Couplings between generalized gauge fields

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Abstract

We analyze the BRST field-antifield construction for generalized gauge fields consisting of massless mixed representations of the Lorentz Group and we calculate all the strictly gauge invariant interactions between them. All these interactions are higher derivative terms constructed out from the derivatives of the curl of field strength.

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1 Introduction

Generalized gauge fields consisting of free massless integer spin mixed representations of the Lorentz group (mixed tensors) can be used as models for higher spin covariant bosonic theories and to study the long standing problem of interactions for massless higher spin particles. A geometric formulation for this type of theories is still an open question.

Another source of interest for this type of theories comes from the formulation of D=11 dimensional supergravity as a gauge theory for the $osp(32|1)$ superalgebra. In addition to the vielbein $e^a_\mu$, the spin connection $\omega^{ab}_\mu$ and the gravitino $\psi_\mu$ the connection for $osp(32|1)$ contains a totally antisymmetric fifth rank Lorentz tensor one form $\omega^{\mu abcde}_\mu$. The antisymmetric part of this tensor could be identified to an abelian six form $A_{[abcdef]}$ but the mixed tensor part does not have any related counterpart in the standard D=11 supergravity theory.

Free consistent massless theories whose field content is this class of mixed tensors have attracted some attention in the past. In particular, possible lagrangians, compatible with gauge invariance, have been proposed in [5, 6, 7] while the complete ghost spectrum and the BRS operator are described in [5, 6, 8].

In this article we extend the BRS analysis to include antifields and calculate the cohomology of the “longitudinal exterior derivative” in order to obtain all the consistent, strictly gauge invariant interactions that can be added to the free theory.

For simplicity, we will concentrate on tensors with three indices which satisfy the following identities:

$$ T_{[ab]c} = -T_{[ba]c}, \quad T_{[ab]c} + T_{[ca]b} + T_{[bc]a} = 0; \quad (1) $$

we comment at the end of the paper on how to generalize our results to other mixed tensors. In terms of Young diagrams the fields (1) are represented by $\begin{array}{c} a \ 0 \ c \\ b \end{array}$.

The paper is organized as follows. First we present the analysis of the model within the Hamiltonian formalism and we rederive some of the results obtained in [7]. Then we present the BRST field-antifield formalism for the theory. After that we obtain the consistent vertices of the theory by calculating the cohomology of the exterior longitudinal derivative. Finally,
we discuss how the method can be applied to other higher rank Lorentz tensors corresponding to more complicated Young diagrams.

2 Hamiltonian formalism

Using the ‘Hook’ formula [9, 10], one easily calculates that the tensors (1) have $\frac{1}{2}D(D-1)(D+1)$ components in $D$ dimensions.

The lagrangian of the theory is,

$$L = -\frac{1}{12} \left( F_{[abc]} F^{[abc]} - 3 F_{[abx]} F^{[abx]} + F_{[ayb]} F^{[ayb]} \right),$$

(2)

where $F_{[abc]} = \partial_a T_{[bc]} + \partial_b T_{[ca]} + \partial_c T_{[ab]}$ and the corresponding action is invariant under the gauge transformations,

$$\delta \epsilon, \delta \eta = \partial_a \epsilon_{bc} - \partial_b \epsilon_{ac} + \partial_a \eta_{bc} - \partial_b \eta_{ac} - 2 \partial_c \eta_{ab},$$

(3)

where $\epsilon_{ab}$ are symmetric and $\eta_{ab}$ are antisymmetric gauge parameters. The Euler-Lagrange equations are,

$$\frac{\delta L}{\delta T_{[abc]}} = \frac{1}{2} E_{[abc]} - \frac{1}{4} (g^{bc} E^a - g^{ac} E^b) = 0,$$

(4)

where $E_{[abc]} = \partial^a F_{[abcd]} - \partial_c F_{[abx]} - 6 F_{[i0j]} F_{[ij]0} + 6 F_{[0ik]} F_{[ijkl]} - 3 F_{[ij]} F_{[ijkl]}.$ By taking the trace of (3) we see that the equations of motion can be equivalently written,

$$E_{[abc]} = 0, \quad E_a = 0.$$  

(5)

These equations of motion satisfy the Noether identities,

$$\partial_a E_{[abc]} + \frac{1}{2} \partial^c E^b \equiv 0, \quad \partial_a E^a \equiv 0.$$  

(6)

To proceed with the Hamiltonian analysis we write (2) as,

$$L = -\frac{1}{12} \left( 3 F_{[0ij]} F^{[0ij]} + F_{[ijk]} F^{[ijk]} - 6 F_{[ij0]} F^{[ij]} + 6 F_{[0ik]} F^{[0ik]} + 3 F_{[ij]} F^{[ijkl]} \right).$$  

(7)
Therefore the momenta are given by,

$$
\pi_{[ij]^k} = \frac{\partial L}{\partial \partial_0 T_{[ij]^k}} = -F_{[0ijkl]} + g^{ik}F_{[0kl]l}, \quad (8)
$$

$$
\pi_{[ij]_0} = \frac{\partial L}{\partial \partial_0 T_{[ij]_0}} = F_{[ijk]}, \quad (9)
$$

where we have used the convention, \( \frac{\partial T_{[ab]c}}{\partial T_{[de]f}} = (\delta_a^d \delta_b^e - \delta_a^e \delta_b^d) \delta_c^f \). The theory has \((D - 1)^2 + (D - 1)\) primary constraints, namely: \( \pi_{[0i]_0} = 0 \) and \( \pi_{[0i]^k} = 0 \).

The canonical Hamiltonian is then given by,

$$
H_c = \frac{1}{2} \pi_{[ab]^c} \partial_0 T_{[ab]c} - L \quad (10)
$$

$$
= \frac{1}{4} \pi_{[ij]^k} \pi_{[ij]_0} - \frac{1}{2(D - 3)} \pi_{[i]^l} \partial_0 \pi_{[i]_0}^l + \frac{1}{12} F_{[ijkl]} F_{[ijkl]}^d + \frac{1}{4} F_{[ijkl]} F_{[ijkl]}^d T_{[j_0]^i} \partial_0 \pi_{[ij]_0} + T_{[j_0]_0} \partial_0 \pi_{[ij]^k}.
$$

Using the Poisson-Bracket,

$$
\{T_{[lm]n}(x), \pi_{[ij]^c}(y)\} = \delta_n^c (\delta_i^l \delta_j^m - \delta_i^m \delta_j^l) \delta(x - y), \quad (11)
$$

one easily obtains the secondary constraints:

$$
\partial_i \pi_{[ij]^k} = 0, \quad \partial_i \pi_{[ij]_0} = 0. \quad (12)
$$

These can also be read out from (11) since \( T_{[j_0]_0} \) and \( T_{[j_0]_0} \) play the role of Lagrange multipliers. The number of secondary class constraints is the same as the primary constraints. However the former provide a reducible set because we have identically,

$$
\partial_j \partial_i \pi_{[ij]^k} \equiv 0, \quad \partial_j \partial_i \pi_{[ij]_0} \equiv 0. \quad (13)
$$

The total number of effective constraints is thus \( 2((D - 1)^2 + (D - 1)) - D = D(2D - 3) \) and therefore the theory possesses \( 2 + D(6D - 1)(D + 1) - 2D(2D - 3) = \frac{1}{3} D(D - 2)(D - 4) \) degrees of freedom (all the constraints are first class). Note that in 4 dimensions the theory is therefore trivial.
The identities (13) induce reducibility identities among the gauge transformations and are responsible for the presence of ghosts of ghosts in the BRS ghost spectrum. Indeed, the gauge variations vanish for the choices, 

\[ \epsilon_{ab} = 3(\partial_a C_b + \partial_b C_a), \]

\[ \eta_{ab} = \partial_a C_b - \partial_b C_a, \]

where \( C_a \) are \( D \) arbitrary functions.

3 BRST field-antifield formalism

According to the general rules of the BRST field-antifield formalism [11], the BRST differential \( s \) is constructed as follows. First, one defines a differential \( \delta \) called the Koszul-Tate differential whose role is to implement the equations of motion in cohomology. Therefore, we first introduce the antifields \( T^{*}_{ab|c} \) which satisfy,

\[ \delta T^{*}_{ab|c} = -\frac{\delta \mathcal{L}}{\delta T_{ab|c}} = -\frac{1}{2} E^{[ab|c]c} + \frac{1}{4}(g^{bc}E^{a} - g^{ac}E^{b}). \] (16)

Because of the presence of Noether identities among the Euler-Lagrange equations, we also need the antifields \( T^{*}_{abc} \) which satisfy,

\[ \delta T^{*}_{abc} = \partial_a T^{*}_{[ab|c]}c. \] (17)

Finally, we introduce the antifields \( T^{*}_{c} \) which satisfy,

\[ \delta T^{*}_{c} = \partial_b T^{*}_{bc}. \] (18)

These are present because the theory is reducible. The roles of \( T^{*}_{bc} \) and \( T^{*}_{c} \) are respectively to eliminate from the cohomology of \( \delta \) the terms \( \partial_a T^{*}_{[ab|c]}c \) and \( \partial_b T^{*}_{abc} \) which would otherwise be present. Each antifield carries a degree called the ‘antighost’ number which is given by, \( \text{antighost}(T^{*}_{[ab|c]}) = 1, \text{antighost}(T^{*}_{bc}) = 2, \text{antighost}(T^{*}_{c}) = 3 \) and \( \text{antighost}(\delta) = -1 \).

Because of the definitions (16), (17) and (18), the cohomology of \( \delta \) in the algebra generated by the fields, the antifields and their derivatives is given by the on-shell functions; in other words, \( \delta \) provides a resolution of the stationary surface.
More important to us in the BRST construction is the longitudinal exterior derivative $\gamma$ which takes into account the gauge invariance of the model. In our case, we first need to introduce the ghosts $S_{ab}$ and $A_{ab}$ in place of each gauge parameter according to the definition,

$$\gamma T_{[ab]c} = \partial_a S_{bc} - \partial_b S_{ac} + \partial_a A_{bc} - \partial_b A_{ac} - 2\partial_c A_{ab},$$

(19)

where $S_{ab}$ and $A_{ab}$ are respectively symmetric and antisymmetric in $ab$.

Because the gauge transformations are reducible we also need the ghosts of ghosts $C_a$ which satisfy,

$$\gamma S_{ab} = 3(\partial_a C_b + \partial_b C_a),$$

(20)

$$\gamma A_{ab} = \partial_a C_b - \partial_b C_a.$$  

(21)

With these definitions, we have $\gamma^2 = 0$. A grading called the ‘pureghost’ number is associated to the ghost fields and we have: $\text{pureghost}(S_{ab}) = \text{pureghost}(A_{ab}) = 1$, $\text{pureghost}(C_a) = 2$ and also $\text{pureghost}(\gamma) = 1$. Note that the fields and their derivatives are of antighost and pureghost number 0 and that $\gamma(\text{antifields}) = \delta(\text{ghosts}) = 0$.

For the model we consider, the full BRST differential is simply given by the sum of the Koszul-Tate differential and the longitudinal exterior derivative: $s = \delta + \gamma$. The grading of $s$ is called the ‘ghost’ number and is given by $\text{ghost} = \text{pureghost} - \text{antighost}$.

It is obvious from the above definitions that a combination of the fields and their derivative will be strictly gauge invariant if and only if it defines an element of the cohomology $H(\gamma)$ of the differential $\gamma$.

It was shown in [12] that the classification of all the consistent interactions that can added to a free action can be obtained by calculating in ghost number zero the cohomology $H(s|d)$ of the BRST differential $s$. These consistent interactions can be grouped in three categories. In the first one, we have the vertices which are strictly gauge invariant; their study only requires the calculation of the $\gamma$-cohomology. The second category consists of interactions which are gauge invariant up to a boundary term. Finally the last category contains the vertices which are gauge invariant up to a boundary term on-shell. These last interactions therefore require a modification of the gauge transformations and correspond to antifield dependent elements of the BRST cohomology whereas the first two categories correspond to antifield independent solutions of $H(s|d)$. 

5
In this article we calculate all the consistent interactions of the first category, i.e., the gauge invariant functions.

4 Gauge invariant functions

In this section we obtain our main result, namely, we calculate all the gauge invariant terms that can be added to the free lagrangian. As we recalled, these are polynomials in the fields and their derivatives which belong to the cohomology of the differential $\gamma$. However, because its study is important in order to obtain the other consistent interactions [13], e.g., to solve the ‘descent equations’, we will calculate $H(\gamma)$ in the full algebra generated by the fields, the ghosts, the antifields and their derivatives.

The procedure we use is based on the following result, sometimes referred to as the “Basic Lemma” [11]:

**Lemma 1** Let $A$ be the polynomial algebra generated by the algebraically independent variables $x^i$, $y^\alpha$, $z^\alpha$ and let $D$ be a differential whose action on the variables is:

$$Dx^i = 0, \quad Dy^\alpha = z^\alpha. \quad (22)$$

The cohomology of $D$ in $A$, $H(D) \equiv \frac{\text{Ker } D}{\text{Im } D}$, is then given by the polynomials in the $x^i$.

In our case the algebra $A$ is the algebra generated by the fields, the antifields, the ghosts, the ghosts of ghosts and all their derivatives. Our task is thus to redefine all our generators in such a way that they obey (22).

First of all, let us note that the antifields and their derivatives are all $\gamma$-closed and do not appear in the $\gamma$ variations of the other fields. This implies that they are automatically part of the $x^i$ variables.

For the other variables, we denote by $V^k$ the vector space spanned by $\partial_{s_1...s_k} C_a$, $\partial_{s_1...s_k-1} A_{ab}$, $\partial_{s_1...s_k-1} S_{ab}$, $\partial_{s_1...s_{k-2}} T_{[abc]}$. Our hole algebra is $A = \bigoplus_k V^k$ and $\gamma$ has a well defined action in each $V^k$. We will therefore look for new coordinates in each $V^k$.

In $V^0$ we have,

$$\gamma C_a = 0. \quad (23)$$

$C_a$ is therefore a variable of type $x^i$. 

6
In $V^1$ the variables split as in (20), (21). $S_{ab}$ and $A_{ab}$ are therefore of type $y^\alpha$ while the symmetrized and antisymmetrized first order derivatives of the $C_a$ are of type $z^{\alpha}$.

For the higher order $V^k$, the analysis proceed in the same fashion but the algebra becomes more involved. The calculus is simplified by the systematic use of Young diagrams in order to decompose the variables into irreducible parts.

Let us first examine the fields and their derivatives. Because partial derivatives commute, the $k$-th order derivatives of the fields can be symbolically represented by the following tensor product of Young diagrams:

$$\partial d_1...d_k T_{[abc]} \equiv \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a & c
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 b
\end{array}
\end{array}
\end{array} \otimes \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 d_1 & ... & d_k
\end{array}
\end{array}
\end{array} .$$

According to the general theory of the representations of the symmetric group, this tensor product decomposes into the following irreducible components under a general invertible transformation,

$$\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a & c
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 b
\end{array}
\end{array}
\end{array} \otimes \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 d_1 & ... & d_k
\end{array}
\end{array}
\end{array} \equiv \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a & c & d_1 & ... & d_k
\end{array}
\end{array}
\end{array} \oplus \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a & c & d_1 & ... & d_k
\end{array}
\end{array}
\end{array} \oplus \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a & c & d_1 & ... & d_k
\end{array}
\end{array}
\end{array} \oplus \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a & c & d_1 & ... & d_k
\end{array}
\end{array}
\end{array} \oplus \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a & c & d_1 & ... & d_k
\end{array}
\end{array}
\end{array} \oplus \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a & c & d_1 & ... & d_k
\end{array}
\end{array}
\end{array} \oplus \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a & c & d_1 & ... & d_k
\end{array}
\end{array}
\end{array} \oplus \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a & c & d_1 & ... & d_k
\end{array}
\end{array}
\end{array} \oplus \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a & c & d_1 & ... & d_k
\end{array}
\end{array}
\end{array} \oplus \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a & c & d_1 & ... & d_k
\end{array}
\end{array}
\end{array} \oplus \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a & c & d_1 & ... & d_k
\end{array}
\end{array}
\end{array} ,$$

where $l = k - 1$.

The combinations of $\partial d_1...d_k T_{[abc]}$ represented by the above diagrams are respectively denoted by $R^T_{abcd_1...d_k}$, $H^T_{abcd_1...d_k}$, $F^T_{abcd_1...d_k}$, and $E^T_{abcd_1...d_k}$. They are obtained by first symmetrizing $\partial d_1...d_k T_{[abc]}$ according to every line and then antisymmetrizing the result according to every column. By convention, for every symmetrization or antisymmetrization, we divide the corresponding sum of terms by a factorial term. For example, $\frac{1}{3!}(Y_{ijk} + Y_{jik} + Y_{kji} + Y_{ikj} + Y_{jki} + Y_{kij})$. The variables $R, H, F$ and $E$ form a new basis for the space spanned by $\partial d_1...d_k T_{[abc]}$.

In exactly the same way, the $(k + 1)$-th order derivatives of the ghosts $A_{ab}$ are decomposed according to,
The diagrams of the rhs of the above decomposition are denoted respectively by $R_{abcd_1...d_k}^A$ and $F_{abcd_1...d_k}^A$. Note that in the case $k = 0$ the above notation is not well adapted because the diagram is missing. In that case, we denote the corresponding combination of $\partial_c A_{ab}$ by $F_A^{abc}$.

For the $(k + 1)$-th order derivatives of the $S_{ac}$ we have,

$$
\begin{align*}
\text{The components of the decomposition are denoted respectively, } &L_S^{abcd_1...d_k}, \\
&\begin{cases}
R_S^{abcd_1...d_k}
\end{cases}, \\
&\begin{cases}
H_S^{abcd_1...d_k}
\end{cases}.
\end{align*}
$$

Finally, the $(k + 2)$-th derivatives of the $C_a$ decompose according to,

$$
\begin{align*}
\text{The two different components are denoted } &L_{C}^{abcd_1...d_k} \text{ and } R_{C}^{abcd_1...d_k}.
\end{align*}
$$

With the above definitions, an explicit calculation shows that we have the following relations among the variables:

\begin{align*}
\gamma R_{abcd_1...d_k}^T &= 3R_{abcd_1...d_k}^A + \frac{k + 3}{2} R_{abcd_1...d_k}^S, \\
\gamma H_{abcd_1...d_k}^T &= \frac{k + 2}{2} H_{abcd_1...d_k}^S, \\
\gamma F_{abcd_1...d_k}^T &= 3F_{abcd_1...d_k}^A, \\
\gamma L_{abcd_1...d_k}^S &= 6L_{abcd_1...d_k}^C, \\
\gamma R_{abcd_1...d_k}^S &= 3R_{abcd_1...d_k}^C, \\
\gamma E_{abcd_1...d_k}^T &= 0, \\
\gamma F_{abc}^A &= 0.
\end{align*}
The variables now obey the relations (22). We see that up to numerical factors, the operator $\gamma$ groups in pairs the various Young diagrams of the same symmetry type. Combinations of the variables corresponding to Young diagrams which don’t belong to a pair have a vanishing $\gamma$-variation and are not $\gamma$-exact. From (23), (24) and (34) we conclude that the cohomology of $\gamma$ is generated by the variables $C_a$, $F^{A}_{abc}$, $E^T_{abcd...d_k}$, the antifields and their derivatives.

Any gauge invariant function made up of the fields and their derivatives is thus a product of $E^T_{abcd...d_k}$ which are derivatives of the curl of the field strength. As a consequence, we note that in order to built gauge invariant interactions we have to use at least second order derivatives of the fields.

5 Conclusions

In this short paper we have calculated the cohomology of the longitudinal exterior derivative $\gamma$ for a theory containing generalized gauge fields represented by the Young diagram $\begin{array}{c} a \\ b \end{array}$. From this study we are able to deduce all the possible gauge invariant interactions that can be constructed from the fields and their derivatives. We have shown that they are at least of second order in derivatives.

The method we use is based on the technic of Young diagrams and the ‘Basic lemma’ in which one tries to associate in pairs diagrams which are of the same symmetry type. This method can be generalized to higher rank mixed tensors. For example, tensors corresponding to Young diagrams with $\begin{array}{c} a \\ b \\ \vdots \end{array}$
an arbitrary number of boxes in the first column, i.e. $\begin{array}{c} a \\ b \\ \vdots \end{array}$, are treated exactly along the lines of this article. For tensor with more boxes on the first line, e.g. $\begin{array}{c} a \\ b \\ \vdots \end{array}$, one has to impose trace conditions in order to build suitable lagrangians. The decomposition into irreducible components performed in section IV then has to be made with respect to $O(n)$ instead of $GL(n)$ to preserve the traces of tensors.

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References

[1] J.W. van Holten and A. van Proyen, *J. Phys. A* **15** (1982) 3763.

[2] P.K. Townsend, [hep-th/9507048](http://arxiv.org/abs/hep-th/9507048).

[3] R. Troncoso and J. Zanelli, [hep-th/9710180](http://arxiv.org/abs/hep-th/9710180).

[4] P.K. Townsend, H. Nicolai and P. van Nieuwenhuizen, *Lett. Nuovo Cimento* **30** (1981) 315.

[5] C.S. Aulakh, I.G. Koh and S. Ouvry, *Phys. Lett.* **B173** (1986) 284.

[6] K.S. Chung, C.W. Han, J.K. Kim and I.G. Koh, *Phys. Rev.* **D37** (1988) 1079.

[7] T. Curtright, *Phys. Lett.* **165B** (1985) 304.

[8] J.M.F. Labastida and T.R. Morris, *Phys. Lett.* **B180** (1986) 101.

[9] N.V. Dragon, *Tensor Algebra and Young Tableaux*, HD-THEP-81-16.

[10] M. Hamermesh, *Group Theory*, Addison Wesley, (1962).

[11] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems*, Princeton University Press, (1992).

[12] G. Barnich and M. Henneaux, *Phys. Lett.* **B311** (1993) 123.

[13] M. Henneaux, *Consistent Interactions Between Gauge Fields: The Cohomological Approach*, [hep-th/9712220](http://arxiv.org/abs/hep-th/9712220) International Conference on *Secondary calculus and cohomological Physics*, Moscow, August 1997.