Lower Bounds on the Distance Domination Number of a Graph

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Abstract

For an integer $k \geq 1$, a (distance) $k$-dominating set of a connected graph $G$ is a set $S$ of vertices of $G$ such that every vertex of $V(G) \setminus S$ is at distance at most $k$ from some vertex of $S$. The $k$-domination number, $\gamma_k(G)$, of $G$ is the minimum cardinality of a $k$-dominating set of $G$. In this paper, we establish lower bounds on the $k$-domination number of a graph in terms of its diameter, radius and girth. We prove that for connected graphs $G$ and $H$, $\gamma_k(G \times H) \geq \gamma_k(G) + \gamma_k(H) - 1$, where $G \times H$ denotes the direct product of $G$ and $H$.

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1 Introduction

Distance in graphs is a fundamental concept in graph theory. Let $G$ be a connected graph. The distance between two vertices $u$ and $v$ in $G$, denoted $d_G(u, v)$, is the length (i.e., the number of edges) of a shortest $(u, v)$-path in $G$. The eccentricity $\text{ecc}_G(v)$ of $v$ in $G$ is the distance between $v$ and a vertex farthest from $v$ in $G$. The minimum eccentricity among all vertices of $G$ is the radius of $G$, denoted by $\text{rad}(G)$, while the maximum eccentricity among all vertices of $G$ is the diameter of $G$, denoted by $\text{diam}(G)$. Thus, the diameter of $G$ is the maximum distance among all pairs of vertices of $G$. A vertex $v$ with $\text{ecc}_G(v) = \text{diam}(G)$ is called a peripheral vertex of $G$. A diametral path in $G$ is a shortest path in $G$ whose length is equal to the diameter of the graph. Thus, a diametral path is a path of length $\text{diam}(G)$ joining two peripheral vertices of $G$. If $S$ is a set of vertices in $G$, then the distance, $d_G(v, S)$, from a vertex $v$ to the set $S$ is the minimum distance from $v$ to a vertex of $S$; that is, $d_G(v, S) = \min\{d_G(u, v) \mid u \in S\}$. In particular, if $v \in S$, then $d(v, S) = 0$.

Domination in graphs is also very well studied in graph theory. A dominating set in a graph $G$ is a set $S$ of vertices of $G$ such that every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in $S$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. The literature on the subject of domination parameters in graphs up to the year 1997 has been surveyed and detailed in the two books [7, 8].

In this paper, we continue the study of distance domination in graphs which combines the concepts of both distance and domination in graphs. Let $k \geq 1$ be an integer and let $G$ be a graph. In 1975, Meir and Moon [15] introduced the concept of a distance $k$-dominating set (called a “$k$-covering” in [15]) in a graph. A set $S$ is a $k$-dominating set of $G$ if every vertex is within distance $k$ from some vertex of $S$; that is, for every vertex $v$ of $G$, we have $d(v, S) \leq k$. The $k$-domination number of $G$, denoted $\gamma_k(G)$, is the minimum cardinality of a $k$-dominating set of $G$. When $k = 1$, the $1$-domination number of $G$ is precisely the domination number of $G$; that is, $\gamma_1(G) = \gamma(G)$. The literature on the subject of distance domination in graphs up to the year 1997 can be found in the book chapter [9]. Distance domination is now widely studied, see, for example, [2, 4-6, 10-11, 14-15, 17, 18, 19].

Definitions and Notation. For notation and graph theory terminology, we in general follow [12]. Specifically, let $G$ be a graph with vertex set $V(G)$ of order $n(G) = |V(G)|$ and edge set $E(G)$ of size $m(G) = |E(G)|$. We assume throughout the paper that all graphs considered are simple graphs, i.e., finite graphs with no directed edges and no loops. A non-trivial graph is a graph on at least two vertices. A neighbor of a vertex $v$ in $G$ is a vertex adjacent to $v$. The open neighborhood of $v$, denoted $N_G(v)$, is the set of all neighbors of $v$ in $G$, while the closed neighborhood of $v$ is the set $N_G[v] = N_G(v) \cup \{v\}$. The closed $k$-neighborhood, denoted $N_k[v]$, of $v$ is defined in [4] as the set of all vertices within distance $k$ from $v$ in $G$; that is, $N_k[v] = \{u \mid d(u, v) \leq k\}$. When $k = 1$, the set
The degree of a vertex \( v \) in \( G \), denoted \( d_G(v) \), is the number of neighbors, \( |N_G(v)| \), of \( v \) in \( G \). The minimum and maximum degree among all the vertices of \( G \) are denoted by \( \delta(G) \) and \( \Delta(G) \), respectively. The subgraph induced by a set \( S \) of vertices of \( G \) is denoted by \( G[S] \). The girth of \( G \), denoted \( g(G) \), is the length of a shortest cycle in \( G \). For sets of vertices \( X \) and \( Y \) of \( G \), the set \( X \) \( k \)-dominates the set \( Y \) if every vertex of \( Y \) is within distance \( k \) from some vertex of \( X \). In particular, if \( X \) \( k \)-dominates the set \( V(G) \), then \( X \) is a \( k \)-dominating set of \( G \).

If the graph \( G \) is clear from context, we simply write \( V, E, d(v), ecc(v), N_G(v) \) and \( N_G[v] \) rather than \( V(G), E(G), d_G(v), ecc_G(v), N_G(v) \) and \( N_G[v] \), respectively. We use the standard notation \([n] = \{1, 2, \ldots, n\}\).

**Known Results.** The \( k \)-domination number of \( G \) is in the class of \( NP \)-hard graph invariants to compute [8]. Because of the computational complexity of computing \( \gamma_k(G) \), graph theorists have sought upper and lower bounds on \( \gamma_k(G) \) in terms of simple graph parameters like order, size, and degree.

In 1975, Meir and Moon [15] established an upper bound for the \( k \)-domination number of a tree in terms of its order. They proved that for \( k \geq 1 \), if \( T \) is a tree of order \( n \geq k+1 \), then \( \gamma_k(T) \leq n/(k+1) \). As a consequence of this result and Observation [5] if \( G \) is a connected graph of order \( n \geq k+1 \), then \( \gamma_k(G) \leq \frac{n}{k+1} \). A short proof of the Meir-Moon upper bound can also be found in [11] (see, also, Proposition 24 and Corollary 12.5 in the book chapter [9]). A complete characterization of the graphs \( G \) achieving equality in this upper bound was obtained by Topp and Volkmann [19]. Tian and Xu [18] improved the Meir-Moon upper bound and showed that for \( k \geq 1 \), if \( G \) is a connected graph of order \( n \geq k+1 \) with maximum degree \( \Delta \), then \( \gamma_k(G) \leq \frac{1}{k} (n - \Delta + k - 1) \). The Tian-Xu bound was further improved by Henning and Lichiardopol [10] who showed that for \( k \geq 2 \), if \( G \) is a connected graph with minimum degree \( \delta \geq 2 \) and maximum degree \( \Delta \) and of order \( n \geq \Delta + k - 1 \), then \( \gamma_k(G) \leq \frac{n+\delta-\Delta}{\delta+k-1} \).

We recall the following well-known lower bound on the domination number of a graph in terms of its diameter.

**Theorem 1** ([8]) If \( G \) is a connected graph with diameter \( d \), then \( \gamma(G) \geq \frac{d+1}{2} \).

The following two results were originally conjectured by the conjecture making program Graffiti.pc (see [1]).

**Theorem 2** ([3]) If \( G \) is a connected graph with radius \( r \), then \( \gamma(G) \geq \frac{2r}{r} \).

**Theorem 3** ([3]) If \( G \) is a connected graph with girth \( g \geq 3 \), then \( \gamma(G) \geq \frac{g}{3} \).
Our Results. In this paper, we establish lower bounds for the $k$-domination number of a graph in terms of its diameter (Theorem 7), radius (Corollary 10), and girth (Theorem 11). These results generalize the results of Theorem 11 Theorem 2 and Theorem 3. A key tool in order to prove our results is the important lemma (Lemma 5) that every connected graph has a spanning tree with equal $k$-domination number. We also prove a key property (Lemma 6) of shortest cycles in a graph that enables us to establish our girth result for the $k$-domination number of a graph. We show that our bounds are all sharp and examples are provided following the proofs.

2 Preliminary Observations and Lemmas

Since every $k$-dominating set of a spanning subgraph of a graph $G$ is a $k$-dominating set of $G$, we have the following observation.

Observation 4 For $k \geq 1$, if $H$ is a spanning subgraph of a graph $G$, then $\gamma_k(G) \leq \gamma_k(H)$.

We shall also need the following lemma.

Lemma 5 For $k \geq 1$, every connected graph $G$ has a spanning tree $T$ such that $\gamma_k(T) = \gamma_k(G)$.

Proof. Let $S = \{v_1, \ldots, v_\ell\}$ be a minimum $k$-dominating set of $G$. Thus, $|S| = \ell = \gamma_k(G)$. We now partition the vertex set $V(G)$ into $\ell$ sets $V_1, \ldots, V_\ell$ as follows. Initially, we let $V_i = \{v_i\}$ for all $i \in [\ell]$. We then consider sequentially the vertices not in $S$. For each vertex $v \in V(G) \setminus S$, we select a vertex $v_i \in S$ at minimum distance from $v$ in $G$ and add the vertex $v$ to the set $V_i$. We note that if $v \in V(G) \setminus S$ and $v \in V_i$ for some $i \in [\ell]$, then $d_G(v, v_i) = d_G(v, S)$, although the vertex $v_i$ is not necessarily the unique vertex of $S$ at minimum distance from $v$ in $G$. Further, since $S$ is a $k$-dominating set of $G$, we note that $d_G(v, v_i) \leq k$. For each $i \in [\ell]$, let $T_i$ be a spanning tree of $G[V_i]$ that is distance preserving from the vertex $v_i$; that is, $V(T_i) = V_i$ and for every vertex $v \in V(T_i)$, we have $d_{T_i}(v, v_i) = d_G(v, v_i)$. We now let $T$ be the spanning tree of $G$ obtained from the disjoint union of the $\ell$ trees $T_1, \ldots, T_\ell$ by adding $\ell - 1$ edges of $G$. We remark that these added $\ell - 1$ edges exist as $G$ is connected. We now consider an arbitrary vertex, $v$ say, of $G$. The vertex $v \in V_i$ for some $i \in [\ell]$. Thus, $d_T(v, v_i) = d_{T_i}(v, v_i) = d_G(v, v_i) \leq k$. Therefore, the set $S$ is a $k$-dominating set of $T$, and so $\gamma_k(T) \leq |S| = \gamma_k(G)$. However, by Observation 5 $\gamma_k(G) \leq \gamma_k(T)$. Consequently, $\gamma_k(T) = \gamma_k(G)$. □

Lemma 6 Let $G$ be a connected graph that contains a cycle, and let $C$ be a shortest cycle in $G$. If $v$ is a vertex of $G$ outside $C$ that $k$-dominates at least $2k$ vertices of $C$, then
then there exist two vertices $u, w \in V(C)$ that are both $k$-dominated by $v$ and such that a shortest $(u, v)$-path does not contain $w$ and a shortest $(v, w)$-path does not contain $u$.

Proof. Since $v$ is not on $C$, it has a distance of at least 1 to every vertex of $C$. Let $u$ be a vertex of $C$ at minimum distance from $v$ in $G$. Let $Q$ be the set of vertices on $C$ that are $k$-dominated by $v$ in $G$. Thus, $Q \subset V(C)$ and, by assumption, $|Q| \geq 2k$. Among all vertices in $Q$, let $w \in Q$ be chosen to have maximum distance from $u$ on the cycle $C$. Since there are $2k - 1$ vertices within distance $k - 1$ from $u$ on $C$, the vertex $w$ has distance at least $k$ from $u$ on the cycle $C$. Let $P_u$ be a shortest $(u, v)$-path and let $P_w$ be a shortest $(v, w)$-path in $G$. If $w \in V(P_u)$, then $d_G(v, w) < d_G(v, u)$, contradicting our choice of the vertex $u$. Therefore, $w \notin V(P_u)$. Suppose that $u \in V(P_w)$. Since $C$ is a shortest cycle in $G$, the distance between $u$ and $w$ on $C$ is the same as the distance between $u$ and $w$ in $G$. Thus, $d_G(u, w) = d_G(u, w)$, implying that $d_G(v, w) = d_G(v, u) + d_G(u, w) \leq 1 + d_G(u, w) = 1 + d_G(u, w) \leq 1 + k$, a contradiction. Therefore, $u \notin V(P_w)$. \(\square\)

3 Lower Bounds

In this section we provide various lower bounds on the $k$-domination number for general graphs. We first prove a generalization of Theorem 1 by establishing a lower bound on the $k$-domination number of a graph in terms of its diameter. We remark that when $k = 1$, Theorem 7 is precisely Theorem 1.

**Theorem 7** For $k \geq 1$, if $G$ is a connected graph with diameter $d$, then

$$
\gamma_k(G) \geq \frac{d + 1}{2k + 1}.
$$

Proof. Let $P: u_0u_1\ldots u_d$ be a diametral path in $G$, joining two peripheral vertices $u = u_0$ and $v = u_d$ of $G$. Thus, $P$ has length $\text{diam}(G) = d$. We show that every vertex of $G$ $k$-dominates at most $2k + 1$ vertices of $P$. Suppose, to the contrary, that there exists a vertex $q \in V(G)$ that $k$-dominates at least $2k + 2$ vertices of $P$. (Possibly, $q \in V(P)$.)

Let $Q$ be the set of vertices on the path $P$ that are $k$-dominated by the vertex $q$ in $G$. By supposition, $|Q| \geq 2k + 2$. Let $i$ and $j$ be the smallest and largest integers, respectively, such that $u_i \in Q$ and $u_j \in Q$. We note that $Q \subseteq \{u_i, u_{i+1}, \ldots, u_j\}$. Thus, $2k + 2 \leq |Q| \leq j - i + 1$. Since $P$ is a shortest $(u, v)$-path in $G$, we therefore note that $d_G(u_i, u_j) = d_P(u_i, u_j) = j - i \geq 2k + 1$. Let $P_i$ be a shortest $(u, q)$-path in $G$ and let $P_j$ be a shortest $(q, v)$-path in $G$. Since the vertex $q$ $k$-dominates both $u_i$ and $u_j$ in $G$, both paths $P_i$ and $P_j$ have length at most $k$. Therefore, the $(u_i, u_j)$-path obtained by following the path $P_i$ from $u_i$ to $q$, and then proceeding along the path $P_j$ from $q$ to $u_j$, has length at most $2k$, implying that $d_G(u_i, u_j) \leq 2k$, a contradiction. Therefore, every vertex of $G$ $k$-dominates at most $2k + 1$ vertices of $P$. 

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Let $S$ be a minimum $k$-dominating set of $G$. Thus, $|S| = \gamma_k(G)$. Each vertex of $S$ $k$-dominates at most $2k + 1$ vertices of $P$, and so $S$ $k$-dominates at most $|S|(2k + 1)$ vertices of $P$. However, since $S$ is a $k$-dominating set of $G$, every vertex of $P$ is $k$-dominated the set $S$, and so $S$ $k$-dominates at most $|S|(2k + 1)$ vertices of $P$. Therefore, $|S|(2k + 1) \geq d + 1$, or, equivalently, $\gamma_k(G) = \frac{d + 1}{2k + 1}$. 

That the lower bound of Theorem 7 is tight may be seen by taking $G$ to be a path, $v_1v_2\ldots v_n$, of order $n = \ell(2k + 1)$ for some $\ell \geq 1$. Let $d = \text{diam}(G)$, and so $d = n - 1 = \ell(2k + 1) - 1$. By Theorem 7 $\gamma_k(G) \geq (d + 1)/(2k + 1) = \ell$. The set

$$S = \bigcup_{i=0}^{\ell-1}\{v_{k+1+i(2k+1)}\}$$

is a $k$-dominating set of $G$, and so $\gamma_k(G) \leq |S| = \ell$. Consequently, $\gamma_k(G) = \ell = (d + 1)/(2k + 1)$. We state this formally as follows.

**Proposition 8** If $G = P_n$ where $n \equiv 0 \mod (2k + 1)$, then $\gamma_k(G) = \frac{\text{diam}(G)+1}{2k+1}$.

More generally, by applying Theorem 7 the $k$-domination number of a cycle $C_n$ or path $P_n$ on $n \geq 3$ vertices is easy to compute.

**Proposition 9** For $k \geq 1$ and $n \geq 3$, $\gamma_k(P_n) = \gamma_k(C_n) = \lceil \frac{n}{2k+1} \rceil$.

By replacing each vertex $v_i$, for $2 \leq i \leq n - 1$, on the path $v_1v_2\ldots v_n$ with a clique (clique $V_i$ corresponds to vertex $v_i$) of size at least $\delta \geq 1$, and adding all edges between $v_1$ and vertices in $V_2$, adding all edges between $v_n$ and vertices in $V_{n-1}$, and adding all edges between vertices in $V_i$ and $V_{i+1}$ for $2 \leq i \leq n - 2$, we obtain a graph with minimum degree $\delta$ achieving the lower bound of Theorem 7.

As a consequence of Theorem 7 we have the following lower bound on the $k$-domination number of a graph in terms of its radius. We remark that when $k = 1$, Corollary 10 is precisely Theorem 2. Therefore, Corollary 10 is a generalization of Theorem 2.

**Corollary 10** For $k \geq 1$, if $G$ is a connected graph with radius $r$, then

$$\gamma_k(G) \geq \frac{2r}{2k + 1}.$$ 

**Proof.** By Lemma 5 the graph $G$ has a spanning tree $T$ such that $\gamma_k(T) = \gamma_k(G)$. Since adding edges to a graph cannot increase its radius, $\text{rad}(G) \leq \text{rad}(T)$. Since $T$ is a tree, we note that $\text{diam}(T) \geq 2\text{rad}(T) - 1$. Applying Theorem 7 to the tree $T$, we have that

$$\gamma_k(G) = \gamma_k(T) \geq \frac{\text{diam}(T) + 1}{2k + 1} \geq \frac{2\text{rad}(T)}{2k + 1} \geq \frac{2\text{rad}(G)}{2k + 1}. \quad \square$$
That the lower bound of Corollary 10 is tight, may be seen by taking $G$ to be a path, $P_n$, of order $n = 2\ell(2k + 1)$ for some integer $\ell \geq 1$. Let $d = \text{diam}(G)$ and let $r = \text{rad}(G)$, and so $d = 2\ell(2k + 1) - 1$ and $r = \ell(2k + 1)$. In particular, we note that $d = 2r - 1$. By Proposition 8, $\gamma_k(G) = \frac{d + 1}{2k + 1} = \frac{2r}{2k + 1}$. As before by replacing each the internal vertices on the path with a clique of size at least $\delta \geq 1$, we can obtain a graph with minimum degree $\delta$ achieving the lower bound of Corollary 10.

We first prove a generalization of Theorem 3 by establishing a lower bound on the $k$-domination number of a graph in terms of its girth. We remark that when $k = 1$, Theorem 11 is precisely Theorem 3.

**Theorem 11** For $k \geq 1$, if $G$ is a connected graph with girth $g$, then

$$\gamma_k(G) \geq \frac{g}{2k + 1}.$$  

**Proof.** The lower bound is trivial if $g \leq 2k + 1$. We may therefore assume that $g \geq 2k + 2$, for otherwise the desired result is immediate. Let $C$ be a shortest cycle in $G$, and so $C$ has length $g$. We note that the distance between two vertices in $V(C)$ is exactly the same in $C$ as in $G$. We consider two cases, depending on the value of the girth.

**Case 1.** $2k + 2 \leq g \leq 4k + 2$. In this case, we need to show that $\gamma_k(G) \geq \left\lceil \frac{g}{2k + 1} \right\rceil = 2$. Suppose, to the contrary, that $\gamma_k(G) = 1$. Then, $G$ contains a vertex $v$ that is within distance $k$ from every vertex of $G$. In particular, $d(u, v) \leq k$ for every vertex $u \in V(C)$. If $v \in V(C)$, then, since $C$ is a shortest cycle in $G$, we note that $d_C(u, v) = d_G(u, v) \leq k$ for every vertex $u \in V(C)$. However, the lower bound condition on the girth, namely $g \geq 2k + 2$, implies that no vertex on the cycle $C$ is within distance $k$ in $C$ from every vertex of $C$, a contradiction. Therefore, $v \notin V(C)$.

By Lemma 6, there exist two vertices $u, w \in V(C)$ such that a shortest $(v, u)$-path does not contain $w$ and a shortest $(v, w)$-path does not contain $u$. We show that we can choose $u$ and $w$ to be adjacent vertices on $C$. Let $w$ be a vertex of $C$ at maximum distance, say $d_w$, from $v$ in $G$. Let $w_1$ and $w_2$ be the two neighbors of $w$ on the cycle $C$. If $d_G(v, w_1) = d_w$, then we can take $u = w_1$, and the desired property (that a shortest $(v, u)$-path does not contain $w$ and a shortest $(v, w)$-path does not contain $u$) holds. Hence, we may assume that $d_G(v, w_1) \neq d_w$. By our choice of the vertex $w$, we note that $d_G(v, w_1) \leq d_w$, implying that $d_G(v, w_1) = d_w - 1$. Similarly, we may assume that $d_G(v, w_2) = d_w - 1$. Let $P_w$ be a shortest $(v, w)$-path. At most one of $w_1$ and $w_2$ belong to the path $P_w$. Renaming $w_1$ and $w_2$, if necessary, we may assume that $w_1$ does not belong to the path $P_w$. In this case, letting $u = w_1$ and letting $P_u$ be a shortest $(v, u)$-path, we note that $w \notin V(P_u)$. As observed earlier, $u \notin V(P_u)$. This shows that $u$ and $w$ can indeed be chosen to be neighbors on $C$.

Let $x$ be the last vertex in common with the $(v, u)$-path, $P_u$, and the $(v, w)$-path, $P_w$. Possibly, $x = v$. Then, the cycle obtained from the $(x, u)$-section of $P_u$ by proceeding
along the edge $uw$ to $w$, and then following the $(w,x)$-section of $P_w$ back to $x$, has length at most $d_G(v,u) + 1 + d_G(v,w) \leq 2k + 1$, contradicting the fact that the girth $g \geq 2k + 2$. Therefore, $\gamma_k(G) \geq 2$, as desired.

Case 2. $g \geq 4k + 3$. Let $S$ be a minimum $k$-dominating set of $G$, and so $|S| = \gamma_k(G)$. Let $K = S \cap V(C)$ and let $L = S \setminus V(C)$. Thus, $S = K \cup L$. If $L = \emptyset$, then $S = K$ and the set $K$ is a $k$-dominating set of $C$, implying by Proposition 9 that $\gamma_k(G) = |S| = |K| \geq \gamma_k(C) = \lceil \frac{g}{2k+1} \rceil$, and the theorem holds. Hence we may assume that $|L| \geq 1$, for otherwise the desired result holds. We wish to show that $|K| + |L| = |S| \geq \lceil \frac{g}{2k+1} \rceil$.

Suppose, to the contrary, that $|K| \leq \left\lfloor \frac{g}{1+2k} \right\rfloor - 1 - |L|.$

As observed earlier, the distance between two vertices in $V(C)$ is exactly the same in $C$ as in $G$. This implies that each vertex of $K$ (recall that $K \subseteq V(C)$) is within distance $k$ from exactly $2k + 1$ vertices of $C$. Thus, the set $K$ $k$-dominates at most

$$|K|(2k + 1) \leq \left( \left\lfloor \frac{g}{2k+1} \right\rfloor - 1 - |L| \right) (2k + 1)$$

$$\leq \left( \frac{g - 2k}{2k+1} - 1 - |L| \right) (2k + 1)$$

$$= g - 1 - |L|(2k + 1)$$

vertices from $C$. Consequently, since $|C(V)| = g$, there are at least $|L|(2k + 1) + 1$ vertices of $C$ which are not $k$-dominated by vertices of $K$, and therefore must be $k$-dominated by vertices from $L$. Thus, by the Pigeonhole Principle, there is at least one vertex, call it $v$, in $L$ that $k$-dominates at least $2k + 2$ vertices in $C$. By Lemma 6 there exist two vertices $u, w \in V(C)$ that are both $k$-dominated by $v$ and such that a shortest $(u,v)$-path, $P_u$ say, (from $u$ to $v$) does not contain $w$ and a shortest $(w,v)$-path, $P_w$ say, (from $w$ to $v$) does not contain $u$. Analogously as in the proof of Lemma 6, we can choose the vertex $u$ to be a vertex of $C$ at minimum distance from $v$ in $G$. Thus, the vertex $u$ is the only vertex on the cycle $C$ that belongs to the path $P_u$. Combining the paths $P_u$ and $P_w$ produces a $(u,w)$-walk of length at most $d_G(u,v) + d_G(v,w) \leq 2k$, implying that $d_G(u,w) \leq 2k$. Since $C$ is a shortest cycle in $G$, we therefore have that $d_C(u,w) = d_G(u,w) \leq 2k$. The cycle $C$ yields two $(u,w)$-paths. Let $P_{wu}$ be the $(w,u)$-path on the cycle $C$ of shorter length (starting at $w$ and ending at $u$). Thus, $P_{wu}$ has length $d_C(u,w) \leq 2k$. Note that the path $P_{wu}$ belongs entirely on the cycle $C$. Let $x \in V(C)$ be the last vertex in common with the $(w,v)$-path, $P_w$, and the $(w,u)$-path, $P_{wu}$. Possibly, $x = w$. However, note that $x \neq u$ since $u \notin V(P_w)$. Let $y$ be the first vertex in common with the $(x,v)$-subsection of the path $P_w$ and with the $(u,v)$-path $P_u$. Possibly, $y = v$. However, note that $y \neq x$ since $x \notin V(P_u)$ and $V(P_u) \cap V(C) = \{u\}$. Using the $(x,u)$-subsection of the path $P_{wu}$, the $(x,y)$-subsection of the path $P_u$, and the $(y,u)$-subsection of the path $P_w$ produces a cycle in $G$ of length at most $d_G(u,v) + d_G(v,w) + d_G(u,w) \leq k + k + 2k = 4k$, contradicting the fact that the girth $g \geq 4k + 3$. Therefore, $\gamma_k(G) = |S| = |K| + |L| \geq \left\lceil \frac{g}{2k+1} \right\rceil$, as desired. □

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4 Direct Product Graphs

The direct product graph, $G \times H$, of graphs $G$ and $H$ is the graph with vertex set $V(G) \times V(H)$ and with edges $(g_1, h_1)(g_2, h_2)$, where $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$. Let $A \subseteq V(G \times H)$. The projection of $A$ onto $G$ is defined as

$$P_G(A) = \{g \in V(G): (g, h) \in A \text{ for some } h \in V(H)\}.$$

Similarly, the projection of $A$ onto $H$ is defined as

$$P_H(A) = \{g \in V(H): (g, h) \in A \text{ for some } h \in V(G)\}.$$

For a detailed discussion on direct product graphs, we refer the reader to the handbook on graph products [3]. There have been various studies on the domination number of direct product graphs. For example, Mekić [16] proved the following lower bound on graph products [5].

**Theorem 12** ([16]) If $G$ and $H$ are connected graphs, then

$$\gamma(G \times H) \geq \gamma(G) + \gamma(H) - 1.$$

Staying within the theme of our previous results, we now prove a projection lemma which will enable us generalize the result of Theorem 12 on the domination number to the $k$-domination number.

**Lemma 13** (Projection Lemma) Let $G$ and $H$ be connected graphs. If $D$ is a $k$-dominating set of $G \times H$, then $P_G(D)$ is a $k$-dominating set of $G$ and $P_H(D)$ is a $k$-dominating set of $H$.

**Proof.** Let $D \subseteq V(G \times H)$ be a $k$-dominating set of $G \times H$. We show firstly that $P_G(D)$ is a $k$-dominating set of $G$. Let $g$ be a vertex in $V(G)$. If $g \in P_G(D)$, then $g$ is clearly $k$-dominated by $P_G(D)$. Hence, we may assume that $g \notin P_G(D)$. Let $h$ be an arbitrary vertex in $V(H)$. Since $g \notin P_G(D)$, the vertex $(g, h) \notin D$. However, the set $D$ is a $k$-dominating set of $G \times H$, and so $(g, h)$ is within distance $k$ from $D$ in $G$; that is, $d_{G \times H}((g, h), D) \leq k$. Let $(g_0, h_0), (g_1, h_1), \ldots, (g_r, h_r)$ be a shortest path from $(g, h)$ to $D$ in $G \times H$, where $(g, h) = (g_0, h_0)$ and $(g_r, h_r) \in D$. By assumption, $1 \leq r \leq k$. For $i \in \{0, \ldots, r - 1\}$, the vertices $(g_i, h_i)$ and $(g_{i+1}, h_{i+1})$ are adjacent in $G \times H$. Hence, by the definition of the direct product graph, the vertices $g_i$ and $g_{i+1}$ are adjacent in $G$, implying that $g_0g_1 \ldots g_r$ is a $(g_0, g_r)$-walk in $G$ of length $r$. This in turn implies that there is a $(g_0, g_r)$-path in $G$ of length $r$. Recall that $g = g_0$ and $1 \leq r \leq k$. Since $(g_r, h_r) \in D$, the vertex $g_r \in P_G(D)$. Hence, there is a path from $g$ to a vertex of
Let $D \subseteq V(G \times H)$ be a minimum $k$-dominating set of $G \times H$. Suppose, to the contrary, that $|D| \leq \gamma_k(G) + \gamma_k(H) - 2$. We will refer to this supposition as (*).

By Lemma 13, $P_G(D)$ is a $k$-dominating set of $G$ and $P_H(D)$ is a $k$-dominating set of $H$. Therefore, we have that $|D| \geq |P_G(D)| \geq \gamma_k(G)$ and $|D| \geq |P_H(D)| \geq \gamma_k(H)$.

If $\gamma_k(G) = 1$, then, by (*), $\gamma_k(H) - 1 \geq |D| \geq \gamma_k(H)$, a contradiction. Therefore, $\gamma_k(G) \geq 2$. Analogously, $\gamma_k(H) \geq 2$. Recall that $|P_G(D)| \geq \gamma_k(G)$. We now remove vertices from the set $P_G(D)$ until we obtain a set, $D_G$, say, of cardinality exactly $\gamma_k(G) - 1$.

Thus, $D_G$ is a proper subset of $P_G(D)$ of cardinality $\gamma_k(G) - 1$. Since $D_G$ is not a $k$-dominating set of $G$, there exists a vertex $g \in V(G)$ that is not $k$-dominated by the set $D_G$ in $G$; that is, $d_G(g, D_G) > k$. Let $D_G = \{g_1, \ldots, g_t\}$, where $t = \gamma_k(G) - 1 \geq 1$. For each $i \in [t]$, there exists a (not necessarily unique) vertex $h_i \in V(H)$ such that $(g_i, h_i) \in D$ (since $D_G = P_G(D)$). We now consider the set

$$D_0 = \{(g_1, h_1), \ldots, (g_t, h_t)\},$$

and note that $D_0 \subset D$ and $|D_0| = \gamma_k(G) - 1$. By (*), we note that

$$|P_H(D \setminus D_0)| \leq |D \setminus D_0| = |D| - |D_0| \leq (\gamma_k(G) + \gamma_k(H) - 2) - (\gamma_k(G) - 1) = \gamma_k(H) - 1 < \gamma_k(H).$$

Hence, there exists a vertex $h \in V(H)$ that is not $k$-dominated by the set $P_H(D \setminus D_0)$ in $H$; that is, $d_H(h, P_H(D \setminus D_0)) > k$. We now consider the vertex $(g, h) \in V(G \times H)$. Since $D$ is a $k$-dominating set of $G \times H$, the vertex $(g, h)$ is $k$-dominated by some vertex, say $(g^*, h^*)$, of $D$ in $G \times H$. An analogous proof as in the proof of Lemma 13 shows that $d_G(g, g^*) \leq k$ and $d_H(h, h^*) \leq k$. If $(g^*, h^*) \in D \setminus D_0$, then $h^* \in P_H(D \setminus D_0)$, implying that $d_H(h, P_H(D \setminus D_0)) \leq d_H(h, h^*) \leq k$, a contradiction. Hence, $(g^*, h^*) \notin D_0$. This in turn implies that $g^* \in P_G(D_0) = G_D$. Thus, $d_G(g, D_G) \leq d_G(g, g^*) \leq k$, contradicting the fact that $d_G(g, D_G) > k$. Therefore, the supposition that $|D| = \gamma_k(G) + \gamma_k(H) - 2$ must be false, and the result follows. \(\square\)
References

[1] E. DeLaViña, Written on the Wall II, Web address: http://cms.dt.uh.edu/faculty/delavinae/research/wowII

[2] J. Cyman, M. Lemańska and J. Raczek, Lower bound on the distance \( k \)-domination number of a tree. Math. Slovaca 56(2) (2006), 235—243.

[3] E. E. DeLaViña, R. Pepper, B. Waller, Lower bounds for the domination number. Discuss. Math. Graph Theory 30(3) (2010), 475–487

[4] P. Fraisse, A note on distance dominating cycles. Discrete Math. 71(1988), 89–92.

[5] R. Hammack, W. Imrich, and S. Klavžar, Handbook of Product Graphs, Second Edition CRC Press (June 3, 2011) ISBN: 9781439813041.

[6] A. Hansberg, D. Meierling, and L. Volkmann, Distance domination and distance irredundance in graphs. Electronic J. Combin. 14 (2007), #R35.

[7] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, Inc. New York, 1998.

[8] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater (eds), Domination in Graphs: Advanced Topics, Marcel Dekker, Inc. New York, 1998.

[9] M. A. Henning, Distance domination in graphs. Domination in Graphs: Advanced Topics, T.W. Haynes, S.T. Hedetniemi, and P.J. Slater (eds), Marcel Dekker, Inc. New York, 1998, 335–365.

[10] M. A. Henning and N. Lichiardopol, Distance domination in graphs with given minimum and maximum degree, manuscript.

[11] M. A. Henning, O. R. Oellermann, and H. C. Swart, Bounds on distance domination parameters. J. Combin. Comput. Inf. Sys. Sciences 16 (1991), 11–18.

[12] M. A. Henning and A. Yeo, Total domination in graphs (Springer Monographs in Mathematics). ISBN-13: 978-1461465249 (2013).

[13] D. Lichtenstein, Planar satisfiability and its uses. SIAM J. Comput. 11 (1982), 329–343.

[14] D. Meierling and L. Volkmann, A lower bound for the distance \( k \)-domination number of trees. Result. Math. 47 (2005), 335–339.

[15] A. Meir and J. W. Moon, Relations between packing and covering number of a tree. Pacific J. Math. 61 (1975), 225–233.

[16] G. Mekiš, Lower bounds for the domination number and the total domination number of direct product graphs. Discrete Math. 310 (2010), 3310–3317.
[17] P. J. Slater, R-domination in graphs. *J. Association Computer Machinery* 23(3) (1976), 446–450.

[18] F. Tian and J. M. Xu, A note on distance domination numbers of graphs. *Australasian J. Combin.* 43 (2009), 181–190.

[19] J. Topp and L. Volkmann, On packing and covering numbers of graphs. *Discrete Math.* 96 (1991), 229–238.