On an integrable deformation of Kapustin-Witten systems

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Abstract

In this article we study an integrable deformation of the Kapustin-Witten equations. Using the Weyl-Wigner-Moyal-Groenewold description an integrable \textit{*}-deformation of a Kapustin-Witten system is obtained. Starting from known solutions of the original equations, some solutions to these deformed equations are obtained.

\textit{Key words:} Kapustin-Witten equations, Integrable systems, Integrable deformations, Self-dual equations in higher dimensions.

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1 Introduction

In a celebrated work, Kapustin and Witten \[1\] described the geometric Langlands program (GLP) in terms of a compactification on a Riemann surface of a certain twisted version of the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) in four dimensions. In such a paper, the authors introduced a set of equations after imposing a BRST-like preservation conditions on a twisted version of $\mathcal{N} = 4$ SYM theory in four dimensions; these equations are now known as the Kapustin-Witten (KW) equations and have been the subject of an intensive work in the last decade in physics as well as in mathematics. In particular, a relation of KW equations with knot theory is also described by Witten in \[2\], where the author describes an approach to Khovanov homology using gauge theory; in that context, the KW equations appear as a localization condition of the $\mathcal{N} = 4$ SYM theory in four dimensions (see \[3\] for a review on this topic). The KW equations are also closed related to another set of equations, recently introduced by Ward \[4\] and usually called the $(2k)$-Hitchin equations; it is important to mention that these equations are a natural generalization of another set of equations introduced by Hitchin \[5\] in a pioneering work in complex geometry; indeed, the article of Hitchin is the origin of the notion of Higgs bundle in mathematics, a notion that plays an important role in the physical interpretation of the GLP developed by Kapustin and Witten.

In this section we fix the notation and review some preliminar notions that will be used in the article. In order to start, let $g$ be a Lie algebra of a Lie group $G$ and let $M$ be a riemannian 4-manifold, with riemannian metric $g$. Let $\phi$ and $A$ be two $g$-valued 1-forms, where $\phi$ is considered as a Higgs field and $A$ is a gauge potential with covariant derivative $D = d + A$. As it is well known, the gauge field of $A$ is a $g$-valued 2-form given by $F = D \wedge D = dA + A \wedge A$. With this data, Kapustin and Witten introduced in \[1\] the following set of equations:

\[
(F - \phi \wedge \phi + t D\phi)^+ = 0, \quad (F - \phi \wedge \phi - t^{-1}D\phi)^- = 0, \quad D^*\phi = 0. \tag{1}
\]

Here the superscript $\pm$ stands for the self-dual and anti-self-dual part, $t$ is a complex parameter and $D^* = *D*$, where $*$ is the Hodge operator on $M$ associated with $g$. Strictly speaking \[1\] is not a system of equations, but a family of equations parametrized by $t$ and certain values of $t$ give rise to equations of interest, e.g., with $t = \pm i$ we obtain a particular case which plays an important role in the physical interpretation of the GLP developed by Kapustin and Witten.

This form of KW equations have been considered recently by Gagliardo and Uhlenbeck \[7\], Dunajski and Hoegner \[8\] and Ward \[4\], in particular in the last two references the

\[\text{To be more precise, in that case there exists a dual parameter } t^\lor \text{ of } t, \text{ and the dual condition } t^\lor = 1 \text{ gives such a set of equations with respect to the dual group } G^\lor.\]
equations (3) are also called the non-abelian Seiberg-Witten equations. From now on, we will refer to (1) and (3) as the KW equations and the non-abelian Seiberg-Witten equations, respectively. At this point, it is important to mention that some solutions to (3) are already proposed by Dunajski and Hoegner in [8], these solutions will be of crucial importance for the purposes of the present paper; indeed, we are going to use such solutions to find solutions of the deformed equations.

Now, KW equations are close related to self-duality in higher dimensions and to the dimensional reduction procedure. Moreover, as we said before such equations appear as equations of motion of a twisted version of $\mathcal{N} = 4$ SYM in four dimensions, which in turn arises as a dimensional reduction to four dimensions of the $\mathcal{N} = 1$ SYM theory in ten dimensions [1]. Also, in Ref. [4] it was shown that KW equations arises as a four dimensional reduction of the self-dual YM equations in eight dimensions. Moreover, the five dimensional extension of KW equations proposed in [2] for a particular case, leads to a set of equations which had already been obtained for general five-manifolds by Haydys in [9]. They are known as Haydys-Witten equations and later on it was found by Cherkis [10] that these equations can be obtained via dimensional reduction from the instanton equation on Spin(7) eight-manifolds and also from the seven-dimensional reductions for any $G_2$ holonomy manifold. All this shows that dimensional reduction and integrability are strongly related to KW equations and play an important role in the understanding of all story about these equations. In fact, since many years ago it is well known that Seiberg-Witten equations have the structure of integrable systems [11] (see [12] for an overview). Moreover, the Hitchin systems [5] are very well known examples of integrable systems that have been studied even in a quantized way [13]. It is also known that self-dual systems are integrable systems and according the Ward conjecture, all integrable systems come from four-dimensional self-dual YM or self-dual gravity equations [14].

In the present paper we perform an integrable deformation of the KW equations via the Weyl-Wigner-Moyal-Groenewold (WWMG) formalism of deformation quantization (for a recent overview see [15]). As it is well known, this procedure does not spoil the integrability of the former equations [16, 17, 18, 19, 20, 21]. For a more recent review containing several of these results the reader would like to consult [23, 24]. Thus, in the present paper we will find integrable deformations of KW equations. In order to find solutions for these deformed equations, we make use of the WWMG correspondence between $su(2)$-valued operators acting on a certain Hilbert space and functions defined in a symplectic surface satisfying similar relations but under the Moyal bracket. It is important to mention that this correspondence has been already explored before in other contexts, indeed in [25, 26, 27] it was employed to study some integrable deformations of the principal chiral model, the Nahm equations and seven-dimensional reductions of the self-dual in eight-dimensions respectively.

This paper is organized as follows: in Section 2 we start with a general overview of the KW equations and we perform the deformation of these equations. In Section 3 we analyze the non-abelian Seiberg-Witten equations, since they have a close relation with our former system KW. In that section we perform the integrable deformation of this system under the same WWMG formalism and found some solutions to it. In Section 4 we close the paper with some final comments. Since we consider this subject
a very important matter, we left to the appendix a unification in terms of language of the Hitchin equations in the believing that it would come in handy to physicists and mathematicians alike.

2 Overview on Kapustin-Witten equations

We begin this section by reviewing the general KW equations. As we mentioned in the introduction, these equations arise as the equations of motion of a topological twisting of $\mathcal{N} = 4$ SYM in four dimensions [1], or also as a set of equations obtained to localize knots in 3-dimensional space [2]. By defining

$$
V^+ = (F - \phi \wedge \phi + tD\phi)^+, \quad V^- = (F - \phi \wedge \phi - t^{-1}D\phi)^-, \quad V^0 = D_\mu \phi^\mu,
$$

where $\pm$ stands for the self-dual and anti-self-dual projections of the 2-forms between parenthesis, the KW equations (1) can be written simply as

$$
V^+ = V^- = V^0 = 0. \tag{4}
$$

As it is explained in [1], these equations are obtained as equations of motion from the action

$$
S = -\int_M d^4 x \sqrt{g} \left[ \frac{t^{-1}}{t + t^{-1}} V^+_{\mu\nu} V^{+\mu\nu} + \frac{t}{t + t^{-1}} V^-_{\mu\nu} V^{-\mu\nu} + (V^0)^2 \right], \tag{5}
$$

where $\sqrt{g}$ is an abbreviation for the square root of the determinant of the metric $g$ of $M$ and Tr is the usual trace. The fields $\phi$ and $F$ are real valued if the parameter $t$ is also real, in the present paper we are interested mainly in the case of real $t$. If we consider $x^\mu$, $\mu = 0, \ldots, 3$, as the coordinates of the four-dimensional manifold $M$ and if we assume it has no boundary (see [1] for details) the action can be rewritten as

$$
S = -\int_M d^4 x \sqrt{g} \left[ \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + D_\mu \phi_\nu D^\mu \phi^\nu + R_{\mu\nu} \phi^\mu \phi^\nu + \frac{1}{2} [\phi_\mu, \phi_\nu] [\phi^\mu, \phi^\nu] \right]
+ \frac{t - t^{-1}}{t + t^{-1}} \int_M \text{Tr} (F \wedge F), \tag{6}
$$

where $R_{\mu\nu}$ is the Ricci tensor of $M$. In this form, the action is given as a sum of two terms in which the dependence on $t$ is reduced to the second one, which is indeed a topological term.

On the other hand, in [2] Witten find solutions to eqs. (2) using the ansatz $A_0 = \phi_3 = 0$, in that case, he showed that the resulting equations can be written nicely in the form

$$
[D_i, D_j] = 0, \quad i, j = 1, \ldots, 3.
$$

$$
\sum_{i=1}^{3} [D_i, D_i^\dagger] = 0, \tag{7}
$$
where the operators $\mathcal{D}_i$ are defined as follows:

$$
\mathcal{D}_1 = \partial_1 + i \partial_2 + [A_1 + i A_2, \cdot], \quad \mathcal{D}_2 = \partial_3 + [A_3 - i \phi_0, \cdot], \quad \mathcal{D}_3 = [\phi_1 - i \phi_2, \cdot].
$$

Now, an important fact to note here is that equations of the same form than (7) appear in the context of complex geometry and are usually called the Hermite-Yang-Mills equations (see [2] for more details). In that context, these equations are defined for a holomorphic vector bundle and the Hitchin-Kobayashi correspondence says that solutions to these equations exists, if and only if, the holomorphic bundle is poly-stable, i.e., it is a direct sum of stable bundles with the same slope (see [28] for details).

In the present paper, we are only focus in gauge configurations with the Lie algebra being $\mathfrak{su}(2)$. Let $t_a, a = 1, \ldots, 3$ be its generators in an anti-hermitian representation. Thus, as it is shown in [2], after complexifying the group and choosing some holomorphic gauge, the ansatz for solving (7) is

$$
\begin{align*}
A_1 + i A_2 &= -\frac{(\partial_1 + i \partial_2)v}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\phi_0 &= -\frac{i \partial_3 v}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\varphi &= z^r e^v \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},
\end{align*}
$$

for an unknown function $v$ and where $r$ is a parameter linked to the spin representation for the complexification of the gauge group $\text{SU}(2)$. The field strength associated with this gauge field is

$$
F_{12} = \frac{i (\partial_1^2 + \partial_2^2)v}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

In order to $v$ solve equations (7) it must be a solution to

$$
\frac{\partial^2}{\partial(x')^2} + \frac{\partial^2}{\partial(x^2)^2} + \frac{\partial^2}{\partial y^2} v + |z|^2 e^{2v} = 0,
$$

which come from the second line of (7), where the definition $y = x^3$ and $z = x^1 + ix^2$ was made. As it is known, the exact solution to this equation is given by

$$
v = -r \log |z| - \log y.
$$

Depending on the class of the solution we are interested, which represents the position where we are inserting the ‘t Hooft operator in the dual description to the D3-NS5 system, $v$ can be redefined in order to give the desired behavior. At the moment, we want to emphasize that the solutions we just found are suitable of being deformed. At this point, let us review the WWMG formalism in order to apply it to the case considered here.

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5This correspondence, also called the Uhlenbeck-Yau-Donaldson-Simpson theorem, plays a fundamental role in complex geometry; in fact, it establishes an equivalence between the algebraic notion of Mumford stability and the differential notion of Hermite-Yang-Mills metric.
2.1 Deforming Kapustin-Witten equations

In order to apply the WWMG formalism \[15\], we promote the fields $A$ and $\phi$ to $\mathfrak{su}(2)$ operator-valued forms acting over a Hilbert space $\mathcal{H} = L^2(\mathbb{R})$. Let us choose $|\psi_n\rangle$, $n = 0, 1, \ldots$ an orthonormal basis of $\mathcal{H}$. As it is well known, we have the closure relations

$$\langle \psi_n | \psi_m \rangle = \delta_{nm}, \quad \sum_n |\psi_n\rangle \langle \psi_n| = \hat{I},$$

where $\hat{I}$ denotes the identity operator in $\mathcal{H}$. We carry out the above-mentioned identification by the correspondence: $A_i \rightarrow \hat{A}_i \in \mathcal{M} \otimes \hat{\mathcal{U}}$, and $\phi_i \rightarrow \hat{\phi}_i \in \mathcal{M} \otimes \hat{\mathcal{U}}$, with $\hat{\mathcal{U}}$ the Lie algebra of anti-self-dual operators acting on $\mathcal{H}$. Also we change the usual Lie algebra brackets by the corresponding commutator $[\cdot, \cdot]$.

This deformation procedure rely on the parameter $\hbar$, and when the limit $\hbar \to 0$ is taken we recover the (undeformed) original system. In order to do this, we perform one further redefinition of the fields in terms of $\hbar$,

$$\hat{A}_i = i\hbar \hat{A}_i, \quad \hat{\phi}_i = i\hbar \hat{\phi}_i.$$

Let $\mathcal{B}$ and $C^\infty(\Sigma, \mathbb{R})$ denote the set of self-adjoint linear operators acting on the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$ and the space of infinite differentiable real functions defined on the two-dimensional phase space $\Sigma$ with coordinates $p, q$, respectively. In general, we define the Weyl correspondence $W^{-1} : \mathcal{B} \to C^\infty(\Sigma, \mathbb{R})$ by

$$A_i(\vec{x}, p, q; \hbar) \equiv W^{-1}(\hat{A}_i) := \int_{-\infty}^{\infty} \left\langle q - \frac{1}{2} \xi | \hat{A}_i(\vec{x}) | q + \frac{1}{2} \xi \right\rangle e^{i\xi \vec{p}} \, d\xi,$$  

for all $\hat{A}_i \in \mathcal{B}$ and $A_i \in C^\infty(M \times \Sigma, \mathbb{R})$. Such a correspondence deforms the product of functions in $C^\infty(\Sigma, \mathbb{R})$ through the Moyal $\star$-product, which is defined by

$$F_i \star F_j := F_i \exp \left( \frac{i\hbar \hat{\rightarrow}}{2} \right) F_j,$$

where $F_j = F_j(\vec{x}, p, q; \hbar) \in C^\infty(M \times \Sigma, \mathbb{R})$ and the operator $\hat{\rightarrow}$ is given by

$$\hat{\rightarrow} := \frac{\partial}{\partial q} \frac{\partial}{\partial p} - \frac{\partial}{\partial p} \frac{\partial}{\partial q}.$$

At the same time, the Lie bracket between operators changes to the Moyal bracket $\{\cdot, \cdot\}_M$ between functions as follows:

$$W^{-1} \left( \frac{1}{i\hbar} [\hat{F}_i, \hat{F}_j] \right) = \frac{1}{i\hbar} (F_i \star F_j - F_j \star F_i) := \{F_i, F_j\}_M.$$  

As we said before, by taking the limit $\hbar \to 0$ we recover the usual product between functions and the Poisson bracket, respectively.
At this point, the deformation of equations \([1]\) can be carry out as follows. First, let us consider the action \([6]\), by promoting \(A\) and \(\phi\) to operator \(\text{su}(2)\)-valued forms \(\hat{A}\) and \(\hat{\phi}\), the action looks like

\[
S_q = -\int d^4x \sqrt{g} \text{Tr} \left[ \frac{1}{2} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} + \hat{D}_\mu \hat{\phi}_\nu \hat{D}^\mu \hat{\phi}^\nu + \hat{R}_{\mu\nu} \hat{\phi}^\mu \hat{\phi}^\nu + \frac{1}{2} \{\hat{\phi}_\mu, \hat{\phi}_\nu\} \{\hat{\phi}^\mu, \hat{\phi}^\nu\} \right] + \frac{t - t^{-1}}{t + t^{-1}} \int_M \text{Tr} \hat{F} \wedge \hat{F}
\]

(13)

where the covariant derivative operator \(\hat{D}\) acting on the operator \(\hat{\phi}\) is given by \(\hat{D} \hat{\phi} = \partial \hat{\phi} + [\hat{A}, \hat{\phi}]\). Even though we are writing the operator of the Ricci tensor \(R_{\mu\nu}\), since its appearance is through the covariant derivative of the metric, \(d = \nabla + A\), it can be treated as a function and not as an operator (in fact, it is proportional to the identity operator). Using by definition that

\[
\text{Tr}(\cdot) = 2\pi \hbar \sum_n \langle \psi_n | (\cdot) | \psi_n \rangle,
\]

(14)

which is the sum of the diagonal elements with respect to the basis, and considering the previous setting, we incorporate these facts into the action \([13]\) with the promoted fields, becoming

\[
S_q = -2\pi \hbar \sum_n \int d^4x \sqrt{g} \left\langle \psi_n \left| \frac{1}{2(i\hbar)^2} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} + \frac{1}{(i\hbar)^2} \hat{D}_\mu \hat{\phi}_\nu \hat{D}^\mu \hat{\phi}^\nu + \frac{1}{(i\hbar)^2} R_{\mu\nu} \hat{\phi}^\mu \hat{\phi}^\nu \right| \psi_n \right\rangle + 2\pi \hbar \frac{t - t^{-1}}{t + t^{-1}} \sum_n \int_M \left\langle \psi_n \left| \frac{1}{(i\hbar)^2} \hat{F} \wedge \hat{F} \right| \psi_n \right\rangle
\]

and hence, making a straight use of the Weyl correspondence \([15]\), the deformed action takes the form

\[
S_M = -2\pi \hbar \int d^4x \; dp \; dq \sqrt{g} \left[ \frac{1}{2} F_{\mu\nu} \star F^{\mu\nu} + D_{M\mu} \Phi_\nu \ast D^\mu_M \Phi^\nu + R_{\mu\nu} \Phi^\nu \ast \Phi^\nu \right.
\]

\[
+ \left. \frac{1}{2} \{\Phi_\mu, \Phi_\nu\}_M \ast \{\Phi^\mu, \Phi^\nu\}_M \right] + 2\pi \hbar \frac{t - t^{-1}}{t + t^{-1}} \int_{M \times \Sigma} \left. dp \; dq \; \frac{1}{\hbar^2} \mathcal{F} \star \mathcal{F} \right.
\]

(15)

We have defined the Moyal covariant derivative \(\mathcal{D}_M = d + \{\mathcal{A}, \cdot\}_M\), where the action on the fields equals \(\text{Tr} \left[ \mathcal{D}_M \Phi = d \Phi + \{\mathcal{A}, \Phi\}_M \right]\) and the deformed field strength is given by \(\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - \{\mathcal{A}_\mu, \mathcal{A}_\nu\}_M\). Note that the deformation process just acts on the functions defining the differential forms and not over the alternating tensors i.e. the deformation is only in the coordinates \(p\) and \(q\) of the symplectic surface \(\Sigma\). Thus, following \([19]\) we define the Moyal wedge product, denoted by \(\star\), as

\[
\omega \star \eta = \frac{1}{p!q!} \omega_{i_1 \ldots i_p} \star \eta_{j_1 \ldots j_q} dx^{i_1} \wedge \cdots \wedge dx^{j_q},
\]

(16)

Here the derivatives are taken with respect to the coordinates of the manifold \(M\).
for two $p$- and $q$-forms $\omega$ and $\eta$. These forms are defined by

$$\omega = \omega_{i_1 \ldots i_p}(\vec{x}, p, q; \hbar)dx^{i_1} \wedge \cdots \wedge dx^{i_p} \in \Omega^p(M, C^\infty(M \times \Sigma, \mathbb{R})).$$  \tag{17}$$

At this point a natural question is: How the localization equations look like under this deformation? It is straightforward to obtain the action (15) from this form

$$S = -2\pi \hbar \int_{M \times \Sigma} d^4x \, dp \, dq \sqrt{g} \frac{1}{\hbar^2} \left[ \frac{t^{-1}}{t + t^{-1}} \mathcal{V}_{M}^{+ \mu \nu} \ast \mathcal{V}_{M}^{+ \mu \nu} + \frac{t}{t + t^{-1}} \mathcal{V}_{M}^{- \mu \nu} \ast \mathcal{V}_{M}^{- \mu \nu} + \mathcal{V}_{M}^{0} \ast \mathcal{V}_{M}^{0} \right]$$

from where we obtain the Moyal localization equations

$$\mathcal{V}_{M}^{+} = \mathcal{V}_{M}^{-} = \mathcal{V}_{M}^{0} = 0,$$  \tag{18}$$

for

$$\mathcal{V}_{M}^{+} = (\mathcal{F} - \Phi \wedge \Phi + td_{A} \Phi)^{+},$$

$$\mathcal{V}_{M}^{-} = (\mathcal{F} - \Phi \wedge \Phi - t^{-1}d_{A} \Phi)^{-},$$

$$\mathcal{V}_{M}^{0} = D_{M} \Phi^{\mu}.$$  \tag{19}$$

As before, taking $t = 1$ and considering the definitions of (anti-)self-dual 2-forms in four dimensions, (18) can be rewritten as

$$\mathcal{F} - \Phi \wedge \Phi + *d_{A} \Phi = 0,$$

$$d_{A} * \Phi = 0.$$  \tag{20}$$

These equations constitutes the Moyal deformation of Kapustin-Witten equations.

### 2.2 Looking for solutions

Having discussed the general framework of the deformation quantization procedure, we are in position to find solutions to the deformed localization equations presented previously. Let $t_i$, $i = 1, \ldots, 3$, be the generators of the Lie algebra $\mathfrak{su}(2)$ in an anti-hermitian representation and denote by $\widehat{\chi}_i$ the corresponding $\mathfrak{su}(2)$ operators. Then we have the correspondence

$$t_1 \rightarrow \widehat{\chi}_1 := i\beta \widehat{q} + \frac{1}{2\hbar}(\widehat{q}^2 - \widehat{1})\widehat{p},$$

$$t_2 \rightarrow \widehat{\chi}_2 := -\beta \widehat{q} + \frac{i}{2\hbar}(\widehat{q}^2 + \widehat{1})\widehat{p},$$

$$t_3 \rightarrow \widehat{\chi}_3 := i\beta \widehat{1} - \frac{1}{\hbar} \widehat{q} \widehat{p},$$

between the generators of $\mathfrak{su}(2)$ and $\mathfrak{su}(2)$-valued operators which act on some Hilbert space. The parameter $\beta$ is due to some choice in the ordering between $\widehat{q}$ and $\widehat{p}$. Under
the Weyl isomorphism these operators correspond to functions defined on the symplectic surface \( \Sigma \) using the formula
\[
\chi_i(\vec{x}, p, q; \hbar) := \int_{-\infty}^{\infty} \left\langle q - \frac{1}{2} \xi | \hat{\chi}_i | q + \frac{1}{2} \xi \right\rangle \exp \left( \frac{i}{\hbar} \xi \cdot p \right) d\xi,
\]
from where we get the corresponding functions
\[
\chi_1(p, q; \hbar) = i \left( \beta - \frac{1}{2} \right) q - \frac{1}{2\hbar} (q^2 - 1)p, 
\]
\[
\chi_2(p, q; \hbar) = - \left( \beta - \frac{1}{2} \right) q - \frac{i}{2\hbar} (q^2 + 1)p, 
\]
\[
\chi_3(p, q; \hbar) = -i \left( \beta - \frac{1}{2} \right) + \frac{1}{\hbar} qp. 
\]
With respect to the Moyal bracket these functions satisfy the \( su(2) \) algebra relations
\[
\{ \chi_1, \chi_2 \}_M = -\frac{1}{i\hbar} \chi_3, \quad \text{(plus cyclic permutations).} 
\]

The solutions we found earlier are promoted to their corresponding operators and by applying the Weyl correspondence, they look like
\[
A_1 + iA_2 = i(\partial_1 + i\partial_2)v_3 \quad \rightarrow \quad A_1 + iA_2 = -\hbar(\partial_1 + i\partial_2)v\chi_3, 
\]
\[
\phi_0 = -\partial_3 v_3 \quad \rightarrow \quad \Phi_0 = -i\hbar\partial_3 v\chi_3, 
\]
\[
\phi = z^r e^v(t_2 - it_1) \quad \rightarrow \quad \mathcal{P} = i\hbar z^r e^v(\chi_2 - i\chi_1), 
\]
where the field strength is given by
\[
\mathcal{F}_{12} = i\hbar(\partial_1^2 + \partial_2^2)v\chi_3, 
\]
and where \( v \) satisfy the differential equation (10). This set of functions fulfill the corresponding deformed system of conditions
\[
\{ \mathcal{D}_{Mi}, \mathcal{D}_{Mj} \}_M = 0, \quad i, j = 1, \ldots, 3, 
\]
\[
\sum_{i=1}^{3} \{ \mathcal{D}_{Mi}, \mathcal{D}^i_{Mj} \}_M = 0, 
\]
for the components
\[
\mathcal{D}_{M1} = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} + \{ A_1 + iA_2, \cdot \}_M, 
\]
\[
\mathcal{D}_{M2} = \frac{\partial}{\partial x_3} + \{ A_3 - i\Phi_0, \cdot \}_M, 
\]
\[
\mathcal{D}_{M3} = \{ \Phi_1 - i\Phi_2, \cdot \}_M. 
\]
3 Non-abelian Seiberg-Witten equations

As we have discussed before, on the localization equations (4) we can impose a further conditions to these equations asking that, for any $t$, $F - \phi \wedge \phi$ be self-dual and $D\phi$ be anti-selfdual. By imposing these conditions we get the so called non-abelian Seiberg-Witten equations \[8\]

\[
\begin{align*}
\mathcal{V}^+ &= (F - \phi \wedge \phi)^+ = 0, \\
\mathcal{V}^- &= (D\phi)^- = 0, \\
\mathcal{V}^0 &= D^*\phi = 0.
\end{align*}
\] (32)

These equations has a striking resemblance to the Hitchin’s equations, which are defined for one complex dimension manifolds. In fact, Ward in \[4\] generalizes these equations to higher dimensions. Similar to the case of (4), the action leading to (32), up to boundary terms, is written as

\[
S = -\int d^4x \sqrt{g} \ Tr \left[ V^+_{\mu\nu} V^{+\mu\nu} + V^-_{\mu\nu} V^{-\mu\nu} + V^2_0 \right]
\]

\[
= -\int d^4x \sqrt{g} \ Tr \left[ \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + D_{\mu}\phi_\nu D^{\nu}\phi_\mu + R_{\mu\nu}\phi_\mu \phi_\nu + \frac{1}{2} \{ \phi_\mu, \phi_\nu \} \{ \phi^\mu, \phi^\nu \} \right].
\]

As we did with (4), we perform a deformation of these equations via the WWMG formalism; provided that we have nearly the same type of equations previously studied in the above section, such deformation follows a similar procedure. In fact, after promoting each field to an operator valued one and applying the Weyl correspondence to (32), we have

\[
\begin{align*}
\mathcal{V}_M^+ &= (F - \Phi^\star \Phi)^+ = 0, \\
\mathcal{V}_M^- &= (D_M\Phi)^- = 0, \\
\mathcal{V}^0 &= D_{M\mu}\Phi^\mu = 0,
\end{align*}
\] (33)

which are the integrable deformation analogous to (19). At the same time, the deformation for the corresponding action is

\[
S_M = \int d^4x \ dp \ dq \sqrt{g} \ \frac{1}{\hbar^2} \left[ \frac{1}{2} F_{\mu\nu}^* F^{\mu\nu} + D_{M\mu} \Phi_\nu^* D_M^{\nu} \Phi^\nu + R_{\mu\nu} \Phi_\mu^* \Phi^\nu \\
+ \frac{1}{2} \{ \Phi_\mu, \Phi_\nu \}_M^* \{ \Phi^{\mu}, \Phi^{\nu} \}_M \right],
\]

which has exactly the form of (15). It is interesting to note that (32) have solutions close in spirit for the generalized Kapustin-Witten equations. As reported in \[8\], we have solutions to the original non-abelian Seiberg-Witten equations for different backgrounds. we will review this in the next subsection.
3.1 Some solutions

In [8] Dunajski and Hoegner found some solutions to the non-abelian Seiberg-Witten equations relying in some ansatz for functions satisfying a certain set of differential equations. From the self-duality equations defined on a Spin(7)-holonomy manifold $M_8$ for some group $G$, splitting this manifold into two four-dimensional hyper-Kähler manifolds $M_4' \times M_4$ with a proper choice of the components of the connection 1-form and a given dimensional reduction suggested by this splitting, the authors obtain (32) with gauge group SU(2). The appearance of such particular group comes from the fact that with this choice of hyper-Kähler manifolds a holonomy reduction is induced, provided that SU(2) $\times$ SU(2) is a proper subgroup of Spin(7). Details of how all this is done can be referred to the original paper [8]. Here, we are mainly concerned with the solutions they found in doing such selection.

Remember that we have chosen $t_a$ as our generators of the Lie algebra $su(2)$ in an anti-hermitian representation. For all the fields involved there is no dependence on the coordinates of $M_4'$. Let $A$ and $\phi$ be again the connection 1-form and a 1-form of scalar fields on $M_4$, respectively, that are $su(2)$-valued. Also on $M_4$, let $e^a$ be our vierbein, and define the two-forms

$$\psi^+_i = e^0 \wedge e^i + \frac{1}{2} \varepsilon_{ijk} e^j \wedge e^k,$$

which are self-dual with respect to the Hodge operator of $M_4$. Let $G$ and $H$ be scalar functions on $M_4$, then the ansatz for $A$ and $\phi$ proposed in [8] reads

$$A = * \left( \sum_i t_i \psi^+_i \wedge dG \right) = \sum_i t_i * (\psi^+_i \wedge dG),$$

$$\phi = * \left( \sum_i t_i \psi^+_i \wedge dH \right) = \sum_i t_i * (\psi^+_i \wedge dH).$$

(34)

Here, we intentionally separate the Lie algebra generator from the two-form in order to make explicit how the WWMG correspondence will be implemented.

In order to satisfy (32) with the proposed $A$ and $\phi$, the scalars functions $G$ and $H$ must be solutions to the next set of differential equations:

$$0 = \Box G + |\nabla G|^2 - |\nabla H|^2,$$

$$0 = (\varepsilon_{e^a} C^a_{bc} \sigma^{bd} - \sigma^{ab} C_{ab}^d) \nabla_d G,$$

$$0 = \tilde{\sigma}_{ac} \sigma^c_b (\nabla^a \nabla^b H - 2 \nabla^a G \nabla^b H),$$

$$0 = \sigma_{ab} (\nabla^a \nabla^b H - 2 \nabla^a G \nabla^b H),$$

(35) - (38)

with $\Box$ and $\nabla$ differential operators on $M_4$; $C_{ab}^c$ are the structure constants defined by $de^a = C_{ab}^c e^b \wedge e^c$.

At this point, we can obtain the functions $G$ and $H$ explicitly for different backgrounds and at the same time we can write down its deformation via the WWMG formalism.
3.1.1 A simple case, flat background

Let \( M_4 = \mathbb{R}^4 \). In this case we have that \( e^i = dx^i \), thus \( C^a_{bc} = 0 \). The connection has the explicit form

\[
A = t_1 \left( \partial_0 G dx^1 - \partial_1 G dx^0 + \partial_2 G dx^3 - \partial_3 G dx^2 \right) \\
+ t_2 \left( \partial_0 G dx^2 - \partial_2 G dx^0 + \partial_3 G dx^1 - \partial_1 G dx^3 \right) \\
+ t_3 \left( \partial_0 G dx^3 - \partial_3 G dx^0 + \partial_1 G dx^2 - \partial_2 G dx^1 \right).
\]

(39)

And a similar expression for \( \phi \). When these 1-forms are inserted in the set of equations (35)-(38) with the given structure constants, the corresponding solutions to the functions \( G \) and \( H \) read

\[
G = -\frac{1}{2} \ln |x^3|, \quad H = \frac{\sqrt{3}}{2} \ln |x^3|.
\]

In order to apply the WWMG correspondence, we promote as before \( A \) and \( \phi \) to operator valued quantities, where the \( \mathfrak{su}(2) \) generators are now \( \mathfrak{su}(2) \)-valued operators, that is \( t_a \to \hat{t}_a \), satisfying the \( \mathfrak{su}(2) \) algebra relations. Thus, the operator form of the connection 1-form is

\[
\hat{A} = \frac{\partial G}{\partial x^3} \left[ \hat{t}_3 dx^0 + \hat{t}_2 dx^1 - \hat{t}_1 dx^2 \right]
\]

with a similar expression for \( \hat{\phi} \). Hence, by applying the Weyl isomorphism, we obtain the deformed 1-form

\[
\hat{A} = \frac{\partial G}{\partial x^3} \left[ \chi_3(p, q; \hbar) dx^0 + \chi_2(p, q; \hbar) dx^1 + \chi_1(p, q; \hbar) dx^2 \right],
\]

(40)

where the functions \( \chi_i \) are functions of the symplectic surface \( \Sigma \) with expressions given by the equations (21)-(23). These functions satisfy the Moyal \( \mathfrak{su}(2) \)-relations

\[
\{\chi_1, \chi_2\}_M = -\frac{1}{i\hbar} \chi_3, \quad \text{(plus cyclic permutations),}
\]

(41)

and we have a similar expression for \( \Phi(\vec{x}, p, q; \hbar) \).

3.1.2 Curved backgrounds

As the previous example shows, the Weyl correspondence acts just on the Lie algebra generators of the 1-forms \( A \) and \( \phi \) in (34). Hence, the \( \mathfrak{su}(2) \) operator-valued 1-forms are

\[
\hat{A} = * \left( \sum_i \hat{t}_i \psi_i^+ \wedge dG \right) = \sum_i \hat{t}_i * (\psi_i^+ \wedge dG),
\]

\[
\hat{\phi} = * \left( \sum_i \hat{t}_i \psi_i^+ \wedge dH \right) = \sum_i \hat{t}_i * (\psi_i^+ \wedge dH).
\]

(42)
Applying the Weyl correspondence to these 1-forms gives the expression

\[ \mathcal{A} = \star \left( \sum \chi_i \psi_i^+ \wedge dG \right) = \sum \chi_i \star (\psi_i^+ \wedge dG), \]

\[ \Phi = \star \left( \sum \chi_i \psi_i^+ \wedge dH \right) = \sum \chi_i \star (\psi_i^+ \wedge dH), \] (43)

with the functions of the phase space \( \chi_i \) previously defined.

Now, let us consider as another example the hyper-Kähler metric

\[ g_4 = V \left[ (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \right] + V^{-1} (dx^0 + \alpha)^2, \]

where the function \( V \) and the 1-form \( \alpha \) depend on \( x^i, i = 1, \ldots, 3 \), and are related by \( \star_3 dV = -d\alpha \), with \( \star_3 \) the Hodge operator in \( \mathbb{R}^3 \). A solution for the corresponding equations is given by

\[ G = -\frac{3}{4} \ln x^3 + \frac{1}{4} \ln 21 - \ln 2, \quad H = -\frac{\sqrt{21}}{3} G. \]

By using the Weyl correspondence once again, we obtain that the next forms are solutions to the non-abelian Seiberg-Witten-Moyal equations

\[ \mathcal{A} = \frac{3}{4} \left( \sigma_2 \otimes \chi_1 - \sigma_1 \otimes \chi_2 + \sigma_0 \otimes \chi_3 \right), \quad \Phi = -\frac{\sqrt{21}}{3} \mathcal{A}, \]

\[ \mathcal{F} = \left( \frac{9}{16} \sigma_0 \wedge \sigma_1 + \frac{3}{4} \sigma_2 \wedge \sigma_3 \right) \otimes \chi_1 \]

\[ + \left( \frac{9}{16} \sigma_0 \wedge \sigma_2 - \frac{3}{4} \sigma_1 \wedge \sigma_3 \right) \otimes \chi_2 + \left( \frac{3}{2} \sigma_0 \wedge \sigma_3 - \frac{3}{16} \sigma_1 \wedge \sigma_2 \right) \otimes \chi_3, \]

where the 1-forms \( \sigma_i \)'s are defined by

\[ \sigma_0 = (x^3)^{-2} (dx^0 + x^2 dx^1), \quad \sigma_1 = (x^3)^{-1} dx^1, \quad \sigma_2 = (x^3)^{-1} dx^2, \quad \sigma_3 = \frac{dx^3}{x^3}. \]

In principle, similar solutions like the ones presented in this section can be obtained when the WWMG formalism is applied to the integral generalization to higher dimensions of the Hitchin equations performed by Ward in [4], something that is under development at the present. Generalizations to the Hitchin equations to higher dimensions can be found also in [28]. In this regard, Ward make emphasis on generalizations which are at the same time integrable; he gave such generalization for dimensions equal to \( 2k \). The model for him is the octonionic self-dual equations defined on an eight-dimensional Spin(7)-holonomy manifold, which are also the case for Dunajski and Hoegner as we already study, and for Cherkis in [10], when he deduce the Haydys-Witten equations from a dimensional reduction of the mentioned octonionic system.
4 Final Comments

Integrability is an aspect of a broad class of system of equations appearing in many contexts of physical and mathematical interest alike. In general, there are conditions to decide when a given system is integrable, but there is no general rule that apply to all systems. Such is the case, for example, of the integrability of the full Yang-Mills equations, which is still an open problem.

In this paper we performed an integrable deformation of the so called Kapustin-Witten equations and the non-abelian Seiberg-Witten equations via the WWMG formalism. It is already known that these set of equations are integrable, thus spoiling nothing when we perform such deformation; also, solutions to these equations together with its corresponding deformations were presented. The possibility of carrying out such deformation in Hitchin’s systems on $\mathbb{R}^2$ [29] and in dimensions greater than two [4] are being explored and will be reported in a future communication.

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A On the Hitchin equations

The Hitchin equations first appear in a classical article by Hitchin [5] as a two dimensional reduction of the selfdual Yang-Mills equations in four dimensions. In geometric terms, such equations can be defined for a $G$-principal bundle over a riemannian 2-manifold $M$ and are given by:

$$F + [\Phi, \Phi^*] = 0, \quad \bar{\partial}_A \Phi = 0. \quad (44)$$

Here $A$ is a connection on the bundle with $F$ its curvature, $\Phi$ is a certain $\mathfrak{g}$-valued holomorphic 1-form with adjoint $\Phi^*$ and $\bar{\partial}_A = D_1 + iD_2$. These equations are close related with the Kapustin-Witten equations and in the literature they usually appear in different forms. For instance, in [1] the eqs. (44) are written as:

$$F - \Phi \wedge \Phi = 0, \quad D\Phi = 0, \quad D^* \Phi = 0, \quad (45)$$

where $D$ is the covariant derivative of $A$ with gauge field $F = dA + A \wedge A$ and $\Phi$ is again a $\mathfrak{g}$-valued 1-form. The systems of equations (44) and (45) are indeed the same, the difference arises because the objects are arranged in a different way. In (44) the form $\Phi = \frac{1}{2}\phi \, dz$, where $\phi = \phi_1 - i\phi_2$ and $z = x^1 + ix^2$. Instead, in (45) the form $\Phi = \phi_1 dx^1 + \phi_2 dx^2$. In both cases $\phi_1$ and $\phi_2$ are the Higgs fields induced by the dimensional reduction procedure and the equivalence is almost evident if we notice that $\phi_1$ and $\phi_2$ are antihermitian and we write the $\mathfrak{g}$-valued form in (44) as $\Phi_c = \frac{1}{2}(\phi_1 - i\phi_2)dz$, where the subscript $c$ remember us that it is a complex Higgs field.
In fact, using this notation $\Phi_c^* = -\frac{1}{2}(\phi_1 + i\phi_2)d\bar{z}$ and we get

$$[\Phi_c, \Phi_c^*] = -\frac{i}{2}[(\phi_1, \phi_2)dz \wedge d\bar{z} = -[(\phi_1, \phi_2)dx^1 \wedge dx^2 = -\Phi \wedge \Phi,$$

which shows that the first eq. in (44) corresponds to the first equation in (45). Now

$$\bar{\partial}_A \Phi_c = \frac{1}{2}(D_1 + iD_2)(\phi_1 - i\phi_2)dz = \frac{1}{2}[D_1\phi_1 + D_2\phi_2 + i(D_2\phi_1 - D_1\phi_2)]dz$$

and hence the second eq. in (44) is equivalent to $D_1\phi_1 + D_2\phi_2 = 0$ and $D_2\phi_1 - D_1\phi_2 = 0$, but these are precisely the expressions in components of $D \ast \Phi = 0$ and $D\Phi = 0$.

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