A Classification and Analysis of Higgs-flavor Models

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November 4, 2011

Abstract
A classification is given of Higgs-flavor models. In these models, there are several Higgs doublets in an irreducible multiplet $R_\Phi$ of a non-abelian symmetry $G_\Phi$, under which the quarks and leptons do not transform (thus giving minimal flavor-changing for the fermions). It is found that different $G_\Phi$ and $R_\Phi$ lead to very distinctive spectra of the extra Higgs doublets, including different numbers of “sequential Higgs” and of “inert Higgs” that could play the role of dark matter, different mass relations, and different patterns of $SU(2)_L$-breaking splittings within the Higgs doublets.

1 Introduction
Just as there is a repetition of the quark and lepton “families,” it is plausible to suppose that there might be a repetition of Higgs doublets [1]. One theoretical difficulty with additional Higgs doublets is that they could exacerbate
the gauge hierarchy problem, since the mass of each doublet might need to be fine-tuned to have masses much less than the “natural” Planck scale or grand unification scale. A second difficulty is that a multiplicity of Higgs doublets tends to lead to unrealistically large flavor-changing neutral current (FCNC) processes. Here we study the recently proposed idea of “Higgs-flavor symmetry” [2]. In models based on this idea, there are no fine-tunings besides that of the Standard Model (SM) Higgs field, and no FCNC effects aside from those caused by CKM mixing.

In these models, the “extra” Higgs doublets and the Standard Model (SM) Higgs doublet together form an irreducible multiplet $\Phi$ of a non-abelian “Higgs-flavor” group $G_\Phi$. Consequently, a single tuning of the mass of the multiplet $\Phi$ (perhaps anthropic [3]) is sufficient to make all the Higgs doublets contained in it have low mass. The symmetry $G_\Phi$, in other words, ties all the Higgs masses together.

Of course, the multiplet $\Phi$ must be split by $G_\Phi$-breaking effects. It is assumed that the breaking of $G_\Phi$ occurs in a sector of superheavy fields that are singlets under the SM group and that the breaking is communicated to the SM fields (including $\Phi$) by a “messenger field” $\eta$ [2]. The vacuum expectation value (VEV) of $\eta$ is assumed to be near the weak scale; but this does not require any fine-tuning, since the mass of $\eta$ is superlarge. To make this concrete, suppose that $G_\Phi$ is spontaneously broken by a fermion-antifermion condensate $\langle \chi \chi \rangle$ to which $\eta$ couples. Then the terms $y\chi\chi\eta+\frac{1}{2}M_\eta^2\eta^2$ give $\langle \eta \rangle = y\langle \chi \chi \rangle / M_\eta^2$. Even if $M_\eta^2$ is superlarge, $\langle \eta \rangle$ can be of any scale without fine-tuning, since the scale of $\langle \chi \chi \rangle$ is dynamically generated by some confining interaction, and its value is determined by the scale at which that interaction becomes strong, which can “naturally” be anything. It should be noted that if the symmetry $G_\Phi$ is local, the gauge bosons associated with it get mass at the scale of the condensate $\langle \chi \chi \rangle$, which is of order $(M_\eta^2\langle \eta \rangle)^{1/3} \gg M_W$. Therefore the FCNC effects produced by their exchange are negligible. At low-energy, there is effectively an approximate global $G_\Phi$ symmetry in the Higgs sector, the pattern of breaking of which is determined by the expectation value of the messenger field.

To avoid dangerous FCNC effects, one makes the crucial assumption that the SM quarks and leptons are singlets under the group $G_\Phi$. The quark and lepton masses thus must come from higher-dimension effective operators. For example, the up quark masses would come from operators of the form
and similarly for the down quarks and leptons, where $m, n$ are quark/lepton family indices, and $\alpha$ is a $G_\Phi$ index. Note that for such a dimension-5 effective Yukawa operator to exist the Higgs multiplet $\Phi$ and the messenger multiplet $\eta$ must transform as the same irreducible representation $R_\Phi$ of $G_\Phi$. The scale $M$ comes from integrating out fermions whose mass is of order $\langle \eta \rangle$ (as was discussed in [2]), so that the quark and lepton masses coming from Eq. (1) need not be much suppressed compared to the weak scale.

From Eq. (1), one sees that only one linear combination of the Higgs doublets (which will be called $\Phi_F$, where $F$ stands for “fermions”) couples to the quarks and leptons:

$$
\Phi_F \propto \langle \eta_\alpha^* \rangle \Phi^\alpha.
$$

This satisfies the well-known conditions for “natural flavor conservation” [4]. This linear combination is not necessarily a mass eigenstate, but may be a mixture of the Standard Model doublet (which we will denote $\Phi_{SM}$) and some of the “extra” Higgs doublets. In that case, those extra doublets will couple to quarks and leptons, but with Yukawa coupling matrices that are proportional to those of $\Phi_{SM}$, so that no FCNC effects result except through CKM mixing. (The idea of an extra Higgs doublet whose Yukawa couplings to quarks and leptons are proportional to those of the Standard Model Higgs has been proposed recently by Pich and Tuzón and further developed by Serôdio [5].)

One could imagine that there are several messenger fields; but then there is a danger of excessive FCNC effects. For example, suppose there were two messengers, $\eta$ and $\eta'$, which were in the same representation of $G_\Phi$ as the Higgs multiplet $\Phi$. Then two kinds of Yukawa term would be present for each type of fermion; for example, for the up quarks one would have $Y^u_{mn} u^c_m u_n (\Phi^\alpha \eta_\alpha^*)/M$ and $Y'^u_{mn} u^c_m u_n (\Phi^\alpha \eta'^*_\alpha)/M$, etc. This would mean that two different linear combinations of the Higgs doublets would couple to quarks and leptons with Yukawa matrices that were not, in general, simultaneously diagonalizable. This would violate the condition for natural flavor conservation [4]. In this paper, therefore, we will consider only the simplest case, namely: there exists only one messenger field, which is in the same irreducible representation $R_\Phi$ of $G_\Phi$ as the Higgs multiplet $\Phi$.  

$$
Y^u_{mn} u^c_m u_n (\Phi^\alpha \eta_\alpha^*)/M,
$$

(1)
As was pointed out in [2], and will be seen in the cases worked out below, the framework we have just sketched leads to models having several typical characteristics: (1) There exist one or more Higgs doublets that are unable to decay to other Higgs doublets or to fermions due to subgroups of $G_{\Phi}$ that are unbroken by the messenger field VEV. The lightest components of such doublets will be absolutely stable and play the role of dark matter. (We will sometimes speak loosely of these doublets as “stable Higgs doublets,” even though only their lightest components are stable. The heavier components can decay into the lighter components plus quarks or leptons by weak interactions.) (2) There exist in many cases extra Higgs doublets that couple to quarks and leptons proportionally to the Standard Model Higgs doublet (and therefore to the masses of the fermions). These obviously are unstable. The constants of proportionality are in some cases constrained by $G_{\Phi}$. (3) There usually exist relationships coming from $G_{\Phi}$ among the masses of the extra Higgs doublets and also between these masses and the constants of proportionality mentioned in (2). (4) There usually exist relationships coming from $G_{\Phi}$ among the $SU(2)_L$-breaking mass splittings within the Higgs doublets.

2 General Discussion and Summary of Results

The breaking of the Higgs-family symmetry $G_{\Phi}$ is communicated to the multiplet of Higgs doublets $\Phi$ by the messenger field $\eta$, and in particular by terms in the Higgs potential that couple $\Phi$ to $\eta$:

$$V_{HFB} \sim \Phi^\dagger \cdot \Phi \eta^\dagger \eta,$$

where here and throughout the dot stands for the contraction of electroweak indices to form an $SU(2)_L$ singlet. The subscript “HFB” stands for “Higgs-flavor breaking.” In some cases, the group theory also allows terms of the form $\Phi^\dagger \cdot \Phi \eta$. Of course, there is always also a $G_{\Phi}$-invariant mass term $M_0^2 \Phi^\dagger \cdot \Phi$. When the VEV of $\eta$ is substituted into $M_0^2 \Phi^\dagger \cdot \Phi + V_{HFB}$, one obtains the mass-squared matrix for the Higgs doublets. It is assumed that $M_0^2$ is tuned so that the lightest eigenstate has a negative mass-squared that is of order the weak scale. This is the Standard Model Higgs doublet, which we will denote $\Phi_{SM}$. The other eigenstates are assumed to have positive
mass-squared. These are the “extra Higgs doublets,” and have no vacuum expectation values. (Technically, since they don’t obtain VEVs, one should not call them “Higgs” fields. We shall, nevertheless, do so for simplicity.)

One linear combination of the Higgs doublets couples to the Standard Model quarks and leptons, namely that shown in Eq. (2). In some cases, this $\Phi_F$ is a mass-squared eigenstate, in which case, it must clearly be identified with $\Phi_{SM}$ if the quarks and leptons are to obtain mass. Frequently however, as will be seen, $\Phi_F$ is a linear combination of $\Phi_{SM}$ and one or more of the “extra Higgs doublets.” Those extra Higgs would therefore couple to the Standard Model quarks and leptons, and therefore be able to decay directly into them. We will call these “sequential Higgs doublets.” Their Yukawa coupling matrices are exactly proportional to those of the Standard Model Higgs, as is clear from Eq. (1). The constant of proportionality for each sequential Higgs doublet is just given by the magnitude of its mixing with $\Phi_F$. That is, if $\Phi_F = c_0 \Phi_{SM} + \sum \kappa c_\kappa \Phi_\kappa$, where $\Phi_\kappa$ stands for the $\kappa^{th}$ sequential Higgs doublet, and $c_0^2 + \sum |c_\kappa|^2 = 1$, then the Yukawa coupling matrix of the $\kappa^{th}$ sequential Higgs doublet is given by $(Y_\kappa)_{mn} = (c_\kappa/c_0)(Y_{SM})_{mn}$. We shall refer to the $c_\kappa$ as the “mixing angles” of the sequential Higgs doublets.

In most cases, there are also Higgs-doublets that are mass-squared eigenstates and do not mix with $\Phi_F$. These do not couple to the quarks and leptons, and we shall call these “inert Higgs doublets.” (Inert Higgs fields have been widely discussed as candidates for dark matter [?], though most of those models posit only one inert Higgs field, whereas the “Higgs-flavor” framework we are discussing here typically leads to the existence of several inert Higgs fields.) An inert Higgs doublet cannot decay directly into quarks and leptons, but in some cases they can decay into another inert Higgs doublet plus some sequential Higgs that can then decay into quarks and leptons. Thus, some inert Higgs doublets are unstable. As will be seen, however, some of the inert Higgs doublets are absolutely stable. More precisely, their lightest components are absolutely stable, since these doublets are split when the weak gauge group $SU(2)_L$ breaks, and their heavier components can beta decay into the lightest one. We shall nevertheless sometimes for simplicity refer to these as “stable Higgs doublets.” We shall assume that the $SU(2)_L$ breaking is such that the stable components of the “stable Higgs doublets” are neutral rather than charged, simply because there are very stringent experimental limits on stable charged particles. This can always be ensured by a choice of signs of certain quartic self couplings of the Higgs doublets.
What makes the “stable Higgs doublets” stable, as will be seen, are subgroups of the Higgs-flavor group that are left unbroken by the messenger field expectation value. These subgroups may be discrete or continuous.

The $SU(2)_L$-breaking splittings within the extra Higgs doublets are produced by the quartic part of the Higgs potential, which has the form

$$V_4 \sim \Phi^\dagger \cdot \Phi \cdot \Phi^\dagger \cdot \Phi.$$ (4)

When expanded out, $V_4$ contains terms that contain two powers of $\Phi_{SM}$ and two powers of “extra Higgs doublets.” (There are, of course, also terms that are higher order in the extra Higgs doublets, but these don’t contribute to splitting those doublets.) In some cases, an extra Higgs doublet only has a splitting between its charged and (complex) neutral components. We shall call this a “CN” splitting. In other cases, an extra Higgs doublet also has a splitting between the pseudoscalar and scalar part of its neutral component. We shall call this a “CPS” splitting.

We shall consider all cases involving continuous Higgs-flavor groups (though they may also have discrete factors) that have six or fewer Higgs doublets. Each case is characterized by a choice of the group $G_\Phi$ and the irreducible representation $R_\Phi$ to which (by assumption) both $\Phi$ and $\eta$ belong. Nearly every case gives a very distinctive spectrum of extra Higgs doublets.

In Table I is displayed, for each case considered, the number of extra Higgs doublets of the following types: (a) sequential Higgs doublets, (b) inert Higgs doublets that are CN-split, and (c) inert Higgs doublets that are CPS-split. If a number $N > 1$ appears as an entry, it refers $N$ degenerate Higgs doublets. So $(1,1,1,1)$ means that there are four Higgs doublets that are not degenerate, $(2,2)$ means that there are two doublets of one mass and two of a different mass, and 4 means that there are four Higgs doublets of the same mass, etc. If there is no superscript on a number, it means that the lightest component(s) of those Higgs doublets are absolutely stable. For a CN-split Higgs doublet, the lightest component is a complex neutral field. For a CPS-split Higgs doublet, it is a real field. So, for example, if there is a 4 in the CN column, it means that there are 4 stable complex neutral fields that are degenerate in mass. If there is a 4 in the CPS column, it means that there are 4 stable real fields that are degenerate. A superscript zero means that the corresponding Higgs doublet does not have a stable component. A superscript asterisk means that whether it does decay or not depends on the
values of certain parameters in the model.

Making use of the isomorphisms $SO(3) \sim SU(2)$, $SO(5) \sim Sp(4)$, $SO(4) \sim SU(2) \times SU(2)$, and $SO(6) \sim SU(4)$, one can see that all cases of representations of dimension less than or equal to 6 have been included in Table 1.
Table I: The number of extra Higgs doublets of various types in different Higgs-flavor schemes. The notation is defined in the text.

| $G_{\Phi}$ | $R_{\Phi}$ | Sequential Higgs | Inert Higgs CN-split | Inert Higgs CPS-split |
|-----------|-----------|-----------------|----------------------|----------------------|
| $SU(2)$   | 2         | -               | 1                    | -                    |
| $SU(3)$   | 3         | -               | 2                    | -                    |
| $SO(3)$   | $3_{R}$   | -               | (1,1)                | -                    |
| $SO(3)$   | $3_{R} (m_{3} = 0)$ | - | - | 2 |
| $SO(3)$   | $3_{C}$   | 1               | -                    | 1                    |
| $SU(4)$   | 4         | -               | 3                    | -                    |
| $SO(4)$   | $4_{R}$   | -               | -                    | 3                    |
| $SO(4)$   | $4_{C}$   | 1               | -                    | (1,1)                |
| $SU(2)$   | 4         | (1,1,1)         | -                    | -                    |
| $SU(2)^{2} \times S_{2}$ | $(2,1) + (1,2)$ | 1 | - | $(1,1^0)$ |
| $SO(5)$   | 4         | 1               | 2                    | -                    |
| $SU(5)$   | 5         | -               | 4                    | -                    |
| $SO(5)$   | $5_{R}$   | -               | -                    | 4                    |
| $SO(5)$   | $5_{C}$   | 1               | -                    | 3                    |
| $SO(3)$   | $5_{R}$   | 1               | -                    | (1,1,1$^*$)          |
| $SO(3)$   | $5_{C}$   | (1,1,1,1)       | -                    | -                    |
| $SU(6)$   | 6         | -               | 5                    | -                    |
| $SO(6)$   | $6_{R}$   | -               | -                    | 5                    |
| $SO(6)$   | $6_{C}$   | 1               | -                    | 4                    |
| $SU(3)$   | 6         | (1,1)           | -                    | (1,1,1$^*$)          |
| $SU(2)$   | 6         | (1,1,1,1,1)     | -                    | -                    |
| $SU(3)^{2} \times S_{2}$ | $(3,1) + (1,3)$ | 1 | (2,2) | - |
| $SO(3)^{2} \times S_{2}$ | $[(3,1) + (1,3)]_{R}$ | 1 | (1,1,1,1) | - |
| $SO(3)^{2} \times S_{2}$ | $[(3,1) + (1,3)]_{C}$ | (1,1,1) | - | (1,1) |
| $SU(3) \times SU(2)$ | $(3,2)$ | 1 | (1,1$^*$) | $(1^*, 1^0)$ |
| $SU(2)^{2}$ | $(3,2)$ | (1,1,1,1,1) | - | - |
| $SU(2)^{3} \times S_{3}$ | $(2,1,1) + (1,2)$ | (1,1) | (1,1,1) | - |

The Higgs-flavor symmetry highly restricts the form of the Higgs potential, and thus gives relations among the masses and mixings of the Higgs
doublets. We analyse the Higgs mass spectrum in two stages, as explained above. First, there is the spectrum in the limit where $SU(2)_L$ breaking is neglected. This is found from the mass-squared matrix that comes from the terms $M^2 \Phi^\dagger \cdot \Phi + V_{HFB}$. The measurable quantities of interest are the values of the masses of the extra Higgs doublets and the values of the mixing angles of the sequential Higgs doublets with $\Phi_F$ (which completely determine their couplings to the Standard Model quarks and leptons). Call the number of extra Higgs doublets $N_{EH}$, the number of distinct masses of the extra Higgs doublets (i.e. treating Higgs doublets that are degenerate with each other as having one mass) $N_M$, and the number of sequential Higgs doublets (and thus of mixing angles) $N_{\theta}$. These $N_M + N_{\theta}$ quantities are determined by the values of two kinds of model parameters: (i) coefficients in the potential $V_{HFB}$ that contribute to splitting among Higgs doublets, and (ii) the number of independent physical quantities in the VEV of the messenger field. Call the total number of these two kinds of model parameters $N_{\text{par}}$. Then there will be $N_{rel} = \max(N_M + N_{\theta} - N_{\text{par}}, 0)$ relations among the masses and mixings of the Higgs doublets.

The second stage of analysis is to find the $SU(2)_L$-breaking splittings within Higgs doublets. Each Higgs doublet (or degenerate set of doublets) that is $CN$-split has one measurable splitting, whereas each that is $CPS$-split has two measurable splittings. Call the total number of measurable splittings $N_{\text{split}}$. These depend on the number $N_4$ of coefficients in $V_4$ that contribute to such splittings. There will therefore be a further $N'_{rel}$ relations where $N'_{rel} = N_{\text{split}} - N_4$ (unless $N_M + N_{\theta} - N_{\text{par}}$ was negative, in which case $N'_{rel}$ is reduced by that amount).

In Table II, we display these numbers for all the cases where $G_\Phi$ contains continuous groups and that have six or fewer Higgs doublets.
**Table II:** This shows the number of distinct masses and mixing angles, the number of model parameters on which they depend, and the number of relations predicted, for each model. The quantities are defined in the text.

| $G_{\Phi}$      | $R_{\Phi}$ | $N_{EH}$ | $N_{M}$ | $N_{\theta}$ | $N_{\text{par}}$ | $N_{\text{rel}}$ | $N_{\text{split}}$ | $N_{4}$ | $N'_{\text{rel}}$ |
|-----------------|------------|----------|---------|--------------|------------------|------------------|---------------------|--------|------------------|
| $SU(2)$         | 2          | 1        | 1       | 0            | 1                | -                | 1                   | 1      | -                |
| $SU(3)$         | 3          | 1        | 0       | 0            | 1                | -                | 1                   | 1      | -                |
| $SO(3)$         | $3_{R}$   | 2        | 2       | 0            | 2                | -                | 2                   | 2      | -                |
| $SO(3)$         | $3_{R} (m_{3} = 0)$ | 2 | 1 | 0 | 1 | - | 2 | 2 |
| $SO(3)$         | $3_{C}$   | 2        | 2       | 1            | 3                | -                | 4                   | 2      | 2                |
| $SU(4)$         | 4          | 3        | 1       | 0            | 1                | -                | 1                   | 1      | -                |
| $SO(4)$         | $4_{R}$   | 3        | 1       | 0            | 1                | -                | 2                   | 2      | -                |
| $SO(4)$         | $4_{C}$   | 3        | 3       | 1            | 4                | -                | 4                   | 3      | 1                |
| $SU(2)$         | 4          | 3        | 3       | 6            | -                | 6                | 3                   | 3      | -                |
| $SU(2)^{2} \times S_{2}$ | (2, 1) + (1, 2) | 3 | 3 | 1 | 5 | - | 4 | 4 |
| $SO(5)$         | 4          | 3        | 2       | 1            | 3                | -                | 3                   | 3      | -                |
| $SU(5)$         | 5          | 4        | 1       | 0            | 1                | -                | 1                   | 1      | -                |
| $SO(5)$         | $5_{R}$   | 4        | 1       | 0            | 1                | -                | 2                   | 2      | -                |
| $SO(5)$         | $5_{C}$   | 4        | 2       | 1            | 3                | -                | 4                   | 2      | 2                |
| $SO(3)$         | $5_{R}$   | 4        | 4       | 1            | 4                | 1                | 8                   | 4      | 4                |
| $SO(3)$         | $5_{C}$   | 4        | 4       | 4            | 9                | -                | 8                   | 4      | 3                |
| $SU(6)$         | 6          | 5        | 1       | 0            | 1                | -                | 1                   | 1      | -                |
| $SO(6)$         | $6_{R}$   | 5        | 1       | 0            | 1                | -                | 2                   | 2      | -                |
| $SO(6)$         | $6_{C}$   | 5        | 2       | 1            | 3                | -                | 4                   | 2      | 2                |
| $SU(3)$         | 6          | 5        | 5       | 2            | 4                | 3                | 10                  | 2      | 8                |
| $SU(2)$         | 6          | 5        | 5       | 5            | 12               | -                | 10                  | 5      | 3                |
| $SU(3)^{2} \times S_{2}$ | (3, 1) + (1, 3) | 5 | 3 | 1 | 4 | - | 4 | 4 |
| $SO(3)^{2} \times S_{2}$ | [(3, 1) + (1, 3)]_{R} | 5 | 5 | 1 | 6 | - | 6 | 4 |
| $SO(3)^{2} \times S_{2}$ | [(3, 1) + (1, 3)]_{C} | 5 | 5 | 3 | 8 | - | 10 | 6 |
| $SU(3) \times SU(2)$          | (3, 2)   | 5        | 5       | 1            | 4                | 2                | 8                   | 3      | 5                |
| $SU(2)^{2}$     | (3, 2)    | 5        | 5       | 5            | 12               | -                | 10                  | 5      | 3                |
| $SU(2)^{3} \times S_{3}$      | (2, 1, 1) + (1, 2, 1) + (1, 1, 2) | 5 | 5 | 2 | 5 | 2 | 7 | 3 | 4 |
3 Analysis of Cases

We now analyse in some detail cases where the number of Higgs doublets is six or fewer. Each case is defined by the Higgs-flavor group $G_\Phi$ and the irreducible representation $R_\Phi$ to which (by our assumption) both $\Phi$ and $\eta$ belong.

3.1 $G_\Phi = SU(N)$ with $R_\Phi = N$.

(a) If the messenger field is in the fundamental representation of $SU(N)$, then its VEV can, without loss of generality, be brought to the form

$$\langle \eta^\alpha \rangle = \begin{pmatrix} \eta \\ 0 \\ . \\ . \\ 0 \end{pmatrix}, \quad (5)$$

where $\eta$ is real. Then

$$\Phi_F \propto \langle \eta^*_\alpha \rangle \Phi^\alpha \Rightarrow \Phi_F = \Phi^1. \quad (6)$$

The breaking of $SU(N)$ is communicated to the Higgs doublets through terms of the form $\Phi^\dagger \cdot \Phi \eta^\dagger \eta$. The $SU(N)$ indices of the product $\eta^\dagger \eta$ can be contracted in two ways, corresponding to the decomposition $\overline{N} \times N = 1 + \text{Adj}$. There are therefore two independent terms in $V_{HF\beta}$:

$$V_{HF\beta} = \sigma_1 \Phi^\dagger_\alpha \cdot \Phi^\alpha \eta^*_\beta \eta^\alpha + \sigma_2 \Phi^\dagger_\alpha \cdot \Phi^\beta \eta^*_\beta \eta^\alpha, \quad (7)$$

where the $\sigma_i$ are real. Substituting in Eq. (7) the VEV of $\eta^\alpha$ given in Eq. (5), one has

$$\begin{pmatrix} \Phi^1 \\ \Phi^2 \\ \ldots \\ \Phi^N \end{pmatrix}^\dagger \cdot \begin{pmatrix} m^2 + \sigma_2 \eta^2 & 0 & 0 \\ 0 & m^2 & 0 \\ \vdots & \vdots & \ddots \\ 0 & 0 & m^2 \end{pmatrix} \begin{pmatrix} \Phi^1 \\ \Phi^2 \\ \ldots \\ \Phi^N \end{pmatrix}, \quad (8)$$

where $m^2 \equiv M_0^2 + \sigma_1 \eta^2$. Since, only $\Phi^1$ couples to quarks and leptons, it must be the Standard Model Higgs doublet $\Phi_{SM}$, and thus the lightest doublet.
One must therefore assume that $\sigma_2 \eta^2 < 0$ and that $M_0^2$ (and thus $m^2$) is fine-tuned (presumably anthropically [3]) so that $m^2 + \sigma_2 \eta^2 = -|\mu|^2$, where $|\mu|^2$ is of order $(100 \text{ GeV})^2$. All the other Higgs doublets $\Phi^\alpha$, $\alpha = 2, \ldots, N$, are degenerate and have mass-squared given by $m^2 = -|\mu|^2 + |\sigma_2| \eta^2$, which we take to be positive and large enough that these extra Higgs doublets have so far evaded detection — how large depends on the size of the messenger VEV $\eta$, which can “naturally” be anything, as noted before.

Given the form of the messenger field VEV in Eq. (5), the Higgs sector of the low-energy effective theory has a residual $SU(N-1)$ global symmetry. Under this symmetry, the extra $N-1$ Higgs doublets form an $(N-1)$-plet, whereas all the Standard Model fields are singlets. This symmetry therefore prevents the decays of the extra Higgs doublets to Standard Model fields. There are, of course, $SU(2)_L$-breaking splittings within the extra Higgs doublets. Consequently, one component of an extra Higgs doublet can beta decay to a lighter component plus a quark-antiquark or lepton-antilepton pair. But the lightest component of each of the $N-1$ extra Higgs doublets is absolutely stable. And these $N-1$ stable particles are all degenerate due to the residual global $SU(N-1)$.

We now turn to the $SU(2)_L$-breaking splittings within the doublets. These come from the quartic terms in the Higgs potential that are of the form $\Phi^\dagger \Phi \cdot \Phi \Phi^\dagger$. The $SU(N)$ indices of the product $\Phi^\dagger \Phi$ can be contracted in two ways, corresponding to the decomposition $\mathbf{N} \times \mathbf{N} = 1 + \mathbf{Adj}$. There are therefore two terms in $V_4$:

$$V_4 = \lambda_1 \Phi_\alpha^\dagger \Phi_\beta^\dagger \Phi_\beta \cdot \Phi_\alpha + \lambda_2 \Phi_\alpha^\dagger \Phi_\beta \Phi_\beta^\dagger \Phi_\alpha,$$  \hspace{1cm} (9)

where the $\lambda_i$ are real. Since $\Phi^\dagger = \Phi_{SM}$, it has a negative mass-squared and its neutral component has a VEV, $\langle (\Phi^\dagger)^0 \rangle = v/\sqrt{2}$, whereas all the other Higgs doublets $\Phi^\alpha$, $\alpha = 2, \ldots, N$ have positive mass-squared and vanishing VEVs. Substituting this expectation value into Eq. (9), it is easy to see that there is a mass term $\lambda_2 v^2 |(\Phi^\beta)^0|^2$, for $\beta = 2, \ldots, N$, which splits the charged components of the extra Higgs doublets from their neutral components, by the same amount for every $\beta$. Note that there is no splitting between the scalar and pseudoscalar components of these doublets, i.e. they are $CN$-split.

Since the dark matter must be neutral, one concludes that $\lambda_2 < 0$. The dark matter fields, therefore, consist of $N-1$ degenerate neutral complex fields.
3.2 $G_\Phi = SO(N)$ with $R_\Phi = N$, $\eta = \text{real.}$

3.2.1 $N > 3.$

The analysis of this case is quite similar to the previous one. We will use latin letters to denote $SO(N)$ indices. Without loss of generality, the VEV of the messenger field can be brought to the form

$$
\langle \eta^i \rangle = \begin{pmatrix}
\eta  \\
0 \\
\vdots \\
0
\end{pmatrix},
$$

(10)

The Higgs doublet that couples to the fermions is therefore again $\Phi_F = \Phi^1$. Since $\eta$ is now a real field, Eq. (3) becomes $V_{HFB} \sim \Phi^\dagger \cdot \Phi \eta \eta$. The product $\eta \eta$ must now be in the symmetric $SO(N)$ product $(N \times N)_S = 1 + \left( \frac{N(N+1)}{2} - 1 \right)$. There are therefore two terms given by

$$
V_{HFB} = \sigma_1 \Phi^\dagger \cdot \Phi^i \eta^i \eta^j + \sigma_2 \Phi^\dagger \cdot \eta^i \eta^j \Phi^j,
$$

(11)

Upon substituting the VEV of the messenger field, one obtains again the mass matrix given in Eq. (8). As in the previous case, one can make $\Phi^1$ the lightest Higgs doublet by choosing $\sigma_2 < 0$, and one can make it have negative mass-squared by tuning $M^2_0$, so that $\Phi_{SM} = \Phi_F = \Phi^1$. The VEV of the messenger field leaves unbroken a global $SO(N-1)$, so that all the extra Higgs doublets are, as before, degenerate. Moreover, this unbroken $SO(N-1)$ causes the lightest components of all the extra doublets to be stable.

The main difference with the previous case lies in the form of $V_4$. Since $\Phi^\dagger \cdot \Phi$ is in the product (not required to be symmetric) $N \times N = 1 + \frac{N(N-1)}{2} + \left( \frac{N(N+1)}{2} - 1 \right)$, there are three independent terms in $V_4$, which can be written

$$
V_4 = \lambda_1 \Phi^\dagger \cdot \Phi^i \Phi^{j\dagger} \Phi^j + \lambda_2 \Phi^\dagger \cdot \Phi^i \Phi^{j\dagger} \Phi^j \Phi^{i\dagger} \Phi^{j\dagger} \Phi^{i\dagger} \Phi^j.
$$

(12)

(One might imagine that $N = 4$ is a special case here, since the existence of a rank-4 epsilon symbol allows one to write a term $\Phi^\dagger \cdot \Phi^i \Phi^{j\dagger} \Phi^k \Phi^{\ell \dagger} \epsilon^{ijk\ell}$.}

13
However, this has no effect on the $SU(2)_L$-breaking splittings, since only $\Phi^1$ has a non-zero VEV.

The second and third terms in Eq. (12) give the following $SU(2)_L$-breaking contributions to the mass-squared of the Higgs doublets: $\lambda_2 v^2 |(\Phi^j)^0|^2 + \lambda_3 v^2 [(\Phi^j)^0]^2 + (\Phi^j)^{0*}]^2$. Note that while the $\lambda_2$ term only splits the charged from the neutral components, as in the $SU(N)$ case, the $\lambda_3$ term now also splits the scalar from the pseudoscalar in the neutral component. So the extra Higgs doublets are $CPS$-split.

The dark matter would be either the neutral scalar or neutral pseudoscalar components of the $\Phi^i$, $i = 2, ..., N$, depending on which is lighter. Thus, the dark matter fields consist of $N - 1$ degenerate real neutral fields.

### 3.2.2 $N = 3$.

The case of $N = 3$ is special, because the existence of the rank-3 epsilon symbol $\epsilon^{ijk}$ allows an additional term in $V_{HFB}$:

$$V_{HFB} = \sigma_1 \Phi^i \eta^j \eta^j + \sigma_2 \Phi^i \eta^j \eta^j + im_3 \Phi^i \cdot \Phi^j \cdot \Phi^k \epsilon^{ijk}. \quad (13)$$

This gives a mass-squared matrix

$$\begin{pmatrix} m^2 + \sigma_2 \eta^2 & 0 & 0 \\ 0 & m^2 & im_3 \eta \\ 0 & -im_3 \eta & m^2 \end{pmatrix} \begin{pmatrix} \Phi^1 \\ \Phi^2 \\ \Phi^3 \end{pmatrix}, \quad (14)$$

where $m^2 = M_0^2 + \sigma_1 \eta^2$. The two extra Higgs doublets are therefore $\Phi_{\pm} = \frac{1}{\sqrt{2}} (\Phi^2 \pm i\Phi^3)$, with mass-squared given by $m_{\pm} = m^2 \pm m_3 \eta$. The VEV of the messenger field breaks the global $SO(3)$ down to $SO(2) = U(1)$, which, being an abelian group, has only singlet representations, and therefore doesn’t cause any degeneracy among the Higgs doublets. Under this $U(1)$ (which is a rotation in the 23 plane), $\Phi_{\pm} \rightarrow e^{\pm i\theta} \Phi_{\pm}$. This symmetry prevents either $\Phi_{\pm}$ from decaying entirely into Standard Model particles, which are all neutral under it. But it does allow decays such as $\Phi_{\pm} \rightarrow \Phi_{\mp}^* + ..., which can happen through quartic scalar couplings. Therefore, only the lighter of the two extra Higgs doublets $\Phi_{\pm}$ has stable components.

The quartic couplings are still described by Eq. (12); but now these lead to a more complicated spectrum. Each of the two extra Higgs doublets has
three types of components: the charged component and the real and imaginary parts of the neutral components, which will be denoted by the subscripts $C$, $R$, and $I$, respectively. One ends up with the following $SU(2)_L$-breaking contributions to the mass-squared matrices of the extra Higgs doublets:

$$
(\phi^+_C, \phi^-_C)^* \begin{bmatrix}
  m_3\eta & 0 \\
  0 & -m_3\eta
\end{bmatrix} \begin{bmatrix}
  \phi^+_C \\
  \phi^-_C
\end{bmatrix},
$$

$$
(\phi^+_R, \phi^-_R, \phi^+_I, \phi^-_I) \begin{bmatrix}
  m_3\eta + \lambda_2 v^2 & \lambda_3 v^2 & 0 & 0 \\
  \lambda_3 v^2 & -m_3\eta + \lambda_2 v^2 & 0 & 0 \\
  0 & 0 & m_3\eta + \lambda_2 v^2 & -\lambda_3 v^2 \\
  0 & 0 & -\lambda_3 v^2 & -m_3\eta + \lambda_2 v^2
\end{bmatrix} \begin{bmatrix}
  \phi^+_R \\
  \phi^-_R \\
  \phi^+_I \\
  \phi^-_I
\end{bmatrix}.
$$

This gives the following masses:

$$
M^2(\phi^+_C) = M_0^2 + m_3\eta,
M^2(\phi^+_R, \phi^+_I) = M_0^2 + \sqrt{(m_3\eta)^2 + (\lambda_3 v^2)^2 + \lambda_2 v^2},
M^2(\phi^-_C) = M_0^2 - m_3\eta,
M^2(\phi^-_R, \phi^-_I) = M_0^2 - \sqrt{(m_3\eta)^2 + (\lambda_3 v^2)^2 + \lambda_2 v^2}.
$$

One sees that the doublets $\Phi^+$ and $\Phi^-$ are both $CN$-split. Moreover, the two splittings are determined by the two parameters $\lambda_2$ and $\lambda_3$. So, there are no relations. Because the dark matter must be neutral, the only stable particle is the (complex) neutral component of the lighter of $\Phi^+$ and $\Phi^-$. That is, the dark matter field consists of one complex neutral field.

### 3.3 $G_\Phi = SO(N)$ with $R_\Phi = N, \eta = \text{complex.}$

#### 3.3.1 $N \neq 4$.

Here we assume that $\eta$ is a complex field in the $N$ representation of $SO(N)$. Besides the $SU(N)$ Higgs-flavor symmetry, there is assumed to be a $U(1)$ (global or local) under which $\eta$ is charged. $\Phi$ must have the same $U(1)$ charge so that effective Yukawa terms for the quarks and leptons can be written down of the form $Y^\eta_{mn}\psi^\dagger_m \psi_n (\Phi^\dagger \eta)$. (Note that if $\eta$ were not charged under a $U(1)$, then two types of effective Yukawa term could be written down,
involving $\Phi$ and $\Phi^*$, whose Yukawa coupling matrices would not be related by any symmetry, thus leading in general to excessive FCNC effects.\)

If $\eta$ is a complex field, then its VEV cannot be brought to the form in Eq.(10) by $SO(N)$ rotations. However, it can be shown that by a combination of $SO(N)$ rotation and rephasing of $\eta$ the VEV can be brought to the following form without any loss of generality:

$$
\langle \eta^i \rangle = \begin{pmatrix}
\eta_1 \\
 i\eta_2 \\
 0 \\
 \vdots \\
 0
\end{pmatrix}, \quad (17)
$$

where $\eta_1$ and $\eta_2$ are real. Therefore, the Higgs doublet that couples to the fermions is $\Phi_F \propto \eta_1 \Phi_1 - i\eta_2 \Phi_2$.

In the terms of $V_{HFB}$, which are of the form $\Phi^\dagger \cdot \Phi \eta^\dagger \eta$, the product $\eta^\dagger \eta$ can be contracted in three ways, corresponding to the decomposition $N \times N = 1 + \frac{N(N-1)}{2} + (\frac{N(N+1)}{2} - 1)$. Thus,

$$
V_{HFB} = \sigma_1 \Phi^{i\dagger} \cdot \Phi^i \eta^i \eta^i + \sigma_2 \Phi^{i\dagger} \cdot \Phi^j \eta^i \eta^j + \sigma_3 \Phi^{i\dagger} \cdot \Phi^j \eta^j \eta^j. \quad (18)
$$

This gives the mass matrix

$$
\begin{pmatrix}
\Phi^1 \\
\Phi^2 \\
\vdots \\
\Phi^N
\end{pmatrix}^\dagger
\begin{pmatrix}
m^2 + (\sigma_2 + \sigma_3)\eta_1^2 & -i(\sigma_2 - \sigma_3)\eta_1\eta_2 & \cdots & 0 \\
i(\sigma_2 - \sigma_3)\eta_1\eta_2 & m^2 + (\sigma_3 + \sigma_2)\eta_2^2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & m^2
\end{pmatrix}
\begin{pmatrix}
\Phi^1 \\
\Phi^2 \\
\vdots \\
\Phi^N
\end{pmatrix}, \quad (19)
$$

where $m^2 \equiv M_0^2 + \sigma_1(\eta_1^2 + \eta_2^2)$. Therefore, the Higgs doublets $\Phi^1$ and $\Phi^2$ mix, and the eigenstates of Eq. (19) are $\Phi \equiv \cos \theta \Phi^1 + i \sin \theta \Phi^2$, and $\Phi' \equiv -\sin \theta \Phi^1 + i \cos \theta \Phi^2$. The two distinct masses of the extra Higgs doublets (that of $\Phi'$ and that of $\Phi^i$, $i = 3, \ldots, N$) and the mixing angle $\theta$ are determined by the three parameters, $\sigma_2 \eta_1^2$, $\sigma_3/\sigma_2$, and $\eta_2/\eta_1$. So, no relation exists among them.

The fields $\Phi^i$, $i = 3, \ldots, N$ are degenerate due to an $SO(N-2)$ that is left unbroken by the messenger VEV. They are prevented by this symmetry from
decaying entirely into Standard Model fields, and so the lightest components of these Higgs doublets are stable. Both $\Phi$ and $\Phi'$ couple to quarks and leptons, and are therefore unstable. The lighter of these (which we can take to be $\Phi$) must be the Standard Model Higgs doublet $\Phi_{SM}$, whose mass-squared is negative (by the tuning of $M_0^2$).

The $SU(2)_L$-breaking splittings of the extra Higgs doublets, $\Phi'$ and $\Phi^i$, $i = 3, ..., N$, are determined by the quartic terms, which have the same form as shown in Eq. (12). As in the case where $\eta$ is real (discussed in section 3.2.1), all the extra Higgs doublets are $CPS$-split. These splittings are the same for the $\Phi^i$, $i = 3, ..., N$, by the residual $SO(N-2)$. It can be shown that these splittings are in general different for the $\Phi'$, however. There are four distinct $SU(2)_L$ splittings, therefore, which are determined by the two parameters $\lambda_2$ and $\lambda_3$, giving two relations.

The dark matter consists of $N-2$ degenerate real neutral fields, which are either the neutral scalar or neutral pseudoscalar components of the $\Phi^i$, $i = 3, ..., N$, depending on which are lighter.

3.3.2 $N = 4$.

For $N = 4$, additional terms are made possible by the existence of the rank-4 epsilon symbol $\epsilon^{ijkl}$. In $V_{HFB}$ there is an additional term besides those shown in Eq. (18), namely $\sigma_4 \Phi^i \cdot \Phi^j \eta^k \eta^l \epsilon^{ijkl}$. (One might think that there are only three terms in $V_{HFB}$, since the product $\eta^l \eta$ must be in $4 \times 4 = 1 + 6 + 9$. However, there are two ways to contract $6 \times 6$ to make a singlet: $T^{[ij]} T^{[ij]}$ and $T^{[ij]} T^{[kl]} \epsilon^{ijkl}$.)

The effect of the extra term is to mix $\Phi^3$ and $\Phi^4$, so that the eigenstates of the mass-squared matrix produced by $V_{HFB}$ are $\Phi \equiv \cos \theta \Phi^1 + i \sin \theta \Phi^2$, $\Phi' \equiv -\sin \theta \Phi^1 + i \cos \theta \Phi^2$, $\Phi_+ = \frac{1}{\sqrt{2}}(\Phi^3 + i \Phi^4)$, and $\Phi_- = \frac{1}{\sqrt{2}}(\Phi^3 - i \Phi^4)$. The masses of the three extra Higgs doublets and the mixing angle $\theta$ are determined by four parameters, $\sigma_2 \eta_1^2$, $\sigma_3/\sigma_2$, $\sigma_4/\sigma_2$, and $\eta_2/\eta_1$; so (as in the $N \neq 4$ case) no relation exists among them.

Under the $SO(2)$ left unbroken by the messenger VEV, which is a rotation in the 34 plane, one has $\Phi_{\pm} \rightarrow e^{\pm i\theta} \Phi_{\pm}$. This symmetry prevents either $\Phi_{\pm}$ from decaying entirely into Standard Model particles. But it does allow decays such as $\Phi_{\pm} \rightarrow \Phi^*_\pm + ...$, which can happen through quartic scalar couplings. Therefore, only the lighter of the two extra Higgs doublets $\Phi_{\pm}$ has stable components.
The $SU(2)_L$-breaking splittings within the extra Higgs doublets are produced by $V_4$, which has, for $N = 4$, an extra term besides those shown in Eq. (12), namely $\lambda_4 \Phi^i j \Phi^k l$. One finds that $\Phi_\pm$ are both $CN$-split. (Note that this is different from the situation for $N \neq 4$. Also note that this means that the stable extra Higgs field is a single complex scalar.) This can be seen from the $4 \times 4$ mass matrix of these neutrals, which has the form

\[
\begin{pmatrix}
\bar{m}_2^2 + \sigma_4 \eta_1 \eta_2 & \lambda_3 \nu^2 & 0 & 0 \\
\lambda_3 \nu^2 & \bar{m}_2^2 - \sigma_4 \eta_1 \eta_2 & 0 & 0 \\
0 & 0 & \bar{m}_2^2 + \sigma_4 \eta_1 \eta_2 & -\lambda_3 \nu^2 \\
0 & 0 & -\lambda_3 \nu^2 & \bar{m}_2^2 \eta - \sigma_4 \eta_1 \eta_2
\end{pmatrix}
\]

where $\bar{m}_2^2 \equiv M_0^2 + \sigma_1 (\eta_1^2 + \eta_2^2) + (\lambda_1 + \lambda_2) \nu^2$, and $\lambda_3 \equiv \lambda_3 (\cos^2 \theta - \sin^2 \theta)$. This has just two distinct eigenvalues: $\bar{m}_2^2 \pm \sqrt{\left(\sigma_4 \eta_1 \eta_2\right)^2 + (\lambda_3 \nu^2)^2}$. The masses of the charged components of $\Phi_\pm$ are $\bar{m}_2^2 - \lambda_2 \nu^2 \pm (\sigma_4 \eta_1 \eta_2 - \lambda_4 \nu^2)$. From this it can be seen that splitting between charged and neutral components is different for $\Phi_+$ and for $\Phi_-$. The Higgs doublet $\Phi'$ is $CPS$-split. Altogether, then, there are four distinct $SU(2)_L$-breaking splittings within the three extra Higgs doublets. These are determined by the three parameters $\lambda_i, i = 2, 3, 4$. There is therefore one relation.

\[3.4 \quad G_\Phi = SU(N) \times SU(N) \times S_2, \quad R_\Phi = (N, 1) + (1, N).\]

Here the group has two $SU(N)$ factors and a discrete symmetry under which they are interchanged. We can write the messenger field as $(\eta^\alpha, \eta'^\alpha)$, in an obvious notation. Without loss of generality, one can bring the VEV of the messenger field to the form

\[
\langle \eta^\alpha \rangle = \begin{pmatrix}
\eta \\
0 \\
\vdots \\
0
\end{pmatrix}, \quad \langle \eta'^\alpha \rangle = \begin{pmatrix}
\eta' \\
0 \\
\vdots \\
0
\end{pmatrix},
\]

where $\eta$ and $\eta'$ are real. The Yukawa terms are of the form $Y_{mn} \psi_m \psi_n (\eta^\alpha \Phi^\alpha + \eta'^\alpha \Phi'^\alpha)$. So that $\Phi_F = \eta \Phi^1 + \eta' \Phi'^1$. The most general form of the Higgs-flavor breaking part of the Higgs potential is
\[ V_{HFB} = \sigma_1 \left( \Phi_1^\dagger \cdot \Phi_1 \eta_1^\dagger \eta_1 + \Phi_2^\dagger \cdot \Phi_2 \eta_2^\dagger \eta_2 + \Phi_3^\dagger \cdot \Phi_3 \eta_3^\dagger \eta_3 \right) \\
+ \sigma_2 \left( \Phi_1^\dagger \cdot \Phi_1 \eta_3^\dagger \eta_3 + \Phi_2^\dagger \cdot \Phi_2 \eta_1^\dagger \eta_1 + \Phi_3^\dagger \cdot \Phi_3 \eta_2^\dagger \eta_2 \right) \\
+ \sigma_3 \left( \Phi_1^\dagger \cdot \Phi_2 \eta_3^\dagger \eta_3 + \Phi_2^\dagger \cdot \Phi_3 \eta_1^\dagger \eta_1 \right) \\
+ \sigma_4 \left( \Phi_1^\dagger \cdot \Phi_3 \eta_2^\dagger \eta_2 \right). \]  

(22)

Substituting the VEV given in Eq. (21) into Eq. (22), one sees that the Higgs doublets \( \Phi_1 \), \( \alpha = 2, ..., N \) all have mass-squared \( \sigma_1 \eta_1^2 + \sigma_2 \eta_2^2 \) and the \( \Phi''_1 \), \( \alpha = 2, ..., N \) all have mass-squared \( \sigma_1 \eta_1^2 + \sigma_2 \eta_2^2 \). This reflects the fact that the form of the messenger VEV leaves a global \( SU(N-1) \times SU(N-1)' \) unbroken in the low-energy Higgs sector, but not the discrete interchange symmetry \( S_2 \). These symmetries prevent any of these \( 2(N-1) \) extra Higgs doublets from decaying entirely into Standard Models particles. In fact, the lightest components of each of these doublets is absolutely stable. The \( \Phi_1 \) and \( \Phi''_1 \) mix:

\[ \left( \begin{array}{c} \Phi_1^\dagger \\ \Phi''_1^\dagger \end{array} \right) \left( \begin{array}{cc} M_0^2 + (\sigma_1 + \sigma_3) \eta_1^2 + \sigma_2 \eta_2^2 & \sigma_4 \eta_1 \eta'_1 \\ \sigma_4 \eta_1 \eta'_1 & M_0^2 + (\sigma_1 + \sigma_3) \eta_2^2 + \sigma_2 \eta_2^2 \end{array} \right) \left( \begin{array}{c} \Phi_1 \\ \Phi''_1 \end{array} \right). \]  

(23)

Thus, we may write \( \Phi = \cos \theta \Phi_1 + \sin \theta \Phi''_1 \) and \( \Phi'' = -\sin \theta \Phi_1 + \sin \theta \Phi''_1 \), and both of these Higgs doublets couple to quarks and leptons (and do so proportionally to each other). The lighter of them (which one can take to be \( \Phi \)) must be identified with the Standard Model Higgs doublet \( \Phi_{SM} \). (Its mass-squared will be the smallest of all the Higgs doublets if, say, \( \sigma_3 \) is sufficiently negative.) The three distinct masses of the extra Higgs doublets and the mixing angle \( \theta \) are determined by the four parameters, \( \sigma_2 \eta_2^2, (\sigma_1 + \sigma_3)/\sigma_2, \sigma_4/\sigma_2, \) and \( \eta'/\eta \); so that there is no relation among them.

The \( SU(2)_L \)-breaking splittings within the Higgs doublets come from

\[ V_4 = \lambda_1 \left( \Phi_1^\dagger \cdot \Phi_1 \Phi_2^\dagger \cdot \Phi_2 + \Phi_3^\dagger \cdot \Phi_3 \Phi_4^\dagger \cdot \Phi_4 \right) \\
+ \lambda_2 \left( \Phi_1^\dagger \cdot \Phi_1 \Phi_3^\dagger \cdot \Phi_3 + \Phi_2^\dagger \cdot \Phi_2 \Phi_4^\dagger \cdot \Phi_4 \right) \\
+ \lambda_3 \left( \Phi_1^\dagger \cdot \Phi_2 \Phi_3^\dagger \cdot \Phi_4 + \Phi_2^\dagger \cdot \Phi_1 \Phi_3^\dagger \cdot \Phi_4 \right) \\
+ \lambda_4 \left( \Phi_1^\dagger \cdot \Phi_3 \Phi_4^\dagger \cdot \Phi_4 + \Phi_2^\dagger \cdot \Phi_4 \Phi_3^\dagger \cdot \Phi_4 \right). \]  

(24)

It is straightforward to show that \( \Phi_\alpha, \alpha = 2, ..., N \) is \( CN \)-split and that the splitting is independent of \( \alpha \) (because of the residual symmetry). The
same is true for $\Phi^{\alpha'}, \alpha' = 2, ..., N$; but the splitting is not the same for $\Phi^\alpha$ and $\Phi^{\alpha'}$. The sequential Higgs doublet $\Psi'$ is $CPS$-split. Altogether, then, there are four distinct $SU(2)_L$-breaking splittings, and they are determined by the four parameters $\lambda_i$, $i = 1, ..., 4$, so that there is no relation among them.

The dark matter would consist of $2(N-1)$ complex neutral fields. Of these, $N - 1$ would be degenerate with one mass, and $N - 1$ would be degenerate with another mass.

3.5 $G_\Phi = SO(N) \times SO(N)' \times S_2$, $R_\Phi = (N, 1) + (1, N)$.

3.5.1 $\eta = \text{real}$.

For $N \neq 3$ this case is very similar to the $SU(N) \times SU(N)' \times S_2$ case considered above. The main difference concerns the $SU(2)_L$-breaking splittings, which in this case depend on six terms (as compared to the four in Eq. (24)):

$$V_4 = \lambda_1 \left( \Phi^i \Phi^i \Phi^{i\dagger} \Phi^j + \Phi^{i\dagger} \Phi^j \Phi^{i\dagger} \Phi^j \right)$$

$$+ \lambda_2 \left( \Phi^i \Phi^i \Phi^{i\dagger} \Phi^{i\dagger} \Phi^j \right)$$

$$+ \lambda_3 \left( \Phi^{i\dagger} \Phi^i \Phi^j \Phi^{i\dagger} \Phi^{i\dagger} \Phi^j \right)$$

$$+ \lambda_4 \left( \Phi^{i\dagger} \Phi^i \Phi^{i\dagger} \Phi^{i\dagger} \Phi^{i\dagger} \Phi^j \right)$$

$$+ \lambda_5 \left( \Phi^{i\dagger} \Phi^{i\dagger} \Phi^{i\dagger} \Phi^j \right)$$

$$+ \lambda_6 \left( \Phi^{i\dagger} \Phi^{i\dagger} \Phi^{i\dagger} \Phi^{i\dagger} \Phi^{i\dagger} \Phi^j \right).$$

The $SU(2)_L$-breaking splittings produced by these terms within the Higgs doublets $\Phi^i$, $i = 2, ..., N$ and $\Phi^{i'}$, $i' = 2, ..., N$, now include the splitting of neutral scalars from neutral pseudoscalars (caused by the $\lambda_4$ and $\lambda_6$ terms, which give mass to $(\Phi^i)^2 + h.c.$), whereas the other terms only give mass to $|\Phi^i|^2$. Thus, all the extra Higgs doublets are $CPS$-split. Altogether, then, there are six distinct $SU(2)_L$-breaking splittings (two within $\Phi^i$; two within $\Phi^{i'}$, $i = 2, ..., N$; and two within $\Phi^{i''}$, $i'' = 2, ..., N$). Since these depend on the six parameters $\lambda_i$, $i = 1, ..., 6$, there are no relations among these splittings.

The dark matter would consist of $2(N-1)$ real neutral fields. Of these, $N - 1$ would be degenerate with one mass, and $N - 1$ would be degenerate with another mass.

The case $N = 3$ is special, as it was for $SO(N)$ (section 3.2.2), because there is an extra term in $V_{HFB}$ due to the rank-3 epsilon symbol:
\[ V_{HFB} = \sigma_1 \left( \Phi^{i\dagger} \cdot \Phi^i \eta^j \eta^j + \Phi^{i'} \cdot \Phi^{i'} \eta^j \eta^j \right) \\
+ \sigma_2 \left( \Phi^{i\dagger} \cdot \Phi^i \eta^j \eta^j \eta^j + \Phi^{i'} \cdot \Phi^{i'} \eta^j \eta^j \right) \\
+ \sigma_3 \left( \Phi^{i\dagger} \cdot \Phi^i \eta^j \eta^j + \Phi^{i'} \cdot \Phi^{i'} \eta^j \eta^j \right) \\
+ \sigma_4 \left( \Phi^{i\dagger} \cdot \Phi^i \eta^j \eta^j \eta^j \eta^j + \Phi^{i'} \cdot \Phi^{i'} \eta^j \eta^j \right) \\
+ i \sigma_5 \left( \Phi^{i\dagger} \cdot \Phi^i \eta^j \epsilon^{ij} \eta^k \epsilon^{jk} + \Phi^{i'} \cdot \Phi^{i'} \eta^j \epsilon^{ij} \epsilon^{jk} \right) \]  

Equation (26)

The last term has the effect of mixing \( \Phi^2 \) with \( \Phi^3 \) and also \( \Phi^{2'} \) with \( \Phi^{3'} \). The mass-squared eigenstates are as follows: 

1. \( \Phi_1 \) and \( \Phi_1' \) mix, similarly to Eq. (23). One linear combination of them is \( \Phi_{SM} \) and the other is a sequential Higgs doublet \( \Phi' \).
2. \( \Phi_2 \) and \( \Phi_3 \) mix, giving eigenstates \( \Phi_{\pm} = \sqrt{2} (\Phi_2 \pm i \Phi_3) \).
3. \( \Phi^{2'} \) and \( \Phi^{3'} \) mix, giving eigenstates \( \Phi'_\pm = \sqrt{2} (\Phi^{2'} \pm i \Phi^{3'}) \). The five masses of the extra Higgs doublets and the mixing angle of \( \Phi' \) are determined by the five coefficients in \( V_{HFB} \) and the ratio of the VEVs of \( \eta^1 \) and \( \eta'^1 \). Thus, there is no relation among them.

The fields \( \Phi_{\pm} \) and \( \Phi'_{\pm} \) are \( CN \)-split, as was the case for \( SO(3) \) in section 3.2.2. \( \Phi' \) is \( CPS \)-split. The six distinct splittings are determined by the six coefficients in \( V_4 \), shown in Eq. (25). So, again, there are no relations among them.

3.5.2 \( \eta = \text{complex} \).

For \( \eta \) complex (and charged under some \( U(1) \)), \( N = 3 \) is not a special case, as there is no extra cubic term involving the epsilon symbol. (It is forbidden by the \( U(1) \).)

The messenger field VEV for complex \( \eta \) cannot in general be brought to the form given in Eq. (21), but can be brought to the form

\[
\langle \eta^i \rangle = \begin{pmatrix}
\eta_1 \\
0 \\
\vdots \\
0
\end{pmatrix}, \quad \langle \eta^{i'} \rangle = \begin{pmatrix}
\eta'_1 \\
0 \\
\vdots \\
0
\end{pmatrix},
\]

Equation (27)

One consequence of this is that \( \Phi_F \) is a linear combination of \( \Phi^1, \Phi^2, \Phi^1', \) and \( \Phi^{2'} \). \( V_{HFB} \) mixes these to give four doublets, \( \Phi_{SM} \), and three sequential Higgs doublets \( \Phi', \Phi'' \), and \( \Phi''' \). The VEV in Eq. (27) leaves unbroken an
$SO(N-2) \times SO(N-2)$. Because of this, the $\Phi^i$, $i = 3, \ldots, N$ are degenerate, as are the $\Phi^{i'}$, $i' = 3, \ldots, N$ (though they are not degenerate with each other). There are thus five distinct masses for the extra Higgs doublets, and three mixing angles (of $\Phi'$, $\Phi''$, and $\Phi'''$ with $\Phi_F$). These are determined by eight parameters: five couplings $\sigma_i$, $i = 2, \ldots, 6$, in $V_{HF_B}$, which has a structure analogous to that shown in Eq. (25), and three ratios of the VEVs given in Eq. (27). So, again, there is no relation among these quantities.

On the other hand, there are now 10 distinct $SU(2)_L$-breaking splittings within the extra Higgs doublets, since all five types of extra Higgs doublet ($\Phi'$, $\Phi''$, $\Phi'''$, $\Phi_i$, and $\Phi_i'$) are CPS-split. These ten quantities are determined by the six parameters $\lambda_i$, $i = 1, \ldots, 6$ shown in Eq. (25). Thus, there are four relations.

The dark matter would consist of $2(N-2)$ real neutral fields. Of these, $N-2$ would be degenerate with one mass, and $N-2$ would be degenerate with another mass.

3.6 $G_{\Phi} = SU(2)$, $R_{\Phi} = N$

The cases $N = 2$ and $N = 3$ have already been covered: the former in subsection 3.1 with $N = 2$, and the latter in subsections 3.2.2 and 3.3.1 with $N = 3$.

3.6.1 $N = 4$ or 6.

In the case $N = 4$ case, both $\Phi$ and $\eta$ are in a 4 of $SU(2)$. The VEV of $\eta$ cannot be brought to a very simple form by a choice of $SU(2)$ basis; rather at least three of its components remain non-zero. It turns out, therefore, that all four Higgs doublets mix, in the sense that $\Phi_F$ is a linear combination of all four mass-squared eigenstates of $V_{HF_B}$. Thus they are all sequential Higgs doublets that are unstable to decay into quarks and leptons. None contain stable components that could be dark matter. The number of masses of extra Higgs doublets is three and the number of mixing angles for the extra Higgs is three. These 6 quantities are determined by 7 parameters (3 coefficients in $V_{HF_B}$ and 4 real parameters from the form of the messenger VEV), so that there are no relations among them. There are testable relations, however, if one also takes into account the $SU(2)_L$-breaking splittings within the Higgs doublets. There are 6 such splittings (two within each extra Higgs doublet,
and three extra Higgs doublets). These are produced by $V_4$, which has four terms, since in $\Phi^\dagger \cdot \Phi \eta^\dagger \eta$ the product $\eta^\dagger \eta$ must be in $4 \times 4 = 1 + 3 + 5 + 7$. Of these four terms, three contribute to splittings. Altogether, then, one has 12 quantities determined by 10 parameters, giving two relations.

The $N = 6$ case is much like the $N = 4$ case. Again, the form of the messenger VEV cannot be made very simple, and all the extra Higgs doublets mix with $\Phi_F$ and are thus sequential Higgs doublets that can decay into quarks and leptons. Again, it turns out that taking into account the $SU(2)_L$-breaking splittings within the five extra Higgs doublets, there are two relations.

### 3.6.2 $N = 5$, $\eta =$ real.

In this case, both $\Phi$ and $\eta$ are in the 5 of $SU(2)$ (or equivalently $SO(3)$). Since this case was studied in detail in [2], here we will only summarize the results. The 5 is the rank-2 symmetric traceless representation of $SO(3)$. Thus we may write the Higgs and messenger fields as $\Phi^{(ij)}$ and $\eta^{(ij)}$, where $i, j = 1, 2, 3$. Since the VEV of the messenger field is a real symmetric traceless matrix, it can be diagonalized by a choice of $SO(3)$ basis:

$$
\langle \eta^{(ij)} \rangle = \begin{pmatrix}
a + \frac{1}{\sqrt{3}} b & 0 & 0 \\
0 & -a + \frac{1}{\sqrt{3}} b & 0 \\
0 & 0 & -\frac{2}{\sqrt{3}} b
\end{pmatrix}.
$$

The Higgs fields $\Phi^{(ij)}$ can also be thought of as a symmetric traceless matrix. In the basis where the messenger VEV has the form in Eq. (28), one can distinguish the three “off-diagonal” Higgs doublets, $\Phi^{(ij)}$, with $i \neq j$, from the two “diagonal” Higgs doublets. The mass-squared matrix produced by $V_{HF}$ leads to the result that the mass-squared eigenstates are $\Phi^{(12)}$, $\Phi^{(23)}$, and $\Phi^{(31)}$ and two linear combinations of the diagonal Higgs doublets, which one may call $\Phi = \Phi_{SM}$ and $\Phi'$. One finds that $\Phi_F = \cos \theta \Phi_{SM} + \sin \theta \Phi'$, where in general $\theta$ is some non-trivial angle. Thus, in general, $\Phi'$ is a sequential Higgs doublet, and is unstable.

In $V_{HF}$ there are three independent terms of the form $\Phi^\dagger \cdot \Phi \eta \eta$, since $\eta \eta$ must be in the symmetric product $[5 \times 5]_S = 1 + 5 + 9$. (Only the two non-singlet contractions contribute to splitting within the 5 of Higgs doublets, however.) There can also be a cubic term of the form $\Phi^\dagger \cdot \Phi \eta$, etc.
since $\Phi^\dagger \cdot \Phi$ is in $[5 \times 5]$, which contains a 5. Thus, the masses of the four extra Higgs doublets and the mixing angle $\theta$ depend on only 4 parameters (three coefficients within $V_{HFB}$ and the ratio $b/a$ in the messenger VEV). Thus, there is one prediction. For example, if one knew the masses of the four extra Higgs doublets, one could predict the angle $\theta$, and thus know the strength of the Yukawa couplings of $\Phi'$ to the quarks and leptons.

The form of the messenger VEV leaves unbroken discrete symmetries that stabilize some of the Higgs fields. Defining $P_{23}$ as a rotation by $\pi$ in the 23 plane (of $SO(3)$), which is equivalent to reflections in the 2 and 3 directions, one has that $\Phi^{(23)}$ is even under it, while $\Phi^{(12)}$ and $\Phi^{(31)}$ are odd. Similarly, one can define $P_{12}$ and $P_{31}$. These symmetries prevent any of the off-diagonal Higgs doublets from decaying into just Standard Model fields (which are all even under them). They do, however, allow the heaviest of the three off-diagonal Higgs doublets to decay into the other two plus Standard Model fields (via terms like $\Phi^{(12)}\dagger \cdot \Phi^{(23)}\Phi^{(31)}\dagger \cdot \Phi^{(11)}$), if that is kinematically allowed. Thus, the two lighter off-diagonal Higgs doublets have absolutely stable components. For certain values of the parameters, all three of the off-diagonal Higgs doublets have stable components.

An interesting special case arises if the cubic term in $V_{HFB}$ vanishes, as may happen due to a symmetry under which the messenger field is odd. In this case, the angle $\theta$ is zero, and the diagonal extra Higgs doublet $\Phi'$ does not couple to quarks and leptons, and has stable components.

The extra Higgs doublets are all $CN$-split. Since there are four extra Higgs doublets, there are altogether eight distinct $SU(2)_L$-breaking splittings. These depend on just four coefficients in $V_4$. (In $V_4$, there are five independent terms, since $\Phi^\dagger \cdot \Phi$ must be in $5 \times 5 = 1 + 3 + 5 + 7 + 9$. The four terms corresponding to non-singlet contractions contribute to the splittings.) Thus there are four relations among the $SU(2)_L$-breaking splittings.

The dark matter consists of two, three, or four real fields of different masses, depending on the values of model parameters.

### 3.6.3 $N = 5$, $\eta = \text{complex}$.

Here we assume that there is a $U(1)$ under which $\Phi$ and $\eta$ have the same non-zero charge. The big difference with the previous case (of $\eta$ real), is that the messenger VEV, being a complex symmetric matrix, can no longer be diagonalized by $SO(3)$ transformations. As a result, $\Phi_F$ is a linear combi-
nation of all five of the Higgs doublets. Consequently, they are all unstable to decay into quarks and leptons. The four mass-squared eigenvalues of the extra Higgs doublets, and their four mixing angles with $\Phi_F$, altogether 8 quantities, are determined by 9 parameters, so that there are no relations among them. (The 9 parameters are 4 coefficients in $V_{HF}$ and 5 from the form of $\langle \eta \rangle$.)

When the 8 $SU(2)_L$-breaking splittings are included, and the 4 parameters in $V_4$, one has altogether 16 quantities determined by 13 parameters. So, overall, there are 3 relations among masses and mixings.

### 3.6.4 $G_\Phi = SO(5), R_\Phi = 4$.

The group $SO(5)$ has a four-component spinor representation. One can show that if the messenger field is in the spinor, then its VEV can be brought to a form that breaks $SO(5)$ down to $SU(2)$, where $SU(2) \subset SU(2) \times SU(2)' \equiv SO(4) \subset SO(5)$. (This breaks seven generators of $SO(5)$, and the rotations associated with these seven generators allow one to eliminate seven of the eight real parameters in $\langle \eta \rangle$.) Under the unbroken $SU(2)$ subgroup, the 4 Higgs doublets decompose into $1 + 1 + 2 = \Phi^{SM} + \Phi' + \Phi^\alpha$, $\alpha = 1, 2$. One finds that $\Phi_F$ is a mixture of $\Phi'$ and $\Phi_{SM}$, so that $\Phi'$ is a sequential Higgs doublet. $\Phi^\alpha$ is a degenerate pair of inert Higgs doublets. The counting of parameters is easy to do, using the methods used above and the fact that $4 \times 4 = 1 + 5 + 9$. One finds that there are no predictions for the masses or mixing angle.

### 3.7 Other Cases

Here we deal with other cases that involve continuous groups, all of which accommodate six Higgs doublets.

#### 3.7.1 $G_\Phi = SU(3), R_\Phi = 6$.

This case is quite similar to that considered in section 3.6.2 ($SO(3)$ with $R_\Phi = 5$). The 6 of $SU(3)$ is a symmetric rank-2 (traceful) tensor, so that we may write the Higgs and messenger fields as $\Phi^{(\alpha\beta)}$ and $\eta^{(\alpha\beta)}$. Without loss of generality, the messenger VEV can be brought to the form
\[ \langle \eta^{(\alpha\beta)} \rangle = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \]

(29)

where \( a, b, \) and \( c \) are real. \( \Phi_F \propto a\Phi^{(11)} + b\Phi^{(22)} + c\Phi^{(33)} \). Since \( \mathbf{6} \times \mathbf{6} = \mathbf{1} + \mathbf{8} + \mathbf{27} \), there are three independent terms in \( V_{HFB} \):

\[
V_{HFB} = \sigma_1 \Phi_{(\alpha\beta)}^\dagger \Phi^{(\alpha\beta)} \eta^{(\gamma\delta)} \eta^{(\gamma\delta)} + \sigma_2 \Phi_{(\alpha\beta)}^\dagger \Phi^{(\beta\gamma)} \eta^{(\gamma\delta)} \eta^{(\delta\alpha)} + \sigma_3 \Phi_{(\alpha\beta)}^\dagger \Phi^{(\gamma\delta)} \eta^{(\gamma\delta)} \eta^{(\alpha\beta)}. \]

(30)

As in the case discussed in section 3.6.2, one can distinguish the “off-diagonal” and “diagonal” Higgs doublets. The off-diagonal Higgs doublets \( \Phi^{(12)}, \Phi^{(23)}, \) and \( \Phi^{(31)} \) are eigenstates of the mass-squared matrix produced by \( V_{HFB} \) with eigenvalues \( m^2 + \sigma_2(a^2 + b^2), m^2 + \sigma_2(b^2 + c^2), \) and \( m^2 + \sigma_2(c^2 + a^2), \) where \( m^2 \equiv M_0^2 + \sigma_1(a^2 + b^2 + c^2) \). The three diagonal Higgs doublets mix with each other, i.e. \( \Phi_F \) is a linear combination of all of them, so that all of them couple to quarks and leptons and are unstable. The five masses of the extra Higgs doublets and the two mixing angles (of the diagonal Higgs doublets with \( \Phi_F \)), are determined by four parameters \( (\sigma_2a^2, \sigma_3/\sigma_2, b/a, \) and \( c/a) \). There are therefore three relations among them.

As in section 3.6.2, there are discrete symmetries (including the \( P_{23}, P_{12}, P_{31} \), defined there) left unbroken by the messenger VEV. These prevent any of the off-diagonal Higgs doublets from decaying entirely to Standard Model fields. However, the heaviest of the off-diagonal Higgs doublets can decay into the lighter ones plus other particles, if this kinematically allowed. Thus, at least two, and possibly all three of the off-diagonal Higgs doublets have absolutely stable components, depending on the values of parameters.

The potential \( V_4 \), which is responsible for the \( SU(2)_L \)-breaking splittings within the extra Higgs doublets, contains three terms (because the decomposition \( \mathbf{6} \times \mathbf{6} = \mathbf{1} + \mathbf{8} + \mathbf{27} \) contains three terms), of which two contribute to splittings. All of the extra Higgs doublets are \( CPS \)-split. (For example, the scalar and pseudoscalar components of \( \Phi^{(12)} \) are split from each other by such terms as \( \langle \Phi_{(11)}^\dagger \rangle \cdot \Phi^{(12)} \langle \Phi_{(22)}^\dagger \rangle \cdot \Phi^{(12)} + h.c. \).) Altogether, there are ten \( SU(2)_L \)-breaking splittings within the five extra Higgs doublets, and these are determined by coefficients in \( V_4 \), so that there are eight relations among these splittings.
The dark matter consists of at least two, and possibly three, real neutral fields that have different masses.

\[ G_{\Phi} = SU(2) \times SU(2)' \times SU(2)'' \times S_3, \quad R_{\Phi} = (2, 1, 1) + (1, 2, 1) + (1, 1, 2). \]

In this case there are three \( SU(2) \) groups and a permutation symmetry that interchanges them. In an obvious notation, the messenger field can be written \( (\eta^{\alpha}, \eta^{\alpha'}, \eta^{\alpha''}) \), with the indices taking the values 1,2. The messenger VEV can be brought to the form in which

\[
\langle \eta^{\alpha} \rangle = \left( \begin{array}{c} a \\ 0 \end{array} \right), \quad \langle \eta^{\alpha'} \rangle = \left( \begin{array}{c} b \\ 0 \end{array} \right), \quad \langle \eta^{\alpha''} \rangle = \left( \begin{array}{c} c \\ 0 \end{array} \right), \quad (31)
\]

This gives \( \Phi_F \propto a \Phi^1 + b \Phi^1' + c \Phi^1''. \)

The potential \( V_{HFB} \) contains three terms: \( \sigma_1 (\Phi^1_\alpha \cdot \Phi^\alpha \eta^{\alpha*}_\beta \eta^{\beta} + \text{perm.}) + \sigma_2 (\Phi^1_\alpha \cdot \Phi^\alpha \eta^{\alpha'}_\beta \eta^{\beta'} + \text{perm.}) + \sigma_3 (\Phi^1_\alpha \cdot \Phi^\alpha \eta^{\alpha''}_\beta \eta^{\beta''} + \text{perm.}) \). This produces a mass-squared matrix, under which the three Higgs doublets \( \Phi^2, \Phi^{2'}, \) and \( \Phi^{2''} \) are eigenstates. The other eigenstates (which we may call \( \Phi = \Phi_{SM}, \Phi', \) and \( \Phi'' \)) are linear combinations of \( \Phi^1, \Phi^1', \) and \( \Phi^1'' \), which all mix, in general, with \( \Phi_F \) and therefore couple to quarks and leptons and are unstable. There are five mass-squareds of the extra Higgs doublets, and two mixing angles (of the extra sequential Higgs with \( \Phi_F \)), which depend on 5 parameters \( (\sigma_1 a^2, \sigma_2/\sigma_1, \sigma_3/\sigma_1, a/b, c/a) \), giving two relations.

The messenger VEV leaves unbroken discrete symmetries \( P_2, P_2', P_2'' \) (where \( P_2 \) is a reflection in the 2 direction in \( SU(2) \), etc.), which make the lightest components of each of the Higgs doublets \( \Phi^2, \Phi^{2'}, \) and \( \Phi^{2''} \) absolutely stable.

There are three terms in \( V_4 \) (whose form is analogous to that of \( V_{HFB} \)). It can be shown that these cause \( \Phi' \) and \( \Phi'' \) to be \( CPS \)-split, but \( \Phi^2, \Phi^{2'}, \) and \( \Phi^{2''} \) to be \( CN \)-split. There are therefore a total of 7 distinct \( SU(2)_L \)-breaking splittings, which are determined by 3 parameters (the coefficients in \( V_4 \)), to give 4 relations.

The dark matter particles would consist of three charged neutral fields of different masses, namely the neutral components of \( \Phi^2, \Phi^{2'}, \) and \( \Phi^{2''} \).
3.7.3 \( G_\Phi = SU(3) \times SU(2), \ R_\Phi = (3, 2). \)

Denoting the \( SU(3) \) indices by \( \alpha, \beta, \) etc., and the \( SU(2) \) indices by \( \lambda, \mu, \) etc., one can write the Higgs and messenger fields as \( \Phi^{\alpha \lambda}, \eta^{\alpha \lambda}. \) Without loss of generality, one can bring the messenger field VEV to the form

\[
\langle \eta^{\alpha \lambda} \rangle = \begin{pmatrix} \eta & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & \kappa \end{pmatrix},
\]

where \( \eta \) and \( \kappa \) are real. Then \( \Phi_F = \eta \Phi^{11} + \kappa \Phi^{22}. \) Since in \( \Phi^\dagger \eta \eta \), the product \( \eta^\dagger \eta \) must be in \((3,2)^* \times (3,2) = (1,1) + (1,3) + (8,1) + (8,3),\) there are four independent terms in \( V_{HF B}: \)

\[
V_{HF B} = \sigma_1 \Phi^\dagger_{\alpha \lambda} \cdot \Phi^{\alpha \lambda} \eta_{\beta \mu}^* \eta^{\beta \mu} \\
+ \sigma_2 \Phi^\dagger_{\alpha \lambda} \cdot \Phi^{\beta \lambda} \eta_{\beta \mu}^* \eta^{\alpha \mu} \\
+ \sigma_3 \Phi^\dagger_{\alpha \lambda} \cdot \Phi^{\alpha \mu} \eta_{\beta \mu}^* \eta^{\beta \lambda} \\
+ \sigma_4 \Phi^\dagger_{\alpha \lambda} \cdot \Phi^{\beta \mu} \eta_{\beta \mu}^* \eta^{\alpha \lambda}.
\]

The mass-squared matrix produced by this has the following eigenstates: \( \Phi = \Phi_{SM} = \cos \theta \Phi^{11} + \sin \theta \Phi^{22}, \Phi' = -\sin \theta \Phi^{11} + \cos \theta \Phi^{22}, \Phi^{12}, \Phi^{21}, \Phi^{31}, \) and \( \Phi^{32}. \) The five masses of the extra Higgs doublets and the angle \( \theta \) are determined by four parameters \( \{\sigma, \eta^2, i = 1, \ldots, 4, \text{ and } \kappa/\eta\}, \) so that there are two relations.

The fields \( \Phi^{31} \) and \( \Phi^{32} \) are \( CN \)-split, whereas the other extra Higgs doublets are all \( CPS \)-split. Thus there are 8 distinct \( SU(2)_L \)-breaking splittings, which are determined by 3 coefficients in \( V_4, \) giving 5 relations among them.

Left unbroken by the messenger VEV is a discrete symmetry \( P_3 \) that is a reflection in 3 direction of \( SU(3). \) Under this symmetry, \( \Phi^{31} \) or \( \Phi^{32} \) are odd, while all other fields are even. So neither \( \Phi^{31} \) nor \( \Phi^{32} \) can decay entirely into Standard Model particles, and the lighter of the two contains an absolutely stable component (a complex neutral field). The quartic terms \( \Phi^\dagger_{31} \Phi_{31} \Phi^\dagger_{21} \Phi_{21} \) and \( \Phi^\dagger_{32} \Phi_{32} \Phi^\dagger_{11} \Phi_{11} \) allow the heavier of \( \Phi^{31} \) and \( \Phi^{32} \) to decay to the lighter of them plus \( \Phi^{21} \) or \( \Phi^{12} \) or \( \Phi^{12} \) or \( \Phi^{21} \) or \( \Phi^{11} \) plus Standard Model fermions, if this is kinematically possible. If none of these decays are kinematically possible (because \( \Phi^{12} \) and \( \Phi^{21} \) are too heavy) then both \( \Phi^{31} \) and \( \Phi^{32} \) contain absolutely stable complex neutral fields.

The quartic term \( \Phi^\dagger_{11} \Phi^{12} \Phi^\dagger_{22} \Phi^{21} \) allows the heavier of \( \Phi^{12} \) and \( \Phi^{21} \) to decay to the lighter one plus Standard Model fermions. If the lighter one
is sufficiently heavy, then it can in turn decay into $\Phi^{31} + \Phi^{*}_{32} + \text{fermions}$ or into $\Phi^{32} + \Phi^{*}_{31} + \text{fermions}$, via the quartic terms mentioned in the previous paragraph. Otherwise, the lighter of $\Phi^{12}$ and $\Phi^{21}$ contains an absolutely stable field, namely a real neutral field.

In short, depending on the values of the masses, there can be the following absolutely stable particles: (i) two complex neutral fields and a real neutral field, (ii) two complex neutral fields, or (iii) one complex neutral field and a real neutral field.

3.7.4 $G_{\Phi} = SU(2) \times SU(2)$, $R_{\Phi} = (3,2)$.

In this case, the VEV of the messenger field cannot be brought to a simple form, so that it turns out that all the extra Higgs fields mix with $\Phi_F$, and all are unstable to decay to Standard Model fermions. However, there are some relations among the masses and mixings.

4 Discrete Higgs-flavor Groups

So far we have only discussed cases where the non-abelian Higgs-flavor group contains continuous symmetries. Another logical possibility is that the Higgs-flavor group has no continuous factors, but is a discrete group $D_{\Phi}$. (For a discussion of non-abelian discrete symmetries in particle physics, see [6].) In the Appendix, we briefly discuss some of the group theory for all the cases where the order of $D_{\Phi}$ is less than 16 and a few other cases. One finds from this analysis that with $o(D_{\Phi}) < 16$ there are relatively few distinguishable cases. The largest $D_{\Phi}$ multiplets are 3-dimensional. And many of the groups give similar structure to the Higgs potential. Aside from this, discrete Higgs-flavor groups are less interesting than continuous ones for two reasons.

First, if the Higgs-flavor group is discrete, the messenger field VEV cannot in general be brought to a simple form by $D_{\Phi}$ transformations. This in practice means that $\Phi_F$ ends up being a linear combination of all the Higgs doublets. Therefore, all the extra Higgs doublets are unstable to decay to quarks and leptons, and none can play the role of dark matter.

Second, by imposing a discrete symmetry we eliminate various terms from the Higgs potential $V_H$ and impose relations on the coupling constants of many of the others. These constraints often result in additional acci-
dental continuous symmetries of $V_H$. However, since $D_\Phi$ and its associated accidental symmetry $\Delta_\Phi$ is not gauged, the breaking of $\Delta_\Phi$ results in pseudo-Goldstone bosons in the spectrum. Moreover, if discrete symmetries are not protected by being gauged, they are subject to being violated by gravity. The Peccei-Quinn solution to the strong CP problem is such an example where gravity spoils the solution [7].

5 Conclusions

We have shown that there are many simple possibilities for the Higgs-flavor symmetry and representations. Moreover, the various possibilities give quite distinctive spectra and properties for the extra Higgs doublets. Indeed, it seems that hardly any two cases are exactly alike. They differ in the number of extra Higgs doublets there are; how many are “sequential” and how many are “inert”; whether the neutral components of the inert Higgs doublets are split (“CPS splitting”) or not (“CN” splitting); how many of the inert Higgs fields are stable against decay to other Higgs fields; and the group-theoretical relations among the masses of the extra Higgs fields. the phenomenological possibilities are clearly rich, not only for collider physics, but also for dark matter.

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Appendix on discrete Higgs-flavor symmetry

For models with discrete Higgs symmetry $D_\Phi$, it is difficult to work out all the cases of extra Higgs doublets in $N$ dimensional irreps of $D_\Phi$, even when $N$ is small. There are an infinite number of such models, so that in order to make the analysis manageable one needs to restrict the order of $D_\Phi$ or else restrict to cases where the group $D_\Phi$ is in a series like the dihedral groups $D_n$ or the dicyclic groups $Q_n$. These series all have two dimensional irreps for arbitrary $n$, and would lead to Higgs potentials that could easily be classified, but which will not be classified here.

We will therefore consider only cases with $o(D_\Phi) \leq 16$. With $o(D_\Phi) < 16$ the maximum dimension on any irrep is 3. Abelian groups need not be considered, since they only have 1-dim irreps. [It should be noted, however, that 1-dim irreps of non-abelian discrete symmetries may play a role if the discrete group has to be “gauged” and anomaly free [8]). For example, to
embed an $S_3$ model in $SO(3)$ would require a singlet since $3 \to 1' + 2$ when $SO(3) \to S_3$, where the irreps of $S_3$ are $1, 1'$ and $2$. Likewise, to embed, say, $\Delta_{27}$ in $SU(3)$ then singlets can arise in the decomposition of the $SU(3)$ irreps, so that singlets are required to construct anomaly free irreps of $\Delta_{27}$.

It is not too difficult to work out the group products and construct the Higgs potentials for all the non-abelian cases with $o(D_8) < 16$. We will not analyze the models, but make some group-theoretical remarks relevant to classifying the models.

$o(G_\Phi) = 6$

$S_3$ is the only non-abelian group of order 6. This group contains a the irreps $1, 1', 2$. The product $2 \times 2$ has the decomposition $2 \times 2 = 1 + 1' + 2$. There are consequently two ways to (symmetrically) contract $\eta^2$ in the quartic terms of form $\Phi^\dagger \cdot \Phi \eta^2$ in $V_{HFB}$. Since $2 \times 2$ contains a singlet, there is also a cubic term of the form $\Phi^\dagger \cdot \Phi \eta$ in $V_{HFB}$.

There is only one group or order 7 and it is abelian (in fact, all groups of prime order are abelian), so the next cases to consider are at order 8.

$o(G_\Phi) = 8$

There are two non-abelian groups to consider at order 8.

$D_4$: The order 8 dihedral group $D_4$ has four 1-dim irreps $(1, 1', 1'', 1'''$) and a single 2-dim irrep, with $(2 \times 2)_A = 1'$ and $(2 \times 2)_S = 1 + 1'' + 1'''$. This means that there are three symmetric ways to contract $\eta^2$ and thus three terms of the form $\Phi^\dagger \cdot \Phi \eta^2$ in $V_{HFB}$. Since $2 \times 2$ contains no singlets, there is no cubic term in $V_{HFB}$. Similar results hold for the $\Phi^4$ quartic terms.

$Q_4$: The other group at order 8 is the first of the dicyclic groups $Q_4$ (also called the group of unit quaternions). Like $D_4$, $Q_4$ has four 1-dim irreps $(1, 1', 1'', 1''')$ and a single 2-dim irrep, but they have different multiplication rules. However, the only difference is in the products of 1-dim irreps, so since we only consider 2-dim irreps for the $\Phi$ and $\eta$ fields, one gets exactly the same model as for $D_4$.

There are only abelian groups at order 9, so the next cases are at order 10.

$o(G_\Phi) = 10$

$D_5$: This is the only non-abelian group at this order. It has irreps $1, 1', 2, 2'$.
(\(D_5\) will give models very much like \(S_3\), which is not surprising since \(S_3\) is also a dihedral group, i.e., \(S_3 = D_3\).)

If the model has only the 2, then since \((2 \times 2)_S = 1 + 2\), and \((2 \times 2)_A = 1'\), there will be two ways to symmetrically contract \(\eta^2\) and thus two terms of the form \(\Phi^\dagger \cdot \Phi \eta^2\) in \(V_{HFB}\). And since \((2 \times 2 \times 2)_S = 1 + \ldots\) there will be a single cubic term in \(V_{HFB}\).

If the model has \(2'\) instead of the 2, then due to the fact that \((2' \times 2')_S = 1 + 2'\), one obtains the same model.

Since 11 is prime, there are only abelian groups at that order, and the next cases are at order 12.

\(o(G_\Phi) = 12\)

There are three nonabelian groups to consider at this order, \(T\), \(D_6\), and \(Q_6\).

\(T\): The tetrahedral group \(T\) has irreps \(1, 1', 1'', 3\), so the Higgs fields and the messenger field must be in 3. As \((3 \times 3)_S = 1 + 1' + 1'' + 3\) and \((3 \times 3)_A = 3\), \(\eta^2\) can be symmetrically contracted in four ways and there will be four terms of the form \(\Phi^\dagger \cdot \Phi \eta^2\) in \(V_{HFB}\), as well as a single cubic term.

\(D_6\): The irreps are \(1, 1_2, 1_3, 1_4, 2, 2'\). There are two possible models.

(i) With \(\Phi\) and \(\eta\) in the 2, one has, since \((2 \times 2)_S = 1 + 2'\), two ways to symmetrically contract \(\eta^2\) and two terms of the form \(\Phi^\dagger \cdot \Phi \eta^2\) in \(V_{HFB}\). Since \(2 \times 2' = 1_2 + 1_3 + 2\) there is no cubic term in \(V_{HFB}\).

(ii) With \(\Phi\) and \(\eta\) in the \(2'\), one has \(\eta^2\) symmetrically contracted in two ways and there will be two terms of the form \(\Phi^\dagger \cdot \Phi \eta^2\) in \(V_{HFB}\). But now, since \(2 \times 2' = 1_2 + 1_3 + 2\), there is a cubic term.

\(Q_6\): The irreps are \(1, 1_2, 1_3, 1_4, 2, 2'\), but with slightly different multiplication rules. Since, however, the doublet sector multiplications are the same as for \(D_6\), one gets two models identical to the (i) and (ii) models of \(D_6\).

Since 13 is prime, the next case with non-abelian groups is at order 14.

\(o(G_\Phi) = 14\)

\(D_7\): \(D_7\) has irreps \(1, 1', 2_1, 2_2, 2_3\). There are potentially three models, but from the symmetry of the multiplication table they are all equivalent. So consider the \(2_1\) case. Since \(2_1 \times 2_1 = (1 + 2)_S + (1')_A\), there are two ways to symmetrically contract \(\eta^2\) and two terms of the form \(\Phi^\dagger \cdot \Phi \eta^2\) in \(V_{HFB}\). Furthermore, since \((2_1 \times 2_1) \times 2_1 = 2_1 + 2_2 + 2_1 + 2_3\), there are no cubic terms.
The next cases are at order 16.

\[ o(G_\Phi) = 16 \]

\[ (Z_4 \times Z_2) \tilde{\times} Z_2: \] (This group is listed as 16/8 in [9]. Here \( \tilde{\times} \) is a twisted product, as opposed to the direct product \( \times \).)

This group has irreps \( 1_1, 1_2, \ldots, 1_8, 2, 2' \). Again, there is a symmetry in the multiplication making the 2 and \( 2' \) models equivalent. Consider, therefore, the case where \( \Phi \) and \( \eta \) are in 2. One has that \( 2 \times 2 = 1_5 + 1_6 + 1_7 + 1_8 = (1_a + 1_b + 1_c)_S + (1_d)_A \), where \( a, b, c, d \) are 5, 6, 7, 8 in some order that will not be relevant. There are thus three terms of the form \( \Phi \eta \) in \( V_{HFB} \). Furthermore, since \( (2 \times 2) \times 2 = 2 + 2 + 2' + 2' \), there can be no cubic terms in \( V_{HFB} \).

\[ Z_4 \tilde{\times} Z_4: \] (16/10 in the notation of [9].) The irreps are \( 1_1, 1_2, \ldots, 1_8, 2, 2' \). Again, there is a symmetry in the multiplication table and only one independent model. Consider then the 2, for which \( 2 \times 2 = 1_1 + 1_3 + 1_5 + 1_7 = (1_1 + 1_a + 1_b)_S + (1_c)_A \), where \( a, b, c \) are 3, 5, 7 in some order. We see that there will be three terms of the form \( \Phi \eta \) in \( V_{HFB} \). Also since \( (2 \times 2) \times 2 = 2 + 2 + 2' + 2' \), there will be no cubic terms.

\[ Z_8 \tilde{\times} Z_2: \] (16/11 in the notation of [9].) This case is similar to the previous example. The irreps are again \( 1_1, 1_2, \ldots, 1_8, 2, 2' \). A symmetry in the multiplication table leaves only one independent model. Consider then the 2, for which \( 2 \times 2 = 1_2 + 1_4 + 1_6 + 1_8 = (1_a + 1_b + 1_c)_S + (1_d)_A \), where \( a, b, c, d \) are 2, 4, 6, 8 in some order that will not be relevant. We see that there will again be three terms of the form \( \Phi \eta \) in \( V_{HFB} \). And since \( (2 \times 2) \times 2 = 2' + 2' + 2' + 2' \), there will be no cubic terms.

\[ (Z_8 \tilde{\times} Z_2)^h: \] (16/13 in the notation of [9].) This group has irreps \( 1_1, 1_2, 1_3, 1_4, 2_1, 2_2, 2_3 \). There is a symmetry between \( 2_1 \) and \( 2_3 \) so they give the same model, but the \( 2_2 \) is different.

(i) For the \( 2_1 \) case one has \( 2_1 \times 2_1 = 1_2 + 1_3 + 2_2 = (1_a + 2_2)_S + (1_b)_A \), and hence two terms of the form \( \Phi \eta \) in \( V_{HFB} \). Here \( a, b \) are 2, 3. Since \( (2_1 \times 2_1) \times 2_1 = 2_3 + 2_3 + 2_1 + 2_3 \) there are no cubic terms in \( V_{HFB} \).

(ii) For the \( 2_2 \) case one has \( 2_2 \times 2_2 = 1_1 + 1_2 + 1_3 + 1_4 = (1_1 + 1_a + 1_b)_S + (1_c)_A \) where \( a, b, c \) are 1, 2, 3 in some order. We therefore have three terms of the form \( \Phi \eta \) in \( V_{HFB} \). Since \( (2_2 \times 2_2) \times 2_2 = 2_2 + 2_2 + 2_2 + 2_2 \), there are no cubic terms in \( V_{HFB} \).
$D_8$: The irreps are $1_1, 1_2, 1_3, 1_4, 2_1, 2_2, 2_3$. There is a symmetry between $2_1$ and $2_3$ so they give the same model, but the $2_2$ is different.

(i) For the $2_1$ case, one has $2_1 \times 2_1 = (1_1 + 2_2)_S + (1_3)_A$ and hence two terms of the form $\Phi^\dagger \cdot \Phi \eta^2$ in $V_{HFB}$. Since $(2_1 \times 2_1) \times 2_1 = 2_1 + 2_1 + 2_3 + 2_1$, there are no cubic terms in $V_{HFB}$.

(ii) For the $2_2$ case, one has $2_2 \times 2_2 = 1_1 + 1_2 + 1_3 + 1_4 = (1_1 + 1_a + 1_b)_S + (1_c)_A$, where $a, b, c$ are $1, 2, 3$ in some order. There are therefore three terms of the form $\Phi^\dagger \cdot \Phi \eta^2$ in $V_{HFB}$. Since $(2_2 \times 2_2) \times 2_2 = 2_2 + 2_2 + 2_2 + 2_2$, there are no cubic terms in $V_{HFB}$.

We see that these two models are the same as the two $(Z_8 \times Z_2)^{''}$ models.

$Q_8$: $Q_8$ has the same irreps and product table as $D_8$, therefore it generates the same two models as $D_8$ and $(Z_8 \times Z_2)^{''}$.