Low Complexity Belief Propagation Polar Code Decoders

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Abstract—Since its invention, polar code has received a lot of attention because of its capacity-achieving performance and low encoding and decoding complexity. Successive cancellation decoding (SCD) and belief propagation decoding (BPD) are two of the most popular approaches for decoding polar codes. SCD is able to achieve good error-correcting performance and is less computationally expensive as compared to BPD. However SCDs suffer from long latency and low throughput due to the serial nature of the successive cancellation algorithm. BPD is parallel in nature and hence is more attractive for high throughput applications. However since it is iterative in nature, the required latency and energy dissipation increases linearly with the number of iterations. In this work, we borrow the idea of SCD and propose a novel scheme based on sub-factor-graph freezing to reduce the average number of computations as well as the average number of iterations required by BPD, which directly translates into lower latency and energy dissipation. Simulation results show that the proposed scheme has no performance degradation and achieves significant reduction in computation complexity over the existing methods.

Keywords— Belief propagation decoding (BPD); successive cancellation decoding (SCD); energy efficiency; iterative decoders; factor graph; polar codes

I. INTRODUCTION

The existence of codes which enable reliable data transmission at the maximum rate, called the channel capacity, was proven to exist by Shannon [1]. Since then, different capacity-approaching codes have been designed, like Turbo codes [2] and LDPC [3] codes. The first provable capacity-achieving code, polar code, was recently invented by Arikan [4]. Polar code is considered to be a major breakthrough in coding theory, since it is the first family of codes known to achieve channel capacity with explicit construction. Besides achieving the capacity for binary-input symmetric memoryless channels [4], polar codes have also been proved to be able to achieve the capacity for any discrete and continuous memoryless channel [5]. Moreover, explicit construction method for polar codes was provided and it was shown that they can be efficiently encoded and decoded with complexity $O(n \log n)$, where $n$ is the code length. Since then, polar codes have become one of the popular topics in information theory and have attracted a lot of attention recently.

Polar code decoding can be roughly divided into two categories: successive cancellation decoding (SCD) and its variants [6]-[13] and belief propagation decoding (BPD) [14]-[20]. SC decoders suffer from long latency and low throughput due to the serial nature of the SC algorithm. However, the SC algorithm requires less computation as compared to BPD. Based on this property, several area-efficient SC decoders have been reported in [9]-[11]. Another advantage of the SC algorithm is its ability to achieve very good error-correcting performance. With the aid of the list-decoding or stack-decoding strategy, improved SC algorithms were introduced in [12]-[13], which can achieve better performance.

On the other hand, polar BP decoders [14]-[20] have unique advantage of parallel processing. Therefore, compared with their SC counterparts, polar BP decoders are more attractive for high-throughput applications. For iterative decoders (such as polar BP decoders), their required latency and energy dissipation increase linearly with the number of iterations. However, the need of a large number of iterations makes BP decoders suffer from high computation complexity, and hence polar BP decoders are still not as attractive as their SC counterparts.

To address the issues of large number of iterations and high computation complexity inherent in BP decoders, Yuan et al. [18] proposed a G-matrix-based early stopping scheme, which is based on the fact that iterative decoders normally converge earlier than reaching a fixed maximum number of iterations. The G-matrix-based stopping criterion can then be used to stop the computation if convergence has been reached. To further reduce the computation complexity, in this paper, we propose a method based on the convergence of the sub-factor-graphs, which is reached at a much earlier stage. Borrowing the idea from SCD, some of the sub-factor-graphs are checked during each iteration and if they are converged, they are frozen and do not need to be computed in the subsequent iterations. Also the freezing of these sub-factor-graphs will help to improve the convergence of the rest of the factor graph. As a result, the computation complexity and also the average number of iteration are reduced. Experimental results show that our proposed method results in about 42 ~ 47 % less computation complexity as well as lower latency when compared to the previously proposed early stopping scheme [18].

Notations

In this paper, the following notation conventions are used. Matrices are denoted in boldface capital letters, and vectors in boldface lowercase letters. The subscript $G_M$ of a matrix represents a $M \times M$ square matrix and $v_m$ denotes a $M \times 1$ vector. $x[i]$ stands for the $i$th element of vector $x$, $x^t$ stands for vector $x$ at the $t$th iteration and $x_{a,b}^t$ represents the sub-vector of $x$ with the starting and ending index of $a$ and $b$. Transpose of a vector is denoted by $x^T$.

II. POLAR CODES OVERVIEW

Polar codes are based on the phenomenon of “channel polarization”. More precisely, by recursively combining and
splitting individual channels, some of these channels become essentially error-free, while others become completely noisy. Furthermore, the fraction of the noiseless channels tends towards the capacity of the underlying binary symmetric channels [4]. Therefore, an \((n, k)\) polar code can be generated in two steps. First, an \(n\)-bit message \(u\) is constructed by assigning the \(k\) reliable and \((n-k)\) unreliable positions as information bits and “0” bits, respectively. \((n-k)\) unreliable positions, which are forced to 0, are called the frozen bits (also known as the frozen set). Then, the \(n\)-bit \(u\) is multiplied with the generator matrix \(G = F^\otimes m\) to generate an \(n\)-bit transmitted codeword \(x\), where \(F^\otimes m\) is the \(m\)th Kronecker power of \(F = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\) and \(m = \log_2 n\). Fig. 1 shows the encoding signal flow graph for \(n = 8\), where the “⊕” sign represents the XOR operation.

### A. Belief Propagation (BP) Algorithm for Polar Code Decoding

As presented in [19], similar to LDPC codes, polar codes can be decoded by applying the belief propagation (BP) algorithm over their factor graphs. For an \((n, k)\) polar code \((n = 2^m)\), its factor graph is an \(m\)-stage network consisting of \(n \times (m+1)\) nodes, where each node is associated with a right-to-left and a left-to-right likelihood message denoted by \(L^t_{i,j}\) and \(R^t_{i,j}\), respectively. \(L^t_{i,j}\) denotes the right to left likelihood message of the \(i^{th}\) node at the \(j^{th}\) stage and the \(t^{th}\) iteration. Fig. 2 (a) shows an example of a 3-stage factor graph for \(n = 8\) polar codes. Here each stage consists of \(n/2 = 4\) processing elements (PEs). During the BP decoding procedure, these messages are propagated and updated among adjacent nodes using the min-sum updating rule as shown by the following equations [18]:

\[
L^t_{i,j} = \min \{ \alpha \cdot \text{sign}(L^t_{i,j+1}) \text{sign}(R^t_{i,j}) \}
\]

\[
R^t_{i,j+1} = \alpha \cdot \text{sign}(L^t_{i,j+1}) \text{sign}(L^t_{i,j+1} + R^t_{i,j+1})
\]

\[
R^t_{i+\frac{n}{2}j+1} = R^t_{i+\frac{n}{2}j} + \alpha \cdot \text{sign}(L^t_{i+\frac{n}{2}j+1}) \text{sign}(R^t_{i+\frac{n}{2}j})
\]

where \(L^t_{i,j}\) and \(R^t_{i,j}\) are the right-to-left and left-to-right logarithmic likelihood ratio (LLR)-based message at the \(t^{th}\) iteration, respectively. \(\alpha\) is a scaling parameter introduced in [20] for the improvement of the decoding performance of a BP decoder. Based on the equations in (1), the likelihood messages are propagated and updated from left to right and right to left iteratively in the factor graph. After the number of iteration reaches the specific maximum number \((\text{max}_\text{iter})\), node \((i, m+1)\) will output the decoded bit \(u[i]\) based on the hard decision of the messages \((R^\text{max}_\text{iter})_{i,m+1}\).

### III. THE PROPOSED SCHEME

Fig. 3(a) shows the scheduling tree of the Successive Cancellation decoding (SCD) of the \((8,4)\) polar code [21]. Fig. 3(b) depicts the equivalent BPD factor graph of the same \((8,4)\) polar code. For the SCD scheduling tree, at each stage it is split into a number of sub-trees, each of which is responsible for decoding a corresponding constituent code [6]. The size of the sub-tree varies at each level and is reduced by half when moving from one stage to another stage.

Before presenting the details of our proposed scheme, we first introduce the notion of connected sub-factor-graph. A connected sub-factor graph (CSFG) is defined as a sub-factor-graph which has the same number of inputs and outputs where the output nodes are at the stage \(m+1\), and each input is connected to each output through some PEs in the sub-factor-graph. Figure 3(b) shows two examples of CSFGs. It can also be seen that each CSFG has a corresponding sub-tree in the scheduling tree of SCD. Fig. 3 (a) and (b) show examples of the corresponding sub-trees and the connected sub-factor-graph of the \((8,4)\) polar code. The number of CSFG at each stage is given by \(2^j\), where \(j\) is the stage number. For the \((8,4)\) polar code, as shown in Figs. 3 (c) and (d), the numbers of CSFG at the stages 1 and 2 are 2 and 4, respectively.

At each iteration \(t\), the nodes at stage \(j\) in the BPD factor graph output left to right LLR-based propagating
messages $r_{1,m,j+1}^t$ and these are the inputs to the $2^j$ CSFGs at stage $j$. $R_{1,2^m-1,j+1}^t$ are the inputs to the first CSFG while $R_{(k-1)2^m-1,(k2^m-1),j+1}^t$ are that for the $k^{th}$ CSFG. Each CSFG is responsible for the decoding of the corresponding constituent code from its respective input messages.

The proposed scheme borrows the idea from the successive interference cancelation (SCD), where the results of the previous-decoded bits are used for the decoding of the current bit. Here we introduce a CSFG freezing concept for a low complexity BPD. At a particular iteration $t$, when the message passing reaches a certain stage $j$, if a CSFG at that stage can correctly decode its corresponding constituent code (i.e. the CSFG has reached convergence), it is frozen and no message passing and updating within the CSFG will be needed in the subsequent iterations. The details of how to check whether a CSFG can be frozen will be presented later.

One important thing is the order of checking for freezing of the CSFG. A CSFG can only be frozen if all the previous CSFGs (in the order of the decoding bits) at that stage have been frozen. If a CSFG is not frozen, that means the message values inside it will still be changed in the subsequent iterations. Similar to the SCD operation, the message values of this CSFG will be used for the decoding of the constituent codes of the subsequent CSFGs. Therefore the freezing of the CSFGs at a stage has to follow the order based on the decoded bit. When a CSFG at a certain stage is checked for freezing, if it cannot correctly decode its constituent code, then it cannot be frozen and the message passing and updating have to be executed for the subsequent PEs at that stage. After that we move to the next stage and check the convergence of the corresponding CSFGs. When we move to the next stage, the number of CSFG will be doubled. This freezing checking procedure will continue from stage to stage until the end of the BPD factor graph is reached.

Next we will present how we can freeze a CSFG. As discussed above, a CSFG is corresponding to a sub-tree in the SCD scheduling tree, which can also be viewed as a constituent code of the original polar code. At the $t^{th}$ iteration and stage $j$, the left to right propagation messages $R_{(k-1)2^m-1,(k2^m-1),j+1}^t$ connected to the $k^{th}$ CSFG can be viewed as the LLR inputs to decode the corresponding constituent code. We can apply Maximum-Likelihood Decoding (MLD) on this constituent code with $R_{(k-1)2^m-1,(k2^m-1),j+1}^t$ as input to obtain a decoded output vector $(u_{(k-1)2^m-1,(k2^m-1)},j+1)$, which is a sub-vector of the source word $(u_m)$ of the original polar code. As will be shown later, if the freezing of CSFGs follows the proposed order, the input messages of CSFG $R_{(k-1)2^m-1,(k2^m-1),j+1}^t$ are reliable enough and MLD $(u_{(k-1)2^m-1,(k2^m-1)},j+1)$, based on these input messages can be taken as decoded result of this constituent code. The freezing order of the CSFG has to follow the decoded bit order, the top CSFGs at each stage will be frozen first.

Fig. 4 shows the SCD scheduling tree and the factor graph of the (8,4) polar code. We can see that the top CSFGs are actually corresponding to the first few sub-trees that follow the depth-first traversal of the SCD scheduling tree. At the first iteration, the input messages to these CSFGs are the same as the input LLR messages of the corresponding SCD sub-trees. Hence if we can decode the input messages of this CSFG using MLD, the decoding performance on the corresponding constituent code will achieve or even exceed that of SCD. If the CSFG cannot be frozen at this iteration, and need further iteration to converge, due to the nature of the iterative decoding, the reliability of the input messages to this CSFG will become better and hence the input LLR messages of this CSFG will be more reliable than input messages to the SCD sub-tree. As a result the MLD performance will not be worse than that of SCD.

The MLD is based on an exhaustive search and hence it has a huge complexity. To this end, a novel checking criteria is suggested to efficiently find the MLD result of the constituent code with much lower complexity. Let $R_{1,2^m-1,j+1}^t$ be the left to right propagation messages of a CSFG at stage $j$, we obtain a...
Given $\hat{x}_{2m-j}$ as input to the CSFG, the decoded bit vector at its output $\hat{u}_{2m-j}$, which is also a sub-vector of the source word of the original polar code $\hat{u}_m$, is obtained by the inverse operation of polar code encoding that is given as:

$$\hat{u}_m = \begin{cases} 0 & \text{if } R_{1:2m-j/i+1} > 0 \\ 1 & \text{if } R_{1:2m-j/i+1} < 0 \end{cases}$$

(2)

Figure 5: Example illustrating the Proposed Scheme

The following lemma shows that if the frozen set criteria (4) is satisfied, the sub-source-word vector $\hat{u}_{2m-j}$ obtained by (3) is indeed the decoding results of the MLD on the corresponding constituent code of the CSFG.

**Lemma 1:** Let $R_{1:2m-j/i+1}$ and $\hat{x}_{2m-j}$ be the input LLR messages and hard decision vector based on (2) for corresponding CSFG at the $j^{th}$ stage. If $\hat{u}_{2m-j}$ is obtained from $\hat{x}_{2m-j}$ based on (3) and it satisfies the frozen-set criteria of (4), then $\hat{u}_{2m-j}$ is the maximum likelihood detection (MLD) result of the corresponding constituent code with input messages $R_{1:2m-j/i+1}$.

**Proof:** The CSFG at the $j^{th}$ stage represents a short polar (constituent) code of length $2^m/j$. Its input and output are related by $x_{2m-j} = u_{2m-j} \otimes (m-j)$. From [21] and [22], given the input LLR $R_{1:2m-j/i+1}$, the likelihood value of an arbitrary source word $u_{2m-j}$ is given by:

$$
\sum_{i=1}^{2^m-j} (1 - 2x_{2m-j}[i])R_{1:2m-j/i+1}[i] \quad \text{where } x_{2m-j} = u_{2m-j} \otimes (m-j).
$$

(3)

If no source word bit is frozen, i.e., $u_{2m-j} \in \{0,1\}^{2^{m-j}}$, the source word $\hat{u}_{2m-j}$ obtained from $\hat{x}_{2m-j}$ has the maximum likelihood value which equals to $\sum_{i=1}^{2^m-j} |R_{1:2m-j/i+1}[i]|$. If certain source word bit is frozen bit, the searching space of valid source word is smaller and $\sum_{i=1}^{2^m-j} |R_{1:2m-j/i+1}[i]|$ may not be achieved. However, if $\hat{u}_{2m-j}$ satisfies (4), this likelihood value is achievable and the source word $\hat{u}_{2m-j}$ is a valid source word. Hence, $\hat{u}_{2m-j}$ is the MLD result.

When a CSFG at stage $j$ is frozen, the corresponding computations and messages updating are not needed for the rest of the iterations. We can also fix its right to left feedback propagating messages ($L_{1:2m-j/i+1}$) for the rest of iterations based on its $\hat{x}_{2m-j}$ since the output decoding decision for this CSFG has already been made and we have

$$L_{1:2m-j/i+1} = (-1)^{\hat{x}_{2m-j}}$$

(4)

During the left to right propagation iteration, for any CSFG, if the frozen set criteria checking (4) is not satisfied, then we cannot freeze this CSFG. We then update the messages at this stage using equation (1) and move to the next stage and repeat the same procedure. Fig. 5(b) shows an example. At the second iteration, we check the bottom CSFG at stage 1. $\hat{u}_4$ does not satisfy the frozen bit criteria (4) and we cannot freeze this CSFG. So the messages are updated at stage 2 and we move to the next stage (stage 2) to check whether the first un-frozen CSFG can be frozen at stage 2 as shown in Fig 5(c). A CSFG can only be considered for freezing, only if all the preceding CSFGs at the same stage have been frozen. This procedure is
repeated until all the CSFGs at a stage are frozen or we reach the maximum number of iterations, which corresponds to the completion of the decoding process.

With the freezing of CSFGs, many computations and message updating operations are not needed to execute in some iterations. Therefore the overall computation complexity and hence the energy consumption are reduced. Moreover the right to left feedback propagating messages ($L_1^{t,m_{n-1}}$) are fixed to either $-\infty$ or $+\infty$ depending on the value of the hard-decision bit when a CSFG is frozen. This boosts up the reliability of the feedback messages and will help the rest of the unfrozen CSFGs to converge faster in the subsequent iterations. This will help to reduce the overall number of iteration for the decoding and increase the average throughput.

IV. SIMULATION RESULTS

To verify the error correcting performance and complexity saving for the proposed frozen-CSFG-based BPD scheme, we carry out simulation on a polar code of length 1024 and rate $\frac{1}{2}$ and compare the result with the original BPD scheme [19] (we denote as the baseline BPD) and the BPD using G-matrix-based stopping criterion [18]. Fig. 6 shows the simulation results over an AWGN channel with BPSK modulation. For a fair comparison, we use the same set of parameters as [18] where min-sum approximation with scaling parameter ($\alpha = 0.9375$) and $max\_iter = 40$ are used. As seen in Fig. 6(a), the proposed method has no performance degradation compared with the other two existing BPD schemes. The average number of iterations required for decoding a code word are compared in Fig. 6(b). It can be seen that the proposed method requires the least number of iterations resulting in lower latency and higher throughput as [18]. At SNR = 3dB, the average number of iterations is reduced by 55% and 33% when compared to the baseline BPD and the G-matrix-based early stop method, respectively.

We also compare the overall computation complexity of the three BPD schemes. For each PE in the factor graph, we count the number of iterations until its operation is frozen in the proposed scheme. We then sum up the number of iteration that the PE is active for all the PEs. For the other two schemes, since every PE needs to be executed in every iteration, the computation complexity just depends on the average number of iteration.

Fig. 6(c) shows the normalized average number of computations required for all the three schemes. It can be observed that the proposed scheme requires the least number of computations, which are translated directly to lower power.
consumption and latency for the overall decoding process. Fig. 6(d) shows the computation savings when compared with the G-matrix based early stop method. It can be seen that at SNR=3dB, the average computation complexity is reduced by 62% and 43% when compared with the baseline scheme and the early-stop scheme, respectively.

V. CONCLUSION

In this work we presented a novel scheme to reduce the average number of computations as well as average latency in belief propagation decoding (BPD) for polar codes based on the concept of frozen connected sub-factor-graph. Simulation results show that there is no performance degradation of the proposed scheme when compared with the original belief propagation algorithm and the G-matrix-based early stopping criterion while enjoying a 42 ~ 62 % reduction in computation complexity, and 35 ~ 53% reduction in latency at SNR=3dB. For future works, the VLSI architecture and the hardware implementation will be developed.

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