Classification and Moduli Kähler Potentials of $G_2$ Manifolds

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Abstract

Compact manifolds of $G_2$ holonomy may be constructed by dividing a seven-torus by some discrete symmetry group and then blowing up the singularities of the resulting orbifold. We classify possible group elements that may be used in this construction and use this classification to find a set of possible orbifold groups. We then derive the moduli Kähler potential for M-theory on the resulting class of $G_2$ manifolds with blown up co-dimension four singularities.

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1 Introduction

Compactification of M-theory on a seven-dimensional space of holonomy $G_2$ provides a way of obtaining a four-dimensional theory with $N = 1$ supersymmetry. It is well known that in order to obtain a realistic compactified theory there must be singularities present on the $G_2$ manifold. Compactification of 11-dimensional supergravity on a smooth $G_2$ manifold results in a theory containing only abelian gauge multiplets and massless uncharged chiral multiplets $[1, 2]$. To be more specific, within this framework, gauge fields descend from dimensional reduction of the three-form field of 11-dimensional supergravity. (There are no continuous symmetries of a manifold of $G_2$ holonomy by which to obtain gauge fields.) Chiral multiplets arise from the metric moduli of the $G_2$ manifold and the associated axions.

Non-abelian gauge symmetry arises when the $G_2$ manifold has co-dimension four $A - D - E$ orbifold singularities. In addition, chiral fermions, that are possibly charged under the gauge multiplets, arise when the locus of such a singularity passes through an isolated conical (co-dimension seven) singularity $[3]$.

In the context of supergravity, one often work with smooth manifolds, but in the present setting we should keep in mind that it is singular limits of the smooth $G_2$ manifolds which are ultimately going to be interesting for phenomenology. (For the manifolds constructed in this paper, the limit in which some or all blow-up moduli are shrunk down to zero.)

There has been much work done on the subject of M-theory on $G_2$ spaces, in which calculations are carried out for some generic, possibly compact manifold of $G_2$ holonomy or else for some specific non-compact $G_2$ manifold $[3]-[23]$. However, it is clear that potentially realistic examples should rely on compact manifolds of $G_2$ holonomy, based on $G_2$ orbifolds, has been constructed in $[26]$. It is an interesting task to pursue this method to construct compact $G_2$ manifolds further and, hence, to classify $G_2$ orbifolds and their associated, blown-up $G_2$ manifolds. In this paper we propose a method for such a classification, and then construct an explicit class of manifolds, many of which appear to be new.

Having obtained a class of manifolds of holonomy $G_2$ it would be useful to be able to compare the four-dimensional effective theories resulting from compactification on different members of the class. An important ingredient for beginning such analysis is the four-dimensional moduli Kähler potential. This has obvious applications to various areas of study, for example, supersymmetry breaking or the cosmological dynamics of moduli fields. For the $G_2$ manifold based on the simplest $G_2$ orbifold $T^7/\mathbb{Z}_3^2$ the Kähler potential has been calculated in Ref. $[24]$. In this paper, we generalize this result by deriving a formula for the moduli Kähler potential valid for the manifolds of our classification.

There are classes of internal manifolds for which a general, explicit formula for the moduli Kähler potential exists (at least at tree level) in terms of certain topological data. For example, Calabi-Yau three-folds have their moduli Kähler potential determined by a cubic polynomial with coefficients given by their triple intersection numbers $[33, 34]$. Key to this result is a quasi topological relation between two-forms and their Hodge duals on the Calabi-Yau space. There appears to be no analogue for three-forms on $G_2$ manifolds. There do exist $[8]$ abstract formulae for the moduli Kähler metric in terms of harmonic three-forms on the $G_2$ manifold, and for the Kähler potential in terms of the volume of the $G_2$ manifold, but these cannot be evaluated generically for all $G_2$ manifolds in an analogous way to Calabi-Yau Kähler moduli spaces. Our approach to compute the Kähler potential...
is to explicitly construct all required objects for all members of our class of manifolds. Concretely, we construct a family of $G_2$ structures $\varphi$ with small torsion, and the associated family of “almost Ricci-flat” metrics $g$. These are used to calculate the volume and the periods which, combined, lead to an explicit expression for the Kähler potential.

Let us give a brief definition of what a $G_2$ manifold is so that we can describe the general idea of how to construct compact $G_2$ manifolds from $G_2$ orbifolds. A $G_2$ manifold is a seven-dimensional Riemannian manifold admitting a globally defined torsion-free $G_2$ structure. A $G_2$ structure is given by a three-form $\varphi$ which can be written locally as

$$
\varphi = dx^1 \wedge dx^2 \wedge dx^3 + dx^1 \wedge dx^4 \wedge dx^5 + dx^1 \wedge dx^6 \wedge dx^7 + dx^2 \wedge dx^4 \wedge dx^6
-dx^2 \wedge dx^5 \wedge dx^7 - dx^3 \wedge dx^4 \wedge dx^7 - dx^3 \wedge dx^5 \wedge dx^6.
$$

(1.1)

The $G_2$ structure is torsion-free if $\varphi$ satisfies $d\varphi = d\ast \varphi = 0$. A $G_2$ manifold has holonomy $G_2$ if and only if its first fundamental group is finite. Our starting point for constructing a compact manifold of $G_2$ holonomy is an arbitrary seven-torus $T^7$. We then take the quotient with respect to a finite group $\Gamma$ contained in $G_2$, such that the resulting orbifold has finite first fundamental group. We shall refer to $\Gamma$ as the orbifold group. The result is a $G_2$ manifold with singularities at fixed loci of elements of $\Gamma$. Smooth $G_2$ manifolds are then obtained by blowing up the singularities. Loosely speaking, this involves removing a patch around the singularity and replacing it with a smooth space of the same symmetry. Note that, following this construction, the independent moduli will come from torus radii and from the radii and orientation of cycles associated with the blow-ups.

Let us also give a brief outline of what is involved in the calculation of the moduli Kähler potential. On a $G_2$ manifold $\mathcal{M}$, a given $G_2$ structure $\varphi$ induces a Riemannian metric. Ricci-flat deformations of the metric can be described by the torsion-free deformations of $\varphi$, and hence by the third cohomology $H^3(\mathcal{M}, \mathbb{R})$. Consequently, the number of independent metric moduli is given by the third Betti number $b^3(\mathcal{M})$. To define these moduli explicitly, we introduce an integral basis $\{C^I\}$ of three-cycles, and a dual basis $\{\Phi_I\}$ of harmonic three forms satisfying

$$
\int_{C^I} \Phi_J = \delta^I_J.
$$

(1.2)

Here $I, J, \ldots = 1, \ldots, b^3(\mathcal{M})$. We can then expand $\varphi$ as

$$
\varphi = \sum_I a^I \Phi_I,
$$

(1.3)

where the coefficients $a^I$ are precisely our metric moduli. Then, by equation (1.2), the $a^I$ can be computed in terms of certain underlying geometrical parameters (on which the $G_2$ structure $\varphi$ depends) by performing the period integrals

$$
a^I = \int_{C^I} \varphi.
$$

(1.4)

In the four-dimensional effective theory the $a^I$ form the real bosonic part of $b^3(\mathcal{M})$ chiral superfields $T^I$. (The corresponding imaginary parts are axionic fields descending from dimensional reduction
of the three-form field of 11-dimensional supergravity.) It is the Kähler potential for these fields \( T^I \) which we wish to compute explicitly. Our strategy for doing this will be to compute the volume of the manifold and to then use the formula

\[
K = -3 \ln \left( \frac{V}{2\pi^2} \right)
\]

from Ref. [8]. With the result for the period integrals this expression for \( K \) can then be re-written in terms of the \( a^I \) and, hence, the superfields \( T^I \).

Ideally one would like to perform the calculation using a torsion-free \( G_2 \) structure. However, such torsion-free structures are not known explicitly on compact \( G_2 \) manifolds. Instead, as in Ref. [26], we write down explicit \( G_2 \) structures with small torsion and compute the Kähler potential in a controlled approximation, knowing that there exist “nearby” torsion-free \( G_2 \) structures.

The plan of the paper is as follows. In the next two sections we describe the classification of orbifold-based compact \( G_2 \) manifolds. The classification shall be in terms of the orbifold group of the manifold, and so in Section 2 we draw up a list of symmetries of seven-tori that may be used as generators of orbifold groups. For a symmetry \( \alpha \) to be suitable for the orbifolding there must exist a \( G_2 \) structure \( \varphi \) on the torus that is preserved by \( \alpha \). Then in Section 3 we look at ways of combining these symmetries to form orbifold groups that give orbifolds of finite first fundamental group. We find a straightforward way of checking when this condition is satisfied. A summary of the results is as follows. There is only one possible abelian orbifold group, \( \mathbb{Z}_3^2 \), with three or less generators. Further, we have been looking for viable examples within the class of orbifold groups formed by three or less generators with co-dimension four singularities subject to an additional technical constraint on the allowed underlying lattices. Within this class we have found all viable examples consisting of ten distinct semi-direct product groups with three generators as well as five exceptional cases built from three generators with a more complicated algebra.

In Section 4 we give a description of a general \( G_2 \) manifold with blown up co-dimension four orbifold fixed points of type \( A \). We present a basis of its third homology, and write down formulae for metrics and \( G_2 \) structures of small torsion. Then in Section 5 having all the machinery in place, we compute the moduli Kähler potential for our class of \( G_2 \) manifolds, valid for sufficiently small blow-up moduli.

To keep the main text more readable we have collected some of the technical details in Appendices A and C. Appendix A contains some results on \( G_2 \) structures useful for the classification of Sections 2 and 3 whilst Appendix C has some of the details of how to blow-up singularities and some calculations on the associated Gibbons-Hawking spaces. Finally, Appendix B contains a table listing the possible orbifold group elements.

### 2 Classification of Orbifold Group Elements

The most general group element \( \alpha \) that we consider acts on seven-dimensional vectors \( \mathbf{x} \) by \( \alpha : \mathbf{x} \mapsto A_{(\alpha)} \mathbf{x} + b_{(\alpha)} \), where \( A_{(\alpha)} \) is an orthogonal matrix and \( b_{(\alpha)} \) is a shift vector. We shall find all such \( \alpha \) that give a consistent orbifolding of some seven-torus, and that preserve some \( G_2 \) structure. Mathematically, by a seven-torus, we mean the fundamental domain of \( \mathbb{R}^7 / \Lambda \), where \( \Lambda \) is some seven-dimensional lattice. In the following we shall use bold Greek letters to denote lattice vectors.
Let us begin by considering the orbifolding. In the following, bold Greek letters denote lattice vectors. For consistency we require that two points \( \mathbf{x} \) and \( \mathbf{y} \) in one unit cell are equivalent under the orbifolding if and only if the corresponding two points \( \mathbf{x} + \lambda \) and \( \mathbf{y} + \lambda \) in another unit cell are equivalent. Suppose

\[
\mathbf{y} = A_{(\alpha)} \mathbf{x} + \mathbf{b}_{(\alpha)} + \mu.
\]  
(2.1)

Then there must exist a \( \nu \) such that

\[
\mathbf{y} + \lambda = A_{(\alpha)}(\mathbf{x} + \lambda) + \mathbf{b}_{(\alpha)} + \nu.
\]  
(2.2)

It follows that \( \alpha \) gives a consistent orbifolding precisely when \( A_{(\alpha)} \) takes lattice vectors to lattice vectors. Note that there is no constraint on \( \mathbf{b}_{(\alpha)} \).

We are interested in classifying orthogonal matrices that preserve a seven-dimensional lattice. A useful starting point is to find out the possible orders of such orthogonal group elements. We first quote a result from Ref. [25] in \( n \)-dimensions and then apply it to \( n = 7 \). Let \( A \in O(n) \) be of order \( m \). Then all its eigenvalues are \( m \)th roots of unity, and so we can choose an orthonormal basis for \( \mathbb{R}^n \) with respect to which

\[
A = \begin{pmatrix}
A_1 & & \\
& A_2 & \\
& & \ddots \\
& & & A_k
\end{pmatrix}
\]  
(2.3)

with \( A_j \in O(d_j) \), the eigenvalues of \( A_j \) being primitive \( m_j \)th roots of unity, \( m_j | m \), and \( m_i \neq m_j \) for \( i \neq j \). The result is that there exists a seven-dimensional lattice preserved by \( A \) if and only if we can write each \( d_j \) in the form

\[
d_j = n_j \varphi(m_j),
\]  
(2.4)

where \( n_j \in \mathbb{N} \) and \( \varphi(m) \) is Euler’s function, the number of integers less than \( m \) that are prime to \( m \). Furthermore, for an \( A \) that satisfies (2.4), each primitive \( m_j \)th root of unity is an eigenvalue of \( A_j \) with geometric multiplicity precisely \( n_j \).

We now consider the values \( m_j \) is allowed to take when \( n = 7 \). Since \( d_j \leq 7 \), \( A_j \) can only be a constituent block of \( A \) if \( \varphi(m_j) \leq 7 \). There is a formula from number theory

\[
\varphi(a) = \prod_i (p_i - 1)p_i^{r_i - 1},
\]  
(2.5)

where now \( a = \prod p_i^{r_i}, r_i \in \mathbb{N} \) is the prime decomposition of \( a \). It is then straightforward to show from Eq. (2.5) that the allowed values of \( m_j \) are \( m_j = 1, 2, \ldots, 10, 12, 14, 18 \). It is also easy to see that this result is the same as for \( n = 6 \), a fact which we will use, since the \( n = 6 \) case is the relevant one for classifying orbifold-based Calabi-Yau spaces.

We now look for conditions on \( A \) to belong in \( G_2 \). According to [26] we must have \( A \in SO(7) \). It therefore has eigenvalues 1 and complex conjugate pairs of modulus one. Hence we can write \( A \) in the canonical form

\[
A = \begin{pmatrix}
1 & R(\theta_1) \\
R(\theta_2) & R(\theta_3)
\end{pmatrix},
\]  
(2.6)
where

\[ R(\theta_i) = \begin{pmatrix} \cos \theta_i & -\sin \theta_i \\ \sin \theta_i & \cos \theta_i \end{pmatrix}. \]  

(2.7)

Accordingly, \( A \) decomposes as

\[ A = 1 \oplus A', \]  

(2.8)

where \( A' \in SO(6) \).

As an aside, let us just mention that naively one may have first derived the decomposition (2.8) and then decided immediately that this implies that the set of possible orders of \( A \) is identical to the set of possible orders of symmetries of six-dimensional lattices. Although this does indeed turn out to be the case, it is not obvious that the \( A \) of (2.8) preserves a seven-dimensional lattice if and only if the corresponding \( A' \) preserves a six-dimensional lattice. Rather, this can be shown by applying Eq. (2.4) which leads to the same allowed values of \( m_j \) for the cases \( n = 6 \) and \( n = 7 \). 

Now \( A \in G_2 \) if and only if it leaves a \( G_2 \) structure invariant. In other words \( A \) must leave \( \varphi \) defined by equation (1.1) invariant, or else there must exist an \( O(7) \) transformation taking \( \varphi \) to a three-form \( \tilde{\varphi} \) that is left invariant by \( A \). It is convenient to recast \( \varphi \) in complex form by taking

\[ x_0 = x_1, \ z_1 = \frac{1}{\sqrt{2}}(x_2 + ix_3), \ z_2 = \frac{1}{\sqrt{2}}(x_4 + ix_5), \ z_3 = \frac{1}{\sqrt{2}}(x_6 + ix_7), \]  

(2.9)

to obtain

\[ \varphi = dx_0 \wedge i(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3) + \sqrt{2}dz_1 \wedge dz_2 \wedge dz_3 \]  

\[ + \sqrt{2}d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3. \]  

(2.10)

We can then see by inspection that \( A \) preserves \( \varphi \) if and only if

\[ \theta_1 + \theta_2 + \theta_3 = 0 \mod 2\pi. \]  

(2.11)

The following is also easily verified. Under a transformation only containing reflections in coordinate axes, asking for the resulting \( \tilde{\varphi} \) to be left invariant by \( A \) imposes one of the following four conditions:

\[ \theta_1 + \theta_2 + \theta_3 = 0 \mod 2\pi, \]  

(2.12)

\[ -\theta_1 + \theta_2 + \theta_3 = 0 \mod 2\pi, \]  

(2.13)

\[ \theta_1 - \theta_2 + \theta_3 = 0 \mod 2\pi, \]  

(2.14)

\[ \theta_1 + \theta_2 - \theta_3 = 0 \mod 2\pi. \]  

(2.15)

Hence that \( A \) satisfies one of (2.12), (2.13), (2.14) or (2.15) is sufficient for \( A \in G_2 \). It turns out that this is also necessary. The proof of this is somewhat technical, and an outline of it is given in Appendix A. It is useful to note that the condition we have on \( A \) to belong in \( G_2 \) is precisely the condition (2.4) on the \( A' \) of (2.8) to belong in \( SU(3) \) under some embedding of \( SU(3) \) in \( SO(6) \).

By combining the above results we see that the classification is in one-to-one correspondence with that of the possible orbifold group elements of a Calabi-Yau space, which is given, for example, in Refs. [24] and [29]. The table in Appendix B gives this classification in terms of the rotation angles \( \theta_i \).
3 Classification of Orbifold Groups

Having obtained a class of possible generators, we now wish to find a class of discrete symmetry groups from which compact manifolds of $G_2$ holonomy may be constructed. Let us state the conditions for $\Gamma$ to be a suitable orbifold group. There must exist both a seven-dimensional lattice $\Lambda$ and a $G_2$ structure $\varphi$ that are preserved by $\Gamma$, and the first fundamental group $\pi_1$ of $(\mathbb{R}^7/\Lambda)/\Gamma$ must be finite.

It is useful to translate the condition on $\pi_1$ into an equivalent condition that is more readily checked. An equivalent condition is that there exist no non-zero vectors $n$ with the property that $A(\alpha)n = n$ for each $\alpha \in \Gamma$. That this condition is sufficient for $\pi_1$ to be finite is shown in Ref. [26]. That this is necessary is demonstrated below.

Let $\{\lambda_j\}$ be a basis of lattice vectors and write $n$ in the form

$$n = \sum_j n_j \lambda_j. \quad (3.1)$$

Let $\{\alpha_1, \ldots, \alpha_k\}$ be the generators of $\Gamma$. Then applying $\alpha_l$ to both sides of (3.1) we have

$$n = \sum_{j,i} n_j a_{ji}^{(l)} \lambda_i, \quad (3.2)$$

where the $a_{ji}^{(l)}$ are matrices with integer coefficients. From (3.1) and (3.2) we obtain

$$\sum_j n_j b_{ji}^{(l)} = 0, \quad \forall i, l, \quad (3.3)$$

where $b_{ji}^{(l)} = a_{ji}^{(l)} - \delta_{ji}$. Now since, by assumption, there exist non-zero solutions to (3.3), in constructing a particular solution we may choose the value of at least one of the $n_i$’s. Let us set the value of this $n_i$ to unity. Now consider another $n_i$. If it is free then let us set it also to unity. If it is constrained then it must be a linear function of other $n_i$’s with rational coefficients. We have hence constructed a solution to (3.3) with each $n_i$ rational. Now, for our solution, write $n_i$ in the form

$$n_i = \frac{p_i}{q_i}, \quad (3.4)$$

with $p_i$ and $q_i$ integers. Then $\text{lcm}\{q_i\}n$ is a lattice vector (where lcm stands for “lowest common multiple”) and so the path that is a straight line from the origin of the orbifold to $w\text{lcm}\{q_i\}n$, with $w$ an integer, is a path of winding number $w$. This establishes the result.

Given the above result, it is clear from (2.8) that a group $\Gamma$ must contain more than one generator if the resulting orbifold $T^7/\Gamma$ is to be of holonomy $G_2$. We now attempt to construct abelian groups of the form $\mathbb{Z}_m \times \mathbb{Z}_n$ for which the corresponding orbifold has holonomy $G_2$. For now we take the generator of the $\mathbb{Z}_n$ symmetry to be a straightforward rotation

$$R = \begin{pmatrix} 1 & R(\theta_1) & R(\theta_2) & R(\theta_3) \\ R(\theta_1) & 1 & R(\theta_2) & R(\theta_3) \\ R(\theta_2) & R(\theta_2) & 1 & R(\theta_3) \\ R(\theta_3) & R(\theta_3) & R(\theta_3) & 1 \end{pmatrix}, \quad (3.5)$$
with \( R(\theta_i) \) as in (2.1) and \((\theta_1, \theta_2, \theta_3)\) one of the triples of the table in Appendix A. We look for a second symmetry, also a pure rotation, with corresponding matrix \( P \) commuting with \( R \) such that the group generated by \( P \) and \( R \) is a suitable orbifold group. To find the constraints on \( P \) coming from commutativity we apply a generalization of Schur’s Lemma, as follows.

Write the reducible representation \( R \) of the group \( G \) as \( R = n_1 R_1 \oplus \cdots \oplus n_r R_r \), where the \( R_i \) are irreducible representations of \( G \) of dimension \( d_i \) and the integers \( n_i \) indicate how often each \( R_i \) appears in \( R \). Then a matrix \( P \) with \([P, R(g)] = 0\) for all \( g \in G \) has the general form

\[
P = P_1 \otimes 1_{d_1} \oplus \cdots \oplus P_r \otimes 1_{d_r},
\]

(3.6)

where the \( P_i \) are \( n_i \times n_i \) matrices.

The table in Appendix B lists the \( n_i \) and \( d_i \) for each \( R \) we are considering. We can therefore simply go through this table and, for each \( R \), see if there are any \( P \)s that are suitable for our construction. It turns out that there are in fact no suitable \( P \)s for any of the \( R \)s. In fact every case fails because \( \pi_1 \) is not finite. Below is one example to provide an illustration.

There is the possibility that \( R \) represents a \( \mathbb{Z}_2 \) symmetry of the lattice and is given by

\[
R = \text{diag}(1,1,1,-1,-1,-1).
\]

(3.7)

Then we have \( n_1 = 3 \), \( d_1 = 1 \) and \( n_2 = 4 \), \( d_2 = 1 \). Applying the lemma (3.6), \( P \) must take the form

\[
P = \begin{pmatrix} P_{3 \times 3}^1 \oplus \cdots \oplus P_{4 \times 4}^4 \\ P_{2 \times 2}^2 \end{pmatrix}.
\]

(3.8)

Now (see Appendix A), any \( G_2 \) structure preserved by \( R \) has \(|\varphi_{123}| = 1\). Under \( P \), \( \varphi_{123} \mapsto \text{det}(P_{3 \times 3}^1)\varphi_{123} \) and so we must have \( \text{det}(P_{3 \times 3}^1) = 1 \). Hence \( P_{3 \times 3}^1 \in SO(3) \), and leaves at least one direction fixed. But \( R \) fixes this direction too, since it fixes all of the 1, 2 and 3 directions. We therefore rule out this case since we can not render \( \pi_1 \) finite by this construction.

Let us now attempt to construct an orbifold group of the form \( \mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_p \). As above, we let \( R \) generate \( \mathbb{Z}_m \), and use the same method as above to find the possibilities for the other generators \( P \) and \( Q \). In looking for \( P \), most possibilities are still ruled out, but we can now relax the condition that there are no non-zero fixed vectors of the group generated by \( R \) and \( P \). There are then three cases we need to consider. Firstly, \( \frac{1}{2\pi} (\theta_1, \theta_2, \theta_3) = (\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}) \). In this case, we find that \( P \) must take the form

\[
P = (-1) \oplus \begin{pmatrix} \cos \phi_1 & \sin \phi_1 \\ \sin \phi_1 & -\cos \phi_1 \end{pmatrix} \oplus \begin{pmatrix} \cos \phi_2 & \sin \phi_2 \\ \sin \phi_2 & -\cos \phi_2 \end{pmatrix} \oplus \begin{pmatrix} \cos \phi_3 & \sin \phi_3 \\ \sin \phi_3 & -\cos \phi_3 \end{pmatrix}.
\]

(3.9)

Here the \( 2 \times 2 \) blocks each represent the most general element of \( O(2) - SO(2) \). It is easily verified that such a matrix can not commute with \( R \), thus ruling out this case. (Note that Eq. (3.8) was a necessary but not sufficient condition for commutativity.) The second case is when \( n_1 = 3 \), \( d_1 = 1 \) and \( n_2 = 2 \), \( d_2 = 2 \), for which \( p = 3, 4 \) or 6. On checking the possibilities for this case we find that there is no way of forming a group with all the correct properties. We are left with one remaining case, the case with \( p = 2 \), and this leads to the group \( \mathbb{Z}_2^3 \). For this case, the matrices \( P, Q \) and \( R \) are given essentially uniquely by

\[
R = \text{diag}(1,1,1,-1,-1,-1),
\]

(3.10)
\[ P = \text{diag}(1, -1, -1, -1, -1, 1, 1), \quad (3.11) \]
\[ Q = \text{diag}(-1, -1, 1, 1, -1, -1, 1). \quad (3.12) \]

The next step is to consider not just pure rotations, but to now allow the group elements to contain translations as well. Let us derive a condition for commutativity. Let two generators of an abelian orbifold group be given by

\[ \alpha : x \mapsto Ax + a, \quad (3.13) \]
\[ \beta : x \mapsto Bx + b. \quad (3.14) \]

Then commutativity requires us to be able to write

\[ (\alpha \circ \beta)x = (\beta \circ \alpha)x + \sum n_j \lambda_j, \quad (3.15) \]

where the \( \lambda_j \) form a basis of lattice vectors and the \( n_j \) are integers. This gives

\[ [A, B]x + Ab - Ba + a - b = \sum n_j \lambda_j. \quad (3.16) \]

Since \( x \) may vary continuously we see that we must still have \([A, B] = 0\) and then we are left with

\[ (A - I)b - (B - I)a = \sum n_j \lambda_j. \quad (3.17) \]

We are now able to write down the most general abelian orbifold group, with at most three generators, from which a \( G_2 \) manifold may be constructed. We apply the constraint (3.17) to any translations added to the matrix transformations of equations (3.10)-(3.12). The result is that the most general set of generators act as follows on a vector \( x = (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \) of the standard seven-torus.

\[ \alpha : x \mapsto \left( x_1 + \frac{m_1}{2}, x_2 + \frac{m_2}{2}, x_3 + \frac{m_3}{2}, -x_4 + a_4, -x_5 + a_5, -x_6 + a_6, -x_7 + a_7 \right), \quad (3.18) \]
\[ \beta : x \mapsto \left( x_1 + \frac{n_1}{2}, -x_2 + b_2, -x_3 + b_3, -x_4 + b_4, -x_5 + b_5, x_6 + \frac{n_6}{2}, x_7 + \frac{n_7}{2} \right), \quad (3.19) \]
\[ \gamma : x \mapsto \left( -x_1 + c_1, -x_2 + c_2, x_3 + \frac{p_3}{2}, x_4 + \frac{p_4}{2}, -x_5 + c_5, -x_6 + c_6, x_7 + \frac{p_7}{2} \right), \quad (3.20) \]

where the \( m_i, n_i \) and \( p_i \) are integers and the \( a_i, b_i \) and \( c_i \) are unconstrained reals. The group generated is always \( \mathbb{Z}_2^3 \).

Our objective was to find a class of orbifold groups and to achieve this, it appears that commutativity is not the most suitable constraint to impose, in spite of the systematic approach it gave
us. Since well-defined procedures to describe the metric on the blow-ups are available for the cases with co-dimension four fixed loci, we now focus on orbifold groups that only lead to these. We thus restrict attention to generators that leave three directions of the torus invariant, namely those whose rotation part is one of \( \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4 \) or \( \mathbb{Z}_6 \) from the table in Appendix B. In fact, for this section we shall consider pure rotations only. We show that a simple constraint on the lattice itself enables us to come up with a substantial class of possible orbifold groups. Let us insist that, for each generator \( \alpha \) of the orbifold group, there exists a partition of our basis of lattice vectors into three sets, spanning the spaces \( U, V \) and \( W \), of dimension 3, 2 and 2 respectively such that

\[
\Lambda = U \perp V \perp W, \tag{3.21}
\]

\[
\alpha U = U, \quad \alpha V = V, \quad \alpha W = W, \tag{3.22}
\]

\[
\alpha|_U = \iota. \tag{3.23}
\]

This seems a sensible condition to impose, since it makes it easy to picture how the orbifold group acts on the lattice. Basically, each generator rotates two two-dimensional sub-lattices.

The classification of orbifold groups, subject to the above constraints and containing three or fewer generators, now goes as follows. We take as the first generator the matrix \( R \), in the canonical form of (3.5) with \( \theta_1 = 0 \) and \( \theta_2 = -\theta_3 = 2\pi/N \), where \( N = 2, 3, 4, \) or 6. We then use coordinate freedom to choose a \( G_2 \) structure \( \varphi \) that is the standard one of Eq. (1.1) up to possible sign differences (see Appendix A).

We are then able to derive the other possible generators of the orbifold group that are distinct up to redefinitions of the coordinates. First we narrow the possibilities using the constraint of \( G_2 \) structure preservation. In particular this imposes the condition that the three fixed directions must correspond precisely to one of the seven terms in the \( G_2 \) structure (see Appendix A). Having done this we look for a preserved lattice. It is straightforward to go through all possibilities having imposed (3.21), (3.22) and (3.23). We find the following distinct generators. Firstly matrices in the canonical form of (3.5) with \( \theta_1 = -\theta_2 = 2\pi/M \) and \( \theta_3 = 0 \), with \( M \) dividing \( N \) or vice-versa (with the exception \( M = 2, N = 3 \)). Secondly \( \mathbb{Z}_2 \) symmetries not in canonical form, for example

\[
Q_0 = \text{diag}(-1,1,-1,1,-1,1), \tag{3.24}
\]

and then, only for the cases \( N = 2 \) or 4, \( \mathbb{Z}_4 \) symmetries not in canonical form, for example

\[
Q_1 : (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mapsto (x_7, x_2, -x_5, x_4, x_3, x_6, -x_1). \tag{3.25}
\]

We can now go about combining these generators together. It is easy to see that, as when we were considering abelian groups, there are no orbifold groups built from just two generators that result in orbifolds with finite first fundamental group. Our class therefore consists solely of orbifold groups containing three generators. The main sub-class has generators \( P \) and \( R \) in the canonical form of (3.5), with \( P \) having \( \theta_1 = -\theta_2 = 2\pi/M \) and \( \theta_3 = 0 \) and \( R \) having \( \theta_1 = 0 \) and \( \theta_2 = -\theta_3 = 2\pi/N \), and
The semi-direct product notation is defined by writing \( G \rtimes H \) if \( G \) and \( H \) are abelian and \([g, h] \in H\) for any \( g \in G \) and \( h \in H \). In fact, within our class of orbifold groups, the stronger condition \([g, h] = h^2\) will be satisfied in each case of a semi-direct product. Note that since we should really be thinking of orbifold group elements as abstract group elements as opposed to matrices, the commutator is defined by \([g, h] = g^{-1}h^{-1}gh\). Some other semi-direct products are attained by using \( R \) as above and then taking \( Q_0 \) and \( \text{diag}(-1, -1, 1, 1, -1, -1, 1) \) as the other generators to obtain the following groups:

\[
Z_2^2 \rtimes Z_N, \; N = 3, 4 \text{ or } 6. \tag{3.27}
\]

There are also some exceptional cases that contain a \( Z_4 \) not in canonical form. These exceptional orbifold groups will contain only \( Z_2 \) and \( Z_4 \) symmetries. We make the observations that \( Z_2 \) and \( Z_4 \) symmetries either commute or give semi-direct products and that two \( Z_4 \) symmetries \( A \) and \( B \) either commute or have the relation \( A^2BA^2 = B^{-1} \). It is then straightforward to find all possible group algebras for our exceptional orbifold groups. Going through the conditions for a \( G_2 \) orbifold, we find some of these algebras can be realised and some can not. Those that can are constructed as follows. Take \( P \) and \( R \) as before with \( M = 2 \) and \( N = 4 \) and use either \( Q_1 \) from above or \( Q_2 \) given by

\[
Q_2 : (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mapsto (x_3, x_2, -x_1, x_4, -x_7, x_6, x_5). \tag{3.28}
\]

These lead to the respective groups

\[
E_1 =: \langle P, Q_1, R \mid P^2 = 1, Q_1^4 = 1, R^4 = 1, [P, Q_1] = 1, [P, R] = 1, Q_1^2RQ_1^2 = R^{-1} \rangle, \tag{3.29}
\]

and

\[
E_2 =: \langle P, Q_2, R \mid P^2 = 1, Q_2^4 = 1, R^4 = 1, [P, Q_2] = Q_2^2, [P, R] = 1, Q_2^2RQ_2^2 = R^{-1} \rangle. \tag{3.30}
\]

Then three more possibilities come about from using \( R \) with \( N = 4, Q_2 \) and one of the \( P_i \) given below.

\[
P_1 : (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mapsto (-x_1, -x_2, x_3, -x_4, x_5, x_6, -x_7), \tag{3.31}
\]

\[
P_2 : (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mapsto (x_1, -x_3, x_2, x_5, -x_4, x_6, x_7), \tag{3.32}
\]

\[
P_3 : (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mapsto (x_1, -x_5, x_4, -x_3, x_2, x_6, x_7). \tag{3.33}
\]
The groups we obtain can be described respectively by

\[ E_3 =: \langle P_1, Q_2, R \mid P_1^2 = 1, Q_2^4 = 1, R^4 = 1, [P_1, Q_2] = Q_2^2, [P_1, R] = R^2, Q_2^2 R Q_2^2 = R^{-1} \rangle, \quad (3.34) \]

\[ E_4 =: \langle P_2, Q_2, R \mid P_2^4 = 1, Q_2^4 = 1, R^4 = 1, P_2^2 Q_2 P_2^2 = Q_2^{-1}, [P_2, R] = 1, Q_2^2 R Q_2^2 = R^{-1} \rangle, \quad (3.35) \]

\[ E_5 =: \langle P_3, Q_2, R \mid P_3^4 = 1, Q_2^4 = 1, R^4 = 1, P_3^2 Q_2 P_3^2 = Q_2^{-1}, P_3^2 R P_3^2 = R^{-1}, Q_2^2 R Q_2^2 = R^{-1} \rangle. \quad (3.36) \]

It is worth briefly summing up what we have found. We have found a class of sixteen distinct groups, each composed of three pure rotations, that may be used as orbifold groups to construct compact manifolds of \( G_2 \) holonomy. If two simple constraints are imposed on the manifolds, we have the complete class of such orbifold groups. The first constraint states that the manifold is to have only co-dimension four fixed points. The second constraint is on the lattice from which the manifold is constructed. It states that the lattice must decompose into an orthogonal sum of smaller lattices, with three constituents of the sum being simple two-dimensional lattices, and the final constituent being the trivial one-dimensional lattice. Furthermore, the action of each orbifold group generator must be to simply rotate two two-dimensional sub-lattices.

There are two obvious ways of extending the classification we have obtained. One is to remove the restriction on the lattice. Perhaps this would give rise to some more complicated examples, in which lattice vectors do not lie precisely in the planes of rotation of the orbifold group generators. A second method would be to allow co-dimension six fixed points and thus allow any of the generators listed in Appendix B.

4 Description of the Manifolds and their \( G_2 \) Structures

In this section we describe in some detail the manifolds for which we shall compute the moduli Kähler potential. We take \( \mathcal{M} \) to be a general smooth \( G_2 \) manifold, constructed from an orbifold \( \mathcal{O} = T^7 / \Gamma \) with co-dimension four fixed points. We assume that points on the torus that are fixed by one generator of the orbifold group are not fixed by other generators. Given an orbifold group, this can always be arranged by incorporating appropriate translations into the generators, and thus all of our previously found examples are relevant. Under this assumption we have a well-defined blow-up procedure.

Let us introduce some notation. We use the index \( \tau \) to label the generators of the orbifold group \( \Gamma \), and write \( N_\tau \) for the order of the generator \( \alpha_\tau \). For \( \Gamma \) to only have co-dimension four fixed points the possible values of \( N_\tau \) are 2, 3, 4 and 6 (see Appendix B). Each generator will have a certain number \( M_\tau \) of fixed points associated with it. A singular point on \( \mathcal{O} \) is therefore labelled by a pair \( (\tau, n) \), where \( n = 1, \ldots, M_\tau \).

Near a singular point \( \mathcal{O} \) takes the approximate form \( T^3_{(\tau, n)} \times \mathbb{C}^2 / \mathbb{Z}_{N_\tau} \), where \( T^3_{(\tau, n)} \) is a three-torus. Blowing up the singularity involves the following. One firstly removes a four-dimensional ball
centred around the singularity times the associated fixed three-torus $T^3_{(r,n)}$. Secondly one replaces the resulting hole by $T^3_{(r,n)} \times U_{(r,n)}$, where $U_{(r,n)}$ is the blow-up of $\mathbb{C}^2/\mathbb{Z}_{N_r}$ as discussed in Appendix C.

Before giving a more detailed description of what $\mathcal{M}$ looks like, let us present a basis of three-cycles. This will be needed to compute the periods and hence the moduli in terms of underlying geometrical parameters. Localised on the blow-up labelled by $(r,n)$ there are $3(N_r - 1)$ three-cycles. These are formed by taking the Cartesian product of one of the $(N_r - 1)$ two-cycles on $U_{(r,n)}$ with one of the three one-cycles on $T^3_{(r,n)}$. Let us label these three-cycles by $C(\tau, n, a, i)$, where $a$ labels the direction on $T^3_{(r,n)}$ and $i$ labels the two-cycles of $U_{(r,n)}$. On the bulk, that is the remaining parts of the torus, we can define three-cycles by setting four of the coordinates $x^A$ to constants (chosen so there is no intersection with any of the blow-ups). The number of these that fall into distinct homology classes is then given by the number of independent terms in the $G_2$ structure on the bulk. Let us explain this statement. The bulk $G_2$ structure can always be chosen so as to contain the seven terms of the standard $G_2$ structure \ref{G2structure}, with positive coefficients multiplying them. If we write $\mathcal{R}^A$ for the coefficient in front of the $A^{th}$ term in Eq. \ref{G2structure}, then by the number of independent terms we mean the number of $\mathcal{R}^A$s that are not constrained by the orbifolding. We then write $C^A$ for the cycle obtained by setting the four coordinates on which the $A^{th}$ term in \ref{G2structure} does not depend to constants, for example,

$$C^1 = \{x^4, x^5, x^6, x^7 = \text{const}\}.$$  \tag{4.1}$$

A pair of $C^A$s for which the corresponding $\mathcal{R}^A$s are independent then belong to distinct homology classes. There is therefore some subset $\mathcal{C}$ of $\{C^A\}$ such that the collection $\{\mathcal{C}, C(\tau, n, a, i)\}$ provides a basis for $H_3(\mathcal{M}, \mathbb{Z})$. We deduce the following formula for the third Betti number of $\mathcal{M}$:

$$b^3(\mathcal{M}) = b(\Gamma) + \sum_{\tau} M_\tau \cdot 3(N_\tau - 1),$$  \tag{4.2}$$

where $b(\Gamma)$ is the number of bulk three-cycles, a positive integer less or equal than seven, and dependent on the orbifold group $\Gamma$. For the class of orbifold groups obtained in the previous section $b(\Gamma)$ takes values as given in Table 1. A description of the derivation is given in the discussion below on constructing the bulk $G_2$ structure.

We now discuss the geometrical structure of $\mathcal{M}$ in more detail, focusing in particular on the blow-up regions. Metrics and $G_2$ structures will be presented on each region of $\mathcal{M}$. Let us begin with the bulk, which is the straightforward part. Assuming that, for a constant metric, only the diagonal components survive the orbifolding, which is certainly the case for the explicit examples constructed in Section 3

$$ds^2 = \sum_{A=1}^{7} (R^A dx^A)^2.$$ \tag{4.3}$$

Here the $R^A$ are precisely the seven radii of the torus. Under a suitable choice of coordinates the $G_2$ structure is obtained from the flat $G_2$ structure \ref{G2structure} by rescaling $x^A \rightarrow R^A x^A$, leading to

$$\varphi = R^1 R^2 R^3 dx^1 \wedge dx^2 \wedge dx^3 + R^1 R^4 R^5 dx^1 \wedge dx^4 \wedge dx^5 + R^1 R^6 R^7 dx^1 \wedge dx^6 \wedge dx^7 + R^2 R^4 R^6 dx^2 \wedge dx^4 \wedge dx^6 - R^2 R^5 R^7 dx^2 \wedge dx^5 \wedge dx^7 - R^3 R^4 R^7 dx^3 \wedge dx^4 \wedge dx^7 - R^3 R^5 R^6 dx^3 \wedge dx^5 \wedge dx^6.$$  \tag{4.4}$$
Now, for the orbifolding to preserve the metric some of the $R^A$ must be set equal to one another. It is straightforward to check that if $\alpha_r$ involves a rotation in the $(A, B)$ plane by an angle not equal to $\pi$, then we must set $R^A = R^B$. Following this prescription it is easy to find the function $b(\Gamma)$ discussed above.

On one of the blow-ups $T^3 \times U$ (for convenience we suppress $\tau$ and $n$ indices) we use coordinates $\xi^a$ on $T^3$ and four-dimensional coordinates $\zeta^\mu$ on $U$. We write $R^a$ to denote the three radii of $T^3$, which will be the three $R^A$ in the directions fixed by $\alpha$. The $G_2$ structure can be written as

$$\varphi = \sum_a \omega^a (w(\zeta), z(\zeta), b_1, \ldots, b_N) \wedge R^a d\xi^a - R^1 R^2 R^3 d\xi^1 \wedge d\xi^2 \wedge d\xi^3. \quad (4.5)$$

Here $w$ and $z$ are complex coordinates on $U$ and the $b_i \equiv (\text{Re } a_i, \text{Im } a_i, b_i)$ are a set of three-vectors, which parameterize the size of the blow-up and its orientation with respect to the bulk. The $\omega^a$ are a triplet of two-forms that constitute a “nearly” hyperkähler structure on $U$, as discussed in Appendix C. We will not need to know explicitly the relation between the two sets of coordinates $\zeta^\mu$ and $w$ and $z$, although we keep in mind that this relation will depend on the four radii $R^\mu$ transverse to the $R^a$. In terms of $w$ and $z$, we can write the $\omega^a$ as

$$\omega^1 = \frac{i}{2} \partial \bar{\partial} K, \quad (4.6)$$

$$\omega^2 = -\text{Re} \left( \frac{dw \wedge dz}{w} \right), \quad \omega^3 = -\text{Im} \left( \frac{dw \wedge dz}{w} \right), \quad (4.7)$$

where $K$ is the Kähler potential for $U$, which interpolates between that for Gibbons-Hawking space in the central region of the blow-up and that for flat space far away from the centre of the blow-up.

| $\Gamma$ | $b(\Gamma)$ |
|----------|--------------|
| $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ | 7 |
| $\mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_3)$ | 5 |
| $\mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_4)$ | 5 |
| $\mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_6)$ | 5 |
| $\mathbb{Z}_2 \times (\mathbb{Z}_3 \times \mathbb{Z}_3)$ | 4 |
| $\mathbb{Z}_2 \times (\mathbb{Z}_3 \times \mathbb{Z}_6)$ | 4 |
| $\mathbb{Z}_2 \times (\mathbb{Z}_4 \times \mathbb{Z}_4)$ | 4 |
| $\mathbb{Z}_2 \times (\mathbb{Z}_6 \times \mathbb{Z}_6)$ | 4 |
| $\mathbb{E}_1$ | 3 |
| $\mathbb{E}_2$ | 3 |
| $\mathbb{E}_3$ | 3 |
| $\mathbb{E}_4$ | 2 |
| $\mathbb{E}_5$ | 1 |

Table 1: Bulk third Betti numbers of the orbifold groups
We clarify this statement in the discussion below, but first we shall describe the central region of the blow-up, where \( U \) looks exactly like Gibbons-Hawking space. For a technical account of this discussion, including how to write \( K \) explicitly, we refer the reader to Appendix C.

Gibbons-Hawking spaces (or gravitational multi-instantons) provide a generalization of the Eguchi-Hanson space and their different topological types are labelled by an integer \( N \) (where the case \( N = 2 \) corresponds to the Eguchi-Hanson case). While the Eguchi-Hanson space contains a single two-cycle, the \( N \)th Gibbons-Hawking space contains a sequence \( \gamma_1, \ldots, \gamma_{N-1} \) of such cycles at the “centre” of the space. Only neighbouring cycles \( \gamma_i \) and \( \gamma_{i+1} \) intersect and in a single point and, hence, the intersection matrix \( \gamma_i \cdot \gamma_j \) equals the Cartan matrix of \( A_{N-1} \). Asymptotically, the \( N \)th Gibbons-Hawking space has the structure \( \mathbb{C}^2 / \mathbb{Z}_N \). Accordingly, we take \( N = N_\tau \) when blowing up \( \mathbb{C}^2 / \mathbb{Z}_{N_\tau} \). The metric on Gibbons-Hawking space can be written

\[
\begin{align*}
\gamma & = \sum_i \frac{1}{r_i}, \\
r_i & = \sqrt{(x - b_i)^2 + 4|z - a_i|^2}, \\
\delta & = \sum_i \frac{x - b_i - r_i}{2(z - \tilde{a}_i)r_i}, \\
w\bar{w} & = \prod_i (x - b_i + r_i).
\end{align*}
\]

Here \( x \) is a real coordinate, given implicitly in terms of \( w \) and \( z \) in the above equations. The sizes and orientations of the two-cycles are determined by the \( N_\tau \) points \( b_i \) in the \( \text{Re} \, z, \text{Im} \, z, x \) hyperplane. Concretely \( \gamma_i \) is parameterized by

\[
\begin{align*}
z & = a_i + \lambda(a_{i+1} - a_i), \\
w & = e^{i\theta}h(\lambda),
\end{align*}
\]

for some function \( h \), as \( 0 \leq \lambda \leq 1, 0 \leq \theta \leq 2\pi \[31\].

We can add a periodic real coordinate \( y \) to \( x, z \) and \( \bar{z} \) to form a well-defined coordinate system on the space. We can also define a radial coordinate \( r \) given by

\[
r = \sqrt{(x - \tilde{b})^2 + 4|z - \tilde{a}|^2},
\]

where tildes denote mean values over the index \( i \). It is precisely this radial coordinate that the interpolating function \( \epsilon \), appearing in the Kähler potential \( K \), depends on. We can now describe the interpolation more precisely. If we set all the \( a \) and \( b \) parameters of Gibbons-Hawking space to zero we have flat space. Therefore the method of constructing \( K \) is to start with the Kähler potential for
Gibbons-Hawking space, and to then place a factor of $\epsilon$ next to every $a$ and $b$ that appears. We can keep $\epsilon$ general, all we require are the following properties:

$$\epsilon(r) = \begin{cases} 
1 & \text{if } r \leq r_0, \\
0 & \text{if } r \geq r_1,
\end{cases}$$

(4.16)

where $r_0$ and $r_1$ are two fixed radii satisfying $|a_i| \ll r_0 < r_1$ and $|b_i| \ll r_0$ for each $i$. Then, as already discussed, $U$ is identical to Gibbons-Hawking space for $r < r_0$. For $r > r_1$ $U$ is identical to the flat space $\mathbb{C}^2/\mathbb{Z}_N$, and we can match the $G_2$ structure (4.5) to the bulk $G_2$ structure (4.1).

Let us briefly discuss the torsion of the $G_2$ structure (4.5). By virtue of the blow-up $U$ being hyperkähler in the regions $r < r_0$ and $r > r_1$, and the $\omega^a$ of equations (4.6) and (4.7) forming the triplet of closed and co-closed Kähler forms expected on such a space, the $G_2$ structure is torsion free in these regions. It departs from non-zero torsion only in the “collar” region $r \in [r_0, r_1]$, where $\omega^2$ and $\omega^3$ fail to be co-closed. However, for sufficiently small blow-ups, $|a_i| \ll 1$, $|b_i| \ll 1$, and a “smooth”, slowly-varying interpolation function $\epsilon$, the deviation from a torsion-free $G_2$ structure is small [24]. Consequently, we can use this $G_2$ structure to reliably compute the Kähler potential to leading non-trivial order in the $a_i$s and $b_i$s.

We end this section by briefly discussing the metric on a blow-up. The metric can be derived directly from the $G_2$ structure, using equations (C.4) and (C.5). Its structure is given by

$$ds^2 = G_0 d\zeta^2 + \sum_{a=1}^3 G_a (d\xi^a)^2,$$

(4.17)

where $d\zeta$ is the line element on the appropriate smoothed Gibbons-Hawking space, which can be derived from equations (C.9) and (C.37), and the $G$s are conformal factors, that may depend on the blow-up moduli and the interpolation function, but whose product must be equal to 1, since they do not appear in the measure (C.11).

5 Periods, Volumes and Kähler Potentials

Having written down a $G_2$ structure of small torsion on our general manifold $M$, we can now compute the periods. The bulk periods

$$a^A = \int_{\gamma^A} \varphi$$

(5.1)

are straightforward to obtain and are given by

$$a^1 = R_1 R_2 R_3, \quad a^2 = R_1 R_4 R_5, \quad a^3 = R_1 R_6 R_7, \quad a^4 = R_2 R_4 R_6,$$

$$a^5 = R_2 R_5 R_7, \quad a^6 = R_3 R_4 R_7, \quad a^7 = R_3 R_5 R_6.$$  

(5.2)

To find the periods associated with the blow-ups, we firstly require the period integrals

$$\int_{\gamma_i} \omega^a,$$

(5.3)

$i = 1, \ldots, N - 1$ on a general Gibbons-Hawking space. Following Ref. [31] we find

$$\int_{\gamma_i} \omega^1 = \frac{\pi}{2} (b_i - b_{i+1}),$$

(5.4)
\[
\int_{\gamma_i} (\omega^2 + i\omega^3) = \pi i (a_i - a_{i+1}).
\] (5.5)

Having obtained these we can write down the periods

\[ A(\tau, n, a, i) = \int_{C(\tau, n, a, i)} \varphi \] (5.6)
on each blow-up \(U(\tau, n) \times T^3_{(\tau, n)}\). We find

\[
A(\tau, n, 1, i) = \frac{\pi}{2} R_{(\tau)}^1 \left( b_{(\tau, n, i)} - b_{(\tau, n, i+1)} \right),
\]
\[
A(\tau, n, 2, i) = \frac{i\pi}{2} R_{(\tau)}^2 \left( a_{(\tau, n, i)} - \bar{a}_{(\tau, n, i)} - a_{(\tau, n, i+1)} + \bar{a}_{(\tau, n, i+1)} \right),
\]
\[
A(\tau, n, 3, i) = \frac{\pi}{2} R_{(\tau)}^3 \left( a_{(\tau, n, i)} + \bar{a}_{(\tau, n, i)} - a_{(\tau, n, i+1)} - \bar{a}_{(\tau, n, i+1)} \right). \tag{5.7}
\]

We remind the reader that \(R_{(\tau)}^a\) denote the three radii of \(T^3_{(\tau, n)}\), consistent with the notation of equation (4.5).

Our next task is to find the total volume of \(\mathcal{M}\). From the bulk metric (4.3), we see that the bulk contribution to the volume is proportional to \(\prod A R_A\). There will be a factor \(f(\Gamma)\) in front of this, dependent on the orbifold group \(\Gamma\). In certain simple cases this is just the inverse of the order of \(\Gamma\), and it is always calculable by obtaining the fundamental domain of the orbifold. For the purposes of most calculations, it is relatively unimportant, since one will be able to absorb it into the normalization of the blow-up moduli fields. The contributions to the volume from the blow-ups are easily obtainable from equations (C.11) and (C.43). Putting everything together we find, to lowest non-trivial order in the blow up moduli,

\[
V = f(\Gamma) \prod_A R_A - \frac{\pi^2}{6} \sum_{\tau, n} N_\tau \left( \text{var}_i \{ b_{(\tau, n, i)} \} + 2 \text{var}_i \{ \text{Re} a_{(\tau, n, i)} \} + 2 \text{var}_i \{ \text{Im} a_{(\tau, n, i)} \} \right) \prod_{\tau} R_{(\tau)}^a. \tag{5.8}
\]

Here var refers to the variance, with the usual definition:

\[
\text{var}_i \{ X_i \} = \frac{1}{N} \sum_i (X_i - \bar{X})^2. \tag{5.9}
\]

Note that this result is independent of the interpolation functions \(\epsilon_{(\tau, n)}\).

We are now ready to compute the Kähler potential. Using the results (5.7) and (5.2) for the periods, we can rewrite the volume (5.8) in terms of \(a^A\) and \(A(\tau, n, a, i)\), which constitute the real, bosonic parts of superfields. We denote these superfields by \(T^A\) and \(U^{(\tau, n, a, i)}\) such that

\[
\text{Re}(T^A) = a^A, \quad \text{Re}(U^{(\tau, n, a, i)}) = A(\tau, n, a, i). \tag{5.10}
\]

Note that in many cases some of the \(T^A\)’s are identical to each other and should be thought of as the same field. As discussed in Section 4 this comes about from requiring some of the radii \(R_A\) to be
Table 2: Values of the index functions $(A(\tau, a), B(\tau, a))$ specifying the bulk moduli $T^A$ by which the blow-up moduli $U^{(\tau,n,a,i)}$ are divided in the Kähler potential.

| Fixed directions of $\alpha_{\tau}$ | $a = 1$ | $a = 2$ | $a = 3$ |
|------------------------------------|--------|--------|--------|
| $(1,2,3)$                          | $(2,3)$| $(4,5)$| $(6,7)$|
| $(1,4,5)$                          | $(1,3)$| $(4,6)$| $(5,7)$|
| $(1,6,7)$                          | $(1,2)$| $(4,7)$| $(5,6)$|
| $(2,4,6)$                          | $(1,5)$| $(2,6)$| $(3,7)$|
| $(2,5,7)$                          | $(1,4)$| $(2,7)$| $(3,6)$|
| $(3,4,7)$                          | $(1,7)$| $(2,4)$| $(3,5)$|
| $(3,5,6)$                          | $(1,6)$| $(2,5)$| $(3,4)$|

equal for a consistent orbifolding of the base torus. The number of distinct $T^A$ is $b(\Gamma)$, which for the orbifold groups $\Gamma$ constructed in Section 3 is given in Table 1. To discover which $T^A$ are equal the procedure is as follows. Let $x^A$ be coordinates in the bulk with respect to which the $G_2$ structure is given by (4.4). Then a generator $\alpha_{\tau}$ of $\Gamma$ acts by simultaneous rotations in two planes $(A, B)$ and $(C, D)$ say. If the order of $\alpha_{\tau}$ is greater than two, then we identify $R^A$ with $R^B$ and $R^C$ with $R^D$. We go through this process for all generators of $\Gamma$ and then use (5.12) to determine which of the $a^A$ and hence $T^A$ are equal. From Eq. (1.5) we find for the Kähler potential

$$K = -\sum_{A=1}^{7} \ln(T^A + \bar{T}^A) - 3 \ln \left( 1 - \frac{2}{3f(\Gamma)} \sum_{n,\tau,a} \frac{1}{N_\tau} \left( \frac{\sum_{k=1}^{j-1} U^{(\tau,n,a,k)} + \bar{U}^{(\tau,n,a,k)}}{T^A(\tau,a) + \bar{T}^A(\tau,a)(T^B(\tau,a) + \bar{T}^B(\tau,a))} \right)^2 \right) + c, \quad (5.11)$$

where the constant $c$ is given by

$$c = 10 \ln 2 - 3 \ln f(\Gamma) + 6 \ln \pi. \quad (5.12)$$

The index functions $A(\tau,a), B(\tau,a) \in \{1, \ldots, 7\}$ indicate by which two of the seven bulk moduli $T^A$ the blow up moduli $U^{(\tau,n,a,i)}$ are divided in the Kähler potential (5.11). Their values depend only on the generator index $\tau$ and the orientation index $a$. They may be calculated from the formula

$$a^{A(\tau,a)} a^{B(\tau,a)} = \left( \frac{R^a_{(\tau)}}{\prod_b R^b_{(\tau)}} \right)^2 \prod_A R^A.$$

The $\tau$ dependence is only through the fixed directions of the generator $\alpha_{\tau}$ and the possible values of the index functions are given in Table 2.

We now state precisely and systematically the scenarios in which (5.11) is valid. Firstly all moduli must be larger than one (in units where the Planck length is set to one) so that the supergravity approximation to M-theory is valid. Secondly, all blow-up moduli $U^{(\tau,n,a,i)}$ must be small compared to the bulk moduli $T^A$ so that corrections of higher order in $U/T$ can be neglected. The action of the generators of the orbifold group $\Gamma$ on the base seven-torus $T^7$ must lead to an orbifold $O$ with co-dimension four singularities. Furthermore, no two generators must fix the same point on the torus. This last requirement is to ensure that the assumption of the structure $T^3 \times \mathbb{C}^2/\mathbb{Z}_N$ around the singularities is correct.
6 Conclusion

Let us summarize what we have found, and mention some possible extensions. Equation (5.11) gives the moduli Kähler potential for a large class of compact manifolds of holonomy $G_2$. This class contains manifolds constructed from a large number of different orbifolds, based on at least sixteen distinct orbifold groups, namely those constructed in Section 3. Moduli fields fall into two categories. Firstly the fields $T^A$, which descend from the seven radii of the manifold, and secondly the fields $U^{(\tau,n,a,i)}$ which descend from geometrical parameters describing the blow-up of singularities of the manifold (namely the radii of two-cycles on the blow-up and their orientation with respect to the bulk). Our formula constitutes the first two terms in an expansion of the Kähler potential in terms of the $U$s. The zeroth order term is simply a consequence of the volume of the manifold being proportional to the product of the seven radii. It is not surprising that the lowest order correction terms arise at second order in the $U$s. Heuristically, one can think of all $U$ dependent terms as being associated with the volume subtracted from the manifold as a result of the presence of two-cycles on the blow-ups. One expects these terms to depend on the two-cycles through their area, and hence on even powers of the $U$s. Dimensional analysis insists that all second-order terms in the $U$s are homogeneous of order minus two in the $T$s, but it did not have to necessarily turn out that the terms took on so simple a form in the general case. This is an attractive feature of our result.

An interesting extension of the work in this paper would be to attempt to include conical, or co-dimension seven, singularities onto the manifolds, thus supporting charged chiral matter. We found a large number of orbifold group elements that lead to co-dimension six singularities. One could attempt to generalize the blow-up procedure and Kähler potential calculation to the case of manifolds with orbifold groups containing such elements. There is also the possibility of a more complicated orbifold fixed point structure. For example, if there exist points on the torus that are fixed by more than one generator of the orbifold group, there can be several topologically distinct ways of blowing up the associated singularity [26]. Questions of moduli stabilization can also now be looked at in a more general context by application of our Kähler potential formula.

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Appendix

A Some results about $G_2$ structures

In this appendix we derive some results about $G_2$ structures that will be used in our calculations.
Let $A$ be an element of $SO(7)$ of the form

$$A = \begin{pmatrix} 1 & R(\theta_1) & R(\theta_2) & R(\theta_3) \\ R(\theta_1) & R(\theta_2) & R(\theta_3) & R(\theta_4) \end{pmatrix}, \quad (A.1)$$

where

$$R(\theta_i) = \begin{pmatrix} \cos \theta_i & -\sin \theta_i \\ \sin \theta_i & \cos \theta_i \end{pmatrix}. \quad (A.2)$$

Then $A$ is in $G_2$ (for some embedding of $G_2$ into $SO(7)$) if and only if one of the following conditions hold on the $\theta_i$:

$$\theta_1 + \theta_2 + \theta_3 = 0 \mod 2\pi, \quad (A.3)$$

$$-\theta_1 + \theta_2 + \theta_3 = 0 \mod 2\pi, \quad (A.4)$$

$$\theta_1 - \theta_2 + \theta_3 = 0 \mod 2\pi, \quad (A.5)$$

$$\theta_1 + \theta_2 - \theta_3 = 0 \mod 2\pi. \quad (A.6)$$

Proof: That this is sufficient has already been demonstrated in Section 2. Now let us assume that this is not necessary and we will find a contradiction. We shall attempt to construct a three-form $\varphi$ that defines a $G_2$ structure and that is left invariant by some $A$ not satisfying one of (A.3), (A.4), (A.5) or (A.6).

When expressed in terms of the coordinates $x_0, z_1, z_2$ and $z_3$, as in (2.9), each non-vanishing component of a three-form imposes a definite constraint on the angles $\theta_i$ if it is to be left invariant by $A$. For example a three-form with a non-vanishing coefficient of $dx_0 \wedge dz_1 \wedge dz_2$ imposes the constraint $\theta_1 + \theta_2 = 0$. We shall use this property whilst attempting to construct our $G_2$ structure.

Since the $G_2$ structure $\varphi$ given in (1.1) satisfies the following tensorial identity, so must the $G_2$ structure that we are attempting to construct.

$$\varphi_{mnr} \varphi_r^{pq} = \phi_{mpnq} + \delta_{mp} \delta_{nq} - \delta_{mq} \delta_{np}, \quad (A.7)$$

where $\phi_{mpnq}$ is the four-form dual to $\varphi$, and $m, n, \ldots = 1, \ldots, 7$ label the seven real dimensions of the manifold. From this equation we obtain

$$\sum_{p=1}^{7} \varphi_{pqr}^2 = 1, \quad (A.8)$$

for $q, r = 1, 2, \ldots, 7$. Now in order for invariance of $\varphi$ to not impose any of (2.12), (2.13), (2.14) or (2.15) on $A$, when expressed in the coordinates of (2.9) it must not contain any of the following terms

$$dz_1 \wedge dz_2 \wedge dz_3, \ dz_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3, \ dz_1 \wedge d\bar{z}_2 \wedge dz_3, \ dz_1 \wedge dz_2 \wedge d\bar{z}_3. \quad (A.9)$$

For this to be the case, all of the following must be zero:

$$\varphi_{246}, \varphi_{247}, \varphi_{256}, \varphi_{257}, \varphi_{346}, \varphi_{347}, \varphi_{356}, \varphi_{357}. \quad (A.10)$$
Bearing this in mind, and taking \((q, r) = (4, 6)\) in equation (A.8) we observe that at least one of the following must be non-zero:

\[
\varphi_{146}, \varphi_{546}, \varphi_{746}, \quad (A.11)
\]

By changing coordinates to those of (2.9), we can spot the constraints this imposes on the \(\theta_i\). We can repeat this process for \((q, r) = (2, 4)\) and \((q, r) = (2, 7)\), and put all the constraints together to obtain the result that \(A\) only leaves \(\varphi\) invariant if at least two of the \(\theta_i\) are zero. By assumption \(\varphi\) is left invariant by some \(A\) not satisfying any of (2.12), (2.13), (2.14) or (2.15), and so assume, by the above, and without loss of generality, that this \(A\) has \(\theta_1 = \theta_2 = 0\), and \(\theta_3 \neq 0 \mod 2\pi\). Then it is easily verified that only the following components (and components with permuted indices of those below) of \(\varphi\) may be non-zero:

\[
\varphi_{123}, \varphi_{145}, \varphi_{167}, \varphi_{124}, \varphi_{125}, \varphi_{134}, \varphi_{135}, \varphi_{234}, \varphi_{235}, \varphi_{245}, \varphi_{345}, \varphi_{267}, \varphi_{267}, \varphi_{467}, \varphi_{567}. \quad (A.12)
\]

Now, using (A.8) we find that

\[
|\varphi_{467}| = |\varphi_{567}| = |\varphi_{267}| = |\varphi_{367}| = 1, \quad (A.13)
\]

by using for example \((q, r) = (4, 6)\). We then invoke the identity

\[
\varphi_{mpq} \varphi_{npq} = 6 \delta_{mn}, \quad (A.14)
\]

which implies

\[
\varphi_{7pq} \varphi_{7pq} = 6. \quad (A.15)
\]

However (A.13) gives

\[
\varphi_{7pq} \varphi_{7pq} \geq 2(\varphi_{746} \varphi_{746} + \varphi_{756} \varphi_{756} + \varphi_{726} \varphi_{726} + \varphi_{736} \varphi_{736}) = 8. \quad (A.16)
\]

We therefore have a contradiction, and hence our result.

Now let \(A\) be as in (A.1), and let it satisfy one of the conditions (A.3)-(A.6) so that it preserves some \(G_2\) structure, but now let us assume that \(\theta_1 = 0\) for simplicity. Then any \(G_2\) structure preserved by \(A\) may be brought to the standard form of equation (1.1), up to possible sign differences, by some redefinition of coordinates that preserves the structure of \(A\) up to some redefinition of \(\theta_2\) and \(\theta_3\).

**Proof:** Following the method used to prove the previous result, we can draw up a list of the components of \(\varphi\) that may be non-zero if it is to be preserved by \(A\). These are

\[
\varphi_{123}, \varphi_{145}, \varphi_{167}, \varphi_{147}, \varphi_{156}, \varphi_{157}, \varphi_{245}, \varphi_{345}, \varphi_{267}, \varphi_{367}, \varphi_{246}, \varphi_{247}, \varphi_{256}, \varphi_{257}, \varphi_{346}, \varphi_{347}, \varphi_{356}, \varphi_{357}. \quad (A.17)
\]

Now using (A.8) and our freedom to choose the orientation of the 1, 2 and 3 directions we see that we have \(\varphi_{123} = 1\). We can then show similarly that, without loss, \(\varphi_{145} = 1\). Then \(\varphi_{167} = \pm 1\) by consistency of the following identity, with \((m, n, q, r, s) = (2, 6, 2, 3, 6)\):

\[
\varphi_{mn}^{pqrs} \varphi_{pqrs} = \delta_{mq} \varphi_{mrs} + \delta_{mr} \varphi_{msq} + \delta_{ms} \varphi_{mqr} - \delta_{mq} \varphi_{mrs} - \delta_{mr} \varphi_{msq} - \delta_{ms} \varphi_{mqr}. \quad (A.18)
\]

Finally, repeated use of (A.8) and remaining coordinate freedom enables us to establish the result.

There is a useful corollary of the above result: Let \(A\) be a rotation matrix with three independent preserved directions \(p, q\) and \(r\). Then \(A\) preserves a given \(G_2\) structure \(\varphi\) only if \(|\varphi_{pqr}| = 1\).
B Table of Possible Orbifold Group Elements of $G_2$ Manifolds

The possible generators $\alpha$ of orbifold groups take the form $\alpha : x \mapsto A(\alpha)x + b(\alpha)$, where $A(\alpha)$ is an orthogonal matrix, which can be put into block diagonal form

$$
\begin{pmatrix}
1 & R(\theta_1) \\
R(\theta_2) & R(\theta_3)
\end{pmatrix},
$$

(B.1)

where

$$
R(\theta_i) = \begin{pmatrix}
\cos \theta_i & -\sin \theta_i \\
\sin \theta_i & \cos \theta_i
\end{pmatrix},
$$

(B.2)

and the possibilities for the $\theta_i$ (up to signs) are listed in Table 3. The $n_i$ and the $d_i$ label how the representation $R$ defined by $A(\alpha)$ decomposes into irreducibles according to $R = n_1 R_1 \oplus \cdots \oplus n_r R_r$, as in Section 3.

| Symmetry | $\frac{1}{2\pi}(\theta_1, \theta_2, \theta_3)$ | $n_1$ | $d_1$ | $n_2$ | $d_2$ | $n_3$ | $d_3$ | $n_4$ | $d_4$ |
|----------|--------------------------------------------|-------|-------|-------|-------|-------|-------|-------|-------|
| $\mathbb{Z}_2$ | $\left( \begin{array}{c}
0, \frac{1}{2}, \frac{1}{2}
\end{array} \right)$ | 3 | 1 | 4 | 1 | - | - | - | - |
| $\mathbb{Z}_3$ | $\left( \begin{array}{c}
0, \frac{1}{3}, \frac{2}{3}
\end{array} \right)$ | 3 | 1 | 2 | 2 | - | - | - | - |
| $\mathbb{Z}_4^*$ | $\left( \begin{array}{c}
\frac{1}{4}, \frac{1}{4}, \frac{1}{4}
\end{array} \right)$ | 1 | 1 | 3 | 2 | - | - | - | - |
| $\mathbb{Z}_8$ | $\left( \begin{array}{c}
0, \frac{1}{8}, \frac{3}{8}
\end{array} \right)$ | 3 | 1 | 2 | 2 | - | - | - | - |
| $\mathbb{Z}_6^*$ | $\left( \begin{array}{c}
\frac{1}{6}, \frac{1}{6}, \frac{1}{6}
\end{array} \right)$ | 1 | 1 | 2 | 2 | 1 | 2 | - | - |
| $\mathbb{Z}_6$ | $\left( \begin{array}{c}
\frac{1}{6}, \frac{1}{6}, \frac{1}{6}
\end{array} \right)$ | 1 | 1 | 2 | 2 | 1 | 2 | - | - |
| $\mathbb{Z}_7^*$ | $\left( \begin{array}{c}
\frac{1}{7}, \frac{2}{7}, \frac{3}{7}
\end{array} \right)$ | 1 | 1 | 1 | 2 | 1 | 2 | 1 | 2 |
| $\mathbb{Z}_8$ | $\left( \begin{array}{c}
\frac{1}{8}, \frac{1}{8}, \frac{1}{8}
\end{array} \right)$ | 1 | 1 | 1 | 2 | 1 | 2 | 1 | 2 |
| $\mathbb{Z}_8^*$ | $\left( \begin{array}{c}
\frac{1}{8}, \frac{3}{8}, \frac{5}{8}
\end{array} \right)$ | 1 | 1 | 1 | 2 | 1 | 2 | 1 | 2 |
| $\mathbb{Z}_{12}$ | $\left( \begin{array}{c}
\frac{1}{12}, \frac{1}{12}, \frac{1}{12}
\end{array} \right)$ | 1 | 1 | 1 | 2 | 1 | 2 | 1 | 2 |
| $\mathbb{Z}_{12}^*$ | $\left( \begin{array}{c}
\frac{1}{12}, \frac{5}{12}, \frac{7}{12}
\end{array} \right)$ | 1 | 1 | 1 | 2 | 1 | 2 | 1 | 2 |

C Blow-up and some Calculations on Gibbons-Hawking Space

In this Appendix we begin by deriving a general volume formula valid on regions of $G_2$ manifolds that take the form $T^3 \times U$, where $T^3$ is some three-torus and $U$ is a four dimensional hyperkähler space. We then describe, for arbitrary $N$, how to blow-up $T^3 \times \mathbb{C}^2 / \mathbb{Z}_N$, and use our formula to compute volumes on blow-ups, as induced by a given $G_2$ structure. We consider in turn cases in which $U$ approaches flat space asymptotically and in which $U$ becomes exactly flat for sufficiently large radius. We follow and generalize results from Ref. [24].

Let us briefly recall the definition of a hyperkähler space. A hyperkähler space is a $4m$-dimensional Riemmanian manifold admitting a triplet $J^a$ of covariantly constant complex structures satisfying the algebra

$$
J^a J^b = -1 \delta^{ab} + \epsilon^{abc} J^c.
$$

(C.1)
Associated with the $J^a$ via
\[ \omega^a_{\mu\nu} = (J^a)^\rho_{\mu} g_{\rho\nu} \] (C.2)
we have a triplet $\omega^a$ of covariantly constant so-called Kähler forms.

If we let $U$ be a hyperkähler space, then we can write down the following $G_2$ structure on $T^3 \times U$:
\[ \phi = \sum_a \omega^a \wedge d\xi^a - d\xi^1 \wedge d\xi^2 \wedge d\xi^3, \] (C.3)
where $\xi^a$ are coordinates on the torus $T^3$. This is torsion free by virtue of $d\omega^a = \star d\omega^a = 0$.

The volume element $\sqrt{\det(g)}$ on $T^3 \times U$ may be found from the $G_2$ structure via the equations
\[ g_{AB} = \det(\gamma)^{-1/9} \gamma_{AB}, \quad \sqrt{\det(g)} = \det(\gamma)^{1/9}, \] (C.4)
where
\[ \gamma_{AB} = \frac{1}{144} \epsilon_{ACDEFGHI} \epsilon^{ACDEFGHI}, \] (C.5)
where $\epsilon$ is the “pure-number” Levi-Civita pseudo-tensor.

We now follow a general method of construction of hyperkähler spaces [30] to derive a formula for the measure. The triplet of Kähler forms is given by
\[ \omega^1 = \frac{i}{2} \partial \bar{\partial} K, \] (C.6)
\[ \omega^2 = \text{Re}(du \wedge dz), \quad \omega^3 = \text{Im}(du \wedge dz), \] (C.7)
where $K$ is the Kähler potential for $U$. These give
\[ \det(\gamma) = \frac{1}{4^9} (\mathcal{K}_{u\bar{z}}\mathcal{K}_{\bar{z}u} - \mathcal{K}_{u\bar{u}}\mathcal{K}_{z\bar{z}})^3. \] (C.8)
Here $u$ and $z$ are complex coordinates on $U$ and $\mathcal{K}_{u\bar{z}} \equiv \partial^2 \mathcal{K} / \partial u \partial \bar{z}$ etc. We can reduce this to a simpler expression if we write $\mathcal{K}$ as the Legendre transform of a real function $\mathcal{F}(x, z, \bar{z})$ with respect to the real coordinate $x$:
\[ \mathcal{K}(u, \bar{u}, z, \bar{z}) = \mathcal{F}(x, z, \bar{z}) - (u + \bar{u})x. \] (C.9)
Here $x$ is a function of $z, \bar{z}, u$ and $\bar{u}$ determined by
\[ \frac{\partial \mathcal{F}}{\partial x} = u + \bar{u}. \] (C.10)
We can then re-express (C.8) in terms of partial derivatives of $\mathcal{F}$ and obtain the rather neat result that
\[ \sqrt{\det(g)} \equiv \frac{1}{4}. \] (C.11)
Note that this is entirely general and valid on any region of a $G_2$ manifold on which the $G_2$ structure can be written as in Eq. (C.3), and on which the Kähler potential can be expressed as in (C.9). Eq. (C.11) gives the measure for integrating over the coordinates $u, z, \bar{u}, \bar{z}$ and $\xi^a$. However it will be more convenient in what follows for us to substitute $u$ and $\bar{u}$ for the real coordinates $x$ and $y$, respectively.
with \(x\) as in \((C.10)\) and \(y\) given by \(y = i(\bar{u} - u)\). Then, assuming for convenience unit volume for \(T^3\), the volume of \(T^3 \times U\) over some compact subspace \(U_0 \subset U\) is given by

\[
\text{vol}(U_0) = \frac{1}{8} \int_{U_0} |F_{xx} dz d\bar{z} dy|.
\] (C.12)

As suggested by the above, the blow-up of \(T^3 \times \mathbb{C}^2 / \mathbb{Z}_N\) will take the form \(T^3 \times U\), where \(U\) is an appropriate hyperkähler space. More specifically, \(U\) will belong to the family of spaces referred to as Gibbons-Hawking spaces or “gravitational multi-instantons” \[31\]. Note that these are generalised versions of Eguchi-Hanson space, which corresponds to the case of \(N = 2\). For each \(N\), we may take \(U\) to be the \(N\)-centred Gibbons-Hawking space, for which the function \(F\) is given by

\[
F = \sum_{i=1}^{N} \left( r_i - x_i \ln(x_i + r_i) + \frac{x_i}{2} \ln(4|z_i\bar{z_i}|) \right),
\] (C.13)

where

\[
x_i = x - b_i, \quad z_i = z - a_i,
\] (C.14)

\[
r_i = \sqrt{x_i^2 + 4|z_i|^2}.
\] (C.15)

We can derive the metric from \(F\) by using the expression \((C.9)\) for the Kähler potential. In addition the change of coordinate

\[
u = -\ln w + \sum_i \frac{1}{2} \ln \left(2(z - a_i)\right),
\] (C.16)

brings the metric into a familiar form \[31\].

\[
ds^2 = \gamma dz d\bar{z} + \gamma^{-1} \left( \frac{dw}{w} + \delta d\bar{z} \right) \left( \frac{dw}{w} + \delta d\bar{z} \right),
\] (C.17)

where

\[
\gamma = \sum_i \frac{1}{r_i},
\] (C.18)

\[
\delta = \sum_i \frac{x - b_i - r_i}{2(z - a_i)r_i},
\] (C.19)

\[
w\bar{w} = \prod_i (x - b_i + r_i).
\] (C.20)

Since we would like to calculate the effect of blow-up on the volume of a ball around the origin of \(\mathbb{C}^2 / \mathbb{Z}_N\), we wish to relate the coordinates \(\{z, \bar{z}, x, y\}\) to the ordinary Cartesian coordinates for flat space. Let us do this from first principles. Consider the “blown-down” version of \(U\), which is actually flat space. This is constructed from

\[
F = N \left( r - x \ln(x + r) + \frac{1}{2} x \ln(4z\bar{z}) \right),
\] (C.21)
where
\[ r = \sqrt{x^2 + 4|z|^2}. \]  
(C.22)

Using (C.9) and (C.10) we find the Kähler potential is simply
\[ K = Nr. \]  
(C.23)

Hence, \( Nr \) corresponds to the square of the usual radius in flat space. We can also derive the relation
\[ u + \bar{u} = \frac{N}{2} \ln \left( \frac{r - x}{r + x} \right), \]  
(C.24)

which leads to
\[ r = 2|z| \cosh \left( \frac{u + \bar{u}}{N} \right), \]  
(C.25)
\[ x = -2|z| \sinh \left( \frac{u + \bar{u}}{N} \right). \]  
(C.26)

Now flat space Cartesian coordinates \( z_1 \) and \( z_2 \) satisfy
\[ K = |z_1|^2 + |z_2|^2 \]  
(C.27)

and so identifying (C.23) and (C.27) we can come up with the following holomorphic transformation relating the two sets of coordinates.
\[ z_1 = \sqrt{Nz}e^{\frac{\bar{u}}{N}}, \quad z_2 = \sqrt{Nz}e^{-\frac{\bar{u}}{N}}. \]  
(C.28)

The coordinates \( z_1 \) and \( z_2 \) are unrestricted, and from this fact and (C.28) we can infer the ranges of the coordinates \( \{x, y, z, \bar{z}\} \). We find that \( x \) and \( z \) are unrestricted, whilst \( y \) is periodic with period \( 4\pi \).

We would like to compute the volume of a ball around the origin of \( U \). First however, we need to define a radial coordinate analogous to \( r \) in equation (C.22). The most sensible choice is to define
\[ r \equiv \sqrt{(x - \bar{b})^2 + 4|z - \bar{a}|^2}, \]  
(C.29)

where tildes denote mean values over the index \( i \). Having done this, we can perform the integration (C.12) over the region \( 0 \leq r \leq R \). Since we will ultimately be interested in the small blow-up limit, let us assume that \( R \) is much larger than the \(|a_i|\) and \(|b_i|\), and derive an answer that is correct to lowest non-trivial order in these blow-up moduli. From (C.12) and (C.13), the contribution from one term in the sum is given by
\[ V = \frac{1}{8} \int \frac{1}{\sqrt{(x - b)^2 + 4(z - a)(\bar{z} - \bar{a})}} \mathrm{d}x \mathrm{d}y |\mathrm{d}z \mathrm{d}\bar{z}|, \]  
(C.30)
dropping the subscript \( i \) for convenience. We make the change of variables \( z = u + iv, \ a = u_0 + iv_0 \) and then \( x' = x, \ u' = 2u, \ v' = 2v, \ c = 2u_0, \ d = 2v_0 \) and carry out the \( y \) integration to obtain
\[ V = \frac{\pi}{4} \int \frac{1}{\sqrt{(x' - b)^2 + (u' - c)^2 + (v' - d)^2}} \mathrm{d}u' \mathrm{d}v' \mathrm{d}x', \]  
(C.31)
with the range \((x' - \tilde{b})^2 + (u' - \tilde{c})^2 + (v' - \tilde{d})^2 \leq R^2\). It is now straightforward to obtain

\[
V = \frac{\pi}{8} \int \left( \sqrt{R^2 + f(\theta, \phi)^2 - b^2} + f(\theta, \phi) \right)^2 \sin \theta d\theta d\phi,
\]

(C.32)

where \(b = (b - \tilde{b}, c - \tilde{c}, d - \tilde{d})\) and

\[
f(\theta, \phi) = (b - \tilde{b}) \sin \theta \cos \phi + (c - \tilde{c}) \sin \theta \sin \phi + (d - \tilde{d}) \cos \theta.
\]

(C.33)

Finally, we can do this to lowest non-trivial order in \(b\) and substitute back for \(a\) to find

\[
V = \frac{\pi^2}{2} \left( R^2 - \frac{1}{3} \left( (b - \tilde{b})^2 + 4|a - \tilde{a}|^2 \right) \right) + \mathcal{O}(|b|^3).
\]

(C.34)

When we sum (C.34) over all moduli we obtain the result that, correct to second order in the \(a_i\) and \(b_i\),

\[
\operatorname{vol}_U(r = 0, R) = \frac{\pi^2}{2} \left( NR^2 - \frac{N}{3} \left( \operatorname{var}_i\{b_i\} + 2 \operatorname{var}_i\{\text{Re }a_i\} + 2 \operatorname{var}_i\{\text{Im }a_i\} \right) \right).
\]

(C.35)

Here \(\operatorname{var}\) refers to the variance, with the usual definition:

\[
\operatorname{var}_i\{X_i\} = \frac{1}{N} \sum_i (X_i - \bar{X})^2.
\]

(C.36)

The Gibbons-Hawking space that we have been discussing approaches flat space asymptotically. However, what we really need for our construction of \(G_2\) manifolds are smoothed versions of this space which become exactly flat for sufficiently large radius. We now describe how our previous results generalise to a space \(U\) which interpolates between Gibbons-Hawking space at small radius and flat space at large radius.

The smoothed version of \(\mathcal{F}\) is given by

\[
\mathcal{F} = \sum_{i=1}^{N} \left( r_i - x_i \ln(x_i + r_i) + \frac{x_i}{2} \ln(4z_i\bar{z}_i) \right),
\]

(C.37)

where now

\[
x_i = x - \epsilon b_i, \quad z_i = z - \epsilon a_i,
\]

(C.38)

\[
r_i = \sqrt{x_i^2 + 4|z_i|^2}.
\]

(C.39)

Here \(\epsilon\) is the smoothing function, dependent on the radius

\[
r \equiv \sqrt{(x - \tilde{b})^2 + 4|z - \tilde{a}|^2},
\]

(C.40)

and satisfying

\[
\epsilon(r) = \begin{cases} 
1 & \text{if } r \leq r_0, \\
0 & \text{if } r \geq r_1.
\end{cases}
\]

(C.41)
Further, $r_0$ and $r_1$ are two characteristic radii satisfying $|a_i| \ll r_0 < r_1$ and $|b_i| \ll r_0$ for each $i$ while $U$ describes Gibbons-Hawking space for $r < r_0$ and the flat space $\mathbb{C}^2/\mathbb{Z}_N$ for $r > r_1$.

Although this space interpolates between two hyperkähler spaces it is not a hyperkähler space by itself. Accordingly the forms $\omega^2$ and $\omega^3$ are no longer co-closed in the “collar” region $r \in [r_0, r_1]$. However, this space can be thought of as being close to hyperkähler as long as the blow-up moduli are sufficiently small compared to one and the function $\epsilon$ is slowly varying [24]. Analogously, the $G_2$ structure on $T^3 \times U$ is not actually torsion free, but has small torsion under the same assumptions.

Let us now work out the volume of the region $r \leq \sigma$, for $\sigma > r_1$. Considering first the contribution up to some radius $R$, much larger than the $|a_i|$ and $|b_i|$ but smaller than $r_0$ so that $\epsilon$ is identically 1 on the region of integration, we have exactly the same result as before (C.35). The corresponding contribution coming from a shell $\rho_1 < r < \rho_2$ in which $\epsilon = 0$ is

$$V = \frac{N \pi^2}{2} (\rho_2 - \rho_1^2).$$

Finally we discuss what happens in the “collar” region $r_0 < r < r_1$. Here $U$ looks locally like Gibbons-Hawking space, except that as one moves outward, away from the origin, the modulus $b$ is decreasing. Therefore local contributions to the volume from $b$ become smaller as one moves away from the origin. We make the observation that the contribution to (C.34) from $b$ is independent of $R$. Hence, the volume of a shell with radii much larger than $|b|$ but smaller than $r_0$ is independent of $b$ to second order. We deduce that at second order there can be no contribution from $b$ to the volume of the collar region. Hence (C.34) also holds for $R > r_0$. Hence, the result is identical to the unsmoothed case, namely that

$$\text{vol}_U(r = 0, \sigma) = \frac{\pi^2}{2} \left( N \sigma^2 - \frac{N}{3} \left( \text{var}_i \{b_i\} + 2 \text{var}_i \{\text{Re }a_i\} + 2 \text{var}_i \{\text{Im }a_i\} \right) \right) + O(|b|^3).$$

Note that this expression is independent of the precise form of the smoothing function $\epsilon$.

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