Sensitivity Analysis of a Stationary Point Set Map under Total Perturbations. Part 2: Robinson Stability

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Abstract In Part 1 of this paper, we have estimated the Fréchet coderivative and the Mordukhovich coderivative of the stationary point set map of a smooth parametric optimization problem with one smooth functional constraint under total perturbations. From these estimates, necessary and sufficient conditions for the local Lipschitz-like property of the map have been obtained. In this part, we establish sufficient conditions for the Robinson stability of the stationary point set map. This allows us to revisit and extend several stability theorems in indefinite quadratic programming. A comparison of our results with the ones which can be obtained via another approach is also given.

Keywords Smooth parametric optimization problem · Smooth functional constraint · Stationary point set map · Robinson stability · Coderivative

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1 Introduction

Appeared at the early stage of optimization theory, smooth programming problems continue to attract common attention of the optimization community due to their importance and beauty. Polynomial optimization problems, including nonconvex quadratic programs, are typical examples of such problems.

The present paper investigates the Lipschitz-like property and the Robinson stability of the stationary point set map of a smooth parametric optimization problem with one smooth functional constraint under total perturbations.

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In Part 1 of the paper [4], we have computed and estimated the Fréchet coderivative and the Mordukhovich coderivative of the stationary point set map by applying some theorems of Levy and Mordukhovich [2] and other related results. From the obtained formulas we derive necessary and sufficient conditions for the local Lipschitz-like property of the stationary point set map. This leads us to new insights into the preceding deep investigations of Levy and Mordukhovich in the just-cited paper and of Qui [3,4].

The reader is referred to Part 1 of this paper [1] for a survey on the local Lipschitz-like property of multifunctions, the Robinson stability of an implicit multifunction, the Mordukhovich criterion for the local Lipschitz-like property of locally closed multifunctions, and some relevant material.

This part of the paper is organized as follows. Section 2 recalls some basic concepts from variational analysis, formulates the problem studied herein, and presents a series of auxiliary results in a unified form. In Section 3, we obtain sufficient conditions for the Robinson stability of the stationary point set map. Section 4 is devoted to several stability theorems in indefinite quadratic programming. A comparison of our results with the ones which can be obtained via Robinson’s theory of strongly regular generalized equations [5] is given in Section 5. The final section contains some concluding remarks.

2 Preliminaries

The scalar product and the norm in a finite-dimensional Euclidean space are denoted respectively by \langle \cdot , \cdot \rangle and \| \cdot \|. The symbols \( B(x, \rho) \) and \( B(x, \rho) \) stand for the open (resp., closed) ball centered at \( x \in X \) with radius \( \rho > 0 \). The distance \( \inf_{u \in A} \| x - u \| \) from \( x \in X \) to a subset \( A \subset X \) is denoted by \( d(x, A) \).

We now recall several basic concepts from variational analysis [6,7] which will be used intensively later on.

The Fréchet normal cone (also called the prenormal cone, or the regular normal cone) to a set \( \Omega \subset \mathbb{R}^n \) at \( \bar{v} \in \Omega \) is given by

\[
\hat{N}_\Omega(\bar{v}) = \left\{ v' \in \mathbb{R}^n \mid \limsup_{v \xrightarrow{D} \bar{v}} \frac{\langle v', v - \bar{v} \rangle}{\|v - \bar{v}\|} \leq 0 \right\},
\]

where \( v \xrightarrow{D} \bar{v} \) means \( v \to \bar{v} \) with \( v \in \Omega \). By convention, \( \hat{N}_\Omega(\bar{v}) := \emptyset \) when \( \bar{v} \notin \Omega \). Provided that \( \Omega \) is locally closed around \( \bar{v} \in \Omega \), one calls

\[
N_\Omega(\bar{v}) = \text{Lim sup } \hat{N}_\Omega(v) = \left\{ v' \in \mathbb{R}^n \mid \exists \text{ sequences } v_k \to \bar{v}, \ v'_k \to v', \right. \left. \text{ with } v'_k \in \hat{N}_\Omega(v_k) \text{ for all } k = 1, 2, \ldots \right\}
\]

the Mordukhovich (or limiting/basic) normal cone to \( \Omega \) at \( \bar{v} \). If \( \bar{v} \notin \Omega \), then one puts \( N_\Omega(\bar{v}) = \emptyset \).

A multifunction \( \Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) is said to be locally closed around a point \( \bar{z} = (\bar{x}, \bar{y}) \) from \( \text{gph} \Phi := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in \Phi(x)\} \) if \( \text{gph} \Phi \) is locally closed around \( \bar{z} \). Here, the product space \( \mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m \) is equipped with the topology generated by the sum norm \( \| (x, y) \| = \| x \| + \| y \| \).

For any \( \bar{z} = (\bar{x}, \bar{y}) \in \text{gph} \Phi \),

\[
\hat{D}^* \Phi(\bar{z})(y') := \left\{ x'' \in \mathbb{R}^n \mid (x'', -y') \in \hat{N}_{\text{gph} \Phi}(\bar{z}) \right\}, \quad (y' \in \mathbb{R}^m)
\]
are called the Fréchet coderivative values of \( \Phi \) at \( \bar{z} \). Similarly, the Mordukhovich coderivative (limiting coderivative) values of \( \Phi \) at \( \bar{z} \) are defined by

\[
D^*\Phi(\bar{z})(y') := \left\{ x' \in \mathbb{R}^n \mid (x', -y') \in N_{\text{gph}}(\bar{z}) \right\} \quad (y' \in \mathbb{R}^m).
\]

Thus, \( \tilde{D}^*\Phi(\bar{z}) \) and \( D^*\Phi(\bar{z}) \) are multifunctions from \( \mathbb{R}^m \) to \( \mathbb{R}^n \). By [6] Theorem 1.38, if \( \Phi \) is strictly Fréchet differentiable at \( \bar{x} \), then

\[
\tilde{D}^*\Phi(\bar{x})(y') = D^*\Phi(\bar{x})(y') = \{ \nabla \Phi(\bar{x})^*(y') \}
\]

for any \( y' \in \mathbb{R}^m \).

Suppose that \( X, Y, \) and \( Z \) are finite-dimensional Euclidean spaces. Consider a function \( \psi : X \to \mathbb{R} \) with \( |\psi(\bar{x})| < \infty \). The set

\[
\partial \psi(\bar{x}) := \{ x' \in X^* \mid (x', -1) \in \text{epi} \psi(\bar{x}, \psi(\bar{x})) \}
\]

is the Mordukhovich subdifferential of \( \psi \) at \( \bar{x} \). We put \( \partial \psi(\bar{x}) = \emptyset \) if \( |\psi(\bar{x})| = \infty \). The set

\[
\partial^\infty \psi(\bar{x}) := \{ x^* \in X^* \mid (x^*, 0) \in \text{epi} \psi(\bar{x}, \psi(\bar{x})) \}
\]

is the singular subdifferential of \( \psi \) at \( \bar{x} \). For a set \( \Omega \subset X \) and a point \( \bar{x} \in \Omega \), we have

\[
N_\Omega(\bar{x}) = \partial \delta_\Omega(\bar{x}) = \partial^\infty \delta_\Omega(\bar{x}),
\]

where \( \delta_\Omega(\bar{x}) \) is the indicator function of \( \Omega \); see [6] Proposition 1.79. If \( \psi \) depends on two variables \( x \) and \( y \), and \( |\psi(\bar{x}, \bar{y})| < \infty \), then \( \partial_x \psi(\bar{x}, \bar{y}) \) denotes the Mordukhovich subdifferential of \( \psi(\cdot, \bar{y}) \) at \( \bar{x} \). For any \( \bar{v} \in \partial \psi(\bar{x}) \),

\[
\partial^2 \psi(\bar{x} | \bar{v})(u) := D^* (\partial \psi)(\bar{x} | \bar{v})(u) \quad (u \in X^{**} = X)
\]

is the limiting second-order subdifferential (or the generalized Hessian).

A multifunction \( G : Y \rightrightarrows X \) is said to be locally Lipschitz-like around \( (\bar{y}, \bar{x}) \in \text{gph} G \) if there exists a constant \( \ell > 0 \) and neighborhoods \( U \) of \( \bar{x} \), \( V \) of \( \bar{y} \) such that

\[
G(y') \cap U \subset G(y) + \ell||y' - y||B_X \quad \forall y, y' \in V,
\]

where \( B_X \) denotes the closed unit ball in \( X \). When \( G \) is locally closed around \( (\bar{y}, \bar{x}) \), the Mordukhovich criterion (see [5], [7] Theorem 9.40, and [6] Theorem 4.10) says that \( G \) is locally Lipschitz-like around \( (\bar{y}, \bar{x}) \) if and only if

\[
D^*G(\bar{y}, \bar{x})(0) = \{0\}.
\]

For a multifunction \( F : X \times Y \rightrightarrows Z \) and a pair \( (\bar{x}, \bar{y}) \in X \times Y \) satisfying \( 0 \in F(\bar{x}, y) \), we say that the implicit multifunction \( G : Y \rightrightarrows X \) given by

\[
G(y) = \{ x \in X \mid 0 \in F(\bar{x}, y) \}
\]

has the Robinson stability at \( \omega_0 := (\bar{x}, \bar{y}, 0) \) if there exist constants \( r > 0, \gamma > 0 \), and neighborhoods \( U \) of \( \bar{x} \), \( V \) of \( \bar{y} \) such that

\[
d(x, G(y)) \leq rd(0, F(x, y))
\]

for any \( (x, y) \in U \times V \) with \( d(0, F(x, y)) < \gamma \). Note that the condition \( d(0, F(x, y)) < \gamma \) can be omitted if \( F \) is inner semicontinuous at \( (\bar{x}, \bar{y}, 0) \); see [9]. Note that, in some cases, the Robinson stability of \( G \) at \( (\bar{x}, \bar{y}, 0) \) implies its local Lipschitz-likeliness around \( (\bar{y}, \bar{x}) \); see, e.g., [10]. For the generalized linear constraint system studied in [9], these properties are equivalent. In the sequel, we will see that the regularity conditions in use guarantee for our stationary point set map to have both properties.
Now, let $f_0$ and $F$ be twice continuously differentiable real-valued functions ($C^2$-functions for brevity) defined on the product $\mathbb{R}^n \times \mathbb{R}^d$ of two Euclidean spaces. For every $w \in \mathbb{R}^d$, we consider the parametric optimization problem

\[(P_w) \quad \text{Minimize } f_0(x, w) \text{ subject to } x \in \mathbb{R}^n \text{ and } F(x, w) \leq 0.\]

The constraint set of $(P_w)$ is $C(w) := \{x \in \mathbb{R}^n \mid F(x, w) \leq 0\}$. The stationary point set of $(P_w)$ is defined by

\[S(w) = \{x \in \mathbb{R}^n \mid 0 \in \nabla_x f_0(x, w) + N_{C(w)}(x)\}. \tag{1}\]

When $w$ varies on $\mathbb{R}^d$, one has a multifunction $S : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ with $S(w)$ being calculated by \(\text{(1)}\). Setting $f(x, w) = g(F(x, w)) = (g \circ F)(x, w)$, where $g(y) = \delta_{\mathbb{R}_+}(y)$, i.e., $g(y) = 0$ for $y \in (-\infty, 0]$ and $g(y) = +\infty$ for $y > 0$, we can rewrite \(\text{(1)}\) as

\[S(w) = \{x \in \mathbb{R}^n \mid 0 \in \nabla_x f_0(x, w) + \partial_x f(x, w)\}. \tag{2}\]

Fix a vector $w = \bar{w} \in \mathbb{R}^d$ and suppose that $x \in S(\bar{w})$. Since $(P_{\bar{w}})$ has a single smooth inequality constraint, the Mangasarian-Fromovitz Constraint Qualification is fulfilled at $x \in C(\bar{w})$ if and only if

\[\text{If } F(x, \bar{w}) = 0, \text{ then } \nabla_x F(x, \bar{w}) \neq 0. \tag{MFCQ}\]

In what follows, we assume that $(\text{MFCQ})$ is valid. To study the stability of the stationary point set map $S$ around the $(\bar{w}, \bar{x})$ in $\text{gph}S$, we compute the Mordukhovich and the Fréchet coderivatives of the partial subdifferential map $\partial_x f : \mathbb{R}^n \times \mathbb{R}^d \rightrightarrows \mathbb{R}^n$. In general, there is no explicit formula for the coderivatives of such maps. However, the results of \cite{2} provide us with some tools which allow us to estimate the coderivative value $D^*S(\bar{w}|\bar{x})(x')$ for every $x' \in \mathbb{R}^n$.

The fulfillment of MFCQ at $(\bar{x}, \bar{w})$ implies that $g(x, w) = g(F(x, w))$ is a strongly amenable in $x$ at $\bar{x}$ with compatible parameterization in $w$ at $\bar{w}$. Then, by \cite{7} Theorem 10.49, for $(x, w)$ near $(\bar{x}, \bar{w})$, we have

\[\partial f(x, w) = \nabla F(x, w)^*(\partial g(F(x, w))) \tag{3}\]

and

\[\partial_x f(x, w) = \nabla_x F(x, w)^*(\partial g(F(x, w))); \tag{4}\]

see \cite{2} formulas (14) and (15)]).

In order to estimate the limiting second-order subdifferential of $f$, we need the following result.

**Lemma 2.1** (see \cite{2} Theorem 3.1) Suppose that $\bar{v} \in \partial f(\bar{x}, \bar{w})$. Then, for any $v' \in \mathbb{R}^n \times \mathbb{R}^d$,

\[
\partial^2 f((\bar{x}, \bar{w})|v')(v') = \bigcup_{\bar{y} \in \partial g(F(\bar{x}, \bar{w})) \text{ with } \nabla F(\bar{x}, \bar{w})^* \bar{y} = \bar{v}} \left(\nabla^2 (\bar{y} \cdot F)(\bar{x}, \bar{w})v' + D^*(\partial g \circ F)(\bar{x}, \bar{w})|\bar{y})(\nabla F(\bar{x}, \bar{w})v')\right),
\]

where the function $\bar{y} \cdot F : \mathbb{R}^{n+d} \to \mathbb{R}$ is defined by $(\bar{y} \cdot F)(x, w) := \bar{y}F(x, w)$.

If, in addition, at every $\bar{y} \in \partial g(F(\bar{x}, \bar{w}))$ with $\nabla F(\bar{x}, \bar{w})^* \bar{y} = \bar{v}$, one has the second-order constraint qualification

\[\partial^2 g(F(\bar{x}, \bar{w})|\bar{y})(0) \cap \ker \nabla F(\bar{x}, \bar{w})^* = \{0\}, \tag{5}\]

then the estimate above for the second-order subdifferential can be refined by replacing the coderivative of the multifunction $\partial g \circ F$ via the inclusion

\[D^*(\partial g \circ F)((\bar{x}, \bar{w})|\bar{y})(\nabla F(\bar{x}, \bar{w})v') \subset \nabla F(\bar{x}, \bar{w})^* \partial^2 g(F(\bar{x}, \bar{w})|\bar{y})(\nabla F(\bar{x}, \bar{w})v').\]
In our problem \((P_w)\), condition (5) can be omitted. Indeed, \(\bar{y} \in \partial g(F(\bar{x}, \bar{w}))\) if and only if \(\bar{y} \in N_{\text{gph}}(F(\bar{x}, \bar{w}))\). Hence, \(\bar{y} \geq 0\). Clearly,

\[ \text{gph} \partial g = (\mathbb{R}_- \times \{0\}) \cup \{\{0\} \times \mathbb{R}_+\}. \]

If \(F(\bar{x}, \bar{w}) < 0\), then \(\bar{y} = 0\) and \(N_{\text{gph}} \partial g(F(\bar{x}, \bar{w}), \bar{y}) = \{0\} \times \mathbb{R}_+\). It follows that

\[
\partial^2 g(F(\bar{x}, \bar{w})| \bar{y})(0) = D^*(\partial g(F(\bar{x}, \bar{w})| \bar{y}))(0) = \{u' \in \mathbb{R} \mid (u', 0) \in N_{\text{gph}} \partial g(F(\bar{x}, \bar{w}), \bar{y})\} = \{0\}.
\]

So (5) is satisfied. If \(F(\bar{x}, \bar{w}) = 0\), then \((\text{MFCQ})\) implies \(\nabla F(\bar{x}, \bar{w}) \neq 0\).

Hence the linear operator \(\nabla F(\bar{x}, \bar{w}) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}\) is surjective. Thus \(\ker \nabla F(\bar{x}, \bar{w})^* = \{0\}\) by \([6, \text{Lemma 1.18}]\), and we see that (5) is fulfilled.

Therefore, applied to \((P_w)\), Remark 2.1 can be reformulated as follows: For any \(\bar{v} \in \partial f(\bar{x}, \bar{w})\) and \(v' \in \mathbb{R}^n \times \mathbb{R}^d\),

\[
\partial^2 f((\bar{x}, \bar{w})|\bar{v})(v') \subset \bigcup_{\bar{y} \in \partial g(F(\bar{x}, \bar{w}))} \left(\nabla^2 (\bar{y} \cdot F)(\bar{x}, \bar{w})v' + \Omega_1(\bar{y}, v')\right),
\]

where

\[
\Omega_1(\bar{y}, v') := \nabla F(\bar{x}, \bar{w})^* \partial^2 g(F(\bar{x}, \bar{w})| \bar{y})(\nabla F(\bar{x}, \bar{w})v').
\]

Remark 2.1 Concerning the paper \([11]\), observe that the set \(\partial^2 f((\bar{x}, \bar{w})|\bar{v})(v')\) in formula (5) is analogous to the set \(\partial^2(\bar{x}, \bar{w}, \bar{y}) (u)\) (a value of the extended partial second-order subdifferential) in formula (3.4) of that work. A careful checking shows that equality (3.4) of \([11]\) implies the upper estimate (6).

In what follows, for any \(\bar{v} = (\bar{v}_x, \bar{v}_u) \in \mathbb{R}^n \times \mathbb{R}^d\), we put \(\text{proj}_1 \bar{v} = \bar{v}_x\). The upper estimation for the coderivative values of the stationary point set map \(S\) given by Levy and Mordukhovich \([2]\) requires the following regularity condition: For any \(v'_1 \in \mathbb{R}^n\),

\[
0 \in \nabla^2 f_0(\bar{x}, \bar{w})^*(v'_1, 0) + \bigcup_{v \in \partial f(\bar{x}, \bar{w})} \partial^2 f((\bar{x}, \bar{w})|v)(v'_1, 0) \quad \Rightarrow \quad v'_1 = 0
\]

(see \([2, \text{formula (11)}]\)). For our problem \((P_w)\), by the assumption \((\text{MFCQ})\) and formula (5), we have \(\partial f(\bar{x}, \bar{w}) = \nabla F(\bar{x}, \bar{w})^* (\partial g(\bar{x}, \bar{w}))\). In addition, it is easy to show that, for every \(\bar{y} \in \partial g(\bar{x}, \bar{w})\), \(\text{proj}_1 (\nabla F(\bar{x}, \bar{w})^* \bar{y}) = \nabla_x F(\bar{x}, \bar{w})^* \bar{y}\).

Hence

\[
\bigcup_{v \in \partial f(\bar{x}, \bar{w})} \partial^2 f((\bar{x}, \bar{w})|v)(v'_1, 0) = \bigcup_{\bar{y} \in \partial g(F(\bar{x}, \bar{w}))} \partial^2 f((\bar{x}, \bar{w})|\nabla F(\bar{x}, \bar{w})^* \bar{y})(v'_1, 0).
\]

So (7) is equivalent to the following condition:

\[
0 \in \nabla^2 f_0(\bar{x}, \bar{w})^*(v'_1, 0) + \Omega_2(v'_1) \quad \Rightarrow \quad v'_1 = 0,
\]

where

\[
\Omega_2(v'_1) := \bigcup_{\bar{y} \in \partial g(F(\bar{x}, \bar{w}))} \partial^2 f((\bar{x}, \bar{w})|\nabla F(\bar{x}, \bar{w})^* \bar{y})(v'_1, 0).
\]

The next result from \([2]\) provides us with an upper estimation for the values of the coderivative map \(D^* S(\bar{w}|\bar{x}) : \mathbb{R}^n \to \mathbb{R}^d\).
Lemma 2.2 (see [2, Corollary 3.1]) If the regularity condition \((C0)\) holds then, for each \(x' \in \mathbb{R}^n\), the coderivative value \(D^* S(\bar{w}|x')(x')\) is contained in the set of \(w' \in \mathbb{R}^d\) for which there exists a vector \(v'_1 \in \mathbb{R}^n\) with
\[
(-x', w') - \nabla^2 f_0(x, \bar{w}) (v'_1, 0) \in \Omega_2(v'_1).
\]

Although it is rather difficult to compute the set \(\Omega_2(v'_1)\), we can still estimate it by using \((3)\).

**Upper estimates** for the limiting coderivative values of \(S\) can be derived from a result of Levy and Mordukhovich [2 Theorem 2.1]. But, a constraint qualification must be imposed to have these estimates (see [12, p. 1020] for details). Interestingly, due to a result of Lee and Yen [12, Theorem 3.4], sharp lower estimates for the Fréchet coderivative values of \(S\) can be given without any condition. Put \(G(x, w) = \nabla_x f_0(x, w)\) and \(M(x, w) = \partial_x f(x, w)\). Then,
\[
S(w) = \{ x \in \mathbb{R}^n \mid 0 \in G(x, w) + M(x, w) \}. \tag{10}
\]

Since \(\bar{x} \in S(\bar{w})\), \(\tau := (\bar{x}, \bar{w}, -\nabla_x f_0(\bar{x}, \bar{w}))\) belongs to \(\text{gph} \ M\). Note that \(\text{gph} \ M\) is locally closed around \(\tau\). The following result combines the lower estimates with the upper estimates mentioned above.

Lemma 2.3 (see [12 Theorem 3.4]) The lower estimates
\[
\hat{\Gamma}(x') \subset D^* S(\bar{w}|\bar{x})(x') \subset D^* S(\bar{w}|\bar{x})(x'),
\]
where
\[
\hat{\Gamma}(x') := \bigcup_{v'_1 \in \mathbb{R}^n} \left\{ w' \in \mathbb{R}^d \mid (-x', w') \in \nabla G(\bar{x}, \bar{w}) v'_1 + \hat{D}^* M(\bar{x})(v'_1) \right\},
\]
hold for any \(x' \in \mathbb{R}^n\). If the constraint qualification
\[
0 \in \nabla G(\bar{x}, \bar{w}) v'_1 + D^* M(\bar{x})(v'_1) \implies v'_1 = 0 \quad \tag{C1}
\]
is satisfied, then the upper estimate
\[
D^* S(\bar{w}|\bar{x})(x') \subset \Gamma(x'),
\]
where
\[
\Gamma(x') := \bigcup_{v'_1 \in \mathbb{R}^n} \left\{ w' \in \mathbb{R}^d \mid (-x', w') \in \nabla G(\bar{x}, \bar{w}) v'_1 + D^* M(\bar{x})(v'_1) \right\},
\]
is valid for any \(x' \in \mathbb{R}^n\). If, in addition, \(M\) is graphically regular at \(\bar{x}\), then
\[
\hat{\Gamma}(x') = \hat{D}^* S(\bar{w}|\bar{x})(x') = D^* S(\bar{w}|\bar{x})(x') = \Gamma(x').
\]

From Lemma 2.3 for any \(x' \in \mathbb{R}^n\), \(\hat{\Gamma}(x') \subset \hat{D}^* S(\bar{w}|\bar{x})(x')\). This implies that \(\hat{\Gamma}(0) \subset \hat{D}^* S(\bar{w}|\bar{x})(0) \subset D^* S(\bar{w}|\bar{x})(0)\). If we put \(M(x, w) = G(x, w) + M(x, w)\), then by the Fréchet coderivative sum rule with equalities [6 Theorem 1.62],
\[
\hat{D}^* M(\bar{w}_0)(v'_1) = \nabla G(\bar{x}, \bar{w}) v'_1 + \hat{D}^* M(\bar{x})(v'_1)
\]
for any \(v'_1 \in \mathbb{R}^n\), where \(\bar{w}_0 := (\bar{x}, \bar{w}, 0) \in \text{gph} \ M\). Therefore, we can write
\[
\hat{\Gamma}(x') = \bigcup_{v'_1 \in \mathbb{R}^n} \left\{ w' \in \mathbb{R}^d \mid (-x', w') \in \hat{D}^* M(\omega_0)(v'_1) \right\}.
\]

Note that \(0 \in \hat{\Gamma}(0)\). According to the Mordukhovich criterion, if \(S\) is locally Lipschitz-like around \((\bar{w}, \bar{x})\), then \(D^* S(\bar{w}|\bar{x})(0) = \{0\}\) and \(\hat{\Gamma}(0) = \{0\}\).
as a result. In addition, if the constraint qualification \( (C1) \) is fulfilled, then Lemma 2.3 yields \( D^*S(\bar{w}|x)(x') \subset \Gamma(x') \) for any \( x' \in \mathbb{R}^n \). In particular, \( D^*S(\bar{w}|x)(0) \subset \Gamma(0) \). Hence, if \( (C1) \) is valid and \( \Gamma(0) = \{0\} \), then
\[
D^*S(\bar{w}|x)(0) = \{0\}.
\]
So, due to the Mordukhovich criterion, \( S \) is locally Lipschitz-like around \( (\bar{w}, \bar{x}) \). This idea has been presented in 12 and we will follow it throughout this paper.

Put \( D = \{(x, w) \in \mathbb{R}^n \times \mathbb{R}^d \mid F(x, w) \leq 0\} \). If \( F(\bar{x}, \bar{w}) < 0 \), then \( (\bar{x}, \bar{w}) \) is an interior point of \( D \). If \( F(\bar{x}, \bar{w}) = 0 \), then \( (\bar{x}, \bar{w}) \) is a boundary point of \( D \).

In the next two sections, we will consider separately these two possibilities of the reference point \( (\bar{x}, \bar{w}) \). Remind that \( \bar{w} \in \mathbb{R}^d \) and \( \bar{x} \in S(\bar{w}) \) are fixed and all the notations of this section are kept unchanged.

### 3 The Robinson Stability of the Stationary Point Set Map

Now we turn attention to the Robinson stability of the stationary point set map \( S \) of the problem \( (P_w) \). As in the preceding sections, we assume the fulfillment of the condition \( (MFCQ) \), which requires that \( \nabla_x F(\bar{x}, \bar{w}) \neq 0 \) whenever \( F(\bar{x}, \bar{w}) = 0 \).

From [6, Theorem 1.62], we have a formula similar to (2):
\[
D^*\tilde{M}(\omega_0)(v'_1) = \nabla G(x, \bar{w})^*v'_1 + D^*M(\tilde{\tau})(v'_1) \tag{11}
\]
for any \( v'_1 \in \mathbb{R}^n \). So, condition \( (C1) \) can be rewritten as
\[
\ker D^*\tilde{M}(\omega_0) = \{0\}.
\]
By [13, Theorem 3.1], \( S \) has the Robinson stability at \( \omega_0 = (\bar{x}, \bar{w}, 0) \in \text{gph} \tilde{M} \) if \( (C1) \) and the condition
\[
\left\{ w' \in \mathbb{R}^d \mid \exists v'_1 \in \mathbb{R}^n \text{ with } (0, w') \in D^*\tilde{M}(\omega_0)(v'_1) \right\} = \{0\}, \tag{C2}
\]
is fulfilled. By (11) we can rewrite (C2) equivalently as
\[
\left\{ w' \in \mathbb{R}^d \mid \exists v'_1 \in \mathbb{R}^n \text{ with } (0, w') \in \nabla G(x, \bar{w})^*v'_1 + D^*M(\tilde{\tau})(v'_1) \right\} = \{0\}.
\]
With \( \Gamma(x') \) defined by (2.3), we can assert that (C2) is equivalent to the requirement \( \Gamma(0) = \{0\} \). In the proof of [2, Corollary 2.2], the authors have commented that the constraint qualification \( (7) \), which is equivalent to \( (C0) \), is stronger than \( (C1) \). Now we go back to three cases considered in Sects. 3 and 4 of Part 1.

First, for the case \( (\bar{x}, \bar{w}) \in \text{int} D \), we have shown in Sect. 3 of Part 1 that \( D^*\tilde{M}(\tilde{\tau})(v'_1) = \{0\} \) for any \( v'_1 \in \mathbb{R}^n \). So, condition (C2) becomes
\[
\left\{ w' \in \mathbb{R}^d \mid \exists v'_1 \in \mathbb{R}^n \text{ with } (0, w') \in \nabla G(x, \bar{w})^*v'_1 \right\} = \{0\}.
\]
Since \( \nabla G(x, \bar{w})^*v'_1 = (\nabla^2_{xx}f_0(x, \bar{w})v'_1, \nabla^2_{wx}f_0(x, \bar{w})v'_1) \), this is equivalent to
\[
\left\{ w' \in \mathbb{R}^d \mid \exists v'_1 \in \mathbb{R}^n \text{ with } \nabla^2_{xx}f_0(x, \bar{w})v'_1 = 0, \ w' = \nabla^2_{wx}f_0(x, \bar{w})v'_1 \right\} = \{0\}.
\]
The latter can be rewritten as
\[
\ker \nabla^2_{xx}f_0(x, \bar{w}) \subset \ker \nabla^2_{wx}f_0(x, \bar{w}). \tag{12}
\]
Besides, if the condition
\[
\ker \nabla^2_{xx}f_0(x, \bar{w}) \cap \ker \nabla^2_{wx}f_0(x, \bar{w}) = \{0\} \tag{13}
\]
is fulfilled, then (C0) is valid and (C1) is also valid as a result. Thus, if (13) and (12) are simultaneously satisfied, then S has the Robinson stability at ω0.

Let us move to the next case where \( F(\bar{x}, \bar{w}) = 0 \) and the Lagrange multiplier \( \lambda \) corresponding to the stationary point \( \bar{x} \in S(\bar{w}) \) is positive. First, it is worth to stress that for \( (P_n) \), the assumptions (i), (ii), and (10) in [2, Proposition 2.1] are fulfilled. So, from [2, Corollary 2.1], for any \( v'_1 \in \mathbb{R}^n \),

\[
D^* M(\bar{w})(v'_1) \subset \bigcup_{v \in \partial f((\bar{x}, \bar{w}))} \partial^2 f((\bar{x}, \bar{w}))(v_1, 0).
\]

With \( \Omega_2(v'_1) \) defined by (11), using (5) we have

\[
\Omega_2(v'_1) = \bigcup_{v \in \partial f((\bar{x}, \bar{w}))} \partial^2 f((\bar{x}, \bar{w}))(v_1, 0).
\]

Hence, \( D^* M(\bar{w})(v'_1) \subset \Omega_2(v'_1) \) for any \( v'_1 \in \mathbb{R}^n \). Therefore, from formula (2.3) and the presentation \( \nabla G(\bar{x}, \bar{w})^*(v'_1) = \nabla^2 f_0(\bar{x}, \bar{w})^*(v'_1, 0) \), we have

\[
\Gamma(\lambda') \subset \{ w' \in \mathbb{R}^d \mid \exists v'_1 \in \mathbb{R}^n \text{ with } (-\lambda', w') - \nabla^2 f_0(\bar{x}, \bar{w})^*(v'_1, 0) \in \Omega_2(v'_1) \}
\]

(14)

for any \( \lambda' \in \mathbb{R}^n \). This implies that \( \Gamma(\lambda') \) is contained in \( \Gamma_2(\lambda') \) which is defined in Subsect. 4.1 of Part 1. In particular, \( \Gamma(0) \subset \Gamma_2(0) \). So, if \( \Gamma_2(0) = \{0\} \), then \( \Gamma(0) = \{0\} \). We have shown that \( \Gamma_2(0) = \{0\} \) if and only if the inclusion

\[
\ker A_1 \cap (\ker \nabla_x F(\bar{x}, \bar{w}) \times \mathbb{R}) \subset \ker A_2.
\]

(15)

is valid. Therefore, if (13) is satisfied, then \( \Gamma(0) = \{0\} \) which implies the fulfillment of (C2). Let

\[
A_1 = \left[ \nabla^2_{xx} f_0(\bar{x}, \bar{w}) + \lambda \nabla^2_{xx} F(\bar{x}, \bar{w}) \nabla_x F(\bar{x}, \bar{w}) \right] \in \mathbb{R}^{n \times (n+1)}
\]

(16)

and

\[
A_2 = \left[ \nabla^2_{ww} f_0(\bar{x}, \bar{w}) + \lambda \nabla^2_{ww} F(\bar{x}, \bar{w}) \nabla_w F(\bar{x}, \bar{w}) \right] \in \mathbb{R}^{d \times (n+1)},
\]

(17)

where \( \nabla_x F(\bar{x}, \bar{w}) \) and \( \nabla_w F(\bar{x}, \bar{w}) \) are interpreted as column vectors. If the equality

\[
\ker A_1 \cap \ker A_2 \cap (\ker \nabla_x F(\bar{x}, \bar{w}) \times \mathbb{R}) = \{(0, 0)\}.
\]

(18)

is satisfied then, as shown in Subsect. 4.1 of Part 1, (C0) is fulfilled; consequently, (C1) is valid. Thus, in the case under our consideration, once (13) and (12) are simultaneously satisfied, S has the Robinson stability at \( \omega_0 \).

Finally, we consider the case where \( F(\bar{x}, \bar{w}) = 0 \) and the Lagrange multiplier \( \lambda \) corresponding to the stationary point \( \bar{x} \in S(\bar{w}) \) equals to zero. In this case, if

\[
\ker A_1' \cap \ker A_2' = \{0\},
\]

(19)

where

\[
A_1' := \left[ \nabla^2_{xx} f_0(\bar{x}, \bar{w}) \nabla_x F(\bar{x}, \bar{w}) \right] \in \mathbb{R}^{n \times (n+1)}.
\]

and

\[
A_2' := \left[ \nabla^2_{ww} f_0(\bar{x}, \bar{w}) \nabla_w F(\bar{x}, \bar{w}) \right] \in \mathbb{R}^{d \times (n+1)},
\]

(20)

is valid, then (C0) holds (see Subsect. 4.2 of Part 1). So, (19) guarantees the validity of (C1). Concerning condition (C2), we will show that if the conditions

\[
\ker A_1' \cap (\ker \nabla_x F(\bar{x}, \bar{w}) \times \mathbb{R}) \subset \ker A_2',
\]

(20)

\[
\ker A_1' \cap \Delta_1 \subset \ker A_2',
\]

(21)
and
\[ \ker \nabla^2_{xx} f_0(x, \bar{w}) \cap \Delta_2 \subset \ker \nabla^2_{ww} f_0(x, \bar{w}). \tag{22} \]
are satisfied, then \((C'2)\) is fulfilled.

Let \( \Gamma_3(x') \) be the set of vectors \( w' \in \mathbb{R}^d \) for which there exists \( v'_1 \in \mathbb{R}^n \) with
\[
\begin{align*}
&\begin{cases}
-x' - \nabla^2_{xx} f_0(x, \bar{w}) v'_1, w' - \nabla^2_{ww} f_0(x, \bar{w}) v'_1 \in \{ \gamma \nabla F(\bar{x}, \bar{w}) \mid \gamma \in \mathbb{R} \}, \\
\nabla_x F(\bar{x}, \bar{w}) v'_1 = 0,
\end{cases}
\end{align*}
\]
or
\[
\begin{align*}
&\begin{cases}
-x' - \nabla^2_{xx} f_0(x, \bar{w}) v'_1, w' - \nabla^2_{ww} f_0(x, \bar{w}) v'_1 \in \{ \gamma \nabla F(\bar{x}, \bar{w}) \mid \gamma \in \mathbb{R}_+ \}, \\
\nabla_x F(\bar{x}, \bar{w}) v'_1 > 0,
\end{cases}
\end{align*}
\]
or
\[
\begin{align*}
&\begin{cases}
-x' - \nabla^2_{xx} f_0(x, \bar{w}) v'_1 = 0, \quad w' - \nabla^2_{ww} f_0(x, \bar{w}) v'_1 = 0, \\
\nabla_x F(\bar{x}, \bar{w}) v'_1 < 0.
\end{cases}
\end{align*}
\]
As it has been proved in Subsect. 4.2 of Part 1,
\[ \Omega_2(v'_1) \subset \begin{cases}
\{ \gamma \nabla F(\bar{x}, \bar{w}) \mid \gamma \in \mathbb{R} \} & \text{if } \nabla_x F(\bar{x}, \bar{w}) v'_1 = 0, \\
\{ \gamma \nabla F(\bar{x}, \bar{w}) \mid \gamma \in \mathbb{R}_+ \} & \text{if } \nabla_x F(\bar{x}, \bar{w}) v'_1 > 0, \\
\{ 0 \} & \text{if } \nabla_x F(\bar{x}, \bar{w}) v'_1 < 0.
\end{cases} \tag{23} \]

From \((13)\) and the inclusion \((23)\) we have \( \Gamma(x') \subset \Gamma_3(x') \) for any \( x' \in \mathbb{R}^n \). In particular, \( \Gamma'(0) \subset \Gamma_3(0) \). Hence, if \( \Gamma_3(0) = \{ 0 \} \), then \( \Gamma'(0) = \{ 0 \} \). We have shown that \( \Gamma_3(0) = \{ 0 \} \) holds if and only if the system \((20) - (22)\) is satisfied. Thus, the validity of \((20) - (22)\) implies \( \Gamma'(0) = \{ 0 \} \) which yields the fulfillment of \((C'2)\). Therefore, if \((13)\) and the system \((20) - (22)\) are simultaneously satisfied, then \( S \) has the Robinson stability at \( \omega_0 \).

We have thus shown that the sufficient conditions for \( S \) being locally Lipschitz-like around \( (\bar{w}, \bar{x}) \) in each case also guarantee for \( S \) having the Robinson stability at \( \omega_0 \).

Our results on the Robinson stability of \( S \) are summarized as follows.

**Theorem 3.1** The stationary point set map \( S \) of \((P_w)\) has the Robinson stability at \( \omega_0 = (\bar{x}, \bar{w}, 0) \) if one of the following is valid:

(a) \( F(\bar{x}, \bar{w}) < 0 \) and the condition
\[ \ker \nabla^2_{xx} f_0(\bar{x}, \bar{w}) = \{ 0 \} \tag{24} \]
holds;

(b) \( F(\bar{x}, \bar{w}) = 0 \), the Lagrange multiplier \( \lambda \) corresponding to the stationary point \( \bar{x} \in S(\bar{w}) \) is positive, and
\[ \ker A_1 \cap (\ker \nabla_x F(\bar{x}, \bar{w}) \times \mathbb{R}) = \{ 0 \}; \tag{25} \]

(c) \( F(\bar{x}, \bar{w}) = 0 \), the Lagrange multiplier \( \lambda \) corresponding to the stationary point \( \bar{x} \in S(\bar{w}) \) equals to zero, and
\[ \begin{cases}
\ker A'_1 \cap \ker A'_2 = \{ 0 \}, \\
\ker A'_1 \cap (\ker \nabla_x F(\bar{x}, \bar{w}) \times \mathbb{R}) \subset \ker A'_2, \\
\ker A'_1 \cap \Delta_1 \subset \ker A'_2, \\
\ker \nabla^2_{xx} f_0(\bar{x}, \bar{w}) \cap \Delta_2 \subset \ker \nabla^2_{ww} f_0(\bar{x}, \bar{w}).
\end{cases} \tag{26} \]

It is worthy to stress that the Robinson stability of \( S \) at \( \omega_0 \) is available for the examples of the previous section where our sufficient conditions for the local Lipschitz-likeness of \( S \) around \( (\bar{w}, \bar{x}) \) are fulfilled.
4 Applications to Quadratic Programming

In this section, the above general results are applied to a class of nonconvex quadratic programming problems. Namely, we will consider the problems of minimizing a linear-quadratic function under one linear-quadratic functional constraint. Special cases of such problems have been considered, e.g., in [14, 15], and [16].

Denote by \( S_n \) the space of \( n \times n \) symmetric matrices. Let \( D, A \in S_n \), \( c \) and \( b \) be vectors in \( \mathbb{R}^n \), and \( \alpha \) a real number. Put \( w = (w_1, w_2) \) with \( w_1 := (D, c) \) and \( w_2 := (A, b, \alpha) \). Denote the problem \( (P_w) \) with \( f_0(x, w) = \frac{1}{2}x^T DX + c^T x \) and \( F(x, w) = \frac{1}{2}x^T Ax + b^T x + \alpha \) by \( (QP_w) \). For convenience, we put \( W_1 = S_n \times \mathbb{R}^n \), \( W_2 = S_n \times \mathbb{R}^n \times \mathbb{R} \), and \( W = W_1 \times W_2 \). Fix a vector \( \bar{w} = (\bar{w}_1, \bar{w}_2) \in W \) with \( \bar{w}_1 = (D, c) \), \( \bar{w}_2 = (A, b, \alpha) \), and suppose that a stationary point \( \bar{x} \in S(\bar{w}) \) is given.

To ease the description of certain second order differential operators, sometimes we will present the matrices \( D \) and \( A \) in the following column forms

\[
D = \begin{pmatrix}
    d_1^T \\
    \vdots \\
    d_n^T
\end{pmatrix}, \quad A = \begin{pmatrix}
    a_1^T \\
    \vdots \\
    a_n^T
\end{pmatrix},
\]

where \( d_i = (d_{i1} \ldots d_{in}) \) and \( a_i = (a_{i1} \ldots a_{in}) \) are, respectively, the \( i \)-th row of \( D \) and the \( i \)-th row of \( A \). We have \( \nabla_x f_0(\bar{x}, \bar{w}) = \bar{D} \bar{x} + \bar{c} \),

\[
\nabla_{w_1} f_0(\bar{x}, \bar{w}) = \left( \frac{1}{2} \bar{x}_1 \bar{x}_1 \ldots \frac{1}{2} \bar{x}_1 \bar{x}_n \ldots \frac{1}{2} \bar{x}_n \bar{x}_1 \ldots \frac{1}{2} \bar{x}_n \bar{x}_n \bar{x}_1 \ldots \bar{x}_n \right)^T,
\]

\[
\nabla_{w_2}^2 f_0(\bar{x}, \bar{w}) = 0_{W_2}, \quad \text{and}
\]

\[
\nabla_{w_1}^2 f_0(\bar{x}, \bar{w}) = \begin{pmatrix}
    \bar{X} & \cdots & 0 \\
    \vdots \\
    0 & \cdots & \bar{X} \\
    1 & \cdots & 0 \\
    \vdots \\
    0 & \cdots & 1
\end{pmatrix},
\]

Here, \( \bar{X} := \begin{pmatrix}
    \bar{x}_1 \\
    \vdots \\
    \bar{x}_n
\end{pmatrix} \) is an \( n \times 1 \) matrix. Similarly, \( \nabla_x F(\bar{x}, \bar{w}) = \bar{A} \bar{x} + \bar{b} \),

\[
\nabla_{w_2} f_0(\bar{x}, \bar{w}) = \left( \frac{1}{2} \bar{x}_1 \bar{x}_1 \ldots \frac{1}{2} \bar{x}_1 \bar{x}_n \ldots \frac{1}{2} \bar{x}_n \bar{x}_1 \ldots \frac{1}{2} \bar{x}_n \bar{x}_n \bar{x}_1 \ldots \bar{x}_n \right)^T,
\]

\[
\nabla_{w_2}^2 F(\bar{x}, \bar{w}) = 0_{W_1}, \quad \text{and}
\]

\[
\nabla_{w_2}^2 F(\bar{x}, \bar{w}) = \begin{pmatrix}
    \bar{X} & \cdots & 0 \\
    \vdots \\
    0 & \cdots & \bar{X} \\
    1 & \cdots & 0 \\
    \vdots \\
    0 & \cdots & 1
\end{pmatrix}.
\]
We have
\[
\nabla^2_{w,x} f_0(\bar{x}, \bar{w}) = (\nabla^2_{w_1,x} f_0(\bar{x}, \bar{w}) \; \nabla^2_{w_2,x} f_0(\bar{x}, \bar{w})) .
\]
Since \(\nabla^2_{w,x} f_0(x, w) = 0\),
\[
\text{ker} \; \nabla^2_{w,x} f_0(x, w) = \{ v_1' \in \mathbb{R}^n \mid \nabla^2_{w_1,x} f_0(x, w)v_1' = 0 \} = \{ 0 \}.
\]

First, we consider the case of interior points \((x, w),\) i.e., \(F(x, w) < 0\). The conditions (13), (12), and (24) are equivalent due to ker \(\nabla^2_{w,x} f_0(x, w) = \{ 0 \}\). Thus, by Theorem 3.1 of Part 1, the stationary point set map \(S\) of \((P_w)\) is locally Lipschitz-like around \((\bar{w}, \bar{x})\) if and only if ker \(\nabla^2_{w,x} f_0(\bar{x}, \bar{w}) = \{ 0 \}\), or ker \(\bar{D} = \{ 0 \}\). In other words, \(S\) is locally Lipschitz-like around \((\bar{w}, \bar{x})\) if and only if matrix \(\bar{D}\) is nonsingular. By Theorem 3.1 this condition is sufficient for \(S\) having the Robinson stability at \(\omega_0\).

Next, consider the second case where \((x, \bar{w})\) is a boundary point of \(D\) and the Lagrange multiplier \(\lambda\) corresponding to \(x \in S(\bar{w})\) is positive. As in Part 1, \(\lambda\) is defined by
\[
\nabla_x f_0(\bar{x}, \bar{w}) + \lambda \nabla_x F(\bar{x}, \bar{w}) = 0,
\]
which is rewritten as
\[
\lambda (\bar{A}x + \bar{b}) = -(\bar{D}x + \bar{c}).
\]
We have
\[
\nabla^2_{w,x} f_0(x, w) + \lambda \nabla^2_{w,x} F(x, w) = \begin{bmatrix} \bar{x} & \cdots & 0 \\ \vdots \\ 0 & \cdots & \bar{x} \\ 1 & \cdots & 0 \\ \vdots \\ 0 & \cdots & 1 \\ \lambda \bar{x} & \cdots & 0 \\ \vdots \\ 0 & \cdots & \lambda \bar{x} \\ \lambda & \cdots & 0 \\ \vdots \\ 0 & \cdots & \lambda \\ 0 & \cdots & 0 \end{bmatrix}.
\]

Now, the matrices \(A_1\) and \(A_2\) defined in Sect. 3 are described as follows
\[
A_1 = [\bar{D} + \lambda \bar{A} \; \bar{A}x + \bar{b}]
\]
and
\[
A_2 = \begin{bmatrix} \bar{x} & \cdots & 0 & 0 \\ \vdots \\ 0 & \cdots & \bar{x} & 0 \\ 1 & \cdots & 0 & 0 \\ \vdots \\ 0 & \cdots & 1 & 0 \\ \lambda \bar{x} & \cdots & 0 & \frac{1}{2} \bar{x}_1 \bar{x}_1 \\ \vdots \\ 0 & \cdots & \lambda \bar{x} & \frac{1}{2} \bar{x}_1 \bar{x}_n \\ \lambda & \cdots & 0 & \bar{x}_1 \\ \vdots \\ 0 & \cdots & \lambda & \bar{x}_n \\ 0 & \cdots & 0 & 1 \end{bmatrix}.
\]
Hence, \( \ker A_2 = \{0\} \). This implies that (15) is automatically satisfied. So, according to Theorem 4.1 of Part 1, \( S \) is locally Lipschitz-like around \((\bar{w}, \bar{x})\) if and only if (15) is fulfilled. Note that

\[
\ker A_1 = \left\{ (v'_1, \gamma) \in \mathbb{R}^n \times \mathbb{R} \mid (D + \lambda \bar{A})v'_1 + \gamma (\bar{A} \bar{x} + \bar{b}) = 0 \right\}
\]

and

\[
\ker \nabla_x F(\bar{x}, \bar{w}) = \left\{ v'_1 \in \mathbb{R}^n \mid (\bar{A} \bar{x} + \bar{b})^T v'_1 = 0 \right\}.
\]

Hence, (15) holds if and only if

\[
\begin{cases}
(D + \lambda \bar{A})v'_1 + \gamma (\bar{A} \bar{x} + \bar{b}) = 0 \\
(\bar{A} \bar{x} + \bar{b})^T v'_1 = 0
\end{cases}
\]

\[
\Rightarrow \begin{cases}
v'_1 = 0 \\
\gamma = 0
\end{cases}
\]

or, equivalently,

\[
\det \begin{pmatrix}
(D + \lambda \bar{A}) & \bar{A} \bar{x} + \bar{b} \\
(\bar{A} \bar{x} + \bar{b})^T & 0
\end{pmatrix} \neq 0.
\]  

Thus, \( S \) is locally Lipschitz-like around \((\bar{w}, \bar{x})\) if and only if (28) is satisfied. Moreover, by Theorem 3.1 of Part 1, (28) guarantees for \( S \) having the Robinson stability at \( \omega_0 \).

Let us consider the last case where \((\bar{x}, \bar{w})\) is a boundary point of \( \mathcal{D} \) and the Lagrange multiplier \( \lambda \) corresponding to \( \bar{x} \in S(\bar{w}) \) equals to zero. The matrices \( A'_1 \) and \( A'_2 \) defined in Sect. 3 are described as \( A'_1 = \left[ \begin{array}{cc} \dot{D} & \bar{A} \bar{x} + \bar{b} \end{array} \right] \) and

\[
A'_2 = \left( \begin{array}{cccc}
\bar{x} & \cdots & 0 & 0 \\
0 & \ddots & \vdots & 0 \\
1 & \cdots & 0 & 0 \\
0 & \cdots & 1 & 0 \\
0 & \cdots & 0 & \frac{1}{2} \bar{x}_1 \bar{x}_1 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & \bar{x}_n \\
0 & \cdots & 0 & 1
\end{array} \right).
\]

Since \( \ker A'_2 = \{0\} \), using the equality \( \ker \nabla^2_{w,x} f_0(\bar{x}, \bar{w}) = \{0\} \) we can rewrite (20) as

\[
\begin{array}{l}
\ker A'_1 \cap (\ker \nabla_x F(\bar{x}, \bar{w}) \times \mathbb{R}) = \{0\}, \\
\ker A'_1 \cap \Delta_1 = \{0\}, \\
\ker \nabla^2_{w,x} f_0(\bar{x}, \bar{w}) \cap \Delta_2 = \{0\}.
\end{array}
\]

This condition holds if and only if the following conditions are simultaneously satisfied:

\[
\begin{cases}
Dv'_1 + \gamma (\bar{A} \bar{x} + \bar{b}) = 0 \\
(\bar{A} \bar{x} + \bar{b})^T v'_1 = 0
\end{cases}
\]

\[
\Rightarrow \begin{cases}
v'_1 = 0 \\
\gamma = 0
\end{cases}
\]

\[
\begin{cases}
\dot{D}v'_1 + \gamma (\bar{A} \bar{x} + \bar{b}) = 0 \\
(\bar{A} \bar{x} + \bar{b})^T v'_1 > 0, \ \gamma \geq 0
\end{cases}
\]

\[
\Rightarrow \begin{cases}
v'_1 = 0 \\
\gamma = 0
\end{cases}
\]
and
\[
\begin{cases}
\dot{D}v'_1 = 0 \\
(\bar{A}x + b)^T v'_1 < 0
\end{cases} \quad \implies \quad v'_1 = 0.
\]
These implications can be rewritten respectively as
\[
\det \left( \begin{array}{cc}
\dot{D} & \bar{A}x + b \\
(\bar{A}x + b)^T & 0
\end{array} \right) \neq 0,
\]
\[
[Dv'_1 + \gamma(\bar{A}x + b) = 0, \; \gamma \geq 0] \quad \implies \quad (\bar{A}x + b)^T v'_1 \leq 0,
\]
and
\[
Dv'_1 = 0 \quad \implies \quad (\bar{A}x + b)^T v'_1 = 0.
\]
Thus, in accordance with Theorem 4.2 of Part 1, $S$ is locally Lipschitz-like around $(\bar{w}, \bar{x})$ if $(29) – (31)$ are satisfied. Moreover, by Theorem 3.1, the fulfillment of $(29) – (31)$ is sufficient for $S$ having the Robinson stability at $\omega_0$. Let us consider the necessary condition
\[
\ker A'_1 \cap \Delta_3 \subset \ker A'_2,
\]
for the local Lipschitz-like property of $S$ around $(\bar{w}, \bar{x})$, which is now reduced to $\ker A'_1 \cap \Delta_3 = \{0\}$. Clearly, this condition is equivalent to
\[
\begin{cases}
\dot{D}v'_1 + \gamma(\bar{A}x + b) = 0 \\
(\bar{A}x + b)^T v'_1 \geq 0, \; \gamma \geq 0
\end{cases} \quad \implies \quad \begin{cases}
v'_1 = 0 \\
\gamma = 0.
\end{cases}
\]
By [1, Theorem 4.1], (32) is a necessary condition for $S$ being locally Lipschitz-like around $(\bar{w}, \bar{x})$.

The obtained results can be formulated as follows.

**Theorem 4.1** The following assertions are true:

(a) If $F(\bar{x}, \bar{w}) < 0$, then $S$ is locally around $(\bar{w}, \bar{x})$ if and only if $\det \bar{D} \neq 0$. Moreover, under this condition, $S$ has the Robinson stability at $\omega_0$;

(b) If $F(\bar{x}, \bar{w}) = 0$ and the Lagrange multiplier $\lambda$ corresponding to $\bar{x} \in S(\bar{w})$ is positive, then $S$ is locally Lipschitz-like around $(\bar{w}, \bar{x})$ if and only if (28) is fulfilled. This condition is sufficient for $S$ having the Robinson stability at $\omega_0$;

(c) If $F(\bar{x}, \bar{w}) = 0$ and the Lagrange multiplier $\lambda$ corresponding to $\bar{x} \in S(\bar{w})$ is zero, then (32) is necessary for $S$ being locally Lipschitz-like around $(\bar{w}, \bar{x})$. Meanwhile, the fulfillment of $(29) – (31)$ is sufficient for the local Lipschitz-like property of $S$ around $(\bar{w}, \bar{x})$, as well as for the Robinson stability of $S$ at $\omega_0$.

To show how these results can work, we revisit some examples from [16].

**Example 4.1** (see [16, Example 4.1]) Consider the problem $(QP_\omega)$ in the case $n = 2$. Choosing $\bar{w} = (\bar{D}, \bar{c}, \bar{A}, \bar{b}, \bar{\alpha})$ with
\[
\bar{D} = \begin{pmatrix} 0 & 0 \\ 0 & -8 \end{pmatrix}, \quad \bar{c} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]
and
\[
\bar{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \bar{\alpha} = -1,
\]
one has $f_0(x, \bar{w}) = -4x_2^2 + x_1$ and $F(x, \bar{w}) = x_1^2 + x_2^2 - 1$. To show that $\bar{x} := (-\frac{1}{8}, \frac{\sqrt{7}}{8})^T$ is a stationary point of $(P_\omega)$, we note by (2) that
\[
S(\bar{w}) = \{x \in \mathbb{R} \mid 0 \in \nabla_x f_0(\bar{x}, \bar{w}) + \partial_x f(\bar{x}, \bar{w})\},
\]
with \( f(x, w) = (g \circ F)(x, w) \) and \( g(y) = \delta_{\mathbb{R}_-}(y) \) for any \( y \in \mathbb{R} \). As \( F(\bar{x}, \bar{w}) = 0 \) and \( \nabla_x F(\bar{x}, \bar{w}) \neq 0 \), condition \( \text{MFCQ} \) is valid. So, from (1) we have
\[
\partial_x f(\bar{x}, \bar{w}) = \nabla_x F(\bar{x}, \bar{w})^* N_{\mathbb{R}_-}(F(\bar{x}, \bar{w})) = \nabla_x F(\bar{x}, \bar{w})^* \mathbb{R}_+ = \left\{ \left( -\frac{1}{\gamma}, \frac{\sqrt{63}}{8} \gamma \right) \mid \gamma \in \mathbb{R}_+ \right\}.
\]

Besides, \( \nabla_x f_0(\bar{x}, \bar{w}) = (1, -\sqrt{63})^T \). Now, it is clear that \( \bar{x} \in S(\bar{w}) \). From (27), the Lagrange multiplier corresponding to \( \bar{x} \) is \( \lambda = 8 \). Hence,
\[
\det \begin{pmatrix} D + \lambda A & \bar{A} \bar{x} + \bar{b} \\ (\bar{A} \bar{x} + \bar{b})^T & 0 \end{pmatrix} = \det \begin{pmatrix} 8 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{63}} \\ -\frac{1}{\sqrt{63}} & 0 & 0 \end{pmatrix} = \frac{63}{8}.
\]

So, (28) is fulfilled. Thus, by Theorem 4.1 the stationary point set map \( S \) of \((P_w)\) not only is locally Lipschitz-like around \((\bar{w}, \bar{x})\) but also has the Robinson stability at \( \omega_0 = (\bar{x}, \bar{w}, 0) \). Similarly, we can show that \( \bar{x} = (-\frac{1}{8}, -\sqrt{63}/8)^T \) and \( \bar{x} = (-1, 0)^T \) belong to \( S(\bar{w}) \) and (28) is also valid for them.

Example 4.2 (see [10, Example 4.2]) Consider the problem \((QP_w)\) in the case \( n = 3 \) and choose \( \bar{w} = (\bar{D}, \bar{c}, \bar{A}, \bar{b}, \bar{\alpha}) \) with
\[
\bar{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -8 \end{pmatrix}, \quad \bar{c} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
\]

and
\[
\bar{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \bar{\alpha} = -1.
\]

Here, \( f_0(x, w) = -4(x_3^2 + x_4^3) + x_1 \) and \( F(x, w) = x_1^2 + x_2^2 + x_3^2 + 1 \). Arguments similar to those in the previous example show that \( \bar{x} := (-1, 0, 0)^T \) is a stationary point of \((P_w)\) with the associated Lagrange multiplier \( \lambda = 1 \). It is easy to check that (28) is satisfied. So, by Theorem 4.1 the stationary point set map \( S \) of \((P_w)\) is locally Lipschitz-like around \((\bar{w}, \bar{x})\) and it has the Robinson stability at \( \omega_0 = (\bar{x}, \bar{w}, 0) \). However, for the stationary points
\[
\bar{x}_1 := \left( -\frac{1}{8}, \frac{\sqrt{63}}{8} \sin t, \frac{\sqrt{63}}{8} \cos t \right)^T,
\]

with \( t \in [0, 2\pi] \), which share the common associated Lagrange multiplier \( \lambda = 8 \), (28) is violated because
\[
\det \begin{pmatrix} D + \lambda A & \bar{A} \bar{x} + \bar{b} \\ (\bar{A} \bar{x} + \bar{b})^T & 0 \end{pmatrix} = \det \begin{pmatrix} 8 & 0 & 0 & -\frac{1}{\sqrt{63}} \sin t \\ 0 & 0 & 0 & -\frac{\sqrt{63}}{8} \sin t \\ 0 & 0 & 0 & -\frac{\sqrt{63}}{8} \cos t \\ -\frac{1}{\sqrt{63}} \sin t & -\frac{\sqrt{63}}{8} \cos t & 0 & 0 \end{pmatrix} = 0.
\]

Thus, by Theorem 4.1 \( S \) is not locally Lipschitz-like around \((\bar{w}, \bar{x})\).
The parametric trust-region subproblem (TRS) considered in [14,12,16] is a special case of our quadratic programming problem (QP), where \( A \) is the unit matrix, \( b = 0 \), and \( \alpha < 0 \).

For (TRS), in the case where \( F(\bar{x}, \bar{w}) = 0 \) and the Lagrange multiplier \( \lambda \) corresponding to \( \bar{x} \in S(\bar{w}) \) is positive, the matrix in [28] coincides with the matrix \( Q(\cdot) \) in [15, Theorem 5.1] and [16, Theorem 4.2], called the stability matrix (see [15, p. 200]). Therefore, Theorem 4.2 in [10], which only discusses the local Lipschitz-like property, is a consequence of the assertions (a) and (b) of Theorem 4.1.

In the case where \( F(\bar{x}, \bar{w}) = 0 \) and the Lagrange multiplier \( \lambda \) corresponding to \( \bar{x} \in S(\bar{w}) \) equals to zero, the matrix in [29] coincides with the stability matrix \( Q(\cdot) \) in [15, Theorem 4.3]. So, condition (4.10) in [16] coincides with our condition (32). The sufficient condition for the local Lipschitz-like property in [16, Theorem 4.3] also requires \( \det \bar{D} \neq 0 \). Under this assumption, (30) and (31) are valid if and only if \( \bar{x}^T \bar{D}^{-1} \bar{x} \geq 0 \). However, our conditions (29)–(31) do not require \( \det \bar{D} \neq 0 \). Thus, for (TRS), the sufficient conditions in Theorem 4.1(c) and in [16, Theorem 4.3(ii)] are independent results. Finally, note that the necessary condition (4.9) in [16] for the local Lipschitz-like property coincides with our condition (32).

5 Results Obtained by Another Approach

Following the detailed hints of one referee of this paper, we will compare our results with those which can be obtained by using the theory of strongly regular generalized equations of Robinson [5].

Suppose that \( \bar{x} \in S(\bar{w}) \) and the condition (MFCQ) is satisfied. It is not difficult to show that, thanks to (MFCQ), there exist a neighborhood \( W_0 \) of \( \bar{w} \) and a neighborhood \( U_0 \) of \( \bar{x} \) such that for every \( (x, w) \in U_0 \times W_0 \) one has

\[ N_{C(w)}(x) = \{ \nabla_x F(x, w) \mid \lambda \geq 0 \} \]

when \( F(x, w) = 0 \) and \( N_{C(w)}(x) = \{ 0 \} \) when \( F(x, w) < 0 \). Hence, for every \( (x, w) \in U_0 \times W_0 \), the condition

\[ 0 \in \nabla_x f_0(x, w) + N_{C(w)}(x) \]

is equivalent to the existence of a Lagrange multiplier \( \alpha \in \mathbb{R} \) such that

\[ 0 \in \left( \nabla_x \mathcal{L}(x, \alpha, w) \right) + N_{\mathbb{R}^n \times \mathbb{R}_+}(x, \alpha), \]

where \( \mathcal{L}(x, \alpha, w) := f_0(x, w) + \alpha F(x, w) \). Setting \( g(x, \alpha, w) = \left( \nabla_x \mathcal{L}(x, \alpha, w) \right) \), we consider the parametric generalized equation

\[ 0 \in g(x, \alpha, w) + N_{\mathbb{R}^n \times \mathbb{R}_+}(x, \alpha) \quad (w \in \mathbb{R}^d) \]

and denote its solution set by \( \hat{S}(w) \). Then,

\[ \hat{S}(w) = \{ (x, \alpha) \in \mathbb{R}^n \times \mathbb{R}^d \mid 0 \in g(x, \alpha, w) + N_{\mathbb{R}^n \times \mathbb{R}_+}(x, \alpha) \} \]

and \( \hat{S}(\cdot) \) is the implicit multifunction defined by [35]. (The writing of the necessary optimality condition of a constrained smooth mathematical programming problem in a form similar to [34] has been used by Robinson [5, p. 54].) From what has been said we have

\[ S(w) \cap U_0 = \{ x \in U_0 \mid \exists \alpha \text{ s.t. } (x, \alpha) \in \hat{S}(w) \} \quad (\forall w \in W_0). \]
As in Part 1 of this paper and in the preceding sections, we will denote by \( \lambda \) the unique multiplier corresponding to \( \bar{x} \in S(\bar{w}) \). Consider the following three cases.

**Case 1:** \( F(\bar{x}, \bar{w}) < 0 \). This case has been analyzed in Remark 3.2 of Part 1 of this paper.

**Case 2:** \( F(\bar{x}, \bar{w}) = 0 \) and \( \lambda > 0 \). In accordance with [5] p. 45, the unperturbed generalized equation

\[
0 \in g(x, \alpha, \bar{w}) + N_{\mathbb{R}^n \times \mathbb{R}_+}(x, \alpha) \tag{35}
\]

is said to be strongly regular at \((\bar{x}, \lambda)\) if there exist a constant \( \ell_0 > 0 \) and neighborhoods \( U \) of the origin in \( \mathbb{R}^n \times \mathbb{R} \) and \( V \) of \((\bar{x}, \lambda)\) such that for every \((x', \alpha') \in U\) one can find a unique vector \((x, \alpha)\) in \( V \), denoted by \( s_0(x', \alpha') \), satisfying

\[
\begin{pmatrix}
x' \\
\alpha
\end{pmatrix} \in g(\bar{x}, \lambda, \bar{w}) + \nabla_{(x, \alpha)}g(\bar{x}, \lambda, \bar{w})(\begin{pmatrix} x \\\n \alpha \end{pmatrix} - (\bar{x}, \lambda)) + N_{\mathbb{R}^n \times \mathbb{R}_+}(x, \alpha)
\]

and the mapping \( s_0 : U \to V \) is Lipschitzian on \( U \) with modulus \( \ell_0 \). Using the condition \( \lambda > 0 \) and the results of Dontchev and Rockafellar [17], one can prove next lemma; see Sect. 6 for details.

**Lemma 5.1** The generalized equation \((35)\) is strongly regular at \((\bar{x}, \lambda)\) iff the matrix

\[
\begin{pmatrix}
\nabla^2_x \mathcal{L}(\bar{x}, \lambda, \bar{w}) & \nabla_x F(\bar{x}, \bar{w}) \\
\nabla_x F(\bar{x}, \bar{w})^T & 0
\end{pmatrix}
\]

is nonsingular.

The condition formulated in Lemma 5.1 is equivalent to condition (23) in Part 1, which was renumbered as condition (25) in Sect. 3. Indeed, by (16) one has

\[
A_1 = \left[ \nabla^2_x \mathcal{L}(\bar{x}, \lambda, \bar{w}) \ \nabla_x F(\bar{x}, \bar{w}) \right].
\]

Hence, \((x', \tau') \in \ker A_1 \cap (\ker \nabla_x F(\bar{x}, \bar{w}) \times \mathbb{R})\) iff

\[
\begin{pmatrix}
\nabla^2_x \mathcal{L}(\bar{x}, \lambda, \bar{w}) & \nabla_x F(\bar{x}, \bar{w}) \\
\nabla_x F(\bar{x}, \bar{w})^T & 0
\end{pmatrix}
\begin{pmatrix}
x' \\
\tau'
\end{pmatrix} = \begin{pmatrix} 0 \\
0 \end{pmatrix}.
\]

Thus, the matrix in \((36)\) is nonsingular if \((25)\) is valid. Now, applying Theorem 2.1 from [5] to the parametric generalized equation \((35)\), we can assert that if \((35)\) is strongly regular at \((\bar{x}, \lambda)\), then the implicit multifunction \( \hat{S}(\cdot) \) has a single-valued localization [18] p. 4] around \( \bar{w} \) for \((\bar{x}, \lambda)\) which is Lipschitz continuous in a neighborhood of \( \bar{w} \). This means that there exist \( \ell > 0 \), a neighborhood \( W \) of \( \bar{w} \), a neighborhood \( U \) of \( \bar{x} \), and neighborhood \( V \) of \( \bar{x} \) such that for each \( w \in W \) there is a unique vector \((x(w), \alpha(w))\), denoted by \( \hat{s}(w) \), in \( U \times V \) satisfying the equation \((35)\) and \( \| \hat{s}(w_2) - \hat{s}(w_1) \| \leq \ell \| w_2 - w_1 \| \) for any \( w_1, w_2 \in W \). Therefore, thanks to \((34)\), we obtain the following result.

**Proposition 5.1** Suppose that \( F(\bar{x}, \bar{w}) = 0 \) and the Lagrange multiplier \( \lambda \) corresponding to the stationary point \( \bar{x} \in S(\bar{w}) \) is positive. If condition \((25)\) is satisfied, then \( S \) has a Lipschitz continuous single-valued localization around \( \bar{w} \) for \( \bar{x} \).

Clearly, Proposition 5.1 encompasses Remark 4.1 of Part 1, which gives a sufficient condition for the local Lipschitz-like property of \( S \) around \((\bar{w}, \bar{x})\).

**Case 3:** \( F(\bar{x}, \bar{w}) = 0 \) and \( \lambda = 0 \). In this case, using the results of Dontchev and Rockafellar [17] one can verify the following lemma; see Sect. 5 for details.
Lemma 5.2 The generalized equation \( (39) \) is strongly regular at \((\bar{x}, \lambda)\) iff the matrix \( \nabla^2_w \mathcal{L}(\bar{x}, \lambda, \bar{w}) = \nabla^2_{xx} f_0(\bar{x}, \bar{w}) \) is nonsingular and
\[
\nabla_v F(\bar{x}, \bar{w})^T \nabla^2_{xx} f_0(\bar{x}, \bar{w})^{-1} \nabla_w F(\bar{x}, \bar{w}) > 0.
\] (37)

The sufficient condition for \( S \) to be locally Lipschitz-like around \((\bar{w}, \bar{x})\) in assertion (b) of Theorem 4.2 in Part 1 is \( (26) \), which reads as
\[
\nabla_x \text{assertion (b) of Theorem 4.2 in Part 1 is (26), which reads as}
\]
\[
\nabla_x \text{assertion (b) of Theorem 4.2 in Part 1 is (26), which reads as}
\]
\[
\text{condition (39) is satisfied. This contradicts (38). If }
\]
\[
\text{Now conditions (4a)-(4c) imply}
\]
\[
\text{ker } A'_1 \cap (\ker \nabla_x F(\bar{x}, \bar{w}) \times \mathbb{R}) = \{0\},
\] (38)
\[
\text{ker } A'_1 \cap \Delta_1 = \emptyset.
\] (39)

Indeed, by (4b) one sees that the linear subspace \( \ker A'_1 \cap (\ker \nabla_x F(\bar{x}, \bar{w}) \times \mathbb{R}) \) is contained in \( \ker A'_1 \cap \Delta_1 \). So, by (4c), the subspace just consists of the origin. This justifies (38). Similarly, by (4c), the set \( \ker A'_1 \cap \Delta_1 \) is contained in \( \ker A'_1 \cap \Delta_1 \). Hence, from (4b) it follows that \( \ker A'_1 \cap \Delta_1 \subset \{0\} \). As \( 0 \notin \Delta_1 \), condition (39) is satisfied.

Condition (38) implies that \( \det \nabla^2_{xx} f_0(\bar{x}, \bar{w}) \neq 0 \). Indeed, if there existed \( x', \lambda' \in \mathbb{R}^n \setminus \{0\} \), then by choosing \( \tau' = 0 \) we would have
\[
(x', \lambda') \in \ker A'_1 \cap (\ker \nabla_x F(\bar{x}, \bar{w}) \times \mathbb{R}).
\]

This contradicts (38).

To see that (38) and (39) yield (37), put \( v' = \nabla^2_{xx} f_0(\bar{x}, \bar{w})^{-1} \nabla_w F(\bar{x}, \bar{w}) \). We have to show that \( \nabla_v F(\bar{x}, \bar{w})^T v' > 0 \). If \( \nabla_v F(\bar{x}, \bar{w})^T v' = 0 \), then
\[
(v', -1) \in \ker A'_1 \cap (\ker \nabla_x F(\bar{x}, \bar{w}) \times \mathbb{R}).
\]

This contradicts (38). If \( \nabla_v F(\bar{x}, \bar{w})^T v' < 0 \), then for \( v'_1 := -v' \) one has \( \nabla_v F(\bar{x}, \bar{w})^T v'_1 > 0 \). Choosing \( \gamma = 1 \), by direct calculation we can verify that \( (v'_1, \gamma) \in \ker A'_1 \cap \Delta_1 \). This is a contraction to (38).

We have thus proved that the conditions \( \det \nabla^2_{xx} f_0(\bar{x}, \bar{w}) \neq 0 \) and (37) follow from (4a)-(4c). Hence, if the conditions (4a)-(4c) are satisfied, then (38) is strongly regular at \((\bar{x}, \lambda) = (\bar{x}, 0)\). Therefore, invoking Theorem 2.1 from [5] to the parametric generalized equation (35), we can assert that if (4a)-(4c) are satisfied, then the implicit multifunction \( \tilde{S}(.) \) has a Lipschitz continuous single-valued localization around \( \bar{w} \) for \((\bar{x}, \lambda) = (\bar{x}, 0)\). Thus, thanks to (34), we have the following result.

Proposition 5.2 Suppose that \( F(\bar{x}, \bar{w}) = 0 \) and the Lagrange multiplier \( \lambda \) corresponding to the stationary point \( \bar{x} \in S(\bar{w}) \) is zero. If (4a)-(4c) are satisfied, then \( S \) has a Lipschitz continuous single-valued localization around \( \bar{w} \) for \( \bar{x} \).

The result stated in Proposition 5.2 is better than assertion (b) of Theorem 4.2 in Part 1, which says that if (20) is fulfilled, i.e., (4a)-(4d) are valid, then \( S \) is locally Lipschitz-like around \((\bar{w}, \bar{x})\).
Proof of Lemma 5.1. By the definition of Robinson [5, p. 45], the strong regularity of (35) at \((\bar{x}, \lambda)\) is identical to the strong regularity of the affine variational inequality

\[
0 \in A \binom{x}{\alpha} + \bar{q} + N_{\mathbb{R}^n \times \mathbb{R}_+}(x, \alpha),
\]

where

\[
A := \nabla_{(x, \alpha)} g(\bar{x}, \lambda, \bar{w}) = \begin{pmatrix}
\nabla^2_{xx} L(\bar{x}, \lambda, \bar{w}) & \nabla_{x} F(\bar{x}, \bar{w})
\end{pmatrix}^T
\]

and

\[
\bar{q} := g(\bar{x}, \lambda, \bar{w}) - A \binom{\bar{x}}{\lambda},
\]

at \((\bar{x}, \lambda)\).

According to [17, Theorem 1], the affine variational inequality (40) is strongly regular at \((\bar{x}, \lambda)\) if and only if the multifunction

\[
L(q) := \left\{ \binom{x}{\alpha} \mid 0 \in A \binom{x}{\alpha} + q + N_{\mathbb{R}^n \times \mathbb{R}_+}(x, \alpha) \right\}
\]

is locally Lipschitz-like around \((\bar{q}, (\bar{x}, \lambda))\). Furthermore, applying [17, Theorem 2], we can assert that the latter is valid iff the critical face condition holds at \((\bar{q}, (\bar{x}, \lambda))\), i.e., for any choice of closed faces \(F_1\) and \(F_2\) of the critical cone \(K_0\) with \(F_1 \supset \ F_2\),

\[
[u \in F_1 - F_2, \ A^T u \in (F_1 - F_2)^*] \implies u = 0.
\]

Here,

\[
K_0 = K((\bar{x}, \lambda), v_0) := \left\{ (x', \alpha') \in T_{\mathbb{R}^n \times \mathbb{R}_+} (\bar{x}, \lambda) \mid (x', \alpha') \perp v_0 \right\},
\]

with

\[
v_0 := -A \binom{\bar{x}}{\lambda} - \bar{q} \in N_{\mathbb{R}^n \times \mathbb{R}_+}(\bar{x}, \lambda).
\]

Recall that a convex subset \(F\) of a convex set \(C \subset \mathbb{R}^p\) is a face of \(C\) if every closed line segment in \(C\) with a relative interior point in \(F\) has both endpoints in \(F\). When \(K_0\) is a linear subspace of \(\mathbb{R}^n \times \mathbb{R}_+\), it has a unique closed face, namely itself. Then, the critical face condition is reduced to

\[
[u \in K_0, \ A^T u \perp K_0] \implies u = 0.
\]

In the case \(\lambda > 0\), the critical face is equivalent to the nonsingularity of the matrix in (39). Indeed, the condition \(\lambda > 0\) implies \(N_{\mathbb{R}^n \times \mathbb{R}_+}(\bar{x}, \lambda) = \{(0, 0)\}\), \(T_{\mathbb{R}^n \times \mathbb{R}_+}(\bar{x}, \lambda) = \mathbb{R}^n \times \mathbb{R}_+\), and \(v_0 = (0, 0)\). It follows that \(K_0 = \mathbb{R}^n \times \mathbb{R}_+\). So, the critical face is reduced to (43), which is

\[
A^T u = 0 \implies u = 0.
\]

The latter means that \(A\) is nonsingular; or, equivalently, the matrix (39) is nonsingular.

Thus, we have proved that the generalized equation (35) is strongly regular at \((\bar{x}, \lambda)\) iff the matrix \(35\) is nonsingular.

\(\Box\)

Proof of Lemma 5.2. The arguments described in the beginning of the proof of Lemma 5.1 show that the generalized equation (35) is strongly regular at \((\bar{x}, \lambda)\) iff the critical face condition holds at \((\bar{q}, (\bar{x}, \lambda))\), i.e., for any choice of
closed faces $F_1$ and $F_2$ of the critical cone $K_0$ with $F_1 \supset F_2$ the condition \textbf{(42)} is fulfilled.

Since $\lambda = 0$, $N_{R^n \times R_+} (\bar{x}, \lambda) = \{0\} \times R_-$, and

$$T_{R^n \times R_+} (\bar{x}, \lambda) = R^n \times R_+.$$  

As $v_0 \in N_{R^n \times R_+} (\bar{x}, \lambda)$, there are two situations: (a) $v_0 = (0, \beta)$ with $\beta < 0$; (b) $v_0 = (0, 0)$. If (a) occurs, then $K_0 = R^n \times \{0\}$. Since $K_0$ is a linear subspace, the critical face condition is reduced to \textbf{(43)}. Using the formula for $A$ in \textbf{(41)}, one can easily show that \textbf{(43)} is equivalent to the requirement that the matrix $\nabla^2_{xx}L(\bar{x}, \lambda, \bar{w})$ is nonsingular. As $\lambda = 0$, one has $\nabla^2_{xx}L(\bar{x}, \lambda, \bar{w}) = \nabla^2_{xx}f_0(\bar{x}, \bar{w})$. So, \textbf{(43)} is also equivalent to the condition saying that the matrix $\nabla^2_{xx}f_0(\bar{x}, \bar{w})$ is nonsingular. Now, suppose that the situation (b) occurs. Then,

$$K_0 = K((\bar{x}, \lambda), v_0) = R^n \times R_+.$$  

Obviously, $K_0$ has only two nonempty faces: $R^n \times \{0\}$ and $R^n \times R_+$.

For $F_1 = F_2 = R^n \times \{0\}$, one has $F_1 - F_2 = R^n \times \{0\}$. Then,

$$(F_1 - F_2)^* = \{0\} \times R$$

and \textbf{(42)} is satisfied iff, for any $u' \in R^n$,

$$\nabla^2_{xx}L(\bar{x}, \lambda, \bar{w})u' = 0 \implies u' = 0.$$  

As $\lambda = 0$, it holds that $\nabla^2_{xx}L(\bar{x}, \lambda, \bar{w}) = \nabla^2_{xx}f_0(\bar{x}, \bar{w})$. Therefore, \textbf{(42)} is valid iff, for any $u' \in R^n$,  

$$\nabla^2_{xx}f_0(\bar{x}, \bar{w})u' = 0 \implies u' = 0.$$  

This is equivalent to saying that $\nabla^2_{xx}f_0(\bar{x}, \bar{w})$ is nonsingular.

For $F_1 = F_2 = R^n \times R_+$, $F_1 - F_2 = R^n \times R$. Then, $(F_1 - F_2)^* = \{0\} \times \{0\}$ and \textbf{(42)} is satisfied iff the matrix

$$A^T = \begin{pmatrix} \nabla^2_{xx}f_0(\bar{x}, \bar{w}) & -\nabla_x F(\bar{x}, \bar{w}) \\ \nabla_x F(\bar{x}, \bar{w})^T & 0 \end{pmatrix}$$

is nonsingular, or, $A$ is nonsingular.

For $F_1 = R^n \times R_+$ and $F_2 = R^n \times \{0\}$, $F_1 - F_2 = R^n \times R_+$ and $(F_1 - F_2)^* = \{0\} \times R_+$. Then, \textbf{(42)} is fulfilled iff

$$\begin{cases} \nabla^2_{xx}f_0(\bar{x}, \bar{w})u' - \gamma \nabla_x F(\bar{x}, \bar{w}) = 0 \\ \nabla_x F(\bar{x}, \bar{w})^Tu' \leq 0 \\ u' \in R^n, \ \gamma \geq 0 \end{cases} \implies \begin{cases} u' = 0 \\ \gamma = 0. \end{cases} \tag{44}$$

The proof of the “necessity part” of Lemma 5.2 will be completed if we can show that \textbf{67} is valid. If \textbf{67} does not hold, then by putting

$$u' = \nabla^2_{xx}f_0(\bar{x}, \bar{w})^{-1}\nabla_x F(\bar{x}, \bar{w}),$$

we have

$$\nabla_x F(\bar{x}, \bar{w})^Tu' = \nabla_x F(\bar{x}, \bar{w})^T\nabla^2_{xx}f_0(\bar{x}, \bar{w})^{-1}\nabla_x F(\bar{x}, \bar{w}) \leq 0.$$  

So, for $\gamma = 1$, one has

$$\begin{cases} \nabla^2_{xx}f_0(\bar{x}, \bar{w})u' - \gamma \nabla_x F(\bar{x}, \bar{w}) = 0 \\ \nabla_x F(\bar{x}, \bar{w})^Tu' \leq 0 \\ u' \in R^n, \ \gamma \geq 0. \end{cases} \tag{66}$$
This contradicts (44). We have thus proved (37) is valid.

To prove the “sufficiency part” of Lemma 5.2, we suppose that the matrix \( \nabla^2_{xx} \mathcal{L} (\bar{x}, \lambda, \bar{w}) = \nabla^2_{xx} f_0 (\bar{x}, \bar{w}) \) is nonsingular and (37) is fulfilled. To verify the fulfillment of the critical face condition at \((\bar{q}, (\bar{x}, \lambda))\), we need only to show that the matrix \( A \) is nonsingular and the implication (44) holds.

To obtain the nonsingularity of \( A \), suppose to the contrary that there exists a pair \((u', \gamma) \neq (0, 0)\) satisfying

\[
\begin{align*}
\nabla^2_{xx} f_0 (\bar{x}, \bar{w}) u' - \gamma \nabla_x F (\bar{x}, \bar{w}) &= 0 \\
\nabla_x F (\bar{x}, \bar{w})^T u' &= 0. 
\end{align*}
\]  

(45)

If \( \gamma = 0 \), then the first equation of (45) forces \( u' = 0 \), because the matrix \( \nabla^2_{xx} f_0 (\bar{x}, \bar{w}) \) is nonsingular. So, we must have \( \gamma \neq 0 \). From the first equation of (45) we deduce that

\[ u' = \gamma \nabla^2_{xx} f_0 (\bar{x}, \bar{w})^{-1} \nabla_x F (\bar{x}, \bar{w}). \]

Hence, by the second equation of (45), we obtain

\[ \gamma \nabla_x F (\bar{x}, \bar{w})^T \nabla^2_{xx} f_0 (\bar{x}, \bar{w})^{-1} \nabla_x F (\bar{x}, \bar{w}) = 0. \]

This obviously contradicts (37). Thus, \( A \) is nonsingular.

Finally, to obtain the implication (44), let \( u' \in \mathbb{R}^n \) and \( \gamma \geq 0 \) be such that

\[
\begin{align*}
\nabla^2_{xx} f_0 (\bar{x}, \bar{w}) u' - \gamma \nabla_x F (\bar{x}, \bar{w}) &= 0 \\
\nabla_x F (\bar{x}, \bar{w})^T u' &= 0. 
\end{align*}
\]  

(46)

Multiplying both sides of the equation in (46) from the left with the \( 1 \times n \) matrix \( \nabla_x F (\bar{x}, \bar{w})^T \nabla^2_{xx} f_0 (\bar{x}, \bar{w})^{-1} \), one obtains

\[ \nabla_x F (\bar{x}, \bar{w})^T u' - \gamma \nabla_x F (\bar{x}, \bar{w})^T \nabla^2_{xx} f_0 (\bar{x}, \bar{w})^{-1} \nabla_x F (\bar{x}, \bar{w}) = 0. \]

(47)

Due to (37) and the condition \( \gamma \geq 0 \),

\[ -\gamma \nabla_x F (\bar{x}, \bar{w})^T \nabla^2_{xx} f_0 (\bar{x}, \bar{w})^{-1} \nabla_x F (\bar{x}, \bar{w}) \leq 0. \]

Combining this with the inequality \( \nabla_x F (\bar{x}, \bar{w})^T u' \leq 0 \) from (46), by (47) one has

\[ -\gamma \nabla_x F (\bar{x}, \bar{w})^T \nabla^2_{xx} f_0 (\bar{x}, \bar{w})^{-1} \nabla_x F (\bar{x}, \bar{w}) = 0. \]

Due to (37), \( \gamma = 0 \). Then, the first equation in (46) implies \( \nabla^2_{xx} f_0 (\bar{x}, \bar{w}) u' = 0 \). As \( \nabla^2_{xx} f_0 (\bar{x}, \bar{w}) \) is nonsingular, one has \( u' = 0 \). Thus, (44) is valid.

The proof is complete. \( \square \)

7 Conclusions

In this paper, we have analyzed the stability of the stationary point set map of a smooth parametric optimization problem with one smooth functional constraint under total perturbations. Not only sufficient conditions for the local Lipschitz-like property of the stationary point set map were given, but also necessary conditions for the latter property have been obtained. Sufficient conditions for the Robinson stability of the stationary point set map were given, in addition, we have revisited several stability theorems in indefinite quadratic programming.

It is still unclear to us whether the Lipschitz continuous single-valued localization mentioned in Propositions 5.1 and 5.2 implies the Robinson stability of the stationary point set map, or not.

Extensions of the obtained results to the case of smooth parametric optimization problem with more than one smooth functional constraint under total perturbations are worthy further investigations.
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