Optimal insurance under maxmin expected utility

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Abstract
We examine a problem of demand for insurance indemnification, when the insured is sensitive to ambiguity and behaves according to the maxmin expected utility model of Gilboa and Schmeidler (J. Math. Econ. 18:141–153, 1989), whereas the insurer is a (risk-averse or risk-neutral) expected-utility maximiser. We characterise optimal indemnity functions both with and without the customary ex ante no-sabotage requirement on feasible indemnities, and for both concave and linear utility functions for the two agents. This allows us to provide a unifying framework in which we examine the effects of the no-sabotage condition, of marginal utility of wealth, of belief heterogeneity, as well as of ambiguity (multiplicity of priors) on the structure of optimal indemnity functions. In particular, we show how a singularity in beliefs leads to an optimal indemnity function that involves full insurance on an event to which the insurer assigns zero probability, while the decision maker assigns a positive probability. We examine several illustrative examples, and we provide numerical studies for the case of a Wasserstein and a Rényi ambiguity set.

Keywords Optimal insurance · Ambiguity · Multiple priors · Maxmin expected utility · Heterogeneous beliefs

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1 Introduction

A foundational problem in the theory of risk exchange is the problem of demand for insurance indemnification. Specifically, an insurance buyer or decision maker (DM) faces a random insurable loss, against which she seeks coverage through the purchase of an insurance policy. The insurance pricing functional is assumed to be known by the DM, and to be given by the certainty equivalent of the insurer’s utility. Although this is a classical problem, it has traditionally been confined to the accustomed framework of expected-utility theory (EUT), going back to the pioneering work of Arrow [6] and Mossin [56]. Arrow [6] shows the optimality of deductible insurance (a zero indemnification below a fixed threshold of loss, and a linear indemnification above) in an EUT framework when the DM is risk-averse, the insurer is risk-neutral and the two parties have the same beliefs about the underlying loss probability distribution. We refer to Gollier [43] and Schlesinger [62] for surveys of the rather large literature on optimal insurance with EU preferences.

1.1 Ambiguity in insurance demand

The vast majority of this literature remains within the confines of the classical EUT. Yet, ever since the major challenges to the foundations of EUT that the work of Allais [3] and Ellsberg [34] has put forward, decision theory has been pulling away from parts of the axiomatic foundations of EUT, in favour of non-EU models that can rationalise behaviour depicted by Allais [3] and Ellsberg [34], as well as other cognitive biases that are not captured by EUT. Arguably, one of the most important achievements of the modern theory of choice under uncertainty is the remarkable development spurred by the work of Ellsberg [34] in the study of what came to be known as ambiguity, or model uncertainty. Two main approaches to the rationalisation of attitudes toward ambiguity have been explored in the literature on axiomatic decision theory: the non-additive prior approach and the multiple additive priors approach. These two approaches intersect, but they are not equivalent. The first category is based on the seminal contributions of Yaari [73] (dual theory, or DT), Quiggin [59] (rank-dependent expected utility, or RDEU), and Schmeidler [63] (Choquet expected utility, or CEU), which encompasses the previous two models. The second category was initiated by Gilboa and Schmeidler [42] (maxmin expected utility, or MEU) and further refined by Ghirardato et al. [39] (the $\alpha$-maxmin model), Klibanoff et al. [50] (the KMM model) and Amarante [4] who provides a unifying framework. Additionally, it is important to note that while MEU and RDEU have a nonempty intersection (e.g. when the distortion function in RDEU is convex), they are vastly different models. In fact, only few MEU models can be mapped to an RDEU model (see e.g. Amarante [5] for a detailed explanation of this point).

While the literature on non-EU preferences in risk-sharing or optimal insurance design problems is considerably thinner than the literature on risk-sharing with EU preferences, behavioural preferences and ambiguity in particular play an increasing role in this literature. Yet, Machina [55] points out that the robustness of standard optimal insurance results under situations of ambiguity is still very much an open ques-
tion, despite a growing literature on the topic. For instance, Bernard et al. [14] and Xu et al. [72] study RDEU preferences of the DM and risk-neutral EU preferences of the insurer, and they derive optimal insurance indemnities. Ghossoub [41] extends the analysis to account for more general premium principles. Recently, Xu [71] reconsiders the problem of optimal insurance under RDEU preferences for the DM and risk-neutral EU preferences of the insurer and provides a novel characterisation of optimal indemnities using an ODE approach. Also within the first category of ambiguity representation as a non-additive prior, Jeleva [48] considers the case of a DM who is a CEU maximiser.

In the second category of ambiguity representation as a collection of additive priors and an aggregation rule, Alary et al. [1] and Gollier [44] consider the case of an ambiguity-averse DM, in the sense of KMM. However, they consider a finite state space and restrict the set of priors to have a given parametric form. More recently, Jiang et al. [49] study a variant of KMM preferences applied to distortion risk measures with a finite set of priors. Under such preferences for the DM, and using an expected-value premium principle, the authors derive an implicit characterisation of optimal indemnity functions.

Despite its appeal, for its ability to provide a separation of the effect of ambiguity aversion from that of risk aversion, as well as for its capability to define the notion of ambiguity neutrality, the KMM model as a model of ambiguity with multiple priors is arguably not as intuitive or popular as the MEU model of Gilboa and Schmeidler [42]. The MEU model gives rise to decision-making problems that can be embedded into a larger class of model uncertainty problems, which lie at the core of the theory of distributionally robust optimisation (DRO). In this framework, a decision-making problem is often modelled via a maxmin formulation: the agent is uncertain about the underlying model (prior), and therefore formulates an objective function using a collection of (additive) priors, also referred to as the ambiguity set. The agent then aims to maximise the objective under the worst-case model (e.g. Bel-Tal et al. [12]). However, the intuitiveness and wide popularity of the MEU model notwithstanding, there has surprisingly been no study of optimal insurance contracting when the DM is an MEU maximiser, to the best of our knowledge. The present paper fills this void. Specifically, we extend the classical setup and results in two ways: (i) the DM is endowed with MEU preferences with a set $C$ of priors, and (ii) the insurer is not necessarily risk-neutral (that is, the premium principle is not necessarily an expected-value premium principle). The main objective of this paper is to determine the shape of the optimal insurance indemnity when the DM is sensitive to ambiguity and behaves according to MEU.

1.2 This paper’s contribution

In the literature on optimal insurance contracting, a popular assumption is the no-sabotage condition, typically imposed as an ex ante condition of feasibility of insurance indemnities. This condition stipulates that the insured (ceded) risk and the retained risk are comonotonic (they are both nondecreasing functions of the underlying loss). Under the no-sabotage condition, the DM has no incentive to under-report the underlying loss, nor does the DM have an incentive to create incremental losses. This condition is also sometimes referred to as incentive compatibility or a condition
that avoids ex post moral hazard, and it is further studied by Huberman et al. [46] and Carlier and Dana [23]. We refer to [23] for a discussion of various notions of ex ante admissible contracts. In this paper, we characterise optimal insurance contracts under MEU, both with and without the no-sabotage condition. In doing so, this paper sheds light on the consequences of the no-sabotage assumption on the construction of optimal insurance indemnities, in the presence of belief heterogeneity and multiple priors for the DM. Furthermore, while the literature on optimal insurance with non-standard preferences assumes risk-neutrality of the insurer, we provide a more general treatment and allow risk-aversion of the insurer, not only of the insured. We do this both with and without the no-sabotage condition.

Our main results are the following. First, we examine in Sect. 3 the general case in which the insurer is a risk-averse EU maximiser, and the DM is an MEU maximiser with a concave utility function, displaying decreasing marginal utility of wealth. Following the vast majority of the literature, we provide an implicit characterisation of optimal indemnity functions, both with and without the no-sabotage condition on feasible indemnities. For instance, under the no-sabotage condition, feasible indemnities are Lipschitz-continuous and hence absolutely continuous. Optimal indemnities are then characterised in implicit form through their derivative, that is, the so-called “marginal indemnification function” (MIF), as in Assa [9]. The vast majority of the literature on optimal insurance with the no-sabotage condition uses the MIF semi-implicit characterisation of the solution (see Xu et al. [72] or Zhuang et al. [76], for example). Optimal indemnity functions can be formulated as a solution to an ordinary differential equation, which can then be easily solved numerically in practice. The implicit characterisation of optimal indemnities is then used to provide closed-form solutions when the relation between the DM’s and insurer’s beliefs is specified. For instance, when the ambiguity set of the DM consists of all models that are absolutely continuous with respect to the insurer’s belief and such that the corresponding Radon–Nikodým derivative is an increasing function as in Furman and Zitikis [36, 37, 38], we characterise the shape of optimal contracts in an explicit way (see Example 4.4). The main technique proposed to solve the problem consists of two main steps. First, the constrained optimisation problem is reformulated as a minimax problem, for which Sion’s minimax theorem can be applied to obtain the existence of a worst-case measure \( P^* \) in the ambiguity set \( \mathcal{C} \). The minimax theorem is an important result and a standard tool that is frequently employed in the robust optimisation and distributionally robust optimisation literature (see for example Žáčková [75], Ben-Tal and Nemirovskí [13] or Shapiro and Kleywegt [64]). In the framework of optimal insurance design, the minimax theorem is used in Cheung et al. [25] to obtain the structure of the optimal insurance contract that minimises a coherent risk measure and under the premium budget constraint. Here, the minimax reformulation of the original problem results from the Kusuoka representation of the coherent risk measures. In the presence of model uncertainty, Jiang et al. [49] adopt the same result to derive an analytical form of the optimal insurance contract under distortion risk measures. In the context of MEU, the minimax theorem is further used in Xu et al. [72] to prove the existence of a Lagrange multiplier that binds the budget constraint. Similarly to Cheung et al. [25], the structure of the worst-case \( P^* \in \mathcal{C} \) is derived numerically by specifying the structure of the ambiguity set. In the second step, once the worst-case
measure $P^*$ is obtained, the problem falls into the literature on belief heterogeneity in insurance contracting, to which we contribute significantly, as discussed below. The saddle point approach is also considered in the work of Birghila and Pflug [16], where the insured’s ambiguity set is a convex hull of a finite number of models that are within some $\epsilon$-distance of a reference/baseline model. The distance between distributions is measured by the Wasserstein distance on the positive real line, with a distorted underlying metric. The saddle point, i.e., the optimal insurance contract and the worst-case distribution, is obtained using a numerical approach.

Additionally, by specifying the structure of the DM’s ambiguity set $C$, we are able to obtain explicitly the worst-case probability measure for the problem analysed in Sect. 3. In particular, we examine the special case in which the DM’s set of priors forms a neighbourhood around the insurer’s probability measure. In a general setting in which both participants are risk-averse, we define $C$ to be a Rényi ambiguity set. In a discretised framework, we use a successive convex programming algorithm to solve the ordinary differential equation obtained in Sect. 3. We then assess the influence of the ambiguity set on the optimal value. In particular, we show numerically that a larger ambiguity set yields a lower certainty equivalent of final wealth for the DM, but increases the willingness to pay for insurance. Moreover, the impact of the no-sabotage condition on the feasible set of insurance indemnities is illustrated.

Second, as a special case of the above setting, we examine in Sect. 4 the situation in which the insurer is risk-neutral, and hence the premium principle is an expected-value premium principle, as is commonly assumed in the literature (e.g. Bernard et al. [14], Xu et al. [72] and Xu [71]). In this case, we provide an explicit, closed-form characterisation of optimal indemnity functions in the absence of the no-sabotage condition, and an implicit characterisation in the presence of the no-sabotage condition. In particular, by doing so, we provide in both cases (with and without the no-sabotage condition) a crisp depiction of the effect of heterogeneity in beliefs between the two parties, showing how a singularity in beliefs leads to an optimal indemnity function that involves full insurance on an event to which the insurer assigns zero probability, but not the DM. This is an important and intuitive feature of our optimal contracts in this case. Similarly to Sect. 3, we conclude this section with a numerical example. When the DM is risk-averse, $C$ is a Wasserstein ambiguity set and the insurer is risk-neutral, we are able to characterise the saddle point of the problem in Sect. 4. In this case, the optimal indemnity is a deductible contract, and the worst-case measure $P^*$ dominates the insurer’s probability measure in the sense of first-order stochastic dominance.

As an application of the results obtained in Sect. 4, one can also examine the situation in which both parties display constant marginal utility of wealth, that is, their utility functions are linear. In that case, it is straightforward to show that if the no-sabotage condition is imposed, then layer insurance is optimal. In the absence of the no-sabotage condition, the optimal indemnity makes use of a partition of the state space into three sets as in Theorem 3.4, providing no insurance for events in the first set, full insurance for events in the second set, and proportional insurance for events in the third set. Moreover, Artzner et al. [7] and Delbaen [32] show that the class of MEU preferences with linear utility is related to the class of coherent risk measures. Therefore, our analysis can be used to derive optimal insurance contracts when the DM is endowed with a general coherent risk measure.
1.3 Other related literature

Broadly speaking, this paper contributes to the literature focusing on incorporating behavioural models of decision-making into the literature on optimal insurance design. While our main focus is on ambiguity-sensitive preferences, it is important to note that other behavioural models of decision-making have been gaining popularity in the theory of insurance demand. For instance, Cheung et al. [26] study disappointment theory (e.g. Bell [11], Loomes and Sugden [53] and Gul [45]), while Chi and Zhuang [29] study the effects of regret theory (e.g. Bell [10] and Loomes and Sugden [52]). Both settings accommodate for a deductible and partial insurance of losses above the deductible as an optimal indemnity function.

By explicitly incorporating model uncertainty into the problem formulation via a set $C$ of priors, the present paper also falls within the DRO framework. In this perspective, insurance contracts can be seen as saddle points of a DRO problem. The benefit of this technique is twofold. First, the worst-case approach ensures that the optimal decision is not sensitive to possible model misspecification. Second, in many situations, there exist tractable reformulations or algorithms to solve these distributionally robust models, even when the corresponding non-ambiguous problem (that is, when there is a unique prior) cannot be efficiently solved. The idea of incorporating multiple models in the decision-making process dates back to the fundamental work of Scarf [61] in the inventory management applications. He considers a robust formulation of the newsvendor problem, where the optimal strategy is constructed over all possible demand functions with known mean and variance. This initial idea is further developed in the work of Ben-Tal et al. [12] and Bertsimas and Sim [15], among others. A key concept in DRO is the structure of the set of priors, known here as the ambiguity set. Clearly, the choice of the ambiguity set $C$ influences the worst-case model, and thus the optimal decision, while it also facilitates a tractable reformulation and efficient algorithm implementation. The existing literature has focused so far on two types of ambiguity sets: those built using the moment-based approach (e.g. Delage and Ye [31], Scarf [61] and Zymler et al. [77]), and those built using the statistical distance-based approach (such as the Kullback–Leibler divergence in Calafiore and El Ghaoui [22], the $L_1$-ball in Thiele [66], or the Wasserstein distance in Esfahani and Kuhn [35]). Each such choice comes with useful structural properties, but also with shortcomings that need to be dealt with. Ultimately, it is the available set of observations and the type of application that would dictate a suitable choice of ambiguity set $C$.

Furthermore, this paper also contributes to the literature on heterogeneity in beliefs between the DM and the insurer. When the DM does not perceive any ambiguity in the assessment of uncertainty, and thereby behaves as an EU maximiser, the set $C$ of priors is a singleton, that is, $C = \{P\}$, for some probability measure $P$ distinct from the insurer’s probability measure $Q$ and potentially exhibiting some singularity with $Q$. Heterogeneity in beliefs has been studied recently in the context of optimal (re)insurance by Ghossoub [40], Boonen and Ghossoub [18, 19], Chi [27], and Yu and Fang [74]. All these studies focus on unambiguous subjective preferences on the side of the DM (that is, a unique subjective prior on the state space), but they differ in the formulation of the objective function that is optimised. As a special case in which...
the set of priors is a singleton, our results provide a unifying treatment of optimal insurance with belief heterogeneity, and extend the existing results in this literature in several ways, as we make no assumption on how the beliefs diverge and allow risk-aversion of the insurer which is not done in the literature. First, while the majority of the existing literature imposes some assumptions on the way the beliefs of the policyholder and the insurer (the measures $P$ and $Q$, respectively) can diverge (e.g. a monotone likelihood ratio, a monotone hazard ratio, or some other assumption), we do not make any such assumption. We allow the measures to truly diverge in any way, and to exhibit singularity. Second, while the existing literature only considers the case of a risk-neutral insurer (a linear utility function $v$), we consider the effect of risk-aversion on the optimal indemnity. We also consider the case of risk-neutrality. Third, when it comes to the choice of the set of ex ante admissible indemnity functions, the existing literature either consider the case of no-sabotage indemnity functions (the set $\hat{I}$ in (2.3)) or the set of general nonnegative indemnity functions that do not exceed the loss (the set $I$ in (2.2)), but not both. We consider both cases, thereby illustrating the impact of the no-sabotage assumption on the structure of optima.

The rest of the paper is organised as follows. Section 2 presents the setup of our problem together with the necessary background. In Sect. 3, we consider the case in which both the insurer and DM have concave utility functions, and the DM is an MEU maximiser. We characterise optimal indemnity functions both in the presence and absence of the no-sabotage condition. The corresponding worst-case measures are obtained numerically by imposing a specific structure of the ambiguity set $C$. Section 4 considers the particular case of a risk-neutral insurer and provides some illustrating examples. Section 5 concludes the paper. Some definitions and technical proofs are provided in the Appendix.

2 Setup and preliminaries

Let $S$ be a nonempty collection of states of the world and equip $S$ with a $\sigma$-algebra $\mathcal{G}$ of events. A DM is facing an insurable state-contingent loss represented by a random variable $X$ on the measurable space $(S, \mathcal{G})$ with values in the interval $[0, M]$ for some $M \in \mathbb{R}_+$. We denote by $\Sigma$ the sub-$\sigma$-algebra $\sigma(X)$ of $\mathcal{G}$ on $S$ generated by the random variable $X$.

Let $B(\Sigma)$ denote the vector space of all bounded, $\mathbb{R}$-valued and $\Sigma$-measurable functions on $(S, \Sigma)$, and let $B_+ (\Sigma)$ be its positive cone. When endowed with the sup-norm $\| \cdot \|_{\sup}$, $B(\Sigma)$ is a Banach space (e.g. Dunford [33, IV.5.1]). By Doob’s measurability theorem (e.g. Aliprantis and Border [2, Theorem 4.41]), for any $Y \in B(\Sigma)$, there exists a bounded, Borel-measurable map $I : \mathbb{R} \to \mathbb{R}$ such that $Y = I \circ X$. Moreover, $Y \in B_+ (\Sigma)$ if and only if the function $I$ is nonnegative.

**Definition 2.1** Two functions $Y_1$ and $Y_2 \in B(\Sigma)$ are said to be *comonotonic* (resp. *anti-comonotonic*) if

\[
(Y_1(s) - Y_1(s'))(Y_2(s) - Y_2(s')) \geq 0 \text{ (resp. } \leq 0) \quad \text{for all } s, s' \in S.
\]
For instance, any \( Y \in B(\Sigma) \) is comonotonic and anti-comonotonic with any \( c \in \mathbb{R} \). Moreover, if \( Y_1, Y_2 \in B(\Sigma) \) and \( Y_2 \) is of the form \( Y_2 = I \circ Y_1 \) for some Borel-measurable function \( I \), then \( Y_2 \) is comonotonic (resp. anti-comonotonic) with \( Y_1 \) if and only if the function \( I \) is nondecreasing (resp. nonincreasing).

Let \( ba(\Sigma) \) denote the linear space of all bounded finitely additive set functions on \( \Sigma \). When endowed with the total variation norm \( \| \cdot \|_v \), \( ba(\Sigma) \) is a Banach space. By a classical result (e.g. Dunford [33, IV.5.1]), \( (ba(\Sigma), \| \cdot \|_v) \) is isometrically isomorphic to the norm-dual of \( B(\Sigma) \) via the duality \( \langle \phi, \lambda \rangle = \int \phi d\lambda \) for \( \lambda \in ba(\Sigma) \) and \( \phi \in B(\Sigma) \). Thus we can endow \( ba(\Sigma) \) with the weak* topology \( \sigma(ba(\Sigma), B(\Sigma)) \).

Let \( ca(\Sigma) \) denote the collection of all countably additive elements of \( ba(\Sigma) \) and \( ca^+(\Sigma) \) its positive cone. Then \( ca(\Sigma) \) is a \( \| \cdot \|_v \)-closed linear subspace of \( ba(\Sigma) \). Hence \( ca(\Sigma) \) is \( \| \cdot \|_v \)-complete, i.e., \( (ca(\Sigma), \| \cdot \|_v) \) is a Banach space. Denote by

\[
ca_1^+(\Sigma) := \{ \mu \in ca^+(\Sigma) : \mu(S) = 1 \}
\]

the collection of probability measures on \((S, \Sigma)\). We endow \( ca_1^+(\Sigma) \) with the weak* topology inherited from \( ba(\Sigma) \).

For any \( Y \in B(\Sigma) \) and \( P \in ca_1^+(\Sigma) \), let \( F_{Y,P}(t) := P\{s \in \Sigma : Y(s) \leq t\} \) denote the cumulative distribution function (CDF) of \( Y \) with respect to the probability measure \( P \), and let \( F_{Y,P}^{-1}(t) \) denote the left-continuous inverse of \( F_{Y,P} \) (i.e., the quantile function of \( Y \)), defined by

\[
F_{Y,P}^{-1}(t) := \inf\{z \in \mathbb{R} : F_{Y,P}(z) \geq t\} \quad \text{for } t \in [0, 1].
\]

### 2.1 The DM’s and the insurer’s preferences

The DM can purchase insurance against the random loss \( X \) in a perfectly competitive insurance market, for a premium set by the insurer. In return for the premium payment, the DM is promised an indemnification against the realisations of \( X \). An indemnity function is a random variable \( Y = I(X) \) on \((S, \Sigma)\) for some bounded, Borel-measurable map \( I : X(S) \to \mathbb{R} \), which pays off the amount \( I(X(s)) \in \mathbb{R} \) in the state \( s \in S \) of the world, corresponding to a realisation \( X(s) \) of \( X \). That is, we can identify the set of indemnity functions with a subset of \( B(\Sigma) \). For each indemnity function \( Y \in B(\Sigma) \), we define the corresponding retention function by \( R := X - Y \in B(\Sigma) \).

The DM has a preference relation over insurance indemnification functions (or over wealth profiles) that admits an MEU representation \( V^{MEU} : B(\Sigma) \to \mathbb{R} \) as in Gilboa and Schmeidler [42], of the form

\[
V^{MEU}(Z) := \min_{\mu \in C} \int u(Z) d\mu \quad \text{for } Z \in B(\Sigma),
\]

where \( u : \mathbb{R} \to \mathbb{R} \) is a concave utility function and \( C \) is a (unique) weak*-compact and convex subset of \( ba_1^+(\Sigma) \). Moreover, we assume that the DM’s preferences satisfy the Arrow–Villegas monotone continuity axiom as in Chateauneuf et al. [24], so that \( C \subseteq ca_1^+(\Sigma) \), i.e., all priors are countably additive. Additionally, the DM’s utility function \( u \) satisfies the following assumption.
Assumption 2.2 The utility function $u : \mathbb{R} \to \mathbb{R}$ is concave, strictly increasing and continuously differentiable.

Let $W_0 \in \mathbb{R}_+$ be the DM’s initial wealth. After purchasing insurance coverage for a premium $\Pi_0 > 0$, the DM’s terminal wealth is a random variable $W \in B(\Sigma)$ given by

$$W := W_0 - X + Y - \Pi_0.$$ 

The insurer’s preference over $B(\Sigma)$ admits an EU representation $V^{\text{Ins}} : B(\Sigma) \to \mathbb{R}$ of the form

$$V^{\text{Ins}}(Z) := \int v(Z) \, dQ \quad \text{for } Z \in B(\Sigma),$$

for a utility function $v : \mathbb{R} \to \mathbb{R}$ satisfying Assumption 2.2 and a probability measure $Q \in \mathcal{CA}^+(\Sigma)$.

The insurer has an initial wealth $W_0^{\text{Ins}}$ and faces an administration cost, often called an indemnification cost, associated with the handling of an indemnity payment. As customary in the literature (e.g. Bernard et al. [14] and Xu et al. [72]), we assume that for a given indemnity function $Y = I \circ X$, this indemnification cost is a proportional cost of the form $\rho Y$, for a given safety loading factor $\rho \geq 0$ specified exogenously and a priori. Hence the insurer’s terminal wealth is the random variable $W^{\text{Ins}} \in B(\Sigma)$ given by

$$W^{\text{Ins}} := W_0^{\text{Ins}} - (1 + \rho)Y + \Pi_0.$$  

2.2 Admissible indemnity functions

In Arrow [6]’s original formulation of the optimal insurance problem under EUT, an ex ante condition of feasibility of indemnity schedules is the requirement that these be nonnegative and no larger than the realisation of the loss in each state of the world. This is often referred to as the indemnity principle, and it translates into the requirement that an admissible set of indemnities be restricted to those $Y \in B(\Sigma)$ that satisfy $0 \leq Y \leq X$. We denote this set of indemnity functions by

$$\mathcal{I} := \{Y = I \circ X \in B_+(\Sigma) : 0 \leq I(x) \leq x, \forall x \in [0, M]\}.$$  

A desirable property of optimal indemnities is that an indemnity function $Y = I \circ X$ and the corresponding retention function $R = X - Y$ be both nondecreasing functions of the loss $X$, that is, both comonotonic with $X$ (and hence $Y$ and $R$ are comonotonic). Indeed, if $Y$ fails to be comonotonic with $X$, the DM has an incentive to under-report the loss, whereas if $R$ fails to be comonotonic with $X$, the DM has an incentive to create additional damage. These situations of ex post moral hazard are not desirable, and one often seeks additional ex ante conditions that would rule out such behaviour from the DM. In the setting of Arrow [6], the optimal indemnity is a deductible contract of the form $Y = \max(X - d, 0)$ for some $d \in \mathbb{R}_+$. For such contracts, both the indemnity and retention functions are comonotonic with
the loss, and optimal indemnities are de facto immune to the kind of ex post moral hazard described above. However, outside of EUT, optimality of deductible contracts does not always hold, and optimal indemnities might suffer from the above type of moral hazard, as in Bernard et al. [14].

In order to rule out ex post moral hazard that might arise from a misreporting of the loss by the DM, an additional condition is often imposed ex ante on the set of feasible indemnity schedules (as in Xu et al. [72]). One such condition is called the no-sabotage condition, and it stipulates that admissible indemnity functions and the corresponding retention functions be comonotonic, hence resulting in the feasibility set

\[
\hat{I} := \{\hat{Y} \in I : \hat{Y} \text{ and } \hat{R} = X - \hat{Y} \text{ are comonotonic}\}.
\]

As \(Y \in I\) is of the form \(Y = I \circ X\) with \(0 \leq I(x) \leq x\) for all \(x \in [0, M]\), we can write \(\hat{I}\) as

\[
\hat{I} = \{\hat{Y} = \hat{I} \circ X \in B_+(\Sigma) : \hat{I}(0) = 0 \text{ and } 0 \leq \hat{I}(x_1) - \hat{I}(x_2) \leq x_1 - x_2 \text{ for all } 0 \leq x_2 \leq x_1 \leq M\}. \tag{2.3}
\]

The no-sabotage condition is also sometimes referred to as incentive compatibility by Xu et al. [72], and it is further studied by Huberman et al. [46] and Carlier and Dana [23]. The latter discuss various classes of ex ante admissible contracts, as well as their implications for optimal indemnities.

**Remark 2.3** Let \(C[0, M]\) denote the set of all continuous (and thus bounded) functions on \([0, M]\), equipped with the sup-norm \(\|\cdot\|_{\text{sup}}\). Note that \(\hat{I}\) is a uniformly bounded subset of \(C[0, M]\) consisting of Lipschitz-continuous functions \([0, M] \to [0, M]\), with common Lipschitz constant \(K = 1\). Therefore, \(\hat{I}\) is equicontinuous, and hence compact by the Arzelà–Ascoli theorem (e.g. Dunford [33, Theorem IV.6.7]).

In this paper, we characterise optimal indemnity functions, both with and without the no-sabotage condition, in order to examine the impact of such an ex ante requirement on feasible indemnity schedules. This is first done in the general setting of an MEU maximising DM with a concave utility and an EU-maximising insurer with concave utility (Sect. 3), and then in a setting where the insurer is risk-neutral (hence uses an expected-value premium principle).

### 3 Optimal indemnity functions

In this section, we investigate the DM’s problem of demand for insurance indemnification when the DM is ambiguity-sensitive and has preferences admitting an MEU representation of the form given in (2.1), whereas the insurer is a risk-averse EU maximiser with a concave utility function \(v\). We first examine in Sect. 3.1 the class \(I\) of indemnities that are nonnegative and cannot exceed the loss \(X\) as in (2.2),
and we provide in Theorem 3.2 a closed-form characterisation of the optimal indemnity in this case. We then consider in Sect. 3.2 the class \( \mathcal{I} \) of indemnities for which both indemnity and retention functions are nondecreasing functions of the loss as in (2.3). In that case, Theorem 3.4 provides an implicit characterisation of the optimal indemnity function.

The DM chooses a premium \( \pi \) and an indemnity function \( Y \) to maximise her MEU preferences. Such a problem can be solved in two steps. In the first step, an optimal indemnification \( Y^* \) is determined for a fixed premium \( \pi \). In the second step, the optimal premium \( \pi^* \) is determined. The second step is a one-dimensional optimisation problem, and the present paper focuses on the first step. That is, we determine the optimal indemnity for a fixed premium \( \pi = \Pi_0 \), as in Bernard et al. [72] and Xu et al. [72].

Let \( \mathcal{F} \) denote the set of admissible indemnity functions, which could be either the set \( \mathcal{I} \) defined in (2.2) or the set \( \mathcal{I} \subseteq \mathcal{I} \) defined in (2.3). For a given insurance premium \( \Pi_0 > 0 \) and a compact and convex set \( C \) of probability measures, the optimal indemnity function is obtained as the solution of the problem

\[
\begin{cases}
\sup_{Y \in \mathcal{F}} \inf_{P \in C} \mathbb{E}_P[u(W_0 - X + Y - \Pi_0)] \\
\text{such that } \mathbb{E}_Q[v(W_0^{\text{Ins}} - (1 + \rho)Y + \Pi_0)] \geq v(W_0^{\text{Ins}}),
\end{cases}
\]

where \( \rho \geq 0 \) is a given safety loading factor. The constraint in (3.1) is interpreted as the insurer’s participation constraint. Observe that for \( P \in C \) and all \( I \in \mathcal{F} \), we have \( \mathbb{E}_P[u(W_0 - X + Y - \Pi_0)] \leq u(W_0 - \Pi_0) \), and thus the value of (3.1) is finite. If \( \mathbb{E}_Q[v(W_0^{\text{Ins}} - (1 + \rho)X + \Pi_0)] \geq v(W_0^{\text{Ins}}) \), we can eliminate the constraint in Problem (3.1) and the optimal indemnity is \( Y^* = X \ P^*-\text{a.s.}, \) where \( P^* \) is the DM’s worst-case belief that attains the infimum in (3.1). In the sequel, we assume \( \mathbb{E}_Q[v(W_0^{\text{Ins}} - (1 + \rho)X + \Pi_0)] < v(W_0^{\text{Ins}}) \). We note that all our results can be derived for any participation constraint of the form \( \mathbb{E}_Q[v(W_0^{\text{Ins}} - (1 + \rho)Y + \Pi_0)] \geq k \) with \( k \leq v(W_0^{\text{Ins}} + \Pi_0) \). To maintain a direct economic interpretation, we choose \( k = v(W_0^{\text{Ins}}) \), i.e., the insurer’s reservation utility.

**Remark 3.1** By the Lebesgue decomposition theorem, for any \( P \in C \), there are finite nonnegative countably additive measures \( P_{ac} \) and \( P_s \) on \( (S, \Sigma) \) such that \( P = P_{ac} + P_s \), where \( P_{ac} \ll Q \) and \( P_s \perp Q \). Hence for each \( P \in C \), there exist some \( A_P \in \Sigma \) and \( h_P : S \to [0, \infty) \) such that \( Q[S \setminus A_P] = P_s[A_P] = 0 \) and \( h_P = dP_{ac}/dQ \). In particular, since \( h_P \) is \( \Sigma \)-measurable, there exists a nonnegative Borel-measurable function \( \xi_P : \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( h_P = \xi_P \circ X \).

### 3.1 Without the no-sabotage condition

The solution \((Y^*, P^*)\) to Problem (3.1) for \( \mathcal{F} = \mathcal{I} \) is given in the following result.

**Theorem 3.2** Suppose the utility functions \( u \) and \( v \) satisfy Assumption 2.2 and are in addition strictly concave with

\[
\lim_{x \to -\infty} u'(x) = \lim_{x \to -\infty} v'(x) = +\infty, \quad \lim_{x \to +\infty} u'(x) = \lim_{x \to +\infty} v'(x) = 0.
\]
Let \( \mathcal{F} = \mathcal{I} \) as in (2.2) be the set of admissible indemnity functions. Then there exists \( P^* \in \mathcal{C} \) such that \( Y^* \in \mathcal{I} \) is optimal for (3.1) if and only if is of the form

\[
Y^* = \tilde{Y}^* 1_{A\setminus A_{h^*}} + Y_{h^*} 1_{A_{h^*}} + X 1_{S\setminus A},
\]

(3.2)

where

(a) \( A \in \Sigma \) is such that \( P^* = P_{ac}^* + P_s^* \), with \( P_s^*[A] = Q[S \setminus A] = 0; \)
(b) \( h^* : S \to [0, \infty) \) is such that \( h^* = dP_{ac}^*/dQ; \)
(c) \( A_{h^*} := \{s \in A : h^*(s) = 0\}; \)
(d) \( \tilde{Y}^* \) and \( Y_{h^*} \) are of the following form:

Case 1: If \( \lambda^* > 0 \), then \( Y_{h^*} = 0 \) and \( \tilde{Y}^* = \max(0, \min(X, Y_0^*)) \), where \( Y_0^* \) solves

\[
u'(W_0 - X(s) + Y(s) - \Pi_0)h^*(s) - \lambda^*(1+\rho)v'(W_0^{\text{Ins}} - (1+\rho)Y(s) + \Pi_0) = 0 \quad \text{for all } s \in A \setminus A_{h^*};
\]

Case 2: If \( \lambda^* = 0 \), then \( \tilde{Y}^* = X \) and \( Y_{h^*} \) solves

\[
\mathbb{E}_Q \left[ v(W_0^{\text{Ins}} - (1+\rho)Y 1_{A_{h^*}} + \Pi_0) \right] 
= v(W_0^{\text{Ins}}) - \mathbb{E}_Q \left[ v(W_0^{\text{Ins}} - (1+\rho)X 1_{A\setminus A_{h^*}} + \Pi_0) \right];
\]

(e) \( \lambda^* \in \mathbb{R}_+ \) defined in (d) is such that

\[
\lambda^* \left( \mathbb{E}_Q \left[ v(W_0^{\text{Ins}} - (1+\rho)Y^* + \Pi_0) \right] - v(W_0^{\text{Ins}}) \right) = 0.
\]

Proof Define the set \( \mathcal{I}_0 := \{Y \in \mathcal{I} : \mathbb{E}_Q[v(W_0^{\text{Ins}} - (1+\rho)Y + \Pi_0)] \geq v(W_0^{\text{Ins}})\}. Observe that for \( Y_1, Y_2 \in \mathcal{I}_0 \) and \( \alpha \in (0, 1) \), we have \( \tilde{Y} := \alpha Y_1 + (1-\alpha)Y_2 \in \mathcal{I} \)
and

\[
\mathbb{E}_Q[v(W_0^{\text{Ins}} - (1+\rho)\tilde{Y} + \Pi_0)] \geq \alpha \mathbb{E}_Q[v(W_0^{\text{Ins}} - (1+\rho)Y_1 + \Pi_0)]
+ (1-\alpha) \mathbb{E}_Q[v(W_0^{\text{Ins}} - (1+\rho)Y_2 + \Pi_0)]
\geq v(W_0^{\text{Ins}}),
\]

and thus the set \( \mathcal{I}_0 \) is convex. The objective function \( \mathbb{E}_P[u(W_0 - X + Y - \Pi_0)] \)
of (3.1) is concave in \( Y \in \mathcal{I}_0 \) and continuous in \( Y \) with respect to the sup-norm \( \| \cdot \|_{\text{sup}} \). Observe that \( \mathbb{E}_P[u(W_0 - X + Y - \Pi_0)] \) is linear in \( P \) and continuous in \( P \in \mathcal{C} \) in the weak* topology. The set \( \mathcal{I}_0 \) is convex and the set \( \mathcal{C} \) is convex and weak*-compact. Therefore (3.1) satisfies the conditions of Sion’s minimax theorem (see Sion [65, Theorem 3.4]), and hence there exists a saddle point \((Y^*, P^*) \in \mathcal{I}_0 \times \mathcal{C}\) such that

\[
\sup_{Y \in \mathcal{I}_0} \inf_{P \in \mathcal{C}} \mathbb{E}_P[u(W_0 - X + Y - \Pi_0)] = \sup_{Y \in \mathcal{I}_0} \min_{P \in \mathcal{C}} \mathbb{E}_P[u(W_0 - X + Y - \Pi_0)]
= \min_{P \in \mathcal{C}} \sup_{Y \in \mathcal{I}_0} \mathbb{E}_P[u(W_0 - X + Y^* - \Pi_0)]
= \mathbb{E}_{P^*}[u(W_0 - X + Y^* - \Pi_0)].
\]

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For $P^* \in \mathcal{C}$, to characterise the optimal indemnity $Y^*$, we focus on the inner problem
\[
\begin{aligned}
\sup_{Y \in \mathcal{I}} \mathbb{E}_{P^*}[u(W_0 - X + Y - \Pi_0)] \\
\text{such that } \mathbb{E}_{Q}[v(W_0^{\text{Ins}} - (1 + \rho)Y + \Pi_0)] \geq v(W_0^{\text{Ins}}).
\end{aligned}
\tag{3.3}
\]
Problem (3.3) is a convex optimisation problem since the constraint can be equivalently written as $\mathbb{E}_{Q}[v_1(W_0^{\text{Ins}} - (1 + \rho)Y + \Pi_0)] \leq v_1(W_0^{\text{Ins}})$, where $v_1 := -v$ is a convex function. For $P^* \in \mathcal{C}$, let $A := A_{P^*}$ and $h^* := h_{P^*}$ be as in Remark 3.1, and consider the two problems
\[
\begin{aligned}
\sup_{Y \in \mathcal{I}} \left\{ \int_{S \setminus A} u(W_0 - X + Y - \Pi_0) dP_s^* : 0 \leq Y \leq X \right\}.
\end{aligned}
\tag{3.4}
\]
\[
\begin{aligned}
\sup_{Y \in \mathcal{I}} \left\{ \int_{A} u(W_0 - X + Y - \Pi_0) h^* dQ : \int_{A} v(W_0^{\text{Ins}} - (1 + \rho)Y + \Pi_0) dQ \geq v(W_0^{\text{Ins}}) \right\}.
\end{aligned}
\tag{3.5}
\]
Observe that $Y := X$ is a feasible solution for (3.4) and that
\[
\int_{S \setminus A} u(W_0 - X + Y - \Pi_0) dP_s^* = u(W_0 - \Pi_0) P_s^* \left[ S \setminus A \right]
\]
\[
\geq \int_{S \setminus A} u(W_0 - X - \Pi_0) dP_s^*
\]
for any feasible solution $Y$ for (3.4). Hence $Y = X$ is optimal for (3.4).

Now let $Y_1^* \in \mathcal{I}$ be a solution for (3.5). We claim that $Y^* := Y_1^* 1_A + X 1_{S \setminus A}$ is optimal for (3.3). To see this, we remark that
\[
\int_{S \setminus A} u(W_0 - X + Y - \Pi_0) dP_s^* = u(W_0 - \Pi_0) P_s^* \left[ S \setminus A \right]
\]
\[
\geq \int_{S \setminus A} u(W_0 - X - \Pi_0) dP_s^*
\]
for any feasible solution $Y$ for (3.4). Hence $Y = X$ is optimal for (3.4).

Next, we focus on the optimal indemnity $Y_1^*$ that solves (3.5). The associated Lagrange function is
\[
\mathcal{L}(Y_1, \lambda) = \int_{A} \left( u(W_0 - X(s) + Y_1(s) - \Pi_0) h^*(s) \right. \]
\[
+ \lambda v(W_0^{\text{Ins}} - (1 + \rho)Y_1(s) + \Pi_0) \left. \right) dQ(s)
\]
\[
- \lambda v(W_0^{\text{Ins}}),
\]
\[\Box\]
where $\lambda \in \mathbb{R}_+$ is the Lagrange multiplier. As the domain $\mathcal{I}$ of $Y_1$ is convex and $\mathcal{L}(Y_1, \lambda)$ is both concave and continuous in $Y_1$ with respect to the sup-norm $\| \cdot \|_{\sup}$ as well as linear in $\lambda$, strong duality holds, i.e.,

$$\text{val}(\mathcal{L}) := \sup_{Y_1 \in \mathcal{I}} \inf_{\lambda \in \mathbb{R}_+} \mathcal{L}(Y_1, \lambda) = \inf_{\lambda \in \mathbb{R}_+} \sup_{Y_1 \in \mathcal{I}} \mathcal{L}(Y_1, \lambda),$$

where the optimal value $\text{val}(\mathcal{L})$ of (3.5) is finite since (3.1) is finite. Moreover, by Sion’s minimax theorem, $(Y_1, \lambda)$ is a saddle point of (3.5).

For a fixed $\lambda \in \mathbb{R}_+$, a necessary and sufficient condition for $Y_1^* \in \mathcal{I}$ to be the solution of (3.5) is

$$\lim_{\theta \to 0^+} \mathcal{L}'((1 - \theta)Y_1^* + \theta Y_1) \leq 0 \quad \text{for all } Y_1 \in \mathcal{I}. \quad (3.6)$$

By direct computation, for all $Y_1 \in \mathcal{I}$, (3.6) becomes

$$\int_A \left( u'(W_0 - X + Y_1^* - \Pi_0)h^* - \lambda(1 + \rho)v'(W_0^{\text{Ins}} - (1 + \rho)Y_1^* + \Pi_0) \right) \times (Y_1 - Y_1^*) \, dQ \leq 0. \quad (3.7)$$

Depending on the Lagrange multiplier $\lambda$, we introduce the function

$$c(s) := u'(W_0 - X(s) + Y_1^*(s) - \Pi_0)h^*(s) - \lambda(1 + \rho)v'(W_0^{\text{Ins}} - (1 + \rho)Y_1^*(s) + \Pi_0)$$

and define the sets

$$A_{\lambda}^+ := \{ s \in A : c(s) > 0 \},$$

$$A_{\lambda}^0 := \{ s \in A : c(s) = 0 \},$$

$$A_{\lambda}^- := \{ s \in A : c(s) < 0 \}.$$

First, observe that on $A_{\lambda}^+$ and $A_{\lambda}^-$, condition (3.7) holds for all $Y_1 \in \mathcal{I}$ only if

$$Y_1^* 1_{A_{\lambda}^+} = X 1_{A_{\lambda}^+} \quad \text{and} \quad Y_1^* 1_{A_{\lambda}^-} = 0. \quad (3.8)$$

Next, define the set $A_{h^*} := \{ s \in A : h^*(s) = 0 \}$. To obtain the structure of $Y_1^*$ in (3.5), we distinguish the following cases, depending on $\lambda$.

**Case A:** If $\lambda > 0$, then $A_{h^*} \subseteq A_{\lambda}^+$ and thus $Y_1^* 1_{A_{h^*}} = 0$. On $A_{\lambda}^+$, $Y_1^*$ satisfies the condition

$$u'(W_0 - X(s) + Y_1^*(s) - \Pi_0)h^*(s) - \lambda(1 + \rho)v'(W_0^{\text{Ins}} - (1 + \rho)Y_1^*(s) + \Pi_0) = 0. \quad (3.9)$$

Let $Y_0^*$ be the solution of (3.9). Considering the observation in (3.8), $Y_1^*$ is thus $Y_1^* 1_{A \setminus A_{h^*}} = \max(0, \min(X, Y_0^*)) 1_{A \setminus A_{h^*}}$, which depends on $s$ only through $h^*$ and $X$.

**Case B:** If $\lambda = 0$, then $A_{h^*} = A_{\lambda}^0$ and $u'(W_0 - X(s) + Y_1^*(s) - \Pi_0)h^*(s) > 0$ on $A \setminus A_{h^*}$. Thus $Y_1^* 1_{A} = Y_1 1_{A_{h^*}} + X 1_{A \setminus A_{h^*}}$ for any feasible $Y \in \mathcal{I}$.
The indemnity \( Y^*_1 := Y^*_1 \), depending on \( \lambda \), is the solution of (3.5) if there exists some \( \lambda^* \geq 0 \) such that \( \mathbb{E}_Q[v(W_0^{\text{Ins}} - (1 + \rho)Y^*_1,\lambda^* + \Pi_0)] = v(W_0^{\text{Ins}}) \).

To see this, define the constant \( \overline{\lambda} := \frac{\text{val}(\mathcal{L}) + \varepsilon}{v(W_0^{\text{Ins}} + \Pi_0) - v(W_0^{\text{Ins}})} \in \mathbb{R}_+ \) for some large \( \varepsilon > |u(W_0 - M - \Pi_0)| \). Then for any \( \lambda > \overline{\lambda} \), we obtain

\[
\sup_{Y_1,\lambda \in \mathcal{I}} \mathcal{L}(Y_1,\lambda, \lambda) \\
\geq \mathcal{L}(0, \lambda) \\
= \int_A u(W_0 - X - \Pi_0)h^* dQ + \lambda \left( \int_A v(W_0^{\text{Ins}} + \Pi_0)dQ - v(W_0^{\text{Ins}}) \right) \\
\geq u(W_0 - M - \Pi_0) P_{\text{at}}^{\gamma}[A] + b(\lambda (v(W_0^{\text{Ins}} + \Pi_0) - v(W_0^{\text{Ins}}))) \\
= u(W_0 - M - \Pi_0) P_{\text{at}}^{\gamma}[A] + \text{val}(\mathcal{L}) + \varepsilon > \text{val}(\mathcal{L}),
\]

where the second inequality follows from the monotonicity of the utility \( u \). Therefore

\[
\inf_{\lambda > \overline{\lambda}} \sup_{Y_1,\lambda \in \mathcal{I}} \mathcal{L}(Y_1,\lambda, \lambda) > \text{val}(\mathcal{L}) \quad \text{for all} \quad \lambda > \overline{\lambda}
\]

and thus

\[
\inf_{\lambda > \overline{\lambda}} \sup_{Y_1,\lambda \in \mathcal{I}} \mathcal{L}(Y_1,\lambda, \lambda) > \text{val}(\mathcal{L}) = \inf_{\lambda \geq 0} \sup_{Y_1,\lambda \in \mathcal{I}} \mathcal{L}(Y_1,\lambda, \lambda).
\]

Hence the feasible set of \( \lambda \) reduces to the compact interval \([0, \overline{\lambda}]\).

Now for \( \lambda \in [0, \overline{\lambda}] \), let \( \text{val}(\mathcal{L}; \lambda) := \sup_{Y \in \mathcal{I}} \mathcal{L}(Y,\lambda) \) be the optimal value as a function of \( \lambda \). We claim that \( \text{val}(\mathcal{L}; \lambda) \) is convex in \( \lambda \). Indeed, let \( \theta \in (0, 1) \) and \( \lambda_1, \lambda_2 \in [0, \overline{\lambda}] \) and consider

\[
\text{val}(\mathcal{L}; \theta \lambda_1 + (1 - \theta)\lambda_2) = \sup_{Y \in \mathcal{I}} \mathcal{L}(Y, \theta \lambda_1 + (1 - \theta)\lambda_2) \\
= \sup_{Y \in \mathcal{I}} \left( \theta \mathcal{L}(Y, \lambda_1) + (1 - \theta)\mathcal{L}(Y, \lambda_2) \right) \\
\leq \theta \sup_{Y \in \mathcal{I}} \mathcal{L}(Y, \lambda_1) + (1 - \theta) \sup_{Y \in \mathcal{I}} \mathcal{L}(Y, \lambda_2),
\]

where the inequality uses that \( \mathcal{L}(Y, \lambda) \) is linear in \( \lambda \) for any given \( Y \). Thus we obtain

\[
\sup_{Y,\lambda \in \mathcal{I}} \min_{\lambda \in [0, \overline{\lambda}]} \mathcal{L}(Y_1, Y_1, \lambda, \lambda) = \sup_{Y,\lambda \in \mathcal{I}} \mathcal{L}(Y_1, \lambda, \lambda) = \mathcal{L}(Y^*_1, \lambda^*) \\
\leq \inf_{\lambda \in [0, \overline{\lambda}]} \text{val}(\mathcal{L}; \lambda) = \text{val}(\mathcal{L}),
\]

where \( \lambda^*(\mathbb{E}_Q[v(W_0^{\text{Ins}} - (1 + \rho)Y^*_1,\lambda^* + \Pi_0)] - v(W_0^{\text{Ins}})) = 0 \).

In consequence, \( Y^*_1 \) is a solution to (3.2) if and only if it is of the form

\[
Y^*_1 = \tilde{Y}^* 1_{A_{h^*}} + Y_{h^*} 1_{A_{h^*}},
\]
where $\bar{Y}^*$ and $Y_h^*$ are defined in Cases A and B, respectively. To see this, let $Y_1 \in \mathcal{I}$ be a feasible solution for (3.2) and consider

$$
\mathcal{L}(Y_1^*, \lambda^*) - \mathcal{L}(Y_1, \lambda^*)
= \int_A \left( u(W_0 - X + Y_1^* - \Pi_0)h^* + \lambda^* v(\text{W}_0^{\text{Ins}} - (1 + \rho)Y_1^* + \Pi_0) \right) dQ \\
- \int_A \left( u(W_0 - X - Y_1 + \Pi_0)h^* + \lambda^* v(\text{W}_0^{\text{Ins}} - (1 + \rho)Y_1 + \Pi_0) \right) dQ \\
= \int_A \left( u(W_0 - X + Y_1^* - \Pi_0) - u(W_0 - X - Y_1 + \Pi_0) \right)h^* dQ \\
- \lambda^* \int_A \left( v(\text{W}_0^{\text{Ins}} - (1 + \rho)Y_1^* + \Pi_0) - v(\text{W}_0^{\text{Ins}} - (1 + \rho)Y_1 + \Pi_0) \right) dQ \\
\geq \int_A \left( u'(W_0 - X + Y_1^* - \Pi_0)h^* - \lambda^* v'(\text{W}_0^{\text{Ins}} - (1 + \rho)Y_1^* + \Pi_0) \right) (Y_1^* - Y_1) dQ \\
= \int_{A_{\lambda^*}^+} \left( u'(W_0 - \Pi_0)h^* - \lambda^* v'(\text{W}_0^{\text{Ins}} - (1 + \rho)X + \Pi_0) \right) (X - Y_1) dQ \\
+ \int_{A_{\lambda^*}^-} \left( u'(W_0 - X - \Pi_0)h^* - \lambda^* v'(\text{W}_0^{\text{Ins}} + \Pi_0) \right) (-Y_1) dQ \geq 0,
$$

where the inequality uses the first-order Taylor approximation for a concave function.

Note that the limit conditions on $u$ and $v$ in Theorem 3.2 are the customary Inada [47] conditions often encountered in the literature. Moreover, note that the optimal indemnity $Y^*$ in (3.2) provides full insurance over the event $S \setminus A$ whenever the DM’s worst-case measure $P^*$ is such that $P^*[S \setminus A] \neq 0$. Note also that $Y^*$ satisfies on the event $A$ an ordinary differential equation for which an analytical expression is difficult to provide in general. However, for particular choices of $\mathcal{C}$, we can obtain numerically the structure of $Y^*$, as well as $P^*$ (see Example 3.6).

### 3.2 With the no-sabotage condition

Next we analyse the case when the set $\mathcal{F}$ of Problem (3.1) is the set of indemnities satisfying the no-sabotage condition. In this case, the feasibility set becomes

$$
\mathcal{I} := \{ \hat{Y} \in \mathcal{I} : \mathbb{E}_Q [v(\text{W}_0^{\text{Ins}} - (1 + \rho)\hat{Y} + \Pi_0)] \geq v(\text{W}_0^{\text{Ins}}) \}.
$$

**Remark 3.3** Since $\mathcal{I}$ is a compact subset of the space $(\mathcal{C}[0, M], \| \cdot \|_{\sup})$ (see Remark 2.3) and $\hat{\mathcal{I}}_0$ is a closed subset of $\hat{\mathcal{I}}$, it follows that $\hat{\mathcal{I}}_0$ is compact.

**Theorem 3.4** Suppose the utility functions $u$ and $v$ satisfy Assumption 2.2. Let $\mathcal{F} = \hat{\mathcal{I}}$ as in (2.3) be the set of admissible indemnity functions. Then there exists $P^* \in \mathcal{C}$ such
that $\hat{Y}^* \in \hat{I}$ is a solution of (3.1) if and only if it is of the form

$$\hat{Y}^* = \hat{Y}_1^* 1_A + X 1_{S \setminus A},$$

where

(a) $A \in \Sigma$ is such that $P^* = P_{\text{ac}}^* + P_s^*$, with $P_s^*[A] = Q[S \setminus A] = 0$;
(b) $h^* : S \to [0, \infty)$ is such that $h^* = dP_{\text{ac}}^*/dQ$;
(c) $\xi^* : \mathbb{R}_+ \to \mathbb{R}_+$ is a Borel-measurable function such that $h^* = \xi^* \circ X$;
(d) $\hat{Y}_1^* = \hat{I}^* \circ X$, where $\hat{I}^*(x) = \int_0^x (\hat{I}^*)'(t) dt$, $\forall x \in [0, M]$, and, with

$$c(x) := \int_{[x, M] \cap X(A)} \tau(t) d(Q \circ X^{-1})(t),$$

we have

$$(\hat{I}^*)'(x) = \begin{cases} 0 & \text{if } c(x) < 0, \\ \kappa(x) & \text{if } c(x) = 0, \\ 1 & \text{if } c(x) > 0, \end{cases}$$

for some Lebesgue-measurable and $[0, 1]$-valued function $\kappa$, and

$$\tau(x) := u'(W_0 - x + \hat{I}^*(x) - \Pi_0)\xi^*(x) - \lambda^*(1 + \rho)u'(W_0^\text{Ins} - (1 + \rho)\hat{I}^*(x) + \Pi_0);$$

(3.11)

(e) $\lambda^* \in \mathbb{R}_+$ is such that $\lambda^* (\mathbb{E}_Q[v(W_0^\text{Ins} - (1 + \rho)\hat{Y}^* + \Pi_0)] - v(W_0^\text{Ins})) = 0$.

Proof Similarly to Theorem 3.2, there exists a saddle point $(\hat{Y}^*, P^*) \in \hat{I}_0 \times C$ such that

$$\sup_{\hat{Y} \in \hat{I}_0} \inf_{P \in C} \mathbb{E}_P[u(W_0 - X + \hat{Y} - \Pi_0)] = \min_{P \in C} \max_{\hat{Y} \in \hat{I}_0} \mathbb{E}_P[u(W_0 - X + \hat{Y} - \Pi_0)]$$

$$= \mathbb{E}_{P^*}[u(W_0 - X + \hat{Y}^* - \Pi_0)],$$

(3.12)

where $\hat{I}_0$ in (3.10) is compact (see Remark 3.3). For $P^* \in C$, the inner optimisation problem in (3.12) becomes

$$\max_{\hat{Y} \in \hat{I}} \int_A u(W_0 - X + \hat{Y} - \Pi_0)h^* dQ + \int_{S \setminus A} u(W_0 - X + \hat{Y} - \Pi_0) dP_s^*$$

such that $\int_A v(W_0^\text{Ins} - (1 + \rho)\hat{Y} + \Pi_0) dQ \geq v(W_0^\text{Ins}),$

where $A := A_{P^*}$ and $h^* := h_{P^*}$, which both depend on $P^*$, are defined in Remark 3.1. Similarly to Theorem 3.2, the optimal indemnity function $\hat{Y}^*$ can be obtained as $\hat{Y}^* = \hat{Y}_1^* 1_A + X 1_{S \setminus A}$, where $\hat{Y}_1^*$ solves

$$\max_{\hat{Y}_1 \in \hat{I}} \left\{ \int_A u(W_0 - X + \hat{Y}_1 - \Pi_0)h^* dQ : \int_A v(W_0^\text{Ins} - (1 + \rho)\hat{Y}_1 + \Pi_0) dQ \geq v(W_0^\text{Ins}) \right\}.$$
The Lagrange function of (3.13) is
\[ L(\hat{Y}_1, \lambda) = \int_A \left( u(W_0 - X + \hat{Y}_1 - \Pi_0) h^* + \lambda v(W_0^{\text{Ins}} - (1 + \rho)\hat{Y}_1 + \Pi_0) \right) dQ - \lambda v(W_0^{\text{Ins}}), \]
where \( \lambda \in \mathbb{R}_+ \) is the Lagrange multiplier. The domain \( \hat{I} \) of \( \hat{Y}_1 \) is convex and \( L(\hat{Y}_1, \lambda) \) is concave in \( \hat{Y}_1 \), continuous in \( \hat{Y}_1 \) with respect to the sup-norm \( \| \cdot \|_{\sup} \) and linear in \( \lambda \). Thus strong duality holds. Therefore, for fixed \( \lambda \in \mathbb{R}_+ \), a necessary and sufficient condition for \( \hat{Y}_1^* \in \hat{I} \) to be the solution of (3.13) is
\[ \lim_{\theta \to 0^+} L'((1 - \theta)\hat{Y}_1^* + \theta \hat{Y}_1) \leq 0 \quad \text{for all } \hat{Y}_1 \in \hat{I}. \] (3.14)
Since \( Q(S \setminus A) = 0 \), we can extend the domain of the \( \hat{Y}_1 = \hat{I}(X) \) in (3.14) over \([0, M]\). By direct computation, (3.14) becomes, for all \( \hat{Y}_1 = \hat{I}(X) \in \hat{I} \),
\[ \int_0^M \left( u'(W_0 - t + \hat{I}^*(t) - \Pi_0) \xi^*(t) - \lambda(1 + \rho)v'(W_0^{\text{Ins}} - (1 + \rho)\hat{I}^*(t) + \Pi_0) \right) \times (\hat{I}(t) - \hat{I}^*(t)) d(Q \circ X^{-1})(t) \leq 0. \] (3.15)
As any \( \hat{Y}_1 = \hat{I}(X) \in \hat{I} \) is absolutely continuous, it is almost everywhere differentiable on \([0, M]\), and hence (3.15) becomes
\[ 0 \geq \int_0^M \int_0^t \left( u'(W_0 - t + \hat{I}^*(t) - \Pi_0) \xi^*(t) \right. \]
\[ \left. - \lambda(1 + \rho)v'(W_0^{\text{Ins}} - (1 + \rho)\hat{I}^*(t) + \Pi_0) \right) \times (\hat{I}'(x) - (\hat{I}^*)'(x)) dx \right) d(Q \circ X^{-1})(t) \]
\[ = \int_0^M \int_0^M \left( u'(W_0 - t + \hat{I}^*(t) - \Pi_0) \xi^*(t) \right. \]
\[ - \lambda(1 + \rho)v'(W_0^{\text{Ins}} - (1 + \rho)\hat{I}^*(t) + \Pi_0) \right) \times \left. d(Q \circ X^{-1})(t)(\hat{I}'(x) - (\hat{I}^*)'(x)) dx \right), \]
for all \( \hat{Y}_1 = \hat{I}(X) \in \hat{I} \). Hence \( \hat{Y}_1^* = \hat{I}^*(X) \) is, with
\[ c(x) := \int_{[x, M] \cap X(A)} \tau(t) d(Q \circ X^{-1})(t), \]
of the form
\[ (\hat{I}^*)'(x) = \begin{cases} 0 & \text{if } c(x) < 0, \\ \kappa(x) & \text{if } c(x) = 0, \\ 1 & \text{if } c(x) > 0, \end{cases} \]
where \( \tau(x) \) is defined in (3.11) and \( \kappa \) is some Lebesgue-measurable and \([0, 1] \)-valued function. The existence of the Lagrange multiplier \( \lambda^* \in \mathbb{R}_+ \) that guarantees the existence of the solution \( \hat{Y}_1^* \) follows similarly to Theorem 3.2.

### 3.3 Numerical example

This section presents a numerical example that illustrates the structure of the optimal indemnity \( \hat{Y}^* \) as well as the worst-case distribution \( P^* \) obtained in Sect. 3 when the ambiguity set \( C \) is constructed as a specific neighbourhood around a reference/base-line distribution. Throughout this analysis, we assume that the underlying space \( S \) is a Polish space with its Borel \( \sigma \)-algebra.

As before, \( X \) is a nonnegative random variable representing the insurable loss, whose true distribution may be unknown. The insurer’s belief \( Q \in \mathcal{C}_1^+(\Sigma) \) regarding the loss \( X \) can be the empirical distribution, derived from experts’ opinion or estimated using standard statistical tools. The DM’s ambiguity regarding the realisations of \( X \) is described by a \( \delta \)-neighbourhood around \( Q \), defined as

\[
C_\delta := \{ P \in \mathcal{C}_1^+(\Sigma) : d(P, Q) \leq \delta \},
\]

where \( d : \mathcal{C}_1^+(\Sigma) \times \mathcal{C}_1^+(\Sigma) \to \mathbb{R}_+ \) is some discrepancy measure between probability measures \( P \) and \( Q \) and \( \delta > 0 \) is a tolerance level/ambiguity radius. The mapping \( d \) satisfies \( d(P, Q) = 0 \) if and only if \( P = Q \). It is worth mentioning that the worst-case distribution \( P^* \) depends not only on the choice of \( d \), but also on the ambiguity radius \( \delta \). In general, the size of \( C_\delta \) is connected to the amount of observations available; if \( \delta \) is close to zero, the impact of ambiguity is negligible, while large values of \( \delta \) indicate high levels of ambiguity. The question of how to optimally choose the ambiguity radius is an ongoing stream of research in robust optimisation. One possible approach is to interpret \( \delta \) as the degree of ambiguity about the reference model and thus argues that this choice depends on the risk preferences of market participants (e.g. Breuer and Csiszar [20] and Wozabal [70]). In Example 3.6, we follow this approach and solve Problem (3.1) for different levels of ambiguity. This allows us to analyse the impact of ambiguity on the optimal indemnity \( \hat{Y}^* \) and the worst-case distribution \( P^* \).

The following observation characterises the change in the DM’s expected utility as a function of the ambiguity radius \( \delta \). This dynamic is later illustrated in Fig. 3 in Example 3.6.

**Remark 3.5** For a fixed premium \( \Pi_0 > 0 \), let \( \hat{\mathcal{Z}}_0 \) (defined in (3.10)) be the feasible set of indemnities in Theorem 3.4. Moreover, for a discrepancy measure \( d \) and some ambiguity radii \( \delta_1 \leq \delta_2 \), let \( \mathcal{C}_{\delta_1} \) and \( \mathcal{C}_{\delta_2} \) be the corresponding ambiguity sets as in (3.16). Let \( (\hat{Y}^*_1, P^*_1) \) and \( (\hat{Y}^*_2, P^*_2) \) be the saddle points of Problems (3.1) for \( \mathcal{C}_{\delta_1} \) and \( \mathcal{C}_{\delta_2} \), respectively. It holds that

\[
\mathbb{E}_{P^*_2}[u(W_0 - X + \hat{Y}^*_2 + \Pi_0)] \leq \mathbb{E}_{P^*_1}[u(W_0 - X + \hat{Y}^*_1 + \Pi_0)] \leq \mathbb{E}_{P^*_1}[u(W_0 - X + \hat{Y}^*_2 + \Pi_0)],
\]

where the first inequality follows from \( \mathcal{C}_{\delta_1} \subseteq \mathcal{C}_{\delta_2} \), as \( \delta_1 \leq \delta_2 \). Hence for increasing values of \( \delta \), the DM’s optimal expected utility decreases.
Example 3.6 For this example, we focus on Problem (3.1) when the admissible set of indemnities is \( \mathcal{F} = \hat{\mathcal{I}} \) as in (2.3). Let the DM’s ambiguity set \( \mathcal{C}_\delta \) be given by

\[
\mathcal{C}_{\delta}^{D_\alpha} := \{ P \ll Q : D_\alpha(P|Q) \leq \delta \},
\]

where \( D_\alpha \) is the Rényi divergence of order \( \alpha \) between \( P \) and \( Q \), i.e.,

\[
D_\alpha(P|Q) := \frac{1}{\alpha - 1} \log \mathbb{E}_Q \left[ \left( \frac{dP}{dQ} \right)^\alpha \right].
\]

We observe that for every \( \alpha > 1 \), \( D_\alpha(P|Q) = 0 \) if and only if \( P = Q \). When \( \alpha \to 1 \), \( D_\alpha \) converges to the well-known Kullback–Leibler divergence. Moreover, since \( \mathcal{S} \) is a Polish space, for any ambiguity radius \( \delta \in [0, \infty) \) and degree \( \alpha \geq 1 \), the set \( \mathcal{C}_{\delta}^{D_\alpha} \) is convex and compact in the topology of weak convergence (e.g. Van Erven and Harremos [68, Theorem 20]). For more on the properties of the divergence \( D_\alpha \), we refer to Rényi [60] and Liese and Vajda [51].

To illustrate our results, we follow the existing literature and consider a discretely distributed loss \( X \). For a sample of size \( n \), we assume without loss of generality that \( x_1 \leq \cdots \leq x_n \), and we denote this \( n \)-sample by \( \hat{x} = (x_1, \ldots, x_n)^\top \). For our example, a random sample \( x \) of size \( n = 100 \) is drawn from a truncated exponential distribution with mean parameter \( \mu = 20 \), and with an upper bound \( M = W_0 - \Pi_1 \). Moreover, the insurer’s belief \( \hat{Q} \) is the empirical distribution of the sample \( x \). Let \( \hat{q} = (\hat{q}_1, \ldots, \hat{q}_n)^\top \) be the insurer’s probability mass function (pmf), where \( \hat{q}_i = \hat{Q}[X = x_i] \), \( \hat{q}_i \geq 0 \), \( i = 1, \ldots, n \), \( 1^\top \hat{q} = 1 \).

Let \( \hat{y} = (\hat{y}_1, \ldots, \hat{y}_n)^\top \in \mathbb{R}^n_+ \) be the indemnification function corresponding to the loss \( x \). Following the approach in Asimit et al. [8], the feasibility constraints

\[
0 \leq \hat{y}_i \leq x_i \quad \text{and} \quad 0 \leq \hat{y}_i - \hat{y}_{i-1} \leq x_i - x_{i-1}, \quad \text{for} \quad i = 1, \ldots, n,
\]

are represented by \( 0 \leq \hat{y} \leq x \) and \( 0 \leq A_{n-1} \hat{y} \leq A_{n-1} x \), where for \( i = 1, \ldots, n - 1 \), the matrix \( A_i \in \mathbb{R}^{(n-1)\times n} \) is defined by

\[
A_i := \begin{pmatrix}
-1 & 1 & 0 & 0 & \cdots & \cdots & 0 \\
0 & -1 & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & -1 & 1 & 0 & \cdots \\
\cdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\( \leftarrow \text{i}th \text{row} \). (3.17)

Moreover, \( p = (p_1, \ldots, p_n)^\top \in [0, 1]^n \) belongs to \( \mathcal{C}_{\delta}^{D_\alpha} \) if it

(i) is a pmf, i.e., \( 1^\top p = 1 \);
(ii) is absolutely continuous with respect to \( \hat{q} \), i.e., if \( \hat{q}_i = 0 \) for some \( i \in \{1, \ldots, n\} \), then \( p_i = 0 \).
(iii) lies in a Rényi ambiguity set around \( \hat{q} \), i.e.,

\[
(p^{\alpha})^\top \hat{q}^{1-\alpha} := \sum_{i=1}^n p_i^{\alpha} \hat{q}_i^{1-\alpha} \leq \bar{\delta},
\]

where \( \bar{\delta} := \exp(\delta(\alpha - 1)) \).

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For ease of notation, let
\[ D = \{ p \in [0, 1]^n : \text{for each } i \in \{1, \ldots, n\}, \text{if } \hat{q}_i = 0 \text{ then } p_i = 0 \}. \]

With the above representations for the variables \( \hat{y} \) and \( p \), Problem (3.1) can be formulated as
\[
\begin{aligned}
\max_{\hat{y} \in \mathbb{R}^n_+} & \min_{p \in D} \sum_{i=1}^n u(W_0 - x_i + \hat{y}_i - \Pi_0) p_i \\
\text{such that} & \quad 0 \leq A_{n-1} \hat{y} \leq A_{n-1} x, \\
& \quad 0 \leq \hat{y} \leq x, \\
& \quad \sum_{i=1}^n -v(W_0^\text{Ins} - (1 + \theta) \hat{y}_i + \Pi_0) \hat{q}_i \leq -v(W_0^\text{Ins}), \\
& \quad (p^\alpha)^\top \hat{q}^{1-\alpha} \leq \delta, \\
& 1^\top p = 1.
\end{aligned}
\]

(3.18)

Observe that (3.18) is a convex optimisation problem, as the objective function is concave in \( \hat{y}_i \) and linear in \( p_i \), for \( i = 1, \ldots, n \), while the constraints are convex in \( \hat{y}_i \) and \( p_i \), for any \( \alpha > 1 \). Problem (3.18) is solved via successive convex programming (SCP, see Pflug and Pichler [57, Sect. 7.2]). The idea is to approximate the infinite dimensional ambiguity set \( C^D_{\delta} \) by a finitely generated set \( \mathcal{P}(m) := \{ \hat{q}, p^{(1)}, \ldots, p^{(m)} \} \), obtained iteratively from solving the inner problem in (3.18). The algorithm starts with \( m = 0, \mathcal{P}(m) := \mathcal{P}(0) = \{ \hat{q} \} \), and solves the outer problem
\[
\begin{aligned}
\max_{\hat{y} \in \mathbb{R}^n_+} & \min_{p \in \mathcal{P}(m)} \sum_{i=1}^n u(W_0 - x_i + \hat{y}_i - \Pi_0) p_i \\
\text{such that} & \quad 0 \leq A_{n-1} \hat{y} \leq A_{n-1} x, \\
& \quad 0 \leq \hat{y} \leq x, \\
& \quad \sum_{i=1}^n -v(W_0^\text{Ins} - (1 + \theta) \hat{y}_i + \Pi_0) \hat{q}_i \leq -v(W_0^\text{Ins}).
\end{aligned}
\]

(3.19)

The solution \( \hat{y}^{(m+1)} \) acts as input for the inner problem
\[
\begin{aligned}
\min_{p \in D} & \sum_{i=1}^n u(W_0 - x_i + \hat{y}_i - \Pi_0) p_i \\
\text{such that} & \quad (p^\alpha)^\top \hat{q}^{1-\alpha} \leq \delta, \\
& 1^\top p = 1.
\end{aligned}
\]

The new \( p^{(m+1)} \) is added to the discrete set, i.e., \( \mathcal{P}(m+1) = \mathcal{P}(m) \cup \{ p^{(m+1)} \} \), and then (3.19) is solved using the updated \( \mathcal{P}(m+1) \). The algorithm stops when no new model is found. The convergence of the algorithm is proved in Pflug and Pichler [57, Proposition B.6].

To obtain an explicit solution, suppose that the DM’s initial wealth is \( W_0 = 200 \), the insurance premium is \( \Pi_0 = 10 \), the safety loading is \( \rho = 0.2 \), and the DM’s utility is given by \( u(x) = x^{1/3} \). The insurer’s initial wealth is \( W_0^\text{Ins} = 400 \), and the utility is
Fig. 1 (Left) The optimal indemnity $\hat{y}^*$ as function of $x$; (Right) The DM’s optimal distribution $F_{X,P^*}$ (red) compared to the insurer’s belief $F_{X,\hat{Q}}$ (black).

Fig. 2 Optimal indemnities $y^*$ and $\hat{y}^*$ for (3.18) when the feasibility sets are $\mathcal{I}$ and $\hat{\mathcal{I}}$, respectively.

$v(x) = x^{1/5}$. For the ambiguity set $C^D_\delta$, we choose the ambiguity radius $\delta = 0.7$ and the order of Rényi divergence $\alpha = 3$. Figure 1 shows the optimal indemnity $\hat{y}^*$ (left) and the worst-case distribution $F_{X,P^*}$ corresponding to $p^*$ (right).

We also solve Problem (3.18) for the same ambiguity set $C^D_\delta$ when the feasibility set is $\mathcal{F} = \mathcal{I}$ as in (2.2). This implies that the constraint $0 \leq A_{n-1} \hat{y} \leq A_{n-1} x$ is removed from (3.18). Figure 2 illustrates the difference between the optimal indemnities corresponding to $\mathcal{I}$ and $\hat{\mathcal{I}}$.

We next investigate the decrease in optimal expected utility when the ambiguity set increases. Certainty equivalence is used to quantify the impact of the ambiguity radii on the value of (3.18). For each $\delta$, let $(\hat{y}^*, p^*)$ be a solution of (3.18) and define the certainty equivalents $CE_1$ and $CE_2$ via

$$
\inf_{P \in C^D_\delta} \mathbb{E}_P[u(W_0 - x + CE_1(\delta))] = \sup_{\hat{y} \in \mathcal{I}} \inf_{P \in C^D_\delta} \mathbb{E}_P[u(W_0 - x + \hat{y} - \Pi_0)],
$$

$$
u(CE_2(\delta)) = \sup_{\hat{y} \in \hat{\mathcal{I}}} \inf_{P \in C^D_\delta} \mathbb{E}_P[u(W_0 - x + \hat{y} - \Pi_0)],$$
Fig. 3 (Left) Certainty equivalent $\text{CE}_1$ as a function of the ambiguity radius $\delta$; (Right) Certainty equivalent $\text{CE}_2$ as a function of the ambiguity radius $\delta$

Fig. 4 (Left) Optimal indemnities $\hat{y}^* \in \hat{I}$ of (3.18) for several values of $\delta$; (Right) The corresponding worst-case probability distributions $F_{X,P}^*$ for several values of $\delta$

where $P$ is the probability measure corresponding to $p$. The constant $\text{CE}_1$ quantifies the marginal benefit of the optimal insurance contract, which we interpret as the willingness to pay for insurance. Moreover, $\text{CE}_2$ measures the certainty equivalent of the DM’s final wealth position. Figure 3 displays the changes in certainty equivalents for increased values of the ambiguity radius. The left panel shows that a larger ambiguity radius yields a higher marginal benefit of the optimal insurance contract. This implies that the DM has a higher willingness to pay for the optimal insurance contract if the ambiguity set gets larger. On the other hand, the certainty equivalent of the final wealth position decreases when the ambiguity set gets larger because the DM is more ambiguity-averse; see the right panel.

Figure 4 (left) provides a closer look at the optimal indemnities $\hat{y}^*$ when the ambiguity set $\mathcal{D}_\alpha$ becomes wider. The right panel shows the worst-case distribution $F_{X,P}^*$ for several values of $\delta$. For increasing values of the ambiguity radius, we observe that each $F_{X,P}^*$ dominates all the previous distributions in the first stochastic order.
4 The case of a risk-neutral insurer

In this section, we examine the case of a risk-neutral insurer, that is, when the utility function \( v \) is linear, and we characterise the optimal indemnity both without and with the no-sabotage condition. In the former case, we obtain a closed-form characterisation of the optimal indemnity (Proposition 4.1), whereas in the latter case, the optimal indemnity is determined implicitly (Proposition 4.3). The results are illustrated in Example 4.4 for a specific ambiguity set \( C \), and closed-form solutions are obtained. We conclude the section with a concrete example to illustrate the structure of the optimal indemnity function when the DM’s ambiguity set is a Wasserstein ball centered around the insurer’s belief \( Q \). By specifying the ambiguity set \( C \), the optimal measures \( P^* \) in Proposition 4.1 and Proposition 4.3 are obtained numerically.

Specifically, we study Problem (3.1) under the assumption of risk neutrality of the insurer, i.e.,

\[
\begin{align*}
\sup_{Y \in \mathcal{F}} \inf_{P \in \mathcal{C}} & \mathbb{E}_P[u(W_0 - X + Y - \Pi_0)] \\
\text{such that } (1 + \rho)\mathbb{E}_Q[Y] & \leq \Pi_0.
\end{align*}
\] (4.1)

If \((1 + \rho)\mathbb{E}_Q[X] \leq \Pi_0\), then we can eliminate the constraint in (4.1). In this case, the optimal indemnity is \( Y^* = X \ P^*\text{-a.s.} \), where \( P^* \) is the DM’s worst-case belief that attains the infimum in (4.1). In the following, we assume that \((1 + \rho)\mathbb{E}_Q[X] > \Pi_0\).

4.1 Without the no-sabotage condition

**Proposition 4.1** Suppose the utility function \( u \) satisfies Assumption 2.2 and is in addition strictly concave and such that \( \lim_{x \to -\infty} u'(x) = +\infty \) and \( \lim_{x \to +\infty} u'(x) = 0 \). Let \( \mathcal{F} = \mathcal{I} \) as in (2.2) be the set of admissible indemnity functions. Then there exists \( P^* \in \mathcal{C} \) such that \( Y^* \in \mathcal{I} \) is a solution of (4.1) if and only if is of the form

\[
Y^* = (X - R^*)1_{A \setminus A_{h^*}} + (X - R_{h^*})1_{A_{h^*}} + X 1_{S \setminus A},
\] (4.2)

where

(a) \( A \in \Sigma \) is such that \( P^* = P_{ac}^* + P_s^* \), with \( P_s^*[A] = Q[S \setminus A] = 0 \);
(b) \( h^* : S \to [0, \infty) \) is such that \( h^* = dP_{ac}^*/dQ \);
(c) \( A_{h^*} := \{ s \in A : h^*(s) = 0 \} \);
(d) \( R^* \) and \( R_{h^*} \) are given as follows:

**Case 1:** If \( (1 + \rho)\mathbb{E}_Q[X 1_{S \setminus A_{h^*}}] > \Pi_0 \), then

\[
R^* = \max \left( 0, \min \left( X, W_0 - \Pi_0 - (u')^{-1}\left(\frac{\lambda^*}{h^*}\right)\right) \right) 1_{A \setminus A_{h^*}}
\]

and \( R_{h^*} = X 1_{A_{h^*}} \), where \( \lambda^* \in \mathbb{R}_+ \) is such that \((1 + \rho)\mathbb{E}_Q[Y^*] = \Pi_0\).

**Case 2:** If \( (1 + \rho)\mathbb{E}_Q[X 1_{S \setminus A_{h^*}}] \leq \Pi_0 \), then \( R^* = 0 \) and \( R_{h^*} = cX 1_{A_{h^*}} \), where \( c \in (0, 1] \) is defined as \( c := \frac{\mathbb{E}_Q[X] - (1 + \rho)\Pi_0}{\mathbb{E}_Q[X 1_{A_{h^*}}]} \).
The above result is a special case of Theorem 3.2.

**Remark 4.2** In the setting of Proposition 4.1, Case 2, an important special case is when \( h^* \equiv 0 \), i.e., \( P^* \perp Q \), where \( P^* \) is the worst-case measure that attains the infimum in (4.1). Thus the optimal indemnity function is \( Y^* = (X - R_{h^*})1_A + X1_{S \setminus A} \), where \( R_{h^*} \in \mathcal{I} \) satisfies \( \int_A R_{h^*}(s) dQ(s) = \Pi_0 \). A possible choice for \( R_{h^*}^* \) is shown in Proposition 4.1.

### 4.2 With the no-sabotage condition

The following result characterises the optimal indemnity \( Y^* \) when the no-sabotage condition is enforced.

**Proposition 4.3** Suppose the utility function \( u \) satisfies Assumption 2.2 and let \( F = \hat{F} \) as in (2.3) be the set of admissible indemnity functions. Then there exists \( P^* \in \mathcal{C} \) such that the solution of (4.1) is \( \hat{Y}^* \in \hat{F} \) if and only if \( \hat{Y}^* \) is of the form

\[
\hat{Y}^* = (X - \hat{R}^*)1_A + X1_{S \setminus A},
\]

where

(a) \( A \in \Sigma \) is such that \( P^* = P_{ac}^* + P_s^* \), with \( P_s^*[A] = Q[S \setminus A] = 0 \);

(b) \( h^* : S \to [0, \infty) \) is such that \( h^* = \frac{dP_{ac}^*}{dQ} \);

(c) \( \xi^* : \mathbb{R}_+ \to \mathbb{R}_+ \) is a Borel-measurable function such that \( h^* = \xi^* \circ X \);

(d) \( \hat{R}^* = \hat{r}^* \circ X \), where \( \hat{r}^*(x) = \int_0^x (\hat{r}^*)'(t) dt \), for all \( x \in [0, M] \), and, with

\[
c(x) := \int_{[x,M] \cap X(A)} \left( u'(W_0 - \hat{r}^*(t) - \Pi_0)\xi^*(t) - \lambda^* \right) d(Q \circ X^{-1})(t),
\]

we have

\[
(\hat{r}^*)'(x) = \begin{cases} 
0 & \text{if } c(x) > 0, \\
\kappa(x) & \text{if } c(x) = 0, \\
1 & \text{if } c(x) < 0,
\end{cases}
\]

for some Lebesgue-measurable and \([0, 1]\)-valued function \( \kappa \);

(e) \( \lambda^* \in \mathbb{R}_+ \) is such that \((1 + \rho)\mathbb{E}_Q[\hat{Y}^*] = \Pi_0\).

Proposition 4.3 is a particular case of Theorem 3.4 when \( v \) is linear.

### 4.3 Example

The following example analyses the structure of the optimal \( Y^* \) and \( \hat{Y}^* \) in Propositions 4.1 and 4.3, respectively, when all probability measures \( P \in \mathcal{C} \) are absolutely continuous with respect to \( Q \), with a particular structure of the Radon–Nikodým derivatives. This is done both with and without the no-sabotage condition. Specifically, we assume that each \( P \in \mathcal{C} \) is such that \( P \ll Q \) with

\[
\frac{dP}{dQ} = \frac{w(X)}{\int w(X) dQ}
\]

(4.3)
for some nonnegative and increasing weight function $w$ satisfying $\int w(X)dQ > 0$. Such measure transformations have a long tradition in insurance pricing, dating back to the Esscher transform (e.g. Bühlmann [21]) in which the function $w$ takes the form $w(x) = e^{bx}$ for a given $b \in (0, \infty)$. More generally, Furman and Zitikis [36, 37, 38] discuss a general class of weighted premium principles where pricing is done via measure transformations as in (4.3).

Suppose the utility function $u$ satisfies Assumption 2.2 and the insurer’s probability measure $Q$ has a continuous CDF over $[0, M]$.

**Example 4.4** Let the DM’s ambiguity set $C$ be defined as

$$C_W := \left\{ P \in ca_1^+(\Sigma) : \frac{dP}{dQ} = \frac{w(X)}{\int w(X)dQ}, w \in W \right\},$$

where $W \subseteq L^1(\mathbb{R}, B(\mathbb{R}), Q \circ X^{-1})$ is a collection of nonnegative increasing weight functions such that $\int w(X)dQ > 0$ for all $w \in W$. The Appendix provides conditions under which the set $C_W$ is convex and weak$^*$-compact.

First we analyse the case when the feasible set of indemnities is $F = I$ as in (2.2). By the definition of $C_W$, any optimal $P^*$ is absolutely continuous with respect to $Q$. Moreover, by monotonicity of $h^* = \frac{dP^*}{dQ} = \xi^*(X)$, there exists some $a \geq 0$ such that $\xi^*(x) = 0$ for $x \in [0, a]$ and $\xi^*(x) > 0$ for $x > a$, i.e., the set $A_{h^*}$ in Proposition 4.1 is precisely $A_{h^*} = X^{-1}([0, a])$.

If $(1 + \rho)\mathbb{E}_Q[X_{1 \setminus A_{h^*}}] = (1 + \rho) \int_a^M x d(Q \circ X^{-1})(x) > \Pi_0$, the optimal indemnity $Y^* = I^*(X)$ in (4.2) is such that

$$I^*(x) = \max \left( 0, \min \left( x, x - W_0 + \Pi_0 + \left( u'\left( W_0 - \hat{r}^*(t) - \Pi_0 \right) \xi^*(t) \right) \right) \right) 1_{[a, M]}(x),$$

where $\lambda^* \in \mathbb{R}_+$ is such that $(1 + \rho)\mathbb{E}_Q[Y^*] = \Pi_0$.

If $(1 + \rho)\mathbb{E}_Q[X_{1 \setminus A_{h^*}}] = (1 + \rho) \int_a^M x d(Q \circ X^{-1})(x) \leq \Pi_0$, then (4.2) becomes

$$I^*(x) = \begin{cases} (1 - c)x & \text{if } x \leq a, \\ x & \text{if } x > a, \end{cases}$$

where the constant $c \in (0, 1]$ is chosen as in Proposition 4.1.

Next, let $F = \hat{I}$ as in (2.3). Following the setting of Proposition 4.3, the utility function $u$ need not be strictly concave, but only concave. According to Proposition 4.3, the optimal retention can be equivalently written, with

$$c(x) = \int_x^M \left( \lambda^* - u'\left( W_0 - \hat{r}^*(t) - \Pi_0 \right) \xi^*(t) \right) dF_{X,Q}(t),$$

as

$$\left( \hat{r}^* \right)'(x) = \begin{cases} 0 & \text{if } c(x) < 0, \\ \kappa(x) & \text{if } c(x) = 0, \\ 1 & \text{if } c(x) > 0. \end{cases}$$
Observe that the function \( \varphi : [0, M] \to \mathbb{R} \), \( \varphi(x) := -u'(W_0 - \hat{r}^*(x) - \Pi_0)\xi^*(x) \) is continuous and decreasing. We distinguish the following cases:

If \( -\lambda^* < \varphi(M) \), then \( \int_t^M (\varphi(x) + \lambda^*) \, dF_{X,Q}(x) > 0 \) for all \( t \in [0, M] \) and thus \((\hat{r}^*)' \equiv 1\).

If \( -\lambda^* > \varphi(0) \), then \( \int_t^M (\varphi(x) + \lambda^*) \, dF_{X,Q}(x) < 0 \) for all \( t \in [0, M] \) and thus \((\hat{r}^*)' \equiv 0\).

If \( -\lambda^* \in [\varphi(M), \varphi(0)] \), there exists some \( d \in (0, M) \) with \( \varphi(x) + \lambda^* \geq 0 \) for all \( x \leq d \) and \( \varphi(x) + \lambda^* < 0 \) for all \( x > d \). This implies that for all \( t > d \), \( \int_t^M (\varphi(x) + \lambda^*) \, dF_{X,Q}(x) < 0 \) and thus \((\hat{r}^*)'(t) = 0 \) for \( t \in (d, M] \). Moreover, for \( t_1 < t_2 \leq d \), it holds that

\[
\int_{t_1}^d (\varphi(x) + \lambda^*) \, dF_{X,Q}(x) + \int_d^M (\varphi(x) + \lambda^*) \, dF_{X,Q}(x) \\
\geq \int_{t_2}^d (\varphi(x) + \lambda^*) \, dF_{X,Q}(x) + \int_d^M (\varphi(x) + \lambda^*) \, dF_{X,Q}(x).
\]

Therefore, there exists some \( d^* \geq 0 \) such that for all \( t \geq d^* \),

\[
\int_t^M (\varphi(x) + \lambda^*) \, dF_{X,Q}(x) < 0.
\]

Hence \((\hat{r}^*)'(t) = 1 \) for all \( t < d^* \) and \((\hat{r}^*)'(t) = 0 \) for all \( t > d^* \). In this case, \( \hat{r}^*(t) = \min(t, d^*) \) and thus \( \hat{I}^*(t) = \max(t - d^*, 0) \).

### 4.4 Numerical example

In this section, we examine the structure of the saddle point \((\hat{Y}^*, P^*)\) in the setting of Problem (4.1) when the insurer is risk-neutral, the admissible set of indemnities is \( \mathcal{F} = \hat{I} \) as in (2.3) and the DM’s ambiguity set \( \mathcal{C} \) is a Wasserstein ball around the insurer’s belief \( Q \). Similarly to Sect. 3.3, we assume that \( S \) is a Polish space and \( X \) is a nonnegative random variable with unknown true distribution. However, compared to the approach in Example 3.6 of selecting the ambiguity radius \( \delta \), we follow this time a data-driven approach to obtain \( \delta \). It consists of estimating \( \delta \) either by evaluating the discrepancy between the empirical model \( \hat{P}_n \) and the calibrated model, or using measure concentration inequalities to target a certain confidence level \( \beta \in (0, 1) \), i.e.,

\[
P[d(P^*, \hat{P}_n) \leq \delta] \geq 1 - \beta \quad \text{(see Esfahani and Kuhn [35, Theorem 3.4 and the discussion afterwards], Blanchet et al. [17, Sect. 5.1]).}
\]

We investigate the former method in Example 4.5 where the ambiguity set is constructed using the Wasserstein metric.

**Example 4.5** In this example, the DM’s ambiguity about the realisations of \( X \) is characterised by the ambiguity set \( \mathcal{C}_\delta \) given by

\[
\mathcal{C}_\delta^{\mathcal{W}_1} := \{ P \in ca_1^+(\Sigma) : \mathcal{W}_1(P \circ X^{-1}, Q \circ X^{-1}) \leq \delta \},
\]
where \( \mathcal{W}_1 \) is the Wasserstein distance on \( \mathbb{R} \), with the \( L_1 \)-norm being the underlying metric (e.g. Vallender [67]), so that

\[
\mathcal{W}_1(P \circ X^{-1}, Q \circ X^{-1}) := \int_{\mathbb{R}} |F_{X,P}(x) - F_{X,Q}(x)| \, dx = \int_0^1 |F_{X,P}^{-1}(t) - F_{X,Q}^{-1}(t)| \, dt.
\]

The Wasserstein distance is a metric (e.g. Villani [69, Ex 6.3]). This induces a metric on \( C^1_\delta \) such that \( C^1_\delta \) is convex and weak*-compact (see [69, Theorem 6.8]). See Villani [69, Chap. 6] for further properties of the Wasserstein distance. We define \( d_1(P, Q) := \mathcal{W}_1(P \circ X^{-1}, Q \circ X^{-1}) \).

Let \( \hat{Q} \) be the insurer’s belief regarding the loss \( X \). In this example, we assume that \( F_{X,Q} \) is a truncated generalised Pareto distribution with an upper bound \( M \) and shape and scale parameters 0.3 and 5, respectively. Next, we simulate from the distribution \( F_{X,Q} \) and obtain the empirical distribution. We then construct a piecewise linear approximation \( F_{X,\hat{Q}} \) of this empirical distribution with given nodes \( x_1, \ldots, x_n \), where the partition \( 0 = x_1 < \cdots < x_n = M \) is chosen arbitrarily, but kept fixed throughout. That is, \( F_{X,\hat{Q}} \) is given by a system \( \{[x_i, x_{i+1}], [F_{X,\hat{Q}}(x_i), F_{X,\hat{Q}}(x_{i+1})]\}_{i=1}^{n-1} \). Note that by construction, \( F_{X,\hat{Q}}(x_n) = 1 \). The corresponding density \( \hat{q} = (\hat{q}_1, \ldots, \hat{q}_{n-1})^\top \) is piecewise constant on each interval \([x_i, x_{i+1}]\) for \( i = 1, \ldots, n - 1 \). More precisely, \( \hat{q}_i = \frac{F_{X,\hat{Q}}(x_{i+1}) - F_{X,\hat{Q}}(x_i)}{x_{i+1} - x_i} \) is the slope of the line passing through the points \((x_i, F_{X,\hat{Q}}(x_i))\) and \((x_{i+1}, F_{X,\hat{Q}}(x_{i+1}))\) for \( i = 1, \ldots, n - 1 \). This representation of \( \hat{Q} \) allows us to compute the Wasserstein distance between \( F_{X,P} \) and \( F_{X,\hat{Q}} \) and thus to characterise the alternative distributions via the values \( F_{X,P}(x_1), \ldots, F_{X,P}(x_n) \), for the same segments \([x_i, x_{i+1}]\), \( i = 1, \ldots, n - 1 \). For two such distributions, the Wasserstein distance is the sum of the areas of the trapezoids with corners given by \( F_{X,\hat{Q}}(x_i), F_{X,\hat{Q}}(x_{i+1}), F_{X,P}(x_i), F_{X,P}(x_{i+1}) \) formed by \( F_{X,P} \) and \( F_{X,\hat{Q}} \) (see the shaded area in Fig. 5), i.e.,

\[
d_1(P, \hat{Q}) = \frac{1}{2} \sum_{i=1}^{n-1} (x_{i+1} - x_i) \phi(F_{X,P}(x_i) - F_{X,\hat{Q}}(x_i), F_{X,P}(x_{i+1}) - F_{X,\hat{Q}}(x_{i+1})),
\]

where the function \( \phi : [-1, 1]^2 \to \mathbb{R}_+ \) defined by

\[
\phi(a, b) = \begin{cases} |a| + |b| & \text{if } ab \geq 0, \\ \frac{a^2 + b^2}{|a| + |b|} & \text{otherwise}, \end{cases}
\]

is convex in each component (e.g. Pflug et al. [58]).

All the alternative measures \( P \) are characterised by \((n - 1)\)-dimensional vectors \( p = (p_1, \ldots, p_{n-1})^\top \), where \( p_i \in [0, 1] \) is the constant forming the piecewise constant density of \( F_{X,P} \). More precisely, the alternative CDF \( F_{X,P} \) is linear on each interval \([x_i, x_{i+1}]\) and differs from \( F_{X,\hat{Q}} \) only in the cumulative probabilities \( F_{X,P}(x_i) \). Thus \( p_i \) is the slope of the line passing through the points \((x_i, F_{X,P}(x_i))\) and \((x_{i+1}, F_{X,P}(x_{i+1}))\). The representation of \( F_{X,P} \) is shown in Fig. 5. Therefore, the variable \( p \) must satisfy \( p^\top A_{n-1} x = 1 \), where \( A_{n-1} \in \mathbb{R}^{(n - 1) \times \tilde{n}} \) is defined in (3.17). Using the matrix \( A_i \) for \( i = 1, \ldots, n - 1 \), \( F_{X,P} \) can also be represented via \( F_{X,P}(x_i) = p^\top A_i x \).
Next, to identify the optimal $\hat{Y}^*$, we follow the equivalent formulation of Problem (4.1) and describe the decision variable in terms of the retention function $\hat{R}$. First, note that since the feasible set of indemnity functions is assumed to be $F = \hat{I}$, it follows that $\hat{R} = \hat{r} \circ X$, where $\hat{r}$ is a $1$-Lipschitz and hence continuous function. For numerical tractability, we further assume that $\hat{r}$ is linear between the segments $[x_i, x_{i+1}]$ and thus piecewise linear and of the form

$$\hat{r}(x) = a_i x + b_i$$

for $x_i \leq x \leq x_{i+1}$, $i = 1, \ldots, n - 1$. Since $\hat{r} \in \hat{I}$, it follows that $a_i \in [0, 1]$ and $b_i \in [-x_i, 0]$ for $i = 1, \ldots, n - 1$.

The error introduced by solving (4.1) in terms of $\hat{Q}$ instead of $Q$ can be used to estimate the ambiguity radius $\delta$. The estimator $\delta = \delta_n$ depends on the number of piecewise linear segments and is thus informed by the data. In particular, we propose to approximate $\delta_n$ as

$$\delta_n := |\mathbb{E}_{\hat{Q}}[X] - \mathbb{E}_Q[X]| \leq d_1(\hat{Q}, Q).$$

The inequality becomes an equality if $F_{X, \hat{Q}}$ dominates $F_{X, Q}$ in the first stochastic order.

With the above representations for $p$ and $\hat{r}$, Problem (4.1) is approximated by

$$\left\{ \begin{array}{ll}
\max_{a \in [0, 1]^n} & \min_{b \in [-x, 0]} \sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1}} u(W_0 - x + a_i x + b_i - \Pi_0) dx p_i \\
\text{such that} & 0 \leq a_i x_i + b_i \leq x_i, \ 0 \leq \sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1}} (x - a_i x + b_i) dx q_i \leq \frac{\Pi_0}{1 + \rho} \\
& d_1(P, \hat{Q}) \leq \delta_n, \ F_{X, \hat{Q}}(x_i) = p^\top A_i x, \ i = 1, \ldots, n, \\
p^\top A_{n-1} x = 1,
\end{array} \right. \quad (4.5)$$

with $d_1(P, \hat{Q})$ computed as in (4.4) and $a = (a_1, \ldots, a_{n-1})^\top$, $b = (b_1, \ldots, b_{n-1})^\top$. The first two constraints in (4.5) specify that the retention function $\hat{r}$ is continuous and linear between the segments $[x_i, x_{i+1}]$, $i = 1, \ldots, n - 1$. The objective function in (4.5) is concave in $a$ and $b$ and linear in $p$, while the constraints are convex in $p$ and linear in $a$ and $b$. Similarly to Example 3.6, (4.5) is solved in a step-wise manner by splitting the initial problem into an inner and outer problem.

For the implementation, we resume the input for (4.5): the DM’s initial wealth is $W_0 = 250$ and the utility is $u(x) = (1 - \exp(-\gamma x))/\gamma$, for $\gamma = 0.03$, while the pre-
Fig. 6 (Left) The optimal indemnity $\hat{y}^* = x - \hat{r}^*$ as a function of $x$; (Right) The DM’s optimal distribution $F_{X,p^*}$ (red) compared to the insurer’s belief $F_{X,\hat{Q}}$ (black). Here, for the sake of presentation, we only display values of $x$ below 50.

Fig. 7 (Left) Optimal retention functions $r^*$ and $\hat{r}^*$ for Problem (4.5) when the feasibility sets are $\mathcal{I}$ and $\hat{\mathcal{I}}$, respectively, with a zoomed-in perspective for small values of the underlying loss $x$; (Right) The corresponding indemnities $y^* = x - r^*$ and $\hat{y}^* = x - \hat{r}^*$.

The premium is $\Pi_0 = 4$ and the safety loading is $\rho = 0.2$. For $n = 200$, we simulate from the distribution $F_{X,\hat{Q}}$ and construct the piecewise linear approximation $F_{X,\hat{Q}}$ of the empirical distribution on the partition $0 = x_1 < x_2 < \cdots < x_n = M = W_0 - \Pi_0 = 246$. The CDF $F_{X,\hat{Q}}$ will play the role of the baseline distribution in (4.5). Finally, the ambiguity radius $\delta_n$ is estimated to be approximately 0.3. Figure 6 shows one of the saddle points of Problem (4.5): the corresponding $F_{X,p^*}$ dominates $F_{X,\hat{Q}}$ in the first stochastic order.

Next, we study a problem related to (4.5), in which the retention function $r$ is required only to be bounded by $x$, i.e., $r \in \mathcal{I}$, where $\mathcal{F} = \mathcal{I}$ as in (2.2). This implies that $a \in \mathbb{R}^n$ in (4.5). In Fig. 7 (left panel), we display the difference between the optimal retention functions of Problem (4.5) for the sets $\mathcal{I}$ and $\hat{\mathcal{I}}$, respectively, and we also display a zoomed-in perspective for small values of $x$. Figure 7 (right panel) provides the corresponding indemnities $y^*$ and $\hat{y}^*$, respectively. In the absence of the no-sabotage condition (the blue lines in Fig. 7), the indemnity $y^*$ can be decreasing with respect to the loss $x$ on some parts of its domain (see the blue line in Fig. 7 (left panel)).
5 Conclusion

The impact of ambiguity on insurance markets in general, and insurance contracting in particular, is by now well documented. One of the most popular and intuitive ways to model the sensitivity of preferences to ambiguity is the maxmin expected utility (MEU) model of Gilboa and Schmeidler [42]. Nonetheless, to the best of our knowledge, none of the theoretical studies of risk sharing in insurance markets in the presence of ambiguity have examined the case in which the decision maker (DM) is an MEU maximiser. The present paper fills this void. Specifically, we extend the classical setup and results in two ways: (i) the DM is endowed with MEU preferences; and (ii) the insurer is an expected-utility maximiser who is not necessarily risk-neutral (that is, the premium principle is not necessarily an expected-value premium principle). The main objective of this paper is to determine the shape of the optimal insurance indemnity in that case.

We characterise optimal indemnity functions both with and without the customary ex ante no-sabotage requirement on feasible indemnities, and for both concave and linear utility functions for the two agents. The no-sabotage condition is shown to play a key role in determining the shape of optimal indemnity functions. An equally important factor in characterising optimal indemnities is the singularity in beliefs between the two agents. We subsequently examine several illustrative examples, and we provide numerical studies for the case of a Wasserstein and a Rényi ambiguity set. Specifically, we provide a successive convex programming algorithm to compute optimal insurance indemnities in a discretised framework. The Wasserstein and Rényi distances are two popular metrics to construct probability ambiguity sets. We show in numerical examples that a larger ambiguity set yields a lower certainty equivalent of final wealth, but increases the willingness to pay for insurance. As a by-product of our analysis, we provide a comprehensive and unifying treatment of optimal insurance design under subjective expected utility theory in the presence of belief heterogeneity, thereby extending many results in the related literature.

An interesting direction for future research would be to give the policyholder the possibility of partially hedging her loss exposure through some hedging instrument that would act as an uninsurable background risk. Specifically, the policyholder initially faces an insurable loss random variable $X$ and is able to partially hedge her loss exposure through another random variable $Z$. For a given hedging investment decision, the latter can be interpreted as a background risk which might be correlated with $X$. Optimal insurance design in the presence of a background risk has been examined in expected-utility theory by Dana and Scarsini [30] (without the no-sabotage condition) and Chi [28] (with the no-sabotage condition).

Appendix: Convexity and compactness of the ambiguity set in Example 4.4

Lemma A.1 For a fixed $Q \in ca_1^+(\Sigma)$, let $C_W$ be the set defined as

$$C_W := \left\{ P \in ca_1^+(\Sigma) : \frac{dP}{dQ} = \frac{w(X)}{\int w(X)dQ}, w \in W \right\},$$
where \( \mathcal{W} \subseteq L^1(\mathbb{R}, B(\mathbb{R}), Q \circ X^{-1}) \) is a collection of nonnegative increasing weight functions such that \( \int w(X) dQ > 0 \) for all \( w \in \mathcal{W} \). Then the following hold:

(i) If \( \mathcal{W} \) is a convex cone, then \( \mathcal{C}_\mathcal{W} \) is convex.

(ii) If \( \mathcal{C}_\mathcal{W} \) is uniformly absolutely continuous with respect to some \( \mu \in ca^+(\Sigma) \), then \( \mathcal{C}_\mathcal{W} \) is weak*-compact.

**Proof** (i) is easy to verify. To show (ii), first note that \( \mathcal{C}_\mathcal{W} \) is norm-bounded. Since \( \mathcal{C}_\mathcal{W} \) is also uniformly absolutely continuous with respect to \( \mu \in ca^+(\Sigma) \), it follows from Dunford [33, Theorem IV.9.2] that \( \mathcal{C}_\mathcal{W} \) is weakly sequentially compact and hence weak*-compact by Maccheroni and Marinacci [54, Theorem 1]. \( \square \)

**Remark A.2** In Lemma A.1, if \( \mathcal{C}_\mathcal{W} \) is countable, that is, of the form

\[
\left\{ P_n \in ca^+_i(\Sigma) : n \in \mathbb{N}, \frac{dP_n}{dQ} = \frac{w_n(X)}{\int w_n(X) dQ}, w_n \in \mathcal{W} \right\},
\]

and if \( \lim_{n \to \infty} P_n[A] \) exists for each \( A \in \Sigma \), then the requirement of uniform absolute continuity of \( \mathcal{C}_\mathcal{W} \) is superfluous by the Vitali–Hahn–Saks theorem (Dunford [33, Theorem III.7.2]).

**Proposition A.3** If \( \mathcal{W} \) is order bounded in the Banach lattice \( L^1(\mathbb{R}, B(\mathbb{R}), Q \circ X^{-1}) \) with a constant upper bound and a nonnegative lower bound having nonzero \( L^1 \)-norm, then \( \mathcal{C}_\mathcal{W} \) is uniformly absolutely continuous with respect to \( Q \).

**Proof** If \( \mathcal{W} \) is order bounded in \( L^1(\mathbb{R}, B(\mathbb{R}), Q \circ X^{-1}) \) with a constant upper bound and a nonnegative lower bound having nonzero \( L^1 \)-norm, then there exist \( M \in \mathbb{R}_+ \) and \( f \in L^1_+(\mathbb{R}, B(\mathbb{R}), Q \circ X^{-1}) \) such that \( \| f \|_1 = \int f d(Q \circ X^{-1}) > 0 \) and \( f \leq w \leq M \) for each \( w \in \mathcal{W} \). Consequently, for each \( P \in \mathcal{C}_\mathcal{W} \),

\[
\frac{dP}{dQ} \leq \frac{M}{\| f \|_1} < \infty.
\]

Hence for each \( P \in \mathcal{C}_\mathcal{W} \) and each \( A \in \Sigma \),

\[
P[A] \leq \frac{M}{\| f \|_1} Q[A].
\]

Consequently, for each \( \varepsilon > 0 \), letting \( \delta := \frac{\| f \|_1}{M} \varepsilon > 0 \), it follows that for each \( A \in \Sigma \) and each \( P \in \mathcal{C}_\mathcal{W} \),

\[
Q[A] < \delta \text{ implies that } P[A] < \frac{M}{\| f \|_1} \delta = \varepsilon.
\]

Hence \( \mathcal{C}_\mathcal{W} \) is uniformly absolutely continuous with respect to \( Q \). \( \square \)

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Declarations

Competing Interests  The authors declare that they have no conflict of interest.

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