Research Article

Pseudodifferential Operators on Weighted Hardy Spaces

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We study two sufficient conditions for the boundedness of a class of pseudodifferential operators $T$ with symbols in the H"{o}lmander class $S^m_{\rho,\delta}(\mathbb{R}^n)$ on weighted Hardy spaces $H^p_w(\mathbb{R}^n)$, where $\omega$ belongs to Muckenhoupt class $A_p$. The first one is an estimate from $H^p_w(\mathbb{R}^n)$ into $L^1_w(\mathbb{R}^n)$. We get a better range of admissible $p$ and $m$. The second one is a weighted version bounded for the operators $T$ on $H^p_w(\mathbb{R}^n)$, and it is an addition to the literature.

1. Introduction

The purpose of this paper is to study some sufficient conditions for the boundedness of pseudodifferential operators $T$ on weighted Hardy space $H^p_w(\mathbb{R}^n)$, where the operators $T$ have symbols in the H"{o}lmander class $S^m_{\rho,\delta}(\mathbb{R}^n)$. As in [1], for $m \in \mathbb{R}$ and $\rho, \delta \in [0,1]$, a symbol $a(x, \xi) \in S^m_{\rho,\delta}(\mathbb{R}^n)$ is a smooth function defined on $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$|\tilde{\partial}_\alpha^\rho \tilde{\partial}_\beta^\delta a(x, \xi)| \leq C_{\alpha,\beta} (1 + |\xi|)^{m-\rho|\beta|+\delta|\alpha|}$$

holds for all multi-indices $\alpha, \beta \in \mathbb{N}^n$, where $C_{\alpha,\beta}$ is independent of $x$ and $\xi$. We now assume that the symbol $a(x, \xi)$ is smooth in both the spatial variable $x$ and the frequency variable $\xi$.

Given $f \in C_0^\infty(\mathbb{R}^n)$, the pseudodifferential operator $T \in S^m_{\rho,\delta}(\mathbb{R}^n)$ associated with the symbol $a(x, \xi) \in S^m_{\rho,\delta}(\mathbb{R}^n)$ is given by

$$Tf(x) = \int_{\mathbb{R}^n} a(x, \xi) e^{2\pi i x \cdot \xi} \tilde{f}(\xi) d\xi,$$

where $\tilde{f}$ denotes the Fourier transform of $f$. Moreover, we can express $T$ by a kernel $K(x, y)$ as (see, e.g., [2])

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy.$$

Pseudodifferential operators play an important role in the theory of partial differential equations. It is well known that the Hardy spaces $H^p(\mathbb{R}^n)$ coincide with the Lebesgue spaces $L^p(\mathbb{R}^n)$ when $p > 1$. The $L^p$ and weighted $L^p$ boundedness of the operator $T \in S^m_{\rho,\delta}(\mathbb{R}^n)$ have been extensively studied. We refer to [1, 2, 3, 4] for the $L^p$ bounds and [5, 6, 7, 8] for the weighted $L^p$ bounds.

For $p \in (0,1]$, there is an estimate from $L^1(\mathbb{R}^n)$ into weak $L^1(\mathbb{R}^n)$ for the pseudodifferential operator $T \in S^m_{\rho,\delta}(\mathbb{R}^n)$ [cf. (5)]. As known, the Hardy space $H^p(\mathbb{R}^n)$ is an advantageous substitute for $L^p(\mathbb{R}^n)$. The behavior of the pseudodifferential operator $T$ on $H^p(\mathbb{R}^n)$ has attracted a lot of interest. For example, Alvarez and Hounie [5] have found that the pseudodifferential operator $T$ with symbol in $S^m_{\rho,\delta}(\mathbb{R}^n)$ is bounded from $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$, where $0 < \rho \leq 1$, $0 \leq \delta < 1$ and $m \leq - (n(1-\rho)/2)$. Hounie and Kapp [9] have shown that the operator $T$ with $0 \leq \delta \leq \rho < 1$ and $m = -(n(1-\rho)/2)$ is bounded from the local Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. Yabuta [10] has proved the operator $T$ involving a modulus of continuity $\omega(t)$ is bounded from $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.

The bounds of the pseudodifferential operator $T$ from the weighted Hardy space $H^p_w(\mathbb{R}^n)$ into the weighted Lebesgue space $L^p_w(\mathbb{R}^n)$ have also been studied. Yabuta [11] has found that the operator $T$ is bounded from $H^p_w(\mathbb{R}^n)$ into $L^p_w(\mathbb{R}^n)$, where $T \in S^m_{\rho,\delta}$ and $\omega \in A_1$. In view of this, it is natural to look for a wide range of operator $T$ in $S^m_{\rho,\delta}$ to study the bounds on the weighted Hardy space $H^p_w(\mathbb{R}^n)$.
In this paper, we establish two estimates for the pseudodifferential operator $T$ with symbols in $S^m_{ρ,δ}(\mathbb{R}^n)$. The first one is an estimate from $H^1_{ω}(\mathbb{R}^n)$ into $L^1_{ω}(\mathbb{R}^n)$. We extend the result in Yabuta [11] to $ω∈A_p$ with $1≤p<1+(ε/n)$ and the operator $T$ with $0<ρ≤1$, $0≤δ<1$, and $−(n+1)<m≤−(n+1)(1−ρ)$. Our first main result is stated as follows.

**Theorem 1.** Let $ε=min\{1,(1+1/m)/n\}$, $p∈[1,1+(ε/n)]$, $ω∈A_p$, and $T∈S^m_{ρ,δ}$ with $0<ρ≤1$, $0≤δ<1$. If $−(n+1)<m≤−(n+1)(1−ρ)$, then $T$ is bounded from $H^1_{ω}(\mathbb{R}^n)$ into $L^1_{ω}(\mathbb{R}^n)$, i.e., there exists a constant $C>0$ such that

$$\|Tf\|_{L^1_{ω}(\mathbb{R}^n)}≤C\|f\|_{H^1_{ω}(\mathbb{R}^n)}. \tag{4}$$

The second one is an estimate on weighted Hardy spaces $H^1_{ω}(\mathbb{R}^n)$ for the pseudodifferential operator $T$. It is well known that under certain conditions of $m, ρ, δ$, the operator $T$ is bounded on $h^1(\mathbb{R}^n)$ (cf. [9, 12]). Alvarez and Hounie [5] have found that the pseudodifferential operator $T$ is bounded on $H^1(\mathbb{R}^n)$, where $T∈S^m_{ρ,δ}$ with $0<ρ≤1$, $0≤δ<1$, and $m≤−λ+min\{0,(n(ρ−δ)/2)\}$ for some $n(1−ρ)/2≤A≤n/2$. It is natural to look for a weighted version estimate on $H^1_{ω}(\mathbb{R}^n)$. We now state our second main result.

**Theorem 2.** Let $μ=(1+n+m)/n−n$, $p∈[1,1+μ/n)$, $ω∈A_p$, and $T∈S^m_{ρ,δ}$ with $0<ρ≤1$, $0≤δ<1$. Assume $pn−(n+1)<m≤−(n+1)(1−ρ)$ and $T^1=0$. Then, $T$ is bounded on $H^1_{ω}(\mathbb{R}^n)$; i.e., there exists a constant $C>0$ such that

$$\|Tf\|_{H^1_{ω}(\mathbb{R}^n)}≤C\|f\|_{H^1_{ω}(\mathbb{R}^n)}. \tag{5}$$

The remainder of this paper is organized as follows. In Section 2, we present some definitions and well-known results we use later. The aim of Section 3 is to set up the estimate from $H^1_{ω}(\mathbb{R}^n)$ into $L^1_{ω}(\mathbb{R}^n)$ for pseudodifferential operators $T$ in $S^m_{ρ,δ}$. We develop a method to handle $‖T‖_{L^1_{ω}(\mathbb{R}^n)}$ (see Proposition 1). The aim of Section 4 is to establish the estimate on weighted Hardy spaces $H^1_{ω}(\mathbb{R}^n)$ for pseudodifferential operators $T$ in $S^m_{ρ,δ}$.

Most of the notations we use are standard. $C$ denotes a constant that may change from line to line and we write $a≤b$ as shorthand for $a≤Cb$. If $a≤b$ and $b≤a$, we mean $a=b$. For a measurable set $A$, $|A|$ denotes the Lebesgue measure of $A$ and $χ_A$ the characteristic function. $B$ will always denote a ball, and $tB(t>0)$ denotes the ball $B$ dilated by $t$.

### 2. Notations and Auxiliary Lemma

In this section, we first present an auxiliary lemma about the pseudodifferential operator $T$ associated with the kernel $K(x,y)$. Let $δ'(\mathbb{R}^n)$ be the class of Schwartz functions and $δ'(\mathbb{R}^n)$ be its dual space. The space of $\mathcal{C}^{∞}_{c}$-function with compact support is denoted by $\mathcal{C}^{∞}_{c}(\mathbb{R}^n)$. Pseudodifferential operators are bounded from $δ'(\mathbb{R}^n)$ to $δ'(\mathbb{R}^n)$ and so possess distribution kernels $K(x,y)∈δ'(\mathbb{R}^n×\mathbb{R}^n)$. Then, the following formula for the kernel is useful (cf. Proposition 3.1 in [9]; see also [5]).

**Lemma 1.** Let $a(x,ξ)∈δ^m_{ρ,δ}(\mathbb{R}^n)$ with $0<ρ≤1$, $0≤δ<1$, and associate with the pseudodifferential operator $T∈S^m_{ρ,δ}$. Then, the distribution kernel $K(x,y)$ of $T$ is smooth away from the diagonal $\{(x,x)\}$ and is given by

$$K(x,y)=\lim_{ξ→0}e^{2πi(x−y)ξ}a(x,ξ)ψ(ξ)dxξ, \tag{6}$$

where $ψ∈\mathcal{C}^{∞}_{c}(\mathbb{R}^n)$ satisfies $ψ(ξ)=1$ for $|ξ|≤1$ and the limit is taken in $δ'(\mathbb{R}^n)$ and independent of the choice of $ψ$. If $M∈\mathbb{N}$ and $(M+m+n)>0$, $K(x,y)$ satisfies the estimates

$$\sup_{|α+β|=M}\|D^α_xD^β_yK(x,y)\|≤C_M1/[x−y]^{(M+m+n)/2}, x≠y. \tag{7}$$

Moreover, for any multi-index $α, β∈\mathbb{N}^n$ and $N∈\mathbb{N},$

$$\sup_{|x−y|<1/2}\|D^α_xD^β_yK(x,y)\|≤C_{α, β, N}. \tag{8}$$

The following useful $L^p_ω$ bound for the pseudodifferential operator $T$ is obtained by Michalowski et al. [7].

**Lemma 2.** Let $T∈S^m_{ρ,δ}$, $0≤δ<1,0<ρ≤1$, and $m<n(1−ρ)$. Then, for each $p∈(1,∞)$ and $ω∈A_p$, there exists a constant $C>0$ such that

$$\|Tf\|_{L^p_ω(\mathbb{R}^n)}≤C\|f\|_{L^p_ω(\mathbb{R}^n)}. \tag{9}$$

**Remark 1.** Obviously, the $L^2_ω$ bounds of pseudodifferential operators $T$ are established automatically.

**Lemma 3.** Let $T∈S^m_{ρ,δ}$, $0≤δ<1,0<ρ≤1$, and $m<min\{0,(n(ρ−δ)/2)\}$. Then, the operator $T$ is bounded on $L^2(\mathbb{R}^n)$.

**Remark 3.** The range of $m, ρ$ here is $m<−n(1−ρ)(1/p)−(1/2)+min\{0,(n(ρ−δ)/2)\}$ and $1<p<∞$, respectively.

**Remark 4.** Obviously, $m≤−(n+1)(1−ρ)$ satisfies the condition of $m$ in Lemma 2.

The following useful $L^2$ bound of the pseudodifferential operator $T∈S^m_{ρ,δ}$ is obtained by Alvarez and Hounie [5].

**Lemma 4.** Let $p∈[1,∞)$. A nonnegative locally integrable function $ω$ belongs to Muckenhoupt class $A_p$, if there exists a constant $C>0$, such that for all balls $B∈\mathbb{R}^n$,

$$\left(\frac{1}{|B|}\int_Bω(x)dx\right)^{1/p−1}\left(\frac{1}{|B|}\int_Bω(x)^{1/(p−1)}dx\right)^{p−1}≤C \text{ if } 1<p<∞,$$

$$\frac{1}{|B|}\int_Bω(x)dx≤C\inf_{x∈B}ω(x) \text{ if } p=1. \tag{10}$$

We denote $A_{∞}=∪_{p=1}^∞A_p$.

It is well known that $ω∈A_p$ implies $ω∈A_q$ for all $q>p$. Also, if $ω∈A_p$, then $ω∈A_q$ for some $q∈[1,p)$. We thus write $d_{ω}=inf\{p≥1: ω∈A_p\}$ to denote the critical index of $ω$. For a measurable set $E$, we denote $ω(E)=∫_Eω(x)dx$. The
following lemma provides a way to compare $|E|$ and $\omega(E)$ of a set $E$ (see [13]).

**Lemma 4.** Let $\omega \in A_p$ and $p \geq 1$. Then, there exists a constant $C > 0$ such that
\[
C \left( \frac{|E|}{|B|} \right)^p \leq \frac{\omega(E)}{\omega(B)^{2}} \tag{11}
\]
for all balls $B$ and measurable subsets $E \subset B$.

Given a weight function $\omega$ on $\mathbb{R}^n$, we denote by $L^p_\omega(\mathbb{R}^n)$ the weighted Lebesgue space of all functions $f$ satisfying
\[
\|f\|_{L^p_\omega(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx \right)^{(1/p)} < \infty. \tag{12}
\]

When $p = \infty$, $L^\infty_\omega(\mathbb{R}^n)$ is $L^\infty(\mathbb{R}^n)$. Analogous to the classical Hardy space, the weighted Hardy space $H^1_\omega(\mathbb{R}^n)$ can be defined in terms of maximal functions.

**Definition 1.** Let $\omega \in A_\infty$. The weighted Hardy space $H^1_\omega(\mathbb{R}^n)$ is defined by
\[
H^1_\omega(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n); \|f\|_{H^1_\omega(\mathbb{R}^n)} = \sup_{r > 0} \left\{ \frac{1}{r} \int_{|x| < r} |\text{supp}(\phi_r * f(x))| \right\} \in L^1_\omega(\mathbb{R}^n) \right\}, \tag{13}
\]
where $\phi \in \mathcal{S}(\mathbb{R}^n)$ is a fixed function with $\int \phi dx \neq 0$ and $\phi_\delta(x) = (1/\delta^n)\phi(x/\delta)$ for any $\delta > 0$. Moreover, we define $\|f\|_{H^1_\omega(\mathbb{R}^n)} = \|\phi_\delta(f)\|_{L^1_\omega(\mathbb{R}^n)}$.

**Remark 5.** Definition 1 is independent of the choice of $\phi$ (see [14]).

**Definition 2.** Let $\omega$ be a weight with the critical index $q_\omega$. An $(1, \omega)$-atom with respect to $\omega$ is a function $a$ satisfying
\[
|a|_{L^\infty} \leq \omega(B)^{-1}, \tag{14}
\]
and $\int a(x)x^\alpha dx = 0$ for every multi-index $\alpha$ with $|\alpha| \leq [q_\omega = 1]$.

The Hardy space $H^1(\mathbb{R}^n)$ is a linear space spanned by all of $(1, \omega)$-atoms with respect to $\omega$. Namely, $f \in H^1(\mathbb{R}^n)$ if and only if $f$ can be written as (see [13])
\[
f = \sum_j \lambda_j a_j, \tag{15}
\]
in the sense of $\mathcal{S}'$, where each $a_j$ is an $(1, \omega)$-atom with respect to $\omega$ and $\lambda_j$ satisfies
\[
\sum_j |\lambda_j| < \infty. \tag{16}
\]

Moreover, $\|f\|_{H^1(\mathbb{R}^n)} = \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| : f = \sum_{j=1}^{\infty} \lambda_j a_j \right\}$.

**Definition 3.** Let $T$ be a pseudodifferential operator in $\mathcal{L}^{m,p}_{\rho,\delta}$. We say $T^* = 0$ if $\int_{\mathbb{R}^n} |Ta(x)| dx = 0$ for all $a \in L^\infty(\mathbb{R}^n)$ with compact support and $\int_{\mathbb{R}^n} a(x) dx = 0$.

**3. The Proof of Theorem 1**

In this section, we prove that the pseudodifferential operators $T$ in $\mathcal{L}^{m,p}_{\rho,\delta}$ are bounded from $H^1_\omega(\mathbb{R}^n)$ into $L^1_\omega(\mathbb{R}^n)$.

**Proposition 1.** Let $\omega \in A_p$, $p \in [1, \infty)$ and $\varepsilon = \min \{1, (1 + m + n/p)\}$. Assume pseudodifferential operator $T \in \mathcal{L}^{m,p}_{\rho,\delta}$ with $0 < \rho \leq 1$, $0 < \delta \leq 1$, and $-\varepsilon \leq m \leq -\delta - n$. Then, there exists a constant $C > 0$ such that
\[
\|Ta\|_{L^1_\omega(B(x_0,r)^c)} \leq C2^{-k(\varepsilon + n(1-p))}, \tag{17}
\]
holds for all $(1, \omega)$-atoms $a$ with respect to $\omega$, where $\text{supp}(a) \subset B = B(x_0, r)$.

**Proof.** Inspired by the proof of Lemma 3.2 in [15], we consider two cases about the radius $r$.

**Case 1.** When $r \geq 1$. For every $x \in 2^{k+1}B \setminus 2^{k}B$ and $y \in B(x_0, r)$, we have
\[
|x - y| \geq |x - x_0| - |y - x_0| \geq 2^k r - r \geq 1. \tag{18}
\]
Hence, by (8) and properties of $(1, \omega)$-atoms with respect to $\omega$, we have
\[
|Ta(x)| = \left| \int_{\mathbb{R}^n} K(x, y) a(y) dy \right| \leq \int_{B} \left| K(x, y) \right| |a(y)| dy \leq C \int_{|x - y| < 2^{-k}r} |a(y)| dy \leq C 2^{-k} \frac{|B|}{|2^{k}B|} \omega(B)^{-1}, \tag{19}
\]
for all $x \in 2^{k+1}B \setminus 2^{k}B$. Thus,
\[
\|Ta\|_{L^1_\omega((2^{k+1}B)^c)} \leq C2^{-k} \frac{|B|}{2^k |B|} \omega(2^{k+1}B) \leq C2^{-k} \omega(2^{k+1}B) \leq C2^{-k(\varepsilon + n(1-p))}. \tag{20}
\]

**Case 2.** When $0 < r < 1$. For every $x \in (2^{k+1}B) \setminus 2^{k}B$ and $y \in B(x_0, r)$, by moment conditions, we have
\[
|Ta(x)| = \left| \int_{\mathbb{R}^n} K(x, y) a(y) dy \right| \leq \int_{B} \left| K(x, y) - K(x, x_0) \right| |a(y)| dy. \tag{21}
\]

By the mean value theorem, $1 + m + n > 0$, and (7), we have
\[
|Ta(x)| \leq C \int_{B} \frac{|y - x_0|}{|x - x_0|^{(1+m)/p}} |a(y)| dy \leq C \frac{r}{(2^{k}r)^{(1+m)/p}} \frac{|B|}{\omega(B)}, \tag{22}
\]
where we take $M = 1$ and use the fact that $|x - \xi| \sim |x - x_0|$ if $\xi \in B(x_0, r)$. Let us now consider two subcases.

**Subcase 1.** If $(2^{k} - 1)r \geq 1$, then, for any $y \in B(x_0, r)$ and $x \in 2^{k+1}B \setminus 2^{k}B$, \[
|x - y| \geq |x - x_0| - |y - x_0| \geq (2^k - 1)r \geq 1. \tag{23}
\]
Similar to the case $r \geq 1$, we get
\[ |Ta(x)| \leq C \frac{B}{|2^k B|} \frac{r}{(2^k r)^{1 + m + n/p}} \omega(B)^{-1}. \]  
(24)

Since \( 0 < r < 1 \) and \( 2^k r > 1 \), we have
\[ r \leq 2^{-k(1 + m + n/p)}, \quad \frac{1 + m + n}{p} \leq 1; \]
(25)
\[ r \geq 2^{-k}, \quad \frac{1 + m + n}{p} \geq 1. \]

Noting \( \epsilon = \min\{1, (1 + m + n/p)\} \), it is easy to see \(|Ta(x)| \leq C 2^{-k(1 + m/n)} B \omega(B)^{-1}\). This implies (17).

**Subcase 2.** If \((2^k - 1) r < 1\). Since \( m \leq - (n + 1)(1 - \rho) \), (22) yields
\[ \|Ta\|_{L^p \omega(2^k B, 2^k B)} \leq C r \frac{\omega(2^k 1 B)}{\omega(B)} \leq C 2^{-k(1 + m/n)} B \omega(B)^{-1}. \]
(26)

In view of (20) and (26), we finish the proof of Proposition 1.

**Proof.** The proof of Theorem 1 is motivated by the atomic decomposition for \( H^1_\omega(\mathbb{R}^n) \). Let \( f \in H^1_\omega(\mathbb{R}^n) \). We obtain an atomic decomposition of \( f \) satisfying (15) and (16). So, to prove that the pseudodifferential operators \( T \) are bounded from \( H^1_\omega(\mathbb{R}^n) \) into \( L^1_\omega(\mathbb{R}^n) \), it suffices to show that for each \((1, \infty)\)-atom \( a \) with respect to \( \omega \), we have \( Ta \in L^1_\omega(\mathbb{R}^n) \) is a function satisfying
\[ \sup \{a \} \subset B, \]
\[ \|a\|_{L^\infty} \leq \omega(B)^{-1}, \]
\[ \int a(x) dx = 0, \]
(27)
for some ball \( B = B(x_0, r) \).

Now, let \( a \) be such an atom and write
\[ \int |Ta| \omega = \int_{2^k B} |Ta| \omega + \int_{(2^k B)} |Ta| \omega = I_1 + I_2. \]
(28)

It is easy to estimate the term \( I_1 \). Using Hölder inequality and \( L^2_\omega\)-boundedness for the pseudodifferential operator \( T \) (see Remark 1), we get
\[ I_1 \leq \left( \int_{2^k B} |Ta|^2 \omega \right)^{1/2} \left( \int_{2^k B} \omega \right)^{1/2} \leq C \|Ta\|_{L^2_\omega(\mathbb{R}^n)} \omega(2B)^{1/2} \leq C \|a\|_{L^2_\omega(\mathbb{R}^n)} \omega(B)^{1/2} \leq C, \]
(29)
where \( C \) is independent of \( a \).

For the term \( I_2 \), we write
\[ I_2 = \int (2^k B) |Ta| \omega \leq \sum_{k=1}^\infty \int (2^k B \cap 2^k B) |Ta| \omega = \sum_{k=1}^\infty \|Ta\|_{L^2(2^k B, 2^k B)} \]
(30)
\[ = \sum_{k=1}^\infty \frac{r}{2^k r} \frac{1 + m + n}{p} \omega(B)^{-1} \leq C 2^{-k(1 + m/n)} B \omega(B)^{-1}. \]
\[ \leq C 2^{-k(1 + m/n)} B \omega(B)^{-1} \]
\[ = C 2^{-k(1 + m/n)} B \omega(B)^{-1}. \]

By Proposition 1, we get
\[ I_2 \leq \sum_{k=1}^\infty C 2^{-k(1 + m/n)} B \omega(B)^{-1} \leq C, \]
(31)
since \( 1 \leq p < 1 + (\epsilon/n) \). Combing (29) and (31), we finish the proof of Theorem 1.

**4. The Proof of Theorem 2**

In this section, we establish the weighted norm inequality on weighted Hardy spaces \( H^1_\omega(\mathbb{R}^n) \) for pseudodifferential operators \( T \) in \( \mathcal{L}_{\rho, \delta}^m \).

**Proof.** Without loss of generality, we assume \( 1 \leq p < 1 + (\mu/n) \), where \( \mu = (1 + m + n/p) - n \). Fix \( \phi \in \mathcal{C}_\mathcal{C} \) (\( \mathbb{R}^n \)) and \( \int_{\mathbb{R}^n} \phi(x) dx \neq 0 \). By (15), it is sufficient to show that for each \((1, \infty)\)-atom \( a \) with respect to \( \omega \), \( \|Ta\|_{L^1_\omega(\mathbb{R}^n)} \leq C \) with \( \psi \) independent of \( a \). In order to do this, one can suppose \( \text{supp}(a) \subset B = B(x_0, r) \) and write
\[ \|Ta\|_{L^1_\omega(\mathbb{R}^n)} = \int_{|x-x_0|<4r} |(Ta)^*(x)| \omega(x) dx \]
(32)
\[ = I_1 + I_2. \]

For the term \( I_1 \), by Hölder inequality, \( L^2_\omega\)-boundedness of the maximal function \( (Ta)^* \), \( L^1_\omega\)-boundedness of the pseudodifferential operator \( T \), and (4), we get
\[ I_1 \leq \left( \int_{|x-x_0|<4r} |(Ta)^*(x)|^2 \omega(x) dx \right)^{1/2} \left( \int_{|x-x_0|<4r} \omega(x) dx \right)^{1/2} \]
\[ \leq C \|Ta\|_{L^2_\omega(\mathbb{R}^n)} \omega(B(x_0, 4r))^{1/2} \leq C \|a\|_{L^2_\omega(\mathbb{R}^n)} \omega(B(x_0, 4r))^{1/2} \leq C, \]
(33)
where \( C \) is independent of \( a \).

To estimate \( I_2 \), we first estimate \( (Ta)^*(x) \) for \(|x-x_0| > 4r \). For any \( t > 0 \), since \( T^* = 0 \) (see Definition 3), we have
\[ |Ta \ast \phi_t(x)| = \left| \int_{\mathbb{R}^n} Ta(y) \frac{1}{t^n} \phi \left( \frac{x-y}{t} \right) dy \right| \]
\[ = \left| \int_{\mathbb{R}^n} Ta(y) \frac{1}{t^n} \left( \phi \left( \frac{x-y}{t} \right) - \phi \left( \frac{x-x_0}{t} \right) \right) dy \right| \]
\[ \leq \frac{1}{t^n} \int_{|y-x_0|<2t} |Ta(y)| \phi \left( \frac{x-y}{t} \right) dy \]
\[ + \frac{1}{t^n} \int_{|y-x_0|>2t} |Ta(y)| \phi \left( \frac{x-x_0}{t} \right) dy \]
\[ = E_1 + E_2 + E_3. \]
(34)

For the term \( E_1 \), by the mean value theorem and Hölder’s inequality, we have
\[ E_1 \leq \frac{1}{t^{m+1}} \| Ta \|_{L^2(\mathbb{R}^n)} \times \left( \int_{|y-x_0| < 2r} \left| \nabla \phi \left( \frac{x-x_0 - y(y-x_0)}{t} \right) \right|^2 \, dy \right)^{1/2} \leq C \frac{\rho^{n+1}}{|x-x_0|^{n+1}} \omega(B) \]

where \( y \in (0,1) \) depends on \( x, y, \) and \( x_0, \) and \( V = ((\partial/\partial x_1), \ldots, (\partial/\partial x_n)). \) Here, we use the inequalities

\[ \left| x-x_0 - y(y-x_0) \right|^{n+1} \leq C, \]

and \( |x-x_0 - y(y-x_0)| \geq |x-x_0| \geq |y-x_0| \geq |x-x_0| - |y-x_0| \geq |x-x_0| / 2, \) and \( L^2 \)-boundedness of the pseudo-differential operator \( T \) (see Lemma 3).

To estimate \( E_2 \) and \( E_3, \) we first estimate \( Ta(y) \) when \( |y-x_0| > 2r \) and consider two cases about \( r. \)

**Case 3.** If \( r > 1, \) then for every \( y \in \mathbb{R}^n \setminus B(x_0, 2r) \) and \( z \in B, \) we have \( |y-z| \geq |y-x_0| - |z-x_0| > 1. \) Hence, by (8), we have

\[ |Ta(y)| = \left| \int_{B} K(y,z) a(z) \, dz \right| \leq \int_{B} |K(y,z)||a(z)| \, dz \leq C \frac{1}{|y-z|^{n+1}} \| a \|_{L^\infty} \, dz \leq C \frac{1}{|y-x_0|^{n+1}} \| a \|_{L^\infty} \, B \leq C \frac{\rho^{n+1}}{|y-x_0|^{n+1}} \omega(B). \]

Case 4. In the case of \( 0 < r < 1, \) we have \( \int_B a(z) \, dz = 0. \) Thus, for every \( y \in \mathbb{R}^n \setminus B(x_0, 2r), \) from \( 1 + n + m > np > 0, \) (7) yields

\[ |Ta(y)| = \left| \int_{\mathbb{R}^n} K(y,z) a(z) \, dz \right| \leq \int_B |K(y,z) - K(y,x_0)| |a(z)| \, dz \leq C \int_B \frac{|z-x_0|}{|y-x_0|^{1+(n+m)p}} \| a \|_{L^\infty} \, dz \leq C \frac{r}{|y-x_0|^{1+(n+m)p}} \| a \|_{L^\infty} |B| \leq C \frac{r^{n+1}}{|y-x_0|^{1+(n+m)p} \omega(B)}. \]

where we use the fact that \(|y-\xi| \sim |y-x_0| \) if \( \xi \in B(x_0, r) \) and \(|y-x_0| \geq 2r. \)

Let us now continue to estimate \( E_2. \) When \( r \geq 1, \) using the mean value theorem and (37), we have

\[ E_2 = \frac{1}{t^{m+1}} \int_{|y-x_0| < 2r} \left| y-x_0 \right| dy \leq C \frac{\rho^{n+1}}{|x-x_0|^{n+1}} \omega(B) \int_{|y-x_0| < 2r} \frac{1}{|y-x_0|^n} dy \leq C \frac{\rho^{n+1}}{|x-x_0|^{n+1}} \ln \left( \frac{|x-x_0|}{4r} \right). \]

Here, we use the fact that \(|x-x_0 - y(y-x_0)| \sim |x-x_0| \) under the condition of \( 2r \leq |y-x_0| < |x-x_0| / 2. \)

Similarly, in the case of \( 0 < r < 1, \) by the moments condition for \( a, \) the mean value condition, and (38), we get

\[ E_2 = \frac{1}{t^{m+1}} \int_{|y-x_0| < 2r} \left| K(y,z) - k(y,x_0) \right| |a(z)| \, dz \leq C \frac{\rho^{n+1}}{|x-x_0|^{n+1}} \omega(B) \int_{|y-x_0| < 2r} \frac{1}{|y-x_0|^n} dy \leq C \frac{\rho^{n+1}}{|x-x_0|^{n+1}} \ln \left( \frac{|x-x_0|}{4r} \right). \]

For the term \( E_3, \) we have

\[ E_3 \leq \frac{1}{t^n} \int_{|y-x_0| < 2r} |Ta(y)| \left( \left| \phi \left( \frac{x-y}{t} \right) \right| + \left| \phi \left( \frac{x-x_0}{t} \right) \right| \right) \, dy. \]

Since \(|y-x_0| \geq (|x-x_0| / 2), \) we have \(|x-y| \geq (|x-x_0| / 2). \) Thus,

\[ \frac{1}{t^n} \left| \phi \left( \frac{x-y}{t} \right) \right| \leq \frac{C}{|x-y|^n} \leq \frac{C}{|x-x_0|^n}. \]

Meanwhile,

\[ \frac{1}{t^n} \left| \phi \left( \frac{x-x_0}{t} \right) \right| \leq \frac{C}{|x-x_0|^n}. \]

So, in the case of \( r \geq 1, \) by (37), (42), and (43), we have
\[ E_3 \leq C \int_{|y-x_0| \geq (|x-x_0|/2)} \frac{r^{n+1}}{|y-x_0|^{\alpha+1} \omega (B)} \left| \frac{1}{|y-x_0|^\alpha} \right| dy \]

\[ = C \frac{r^{n+1}}{|x-x_0|^{\alpha+1} \omega (B)} \int_{|y-x_0| \geq (|x-x_0|/2)} \frac{1}{|y-x_0|^{\alpha+1}} dy \]

\[ = C \frac{r^{n+1}}{|x-x_0|^{\alpha+1} \omega (B)} \]

(44)

In the case of 0 < r < 1, by (38), (42), and (43), we have

\[ E_3 \leq C \int_{|y-x_0| \geq (|x-x_0|/2)} \frac{r^{n+1}}{|y-x_0|^{(1+n+\rho)} \omega (B)} \left| \frac{1}{|y-x_0|^{\alpha+1}} \right| dy \]

\[ = C \frac{r^{n+1}}{|x-x_0|^{\alpha+1} \omega (B)} \int_{|y-x_0| \geq (|x-x_0|/2)} \frac{1}{|y-x_0|^{(1+n+\rho)}} dy \]

\[ = C \frac{r^{n+1}}{|x-x_0|^{\alpha+1} \omega (B)} \]

\[ \leq C \frac{r^{\alpha+\mu}}{|x-x_0|^{\alpha \mu} \omega (B)} \]

(45)

since \( \rho n < 1 + n + m \leq \rho (n + 1) \).

Let \( x \notin B(x_0, 4r) \). In view of (35), (39), (40), (44), and (45), we shall unify these formulas. Firstly, \( pn - (n+1) < m \leq (\rho - 1)(n + 1) \) implies \( \mu \in [0, 1] \). Secondly, \( x \notin B(x_0, 4r) \) implies \( (r/|x-x_0|) < 1 \). Therefore,

\[ \frac{r^{\alpha+\mu}}{|x-x_0|^{\alpha \mu} \omega (B)} \leq \frac{r^{\alpha+\mu}}{|x-x_0|^{\alpha \mu} \omega (B)} \]

holds for any \( \mu \in (0, 1) \). Finally, since \( \ln x \leq (1/\alpha)x^\alpha \) holds for \( x \geq 1 \) and \( \alpha \in (0, 1) \), we have

\[ \frac{r^{\alpha+\mu}}{|x-x_0|^{\alpha \mu} \omega (B)} \leq \frac{C}{|x-x_0|^{(1+\alpha)(1-\mu)/4r}} \frac{r^{\alpha+\mu}}{|x-x_0|^{\alpha \mu} \omega (B)} \]

\[ = C \frac{r^{\alpha+\mu}}{|x-x_0|^{\alpha \mu} \omega (B)} \]

(46)

Using these three facts, we have

\[ |(Ta)^*(x)| \leq C \frac{r^{\alpha+\mu}}{|x-x_0|^{\alpha \mu} \omega (B)} \]

(47)

Note that \( p \leq 1 + (\mu/n) \). Then, in all cases, we have \( n + \mu > np \) and

\[ I_2 \leq C \int_{|x-x_0| > 4r} \frac{r^{\alpha+\mu}}{|x-x_0|^{\alpha \mu} \omega (B)} \omega (x) dx \]

\[ = C \frac{r^{\alpha+\mu}}{\omega (B)} \sum_{l=0}^{\infty} \int_{2^{l+1}r \leq |x-x_0| < 2^{l+2}r} \frac{\omega (x)}{|x-x_0|^{\alpha \mu} \omega (B)} \]

\[ \leq C \sum_{l=0}^{\infty} \frac{2^{l(\alpha \mu) - np}}{2^{2l+3}} \leq C. \]

This concludes the proof of Theorem 2.

\[ \square \]

Data Availability

The authors confirm that no data were used to support this study. All References used were listed.

Disclosure

This study is a part of research work done by Yu-long Deng, a Ph.D student, under the supervision of the second author.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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