Exotic glueball $0^{\pm-}$ states in QCD Sum Rules

Alexandr Pimikov,$^1$† Hee-Jung Lee,$^2$ Nikolai Kochelev,$^{1,3}$ Pengming Zhang,$^1$ and Viachaslau Khandramai$^4$

$^1$Institute of Modern Physics, Chinese Academy of Science, Lanzhou 730000, China
$^2$Department of Physics Education, Chonbuk National University, Cheongju, Chungbuk 28644, Korea
$^3$Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Moscow Region, 141980 Russia
$^4$International Center for Advanced Studies, Technical University, Gomel, 246746, Belarus

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The lowest dimension three-gluon currents that couple to the exotic $0^{\pm-}$ glueballs have been constructed using the helicity formalism. Based on the constructed currents, we obtain new QCD SRs that have been used to extract the masses and the decay constants of the scalar exotic $0^{\pm-}$ glueballs. We estimate the masses for the scalar state and for the pseudoscalar state to be $m_+ = 9.8^{+1.4}_{-1.1}$ GeV and $m_- = 6.8^{+1.1}_{-1.2}$ GeV.

I. INTRODUCTION

Glueballs are composite particles that contain gluons and no valence quarks. Theoretically, glueball should exist because of the non-Abelian and confinement properties of Quantum Chromodynamics (QCD), due to the gluon self-interaction and strong “dressing” through vacuum fluctuations. However, there is no clear experimental evidence and glueballs remain undiscovered. Their mixing with ordinary meson states makes it difficult to discover glueballs in an experimental search. Glueball studies are important for phenomenology both at the running and projected large-scale experiments in many research centers: Belle (Japan), BESIII (Beijing, China), LHC (CERN), GlueX (JLAB, USA), NICA (Dubna, Russia), HIAF (China) and FAIR (GSI, Germany).

Theoretical studies of glueballs are only performed within nonperturbative approaches. The bound states of gluons were considered within the lattice QCD, the flux tube model, constituent models, and in the holographic approach. The first study of glueballs in the framework of QCD Sum Rules (SRs) considered a pseudo-scalar $0^{-+}$ state with an obtained mass of $\sim 1$ GeV. Later the same group applied this method to a scalar $0^{++}$ glueball state and estimated its mass to be $\sim 0.7$ GeV. Two-gluon glueballs have been broadly studied using QCD SRs. In further studies, these QCD SRs for the scalar and pseudoscalar glueballs was improved by calculating the direct instanton contribution and the radiative corrections to the perturbative and nonperturbative parts of the correlator. Three-gluon glueballs were considered for a $0^{++}$-state and later in works the application of QCD SRs was extended to the $0^{-+}$ scalar, vector and tensor states. Further reviews of glueball physics can be found in.

A way to avoid problems related to the mixing of glueballs with ordinary mesonic states would be to study glueballs with exotic quantum numbers ($0^{\pm-}$, $1^{-+}$, $2^{++}$,...) which are not allowed in quark-antiquark systems. In our recent study we proposed the $0^{-+}$ glueball current of dimension-12, which was used to obtain estimations of the mass, the decay constant and the width of the $0^{-+}$ glueball.

In this paper, we present for the first time a detailed procedure for the construction of the three-gluon glueball currents based on the helicity formalism following. This procedure is applied to construct the $0^{\pm-}$ glueball currents of the lowest possible dimension. Using these constructed currents, the QCD SRs have been obtained and analyzed to extract masses and decay constants of $0^{\pm-}$ glueballs.

The search for the lowest dimension currents has been motivated by the necessity to improve the reliability of QCD SRs. In comparison with our previous study we presented here have the following improvements: the Operator Product Expansion (OPE) starts from the condensates of lower dimensions so that the uncertainties in the OPE can be reduced; the current of lower dimension leads to a larger coupling with the glueball state; the first resonance contribution to SRs is larger for the current of lower dimension. In fact, in the new QCD SR for the $0^{-+}$ state, the leading nonperturbative contribution comes from the 3-gluon condensate $< G^3 >$, while in our previous study the OPE starts from 4-gluon condensates. From the new SR, we have found that the mass of the $0^{-+}$ glueball is very close to our previous result. At the same time, the coupling of the new current to the glueball state has been found to be significantly larger compared to the dimension-12 current suggested in. Therefore we conclude that the new current better represents the glueball state.

The paper is organized as follows. In Sec. II we present the procedure for constructing of the three-gluon current using the helicity formalism. We construct the currents that couple to the exotic $0^{\pm-}$ glueball helicity states. In Sec. III we present the OPE of correlators of the new currents and present the detailed theoretical scheme of QCD SRs. The masses and decay constant...
of the $0^\pm$ glueballs are extracted then from QCD SRs. Section IV contains the discussion of our results.

II. THREE-GLUON CURRENTS

Here we provide the application of the helicity formalism to the construction of three-gluon currents in general form. The described technique is applied to construct the gauge invariant colorless currents that couple to $0^\pm$ glueball states.

A. Three-gluon helicity states

The gluon field tensor $G_{\mu\nu}$ corresponds to $(1,0)\oplus(0,1)$ representation of the Lorentz group and can be decomposed to positive and negative helicity parts $G_{\mu\nu} = G^+_{\mu\nu} + G^-_{\mu\nu}$, where $G^\pm_{\mu\nu} = (G_{\mu\nu} \pm G_{\nu\mu})/2$ and dual tensor $\tilde{G}_{\mu\nu} = -iG_{\mu\nu3}G^{3\beta}/2$. The negative helicity strength tensor $G^-$ is in the $(1,0)$ representation, and the positive-helicity strength tensor $G^+$ is in the $(0,1)$ representation, thus the different helicity tensors are not mixed under Lorentz transformations. Therefore using helicity strength tensor $G^\pm_{\mu\nu}$ as building blocks allows to decompose the glueball currents into irreducible representations of the Lorentz group. To consider the three gluon helicity current in a general form, we define the generating current as:

$$J(3) \sim 1/3! S_{123}g_3^a S_{G1G2G3}(O_1G_{1\mu\nu}O_2G_{2\mu\nu}O_3G_{3\mu\nu})^a,$$

where $G_i$ with $i = 1,2,3$ stands for the gluon field strength tensor in one of the following forms: the strength tensor $G$, the dual tensor $\tilde{G}$, the positive helicity tensor $G^+$ or the negative helicity tensor $G^-$. The operator of symmetrization $S_{G1G2G3}$ ensures that the current is symmetrical with respect to gluon interchange. The operators $O_i$ with $i = 1,2,3$ are the product of covariant derivatives to respect the gauge invariance of the constructed currents:

$$O_iG_{\mu\nu} = D_{\tau_1}D_{\tau_2}\cdots D_{\tau_n}G_{\mu\nu}. \quad (2)$$

In order to consider both C-parities, we omit here the trace $Tr$ in the color space that will be recovered later to construct colorless currents and insure the gauge invariance. Taking various $O_i$ and ways for the contraction of the Lorentz indices, the currents of various quantum numbers will be generated.

There are two possible combinations to construct helicity-$\lambda$ current $J_3^\lambda$ of the parity-$P$ that are symmetrical with respect to the gluon exchanges: the maximal helicity ($\lambda = 3$) state with the parities $P = \pm 1$:

$$J_3^\pm = J_{+++} \pm J_{--} ,$$

and the minimal helicity ($\lambda = 1$) current with the parities $P = \pm 1$:

$$J_1^\pm = J_{++-} \pm J_{--},$$

where the indices in the currents on the right-hand side of the equation mean the helicities of gluons as $J_{h_1h_2h_3} = J(G_{h_1}G_{h_2}G_{h_3})$ in the general form, see Eq. (11). In the definitions of the maximal and minimal helicity current we have omitted for simplicity the sign C of the arbitrary charge parity $J_3^\pm = J_3^\lambda C$. Expanding the helicity currents in terms of the gluon strength tensor and its dual tensor one finds:

$$J_3^+ = 1/4 \left( J(GGG) + J(\tilde{G}\tilde{G}\tilde{G}) + J(G\tilde{G}G) + J(\tilde{G}GG) \right) ;$$

$$J_3^- = -1/4 \left( J(\tilde{G}\tilde{G}\tilde{G}) + J(\tilde{G}GG) + J(G\tilde{G}G) + J(\tilde{G}GG) \right) ;$$

$$J_1^+ = 1/12 \left( 3J(\tilde{G}\tilde{G}\tilde{G}) - J(GGG) - J(\tilde{G}GG) + J(G\tilde{G}G) \right) ;$$

$$J_1^- = -1/12 \left( 3J(\tilde{G}\tilde{G}\tilde{G}) - J(GGG) - J(\tilde{G}GG) - J(G\tilde{G}G) \right) .$$

In this consideration the three-gluon $0^{++}$ glueball currents represent the maximal ($\lambda = 3$) helicity states $J_3^\pm = J_3^\pm$ while all minimal ($\lambda = 1$) helicity states have $J_1^\pm = 0$. By introducing arbitrary linear operators $O_i$ these currents, Eqs. (3) and (4), can be generalized in following form

$$J(GG\tilde{G}) = J(\tilde{G}GG) = J(G\tilde{G}G) = J(\tilde{G}GG) ,$$

This form of the current has been used in the first QCD SR based study of negative charge parity $0^-\leftrightarrow$ scalar glueballs [31]. One can see that the contraction of the Lorentz indices leads to the following property for this type of currents, Eq. (5):

$$J(\tilde{G}GG) = J(\tilde{G}GG) = J(G\tilde{G}G) = J(\tilde{G}GG) .$$

Therefore such a currents represents maximal helicity states.

B. Three-gluon helicity states of $0^{\pm-}$ glueballs

In order to construct the gauge invariant currents that couple to $0^{\pm-}$ glueballs, we are looking for scalar or unconserved vector local currents. The conserved vector currents correspond to the spurious state and do not couple to the scalar state [3]. Another important requirement to the current is having the nonzero Leading Order (LO) perturbative contribution to the spin-0 part of the correlator. In configuration space, the spin-0 projector in
the correlator is a partial derivative. Therefore, the conserved vector currents have no spin-0 contribution. To eliminate possible ambiguity in the construction of the current and to avoid spurious states, we consider only the currents that are defined by the helicity gluons field strength tensor adopting the helicity formalism [13–33]. To construct the lowest dimension currents from helicity gluons that couple to 0± vector states, we propose the generating current that respects all requirements described above:

\[ J_\alpha (G_1 G_2 G_3) = \frac{2g_3^3}{3!} S_{123} \text{Tr} \left( (D_\mu G_{1\mu}), (D_\nu G_{2\nu}) (D_\sigma G_{3\sigma}) \right), \tag{6} \]

where the factor 2g_3^3 was introduced to have at LO

\[ J_\alpha (GGG) \overset{\text{LO}}{=} g_3^3 d^{abc} (\partial_\mu G_{\mu a})(\partial_\nu G_{\nu b})(\partial_\sigma G_{\sigma c}), \]

that can be easily compared to the currents, Eq. (6), suggested in [26, 28] for 0± glueball states.

The currents Eq. (6) of the maximal (\lambda = 3) helicity state appear to be conserved in LO \partial_\alpha J_{3,\alpha}^{\pm} = 0 and therefore the maximal helicity current does not respect the nonzero LO term condition. While the minimal (\lambda = 1) helicity 0± currents based on the generating current Eq. (6) are non-conserved currents and have all desired properties:

\[ J_\alpha^{\pm} = g_3^3 \text{Tr} \left( (D_\tau G_{\tau\mu}), (D_\lambda G_{\lambda\sigma}) (D_\rho G_{\rho\alpha}) \right), \]

\[ J_\alpha^{-} = g_3^3 \text{Tr} \left( (D_\tau G_{\tau\mu}), (D_\lambda G_{\lambda\sigma}) (D_\rho \tilde{G}_{\rho\alpha}) \right). \]

We propose these currents to study 0± states. The new current for the 0− state has significantly lower dimension than the current suggested recently in [31]. As we see in the next section and discuss in the introduction, the reduction of the dimension leads to improvements of the reliability of QCD SRs: it reduces the OPE uncertainty and increases the coupling with the state and the first resonance contribution to SR.

Any other choice of the dimension-9 generating current Eq. (6) leads to the zero current or to the alternative current \( J_{\alpha,\text{alt}}^{\pm} \) that has identical coupling to spin-0 state in LO: \( \partial_\alpha J_{\alpha,\text{alt}}^{\pm} \overset{\text{LO}}{=} \partial_\alpha J_\alpha^{\pm} \). Using the gluon field tensors and covariant derivatives to ensure a gauge invariance of the current, we did not find any three-gluon current of dimension-7 and dimension-8 that respect above requirement. Applying the helicity formalism to the four-gluon states leads us to the conclusion that there is no nonzero helicity current of dimension-8 which couples to the exotic 0− glueballs.

### III. SUM RULES

#### A. OPE of correlators

Here we present the result for OPE of correlators that is the theoretical basis of QCD SRs approach [19]:

\[ \Pi_{\mu\nu}^{\pm}(q) = i \int d^4x \, e^{i x \cdot q} \langle J_\mu^{\pm}(x) J_\nu^{\pm}(0) \rangle \]

where the proposed current \( J_\mu^{\pm} \) is given by Eq. (7) and couples to the gluonic bound state \( |G(0^{\pm})\rangle \) with the mass \( m_{\pm} \) and the decay constant \( f_{\pm} \) through the relation:

\[ \langle 0 | J_\mu^{\pm} | G(0^{\pm}) \rangle = p_\alpha f_{\pm} m_{\pm}^6. \]

The correlators of the vector currents have two components:

\[ \Pi_{\mu\nu}^{\pm}(q) = \Pi_{}^{(1)\pm}(q^2)\langle|q_\mu q_\nu - q^2 g_{\mu\nu}\rangle + \Pi_{}^{(0)\pm}(q^2)q_\mu q_\nu, \]

where \( \Pi_{}^{(0)} \) and \( \Pi_{}^{(1)} \) are spin-0 and spin-1 contributions, respectively.

Here we consider only the spin-0 part of the correlator OPE up to dimension-8 condensates:

\[ \Pi_{}^{(0)\pm}(\text{OPE}) = \Pi_{}^{(0)\pm}(\text{pert}) + \Pi_{}^{(0)\pm}(\text{G2}) + \Pi_{}^{(0)\pm}(\text{G3}) + \Pi_{}^{(0)\pm}(\text{G4}) + \cdots, \]

where the following terms are considered: the LO perturbative term (pert), the dimension-4 (G2), the dimension-6 (G3), and dimension-8 (G4) nonperturbative terms. The terms of the correlator OPE have been calculated and are given as follows:

\[ \Pi_{}^{(0)\pm}(\text{pert}) = -\frac{5\alpha_s^3}{918\pi^2} g_3^3 \ln\frac{-q^2}{\mu^2}, \quad \Pi_{}^{(0)\pm}(\text{G2}) = 0, \]

\[ \Pi_{}^{(0)\pm}(\text{G2}) = \pm \frac{5\alpha_s^2}{2^{33/2} 3^1} G_3^2 \cdot q^6 \ln\frac{-q^2}{\mu^2}, \]

\[ \Pi_{}^{(0)\pm}(\text{G4}) = \pm \frac{\alpha_s^2}{2^{33/2}} \langle\alpha_s^2 G^4\rangle \pm q^4 \ln\frac{-q^2}{\mu^2}, \]

where \( \alpha_s = g_3^2/(4\pi) \) is the coupling constant, \( \mu \) is the renormalization scale. The contributions \( G_3^2 \) and \( \langle\alpha_s^2 G^4\rangle \) are linear combinations of the dimension-6 and dimension-8 condensates described below. We adopt Mathematica package FEYNCALC [37] to handle the algebraic manipulation. The LO perturbative term is represented by the two-loop sunset diagram (the first diagram in Fig. 1), therefore for any scalar three-gluon current the largest prime divisor of denominator must be less than the dimension of the current. The leading nonperturbative contribution from the nonlocal two gluon condensate [38–41], represented by second diagram in Fig. 1, is defined by the dimension-6 local condensates thanks to the derivatives in the currents:

\[ G_3^2 = 9(g_3^3 G^3) - 88\pi\alpha_s\langle j^2 \rangle, \]

\[ G_3^4 = 9(g_3^3 G^3) - 20\pi\alpha_s\langle j^2 \rangle, \]
where notations for condensates of dimension-6 are $\langle g^3 G^3 \rangle = \langle g^3 f^{abc} G^a_{\mu\nu} G^b_{\nu\rho} G^c_{\rho\mu} \rangle$ and $\langle J^2 \rangle = \langle J^a J^a \rangle$ with the quark current $J^a_{\mu} = q^a_{\mu} \bar{q} q$. For the same reason, the leading term of the third and fourth diagrams shown in Fig. 1 is dimension-8 contribution. While the last diagram in Fig. 1 starts from dimension-12 condensate and therefore has been included in the QCD SRs analysis. Therefore, the quarks contribute only perturbatively due to the strong coupling evolution as it is discussed below (see Eq. (12)).

The total dimension-8 condensate contribution to the correlator is presented by the four-gluon condensates:

$$\langle \alpha_s^2 G^4 \rangle_+ = \frac{155}{2} \langle \alpha_s f^{abc} G^a_{\mu\nu} G^b_{\nu\rho} G^c_{\rho\mu} \rangle,$$

$$\langle \alpha_s^2 G^4 \rangle_- = \frac{845}{2} \langle \alpha_s f^{abc} G^a_{\mu\nu} G^b_{\nu\rho} G^c_{\rho\mu} \rangle,$$

where quark-gluon condensates have been omitted. As expected, the nonperturbative terms in the approximation of self-dual (SD) gluon fields are in equal value and have different signs (see Eq. (9)) for the parity $P = \pm 1$:

$$G_{G3}^{\pm} \stackrel{SD}{=} 9 \langle \alpha_s f^{abc} G^a_{\mu\nu} G^b_{\nu\rho} G^c_{\rho\mu} \rangle,$$

$$\langle \alpha_s^2 G^4 \rangle_+ \stackrel{SD}{=} \frac{2}{5} \frac{2}{32} \langle \alpha_s f^{abc} G^a_{\mu\nu} G^b_{\nu\rho} G^c_{\rho\mu} \rangle.$$

For QCD SRs analysis we apply the hypothesis of vacuum dominance (HVD) to estimate the dimension-8 condensate:

$$\langle \alpha_s^2 G^4 \rangle_+ \stackrel{HVD}{=} k_{HVD} \frac{3}{2},$$

$$\langle \alpha_s^2 G^4 \rangle_- \stackrel{HVD}{=} k_{HVD} \frac{3}{2},$$

where $k_{HVD}$ denotes the coefficient of the HVD factorization violation. We vary this coefficient in the range $k_{HVD} \in [0.25, 4]$ to include the HVD-related uncertainty. Evaluating QCD SRs, we apply the results of recent studies \cite{42, 43} where the charmonium moments sum rules has been used to obtain the gluon condensate estimations:

$$\langle g^3 G^3 \rangle = (8.2 \pm 2.0) \text{GeV}^2 \langle \alpha_s G^2 \rangle,$$

$$\langle \alpha_s G^2 \rangle = 0.07(2) \text{ GeV}^4.$$

The ratio between the three-gluon and the two-gluon condensates agrees well with the instanton model \cite{42} for the instanton radius $\rho_c = 1/(600 \text{ MeV})$:

$$\langle g^3 G^3 \rangle = \frac{48\pi}{3\rho_c^2} \langle \alpha_s G^2 \rangle.$$

Due to the large value of the Borel parameter $M^2$ in QCD SRs (see bellow) for exotic glueballs the possible direct instanton contributions to the correlators are expected to be strongly suppressed in comparison to OPE terms and therefore are not considered here.

### B. QCD SRs

We analyze the constructed QCD SRs for the $0^{\pm -}$ states on the same footing. Therefore here and below for simplicity we omit the parity and the spin signs $\Pi_t^{(0)^+} \rightarrow \Pi_t$, where $t$ denotes the different contributions to OPE of the correlator as explained above Eqs. (8). In simplified notation the truncated OPE of correlator has a form:

$$\Pi_{\text{OPE}} = \Pi_{(\text{pert})} + \Pi_{(G3)} + \Pi_{(G4)}.$$

The phenomenological part of QCD SR is based on the modeling of spectral density. For the phenomenological description of the correlator, we use the one-resonance model with the continuum contribution modeled by Impart of the correlator OPE:

$$\text{Im}\Pi_{(\text{ph})}(s) = \pi m^{12} f^2 \delta(s - m^2) + \Theta(s - s_0) \text{Im}\Pi_{(\text{OPE})}(s),$$

where $m$ is the mass of a resonance and $s_0$ is the continuum threshold. Then QCD SR reads

$$\frac{1}{\pi} \int_{0}^{s_0} \frac{\text{Im}\Pi_{(\text{OPE})}(s)}{s + Q^2} ds = \frac{f^2 m^{12}}{m^2 + Q^2}.$$
In the framework of QCD SRs [19], the Borel transform \(\hat{B}\)
\[
\hat{B}_{Q^{2} \to M^{2}}[\Pi(Q^{2})] = \lim_{n \to \infty} \frac{(-Q^{2})^{n}}{\Gamma(n)} \left[ \frac{d^{n}}{dQ^{2n}} \Pi(Q^{2}) \right]_{Q^{2}=nM^{2}},
\]
is applied to both sides of the SR, Eq. (11) in order to reduce the SR uncertainties by suppressing the contributions from excited resonances and higher order OPE terms. The Borel transformation modifies the components of the sum rule:
\[
R_{0}^{t}(M^{2}, s_{0}) = \frac{1}{\pi} \int_{0}^{s_0} ds \text{Im}\Pi_{t}(-s) e^{-s/M^{2}},
\]
\[
R_{0}^{(\text{res})}(M^{2}) = m^{2} f^{2} e^{-m^{2}/M^{2}}.
\]
Here we follow common practice of renormalization group improvement after Borel transformation, therefore in \(\text{Im}\Pi_{t}(-s)\) all coupling constant are replaced by running constant \(\alpha_{s} \to \alpha_{s}(M^{2})\):
\[
\alpha_{s}(Q^{2}) = \frac{4\pi}{b_{0} \ln(Q^{2}/\Lambda_{\text{QCD}}^{2})},
\]
where the beta-function LO coefficient \(b_{0} = 11 - 2N_{f}/3\), the QCD scale \(\Lambda_{\text{QCD}} = 350\) MeV, and number of the flavors \(N_{f} = 4\).

The mass is extracted from the family of the derivative SRs defined by
\[
R_{k}^{(\text{SR})}(M^{2}, s_{0}) = M^{4} \frac{d}{dM^{2}} R_{k-1}^{(\text{pert})}(M^{2}, s_{0}).
\]
Denoting by \(R^{(\text{SR})}\) the difference of the OPE result and the continuum contribution for any \(k \geq 0\):
\[
R_{k}^{(\text{SR})}(M^{2}, s_{0}) = R_{k}^{(\text{pert})}(M^{2}, s_{0}) + R_{k}^{(G3)}(M^{2}, s_{0}) + R_{k}^{(G4)}(M^{2}, s_{0}).
\]
we define the master sum rule \((k = 0)\) and the derivative SRs \((k > 0)\) by the following equations:
\[
R_{k}^{(\text{SR})}(M^{2}, s_{0}) \approx R_{k}^{(\text{res})}(M^{2}, s_{0}).
\]

The high dimension of the considered currents leads to the dependency of the continuum spectral density on \(s\) as \(\text{Im}\Pi_{(\text{OPE})} \sim s^{\alpha}\). Therefore, having in mind that the continuum contribution could give the large contribution \([43, 47]\), we define the upper boundary \(M^{2}_{\text{up}} < M^{2}_{\text{SR}}(s_{0})\) of the fiducial window by the following condition which is less restrictive than the condition for the low-dimension correlators suggested in [13]:
\[
\frac{\mathcal{R}_{k}^{(\text{res})}(M^{2})}{\mathcal{R}_{k}^{(\text{pert})}(M^{2}, \infty)} \approx \frac{\mathcal{R}_{k}^{(\text{SR})}(M^{2}, s_{0})}{\mathcal{R}_{k}^{(\text{SR})}(M^{2}, \infty)} > \frac{1}{10}.
\]

This condition influences the definition of the SR uncertainty, while the central values of predictions appear to be insensitive to it. The lower boundary \(M^{2}_{\text{down}}\) of the fiducial window \(M^{2} \in [M^{2}_{\text{down}}, M^{2}_{\text{up}}(s_{0})]\) is limited by the conditions
\[
\frac{|\mathcal{R}_{k}^{(G3)}(M^{2}, \infty)|}{\mathcal{R}_{k}^{(\text{pert})}(M^{2}, \infty)} < \frac{2}{3}, \quad \frac{|\mathcal{R}_{k}^{(G4)}(M^{2}, \infty)|}{\mathcal{R}_{k}^{(\text{pert})}(M^{2}, \infty)} < \frac{1}{3},
\]
that insure the OPE reliability.

The values of mass and the decay constant can be extracted from QCD SRs, Eq. (13), as:
\[
m_{k}(M^{2}, s_{0}) = \sqrt{\mathcal{R}_{k+1}^{(\text{SR})}(M^{2}, s_{0}) / \mathcal{R}_{k}^{(\text{SR})}(M^{2}, s_{0})},
\]
\[
f_{k}^{2}(M^{2}, s_{0}) = e^{M_{G}^{2}/2M^{2}} R_{k}^{(\text{SR})}(M^{2}, s_{0}) / M_{G}^{2(6+k)}.
\]

We define the mass and the decay constant by keeping the \(M^{2}\)-stability criteria \(\delta_{k}\) below \(10% \sim 1/3^{2}\) that is the assumed OPE accuracy related to the condition Eq. (15):
\[
\delta_{k} = \frac{\max f_{k}^{2}(M^{2}, s_{0}) - \min f_{k}^{2}(M^{2}, s_{0})}{\max f_{k}^{2}(M^{2}, s_{0}) + \min f_{k}^{2}(M^{2}, s_{0})} < 1/10.
\]
TABLE I: QCD SRs results on the masses, the decay constants of $0^{\pm -}$ glueballs for the $k = 0$ case are presented together with corresponding fiducial intervals of the SR parameters: the Borel parameter $M^2$ (the upper limit is given for the central value of the threshold) and the threshold value $s_0$. The fiducial intervals have been defined through the conditions Eqs. (14,15,16). Three given uncertainties of the mass and the decay constant values are estimated by the Borel parameter $M^2$ variation, the threshold $s_0$ dependence and the gluon condensates uncertainties.

| state | mass, GeV | decay constant, MeV | $M^2$, GeV$^2$ | $s_0$, GeV$^2$ |
|-------|-----------|---------------------|---------------|---------------|
| $0^{-}$ | $6.84 \pm 0.36 \pm 0.44 \pm 0.26$ | $1.34 \pm 0.04 \pm 0.07 \pm 0.02$ | [9.2, 21.5] | 78 $\pm 10$ |
| $0^{+-}$ | $9.23 \pm 0.56 \pm 0.38 \pm 0.40$ | $0.93 \pm 0.02 \pm 0.16 \pm 0.02$ | [9.2, 30.0] | 120 $\pm 14$ |

This condition puts limits on the continuum threshold value $s_0$. The conditions Eqs. (14,15,16) define the fiducial set of $(M^2, s_0)$-values. Finally we define the prediction for the mass and the decay constant as an average of the maximal and the minimal values on the fiducial interval of $M^2$ with the fixed central value of threshold given in the last column of Table I

$$m_k = \frac{\max m_k(M^2, s_0) + \min m_k(M^2, s_0)}{2},$$

$$f_k^2 = \frac{\max f_k^2(M^2, s_0) + \min f_k^2(M^2, s_0)}{2}.$$

The variation of the mass and the decay constant in the fiducial $(M^2, s_0)$-set defines uncertainties coming from the OPE truncation and the spectral function modeling.

C. QCD SRs results for the $0^{\pm -}$ glueball states

Performing the QCD SRs analysis described above, we obtain predictions for the masses and decay constants of the $0^{\pm -}$-states. These are presented in Table I for the $k = 0$ case together with the fiducial intervals of the SR parameters: the Borel parameter $M^2$ and the threshold value $s_0$.

There are three sources of errors for mass and decay constant presented in Table I the first error represents the SR stability triggering Borel parameter $M^2$ dependence, the second represents the threshold $s_0$ dependence and the third is the uncertainty related to the variations of the gluon condensates $(G^0)$ and $(G^1)$. The first two errors, which originate from OPE truncation and continuum modeling, are defined by variation of results on the fiducial $(M^2, s_0)$-set that represents the conditions Eqs. (14,15,16). The variation of the $(G^0)$ condensate comes from [12, 43] (see Eq. (10)). The uncertainties of the $(G^1)$ have been estimated from the variation of the HVD violation coefficient (see Eq. (39)) and the variation of the two-gluon condensate $(G^2)$ was estimated in [42, 43]. In Fig. 2 we present the $k = 0$ results for the glueball mass and the decay constant as a function of the Borel parameter for various values of the threshold parameter. As one can see, there is rather a good stability plateau for both quantities which is ensured by the condition in Eq. (10). The masses and decay constants estimated with the higher values of $k = 1, 2, 3$ are in agreement with the $k = 0$ case.

IV. SUMMARY

We have performed a study of C-odd scalar and pseudoscalar exotic glueball states within the framework of QCD SRs. The constructed QCD SRs include LO perturbative term and the nonperturbative contributions up to dimension-8 gluon condensates. The results from the QCD SRs analysis on the masses $m_\pm$ and decay constants $f_\pm$ of the $0^{\pm -}$ glueballs are given as follows: for the pseudoscalar state

$$m_-= 6.8^{+1.1}_{-1.0} \text{ GeV},$$

$$f_- = 1.3 \pm 0.1 \text{ MeV}, \quad (17)$$

and for the scalar state

$$m_+= 9.2^{+1.3}_{-1.4} \text{ GeV},$$

$$f_+= 0.9 \pm 0.1 \text{ MeV}.$$

The construction of the three-gluon currents has been addressed in general form on the basis of the helicity formalism. The developed techniques of the helicity based current construction have been used to build new three-gluon currents of minimal dimension that couple to the $0^{\pm -}$ glueball states.

Our previous QCD SRs results [31] on the $0^{-+}$-glueball mass using a dimension-12 current, $M_G = 6.3^{+0.8}_{-1.1} \text{ GeV}$, is in good agreement with our new estimation, Eq. (17). As one would expect, the current with higher dimension leads to a smaller coupling to the dimension-12 current with the glueball state. [31]: $F_G = 67 \pm 6 \text{ keV}$. Therefore, the new current of minimal dimension represents the most possible configuration of $0^{-+}$ glueball.

The Belle Collaboration [1] has performed a search in the range of masses lower than our predicted mass and found no evidence for the exotic $0^{-+}$ glueball. Our result on the mass of the exotic $0^{-+}$ glueball is in qualitative agreement with the result of lattice QCD [8]. On the other hand, the obtained mass of the $0^{-+}$ glueball state is noticeably larger than the lattice results [4, 5]. Unfortunately, the status of exotic glueball masses calculated using lattice QCD not clear at the present time (see the discussion in [8] and Table 3 therein). Some lattice groups have seen exotic glueball signals, while others found no indication of any signals for the same exotic states. Furthermore, in [8] it was emphasized that lattice QCD calculations using heavy glueball degrees of
freedom should use improved techniques to assign $J^{PC}$ quantum numbers. Due to these issues in lattice QCD, it is not a problem that our calculations do not match theirs.

A recent study [40] within QCD SR for the exotic $0^{−−}$ tetraquark with light quark content predicted a small mass, $M_{\text{tetra}} = 1.66 \pm 0.14$ GeV. Therefore, the large mass difference should lead to a very small mixing between this light tetraquark state and the heavy exotic $0^{−−}$ glueball. However, one cannot avoid the discussion about possible mixing between the exotic glueball states and the heavier tetraquarks of the same exotic quantum number, if such heavy tetraquarks exist. But we would like to point out to our knowledge, all estimations within various models give the value of the mass for the hidden-charm tetraquarks to be around 4 GeV (see review [48]), which is rather small in comparison to our glueball masses. In principle, the exotic glueballs can also mix with the hidden-charm hybrid which has the same quantum numbers. The recent lattice calculation for $0^{++}$ hybrid predicts the mass to be around 4.4 GeV [49]. Since glueball masses. In principle, the exotic glueballs can also mix with the hidden-charm hybrid which has the same quantum numbers. The recent lattice calculation for $0^{++}$ hybrid predicts the mass to be around 4.4 GeV [49]. Since there is a large mass gap between the $0^{−−}$ glueball and the exotic hadrons with hidden-charm, the mixing of the exotic glueballs with hidden-charm states is expected to be small. In any case, the calculation of the mixing between different exotic states is very complicated due to contributions coming from both the perturbative and nonperturbative sectors of QCD and such kinds of studies are out of the scope of the present work.

The decay of the three-gluon state to the hadrons is suppressed by the large power of the strong coupling at the virtuality of the glueball’s gluons $Q^2 \sim 4$ GeV$^2$, where we assume that the gluons carry equal momenta. One of the allowed channels includes charmonium in the final state. In particular, we consider the S-wave decay of glueball $G(0^{−−}) \rightarrow f_1(1285)+J/\Psi$ to be the most preferable due to the large glueball mass and the small widths of the final particles. Additionally, this channel could be enhanced by the decay of the hidden charm tetraquark. Therefore, charmonium data could be a good place to search for experimental evidence of exotic glueballs.

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