On the Gauss map of embedded minimal tubes

I.M. Reshetnikova and V.G. Tkachev

Abstract. A surface is called a tube if its level-sets with respect to some coordinate function (the axis of the surface) are compact. Any tube of zero mean curvature has an invariant, the so-called flow vector. We study how the geometry of the Gaussian image of a higher-dimensional minimal tube $M$ is controlled by the angle $\alpha(M)$ between the axis and the flow vector of $M$. We prove that the diameter of the Gauss image of $M$ is at least $2\alpha(M)$ if the angle $\alpha(M)$ is positive. As a consequence we derive an estimate on the length of a two-dimensional minimal tube $M$ in terms of $\alpha(M)$ and the total Gaussian curvature of $M$.

Mathematical Subject Classification (1991): 53A10, 53A55

Key words: Minimal surfaces; minimal tubes; Gaussian map; total Gaussian curvature; flow vector.

1. Introduction

Let $M$ be a Riemannian orientable manifold of dimension $(n-1)$ with $n \geq 3$ and $x = x(m) : M \to \mathbb{R}^n$ be an embedded surface $M$. Further we identify $M$ with $x(M)$. The following definition is due [9]

**DEFINITION.** Let $t(M) \subset \mathbb{R}^1$ be an open interval. A surface $M$ is called a tube (tubular) with the axis $Ox_n$ (or a tube in the $e_n$-direction) with the projection interval $t(M)$ if

1) $\forall t \in t(M)$ the cross-sections $\Sigma_t = M \cap \Pi_t$, $\Pi_t = \{x \in \mathbb{R}^n : x_n = t\}$, are nonempty compact sets lying in the interior of $M$ (other words, the preimage $x^{-1}(\Sigma_t)$ is a compact subset of $M$);

2) $\forall t_1, t_2 \in t(M)$ any portion of $M$ situated between $\Pi_{t_1}$ and $\Pi_{t_2}$ is compact.

In this case, the length $|t(M)|$ of the projection interval $t(M)$ is called the life-time of $M$.

In the present paper we study the geometry of the Gaussian image of the minimal tubes (that is tubes having zero mean curvature). We notice that the previous definition does not impose a priori restrictions on the topological structure of $M$. Important results on two-dimensional minimal tubes have been obtained by J.C.C.Nitsche in [14], [15]. We should mention also the substantial papers of W.H.Meeks and B.White [7], [8] and Y.Fang [3] where minimal tubes with convex sections $\Sigma_t$ have been studied.

The simplest example of a minimal tube in higher dimensions are the rotationally symmetric minimal surfaces (the so-called $(n-1)$-dimensional catenoids). In [12] (see also [11]) V.M.Miklyukov and A.D.Vedenyapin obtained finiteness of the life-time of every minimal $(n-1)$-dimensional tube in $\mathbb{R}^n$ in the case $n \geq 4$.

---

1This paper was supported by Sankt-Peterburg University grant, project 95-0-1.9-34
The situation changes in the two-dimensional case. An example of a tube with infinite life-time is the standard catenoid. On the other hand, there are properly embedded singly periodic minimal surfaces $M$ constructed by B. Riemann in [16] and their generalizations given in [2] which produce by quotient the minimal tube $M_1 = M/\mathbb{Z}$ of finite life-time. The latter means that $M_1$ can not be a proper part of any larger minimal tube.

In the recent paper [18] the second author proposed a new approach to the problem of determining whether given a minimal tube be of finite or infinite life-time. The main tool is the notion of the flow vector of a minimal tube $M$ which is defined below. In the present paper we show (Theorem 1 below) that the length of the Gaussian image of section $\Sigma_t$ can be described in terms of slope of the flow vector to the axis of $M$.

Let $t$ be a regular value of the coordinate function $x_n$. Then $\Sigma_t$ splits in finite union of compact $(n-2)$-dimensional connected submanifolds of $M$. Let $\nu$ be the unit exterior normal to $\Sigma_t$ as a boundary of $M \cap \{x_n < t\}$. In particularly, we have $\langle \nu, e_n \rangle > 0$ everywhere in $\Sigma_t$.

DEFINITION. A union $\Sigma = \bigcup_{i=1}^k \Sigma^i$ of components of $\Sigma_t$ with the induced orientation is called a cycle. The linear functional $F(e)$ defined by

$$F_\Sigma(e) = \int_\Sigma \langle e, \nu \rangle : \mathbb{R}^n \to \mathbb{R}^1$$

generates the dual element $J(\Sigma) \in \mathbb{R}^n$ such that

$$F_\Sigma(e) = \langle e, J(\Sigma) \rangle .$$

Then

$$J_n(\Sigma) \equiv \langle J(\Sigma), e_n \rangle = \int_\Sigma \langle e_n, \nu \rangle = \int_\Sigma |e_n^\top| > 0,$$

and it follows that $J(\Sigma) \neq 0$. Here and subsequently we use the notation $a^V$ for the orthogonal projection of $a \in \mathbb{R}^n$ onto a subspace $V \subset \mathbb{R}^n$ and by $T = T_m.M$ we denote the tangent space of the surface $M$ at $m \in M$.

DEFINITION. Two oriented cycles $\Sigma'$ and $\Sigma''$ are equivalent in $M$, or $\Sigma' \sim M \Sigma''$, if there exists an open subset $D \subset M$ such that $\partial D = (-\Sigma') \cup \Sigma''$ (here $-\Sigma'$ is the opposite oriented cycle to $\Sigma$). This notion is actually the oriential bordism equivalence (see [5, § 7]). A connected cycle $\Sigma$ is called simple if it is equivalent to zero cycle in the hyperplane $\Pi_t$. Other words, it is a boundary of an open subset of $\Pi_t$.

Proposition 1. We have for $\Sigma = \bigcup_{i=1}^k \Sigma^i$

$$J(\Sigma) = J(\Sigma^1) + \ldots + J(\Sigma^k).$$

Moreover, if $\Sigma' \sim M \Sigma''$ then $J(\Sigma') = J(\Sigma'')$. 
Proof. The first property is direct consequence of the above definitions. To prove the second one we recall that all coordinate functions of minimal immersion are harmonic [6]. Let $D \subset M$ be an open set in the Definition 2 such that $\partial D = (-\Sigma') \cup \Sigma''$. Then for arbitrary coordinate vector $e_k \in \mathbb{R}^n$

$$
\langle J(\Sigma''), e_k \rangle - \langle J(\Sigma'), e_k \rangle = \int_{\partial D} \langle e_k, \nu \rangle = \int_{\partial D} \langle \nabla f_k, \nu \rangle = \int_{D} \Delta f_k = 0,
$$

where $\nabla f_k = e_k^\top$ is the gradient of $f_k = \langle e_k, x(m) \rangle$.

Definition. We call $J(M) = J(\Sigma_t)$ to be the flow vector of the tube $M$.

Remark 1. It follows from Proposition 1 that the flow vector $J(\Sigma_t)$ does not depend on a choice of $t \in t(M)$. One easy to see also that both the angle $\alpha(M)$ between $J(M)$ and $e_n$ and the norm $\|J(M)\|$ are invariants under the action of the orthogonal subgroup of $\mathbb{R}^n$ preserving the axis $Ox_n$.

Moreover, we emphasize that the flow vector of $M$ is a local characteristic of $M$ in the sense that it can be computed if we consider only a portion of $M$ situated between $\Pi_{t_1}$ and $\Pi_{t_2}$ for $t_1$ and $t_2$ arbitrarily close.

Let $S^{n-1}$ be the unit sphere in the Euclidean space $\mathbb{R}^n$ and $d(E)$ be the spherical diameter of a set $E \subset S^{n-1}$. By $\gamma : M \to S^{n-1}$ we denote the Gaussian map of $M$, where $\gamma(m)$ is the unit normal at $m \in M$; by $\gamma(E)$ we denote the Gaussian image of a set $E \subset M$.

Our main result is the following lower estimate of the diameter of the Gaussian image.

Theorem 1. Let $M$ be an embedded minimal tube in $\mathbb{R}^n$; $\Sigma \subset \Sigma_t$ be a simple cycle with the flow vector $J(\Sigma)$. Then the diameter of $\gamma(\Sigma)$ satisfies

$$
d(\gamma(\Sigma)) \geq 2 \alpha(\Sigma),
$$

where $\alpha(\Sigma)$ is the angle between $J(M)$ and $e_n$.

As a consequence in Section 4 we obtain the upper estimate on the life-time of minimal tubes of finite total Gaussian curvature.

Theorem 2. Let $M$ be a two-dimensional minimal tube in $\mathbb{R}^3$ of finite total Gaussian curvature $-G(M)$. If $\alpha(M) > 0$ then $M$ has finite life-time and

$$
|t(M)| \leq \|J(M)\| G(M) \frac{\cos \alpha(M)}{16 \alpha^2(M)}.
$$

Corollary 1. Let $M$ be a two-dimensional minimal tube in $\mathbb{R}^3$ with univalent Gaussian map. Then $M$ has finite life-time provided that $\alpha(M) > 0$.

Now we indicate the main idea of the proof of Theorem 2. In this case $\dim \Sigma = 1$ and it follows that all one-dimensional cycles are simple (see [5, § 7]). Moreover, (3) implies that the Gaussian image of every section $\Sigma_t$ is uniformly ‘large’ provided the angle between $J(M)$ and $e_n$ is strictly positive. On the other hand,
in the two-dimensional case \( \dim \mathcal{M} = 2 \) the Gaussian map is conformal and (3) yields that \( \mathcal{M} \) must be a surface of hyperbolic conformal type. The final step is to use the connection between the conformal module of minimal tube and its life-time value.

We notice that Theorem 2 fails if we drop the requirements of finiteness of the total Gaussian curvature. Really, in the previous paper [19] we have constructed the corresponding examples by using the suitable Weierstrass representation for minimal tubes. Namely, given arbitrary \( \alpha(M) > 0 \) there exists a properly embedded minimal tube of infinite life-time.

The author thanks Yi Fang for helpful discussions concerning the subject of the paper.

2. Preliminary facts

By \( \Lambda(\mathbb{R}^n) \) and \( \Lambda^k(\mathbb{R}^n) \) we denote the exterior algebra of \( \mathbb{R}^n \) and the subspace of all \( k \)-form respectively. We specify an orthonormal basis \( \{e_k\}_{k=1}^n \) in \( \mathbb{R}^n \) and by \( \omega = e_1 \wedge \ldots \wedge e_n \) we denote the volume-form of \( \mathbb{R}^n \). We write \( a \simeq b \) if \( a = \pm b \).

Given \( u \in \Lambda(\mathbb{R}^n) \) we define the inner product \( u \sqcup \cdot \) on \( \Lambda(\mathbb{R}^n) \) by

\[
(x, u \sqcup y) \equiv \langle u \wedge x, y \rangle, \quad \forall x, y \in \Lambda(\mathbb{R}^n).
\]

Then the Hodge \(*\)-operator: \( \Lambda^k(\mathbb{R}^n) \rightarrow \Lambda^{n-k}(\mathbb{R}^n) \), can be written for every \( k \)-form \( x \) by

\[
* x = x \sqcup \omega.
\]

The following facts are elementary and can be found in [17].

(i) \( ** x = (-1)^{k(n-k)} x \simeq x \), \( \forall x \in \Lambda^k(\mathbb{R}^n) \);
(ii) \( x \wedge * y = \langle x, y \rangle \omega \), \( \forall x \in \Lambda^k(\mathbb{R}^n), \forall y \in \Lambda^{n-k}(\mathbb{R}^n) \);
(iii) \( \langle * x, * y \rangle = \langle x, y \rangle \), \( \forall x, y \in \Lambda^k(\mathbb{R}^n) \).

Let \( V \subset \mathbb{R}^n \) be an oriented \( k \)-dimensional subspace and \( v_1, \ldots, v_k \) be an orthonormal basis of \( V \). By

\[
\sigma(V) \equiv v_1 \wedge \ldots \wedge v_k
\]

we denote the volume form of \( V \). Further we use the operator

\[
\pi_V(\xi) = * (\sigma(V) \wedge \xi).
\]

Let \( V \subset \mathbb{R}^n \) be an oriented hyperspace, \( \dim V = n - 1 \). If \( \xi \) is a unit vector orthogonal to \( V \) (i.e. \( \xi \in V^\perp \)) then it follows from (ii)

\[
\xi \simeq * \sigma(V).
\]

**Lemma 1.** For all \( a \in \Lambda^r(\mathbb{R}^n), b \in \Lambda^k(\mathbb{R}^n) \) one holds

\[
a \sqcup (sb) = *(b \wedge a).
\]
Proof. Let $\xi \in \Lambda^{n-k-r}(\mathbb{R}^n)$ be chosen arbitrary. Then
\[\langle a \wedge (\ast b), \xi \rangle = \langle \ast b, a \wedge \xi \rangle = \langle b \wedge \omega, a \wedge \xi \rangle = \langle (b \wedge a) \wedge \omega, \xi \rangle = \langle \omega, (b \wedge a) \wedge \xi \rangle = \langle \ast (b \wedge a), \xi \rangle.\]
Then by duality we have $\ast (b \wedge a) = a \wedge (\ast b)$ and the lemma is proved. \hfill \square

Lemma 2. Let $V \subset \mathbb{R}^n$ be a subspace, $\dim V = k$, and $V^\perp$ be its orthogonal complement. Then for all $x \in \mathbb{R}^n$
\[(8)\quad x^V = \sigma(V) \wedge (\sigma(V) \wedge x).\]

Proof. We choose $v_1, \ldots, v_k$ to be an orthonormal basis of $V$ and consider its complement in $\mathbb{R}^n$: $w_1, \ldots, w_{n-k}$. Given arbitrary $x \in \mathbb{R}^n$ we have $x = x_1v_1 + \ldots + x_kw_k + y_1w_1 + \ldots + y_{n-k}w_{n-k}$. Then for every $v_i$
\[\langle \sigma(v) \wedge (\sigma(v) \wedge x), v_i \rangle = \langle \sigma(v) \wedge x, \sigma(v) \wedge v_i \rangle = 0.\]
On the other hand,
\[\langle \sigma(v) \wedge (\sigma(v) \wedge x), w_\alpha \rangle = \langle \sigma(v) \wedge x, \sigma(v) \wedge w_\alpha \rangle = \sum_{j=1}^{n-k} \langle \sigma(v) \wedge w_j, \sigma(v) \wedge w_\alpha \rangle = \sum_{j=1}^{n-k} y_j \delta_{j\alpha} = y_\alpha.\]
Hence, by the definition we have the identity
\[\sigma(v) \wedge (\sigma(v) \wedge x) = y_1w_1 + \ldots + y_{n-k}w_{n-k} = x^V,\]
which proves the lemma. \hfill \square

3. Proof of Theorem 1

In this section by $\Pi = \Pi_0$ we denote a hyperspace $x_n = 0$ in $\mathbb{R}^n$ and by $T \equiv T_m\Sigma$ — the tangent space to the section $\Sigma$ being considered as submanifold of $\Pi$. Let $\gamma = \gamma(m)$ be the unit normal to $\mathcal{M}$ at $m$. We specify an orthonormal basis $\tau_1, \ldots, \tau_{n-2}$ in $T$ and by $\tau \equiv \sigma(T)$ denote the volume form of $T_m\Sigma$.

We need the following auxiliary assertion

Lemma 3. Let $\xi$ and $\eta$ be two unit vectors such that $\xi, \eta, \tau_1, \ldots, \tau_{n-2}$ form oriented orthonormal basis of $\mathbb{R}^n$. Then for every $q \in \mathbb{R}^n$
\[\langle q, \xi \rangle \simeq \langle \eta, \pi_T(q) \rangle.\]
Proof. We have from (7) that
\[ \xi \simeq \ast (\eta \wedge \tau), \]
and by virtue of (iii) and Lemma 1 we obtain
\[ \langle q, \xi \rangle \simeq \langle q, \ast (\eta \wedge \tau) \rangle \simeq \langle q, \ast (\eta \wedge \tau) \rangle \simeq \langle q, \ast \gamma \rangle \simeq \langle q, \ast \gamma \rangle \simeq \langle \eta, \ast \gamma \rangle \simeq \langle \eta, \ast \gamma \rangle \simeq \langle \eta, \ast \gamma \rangle \simeq \langle \xi, \pi_T(q) \rangle \]
and the lemma is proved.

By the Sard’s theorem and regularity of \( t \) we conclude that \( \Sigma \) is a smooth submanifold of \( \Pi_t \). Assume that \( \eta = \eta(m) \) is the unit normal vector field to \( \Sigma \) in \( \Pi_t \) oriented such that the pair \( (T; \eta) \) is an oriented basis of \( \Pi \). Then by Lemma 2 we have for every \( q \in \mathbb{R}^n \)
\[ \int_{\Sigma} \langle \pi_T(q), e_n \rangle = \int_{\Sigma} \langle \ast (\tau \wedge q), e_n \rangle \simeq \int_{\Sigma} \langle \tau \wedge q, \ast e_n \rangle \]
\[ \simeq \int_{\Sigma} \langle \tau \wedge q, \ast \gamma \rangle \simeq \int_{\Sigma} \langle q, \ast \gamma \rangle \simeq \int_{\Sigma} \langle q, \gamma \rangle \]
\[ \simeq \]
(9)

To show that in fact the last integral vanishes we observe that by the definition, the simple cycle \( \Sigma \) is the boundary of some open subset \( \Omega \subset \Pi_t \). The by the Stokes’ formula we obtain
\[ \int_{\partial \Omega} \langle q, \eta \rangle = \int_{\Omega} \text{div } q = 0. \]
Thus (9) yields the following identity
\[ \int_{\Sigma} \langle \pi_T(q), e_n \rangle = 0. \]

Choose \( q \neq 0 \) arbitrarily such that the equality \( \langle q, J(\Sigma) \rangle = 0 \) holds. Then taking into account the mutual orthogonality of \( \gamma, \nu \) and the tangent space \( T_m \Sigma \), we obtain from (1) and Lemma 3
\[ 0 = \int_{\Sigma} \langle q, \nu \rangle = \int_{\Sigma} \langle \pi_T(q), \gamma \rangle. \]
Hence, we conclude from (11) that
\[ \int_{\Sigma} \langle \pi_T(q), \gamma \pm e_n \rangle = 0. \]

By virtue of regularity of \( t \), the expressions \( \gamma \pm e_n \) does not vanish everywhere in \( \Sigma \). It follows that along \( \Sigma \) the vector fields
\[ v_\pm(m) = \frac{e_n \pm \gamma(m)}{\|e_n \pm \gamma(m)\|} \]
are well-defined. Using the mean value theorem we deduce from (12) that there exist two points $m_-$ and $m_+$ in $\Sigma$ such that

\begin{equation}
\langle \pi_{T_\pm}(q), v_\pm \rangle = 0,
\end{equation}

where $T_\pm = T_{m_\pm}M$ and $v_\pm = v_\pm(m_\pm)$.

Now we observe that the set $(v_-, v_+, \tau_1, \ldots, \tau_n)$ forms an orthonormal basis $\mathbb{R}^n$. Thus, applying Lemma 3 to (13) we obtain

\begin{equation}
0 = \langle \pi_{T_+}(q), v_+ \rangle \simeq \langle v_-, q \rangle
\end{equation}
at $m_-$ and similarly at $m_+$:

\begin{equation}
0 = \langle \pi_{T_-}(q), v_- \rangle \simeq \langle v_+, q \rangle.
\end{equation}

By getting rid of the denominator in the definition of the vectors $v_\pm$ we arrive at

\begin{equation}
\langle q, e_n \pm \gamma(m_\pm) \rangle = 0.
\end{equation}

It follows that

\begin{equation}
\langle \gamma(m_+) - \gamma(m_-), q \rangle = 2q_n,
\end{equation}

where $q_n = \langle q, e_n \rangle$. Finally, applying the Cauchy’s integral inequality in the last identity yields

\begin{equation}
\| \gamma(m_+ - \gamma(m_-) \| \geq \frac{2q_n}{\| q \|}.
\end{equation}

Taking into account that $\gamma(m_\pm)$ are points on the unit sphere lying in the Gaussian image of $\Sigma$, we obtain $\| \gamma(m_+) - \gamma(m_-) \| = 2\sin \frac{\beta}{2}$, where $\beta$ is the angle between $\gamma(m_+)$ and $\gamma(m_-)$. It follows by the definition of the spherical diameter of $\gamma(\Sigma) \subset S^{n-1}$ that $d(\gamma(\Sigma)) \geq \beta$. Thus, by virtue of (14) we conclude that

\begin{equation}
d(\gamma(\Sigma)) \geq 2\sin \frac{q_n}{\| q \|}.
\end{equation}

To find the maximum of the right part of the last expression we assume $\alpha(\Sigma)$ to be equal the angle between $J(\Sigma)$ and $e_n$. Then, by orthogonality of $q$ to the flow vector $J(\Sigma)$ one easily sees that

\begin{equation}
\max_{q \perp J(\Sigma)} \frac{q_n}{\| q \|} = \sin \alpha(\Sigma).
\end{equation}

Hence,

\begin{equation}
d(\gamma(\Sigma)) \geq 2\alpha(\Sigma),
\end{equation}

and Theorem 1 is proved completely.

**Remark 2.** We notice that in the two-dimensional case the assertion of Theorem 1 is still true provided $\mathcal{M}$ is properly immersed minimal tube. To check this fact we observe that the unique place in the proof of the theorem where we essentially needed the embeddedness hypothesis is formula (10). The validity of this formula in the two-dimensional immersed case is a direct consequence of the Green integration formula.
4. Applications to two-dimensional minimal tubes

To prove Theorem 2 we need some terminology from potential theory.

Let us consider an embedded minimal hypersurface $M$ in $\mathbb{R}^n$ which is a tube in $e_n$-direction. Given $t_1$, $t_2$ from the interval $t(M)$ we notice by $\mathcal{M}(t_1; t_2)$ the portion of $\mathcal{M}$ situated in the slab $t_1 < x_n < t_2$. Then the quantity

$$\text{cap} \mathcal{M}(t_1; t_2) = \inf \int_{\mathcal{M}(t_1; t_2)} |\nabla \varphi|^2,$$

where the infimum is taken over all Lipschitzian functions $\varphi(m)$ on $\mathcal{M}(t_1; t_2)$ such that $\varphi(m) = 0$ on $\Sigma_{t_1}$ and $\varphi(m) = 1$ on $\Sigma_{t_2}$ is called the capacity of $\mathcal{M}(t_1; t_2)$.

Let $\Gamma$ be a family of locally rectifiable curves $\gamma \subset \mathcal{M}$ and $\rho(m) \geq 0$ be a Baire function with the property

$$\int_{\gamma} \varphi(z) \, ds \geq 1,$$

for every $\gamma \in \Gamma$. The infimum

$$\text{mod} \Gamma = \inf \int_{\mathcal{M}(t_1; t_2)} \rho^2(m)$$

over all such $\rho(m)$ is called the module of the family $\Gamma$.

If $\dim \mathcal{M} = 2$ the following connection between the capacity of $\mathcal{M}(t_1; t_2)$ and the module of the family $\Gamma(t_1; t_2)$ of all curves which connect two boundary components of $\mathcal{M}(t_1; t_2)$ is well-known

$$\text{mod} \Gamma(t_1; t_2) = \text{cap} \mathcal{M}(t_1; t_2)$$

(see for the Euclidean case [4] and for the Riemannian case [10] respectively).

In his paper [9] V.M.Miklyukov has studied the higher-dimensional minimal tubes in $\mathbb{R}^n$ and has established the following connection between the capacity of $\mathcal{M}(t_1; t_2)$ and its life-time

$$(15) \quad \text{cap} \mathcal{M}(t_1; t_2) = \frac{t_2 - t_1}{\langle J(\mathcal{M}), e_n \rangle}.$$

**STEP 1.** Let us first assume that $\mathcal{M}$ be a two-dimensional embedded minimal tube to be homeomorphic to an annulus. Then from (15) we obtain

$$(16) \quad \text{mod} \Gamma(t_1; t_2) = \frac{t_2 - t_1}{\langle J(\mathcal{M}), e_3 \rangle}.$$

Since the Gaussian map of a minimal surface is conformal, we have for every tangent vector $X \in T_m \mathcal{M}$

$$\|d\gamma_m(X)\| = \lambda(m)\|X\|,$$

and the Gaussian curvature is $K(m) = -\lambda^2(m)$.

Given an arbitrary $t \in t(\mathcal{M})$ we write by change coordinates formula

$$\int_{\Sigma_t} \lambda \, ds \geq \int_{\gamma(\Sigma_t)} ds_1 \geq 2d(\gamma(\Sigma_t)),$$
where $ds_1$ is the metric element on the unit sphere. By virtue of Theorem 1 we obtain

$$\int_{\Sigma_1} \lambda ds \geq 4\alpha(M),$$

where $\alpha(M)$ is the angle between the flow vector $J(M)$ and $e_3$.

Substituting $\rho(m) = \lambda(m)$ in the definition of the module, we get from (17)

$$\text{mod} \Gamma(t_1; t_2) \leq \frac{1}{16\alpha^2(M)} \int_{M(t_1; t_2)} (-K),$$

and by (16) we arrive at

$$t_2 - t_1 \leq \frac{J_3(M)G(M(t_1; t_2))}{16\alpha^2(M)}.$$

Then the required estimate in the annulus case yields from the arbitrariness of $t_1$ and $t_2$.

In order to prove the general case we need the following elementary fact.

**Lemma 4.** Let $v_1, \ldots, v_l$ be a system of nonzero vectors from $\mathbb{R}^n$ such that for some $e \in \mathbb{R}^n$ one holds $\alpha_i \leq \pi/2$, where $\alpha_i$ is the angle between $v_i$ and $e$. Let $v = \sum_{i=1}^l v_i$. Then we have for the angle $\alpha$ between $v$ and $e$:

$$\alpha \leq \max\{\alpha_1, \ldots, \alpha_l\}.$$

**Proof.** By virtue of $\langle v_i, e \rangle \geq 0$ we notice that $\alpha_i \leq \pi/2$ and by the triangle inequality obtain

$$\cos \alpha = \frac{\langle v, e \rangle}{\|v\|} \geq \frac{\sum_{i=1}^l \langle v_i, e \rangle}{\sum_{i=1}^l \|v_i\|} = \frac{\sum_{i=1}^l \|v_i\| \cos \alpha_i}{\sum_{i=1}^l \|v_i\|} \geq \min_{1 \leq j \leq l} \cos \alpha_i = \cos(\max_{1 \leq j \leq l} \alpha_j),$$

as required.

**STEP 2.** Let $M$ be a properly embedded minimal tube in $\mathbb{R}^3$ of general topological structure. First we notice that for arbitrary closed subinterval $[\alpha; \beta] \subset t(M)$ there exist at most finitely many points $m \in M(\alpha; \beta)$ such that $\gamma(m) = \pm e_3$. Really, the coordinate function $f_3(m) = \langle e_3, x(m) \rangle$ is harmonic on $M$ and it follows that the critical set $H = \{m \in M : \nabla f_3 \equiv e_3 = 0\}$ has no accumulation points inside of $M$. Other words, for any compact part $M(\alpha; \beta)$ the set $H$ is finite.

Let $c_1 < c_2 < \ldots < c_{k-1}$ are all the values of $f_3(m)$ when $m$ runs $H$ and $\alpha = c_0$, $\beta = c_k$. Then by the Morse theory, every part of $M_i = M(c_{i-1}, c_i)$, $i = 1, \ldots, k$, is a union of annuli $M_1^i, \ldots, M_k^i$. 
From positiveness of the third coordinate of the flow vector (2) and Proposition 1 we have $J_3(\mathcal{M}_i) \leq J_3(\mathcal{M})$. Applying Step 1 we obtain

\begin{equation}
 c_i - c_{i-1} \leq \frac{J_3(\mathcal{M})G(\mathcal{M}_i)}{16\alpha^2(\mathcal{M})}, \quad 1 \leq j \leq \ell_i.
\end{equation}

On the other hand, by virtue of Proposition 1

\[ J(\mathcal{M}) = J(\mathcal{M}_i) = J(\mathcal{M}_1) + \ldots + J(\mathcal{M}_\ell). \]

Let the index $j_0$ corresponds to the maximum angle $\alpha(\mathcal{M}_{j_0})$ when $i$ is fixed. Then applying Lemma 4 to $e = e_3$ and $v_j = J(\mathcal{M}_j)$ we obtain

\[ \alpha(\mathcal{M}_{j_0}) \geq \alpha(\mathcal{M}_i) = \alpha(\mathcal{M}). \]

Hence, by (18) and positiveness of the absolute total Gaussian curvature $G$

\[ c_i - c_{i-1} \leq \frac{J_3(\mathcal{M})G(\mathcal{M}_{j_0})}{16\alpha^2(\mathcal{M})} \leq \frac{J_3(\mathcal{M})G(\mathcal{M})}{16\alpha^2(\mathcal{M})}. \]

Summing of the last inequalities over all $i = 1, \ldots, k$ we arrive at

\[ \beta - \alpha \leq \frac{J_3(\mathcal{M})G(\mathcal{M}(c_0; c_k))}{16\alpha^2(\mathcal{M})} \leq \frac{J_3(\mathcal{M})G(\mathcal{M})}{16\alpha^2(\mathcal{M})} = \|J(\mathcal{M})\|G(\mathcal{M}) \cos \alpha(\mathcal{M}) \frac{\cos \alpha(\mathcal{M})}{16\alpha^2(\mathcal{M})}. \]

By arbitrariness of subinterval $[\alpha; \beta] \subset t(\mathcal{M})$ we obtain the assertion of Theorem 2.

References

[1] Ahlfors, L.: Lectures on quasiconformal mappings, New York, Van Nostrand Math. Studies, 1966.
[2] Callahan, M., Hoffman, D. and Meeks W. H.: Embedded minimal surfaces with an infinite number of ends, Invent. math. 96(1989), 459–505.
[3] Fang, Y.: Minimal annuli in $\mathbb{R}^3$ bounded by non-compact complete convex curves in parallel planes, J. Austral. Math. Soc. (Series A) 60(1996), 171–219.
[4] Fuglede, B.: Extremal length and functional completionetion, Acta Math., 98(1957), N 3–4, 171–219.
[5] Hirsch, M. W.: Differential topology, Springer-Verlag, New York Heidelberg Berlin, 1976.
[6] Kobayashi, Sh. and Nomizu, K.: Foundations of differential geometry. Vol.2, Inter-science, 1969.
[7] Meeks, W. H., White, B.: Minimal surfaces bounded by convex curves in parallel planes, Comment. Math. Helv. 66(1991) 263–278.
[8] Meeks, W. H., White, B.: The space of minimal annuli bounded by an extremal pair of planar curves, Comm. Anal. and Geom. 1(1993), No 3, 415–437.
[9] Miklyukov, V. M.: On some properties of tubular minimal surfaces in $\mathbb{R}^n$, Soviet Math. Dokl. 247(1979), N 3, 549-552.
[10] Miklyukov, V. M.: Some parabolicity and hyperbolicity criteria for boundary sets of surfaces, Izv. Ross. Acad. Nauk Ser. Mat. 60(1996), no. 4, 111–158.
[11] Miklyukov, V. M. and Tkachev, V. G.: Some properties of tubular minimal surfaces of arbitrary codimension, *Math. USSR-Sb.* 68 (1991), no. 1, 133–150.

[12] Miklyukov, V. M. and Vedenyapin, V. D.: Extrinsic dimension of tubular minimal hypersurfaces, *Math. USSR-Sb.* 131 (1986), 240–250.

[13] Miklyukov, V. M. and Tkachev, V. G.: Denjoy-Ahlfors Theorem for Harmonic Functions on Riemannian Manifolds and External Structure of Minimal Surfaces, *Commun. in Anal. and Geometry*, 4 (1996), N4, 547–587.

[14] Nitsche, J. C. C.: A characterization of the catenoid, *J. Mathem. Mech.*, 11 (1962), 293–302.

[15] Nitsche, J. C. C.: *Lectures on minimal surfaces. Vol 1.*, Cambridge Univ. Press, Cambridge New York Melbourne Sydney, 1989.

[16] Riemann, B.: *Œuvres Mathematiques de Riemann*, Paris, Gautheriers-Villars, 1898.

[17] Sternberg, S.: *Lectures on differential geometry*, Prentice Hall, Inc. Englewood Cliffs, New York, 1964.

[18] Tkachev, V. G.: Minimal tubes and coefficients of holomorphic functions in annulus, *Bull. de la Soc. Sci. de Lódz.*, Recherches sur deform., Paris, 1995, V. XX, 19-26.

[19] Tkachev, V. G.: Minimal tubes of finite total curvature, *Siberian Math. J.*, 1997 (to be appear).

Department of Mathematics
Volgograd State University