Existence of weak solutions for unsteady motions of micro-polar electrorheological fluids

E. Bäumle\textsuperscript{a}, M. Růžička\textsuperscript{b,∗}

\textsuperscript{a}Blütenweg 12, D-77656 Offenburg, GERMANY
\textsuperscript{b}Institute of Applied Mathematics, Albert-Ludwigs-University Freiburg, Eckerstr. 1, D-79104 Freiburg, GERMANY.

Abstract

In this paper we study the existence of weak solutions to an unsteady system describing the motion of micro-polar electrorheological fluids. The constitutive relations for the stress tensors belong to the class of generalized Newtonian fluids. Using the Lipschitz truncation and the solenoidal Lipschitz truncation we establish the existence of global solutions for shear exponents \( p > \frac{6}{5} \) in three-dimensional domains.

Keywords: Existence of solutions, Lipschitz truncation, solenoidal Lipschitz truncation, micro-polar electrorheological fluids.

1. Introduction

In this paper we investigate the existence of solutions of the system\textsuperscript{1}

\[
\begin{align*}
\partial_t \mathbf{v} + \text{div} (\mathbf{v} \otimes \mathbf{v}) - \text{div} \mathbf{S} + \nabla \pi &= \mathbf{f} \quad \text{in } \Omega_T, \\
\text{div} \mathbf{v} &= 0 \quad \text{in } \Omega_T, \\
\partial_t \mathbf{\omega} + \text{div} (\mathbf{\omega} \otimes \mathbf{v}) - \text{div} \mathbf{N} &= \ell - \varepsilon : \mathbf{S} \quad \text{in } \Omega_T,
\end{align*}
\]

(1.1)

completed with homogeneous Dirichlet boundary conditions

\[
\mathbf{v} = \mathbf{0}, \quad \mathbf{\omega} = \mathbf{0} \quad \text{on } I \times \partial \Omega,
\]

(1.2)

and initial conditions

\[
\mathbf{v}(0) = \mathbf{v}_0, \quad \mathbf{\omega}(0) = \mathbf{\omega}_0 \quad \text{in } \Omega.
\]

(1.3)

Here \( \Omega \subset \mathbb{R}^3 \) is a bounded domain and \( I = (0, T) \) with \( T \in (0, \infty) \) a given finite time interval. The three equations in (1.1) are the balance of momentum, mass and angular momentum for an incompressible, micro-polar electrorheological fluid. In these equations \( \mathbf{v} \) denotes the velocity, \( \mathbf{\omega} \) the micro-rotation, \( \pi \) the pressure, \( \mathbf{S} \) the mechanical extra stress tensor, \( \mathbf{N} \) the couple stress tensor, \( \ell \) the

\textsuperscript{*}Corresponding author

Email addresses: Erik.Baeumle@gmx.de (E. Bäumle),
rose@mathematik.uni-freiburg.de (M. Růžička)

\textsuperscript{1}We denote by \( \varepsilon \) the isotropic third order tensor and by \( \varepsilon : \mathbf{S} \) the vector having the components \( \varepsilon_{ijk} S_{jk}, i = 1, \ldots, d \), where the summation convention is used.

October 2, 2015
electromagnetic couple force, \( f = \tilde{f} + \chi^E \text{div}(E \otimes E) \) the body force, where \( \tilde{f} \) is the mechanical body force, \( \chi^E \) the dielectric susceptibility and \( E \) the electric field. The electric field \( E \) solves quasi-static Maxwell’s equations
\[
\begin{align*}
\text{div} E &= 0 \quad \text{in } \Omega_T, \\
\text{curl} E &= 0 \quad \text{in } \Omega_T, \\
E \cdot n &= E_0 \cdot n \quad \text{on } I \times \partial \Omega,
\end{align*}
\] (1.4)

where \( n \) is the outer normal vector of the boundary \( \partial \Omega \) and \( E_0 \) is a given time-dependent electric field. The model (1.1)-(1.4) is derived in [10]. It contains a more realistic description of the dependence of the electrorheological effect on the direction of the electric field compared to the previous model in [21], [23]. Nevertheless, we concentrate in this paper on the investigation of the mechanical properties of electrorheological fluids governed by (1.1). This is possible due to the fact that Maxwell’s equations (1.4) are separated from the balance laws (1.1) and that there exists a well developed existence theory for Maxwell’s equations. Thus, we will assume throughout the paper that an electric field \( E \) with appropriate properties is given (cf. Assumption 2.15).

A representative example for a constitutive relation for the stress tensors in (1.1) reads (cf. [10], [23])
\[
S = (\alpha_{31} + \alpha_{33}|E|^2)(1 + |D|)^{p-2}D + \alpha_{51}(1 + |D|)^{p-2}(DE \otimes E + E \otimes DE) \\
+ \alpha_{71}|E|^2(1 + |R|)^{p-2}R + \alpha_{91}(1 + |R|)^{p-2}(RE \otimes E + E \otimes RE),
\]
\[
N = (\beta_{31} + \beta_{33}|E|^2)(1 + |\nabla \omega|)^{p-2}\nabla \omega \\
+ \beta_{51}(1 + |\nabla \omega|)^{p-2}(\nabla \omega)E \otimes E + E \otimes (\nabla \omega)E,
\] (1.5)

with constants \( \alpha_{31}, \alpha_{33}, \alpha_{71}, \beta_{31}, \beta_{51} > 0 \) and \( \beta_{31} \geq 0 \). The constants \( \alpha_{51}, \alpha_{91}, \beta_{51} \) have to satisfy certain restrictions (cf. [10], [23]), which ensure the validity of the second law of thermodynamics. In (1.5) we used the notation\(^2\) \( D = (\nabla v)^{sym}, \)
\( R = Wv + \varepsilon \cdot \omega, \) with \( Wv = (\nabla v)^{skew} \). In the present paper we refrain from considering concrete constitutive relations for the stress tensors, but we make general assumptions covering prototypical situations (cf. Assumption 2.6 and Assumption 2.10).

Micro-polar fluids have been introduced by Eringen in the sixties (cf. [11] for an exhaustive treatment). Electrorheological fluids can be modelled in various ways, see e.g. [1], [22], [26], [21], [10]. While there exists many investigations of micro-polar as well as of electrorheological fluids (cf. [17], [23]), there exists to our knowledge no investigations of micro-polar electrorheological fluids except [13], which is based on the PhD thesis [12] and the diploma thesis [25]; and the diploma thesis [2]. The present paper is based on the latter thesis.

In the next section we introduce the notation, the functional setting, give assumptions for the stress tensors and collect some auxiliary results. In particular, the properties of the solenoidal unsteady Lipschitz truncation are stated and a generalization of the unsteady Lipschitz truncation is discussed. In Section 3 we present the analysis of our problem in the context of pseudomonotone operator theory, which applies for shear exponents \( p \geq 11/5 \). With the same

\(^2\)Here \( \varepsilon \cdot \omega \) denotes the tensor with components \( \epsilon_{ijk}\omega_k, i, j = 1, \ldots, d. \)
tools we construct approximate solutions in the more interesting case $p < 11/5$ in Section 4. Using the different Lipschitz truncations we prove our main result in Section 5.

2. Preliminaries

2.1. Notation and function spaces

We denote by $c$ generic constants, which may change from line to line. Scalar-valued functions will be written in normal font, e.g., $f, \zeta$ while vector-valued functions will be denoted by boldfaced letters, e.g., $u, \varphi$. Capital boldface letters will be used for tensor-valued functions\footnote{The only exception will be the electric field which is denoted as usual by $E$.}, e.g., $S$. The standard scalar product for vectors is denoted by $\mathbf{v} \cdot \mathbf{w}$, while the standard scalar product for tensors is denoted by $\mathbf{A} : \mathbf{B}$. We use the usual Lebesgue and Sobolev spaces $L^p(\Omega)$, $W^{k,p}(\Omega)$, $1 \leq p \leq \infty$, $k \in \mathbb{N}$, where $\Omega \subset \mathbb{R}^3$ is bounded domain with Lipschitz-boundary. For given $T \in (0, \infty)$ we use the notation $\Omega_T := (0, T) \times \Omega$. By $W^{1,p}_0(\Omega)$ we denote the completion of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$ which will be equipped with the gradient norm $\| \nabla \cdot \|_{L^p}$. In the notation of function spaces we do not distinguish between scalar, vector-valued and tensor-valued spaces. For $v \in W^{1,p}(\Omega)$ we denote by $Dv$ the symmetric and by $Wv$ the skew-symmetric part of the gradient, i.e. $Dv = \frac{1}{2}(\nabla v + \nabla v^\top)$ and $Wv = \frac{1}{2}(\nabla v - \nabla v^\top)$. Further we define for vectors $v, \omega$ the tensor $R(v, \omega) := Wv + \varepsilon \cdot \omega$. These definitions imply the equality

$$S : \nabla v + (\varepsilon : S) \cdot \omega = S : (Dv + R(v, \omega)).$$

(2.1)

Moreover, for any $\omega \in \mathbb{R}^d$, $S \in \mathbb{R}^{d \times d}$ there holds

$$|\varepsilon : S| \leq c|S|, \quad |\varepsilon \cdot \omega| \leq c|\omega|. \quad (2.2)$$

Additionally we need some completions of $V(\Omega) := \{u \in C_0^\infty(\Omega) \mid \text{div } u = 0\}$. We define $H(\Omega) := \overline{V(\Omega)}^{L^2}$, $V_p(\Omega) := \overline{V(\Omega)}^{W^{1,p}}$, $V^3(\Omega) := \overline{V(\Omega)}^{W^{3,2}}$. While we will use the usual $L^2$-norm on $H(\Omega)$ and the usual $W^{3,2}$-norm on $V^3(\Omega)$, we will equip the space $V_p(\Omega)$ for $1 < p < \infty$ with the norm $\| \cdot \|_{V_p} := \| D \cdot \|_{L^p}$, which defines an equivalent norm due to Korn’s inequality (cf. [6]). The duality pairing between a Banach space $V$ and its dual $V^*$ will be denoted by $\langle \cdot, \cdot \rangle_V$. We use the usual notation for Bochner spaces (cf. [14], [29]) and denote by $\frac{du}{dt}$ the generalized derivative, i.e. let $V, W$ be Banach spaces with a dense embedding $V \to W$ and assume that for $u \in L^p(0, T; V)$ there exists $w \in L^q(0, T; W)$ such that $\int_0^T \varphi'(t)u(t) \, dt = -\int_0^T \varphi(t)uw(t) \, dt$ holds for any $\varphi \in C_0^\infty((0, T))$, then we set $\frac{du}{dt} := w$. We introduce the Bochner-Sobolev space

$$W^{1,p,q}(0, T; V, W) := \left\{ u \in L^p(0, T; V) \mid \frac{du}{dt} \in L^q(0, T; W) \right\}.$$

For details concerning these spaces we refer to [6]. Let $(V, H, V^*)$ be a Gelfand-triple, i.e. $V$ is a Banach space and $H$ is a Hilbert space, such that $V$ embeds densely into $H$. Then we define the Bochner-Sobolev space

$$W_p^1(0, T; V, H) := \left\{ u \in L^p(0, T; V) \mid \frac{du}{dt} \in L^p(0, T; V^*) \right\}.$$
It is well known that (cf. [6], [29]) we have the continuous embeddings

\[
W^{1,p,q}(0,T;V,W) \hookrightarrow C(0,T;W), \\
W^{1,q}(0,T;V,H) \hookrightarrow C(0,T;H),
\]

and that for \(u,v \in W^{1,q}_0(0,T;V,H)\) there holds the integration by parts formula, i.e. for any \(0 \leq s,t \leq T\) there holds (cf. [29])

\[
(u(t),v(t))_H - (u(s),v(s))_H = \int_s^t \left( \frac{du(\tau)}{dt}, v(\tau) \right) + \left( \frac{dv(\tau)}{dt}, u(\tau) \right) \, dt.
\]

In the Sections 3, 4 and 5 we will renounce to mark the integration variables to ensure a better readability.

2.2. The stress tensor, the couple stress tensor and the electric field

We denote the symmetric and the skew-symmetric part, resp., of a tensor \(A\) by \(A^\text{sym} := \frac{1}{2}(A + A^\top)\) and \(A^\text{skew} := \frac{1}{2}(A - A^\top)\), respectively. Moreover, we set \(\mathbb{R}^{3 \times 3}_\text{sym} := \{ A \in \mathbb{R}^{3 \times 3} \mid A = A^\text{sym} \}\) and \(\mathbb{R}^{3 \times 3}_\text{skew} := \{ A \in \mathbb{R}^{3 \times 3} \mid A = A^\text{skew} \}\). Motivated by the typical example for the extra stress tensor \(S\) and for the couple stress tensor \(N\) in (1.5) and the residual entropy inequality (cf. [10, (2.30)])

\[
S : D + S : R + N : \nabla \omega \geq 0
\]

we make the following assumptions:

**Assumption 2.6.** The stress tensor \(S = S(D,R,E)\) belongs to the space \(C^0(\mathbb{R}^{3 \times 3}_\text{sym}, \mathbb{R}^{3 \times 3}_\text{skew}; \mathbb{R}^3; \mathbb{R}^{3 \times 3})\) and fulfills the following assumptions:

1. coercivity: for all \(D \in \mathbb{R}^{3 \times 3}_\text{sym}\), \(R \in \mathbb{R}^{3 \times 3}_\text{skew}\) and \(E \in \mathbb{R}^3\) we have

\[
S(D,R,E) : D \geq c (1 + |E|^2) (|D|^p - c), \\
S(D,R,E) : R \geq c |E|^2 (|R|^p - c),
\]

2. boundedness: for all \(D \in \mathbb{R}^{3 \times 3}_\text{sym}\), \(R \in \mathbb{R}^{3 \times 3}_\text{skew}\) and \(E \in \mathbb{R}^3\) we have

\[
|S^\text{sym}(D,R,E)| \leq c (1 + |E|^2) (1 + |D|^{p-1}), \\
|S^\text{skew}(D,R,E)| \leq c |E|^2 (1 + |R|^{p-1}),
\]

3. strict monotonicity: for all \(D_1,D_2 \in \mathbb{R}^{3 \times 3}_\text{sym}, R_1,R_2 \in \mathbb{R}^{3 \times 3}_\text{skew}\) and \(E \in \mathbb{R}^3\) such that \(D_1,|E|R_1 \neq (D_2,|E|R_2)\) we have

\[
(S(D_1,R_1,E) - S(D_2,R_2,E)) : (D_1 - D_2 + R_1 - R_2) > 0.
\]

**Assumption 2.10.** The couple stress tensor \(N = N(L,E)\) belongs to the space \(C^0(\mathbb{R}^{3 \times 3}, \mathbb{R}; \mathbb{R}^{3 \times 3})\) and fulfills the following assumptions:

1. coercivity: for all \(L \in \mathbb{R}^{3 \times 3}\) and \(E \in \mathbb{R}^3\) we have

\[
N(L,E) : L \geq c (1 + |E|^2) (|L|^p - c),
\]
2. boundedness: for all \( \mathbf{L} \in \mathbb{R}^{3 \times 3} \) and \( \mathbf{E} \in \mathbb{R}^3 \) we have
\[
|\mathbf{N} ( \mathbf{L}, \mathbf{E} )| \leq c \left( 1 + |\mathbf{E}|^2 \right) (1 + |\mathbf{L}|^{p-1}),
\] (2.12)

3. strict monotonicity: for all \( \mathbf{L}_1, \mathbf{L}_2 \in \mathbb{R}^{3 \times 3} \) and \( \mathbf{E} \in \mathbb{R}^3 \) with \(|\mathbf{E}| > 0\) and \( \mathbf{L}_1 \neq \mathbf{L}_2 \) we have
\[
(\mathbf{N} (\mathbf{L}_1, \mathbf{E}) - \mathbf{N} (\mathbf{L}_2, \mathbf{E})) : (\mathbf{L}_1 - \mathbf{L}_2) > 0.
\] (2.13)

Remark 2.14. We could also have adapted the notion of \((p, \delta)\)-structure, used e.g. in [19], [8], [5] and [4], to the present situation. In fact, all results remain valid also under that assumption. Moreover, all estimates would depend on \( \delta \in [0, \delta_1] \) only through \( \delta_0 > 0 \) and \( \delta_1 \).

In the steady case the quasi-static Maxwell-equations possess very regular solutions. Following [23], [12], [13] and the references therein, we know that the electric field is a real-analytic function and that and the set \(|\mathbf{E}|^{-1}(0)\) is a finite union of lower-dimensional \(C^1\)-manifolds. Especially \(|\mathbf{E}|^{-1}(0)\) is a set of measure zero. If we consider the time-dependent case and assume the data to be regular, the solution also possesses good regularity properties (cf. [23] for more details). By using Fubini’s Theorem we conclude
\[
\int_{\Omega_T} \chi_{\mathbf{E} = 0} = \int_0^T |\mathbf{E}(t)|^{-1} dt = 0.
\]
Therefore we make the following assumption.

Assumption 2.15. The electric field \( \mathbf{E} \) belongs to the space \( L^\infty(0,T; L^\infty(\Omega)) \) and a.e. in \( \Omega_T \) there holds \(|\mathbf{E}| > 0\).

Throughout the paper we assume that there exists \( p \in (1, \infty) \) and such that \( \mathbf{S} \) satisfies Assumption 2.6 and \( \mathbf{N} \) satisfies Assumption 2.10. Moreover, the electric field satisfies Assumption 2.15.

2.3. Auxiliary results

In this section we want to present two Lipschitz truncation methods for unsteady problems as well as an existence result for parabolic PDEs which will be used to solve the easy case \( p \geq \frac{11}{5} \) and the approximation of our system. The first result is a solenoidal Lipschitz truncation which was established in [7].

Theorem 2.16 (Solenoidal Lipschitz truncation). Let \( Q_0 = I_0 \times B_0 \) be a space-time cylinder with a finite time interval \( I_0 \subset \mathbb{R} \) and a ball \( B_0 \subset \mathbb{R}^3 \). Let \( \frac{2}{5} < p < \infty \) and \( 1 < \sigma < \min\{p, p'\} \). Let \( (\mathbf{u}_m) \) and \( (\mathbf{G}_m) \) satisfy
\[
-\int_{Q_0} \mathbf{u}_m \cdot \partial_t \xi \, dx \, dt = \int_{Q_0} \mathbf{G}_m : \nabla \xi \, dx \, dt \quad \text{for all } \xi \in C^\infty_{0, \text{div}}(Q_0).
\]
Assume \( (\mathbf{u}_m) \) is a weak null sequence in \( L^p(I_0; W^{1,p}(B_0)) \), a strong null sequence in \( L^p(Q_0) \) and bounded in \( L^\infty(I_0; L^p(B_0)) \). Further assume that \( \mathbf{G}_m = \mathbf{G}_{1,m} + \mathbf{G}_{2,m} \) is such that \( (\mathbf{G}_{1,m}) \) is a weak null sequence in \( L^p(Q_0) \) and \( (\mathbf{G}_{2,m}) \) converges strongly to zero in \( L^p(Q_0) \). Then there exist double sequences \( (\lambda_{m,k}) \subset \mathbb{R}^+, \ (\mathbf{C}_{m,k}) \subset \mathbb{R} \times \mathbb{R}^3 \), \( (\mathbf{u}_{m,k}) \subset L^1(Q_0) \) and \( k_0 \in \mathbb{N} \) such that for all \( k \geq k_0 \):
(a) $2^{2^s} \leq \lambda_{m,k} \leq 2^{2^{s+1}}$.

(b) $(u_{m,k}) \subset L^s\left(\frac{1}{t}B_0; W^{1,s}_{0,\text{div}}\left(\frac{1}{t}B_0\right)\right)$ for all $s < \infty$ and $\supp(u_{m,k}) \subset \frac{1}{t}Q_0$.\footnote{Let $I_0 = (t_0 - \rho, t_0 + \rho)$ and $B_0 = B_r(x_0)$. Then we define the scaled space-time cylinder $\frac{1}{t}Q_0$ by $(t_0 - \frac{1}{t}\rho, t_0 + \frac{1}{t}\rho) \times B_{\frac{1}{t}\rho}(x_0)$.}

c) $u_{m,k} = u_m$ a.e. on $\frac{1}{t}Q_0 \setminus \Omega_{m,k}$.

d) $\|\nabla u_{m,k}\|_{L^\infty(\frac{1}{t}Q_0)} \leq c\lambda_{m,k}$.

e) $u_{m,k} \to 0$ in $L^\infty(\frac{1}{t}Q_0)$ for $m \to \infty$ and $k$ fixed.

(f) $\nabla u_{m,k} \to 0$ in $L^s(\frac{1}{t}Q_0)$ for $m \to \infty$ and $k$ fixed.

g) $\limsup_{m \to \infty} \lambda^p_{m,k}|\Omega_{m,k}| \leq c2^{-k}$.

(h) $\limsup_{m \to \infty} \int G_m : \nabla u_{m,k} \, dx \, dt \leq c\lambda^p_{m,k}|\Omega_{m,k}|$.

**Proof:** This is Theorem 2.2 in [7]. Another proof can be found in [2] where the result of [7, Lemma 2.6] is proved differently.

We also cite a useful corollary of this theorem, cf. [7, Corollary 2.4].

**Corollary 2.17.** Let all the assumptions of Theorem 2.16 be satisfied and let $\zeta \in C_0^\infty(\frac{1}{t}Q_0)$ satisfy $\chi_{\frac{1}{t}Q_0} \leq \zeta \leq \chi_{\frac{1}{t}Q_0}$. Then for every $K \in L^r(\frac{1}{t}Q_0)$

$$\limsup_{m \to \infty} \int \left((G_{1,m} + K) : \nabla u_m\right) \zeta \chi_{\Omega_{m,k}} \, dx \, dt \leq c2^{-\frac{k}{r}}.$$ 

In [9] a Lipschitz truncation method for non-solenoidal functions was developed. Since we need a small generalization of [9, Theorem 3.9] we sketch the proof. Let us start with some notation. For $\alpha > 0$ we define the anisotropic parabolic metric $d_\alpha$ in $\mathbb{R} \times \mathbb{R}^3$ by $d_\alpha((t,x),(s,y)) := \max\{|x-y|, |\alpha^{-1}(t-s)|^\alpha\}$. For $(t,x) \in \mathbb{R} \times \mathbb{R}^3$ and $r > 0$ we define $\alpha$-parabolic cylinders $Q^\alpha_r((t,x)) := \{(s,y) \in \mathbb{R} \times \mathbb{R}^3 | d_\alpha((t,x),(s,y)) < r\}$. Note that $Q^\alpha_r((t,x)) = (t - \alpha^2r, t + \alpha^2r) \times B_r(x)$. Let $E \subset \mathbb{R} \times \mathbb{R}^3$ be open and bounded, where $\mathbb{R} \times \mathbb{R}^3$ is equipped with the anisotropic metric $d_\alpha$ for some $\alpha > 0$. According to [9, Lemma 3.1] there exists a Whitney type covering $(Q^\alpha_{1,n})_{n \in \mathbb{N}}$ of $E$ with $\alpha$-parabolic cylinders $Q^\alpha_1 := Q^\alpha_{1,1}((t_1,x_1))$. There also exists a subordinate partition of unity $(\psi_n)_{n \in \mathbb{N}}$ to this Whitney type covering $(Q^\alpha_{1,n})_{n \in \mathbb{N}}$ (cf. [9, (3.2)]). So we are able to define the truncation operator.

**Definition 2.18.** Let $E \subset \Omega_T$ be open. For $u \in L^1_{\text{loc}}(\Omega_T)$ we define

$$(T^\alpha_E u)(t,x) := \begin{cases} u(t,x) & \text{if } (t,x) \in \Omega_T \setminus E \\ \sum_{i \in \mathbb{N}} \psi_i(t,x)u_{Q^\alpha_i} & \text{if } (t,x) \in E, \end{cases}$$

where $u_{Q^\alpha_i} := \int_{Q^\alpha_i} u \, dx \, dt$ is the mean value over $Q^\alpha_i$.

With this definition we get the first continuity result.
Lemma 2.19. There exists a constant c such that for every $1 \leq p \leq \infty$

$$\|T_E^\alpha u\|_{L^p(\Omega_T)} \leq c \|u\|_{L^p(\Omega_T)}.$$ 

Proof: cf. [9, Lemma 3.5]. □

Before we state the main theorem of chapter 3 in [9] in a generalized version, we need some results for the maximal operator. For $g \in L^p(\mathbb{R}^{1+3}) \ (p > 1)$ we denote by

$$M_\varepsilon(f)(t,x) := \sup_{0 < r < \infty} \int_{B_r(x)} |f(t,y)| \, dy,$$

$$M_r(f)(t,x) := \sup_{0 < \rho < \infty} \int_{I_{\rho}(t)} |f(s,x)| \, ds,$$

the usual maximal operators in the space and time variables. Here $I_\rho(t) := (t - \rho, t + \rho)$. More details concerning maximal operators can be found in [24].

We want to use the composition of these two maximal operators

$$M^*(g) := M_r(M_\varepsilon(f)).$$

This maximal operator satisfies strong and weak type estimates and for all $(t,x) \in \mathbb{R}^{1+3}$ and $r, \rho > 0$ there holds (cf. [9, Appendix A])

$$\int_{I_\rho(t)} \int_{B_r(x)} |g(s,y)| \, dy \, ds \leq M^*(g)(t,x). \quad (2.20)$$

Theorem 2.21. For $1 < p, \sigma < \infty$ let $u \in L^\infty(I; L^2(\Omega)) \cap L^p(I; W^{1,p}(\Omega))$, $H \in L^\sigma(\Omega_T), \ k \in L^p(\Omega_T)$ satisfy

$$- \int_{\Omega_T} u \cdot \partial_t \phi \, dx \, dt = \int_{\Omega_T} H : \nabla \phi \, dx \, dt + \int_{\Omega_T} k \cdot \phi \, dx \, dt \quad (2.22)$$

for all $\phi \in C_0^\infty(\Omega_T)$. We define $(\Lambda > 0)$

$$O_\Lambda := \{(t,x) \in \mathbb{R}^{1+3} \mid M^*(|\nabla u|)(t,x) + \alpha M^*(|H|)(t,x) + \alpha M^*(|k|)(t,x) > \Lambda\},$$

$$U_1 := \{(t,x) \in \mathbb{R}^{1+3} \mid M^*(|u|)(t,x) > 1\}.$$

Let $E$ be an open, bounded set such that $\Omega_T \cap (O_\Lambda \cup U_1) \subset E \subset \Omega_T$. Let $K \subset \Omega_T$ be a compact set, then there holds:

(i) The Lipschitz truncation $T_E^\alpha u$ belongs to $C_0^{0,1}(K)$, the space of Lipschitz continuous functions with respect to $d_\alpha$ on $K$, where the norm depends on $K, \Lambda, \alpha, \|u\|_{L^1(E)}$ and $\|u\|_{L^1(\Omega_T)}$. In particular $T_E^\alpha u, \nabla T_E^\alpha u \in L^\infty(K)$. (ii) The Lipschitz truncation $T_E^\alpha u$ satisfies the estimates

$$\|\nabla T_E^\alpha u\|_{L^\infty(K)} \leq c(\Lambda + \alpha^{-1}_\delta \alpha^{-3-3}_K \|u\|_{L^1(E)}),$$

$$\|T_E^\alpha u\|_{L^\infty(K)} \leq c(1 + \alpha^{-1}_\delta \alpha^{-3-2}_K \|u\|_{L^1(E)}),$$

where $\delta_\alpha_K := d_\alpha(K, \partial \Omega_T)$.

(iii) The function $(\partial_t T_E^\alpha u) \cdot (T_E^\alpha u - u)$ belongs to $L^1(K \cap E)$ and we have

$$\|(\partial_t T_E^\alpha u) \cdot (T_E^\alpha u - u)\|_{L^1(K \cap E)} \leq c \alpha^{-1} |E| (\Lambda + \alpha^{-1}_\delta \alpha^{-3-3}_K \|u\|_{L^1(E)})^2.$$
(iv) For all $\zeta \in C_0^\infty(\Omega_T)$ there holds the identity
\[
\int_1 \left( \frac{d}{dt}(t), (T_E^\alpha u(t))\zeta(t) \right) dt \\
= \frac{1}{2} \int_{\Omega_T} \left( |T_E^\alpha u|^2 - 2u \cdot T_E^\alpha u \right) \partial_t \zeta \, dx \, dt + \int_{E} \left( \partial_t T_E^\alpha u \cdot (T_E^\alpha u - u) \right) \zeta \, dx \, dt,
\]
where $\langle \cdot, \cdot \rangle$ denotes the usual duality pairing with respect to $\Omega$.

In [9] this Theorem is proved only with $k \equiv 0$. In order to deal with $k \neq 0$ we changed the definition of $O_k$ and hence the general result of this lemma follows from a transformation of coordinates. We only proof the special case $(0,1) = (0,0)$.

Proof: We only proof the special case $(t,x) = (0,0)$ since the general result of this lemma follows from a transformation of coordinates. We abbreviate $Q_1 := Q_1\{(0,0)\}$ and $B_1 := B_1(0)$. With the same arguments as in [9] we get
\[
\int_{Q_1} |u(t,x) - u_{Q_1}| \, dx \, dt \leq c \int_{Q_1} |\nabla u| \, dx \, dt + \frac{|B_1|}{2} \int_{-1}^{1} \int_{-1}^{1} |g(u(t) - u(s))| \, ds \, dt
\]
where $g : L^1(B_1) \to \mathbb{R}^3$ is defined for $\zeta \in C_0^\infty(B_1)$ satisfying $\chi_{\frac{1}{4}B_1} \leq \zeta \leq \chi_{B_1}$, by
\[
g(v) = \frac{1}{\int_{B_1} \zeta \, dx} \int_{B_1} \zeta v \, dx.
\]
For arbitrary $\xi \in C_0^\infty(B_1)$ and $\gamma \in C_0^\infty(-1,1)$ the function $\varphi(t,x) := \xi(x)\gamma_h(t)$, where $\gamma_h(t) := \frac{1}{h} \int_{t-h}^{t-h} \gamma(s) \, ds$ is the Steklov average, is an admissible testfunction for equation (2.24). By standard arguments concerning the Steklov average

\[\text{cf. [9, Appendix B]}, \text{Lemma B.3}] \text{ which allows to control mean values of the form}\]

\[
\int_{Q_1} |\nabla u| \, dx \, dt \leq c \left( \left\| \nabla u \right\|_{L^1(Q_1)} + \|H\|_{L^1(Q_1)} + \alpha r \|k\|_{L^1(Q_1)} \right).
\]

Lemma 2.23 (Poincaré-type inequality). For $\alpha, r > 0$ let $Q_1^\alpha := Q_1^\alpha((t,x))$ be a $\alpha$-parabolic cylinder. Assume $u \in L^1(Q_1^\alpha)$ with $\nabla u \in L^1(Q_1^\alpha)$, $H \in L^1(Q_1^\alpha)$ and $k \in L^1(Q_1^\alpha)$ satisfy
\[
- \int_{Q_1^\alpha} u \cdot \partial_t \varphi \, dx \, dt = \int_{Q_1^\alpha} H : \nabla \varphi \, dx \, dt + \int_{Q_1^\alpha} k \cdot \varphi \, dx \, dt
\]
for all $\varphi \in C_0^\infty(Q_1^\alpha)$. Then there exists a constant $c$ independent of $\alpha$, $r$ and $(t,x)$ such that
\[
\int_{Q_1^\alpha} |u - u_{Q_1^\alpha}| \, dx \, dt \leq c r \left( \left\| \nabla u \right\|_{L^1(Q_1^\alpha)} + \|H\|_{L^1(Q_1^\alpha)} + \alpha r \|k\|_{L^1(Q_1^\alpha)} \right).
\]

Proof: We only proof the special case $(t,x) = (0,0)$, $1 = \alpha = r$ since the general result of this lemma follows from a transformation of coordinates. We abbreviate $Q_1 := Q_1\{(0,0)\}$ and $B_1 := B_1(0)$. With the same arguments as in [9] we get
\[
\int_{Q_1} |u(t,x) - u_{Q_1}| \, dx \, dt \leq c \int_{Q_1} |\nabla u| \, dx \, dt + \frac{|B_1|}{2} \int_{-1}^{1} \int_{-1}^{1} |g(u(t) - u(s))| \, ds \, dt
\]
where $g : L^1(B_1) \to \mathbb{R}^3$ is defined for $\zeta \in C_0^\infty(B_1)$ satisfying $\chi_{\frac{1}{4}B_1} \leq \zeta \leq \chi_{B_1}$, by
\[
g(v) = \frac{1}{\int_{B_1} \zeta \, dx} \int_{B_1} \zeta v \, dx.
\]

For arbitrary $\xi \in C_0^\infty(B_1)$ and $\gamma \in C_0^\infty(-1,1)$ the function $\varphi(t,x) := \xi(x)\gamma_h(t)$, where $\gamma_h(t) := \frac{1}{h} \int_{t-h}^{t-h} \gamma(s) \, ds$ is the Steklov average, is an admissible testfunction for equation (2.24). By standard arguments concerning the Steklov average

\[\text{cf. [9, Appendix B]}, \text{Lemma B.3}] \text{ which allows to control mean values of the form}\]

\[
\int_{Q_1} |\nabla u| \, dx \, dt \leq c r \left( \left\| \nabla u \right\|_{L^1(Q_1^\alpha)} + \|H\|_{L^1(Q_1^\alpha)} + \alpha r \|k\|_{L^1(Q_1^\alpha)} \right).
\]
we conclude
\[ \int_{-1}^{1} \int_{B_1} \partial_t u_h \cdot \zeta dx \, dt = \int_{-1}^{1} \int_{B_1} H_h : \nabla \zeta dx \, dt + \int_{-1}^{1} \int_{B_1} k_h \cdot \zeta dx \, dt \]
and with the fundamental lemma of the calculus of variations we get
\[ \int_{B_1} \partial_t u_h(\tau) \cdot \zeta dx = \int_{B_1} H_h(\tau) : \nabla \zeta dx + \int_{B_1} k_h(\tau) \cdot \zeta dx \]
for all \( \tau \in (-1,1) \). Splitting this equation into the components \( u_h^i, k_h^i \) and \( H_h^i \), where \( H_h^i \) is the \( i \)-th line of the tensor \( H_h \), by using a test function which is 0 in all components except the \( i \)-th component, we get
\[ \int_{B_1} \partial_t u_h^i(\tau) \zeta dx = \int_{B_1} H_h^i(\tau) : \nabla \zeta dx + \int_{B_1} k_h^i(\tau) \zeta dx \quad \text{for all } \zeta \in C_0^\infty(B_1). \]
This equation, the properties of the Steklov average, the definition of \( g \) and the properties of \( \zeta \) imply
\[
|g(u(t) - u(s))| = \lim_{h \to 0} |g(u_h(t) - u_h(s))| \\
= \lim_{h \to 0} \frac{1}{\int_{B_1} \zeta dx} \int_s^t \int_{B_1} \partial_t u_h \zeta dx \, d\tau \\
\leq \lim_{h \to 0} \frac{c}{\int_{B_1} \zeta dx} \sum_{i=1}^3 \int_s^t \int_{B_1} \partial_t u_h^i \zeta dx \, d\tau \\
\leq \lim_{h \to 0} \frac{c}{\int_{B_1} \zeta dx} \sum_{i=1}^3 \int_s^t \int_{B_1} H_h^i : \nabla \zeta + k_h^i \zeta dx \, d\tau \\
\leq \lim_{h \to 0} \|H_h\|_{L^1(Q_1)} + \|k_h\|_{L^1(Q_1)} \\
= \|H\|_{L^1(Q_1)} + \|k\|_{L^1(Q_1)}
\]
This together with (2.25) proves the special case of this lemma.

With this Poincaré-type inequality we can prove [9, Lemma 3.11] which is important for the proof of [9, Theorem 3.9]

**Lemma 2.26.** Under the assumptions of Theorem 2.21 we have for all \( Q_0^i \) belonging to the Whitney covering of \( E \) such that \( Q_0^i \cap K = \emptyset \)
\[
\int_{4Q_0^i} |u - u_{4Q_0^i}| \, dx \, dt \leq cr_i(\Lambda + \alpha^{-1} \delta_{a,K}^{-3/2} \|u\|_{L^1(E)}), \tag{2.27}
\]
where the constant depends on the diameter of \( \Omega \).

**Proof:** The properties of the Whitney covering imply (i) \( 16Q_0^i \cap ((\mathbb{O}_0)^c \cap (U_1)^c) \) or (ii) \( 16Q_0^i \cap Q \neq \emptyset \). In case (i) we use the new Poincaré-type inequality in Lemma 2.23 to estimate
\[
\int_{4Q_0^i} |u - u_{4Q_0^i}| \, dx \, dt \leq cr_i \int_{4Q_0^i} |\nabla u| + \alpha |H| + \alpha r_i |k| \, dx \, dt,
\]
where we used $4Q_{r_i}^a \subset E \subset Q$, which also implies $0 \leq r_i \leq \text{diam} \Omega$. Since $16Q_{r_i}^a \cap (O_\Lambda)^c \neq \emptyset$ we find a $(\tilde{t}, \tilde{x}) \in 16Q_{r_i}^a \cap (O_\Lambda)^c$. The definition of the $\alpha$-parabolic cylinders imply $4Q_{r_i}^a \subset Q_{20t_i}(t_0, x_0)$. Now from the above inequality, the definition of the maximal operator $M^*$, (2.20) and the new definition of $O_\Lambda$ in Theorem 2.21 we get

$$
\int_{4Q_{r_i}^a} |u - u_{4Q_{r_i}^a}| \, dx \, dt \leq c r_i \int_{4Q_{r_i}^a} |\nabla u| + \alpha |H| + \alpha r_i |k| \, dx \, dt \\
\leq c(\Omega) r_i \int_{Q_{20t_i}(\tilde{t}, \tilde{x})} |\nabla u| + \alpha |H| + \alpha |k| \, dx \, dt \\
\leq c(\Omega) r_i \left( M^*(\nabla u)(\tilde{t}, \tilde{x}) + \alpha M^*(H)(\tilde{t}, \tilde{x}) + \alpha M^*(k)(\tilde{t}, \tilde{x}) \right) \\
\leq c(\Omega) r_i \Lambda.
$$

Case (ii) can be treated exactly as in [9].

Now the proof of Theorem 2.21 is exactly the same as the proof of [9, Theorem 3.9] we just have to use Lemma 2.26 whenever [9, Lemma 3.11] is used.

Finally we quote an existence result for parabolic PDEs (cf. [3]).

**Theorem 2.28.** Let $(V, H, V^*)$ be a Gelfand-Triple. Assume $Z$ is another reflexive, separable Banach space such that $Z \hookrightarrow V$ with a continuous and dense embedding. Moreover, assume that there exists an increasing sequence of finite dimensional subspaces $V_n \subseteq Z$, such that $\bigcup_{n \in \mathbb{N}} V_n$ is dense in $V$. Additionally, there exists self-adjoint projections $P_n : H \rightarrow H$, such that $P_n(V) = V_n$ and $\|P_n(\cdot)\|_{L(Z,Z)} \leq c$ with a constant $c$ independent of $n \in \mathbb{N}$. Finally, let $\{A(t) \mid 0 \leq t \leq T\}$ be a family of operators from $V$ to $V^*$ with the following properties:

(A1) $A(t) : V \rightarrow V^*$ is pseudomonotone for almost every $t \in [0, T]$.

(A2) For every $u \in L^p(0, T; V) \cap L^\infty(0, T; H)$ the mapping $t \mapsto A(t)u(t)$ from $[0, T]$ to $V^*$ is Bochner-measurable.

(A3) There exists a positive constant $c_1$ and a nonnegative function $C_2 \in L^1(0, T)$, such that

$$
\langle A(t)x, x \rangle_V \geq c_1 \|x\|^p_V - C_2(t)
$$

for almost every $t \in [0, T]$ and all $x \in V$.

(A4) There exists $0 < q < \infty$, as well as constants $c_3 > 0, c_4 \geq 0$ and a nonnegative function $C_5 \in L^q(0, T)$, such that

$$
\|A(t)x\|_{V^*} \leq c_3 \|x\|^{p-1}_V + c_4 \|x\|^q_H \|x\|_{V^*}^{p-1} + C_5(t)
$$

for almost every $t \in [0, T]$ and all $x \in V$.

Then for every $u_0 \in H, f \in L^p(0, T; V^*)$ there exists a function $u \in W^1_p(0, T; V, H)$ such that

$$u'(t) + A(t)u(t) = f(t) \quad \text{in } V^* \quad \text{for a.e. } t \in [0, T],$$

$$u(0) = u_0 \quad \text{in } H.$$
3. Easy case \( p \geq \frac{4}{3} \)

**Theorem 3.1.** Let \( p \in [\frac{4}{3}, \infty) \), \( T \in (0, \infty) \) and \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with Lipschitz-boundary. Assume that \( S \) satisfies Assumption \( 2.6 \), that \( N \) satisfies Assumption \( 2.10 \) and that \( E \) satisfying Assumption \( 2.15 \) is given. Then there exists for all \( \psi \in W^1_p(\Omega) \times W^{1, p}_0(\Omega) \) a weak solution \((v, \omega) \in W^1_p(0, T; V_p(\Omega)) \times W^{1, p}_0(0, T; W^1_p(\Omega), L^2(\Omega)) \) of the problem (1.1)–(1.3) satisfying for all \((\varphi, \psi) \in L^p(0, T; V_p(\Omega)) \times L^p(0, T; W^1_p(\Omega)) \)

\[
\int_0^T \langle \frac{dv}{dt}(t), \varphi(t) \rangle_{V_p} + \int_{\Omega_T} S(\nabla v, R(v, \omega), E) : \nabla \varphi - \int_{\Omega_T} v \otimes v : \nabla \varphi \\
+ \int_0^T \langle \frac{d\omega}{dt}(t), \psi(t) \rangle_{W^{1, p}_0} + \int_{\Omega_T} N(\nabla \omega, E) : \nabla \psi - \int_{\Omega_T} \omega \otimes \omega : \nabla \psi \geq 0
\]

(3.2)

as well as \( v(0) = v_0 \) and \( \omega(0) = \omega_0 \).

**Proof:** We want to use Theorem 2.28. In view of (2.5), the identity (2.1) and the assumptions on the stress tensors it is natural to view the system (1.1) as a unit. Thus we are searching two unknown functions \( v \) and \( \omega \) as elements \((v, \omega)\) of the product space \( V_p(\Omega) \times W^{1, p}_0(\Omega) \). To simplify the notation we set

\[
\mathcal{V}_p := V_p(\Omega) \times W^{1, p}_0(\Omega), \quad \| (u, w) \|_{\mathcal{V}_p} := \left( \| u \|_{V_p}^2 + \| w \|_{W^{1, p}_0}^2 \right)^{\frac{1}{2}},
\]

\[
\mathcal{H} := H(\Omega) \times L^2(\Omega), \quad \| (u, w) \|_{\mathcal{H}} := \left( \| u \|_H^2 + \| w \|_{L^2}^2 \right)^{\frac{1}{2}}.
\]

Since \((V_p(\Omega), H(\Omega), V_p(\Omega)^*)\) and \((W^{1, p}_0(\Omega), L^2(\Omega), W^{1, p}_0(\Omega)^*)\) form Gelfand-triples, it is obvious that \((\mathcal{V}_p, \mathcal{H}, (\mathcal{V}_p)^*)\) forms a Gelfand-triple as well. We set \( \mathcal{Z} := V^3(\Omega) \times W^{1, 2}_0(\Omega) \). Then, according to [18, Appendix 4.11 and 4.14], we know that \( \mathcal{Z} \), \( \mathcal{V}_p \), \( \mathcal{H} \) satisfy all assumptions in Theorem 2.28. Next we define operators \( A(t), A_i(t) : \mathcal{V}_p \to (\mathcal{V}_p)^*, \ i = 1, \ldots, 4 \), by

\[
\langle A_1(t)(u, w), (\varphi, \psi) \rangle := \int_{\Omega} S(\nabla u, R(u, w), E(t)) : (\nabla \varphi + R(\varphi, \psi)),
\]

\[
\langle A_2(t)(u, w), (\varphi, \psi) \rangle := -\int_{\Omega} u \otimes u : \nabla \varphi,
\]

\[
\langle A_3(t)(u, w), (\varphi, \psi) \rangle := \int_{\Omega} N(\nabla w, E(t)) : \nabla \psi,
\]

\[
\langle A_4(t)(u, w), (\varphi, \psi) \rangle := -\int_{\Omega} w \otimes u : \nabla \psi,
\]

\[
A(t) := A_1(t) + A_2(t) + A_3(t) + A_4(t),
\]

where \( \langle \cdot, \cdot \rangle \) denotes the duality pairing between \( \mathcal{V}_p \) and \((\mathcal{V}_p)^*\). In order to apply Theorem 2.28 we have to check that the family \( A(t) \) satisfies (A1)–(A4).

(A1): Using (2.8), (2.12) as well as \( E(t) \in L^\infty(\Omega) \) for almost every \( t \in [0, T] \) we conclude with the theory of Nemyckii operators that \( A_1(t) \) and \( A_3(t) \) are continuous operators. Moreover, from (2.9) and (2.13) we get the monotonicity of \( A_1(t) \) and \( A_3(t) \). Using the compact embedding \( \mathcal{V}_p \hookrightarrow L^s(\Omega) \times L^s(\Omega) \) for any
1 \leq s < \frac{9}{2p}$ and Hölder’s inequality it is easy to show that $A_2(t)$ and $A_4(t)$ are strongly continuous operators for $p > \frac{9}{7}$. Since monotone, continuous operators and strongly continuous operators are pseudomonotone and since summation maintains pseudomonotonicity, we conclude that $A(t)$ is pseudomonotone for almost every $t \in [0, T]$.

(A2): Fix $(u, w) \in L^p(0, T; \mathcal{Y}_p) \cap L^{\infty}(0, T; \mathcal{M})$. The argumentation will be the same as in [3]. Since $\mathcal{Y}_p$ is separable we are able to use Pettis’ Theorem. Therefore it is enough to prove that $t \mapsto \langle A(t)(u(t), w(t)), (\varphi, \psi) \rangle_{\mathcal{Y}_p}$ is Lebesgue measurable for arbitrary $(\varphi, \psi) \in \mathcal{Y}_p$. Using (2.8), (2.12), (2.15) it is clear that every function appearing in the definitions of $A_i$ is an element of $L^1(\Omega_T)$ so that Fubini’s theorem yields the assertion.

(A3) For arbitrary $(u, w) \in \mathcal{Y}_p$, $p \geq \frac{9}{7}$, there holds

$$
\langle A_2(t)(u, w), (u, w) \rangle = - \int_{\Omega} u \otimes u : u = 0, \\
\langle A_4(t)(u, w), (u, w) \rangle = - \int_{\Omega} w \otimes u : w = 0,
$$

due to $\text{div} u = 0$ and integration by parts. Using (2.7) and (2.11) we immediately estimate

$$
\langle A_1(t)(u, w), (u, w) \rangle + \langle A_3(t)(u, w), (u, w) \rangle \\
\geq c \int_{\Omega} |Du|^p + c \int_{\Omega} |\nabla w|^p - c(E, \Omega)
$$

$$
= c\|u\|_{W^{1,p}}^p + c\|w\|_{W^{1,p}}^p - c(E, \Omega) \geq c\|(u, w)\|_{W^{1,p}}^p - c(E, \Omega),
$$

where the constant $c(E, \Omega)$ depends only on $\|E\|_{L^\infty(\Omega_T)}$ and $\Omega$ because of Assumption 2.15.

(A4) Let us start with the operator $A_1$. Using (2.8), Hölder’s inequality and the continuous embedding $L^p \hookrightarrow L^1$ we get

$$
\|A_1(t)(u, w)\|_{\mathcal{Y}_p} \leq \sup_{\|\varphi, \psi\|_{\mathcal{Y}_p} \leq 1} \int_{\Omega} c(1 + |E(t)|^2)(1 + |Du|^{p-1})|D\varphi|
$$

$$
+ \sup_{\|\varphi, \psi\|_{\mathcal{Y}_p} \leq 1} \int_{\Omega} c|E(t)|^2(1 + |R(u, w)|^{p-1})|R(\varphi, \psi)|
$$

$$
\leq \sup_{\|\varphi, \psi\|_{\mathcal{Y}_p} \leq 1} c_{E_p, \Omega}(1 + \|Du\|_{L^p}^{p-1})\|D\varphi\|_{L^p}
$$

$$
+ \sup_{\|\varphi, \psi\|_{\mathcal{Y}_p} \leq 1} c_{E_p, \Omega}(1 + \|R(u, w)\|_{L^p}^{p-1})\|R(\varphi, \psi)\|_{L^p}.
$$

Here the constant $c_{E_p, \Omega}$ is again independent of $t \in [0, T]$ because of Assumption 2.15. Due to (2.2) we are able to estimate that for any $(\varphi, \psi) \in \mathcal{Y}_p$ and a.e. $x \in \Omega$

$$
|R(\varphi(x), \psi(x))| \leq |\nabla \varphi(x)| + c|\psi(x)|.
$$

This, together with Poincaré’s and Korn’s inequality, implies

$$
\|R(\varphi, \psi)\|_{L^p} \leq c\|D\varphi\|_{L^p} + \|\nabla \psi\|_{L^p} = c\|\varphi\|_{V_p} + \|\psi\|_{W^{1,p}_T}
$$

$$
\leq c\|(\varphi, \psi)\|_{\mathcal{Y}_p}.
$$
Now (3.6) and (3.7) immediately imply
\[ \|A_1(t)(u, w)\|_{(\mathcal{Y}_p)}^r \leq c_{\mathbf{E}, p, |\Omega|} (1 + \|u, w\|_{\mathcal{Y}_p}^{p-1}) \] (3.8)

From (2.12) and Hölder’s inequality we are able to derive that
\[ \|A_3(t)(u, w)\|_{(\mathcal{Y}_p)}^r \leq \sup_{\|\rho, \psi\|_{\mathcal{Y}_p} \leq 1} c_E (1 + \|w\|_{W^{1, p}_0})^r \|\psi\|_{W^{1, p}_0} \]
\[ \leq c_E (1 + \|(u, w)\|_{\mathcal{Y}_p}^{p-1}) \] (3.9)

To treat the convective terms, we argue the same way as in [3]. Using Hölder’s inequality we immediately get
\[ \|A_2(t)(u, w)\|_{(\mathcal{Y}_p)} \leq c \|u\|_{L^{2p'}}^2, \]
\[ \|A_4(t)(u, w)\|_{(\mathcal{Y}_p)} \leq c \|u\|_{L^{2p'}}^2 \|w\|_{L^{2p'}} \leq c \max \left\{ \|u\|_{L^{2p'}}^2, \|w\|_{L^{2p'}}^2 \right\}. \]

If \( p \in \left[ \frac{6}{5}, 3 \right) \) we use for \( r = \frac{12p}{5p' + 1 + 6p - 6} \) the Hölder interpolation
\[ \|u\|_{L^r} \leq \|u\|_{L^{\frac{6p'}{5p' + 1 + 6p - 6}}} \|u\|_{L^{\frac{6p}{5p + 1}}} \]. (3.10)

Since for these \( p \) there holds \( 2p' \leq r \), we conclude that
\[ \|A_2(t)(u, w)\|_{(\mathcal{Y}_p)} \leq c \|u\|_{L^{2p'}}^3 \|u\|_{V_p}^{p-1} \leq c \|(u, w)\|_{W^{1, p}}^{3-p} \|(u, w)\|_{\mathcal{Y}_p}^{p-1}. \] (3.11)

Moreover, (3.10) also holds for \( w \), so we also get
\[ \|A_4(t)(u, w)\|_{(\mathcal{Y}_p)} \leq c \|(u, w)\|_{W^{1, p}}^{3-p} \|(u, w)\|_{\mathcal{Y}_p}^{p-1}. \] (3.12)

If, on the other hand, \( p \geq 3 \), we get \( p' \leq \frac{3}{2} \). This implies that \( 2 < 2p' \leq 3 \) always holds and therefore it is sufficient to use the interpolation
\[ \|u\|_{L^3} \leq \|u\|_{L^{\frac{6p}{5p + 1}}} \|u\|_{L^{\frac{6p'}{5p' + 1} + (3-p)}} \] (3.13)

to get (3.11) and (3.12) also for \( p \geq 3 \). Now (3.8), (3.9), (3.11) and (3.12) imply
\[ \|A(t)(u, w)\|_{(\mathcal{Y}_p)} \leq C_{\mathbf{E}, p, |\Omega|} (1 + \|(u, w)\|_{\mathcal{Y}_p}^{p-1}) + c \|(u, w)\|_{W^{1, p}}^{3-p} \|(u, w)\|_{\mathcal{Y}_p}^{p-1}. \]
so that all the assumptions of Theorem 2.28 are satisfied. \(\blacksquare\)

4. Approximative solutions of the system for \( \frac{6}{5} < p \leq \frac{11}{5} \)

In this section we prove the solvability of an appropriate approximation of our system (1.1). The approximate system arises by adding two terms which are monotone but provide a better coercivity than the terms induced by \( S \) and \( N \). To solve this problem we again use Theorem 2.28.

Theorem 4.1. Let \( p \in \left( \frac{6}{5}, \frac{11}{5} \right] \), \( T \in (0, \infty) \) and \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with Lipschitz-boundary. Assume that \( S \) satisfies Assumption 2.6, that \( N \) satisfies Assumption 2.10 and that \( \mathbf{E} \) satisfying Assumption 2.15 is given. Let \( \mathbf{v}_0 \in H(\Omega), \omega_0 \in L^2(\Omega) \) and \( f, \ell \in L^p(\Omega_T) \) be given. Then for any \( q \in \left( \frac{11}{5}, 3 \right) \),
$M \in \mathbb{N}$ there exists $(v, \omega) \in W^1_q(0, T; V_q(\Omega), H(\Omega)) \times W^1_q(0, T; W^{1,q}_0(\Omega), L^2(\Omega))$ satisfying for all $(\varphi, \psi) \in L^q(0, T; V_q(\Omega)) \times L^q(0, T; W^{1,q}_0(\Omega))$

$\int_0^T \langle \frac{dv}{dt}(t), \varphi(t) \rangle_{V_q} + \int_{\Omega_T} S(Dv, R(v, \omega), E) : \nabla \varphi - \int_{\Omega_T} v \otimes v : \nabla \varphi$

$+ \int_0^T \langle \frac{d\omega}{dt}(t), \psi(t) \rangle_{W^{1,q}_0} + \int_{\Omega_T} N(\nabla \omega, E) : \nabla \psi - \int_{\Omega_T} \omega \otimes v : \nabla \psi$

$+ \frac{1}{M} \int_{\Omega_T} |Dv|^{q-2} Dv : D\varphi + \frac{1}{M} \int_{\Omega_T} |\nabla \omega|^{q-2} \nabla \omega : \nabla \psi$

$= \int_{\Omega_T} f \cdot \varphi + \int_{\Omega_T} \ell \cdot \psi - \int_{\Omega_T} (\epsilon : S(Dv, R(v, \omega), E)) : \psi$

as well as $v(0) = v_0$ and $\omega(0) = \omega_0$.

**Proof:** To apply Theorem 2.28 we again have to work on a product space. Let $\mathcal{V}_q := V_q(\Omega) \times W^{1,q}_0(\Omega)$, $\mathcal{H} := H(\Omega) \times L^2(\Omega)$ and $\mathcal{E} := V^3(\Omega) \times W^{3,3}_0(\Omega)$. Since $q \in (\frac{4}{3}, 3)$ we get similarly to Section 3 that $\mathcal{V}_q, \mathcal{H}, \mathcal{E}$ satisfy the assumptions on the function spaces, which are required in Theorem 2.28. The operators $A(t), A_i(t) : \mathcal{V}_q \to (\mathcal{V}_q)^*$, $i = 1, \ldots, 6$, are defined in (3.3)1–4 and by

$$
(A_5(t)(u, w), (\varphi, \psi)) := \frac{1}{M} \int_{\Omega_T} |Du|^{q-2} Du : D\varphi,
$$

$$
(A_6(t)(u, w), (\varphi, \psi)) := \frac{1}{M} \int_{\Omega_T} |\nabla w|^{q-2} \nabla w : \nabla \psi,
$$

$$
A(t) := \sum_{i=1}^6 A_i(t),
$$

where $\langle \cdot, \cdot \rangle$ now denotes in all cases the duality pairing between $\mathcal{V}_q$ and $(\mathcal{V}_q)^*$. Again we have to verify that (A1)–(A4) in Theorem 2.28 are satisfied.

(A1) From the theory of Nemyckii operators as well as (2.8), (2.12) and $E(t) \in L^\infty(\Omega)$ we immediately derive that $A_1(t), A_3(t), A_5(t)$ and $A_6(t)$ are continuous operators. The monotonicity of $A_1(t)$ and $A_3(t)$ is again provided by (2.9) and (2.13), whereas $A_5(t)$ and $A_6(t)$ are classical examples of monotone operators (cf. [16], [14]). Since $A_2(t)$ and $A_4(t)$ are again strongly continuous operators, we see that $A(t)$ is pseudomonotone for almost every $t \in [0, T]$.

(A2) This follows in the same way as in the proof of Theorem 3.1.

(A3) From (3.4) and (3.5) follows

$$
\langle A_2(t)(u, w), (u, w) \rangle = \langle A_4(t)(u, w), (u, w) \rangle = 0,
$$

$$
\langle A_1(t)(u, w), (u, w) \rangle + \langle A_3(t)(u, w), (u, w) \rangle \geq -c(E, \Omega).
$$

The definitions of $A_5(t)$ and $A_6(t)$ immediately imply

$$
\langle A_5(t)(u, w), (u, w) \rangle_{\mathcal{V}_q} + \langle A_6(t)(u, w), (u, w) \rangle_{\mathcal{V}_q} = \frac{1}{M} \|u\|_{V_q}^q + \frac{1}{M} \|w\|_{W^{1,q}_0}^q
$$

$$
\geq \frac{c}{M} \|(u, w)\|_{\mathcal{V}_q}^q,
$$

so that (A3) is satisfied.
(A4) To treat $A_2(t)$ and $A_4(t)$ we can use the same argumentation as in Section 3, if we replace $p$ by $q$. Therefore we get
\[
\|A_2(t)(u, w)\|_{(\mathcal{Y}_q)^*} + \|A_4(t)(u, w)\|_{(\mathcal{Y}_q)^*} \leq c \| (u, w) \|_{(\mathcal{Y}_q)}^{3-q}. \tag{4.3}
\]
Using (3.8), (3.9), the continuous embedding $W_0^{1, q} \hookrightarrow W_0^{1, p}$ and Young’s inequality, we get
\[
\|A_1(t)(u, w)\|_{(\mathcal{Y}_q)^*} \leq c_E, p, q; |\Omega| (1 + \| (u, w) \|_{(\mathcal{Y}_q)}^{q-1}).
\]
\[
\|A_3(t)(u, w)\|_{(\mathcal{Y}_q)^*} \leq c_E, p, q; |\Omega| (1 + \| (u, w) \|_{(\mathcal{Y}_q)}^{q-1}). \tag{4.4}
\]
Hölder’s inequality yields
\[
\|A_5(t)(u, w)\|_{(\mathcal{Y}_q)^*} + \|A_6(t)(u, w)\|_{(\mathcal{Y}_q)^*} \leq \frac{1}{M} \| D(u) \|_{L^q}^{q-1} + \frac{1}{M} \| \nabla w \|_{L^r}^{q-1} \leq \frac{c}{M} \| (u, w) \|_{(\mathcal{Y}_q)}^{q-1}. \tag{4.5}
\]
Altogether (4.3), (4.4), (4.5) imply that (A4) also holds and Theorem 2.28 reveals the existence of a solution of the problem in Theorem 4.1. \hfill \Box

5. Proof of the main theorem

**Theorem 5.1** (Main Theorem). Let $p \in \left(\frac{3}{5}, \frac{11}{5}\right]$, $T \in (0, \infty)$ and $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz-boundary. Assume that $S$ satisfies Assumption 2.6, that $N$ satisfies Assumption 2.10 and that $E$ satisfying Assumption 2.15 is given. Then there exists for all $v_0 \in H(\Omega)$, $\omega_0 \in L^2(\Omega)$ and $f, \ell \in L^q(\Omega_T)$ a weak solution $(v, \omega) \in \left(\bigcap_{p \geq 1} (0; L^p(0, T; V_p(\Omega)))) \cap L^\infty(0; T; H(\Omega))\right) \times \left(\bigcap_{p \geq 1} (0; V_0^q(\Omega)))) \cap L^\infty(0; T; L^2(\Omega))\right)$ of the problem (1.1)–(1.3) satisfying for all $(\varphi, \psi) \in C_0^{\infty}(0, T) \times \Omega)
\]
\[
\begin{align*}
- \int_{\Omega_T} v \cdot \partial_t \varphi + \int_{\Omega_T} S(Dv, R(v, \omega), E) : \nabla \varphi - \int_{\Omega_T} v \otimes v : \nabla \varphi

- \int_{\Omega_T} \omega \cdot \partial_t \psi + \int_{\Omega_T} N(\nabla \omega, E) : \nabla \psi - \int_{\Omega_T} \omega \otimes v : \nabla \psi

+ \int_{\Omega_T} \left( \varepsilon : S(Dv, R(v, \omega), E) \right) : \varphi

= \int_{\Omega_T} f \cdot \varphi + \int_{\Omega_T} \ell \cdot \psi + \int_{\Omega_T} v_0 \cdot \varphi(0) + \int_{\Omega_T} \omega_0 \cdot \psi(0). \tag{5.2}
\end{align*}
\]

**Proof:** For any fixed $q \in \left(\frac{11}{3}, 3\right]$ Theorem 4.1 yields that for any $M \in \mathbb{N}$ there exist solutions $(\nu^M, \omega^M) \in W_0^q(0, T; V_0^q(\Omega), H(\Omega)) \times W_0^1(0, T; W_0^q(\Omega), L^2(\Omega))$ with $v^M(0) = v_0$ and $\omega(0) = \omega_0$ which solve (4.2). Now for any $t \in [0, T]$ we are allowed to test (4.2) with $\varphi = v^M \chi_{[0, t]}$ and $\psi = \omega^M \chi_{[0, t]}$. Due to div $v^M = 0$ the convective terms vanish and by using the integration by parts formula (2.4)
together with (2.1) we obtain
\[
\frac{1}{2}\left(\|\mathbf{v}^M(t)\|_{H^2(\Omega)}^2 - \|\mathbf{v}_0\|_{H^2(\Omega)}^2 + \|\omega^M(t)\|_{L^2(\Omega)}^2 - \|\omega_0\|_{L^2(\Omega)}^2\right) \\
+ \frac{1}{M} \int_0^t \int_\Omega |\mathbf{D}(\mathbf{v}^M)|^q + \frac{1}{M} \int_0^t \int_\Omega |\nabla \omega^M|^q + \int_0^t \int_\Omega \mathbf{N}(\nabla \omega^M, \mathbf{E}) : \nabla \omega^M \\
+ \int_0^t \int_\Omega \mathbf{S}(\mathbf{Dv}^M, \mathbf{R}(\mathbf{v}^M, \omega^M), \mathbf{E}) : (\mathbf{Dv}^M + \mathbf{R}(\mathbf{v}^M, \omega^M)) \\
= \int_0^t \int_\Omega f \cdot \mathbf{v}^M + \int_0^t \int_\Omega \mathbf{e} \cdot \omega^M.
\]  
(5.3)

We use the coercivity of $\mathbf{S}$ in (2.7) and $\mathbf{N}$ in (2.11), treat the right-hand side of (5.3) with Hölder’s, Poincaré’s, Korn’s and Young’s inequality and absorb the resulting terms with $\mathbf{v}^M$ and $\omega^M$ in the left-hand side of (5.3) to get the a priori estimate
\[
\|\mathbf{v}^M\|_{L^2(0,T;H^2(\Omega))}^2 + \|\omega^M\|_{L^2(0,T;L^2(\Omega))}^2 + \|\mathbf{v}^M\|_{L^p(0,T;V_p(\Omega))}^p \\
+ \|\omega^M\|_{L^p(0,T;W_0^{1,q}(\Omega))}^p + \frac{1}{M} \|\mathbf{v}^M\|_{L^q(0,T;V_q(\Omega))}^q + \frac{1}{M} \|\omega^M\|_{L^q(0,T;W_0^{1,q}(\Omega))}^q \\
\leq c(f, \mathbf{e}, \mathbf{E}, \Omega, \mathbf{v}_0, \omega_0).
\]  
(5.4)

The growth condition of $\mathbf{S}$ and $\mathbf{N}$ ((2.8) and (2.12)) together with the theory of Nemyckii operators and (5.4) yield
\[
\|\mathbf{S}(\mathbf{Dv}^M, \mathbf{R}(\mathbf{v}^M, \omega^M), \mathbf{E})\|_{L^{p'}(\Omega)} + \|\mathbf{N}(\nabla \omega^M, \mathbf{E})\|_{L^{p'}(\Omega)} \leq c.
\]  
(5.5)

The parabolic interpolation $L^p(0,T;W_0^{1,q}(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \hookrightarrow L^{2\sigma}(\Omega_T)$ implies
\[
\|\mathbf{v}^m \otimes \mathbf{v}^M\|_{L^{2\sigma}(\Omega_T)} + \|\omega^M \otimes \mathbf{v}^M\|_{L^{2\sigma}(\Omega_T)} \leq c.
\]  
(5.6)

Furthermore there holds
\[
\frac{1}{M} \|\mathbf{Dv}^M\|_{L^q(\Omega_T)} \rightarrow 0 \quad \text{in } L'^p(\Omega_T), \\
\frac{1}{M} \|\nabla \omega^M\|_{L^q(\Omega_T)} \rightarrow 0 \quad \text{in } L'^q(\Omega_T).
\]  
(5.7)

Now we choose $\sigma \in \mathbb{R}$ which satisfies
\[
1 < \sigma < \min\left\{\frac{5p}{6}, q'\right\} \text{ and } 2\sigma > p.
\]  
(5.8)

This is always possible, since the second inequality in (5.8) only provides a restriction if $p > 2$ and in this case we have $\min\left\{\frac{5p}{6}, q'\right\} > \frac{5}{2} \Leftrightarrow \sigma = \frac{5}{2}$ satisfies (5.8). The next step in our proof is to show the boundedness of $(\mathbf{v}^M)$ in $W^{1,p,\sigma}(0,T;V_p(\Omega), V_\sigma(\Omega)^*)$ and of $(\omega^M)$ in $W^{1,p,\sigma}(0,T;W_0^{1,p}(\Omega), W_0^{1,\sigma'}(\Omega)^*)$. Let us firstly treat $(\mathbf{v}^M)$. First due to $\sigma < \frac{5p}{6}$ and $p \leq \frac{11}{3}$ we have $\sigma' > p$. This together with $p > \frac{6}{5}$ ensures
\[
V_p(\Omega) \hookrightarrow H(\Omega) \hookrightarrow V_p(\Omega)^* \hookrightarrow V_{\sigma'}(\Omega)^*.
\]
with continuous and dense embeddings. Therefore $W^{1,p,\sigma}(0,T; V_\sigma(\Omega), V_{\sigma'}(\Omega)^*)$ is a Bochner-Sobolev space as introduced in Section 2.1. From (5.4) we already know that $\|v^M\|_{L^p(0,T; V_\sigma(\Omega))}$ is bounded and because of $\sigma < q'$ we have that $L^p(0,T; V_\sigma(\Omega)^*) \hookrightarrow L^q(0,T; V_{\sigma'}(\Omega)^*)$, which allows us to interpret $\frac{dv^M}{dt}$ as an element of $L^q(0,T; V_{\sigma'}(\Omega)^*)$. To show the boundedness we use arbitrary $\varphi \in L^q(0,T; V_{\sigma'}(\Omega))$ and $\psi = 0$ in (4.2). With the help of Hölder’s inequality we get

$$\|d\varphi\|_{L^q(0,T; V_{\sigma'}(\Omega)^*)} \leq c. \quad (5.9)$$

For $\omega^M$ we proceed analogously. The choice of $\sigma$ ensures that

$$W^{1,p,\sigma}_0(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow W^{1,p}_0(\Omega)^* \hookrightarrow W^{1,\sigma'}_0(\Omega)^*$$

with continuous and dense embeddings. In (5.4) we already proved the boundedness of $\omega^M$ in $L^p(0,T; W^{1,\sigma'}_0(\Omega))$ and to get an estimate for the time derivative we test (4.2) with $\varphi = 0$ and arbitrary $\psi \in L^\sigma(0,T; W^{1,\sigma'}_0(\Omega))$. Then again Hölder’s inequality, the choice of $\sigma$ and (5.5)–(5.8) imply

$$\|d\omega^M\|_{L^\sigma(0,T; W^{1,\sigma'}_0(\Omega)^*)} \leq c. \quad (5.10)$$

From (5.4), (5.9), (5.10) we get

$$\|v^M\|_{W^{1,p,\sigma}(0,T; V_\sigma(\Omega), V_{\sigma'}(\Omega)^*)} + \|\omega^M\|_{W^{1,p,\sigma}(0,T; W^{1,\sigma'}_0(\Omega), W^{1,\sigma'}_0(\Omega)^*)} \leq c. \quad (5.11)$$

Now (5.4), (5.5), (5.11), the Aubin-Lions lemma and parabolic interpolation lead to the following convergence results after choosing appropriate subsequences:

$$v^M \rightharpoonup v \quad \text{in } L^\infty(0,T; H(\Omega)),$$
$$\omega^M \rightharpoonup \omega \quad \text{in } L^\infty(0,T; L^2(\Omega)),$$
$$v^M \rightarrow v \quad \text{in } W^{1,p,\sigma}(0,T; V_\sigma(\Omega), V_{\sigma'}(\Omega)^*),$$
$$\omega^M \rightarrow \omega \quad \text{in } W^{1,p,\sigma}(0,T; W^{1,\sigma'}_0(\Omega), W^{1,\sigma'}_0(\Omega)^*),$$
$$v^M \rightarrow v \quad \text{in } L^{2\sigma}(\Omega_T),$$
$$\omega^M \rightarrow \omega \quad \text{in } L^{2\sigma}(\Omega_T),$$
$$v^M \otimes v^M \rightarrow v \otimes v \quad \text{in } L^{\sigma}(\Omega_T),$$
$$\omega^M \otimes \omega^M \rightarrow \omega \otimes \omega \quad \text{in } L^{\sigma}(\Omega_T),$$
$$S(Dv^M, R(v^M, \omega^M), E) \rightarrow \tilde{S} \quad \text{in } L^{\nu'}(\Omega_T),$$
$$N(\nabla \omega^M, E) \rightarrow \tilde{N} \quad \text{in } L^{\nu'}(\Omega_T).$$

Here we also made use of our choice of $\sigma$ since the Aubin-Lions lemma and parabolic interpolation imply $v^M \rightarrow v$, $\omega^M \rightarrow \omega$ in $L^s(\Omega_T)$ for any $1 \leq s < \frac{2p}{q'}$. In particular $v$ and $\omega$ belong to the required function spaces in Theorem 5.1.

To derive our limit equation we test (4.2), which is solved for any $M \in \mathbb{N}$ by $v^M$ and $\omega^M$, with arbitrary $\varphi \in C^\infty_{0, \text{div}}([0,T) \times \Omega)$ and $\psi \in C^\infty_{0, \text{div}}([0,T) \times \Omega)$.
and use the integration by parts formula. We get

\[
- \int_{\Omega_T} v^M \cdot \partial_t \varphi + \int_{\Omega_T} S(Dv^M, R(v^M, \omega^M), E) : \nabla \varphi - \int_{\Omega_T} v^M \otimes v^M : \nabla \varphi \\
- \int_{\Omega_T} \omega^M \cdot \partial_t \psi + \int_{\Omega_T} N(\nabla \omega^M, E) : \nabla \psi - \int_{\Omega_T} \omega^M \otimes v^M : \nabla \psi \\
+ \frac{1}{M} \int_{\Omega_T} |Dv^M|^{-2} Dv^M : D\varphi + \frac{1}{M} \int_{\Omega_T} |\nabla \omega^M|^{-2} \nabla \omega^M : \nabla \psi \\
+ \int_{\Omega_T} (\varepsilon : S(Dv^M, R(v^M, \omega^M), E)) : \psi
\]

(5.13)

Since for any $M \in \mathbb{N}$ there holds $v^M(0) = v_0$ in $H(\Omega)$ and $\omega^M(0) = \omega_0$ in $L^2(\Omega)$ the convergences in (5.7) and (5.12) allow us to pass to the limit in every term and we conclude

\[
- \int_{\Omega_T} v \cdot \partial_t \varphi + \int_{\Omega_T} \hat{S} : \nabla \varphi - \int_{\Omega_T} v \otimes v : \nabla \varphi \\
- \int_{\Omega_T} \omega \cdot \partial_t \psi + \int_{\Omega_T} \hat{N} : \nabla \psi - \int_{\Omega_T} \omega \otimes v : \nabla \psi + \int_{\Omega_T} (\varepsilon : \hat{S}) : \psi
\]

(5.14)

holds for all $\varphi \in C^{2,2}_0([0, T) \times \Omega)$ and all $\psi \in C^\infty_0([0, T) \times \Omega)$. So it remains to prove that $\hat{S} = S(Dv, R(v, \omega), E)$ and $\hat{N} = N(\nabla \omega, E)$ a.e. in $\Omega_T$ to finish the proof of Theorem 5.1. To this end we use the two Lipschitz truncation results in Section 2.3.

We start with the proof of $\hat{S} = S(Dv, R(v, \omega), E)$ a.e. in $\Omega_T$. We set $\psi = 0$ and choose $\varphi \in C^{2,2}_0(\Omega_T)$ arbitrarily in (5.13) and (5.14) and subtract these two equations. Thus we obtain for any $\varphi \in C^\infty_0(\Omega_T)$

\[
- \int_{\Omega_T} (v^M - v) \cdot \partial_t \varphi = \int_{\Omega_T} \left( \hat{S} - S(Dv^M, R(v^M, \omega^M), E) \right) : \nabla \varphi
\]

(5.15)

Now we argue by contradiction. Assume that there exists a set $M \subseteq \Omega_T$, which satisfies $|M| \geq 2\delta$ for some $\delta > 0$, so that almost everywhere in $M$ there holds $\hat{S} \neq S(Dv, R(v, \omega), E)$. For $\epsilon > 0$ we define

$$
\Omega_\epsilon := \{ x \in \Omega \mid \text{dist}(\partial \Omega, x) \geq \epsilon \} \quad \text{and} \quad \Omega_{T, \epsilon} := \{ \epsilon, T - \epsilon \} \times \Omega_\epsilon.
$$

Clearly we can choose $\epsilon > 0$ sufficiently small so that $|M \cap \Omega_{T, \epsilon}| \geq \delta$. Since $\Omega_{T, \epsilon}$ is compact there exists $n \in \mathbb{N}$ and $(t_i, x_i)_{1 \leq i \leq n} \subseteq \Omega_{T, \epsilon}$ such that

$$
\Omega_{T, \epsilon} \subset \bigcup_{i=1}^n (t_i - \frac{\epsilon}{2}, t_i + \frac{\epsilon}{2}) \times B_{\frac{\epsilon}{2}}(x_i).
$$
This in turn implies the existence of \( j \in \{1, \ldots, n\} \) so that
\[
|\mathcal{M} \cap ((t_j - \frac{\epsilon}{2}, t_j + \frac{\epsilon}{2}) \times B_2(x_j))| \geq \frac{\epsilon}{2} > 0.
\]
Therefore it is sufficient to prove that \( \hat{\mathcal{S}} = \mathbf{S}(\nabla v, \mathbf{R}(v, \omega), \mathbf{E}) \) holds a.e. in \((t_j - \frac{\epsilon}{2}, t_j + \frac{\epsilon}{2}) \times B_2(x_j)\) to achieve a contradiction. To this end we set \( I_0 := (t_j - \epsilon, t_j + \epsilon), B_0 := B_2(x_j), Q_0 := I_0 \times B_0 \) and define
\[
\begin{align*}
\mathbf{u}_M & := (v^M - v)\chi_{Q_0}, \\
G_{1,M} & := (\hat{\mathcal{S}} - \mathbf{S}(\nabla v^M, \mathbf{R}(v^M, \omega^M), \mathbf{E}))\chi_{Q_0}, \\
G_{2,M} & := (v^M \otimes v^M - v \otimes v - \frac{I}{M} \nabla v^M |_{g^{-2}} \nabla v^M)\chi_{Q_0}, \\
G_M & := G_{1,M} + G_{2,M}.
\end{align*}
\]

With these definitions we get from (5.15), by restricting the test functions, that for any \( \xi \in C_{0,\text{div}}(Q_0) \)
\[
- \int_{Q_0} \mathbf{u}_M \cdot \partial_t \xi = \int_{Q_0} G_M : \nabla \xi. \tag{5.16}
\]

The functions \( \mathbf{u}_M \) and \( G_M \) satisfy the assumptions of Theorem 2.16 because of (5.7) and (5.12). Thus Theorem 2.16 yields that there exist double-sequences \((\lambda_{M,k})\) and \((Q_{M,k})\) such that for every \( k \geq k_0 \) there holds \( 2^{2k} \leq \lambda_{M,k} \leq 2^{2k+1} \) and \(|Q_{M,k}| \leq 2^{-k} \). If we choose \( \zeta \in C_0^\infty(\overline{Q_0}) \) with \( \chi_{\overline{Q_0}} \leq \zeta \leq \chi_{\frac{1}{2}Q_0} \) as well as \( K := \mathbf{S}(\nabla v, \mathbf{R}(v, \omega), \mathbf{E}) - \hat{\mathcal{S}} \in L^p(\frac{1}{p}Q_0) \) we get from Corollary 2.17 that
\[
\limsup_{M \to \infty} \left| \int_{\frac{1}{2}Q_0} \left( \mathbf{S}(\nabla v^M, \mathbf{R}(v^M, \omega^M), \mathbf{E}) - \mathbf{S}(\nabla v, \mathbf{R}(v, \omega), \mathbf{E}) \right) : \nabla (v^M - v) \zeta \chi_{Q_{M,k}} \right| \leq 2^{1/p}. \tag{5.17}
\]

Unfortunately the integrand has no sign, since we can’t use the monotonicity of \( \mathbf{S} \) in (2.9). On the other hand \( \mathbf{S}(\nabla v^M, \mathbf{R}(v^M, \omega^M), \mathbf{E}) - \mathbf{S}(\nabla v, \mathbf{R}(v, \omega), \mathbf{E}) \) is bounded in \( L^p(\Omega_T) \) and due to the choice of \( \sigma \) in (5.8), (5.12) and (2.2) we can deduce that \( \varepsilon \cdot \omega^M \to \varepsilon \cdot \omega \) in \( L^p(\Omega_T) \). Therefore
\[
\limsup_{M \to \infty} \left| \int_{\frac{1}{2}Q_0} \left( \mathbf{S}(\nabla v^M, \mathbf{R}(v^M, \omega^M), \mathbf{E}) - \mathbf{S}(\nabla v, \mathbf{R}(v, \omega), \mathbf{E}) \right) : \varepsilon \cdot (\omega^M - \omega) \zeta \chi_{Q_{M,k}} \right| = 0. \tag{5.18}
\]

Form (5.17), (5.18) and the definition of \( \mathbf{R} \) we conclude that
\[
\limsup_{M \to \infty} \left| \int_{\frac{1}{2}Q_0} \left( \mathbf{S}(\nabla v^M, \mathbf{R}(v^M, \omega^M), \mathbf{E}) - \mathbf{S}(\nabla v, \mathbf{R}(v, \omega), \mathbf{E}) \right) : \varepsilon \cdot \omega^M - \varepsilon \cdot \omega \right| \chi_{Q_{M,k}} \right| \leq c 2^{1/p}. \tag{5.19}
\]

\(^6\)Since \( v \in L^p(0, T; V_p(\Omega)) \) and \( \omega \in L^p(0, T; W^{1,p}_0(\Omega)) \) the growth condition of \( \mathbf{S} \) ensures that \( \mathbf{S}(\nabla v, \mathbf{R}(v, \omega), \mathbf{E}) \in L^p(\Omega_T) \).
Now this term has a sign due to (2.9). From now on we use the abbreviation
\[ S^M := (S(Dv^M, R(v^M, \omega^M), E) - S(Dv, R(v, \omega), E)) : (Dv^M + R(v^M, \omega^M) - Dv - R(v, \omega)). \]

Note that from (5.12) we are able to conclude that \( S^M \) is bounded in \( L^1(\Omega_T) \). Since \( S^M \geq 0 \) the expression \( (S^M)_{\frac{1}{2}} \) is well defined. From Hölder’s inequality, \( |\xi| \leq 1 \) and \( |M, k| \leq c 2^{-k} \) we easily get
\[
\lim_{M \to \infty} \int_{\frac{1}{k}Q_0} (S^M)_{\frac{1}{2}} \zeta \chi_{M,k} \leq \lim_{M \to \infty} \|S^M\|_{L^1(\frac{1}{k}Q_0)}^{\frac{1}{2}} \leq c 2^{-k};
\]
which together with (5.19) implies
\[
\lim_{M \to \infty} \int_{\frac{1}{k}Q_0} \min \left\{ S^M, (S^M)_{\frac{1}{2}} \right\} \zeta \leq c \min \left\{ 2^{-k}, 2^{-\frac{3}{2}} \right\}. \tag{5.20}
\]
Since this holds for any \( k \geq k_0 \) and since \( \zeta = 1 \) on \( \frac{1}{k}Q_0 \) we get
\[
S^M \to 0 \quad \text{almost everywhere in } \frac{1}{k}Q_0, \tag{5.21}
\]
at least for a not relabeled subsequence. Now we define \( T : \mathbb{R}^{3 \times 3} \to \mathbb{R}^{3 \times 3} \) by
\[
T(A) = S(A_{\text{sym}}, A_{\text{skew}}). \tag{5.22}
\]
Due to our Assumptions 2.6 and 2.15 on \( S \) and \( E \) it is clear that \( T \) is continuous and strictly monotone. So if we set \( \eta^M(t,x) := Dv^M(t,x) + R(v^M, \omega^M)(t,x) \) and \( \eta(t,x) := Dv(t,x) + R(v, \omega)(t,x) \) the equation (5.21) reads
\[
(T(\eta^M(t,x)) - T(\eta(t,x))) : (\eta^M(t,x) - \eta(t,x)) \to 0
\]
almost everywhere in \( \frac{1}{k}Q_0 \). Thus [20, Lemma 6] implies
\[
\eta^M(t,x) \to \eta(t,x)
\]
a.e. in \( \frac{1}{k}Q_0 \). Since we already know from (5.12) that \( \varepsilon \cdot \omega^M \to \varepsilon \cdot \omega \) holds a.e. in \( \frac{1}{k}Q_0 \), we conclude that \( Dv^M \to Dv \) and \( Wv^M \to Wv \) also holds a.e. in \( \frac{1}{k}Q_0 \). Since \( S \) is continuous this implies
\[
S(Dv^M, R(v^M, \omega^M), E) \to S(Dv, R(v, \omega), E) \quad \text{a.e. in } \frac{1}{k}Q_0. \tag{5.22}
\]
Since weak and a.e. limits coincide (cf. [15]) we conclude from (5.12) and (5.22) that \( S(Dv, R(v, \omega), E) = S \) a.e. in \( \frac{1}{k}Q_0 = (t_j - \delta, t_j + \delta) \times B_{\delta}(x_j) \), which gives the desired contradiction and proofs
\[
S(Dv, R(v, \omega), E) = S \quad \text{almost everywhere in } \Omega_T.
\]
So the remaining step in the proof of the main Theorem 5.1 is to prove that \( \widetilde{N} = N(\nabla \omega, E) \) holds a.e. in \( \Omega_T \). We start by subtracting the equations (5.13) and (5.14) one from another. If we set \( \varphi = 0 \) we get for any \( \psi \in C_0^\infty(\Omega_T) \)
\[
- \int_{\Omega_T} (\omega^M - \omega) \cdot \partial_t \psi = \int_{\Omega_T} (\widetilde{N} - N(\nabla \omega^M, E)) : \nabla \psi \tag{5.23}
+ \int_{\Omega_T} (\omega^M \otimes v^M - \omega \otimes v - \frac{1}{M} |\nabla \omega^M|^2 \nabla \omega^M) : \nabla \psi
+ \int_{\Omega_T} \left( \varepsilon : (S(Dv^M, R(v^M, \omega^M), E) - \tilde{S}) \right) \cdot \psi.
\]
The ideas we want to use have been developed in [9] but there are some differences. The first difference is that (5.23) holds for any \( \psi \in C_0^{\infty}(\Omega_T) \) not just for \( \psi \in C_0^\infty(\Omega_T) \) as in [9]. Therefore we don’t need to construct a local pressure which had to be done in [9]. The second difference is that in our equation (5.23) a term of lower order, namely \( \int_{\Omega_T} (\varepsilon : (S(Dv_M, R(v^M, \omega^M), E) - \hat{S})) \cdot \psi \), appears. This is why we had to prove a slightly generalized Lipschitz Truncation in Theorem 2.21. We define

\[
\begin{align*}
  u_M &:= \omega^M - \omega, \\
  H_{1,M} &:= \hat{N} - N(\nabla \omega^M, E), \\
  H_{2,M} &:= \omega^M \otimes v^M - \omega \otimes v - \frac{1}{M} |\nabla \omega^M|^{q-2} \nabla \omega^M, \\
  H_M &:= H_{1,M} + H_{2,M}, \\
  k_M &:= \varepsilon : (S(Dv_M, R(v^M, \omega^M), E) - \hat{S}),
\end{align*}
\]

so that (5.23) reads

\[
- \int_{\Omega_T} u_M \cdot \partial_t \psi = \int_{\Omega_T} H_M : \nabla \psi + \int_{\Omega_T} k_M \cdot \psi
\]

for any \( \psi \in C_0^{\infty}(\Omega_T) \) just as in (2.22) in Theorem 2.21. Using a density argument, we conclude that for any \( \psi \in L^{q^*}(0, T; W_0^{1,q^*}(\Omega)) \)

\[
\int_0^T \left( \frac{d u_M}{dt}(\tau), \psi(\tau) \right)_{W_0^{1,q^*}(\Omega)} = \int_{\Omega_T} H_M : \nabla \psi + \int_{\Omega_T} k_M \cdot \psi.
\]

We also already know from (5.7) and (5.12) that

\[
\begin{align*}
  u_M &\to 0 \quad \text{in } W^{1,p}(0, T; W_0^{1,p}(\Omega), W_0^{1,q^*}(\Omega)^*), \\
  u_M &\to 0 \quad \text{in } L^{q^*}(\Omega_T), \\
  H_{1,M} &\to 0 \quad \text{in } L^{p^*}(\Omega_T), \\
  H_{2,M} &\to 0 \quad \text{in } L^{q^*}(\Omega_T), \\
  k_M &\to 0 \quad \text{in } L^{p^*}(\Omega_T).
\end{align*}
\]

Now we have to choose similar to [9] a double sequence \( (\lambda_{M,k}, E_{M,k}) \) and exceptional sets \( (E_{M,k}) \) for which we want to apply Theorem 2.21. To this end we set

\[
g_M := M^*(|\nabla u_M|) + M^*((H_{1,M})^{\frac{1}{p'}}) + M^*((k_M)^{\frac{1}{p'}}).
\]

Here we extended all appearing functions by zero to the whole space. Due to the strong-type estimate of the maximal operator \( M^* \) we obtain

\[
\begin{align*}
  \|g_M\|_{L^p} &\leq \|M^*(|\nabla u_M|)\|_{L^p} + \|M^*((H_{1,M})^{\frac{1}{p'}})\|_{L^{p'}} + \|M^*((k_M)^{\frac{1}{p'}})\|_{L^{p'}} \\
  &\leq c(p, p') \left( \|\nabla u_M\|_{L^p} + \|H_{1,M}\|_{L^{p'}} + \|k_M\|_{L^{p'}} \right) \leq c.
\end{align*}
\]

21
Therefore we have for any $k \in \mathbb{N}$

$$c^p \geq \int_{2^k}^{2^{k+1}} p \lambda^{p-1}(\{ |g_M| > \lambda \}) d\lambda \geq p \int_{2^k}^{2^{k+1}} \lambda^{-1} d\lambda \inf_{2^k \leq \gamma \leq 2^{k+1}} \gamma^p |\{ |g_M| > \gamma \}|$$

$$\geq p \cdot 2^k \ln(2) \inf_{2^k \leq \gamma \leq 2^{k+1}} \gamma^p |\{ |g_M| > \gamma \}|$$

so that we are able to choose

$$\lambda_{M,k} \in \left[2^k, 2^{k+1}\right]$$

(5.28)

such that

$$\lambda_{M,k}^p |\{ |g_M| > \lambda_{M,k} \}| \leq c \cdot 2^{-k}$$

(5.29)

holds for any $k, M \in \mathbb{N}$. For any $k, M \in \mathbb{N}$ we define $G_{M,k} := \{ |g_M| > \lambda_{M,k} \}$ and $\alpha_{M,k} := \lambda_{M,k}^{2-p}$. Now (5.29) reads

$$\lambda_{M,k}^p |G_{M,k}| \leq c \cdot 2^{-k}. \quad (5.30)$$

Since $\lambda_{M,k} = \lambda_{M,k}^{2-p} = \lambda_{M,k}^{p-1}$, we have

$$G_{M,k} = \{ \mathcal{M}^*(|\nabla u_M|) + \mathcal{M}^*(|H_{1,M}|) + \mathcal{M}^*(|k_M|) > \lambda_{M,k} \}$$

$$\supset \{ \mathcal{M}^*(|\nabla u_M|) > \lambda_{M,k} \} \cup \{ \alpha_{M,k} \mathcal{M}^*(|H_{1,M}|) > \lambda_{M,k} \} \cup \{ \alpha_{M,k} \mathcal{M}^*(|k_M|) > \lambda_{M,k} \}. \quad (5.31)$$

Next we define

$$F_{M,k} := \{ \mathcal{M}^*(|H_{2,M}|) > \lambda_{M,k}^{p-1} \} = \{ \alpha_{M,k} \mathcal{M}^*(|H_{2,M}|) > \lambda_{M,k} \}. \quad (5.32)$$

Then the weak-type estimate of $\mathcal{M}^*$ and (5.27) imply

$$|F_{M,k}| \leq c(\lambda_{M,k}^{p-1})^{-\sigma} ||H_{2,M}||^\sigma_{L^\sigma} \overset{M \to \infty}{\to} 0 \quad (5.33)$$

for any fixed $k \in \mathbb{N}$. Since $\mathcal{M}^*$ is subadditive we conclude from (5.31) and (5.32) that

$$G_{M,k} \cup F_{M,k} \supset \{ \mathcal{M}^*(|\nabla u_M|) + \alpha_{M,k} \mathcal{M}^*(|H_M|) + \alpha_{M,k} \mathcal{M}^*(|k_M|) > 4\lambda_{M,k} \} \quad (5.34)$$

Moreover, from the fact that $\{ \mathcal{M}(f) > \beta \}$ is an open set for any $f \in L^\sigma$ and $\beta > 0$ it is easy to prove that $G_{M,k}$ and $F_{M,k}$ are open sets as well. We also define for each $M, k \in \mathbb{N}$ the set

$$H_{M,k} := \{ \mathcal{M}^*(|u_M|) > 1 \} \quad (5.35)$$

so that the weak-type estimate for $\mathcal{M}^*$ and (5.27) imply

$$|H_{M,k}| \leq c ||u_M||^\sigma_{L^\sigma} \overset{M \to \infty}{\to} 0. \quad (5.36)$$
Now we can define our exceptional set
\[ E_{M,k} := (G_{M,k} \cup E_{M,k} \cup H_{M,k}) \cap \Omega_T. \]  
(5.37)

Clearly from (5.30), (5.33) and (5.36) we get for any fixed \( k \in \mathbb{N} \)
\[ \limsup_{M \to \infty} \lambda_{M,k}^p |E_{M,k}| \leq c 2^{-k}. \]  
(5.38)

We choose an arbitrary cut-off function \( \zeta \in C_0^\infty(\Omega_T) \) and define the compact set \( K := \text{supp} \zeta. \) Due to (5.34) and (5.35) the set \( E_{M,k} \) satisfies
\[ \Omega_T \cap (O_{\lambda_{M,k}} \cup \mathcal{U}_1) \subset E_{M,k} \subset \Omega_T \]  
(5.39)

so that we are able to apply Theorem 2.21 with \( \Lambda = 4 \lambda_{M,k}, \alpha = \alpha_{M,k}, u = u_M, \ H = H_M, k = k_M, E = E_{M,k} \) and \( K = \text{supp} \zeta \) for any \( M, k \in \mathbb{N}. \) To ensure a better readability we denote \( \mathcal{T}_{M,k} := \mathcal{T}_{E_{M,k}}^{\alpha_{M,k}}. \) Due to Theorem 2.21 (i) the function \( (\mathcal{T}_{M,k} u_M)\zeta \) is an admissible test function for (5.26) and due to Theorem 2.21 (iv) we have
\[
\int_0^T \left( \frac{d u_M(t)}{dt}, (\mathcal{T}_{M,k} u_M(t))\zeta(t) \right) W^p_{\alpha,k} \, dt
= \frac{1}{2} \int_{\Omega_T} (|\mathcal{T}_{M,k} u_M|^2 - 2 u_M \cdot \mathcal{T}_{M,k} u_M) \partial_t \zeta
+ \int_{E_{m,k}} (\partial_t \mathcal{T}_{M,k} u_M) \cdot (\mathcal{T}_{M,k} u_M - u_M) \zeta.
\]

This, (5.26) and the product-rule leads to
\[
\int_{\Omega_T} \left( N(\nabla u^M, E) - \hat{N} \right) : (\nabla \mathcal{T}_{M,k} u_M) \zeta
= \int_{\Omega_T} (\hat{N} - N(\nabla u^M, E)) : (\mathcal{T}_{M,k} u_M \otimes \nabla) \zeta
+ \int_{\Omega_T} H_{2,M} : \nabla ((\mathcal{T}_{M,k} u_M)\zeta) + \int_{\Omega_T} k_M \cdot (\mathcal{T}_{M,k} u_M) \zeta
+ \frac{1}{2} \int_{\Omega_T} (2 u_M \cdot \mathcal{T}_{M,k} u_M - |\mathcal{T}_{M,k} u_M|^2) \partial_t \zeta
+ \int_{E_{m,k}} (\partial_t \mathcal{T}_{M,k} u_M) \cdot (u_M - \mathcal{T}_{M,k} u_M) \zeta
=: 1_{M,k} + 2_{M,k} + 3_{M,k} + 4_{M,k} + 5_{M,k}.
\]  
(5.40)

From now on we are able to use exactly the same ideas, which have been used in [9] to finish the proof. For the convenience of the reader we sketch them here. For any fixed \( k \in \mathbb{N} \) we will pass to the limit in \( M \to \infty \) in every integral of (5.40) separately.

(i) \( \limsup_{M \to \infty} (|1_{M,k}| + |3_{M,k}|) = 0. \)

Since \( \hat{N} - N(\nabla u^M, E) \) and \( k_M \) are bounded in \( L^p(\Omega_T) \) by (5.27), we only need Hölder’s inequality, the continuity result in Lemma 2.19 and (5.27) to prove
\[
|1_{M,k}| \leq \|\zeta\|_{L^\infty(\Omega_T)} \|\hat{N} - N(\nabla u^M, E)\|_{L^{p'}(\Omega_T)} \|\mathcal{T}_{M,k} u_M\|_{L^p(\Omega_T)}
\leq c \|u_M\|_{L^p(\Omega_T)} \leq c \|u_M\|_{L^{2p}(\Omega_T)} \to 0
\]
and

\[ |3_{M,k}| \leq \| \zeta \|_{L^\infty(\Omega_T)} \| k_M \|_{L^p(\Omega_T)} \| T_{M,k} \mathbf{u}_M \|_{L^p(\Omega_T)} \leq c \| \mathbf{u}_M \|_{L^p(\Omega_T)} \leq c \| \mathbf{u}_M \|_{L^{2p}(\Omega_T)} \to 0. \]

(ii) \( \limsup_{M \to \infty} |2_{M,k}| = 0. \)

We estimate

\[ |2_{M,k}| \leq \| H_{2,M} \|_{L^1(\Omega_T)} \| \nabla((T_{M,k} \mathbf{u}_M) \zeta) \|_{L^\infty(\Omega_T)}. \]

The boundedness of \( \Omega_T \) and (5.27) implies \( H_{2,M} \to 0 \) in \( L^1(\Omega_T) \), so that it remains to prove that for fixed \( k \in \mathbb{N} \) the sequence \( (\nabla((T_{M,k} \mathbf{u}_M) \zeta))_{M \in \mathbb{N}} \) is bounded in \( L^\infty(\text{supp}(\zeta)) \). In view of (5.28) we conclude that \( \alpha_{M,k}, \lambda_{M,k} = \lambda_{M,k}^{-2-p}, \lambda_{M,k}^{-1} \) and \( \alpha_{M,k}^{-1} \) are all bounded in \( M \) for fixed \( k \in \mathbb{N} \). Additionally

\[ \inf_{M \in \mathbb{N}} \delta_{\alpha_{M,k},K} = d_{\alpha_{M,k}}(K, \partial \Omega_T) > 0 \]

holds for fixed \( k \in \mathbb{N} \) as well. According to Theorem 2.21(ii) we are able to estimate

\[ \| \nabla((T_{M,k} \mathbf{u}_M) \zeta) \|_{L^\infty(K)} \leq c(\lambda_{M,k} + \alpha_{M,k}^{-1} \delta_{\alpha_{M,k},K} \| \mathbf{u}_M \|_{L^1(E_{M,k})}) + c(1 + \alpha_{M,k}^{-1} \delta_{\alpha_{M,k},K} \| \mathbf{u}_M \|_{L^1(E_{M,k})}) \leq c(k), \]

so that altogether \( \limsup_{M \to \infty} |2_{M,k}| = 0 \) holds.

(iii) \( \limsup_{M \to \infty} |4_{M,k}| = 0. \)

Using again Hölder’s inequality, the continuity result in Lemma 2.19 and (5.27) we estimate

\[ \limsup_{M \to \infty} |4_{M,k}| \leq \limsup_{M \to \infty} c(1 + \| \partial_t \zeta \|_{L^\infty(\Omega_T)}) \| \mathbf{u}_M \|_{L^2(\Omega_T)}^2 = 0. \]

(iv) \( \limsup_{M \to \infty} |5_{M,k}| \leq c 2^{-k}. \)

Using supp \( \zeta = K \) and Theorem 2.21(iii) we estimate

\[ |5_{M,k}| \leq \| (\partial_t T_{M,k} \mathbf{u}_M) \cdot (\mathbf{u}_M - T_{M,k} \mathbf{u}_M) \|_{L^1(E_{M,k} \cap K)} \leq c \alpha_{M,k}^{-1} |E_{M,k}| (\lambda_{M,k} + \alpha_{M,k}^{-1} \delta_{\alpha_{M,k},K} \| \mathbf{u}_M \|_{L^1(E_{M,k})})^2. \]

Since \( \alpha_{M,k}^{-1} |E_{M,k}|, \lambda_{M,k} \) and \( \delta_{\alpha_{M,k},K} \) are all bounded in \( M \) for fixed \( k \in \mathbb{N} \) and since

\[ \| \mathbf{u}_M \|_{L^1(E_{M,k})} \leq \| \mathbf{u}_M \|_{L^1(\Omega_T)} \xrightarrow{M \to \infty} 0 \]

holds due to (5.39) and (5.27), we conclude that

\[ \limsup_{M \to \infty} |5_{M,k}| \leq \limsup_{M \to \infty} c \alpha_{M,k}^{-1} |E_{M,k}| \lambda_{M,k}^2 = \limsup_{M \to \infty} c \lambda_{M,k}^2 |E_{M,k}| \xrightarrow{M \to \infty} c 2^{-k}. \]
Altogether, (i)–(iv) imply
\[
\limsup_{M \to \infty} \int_{\Omega_T} \left| \left( N(\nabla \omega^M, E) - \nabla \omega \right) : (\nabla \mathcal{T}_{M,k} u_M) \zeta \right| \leq c 2^{-k}. \tag{5.41}
\]

On the other hand, from (5.5), (5.27), Theorem 2.21 (ii) and the boundedness of \( \sigma_{M,k}^{-1}, \delta_{M,k}^{-1} \) for fixed \( k \in \mathbb{N} \), we conclude
\[
\limsup_{M \to \infty} \int_{E_{M,k}} \left| \left( N(\nabla \omega^M, E) - \nabla \omega \right) : (\nabla \mathcal{T}_{M,k} u_M) \zeta \right| \\
\leq \limsup_{M \to \infty} \| N(\nabla \omega^M, E) - \nabla \omega \|_{L^p(\Omega_T)} \| \nabla \mathcal{T}_{M,k} u_M \|_{L^\infty(K)} E_{M,k}^\frac{1}{p} \leq 2^{-\frac{k}{\beta}}. \tag{5.42}
\]

Since \( \nabla \mathcal{T}_{M,k} u_M = \nabla u_M = \nabla \omega^M - \nabla \omega \) holds on \( \Omega_T \setminus E_{M,k} \) due to Definition 2.18, the estimates (5.41) and (5.42) imply
\[
\limsup_{M \to \infty} \int_{\Omega_T \setminus E_{M,k}} \left| \left( N(\nabla \omega^M, E) - \nabla \omega \right) : (\nabla \omega^M - \nabla \omega) \zeta \right| \leq c 2^{-\frac{k}{\beta}}. \tag{5.43}
\]

Due to (5.30), (5.33), (5.36) and (5.43) we are able to to find for each \( k \in \mathbb{N} \) a number \( M_k \in \mathbb{N} \) such that
\[
\left| \int_{\Omega_T \setminus E_{M_k,k}} \left( N(\nabla \omega^{M_k}, E) - \nabla \omega \right) : (\nabla \omega^{M_k} - \nabla \omega) \zeta \right| \leq c 2^{-\frac{k}{\beta}}, \tag{5.44}
\]
\[
|G_{M_k,k}| \leq c 2^{-k}, \quad |F_{M_k,k}| \leq c 2^{-k}, \quad |H_{M_k,k}| \leq c 2^{-k}.
\]

Now we set \( \zeta_k := \zeta|_{\Omega_T \setminus E_{M_k,k}} \). Clearly we have
\[
\zeta_k \to \zeta \quad \text{pointwise in} \quad \bigcup_{k=1}^{\infty} \bigcap_{\ell=k}^{\infty} (\Omega_T \setminus E_{M_\ell,\ell}) = \Omega_T \setminus \bigcap_{k=1}^{\infty} \bigcup_{\ell=k}^{\infty} E_{M_\ell,\ell}.
\]

From (5.44) we conclude that \( |\bigcap_{k=1}^{\infty} \bigcup_{\ell=k}^{\infty} E_{M_\ell,\ell}| = 0 \), so that
\[
\zeta_k \to \zeta \quad \text{holds a.e. in} \quad \Omega_T.
\]

This in turn implies the strong convergence of \( \nabla \zeta_k \to \nabla \zeta \) in \( L^p(\Omega_T) \) and \( (\nabla \omega)_k \to (\nabla \omega) \zeta \) in \( L^p(\Omega_T) \). Now this together with (5.44) implies
\[
\lim_{k \to \infty} \int_{\Omega_T} N(\nabla \omega^{M_k}, E) : (\nabla \omega^{M_k}) \zeta_k - \nabla \omega : (\nabla \omega) \zeta_k
\leq \lim_{k \to \infty} \int_{\Omega_T} \left( N(\nabla \omega^{M_k}, E) - \nabla \omega \right) : (\nabla \omega^{M_k} - \nabla \omega) \zeta_k
+ \lim_{k \to \infty} \int_{\Omega_T} N(\nabla \omega^{M_k}, E) : (\nabla \omega) \zeta_k + \nabla \omega : (\nabla \omega^{M_k,k}) \zeta_k - 2 \nabla \omega : (\nabla \omega) \zeta_k
= 0,
\]
so that we have
\[
\lim_{k \to \infty} \int_{\Omega_T} N(\nabla \omega^{M_k}, E) : (\nabla \omega^{M_k}) \zeta_k = \int_{\Omega_T} \nabla \omega : (\nabla \omega) \zeta.
\]

The local Minty trick (cf. [28, Lemma A.2]) implies \( \nabla \zeta = N(\nabla \omega, E) \zeta \) a.e. in \( \Omega_T \). \( \blacksquare \)
Remark 5.45. It is also possible to derive the weak continuity of the solutions $v$ and $\omega$. Due to (5.12) and (2.3) we have that $v \in C(0,T;\mathcal{V}_\sigma'(\Omega)^*)$ and $\omega \in C(0,T;W^{1,\sigma'}_0(\Omega)^*)$. Since the solutions belong to the spaces $L^\infty(0,T;H(\Omega))$ and $L^\infty(0,T;L^2(\Omega))$, respectively, we are able to conclude (cf. [6, Lemma II.5.9]) that $v$ is weakly continuous with values in $H(\Omega)$, while $\omega$ is weakly continuous with values in $L^2(\Omega)$. Let us have a quick look why the equality $v(0) = v_0$ also holds in the sense of this weak continuity. Since the approximative solutions $v^M$ possess the required initial values, we are able to derive from the integration by parts formula in (2.4) that

$$(v_0, \eta)_H = \int_0^T (\frac{dv^M}{dt}, \eta)_{V_\sigma'} + \int_{\Omega_T} v^M \cdot \eta \partial_t \zeta$$

holds for any $\eta \in \mathcal{V}(\Omega)$ and $\zeta \in C^\infty([0,T])$ with $\zeta(0) = -1$ and $\zeta(T) = 0$. With this $\zeta$ we have $\zeta v \in W^{1,p,\sigma}(0,T;V_\sigma'(\Omega),\mathcal{V}_\sigma'(\Omega)^*)$ which implies

$$(v(0), \eta)_H = (v(0), \eta)_{V_\sigma'} = \int_0^T (\frac{dv}{dt}, \eta)_{V_\sigma'} + \int_{\Omega_T} v \cdot \eta \partial_t \zeta,$$

so that we get $(v_0, \eta)_H = (v(0), \eta)_H$ due to (5.12). This implies $v_0 = v(0)$ in $H(\Omega)$. The argumentation for $\omega_0 = \omega(0)$ in $L^2(\Omega)$ is the same.

References

[1] R.J. Atkin, X. Shi, and W.A. Bulloch, Solutions of the Constitutive Equations for the Flow of an Electrorheological Fluid in Radial Configurations, J. Rheology 35 (1991), 1441–1461.

[2] E. Bäumle, Existenz schwacher Lösungen für instationäre mikropolare elektroreologische Flüssigkeiten, 2014, Diplomarbeit, Universität Freiburg.

[3] E. Bäumle and M. Růžička, Note on the existence theory for evolution equations with pseudo-monotone operators.

[4] L. Belenki, L. C. Berselli, L. Diening, and M. Růžička, On the Finite Element Approximation of $p$-Stokes Systems, SIAM J. Numer. Anal. 50 (2012), no. 2, 373–397.

[5] L. C. Berselli, L. Diening, and M. Růžička, Existence of strong solutions for incompressible fluids with shear dependent viscosities, J. Math. Fluid Mech. 12 (2010), no. 1, 101–132.

[6] F. Boyer and P. Fabrie, Eléments d’analyse pour l’étude de quelques modèles d’écoulements de fluides visqueux incompressibles, Mathématiques & Applications (Berlin) [Mathematics & Applications], vol. 52, Springer-Verlag, Berlin, 2006.

[7] D. Breit, L. Diening, and S. Schwarzacher, Solenoidal Lipschitz Truncation For Parabolic PDEs, M3AS 23 (2013), 2671–2700.
[8] L. Diening, C. Ebmeyer, and M. Růžička, Optimal Convergence for the Implicit Space-Time Discretization of Parabolic Systems with $p$-Structure, SIAM J. Numer. Anal. 45 (2007), 457–472.

[9] L. Diening, M. Růžička, and J. Wolf, Existence of weak solutions for unsteady motions of generalized Newtonian fluids, Ann. Scuola Norm. Sup. Pisa Cl. Sci. V IX (2010), 1–46.

[10] W. Eckart and M. Růžička, Modeling Micropolar Electrorheological Fluids, Int. J. Appl. Mech. Eng. 11 (2006), 813–844.

[11] A.C. Eringen, Microcontinuum field theories. I,II., Springer-Verlag, New York, 1999.

[12] F. Ettwein, Mikropolare Elektorheologische Flüssigkeiten, Tech. Report, University Freiburg, 2007, PhD thesis.

[13] F. Ettwein, M. Růžička, and B. Weber, Existence of steady solutions for micropolar electrorheological fluid flows, Nonlin. Anal. TMA 125 (2015), 1–29.

[14] H. Gajewski, K. Gröger, and K. Zacharias, Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen, Akademie-Verlag, Berlin, 1974.

[15] E. Hewitt and K. Stromberg, Real and abstract analysis. A modern treatment of the theory of functions of a real variable, Springer-Verlag, New York, 1965.

[16] J.L. Lions, Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires, Dunod, Paris, 1969.

[17] G. Lukaszewicz, Micropolar Fluids. Theory and applications, Birkhäuser Boston Inc., Boston, MA, 1999.

[18] J. Málek, J. Nečas, M. Rokyta, and M. Růžička, Weak and measure-valued solutions to evolutionary PDEs, Applied Mathematics and Mathematical Computation, vol. 13, Chapman & Hall, London, 1996.

[19] J. Málek, M. Růžička, and V.V. Shelukhin, Herschel–Bulkley Fluids: Existence and Regularity of Steady Flows, M3AS 15 (2005), no. 12, 1845–1861.

[20] G. Dal Maso and F. Murat, Almost Everywhere Convergence of Gradients of Solutions to Nonlinear Elliptic Systems, Nonlin. Anal. TMA 31 (1998), 405–412.

[21] K.R. Rajagopal and M. Růžička, Mathematical Modeling of Electrorheological Materials, Contin. Mech. Thermodyn. 13 (2001), 59–78.

[22] K.R. Rajagopal and A.S. Wineman, Flow of Electrorheological Materials, Acta Mechanica 91 (1992), 57–75.

[23] M. Růžička, Electrorheological Fluids: Modeling and Mathematical Theory, Lecture Notes in Math., vol. 1748, Springer, Berlin, 2000.
[24] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, N.J., 1970.

[25] B. Weber, *Existenz sehr schwacher Lösungen für mikropolare elektrorheologische Flüssigkeiten*, 2011, Diplomarbeit, Universität Freiburg.

[26] A. S. Wineman and K. R. Rajagopal, *On Constitutive Equations for Electrorheological Materials*, Cont. Mech. and Thermodynamics 7 (1995), 1–22.

[27] J. Wolf, *Regularität schwacher Lösungen elliptischer und parabolischer Systeme partieller Differentialgleichungen mit Entartung. Der Fall $1 < p < 2$*, 2001, Dissertation, Humboldt Universität.

[28] J. Wolf, *Existence of weak solutions to the equations of nonstationary motion of non–Newtonian fluids with shear–dependent viscosity*, J. Math. Fluid Mech. 9 (2007), 104–138.

[29] E. Zeidler, *Nonlinear functional analysis and its applications. II/A*, Springer, New York, 1990, Linear monotone operators.