BOUNDS ON THE CLOSENESS CENTRALITY OF A GRAPH

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Abstract. We present new values and bounds on the (normalised) closeness centrality $\bar{C}$ of connected graphs and on its product $\bar{l}\bar{C}$ with the mean distance $\bar{l}$ of these graphs. Our main result presents the fundamental bounds $1 \leq \bar{l}\bar{C} < 2$. The lower bound is tight and the upper bound is asymptotically tight. Combining the lower bound with known upper bounds on the mean distance, we find ten new lower bounds for the closeness centrality of graphs. We also present explicit expressions for $\bar{C}$ and $\bar{l}\bar{C}$ for specific families of graphs. Elegantly and perhaps surprisingly, the asymptotic values $\bar{C}(P_n)$ and $\bar{C}(L_n)$ both equal $\pi$, and the asymptotic limits of $\bar{l}\bar{C}$ for these families of graphs are both equal to $\pi/3$. We conjecture that the set of values $\bar{l}\bar{C}$ for all connected graphs is dense in the interval $[1, 2)$.

1. Introduction

Let $G = (V, E)$ be a connected graph on $|V| = n$ vertices. Defined in 1950 by Bavelas [5], the (normalised) closeness centrality of $G$ is defined as the average of the (normalised) closeness centralities $\bar{C}_C(v)$ for each vertex $v$ in $G$:

$$\bar{C}_C(G) := \frac{1}{n} \sum_{v \in V} \bar{C}_C(v) \quad \text{where} \quad \bar{C}_C(v) = \frac{n-1}{\sum_{w \in V} d(v, w)}$$

and where $d(v, w)$ is the distance between $v$ and $w$; that is, the shortest length of a path between $v$ and $w$ in $G$.

The purpose of this paper is to provide values and bounds on $\bar{C}_C := \bar{C}_C(G)$ and on the product $\bar{l}\bar{C}$ where $\bar{l} := \bar{l}(G)$ is the mean distance of $G$, defined as

$$\bar{l}(G) = \frac{1}{n(n-1)} \sum_{u,v \in V} d(u, v).$$

In Section 2, Theorem 1 presents the fundamental and elegant new bounds $1 \leq \bar{l}\bar{C} < 2$. The lower bound is tight and the upper bound is asymptotically tight. Although these bounds are simple, they appear to have been overlooked in previous literature. They show that the closeness centrality $\bar{C}_C(G)$ and the mean distance $\bar{l}(G)$ cannot both be small nor both be large. These bounds are also useful: the lower bound combined with known upper bounds on $\bar{l}(G)$, the yields ten new lower bounds for $\bar{C}_C(G)$; see Corollary 2.

In Section 3, we prove four new upper bounds on $\bar{C}_C(G)$ that depend on different graph parameters; see Theorems 6, 9 and 10.

In Section 4, we provide lower bounds for three infinite families of graphs, namely the self-complementary graphs, the 2-connected graphs, and trees; see Theorem 12. We also provide explicit expressions for the closeness centralities $\bar{C}_C(v)$ for the vertices in twelve specific infinite families of graphs, as well as the closeness centralities $\bar{C}_C(G)$ for ten of these graphs; see Proposition 13 and Corollary 14. For the remaining two families, namely the path graphs $P_n$ and the ladder graphs $L_n$, we provide asymptotically tight lower and upper bounds; see
Conversely, these upper bounds on $\bar{C}$ were introduced by Handa [15] as transmission regular graphs, and all other distance-regular graphs but which, as shown in [2, 3], are not necessarily so-called transmission regular graphs. For more information on distance-regular graphs, see, for instance [1, 4].

Our first main result are elegantly interlinked lower and upper bounds for $\bar{l}$ and $\bar{C}$.

**Theorem 1.**

$$1 \leq \bar{l} \bar{C} < 2.$$  

The lower bound is tight and the upper bound is asymptotically tight.

This theorem will be proved in Section 6. The proof shows that $G$ achieves equality $\bar{l} \bar{C} = 1$ exactly when the terms $\bar{C}(v)$ are identical for all vertices $v$; equivalently, this is when the so-called transmission $\sum_w d(v, w)$ is independent of the vertex $v$. Graphs with this property were introduced by Handa [15] as transmission regular graphs and include cycles, complete graphs, and all other distance-regular graphs but which, as shown in [2, 8], are not necessarily regular. For more information on distance-regular graphs, see, for instance [1, 4].

Theorem 1 provides the lower bound $\bar{l} \geq \bar{C}^{-1}$ on the mean distance $\bar{l}$. This complements the many upper bounds on $\bar{l}$ found in the literature; see, for instance [1, 13, 20, 22, 23]. Conversely, these upper bounds on $\bar{l}$ may be combined with the general lower bound $\bar{C} \geq \bar{l}^{-1}$ from Theorem 1 to provide new lower bounds on the closeness centrality $\bar{C}$, as described in the following corollary.

**Corollary 2.** Let $G = (V, E)$ be a connected graph on $n$ vertices and $m$ edges with minimum vertex degree $\delta$, maximum vertex degree $\Delta$ and diameter $D = \max_{u,v \in V} d(u, v)$. Then

1. $\bar{C} \geq (n - 1)\frac{n - \Delta - 1}{n} - \frac{2(m - n + 1)}{n(n - 1)} + 1$ ;
2. $\bar{C} \geq 6n(n - 1)(3n(n - 4)D + 6n(n + 2) - 12(m + 1) - D^2(D - 6) + h(D))^{-1}$ ;
3. $\bar{C} \geq (n - 1)\left(\sum_{k : Q_k(0) \leq n - 1} \frac{n(n - 1)}{Q_k^2(0) + n - 1}\right)^{-1}$ ;
4. $\bar{C} \geq (n - 1)n(\delta n(n - 1) - 2(b - 1)m)^{-1}$ ;
5. $\bar{C} \geq \frac{n - 1}{n}\left(\frac{\Delta}{\frac{\theta_1}{\theta_2} + \frac{\theta_2}{\theta_3} + \cdots + \frac{\theta_{n-1}}{\theta_n}}\right)^{-1}$ ;
6. $\bar{C} \geq (n - 1)\frac{2(m - n - 2)}{\theta_2^{-1} + \cdots + \theta_n^{-1}}$ ;
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(7) \( \bar{C} \geq \frac{3}{n+1} \);

(8) \( \bar{C} \geq n(n-1) \left( \frac{(n+1)(n-1) - 2m}{\delta + 1} \right)^{-1} \);

(9) \( \bar{C} \geq \alpha^{-1} \);

(10) \( \bar{C} \geq D^{-1} \);

where \( h(D) = 6 - D \) if \( n - D \) is odd and \( h(D) = 2D \) when \( n - D \) is even; \( \theta_i \) is the \( i \)th largest eigenvalue of the Laplacian matrix for \( G \); \( b \) is the number of these Laplacian eigenvalues; \( Q_k \) is the \( k \)-alternating polynomial defined over them; and \( \alpha \) is the independence number for \( G \).

The first two bounds in Corollary 2 follow from upper bounds on \( \bar{l} \) proved by Gago, Hurajová and Madaras [13, 14]; the third and fourth bounds follow from upper bounds on \( \bar{l} \) by Rodriguez and Yebra [22]; and the last six bounds follow from upper bounds on \( \bar{l} \) by Mohar [20], Teranishi [23], Entringer, Jackson and Snyder [10], Beezer, Riegsecker, and Smith [6], Chung [8], and Doyle and Graver [9], respectively. The last three of these bounds are tight and are attained exactly when \( G \) is the complete graph \( K_n \).

The lower bound in Theorem 1 was first discovered by the second author in his thesis [17], expressed in terms of the betweenness centrality of \( G \). This centrality was defined in 1977 by Freeman [11] as

\[
\bar{C}_B := \bar{C}_B(G) = \frac{1}{n} \sum_{v \in V} C_B(v) \quad \text{where} \quad C_B(v) = \sum_{s \neq t} \frac{\sigma_{st}(v)}{\sigma_{st}},
\]

where \( \sigma_{st} \) is the number of shortest paths between vertices \( s \) and \( t \) and where \( \sigma_{st}(v) \) is the number of such paths that pass through vertex \( v \neq s, t \). The betweenness centrality \( \bar{C}_B \) and the mean distance \( \bar{l} \) are related bijectively via the following identity first observed by Gago [12].

**Theorem 3.** \( \bar{C}_B = \frac{1}{2}(n-1)(\bar{l} - 1) \).

The bound in Theorem 1 thus reflects that the betweenness centrality \( \bar{C}_B \) and the closeness centrality \( \bar{C}_C \) can be interpreted as dual concepts, as noted in [7]. The upper bound in Theorem 1 can be used together with known lower bounds on \( \bar{l} \) to give upper bounds for \( \bar{C}_C \). However, we will instead present upper bounds in the following section that do not rely on Theorem 1.

### 3. Upper bounds on closeness centrality

In Section 2, we proved lower bounds on the closeness centrality \( \bar{C}_C(G) \) of a graph \( G = (V,E) \). In this section, we prove upper bounds on \( \bar{C}_C(G) \). To derive the first bound, we make a trivial but useful observation.

**Remark 4.** If \( e \) is an edge of a connected graph \( G \) for which \( G - e \) is also connected, then \( \bar{C}_C(G - e) < \bar{C}_C(G) \).

A complete graph is obtained by adding edges to \( G \), and it is easy to see that complete graphs have closeness centrality \( \bar{C}_C(K_n) = 1 \). Remark 4 therefore implies the following simple upper bound on \( \bar{C}_C(G) \).

**Lemma 5.** Let \( G \) be a connected graph. Then \( \bar{C}_C(G) \leq 1 \), and equality is attained exactly when \( G \) is a complete graph.

We can improve Remark 4 by estimating the reduction of the closeness centrality.
Theorem 6. Let $G$ be a connected graph on $n$ vertices and $k$ edges. Suppose that $e$ is an edge of $G$ such that $G - e$ is also connected. Then
\[
\tilde{C}_C(G - e) \leq \tilde{C}_C(G) - \frac{2n - 2}{n} \left( \frac{2}{(n - 1)(n + 2) - 2} - \frac{2}{(n - 1)(n + 2) + 1 - 2k} \right).
\]

Proof. Let $u$ and $v$ be the end-vertices of $e$. Then $d_{G-e}(u, v) = d_{G-e}(y, x) \geq 2 = d_G(x, y) + 1 = d_G(y, x) + 1$. Thus,
\[
\tilde{C}_C(G - e) \leq \frac{n}{n - 1} - \frac{1}{1 + \sum_{v \in V} d_G(x, v) - \sum_{v \in V} d_G(x, v)} + \frac{1}{1 + \sum_{v \in V} d_G(y, v) - \sum_{v \in V} d_G(y, v)} - \frac{1}{\sum_{u \in V} \sum_{v \in V} d_G(u, v)}.
\]

The proof follows from $\sum_{v \in V} d_G(w, v) \leq \frac{1}{2} (n - 1)(n + 2) - k$ for $w \in \{x, y\}$. \(\square\)

Remark 7. We can replace $\frac{1}{2} (n - 1)(n + 2) - k$ with $\max_{u \in V} \sum_{v \in V} d_G(w, v)$ for particular classes of graphs to obtain tighter bounds for those classes.

Corollary 8. If $G$ is a connected graph on $n$ vertices and $e$ edges, then
\[
\tilde{C}_C(G) \leq 1 - \frac{2}{n} \frac{4(n - 1)}{n((n - 1)(n + 2) - 2e)}.
\]

Proof. Obtain $G$ by removing exactly $\binom{n}{2} - e$ edges from $K_n$ and apply Theorem 6. \(\square\)

The eccentricity $\epsilon(v)$ of a vertex $v$ of a graph $G$ is the greatest distance between $v$ and any other vertex in $G$. The radius $r(G)$ is the smallest eccentricity $\epsilon(v)$ of a vertex $v$ of $G$. For each vertex $v \in V$ and integer $i = 1, \ldots, \epsilon(v)$, define $d_i(v) = \{u \in V : d(u, v) = i\}$.

Theorem 9. Let $G = (V, E)$ be a connected graph on $n$ vertices with radius $r \geq 1$. Then
\[
\tilde{C}_C(G) \leq \frac{n - 1}{n - 1 + \binom{r}{2}}.
\]

Proof. If $r = 1$, then $\binom{r}{2} = 0$, so we may apply Lemma 5. Suppose then that $r \geq 2$. Since the identity $d_1(v) + \cdots + d_{\epsilon(v)}(v) = n - 1$ is true for each vertex $v \in V$, the sum
\[
\sum_{u \in V} d(u, v) = \sum_{i=1}^{\epsilon(v)} id_i(v)
\]
is minimal exactly when $d_1(v) = n - \epsilon(v)$ and $d_i(v) = 1$ for each $i = 2, \ldots, \epsilon(v)$. Therefore,
\[
\tilde{C}_C(G) = \frac{1}{n} \sum_{v \in V} \tilde{C}_C(v) \leq \frac{1}{n} \sum_{v \in V} \frac{n - 1}{n - \epsilon(v) + 2 + \cdots + \epsilon(v)} = \frac{1}{n} \sum_{v \in V} \frac{n - 1}{n - 1 + \binom{\epsilon(v)}{2}}.
\]

Since $\epsilon(v) \geq r$ for each vertex $v \in V$, the theorem follows. \(\square\)

The next theorem provides a similar bound to that in Theorem 9 but also takes into consideration the maximal vertex degree $\Delta$ of $G$. 
Theorem 10. Let \( G = (V, E) \) be a connected graph on \( n \) vertices with radius \( r \geq 2 \) and maximum vertex degree \( \Delta \). Then

\[
\bar{C}_C(G) \leq \frac{n-1}{2n-1-\Delta + r(r-3)/2}.
\]

Proof. Since \( d_1(v) \leq \Delta \) for each \( v \in V \), the sum (1) is minimal when \( d_1(v) = \Delta \), \( d_2(v) = n-1-\Delta - (\epsilon(v) - 2) \) and \( d_i(v) = 1 \) for each \( 3 \leq i \leq \epsilon(v) \). Therefore,

\[
\bar{C}_C(G) = \frac{1}{n} \sum_{v \in V} \bar{C}_C(v) \leq \frac{1}{n} \sum_{v \in V} \frac{n-1}{2n-1-\Delta + \epsilon(v)(\epsilon(v) - 3)/2}.
\]

Since \( \epsilon(v) \geq r \) for each vertex \( v \in V \), the theorem follows. \( \square \)

Remark 11. Note that the lower bounds on \( \bar{C}_C(G) \) in Corollary 2 can be combined with the upper bounds on \( \bar{C}_C(G) \) in Corollary 8 and Theorems 9 and 10 to form potentially interesting bounds of the form \( A \leq B \) where \( A \) is one of the right-hand side expressions in the ten bounds of Corollary 2 and where \( B \) is one of the three expressions

\[
1 - \frac{2}{n} + \frac{4n-4}{n((n-1)(n+2) - 2e)}, \quad \frac{n-1}{n-1+\binom{r}{2}}, \quad \text{and} \quad \frac{n-1}{2n-1-\Delta + r(r-3)/2}.
\]

4. Closeness Centralities for Particular Families of Graphs

The previous two sections presented lower and upper bounds on the closeness centrality \( \bar{C}_C(G) \) of a graph \( G \). This section gives lower and upper bounds on \( \bar{C}_C(G) \) for particular classes of graphs.

Lower bounds on \( \bar{C}_C \) are given for three particular classes of graphs in the following proposition. The bounds are attained by applying Theorem 1 to upper bounds on \( l \) proved by Hendry [16], Plesnik [21], and Gago, Hujarová and Madaras [13], respectively.

Theorem 12. Let \( G = (V, E) \) be a connected graph on \( n \) vertices. Then

1. if \( G \) is self-complementary, then \( n \equiv 0 \) or \( 1 \) (mod 4) and

\[
\bar{C}_C(G) \geq \begin{cases} 
8n - 8, & n \equiv 0 \pmod{4} \\
13n - 12, & n \equiv 1 \pmod{4} \\
8n, & n \equiv 3 \pmod{4} \\
13n - 1, & n \equiv 2 \pmod{4} 
\end{cases}.
\]

2. if \( G \) is 2-vertex-connected or 2-edge-connected, then \( \bar{C}_C(G) \geq (n-1)\left[\frac{1}{4}n^2\right]^{-1} \).

3. if \( T \) is a tree with maximal degree \( \Delta \), then \( \bar{C}_C(T) \geq \frac{n\Delta}{2(n-\Delta)(\Delta-1)(n-1)+2} \).

The second bound is tight and is achieved when \( G \) is the cycle \( C_n \).

Bounds such as those in Theorem 12 and in the preceding sections are particularly useful if it is not possible to calculate a formula for the closeness centrality. For some classes of graphs, however, such calculations are possible. Proposition 13 below presents explicit expressions for the closeness centrality \( \bar{C}_C(v) \) for the vertices in twelve particular infinite families of graphs. The proofs of these expressions are easy and straightforward, and are therefore omitted. The expressions and their derivations first appeared in the second and third authors’ works [17, 18], along with expressions for the closeness and betweenness centralities of similar graphs. The twelve families of graphs also featured in [24] where their betweenness centralities were calculated. Each of the families is well-known, except possibly the cocktail party graph \( CP(n) \).
and the crown graph $S_0^n$: these are the graphs obtained by deleting $n$ vertex-disjoint edges from $K_{2n}$ and $K_{n,n}$, respectively.

**Proposition 13.** The vertex closeness centralities $\bar{C}_C(v)$ for all $v \in V$ are given below for the families of graphs $G = (V, E)$ shown in Figure 1.

**Complete graph $K_n$:** $\bar{C}_C(v) = 1$

**Cycle graph $C_n$:** $\bar{C}_C(v) = \frac{n-1}{\lfloor n^2/4 \rfloor}$

**Wheel graph $W_n$:** $\bar{C}_C(v) = 1$ if $v$ is the central vertex and $\frac{n-1}{2n-5}$ otherwise

**Star graph $S_n$:** $\bar{C}_C(v) = 1$ if $v$ is the central vertex and $\frac{n}{2(n-1)}$ otherwise

**Near-complete graph $K_n - e$:** $\bar{C}_C(v) = \frac{n-1}{n}$ if $v$ is adjacent to $e$ and $1$ otherwise

**Cocktail party graph $CP(n)$:** $\bar{C}_C(v) = \frac{n-1}{2n}$

**Crown graph $S_0^n$:** $\bar{C}_C(v) = \frac{n-1}{2n}$

**Path graph $P_n$:** $\bar{C}_C(v) = \frac{4(n-1)}{(2k-n+1)^2+n^2-1}$ for $k = 0, \ldots, n-1$

**Ladder graph $L_n$:** $\bar{C}_C(v) = \frac{4n-2}{(2k-n+1)^2+n^2+2n-1}$ for $k = 0, \ldots, n-1$

**Circular ladder graph $CL_n$:** $\bar{C}_C(v) = \frac{2(n^2-4)+n}{2k-2}$

**Hypercube graph $Q_k$:** $\bar{C}_C(v) = \frac{2k-1}{2^k}$

The expressions for $\bar{C}_C(v)$ in Proposition 13 can easily be used to find simple expressions for the closeness centrality $\bar{C}_C(G)$ of ten of the twelve families of graphs $G$. 

**Figure 1.** Twelve families of graphs
Consider the Corollaries 4 and 5 from [14]. For the graphs $G \in \{P_n, L_n\}$, closed-form expressions for the closeness centrality $\bar{C}_c(G)$ can be found, for instance by using Maple [19].

**Corollary 15.** For the graphs $G \in \{P_n, L_n\}$,\[ \bar{C}_c(G) = \frac{1}{\binom{n}{2}} \frac{abc}{\psi\left(\frac{1 + an + bn}{2}\right)} \]
where $\psi(x)$ is the digamma function and where $\hat{n} = \sqrt{1 - n^2}$ and $c = \frac{1}{n}$ if $G = P_n$, and $\hat{n} = \sqrt{1 - 2n - n^2}$ and $c = \frac{2n - 1}{4n}$ if $G = L_n$.

Propositions 16 and 17 below provide approximations for $\bar{C}_c(P_n)$ and $\bar{C}_c(L_n)$ by way of lower and upper bounds which converge asymptotically as $n$ grows large.

**Proposition 16.** \[ \frac{4}{n} \sqrt{\frac{n - 1}{n + 1}} \tan^{-1} \left( \sqrt{\frac{n - 1}{n + 1}} \right) \leq \bar{C}_c(P_n) \leq \frac{\pi}{n} \sqrt{\frac{n - 1}{n + 1}} + \frac{n - 1}{n \left[ \frac{n}{2} \right] \left[ \frac{n}{2} \right]}. \]

**Proof.** By Proposition 13, \[ \frac{n}{n - 1} \bar{C}_c(P_n) = \sum_{k=0}^{n-1} f(k) \quad \text{where} \quad f(x) = \frac{4}{(2x - n + 1)^2 + n^2 - 1}. \]

Since $f$ is non-decreasing for all $x \leq \frac{n-1}{2}$ and is symmetric in $x = \frac{n-1}{2}$, \[ \frac{n}{n - 1} \bar{C}_c(P_n) - f \left( \left\lfloor \frac{n}{2} \right\rfloor \right) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} f(k) + \sum_{k=\left\lfloor \frac{n}{2} \right\rfloor + 1}^{n-1} f(k) \]
\[ \leq \int_0^{(n-1)/2} f(x)dx + \int_{(n-1)/2}^{n-1} f(x)dx = \int_0^{n-1} f(x)dx. \]

Therefore by integration, \[ \frac{n}{n - 1} \bar{C}_c(P_n) - f \left( \left\lfloor \frac{n}{2} \right\rfloor \right) \leq 2 \tan^{-1} \left( \frac{2x - n + 1}{\sqrt{n^2 - 1}} \right) \bigg|_0^{n-1} = \frac{4}{\sqrt{n^2 - 1}} \tan^{-1} \left( \frac{n - 1}{\sqrt{n^2 - 1}} \right). \]

Hence, \[ \bar{C}_c(P_n) - \frac{n-1}{n} f \left( \left\lfloor \frac{n}{2} \right\rfloor \right) \leq 4 \sqrt{\frac{n-1}{n+1}} \tan^{-1} \left( \sqrt{\frac{n-1}{n+1}} \right). \]
Since \(\sqrt{\frac{n-1}{n+1}} \leq 1\),
\[
\bar{C}(P_n) \leq \frac{\pi}{n} \sqrt{\frac{n-1}{n+1}} \cdot \varliminf_{n \to \infty} \frac{n}{2} = \frac{\pi}{n} \sqrt{\frac{n-1}{n+1}} \cdot \frac{n}{2} \leq \frac{\pi}{n} \sqrt{\frac{n-1}{n+1} + \frac{n-1}{n}}.
\]

The lower bound is proved similarly. \(\square\)

The proof of Proposition 17 below is similar to that above.

Proposition 17.
\[
\frac{4n - 2}{n \sqrt{n^2 + 2n - 1}} \tan^{-1} \left( \frac{n-1}{\sqrt{n^2 + 2n - 1}} \right) + \frac{2}{n^3} \left( \frac{2n-1)^2}{n^2 + 2n - 1} \right) \leq \bar{C}(L_n) \leq \frac{\pi}{n} \sqrt{\frac{2n-1}{2n + 5} + \frac{2}{n(2n + 5)}}.
\]
Interestingly, Propositions 16 and 17 imply that the closeness centrality measures for paths and ladders both converge to \(\bar{C}(L_n) = \frac{\pi}{n} \) as \(n\) grows large:

Corollary 18.
\[
\lim_{n \to \infty} n\bar{C}(P_n) = \lim_{n \to \infty} n\bar{C}(L_n) = \pi.
\]

5. Product of Closeness Centralities and Mean Distances for Graph Families

By Theorem 1, the product of the closeness centrality and the mean distance of each graph \(G\) satisfies the inequality \(1 \leq \bar{C}(G) < 2\). In this section, we will calculate the product \(\bar{C}(G)\) explicitly for specific classes of graphs. To do so, we first present some easily-calculated mean distances \(\bar{l}(G)\).

Proposition 19. The mean distances \(\bar{l}(G)\) for the families of graphs in Figure 4 are:
\[
\begin{align*}
\bar{l}(K_n) &= 1, & \bar{l}(S_n) &= \frac{2n}{n+1}, & \bar{l}(K_{n,k}) &= \frac{m(k+2m-2)+k(m+2k-2)}{(m+k)(m+k-1)}, & \bar{l}(L_n) &= \frac{n+2}{k}, \\
\bar{l}(C_n) &= \frac{n^2 - 1}{n - 1}, & \bar{l}(K_n - e) &= \frac{n(n-1)+2}{n(n-1)}, & \bar{l}(S_n^0) &= \frac{3n}{2n-1}, & \bar{l}(L_n) &= \frac{2n^2 - 4k + n}{n - 1}, \\
\bar{l}(W_n) &= \frac{2n-4}{n}, & \bar{l}(CP_n) &= \frac{2n}{2n-1}, & \bar{l}(P_n) &= \frac{n+1}{3}, & \bar{l}(Q_k) &= \frac{k^2 - 1}{2k - 1}.
\end{align*}
\]

For several well-known families of graphs, the values of \(\bar{C}(G)\) attain the lower bound 1 in Theorem 1, as the following corollary to Proposition 19 shows.

Corollary 20. \(\bar{C}(G) = 1\) for \(G \in \{K_n, C_n, CP_n, S_n^0, CL_n, Q_k\}\).

For other families, \(\bar{C}(G)\) is always greater than 1 but converges towards 1 as \(n\) grows large.

Corollary 21. \(\lim_{n \to \infty} \bar{C}(G) = 1\) for \(G \in \{S_n, K_n - e, W_n\}\).

Meanwhile, the values of \(\bar{C}(G)\) converge to unexpectedly interesting constants for path and ladder graphs.

Corollary 22. \(\lim_{n \to \infty} \bar{C}(G) = \frac{\pi}{3}\) for \(G \in \{P_n, L_n\}\).

Proof. Straightforward from Corollary 18 and Proposition 19. \(\square\)
Corollary 23. For $G = K_{n-k,k}$,

$$\lim_{n \to \infty} \max_{1 \leq k \leq n-1} \bar{\lambda} \mathcal{C}_C \leq 18 - 12\sqrt{2}.$$ 

Indeed,

$$\lim_{n \to \infty} \bar{\lambda} \mathcal{C}_C = 18 - 12\sqrt{2}$$

for

$$k = \left\lfloor \frac{n}{2} - \frac{\sqrt{(3n-4)^2 - 2(3n-4)\sqrt{2(n-1)(n-2)}}}{2} \right\rfloor.$$ 

Proof. Apply simple calculus and Maple [19] to Corollary 14 and Proposition 19. □

We also consider another family of graphs that provides another unexpectedly interesting asymptotic limit. For any positive integer $n, k$, let $G_{n,k}$ denote the graph obtained by attaching a path of $k$ vertices to each vertex of the complete graph $K_n$; see Figure 2 for an example.

![Figure 2. $G_{5,2}$](image)

Proposition 24.

$$\bar{\lambda}(G_{n,k}) = \frac{(k+1)^2 n - \frac{2}{3}k^2 - \frac{4}{3}k - 1}{(k+1)n-1}$$

and

$$\bar{\mathcal{C}}_C(G_{n,k}) = \frac{(k+1)n-1}{k+1} \sum_{j=0}^{k} j(j+1) + (k-j)(k-j+1) + (n-1)(k+1)(2j + k + 2).$$

Proof. Let $G_{n,k}$ be written as $\bigcup_{i=1}^{n} P_{k+1}^i + E$ where $P_{k+1}^i = v_0^i v_1^i \ldots v_k^i$ is a path and $E = \{ v_0^i, v_0^j \} : i, j \in \{1, \ldots, n\}, i \neq j$. For any $i \in \{1, \ldots, n\}$ and $j \in \{0, \ldots, k\}$,

$$\sum_{v \in V} d(v_j, v) = \sum_{r=1}^{j-1} d(v_j^r, v_i^r) + \sum_{r=j+1}^{k} d(v_j^r, v_i^r) + \sum_{i \neq j} \sum_{r=0}^{k} d(v_j^r, v_i^r)$$

$$= \sum_{r=1}^{j-1} r + \sum_{r=j+1}^{k} (r-j) + (n-1) \sum_{r=0}^{j} (j+r+1)$$

$$= \frac{j(j+1)}{2} + \frac{(k-j)(k-j+1)}{2} + \frac{(n-1)(k+1)(2j + k + 2)}{2}.$$ 

The values of $\bar{\lambda}$ and $\bar{\mathcal{C}}_C$ can now be obtained directly from their definition. □
Theorem 25. Let $G = G_{n,k}$. For each integer $k \geq 1$, $\ln \left(3 - \frac{2}{k+2}\right) \leq \lim_{n \to \infty} \bar{\mathcal{C}}_C \leq \ln \left(3 + \frac{2}{k}\right)$. Moreover, 

$$\lim_{k \to \infty} \left( \lim_{n \to \infty} \bar{\mathcal{C}}_C \right) = \ln 3.$$ 

Proof. It is easy to check that $\lim_{n \to \infty} \bar{\mathcal{C}}_C = \sum_{j=0}^{k} \frac{1}{2j + k + 2}$. The bounds follow from the inequalities 

$$\int_{0}^{k+1} f(x) \, dx \leq \sum_{j=0}^{k} f(j) \leq \int_{0}^{k+1} f(x-1) \, dx$$ 

where $f(x) = \frac{1}{2x + k + 2}$. \hfill $\Box$

6. Proof of Theorem 26

In order to prove Theorem 26, we first prove the lower bound therein.

Theorem 26. Let $G$ be a connected graph. Then 

$$\bar{\mathcal{C}}_C \geq 1.$$ 

Proof. Note that 

$$\sum_{v \in V} \frac{1}{\mathcal{C}_C(v)} = \frac{1}{n-1} \sum_{v \in V} \sum_{u \in V} d(u,v) = n \bar{l},$$ 

so, by the Arithmetic-Harmonic Inequality, 

$$\bar{\mathcal{C}}_C = \frac{n \bar{l}}{\sum_{v \in V} \frac{1}{\mathcal{C}_C(v)}} \geq \frac{n}{\sum_{v \in V} \mathcal{C}_C(v)} = 1.$$ \hfill $\Box$

Next, we prove an upper bound on $\bar{\mathcal{C}}_C$ that is stronger than that in Theorem 25.

Theorem 27. Let $G$ be a connected graph on $n$ vertices. Then 

$$\bar{\mathcal{C}}_C < 2 \left(1 - \frac{1}{n}\right)^2 + \frac{2}{n^2}.$$ 

Proof. Since $d(x,u) + d(u,y) \geq d(x,y)$ for all $x,y,u \in V$, it holds that, for each $u \in V$, 

$$\sum_{x,y \in V\setminus\{u\}, x \neq y} d(x,u) + d(u,y) \geq \sum_{x,y \in V\setminus\{u\}} d(x,y).$$ 

Adding $\sum_{x \in V} d(x,u) + \sum_{y \in V} d(u,y)$ to both sides yields a nice inequality for each $u \in V$: 

$$(2n - 2) \sum_{v \in V} d(u,v) \geq \sum_{x,y \in V} d(x,y).$$ 

Finally, $\bar{\mathcal{C}}_C$ is bounded as follows:

$$\bar{\mathcal{C}}_C = \frac{1}{n^2} \sum_{x \neq y \in V} d(x,y) \leq \frac{2}{n} + \frac{1}{n^2} \sum_{x \neq y \in V\setminus\{u\}} d(x,y) \leq \frac{2}{n} + \frac{2n - 2}{n^2} \sum_{u \in V} \sum_{x,y \in V\setminus\{u\}} d(x,y).$$
Since equality is not possible here, and since
\[ \sum_{u \in V} \sum_{x,y \in V - \{u\}} d(x,y) = (n-2) \sum_{x,y \in V} d(x,y), \]
the proof follows. \(\square\)

**Corollary 28.** Let \(G\) be a connected graph. Then
\[ \bar{l}C_C < 2. \]

Next, we consider one more class of graphs, as follows. For all positive integers \(k, m, n\), let \(H_{k,m,n}\) be the graph obtained by joining edges between all vertices of the complete graph \(K_n\) to one endpoint of each of \(m\) copies of the path \(P_{k+1}\); see Figure 3 for an example.

\[ \text{Figure 3. } H_{3,2,5} \]

When \(m = 1\), this family coincides with the graphs mentioned by Entringer, Jackson and Snyder [10, Figure 2.1]. Below are semi-explicit expressions for \(\bar{l}(H_{k,m,n})\) and \(C_C(H_{k,m,n})\); we omit the messy but easily-derived explicit forms. Write \(H_{k,m,n}\) as \(\bigcup_{i=1}^{m} P_{k+1} \cup K_n\) + \(E\), where \(P_{k+1}^i = v_0^i v_1^i \ldots v_k^i\) is a path and \(E = \{\{v_0^i, v\} : i \in \{1, \ldots, m\}, v \in K_n\}\).

**Proposition 29.** Let \(V = V(H_{k,m,n})\). Then

\[ \bar{l}(H_{k,m,n}) = \frac{2n(n-1) + nm(k+1)(k+2) + 2m \sum_{j=0}^{k} \sum_{v \in V} d(v_j^i, v)}{2((k+1)m + n)((k+1)m + n - 1)}, \]
\[ C_C(H_{k,m,n}) = \frac{(k+1)m + n - 1}{(k+1)m + n} \left( \frac{2n}{2n - 2 + m(k+1)(k+2)} + \sum_{j=0}^{k} \sum_{v \in V} d(v_j^i, v) \right). \]

**Proof.** For each \(u \in V(K_n)\),
\[ \sum_{v \in V} d(u, v) = \sum_{v \in V(K_n)} d(u, v) + \sum_{i=1}^{m} \sum_{j=0}^{k} d(v, v_j^i) = n - 1 + m \sum_{j=0}^{k} (j + 1) \]
\[ = n - 1 + \frac{m(k+1)(k+2)}{2}. \]
For any $i \in \{1, \ldots, m\}$ and $j \in \{0, \ldots, k\}$, we have

$$\sum_{v \in V} d(v^i_j, v) = \sum_{v \in V(K_n)} d(v^i_j, v) + \sum_{r=1}^{j-1} d(v^i_j, v^i_r) + \sum_{r=j+1}^{k} d(v^i_j, v^i_r) + \sum_{s \neq j, r=0}^{k} d(v^i_j, v^i_s) \quad \text{for all connected graphs} \quad H \in \mathcal{H}(3)$$

$$= \sum_{v \in V(K_n)} (j+1) + \sum_{r=1}^{j-1} r + \sum_{r=j+1}^{k} (r-j) + (m-1) \sum_{r=0}^{k} (j+r+2) \quad \text{for all connected graphs} \quad H \in \mathcal{H}(3)$$

$$= (n-1)(j+1) + \frac{j(j+1)}{2} + \frac{(k-j)(k-j+1)}{2} + \frac{(m-1)(k+1)(2j+k+4)}{2} \quad \text{for all connected graphs} \quad H \in \mathcal{H}(3)$$

Since $H_{k,m,n}$ has $(k+1)m+n$ vertices, the values of $\bar{l}$ and $\bar{C}_C$ follow from their definition. \hfill \Box

Consider the examples below, calculated by using Maple [19]:

$$H_{10,2,80} : \bar{l} \bar{C}_C \approx 1.2688, \quad H_{10^5,2,10^6} : \bar{l} \bar{C}_C \approx 1.9547, \quad H_{10^6,3,10^6} : \bar{l} \bar{C}_C \approx 1.9956.$$  

These values suggest that, for some large values of $n$ and $k$, the value of $\bar{l} \bar{C}_C$ for some members of this family will get very close to 2. Indeed, this is true, as the following theorem shows.

**Theorem 30.** For $G = H_{r^2,1,r^3}$,

$$\lim_{r \to \infty} \bar{l} \bar{C}_C = 2.$$

**Proof.** By Proposition 29

$$\bar{l}(H_{r^2,1,r^3}) = \frac{3r^5 + 4r^4 + 9r^3 + 3r^2 + 3r + 2}{3(r^3 + r^2 + 1)(r + 1)}$$

and

$$\bar{C}_C(H_{r^2,1,r^3}) = \frac{r^2(r+1)}{r^3 + r^2 + 1} \left( \frac{2r}{r^2 + 2r + 3} + \sum_{j=0}^{k} \frac{1}{(j + \frac{r^3 - r^2 - 2 - 4r}{2})^2} - \frac{r^2}{4} \right).$$

Since $\sum_{j=0}^{k} (j + \frac{r^3 - r^2 - 2 - 4r}{2})^2 - \frac{r^2}{4}$ is always positive,

$$\bar{l} \bar{C}_C(H_{r^2,1,r^3}) > \bar{l}(H_{r^2,1,r^3}) \frac{r^2(r+1)}{r^3 + r^2 + 1} \frac{2r}{r^2 + 2r + 3} = 2$$

The proof now follows since $\lim_{r \to \infty} \bar{l}(H_{r^2,1,r^3}) \frac{r^2(r+1)}{r^3 + r^2 + 1} \frac{2r}{r^2 + 2r + 3} = 2$ and since $\bar{l} \bar{C}_C < 2$ by Corollary 28. \hfill \Box

We are now able to prove Theorem 1.

**Proof of Theorem 1.** Apply Theorems 26, 27 and 30 and Corollary 20. \hfill \Box

Finally, note that Theorem 30 has the following corollary.

**Corollary 31.** For all $\epsilon > 0$, there exists a connected graph $G$ for which $\bar{l} \bar{C}_C > 2 - \epsilon$.

Partly inspired by this corollary, we suggest the following conjecture.

**Conjecture 32.** The numbers $\bar{l} \bar{C}_C$ for all connected graphs $G$ form a dense subset of the interval $[1, 2)$. 
BOUNDS ON THE CLOSENESS CENTRALITY OF A GRAPH

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DECLARATION OF COMPETING INTEREST

There is no conflict of interest.

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