Incentives in Dominant Resource Fair Allocation under Dynamic Demands

Giannis Fikioris*  
Cornell University  
gfikioris@cs.cornell.edu  

Rachit Agarwal†  
Cornell University  
ragarwal@cornell.edu  

Éva Tardos‡  
Cornell University  
eva.tardos@cornell.edu  

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Abstract

Every computer system—from schedulers in clouds (e.g., Amazon, Google, Microsoft, etc.) to computer networks to hypervisors to operating systems—performs resource allocation across system users. The defacto allocation policies used in most of these systems are max-min fairness for single resource settings and dominant resource fairness for multiple resources. These allocation schemes guarantee many desirable properties like incentive compatibility, envy-freeness, and Pareto efficiency, assuming user demands are static (time-independent). However, in modern real-world production systems, user demands are dynamic, that is, vary over time. As a result, there is now a fundamental mismatch between the resource allocation goals of computer systems, and the properties enabled by classical resource allocation policies. This paper aims to bridge this mismatch. When demands are dynamic, instant-by-instant max-min fairness can be extremely unfair over a longer period of time, i.e., lead to unbalanced user allocations, as previous large allocations have no effect in the current time step. We consider a natural generalization of the classic algorithm for max-min fair allocation and dominant resource fairness for multiple resources when users have dynamic demands: this algorithm guarantees Pareto optimality while ensuring that resources allocated to users are as max-min fair as possible up to any time instant, given the allocation in previous periods. While this dynamic allocation scheme remains Pareto optimal and envy free, unfortunately, it is not incentive compatible. We study the strength of the incentive to misreport; our results show that the possible increase in utility by misreporting demand is bounded and, since this misreporting can lead to significant decrease in overall useful allocation, this suggests that it is not a useful strategy. Our main result is to show that our dynamic version of the dominant resource fairness algorithm is $(1 + \rho)$-incentive compatible, where $\rho$ quantifies the relative importance of a resource for different users; we also show that this factor is tight even with only two resources. We also present a $3/2$ upper bound and a $\sqrt{2}$ lower bound for the incentive compatibility when there is only one resource. We also offer extensions of the results for the case when the mechanism uses weights to prioritize every user differently. Our results indicate a big disparity between single resource and multiple resources even with just two resources showing a surprising difference in the users’ incentive to deviate, with significantly less incentive to deviate in the case of one resource, and this is especially true in the case when the mechanism uses weights to prioritize every user differently.

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1 Introduction

Resource allocation is a fundamental problem in computer systems. Companies like Google [Ver+15] and Microsoft [Gra+14] use schedulers in private clouds to allocate a limited and divisible amount of resources (e.g., CPU, memory, servers, etc.) among a number of selfish and possibly strategic users that want to maximize their allocation; the goal of the scheduler is to maximize resource utilization while achieving fairness in resource allocation. The defacto allocation policies used in many of these systems are the classic max-min fairness (MMF) and its relatively recent generalization, dominant resource fairness (DRF) [Gho+11] policies, for single and multiple resource settings, respectively. For instance, these policies are used in most schedulers in private clouds [Bou+14; Gho+11; Gra+14; Gra+16a; Gra+16b; Hin+11; SFS12; Vav+13; Ver+15]; they are deeply entrenched in congestion control protocols like TCP and its variants [Ali+10; CJ89]; and are the default policies for resource allocation in most operating systems and hypervisors [Cha08; KVM16]. DRF has also attracted a lot of attention in the economics and computing community, starting with Parkes, Procaccia, and Shah [PPS12] and with followup work [FPV15; FPV17; Im+20; KPS13; LLL18].

Such prevalence of MMF and DRF is rooted in the properties they guarantee: Pareto-efficiency, sharing incentives (users are not better off by getting their fair share of resources every round), incentive compatibility, envy-freeness (no user envies the allocation of another user), and fairness. However, to guarantee these properties, both MMF and DRF policies assume that user demands do not change over time. This assumption is far from realistic in modern real-world deployments: several recent studies in the systems community have shown that user demands have become highly dynamic, that is, vary over time [CAD18; Rei+12; Vup+20a; YYR20]. For such dynamic user demands, naively using these policies (e.g., to perform a new instantaneously max-min fair allocation every unit of time) can result in vastly disparate user allocations over time—intuitively, since MMF does not take past allocations into account, dynamic user demands can result in increasingly unfair user allocations over time.

Motivated by the realistic case of dynamic user demands over divisible resources, we study Dynamic DRF\(^1\), a mechanism that generalizes DRF for dynamic demands; just like DRF generalizes MMF for multi-resource allocations, Dynamic DRF generalizes Dynamic MMF [Fre+18] for multi-resource allocations over dynamic demands by taking past allocations into account. Our model is the same as the original DRF paper [Gho+11]: every round, each user specifies a vector of ratios (the proportions according to which the different resources are used by the user, e.g., for her application) and a demand (the maximum allocation of resources that would be useful to her). Users have Leontief preferences—as they are known in economics, i.e., their utility each round is equal to the minimum over the amount of every resource they receive divided by their ratio for it, up to their demand. However, in contrast to the DRF paper, we consider scenarios where the ratios and the demands can vary over time and users want to maximize the sum of their utilities across rounds.

In every round, Dynamic DRF allocates resources while being as fair as possible given the past allocations: first the minimum total utility of any user is maximized, then the second minimum, etc. Besides being fair, Dynamic DRF is also Pareto-efficient by construction: every round, either every user’s demand is satisfied or for every user a resource she wants to use is saturated. However, similar to Dynamic MMF [Fre+18], Dynamic DRF is not incentive compatible, i.e., it is possible that a user can misreport her demand or her ratios on one round to increase her total useful allocation in the future (in fact, our study leads to stronger lower bounds on incentive compatibility of Dynamic MMF; see Theorems 3.2 and 4.7). Nevertheless, studying Dynamic DRF is both important and interesting. First, similar to the widely-used classic MMF and DRF (also referred to as static MMF and DRF), Dynamic DRF is simple and easy to understand; thus, it has the potential for real-world adoption (similar to many other non-incentive compatible mechanisms used in practice, e.g., non-incentive compatible auctions used by U.S. Treasury to sell treasury bills since 1929 and by the U.K to sell electricity [Har18; Kri09; Par18]). Second, our results show that Dynamic DRF is approximately incentive

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\(^1\)The name Dynamic DRF has also been used by [KPS13] for the extension of DRF when users arrive and depart sequentially. See more about the difference in Section 1.1.
compatible, that is, strategic users can increase their allocation by misreporting their demands but this increase is bounded by a relatively small constant factor, independent of the number of users and the number of resources; moreover, effective misreporting not only requires knowledge of future demands, but can lead to a significant decrease in overall useful allocation, suggesting that misreporting is unlikely to be a useful strategy for any user.

**Our Contribution.** Our goal is to study Dynamic MMF and Dynamic DRF and the incentive to misreport in them. A popular relaxation of incentive compatibility is $\gamma$-incentive compatibility \cite{Arc+03; AB18; BSV19; DFP08; Dütt+12; KPS03; MS14}, which requires that the possible increase in utility by untruthful reporting must be bounded by a factor of $\gamma \geq 1$ ($\gamma$ is referred to as incentive compatibility ratio). Using this notion we show that users have limited incentive to be untruthful:

- We start the technical part of this paper in Section 3 by considering the simpler problem of single resource environments, Dynamic MMF. We show that users have no incentive to over-report their demand (Theorem 3.5), that Dynamic MMF is $3/2$-incentive compatible (Theorem 3.6), and give an almost matching lower bound of $\sqrt{2}$ (Theorem 3.2). We also show that Dynamic MMF is envy-free (Corollary 4.9) and the variant where every user is guaranteed an $\alpha$ fraction of her fair share of resources (for $\alpha \in [0, 1]$) satisfies $\alpha$-sharing incentives while maintaining the incentive compatibility ratio, as well as every one of the aforementioned properties.

- Our main results are presented in Section 4, where we focus on the setting of multiple resources. In the case that every user demands every resource when using the system, we show that in Dynamic DRF user $i$ cannot increase her utility more than a factor of $(1 + \rho_i)$ (Theorem 4.6) and give a matching lower bound (Theorem 4.7) where $\rho_i$ is a parameter that quantifies the relative importance of every resource between user $i$ and the other users. We also show that in this case users have no incentive to over-report their demand or misreport their ratios (Theorem 4.4); this guarantees that resources allocated to the users are always in use. The assumption that every user demands every resource when using the system, even if in different ratios, is quite natural; in computer systems where the resources shared are CPU, memory, storage, etc. the users run applications that use every resource. This assumption is also used by \cite{KPS13}, where they extend DRF to the case of users arriving and leaving sequentially resulting in dynamically changing the allocations in the system.

Additionally, we show that Dynamic DRF is envy-free (Theorem 4.8), and the variant where every user is guaranteed an $\alpha$ fraction of her fair share satisfies $\alpha$-sharing incentives while retaining every one of the aforementioned properties.

- We also consider some generalizations. First, we consider the case where every user $i$ is associated with a positive weight $w_i$ indicating their priority. This extra assumption does not change our results in Section 4, other than slightly altering the definition of $\rho_i$ affected by the weights $w$—the bound of $(1 + \rho_i)$ remains tight for the new $\rho_i$. We also study weighted users in single resource settings in Section 5.1, along with the assumption that users can collude. In this case, Dynamic weighted MMF is $2$-incentive compatible (Theorem 5.3) and demand over-reporting does not increase utility (Theorem 5.2). The former of these results strikes a big contrast between single and multiple resource settings: assuming that users do not collude and directly applying our results from Dynamic DRF in Dynamic weighted MMF proves a $1 + \rho_i = 1 + \max_{k \neq i}(w_i/w_k)$ upper bound for the incentive compatibility of user $i$, which is possibly unbounded; in contrast, we manage to prove a $2$-upper bound.

### 1.1 Related Work

The simplest algorithm for resource allocation is strict partitioning \cite{Ver+17; Vup+20b}, that allocates a fixed amount of resources to each user independent of their demands. While incentive compatible, strict partitioning
can have arbitrarily bad efficiency. Static MMF and DRF [Fre+18; Gho+11; Gra+14; Gra+16b; PPS12; SFS12] are Pareto-efficient, incentive compatible, envy-free, and satisfy sharing incentives, but fair only when user demands are static. Freeman et al. [Fre+18] prove that Dynamic MMF is not incentive compatible under the same utility model as ours. The papers [Fre+18; Hos19] study resource allocation for single resource settings with dynamic demands focusing on the case when users have utility for resources above their demand, only at a lower rate. They offer alternate mechanisms where past allocations have some effect on the current ones (unlike static MMF) while maintaining incentive compatibility, but the mechanisms they consider are closer to MMF separately in each epoch, and less aim to be fair overall. Under this model, they present two mechanisms that are incentive compatible but either satisfy sharing incentives and have no Pareto-efficiency guarantees, or approximately satisfy sharing incentives and are approximately Pareto-efficient under strong assumptions (user demands being i.i.d. random variables and number of rounds growing large).

Sadok, Campista, and Costa [SCC21] present minor improvements in fairness over static DRF for dynamic demands while maintaining incentive compatibility. Their mechanism allocates resources in an incentive compatible way according to DRF while marginally penalizing users with larger past allocations using a parameter $\delta \in [0, 1)$. Specifically, if $t$ rounds ago a user received an allocation of $r$, that allocation penalizes the user in the current round by $r(1 - \delta)^t$. This means that the penalty of $(1 - \delta)^t \leq 0.25$ reduces exponentially fast with time for any fixed $\delta < 1$ and as $\delta \to 1$ the penalty tends to 0. Thus, for every $\delta$ (and, especially for $\delta = 0$ and $\delta \to 1$), their mechanism suffers from similar problems as static DRF: past allocations are barely taken into account.

Several other papers study resource allocation where user demands can be dynamic, but with significantly different setting than ours. [AW19; ZP20] examine the setting where indivisible items arrive over time and have to be allocated to users whose utilities are random; however Aleksandrov and Walsh [AW19] study a very weak version of incentive compatibility in which a mechanism is incentive compatible if misreporting cannot increase a user’s utility in the current round and Zeng and Psomas [ZP20] do not consider strategic users. Kandasamy et al. [Kan+20] study single resource allocation and assume the users do not know their exact demands every round and need to provide feedback to the mechanism after each round of allocation to allow the mechanism to learn. The goal of the paper is to offer a version of MMF that approximately satisfies incentive compatibility, sharing incentives, and Pareto-efficiency, despite the lack of information, but is not considering the long-term fairness that is the focus of our mechanisms.

Another series of work study the setting where users arrive consecutively and potentially leave after some period of time: [FPV15; LLL18] focus on single resource settings, while [FPV17; Im+20; KPS13] also study the allocation of multiple resources. Even though their setting is dynamic, users have constant demands and cannot re-arrive after leaving, making the user demands static. After every arrival or departure of a user, resources need to be re-distributed while maintaining some sort of fairness, e.g., the users’ utilities need to be approximately similar. Kash, Procaccia, and Shah [KPS13] focus on never decreasing a user’s allocation when re-distributing resources, thus creating a mechanism that allocates at most $k/n$ fraction of every resource when $k$ out of $n$ users are present in the system. They offer a mechanism for this setting that they also call dynamic DRF, however, in their setting the dynamic nature of the problem comes from churn in users, as well as the corresponding changes in total resources, and not from dynamic demands. The papers [FPV15; FPV17; Im+20; LLL18] consider resource allocation, but incentive compatibility is not taken into account and the focus is to maximize fairness while bounding the “disruptions” of the system every time a user enters or leaves, which is how many users’ allocations are altered.

There are many reasons why mechanisms used in practice are often not incentive compatible, including the relative simplicity of these mechanisms that makes it easier for users to understand and use them and the fact that incentive compatible mechanisms are not actually incentive compatible depending on the information structure, e.g., when participants collude (see Balcan, Sandholm, and Vitercik [BSV19] for a nice discussion of a long
list of other reasons). In most settings, even if the mechanism is manipulable, finding a profitable manipulation is hard. In our setting, finding such manipulation requires knowledge of all users’ future demands, and while under-reporting demand can lead to increased future utility, it may also lead to decreased utility. When using a manipulable mechanism, it is important to understand how large is the incentive to manipulate. γ-incentive compatibility has been considered in many settings. As we mentioned before, Kandasamy et al. [Kan+20] study a setting similar to ours, but where long-term fairness is not their focus and they try to learn users’ demands; to achieve their results they propose several mechanisms most of which are approximate incentive compatible. [Arc+03; Düt+12; KPS03; MS14] study combinatorial auctions that are almost incentive compatible. Dekel, Fischer, and Procaccia [DFP08] studies approximate incentive compatibility in machine learning, when users are asked to label data. Azevedo and Budish [AB18] examines approximate incentive compatibility in large markets, where the number of users grows to infinity. Balcan, Sandholm, and Vitercik [BSV19] develops algorithms that can estimate how incentive compatible mechanisms for buying, matching, and voting are. Hartline and Taggart [HT19] develop auctions that use samples from past non-incentive compatible auctions to improve social welfare or revenue guarantees. In the same spirit, Dynamic DRF is aimed at improving fairness.

2 Preliminaries

We first make some definitions that apply to all sections. We use \([n]\) to denote the set \(\{1, 2, \ldots, n\}\) for any \(n \in \mathbb{N}\). Additionally, we define \(x^+ = \max(x, 0)\) and denote with \(\mathbbm{1}[\cdot]\) the indicator function.

There are \(n\) users, where \(n \geq 2\). The set of users is denoted with \([n]\). In some settings, every user \(i\) is associated with a weight, \(w_i > 0\) which indicates each user’s priority and in this case we view the allocation fair, if user \(i\) has (approximately) \(w_i/w_j\) more utility than user \(j\). Additionally, we sometimes assume that user \(i\) has initial endowment or \emph{fair share} a \(w_i/\sum_{j \in [n]} w_j\) fraction of the total resources. The game is divided into \emph{epochs} \(1, 2, \ldots, t, \ldots\)

A mechanism is called \emph{envy-free} if for any pair of users \(i\) and \(j\), and if user \(i\) is truthful, she would not have gained utility if she had been allocated the resources user \(j\) was allocated. Similarly, we define \emph{weighted envy-freeness}: if every user \(i\) is associated with a weight \(w_i > 0\), a mechanism is envy-free if for any pair of users \(i\) and \(j\), user \(i\) would not have gained utility if she had been allocated the resources user \(j\) was allocated, scaled by \(w_i/w_j\) (note that this scaling sometimes results in comparing to usage with more resources than what is available).

A mechanism satisfies \emph{sharing incentives} if every truthful user’s utility is not less than her utility if she had been allocated her fair share every epoch, i.e., a \(\frac{1}{n}\) fraction of the total resources. For weighted users, a mechanism satisfies sharing incentives if the utility of user \(i\) is not less than her utility if she had been allocated her fair share every epoch, i.e., a \(w_i/\sum_{j \in [n]} w_j\) fraction of the total resources. For \(\alpha \in [0, 1]\), there is also the notion of \emph{\(\alpha\)-sharing incentives}, in which user \(i\)’s utility must be at least \(\alpha\) times her utility if she had been allocated her fair share every epoch.

3 Single Resource Setting

Before we present our results on the multiple resource setting in Section 4, we first present results on the simpler setting of dynamic max-min fairness (Dynamic MMF) where there is only one resource and users are unweighted (\(w_i = w_j\) for all users \(i, j\)).

Notation. Every epoch there is a fixed amount of a resource shared amongst the users. We denote the total amount of that resource with \(\mathcal{R}\).

We denote with \(r_i^t\) the \emph{allocation} of user \(i\) in epoch \(t\), i.e., the amount of resources the user receives. We also denote with \(R_i^t\) the cumulative allocation of user \(i\) up to round \(t\), i.e., \(R_i^t = \sum_{\tau=1}^{t} r_i^\tau\). By definition, \(R_i^0 = 0\).
Every epoch \( t \), each user \( i \) has a demand, denoted with \( d_i^t \). This represents the maximum allocation that is useful for that user in that epoch, i.e., a user is indifferent between getting an allocation equal to her demand and an allocation higher than her demand, so the utility of user \( i \) in epoch \( t \) is \( u_i^t = \min(r_i^t, d_i^t) \). The total utility of user \( i \) after epoch \( t \) equals the sum of utilities up to that round, i.e., \( U_i^t = \sum_{r=1}^t u_i^r = \sum_{r=1}^t \min(r_i^r, d_i^r) \).

**Dynamic Max-min Fairness**  In MMF the resources are allocated such that the minimum amount of resources is maximized, then the second minimum is maximized, etc., as long as every user gains an amount of resources that does not exceed her demand. If for example, we have \( R = 1 \) total resource and three users with demands 1/4, 3/8, and 1, then the first user gets 1/4 resources and the other two get 3/8 resources each. In Dynamic MMF, every epoch the MMF algorithm is applied to the users’ cumulative allocations constrained by what they have already been allocated in previous iterations. Our mechanism will have an additional parameter \( \alpha \in [0, 1] \) that guarantees a fraction of the fair share of each user every single epoch, independent of allocations in previous epochs: we guarantee every user at least \( \alpha R/n \) resources (assuming they have big enough demand to use it). Formally, given an epoch \( t \) and that every user \( i \) has cumulative allocation \( R_{i}^{t-1} \), Dynamic MMF solves the following problem

\[
\text{choose } r_1^t, r_2^t, \ldots, r_n^t \\
\text{applying max-min fairness on } R_1^{t-1} + r_1^t, R_2^{t-1} + r_2^t, \ldots, R_n^{t-1} + r_n^t \\
given the constraints \sum_{i \in [n]} r_i^t \leq R \quad \text{and } \forall i \in [n], \min \left\{ d_i^t, \frac{\alpha R}{n} \right\} \leq r_i^t \leq d_i^t
\]

We are going to call the value \( \min \{ d_i^t, \frac{\alpha R}{n} \} \) the guarantee of user \( i \) in epoch \( t \), and denote it with \( g(d_i^t) = \min \{ d_i^t, \frac{\alpha R}{n} \} \). Note that \( g(\cdot) \) is a non-decreasing function. Because of the guarantee of every user it is easy to see that Dynamic MMF satisfies \( \alpha \)-sharing incentives.

**Theorem 3.1.** When every user is guaranteed an \( \alpha \)-fraction of their fair share, Dynamic MMF satisfies \( \alpha \)-sharing incentives.

**Incentives in Dynamic MMF**  It is easy to see [PPS12] that applying MMF when there is a single epoch is incentive compatible, i.e., users can never increase their utility by misreporting their demand. In dynamic settings, however, this is not the case. As was shown by [Fre+18], a user can increase her allocation by misreporting. See also the improved lower bound (Theorem 3.2).

### 3.1 Bounds on Incentive Compatibility

We will now focus in how far Dynamic MMF is from incentive compatible. W.l.o.g. we are usually going to study the possible deviations of user 1, i.e., how much user 1 can increase her allocation by lying about her demand. We use the symbols \( d_1^t, r_1^t, R_1^t, \bar{u}_1^t, \bar{U}_1^t \) or \( d_1^t, r_1^t, R_1^t, \hat{u}_1^t, \hat{U}_1^t \) to denote the claimed demand and resulting outcome of some deviation of user 1. We want to prove that for some \( \gamma \geq 1 \), Dynamic MMF is always \( \gamma \)-incentive compatible, i.e., for every users’ true demands \( \{ d_i^t \}_{i \in [n], t \in \mathbb{N}} \), for every deviation of user 1 \( \{ d_1^t \}_{t \in \mathbb{N}} \), and for every epoch \( t \), to prove that \( \hat{U}_1^t \leq \gamma \bar{U}_1^t \). \( \gamma \) is often referred to as the incentive compatibility ratio.

**Lower bound on incentive compatibility**  As mentioned before, Dynamic MMF does not guarantee incentive compatibility. This is demonstrated in the following theorem, where user 1 can misreport her demand to increase her utility by a factor of almost \( \sqrt{2} \).

**Theorem 3.2.** For any value of \( \alpha \in [0, 1] \), there is an instance with \( n \) users, where in Dynamic MMF a user can under-report her demand to increase her utility by a factor of \( \sqrt{2} \) as \( n \to \infty \).
We defer the proof of the theorem to Appendix A. To provide intuition about how a user can increase her utility by misreporting, we include here the example of [Fre+18], where user 1 can increase her utility by a factor of 10/9.

**Example 1 ([Fre+18]).** There are 3 users, 3 epochs, and $\alpha = 0$. Every epoch the available amount of the resource is $R = 8$ and the real demands of the users are shown in Table 1, as well as their allocations when user 1 is truthful and when she misreports.

| Users | Epoch 1 | Epoch 2 | Epoch 3 |
|-------|---------|---------|---------|
| User 1 | 8 4 0   | 8 2 4   | 8 3 6   |
| User 2 | 8 4 8   | 0 0 0   | 8 5 2   |
| User 3 | 0 0 0   | 8 6 4   | 0 0 0   |

Table 1: Every epoch there is a total of 8 resources. The black numbers are the users’ demands, the blue ones are the users’ allocations when user 1 is truthful, and the red ones are the allocations when user 1 misreports her demand in epoch 1 by demanding 0.

Because user 1 under-reports her demand in epoch 1, in epoch 2 she manages to “steal” some of user 3’s resources. Then, in epoch 3 the allocation mechanism equalizes the total allocations of users 1 and 2, making user 1 get back some of the resources she lost in epoch 1. This results in user 1 having a total allocation of 10 instead of 9, i.e., her utility increases by a factor of 10/9.

Both in Theorem 3.2 and Example 1, it is important to note that user 1 can increase her utility only by a bounded constant factor. Additionally, this is done by user 1 under-reporting her demand, not over-reporting; this is important because it implies that the resources allocated are always used by the users and hence the allocation remains Pareto optimal. As we will show next, both of these facts are true in general.

**Upper bound on incentive compatibility ratio**

To prove the above, first we show a lemma offering a simple condition on which pair of users can gain overall allocations from one another. When users’ demands are not satisfied, for a user to get more resources someone else needs to get less. The lemma will allow us to reason about how a deviation by user 1 can lead to a user $i$ getting more resources and another user $j$ getting less (possibly $i = 1$ or $j = 1$).

**Lemma 3.3.** Fix an epoch $t$ and the total allocations up to epoch $t - 1$ of any two outcomes $\{\hat{R}_k^{t-1}\}_{k \in [n]}$ and $\{\bar{R}_k^{t-1}\}_{k \in [n]}$. Let $i, j$ be two different users. If the following conditions hold

- For $i$, $\hat{r}_i^t < \tilde{r}_i^t$ and $\hat{d}_i^t \leq \tilde{d}_i^t$.
- For $j$, $\hat{r}_j^t > \tilde{r}_j^t$ and $\bar{d}_j^t \leq \tilde{d}_j^t$.

then, for any value of $\alpha \in [0, 1]$, in Dynamic MMF it holds that $\hat{R}_i^t \geq \bar{R}_i^t$ and $\hat{R}_j^t \leq \bar{R}_j^t$, implying

$$\hat{R}_i^t - \bar{R}_i^t \leq \bar{R}_j^t - \hat{R}_j^t$$

It should be noted that the second condition for $i$ (similarly for $j$) is not needed when $i$ is not the user who misreports her demand, i.e., $i \neq 1$.

**Proof.** Because of the conditions, we notice that $\hat{r}_i^t < \tilde{r}_i^t$ (since $\hat{r}_i^t \leq \tilde{d}_i^t$) and $\hat{r}_j^t > \tilde{r}_j^t$ (since $\hat{r}_j^t > \tilde{d}_j^t \geq g(\tilde{d}_j^t)$). These two inequalities imply that it would have been feasible in (1) to increase $\hat{r}_i^t$ by decreasing $\hat{r}_j^t$. This implies that $\hat{R}_i^t \geq \bar{R}_j^t$, otherwise it would have been more fair to give some of the resources user $j$ got to user $i$. With the analogous inverse argument (we can increase $\hat{r}_j^t$ by decreasing $\hat{r}_i^t$) we can prove that $\hat{R}_i^t \leq \bar{R}_j^t$. This completes the proof.
The main technical tool in our work is the following lemma bounding the total amount all the users can “win” because of user 1 deviating, i.e., \( \sum_k (\hat{R}_k^t - R_k^t)^+ \). Rather than bounding the deviating user 1’s gain directly, it is better to consider the overall increase of all users. Specifically, the lemma upper bounds the increase of that amount after any epoch, given that user 1 does not over-report her demand (which as we are going to show later in Theorem 3.5 users have no benefit in doing). The bound on the total over-allocation then follows by summing over the epochs. The bound on the increase of \( \sum_k (\hat{R}_k^t - R_k^t)^+ \) after any epoch is different according to two different cases:

- If user 1 is truthful, the increase is at most 0: in these steps over-allocation can move between users but cannot increase. This is the reason studying the total over-allocation is so helpful.

- If user 1 under-reports, the increase is bounded by the amount of resources she receives when she is truthful.

**Lemma 3.4.** Fix an epoch \( t \) and the total allocations of any two outcomes \( \{\hat{R}_k^{t-1}\}_{k \in [n]} \) and \( \{\hat{R}_k^{t-1}\}_{k \in [n]} \). Assume that \( \{d_k^t\}_{k \in [n]} \) are some user demands and that \( \{d_k^t\}_{k \in [n]} \) are the same demands except user 1’s, who deviates but not by over-reporting, i.e., \( d_1^t \leq d_1^t \). Then, for any \( \alpha \in [0, 1] \), it holds that

\[
\sum_{k \in [n]} (\hat{R}_k^t - R_k^t)^+ - \sum_{k \in [n]} (\hat{R}_k^{t-1} - R_k^{t-1})^+ \leq \alpha \left[ d_1^t - d_1^t \right]
\]

We will use Lemma 3.3 to show that if user 1 is truthful in epoch \( t \), then the l.h.s. of (2) is at most 0: Dynamic MMF allocates resources such that the large \( \hat{R}_k^t - R_k^t \) are decreased and the small \( \hat{R}_k^t - R_k^t \) are increased. Finally, if user 1 reports a lower demand, then the (at most) \( \hat{r}_1^t \) resources user 1 does not get might increase the total over-allocation by the same amount.

**Proof.** First we define \( P_1^t = \{k \in [n] : \hat{R}_k^t > \hat{R}_k^t \} \) for all \( t \). Suppose by contradiction:

\[
\sum_{k \in P_1^t} (\hat{R}_k^t - \hat{R}_k^t) - \sum_{k \in P_1^{t-1}} (\hat{R}_k^{t-1} - \hat{R}_k^{t-1}) > \hat{r}_1^t \left[ d_1^t - d_1^t \right]
\]

Because \( \sum_{k \in P_1^t} (\hat{R}_k^{t-1} - \hat{R}_k^{t-1}) \leq \sum_{k \in P_1^{t-1}} (\hat{R}_k^{t-1} - \hat{R}_k^{t-1}) \), the above inequality implies

\[
\sum_{k \in P_1^t} (\hat{r}_k^t - \hat{r}_k^t) > \hat{r}_1^t \left[ d_1^t - d_1^t \right]
\]

Since user 1 reports a lower demand \( \hat{d}_1^t \leq d_1^t \), it holds that \( \sum_{k \in P_1^t} \hat{r}_k^t \geq \sum_{k \in P_1^t} \hat{r}_k^t \), i.e., the total resources allocated to the users do not increase when user 1 reports a lower demand. Combining this fact with (3) we get that

\[
\sum_{k \in P_1^t} (\hat{r}_k^t - \hat{r}_k^t) > \hat{r}_1^t \left[ d_1^t - d_1^t \right]
\]

We notice that because of (3), there exists a user \( i \in P_1^t \) for whom \( \hat{r}_i^t > \hat{r}_i^t \); because of (4), there exists a user \( j \notin P_1^t \) for whom \( \hat{r}_j^t > \hat{r}_j^t \). Additionally for that \( j \) we can assume that \( d_j^t = d_j^t \) because:

- If user 1 does not deviate then for all \( k \), \( \hat{d}_k^t = \hat{d}_k^t \),

- If \( \hat{d}_1^t < d_1^t \), then (4) becomes \( \sum_{k \notin P_1^t, k \neq 1} (\hat{r}_k^t - \hat{r}_k^t) > 0 \), implying \( j \neq 1 \) which entails \( d_j^t = d_j^t \).

Thus we have \( \hat{d}_1^t \leq d_1^t \) (since no user over-reports), \( \hat{d}_j^t = d_j^t \), \( \hat{r}_i^t > \hat{r}_i^t \), and \( \hat{r}_j^t < \hat{r}_j^t \). These, using Lemma 3.3, prove that \( \hat{R}_i^t - R_i^t \leq R_j^t - R_j^t \). This is a contradiction because \( i \in P_1^t \) and \( j \notin P_1^t \).
Before proving the upper bound on the incentive compatibility ratio, first we will prove that users have no incentive to over-report their demand. The immediate effect of over-reporting is allocating resources to user 1 in excess of her demand, which do not contribute to her utility. Intuitively, this suggests that user 1 is put into a disadvantageous position: other users get less resources which makes them be favored by the allocation algorithm in the future, while user 1 becomes less favored. However, a small change in the users’ resources causes a cascading change in future allocations making the proof of this theorem hard. We will see in Section 4 that this is no longer the case with multiple resources when users only use a subset of them. We defer the proof of the theorem to the end of this section.

**Theorem 3.5.** Users have nothing to gain in Dynamic MMF by declaring a demand higher than their actual demand, for any value \( \alpha \in [0,1] \).

Now we show that using this theorem and Lemma 3.4 allows us to bound the incentive to deviate. A bound of 2 is easy to get by summing (2) for all \( t \) up to some certain epoch and assuming that \( \hat{d}_t \) are users’ true demands \((\hat{d} = \hat{d})\) and that \( \hat{d}_1 \) is any under-reporting of user 1. We give an incentive compatibility bound of \( 3/2 \) by using the same lemma, but arguing that some other user \( j \) must also share the same increased allocation of resources as user 1 using Lemma 3.3.

**Theorem 3.6.** In Dynamic MMF for any value of \( \alpha \in [0,1] \), no user can misreport her demand to increase her utility by a factor larger than \( 3/2 \), i.e., for any user \( i \) and demand misreporting user \( i \) makes, \( \hat{U}_i^t \leq \frac{3}{2} U_i^t \), for all epochs \( t \).

**Proof.** We will prove the theorem for \( i = 1 \). Theorem 3.5 implies that it is of no loss of generality to assume that user 1 does not over-report her demand, since any benefit gained by over-reporting can be gained by changing every over-report to a truthful one. This means that instead of \( \hat{U}_1^t \leq \frac{3}{2} U_1^t \) we can show \( \hat{R}_1^t \leq \frac{3}{2} R_1^t \). Towards a contradiction, let \( t \) be the first epoch when user 1 gets more than \( 3/2 \) more resources by some deviation of demands, i.e., \( \hat{R}_1^t > \frac{3}{2} R_1^t \) and \( \hat{R}_1^{t-1} \leq \frac{3}{2} R_1^{t-1} \). This implies that \( \hat{r}_1^t > r_1^t \), which in turn entails that there exists a user \( j \) for whom \( \hat{r}_j^t < r_j^t \), since the total resources allocated when user 1 is under-reporting cannot be less than those when 1 is truthful. Because \( \hat{r}_1^t > r_1^t \), \( \hat{r}_j^t < r_j^t \), \( \hat{d}_1^t \leq d_1^t \), and \( \hat{d}_j^t = d_j^t \), we can use Lemma 3.3 and get \( \hat{R}_1^t - R_1^t \leq \hat{R}_j^t - R_j^t \). This inequality, \( \hat{R}_1^t - R_1^t \geq 0 \), and Lemma 3.4 by summing (2) for every epoch up to \( t \), implies

\[
2 \left( \hat{R}_1^t - R_1^t \right) \leq \left( \hat{R}_1^t - R_1^t \right) + \left( \hat{R}_j^t - R_j^t \right) \leq \sum_{k \in [n]} \left( \hat{R}_k^t - R_k^t \right)^+ \leq \sum_{\tau=1}^t r_\tau^t = R_1^t
\]

The above inequality leads to \( \hat{R}_1^t \leq \frac{3}{2} R_1^t \), a contradiction. \( \blacksquare \)

Finally, we prove Theorem 3.5, that users have no incentive to over-report. To do that, we use the following lemma, which says that if a user wants to increase her utility in epoch \( T \), she has nothing to gain by over-reporting her demand in epoch \( T_0 \), given that she does not over-report her demand in epochs \( T_0 + 1 \) to \( T \).

**Lemma 3.7.** Fix an epoch \( T_0 \) and the allocations of an outcome \( \{\hat{R}^t_k\}_{k \in [n]} \). Fix another epoch \( T \geq T_0 \) and assume that in epochs \( t = T_0 + 1, T_0 + 2, \ldots , T \) user 1 does not over-report her demand, i.e., \( \hat{d}_1^t \leq d_1^t \). Then user 1 cannot increase her utility in round \( T \) by over-reporting her demand in epoch \( T_0 \), for any \( \alpha \in [0,1] \).

By over-reporting her demand in \( T_0 \), user 1 (potentially) gains some resources that do not contribute to her utility, while others get less resources. This puts user 1 in a disadvantage entailing that in the epochs after \( T_0 \) she cannot increase her total allocation further (even though it is possible that her allocation in a single epoch can increase). Since she doesn’t over-report her demand in epochs after \( T_0 \) her allocation is the same as her utility, hence her total utility also does not increase.
Proof. To prove the lemma we are going to create another outcome in which user 1 changes her over-report in epoch $T_0$ to a truthful one which does not decrease her total utility in epoch $T$.

For all users $k$ and epochs $t$, let $\hat{d}_{ik}^t = d_{ik}^t$, except for $\hat{d}_{i0}^0$, which is user 1’s actual demand: $\hat{d}_{i0}^0 = d_{i0}^0$. This means that the two outcomes are the same before epoch $T_0$, i.e., for all users $k$, $\hat{R}_k^{T_0-1} = \hat{R}_k^{T_0-1}$ and $\hat{U}_k^{T_0-1} = U_k^{T_0-1}$. Because $\hat{d}_{i0}^0 > d_{i0}^0$ and for $k \neq 1$, $\hat{d}_{i0}^0 = d_{i0}^0$, user 1 may earn some additional resources on $T_0$, i.e., $\hat{r}_{10}^{T_0} - \hat{r}_{i0}^{T_0} = \hat{R}_{10}^{T_0} - \hat{R}_{i0}^{T_0} = x$, for some $x \geq 0$, while other users $k \neq 1$ get less resources: $\hat{r}_{k0}^{T_0} - \hat{r}_{k0}^{T_0} = \hat{R}_{k0}^{T_0} - \hat{R}_{k0}^{T_0} \leq 0$. We first note that the $x$ additional resources that user 1 gets are in excess of 1’s true demand, meaning they do not contribute towards her utility:

$$\hat{U}_{10}^{T_0} - U_{10}^{T_0} = \hat{R}_{10}^{T_0} - \hat{R}_{10}^{T_0} - x = 0 \tag{5}$$

Additionally, because user 1 does not over-report $\hat{d}_{1}^t$ or $\hat{d}_{1}^t$ in epochs $t \in [T_0 + 1, T]$, it holds that for $t \in [T_0 + 1, T]$ user 1’s utility is the same as the resources she receives: $\hat{u}_{1}^t = \tilde{r}_{1}^t$ and $\hat{u}_{1}^t = \hat{r}_{1}^t$. This fact, combined with (5) proves that

$$\forall t \in [T_0, T], \hat{U}_{1}^t - \hat{U}_{1}^t = \hat{R}_{1}^t - \hat{R}_{1}^t - x$$

Thus, in order for this over-reporting to be a strictly better strategy, it must hold that $\hat{R}_{1}^t - \hat{R}_{1}^t > x$. We will complete the proof by proving that the opposite holds. Since in epochs $t \in [T_0 + 1, T]$ it holds that $\hat{d}_{1}^t = \hat{d}_{1}^t$, we can use Lemma 3.4 to sum (2) for all $t \in [T_0 + 1, T]$ and get that

$$\sum_{k} \left( \hat{R}_k^T - \hat{R}_k^T \right)^+ - \sum_{k} \left( \hat{R}_k^{T_0} - \hat{R}_k^{T_0} \right)^+ \leq 0$$

The above inequality, because $(\hat{R}_k^T - \hat{R}_k^T)^+ \geq 0$, $\hat{R}_k^{T_0} - \hat{R}_k^{T_0} \leq 0$ for $k \neq 1$, and $\hat{R}_{1}^{T_0} - \hat{R}_{1}^{T_0} = x \geq 0$, proves that $\hat{R}_1^T - \hat{R}_1^T \leq x$. This completes the proof. \blacksquare

Proof of Theorem 3.5. Fix an epoch $T$ and let $T_0 \leq T$ be the last epoch where user 1 over-reported. Lemma 3.7 allows us to change user 1’s over-report in $T_0$ to a truthful one without decreasing her total utility in $T$. Doing this inductively for every such $T_0$ creates a demand profile with no over-reporting that does not decrease user 1’s total utility in $T$. \blacksquare

**Envy-freeness in Dynamic MMF.** We finally note that Dynamic MMF is envy-free, for any value of the parameter $\alpha$. This is a corollary of the fact that Dynamic DRF, the generalization of Dynamic MMF for multiple resources, is envy-free (see details in Theorem 4.8 and Corollary 4.9).

## 4 Multiple Resources Setting

In this section we turn to the main focus of our paper, analyzing the generalization of Dynamic MMF for multiple resources, Dynamic Dominant Resource Fairness (Dynamic DRF).

**Notation and User Utilities.** We consider users that have varying demand for a set of $m \geq 1$ different resources over time. We use $1, 2, \ldots, q, \ldots, m$ to denote the $m$ resources, and w.l.o.g., we assume that for every resource the amount available is the same, $R$. A typical example of such a system may focus on users running applications that use resources such as CPU, memory, storage, etc.

Every epoch, each user demands a non-negative amount of every resource which they report to the mechanism. With the multidimensional nature of demand, users have more complex ways to misreport their demand. Throughout this paper we will assume that users have *Leontief preferences*, which we define next. Leontief preferences have been considered by much of the previous work in resource sharing, including [Gho+11; KPS13; PPS12].
Formally, with Leontief preferences a user $i$’s demand in an epoch $t$ is characterized by the ratios $a_i^t = (a_{i1}^t, \ldots, a_{im}^t)$ she needs for the resources and a demand $d_i^t$. The ratios indicate that for some $\xi \geq 0$, user $i$’s application in that epoch is going to use $\xi a_{iq}^t$ amount of every resource $q \in [m]$. The resource $q$ with the maximum ratio $a_{iq}^t$ is called the dominant resource of user $i$ in epoch $t$. We assume that ratios are normalized: for every epoch $t$, $\max_q a_{iq}^t = 1$ for all users. The demand $d_i^t$ of user $i$ in epoch $t$ represents the maximum fraction $\xi$ (possibly, $\xi > 1$) of the ratios the user can take advantage of. More specifically, user $i$ demands (or is asking for) $d_i^t a_{iq}^t$ amount of every resource $q$. If a user $i$ receives $x_{i1}^t, \ldots, x_{im}^t$ of every resource, respectively, her utility that epoch is $u_i^t = \min \{ d_i^t, \min_{q, a_{iq} > 0} \{ \frac{x_{iq}^t}{a_{iq}^t} \} \}$. In each epoch, users will be asked to report both their ratios for the epoch as well as their demand. Note that users can now misreport their type in two different ways: they can either request less or more from all the resources or they can demand the resources in different proportions (or they can do both).

**Dynamic Dominant Resource Fairness** DRF is the generalization of MMF for the case of multiple resources, where the fairness criterion is applied to each user’s dominant resource and the rest of the resources are distributed according to the users’ ratios. If $r_i^t$ is the amount that user $i$ receives of her dominant resource in epoch $t$, then she receives $r_i^t a_{iq}^t$ of every resource $q$ (recall that $\max_q a_{iq}^t = 1$). We call $r_i^t$ the allocation of user $i$ in epoch $t$.

Dynamic DRF extends to dynamic user demands the weighted DRF policy, the generalization of DRF where the fairness criterion is applied to the users’ allocation normalized by some weights, indicating users’ priorities. More specifically, if every user $i$ is associated with a weight $w_i > 0$, then the mechanism tries to give to each user $i$ an allocation proportional to her weight $w_i$.

Similar to Dynamic MMF, Dynamic DRF has an additional parameter $\alpha \in [0, 1]$ that guarantees a fraction of fair share of each resource to each user in every epoch, independent of allocations in previous ones. User $i$’s fair share of a resource $q$ is $\mathcal{R} w_i / \sum_j w_j$. When $\alpha = 1$, we guarantee at least the fair share of their dominant resource (assuming a big enough demand to use it), when $\alpha < 1$ we guarantee a smaller share. Beyond this guarantee, the goal of the mechanism in epoch $t$ is to be as fair as possible to the cumulative allocation of every user, normalized by their weights. We use $R_i^t = \sum_{\tau=1}^t r_i^\tau$ for the sum of allocations till time $t$. Using this notation, Dynamic DRF is easy to describe. For a given epoch $t$, assuming that every user $i$ has cumulative allocation $R_i^{t-1}$:

$$
\begin{align*}
\text{choose } & \mathcal{R}w_i\text{-min fairness on } \frac{R_i^{t-1} + r_i^1}{w_1}, \frac{R_i^{t-1} + r_i^2}{w_2}, \ldots, \frac{R_i^{t-1} + r_i^n}{w_n} \\
\text{given the constraints } & \forall i \in [n], \min \left\{ d_i^t, \alpha \frac{\mathcal{R} w_i}{\sum_{k \in [n]} w_k} \right\} \leq r_i^t \leq d_i^t, \\
& \forall q \in [m], \sum r_i^t a_{iq}^t \leq \mathcal{R}
\end{align*}
$$

We define $g_i(d_i^t) = \min \left\{ d_i^t, \alpha \frac{\mathcal{R} w_i}{\sum_{k \in [n]} w_k} \right\}$ to be the guaranteed amount that every user receives every epoch. Note that $g_i(\cdot)$ is a non-decreasing function.

We note a few properties that immediately follow from the description. If all users share the same dominant resource $q^t$ in each epoch and have equal weights, then the mechanism will become identical to Dynamic MMF, as at each epoch the allocation of the shared dominant resource is the bottleneck for all users.

Second, if all the users share their dominant resource $q^t$, $\alpha = 1$, and demands are high, then the minimum guarantee becomes $g(d_i^t) = \mathcal{R} w_i / \sum_k w_k$ which will become user $i$’s allocation, as this saturates resource $q^t$ (the guaranteed total use of $q^t$ is $\sum_i g(d_i^t) a_{iq}^t = \sum_i g(d_i^t) = \mathcal{R}$). However, even with large demands
each iteration and $\alpha = 1$, the dynamic fair sharing nature of our allocation will play an important role when applications (users) do not always share their dominant resource.

Third, because of the guarantee of every user, Dynamic DRF satisfies $\alpha$-sharing incentives.

**Theorem 4.1.** When every user is guaranteed an $\alpha$-fraction of their fair share, Dynamic DRF satisfies $\alpha$-sharing incentives.

**User’s utility when misreporting ratios** In defining the Dynamic DRF mechanism, we have not considered the difference of truthful reporting and misreporting one’s demands or ratios to the mechanism. The main topic of this section is understanding how a user’s utility behaves in these two scenarios for Dynamic DRF.

When user $i$ truthfully reports her demand $d_i^t$ and ratios $a_i^t$, and Dynamic DRF gives her an allocation of $r_i^t$, her utility in that epoch is $u_i^t = \min\{d_i^t, \min_{q:a_{iq}^t > 0} r_i^t a_{iq}^t / a_i^t\} = r_i^t$, since Dynamic DRF guarantees that $r_i^t \leq d_i^t$.

When user $i$ misreports $\hat{a}_i^t$ and $\hat{d}_i^t$, and Dynamic DRF gives her an allocation $\hat{r}_i^t$ based on the reported values, let $\hat{u}_i^t$ denote the user’s true utility in epoch $t$ under that reporting. In this case she receives $x_i^t = \hat{a}_i^t \hat{r}_i^t$ of every resource $q$ and thus she gets true utility

$$\hat{u}_i^t = \min\left\{d_i^t, \min_{q:a_{iq}^t > 0} \frac{x_i^t}{a_{iq}^t}\right\} = \min\left\{d_i^t, \hat{r}_i^t \min_{q:a_{iq}^t > 0} \frac{\hat{a}_i^t}{a_{iq}^t}\right\}$$

We define $\hat{\lambda}_i^t = \min_{q:a_{iq}^t > 0} \{\hat{a}_i^t / a_{iq}^t\}$ making the above expression equal to $\hat{u}_i^t = \min\{d_i^t, \hat{r}_i^t \hat{\lambda}_i^t\}$. We should note that if the user reports ratios truthfully ($\hat{a}_i^t = a_i^t$), then $\hat{\lambda}_i^t = 1$. Additionally, because each user is constrained to declare $\max_q \hat{a}_i^t a_{iq}^t = \max_q a_i^t a_{iq}^t = 1$, it must hold that $\hat{\lambda}_i^t \leq 1$ for any $\hat{a}_i^t$.

Similar to Section 3, the total utility of user $i$ by epoch $t$ is $U_i^t = \sum_{\tau=1}^t u_i^\tau$.

### 4.1 Incentives assuming positive ratios for all resources

Our main results in this paper consider Dynamic DRF resource allocation with multiple resources under the assumption that when users want to use the system, they demand at least some of each of the resources, that is $a_{iq}^t > 0$ for all $i, q, t$. With typical system resources, such as CPU, memory, RAM, etc., it is indeed the likely scenario. While different applications have different dominant resources (e.g., some have heavy use of compute power, while in others the main bottleneck is bandwidth), each uses at least some of each one of these basic resources.

As already observed by Parkes, Procaccia, and Shah [PPS12], having zero ratios for some resources significantly changes the problem from having a tiny $\epsilon > 0$ ratio, no matter how small $\epsilon$ is. In Appendix B we show that with zero ratios users do have an incentive to over-report their demand, which will turn out to not be useful with positive ratios. Further, the benefit of such over-reporting can increase the user’s utility by a factor of $\Omega(m)$, increasing with the number of resources in the system.

In contrast, the main results of this section are that, assuming users have positive ratios, misreporting them, as well as over-reporting demand, is not beneficial (Theorem 4.4). Further, the approximate incentive compatibility ratio for user 1 is bounded by $1 + \rho_1$, where $\rho_1 = \max_{k \neq 1, q, t} \{w_1 a_{iq}^t / w_k a_{iq}^t\}$ (Theorem 4.6), and this bound is tight (Theorem 4.7) even with just two resources and static ratios, extending the results of Section 3 to multiple resources.

**Upper bound on incentive compatibility ratio** We start by pointing out why positive ratios are so different from zero ones. With positive ratios, users are bottlenecked by the same resource being saturated. In contrast, when ratios are zero, different users are bottlenecked by different resources, resulting in users’ allocations being
almost independent from one another (which is the case if users do not share resources for which they both have positive ratios). The difficulty in adapting the results of Section 3 to the case of multiple resources comes from the fact that different users use the resources in different proportions. For example, consider the users’ allocations being bottlenecked by a single saturated resource \( q \) for which user 1 has a ratio of 1 and user \( i \) has a ratio of \( \epsilon < 1 \). User 1 can free a \( \delta \) amount of resource \( q \) by under-reporting her demand so that her allocation gets smaller by \( \delta \), for a small \( \delta > 0 \). If this available amount of resources goes to user \( i \), it has the potential to raise her allocation by \( \delta / \epsilon \) (recall that user \( i \) has a ratio of \( \epsilon \) for resource \( q \), which is disproportionately bigger than the \( \delta \) utility that user 1 lost.

The assumption that all users are using each resource allows us to prove a lemma analogous to Lemma 3.3, as with this assumption, we can increase any user’s allocation by decreasing another user’s allocation, allowing the proof to be identical to the proof of Lemma 3.3. In contrast, if \( a_{iq}^t = 0 \) and \( a_{jq}^t > 0 \), then decreasing the allocation of user \( i \) does not free any amount of resource \( q \) for user \( j \). Proving this lemma will lead to results similar to Theorems 3.5 and 3.6.

**Lemma 4.2.** Fix an epoch \( t \) and the total allocations up to epoch \( t - 1 \) of any two outcomes \( \{\hat{R}_k^t\}_{k \in [n]} \) and \( \{\check{R}_k^t\}_{k \in [n]} \). Let \( i, j \) be two different users. If the following conditions hold

- For \( i \), \( \bar{r}_i^t < r_i^t \), \( \bar{d}_i^t \leq \hat{d}_i^t \), and \( \hat{d}_{iq}^t > 0 \) for all \( q \).
- For \( j \), \( \bar{r}_j^t > r_j^t \), \( \check{d}_j^t \leq \check{d}_j^t \), and \( \check{d}_{jq}^t > 0 \) for all \( q \).

then, for any \( \alpha \in [0, 1] \) used by Dynamic DRF, it holds that \( \check{R}_i^t / w_i \geq \hat{R}_j^t / w_j \) and \( \hat{R}_i^t / w_i \leq \check{R}_j^t / w_j \), implying

\[
\frac{\hat{R}_i^t - \check{R}_i^t}{w_i} \leq \frac{\check{R}_j^t - \hat{R}_j^t}{w_j}
\]

The proof of this lemma is identical to the one in Lemma 3.3, so it’s deferred in Appendix B.

Now we present an auxiliary lemma, similar to Lemma 3.4, but this time we bound the increase of any user’s allocation when user 1 deviates, i.e., \( \hat{R}_k^t - \bar{R}_k^t \) for each \( k \). This lemma shows the key difficulty in extending the results to the multiple resource case: reporting low demand in one epoch can result in a different user being able to get much higher amounts, even when the two users have the same dominant resource. Unfortunately, as a result of this difficulty the resulting bound is a weaker version of Lemma 3.4, involving the parameter \( \rho_1 \).

If the users ratios are the same and users have same priority, i.e., \( \hat{a}_i^t = \bar{a}_i^t = \check{a}_i^t \) and \( \bar{d}_i^t \leq \hat{d}_i^t \) for every \( t \) and \( i \neq j \), then \( \rho_1 = 1 \) and the following lemma would allow us to prove a 2-incentive compatibility ratio upper bound. In general however, the incentive compatibility ratio depends on \( \rho_1 \) and it becomes larger the larger \( \rho_1 \) is.

**Lemma 4.3.** Fix an epoch \( t \) and the allocations of two different outcomes \( \{\hat{R}_k^t\}_{k \in [n]} \) and \( \{\check{R}_k^t\}_{k \in [n]} \) for which it holds that for some \( X \geq 0 \),

\[
\forall k \in [n], \quad \frac{\hat{R}_k^t - \check{R}_k^t}{w_k} \leq X
\]

(7)

If in epoch \( t \) users have positive ratios, user 1 reports her ratios truthfully, i.e., \( \hat{a}_i^t = \bar{a}_i^t = \check{a}_i^t \), and \( \bar{d}_i^t \leq \hat{d}_i^t \) then, for all \( \alpha \in [0, 1] \) used by Dynamic DRF it holds that

\[
\forall k \in [n], \quad \frac{\hat{R}_k^t - \check{R}_k^t}{w_k} \leq X + 1 \left[ \bar{d}_1^t < \hat{d}_1^t \right] \rho_1^t \frac{\bar{r}_1^t}{w_1}
\]

where \( \rho_1^t = \max_{k \neq 1, q \in [m]} \frac{w_1 a_{1q}^t}{w_k a_{kq}^t} \).
The above lemma, by assuming truthful reporting of ratios and that user 1 does not over-report her demand \((\tilde{d}_1^t \leq d_1^t)\), inductively proves a \((1 + \max_i \{\rho_i^t\})\) incentive compatibility ratio.

The proof of this lemma is similar to the one in Lemma 3.4. If \(\tilde{d}_1^t = d_1^t\), then users with larger \(\tilde{R}_k^t-\tilde{R}_k^{t-1}/w_k\) will not gain additional resources. If \(\tilde{d}_1^t < d_1^t\), user 1 can increase the allocation of some other user \(k\), but by at most \(\rho_1^t\tilde{d}_1^t/w_k\); note that this is much larger than \(r_1^t\), the guarantee of Lemma 3.3.

**Proof.** Assume that there exists a user \(i\) such that

\[
\frac{\tilde{R}_i^t - \tilde{R}_i^{t-1}}{w_i} > X + 1 \left[ \tilde{d}_i^t < d_i^t \right] \rho_1 w_i \tag{8}
\]

Subtracting (7) from (8) implies that

\[
\frac{\tilde{r}_i^t - \tilde{r}_i^{t-1}}{w_i} > 1 \left[ \tilde{d}_i^t < d_i^t \right] \rho_1 w_i \tag{9}
\]

Note that (9) implies that user \(i\) satisfies the requirements of user \(i\) in Lemma 4.2 (since for all users \(k\) and resources \(q\), \(\tilde{d}_k^t \leq d_k^t\) and \(a_{kq}^t > 0\)). Now note that because for all users \(k\), \(\tilde{d}_k^t \leq d_k^t\), there must exist a resource \(q^*\) for which

\[
\sum_k a_{kq^*}^t \tilde{r}_k^t \geq \sum_k a_{kq^*}^t \tilde{r}_k^t
\]

since otherwise no resources would be saturated in allocations \(\{\tilde{r}_k^t\}_k\), which would imply that every user gets her demand in that allocation, which would imply the same in allocations \(\{r_i^t\}_k\) (since ratios are the same and demands are not higher); the last fact leads to a contradiction. Multiplying (9) with \(a_{iq^*}^t w_i\) and adding it with the above inequality yields

\[
\sum_{k \neq i} a_{kq^*}^t (\tilde{r}_k^t - \tilde{r}_k^{t-1}) > w_i a_{iq^*}^t \left[ \tilde{d}_i^t < d_i^t \right] \rho_1 \frac{\tilde{r}_i^t}{w_i} \geq 0 \tag{10}
\]

(10) proves that there exists a user \(j\) such that \(\tilde{r}_j^t > \tilde{r}_j^t\). Additionally, for that \(j\) we can assume that \(\tilde{d}_j^t = d_j^t\); to see why that is we distinguish three cases:

- If \(\tilde{d}_1^t = d_1^t\) then for all users \(k\), \(\tilde{d}_k^t = d_k^t\).
- If \(i = 1\), then \(j \neq 1\) since \(\tilde{r}_i^t > \tilde{r}_i^t\) and \(\tilde{r}_j^t > \tilde{r}_j^t\). Because \(j \neq 1\), \(\tilde{d}_j^t = d_j^t\).
- If \(\tilde{d}_1^t < d_1^t\) and \(i \neq 1\), then, (10) becomes

\[
\sum_{k \neq i} a_{kq^*}^t (\tilde{r}_k^t - \tilde{r}_k^{t-1}) > w_i a_{iq^*}^t \rho_1 w_i \tilde{r}_i^t \geq 1 \rightarrow a_{iq^*}^t \tilde{r}_i^t
\]

which implies that \(\sum_{k \neq i} a_{kq^*}^t (\tilde{r}_k^t - \tilde{r}_k^{t-1}) > 0\). This proves that we can assume that \(j \neq 1\), entailing \(\tilde{d}_j^t = d_j^t\).

This proves that exists a user \(j\) that satisfies the requirements of user \(j\) from Lemma 4.2. Using the lemma on users \(i\) and \(j\) we get \(\tilde{R}_j^t - \tilde{R}_j^{t-1} \leq \tilde{R}_j^t - \tilde{R}_j^{t-1}/w_j\). This is a contradiction, because it holds that \(\tilde{R}_j^t - \tilde{R}_j^t < \tilde{R}_j^{t-1} - \tilde{R}_j^{t-1} \leq w_j X\) (from (7) and \(\tilde{r}_j^t > \tilde{r}_j^t\)) and \(\tilde{R}_j^t - \tilde{R}_j^t > w_j X\) (from (8)).

Similarly to Theorem 3.5, we can now prove that there is no benefit to over-reporting or misreporting ratios. As mentioned previously, this is a very important property because it guarantees that every resource allocated is utilized and that, even when user 1 is not truthful, her utility is equal to her allocation: \(\tilde{u}_i^t = \tilde{r}_i^t\).
Theorem 4.4. Assume that the users’ true ratios are positive, i.e., for all users \( i \in [n] \), resources \( q \in [m] \), and epochs \( t \), it holds that \( \hat{a}^t_{iq} > 0 \). Then, for any \( \alpha \in [0, 1] \) used by Dynamic DRF, the users have nothing to gain by declaring a demand higher than their actual demand, and any gain achievable by misreporting ratios can also be obtained by under-reporting demand.

The proof of this theorem is quite similar to the one in Theorem 3.5: first we prove the following auxiliary lemma, that makes the proof of the theorem easy. The lemma is similar to Lemma 3.7: if in epochs \( T_0 + 1 \) to \( T \) user 1 does not over-report her demand and reports her ratios truthfully, she cannot increase her utility in epoch \( T \) by over-reporting demand or misreporting ratios in \( T_0 \).

Lemma 4.5. Fix an epoch \( T_0 \) and the allocations of an outcome \( \{ \hat{R}_k^{T_0-1} \}_{k \in [n]} \). Fix another epoch \( T \geq T_0 \) and assume that in epochs \( T_0 + 1, T_0 + 2, \ldots, T \) user 1 reports her ratios truthfully and does not over-report her demand. Then, for any \( \alpha \in [0, 1] \) used by Dynamic DRF, any increase in user 1’s utility in epoch \( T \) gained by over-reporting demand or misreporting ratios in \( T_0 \) can be achieved with truthful ratio reporting and no demand over-reporting in epoch \( T_0 \).

The above proves that over-reporting demand or misreporting ratios is not useful by applying the lemma inductively, similar to the proof of Theorem 3.5.

The lemma’s proof is based on an alternative reporting that both reports ratios truthfully and guarantees no demand over-reporting. With this reporting user 1 is guaranteed the same utility without increasing her allocation or decreasing the other user’s allocations; this entails a more advantageous position for her in the following epochs.

Proof of Lemma 4.5. We are going to consider another outcome that is the same as the one in the lemma up to epoch \( T_0 - 1 \). All users \( k \neq 1 \) report the same type in all epochs: \( d^t_k = \hat{d}^t_k \) and \( a^t_k = \hat{a}^t_k \) for all \( t \). In epoch \( T_0 \) user 1 deviates from reporting \( \hat{a}^T_1 \) and \( \hat{d}^T_1 \), and instead declares

- her ratios truthfully: \( \hat{a}^T_1 = a^T_1 \),
- her demand: \( \hat{d}^T_1 = \min \{ d^T_1, \hat{\lambda}^T_1 \} \); recall that \( \hat{\lambda}^T_1 = \min_q \frac{a^T_{1q}}{\hat{a}^T_{1q}} \leq 1 \).

Note that the above reporting satisfies what we want: it reports the ratios truthfully and does not over-report the demand. In later epochs \( t \in [T_0 + 1, T] \) user 1 does not deviate from the other reporting: \( \hat{a}^t_1 = a^t_1 \) and \( \hat{d}^t_1 = \hat{d}^t_1 \leq d^t_1 \).

Now we notice that in epoch \( T_0 \), for every resource \( q \) it holds that

\[
\frac{\hat{a}^T_1}{\hat{d}^T_1} = \frac{\hat{a}^T_1}{\hat{d}^T_1} \leq \frac{\hat{a}^T_1}{\hat{d}^T_1} \leq \frac{\hat{a}^T_1}{\hat{d}^T_1} \leq \frac{\hat{a}^T_1}{\hat{d}^T_1} = \frac{\hat{a}^T_1}{\hat{d}^T_1}
\]

\((11)\) proves that in allocation \( \{ \hat{r}^T_k \}_{k} \) user 1 uses less of every resource compared to allocation \( \{ \hat{r}^T_k \}_{k} \). This proves that the following allocation is feasible: \( \hat{r}^T_1 \leftarrow \hat{d}^T_1 \) and for \( k \neq 1 \), \( \hat{r}^T_k \leftarrow \hat{r}^T_k \). Because \( \hat{a}^T_1 \leq \hat{d}^T_1 \) (which follows from \( \hat{\lambda}^T_1 \leq 1 \)) we have that

\[
\hat{r}^T_1 = \hat{d}^T_1 \quad \text{and} \quad \forall k \neq 1, \hat{r}^T_k \geq \hat{r}^T_k
\]

\((12)\) Now we notice that

\[
\frac{\hat{a}^T_1}{\hat{d}^T_1} \quad \text{definition of utility} \quad \min \left\{ \frac{\hat{a}^T_1}{\hat{r}^T_1}, \frac{\hat{r}^T_1}{\lambda^T_1} \right\} \quad \text{definition of } \hat{d}^T_1 \quad \frac{\hat{a}^T_1}{\hat{d}^T_1} = \frac{\hat{a}^T_1}{\hat{d}^T_1} \quad \frac{\hat{a}^T_1}{\hat{d}^T_1} = \frac{\hat{a}^T_1}{\hat{d}^T_1}
\]
The difference in total utility in epoch $T$ is

$$
\hat{U}_1^T - \hat{U}_1^T = \hat{U}_1^{T_0-1} - \hat{U}_1^{T_0-1} + \hat{u}_1^{T_0} - \hat{u}_1^{T_0} + \sum_{\tau=T_0+1}^{T} (\hat{u}_1^\tau - \hat{u}_1^\tau)
$$

same outcomes up to $T_0-1$ 

$$
\hat{u}_1^0 = \bar{u}_1^{T_0}
$$

no demand over-reporting and true ratio reporting in $\tau$

$$
\sum_{\tau=T_0+1}^{T} (\hat{r}_1^\tau - \bar{r}_1^\tau) \leq \hat{R}_1^{T_0-1} - \bar{R}_1^{T_0-1} 
$$

Let $X = \hat{r}_1^{T_0} - \bar{r}_1^{T_0} \geq 0$. To prove that $\hat{U}_1^T \leq \hat{U}_1^T$ (thus proving the lemma) all we need to prove is that $\hat{R}_1^T - \bar{R}_1^T \leq X$. For every epoch $t \in [T_0+1, T]$ the requirements of Lemma 4.3 are true in $t$ and $\hat{d}_i^t = d_i^t$, which proves inductively that if

$$
\forall k \in [n]: \frac{\hat{R}_k^T - \bar{R}_k^T}{w_k} \leq \frac{X}{w_1}
$$

then for all $t \in [T_0, T]$

$$
\forall k \in [n]: \frac{\hat{R}_k^t - \bar{R}_k^t}{w_k} \leq \frac{X}{w_1}
$$

We can prove (13) by noticing that for $k \neq 1$, it holds $\hat{R}_k^t - \bar{R}_k^t \leq 0 \leq X$ (from (12) and $\hat{R}_k^{T_0-1} = \bar{R}_k^{T_0-1}$) and for $k = 1$ it holds that $\hat{r}_1^{T_0} - \bar{r}_1^{T_0} = X$ and $\hat{R}_1^{T_0-1} = \bar{R}_1^{T_0-1}$. This completes the proof.

Now we prove the upper bound for the incentive compatibility of Dynamic DRF with a simple proof utilizing Lemma 4.3 and Theorem 4.4.

**Theorem 4.6.** Assume that all users have positive ratios: $d_{iq}^t > 0$ for all users $i$, resources $q$, and epochs $t$. Then for any user $i$ and $\alpha \in [0, 1]$ used by Dynamic DRF, user $i$ cannot misreport her demand or ratios to increase her utility by a factor larger than $(1 + \rho_i)$, where $\rho_i = \max_{k \neq i, q, t} \left\{ \frac{w_k d_{iq}^t}{w_q d_{iq}^t} \right\}$.

**Proof.** W.l.o.g. we are going to prove the theorem for $i = 1$. Because of Theorem 4.4 we can assume that user 1 does not over-report her demand or misreport her ratios and thus we can bound $\hat{R}_1^t$ instead of $\hat{U}_1^t$. Lemma 4.3 inductively implies that for any $t$, $\hat{R}_1^t - \bar{R}_1^t \leq \rho_1 \hat{R}_1^t$, which proves the theorem.

**Lower bound on incentive compatibility ratio** In the last theorem we proved that the incentive compatibility ratio of user 1 is upper bounded by $(1 + \rho_1)$. We now show that if the only constraints on users’ ratios and weights is that they are positive and $\rho_1$ is fixed, then it is possible for the incentive compatibility ratio of user 1 to be $(1 + \rho_1)$, even if the users’ ratios do not change over time.

**Theorem 4.7.** For any $\epsilon \in (0, 1)$, $w_1, w_2 > 0$, and $\alpha \in [0, 1]$ used by Dynamic DRF, there is an instance where the users’ ratios are constant every epoch, user 1 has weight $w_1$ and another user has weight $w_2$, $\rho_1 = \frac{w_1}{w_2^\epsilon}$, and user 1 can under-report her demand to increase her utility by a factor of $1 + \rho_1$.

We briefly sketch the proof of the theorem when users have time-varying ratios, there is only one resource, users have equal weights, and it does not necessarily hold that $\max_{q \in [m]} a_{iq}^t = 1$. With some complicated details the full proof reduces to the same instance as the one we present below.

**Proof sketch.** There are two users, one resource, two epochs, $w_1 = w_2 = 1$, and $\alpha = 0$. 

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1. In epoch 1, user 1 has a ratio of $a_1^1 = 1$, user 2 has a ratio of $a_2^1 = \epsilon$, the amount available of the resource is 1, and both users have infinite demand. Dynamic DRF allocates the resource such that $r_1^1 = r_2^1 = \frac{1}{1+\epsilon}$.

2. In epoch 2, user 1 has a ratio of $a_1^2 = \delta \leq 1$, user 2 has a ratio of $a_2^2 = 1$, the amount available of the resource is $\delta/\epsilon$, and both have infinite demand. Dynamic DRF, since the users’ past allocations are the same, allocates the resource such that $r_1^2 = r_2^2 = \frac{\delta}{\epsilon} + \frac{1}{1+\delta}$.

This results in $R_1^2 = R_2^2 = \frac{1}{1+\epsilon} + \frac{\delta}{1+\delta}$. We note that $\rho_1 = \max \left\{ \frac{a_1^1}{a_2^1}, \frac{a_2^1}{a_2^1} \right\} = \max \left\{ \frac{1}{\epsilon}, \frac{\delta}{1} \right\} = \frac{1}{\epsilon}$.

Now we study what happens when user 1 deviates:

1. If in epoch 1 user 1 demands 0 resources, then Dynamic DRF allocates $\hat{r}_1^1 = 0$ and $\hat{r}_2^1 = 1/\epsilon$.

2. In epoch 2, the previous misreporting results in user 1 having a lower total allocation, making Dynamic DRF allocate $\hat{r}_1^2 = 1/\epsilon$ and $\hat{r}_2^2 = 0$.

This results in $\hat{R}_1^2 = \hat{R}_2^2 = 1/\epsilon$. The incentive compatibility ratio is

$$\frac{\hat{R}_2^2}{R_1^2} = \frac{1/\epsilon}{1+\epsilon} + \frac{\delta}{1+\delta} \xrightarrow{\delta \to 0} 1 + \frac{1}{\epsilon} = 1 + \rho_1$$ (14)

We defer the full proof in Appendix B. We should note that in the above proof sketch and in the full proof of Theorem 4.7, even though $\rho_1 = 1/\epsilon$, a ratio of user 1 is $\delta$, which is much less than $\epsilon$. If we wanted a bound in terms of $\rho = \max_{i\in[n]} \rho_i$, the proof of Theorem 4.7 for $w_1 = w_2$ (and the proof sketch above, see (14) by taking $\delta = \epsilon$) easily yields a $\frac{1+\rho}{2}$ lower bound.

### 4.2 Envy-freeness

In this section we prove that Dynamic DRF is envy-free.

**Theorem 4.8.** For every $\alpha \in [0, 1]$, Dynamic DRF is envy-free according to the weights $w_1, \ldots, w_n$, i.e., for every epoch $t$, no user $i$ envies the total allocation of user $j$ scaled by $w_i/w_j$:

$$U_i^t \geq \sum_{\tau=1}^{t} \min \left\{ a_i^\tau, \frac{w_i}{w_j} r_j^\tau \min_{q: a_{ij}^\tau > 0} \frac{a_{jq}^\tau}{a_{ij}^\tau} \right\}$$

The key to this proof is to study the (potential) first epoch $t$ user $i$ envies user $j$. In the simple case that $w_i = w_j$, that proves that $r_j^t > r_i^t$, which leads to a reasoning similar to Lemma 4.2 proving that $R_j^t \leq R_i^t$. However, since user $j$ has an allocation smaller than $i$’s it is impossible for user $i$ to envy her.

**Proof.** Towards a contradiction assume that in epoch $t$ user $i$ envies user $j$ for the first time, i.e.,

$$U_i^t = R_i^t < \sum_{\tau=1}^{t} \min \left\{ a_i^\tau, \frac{w_i}{w_j} r_j^\tau \min_{q: a_{ij}^\tau > 0} \frac{a_{jq}^\tau}{a_{ij}^\tau} \right\}$$ (15)

We can assume that in every epoch there is a different total amount of resources without loss of generality: we can split an epoch into $x$ sub-epochs where in each there are $1/x$ available resources.
and
\[ U^{t-1}_i = R^{t-1}_i \geq \sum_{\tau=1}^{t-1} \min \left\{ d^{t\tau}_i, \frac{w_{ij}}{w_j} r^{\tau}_j \min_{q:a_{iq} > 0} \frac{a^\tau_{jq}}{a_{iq}} \right\} \]

Subtracting the two inequalities above we get
\[ r^{t}_i < \min \left\{ d^{t}_i, \frac{w_{ij}}{w_j} r^{\tau}_j \min_{q:a_{iq} > 0} \frac{a^\tau_{jq}}{a_{iq}} \right\} \]

Using the above inequality we can prove that
- \( \min_{q:a_{iq} > 0} \frac{a^t_{iq}}{a_{iq}} > 0 \), i.e., user \( j \) uses every resource that user \( i \) uses.
- \( r^t_i < d^t_i \). This inequality proves that \( g_i(d^t_i) = \alpha R_i w_i / (\sum_k w_k) \), since otherwise \( g_i(d^t_i) = d^t_i = r^t_i \).
- \( r^t_i / w_i < r^t_j / w_j \), since \( \max_q a^t_{iq} = \max_q a^t_{jq} = 1 \). This inequality, since \( r^t_i \geq g_i(d^t_i) = \alpha R_i w_i / (\sum_k w_k) \), proves that \( r^t_j > \alpha R_j w_j / (\sum_k w_k) \geq g_j(d^j_j) \).

These three facts, \( r^t_i < d^t_i \), \( r^t_j > g_j(d^j_j) \), and \( \forall q : a^t_{iq} > 0 \Rightarrow a^t_{jq} > 0 \), prove that it would have been feasible to increase \( r^t_i \) (thus increasing \( R^t_i \)) by decreasing \( r^t_j \) (thus decreasing \( R^t_j \)). This proves that, because our mechanism allocates resources max-min fairly, \( R^t_i / w_i \geq R^t_j / w_j \). The last inequality contradicts with (15), which proves that \( R^t_i / w_i < R^t_j / w_j \) since \( \max_q a^t_{iq} = \max_q a^t_{jq} = 1 \).

**Corollary 4.9.** Theorem 4.8 also proves that Dynamic MMF is envy-free (by restricting \( m = 1 \) and \( w_i = a^t_{i1} = 1 \) for every user \( i \) and epoch \( t \)) and also that Dynamic weighted MMF, which will be studied in Section 5.1, is envy-free (by restricting \( m = 1 \) and \( a^t_{i1} = 1 \) for every user \( i \) and epoch \( t \)).

## 5 Generalizations in Single Resource Settings

### 5.1 Coalitions and Dynamic Weighted Max-min Fairness

In this section we present a generalized version of our previous results: we extend the results of Section 3 to Dynamic weighted MMF and when users collude to increase their overall utility.

**Notation.** We begin with the necessary definitions for this section.

**Dynamic weighted Max-min Fairness** Dynamic weighted MMF is a special case of Dynamic DRF where there is only one resource, \( m = 1 \), i.e., every epoch \( t \) the following problem is solved

choose \( r^t_1, r^t_2, \ldots, r^t_n \)

applying max-min fairness on \( \frac{R^{t-1}_1 + r^t_1}{w_1}, \frac{R^{t-1}_2 + r^t_2}{w_2}, \ldots, \frac{R^{t-1}_n + r^t_n}{w_n} \)

given the constraints \( \sum_{i \in [n]} r^t_i \leq R \) and \( \min \left\{ d^t_i, \alpha \frac{R_i w_i}{\sum_k w_k} \right\} \leq r^t_i \leq d^t_i, \forall i \)

where \( \alpha \in [0, 1] \) and \( w_1, \ldots, w_n \) are positive numbers, similar to the definition of Dynamic DRF. Similar to previous sections, for a given \( \alpha \), we denote \( g_i(d^t_i) = \min \left\{ d^t_i, \alpha \frac{R_i w_i}{\sum_k w_k} \right\} \) the guarantee of user \( i \) in epoch \( t \). It is easy to see that Dynamic weighted MMF satisfies \( \alpha \)-sharing incentives.

**Theorem 5.1.** When every user is guaranteed an \( \alpha \)-fraction of their fair share, Dynamic weighted MMF satisfies \( \alpha \)-sharing incentives.
Coalitions When users form coalitions they try to increase the sum of their utilities by each member of the coalition deviating. We bound that increase, i.e., if the set \( I \subset [n] \) of users forms a coalition and report demands \( \{d_i^t\}_{i \in I, t} \) instead of \( \{d_i^0\}_{i \in I, t} \), then for some \( \gamma \geq 1 \) and for all \( t \) we want to prove that \( \sum_{i \in I} \hat{U}_i^t \leq \gamma \sum_{i \in I} U_i^t \). Proofs of the results in this section are included in Appendix C.

5.1.1 Over-reporting and Incentive Compatibility Upper Bound

Analogously to Theorem 3.5 users have nothing to gain in Dynamic weighted MMF by over-reporting, even when colluding.

**Theorem 5.2.** Let \( I \subset [n] \) be a set of users that form a coalition. Then, for any value of \( \alpha \in [0, 1] \) used by Dynamic weighted MMF, the users in \( I \) have nothing to gain over-reporting their demand.

Next we show a generalized version of Theorem 5.6. More specifically, that users in the coalition cannot misreport their demands to increase their total utility by a factor more than 2 and if there is no coalition (\( I = \{i\} \)), user \( i \) cannot increase her utility by a factor larger than \( 1 + \max_{j \neq i} w_i/w_j \) (which is strictly less than 2).

We should note that this result is much different than the results of Section 4: using the notation of that section, here we have that \( \rho_i = \max_{j \neq i} \{w_i/w_j\} \). Even though \( 1 + \rho_i \) is possibly unbounded, the incentive compatibility ratio here is at most 2.

**Theorem 5.3.** Let \( I \subset [n] \) be a set of users that form a coalition and \( w_1, \ldots, w_n \) be any weights, according to which Dynamic weighted MMF allocates resources. Then, for any \( \alpha \in [0, 1] \), any deviation of the users in \( I \), and any epoch \( t \) it holds that

\[
\sum_{i \in I} \hat{U}_i^t \leq 2 \sum_{i \in I} U_i^t
\]

Additionally, when \( I = \{i\} \) for any user \( i \), it holds that

\[
\hat{U}_i^t \leq \left(1 + \max_{j \neq i} \frac{w_i}{w_i + w_j}\right) U_i^t.
\]

Note that if \( I = \{i\} \) and users have the same weights the above theorem is similar to Theorem 3.6. To prove Theorem 5.3 we prove Lemma C.2, a generalization of Lemma 3.4 bounding the over-allocation of all users, where the \( r_i^t \) in the right hand side of the inequality is replaced with \( \sum_{i \in I} r_i^t \).

5.2 Getting more utility multiple times

In this section we study what happens when user 1 deviates and gets more utility over an extended time period, or multiple times. More specifically, assuming that there are alternating intervals where either \( \hat{R}_1 > R_1 \) or \( \hat{R}_1 < R_1 \) we study the length of those intervals and the duration between them. We first make the following definitions:

- For \( \ell = 0, 1, 2, \ldots \) let \( s_\ell \) be distinct and ordered times (i.e., \( s_{\ell-1} < s_\ell \)) when user 1 begins having more resources by misreporting, i.e., \( \hat{R}_1^{s_{\ell-1}} \leq R_1^{s_{\ell-1}} \) and \( \hat{R}_1^{s_\ell} > R_1^{s_\ell} \).

- For \( \ell = 0, 1, 2, \ldots \) let \( e_\ell \) be the first time after epoch \( s_\ell \) when user 1 begins having less resources by misreporting, i.e., \( \hat{R}_1^{e_\ell} \geq R_1^{e_\ell} \) and \( \hat{R}_1^{e_\ell} < R_1^{e_\ell} \).

Note that \( 0 < s_0 < e_0 < s_1 < e_1 < \ldots \) by definition. Using the above notation we prove that if during every interval \([s_\ell, e_\ell]\) user 1 got a factor of \( \gamma \) more resources in epoch \( t_\ell \) for some \( t_\ell \in [s_\ell, e_\ell] \) by misreporting, then \( t_\ell \) cannot be much larger than \( s_\ell \) and that also \( t_\ell \) scales exponentially with \( \ell \). The proof of the theorem is presented in Appendix D.
**Theorem 5.4.** Assume that for every $t$, $R^t_1 \in \Theta(t)$ and for every $\ell = 0, 1, \ldots$ there exists an epoch $t_\ell \in [s_\ell, e_\ell)$ for which $R^\ell_1 \geq \gamma R^\ell_1$, for some $\gamma > 1$. Then, in Dynamic MMF for any $\alpha \in [0, 1]$, any $\ell = 0, 1, \ldots$, and any $t_\ell \in [s_\ell, e_\ell)$ such that $R^\ell_1 \geq \gamma R^\ell_1$, it holds that

$$t_\ell = O(s_\ell) \quad \text{and} \quad t_\ell = \left(\frac{2 - \gamma}{3 - 2\gamma}\right)^\ell \Omega(t_0)$$

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A Deferred Proofs of Section 3

In this section we prove Theorem 3.2, the lower bound for Dynamic MMF.

**Theorem 3.2.** For any value of $\alpha \in [0, 1]$, there is an instance with $n$ users, where in Dynamic MMF a user can under-report her demand to increase her utility by a factor of $\sqrt{2}$ as $n \to \infty$.

**Proof.** We are going to prove the theorem for $\alpha = 0$. We can do that without loss of generality because for a fixed $R$ we can add users with zero demands (that do not affect the allocation) to make the guarantee $g(d_i) = \min\{d_i, \alpha R/n\}$ tend to 0 for any $\alpha \in [0, 1]$.

The example in which a user (here called Alice) can deviate to increase her allocation by a factor of almost $\sqrt{2}$ is the following (the details of the example are also in Tables 2 and 3).

- There are $1 + m + k$ users: Alice, users $B_1$ to $B_m$, and users $C_1$ to $C_k$.

- The epochs are divided into 4 phases:

  1. In the first phase there are $m$ epochs. For $i \in [m]$, in the $i$-th of these epochs only user $B_i$ has a demand, and she demands and gets an allocation of $F_i$, which will be defined later. We are going to assume that $F_{i+1} \geq F_i$ and also define $f_i = F_i - F_{i-1}$ with $F_0 = 0$.

We can assume that in every epoch there is a different total amount of resources without loss of generality: we can split an epoch into $x$ sub-epochs where in each there are $1/x$ available resources.
2. In the second phase there are again $m$ epochs. For $i \in [m]$, in the $i$-th of these epochs there are $f_i$ resources and only users Alice and $B_i$ have demand, each demanding all the resources.

3. In the third phase there are $k$ epochs. For $i \in [k]$, in the $i$-th of these epochs there are $F_m$ resources and only users Alice and $C_i$ have demand, each demanding all the resources.

4. In the fourth phase there are $m$ epochs. In the 1st of these epochs there are $f_m$ resources and only Alice and user $B_m$ have demands, demanding all the resources. For $i \in [m-1]$, in the $(m-1+i)$-th of these epochs there are

$$F_m - f_i + \frac{f_m}{2} = (F_i + f_i) - (F_{i+1} + f_{i+1}) \quad (17)$$

resources (the above equation defines $F_i$) and only users Alice and $B_i$ have demand, each demanding all the resources.

| Users       | Total allocations after phase 1 | Epoch $i$, phase 2, Alice and $B_i$, $i \in [m]$ | Epoch $i$, phase 3, Alice and $C_i$, $i \in [k]$ | Total allocations after phase 3 |
|-------------|--------------------------------|-----------------------------------------------|-----------------------------------------------|--------------------------------|
| Alice       | 0                              | $f_i$                                          | $F_m$                                          | $F_m$                          |
| $B_i$, $i \in [m]$ | $F_i$                           | $f_i$                                          | $F_m$                                          | $F_m 2^{-i}$                  |
| $C_i$, $i \in [k]$ | 0                              | 0                                              | $F_m$                                          | $F_m (1 - 2^{-k})$            |

Table 2: The allocation of resources for the first 3 phases. The black numbers denote the users’ demands, the blue numbers are the users’ allocations in each epoch when Alice is truthful, and the red numbers are the allocations in each epoch when Alice misreports a demand of 0 for every epoch of phase 2. The total resources in each epoch is equal to the maximum demand of any user.

| Users       | Epoch 1 phase 1 | Epoch 2 phase 4 | ... | Epoch $m - i + 1$ phase 4 | ... | Epoch $m$ phase 4 |
|-------------|-----------------|-----------------|-----|---------------------------|-----|------------------|
| Alice       | $f_m$           | $f_m/2$         | ... | $F_m + f_m/2$             | ... | $F_m + f_m/2$    |
| $B_1$       | 0               | 0               | ... | $F_m - f_m/2$             | ... | $F_m - f_m/2$    |
| ...         | ...             | ...             | ... | ...                       | ... | ...              |
| $B_i$       | 0               | 0               | ... | $F_m + f_m/2$             | ... | $F_m + f_m/2$    |
| ...         | ...             | ...             | ... | ...                       | ... | ...              |
| $B_{m-1}$   | 0               | 0               | ... | $F_m - f_m/2$             | ... | $F_m - f_m/2$    |
| $B_m$       | $f_m$           | $f_m/2$         | ... | 0                         | 0   | 0                |

Table 3: The allocation of resources for phase 4. The top part of the table is for when Alice is truthful, and the bottom part is for when Alice misreports a demand of 0 for every epoch of phase 2. The total resources in each epoch is equal to the maximum demand of any user.

If Alice is truthful she will end up with $F_m + f_m/2$ resources after phase 4. If she misreports a demand of 0 in every epoch of phase 2 she will get an allocation of $F_1 + f_1 - F_m 2^{-k}$ after phase 4 (both cases are depicted in
Solving the recursive equations (17) and taking \( m \to \infty \) and \( k \to \infty \) we get that

\[
\frac{F_1 + f_1 - F_m 2^{-k}}{F_m + f_m/2} \to \sqrt{2}
\]

which is the desired incentive compatibility ratio. For completeness, the solution for \( F_i \) when we set \( F_m + f_m/2 = 1 \) (for normalization) is

\[
F_i = \frac{(\sqrt{2} + 1)^{m+1} \left( 1 - \left( \frac{2 - \sqrt{2}}{2} \right)^i \right) + (\sqrt{2} - 1)^{m+1} \left( 1 - \left( \frac{2 + \sqrt{2}}{2} \right)^i \right)}{(\sqrt{2} - 1)^{m+1} + (\sqrt{2} + 1)^{m+1}}
\]

One can check that our initial conditions are true: for any \( m \geq 1 \), \( F_{i+1} \geq F_i \) and the quantities in (17) are positive.

\[\blacksquare\]

B Deferred Proofs of Section 4

B.1 Incentives when users have zero ratios

In this section we include the discussion about Dynamic DRF when some users have zero ratios. This special case is much different, even when there is only one epoch. To illustrate this, consider the example of [PPS12]: there are two resources and three users with infinite demands and ratios \((1, \epsilon), (\epsilon, 1)\), and \((\epsilon, 1)\), for a small but positive \( \epsilon \). If the total amount available of every resource is \( R = 1 \), then DRF gives every user an allocation of almost \( 1/2 \). If instead \( \epsilon = 0 \), it is easy to check that the allocations are much different: the first user receives an allocation of \( 1 \), while the other two users still receive an allocation of \( 1/2 \).

For Dynamic DRF, we prove that a user can over-report her demand to increase her allocation, even when the ratios stay fixed over time. In the next section, we will show that this is no longer possible once all resources are used by all users, even if the ratios are very different and change over time.

**Theorem B.1.** For any \( \alpha \in [0, 1] \), there is an instance were all the users have the same weights \((w_i = w_j \text{ for every users } i, j)\), there are \( m \) resources, and users have static ratios some of which are zero where a user can increase her utility by a factor of \( \Theta(m) \) by only over-reporting her demand.

**Proof.** Similar to the proof of Theorem 3.2, we are going to prove the theorem for \( \alpha = 0 \). We can do that without loss of generality because for a fixed \( R \) we can add users with zero demands (that do not affect the allocation) to make the guarantee \( g(d_i^t) = \min \{ d_i^t, \alpha \frac{R w_i}{\sum_k w_k} \} \) tend to 0 for any \( \alpha \in [0, 1] \).

Table 4 presents the example graphically. In detail, the example has \( n^2 + m - 1 \) users and \( m \geq 3 \) resources:

- Alice uses every resource with a ratio of 1, except for the second resource which she does not use.
- There are \( n^2 \) copies of a user, Bob, where each uses the first resource with a ratio of 1, the second one with a ratio of \( 1 - \frac{1}{n} \), and does not want any other resource.
- User \( i \), for all \( i \in [m-2] \), uses the second resource with a ratio of \( \frac{1}{n(m-2)} \), the \( (i+2) \)-th one with a ratio of 1, and does not use any other resources.
- In epoch 1 the total available amount of every resource is 1. On the other epochs, 2, 3, \ldots, \( m-1 \), the total available amount for each resource is \( \frac{1}{m^4} \).
If Alice lies on the first epoch and requests resources instead of requesting zero, she can take resources from every Bob, which increases the allocation of the other users by a factor of \(\frac{1}{n^2}\). This allows us to set \(m = \Theta(n)\). In total, Alice gets an allocation of \(\Theta(n)\) if she is truthful and a (useful) allocation of \(\frac{m-2}{n^2}\) if she is untruthful, i.e., she increases her utility by a factor of at least \(\frac{m-2}{n^2}\). This proves the theorem.

### B.2 Deferred Proofs of Section 4.1

First we restate and prove Lemma 4.2.

**Lemma 4.2.** Fix an epoch \(t\) and the total allocations up to epoch \(t-1\) of any two outcomes \(\{\hat{R}_k^{t-1}\}_{k\in[n]}\) and \(\{\bar{R}_k^{t-1}\}_{k\in[n]}\). Let \(i, j\) be two different users. If the following conditions hold

- For \(i\), \(\bar{r}_i^t < \hat{r}_i^t\), \(\bar{d}_i^t \leq \hat{d}_i^t\), and \(\bar{a}_{iq}^t > 0\) for all \(q\).
- For \(j\), \(\bar{r}_j^t > \hat{r}_j^t\), \(\bar{d}_j^t \leq \hat{d}_j^t\), and \(\bar{a}_{jq}^t > 0\) for all \(q\).

then, for any \(\alpha \in [0, 1]\) used by Dynamic DRF, it holds that \(\bar{R}_i^t/w_i \geq \hat{R}_j^t/w_j\) and \(\bar{R}_i^t/w_i \leq \hat{R}_j^t/w_j\), implying

\[
\frac{\hat{R}_i^t - \bar{R}_i^t}{w_i} \leq \frac{\hat{R}_j^t - \bar{R}_j^t}{w_j}
\]

**Proof.** We notice that

- Because \(\bar{r}_i^t > g(\bar{d}_i^t)\) (since \(\bar{r}_i^t > \hat{r}_i^t \geq g(\hat{d}_i^t) \geq g(\bar{d}_i^t)\)) and \(\bar{a}_{jq}^t > 0\) for all \(q\), decreasing \(\bar{r}_i^t\) would free a positive amount of every resource.

---

\(^{4}\)We can make that assumption w.l.o.g. because we can split the first epoch where there is 1 amount of every resource into \(n^2\) epochs where the available amount of every resource is \(n^{-2}\) every round.
Because \( \hat{r}_j^i < d_j^i \) it is feasible to increase \( \hat{r}_j^i \) in order to increase \( \hat{R}_j^i \).

This implies that \( \hat{R}_i^i/w_i \leq \hat{R}_j^j/w_j \); otherwise it would have been more fair to give some of the resources user \( i \) got to user \( j \). With the analogous inverse argument (we can increase \( \hat{r}_i^i \) by decreasing \( \hat{r}_j^j \)) we can prove that \( \hat{R}_i^i/w_i \geq \hat{R}_j^j/w_j \). This completes the proof.

Finally we prove the lower bound for the incentive compatibility ratio of Dynamic DRF.

**Theorem 4.7.** For any \( \epsilon \in (0, 1) \), \( w_1, w_2 > 0 \), and \( \alpha \in [0, 1] \) used by Dynamic DRF, there is an instance where the users’ ratios are constant every epoch, user 1 has weight \( w_1 \) and another user has weight \( w_2 \), \( \rho_1 = \frac{w_1}{w_2 \epsilon^\alpha} \), and user 1 can under-report her demand to increase her utility by a factor of \( 1 + \rho_1 \).

**Proof.** We are going to prove the theorem for \( \alpha = 0 \). We can make the assumption that \( \alpha = 0 \) without loss of generality because for a fixed \( R \) we can add users with zero demands and same ratios and weights as some user other that user 1 (thus not affecting the allocation nor the definition of \( \rho_1 \)) to make the guarantee \( g(d_i^j) = \min \{d_i^j, \alpha \frac{R_{w_i}}{\sum_k w_k} \} \) tend to 0 for any \( \alpha \in [0, 1] \).

There are 2 epochs, 2 resources, and 4 groups of users where the users in each group have the same ratios and demands. Groups 1 and 2 have one user each, user 1 and user 2 respectively, while groups 3 and 4 have \( n_1 \) and \( n_2 \) users, respectively.

User 1 is the one who will under-report her demand to increase her utility and w.l.o.g. we assume that her weight is \( w_1 = 1 \). All other users have weight \( w > 0 \) (this corresponds to the weight \( w_2 \) in the theorem’s statement). Their ratios are depicted in Table 5, along with a summary of the whole example. We assume that \( \delta \) and \( \epsilon \) are fixed. We note that

\[
\rho_1 = \max_{k \neq 1, q} \frac{w_1 a_{1q}}{w_k a_{kq}} = \max \left\{ \frac{1}{w \epsilon}, \frac{1}{w}, \frac{\delta}{w \epsilon}, \frac{\delta}{w}, \frac{\delta}{w} \right\} = \frac{1}{w \epsilon}
\]

as needed by the theorem.

|          | Res. 1 | Res. 2 | Ep. 1 \((R_1^1 = 1 + \frac{n_1 w}{1 + w \epsilon})\) | Ep. 2 \((R_2^2 = \frac{\delta}{w \epsilon} + \frac{n_2 \delta}{\epsilon(w+\delta)})\) |
|----------|--------|--------|---------------------------------|---------------------------------|
| User 1 \((w_1 = 1)\) | 1      | \(\delta\) | \(\infty\) | \(\frac{1}{1 + w \epsilon}\) |
| User 2 \((w_2 = w)\) | \(\epsilon\) | 1      | \(\infty\) | \(\frac{1}{1 + w \epsilon}\) |
| Group 3 \((n_1\) users, \(w_3 = w)\) | 1      | \(\epsilon\) | \(\frac{w}{1 + w \epsilon}\) | \(\frac{w}{1 + w \epsilon}\) |
| Group 4 \((n_2\) users, \(w_4 = w)\) | \(\epsilon\) | 1      | 0      | 0      |

Table 5: Summary of the example of Theorem 4.7. The 2nd and 3rd columns depict the users’ ratios, where \( 0 \leq \delta \leq \epsilon < 1 \). The last two columns for each epoch show the total available resources for that round in the parentheses of the first row, the demands of each group (black numbers), their allocations when the user in group 1 is truthful (blue numbers), and their allocations when the user in group 1 misreports a demand of 0 in epoch 1 (red numbers).

**Epoch 1.** In the first epoch, users 1 and 2 demand \( \infty \) resources, group 3 demand \( \frac{w}{1 + w \epsilon} \), and group 4 demand 0. The total amount available for every resource is \( 1 + \frac{n_1 w}{1 + w \epsilon} \).

We first show that with these demands, user 1 gets \( r_1^1 = \frac{1}{1 + w \epsilon} \), while user 2 and users in group 3 get an allocation of \( \frac{w}{1 + w \epsilon} \). Note that this is fair, since every user with positive demand gets an allocation proportional to their weight. Because this allocation is fair, to prove its validity we need that one resource is saturated and the other is not over-used. The total amount of resource 1 used is

\[
\frac{1}{1 + w \epsilon} + \epsilon \frac{w}{1 + w \epsilon} + n_1 \frac{w}{1 + w \epsilon} = 1 + \frac{n_1 w}{1 + w \epsilon} = R_1^1
\]
meaning that resource 1 is saturated, proving that we cannot increase the users’ allocations. The total amount of resource 2 used is
\[
\delta \frac{1}{1 + w\epsilon} + \frac{w}{1 + w\epsilon} + n_1 \epsilon \frac{w}{1 + w\epsilon} = \delta + \frac{w}{1 + w\epsilon} + n_1 \frac{w\epsilon}{1 + w\epsilon} \leq 1 + n_1 \frac{w}{1 + w\epsilon} = R^1
\]
where the inequality holds because \(\epsilon < 1\) and we assume that \(n_1\) is large enough. This proves that we do not over-use resource 2.

Now we will prove that if the user in group 1 misreports her demand and asks for 0 instead, then the allocations are as follows: user 1 gets \(\hat{r}_1^1 = 0\), user 2 gets an allocation of \(\frac{1}{\epsilon}\) and users in group 3 get an allocation of \(\frac{w\epsilon}{1 + w\epsilon}\).

Note that this is fair, since they have the same weight and users in group 3 get their demand, while user 2 gets more than. The total amount of resource 1 used is
\[
0 + \frac{1}{\epsilon} + n_1 \frac{w}{1 + w\epsilon} = 1 + \frac{n_1 w}{1 + w\epsilon} = R^1
\]
meaning that resource 1 is saturated, proving that we cannot increase the users’ allocations. The total amount of resource 2 used is
\[
0 + \frac{1}{\epsilon} + n_1 \epsilon \frac{w}{1 + w\epsilon} \leq 1 + n_1 \frac{w}{1 + w\epsilon} = R^1
\]
where the inequality holds because \(\epsilon < 1\) and we assume that \(n_1\) is large enough. This proves that we do not over-use resource 2.

**Epoch 2.** In the second epoch, users 1 and 2 demand \(\infty\) resources, group 3 demand 0, and group 4 demand \(\frac{\delta}{\epsilon(w + \sigma)}\). The total amount available for every resource is \(\frac{\delta}{\epsilon w} + \frac{n_2 \delta}{\epsilon(w + \sigma)}\).

We first show that with these demands, when user 1 was truthful, user 1 gets \(\hat{r}_1^1 = 0\), while user 2 and users in group 4 get an allocation of \(\frac{\delta}{\epsilon(w + \sigma)}\). Note that this is fair, since user 2’s total allocation is the same as user 1’s times \(w\) and users in group 4 get their demand which is less than the total allocation of user 2 with which they have the same weight. The total amount of resource 1 used is
\[
\frac{\delta}{w\epsilon(w + \delta)} + \frac{\delta}{\epsilon(w + \delta)} + n_2 \epsilon \frac{\delta}{\epsilon(w + \delta)} \leq \frac{\delta}{w\epsilon} + \frac{n_2 \delta}{\epsilon(w + \delta)} = R^2
\]
where the inequality holds because \(\epsilon < 1\) and by assuming that \(n_2\) is large enough. This proves that resource 1 is not over-used. The total amount of resource 2 used is
\[
\frac{\delta}{w\epsilon(w + \delta)} + \frac{\delta}{\epsilon(w + \delta)} + n_2 \frac{\delta}{\epsilon(w + \delta)} = \frac{\delta}{w\epsilon} + \frac{n_2 \delta}{\epsilon(w + \delta)} = R^2
\]
meaning that resource 1 is saturated, proving that we cannot increase the users’ allocations.

Now we will prove that when the user 1 misreported in the first epoch, the allocations are as follows: user 1 gets \(\hat{r}_1^1 = \frac{1}{\epsilon w}\), user 2 gets 0, and group 4 gets \(\frac{\delta}{\epsilon(w + \sigma)}\). Note that this is fair; the total resource of user 1 and user 2 are now \(\frac{1}{\epsilon w}\) and \(\frac{1}{\epsilon}\) (which are proportional to users’ weights) and users in group 4 get their demand which is not more than what user 2 gets in total who has the same weight. The total amount of resource 1 used is
\[
\frac{1}{\epsilon w\epsilon} + \epsilon 0 + n_2 \epsilon \frac{\delta}{\epsilon(w + \delta)} \leq \frac{\delta}{\epsilon w\epsilon} + n_2 \frac{\delta}{\epsilon(1 + \delta)} = R^2
\]
where the inequality holds because \(\epsilon < 1\) and by assuming that \(n_2\) is large enough. This proves that resource 1 is not over-used. The total amount of resource 2 used is
\[
\frac{\delta}{\epsilon w\epsilon} + 0 + n_2 \frac{\delta}{\epsilon(w + \delta)} = \frac{\delta}{\epsilon w\epsilon} + n_2 \frac{\delta}{\epsilon(w + \delta)} = R^2
\]
meaning that resource 1 is saturated, proving that we cannot increase the users’ allocations.

This proves that the misreporting of user 1 increases her resources by

\[
\frac{r_1^1 + r_1^2}{r_1^1 + r_1^2} = \frac{0 + \frac{1}{we} \frac{\delta}{\delta \to 0}}{1 + \frac{1}{we}} = 1 + \rho_1
\]  (18)

the above proves that as \( \delta \to 0 \) we get an incentive compatibility ratio of \( 1 + \rho_1 \), as needed.

It is worth pointing out what we mentioned in Section 4: if we want a lower bound that depends on \( \rho = \max_i \rho_i \) instead of \( \rho_1 \), we get an interesting result when \( w = 1 \): if we set \( \delta = \epsilon \), making \( \rho = 1/\epsilon \), (18) now proves an incentive compatibility bound of \( \frac{1 + 1/\epsilon}{2} = \frac{1 + \rho}{2} \).

C  Deferred Proofs of Section 5.1

First we state a more specific version of Lemma 4.2.

**Lemma C.1.** Fix an epoch \( t \) and the allocations of two different outcomes \( \{\hat{R}_k^{t-1}\}_{k \in [n]} \) and \( \{\hat{R}_k^{t-1}\}_{k \in [n]} \). Let \( i, j \) be two different users. If the following conditions hold

- For \( i \), \( r_i^1 < r_i^1 \) and \( \bar{d}_i^1 \leq \hat{d}_i^1 \),
- For \( j \), \( r_j^1 > r_j^1 \) and \( \bar{d}_j^1 \leq \hat{d}_j^1 \),

then, for any \( \alpha \in [0, 1] \),

\[
\frac{\hat{R}_i^t}{w_i} \geq \frac{\hat{R}_j^t}{w_j} \quad \text{and} \quad \frac{\hat{R}_i^t}{w_i} \leq \frac{\hat{R}_j^t}{w_j}
\]

and consequentially

\[
\frac{\hat{R}_i^t - \hat{R}_i^t}{w_i} \leq \frac{\hat{R}_j^t - \hat{R}_j^t}{w_j}
\]  (19)

Next we generalize Lemma 3.4. Note that this is much different and stronger than what is stated in Lemma 4.3; even without the assumption that users collude Lemma 4.3 proves that in Dynamic weighted MMF, for all \( i, t \):

\[
\hat{R}_i^t - \hat{R}_i^t \leq \hat{R}_i^t \max_k \frac{w_i}{w_k}
\]

**Lemma C.2.** Fix an epoch \( t \) and the allocations of two different outcomes \( \{\hat{R}_k^{t-1}\}_{k \in [n]} \) and \( \{\hat{R}_k^{t-1}\}_{k \in [n]} \). Assume that \( \{d_i^1\}_{i \in [n]} \) are some users’ demands and that \( \{d_i^1\}_{i \in [n]} \) are the same demands except users in \( I \), who deviate but not by over-reporting, i.e., \( d_i^1 \leq \hat{d}_i^1 \) for \( i \in I \). Then, for any \( \alpha \in [0, 1] \), it holds that

\[
\sum_{k \in [n]} \left( \hat{R}_k^t - \hat{R}_k^t \right)^+ - \sum_{k \in [n]} \left( \hat{R}_k^{t-1} - \hat{R}_k^{t-1} \right)^+ \leq \left[ \sum_{k \in I} \hat{d}_k^1 < \sum_{k \in I} \hat{d}_k^1 \right] \sum_{k \in I} \hat{r}_k^t
\]  (20)

**Proof.** First we define \( P_t = \{k \in [n] : \hat{R}_k^t > \hat{R}_k^t\} \) for all \( t \). Suppose by contradiction:

\[
\sum_{k \in P_t} \left( \hat{R}_k^t - \hat{R}_k^t \right) - \sum_{k \in P_t^{-1}} \left( \hat{R}_k^{t-1} - \hat{R}_k^{t-1} \right) > 1 \left[ \sum_{k \in I} \hat{d}_k^1 < \sum_{k \in I} \hat{d}_k^1 \right] \sum_{k \in I} \hat{r}_k^t
\]
Because \( \sum_{k \in P^t}(\hat{R}_k^{t-1} - \hat{R}_k^t) \leq \sum_{k \in P^{t-1}}(\hat{R}_k^{t-1} - \hat{R}_k^t) \), the above inequality implies

\[
\sum_{k \in P^t} (\hat{r}_k^t - \hat{r}_k^t) > 1 \left[ \sum_{i \in I} \hat{d}_k^t < \sum_{i \in I} \hat{d}_k^t \right] \sum_{k \in I} \hat{r}_k^t
\]  

(21)

Because users only under-report their demand, \( \hat{d}_k^t \leq \hat{d}_k^t \), it holds that \( \sum_{k \in P^t} \hat{r}_k^t \geq \sum_{k \in P^t} \hat{r}_k^t \), i.e., the total resources allocated to the users does not increase when user 1 deviates. Combining this fact with (21) we get that

\[
\sum_{k \notin P^t} (\hat{r}_k^t - \hat{r}_k^t) > 1 \left[ \sum_{i \in I} \hat{d}_k^t < \sum_{i \in I} \hat{d}_k^t \right] \sum_{k \in I} \hat{r}_k^t
\]

(22)

We notice that because of (21), there exists a user \( i \in P^t \) for whom \( \hat{r}_i^t > \hat{r}_i^t \); because of (22), there exists a user \( j \notin P^t \) for whom \( \hat{r}_j^t > \hat{r}_j^t \). Additionally for that \( j \) we can assume that \( \hat{d}_j^t = \hat{d}_j^t \) because:

- If users in \( I \) do not deviate then for all \( k \), \( \hat{d}_k^t = \hat{d}_k^t \).
- If \( \sum_{k \in I} \hat{d}_k^t < \sum_{k \in I} \hat{d}_k^t \), then (22) implies \( \sum_{k \notin P^t \cap I} \hat{r}_k^t > 0 \), i.e., \( j \notin I \) and we assumed that only users in \( I \) deviate.

Thus we have \( \hat{d}_i^t \leq \hat{d}_i^t \) (since no user over-reports), \( \hat{d}_j^t = \hat{d}_j^t \), \( \hat{r}_i^t > \hat{r}_i^t \), and \( \hat{r}_j^t < \hat{r}_j^t \). Now Lemma C.1 proves that \( (\hat{R}_i^t - \hat{R}_i^t)/\bar{w}_i \leq (\hat{R}_j^t - \hat{R}_j^t)/\bar{w}_j \). This leads to a contradiction, because \( i \in P^t \) and \( j \notin P^t \), i.e., \( \hat{R}_i^t - \hat{R}_i^t \geq 0 > \hat{R}_j^t - \hat{R}_j^t \).

Next we prove that users do not want to over-report their demand.

**Theorem 5.2.** Let \( I \subset [n] \) be a set of users that form a coalition. Then, for any value of \( \alpha \in [0, 1] \) used by Dynamic weighted MMF, the users in \( I \) have nothing to gain over-reporting their demand.

Similar to Theorem 3.5, we first prove an auxiliary lemma.

**Lemma C.3.** Fix an epoch \( T_0 \) and the allocations of an outcome \( \{\hat{R}_k^{T_0-1}\}_{k \in [n]} \). Fix another epoch \( T \geq T_0 \) and assume that in epochs \( T_0 + 1, T_0 + 2, \ldots, T \) users in \( I \) do not over-report their demand. Then, for any \( \alpha \in [0, 1] \), the users in \( I \) cannot increase their utility in round \( T \) by over-reporting their demand in epoch \( T_0 \).

**Proof.** To prove the lemma we are going to create another outcome in which the over-reports in epoch \( T_0 \) are changed to a truthful ones and prove that this does not decrease \( I \)'s total utility in epoch \( T \).

For all \( k, t \), let \( \hat{d}_k^t = \hat{d}_k^t \), except for epoch \( T_0 \) and users \( i \in I \), where \( \hat{d}_i^{T_0} \) are the actual demands if they were over-reports: \( \hat{d}_i^{T_0} = \min\{\bar{d}_i^{T_0}, \hat{d}_i^{T_0}\} \) for \( i \in I \). This means that the two outcomes are the same before epoch \( T_0 \), i.e., for all \( k \), \( \hat{R}_k^{T_0-1} = \bar{R}_k^{T_0-1} \) and \( \hat{U}_k^{T_0-1} = \bar{U}_k^{T_0-1} \). In epoch \( T_0 \) users’ allocations are different. First, let \( I' \subset I \) be the users who over-report their demand in \( T_0 \), i.e., the users \( i \) for whom \( \hat{d}_i^{T_0} = \hat{d}_i^{T_0} < \hat{d}_i^{T_0} \). Now we prove that for any user \( i \) who got more resources, \( \hat{r}_i^{T_0} > \hat{r}_i^{T_0} \), that user must be in \( I' \) and that additional resources must be in excess of their true demand:

\[
\text{if } \hat{r}_i^{T_0} - \hat{r}_i^{T_0} = \hat{R}_i^{T_0} - \bar{R}_i^{T_0} > 0 \text{ then } \hat{r}_i^{T_0} = \hat{d}_i^{T_0} \text{ and } i \in I'
\]

(23)

To prove (23) we study two cases:

- If \( \hat{r}_i^{T_0} - \hat{r}_i^{T_0} = \hat{R}_i^{T_0} - \hat{R}_i^{T_0} > 0 \) and \( \hat{r}_i^{T_0} < \hat{d}_i^{T_0} \), then \( \hat{r}_i^{T_0} < \hat{d}_i^{T_0} \). This means user \( i \)'s demands are not satisfied both in \( \hat{d}_i^{T_0} \) and \( \hat{d}_i^{T_0} \), which implies that, since \( \hat{r}_i^{T_0} > \hat{r}_i^{T_0} \) and for all users \( k \), \( \hat{d}_k^{T_0} \leq \hat{d}_k^{T_0} \), for some other user \( j \), \( \hat{r}_j^{T_0} - \hat{r}_j^{T_0} = \hat{R}_j^{T_0} - \hat{R}_j^{T_0} < 0 \). This, because of Lemma C.1 and \( \hat{d}_j^{T_0} \leq \hat{d}_j^{T_0} \), entails that \( (\hat{R}_i^{T_0} - \hat{R}_i^{T_0})/\bar{w}_i \leq (\hat{R}_j^{T_0} - \hat{R}_j^{T_0})/\bar{w}_j \), a contradiction.
• If \( r^T_i - \hat{r}^T_i = \hat{R}^T_i - R^T_i > 0 \) and \( i \notin I' \), then, because user \( i \) got more resources in \( \hat{d}^T \) with the same demand, for some user \( j \), \( r^T_j - \hat{r}^T_j = R^T_j - \hat{R}^T_j < 0 \), which because of Lemma C.1, \( \hat{d}^T_i = d^T_i \), and \( d^T_i \leq d^T_j \) entails that \((R^T_i - \hat{R}^T_i)/w_i \leq ( \hat{R}^T_j - R^T_j )/w_j \), a contradiction.

Because of (23) we see that any additional resources that users in \( I \) get in \( T_0 \) when they over-report are in excess of their demand (because for \( i \in I' \), \( d^T_i \) is the true demand). This means that \( \sum_{i \in I} (R^T_i - \hat{R}^T_i)^+ = x \geq 0 \) is in excess of their demand and therefore

\[
\sum_{i \in I} (\hat{U}^T_i - U^T_i) = \sum_{i \in I} (\hat{R}^T_i - \hat{R}^T_i) - x \leq 0
\]  

(24)

Additionally, because users in \( I \) do not over-report \( \hat{d} \) or \( \bar{d} \) in epochs \( T_0 + 1 \) to \( T \), it holds that for \( t \in [T_0 + 1, T] \) \( I' \) utility is the same as the resources they receive: \( \bar{u}_i^t = \bar{r}_i^t \) and \( \hat{u}_i^t = \hat{r}_i^t \) for \( i \in I \). This fact, combined with (24) proves that

\[
\forall t \in [T_0, T], \sum_{i \in I} (\hat{U}_i^t - \bar{U}_i^t) = \sum_{i \in I} (\hat{R}_i^t - \hat{R}_i^t) - x
\]

Thus, in order for this over-reporting to be a strictly better strategy, it must hold that \( \sum_{i \in I} (\hat{R}_i^t - \hat{R}_i^t) > x \). We will complete the proof by proving that the opposite holds. Since in epochs \( t \in [T_0 + 1, T] \) it holds that \( \bar{d}_i^t = \bar{d}_i^t \) for \( i \in I \), we can use Lemma C.2 to sum (20) for all \( t \in [T_0 + 1, T] \) and get that

\[
\sum_k (\hat{R}_k^t - \hat{R}_k^t)^+ - \sum_k (\hat{R}_k^t - \hat{R}_k^t)^+ \leq 0
\]

The above, because \( (\hat{R}_k^t - \hat{R}_k^t)^+ \geq 0 \), \( \hat{R}_k^t - \hat{R}_k^t \leq 0 \) for \( k \notin I \) (from (23)), and \( \sum_{i \in I} (\hat{R}_i^t - \hat{R}_i^t)^+ = x \), proves that \( \sum_{i \in I} (\hat{R}_i^t - \hat{R}_i^t) \leq x \). This completes the proof.

Now Theorem 5.2 can be viewed as a corollary of the above lemma. Finally, we prove the inventive compatibility bound for adversarial demands.

**Theorem 5.3.** Let \( I \subseteq [n] \) be a set of users that form a coalition and \( w_1, \ldots, w_n \) be any weights, according to which Dynamic weighted MMF allocates resources. Then, for any \( \alpha \in [0, 1] \), any deviation of the users in \( I \), and any epoch \( t \) it holds that

\[
\sum_{i \in I} \hat{U}_i^t \leq 2 \sum_{i \in I} U_i^t
\]

Additionally, when \( I = \{i\} \) for any user \( i \), it holds that \( \hat{U}_i^t \leq \left( 1 + \max_{j \neq i} \frac{w_j}{w_i + w_j} \right) U_i^t \).

**Proof.** Because of Theorem 5.2 we assume without loss of generality that users do not over-report their demand. This entails that we can focus on the users’ allocations instead of their utilities.

Fix an epoch \( t \) and let \( T \leq t \) be the last time where \( \sum_{i \in I} \alpha_i > \sum_{i \in I} r_i^t \). We notice that

\[
\sum_{i \in I} (\hat{R}_i^t - \hat{R}_i^t) = \sum_{i \in I} (\hat{R}_i^t - \hat{R}_i^t) + \sum_{\tau = T + 1}^t \sum_{i \in I} (r_i^\tau - r_i^\tau) \leq \sum_{i \in I} (\hat{R}_i^t - \hat{R}_i^t)
\]  

(25)

Because \( \sum_{k \in I} \bar{r}_k^T \geq \sum_{k \in I} \bar{r}_k^T \) and \( \sum_{k \in [n]} \bar{r}_k^T \leq \sum_{k \in [n]} r_k^T \) (since users do not over-report their demand) there must exist a \( i \in I \) such that \( \bar{r}_i^T > r_i^T \) and a \( j \notin I \) such that \( \bar{r}_j^T < r_j^T \). Because \( d_i^T \leq \bar{d}_i^T \) and \( \bar{d}_j^T = \bar{d}_j^T \) we can use Lemma C.2 to get

\[
\frac{w_i}{w_i} (\hat{R}_i^t - \bar{R}_i^t) \leq \hat{R}_j^T - R_j^T
\]  

(26)
If we use Lemma C.2 and sum (20) for epochs up to $T$ we get

$$\sum_{k \in [n]} (\hat{R}_k^T - R_k^T)^+ \leq \sum_{k \in I} R_k^T$$

which combined with (26) gives

$$\frac{w_i}{w_j} \left( \hat{R}_i^T - R_i^T \right)^+ + \sum_{k \in I} (\hat{R}_k^T - R_k^T) \leq \sum_{k \in I} R_k^{T \geq T} \leq \sum_{k \in I} R_k^T \tag{27}$$

(25) and (27) prove that $\sum_{k \in I} \hat{R}_k^t \leq 2 \sum_{k \in I} R_k^t$, which proves the first part of the lemma.

If $|I| = 1$, then $I = \{i\}$ where $i$ is the user appearing in (27). If $\hat{R}_i^t - R_i^t < 0$ the desired bound is true; otherwise, (25) and (27) prove that

$$\min_{j \neq i} \frac{w_i}{w_j} \left( \hat{R}_i^t - R_i^t \right)^+ + \hat{R}_i^t - R_i^t \leq R_i^t$$

which proves the desired bound: $\hat{R}_i^t \leq \left( 1 + \max_{j \neq i} \frac{w_i}{w_i + w_j} \right) R_i^t$. \hfill \blacksquare

## D Deferred Proof of Section 5.2

We first prove a more general version of Lemma 3.4.

**Lemma D.1.** Fix an epoch $t$ and the allocations of two different outcomes $\{\hat{R}_k^{t-1}\}_{k \in [n]}$ and $\{\bar{R}_k^{t-1}\}_{k \in [n]}$. Assume that $\{d_k\}_{k \in [n]}$ are some users’ demands and that $\{\bar{d}_k\}_{k \in [n]}$ are the same demands except user 1’s, who deviates but not by over-reporting, i.e., $\bar{d}_1 \leq d_1$. Then, in Dynamic MMF for any $\alpha \in [0, 1]$, it holds that

$$\sum_{k \in [n]} \left( \hat{R}_k^t - R_k^t \right)^+ - \sum_{k \in [n]} \left( \bar{R}_k^{t-1} - R_k^{t-1} \right)^+ \leq f^t \tag{28}$$

where $f^t = \min \left( (r_1^t - \hat{r}_1^t)^+, (R_1^t - \bar{R}_1^t)^+ \right)$.

**Proof.** Let $P^t = \{k : \hat{R}_k^t > R_k^t\}$ and suppose by contradiction that

$$\sum_{k \in P^t} \left( \hat{R}_k^t - R_k^t \right)^+ - \sum_{k \in P^{t-1}} \left( \bar{R}_k^{t-1} - R_k^{t-1} \right)^+ > f^t \tag{29}$$

In order to get a contradiction we are going to show that the following two conditions are implied by (29):

(I) There exists a user $i \in P^t$, such that $\hat{r}_i^t > r_i^t$.

(II) There exists a user $j \notin P^t$, such that $j \neq 1$ and $\hat{r}_j^t < r_j^t$.

The reason conditions (I) and (II) lead to a contradiction is the following: users $i$ and $j$ satisfy the conditions of Lemma 3.3, meaning that $\hat{R}_i^t - R_i^t \leq \hat{R}_j^t - R_j^t$. However this is a contradiction due to the facts that $i \in P^t$ and $j \notin P^t$.

To prove that (29) implies conditions (I) and (II), we will now distinguish two cases; these are shown in Propositions D.2 and D.3.

**Proposition D.2.** If $\hat{r}_1^t \geq r_1^t$ or $\hat{R}_1^t \geq R_1^t$, (29) implies conditions (I) and (II).
**Proof of Proposition D.2.** In this case we have that $f^t = 0$. Because of the definition of $P^t$, we can re-write (29)

$$\sum_{k \in P^t} \hat{r}_k^t - r_k^t > 0$$  

(30) implies (I). (30) and the fact that $\sum_k r_k^t \geq \sum_k \hat{r}_k^t$ (which comes from user 1 not over-reporting her demand) prove that

$$\sum_{k \in P^t} r_k^t - \hat{r}_k^t > 0$$

The above implies condition (II): there exists a $j \notin P^t$ such that $\hat{r}_j^t < r_j^t$. The reason that the aforementioned $j$ cannot be user 1 is because of our assumptions: either $\hat{r}_1^t \geq r_1^t$ or $\hat{R}_1^t \geq R_1^t$. ■

**Proposition D.3.** If $\hat{r}_1^t < r_1^t$ and $\hat{R}_1^t < R_1^t$, (29) implies conditions (I) and (II).

**Proof of Proposition D.3.** Because $\hat{R}_1^t < R_1^t$, it holds that $1 \notin P^t$. Consider two cases:

- **If $\hat{R}_1^{t-1} < R_1^{t-1}$**, then $f^t = r_1^t - \hat{r}_1^t$. In this case (29) implies

  $$\sum_{k \in P^t} \hat{r}_k^t - r_k^t > r_1^t - \hat{r}_1^t > 0$$

- **If $\hat{R}_1^{t-1} \geq R_1^{t-1}$**, then $f^t = R_1^t - \hat{R}_1^t$ and $1 \in P^{t-1}$. In this case (29) implies

  $$\sum_{k \in P^t} (\hat{R}_k^t - R_k^t) - \sum_{k \in P^{t-1}} (\hat{R}_k^{t-1} - R_k^{t-1}) > R_1^t - \hat{R}_1^t$$

  $$\sum_{k \in P^t} (\hat{R}_k^t - R_k^t) - \sum_{k \in P^{t-1} \setminus \{1\}} (\hat{R}_k^{t-1} - R_k^{t-1}) > r_1^t - \hat{r}_1^t$$

  $$\sum_{k \in P^t} \hat{r}_k^t - r_k^t > r_1^t - \hat{r}_1^t > 0$$

  where to get the last inequality we use the fact that $1 \notin P^t$.

Thus in both cases the following inequality holds:

$$\sum_{k \in P^t} \hat{r}_k^t - r_k^t > r_1^t - \hat{r}_1^t > 0$$  

(31) implies (I). (31) and $\sum_k r_k^t \geq \sum_k \hat{r}_k^t$ (which is true because user 1 does not over-report) prove that

$$\sum_{k \notin P^t} r_k^t - \hat{r}_k^t > r_1^t - \hat{r}_1^t$$

$$\sum_{k \notin P^t \setminus \{1\}} r_k^t - \hat{r}_k^t > 0$$

The above implies condition (II), which completes the proposition’s proof. ■

Due to Propositions D.2 and D.3 we have proven that (29) always leads to a contradiction. This proves the lemma. ■
We now use the above lemma to prove a corollary that directly bounds $\hat{R}_1^t - R_1^t$.

**Corollary D.4.** Let $f^t = \min \left( (r_1^t - \hat{r}_1^t)^+, (R_1^t - \hat{R}_1^t)^+ \right)$ and assume that user 1 does not over-report her demand. Then, in Dynamic MMF for any $\alpha \in [0, 1]$ and every epoch $t$,

$$2 \left( \hat{R}_1^t - R_1^t \right) \leq \sum_{\tau=1}^{t} f^\tau \quad (32)$$

**Proof.** Fix an epoch $t$ and let $t' \leq t$ be the last epoch before $t$ where $r_1^{t'} > r_1^{t'}$. If no such epoch exists, then $R_1^t \leq R_1^{t'}$, in which case the lemma holds. We notice that $R_1^t - R_1^{t'} \leq \hat{R}_1^t - R_1^{t'}$.

Using Lemma D.1 and summing (28) for $t$ from 1 to $t'$ we get

$$\sum_{k} \left( \hat{R}_k^t - R_k^t \right)^+ \leq \sum_{\tau=1}^{t'} f^\tau$$

Because $\hat{r}_1^t > r_1^{t'}$ and $\sum_i \gamma_i \hat{r}_1^t \leq \sum_i \gamma_i r_1^{t'}$ (since user 1 does not over-report) it holds that for some user $j \neq 1$, $\hat{r}_j^{t'} < r_j^{t'}$. This means we can use Lemma 3.4 to prove that $\hat{R}_1^t - R_1^{t'} \leq \hat{R}_1^t - R_1^{t'}$, which makes the above inequality

$$2(\hat{R}_1^t - R_1^t) \leq 2(\hat{R}_1^t - R_1^{t'}) \leq \hat{R}_1^t - R_1^{t'} + \hat{R}_j^t - R_j^t \leq \sum_{\tau=1}^{t'} f^\tau \leq \sum_{\tau=1}^{t} f^\tau$$

This completes the proof. □

Now we prove a series of lemmas with the notation introduced in Section 5.2, in order to prove Theorem 5.4, which we restate for completeness.

**Theorem 5.4.** Assume that for every $t$, $R_1^t \in \Theta(t)$ and for every $\ell = 0, 1, \ldots$ there exists an epoch $t_\ell \in [s_\ell, e_\ell)$ for which $\hat{R}_1^{t_\ell} \geq \gamma R_1^{t_\ell}$, for some $\gamma > 1$. Then, in Dynamic MMF for any $\alpha \in [0, 1]$, any $\ell = 0, 1, \ldots$, and any $t_\ell \in [s_\ell, e_\ell)$ such that $\hat{R}_1^{t_\ell} \geq \gamma R_1^{t_\ell}$, it holds that

$$t_\ell = O(s_\ell) \quad \text{and} \quad t_\ell = \left( \frac{2 - \gamma}{3 - 2\gamma} \right) \Omega(t_0)$$

**Lemma D.5.** If user 1 does not over-report her demand, for any $\ell = 0, 1, 2, \ldots$ and any $t_\ell \in [s_\ell, e_\ell)$ it holds that

$$2 \left( \hat{R}_1^{t_\ell} - R_1^{t_\ell} \right) \leq R_1^{s_\ell - 1} - \sum_{k=0}^{t_\ell - 1} \left( \hat{R}_1^{e_k} - R_1^{s_k - 1} \right)$$

**Proof.** We are going to use Corollary D.4: for every $k \in [0, \ell - 1]$ and $t \in [s_k, e_k)$ it holds that the r.h.s. of (32) is $f^t = 0$, because $\hat{R}_1^t \geq R_1^t$. Fix a $t_\ell \in [s_\ell, e_\ell)$ and we notice that

$$2 \left( \hat{R}_1^{t_\ell} - R_1^{t_\ell} \right) \leq \sum_{\tau=1}^{t_\ell} f^\tau = \sum_{\tau=1}^{s_0 - 1} f^\tau + \sum_{\tau=s_0}^{s_1 - 1} f^\tau + \ldots + \sum_{\tau=s_{\ell - 1}}^{s_\ell - 1} f^\tau$$

$$\leq R_1^{s_0 - 1} + \left( R_1^{s_1 - 1} - R_1^{e_0} + f_0 \right) + \ldots + \left( R_1^{s_\ell - 1} - R_1^{e_{\ell - 1}} + f_{\ell - 1} \right)$$

$$= R_1^{s_\ell - 1} - \sum_{k=0}^{\ell - 1} \left( \hat{R}_1^{e_k} - g_{e_k} - R_1^{s_k - 1} \right)$$

Now all that is left to complete the proof is to prove that for every $k$, $R_1^{e_k} - f_{e_k} \geq \hat{R}_1^{e_k}$. This actually holds with an equality: due to the definition of $e_k$, $\hat{R}_1^{e_k - 1} > R_1^{e_k - 1}$ and $\hat{R}_1^{e_k} \leq R_1^{e_k}$, meaning that $f_{e_k} = R_1^{e_k} - \hat{R}_1^{e_k}$. □
Lemma D.6. Assume that for every $\ell = 0, 1, \ldots$ there exists a $t_\ell \in [s_\ell, e_\ell)$ such that $R_\ell^t \geq \gamma R_\ell^t$ for some $\gamma \in [1, 3/2)$. Then for any such $\{t_\ell\}$ and any $\ell \geq 1$:

$$R_\ell^t \geq \frac{\gamma - 1}{3 - 2\gamma} \sum_{k=0}^{\ell-1} R_1^{t_k}$$

Proof. Fix an $\ell \geq 1$. We use Lemma D.5 and get that

$$2 \left( \hat{R}_\ell^t - R_\ell^t \right) \leq R_1^{s_{t-1}} - \sum_{k=0}^{\ell-1} \left( \hat{R}_1^{e_{k-1}} - R_1^{s_{k-1}} \right) \leq R_1^{s_{t-1}} - \sum_{k=0}^{\ell-1} \left( \hat{R}_1^{e_k} - R_1^{s_k} \right)$$

Using the fact that for every $k = 0, \ldots, \ell$, $\hat{R}_1^{e_k} \geq \gamma R_1^{e_k}$, the above inequality becomes

$$(2\gamma - 3)R_1^t \leq -(\gamma - 1) \sum_{k=0}^{\ell-1} R_1^{t_k}$$

which proves the lemma.

Corollary D.7. If the conditions of Lemma D.6 hold, then for all $\ell \geq 1$,

$$R_\ell^t \geq \frac{\gamma - 1}{2 - \gamma} \left( \frac{2 - \gamma}{3 - 2\gamma} \right) \gamma \sum_{k=0}^{\ell-1} R_1^{t_k}$$

(33)

Proof. We will prove the corollary with induction on $\ell$. For $\ell = 1$, (33) follows from Lemma D.6. Assume that for some $L$, (33) holds for all $\ell = 1, 2, \ldots, L$. Using Lemma D.6 we have that

$$R_\ell^{L+1} \geq \frac{\gamma - 1}{3 - 2\gamma} \sum_{k=0}^{L} R_1^{t_k} \geq R_1^{t_0} \frac{\gamma - 1}{3 - 2\gamma} \left( 1 + \sum_{k=1}^{L} \frac{\gamma - 1}{2 - \gamma} \left( \frac{2 - \gamma}{3 - 2\gamma} \right)^k \right) = R_1^{t_0} \frac{\gamma - 1}{2 - \gamma} \left( \frac{2 - \gamma}{3 - 2\gamma} \right)^{L+1}$$

Corollary D.8. If the conditions of Lemma D.6 hold and for all $t$, $R_1^t \in \Theta(t)$, then for all $\ell \geq 1$

$$t_\ell = \left( \frac{2 - \gamma}{3 - 2\gamma} \right)^\ell \Omega(t_0)$$

This corollary proves the second part of Theorem 5.4. Now we are going to prove the first part.

Lemma D.9. Fix an $\ell \in \{0, 1, \ldots\}$ and assume that for some $t_\ell \in [s_\ell, e_\ell)$ it holds that $\hat{R}_\ell^t \geq \gamma R_\ell^t$, for some constant $\gamma > 1$. If $R_1^t \in \Theta(t)$ for all $t$, then for all $\ell \in \{0, 1, \ldots\}$

$$t_{\ell} \in O(s_\ell)$$

Proof. Using Lemma D.5 and $\hat{R}_\ell^t \geq \gamma R_\ell^t$ we can easily prove that

$$2(\gamma - 1)R_1^t \leq R_1^t$$

The lemma follows from the facts that $R_1^t = \Theta(t_\ell)$ and $R_1^{s_{t-1}} = \Theta(s_\ell)$.