Power law distribution of Rényi entropy for equilibrium systems having nonadditive energy

Qiuping A. Wang
Institut Supérieur des Matériaux du Mans,
44, Avenue F.A. Bartholdi, 72000 Le Mans, France

Abstract

Using Rényi entropy, a possible thermostatistics for nonextensive systems is discussed. We show that it is possible to have the $q$-exponential distribution function for nonextensive systems having nonadditive energy but additive entropy. It is also shown that additive energy, as an approximation within nonextensive statistics, is not suitable for discussing fundamental problems for interacting systems.

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1 Introduction

Rényi entropy[1]

$$S^R = \frac{\ln \sum_{i=1}^w p_i^q}{1 - q}, \quad q \geq 0 \tag{1}$$

often applied in the study of multi-fractal and chaotic systems[2], where $p_i$ is the probability that the system is at the state labelled by $i$ (Boltzmann constant $k=1$) and $w$ is the total number of the states. In these studies, Rényi entropy was associated with the exponential probability distributions of BGS[2] which as a matter of fact are not the distributions derived from this entropy. Since the proposal[3] of the nonextensive statistical mechanics (NSM), there is a growing interest in Rényi entropy which has been compared to Tsallis entropy[3] $S^T = \frac{\sum_{i=1}^w p_i^{q-1}}{1-q}$, associated with a $q$-exponential distribution $exp_q(x) = [1 + (1 - q)x]^{1/(1-q)}$, in the discussions of possible
nonextensive statistics and the relative fundamental problems such as thermodynamic equilibrium and stability\cite{4, 5, 6, 7, 8, 9, 10, 11, 12} for systems having additive energy or infinite number of states\cite{6}.

Regarding the possible statistics of Rényi entropy, some questions should be asked: $S^R$ is additive just as, e.g., the entropies of Boltzmann-Gibbs-Shannon statistics (BGS) for systems having product joint probability\cite{13}, but should it be associated to independent systems having additive energy just as in BGS? Should it be associated to exponential probability distributions as has been done in many works? Are there alternative distributions intrinsic to Rényi entropy if it can be maximized to get thermodynamic equilibrium? If the Rényi statistics is not extensive, what are its possible nonextensive properties? Due to the importance of $S^R$ in the study of chaos and fractals and since $S^R$ is identical to Boltzmann entropy $S = \ln W$ for the fundamental microcanonical ensemble\cite{2}, the possible answers to the above questions would be interesting for both BGS and its possible extensions based on $S^R$.

In this paper, I will present a thermo-statistics derived from Rényi entropy for equilibrium systems with nonadditive energy required by the existence of thermodynamic equilibrium\cite{14}. This formalism may be considered as an alternative to BGS and to NSM for canonical systems with additive entropy but nonadditive energy\cite{15}.

## 2 Rényi and Tsallis entropies

There is a monotonic relationship between these two entropies:

$$S^R = \frac{\ln[1 + (1 - q)S^T]}{1 - q} \quad \text{or} \quad S^T = e^{\frac{(1-q)S^R - 1}{1 - q}},$$

(2)

and, for complete probability distribution ($\sum p_i = 1$) in microcanonical ensemble, $S^R$ is identical to the Boltzmann entropy $S$:

$$S^R = S = \ln w$$

(3)

since $\sum p_i^q = w^{1-q}$. Other properties of $S^R$ can be found in \cite{1, 2, 4}.

The concavity of $S^R$ and $S^T$ for $q > 1$ is shown in Figure 1 and 2, respectively (they are convex and have minimums for $q < 0$, so we do not consider this case in this paper). It is worth noticing that the two entropies
get their maximum at the same time for any \( q \). But the maximum of \( S^R \) is \( \ln W \) and independent of \( q \), while the maximum of \( S^T \) is \( \frac{W^{1-q-1}}{1-q} \) and decreases down to zero when \( q \to \infty \) and increases up to \( W - 1 \) when \( q \to 0 \). It should be mentioned that \( S^R \) is not always concave for \( q > 1 \), as shown in Figure 1.

![Figure 1: The concavity of Rényi entropy \( S^R \) for \( q > 0 \) with a two probability distribution \( p_1 = x \) and \( p_2 = 1 - x \). Note that the maximal value does not change with \( q \). The maximum becomes more sharp for larger \( q \). The curve for \( q > 1 \) shows that \( S^R \) is not always concave, but the maximum remains the unique extremum.](image)

Due to the fact that \( S^R \) is a monotonically increasing function of \( S^T \), they reach the extremum together (see Figures 1 and 2). One can hope that the maximum entropy (for \( q > 0 \)) will give same results with same constraints. Indeed, Rényi entropy has been used to derive, by maximum entropy method, the Tsallis \( q \)-exponential distribution within an additive energy formalism\cite{16, 17}. In what follows, we present a thermostatistics based on \( S^R \) for nonextensive systems having additive entropy and nonadditive energy.
Figure 2: The concavity of Tsallis entropy $S^T$ for $q > 0$ with a two probability distribution $p_1 = x$ and $p_2 = 1 - x$. Note that the maximal value increases from zero to unity with $q$ decreasing from infinity to zero.

3 Canonical distribution of Rényi

We suppose complete distribution $\sum_{i=1}^{w} p_i = 1$ and $U = \sum_{i=1}^{w} p_i E_i$ where $U$ is the internal energy and $E_i$ the energy of the system at the state $i$. We will maximize as usual the following functional :

$$F = \ln \sum_{i} p_i^q = + \alpha \sum_{i=1}^{w} p_i - \gamma \sum_{i=1}^{w} p_i E_i.$$  \hspace{1cm} (4)

We get

$$p_i \propto \left[ \alpha - \gamma E_i \right]^{1/(q-1)}.$$  \hspace{1cm} (5)

Since $S^R$ recovers Boltzmann-Gibbs entropy $S = - \sum_{i=1}^{w} p_i \ln p_i$ when $q = 1$, it is logical for us to require that Eq.(5) recovers the conventional exponential distribution for $q = 1$. This leads to

$$p_i = \frac{1}{Z} \left[ 1 - (q - 1) \beta E_i \right]^{1/(q-1)}.$$  \hspace{1cm} (6)
where \((q - 1)\beta = \gamma / \alpha\) and \(Z = \sum_{i=1}^{w} [1 - (q - 1)\beta E_i]^{1/(q-1)}\). We will show the physical meaning of \(\beta\) later.

Note that the second derivative \(\frac{d^2 F}{dp_i^2}\) is negative for any distribution only when \(q < 1\). In general \(\frac{d^2 F}{dp_i^2}\) may be positive for \(q > 1\) so that the above distribution Eq. (6) is not stable.

This formalism can be applied to an important family of chaotic behaviors described by Lévy flight if we assume \(0 \leq q \leq 2/3\). In this case, the first moment \(\bar{x}\) calculated with the usual expectation is convergent for \(1/2 \leq q \leq 2/3\) and can be used as a constraint of maximum entropy to obtain \(p(x) \sim [1 + (1 - q)\beta x]^{-1/(1-q)}\). For Lévy flight distribution with large \(x\), \(p \sim x^{-1-\gamma}\), one obtain: \(q = \gamma / (1 + \gamma)\) (\(0 < \gamma < 2\)).

4 Mixte character: nonadditive energy and additive entropy

It has been shown that, for thermal equilibrium to take place in nonextensive systems, the internal energy of the composite system \(A + B\) containing two subsystems \(A\) and \(B\) must satisfy

\[
U(A + B) = U(A) + U(B) + \lambda U(A)U(B) \tag{7}
\]

which means

\[
E_{ij}(A + B) = E_i(A) + E_j(B) + \lambda E_i(A)E_j(B) \tag{8}
\]

where \(\lambda\) is a constant. Applying Eq. (8) to Eq. (6), we straightforwardly get the product joint probability:

\[
p_{ij}(A + B) = p_i(A)p_j(B) \tag{9}
\]

and the additivity of \(S^R\):

\[
S^R(A + B) = S^R(A) + S^R(B) \tag{10}
\]

if \(\lambda = (1 - q)\beta\). So \(S^R\) is essentially different from \(S^T\), because in this case \(S^T\) is nonadditive with \(S^T(A + B) = S^T(A) + S^T(B) + (1 - q)S^T(A)S^T(B)\). Note that we do not need independent or noninteracting or weakly interacting subsystems for establishing the additivity of \(S^R\) or the nonadditivity of \(S^T\), as discussed in \([14, 19, 20]\)). So in this formalism, we can deal with interacting systems with nonadditive energy but additive entropy.
5 Zeroth law and temperature

It is easy to show that, from Eq. (6),

$$S^R = \ln Z + \ln[1 + (1 - q)\beta U]/(1 - q).$$

(11)

So we have

$$\beta = [1 + (1 - q)\beta U] \frac{\partial S^R}{\partial U}$$

(12)

or

$$\frac{1}{\beta} = \frac{\partial U}{\partial S^R} - (1 - q)U.$$  

(13)

Since $[1 + (1 - q)\beta U]$ is always positive ($q$-exponential probability cutoff), $\beta$ has always the same sign as $\frac{\partial S^R}{\partial U}$. $\beta$ can be proved to be the effective inverse temperature if we consider the zeroth law of thermodynamics. Let $\delta S^R$ be a small change of $S^R$ of the isolated composite system $A + B$. Equilibrium means $\delta S^R = 0$. From Eq. (10), we have $\delta S^R(A) = -\delta S^R(B)$. However, from Eq. (7), the energy conservation of $A + B$ gives $\frac{1}{[1 + (1 - q)\beta U(A)]}\delta U(A) = -\frac{1}{[1 + (1 - q)\beta U(B)]}\delta U(B)$. That leads to

$$[1 + (1 - q)\beta U(A)]\frac{\partial S^R(A)}{\partial U(A)} = [1 + (1 - q)\beta U(B)]\frac{\partial S^R(B)}{\partial U(B)}$$

(14)

or $\beta(A) = \beta(B)$ which characterizes the thermal equilibrium.

6 Some “additive” thermodynamic relations

Due to the mixte character of this formalism with additive entropy and non-additive energy, all the thermodynamic relations become nonlinear. In what follows, we will try to simplify this formal system and to give a linear form to this nonlinearity.

Using the same machinery as in [20] which gives an extensive form to the nonextensive Tsallis statistics, we define an additive deformed energy $E$ as follows :

$$E = \ln[1 + (1 - q)\beta U]/(1 - q)\beta$$

(15)

which is identical to $U$ whenever $q = 1$. Note that $E(A + B) = E(A) + E(B)$. In this way, Eq. (11) can be recast into

$$S^R = \ln Z + \beta E.$$

(16)
So that $\beta = \frac{\partial S^R}{\partial E}$. The first law can be written as

$$\delta E = T \delta S^R + Y \delta X \quad (17)$$

where $Y$ is the deformed pressure and $X$ the coordinates (volume, surface ... ) and $T = 1/\beta$. The deformed free energy can be defined by

$$F = E - TS^R = -T \ln Z, \quad (18)$$

so that $Y = \frac{\partial F}{\partial X}$. The real pressure is $Y^R = [1 + (1 - q)\beta U]Y$ and the work is $\delta W = Y^R \delta X$. The deformed heat is $\delta Q = T \delta S^R$ and the real heat is $\delta Q^R = [1 + (1 - q)\beta U] \delta Q$.

7 Grand-canonical distributions

It is easy to get the grand-canonical ensemble distribution given by

$$p_i = \frac{1}{Z}[1 - (q - 1)\beta(E_i - \mu N_i)]^{1/(q-1)}, \quad (19)$$

which gives

$$S^R = \ln Z + \ln[1 + (1 - q)\beta U]/(1 - q) + \ln[1 - (1 - q)\beta \omega N]/(1 - q). \quad (20)$$

Let $M$ be the deformed particle number : $M = \ln[1 - (1 - q)\beta \omega N]/(1 - q)\beta \omega$, Eq.(20) becomes

$$S^R = \ln Z + \beta E + \beta \omega M. \quad (21)$$

$M$ must be additive, so that $N$ is nonadditive satisfying

$$N(A + B) = N(A) + N(B) + (1 - q)\beta \omega N(A)N(B) \quad (22)$$

Due to the distribution function of Eq.(19), the quantum distributions will be identical to those in NSM[21].

8 Rényi statistics with additive energy?

Now we know that $S^R$ should be intrinsically associated with the $q$-exponential distributions. Only when $q = 1$ this statistics recovers BGS and the $q$-exponential becomes the usual exponential function. Then an interesting
question is whether or not this statistics may be associated with additive energy when \( q \neq 1 \).

Supposing \( A \) and \( B \) are independent, i.e., \( E_{ij}(A + B) = E_i(A) + E_j(B) \), let us see the probability of the system \( A + B \) for a joint state \( ij \):

\[
p_{ij}(A + B) = \frac{1}{Z(A + B)}[1 - (q - 1)\beta(E_i(A) + E_j(B))]^{1/(q-1)}
\]

\[
= p_i(A)p_{ji}(B \mid A)
\]

where

\[
p_i(A) = \frac{1}{Z(A)}[1 - (q - 1)\beta E_i(A)]^{1/(q-1)}
\]

is the probability for \( A \) to be at the state \( i \) and

\[
p_{ji}(B \mid A) = \frac{1}{Z_i(B \mid A)}[1 - (q - 1)\beta e_{ji}(A \mid B)]^{1/(q-1)}
\]

is a conditional probability for \( B \) to be at a state \( j \) with energy \( e_{ji}(A \mid B) = E_j(B)/[1 - (q - 1)\beta E_i(A)] \) if \( A \) is at \( i \) with energy \( E_i(A) \). In this case, the total entropy is given by

\[
S^R(A + B) = \frac{\ln[\sum_i p_i(A)^q \sum_j p_{ji}(B \mid A)^q]}{1 - q}
\]

\[
\neq S^R(A) + S^R(B).
\]

This is in contradiction with Eq.(10). So \( S^R \) is no more additive with additive energy. As a matter of fact, Eq.(26) is wrong because Eq.(23) does not hold if we consider the product probability Eq.(9). We would get

\[
p_j(B) = \frac{1}{Z(B)}[1 - (q - 1)\beta E_j(B)]^{1/(q-1)}
\]

\[
= \frac{1}{Z_i(B \mid A)}[1 - (q - 1)\beta e_{ji}(A \mid B)]^{1/(q-1)}
\]

which implies \( E_j(B) = e_{ji}(A \mid B) = E_j(B)/[1 - (q - 1)\beta E_i(A)] \) and is valid only when \( q = 1 \). This means that additive energy may force back the nonextensive statistics to BGS if the product joint probability applies.

We would like to discuss in passing the problem of thermodynamic instability of Rényi entropy which has been shown[6] to be non-observable and
instable because an arbitrarily small variation $\delta$ in probability distribution may lead to an important variation in $S^R$ for a system having infinite number of states $w$. According to this analysis, Rényi entropy can be physically useful for finite systems having finite number of states.

However, it should be noted that the Lesche’s conclusion for infinite $w$ is reached with the asymptotic behavior of $S^R$ for the case of $w \to \infty$ and of finite variation $\delta$ of probability distributions, i.e., $1/w$ is negligible compared to $\delta$. This is a very harsh condition if we consider that $\delta$ must be arbitrarily small for observability condition. It should be noted that the asymptotic behavior of $S^R$ for finite $\delta$ and $w \to \infty$ is different from the asymptote for arbitrarily small $\delta$ and arbitrary $w$ which can be very large, e.g., $\delta$ is smaller than or of same order of magnitude as $1/w$. This second asymptotic behavior should be more general to our opinion because it applies for any system. Taking the probability distributions proposed by Lesche and making the same calculations without neglecting anything, one gets, for both $q > 1$ and $q < 1$, $\Delta S^R(\delta, w)/S_{max} \propto (\delta/2)^q$ for arbitrarily small $\delta \to 0$. The observability condition[6] is satisfied. This result is in addition consistent with the fact that $S^R$ is a monotonic function of $S^T$ which is observable according to the same analysis[7]. We indeed have $dS^R = \frac{dS^T}{1+(1-q)S^T} = \sum_i \rho_i$. So if $dS^T/S^T \to 0$, we also have $dS^R/S^R$ for finite $S^R$ and $S^T$.

9 Conclusion

In summary, the additive Rényi entropy is associated with nonadditive energy to give an nonextensive thermostatistics characterized by $q$-exponential distributions which have been proved to be useful for many systems showing non Gaussian and power law distributions[22]. This formalism would be useful for interacting nonextensive systems whose information and entropy may be additive or approximately additive. The problem of the instability of Renyi entropy is reviewed. We think that this entropy may be physically useful for any number of states.

A important point should be underlined following the result of the present work. Rényi entropy is identical to Boltzmann one for microcanonical ensemble. So, theoretically, its applicability to systems with nonadditive energy means that Boltzmann entropy may also be applied to nonextensive microcanonical systems as indicated by Gross[15].
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