \ \alpha \to \infty).
\]

It follows that

\[
I_\alpha(\gamma, \mu, s) = \alpha^{4/(4-N)}(\Sigma_0(\gamma, \mu, s) + o(1)) \quad (\text{as} \ \alpha \to \infty).
\]

Next we show (ii). We assume \( N \leq 2 \). Let \( \{\alpha_n\}_{n=1}^{\infty} \subset (0, \infty) \) with \( \alpha_n \to \infty \) (as \( n \to \infty \)) and let \( \vec{u}_n \) a minimizer for \( I_{\alpha_n}(\gamma, \mu, s) \). In addition, we set

\[
\vec{v}_n(x) := \alpha_n^{-N/(4-N)}\vec{u}_n(\alpha_n^{-2/(4-N)}x).
\]

In a similar way to Lemma 6.9, we have

\[
I_{\alpha_n}(\gamma, \mu, s) = \alpha_n^{4/(4-N)}(\Sigma_0(\gamma, \mu, s) + o(1))
\]

\[
\geq \alpha_n^{4/(4-N)}(\Sigma_0(\gamma, \mu, s) + o(1)) \quad (\text{as} \ n \to \infty).
\]

Combining this and Lemma 6.8, we have

\[
\alpha_n^{4/(4-N)}\Sigma_0(\gamma, \mu, s)
\]

\[
\geq \alpha_n^{4/(4-N)}(\Sigma_0(\gamma, \mu, s) + o(1))
\]

\[
\geq \alpha_n^{4/(4-N)}(\Sigma_0(\gamma, \mu, s) + o(1)) \quad (\text{as} \ n \to \infty).
\]

Thus it follows that

\[
\lim_{n \to \infty} E^0(\vec{v}_n) = \Sigma_0(\gamma, \mu, s).
\]

This implies that \( \{\vec{v}_n\}_{n=1}^{\infty} \) is a minimizing sequence for \( \Sigma_0(\gamma, \mu, s) \). From Proposition 6, taking a subsequence, there exist \( \{y_n\}_{n=1}^{\infty} \subset \mathbb{R}^N \) and a minimizer \( \vec{v} \) for \( \Sigma_0(\gamma, \mu, s) \) such that

\[
||v_{j,n}(\cdot + y_n) - v_j||_{H^1} \to 0 \quad (j = 1, 2, 3, \text{ as} \ n \to \infty).
\]
\( \square \)
Appendix A. Proof of Lemma 2.6. We give the proof of Lemma 2.6.

Proof of Lemma 2.6. Let \( \gamma_0, \mu_0, s_0 \geq 0 \). We prove that \( I \) is continuous at \((\gamma_0, \mu_0, s_0)\). Let \( \{\gamma_n\}_{n=1}^{\infty}, \{\mu_n\}_{n=1}^{\infty}, \{s_n\}_{n=1}^{\infty} \subset (0, \infty) \) such that
\[
\gamma_n \to \gamma_0, \quad \mu_n \to \mu_0, \quad s_n \to s_0 \quad (n \to \infty).
\]

Claim 1.
\[
I(\gamma_n, \mu_n, s_n) - I(\gamma_0, \mu_0, s_0) \leq o(1) \quad (n \to \infty).
\]

From the definition of \( I(\gamma_0, \mu_0, s_0) \), it holds that for all \( n \in \mathbb{N} \), there exists \( \bar{u}_n \in H^1(\mathbb{R}^N; \mathbb{C}^3) \) such that
\[
\|u_{1,n}\|_2^2 = \gamma_n, \quad \|u_{2,n}\|_2^2 = \mu_n, \quad \|u_{3,n}\|_2^2 = s_n, \quad E(\bar{u}_n) < I(\gamma_0, \mu_0, s_0) + \frac{1}{n}. \tag{A.1}
\]

Then from Lemma 2.3 and (A.1), \( \{u_{j,n}\}_{n=1}^{\infty} \) is bounded in \( H^1(\mathbb{R}^N) \).

Here, we define \( \{\bar{v}_n\}_{n=1}^{\infty} \) as the following:
\[
v_{1,n} := \begin{cases} \sqrt{\gamma_n} u_{1,n} & (\gamma_0 > 0) \\ \sqrt{\gamma_0} \phi & (\gamma_0 = 0) \end{cases}, \quad v_{2,n} := \begin{cases} \sqrt{\mu_n} u_{2,n} & (\mu_0 > 0) \\ \sqrt{\mu_0} \phi & (\mu_0 = 0) \end{cases},
\]
\[
v_{3,n} := \begin{cases} \sqrt{s_n} u_{3,n} & (s_0 > 0) \\ \sqrt{s_0} \phi & (s_0 = 0) \end{cases},
\]
where \( \phi \in H^1(\mathbb{R}^N) \) and \( \|\phi\|_2 = 1 \). Since
\[
\|v_{1,n}\|_2^2 = \gamma_n, \quad \|v_{2,n}\|_2^2 = \mu_n, \quad \|v_{3,n}\|_2^2 = s_n,
\]
we have
\[
I(\gamma_n, \mu_n, s_n) \leq E(\bar{v}_n). \tag{A.2}
\]

Furthermore, it follows that
\[
\|v_{j,n} - u_{j,n}\|_{H^1} \to 0 \quad (n \to \infty, \ j = 1, 2, 3). \tag{A.3}
\]

Indeed, if \( \gamma_0 > 0 \), then it holds that \( \|v_{1,n} - u_{1,n}\|_{H^1} \to 0 \ (n \to \infty) \) since \( \sqrt{\gamma_n}/\sqrt{\gamma_0} \to 1 \ (n \to \infty) \) and \( \{u_{1,n}\}_{n=1}^{\infty} \) is bounded in \( H^1(\mathbb{R}^N) \). Also, if \( \gamma_0 = 0 \), then it holds that \( \|v_{1,n} - u_{1,n}\|_{H^1} = \sqrt{s_n}\|\phi\|_{H^1} \to 0 \ (n \to \infty) \) since \( u_{1,n} = 0 \) and \( \gamma_n \to 0 \ (n \to \infty) \).

We can prove similarly \( \|v_{j,n} - u_{j,n}\|_{H^1} \to 0 \ (n \to \infty) \) for \( j = 2, 3 \).

Since \( \{u_{j,n}\}_{n=1}^{\infty} \) is bounded in \( H^1(\mathbb{R}^N) \), \( \{v_{j,n}\}_{n=1}^{\infty} \) is also bounded in \( H^1(\mathbb{R}^N) \). Thus, there exists \( M > 0 \) such that
\[
\sum_{j=1}^{3} (\|u_{j,n}\|_{H^1}^2 + \|v_{j,n}\|_{H^1}^2) \leq M^2 \quad \text{for all } n \in \mathbb{N}.
\]

From Lemma 2.5, there exists \( C(M) > 0 \) such that
\[
|E(\bar{v}_n) - E(\bar{u}_n)| \leq C(M) \sum_{j=1}^{3} \|v_{j,n} - u_{j,n}\|_{H^1} \quad \text{for all } n \in \mathbb{N}. \tag{A.4}
\]

From (A.3) and (A.4), it follows that
\[
|E(\bar{v}_n) - E(\bar{u}_n)| \to 0 \quad (n \to \infty). \tag{A.5}
\]
Therefore, from (A.1), (A.2) and (A.5), we have
\[ I(\gamma_n, \mu_n, s_n) - I(\gamma_0, \mu_0, s_0) \leq E(\vec{v}_n) - E(\vec{u}_n) + \frac{1}{n} = o(1) \quad (\text{as } n \to \infty). \]
Thus Claim 1 holds.

**Claim 2.**

\[ I(\gamma_0, \mu_0, s_0) - I(\gamma_n, \mu_n, s_n) \leq o(1) \quad (\text{as } n \to \infty). \]

From the definition of \( I(\gamma_n, \mu_n, s_n) \), we have for all \( n \in \mathbb{N} \), there exists \( \vec{v}_n \in H^1(\mathbb{R}^N; \mathbb{C}^3) \) such that
\[ \|v_{1,n}\|^2 = \gamma_n, \quad \|v_{2,n}\|^2 = \mu_n, \quad \|v_{3,n}\|^2 = s_n, \quad E(\vec{v}_n) < I(\gamma_n, \mu_n, s_n) + \frac{1}{n}. \tag{A.6} \]
From Lemma 2.3, (A.6) and \( I(\gamma_n, \mu_n, s_n) < 0 \), \( \{v_{j,n}\}_{n=1}^\infty \) is bounded in \( H^1(\mathbb{R}^N) \).

Here, we define \( \{\vec{u}_n\}_{n=1}^\infty \) as the following:
\[
\begin{align*}
v_{1,n} &:= \begin{cases}
\frac{\sqrt{\gamma_0}}{\sqrt{\gamma_n}} v_{1,n} & (\gamma_0 > 0), \\
0 & (\gamma_0 = 0)
\end{cases}, \\
u_{2,n} &:= \begin{cases}
\frac{\sqrt{\mu_0}}{\sqrt{\mu_n}} v_{2,n} & (\mu_0 > 0), \\
0 & (\mu_0 = 0)
\end{cases}, \\
v_{3,n} &:= \begin{cases}
\frac{\sqrt{s_0}}{\sqrt{s_n}} v_{3,n} & (s_0 > 0), \\
0 & (s_0 = 0)
\end{cases}.
\end{align*}
\]
Since
\[ \|v_{1,n}\|^2 = \gamma_0, \quad \|v_{2,n}\|^2 = \mu_0, \quad \|v_{3,n}\|^2 = s_0, \]
we have
\[ I(\gamma_0, \mu_0, s_0) \leq E(\vec{u}_n). \tag{A.7} \]
Furthermore, it holds that
\[ \|u_{j,n} - v_{j,n}\|_{H^1} \to 0 \quad (\text{as } n \to \infty). \tag{A.8} \]
Indeed, if \( \gamma_0 > 0 \), then \( \|v_{1,n} - v_{1,n}\|_{H^1} \to 0 \) (as \( n \to \infty \)) since \( \sqrt{\gamma_0}/\sqrt{\gamma_n} \to 1 \) (as \( n \to \infty \)) and \( \{v_{1,n}\}_{n=1}^\infty \) is bounded in \( H^1(\mathbb{R}^N) \). When \( \gamma_0 = 0 \), we can prove \( \|u_{j,n} - v_{j,n}\|_{H^1} \to 0 \) (as \( n \to \infty \)) as the following. If \( \gamma_0 = 0 \), then \( \|v_{1,n}\|^2 = \gamma_n \to 0 \) (as \( n \to \infty \)) and \( \{v_{j,n}\}_{n=1}^\infty \) \( (j = 1, 2, 3) \) is bounded in \( H^1(\mathbb{R}^N) \). Therefore, from Gagliardo–Nirenberg’s inequality, we have
\[ \|v_{1,n}\|_{p+1}^{p+1} \to 0, \quad \alpha \text{Re} \int_{\mathbb{R}^N} v_{1,n} v_{2,n} \overline{v}_{3,n} \, dx \to 0 \quad (n \to \infty). \]
Thus, it holds that
\[ E(\vec{v}_n) \geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_{1,n}|^2 \, dx + S_{V_2}(\mu_n) + S_{V_3}(s_n) + o(1) \quad (\text{as } n \to \infty). \tag{A.9} \]
Moreover, note that (A.6), Lemma 4.1 and \( S_{V_2}(\gamma_n) \leq 0 \), we have
\[ E(\vec{v}_n) < I(\gamma_n, \mu_n, s_n) + \frac{1}{n} \leq S_{V_2}(\mu_n) + S_{V_3}(s_n) + \frac{1}{n}. \tag{A.10} \]
Therefore, from (A.9) and (A.10), it follows that
\[ \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_{1,n}|^2 \, dx \leq o(1) \quad (n \to \infty). \]
combining this and \( \|v_{1,n}\|_{L^2}^2 \to 0 \) (as \( n \to \infty \)), we have \( \|v_{1,n}\|_{H^1} \to 0 \) (as \( n \to \infty \)). Since \( \gamma_0 = 0 \), \( u_1 = 0 \). It holds that \( \|u_{1,n} - v_{1,n}\|_{H^1} \to 0 \) (as \( n \to \infty \)). Similarly, we can prove that \( \|u_{j,n} - v_{j,n}\|_{H^1} \to 0 \) (as \( n \to \infty \)) for \( j = 2, 3 \).

Since \( \{v_{j,n}\}_{n=1}^{\infty} \) is bounded in \( H^1(\mathbb{R}^N) \) and (A.8), \( \{u_{j,n}\}_{n=1}^{\infty} \) is also bounded in \( H^1(\mathbb{R}^N) \). Therefore, there exists \( M > 0 \) such that

\[
\sum_{j=1}^{3} (\|u_{j,n}\|_{H^1}^2 + \|v_{j,n}\|_{H^1}^2) \leq M^2 \quad \text{for all } n \in \mathbb{N}.
\]

From Lemma 2.5, there exists \( C(M) > 0 \) such that

\[
|E(\bar{u}_n) - E(\bar{v}_n)| \leq C(M) \sum_{j=1}^{3} \|u_{j,n} - v_{j,n}\|_{H^1}, \quad \text{for all } n \in \mathbb{N}. \tag{A.11}
\]

From (A.8) and (A.11), we see that

\[
|E(\bar{u}_n) - E(\bar{v}_n)| \to 0 \quad (n \to \infty). \tag{A.12}
\]

Thus, since (A.6), (A.7) and (A.12), we have

\[
I(\gamma_0, \mu_0, s_0) - I(\gamma, \mu, s) \leq E(\bar{u}_n) - E(\bar{v}_n) + \frac{1}{n} = o(1) \quad (as \ n \to \infty).
\]

Hence, Claim 2 follows.

From Claim 1 and Claim 2, we have

\[
o(1) \leq I(\gamma, \mu, s) - I(\gamma_0, \mu_0, s_0) \leq o(1) \quad (as \ n \to \infty).
\]

Thus it follows that

\[
\lim_{n \to \infty} I(\gamma, \mu, s) = I(\gamma_0, \mu_0, s_0).
\]

Since \( \{\gamma_n\}_{n=1}^{\infty}, \{\mu_n\}_{n=1}^{\infty} \) and \( \{s_n\}_{n=1}^{\infty} \subset (0, \infty) \) is arbitrary sequence such that \( \gamma_n \to \gamma_0, \mu_n \to \mu_0 \) and \( s_n \to s_0 \) (\( n \to \infty \)), we have

\[
\lim_{(\gamma, \mu, s) \to (\gamma_0, \mu_0, s_0)} I(\gamma, \mu, s) = I(\gamma_0, \mu_0, s_0).
\]

Thus, \( I \) is continuous at \( (\gamma_0, \mu_0, s_0) \).

Since \( (\gamma_0, \mu_0, s_0) \in [0, \infty) \times [0, \infty) \times [0, \infty) \) is arbitrary, \( I \) is continuous on \( [0, \infty) \times [0, \infty) \times [0, \infty) \) \( . \)

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