LATTICES IN A PRODUCT OF TREES, HIERARCHICALLY
HYPERBOLIC GROUPS, AND VIRTUAL TORSION-FREENESS

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Abstract. We construct cocompact lattices in a product of trees which are not virtually
torsion-free. This gives the first examples of hierarchically hyperbolic groups which are
not virtually torsion-free.

1. INTRODUCTION

Hierarchically hyperbolic groups (HHGs) and spaces (HHSs) were introduced by Behr-
stock, Hagen and Sisto in [BHS17a]. Hierarchically hyperbolic groups are known to
have a number of properties such as having finite asymptotic dimension [BHS17b, The-
orem A], having a uniform bound on the conjugator length of Morse elements [AB19],
and for virtually torsion-free HHGs, their uniform exponential growth is well understood
[ANS19]. HHGs belong to the class of semihyperbolic groups [HHP22, Corollary F] (see
also [DMS20]). In particular, their finitely generated abelian subgroups are undistorted,
they have solvable conjugacy problem, finitely many conjugacy classes of finite subgroups,
and are of type $FP_{\infty}$.

That HHGs have only many finitely many conjugacy classes of finite subgroups implies
that every residually finite HHG is in fact virtually torsion-free. This motivates the
question of whether there exist any HHGs which are not virtually torsion-free. The
question is of considerable interest to specialists since, for example, a number of theorems
about HHGs require the assumption of virtual torsion-freeness (see for instance [ANS19,
Theorem 1.1] and [RS20, Theorem 1.2(3')]).

In this paper we construct an infinite family of $CAT(0)$ lattices acting faithfully and
geometrically on a product of trees. We then prove that each lattice $\Gamma$ is a hierarchically
hyperbolic group and has no finite index torsion-free subgroups. This appears to be the
first examples in the literature of cocompact lattices in a product of trees which are not
virtually torsion-free (non-cocompact examples were given by Caprace and Rémy [CR09]).

Theorem A (Theorem 4.3). There exist uniform lattices in products of trees which are
hierarchically hyperbolic groups and which are not virtually torsion-free.
To the author’s knowledge this is the first explicit example of an HHG which is not virtually torsion-free. The author suspects that it is possible to apply the results of Hagen–Susse [HS20] to Wise’s examples in [Wis07] to obtain an HHG which is not virtually torsion-free; however, the construction presented here is much more elementary and gives an explicit HHG structure.

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2. Definitions

In this section we will give the relevant background on HHSs and HHGs for our endeavours. The definitions are rather technical so we will only focus on what we need, for a full account the reader should consult [BHS19, Definition 1.1, 1.21]. We will follow the treatment in [PS22, Section 2]. To this end, a hierarchically hyperbolic space (HHS) is pair \((X, \mathcal{G})\) where \(X\) is an \(\epsilon\)-quasigeodesic space and \(\mathcal{G}\) is a set with some extra data which essentially functions as a coordinate system on \(X\) where each coordinate entry is a hyperbolic space. The relevant parts of the axiomatic formalisation are described as follows:

- For each domain \(U \in \mathcal{G}\), there is a hyperbolic space \(CU\) and projection \(\pi_U : X \to CU\) that is coarsely Lipschitz and coarsely onto [BHS19, Remark 1.3].
- \(\mathcal{G}\) has a partial order \(\sqsubseteq\), called nesting. Nesting chains are uniformly finite, and the length of the longest such chain is called the complexity of \((X, \mathcal{G})\).
- \(\mathcal{G}\) has a symmetric relation \(\perp\), called orthogonality. The complexity bounds pairwise orthogonal sets of domains.
- The relations \(\sqsubseteq\) and \(\perp\) are mutually exclusive. The complement of \(\sqsubseteq\) and \(\perp\) is called transversality and denoted \(\not\sqsubseteq\).
- Whenever \(U \not\sqsubseteq V\) or \(U \sqsubseteq V\) there is a bounded set \(\rho_U^V \subset CV\). These sets, and projections of elements \(x \in X\), are consistent in the following sense:
  - \(\rho\)-consistency: Let \(U, V, W \in \mathcal{G}\) such that \(U \sqsupseteq V\) and \(\rho_W^V\) is defined, then \(\rho_W^V\) coarsely agrees with \(\rho_W^V\);
hierarchically hyperbolic group structure (HHG)

Definition 1.1, 1.21]. The following definition of an HHG is however complete. We recall the structure we will need. For the full definition the reader should consult [BHS19, Remark 1.6]. All coarseness may taken to be uniform so we can and will fix a uniform constant $\epsilon$ [BHS19, Remark 1.6].

We remind the reader that these axioms for an HHS are not a complete set but only require the equivariance to be exact rather than coarse and so imply the original axioms. However, by [DHS20, Section 2.1] if the axioms given in [BHS19] are satisfied then one can modify the HHG structure to satisfy the axioms given here. Note that this is not the original definition of an HHG as given in [BHS19]. Instead, we have adopted the simpler axioms from [PS22]. Specifically, the axioms we have given on a product of isometric factors is an HHG. Let $\Gamma$ be a discrete subgroup, where each $\Gamma$ is a hierarchically hyperbolic group.

3. Hierarchical hyperbolicity and products

In this section we provide a proof of the folklore result that a group acting geometrically on a product of $\delta$-hyperbolic spaces with equivariant projections and without permuting isometric factors is an HHG. Let $X$ be a proper metric space and let $H = \text{Isom}(X)$, then $H$ is a locally compact group with the topology given by uniform convergence on compacta. Let $\Gamma$ be a discrete subgroup of $H$. We say $\Gamma$ is a uniform lattice if $X/\Gamma$ is compact.

**Proposition 3.1.** Let $m > 0$, $n \geq 0$ and let $H \leq \text{Isom}(\mathbb{E}^n) \times \prod_{i=1}^{m} \text{Isom}(X_i)$ be a closed subgroup, where each $X_i$ is a proper non-elementary $\delta$-hyperbolic space. Let $\Gamma$ be a uniform $H$-lattice. Suppose the projection $\pi_{O(n)} : \Gamma \to O(n) < \text{Isom}(\mathbb{E}^n)$ is trivial, then $\Gamma$ is a hierarchically hyperbolic group.

**Proof.** Let $g$ be a $\Gamma$-equivariant quasi-isometry $\text{Cay}(\Gamma, A) \to X$ given by the Švarc-Milnor Lemma [BH99, p. 1.8.19]. If $n > 0$, then for $j \in \{1 - n, \ldots, 0\}$ let $X_j = \mathbb{E}$ and $H_j = \text{Isom}(\mathbb{E})$. If $n > 0$, then let $i \in \{1 - n, \ldots, m\}$, otherwise let $i \in \{1, \ldots, m\}$. Let $\mathcal{S}$ be the HHS structure for the product $X = \prod_{i=1}^{m} X_i$ given by [BHS19, Proposition 8.27]. As explained in the proof thereof every domain of $\mathcal{S}$ is either some $X_i$ for $i \in \{1 - n, \ldots, m\}$ or bounded (in fact a point) and labelled by $I \subseteq \{1 - n, \ldots, m\}$ corresponding to some non-trivial subproduct of $X$ with at least two factors. The transversality relation is given by pairs $\{J, K\}$ of subsets of $I$ with $|J|, |K| \geq 2$, and $J \cap K \neq \emptyset$. The nesting relation
is given by inclusions of subproducts of $X$, and every distinctly labelled pair of domains which are not nested are orthogonal.

Note that $\mathcal{G}$ is finite and the action on $\mathcal{G}$ is trivial because $\Gamma$ does not permute isometric factors of $X$. Indeed, $H$ which contains $\Gamma$ preserves the decomposition of $X$ and $\pi_{O(n)}(\Gamma)$ is trivial. Every domain of the structure is a point or one of the $X_i$. In the first case the $\Gamma$ action is trivial and in the second case $\Gamma$ acts via $\pi_H : \Gamma \to \text{Isom}(X_i)$. This immediately yields the first axiom because $\pi_H$ is a homomorphism. The other $\rho$-consistency equivariance condition is established immediately since any two domains that are not points are orthogonal to each other.

For the second axiom consider the following diagram where the vertical arrows are given by applying the obvious group action:

$$
\begin{array}{ccc}
\Gamma \times \text{Cay}(\Gamma, A) & \xrightarrow{(\pi_H, \pi_{X_i} \circ q)} & \pi_H(\Gamma) \times X_i \\
\downarrow & & \downarrow \\
\text{Cay}(\Gamma, A) & \xrightarrow{\pi_{X_i} \circ q} & X_i.
\end{array}
$$

We will verify the diagram commutes. Let $x \in \text{Cay}(\Gamma, A)$ and $g \in \Gamma$. First, we evaluate the composite map going down then across, we have

$$(g, x) \mapsto gx \mapsto \pi_{X_i}(q(gx)).$$

Going the other way we have

$$(g, x) \mapsto (\pi_H(g), \pi_{X_i}(q(x))) \mapsto \pi_{X_i}(ggx) = \pi_{X_i}(q(gx))$$

where the last equality is given by the $\Gamma$-equivariance of $q$. In particular, $g\pi_{X_i}(x) = \pi_{X_i}(gx)$.

**Lemma 3.2.** If $\Gamma$ is a finite-by-(hierarchically hyperbolic group), then $\Gamma$ is a hierarchically hyperbolic group.

**Proof.** The group $\Gamma$ splits as a short exact sequence

$$(1) \quad \{1\} \to F \to \Gamma \to \Lambda \to \{1\},$$

where $\Lambda$ is a hierarchically hyperbolic group and $F$ is the finite kernel of the action on the HHS $(\Lambda, \mathcal{G})$. Since $F$ acts trivially on $X$, it acts trivially on the HHG structure $\mathcal{G}$ for $\Lambda$. The epimorphism $\varphi : \Gamma \to \Lambda$ induces an equivariant quasi-isometry $\psi$ on the associated Cayley graphs. Thus, we may precompose every map in the HHG structure with $\varphi$ or $\psi$ to endow $\Gamma$ with the structure of an HHG. \qed

We restate Proposition 3.1 in terms of groups acting geometrically on products of $\text{CAT}(-1)$ spaces. For an introduction to $\text{CAT}(\kappa)$ groups and spaces see [BH99]. We will assume some non-degeneracy conditions on the $\text{CAT}(0)$ spaces to avoid many technical difficulties associated with the $\text{CAT}(0)$ condition (see [CM09, Section 1.B] for a thorough
A group $H$ acting on a CAT(0) space $X$ is minimal if there is no $H$-invariant closed convex subset $X' \subset X$. If $\text{Isom}(X)$ is minimal, then we say $X$ is minimal.

**Corollary 3.3.** Let $\Gamma$ be a group acting properly cocompactly by isometries on a finite product of proper minimal CAT$(-1)$-spaces. If $\Gamma$ does not permute isometric factors, then $\Gamma$ is a hierarchically hyperbolic group.

**Proof.** The group $\Gamma$ splits as in (1) where $\Lambda$ acts geometrically on a finite product of proper minimal CAT($-1$)-spaces and $F$ is the finite kernel of the action. By Proposition 3.1 we see that $\Lambda$ is an HHG and so by Lemma 3.2 $\Gamma$ is an HHG as well. \qed 

The author suspects it is possible to strengthen the corollary to allow for permuting isometric factors provided the projection of $\Gamma$ to $O(p,q) \subset \text{Isom}(\mathbb{E}^n)$ is contained in $O(p)$. To prove a converse to this corollary one may need to investigate the commensurators of maximal abelian subgroups of a hierarchically hyperbolic group $\Gamma$. Indeed, the CAT(0) not biautomatic groups introduced by Leary–Minasyan [LM21] and the groups constructed by the author in [Hug21b] (see also [Hug22] and [Hug21a]) have undistorted maximal abelian subgroups which have infinite index in their commensurator and are not virtually normal. All of these groups have a non-discrete projection to $O(n)$.

**Question 3.4.** Is a maximal abelian subgroup $A$ of a hierarchically hyperbolic group $\Gamma$ either finite index in its commensurator $\text{Comm}_\Gamma(A)$ or virtually normal?

## 4. Non-virtually torsion-free lattices

In this section we will construct a cocompact lattice in a product of trees which is not virtually torsion-free.

Let $\Lambda$ be a Burger-Mozes simple group [BM97; BM00a; BM00b] acting on $T_1 \times T_2$ splitting as an amalgamated free product $F_n \ast_{F_m} F_n$ with embeddings $i, j : F_m \to F_n$. This defines a group $\Lambda$ which embeds discretely into the product of $T_1 = \text{Aut}(T_1)$ and $T_2 = \text{Aut}(T_2)$ with compact quotient. For instance one may take Rattaggi’s example of a lattice in the product of an 8-regular and 12-regular tree which splits as $F_7 \ast_{F_{73}} F_7$ [Rat07b] (see also [Rat07a]) or one of Radu’s examples [Rad20].

Define $A = \mathbb{Z}_p \rtimes F_n$ for $p$ prime such that the $F_n$-action is non-trivial. Consider the embeddings $\bar{i}, \bar{j} : F_m \hookrightarrow F_n \hookrightarrow A$ given by the composition of $i$ or $j$ with the obvious inclusion. Now, we build a group $\Gamma$ as an amalgamated free product $A \ast_{F_m} A$, note that $\Gamma$ surjects onto the original Burger-Mozes group $\Lambda$ with kernel the normal closure of the torsion elements. Let $T_3$ denote the Bass-Serre tree of $\Gamma$ and let $T_3$ denote the corresponding automorphism group.

**Proposition 4.1.** $\Gamma$ is a uniform $(T_1 \times T_3)$-lattice which does not permute the factors.

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1 The key point here is that $F_n$ will normally generate $A$, in particular, other finite groups with non-trivial $F_n$-action could be used here.
This can be easily deduced by endowing $\Gamma$ with a graph of lattices structure in the sense of [Hug21b, Definition 3.2] and then applying [Hug21b, Theorem A]. Instead we will provide a direct proof.

**Proof.** The group $\Gamma$ acts on its Bass-Serre tree $T_3$ and also on $T_1$ via the homomorphism $\psi: \Gamma \to T_1$ defined by taking the composition of the surjection $\Gamma \to \Lambda$ and the projection $T_1 \times T_2 \to T_2$. The diagonal action on the product space $T_1 \times T_3$ is properly discontinuous cocompact and by isometries. Indeed, the action is clearly cocompact since $T_1 \{ \Gamma \}$ is a finite graph and $T_3 \{ \Gamma \}$ is a finite graph by construction. The action is properly discontinuous since by construction the only elements which fix a point in $T_1 \times T_3$ are finite order and every torsion subgroup is finite. The kernel of the action is trivial, since the only elements which could act trivially are the torsion elements. However, these all clearly act non-trivially on $T_3$ by elementary Bass-Serre theory. Thus, the action is faithful. We conclude that $\Gamma$ is a uniform $(T_1 \times T_3)$-lattice. □

It remains to show $\Gamma$ is not virtually torsion-free.

**Proposition 4.2.** The group $\Gamma$ has no proper finite-index subgroups and contains torsion. In particular, $\Gamma$ is not virtually torsion-free.

The author thanks Yves de Cornulier for the following argument.

**Proof.** Note that since $F_n$ acts non-trivially on $\mathbb{Z}_p$ it follows that $F_n$ normally generates $A$. Because the Burger–Mozes subgroup $\Lambda < \Gamma$ is a simple group, every finite index normal subgroup of $\Gamma$ contains it. Thus, their intersection $\Gamma^{(x)} = \bigcap_{[\Gamma: \Gamma']} < \infty \Gamma'$ contains $\Lambda$. It follows, both copies of $F_n$ are contained in $\Gamma^{(x)}$. Now, $F_n$ normally generates $A$, so $\Gamma^{(x)} = \Gamma$. In particular, $\Gamma$ has no proper finite-index subgroups. Since, $A$ is not torsion-free, we conclude that $\Gamma$ is not virtually torsion-free. □

To summarise we have the following theorem.

**Theorem 4.3 (Theorem A).** The group $\Gamma$ is a cocompact lattice in a product of trees, is a hierarchically hyperbolic group, and is not virtually torsion-free.

**Proof.** By Proposition 4.1 and Corollary 3.3 we see that $\Gamma$ is a hierarchically hyperbolic group. By Proposition 4.2 we see $\Gamma$ is not virtually torsion-free. □

**Remark 4.4.** In [Hug21b, Corollary 8.7] the author gave a way to use A. Thomas’s construction in [Tho06] to promote lattices in products of trees to lattices in products of “sufficiently symmetric” right-angled buildings. Applying [Hug21b, Corollary 8.7] to one of the non-virtually torsion-free lattices $\Gamma$ we obtain a non-virtually torsion-free lattice $\Lambda$ acting on a product of “sufficiently symmetric” right-angled hyperbolic buildings each not quasi-isometric to a tree. Moreover, by Corollary 3.3 $\Lambda$ is hierarchically hyperbolic.
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