GLOBAL WELL-POSEDNESS FOR THE $L^2$-CRITICAL NONLINEAR
SCHRÖDINGER EQUATION IN HIGHER DIMENSIONS

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ABSTRACT. The initial value problem for the $L^2$ critical semilinear Schrödinger equation in $\mathbb{R}^n$, $n \geq 3$ is considered. We show that the problem is globally well posed in $H^s(\mathbb{R}^n)$ when $1 > s > \frac{1}{3}$ for $n = 3$, and when $1 > s > \frac{(n-2) - \sqrt{(n-2)^2 + 8(n-2)}}{4}$ for $n \geq 4$. We use the “I-method” combined with a local in time Morawetz estimate.

1. INTRODUCTION

In this paper we study the $L^2$ critical defocusing Cauchy problem

\begin{align}
    (1.1) & \quad iu_t + \Delta u - |u|^{\frac{4}{n}} u = 0, \quad x \in \mathbb{R}^n, \ t \geq 0, \\
    (1.2) & \quad u(x, 0) = u_0(x) \in H^s(\mathbb{R}^n), \ n \geq 3,
\end{align}

where $u(t, x)$ is a complex-valued function in space-time $\mathbb{R}^+ \times \mathbb{R}^n$. Here $H^s(\mathbb{R}^n)$ denotes the usual inhomogeneous Sobolev space.

The local well-posedness definition that we use here reads as follow: for any choice of initial data $u_0 \in H^s$, there exists a positive time $T = T(\|u_0\|_{H^s})$ depending only on the norm of the initial data, such that a solution to the initial value problem exists on the time interval $[0, T]$, it is unique in a certain Banach space of functions $X \subset C([0, T], H^s_x)$, and the solution map from $H^s_x$ to $C([0, T], H^s_x)$ depends continuously on the initial data on the time interval $[0, T]$. If $T = \infty$ we say that a Cauchy problem is globally well-posed.

Although here we only study the case $n \geq 3$, we recall some known results for general dimensions that highlight, in a certain way, the differences that arise when the nonlinearity is not smooth. It is known, for example, that the local and global theory for (1.1) - (1.2), if one considers general smooth data, depend on the smoothness of the nonlinearity in a crucial way. In the case when the nonlinearity is smooth enough (for example, for $n = 1$ and $n = 2$, when the nonlinearity is algebraic and thus $C^\infty$) regularity properties of solutions to the above initial value problem are very well understood. However in the general case, certain restrictions on $s$ are needed in order to answer the questions of local/global well-posedness, regularity and others, as clearly as in the algebraic case. For more information the reader should consult [1].

For our purposes we restrict ourselves to initial data in $H^s$ with $0 < s < 1$, and make some comments for the limiting cases where $s = 0, 1$. It is known, for example, that the initial value problem (1.1) - (1.2) is locally well-posed in $H^s$ when $s > 0$, and local in time solutions enjoy mass conservation

\begin{align}
    (1.3) & \quad \|u(\cdot, t)\|_{L^2(\mathbb{R}^n)} = \|u_0(\cdot)\|_{L^2(\mathbb{R}^n)}.
\end{align}

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Moreover, $H^1$ solutions enjoy conservation of the energy

$$
E(u)(t) = \frac{1}{2} \int |\nabla u(t)|^2 dx + \frac{n}{2n+4} \int |u(t)|^{2n+4} \frac{n}{n+4} dx = E(u)(0),
$$

which together with (1.3) and the local theory immediately yields global in time wellposedness for (1.1)-(1.2) with initial data in $H^1$. Actually T. Kato proved local and global well posedness at the energy level for nonlinearities that are more general than ours. In both these cases no scattering has been proved. In higher dimensions a first result [13] was presented in [10]. There the authors proved that for data in $L^2$ the IVP (1.1)-(1.2) is well-posed in an interval $[0,T]$, but in this case $T = T(u_0)$, making the conservation of the mass not of immediate use in order to obtain global well-posedness.

The main purpose of this paper is to partially extend the techniques that have been developed so far to obtain local or global well-posedness for the $L^2$ critical problem to the non algebraic case.

The question of global well-posedness of (1.1)-(1.2) in the case $0 \leq s < 1$ is at this point only partially answered. Previous work establishes that in one dimension the problem is globally well-posed for $s > 4/9$ (see [7], [19]) and in two dimensions for $s \geq 1/2$ (see [8], [13]). In both these cases no scattering has been proved. In higher dimensions a first result on global well-posedness below the energy norm was presented in [10]. There the authors obtained $s > 1 - \epsilon$, with $\epsilon > 0$, but not explicitly quantifiable. Recently T. Tao and M. Visan announced that (1.1)-(1.2) is globally well-posed in $L^2$ provided $n = 4$ and the data are radially symmetric. Still in higher dimensions a partial result is included in [22] without being explicitly stated. In fact there the authors are considering the $L^2$ critical focusing problem. However, a byproduct of their analysis using the “I-method” gives global well-posedness for some $s < 1$ in dimensions $n \geq 3$. In this paper we extend this result. The precise statement of what we prove is contained in the following theorem.

**Theorem 1.1.** The initial value problem (1.1)-(1.2) is globally well-posed in $H^s(\mathbb{R}^n)$, for any $1 > s > \frac{\sqrt{n^2-1}}{3}$ when $n = 3$, and for any $1 > s > \frac{-(n-2)+\sqrt{(n-2)^2+8(n-2)}}{4}$ when $n \geq 4$.

We notice that our best result, which is obtained for $n = 3$, gives $s > 0.55$ and as $n \to \infty$, $s \to 1$.

The proof of Theorem 1.1 was inspired by a recent paper of Fang and Grillakis [13] in which the case $n = 2$ is considered. It relies on two main ingredients. The first one is the so-called $I$-method introduced by J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao (see, for example [3], [8], [9], [11]) which is based on the almost conservation of a certain modified energy functional. The idea is to replace the conserved quantity $E(u)$ which is no longer available for $s < 1$, with an “almost conserved” variant $E(Iu)$ where $I$ is a smoothing operator of order $1 - s$ which behaves like the identity for low frequencies and like a fractional integral operator for high frequencies. Thus the operator $I$ maps $H^s_x$ to $H^1_x$. Notice that $Iu$ is not a solution to (1.1) and hence we expect an energy increment. This increment is in fact quantifying $E(Iu)$ as an “almost conserved” energy. The key is to prove that on intervals of fixed length, where local well-posedness is satisfied, the increment of the modified energy $E(Iu)$ decays with respect to a large parameter $N$. (For the precise definition of $I$ and $N$ we refer the reader to Section 2.) This requires delicate estimates on the commutator between $I$ and the nonlinearity. In dimensions 1 and 2, where the nonlinearity is algebraic, one can write the commutator explicitly using the Fourier transform, and control it by multi-linear analysis and bilinear estimates. The analysis above can be carried out in the $X^{s,b}$ spaces setting, where one can use the smoothing bilinear Strichartz estimate of Bourgain (see e.g. [2]) along with Strichartz estimates to demonstrate the existence of global rough solutions.
Then we partition the arbitrarily large interval \([0, \lambda T_0]\) so that 

\[
\lambda > \|u\|_{L^\infty_t H^\frac{n}{n-2}(J \times \mathbb{R}^n)}^{\frac{n}{n-2}}
\]

for some suitable \(\nu > 0\) that if \((1.5)\) holds. We first control the growth of \((1.5)\) in \([0, \lambda T_0]\). A little analysis and interpolation we obtain for any compact interval \(J = [a, b]\) a priori control of the 

\[
\|u\|_{L^{\frac{4(n-1)}{n}}_t L^{\frac{2(n-1)}{n}}_x} \lesssim (b-a)^{\frac{n-2}{n}} \|u_0\|_{L^\frac{r}{2}_x} \|u\|_{L^{\frac{n}{n-2}}_t H^\frac{\mu}{2}(J \times \mathbb{R}^n)}^{\frac{n}{n-2}}.
\]

Notice that the above norm is Strichartz admissible (again consult Section 2 for a definition) a fact that will be very important in our argument.

Combining the two ingredients above, the idea of the proof is as follows. Fix a large value of time \(T_0\). We observe that if \(u\) is a solution to \((1.1)\) in the time interval \([0, T_0]\), then 

\[
uu(x) = \frac{1}{\lambda} u(\frac{x}{\lambda}, T_0)
\]

is a solution to the same equation in \([0, T_0]\). We choose the parameter \(\lambda > 0\) so that 

\[
E(Iu_\lambda^0) = O(1).
\]

Using Strichartz estimates we show (see Proposition 3.3) that if \(J = [a, b]\) and 

\[
\|u\|_{L^{\frac{4(n-1)}{n}}_t L^{\frac{2(n-1)}{n}}_x} < \mu,
\]

where \(\mu\) is a small universal constant, then 

\[
Z_I(J) := \sup_{\langle q, r \rangle \text{ admissible}} \|\langle \nabla \rangle Iu_\lambda\|_{L^q_t L_x^r(J \times \mathbb{R}^n)} \lesssim \|Iu_\lambda(a)\|_{H^1}.
\]

Moreover in this same time interval where the problem is then well-posed, we can prove the “almost conservation law” (see Proposition 4.1) 

\[
|E(Iu_\lambda^0)(b) - E(Iu_\lambda^0)(a)| \lesssim N^{-1+s-\nu} Z_I(J)^{2+\frac{\mu}{2n}} \lesssim N^{-1+s-\nu} \|Iu_\lambda^0(a)\|_{H^1}^{2+\frac{\mu}{2n}},
\]

for some suitable \(\nu > 0\).

Of course for the arbitrarily large interval \([0, \lambda^2 T_0]\) we do not have that 

\[
\|u_\lambda\|_{L^{\frac{4(n-1)}{n}}_t L^{\frac{2(n-1)}{n}}_x([0, \lambda^2 T_0] \times \mathbb{R}^n)} < \mu.
\]

This is where we use \((1.5)\). We first control the growth of \((1.5)\) in \([0, \lambda^2 T_0]\). A little analysis shows that 

\[
\|u_\lambda\|_{L^{\frac{4(n-1)}{n}}_t L^{\frac{2(n-1)}{n}}_x([0, \lambda^2 T_0] \times \mathbb{R}^n)} \lesssim (\lambda^2 T_0)^{\frac{n-2}{4(n-1)}} \|u_0\|_{L^\frac{r}{2}_x} \sup_{[0, \lambda^2 T_0]} \left(\|u_0\|_{L^\frac{r}{2}_x} \|Iu_\lambda(t)\|_{H^\frac{\mu}{2}}^{\frac{1}{2}} + \|Iu_\lambda(t)\|_{H^1}^{\frac{1}{2}}\right)\left(\frac{\mu}{n}\right)^{\frac{n-2}{n}}.
\]

Now suppose we knew that for any \(t \in [0, \lambda^2 T_0]\),

\[
\|Iu_\lambda(t)\|_{H^1} = O(1).
\]

Then we partition the arbitrarily large interval \([0, \lambda^2 T_0]\) into \(L\) intervals where the local theory uniformly applies. From \((1.7)\) we have 

\[
L \sim \frac{C(\lambda^2 T_0)^{\frac{n-2}{2}}}{\mu},
\]
with $C$ a large constant that will depend only on the $\|u_0\|_{L^2}$ norm. $L$ is of course finite and defines the number of the intervals in the partition that will make the Strichartz $\mathcal{L}^{\frac{4(n-1)}{2(n-1)-1}}_t \mathcal{L}^{\frac{n-2}{n-1}}_x$ norm less than $\mu$.

In order to obtain (1.8) we observe that
\[
\|Iu^\lambda(t)\|_{H^1} \leq E(1)^2 (Iu^\lambda(t)) + \|u^\lambda(t)\|_{L^2}.
\]
Hence by the fact that the $L^2$ norm is scaling invariant and conserved, we only have to show that
\[
E(Iu^\lambda(t)) \leq O(1)
\]
for all $t \in [0, \lambda^2 T_0]$. By the “almost conservation” law (1.6) we then require that $L \sim N^{1-s+\nu}$, for suitable $\nu$. Since this restriction needs to be compatible with (1.9), we obtain the conditions $1 > s > \frac{\sqrt{7}-1}{3}$ for $n = 3$ and $1 > s > -\frac{(n-2)+\sqrt{(n-2)^2+8(n-2)}}{4}$ for any $n \geq 4$. For a more detailed proof the reader should check Section 5.

The rest of the paper is organized as follows. In Section 2 we introduce some notation and state important propositions that we will use throughout the paper. There we also present as in [12], [18] the estimate (1.5). In Section 3 we prove the local well-posedness theory for $Iu$, and the main estimates that we use to prove the decay of the increment of the modified energy. The decay itself is obtained in Section 4. Finally in Section 5 we give the details of the proof of global well-posedness stated in Theorem 1.1.

2. Preliminaries

In what follows we use $A \lesssim B$ to denote an estimate of the form $A \leq CB$ for some absolute constant $C$. If $A \lesssim B$ and $B \lesssim A$ we say that $A \sim B$. We write $A \ll B$ to denote an estimate of the form $A \leq cB$ for some small constant $c > 0$. In addition $\langle a \rangle := 1 + |a|$ and $a \pm := a \pm \epsilon$ with $0 < \epsilon << 1$.

2.1. Norms and Strichartz estimates. We use $L^r_x(\mathbb{R}^n)$ to denote the Lebesgue space of functions $f : \mathbb{R}^n \to \mathbb{C}$ whose norm
\[
\|f\|_{L^r_x} := \left( \int_{\mathbb{R}^n} |f(x)|^r \, dx \right)^{\frac{1}{r}}
\]
is finite, with the usual modification in the case $r = \infty$. We also define the space-time spaces $L^q_tL^r_x$ by
\[
\|u\|_{L^q_tL^r_x} := \left( \int_J \|u(t)\|_{L^r_x}^q \, dt \right)^{\frac{1}{q}}
\]
for any space-time slab $J \times \mathbb{R}^n$, with the usual modification when either $q$ or $r$ are infinity. When $q = r$ we abbreviate $L^q_tL^q_x$ by $L^q_t$.\r

Definition 2.1. A pair of exponents $(q, r)$ is called admissible in $\mathbb{R}^n$ if
\[
\frac{2}{q} + \frac{n}{r} = \frac{n}{2}, \quad 2 \leq q, r \leq \infty, \quad (q, r, n) \neq (2, \infty, 2).
\]

We recall the following Strichartz estimate [14], [16].

Proposition 2.2. Let $(q, r)$ and $(\tilde{q}, \tilde{r})$ be any two admissible pairs. Suppose that $u$ is a solution to
\[
\begin{align*}
iu_t + \Delta u - G(x, t) &= 0, \quad x \in J \times \mathbb{R}^n, \\
u(x, 0) &= u_0(x).
\end{align*}
\]
Then we have the estimate
\begin{equation}
\|u\|_{L^q_t L^r_x} \lesssim \|u_0\|_{L^2} + \|G\|_{L^q_t L^{r'}_x(J \times \mathbb{R}^n)}
\end{equation}
with the prime exponents denoting Hölder dual exponents.

We now define the spatial Fourier transform on \( \mathbb{R}^n \) by
\[
\hat{f}(\xi) := \int e^{-2\pi i x \cdot \xi} f(x) dx.
\]
We also define the fractional differentiation operator \( |\nabla|^\alpha \) for any real \( \alpha \) by
\[
|\nabla|^\alpha u(\xi) := |\xi|^\alpha \hat{u}(\xi)
\]
and analogously
\[
(\nabla)^\alpha u(\xi) := (\xi)^\alpha \hat{u}(\xi).
\]
We then define the inhomogeneous Sobolev space \( H^s \) and the homogeneous Sobolev space \( \dot{H}^s \) by
\[
\|u\|_{H^s} = \|(|\nabla|^s u)\|_{L^2}; \quad \|u\|_{\dot{H}^s} = \|\nabla|^s u\|_{L^2}.
\]

2.2. Nonlinearity. As in [21, 22] we use the notation \( F(z) = |z|^p z, p = \frac{4}{n} \), for the function that defines the nonlinearity in (1.1). We compute the derivatives
\[
F_z(z) = \frac{p+2}{2} |z|^p, \quad \text{and} \quad F_z(z) = \frac{p}{2} z |z|^{p-2}.
\]
We denote by \( F' \) the vector \( (F_z, F_w) \). Also we adopt the notation
\[
w \cdot F'(z) = w F(z) + \overline{w} F_z(z).
\]
In particular, the following chain rule is valid
\[
\nabla F(u) = \nabla u \cdot F'(u).
\]
Clearly \( F'(z) = O(|z|^p) \) and we can estimate the modulus of continuity of \( F' \) as follows
\begin{equation}
|F'(z) - F'(w)| \lesssim |z - w|^{\min\{1,p\}} (|z| + |w|)^{p - \min\{1,p\}},
\end{equation}
for all \( z, w \in \mathbb{C} \). By the fundamental theorem of calculus we have that
\[
F(z + w) - F(z) = \int_0^1 w \cdot F'(z + \theta w) d\theta
\]
and thus the following estimate holds true
\[
F(z + w) = F(z) + O(|w||z|^p) + O(|w|^{p+1})
\]
for all complex values \( z, w \).

We notice that in the case \( n = 3,4 \) the nonlinearity \( F \) is in \( C^{1,1}(\mathbb{C}) \), while in the case \( n \geq 5 \), \( F \in C^{1,4/n}(\mathbb{C}) \), that is \( F' \) is only Hölder continuous. Hence, to estimate our nonlinearity, we will need the following fractional chain rules\(^1\).

**Lemma 2.3.** Suppose that \( G \in C^{0,1}(\mathbb{C}) \), and \( \alpha \in (0,1) \). Then for \( \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r} \), with \( 1 < r \leq r_2 < \infty \) and \( 1 < r_1 \leq \infty \) we have
\[
\||\nabla|^\alpha G(u)\|_{L^r_x} \lesssim \|G'(u)\|_{L^{r_1}_x} \|\nabla|^\alpha u\|_{L^{r_2}_x}.
\]

The proof of this lemma when \( 1 < r_1 < \infty \) can be found in [3] and when \( r_1 = \infty \) in [17].

\(^1\)The reader should keep in mind that these rules will be used with \( G = F' \).
Lemma 2.4. Suppose that $G \in C^{0, \alpha}(\mathbb{C}), \alpha \in (0, 1)$. Then, for every $0 < \sigma < \alpha, 1 < r < \infty$, and $\sigma / \alpha < \rho < 1$ we have

$$\|\nabla |^\sigma G(u)\|_{L^r_x} \lesssim \|u|^{\alpha - \frac{\sigma}{\rho}}\|_{L^r_x} \|\nabla |^\rho u\|_{L^r_x}^\frac{\sigma}{\rho},$$

provided $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$, and $(1 - \frac{\sigma}{\rho})r_1 > 1$.

The proof of this lemma can be found in [20].

Also the following estimates can be found in [22]. We notice that for these estimates to hold, it suffices to require that $F \in C^1(\mathbb{C})$.

Lemma 2.5. Let $1 < r, r_1, r_2 < \infty$ be such that $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. Then, for any $0 < \nu < s$ we have

$$\|\nabla IF(u) - (I\nabla u)F'(u)\|_{L^r_x} \lesssim N^{-1+s-\nu}\|\nabla Iu\|_{L^r_x} \||\nabla |^{1-s+\nu} F'(u)\|_{L^r_x},$$

$$\|\nabla IF(u)\|_{L^r_x} \lesssim \|\nabla Iu\|_{L^r_x} \|F'(u)\|_{L^r_x} + N^{-1+s-\nu}\|\nabla Iu\|_{L^r_x} \||\nabla |^{1-s+\nu} F'(u)\|_{L^r_x}. $$

2.3. Littlewood-Paley Theory and the $I$-operator. We shall also need some Littlewood-Paley theory. In particular, let $\eta(\xi)$ be a smooth bump function supported in the ball $|\xi| \leq 2$, which is equal to one on the unit ball. Then, for each dyadic number $M$ we define the Littlewood-Paley operators

$$\widetilde{P_{\leq M}} f(\xi) = \eta(\xi/M) \hat{f}(\xi),$$

$$\widetilde{P_{> M}} f(\xi) = (1 - \eta(\xi/M)) \hat{f}(\xi),$$

$$\widetilde{P_{M}} f(\xi) = (\eta(\xi/M) - \eta(2\xi/M)) \hat{f}(\xi).$$

Similarly, we can define $P_{< M}, P_{\geq M}$.

Finally, we introduce the $I$-operator. For $s < 1$ and a parameter $N >> 1$ let $m(\xi)$ be the following smooth monotone multiplier:

$$m(\xi) := \begin{cases} 1 & \text{if } |\xi| < N, \\ \left(\frac{|\xi|}{N}\right)^s - 1 & \text{if } |\xi| > 2N. \end{cases}$$

We define the multiplier operator $I : H^s \to H^1$ by

$$\hat{I}u(\xi) = m(\xi) \hat{u}(\xi).$$

Some basic properties of this operator are collected in the following Lemma.

Lemma 2.6. Let $1 < r < \infty$ and $0 < s < 1$. Then,

$$\|\langle \nabla |^\rho P_{> N} f\|_{L^r_x} \lesssim N^{\rho - 1}\|\nabla I f\|_{L^r_x},$$

$$\|\langle \nabla |^\rho f\|_{L^r_x} \lesssim \|\langle \nabla | I f\|_{L^r_x},$$

$$\|f\|_{H^s_x} \lesssim \|I f\|_{H^1_x} \lesssim N^{1-s}\|f\|_{H^s_x}.$$ 

for all $0 \leq \rho \leq s$.

Proof. We write,

$$\|\langle \nabla |^\rho P_{> N} f\|_{L^r_x} = \|P_{> N} \langle \nabla |^\rho (\nabla I)^{-1} \nabla I f\|_{L^r_x},$$

and the claim (2.5) follows from Hörmander’s multiplier theorem.
In order to get (2.6) we write
\[ \| \langle \nabla \rangle^{\delta} f \|_{L^2_x} \leq \| P_{\leq N} \langle \nabla \rangle^{\delta} f \|_{L^2_x} + \| P_{> N} \langle \nabla \rangle^{\delta} f \|_{L^2_x} \leq \| \langle \nabla \rangle^s \|_{L^2} + \| P_{> N} \langle \nabla \rangle^{\delta} f \|_{L^2_x} \]
and again the claim follows from Hörmander’s multiplier theorem and (2.5).

Finally, to show (2.7) we observe that by the definition of the I-operator and (2.6) we get
\[ \| f \|_{H^s_x} \lesssim \| P_{\leq N} f \|_{H^s_x} + \| \langle \nabla \rangle^s P_{> N} f \|_{L^2_x} \lesssim \| IP_{\leq N} f \|_{H^s_x} + N^{s-1} \| \nabla I f \|_{L^2_x} \lesssim \| I f \|_{H^1_x}. \]
Furthermore,
\[ \| I f \|_{H^1_x} = \| \langle \nabla \rangle^{1-s} I \langle \nabla \rangle^s f \|_{L^2_x} \lesssim N^{1-s} \| \langle \nabla \rangle^s f \|_{L^2_x} \lesssim N^{1-s} \| f \|_{H^2_x}, \]
which concludes the proof of (2.7). □

2.4. Interaction Morawetz estimates. We conclude this section with some interaction Morawetz estimates. In [12] it was proved that when \( n = 3 \), on any space-time slab \( J \times \mathbb{R}^n \) on which the solution \( u \) to (1.1)-(1.2) exists, the following \textit{a priori} bound is satisfied:
\[ \left( \int_J \int_{\mathbb{R}^n} |u(x, t)|^4 dx dt \right)^{\frac{1}{4}} \lesssim \left( \int_J \int_{\mathbb{R}^n} |u(x, t)|^2 L^2_x L^\infty_t (J \times \mathbb{R}^3) \right)^{\frac{1}{2}} \lesssim \| u_0 \|_{L^2_x}^2. \]
By Hölder’s inequality in time we get
\[ \| u \|_{L^\frac{8}{5} L^2_x} \lesssim T^\frac{1}{4} \| u_0 \|_{L^2_x}^2. \]
A generalization of (2.11) in any dimension \( n \geq 4 \) was proved in [18]:
\[ \left( \int_J \int_{\mathbb{R}^n} |u(x, t)|^2 |u(y, t)|^2 dx dy dt \right)^{\frac{1}{2}} \lesssim \left( \int_J \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x, t)|^2 |u(y, t)|^{\frac{4}{n} + 2} |x - y|}{|x - y|^3} dx dy dt \right)^{\frac{1}{2}} \lesssim \| u_0 \|_{L^2_x}^2. \]
As a consequence of (2.13) and some harmonic analysis [18], gives the following \textit{a priori} estimate for the solution to (1.1)-(1.2) for \( n \geq 4 \),
\[ \| \langle \nabla \rangle^{-\frac{n-3}{4}} u \|_{L^4_{t,x} (J \times \mathbb{R}^n)} \lesssim \| u_0 \|_{L^2_x}^\frac{3}{2} \| u \|_{L^\infty_t H^\frac{1}{2}_x (J \times \mathbb{R}^n)}. \]
Interpolating between (2.14) and the trivial estimate
\[ \| \langle \nabla \rangle^s u \|_{L^2_x L^\infty_t} \lesssim \| u \|_{L^\infty_t H^\frac{s}{2}_x}, \]
we have that
\[ \| u \|_{L^{\frac{2(n-1)}{n}} L^{\frac{2(n-1)}{n-2}}_t} \lesssim \| u_0 \|_{L^2_x}^\frac{1}{2} \| u \|_{L^{\frac{n-2}{n}}_t H^\frac{1}{2}_x}. \]
Finally, applying Hölder inequality in time, for \( J = [0, T] \), we obtain
\[ \| u \|_{L^{\frac{4(n-1)}{n}} L^{\frac{4(n-1)}{n-2}}_t} \lesssim \| u_0 \|_{L^2_x}^\frac{1}{2} \| u \|_{L^{\frac{n-2}{n}}_t H^\frac{1}{2}_x}. \]
Notice that (2.17) coincides with the inequality obtained by formally substituting \( n = 3 \) into (2.17). Thus from now on we may use (2.17), for every \( n \geq 3 \).
Also we remark that the pair \( \left( \frac{2(n-1)}{n}, \frac{2(n-1)}{n-2} \right) \) is admissible.
3. The main estimates and the local well-posedness for the \( I \)-system.

Assume \( J = [a,b] \). We denote by

\[
Z_I(J) := \sup_{(q,r)} \parallel \langle \nabla \rangle I u \parallel_{L^q_t L^r_x(J \times \mathbb{R}^n)}.
\]

Often we will drop the dependence on the time interval \( J \), and we will write \( Z_I \).

Our main estimates for \( F(u) = |u|^\frac{4}{n} u \) read as follow\(^2\)

**Proposition 3.1.** Let \( 1 > s > \frac{1}{1+\min(1,\frac{4}{n})} \), and let \( 0 < \nu \leq \min\{1,\frac{4}{n}\} s - s - 1 \). Then,

\[
\begin{align}
\| \nabla IF(u) - (I \nabla u) F'(u) \|_{L^s_t L^{2(n-1)}_x L^{n-2}_x}^2 & \lesssim N^{-1+s-\nu} Z_I^{\frac{4}{n}} , \\
\| \nabla IF(u) \|_{L^s_t L^{2(n-1)}_x L^{n-2}_x} & \lesssim \| u \|_{L^{\frac{4}{n}}_t L^{\frac{n+4}{n}}_x} \| \langle \nabla \rangle^{1-s+\nu} F'(u) \|_{L^{\frac{2(n-1)}{n-2}}_t L^{n-1}_x} , \\
\| \nabla IF(u) \|_{L^2_t L^{2(n-1)}_x L^{n-2}_x} & \lesssim N^{-1+s-\nu} Z_I^{\frac{4}{n}} , \\
\| \nabla IF(u) \|_{L^2_t L^{2(n-1)}_x L^{n-2}_x} & \lesssim Z_I^{\frac{4}{n}} .
\end{align}
\]

**Proof.** First we prove the estimate (3.2). In order to do that we apply (2.3) to the left hand side of (3.2) with \( r = 2(n-1)/n \), \( r_1 = 2(n-1)/(n-2) \) and \( r_2 = n-1 \). Performing Hölder’s inequality in time, we then obtain

\[
\begin{align}
\| \nabla IF(u) - (I \nabla u) F'(u) \|_{L^s_t L^{2(n-1)}_x L^{n-2}_x} & \lesssim N^{-1+s-\nu} \| \nabla I u \|_{L^{\frac{4(n-1)}{n}}_t L^{\frac{2(n-1)}{n-2}}_x} \| \langle \nabla \rangle^{1-s+\nu} F'(u) \|_{L^{\frac{2(n-1)}{n-2}}_t L^{n-1}_x} .
\end{align}
\]

Notice that the pair \( (\frac{4(n-1)}{n}, \frac{2(n-1)}{n-2}) \) is admissible, hence

\[
\begin{align}
\| \nabla IF(u) - (I \nabla u) F'(u) \|_{L^s_t L^{2(n-1)}_x L^{n-2}_x} & \lesssim N^{-1+s-\nu} Z_I \| \langle \nabla \rangle^{1-s+\nu} F'(u) \|_{L^{\frac{2(n-1)}{n-2}}_t L^{n-1}_x} .
\end{align}
\]

Thus, we need to control \( \| \langle \nabla \rangle^{1-s+\nu} F'(u) \|_{L^{\frac{2(n-1)}{n-2}}_t L^{n-1}_x} \). Towards this aim, we need to distinguish the cases \( n = 3, 4 \) and the case \( n \geq 5 \).

**Case n = 3.** We first write

\[
\begin{align}
\| \langle \nabla \rangle^{1-s+\nu} F'(u) \|_{L^3_t L^6_x} & \lesssim \| \nabla I^{1-s+\nu} F'(u) \|_{L^3_t L^6_x} + \| F'(u) \|_{L^3_t L^6_x} .
\end{align}
\]

Since \( F'(u) = O(|u|^\frac{4}{n}) \) we have

\[
\begin{align}
\| F'(u) \|_{L^3_t L^6_x} & \lesssim \| u \|_{L^{\frac{16}{3}}_t L^{\frac{8}{3}}_x}^{\frac{4}{n}} \lesssim Z_I^{\frac{4}{n}} ,
\end{align}
\]

where the last inequality follows from the fact that the pair \( (\frac{16}{3}, \frac{8}{3}) \) is admissible together with Lemma 2.6. To bound the first term on the right-hand side of (3.8) we use Lemma

\[\text{(3.8)}\]

\[\text{(3.9)}\]
and obtain for $1/q_1 + 1/q_2 = 1/4$ and $1/r_1 + 1/r_2 = 1/2$,

$$\|\nabla|^{s+\nu}F'(u)\|_{L^4_t L^2_x} \lesssim \|u\|^{1/4}_{L^4_t L^2_x} \cdot \|\nabla|^{s+\nu}u\|_{L^{q_1}_{t,x} L^{r_1}_x}.$$  

(3.10)

Now if the pairs $(\frac{q_1}{4}, \frac{r_1}{4})$ and $(q_2, r_2)$ are admissible then by Lemma 2.3 the expression (3.10) implies

$$\|\nabla|^{s+\nu}F'(u)\|_{L^4_t L^2_x} \lesssim Z^4_I,$$

(3.11)

as long as $0 \leq \nu \leq 2s - 1$. This is guaranteed by the assumption $0 < \nu \leq \min\{1, \frac{4}{n}\}s + s - 1$. The above pairs are admissible if

$$\frac{2}{q_2} + \frac{3}{r_2} = \frac{3}{2}, \quad \frac{2}{q_1} + \frac{3}{r_1} = \frac{1}{2}.$$  

If we add these two equalities we get

$$\frac{2}{q_1} + \frac{2}{q_2} + \frac{3}{r_1} + \frac{3}{r_2} = 2,$$

which is exactly the condition on the Hölder’s exponents. Thus by combining (3.7) with (3.8), (3.9) and (3.11), we obtain the desired estimate (3.3).

**Case $n = 4$.** Now $F'(u) = O(|u|)$, so

$$\|\nabla|^{s+\nu}F'(u)\|_{L^4_t L^2_x} \lesssim \|\nabla|^{s+\nu}F'(u)\|_{L^4_t L^2_x} + \|F'(u)\|_{L^4_t L^2_x}.$$  

(3.12)

Since the pair $(3, 3)$ is admissible, again using Lemma 2.6 we have that

$$\|F'(u)\|_{L^4_t L^2_x} \lesssim \|u\|_{L^4_t L^2_x} \lesssim Z_I.$$  

(3.13)

To bound the first term on the right-hand side of (3.12) we use Lemma 2.3 with $r, r_2 = 3$, and $r_1 = \infty$. Hence,

$$\|\nabla|^{s+\nu}F'(u)\|_{L^4_t L^2_x} \lesssim \|\nabla|^{s+\nu}u\|_{L^4_t L^2_x} \lesssim Z_I,$$

(3.14)

where the last inequality follows from Lemma 2.6 as long as $0 \leq \nu \leq 2s - 1$. This is guaranteed by the assumption $0 < \nu \leq \min\{1, \frac{4}{n}\}s + s - 1$. Thus by combining (3.7) with (3.8), (3.9) and (3.11), we obtain the desired estimate (3.3).

**Case $n \geq 5$.** First we bound $\|F'(u)\|_{L^2_t L^\frac{2(n-1)}{n} L^\frac{4(n-1)}{n-1}_x}$, where $F'(u) = O(|u|^{\frac{4}{n}})$. Hence

$$\|F'(u)\|_{L^2_t L^\frac{2(n-1)}{n} L^\frac{4(n-1)}{n-1}_x} \lesssim \|u\|^{4/n}_{L^{2(n-1)/n}_{t,x}} \cdot \|u\|^{\frac{4(n-1)}{n}}_{L^{4(n-1)/n}_x},$$

where the pair $(\frac{8(n-1)}{n(n-2)}, \frac{4(n-1)}{n})$ is admissible. Thus we immediately have that

$$\|F'(u)\|_{L^2_t L^\frac{2(n-1)}{n} L^\frac{4(n-1)}{n-1}_x} \lesssim Z^4_I.$$  

(3.15)

To bound the homogeneous derivative, since $4/n < 1$, we apply Lemma 2.4 with $\alpha = 4/n, \sigma = 1 - s + \nu, r = n - 1$ and $r_1, r_2$ satisfying

$$\left(\frac{4}{n} - \frac{\sigma}{\rho}\right)r_1 = \frac{4(n-1)}{n}, \quad \frac{\sigma r_2}{\rho} = \frac{4(n-1)}{n},$$

where $(\frac{8(n-1)}{n(n-2)}, \frac{4(n-1)}{n})$ is admissible.
with \( \sigma \frac{4}{n} < \rho < 1 \) to be chosen later. Notice that in order to apply such lemma, we need \( (1 - \frac{\sigma}{\rho}) r_1 > 1 \). For our choices of values, this quantity equals \( n - 1 \), hence the required assumption is satisfied. Then,

\[
\| |\nabla|^{1-s+\nu} F'(u) \|_{L_t^{n-1}} \lesssim \| u \|_{L_t^{\frac{n}{n-2}}}^{\frac{4}{n-2}} \| |\nabla|^{\rho} u \|_{L_t^{\frac{4(n-1)}{n}}}^{\frac{n}{n-1}}
\]

where

\[
\epsilon_1 = \frac{4}{n} - \sigma, \quad \epsilon_2 = \frac{\sigma}{\rho}, \quad \text{hence} \quad \epsilon_1 + \epsilon_2 = \frac{4}{n}.
\]

Thus, applying Hölder’s inequality in time, we obtain

\[
\| |\nabla|^{1-s+\nu} F'(u) \|_{L_t^{\frac{2(n-1)}{n}}} \lesssim \| u \|_{L_t^{\frac{1}{q_1+1}}}^{\frac{4(n-1)}{n}} \| |\nabla|^{\rho} u \|_{L_t^{\frac{2(n-1)}{n}}}^{\frac{n}{n-1}}
\]

where

\[
q_i = \frac{8(n-1)}{n(n-2)} \epsilon_i, \quad i = 1, 2.
\]

Finally, for any \( \rho \) such that \( (1-s+\nu) \frac{4}{n} < \rho < 1 \), which exists by our assumptions on \( \nu \) since \( \nu \leq \frac{4}{n} s + s - 1 \), we have

\[
\| |\nabla|^{1-s+\nu} F'(u) \|_{L_t^{\frac{2(n-1)}{n}}} \lesssim \| u \|_{L_t^{\frac{8(n-1)}{n(n-2)}}}^{\frac{4(n-1)}{n}} \| |\nabla|^{\rho} u \|_{L_t^{\frac{8(n-1)}{n(n-2)}}}^{\frac{n}{n-1}}
\]

\[
\quad \lesssim \| u \|_{L_t^{\frac{8(n-1)}{n(n-2)}}}^{\frac{4}{n}} \| \langle \nabla \rangle^{\rho} u \|_{L_t^{\frac{8(n-1)}{n(n-2)}}}^{\frac{4}{n}}.
\]

The pair \( \left( \frac{8(n-1)}{n(n-2)}, \frac{4(n-1)}{n} \right) \) is admissible. Hence, the expression (3.16) implies by Lemma 2.6 that

\[
\| \langle \nabla \rangle^{1-s+\nu} F'(u) \|_{L_t^{\frac{2(n-1)}{n}}} \lesssim Z_I^{\frac{4}{n}}
\]

as long as \( 0 \leq \rho \leq s \). By our assumption on \( \nu \), there exists \( \rho \) such that \( (1-s+\nu) \frac{4}{n} < \rho \leq s \). Then, combining (3.7), (3.15) and (3.17), we obtain

\[
\| \nabla IF(u) - (I \nabla u) F'(u) \|_{L_t^{\frac{4(n-1)}{n}} L_x^{\frac{2(n-1)}{n}}} \lesssim N^{-1+s-\nu} Z_I^{1+\frac{4}{n}}
\]

which is the desired estimate.

We now proceed with the proof of (3.3). By the triangle inequality, and the estimate (3.2), we obtain

\[
\| \nabla IF(u) \|_{L_t^{\frac{4(n-1)}{n}} L_x^{\frac{2(n-1)}{n}}} \lesssim \| (I \nabla u) F'(u) \|_{L_t^{\frac{4(n-1)}{n}} L_x^{\frac{2(n-1)}{n}}} + \| \nabla IF(u) - (I \nabla u) F'(u) \|_{L_t^{\frac{4(n-1)}{n}} L_x^{\frac{2(n-1)}{n}}}
\]

\[
\lesssim \| (I \nabla u) F'(u) \|_{L_t^{\frac{4(n-1)}{n}} L_x^{\frac{2(n-1)}{n}}} + N^{-1+s-\nu} Z_I^{1+\frac{4}{n}}.
\]
In order to conclude the proof of (3.3), we need to estimate \( \|(I\nabla u)F'(u)\|_{L_t^{(4(n-1))}\over L_x^{(2n-8)}} \). Hölder’s inequality gives,

\[
\|(I\nabla u)F'(u)\|_{L_t^{(4(n-1))}\over L_x^{(2n-8)}} \lesssim \|u\|_{L_t^{(n-1)}\over L_x^{(2n-2)}} \|I\nabla u\|_{L_t^{(4(n-1))}\over L_x^{(2n-8)}} \|\nabla\|^\nu_{L_t^{(2n-8)}\over L_x^{(2n-8)}}
\]

(3.18)

where the last inequality follows from the fact that \((\frac{4(n-1)}{3n-8}, \frac{2n(n-1)}{n^2-4n+8})\) is admissible. This concludes the proof of (3.3).

The proof of (3.4) is along the lines of (3.2). Indeed, by (2.3) we get,

\[
\|\nabla IF(u) - (I\nabla u)F'(u)\|_{L_t^{(2n)\over L_x^{(n+2)}}} \lesssim N^{-1+s-\nu}\|\nabla Iu\|_{L_t^{(n)\over L_x^{(n)}}}\|\langle\nabla\rangle^{1-s+\nu}F'(u)\|_{L_t^{(q)\over L_x^{(r)}}},
\]

where

\[
\frac{1}{2} = \frac{1}{q_1} + \frac{1}{q_2}, \quad \frac{n+2}{2n} = \frac{1}{r_1} + \frac{1}{r_2}, \quad \text{and} \quad \frac{2}{q_2} + \frac{n}{r_2} = 2.
\]

The last condition follows from choosing the pair \((q_1, r_1)\) to be admissible. Hence, setting

\[ r_2 = n - 1, \quad \text{and} \quad q_2 = \frac{2(n-1)}{n-2}, \]

we obtain

\[
\|\nabla IF(u) - (I\nabla u)F'(u)\|_{L_t^{(2n)\over L_x^{(n+2)}}} \lesssim N^{-1+s-\nu}Z_I\|\langle\nabla\rangle^{1-s+\nu}F'(u)\|_{L_t^{(2n)\over L_x^{(n)}}}.
\]

(3.21)

Notice that the right-hand side of (3.21) was estimated in the proof of (3.2). This concludes the proof of (3.4).

We now turn to (3.5). The triangle inequality and (3.3) imply,

\[
\|\nabla IF(u)\|_{L_t^{(2n)\over L_x^{(n+2)}}} \lesssim \|(\nabla Iu)F'(u)\|_{L_t^{(2n)\over L_x^{(n+2)}}} + N^{-1+s-\nu}Z_I^{1+\frac{4}{n}}.
\]

(3.22)

Let \((q_1, r_1)\) be an admissible pair, and let \((q_2, r_2)\) be as in (3.20). Then Hölder’s inequality yields,

\[
\|(\nabla Iu)F'(u)\|_{L_t^{(2n)\over L_x^{(n+2)}}} \lesssim \|\nabla Iu\|_{L_t^{(q_1)\over L_x^{(r_1)}}}\|F'(u)\|_{L_t^{(q_2)\over L_x^{(r_2)}}} \lesssim Z_I\|u\|_{L_t^{(4q_2)\over L_x^{(4r_2)}}} \lesssim Z_I^{1+\frac{4}{n}},
\]

where the last inequality follows from the fact that \((\frac{4q_2}{n}, \frac{4r_2}{n})\) is admissible.

**Remark 3.2.** We notice that the estimates in Proposition 3.1 hold true for any choice of dual admissible pair \((q', r')\) on the right-hand side. This follows immediately by a simple modification of the proof above.

We now turn to the proof of a “modified” local existence theory, that is a local existence involving norms of \(Iu\) instead of \(u\). Following for example the argument in [13], the proof can be reduced to showing the following:
Proposition 3.3. Let $1 > s > \frac{1}{1 + \min\{1, \frac{4}{n}\}}$ and assume that if $u$ is a solution to (1.1) on $J = [a, b]$, the a priori bound

$$\|u\|_{L_t^{\frac{4(n-1)}{n-1}}L_x^{\frac{2(n-1)}{n}}} < \mu,$$

with $\mu$ a small universal constant, is satisfied. Then $Z_I(J) \lesssim \|u(a)\|_{H^1}$.

Proof. Applying $I\langle \nabla \rangle$ to (1.1), and using the Strichartz estimate in (2.1), for any pair of admissible exponents $(q, r)$ we have

$$\|I\langle \nabla \rangle u\|_{L_t^q L_x^r} \lesssim \|u(a)\|_{H^1} + \|I\langle \nabla \rangle IF(u)\|_{L_t^{\frac{4(n-1)}{4n-4}} L_x^{\frac{2(n-1)}{n}}},$$

(3.24)

where the last inequality follows from (3.3), and $0 < \nu \leq \min\{1, \frac{4}{n}\} s + s - 1$.

We now need to control $\|IF(u)\|_{L_t^{\frac{4(n-1)}{4n-4}} L_x^{\frac{2(n-1)}{n}}}$. We compute,

$$\|IF(u)\|_{L_t^{\frac{4(n-1)}{4n-4}} L_x^{\frac{2(n-1)}{n}}} \leq \|IP_{\leq N} F(u)\|_{L_t^{\frac{4(n-1)}{4n-4}} L_x^{\frac{2(n-1)}{n}}} + \|IP_{> N} F(u)\|_{L_t^{\frac{4(n-1)}{4n-4}} L_x^{\frac{2(n-1)}{n}}}.$$  

Using standard harmonic analysis we can bound

$$\|IP_{> N} F(u)\|_{L_t^{\frac{4(n-1)}{4n-4}} L_x^{\frac{2(n-1)}{n}}} \leq \frac{1}{N} \|\nabla IF(u)\|_{L_t^{\frac{4(n-1)}{4n-4}} L_x^{\frac{2(n-1)}{n}}},$$

which in turn is bounded in (3.3). Moreover, by Hölder’s inequality we have,

$$\|IP_{\leq N} F(u)\|_{L_t^{\frac{4(n-1)}{4n-4}} L_x^{\frac{2(n-1)}{n}}} = \|F(u)\|_{L_t^{\frac{4(n-1)}{4n-4}} L_x^{\frac{2(n-1)}{n}}} \lesssim \|u\|_{L_t^{\frac{4(n-1)}{4n-4}} L_x^{\frac{2(n-1)}{n}}} \|u\|_{L_t^{\frac{4(n-1)}{4n-4}} L_x^{\frac{2(n-1)}{4n-8}}} \lesssim \mu^{\frac{1}{n-1}} Z_I,$$

where the last inequality follows from Lemma 2.6 together with the fact that the pair $(\frac{4(n-1)}{4n-4}, \frac{2n(n-1)}{n^2-4n+8})$ is admissible. Thus,

$$\|IF(u)\|_{L_t^{\frac{4(n-1)}{4n-4}} L_x^{\frac{2(n-1)}{n}}} \lesssim N^{-1+s+\nu} Z_I^{\frac{1}{n}} + \mu^{\frac{1}{n-1}} Z_I,$$

(3.26)

which combined with (3.24) gives

$$Z_I \lesssim \|u(a)\|_{H^1} + N^{-1+s+\nu} Z_I^{\frac{1}{n}} + \mu^{\frac{1}{n-1}} Z_I.$$  

(3.27)

A standard continuity argument finishes the proof if we pick $\mu$ sufficiently small and $N$ sufficiently large. \hfill \square

4. Almost conservation of the modified energy.

In this section we will prove that the modified energy functional $E(Iu)$ is almost conserved. We recall that,

$$E(u)(t) = \frac{1}{2} \int |\nabla u(t)|^2 dx + \frac{n}{2n+4} \int |u(t)|^{\frac{2n+4}{n}} dx = E(u)(0).$$
for any smooth solution $u$. We wish to prove the following decay for the increment of the modified energy$^3$.

**Proposition 4.1.** Let $1 > s > \frac{1}{1 + \min\{1, \frac{4}{n}\}}$ and let $0 < \nu \leq \min\{1, \frac{4}{3}\} s + s - 1$. Assume that $u \in C_0^\infty(\mathbb{R}^n)$ is a solution to (1.1) on a time interval $[0, T_0]$. Then, for any $J = [a, b] \subset [0, T_0]$

\begin{equation}
|E(Iu)(b) - E(Iu)(a)| \lesssim N^{-1 + s - \nu} Z_t^{2 + \frac{4}{n}} (J).
\end{equation}

**Proof.** Without loss of generality let us assume that $J = [0, T]$. By the Fundamental Theorem of Calculus

\begin{equation}
E(Iu)(T) - E(Iu)(0) = \int_0^T \frac{\partial}{\partial t} E(Iu)(t) \, dt
\end{equation}

(4.2)

\begin{equation}
= \text{Re} \int_0^T \int_{\mathbb{R}^n} \overline{Iu}(-\Delta Iu + F(Iu)) \, dx \, dt.
\end{equation}

(4.3)

Since $Iu_t = i\Delta Iu - iIF(u)$, we get

\begin{equation}
\text{Re} \int_0^T \int_{\mathbb{R}^n} \overline{Iu}(-\Delta Iu + IF(u)) \, dx \, dt = 0.
\end{equation}

Hence, after integration by parts we obtain

\begin{align*}
E(Iu)(T) - E(Iu)(0) &= -\text{Im} \int_0^T \int_{\mathbb{R}^n} \nabla Iu \cdot \nabla [F(Iu) - IF(u)] \, dx \, dt \\
& \quad - \text{Im} \int_0^T \int_{\mathbb{R}^n} |F(u)| \cdot [F(Iu) - IF(u)] \, dx \, dt \\
& = -\text{Im} \int_0^T \int_{\mathbb{R}^n} \nabla Iu \cdot \nabla [F(Iu) - F'(u)] \, dx \, dt \\
& \quad - \text{Im} \int_0^T \int_{\mathbb{R}^n} \nabla Iu \cdot [F'(u)F(u) - \nabla IF(u)] \, dx \, dt \\
& \quad - \text{Im} \int_0^T \int_{\mathbb{R}^n} IF(u) \cdot [F(Iu) - IF(u)] \, dx \, dt.
\end{align*}

(4.4)

(4.5)

(4.6)

We start by estimating (4.4). By Hölder’s inequality we get,

\begin{equation}
|4.4| \lesssim \|\nabla Iu\|_{L^2_t L^{2n}_{x,S^{n-1}}} \|F(Iu) - F'(u)\|_{L^2_t L^{\frac{2n}{n-2}}_{x}} \\
\lesssim Z_t \|\nabla Iu\| \|F'(u) - F'(u)\|_{L^2_t L^{\frac{2n}{n-2}}_{x}},
\end{equation}

(4.7)

where to obtain (4.7) we use the fact that the pair $(2, \frac{2n}{n-2})$ is admissible. Now we proceed to bound \|\nabla Iu\| \|F'(u) - F'(u)\|_{L^2_t L^{\frac{2n}{n-2}}_{x}}. By Hölder’s inequality we have

\begin{equation}
\|\nabla Iu\| \|F'(u) - F'(u)\|_{L^2_t L^{\frac{2n}{n-2}}_{x}} \lesssim \|\nabla Iu\|_{L^2_t L^{\frac{2n}{n-2}}_{x}} \|F'(Iu) - F'(u)\|_{L^\infty_t L^\frac{2n}{n-2}_x}
\end{equation}

(4.8)

\begin{equation}
\lesssim Z_t \|Iu - u\|_{L^\infty_t L^{\frac{2n}{n-2}}_x} \|Iu\|_{L^\infty_t L^\frac{2n}{n-2}_x} \|u\|_{L^\infty_t L^\frac{2n}{n-2}_x}
\end{equation}

(4.9)

$^3$A similar estimate was also proved in [21, 22].
where to obtain (4.8) we use the fact that the pair \((2, \frac{2n}{n-2})\) is admissible and (2.2), while to obtain (4.9) we distinguish the cases depending on \(\min\{1, \frac{4}{n}\}\) and then use Hölder’s inequality in case \(n \leq 4\). However (4.5) implies

\[
\|P_{>N}u\|_{L_t^\infty L_x^2} \lesssim N^{-\frac{1}{2}}\|\nabla Iu\|_{L_t^\infty L_x^2}.
\]

Thus

\[
\|P_{>N}u\|_{L_t^\infty L_x^2}^{\min\{1, \frac{4}{n}\}} \lesssim N^{-\min\{1, \frac{4}{n}\}}\|\nabla Iu\|_{L_t^\infty L_x^2}^{\min\{1, \frac{4}{n}\}}
\]

(4.10)

where (4.10) follows from the fact that the pair \((\infty, 2)\) is admissible. On the other hand by splitting \(u\) into high and low frequencies we obtain

\[
\|u\|_{L_t^\infty L_x^2} \leq \|P_{<N}u\|_{L_t^\infty L_x^2} + \|P_{>N}u\|_{L_t^\infty L_x^2}
\]

(4.11)

\[
\lesssim \|IP_{<N}u\|_{L_t^\infty L_x^2} + N^{-1}\|\nabla Iu\|_{L_t^\infty L_x^2}
\]

(4.12)

where to obtain (4.11) we used the definition of the operator \(I\) and (2.5), while to obtain (4.12) we used the fact that the pair \((\infty, 2)\) is admissible. However (4.9), (4.10) and (4.12) imply that

\[
\|(\nabla Iu)\ [F'(Iu) - F'(u)]\|_{L_t^2 L_x^{4n}} \lesssim N^{-\min\{1, \frac{4}{n}\}}Z_I^{1+\frac{4}{n}}.
\]

(4.13)

Now we combine (4.7) and (4.13) to conclude

\[
(4.4) \lesssim N^{-\min\{1, \frac{4}{n}\}}Z_I^{2+\frac{4}{n}}
\]

which for \(1 > s\) and \(0 < \nu \leq \min\{1, \frac{4}{n}\}s + s - 1\) implies

\[
(4.1) \lesssim N^{-1+s-\nu}Z_I^{2+\frac{4}{n}}.
\]

We now proceed to bound (4.5). Applying Hölder’s inequality we get

\[
\|\nabla Iu\|_{L_t^\infty L_x^{4(n-1)}} \lesssim \|\nabla Iu\|_{L_t^{4(n-1)} L_x^{2(n-1)}}\|F'(u) - \nabla IF(u)\|_{L_t^{4(n-1)} L_x^{2(n-1)}}
\]

(4.14)

\[
\lesssim Z_I\|\nabla Iu\|_{L_t^{4(n-1)} L_x^{2(n-1)}}\|F'(u) - \nabla IF(u)\|_{L_t^{4(n-1)} L_x^{2(n-1)}}^{\frac{4(n-1)}{n}}\|L_x^{\frac{2(n-1)}{n}}
\]

(4.15)

where to obtain (4.14) we used that the pair \((\frac{4(n-1)}{n}, \frac{2(n-1)}{n-2})\) is admissible, while to obtain (4.15) we used (3.2).

Finally, we bound (4.6).

\[
\|\nabla^{-1}I(F(u))\|_{L_t^2 L_x^{2n}} \lesssim \|\nabla I(F(u))\|_{L_t^2 L_x^{2n}}\|\nabla [F(Iu) - IF(u)]\|_{L_t^2 L_x^{2n}}
\]

(4.16)

\[
\lesssim \|\nabla I(F(u))\|_{L_t^2 L_x^{2n}}\|\nabla [F(Iu) - IF(u)]\|_{L_t^2 L_x^{2n}}^{\frac{4(n-1)}{n}}\|L_x^{\frac{2(n-1)}{n}}
\]

(4.17)

\[
\lesssim Z_I^{1+\frac{4}{n}}\left(\|(\nabla Iu)\ [F'(Iu) - F'(u)]\|_{L_t^2 L_x^{2n}} + \|(\nabla Iu)\ [F'(u) - \nabla IF(u)]\|_{L_t^2 L_x^{2n}}\right)
\]

(4.18)
where in order to obtain (4.16) we used Sobolev’s embedding Theorem, while to obtain (4.17) we used (3.5) and the triangle inequality, and to obtain (4.18) we used (4.13) and (4.14). Finally, for $1 > s$ and $0 < \nu \leq \min\{1, \frac{4}{n}\} s + s - 1 \{4.18\}$ implies

\[
\|Iu\| \lesssim N^{-1+s-\nu} Z_1^{2+s/n},
\]

which concludes our proof.

\[\square\]

5. Global-well posedness.

In this section we present the proof of Theorem 1.1. We recall once again that this proof was inspired by the arguments in [13].

Proof. We start by observing that the assumption on $s$ guarantees that $s > \frac{1}{1+\min\{1, \frac{4}{n}\}}$, thus we can apply the results in the previous sections.

Now, suppose that $u(t,x)$ is a global in time solution to (1.1), with initial data $u_0 \in C_0^0(\mathbb{R}^n)$. Set $u^\lambda(x) = \frac{1}{\lambda^2} u\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right)$. We choose the parameter $\lambda$ so that $\|Iu_0^\lambda\|_{H^1} = O(1)$, that is

\[
\lambda \sim N^{-\frac{1-s}{\nu}}.
\]

Next, let us pick a time $T_0$ arbitrarily large, and inspired by (2.17) let us define

\[
S := \{0 < t < \lambda^2 T_0 : \|u^\lambda\|_{L_t^{4(n-1)/n} L_x^{2(n-1)/n}([0,t] \times \mathbb{R}^n)} \leq KT^{\frac{n-2}{4(n-1)}}\},
\]

with $K$ a constant to be chosen later. We claim that $S$ is the whole interval $[0, \lambda^2 T_0]$. Indeed, assume by contradiction that it is not so, then since

\[
\|u^\lambda\|_{L_t^{4(n-1)/n} L_x^{2(n-1)/n}([0,t] \times \mathbb{R}^n)}
\]

is a continuous function of time, there exists a time $T \in [0, \lambda^2 T_0]$ such that

\[
\|u^\lambda\|_{L_t^{4(n-1)/n} L_x^{2(n-1)/n}([0,T] \times \mathbb{R}^n)} > KT^{\frac{n-2}{4(n-1)}} \quad (5.3)
\]

and

\[
\|u^\lambda\|_{L_t^{4(n-1)/n} L_x^{2(n-1)/n}([0,T] \times \mathbb{R}^n)} \leq 2KT^{\frac{n-2}{4(n-1)}} \quad (5.4)
\]

From the interaction Morawetz estimate (2.17), we have that

\[
\|u^\lambda\|_{L_t^{4(n-1)/n} L_x^{2(n-1)/n}([0,T] \times \mathbb{R}^n)} \lesssim T^{\frac{n-2}{4(n-1)}} \|u_0\|_{L_x^\infty}^{\frac{1}{2}} \|L_t^\frac{n-2}{2} u^\lambda\|_{L_t^\infty H^\frac{n}{2}}^{\frac{1}{2}} ([0,T] \times \mathbb{R}^n). \quad (5.5)
\]

We proceed to estimate the right-hand side of the inequality above in terms of norms involving $Iu$ instead of $u$,

\[
\|u^\lambda(t)\|_{H^\frac{n}{2}} \leq \frac{1}{H^\frac{n}{2}} \left\| P_{< N} u^\lambda(t) \right\|_{H^\frac{n}{2}} + \frac{1}{H^\frac{n}{2}} \left\| P_{\geq N} u^\lambda(t) \right\|_{H^\frac{n}{2}} \quad (5.6)
\]

\[
\leq \left\| P_{< N} u^\lambda(t) \right\|_{L^2} \left\| P_{< N} u^\lambda(t) \right\|_{H^\frac{n}{2}} \left\| P_{< N} u^\lambda(t) \right\|_{H^\frac{n}{2}} + \frac{1}{H^\frac{n}{2}} \left\| I u^\lambda(t) \right\|_{H^\frac{n}{2}} \quad (5.6)
\]

\[
\leq \left\| u_0 \right\|_{L^2} \left\| I u^\lambda(t) \right\|_{H^\frac{n}{2}} \left\| I u^\lambda(t) \right\|_{H^\frac{n}{2}},
\]
where to obtain (5.6) we used an interpolation and (2.5). Hence,

\[
\|u^\lambda\|_{L^4_t L^{(n-1)}_x([0,T] \times \mathbb{R}^n)} \lesssim T^{\frac{n-2}{4(n-1)}} \|u_0\|_{L^2_t L^{(n-1)}_x([0,T])}^{\frac{1}{2}} \sup_{t \in [0,T]} (\|u^\lambda(t)\|_{H^1_x}^{\frac{1}{2}} + \|Iu^\lambda(t)\|_{H_x^2}^{\frac{1}{2}})^{\frac{n-2}{n-1}}.
\]

We now split the interval \([0, T]\) into subintervals \(J_k, k = 1, \ldots, L\) in such a way that

\[
\|u^\lambda\|_{L^4_t L^{(n-1)}_x(J_k \times \mathbb{R}^n)} \leq \mu,
\]

with \(\mu\) as in Proposition 3.3. This is possible because of (5.4). The number \(L\) of possible subinterval must satisfy

\[
L \sim (2K)^{\frac{4(n-1)}{n} T^{\frac{n-2}{n}}} / \mu.
\]

From Proposition 3.3 and Proposition 4.1 we know that, for any \(\nu\) such that \(0 < \nu \leq \min\{1, \frac{4}{n}\} s + s - 1\)

\[
\sup_{[0, T]} E(Iu^\lambda(t)) \lesssim E(Iu^\lambda_0) + \frac{L}{N^{1-s+\nu}}
\]

and by our choice of \(\lambda\), \(E(Iu^\lambda_0) \lesssim 1\). Hence, in order to guarantee that

\[
E(Iu^\lambda) \lesssim 1
\]

holds for all \(t \in [0, T]\) we need to require that

\[
L \lesssim N^{1-s+\nu}.
\]

Since \(T \leq \lambda^2 T_0\), according to (5.9), this is fulfilled as long as

\[
(2K)^{\frac{4(n-1)}{n} (\lambda^2 T_0)^{\frac{n-2}{n}}} / \mu \sim N^{1-s+\nu},
\]

From our choice of \(\lambda\), the expression (5.12) implies that

\[
N^{1-s+\nu} \sim (2K)^{\frac{4(n-1)}{n} N^{\frac{2(1-s)}{s}} (\frac{n-2}{n}) T_0^{\frac{n-2}{n}}},
\]

where

\[
0 < \nu \leq \min\{1, \frac{4}{n}\} s + s - 1.
\]

We pick

\[
\nu = \left(2 \frac{1-s}{s} \frac{n-2}{n} - 1 + s\right) + .
\]

The choice of \(\nu\) given by (5.14) is permissible for \(n = 3\) as long as

\[
\frac{2(1-s)}{s} \frac{1}{3} - 1 + s < 2s - 1,
\]

which is possible as long as \(s > \frac{\sqrt{7}-1}{3}\).
For dimension $n \geq 4$, (5.14) is permissible as long as
\[
\frac{2(1-s)}{s} \left( \frac{n-2}{n} \right) - 1 + s < \frac{n+4}{n} s - 1,
\]
i.e.
\[
s > \frac{-\left(n-2\right) + \sqrt{(n-2)^2 + 8(n-2)}}{4}.
\]
With this choice of $\nu$, we have that $N$ is a large number, for $T_0$ large. Then, from (5.7) and (5.11) we obtain
\[
\|u^\lambda\|_{L^4(\mathbb{R}^n)} \lesssim CT^{\frac{n-2}{4(n-1)}}
\]
for some constant $C > 0$. This contradicts (5.3) for an appropriate choice of $K$. Hence $S = [0, \lambda^2 T_0]$, and $T_0$ can be chosen arbitrarily large. In addition, we have also proved that
\[
\|Iu^\lambda(\lambda^2 T_0)\|_{H^s} = O(1).
\]
But then,
\[
\|u(T_0)\|_{H^s} \lesssim \|u(T_0)\|_{L^2} + \|u(T_0)\|_{H^s} = \|u_0\|_{L^2} + \lambda^s \|u^\lambda(\lambda^2 T_0)\|_{H^s}
\]
\[
\lesssim \lambda^s \|Iu^\lambda(\lambda^2 T_0)\|_{H^2_\alpha} \lesssim \lambda^s \lesssim N^{1-s} \lesssim T_0^{\alpha(s,n)}
\]
where $\alpha(s,n)$ is a positive number that depends on $s$ and $n$. Since $T_0$ is arbitrarily large, the apriori bound on the $H^s$ norm concludes the global well-posedness in the range of $s$ that we summarize below.

$$1 > s_3 > \frac{\sqrt{7} - 1}{3},$$

$$1 > s_n > \frac{-(n-2) + \sqrt{(n-2)^2 + 8(n-2)}}{4}, \quad n \geq 4,$$

where $s_n$ denotes the Sobolev index $s$ corresponding to the space $H^s_\alpha(\mathbb{R}^n)$.

\[\square\]

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