QCD and relativistic $O(\alpha_s v^2)$ corrections to hadronic decays of spin-singlet heavy quarkonia $h_c, h_b$ and $\eta_b$

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Abstract

We calculate the annihilation decay widths of spin-singlet heavy quarkonia $h_c, h_b$ and $\eta_b$ into light hadrons with both QCD and relativistic corrections at order $O(\alpha_s v^2)$ in nonrelativistic QCD. With appropriate estimates for the long-distance matrix elements by using the potential model and operator evolution method, we find that our predictions of these decay widths are consistent with recent experimental measurements. We also find that the $O(\alpha_s v^2)$ corrections are small for $b\bar{b}$ states but substantial for $c\bar{c}$ states. In particular, the negative contribution of $O(\alpha_s v^2)$ correction to the $h_c$ decay can lower the decay width, as compared with previous predictions without the $O(\alpha_s v^2)$ correction, and thus result in a good agreement with the recent BESIII measurement.

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I. INTRODUCTION

The inclusive annihilation decay of heavy quarkonium is one of the important issues in heavy quarkonium physics. It is widely accepted that the heavy quarkonium inclusive annihilation decay can be described by nonrelativistic QCD (NRQCD) factorization \cite{1}. In this framework, the long-distance effects that cannot be calculated perturbatively are described by the long-distance matrix elements (LDMEs), which are classified in the order of $v$, the relative velocity of heavy quarks in quarkonium. As $v$ is small in heavy quarkonium system, we need to keep only a few number of LDMEs in the calculation. Recently, more precise measurements for heavy quarkonium decay widths and branching ratios are available \cite{2–12}. Thus, it is necessary to provide more precise theoretical predictions to compare with the data.

For charmonium, the $c\bar{c}$ system, the inclusive annihilation hadronic decay (into gluons and light quark pairs) widths for $S$-, $P$-, and $D$-wave states are all calculated up to $O(\alpha_s)$ in NRQCD \cite{13–19}. Particularly, for the $S$-wave state $\eta_c$, the $O(\alpha_s v^2)$ corrections have recently been carried out \cite{20}, which means the short-distance coefficients of $O(v^2)$ LDMEs are calculated perturbatively to next-to-leading order (NLO) in $\alpha_s$. After taking the $O(\alpha_s v^2)$ corrections into account, the measurements of $\eta_c$ decay can be described much better in NRQCD. For the $P$-wave state $h_c$, the earlier theoretical result at $O(\alpha_s)$ predicts the hadronic decay width of $h_c$ to be about 0.72 MeV \cite{17}, which is a factor of 2 larger than the latest measurements by BESIII, where the central value of the total width is about 0.73 MeV and the hadronic decay branching ratio is about 50\% \cite{5}. Thus it is needed to study higher order in $v$ corrections to examine whether the gap between theoretical predictions and experimental measurements can be explained. It will be an interesting test for the validity of NRQCD factorization for charmonium system.

For bottomonium, the $b\bar{b}$ system, the value of $v^2$ is about 0.1, which is much smaller than $v^2 \approx 0.3$ for charmonium. It is then expected that the $v^2$ expansion should be better for bottomonium, thus the study of bottomonium is more solid to check NRQCD factorization. Recently, the process $h_b(1P) \to \eta_b(1S)\gamma$ was measured by the Belle Collaboration \cite{8}. It was found that the $\eta_b$ decay width was about 12.4 MeV and the decay branching fraction of $\mathcal{B}[h_b(1P) \to \eta_b(1S)\gamma] = 49.2 \pm 5.7^{+5.6}_{-3.3}\%$. It is tempting to try to explain these data in NRQCD.
In this paper, we will perform the $O(\alpha_s v^2)$ calculations for the spin-singlet $P$-wave charmonium $h_c$ and bottomonium $h_b$, and also for the spin-singlet $S$-wave bottomonium $\eta_b$. We find these corrections are important to understand the measured data. The rest of this paper is organized as follows. In Sec. II we briefly introduce the NRQCD factorization formulism in heavy quarkonium annihilation decays. Then we describe some technical method in calculating $O(\alpha_s v^2)$ short-distance coefficients in Sec. III. The results for $S$-wave and $P$-wave states including real and virtual contributions are presented in Sec. IV. With these results and appropriate estimates of the LDMEs, we discuss the related phenomenology in Sec. V. In the Appendix A, we calculate the evolution of LDMEs at $O(\alpha_s v^2)$. In the Appendix B, we describe our factorization scheme choice and show how to eliminate higher twist operators. Finally, we give a brief summary in Sec. VI.

II. NRQCD FACTORIZATION FOR QUARKONIUM DECAY

In this section, we introduce the NRQCD factorization formula for the rates of spin-singlet heavy quarkonium ($\eta_{c,b}$ and $h_{c,b}$) decays to light hadrons. The inclusive annihilation decay width of heavy quarkonium can be factorized by the following formula [1]

$$\Gamma(H) = \sum_n \frac{2}{m_Q^{a_n-4}} \text{Im} f_n(\mu_A) (H|O_n(\mu_A)|H),$$

where $\text{Im} f_n(\mu_A)$ is the short-distance (SD) coefficient that can be perturbatively calculated using full QCD Lagrangian. The LDMEs $(H|O_n(\mu_A)|H)$ involve non-perturbative effects and are classified by the relative velocity $v$ between $Q$ and $Q'$, according to power counting in Refs. [1, 21–24].

The NRQCD Lagrangian can be derived by integrating out the degrees of freedom of order $m_Q$, the mass of the heavy quark, from the QCD Lagrangian, which gives

$$\mathcal{L}_{\text{NRQCD}} = \mathcal{L}_{\text{light}} + \mathcal{L}_{\text{heavy}} + \delta\mathcal{L}. \quad (2)$$

The heavy part of the Lagrangian describes the motions of (anti-)heavy quark in spacetime and is given by

$$\mathcal{L}_{\text{heavy}} = \bar{\psi}(iD_t + \frac{D^2}{2m_Q})\psi + \bar{\chi}(iD_t - \frac{D^2}{2m_Q})\chi \quad (3)$$

where $\psi(\chi)$ denotes the Pauli spinor field that annihilates (creates) a heavy (anti-)quark, and $D_t(D)$ is the time(space) component of the gauge-covariant derivative $D^\mu$. The light
The piece of the Lagrangian reads

\[ \mathcal{L}_{\text{light}} = -\frac{1}{2} \text{Tr} \, G^{\mu\nu} G_{\mu\nu} + \sum_{n_f} \bar{q} i \not{D} q \] (4)

where \( G^{\mu\nu} \) is the gluon field strength tensor, \( q \) is the Dirac spinor field of light quarks and \( n_f \) is the number of light flavors. The bilinear Lagrangian term which contains the order \( v^2 \) correction is

\[ \delta \mathcal{L}_{\text{bilinear}} = \left( \frac{c_1}{8m_Q^2} \psi^\dagger (D^2)^2 \psi + \frac{c_2}{8m_Q^2} \psi^\dagger (D \cdot gE - gE \cdot D) \psi \right) + \frac{c_3}{8m_Q^2} \psi^\dagger (iD \times gE - gE \times iD) \cdot \sigma \psi + \frac{c_4}{2m_Q} \psi^\dagger (gB \cdot \sigma) \psi + \text{charge conjugate terms}, \] (5)

where \( E^i = G^{0i} \) and \( B^i = \frac{1}{2} \epsilon^{ijk} G^{jk} \) are the electric and magnetic components of the gluon field strength tensor \( G^{\mu\nu} \), and \( c_i = 1 + O(\alpha_s) \), \( i = 1, 2, 3, 4 \) are the dimensionless coefficients corresponding to each operator.

In order to describe the annihilation decay of quarkonium, a set of local four-fermion operators \( \mathcal{O}_i \) which appear in Eq. (4) are needed. For example, the operator \( \psi^\dagger \chi \chi^\dagger \psi \) can annihilate a \( \bar{Q}Q \) pair in the \( ^1S_0^1 \) configuration. In our case, for the \( O(\alpha_s v^2) \) calculation of spin-singlet quarkonium decay, the power counting rules \([1]\) give the following seven operators and LDMEs in Eq. (4): for \( S \)-wave quarkonium,

\[ \mathcal{O}(^1S_0^1) = \psi^\dagger \chi \chi^\dagger \psi, \] (6a)
\[ \mathcal{P}(^1S_0^1) = \frac{1}{2} \psi^\dagger \chi \chi^\dagger \left( -\frac{i \not{D}}{2} \right)^2 \psi + \text{h.c.}, \] (6b)

for \( P \) wave quarkonium,

\[ \mathcal{O}(^1S_0^8) = \psi^\dagger T^a \chi \chi^\dagger T^a \psi, \] (7a)
\[ \mathcal{P}(^1S_0^8) = \frac{1}{2} \psi^\dagger T^a \chi \chi^\dagger T^a \left( -\frac{i \not{D}}{2} \right)^2 \psi + \text{h.c.}, \] (7b)
\[ \mathcal{O}(^1P_1^1) = \psi^\dagger \left( -\frac{i \not{D}}{2} \right) \chi \chi^\dagger \left( -\frac{i \not{D}}{2} \right) \psi, \] (7c)
\[ \mathcal{P}(^1P_1^1) = \frac{1}{2} \psi^\dagger \left( -\frac{i \not{D}}{2} \right) \chi \chi^\dagger \left( -\frac{i \not{D}}{2} \right)^2 \psi + \text{h.c.}, \] (7d)
\[ T_{1-S}(^1S_0, ^1P_1) = \frac{1}{2} \psi^\dagger gE \chi \chi^\dagger \not{D} \psi + \text{h.c.}, \] (7e)
and

$$\langle O^{(2S+1)L_j^{[1,8]}} \rangle_H \equiv \langle H|O^{(2S+1)L_j^{[1,8]}}|H\rangle, \quad (8a)$$

$$\langle P^{(2S+1)L_j^{[1,8]}} \rangle_H \equiv \langle H|P^{(2S+1)L_j^{[1,8]}}|H\rangle. \quad (8b)$$

Note that, choosing different power counting rules, one may get a different set of operators. For example, in the power counting rule of Ref. [24], \( m_Q \) and \( v \) are homogeneous, which gives that the chromomagnetic field \( gB \) scales as \( (m_Qv)^2 \). While that field scales as \( m_Q^2v^4 \) in Ref. [1], which is further suppressed by \( v^2 \). As a result, many operators considered in Ref. [24] disappear in our calculation, leaving the above seven. These seven matrix elements are all independent with each other, i.e. they cannot be eliminated by field redefinition or Poincare invariance [24].

Using the seven operators, we give the explicit form of Eq. (1) for \( ^1S_0 \) and \( ^1P_1 \) states,

$$\Gamma(H(^1S_0) \to LH) = \frac{F(^1S_0[^1l])}{m_Q^2} \langle O(^1S_0[^1l]) \rangle_{^1S_0} + \frac{G(^1S_0[^1l])}{m_Q^4} \langle P(^1S_0[^1l]) \rangle_{^1S_0}, \quad (9a)$$

$$\Gamma(H(^1P_1) \to LH) = \frac{F(^1S_0[^8])}{m_Q^2} \langle O(^1S_0[^8]) \rangle_{^1P_1} + \frac{G(^1S_0[^8])}{m_Q^4} \langle P(^1S_0[^8]) \rangle_{^1P_1}$$

$$+ \frac{F(^1P_1[^1l])}{m_Q^2} \langle O(^1P_1[^1l]) \rangle_{^1P_1} + \frac{G(^1P_1[^1l])}{m_Q^4} \langle P(^1P_1[^1l]) \rangle_{^1P_1}. \quad (9b)$$

Note that, we omit a term of \( \frac{T(^1S_0[^1l])}{m_Q^2} \langle T_{-8}(^1S_0, ^1P_1) \rangle_{^1P_1} \) in Eq. (9b) to simplify our theoretical framework, although the LDME \( \langle T_{-8}(^1S_0, ^1P_1) \rangle_{^1P_1} \) is of the same order in \( v \) as \( \langle P(^1P_1[^1l]) \rangle_{^1P_1} \). There are two reasons that lead us to do this simplification. Numerically, this contribution is small, which is because \( T(^1S_0, ^1P_1) \) vanishes at leading order (LO) in \( \alpha_s \) due to the charge parity conservation. Theoretically, and more importantly, this contribution is finite, that is, no infrared (IR) poles are needed to cancel between this channel and other four channels in Eq. (9b). It is then impossible to distinguish this finite contribution from the renormalization scheme or factorization scheme choice of other operators, such as \( \langle O(^1P_1[^1l]) \rangle_{^1P_1} \) or \( \langle O(^1S_0[^8]) \rangle_{^1P_1} \). Therefore, by ignoring this operator in the hadronic decay width, it is equivalent that we choose a specific renormalization scheme or factorization scheme for other operators. In Appendix B, we will give an explicit definition of our factorization scheme to absorb the term \( \frac{T(^1S_0[^1l])}{m_Q^2} \langle T_{-8}(^1S_0, ^1P_1) \rangle_{^1P_1} \). Although our scheme is in principle distinguished from \( \overline{\text{MS}} \) scheme, as we will discussed in Appendix E, there is no difference between
these two schemes for our purpose in this work. As a result, we will pretend to use \( \overline{\text{MS}} \) scheme in the following.

Through the above factorization formula, one can match full QCD with NRQCD to get the short-distance (SD) coefficients \( F \) and \( G \) perturbatively. The skeleton of the matching procedure is given by

\[
\text{Im} \mathcal{M}(Q\overline{Q} \rightarrow Q\overline{Q})|_{\text{pert QCD}} = \sum_n 2 \frac{\text{Im} f_n(\mu_A)}{m_Q^{n-4}} \langle Q\overline{Q}| \mathcal{O}_n(\mu_A) | Q\overline{Q} \rangle \bigg|_{\text{NRQCD}},
\]

(10)

The determination of SD coefficients will be discussed in detail in the next section.

III. DETAILS IN FULL QCD CALCULATION

A. Kinematics

We work in the rest frame of the heavy quarkonium. It is customary to decompose the momenta of \( Q \) and \( \overline{Q} \) as

\[
p_Q = \frac{1}{2} P + q,
\]

(11a)

\[
p_{\overline{Q}} = \frac{1}{2} P - q,
\]

(11b)

where \( P \) is the total momentum and \( q \) is half of the relative momentum, which satisfies the relation \( P \cdot q = 0 \). The explicit four-vector form of \( P \) and \( q \) in the rest frame are

\[
P = \left( 2E_Q, 0 \right),
\]

(12a)

\[
q = \left( 0, q \right),
\]

(12b)

with \( E_Q = \sqrt{m_Q^2 + q^2} \).

The treatment of final state phase space integration at \( O(\alpha_s v^2) \) level is slightly different from ordinary calculations (i.e. leading order of \( v \) calculation). To make it simpler, we use the following rescaling transformation for all external momenta \( \overline{\text{[20, 25]}} \),

\[
P \rightarrow P' \frac{E_Q}{m_Q},
\]

(13a)

\[
k_f \rightarrow k_f' \frac{E_Q}{m_Q},
\]

(13b)

but keep the relative momentum \( q \) and loop integral momentum \( l \) unchanged. Once we take such trick, the \( q^2 \) dependence in both phase space and current factor [i.e. \( 1/(2M) \) where
\( M \) is the quarkonium mass] can be absorbed into the amplitude, then we can safely take \( q \rightarrow 0 \) in these terms and only expand \( q, q' \) at the amplitude level, where \( q' \) is half of the relative momentum between \( Q\overline{Q} \) pair on the complex conjugate side. (Note that \( |q| = |q'| \) but their direction does not need to be the same, so in general \( q \neq q' \)). It should be kept in mind that this trick can only work in the case where all final state partons are massless (i.e. gluons and light quarks), because, in the case of massive partons, the on-shell relation does not hold under rescaling, which will break the QCD gauge invariance.

### B. Covariant Projection Method in D-Dimension

Instead of using matching method directly, we use an equivalent but more efficient method, i.e., the covariant projection method, to calculate the imaginary part of the SD coefficients in Eqs. \([1a] \) and \([11] \). In order to get spin-singlet \( Q\overline{Q} \) decay amplitudes, we take the following spin and color projectors onto \( Q\overline{Q} \) quark lines \([20] \):

\[
\Pi_0 = \frac{1}{2\sqrt{2}(E_q + m_Q)}(\frac{\not{p}}{2} + \not{q} + m_Q)(\frac{\not{p} + 2E_q}{8E_q^2})\gamma_5(\frac{\not{p}}{2} - \not{q} - m_Q),
\]

and

\[
C_1 = \frac{1}{\sqrt{N_c}},
\]

\[
C_8 = \sqrt{2}T^a.
\]

We do Taylor expansion of the projected amplitudes in powers of \( q \) to the required order,

\[
\mathcal{M}(q) = \mathcal{M}(0) + \frac{\partial \mathcal{M}(q)}{\partial q^\alpha}|_{q=0} q^\alpha + \frac{1}{2!} \frac{\partial^2 \mathcal{M}(q)}{\partial q^\alpha \partial q^\beta}|_{q=0} q^\alpha q^\beta
\]

\[
+ \frac{1}{3!} \frac{\partial^3 \mathcal{M}(q)}{\partial q^\alpha \partial q^\beta \partial q^\gamma}|_{q=0} q^\alpha q^\beta q^\gamma + \cdots,
\]

and then make the replacement:

\[
q_\alpha q_\beta \rightarrow \frac{\not{q}^2}{D-1} \Pi_{\alpha\beta},
\]

\[
q_\alpha q'_\beta \rightarrow \frac{\not{q} \cdot \not{q}'}{D-1} \Pi_{\alpha\beta},
\]

\[
q_\alpha q_\gamma q'_\lambda \rightarrow \frac{\not{q}^2 \not{q} \cdot \not{q}'}{D+1}(\Pi_{\alpha\beta} \Pi_{\gamma\lambda} + \Pi_{\alpha\gamma} \Pi_{\beta\lambda} + \Pi_{\alpha\lambda} \Pi_{\gamma\beta}),
\]

to project them to definite states, where

\[
\Pi_{\alpha\beta} = -g_{\alpha\beta} + \frac{P^\alpha_{\alpha'} P^\beta_{\beta'}}{4m_Q^2}.
\]
with $P'$ the rescaled heavy quarkonium momentum. For example, the third derivative term of $\mathcal{M}$ convolutes with the first derivative term of $\mathcal{M}^\dagger$ giving the squared amplitudes term,

$$\frac{1}{3!} \frac{\partial^3 \mathcal{M}(q)}{\partial q^\alpha \partial q^\beta \partial q^\gamma} \bigg|_{q=0} \frac{\partial \mathcal{M}^\dagger(q')}{\partial q'^\lambda} \bigg|_{q'=0} q^\alpha q^\beta q^\gamma q'^\lambda$$

$$\rightarrow \frac{1}{3! D + 1} (\Pi_{\alpha\beta} \Pi_{\gamma\lambda} + \Pi_{\alpha\gamma} \Pi_{\beta\lambda} + \Pi_{\alpha\lambda} \Pi_{\gamma\beta}) \frac{\partial^3 \mathcal{M}(q)}{\partial q^\alpha \partial q^\beta \partial q^\gamma} \bigg|_{q=0} \frac{\partial \mathcal{M}^\dagger(q')}{\partial q'^\lambda} \bigg|_{q'=0}$$

which contributes to the SD coefficient of $G(1P[1])$ in Eq. (19).

**IV. PERTURBATIVE QCD RESULTS OF SHORT-DISTANCE COEFFICIENTS**

We generate Feynman diagrams and amplitudes by **FeynArts** \cite{27,28}, and then calculate the squared amplitudes by self-written **Mathematica** codes. The phase space integrals are done analytically using the method presented in Ref. \cite{16}. Ultra-violet (UV) and IR divergences are both regularized by dimensional regularization. The renormalizations for heavy quark mass $m_Q$, heavy quark field $\psi_Q$, light quark field $\psi_q$ and gluon field $A_\mu$ are in the on-mass-shell scheme (OS), and that for the QCD coupling constant $g_s$ is in the $\overline{\text{MS}}$ scheme,

\begin{align*}
\delta Z^\text{OS}_{m_Q} &= -3C_F \frac{\alpha_s}{4\pi} N_e \left[ \frac{1}{\epsilon_{\text{UV}}} + \frac{4}{3} \right], \quad (20a) \\
\delta Z^\text{OS}_2 &= -C_F \frac{\alpha_s}{4\pi} N_e \left[ \frac{1}{\epsilon_{\text{UV}}} + \frac{2}{\epsilon_{\text{IR}}} + 4 \right], \quad (20b) \\
\delta Z^\text{OS}_{2l} &= -C_F \frac{\alpha_s}{4\pi} N_e \left[ \frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right], \quad (20c) \\
\delta Z^\text{OS}_3 &= \frac{\alpha_s}{4\pi} N_e \left[ (\beta_0 - 2C_A) \left( \frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right) \right], \quad (20d) \\
\delta Z^\text{MS}_g &= -\frac{\beta_0 \alpha_s}{2} \frac{\alpha_s}{4\pi} N_e \left[ \frac{1}{\epsilon_{\text{UV}}} + \ln \frac{m_Q^2}{\mu_r^2} \right], \quad (20e)
\end{align*}

where $N_e(m_Q) = \left(\frac{4\pi^2}{m_Q}\right)\Gamma(1 + \epsilon)$ is an overall factor, and $\mu_r$ is the renormalization scale. $\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F n_f$ is the one-loop coefficient of the $\beta$ function, $n_f$ is the active quark flavors, which we set to be 3 for charmonium and 4 for bottomonium.
FIG. 1: Born level Feynman diagrams for \(1^S_0 \rightarrow gg\).

A. Short-Distance Coefficients of S-Wave Quarkonium Hadronic Decay

Leading order in \(\alpha_s\) calculations give the Born level decay width and its relativistic correction, respectively, as

\[
\Gamma_{\text{Born}}(1^S_0 \rightarrow gg) = \frac{4}{3} (4\pi\alpha_s)^2 \frac{\mu_r^4}{m_Q^2} \Phi(2) (1 - \epsilon)(1 - 2\epsilon) \frac{\langle O(1^S_0) \rangle_{\text{Born}}}{2N_c},
\]

\[
\Gamma_{\text{Born}}^{(v^2)}(1^S_0 \rightarrow gg) = -\frac{2(2 - \epsilon)}{3 - 2\epsilon} \frac{q^2}{m_Q^2} \Gamma_{\text{Born}}(1^S_0 \rightarrow gg),
\]

where \(\Phi(2) = \frac{1}{8\pi} \left( \frac{4\pi}{M^2} \right)^2 \frac{\pi(1-\epsilon)}{\pi(2-2\epsilon)}\) is the total two-body phase space in \(D\) dimension and \(M = 2m_Q \sqrt{1 + \frac{q^2}{m_Q^2}}\) is the quarkonium mass including the relativistic correction. The two Born diagrams are illustrated in Fig. 1.

The next-to-leading order calculations include real and virtual corrections. For \(S\)-wave Fock states (i.e. \(1^S_0^{[1]}\) and \(1^S_0^{[8]}\)), UV divergences will be canceled by counterterm diagrams, and most IR divergences will be canceled between real and virtual corrections, leaving some residue divergences at \(O(v^2)\). The cancelation of such residue divergences will be presented in the next section by calculating NRQCD LDMEs at one-loop level. The contribution of virtual plus counterterm corrections is

\[
\Gamma_{\text{Virtual}}(1^S_0^{[1]} \rightarrow gg) = \frac{3\alpha_s}{\pi} \Gamma_{\text{Born}}(1^S_0^{[1]} \rightarrow gg) f_e(m_Q) \left\{ \left[ -\frac{1}{\epsilon^2} - \frac{1}{6}\beta_0 \frac{1}{\epsilon} \right. \right.
\]

\[
\left. + \frac{1}{36} (-6\beta_0 \ln\left( \frac{4m_Q^2}{\mu_r^2} \right) + 19\pi^2 - 44) \right]
\]

\[
+ \frac{q^2}{m_Q^2} \left[ \frac{4}{3\epsilon^2} - \frac{4n_f - 97}{27} \right]
\]

\[
- \frac{1}{324} (-72\beta_0 \ln\left( \frac{4m_Q^2}{\mu_r^2} \right) + 8n_f + 267\pi^2 - 280) \right\},
\]

where \(f_e(m_Q) = (\frac{\pi m_Q^2}{\mu_r^2})^{\Gamma(1 + \epsilon)}\) is some selected Feynman diagrams are shown in Fig. 1.

The real correction contains two sets, where one set is the final states with \(ggg\) and the other one with \(qgq\). Some typical Feynman diagrams are shown in Fig. 2 and Fig. 3 and the
FIG. 2: Virtual correction Feynman diagrams for $^1S_0^{[1]}$, $^1S_0^{[8]} \to gg$. The crossed diagrams have been suppressed.

FIG. 3: Real correction Feynman diagrams for $^1S_0^{[1]}$, $^1S_0^{[8]}$, $^1P_1^{[1]} \to ggg$. The crossed diagrams have been suppressed. The second diagram vanishes in $^1P_1^{[1]}$.

FIG. 4: Real correction Feynman diagrams for $^1S_0^{[1]}$, $^1S_0^{[8]} \to q\bar{q}g$. The crossed diagrams have been suppressed.

The contributions to decay width are

$$\Gamma(^1S_0^{[1]} \to ggg) = \frac{3\alpha_s}{\pi} \Gamma_{\text{Born}}(^1S_0^{[1]} \to gg) f_r(m_Q) \left\{ \left[ \frac{1}{\epsilon^2} + \frac{111}{6} \frac{1}{\epsilon} + \frac{1}{72} (724 - 69\pi^2) \right] + \frac{q^2}{m_Q^2} \left[ -\frac{4}{3} \frac{1}{\epsilon^2} - \frac{3}{\epsilon} - \frac{437 - 42\pi^2}{27} \right] \right\}.$$  \hspace{1cm} (23a)

$$\Gamma(^1S_0^{[1]} \to q\bar{q}g) = \frac{n_f \alpha_s}{2\pi} \Gamma_{\text{Born}}(^1S_0^{[1]} \to gg) \frac{f_r(m_Q)}{\Gamma(1+\epsilon)\Gamma(1-\epsilon)} \left\{ -\frac{21}{3} \frac{1}{\epsilon} - \frac{16}{9} + \frac{q^2}{m_Q^2} \left( \frac{81}{9} + \frac{86}{27} \right) \right\}. \hspace{1cm} (23b)$$

Combining Eqs. (21), (22) and (23), we obtain the hadronic decay width with both QCD
radiative and relativistic corrections at NLO of $^1S_0$ heavy quarkonium,

$$\Gamma_{\text{QCD}}(^1S_0 \rightarrow \text{LH}) = \Gamma_{\text{Born}}(^1S_0[1] \rightarrow gg) \left\{ \left[ 1 + \frac{\alpha_s}{\pi} f_c(m_Q) \right] \frac{1}{72} (-36\beta_0 \ln \left( \frac{4m_Q^2}{\mu_r^2} \right) 

- 64n_f - 93\pi^2 + 1908) \right\} - \frac{4}{3} \frac{q^2}{m_Q^2} \left[ 1 + \frac{\alpha_s}{\pi} f_c(m_Q) \left( -\frac{4}{3} \epsilon \right) \right] \right\} \right\} \right\}.

We note that our results agree with the previous work for $O(\alpha_s v^2)$ correction [20] and $O(\alpha_s)$ correction [16, 19]. Comparing our results with Ref. [20], a slight difference of two body phase space $\Phi_2$ between them can be found. In Ref. [20] $\Phi_2$ is defined so as to remove the $q^2$ dependence into the coefficients, so our individual virtual and real parts, Eqs. (22) and (23), look different from the results in Ref. [20], but essentially they are equivalent. The total NLO result Eq. (24) is explicitly the same, independent of the definition of $\Phi_2$. The correct repetition of the hadronic decay SD coefficients of $^1S_0$ heavy quarkonium enables us to extend discussion from charm quark system to bottom quark system (i.e. $\eta_b$) and also partly checks our codes when dealing with $P$-wave heavy quarkonium.

**B. Short-Distance Coefficients of P-Wave Quarkonium Hadronic Decay**

The procedure in calculating the $^1P_1$ heavy quarkonium is similar to $^1S_0$, although more complicated. Additional simplification can be taken by imposing $C$ (charge) parity conservation of QCD to constrain Feynman diagrams. A straightforward result is that $C$ parity conservation prohibits $^1P_1[1]$ Fock state, which has $C = -1$, to decay to two gluons, whose $C = +1$, no matter they are real or virtual. By tedious but straightforward calculation, we get the results as follows.

At the Born level,

$$\Gamma_{\text{Born}}(^1S_0[8] \rightarrow gg) = \frac{5}{12} (4\pi \alpha_s)^2 \frac{\mu_r^4}{m_Q^2} \Phi_{(2)}(1 - \epsilon)(1 - 2\epsilon) \langle O(^1S_0[8]) \rangle_{1P_1}^{\text{Born}}, \quad (25a)$$

$$\Gamma_{\text{Born}}^{(v^2)}(^1S_0[8] \rightarrow gg) = -\frac{2(2 - \epsilon)}{3 - 2\epsilon} \frac{q^2}{m_Q^2} \Gamma_{\text{Born}}(^1S_0[8] \rightarrow gg), \quad (25b)$$
For NLO corrections,

\[
\Gamma_{\text{Virtual}}(^{1}S_0^{[8]} \rightarrow gg) = \frac{3\alpha_s}{\pi} \Gamma_{\text{Born}}(^{1}S_0^{[8]} \rightarrow gg) f_\epsilon(m_Q) \left\{ \left[ -\frac{1}{\epsilon^2} + \frac{n_f - 21}{9} \right] \right.
\]
\[
+ \frac{1}{72} \left( -12\beta_0 \ln \left( \frac{4m_Q^2}{\mu_r^2} \right) + 29\pi^2 - 16 \right) \left. \right\} + \frac{q^2}{m_Q^2} \left[ \frac{4}{3} \frac{1}{\epsilon^2} - \frac{115}{27} \frac{1}{\epsilon} \right] - \frac{1}{628} \left( -144\beta_0 \ln \left( \frac{4m_Q^2}{\mu_r^2} \right) + 16n_f + 345\pi^2 - 992 \right) \} ,
\]

\[(26)\]

\[
\Gamma(1^{1}S_0^{[8]} \rightarrow ggg) = \frac{3\alpha_s}{\pi} \Gamma_{\text{Born}}(1^{1}S_0^{[8]} \rightarrow gg) f_\epsilon(m_Q) \left\{ \left[ \frac{1}{\epsilon^2} + \frac{7}{3} \frac{1}{\epsilon} - \frac{288}{27} \right] \right.
\]
\[
+ \frac{q^2}{m_Q^2} \left[ \frac{1}{3} \frac{1}{\epsilon^2} - \frac{1}{\epsilon} - \frac{554 - 45\pi^2}{27} \right] \} ,
\]

\[(27)\]

\[
\Gamma(1^{1}S_0^{[8]} \rightarrow qg) = \frac{n_f \alpha_s}{2\pi} \Gamma_{\text{Born}}(1^{1}S_0^{[8]} \rightarrow gg) f_\epsilon(m_Q) \left\{ \left[ \frac{1}{\epsilon^2} + \frac{7}{3} \frac{1}{\epsilon} - \frac{288}{27} \right] \right.
\]
\[
+ \frac{q^2}{m_Q^2} \left[ \frac{1}{3} \frac{1}{\epsilon^2} - \frac{1}{\epsilon} - \frac{554 - 45\pi^2}{27} \right] \} ,
\]

\[(28)\]

\[
\Gamma(1^{1}P_1^{[1]} \rightarrow ggg) = \frac{40\alpha_s^3}{27} f_\epsilon(m_Q)(8\pi\Phi_2) \left\{ \left[ \frac{1}{\epsilon^2} + \frac{7}{24} \frac{1}{\epsilon} - \frac{5}{3} \right] \right.
\]
\[
+ \frac{q^2}{m_Q^2} \left[ \frac{29}{15} \frac{1}{\epsilon} + \frac{4216 - 555\pi^2}{900} \right] \} \langle O(1^{1}P_1^{[1]}) \rangle_{\text{Born}}^{1^{1}P_1^{[1]}} ,
\]

\[(29)\]

Summing over the above results, we get the total hadronic decay width,

\[
\Gamma_{\text{QCD}}(1^{1}P_1 \rightarrow LH) = \Gamma_{\text{Born}}(1^{1}S_0^{[8]} \rightarrow gg) \left\{ \left[ 1 + \frac{\alpha_s}{\pi} f_\epsilon(m_Q)(-\frac{1}{2} \beta_0 \ln \left( \frac{4m_Q^2}{\mu_r^2} \right) \right.ight.
\]
\[
- \frac{8}{9} n_f - \frac{43\pi^2}{24} + 34 \right) \left. \right\} + \frac{q^2}{3m_Q^2} \left( 1 + \frac{\alpha_s}{\pi} f_\epsilon(m_Q)(-\frac{7}{12} \right.
\]
\[
\left. \right] \right\} + \frac{1}{288} \left( -144\beta_0 \ln \left( \frac{4m_Q^2}{\mu_r^2} \right) - 328n_f - 735\pi^2 - 12304 \right) \} \}
\]

\[(30)\]

C. Evaluating NRQCD LDMEs And Matching Full QCD Results

In Eqs. (24) and (30), there exist explicit IR divergences. To cancel these divergence, we need to evaluate LDMEs at the loop level. By replacing all the Born LDMEs appearing in
Eqs. (24) and (30) by one-loop LDMEs, all IR divergences should be canceled and the final results will be infra-red safe quantities.

The self-energy contributions that connect Born LDMEs to their corresponding relativistic ones are first calculated in Ref. [1]. The intersecting diagrams that describe the E1 transition between $^{1}S_{0}^{[8]}$ and $^{1}P_{1}^{[1]}$ states at $O(\alpha_{s}v^{2})$ in this work are new. The detailed calculation is presented in Appendix A. Here we give the relevant results in dimensional regularization with $\overline{MS}$ renormalization scheme,

\begin{align}
\langle O(1S_{0}^{[1]}) \rangle_{1S_{0}}^{\text{Born}} & \rightarrow \langle O(1S_{0}^{[1]}) \rangle_{1S_{0}}^{(\mu_{A})} \left\{ 1 - \frac{4}{3} \frac{q^{2}}{m_{Q}^{2}} \frac{4\alpha_{s}}{3\pi} f_{\epsilon}(m_{Q}) \left[ \frac{1}{\epsilon} - \ln\left( \frac{\mu_{A}^{2}}{4m_{Q}^{2}} \right) \right] \right\}, \\
\langle O(1S_{0}^{[8]}) \rangle_{1P_{1}}^{\text{Born}} & \rightarrow \langle O(1S_{0}^{[8]}) \rangle_{1P_{1}}^{(\mu_{A})} \left\{ 1 - \frac{4}{3} \frac{q^{2}}{m_{Q}^{2}} \frac{7\alpha_{s}}{12\pi} f_{\epsilon}(m_{Q}) \left[ \frac{1}{\epsilon} - \ln\left( \frac{\mu_{A}^{2}}{4m_{Q}^{2}} \right) \right] \right\} \\
& \quad + \frac{16\alpha_{s}}{9\pi} f_{\epsilon}(m_{Q}) \left\{ \frac{1}{\epsilon} - \ln\left( \frac{\mu_{A}^{2}}{4m_{Q}^{2}} \right) \right\}, \\
\langle P(1S_{0}^{[1]}) \rangle_{1S_{0}}^{\text{Born}} & \rightarrow \langle P(1S_{0}^{[1]}) \rangle_{1S_{0}}^{(\mu_{A})} + \frac{16\alpha_{s}}{9\pi} f_{\epsilon}(m_{Q}) \left[ \frac{1}{\epsilon} - \ln\left( \frac{\mu_{A}^{2}}{4m_{Q}^{2}} \right) \right] \frac{\langle P(1P_{1}^{[1]}) \rangle_{1P_{1}}^{\text{Born}}}{2N_{c}m_{Q}^{2}},
\end{align}

where $\mu_{A}$ is the factorization scale. Substituting them into Eqs. (24) and (30), and considering the relation

\begin{align}
\langle P(1S_{0}^{[1]}) \rangle_{1S_{0}}^{\text{Born}} & = q^{2} \langle O(1S_{0}^{[1]}) \rangle_{1S_{0}}^{\text{Born}}, \\
\langle P(1S_{0}^{[8]}) \rangle_{1P_{1}}^{\text{Born}} & = q^{2} \langle O(1S_{0}^{[8]}) \rangle_{1P_{1}}^{\text{Born}}, \\
\langle P(1P_{1}^{[1]}) \rangle_{1P_{1}}^{\text{Born}} & = q^{2} \langle O(1P_{1}^{[1]}) \rangle_{1P_{1}}^{\text{Born}},
\end{align}

we get the SD coefficients for heavy quarkonium hadronic decay of $S$-wave and $P$-wave states
by matching full QCD and NRQCD,

\[ F(1S^0) = \frac{4\pi\alpha_s^2}{9} \left[ 1 - \frac{\alpha_s}{\pi} \frac{1}{72} \left( 36\beta_0 \ln(\frac{4m_Q^2}{\mu_r^2}) + 64n_f + 93\pi^2 - 1908 \right) \right], \]  

(33a)

\[ G(1S^0) = -\frac{4\pi\alpha_s^2}{3} \left[ 1 - \frac{\alpha_s}{\pi} \frac{1}{144} \left( 192\ln(\frac{\mu_A^2}{4m_Q^2}) + 72\beta_0 \ln(\frac{4m_Q^2}{\mu_r^2}) \right) \right] + 164n_f + 237\pi^2 - 4964, \]  

(33b)

\[ F(1S^8) = \frac{5\pi\alpha_s^2}{6} \left[ 1 - \frac{\alpha_s}{\pi} \frac{1}{72} \left( 36\beta_0 \ln(\frac{4m_Q^2}{\mu_r^2}) + 64n_f + 129\pi^2 - 2448 \right) \right], \]  

(33c)

\[ G(1S^8) = -\frac{5\pi\alpha_s^2}{3} \left[ 1 - \frac{\alpha_s}{\pi} \frac{1}{288} \left( 168\ln(\frac{\mu_A^2}{4m_Q^2}) + 144\beta_0 \ln(\frac{4m_Q^2}{\mu_r^2}) \right) \right] + 328n_f + 735\pi^2 - 12304, \]  

(33d)

\[ F(1P^1) = \frac{5\alpha_s^3}{486} \left[ 7(\pi^2 - 16) - 24\ln(\frac{\mu_A^2}{4m_Q^2}) \right], \]  

(33e)

\[ G(1P^1) = \frac{\alpha_s^3}{3645} \left[ 1740\ln(\frac{\mu_A^2}{4m_Q^2}) - 555\pi^2 + 9236 \right], \]  

(33f)

where \( F \)'s and \( G \)'s are defined in Eqs. (9a) and (9b).

The SD coefficients of \( 1S^0 \) agree with those in Refs. [1, 16, 19, 20, 25], that of \( 1S^8 \) and \( 1P^1 \) at leading order in \( v^2 \) are also agree with previous results in Ref. [16]. The relativistic corrections \( G(1S^8) \) and \( G(1P^1) \) are primarily new results in this work. Based on these results, we will analyze the decay of \( 1S_0 \) and \( 1P_1 \) heavy quarkonium into light hadrons.

V. PHENOMENOLOGICAL DISCUSSIONS

A. Estimating NRQCD LDMEs

To get the numerical result, we also need to know the value of LDMEs. For \( 1S_0 \) quarkonium there are two LDMEs, and for \( 1P_1 \) there are four. In Ref. [20] the LDMEs of \( \eta_c \) are determined by combining the Cornell potential[29] with one experimental measurement, \( \Gamma^{LH}(\eta_c) \) or \( \Gamma^{\gamma\gamma}(\eta_c) \) [30], and then one can predict other quantities. In the present work, since there are not enough experimental inputs to determine all involved LDMEs, we will estimate them by other methods.

For \( \eta_b \), the situation is similar to Ref. [21], but lacking the experiment input of the decay width to two photons \( \Gamma^{\gamma\gamma}(\eta_b) \). In this case we will determine \( \langle O(1S^0) \rangle_{\eta_b} \) from the potential model. Here we use the Buchmüller-Tye(B-T) potential model [31] and Cornell(Corn)
potential model \[23\] results as input, which give \[32, 33\]
\[
\langle \mathcal{O}(1S_0^{[1]}) \rangle_{\eta_b}^{\text{B-T}} = \frac{N_c}{2\pi} |R_{S}^{\text{B-T}}(0)|^2 = 3.093 \text{ GeV}^3, \tag{34a}
\]
\[
\langle \mathcal{O}(1S_0^{[1]}) \rangle_{\eta_b}^{\text{Corn}} = \langle \mathcal{O}(3S_1^{[1]}) \rangle_{\Upsilon(1S)}^{\text{Corn}} = 3.07^{+0.21}_{-0.19} \text{ GeV}^3. \tag{34b}
\]

In the Eq. (34b) we use the heavy quark spin symmetry (HQSS) to relate LDMEs of \(\eta_b\) with that of \(\Upsilon(1S)\). As the B-T model and Cornell model give almost the same result, we will only use B-T model in the following.

In order to determine \(\langle \mathcal{P}(1S_0^{[1]}) \rangle_{\eta_b}\), we define \[20, 22\]
\[
\langle \mathbf{v}^2 \rangle_{\eta_b} = \frac{\langle \mathcal{P}(1S_0^{[1]}) \rangle_{\eta_b}}{m_b^2/\langle \mathcal{O}(1S_0^{[1]}) \rangle_{\eta_b}}. \tag{35}
\]

Although \(\langle \mathbf{v}^2 \rangle_{\eta_b}\) can not be understood as the expectation value of \(\mathbf{v}^2\) in potential model, it can be estimated from the Gremm-Kapustin relation \[34\]
\[
\langle \mathbf{v}^2 \rangle_{\eta_b}^{\text{G-K}} = \frac{m_{\eta_b} - 2m_{\text{pole}}}{m_{\text{pole}}}. \tag{36}
\]

Choosing \(m_{\text{pole}} = 4.6 \text{ GeV}\) for \(b\) quark and \(m_{\eta_b} = 9.391 \text{ GeV}\) \[30\], we get \(\langle \mathbf{v}^2 \rangle_{\eta_b} = 0.042\), which is close to the potential model estimated value \(\mathbf{v}^2 \sim 0.05 - 0.1\). Combining these results, we get the value of redefined LDMEs in B-T model as
\[
\langle \mathcal{O}(1S_0^{[1]}) \rangle_{\eta_b} = \frac{\langle \mathcal{O}(1S_0^{[1]}) \rangle_{\eta_b}}{2N_c m_b^2} = 24.36^{+1.09}_{-1.03} \text{ MeV}, \tag{37}
\]
\[
\langle \mathcal{P}(1S_0^{[1]}) \rangle_{\eta_b} = \frac{\langle \mathcal{P}(1S_0^{[1]}) \rangle_{\eta_b}}{2N_c m_b^4} = \langle \mathbf{v}^2 \rangle_{\eta_b} \langle \mathcal{O}(1S_0^{[1]}) \rangle_{\eta_b} = 1.01^{+0.05}_{-0.04} \text{ MeV}. \tag{38}
\]

where the uncertainties are introduced by choosing \(m_b = 4.6 \pm 0.1 \text{ GeV}\).

For \(h_c\), we need to determine four LDMEs \(\langle \mathcal{O}(1P_1^{[1]}) \rangle_{h_c}, \langle \mathcal{O}(1S_0^{[8]}) \rangle_{h_c}, \langle \mathcal{P}(1P_1^{[1]}) \rangle_{h_c}\) and \(\langle \mathcal{P}(1S_0^{[8]}) \rangle_{h_c}\). \(\langle \mathcal{O}(1P_1^{[1]}) \rangle_{h_c}\) is determined by the B-T potential model \[32\] and
\[
\langle \mathcal{P}(1P_1^{[1]}) \rangle_{h_c} \equiv \langle \mathbf{v}^2 \rangle_{h_c} m_c^2 \langle \mathcal{O}(1P_1^{[1]}) \rangle_{h_c} \approx \langle \mathbf{v}^2 \rangle_{\eta_c} m_c^2 \langle \mathcal{O}(1P_1^{[1]}) \rangle_{h_c}, \tag{39}
\]

where \(\langle \mathbf{v}^2 \rangle_{\eta_c} = 0.228\) is taken from Ref. \[20\]. Here we have tentatively assumed \(\langle \mathbf{v}^2 \rangle_{h_c} \approx \langle \mathbf{v}^2 \rangle_{\eta_c}\). The remaining two color-octet LDMEs are determined by the operator evolution method (OEM) \[1, 24, 25\]. From Eq. (A10) we get the evolution equations,

\[
\mu^2 \frac{d\langle \mathcal{O}(1S_0^{[8]}) \rangle_{\eta_b}}{d\mu^2} = -\frac{7\alpha_s \langle \mathcal{P}(1S_0^{[8]}) \rangle}{9\pi} m_Q^2 + \frac{16\alpha_s}{9\pi} \frac{\langle \mathcal{O}(1P_1^{[1]}) \rangle}{2N_c m_Q^2} - \frac{16\alpha_s}{15\pi} \frac{\langle \mathcal{P}(1P_1^{[1]}) \rangle}{2N_c m_Q^4},
\]
\[
\mu^2 \frac{d\langle \mathcal{P}(1S_0^{[8]}) \rangle_{\eta_b}}{d\mu^2} = \frac{16\alpha_s}{9\pi} \frac{\langle \mathcal{P}(1P_1^{[1]}) \rangle}{2N_c m_Q^2}.
\]
Knowing the values of \( \langle O^{(1)} \rangle \) and \( \langle P^{(1)} \rangle \), the above differential equations will determine the values of \( \langle O^{(8)} \rangle \) and \( \langle P^{(8)} \rangle \) by evolving from initial values at \( \mu_A = \mu_{A0} \). Using two-loop running of \( \alpha_s \), we get

\[
\langle O^{(8)} \rangle^{(\mu_A)} = \frac{64}{9\beta_0} A \langle O^{(1)} \rangle^{(\mu_A)} + \frac{28}{9\beta_0} A \langle P^{(8)} \rangle^{(\mu_A)} \hspace{1cm} (40)
\]

where \( A \equiv \frac{\alpha_s(\mu_{A0})}{\alpha_s(\mu_A)} - \ln \frac{1 + \alpha_s(\mu_{A0})/\beta_0}{1 + \alpha_s(\mu_A)/\beta_0} \) with \( \beta_4 = (17C_A^2 - n_fT_R(10C_A + 6C_F))/(6\pi) \).

Choosing \( \mu_{A0} = m_c v \approx 0.8 \pm 0.2 \text{ GeV} \), the OEM assumes that the the values of \( \langle O^{(8)} \rangle \) and \( \langle P^{(8)} \rangle \) evaluated at \( \mu_A \approx 2m_c \) can be estimated by the evolution term only, i.e., neglecting initial values at \( \mu_{A0} \). Set \( m_c \) to be its pole mass, \( 1.5 \pm 0.1 \text{ GeV} \), LDMEs at \( \mu_A = 2m_c \) are

\[
\langle O^{(1)} \rangle^{(h_c)} = \frac{\langle O^{(1)} \rangle^{(h_c)}}{2N_cm_c^2} = 3.537^{+1.124}_{-0.805} \text{ MeV},
\]

\[
\langle P^{(1)} \rangle^{(h_c)} = \frac{\langle P^{(1)} \rangle^{(h_c)}}{2N_cm_c^2} = 0.806^{+0.256}_{-0.183} \text{ MeV},
\]

\[
\langle O^{(8)} \rangle^{(h_c)} = \frac{\langle O^{(8)} \rangle^{(h_c)}}{m_c^2} = 2.040^{+1.208}_{-0.704} \text{ MeV},
\]

\[
\langle P^{(8)} \rangle^{(h_c)} = \frac{\langle P^{(8)} \rangle^{(h_c)}}{m_c^2} = 0.561^{+0.350}_{-0.192} \text{ MeV}.
\]

The errors are estimated by varying \( m_c \) and \( \mu_{A0} \), among which, the uncertainty of \( \mu_{A0} \) dominates the errors for the two S-wave LDMEs.

Using the same method we can determine the LDMEs for \( h_b \),

\[
\langle O^{(1)} \rangle^{(h_b)} = 0.7555^{+0.0694}_{-0.0623} \text{ MeV}, \quad \langle P^{(1)} \rangle^{(h_b)} = 0.0314^{+0.0029}_{-0.0026} \text{ MeV},
\]

\[
\langle O^{(8)} \rangle^{(h_b)} = 0.3959^{+0.0611}_{-0.0503} \text{ MeV}, \quad \langle P^{(8)} \rangle^{(h_b)} = 0.0169^{+0.0026}_{-0.0026} \text{ MeV}.
\]

Here we choose \( m_b = 4.6 \pm 0.1 \text{ GeV}, \mu_{A0} = m_b v \approx 1.5 \pm 0.2 \text{ GeV} \) and set \( \langle v^2 \rangle_{h_b} \approx \langle v^2 \rangle_{h_c} \), similar to the assumption for \( h_c \).

Note that, another method to determine the value of the color-octet LDME \( \langle O^{(8)} \rangle^{(h_c)} \) at leading order in \( v \) is provided in Ref. [30], where LDMEs are further factorized by potential-NRQCD factorization, and they are then expressed in terms of gluonic vacuum condensation.
factor $\mathcal{E}(\mu)$. In Ref. [36] they gave both its evolution equation and the initial value at the scale $\mu_0 = 1\text{GeV}$. We evolve this factor from the initial scale to $2m_c$ and find that the value of $(\overline{\mathcal{O}}(1S_0^8))_{hc}$ through this method is about 3.5 MeV, which is a little larger than our result. However, the derivation is reasonable since we include the relativistic corrections which essentially decrease the value at leading order in $v$ [see the second term at the right-hand of the first line in Eq. (40)].

B. $\Gamma(\eta_b \to \text{LH})$

We now discuss the hadronic decay width of $\eta_b$ based on the values of LDMEs given above. Let’s first fix both the renormalization scale $\mu_r$ and factorization scale $\mu_\Lambda$ to be $2m_b$ and consider the uncertainty introduced by LDMEs. For this choice of scales, the decay width can be written as

$$
\Gamma(\eta_b \to \text{LH}) = 427.4^{+5.9}_{-5.7} \times 10^{-3} \langle \overline{\mathcal{O}}(1S_0^{[1]}) \rangle_{\eta_b} - 641.4^{+8.8}_{-9.2} \times 10^{-3} \langle \overline{\mathcal{P}}(1S_0^{[1]}) \rangle_{\eta_b},
$$

(43)

where errors are estimated by varying $m_b = 4.6 \pm 0.1\text{GeV}$, and LDMEs $\langle \overline{\mathcal{O}}(1S_0^{[1]}) \rangle_{\eta_b}$ and $\langle \overline{\mathcal{P}}(1S_0^{[1]}) \rangle_{\eta_b}$ are given by Eq. (37). As the coefficients for $\langle \overline{\mathcal{O}}(1S_0^{[1]}) \rangle_{\eta_b}$ and $\langle \overline{\mathcal{P}}(1S_0^{[1]}) \rangle_{\eta_b}$ are at the same order, the smallness of $\langle \overline{\mathcal{P}}(1S_0^{[1]}) \rangle_{\eta_b}$ means the relativistic correction can only change the total decay width by about 5%, which is not important as expected. Considering also the correlation between errors, we get the hadronic decay width of $\eta_b$ with the choice of $\mu_r = \mu_\Lambda = 2m_b$.

$$
\Gamma(\eta_b \to \text{LH}) = 9.76^{+0.58}_{-0.54} \text{MeV}.
$$

(44)

We find the $\mu_\Lambda$ dependence is much weaker than the $\mu_r$ dependence, thus we only discuss the $\mu_r$ dependence here. By varying the $\mu_r$, we get the $\mu_r$ dependence of hadronic decay width in FIG. [3]. It is clear that the NLO calculation significantly reduces the $\mu_r$ dependence. Varying $\mu_r$ from $m_b$ to $2m_b$, we get the decay width $\Gamma(\eta_b \to \text{LH}) \approx \Gamma^\text{total}(\eta_b) \sim 9.5 - 12 \text{MeV}$. This value is consistent with the experimental data $\Gamma^\text{exp}(\eta_b) = 10.8^{+4.0+4.5}_{-3.7-2.0} \text{MeV}$ [8].

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FIG. 5: $\mu_r$ dependence of $\Gamma(\eta_b \rightarrow \text{LH})$. LO represents values without QCD and relativistic corrections, NLO* includes QCD corrections but only at leading order in $v$, and NLO takes into account all contributions up to $O(\alpha_s v^2)$. The LDMEs are taken from the B-T potential model and the Gremm-Kapustin relation. Here we set $\mu_\Lambda = 2m_b$, and $m_b = 4.6$ GeV.

C. $\Gamma(h_c \rightarrow \text{LH})$

The numerical values of SD coefficients for hadronic decay width of $h_c$ are

$$\Gamma(h_c \rightarrow \text{LH}) = 328.7^{+26.1}_{-21.8} \times 10^{-3}\langle \mathcal{O}(^{1}\text{S}_0[^8]) \rangle_{h_c} - 39.6^{+3.1}_{-3.8} \times 10^{-3}\langle \mathcal{O}(^{1}\text{P}_1[^1]) \rangle_{h_c} - 446.0^{+29.7}_{-35.5} \times 10^{-3}\langle \mathcal{P}(^{1}\text{S}_0[^8]) \rangle_{h_c} + 92.4^{+8.8}_{-7.3} \times 10^{-3}\langle \mathcal{P}(^{1}\text{P}_1[^1]) \rangle_{h_c},$$

(45)

where both the renormalization scale $\mu_r$ and factorization scale $\mu_\Lambda$ are set to be $2m_c$. The redefined LDMEs and their values are given in Eq. (41). With these results we then investigate the effects of the QCD corrections and relativistic corrections.

Let us first analysis the partial widths of the four channels in Table I. Among the four, the $\langle \mathcal{O}(^{1}\text{S}_0[^8]) \rangle_{h_c}$ channel is positive and it dominates the total width. Contributions of the $\langle \mathcal{O}(^{1}\text{P}_1[^1]) \rangle_{h_c}$ channel and $\langle \mathcal{P}(^{1}\text{S}_0[^8]) \rangle_{h_c}$ channel are negative and compatible, although the latter one is suppressed by $v^2$. This is because, as we mentioned before, the $^{1}\text{P}_1[^1]$ Fock state cannot couple with two gluons, and its SD coefficient is suppressed by $\alpha_s$. It is the balance between $\alpha_s$ and $v^2$ that results in the two partial decay widths being compatible. The last term, $\langle \mathcal{P}(^{1}\text{P}_1[^1]) \rangle_{h_c}$ channel, is suppressed by both $\alpha_s$ and $v^2$, and it gives the smallest
contribution. Summing up the first two channels we get the decay width at leading order in $v$, $\Gamma(v^0) = 0.53^{+0.40}_{-0.23}\text{MeV}$, which is consistent with the previous work [7]. However, we will show later that the experimental data favor a smaller value. Including also the relativistic corrections, the total decay width will decrease by about $1/3$. Next we list the partial widths

| TABLE I: $\Gamma(h_c \to \text{LH})$ expressed with the contributions of each LDME. |
|-----------------------------------------------|
| $\langle O(1S_0^{[8]})\rangle_{h_c}$ | $\langle O(1P_1^{[1]})\rangle_{h_c}$ | $\langle P(1S_0^{[8]})\rangle_{h_c}$ | $\langle P(1P_1^{[1]})\rangle_{h_c}$ | Total |
| $\Gamma(2S+1L_J^{[c]} \to \text{LH})$(MeV) | $0.67^{+0.43}_{-0.25}$ | $-0.14^{+0.04}_{-0.06}$ | $-0.25^{+0.10}_{-0.17}$ | $0.07^{+0.03}_{-0.02}$ | $0.35^{+0.25}_{-0.15}$ |

order by order in $\alpha_s$ and $v$ in Table I. We find the QCD correction, $\alpha_s^1v^0$ contribution, is as large as the leading order contribution. Detailed study reveals that the large correction mainly comes from the $1S^0_0$ channel. In Ref. [37], the authors pointed out that the large correction for $1S^0_0$ channel, similar to the $1S^0_8$ channel, is due to the existence of renormalons, and they also proposed a resummation method to deal with the renormalons. Nevertheless, resummation of this kind for $1S^0_0$ channel is beyond the scope of this work, and we will leave it as a future study. In our work, as both of the $\alpha_s^0v^2$ contribution and the $\alpha_s^1v^2$ contribution are negative, they can balance the enhancement by QCD correction of $1S^0_0$ channel. Moreover, we find our complete NLO correction improves the normalization and factorization scale dependence compared with the NLO* result, which are shown in FIG. 3.

| TABLE II: $\Gamma(h_c \to \text{LH})$ expressed with contributions at various orders of $\alpha_s$ and $v$. |
|-----------------------------------------------|
| $\alpha_s^0v^0$ | $\alpha_s^1v^0$ | $\alpha_s^0v^2$ | $\alpha_s^1v^2$ | Total |
| $\Gamma(h_c \to \text{LH})$(MeV) | $0.32^{+0.21}_{-0.12}$ | $0.21^{+0.20}_{-0.11}$ | $-0.13^{+0.04}_{-0.06}$ | $-0.06^{+0.04}_{-0.08}$ | $0.35^{+0.25}_{-0.15}$ |

In order to compare with the experiment data [6], we also need the E1 transition decay width $\Gamma(h_c \to \eta_c + \gamma)$ up to the $v^2$ order, because this is another important decay channel of $h_c$. Ref. [17] estimated the transition decay widths but only at leading order in $v$ by using HQSS between the spin-singlet and triplet P-wave charmonia,

$$\Gamma(h_c \to \gamma\eta_c) = \frac{(E_{\eta_c}^h)^3}{9} \sum_{J=0}^2 (2J+1) \frac{\Gamma(\chi_{cJ} \to \gamma J/\psi)}{(E_{\chi_c J}^h)^3}. \quad (46)$$

And the obtained E1 width is $615 \pm 29$ keV using the PDG Data [30]. This result is consistent with the potential model calculations at leading order in $v$ [38]. However, if the $v^2$ corrections are considered, HQSS will not hold any more. Ref. [38] showed that
FIG. 6: $\mu_r$ and $\mu_\Lambda$ dependence of $\Gamma(h_c \to \text{LH})$. The upper plots are for $\mu_r$ and lower ones for $\mu_\Lambda$. From left to right the plots are shown for LO, NLO* and NLO respectively, where NLO* includes $O(\alpha_s)$ but excludes $O(\alpha_s v^2)$ corrections.

the width of $h_c \to \gamma \eta_c$ can be reduced from 650 KeV to 385 KeV by relativistic effects. Subsequent studies using various potential models \cite{39,41} also observed similar relativistic effects, resulting in E1 transition width at the range of 354-323 KeV. In this paper we choose the value $\Gamma(h_c \to \gamma \eta_c) = 385$ keV from Ref. \cite{38}.

Combining the LH and $\gamma \eta_c$ decay channels of $h_c$, we get the predictions for total decay width $\Gamma^{\text{th}}(h_c) = 0.74^{+0.25}_{-0.15}$ MeV and the branching ratio $\mathcal{B}^{\text{th}}(h_c \to \eta_c + \gamma) = 52 \pm 13\%$.

Our predictions are consistent with the new experimental data $\Gamma^{\text{exp}}(h_c) = 0.73^{+0.45}_{-0.28}$ MeV and $\mathcal{B}^{\text{exp}}(h_c \to \eta_c + \gamma) = 54.3 \pm 6.7 \pm 5.2\%$ measured by the BESIII Collaboration \cite{5}. However, if we ignore the relativistic corrections to the hadronic decay width, the total width will increase to 0.92 MeV and the E1 transition branching ratio will be decreased to 42\%. Therefore, it is evident that the relativistic corrections play an important role in the $h_c$ decay and they can lead to a better agreement between theoretical prediction and the experimental data.
D. $\Gamma(h_b \rightarrow LH)$

Similar to $h_c$, we get the decay width for $h_b$,

$$
\Gamma(h_b \rightarrow LH) = 145.9^{+2.1}_{-2.0} \times 10^{-3} \langle \mathcal{O}(1S_{0}^{[8]}) \rangle_{h_b} - (15.3 \pm 0.3) \times 10^{-3} \langle \mathcal{O}(1P_{1}^{[1]}) \rangle_{h_b} \\
- (196.0 \pm 3.0) \times 10^{-3} \langle \mathcal{P}(1S_{0}^{[8]}) \rangle_{h_b} + (35.8 \pm 0.6) \times 10^{-3} \langle \mathcal{P}(1P_{1}^{[1]}) \rangle_{h_b}.
$$

(47)

The $\mu_r$ and $\mu_A$ dependence are plotted in Fig. 7, where again we find the complete NLO correction largely reduces the scale dependence. From partial decay width of each contribution in Tables III and IV, it is clear that the $v^2$ correction effect is much smaller for $h_b$ than that for $h_c$, while QCD correction is still important. The E1 transition decay width for $h_b$ is evaluated in the NR [42], GI [41] and Screened-potential models [43], and the results are listed in Table V. Compared with the experiment data $B_{\exp}(h_b(1P) \rightarrow \eta_b(1S) \gamma) = 49.2^{+5.7}_{-3.3}\%$ [8], our prediction using NR model fits it very well, and predictions using other three models are also within the error band.

| TABLE III: $\Gamma(h_b \rightarrow LH)$ expressed with contributions of each LDME. |
| --- |
| $\langle \mathcal{O}(1S_{0}^{[8]}) \rangle_{h_b}$ | $\langle \mathcal{O}(1P_{1}^{[1]}) \rangle_{h_b}$ | $\langle \mathcal{P}(1S_{0}^{[8]}) \rangle_{h_b}$ | $\langle \mathcal{P}(1P_{1}^{[1]}) \rangle_{h_b}$ | Total |
| $\Gamma(2S+1L^{[c]} \rightarrow LH)$ (keV) | 57.78$^{+9.42}_{-7.79}$ | $-11.58^{+1.13}_{-1.29}$ | $-3.32^{+0.45}_{-0.54}$ | $1.12^{+0.12}_{-0.11}$ | 44.00$^{+8.23}_{-6.73}$ |

| TABLE IV: $\Gamma(h_b \rightarrow LH)$ expressed with various orders of $\alpha_s$ and $v$. |
| --- |
| $\Gamma(h_b \rightarrow LH)$ (keV) | $\alpha_s^0 v^0$ | $\alpha_s^1 v^0$ | $\alpha_s^0 v^2$ | $\alpha_s^1 v^2$ | Total |
| | 33.41$^{+5.39}_{-4.46}$ | 12.78$^{+3.39}_{-2.72}$ | $-1.91^{+0.26}_{-0.31}$ | $-0.29^{+0.15}_{-0.19}$ | 44.00$^{+8.23}_{-6.73}$ |

| TABLE V: $\Gamma(h_b \rightarrow \eta_b + \gamma)$ and $B(h_b \rightarrow \eta_b + \gamma)$ in NR, GI and Screened-potential models(SNR$0$ is calculated using the zeroth-order wave functions while SNR$1$ using the first-order relativistically corrected wave functions) |
| --- |
| $\Gamma(h_b \rightarrow \eta_b + \gamma)$ (keV) | NR | GI | SNR$0$ | SNR$1$ |
| 41.8 | 37.0 | 55.8 | 36.3 |
| $\Gamma_{\text{total}}(h_b)$ (keV) | 85.8 | 81.0 | 100.0 | 80.3 |
| $B(h_b \rightarrow \eta_b + \gamma)$ | 48.7% | 45.7% | 55.9% | 45.2% |
FIG. 7: $\mu_r$ and $\mu_A$ dependence of $\Gamma(h_b \to \text{LH})$. From left to right the three plots represent LO, NLO$^*$ and NLO respectively, where NLO$^*$ includes $O(\alpha_s)$ but excludes $O(\alpha_s v^2)$ corrections.

VI. SUMMARY

We have calculated order $\alpha_s v^2$ corrections for the annihilation hadronic decay widths of spin-singlet heavy quarkonia $\eta_b$, $h_c$ and $h_b$ within the framework of NRQCD. The short-distance coefficients are calculated by covariant projection method, and the LDMEs are estimated by using the potential model and operator evolution methods. For the $h_c$ decay, we find that $O(v^2)$ and $O(\alpha_s v^2)$ corrections contribute large and negative values to the decay width, which substantially reduce the decay width calculated in the leading order in $v^2$. It shows that relativistic corrections play an important role in hadronic decays of $c\bar{c}$ system, and can improve the theoretical results as compared with experimental data. Our calculated total decay width $\Gamma^\text{th}(h_c) = 0.74^{+0.25}_{-0.15}$ MeV and branching ratio $B^\text{th}(h_c \to \eta_c + \gamma) = 52 \pm 13\%$ are consistent with the measurements by BESIII $[5]$. For $\eta_b$ and $h_b$ decays, we have calculated their hadronic decay widths and found that $\Gamma(\eta_b \to \text{LH}) = 9.76^{+0.58}_{-0.54}$ MeV and $\Gamma(h_b \to \text{LH}) = 44.00^{+8.23}_{-6.73}$ keV. We conclude that for the $b\bar{b}$ system $O(\alpha_s v^2)$ corrections are not as important as in the $c\bar{c}$ system. We have also compared our theoretical results with experimental data $[5, 8]$ and found that in general our calculations are consistent with data within theoretical and experimental uncertainties.
FIG. 8: The one-loop NRQCD diagrams which involve the Feynman rules up to $O(v^2)$. The Coulomb interactions and the cross diagrams have been suppressed.

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Appendix A: EVOLUTION OF NRQCD MATRIX ELEMENTS $O(1S_0^{[8]})$ and $P(1S_0^{[8]})$ AT $O(\alpha_s v^2)$

In order to cancel the infrared divergence in short-distance coefficients of $1P^{[1]}$ Fock state, we need to evaluate the NRQCD four-fermion operators $O(1S_0^{[8]})$ and $P(1S_0^{[8]})$ to sufficient order.

The $O(\alpha_s)$ correction diagrams include three sets: self-energy diagrams which are related to self-energy corrections of external heavy (anti-)quarks; Coulomb diagrams where the gluon is connected with both initial or final heavy quark and anti-quark; and the intersecting diagrams where the gluon is related to an initial heavy (anti-)quark and a final (anti-)quark. The results of the first two sets have been given in Refs. [20, 44], and here we only calculate the intersecting diagrams which relate to the transition from $S$ wave to $P$ wave.

Using the Lagrangian shown in Eqs. (3) and (5), we can write the amplitudes of diagrams in Fig. 8 as (other crossed diagrams are not shown)
This integral can be reduced by taking the following substitution,

\[ I_{a+b+c} = ig_s^2 \int \frac{d^Dl}{(2\pi)^D} \frac{\mathbf{q} \cdot \mathbf{q}' - (\mathbf{q} \cdot \mathbf{l})(\mathbf{q}' \cdot \mathbf{l})/l^2}{m_Q^2(l_0^2 - l^2 + i\epsilon)} [1 - \frac{q^2}{2m_Q^2} - \frac{q'^2}{2m_Q^2}] (q_0 - l_0 - \frac{(q-l)^2}{2m_Q} + i\epsilon) [q_0' - l_0 - \frac{(q'-l)^2}{2m_Q} + i\epsilon], \]  

(A1a)

\[ I_d = ig_s^2 \int \frac{d^Dl}{(2\pi)^D} \frac{1}{[q_0 - l_0 - \frac{(q-l)^2}{2m_Q} + i\epsilon][q_0' - l_0 - \frac{(q'-l)^2}{2m_Q} + i\epsilon]}, \]  

(A1b)

where \( q = (q_0, \mathbf{q}) \) is the heavy quark external momentum and \( l = (l_0, \mathbf{l}) \) is loop integral momentum. Since there is no pole on the upper half of \( l_0 \)'s complex plane, the second integral \( I_d \) yields zero. Contour integrating the first integral over \( l_0 \) around the \( l_0 = |\mathbf{l}| - i\epsilon \) pole, we find

\[ I_{a+b+c} = g_s^2 \int \frac{d^{D-1}l}{(2\pi)^{D-1}} \frac{\mathbf{q} \cdot \mathbf{q}' - (\mathbf{q} \cdot \mathbf{l})(\mathbf{q}' \cdot \mathbf{l})/l^2}{2m_Q^2|\mathbf{l}|} [-|\mathbf{l}| - \frac{l^2}{2m_Q} + \frac{q_0^2}{m_Q} + i\epsilon][-|\mathbf{l}| - \frac{l^2}{2m_Q} + \frac{q_0'^2}{m_Q} + i\epsilon]. \]  

(A2)

Before further performing the integration, we will expand the relative momentum in the denominator \( [D] \). Assuming that \( \mathbf{q} \cdot \mathbf{m}_Q, \mathbf{q}' \cdot \mathbf{m}_Q \) and \( l^2/m_Q \) are far smaller than \( |\mathbf{l}| \), we get the required expansion,

\[ I_{a+b+c} = \frac{g_s^2}{2m_Q^2} \int \frac{d^{D-1}l}{(2\pi)^{D-1}} \frac{\mathbf{q} \cdot \mathbf{q}' - (\mathbf{q} \cdot \mathbf{l})(\mathbf{q}' \cdot \mathbf{l})/l^2}{|\mathbf{l}|^3} (1 - \frac{q^2}{2m_Q^2} - \frac{q'^2}{2m_Q^2}) \times \left( 1 + \frac{(\mathbf{q} \cdot \mathbf{l})^2}{|\mathbf{l}|^2 m_Q^2} \right) + \text{(high order or irrelevant expansions)}. \]  

(A3)

This integral can be reduced by taking the following substitution,

\[ \mathbf{l}^i \mathbf{l}^j \rightarrow \frac{1}{D-1} \delta^{ij} \mathbf{l}^2, \]  

(A4a)

\[ \mathbf{l}^i \mathbf{l}^k \mathbf{l}^r \rightarrow \frac{1}{(D-1)(D+1)} \left( \delta^{ij} \delta^{kr} + \delta^{ik} \delta^{jr} + \delta^{ir} \delta^{kj} \right) \mathbf{l}^4, \]  

(A4b)

where \( \delta^{ij} \) is \( D - 1 \) dimensional Euclidean delta symbol. The integral yields

\[ I_{a+b+c} = \frac{\pi \alpha_s^{(b)}}{2m_Q^2} \frac{D-2}{\pi^2} \frac{D-1}{D+1} \left( \frac{1}{D+1} \frac{1}{2} \left( \frac{q^2 + q'^2}{D^2} \right) \left( \frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} \right) \right). \]  

(A5)

Summing up all the diagrams we get

\[ I = \frac{2\alpha_s^{(b)}}{\pi m_Q^2} \frac{D-2}{D-1} \mathbf{q} \cdot \mathbf{q}' \left( \frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} \right) \left( 1 - \frac{D-1}{D+1} \frac{1}{2} \left( \frac{q^2 + q'^2}{D^2} \right) \right) \times \left[ C_F \frac{1}{2N_c} + B_F T^a \otimes T^a \right] O(\mathcal{S}_6^{[8]}). \]  

(A6)
Recalling the definitions of $\mathcal{O}(^1P_1^{[1]})$ and $\mathcal{P}(^1P_1^{[1]})$, we can write

$$
\langle H|\mathcal{O}(^1S_0^{[8]})|H\rangle = \langle H|\mathcal{O}(^1S_0^{[8]})|H\rangle_{\text{Born}} + \frac{2(D-2)\alpha_s^{(b)}}{(D-1)\pi m_Q^2} \left( \frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right) \times \left[ C_F \frac{\langle H|\mathcal{O}(^1P_1^{[1]})|H\rangle}{2N_c} - \frac{D-1}{(D+1)\pi m_Q^2} C_F \frac{\langle H|\mathcal{P}(^1P_1^{[1]})|H\rangle}{2N_c} \right],
$$

(A7a)

$$
\langle H|\mathcal{P}(^1S_0^{[8]})|H\rangle = \langle H|\mathcal{P}(^1S_0^{[8]})|H\rangle_{\text{Born}} + \frac{2(D-2)\alpha_s^{(b)}}{(D-1)\pi m_Q^2} \left( \frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right) C_F \frac{\langle H|\mathcal{P}(^1P_1^{[1]})|H\rangle}{2N_c},
$$

(A7b)

where we have omitted terms for $\mathcal{O}(^1P_1^{[8]})$ and $\mathcal{P}(^1P_1^{[8]})$ since they are irrelevant in our work. The presence of UV divergence indicates that the LDMEs need renormalization. The relevant counter-term in the $\overline{MS}$ scheme can be chosen as

$$
\langle H|\mathcal{O}(^1S_0^{[8]})|H\rangle = \mu^2 \left\{ \langle H|\mathcal{O}(^1S_0^{[8]})|H\rangle^{(\mu)} + \frac{4\alpha_s}{3\pi m_Q^2} \left( \frac{1}{\epsilon_{\text{UV}}} + \ln 4\pi - \gamma_E \right) \right\} \times \left[ C_F \frac{\langle H|\mathcal{O}(^1P_1^{[1]})|H\rangle}{2N_c} - \frac{3}{5m_Q^2} C_F \frac{\langle H|\mathcal{P}(^1P_1^{[1]})|H\rangle}{2N_c} \right],
$$

(A8a)

$$
\langle H|\mathcal{P}(^1S_0^{[8]})|H\rangle = \mu^2 \left\{ \langle H|\mathcal{P}(^1S_0^{[8]})|H\rangle^{(\mu)} + \frac{4\alpha_s}{3\pi m_Q^2} \left( \frac{1}{\epsilon_{\text{UV}}} + \ln 4\pi - \gamma_E \right) \right\} \times C_F \frac{\langle H|\mathcal{P}(^1P_1^{[1]})|H\rangle}{2N_c},
$$

(A8b)

where $\mu$ is the NRQCD renormalization scale. Combining Eqs. (A7) and (A8), we find

$$
\langle H|\mathcal{O}(^1S_0^{[8]})|H\rangle_{\text{Born}} = \mu^2 \langle H|\mathcal{O}(^1S_0^{[8]})|H\rangle^{(\mu)} + \frac{4\alpha_s}{3\pi m_Q^2} \left( \frac{1}{\epsilon_{\text{IR}}} + \ln 4\pi - \gamma_E \right) \times \left( \frac{\mu}{\mu} \right)^2 \left[ C_F \frac{\langle H|\mathcal{O}(^1P_1^{[1]})|H\rangle}{2N_c} - \frac{3}{5m_Q^2} C_F \frac{\langle H|\mathcal{P}(^1P_1^{[1]})|H\rangle}{2N_c} \right],
$$

(A9a)

$$
\langle H|\mathcal{P}(^1S_0^{[8]})|H\rangle_{\text{Born}} = \mu^2 \langle H|\mathcal{P}(^1S_0^{[8]})|H\rangle^{(\mu)} + \frac{4\alpha_s}{3\pi m_Q^2} \left( \frac{1}{\epsilon_{\text{IR}}} + \ln 4\pi - \gamma_E \right) \times C_F \frac{\langle H|\mathcal{P}(^1P_1^{[1]})|H\rangle}{2N_c}.
$$

(A9b)

Considering also the self-energy contribution [see Eq. (B14) in Ref. [1]], we get the total
loop corrections of NRQCD LDMEs,

\[
\langle H|\mathcal{O}(^{1S_0^{[8]}[H]}|H)_{\text{Born}} = \mu_A^{-2e} \langle H|\mathcal{O}(^{1S_0^{[8]}[H]}|H)^{\mu_A} + \frac{4\alpha_s}{3\pi m_Q^2} \left( \frac{1}{\epsilon_{1R}} + \ln 4\pi - \gamma_E \right) \\
\times \left( \frac{\mu}{\mu_A} \right)^{2e} \left[ C_F \frac{\langle H|\mathcal{O}(^{1P_1^{[1]}[H]}|H)_{2N_c}}{2} - \frac{3}{5m_Q^2} C_F \langle H|\mathcal{P}(^{1P_1^{[1]}[H]}|H)_{2N_c} \right] \\
- \frac{N_c^2 - 2}{4N_c} \langle H|\mathcal{P}(^{1S_0^{[8]}[H]}|H) \right], \tag{A10a}
\]

\[
\langle H|\mathcal{P}(^{1S_0^{[8]}[H]}|H)_{\text{Born}} = \mu_A^{-2e} \langle H|\mathcal{P}(^{1S_0^{[8]}[H]}|H)^{\mu_A} + \frac{4\alpha_s}{3\pi m_Q^2} \left( \frac{1}{\epsilon_{1R}} + \ln 4\pi - \gamma_E \right) \\
\times \left( \frac{\mu}{\mu_A} \right)^{2e} C_F \frac{\langle H|\mathcal{P}(^{1P_1^{[1]}[H]}|H)_{2N_c}}{2}. \tag{A10b}
\]

Appendix B: Scheme choice and absorption of \((\mathcal{T}_{1-8}^{(1S_0,1P_1)})_{1P_1}\)

In this appendix, we define the factorization scheme that we use in this work, and we will show that there is no contribution from \((\mathcal{T}_{1-8}^{(1S_0,1P_1)})_{1P_1}\) in our scheme. Let’s begin with the factorization formula for \(\Gamma(H^{1P_1} \rightarrow \text{LH})\) in \(\overline{\text{MS}}\) scheme,

\[
\Gamma(H^{1P_1} \rightarrow \text{LH}) = \frac{F(1S_0^{[8]}\overline{\text{MS}})}{m_Q^2}(\mathcal{O}(1S_0^{[8]}))_{1P_1} + \frac{G(1S_0^{[8]}\overline{\text{MS}})}{m_Q^4}(\mathcal{P}(1S_0^{[8]}))_{1P_1} \\
+ \frac{F(1P_1^{[1]}\overline{\text{MS}})}{m_Q^4}(\mathcal{O}(1P_1^{[1]}))_{1P_1} + \frac{G(1P_1^{[1]}\overline{\text{MS}})}{m_Q^6}(\mathcal{P}(1P_1^{[1]}))_{1P_1} \\
+ \frac{T(1S_0^{[8]}\overline{\text{MS}})}{m_Q^5}(\mathcal{T}_{1-8}^{(1S_0,1P_1)})_{1P_1}, \tag{B1}
\]

where an explicit \(\overline{\text{MS}}\) is marked for any LDME and SD coefficient. There are many scheme choices to eliminate the last term in Eq. (B1). Our choice is to define the factorization scheme of \((\mathcal{O}(1S_0^{[8]}))_{1P_1}\) by the following relation

\[
\Gamma(H^{1P_1} \rightarrow \text{LH}) = \frac{F(1S_0^{[8]}\overline{\text{MS}})}{m_Q^2}(\mathcal{O}(1S_0^{[8]}))_{\text{LT}} + \frac{G(1S_0^{[8]}\overline{\text{MS}})}{m_Q^4}(\mathcal{P}(1S_0^{[8]}))_{1P_1} \\
+ \frac{F(1P_1^{[1]}\overline{\text{MS}})}{m_Q^4}(\mathcal{O}(1P_1^{[1]}))_{1P_1} + \frac{G(1P_1^{[1]}\overline{\text{MS}})}{m_Q^6}(\mathcal{P}(1P_1^{[1]}))_{1P_1}, \tag{B2}
\]

where, to distinguish from \(\overline{\text{MS}}\) scheme, we denote it as the leading twist scheme (LT). Note that the relation in Eq. (B2) should be understood to be valid only at \(\alpha_s\) order, that is, \(T(1S_0^{[8]}\overline{\text{MS}})_{\text{LT}}\) can be nonzero at higher order in \(\alpha_s\). From Eqs. (B1) and (B2), we get the
scheme transformation relation,
\[ \langle O(1S_0^{[8]}) \rangle_{iP_1}^{LT} - \langle O(1S_0^{[8]}) \rangle_{iP_1}^{\overline{MS}} = \frac{T(1S_0,1P_1)^{\overline{MS}}}{m_Q^3 F(1S_0^{[8]})_{\overline{MS}}} \langle T_{1-s}(1S_0,1P_1) \rangle_{iP_1}^{\overline{MS}}. \] (B3)

According to the \( \alpha_s \) expansion of SD coefficients,
\[ F(1S_0^{[8]})^{\overline{MS}} = F(1S_0^{[8]})^{(0)} + \alpha_s F(1S_0^{[8]})^{(1)\overline{MS}} + O(\alpha_s^2), \] (B4a)
\[ T(1S_0,1P_1)^{\overline{MS}} = \alpha_s T(1S_0,1P_1)^{(1)\overline{MS}} + O(\alpha_s^2), \] (B4b)
we rewrite the difference as
\[ \langle O(1S_0^{[8]}) \rangle_{iP_1}^{LT} - \langle O(1S_0^{[8]}) \rangle_{iP_1}^{\overline{MS}} = \alpha_s \frac{T(1S_0,1P_1)^{(1)\overline{MS}}}{m_Q^3 F(1S_0^{[8]})^{(0)}} \langle T_{1-s}(1S_0,1P_1) \rangle_{iP_1}^{\overline{MS}} + O(\alpha_s^2). \] (B5)

It is clear that the difference is suppressed by \( O(\alpha_s v^2) \). Eq. (B2) does not determine the scheme choice of \( \langle T_{1-s}(1S_0,1P_1) \rangle_{iP_1} \), and one can still choose \( \overline{MS} \) or other schemes. The reason is that the scheme dependence of \( \langle T_{1-s}(1S_0,1P_1) \rangle_{iP_1} \) is at higher order in \( \alpha_s \), which is irrelevant to our calculation. Note that, the relation between our scheme and \( \overline{MS} \) scheme here is similar to the relation between DIS scheme and \( \overline{MS} \) scheme definition for the \( F_2 \) structure function of virtual \( \gamma \) deep inelastic scattering (see Refs. [46, 47], for example).

An important consequence of Eq. (B5) is that, the evolution equations for \( \langle O(1S_0^{[8]}) \rangle_{iP_1} \) in both \( \overline{MS} \) and LT scheme at \( O(\alpha_s) \) are exactly the same, which follows from the fact that the factorization scale dependence of both \( T(1S_0,1P_1)^{(1)\overline{MS}} \) and \( \langle T_{1-s}(1S_0,1P_1) \rangle_{iP_1}^{\overline{MS}} \) are at \( O(\alpha_s) \). Therefore, although we calculate evolution equations for LDMEs in \( \overline{MS} \) scheme in Appendix \( \text{Appendix A} \), these results are unchanged for the LT scheme.

Especially, the estimated \( \langle O(1S_0^{[8]}) \rangle_{iP_1} \) in Sec. VA using OEM is the same for both LT scheme and \( \overline{MS} \) scheme. This seems to be questionable at first glance, as Eq. (B3) may imply its value is different under the two different schemes. However, remember that the OEM picks up only the evolution terms in the LDMEs and disregards all other terms. Although Eq. (B3) tells us that \( \langle O(1S_0^{[8]}) \rangle_{iP_1} \) is different under the two schemes, the difference only changes the initial value, which is ignored in the OEM. As a result, in the OEM this difference is ignored.

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