\textbf{W-algebras associated with centralizers in type A}

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Abstract

We introduce a new family of affine $\mathcal{W}$-algebras $\mathcal{W}^k(a)$ associated with the centralizers of arbitrary nilpotent elements in $\mathfrak{gl}_N$. We define them by using a version of the BRST complex of the quantum Drinfeld–Sokolov reduction. A family of free generators of $\mathcal{W}^k(a)$ is produced in an explicit form. We also give an analogue of the Fateev–Lukyanov realization for the new $\mathcal{W}$-algebras by applying a Miura-type map.

1 Introduction

The affine $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g})$ at the level $k \in \mathbb{C}$ associated with a simple Lie algebra $\mathfrak{g}$ is a vertex algebra defined by a quantum Drinfeld–Sokolov reduction [8]. These algebras originate in conformal field theory and first appeared in the work of Zamolodchikov [20] and Fateev and Lukyanov [7]. They were intensively studied both in mathematics and physics literature; see e.g. [1], [2], [5], [11, Ch. 15] for detailed reviews. More general $\mathcal{W}$-algebras $\mathcal{W}^k(\mathfrak{g}, f)$ were introduced in [13], which depend on a simple Lie (super)algebra $\mathfrak{g}$ and an (even) nilpotent element $f \in \mathfrak{g}$ so that $\mathcal{W}^k(\mathfrak{g})$ corresponds to a principal nilpotent element $f$. Their counterparts for odd nilpotents $f$ were studied in [14] and [18] from the viewpoint of quantum hamiltonian reduction.

In an important particular case, where the level $k$ takes the critical value, the vertex algebra $\mathcal{W}^{\text{cri}}(\mathfrak{g})$ is commutative. It is isomorphic to the center $\mathfrak{z}(\hat{\mathfrak{g}})$ of the affine vertex algebra $V^{\text{cri}}(\mathfrak{g})$ and is known as the Feigin–Frenkel center following the paper [9], where the structure of $\mathfrak{z}(\hat{\mathfrak{g}})$ was described; see also [10] for detailed arguments and [15] for explicit constructions of generators of $\mathfrak{z}(\hat{\mathfrak{g}})$ for the Lie algebras $\mathfrak{g}$ of classical types. The arguments rely on a basic property of the simple Lie algebras stating that the subalgebra of $\mathfrak{g}$-invariants of the symmetric algebra $S(\mathfrak{g})$ is a polynomial algebra. It was shown in [19], that this property is shared by a wide class of non-reductive Lie algebras $\mathfrak{a}$ which are centralizers of nilpotent elements in $\mathfrak{g}$. This served as a starting point for the work [4], where the Feigin–Frenkel theorem was extended to the affine vertex algebras $V^{\text{cri}}(\mathfrak{a})$ associated with the centralizers.

These results motivate the question whether analogues of the $\mathcal{W}$-algebras can be associated with the underlying Lie algebras $\mathfrak{a}$. Our goal in this paper is to introduce and describe some basic properties of $\mathcal{W}$-algebras $\mathcal{W}^k(\mathfrak{a})$, where $\mathfrak{a}$ is the centralizer of a nilpotent element $e$ in $\mathfrak{gl}_N$. In the case $e = 0$ the corresponding algebra coincides with the principal $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{gl}_N)$.

Since the Lie algebra $\mathfrak{a}$ is not semisimple, the construction depends on a choice of the invariant symmetric bilinear form on $\mathfrak{a}$. We will follow [4], where a natural form was used to
introduce the corresponding affine Kac–Moody algebra \( \hat{\alpha} \). Its vacuum module \( V^k(\alpha) \) at the level \( k \) is a vertex algebra. The Lie algebra \( \alpha \) admits a triangular decomposition \( \alpha = n_- \oplus h \oplus n_+ \) which gives rise to a Clifford algebra associated with \( n_+ \) and we let \( \mathcal{F} \) be its vacuum module. As with the case of simple Lie algebras \([11, \text{Ch. 15}]\), the vertex algebra \( C^k(\alpha) = V^k(\alpha) \otimes \mathcal{F} \) acquires a structure of a BRST complex of the quantum Drinfeld–Sokolov reduction. We show that its cohomology \( H^k(\alpha)^i \) is zero for all degrees \( i \neq 0 \) and define the \( \mathcal{W} \)-algebra by setting \( \mathcal{W}^k(\alpha) = H^k(\alpha)^0 \).

Furthermore, we give an explicit construction of free generators of the \( \mathcal{W} \)-algebra \( \mathcal{W}^k(\alpha) \). In the particular case \( e = 0 \) they coincide with those previously found in \([3]\). A version of the quantum Miura map yields an embedding of \( \mathcal{W}^k(\alpha) \) into the vertex algebra \( V^k(\alpha) \otimes n_+ \) and we let \( \mathcal{F} \) be its vacuum module. As with this particular case, the vertex algebra \( \mathcal{W}^{-N}(\alpha) \) at the critical level \( k = -N \) turns out to be commutative. It is isomorphic to the classical \( \mathcal{W} \)-algebra \( \mathcal{W}(\alpha) \) introduced in \([17]\) and to the center of the vertex algebra \( V^{-N}(\alpha) \), as described in \([4]\) and \([16]\); cf. \([9]\).

Note that if all Jordan blocks of the nilpotent element \( e \) are of the same size, the Lie algebra \( \alpha \) is isomorphic to a truncated polynomial current algebra of the form \( \mathfrak{gl}_n[v]/(v^p = 0) \), which is also known as the Takiff algebra. This leads to a natural generalization of our definition of the \( \mathcal{W} \)-algebras to the class of Takiff algebras \( \mathfrak{gl}_n[v]/(v^p = 0) \) associated with an arbitrary simple Lie algebra \( \mathfrak{g} \).

2 BRST cohomology for centralizers

Here we adapt the well-known BRST construction of vertex algebras to the case of centralizers in type \( A \). We generally follow \([1, \text{Sec. 4}]\) and \([11, \text{Ch. 15}]\) with some straightforward modifications.

Let \( e \in \mathfrak{gl}_N \) be a nilpotent matrix and let \( \alpha \) be the centralizer of \( e \) in \( \mathfrak{gl}_N \). Suppose that the Jordan canonical form of \( e \) has Jordan blocks of sizes \( \lambda_1, \ldots, \lambda_n \), where \( \lambda_1 \leq \cdots \leq \lambda_n \) and \( \lambda_1 + \cdots + \lambda_n = N \). The corresponding pyramid is a left-justified array of rows of unit boxes such that the top row contains \( \lambda_1 \) boxes, the next row contains \( \lambda_2 \) boxes, etc. Denote by \( q_1 \geq \cdots \geq q_l \) the column lengths of the pyramid (with \( l = \lambda_n \)). The row-tableau is obtained by writing the numbers \( 1, \ldots, N \) into the boxes of the pyramid consecutively by rows from left to right. For instance, the row-tableau

\[
\begin{array}{cccc}
1 & 2 & & \\
3 & 4 & 5 & \\
6 & 7 & 8 & 9
\end{array}
\]

corresponds to the pyramid with the rows of lengths 2, 3, 4; its column lengths are 3, 3, 2, 1. We let \( \text{row}(a) \) and \( \text{col}(a) \) denote the row and column number of the box containing the entry \( a \).

Denote by \( e_{ab} \) the standard basis elements of the Lie algebra \( \mathfrak{gl}_N \). For any \( 1 \leq i, j \leq n \) and
any integral values of $r$ with $\lambda_j - \min(\lambda_i, \lambda_j) \leq r < \lambda_j$ set
\[
E^{(r)}_{ij} = \sum_{\text{row}(a) = i, \text{row}(b) = j, \text{col}(b) - \text{col}(a) = r} e_{ab},
\]
(2.1)
summed over $a, b \in \{1, \ldots, N\}$. It is well-known that the elements $E^{(r)}_{ij}$ form a basis of the Lie algebra $\mathfrak{a}$; see e.g. [6] and [19]. The commutation relations are given by
\[
\left[ E^{(r)}_{ij}, E^{(s)}_{kl} \right] = \delta_{hj} E^{(r+s)}_{il} - \delta_{il} E^{(r+s)}_{hj},
\]
assuming that $E^{(r)}_{ij} = 0$ for $r \geq \lambda_j$.

### 2.1 Affine vertex algebra

The Lie algebra $\mathfrak{g} = \mathfrak{gl}_N$ acquires a $\mathbb{Z}$-gradation $\mathfrak{g} = \bigoplus_{r \in \mathbb{Z}} \mathfrak{g}_r$ determined by $e$ such that the degree of the basis element $e_{ab}$ equals $\text{col}(b) - \text{col}(a)$. We thus get an induced $\mathbb{Z}$-gradation $\mathfrak{a} = \bigoplus_{r \in \mathbb{Z}} \mathfrak{a}_r$ on the Lie algebra $\mathfrak{a}$, where $\mathfrak{a}_r = \mathfrak{a} \cap \mathfrak{g}_r$. Note that the element (2.1) is homogeneous of degree $r$. The subalgebra $\mathfrak{g}_0$ is isomorphic to the direct sum
\[
\mathfrak{g}_0 \cong \mathfrak{gl}_{q_1} \oplus \cdots \oplus \mathfrak{gl}_{q_l}.
\]
(2.2)
Equip this subalgebra with the normalized Killing form
\[
\langle X, Y \rangle = \frac{1}{2N} \text{tr}(\text{ad}X \text{ad}Y), \quad X, Y \in \mathfrak{g}_0.
\]
(2.3)
Now define an invariant symmetric bilinear form on $\mathfrak{a}$ following [4]. The value $\langle X, Y \rangle$ for homogeneous elements $X, Y \in \mathfrak{a}$ is found by (2.3) for $X, Y \in \mathfrak{a}_0$, and is zero otherwise. Writing $X = X_1 + \cdots + X_l$ and $Y = Y_1 + \cdots + Y_l$ in accordance with the decomposition (2.2), we get
\[
\langle X, Y \rangle = \frac{1}{N} \sum_{i=1}^l (q_i \text{tr}X_i Y_i - \text{tr}X_i \text{tr}Y_i).
\]
Therefore, if $\lambda_i = \lambda_j$ for some $i \neq j$ then
\[
\langle E^{(0)}_{ij}, E^{(0)}_{ji} \rangle = \frac{1}{N} (q_1 + \cdots + q_{\lambda_i}) = \frac{1}{N} \left( \lambda_1 + \cdots + \lambda_{i-1} + (n - i + 1)\lambda_i \right),
\]
and for all $i$ and $j$ we have
\[
\langle E^{(0)}_{i\bar{i}}, E^{(0)}_{j\bar{j}} \rangle = \frac{1}{N} \left( \delta_{ij} (\lambda_1 + \cdots + \lambda_{i-1} + (n - i + 1)\lambda_i) - \min(\lambda_i, \lambda_j) \right),
\]
whereas all remaining values of the form on the basis vectors are zero.
The affine Kac–Moody algebra $\hat{\mathfrak{a}} = \mathfrak{a}[t, t^{-1}] \oplus \mathbb{C}K$, where $\mathfrak{a}[t, t^{-1}]$ is the Lie algebra of Laurent polynomials in $t$ with coefficients in $\mathfrak{a}$. For any $r \in \mathbb{Z}$ and $X \in \mathfrak{g}$ we will write $X[m] = X t^m$. The commutation relations of the Lie algebra $\hat{\mathfrak{a}}$ have the form

$$[X[m], Y[p]] = [X, Y][m + p] + m \delta_{m,-p} \langle X, Y \rangle K, \quad X, Y \in \mathfrak{a},$$

and the element $K$ is central in $\hat{\mathfrak{a}}$. The vacuum module at the level $k \in \mathbb{C}$ over $\hat{\mathfrak{a}}$ is the quotient $V^k(\mathfrak{a}) = \mathcal{U}(\hat{\mathfrak{a}})/I$, where $I$ is the left ideal of $\mathcal{U}(\hat{\mathfrak{a}})$ generated by $\mathfrak{a}[t]$ and the element $K - k$. This module is equipped with a vertex algebra structure and is known as the (universal) affine vertex algebra associated with $\mathfrak{a}$ and the form $\langle , \rangle$; see [11], [12]. The vacuum vector is the image of the element 1 in the quotient and we will denote it by $|0\rangle$. Furthermore, introduce the fields

$$E_{ij}^{(r)}(z) = \sum_{m \in \mathbb{Z}} E_{ij}^{(r)}[m] z^{-m-1} \in \text{End} V^k(\mathfrak{a})[[z, z^{-1}]]$$

so that under the state-field correspondence map we have

$$Y : E_{ij}^{(r)}[-1]|0\rangle \mapsto E_{ij}^{(r)}(z).$$

The map $Y$ extends to the whole of $V^k(\mathfrak{a})$ with the use of normal ordering. The translation operator $T$ on $V^k(\mathfrak{a})$ is determined by the properties

$$T : |0\rangle \mapsto 0 \quad \text{and} \quad [T, X[m]] = -m X[m - 1], \quad X \in \mathfrak{a}, \quad m < 0, \quad \text{(2.4)}$$

where $X[m]$ is understood as the operator of left multiplication by $X[m]$.

### 2.2 Affine Clifford algebra

Consider the following triangular decomposition of the Lie algebra $\mathfrak{a}$,

$$\mathfrak{a} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+,$$

where the subalgebras are defined by

$$\mathfrak{n}_- = \text{span of} \{ E_{ij}^{(r)} | i > j \}, \quad \mathfrak{n}_+ = \text{span of} \{ E_{ij}^{(r)} | i < j \} \quad \text{and} \quad \mathfrak{h} = \text{span of} \{ E_{ii}^{(r)} \},$$

with the superscript $r$ ranging over all admissible values. Denote by $\mathcal{C}l(t)$ the Clifford algebra associated with $\mathfrak{n}_+[t, t^{-1}]$, so it is generated by odd elements $\psi_{ij}^{(r)}[m]$ and $\psi_{ij}^{(r)*}[m]$ with the parameters satisfying the conditions $1 \leq i < j \leq n$ together with $\lambda_j - \lambda_i \leq r \leq \lambda_j - 1$ and $m \in \mathbb{Z}$. The defining relations are given by the anti-commutation relations

$$[\psi_{ij}^{(r)}[m], \psi_{ij}^{(r)*}[-m]] = 1,$$
while all other pairs of generators anti-commute. Let $\mathcal{F}$ be the Fock representation of $\mathcal{C}l$ generated by a vector $1$ such that

$$
\psi^{(r)}_{ij} [m] 1 = 0 \quad \text{for} \quad m \geq 0 \quad \text{and} \quad \psi^{(r)*}_{ij} [m] 1 = 0 \quad \text{for} \quad m > 0.
$$

The space $\mathcal{F}$ is a vertex algebra with the vacuum vector $1$, and the translation operator $T$ is determined by the properties

$$
[T, \psi^{(r)}_{ij} [m]] = -m \psi^{(r)}_{ij} [m - 1], \quad [T, \psi^{(r)*}_{ij} [m]] = -(m - 1) \psi^{(r)*}_{ij} [m - 1].
$$

The fields are defined by

$$
\psi^{(r)}_{ij} (z) = \sum_{m \in \mathbb{Z}} \psi^{(r)}_{ij} [m] z^{-m - 1} \quad \text{and} \quad \psi^{(r)*}_{ij} (z) = \sum_{m \in \mathbb{Z}} \psi^{(r)*}_{ij} [m] z^{-m}
$$

so that

$$
Y : \psi^{(r)}_{ij} [-1] 1 \mapsto \psi^{(r)}_{ij} (z) \quad \text{and} \quad Y : \psi^{(r)*}_{ij} [0] 1 \mapsto \psi^{(r)*}_{ij} (z).
$$

The vertex algebra $\mathcal{F}$ has a $\mathbb{Z}$-gradation $\mathcal{F} = \bigoplus_{i \in \mathbb{Z}} \mathcal{F}^i$, defined by

$$
\deg 1 = 0, \quad \deg \psi^{(r)}_{ij} [m] = -1 \quad \text{and} \quad \deg \psi^{(r)*}_{ij} [m] = 1.
$$

### 2.3 BRST complex

Introduce the vertex algebra $C^k (a)$ as the tensor product

$$
C^k (a) = V^k (a) \otimes \mathcal{F}.
$$

We will use notation $|0\rangle$ for its vacuum vector $|0\rangle \otimes 1$. The vertex algebra $C^k (a)$ is $\mathbb{Z}$-graded, its $i$-th component has the form

$$
C^k (a)^i = V^k (a) \otimes \mathcal{F}^i.
$$

Consider the fields $Q(z)$ and $\chi(z)$ defined by

$$
Q(z) = \sum_{i<j} E^{(a)}_{ij} (z) \psi^{(a)*}_{ij} (z) - \sum_{i<j<h} \psi^{(a)*}_{ij} (z) \psi^{(b)*}_{jh} (z) \psi^{(a+b)*}_{ih} (z), \quad (2.6)
$$

and

$$
\chi(z) = \sum_{i=1}^{n-1} \psi^{(\lambda_{i+1}-1)*}_{i+1} (z). \quad (2.7)
$$

To simplify the formulas, here and throughout the paper we use the convention that summation over all admissible values of repeated superscripts of the form $a, b, c$ is assumed. For instance, summation over $a$ running over the values $\lambda_j - \lambda_i, \ldots, \lambda_j - 1$ is assumed within the first sum in (2.6). Define the odd endomorphisms $d_{st}$ and $\chi$ of $C^k (a)$ as the residues (coefficients of $z^{-1}$) of the fields (2.6) and (2.7),

$$
d_{st} = \text{res} Q(z) \quad \text{and} \quad \chi = \sum_{i=1}^{n-1} \psi^{(\lambda_{i+1}-1)*}_{i+1} [1].
$$
Lemma 2.1. We have the relations
\[ d_{st}^2 = \chi^2 = [d_{st}, \chi] = 0. \]

Proof. The relations are verified by the standard OPE calculus with the use of the Taylor formula and Wick theorem [12]. Using the basic OPEs
\[ E^{(r)}_{ij}(z)E^{(s)}_{hl}(w) \sim \frac{1}{z-w}\left(\delta_{hj}E^{(r+s)}_{il}(w) - \delta_{il}E^{(r+s)}_{hj}(w)\right) + \frac{k\langle E^{(r)}_{ij}, E^{(s)}_{hl}\rangle}{(z-w)^2}, \]

and
\[ \psi^{(r)}_{ij}(z)\psi^{(r)*}_{ij}(w) \sim \frac{1}{z-w}, \quad \psi^{(r)*}_{ij}(z)\psi^{(r)}_{ij}(w) \sim \frac{1}{z-w}, \]
we find that the OPE \( Q(z)Q(w) \) is regular, thus implying that \( d_{st}^2 = 0 \). The remaining relations are straightforward to verify. \( \square \)

By Lemma 2.1, the odd endomorphism \( d = d_{st} + \chi \) of \( C^k(a) \) has the properties \( d^2 = 0 \) and \( d : C^k(a)^i \to C^k(a)^{i+1} \). We thus get an analogue \((C^k(a)^*, d)\) of the BRST complex of the quantum Drinfeld–Sokolov reduction, associated with the Lie algebra \( a \); cf. [11, Ch. 15]. Since \( d \) is the residue of a vertex operator, the cohomology \( H^k(a)^* \) of the complex is a vertex algebra which we will use to define and describe the \( \mathcal{W} \)-algebras \( \mathcal{W}^k(a) \).

3 \( \mathcal{W} \)-algebras \( \mathcal{W}^k(a) \)

Introduce another \( \mathbb{Z} \)-gradation on \( C^k(a)^* \) by defining the (conformal) degrees by
\[ \deg E^{(r)}_{ij}[m] = \deg \psi^{(r)}_{ij}[m] = -m + i - j \quad \text{and} \quad \deg \psi^{(r)*}_{ij}[m] = -m + j - i. \]
Observe that the differential \( d \) has degree 0 and so it preserves this gradation thus defining a \( \mathbb{Z} \)-gradation on the cohomology \( H^k(a)^* \).

Definition 3.1. The \( \mathbb{Z} \)-graded vertex algebra \( H^k(a)^0 \) is called the \( \mathcal{W} \)-algebra associated with the centralizer \( a \) at the level \( k \) and denoted by \( \mathcal{W}^k(a) \). \( \square \)

Note that the definition also depends on the chosen bilinear form \( \langle \cdot , \cdot \rangle \) on \( a \). Our next goal is to prove the following analogue of [11, Thm 15.1.9] which describes the structure of principal \( \mathcal{W} \)-algebras associated with simple Lie algebras.

Theorem 3.2. The \( \mathcal{W} \)-algebra \( \mathcal{W}^k(a) \) is strongly and freely generated by elements \( w_1, \ldots, w_N \) of the respective degrees
\[ \frac{1}{\lambda_n}, \ldots, \frac{1}{\lambda_n}, \frac{2}{\lambda_{n-1}}, \ldots, \frac{2}{\lambda_{n-1}}, \ldots, \frac{n}{\lambda_1}, \ldots, \frac{n}{\lambda_1}. \]

Hence, the Hilbert–Poincaré series of the algebra \( \mathcal{W}^k(a) \) is given by
\[ \prod_{s=0}^{\infty} \prod_{l=1}^{n} (1 - q^{l+s})^{-\lambda_{n-l+1}}. \]

Moreover, \( H^k(a)^i = 0 \) for all \( i \neq 0 \).
The proof relies on essentially the same arguments as in [11, Ch. 15] (see also [1, Sec. 4]) which we will outline in the rest of this section. A family of generators $w_1, \ldots, w_N$ will be produced in Sec. 4.

For all $1 \leq i < j \leq n$ and $r = \lambda_j - \lambda_i, \ldots, \lambda_j - 1$ introduce the fields

$$e_{ij}^{(r)}(z) = E_{ij}^{(r)}(z) + \sum_{h>j} \psi_{ih}^{(a)}(z) \psi_{jh}^{(a-r)*}(z) - \sum_{h<i} \psi_{hj}^{(a)}(z) \psi_{hi}^{(a-r)*}(z), \quad (3.1)$$

where we keep using the convention on the summation over $a$ as in (2.6). Similarly, for $i \geq j$ and $r = 0, 1, \ldots, \lambda_j - 1$ set

$$e_{ij}^{(r)}(z) = E_{ij}^{(r)}(z) + \sum_{h>i} \psi_{ih}^{(a)}(z) \psi_{jh}^{(a-r)*}(z) : - \sum_{h<j} \psi_{hj}^{(a)}(z) \psi_{hi}^{(a-r)*}(z) : . \quad (3.2)$$

Note that by the defining relations in the Clifford algebra $\mathcal{C}$, the normal ordering is necessary only for the case where $i = j$ and $r = 0$. Introduce Fourier coefficients $e_{ij}^{(r)}[m]$ of the fields (3.1) and (3.2) by setting

$$e_{ij}^{(r)}(z) = \sum_{m \in \mathbb{Z}} e_{ij}^{(r)}[m] z^{-m-1}.$$ 

In the formulas of the next lemmas we assume that the fields with out-of-range parameters are equal to zero.

**Lemma 3.3.** (i) For $i \geq j$ and $h < l$ we have

$$[e_{ij}^{(r)}[m], \psi_{hl}^{(s)*}[p]] = \delta_{ij} \psi_{hl}^{(s-r)*}[m+p] - \delta_{hi} \psi_{jl}^{(s-r)*}[m+p]. \quad (3.3)$$

Moreover, if $i \geq j$ and $h \geq l$ then

$$[e_{ij}^{(r)}[m], e_{hl}^{(s)}[p]] = \delta_{hj} e_{il}^{(r+s)}[m+p] - \delta_{il} e_{hj}^{(r+s)}[m+p] + m \delta_{m,-p}(k + N) \langle E_{ij}^{(r)}, E_{hl}^{(s)} \rangle.$$ 

(ii) For $i < j$ and $h < l$ we have

$$[e_{ij}^{(r)}[m], \psi_{hl}^{(s)}[p]] = \delta_{hj} \psi_{il}^{(r+s)}[m+p] - \delta_{il} \psi_{kj}^{(r+s)}[m+p]$$

and

$$[e_{ij}^{(r)}[m], e_{hl}^{(s)}[p]] = \delta_{hj} e_{il}^{(r+s)}[m+p] - \delta_{il} e_{hj}^{(r+s)}[m+p].$$

**Proof.** All relations are easily verified with the use of the OPEs (2.8) and (2.9). \qed

For all $i = 1, \ldots, n$ set

$$\alpha_i = -\lambda_i + \frac{k + N}{N} \left( \lambda_1 + \cdots + \lambda_{i-1} + (n - i + 1) \lambda_i \right). \quad (3.4)$$
Lemma 3.4. The following relations hold for all \( i \geq j \):

\[
[d_{st}, e_{ij}^{(r)}(z)] = \sum_{h=j}^{i-1} :e_{hj}^{(a+r)}(z)\psi_{hi}^{(a)}(z) : - \sum_{h=j+1}^{i} :\psi_{jh}^{(a)}(z)e_{ih}^{(a+r)}(z) : + \alpha_j \delta_{r0} \partial_z \psi_{ji}^{(0)}(z),
\]

\[
[\chi, e_{ij}^{(r)}(z)] = \psi_{ji}^{(\lambda_i-1)}(z) - \psi_{ji}^{(\lambda_j-1)}(z) - \psi_{ji}^{(\lambda_j-1)}(z).
\] (3.5)

Moreover, for all \( i < j \) we have

\[
[d_{st}, e_{ij}^{(r)}(z)] = 0, \quad [\chi, e_{ij}^{(r)}(z)] = 0,
\]

\[
[d_{st}, \psi_{ij}^{(r)}(z)] = e_{ij}^{(r)}(z), \quad [\chi, \psi_{ij}^{(r)}(z)] = \delta_{ij} \delta_r \lambda_{j-1},
\]

and

\[
[d_{st}, \psi_{ij}^{(r)}(z)] = - \sum_{i<h<j} \psi_{ih}^{(a)}(z) \psi_{hj}^{(r-a)}(z), \quad [\chi, \psi_{ij}^{(r)}(z)] = 0.
\]

Proof. All relations are verified by using the OPEs (2.8) and (2.9). We give some details for the proof of the first relation. As a first step, by a direct computation with the use of the Wick theorem we get the OPE

\[
Q(z) e_{ij}^{(r)}(w) \sim \frac{1}{z-w} \left( \sum_{h=j}^{i-1} :e_{hj}^{(a+r)}(w)\psi_{hi}^{(a)}(w) : - \sum_{h=j+1}^{i} :\psi_{jh}^{(a)}(w)e_{ih}^{(a+r)}(w) : \right)
\]

\[
+ \frac{1}{(z-w)^2} \delta_{r0} \left( k\langle E_{ij}^{(0)} , E_{ji}^{(0)} \rangle + \lambda_1 + \cdots + \lambda_{j-1} + (n-i)\lambda_j \right) \psi_{ji}^{(0)}(z),
\]

where the term \( \psi_{ji}^{(0)}(z) \) is nonzero only if \( j < i \) and \( \lambda_i = \lambda_j \). Relation (3.3) of Lemma 3.3 implies (assuming summation over \( a \)) that

\[
:e_{hj}^{(a+r)}(w)\psi_{jh}^{(a)}(w) : = :\psi_{jh}^{(a)}(w)e_{ih}^{(a+r)}(w) : + \delta_{r0} \lambda_j \partial_w \psi_{ji}^{(0)}(w).
\]

The required relation now follows by applying the Taylor formula to \( \psi_{ji}^{(0)}(z) \) to write

\[
\psi_{ji}^{(0)}(z) = \psi_{ji}^{(0)}(w) + (z-w) \partial_w \psi_{ji}^{(0)}(w) + \ldots,
\]

and then by taking the residue over \( z \) in the resulting expressions. \( \square \)

Denote by \( C^k(a)_0 \) the subspace of \( C^k(a) \) spanned by all vectors of the form

\[
e_{i_{j_1}j_1}[m_{i_1}] \ldots e_{i_{q}j_q}[m_{i_q}] \psi_{h_{i_1}l_1}^{(s_1)}[p_{i_1}] \ldots \psi_{h_{i_q}l_q}^{(s_q)}[p_{i_q}] [0], \quad i_a \geq j_a, \quad h_a < l_a,
\]

and by \( C^k(a)_+ \) the subspace of \( C^k(a) \) spanned by all vectors of the form

\[
e_{i_{j_1}j_1}[m_{i_1}] \ldots e_{i_{q}j_q}[m_{i_q}] \psi_{h_{i_1}l_1}^{(s_1)}[p_{i_1}] \ldots \psi_{h_{i_q}l_q}^{(s_q)}[p_{i_q}] [0], \quad i_a < j_a, \quad h_a < l_a.
\]
By Lemma 3.3, both $C^k(a)_0$ and $C^k(a)_+$ are vertex subalgebras of $C^k(a)$. Furthermore, by Lemma 3.4 each of the subalgebras is preserved by the differential $d = d_{st} + \chi$. This implies the tensor product decomposition of complexes

$$C^k(a)^* \cong C^k(a)^*_0 \otimes C^k(a)^*_+.$$

Hence the cohomology of $C^k(a)^*$ is isomorphic to the tensor product of the cohomologies of $C^k(a)^*_0$ and $C^k(a)^*_+$. 

By Lemma 3.4, for $i < j$ we have

$$[d, e_{ij}^{(r)}[m]] = 0, \quad [d, \psi_{ij}^{(r)}[m]] = e_{ij}^{(r)}[m] + \delta_{i,j-1} \delta_{r,\lambda_j-1} \delta_{m,-1}.$$ 

Therefore, the complex $C^k(a)^*_+$ has no higher cohomologies, while its zeroth cohomology is one-dimensional; see [11, Sec 15.2.6]. So the cohomology of $C^k(a)^*$ is isomorphic to the cohomology of the complex $C^k(a)^*_0$. To calculate the latter, equip this complex with a double gradation by setting

$$\text{bideg } e_{ij}^{(r)}[m] = (i - j, j - i), \quad \text{bideg } \psi_{ij}^{(r)}[m] = (j - i, i - j + 1).$$ 

Then $C^k(a)^*_0$ acquires a structure of bicomplex with bideg $\chi = (1,0)$ and bideg $d_{st} = (0,1)$. Take $\chi$ as the zeroth differential of the associated spectral sequence and $d_{st}$ as the first. Next we compute the cohomology of $C^k(a)^*_0$ with respect to $\chi$. 

Consider the linear span of all fields $e_{ij}^{(r)}(z)$ with $i \geq j$ and $r = 0, 1, \ldots, \lambda_j - 1$. We will choose a new basis of this vector space which is formed by the fields

$$P_l^{(r)}(z) = e_{n-l+1}^{(r)}(z) + e_{n-1-n-l}^{(r+\lambda_n-\lambda_2)}(z) + \cdots + e_{l-1}^{(r+\lambda_n+\cdots+\lambda_{l+1}-\lambda_{n-l+1}-\cdots-\lambda_2)}(z)$$

for $l = 1, \ldots, n$ and $r = 0, 1, \ldots, \lambda_n-l+1 - 1$ together with

$$I_{ij}^{(r)}(z) = \sum_{h=1}^{i} e_{j-h-1-h+1}^{(r+\lambda_{j-1}+\cdots+\lambda_{h-1}-\lambda_{h+1}-\cdots-\lambda_{l-h+2})}(z)$$

for $i < j$ and $r = 0, 1, \ldots, \lambda_i - 1$. The following properties of the new basis vectors are immediate from (3.5).

**Lemma 3.5.** We have the relations

$$[\chi, P_l^{(r)}(z)] = 0 \quad \text{and} \quad [\chi, I_{ij}^{(r)}(z)] = \psi_{ij}^{(\lambda_j-r-1)}(z).$$

**Lemma 3.5** allows us to apply the arguments of [11, Sec. 15.2.9] to conclude that all higher cohomologies of the complex $C^k(a)^*_0$ with respect to $\chi$ vanish, while the zeroth cohomology is the commutative vertex subalgebra of $C^k(a)_0$ spanned by all monomials

$$P_{l_1}^{(r_1)}[m_1] \cdots P_{l_q}^{(r_q)}[m_q][0],$$

(3.6)
where we use the Fourier coefficients $P_l^{(r)}[m]$ defined by
\begin{equation}
P_l^{(r)}(z) = \sum_{m \in \mathbb{Z}} P_l^{(r)}[m] z^{-m-1}.
\end{equation}

By a standard procedure outlined in [11, Sec. 15.2.11], each element of this subalgebra gives rise to a unique cocycle in the complex $C^k(a)_0^*$ with the differential $d$. Moreover, the cocycles $W_l^{(r)}$ corresponding to the vectors $P_l^{(r)}[-1]|0\rangle$ with $l = 1, \ldots, n$ and $r = 0, 1, \ldots, \lambda_{n-l+1} - 1$ strongly and freely generate the $\mathcal{W}$-algebra $\mathcal{W}^k(a)$. More precisely, introduce the Fourier coefficients $W_l^{(r)}[m]$ by
\begin{equation}
W_l^{(r)}(z) = \sum_{m \in \mathbb{Z}} W_l^{(r)}[m] z^{-m-1}.
\end{equation}

Define a linear ordering on the set of coefficients $W_l^{(r)}[m]$ by positing that the corresponding triples $(l, r, m)$ are ordered lexicographically. Then the ordered monomials
\begin{equation}
W_{l_1}^{(r_1)}[m_1] \ldots W_{l_q}^{(r_q)}[m_q] |0\rangle,
\end{equation}
where all $m_i < 0$, form a basis of $\mathcal{W}^k(a)$. The proof of Theorem 3.2 is completed by the observation that the conformal degree of the element $W_l^{(r)}[m]$ equals $l - m - 1$.

## 4 Generators of $\mathcal{W}^k(a)$

For an $n \times n$ matrix $A = [a_{ij}]$ with entries in a ring we will consider its column-determinant defined by
\begin{equation}
c\text{det} A = \sum_{\sigma \in S_n} \text{sgn} \sigma \cdot a_{\sigma(1)} \cdots a_{\sigma(n)} n.
\end{equation}

We will produce generators of the $\mathcal{W}$-algebra $\mathcal{W}^k(a)$ as elements of the vertex algebra $C^k(a)_0$. Combine the Fourier coefficients $e_{ij}^{(r)}[-1] \in \text{End} C^k(a)_0$ into polynomials in a variable $u$ by setting
\begin{equation}
e_{ij}(u) = \sum_{r=0}^{\lambda_j - 1} e_{ij}^{(r)}[-1] u^r, \quad i \geq j.
\end{equation}

Let $x$ be another variable and consider the matrix
\begin{equation}
\mathcal{E} = \begin{bmatrix}
x + \alpha_1 T + e_{11}(u) & -u^{\lambda_2 - 1} & 0 & \ldots & 0 \\
e_{21}(u) & x + \alpha_2 T + e_{22}(u) & -u^{\lambda_3 - 1} & \ldots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
e_{n1}(u) & e_{n2}(u) & \ldots & \ldots & x + \alpha_n T + e_{nn}(u)
\end{bmatrix},
\end{equation}
where the constants $\alpha_i$ are defined in (3.4). Its column-determinant is a polynomial in $x$ of the form
\begin{equation}
c\text{det} \mathcal{E} = x^n + w_1(u) x^{n-1} + \cdots + w_n(u), \quad w_l(u) = \sum_r w_l^{(r)} u^r, \quad (4.1)
\end{equation}
so that the coefficients $w_{i}^{(r)}$ are endomorphisms of $C^{k}(a)_{0}$.

The particular case $e = 0$ of the following theorem (that is, with $\lambda_{1} = \cdots = \lambda_{n} = 1$) is contained in [3, Thm 2.1].

**Theorem 4.1.** All elements $w_{i}^{(r)}|0\rangle$ with $l = 1, \ldots , n$ and

$$\lambda_{n-l+2} + \cdots + \lambda_{n} < r + l \leq \lambda_{n-l+1} + \cdots + \lambda_{n}$$

(4.2)

belong to the $\mathcal{W}$-algebra $\mathcal{W}^{k}(a)$. Moreover, the $\mathcal{W}$-algebra is strongly and freely generated by these elements.

**Proof.** The first part of the theorem will follow if we show that the elements $w_{i}^{(r)}|0\rangle \in C^{k}(a)_{0}$ are annihilated by the differential $d$. To verify this property, it will be convenient to identify $C^{k}(a)_{0}$ with an isomorphic vertex algebra $\bar{\mathcal{V}}^{k}(a)$ defined as follows; cf. [3]. Consider the Lie superalgebra

$$(b[t, t^{-1}] \oplus \mathbb{C} K) \oplus m[t, t^{-1}],$$

(4.3)

where the Lie algebra $b$ is spanned by the vectors $e_{ij}^{(r)}$ with $i \geq j$ and $r = 0, 1, \ldots , \lambda_{j} - 1$ understood as basis elements of the low triangular part $n_{-} \oplus h$ in the decomposition (2.5) via the identification $e_{ij}^{(r)} \sim E_{ij}^{(r)}$, the even element $K$ is central and $m$ is the supercommutative Lie superalgebra spanned by (abstract) odd elements $\psi_{ij}^{(r)*}$ with $i < j$ and $r = \lambda_{j} - \lambda_{i}, \ldots , \lambda_{j} - 1$. The even component of the Lie superalgebra (4.3) is the Kac–Moody affinization $b[t, t^{-1}] \oplus \mathbb{C} K$ of $b$ with the commutation relations given by

$$[e_{ij}^{(r)}[m], e_{hl}^{(s)}[p]] = \delta_{hj} e_{il}^{(r+s)}[m + p] - \delta_{il} e_{hl}^{(r+s)}[m + p] + m \delta_{m,-p} K \langle E_{ij}^{(r)}, E_{hl}^{(s)} \rangle,$$

where the element $e_{ij}^{(r)}[m]$ is now understood as the vector $e_{ij}^{(r)}t^{m}$. The remaining commutation relations coincide with those in (3.3), where $\psi_{ij}^{(r)*}[m]$ is understood as the vector $\psi_{ij}^{(r)*}t^{m-1}$. Now define $\bar{\mathcal{V}}^{k}(a)$ as the representation of the Lie superalgebra (4.3) induced from the one-dimensional representation of $(b[t] \oplus \mathbb{C} K) \oplus m[t]$ on which $b[t]$ and $m[t]$ act trivially and $K$ acts as $k + N$. Then $\bar{\mathcal{V}}^{k}(a)$ is a vertex algebra isomorphic to $C^{k}(a)_{0}$ so that the fields with the same names respectively correspond to each other. Moreover, the cyclic span of the vacuum vector over the Lie algebra $b[t, t^{-1}] \oplus \mathbb{C} K$ is a subalgebra of the vertex algebra $\bar{\mathcal{V}}^{k}(a)$ isomorphic to the vacuum module $\bar{\mathcal{V}}^{k+N}(b)$.

Observe that the coefficients $w_{i}^{(r)}$ defined in (4.1) can now be understood as elements of the universal enveloping algebra $U(t^{-1}b[t^{-1}])$. As a vertex algebra, $\bar{\mathcal{V}}^{k}(a)$ is equipped with the $(-1)$-product, and each Fourier coefficient $e_{ij}^{(r)}[m]$ with $m < 0$ can be regarded as the operator of left $(-1)$-multiplication by the vector $e_{ij}^{(r)}[m]|0\rangle$, and which is the same as the left multiplication by the element $e_{ij}^{(r)}[m]$ in the algebra $U(t^{-1}b[t^{-1}])$. Therefore, the monomials in the elements $e_{ij}^{(r)}[m]$ which occur in the expansion of the column-determinant $\text{cdet} \mathcal{E}$ can be regarded as the corresponding $(-1)$-products calculated consecutively from right to left, starting from the vacuum vector.
By Lemma 3.4, for $i \geq j$ we have the relations
\[
[d, e_{ij}^{(r)}[-1]] = \sum_{h=j}^{i-1} E_{hj}^{(a_{i-1})}[-1] \psi_{hi}^{(a)}[0] - \sum_{h=j+1}^{i} \psi_{jh}^{(a)}[0] e_{ih}^{(a_r)}[-1] \\
+ \psi_{j+1}^{(\lambda j-1-r)}[0] - \psi_{j-1}^{(\lambda j-1-r)}[0] + \alpha_j \delta_r \psi_{ji}^{(0)}[-1].
\]
Introducing the Laurent polynomials
\[
\phi_{ij} = \sum_{r=\lambda_i-\lambda_j}^{\lambda_i-1} \psi_{ji}^{(r)}[0] u^{-r}, \quad i > j,
\]
we can write the relations in the form
\[
[d, e_{ij}(u)] = \left\{ \sum_{h=j}^{i-1} E_{hj} \phi_{ih} - \sum_{h=j+1}^{i} \phi_{hj} e_{ih}(u) + \phi_{i+1} j u^{\lambda_{i+1} - 1} - \phi_{i-1} j u^{\lambda_{i-1} - 1} + \alpha_j T \phi_{ij} \right\}_+,
\]
where the symbol $\{ \ldots \}_+$ indicates the component of a Laurent polynomial containing only nonnegative powers of $u$,
\[
\left\{ \sum_{i} c_i u^i \right\}_+ = \sum_{i \geq 0} c_i u^i.
\]
Let $E_{ij}$ denote the $(i, j)$ entry of the matrix $E$. Since $d$ commutes with the translation operator $T$, we come to the commutation relations
\[
[d, E_{ij}] = \left\{ \sum_{h=j}^{i-1} E_{hj} \phi_{ih} - \sum_{h=j+1}^{i} \phi_{hj} E_{ih} + \phi_{i+1} j u^{\lambda_{i+1} - 1} - \phi_{i-1} j u^{\lambda_{i-1} - 1} \right\}_+,
\] (4.4)
which hold for $i > j$. The column-determinant of $E$ can be written explicitly in the form\(^1\)
\[
cdet \ E = \sum_{p=0}^{n-1} \sum_{0=i_0 < i_1 < \cdots < i_p < i_{p+1} = n} E_{i_1 i_0 + 1} E_{i_2 i_1 + 1} \cdots E_{i_{p+1} i_p + 1} u^{\lambda_{i_1} - 1 + \cdots + \lambda_{i_p} - 1},
\]
where $\{j_1, \ldots, j_q\}$ is the complement to the subset $\{i_0 + 1, \ldots, i_p + 1\}$ in the set $\{1, \ldots, n\}$. Since $d$ is the residue of a vertex operator, $d$ is a derivation of the $(−1)$-product on $V^k(a)$. Hence, using (4.4), we get
\[
[d, \ cdet \ E] = \sum_{p=0}^{n-1} \sum_{0=i_0 < i_1 < \cdots < i_p < i_{p+1} = n} \sum_{s=0}^{p} E_{i_1 i_0 + 1} \cdots E_{i_s i_{s-1} + 1} \\
\times \left\{ \sum_{i_s < i_{s+1} < i_{s+1}} E'_{i_{s+1} i_s + 1} \phi_{i_s + 1 i_{s+1} e_{i_{s+1} e_{i_{s+1}}} + 1} - \sum_{i_s < i_{s+1} < i_{s+1}} \phi_{i_{s+1} i_{s+1} e_{i_{s+1} e_{i_{s+1}}} + 1} E_{i_s + 1 i_{s+1} e_{i_{s+1} e_{i_{s+1}}} + 1} \\
\times E_{i_s + 2 i_{s+1} + 1} \cdots E_{i_{p+1} i_p + 1} u^{\lambda_{i_s} - 1 + \cdots + \lambda_{i_p} - 1} \right\}_+.
\]

\(^1\)This also shows that it coincides with the row-determinant of $E$ defined in a similar way.
Now apply the quasi-associativity property of the \((-1)\)-product [12, Ch. 4],

\[
(a_{-1} b)_{-1} c = a_{-1} (b_{-1} c) + \sum_{j \geq 0} a_{-j-2} (b_{-j} c) + \sum_{j \geq 0} b_{-j-2} (a_{-j} c),
\]

to bring the expression to the right-normalized form, where the consecutive \((-1)\)-products are calculated from right to left. Note that by Lemma 3.3(i) the additional terms coming from the sums over \(j \geq 0\) annihilate the vacuum vector because all arising commutators involve elements with distinct subscripts.

Regarding the above expansion of \([d, \text{cdet } E]\) as written in the right-normalized form, observe that if we ignore all symbols \(\{ \ldots \}^+\), then it would turn into a telescoping sum and so would be identically zero.

As a next step, for a fixed value \(l \in \{1, \ldots, n\}\) consider the terms in the expansion of \([d, \text{cdet } E]\) containing the variable \(x\) with the powers at least \(n - l\). Such terms can occur only in those summands where the cardinality of the subset \(\{i_0 + 1, \ldots, i_p + 1\}\) is at least \(n - l + 1\). Therefore, the maximum value of the powers \(\lambda_{j_1} - 1 + \cdots + \lambda_{j_q} - 1\) of the variable \(u\) which occur in these terms in the expansion, equals \(\lambda_{n-l+2} + \cdots + \lambda_{n-l+1}\). This means that the coefficients of the powers of \(u\) exceeding \(\lambda_{n-l+2} + \cdots + \lambda_{n-l}\) can be calculated from the expansion \([d, \text{cdet } E]\) with all symbols \(\{ \ldots \}^+\) omitted. However, as we observed above, this expansion is identically zero. It is clear from (4.1) that the degree of the polynomial \(w_l(u)\) equals \(\lambda_{n-l+1} + \cdots + \lambda_{n-l}\) so that the relations \(d w_l^{(r)} |0\rangle = 0\) hold for the parameters \(r\) and \(l\) satisfying the conditions of the theorem.

To show that the vectors \(w_l^{(r)} |0\rangle\) are strong and free generators of \(\mathcal{W}^k(\mathfrak{a})\), consider the gradation on \(U(t^{-1} b[t^{-1}]\) defined by setting the degree of \(e_{ij}^{(r)} [m]\) equal to \(j - i\). It is clear from the formulas for the column-determinant \(\text{cdet } E\) that the lowest degree component of the vector \(w_l^{(r)} |0\rangle\) with \(r = r' + \lambda_{n-l+2} + \cdots + \lambda_{n-l+1}\) coincides with \(P_l^{(r')} [-1] |0\rangle\) for all \(r' = 0, 1, \ldots, \lambda_{n-l+1} - 1\), as defined in (3.7). Therefore, by the argument completing the proof of Theorem 3.2 at the end of Sec. 3, the vector \(w_l^{(r)} |0\rangle\) coincides with the respective cocycle \(W_l^{(r')}\).

\section{Miura map and Fateev–Lukyanov realization}

Consider the affine Kac–Moody algebra \(\hat{\mathfrak{h}} = \mathfrak{h}[t, t^{-1}] \oplus \mathbb{C} K\) associated with \(\mathfrak{h}\) and the bilinear form defined in Sec. 2.1. Denote its generators by \(e_{ji}^{(r)} [m]\) with \(i = 1, \ldots, n\), where \(r\) runs over the set \(0, 1, \ldots, \lambda_i - 1\) and \(m\) runs over \(\mathbb{Z}\). The element \(K\) is central and the commutation relations are given by the OPEs

\[
e_{ii}^{(r)} (z) e_{jj}^{(s)} (w) \sim \frac{K \langle E_{ii}^{(r)}, E_{jj}^{(s)} \rangle}{(z-w)^2},
\]

where we set

\[
e_{ii}^{(r)} (z) = \sum_{m \in \mathbb{Z}} e_{ii}^{(r)} [m] z^{-m-1}.
\]
Define the vacuum module \( V^{k+N}(\mathfrak{h}) \) over the Lie algebra \( \hat{\mathfrak{h}} \) as the representation induced from the one-dimensional representation of \( \mathfrak{h}[t] \oplus \mathbb{C} K \) on which \( \mathfrak{h}[t] \) acts trivially and \( K \) acts as \( k + N \). Then \( V^{k+N}(\mathfrak{h}) \) is a vertex algebra with the vacuum vector \( |0\rangle \) and translation operator \( T \) defined as in (2.4) for \( X \in \mathfrak{h} \). Recalling the constants \( \alpha_i \) introduced in (3.4), expand the product

\[
(x + \alpha_1 T + e_{11}(u)) \ldots (x + \alpha_n T + e_{nn}(u)) = x^n + v_1(u)x^{n-1} + \cdots + v_n(u)
\]

and define the coefficients \( v_i^{(r)} \) by writing \( v_i(u) = \sum_r v_i^{(r)} u^r \).

The particular case \( e = 0 \) of the following proposition is the realization of the \( \mathcal{W} \)-algebra \( \mathcal{W}^k(\mathfrak{gl}_n) \) given by Fateev and Lukyanov [7]; see also [3].

**Proposition 5.1.** The elements \( v_i^{(r)}|0\rangle \) with \( l = 1, \ldots, n \) and \( r \) satisfying (4.2) generate a subalgebra of the vertex algebra \( V^{k+N}(\mathfrak{h}) \), isomorphic to the \( \mathcal{W} \)-algebra \( \mathcal{W}^k(\mathfrak{a}) \).

**Proof.** The Lie algebra projection \( \mathfrak{b} \rightarrow \mathfrak{h} \) with the kernel \( \mathfrak{n}_- \) induces the vertex algebra homomorphism \( V^{k+N}(\mathfrak{b}) \rightarrow V^{k+N}(\mathfrak{h}) \). As we have seen in the previous section, the \( \mathcal{W} \)-algebra \( \mathcal{W}^k(\mathfrak{a}) \) can be regarded as a subalgebra of the vertex algebra \( V^{k+N}(\mathfrak{b}) \). Hence, we get a vertex algebra homomorphism

\[
\Upsilon : \mathcal{W}^k(\mathfrak{a}) \rightarrow V^{k+N}(\mathfrak{h}),
\]

obtained by restriction, which we can call the Miura map; cf. [2, Sec. 5.9], [11, Sec. 15.4]. For the image of the column-determinant (4.1) we have

\[
\Upsilon : \text{cdet} \mathcal{E} \mapsto \left(x + \alpha_1 T + e_{11}(u)\right) \ldots \left(x + \alpha_n T + e_{nn}(u)\right).
\]

Therefore, the Miura images of the generators of \( \mathcal{W}^k(\mathfrak{a}) \) provided by Theorem 4.1 are found by

\[
\Upsilon : v_i^{(r)}|0\rangle \mapsto v_i^{(r)}|0\rangle.
\]

It remains to verify that the Miura map is injective. For \( l = 1, \ldots, n \) and \( r \) satisfying (4.2) introduce the Fourier coefficients \( w_i^{(r)}[m] \) by

\[
w_i^{(r)}(z) = \sum_{m \in \mathbb{Z}} w_i^{(r)}[m] z^{-m-1}.
\]

By Theorem 4.1, the monomials

\[
w_i^{(r_1)}[m_1] \ldots w_i^{(r_q)}[m_q]|0\rangle, \quad m_i < 0,
\]

with the factors ordered as in (3.8), form a basis of \( \mathcal{W}^k(\mathfrak{a}) \). Suppose that a certain linear combination \( w \) of these monomials belongs to the kernel of \( \Upsilon \). That is, the corresponding linear combination of their images is zero in the vertex algebra \( V^{k+N}(\mathfrak{h}) \). Then so is the linear combination of the components of the images of the top degree, regarded as elements of the symmetric algebra \( S(t^{-1}\mathfrak{h}[t^{-1}]) \cong \text{gr} V^{k+N}(\mathfrak{h}) \), where we equip the vacuum module \( V^{k+N}(\mathfrak{h}) \) with the canonical filtration of the universal enveloping algebra. However, it follows from the proof of [17, Prop. 4.3] that the top degree components of the elements \( T^s v_i^{(r)} \), where \( s \geq 0 \) and \( l = 1, \ldots, n \) with \( r \) satisfying conditions (4.2), are algebraically independent. This implies that the linear combination \( w \) is zero. \( \square \)
At the critical level $k = -N$ we have $\alpha_i = -\lambda_i$ for all $i = 1, \ldots, n$ so that the change of signs $e_{ii}(u) \mapsto -e_{ii}(u)$ allows up to identify the coefficients in the expansion (5.1) with the generators of the classical $\mathcal{W}$-algebra $W(a)$ and with the generators of the center $\mathfrak{z}(\hat{\alpha})$ of the vertex algebra $V^{-N}(a)$ found in [16] and [17, Sec. 4].

**Corollary 5.2.** The $\mathcal{W}$-algebra $\mathcal{W}^{-N}(a)$ is a commutative vertex algebra and we have isomorphisms

$$\mathcal{W}^{-N}(a) \cong \mathfrak{z}(\hat{\alpha}) \cong W(a).$$

In the case $e = 0$ we recover the corresponding result of [9] in type $A$.

**References**

[1] T. Arakawa, *Representation theory of $\mathcal{W}$-algebras*, Invent. Math. 169 (2007), 219–320.

[2] T. Arakawa, *Introduction to $\mathcal{W}$-algebras and their representation theory*, in “Perspectives in Lie theory”, pp. 179–250, Springer INdAM Ser., 19, Springer, Cham, 2017.

[3] T. Arakawa and A. Molev, *Explicit generators in rectangular affine $\mathcal{W}$-algebras of type $A$*, Lett. Math. Phys. 107 (2017), 47–59.

[4] T. Arakawa and A. Premet, *Quantizing Mishchenko–Fomenko subalgebras for centralizers via affine $\mathcal{W}$-algebras*, Trans. Moscow Math. Soc. 78 (2017), 217–234.

[5] P. Bouwknegt and K. Schoutens, *$\mathcal{W}$-symmetry in conformal field theory*, Phys. Rep. 223 (1993), 183–276.

[6] J. Brown and J. Brundan, *Elementary invariants for centralizers of nilpotent matrices*, J. Aust. Math. Soc. 86 (2009), 1–15.

[7] V. A. Fateev and S. L. Lukyanov, *The models of two-dimensional conformal quantum field theory with $\mathbb{Z}_n$ symmetry*, Internat. J. Modern Phys. A 3 (1988), 507–520.

[8] B. Feigin and E. Frenkel, *Quantization of the Drinfeld–Sokolov reduction*, Phys. Lett. B 246 (1990), 75–81.

[9] B. Feigin and E. Frenkel, *Affine Kac–Moody algebras at the critical level and Gelfand–Dikii algebras*, Internat. J. Modern Phys. A 7, Suppl. 1A (1992), 197–215.

[10] E. Frenkel, *Langlands correspondence for loop groups*, Cambridge Studies in Advanced Mathematics, 103. Cambridge University Press, Cambridge, 2007.

[11] E. Frenkel and D. Ben-Zvi, *Vertex algebras and algebraic curves*, Mathematical Surveys and Monographs, vol. 88, Second ed., American Mathematical Society, Providence, RI, 2004.
[12] V. Kac, *Vertex algebras for beginners*, University Lecture Series, 10. American Mathematical Society, Providence, RI, 1997.

[13] V. Kac, Shi-Shyr Roan and M. Wakimoto, *Quantum reduction for affine superalgebras*, Comm. Math. Phys. **241** (2003), 307–342.

[14] J. O. Madsen and E. Ragoucy, *Quantum hamiltonian reduction in superspace formalism*, Nuclear Phys. B **429** (1994), 277–290.

[15] A. Molev, *Sugawara operators for classical Lie algebras*. Mathematical Surveys and Monographs 229. AMS, Providence, RI, 2018.

[16] A. Molev, *Center at the critical level for centralizers in type A*, arXiv:1904.12520.

[17] A. Molev and E. Ragoucy, *Classical W-algebras for centralizers*, Comm. Math. Phys. **378** (2020), 691–703.

[18] A. Molev, E. Ragoucy and U.R. Suh, *Supersymmetric W-algebras*, arXiv:1901.06557.

[19] D. Panyushev, A. Premet and O. Yakimova, *On symmetric invariants of centralisers in reductive Lie algebras*, J. Algebra **313** (2007), 343–391.

[20] A. B. Zamolodchikov, *Infinite extra symmetries in two-dimensional conformal quantum field theory*, Teoret. Mat. Fiz. **65** (1985), 347–359.

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