Duality results for Iterated Function Systems with a general family of branches

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Abstract

For $X, Y, Z$ and $W$ compact metric spaces, consider two uniformly contractive IFS $\{\tau_x : Z \to Z, x \in X\}$ and $\{\tau_y : W \to W, y \in Y\}$. For a fixed $\alpha \in \mathcal{P}(X)$ with $\text{supp}(\alpha) = X$ we define the entropy of a holonomic measure $\pi \in \mathcal{P}(X \times Z)$ relative to $\alpha$, the pressure of a continuous cost function $c(x, z)$ and show that for $c$ Lipschitz this pressure coincides with the spectral radius of the associated transfer operator.

The same approach can be applied to the pair $Y, W$.

For fixed probabilities $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$ we define $\Pi(\mu, \nu, \tau)$ as the set of probabilities $\pi \in \mathcal{P}(X \times Y \times Z \times W)$ with $X$-marginal $\mu$, $Y$-marginal $\nu$, $(X, Z)$-marginal holonomic to $\tau_x$ and $(Y, W)$-marginal holonomic to $\tau_y$. We denote also by $\Pi(\cdot, \cdot, \cdot)$ the set of plans without restriction on the $X$ and $Y$ marginals. For fixed probabilities $\alpha \in \mathcal{P}(X)$ and $\beta \in \mathcal{P}(Y)$ with $\text{supp}(\alpha) = X, \text{supp}(\beta) = Y$ we denote by $H_{\alpha}(\pi), \pi \in \Pi(\cdot, \cdot, \cdot)$, the entropy of the $(X, Z)$-marginal of $\pi$ relative to $\alpha$ and denote by $H_{\beta}(\pi)$, the entropy of the $(Y, W)$-marginal of $\pi$ relative to $\beta$. The marginal pressure of a continuous cost function $c \in C(X \times Y \times Z \times W)$ relative to $(\alpha, \beta)$ will be defined by $P_m(c) = \sup_{\pi \in \Pi(\cdot, \cdot, \cdot)} \int c d\pi + H_{\alpha}(\pi) + H_{\beta}(\pi)$ and we will show the following duality result:

$$\inf_{P_m(c-\varphi(x)-\psi(y))=0} \int \varphi(x) d\mu + \int \psi(y) d\nu = \sup_{\pi \in \Pi(\mu, \nu, \tau)} \int c d\pi + H_{\alpha}(\pi) + H_{\beta}(\pi).$$

When $Z$ and $W$ have only one point and the entropy is unconsidered this equality can be rewritten as the Kantorovich Duality for compact spaces $X, Y$ and continuous cost $-c$:

$$\inf_{c-\varphi(x)-\psi(y)\leq0} \int \varphi(x) d\mu + \int \psi(y) d\nu = \sup_{\pi \in \Pi(\mu, \nu)} \int c d\pi.$$

1 Introduction

Let $X$ and $Z$ be compact metric spaces and for each $x \in X$ associate a continuous map $\tau_x : Z \to Z$. We denote by $\Pi(\tau)$ the set of holonomic probabilities in $\mathcal{P}(X \times Z)$, that means the probabilities $\pi$ satisfying

$$\int g(\tau_x(z)) d\pi(x, z) = \int g(z) d\pi(x, z) \text{ for any } g \in C(Z).$$
The holonomic constraint appears in several contexts, not just in discrete dynamic and thermodynamic formalism. For instance, we can find an analogous for Lagrangian mechanics, the Aubry-Mather theory. In [4], [7] is studied the minimizing measures for a Lagrangian (Mather’s Measures) under the constraint

\[ \int_{TM} L(x, v) \, d\mu(x, v), \quad \text{such that} \quad \int_{TM} v d\varphi \, d\mu(x, v) = 0, \quad \forall \varphi \in C^1. \]

The discrete version of the holonomic constraint is

\[ \int_{TM} \varphi(x + v) \, d\mu(x, v) = \int_{TM} \varphi(x) \, d\mu(x, v). \]

Also, in [3] - Section 6, is used a holonomic constraint to study the Monge transportation problem when the cost is the action associated to a Lagrangian function on a compact manifold. See [8], [2] and [15], for general properties and foundations on iterated function systems (IFS) theory.

We will suppose that the family of applications \( \{\tau_x\}_{x \in X} \) is a uniformly contractive IFS, that means, there exists some constant \( 0 < \gamma < 1 \) such that \( d(\tau_x(z_1), \tau_x(z_2)) \leq \gamma[d(x_1, x_2) + d(z_1, z_2)] \) for any \( x_i \in X \) and \( z_i \in Z \).

For each point \( x_0 \in X \) there exists a unique point \( z(x_0) \in Z \) fixed for \( \tau_{x_0} \). The probability \( \pi = \delta_{x_0,z(x_0)} \) is holonomic because for \( g \in C(Z) \)

\[ \int g \, \delta_{x_0,z(x_0)} = g(z(x_0)) = g(\tau_{x_0}(z(x_0))) = \int g(\tau_x(z)) \, \delta_{x_0,z(x_0)}. \]

In the way of transport theory [16] is natural to try to impose some condition on the \( X \)-marginal of \( \pi \). For a fixed \( \mu \in \mathcal{P}(X) \) we denote by \( \Pi(\mu, \tau) \) the set of holonomic probabilities that also satisfy

\[ \int f(x) \, d\pi(x, z) = \int f(x) \, d\mu \]

for any \( f \in C(X) \), that means the set of holonomic probabilities with \( X \)-marginal equal to \( \mu \).

Following [1], [10], for a fixed \( \alpha \) in \( \mathcal{P}(X) \) with \( \text{supp} \alpha = X \) and a Lipschitz cost function \( c(x, z) \) we define an operator \( L_{c,\alpha} : C(Z) \to C(Z) \) (denoted also by \( L \) or \( L_c \)) from

\[ L_{c,\alpha}(\psi)(z) = \int e^{c(x,z)} \psi(\tau_x(z)) \, d\alpha(x). \]

We observe that for countable or finite \( X \) the probability \( \alpha(x) \) represents the summation over the branches of the weighted IFS (see [8], [12], [15]), for instance if \( \alpha(x) = \sum_i \delta_{x_i} \), then we recover the transfer operator

\[ L_{c,\alpha}(\psi)(z) = \sum_i e^{c(x_i,z)} \psi(\tau_{x_i}(z)). \]

This kind of Ruelle-Perron-Frobenius operators, has been studied by several authors in the last few years under different hypothesis on the weights \( e^{c(x_i,z)} \) and on the IFS. For
example in [14] is shown, under suitable hypothesis, the uniqueness of invariant measures for the operator

$$T(f)(x) = \int_S f(w_s(x))d\mu(s)$$

where \{(X, d), w_s : X \to X, s \in S\} is an IFS and \(\mu\) is an a priori distribution of \(s\).

Also, in [15], is studied an hyperbolic IFS with countable many branches, \(\{\phi_i : X \to X, i \in I\}\) and considered thermodynamic formalism for the operator

$$L(\psi)(x) = \sum_{i \in I} e^{\phi(i)}\psi(\phi_i(x))$$

where the family of functions \(\{\phi^i\}\) is a Holder system of functions of order \(\beta\). In this way is defined the topological pressure \(P(\phi) = \lim_{n \to \infty} \frac{1}{n}Z_n(\phi)\), where \(Z_n(\phi)\) is a subadditive partition function associated, and shown that the dual operator \(L^*\) acting in probabilities has an eigenmeasure with eigenvalue \(\lambda = e^{P(\phi)}\). Under the normalization of \(L^*\) there is an unique Gibbs state.

We will briefly show that some results about XY model [1] and Spin Lattice Systems [10] can be applied on the present setting. In the chapter 2 we study the operator \(L_c\) and the dual operator \(L_c^*\). We also define the entropy of a holonomic probability \(\pi \in \Pi(\tau)\) relative to \(\alpha\) by

$$H_\alpha(\pi) = - \sup_{L_\alpha(\tau)=1} \int c(x,z)\,d\pi,$$

the pressure of a continuous cost function \(c\) by

$$P_\alpha(c) = \sup_{\pi \in \Pi(\tau)} \int c\,d\pi + H(\pi)$$

and show that \(e^{P(c)}\) is equal to the spectral radius of \(L_c\) if \(c\) is Lipschitz (related results can be found in [12], [10]).

In the chapter 3 we consider the marginal problem. In this case we fix two other compact metric spaces \(Y\) and \(W\) and an IFS \(\{\tau_y : W \to W, y \in Y\}\) uniformly contractive as \(\{\tau_x\}\) above. By \(\Pi(\cdot, \cdot, \tau)\) we denote the set of probabilities \(\pi \in \mathcal{P}(X \times Y \times Z \times W)\) satisfying

$$\int f(\tau_x(z))\,d\pi = \int f(z)\,d\pi, f \in C(Z) \text{ and } \int g(\tau_y(w))\,d\pi = \int g(w)\,d\pi, g \in C(W).$$

This is the set of plans with \((X, Z)\) and \((Y, W)\) marginals holonomies to \(\{\tau_x\}\) and \(\{\tau_y\}\) respectively. For probabilities \(\alpha \in \mathcal{P}(X)\), \(\beta \in \mathcal{P}(Y)\), with \(\text{supp}(\alpha) = X\), \(\text{supp}(\beta) = Y\) and \(\pi \in \Pi(\cdot, \cdot, \tau)\) we denote by \(H_\alpha(\pi)\) the entropy of the holonomic \((X, Z)\)-marginal of \(\pi\) relative to \(\alpha\). Analogously we denote by \(H_\beta(\pi)\) the entropy of the \((Y, W)\)-marginal of \(\pi\) relative to \(\beta\). The marginal pressure of a continuous cost function \(c(x, y, z, w)\) relative to \((\alpha, \beta)\) will be defined by

$$P^m(c) = \sup_{\pi \in \Pi(\cdot, \cdot, \tau)} \int c\,d\pi + H_\alpha(\pi) + H_\beta(\pi).$$

Using ideas of transport theory [16], [9], [11] we fix two probabilities \(\mu \in \mathcal{P}(X)\) and \(\nu \in \mathcal{P}(Y)\) and denote by \(\Pi(\mu, \nu, \tau)\) the set of probabilities \(\pi \in \Pi(\cdot, \cdot, \tau)\) satisfying

$$\int f(x)\,d\pi = \int f(x)\,d\mu(x), f \in C(X) \text{ and } \int g(y)\,d\pi = \int g(y)\,d\nu(y), g \in C(Y).$$
Applying the Fenchel-Rockafellar Duality Theorem we will prove in Theorem 11 that
\[
\inf_{P_m(c - \varphi(x) - \psi(y)) = 0} \int \varphi(x) \, d\mu + \int \psi(y) \, d\nu = \sup_{\pi \in \Pi(\mu, \nu, \tau)} \int c \, d\pi + H_\alpha(\pi) + H_\beta(\pi).
\]

When \(Z\) and \(W\) have only one point and the entropy is unconsidered (zero temperature in spin lattice system) \(P_m(c - \varphi - \psi) = 0\) is equivalent to \(\sup_{x,y} \{c(x, y) - \varphi(x) - \psi(y)\} = 0\). In this case, the duality above can be rewritten as the Kantorovich Duality for compact spaces \(X, Y\) and continuous cost \(-c:\)
\[
\inf_{c - \varphi(x) - \psi(y) \leq 0} \int \varphi(x) \, d\mu + \int \psi(y) \, d\nu = \sup_{\pi \in \Pi(\mu, \nu)} \int c \, d\pi.
\]

We present this result in Theorem 12.

On the other hand, if we suppose that \(Y\) and \(W\) have only one point, then this result (Theorem 11) can be rewritten in the form
\[
\inf_{P_m(c - \varphi(x)) = 0} \int \varphi(x) \, d\mu = \sup_{\pi \in \Pi(\mu, \tau)} \int c \, d\pi + H_\alpha(\pi).
\]

This result can be interpreted as a kind of variational principle using the operator \(L_{c,\alpha}\) where we change the concept of eigenvalue from number to a function on the variable \(x\).

Although this result seems to remember the main result containing in [11], we have different situations. As an example of application in Thermodynamic Formalism we can suppose \(X = \{0, 1\}\) and \(Z = \{0, 1\}^N\). Let \(\tau_x(z_1, z_2, \ldots) = (x, z_1, z_2, \ldots)\) and for a fixed measure \(\mu = (p_1, p_2)\) over \(\{0, 1\}\) consider the set of invariant probabilities \(\pi\) for the shift map \(\sigma\) acting over \(Z = \{0, 1\}^N\) such that \(\pi([0]) = p_0\) and \(\pi([1]) = p_1\) where \([0]\) and \([1]\) are the cylinders of length one. Denote this set by \(\Pi(\mu)\). Consider the variational principle
\[
\sup_{\pi \in \Pi(\mu)} \int A \, d\pi + h(\pi)
\]
where \(A\) is a Holder potential and \(h\) is the Kolmogorov entropy. In this case we are interested only on invariant probabilities \(\pi\) such that \(\pi([0]) = p_0\). This supremum is equal to
\[
\inf \{\varphi_0 p_0 + \varphi_1 p_1\}
\]
where \((\varphi_0, \varphi_1)\) satisfy \(\sum_{i=1,2} e^{A(i) - \varphi_i} h(i) = h(z)\), for some positive function \(h\) and any \(z\).

## 2 Entropy and pressure

In this section we study the operator \(L_c = L_{c,\alpha}\) and the dual operator \(L^*_c\). We use this operator to define the entropy of an holonomic probability (see [12], [10] for related results). This entropy will be used in the next section when we study the marginal problem.

**Lemma 1.** There exists a positive eigenfunction \(h(z)\) associated to a positive eigenvalue \(\lambda\) for \(L_c\).
Proof. This proof follows [5] (see also [1]). For each 0 < s < 1, we define the operator $T_s$ on $C(Z)$, from

$$T_s(u)(z) = \log \int e^{c(x,z) + su(\tau_z(z))} \, d\alpha(x).$$

We have $|T_s(u) - T_1(v)|_\infty \leq s|u - v|_\infty$, that means, $T$ is an uniform contraction map on $C(Z)$ for the supremum norm ($|g|_\infty = \sup |g(z)|$). Let $u_s$ be the unique fixed point for $T_s$. We claim that $u_s$ are Lipschitz functions with a common constant. Indeed, suppose $u_s(z_1) > u_s(z_2)$, then

$$u_s(z_1) - u_s(z_2) \leq \max_x \{ c(x, z_1) - c(x, z_2) + s(u_s(\tau_x z_1) - u_s(\tau_x z_2)) \}.$$ 

On this way

$$u_s(z_1) - u_s(z_2) \leq \max_{(x_0, x_1, \ldots)} \left( \sum_{n=0}^{\infty} s^n [c(x_n, \tau_{x_n-1} \circ \ldots \circ \tau_{x-1}(z_1)) - c(x_n, \tau_{x_n-1} \circ \ldots \circ \tau_{x-1}(z_2))]) \right)$$

$$\leq \sum_{n=0}^{\infty} s^n \text{Lip}(c) \gamma^n = \frac{\text{Lip}(c)}{1 - s\gamma} \leq \frac{\text{Lip}(c)}{1 - \gamma}.$$

Note that

$$-|c| + s \min u_s \leq u_s(x) \leq |c| + s \max u_s.$$

Then $-|c| \leq (1 - s) \min u_s \leq (1 - s) \max u_s \leq |c|$, for any 0 < s < 1. The family $\{u_n = u_s - \max u_s\}_{0 < s < 1}$ is equicontinuous and uniformly bounded. Then there exists a subsequence $s_n \to 1$ such that $[(1 - s_n) \max u_{s_n}] \to k$, and such that (using Arzela-Ascoli theorem) $\{u_n\}_{n \geq 1}$ has an accumulation point $u \in C(Z)$.

For any $s$

$$e^{u_n(z)} = e^{u_n(z) - \max u_s} = e^{-\max u_s + u_n(z)} \max u_s$$

$$= e^{-\max u_s} \int e^{c(x,z) + (su_n(\tau_x z)) - \max u_s} \, d\alpha(x).$$

Taking the limit we conclude that $u$ satisfies

$$e^{u(z)} = e^{-k} \int e^{c(x,z) + u(\tau_x z)} \, d\alpha(x) = e^{-k} L_{c,\alpha}(e^u)(z).$$

This shows that $h = e^u$ is a positive Lipschitz eigenfunction associated to the eigenvalue $\lambda = e^k$.

Let $\overline{c}(x, z) = c(x, z) + \log(h(\tau_x(z))) - \log(h(z)) - \log(\lambda)$. We can do this because $\lambda > 0$ and $h$ is strictly positive. Then $L_{\overline{c},\alpha}(1) = 1$. Note that $\overline{c}$ is Lipschitz. Indeed, using that $c$ and $h$ are Lipschitz, $h > 0$ and $\log(\cdot)$ is an analytic function we only need to analyze $h(\tau_x(z))$. In this way we have for $(x_1, z_1)$ and $(x_2, z_2)$, from the contraction hypothesis on $\tau$:

$$|h(\tau_{x_1}(z_1)) - h(\tau_{x_2}(z_2))| \leq C d(\tau_{x_1}(z_1), \tau_{x_2}(z_2)) \leq C \gamma (d(x_1, x_2) + d(z_1, z_2)).$$

We say that $c$ is a normalized (cost) function if $L_c(1) = 1$. When we write this, or consider the operator $L_c$, we always suppose $c$ Lipschitz. The function $\overline{c}$ is the normalized
cost function associated to $c$. We have that $\int \tau d\pi = \int c\,d\pi - \log(\lambda)$ for any holonomic probability $\pi$.

**Example:** Different from the Thermodynamic Formalism setting, we can have more than one eigenfunction $h \geq 0$ with positive eigenvalue associated. In Thermodynamic formalism, if $h(x_0) = 0$ in some point $x_0$, then $h = 0$ and this result can be used to prove that the main eigenvalue $\lambda$ has multiplicity equal to one [13].

Consider $X = \{1\}$, $Z = [0, 1]$, $\tau_x(z) = z/2$ and $c = 0$. Then $\tau_x$ is a contraction map with $\gamma = 1/2$.

$$L_{c,\alpha}(\psi)(z) = L(\psi)(z) = \psi(z/2).$$

In this way we have for example two eigenfunctions greater or equal than zero, with positive eigenvalues:

$$h_1(z) = z \quad h_2(z) = 1.$$

Indeed

$$L(h_1)(z) = z/2 = \frac{1}{2} h_1(z), \quad L(h_2)(z) = 1 = 1 \cdot h_2(z).$$

$h_2$ is strictly positive but $h_1 \geq 0$ and equal to zero in $z = 0$. ♦

**Lemma 2.** There exists an unique positive eigenvalue $\lambda$ associated to a strictly positive continuous eigenfunction for $L_{c,\alpha}$ and this $\lambda$ is equal to the spectral radius of $L_{c,\alpha}$ over $C(Z)$. The eigenfunction $h$ associated to $\lambda$ is Lipschitz and unique except by multiplication for constant.

**Proof.** We can consider the function $\tau$, from an automorphism of $C(Z)$ given by the product by $h$ getting above. Suppose that $L_{c,\alpha}$ has a positive eigenvalue $\lambda_2$ associated to a strictly positive continuous eigenfunction $h_2$. We want to show that $\lambda_2 = 1$. Let $h_2(z_0) = \min\{h_2(z)\}$ and $h_2(z_1) = \max\{h_2(z)\}$. Then

$$\lambda_2 h_2(z_1) = \int e^{\tau(x,z)} h_2(\tau_x(z_1)) \, d\alpha \leq \int e^{\tau(x,z)} h_2(z_1) \, d\alpha = h_2(z_1)$$

(this shows that $\lambda_2 \leq 1$) and

$$\lambda_2 h_2(z_0) = \int e^{\tau(x,z)} h_2(\tau_x(z_0)) \, d\alpha \geq \int e^{\tau(x,z)} h_2(z_0) \, d\alpha = h_2(z_0)$$

(this shows that $\lambda_2 \geq 1$). In order to show that 1 is equal to the spectral radius of $L_{\tau}$ note that if $|u|_{\infty} = \sup_z \{|u(z)|\} \leq 1$ then

$$|L_{\tau}(u)|_{\infty} \leq \int e^{\tau(x,z)} |u|_{\infty} \, d\alpha(x) = |u|_{\infty}.$$ 

Therefore

$$|L_{\tau}^n(u)|_{\infty}^{1/n} \leq 1.$$

Now we will show that the constant function is the unique eigenfunction for the normalized operator associated to the eigenvalue 1. In order to show this result we suppose (in contradiction) that there exists an eigenfunction $h$ non constant. We can suppose $h \geq 0$
Let $\rho$ below.

Suppose that Lemma 3.

This show that max\{h\} = $h(\tau_{x_n} \ldots \tau_{x_1} z_1)$ for $\alpha^n$-a.e. $(x_1, \ldots, x_n)$. From the contraction hypothesis on $\tau$ we have

$$d(\tau_{x_n} \ldots \tau_{x_1} z_1, \tau_{x_n} \ldots \tau_{x_1} z_0) < \gamma^n \text{diam}(Z).$$

The function $h$ is uniformly continuous (because $Z$ is compact) and then for $n$ sufficiently large we have $|h(\tau_{x_n} \ldots \tau_{x_1} z_1) - h(\tau_{x_n} \ldots \tau_{x_1} z_0)| < \epsilon$. Consequently

$$h(\tau_{x_n} \ldots \tau_{x_1} z_0) > \max\{h\} - \epsilon > \min\{h\} = h(z_0), \quad \alpha^n$-a.e. $(x_1, \ldots, x_n)$.

This is a contradiction because from

$$h(z_0) = \int_{X^n} e^{c(x_n, \tau_{x_{n-1}} \ldots \tau_{x_1} z_0)} \ldots e^{c(x_2, \tau_{x_1} z_0)} e^{c(x_1, z_0)} h(\tau_{x_n} \ldots \tau_{x_1} z_0) \, d\alpha^n(x_1, \ldots, x_n),$$

we conclude that $h(\tau_{x_n} \ldots \tau_{x_1} z_0) = h(z_0)$, $\alpha^n$-a.e. $(x_1, \ldots, x_n)$. \hfill $\square$

From the above lemma there exists a unique way of associate a normalized function to a Lipschitz function $c(x, z)$ adding a constant and a continuous function (that is also Lipschitz) in the form $g(\tau_{x}(z)) - g(z)$.

If $L_{c,\alpha}(1) = 1$, the dual operator $L_{c,\alpha}^*$ (denoted also by $L^*$) acting over probabilities in $\mathcal{P}(Z)$ is defined from

$$\int \psi(z) \, dL^*(P) = \int L(\psi) \, dP.$$ 

Let $\rho$ be a fixed probability for the dual operator. We want to prove the uniqueness of $\rho$ below.

We denote by $|u|_{\infty}$ the supremum norm of a function $u$ and by $|u|_{lip}$ the Lipschitz constant of $u$.

**Lemma 3.** Suppose that $c$ is normalized and $u$ is a Lipschitz function. There exists a constant $C$ that does not depends of $u$ such that

$$|L_c^nu|_{lip} \leq C |u|_{\infty} + \gamma^n |u|_{lip}.$$

**Proof.** We Follow the proof contained in [13] prop. 2.1.

For $n = 1$ we have

$$|(L_c u)(z_1) - (L_c u)(z_2)| = \int |e^{c(x,z_1)} u(\tau_{x}z_1) - e^{c(x,z_2)} u(\tau_{x}z_2) \, d\alpha(x)|$$

$$\leq \int |e^{c(x,z_1)} - e^{c(x,z_2)}| |u(\tau_{x}z_1)| + |e^{c(x,z_2)}| |u(\tau_{x}z_2) - u(\tau_{x}z_1)| \, d\alpha(x)$$
\[
|e^{c}|_{lip}d(z_1, z_2)|u|_{\infty} + \int e^{c(x, z_2)}|u|_{lip}d(z_1, z_2) \, d\alpha(x)
= (|e^{c}|_{lip}|u|_{\infty} + \gamma|u|_{lip})d(z_1, z_2).
\]

Suppose by induction that

\[
|L^n_{c}u|_{lip} \leq C_n|u|_{\infty} + \gamma^n|u|_{lip},
\]

Then

\[
|L^{n+1}_{c}u|_{lip} \leq C_n|L_{c}u|_{\infty} + \gamma^n|L_{c}u|_{lip} \leq C_n|u|_{\infty} + \gamma^n(C_0|u|_{\infty} + \gamma|u|_{lip})
= (C_n + \gamma^nC_0)|u|_{\infty} + \gamma^{n+1}|u|_{lip}.
\]

We can assume \(C_0 = |e^{c}|_{lip}\) and \(C_{n+1} = C_n + \gamma^nC_0 = \sum_{k=0}^{n}\gamma^kC_0 \leq \frac{1}{\gamma}C_0\). We conclude the proof taking \(C = \frac{1}{\gamma}|e^{c}|_{lip}\).

For each sequence \(x_1, x_2, ...,\) of elements in \(X\) and any points \(z_1, z_2\) in \(Z\) we have

\[
d(\tau_{x_n}...\tau_{x_1}z_1, \tau_{x_n}...\tau_{x_1}z_2) < \gamma^n diam(Z).
\]

Let \(\hat{Z}\) be the set of points \(z \in Z\) satisfying that there exists elements \(x_1, x_2, ...\) in \(X\) and \(z_0 \in Z\) such that \(z\) is an accumulation point of \(\{\tau_{x_n}...\tau_{x_1}z_0\}_{n \geq 1}\). From the above computation the point \(z_0\) is not relevant, but \(x_1, x_2, ....\). If for each \(k\) there exists a sequence \(x_1^k, ..., x_k^k\) and a point \(z_k \in Z\) such that

\[
d(z, \tau_{x_k^k}...\tau_{x_1^k}z_k) < 1/k
\]

then \(z \in \hat{Z}\) because is an accumulation point for the sequence

\[
x_1^1, x_1^2, x_2^3, x_2^3, x_3^3, ...
\]

**Proposition 4.** There exists an unique probability \(\rho \in P(Z)\) fixed for the dual operator \(L^n_{c}\). The support of \(\rho\) is a subset of \(\hat{Z}\) and for any Lipschitz function \(u : Z \to \mathbb{R}\)

\[
L^n_{c, \alpha}(u) \to \int u \, d\rho
\]

uniformly in \(Z\).

**Proof.** First we show that for any possible fixed probability \(\rho\), we have \(supp(\rho) \subseteq \hat{Z}\). Fix a point \(z_1 \in Z - \hat{Z}\). Then there exist \(\epsilon > 0\) and \(n_0\) such that for any \(z_0 \in Z\) and \(x_1, ..., x_n\) in \(X\) with \(n \geq n_0\)

\[
d(z_1, \tau_{x_n}...\tau_{x_1}z_0) > \epsilon.
\]

Let \(u : Z \to [0, +\infty)\) be a continuous function satisfying: \(u = 1\) in \(B(z_1, \epsilon/2)\) and \(u = 0\) out of \(B(z_1, \epsilon)\). Then, for \(n \geq n_0\):

\[
\int u \, d\rho = \int L^n_{c}u \, d\rho
= \int_{Z} \int_{X^n} e^{c(x_n, \tau_{x_{n-1}}...\tau_{x_1}z)} \cdots e^{c(x_2, \tau_{x_1}z)} e^{c(x_1, z)} u(\tau_{x_n}...\tau_{x_1}z) \, d\alpha^n(x_1, ..., x_n) \, d\rho(z) = 0.
\]

This show that \(supp(\rho) \subseteq \hat{Z}\).
Now we show the other parts of the proposition. Fix a Lipschitz function \( u : Z \to [0, \infty) \). From the above lemma

\[
|L^n_c(u)|_{l^p} \leq C|u|_{\infty} + \gamma^n|u|_{l^p}.
\]

We have also

\[
|L^n_c(u)|_{\infty} \leq |u|_{\infty}.
\]

then \( \{L^n_c u\} \) is an equicontinuous family. From the Arzela-Ascoli theorem there exists a convergent sub-sequence \( L^n_c u \to w \) uniformly. Using that \( u \geq 0 \) we have \( w \geq 0 \). From the inequalities \( \sup u \geq \sup \{L_c(u)\} = \sup\{L^2_c(u)\} \) and \( \inf u \leq \inf \{L_c(u)\} \leq \inf \{L^2_c(u)\} \) we conclude that \( \sup w = \sup \{L^n_c(w)\} \), \( n \geq 1 \) and \( \inf\{w\} = \inf \{L^n_c(w)\} \), \( n \geq 1 \).

We want to show that \( w \) is a constant function. In this way, we suppose in contradiction that \( \sup\{w\} = \inf\{w\} > \epsilon > 0 \). Let \( \{z^n_1\} \) and \( \{z^n_2\} \) be such that \( \sup\{w\} = (L^n_c w)(z^n_1) \) and \( \inf\{w\} = (L^n_c w)(z^n_2) \). Then

\[
\sup\{w\} = \int_{X^n} e^{c(x, x_{-1}, \ldots, x_2) + \ldots + c(x_1, x^n_2)} w(x, w_{x_1}, \ldots, w_{x_2}) d\alpha^n(x_1, \ldots, x_n).
\]

This shows that \( w(x_{x_1}, \ldots, x_2) = (L^n_c w)(z^n_1) \), \( \alpha^n\)-a.e. \( (x_1, \ldots, x_n) \). From the contraction hypothesis on \( \tau \) we have

\[
dr \tau x_{x_1}, \ldots, x_2 < \gamma^n \text{diam}(Z).
\]

The function \( w \) is uniformly continuous and then for \( n \) sufficiently large we have \( |w(x, \ldots, x_2) - w(x_{x_1}, \ldots, x_2)| < \epsilon \). Consequently

\[
w(x_{x_1}, \ldots, x_2) > \sup\{w\} - \epsilon > \inf\{w\}, \ \alpha^n\)-a.e. \( (x_1, \ldots, x_n) \).
\]

This is a contradiction because from

\[
\inf\{w\} = \int_{X^n} e^{c(x, x_{-1}, \ldots, x_2) + \ldots + c(x_1, x^n_2)} w(x, w_{x_1}, \ldots, w_{x_2}) d\alpha^n(x_1, \ldots, x_n)
\]

we conclude that \( w(x_{x_1}, \ldots, x_2) = \inf\{w\}, \ \alpha^n\)-a.e. \( (x_1, \ldots, x_n) \).

The partial conclusion is that \( w \) is constant. Let \( \rho \) be a fixed probability for the dual operator. Then \( \text{supp}(\rho) \subseteq Z \) and

\[
\int u \d \rho = \lim_{i \to \infty} \int L^n_c u \d \rho = \int w \d \rho = w.
\]

This can be applied for any sub-sequence and shows that \( L^n_c u \) converges uniformly to \( \int u \d \rho \). If we consider any possible Lipschitz function \( u \) we obtain that \( \rho \) is unique and satisfy

\[
\int u \d \rho = \lim_{n \to \infty} (L^n_c u), \ u \text{ lipschitz}.
\]
The probability \( \pi \in \mathcal{P}(X \times Z) \) defined by
\[
\int g(x, z) \, d\pi = \int \int e^{c(x,z)} g(x, z) \, d\alpha(x) \, d\rho(z)
\] (1)
is holonomic. In fact, if \( g \in C(Z) \):
\[
\int g \, d\pi = \int g \, d\rho = \int L(g) \, d\rho = \int \int e^{c(x,z)} g(\tau_x(z)) \, d\alpha(x) \, d\rho(z) = \int g(\tau_x(z)) \, d\pi.
\]

We call \( \pi \) the holonomic probability associated to the pair \( c, \alpha \). If \( c \) is not normalized there exists a unique normalized \( \overline{c} \) associated to \( c \) (following the above discussion) and the holonomic probability associated to \( c \) will be (by convention) the same holonomic probability associated to \( \overline{c} \).

Following [10] we define the entropy of a holonomic probability \( \pi \in \mathcal{P}(X \times Z) \) relative to the probability \( \alpha \in \mathcal{P}(X) \) by
\[
H_\alpha(\pi) = -\sup_{L_{c,\alpha}1=1} \int c(x, z) \, d\pi(x, z).
\]
The pressure of a continuous function \( c \) relative to \( \alpha \) is defined by
\[
P_\alpha(c) = \sup_{\pi \in \Pi(\tau)} \int c \, d\pi + H_\alpha(\pi).
\]

The relative entropy is not positive because \( c = 0 \) belongs to the set of normalized functions. There are no problem in this notes if we try to change the sign of the entropy relative defining
\[
I_\alpha(\pi) = \sup_{L_{c,\alpha}1=1} \int c(x, z) \, d\pi(x, z)
\]
and then
\[
P_\alpha(c) = \sup_{\pi \in \Pi(\tau)} \int c \, d\pi - I_\alpha(\pi).
\]
In this case the entropy will be non negative. See [10] for an interesting discussion about the relative entropy theme.

**Example:** Suppose that \( Z = \{z\} \) has only one point and \( X = \{1, ..., d\} \) is a finite set. Fix a probability \( \alpha = (p_1, p_2, ..., p_d) \) where \( p_i > 0 \) is the mass of the point \( i \in X \). Any probability \( q = (q_1, ..., q_d) \), \( q_i > 0 \), over \( X \) can be identified with a probability \( \pi = \pi(q) = q \) on \( X \times Z \). We have \( \tau_x(z) = z \), \( x = 1, ..., d \). Note that \( \pi = q \) is holonomic because if \( g \) is a function on the variable \( z \) then \( g \) is constant and \( g(z) = g(\tau_x(z)) \). Any function \( c(x, z) \) is identified with a function \( c(x) \) and will be normalized if
\[
\sum_{i=1}^{d} e^{c(i)} p_i = \sum_{i=1}^{d} e^{c(i)+\log(p_i)} = 1.
\]
Note that \( c(x) = \log(\frac{q_i}{p_i}) \) is normalized because
\[
\sum_{i=1}^{d} e^{\log(\frac{q_i}{p_i})+\log(p_i)} = \sum_{i=1}^{d} e^{\log(q_i)} = \sum_{i=1}^{d} q_i = 1.
\]
From the Jensen’s inequality we have

\[ I_\alpha(q) = \sum_{i=1}^{d} \log\left( \frac{q_i}{p_i} \right) q_i = \sum_{i=1}^{d} \log(q_i) q_i - \sum_{i=1}^{d} \log(p_i) q_i \geq 0. \]

This is the Kullback-Leibler entropy of \( q \) relative to \( p \). We will prove the first equality above as follows in the general case because \( q \) is the holonomic probability associated to the normalized function \( \log\left( \frac{q_x}{p_x} \right) \).

Note that

\[ H_\alpha(q) = -\sum_{i=1}^{d} \log(q_i) q_i = -\sum_{i=1}^{d} \log(q_i) q_i + \sum_{i=1}^{d} \log(p_i) q_i \leq 0. \]

If \( p_i = 1/d \) then

\[ H_\alpha(q) = -\sum_{i=1}^{d} \log(q_i) q_i - \log(d) = h(q) - \log(d) \]

where \( h(q) \) is the Shannon entropy of \( q \). ♦

Consider, for \( c \) Lipschitz, the operator \( \hat{L}_c : C(X \times Z) \to C(Z) \) defined by

\[ \hat{L}_c(g)(z) = \int e^{c(x,z)} g(x,z) \, d\alpha(x). \]

If \( \pi \) is the holonomic probability associated to the normalized Lipschitz function \( c \) then from (Π) we obtain that for any \( g \in C(X \times Z) \):

\[ \int g(x,z) \, d\pi = \int \int e^{c(x,z)} g(x,z) \, d\alpha(x) d\pi(x,z) = \int \hat{L}_c(g) d\pi(x,z) \quad (2) \]

where we use that

\[ \int \int e^{c(x,z)} g(x,z) \, d\alpha(x) d\rho(z) = \int \int e^{c(x,z)} g(x,z) \, d\alpha(x) d\pi(x,z) \]

because \( \int e^{c(x,z)} g(x,z) \, d\alpha(x) \) does not depend on \( x \) and the \( z \) marginal of \( \pi \) is \( \rho \).

**Lemma 5.** Given a normalized function \( c_0 \) with associated holonomic probability \( \pi_0 \) we have that for any normalized function \( c \):

\[ \int c \, d\pi_0 \leq \int c_0 \, d\pi_0. \]

**Proof.** Using that \( e^{-c+c_0} \) is a positive function, we can write

\[ 1 = u(x,z) e^{-c(x,z)+c_0(x,z)}, \]

where \( u = e^{c(x,z)-c_0(x,z)} \) is also positive. Note that, in this case,

\[ 1 = \hat{L}_c 1 = \hat{L}_{c_0} u. \]
Hence,
\[
0 = \log \left( \frac{\hat{L}_c 1}{1} \right) = \log \left( \frac{\hat{L}_{c_0} u}{u(x, z)e^{-c(x, z)+c_0(x, z)}} \right) = \log(\hat{L}_{c_0} u) - \log u + c - c_0,
\]
therefore,
\[
0 = \int \log(\hat{L}_{c_0} u) \, d\pi_0 - \int \log u \, d\pi_0 + \int c \, d\pi_0 - \int c_0 \, d\pi_0.
\]

It follows from (2) and of Jensen’s inequality that
\[
\int c_0 \, d\pi_0 = \int \log(\hat{L}_{c_0} u) \, d\pi_0 - \int \log u \, d\pi_0 + \int c \, d\pi_0 = \int \log(e^{c_0} u \, d\alpha) \, d\pi_0 - \int \int e^{c_0} \log u \, d\alpha \, d\pi_0 + \int c \, d\pi_0 \geq \int c \, d\pi_0.
\]

**Corollary 6.** If \( \pi \) is the holonomic probability associated to the normalized cost \( c \), then
\[
H_\alpha(\pi) = -\int c \, d\pi.
\]

**Theorem 7** (Duality). If \( c \) is Lipschitz, then \( P_\alpha(c) \) is equal to \( \log(\lambda) \) where \( \lambda = \lambda_c \) is the unique positive eigenvalue associated to a positive eigenfunction \( h \) for \( L_{c,\alpha} \). If \( \pi \) is the holonomic probability associated to the normalized \( \overline{c} := c + \log(h \circ \tau_x) - \log(h) - \log(\lambda_c) \), then
\[
P_\alpha(c) = \int \overline{c} \, d\pi + H(\pi).
\]

**Proof.** Let \( \lambda_c > 0 \) be the eigenvalue and \( h > 0 \) the eigenfunction associated to \( c \), then \( \overline{c}(x, z) := c(x, z) + \log(h \circ \tau_x)(z) - \log(h)(z) - \log(\lambda_c) \) is the normalized cost associated to \( c \). As \( h \) depends only on \( z \), for any \( \pi \in \Pi(\tau) \) we have that \( \int \overline{c} \, d\pi = \int c \, d\pi - \log(\lambda_c). \) Note that from the definition of entropy, we obtain for any \( \pi \in \Pi(\tau) \) that \( H(\pi) \leq -\int \overline{c} \, d\pi. \) Then
\[
P(c) = \sup_{\pi \in \Pi(\tau)} \left( \int c \, d\pi + H(\pi) \right) \leq \sup_{\pi \in \Pi(\tau)} \left( \int c \, d\pi - \int \overline{c} \, d\pi \right) = \log(\lambda_c). \quad (3)
\]

In order to show the other inequality, let \( \pi_{\overline{c}} \) be the holonomic probability associated to \( \overline{c} \). Then from the above corollary
\[
P(c) \geq \int c \, d\pi_{\overline{c}} + H(\pi_{\overline{c}}) = \int c \, d\pi_{\overline{c}} - \int \overline{c} \, d\pi_{\overline{c}} = \log(\lambda_c).
\]

**Corollary 8.** For a holonomic probability \( \pi \):
\[
H_\alpha(\pi) = -\sup_{c \text{ continuous}} \left[ \int c \, d\pi - P_\alpha(c) \right] = -\sup_{P_\alpha(c) = 0, c \text{ continuous}} \int c \, d\pi.
\]
Proof. We have

$$I_\alpha(\pi) = \sup_{c \text{Lipschitz}} \left[ \int c \, d\pi - P_\alpha(c) \right] \leq \sup_{c \text{continuous}} \left[ \int c \, d\pi - P_\alpha(c) \right].$$

Using that any continuous function $c$ can be approximated by Lipschitz functions in the uniform topology, and that $P_\alpha$ is continuous we get the equality. \qed

3 Duality results

This section follows ideas from [9], [11]. We are going to use the following result [16]:

**Theorem 9 (Fenchel-Rockafellar duality).** Suppose that $E$ is a normed vector space, $\Theta$ and $\Xi$ are two convex functions defined on $E$ taking values in $\mathbb{R} \cup \{+\infty\}$. Denote $\Theta^*$ and $\Xi^*$, respectively, the Legendre-Fenchel transform of $\Theta$ and $\Xi$. Suppose there exists $v_0 \in E$, such that $\Theta(v_0) < +\infty$, $\Xi(v_0) < +\infty$ and that $\Theta$ is continuous on $v_0$.

Then,

$$\inf_{v \in E} [\Theta(v) + \Xi(v)] = \sup_{f \in E^*} [-\Theta^*(-f) - \Xi^*(f)] \quad (4)$$

Moreover, the supremum in (4) is attained in at least one element in $E^*$.

In the above section for a Lipschitz function $c(x,z)$ we make a normalization

$$\bar{c}(x,z) = c(x,z) + \log(h(\tau_x z)) - \log(h(z)) - \log(\lambda).$$

This normalization is very related with the variational principle given in Theorem 7. Different ways of make the normalization imply in different duality results for the variational principles. Below we show some examples concerning this fact. Actually we can replace the equality in the normalization above for an inequality.

Now we consider two more compact spaces $Y, W$ and applications $\tau_y : W \to W$ defining an uniformly contractive IFS in the same way that $\tau_x : Z \to Z$. Then the results contained in the above section can be applied if we replace $X$ for $Y$ and $Z$ for $W$. We have now the spaces $X, Y, Z, W$ and two IFS $\{\tau_x(z)\}, \{\tau_y(w)\}$.

Denote by $\Pi(\cdot, \cdot, \tau)$ the set of probabilities $\pi \in \mathcal{P}(X \times Y \times Z \times W)$ satisfying

$$\int g(\tau_x(z)) \, d\pi = \int g(z) \, d\pi, \quad g \in C(Z) \text{ and } \int g(\tau_y(w)) \, d\pi = \int g(w) \, d\pi, \quad g \in C(W) \quad (5)$$

that are the probabilities with $(X,Z)$—marginal holonomic for $\tau_x$ and $(Y,W)$—marginal holonomic for $\tau_y$.

Fix probabilities $\alpha(x)$ and $\beta(y)$ satisfying $\text{supp}(\alpha) = X$ and $\text{supp}(\beta) = Y$. For $\pi \in \Pi(\cdot, \cdot, \tau)$, denote by $H_\alpha(\pi)$ the entropy of the $(X,Z)$—marginal of $\pi$ relative to $\alpha$. In the same way denote by $H_\beta(\pi)$ the entropy of the $(Y,W)$—marginal of $\pi$ relative to $\beta$. Define the marginal Pressure of a continuous cost $c(x,y,z,w)$ relative to $(\alpha, \beta)$ by

$$P^m(c) = \sup_{\pi \in \Pi(\cdot, \cdot, \tau)} \int c(x,y,z,w) \, d\pi + H_\alpha(\pi) + H_\beta(\pi).$$
Proposition 10. For a continuous cost $c = c(x, y, z, w)$, consider the set $\Phi$ containing the numbers $\lambda$ such that

$$c(x, y, z, w) - \lambda \leq b(x, z) + d(y, w)$$

for some continuous functions $b(x, z)$ and $d(y, w)$ with $P_\alpha(b) = P_\beta(d) = 0$. Then

$$P^m_\alpha(c) = \inf \{\lambda : \lambda \in \Phi\}.$$ 

This proposition will be proved below.

Following ideas of Transport Theory, we fix probabilities $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$ and consider the set of probabilities $\Pi(\mu, \nu, \tau) \subset \Pi(\cdot, \cdot, \tau)$ containing the plans $\pi$ that also satisfy

$$\int f(x) \, d\pi = \int f(x) \, d\mu, \quad f \in C(X) \quad \text{and} \quad \int g(y) \, d\pi = \int g(y) \, d\nu, \quad g \in C(Y). \quad (6)$$

This is the set of probabilities $\pi \in \mathcal{P}(X \times Y \times Z \times W)$ with $x$-marginal $\mu$, $y$-marginal $\nu$, $(x, z)$-marginal holonomic for $\tau_x$ and $(y, w)$-marginal holonomic for $\tau_y$.

In order to show that this set is not empty consider for each $x \in X$ and $y \in Y$ the points $z_x$ and $w_y$ that are the fixed points for the contractions $\tau_x$ and $\tau_y$ respectively. Let $\pi$ be a plan defined from $d\pi = (d\delta_{x,z_x} \, d\mu(x))(d\delta_{y,w_y} \, d\nu(y))$, that means

$$\int g(x, y, z, w) \, d\pi = \int \int g(x, y, z_x, w_y) \, d\mu(x) \, d\nu(y).$$

If $g \in C(Z)$ we have

$$\int g(z) \, d\pi = \int g(z_x) \, d\mu(x) = \int g(\tau_x(z_x)) \, d\mu(x) = \int g(\tau_x(z)) \, d\pi.$$ 

If $g \in C(W)$ we have

$$\int g(w) \, d\pi = \int g(w_y) \, d\nu(y) = \int g(\tau_y(w_y)) \, d\nu(y) = \int g(\tau_y(w)) \, d\pi.$$ 

This shows that $\pi$ satisfies (5), and is clear that $\pi$ satisfies (6).

Theorem 11. (Duality) For a continuous cost $c(x, y, z, w)$ we have

$$\inf_{P^m(c-\varphi(x)-\psi(y))=0} \int \varphi(x) \, d\mu + \int \psi(y) \, d\nu = \sup_{\pi \in \Pi(\mu, \nu, \tau)} \int c(x, y, z, w) \, d\pi + H_\alpha(\pi) + H_\beta(\pi).$$

Proof. In order to make the computations let $E = C(X \times Y \times Z \times W)$. We can suppose that $c \leq 0$, because if we add a constant in $c$ then we change the booth sides in the same form.

Define $\Theta, \Xi : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ from

$$\Theta(u) = \begin{cases} 
0, & \text{if } u(x, y, z, w) \geq c(x, y, z, w) - b(x, z) - d(y, w) \forall (x, y, z, w) \text{ for some } b, d \text{ continuous with } P_\alpha(b) = 0 = P_\beta(d), \\
+\infty, & \text{in the other case}
\end{cases}$$
and
\[
\Xi(u) = \begin{cases} 
\int_X \varphi \, d\mu + \int_Y \psi \, d\nu, & \text{if } u = \varphi(x) + \psi(y) - g(\tau_x(z)) + g(z) - f(\tau_y(w)) + f(w), \\
+\infty, & \text{in the other case,}
\end{cases}
\]
where the functions are continuous,

\Xi is well defined because if is not \(+\infty\), coincides with \(\int u \, d\pi, \pi \in \Pi(\mu, \nu, \tau)\). The hypothesis in Theorem 9 are satisfied. Indeed taking \(u\) constant sufficiently large then \(\Theta\) is continuous in \(u\), \(\Theta(u) < \infty\) and \(\Xi(u) < \infty\). Clearly \(\Xi\) is convex. In order to show that \(\Theta\) is convex suppose that \(\Theta(u_1) = \Theta(u_2) = 0\). Then we can write \(u_1 \geq c - b_1 - d_1, \ u_2 \geq c - b_2 - d_2, \ P(b_1) = P(d_1) = 0\) and \(\lambda u_1 + (1 - \lambda) u_2 \geq c - (\lambda b_1 + (1 - \lambda) b_2) - (\lambda d_1 + (1 - \lambda) d_2)\). From the convexity of the pressure we get \(P_\alpha(\lambda b_1 + (1 - \lambda) b_2) \leq 0\), so there exists a constant \(a_1 \geq 0\) such that \(P_\alpha(\lambda b_1 + (1 - \lambda) b_2 + a_1) = 0\). In the same way there exists a constant \(a_2 \geq 0\) such that \(P_\beta(\lambda d_1 + (1 - \lambda) d_2 + a_2) = 0\), and we have
\[
\lambda u_1 + (1 - \lambda) u_2 \geq c - (\lambda b_1 + (1 - \lambda) b_2 + a_1) - (\lambda d_1 + (1 - \lambda) d_2 + a_2),
\]
showing that \(\Xi(\lambda u_1 + (1 - \lambda) u_2) = 0\).

For any \(\pi \in \mathcal{E}^*\), we get
\[
\Theta^*(-\pi) = \sup_{u \in \mathcal{E}} \{\langle -\pi, u \rangle - \Theta(u)\}
= \sup_{u \in \mathcal{E}} \{\langle \pi, u \rangle : u \leq -c + b + d, \ P_\alpha(b) = 0 = P_\beta(d)\}
= \begin{cases} 
\langle \pi, -c \rangle + \sup_{b : P_\alpha(b) = 0} \langle \pi, b \rangle + \sup_{d : P_\beta(d) = 0} \langle \pi, d \rangle, & \text{if } \pi \in \mathcal{M}^+ \\
+\infty, & \text{in the other case.}
\end{cases}
\]

In the above computation we use that if \(\pi \notin \mathcal{M}^+(X \times Y \times Z \times W)\), there exists \(u \leq 0\) such that \(\langle \pi, u \rangle > 0\). From the hypothesis of that \(-c \geq 0\) we get \(u \leq -c + 0 + 0\) where \(b = 0\) and \(d = 0\) have zero pressure. The same can be applied for \(\lambda u\), \(\lambda \to +\infty\), showing that \(\Theta^*(-\pi) = +\infty\).

Analogously
\[
\Xi^*(\pi) = \sup_{u \in \mathcal{E}} \{\langle \pi, u \rangle - \Xi(u)\}
= \sup_{(\varphi, \psi, f, g)} \left\{\langle \pi, \varphi(x) + \psi(y) - g(\tau_x(z)) + g(z) - f(\tau_y(w)) + f(w)\rangle - \int_X \varphi \, d\mu - \int \psi \, d\nu\right\}
= \begin{cases} 
0, & \text{if } \pi \text{ satisfies } [5], [6] \\
+\infty, & \text{in the other case.}
\end{cases}
\]

We observe that if \(\Theta^*(-\pi) < +\infty\) and \(\Xi^*(\pi) < +\infty\) then \(\pi \in \Pi(\mu, \nu, \tau)\). In this case we get
\[
- \sup_{b : P_\alpha(b) = 0} \pi(b) = H_\alpha(\pi)
\]
and
\[ - \sup_{d : P_d = 0} \pi(d) = H_\beta(\pi). \]

Let \( \Phi_c \) be the set of continuous functions \( \varphi(x), \psi(y) \) such that
\[ c(x, y, z, w) - \varphi(x) - \psi(z) + g(\tau_x(z)) - g(z) + f(\tau_y(w)) - f(w) \leq b(x, z) + d(y, w) \]
for some continuous functions \( f, g, b, d \) with \( P_a(b) = P_\beta(d) = 0 \).

From (1) we get
\[ \inf_{(\varphi, \psi) \in \Phi_c} \int_X \varphi \, d\mu + \int_Y \psi \, d\nu = \inf_{u \in E} \left[ \Theta(u) + \Xi(u) \right] \]
\[ = \max \left[ -\Theta^*(-\pi) - \Xi^*(\pi) \right] = \max \left\{ \pi(c) + H_\alpha(\pi) + H_\beta(\pi), \text{ if } \pi \in \Pi(\mu, \nu, \tau) \right\} \]
\[ = \max_{\pi \in \Pi(\mu, \nu, \tau)} \left\{ \pi(c) + H_\alpha(\pi) + H_\beta(\pi) \right\} = \max_{\pi \in \Pi(\mu, \nu, \tau)} \int c \, d\pi + \int H_\alpha(\pi) + H_\beta(\pi). \]

Now we are going to show that
\[ \inf_{P^m(c - \varphi - \psi) = 0} \int_X \varphi \, d\mu + \int_Y \psi \, d\nu = \inf_{(\varphi, \psi) \in \Phi_c} \int_X \varphi \, d\mu + \int_Y \psi \, d\nu \]
\[ = \sup_{\pi \in \Pi(\mu, \nu, \tau)} \int c \, d\pi + \int H_\alpha(\pi) + H_\beta(\pi). \]

The second equality was proved above. If \((\varphi, \psi) \in \Phi_c\) then there exist \( f, g, b, d \) such that
\[ c(x, y, z, w) - \varphi(x) - \psi(y) + g(\tau_x(z)) - g(z) + f(\tau_y(w)) - f(w) \leq b(x, z) + d(y, w) \]
and for any \( \pi \in \Pi(\cdot, \cdot, \tau) \) we have
\[ \int c(x, y, z, w) - \varphi(x) - \psi(y) \, d\pi + H_\alpha(\pi) + H_\beta(\pi) \leq P_a(b) + P_\beta(d) = 0. \]

Therefore \( P^m(c - \varphi - \psi) \leq 0 \). This shows that
\[ \inf_{P^m(c - \varphi - \psi) \leq 0} \int_X \varphi \, d\mu + \int_Y \psi \, d\nu \leq \inf_{(\varphi, \psi) \in \Phi_c} \int_X \varphi \, d\mu + \int_Y \psi \, d\nu. \]

If \( P^m(c - \varphi - \psi) < 0 \) there exists a number \( a > 0 \) such that \( P^m(c - \varphi - \psi + a) = 0 \). If we call \( \hat{\psi} = \psi - a \) we get \( P^m(c - \varphi - \hat{\psi}) = 0 \) and
\[ \int_X \varphi \, d\mu + \int_Y \hat{\psi} \, d\nu = \int_X \varphi \, d\mu + \int_Y \psi \, d\nu - a < \int_X \varphi \, d\mu + \int_Y \psi \, d\nu. \]

This shows that
\[ \inf_{P^m(c - \varphi - \psi) \leq 0} \int_X \varphi \, d\mu + \int_Y \psi \, d\nu = \inf_{P^m(c - \varphi - \psi) = 0} \int_X \varphi \, d\mu + \int_Y \psi \, d\nu. \]
Therefore we conclude that
\[
\inf_{P_m(c-\varphi-\psi) = 0} \int_X \varphi \, d\mu + \int_Y \psi \, d\nu \leq \sup_{\pi \in \Pi(\mu, \nu, \tau)} \int c \, d\pi + H_\alpha(\pi) + H_\beta(\pi).
\]
Note now that if \(P_m(c-\varphi-\psi) = 0\) and \(\pi \in \Pi(\mu, \nu, \tau)\) then
\[
\int \varphi \, d\mu - \int \psi \, d\mu - \int \psi \, d\nu + H_\alpha(\pi) + H_\beta(\pi) \leq 0
\]
that means
\[
\int \varphi \, d\mu + \int \psi \, d\nu \geq \int c \, d\pi + H_\alpha(\pi) + H_\beta(\pi).
\]
This shows that
\[
\inf_{P_m(c-\varphi-\psi) = 0} \int_X \varphi \, d\mu + \int_Y \psi \, d\nu \geq \sup_{\pi \in \Pi(\mu, \nu, \tau)} \int c \, d\pi + H_\alpha(\pi) + H_\beta(\pi).
\]

**Proof.** (Proposition 10)
It follows similar arguments defining now \(\Xi : E \rightarrow \mathbb{R} \cup \{+\infty\}\) from
\[
\Xi(u) = \begin{cases} 
\lambda & \text{if } u = \lambda - g(\tau_x(z)) + g(z) - f(\tau_y(w)) + f(w), \\
+\infty & \text{in the other case.}
\end{cases}
\]
In this case
\[
\Xi^*(\pi) = \begin{cases} 
0, & \text{if } \pi \text{ satisfy (5)} \\
+\infty, & \text{in the other case.}
\end{cases}
\]
If \(\Theta^*(-\pi) < +\infty\) and \(\Xi^*(\pi) < +\infty\) then \(\pi \in \Pi(\cdot, \cdot, \tau)\). Let \(\Phi\) be the set of numbers \(\lambda\) such that
\[
c(x, y, z, w) - \lambda - g(\tau_x(z)) + g(z) - f(\tau_y(w)) + f(w) \leq b(x, z) + d(y, w)
\]
for some continuous functions \(f, g, b, d\) with \(P_\alpha(b) = P_\beta(d) = 0\).
From (4) we get
\[
\inf_{\lambda \in \Phi} \lambda = \sup_{\pi \in \Pi(\cdot, \cdot, \tau)} \int c \, d\pi + H_\alpha(\pi) + H_\beta(\pi).
\]
In order to finish the proof note that in the inequality
\[
c(x, y, z, w) - \lambda - g(\tau_x(z)) + g(z) - f(\tau_y(w)) + f(w) \leq b(x, z) + d(y, w)
\]
we have \(P_\alpha(b(x, z) + g(\tau_x(z)) - g(z)) = 0\) and \(P_\beta(d(y, w) + f(\tau_y(w)) - f(w)) = 0\).

The next result is related with the zero temperature case in Spin Lattice Systems (when the temperature is unconsidered). This result corresponds to the Kantorovich Duality for compact spaces and continuous cost \(-c\) if \(Z\) and \(W\) have only one element.
Theorem 12. Let $\Phi_c$ the set of continuous functions $\varphi(x), \psi(y)$ satisfying  
\[ c(x, y, z, w) + g(\tau_x(z)) - g(z) + f(\tau_y(w)) - f(w) \leq \varphi(x) + \psi(z) \]
for some functions $f \in C(W)$ and $g \in C(Z)$. Then  
\[ \inf_{(\varphi, \psi) \in \Phi_c} \int \varphi(x) \, d\mu + \int \psi(y) \, d\nu = \sup_{\pi \in \Pi(\mu, \nu, \tau)} \int c \, d\pi. \]

Proof. Follows the same arguments presented above considering
\[
\Theta(u) = \begin{cases} 
0, & \text{if } u(x, y, z, w) \geq c(x, y, z, w), \forall (x, y, z, w) \\
+\infty, & \text{in the other case}
\end{cases}
\]
and
\[
\Xi(u) = \begin{cases} 
\int_X \varphi \, d\mu + \int_Y \psi \, d\nu, & \text{if } u = \varphi(x) + \psi(y) - g(\tau_x(z)) + g(z) - f(\tau_y(w)) + f(w), \\
+\infty, & \text{in the other case},
\end{cases}
\]
where the functions are continuous.

\[ \square \]

If we suppose in Theorem 11 that $Y = \{y_0\}$ and $W = \{w_0\}$ we obtain the following result

Corollary 13. For a fixed $\mu \in \mathcal{P}(X)$ and $c(x, z)$ continuous we have
\[ \inf_{P_\alpha(c-\varphi(x))=0} \int \varphi \, d\mu = \sup_{\pi \in \Pi(\mu, \tau)} \int c \, d\pi + H_\alpha(\pi). \tag{7} \]

In some sense in this result the concept of eigenvalue was changed. For the propose of the result above we can try to think that the eigenvalue is a function on the $x$ variable. The equation $L(h) = \lambda h$ should be changed for the existence of functions $h(z)$ and $\varphi(x)$ such that $L(h) = \varphi \cdot h$. But the left hand side is a function on the variable $z$ and the right hand side is a function (product of functions) on the variables $x$ and $z$. Then we return to the original equation $L(h) = \lambda h$ and rewrite this in the form $L(\frac{\varphi}{\lambda}) = h$. In this way we can try to find functions $\varphi(x)$ and $h(z)$ such that
\[ \int e^{c(x,z) - \varphi(x)} h(\tau_x(z)) = h(z). \]

We remark that there exist too many pair of solutions. Indeed, for each fixed $\varphi(x)$ we can apply the result of the Lemma 1 for $L_{\varphi - \varphi}$ and determine $\lambda > 0$ and $h > 0$ such that $L_{\varphi - \varphi} h = \lambda h$. This can be rewritten in the form
\[ \int e^{c(x,z) - \varphi(x) - \log(\lambda)} h(\tau_x(z)) = h(z). \]
Then there exists a function $\hat{\varphi}(x) = \varphi(x) + \log(\lambda)$ and a function $h > 0$ such that
\[ \int e^{c(x,z) - \hat{\varphi}(x)} h(\tau_x(z)) = h(z). \]
Therefore any function $\varphi(x)$ plays the role of an “eigenvalue” except by the addition of a constant. For a given cost $c$ there exist too many ways of get a normalization adding a function $\varphi(x)$ and a function in the form $g(z) - g(\tau_x(z))$. In the above section we make the normalization adding a constant and not a function $\varphi(x)$.

The next result can be interpreted as a kind of Slackness Condition in the present setting

**Proposition 14.** Let $c(x, z)$ and $\varphi(x)$ be Lipschitz functions and $\pi \in \Pi(\mu, \tau)$. If $P(c - \varphi) = 0$ and $\pi$ is the holonomic probability associated to $c - \varphi$, then $\varphi$ and $\pi$ realize the infimum and the supremum in $\mathbb{R}$.

**Proof.**

$$0 = P(c - \varphi) = \int c - \varphi \, d\pi + H(\pi)$$

Then

$$\int \varphi \, d\mu = \int c \, d\pi + H(\pi).$$

This is only possible if $\varphi$ realizes the infimum and $\pi$ realizes the supremum in $\mathbb{R}$. \hfill \Box

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