SHARP BOUNDS FOR NEUMAN MEANS IN TERMS OF GEOMETRIC, ARITHMETIC AND QUADRATIC MEANS

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Abstract. In this paper, we find the greatest values \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \) and the least values \( \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8 \) such that the double inequalities

\[
\begin{align*}
A^\alpha_1 (a, b) G^{1-\alpha_1} (a, b) &< N_{GA}(a, b) < A^\beta_1 (a, b) G^{1-\beta_1} (a, b), \\
\frac{\alpha_2}{G(a,b)} + \frac{1 - \alpha_2}{A(a,b)} &< \frac{1}{N_{GA}(a,b)} < \frac{\beta_2}{G(a,b)} + \frac{1 - \beta_2}{A(a,b)}, \\
A^\alpha_3 (a, b) G^{1-\alpha_3} (a, b) &< N_{AG}(a, b) < A^\beta_3 (a, b) G^{1-\beta_3} (a, b), \\
\frac{\alpha_4}{G(a,b)} + \frac{1 - \alpha_4}{A(a,b)} &< \frac{1}{N_{AG}(a,b)} < \frac{\beta_4}{G(a,b)} + \frac{1 - \beta_4}{A(a,b)}, \\
Q^\alpha_5 (a, b) A^{1-\alpha_5} (a, b) &< N_{AQ}(a, b) < Q^\beta_5 (a, b) A^{1-\beta_5} (a, b), \\
\frac{\alpha_6}{A(a,b)} + \frac{1 - \alpha_6}{Q(a,b)} &< \frac{1}{N_{AQ}(a,b)} < \frac{\beta_6}{A(a,b)} + \frac{1 - \beta_6}{Q(a,b)}, \\
Q^\alpha_7 (a, b) A^{1-\alpha_7} (a, b) &< N_{QA}(a, b) < Q^\beta_7 (a, b) A^{1-\beta_7} (a, b), \\
\frac{\alpha_8}{A(a,b)} + \frac{1 - \alpha_8}{Q(a,b)} &< \frac{1}{N_{QA}(a,b)} < \frac{\beta_8}{A(a,b)} + \frac{1 - \beta_8}{Q(a,b)}
\end{align*}
\]

hold for all \( a, b > 0 \) with \( a \neq b \), where \( G, A \) and \( Q \) are respectively the geometric, arithmetic and quadratic means, and \( N_{GA}, N_{AG}, N_{AQ} \) and \( N_{QA} \) are the Neuman means derived from the Schwab-Borchardt mean.

1. Introduction

For \( a, b > 0 \) with \( a \neq b \), the Schwab-Borchardt mean \( SB(a, b) \) [1-3] of \( a \) and \( b \) is given by

\[
SB(a, b) = \begin{cases} 
\frac{\sqrt{a^2 - x^2}}{\cos^{-1}(a/b)} & a < b, \\
\frac{\sqrt{x^2 - b^2}}{\cosh^{-1}(a/b)} & a > b,
\end{cases}
\]

where \( \cos^{-1}(x) \) and \( \cosh^{-1}(x) = \log(x + \sqrt{x^2 - 1}) \) are the inverse cosine and inverse hyperbolic cosine functions, respectively. Recently, the Schwab-Borchardt mean has been the subject of intensive research. In particular, many remarkable inequalities for Schwab-Borchardt mean can be found in the literature [1-7]. Very recently, the Neuman mean \( N(a, b) = (a + b^2/\sqrt{SB(a, b)})/2 \) derived from the Schwab-Borchardt was introduced and researched by Neuman in [8].

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Let $N_{AG}(a, b) = N(A(a, b), G(a, b))$, $N_{GA}(a, b) = N(G(a, b), A(a, b))$, $N_{QA}(a, b) = N(Q(a, b), A(a, b))$ and $N_{AQ}(a, b) = N(A(a, b), Q(a, b))$ be the Neuman means, where $G(a, b) = \sqrt{ab}$, $A(a, b) = (a + b)/2$ and $Q(a, b) = \sqrt{(a^2 + b^2)/2}$ are the classical geometric, arithmetic and quadratic means of $a$ and $b$, respectively. Then Neuman [8] proved that the inequalities

$$G(a, b) < N_{AG}(a, b) < N_{GA}(a, b) < A(a, b) < N_{QA}(a, b) < N_{AQ}(a, b) < Q(a, b)$$

hold for all $a, b > 0$ with $a \neq b$.

Let $a > b > 0$ and $v = (a - b)/(a + b) \in (0, 1)$. Then we clearly see that

$$G(a, b) = A(a, b)\sqrt{1 - v^2}, \quad Q(a, b) = A(a, b)\sqrt{1 + v^2},$$

and the following explicit formulas for $N_{AG}(a, b), N_{GA}(a, b), N_{QA}(a, b)$ and $N_{AQ}(a, b)$ are given in [8]

\[
\begin{align*}
N_{AG}(a, b) &= \frac{1}{2}A(a, b) \left[ 1 + (1 - v^2)\frac{\tanh^{-1} v}{v} \right], \\
N_{GA}(a, b) &= \frac{1}{2}A(a, b) \left[ \sqrt{1 - v^2} + \frac{\sin^{-1} v}{v} \right], \\
N_{QA}(a, b) &= \frac{1}{2}A(a, b) \left[ \sqrt{1 + v^2} + \frac{\sinh^{-1} v}{v} \right], \\
N_{AQ}(a, b) &= \frac{1}{2}A(a, b) \left[ 1 + (1 + v^2)\frac{\tan^{-1} v}{v} \right],
\end{align*}
\]

where $\tanh^{-1}(x) = \log((1 + x)/(1 - x))/2$, $\sin^{-1}(x)$, $\sinh^{-1}(x) = \log(x + \sqrt{1 + x^2})$ and $\tan^{-1}(x)$ are the inverse hyperbolic tangent, inverse sine, inverse hyperbolic sine and inverse tangent functions, respectively.

In [8], Neuman also proved that the double inequalities

\[
\begin{align*}
\alpha_1A(a, b) + (1 - \alpha_1)G(a, b) &< N_{GA}(a, b) < \beta_1A(a, b) + (1 - \beta_1)G(a, b), \\
\alpha_2Q(a, b) + (1 - \alpha_2)A(a, b) &< N_{AQ}(a, b) < \beta_2Q(a, b) + (1 - \beta_2)A(a, b), \\
\alpha_3A(a, b) + (1 - \alpha_3)G(a, b) &< N_{AG}(a, b) < \beta_3A(a, b) + (1 - \beta_3)G(a, b), \\
\alpha_4Q(a, b) + (1 - \alpha_4)A(a, b) &< N_{QA}(a, b) < \beta_4Q(a, b) + (1 - \beta_4)A(a, b)
\end{align*}
\]

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 2/3$, $\beta_1 \geq \pi/4$, $\alpha_2 \leq 2/3$, $\beta_2 \geq (\pi - 2)/(4(\sqrt{2} - 1)) = 0.689 \ldots$, $\alpha_3 \leq 1/3$, $\beta_3 \geq 1/2$, $\alpha_4 \leq 1/3$ and $\beta_4 \geq \log(1 + \sqrt{2}) + \sqrt{2 - 2}/[2(\sqrt{2} - 1)] = 0.356 \ldots$.

In [9], the authors presented the best possible parameters $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0, 1/2]$ and $\alpha_3, \alpha_4, \beta_3, \beta_4 \in [1/2, 1]$ such that the double inequalities

\[
\begin{align*}
G(\alpha_1a + (1 - \alpha_1)b, \alpha_1b + (1 - \alpha_1)a) &< N_{AG}(a, b) < G(\beta_1a + (1 - \beta_1)b, \beta_1b + (1 - \beta_1)a), \\
G(\alpha_2a + (1 - \alpha_2)b, \alpha_2b + (1 - \alpha_2)a) &< N_{GA}(a, b) < G(\beta_2a + (1 - \beta_2)b, \beta_2b + (1 - \beta_2)a), \\
Q(\alpha_3a + (1 - \alpha_3)b, \alpha_3b + (1 - \alpha_3)a) &< N_{QA}(a, b) < Q(\beta_3a + (1 - \beta_3)b, \beta_3b + (1 - \beta_3)a), \\
Q(\alpha_4a + (1 - \alpha_4)b, \alpha_4b + (1 - \alpha_4)a) &< N_{AQ}(a, b) < Q(\beta_4a + (1 - \beta_4)b, \beta_4b + (1 - \beta_4)a)
\end{align*}
\]

hold for all $a, b > 0$ with $a \neq b$. 
The main purpose of this paper is to find the greatest values $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8$ and the least values $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8$ such that the double inequalities

$$A^\alpha(a,b)G^{1-\alpha}(a,b) < N_{GA}(a,b) < A^\beta(a,b)G^{1-\beta}(a,b),$$

$$\alpha_2 \frac{1 - \alpha_2}{G(a,b)} < \frac{1}{N_{GA}(a,b)} < \beta_2 \frac{1 - \beta_2}{G(a,b)},$$

$$A^\alpha(a,b)G^{1-\alpha}(a,b) < N_{AG}(a,b) < A^\beta(a,b)G^{1-\beta}(a,b),$$

$$\alpha_4 \frac{1 - \alpha_4}{A(a,b)} < \frac{1}{N_{AG}(a,b)} < \beta_4 \frac{1 - \beta_4}{A(a,b)},$$

$$Q^\alpha(a,b)A^{1-\alpha}(a,b) < N_{AQ}(a,b) < Q^\beta(a,b)A^{1-\beta}(a,b),$$

$$\alpha_6 \frac{1 - \alpha_6}{Q(a,b)} < \frac{1}{N_{AQ}(a,b)} < \beta_6 \frac{1 - \beta_6}{Q(a,b)},$$

$$Q^\alpha(a,b)A^{1-\alpha}(a,b) < N_{QA}(a,b) < Q^\beta(a,b)A^{1-\beta}(a,b),$$

$$\alpha_8 \frac{1 - \alpha_8}{Q(a,b)} < \frac{1}{N_{QA}(a,b)} < \beta_8 \frac{1 - \beta_8}{Q(a,b)}$$

hold for all $a,b > 0$ with $a \neq b$.

2. Lemmas

In order to prove our main results we need several lemmas, which we present in this section.

**Lemma 2.1.** (See [10, Theorem 1.25]) Let $-\infty < a < b < \infty$, $f,g : [a,b] \to (-\infty, \infty)$ be continuous on $[a,b]$ and differentiable on $(a,b)$, and $g'(x) \neq 0$ on $(a,b)$. If $f'(x)/g'(x)$ is increasing (decreasing) on $(a,b)$, then so are

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, the the monotonicity in the conclusion is also strict.

**Lemma 2.2.** (See [11, Lemma 1.1]) Suppose that the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ have the radius of convergence $r > 0$ and $b_n > 0$ for all $n \geq 0$. If the sequence $\{a_n/b_n\}$ is (strictly) increasing (decreasing) for all $n \geq 0$, then the function $f(x)/g(x)$ is also (strictly) increasing (decreasing) on $(0,r)$.

**Lemma 2.3.** The function

$$f_1(x) = \frac{\log[\sin(2x)] - \log[2x + \sin(2x)] + \log 2}{\log(\cos x)}$$

is strictly increasing from $(0, \pi/2)$ onto $(2/3, 1)$. 
Proof. It follows from (3.1) that

\[(2.2)\quad f_1(0) = \frac{2}{3},\]

\[(2.3)\quad f_1\left(\frac{\pi}{2}\right) = 1.\]

Let \(g_1(x) = \log\sin(2x) - \log[2x + \sin(2x)] + \log 2,\) \(h_1(x) = \log\cos(x),\) \(g_2(x) = \sin(2x) - 2x \cos(2x)\) and \(h_2(x) = [2x + \sin(2x)]\sin^2 x.\) Then simple computations lead to

\[(2.4)\quad g_1(0) = h_1(0) = g_2(0) = h_2(0) = 0,\]

\[(2.5)\quad f_1(x) = \frac{g_1(x)}{h_1(x)}, \quad \frac{g_1'(x)}{h_1'(x)} = \frac{g_2(x)}{h_2(x)}.
\]

\[(2.6)\quad \frac{g_2'(x)}{h_2'(x)} = \frac{1}{2 + \frac{\sin(2x)}{2x}}.
\]

It is well known that the function \(\sin x/x\) is strictly decreasing on \((0, \pi),\) hence equation (2.6) leads to the conclusion that the function \(\frac{g_2'(x)}{h_2'(x)}\) is strictly increasing on \((0, \pi/2).\) Therefore, Lemma 2.3 follows from Lemma 2.1 and (2.2)-(2.5) together with the monotonicity of \(\frac{g_2'(x)}{h_2'(x)}.\)

Lemma 2.4. The function

\[(2.7)\quad f_2(x) = \frac{\log[2x + \sinh(2x)] - \log[\sinh(x)] - 2 \log 2}{\log[\cosh(x)]}\]

is strictly increasing from \((0, \infty)\) onto \((1/3, 1).\)

Proof. It follows from (2.7) that

\[(2.8)\quad f_2(0^+) = \frac{1}{3},\]

\[(2.9)\quad \lim_{x \to \infty} f_2(x) = 1.\]

Let \(g_3(x) = \log[2x + \sinh(2x)] - \log[\sinh(x)] - 2 \log 2\) and \(h_3(x) = \log[\cosh(x)].\) Then simple computations lead to

\[(2.10)\quad f_2(x) = \frac{g_3(x)}{h_3(x)}, \quad g_3(0^+) = h_3(0) = 0,\]

\[(2.11)\quad \frac{g_3'(x)}{h_3'(x)} = \frac{\sinh(4x) - 4x \cosh(2x) + 2 \sinh(2x) - 4x}{\sinh(4x) + 4x \cosh(2x) - 2 \sinh(2x) - 4x}
\]

\[= \sum_{n=1}^{\infty} \frac{(2^n - 2n)^{2n+2}}{(2n+1)!} x^{2n+1}
\]

\[= \sum_{n=1}^{\infty} \frac{(2^n + 2n)^{2n+2}}{(2n+1)!} x^{2n+1}
\]

\[= \sum_{n=0}^{\infty} \frac{(2^n + 2n - 2n)^{2n+4}}{(2n+3)!} x^{2n}
\]

\[= \sum_{n=0}^{\infty} \frac{(2^n + 2n + 2)^{2n+4}}{(2n+3)!} x^{2n}.
\]
Let
\[ a_n = \frac{(2^{2n+2} - 2n - 2) 2^{2n+4}}{(2n + 3)!}, \quad b_n = \frac{(2^{2n+2} + 2n + 2) 2^{2n+4}}{(2n + 3)!}. \]
Then
\[ b_n > 0, \quad \frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} = \frac{(3n + 2) 2^{2n+2}}{(2^{2n+3} + n + 2) (2^{2n+1} + n + 1)} > 0 \]
for all \( n \geq 0. \)

It follows from Lemma 2.2 and (2.11)-(2.13) that the function \( g_3'(x)/h_3'(x) \) is strictly increasing on \((0, \infty)\). Therefore, Lemma 2.4 follows from Lemma 2.1 and (2.8)-(2.10) together with the monotonicity of \( g_3'(x)/h_3'(x) \).

**Lemma 2.5.** The function
\[ f_3(x) = \frac{2x - \sin(2x)}{(1 - \cos x)[2x + \sin(2x)]} \]
is strictly increasing from \((0, \pi/2)\) onto \((2/3, 1)\).

**Proof.** It follows from (2.14) that
\[ f_3(0^+) = \frac{2}{3}, \]
\[ f_3(\pi/2) = 1. \]

Let \( g_4(x) = 2x - \sin(2x) \) and \( h_4(x) = (1 - \cos x)[2x + \sin(2x)] \). Then simple computations lead to
\[ f_3(x) = \frac{g_4(x)}{h_4(x)}, \quad g_4(0) = h_4(0) = 0,\]
\[ g_4'(x) = 4 \sin^2 x, \]
\[ h_4'(x) = 2 \sin^2 x \cos x - 4 \cos^3 x + 4 \cos^2 x + 2x \sin x, \]
\[ g_4'(0) = h_4'(0) = 0, \]
\[ \frac{g_4''(x)}{h_4''(x)} = \frac{4}{9 \cos x + \frac{x}{\sin x} - 4}, \]
\[ (9 \cos x + \frac{x}{\sin x})' = -8 \sin x - \frac{[2x - \sin(2x)] \cos x}{2 \sin^2 x} < 0 \]
for \( x \in (0, \pi/2) \).

Therefore, Lemma 2.5 follows easily from Lemma 2.1 and (2.15)-(2.20).

**Lemma 2.6.** The function
\[ f_4(x) = \frac{\sinh(x) \cosh^2(x) - 2 \sinh(x) \cosh(x) + x \cosh(x)}{\sinh(x) \cosh^2(x) - \sinh(x) \cosh(x) + x \cosh(x) - x} \]
is strictly increasing from \((0, \infty)\) onto \((1/3, 1)\).
Proof. It follows from (2.21) that
\[
(2.22) \quad f_4(x) = \frac{\frac{1}{3} \sinh(3x) + \frac{2}{3} \sinh(x) - \sinh(2x) + x \cosh(x)}{\frac{7}{3} \sinh(3x) + \frac{2}{3} \sinh(2x) + x \cosh(x) - x}
= \sum_{n=1}^{\infty} \frac{3^{2n+1} - 2^{2n+3} + 8n + 5}{4[(2n+1)!]} x^{2n+1}
= \sum_{n=1}^{\infty} \frac{3^{2n+1} - 2^{2n+3} + 8n + 5}{4[(2n+1)!]} x^{2n+1}
= \sum_{n=1}^{\infty} \frac{3^{2n+3} - 2^{2n+5} + 8n + 13}{x^{2n+1}} x^{2n}
= \sum_{n=0}^{\infty} \frac{3^{2n+3} - 2^{2n+5} + 8n + 13}{4[(2n+3)!]} x^{2n}.
\]
Let
\[
(2.23) \quad a_n = \frac{3^{2n+3} - 2^{2n+5} + 8n + 13}{4[(2n+3)!]}, \quad b_n = \frac{3^{2n+3} - 2^{2n+4} + 8n + 13}{4[(2n+3)!]}.
\]
Then simple computations lead to
\[
(2.24) \quad b_n > \frac{3^{2n+3} - 2^{2n+4}}{4[(2n+3)!]} = \frac{2^{2n+3} \left( \left( \frac{3}{7} \right)^{2n+3} - 2 \right)}{4[(2n+3)!]} > 0
\]
\[
(2.25) \quad \frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} = \frac{(135 \times 3^{2n} - 24n - 31) 2^{2n+4}}{(3^{2n+3} - 2^{2n+5} + 8n + 13) (3^{2n+5} - 2^{2n+6} + 8n + 21)} > 0
\]
for all \( n \geq 0 \).

Note that
\[
(2.26) \quad f_4(0^+) = \frac{1}{3}, \quad \lim_{x \to -\infty} f_4(x) = \lim_{n \to \infty} \frac{a_n}{b_n} = 1.
\]

Therefore, Lemma 2.6 follows easily from Lemma 2.2 and (2.22)-(2.26).

3. Main Results

**Theorem 3.1.** The double inequalities
\[
(3.1) \quad A^{\alpha_1}(a,b) G^{1-\alpha_1}(a,b) < N_{G_A}(a,b) < A^{\beta_1}(a,b) G^{1-\beta_1}(a,b),
\]
\[
(3.2) \quad \frac{\alpha_2}{G(a,b)} + \frac{1 - \alpha_2}{A(a,b)} < \frac{1}{N_{G_A}(a,b)} < \frac{\beta_2}{G(a,b)} + \frac{1 - \beta_2}{A(a,b)}
\]
holds for all \( a,b > 0 \) with \( a \neq b \) if and only if \( \alpha_1 \leq 2/3, \beta_1 \geq 1, \alpha_2 \leq 0 \) and \( \beta_2 \geq 1/3 \).

*Proof.* We clearly see that inequalities (3.1) and (3.2) can be rewritten as
\[
(3.3) \quad \left( \frac{A(a,b)}{G(a,b)} \right)^{\alpha_1} < \frac{N_{G_A}(a,b)}{G(a,b)} < \left( \frac{A(a,b)}{G(a,b)} \right)^{\beta_1}
\]
and
\[
(3.4) \quad 1 - \beta_2 < \frac{1}{\frac{1}{G(a,b)} - \frac{1}{N_{G_A}(a,b)}} < 1 - \alpha_2,
\]
respectively.

Since both the geometric mean \( G(a,b) \) and arithmetic mean \( A(a,b) \) are symmetric and homogeneous of degree 1, without loss of generality, we assume that \( a > b \).
Let \( v = (a - b)/(a + b) \in (0, 1) \). Then from (1.1) and (1.3) we know that inequalities (3.3) and (3.4) are equivalent to

\[
\alpha_1 < \frac{\log \left[ \frac{1}{2} \left( 1 + \frac{\sin^{-1}(v)}{\sqrt{1 - v^2}} \right) \right]}{\log \sqrt{1 - v^2}} < \beta_1
\]

and

\[
1 - \beta_2 < \frac{\sin^{-1} v - v \sqrt{1 - v^2}}{(1 - \sqrt{1 - v^2})(v \sqrt{1 - v^2} + \sin^{-1} v)} < 1 - \alpha_2,
\]

respectively.

Let \( x = \sin^{-1}(v) \). Then \( x \in (0, \pi/2) \),

\[
\frac{\log \left[ \frac{1}{2} \left( 1 + \frac{\sin^{-1}(v)}{\sqrt{1 - v^2}} \right) \right]}{\log \sqrt{1 - v^2}} = \frac{\log[\sin(2x)] - \log[2x + \sin(2x)] + \log 2}{\log(\cos x)},
\]

\[
\frac{\sin^{-1} v - v \sqrt{1 - v^2}}{(1 - \sqrt{1 - v^2})(v \sqrt{1 - v^2} + \sin^{-1} v)} = \frac{2x - \sin(2x)}{(1 - \cos x)[2x + \sin(2x)]}.
\]

Therefore, inequality (3.1) holds for all \( a, b > 0 \) with \( a \neq b \) follows from (3.5) and (3.7) together with Lemma 2.3, and inequality (3.2) holds for all \( a, b > 0 \) with \( a \neq b \) follows from (3.6) and (3.8) together with Lemma 2.5.

**Theorem 3.2.** The double inequalities

\[
A^{\alpha_3}(a, b)G^{1-\alpha_3}(a, b) < N_{AG}(a, b) < A^{\beta_3}(a, b)G^{1-\beta_3}(a, b),
\]

\[
\frac{\alpha_4}{G(a, b)} + 1 - \frac{\alpha_4}{A(a, b)} < \frac{1}{N_{AG}(a, b)} < \frac{\beta_4}{G(a, b)} + 1 - \frac{\beta_4}{A(a, b)}
\]

hold for all \( a, b > 0 \) with \( a \neq b \) if and only if \( \alpha_3 \leq 1/3, \beta_3 \geq 1, \alpha_4 \leq 0 \) and \( \beta_4 \geq 2/3 \).

**Proof.** We clearly see that inequalities (3.9) and (3.10) can be rewritten as

\[
\left( \frac{A(a, b)}{G(a, b)} \right)^{\alpha_3} < N_{AG}(a, b) < \left( \frac{A(a, b)}{G(a, b)} \right)^{\beta_3}
\]

and

\[
1 - \beta_4 < \frac{1 - \frac{A(a, b)}{G(a, b)}}{1 - \frac{A(a, b)}{G(a, b)}} < 1 - \alpha_4,
\]

respectively.

Without loss of generality, we assume that \( a > b \). Let \( v = (a - b)/(a + b) \in (0, 1) \). Then it follows from (1.1) and (1.2) that inequalities (3.11) and (3.12) are equivalent to

\[
\alpha_3 < \frac{\log \left[ \frac{1}{v \sqrt{1 - v^2}} + \frac{\sqrt{1 - v^2}}{v} \tanh^{-1}(v) \right] - \log 2}{\log \sqrt{1 - v^2}} < \beta_3
\]

and

\[
1 - \beta_4 < \frac{v + (1 - v^2) \tanh^{-1}(v) - 2v \sqrt{1 - v^2}}{(1 - \sqrt{1 - v^2})(v + (1 - v^2) \tanh^{-1}(v))} < 1 - \beta_4,
\]
respectively. Let \( x = \tanh^{-1}(v) \in (0, \infty) \). Then simple computations lead to

\[
\log \left[ \frac{1}{\sqrt{1-v^2}} + \frac{\sqrt{1-v^2}}{v} \tanh^{-1}(v) \right] - \log 2
\]

(3.15)

\[
= \log[2x + \sinh(2x)] - \log[\sinh(x)] - 2 \log 2
\]

and

\[
\frac{v + (1-v^2) \tanh^{-1}(v) - 2v \sqrt{1-v^2}}{(1-\sqrt{1-v^2})(v + (1-v^2) \tanh^{-1}(v))}
\]

(3.16)

\[
= \frac{\sinh(x) \cosh^2(x) - 2 \sinh(x) \cosh(x) + x \cosh(x)}{\sinh(x) \cosh^2(x) - \sinh(x) \cosh(x) + x \cosh(x) - x}
\]

Therefore, inequality (3.9) holds for all \( a, b > 0 \) with \( a \neq b \) if and only if \( \alpha_3 \leq 1/3 \) and \( \beta_1 \geq 1 \) follows from (3.13) and (3.15) together with Lemma 2.4, and inequality (3.10) holds for all \( a, b > 0 \) with \( a \neq b \) if and only if \( \alpha_4 \leq 0 \) and \( \beta_4 \geq 2/3 \) follows from (3.14) and (3.16) together with Lemma 2.6.

**Theorem 3.3.** The double inequalities

(3.17) \( Q^{\alpha_5}(a, b)A^{1-\alpha_5}(a, b) < N_{AQ}(a, b) < Q^{\beta_5}(a, b)A^{1-\beta_5}(a, b) \),

(3.18) \( \frac{\alpha_6}{A(a, b)} + \frac{1}{Q(a, b)} \quad \frac{1}{N_{AQ}(a, b)} \quad \frac{1}{A(a, b)} + \frac{1}{Q(a, b)} \quad \frac{1}{N_{AQ}(a, b)} < \frac{1}{A(a, b)} \quad \frac{1}{Q(a, b)} < \frac{1}{A(a, b)} \quad \frac{1}{Q(a, b)} \)

hold for all \( a, b > 0 \) with \( a \neq b \) if and only if \( \alpha_5 \leq 2/3, \beta_5 \geq 2 \log(\pi + 2)/\log 2 - 4 = 0.7244\ldots, \alpha_6 \leq [6 + 2\sqrt{2} - (1 + \sqrt{2})\pi]/(\pi + 2) = 0.2419\ldots \) and \( \beta_6 \geq 1/3 \).

**Proof.** We clearly see that inequalities (3.17) and (3.18) can be rewritten as

(3.19) \( \left( \frac{Q(a, b)}{A(a, b)} \right)^{\alpha_5} < \frac{N_{AQ}(a, b)}{A(a, b)} < \left( \frac{Q(a, b)}{A(a, b)} \right)^{\beta_5} \)

and

(3.20) \( 1 - \beta_6 < \frac{1}{A(a, b)} - \frac{1}{N_{AQ}(a, b)} < 1 - \alpha_6 \),

respectively.

Without loss of generality, we assume that \( a > b \). Let \( v = (a-b)/(a+b) \in (0, 1) \).

Then from (1.1) and (1.5) we clearly see that inequalities (3.19) and (3.20) are equivalent to

(3.21) \( \alpha_5 < \frac{2 \log(1 + \frac{\sqrt{1+v^2}}{v} \tanh^{-1}(v)) - 2 \log 2}{\log(1+v^2)} \quad < \beta_5 \)

and

(3.22) \( 1 - \beta_6 < \frac{[(1+v^2) \tanh^{-1}(v) - v] \sqrt{1+v^2}}{[(1+v^2) \tanh^{-1}(v) + v] (\sqrt{1+v^2} - 1)} \quad < 1 - \alpha_6 \),

respectively.

Let \( x = \tan^{-1}(v) \). Then \( x \in (0, \pi/4) \),

(3.23) \( \frac{2 \log(1 + \frac{\sqrt{1+v^2}}{v} \tanh^{-1}(v)) - 2 \log 2}{\log(1+v^2)} \)


\[
\frac{\log[\sin(2x)] - \log[2x + \sin(2x)] + \log 2}{\log(\cos x)} = f_1(x)
\]
and
\[
\frac{\left[1 + v^2\right] \tan^{-1}(v) - v \sqrt{1 + v^2}}{\left[1 + v^2\right] \tan^{-1}(v) + v \sqrt{1 + v^2} - 1} = f_3(x).
\]

Note that
\[
(3.32)\quad 1 - \frac{\alpha}{\sqrt{\beta}}.
\]

Therefore, inequality (3.17) holds for all \(a, b > 0\) with \(a \neq b\) if and only if \(\alpha_5 \leq 2/3\) and \(\beta_5 \geq 2\log(\pi + 2)/\log 2 - 4\) follows from (3.21), (3.23), (3.25) and Lemma 2.3, and inequality (3.18) holds for all \(a, b > 0\) with \(a \neq b\) if and only if \(\alpha_6 \leq [6 + 2\sqrt{2} - (1 + \sqrt{2})\pi]/(\pi + 2)\) and \(\beta_6 \geq 1/3\) follows from (3.22), (3.24), (3.26) and Lemma 2.5.

**Theorem 3.4.** The double inequalities
\[
Q^{\alpha_7}(a, b)A^{1-\alpha_7}(a, b) < N_{QA}(a, b) < Q^{\beta_7}(a, b)A^{1-\beta_7}(a, b),
\]
\[
\frac{\alpha_8}{Q(a, b)} + \frac{1 - \alpha_8}{A(a, b)} < \frac{1}{N_{QA}(a, b)} < \frac{\beta_8}{A(a, b)} + \frac{1 - \beta_8}{Q(a, b)}
\]
hold for all \(a, b > 0\) with \(a \neq b\) if and only if \(\alpha_7 \leq 1/3\), \(\beta_7 \geq 2\log[\sqrt{2} + \log(1 + \sqrt{2})]/\log 2 - 2 = 0.3977\ldots\), \(\alpha_8 \leq [2 + \sqrt{2} - (1 + \sqrt{2})\log(1 + \sqrt{2})]/[\sqrt{2} + \log(1 + \sqrt{2})] = 0.5603\ldots\) and \(\beta_8 \geq 2/3\).

**Proof.** We clearly see that inequalities (3.27) and (3.28) can be rewritten as
\[
\left(\frac{Q(a, b)}{A(a, b)}\right)^{\alpha_7} < N_{QA}(a, b) < \left(\frac{Q(a, b)}{A(a, b)}\right)^{\beta_7}
\]
and
\[
1 - \beta_8 < \frac{1}{A(a, b)} - \frac{1}{N_{QA}(a, b)} < 1 - \alpha_8,
\]
respectively.

Without loss of generality, we assume that \(a > b\). Let \(v = (a - b)/(a + b) \in (0, 1)\). Then from (1.1) and (1.4) we clearly see that inequalities (3.29) and (3.30) are equivalent to
\[
\alpha_7 < \frac{2 \log \left[\sqrt{1 + v^2} + \sinh^{-1}(v)\right] - 2 \log 2 \log(1 + v^2)}{\log(1 + v^2)} < \beta_7
\]
and
\[
1 - \beta_8 < \frac{[v(1 + v^2) + \sqrt{1 + v^2} \sinh^{-1}(v)] - 2v\sqrt{1 + v^2}}{(\sqrt{1 + v^2} - 1)[v\sqrt{1 + v^2} + \sinh^{-1}(v)]} < 1 - \alpha_8,
\]
respectively. 

Let \( x = \sinh^{-1}(v) \). Then \( x \in (0, \log(1 + \sqrt{2})) \),

\[
2 \log \left[ \sqrt{1 + v^2 + \sinh^{-1}(v)} \right] - 2 \log 2 \\
= \frac{\log(2x + \sinh(2x)) - \log(\sinh(x)) - 2 \log 2}{\log[\cosh(x)]} = f_2(x),
\]

\[
(3.33)
\]

\[
\frac{[v(1 + v^2) + \sqrt{1 + v^2} \sinh^{-1}(v)] - 2v\sqrt{1 + v^2}}{(\sqrt{1 + v^2} - 1)[v\sqrt{1 + v^2} + \sinh^{-1}(v)]}
= \frac{\sinh(x) \cosh^2(x) - 2 \sinh(x) \cosh(x) + x \cosh(x)}{\sinh(x) \cosh^2(x) - \sinh(x) \cosh(x) + x \cosh(x) - x} = f_4(x).
\]

Note that

\[
f_2[\log(1 + \sqrt{2})] = \frac{2 \log[\sqrt{2} + \log(1 + \sqrt{2})]}{\log 2} - 2,
\]

\[
f_4[\log(1 + \sqrt{2})] = \frac{(2 + \sqrt{2}) \log(1 + \sqrt{2}) - 2}{\sqrt{2} + \log(1 + \sqrt{2})}
= 1 - \frac{2 + \sqrt{2} - (1 + \sqrt{2}) \log(1 + \sqrt{2})}{\sqrt{2} + \log(1 + \sqrt{2})}.
\]

Therefore, inequality (3.27) holds for all \( a, b > 0 \) with \( a \neq b \) if and only if \( \alpha_7 \leq 1/3 \) and \( \beta_7 \geq 22\log[\sqrt{2} + \log(1 + \sqrt{2})/\log 2 - 2 \) follows from (3.31), (3.33), (3.35) and Lemma 2.4, and inequality (3.28) holds for all \( a, b > 0 \) with \( a \neq b \) if and only if \( \alpha_8 \leq \frac{[2 + \sqrt{2} - (1 + \sqrt{2}) \log(1 + \sqrt{2})/\sqrt{2} + \log(1 + \sqrt{2})]}{\sqrt{2} + \log(1 + \sqrt{2})} \) and \( \beta_8 \geq 2/3 \) follows from (3.32), (3.34), (3.36) and Lemma 2.6.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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