ABSTRACT. We consider stochastic spin-flip dynamics for: (i) monotone discrete surfaces in $\mathbb{Z}^3$ with planar boundary height and (ii) the one-dimensional discrete Solid-on-Solid (SOS) model confined to a box. In both cases we show almost optimal bounds $O(L^2\text{polylog}(L))$ for the mixing time of the chain, where $L$ is the natural size of the system. The dynamics at a macroscopic scale should be described by a deterministic mean curvature motion such that each point of the surface feels a drift which tends to minimize the local surface tension [17]. Inspired by this heuristics, our approach consists in bounding the dynamics with an auxiliary one which, with very high probability, follows quite closely the deterministic mean curvature evolution. Key technical ingredients are monotonicity, coupling and an argument due to D. Wilson [18] in the framework of lozenge tiling Markov Chains. Our approach works equally well for both models despite the fact that their equilibrium maximal height fluctuations occur on very different scales ($\log L$ for monotone surfaces and $\sqrt{L}$ for the SOS model). Finally, combining techniques from kinetically constrained spin systems [2] together with the above mixing time result, we prove an almost diffusive lower bound of order $1/L^2\text{polylog}(L)$ for the spectral gap of the SOS model with horizontal size $L$ and unbounded heights.

2000 Mathematics Subject Classification: 60K35, 82C20
Keywords: Mixing time, Lozenge tilings, Solid-on-Solid model, Monotone surfaces, Glauber dynamics, Mean curvature motion.
challenge. Similar questions arise also in combinatorics and computer science when the interface configurations can be put in correspondence with the dimer coverings of a planar graph: the main focus there is to evaluate the running time of Markov Chain algorithms which sample uniformly among such combinatorial structures (cf. in particular [18, Section 5] for background and motivations in this direction). In this paper we address this question for two natural and widely studied models and obtain essentially optimal bounds on the equilibration time.

The first classical example is the continuous-time single spin-flip dynamics of discrete monotone surfaces with fixed boundary, cf. for instance [11, 18]. Monotone surfaces can be visualized as a stack of unit cubes centered around the vertices of $\mathbb{Z}^3$, which is decreasing in both the $x$ and the $y$ direction (see Figure 1); see Section 2 for the precise definition. This is equivalent to the zero-temperature dynamics of the three-dimensional Ising model with boundary conditions that enforce the presence of an interface; alternatively, it can be seen as a stochastic dynamics for lozenge tilings of a finite region of the plane, or of dimer coverings of a finite region of the honeycomb lattice. The invariant measure is uniform over all discrete surfaces compatible with the monotonicity constraints and with the boundary conditions. The conjectured behavior of the mixing time $T_{\text{mix}}$ is of order $L^2 \log L$ if $L$ is the linear size of the region under consideration. The monotonicity condition produces dynamical constraints which prevent the application of standard tools to obtain non-trivial bounds on $T_{\text{mix}}$. To overcome this difficulty, a modified “non-local” version of the dynamics, whose moves involve adding or removing piles of unit cubes stacked one on top of the other, was introduced in [11]. Using this device, a polynomial (in $L$) upper bound on $T_{\text{mix}}$ was proven in [11]. An important breakthrough was obtained by D. Wilson in [18], where the mixing time of the non-local dynamics was sharply analysed and shown to be of order $L^2 \log L$ (from below and above). Via classical comparison arguments, this implies [15] that $T_{\text{mix}} = \tilde{O}(L^p)$ for the single-site dynamics - here and below we use the notation $\tilde{O}(L^p)$ for any quantity that is bounded above by $L^p$ up to polylog($L$) factors. An improved

Figure 1. An example of a monotone surface with planar boundary conditions on the plane $x + y + z = 0$. The picture is taken from [8].
comparison, relying also on the so-called Peres-Winkler censoring inequalities [13], shows that $T_{\text{mix}} = \tilde{O}(L^4)$; see [5, Section 4.1].

The second example is the continuous-time stochastic one-dimensional SOS model, described by a set of integer-valued heights $\eta_1, \ldots, \eta_L$ such that each $\eta_i$ is confined in an interval whose size is of order $L$ (see Section 3 for the precise definition and Figure 2 for an illustration) and the heights $\eta_0, \eta_{L+1}$ are fixed boundary conditions. The dynamics involves local moves where the discrete heights can change by $\pm 1$ at each step. The invariant measure is the Gibbs distribution corresponding to a potential given by the absolute value of the height gradients. Due to the non-strictly convex character of the interaction, even obtaining a diffusive spectral gap bound (of order $L^{-2}$) for zero boundary conditions has been a long-standing open problem. The analysis of an auxiliary non-local dynamics, in the spirit of [18], plus a judicious use of the Peres-Winkler inequalities were recently combined to obtain a mixing time upper bound $\tilde{O}(L^{5/2})$ [12], while again the conjectured behavior is $O(L^2 \log L)$.

The main contribution of the present paper is a proof that, for both problems, $T_{\text{mix}} = \tilde{O}(L^2)$; see Theorem 2 for monotone surfaces and Theorem 3 for SOS. In the monotone surface case, we must restrict our analysis to the case where the boundary condition is approximately planar (see e.g. Figure 1), so that on a macroscopic scale, at equilibrium, the surface is flat, cf. Theorem 1. At the microscopic scale this corresponds to the zero temperature limit of the 3D Ising model with so-called Dobrushin boundary conditions, i.e. the boundary spins take the value $\pm 1$ according to whether they lie above or below a given plane. In any finite volume, this leads to the uniform distribution over all monotone surfaces compatible with the given “planar” boundary conditions. In the infinite volume limit $L \to \infty$, such uniform distribution is related to a translation invariant, ergodic Gibbs measure on dimer coverings of the infinite honeycomb lattice [9], described by a determinantal point processes whose kernel is known explicitly. Our method relies crucially on the Gaussian Free Field-like fluctuation properties of such infinite volume state.
Theorem 3, together with techniques developed in the context of kinetically constrained spin models [2], allows us to implement a recursive analysis whose output is an almost diffusive lower bound $\text{gap} \geq 1/L^2$ up to polylog($L$) factors for the spectral gap of the SOS model with unbounded heights and zero boundary conditions $\eta_0 = \eta_{L+1} = 0$; see Theorem 4.

Our approach can be roughly described as follows. At equilibrium the interface is macroscopically flat, with maximal height fluctuations much smaller than $L$ (logarithmic in the monotone surface case and of order $\sqrt{L}$ for SOS). The main step in the proof of the mixing time upper bounds is to show that an initial macroscopically non-flat profile approaches the flat equilibrium profile within the correct time. Heuristically, the interface evolves by minimizing the surface tension and therefore feels a drift proportional to the local mean curvature. A major difficulty encountered in previous approaches is the possible appearance of very large (up to order $L$) gradients of the surface height. Our method consists in introducing an auxiliary “mesoscopic” dynamics which approximately follows the mean curvature flow, such that large gradients are absent in the initial condition and are very unlikely to be created at later times. The key point is that, thanks to monotonicity and coupling considerations, the original dynamics converges to equilibrium faster than the auxiliary one.

The study of the mesoscopic dynamics involves a local analysis of the relaxation of a mesoscopic portion of the interface, whose size $\ell$ depends i) on the current local mean curvature $1/R_t$ of the surface at time $t$ and ii) on the size of equilibrium height fluctuations. For the monotone surface case it turns out that one has to choose $\ell \simeq R_t^{1/2}$ and for the SOS model $\ell \simeq R_t^{2/3}$ (in both cases modulo polylogarithmic factors). We refer to Section 4 for a more detailed explanation of the strategy which leads to the choice of the different scales in the two models. Note that, since the evolution tends to a profile with vanishing mean curvature, $R_t$ grows with time and becomes much larger than the initial value $R_0 \simeq L$ when equilibrium is approached. One key input is to prove that the equilibration time within such mesoscopic regions is of the correct order $\tilde{O}(\ell^2)$: this result can be obtained following recent important progress in [5] for monotone surfaces and in [12] for SOS (both use the Peres-Winkler inequalities and Wilson’s argument [18] in an essential way). The other key point, which is the main contribution of this work, is to show that the mesoscopic dynamics is dominated with high probability by the deterministic mean curvature evolution of a macroscopic smooth profile with the appropriate boundary conditions.

In this paper we do not focus on mixing time lower bounds. Let us however mention that, for the monotone surface dynamics, under natural assumptions mentioned in Remark 1 below, one can apply [5, Theorem 3.1] to obtain a lower bound of order $L^2/\log L$ for the mixing time. Similarly, for the SOS model it is known that the spectral gap is at most of order $L^{-2}$ [12], which by standard inequalities directly implies a lower bound of order $L^2$ for $T_{\text{mix}}$.

We believe our method could potentially work for a wide range of stochastic interface dynamics models where mean curvature motion is expected to occur macroscopically. An example that comes naturally to mind is the dynamics of domino tilings of the plane, for which at present only non-optimal polynomial upper bounds on $T_{\text{mix}}$ are available [11].

A real challenge is on the other hand to prove that $T_{\text{mix}} = \tilde{O}(L^2)$ for the monotone surface dynamics when the boundary height is not approximately planar, in which case the equilibrium shape is not macroscopically flat and arctic circle-type phenomena can occur [6]. While in principle our idea of mesoscopic auxiliary dynamics could be adapted to this case too, what is missing here are precise, finite-$L$ equilibrium estimates on height fluctuations and on the rate of convergence of the equilibrium average height to its macroscopic limit.
Concerning the SOS model, a big challenge is the analysis of the dynamics for the model in dimension $2 + 1$, for which not even crude polynomial bounds on $T_{\text{mix}}$ are available, while on general grounds one can expect once more the “diffusive” scaling $L^2 \log L$.

1.1. Generalities and notation. Let us recall some standard definitions for continuous time reversible Markov chains (see e.g. [10]). We will mostly work in the case where the state space $\Omega$ is finite and the Markov chain is irreducible. In particular, there is a unique reversible invariant measure $\pi$. For $\xi \in \Omega$ and $t \geq 0$, $\mu_t^\xi$ denotes the law of the configuration at time $t$ started from the initial configuration $\xi$. The law of the chain is denoted by $\mathbb{P}$.

Given two laws $\mu, \nu$ on $\Omega$, we let $\|\mu - \nu\| = \sup_{A \subset \Omega} |\mu(A) - \nu(A)|$ denote their total variation distance. The mixing time $T_{\text{mix}}$, defined as

$$T_{\text{mix}} = \inf \left\{ t > 0 : \sup_{\xi \in \Omega} \|\mu_t^\xi - \pi\| \leq \frac{1}{2} \right\}, \quad (1.1)$$

measures the time it takes for the dynamics to be close in total variation to equilibrium, uniformly in the initial condition. It is well known that

$$\sup_{\xi \in \Omega} \|\mu_t^\xi - \pi\| \leq e^{-t/T_{\text{mix}}}, \quad (1.2)$$

i.e., the worst-case variation distance from equilibrium decays exponentially with rate $1/T_{\text{mix}}$.

If $L$ denotes the infinitesimal generator of the reversible Markov chain, the spectral gap is defined as the lowest nonzero eigenvalue of $-L$. Equivalently, if $E(f,g) = \pi[f(-Lg)]$ denotes the associated Dirichlet form, one has

$$\text{gap} = \inf_{f} \frac{E(f,f)}{\text{Var}(f)}, \quad (1.3)$$

where $\text{Var}(f)$ stands for the variance $\pi[f^2] - \pi[f]^2$ and the infimum ranges over all functions $f : \Omega \mapsto \mathbb{R}$ such that $\text{Var}(f) \neq 0$. This definition makes sense also in the case where $\Omega$ is countably infinite, as will be the case for the unbounded SOS model to be considered in Theorem 4 below.

Throughout the paper, we will adopt the following conventions:

(i) if $x, y \in \mathbb{R}^n$, then $d(x, y)$ denotes their Euclidean distance;
(ii) if $x \in \mathbb{R}^n$, then we write $x^{(a)}$, $a = 1, \ldots, n$ for its components;
(iii) if $U \subset \mathbb{R}^n$, then $\text{diam}(U) = \max\{d(x, y) : x, y \in U\}$ denotes its diameter;
(iv) if $U \subset \mathbb{Z}^n$, then $\partial U = \{x \in \mathbb{Z}^n \setminus U \text{ such that } \exists y \in U \text{ with } d(x, y) = 1\}$. If on the other hand $U$ is a smooth subset of $\mathbb{R}^n$, then $\partial U$ denotes its usual boundary.

2. Monotone surfaces with “planar” boundary conditions

Definition 1 (Monotone surfaces). A function $\phi : \mathbb{Z}^2 \mapsto (\mathbb{Z} \cup \{\pm \infty\})$ defines a (discrete) monotone surface if $\phi_x \geq \phi_y$ whenever $x^{(a)} \leq y^{(a)}$, $a = 1, 2$. The collection of all monotone surfaces is denoted by $\Omega$.

On $\Omega$ there is a natural partial order: we say that $\phi \leq \phi'$ if $\phi_x \leq \phi'_x$ for every $x \in \mathbb{Z}^2$. Analogously, for $U \subset \mathbb{Z}^2$ we write $\phi_U \leq \phi'_U$ if $\phi_x \leq \phi'_x$ for every $x \in U$.

1This model is actually quite tricky and can hide surprises: one can for instance show [4] that, if one adds a hard-wall floor constraint at height zero, the boundary height being also fixed to zero, the dynamics is slowed down by the presence of a bottleneck which causes the relaxation time to be exponentially large in $L$. This phenomenon, related to the so-called entropic repulsion, should not be present in absence of the wall.
2.1. **Heat bath dynamics.** We define a dynamics \( \{\phi^\xi(t)\}_{t \geq 0} \) on monotone surfaces with initial condition \( \xi \) and fixed boundary conditions (b.c.) outside a finite region. Let \( U \) be a finite connected subset of \( \mathbb{Z}^2 \) (the finite region) and \( \eta \in \Omega \) (the boundary condition). Without loss of generality we will always assume that \( U \) contains the origin.

Given \( \phi \in \Omega \) and \( x \in \mathbb{Z}^2 \), let \( \phi^{(x,+)}, \phi^{(x,-)} \in \Omega \) be defined by

\[
\phi_y^{(x,+)} = \begin{cases} 
\phi_y & \text{if } x \neq y \\
\min\{\phi_x + 1, \phi_{(x(1)-1,x(2))}, \phi_{(x(1),x(2)-1)}\} & \text{if } x = y
\end{cases}
\]

and

\[
\phi_y^{(x,-)} = \begin{cases} 
\phi_y & \text{if } x \neq y \\
\max\{\phi_x - 1, \phi_{(x(1)+1,x(2))}, \phi_{(x(1),x(2)+1)}\} & \text{if } x = y
\end{cases}
\]

The dynamics is a continuous-time Markov chain on the set

\[\Omega_{\eta,U} := \{\phi \in \Omega : \phi_x = \eta_x \text{ for } x \notin U\}.\]

The initial condition at time zero is some given \( \xi \in \Omega_{\eta,U} \). To each \( x \in U \) is assigned an i.i.d. exponential clock of rate 1. If the clock labeled \( x \) rings at time \( t \), we replace \( \phi(t) \) with \( [\phi(t)]^{(x,+)} \) or \( [\phi(t)]^{(x,-)} \) with equal probabilities. It is immediate to check that such Markov chain is irreducible and reversible with respect to the uniform measure on \( \Omega_{\eta,U} \), which we denote \( \pi^n_U \) or simply \( \pi \). The mixing time is then defined as in (1.1) where the supremum is taken over \( \xi \in \Omega_{\eta,U} \).

2.2. **Monotonicity.** A function \( f \) on \( \Omega \) is said to be increasing (resp. decreasing) if \( f(\phi) \leq f(\phi') \) (resp. \( f(\phi) \geq f(\phi') \)) whenever \( \phi \leq \phi' \). Given two laws \( \mu, \nu \) on \( \Omega \), we write \( \mu \leq \nu \) (\( \nu \) dominates stochastically \( \mu \)) if \( \mu(f) \leq \nu(f) \) for every increasing function \( f \). The heat-bath dynamics is monotone (or attractive) with respect to the partial ordering “\( \leq \)”, in the following sense. If \( \mu_{\xi,\eta}^t \) denotes the law of \( \phi^\xi_{\eta}(t) \), the dynamics at time \( t \) started from \( \xi \) and evolving with b.c. \( \eta \), one has the following property (cf. for instance the discussion in [5, Sec. 2.1]):

\[\mu_{\xi,\eta}^t \preceq \mu_{\xi',\eta'}^t \text{ if } \xi \leq \xi' \text{ and } \eta \leq \eta'.\]

In particular, letting \( t \to \infty \), one has \( \pi_{\eta,U}^n \preceq \pi_{\eta',U}^n \). It is possible to realize on the same probability space the trajectories of the Markov chain corresponding to distinct initial conditions \( \xi \) and/or distinct boundary conditions \( \eta \) in such a way that, with probability one,

\[\phi_{\eta}^\xi(t) \leq \phi_{\eta'}^\xi(t) \text{ for every } t \geq 0, \text{ if } \xi \leq \xi' \text{ and } \eta \leq \eta'.\]

Such a construction takes the name of **global monotone coupling**. Throughout the paper we will apply several times the above monotonicity properties: for brevity, we will simply say “by monotonicity...”

2.3. **Mixing time upper bound.** As we mentioned in the introduction, it is expected that \( T_{\text{mix}} = O(L^2 \log L) \), where \( L \) is the diameter of the region \( U \). The next result proves such conjecture, up to logarithmic corrections, under the assumption that the boundary conditions are “approximately planar” (cf. condition (2.4) below). Such “planar” case is rather natural in terms of the three-dimensional Ising model: indeed, it corresponds to the zero-temperature limit of a system defined in the cylinder \( U \times \mathbb{Z} \), with Dobrushin-type boundary conditions which are, say, “\( + \)” above some plane and “\( - \)” below.
Definition 2. Given \( n \in \mathbb{R}^3 \) with \( \| n \| = 1 \), we write \( n > 0 \) if \( n^{(i)} > 0 \), \( i = 1, 2, 3 \). We let \( \tilde{\phi}^n \in \Omega \) be the discrete monotone surface with slope \( n \):

\[
\mathbb{Z}^2 \ni x \mapsto \tilde{\phi}_x^n = \max\{ z \in \mathbb{Z} : x^{(1)}(1)n^{(1)} + x^{(2)}(2)n^{(2)} + zn^{(3)} \leq 0 \}
\]

and \( \Pi^n \) denotes the plane

\[
\Pi^n = \{ z \in \mathbb{R}^3 : n \cdot z = 0 \}.
\] (2.3)

For \( x \in \mathbb{Z}^2 \) we let \( \Pi^n(x) \) denote the vertical coordinate of the point in the plane \( \Pi^n \) with horizontal coordinates \((x^{(1)}, x^{(2)})\). For \( h > 0 \) (resp. \( h < 0 \)), \( \Pi^n_h \) is the plane obtained translating \( \Pi^n \) upwards (resp. downwards) by \( |h| \) along the vertical direction.

The planarity condition on the boundary conditions is specified as follows:

Definition 3. Let \( n > 0 \). We say that \( \eta \) is a good planar boundary condition with slope \( n \) if there exists \( C > 0 \) such that

\[
|\eta_x - \tilde{\phi}_x^n| \leq C \log(|x| + 1)
\] (2.4)

for every \( x \in \mathbb{Z}^2 \).

As we mentioned in the introduction, under such boundary conditions the surface at equilibrium is essentially flat:

Theorem 1. Let \( n > 0 \) and let \( \eta \) be a good planar boundary condition with slope \( n \). For every \( \epsilon > 0 \) there exists \( c > 0 \) such that for every \( a > 0 \) the following holds. Let \( U \) be a finite, connected subset of \( \mathbb{Z}^2 \) containing the origin and let \( L := \text{diam}(U) \). Then, for any \( L \) large enough,

\[
\pi_U^n (\exists y \in U : |\phi_y - \tilde{\phi}_y^n| \geq a(\log L)^{1+\epsilon}) \leq (1/c)e^{-c a(\log L)^{1+\epsilon}}.
\] (2.5)

We can finally formulate our mixing time upper bound:

Theorem 2. In the same assumption of Theorem 1, for \( L \) sufficiently large one has

\[
\tau_{\text{mix}} \leq L^2 (\log L)^{12}.
\] (2.6)

With some technical effort (but no need of new ideas) one can improve the exponent 12 to 6 but we will not do so, since neither is close to the conjectured optimal value 1.

Remark 1. Concerning lower bounds: using the same idea of the proof of the lower bound on the mixing time of the three-dimensional zero-temperature Ising model with “+” boundary conditions in [5, Theorem 3.1], it is not hard to see that \( \tau_{\text{mix}} \geq L^2/(c \log L) \) for a suitable \( c > 0 \) for instance when \( U = \mathcal{U}_L \cap \mathbb{Z}^2 \), with \( \mathcal{U}_L \) a smooth open set of \( \mathbb{R}^2 \) expanded by a factor \( L \).

Remark 2. Concerning the assumption \( n > 0 \) a first obvious observation is that, using lattice symmetries, one could replace it with the condition \( n^{(i)} \neq 0 \), \( i = 1, 2, 3 \). The main reason why we excluded the case in which one or two components of \( n \) vanish is that, in these cases, the fluctuations of the surface in \( U \) around \( \tilde{\phi}^n \) are deterministically upper bounded by \( C' \log L \), where \( C' \) depends on the constant \( C \) in (2.4). As a consequence Theorem 1 becomes trivial and one can appeal to Proposition 3 below to get immediately Theorem 2. Thus, the really interesting and non-trivial case is \( n > 0 \).
2.4. Dynamics with “floor” and “ceiling”. In the course of the proof of Theorem 2 we need an auxiliary restricted dynamics for an interface constrained between a floor and a ceiling. Let $U$ and $\eta$ be as in the previous section; fix some $\phi^+, \phi^- \in \Omega_{\eta,U}$ with $\phi^- \leq \phi^+$ and let

$$\Omega_{\eta,U}^{\phi^\pm} = \{ \phi \in \Omega_{\eta,U} : \phi^(-) \leq \phi \leq \phi^+ \}.$$  \hfill (2.7)

One can define a dynamics restricted to $\Omega_{\eta,U}^{\phi^\pm}$ simply by choosing an initial condition $\xi \in \Omega_{\eta,U}^{\phi^\pm}$ and redefining

$$\phi(x,+)(y) = \left\{ \begin{array}{ll}
\phi(y) & \text{if } x \neq y \\
\min[\phi_x + 1, \phi_{(x(1)-1,x(2))}, \phi_{(x(1),x(2)-1)}, \phi_x^+] & \text{if } x = y
\end{array} \right.$$  \hfill (2.8)

and

$$\phi(x,-)(y) = \left\{ \begin{array}{ll}
\phi(y) & \text{if } x \neq y \\
\max[\phi_x - 1, \phi_{(x(1)+1,x(2))}, \phi_{(x(1),x(2)+1)}, \phi_x^-] & \text{if } x = y
\end{array} \right.$$  \hfill (2.9)

(compare with Eqs. (2.1), (2.2)). The dynamics is again monotone in the sense of Section 2.2, but this time the invariant measure $\pi$ is the uniform measure on $\Omega_{\eta,U}^{\phi^\pm}$.

3. Solid-on-Solid model

We turn to the study of the mixing time and spectral gap of a one-dimensional interface of Solid-on-Solid (SOS) type. The generic configuration (height function) of the standard SOS model is $\eta = (\eta_1, \ldots, \eta_L) \in \mathbb{Z}^L$ and its equilibrium measure $\tilde{\pi} = \tilde{\pi}_{L,h}$ corresponding to boundary conditions $0, h \in \mathbb{Z}$ is

$$\tilde{\pi}_{L,h}(\eta) \propto \exp \left( -\sum_{i=0}^{L} |\eta_{i+1} - \eta_i| \right)$$  \hfill (3.1)

with $\eta_0 = 0$ and $\eta_{L+1} = h$. There is no inverse temperature parameter $\beta$ in (3.1) since in this one-dimensional model its numerical value does not affect the qualitative behavior of the system and there is no loss of generality in fixing its value to unity. It is well known that $\tilde{\pi}_{L,h}$ describes the law of the unique open contour in the two-dimensional Ising model in the box $\{1, \ldots, L\} \times \mathbb{Z}$ with Dobrushin boundary conditions (boundary spins “+$+$” under the line which joins $(0,0)$ to $(L+1,h)$ and “$-$” below it), in the limit where the couplings on vertical edges tend to infinity.

Since the mixing time (1.1) deals with relaxation to equilibrium from an arbitrary initial condition it is necessary to introduce the following bounded version [12] of the SOS model, enclosed in a rectangular box of sides of order $L$. Thanks to standard equilibrium estimates, see also Lemma 1 below, the behavior at equilibrium of this bounded version of the model is essentially the same as the usual unbounded one defined above. We come back to the unbounded model in Theorem 4, which deals with the spectral gap.

For nonnegative integers $L$ and $h \leq L$, consider the configuration space $\Omega_{L,h}$ defined by

$$\Omega_{L,h} = \{ \eta = (\eta_1, \ldots, \eta_L), \eta_i \in \mathbb{Z} \cap [-L,L+h] \}.$$  \hfill (3.2)

The equilibrium measure on $\Omega_{L,h}$ is then given by $\pi = \pi_{L,h} = \tilde{\pi}(\cdot | \Omega_{L,h})$.

Occasionally we will consider the SOS model with further hard wall constraints, obtained by conditioning $\pi_{L,h}$ to the event $\xi^1 \leq \eta \leq \xi^2$, where $\xi^i \in \Omega_{L,h}$, $i = 1, 2$ are two configurations such that $\xi^1 \leq \xi^2$. Here, and below, we use the notation $\xi \leq \sigma$, for the natural partial order in $\Omega_{L,h}$ defined via $\xi_i \leq \sigma_i$, for all $i = 1, \ldots, L$. We refer to $\xi^1, \xi^2$ as the floor and the ceiling, respectively, and write $\pi_{L,h}^{\xi^1,\xi^2}$ for the corresponding equilibrium measure. If $\wedge$ denotes the maximal configuration in $\Omega_{L,h}$, i.e. $\wedge_i \equiv L + h$, we sometimes consider the model with $\xi^2 = \wedge$ and $\xi^1 = \bar{\eta}$ where $\bar{\eta}_i := [ih/(L+1)]$ for all $i = 1, \ldots, L$, i.e. the interface is above the straight
line connecting the two boundary values, cf. Figure 2 (b). In this case one speaks simply of an interface above the wall. Note that $\bar{\eta}$ is the SOS equivalent of the “monotone surface $\phi^n$ with fixed slope”, cf. Definition 2.

In the following, whenever we do not explicitly mention floor and ceiling, it is understood that we are talking about the bounded model where $\xi^1 = \wedge$ and $\xi^2 = \vee$, where $\vee$ is the minimal configuration in $\Omega_{L,h}$: $\forall i \equiv -L$.

3.1. Dynamics. The evolution of the interface is given by the standard heat bath dynamics, i.e. single-site Glauber dynamics described as follows. There are independent Poisson clocks with mean 1 at each site $i \in \{1, \ldots, L\}$. When site $i$ rings, the height $\eta_i$ is updated to the new value $\max\{\eta_i - 1, -L\}$ or $\min\{\eta_i + 1, L + h\}$ with probabilities $p_{i,-}(\eta)$, $p_{i,+}(\eta)$ respectively, determined by:

$$p_{i,-}(\eta) = \frac{e^{-2}}{1 + e^{-2}} 1_{\{\eta_i \leq a\}} + \frac{1}{2} 1_{\{b \geq \eta_i > a\}} + \frac{1}{1 + e^{-2}} 1_{\{\eta_i > b\}}$$

$$p_{i,+}(\eta) = \frac{e^{-2}}{1 + e^{-2}} 1_{\{\eta_i \geq b\}} + \frac{1}{2} 1_{\{b > \eta_i \geq a\}} + \frac{1}{1 + e^{-2}} 1_{\{\eta_i < a\}}$$

(3.3)

where $a := \min\{\eta_{i-1}, \eta_{i+1}\}$ and $b := \max\{\eta_{i-1}, \eta_{i+1}\}$. With the remaining probability $1 - (p_{i,-}(\eta) + p_{i,+}(\eta))$, $\eta_i$ stays at its current value. It is not hard to check that this defines a continuous time Markov chain with state space $\Omega_{L,h}$ and stationary reversible measure given by $\pi_{L,h}$. In the sequel we will write $\eta^t(t)$ for the random variable describing the state of the Markov chain at time $t$ with initial state $\xi$ and $\mu^t$ for its distribution. Let $T_{mix} = T_{mix}(L, h)$ denote the mixing time of this Markov chain.

We may consider the evolution of the system under hard wall constraints $\xi^1, \xi^2$ as above. This amounts to the same dynamics except that any update which would violate the constraints is rejected. The dynamics is then reversible w.r.t. the equilibrium measure $\pi_{L,h}^{\xi^1,\xi^2}$ associated to the floor $\xi^1$ and the ceiling $\xi^2$, see also Section 2.4 above for the analogous constrained dynamics in the monotone surface case. In either case, with or without hard walls, the monotonicity considerations recalled in Section 2.2 apply here without modifications, with the natural partial order on configurations introduced above.

Our main result about the mixing time of the SOS interface is:

**Theorem 3.** There exists $\alpha > 0$ such that for any $L$ sufficiently large, uniformly in $0 \leq h \leq L$ one has

$$T_{mix} \leq L^2 (\log L)^\alpha.$$  

(3.4)

The same bound holds for the interface constrained to stay above the wall $\bar{\eta}$.

**Remark 3.**

1. It will be clear from the proof that the above bound is satisfied as soon as $\alpha > 21$. With some effort this power of $\log L$ can be considerably improved. However, the method we use does not seem to be capable of reaching the presumably optimal bound $T_{mix} = O(L^2 \log L)$.

2. Minor modifications of the proof show that the result of Theorem 3 continues to hold as it is for the dynamics constrained between a floor $\xi^1$ and a ceiling $\xi^2$ for any $\xi^1, \xi^2 \in \Omega_{L,h}$ such that $\xi^1 \leq \bar{\eta} \leq \xi^2$.

3. A further extension is obtained by letting the constraints $-L \leq \eta_i \leq L + h$ be replaced by $-M \leq \eta_i \leq M + h$, where $M$ is an independent parameter, possibly much larger than $L$. It is possible to extend the proof of Theorem 3 to get the essentially sharp bound $T_{mix} = O(L \max(L, M))$. 


3.2. Spectral gap. The unbounded version of the SOS model is given in (3.1). We write again \( \tilde{\pi} = \tilde{\pi}_{L,h} \) for the corresponding Gibbs measure. The dynamics is the same as above except for the absence of the constraints \( \vee \leq \eta \leq \wedge \), i.e. when the clock labeled \( i \) rings, \( \eta_i \) is updated to the new value \( \eta_i \pm 1 \) with probability \( p_{i,\pm} \) given by (3.1). The infinitesimal generator is given by

\[
\mathcal{L} f(\eta) = \sum_{i=1}^{L} \left\{ p_{i,+}(\eta) \nabla_{i,+} f(\eta) + p_{i,-}(\eta) \nabla_{i,-} f(\eta) \right\},
\]

where \( \nabla_{i,\pm} f(\eta) = f(\eta^{i,\pm}) - f(\eta) \), and \( \eta^{i,\pm} \) is the configuration coinciding with \( \eta \) everywhere except that at site \( i \) the value of \( \eta_i \) is replaced by \( \eta_i \pm 1 \). The Dirichlet form is given by

\[
\mathcal{E}(f,f) = \frac{1}{2} \sum_{i=1}^{L} \tilde{\pi} [p_{i,+}(\nabla_{i,+} f)^2 + p_{i,-}(\nabla_{i,-} f)^2]
\]

where \( \tilde{\pi}[\cdot] \) denotes expectation w.r.t. \( \tilde{\pi} \). Note that, for each finite \( L \), \( \mathcal{L} \) defines a bounded self-adjoint operator in \( L^2(\mathbb{Z}^L, \tilde{\pi}) \). The associated spectral gap is defined by (1.3), where \( f \) ranges over all \( f \in L^2(\mathbb{Z}^L, \tilde{\pi}) \) with nonzero variance.

**Theorem 4.** Let \( \alpha > 0 \) be as in Theorem 3. For some constant \( c > 0 \), for all \( 0 \leq h \leq L \), the spectral gap of the unbounded SOS dynamics satisfies

\[
gap(L) \geq c L^{-2}(\log L)^{-\alpha}.
\]

The same bound holds for the interface constrained to stay above the wall \( \bar{\eta} \).

This estimate is optimal, modulo the logarithmic factor: an upper bound \( O(1/L^2) \) is given e.g. in [12]. The lower bound \( \gap(L) \geq c/L^2 \) was proven in [14] for a modified version of the SOS model with weak boundary couplings; such modified model is much less sensitive to the boundary conditions and has a genuinely different dynamical behavior. If instead the absolute value interaction potential were replaced by one with strictly convex behavior at infinity, then the correct lower bound \( \gap(L) \geq c/L^2 \) would follow by well-established recursive methods, see e.g. [3].

4. Strategy of the proof

As already announced in the introduction, despite the fact that the equilibrium fluctuations of the interface in the two models are very different, our bound \( T_{\text{mix}} = O(L^2) \) is proved following a common strategy that we sketch here.

The crucial step is the following (see Propositions 1 and 6 below for a precise formulation in the case of monotone surfaces and SOS model):

**Step 1.** Starting from any configuration, after time \( \tilde{O}(L^2) \) the distance between the interface and the flat profile is not larger than its typical equilibrium value \( \chi_L \).

At that point one can conclude \( T_{\text{mix}} = \tilde{O}(L^2) \) provided that a result of the following type is available (see e.g. Proposition 5 below in the case of the SOS model):

**Step 2.** If the initial condition \( \xi \) is at distance \( \chi_L \) from the flat profile, then \( \| \mu_T^{\xi} - \pi \| \ll 1 \) for some \( T = \tilde{O}(L^2) \).

The proof of such result is model-dependent: for the SOS model it was given in [12] and for monotone surfaces it follows from results in [5] (see Propositions 2 and 3 below).

In turn, Step 1 follows if one proves that, with high probability, the interface started from the maximal configuration stays below a deterministic interface evolution which after time \( \tilde{O}(L^2) \) is
at the correct distance $O(\chi_L)$ from the flat profile. It turns out that it is actually sufficient to define the deterministic interface evolution along a sequence of deterministic times $t_n, 0 \leq n \leq M$. At all times $t_n$, the deterministic interface is the boundary of $C_{u_n}$ where, given $u > 0$, $C_u$ is a spherical cap (if we are considering two-dimensional interfaces like SOS) of height $u$ and base of linear size $\rho_L$ roughly of order $L$, see Figure 3. The base of $C_u$ lies on the plane/line which contains the macroscopic flat profile. The evolution of $C_{u_n}$ by a kind of “flattening process”, in the time interval $(t_{n-1}, t_n)$ transforms $C_{u_{n-1}}$ into $C_{u_n}$. The sequence of increasing times $\{t_n\}_{0 \leq n \leq M}$ and of decreasing heights $\{u_n\}_{0 \leq n \leq M}$ will be introduced in a moment. The “domination statement” then is of the following type (see Propositions 4 and 7):

**Claim 1.** For all $0 \leq n \leq M$, with high probability the following holds. For all times in $[t_n, L^3]$ the evolution started from the maximal configuration stays below the boundary of $C_{u_n}$.

The initial height $u_0$ is taken to be proportional to $L$ and one sets $t_0 = 0$; this guarantees that the statement of Claim 1 holds trivially for $n = 0$. In order to choose $u_{n+1}$ given $u_n$, one uses the following procedure. Consider the spherical cap/circular segment $C_{u_n}$ and choose a point on its curved boundary (e.g. the highest one). Move inward (i.e. inside $C_{u_n}$) the tangent plane/line at the chosen point by an amount $\Delta$ and call $d_\Delta$ the diameter of the intersection between the plane/line with $C_{u_n}$, see Figure 3.

![Figure 3](image_url)

**Figure 3.** The spherical cap/circular segment $C_{u_n}$. The deterministic evolution at time $t_n$ coincides with the curved portion of the boundary of $C_{u_n}$. At that time, the true stochastic evolution (wiggled line) stays with high probability below it. Elementary geometry shows that $\Delta \simeq d_\Delta^3 (u_n/\rho_L^3)$. The requirement $u_n - u_{n+1} = \Delta_n \simeq d_{\Delta_n}^3$ then leads to $u_n - u_{n+1} \simeq (\rho_L^2/u_n)^{\gamma/(2-\gamma)}$.

Then $u_n - u_{n+1}$ is chosen as the critical value $\Delta_n$ such that the equilibrium fluctuations on scale $d_{\Delta_n}$ are of order $\Delta_n$ (apart from logarithmic corrections), i.e. $\Delta_n = O(\chi_{d_{\Delta_n}})$. Also, $M$ is the smallest index such that $u_M \leq \chi_L$, i.e. $u_M$ is of the order of the equilibrium height fluctuations on scale $L$. As for the time sequence $\{t_n\}_{0 \leq n \leq M}$, one sets $t_{n+1} - t_n$ to be of order $d_{\Delta_n}^3$ (again neglecting logarithmic corrections): that this is the correct choice is guaranteed by a careful use of Step 2, applied with $L = d_{\Delta_n}$. It is not difficult to realize that $t_M = \tilde{O}(L^2)$. Indeed, assume for definiteness that $\chi_L \sim L^\gamma$ for some $0 \leq \gamma < 1$, where if $\gamma = 0$ we mean that $\chi_L \sim \text{polylog}(L)$. Then, simple geometric considerations show that

$$u_n - u_{n+1} = \tilde{O}\left(\left(\frac{\rho_L^2}{u_n}\right)^{\gamma/(2-\gamma)}\right), \quad t_{n+1} - t_n = \tilde{O}\left(\left(\frac{\rho_L^2}{u_n}\right)^{2/(2-\gamma)}\right).$$
Approximating the recursion for $u_n$ with a differential equation gives
\[ u_n^{2/(2-\gamma)} \simeq u_0^{2/(2-\gamma)} - \rho_L^{2\gamma/(2-\gamma)} n \simeq L^{2\gamma/(2-\gamma)} \left[ L^{(2-2\gamma)/(2-\gamma)} - n \right] \]
since both $u_0$ and $\rho_L$ are of order $L$. In particular, one has roughly $M = O(L^{(2-2\gamma)/(2-\gamma)})$. Then,
\[ t_M = \sum_{n=0}^{M-1} (t_{n+1} - t_n) \simeq \rho_L^{4/(2-\gamma)} L^{4/(2-\gamma)} \sum_{n=0}^{M-1} \frac{1}{u_n^{2/(2-\gamma)} L^{2\gamma/(2-\gamma)}} \sum_{n=0}^{M-1} \frac{1}{L^{(2-2\gamma)/(2-\gamma)} - n} = \tilde{O}(L^2) \]
since the last sum is of order $\log L$. Remarkably, the order of magnitude of $t_M$ does not depend on the fluctuation exponent $\gamma$, while the sequence $(t_n, u_n)$ and the value of $M$ do. The statement of Claim 1 for $n = M$ allows to conclude Step 1: the evolution started from the maximal configuration, at time $t_M = \tilde{O}(L^2)$, is below the deterministic evolution, which is within distance $\chi_L$ from the flat profile.

Another way to understand the choice of the time-scales $t_n$ is the following. If one imagines that the boundary of $C_u$ evolves by “mean curvature”, i.e. feeling a inward drift proportional to the inverse of its instantaneous radius of curvature, then the time $t_{n+1} - t_n$ to transform $C_{u_n}$ into $C_{u_{n+1}}$ must be $O(R_n \times (u_n - u_{n+1}))$, where $R_n$ is the radius of curvature of $C_{u_n}$. One can easily check that, apart from logarithmic corrections, this coincides with the requirement $t_{n+1} - t_n \simeq d_{\Delta n}^2$.

5. MONOTONE SURFACES: PROOF OF THEOREM 1

An essential tool in the proof of Theorem 1 are the translation invariant, ergodic Gibbs measures, with given slope $\mathbf{n}$, on the set $\Omega$ of monotone surfaces [9, 16]. The trick of using the properties of such infinite-volume states to obtain fluctuation bounds for surfaces with fixed boundary conditions around a finite region was already crucial in [5]. The most relevant result for this purpose is the following.

Theorem 5. [9, 16] Given $n > 0$, $h \in \mathbb{Z}$ and $\bar{x} \in \mathbb{Z}^2$, there exists a unique law $\mu_{n,h,\bar{x}}$ on $\Omega$ such that

(i) $\mu_{n,h,\bar{x}}(\phi_{\bar{x}} = h) = 1$;
(ii) for every $x \in \mathbb{Z}^2$, $\mu_{n,h,\bar{x}}(\phi_{\bar{x}}) = h + \Pi \mathbf{n}(x - \bar{x})$ (cf. Definition 2);
(iii) for every $x_1, \ldots, x_k \in \mathbb{Z}^2$ and $v \in \mathbb{Z}^2$ one has that the joint law of \{\phi_{x_i} - \phi_{x_j}\}_{1 \leq i,j \leq k}$ is the same as the joint law of \{\phi_{x_i+v} - \phi_{x_j+v}\}_{1 \leq i,j \leq k}$ (translation invariance of the law of the height gradients);
(iv) for every finite $U \subset \mathbb{Z}^2$ such that $\bar{x} \notin U$ and every monotone surface $\xi \in \Omega$ such that $\xi_{\bar{x}} = h$, the measure $\mu_{n,h,\bar{x}}$ conditioned to $\{\phi_{\bar{x}} = \xi_{\bar{x}}$ for every $x \notin U\}$ is $\pi_\xi^U$, i.e. the uniform measure over all $\phi \in \Omega$ which coincide with $\xi$ outside $U$ (DLR property).

Concerning height fluctuations under these Gibbs measures one can prove that for every $\epsilon > 0$ there exists a positive constant $c$ such that, for every $a > 0$ and $L$ large enough,
\[ \mu_{n,h,\bar{x}}(\exists y \in \mathbb{Z}^2 : d(\bar{x}, y) \leq L \text{ and } |\phi_y - \mu_{n,h,\bar{x}}(\phi_y)| \geq a(\log L)^{1+\epsilon} \leq \frac{1}{c} e^{-c a(\log L)^{1+\epsilon}}, \] (5.1)
(see [5, Proposition 5.7]; the proof is given there for $\epsilon = 1/2$ and $a = 1/8$ but it works identically for $\epsilon > 0, a > 0$). In other words, the surface lies on average on a plane of slope $\mathbf{n}$ and the height difference between two points $x, y$ does not differ from the average height difference by much more than the logarithm of $d(x, y)$. The estimate (5.1) is obtained via the well-known fact [8] that height differences between two points can be written as the number of points of a determinantal point process whose kernel is explicitly known.
The connection between the measure $\mu_{n,h,x}$ and the “infinite volume states” for dimer coverings of the infinite honeycomb lattice defined in [9] is well known and is discussed for instance in [5, Section 5]. Let us just recall that the components $n^{(a)}$, $a = 1, 2, 3$ are directly related to the fractions of dimers of the three possible types (horizontal or rotated by $\pi/3$ or by $(2/3)\pi$).

We now turn to the proof of (2.5). By symmetry it is enough to show that

$$\pi_n^U (\exists y \in U : \phi_y \geq \phi_n^x + a(\log L)^{1+\epsilon}) \leq (1/e) e^{-a(\log L)^{1+\epsilon}}. \quad (5.2)$$

By monotonicity the probability in the left-hand side of (5.2) is increased whenever $\eta$ is replaced by $\eta'$ such that $\eta_{\partial U} \leq \eta_{\partial U}'$. In particular, let $\eta'$ be sampled from $\mu_{n,h,x}(|A|$) where $x$ is a given element of $\partial U$, $h$ is such that $\mu_{n,h,x}(\phi_x) = \eta_x + (a/2)(\log L)^{1+\epsilon}$ and $A$ is the event $A = A_{n,U} = \{\phi \in \Omega : \phi_{\partial U} \geq \eta_{\partial U}\}$. From (5.1) and the fact that $\text{diam}(U) = L$ one sees that

$$\|\mu_{n,h,x}(|A) - \mu_{n,h,x}\| = O \left[ \exp \left( -\frac{a}{3} c (\log L)^{1+\epsilon} \right) \right].$$

This is because, if $y \in \partial U$, $\phi_y < \eta_y$ implies $\phi_y < \mu_{n,h,x}(\phi_y) - (a/2 + o(1))(\log L)^{1+\epsilon}$, recall Definition 3, so that $\mu_{n,h,x}(A^c) = O(\exp(-O(3) c (\log L)^{1+\epsilon}))$. Therefore, the probability in the left-hand side of (5.2) is upper bounded by

$$\int \mu_{n,h,x}(d\eta') \pi_n^U (\exists y \in U : \phi_y - \phi_n^x \geq a(\log L)^{1+\epsilon}) + O \left[ \exp \left( -\frac{a}{3} c (\log L)^{1+\epsilon} \right) \right]$$

$$= \mu_{n,h,x} (\exists y \in U : \phi_y - \phi_n^x \geq a(\log L)^{1+\epsilon}) + O \left[ \exp \left( -\frac{a}{3} c (\log L)^{1+\epsilon} \right) \right]$$

$$= O \left[ \exp \left( -\frac{a}{3} c (\log L)^{1+\epsilon} \right) \right]$$

where in the first step we used the DLR property and in the second one the fluctuation bound (5.1), together with the fact that with our choice of $h$ one has $\mu_{n,h,x}(\phi_y) = \phi_n^x + (a/2 + o(1))(\log L)^{1+\epsilon}$.

6. MONOTONE SURFACES: PROOF OF THEOREM 2

In this section the slope $n > 0$, the region $U$ and the good planar boundary condition $\eta$ with slope $n$ are fixed as in Theorems 1, 2. Denote $\wedge$ (resp. $\vee$) the maximal (resp. minimal) configuration of $\Omega_{n,U}$ with respect to the partial ordering “$\leq$” and recall that $\phi_{\xi}(t)$ denotes the configuration at time $t$ started from $\xi$.

The first key ingredient is a result saying that, after time of order $L^2(\log L)^{12}$, the surface is at most at distance $(\log L)^{3/2}$ away from the plane $\Pi^n$ of slope $n$ (cf. Definition 2). It is here that the new ideas of mimicking the evolution by mean curvature play a crucial role.

Proposition 1. Let $T = (1/2)L^2(\log L)^{12}$. Then, there exists $c > 0$ such that

$$\mathbb{P} \left( \max_{x \in U} (\phi_x^\wedge(T) - \phi_n^x) > (\log L)^{3/2} \right) = O \left( e^{-c(\log L)^{3/2}} \right) \quad (6.1)$$

and similarly

$$\mathbb{P} \left( \min_{x \in U} (\phi_x^\vee(T) - \phi_n^x) < -(\log L)^{3/2} \right) = O \left( e^{-c(\log L)^{3/2}} \right). \quad (6.2)$$

The second result says that once the surface is within distance $(\log L)^{3/2}$ from $\Pi^n$, it does not go much farther than that for a time much longer than $L^2(\log L)^{12}$. This second step is much more standard and its proof combines monotonicity and reversibility together with the fluctuation bounds of Theorem 1.
Definition 4. Let $\Phi^+$ (resp. $\Phi^-$) be the maximal (resp. minimal) configuration in the set
\[ \{ \phi \in \Omega_{\eta,U} : \max_{x \in U} |\phi_x - \bar{\phi}^n_x| \leq 2(\log L)^{3/2} \}. \]

Proposition 2. Let $\xi \in \Omega_{\eta,U}$ be such that
\[ \max_{x \in U} |\xi_x - \bar{\phi}^n_x| \leq (\log L)^{3/2}. \]  
Then, there exists $c > 0$ such that
\[ \mathbb{P} \left( \exists t < L^{10} : \phi^\xi(t) \notin \Omega^\Phi_{\eta,U} \right) = O \left( e^{-c(\log L)^{3/2}} \right) \]
with $\Omega^\Phi_{\eta,U}$ given in (2.7) and $\Phi^\pm$ as in Definition 4.

Finally, the last step shows that if the surface evolves constrained between a ceiling and a floor which are within distance $O((\log L)^{3/2})$ from $\Pi^n$, then mixing occurs within a time $\tilde{O}(L^2)$.

Proposition 3. [5, Theorem 4.3] Let $\phi^\pm \in \Omega_{\eta,U}$ with $\phi^- \leq \phi^+$. For the dynamics restricted to $\Omega^\Phi_{\eta,U}$ (cf. Section 2.4) one has
\[ T_{\text{mix}} = O(L^2(\log L)^2H^2) \]
where $H = \max_{x \in U}(\phi^+_x - \phi^-_x)$.

We can now easily put together Propositions 1 to 3 to obtain the desired upper bound (2.6) on the mixing time:

**Proof of Theorem 2.** It is a standard fact that
\[ \max_{\xi \in \Omega_{\eta,U}} \| \mu^\xi_t - \pi \| \leq L^3 \max(\| \mu^\Lambda_t - \pi \|, \| \mu^\nu_t - \pi \|) \]
(see e.g. [5, Lemma 6.2] for a similar statement) so it is sufficient to prove that
\[ \max(\| \mu^\Lambda_t - \pi \|, \| \mu^\nu_t - \pi \|) \leq 1/(2eL^3) \]
for $t = L^2(\log L)^{12} = 2T$. Let us consider e.g. the case of the maximal initial condition $\Lambda$, the other case being analogous.

Define $\Omega' = \{ \phi \in \Omega_{\eta,U} : \max_{x \in U} |\phi_x - \bar{\phi}^n_x| \leq (\log L)^{3/2} \}$ and, for $\xi \in \Omega'$,
\[ \tau = \inf \{ t > 0 : \max_{x \in U} |\phi^\xi_x(t) - \bar{\phi}^n_x| \geq 2(\log L)^{3/2} - 1 \}. \]

Let $A$ be a subset of $\Omega_{\eta,U}$. Then, using Proposition 1,
\[ \mu^\Lambda_{\tau T}(A) = \mu^\xi_{\tau T}(A) |\xi \in \Omega' \right) + O \left( e^{-c(\log L)^{3/2}} \right). \]

Next, from Proposition 2, one has for every $\xi \in \Omega'$
\[ \mu^\xi_T(A) = \mathbb{P}(\phi^\xi(T) \in A; \tau > T) + O \left( e^{-c(\log L)^{3/2}} \right) = \mathbb{P}^{\Phi^\pm}(\phi^\xi(T) \in A; \tau > T) + O \left( e^{-c(\log L)^{3/2}} \right) \]
where $\mathbb{P}^{\Phi^\pm}$ denotes the law of the dynamics restricted to the set $\Omega^\Phi_{\eta,U}$. Indeed, up to the random time $\tau$ the two dynamics $\mathbb{P}$ and $\mathbb{P}^{\Phi^\pm}$ can be perfectly coupled so that they coincide. In particular, $\tau$ has the same law under $\mathbb{P}$ and $\mathbb{P}^{\Phi^\pm}$. Finally, thanks to Proposition 3, $T$ is at least $(\log L)^2$ times
the mixing time of the restricted dynamics (which is \( O(L^2(\log L)^5) \)). Therefore, from (1.2) and the fact that the invariant measure of the restricted dynamics is \( \pi(\cdot|\Omega_{\eta,U}) \), one has
\[
|\mathbb{P}^\Phi(\phi^\Phi_T(\mathcal{T}) \in A) - \pi(A|\Omega_{\eta,U}^\Phi)| \leq e^{-(\log L)^2}.
\] (6.6)
Thanks to Theorem 1 one has \( \pi(\Omega_{\eta,U}^\Phi) \geq 1 - O(\exp(-c(\log L)^{3/2})) \) and finally
\[
|\mu^{\Delta}_{2T}(A) - \pi(A)| = O\left(e^{-c(\log L)^{3/2}}\right)
\]
for every event \( A \in \Omega_{\eta,U} \), which implies \( \|\mu^{\Delta}_{2T} - \pi\| \leq 1/(2eL^3) \) for \( L \) large enough. □

As a warm-up, we start by proving the easier Proposition 2.

Proof of Proposition 2. Let \( \xi \in \Omega_{\eta,U} \) satisfy (6.3). By monotonicity, if \( \eta' \in \Omega \) and \( \xi' \in \Omega_{\eta',U} \) are such that \( \xi \leq \xi' \leq \eta \), then
\[
\mathbb{P}\left( \exists t < L^{10} : \max_{x \in U} (\phi^\xi_x(t) - \phi^\eta_x) \geq 2(\log L)^{3/2} \right) \leq \mathbb{P}'\left( \exists t < L^{10} : \max_{x \in U} (\phi^\xi_x(t) - \phi^\eta_x) \geq 2(\log L)^{3/2} \right)
\] (6.7)
where \( \mathbb{P}' \) denotes the evolution with boundary condition \( \eta' \) (instead of \( \eta \)). In particular, this is the case if we set \( \eta'_x = \eta_x + \lfloor (3/2)(\log L)^{3/2} \rfloor \) and \( \xi' \) is sampled from the measure \( \pi_{U}^{\eta'}(\cdot|A) \), where \( A \) is the event \( A_{\eta',\xi,U} = \{ \phi \in \Omega_{\eta',U} : \phi_U \geq \xi_U \} \). From Theorem 1 (applied with \( \epsilon = a = 1/2 \)) one sees that
\[
\|\pi_{U}^{\eta'}(\cdot|A) - \pi_{U}^{\eta'}\| = O\left(e^{-c(\log L)^{3/2}}\right)
\] (6.9)
for some \( c > 0 \). This is because \( \eta' \) is within distance \( C\log L \) from the plane \( \Pi_u^{(3/2)(\log L)^{3/2}} \) (cf. Definition 2) while \( \xi \) is within distance \( (\log L)^{3/2} \) from the plane \( \Pi_u \). Therefore, the probability in (6.7) is upper bounded by
\[
\int \pi_{U}^{\eta'}(d\xi') \mathbb{P}'\left( \exists t < L^{10} : \max_{x \in U} (\phi^\xi_x(t) - \phi^\eta_x) \geq 2(\log L)^{3/2} \right) + O\left(e^{-c(\log L)^{3/2}}\right).
\] (6.10)
The initial condition \( \xi' \) in (6.10) is sampled from \( \pi_{U}^{\eta'} \), which is the invariant measure of the dynamics \( \mathbb{P}' \), so that the distribution of \( \phi^\xi_x(t) \) coincides with \( \pi_{U}^{\eta'} \) at all later times. Via a union bound over times and recalling the relation between \( \eta \) and \( \eta' \), the first term in (6.10) is upper bounded by
\[
\pi_{U}^{\eta}\left( \max_{x \in U} (\phi_x - \phi^\eta_x) \geq 2(\log L)^{3/2} \right) \times O(L^{10} \times L^2)
\] (6.11)
\[
\leq \pi_{U}^{\eta}\left( \max_{x \in U} (\phi_x - \phi^\eta_x) \geq (1/2)(\log L)^{3/2} - 1 \right) \times O(L^{10} \times L^2)
\] (6.12)
which is of order \( \exp(-c(\log L)^{3/2}) \), see Theorem 1. The factor \( O(L^{10} \times L^2) \) is just the average number of Markov chain moves within time \( L^{10} \), since there are order of \( L^2 \) lattice points in \( U \). Similarly one bounds the probability that \( \min_{t \in U}(\phi^\xi_x(t) - \phi^\eta_x) < -2(\log L)^{3/2} \) for some \( t \leq L^{10} \) and claim (6.4) is proven. □
Proof of Proposition 1. We prove only Eq. (6.1) since (6.2) is obtained essentially in the same way. Let \( W \) be a disk of radius 
\[
\rho_L = L \times \log L
\]
on the plane \( \Pi^0_{C(\log L)} \) of slope \( n \) (cf. Definition 2, with \( C \) the same constant as in (2.4)) such that its projection \( V \) on the horizontal plane contains \( U \) and moreover the distance between \( \partial U \) and \( \partial V \) is at least \( \rho_L/2 \) (recall that \( U \) has diameter \( L \)).

Given \( u > 0 \), let \( C_u \) be the spherical cap whose base is the disk \( W \) and whose height is \( u \). The radius of curvature \( R \) is related to \( u \) and \( \rho_L \) by
\[
(2R - u)u = \rho_L^2
\]
and, since we will always work under the condition \( u \ll \rho_L \ll R \), we have \( R = \rho_L^2/(2u)(1+o(1)) \).

For a point \( v \) on the curved portion of the boundary of \( C_u \), let \( n_v \) be the normal at \( v \) directed towards the exterior of \( C_u \). It is clear that, if \( u \ll \rho_L \), one has \( n_v = n + o(1) \); in particular, \( n_v > 0 \) with the convention of Definition 2. Finally the height (w.r.t. the horizontal plane) of the spherical cap at horizontal coordinates \( x \in V \) is denoted by
\[
\psi_u(x) = \sup\{z \in \mathbb{R} : (x^{(1)}, x^{(2)}, z) \in C_u\}.
\]

We now define a sequence of spherical caps \( \{C_{u_n}\}_{n=0}^M \) with constant base \( W \), decreasing height \( u_n \) and increasing radius of curvature \( R_n \). More precisely, let \( u_0 = 2L \) and \( M := 2L - (\log L)^{3/4} \). Then we let \( u_n = u_{n-1} - 1 = 2L - n \) and \((2R_n - u_n)u_n = \rho_L^2 \). For later purposes we also define
\[
t_n = t_{n-1} + R_n(\log L)^{17/2}, \quad t_0 = 0.
\]

Remark 4. It is worth noting that \( R_0 \sim L(\log L)^2/4 \), \( R_M \sim L^2/(2(\log L)^{3/4}) \) and \( R_{n+1}/R_n = 1 + o(1) \) uniformly in the whole range \( n = 0, \ldots, M \).

Recalling that \( R_n = \rho_L^2/(2u_n)(1+o(1)) \), where \( o(1) \) is small uniformly in \( 1 \leq n \leq M \), it is immediate to deduce that
\[
t_M = \rho_L^2(\log L)^{17/2} \times O \left( \sum_{n=1}^M \frac{1}{u_n} \right) = O(L^2(\log L)^{23/2}). \tag{6.14}
\]

With this notation the key step is represented by the next Proposition.

Proposition 4. There exists a positive constant \( c' \) such that the following holds for \( L \) large enough. For every \( 0 \leq n \leq M \) one has, with probability at least \( 1 - n \exp(-c'(\log L)^{3/2}) \),
\[
\phi_\wedge(t) \leq \psi_{u_n}(x) \text{ for every } x \in U \text{ and every } t \in [t_n, L^3]. \tag{6.15}
\]

Since \( u_M \ll (\log L)^{3/2} \), Proposition 4 together with (6.14) imply the desired inequality (6.1).

Proof of Proposition 4. We prove the claim by induction on \( n \). For \( n = 0 \) this is trivial since we chose \( u_0 = 2L \) such as to guarantee that the maximal configuration \( \wedge \in \Omega_{\eta,U} \) is below the function \( U \ni x \mapsto \psi_{u_0}(x) \).

Assume the claim for some \( n \). For \( x \in U \) define the event
\[
A_x = \{ t \in [t_{n+1}, L^3] : \phi_\wedge(t) > \psi_{u_{n+1}}(x) \}
\]
so we need to prove \( \mathbb{P}(\cup_{x \in U} A_x) \leq (n + 1) \exp(-c'(\log L)^{3/2}) \). We have
\[
\mathbb{P}(\cup_{x \in U} A_x) \leq \sum_{x \in U} \mathbb{P}(A_x; \phi_\wedge(s) \leq \psi_{u_n} \text{ for every } s \in [t_n, L^3]) + n \exp(-c'(\log L)^{3/2}) \tag{6.16}
\]
where we write \( \phi_\wedge(s) \leq \psi_{u_n} \) to mean that \( \phi_\wedge(s) \leq \psi_{u_n}(y) \) for every \( y \in U \).
Given $x \in U$, consider the plane $\tilde{\Pi}$ tangent to $C_{u_n}$ at the point $(x^{(1)}, x^{(2)}, \psi_{u_n}(x))$ and the plane $\tilde{\Pi}'$ obtained by translating downwards $\tilde{\Pi}$ by $(\log L)^{3/2}$. The intersection of $\tilde{\Pi}'$ with $C_{u_n}$ is a disk $\tilde{\mathcal{W}}$ of radius $O((\log L)^{3/4})$, whose projection on the horizontal plane we call $Z$. Let $\tilde{U} \subset \mathbb{R}^2$ be such that $\tilde{U} \subset Z$ and $\partial \tilde{U}$ is at distance of order $1$ from $\partial Z$, so that of course $\text{diam} (\tilde{U}) = O((\log L)^{3/4})$.

Let $\wedge^{(n)} \in \Omega$ be the maximal monotone surface such that $\wedge_x^{(n)} \leq \psi_{u_n}(x)$ for every $x \in U$. Let $\tilde{\mathbb{P}}$ denote the law of the auxiliary monotone surface dynamics in $\tilde{U}$, starting at time $t_n$ from $\wedge^{(n)}$ and with boundary conditions $\wedge_{\partial \tilde{U}}^{(n)}$. By monotonicity and the definition of $\wedge^{(n)}$ we have

$$\mathbb{P}(A_x; \phi^x(s) \leq \psi_{u_n} \text{ for every } s \in [t_n, L^3]) \leq \tilde{\mathbb{P}} \left( \exists t \in [t_{n+1}, L^3] \text{ such that } \phi_x^{\wedge(n)}(t) \geq \psi_{u_{n+1}}(x) \right).$$

As in the proof of Theorem 2, Propositions 2 and 3 show that after time

$$t_{n+1} - t_n = R_{n+1}(\log L)^{17/2} \geq \text{const} \times \text{diam}(\tilde{U})^2(\log L)^7$$

the dynamics $\tilde{\mathbb{P}}$ has a variation distance of order $\exp(-c(\log L)^{3/2})$ for some $c > 0$ from its equilibrium $\pi_{U}^{\wedge(n)}$ (recall that $c - \log L \leq \log R_n \leq c_+ \log L$, cf. Remark 4). Theorem 1 gives that

$$\pi_{U}^{\wedge(n)}[\phi_x \leq \psi_{u_{n+1}}(x)] \geq 1 - O(e^{-c(\log L)^{3/2}}).$$

Indeed, the point $(x, \psi_{u_{n+1}})$ is at distance $(1 + o(1))(\log L)^{3/2}$ from the plane $\tilde{\Pi}'$ containing the “planar” boundary condition $\wedge_{\partial \tilde{U}}^{(n)}$.

Putting everything together (plus a union bound over times $t \in [t_{n+1}, L^3]$ as in (6.11)) one gets

$$\mathbb{P}(A_x; \phi^x(s) \leq \psi_{u_n} \text{ for every } s \in [t_n, L^3]) = O(e^{-c(\log L)^{3/2}}) \times O(L^3 \times L^2) = O(e^{-\frac{c}{2}(\log L)^{3/2}}).$$

Finally, provided that we choose $c' = c/2$, from (6.16) we get

$$\mathbb{P}(\bigcup_{x \in U} A_x) \leq (n+1)e^{-c'(\log L)^{3/2}}$$

(again union bound over $x \in U$ gives just an extra $O(L^2)$) which is the desired claim.

7. SOS model: Proof of Theorem 3

The proof of Theorem 3 is based on the following crucial results. The first is essentially an application of the results in [12]2, which can be formulated as follows. Let $L$ and $0 \leq h \leq L$ be fixed and define the set $\Omega_{L,h}^\kappa$ of configurations $\xi \in \Omega_{L,h}$ satisfying

$$|\xi_i - \bar{\eta}_i| \leq \sqrt{L}(\log L)\kappa, \quad i = 1, \ldots, L,$$

where, as before, $\bar{\eta}_i = \lfloor ih/(L + 1) \rfloor$.

Proposition 5. There exist $\alpha_1 > 0$ and $c > 0$ such that the following holds for every $\kappa \geq 1$. For any $L$ sufficiently large (large depending on $\kappa$), uniformly in $0 \leq h \leq L$ and $\xi \in \Omega_{L,h}^\kappa$:

$$\|\mu_t^\xi - \pi\| \leq e^{-c(\log L)^2}, \quad \forall t \geq L^2(\log L)^{\alpha_1}.$$  

Moreover, the same statement holds for the evolution constrained to stay above the wall $\bar{\eta}$ (in this case of course one must require also that $\xi \geq \bar{\eta}$).

---

2The results in [12] are stated for the discrete time chain, and thus a trivial overall factor of $L$ must be taken into account when comparing our results with theirs. Moreover, [12] only deals with the case $h = 0$. 


From the proof it will follow that one can actually choose $\alpha_1 = 17$ (with room to spare). The second crucial result deals with the relaxation of the extremal evolutions (the analogous result for monotone surfaces is Proposition 1). Let $\eta^{\wedge,+}(t)$ denote the evolution of the maximal initial configuration $\wedge \equiv L + h$, with the wall constraint $\eta_i \geq \bar{\eta}_i$, $i = 1, \ldots, L$. Similarly, let $\eta^{\vee,-}(t)$ denote the evolution of the minimal initial configuration $\vee \equiv -L$, with the wall constraint $\eta_i \leq \bar{\eta}_i$, $i = 1, \ldots, L$.

**Proposition 6.** Let $\alpha_1 > 0$ be as in Proposition 5. For all $L$ sufficiently large, and for some time $T_1 = O(L^2 (\log L)^{4+\alpha})$:

$$\mathbb{P}(\exists i \in \{1, \ldots, L\}, \eta^{\wedge,+}(T_1) > \bar{\eta}_i + \sqrt{L} (\log L)^{5}) \leq L^{-3},$$  \hfill (7.3)

and similarly,

$$\mathbb{P}(\exists i \in \{1, \ldots, L\}, \eta^{\vee,-}(T_1) < \bar{\eta}_i - \sqrt{L} (\log L)^{5}) \leq L^{-3}.$$  \hfill (7.4)

Once Proposition 5 and Proposition 6 are established, it is an easy task to complete the proof of Theorem 3:

**Proof of Theorem 3.** From a standard comparison estimate, cf. also (6.5) above, one has

$$\max_{\eta} \|\mu^\eta_i - \pi\| \leq C L^2 \max \left( \|\mu^\wedge_i - \pi\|, \|\mu^\vee_i - \pi\| \right),$$

where $\mu^\wedge_i, \mu^\vee_i$ denote the law of evolutions $\eta^{\wedge}(t), \eta^{\vee}(t)$ from maximal and minimal initial condition respectively, with no wall constraint. Let $G$ denote the event that $|\eta_i - \bar{\eta}_i| \leq \sqrt{L} (\log L)^5$ for all $i$. Observe that for any event $A$, for $t > T_1$ (where $T_1$ is as in Proposition 6)

$$|\mu^\wedge_i(A) - \pi(A)| \leq \mathbb{P}(\eta^{\wedge}(t) \in A; \eta^{\wedge}(T_1) \in G) - \pi(A) + \mathbb{P}(\eta^{\wedge}(T_1) \notin G) \leq \max_{\xi \in G} \|\mu^\xi_{t-T_1} - \pi\| + 2 \mathbb{P}(\eta^{\wedge}(T_1) \notin G).$$

By monotonicity one can couple the dynamics in such a way that $\eta^{\vee,-}(T_1) \leq \eta^{\wedge}(T_1) \leq \eta^{\wedge,+}(T_1)$. Thus, from Proposition 6, $\mathbb{P}(\eta^{\wedge}(T_1) \notin G) \leq L^{-3}$ for all $L$ large enough. On the other hand, from Proposition 5 one has $\max_{\xi \in G} \|\mu^\xi_i - \pi\| \leq e^{-c(\log L)^2}$, for $T := L^2 (\log L)^{\alpha_1}$. Thus, taking $t = T_1 + T$, it follows that $\|\mu^\wedge_i - \pi\| \leq 3L^{-3}$, for all $L$ large enough. The same bound, by symmetry, can be obtained for $\|\mu^\vee_i - \pi\|$. Therefore,

$$\max_{\eta} \|\mu^\eta_i - \pi\| \leq 3C L^{-1},$$

which concludes the proof of Theorem 3, with any power $\alpha > \alpha_1 + 4$. \hfill \qed

### 7.1. Proof of Proposition 6

This and the next subsection are devoted to the proof of Proposition 6 (by symmetry it is sufficient to prove (7.3) only) assuming the validity of Proposition 5. The latter is proved in Section 7.3 below. We first recall a basic equilibrium estimate.

**Lemma 1.** There exists some constant $C > 0$ such that, uniformly in $0 \leq h \leq L$ and $H = \sqrt{L} \log L$:

$$\pi \left[ \exists i = 1, \ldots, L: |\xi_i - \bar{\eta}_i| > H \right] \leq C \exp \left( -C^{-1} \min\{H^2/L, H\} \right).$$  \hfill (7.5)

Moreover, (7.5) continues to hold as it is for the interface conditioned to stay above the wall, as well as for the unbounded SOS model defined in (3.1).

The proof of Lemma 1 can be obtained by standard large deviation arguments as in e.g. [12, Appendix C].

We give a detailed proof of Proposition 6 in the case $h = 0$ only. The reader may verify that the same approach with minor modifications gives a proof for all $0 \leq h \leq L$. In order to underline the similarities between the proof of Proposition 6 with that of the companion result Proposition 1 in the monotone surface case, we follow closely the notation introduced there.
Let \( \rho_L = L \log L \) and, for \( u > 0 \), let \( C_u \subset \mathbb{R}^2 \), denote the circular segment of height \( u \) and base a segment on the horizontal axis, with length \( 2\rho_L \) and centered at \( L/2 \). We shall use the notation \( \eta \in C_u \) for any configuration \( \eta \) such that \( 0 \leq \eta \leq \psi_u(i), i = 1, \ldots, L \) where \( \psi_u(i) = \sup \{ y \in \mathbb{R} : (i, y) \in C_u \} \). Note that the radius of curvature \( R \) of the circular segment \( C_u \) satisfies \( (2R - u)u = \rho_L^2 \). As in Section 6, in what follows we will always have \( u \ll \rho_L \ll R \) so that \( R = \rho_L^2/(2u)(1 + o(1)) \). Define recursively \( u_0 = 2L, t_0 = 0 \) and
\[
 u_{n+1} = u_n - (\rho_L^2/u_n)^{1/3}(\log L)^2, \quad t_{n+1} = t_n + (\rho_L^2/u_n)^{4/3}(\log L)^{3+\alpha_1}
\]
and let \( M = \min \{ n : u_n \leq \sqrt{L}(\log L)^4 \} \). Clearly, if \( R_n \) denotes the radius of curvature of the circular segment \( C_{u_n} \), then \( R_n = \rho_L^2/(2u_n)(1 + o(1)) \).

Crucially, as in the monotone surface case, \( t_M = \tilde{O}(L^2) \).

**Lemma 2.** For any \( L \) large enough \( t_M \leq L^2 (\log L)^{4+\alpha_1} \).

**Proof.** We write
\[
 t_M = \sum_{n=0}^{M-1} (\rho_L^2/u_n)^{4/3}(\log L)^{3+\alpha_1}.
\]  
Next observe that, by convexity, \( a^{4/3} - b^{4/3} \geq b^{1/3}(a - b) \) for all \( a > b > 0 \). Taking \( a = u_n \) and \( b = u_{n+1} \), and using \( u_n - u_{n+1} = (\rho_L^2/u_n)^{1/3}(\log L)^2 \) one has
\[
 u_n^{4/3} - u_{n+1}^{4/3} \geq u_n^{1/3}(u_n - u_{n+1}) \geq \frac{1}{2}L^{2/3}(\log L)^{8/3},
\]
where the last bound follows from \( u_{n+1} \geq \frac{1}{2}u_n \) which is easily seen to be implied by the assumption \( u_n \geq \sqrt{L}(\log L)^4 \) for all \( 0 \leq n \leq M - 1 \). Therefore,
\[
 u_n^{4/3} = u_n^{4/3} + \sum_{m=n}^{M-2} (u_m^{4/3} - u_{m+1}^{4/3})
\geq u_{M-1}^{4/3} + \frac{1}{2}(M - n - 1)L^{2/3}(\log L)^{8/3} \geq \frac{1}{2}L^{2/3}(\log L)^{8/3},
\]
the last bound following from \( u_{M-1}^{4/3} \geq L^{2/3}(\log L)^{8/3} \). In conclusion, using this bound in (7.6),
\[
 t_M \leq 2L^2 (\log L)^{3+\alpha_1} \sum_{n=0}^{M-1} \frac{1}{M - n} \leq 2L^2 (\log L)^{4+\alpha_1},
\]
whenever \( L \) is large enough. \( \square \)

**Remark 5.** As the careful reader has noticed, the length and time scales \( (u_n, t_n) \) are quite different from their analogue in the monotone surface case. The main reason is the different order of magnitude of the maximal equilibrium fluctuations in the two models: \( \log L \) versus \( \sqrt{L} \). However their value is determined by the common recipe which was described in Section 4.

The key step in the proof of (7.3) is analogous to Proposition 4.

**Proposition 7.** For any \( L \) large enough and for every \( 0 \leq n \leq M \), with probability at least \( 1 - n e^{-(\log L)^{3/2}} \),
\[
 \eta(t) \in C_{u_n} \text{ for every } t \in [t_n, t_{n+1}].
\]

If we now apply Proposition 7 with \( n = M \) and use the fact that \( t_M = O(L^2 (\log L)^{4+\alpha_1}) \) and \( u_M \leq \sqrt{L}(\log L)^{5} \), we obtain the desired claim (7.3). \( \square \)
7.2. **Proof of Proposition 7.** As in the proof of Proposition 4 we proceed by induction in \( n \leq M \). The initial step \( n = 0 \) is obvious because the maximal configuration \( \Lambda \) is inside \( C_{u_0} \). Thus let us assume the statement true for \( n < M \) and let us prove it for \( n + 1 \).

For \( 1 \leq i \leq L \) define the event

\[
A_i = \{ \exists t \in [t_{n+1}, L^3] : \eta^\wedge_i(t) > \psi_{u_{n+1}}(i) \}.
\]

Using the inductive assumption we may write

\[
\mathbb{P}(\bigcup_{i=1}^L A_i) \leq \sum_{i=1}^L \mathbb{P}(A_i; \eta^\wedge(s) \in C_{u_n} \text{ for every } s \in [t_n, L^3]) + n e^{-(\log L)^{3/2}}. \tag{7.7}
\]

Fix \( i = 1, \ldots, L \) and consider the line \( L \) tangent to \( C_{u_n} \) at the point \( (i, \psi_{u_n}(i)) \) and the line \( L' \) obtained by translating downwards (i.e. in the \(-y\) direction) \( L \) by

\[
2(u_n - u_{n+1}) = 2(\rho_L^2/u_n)^{1/3}(\log L)^2.
\]

Let us denote by \( x_- , x_+ \) the horizontal coordinates of the leftmost and rightmost points of \( L \cap C_{u_n} \) and let \( I \) be the set of integers in \([x_- , x_+]\). Clearly \(|I| = O((\rho_L^2/u_n)^{2/3} \log L)\).

Consider now the SOS dynamics in the interval \( I \) with boundary conditions equal to \([\psi_{u_n}(i))\) at the left and right boundary \( i_\pm \) of \( I \) and floor at zero height. This auxiliary evolution starts at time \( t_n \) from the maximal configuration \( \wedge^{(n)} \) in the set of \( \eta \in [0, L]^I \) such that \( \eta_i \leq \psi_{u_n}(i) \) for any \( i \in I \). We denote by \( \mathbb{P}' \) the law of this auxiliary chain. Observe that \( \wedge^{(n)} \) is within distance

\[
2(u_n - u_{n+1}) = O(\sqrt{|I|}(\log |I|)^{3/2})
\]

from the line \( L' \) so that Proposition 5 will be applicable with \( \kappa = 3/2 \). By monotonicity we have

\[
\mathbb{P}(A_i; \eta^\wedge(s) \in C_{u_n} \text{ for every } s \in [t_n, L^3]) \leq \mathbb{P}'(\exists t \in [t_{n+1}, L^3] \text{ such that } \eta^\wedge_i(t) \geq \psi_{u_{n+1}}(i)).
\]

Because of Proposition 5, after time \(|I|^2(\log |I|)^{\alpha_1} \leq t_{n+1} - t_n \) the dynamics \( \mathbb{P}' \) has a variation distance of order \( \exp(-c(\log L)^2) \) from its equilibrium which we denote by \( \pi^{(n)}_{\wedge} \). Since the distance between the point \((i, \psi_{u_{n+1}}(i))\) and the line \( L' \) is at least \( c\sqrt{|I|}(\log |I|)^{3/2} \) for some \( c > 0 \), Lemma 1 gives that

\[
\pi^{(n)}_{\wedge}(\eta_i \leq \psi_{u_{n+1}}(i)) \geq 1 - O\left(e^{-c(\log |I|)^3}\right) \geq 1 - e^{-c'(\log L)^3}.
\]

As in the proof of Proposition 4, simple union bounds over \( i \in [1, L] \) and \( t \in [t_{n+1}, L^3] \) give

\[
\sum_{i=1}^L \mathbb{P}(A_i; \eta^\wedge(s) \in C_{u_n} \text{ for every } s \in [t_n, L^3]) \leq L^5 e^{-c''(\log L)^2} \leq e^{-(\log L)^{3/2}}
\]

which finishes the proof of the inductive step.

\[\square\]

7.3. **Proof of Proposition 5.** We describe the main steps in the absence of the wall. At the end we will comment on the case with the wall.

Let \( \xi^{\wedge}(t), \xi^{\vee}(t) \) denote the Glauber dynamics at time \( t \) starting from the maximal \( \wedge \) and minimal \( \vee \) initial condition in \( \Omega^\wedge_{L,h}, \) respectively, and let \( \mu^{\wedge}_{\xi}, \mu^{\vee}_{\xi} \) denote their distribution. Notice that \( \wedge, \vee \) are different from the maximal \( \wedge \) and minimal \( \vee \) configuration in \( \Omega_{L,h} \). For any \( \xi \in \Omega^\wedge_{L,h}, \) for any event \( A \subset \Omega_{L,h} \) one has:

\[
\mu^{\xi}(A) - \pi(A) = \sum_{\eta \in \Omega_{L,h}} \pi(\eta)(\mu^{\xi}(A) - \mu^{\eta}(A)).
\]
Therefore, 
\[
\max_{\xi \in \Omega_{L,h}} \|\mu_t^\xi - \pi\| \leq \pi(\Omega_{L,h} \setminus \Omega_{L,h}^* ) + \max_{\xi, \eta \in \Omega_{L,h}} \|\mu_t^\xi - \mu_t^\eta\|.
\]

By monotonicity, \(\max_{\xi, \eta \in \Omega_{L,h}} \|\mu_t^\xi - \mu_t^\eta\| \leq \mathbb{P}(\xi^{\wedge \kappa}(t) \neq \xi^{\vee \kappa}(t))\), where \(\mathbb{P}(\cdot)\) denotes the global monotone coupling (see Section 2.2). Using (7.5) to estimate \(\pi(\Omega_{L,h} \setminus \Omega_{L,h}^* )\) for any \(\kappa \geq 1\), we arrive at 
\[
\max_{\xi \in \Omega_{L,h}} \|\mu_t^\xi - \pi\| \leq Ce^{-(\log L)^2/C} + \mathbb{P}(\xi^{\wedge \kappa}(t) \neq \xi^{\vee \kappa}(t)).
\]

Let now \(\nu_+ := \pi(\cdot | \xi \geq \wedge \kappa)\) and \(\nu_- := \pi(\cdot | \xi \leq \vee \kappa)\). Monotonicity implies that 
\[
\mathbb{P}(\xi^{\wedge \kappa}(t) \neq \xi^{\vee \kappa}(t)) \leq \sum_{i=1}^L \sum_{j=-L}^{L+h} \mathbb{P}(\xi^{\wedge \kappa}(t) \geq j ; \xi^{\vee \kappa}(t) < j) = \sum_{i=1}^L \sum_{j=-L}^{L+h} \left[ \mu_t^{\wedge \kappa}(\xi_i \geq j) - \mu_t^{\vee \kappa}(\xi_i \geq j) \right]
\]
\[
\leq 2L(L + h) (\|\mu_t^{\nu_+} - \pi\| + \|\mu_t^{\nu_-} - \pi\|) = 4L(L + h)\|\mu_t^{\nu_+} - \pi\|
\]
where \(\mu_t^{\nu_\pm}\) is the law of the process at time \(t\) starting from the distribution \(\nu_\pm\). Above we used symmetry to conclude that \(\|\mu_t^{\nu_+} - \pi\| = \|\mu_t^{\nu_-} - \pi\|\).

Since the event \(\{\xi \geq \wedge \kappa\}\) is increasing, one has \(\pi \preceq \nu_+\) and the relative density \(\nu_+ / \pi\) is an increasing function. The Peres-Winkler censoring argument [13] (see also Lemma 2.2 in [12]) proves that the same holds if we replace \(\nu_+\) with \(\mu_t^{\nu_+}\). Thus the event \(\{\mu_t^{\nu_+}(\xi) \geq \pi(\xi)\}\) is increasing and \(\|\mu_t^{\nu_+} - \pi\| = \mu_t^{\nu_+}(F) - \pi(F)\).

Next, we claim that, for some constants \(c_1 \leq 17\) and \(c > 0\), for all \(t \geq L^2 (\log L)^{c_1}\) and for any increasing event \(A\),
\[
\mu_t^{\nu_+}(A) - \pi(A) \leq e^{-(\log L)^2}.
\]

Clearly (7.8) finishes the proof of Proposition 5. In turn, the proof of (7.8) is adapted from the proof of a similar result in [12] (see Lemma 4.5 there). For the reader’s convenience, we now describe its main steps.

**Step 1.** One first introduces an auxiliary parallel column dynamics which consists in replacing the single-site moves of the Glauber dynamics (3.1) with non-local moves as follows. At each integer time \(s = 1, 2, \ldots\) firstly all even-numbered heights \(\{\xi_{2i}\}, i = 1, 2, \ldots\), are re-sampled from their equilibrium distribution given the neighboring odd-numbered heights \(\xi_{2i+1}, i = 0, 1, \ldots\), and then vice versa with the role of even/odd columns reversed. By construction the column chain is ergodic and reversible w.r.t. \(\pi\). Moreover, and that is the main reason for introducing it, Wilson’s argument [18] applies to the new chain. Namely, by an almost exact computation, one can prove that under the column dynamics the maximal and minimal configuration in \(\Omega_{L,h}\) couple in a time \(O(L^2 \log L)\) (see Section 3 in [12]).

**Step 2.** Next, one relates the single site Glauber dynamics started from \(\nu_+\) to the column dynamics described above by means of the Peres-Winkler censoring idea. More precisely, given a time-lag \(T\), one splits the time interval \([0, t]\) into \(N = t / 2T\) epochs each of length \(2T\) (we assume for simplicity \(N \in \mathbb{N}\)) and in each epoch one first censors (i.e. freezes) all the Poisson clocks at odd sites for the first half of the epoch and then all the even sites for the second half. If \(\mu_t^{\nu_+}\) denotes the distribution of the censored chain, then the censoring inequality of [13] says that \(\mu_t^{\nu_+} \preceq \tilde{\mu}_t^{\nu_+}\) so that, for any increasing event \(A\),
\[
\mu_t^{\nu_+}(A) - \pi(A) \leq \tilde{\mu}_t^{\nu_+}(A) - \pi(A) \leq \|\tilde{\mu}_t^{\nu_+} - \pi\|.
\]
If the free parameter \( T \) is chosen in such a way that each epoch simulates very closely one step of the column chain then, thanks to Step 1, 
\[
\| \tilde{\mu}_t^\nu - \pi \| \leq e^{-c(\log L)^2}
\]
provided that \( N \geq L^2(\log L)^3 \).

As explained in \cite{12}, the overhead \( T \) introduced in the mixing time by the censoring is essentially the square of the maximum gradient \( |\nabla \xi|_\infty := \max_i |\xi_{i+1} - \xi_i| \) that may arise in the censored dynamics. Since gradients of the equilibrium interface are i.i.d. random variables with exponential tail, conditioned to their sum being equal to \( h \), well known estimates (see e.g. Lemma 1 above or \cite{12, Appendix C}) imply that for any \( i = 0, \ldots, L \), uniformly in \( 0 \leq h \leq L \), and \( \ell \geq (\log L)^2 \):
\[
\pi (|\nabla \xi|_\infty \geq \ell) \leq C e^{-\ell/C}.
\]
Therefore, for any event \( A \) and starting from the distribution \( \nu := \pi(\cdot | A) \), the invariance of \( \pi \) implies that
\[
\mu_t^\nu (|\nabla \xi|_\infty \geq \ell) \leq \frac{\pi (|\nabla \xi|_\infty \geq \ell)}{\pi(A)} \leq C \frac{e^{-\ell/C}}{\pi(A)},
\]
for \( \ell \geq (\log L)^2 \). A similar bound holds for the censored dynamics \( \tilde{\mu}_t^\nu \). The above observations lead to the following lemma\(^3\).

**Lemma 3** (see Lemma 4.2 in \cite{12}). Let \( A \) be any increasing event, and consider the censored single-site dynamics started from \( \nu := \pi(\cdot | A) \). Let \( D := 1 + \log(\frac{L}{\pi(A)}) \) and choose \( T = D^2(\log L)^9 \) and \( t = TL^2(\log L)^3 \). Then, for some constant \( c > 0 \) and for all \( L \) large enough
\[
\| \tilde{\mu}_t^\nu - \pi \| \leq e^{-c(\log L)^2}.
\]

**Step 3.** The above lemma cannot be applied directly to our case \( A = \{ \xi \geq \land \} \) because at time \( t = 0 \) the maximum gradient is too large, of order \( \sqrt{L} (\log L)^\alpha \), and therefore \( \pi(A) = O(e^{-c\sqrt{L}(\log L)^\alpha}) \) is very small. This makes the corresponding overhead \( T \) too large. The solution to this problem proposed in \cite{12} goes as follows (see also Lemma 4.5 and Corollary 4.6 there).

**Lemma 4.** Fix \( \kappa \geq 1 \). Let \( (\log L)^2 \leq H \leq \sqrt{L} (\log L)^\alpha \) and let \( A_H = \{ \xi_i \geq \bar{\eta}_i + H, \ i = 1, \ldots L \} \). Let \( \nu_H := \pi(\cdot | A_H) \) and let \( t = L^2(\log L)^3 \). Then, for some \( c > 0 \), for \( L \) large enough,
\[
\mu_t^{\nu_H}(f) \leq \nu_H/2(f) + e^{-c(\log L)^2}, \tag{7.9}
\]
for any non-negative, increasing function \( f \) with \( \| f \|_\infty \leq 1 \).

**Proof.** A sketch of the proof is as follows. Choose \( \ell = H^2/(\log L)^2 \) and consider the Glauber dynamics on the enlarged state space \( \Omega_{t,L,H} \) of SOS interfaces \( \{\eta_i\}_{i=\ell}^{L+\ell} \) such that
\[
\begin{align*}
\eta_i \in [-L, L + h] & \quad \text{if } i \in [1, L] \\
\eta_i \geq \lceil i h/(L + 1) \rceil & \quad \text{if } i \in [-\ell] \cup [L + 1, L + \ell]
\end{align*}
\]
and with boundary condition
\[
\eta_{-(\ell+1)} = -\lceil (\ell + 1) h/(L + 1) \rceil, \quad \eta_{L+\ell+1} = \lceil (L + \ell + 1) h/(L + 1) \rceil.
\]
Let \( \pi^{(\ell)} \) be the corresponding SOS reversible equilibrium measure and let \( \mu_t^{\nu_t^{(\ell)}} \) be the distribution at time \( t \) starting from \( \nu_t^{(\ell)} := \pi^{(\ell)}(\cdot | A_H) \). Clearly the marginals of \( \mu_t^{\nu_t^{(\ell)}} \) and of \( \pi^{(\ell)} \) on

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\(^3\) Actually in \cite{12} the bound is given as \( o(1/L^b) \) for any \( b > 0 \) but it is easily seen to hold in the form stated here.
\( \Omega_{L,h} \) are stochastically larger than \( \mu_t^{\nu_H} \), and \( \pi \) respectively. Moreover, from standard local limit theorem estimates (see e.g. [12, Appendix C]) one has
\[
\pi^{(t)}(A_H) \geq \frac{1}{C L^2} e^{-CH^2/\ell} \geq e^{-C'(\log L)^2}
\] (7.10)
for suitable constants \( C, C', \gamma > 0 \). Then, we apply Lemma 3 to the enlarged dynamics, with \( D = O((\log L)^2) \), and get that at time \( t = L^2(\log L)^b \), with \( b = 16 \),
\[
\|\mu_t^{\nu_H} - \pi^{(t)}\| \leq e^{-c(\log L)^2}.
\]
Thus, for any \( f \) as in the lemma,
\[
\mu_t^{\nu_H}(f) \leq \mu_t^{\nu_{H_0}}(f) \leq \pi^{(t)}(f) + e^{-c(\log L)^2}.
\]
Finally, let \( E_H = \{ \eta \in \Omega_{\ell,L,h} : \{ \eta_1 \leq H \} \cap \{ \eta_L \leq h + H \} \} \). Then, using the positivity of \( f \),
\[
\pi^{(t)}(f) \leq \pi^{(t)}(f | E_{H/2}) + \pi^{(t)}(E_{H/2}') \leq \pi(f | A_{H/2}) + \pi^{(t)}(E_{H/2}').
\]
The Gaussian bounds of Lemma 1 show that
\[
\pi^{(t)}(E_{H/2}') \leq C e^{-H^2/C\ell} \leq e^{-c(\log L)^2}.
\]
In conclusion, the desired estimate (7.9) holds with \( t = L^2(\log L)^{16} \). \( \Box \)

Using the semigroup property, one can iterate the bound of Lemma 4 to go from the initial height \( H_0 := \sqrt{A} (\log L)^{\alpha} \) to the final height \( H_n := (\log L)^2 \), with \( n = O(\log L) \) steps. Therefore, at time \( t = c' L^2(\log L)^{17} \), one has, for all \( f \) as in Lemma 4:
\[
\mu_{2t}^{\nu_{H_0}}(f) \leq \mu_t^{\nu_{H_n}}(f) + e^{-c''(\log L)^2}, \quad (7.11)
\]
for some constant \( c', c'' > 0 \). To finish the proof Proposition 5, cf. (7.8), observe that \( \nu_{H_0} = \nu_\ell \) and therefore, by (7.11), it is sufficient to establish
\[
\mu_t^{\nu_{H_n}}(f) \leq \pi(f) + e^{-c_1(\log L)^2}, \quad (7.12)
\]
for some \( c_1 > 0 \), for all \( f \) as above. However, from the censoring inequality we know that \( \mu_t^{\nu_{H_n}}(f) \leq \tilde{\mu}_t^{\nu_{H_n}}(f) \). Moreover, a shift of the boundary height by \( O((\log L)^2) \) shows that
\[
\pi(A_{H_n}) \geq e^{-C(\log L)^2}.
\]
Therefore, (7.12) follows from Lemma 3, with \( D = O((\log L)^2) \), for any \( t \geq D^2 L^2(\log L)^{12} = O(L^2(\log L)^{16}) \). This ends the proof of Proposition 5 without the wall, with the choice of the constant \( c_1 = 17 \).

When the wall is present, the symmetry between the maximal and minimal configuration breaks down and one has the new problem of proving convergence to equilibrium within the correct time scale starting from the minimal configuration \( \eta \). This problem has been solved in [12] when \( h = 0 \) but the same proof applies to \( h \in [0, L] \). \( \Box \)

8. SOS model: Proof of Theorem 4

Let \( \text{gap}(L, H) \) denote the spectral gap of the SOS model with zero boundary conditions, floor at \(-H\) and ceiling at \(+H\). Similarly, let \( \text{gap}^w(L, H) \) denote the same quantity for the system with the wall, i.e. the spectral gap of the SOS model with zero boundary conditions, floor at 0 and ceiling at \(+H\).

**Proposition 8.** There exists \( c > 0 \) such that for all \( L \in \mathbb{N} \) sufficiently large, and for all \( H \in \mathbb{N} \):
\[
\text{gap}(L, H) \geq c L^{-2}(\log L)^{-\alpha}, \quad (8.1)
\]
where \( \alpha > 0 \) is the same constant appearing in Theorem 3. The same bound holds for \( \text{gap}^w(L, H) \).
The first observation is that Proposition 8 immediately implies Theorem 4. Indeed, this follows from an elementary approximation since the bound (8.1) is uniform in $H$: both $\mathcal{E}(f, f)$ and $\text{Var}(f)$ can be seen as limit as $H \to \infty$ of the corresponding quantities for the model with floor and ceiling at $\pm H$, so that for fixed $L$ one has $\text{gap}(L) \geq \liminf_{H \to \infty} \text{gap}(L, H)$.

8.1. Proof of Proposition 8. To prove Proposition 8 we use a recursive approach together with a powerful idea borrowed from the mathematical theory of kinetically constrained spin systems (see section 4 in [2] and the proof of Proposition 9 below). We give the details of the proof of the lower bound on $\text{gap}^w(L, H)$, and will later illustrate the minor modifications necessary to cover the case without the wall; see Section 8.3 below.

For any rectangle $\Lambda$ in $\mathbb{Z}^2$, with base $\ell_\Lambda$ (horizontal side) and height $h_\Lambda$ (vertical side), we write $\pi_\Lambda$ for the equilibrium measure of the SOS model in the rectangle $\Lambda$ with zero boundary conditions at the endpoints of the bottom horizontal side of length $\ell_\Lambda$ (the lower horizontal side of the rectangle has vertical coordinate zero). We call $\Omega_\Lambda$ the set of all allowed configurations, i.e. $\Omega_\Lambda = \{0, \ldots, h_\Lambda\}^{\ell_\Lambda}$. Let also

$$\text{gap}(\Lambda) = \inf_f \frac{\mathcal{E}_\Lambda(f, f)}{\text{Var}_\Lambda(f)} , \tag{8.2}$$

where we use the notation $\mathcal{E}_\Lambda$, $\text{Var}_\Lambda$ for the Dirichlet form and the variance associated to $\pi_\Lambda$, and $f$ ranges over all functions such that $\text{Var}_\Lambda(f) \neq 0$. Note that, with this notation, one has $\text{gap}(\Lambda) = \text{gap}^w(\ell_\Lambda, h_\Lambda)$. Define

$$\gamma(L,n) = \max_{\Lambda \in \Omega_n} \text{gap}(\Lambda)^{-1} ,$$

where $\Omega_n$ is the class of all rectangles $\Lambda$ in $\mathbb{Z}^2$ such that $\ell_\Lambda \leq L$ and $h_\Lambda \leq (3/2)^n$. The crucial recursive estimate reads as follows.

**Proposition 9.** Let $n_0 = n_0(L) \in \mathbb{N}$ be such that $(3/2)^{n_0} \leq L < (3/2)^{n_0+1}$. Then for any $L$ large enough, for any $n \geq n_0$,

$$\gamma(L,n) \leq (1+\delta_n) \gamma(L,n-1) \tag{8.3}$$

where $\delta_n = (3/2)^{-n/6}$.

Let us finish the proof of Proposition 8, assuming the validity of Proposition 9. Iterating the bound (8.3) one has

$$\gamma(L,n) \leq C \gamma(L,n_0) , \tag{8.4}$$

and

$$C = \prod_{n=n_0}^{\infty} (1+\delta_n) < \infty ,$$

uniformly in $L$. Proposition 8 now follows from (8.4) and the following lemma.

**Lemma 5.** There exists $C < \infty$ such that for any $L$ large enough

$$\gamma(L,n_0) \leq CL^2 (\log L)^\alpha ,$$

with $n_0 = n_0(L)$ as in Proposition 9.

**Proof.** Note that, thanks to the straightforward bound $\text{gap}^{-1} \leq T_{\text{mix}}$ (which holds for any reversible Markov chain [10]), if $h_\Lambda \leq (3/2)^{n_0} \leq \ell_\Lambda \leq CL$ for some $C > 0$, from Theorem 3 one has

$$\text{gap}(\Lambda)^{-1} = O(L^2 (\log L)^\alpha) .$$

(Strictly speaking Theorem 3 is stated only for $h_\Lambda \leq \ell_\Lambda$, but the proof works as it is as soon as $h_\Lambda \leq C\ell_\Lambda$ for some $C$). Therefore, the same bound applies to every $\Lambda \in \Omega_{n_0} \in \mathbb{F}_{n_0,L}$ such that
\( \ell_A > L/8 \). Thus, one has to estimate only \( \max\{\text{gap}(\Lambda)^{-1}, \Lambda \in \mathcal{F}_{n_0L} : \ell_A \leq L/8\} \). In order to deal with this term, define \( f(L) = \gamma(L, n_0) \), for all \( L \) and \( n_0 = n_0(L) \) as above. From Proposition 9, one has
\[
\max\{\text{gap}(\Lambda)^{-1}, \Lambda \in \mathcal{F}_{n_0L} : \ell_A \leq L/8\} \leq C_1 f(L/8),
\]
where the constant \( C_1 = \prod_{n=n_0-3}^{n_0} (1 + \delta_n) \leq 2 \) for \( n_0 \) large enough. In conclusion,
\[
f(L) \leq \max\left\{2f(L/8), CL^2(\log L)\alpha\right\} \leq 2f(L/8) + CL^2(\log L)\alpha.
\]
A simple induction now proves that \( f(L) \leq C'L^2(\log L)^\alpha \) for some new constant \( C' \).

8.2. Proof of Proposition 9. Take \( n \in \mathbb{N} \) and \( \Lambda \in \mathcal{F}_{nL}. \) Without loss of generality we write \( \Lambda = [1, \ell_A] \times [0, h_A] \). We need to show that
\[
\text{gap}(\Lambda)^{-1} \leq (1 + \delta_n)\gamma(L, n-1).
\]
We use a geometric construction analogous to that of [1, Proposition 3.2]. If \( h_A \leq (3/2)^{n-1} \) then actually \( \Lambda \in \mathcal{F}_{n-1L} \) and therefore \( \text{gap}(\Lambda)^{-1} \leq \gamma(L, n-1) \). If instead \( h_A > (3/2)^{n-1} \), then it is easily seen that we can write \( \Lambda = \Lambda_1 \cup \Lambda_2 \) where \( \Lambda_1, \Lambda_2 \) are such that:
(a) \( \Lambda_i \in \mathcal{F}_{n-1L}, i = 1, 2; \)
(b) \( \Lambda_i, i = 1, 2, \) has the same base \([1, \ell_A]\) of \( \Lambda; \)
(c) the overlap rectangle \( I = \Lambda_1 \cap \Lambda_2 \neq \emptyset \) has base length \( \ell_A \) and height at least \( \Delta_n := (3/2)^{3n/4} \).

Moreover, there are at least \( s_n := (3/2)^{n/5} \) such decompositions \( \{(\Lambda_1^{(i)}, \Lambda_2^{(i)})\}_{i=1}^{s_n} \) with the property that the overlap rectangles \( I^{(i)} = \Lambda_1^{(i)} \cap \Lambda_2^{(i)} \) are disjoint, i.e. \( I^{(i)} \cap I^{(j)} = \emptyset \) for all \( i \neq j \).

Next, fix one of the \( s_n \) decompositions mentioned above, say \( \Lambda_1 = [1, \ell_A] \times [0, h_{A_1}] \) and \( \Lambda_2 = [1, \ell_A] \times [h_{A_1} - |I|, h_A], \) where \( h_A = h_{A_1} + h_{A_2} - h_I \), with \( h_I \geq \Delta_n \) denoting the height of the overlap rectangle \( I \). Consider the distribution \( \pi_{A_2}^{\eta} \) corresponding to the equilibrium measure in the region \( \Lambda_2 \) conditioned to the value of the configuration \( \eta \) in the region \( \Lambda \setminus \Lambda_2 = [1, \ell_A] \times [0, h_{A_1} - h_I] \). Let
\[
C(\eta) := \{i \in [1, \ell_A] : \eta_i > h_{A_1} - h_I\}.
\]
Note that, if \( C(\eta) \neq \emptyset \), then \( C(\eta) \) is the disjoint union of intervals \( B_1, B_2, \ldots, B_k \), and one has that \( \pi_{A_2}^{\eta} \) is a product measure \( \otimes_{i=1}^{k} \pi_{R_i} \) on the rectangles \( R_i = B_i \times [h_{A_1} - h_I + 1, h_A], \) such that each \( \pi_{R_i} \) is the SOS equilibrium measure with floor at height \( h_{A_1} - h_I + 1 \) and ceiling at height \( h_A \). If, on the other hand, \( C(\eta) = \emptyset \), then \( \eta \) is entirely contained inside \( \Lambda \setminus \Lambda_2 \), and \( \pi_{A_2}^{\eta} \) is trivial, i.e. it gives full mass to the empty configuration in the region \( \Lambda_2 \).

Define a constrained block dynamics as follows. With rate one the current configuration \( \eta \) is re-sampled inside \( \Lambda_2 \) from the distribution \( \pi_{A_2}^{\eta} \) described above. Moreover, if \( \eta \subset \Lambda_1, \) that is \( \eta_i \leq h_{A_1} \) for all \( i = 1, \ldots, \ell_A \), then with rate one, the current configuration \( \eta \) is re-sampled inside \( \Lambda_1 \) from the distribution \( \pi_{A_1}^{\eta} \) (defined just before (8.2)). This process is kinetically constrained in that the region \( \Lambda_1 \) is updated only if \( \eta \subset \Lambda_1 \).

It is not hard to check that the above dynamics is reversible w.r.t. the SOS equilibrium measure \( \pi_{A_1} \), and that its Dirichlet form is equal to:
\[
\mathcal{E}_{\text{block}}(f, f) = \pi_A \left( 1_{\{\eta \subset \Lambda_1\}} \text{Var}_{A_1}(f) + \text{Var}_{A_2}^{\eta}(f) \right),
\]
where \( \text{Var}_{A_1} \) stands for the variance w.r.t. \( \pi_{A_1} \), while \( \text{Var}_{A_2}^{\eta} \) stands for the variance w.r.t. \( \pi_{A_2}^{\eta} \). Notice that, even starting from the maximal configuration \( \Lambda \) in \( \Lambda \), if we first update \( \Lambda_2 \) and then \( \Lambda_1 \), with very high probability we reach the distribution \( \pi_{A_1} \). Indeed, thanks to the fact that \( h_I \geq \Delta_n \gg \sqrt{\ell_A} \), the first update in \( \Lambda_2 \) will produce with very high probability a new configuration entirely contained in \( \Lambda_1 \). In turn, since \( \|\pi_{A_1} - \pi_A\| \) is very small (because \( h_{A_1} \gg \sqrt{\ell_A} \)), the above
example suggests that the spectral gap of the constrained block dynamics should be very close to one. This is quantified in the next lemma.

**Lemma 6.** For all $L$ large enough, and $n \geq n_0(L)$, for all functions $f$:

$$\text{Var}_\Lambda(f) \leq (1 + e^{-(3/2)n/4}) \mathcal{E}_\text{block}(f, f).$$

Assuming the validity of Lemma 6, the proof of Proposition 9 can be completed as follows. From (3.6) and (8.2), for all $f : \Omega_\Lambda \to \mathbb{R}$:

$$\pi_\Lambda \left( \{ \eta \subset \Lambda_1 \} \text{Var}_{\Lambda_1}(f) \right)$$

$$\leq \gamma(L, n - 1) \frac{1}{2} \sum_{i=1}^{\ell_\Lambda} \pi_\Lambda \left[ p_{i,+}(\nabla i,+f)^2 1_{\{ \eta_i \leq h_{\Lambda_1} \}} + p_{i,-}(\nabla i,-f)^2 1_{\{ \eta_i \leq h_{\Lambda_1} \}} \right]. \tag{8.5}$$

Similarly,

$$\pi_\Lambda (\text{Var}_{\Lambda_2}^0(f))$$

$$\leq \gamma(L, n - 1) \frac{1}{2} \sum_{i=1}^{\ell_\Lambda} \pi_\Lambda \left[ p_{i,+}(\nabla i,+f)^2 1_{\{ \eta_i \geq h_{\Lambda_1} - h_I \}} + p_{i,-}(\nabla i,-f)^2 1_{\{ \eta_i \geq h_{\Lambda_1} - h_I \}} \right]. \tag{8.6}$$

From Lemma 6, (8.5) and (8.6) we have

$$\text{Var}_\Lambda(f) \leq (1 + e^{-(3/2)n/4}) \gamma(L, n - 1) (\mathcal{E}_\Lambda(f, f) + \mathcal{E}_\text{I}(f, f)) \tag{8.7}$$

where $\mathcal{E}_\Lambda(f, f)$ is the usual Dirichlet form in $\Lambda$, while

$$\mathcal{E}_\text{I}(f, f) := \frac{1}{2} \sum_{i=1}^{\ell_\Lambda} \pi_\Lambda \left[ p_{i,+}(\nabla i,+f)^2 1_{\{ h_{\Lambda_1} - h_I \leq \eta_i \leq h_{\Lambda_1} \}} + p_{i,-}(\nabla i,-f)^2 1_{\{ h_{\Lambda_1} - h_I \leq \eta_i \leq h_{\Lambda_1} \}} \right].$$

Since (8.7) is valid for every one of the $s_n$ choices of the rectangles $(\Lambda_1^{(j)}, \Lambda_2^{(j)})_{j=1}^{s_n}$, one can average this estimate to obtain, with $I^{(j)} = \Lambda_1^{(j)} \cap \Lambda_2^{(j)}$:

$$\text{Var}_\Lambda(f) \leq (1 + e^{-(3/2)n/4}) \gamma(L, n - 1) (\mathcal{E}_\Lambda(f, f) + s_n^{-1} \sum_{j=1}^{s_n} \mathcal{E}_\text{I}^{(j)}(f, f)).$$

Since the overlaps $I^{(j)}$ are disjoint, one has the obvious bound $\sum_{j=1}^{s_n} \mathcal{E}_\text{I}^{(j)}(f, f) \leq \mathcal{E}_\Lambda(f, f)$. It follows that

$$\gamma(L, n) \leq (1 + e^{-(3/2)n/4}) (1 + s_n^{-1}) \gamma(L, n - 1).$$

By construction, $s_n^{-1} = O((3/2)^{-n/5})$, so that $(1 + e^{-(3/2)n/4}) (1 + s_n^{-1}) \leq 1 + \delta_n$ for all $n$ large enough, with $\delta_n = (3/2)^{-n/6}$. This concludes the proof of Proposition 9.

**Proof of Lemma 6.** The proof is similar to that of [2, Proposition 4.4]. Let $G = \{ \eta \in \Omega_\Lambda : \eta \subset \Lambda_1 \}$. Note that the event $G$ is very likely to occur under the equilibrium $\pi_\Lambda$. In particular, from (7.5), one has

$$\epsilon := \pi_\Lambda(G^c) \leq C \exp(-C^{-1}(3/2)^n). \tag{8.8}$$

Moreover, $G$ is very likely to occur even under the equilibrium $\pi_\Lambda^n$, uniformly in $\eta \in \Omega_\Lambda$. Indeed, by monotonicity, the smallest value of $\pi_\Lambda^n(G)$ is achieved at configurations $\eta \in \Omega_\Lambda$ such that $\eta_i > h_{\Lambda_1} - h_I$ for all $i = 1, \ldots, \ell_\Lambda$ and in that case $\pi_\Lambda^n(G)$ is the $\pi_\Lambda^\text{I}$-probability in the rectangle
\[ \Lambda^*_2 := [1, \ell_A] \times [0, h_{A_2}] \] that the height does not exceed \( h_f \) at any point. Therefore, using \( h_f^2/\ell_A \geq \Delta^2_n/2 \) and the bounds (7.5) one has
\[
\delta := \max_{\eta} \pi^{\eta}_{A_2}(G^c) \leq C \exp (-C^{-1} (3/2)^{n/2}).
\] (8.9)

On the other hand, reasoning as in (7.10) (and using \( h_{A_2} \gg \sqrt{\lambda_A} \)), \( \delta \) also satisfies
\[
\delta \geq C^{-1} \epsilon^{-C(3/2)^{n/2}},
\]
and therefore in particular \( \delta \geq \sqrt{\epsilon/(1 - \epsilon)} \) for all \( n \) large enough.

The infinitesimal generator of the block dynamics acts on functions \( f : \Omega_A \mapsto \mathbb{R} \) as
\[
\mathcal{L}_{\text{block}} f(\eta) = 1_G(\eta) \left( \pi_{A_1}(f) - f(\eta) \right) + \pi^{\eta}_{A_2}(f) - f(\eta).
\]

Let \( \lambda \) be the spectral gap of the block dynamics and let \( f \) be an eigenfunction of \( \mathcal{L}_{\text{block}} \) with eigenvalue \(-\lambda\). Thus, \( f \) must satisfy, for all \( \eta \in \Omega_A \):
\[
(1 - \lambda) f(\eta) = 1_G(\eta) \left( \pi_{A_1}(f) - f(\eta) \right) + \pi^{\eta}_{A_2}(f).
\] (8.10)

If we average both sides w.r.t. \( \pi_{A_1} = \pi_{A}(-f) \) we get
\[
(1 - \lambda) \pi_{A_1}(f) = \pi_{A_1}(\pi^{\eta}_{A_2}(f)) = \pi_{A}(\pi^{\eta}_{A_2}(f) | G) = (1 - \epsilon)^{-1} \text{Cov}_{A}(1_G, \pi^{\eta}_{A_2}(f))
\]
where \( \text{Cov}_{A}(\cdot, \cdot) \) denotes the covariance w.r.t. \( \pi_A \), and we have used \( \pi_{A}(\pi^{\eta}_{A_2}(f)) = \pi_{A}(f) = 0 \).

Assume now \( 1 - \lambda > 0 \), otherwise there is nothing to be proved. Schwarz' inequality gives
\[
\text{Cov}_{A}(1_G, \pi^{\eta}_{A_2}(f)) \leq \text{Var}_{A}(1_G)^{1/2} \| \pi^{\eta}_{A_2}(f) \|_{\infty}.
\]
Therefore, from (8.11), one has
\[
\| \pi_{A_2}(f) \|_{\infty} \leq \frac{1}{1 - \lambda} \sqrt{\frac{\epsilon}{1 - \epsilon}} \| \pi^{\eta}_{A_2}(f) \|_{\infty} \leq \frac{\delta}{1 - \lambda} \| \pi^{\eta}_{A_2}(f) \|_{\infty},
\] (8.12)
where we use \( \delta \geq \sqrt{\epsilon/(1 - \epsilon)} \), for \( n \) large enough and \( \| \pi^{\eta}_{A_2}(f) \|_{\infty} = \sup_{\eta} | \pi^{\eta}_{A_2}(f) | \). On the other hand we can rewrite (8.10) as
\[
f(\eta) = \frac{1_G(\eta) \pi_{A_1}(f)}{1 - \lambda + 1_G(\eta)} + \frac{\pi^{\eta}_{A_2}(f)}{1 - \lambda + 1_G(\eta)}.
\]

Therefore, applying \( \pi^{\eta}_{A_2} \) to both sides, we get
\[
\pi^{\eta}_{A_2}(f) = \frac{\pi^{\eta}_{A_2}(G) \pi_{A_1}(f)}{2 - \lambda} + \frac{\pi^{\eta}_{A_2}(G) \pi^{\eta}_{A_2}(f)}{2 - \lambda} + \frac{\pi^{\eta}_{A_2}(G^c) \pi^{\eta}_{A_2}(f)}{1 - \lambda}.
\]

Then, using (8.12), one has
\[
\| \pi^{\eta}_{A_2}(f) \|_{\infty} \leq \left( \frac{\delta}{1 - \lambda} \left( \frac{1}{2 - \lambda} + 1 \right) + \frac{1}{2 - \lambda} \right) \| \pi^{\eta}_{A_2}(f) \|_{\infty} =: \chi(\delta, \lambda) \| \pi^{\eta}_{A_2}(f) \|_{\infty}.
\]
The latter bound is possible only if: (a) \( \pi^{\eta}_{A_2}(f) \equiv 0 \) or (b) \( \chi(\delta, \lambda) \geq 1 \). It is easy to exclude (a). Suppose in fact that \( \pi^{\eta}_{A_2}(f) \equiv 0 \). Then also \( \pi_{A_1}(f) = 0 \) by (8.12), and (8.10) would reduce to \( (1 - \lambda + 1_G)f = 0 \) which is possible iff \( f \equiv 0 \) because of the assumption \( 1 - \lambda > 0 \). On the other hand it is straightforward to check that \( \chi(\delta, \lambda) \geq 1 \) implies that \( 1 - \lambda = O(\sqrt{\delta}) \leq e^{-(3/2)^n/4} \) for all \( n \) large enough. The proof of the Lemma is complete. \( \square \)
8.3. Spectral gap lower bound without the wall. The proof of Proposition 8 in the case without the wall is very similar. The analogue of Proposition 9 is described as follows. Given a rectangle $\Lambda = [1, \ell_A] \times [-h_A, h_A]$, write $\text{gap}(\Lambda) = \text{gap}(\ell_A, h_A)$ for the spectral gap of the SOS model in $\Lambda$ with zero boundary conditions at 0 and $\ell_A + 1$. Set $\beta(L, n) = \max_{\Lambda \in G_{n,L}} \text{gap}(\Lambda)^{-1}$, where $G_{n,L}$ stands for the set of rectangles $\Lambda = [1, \ell_A] \times [-h_A, h_A]$ such that $\ell_A \leq L$ and $h_A \leq (3/2)^n$. The decompositions presented in items (a)-(b)-(c) in Section 8.2 can now be obtained as follows. Let $\Lambda_1 = [1, \ell_A] \times [-h_{A_1}, h_{A_1}]$ and $\Lambda_2$ be given by the union of two rectangles $\Lambda_2 = \Lambda_2^b \cup \Lambda_2^t$, where the bottom rectangle is $\Lambda_2^b = [1, \ell_A] \times [-h_A, -h_A + h_{A_2}]$ while the top rectangle is $\Lambda_2^t = [1, \ell_A] \times [h_A - h_{A_2}, h_A]$. Note that if $h_{A_1} + h_{A_2} > h_A$ then there are two overlap regions $I^b, I^t$. As in Section 8.2, if $\Lambda \in G_{n,L}$ is such that $h_A > (3/2)^n - 1$ then one can find at least $s_n$ decompositions of the form $\Lambda = \Lambda_1 \cup \Lambda_2$ where $\Lambda_i, i = 1, 2$ are as above with $h_{A_1} + h_{A_2} - h_A \geq 2\Delta_n$, with $\Lambda_1 \in G_{n-1,L}$ and $\Lambda_2^b, \Lambda_2^t \in G_{n-1,L}$, and such that all the overlaps $I = I^b \cup I^t$ corresponding to distinct decompositions are disjoint. Let $\pi_{\Lambda}, \pi_{\Lambda_1}$ be now the equilibrium distribution in $\Lambda, \Lambda_1$ with zero boundary conditions and $\pi_{\Lambda_2}$ be the distribution in the region $\Lambda_2$ conditioned to the value of $\eta \in \Lambda \setminus \Lambda_2$. With this notation, it is not hard to check that the result of Lemma 6 remains true as it stands. Then, the same argument of Section 8.2 proves that

$$\beta(L, n) \leq (1 + \delta_n) \tilde{\beta}(L, n - 1),$$

where we define $\tilde{\beta}(L, n) := \max\{\beta(L, n), \gamma(L, n)\}$. Since we know the results for the constants $\gamma(L, n)$ (cf. Proposition 9), it is now simple to infer the desired conclusion: $\tilde{\beta}(L, n) \leq CL^2(\log L)^n$. This is sufficient to end the proof in the case without the wall.

**Acknowledgements**

This work has been carried out while FLT was visiting the Department of Mathematics of the University of Roma Tre under the ERC Advanced Research Grant “PTRELSS”. FLT acknowledges partial support by ANR through grant SHEPI.

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