Gelfand–Tsetlin algebras, expectations, inverse limits, Fourier analysis

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Abstract

This text mainly follows my talk at the conference “Unity of Mathematics” devoted to the 90th anniversary of I. M. Gelfand (Harvard, September 2003). I introduce some new notions that are related to several old ideas of I. M. and try to give a draft of the future development of this area, which includes the representation theory of inductive families of groups and algebras and Fourier analysis on such groups. I also include a few reminiscences about I. M. as my guide.

0 Historical excursus: I. M. Gelfand as my correspondence advisor

The first substantial series of mathematical works that I had studied being a student was the series of papers by Gelfand, Raikov, and Shilov (GIMDARGESH, as I called it to myself) on commutative normed rings and subsequent papers on the generalized Fourier analysis. This theory became a mathematical shock for me, I was struck by its beauty and naturalness, universality and depth.

Before this I hesitated whether I should join the Chair of Algebra — I attended the course of Z. I. Borevich on group theory and the course of D. K. Faddeev on Galois theory — or the Chair of Mathematical Analysis, where my first advisor G. P. Akilov worked; in the latter case I could choose the complex analysis (V. I. Smirnov, N. A. Lebedev) or the real and functional analysis (G. M. Fikhtengolts, L. V. Kantorovich). But now the choice was clear: the functional analysis. At the same time I was more interested in the works of the
Moscow (Gelfand’s) school of functional analysis focused on noncommutative problems than in the works of the Leningrad school that was oriented rather towards the theory of functions and operator theory.

Since then the works of I. M. Gelfand and his school in various fields became a kind of mathematical guidebook for me. My master thesis was devoted to the theory of generalized functions; this topic equally interested the Leningrad mathematicians (L. V. Kantorovich, G. P. Akilov). Later, following G. P. Akilov’s advice, I began to study the representation theory, which was at the time absolutely unrepresented in Leningrad. While my being a postgraduate student, I. M. Gelfand popularized problems concerning measure theory in infinite-dimensional spaces, inspired by the theory of distributions, the notion of generalized random processes, and quantum physics. These problems were communicated to us by D. A. Raikov, who, following I. M. Gelfand’s advice, worked at the new theory of locally convex and nuclear spaces, which we also studied in G. P. Akilov’s seminar. In Leningrad, the measure theory in linear topological spaces was being studied in the late 50s – early 60s by V. N. Sudakov and me. At the time everybody believed that the theory of generalized functions and measure theory in infinite-dimensional spaces would require to overstep the limits of the conventional Banach functional analysis, which would be replaced by the theory of nuclear spaces (Minlos–Sazonov and Gelfand–Kostyuchenko theorems, quasi-invariant measures, etc.). However, soon it became clear that the measure theory in linear spaces is a natural part of the general measure theory, and the Banach analysis continued to be the traditional language of functional analysis. After all, the interest to all these problems gradually died away.

V. A. Rokhlin’s arrival at Leningrad thoroughly changed the mathematical landscape at the Department of Mathematics. In particular, he organized seminars on ergodic theory and topology. V. A. became my principal advisor during my postgraduate studies and several subsequent years. I seriously studied the theory of dynamical systems and general measure theory, and both my dissertations were devoted to these problems. But representation theory continued to fascinate me equally. Even earlier, in his talks on problems of functional analysis at the All-Union Conference on Functional Analysis and the 3rd Mathematical Congress (1956), I. M. spoke about the von Neumann factors and Wiener measure as about subjects that were possibly related and underestimated at the time. Later, in the 60s, I began to study factors and relations of the theory of $C^*$-algebras, introduced
by Gelfand and Naimark, with the theory of dynamical systems; this became the subject of my research for several years.

But for several short discussions with I. M. in the mid and late 60s and the correspondence acquaintance via V. A. Rokhlin (and possibly via Yu. V. Linnik), our close acquaintance took place in the spring 1972. After a session of his seminar, I began to talk to him about my work (joint with A. A. Shmidt) on the limit statistics of cycles of random permutations; and the next day, at his home, about my plans to study the representations of the symmetric groups. Though first he said that with them everything is clear and started to talk enthusiastically about the theory of symmetric functions, but later he agreed that not everything is that clear and advised me to look at the paper by E. Thoma on the characters of the infinite symmetric group, which was of most interest for me. This paper played an important role in our subsequent studies of this group with my pupil S. V. Kerov. One of our principal contributions was an explanation and a new proof of Thoma’s result in terms of representation theory (asymptotics of Young diagrams). And in that conversation I. M. approved wholeheartedly of my ideas, which I later called the asymptotic representation theory; and even when he retold them to D. Kazhdan, who had appeared a little later, he referred to the theorems on the asymptotic behavior of Young diagrams, characters, etc., which were only conjectured at the time (many of them were proved later in joint papers with S. V. Kerov), as to results already obtained. Those of the results I talked about that were already proved related rather to probability theory (Poisson–Dirichlet measures) and theory of dynamical systems than to representation theory. Other groups besides the symmetric groups and their representations were not discussed in those conversations. I took leave of him and was going to depart for Leningrad.

Suddenly, at the day of my departure, I. M., having found out, in a rather complicated way, the phone number of my friends with whom I stayed at Moscow, called me and asked to come to him immediately. He also invited M. I. Graev, and during our long walk told me about the problem of constructing the noncommutative integral of representations for semisimple groups, and especially for SL(2, \( \mathbb{R} \)). He had earlier offered this problem to other his pupils, but he said that he had no doubt that it “fitted” me. I was slightly surprised, because I supposed that he could not know to what extent I was acquainted with the representation theory of Lie groups, and in par-
ticular that of SL(2); as I have mentioned above, we did not discuss these matters at all. But I. M. was right — this problem was offered to me at a very appropriate moment. Several years before this conversation, at the youth seminar organized by L. D. Faddeev and me, we studied Gelfand’s volumes on generalized functions and other useful things, which were not widespread in Leningrad. And in the early 70s, apart from my studies of the ergodic theory, I gave a course and seminars just on the representation theory of groups and algebras, tensor products, and factors. Apparently, I. M. had heard about it, but I did not ask him. Thus his problem appeared at an appropriate moment. We coped with it within several months (the end 1972 — the beginning 1973). The first paper in “Uspekhi” (“Russian Math. Surveys”) appeared in a volume dedicated to Kolmogorov in 1973, and this was the beginning of our collaboration with I. M. and M. I. Graev, which lasted with intervals about ten years and which I am going to describe one day in more detail. That first (the best, in I. M.’s and my opinion) paper of this series was devoted to the “integral” of representations of SL(2, \mathbb{R}) and touched upon many topics that are still actual; in that paper we rediscovered several constructions that had recently appeared (Araki’s Gaussian construction, cohomologies in groups without Kazhdan’s property, etc.), gave the first explicit formulas for nonzero cohomologies of semisimple group of rank 1, and constructed irreducible nonlocal representations of current groups with values in finite-dimensional Lie groups. I. M. repeatedly (and the last time — at this conference (Harvard 2003)) expressed his wish to continue our joint work in this direction. We had no doubt that this series of papers has various applications, which has already been repeatedly confirmed, and the work would be continued.

This paper is devoted to a subject from another line, which also goes back to I. M.’s works. Having worked for many years with inductive families of semisimple algebras, S. V. Kerov and the author at once appreciated the importance of the notion that we called the Gelfand–Tsetlin algebras; this notion is a generalization of the well-known and still popular construction of the Gelfand–Tsetlin bases for the unitary and orthogonal groups. These algebras serve as a basis for the harmonic analysis and Fourier analysis on noncommutative groups. They play an especially important role in the representation theory of locally finite groups, symmetric groups, and, more generally, inductive limits of groups and algebras. Our joint works with
A. Okounkov (see [1] and [2]) show how applying these algebras, and especially a natural basis in them (the Young–Jucys–Murphy basis), allows one to reconstruct the representation theory of the symmetric groups at a completely different basis. In my talk and in this paper I draw attention to yet another idea, closely related to the previous one; namely, to the idea of inverse limits of algebras with respect to conditional expectations. For the symmetric group, this question will be considered in detail in the joint work with N. V. Tsilevich (under preparation). On the other hand, inverse limits of finite-dimensional algebras generalize von Neumann’s theory of complete and noncomplete tensor products [3], and I remember one of my first visits to Gelfand’s seminar in the late 50s, when this von Neumann’s paper was being discussed and commented by the head of the seminar. In this paper I do not touch upon one subject that I mentioned in the talk, namely, the results on representations of the group of infinite matrices over a finite field, which we intensively studied with S. V. Kerov during several last years. It will be considered in other publications under preparation.

1 Definition of a generalized expectation on a subalgebra

Let $A$ be a $C^*$-algebra over $\mathbb{C}$ with involution $*$, and let $B$ be its involution $C^*$-subalgebra. All algebras in the paper are supposed to be algebras with identity, and all subalgebras are supposed to contain this identity. Here we mainly consider finite-dimensional algebras, but the definitions below are valid for the general case.

**Definition 1.** A linear operator $P : A \to B$ is called a conditional mathematical expectation, or expectation\footnote{The word “conditional” is the traditional one, but I prefer to omit it below, as well as the word “mathematical,” violating the old tradition. The reason is that the “unconditional expectation” is simply the “conditional expectation” onto the algebra of scalars $\mathbb{C}$ (conditions are trivial), thus if we fix a subalgebra $B$, we do not need to use the word “conditional,” because it is clear what are the “conditions.”} for short, of the algebra $A$ onto the subalgebra $B$ if

1. $P(b) = b$ and $P(b_1ab_2) = b_1P(a)b_2$ for all $a \in A$ and $b, b_1, b_2 \in B$.
2. $P(a^*) = a^*$, $P1 = 1$.

and
3. $P \geq 0$, which means that for all $a \in A$, $P(aa^*)$ is positive, i.e., belongs to the real cone in $B$ generated by the elements of the form $bb^*$.

We will say that $P$ is a generalized expectation if only the first and second conditions hold, and $P$ is a true expectation, or expectation, if condition 3 also holds.

The notion of (“conditional”!) expectation is well known and has been used in many situations; for commutative algebras, it coincides with the ordinary notion of (mathematical) conditional expectation on sigma-subfield or subalgebra. A fruitful example of generalized, i.e., nonpositive expectation appeared, I believe, only recently, in the very concrete situation of the group algebra of the symmetric group (see below), and this is the reason for considering this notion in full generality. Sometimes people require that an expectation $P$ should be not only positive but even totally positive, but we will not put emphasis on this.

Note also that it is clear from definition that the set of all expectations in an algebra $A$ to a subalgebra $B$ is a convex set.

In the main part of the paper, our attention will be focused not on a single generalized expectation for some pair $B \subset A$, but on sequences of generalized expectations in an inductive family of algebras.

It is not difficult to describe all expectations for finite-dimensional semisimple $\mathbb{C}^*$-algebras over $\mathbb{C}$, which are the sums of several copies of full matrix algebras $M_n(\mathbb{C})$, as well as to describe generalized conditional expectations for these algebras. Recall that for a general pair $(A, B)$, where $A = \sum_{j=1}^{m} A_j$ is a finite-dimensional $\mathbb{C}^*$-algebra, $B = \sum_{i=1}^{k} B_i$ is its $\mathbb{C}^*$-subalgebra, and $A_j = M_{k_j}(\mathbb{C})$, $j = 1, \ldots, m$, $B_i = M_{n_i}(\mathbb{C})$, $i = 1, \ldots, k$, are their decompositions into simple algebras, one can define a bipartite multigraph in which the first (upper) part of vertices is indexed by the subalgebras $B_i$, $i = 1, \ldots, k$, and the second (lower) part of vertices is indexed by the subalgebras $A_j$, $j = 1, \ldots, m$, and the multiplicity of an edge $(i, j)$ is equal to the number of copies of the subalgebra $B_i$ as a subalgebra of $A_j$. We will use this construction in the below theorem (claim 2). For the sake of clarity, we consider the multiplicity-free case when each $B_i$ belongs to at most one $A_j$; a pair $(i, j)$ is called admissible if it is an edge, or $B_i \subset A_j$. In order to determine the pair $(A, B)$ uniquely up to isomorphism, we must fix this bipartite multigraph and positive integers in each upper vertex (the dimensions of $B_i$).
Theorem 1. 1. First suppose that $A = M_n(C)$ and its subalgebra $B$ is also a full matrix algebra $B = M_m(C)$ (that is, the multigraph reduces to two vertices and one edge). Then there exists a unique expectation $P(a) = pap,$ where $a \in A$ and $p$ is the natural orthogonal projection determined by the identity of the algebra $B.$

2. Suppose that $A$ is a finite-dimensional semisimple algebra and $B$ is its semisimple subalgebra as above. Then every conditional expectation $P: A \rightarrow B$ is the sum

$$P = \sum_{i,j} P_{i,j}$$

over all admissible pairs $(i, j)$ of generalized expectations from claim 1: $P_{i,j}: A_j \rightarrow B_i, P_{i,j}(a) = \lambda_{i,j}p_{i,j}a p_{i,j},$ where $\lambda_{i,j}$ are real numbers (for a true expectation, nonnegative real numbers) such that $\sum_j \lambda_{i,j} = 1$ for every $i.$

The proof of claim 1 is obvious; in order to prove claim 2, it suffices to separate the restrictions of $P$ to each $A_j$ by linearity of $P$ and then apply claim 1 and condition 2 from the definition of expectation ($P1 = 1$).

Thus a real matrix $\{\lambda_{i,j}\}$ satisfying the condition $\sum_j \lambda_{i,j} = 1$ for every $i$ is a parameter on the set of generalized conditional expectations for a fixed semisimple finite-dimensional algebra and its subalgebra; for true expectations, we have an additional condition $\lambda_{i,j} \geq 0,$ and $\{\lambda_{i,j}\}$ is a Markovian matrix on the bipartite graph. For this reason, in the general case we will say that the matrix $\lambda_{i,j}$ is a generalized Markovian matrix. It is clear that the set of (generalized) expectations for a finite-dimensional pair $B \subset A$ is always nonempty.

The conjugate operator to a generalized expectation $P$ is an operator $P^*$ from the space $A^*$ conjugate to $A$ to $B^*.$ If $P$ is a true (positive) expectation, then $P^*$ maps each state (= positive normalized functional) on $B$ to some state on $A.$ But since $P$ is not a homomorphism of algebras, it does not map traces (characters) to traces. We may consider more refined properties of expectations in regard to this fact, e.g., call an expectation central if the image of each trace is a trace, etc. We will not discuss this topic here.

The following natural question arises. Suppose that $P_B$ is an expectation for a pair of finite-dimensional algebras $A, B.$ Let us regard $A$ as a vector space. The problem is to describe the $*$-algebra $E = \langle A, P_B \rangle$ generated by the left action of $A$ and $P_B$ in $\text{END}(A).$
We give the answer to this question in terms of the decomposition of \( E \) into simple algebras.

**Theorem 2.** Let \( \Gamma(L_B, L_A) \) be the bipartite graph corresponding to the pair \((A, B)\), where \( L_A \) (\( L_B \)) are the vertices of \( \Gamma \) corresponding to the decomposition of \( A \) (\( B \)), respectively. Then the diagram of the triple of algebras \((B \subset A \subset E)\) is the graph \( \Gamma(L_B, L_A, L_E) \), where the bipartite part \( \Gamma(L_A, L_E) \) is the reflection of \( \gamma(L_B, L_A) \), which means that \( L_D \equiv L_B \) and the edges between the vertices of \((L_A, L_E)\) are the same as the corresponding edges in \( \Gamma(L_B, L_A) \). This means, in particular, that the algebra \( E = \langle A, P_B \rangle \) does not depend on the choice of the expectation \( P_B \), but only on the subalgebra \( B \) itself, so that we can denote it by \( E(A, B) \).

The proof of this theorem uses Theorem 1 (the structure of expectations); we will not give examples and details here.

## 2 Two classes of examples for group algebras

For the group algebras (over \( \mathbb{C} \)) of finite groups, we present two types of expectations related to the group structure. Since a linear map in the group algebra is determined by its values on the group, we can state the question in terms of the group.

1. The first type of examples relates to the case when the value of the expectation at an element of the group (regarded as a subset of the group algebra) belongs to the group again.

   In this case we can formulate a pure group-theoretical question concerning a group analog of expectation.

   Assume that \( G \) is a finite group and \( H \) is its subgroup. When there exists a map \( p \) from \( G \) onto \( H \) such that

   \[
   p(h) = h, \quad p(h_1gh_2) = h_1p(g)h_2, \quad p(e_G) = e_H
   \]

   for all \( h, h_1, h_2 \in H \) and \( g \in G \), where \( e_H \) and \( e_G \) are the identity elements in \( G \) and \( H \), respectively?

   If such a map \( p \) exists, we say that it is a *virtual projection* of the group \( G \) to the subgroup \( H \).

   **Theorem 3.** The following two conditions are equivalent:
1. There exists a virtual projection $p : G \to H$.

2. There exists a set $K \subset G$ such that
   
   a) $K$ is invariant under the inner automorphisms generated by the elements of $H$, that is, for every $h \in H$, for every $k \in K$, $hkh^{-1} \in K$;

   b) the intersection of the set $K$ with any left (equivalently, right) coset of $H$ in $G$ has only one element; in other words, for all $k, k' \in K$, $k \neq k'$, we have $k^{-1}k' \notin H$.

Proof. The proof is straightforward, and we only supplement it with some comments. Condition b) means that the group $G$ can be partitioned into left cosets of the subgroup $H$, each of them containing exactly one element of the set $K$; thus $G \cong H \times K$, and for every $g \in G$ there is a unique left decomposition $g = hk$ with $h \in H$, $k \in K$; condition a) gives the right decomposition with the same $h$ but another $k' \in K : g = k'h$. We assert that there is a bijection between the set of all virtual projections $p : G \to H$ and the set of all subsets $K$ that satisfy these conditions. Namely, if $K$ enjoys properties a), b) above, then the corresponding virtual projection $p$ is given by the formula

$$p(g) = h$$

for the element $g = hk = k'h$; and vice versa: if $p$ is a virtual projection, then the set $K = p^{-1}(e_H) \subset G$ enjoys properties a), b).

Remark 1. It is clear from construction that the set $K$ is the union of orbits of the group of inner automorphisms of $H$. If $O$ is one of such orbits in $K$, then its characteristic function commutes with $H$. In the case of the symmetric group, $K$ is a single orbit.

Remark 2. The set $K$ above can be described in the following terms (E. Vinberg’s observation): $\bar{K} = \{ k \in G : k \text{ belongs to the center of the group } H \cap k^{-1}Hk \}$. Then our $K$ is a subset of $\bar{K}$, which is $H$-invariant and intersects each left (and, automatically, right) coset of the subgroup $H$ at one point.

For different groups, it may happen that such a set $K$ either is nonunique, or does not exist at all.

It is an interesting question for what pairs $H \subset G$ a virtual projection (in terms of Theorem 3, a set $K$ with properties a), b)) does exist. In the trivial example $G$ is the direct product of two groups: $G = H \times K$. 

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As a nontrivial example, consider the symmetric groups $G = S_n$ and $H = S_{n-1}$ with the ordinary embedding; then $K$ is the set of transpositions $(i, n)$, where $i$ runs over $1, 2, \ldots, n$. The map $p : G \to H$ determined by this decomposition is a virtual projection; it simply deletes the element $n$ from a permutation. This projection was defined in [1] (see also [5]) and called the virtual projection. It is easy to check that for $n > 4$, the virtual projection and the corresponding set $K$ are unique; for $n = 3, 4$, there are several possibilities to choose such a set $K$.

Let us extend a virtual projection by linearity to an operator $P$ in the group algebra:

$$P : C(G) \rightarrow C(H) \subset C(G).$$

**Lemma 1.** The linear operator $P$ defined above is a generalized expectation of the algebra $C(G)$ to $C(H)$ in the sense of Definition 1.

An important remark: in general, the generalized expectation $P$ does not satisfy the positivity condition 3 from Definition 1; for example, in the case of the symmetric group (see above), this operator is not positive, because, e.g., the signature of a permutation can change under this projection. Thus $P$ is not an expectation, but a generalized expectation.

Thus we have defined a particular class of generalized expectations on group algebras, which arise from virtual projections on groups. A very interesting problem is to describe pairs $(G, H)$ for which a virtual projection, and hence the corresponding generalized expectation on the group algebra, does exist. For an abelian group, it is easy to describe all virtual projections (they exist for all pairs $(G, H)$ and determine true expectations), but even for metabelian groups, I do not know the answer.

For some classes of groups, such as free groups, “local groups” (see [6]), Coxeter groups, presumably the following recipe works: suppose that $G_n = \langle \sigma_1, \sigma_2, \ldots, \sigma_n \rangle$ and $G_n \supset G_{n-1} = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \rangle$. There exists a normal form of each element of $G_n$ as a word in the alphabet $\sigma_1, \ldots, \sigma_n$ such that the deletion of the letter $\sigma_n$ in this normal form is a virtual projection of $G_n$ onto $G_{n-1}$. This is true for free, locally free, and symmetric groups (such a normal form does exist).

2. The second type of examples is closer to the classical definitions, because it leads to true (positive) expectations. Again let $G$ and $H$ be
a finite group and its subgroup, respectively; now we allow the values of expectations at the elements of the group not only to belong to the group, but also to be equal to zero. Define a projection

\[ P : C(G) \longrightarrow C(H) \subset C(G) \]

as follows: \( P \) is the linear extension to the whole group algebra of the following map on the group: \( P(h) = h \) for all \( h \in H \), and \( P(g) = 0 \) if \( g \in G, g \notin H \). This definition makes sense for an arbitrary group and its subgroup. Obviously, \( P \) is a (positive) expectation. For some reason, we call it the Plancherel expectation. This definition leads, in particular, to the Fourier analysis on the symmetric groups, which will be the subject of the joint paper with N. Tsilevich, which is now in preparation.

It is easy to formulate the analog of Lemma 2 for algebras: the set of all generalized expectations \( P : A \longrightarrow B \) is in a one-to-one correspondence with the set of subspaces \( T \) of \( A \) satisfying the following properties:

1. \( T \) is a closed complement to the subspace \( B \) of the vector space \( A \);
2. \( B T B \subset T \).

The correspondence is as follows: \( T = \ker P \).

Because of the convexity of the set of expectations, we can consider convex combinations of these two types of examples. For the symmetric group, such deformations are related to the content of the papers [4, 5].

### 3 Gelfand–Tsetlin (GZ-) algebras

Now we introduce the central notion of the theory of inductive families of algebras (not only finite-dimensional). This notion follows the idea of the classical papers by Gelfand and Tsetlin [7, 8], in which a particular basis was defined for the orthogonal \( SO(n) \) and unitary \( SU(n) \) groups. This basis appears only if we consider not just one group, say \( SO(n) \) or \( SU(n) \), but the whole inductive family \( SO(2) \subset SO(3) \subset \cdots \subset SO(n) \) or \( SU(1) \subset SU(2) \subset \cdots \subset SU(n) \) simultaneously. Since the restrictions of irreducible representations of the group \( SO(n) \) to the subgroup \( SO(n-1) \) (and similarly with \( SU \)) are multiplicity-free, this inductive family determines a basis.
(Gelfand–Tsetlin basis), which is unique up to scalar multipliers (see below). But even more important is the notion of Gelfand–Tsetlin algebras, which was introduced for a general inductive family of algebras in our papers with S. Kerov (a detailed exposition is given in [9]) and independently, but not in the same spirit, in [10]. I do not know any papers about the Gelfand–Tsetlin algebras even in the classical case (that of the universal enveloping algebras of semisimple Lie algebras) apart from the paper [11], which concerns a completely different problem. The most important problem is to define reasonable multiplicative generators of the Gelfand–Tsetlin algebras in terms of the initial algebras; having such generators, one can create the representation theory of the inductive family of algebras in a very natural way. The realization of this plan allows one to define an analog of the Fourier transform for algebras with inductive family of subalgebras inside it. For the symmetric group, these generators were defined (independently of GZ-algebras) by A. Young and in more recent times by Jucys and Murphy (YJM-generators). The consistent development of the representation theory of the symmetric groups was given in [1, 2]. For other groups (even for the orthogonal and unitary groups), this is still not done. Below we consider only complex $\ast$-representations of algebras over $\mathbb{C}$.

**Definition 2 (Gelfand–Tsetlin algebra).** Suppose we are given a finite or infinite family $A_k$, $k = 0, \ldots, n$ (here $n$ can be finite or infinite), of semisimple algebras over $\mathbb{C}$, $A_0 = \mathbb{C}$, $A_k \subset A_{k+1}$. Assume for the sake of clarity that the multiplicity of the restriction of an irreducible representation of $A_k$ to $A_{k-1}$ for $k = 1, \ldots, n-1$ is equal to one or zero (the so-called simple spectrum). By definition, the Gelfand–Tsetlin algebra $GZ_n$ is the algebra generated by the centers, which we denote by $\zeta(A_k) \subset A_k$, $k = 0, \ldots, n$:

$$GZ_n = \langle \zeta(A_1), \ldots, \zeta(A_n) \rangle$$

(the notation $\langle \ldots \rangle$ stands for the subalgebra of $A_n$ generated by the contents of the brackets).

It is clear from this definition that all $GZ_k$ are abelian algebras and the family of algebras $\{GZ_k\}_{k=1}^n$ is an inductive family of subalgebras in $A_n$ (the centers do not form an inductive family); the definition and the assumption on the simplicity of the spectrum also imply that $GZ_n$ is a maximal abelian subalgebra of $A_n$. Moreover, from the definition we can conclude that there is a particular basis (defined up to scalars)
in the algebra $GZ_n$, which we call the GZ-basis; and, consequently, there is a particular basis in each irreducible representation of $A_n$ — this is what people usually called the Gelfand–Tsetlin basis. In the case of the groups $SO(n)$ and $SU(n)$, this is just the classical Gelfand–Tsetlin basis [7, 8]. It leads to the well-known notion of Gelfand–Tsetlin patterns.

The elements of the GZ-basis of the algebra $GZ_n$ in the general case are defined as such elements that each of them has a nonzero image in only one irreducible representation. All such elements are defined uniquely (up to scalar). We may say that there is a bijection between this basis and paths in the graph of the Bratteli diagram of the algebra $A_n$ (see below). As we have mentioned above, a nontrivial problem is to describe the GZ-algebra, as well as the GZ-basis, using some multiplicative generators of $GZ(A_n)$, not in terms of representations, but in intrinsic terms of the initial definition of the algebras $A_n$ (or groups in the case when $A_n$ is a group algebra). This problem leads to what we called the Fourier analysis of inductive families of algebras (groups).

We want to emphasize that the notion of $GZ_n$-subalgebra of an algebra $A_n$ does depend on the structure of the inductive family $A_i$, $i = 1, \ldots, n$, and not only on the algebra $A_n$ itself; so if we choose another inductive family inside $A_n$, then $GZ_n$ also can change. The development of these ideas for the symmetric groups can be found in [1, 2]. The assumption on the simplicity of the spectrum is supposed to be satisfied in all further considerations.

The analysis of examples of Gelfand–Tsetlin algebras in the case of groups, and especially of the $GZ_n$ subalgebras of $C(S_N)$, allows us to formulate the following theorem.

**Theorem 4.** Suppose that $G_1 \subset G_2 \subset \cdots \subset G_n$ is a finite sequence of finite groups. Suppose that the restriction of irreducible representations of $G_k$ to $G_{k-1}$, $k = 1, \ldots, n$, is multiplicity-free and there exists a virtual projection of $G_k$ to $G_{k-1}$, $k = 1, \ldots, n$. Then the family of sets $\{X_k = \ker P_k, \ k = 1, \ldots, n\}$ generates (as multiplicative generators) the subalgebra $GZ_n$: here $P_k$ is the generalized expectation $C(G_k) \to C(G_{k-1})$ corresponding to the virtual projection $p_k : G_k \to G_{k-1}$ (see the previous section).

**Proof.** Using Remark 1 after Theorem 2, we can prove that the center of $C(G_k)$ belongs to the algebra generated by $GZ_{k-1}$ and the set $X_k$. \[\square\]
In the case of the symmetric group, the set $X_k$ is determined by the YJM-elements.

4 The inverse limit of an inductive family of algebras and GZ-algebras, and martingales

Suppose now we have a countable sequence $A_n$, $n = 0, 1, \ldots, A_0 = \mathbb{C}$, $A_n \subset A_{n+1}$, of $C^*$-algebras that form an inductive family of algebras and define the inductive limit

$$A_\infty = \lim \text{ind } A_i$$

with respect to the embedding of algebras.

In the same spirit we can define the inductive limit of the Gelfand–Tsetlin algebras

$$GZ_\infty = \lim \text{ind } GZ_n;$$

under our assumptions, it is again a maximal abelian subalgebra of $A_\infty$.

Using Theorem 4 from the previous section, we can define multiplicative generators of $GZ_\infty = \lim \text{ind } GZ_n$ for the case of group algebras. In particular, this gives a description of a multiplicative basis for the $GZ$-algebra of the infinite symmetric group.

An inductive family $\{A_n\}$ of finite-dimensional algebras determines a $\mathbb{Z}_+$-graded graph $Y$ (the Bratteli diagram). The vertices of level $n \geq 0$ correspond to the simple subalgebras of the algebra $A_n$ (at the zero level we have one vertex $0$), and two adjacent levels $Y_n$ and $Y_{n+1}$ form precisely the bipartite graph that was mentioned in Sec. 1. The set of all maximal paths (finite if the number of algebras is finite, or infinite) from the vertex $0$ to the end is called the set of tableaux and denoted by $T(Y)$ (recall that a path is a sequence of edges, and in the multiplicity-free case a path is also a sequence of vertices). Now let us choose a sequence of generalized expectations of these algebras at each level:

$$P_n : A_n \rightarrow A_{n-1}, \quad n = 1, 2, \ldots$$

Lemma 2. The restriction of the generalized expectation $P_n$ to the Gelfand–Tsetlin algebra $GZ_n$ sends it to $GZ_{n-1}$; thus this restriction is an expectation of $GZ_n$ to $GZ_{n-1}$. 
Proof. Each expectation sends the center of the algebra onto the center of the subalgebra: \( P_n(\zeta(A_n)) = \zeta(A_{n-1}) \). Indeed, let \( z \in \zeta(A_n) \) and \( b \in A_{n-1} \); then \( P_n(zb) = P_n(z)b = P_n(bz) = bP_nz \). At the same time \( P_n(\zeta(A_{n-1})) = \zeta(A_{n-1}) \). Consequently, \( P_n(GZ_n) = GZ_{n-1} \) by definition.

Now let us define the projective limit

\[
A^\infty = \limproj\{A_n, P_n\}
\]

with respect to the sequence of generalized expectations. It is obvious from definition that the following lemma holds.

**Lemma 3.** \( A^\infty \) is a left and right \( A_\infty \)-bimodule (but not an algebra in general).

Indeed, all algebras \( A_n \) act from the left and from the right on all \( A_m, m > n \); thus these actions extend to the projective limit. This definition makes sense for a general inductive family with an arbitrary system of expectations.

By Lemma 3, we can also correctly define the inverse (projective) limit of the algebras \( \{GZ_n\} \):

\[
GZ^\infty = \limproj\{GZ_n, P_n\}.
\]

This is not an algebra either, but a module over \( GZ_\infty \). The interpretation of this limit will be given below.

Suppose now that all algebras \( A_n, n = 1, 2, \ldots \), are finite-dimensional semisimple algebras. Since (generalized) expectations are determined by systems of (generalized) Markovian matrices, the projective module is determined by the system of matrices \( \Lambda_n, n = 1, 2, \ldots \), where \( \Lambda_n \) determines the expectation of \( A_n \) to \( A_{n-1} \).

Let us fix such a system of generalized (or true) Markovian matrices \( \Lambda_n, n = 1, 2, \ldots \). The size of \( \Lambda_n \) is \( m_n \times m_{n-1} \), where \( m_k \) is the number of simple subalgebras in the algebra \( A_k \). We denote this system of matrices by \( \mathbf{L} = \{\Lambda_n, n = 1, 2, \ldots \} \), and in order to emphasize the dependence of the projective limit on the expectations, we will sometimes write

\[
A^\infty_{\mathbf{L}} = \limproj\{A_n, P_n\}
\]

and

\[
GZ^\infty_{\mathbf{L}} = \limproj\{A_n, P_n\}.
\]
In the case of abelian algebras, as well as in the case of GZ-algebras, such an inverse limit is well known by another name, at least when all matrices $\Lambda_n$ are true Markovian matrices. We will shortly explain this link. First of all, as usual, the system of Markovian matrices $L$ determines a Markov measure $\mu_L$ on the space of tableaux $T(Y)$ (see above). Thus we have a measure space (or precisely, a Lebesgue space) $(T(Y), A_{\mu_L})$, where $A$ is the sigma-field generated by elementary cylindric sets (elementary cylindric set of order $n$ is the set of all paths with common fragment of length $n$). Second, in $A$ we have an increasing sequence of finite sigma-subfields of cylindric sets of order $n$. Following the general definition of martingales, we can now define the vector space $M_L$ of martingales over this increasing sequence of sigma-subfields, each of them being a sequence $\{f_n\}_n$ of measurable functions such that $f_n$ is $A_n$-measurable and the expectation of $f_n$ on the sigma-field $A_{n-1}$ is equal to $f_{n-1}$.

It is clear from definition that this space of martingales is exactly the inverse limit $GZ_L^\infty$ defined above.

This is the reason for calling the elements of the inverse limit $A_L^\infty$ of algebras noncommutative martingales. This opens a wide range of generalizations of the martingale theory to this noncommutative case. If we have a generalized expectation, then we need to consider martingales with respect to non-positive measures, which, as far as I know, were never considered.

In the group case there is a distinguished Markov measure — the so-called Plancherel measure on the space of tableaux $T(Y)$; namely, if $G = \lim \text{ind } G_n$ is a locally finite group with simple spectrum (like $S_\infty = \lim \text{ind } S_n$), then, using one of the expectations defined in the previous section, we obtain the Plancherel measure on $T(Y)$, which is the inverse limit of the Plancherel measures on the spaces of finite tableaux. Martingales with respect to the Plancherel measure play an important role as a special kind of modules over the group algebras of the group $G$.

Our last remark concerns the link with von Neumann’s theory of infinite tensor products: if our algebra $A_\infty$ is the infinite tensor product of algebras of matrices (e.g., of order two), the so-called Glimm algebras, then each incomplete tensor product of Hilbert spaces in the sense of [3] is generated by the inverse limit of algebras with respect to some sequence of expectations. In this spirit, the scheme of this section allows us to generalize von Neumann’s theory to an arbitrary inductive limit of finite-dimensional algebras instead of Glimm algebras.
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