Dynamics of Learning with Restricted Training Sets

I. General Theory

A.C.C. Coolen
Department of Mathematics
King’s College London
Strand, London WC2R 2LS, UK

D. Saad
The Neural Computing Research Group
Aston University
Birmingham B4 7ET, UK

August 7th 1999

Abstract

We study the dynamics of supervised learning in layered neural networks, in the regime where the size \( p \) of the training set is proportional to the number \( N \) of inputs. Here the local fields are no longer described by Gaussian probability distributions and the learning dynamics is of a spin-glass nature, with the composition of the training set playing the role of quenched disorder. We show how dynamical replica theory can be used to predict the evolution of macroscopic observables, including the two relevant performance measures (training error and generalization error), incorporating the old formalism developed for complete training sets in the limit \( \alpha = p/N \to \infty \) as a special case.

For simplicity we restrict ourselves in this paper to single-layer networks and realizable tasks.

PACS: 87.10.+e, 02.50.-r, 05.20.-y

Contents

1 Introduction 2

2 From Microscopic to Macroscopic Laws 4

2.1 Definitions 4

2.2 Derivation of Macroscopic Fokker-Planck Equation 6

2.3 Choice and Properties of Canonical Observables 9

2.4 Derivation of Deterministic Dynamical Laws 11

2.5 Closure of Macroscopic Dynamical Laws 15

3 Summary of the Theory and Connection with \( \alpha \to \infty \) Formalism 17

3.1 Summary of the Theory 17

3.2 Uniqueness and Iterative Calculation of the Functional Saddle-Point 18

3.3 Fourier Representation and Conditionally-Gaussian Solutions 22

3.4 Link with the Formalism for Complete Training Sets 24

4 Discussion 25

A Replica Calculation of the Green’s Function 27

A.1 Disorder Averaging 27

A.2 Derivation of Saddle-Point Equations 32

A.3 Replica-Symmetric Saddle-Points 34

A.4 Explicit Expression for the Green’s Function 36
1 Introduction

In the last few years much progress has been made in the analysis of the dynamics of supervised learning in layered neural networks, using the strategy of statistical mechanics: by deriving from the microscopic dynamical equations of the learning process a set of closed laws describing the evolution of suitably chosen macroscopic observables (dynamic order parameters), in the limit of an infinite system size (e.g. [1, 2, 3, 4, 5]). A recent review and more extensive guide to the relevant references can be found in [6, 7]. [7] also contains a preliminary presentation of some of the results in the present paper, without proofs or derivations. The main successful procedure developed so far is built on the following four cornerstones:

- **The task to be learned by the network is defined by a (possibly noisy) ‘teacher’, which is itself a layered neural network.** This induces a canonical set of dynamical order parameters, typically the (rescaled) overlaps between the various student weight vectors and the corresponding teacher weight vectors.

- **The number of network inputs is (eventually) taken to be infinitely large.** This ensures that fluctuations in mean-field observables will vanish, and creates the possibility of using the central limit theorem.

- **The number of ‘hidden’ neurons is finite.** This prevents the number of order parameters from being infinite, and ensures that the cumulative impact of their fluctuations is insignificant.

- **The size of the training set is much larger than the number of weight updates made.** Each example presented to the system is now different from those that have already been seen, such that the local fields will have Gaussian probability distributions, which leads to closure of the dynamic equations.

These are not ingredients to simplify the calculations, but vital conditions, without which the standard method fails. Although the assumption of an infinite system size has been shown not to be too critical [8], the other assumptions do place serious restrictions on the degree of realism of the scenarios that can be analyzed, and have thereby, to some extent, prevented the theoretical results from being used by practitioners.

Here we study the dynamics of learning in layered neural networks with restricted training sets, where the number \( p \) of examples (‘questions’ with corresponding ‘answers’) scales linearly with the number \( N \) of inputs, i.e. \( p = \alpha N \) with \( 0 < \alpha < \infty \). Here individual questions will re-appear during the learning process as soon as the number of weight updates made is of the order of the size of the training set. In the traditional models, where the duration of an individual update is defined as \( N^{-1} \), this happens as soon as \( t = \mathcal{O}(\alpha) \). At that point correlations develop between the weights and the questions in the training set, and the dynamics is of a spin-glass type, with the composition of the training set playing the role of ‘quenched disorder’. The main consequence of this is that the central limit theorem no longer applies to the student’s local fields, which are now indeed described by non-Gaussian distributions. To demonstrate this we trained (on-line) a perceptron with weights \( J \) on noiseless examples generated by a teacher perceptron with weights \( B \), using the Hebb and AdaTron rules. We plotted in Fig. 1 the student and teacher fields, \( x = J \cdot \xi \) and \( y = B \cdot \xi \), respectively, where \( \xi \) is the input vector, for \( p = N/2 \) examples and at time \( t = 50 \). The marginal distribution \( P(x) \) for \( p = N/4 \), at times \( t = 10 \) for the Hebb rule and \( t = 20 \) for the AdaTron rule, is shown in Fig. 2. The non-Gaussian student field distributions observed in Figs. 1 and 2 induce a deviation between
the training- and generalization errors, which measure the network performance on training and test examples, respectively. The former involves averages over the non-Gaussian field distribution, whereas the latter (which is calculated over all possible examples) still involves Gaussian fields.

The appearance of non-Gaussian fields leads to a complete breakdown of the standard formalism, based on deriving closed equations for a finite number of observables: the field distributions can no longer be characterized by a few moments, and the macroscopic laws must now be averaged over realizations of the training set. One could still try to use Gaussian distributions as large $\alpha$ approximations, see e.g. [9], but it will be clear from Figs. 1 and 2 that a systematic theory will have to give up Gaussian distributions entirely. The first rigorous study of the dynamics of learning with restricted training sets in non-linear networks, via the calculation of generating functionals, was carried out in [10] for perceptrons with binary weights. The only cases where explicit and relatively simple solutions can be obtained, even for restricted training sets, are those where linear learning rules are used, such as [11] or [12].

In this paper we show how the formalism of dynamical replica theory (see e.g. [13]) can be used successfully to predict the evolution of macroscopic observables for finite $\alpha$, incorporating the infinite training set formalism as a special case, for $\alpha \to \infty$. Central to our approach is the derivation of a diffusion equation for the joint distribution $P[x, y]$ of the student and teacher fields, which will be found to have Gaussian solutions only for $\alpha \to \infty$. For simplicity and transparency we restrict ourselves in the present paper to single-layer systems and noise-free teachers. Application and generalization of our methods to multi-layer systems [14] and learning scenarios involving ‘noisy’ teachers [15] are presently under way.

Our paper is organized as follows. In section 2 we first derive a Fokker-Planck equation describing the evolution of arbitrary mean-field observables for $N \to \infty$. This allows us to identify the conditions for the latter to be described by closed deterministic laws. We then choose as our observables the joint

Figure 1: Student and teacher fields $(x, y) = (J \cdot \xi, B \cdot \xi)$ as observed during numerical simulations of on-line learning (learning rate $\eta = 1$) in a perceptron of size $N = 10,000$ at $t = 50$, using ‘questions’ from a restricted training set of size $p = N/2$. Left: Hebbian learning. Right: AdaTron learning. Note: in the case of Gaussian field distributions one would have found spherically shaped plots.
Figure 2: Distribution $P(x)$ of student fields as observed during numerical simulations of on-line learning (learning rate $\eta = 1$) in a perceptron of size $N = 10,000$, using ‘questions’ from a restricted training set of size $p = N/4$. Left: Hebbian learning, measured at $t = 10$. Right: AdaTron learning, measured at $t = 20$. Note: not only are these distributions distinctively non-Gaussian, they also appear to vary widely in their basic characteristics, depending on the learning rule used.

Field distribution $P[x, y]$, in addition to (the traditional ones) $Q$ and $R$, and show that this set $\{Q, R, P\}$ obeys deterministic laws. In order to close these laws we use the tools of dynamical replica theory. Details of the replica calculation are given in an Appendix, so that they can be skipped by those primarily interested in results. In section 3 we summarize the final replica-symmetric macroscopic theory and its notational conventions, discuss some of its general properties, and show how in the limit $\alpha \to \infty$ (infinite training sets) the equations of the conventional theory are recovered. In a subsequent paper we will work out and apply our equations explicitly for several types of learning rules, and compare the predictions of our theory with exact results (derived directly from the microscopic equations, for Hebbian learning [12]) and with numerical simulations.

2 From Microscopic to Macroscopic Laws

2.1 Definitions

A student perceptron operates the following rule, which is parametrised by a weight vector $J \in \mathbb{R}^N$:

$$S : \{-1, 1\}^N \to \{-1, 1\} \quad S(\xi) = \text{sgn}[J \cdot \xi]$$

(1)

It tries to emulate the operation of a teacher perceptron, which is assumed to operate a similar rule, characterized by a given (fixed) weight vector $B \in \mathbb{R}^N$:

$$T : \{-1, 1\}^N \to \{-1, 1\} \quad T(\xi) = \text{sgn}[B \cdot \xi]$$

(2)

In order to improve its performance, the student perceptron modifies its weight vector $J$ according to an iterative procedure, using examples of input vectors (or ‘questions’) $\xi$, drawn at random from a fixed training set $\hat{D} \subseteq D = \{-1, 1\}^N$, and the corresponding values of the teacher outputs $T(\xi)$. 

4
We will analyze the following two classes of learning rules:

\[ \hat{D} = \{\xi^1, \ldots, \xi^p\} \quad \quad p = \alpha N \quad \quad \xi^\mu \in D \quad \text{for all} \ \mu \]

We will denote averages over the training set \( \hat{D} \) and averages over the full question set \( D \) in the following way:

\[ \langle \Phi(\xi) \rangle_{\hat{D}} = \frac{1}{|\hat{D}|} \sum_{\xi \in \hat{D}} \Phi(\xi) \quad \text{and} \quad \langle \Phi(\xi) \rangle_D = \frac{1}{|D|} \sum_{\xi \in D} \Phi(\xi) . \]

We will analyze the following two classes of learning rules:

- **on-line**: \( J(m+1) = J(m) + \frac{\eta}{N} \xi(m) \mathcal{G} [J(m) \cdot \xi(m), B \cdot \xi(m)] \)
- **batch**: \( J(m+1) = J(m) + \frac{\eta}{N} \langle \xi \mathcal{G} [J(m) \cdot \xi, B \cdot \xi] \rangle_{\hat{D}} \)  

In on-line learning one draws at each iteration step \( m \) a question \( \xi(m) \in \hat{D} \) at random, the dynamics is thus a stochastic process; in batch learning one iterates a deterministic map. The function \( \mathcal{G}[x, y] \) is assumed to be bounded and not to depend on \( N \), other than via its two arguments.

Our most important observables during learning are the training error \( E_t(J) \) and the generalization error \( E_g(J) \), defined as follows:

\[ E_t(J) = \langle (J \cdot \xi)(B \cdot \xi) \rangle_{\hat{D}} \quad \quad E_g(J) = \langle (J \cdot \xi)(B \cdot \xi) \rangle_D . \]

Only if the training set \( \hat{D} \) is sufficiently large, and if there are no correlations between \( J \) and the questions \( \xi \in \hat{D} \), will these two errors will be identical.

We next convert the dynamical laws (4) into the language of stochastic processes. We introduce the probability \( \hat{p}_m(J) \) to find weight vector \( J \) at discrete iteration step \( m \). In terms of this microscopic probability distribution the processes (4) can be written in the general Markovian form

\[ \hat{p}_{m+1}(J) = \int dJ' W[J; J'] \hat{p}_m(J') , \]

with the transition probabilities

- **on-line**: \( W[J; J'] = \langle \delta [J - J' - \frac{\eta}{N} \xi \mathcal{G} [J' \cdot \xi, B \cdot \xi]] \rangle_{\hat{D}} \)
- **batch**: \( W[J; J'] = \delta [J - J' - \frac{\eta}{N} \langle \xi \mathcal{G} [J' \cdot \xi, B \cdot \xi] \rangle_{\hat{D}}] \)

We make the transition to a description involving real-valued time labels by choosing the duration of each iteration step to be a real-valued random number, such that the probability that at time \( t \) precisely \( m \) steps have been made is given by the Poisson expression

\[ \pi_m(t) = \frac{1}{m!} (Nt)^m e^{-Nt} . \]

For times \( t \gg N^{-1} \) we find \( t = m/N + \mathcal{O}(N^{-\frac{1}{2}}) \), the usual time unit. Due to the random durations of the iteration steps we have to switch to the following microscopic probability distribution:

\[ p_t(J) = \sum_{m \geq 0} \pi_m(t) \hat{p}_m(J) . \]
This distribution obeys a simple differential equation, which immediately follows from the pleasant properties of $\{\Omega\}$ under temporal differentiation:

$$\frac{d}{dt} p_t(J) = N \int dJ' \left\{ W[J; J'] - \delta[J - J'] \right\} p_t(J').$$  \hfill (10)

So far no approximations have been made, equation (10) is exact for any $N$. It is the equivalent of the master equation often introduced to define the dynamics of spin systems.

2.2 Derivation of Macroscopic Fokker-Planck Equation

We now wish to investigate the dynamics of a number of as yet arbitrary macroscopic observables $\Omega[J] = (\Omega_1[J], \ldots, \Omega_k[J])$. To do so we introduce a macroscopic probability distribution

$$P_t(\Omega) = \int dJ \, p_t(J) \delta[\Omega - \Omega[J]]$$ \hfill (11)

Its time derivative immediately follows from that in (10):

$$\frac{d}{dt} P_t(\Omega) = N \int dJ dJ' \delta[\Omega - \Omega[J]] \left\{ W[J; J'] - \delta[J - J'] \right\} p_t(J')$$

$$= N \int d\Omega' \int dJ dJ' \delta[\Omega - \Omega[J]] \delta[\Omega' - \Omega[J']] \left\{ W[J; J'] - \delta[J - J'] \right\} p_t(J')$$

This then can be written in the standard form

$$\frac{d}{dt} P_t(\Omega) = \int d\Omega' \, \mathcal{W}_t[\Omega; \Omega'] P_t(\Omega')$$ \hfill (12)

where

$$\mathcal{W}_t[\Omega; \Omega'] = \int \frac{dJ' \, p_t(J') \delta[\Omega' - \Omega[J']]}{\int \frac{dJ' \, p_t(J') \delta[\Omega' - \Omega[J']]}{\int \frac{dJ \, \delta[\Omega - \Omega[J]] \left\{ W[J; J'] - \delta[J - J'] \right\}}{\int \frac{dJ \, \delta[\Omega - \Omega[J]] \left\{ W[J; J'] - \delta[J - J'] \right\}}$$

If we now insert the relevant expressions (10) for $W[J; J']$ we can perform the $J$-integrations, and obtain results given in terms of so-called sub-shell averages, which are defined as

$$\langle f(J) \rangle_{\Omega,t} = \frac{\int dJ \, p_t(J) \delta[\Omega - \Omega[J]] f(J)}{\int dJ \, p_t(J) \delta[\Omega - \Omega[J]]}$$

For the two classes of learning rules at hand we obtain:

$$\mathcal{W}_t^{\text{som}}[\Omega; \Omega'] = N \left\langle \delta \left[ \Omega - \Omega[J + \frac{\eta}{N} \xi G[J, \xi, B, \xi]] \right] \right\rangle_{\Omega,t}$$

$$\mathcal{W}_t^{\text{hat}}[\Omega; \Omega'] = N \left\langle \delta \left[ \Omega - \Omega[J + \frac{\eta}{N} \xi G[J, \xi, B, \xi]] \right] \right\rangle_{\Omega,t}$$

We now insert integral representations for the $\delta$-distributions. The observables $\Omega[J] \in \mathbb{R}^k$ are assumed to be $O(1)$ each, and finite in number (i.e. $k \ll N$):

$$\delta[\Omega - Q] = \frac{d\Omega}{(2\pi)^k} e^{i\Omega \cdot [\Omega - Q]}$$ \hfill (13)
which gives for our two learning scenario’s:

\[
W_t^{\text{out}}[\Omega; \Omega'] = \int \frac{d\hat{\Omega}}{(2\pi)^{N}} e^{i\hat{\Omega} \cdot \Omega} N \left\langle \left\{ e^{-i\hat{\Omega} \cdot \Omega} J_{\xi} B \right\} D - e^{-i\hat{\Omega} \cdot \Omega} J_{\xi} B \right\rangle \Omega'; t \tag{14}
\]

\[
W_t^{\text{hat}}[\Omega; \Omega'] = \int \frac{d\hat{\Omega}}{(2\pi)^{N}} e^{i\hat{\Omega} \cdot \Omega} N \left\langle \left\{ e^{-i\hat{\Omega} \cdot \Omega} J_{\xi} B \right\} D - e^{-i\hat{\Omega} \cdot \Omega} J_{\xi} B \right\rangle \Omega'; t \tag{15}
\]

Still no approximations have been made. The above two expressions differ only in at which stage the averaging over the training set occurs.

In expanding equations (14,15) for large \(N\) and finite \(t\) we have to be careful, since the system size \(N\) enters both as a small parameter to control the magnitude of the modification of individual components of the weight vector, but also determines the dimensions and lengths of various vectors that occur. We therefore inspect more closely the usual Taylor expansions:

\[
F[J + k] - F[J] = \sum_{\ell \geq 1} \frac{1}{\ell!} \sum_{i_1 = 1}^{N} \cdots \sum_{i_\ell = 1}^{N} k_{i_1} \cdots k_{i_\ell} \frac{\partial^\ell F[J]}{\partial J_{i_1} \cdots \partial J_{i_\ell}}.
\]

If we assess how derivatives with respect to individual components \(J_i\) scale for mean-field observables such as \(Q[J] = J^2\) and \(R[J] = B \cdot J\), we find the following scaling property which we will choose as our definition of simple mean-field observables:

\[
F[J] = O(N^0), \quad \frac{\partial^\ell F[J]}{\partial J_{i_1} \cdots \partial J_{i_\ell}} = O \left( |J|^{-\ell} N^{\frac{d}{2} - \ell} \right) \quad (N \to \infty) \tag{16}
\]

in which \(d\) is the number of different elements in the set \(\{i_1, \ldots, i_\ell\}\). For simple mean-field observables we can now estimate the scaling of the various terms in the Taylor expansion. However, we will find that for restricted training sets not all relevant observables will have the properties (16). In particular, the joint distribution of student and teacher fields will, for on-line learning, have a contribution for which all terms in the Taylor series will have to be summed, giving rise to an additional term \(\Delta[J; k]\).

The latter type of more general mean-field observables will have to be defined via the identities

\[
F[J + k] - F[J] = \Delta[J; k] + \sum_{i} k_i \frac{\partial F[J]}{\partial J_i} + \frac{1}{2} \sum_{ij} k_i k_j \frac{\partial^2 F[J]}{\partial J_i \partial J_j} + \sum_{\ell \geq 3} O \left( \frac{|k|^\ell}{|J|^\ell} \right) \tag{17}
\]

\[
F[J] = O(N^0), \quad \Delta[J; k] = O \left( \frac{|k|^2}{|J|^2} \right) \tag{18}
\]

(in the assessment of the order of the remainder terms of (17) we have used \(\sum_i k_i = O(\sqrt{N} |k|)\)). Simple mean-field observables correspond to \(\Delta[J; k] = 0\).

We expand our macroscopic equations (14,15) for large \(N\) and finite times, restricting ourselves from now on to mean-field observables in the sense of (17,18). One of our observables we choose to be \(J^2\). In the present problem the shifts \(k\), being either \(\frac{1}{\sqrt{N}} \xi G[J \cdot \xi, B \cdot \xi]\) or \(\frac{1}{\sqrt{N}} \xi G[J \cdot \xi, B \cdot \xi]_D\), scale as \(|k| = O(N^{-\frac{d}{2}})\). Consequently:

\[
e^{-i\hat{\Omega} \cdot \Omega} J_{\xi} B = e^{-i\hat{\Omega} \cdot \Omega} J_{\xi} B \left\{ 1 - i\Omega \cdot \Delta[J; k] - i \sum_i k_i \frac{\partial}{\partial J_i} (\hat{\Omega} \cdot \Omega[J]) - \frac{i}{2} \sum_{ij} k_i k_j \frac{\partial^2}{\partial J_i \partial J_j} (\hat{\Omega} \cdot \Omega[J]) \right\}
\]

\footnote{We are grateful to Dr. Yuan-sheng Xiong for alerting us to this important point.}
with transition probability densities of the general form

\[ \frac{1}{2} \left[ \sum_i k_i \frac{\partial}{\partial J_i} (\Omega \cdot \Omega[J]) \right]^2 \] + \mathcal{O}(N^{-\frac{3}{2}}). \]

This, in turn, gives

\[ \int \frac{d\Omega}{(2\pi)^N} e^{i\Omega \cdot \Omega[J]} \mathcal{N} \left[ e^{-i\Omega \cdot \Omega[J] - e^{-i\Omega \cdot \Omega[J]}} \right] \]

\[ = -N \left\{ \sum_{\mu} \frac{\partial}{\partial \Omega_{\mu}} \left[ \Delta_{\mu} [J, k] + \sum_i k_i \frac{\partial \Omega_{\mu} [J]}{\partial J_i} + \frac{1}{2} \sum_{ij} k_i k_j \frac{\partial^2 \Omega_{\mu} [J]}{\partial J_i \partial J_j} \right] \right. \]

\[ - \frac{1}{2} \sum_{\mu \nu} \frac{\partial^2}{\partial \Omega_{\mu} \partial \Omega_{\nu}} \sum_{ij} k_i k_j \frac{\partial \Omega_{\mu} [J]}{\partial J_i} \frac{\partial \Omega_{\nu} [J]}{\partial J_j} \left\} \delta [\Omega - \Omega[J]] + \mathcal{O}(N^{-\frac{3}{2}}) \]

It is now evident, in view of (14,15), that both types of dynamics are described by macroscopic laws with transition probability densities of the general form

\[ \mathcal{W}_t^{**} [\Omega; \Omega'] = \left\{ - \sum_{\mu} F_{\mu} [\Omega'; t] \frac{\partial}{\partial \Omega_{\mu}} + \frac{1}{2} \sum_{\mu \nu} G_{\mu \nu} [\Omega'; t] \frac{\partial^2}{\partial \Omega_{\mu} \partial \Omega_{\nu}} \right\} \delta [\Omega - \Omega'] + \mathcal{O}(N^{-\frac{3}{2}}) \]

which, due to (12) and for \( N \to \infty \) and finite times, leads to a Fokker-Planck equation:

\[ \frac{d}{dt} P_t (\Omega) = - \sum_{\mu=1}^{4} \frac{\partial}{\partial \Omega_{\mu}} \left\{ F_{\mu} [\Omega; t] P_t (\Omega) \right\} + \frac{1}{2} \sum_{\mu \nu=1}^{4} \frac{\partial^2}{\partial \Omega_{\mu} \partial \Omega_{\nu}} \left\{ G_{\mu \nu} [\Omega; t] P_t (\Omega) \right\}. \] (19)

The differences between the two types of dynamics are in the explicit expressions for the flow- and diffusion terms:

\[ F_{\mu}^{\text{sonl}} [\Omega; t] = \lim_{N \to \infty} \left( N \langle \Delta_{\mu} [J, \frac{\eta}{N} \xi \cdot G[J, \xi, B, \xi]] \rangle_D + \eta \sum_i \langle \xi_i G[J, \xi, B, \xi] \rangle_D \frac{\partial \Omega_{\mu} [J]}{\partial J_i} \right. \]

\[ + \frac{\eta^2}{2N} \sum_{ij} \langle \xi_i \xi_j G^2 [J, \xi, B, \xi] \rangle_D \frac{\partial^2 \Omega_{\mu} [J]}{\partial J_i \partial J_j} \left\} \Omega; t \]

\[ G_{\mu \nu}^{\text{sonl}} [\Omega; t] = \lim_{N \to \infty} \frac{\eta^2}{N} \left( \sum_{ij} \langle \xi_i \xi_j G^2 [J, \xi, B, \xi] \rangle_D \left[ \frac{\partial \Omega_{\mu} [J]}{\partial J_i} \right] \left[ \frac{\partial \Omega_{\nu} [J]}{\partial J_j} \right] \right) \Omega; t \]

\[ F_{\mu}^{\text{bat}} [\Omega; t] = \lim_{N \to \infty} \left( N \Delta_{\mu} [J, \frac{\eta}{N} \xi \cdot G[J, \xi, B, \xi]] \rangle_D + \eta \sum_i \langle \xi_i G[J, \xi, B, \xi] \rangle_D \frac{\partial \Omega_{\mu} [J]}{\partial J_i} \right. \]

\[ + \frac{\eta^2}{2N} \sum_{ij} \langle \xi_i \xi_j G^2 [J, \xi, B, \xi] \rangle_D \langle \xi_j G[J, \xi, B, \xi] \rangle_D \frac{\partial^2 \Omega_{\mu} [J]}{\partial J_i \partial J_j} \left\} \Omega; t \]

\[ G_{\mu \nu}^{\text{bat}} [\Omega; t] = \lim_{N \to \infty} \frac{\eta^2}{N} \left( \sum_{ij} \langle \xi_i G[J, \xi, B, \xi] \rangle_D \langle \xi_j G[J, \xi, B, \xi] \rangle_D \left[ \frac{\partial \Omega_{\mu} [J]}{\partial J_i} \right] \left[ \frac{\partial \Omega_{\nu} [J]}{\partial J_j} \right] \right) \Omega; t \]

Equation (19) allows us to define the goal of our exercise in more explicit form. If we wish to arrive at closed deterministic macroscopic equations, we have to choose our observables such that
1. \( \lim_{N \to \infty} G_{\mu \nu}[\Omega; t] = 0 \) (this ensures determinism)
2. \( \lim_{N \to \infty} \frac{\partial}{\partial t} F_{\mu}[\Omega; t] = 0 \) (this ensures closure)

In the case of having time-dependent global parameters, such as learning rates or decay rates, the latter condition relaxes to the requirement that any explicit time-dependence of \( F_{\mu}[\Omega; t] \) is restricted to these global parameters.

2.3 Choice and Properties of Canonical Observables

We next apply the general results obtained so far to a specific set of observables, \( \Omega \to \{ Q, R, P \} \), which are tailored to the problem at hand (note that we restrict ourselves to \( J^2 = O(1) \) and \( B^2 = 1 \)):

\[
Q[J] = J^2, \quad R[J] = J \cdot B, \quad P[x, y; J] = \langle \delta[x - J \cdot \xi] \delta[y - B \cdot \xi] \rangle_D
\]

with \( x, y \in \mathbb{R} \). This choice is motivated by the following considerations: (i) in order to incorporate the standard theory in the limit \( \alpha \to \infty \) we need at least \( Q[J] \) and \( R[J] \), (ii) we need to be able to calculate the training error, which involves field statistics calculated over the training set \( \tilde{D} \), as described by \( P[x, y; J] \), and (iii) for finite \( \alpha \) one cannot expect closed macroscopic equations for just a finite number of order parameters, the present choice (involving the order parameter function \( P[x, y; J] \)) represents effectively an infinite number \( \bar{\Omega} \).

In subsequent calculations we will, however, assume the number of arguments \( (x, y) \) for which \( P[x, y; J] \) is to be evaluated (and thus our number of order parameters) to go to infinity only after the limit \( N \to \infty \) has been taken. This will eliminate many technical subtleties and will allow us to use the Fokker-Planck equation (14).

The observables (20) are indeed of the general mean-field type in the sense of (17-18). Insertion into the stronger condition (16) immediately shows this to be true for the scalar observables \( Q[J] \) and \( R[J] \) (they are simple mean field observables, for which the term (18) is absent). Verification of (17,18) for the function \( P[x, y; J] \) is less trivial. We denote with \( \mathcal{I} \) the set of all different indices in the list \( (i_1, \ldots, i_\ell) \), with \( n_k \) giving the number of times a number \( k \) occurs, and with \( \mathcal{I}^+ \subseteq \mathcal{I} \) defined as the set of all indices \( k \in \mathcal{I} \) for which \( n_k \) is even (+), or odd (−). Note that with these definitions \( \ell = \sum_{k \in \mathcal{I}^+} n_k + \sum_{k \in \mathcal{I}^-} n_k \geq 2|\mathcal{I}^+| + |\mathcal{I}^-| \). We then have:

\[
\frac{\partial^\ell P[x, y; J]}{\partial J_{i_1} \ldots \partial J_{i_\ell}} = (-1)^{\ell} \frac{\partial^\ell}{\partial x^\ell} \int \frac{d\hat{x} \, d\hat{y}}{(2\pi)^2} e^{i(x\hat{x} + y\hat{y})} \left( \prod_{k \in \mathcal{I}} c_{n_k}^{n_k} e^{-i\xi_k[x_j \hat{J}_k + y_B \hat{B}_k]} \prod_{k \not\in \mathcal{I}} c_{n_k}^{n_k} e^{-i\xi_k[x_j \hat{J}_k + y_B \hat{B}_k]} \right)_{\tilde{D}}
\]

Upon writing averaging over all training sets of size \( p = \alpha N \) (where each realization of \( \tilde{D} \) has equal probability) as \( \langle \ldots \rangle_{\text{sets}} \), this allows us to conclude

\[
\left\langle \frac{\partial^\ell P[x, y; J]}{\partial J_{i_1} \ldots \partial J_{i_\ell}} \right\rangle_{\text{sets}} = O\left(N^{-\frac{3}{2}|\mathcal{I}^-|}\right)
\]

Since \( \frac{3}{2} - |\mathcal{I}^-| - \frac{1}{2} |\mathcal{I}^+| \geq 0 \), the average over all training sets of the function \( P[x, y; J] \) is found to be a simple mean-field observable in the sense of (14).

The scaling properties of expansions or derivations of \( P[x, y; J] \) for a given training set \( \tilde{D} \), however, need not be identical to those of its average over all training sets \( \langle P[x, y; J] \rangle_{\text{sets}} \). Here we have to use

\[\text{A simple rule of thumb is the following: if a process requires replica theory for its stationary state analysis, as does learning with restricted training sets, its dynamics is of a spin-glass type and cannot be described by a finite set of closed dynamic equations.}\]
the fact that $\hat{D}$ has been composed in a random manner, as well as the specific form of the shifts $k$ in $P[x, y; J + k]$ that occur for the two types of dynamics under consideration:

$$P[x, y; J + k] - P[x, y; J] = \int \frac{dx}{(2\pi)^2} e^{-i\hat{x}^2/2} \sum_{\mu=1}^{p} e^{-i\hat{x}J\xi^\mu - i\eta_B \xi^\mu} \left[ e^{-i\hat{x}B\xi^\mu} - 1 \right]$$

All complications are caused by the dependence of $k$ on the composition of the training set $\hat{D}$, and would therefore have been absent in the $\alpha \to \infty$ case. This dependence will turn out to be harmless in the case of batch learning, where $k = \frac{1}{N} \langle \xi G[J \cdot \xi, B \cdot \xi] \rangle_D$ is an average over $\hat{D}$, but will have a considerable impact in the case of on-line learning, where $k = \frac{1}{N} \xi G[J \cdot \xi, B \cdot \xi]$ is proportional to an individual member of $\hat{D}$. Working out the relevant expression for on-line learning gives

$$P[x, y; J + k^{onl}] - P[x, y; J] = \int \frac{dx}{(2\pi)^2} e^{-i\hat{x}^2/2} \sum_{\mu=1}^{p} e^{-i\hat{x}J\xi^\mu - i\eta_B \xi^\mu} \left[ e^{-i\eta_B \xi^\mu} - 1 \right]$$

$$= \frac{1}{p} \int \frac{dx}{(2\pi)^2} e^{-i\hat{x}^2/2} \sum_{\mu=1}^{p} e^{-i\hat{x}J\xi^\mu - i\eta_B \xi^\mu} \left[ e^{-i\eta_B \xi^\mu} - 1 \right] + \frac{1}{2} \sum_{ij} k_{ij}^{onl} \frac{\partial^2}{\partial J_i \partial J_j} P[x, y; J] + O(N^{-\frac{3}{2}})$$

We conclude that, at least for the purpose of the expansions relevant to on-line learning, $P[x, y; J]$ is a mean field observable in the sense of (18), with the non-trivial contribution of (18) given by

$$\Delta[J; k^{onl}] = \frac{1}{p} \left[ \sum_{\mu=1}^{p} e^{-i\hat{x}J\xi^\mu - i\eta_B \xi^\mu} \left[ e^{-i\eta_B \xi^\mu} - 1 \right] + \frac{1}{2} \sum_{ij} k_{ij}^{onl} \frac{\partial^2}{\partial J_i \partial J_j} P[x, y; J] + O(N^{-\frac{3}{2}}) \right]$$

Note that $\lim_{N \to \infty} N \Delta[J; k^{onl}] = O(\eta^3/\alpha)$, so that for small learning rates or large training sets this non-trivial term will vanish. Working out the relevant expression for batch learning, on the other hand, gives

$$P[x, y; J + k^{bat}] - P[x, y; J] = \int \frac{dx}{(2\pi)^2} e^{-i\hat{x}^2/2} \sum_{\mu=1}^{p} e^{-i\hat{x}J\xi^\mu - i\eta_B \xi^\mu} \left[ e^{-i\hat{x}B\xi^\mu} - 1 \right]$$

$$= \sum_i k_i^{bat} \frac{\partial}{\partial J_i} P[x, y; J] + \frac{1}{2} \sum_{ij} k_i^{bat} k_j^{bat} \frac{\partial^2}{\partial J_i \partial J_j} P[x, y; J] + O(N^{-\frac{3}{2}})$$

Here the term $\Delta[J; k^{bat}]$ is absent. In fact also the quadratic contribution $\sum_{ij} k_i^{bat} k_j^{bat} \ldots$ in the above expansion will turn out to be of insignificant order in $N$. For the purpose of the expansions relevant to batch learning, $P[x, y; J]$ is apparently a simple mean field observable in the sense of (10). This could have been anticipated, since one should ultimately obtain the batch learning equations upon expanding those of on-line learning for small learning rate $\eta$, and retaining only the leading order $\eta^1$ in this expansion.
2.4 Derivation of Deterministic Dynamical Laws

Having defined our order parameters \( Q, R \) and \( \{P[x, y]\} \), from this stage onwards the notation \( \langle \cdots \rangle_{Q,R,t} \) will be used to denote sub-shell averages defined with respect to these order parameters, at time \( t \).

With a modest amount of foresight we define the complementary Kronecker delta \( \delta_{ab} = 1 - \delta_{ab} \), and the following key functions:

\[
A[x, y; x', y'] = \lim_{N \to \infty} \left\langle \langle \vec{\delta}_{\xi' \xi} \delta[x - J \cdot \xi] \delta[y - B \cdot \xi] \delta[x' - J \cdot \xi'] \delta[y' - B \cdot \xi'] \rangle_{\tilde{D}} \right\rangle_{Q,R,t} \tag{22}
\]

\[
B[x, y; x', y'] = \lim_{N \to \infty} \left\langle \frac{1}{N} \sum_{i \neq j} \langle \langle \vec{\delta}_{\xi' \xi} \delta[x - J \cdot \xi] \delta[y - B \cdot \xi] \delta[x' - J \cdot \xi'] \delta[y' - B \cdot \xi'] \rangle_{\tilde{D}} \rangle_{\tilde{D}} \right\rangle_{Q,R,t} \tag{23}
\]

\[
C[x, y; x', y'; x'', y''] = \lim_{N \to \infty} \left\langle \langle \langle \vec{\delta}_{\xi''} \delta_{\xi' \xi''} \rangle_{\tilde{D}} \rangle_{\tilde{D}} \rangle_{\tilde{D}} \right\rangle_{Q,R,t} \tag{24}
\]

We will eventually show in a subsequent section that (23) and (24) are zero. The function (22), on the other hand, will contain all the interesting physics of the learning process, and its calculation will turn out to be our central problem.

We next show that for the observables (20) the diffusion matrix elements \( G^{***} \) in the Fokker-Planck equation (19) vanish for \( N \to \infty \). Our observables will consequently obey deterministic dynamical laws. Calculating diffusion terms associated with \( Q[J] \) and \( R[J] \) is trivial:

\[
\begin{bmatrix}
G_{QQ}^{onl}[\cdots] \\
G_{QR}^{onl}[\cdots] \\
G_{RR}^{onl}[\cdots]
\end{bmatrix} = \lim_{N \to \infty} \frac{\eta^2}{N} \int dxdy \ P[x, y] \ G^2[x, y] 
\begin{bmatrix}
4x^2 \\
2xy \\
y^2
\end{bmatrix} = 0
\]

\[
\begin{bmatrix}
G_{QQ}^{bat}[\cdots] \\
G_{QR}^{bat}[\cdots] \\
G_{RR}^{bat}[\cdots]
\end{bmatrix} = \lim_{N \to \infty} \frac{\eta^2}{N} 
\begin{bmatrix}
4 \left\{ \int dxdy \ P[x, y] \ xG[x, y] \right\}^2 \\
2 \left\{ \int dxdy \ P[x; y] \ xG[x, y] \right\} \left\{ \int dxdy \ P[x; y] \ yG[x, y] \right\} \\
\left\{ \int dxdy \ P[x; y] \ yG[x, y] \right\}^2
\end{bmatrix} = 0
\]

We next turn to diffusion terms with one occurrence of \( P[x, y; J] \). Here we repeatedly build on the cornerstone assumption that all fields \( J \cdot \xi \) and \( B \cdot \xi \) are of order unity (which is clear from numerical simulations, and will be supported self-consistently by the equations resulting from our theory), in combination with two simple scaling consequences of the random composition of \( \tilde{D} \), as \( N \to \infty \):

\[
\xi \in \tilde{D}: \quad \frac{1}{p} \sum_{\xi' \in \tilde{D}} \delta_{\xi' \xi} = p^{-1} + \mathcal{O}(p^{-2}) \quad \frac{1}{p^2} \sum_{\xi \in \tilde{D}} \sum_{\xi' \in \tilde{D}} [1 - \delta_{\xi' \xi}] |\xi \cdot \xi'| = \mathcal{O}(N^{1/2}) \tag{25}
\]

For on-line learning we find:

\[
\begin{bmatrix}
G_{Q,P[x,y]}^{onl}[\cdots] \\
G_{R,P[x,y]}^{onl}[\cdots]
\end{bmatrix} = - \lim_{N \to \infty} \frac{\eta^2}{N} \frac{\partial}{\partial x} \left\langle \langle \ G^{2}[J \cdot \xi; B \cdot \xi] \left[ \begin{array}{c}
2J \cdot \xi \\
B \cdot \xi
\end{array} \right] (\xi \cdot \xi') \delta[x - J \cdot \xi'] \delta[y - B \cdot \xi'] \rangle_{\tilde{D}} \right\rangle_{Q,R,t}
\]
\[-\eta^2 \frac{\partial}{\partial x} \lim_{N \to \infty} \left( \frac{1}{N} \langle [1 - \delta \xi''] \rangle G^2[\xi''] \right) \right]_{\xi''; t} + G^2[x, y] \left[ \frac{2x}{y} \right] \left( \langle \delta \xi'' \delta y - B \cdot \xi' \rangle \right)_{\xi''; t} = 0 \]

For batch learning we find:

\[ G_{Q, P[x, y], t}^{\text{bat}} = -\eta^2 \frac{\partial}{\partial x} \lim_{N \to \infty} \left( \frac{1}{N} \langle G[x, y] \rangle \right)_{\xi''; t} \]

\[ = -\eta^2 \frac{\partial}{\partial x} \lim_{N \to \infty} \left( \frac{1}{N} \langle G[x, y] \rangle \right)_{\xi''; t} \]

For on-line learning we find:

\[ G_{P[x, y], t}^{\text{onl}} = \lim_{N \to \infty} \eta^2 \frac{\partial^2}{\partial x \partial x'} \left( \frac{1}{N} \langle G^2[x, y] \rangle \right)_{\xi''; t} \]

The difficult terms are those where two derivatives of the order parameter function \( P[x, y, J] \) come into play. Here we have to deal separately with four distinct contributions, defined according to which of the \( \{ \xi, \xi', \xi'' \} \) are identical. For on-line learning we find:

\[ G_{P[x, y], P[x', y'], t}^{\text{onl}} = \lim_{N \to \infty} \eta^2 \frac{\partial^2}{\partial x \partial x'} \left( \frac{1}{N} \langle G^2[x, y] \rangle \right)_{\xi''; t} \]

\[ = \eta^2 \frac{\partial^2}{\partial x \partial x'} \left( \frac{1}{N} \langle G^2[x, y] \rangle \right)_{\xi''; t} \]

\[ = \eta^2 \frac{\partial^2}{\partial x \partial x'} \left( \frac{1}{N} \langle G^2[x, y] \rangle \right)_{\xi''; t} \]

\[ = \eta^2 \frac{\partial^2}{\partial x \partial x'} \left( \frac{1}{N} \langle G^2[x, y] \rangle \right)_{\xi''; t} \]

\[ = \eta^2 \frac{\partial^2}{\partial x \partial x'} \left( \frac{1}{N} \langle G^2[x, y] \rangle \right)_{\xi''; t} \]
Similarly:

\[
G^\text{bat}_{P[x,y], P[x',y']}[\cdots] = \lim_{N \to \infty} \frac{\eta^2}{N} \frac{\partial^2}{\partial x \partial x'}
\]

\[
\left\langle \langle [G(J \cdot \xi', B \cdot \xi')] (\xi \cdot \xi') \delta[x - J \cdot \xi] \delta[y - B \cdot \xi] \rangle \right\rangle_D
\]

\[
\times \left\langle \langle [G(J \cdot \xi, B \cdot \xi)] (\xi \cdot \xi) \delta[x' - J \cdot \xi'] \delta[y' - B \cdot \xi'] \rangle \right\rangle_D
\]

\[
= \eta^2 \frac{\partial^2}{\partial x \partial x'} \lim_{N \to \infty} \left\langle \left\{ \mathcal{O}(N^{-1}) + \mathcal{O}(N^{-\frac{1}{2}}) \right\} \left\{ \mathcal{O}(N^{-1}) + \mathcal{O}(N^{-\frac{1}{2}}) \right\} \right\}_{\text{app} ; t} = 0
\]

For batch learning all diffusion matrix elements of (13) vanish in a straightforward manner. For on-line learning all diffusion terms vanish provided we can prove that the function \( C[\ldots] \) of (23) is zero. This is indeed the case within the present theory, as will be verified in the Appendix. The Fokker-Planck equation (13) now reduces to the Liouville equation \( \frac{d}{dt} P(t) = -\sum \frac{\partial}{\partial x} \left\{ F_\mu [\Omega; t] P(t) \right\} \), describing deterministic evolution for our macroscopic observables: \( \frac{d}{dt} \Omega = F[\Omega; t] \). These deterministic equations we will now work out explicitly.

**On-Line Learning**

First we deal with the scalar observables \( Q \) and \( R \):

\[
\frac{d}{dt} Q = \lim_{N \to \infty} \left\{ 2\eta \left\langle \langle [J \cdot \xi] G(J \cdot \xi) \rangle \right\rangle_D + \eta^2 \left\langle \langle [G(J \cdot \xi, B \cdot \xi)] \rangle \right\rangle_D \right\}_{\text{app} ; t}
\]

\[
= 2\eta \int dx dy \ P[x, y] \ G[x, y] + \eta^2 \int dx dy \ P[x, y] \ G^2[x, y]
\]

\[
\frac{d}{dt} R = \lim_{N \to \infty} \eta \left\langle \langle [B \cdot \xi] G(J \cdot \xi, B \cdot \xi) \rangle \right\rangle_D = \eta \int dx dy \ P[x, y] \ G[x, y]
\]

These equations are identical to those found in the \( \alpha \to \infty \) formalism. The difference is in the function to be substituted for \( P[x, y] \), which here is the solution of

\[
\frac{\partial}{\partial t} P[x, y] = \lim_{N \to \infty} \left\{ -\eta \frac{\partial}{\partial x} \left\langle \langle [G(J \cdot \xi', B \cdot \xi')] (\xi \cdot \xi') \delta[x - J \cdot \xi] \delta[y - B \cdot \xi] \rangle \right\rangle_D \right\}_{\text{app} ; t}
\]

\[
+ \eta^2 \frac{\partial^2}{\partial x \partial x'} \left\langle \langle [G^2(J \cdot \xi', B \cdot \xi')] (\xi \cdot \xi')^2 \delta[x - J \cdot \xi] \delta[y - B \cdot \xi] \rangle \right\rangle_D \right\}_{\text{app} ; t}
\]

\[
+ \frac{1}{\alpha} \left\langle \delta[x - J \cdot \xi] \eta \delta[y - B \cdot \xi] - \delta[x - J \cdot \xi] \eta \delta[y - B \cdot \xi] \right\rangle_D
\]

\[
+ \eta \frac{\partial}{\partial x} \left[ G[x, y] \langle \delta[x - J \cdot \xi] \delta[y - B \cdot \xi] \rangle \right] - \frac{1}{2} \eta^2 \frac{\partial^2}{\partial x \partial x'} \left[G^2[x, y] \langle \delta[x - J \cdot \xi] \delta[y - B \cdot \xi] \rangle \right] \right\}_{\text{app} ; t}
\]

\[

\text{13}
\]
Finally we calculate the temporal derivative of the joint field distribution:

\[
\frac{d}{dt} Q = 2\eta \int dxdy \ P[x, y] \ x \ G[x, y] + \eta^2 \int dxdy \ P[x, y] \ G^2[x, y]
\]

(26)

\[
\frac{d}{dt} R = \eta \int dxdy \ P[x, y] \ y \ G[x, y]
\]

(27)

\[
\frac{d}{dt} P[x, y] = \frac{1}{\alpha} \left\{ \int dxdy \ P[x', y'] \delta[x - x' - \eta \xi]\right\} - \eta \partial_{x} \left\{ \int dxdy' \ A[x, y; x', y'] \xi\left[d\xi[x', y'] - P[x, y]\right] + \frac{1}{2}\eta^2 \int dxdy' \ P[x', y'] G^2[x', y'] \frac{\partial^2}{\partial x^2} P[x, y]
\]

(28)

**Batch Learning**

For \( Q \) and \( R \) one again finds simple equations:

\[
\frac{d}{dt} Q = \lim_{N \to \infty} \left\{ 2\eta \left\langle \langle (J \cdot \xi) G[J \cdot \xi, B \cdot \xi] \rangle \right\rangle_{G^2,t} + \frac{\eta^2}{N} \left\langle \sum_{i} \langle \xi_i G[J \cdot \xi, B \cdot \xi] \rangle^2 \right\rangle_{G^2,t} \right\}
\]

\[
= 2\eta \int dxdy \ P[x, y] \ x \ G[x, y]
\]

\[
\frac{d}{dt} R = \lim_{N \to \infty} \eta \left\langle \langle B \cdot \xi) G[J \cdot \xi, B \cdot \xi] \rangle \right\rangle_{G^2,t} = \eta \int dxdy \ P[x, y] \ y \ G[x, y]
\]

Finally we calculate the temporal derivative of the joint field distribution:

\[
\frac{\partial}{\partial t} P[x, y] = \lim_{N \to \infty} \left\{ -\frac{\eta}{\alpha} \left\langle \langle G[J \cdot \xi', B \cdot \xi'] \xi \rangle \right\rangle_{G^2,t} \right\} + \frac{\eta^2}{2N} \frac{\partial^2}{\partial x^2} \left\langle \langle G[J \cdot \xi', B \cdot \xi'] G[J \cdot \xi'', B \cdot \xi''] \xi \rangle \right\rangle_{G^2,t}
\]

\[
= -\frac{\eta}{\alpha} \frac{\partial}{\partial x} [G[x, y] P[x, y]] - \frac{\partial}{\partial x} \left\langle \int dxdy' \ A[x, y; x', y'] G[x', y'] \right\rangle + \frac{1}{2}\eta^2 \frac{\partial^2}{\partial x^2} \int dxdy'dxdy'' \xi G[x', y'] G[x'', y'']
\]
Anticipating the term $C[...]$ to be zero (to be demonstrated in the Appendix) we thus arrive at the following coupled deterministic macroscopic equations:

\begin{align}
\frac{d}{dt} Q &= 2\eta \int dx dy \ P[x, y] \ x \ G[x; y] \\
\frac{d}{dt} R &= \eta \int dx dy \ P[x, y] \ y \ G[x; y] \\
\frac{d}{dt} P[x, y] &= -\eta \frac{\partial}{\partial x} [G[x, y] P[x, y]] - \eta \frac{\partial}{\partial x} \int dx' dy' \ A[x, y; x', y'] \ G[x', y']
\end{align}

The difference between the macroscopic equations for batch and on-line learning is merely the presence (on-line) or absence (batch) of those terms which are not linear in the learning rate $\eta$ (i.e. of order $\eta^2$ or higher).

### 2.5 Closure of Macroscopic Dynamical Laws

The complexity of the problem is fully concentrated in the Green’s function $A[x, y; x', y']$ defined in (22). Our macroscopic laws are exact for $N \to \infty$ but not yet closed due to the appearance of the microscopic probability density $p_t(J)$ in the sub-shell average of (22). We now close our macroscopic laws by making, for $N \to \infty$, the two key assumptions underlying dynamical replica theories:

1. Our macroscopic observables $\{Q, R, P\}$ obey closed dynamic equations.

2. These macroscopic equations are self-averaging with respect to the disorder, i.e. the microscopic realisation of the training set $\tilde{D}$.

Assumption 1 implies that all microscopic probability variations within the $\{Q, R, P\}$ sub-shells of the $J$-ensemble are either absent or irrelevant to the evolution of $\{Q, R, P\}$. We may consequently make the simplest self-consistent choice for $p_t(J)$ in evaluating the macroscopic laws, i.e. in (22): microscopic probability equipartitioning in the $\{Q, R, P\}$-subshells of the ensemble, or

$$p_t(J) \to w(J) \sim \delta[Q - Q(J)] \delta[R - R(J)] \prod_{xy} \delta[P[x, y] - P[x, y, J]]$$

This new microscopic distribution $w(J)$ depends on time via the order parameters $\{Q, R, P\}$. Note that (22) leads to exact macroscopic laws if our observables $\{Q, R, P\}$ for $N \to \infty$ indeed obey closed equations, and is true in equilibrium for detailed balance models in which the Hamiltonian can be written in terms of $\{Q, R, P\}$. It is an approximation if our observables do not obey closed equations. Assumption 2 allows us to average the macroscopic laws over the disorder; for mean-field models it is usually convincingly supported by numerical simulations, and can be proven using the path integral formalism (see e.g. [10]). We write averages over all training sets $\tilde{D} \subseteq \{-1, 1\}^N$, with $|\tilde{D}| = p$, as $\langle \ldots \rangle_\Xi$. Our assumptions result in the closure of the two sets (22), (27) and (28), since now the function $A[x, y; x', y']$ is expressed fully in terms of $\{Q, R, P\}$:

$$A[x, y; x', y'] = \lim_{N \to \infty} \frac{\int dJ \ w(J) \ \langle \delta[x - J \cdot \xi] \ \delta[y - B \cdot \xi] \ (\xi \cdot \xi') \ \delta[x' - J \cdot \xi'] \ \delta[y' - B \cdot \xi'] \rangle_{\tilde{D}}}{\int dJ \ w(J)}$$
The final ingredient of dynamical replica theory is the realization that averages of fractions can be calculated with the replica identity

\[
\left\langle \frac{\int dJ \ W[J, z] G[J, z]}{\int dJ \ W[J, z]} \right\rangle_z = \lim_{n \to 0} \int dJ^1 \ldots dJ^n \langle G[J^1, z] \prod_{\alpha=1}^n W[J^\alpha, z] \rangle_z
\]

Since each weight component scales as \( J_i^\alpha = O(N^{-\frac{1}{2}}) \) we transform variables in such a way that our calculations will involve \( O(1) \) objects:

\[
(\forall i)(\forall \alpha) : \quad J_i^\alpha = (Q/N)^{\frac{1}{2}} \sigma_i^\alpha, \quad B_i = N^{-\frac{1}{2}} \tau_i
\]

This ensures \( \sigma_i^\alpha = O(1), \tau_i = O(1) \), and reduces various constraints to ordinary spherical ones: \( (\sigma^\alpha)^2 = \tau^2 = N \) for all \( \alpha \). Overall prefactors generated by these transformations always vanish due to \( n \to 0 \). We find a new effective measure:

\[
\prod_{\alpha=1}^n w(J^\alpha) \ dJ^\alpha \to \prod_{\alpha=1}^n \tilde{w}(\sigma^\alpha) \ d\sigma^\alpha,
\]

with

\[
\tilde{w}(\sigma) \sim \delta \left[ N - \sigma^2 \right] \delta \left[ N R Q^{-\frac{1}{2}} - \mathbf{\tau} \cdot \mathbf{\sigma} \right] \prod_{xy} \delta \left[ P[x, y] - P[x, y; (Q/N)^{\frac{1}{2}} \sigma] \right]
\]

We thus arrive at

\[
A[x, y; x', y'] = \lim_{N \to \infty} \int \prod_{\alpha=1}^n \tilde{w}(\sigma^\alpha) d\sigma^\alpha \left\langle \delta \left[ x - \frac{\sqrt{Q} \sigma^1 \cdot \xi}{\sqrt{N}} \right] \delta \left[ y - \frac{\mathbf{\tau} \cdot \xi}{\sqrt{N}} \right] \right\rangle \xi
\]

In the same fashion one can also express \( P[x, y] \) in replica form (which will prove useful for normalization purposes and for self-consistency tests):

\[
P[x, y] = \lim_{N \to \infty} \int \prod_{\alpha=1}^n \tilde{w}(\sigma^\alpha) d\sigma^\alpha \left\langle \delta \left[ x - \frac{\sqrt{Q} \sigma^1 \cdot \xi}{\sqrt{N}} \right] \delta \left[ y - \frac{\tau \cdot \xi}{\sqrt{N}} \right] \right\rangle \xi
\]

Finally we will have to demonstrate that the two functions \( B[\ldots] \) and \( C[\ldots] \), as defined in (33), do indeed vanish self-consistently, as claimed. To achieve this we again express them in replica form:

\[
B[x, y; x', y'] = \lim_{N \to \infty} \int \prod_{\alpha=1}^n \tilde{w}(\sigma^\alpha) d\sigma^\alpha \left\langle \delta \left[ x - \frac{1}{N} \sum_{i \neq j} \xi_i \xi_j \xi_i' \xi_j' \right] \delta \left[ y - \frac{\sqrt{Q} \sigma^1 \cdot \xi}{\sqrt{N}} \right] \delta \left[ y' - \frac{\tau \cdot \xi'}{\sqrt{N}} \right] \right\rangle \xi
\]

and

\[
C[x, y; x', y'; x'', y''] = \lim_{N \to \infty} \int \prod_{\alpha=1}^n \tilde{w}(\sigma^\alpha) d\sigma^\alpha \left\langle \delta \left[ x - \frac{1}{N} \sum_{i \neq j} \xi_i \xi_j \xi_i' \xi_j' \right] \delta \left[ y - \frac{\sqrt{Q} \sigma^1 \cdot \xi'' \cdot (\xi' \cdot \xi'')}{N} \right] \delta \left[ y - \frac{\tau \cdot \xi''}{\sqrt{N}} \right] \right\rangle \xi
\]
At this stage the physics is over, what remains is to perform the summations and integrations in (34, 35, 36, 37) in the limit $N \to \infty$. Full details of this exercise are given in Appendix A, where we show that (36) and (37) are indeed zero, and where we derive, in replica symmetric ansatz, an expression for the Green’s function (34). It turns out that to calculate this Green’s function $A[\ldots]$ one has to solve two coupled saddle-point equations at each time-step, one scalar equation relating to a spin-glass order parameter $q$, and one functional saddle-point equation relating to an effective single-spin measure.

3 Summary of the Theory and Connection with $\alpha \to \infty$ Formalism

In this section we summarize the results obtained so far (including the replica calculation in Appendix A), and we show that our general theory has the satisfactory property that it incorporates the standard formalism developed for infinite training sets (with Gaussian joint field distributions $P[x, y]$ at any time) as a special case, recovered in the limit $\alpha \to \infty$. In addition we provide a proof of the uniqueness of the RS functional saddle-point equation and show that it can be found as the fixed-point of an iterative map.

3.1 Summary of the Theory

Our theory can be summarized in the following compact way:

Dynamic Equations for Observables

Our observables are $Q = J^2$, $R = J \cdot B$, and the joint distribution of student and teacher fields $P[x, y] = \langle \delta[x-J \cdot \xi] \delta[y-B \cdot \xi]\rangle_D$. For $N \to \infty$ these quantities obey closed, deterministic, and self-averaging macroscopic dynamic equations. One always has $P[x, y] = P[x|y]P[y]$ with $P[y] = (2\pi)^{-\frac{1}{2}} \epsilon^{-\frac{1}{2}} y^2$. We define $\langle f[x,y] \rangle = \int dx dy \ P[x|y] f[x, y]$, with the familiar short-hand $Dy = (2\pi)^{-\frac{1}{2}} \epsilon^{-\frac{1}{2}} y^2 dy$, and the following four averages (the function $\Phi[x, y]$ will be given below):

$$
U = \langle \Phi[x, y] G[x, y] \rangle \quad V = \langle x G[x, y] \rangle \quad W = \langle y G[x, y] \rangle \quad Z = \langle G^2[x, y] \rangle
$$

For on-line learning our macroscopic laws are

$$
\frac{d}{dt} Q = 2 \eta V + \eta^2 Z \quad \frac{d}{dt} R = \eta W
$$

$$
\frac{d}{dt} P[x|y] = \frac{1}{\alpha} \int dx' P[x'|y] \left[ \delta[x-x'-\eta G'] - \delta[x-x'] \right] - \eta \frac{\partial}{\partial x} \left\{ P[x|y] \left[ U(x-Ry)+Wy \right]\right\}
$$

$$
+ \frac{1}{2} \eta^2 Z \frac{\partial^2}{\partial x^2} P[x|y] - \eta \left[ V RW -(Q-R^2) U \right] \frac{\partial}{\partial x} \left\{ P[x|y] \Phi[x, y] \right\}
$$

For batch learning one has:

$$
\frac{d}{dt} Q = 2 \eta V \quad \frac{d}{dt} R = \eta W
$$

$$
\frac{d}{dt} P[x|y] = - \frac{\eta}{\alpha} \frac{\partial}{\partial x} [P[x|y] G[x, y]] - \eta \frac{\partial}{\partial x} \left\{ P[x|y] \left[ U(x-Ry)+Wy \right]\right\}
$$
Note that the batch equations follow from the on-line ones by retaining only terms which are linear in the learning rate. From the solution of the above equations follow, in turn, the training- and generalization errors:

\[ E_t = \langle \theta[-xy] \rangle \quad E_g = \frac{1}{\pi} \arccos \left( \frac{R}{\sqrt{Q}} \right) \]  

3.2 Uniqueness and Iterative Calculation of the Functional Saddle-Point

The uniqueness proof is more easily set up in terms of the original order parameter function \( \chi[x, y] \), rather than the new (normalised) measure \( M[x|y] \) (see the Appendix). For a given state \( \{Q, R, P\} \)
and a given value for \( q \in [R^2/Q, 1] \) we have to find the functional saddle-points of the functional \( \Psi[\chi] \), defined as:

\[
\Psi[\chi] = \alpha \int DyDz \log \int dx \ e^{-\frac{x^2}{2\sigma^2}} - x[Ay+Bz] + \alpha^{-1} \chi[x,y] - \int DydxP[x,y] \chi[x,y] \tag{49}
\]

Our proof will carry the existence of the various integrals as an implicit condition for validity. To reduce notational ballast we define

\[
w(x, y, z) = \frac{e^{\frac{x^2}{2\sigma^2}} + x[Ay+Bz] + \alpha^{-1} \chi[x,y]}{\int dx' \ e^{-\frac{x'^2}{2\sigma^2}} + x'[Ay+Bz] + \alpha^{-1} \chi[x',y]}, \quad \langle f[x, y, z] \rangle_* = \int dx \ w(x, y, z)f[x, y, z]
\]

Note: \( w(x, y, z) = M[x,y]e^{Bxz} / \int dx' \ M[x'|y]e^{Bxz} \). The function \( w(u, v, z) \) obeys

\[
\frac{\delta w(u, v, z)}{\delta \chi[u', v']} = \alpha^{-1} \delta[u-u'] [\delta[u-u']w(u, v, z) - w(u, v, z)w(u', v, z)]
\]

The functional saddle-point equation is obtained by requiring the first functional derivative of \( \Psi[\chi] \) with respect to \( \chi[u, v] \) to be zero for all \( u, v \in \Re \), where

\[
\frac{\delta \Psi}{\delta \chi[u, v]}|_\chi = e^{-\frac{1}{2}v^2} \left\{ \int Dz \ w(u, v, z) - P[u, v] \right\}
\]

Clearly, if the function \( \chi[x, y] \) is a saddle-point, then also the function \( \chi[x, y] + \rho(y) \) for any \( \rho(y) \). This degree of freedom is irrelevant because such terms \( \rho(y) \) will drop out of the measure \( \langle \ldots \rangle_* \). Furthermore, one immediately verifies that transformations of the form \( \chi[x, y] \rightarrow \chi[x, y] + \rho(y) \) leave the functional \( \Psi[\ldots] \) (49) invariant. Next we calculate the Hessian (or curvature) operator \( H[u, v; u', v'; \chi] \), using (50):

\[
H[u, v; u', v'; \chi] = \frac{\delta^2 \Psi}{\delta \chi[u, v] \delta \chi[u', v']}|_\chi = e^{-\frac{1}{2}v^2} \left\{ \int Dz \ \frac{\delta w(u, v, z)}{\delta \chi[u', v']} \right\}
\]

\[
= \delta[v-v'] e^{-\frac{1}{2}v^2} \frac{1}{\alpha \sqrt{2\pi}} \int Dz \left[ \delta[u-u']w(u, v, z) - w(u, v, z)w(u', v, z) \right]
\]

\( H[u, v; u', v'; \chi] \) is non-negative definite for each \( \chi \), and thus the functional \( \Psi \) is convex, since for any function \( \phi[u, v] \) for which the relevant integrals exist we find

\[
\int dudvdudv' \phi[u, v] H[u, v; u', v'; \chi] \phi[u', v'] = \frac{1}{\alpha} \int DvDz \left[ \langle \phi^2[u, v] \rangle_* - \langle \phi[u, v] \rangle_*^2 \right] \geq 0
\]

The kernel of \( H[u, v; u', v'; \chi] \), for a given ‘point’ \( \chi \) in \( \chi \)-space, is determined by requiring equality in the above inequality, i.e.

for each \( v, z \in \Re \):

\[
\langle \phi[u, v] - \langle \phi[u, v] \rangle_* \rangle^2 = 0 \quad \text{so} \quad \frac{\partial}{\partial u} \phi[u, v] = 0
\]

For each \( \chi \) the kernel of the second functional derivative \( H[x, y; x', y'; \chi] \) thus consists of the set of all (integrable) functions \( \phi[x, y] \) which depend on \( y \) only.

We now find that, if \( \chi_0[x, y] \) and \( \chi_1[x, y] \) are both functional saddle-points of \( \Psi[\chi] \), then \( \chi_1[x, y] - \chi_0[x, y] = \rho(y) \) for some function \( \rho(y) \). In other words: apart from the aforementioned irrelevant degree
of freedom, the solution of the functional saddle-point equation \([13]\) is unique. To show this, consider two functions \(\chi_0[x, y]\) and \(\chi_1[x, y]\) which are both functional saddle-points of \(\Psi\), i.e. corresponding to solutions of \([15]\). Define a path \(\{\chi_t\}\) through \(\chi\)-space, connecting these two functions:

\[
\chi_t[x, y] = \chi_0[x, y] + t \left\{ \chi_1[x, y] - \chi_0[x, y] \right\}, \quad t \in [0, 1]
\]

Integration along this path will bring us from \(\chi_0\) to \(\chi_1\). Thus for any functional \(L[\chi]\) one has

\[
L[\chi_1] - L[\chi_0] = \int_{x_0}^{x_1} dL[\chi] = \int \int \frac{\delta L}{\delta \chi[u, v]} \left( \chi_0[u, v] - \chi_1[u, v] \right) d\chi[u, v] = 0
\]

For the functional \(L[\chi]\) we now choose a functional first derivative of \(\Psi[\chi]\), i.e. \(L[\chi] = \delta \Psi / \delta \chi[x, y]\) for some \(x, y \in \mathbb{R}\). Since both \(\chi_0\) and \(\chi_1\) are saddle-points one finds \(L[\chi_0] = L[\chi_1] = 0\). Thus

\[
\int \int \frac{\delta^2 \Psi}{\delta \chi[u, v] \delta \chi[x, y]} \left( \chi_0[u, v] - \chi_1[u, v] \right) d\chi[u, v] = 0
\]

Multiply both sides by \(\chi_1[x, y] - \chi_0[x, y]\) and integrate the result over \(x, y \in \mathbb{R}\):

\[
\int_0^1 dt \int \int \chi_1[x, y] - \chi_0[x, y] H[u, v; x, y; \chi_t] \left[ \chi_1[x, y] - \chi_0[x, y] \right] = 0
\]

One concludes (since the Hessian is a symmetric non-negative operator):

\[
\text{for all } t \in [0, 1], u, v \in \mathbb{R} : \quad \int dy H[u, v; x, y; \chi_t] \left[ \chi_1[x, y] - \chi_0[x, y] \right] = 0
\]

The function \(\chi_1[x, y] - \chi_0[x, y]\) is in the kernel of \(H|_{\chi_t}\) for any \(t \in [0, 1]\). The kernel of \(H\) was already determined to be the set of all integrable functions which depend on \(y\) only, whatever the point \(\chi\) where one chooses to evaluate \(H\). Hence \(\chi_1[x, y] - \chi_0[x, y] = \rho(y)\) for some function \(\rho(y)\). Finally, the remaining freedom in choosing a function \(\rho\) is eliminated by our normalisation \(\int dx M[x|y] = 1\) (for each \(y\)), so that the solution \(M[x|y]\) is indeed truly unique.

Next we will show how for any given value of the scalar order parameter \(q\) and the observables \(\{Q, R, P\}\) (and thus of \(B\)), for which the relevant integrals exist, the unique solution \(M[x|y]\) of the functional saddle-point equation \([15]\) can be constructed as the stable fixed-point of the following functional map:

\[
M_{\ell+1}[x|y] = \frac{P[x|y]}{\int du \ P[u|y]} \left\{ \int Dz \left[ \int dx' \ e^{Bz(x' - x)} M_{\ell}[x'|y] \right]^{-1} \right\}^{-1}
\]

Clearly all fixed-points of this map correspond to normalised solutions \(M[x|y]\) of a functional saddle-point equation \([13]\), of which there can be only one. Thus we only need to verify the convergence of \([12]\), which can be done most efficiently using an appropriate Lyapunov functional. Note that the functional \([19]\) can be written as

\[
\Psi[M|y] = \alpha \int Dy \ \tilde{\Psi}[M|y] + \text{terms independent of } M[\ldots]
\]
with
\[ \tilde{\Psi}[M|y] = \int Dz \log \int dx \ M[x|y]e^{Bxz} - \int dx P[x|y] \log M[x|y] \] (53)

For any given \( y \in \mathbb{R} \) we will show (53) to be a Lyapunov functional for the mapping (52), i.e. \( \tilde{\Psi}[M|y] \) is bounded from below and monotonically increasing during the iteration of (52) with stationarity obtained only when \( M[\ldots] \) is the (unique) fixed-point of (52). First we prove that a lower bound for \( \tilde{\Psi} \) is given by the entropy of the conditional distribution \( P[x|y] \):

for any \( M[\ldots] \) and any \( y \in \mathbb{R} : \quad \tilde{\Psi}[M|y] \geq - \int dx \ P[x|y] \log P[x|y] \) (54)

The proof is elementary (using Jenssen’s inequality):

\[ \tilde{\Psi}[M|y] = \int Dz \ \log \left\{ \int dx \ P[x|y]e^{Bxz+\log M[x|y]-\log P[x|y]} \right\} - \int dx \ P[x|y] \log M[x|y] \]
\[ \geq \int Dz \int dx \ P[x|y] \{Bxz + \log M[x|y] - \log P[x|y]\} - \int dx \ P[x|y] \log M[x|y] \]
\[ = - \int dx \ P[x|y] \log P[x|y] \]

Secondly we show that (53) indeed decreases monotonically under (52) until the fixed-point of (52) is reached. To do so we introduce the short-hand notations \( \lambda_\ell(x, y, z) = Bxz + \log M_\ell[x|y] - \log P[x|y] \), \( \langle f[x] \rangle = \int dx \ P[x|y]f[x] \), and

\[ v_\ell(x, y) = \left\{ \int Dz \ e^{\lambda_\ell(x,y,z)}\langle e^{\lambda_\ell(x',y,z)} \rangle^{-1} \right\}^{-1} \]

The iterative map can now be written as

\[ M_{\ell+1}[x|y] = \frac{M_\ell[x|y]v_\ell(x, y)}{\int du \ M_\ell[u|y]v_\ell(u, y)} \]

This gives for the change in \( \tilde{\Psi}[\ldots] \) during one iteration of the mapping, again with Jenssen’s inequality:

\[ \tilde{\Psi}[M_{\ell+1}|y] - \tilde{\Psi}[M_\ell|y] = \int Dz \ \log \left\{ \int dx \ M_{\ell+1}[x|y]e^{Bxz} \right\} - \int dx P[x|y] \log \left\{ \frac{M_{\ell+1}[x|y]}{M_\ell[x|y]} \right\} \]
\[ = \int Dz \ \left\{ \log \frac{\langle e^{\lambda_\ell(x,y,z)}v_\ell(x, y) \rangle}{\langle e^{\lambda_\ell(x',y,z)} \rangle} \right\} - \langle \log v_\ell(x, y) \rangle \]
\[ \leq \log \left\{ \langle v_\ell(x, y) \int Dz \ e^{\lambda_\ell(x,y,z)}\langle e^{\lambda_\ell(x',y,z)} \rangle^{-1} \rangle \right\} - \langle \log v_\ell(x, y) \rangle \]
\[ = -\langle \log v_\ell(x, y) \rangle + \langle \log \int Dz \ e^{\lambda_\ell(x,y,z)}\langle e^{\lambda_\ell(x',y,z)} \rangle^{-1} \rangle \]
\[ \leq \log \int Dz \ \langle e^{\lambda_\ell(x,y,z)} \rangle \langle e^{\lambda_\ell(x',y,z)} \rangle^{-1} = 0 \]

Finally we round off our argument by inspecting the implications of having strict equality in the above inequality. Equality can only occur if at both instances where Jenssen’s inequality was used in
replacements of the form $\langle \log(X) \rangle \leq \log(X)$ the relevant stochastic variable $X$ was a constant. In our problem this gives the two conditions

$$\frac{\partial}{\partial z} \frac{\langle e^{\lambda(x,y,z) v_\ell(x,y) \rangle}}{\langle e^{\lambda(x,y,z)} \rangle} = 0, \quad \frac{\partial}{\partial x} v_\ell(x,y) = 0$$

If the second condition is met, the first immediately follows. Working out the second condition gives, in combination with the property that $P[x|y]$ is normalised:

$$\int Dz M_{\ell}[x|y] e^{Bxz} \int dx' M_{\ell}[x'|y] e^{Bx'z} = P[x|y]$$

Thus we have confirmed that $\tilde{\Psi}[M_{\ell+1}|y] = \tilde{\Psi}[M_\ell|y]$ if and only if $M_{\ell}[\ldots]$ is the (unique) fixed-point of (52).

As a consequence of the above we may now write the normalised solution of our functional saddle-point equation (45) in terms of repeated execution of the mapping (52) following an arbitrary initialisation:

$$\text{for all } y \in \mathbb{R} : \quad M[x|y] = \lim_{\ell \to \infty} M_{\ell}[x|y], \quad M_0[x|y] = P[x|y]$$

This property simplifies the numerical solution of our equations drastically.

### 3.3 Fourier Representation and Conditionally-Gaussian Solutions

There are two potential advantages of rewriting our equations in Fourier representation. Firstly, after a Fourier transform the functional saddle-point equation (45) will acquire a much simpler form. Secondly, in those cases where we expect $P[x|y]$ to be of a Gaussian shape in $x$ this would simplify solution of the diffusion equations (40,42). Clearly, $P[x,y]$ being Gaussian in $(x,y)$ is not equivalent to $P[x|y]$ being Gaussian in $x$ only. The former requires

$$\frac{\partial^2}{\partial y^2} \int dx xP[x|y] = \frac{\partial}{\partial y} \left\{ \int dx x^2 P[x|y] - \left[ \int dx x P[x|y] \right]^2 \right\} = 0,$$

which only will turn out to happen for $\alpha \to \infty$. A Gaussian $P[x|y]$ with moments which depend on $y$ in a non-trivial way, on the other hand, is found to occur also for $\alpha < \infty$, provided we consider simple learning rules and small $\eta$ (see [17]). To avoid ambiguity we will call solutions of the latter type ‘conditionally-Gaussian’.

We introduce the Fourier transforms

$$\hat{P}[k|y] = \int dx e^{-ikx} P[x|y] \quad \hat{M}[k|y] = \int dx e^{-ikx} M[x|y]$$

The transformed functional saddle-point equation thereby acquires a very simple form

$$\hat{P}[k|y] = \int Dz \frac{\hat{M}[k+iBz|y]}{\hat{M}[iBz|y]}$$
Note that, in contrast to the original equation (43), the transformed equation (54) need not have a unique solution (it could allow for solutions corresponding to non-integrable functions in the original problem). Consider, for instance, the transformation

$$ P_M[k|y] \rightarrow \hat{M}[k|y] = \frac{e^{\frac{i}{2}k^2/B^2}}{M[-k|y]} $$

with the property (verified by a simple transformation of variables):

$$ \int Dx \frac{M[k+iBz|y]}{M[iBz|y]} = \int_{ik/B-\infty}^{ik/B+\infty} Dz \frac{\hat{M}[k+iBz|y]}{M[iBz|y]} $$

If \( \hat{M}[k] \), which by definition cannot have poles, is sufficiently well behaved, a simple deformation of the integration path (via contour integration) leads to the statement that if \( \hat{M}[k|y] \) is a solution of (56), then so is \( \hat{M}[k|y] \).

Transformation of the dynamical on-line equation (40) for \( P[x|y] \) (from the which the batch equation (12) can be obtained by expansion in \( \eta \)) gives:

$$ \frac{d}{dt} \log \hat{P}[k|y] = \frac{1}{\alpha} \left\{ \int dk' \frac{\hat{P}[k'|y]}{\hat{P}[k|y]} \int \frac{dx'}{2\pi} e^{ix'(k'-k)-i\eta k G[x',y]} - 1 \right\} - i\eta k (W-UR)y $$

$$ + \eta U \frac{\partial}{\partial k} \log \hat{P}[k|y] - \frac{1}{2} \eta^2 k^2 Z - i\eta k \left[ \frac{V-RW-(Q-R^2)U}{\sqrt{Q-R^2}P[k|y]} \right] \int Dz \frac{\hat{M}[k+iBz|y]}{M[iBz]} \quad (57) $$

We now determine the conditions for equation (57) to have conditionally-Gaussian solutions. If \( P[x|y] \) is Gaussian in \( x \) we can solve the functional saddle-point equation (43) (whose solution is unique), and find the resulting pair of measures

$$ P[x|y] = e^{-\frac{1}{2}(x-\bar{x}(y))^2/\Delta^2(y)}/\Delta(y)\sqrt{2\pi} \quad M[x|y] = \frac{e^{-\frac{1}{2}(x-\bar{x}(y))^2/\sigma^2(y)}}{\sigma(y)\sqrt{2\pi}} $$

$$ \Delta^2(y) = \sigma^2(y) + B^2\sigma^4(y) $$

with their Fourier transforms \( \hat{P}[k|y] = \exp \left[ -ik\bar{x}(y) - \frac{1}{2} k^2 \Delta^2(y) \right] \) and \( \hat{M}[k|y] = \exp \left[ -ik\bar{x}(y) - \frac{1}{2} k^2 \sigma^2(y) \right] \).

Insertion of these expressions as an Ansatz into (57), using the identity

$$ \int Dz \frac{\hat{M}[k+iBz|y]}{M[iBz]} = ikB\sigma^2(y)\hat{P}[k|y] $$

and performing some simple manipulations, gives the following simplified equation:

$$ -ik \frac{d}{dt} \bar{x}(y) - \frac{1}{2} k^2 \frac{d}{dt} \Delta^2(y) = \frac{1}{\alpha} \left\{ \int \frac{du}{\sqrt{2\pi}} e^{\frac{1}{2}[u-i\Delta(y)]^2-i\eta u\bar{x}(y)+u\Delta(y),y] - 1 \right\} - i\eta k \{Wy+U[\bar{x}(y)-Ry]\} $$

$$ - \frac{1}{2} k^2 \left\{ \eta^2 Z + 2\eta U \Delta^2(y) + 2\eta^2 \sigma^2(y) \left[ \frac{V-RW-(Q-R^2)U}{Q(1-q)} \right] \right\} \quad (60) $$
From this it follows that conditionally-Gaussian solutions can occur in two situations only:

$$\alpha \to \infty \quad \text{or} \quad \frac{\partial^3}{\partial k^3} \int \frac{du}{\sqrt{2\pi}} e^{-\frac{1}{2}[u-ik\Delta(y)]^2-ik\eta_G[\Phi(y)+u\Delta(y),y]} = 0$$

(61)

The first case corresponds to the familiar theory of infinite training sets (see next section). The second case occurs for sufficiently simple learning rules \( G[x,y] \), in combination either with batch execution (so that of (61) we retain only the term linear in \( \eta \)) or with on-line execution for small \( \eta \) (retaining in (61) only \( \eta \) and \( \eta^2 \) terms). The latter cases will be dealt with in more detail in [17].

### 3.4 Link with the Formalism for Complete Training Sets

The very least we should require of our theory is that it reduces to the simple \((Q,R)\) formalism of complete training sets [4, 3] in the limit \( \alpha \to \infty \). Here we will show that this indeed happens. In the previous section we have seen that for \( \alpha \to \infty \) our driven diffusion equation for the conditional distribution \( P[x|y] \) has conditionally-Gaussian solutions, with \( \int dx\ xP[x|y] = \overline{\Phi}(y) \) and \( \int dx\ |x-\overline{\Phi}(y)|^2P[x|y] = \Delta^2(y) \). Note that for such solutions we can calculate objects such as \( \langle x \rangle \) and the function \( \Phi[x,y] \) [17] directly, giving

$$\langle x \rangle = \overline{\Phi}(y) + zB\sigma^2(y), \quad \Phi[x,y] = \frac{x-\overline{\Phi}(y)}{Q(1-Q)[1+B^2\sigma^2(y)]}$$

with \( \Delta^2(y) = \sigma^2(y) + B^2\sigma^4(y) \) and \( B = \sqrt{Q-R^2}/Q(1-Q) \). The remaining dynamical equations to be solved are those for \( Q \) and \( R \), in combination with dynamical equations for the \( y \)-dependent cumulants \( \overline{\Phi}(y) \) and \( \Delta^2(y) \). These equations reduce to:

$$\frac{d}{dt}Q = \begin{cases} 2\eta\langle xG[x,y] \rangle + \eta^2\langle G^2[x,y] \rangle \quad & \text{(on-line)} \\ 2\eta\langle xG[x,y] \rangle \quad & \text{(batch)} \end{cases}$$

$$\frac{d}{dt}R = \eta\langle yG[x,y] \rangle$$

(62)

$$\frac{1}{\eta} \frac{d}{dt} \left[ \overline{\Phi}(y) - R \right] = \frac{\overline{\Phi}(y) - R \rangle}{\langle \Phi(x',y')|G[x',y']}$$

(63)

$$\Delta^2(y) = \frac{\sigma^2(y)}{Q(1-Q)} - 1 + \langle \Phi(x',y')|G[x',y'] \rangle \left[ \Delta^2(y) - \frac{Q-R^2}{Q(1-Q)\sigma^2(y)} \right]$$

(64)

with one remaining saddle-point equation to determine \( q \), obtained upon working out [44] for conditionally-Gaussian solutions:

$$\int dy \left[ \Delta^2(y) + \overline{\Phi}(y) - R \right] + qQ - R^2 = \left[ \frac{qQ-R^2}{Q(1-Q)} + 1 \right] \int dy \sigma^2(y)$$

(65)

We now make the Ansatz that \( \overline{\Phi}(y) = R \) and \( \Delta^2(y) = Q - R^2 \), i.e.

$$P[x|y] = e^{-\frac{1}{2}[x-Ry]^2/(Q-R^2)}$$

(66)

Insertion into the dynamical equations shows that (63) is now immediately satisfied, that (64) reduces to \( \sigma^2(y) = Q(1-Q) \), and that as a result the saddle-point equation (65) is automatically satisfied. Since (66) is parametrized by \( Q \) and \( R \) only, this leaves us with the closed equations

$$\frac{d}{dt}Q = \begin{cases} 2\eta\langle xG[x,y] \rangle + \eta^2\langle G^2[x,y] \rangle \quad & \text{(on-line)} \\ 2\eta\langle xG[x,y] \rangle \quad & \text{(batch)} \end{cases} \quad \frac{d}{dt}R = \eta\langle yG[x,y] \rangle$$

(67)
These are the equations found in e.g. [2, 3]. From our general theory for restricted training sets we thus indeed recover in the limit $\alpha \to \infty$ the standard formalism [66, 67] describing learning with complete training sets, as claimed.

4 Discussion

In this paper we have shown how the formalism of dynamical replica theory (see e.g. [13]) can be successfully employed to construct a general theory which enables one to predict the evolution of the relevant macroscopic performance measures for supervised (on-line and batch) learning in layered neural networks, with randomly chosen but restricted training sets, i.e. for finite $\alpha = p/N$ where weight updates are carried out by sampling with repetition. In this case the student nodes local fields are no longer described by (multivariate) Gaussian distributions and the traditional and familiar statistical mechanical formalism consequently breaks down. For simplicity and transparency we have restricted ourselves to single-layer systems and realizable tasks.

In our approach the joint field distribution $P[x, y]$ for the student and teacher local fields is itself taken to be a dynamical order parameter, in addition to the conventional observables $Q$ and $R$ representing overlaps between the student-student and student-teacher vectors respectively. The new order parameter set $\{Q, R, P\}$, in turn, enables one to monitor the generalization error $E_g$ as well as the training error $E_t$. This then results, following the prescriptions of dynamical replica theory in a diffusion equation for $P[x, y]$, which we have evaluated by making the replica-symmetric ansatz in the saddle-point equations. This diffusion equation is generally found to have Gaussian solutions only for $\alpha \to \infty$; in the latter case we indeed recover correctly from our theory the more familiar formalism of infinite training sets (in the $N \to \infty$ limit), providing closed equations for $Q$ and $R$ only. For finite $\alpha$ our theory is by construction exact if for $N \to \infty$ the dynamical order parameters $\{Q, R, P\}$ obey closed deterministic equations, which are self-averaging (i.e. independent of the microscopic realization of the training set). If this is not the case, our theory can be interpreted as employing a maximum entropy approximation. In a sequel paper [17] we will work out our equations explicitly for various choices of learning rules, and compare our theoretical predictions both to exact solutions, derived for special cases directly from the microscopic equations, and with numerical simulations. We will also construct a number of simple but effective approximations to our full equations. As it will turn out, our theory describes the various learning processes examined highly accurately.

The present study represents only a first step in understanding on-line learning with restricted training sets. It opens up many extensions, applications and generalizations that can be carried out (some of which are already under way). Firstly, our theory would simplify significantly if one could find a more explicit solution of the functional saddle-point equation (96), enabling us to express the function $\Phi[x, y]$ directly in terms of our order parameters. The benefits of such a solution will become even greater as we apply our theory to more sophisticated learning rules, such as to perceptron or AdaTron learning, or to learning in multi-layer networks (which run the risk of requiring a serious amount of CPU time). Secondly, this theory opens up new possibilities for considering unrealizable learning scenarios, either due to structural limitations or due to noise, which require some sort of

3 The reason why the replica formalism is inevitable (unless we are willing to pay the price of having observables with two time arguments, and turn to path integrals) is the necessity, for finite $\alpha$, to average the macroscopic equations over all possible realizations of the training set.

4 Such exact results can only be obtained for Hebbian-type rules, where the dependence of the updates $\Delta J(t)$ on the weights $J(t)$ is trivial or even absent (a decay term at most), whereas our present theory generates macroscopic equations for arbitrary learning rules.
regularization. The examination of regularization techniques in such scenarios, which is of great practical significance, was out of reach so far as they come into effect only where the error-surface is fixed by having a fixed example set. Thirdly, at a more fundamental level one could explore the effects of (dynamic) replica symmetry breaking (by calculating the AT-surface, signalling instability of the replica symmetric solution with respect to replicon fluctuations), or one could improve the built-in accuracy of our theory by adding new observables to the present set (such as the Green’s function $\mathcal{A}[x, y; x', y']$ itself). Finally it would be interesting to see the connection between the present formalism and a suitable adaptation of the generating functional methods, as applied in [10] to networks with binary weights, to the learning processes studied in this paper.

Acknowledgements

It is our pleasure to thank Yuan-sheng Xiong and Charles Mace for valuable discussions. DS acknowledges support by EPSRC Grant GR/L52093 and the British Council grant: British-German Academic Research Collaboration Programme project 1037.

References

[1] Kinzel W and Rujan P 1990 Europhys. Lett. 13 473
[2] Kinouchi O and Caticha N 1992 J. Phys. A: Math. Gen. 25 6243
[3] Biehl M and Schwarze H 1992 Europhys. Lett. 20 733
[4] Biehl M and Schwarze H 1995 J. Phys. A: Math. gen. 28 643
[5] Saad D and Solla S 1995 Phys. Rev. E52 4225; Phys. Rev. L74 4337
[6] Mace CWH and Coolen ACC 1998 Statistics and Computing 8 55
[7] Saad D (Ed) 1998 On-line Learning in Neural Networks (Cambridge: Cambridge University Press)
[8] Barber D, Saad D and Sollich P 1996 Europhys. Lett. 34 151
[9] Sollich P and Barber D 1998 in Advances in Neural Information Processing Systems Jordan MI, Kearns MJ and Solla SA (Eds) (MIT Press, Cambridge, MA) Vol. 10, p 385
[10] Horner H 1992 Z. Phys. B86 291; Z. Phys. B87 371
[11] Krogh A and Hertz JA 1992 J. Phys. A: Math. Gen. 25 1135
[12] Rae HC, Sollich P and Coolen ACC 1999 J. Phys. A: Math. Gen. 32 3321
[13] Coolen ACC, Laughton SN and Sherrington D 1996 Phys. Rev. B53 8184
[14] Coolen ACC, Saad D and Xiong YS 1999 in preparation
[15] Mace CWH and Coolen ACC 1999 submitted to NIPS*99
[16] Mézard M, Parisi G and Virasoro MA 1987 Spin Glass Theory and Beyond (Singapore: World Scientific)
[17] Coolen ACC and Saad D 1999 King’s College London preprint KCL-MTH-99-33
A Replica Calculation of the Green’s Function

The main objective of this Appendix is to calculate the Green’s function $A[...]$, with which we obtain our macroscopic dynamic equations in explicit form. We first carry out the disorder averages, leading to an effective single-spin problem. The integrations are done by steepest descent, giving a saddle-point problem for replicated order parameters at each time step. In the saddle point equations we then make the replica symmetry (RS) ansatz, so that the limit $n \to 0$ can be taken. In addition we show that the two functions $B[...]$ and $C[...]$ do indeed vanish, as claimed.

A.1 Disorder Averaging

The fundamental quantities $A[x, y; x', y']$, $B[x, y; x', y']$, $C[x, y; x', y'; x'', y'']$, and $P[x, y]$, which control the macroscopic equations can be written as

$$
\begin{align*}
&\left\{ \begin{array}{l}
P[x, y] \\
A[x, y; x', y'] \\
B[x, y; x', y'] \\
C[x, y; x', y'; x'', y'']
\end{array} \right\} = \lim_{N \to \infty} \left\{ \begin{array}{l}
P[x, y] \\
A[x, y; x', y'] \\
B[x, y; x', y'] \\
C[x, y; x', y'; x'', y'']
\end{array} \right\} \\
&\int \prod_{\alpha} \left\{ \delta \left[ N - (\sigma^2) \right] \delta \left[ \frac{N R}{\sqrt{Q}} - \tau \cdot \sigma^\alpha \right] d\sigma^\alpha \prod_{x, y, \alpha} \delta \left[ P[x, y] - P[x, y; \frac{\sqrt{Q}\sigma^\alpha}{\sqrt{N}}] \right] \right\} \delta \left[ x - \frac{\sqrt{Q}\sigma^\alpha}{\sqrt{N}} \right] \delta \left[ y - \frac{\tau \cdot \xi}{\sqrt{N}} \right] \\
&\cdot \left\{ \begin{array}{l}
(\xi' \cdot \xi) \delta \left[ x' - \frac{\sqrt{Q}\sigma^\alpha}{\sqrt{N}} \right] \delta \left[ y' - \frac{\tau \cdot \xi}{\sqrt{N}} \right] \\
\left[ \frac{1}{N} \sum_{i \neq j} \xi_i \xi_j \xi_i'^\alpha \xi_j'^\alpha \right] \sum_{\alpha} \delta \left[ x' - \frac{\sqrt{Q}\sigma^\alpha}{\sqrt{N}} \right] \delta \left[ y' - \frac{\tau \cdot \xi}{\sqrt{N}} \right] \\
\frac{1}{N} (\xi \cdot \xi'') (\xi' \cdot \xi'') \delta \left[ x' - \frac{\sqrt{Q}\sigma^\alpha}{\sqrt{N}} \right] \delta \left[ y' - \frac{\tau \cdot \xi}{\sqrt{N}} \right] \delta \left[ x'' - \frac{\sqrt{Q}\sigma^\alpha}{\sqrt{N}} \right] \delta \left[ y'' - \frac{\tau \cdot \xi''}{\sqrt{N}} \right]
\end{array} \right\}
\end{align*}
$$

We next use the definition of $P[x, y; J]$, introduce integral representations for the $\delta$-distributions involving $P[x, y]$, and obtain

$$
\begin{align*}
&\left\{ \begin{array}{l}
P[x, y] \\
A[x, y; x', y'] \\
B[x, y; x', y'] \\
C[x, y; x', y'; x'', y'']
\end{array} \right\} = \lim_{N \to \infty} \left\{ \begin{array}{l}
P[x, y] \\
A[x, y; x', y'] \\
B[x, y; x', y'] \\
C[x, y; x', y'; x'', y'']
\end{array} \right\} \\
&\cdot \left\{ \begin{array}{l}
\int \prod_{\alpha} \left\{ \delta \left[ N - (\sigma^2) \right] \delta \left[ \frac{N R}{\sqrt{Q}} - \tau \cdot \sigma^\alpha \right] d\sigma^\alpha \prod_{x, y, \alpha} \delta \left[ P[x, y] - P[x, y; \frac{\sqrt{Q}\sigma^\alpha}{\sqrt{N}}] \right] \right\} \delta \left[ x - \frac{\sqrt{Q}\sigma^\alpha}{\sqrt{N}} \right] \delta \left[ y - \frac{\tau \cdot \xi}{\sqrt{N}} \right] \\
\end{array} \right\}
\end{align*}
$$

$$
\begin{align*}
&\times \left\{ \begin{array}{l}
e^{-iN \sum_{x, y} \sum_{\alpha} \pi_{x, y} (x, y; [x, y] - \frac{\sqrt{Q}\sigma^\alpha}{\sqrt{N}}; [y, y] - \frac{\tau \cdot \xi}{\sqrt{N}}) \delta \left[ x - \frac{\sqrt{Q}\sigma^\alpha}{\sqrt{N}} \right] \delta \left[ y - \frac{\tau \cdot \xi}{\sqrt{N}} \right] \\
\end{array} \right\}
\end{align*}
$$
The summations involving \((x_\alpha, y_\alpha)\) automatically lead to integrals, which can be performed due to the \(\delta\)-distributions involved. We define new conjugate functions \(\hat{P}_\alpha[x, y]\) via

\[
\sum_{x_\alpha y_\alpha} \pi_\alpha[x_\alpha, y_\alpha] f[x_\alpha, y_\alpha] \to \int dx'' dy'' \hat{P}_\alpha[x'', y''] f[x'', y'']
\]

We write averages over the training set explicitly in terms of the \(p = \alpha N\) constituent vectors \(\{\xi^\mu\}\). Finally we introduce integrals representations for the remaining delta-distributions, and obtain the following expressions (at this stage we will have to separate the various structurally different cases):

\[
P[x, y] = \int \frac{d\hat{x} d\hat{y}}{(2\pi)^2} e^{i[x_\hat{x} + y_\hat{y}]} \lim_{N \to \infty} \frac{\prod_{\alpha} \left\{ \delta [N - (\sigma^\alpha)^2] \delta \left[ \frac{NR}{\sqrt{Q}} - \tau \cdot \sigma^\alpha \right] d\sigma^\alpha e^{iN} \int dx'' dy'' \hat{P}_\alpha[x'', y''] P[x'', y''] \prod_{x'' y''} d\hat{P}_\alpha[x'', y''] \right\}}{P} \sum_{\mu=1}^{p} \frac{1}{P^{\mu=1}} \left( e^{-\frac{1}{\alpha} \sum_{\lambda} \sum_{\nu} \hat{P}_\alpha \left( \frac{\sqrt{\sigma^\alpha} \xi^\lambda}{\sqrt{N}} - \frac{\tau \xi^\lambda}{\sqrt{N}} \right) - i[x_\hat{x} \sqrt{Q} \sigma^1 \xi^\mu + y_\hat{y} \sqrt{N} \sigma^1 \xi^\mu] / \sqrt{N} \right) \Xi \tag{68}
\]

\[
A[x, y; x', y'] = \int \frac{d\hat{x} d\hat{x}' d\hat{y} d\hat{y}'}{(2\pi)^4} e^{i[x_\hat{x} + x'_\hat{x} + y_\hat{y} + y'_\hat{y}]} \lim_{N \to \infty} \frac{\prod_{\alpha} \left\{ \delta [N - (\sigma^\alpha)^2] \delta \left[ \frac{NR}{\sqrt{Q}} - \tau \cdot \sigma^\alpha \right] d\sigma^\alpha e^{iN} \int dx'' dy'' \hat{P}_\alpha[x'', y''] P[x'', y''] \prod_{x'' y''} d\hat{P}_\alpha[x'', y''] \right\}}{P^{\mu=1} \sum_{\mu=1}^{p}} \left( e^{-\frac{1}{\alpha} \sum_{\lambda} \sum_{\nu} \hat{P}_\alpha \left( \frac{\sqrt{\sigma^\alpha} \xi^\lambda}{\sqrt{N}} - \frac{\tau \xi^\lambda}{\sqrt{N}} \right) - i[x_\hat{x} \sqrt{Q} \sigma^1 \xi^\mu + y_\hat{y} \sqrt{N} \sigma^1 \xi^\mu] / \sqrt{N} \right) \Xi \tag{69}
\]

\[
C[x, y; x', y'; x'', y''] = \int \frac{d\hat{x} d\hat{x}' d\hat{y} d\hat{y}'}{(2\pi)^6} e^{i[x_\hat{x} + x'_\hat{x} + x''_\hat{x} + y_\hat{y} + y'_\hat{y} + y''_\hat{y}]} \lim_{N \to \infty} \frac{\prod_{\alpha} \left\{ \delta [N - (\sigma^\alpha)^2] \delta \left[ \frac{NR}{\sqrt{Q}} - \tau \cdot \sigma^\alpha \right] d\sigma^\alpha e^{iN} \int dx'' dy'' \hat{P}_\alpha[x'', y''] P[x'', y''] \prod_{x'' y''} d\hat{P}_\alpha[x'', y''] \right\}}{P^{\mu=1} \sum_{\mu=1}^{p}} \left( e^{-\frac{1}{\alpha} \sum_{\lambda} \sum_{\nu} \hat{P}_\alpha \left( \frac{\sqrt{\sigma^\alpha} \xi^\lambda}{\sqrt{N}} - \frac{\tau \xi^\lambda}{\sqrt{N}} \right) - i[x_\hat{x} \sqrt{Q} \sigma^1 \xi^\mu + y_\hat{y} \sqrt{N} \sigma^1 \xi^\mu] / \sqrt{N} \right) \Xi \tag{70}
\]
The averages over the training sets \( \langle \ldots \rangle_\Xi \) in (68), (69), (70) will now be done separately. First we define some relevant objects:

\[
D[u, v] = \left\langle e^{-\frac{1}{\alpha} \sum \alpha \hat{P}_\alpha \left( \frac{\sqrt{\sigma^\alpha} \xi \xi}{\sqrt{N}} \right) - i [x, y] \sqrt{\sigma^1} \xi y / \sqrt{N}} \right\rangle \xi
\]

(71)

\[
E_j[u, v] = \left\langle \sqrt{N} e^{-\frac{1}{\alpha} \sum \alpha \hat{P}_\alpha \left( \frac{\sqrt{\sigma^\alpha} \xi \xi}{\sqrt{N}} \right) - i [x, y'] \sqrt{\sigma^1} \xi y' / \sqrt{N}} \right\rangle \xi
\]

(72)

\[
E_{ij}[u, v] = \left\langle N \xi_j e^{-\frac{1}{\alpha} \sum \alpha \hat{P}_\alpha \left( \frac{\sqrt{\sigma^\alpha} \xi \xi}{\sqrt{N}} \right) - i [x, y'] \sqrt{\sigma^1} \xi y' / \sqrt{N}} \right\rangle \xi \quad (i \neq j)
\]

(73)

As we will see, all are of order \( \mathcal{O}(N^0) \) as \( N \to \infty \). We next use the permutation invariance of our integrations and summations with respect to pattern labels. First we calculate the first training sets average occurring in (68):

\[
\frac{1}{p} \sum_{\mu=1}^p \left\langle \left( \xi^\mu \cdot \xi^\nu \right)^{e^{-\cdots}} \right\rangle_\Xi = \frac{p-1}{p} \left\langle \left( \xi_1 \cdot \xi^2 \right)^{e^{-\cdots}} \right\rangle_\Xi
\]

(74)

The prefactor \( e^{p \log D[0,0]} \), will turn out to take care of appropriate normalisation, and will drop out of the final result for all four functions \( P[x, y], A[x, y; x', y'], B[x, y; x', y'], C[x, y; x', y'; x'', y''] \). Secondly we evaluate the training sets average of the expression for \( A[\ldots] \) in (69):

\[
\frac{1}{p^2} \sum_{\mu \neq \nu} \left\langle \left( \xi^\mu \cdot \xi^\nu \right)^{e^{-\cdots}} \right\rangle_\Xi = \frac{p-1}{p} \left\langle \left( \xi_1 \cdot \xi^2 \right)^{e^{-\cdots}} \right\rangle_\Xi
\]

(75)

(provided we indeed show that \( E_j[u, v] = \mathcal{O}(N^0) \) as \( N \to \infty \)). Secondly, the training sets average of the expression for \( B[\ldots] \) in (69) is given by:

\[
\frac{1}{p^2} \sum_{\mu \neq \nu} \left\langle \left( \xi^\mu \xi^\nu \xi^\nu \xi^\nu \right)^{e^{-\cdots}} \right\rangle_\Xi = \frac{p-1}{pN} \sum_{i \neq j} \left\langle \left( \xi_1 \xi_j \right)^2 \left( \xi^1 \xi^2 \right)^2 \right\rangle_\Xi
\]

(76)

\[
\times \left\langle \xi_i \xi_j e^{\frac{-1}{\sqrt{N}} \sum \alpha \dot{P}_\alpha (\sqrt{\sqrt{\frac{\sigma_0^\alpha \xi \cdot \tau \xi}{\sqrt{N}}}) - i [\dot{\xi}_i, \dot{\xi}_j, \dot{\tau}_i, \dot{\tau}_j] / \sqrt{N}} \right\rangle = e^{p \log D[0,0]} \left\{ \frac{1}{N^3} \sum_{i \neq j = 1}^N \mathcal{E}_{ij}[\dot{x}, \dot{y}, \dot{y}'] + O(N^{-\frac{1}{2}}) \right\} = e^{p \log D[0,0]} \left\{ O(N^{-1}) \right\}
\]

(76)

(provided we indeed show that \( \mathcal{E}_{ij} = O(N^0) \) as \( N \to \infty \)). Finally, we also obtain for the training sets average in (70), in a similar fashion:

\[
\frac{1}{p^3} \sum_{\rho = 1}^p \sum_{\mu, u \neq \mu} \left\langle \frac{1}{N} (\xi^{\mu} \cdot \xi^{\mu^*}) (\xi^{u} \cdot \xi^{u}) e^{\cdot \cdot} \right\rangle \infty = \frac{p-1}{p^2 N} \sum_{ij} \left\langle \xi_i^1 \xi_j^2 \xi_i^3 \xi_j^3 \cdot \cdot \cdot \right\rangle \infty + \frac{(p-1)(p-2)}{p^2 N} \sum_{ij} \left\langle \xi_i^1 \xi_j^2 \xi_j^3 \cdot \cdot \cdot \right\rangle \infty
\]

\[
= \sum_{i \neq j} \left\langle \xi_i^1 \xi_j^2 \xi_j^3 \cdot \cdot \cdot \right\rangle \infty O(N^{-2}) + \left\langle \cdot \cdot \cdot \right\rangle \infty O(N^{-1}) + \sum_{i \neq j} \left\langle \xi_i^1 \xi_j^2 \xi_j^3 \cdot \cdot \cdot \right\rangle \infty O(N^{-1}) + \sum_i \left\langle \xi_i^1 \xi_i^2 \cdot \cdot \cdot \right\rangle \infty O(N^{-1})
\]

\[
= D[0,0]^p \left\{ \sum_{i \neq j} \mathcal{D}[\dot{x}^n, \dot{y}^n] \mathcal{E}_{ij} \dot{x}^n \mathcal{E}_{ij} \dot{y}^n \cdot \cdot \cdot \right\} \infty O(N^{-2}) + O(N^{-4}) + O(N^{-1}) + \sum_{i \neq j} \mathcal{E}_i \dot{x}^n \mathcal{E}_j \dot{x}^n \cdot \cdot \cdot \right\} \infty O(N^{-1})
\]

\[
= e^{p \log D[0,0]} \left\{ O(N^{-1}) \right\}
\]

(77)

We now work out (72) and we show that it is of order \( N^0 \). This is achieved by separating in the exponent the terms with site label \( i = j \) from those with site labels \( i \neq j \), followed by expansion in powers of the (relatively small) \( i = j \) terms, and will involve the following two functions:

\[
\mathcal{F}_1^\alpha[u, v] = \left\langle \partial_x \dot{P}_\alpha (\sqrt{\sqrt{\frac{\sigma_0^\alpha \xi \cdot \tau \xi}{\sqrt{N}}}) e^{\frac{-1}{\sqrt{N}} \sum \alpha \dot{P}_\alpha (\sqrt{\sqrt{\frac{\sigma_0^\alpha \xi \cdot \tau \xi}{\sqrt{N}}}) - i [\dot{u}, \dot{v}, \dot{\tau}_1, \dot{\tau}_j] / \sqrt{N}} \right\rangle \xi
\]

(78)

\[
\mathcal{F}_2^\alpha[u, v] = \left\langle \partial_y \dot{P}_\alpha (\sqrt{\sqrt{\frac{\sigma_0^\alpha \xi \cdot \tau \xi}{\sqrt{N}}}) e^{\frac{-1}{\sqrt{N}} \sum \alpha \dot{P}_\alpha (\sqrt{\sqrt{\frac{\sigma_0^\alpha \xi \cdot \tau \xi}{\sqrt{N}}}) - i [\dot{u}, \dot{v}, \dot{\tau}_1, \dot{\tau}_j] / \sqrt{N}} \right\rangle \xi
\]

(79)

Note that there is no need to calculate the auxiliary functions (73); we only need to verify their magnitude to scale as \( O(N^0) \) for \( N \to \infty \).

\[
\mathcal{E}_j[u, v] = \left\langle \sqrt{N} \xi_j e^{\frac{-1}{\sqrt{N}} \sum \alpha \dot{P}_\alpha (\sqrt{\sqrt{\frac{\sigma_0^\alpha \xi \cdot \tau \xi}{\sqrt{N}}}) - i [\dot{u}, \dot{v}, \dot{\tau}_1, \dot{\tau}_j] / \sqrt{N}} \right\rangle \xi
\]

\[
= \left\langle \sqrt{N} \xi_j e^{\frac{-1}{\sqrt{N}} \sum \alpha \dot{P}_\alpha (\sqrt{\sqrt{\frac{\sigma_0^\alpha \xi \cdot \tau \xi}{\sqrt{N}}}) - i [\dot{u}, \dot{v}, \dot{\tau}_1, \dot{\tau}_j] / \sqrt{N}} \right\rangle \xi
\]

\[
= e^{\frac{1}{\sqrt{N}} \sum \alpha \sigma_0^\alpha \xi_j \dot{P}_\alpha (\sqrt{\sqrt{\frac{\sigma_0^\alpha \xi \cdot \tau \xi}{\sqrt{N}}}) - i [\dot{u}, \dot{v}, \dot{\tau}_1, \dot{\tau}_j] / \sqrt{N}} \right\rangle \xi
\]

\[
\times e^{-i [\dot{u}, \dot{v}, \dot{\tau}_1, \dot{\tau}_j] / \sqrt{N}} \right\rangle \xi
\]

\[
\text{30}
\]
where we have used the built-in properties

\[
\frac{1}{i} \left\{ \frac{\sqrt{Q}}{\alpha} \sum_{\alpha} \sigma_{ij}^\alpha \mathcal{P}_\alpha \left( \frac{\sqrt{Q}}{N} \sum_{i \neq j} \sigma_{ij}^\alpha \xi_{ij} + \frac{1}{\sqrt{N}} \sum_{i \neq j} \tau_{ij} \xi_{ij} \right) - i \frac{\sqrt{Q}}{\alpha} \sum_{i \neq j} \sigma_{ij}^\alpha \xi_{ij} + \frac{1}{\sqrt{N}} \sum_{i \neq j} \tau_{ij} \xi_{ij} \right\} \times 
\]

so that

\[
\mathcal{E}_j[u,v] = -iu \sqrt{Q} \sigma_j^\alpha \mathcal{D}[u,v] - iv \tau_j \mathcal{D}[u,v] - \frac{i}{\alpha} \frac{\sqrt{Q}}{\alpha} \sum_{\alpha} \sigma_{ij}^\alpha \mathcal{F}_1^\alpha[u,v] - \frac{i}{\alpha} \tau_j \sum_{\alpha} \mathcal{F}_2^\alpha[u,v] + O(N^{-\frac{1}{2}})
\]

(80)

Repetition/extension of this argument, by separating in the exponent terms with two special indices \((i,j)\) rather than one, and by subsequent expansion (whereby each term brings down a factor \(N^{-\frac{1}{2}}\)), immediately shows that terms of the form \((N \xi_i \xi_j e^{-\alpha})\) with \(i \neq j\) will be of order \(O(N^0)\). This confirms that \(\mathcal{E}_j[u,v] = O(N^0)\) and that (73) indeed scales as indicated. Note that the relevant combination of intensive terms in (73) can be abbreviated as

\[
\mathcal{L}[u,v;u',v'] = \frac{1}{N} \sum_j \mathcal{E}_j[u,v] \mathcal{E}_j[u',v']
\]

where we have used the built-in properties \(\frac{1}{N} \sigma \cdot \sigma^\alpha = R/\sqrt{Q}\) and \(\sigma^2 = N\), and in which we find the spin-glass order parameters

\[
q_{\alpha\beta}(\{\sigma\}) = \frac{1}{N} \sum_i \sigma_i^\alpha \sigma_i^\beta
\]

(82)

Let us finally work out further the remaining fundamental objects \(\mathcal{D}[]\ldots\) and \(\mathcal{F}_1^\alpha[]\ldots\). The basic property to be used is that for large \(N\) the \(n+1\) quantities \(\{x_\alpha = \sigma_\alpha \cdot \xi / \sqrt{N}, y = \sigma \cdot \xi / \sqrt{N}\}\) inside averages of the form \((\ldots)\xi\) will become (zero average but correlated) Gaussian variables, with probability distribution

\[
P(x_1, \ldots, x_n, y) = \frac{\det^\frac{1}{2} \mathbf{A}}{(2\pi)^{(n+1)/2}} e^{-\frac{1}{2} \left( \begin{array}{c} x_1 \\ \vdots \\ x_n \\ y \end{array} \right)^T \mathbf{A}^{-1} \left( \begin{array}{c} x_1 \\ \vdots \\ x_n \\ y \end{array} \right)}
\]

\[
\mathbf{A}^{-1} = \begin{pmatrix}
q_{11} & \cdots & q_{1n} & R/\sqrt{Q} \\
\vdots & \ddots & \vdots & \vdots \\
q_{n1} & \cdots & q_{nn} & R/\sqrt{Q} \\
R/\sqrt{Q} & \cdots & R/\sqrt{Q} & 1
\end{pmatrix}
\]
This allows us to write

\[
\mathcal{D}[u, v] = \frac{\det \frac{\partial}{\partial \bar{\alpha}} A}{(2\pi)^{(n+1)/2}} \int dx dy e^{-\frac{i}{2} \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ y \end{pmatrix} \cdot A \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ y \end{pmatrix} - \frac{i}{2} \sum_\alpha P_\alpha(\sqrt{Q} x_\alpha, y) - i[u\sqrt{Q} x_1 + vy]} 
\]

(83)

\[
\mathcal{F}^{1,2}[u, v] = \frac{\det \frac{\partial}{\partial \bar{\alpha}} A}{(2\pi)^{(n+1)/2}} \int dx dy \partial_{1,2} P_\alpha(\sqrt{Q} x_\alpha, y) e^{-\frac{i}{2} \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ y \end{pmatrix} \cdot A \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ y \end{pmatrix} - \frac{i}{2} \sum_\alpha P_\alpha(\sqrt{Q} x_\alpha, y) - i[u\sqrt{Q} x_1 + vy]} 
\]

Note that these quantities depend on the microscopic variables \(\sigma^\alpha\) only through the macroscopic observables \(q_{\alpha\beta}(\{\sigma\})\).

**A.2 Derivation of Saddle-Point Equations**

We will now combine the results [74, 75, 76, 77] and (81) with the expressions [68, 69, 70]. We use integral representations for the remaining delta functions, and isolate the observables \(q_{\alpha\beta}\), by inserting

\[
1 = \int \frac{dq d\bar{q} dR d\bar{R}}{(2\pi)^{n^2+2n}} e^{i \sum_\alpha (\bar{Q}_\alpha + \bar{R}_\alpha R/\sqrt{Q}) + \sum_\alpha \bar{q}_{\alpha\beta} q_{\alpha\beta}} \prod_{\alpha} \frac{D}[\bar{x}, \bar{y}] + iN \log \mathcal{D}[0, 0] 
\]

and

\[
P[x, y] = \int \frac{d\bar{q} d\bar{R}}{(2\pi)^2} e^{i [\bar{q} x + \bar{R} y]} \lim_{n \to 0} \lim_{N \to \infty} \int dq d\bar{q} d\bar{R} \prod_{\alpha} \frac{D}[\bar{x}, \bar{y}] + iN \log \mathcal{D}[0, 0] 
\]

Both can be written in the form of an integral dominated by saddle-points:

\[
A[x, y; x', y'] = \int \frac{d\bar{x} d\bar{y} d\bar{x}' d\bar{y}'}{(2\pi)^4} e^{i [\bar{x} x + \bar{y} y + \bar{x}' x' + \bar{y}' y']} 
\]

\[
\lim_{n \to 0} \lim_{N \to \infty} \int dq d\bar{q} d\bar{R} \prod_{\alpha} \frac{D}[\bar{x}, \bar{y}] + iN \log \mathcal{D}[0, 0] 
\]
and

\[ P[x, y] = \int \frac{d\hat{x}d\hat{y}}{(2\pi)^2} e^{i[x\hat{x} + y\hat{y}]} \lim_{n \to 0 \ N \to \infty} \int dq d\hat{q} d\hat{R} \prod_{\alpha x'' y''} d\hat{P}_\alpha(x'', y'') \ e^{N \Psi[q, \dot{q}, \dot{Q}, \dot{R}, (\dot{\hat{P}})]} \frac{D[\hat{x}, \hat{y}]}{D[0, 0]} \]

with

\[ \Psi[\ldots] = i \sum_\alpha (\dot{Q}_\alpha + \dot{R}_\alpha R/\sqrt{Q}) + i \sum_\alpha \dot{q}_{\alpha \beta} q_{\alpha \beta} + i \sum_\alpha \int dx'' dy'' \ \hat{P}_\alpha(x'', y'') P[x'', y''] \]

\[ + \alpha \log D[0, 0] + \lim_{N \to \infty} \frac{1}{N} \sum_i \log \int d\sigma \ e^{-i \sum_\alpha [\dot{q}_{\alpha \beta} + \dot{R}_\alpha \tau_i] - i \sum_\alpha \dot{q}_{\alpha \beta} \sigma_i} \]

Finally we use that fact that the above expressions will be given by the intensive parts evaluated in the dominating saddle-point of \( \Psi \). We can use the expression for \( P[x, y] \) and its property \( \int dxdy P[x, y] = 1 \) to verify that all expressions are properly normalised (no overall prefactors are to be taken into account). We perform a simple transformation on some of our integration variables:

\[ \dot{q}_{\alpha \beta} \to \dot{q}_{\alpha \beta} - \dot{Q}_\alpha \delta_{\alpha \beta} \quad \dot{R}_\alpha \to \sqrt{Q} \dot{R}_\alpha \]

and finally we get

\[ A[x, y; x', y'] = \int \frac{d\hat{x}d\hat{y}'d\hat{y}''}{(2\pi)^4} e^{i[x\hat{x} + x'\hat{x}' + y'\hat{y} + y''\hat{y}']} \lim_{n \to 0 \ \hat{N} \to \infty} \frac{L[\hat{x}, \hat{y}; \hat{x}', \hat{y}']}{D^2[0, 0]} \]

\[ P[x, y] = \int \frac{d\hat{x}d\hat{y}}{(2\pi)^2} e^{i[x\hat{x} + y\hat{y}]} \lim_{n \to 0} \frac{D[\hat{x}, \hat{y}]}{D[0, 0]} \]

in which all functions are to be evaluated upon choosing for the order parameters the appropriate saddle-points of \( \Psi \) (variation with respect to \( q, \dot{q}, Q, \dot{R} \) and \{\dot{P}\}), which itself takes the form:

\[ \Psi[\ldots] = i \sum_\alpha (\dot{Q}_\alpha - q_{\alpha \alpha}) + i \dot{R} \sum_\alpha \dot{R}_\alpha + i \sum_\alpha \dot{q}_{\alpha \beta} q_{\alpha \beta} + i \sum_\alpha \int dx'' dy'' \ \hat{P}_\alpha(x'', y'') P[x'', y''] \]

\[ + \alpha \log D[0, 0] + \lim_{N \to \infty} \frac{1}{N} \sum_i \log \int d\sigma \ e^{-i \sum_\alpha \dot{R}_\alpha \sigma_i - i \sum_\alpha \dot{q}_{\alpha \beta} \sigma_i} \]

With \( D[\ldots] \) given by (83), which depends on the variational parameters \{\dot{P}\} and \( q_{\alpha \beta} \) only. The function \( L[\ldots] \) is given by (81). The order parameters \( q_{\alpha \beta} \) have the usual interpretation in terms of the average probability density for finding a mutual overlap \( q \) of two independently evolving weight vectors \( (J^a, J^b) \), in two systems \( a \) and \( b \) with the same realization of the training set (see e.g. [11]):

\[ \left< P(q) \right> \bigg|_{\Xi} = \left< \left< \left< \left< \delta \left[ q - \frac{J^a \cdot J^b}{|J^a||J^b|} \right] \right> \right> \right> \bigg|_{\Xi} = \lim_{n \to 0} \frac{1}{n(n-1)} \sum_{\alpha \neq \beta} \delta[q - q_{\alpha \beta}] \]

(88)

Note that upon applying the above procedure to the functions \( B[\ldots] \) and \( C[\ldots] \) in (62, 71) we find again integrals dominated by the dominant saddle-point of \( \Psi \); here, in view of (70) and (77), the intensive parts are zero, and thus

\[ B[x, y; x', y'] = C[x, y; x', y'; x'', y''] = 0 \]

as anticipated earlier.
A.3 Replica-Symmetric Saddle-Points

We now make the replica symmetric (RS) ansatz in the extremisation problem, which according to \cite{SS} is equivalent to assuming ergodicity. With a modest amount of foresight we put

\[ q_{\alpha \beta} = q_0 \delta_{\alpha \beta} + q[1 - \delta_{\alpha \beta}], \quad \hat{q}_{\alpha \beta} = \frac{1}{2} i [r - r_0 \delta_{\alpha \beta}], \quad \hat{R}_\alpha = i \rho, \quad \hat{Q}_\alpha = i \phi, \quad \hat{P}_\alpha[u, v] = i \chi[u, v] \]

This converts the quantity \( \Psi \) of equation (87) for small \( q \)

\[ \lim_{n \to 0} \frac{1}{n} \Psi[\ldots] = -\phi(1 - q_0) - \rho R + \frac{1}{2} q r - \frac{1}{2} q_0 (r - r_0) - \int dx'' dy'' \chi[x'', y''] P[x'', y''] \]

\[ \quad + \lim_{n \to 0} \frac{\alpha}{n} \log \mathcal{D}[0, 0] + \lim_{n \to 0} \frac{1}{nn_i} \log \int Dz \int d\sigma \ e^{\tau_i \rho \nu_1 \sum \sigma_\alpha - \frac{1}{2} r_0 \sum \sigma_\alpha^2 + z \sqrt{\tau} \sum \sigma_\alpha} \]

with the abbreviation \( Dz = (2\pi)^{-\frac{3}{2}} e^{-\frac{1}{4} \chi^2} dz \). We do the Gaussian integral in the last term, and expand the result for small \( n \):

\[ \lim_{n \to 0} \frac{1}{n} \Psi[\ldots] = -\phi(1 - q_0) - \rho R + \frac{1}{2} q r - \frac{1}{2} q_0 (r - r_0) - \frac{1}{2} \log r_0 + \frac{1}{2} r_0 (r + \rho^2 Q) \]

\[ - \int dx'' dy'' \chi[x'', y''] P[x'', y''] + \lim_{n \to 0} \frac{\alpha}{n} \log \mathcal{D}[0, 0] + \text{const} \]  

(90)

Note that ‘const’ refers to terms which do not depend on the order parameters to be varied, and will thus not show up in saddle-point equations; such terms can, however, depend on time via quantities such as \((Q, R)\). At this stage it is useful to work out four of our saddle-point equations:

\[ \frac{\partial \Psi}{\partial \phi} = \frac{\partial \Psi}{\partial r} = \frac{\partial \Psi}{\partial \rho} = \frac{\partial \Psi}{\partial r_0} = 0 : \quad q_0 = 1, \quad r_0 = \frac{1}{1 - q}, \quad \rho = \frac{R}{Q(1 - q)}, \quad r = \frac{qQ - R^2}{Q(1 - q)^2} \]

These allow us to eliminate most variational parameters, leaving a saddle-point problem involving only the function \( \chi[x, y] \) and the scalar \( q \):

\[ \lim_{n \to 0} \frac{1}{n} \Psi[q, \{\chi\}] = \frac{1 - R^2/Q}{2(1 - q)} + \frac{1}{2} \log(1 - q) - \int dx' dy' \chi[x', y'] P[x', y'] + \lim_{n \to 0} \frac{\alpha}{n} \log \mathcal{D}[0, 0; q, \{\chi\}] + \text{const} \]  

(91)

Finally we have to work out the RS version of \( \mathcal{D}[u, v; q, \{\chi\}] \):

\[ \mathcal{D}[u, v; \chi, q, 1] = \frac{\det \frac{1}{2} A}{(2\pi)^{(n+1)/2}} \int dx dy e^{-\frac{1}{2} \left( \begin{array}{ccc} x_1 & \cdots & q \\ \vdots & \ddots & \vdots \\ q & \cdots & 1 \end{array} \right) \left( \begin{array}{cccc} x_1 & \cdots & q & R/\sqrt{Q} \\ \vdots & \ddots & \vdots & \vdots \\ q & \cdots & 1 & R/\sqrt{Q} \\ R/\sqrt{Q} & \cdots & R/\sqrt{Q} & 1 \end{array} \right)} \]  

(92)

with

\[ A^{-1} = \left( \begin{array}{cccc} 1 & \cdots & q & R/\sqrt{Q} \\ \vdots & \ddots & \vdots & \vdots \\ q & \cdots & 1 & R/\sqrt{Q} \\ R/\sqrt{Q} & \cdots & R/\sqrt{Q} & 1 \end{array} \right) \]
The inverse of the above matrix is found to be

\[
A = \begin{pmatrix}
C_{11} & \cdots & C_{1n} & \gamma \\
\vdots & \ddots & \vdots & \vdots \\
C_{n1} & \cdots & C_{nn} & \gamma \\
\gamma & \cdots & \gamma & b
\end{pmatrix}
\]

\[C_{\alpha\beta} = \frac{\delta_{\alpha\beta}}{1-q} - d\]

\[
\gamma = -\frac{R\sqrt{Q}}{Q(1-q)} + \mathcal{O}(n)
\]

\[b = 1 + \mathcal{O}(n)\]

\[d = \frac{q-R^2/Q}{(1-q)^2} + \mathcal{O}(n)\]

With this expression, and upon linearising the terms in the exponents which are quadratic in \(x\) in the usual manner with Gaussian integrals, we obtain

\[
\mathcal{D}[u, v; q, \chi] = \int dx dy e^{-\frac{i}{\hbar} x A - \frac{b}{2q(1-q)} + \frac{1}{\alpha} \frac{Q}{4}(x^2 + y^2)} d\beta d\alpha
\]

\[
\gamma = \frac{1}{\alpha} \log n
\]

For the saddle-point problem we only need to calculate \(\lim_{n \to 0} \frac{\alpha}{n} \log \mathcal{D}[0, 0; q, \chi]\):

\[
\lim_{n \to 0} \frac{\alpha}{n} \log \mathcal{D}[0, 0; q, \chi] = \lim_{n \to 0} \frac{1}{n} \left\{ \log \int DzDy \left[ \int dx \ e^{-\frac{x^2}{2(1-q)} + \frac{1}{\alpha} \frac{Q}{4}(x^2 + y^2)} \right]^n - \log \int DzDy \left[ \int dx \ e^{-\frac{x^2}{2(1-q)} + \frac{1}{\alpha} \frac{Q}{4}(x^2 + y^2)} \right]^n \right\}
\]

\[
= \alpha \int DzDy \log \left\{ \frac{\int dx \ e^{-\frac{x^2}{2q(1-q)} + \frac{1}{\alpha} \frac{Q}{4}(x^2 + y^2)} \chi[x, y]} {\int dx \ e^{-\frac{x^2}{2q(1-q)} + \frac{1}{\alpha} \frac{Q}{4}(x^2 + y^2)} \chi[x, y]} \right\}
\]

with \(\gamma\) and \(d\) evaluated in the limit \(n \to 0\). Equivalently we can define

\[
A = R/Q(1-q) \quad B = \sqrt{qQ-R^2/Q(1-q)}
\]

which gives

\[
\lim_{n \to 0} -\frac{\alpha}{n} \log \mathcal{D}[0, 0; q, \chi] = \alpha \int DzDy \log \left\{ \frac{\int dx \ e^{-\frac{x^2}{2q(1-q)} + \frac{1}{\alpha} \frac{Q}{4}(x^2 + y^2)} \chi[x, y]} {\int dx \ e^{-\frac{x^2}{2q(1-q)} + \frac{1}{\alpha} \frac{Q}{4}(x^2 + y^2)} \chi[x, y]} \right\}
\]

Upon doing the \(x\)-integration in the denominator of this expression we can write the explicit expression for the surface \(\Psi\) to be extremised with respect to \(q\) and the function \(\chi[x, y]\), apart from irrelevant constants, in the surprisingly simple form (with the short-hand (\ref{94})):

\[
\lim_{n \to 0} \frac{1}{n} \Psi[q, \chi] = -\frac{1}{2(1-q)} + \frac{1}{2} (1-\alpha) \log(1-q) - \int dx dy \ \chi[x, y] P[x, y]
\]

\[
+ \alpha \int DzDy \ \log \left\{ \frac{\int dx \ e^{-\frac{x^2}{2q(1-q)} + \frac{1}{\alpha} \frac{Q}{4}(x^2 + y^2)} \chi[x, y]} {\int dx \ e^{-\frac{x^2}{2q(1-q)} + \frac{1}{\alpha} \frac{Q}{4}(x^2 + y^2)} \chi[x, y]} \right\}
\]

(95)
Note that (95) is to be minimised, both with respect to \( q \) (which originated as an \( n(n-1) \) fold entry in a matrix, leading to curvature sign change for \( n < 1 \)) and with respect to the function \( \chi[x, y] \) (obtained from the \( n \)-fold occurrence of the original function \( P \), multiplied by \( i \), which also leads to curvature sign change).

The remaining saddle point equations are obtained by variation of (95) with respect to \( \chi \) and \( q \). Functional variation with respect to \( \chi \) gives:

\[
\text{for all } x, y : \quad P[x, y] = \frac{e^{-\frac{1}{2} y^2}}{\sqrt{2\pi}} \int Dz \left\{ e^{-\frac{1}{2q(1-q)}x + x[AY + Bz] + \frac{1}{\alpha} \chi[x, y]} \right\} \tag{96}
\]

Note that \( P[x, y] = P[x|y]P[y] \) with \( P[y] = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2} y^2} \), as could have been expected. Next we vary \( q \), and use (96) wherever possible:

\[
\frac{1-\alpha-R^2/Q}{2(1-q)^2} = \frac{1-\alpha}{2(1-q)} = \alpha \int Dz Dy \left\{ \int dx e^{-\frac{x^2}{2q(1-q)}x + x[AY + Bz] + \frac{1}{\alpha} \chi[x, y]} \left[ \frac{x^2}{2q(1-q)} - x \left[ y \frac{\partial A}{\partial q} + z \frac{\partial B}{\partial q} \right] \right] \right\}
\]

giving

\[
\int dx dy P[x, y](x-Ry)^2 + (R^2-qQ)\frac{1}{\alpha} - 1
\]

\[
= \left[ 2\sqrt{qQ-R^2} + \frac{Q(1-q)}{\sqrt{qQ-R^2}} \right] \int Dz Dy \left\{ \int dx e^{-\frac{x^2}{2q(1-q)} + x[AY + Bz] + \frac{1}{\alpha} \chi[x, y]} \right\}
\]

\[
\int dx e^{-\frac{x^2}{2q(1-q)} + x[AY + Bz] + \frac{1}{\alpha} \chi[x, y]} - x \left[ y \frac{\partial A}{\partial q} + z \frac{\partial B}{\partial q} \right] \right\}
\]

\[
\int dx dy P[x, y](x-Ry)^2 + (R^2-qQ)\frac{1}{\alpha} - 1\right]
\]

A.4 Explicit Expression for the Green’s Function

In order to work out the Green’s function (95) we need the function \( L[u, v; u', v'] \) as defined in (81) which, in turn, is given in terms of the integrals (83-84). First we calculate the \( n \to 0 \) limit of \( D[u, v; q, \{ \chi \}] \) (93), and simplify the result with the saddle-point equation (96):

\[
\lim_{n \to 0} D[u, v; q, \{ \chi \}] = \int Dz Dy e^{-iv \chi} \left\{ \int dx e^{-\frac{x^2}{2q(1-q)} + x[AY + Bz] + \frac{1}{\alpha} \chi[x, y]} \right\}
\]

\[
= \int dx dy P[x, y] e^{-iv \chi} \int dx e^{-\frac{x^2}{2q(1-q)} + x[AY + Bz] + \frac{1}{\alpha} \chi[x, y]} \right\}
\]

Next we work out the quantities \( F_{1,2}^n [u, v] \) of equation (84) in RS ansatz, using Gaussian linearizations:

\[
\lim_{n \to 0} F_{1,2}^n [u, v] = \lim_{n \to 0} \int dx dy \left\{ \frac{\partial_1 \chi[\sqrt{Q} x]\sqrt{Q} x, y]}{e} \right\}
\]

\[
= \lim_{n \to 0} \int dx dy \left\{ \frac{\partial_1 \chi[\sqrt{Q} x]\sqrt{Q} x, y]}{e} \right\}
\]

\[
= \lim_{n \to 0} \int dx dy \left\{ \frac{\partial_1 \chi[\sqrt{Q} x]\sqrt{Q} x, y]}{e} \right\}
\]
With the Fourier transforms of the functions $G$, we obtain, upon performing the summations over replica indices in (81):

$$G^2 = \int \sum_\beta \left[ \frac{1}{2} \hat{x}_\beta^2 + |z\sqrt{\Delta - \gamma}|x_\beta + \frac{1}{2} \chi[\sqrt{Q}x_\beta, y] \right] - iux_1 \sqrt{Q} \partial_\lambda \chi[\sqrt{Q}x_\alpha, y]$$

The replica permutation symmetries of this expression allow us to conclude

$$\lim_{n \to 0} F^1_\lambda[u, v] = \delta_{\alpha 1} F^1_\lambda[u, v] + (1 - \delta_{\alpha 1}) F^2_\lambda[u, v]$$

where

$$F^1_{1,2}[u, v] = i \int dx dy P[x, y] e^{-ivy - iux_1 \partial_1 \chi[x, y]}$$

$$F^2_{1,2}[u, v] = i \int Dz e^{-iyv} \left[ \frac{1}{2} \partial_1^2 + x[Ay + Bz] + \frac{1}{2} \chi[x, y] \right] \left( \frac{1}{2} \partial_1^2 + x[Ay + Bz] + \frac{1}{2} \chi[x, y] \right)$$

We can now proceed to the calculation of (81). First we note that the basic building blocks of (81) are most easily expressed in terms of the functions

$$G_1[u, v] = \frac{1}{\alpha} F^1_1[u, v] + u D[u, v]$$

$$G_1[u, v] = \frac{1}{\alpha} F^2_1[u, v]$$

$$G_2[u, v] = \frac{1}{\alpha} F^1_2[u, v] + v D[u, v]$$

$$G_2[u, v] = \frac{1}{\alpha} F^2_2[u, v]$$

With these short-hands we obtain, upon performing the summations over replica indices in (81):

$$\mathcal{L}[u, v; u', v'] = -Q(1 - q) G_1[u, v] G_1[u', v'] - Q(1 - q)(n - 1) \tilde{G}_1[u, v] \tilde{G}_1[u', v']$$

$$-Qq \left[ G_1[u, v] + (n - 1) \tilde{G}_1[u, v] \right] \left[ G_1[u', v'] + (n - 1) \tilde{G}_1[u', v'] \right]$$

$$-R \left[ G_1[u, v] + (n - 1) \tilde{G}_1[u, v] \right] \left[ G_2[u', v'] + (n - 1) \tilde{G}_2[u', v'] \right]$$

$$-R \left[ G_1[u', v'] + (n - 1) \tilde{G}_1[u', v'] \right] \left[ G_2[u, v] + (n - 1) \tilde{G}_2[u, v] \right]$$

$$- \left[ G_2[u, v] + (n - 1) \tilde{G}_2[u, v] \right] \left[ G_2[u', v'] + (n - 1) \tilde{G}_2[u', v'] \right]$$

and so

$$\lim_{n \to 0} \mathcal{L}[u, v; u', v'] = -Q(1 - q) \left[ G_1[u, v] G_1[u', v'] - \tilde{G}_1[u, v] \tilde{G}_1[u', v'] \right]$$

$$-Qq \left[ G_1[u, v] - \tilde{G}_1[u, v] \right] \left[ G_1[u', v'] - \tilde{G}_1[u', v'] \right]$$

$$-R \left[ G_1[u, v] - \tilde{G}_1[u, v] \right] \left[ G_2[u', v'] - \tilde{G}_2[u', v'] \right] - R \left[ G_1[u', v'] - \tilde{G}_1[u', v'] \right] \left[ G_2[u, v] - \tilde{G}_2[u, v] \right]$$

$$- \left[ G_2[u, v] - \tilde{G}_2[u, v] \right] \left[ G_2[u', v'] - \tilde{G}_2[u', v'] \right]$$

With the Fourier transforms of the functions $G[\ldots]$, given by

$$\hat{G}_1[u, v] = \int \frac{dudv}{(2\pi)^2} e^{iuv - iv\hat{v}} \left[ \frac{1}{\alpha} F^1_1[u, v] + u D[u, v] \right]$$

$$\bar{G}_1[u, v] = \frac{1}{\alpha} \int \frac{dudv}{(2\pi)^2} e^{iu\hat{u} + iv\hat{v}} F^1_1[u, v]$$

$$\hat{G}_2[u, v] = \int \frac{dudv}{(2\pi)^2} e^{iuv - iv\hat{v}} \left[ \frac{1}{\alpha} F^1_2[u, v] + v D[u, v] \right]$$

$$\bar{G}_2[u, v] = \frac{1}{\alpha} \int \frac{dudv}{(2\pi)^2} e^{iu\hat{u} + iv\hat{v}} F^2_1[u, v]$$
the Green’s function $\mathcal{A}[x, y; x', y']$ (34) can now be written in explicit form as

$$\mathcal{A}[x, y; x', y'] = -Q(1-q) \left[ \hat{G}_1[x, y] \hat{G}_1[x', y'] - \overline{G}_1[x, y] \overline{G}_1[x', y'] \right]$$

$$-Qq \left[ \hat{G}_1[x, y] - \overline{G}_1[x, y] \right] \left[ \hat{G}_1[x', y'] - \overline{G}_1[x', y'] \right]$$

$$-R \left[ \hat{G}_1[x, y] - \overline{G}_1[x, y] \right] \left[ \hat{G}_2[x', y'] - \overline{G}_2[x', y'] \right] - R \left[ \hat{G}_1[x', y'] - \overline{G}_1[x', y'] \right] \left[ \hat{G}_2[x, y] - \overline{G}_2[x, y] \right]$$

$$- \left[ \hat{G}_2[x, y] - \overline{G}_2[x, y] \right] \left[ \hat{G}_2[x', y'] - \overline{G}_2[x', y'] \right]$$

Finally, working out the four relevant Fourier transforms, using (108-110), gives:

$$\hat{G}_1[x, y] = iP[y] \left[ \frac{1}{\alpha} \frac{\partial}{\partial x} \chi[x, y] - \frac{\partial}{\partial x} \log P[x, y] \right]$$

$$\hat{G}_2[x, y] = iP[y] \left[ \frac{1}{\alpha} \frac{\partial}{\partial y} \chi[x, y] - \frac{\partial}{\partial y} \log P[x, y] \right]$$

$$\overline{G}_1[x, y] = \frac{i}{\alpha} P[y] \int Dz \frac{\int dx' e^{-\frac{x'^2}{2Q(1-q)} + x'\left[Ay+Bz + \frac{1}{\alpha} \chi[x', y]\right]} \partial_1 \chi[x', y]}{\int dx' e^{-\frac{x'^2}{2Q(1-q)} + x'\left[Ay+Bz + \frac{1}{\alpha} \chi[x', y]\right]}^2}$$

$$\overline{G}_2[x, y] = \frac{i}{\alpha} P[y] \int Dz \frac{\int dx' e^{-\frac{x'^2}{2Q(1-q)} + x'\left[Ay+Bz + \frac{1}{\alpha} \chi[x', y]\right]} \partial_2 \chi[x', y]}{\int dx' e^{-\frac{x'^2}{2Q(1-q)} + x'\left[Ay+Bz + \frac{1}{\alpha} \chi[x', y]\right]}^2}$$

with $P[y] = (2\pi)^{-\frac{3}{2}} e^{-\frac{y^2}{2}}$.

Since the distribution $P[x, y]$ obeys $P[x, y] = P[x]\, P[y]$ with $P[y] = (2\pi)^{-\frac{3}{2}} e^{-\frac{y^2}{2}}$, our equations can be simplified by choosing as our order parameter function the conditional distribution $P[x|y]$. We also replace the conjugate order parameter function $\chi[x, y]$ by the effective measure $M[x, y]$, and we introduce a compact notation for the relevant averages in our problem:

$$M[x, y] = e^{-\frac{x^2}{2Q(1-q)} + Ay + \frac{1}{\alpha} \chi[x, y]} \quad (f[x, y, z])_* = \frac{\int dx \, M[x, y] e^{By} f[x, y, z]}{\int dx \, M[x, y] e^{By}}$$

Instead of the original Green’s function $\mathcal{A}[x, y; x', y']$ we turn to the transformed Green’s function $\tilde{\mathcal{A}}[x, y; x', y']$, defined as

$$\tilde{\mathcal{A}}[x, y; x', y'] = P[x, y] \tilde{\mathcal{A}}[x, y; x', y'] P[x', y']$$

With these notational conventions one finds that (106) translates into the following expression:

$$\tilde{\mathcal{A}}[x, y; x', y'] = Q(1-q) \left[ J_1[x, y] J_1[x', y'] - \tilde{J}_1[x, y] \tilde{J}_1[x', y'] \right] + Qq \left[ J_1[x, y] - \tilde{J}_1[x, y] \right] \left[ J_1[x', y'] - \tilde{J}_1[x', y'] \right]$$

$$+ R \left[ J_1[x, y] - \tilde{J}_1[x, y] \right] J_2[x', y'] + R \left[ J_1[x', y'] - \tilde{J}_1[x', y'] \right] J_2[x, y] + J_2[x, y] J_2[x', y']$$

with

$$J_1[X, Y] = \frac{\partial}{\partial X} \log \frac{M[X, Y]}{P[X\, Y]} + \frac{X - RY}{Q(1-q)}$$

38
\[
J_1[X, Y] = P[X|Y]^{-1} \int Dz \left( \frac{\partial}{\partial x} \log M[x, Y] + \frac{x - RY}{Q(1-q)} \right) \langle \delta[X - x] \rangle_{\ast} \\
J_2[X, Y] = \frac{\partial}{\partial Y} \log \frac{M[X, Y]}{P[X|Y]} - \frac{RX}{Q(1-q)} + Y - P[X|Y]^{-1} \int Dz \left( \frac{\partial}{\partial Y} \log M[x, Y] - \frac{Rx}{Q(1-q)} \right) \langle \delta[X - x] \rangle_{\ast}
\]

It turns out that significant simplification of the result (11.2) is possible, upon using the following two identities to rewrite the functions \(J_1[\ldots], \tilde{J}_1[\ldots]\) and \(J_2[\ldots] \):

\[
\langle \frac{\partial}{\partial x} \log M[x, y] \rangle_{\ast} = -Bz \quad (113)
\]
\[
\langle \frac{\partial}{\partial y} \log M[x, y] \rangle_{\ast} = \frac{\partial}{\partial y} \log \int dx e^{Bxz} M[x, y] \quad (114)
\]

Identity (113) results upon integrating by parts with respect to \(x\), whereas identity (114) is a direct consequence of \(y\) dependencies occurring in \(M[x, y]\) only. Note that \(B = \sqrt{qQ - R^2/Q(1-q)}\). To achieve the desired simplification of \(\tilde{A}[x, y; x', y']\) we define the following object:

\[
\Phi[X, y] = \left\{ Q(1-q)P[X|y] \right\}^{-1} \int Dz \langle X - x \rangle_{\ast} \langle \delta[X - x] \rangle_{\ast} \\
\]

(115)

We can now, after additional integration by parts with respect to \(z\), simplify the above expressions for \(J_1[\ldots], \tilde{J}_1[\ldots]\) and \(J_2[\ldots]\) to

\[
J_1[X, Y] = \frac{X - RY}{Q(1-q)} - \frac{qQ - R^2}{Q(1-q)} \Phi[X, Y] \\
\tilde{J}_1[X, Y] = J_1[X, Y] - \Phi[X, Y]
\]
\[
J_2[X, Y] = Y - R\Phi[X, Y]
\]

and consequently

\[
A[x, y; x', y'] = P[x, y] \tilde{A}[x, y; x', y'] P[x', y'] \\
\tilde{A}[x, y; x', y'] = yy' + (x - Ry)\Phi[x', y'] + (x' - Ry')\Phi[x, y] - (Q - R^2)\Phi[x, y] \Phi[x', y'] \\
\]

with \(\Phi[x, y]\) as given in (115).