We study a massive cosmic strings with BII symmetries cosmological models in two contexts. The first of them is the standard one with a barotropic equation of state. In the second one we explore the possibility of taking into account variable “constants” ($G$ and $\Lambda$). Both models are studied under the self-similar hypothesis. We put special emphasis in calculating the numerical values for the equations of state. We find that for $\omega \in (0, 1]$, $G$, is a growing time function while $\Lambda$, behaves as positive decreasing time function. If $\omega = 0$, both “constants”, $G$ and $\Lambda$, behave as true constants.

I. INTRODUCTION

The exponential expansion of the Universe (inflationary era) causes the Universe to heat up to a very high temperature so the subsequent evolution of the Universe is exactly as in hot BB model. Hence, the phase transition (as the temperature falls below some critical temperature) in the early universe causes topologically stable defects: vacuum domain walls, strings and monopoles (see [1] and [2]). But domain walls and monopoles are disastrous for the cosmological models. Strings, on the other hand, causes no harm, but can lead to very interesting astrophysical consequences (see [3]). Also the existence of a large scale network of strings the early universe does not contradict the present-day observations. The vacuum strings may generate density fluctuations sufficient to explain the galaxy formation (see [4]).

The relativistic treatment of strings was initiated by Letelier (see [5]-[9]). Here we have considered gravitational effects, arisen from strings by coupling of stress energy of strings to the gravitational field. Letelier (see [3]) defined the massive strings as the geometric strings (massless) with particles attached along its expansions.

The strings that form the cloud are massive strings instead of geometrical strings. Each massive string is formed by a geometrical string with particles attached along its extension. Hence, the string that form the cloud are the generalization of Takabayasi’s relativistic model of strings (called $p-$strings). This is simplest model wherein we have particles and strings together. In principle we can eliminate the strings and end up with a cloud of particles. This a desirable property of a model of a string cloud to be used in cosmology since strings are not observed at the present time of evolution of the universe (see [5]-[6]).

In modern cosmological theories, the cosmological constant remains a focal point of interest (see [11]-[14] for reviews of the problem). A wide range of observations now compellingly suggest that the universe possesses a non-zero cosmological constant. Some of the recent discussions on the cosmological constant “problem” and on cosmology with a time-varying cosmological constant point out that in the absence of any interaction with matter or radiation, the cosmological constant remains a “constant”. However, in the presence of interactions with matter or radiation, a solution of Einstein equations and the assumed equation of covariant conservation of stress-energy with a time-varying $\Lambda$ can be found. This entails that energy has to be conserved by a decrease in the energy density of the vacuum component followed by a corresponding increase in the energy density of matter or radiation. Recent observations strongly favour a significant and a positive value of $\Lambda$ with magnitude $\Lambda(G\hbar/c^3) \approx 10^{-123}$. These observations suggest on accelerating expansion of the universe, $q < 0$.

Our current understanding of the physical universe is anchored on the analysis of expanding, isotropic and homogeneous models with a cosmological constant, and linear perturbations thereof. This model successfully accounts for the late time universe, as is evidenced by the observation of large scale cosmic microwave background observations. Parameter determination from the analysis of CMB fluctuations appears to confirm this picture. However, further analyses seem to suggests some inconsistency. In particular, it appears that the universe could have a preferred direction. Followup analyses of various sets of WMAP data sets, with different techniques seem to lead to the same conclusion. It is still unclear whether or not the directional preference is intrinsic to the underlying model, and what implications this has on our understanding of cosmology. For this reason Bianchi models are important in the study of anisotropies.

The study of SS models is quite important since a large class of orthogonal spatially homogeneous models are asymptotically self-similar at the initial singularity and are approximated by exact perfect fluid or vacuum self-similar power law models. Exact self-similar power-law models can also approximate general Bianchi models at intermediate stages of their evolution. This last point is of particular importance in relating Bianchi models to the real Universe. At the same time, self-similar solutions can describe the behaviour of Bianchi models at late times i.e. as $t \to \infty$ (see [13]). A particular interest is in determining the exact value of the equations of state (see [20]). The geometry and physics at different points on an integral curve of a homothetic vector field (HVF) differ only by a change in the overall length scale and in particular any dimensionless scalar will be constant along the integral curves. In this sense the existence of a
HVF is a weaker condition than the existence of a KVF since the geometry and physics are completely unchanged along the integral curves of a Killing vector field (KVF). However, the existence of a non-trivial HVF leads to restrictions on the equations of state. Therefore the paper is organized as follows. In section II we outline the model as well as the self-similar solution. In section III we shall study a massive cosmic string model with barotropic equation of state under the self-similar hypothesis. We put special emphasis in calculating the values of the equation of state which made that the solution follows a power law. In section IV we shall study a massive cosmic string model with barotropic equation of state under the self-similar hypothesis but allowing that “constants” $G$ and $\Lambda$ may vary. Section V is devoted to study the curvature behaviour of the model taking into account the obtained solutions in the above sections. We end summarizing the main results.

II. THE MODEL.

In synchronous co-ordinates the metric is:

$$ds^2 = -c^2 dt^2 + a^2(t) dx^2 + (b^2(t) + K^2 z^2 a^2(t)) dy^2 + 2Ka^2(t)z dx dy + d^2(t)dz^2,$$

where the metric functions $a(t), b(t), d(t)$ are functions of the time co-ordinate only and $K \in \mathbb{R}$. The introduction of this constant is essential since if we set $K = 1$, as it is the usual way, then there is not SS solution for the outlined field equations. In this paper we are interested only in Bianchi II space-times, hence all metric functions are assumed to be different and the dimension of the group of isometries acting on the spacelike hypersurfaces is three.

Once we have defined the metric and we know which are its Killing vectors, then we calculate the four velocity. It must verify, $\mathcal{L}_\xi u_i = 0$, so we may define the four velocity as follows:

$$u^i = \left(\frac{1}{c}, 0, 0, 0\right),$$

in such a way that it is verified, $g(u^i, u^i) = -1$.

From the definition of the 4-velocity we find that:

$$\theta = u^i, \quad \frac{1}{c} \left(\frac{a'}{a} + \frac{b'}{b} + \frac{d'}{d}\right) = \frac{1}{c} H,$$

$$H = \sum_{i=1}^{3} H_i,$$

$$q = \frac{1}{c} \left(\frac{d}{dt} \left(\frac{1}{H}\right)\right) - 1,$$

$$\sigma^2 = \frac{1}{3c^2} \left(H_1^2 + H_2^2 + H_3^2 - H_1H_2 - H_1H_3 - H_2H_3\right).$$

From equation $\mathcal{L}_V g_{ij} = 2g_{ij}$, where $V \in \mathfrak{X}(M)$, we find the following homothetic vector field for the BH metric

1:

$$V = t\partial_t + \left(1 - t\frac{a'}{a}\right) x\partial_x + \left(1 - t\frac{b'}{b}\right) y\partial_y + \left(1 - t\frac{d'}{d}\right) z\partial_z,$$

with the following constrains for the scale factors:

$$a(t) = a_0 t^{a_1}, \quad b(t) = b_0 t^{a_2}, \quad d(t) = d_0 t^{a_3},$$

with $a_1, a_2, a_3 \in \mathbb{R}$, in such a way, that the constants $a_i$ must verify the following restriction (see [22]):

$$a_2 + a_3 - a_1 = 1.$$  

Taking into account the field equations (FE)

$$R_{ij} - \frac{1}{2} R g_{ij} = \frac{8\pi G}{c^4} T_{ij} - \Lambda g_{ij},$$

where $\Lambda$ is the cosmological constant, and the energy-momentum tensor, $T_{ij}$, for a cloud of massive strings is given by

$$T^j_i = (\rho + p) u^i u_j + p g^j_i - \lambda x^j x_i,$$

where $\rho(t)$ is the rest energy density, i.e. is the rest energy density of the cloud of strings with particles attached to them ($p$-strings), $\lambda(t)$ is the string tension density, which may be positive or negative, $u^i$ is the four-velocity for the cloud particles. $x^i$ is the four-vector which represents the strings direction which is the direction of anisotropy and $\rho = \rho_p + \lambda$, where $\rho_p$ denotes the particle energy density (is the cloud rest energy density), i.e. the string tension density is connected to the rest energy $\rho$ for a cloud of strings ($p$-strings) with particle attached to them by this relation (see [8] and [10]).

Since there is no direct evidence of strings in the present-day universe, we are in general, interested in constructing models of a Universe that evolves purely from the era dominated by either geometric string or massive strings and ends up in a particle dominated era with or without remnants of strings.

Moreover the direction of strings satisfy the standard relations:

$$u^i u_i = -x^j x_j = -1,$$

$$u^i x_i = 0,$$

$$x^i = \left(0, a^{-1}, 0, 0\right).$$

The Raychaudhuri equation reads

$$\theta' = -\frac{1}{3} \theta^2 - 2\sigma^2 + R_{ij} u^i u^j,$$

with

$$R_{ij} u^i u^j = -\frac{1}{2} \rho_p.$$  

It is customary to assume a relation between $\rho$ and $\lambda$ in accordance with the state equation for strings. The
III. BAROTROPIC EQUATION OF STATE

We shall consider the Takabayasi’s equation of state, i.e. $\rho = \alpha \lambda$, with $\alpha = 1 + W$, so the resulting FE are:

$$
\frac{a'}{a} + \frac{b'}{b} - \frac{K^2 a c^2}{4 b^2 d^2} = \frac{8 \pi G}{c^2} \rho, \quad (12)
$$

$$
\frac{b''}{b} + \frac{a'}{a} - \frac{3 K^2 a c^2}{4 b^2 d^2} = -\frac{8 \pi G}{c^2} (\omega \rho - \lambda), \quad (13)
$$

$$
\frac{d''}{d} + \frac{a'}{a} + \frac{a' d'}{d} = -\frac{K^2 a c^2}{4 b^2 d^2} = -\frac{8 \pi G}{c^2} \omega \rho, \quad (14)
$$

$$
\rho' + \rho (\omega + 1) \left( \frac{a'}{a} + \frac{b'}{b} + \frac{d'}{d} \right) - \lambda \frac{a'}{a} = 0. \quad (16)
$$

noting that $b = d$. So once again the model collapses to a LRS BII model (see [23]) where

$$
\rho = \rho_0 t^{\alpha_1}, \quad b(t) = b_0 t^{\alpha_2} = d(t),
$$

and the constrain $a_2 + a_3 - a_1 = 1$, reads now $2a_2 - a_1 = 1$. Therefore the FE yield:

$$
\frac{2 a' b' b^2}{a b} + \frac{b^2}{b} - \frac{K^2 c^2 a^2}{4 b^4} = \frac{8 \pi G}{c^2} \rho, \quad (17)
$$

$$
\frac{b''}{b} + \frac{a'}{a} - \frac{3 K^2 c^2 a^2}{4 b^4} = -\frac{8 \pi G}{c^2} (\omega \rho - \lambda), \quad (18)
$$

$$
\rho' + \rho \left( \frac{a'}{a} + \frac{2 b'}{b} \right) - \lambda \frac{a'}{a} = 0, \quad (20)
$$

In this way, following the standard procedure (see [23]) we find that

$$
\rho = \rho_0 t^{-\beta}, \quad \beta = (4a_2 - 1) (\omega + 1) - \frac{(2a_2 - 1)}{1 + W},
$$

and hence we need only to solve the system of algebraic equations

$$
3a_2^2 - 2a_2 - \frac{3K^2}{4} = -A \left( \omega - \frac{1}{1 + W} \right), \quad (21)
$$

$$
7a_2^2 - 8a_2 + 2 + \frac{K^2}{4} = -\omega A, \quad (22)
$$

$$
(4a_2 - 1) (\omega + 1) - \frac{(2a_2 - 1)}{1 + W} = 2, \quad (23)
$$

with

$$
A = 5a_2^2 - 2a_2 - \frac{K^2}{4}, \quad \rho_0 = \frac{A e^2}{8 \pi G}, \quad \lambda = \frac{\rho}{1 + W}.
$$

By solving the system (21-23) we obtain the following results:

$$
a_2 = 2 + \omega (1 + W) + 3W \left( 2 + 2W + 2\omega (1 + W) \right), \quad (24)
$$

and

$$
K = \sqrt{\left( W - 2 \right)^2 + 2 \omega (W^2 + 2W + 2) - 3 \omega^2 (1 + W)^2 - 8} / \left( 1 + 2W + 2\omega (1 + W) \right). \quad (25)
$$

We may list the physical solutions for different values of $\omega$:

1. If we set $\omega = 0$, then we get:

$$
K = \sqrt{W^2 - 4 (W + 1) \over 1 + 2W}, \quad a_2 = \frac{2 + 3W}{2 (1 + 2W)},
$$

and the following restriction: $W > 2 + 2\sqrt{2}$. Therefore $K \in (0,0.5)$, while $a_2 \in (0.75,0.77)$ and $a_1 \in (0.5,0.54)$.

2. If $\omega = 1/3$,

$$
K = \sqrt{-27 + 24W + 12W^2} / 5 + 8W \implies K \in (0,0.43301),
$$

with $W > 1 + \sqrt{10} = 2.8028$, therefore the scale factor $a_2$ behaves as:

$$
a_2 = 7 + 10W / (5 + 8W) \implies a_2 \in (0.625,0.638),
$$

therefore $a_1 \in (0.25,0.277)$.

In the following table we summarize the behavior of the main quantities for this model
and the energy density behaves as:
\[
\rho = \rho_0 t^{-\beta}, \quad \beta = (4a_2 - 1)(\omega + 1) - \frac{(2a_2 - 1)}{1 + W} \approx 2.
\]

Therefore we are only able to obtain solutions for \(\omega = 1/3\) and \(\omega = 0\). For the rest of equations of state the obtained solution is unphysical since in all these cases we have obtained \(W < 0\), in order to get \(K \in \mathbb{R}\). But we have ruled out the equation of state \(W < 0\).

With the obtained results we can see that
\[
H = \frac{4a_2 - 1}{t}, \quad q = \frac{2 - 4a_2}{4a_2 - 1} < 0,
\]

noting that \(2 - 4a_2 < 0\), \(\forall \omega \in [0, 1/3]\), while the shear behaves as:
\[
\sigma^2 = \frac{(a_2 - 1)^2}{3c^2t^2} \to 0.
\]

As it is observed, these quantities fit perfectly with the current observations obtained by High-Z Supernova Team and Supernova Cosmological Project (see for example [15], [16], [17] and [18]).

### IV. VARIABLE CONSTANTS.

In this model the resulting FE are:

\[
\begin{align*}
\frac{a'}{a} + \frac{d}{d} \frac{b'}{b} + \frac{d'}{d} \frac{b'}{b} - \frac{K^2 c^2 a^2 c^2}{4 b^2 d^2} &= \frac{8\pi G}{c^2} \rho + \Lambda c^2, \\
\frac{b''}{b} + \frac{d''}{d} \frac{b'}{b} - \frac{3K^2 c^2 a^2 c^2}{4 b^2 d^2} &= -\frac{8\pi G}{c^2} (\omega \rho - \lambda) + \Lambda c^2,
\end{align*}
\]

\[
\begin{align*}
\frac{d''}{d} + \frac{d''}{d} \frac{a'}{a} + \frac{d'}{d} \frac{a'}{a} + \frac{K^2 c^2 a^2 c^2}{4 b^2 d^2} &= -\frac{8\pi G}{c^2} \omega \rho + \Lambda c^2, \\
\frac{b''}{b} + \frac{d''}{d} \frac{b'}{b} + \frac{d'}{d} \frac{b'}{b} &= -\frac{8\pi G}{c^2} \omega \rho + \Lambda c^2,
\end{align*}
\]

\[
\rho' + \rho(\omega + 1) \left( \frac{a'}{a} + \frac{b'}{b} + \frac{d'}{d} \right) = \lambda \frac{a'}{a},
\]

\[
\Lambda' = -\frac{8\pi G'}{c^4} \rho.
\]

noting that \(b = d\). So once again the model collapses to a LRS BII model (see [23]), where
\[
a(t) = a_0 t^{a_1}, \quad b(t) = b_0 t^{a_2} = d(t),
\]

and the constrain \(a_2 + a_3 - a_1 = 1\), reads now \(2a_2 - a_1 = 1\), as above. Therefore the FE yield:

\[
\begin{align*}
\frac{2a'}{a} + \frac{b'}{b} &= \frac{K^2 c^2 a^2}{4 b^2} - \frac{8\pi G}{c^2} \rho + \Lambda c^2, \\
\frac{b'}{b} + \frac{c'}{c} &= -\frac{8\pi G}{c^2} (\omega \rho - \lambda) + \Lambda c^2,
\end{align*}
\]

\[
\rho' + \rho \left( \frac{a'}{a} + \frac{b'}{b} \right) - \lambda \frac{a'}{a} = 0,
\]

\[
\Lambda' = -\frac{8\pi G'}{c^4} \rho.
\]

with
\[
\rho = \alpha \lambda, \quad \alpha = 1 + W.
\]

From eqs. (35) and (37) we get
\[
\rho = \rho_0 t^{-\gamma}, \quad \gamma = (4a_2 - 1)(\omega + 1) - \frac{(2a_2 - 1)}{1 + W},
\]

where \(\gamma = (\omega + 1) - \frac{2a_2 - 1}{1 + W}\) and \(a = (a_1 + 2a_2), \ 2a_2 - a_1 = 1\), and we shall assume that \(\rho_0 > 0\).

From eq. (32) we obtain:
\[
\Lambda = \frac{1}{c^2} \left[ A t^{-\gamma} - \frac{8\pi G}{c^2} \rho_0 t^{-\gamma} \right],
\]

where \(A = 2a_1 a_2 + a_2^2 - \frac{K^2}{4}\).

Now, taking into account eq. (36) and eq. (39), algebra brings us to obtain
\[
G = G_0 t^{-\gamma}, \quad G_0 = \frac{c^2 A}{4\pi \rho_0 \gamma},
\]

and therefore the cosmological “constant” behaves as:
\[
\Lambda = \frac{A}{c^2} \left( 1 - \frac{2}{\gamma} \right) t^{-\gamma} = \Lambda_0 t^{-2}.
\]
With all these results, we find that the system to solve is the following:

\[ 3a_2^2 - 2a_2 - \frac{3K^2}{4} = -\frac{2A}{\gamma} \left( \omega + 1 - \frac{\gamma}{2} - \frac{1}{1+W} \right), \]

\[ 7a_2^2 - 8a_2 + 2 + \frac{K^2}{4} = -\frac{2A}{\gamma} \left( \omega + 1 - \frac{\gamma}{2} \right), \]

and whose solutions may be listed as follows:

1. If \( \omega = 0 \),

\[ a_1 = 2a_2 - 1, \]

and

\[ K = -\frac{\sqrt{W(W^2 - 4(W + 1))}}{W (1 + 2W)} \in \left( -\frac{1}{2}, 0 \right), \]

therefore this solution has only sense if \( W \geq 2 + 2\sqrt{2} \approx 4.8284 \). In this way we find that

\[ a_2 = \frac{2 + 3W}{2(1 + 2W)} \in (0.75, 0.77), \]

and the behaviour for the rest quantities is:

\[ \rho \approx t^2, \quad \gamma = \left(4a_2 - 1\right) - \frac{\left(2a_2 - 1\right)}{1+W} \approx 2, \]

\[ G \approx t^{-2}, \quad G_0 = \frac{c^2 A \frac{1}{4\pi \rho_0 \gamma}}{\gamma} = \text{const.} > 0, \]

\[ \Lambda \approx t^{-2}, \quad \Lambda_0 = \frac{A}{c^2} \left(1 - \frac{2}{\gamma}\right) = 0, \]

with \( A = a_2 \left(5a_2 - 2\right) - \frac{K^2}{4} > 0 \). Note that this solution coincides with the above one where the constants behaves as true constants and \( \Lambda \) vanishes.

2. If \( \omega = 1 \)

\[ K = -\frac{\sqrt{-27W^2 - 22W - 7 - 12W^3 + 4W^4}}{(-1 + W)(3 + 4W)}, \]

so \( K \in (-\frac{1}{2}, 0) \) with \( W > 4.7 \), and

\[ a_2 = \frac{-1 + 3W + 6W^2}{2(-5W - 3 + 4W^2)} \in (0.75, 0.90), \]

hence \( a_1 \in (0.5, 0.80) \), with \( A = a_2 \left(5a_2 - 2\right) - \frac{K^2}{4} \approx 1.25 > 0 \), and therefore

\[ \rho \approx t^7, \quad \gamma = 2(4a_2 - 1) - \frac{(2a_2 - 1)}{1+W} \in (4, 5.06), \]

\[ G \approx t^{-2}, \quad G_0 = \frac{c^2 A \frac{1}{4\pi \rho_0 \gamma}}{\gamma} > 0, \]

\[ \Lambda \approx t^{-2}, \quad \Lambda_0 = \frac{A}{c^2} \left(1 - \frac{2}{\gamma}\right) \in \left(\frac{1}{2}, \frac{3}{5}\right). \]

3. If \( \omega = 1/3 \), we get the following solution:

\[ K = -\frac{\sqrt{-519W^2 - 78W - 15 - 456W^3 + 144W^4}}{(-1 + W)(5 + 8W)}, \]

so \( K \in (-\frac{1}{2}, 0) \), with \( W > 4.1 \), and

\[ a_2 = \frac{-1 + 23W + 36W^2}{2(7W - 5 + 24W^2)} \in (0.75, 0.81), \]

hence \( a_1 \in (0.5, 0.63) \), therefore

\[ \rho \approx t^7, \quad \gamma = \frac{4}{3} \left(4a_2 - 1\right) - \frac{(2a_2 - 1)}{1+W} \in (2.66, 2.86), \]

\[ G \approx t^{-2}, \quad G_0 = \frac{c^2 A \frac{1}{4\pi \rho_0 \gamma}}{\gamma} > 0, \]

\[ \Lambda \approx t^{-2}, \quad \Lambda_0 = \frac{A}{c^2} \left(1 - \frac{2}{\gamma}\right) > 0. \]

4. If \( \omega = -1/3 \), then

\[ K = -\frac{\sqrt{-243W^2 - 6W + 9 - 228W^3 + 36W^4}}{(1 + 3W)(1 + 4W)}, \]

so \( K \in (-\frac{1}{2}, 0) \) with \( W > 6.6 \), and

\[ a_2 = \frac{-1 + 11W + 18W^2}{2(7W + 1 + 12W^2)} \in (0.75, 0.7507), \]

hence \( a_1 \in (0.5, 0.501) \), therefore

\[ \rho \approx t^7, \quad \gamma = \frac{2}{3} \left(4a_2 - 1\right) - \frac{(2a_2 - 1)}{1+W} \in (1.26, 1.33), \]

\[ G \approx t^{-2}, \quad G_0 = \frac{c^2 A \frac{1}{4\pi \rho_0 \gamma}}{\gamma} > 0, \]

\[ \Lambda \approx t^{-2}, \quad \Lambda_0 = \frac{A}{c^2} \left(1 - \frac{2}{\gamma}\right) < 0. \]

5. If \( \omega = -2/3 \),

\[ K = -\frac{\sqrt{-132W^2 - 48W + 48 - 132W^3 + 9W^4}}{(2 + 3W)(-1 + 2W)}, \]

in such a way that \( K \in (-\frac{1}{2}, 0) \) with \( W > 15.2 \), and

\[ a_2 = \frac{-4 + 2W + 9W^2}{2(W - 2 + 6W^2)} \in (0.75, 0.752), \]

hence \( a_1 \in (0.5, 0.504) \), therefore

\[ \rho \approx t^7, \quad \gamma = \frac{1}{3} \left(4a_2 - 1\right) - \frac{(2a_2 - 1)}{1+W} \in (0.63, 0.66), \]

\[ G \approx t^{-2}, \quad G_0 = \frac{c^2 A \frac{1}{4\pi \rho_0 \gamma}}{\gamma} > 0, \]

\[ \Lambda \approx t^{-2}, \quad \Lambda_0 = \frac{A}{c^2} \left(1 - \frac{2}{\gamma}\right) < 0. \]

In the following table we have summarized the behavior of all the quantities.
Therefore, as we can see, and comparing this table with the above one obtained in the last section we may conclude that to allow that the “constants” may vary enlarge the set of solutions. In this case we have obtained a solution for equation of state $\omega \in (-1, 1)$ while in the standard solution we only have a solution for $\omega = 0$, and $\omega = 1/3$. In the same way it is interesting to emphasize that for $\omega > 0$ i.e. $\omega \in (0, 1]$, the “constant”, $G$, is a growing time function while the cosmological “constant”, $\Lambda$, behaves as positive decreasing time function (see for example [24]–[27]).

In fact we have obtained the same solution as in the above section. If $\omega < 0$ i.e. $\omega \in (-1, 0)$, the “constant”, $G$, is a decreasing time function while the cosmological “constant”, $\Lambda$, behaves as negative decreasing time function, so, from the observational data we may rule out these solutions.

With the obtained results we can see that

$$H = \frac{4a_2 - 1}{t}, \quad q = 2 - 4a_2 < 0,$$

note that $2 - 4a_2 < 0$, and $4a_2 - 1 > 0$, $\forall \omega \in [0, 1]$, while the shear behaves as:

$$\sigma^2 = \frac{(a_2 - 1)^2}{3c^{-2}t^2} \rightarrow 0.$$

As it is observed, these quantities fit perfectly with the current observations obtained by High-Z Supernova Team and Supernova Cosmological Project (see for example [13], [16], [17] and [18]).

V. CURVATURE BEHAVIOUR.

With all these results, we find the following behaviour for the curvature invariants (see for example [24]–[27]).

Ricci Scalar, $I_0$, yields

$$I_0 = \frac{2}{c^4 t^2} \left( 11a_2^2 - 10a_2 + 2 - \frac{K^2 c^2}{4} \right),$$

while Krestchmann scalar, $I_1 := R_{ijkl} R^{ijkl}$, yields:

$$I_1 = \frac{1}{4c^4 t^4} \left[ 432a_2^4 - 960a_2^3 + 896a_2^2 - 384a_2 + 64 + 11K^2 c^2 \left( 11K^2 c^2 - 40a_2^2 + 80a_2 - 48 \right) \right].$$

The full contraction of the Ricci tensor, $I_2 := R_{ij} R^{ij}$, is:

$$I_2 = \frac{1}{4c^4 t^4} \left( 528a_2^4 - 1024a_2^3 + 768a_2^2 - 256a_2 + 32 \right) + K^2 c^2 \left( 3K^2 c^2 - 16a_2^2 + 8 \right),$$

this means that the model is singular.

The non-zero components of the Weyl tensor. The following components of the Weyl tensor run to $\pm \infty$ when $t \rightarrow 0$,

$$C_{txtx} \approx C_{ttxy} \approx t^{4(a_2 - 1)}, \quad C_{txty} \approx t^{2(a_2 - 1)},$$

and

$$C_{tyty} \approx t^{2(a_2 - 1)} + z^2 t^{4(a_2 - 1)},$$

these others run to zero when $t \rightarrow 0$,

$$C_{xxyy} \approx C_{zxxx} \approx C_{xyyz} \approx t^{2(3a_2 - 2)},$$

$$C_{yzyy} \approx t^{2(2a_2 - 1)} + z^2 t^{2(3a_2 - 2)}.$$
where
\[ W^2 = \frac{1}{36c^4t^4} \left( (a_2 - 1)^2 + K^2c^2 \right)^2 \left( (2a_2 - 2)^2 + K^2c^2 \right)^2, \]

note that the value of \( W^2 \) is really small. We may calculate the quantity
\[ \mathcal{W}^2 = \frac{\left( (a_2 - 1)^2 + K^2c^2 \right)^2 \left( (2a_2 - 2)^2 + K^2c^2 \right)^2}{36c^4(4a_2 - 1)^4} \approx const. \]

but as it is observed \( \mathcal{W} \rightarrow 0 \). \( \mathcal{W} \) can be regarded as describing the intrinsic anisotropy in the gravitational field. Hence we may conclude that the model is close to being isotropic. (see [29]) since we have that it is verified the Weyl isotropization criterion i.e. \( \mathcal{W} \rightarrow 0 \).

The gravitational entropy (see [25])
\[ P^2 = \frac{I_3}{I_2} = \frac{I_1 - 2I_2 + \frac{1}{3}I_3^2}{I_2} = \frac{I_1}{I_2} + \frac{1}{3}I_3^2 - 2 = const., \]

since we are working with a SS solution (see [30] and [31] for a discussion), and all the dimensionless quantities remain constant along timelike homothetic trajectories as the Weyl parameter.

VI. CONCLUSIONS

We have studied two massive cosmic string Bianchi type II cosmological models under the self-similar hypothesis. In the first of the studied models (those with “constant” constants) we have obtained that the self-similar solution is only valid if the equation of state is \( \omega = 0 \) and \( \omega = 1/3 \). For the rest of possible values of \( \omega \) the solution is unphysical. In the same way we have shown that such solutions are only valid if \( W \) (the parameter in the equation of state for the strings) is \( W \geq 2.8 \) (if \( \omega = 1/3 \), and \( W \geq 4.8 \) (if \( \omega = 0 \)). So may conclude that the self-similar solution is quite restrictive. Furthermore the obtained solution collapses to a LRS BII solution since two of the scale factors are equal.

In the second of the studied models (those with “varying” constants) we have shown that considering a time varying constants we may enlarge the set of solutions. In this case we have obtained a mathematical solution for \( \omega \in (-1,1] \), and with similar restrictions for \( W \). In the same way it is interesting to emphasize that for \( \omega > 0 \) i.e. \( \omega \in (0,1] \), the “constant”, \( G \), is a growing time function while the cosmological “constant”, \( \Lambda \), behaves as positive decreasing time function. This fact fits perfectly with the current observational data. For \( \omega = 0 \), we have that both “constants”, \( G \) and \( \Lambda \), behave as true constants, in fact we have obtained the same solution as in the first of the studied models. If \( \omega < 0 \) i.e. \( \omega \in (-1,0] \), the “constant”, \( G \), is a decreasing time function while the cosmological “constant”, \( \Lambda \), behaves as negative decreasing time function, so, from the observational data we may rule out these solutions.

[1] Ya. B. Zel’dovich, et al., Sov. Phys.-JETP 40, 1, (1975).
[2] A. Vilenkin, Phys. Rev. D24, 2082, (1981). A. Vilenkin, Phys. Rep. 121, 263, (1985).
[3] T. W. S. Kibble, J. Phys. A9, 1387, (1976)
[4] Ya. B. Zeldovich, MNRAS 192, 663, (1980).
[5] P. S. Letelier, Phys. Rev. D20, 1924, (1979)
[6] P. S. Letelier, Phys. Rev. D28, 2414, (1983)
[7] J. Stachel, Phys. Rev. D21, 2171, (1980)
[8] A. Banerjee, et al Pramana 34.1, (1990).
[9] M.K. Yadav et al. arXiv:061032v2.
[10] B.Saha and M. Visinescu. arXiv:0805.2413v1. B. Saha et al arXiv:0812.1443v1.
[11] V. Sahni and A. Starobinsky, Int. J. Mod. Phys. D9, 373 (2000).
[12] P. J. E. Peebles, Rev. Mod. Phys. 75, 559 (2003).
[13] T. Padmanabhan, Phys. Rep. 380, 235 (2003).
[14] T. Padmanabhan, gr-qc/0705.2533 (2007).
[15] Garnavich, P.M. et al., Astrophys. J. 493, L53,(1998); Ibid Astrophys. J. 509, 74, (1998).
[16] Perlmutter, S. et al.; Astrophys. J. 483, 565, (1997). Perlmutter, S. et al: Nature 391, 51,(1998). Perlmutter, S. et al: Astrophys. J. 517, 565,(1999)
[17] Riess, A.G. et al.; Astron. J. 116, 1009,(1998) astro-ph/9805201.
[18] Schmidt, B. P. et al.; Astrophys. J. 507, 46, (1998) astro-ph/9805200.
[19] A.A. Coley. “Dynamical Systems and Cosmology”. Kluwer Academic Publishers (2003).
[20] J. Wainwright, “Self-Similar Solutions of Einstein’s Equations”. Published in Galaxies, Axisymmetric Systems & Relativity, ed M.A.H MacCallum CUP (1985).
[21] J. Wainwright, Gen. Rel. Grav. 16, 657 (1984)
[22] K. Rosquist and R. Jantzen, Class. Quantum Grav., 2, L129, (1985). K. Rosquist and R. Jantzen, “Transitively Self-Similarity-Space-Time”, Proc. Marcel Grossmann Meeting on General Relativity. Ed Ruffini Elsevier S.P. (1986). pg 1033.
[23] J.A. Belinchón, gr-qc/9901057.
[24] J. Caminati and R.G. Mcleanagh. J. Math. Phys. 32, 3135, (1991).
[25] O. Rudjord and Ø. Grøn, gr-qc/0607064.
[26] Ø. Grøn and S. Hervik. gr-qc/0205026.
[27] J.D. Barrow and S. Hervik. Class. Quant. Grav. 19, 5173, (2002).
[28] W.C. Lim, A.A. Coley, S. Hervik. Class. Quant. Grav. 24, 595, (2006).
[29] J. Wainwright et al Class. Quant. Grav. 16, 2577, (2004).
[30] N. Pelavas and K. Lake. Phys. REv. D62, 044009, (2000).
[31] J.A. Belinchón IJMP A 23, 5021, (2008).