ON THE REFLECTION PRINCIPLE IN $\mathbb{C}^n$

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Abstract. We propose a reflection principle for holomorphic objects in $\mathbb{C}^n$. Our construction generalizes the classical principle of H.Lewy, S.Pinchuk and S.Webster.

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INTRODUCTION

The classical Schwarz reflection principle asserts that a conformal isomorphism between bounded domains $D \subset \mathbb{C}$ and $D' \subset \mathbb{C}$ with real analytic boundaries extends holomorphically to a neighborhood of the closure $\overline{D}$. In several variables, this phenomenon was first investigated by H.Lewy [Le] and S.Pinchuk [Pi] for the case of strictly pseudoconvex boundaries. They introduced a multidimensional reflection principle. Far reaching generalizations of their result were obtained by several authors (for instance, see [BJT, DW, DF], etc.). Especially we would like to note the important paper of S.Webster [We1]. He discovered a new reflection principle that differs from the reflection principle of H.Lewy and S.Pinchuk. Using his technique S.Webster explored the extension phenomenon for holomorphic mappings between domains with piecewise smooth real analytic boundaries [We2].

There is another way to develop the reflection principle in several complex variables. Let $D$ and $D'$ be strictly pseudoconvex domains with real analytic boundaries in $\mathbb{C}^n$. Suppose that $f : D \to D'$ is a biholomorphic mapping. One can assume that $f$ extends smoothly to the boundary $\partial D$ of $D$ by Fefferman’s theorem [Fe]. Let $A = \Gamma_f$ be the graph of $f$ in $\mathbb{C}^{2n}$. Then $A$ is a complex $n$-dimensional manifold with smooth boundary $\partial A \subset M$ and $M = \partial D \times \partial D'$ is a generic real analytic manifold. The extension theorem of H.Lewy and S.Pinchuk means that $A$ continues to a complex manifold in a neighborhood of $M$. Therefore, the following natural question arises. Let $\Omega$ be a domain in $\mathbb{C}^N$ and let $M$ be a generic real analytic manifold in $\Omega$. Assume that $A$ is a complex $p$-dimensional manifold in $\Omega \setminus M$ with smooth boundary $\partial A \subset M$. Under what conditions does $A$ continue analytically through $M$?

H.Alexander [Al], B.Shiffman [Sh] and E.Chirka [Ch1] investigated the case where $A$ is a complex 1-dimensional analytic set and $M$ is a totally real manifold. In the general case an affirmative answer depends on the Levi form of $M$. Also, a direction of the approach of $A$ to $M$ is essential. In this connection the notion of Levi transversality was introduced [PiCh,Su1,Su2,Su3]. The condition of Levi transversality means that $M$ has a sufficiently non-degenerate Levi form and the tangent space of $A$ at a boundary point is in the general position with respect...
to the Levi form of $M$. Using the concept of Levi transversality one can generalize
the extension theorem of H.Levi and S.Pinchuk [PCh,Su1,Su2,Su3].

However, this version of the reflection principle does not work if $A$ is the graph of
a holomorphic mapping between wedges with generic edges. Indeed, the condition
$bA \subset M$ presupposes that (locally) $A$ is the graph of a holomorphic mapping
over a domain with smooth boundary. Hence, it seems natural to consider the case
where the intersection $\overline{A} \cap M$ has an arbitrary real dimension. The present paper
is devoted to this problem.

We shall consider a mapping $f$ defined and holomorphic on a wedge $W \subset \mathbb{C}^p$
with a smooth (generic) edge $E$. We suppose that $f$ is smooth on $W \cup E$. Let
$A \subset \mathbb{C}^N$ be the graph of $f$ over $W$ and let $A_E$ be the graph of the restriction $f|E$
( if $E$ is a hypersurface, then $A_E = bA$ ). We suppose that $A_E$ is contained in a
generic real analytic manifold $M$.

We introduce the notion of Levi transversality of $A$ and $M$ at $a \in A_E$ by analogy
with [Su2]. This condition means that the Levi form of $M$ and the tangent space
$T_a(A_E)$ are in the general position. The important example arises when $A$ is the
graph of a holomorphic mapping $f$ between wedges $W$ and $W'$ with real analytic
edges $E$ and $E'$, $f(E)$ is contained in $E'$ and $M = E \times E'$. Our first result is
Theorem 1 that establishes the holomorphic extendability of $f$ to a neighborhood
of $E$ under the condition of Levi transversality. Moreover, Theorem 1 asserts that in
fact $E$ is a real analytic manifold. This theorem generalizes the above-mentioned
results on the extension of holomorphic mappings and complex manifolds. For
instance, the graph of a biholomorphism $f : D \to D'$ between strictly pseudoconvex
domains is Levi transverse to $M = bD \times bD'$; a complex 1-dimensional manifold $A$
with smooth boundary $bA \subset M$ is Levi transverse to $M$ if $M$ is totally real, etc.
(see Propositions 2, 3 and Corollaries 1-4).

Our main tool is the generalized Lewy - Pinchuk - Webster reflection principle
that was developed in [PCh,Su1,Su2]. We use our technique to push further an
interesting recent result of A.Nagel and J.-P.Rosay [NR] on maximum modulus
sets (subsets of the boundary of a domain $\Omega$ in $\mathbb{C}^n$, where a holomorphic function
takes its maximum modulus). They proved that if a maximum modulus set $E$ is a
smooth real n-dimensional submanifold in $b\Omega$, and if $b\Omega$ is strictly pseudoconvex and
real analytic near $E$, then $E$ is real analytic (and totally real). Our second result
(Theorem 2) shows that both the extension theorem of H.Lewy and S.Pinchuk and
the theorem of A.Nagel and J.-P.Rosay (under the slight additional assumption
for $E$ to be totally real) are consequences of a general reflection principle.

All our considerations are purely local.

This paper is organized as follows. In section 1 we introduce notations, give
precise definitions and statements of our results. In sections 2-4 we develop the
reflection principle and prove Theorem 1. In section 5 we prove certain corollaries
of Theorem 1 (Proposition 1, Corollaries 1, 4). In section 6 we prove Theorem 2.

1. NOTATIONS, DEFINITIONS AND RESULTS

Let $D$ be a domain in $\mathbb{C}^p$ and let $E \subset D$ be a generic manifold of class $C^2$
and of real codimension $d$. Then

\begin{equation}
E = \{ x \in D : r_j(x) = 0, j = 1, \ldots, d \} \end{equation}
where the functions \( r_j : D \to \mathbb{R} \) are of class \( C^2(D) \) and \( \partial r_1 \wedge \ldots \wedge \partial r_d \neq 0 \) on \( D \).

Let \( W(E, D) \) be a wedge with the edge \( E \):

\[
W(E, D) = \{ x \in D : r_j(x) < 0 \}
\]

Let \( G \) be a domain in \( \mathbb{C}^n \), and \( f : W(E, D) \to G \) be a holomorphic mapping of class \( C^2 \) on \( W(E, D) \cup E \). Let us consider the graph \( A \) of \( f \):

\[
A = \Gamma_f = \{ z = (x, y) \in \mathbb{C}^p \times \mathbb{C}^n = \mathbb{C}^N : y = f(x), x \in W(E, D) \}
\]

that is a complex \( p \)-dimensional manifold in a domain \( \Omega = D \times G \). We denote by \( A_E \) the graph of the restriction \( f|_E \):

\[
A_E = \{ z = (x, y) : y = f(x), x \in E \}
\]

Then \( A_E \) is a real \((2p - d)\)-dimensional manifold (if \( d = 1 \), then \( A_E = bA = (A \cap \Omega) \setminus A \)).

Let \( M \) be a generic real analytic (closed) manifold of real codimension \( m \) in a domain \( \Omega \subset \mathbb{C}^N \):

\[
M = \{ z \in \Omega : \rho_j(z, \overline{z}) = 0, j = 1, \ldots, m \},
\]

where the functions \( \rho_j : \Omega \to \mathbb{R} \) are real analytic on \( \Omega \) and \( \partial \rho_1 \wedge \ldots \wedge \partial \rho_m \neq 0 \) on \( \Omega \). We suppose that \( A_E \subset M \). Our main question: under what conditions does \( A \) continue analytically across \( M \)?

We recall certain basic definitions of the theory of Cauchy-Riemann manifolds (for instance, see [Ch2]). Let \( T_aM \) be the real tangent space of \( M \) at \( a \in M \). We denote by \( T_a^cM \) the complex tangent space. We recall that \( T_a^cM = T_aM \cap i(T_aM) \).

For \( M \) of the form (1.5) we have

\[
T_aM = \{ t \in \mathbb{C}^N : \Re \sum_{\nu=1}^{N} \frac{\partial \rho_j}{\partial z_\nu}(a)t_\nu = 0, j = 1, \ldots, m \},
\]

\[
T_a^cM = \{ t \in \mathbb{C}^N : \sum_{\nu=1}^{N} \frac{\partial \rho_j}{\partial \overline{z}_\nu}(a)t_\nu = 0, j = 1, \ldots, m \}.
\]

The complex dimension of \( T_a^cM \) is equal to \( N - m \). It is called a CR dimension of \( M \). We denote the CR dimension of \( M \) by \( CRdimM \).

For \( u, v \in \mathbb{C}^N \) and \( a \in M \) we denote by \( H_a(\rho_j, u, v) \) the Levi form of \( \rho_j \):

\[
H_a(\rho_j, u, v) = \sum_{\nu=1}^{N} \frac{\partial^2 \rho_j}{\partial z_\nu \partial \overline{z}_\mu}(a)u_\nu \overline{v}_\mu.
\]

Fix a hermitian scalar product \( <,> \) in \( \mathbb{C}^N \). We associate with each Levi form a hermitian \( \mathbb{C} \)-linear operator \( L^j_a \) defined on \( T_a^cM \) by the condition \( H_a(\rho_j, u, v) = < L_a^j(u), v > \) for any \( u, v \in T_a^cM \). It is called the Levi operator of \( \rho_j \).
Assume that $A_E \subset M$ and $m > d$. We shall consider the case when the restrictions $d\rho_j(a)|_{T_a A}, j = m - d + 1, \ldots, m$ are linearly independent (perhaps, after a renumeration of the functions $\rho_j$). Here $a \in A_E \subset M$. One can consider this requirement as a condition of the "partial" transversality of the tangent spaces $T_a A$ and $T_a M$ (taking into account the inclusion $A_E \subset M$). Since $A_E \subset M$ and $\text{dim}_R A_E = 2p - d$, the restrictions $d\rho_j(a)|_{T_a A}, j = k, m - d + 1, \ldots, m$ are linearly dependent for any $k = 1, \ldots, m - d$. Hence, after the replacement of $\rho_k$ by $\rho_k - \sum_{j=m-d+1}^{m} \lambda_{kj} \rho_j$, one can assume that

\begin{equation}
(1.8) \quad d\rho_k(a)|_{T_a A} = 0, k = 1, \ldots, m - d.
\end{equation}

The following definition is a basic point of our approach.

**Definition 1.** Let $A$ and $M$ be given as above. We say that $A$ and $M$ are Levi transverse at $a \in A_E \subset M$ if the following conditions hold:

(i) $m > d$;

(ii) the restrictions $d\rho_j(a)|_{T_a A}, j = m - d + 1, \ldots, m$ are linearly independent (perhaps, after a renumeration of the functions $\rho_j$);

(iii) assume that the functions $\rho_j$ are chosen so that (1.8) holds. We suppose that

\begin{equation}
(1.9) \quad T_a^c(A_E) + \sum_{j=1}^{m-d} L_a^j(T_a^c(A_E)) = T_a^c M.
\end{equation}

**Remark 1.** We shall show that (1.9) does not depend on the choice of a hermitian scalar product that defines the Levi operators $L_a^j$.

**Remark 2.** Formally one can treat (i) as a consequence of (iii). Indeed, in the case $m \leq d$ the condition (1.9) has the form $T_a^c(A_E) = T_a^c M$ or $p - d = N - m$. Since $N - p > 0$, we obtain (i). We prefer to impose (i) explicitly.

**Remark 3.** We emphasize that one requires (1.9) to be true at least for a certain collection of the defining functions satisfying (ii) and (1.8) (not necessarily for all such collections).

Our first result is the following

**Theorem 1.** Let $\Omega \subset \mathbb{C}^N = \mathbb{C}^p \times \mathbb{C}^n$ be a domain of the form $\Omega = D \times G$, where $D \subset \mathbb{C}^p$ and $G \subset \mathbb{C}^n$. Let $M \subset \Omega$ be a generic real analytic manifold. Suppose that $E \subset D$ is a generic manifold of class $C^2$ and $f : W(E, D) \to G$ is a mapping holomorphic on the wedge $W(E, D)$ and of class $C^2$ on $W(E, D) \cup E$. Assume that $A \ (= \text{the graph of } f)$ and $M$ are Levi transverse at a point $\tilde{a} = (a, f(a)) \in A_E \subset M$. Then $f$ extends holomorphically to a neighborhood of the point $a \in E$ and $E$ is a real analytic manifold near $a$.

This theorem generalizes well-known results connected with the reflection principle. We start from the applications of Theorem 1 to the mapping problem.

Let $\Omega$ be a domain in $\mathbb{C}^p$ and let $S \in \Omega$ be a generic real analytic manifold of the form

\begin{equation}
(1.10) \quad S = \{x \in \Omega : r_j(x, \overline{x}) = 0, j = 1, \ldots, d\},
\end{equation}
where the functions $r_j : \Omega \to \mathbb{R}$ are real analytic on $\Omega$ and $\partial r_1 \wedge ... \wedge \partial r_d \neq 0$ on $\Omega$. Let

\begin{equation}
W(S, \Omega) = \{ x \in \Omega : r_j(x, \overline{x}) < 0, j = 1, ..., d \},
\end{equation}

be a wedge in $\Omega$ with the edge $S$. Similarly, let us consider a domain $\Omega' \subset \mathbb{C}^{d'}$ and a generic real analytic manifold $S' \subset \Omega'$ of the form

\begin{equation}
S' = \{ x' \in \Omega' : r'_j(x', \overline{x'}) = 0, j = 1, ..., d' \},
\end{equation}

where the functions $r'_j : \Omega' \to \mathbb{R}$ are real analytic on $\Omega'$ and $\partial r'_1 \wedge ... \wedge \partial r'_{d'} \neq 0$ on $\Omega'$. Fix a hermitian scalar product on $\mathbb{C}^{d'}$. We denote by $L_{a'}^\mu$ the Levi operators of the functions $r'_j$ at $a' \in S'$.

The first consequence of Theorem 1 is the following

**Proposition 1.** Suppose that $\Omega$ is a domain in $\mathbb{C}^p$, $S$ is a generic real analytic manifold of the form (1.10) in $\Omega$, $\Omega'$ is a domain in $\mathbb{C}^{d'}$, $S' \subset \Omega'$ is a generic real analytic manifold of the form (1.12). Suppose further that $f : W(S, \Omega) \to \mathbb{C}^{d'}$ is a holomorphic mapping of class $C^2$ on $W(S, \Omega) \cup S$ and $f(S) \subset S'$. Assume that for certain $a \in S$ the following condition holds:

\begin{equation}
\sum_{j=1}^{d'} L_{a'}^\mu(df_a(T_a^\mu S)) = T_{a'}^\mu(S'),
\end{equation}

where $a' = f(a)$ and $df_a$ is the tangent mapping. Then $f$ extends holomorphically to a neighborhood of $a$.

In fact this statement is true for $f \in C^1$. This result is due to the author [Su4,Su5,Su6]. A similar result (in another form) was obtained by M.Deridj [Der]. We shall show that the graph of $f$ is Levi transverse to $M = S \times S'$. Proposition 1 generalizes the classical reflection principle of H.Lewy, S.Pinchuk and S.Webster. To show this fact let us consider the Levi form $Levi^S_{a'}(u, v)$ of $S'$ at $a'$:

\[ Levi^S_{a'}(u, v) = (H_{a'}(r'_1, u, v), ..., H_{a'}(r'_{d'}, u, v)). \]

Recall, that manifold $S'$ is called Levi non-degenerate at $a' \in S'$ if the equality $Levi^S_{a'}(u, v) = 0$ for any $v \in T_{a'}^\mu(S')$ implies $u = 0$ (see [We2]).

**Corollary 1.** Suppose that $\Omega$ is a domain in $\mathbb{C}^p$, $S \subset \Omega$ is a generic real analytic manifold of the form (1.10), $\Omega'$ is a domain in $\mathbb{C}^{d'}$, $S' \subset \Omega'$ is a generic real analytic manifold of the form (1.12). Suppose further that $f : W(S, \Omega) \to \mathbb{C}^{d'}$ is a holomorphic mapping of class $C^2$ on $W(S, \Omega) \cup S$ and $f(S) \subset S'$. Assume that the tangent mapping $df_a : T_a^\mu S \to T_{a'}^\mu(S')$ is surjective and $S'$ is Levi non-degenerate at $a' = f(a)$. Then $f$ extends holomorphically to a neighborhood of $a \in S$.

This result (with $f$ of class $C^1$) is due to S.Webster [We2] (see also [TH]). We shall show in section 5 that Corollary 1 is an immediate consequence of Proposition 1. We would like to emphasize that Proposition 1 is a considerably more general
statement. Indeed, the surjectivity of $d\sigma$ implies $CRdimS' \leq CRdimS$. From the other side (1.13) is valid if $d'CRdimS \geq CRdimS'$. In particular, the difference $CRdimS' - CRdimS$ can be arbitrarily large.

We point out that Proposition 1 and Corollary 1 deal with a very special case of Theorem 1, because of in the hypotheses of Theorem 1 $M$ is not obliged to be the cartesian product.

If both $S$ and $S'$ are real hypersurfaces in $\mathbb{C}^p$, we obtain from Corollary 1 the classical theorem of H. Lewy and S. Pinchuk.

The next interesting special case of Theorem 1 arises if $d = 1$. Then $A_E = bA$, the condition (i) of Definition 1 means that $m \geq 2$ and (ii) is equivalent to the requirement that $T_0A$ is not contained in $T_0M$. Thus, we obtain the following

Proposition 2. Let $M$ be a generic real analytic manifold of real codimension $\geq 2$ in a domain $\Omega \subset \mathbb{C}^N$. Suppose that $A$ is a complex $p$-dimensional manifold in $\Omega \setminus M$ and $(A, bA)$ is a $C^2$-manifold with boundary $bA \subset M$. Assume that $A$ and $M$ are Levi transverse at $a \in bA$. Then $A$ continues analytically to a complex manifold in a neighborhood of $a$.

This result was obtained by the author [Su2]. In fact this assertion is true for $(A, bA) \in C^1$ (see [Su2]). Proposition 2 generalizes the Lewy - Pinchuk extension theorem as well. We obtain their statement if $M$ is the cartesian product of strictly pseudoconvex hypersurfaces $\Lambda_j, j = 1, 2$ and $A$ is the graph of a mapping $f : \Lambda_1 \to \Lambda_2$ of class $C^1$; $f$ is holomorphic on a domain $D$ with $bD = \Lambda_1$ and $df$ is non-degenerate on $\Lambda_1$.

Corollary 1. Let $M$ be a real analytic totally real $N$-dimensional manifold in a domain $\Omega \subset \mathbb{C}^N$. Assume that $A \subset \Omega \setminus M$ is a complex 1-dimensional manifold and $(A, bA)$ is a $C^1$-manifold with boundary $bA \subset M$. Then $A$ continues analytically to a complex manifold in a neighborhood of $bA$.

For complex 1-dimensional analytic sets similar results were obtained by H. Alexander [Al], B. Shiffman [Sh] and E. Chirka [Ch1]. One can consider Corollary 2 as a generalization of the classical Schwarz reflection principle. Indeed, if $f : D \to D'$ is a conformal isomorphism between domains in $\mathbb{C}$ then the graph of $f$ is a complex 1-dimensional manifold whose boundary is contained in the totally real manifold $M = bD \times bD' \subset \mathbb{C}^2$.

Corollary 3. Let $M$ be a generic real analytic manifold in a domain $\Omega \subset \mathbb{C}^N$ and let $A \subset \Omega \setminus M$ be a complex $p$-dimensional manifold; we suppose that $(A, bA)$ is a $C^1$-manifold with boundary $bA \subset M$. Assume that $CRdimM = p - 1$. Then $A$ continues to a complex manifold in a neighborhood of $bA$.

E. Chirka [Ch1] obtained a similar result for analytic sets, see also [Fo].

Theorem 1 also implies the next

Corollary 4. Let $\Omega \subset \mathbb{C}^N = \mathbb{C}^p \times \mathbb{C}^n$ be a domain of the form $\Omega = D \times G$, where $D \subset \mathbb{C}^p$ and $G \subset \mathbb{C}^n$. Let $M \subset \Omega$ be a real analytic totally real $N$-dimensional manifold, $E \subset D$ be a totally real $p$-dimensional $C^1$-manifold. Suppose that $f : W(E, D) \to G$ is a holomorphic mapping of class $C^1$ on $W(E, D) \cup E$. Assume that $A_E \subset M$ (here $A_E$ is the graph of the restriction $f|E$). Then $f$ extends holomorphically to a neighborhood of $E$ and $E$ is a real analytic manifold.

We shall use Corollary 4 to prove our second main result.
Theorem 2. Let \( \Omega \) be a domain in \( \mathbb{C}^n \) and let \( E \subset b\Omega \) be a totally real \( n \)-dimensional \( C^1 \)-manifold. Assume that \( b\Omega \) is strictly pseudoconvex and real analytic near \( E \). Let \( E' \) be a totally real \( n' \)-dimensional real analytic manifold that is contained in the boundary of a domain \( \Omega' \subset \mathbb{C}^{n'} \). Suppose that for a neighborhood \( V \supset E' \in \mathbb{C}^{n'} \) one has \( \Omega' \cap V = \{ x' \in V : \rho(x', \overline{x'}) < 0 \} \), where \( \rho \) is a real analytic plurisubharmonic function on \( V \), \( \rho \neq 0 \). Assume that \( U \) is a neighborhood of \( E \) in \( \mathbb{C}^n \) and \( f : \Omega \cap U \rightarrow \Omega' \cap V \) is a holomorphic mapping of class \( C^2 \) on \( \overline{\Omega} \cap U \) such that \( f(E) \subset E' \). Then \( f \) extends holomorphically to a neighborhood of \( E \) and \( E \) is a real analytic manifold.

We would like to emphasize that we do not require the inclusion \( f(b\Omega) \subset b\Omega' \). Theorem 2 implies the following well-known assertion that is a version of the extension theorem of H. Lewy and S. Pinchuk.

Corollary 5. Let \( \Omega \) and \( \Omega' \) be bounded strictly pseudoconvex domains with real analytic boundaries and let \( f : \Omega \rightarrow \Omega' \) be a biholomorphic mapping. Then \( f \) extends holomorphically to a neighborhood of the closure \( \overline{\Omega} \).

Indeed, Fefferman’s theorem [Fe] implies that \( f \in C^\infty(\overline{\Omega}) \) and \( f : \overline{\Omega} \rightarrow \overline{\Omega'} \) is a diffeomorphism. Let us consider a point \( a \in b\Omega \) and a totally real \( n \)-dimensional real analytic manifold \( E' \subset b\Omega' \) through \( f(a) \). Then \( f^{-1}(E') = E \ni a \) is a smooth totally real \( n \)-dimensional manifold in \( b\Omega \) and one can apply Theorem 2.

Another application of Theorem 2 considers maximum modulus sets. Let \( \Omega \) be a strictly pseudoconvex (bounded) domain in \( \mathbb{C}^n \) with \( C^2 \)-boundary. A subset \( E \subset b\Omega \) is called a maximum modulus set if for every \( a \in E \) there exists a neighborhood \( U \) of \( a \) and a function \( f \) defined and continuous on \( \overline{\Omega} \cap U \), holomorphic on \( \Omega \cap U \), so that \( |f| < 1 \) on \( \Omega \cap U \) and \( |f| = 1 \) on \( E \cap U \).

In the case the function \( f \) in the definition of maximum modulus set can be chosen of class \( C^k \) on \( \overline{\Omega} \cap U \), \( E \) is called a \( C^k \)-maximum modulus set (see [NR]).

Theorem 2 immediately implies the following

Corollary 6. Let \( \Omega \subset \mathbb{C}^n \) be a domain and let \( E \subset b\Omega \) be a totally real \( n \)-dimensional manifold of class \( C^1 \). If \( b\Omega \) is strictly pseudoconvex and real analytic near \( E \), and if \( E \) is a \( C^2 \)-maximum modulus set, then \( E \) is real analytic.

This result was obtained by A. Nagel and J.-P. Rosay [NR]. In fact they proved it without the assumption that \( E \) is totally real. This requirement is not essential for our proof and we claim it for the convenience.

There is the firm confidence that Theorem 1 is true if both \( f \) and \( E \) are of class \( C^1 \). To avoid technical complications we consider here only the \( C^2 \) case.

2. Basic Properties of the Levi Transversality.

The concept of Levi transversality is obviously invariant under biholomorphic transformations. Hence, without loss of generality one can assume that \( A \) and \( M \) are Levi transverse at \( 0 \in A_E \subset M \).

The condition of Levi transversality imposes the restrictions on the dimensions of \( A \) and \( M \). Indeed, the inclusion \( A_E \subset M \) implies

\[
dim T_0^\delta(A_E) = p - d \leq \dim T_0^\delta M = N - m
\]
From the other side we have
\[
\dim[T_0^c(AE) + \sum_{j=1}^{m-d} L_j^j(T_0^c(AE))] \leq (m - d + 1)(p - d)
\]

and (1.9) implies
\[
(m - d + 1)(p - d) \geq N - m.
\]

**Lemma 2.1.** There exists a non-degenerate $\mathbb{C}$-linear change of coordinates in $\mathbb{C}^N$ such that in the new coordinates (perhaps, after a replacement of the defining functions by their linear combinations) we have
\[
\rho_j(z, \overline{z}) = z_{N-m+j} + \overline{z}_{N-m+j} + o(|z|), j = 1, ..., m,
\]

(2.1)

\[
T_0A = \{ t \in \mathbb{C}^N : t_{p-d+1} = ... = t_{N-d} = 0 \},
\]

\[
T_0^c(AE) = \{ t \in \mathbb{C}^N : t_{p-d+1} = ... = t_N = 0 \}.
\]

The condition (1.9) holds for the new defining functions.

**Proof.** Since $AE \subset M$ and $d\rho_{m-d+1} \wedge ... \wedge d\rho_m |T_0A \neq 0$, we have $T_0(AE) = T_0A \cap T_0M$. Also, we have $d\rho_j(0)|T_0A = 0, j = 1, ..., m-d$. Therefore, taking into account the equality $dimT_0^c(A) = p - d$, we conclude that $\partial\rho_{m-d+1} \wedge ... \wedge \partial\rho_m |T_0A \neq 0$.

After a non-degenerate $\mathbb{C}$-linear change of coordinates we obtain
\[
T_0^cM = (e_1, ..., e_{N-m})_\mathbb{C}, T_0^c(AE) = (e_1, ..., e_{p-d})_\mathbb{C}
\]

where $e_j, j = 1, ..., N$ is the standard basis in $\mathbb{C}^N$ and ()$_\mathbb{C}$ is the $\mathbb{C}$-linear hull. Since $T_0^c(AE) = T_0^cM \cap T_0A$, we have $T_0A = T_0^c(AE) \oplus (v_1, ..., v_d)_\mathbb{C}$ and the vectors $e_1, ..., e_{N-m}, v_1, ..., v_d$ are linearly independent. Let us consider $v_1, ..., v_d$ as the vectors $e_{N-d+1}, ..., e_N$ of a new basis in $\mathbb{C}^N$ (we do not change other basis vectors). Then (we omit the primes)

\[
T_0A = (e_1, ..., e_{p-d})_\mathbb{C} \oplus L, L = (e_{N-d+1}, ..., e_N)_\mathbb{C}.
\]

One can represent $T_0M$ in the form $T_0M = T_0^cM \oplus V$, where $V$ is the orthogonol complement of $T_0^cM$ in $T_0M$ with respect to the standard real scalar product on $\mathbb{C}^N \cong \mathbb{R}^{2N}$. Then $V \subset \mathbb{C}^m = (e_{N-m+1}, ..., e_N)_\mathbb{C}$. Moreover, $dim_{\mathbb{R}} V = m, (V)_\mathbb{C} = \mathbb{C}^m, \dim_{\mathbb{R}} V \cap L = d$ and $(V \cap L)_\mathbb{C} = L$. One can represent $\mathbb{C}^m$ in the form $\mathbb{C}^m = L + P$, where $P$ is a complex $(m-d)$-dimensional linear space, $V = (V \cap L) \oplus (V \cap P)$, $\dim_{\mathbb{R}} V \cap P = m - d$ and $(V \cap P)_\mathbb{C} = P$. After a non-degenerate $\mathbb{C}$-linear change of coordinates in $\mathbb{C}^m$ we obtain $L = \mathbb{C}^d, P = \mathbb{C}^{m-d}, V \cap L = i\mathbb{R}^d, V \cap P = i\mathbb{R}^{m-d}$ and $V = i\mathbb{R}^d + i\mathbb{R}^{m-d} = i\mathbb{R}^m$. Thus, we have $L = (e_{N-d+1}, ..., e_N)_\mathbb{C}$ and $V = i\mathbb{R}^m \subset (e_{N-m+1}, ..., e_N)_\mathbb{C}$. Then $T_0M = \{ t \in \mathbb{C}^N : t_j + \overline{t}_j = 0, j = N - m + 1, ..., N \}$. After a replacement of the functions $\rho_j$ by their (real) linear combinations, we obtain
\[ \rho_j' = z_{N-m+j} + m_{N-m+j} + o(|z|), j = 1, ..., m. \]

Since \( dp_{m-d+1} \wedge \ldots \wedge dp_m T_0A \neq 0 \), we have \( \rho_j' = \sum_{\nu=1}^{m-d} \lambda_{j\nu} \rho_{\nu} \) for \( j = 1, ..., m-d \) and \( \text{det}(\lambda_{j\nu}) \neq 0 \). Hence, for any complex linear space \( S \subset T_0^*M \) we obviously have \( \sum_{j=1}^{m-d} L_{0j}'(S) = \sum_{j=1}^{m-d} L_{0j}'(S) \), where \( L_{0j}' \) and \( L_{0j}' \) are the Levi operators of \( \rho_j' \) and \( \rho_j \) respectively. \( \text{Q.E.D.} \)

It is convenient to represent \( A \) as a graph over \( T_0A \cong \mathbb{C}^p \) in the coordinates (2.1). We set \( z' = (z_1, ..., z_{p-d}), z'' = (z_{p-d+1}, ..., z_{N-d}), z''' = (z_{N-d+1}, \ldots, z_N) \).

**Lemma 2.2.** Assume that the coordinates (2.1) are chosen. Then (in a neighborhood of the origin) \( A \) can be represented in the form \( z''' = g(z', z'''') \). Here \( g \) is a mapping holomorphic on a wedge \( W(\overline{E}, \overline{D}) \subset \mathbb{C}^p = T_0A \) with a \( C^2 \)-edge \( \overline{E} \) and \( g \) is of class \( C^2 \) on \( W(\overline{E}, \overline{D}) \cup \overline{E} \). The graph of the restriction \( g| E \) coincides with \( A_E \).

This is a simple consequence of the implicit function theorem. We leave details to the reader.

In what follows we omit the tilde and assume that

\[ E = \{(z', z''') \in D : r_j(z', z''') = 0, j = 1, ..., d\}, \]

\[ W(E, D) = \{(z', z''') \in D : r_j(z', z''') < 0, j = 1, ..., d\}. \]

**Lemma 2.3.** One has \( H_0(\rho_j, u, u) = 0 \) for \( u \in T_0^*(A_E) \) and \( j = 1, ..., m-d \).

**Proof.** Since \( A_E \subset M \), we have \( \rho_j = \rho_j(z', g(z', z'''), z''', \overline{f}(z', z'''), \overline{\tau}) = 0 \), for \( j = 1, ..., m \) and \( (z', z''') \in E \). By the division lemma we obtain \( \rho_j = \sum \lambda_{j\nu} r_{\nu}(z', z''') \), where the functions \( \lambda_{j\nu} \) are of class \( C^1 \) in a neighborhood of the origin in \( \mathbb{C}^p \). Since \( dp_j(0)|T_0A = 0, j = 1, ..., m-d \), we conclude that \( \lambda_{j\nu}(0) = 0 \) for \( j = 1, ..., m-d, \nu = 1, ..., d \). Hence, \( \rho_j = o(|z'|^2 + |z'''|^2) \) for \( (z', z''') \in T_0^*E \). This implies our statement. \( \text{Q.E.D.} \)

Let us show that (1.9) does not depend on the choice of the hermitian scalar product which defines the Levi operators. Let \( <, >' \) be another hermitian scalar product and let the Levi operators \( L_{0j}' \) be defined by the condition \( < L_{0j}'(u), v >' = H_0(\rho_j, u, v) \) for \( u, v \in T_0^*M \). Lemma 2.3 implies

\[ H_0(\rho_j, u, v) = < L_{0j}'(u), v >= < L_{0j}'(u), v >' = 0 \]

for \( j = 1, ..., m-d \) and \( u, v \in T_0^*(A_E) \). Also, it is easy to show that \( L_{0j}' = RL_{0j}' \), where \( R \) is a non-degenerate \( \mathbb{C} \)-linear operator on \( T_0^*M \). Hence, the spaces \( S = \sum_{j=1}^{m-d} L_{0j}'(T_0^*A_E) \) and \( S' = \sum_{j=1}^{m-d} L_{0j}'(T_0^*A_E) \) have the same dimension. But \( S \) (resp. \( S' \)) is contained in the orthogonal complement of \( T_0^*(A_E) \) with respect to \( <, > \) (resp. \( <, >' \)). Thus, if (1.9) holds for the operators \( L_{0j}' \) then (1.9) also holds for \( L_{0j}' \).

Now we are going to investigate the condition (1.9) in the coordinates (2.1). Without loss of generality one can assume that \( < z, w >= \sum z_{j} \overline{w}_j \) in the coordinates (2.1) and this hermitian scalar product defines the Levi operators \( L_{0j}' \).

For positive integers \( j, \nu_1, ..., \nu_k \) we consider the matrices of the following form
\[ H(j, \nu_1, ..., \nu_k) = \left( \frac{\partial^2 \rho_j}{\partial z_\nu \partial z_\mu} (0) \right)_{\nu=\nu_1, ..., \nu_k, \mu=p-d+1, ..., N-m}, \]  
(there are \( k \) rows and \((N - m - p + d)\) columns in (2.2)).

**Lemma 2.4.** Assume that the coordinates (2.1) are chosen. Then the condition of Levi transversality is equivalent to the following: there exist collections \( (j(n), \nu_1(n), ..., \nu_k(n)) \), \( n = 1, ..., s \) of positive integers so that \( 1 \leq j(1) < ... < j(s) \leq m - d \), \( 1 \leq \nu_1(n) < ... < \nu_k(n) \leq p - d \) for \( n = 1, ..., s \), \( \sum_{n=1}^{s} k(n) = N - m - p + d \) and

\[ \det H \neq 0, \]

where \( H \) is a \((N - m - p + d) \times (N - m - p + d)\)-matrix of the form

\[ H = \begin{bmatrix} H(j(1), \nu_1(1), ..., \nu_k(1)) \\ \vdots \\ H(j(s), \nu_1(s), ..., \nu_k(s)) \end{bmatrix}. \]  

**Proof.** Let \( e_j, j = 1, ..., N \) be the standard basis in \( \mathbb{C}^N \). Since the spaces \( L_j^0(T_0^c A_E) \) are contained in the orthogonal complement of \( T_0^c A_E \) (see Lemma 2.3), the condition (1.9) is equivalent to

\[ \dim \left[ \sum_{j=1}^{m-d} L_j^0(T_0^c A_E) \right] = N - m - p + d. \]

Hence, there are \((N - m - p + d)\) \( \mathbb{C} \)-linearly independent vectors among the vectors \( L_j^0(e_k), j = 1, ..., m - d, k = 1, ..., p - d \).

Let \( H_j \) be the matrix of the restriction of \( H_0(\rho_j, u, u) \) on \( T_0^c M \) (with respect to the basis \( e_\nu, \nu = 1, ..., N - m \)). Each Levi operator has the matrix (with respect to this basis) that is transpose of \( H_j \). But the coordinates of the vector \( L_j^0(e_k) \) form the \( \nu \)-th column of the matrix of \( L_j^0 \). Let us consider the system of vectors that is the union over \( j = 1, ..., m - d \) of the first \((p-d)\) rows of \( H_j \). It was just shown that there are \((N - m - p + d)\) \( \mathbb{C} \)-linearly independent vectors in this system. Lemma 2.3 implies that

\[ \frac{\partial^2 \rho_j}{\partial z_\nu \partial z_\mu} (0) = 0, \nu, \mu = 1, ..., p - d, j = 1, ..., m - d. \]

We consider the matrix \( C \) which is formed by the above-mentioned \((N - m - p + d)\) independent rows. Then \( C \) has the form \( C = (O|H) \), where \( H \) is the matrix (2.4). Hence, the linear independence of the rows of \( C \) is equivalent to (2.3), \( Q.E.D. \).
3. COMPLEXIFICATION.

Assume the coordinates (2.1) are chosen. According to Lemma 2.2 we have in a neighborhood $U \ni 0$ such that

$$A \cap U = \{ z \in U : z'' = g(z', z''') \},$$

where $g$ is a mapping holomorphic on $W(E, D)$ and of class $C^2$ on $W(E, D) \cup E$. Moreover, $g(0) = 0$ and $dg(0) = 0$. We set

$$\rho^i(z', z''') = (\rho^i|A) = \rho_j(z', g(z', z'''), z''', \overline{g(z', z'''}, z'''), \overline{z''').$$

Since $A_E \subset M$, one has $\rho^i|E = 0$ for $j = 1, ..., m$. By $d\rho^{m-d+1} \wedge ... \wedge d\rho^m \neq 0$, we get

$$(3.1) \quad E = \{(z', z''') \in \mathbb{C}^p : \rho^j = 0, j = m - d + 1, ..., m\},$$

in a neighborhood of the origin.

Let us consider the vector fields $T_q, q = 1, ..., p - d$ (the sections of the complex tangent bundle $T^eE$) of the form

$$(3.2) \quad T_q = \frac{\partial}{\partial z_q} - \sum_{j=1}^{d} a_{jq}(z', z''', \overline{z}', \overline{z'''}) \frac{\partial}{\partial z_{N-d+j}},$$

where

$$(3.3) \quad a_{jq} = \sum_{s=1}^{d} b_{js} \frac{\partial \rho^{s+m-d}}{\partial z_q}.$$ 

Here $(b_{js})$ is the inverse of the Jacobian matrix

$$(3.4) \quad S = \left( \frac{\partial \rho^k}{\partial z_l} \right)_{k=m-d+1, ..., m \atop l=N-d+1, ..., N}. $$

It is easy to see that the vector fields $T_q, q = 1, ..., p - d$ form a basis of the bundle $T^eE$ over a neighborhood of the origin in $E$.

**Lemma 3.1.** For $(z', z''') \in E$ we have

$$T_{\nu} \rho^j = 0, \nu = 1, ..., p - d, j = 1, ..., m - d$$

in a neighborhood of the origin.

**Proof.** Since $\rho^i|E = 0$, there are $C^1$-functions $\lambda_{jk}(z', z''')$ such that

$$\rho^j = \sum_{k=m-d+1}^{m} \lambda_{jk} \rho^k, j = 1, ..., m - d,$$
where \((z', z'')\) belongs to a neighborhood of the origin in \(C^p\). Hence, for \((z', z'') \in E\) we have

\[
\frac{\partial \rho^j}{\partial z_q} = \sum_{k=m-d+1}^m \lambda_{jk} \frac{\partial \rho^k}{\partial z_q},
\]

for \(j = 1, \ldots, m - d, q = 1, \ldots, p - d\). Fix an arbitrary \(j\). We assume that \(q = N - d + 1, \ldots, N\) in (3.5) and consider (3.5) as a system of linear equations in \(\lambda_{jk}, k = m - d + 1, \ldots, m\). Then the matrix of this system is transpose of \(S\) of the form (3.4). Let us consider the rows

\[
U_q = \begin{pmatrix} \frac{\partial \rho^{m-d+1}}{\partial z_q} & \cdots & \frac{\partial \rho^m}{\partial z_q} \end{pmatrix},
\]

and the columns

\[
\Lambda_j = \begin{pmatrix} \lambda_{jm-d+1} \\ \vdots \\ \lambda_{jm} \end{pmatrix}, \quad V_j = \begin{pmatrix} \frac{\partial \rho^j / \partial z_{N-d+1}}{\partial z_q} & \cdots & \frac{\partial \rho^j / \partial z_N}{\partial z_q} \end{pmatrix}.
\]

Since the matrix (3.4) is non-degenerate in the origin, one has \(\Lambda_j = (S)^{-1} V_j\), where \((S)^t\) denotes the transposed matrix. Then (3.5) implies

\[
\frac{\partial \rho^j}{\partial z_q} = U_q \Lambda_j = U_q (S)^{-1} V_j,
\]

for \(j = 1, \ldots, m - d, q = 1, \ldots, p - d\). Hence,

\[
\frac{\partial \rho^j}{\partial z_q} - (V_j)^t (S)^{-1} (U_q)^t = 0
\]

In view of (3.2), (3.3), (3.4), (3.6), (3.7) the last equality is exactly equivalent to the assertion of Lemma 3.1, \(Q.E.D\).

Let us consider the functions \(\tilde{\rho}_j(z, \zeta)\) that we obtain after the substitution of \(\zeta \in C^N\) instead of \(\bar{\tau}\) in the expansion of \(\rho_j(z, \bar{\tau})\) at the origin. Then the functions \(\tilde{\rho}_j(z, \zeta)\) are holomorphic on a neighborhood of the origin in \(C^{2N}\) and \(\tilde{\rho}_j(z, \zeta)|\{\zeta = \bar{\tau}\} = \rho_j(z, \bar{\tau})\). We set

\[
M^c = \{(z, \zeta) \in C^{2N} : \tilde{\rho}_j(z, \zeta) = 0, j = 1, \ldots, m\}.
\]

Then \(M^c\) is a complex \((2N - m)\)-dimensional manifold in a neighborhood of the origin in \(C^{2N}\). It is called a complexification of \(M\). Also, we introduce a real manifold \(\tilde{A}_E = \{(z, \bar{\tau}) \in C^{2N} : z \in A_E\}\). Then \(\tilde{A}_E \subset M^c \cap \{\zeta = \bar{\tau}\}\) and \(\dim_{\mathbb{R}} \tilde{A}_E = 2p - d\).

Set \(\tilde{\rho}^j(z', z'', \zeta) = \tilde{\rho}_j(z', g(z', z''), \zeta)\). For \(j = 1, \ldots, m - d, \nu = 1, \ldots, p - d\) we define the functions
(3.9) \[ \varphi_j(z', z'', \zeta) = \frac{\partial \tilde{\rho}_j}{\partial z_{\nu}} - \sum_{t=1}^{d} \tilde{a}_{t\nu} \frac{\partial \tilde{\rho}_j}{\partial z_{n-d+t}}, \]

where

\[ \tilde{a}_{t\nu} = \sum_{s=1}^{d} b_{ts} \frac{\partial \tilde{\rho}^{s+m-d}}{\partial z_{\nu}}, \]

and \((b_{ts})\) is the inverse of the matrix

\[ \left( \frac{\partial \tilde{\rho}^k}{\partial z_l} \right)_{k=m-d+1, \ldots, m}^{l=N-d+1, \ldots, N} \]

Then the functions (3.9) are holomorphic on \(W(E,D) \times U\) and continuous on \((W(E,D) \cup E) \times U\), where \(U\) is a neighborhood of the origin in \(\mathbb{C}^N\).

Let \((j(n), \nu_1(n), \ldots, \nu_k(n))\), \(n = 1, \ldots, s\) be the collections of positive integers from Lemma 2.4. We define a set \(X \subset \mathbb{C}^{2N}\) of the form

(3.10) \[ X = X_0 \cap X_1 \cap \ldots \cap X_s \cap M^c, \]

where

\[ X_0 = \{(z, \zeta) \in \mathbb{C}^{2N} : z'' = g(z', z'')\}, \]

and for \(n = 1, \ldots, s\)

(3.11) \[ X_n = \{(z, \zeta) \in \mathbb{C}^{2N} : \varphi_j^{(n)}(z', z'', \zeta) = 0, \nu = \nu_1(n), \ldots, \nu_k(n)\}. \]

Lemma 3.2. \(X\) can be represented in the following form (near the origin)

(3.12) \[ z'' = g(z', z'''), \]
\[ \zeta_\nu = \psi_\nu(z', z'''', \zeta'), \nu = p-d+1, \ldots, N, \]

Here \(\zeta' = (\zeta_1, \ldots, \zeta_{p-d})\) and the functions \(\psi_\nu\) are holomorphic on \(W(E,D) \times U'\) and of class \(C^1\) on \((W(E,D) \cup E) \times U'\), where \(U'\) is a neighborhood of the origin in \(\mathbb{C}^{p-d}\).

Proof. Let us consider the set \(X_1 \cap \ldots \cap X_s \cap M^c\) that is defined by equations (3.11) (with \(n = 1, \ldots, s\)) and (3.8). We compute the Jacobian matrix \(J\) of this united system (which contains \((N-p+d)\) equations) with respect to the variables \(\zeta_{p-d+1}, \ldots, \zeta_N\) at the origin. One has

\[ \frac{\partial \tilde{\rho}_j}{\partial z_{\nu}}(z', z'', \zeta) = (\frac{\partial \tilde{\rho}_j}{\partial z_{\nu}})(z, \zeta) + \sum_{q=p-d+1}^{N-d} (\frac{\partial \tilde{\rho}_j}{\partial z_q})(z, \zeta) \frac{\partial g_q}{\partial z_{\nu}}. \]
where we set \( z = (z', g(z'), z'', z''') \). We recall that \( g(0) = 0 \) and \( \partial g_q/\partial z_\nu(0) = 0 \) for any \( q, \nu \). Note also that \( p - d < N - m + 1 \) (see section 2). Hence, (2.1) implies \( \partial \rho_j/\partial z_\nu(0) = 0 \) for \( \nu = 1, \ldots, p - d \) and each \( j \). Moreover, \( \partial \rho_j/\partial z_{N-d+t}(0) = 0, j = 1, \ldots, m - d, t = 1, \ldots, d \). Therefore,

\[
\frac{\partial^2 \rho_j}{\partial \xi_\mu \partial \xi_\nu}(0) = \frac{\partial^2 \rho_j}{\partial z_\nu \partial z_\mu}(0)
\]

for \( \nu = 1, \ldots, p - d, \mu = p - d + 1, \ldots, N - m, j = 1, \ldots, m - d \). Hence, \( J \) has the form

\[
J = \begin{pmatrix}
H & 0 \\
0 & I_m
\end{pmatrix},
\]

where \( H \) is the matrix (2.4), \( I_m \) is the identity \((m \times m)\)-matrix. Now (2.3) implies \( \det J \neq 0 \). By the implicit function theorem we get our statement, \( \text{Q.E.D.} \)

We denote by \( X_E \) the set defined by the equations (3.12) under the condition \( (z', z'', \zeta') \in E \times U' \).

**Lemma 3.3.** One has the inclusion \( \hat{A}_E \subset X_E \).

**Proof.** Since \( A_E = \{ z \in \mathbb{C}^N : z'' = g(z', z'''), (z', z''') \in E \} \), one can conclude that \( (\partial g^2/\partial z_\nu)(\zeta = \mathbf{z}) = \partial g^2/\partial z_\nu \) for \( z \in A_E \). Hence, (3.9) and Lemma 3.1 imply \( \varphi_\nu'(\zeta = \mathbf{z}) = 0 \) for \( z \in A_E \). Since \( A_E \subset M \), one has \( \varphi_\nu'|A_E = 0 \). Taking into account the equivalence of (3.10) and (3.12), we obtain our assertion, \( \text{Q.E.D.} \)

**4. THE REFLECTION PRINCIPLE.**

We consider the coordinates \( (z, \zeta) \in \mathbb{C}^{2N} = \mathbb{C}^N \times \mathbb{C}^N \). Set \( z = (\alpha, \beta) \), where \( \alpha = (z_1, \ldots, z_n) \), \( \beta = (z_{n+1}, \ldots, z_N) \).

**Lemma 4.1.** Let \( S \) be a generic \( C^1 \)-manifold in \( \mathbb{C}_\alpha \) and let \( W \) be a wedge in \( \mathbb{C}^n \) with the edge \( S \). Suppose that \( F : W \to \mathbb{C}^{N-n} \times \mathbb{C}_\xi^N \) is a holomorphic mapping of class \( C^1 \) on \( W \cup S \). Let \( Y \subset \mathbb{C}^{2N} \) be the graph of \( F \). Assume that there exists a real \( n \)-dimensional \( C^1 \)-manifold \( S' \subset \mathbb{C}^{2N} = \mathbb{C}_\zeta^N \times \mathbb{C}_\xi^N \) such that \( S' \subset Y_S \cap \{ \zeta = \mathbf{z} \} \) (here \( Y_S = \{ (z, \zeta) : (\beta, \zeta) = F(\alpha), \alpha \in S \}) \). Then \( F \) extends holomorphically to \( S \).

**Proof.** One can assume \( 0 \in S' \). We make a change of coordinates of the form \( z = \xi' + i\xi'', \zeta = \xi' - i\xi'' \) (here \( \xi' = (\xi_1, \ldots, \xi_N), \xi'' = (\xi_{N+1}, \ldots, \xi_{2N}) \)). In the new coordinates the diagonal \( \{ \zeta = \mathbf{z} \} \) coincides with \( \mathbb{R}^{2N} = \{ \xi = (\xi', \xi'') : \exists \xi = 0 \} \). Since \( T_0Y \) is a complex \( n \)-dimensional linear space in \( \mathbb{C}^{2N} \), there exists a \( n \)-dimensional coordinate plane \( \Pi \) in \( \mathbb{C}^{2N} \) such that \( \pi : T_0Y \to \Pi \) is an isomorphism (here \( \pi : \mathbb{C}^{2N} \to \Pi \) is the natural projection). One can assume that \( \Pi \) is the plane of the variables \( \xi_1, \ldots, \xi_N \). The restriction \( \pi : Y \to \Pi \) is a local biholomorphism and \( \tilde{W} = \pi(Y) \) is a wedge in \( \Pi \) with the edge \( Q = \pi(Y_S) \) that is generic in a neighborhood of the origin. Since \( S' \subset \mathbb{R}^{2N} \) and the restriction \( \pi|S' \) is a local diffeomorphism, \( \pi(S') \) coincides with the plane of the variables \( \Re \xi_1, \ldots, \Re \xi_n \) in a neighborhood of the origin. We denote this plane by \( \mathbb{R}^n \). Since \( S' \subset Y_S \), we conclude that \( \mathbb{R}^n \subset Q \).

Let \( G \) be a mapping holomorphic on \( \tilde{W} \) and of class \( C^1 \) on \( \tilde{W} \cup Q \) so that \( Y \) is the graph of \( G \). Then \( S' \) is the graph of the restriction \( G|\mathbb{R}^n \). Since \( S' \subset \mathbb{R}^{2N} \), the restriction \( G|\mathbb{R}^n \) is a real valued mapping. Therefore, one can apply the edge
of the wedge theorem [Ru] to the mappings $G$ and $G^* = \overline{G}(\xi_1, ..., \xi_n)$. Thus, $G$ extends holomorphically to a neighborhood of the origin. Therefore, $Y$ extends to a complex manifold in a neighborhood of the origin. This implies our assertion, Q.E.D.

**Completion of the proof of Theorem 1.** Setting $S = E \times U'$ and $S' = \tilde{A}S$, we apply Lemma 4.1 to the mapping $F = (g, \psi_{p-d+1}, ..., \psi_N)$ (Lemmas 3.2 and 3.3 show that one can do it). We get that $F$ extends holomorphically to a neighborhood of the origin. Hence, $g$ extends holomorphically. But this means that $f$ extends holomorphically to a neighborhood of the origin in $\mathbb{C}^p$. Thus, the mapping $\Phi(x) = (x, f(x))$ is holomorphic near the origin and the condition (ii) of Definition 1 implies that $E = \{x : \rho_j \circ \Phi(x) = 0, j = m - d + 1, ..., m\}$. Hence, $E$ is real analytic, Q.E.D.

5. CERTAIN SPECIAL CASES.

**Proof of Proposition 2.** Without loss of generality we assume $a = 0, a' = f(a) = 0$. It is sufficient to show that $A = \Gamma_f$ and $M = S \times S'$ are Levi transverse at the origin. Let $r_j', j = 1, ..., d'$ be the defining functions of $S'$ and let $r_j, j = 1, ..., d$ be the defining functions of $S$. Considering $\rho_j = r_j' - \sum \lambda_{j\nu} r_{\nu}$, we get the defining functions of $M$ with the condition $d\rho_j[T_0a] = 0$. The matrices of the Levi operators $L_0^j$ of the functions $\rho_j$ (on $T_0^\flat M$) have the form

$$\begin{pmatrix} R^j & 0 \\ 0 & L_0^j \end{pmatrix}$$

where $R^j$ being the operators on $T_0^\flat S$, and $L_0^j$ being the Levi operators of $r_j'$ (we identify operators and their matrices). Since $E = S$ and $T_0^\flat A_E = \{(v, df_0(v)) : v \in T_0^\flat S\}$, one can conclude by (1.13) that the dimension of the space $L = \sum_{j=1}^d L_0^j(T_0^\flat S)$ is equal to $p' - d'$. Since $L$ and $T_0^\flat (A_E)$ are orthogonal by Lemma 2.3, we get (1.9), Q.E.D.

Let $<,>$ be a hermitian scalar product on $\mathbb{C}^{n'}$ defining the Levi operators. Assume that the conditions of Corollary 1 hold. We show that in this case (1.13) also holds. Assume that this is not true. Then there exists a vector $\xi \in T_0^\flat S \setminus \{0\}$ orthogonal (with respect to $<,>$) to the space $\sum_{j=1}^d L_0^j(df_0(T_0^\flat S))$. Since $df_0$ is surjective, one has $df_0(T_0^\flat S) = T_0^\flat S'$. Therefore, $<L_0^\flat(\xi), \eta> = <L_0^\flat(\eta), \xi> = 0$ for each $\eta \in T_0^\flat S', j = 1, ..., d'$ ($L_0^\flat$ is hermitian). We obtain a contradiction to the condition of Levi non-degeneracy of $S'$. This proves Corollary 1.

**Proof of Corollary 4.** The condition (i) of the Levi transversality is trivial. Since the intersection $T_0A \cap T_0M$ is a totally real space in $T_0A$, the condition (ii) holds. Since $T_0^\flat A_E = T_0^\flat M = \{0\}$, the condition (iii) also is trivial. In this case we do not need the functions (3.9) to define the set $X$. Hence, the proof of Theorem 1 is valid for $C^1$ - wedges and $C^1$ - mappings in this special case, Q.E.D.

Corollary 2 is a special case of Corollary 4; the proof of Corollary 3 is quite similar.

6. PROOF OF THEOREM 2.

Set $E = \{x \in U : r_j(x) = 0, j = 1, ..., n\}$, $\partial r_1 \land ... \land \partial r_n \neq 0$, $E' = \{x' \in V : \psi_j(x', \overline{x'}) = 0, j = 1, ..., n\}$, $\partial \psi_1 \land ... \land \partial \psi_{n'} \neq 0$. Let $\varphi(x, \overline{x})$ be a real analytic
Lemma 6.1. We have
\[ E \cap U = (6.1) \]
for \( x \in U \). Here \( \lambda_j \in C^1(U) \).

Proof. Since \( f(\Omega \cap U) \subset \Omega' \cap V \), we get that \( \rho \circ f \) is a negative plurisubharmonic function on \( \Omega \cap U \). By Hopf’s lemma we conclude \(|\rho \circ f(x)| \geq Cdist(x, b\Omega)\) for each \( x \in \Omega \cap U \) and a positive constant \( C \) (here \( dist \) is the Euclidean distance).

Hence, \( d(\rho \circ f)(0) \neq 0 \). Since the domain \( \Omega \cap U = \{ x \in U : \varphi(x, \overline{\varphi}) < 0 \} \) is contained in the domain \( D = \{ x \in U : (\rho \circ f) < 0 \} \), the tangent planes of their boundaries coincide at each point of \( E \subset b\Omega \cap bD \). This implies the desired statement, Q.E.D.

Lemma 6.1 and (6.1) imply
\[ \frac{\partial}{\partial x_\nu} (\rho \circ f)(x) = (6.2) \]
for \( x \in E \cap U \). We get
\[ \frac{\partial \varphi}{\partial x_n} \frac{\partial (\rho \circ f)}{\partial x_\nu} - \frac{\partial \varphi}{\partial x_\nu} \frac{\partial (\rho \circ f)}{\partial x_n} = 0, \nu = 1, ..., n - 1, \]
for \( x \in E \cap U \). Set \( y \in \mathbb{C}^n \), \( y' \in \mathbb{C}^{n'} \), \( \mathbb{C}^N = \mathbb{C}^n \times \mathbb{C}^{n'} \). We define the functions
\[ h_\nu(x, y, y') = (6.4) \]
for \( \nu = 1, ..., n - 1, \)
and
\[ x' = f(x), \]
where \( x \in \overline{\Omega} \cap U \) and \( (y, y') \) belongs to a neighborhood of the origin in \( \mathbb{C}^N \).
Lemma 6.2. The set $A$ defined by (6.5), (6.6), is a complex $n$-dimensional manifold with $C^1$-boundary in a neighborhood of the origin in $\mathbb{C}^{2N}$.

Proof. We compute the Jacobian matrix $J$ of (6.5) with respect to the variables $(y, y')$ at 0. For $\mu = 1, \ldots, n-1$ we have

$$\frac{\partial h_\nu}{\partial y_\mu}(0) = \frac{\partial^2 \varphi}{\partial x_\mu \partial y_\nu}(0,0) + \frac{\partial \varphi}{\partial x_\mu}(0,0) \frac{\partial^2 \rho(f(x), y')}{\partial y_\mu \partial x_\nu}(0,0) - \frac{\partial^2 \varphi}{\partial x_\nu \partial y_\mu}(0,0) \frac{\partial \rho(f(x), y')}{\partial x_\mu}(0,0) - \frac{\partial \varphi}{\partial x_\nu}(0,0) \frac{\partial^2 \rho(f(x), y')}{\partial y_\mu \partial x_\nu}(0,0).$$

One has

$$\frac{\partial^2 \varphi}{\partial x_\mu \partial y_\nu}(0,0) = \frac{\partial^2 \rho(f(x), y')}{\partial y_\mu \partial x_\nu}(0,0) = \frac{\partial \varphi}{\partial x_\nu}(0,0) = 0,$$

for $\mu, \nu = 1, \ldots, n-1, k = 1, \ldots, n$. Also,

$$\frac{\partial^2 \varphi}{\partial x_\nu \partial y_\mu}(0,0) = \delta_{\nu \mu},$$

( the Kronecker symbol ). Lemma 6.1 implies

$$\frac{\partial \rho(f(x), y')}{\partial x_\mu}(0,0) = \alpha \neq 0$$

. Thus,

$$J = \begin{pmatrix} -\alpha I_{n-1} & 0 & 1 \\ 0 & 1 \\ 0 & 0 & I_n' \end{pmatrix}$$

and $\det J \neq 0$. Hence, the implicit function theorem implies that $A$ can be represented in the following form :

(6.7) \hspace{1cm} x' = f(x), y = g(x), y' = p(x).

and the mapping $(f, g, p)$ is holomorphic on $\Omega \cap U$ and of class $C^1$ on $\overline{\Omega} \cap U$, Q.E.D.

Let $W$ be a wedge in $\Omega \cap U$ with the edge $E$. In view of (6.7) one can assume that $A$ is the graph of the mapping $F = (f, g, p)$ over $W$.

Lemma 6.3. The set

$$A_E = \{(x, y, x', y') : (x', y, y') = F(x), x \in E\}$$

is contained in the diagonal

$$M = \{(x, x', y, y') : x = y, x' = y'\}.$$  

Proof. We set $x = y, x' = y', x \in E$ in (6.5), (6.6). Since $E \subset b\Omega$ and $F(E) \subset E'$, we get $\varphi(x, x') = 0$ and $\psi(x', x') = 0$. In view of (6.3) and (6.4) we obtain

$$h_\nu(x, x', y, y') = h_\nu(x, x', f(x)) = 0.$$  

Hence, the equivalence of (6.5), (6.6) and (6.7) implies that $A_E$ and $A_E \cap M$ have the same real dimension. This gives the desired assertion, Q.E.D.

Thus, one can apply Corollary 4 to $A$ and $M$. We conclude that the mapping $F = (f, g, p)$ ( and, certainly, $f$ ) extends holomorphically to a neighborhood of the origin and $E$ is real analytic, Q.E.D.
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