PERSISTENCE PROPERTIES AND WAVE-BREAKING CRITERIA FOR A GENERALIZED TWO-COMPONENT ROTATIONAL B-FAMILY SYSTEM

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Abstract. In this paper, we investigate a generalized two-component rotational b-family system arising in the rotating fluid with the effect of the Coriolis force. First, we study the persistence properties of the system in weighted $L^p$-spaces, for a large class of moderate weights. Secondly, in order to overcome the difficulty arising from higher order nonlinearity and no conservation law, we take the advantage of the specially intrinsic structure of the system and make use of commutator estimate, and then derive two blow-up results for the strong solutions to the system.

1. Introduction. This paper is to study the following generalized two-component rotational b-family (R-b-family) system

$$
\begin{cases}
    u_t - u_{xxt} - Au_x + (b + 1)uu_x = \sigma(bu_x u_{xx} + uu_{xxx}) \\
    \quad - \mu u_{xxx} - (1 - 2\Omega A)\rho p_x + 2\Omega \rho (pu)_{xx}, \\
    \rho_t + (\rho u)_x = 0, \\
    u(0, x) = u_0(x), \quad \rho(0, x) = \rho_0(x),
\end{cases}
$$

for $t > 0$ and $x \in \mathbb{R}$. In the above system, $u(x, t)$ is a horizontal velocity, $\rho(t, x)$ is related to the free surface elevation from equilibrium, the parameter $A$ describes a linear underlying shear flow, $\sigma$ is a real dimensionless constant, providing the competition or balance, in fluid convection between nonlinear steepening and amplification due to stretching, $\mu$ is a non-dimensional parameter, and $\Omega$ is a real number characterizing the constant rotational speed of the Earth [19]. As we all

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known, the Earth’s rotation profoundly affects the dynamics of the atmosphere and of the ocean. This model refers to equatorial geophysical waves, that is, ocean waves with Coriolis effect accounted for. In [11, 12, 13], the authors give a detailed description of the salient physical features and the strongly nonlinear effects of the ocean flow in the equatorial Pacific. In equatorial regions, due to the peculiar feature of the Earth’s rotation in this region (see the discussion in [11, 12]), the flow is mainly one-directional and two-dimensional, but with underlying non-uniform currents. Direct measurements also confirm the nonlinear dynamics of the Pacific equatorial flow (see [32]). According to the derivation of the model (see [37]), noticing that \( \Omega = 7.3 \times 10^{-5} \text{ rad/s} \), and \( A \) is of order \( 10^{-2} \), one can assume that \( 1 - 2\Omega A > 0 \).

In fact, system (1) has significant relationship to several models describing the motion of waves on the free surface of shallow water under the influence of gravity. When \( b = 2 \), it becomes the following rotational-two-component Camassa-Holm (R2CH) system

\[
\begin{cases}
  u_t - u_{xxt} - Au_x + 3uu_x = \sigma(2u_xu_{xx} + uu_{xxx}) - \mu u_{xxx} - (1 - 2\Omega A)\rho \rho_x + 2\Omega \rho (\rho u)_x, \\
  \rho_t + (\rho u)_x = 0,
\end{cases}
\]

for \( t > 0 \) and \( x \in \mathbb{R} \). The R2CH system is derived by Fan-Gao-Liu [19] by using the approach in the spirit of Ivanov’s asymptotic perturbation analysis for the governing equations of two-dimensional rotational gravity water waves [28]. There are at least three conservation laws for the system (2)

\[
E(u, \rho) = \int_{\mathbb{R}} \frac{1}{2}(u^2 + u_x^2 + (1 - 2\Omega A)(\rho - 1)^2) \, dx,
\]

\[
I_1(u, \rho) = \int_{\mathbb{R}} (u - \Omega(\rho - 1)^2) \, dx,
\]

and

\[
I_2(u, \rho) = \int_{\mathbb{R}} (\rho - 1) \, dx.
\]

Let \( \sigma = 1, \mu = 0 \) in (2), then the system reads as

\[
\begin{cases}
  u_t - u_{xxt} - Au_x + 3uu_x = 2u_xu_{xx} + uu_{xxx} - (1 - 2\Omega A)\rho \rho_x + 2\Omega \rho (\rho u)_x, \\
  \rho_t + (\rho u)_x = 0,
\end{cases}
\]

where the linear dispersion is absent. In [19], the authors provide an initial condition, which guarantees that the wave-breaking phenomena of (3) occur in a finite time. In [37], the authors also derive two finite-time blow-up results regarding the periodic case of (3).

If the system (1) has no the effect of the Earth’s rotation, i.e. \( \Omega = 0 \), then it becomes the following generalized Dullin-Gottwald-Holm (DGH) system [5, 26]

\[
\begin{cases}
  u_t - u_{xxt} - Au_x + (b + 1)uu_x = \sigma(bu_xu_{xx} + uu_{xxx}) - \mu u_{xxx} - \rho \rho_x, \\
  \rho_t + (\rho u)_x = 0,
\end{cases}
\]

which recovers the standard two-component Camassa-Holm (2CH) system with \( b = 2 \) and \( \mu = 0 \):

\[
\begin{cases}
  u_t - u_{xxt} - Au_x + 3uu_x + \rho \rho_x = 2u_xu_{xx} + uu_{xxx}, \\
  \rho_t + (\rho u)_x = 0.
\end{cases}
\]
In [10], the authors derive the system (4) in the context of shallow water theory. It is shown that while small initial data develop into global solutions, the wave breaking occurs for some initial data. The solitary wave solutions and peakon solutions are also addressed in [10]. Thereafter many papers are devoted to the study of 2CH. For instance, the local well-posedness in Besov spaces for (4), another wave-breaking mechanism, and the exact blow-up rate are established in [24]. Guan and Yin refine the results obtained in [10] by presenting a new global existence and several new blow-up results in [23]. The wave-breaking criteria of the system (4) in the lowest Sobolev spaces $H^s(s > 3/2)$ has been addressed by Gui and Liu [25]. Moreover, they essentially achieve the improved result of global solutions with only a nonzero initial profile of the free surface for the component system.

In addition, in the case of $\rho = 0$ and $b = 2$, the system (4) becomes the Camassa-Holm (CH) equation:

$$u_{t} - u_{xxt} - Au_{x} + 3uu_{x} = 2u_{x}u_{xx} + uu_{xxx}.$$  

The CH equation was originally implied in Fokas and Fuchssteiner work [21] of studying completely integrable generalizations of the KdV equation with bi-Hamiltonian structures. Later it was found by Camassa and Holm to describe the unidirectional propagation of shallow water wave over a flat bottom. The CH equation has many remarkable properties, such as a bi-Hamiltonian structure, complete integrability, and thus infinitely many conservation laws, peakons, wave-breaking, etc. There have been numerous interesting results on the properties of solutions of the CH equation, such as local well-posedness [2, 6, 7, 16, 17, 35], global strong solutions and blow-up phenomenon [6, 7, 8, 9], weak solutions [14, 36], asymptotic behaviors of solutions [3, 15, 27], and the algebro-geometric solutions on a symplectic submanifold [34].

In this paper, we focus our attention on the generalized two-component rotational b-family (R-b-family) system (1). So far, it seems that there is no any persistence results for system (1). Motivated by the work of Brandolese on the CH equation in the weighted Sobolev spaces [3], we are going to establish the persistence properties for the system (1) in weighted $L^p$-spaces, for a large class of moderate weights. The second part in this paper is to discuss the blow-up scenario. However, due to the system (1) possessing a cubic nonlinearity “$\rho(\rho u)_x$” and no conservation law found, seeking a good proof for several required nonlinear estimates seems somehow delicate. These difficulties are nevertheless overcome by careful estimates, commutator configuration, and the Gagliardo-Nirenberg inequality (see our Lemma 4.3). To be specific, thanks to the intrinsic but special structure of the equation, through performing an energy method on the system, we can figure out the estimates of $\|u(t, \cdot)\|_{L^2}$ and $\|u_{x}(t, \cdot)\|_{L^2}$. Then we are able to provide the $\|u(t, \cdot)\|_{L^\infty}$-estimate by using Gagliardo-Nirenberg inequality and therefore to complete the blow-up proof.

The precise statements of the main results in this paper are described below.

**Theorem 1.1.** Suppose $T > 0$, $s \geq 2$ and $2 \leq p \leq \infty$. Let $(u, \rho) \in C([0, T], H^s(\mathbb{R})) \times C([0, T], H^{s-1}(\mathbb{R}))$ be a strong solutions of the Cauchy problem (1) with the following assumptions

$$u_0\phi \in L^p(\mathbb{R}), \quad (\partial_x u_0)\phi \in L^p(\mathbb{R}), \quad \rho_0\phi \in L^p(\mathbb{R}),$$
where $\phi$ is an admissible weight function (see Definition 2.1 in Section 2) for the system (1). Then, for all $t \in [0, T]$, the following estimate holds
\[
\|u(t)\phi\|_{L^p} + \|\partial_x u(t)\phi\|_{L^p} + \|\rho(t)\phi\|_{L^p} \\
\leq \left(\|u_0\phi\|_{L^p} + \|\partial_x u_0\phi\|_{L^p} + \|\rho_0\phi\|_{L^p}\right)\exp\left\{C(1 + M)^2t\right\}
\]
for the real constant $C > 0$, only depending on $v, \phi$, through the constants $A, C_0, \inf v$ and $\int_{\mathbb{R}} \frac{\phi(x)}{|x|^2} dx$. In addition, $M$ is defined by
\[
M = \sup_{t \in [0, T]} \left(\|u(t)\|_{L^\infty} + \|\partial_x u(t)\|_{L^\infty} + \|\rho(t)\|_{L^\infty}\right) < \infty.
\]

**Remark 1.** In Theorem 1.1, the basic example of application is to choose the standard weight function $\phi(x) = \phi_{a,b,c,d}(x) = e^{a|x|^b}(1 + |x|)^c \log(e + |x|)^d$ with the conditions
\[
a \geq 0, \quad c, d \in \mathbb{R}, \quad 0 \leq b \leq 1, \quad ab < 1.
\]
Considering different restricted conditions of $a, b, c,$ and $d$, we have the following two special persistence properties.

(i) Let $\phi = \phi_{0,1,0,0} = (1 + |x|)^c$ with $c > 0$, and choose $p = \infty$. In this case, Theorem 1.1 gives the following result: under the condition
\[
|u_0(x)|, \quad |\partial_x u_0(x)|, \quad |\rho_0(x)| \leq C(1 + |x|)^{-c},
\]
the uniformly algebraic decay
\[
|u(t, x)| + |\partial_x u(t, x)| + |\rho(t, x)| \leq C(1 + |x|)^{-c}, \quad \text{in } t \in [0, T]
\]
can be obtained. Here $C$ depends on the initial data $u_0, \partial_x u_0, \rho_0$ and $T$.

(ii) Let $\phi = \phi_{a,1,0,0} = e^{ax}$ with $0 \leq a < 1$ if $x \geq 0$ and $\phi = 1$ if $x \leq 0$. We can verify that such a weight function clearly satisfies the admissibility conditions of Definition 2.1. Applying $p = \infty$ in Theorem 1.1 leads to the point-wise decay of the system (1). If the initial data satisfy
\[
|u_0(x)|, \quad |\partial_x u_0(x)|, \quad |\rho_0(x)| \sim O(e^{-ax}) \quad \text{as } x \to +\infty,
\]
then the solution admits
\[
|u(t, x)|, \quad |\partial_x u(t, x)|, \quad |\rho(t, x)| \sim O(e^{-ax}) \quad \text{as } x \to +\infty
\]
uniformly in the time interval $[0, t]$. Similar arguments hold for the case of the decay $O(e^{-ax})$ as $x \to -\infty$.

**Remark 2.** As we all know, there is no persistence result about the system (1). However, our result could apply to the two-component Camassa-Holm (2CH) system without the effect of the Earth’s rotation. When $\rho = 0$, it coincides with the result of [3].

The blow-up criteria reads:

**Theorem 1.2.** Let $(u_0, \rho_0) \in H^s \times H^{s-1}$, $s \geq 2$ be given, and assume $T$ is the maximal existence time of the corresponding solution $(u, \rho)$ to the system (1) guaranteed by Theorem 3.1. If there exists $\widetilde{M}$ such that
\[
\sup_{t \in [0, T]} \left(\|u(t)\|_{L^\infty}, \quad \|\partial_x u(t)\|_{L^\infty}, \quad \|\rho(t)\|_{L^\infty}, \quad \|\partial_x \rho(t)\|_{L^\infty}\right) \leq \widetilde{M},
\]
then the $H^s \times H^{s-1}$-norm of $(u, \rho)$ does not blow up on $[0, T)$. 

Theorem 1.3. Assume that $1 - 2\Omega A > 0$, and $\sigma, b$ satisfy the following condition

$$\sigma = 1, b \geq 2; \quad \text{or} \quad b = -1, \sigma < 0.$$  \hfill (5)

Let $(u_0, \rho_0) \in H^s \times H^{s-1}$, $s \geq 2$, and $T$ be the maximal existence time of the corresponding solution $(u, \rho)$ to the system (6) guaranteed by Theorem 3.1. Then the corresponding solution blows up in a finite time if and only if

$$\lim_{t \to T} \inf_{x \in \mathbb{R}} \{ u_x(t, x) \} = -\infty, \quad \text{or} \quad \limsup_{t \to T} \{ \| \rho_x(t, x) \|_{L^\infty} \} = \infty.$$

Remark 3. The proof of Theorem 1.3 is placed in the end of Section 4. To obtain the estimate of $\| u(t, \cdot) \|_{L^\infty}$ a little manipulation is needed (see Lemma 4.3 for more details). It should be noted that the $\sigma, b$ condition (5) is presented as per the proof need. The condition (5) already covers the special case $b = 2, \sigma = 1$ and $\mu = 0$, which has been investigated in [33] and [37].

The rest of our paper is organized as follows. In Section 2, some materials regarding weight functions and useful lemmas are recalled, and also the commutator estimate is provided. Section 3 is devoted to the proof of Theorem 1.1. Section 4 is denoted to the study of the blow-up scenario for strong solutions. We conclude the paper by giving the proof of Theorem 1.2 and Theorem 1.3.

Notation. We have the following formulations:

$$|f(x)| \sim O(g(x)), \quad \text{as} \quad x \to \infty, \quad \text{if} \quad \lim_{x \to \infty} \frac{f(x)}{g(x)} = L \text{ (constant)}.$$ 

All function spaces are considered over the real field $\mathbb{R}$, and we shall omit it in our context if there is no confusion.

2. Preliminaries. Here we recall some basic materials and lemmas, including some properties of weight function and commutator estimate.

First, we turn to some standard definitions and some lemmas concerning weight function, which can be found in [3, 22, 29]. In general, a weight function is a simply non-negative function. A weight function $v : \mathbb{R}^n \to \mathbb{R}$ is called sub-multiplicative, if

$$v(x + y) \leq v(x)v(y), \quad \forall x, y \in \mathbb{R}^n.$$ 

Moreover, for a given sub-multiplicative weight function, by definition a positive function $\phi$ is $v$-moderate if and only if

$$\exists C_0 > 0, \quad \text{s.t.} \quad \phi(x + y) \leq C_0 v(x)\phi(y), \quad \forall x, y \in \mathbb{R}^n.$$ 

Briefly speaking, if $\phi$ is $v$-moderate for some sub-multiplicative function $v$, then we always say: $\phi$ is moderate. This is the usual terminology in time-frequency analysis papers [1, 22].

In [3], the author gives a standard example of such weights (also see [20, 22]):

$$\phi(x) = \phi_{a,b,c,d}(x) = e^{\alpha|x|^b} (1 + |x|)^c \log(e + |x|)^d$$

(i) For $a, b, d \geq 0$ and $0 \leq b \leq 1$ such weight is sub-multiplicative.

(ii) If $a, c, d \in \mathbb{R}$ and $0 \leq b \leq 1$, then $\phi$ is moderate. More precisely, $\phi_{a,b,c,d}$ is $\phi_{\alpha,\beta,\gamma,\delta}$-moderate for $|a| \leq \alpha, b \leq \beta, |c| \leq \gamma$ and $|d| \leq \delta$.

For the Camassa-Holm type equation, we will specify the class of admissible weight functions.
Definition 2.1. An admissible weight function for the Camassa-Holm type equation is a locally absolutely continuous function $\phi : \mathbb{R} \to \mathbb{R}$ satisfying
(i) For some $R > 0$, $|\phi'(x)| \leq R|\phi(x)|$, a.e. $x \in \mathbb{R}$;
(ii) $\phi$ is $v$-moderate for some sub-multiplicative weight function $v$, which is
\[
\inf_{x \in \mathbb{R}} v(x) > 0,
\]
\[
\int_{\mathbb{R}} \frac{v(x)}{e^{x^2}} \, dx < \infty.
\]

The importance of imposing the sub-multiplicativity condition and moderateness on a weight function can be seen clearly by the following two propositions.

Proposition 1. ([3]) Let $v : \mathbb{R}^n \to \mathbb{R}^+$ and $C_0 > 0$. Then the following conditions are equivalent:
(i) For all $1 \leq p,q,r \leq \infty$ and for any measurable functions $f_1, f_2 : \mathbb{R}^n \to \mathbb{C}$, the weighted Young inequalities hold:
\[
\|f_1 * f_2\|_{L^r} \leq C_0 \|f_1\|_{L^p} \|f_2\|_{L^q}, \quad 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.
\]

Proposition 2. ([3]) Let $1 \leq p \leq \infty$ and $v$ be a sub-multiplicative weight on $\mathbb{R}^n$. Then the following conditions are equivalent:
(i) $\phi$ is a $v$-moderate weight function (with constant $C_0 > 0$).
(ii) For all measurable functions $f_1, f_2 : \mathbb{R}^n \to \mathbb{C}$, the weighted Young estimate holds
\[
\|(f_1 * f_2)\phi\|_{L^p} \leq C_0 \|f_1\|_{L^1} \|f_2\|_{L^p}.
\]

We recall some lemmas, which will be frequently used in the next sequel. Their proofs can be found in [4, 31].

Lemma 2.2. (i) If $r > 0$, then $H^r \cap L^\infty$ is an algebra. Moreover, there exists a positive constant $C$ only dependent on $r$ such that
\[
\|fg\|_{H^r} \leq C(\|f\|_{L^\infty} \|g\|_{H^r} + \|g\|_{L^\infty} \|f\|_{H^r}).
\]
(ii) If $r > 0$, there exists a positive constant $C$ only dependent on $r$ such that
\[
\|f \partial_x g\|_{H^r} \leq C(\|g\|_{L^\infty} \|f\|_{H^{r+1}} + \|f\|_{L^\infty} \|\partial_x g\|_{H^r}).
\]

Lemma 2.3. If $r > 0$, there exists a positive constant $C$ only dependent on $r$ such that
\[
\|\Lambda^r f g\|_{L^2} \leq C(\|\partial_x f\|_{L^\infty} \|\Lambda^{r-1} g\|_{L^2} + \|\Lambda^r f\|_{L^2} \|g\|_{L^\infty}),
\]
where $[A, B]$ is commutator representing $AB - BA$.

3. The persistence properties. In this section, we will focus on the investigation of the persistence properties of the generalized two-component rotation b-family system (1).

Let $G(x) = \frac{1}{2}e^{-|x|}$ be the fundamental solution to $u - u_{xx} = \delta$. Then for any $f \in L^2(\mathbb{R})$, $G(x) * f = (1 - \partial_x)^{-1} f$. Thus system (1) can be rewritten as
\[
\begin{cases}
\frac{\partial u}{\partial t} + (\sigma u - \mu)u_x = -\partial_x G * F(u, \rho) + G * H(u, \rho), \\
\frac{\partial \rho}{\partial t} + u \rho_x = -\rho u_x, \\
u(0, x) = u_0(x), \quad \rho(0, x) = \rho_0(x),
\end{cases}
\]

(6)
where

\[ F(u, \rho) = (\mu - A)u + \frac{b + 1 - \sigma}{2} u^2 + \frac{(b - 1)\sigma}{2} u_x^2 + \frac{1 - 2\Omega A}{2} \rho^2 - \Omega \rho^2 u, \]

\[ H(u, \rho) = \Omega \rho^2 u_x. \]

Similar with system (2), the generalized two-component rotation b-family system (6) (or equivalent system (1)) is suitable for applying Kato’s theory [30], and following the analogous proof to [18, 19] (with some slight modifications), we thus have the well-posedness theorem.

**Theorem 3.1.** Given \( z_0 = (u_0, \rho_0) \in H^s \times H^{s-1}, s \geq 2 \), there exists a \( T = T(z_0) > 0 \) and a unique solution \( z = (u, \rho) \) to system (6) such that

\[ z = z(\cdot, z_0) \in C([0, T], H^s \times H^{s-1}) \cap C^1([0, T], H^{s-1} \times H^{s-2}). \]

Moreover, the solution depends continuously on the initial data, i.e., the solution mapping

\[ z_0 \mapsto z : H^s \times H^{s-1} \to C([0, T], H^s \times H^{s-1}) \cap C^1([0, T], H^{s-1} \times H^{s-2}) \]

is continuous.

To derive the persistence properties of the solution, we have to resort to the estimates with nonlinear term. We first give the following lemma.

**Lemma 3.2.** For \( 1 \leq p \leq \infty \), \( (u, \rho) \in C([0, T], H^s \times H^{s-1}), s \geq 2 \) is a strong solutions of the Cauchy problem for system (6) guaranteed by Theorem 3.1. Then, we have the following estimate

\[ \|F(u, \rho)f\|_{L^p} + \|H(u, \rho)f\|_{L^p} \leq c(1 + M)^2 \left( \|uf\|_{L^p} + \|uf_x\|_{L^p} + \|\rho f\|_{L^p} \right), \]

where \( F(u, \rho) \) and \( H(u, \rho) \) are defined above, \( c \) depends on parameters \( \mu, A, b, \sigma \) and \( \Omega \), and

\[ M = \sup_{t \in [0, T]} \left( \|u(t)\|_{L^\infty} + \|\partial_x u(t)\|_{L^\infty} + \|\rho(t)\|_{L^\infty} \right) < \infty. \]

**Proof.** Recall the expression of \( F(u, \rho) \):

\[ F(u, \rho) = (\mu - A)u + \frac{b + 1 - \sigma}{2} u^2 + \frac{(b - 1)\sigma}{2} u_x^2 + \frac{1 - 2\Omega A}{2} \rho^2 - \Omega \rho^2 u. \]

By using Hölder inequality, we have

\[ \left\| \left( \mu - A \right) u + \frac{b + 1 - \sigma}{2} u^2 + \frac{(b - 1)\sigma}{2} u_x^2 + \frac{1 - 2\Omega A}{2} \rho^2 \right\|_{L^p} \]

\[ \leq \left\| (\mu - A) u \right\|_{L^p} + \left\| \frac{b + 1 - \sigma}{2} u^2 \right\|_{L^p} + \left\| \frac{(b - 1)\sigma}{2} u_x^2 \right\|_{L^p} + \left\| \frac{1 - 2\Omega A}{2} \rho^2 \right\|_{L^p} \]

\[ \leq c' \left( \|uf\|_{L^p} + \|uf_x\|_{L^p} \right) \left( \|u_x\|_{L^\infty} + \|\rho\|_{L^\infty} \right) \]

where \( c' \) depends on \( \mu, A, b, \sigma \) and \( \Omega \).

Similarly, to bound the last term of \( F(u, \rho) \) and \( H(u, \rho) \), we have

\[ \|\Omega^2 u f + \Omega^2 u_x f\|_{L^p} \]

\[ \leq \Omega \left( \|uf\|_{L^p} \|u_x\|_{L^\infty} \right) + \|uf\|_{L^p} \|\rho\|_{L^\infty} \|u_x\|_{L^\infty} \|\rho\|_{L^\infty} \]

\[ \leq \Omega M^2 \|\rho f\|_{L^p}. \]

Via above two inequalities, we obtain the desired estimate. □
Proof of Theorem 1.1. For any $N \in \mathbb{N} \setminus \{0\}$, consider the $N$-truncation
\[ f(x) = f_N(x) = \min\{\phi, N\}. \]

An obvious induction gives that $f : \mathbb{R} \to \mathbb{R}$ is a locally absolutely continuous function, such that
\[ \|f\|_{L^\infty} \leq \infty, \quad |f'(x)| \leq R|f(x)|. \]

What’s more, let $\alpha = \inf_{x \in \mathbb{R}} v(x) > 0$. Then
\[ f(x + y) \leq C_1 v(x)f(y), \quad \forall x, y \in \mathbb{R}, \]
where $C_1 = \max\{C_0, \alpha^{-1}\}$. Namely, $f$ is $v$-moderate weight function with constant $C_1 > 0$. In addition, the constant $C_1$ being independent on $N$, which indicates that $f$ is uniformly $v$-moderate with respect to $N$.

Let us start with the case $2 \leq p < \infty$. Multiplying the first equation in (6) by $f$, next by $|uf|^{p-2}(uf)$, integrating over $\mathbb{R}$, then we arrive at
\[
\frac{1}{p} \frac{d}{dt} \|uf\|_{L^p}^p + \int_{\mathbb{R}} |uf|^{p-1}(\sigma u - \mu) u_x f \, dx \\
= -\int_{\mathbb{R}} (f \partial_x G * F(u, \rho))(uf)^{p-2} uf \, dx \\
+ \int_{\mathbb{R}} (fG * H(u, \rho))(uf)^{p-2} uf \, dx.
\]

Note that $f$ is $v$-moderate with constant $C_1$, applying Proposition 2, the right hand side can be estimated by
\[
-\int_{\mathbb{R}} (f \partial_x G * F(u, \rho))(uf)^{p-1} dx + \int_{\mathbb{R}} (fG * H(u, \rho))(uf)^{p-1} dx \\
\leq \|uf\|_{L^p}^{p-1} \left( \left\| f(\partial_x G * F(u, \rho)) \right\|_{L^p} + \left\| f(G * H(u, \rho)) \right\|_{L^p} \right) \\
\leq C_1 \|uf\|_{L^p}^{p-1} \left( \left\| (\partial_x G)v \right\|_{L^p} + \|F(u, \rho)f\|_{L^p} + \|Gv\|_{L^p} \|H(u, \rho)f\|_{L^p} \right) \\
\leq C_2 \|uf\|_{L^p}^{p-1} \left( \|F(u, \rho)f\|_{L^p} + \|H(u, \rho)f\|_{L^p} \right).
\]

In the last inequality we use the fact that $|\partial_x G(x)| \leq \frac{1}{2} e^{-|x|}$, and $\int_{\mathbb{R}} \frac{v(x)}{\varepsilon^\sigma} dx < \infty$.

Estimating the integral in left hand side by
\[
\int_{\mathbb{R}} |uf|^{p-1}(\sigma u - \mu) u_x f \, dx \\
\leq C_3 \|uf\|_{L^p}^{p-1} \left( \|uxf\|_{L^p} \|u\|_{L^\infty} + \|uxf\|_{L^p} \right) \\
\leq C_3 (1 + M) \|uf\|_{L^p}^{p-1} \|uxf\|_{L^p},
\]
where $C_3$ depends on $\sigma$ and $\mu$.

Therefore, from (7)–(9) and Lemma 3.2, we obtain
\[
\frac{d}{dt} \|uf\|_{L^p} \leq C_2 \left( \|F(u, \rho)f\|_{L^p} + \|H(u, \rho)f\|_{L^p} \right) + C_3 (1 + M) \|uxf\|_{L^p} \\
\leq C_4 (1 + M)^2 \left( \|uf\|_{L^p} + \|uxf\|_{L^p} + \|\rho f\|_{L^p} \right).
\]

The constant $C_4$ depends on the parameters $\mu, A, b, \sigma, \Omega$ and $v, \phi$.

Now we reduce the estimate of $uxf$. Differentiating equation the first equation in (6) with respect to $x$, and using the fact that $-\partial_{xx} G * f = f - G * f$, we obtain
\[
\partial_t (ux) + (\sigma u - \mu) u_{xx} + \sigma u_x^2 = F(u, \rho) - G * F(u, \rho) + \partial_x G * H(u, \rho).
\]
Before we multiply (11) by \(|u_x f|^p(u_x f)f\) and integrate, we observe that
\[
\int_{\mathbb{R}} (\sigma u - \mu) u_x f \cdot |u_x f|^p(\sigma u_x f) dx \\
= \int_{\mathbb{R}} (\sigma u - \mu) \left[ \partial_x (u_x f) - u_x f_x \right] |u_x f|^p(u_x f) dx \\
= \int_{\mathbb{R}} (\sigma u - \mu) \partial_x \left( \frac{|u_x f|^p}{p} \right) dx - \int_{\mathbb{R}} (\sigma u - \mu) |u_x f|^p(u_x f) u_x f_x dx.
\]
Note that \(f'(x) \leq R|f(x)|\) for some \(R > 0\), we have
\[
\int_{\mathbb{R}} (\sigma u - \mu) u_x f \cdot |u_x f|^p(u_x f) dx \\
\leq C_5 \|u\|_{L^\infty} \|u_x f\|_{L^p}^p + C_6 \|u_x f\|_{L^p}^{p-1} (\|u_x f\|_{L^p} \|u\|_{L^\infty} + \|u_x f\|_{L^p}) \\
\leq C_7 (1 + M) \|u_x f\|_{L^p}^p.
\]

Now after the multiplying and integration on (11), in view of (12) and (13), on account of Proposition 2, we have
\[
\frac{d}{dt} \|u_x f\|_{L^p} \\
\leq C_7 (1 + M) \|u_x f\|_{L^p} + \|u_x\|_{L^\infty} \|u_x f\|_{L^p} + \|F(u, \rho)f\|_{L^p} + \|H(u, \rho)f\|_{L^p} \\
\leq C_8 (1 + M)^2 (\|uf\|_{L^p} + \|u_x f\|_{L^p} + \|\rho f\|_{L^p}).
\]

In the last inequality, we take advantage of Lemma 3.2.

Now it is our condition to obtain the estimate of \(\rho f\). Multiplying the second equation in (6) by \(f\), we obtain
\[
\partial_t (\rho f) + f u_x \rho = -f \rho u_x. \tag{15}
\]
We can see \(f u_x \rho\) is not a good term because of the low regularity of \(\rho\). Before we multiply this equation by \(|\rho f|^p(\rho f)|\) and integrate, let us study the second term:
\[
\int_{\mathbb{R}} u \partial_x (\rho f) \cdot |\rho f|^p(\rho f) dx \\
= \int_{\mathbb{R}} u [\partial_x (\rho f) - \rho f_x] |\rho f|^p(\rho f) dx \\
= \int_{\mathbb{R}} u \partial_x \left( \frac{|\rho f|^p}{p} \right) dx - \int_{\mathbb{R}} u (\rho f_x) |\rho f|^p(\rho f) dx \\
\leq \frac{1}{p} \|\rho f\|_{L^p}^p \|u\|_{L^\infty} + R \|\rho f\|_{L^p}^p \|u\|_{L^\infty}.
\]

In the last inequality, we use the fact that \(f'(x) \leq R|f(x)|\) for some \(R > 0\).
Then it follows (15) and (16) that
\[
\frac{d}{dt} \|\rho f\|_{L^p} \leq \frac{1}{p} \|\rho f\|_{L^p} \|u\|_{L^\infty} + R \|\rho f\|_{L^p} \|u\|_{L^\infty} + \|\rho f\|_{L^p} \|u_x\|_{L^\infty} \\
\leq C \|\rho f\|_{L^p} (\|u\|_{L^\infty} + \|u_x\|_{L^\infty}) \\
\leq C M \|\rho f\|_{L^p}. \tag{17}
\]
Together with (10), (14) and (17), we have
\[
\frac{d}{dt} \|(uf)\|_{L^p} + \|u_x f\|_{L^p} + \|\rho f\|_{L^p} \\
\leq C(1 + M)^2 (\|uf\|_{L^p} + \|u_x f\|_{L^p} + \|\rho f\|_{L^p}).
\]
By using Gronwall’s inequality, for all \( t \in [0, T] \),
\[
\|uf\|_{L^p} + \|u_xf\|_{L^p} + \|\rho f\|_{L^p} \\
\leq (\|u_0f\|_{L^p} + \|\partial_xu_0f\|_{L^p} + \|\rho_0f\|_{L^p}) e^{C(1+M)^2t},
\]
where \( C > 0 \) depends on parameters \( \mu, A, b, \sigma \) and \( \Omega, v, \phi \). But for a.e. \( x \in \mathbb{R} \), \( f(x) = f_N(x) \to \phi(x) \) as \( N \to \infty \). By our assumptions \( \partial_xu_0\phi \in L^p(\mathbb{R}), u_0\phi \in L^p(\mathbb{R}) \) and \( \rho_0\phi \in L^p(\mathbb{R}) \), we arrive at
\[
\|uf\|_{L^p} + \|u_xf\|_{L^p} + \|\rho f\|_{L^p} \\
\leq (\|u_0f\|_{L^p} + \|\partial_xu_0f\|_{L^p} + \|\rho_0f\|_{L^p}) e^{C(1+M)^2t}.
\]

It remains to treat the case \( p = \infty \). We have \( u_0, \partial_xu_0, \rho_0 \in L^2 \cap L^\infty \) and \( f = f_N \in L^\infty \). Therefore, for all integrable index \( 2 \leq q < \infty \) we have as before
\[
\|uf\|_{L^q} + \|u_xf\|_{L^q} + \|\rho f\|_{L^q} \\
\leq (\|u_0f\|_{L^q} + \|\partial_xu_0f\|_{L^q} + \|\rho_0f\|_{L^q}) e^{C(1+M)^2t}.
\]
However, the exponential part in the right-hand side is independent on \( q \). Letting \( q \to 0 \), it leads to
\[
\|uf\|_{L^\infty} + \|u_xf\|_{L^\infty} + \|\rho f\|_{L^\infty} \\
\leq (\|u_0f\|_{L^\infty} + \|\partial_xu_0f\|_{L^\infty} + \|\rho_0f\|_{L^\infty}) e^{C(1+M)^2t}.
\]
Taking \( N \to \infty \) implies the estimate (18) remains valid for \( 2 \leq p < \infty \) and \( p = \infty \). □

4. The blow-up scenario. In this section, we present the blow-up scenario of the generalized two-component rotation b-family system (1). For convenience, we rewrite system (1) as
\[
\begin{cases}
u_t + (\sigma u - \mu)u_x = -\partial_x(1 - \partial_{xx})^{-1}[(\mu - A)u + \frac{b + 1 - \sigma}{2}u^2] \\
\quad + \frac{(b - 1)\sigma}{2}u_x^2 + \frac{1 - 2\Omega A}{2}\rho^2 - \Omega \rho^2 u_x + (1 - \partial_{xx})^{-1}(\Omega \rho^2 u_x), \\
\rho_t + u\rho_x = -\rho u_x, \\
u(0,x) = u_0(x), \quad \rho(0,x) = \rho_0(x).
\end{cases}
\]

First, we will prove Theorem 1.2 which have been stated in Section 1.

Proof of Theorem 1.2. Throughout the proof, \( \Lambda \) stands for \( (1 - \partial_{xx})^{1/2} \). Applying the operator \( \Lambda^* \) to the first equation in system (19), multiplying \( \Lambda^* u \), and integrating
over $\mathbb{R}$, we obtain
\[
\frac{1}{2} \frac{d}{dt} \|u\|_{H^s} = - \int_{\mathbb{R}} \Lambda^s((\sigma u - \mu)u_x)\Lambda^s u \, dx \\
- \int_{\mathbb{R}} \Lambda^s(\partial_x(1 - \partial_{xx})^{-1}(\mu - A)u)\Lambda^s u \, dx \\
- \int_{\mathbb{R}} \Lambda^s(\partial_x(1 - \partial_{xx})^{-1}\left(\frac{b + 1 - \sigma}{2}u^2\right)\Lambda^s u \, dx \\
- \int_{\mathbb{R}} \Lambda^s(\partial_x(1 - \partial_{xx})^{-1}\left(\frac{b - 1}{2}\sigma u^2\right)\Lambda^s u \, dx \\
- \int_{\mathbb{R}} \Lambda^s(\partial_x(1 - \partial_{xx})^{-1}\left(\frac{1 - 2\Omega A}{2}\rho^2\right)\Lambda^s u \, dx \\
+ \int_{\mathbb{R}} \Lambda^s(1 - \partial_{xx})^{-1}(\Omega \rho^2 u)\Lambda^s u \, dx \\
+ \int_{\mathbb{R}} \Lambda^s(1 - \partial_{xx})^{-1}(\Omega \rho^2 u_x)\Lambda^s u \, dx
\]
\[(20)\]

Let us estimate the terms in right hand side of (20) term by term. In order to deal with the high order $\|u_x\|$ in the first term of right hand side, we try to introduce the commutator as following:
\[
|J_1| \\
= \left| \int_{\mathbb{R}} \Lambda^s((\sigma u - \mu)u_x)\Lambda^s u \, dx \right| \\
= \left| \sigma \int_{\mathbb{R}} [(\Lambda^s, u]\partial_x u)\Lambda^s u \, dx + \int_{\mathbb{R}} \mu \Lambda^s u_x \Lambda^s u \, dx \right| \\
\leq |\sigma| \left( \|\Lambda^s, u\|_{L^2} \|\Lambda^s u\|_{L^2} + \frac{\sigma}{2} \int_{\mathbb{R}} u_x(\Lambda^s u)^2 \, dx \right) \\
\leq |\sigma| \left( \|u_x\|_{L^2} \|\Lambda^s u\|_{L^2} + \|\Lambda^s u\|_{L^2} \|u_x\|_{L^2} \right) \|u\|_{H^s} + \frac{\sigma}{2} \|u\|_{L^2} \|u_x\|_{H^s} \\
\leq |\sigma| \|u_x\|_{L^2} \|u\|_{H^s}^2.
\]

In the first inequality, we make use of Lemma 2.3 and integration by parts. According to the fact that the operator $\partial_x(1 - \partial_{xx})^{-1}$ is continuous from $H^{r-1}$ to $H^r$, as well as Lemma 2.2, we get
\[
|J_2| = \left| \int_{\mathbb{R}} \Lambda^s(\partial_x(1 - \partial_{xx})^{-1}(\mu - A)u)\Lambda^s u \, dx \right| \\
\leq \|\partial_x(1 - \partial_{xx})^{-1}(\mu - A)u\|_{H^s} \|u\|_{H^s} \\
\leq |\mu - A| \|u\|_{H^{r-1}} \|u\|_{H^s} \\
\leq |\mu - A| \cdot 1 \cdot \|u\|_{H^s}.
\]

For the same trike, we have the following estimates:
\[
|J_3| \leq \left| \frac{b + 1 - \sigma}{2} \|u_x\|_{H^{r-1}} \|u\|_{H^s} \right| \leq \left| \frac{b + 1 - \sigma}{2} \|u_x\|_{H^s} \|u\|_{H^s} \right| \\
|J_4| \leq \left| \frac{(b + 1)\sigma}{2} \|u_x\|_{L^2} \|u\|_{H^s} \right| \leq \left| \frac{(b + 1)\sigma}{2} \|u_x\|_{L^2} \|u\|_{H^s} \right| \\
|J_5| \leq \left| \frac{1 - 2\Omega A}{2} \|\rho^2\|_{H^{r-1}} \|u\|_{H^s} \right| \leq \left| \frac{1 - 2\Omega A}{2} \|\rho\|_{L^2} \|\rho\|_{H^{r-1}} \|u\|_{H^s} \right|
\]
\[(23)\]
Then the last two terms with \( J_6 \) and \( J_7 \) can be handled like above as the following:
\[
|J_6| \leq \|\rho^2 u\|_{H^{s-1}} \|u\|_{H^s} \leq \Omega \left( \|\rho^2\|_{L^\infty} \|u\|_{H^{s-1}} + \|\rho^2\|_{H^{s-1}} \|u\|_{L^\infty} \right) \|u\|_{H^s} \tag{24}
\]
\[
|J_7| \leq \Omega \|\rho^2 u_x\|_{H^{s-1}} \|u\|_{H^s} \leq \Omega \left( \|\rho^2\|_{L^\infty} \|u_x\|_{H^{s-1}} + \|\rho^2\|_{H^{s-1}} \|u_x\|_{L^\infty} \right) \|u\|_{H^s} \leq \Omega \left( \|\rho\|_{L^\infty} \|u\|_{H^s} + \|\rho\|_{L^\infty} \|u_x\|_{L^\infty} \right) \|\rho\|_{H^{s-1}} \|u\|_{H^s}. \tag{25}
\]
As we have the operator \((1 - \partial_{xx})^{-1}\) is continuous from \(H^{r-2}\) to \(H^r\), one can conclude exactly as for the previous term that
\[
|J_7| \leq \Omega \|\rho^2 u_x\|_{H^{s-1}} \|u\|_{H^s} \leq \Omega \left( \|\rho^2\|_{L^\infty} \|u_x\|_{H^{s-1}} + \|\rho^2\|_{H^{s-1}} \|u_x\|_{L^\infty} \right) \|u\|_{H^s} \leq \Omega \left( \|\rho\|_{L^\infty} \|u\|_{H^s} + \|\rho\|_{L^\infty} \|u_x\|_{L^\infty} \right) \|\rho\|_{H^{s-1}} \|u\|_{H^s}.
\]
In virtue of (20), and \(ab \leq \frac{1}{2}(a^2 + b^2)\), adding up inequalities (21)-(25), we arrive at
\[
\frac{d}{dt} \|u\|_{H^s}^2 \leq C(1 + \tilde{M} + \tilde{\Omega}^2) \left( \|u\|_{H^s}^2 + \|\rho\|_{H^{s-1}}^2 \right), \tag{26}
\]
where \(C\) depends on parameters \(\mu, A, b, \sigma\) and \(\Omega\).

In order to derive a similar estimate for the second component \(\rho\), we apply operator \(\Lambda^{s-1}\) to the second equation of (19), multiply \(\Lambda^{s-1}\rho\), and integrate over \(\mathbb{R}\). Thus we have
\[
\frac{1}{2} \frac{d}{dt} \|\rho\|_{H^{s-1}}^2 = -\int u \Lambda^{s-1}(\rho u_x)\Lambda^{s-1}\rho \, dx - \int \Lambda^{s-1}(\rho u_x) \Lambda^{s-1}\rho \, dx. \tag{27}
\]
Because of the badness of \(\rho u_x\), we begin with the first term of (27) by introducing the commutator:
\[
\int \Lambda^{s-1}(u \partial_x \rho) \Lambda^{s-1}\rho \, dx
\]
\[
= \int \Lambda^{s-1}(u) \partial_x \rho \Lambda^{s-1}\rho \, dx + \int u \Lambda^{s-1} \partial_x \rho \Lambda^{s-1}\rho \, dx
\]
\[
\leq \left\| \Lambda^{s-1}(u) \partial_x \rho \right\|_{L^2} \|\rho\|_{H^{s-1}} + \frac{1}{2} \int u_x \Lambda^{s-1}(\rho)^2 \, dx \tag{28}
\]
\[
\leq \left( \|u_x\|_{L^\infty} \|\Lambda^{s-2}\rho_x\|_{L^2} + \|\Lambda^{s-1}u\|_{L^2} \|\rho\|_{L^\infty} \right) \|\rho\|_{H^{s-1}} + \frac{1}{2} \|u_x\|_{L^\infty} \|\rho\|_{H^{s-1}}^2
\]
\[
\leq \left( \|u_x\|_{L^\infty} + \|\rho\|_{L^\infty} \right) \left( \|u\|_{H^{s-1}}^2 + \|\rho\|_{H^{s-1}}^2 \right).
\]
Here we make use of Lemma 2.3 with \(r = s - 1\). As well as we estimate
\[
\int \Lambda^{s-1}(\rho u_x) \Lambda^{s-1}\rho \, dx \leq \left( \|\rho\|_{L^\infty} + \|u_x\|_{H^{s-1}} + \|u_x\|_{L^\infty} \|\rho\|_{H^{s-1}} \right) \|\rho\|_{H^{s-1}}. \tag{29}
\]
Combining inequalities (28) and (29) with (27), we obtain
\[
\frac{d}{dt} \|\rho\|_{H^{s-1}}^2 \leq C \left( \|\rho\|_{L^\infty} + \|u_x\|_{L^\infty} \right) \left( \|\rho\|_{H^{s-1}}^2 + \|u\|_{H^s}^2 \right). \tag{30}
\]
Together with (26) and (30), we have
\[
\frac{d}{dt} \left( \|u\|_{H^s}^2 + \|\rho\|_{H^{s-1}}^2 \right) \leq C(1 + \tilde{M})^2 \left( \|u\|_{H^s}^2 + \|\rho\|_{H^{s-1}}^2 \right).
\]
An application of Gronwall’s inequality leads to
\[
\|u\|_{H^s}^2 + \|\rho\|_{H^{s-1}}^2 \leq \left( \|u_0\|_{H^s}^2 + \|\rho_0\|_{H^{s-1}}^2 \right) \exp \{ C(1 + \tilde{M})^2 t \}.
\]
This completes the proof of the theorem.

Consider now the following equation

\[
\begin{aligned}
\frac{dq(t,x)}{dt} &= u(t,q(t,x)), \quad t \in [0,T), \ x \in \mathbb{R}, \\
q(0,x) &= x, \quad x \in \mathbb{R}.
\end{aligned}
\]

where \(u\) denotes the first component of the solution to (19). Applying classical results in the theory of ODE, one can derive that problem (31) has a unique solution \(q(t,x) \in C^1([0,T) \times \mathbb{R})\). Moreover, we have the following lemmas.

**Lemma 4.1.** The map \(q(t,\cdot)\) is an increasing diffeomorphism of \(\mathbb{R}\) with

\[
q_x(t,x) = \exp \left( \int_0^t u(\tau,q(\tau,x)) \, d\tau \right) > 0, \quad \forall (x,t) \in [0,T) \times \mathbb{R}.
\]

**Proof.** Differentiating (31) with respect to \(t\) yields

\[
\begin{aligned}
\frac{d}{dt} q_x(t,x) &= u_x(t,q(t,x)) q_x(t,x), \quad t \in [0,T), \ x \in \mathbb{R}, \\
q_x(0,x) &= 1, \quad x \in \mathbb{R},
\end{aligned}
\]

which leads to

\[
q_x(t,x) = \exp \left( \int_0^t u_x(\tau,q(\tau,x)) \, d\tau \right) > 0.
\]

This yields the desired result.

**Lemma 4.2.** We have the following identical equation

\[
\rho(t,q(t,x)) q_x(t,x) = \rho_0(x), \quad \text{for all} \ (t,x) \in [0,T) \times \mathbb{R}.
\]

Moreover, if there exists \(M_1 > 0\) such that \(u_x(t,x) \geq -M_1\) for all \((t,x) \in [0,T) \times \mathbb{R}\), then

\[
\|\rho(t,\cdot)\|_{L^\infty} \leq e^{M_1 T} \|\rho_0(\cdot)\|_{L^\infty}, \quad \forall t \in [0,T).
\]

**Proof.** In virtue of system (1), by direct computation, we discover that

\[
\begin{aligned}
\frac{d}{dt} \left[ \rho(t,q(t,x)) q_x(t,x) \right] &= \rho_t(t,q(t,x)) q_x(t,x) + \rho_x(t,q(t,x)) q_x(t,x) + \rho(t,q(t,x)) q_{xt}(t,x) \\
&= 0
\end{aligned}
\]

which implies \(\rho(t,q(t,x)) q_x(t,x) = \rho_0(x)\) for all \((t,x) \in [0,T) \times \mathbb{R}\).

By using Lemma 4.1, in view of (32) and the assumption of \(u_x\), we obtain

\[
\begin{aligned}
\|\rho(t,\cdot)\|_{L^\infty} &= \|\rho(t,q(t,\cdot))\|_{L^\infty} \\
&= \|\rho_0(\cdot)e^{\int_0^t u_x(\tau,q(\tau,x)) \, d\tau}\|_{L^\infty} \\
&\leq e^{M_1 T} \|\rho_0(\cdot)\|_{L^\infty}, \quad \forall t \in [0,T).
\end{aligned}
\]

In order to finish our topics, we provide the following lemma, which plays a pivotal role in the proof of the blow-up scenario, especially in Theorem 1.3.
Lemma 4.3. Let \((u, \rho)\) be the corresponding solution \((u, \rho)\) to system (1) with initial data \((u_0, \rho_0)\) guaranteed by Theorem 3.1. Assume that \(1 - 2\Omega A > 0\), and \(\sigma, b\) satisfy the condition
\[
\sigma = 1, b \geq 2; \quad \text{or} \quad b = -1, \sigma < 0.
\] (33)
and there exists \(M_1 > 0\) such that \(u_x(t, x) \geq -M_1\) for all \((t, x) \in [0, T) \times \mathbb{R}\).

(i) Then we have the following \(L^2\)-norm estimate of \(u(t, \cdot), u_x(t, \cdot)\)
\[
\|u(t, \cdot)\|_{L^2} \leq e^{M_1 \sigma(b-2)T/2}\left(\|u_0\|_{L^2} + \|u_0 x\|_{L^2} + (1 - 2\Omega A)\|\rho_0\|_{L^2}\right),
\]
\[
\|u_x(t, \cdot)\|_{L^2} \leq e^{M_1 \sigma(b-2)T/2}\left(\|u_0\|_{L^2} + \|u_0 x\|_{L^2} + (1 - 2\Omega A)\|\rho_0\|_{L^2}\right).
\]

(ii) With the condition (33) and the assumption of \(u_x\), the following estimate holds:
\[
\|u(t, \cdot)\|_{L^\infty} \leq Ce^{M_1 \sigma(b-2)T/2}\left(\|u_0\|_{L^2} + \|u_0 x\|_{L^2} + (1 - 2\Omega A)\|\rho_0\|_{L^2}\right).
\]

Proof. In virtue of system (1), multiplying the second equation with \(u\) and the first equation with \(\rho\) and integrating over \(\mathbb{R}\) leads to
\[
\frac{1}{2} \frac{d}{dt} \|\rho\|_{L^2}^2 = \int_{\mathbb{R}} \rho \rho_x u \, dx
\]
and
\[
\frac{1}{2} \frac{d}{dt} \left(\|u\|_{L^2}^2 + \|u_x\|_{L^2}^2\right)
= \int_{\mathbb{R}} \sigma(buu_x u_{xx} + u^2 u_{xxx}) \, dx - (1 - 2\Omega A) \int_{\mathbb{R}} \rho \rho_x u \, dx.
\]
Thus, by performing integration by parts we arrive at
\[
\frac{d}{dt} \left(\|u\|_{L^2}^2 + \|u_x\|_{L^2}^2 + (1 - 2\Omega A)\|\rho\|_{L^2}^2\right)
= \int_{\mathbb{R}} 2\sigma(buu_x u_{xx} + u^2 u_{xxx}) \, dx
= -\sigma(b - 2) \int_{\mathbb{R}} u_x^2 \, dx.
\]
Under the condition of (33), we can verify that \(\sigma(b - 2) \geq 0\). In view of the assumption of \(u_x\), we obtain
\[
\frac{d}{dt} \left(\|u\|_{L^2}^2 + \|u_x\|_{L^2}^2 + (1 - 2\Omega A)\|\rho\|_{L^2}^2\right)
\leq M_1 \sigma(b - 2) \int_{\mathbb{R}} u_x^2 \, dx
\leq M_1 \sigma(b - 2) \int_{\mathbb{R}} (u^2 + u_x^2 + (1 - 2\Omega A)\rho^2) \, dx.
\]
By means of Gronwall’s inequality, we have
\[
\|u\|_{L^2}^2 + \|u_x\|_{L^2}^2 + (1 - 2\Omega A)\|\rho\|_{L^2}^2
\leq e^{M_1 \sigma(b-2)T}\left(\|u_0\|_{L^2}^2 + \|u_0 x\|_{L^2}^2 + (1 - 2\Omega A)\|\rho_0\|_{L^2}^2\right).
\]
This yields the desired result for (1) by applying Cauchy-Schwarz inequality.
For (2), we take advantage of the Gagliardo-Nirenberg inequality
\[
\|D^j f\|_{L^r(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}^{1-\theta} \|D^m f\|_{L^q(\mathbb{R}^n)}^\theta, \quad \frac{n}{r} - j = (1 - \theta) \frac{n}{p} + \theta \left(\frac{n}{q} - m\right)
\]
for all
\[
\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{\theta}{m}, \quad 0 < \theta < 1.
\]
with \( n = 1, j = 0, m = 1 \) and \( r = \infty, p = q = 2 \). Thus
\[
\|u\|_{L^\infty} \leq C\|u\|_{L^2}^{1/2}\|u_x\|_{L^2}^{1/2}
\]
\[
\leq e^{M_1\sigma(\beta-2)/2}\left(\|u_0\|_{L^2} + \|u_{0x}\|_{L^2} + (1 - 2\Omega A)\|\rho_0\|_{L^2}\right).
\]
This completes the proof of the lemma. \( \square \)

We end this section with the proof of Theorem 1.3. The crucial point is to make use of Lemma 4.3, looking at the nonlinearity estimate. More concretely, we prove it as following.

**Proof of Theorem 1.3.** The starting point is to rewrite system (1) as
\[
\begin{cases}
m_t + (\sigma u - \mu)m_x = (A - \mu)u_x \\
+ (\sigma - b - 1)u_x m - (1 - 2\Omega A)\rho \rho_x + 2\Omega \rho (\rho u)_x, \\
\rho_t + (\rho u)_x = 0, \\
m(0, x) = m_0(x), \quad \rho(0, x) = \rho_0(x),
\end{cases}
\] (34)
under the condition of \( \sigma, b \) (33), where \( m = u - u_{xx} \).

Let \((u, \rho)\) be the solution to (1) with the initial data \((u_0, \rho_0) \in H^s \times H^{s-1}\), and \( T \) be the maximal existence time of the solution guaranteed by Theorem 3.1.

Multiplying the second equation in (34) by \( \rho \) and integrating by parts, we receive
\[
\frac{d}{dt}\|\rho\|_{L^2}^2 = -\int_R u_x \rho^2 \, dx.
\] (35)
Differentiating the second equation in (34) with respect to \( x \), multiplying the result equation by \( \rho_x \) and integrating by parts leads to
\[
\frac{d}{dt}\|\rho_x\|_{L^2}^2 = -3\int_R u_x \rho_x^2 \, dx + \int_R u_{xxx} \rho^2 \, dx.
\] (36)
Differentiating the second equation in (34) with respect to \( x \) for twice, multiplying the obtained equation by \( \rho_{xx} \) and integrating by parts, we arrive at
\[
\frac{d}{dt}\|\rho_{xx}\|_{L^2}^2 = -5\int_R u_x \rho_x^2 \, dx + \int_R u_{xxx}(3\rho_x^2 - 2\rho\rho_{xx}) \, dx.
\] (37)
Summing (35)–(37), we obtain
\[
\frac{d}{dt}(\|\rho\|_{L^2}^2 + \|\rho_x\|_{L^2}^2 + \|\rho_{xx}\|_{L^2}^2) = -\int_R u_x (\rho^2 + 3\rho_x^2 + 5\rho_{xx}^2) \, dx + \int_R u_{xxx}(\rho^2 + 3\rho_x^2 - 2\rho\rho_{xx}) \, dx.
\] (38)

Now we turn to the estimate of \( m \). Multiplying the first equation in (34) by \( m \) and integrating by parts, by some calculations, we get
\[
\frac{d}{dt}\|m\|_{L^2}^2
\]
\[
= (3\sigma - 2b - 2)\int_R u_x m^2 \, dx + (1 - 2\Omega A)\int_R u_x \rho^2 \, dx
\]
\[
- (1 - 2\Omega A)\int_R u_{xxx} \rho^2 \, dx - \Omega \int_R \rho^2 u_x u_{xx} \, dx + \Omega \int_R \rho^2 u_{xx} \, dx.
\] (39)
where we use the relationship that \( \int_{\mathbb{R}} u_x m \, dx = 0 \). After differentiating the second equation in (34) with respect to \( x \), we get the following equation

\[
m_{tx} + (\sigma u - \mu)m_{xx} + \sigma u_x m_x = (A - \mu)u_{xx} + (\sigma - b - 1)(u_x m)_x - (1 - \Omega A)(\rho \rho_x)_x + 2\Omega \rho (\rho u)_x.
\]

(40)

Because of the high order of the nonlinear term, the most difficult work is to handle with the last term \( "2\Omega (\rho (\rho u)_x)_x" \). Before we do the multiplying and integrating process, we first estimate this term. Assume that there exists \( M_1 > 0 \) and \( M_2 > 0 \) such that

\[
u_x(t, x) \geq -M_1, \quad \forall (t, x) \in [0, T) \times \mathbb{R}, \quad \text{and}
\]

\[
\|\rho_x(t, \cdot)\|_{L^\infty} \leq M_2, \quad \forall t \in [0, T).
\]

By using Lemma 4.2 and Lemma 4.3, there are

\[
\|\rho(t, \cdot)\|_{L^\infty} \leq e^{M_1 T} \|\rho_0(\cdot)\|_{L^\infty},
\]

\[
\|u(t, \cdot)\|_{L^\infty} \leq C e^{M_1 \sigma(b - 2)T/2} \left( \|u_0\|_{L^2} + \|u_0\|_{L^2} + (1 - 2\Omega A)\|\rho_0\|_{L^2} \right),
\]

for all \( t \in [0, T) \). Let \( Z(0) \) denote the part \( \|u_0\|_{L^2} + \|u_0\|_{L^2} + (1 - 2\Omega A)\|\rho_0\|_{L^2} \). Thus we obtain

\[
2\Omega \int_{\mathbb{R}} [u_x (\rho u)_x]_x \, dx
\]

\[
= \Omega \int_{\mathbb{R}} \left( -\rho^2 u_x u_{xx} + \rho^2 u u_{xxx} - 2\rho^2 u_x u_{xxx} \right) \, dx
\]

\[
- \Omega \int_{\mathbb{R}} (2\rho_x^2 u + 6\rho_x u_x u_x + 2\Omega u_x u_{xxx}) \, dx
\]

\[
\leq 2\Omega \|\rho_0\|_{L^\infty}^2 e^{2M_1 T} \int_{\mathbb{R}} (u_x^2 + u_{xx}^2 + u_{xxx}^2) \, dx
\]

\[
+ 6\Omega (M_1^2 + M_1 M_2) \int_{\mathbb{R}} (u^2 + \rho^2 + u_{xxx}^2) \, dx
\]

\[
+ 2\Omega C \|\rho_0\|_{L^\infty} e^{M_1 T} e^{M_1 \sigma(b - 2)T/2} Z(0) \int_{\mathbb{R}} (\rho_x^2 + u_{xxx}^2) \, dx.
\]

(41)

In the next moment, multiplying (40) by \( m_x \) and performing the integration by parts, we arrive at

\[
\frac{d}{dt} \|m_x\|_{L^2}^2
\]

\[
= (\sigma - 2b - 2) \int_{\mathbb{R}} u_x m_x^2 \, dx - (\sigma - b - 1) \int_{\mathbb{R}} u_x m^2 \, dx
\]

\[
- (1 - 2\Omega A) \int_{\mathbb{R}} u_{xxx} (\rho^2 - 2\rho_x^2 - 2\rho_x \rho_{xx}) \, dx + 4\Omega \int_{\mathbb{R}} [(\rho u)_x]_x \, dx.
\]

(42)

Here we use the fact that \( \int_{\mathbb{R}} u_{xx} m^2 \, dx = 0 \) which implies

\[
\int_{\mathbb{R}} u_{xxx} m^2 \, dx = \int_{\mathbb{R}} u_x m^2 \, dx.
\]
It is observed from (38), (39), and (42) that
\[
\frac{d}{dt}(\|m\|^2_{L^2} + \|m_x\|^2_{L^2} + \|\rho\|^2_{L^2} + \|\rho_x\|^2_{L^2} + \|\rho_{xx}\|^2_{L^2}) \\
= - \int_{\mathbb{R}} u_x (\rho^2 + 3\rho_x^2 + 5\rho_{xx}^2) dx \\
+ \int_{\mathbb{R}} (\rho^2 + 3\rho_x^2 - 2\rho_{xx}^2 - (1 - 2\Omega A)\rho^2) u_{xxx} dx \\
- (-2\sigma + b + 1) \int_{\mathbb{R}} u_x m^2 dx - (-\sigma + 2b + 2) \int_{\mathbb{R}} u_x m_x^2 dx \\
+ \int_{\mathbb{R}} (\rho^2 u_{xxx} - 2\rho^2 u_x u_{xxx}) dx + 4\Omega \int_{\mathbb{R}} [\rho(u_x)]^2 dx.
\] (43)

Under the $\sigma, b$ condition (that is equation (33)), we can check that
\[-2\sigma + b + 1 > 0, \quad \text{and} \quad -\sigma + 2b + 2 > 0.
\]
We will denote by $C_{\sigma,b}$ the maximal of $(-2\sigma + b + 1), (-\sigma + 2b + 2)$. It then follows from (41) and (43) that
\[
\frac{d}{dt}(\|m\|^2_{L^2} + \|m_x\|^2_{L^2} + \|\rho\|^2_{L^2} + \|\rho_x\|^2_{L^2} + \|\rho_{xx}\|^2_{L^2}) \\
\leq (5M_1 + C_{\sigma,b}M_2) \int_{\mathbb{R}} (m^2 + m_x^2 + \rho^2 + \rho_x^2 + \rho_{xx}^2) dx \\
+ (3M_2 + (2 - 2\Omega A)||\rho_0||_{L^\infty} e^{M_1 T}) \int_{\mathbb{R}} (\rho^2 + \rho_x^2 + \rho_{xx}^2 + u_x^2 + u_{xxx}^2) dx \\
+ 6\Omega||\rho_0||_{L^\infty} e^{2M_1 T} \int_{\mathbb{R}} (u_x^2 + u_{xx}^2 + u_{xxx}^2) dx \\
+ 12\Omega (M_2^2 + M_1 M_2) \int_{\mathbb{R}} (u^2 + \rho^2 + u_{xxx}^2) dx \\
+ 4\Omega C||\rho_0||_{L^\infty} e^{M_1 T} e^{M_1 \sigma(b-2)/T} Z(0) \int_{\mathbb{R}} (\rho_x^2 + u_{xxx}^2) dx
\]
where $C(\sigma, b, \Omega, A, M_1, M_2, T, u_0, u_{0x}, \rho_0)$ represents the constant depending on $\sigma$, $b$, $\Omega$, $A$, $M_1$, $M_2$, $T$, and initial data $u_0$, $u_{0x}$, $\rho_0$. To be more precise, it should be
\[3M_2 + (5 + C_{\sigma,b}) M_1 + 6(M_1 + M_2)^2 \\
+ (2 - 2\Omega A)||\rho_0||_{L^\infty} e^{M_1 T} + ||\rho_0||_{L^\infty}^2 e^{2M_1 T} \\
+ 4\Omega C||\rho_0||_{L^\infty} e^{M_1 T} e^{M_1 \sigma(b-2)/T} Z(0).
\]
According to the relationship between $u$ and $m$, as well as the Gronwall’s inequality, we deduce
\[
(\|u(t, \cdot)\|_{H^3} + \|\rho(t, \cdot)\|_{H^2}) \leq (\|m(t, \cdot)\|_{H^1} + \|\rho(t, \cdot)\|_{H^2}) \\
\leq (\|m_0\|_{H^1} + ||\rho_0||_{H^2}) \exp \left\{ C(\sigma, b, \Omega, A, M_1, M_2, T, u_0, u_{0x}, \rho_0) T \right\}
\].
With the above inequality, on account of Theorem 1.2, we conclude that the solution do not blow up in finite time.

On the other hand, if
\[
\lim_{t \to T} \inf_{x \in \mathbb{R}} \{ u_x(t, x) \} = -\infty, \quad \text{or} \quad \lim_{t \to T} \sup_{t \in \mathbb{R}} \{ \|\rho_x(t, \cdot)\|_{L^\infty} \} = \infty
\]
holds, the Sobolev embedding theorem $H^s(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ with $s > 1/2$ then implies that the corresponding solution blows up in finite time. This completes the proof of the theorem.

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