BIFLIPPERS AND HEAD TO TAIL COMPOSITION RULES

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ABSTRACT. A new graphical calculus for operating with isometries of low dimensional spaces of classical geometries is proposed. It generalizes a well-known graphical representation for vectors and translations in an affine space. Instead of arrows, we use biflippers, which are arrows framed at the end points with subspaces. The head to tail addition of vectors and composition of translations is generalized to head to tail composition rules for isometries.

Introduction

Any isometry of Euclidean space is a composition of two symmetries of order two. A symmetry of order two is determined by its set of fixed point. Drawing of the fixed point sets gives an easy and effective graphical way to present isometries and operate with them. Below a new technique for this is proposed.

It is naturally limited to low dimensions (up to 3), because in the higher dimensions the situation becomes more complicated on the one hand and drawing less practical, on the other hand. In the low dimensions, the same technique works for other classical geometries: elliptic (both in spheres and projective spaces), hyperbolic, and conformal.

1. Portraits of isometries

1.1. Flips and flippers. Let $X$ be an Euclidean space (say, a plane or the 3-space) or a subset of a Euclidean space. Let $A$ be a non-empty subset of $X$. If

- there exists an isometry $F : X \to X$ (i.e., a map which preserves distances between points),
• which is an involution
  (i.e., $F \circ F = \text{id}$),
• $A$ is the fixed point set of $F$
  (i.e., the set of all points of $X$ that are not moved by $F$),
• and if such $F$ is unique,

then $A$ is called a flipper in $X$, and $F$ is called a flip in $A$ and is denoted by $F_A$.

Flipper $A$ and flip $F_A$ determine each other. In particular, $A$ encodes $F_A$. In low dimensional cases, if $A$ is clearly drawn, the picture of $A$ is a portrait for $F_A$.

**Examples.** Obviously, in any $X$ the whole $X$ is a flipper, the corresponding flip is the identity map.

On the plane, a set is a flipper iff it is either a line or a point. More generally, in a Euclidean space of any dimension, a set is a flipper iff it is a non-empty affine subspace. In a Euclidean space of any dimension, if an isometry is involution, then it is a flip in a non-empty affine subspace.

On the 2-sphere there are two kinds of flippers: great circles and pairs of antipodal points. The corresponding flips are the restrictions of the flips of the ambient 3-space with the flippers passing through the origin.

**Generalizations.** The notions introduced above can be considered in much more general setup: $X$ may be a whatever space with whatever structure, and the rôle of isometries would play automorphisms of this structure. In the last section we will consider the 2-sphere with conformal structure.

**About uniqueness.** In all the examples above uniqueness of $F$ was not a restriction: if $F$ existed, then it was unique. The uniqueness condition was included in order to ensure one-to-one correspondence between flips and flippers. There are sets in more complicated figures, which are fixed point sets of several different involutions-isometries.

One of the simplest examples: $X$ is the union of two perpendicular lines $L$ and $M$ on the plane, and $A$ is the intersection point of the lines. Then $A$ is the fixed point set for three involutions-isometries. They are restrictions to $X$ of three flips of the ambient plane: the central symmetry in $A$ and the reflections in the two lines bisecting the angles between $L$ and $M$. Therefore $A$ is not qualified to be a flipper of $X$, although $A$ is a flipper of the ambient plane.

**Terminological remark.** The words flip and flipper seems to be new in this context. Old words that are used in mathematics with a close meanings are symmetry, reflection and inversion.
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The closest is *reflection*, but it has a well-established meaning. Most mathematicians believe that the fixed point set of a reflection must have codimension one. This kind of reflections plays a special role in geometry and group theory, but is too narrow for our purposes. A use of the word reflection in the sense above (i.e., for what we call a flip) would be considered a language abuse, see Wikipedia article *Point reflection*.

The word *inversion* is used for several different notions including the symmetry with respect to a point and the symmetry with respect to a sphere. The latter seems to be commonly used and loading yet another meaning would be fairly criticized.

The word *symmetry*, is commonly used in much broader sense than we need, often just as a synonym for automorphism.

1.2. Flip-flop decompositions. A presentation of an isometry $T$ as a composition of two flips $F_B \circ F_A$ is called a *flip-flop decomposition* of $T$. An ordered pair $(A, B)$ of flippers $A$ and $B$ is called a *biflipper*. The biflipper $(A, B)$ encodes the flip-flop decomposition $T = F_B \circ F_A$ and hence the isometry $T$.

If $A$ is a flipper in a space $X$, then the flip $F_A$ has a flip-flop decompositions $F_A \circ F_X$ and $F_X \circ F_A$, because $X$ is a flipper in itself and $F_X = \text{id}$. However, often a flip has other flip-flop decompositions. Flip-flop decompositions $F_A = F_X \circ F_A = F_A \circ F_X$ are said to be *improper*, all other flip-flop decompositions are said to be *proper*. A biflipper is said to be *proper*, if none of its two flippers is the whole space.

A flip-flop decomposition of an isometry gives an opportunity to draw a picture, which completely describes the isometry. However, a picture showing just $A$ and $B$ would be incomplete. It does not show a crucial bit of information: which of the flipper is the first in the biflipper and which is the second. In other words, it does not show the order of $F_A$ and $F_B$ in the flip-flop decomposition. The order distinguishes an isometry from the inverse isometry. Indeed, since flips are involutions, $F_A \circ F_B = F_B^{-1} \circ F_A^{-1} = (F_B \circ F_A)^{-1}$.

1.3. What an arrow may be for. Often, an arrow in a mathematical picture portrays nothing but an ordered pair of points, the arrow tail and arrowhead. Individual points are difficult to discern on a picture, and, in order to make the points more visible, they are connected with a line segment, and, in order to show which point is first and which is the second, the segment is turned into an arrow directed from the first of the points to the second.
1.4. Arrows between flippers. We need to portray an ordered pair of flippers. The flippers may happen to be points in an Euclidean space, and then we just draw the arrow between them, as usual. In a general situation, we draw the flippers and connect them with an arrow, in order to show the ordering. If the flippers are disjoint affine subspaces in a Euclidean space, then we choose an arrow along a common perpendicular to the both subspaces. If the flippers intersect, we still need to show their ordering, and, to this end, we connect them with an arc-arrow.

In order to distinguish the first and the second flippers in words, we will call the first flipper in a biflipper the tail flipper and the second one, the head flipper, according to the place occupied by them in a picture: at the arrow tail or arrow head, respectively.

A pair of flippers $A$ and $B$ together with an arrow, which connects $A$ to $B$ and specifies the ordering, is called a arrowed biflipper (or just a biflipper or just an arrow, when there is no danger of confusion) and denoted by $\vec{AB}$. Thus an arrow $\vec{AB}$ provides a graphical encoding for the flip-flop decomposition $T = F_B \circ F_A$ and, in particular, a graphical encoding, a portrait for $T$.

1.5. Biflippers on the plane. It is easy to list all kinds of biflippers that exist on the plane. Take all possible mutual positions of two different flippers (recall that a proper flipper on the plane is either a points or a line), and connect the flippers with an arrow. Proper biflippers look as follows:

These biflippers encode all the plane isometries (except the identity). Let us consider them one by one, from left to right:

1.6. Translations. biflippers $\vec{\ }$ and $\bullet \longrightarrow \bullet$ encode the translation in the direction of the arrow by the distance twice the length of the arrow. Here are sketches of the well-known proofs:
1.7. Rotations. Biflipper encodes the rotation about the intersection point of the flippers, in the direction from the tail flipper to the head flipper, by the angle twice greater than the angle between them. A sketch of proof:.

**Warning:** often a rotation is shown by a similar picture, but with the angle between the lines twice greater and the direction of the arc-arrow indicating if the rotation is clockwise or counter-clockwise. Recall that in a biflipper the arc-arrow points from the tail flipper to head flipper. The following two biflippers and a traditional picture show rotations in the same clockwise direction by $151^\circ$: 

1.8. Glide reflections. Biflippers and encode glide reflections. This is the reflection in the line which contains the arrow, followed by the translation in the direction of the arrow by the distance twice greater than the length of the arrow. Sketches of proofs:

1.9. Reflections. Biflippers and encode reflections; specifically, the reflections in the line which is perpendicular to the line-flipper, and erected from the point-flipper. Sketches of proofs: 

These are degenerate versions of the biflippers that encode glide reflections, in the same sense as reflections in lines are degenerate glide reflections.

1.10. The identity. The identity is also an isometry and it is encoded by any biflipper consisting of two identical flippers. The identity is a special case (i.e., a degeneration) for translations as well as for rotations.

2. Biflippers of the same isometry

2.1. Equivalence of biflippers. Biflippers $AB$ and $CD$ are said to be *equivalent* if $F_B \circ F_A = F_D \circ F_C$. 

Below we describe two easy ways to change a biflipper into a different, but equivalent biflipper.

**Theorem 1.** Let $A$, $B$ and $C$ be flippers such that $F_C$ commutes both with $F_A$ and $F_B$. Then $F_A \circ F_C$ and $F_B \circ F_C$ are involutions. If these involutions are flips in flippers $A'$ and $B'$ (i.e., $F_A \circ F_C = F_{A'}$ and $F_B \circ F_C = F_{B'}$), then $\overrightarrow{AB}$ is equivalent to $\overrightarrow{A'B'}$.

**Proof.** If two involutions, say $\sigma$ and $\tau$, commute, then their product is an involution. Indeed, by multiplying the equality $\sigma \tau = \tau \sigma$ by $\sigma \tau$ from the right, we obtain $(\sigma \tau)^2 = \tau \sigma \sigma \tau = 1$. Further,

$$F_{B'} \circ F_{A'} = F_B \circ F_C \circ F_A \circ F_C = F_B \circ F_C \circ F_C = F_B \circ F_A.$$

Thus $\overrightarrow{AB}$ is equivalent to $\overrightarrow{A'B'}$. □

In a Euclidean space any isometry of order two is a reflection in a subspace. Thus for Euclidean space the second assumption of Theorem 1 is nothing but just a definition for $A'$ and $B'$.

**Examples.** Let $A$, $B$ and $C$ be lines on the plane, $A \perp C \perp B$, so $A \parallel B$. Let $A' = A \cap C$ and $B' = B \cap C$. Then

1. $\overrightarrow{AB}$ and $\overrightarrow{A'B'}$ are equivalent;
2. $\overrightarrow{AB'}$ and $\overrightarrow{A'B}$ are equivalent.

As we saw above, in the first example the arrows encode the same translation, in the second - the same glide reflection.

**Theorem 2.** Let $S$ be the isometry encoded by $\overrightarrow{AB}$ and let $T$ belong to the centralizer of $S$ in the isometry group (i.e., $T$ be an isometry, which commutes with $S$). Then $\overrightarrow{T(A)T(B)}$ is equivalent to $\overrightarrow{AB}$.

**Proof.** Notice, that if $C$ is a flipper and $T$ is any isometry, then $T(C)$ is a flipper, and $F_{T(C)} = T \circ F_C \circ T^{-1}$. Therefore,

$$F_{T(B)} \circ F_{T(A)} = T \circ F_B \circ T^{-1} \circ T \circ F_A \circ T^{-1}$$

$$= T \circ F_B \circ F_A \circ T^{-1} = TST^{-1} = S.$$ □

### 2.2. Equivalence of plane biflippers.

Recall that biflippers, which encode the same isometry, are said to be equivalent. For different biflippers equivalence may look quite different.

Arrowed biflippers, that encode translations, are equivalent iff the underlying arrows (i.e., the bare arrow, with the flippers removed), can be obtained from each other by a translation.
Arrowed biflippers, that define glide reflections, are equivalent iff the underlying arrows lie on the same line and have the same length and direction. In other words, they can be obtained from each other by a translation along this line. The equivalence classes of the underlying arrows are known as sliding vectors. By Theorem 1, it does not matter whether the tail flipper or the head flopper is a line. Simultaneous change of the dimensions of the tail and head flippers does not change the gliding reflection.

Arrowed biflippers, that define rotations, are equivalent iff they can be obtained from each other by a rotation about the intersection points of the flippers (which is the center of the rotation).

Arrowed biflippers, that define reflections, are equivalent iff they can be obtained from each other by a translation in the direction perpendicular to the line-flipper (this is the direction of the reflection mirror).

3. Head to tail composition method

3.1. Generalities. If $S = I \circ J$, $T = J \circ K$, and $J$ is an involution, then, obviously, $S \circ T = I \circ J^2 \circ K = I \circ K$.

Therefore if isometries $T$ and $S$ are encoded by biflippers $\overrightarrow{AB}$ and $\overrightarrow{BC}$, respectively, then $\overrightarrow{AC}$ encodes $S \circ T$. Here it is important that the head flipper of the first biflipper coincides with the tail flipper of the second.

If isometries $T$ and $S$ can be represented by such biflippers, we say that they admit head to tail composition. An algorithm for finding biflippers of this kind is called a head to tail composition method.

The head to tail composition methods depend on the types of the composed isometries. It’s remarkable that a head to tail composition method exists for any pair of isometries of the Euclidean plane and in many other similar setups. However, as we show below, for some pairs of isometries of the hyperbolic plane and Euclidean 3-space it does not exist.

The problem of finding a biflipper, which encodes a composition of isometries, given biflippers, that encode these isometries, is solved easily in those two situations by slightly more complicated methods. See sections 8 and 10 below.

3.2. Translations. Choose biflippers, which encode the translations, with 0-dimensional flippers and move one of them by an appropriate translation so that the head flipper of the first biflipper would coincide with the tail flipper of the second biflipper. After that this looks as the
usual head to tail addition of vectors.

3.3. Rotations. Take any biflippers encoding the rotations. Turn the biflippers around the centers in order to make the head flipper in the first of them coinciding with the tail flipper in the second. If after this the other two lines are not parallel, then draw an arc-arrow connecting the tail flipper of the first biflipper to the head flipper of the second biflipper. Erase the coinciding lines and the old arc-arrows. The composition is a rotation. Notice that this head to tail construction gives the center of the rotation composition.

If the tail flipper of the first biflipper is parallel to the head flipper of the second biflipper, then after erasing everything besides these two parallel lines and connecting them with an arrow perpendicular to them, we obtain a biflipper for a translation.

3.4. Translation followed by rotation. Choose a biflipper representing the translation to be formed by lines. Turn a biflipper representing the rotation to make the flipper perpendicular to the direction of the translation (i.e., parallel to the lines which form the biflipper representing the translation). By a translation of the biflipper representing the translation, superimpose the appropriate lines.

3.5. Translation followed by glide reflection. The composition is a glide reflection. Here is how to find its biflippers.

3.6. Two glide reflections with non-parallel axes. The composition is a rotation. Here is how to find its biflipper.
3.7. **Two glide reflections with parallel axes.** The composition is a translation. Here is how to find its biflipper.

3.8. **Generators and relations for the plane isometry group.**

Non-uniqueness of flip-flop decompositions of a translation implies that the composition $F_l \circ F_m$ of reflections in parallel lines $l$ and $m$ does not change if one replaces $l$ and $m$ by their images $l'$ and $m'$ under any translation. Thus we have a relation $F_l \circ F_m = F_{l'} \circ F_{m'}$.

Similar relations follow from non-uniqueness of flip-flop decompositions of a rotation. Namely, if lines $l$ and $m$ intersect at a point $A$ and lines $l'$ and $m'$ are their images under some rotation about $A$, then $F_l \circ F_m$ and $F_{l'} \circ F_{m'}$ are flip-flop decompositions of the same rotation, and hence $F_l \circ F_m = F_{l'} \circ F_{m'}$.

Observe that in both situations the lines $l$, $m$, $l'$ and $m'$ belong to a pencil: in the first situation they are all parallel to each other, in the second, they have common point $A$. We will call them **pencil relations**.

Besides, any flip is an involution. Thus $F_l^2 = \text{id}$.

**Theorem 3 (O.Viro [5]).** Any relation among reflections in the group of isometries of Euclidean plane is a corollary of the pencil and involution relations.

**Lemma.** Any composition of four reflections can be converted by pencil and involution relations into a composition of two reflections.

**Proof.** Consider a composition $F_n \circ F_m \circ F_l \circ F_k$ of four reflections. If any two consecutive lines coincide (i.e., $k = l$, or $l = m$, or $m = n$), then, by applying the involution relation, we can eliminate the corresponding reflection from the composition. So, in the rest of the proof, we assume that none of consecutive lines coincide.

Compositions $F_n \circ F_m$ and $F_l \circ F_k$ are flip-flop decompositions of translations or rotations. If at least one of them is a rotation, we can apply head to tail rule for identification of their composition. Notice that all the transformations used in this method are applications of the pencil and involution relations.

Let both $F_n \circ F_m$ and $F_l \circ F_k$ be translations. Then $k \parallel l$ and $m \parallel n$. If $l \cap m \neq \emptyset$, then by rotating the middle pair of lines $l$, $m$ by right angle we replace them by lines $l'$ and $m'$ such that $k \perp l'$ and $m' \perp n$.
and obtain the situation of two rotations for which head to tail method works.

If all the lines are parallel, then by a translation of $k \cup l$ such that the image of $l$ would coincide with $m$ and applying pencil and involution relations as above, we can reduce the number of reflections. □

Proof of Theorem 3. Take any relation $F_1 \circ F_2 \circ \cdots \circ F_n = \text{id}$ among reflections. By Lemma we may reduce by applying pencil and involution relations its length $n$ to a number which is less than four. It cannot be three, because a composition of an odd number of reflections reverses the orientation, and hence cannot be equal to the identity. The only composition of two reflections which is the identity is an involution relation. □

Theorem 3 gives a presentation by generators and relations for the group of plane isometries. The generators in this presentation are reflections in all the lines. This is an uncountable set. Since the group is uncountable, the set of generators must be uncountable, too. Though, this set has quite a simple structure. The set of relations has cardinality of the continuum, too. It consists of all pencil and involution relations. Geometrically the relations are easy to deal with. It is so powerful that allows to present any isometry as a product of at most three generators.

Enlarging the system of generators by inclusion all the flips (i.e., by addition of symmetries about points) makes it even more powerful: any plane isometry admits a presentation as a product of two flips. The set of relations has to be enlarged by expressions of symmetries about points as compositions of reflections: the symmetry about a point $A$ (being a rotation by $\pi$ about $A$) is the composition of reflections in any two perpendicular lines intersecting at $A$.

The following theorem ensures an extra comfort in graphical operations in this presentation of the plane isometry group.

Theorem 4. Any pair of plane isometries admits a head to tail composition. In other words, any isometries $S$ and $T$ of Euclidean plane are presented by biflippers $\overrightarrow{AB}$ and $\overrightarrow{BC}$, respectively.

Proof. For any plane isometry, besides a rotation, the tail flipper in a biflipper can be chosen to be either a point or a line, as we like, or the
head flipper can be chosen to be either a point or a line. (We cannot control the dimensions of both of them simultaneously.)

Assume first that none of the two isometries $S$ and $T$ is a rotation. Then there exist biflippers $\overrightarrow{KL}$ and $\overrightarrow{MN}$ representing $T$ and $S$, respectively, such that both $L$ and $M$ are points, or both of them are lines. By Theorem 2, we can move each of the biflippers either freely or along a line. If at least one of them can be moved freely (the corresponding isometry is a translation) or the lines meet, one can choose $L$ and $M$ to be points, superimpose them, and we are done. If the lines along which the points $L$ and $M$ can move are parallel, then we choose $L$ and $M$ to be lines, these lines are parallel, and now, by moving the biflippers, we can make the lines superimposed.

The case of two rotations was considered above.

If one of the isometries is a rotation, then, by rotating about the center of the rotation, we can make the lines $L$ and $M$ parallel, and by a move of the other biflipper superimpose $L$ and $M$. \hfill \Box

**Exercise.** Find head to tail composition rules for the pairs of plane isometries that have not been discussed above.

4. On the 2-sphere and projective plane

4.1. On the 2-sphere. The group of isometries of the 2-sphere $S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ coincides with the orthogonal group $O(3)$. The isomorphism maps a linear orthogonal transformation of $\mathbb{R}^3$ to its restriction.

As was mentioned in Section 1, there are two kinds of flippers on $S^2$: great circles and pairs of antipodal points. Proper biflippers look as follows:

![rotations](image1) ![rotary reflections](image2) ![reflections](image3)

These biflippers encode all the non-identity isometries of $S^2$.

4.2. Rotations. A biflipper, which is formed by two great circles, defines the rotation about the intersection points of the circles by the angle twice the angle between the circles. This becomes clear, if we recall that the whole picture is cut on $S^2$ by the picture in $\mathbb{R}^3$ and the
corresponding biflipper in $\mathbb{R}^3$ is a pair of planes.

A transition from a biflipper, which is made of great circles $A$ and $B$, to a biflipper made of two pairs of antipodal points is described by Theorem 2. For $C$ one has to take the great circle perpendicular to $A$ and $B$.

Biflippers, that encode rotations and are formed of flippers of the same dimension, are equivalent iff they can be obtained from each other by a rotation of the sphere about the same axis. This follows from Theorem 2.

Head to tail rule is very simple:

It has been recently introduced in the author’s preprint [5].

4.3. Rotary reflections. An equivalence class of biflippers for a rotary reflection is determined by an arc-arrow on a great circle, as for a rotation. The only difference is that in the case of rotation, at both end points of the arc-arrow we have flippers of the same kind: either the end-points (or, rather, two pairs of antipodal points), or two great circles perpendicular to the arc-arrow, while for a rotary reflection the flippers are of different kinds: one is a point, the other is a great circle. It does not matter, which of them is located at arrowhead and which at the arrow tail. Compare to the relation between biflippers for glide reflection and translations on the plane.

Rotary reflections and rotations of $S^2$ are related even closer than glide reflections and translations of $\mathbb{R}^2$. The equivalences for biflippers in the former case coincide and can be described as obtaining arc-arrows from each other by a rotation along the great circle containing the arc-arrows. In the latter case (on $\mathbb{R}^2$), for translations an equivalence class of biflippers is a free vector, while for glide symmetries the equivalence class of biflippers is a sliding vector.

Head to tail methods involving rotary reflections are similar to head to tail methods involving glide reflections. For example, composition
of two rotary reflections is a rotation calculated as follows:

![Diagram of rotations and rotary reflections](image)

### 4.4. Degenerations

Rotations and rotary reflections together fill open everywhere dense subset of $O(3)$. Reflections in great circles appear as degenerations of rotary reflections, as the rotation angle vanishes. These degenerations can be lifted to degenerations of biflipper under which the flippers moves towards each other and the pair of antipodal points comes to the great circle.

A rotation by $\pi$ is a reflection in two antipodal points (the restriction of the reflection about a line). The corresponding biflippers are either a pair of two orthogonal great circles, or two pairs of antipodal points dividing the great circle passing through them into four congruent arcs.

The antipodal symmetry $x \mapsto -x$ is a rotary reflection, in which the angle of the rotation part is $\pi$. The corresponding biflippers consist of a great circle and a pair of antipodal points polar to the circle. Polarity means here that the great circle is the intersection of $S^2$ with a plane orthogonal to the line which intersects $S^2$ at the pair of points.

In the first two degenerations, the equivalence of biflippers and head to tail methods appear as limits and do not require additional considerations. In the last degeneration there is only one isometry and all the biflippers formed by a great circle and a pair of antipodal points polar to each other are equivalent.

### 4.5. On the projective plane

Projective plane considered as the quotient space of $S^2$ by the antipodal symmetry inherits a metric from $S^2$. Isometries of the projective plane are covered by isometries of $S^2$. In particular, flips are covered by flips. However, flips of the two kinds are covered by flips of the same type, with the flipper consisting of a projective line and its polar - the point the most distant from the projective line.

The whole flipper is determined by this point - the rest of the flipper can be recovered as the polar of the point. Any isometry is determined by a biflipper, which, in turn, is determined by an arrow connecting the 0-dimensional parts of the mirrors.

Sliding of an arrow along its line does not change the equivalence class of the corresponding biflipper. Therefore for composing of isometries, one can use the obvious head to tail rule.
5. Concise self-contained digression on quaternions

5.1. Quaternions. Any story about rotations of the 3-space would be incomplete without quaternions. Unit quaternions provide a convenient parametrization for the group of rotations. Biflippers give another presentation of the same group. The quaternion parametrization is more convenient for calculations. The biflippers seems to be more visual and intuitive. We will relate the two pictures below. In this section, all the information about quaternions, that is needed for this, is presented in all the details with brief, but complete proofs.

Quaternions form a 4-dimensional associative algebra over the field of real numbers. The algebra of quaternions is denoted by $H$. As a vector space over $\mathbb{R}$, it has the standard basis $1, i, j, k$. The generators are subject to relations $i^2 = j^2 = k^2 = ijk = -1$. A quaternion expanded in the standard basis is $a + bi + cj + dk$, where $a, b, c, d \in \mathbb{R}$. The quaternion addition is component-wise. The quaternion multiplication is associative and the products of the generators are calculated according to formulas $ij = k$, $ji = -k$, $jk = i$, $kj = -i$, $ki = j$ and $ik = -j$ (these formulas follow from the relations $i^2 = j^2 = k^2 = ijk = -1$).

5.2. Scalars and vectors. The field $\mathbb{R}$ is contained in $H$ as $\{a + 0i + 0j + 0k \mid a \in \mathbb{R}\}$. A quaternion of the form $a + 0i + 0j + 0k$, is called real. A quaternion of the form $0 + bi + cj + dk$, where $b, c, d \in \mathbb{R}$ is called pure imaginary. If $q = a + bi + cj + dk$ is any quaternion, then $a$ is called its scalar part and denoted by $q_s$ and $bi + cj + dk$ is called its vector part and denoted by $q_v$. The set of purely imaginary quaternions $bi + cj + dk$ is identified with the real 3-space $\mathbb{R}^3$.

5.3. Multiplication of quaternions. The set of real quaternions is the center of $H$. Multiplication of quaternions is composed of all the standard multiplications of factors which are real numbers and vectors: multiplications of real numbers, multiplication of a vector by a real number and dot and cross products of vectors. It is not accident: the very notion of vector and all the operations with vectors were introduced by Hamilton after invention of quaternions.

Quaternion product of vectors. Let $p = ui +vj +wk$ and $q = xi + yj + zk$ be vector quaternions. Then $pq = -p \cdot q + p \times q$. Indeed,

$$pq = (ui +vj +wk)(xi + yj + zk)$$
$$= -(ux + vy + wz) + (vz - wy)i + (wx - uz)j + (uy - vx)k$$
$$= -p \cdot q + p \times q \quad \square$$
Product of quaternions via other products. For any \( p, q \in \mathbb{H} \)
\[
pq = (p_s + p_v)(q_s + q_v) = p_sq_s + p_sq_v + p_vq_s + p_vq_v \\
= p_sq_s + p_sq_v + q_sp_v - p_v \cdot q_v + p_v \times q_v \\
= p_sq_s - p_v \cdot q_v + p_sq_v + q_sp_v + p_v \times q_v \quad \square
\]

5.4. Conjugation. The map \( \mathbb{H} \to \mathbb{H} : q \mapsto q^* = q_s - q_v \) is called conjugation. The conjugation is an antiautomorphism of \( \mathbb{H} \) in the sense that it is an automorphism of \( \mathbb{H} \) as a real vector space and \( (pq)^* = q^*p^* \).

The latter is verified as follows:
\[
(pq)^* = (p_sq_s - p_v \cdot q_v + p_sq_v + q_sp_v + p_v \times q_v)^* \\
= p_sq_s - p_v \cdot q_v - p_sq_v - q_sp_v - p_v \times q_v \\
= p_sq_s - (p_v) \cdot (q_v) - q_sp_v + q_sp_v + p_v \times q_v \\
= (q_s - q_v)(p_s - p_v) = q^*p^*.
\]

5.5. Norm. The product \( q^*q \) is a non-negative real number for any non-zero quaternion \( q \). Indeed, \( (q^*q)^* = (q^*)^* = q^*q \).

If \( q = a + bi + cj + dk \), then \( q^*q = a^2 + b^2 + c^2 + d^2 \). Indeed, \( q^*q = q_s^2 - (q_v) \cdot q_v + q_sq_v + q_v(q_v) + (q_v) \times q_v = q_s^2 + a^2 + b^2 + c^2 + d^2 \).

The number \( \sqrt{q^*q} \) is called the norm of \( q \) and denoted by \( |q| \). It is the Euclidean distance from \( q \) to the origin in \( \mathbb{R}^4 \).

The norm is a multiplicative homomorphism \( \mathbb{H} \to \mathbb{R} \). Indeed, \( |pq| = \sqrt{pq(pq)^*} = \sqrt{pqq^*p^*} = \sqrt{pp^*qq^*} = |p||q| \).

6. The group of unit quaternions.

6.1. Unit quaternions. The sphere \( S^3 = \{ q \in \mathbb{H} \mid |q| = 1 \} \), being the kernel of the multiplicative homomorphism \( \mathbb{H} \to \mathbb{R} : q \mapsto |q| \), is a multiplicative subgroup of \( \mathbb{H} \). The inverse to a quaternion \( q \in S^3 \) coincides with \( q^* \). Indeed, \( |q| = \sqrt{q^*q} = 1 \), hence \( qq^* = 1 \).

Unit vector quaternions form the unit 2-sphere \( S^2 \) in \( \mathbb{R}^3 \). It is contained in \( S^3 \) as an equator. The unit vectors are very special quaternions.

**Theorem 5.** Each unit quaternion can be presented as a product of two unit vectors. Moreover, if \( q = q_u \) is a unit quaternion and \( v \) is a unit vector perpendicular to \( q_u \), then there exist unit vectors \( w_+ \) and \( w_- \) such that \( q = vw_+ = w_-v \).

In particular, unit vectors generate the group of unit quaternions.
Proof. Let \( q \in S^3 \) be a unit quaternion. Then \( q = q_s + q_v \) with 
\[
1 = |q|^2 = q_s^2 + |q_v|^2.
\]
Choose \( \alpha \in [0, \pi] \) such that \( q_s = \cos \alpha \) and \( |q_v| = \sin \alpha \). Then \( q = \cos \alpha + u \sin \alpha \) for some unit vector \( u \).

Take any unit vector \( v \) perpendicular to \( u \). Then \( w_+ = -v \cos \alpha + (u \times v) \sin \alpha \) and \( w_- = -v \cos \alpha - (u \times v) \sin \alpha \) are also unit vectors perpendiculars to \( u \), with the required properties: \( vw_+ = q \) and \( w_-v = q \).

\[
vw_+ = v(-v \cos \alpha + (u \times v) \sin \alpha) = -v \cdot (-v \cos \alpha + v \times (u \times v)) \sin \alpha = \cos \alpha + u \sin \alpha = q \quad \text{and} \quad w_-v = (-v \cos \alpha - (u \times v) \sin \alpha)v = -(v \cos \alpha) \cdot v - (u \times v) \times v \sin \alpha = \cos \alpha + u \sin \alpha = q
\]
\( \square \)

Remark. Any unit vector quaternion \( u \) has order four, its multiplicative inverse coincides with the additive inverse: \( u^{-1} = -u \).

Indeed, let \( u \) be unit vector. Then \( u^2 = -u \cdot u + u \times u = -1 \), hence \( u^3 = -u \) and \( u^4 = (u^2)^2 = (-1)^2 = 1 \). \( \square \)

6.2. Unit quaternions as fractions of unit vectors. By Theorem 5 any unit quaternion \( q \) admits presentation as product of two unit vector quaternions: \( q = vw \). If the factors \( v \) and \( w \) were of order two, we could use this presentation for head to tail rule. However the unit vector quaternions have order four, and we need to modify the presentation slightly, by replacing the product with a quotient.

Namely, let us present a unit quaternion as a sort of quotient of two unit vectors: \( q = v^{-1}w = -vw \). By the way, this goes back to W.R. Hamilton, the inventor of quaternions. In his book [4], Hamilton introduced quaternions as quotients of vectors.

6.3. Hamilton’s model for the group of unit quaternions. The underlying space is the 2-sphere \( S^2 \). Unit quaternions are interpreted as classes of equivalent vector-arcs on \( S^2 \). A vector-arc on \( S^2 \) is an ordered pair of points of \( S^2 \) connected by an arc of great circle and equipped with an arrowhead pointing to the second of the points. Two vector-arcs are equivalent iff they either lie on the same great circle and can be obtained from each other by an (orientation preserving) rotation of the great circle, or connect a pair of antipodal points.

Multiplication is defined by the head to tail rule: given two equivalence classes of vector-arcs, find the intersection point of two great circles containing vector-arcs of the classes, find the representatives in the head to tail position and draw the vector-arc connecting the tail of the first representative with the head of the second. See Hamilton [4], Book II, Chapter I, Section 9.
The unit with respect to this multiplication is the class of all vector-arcs of the zero length.

The vector-arc connecting points $v$ and $w$ on $S^2$ represents the unit quaternion $-vw$. Equivalent vector-arcs represent the same unit quaternion. The product of the quaternions $-vw, -wx$ representing vector-arcs, which are in head to tail position, equals $(-vw)(-wx) = vw^2x = v(-1)x = -vx$, the quaternion corresponding to the vector-arc obtained by the head to tail rule.

**Remark 1.** Our model of the group $S^3$ is not based on flip-flop decomposition for element of $S^3$ acting in itself. Indeed, $S^3$ is not generated by involutions. There is only one involution in $S^3$, the quaternion $-1$. It belongs to the center. A single point, i.e., an end point of an arrow in the model of $S^3$ corresponds to an element of order 4. In the Hamilton model, such elements are represented by vector-arcs occupying a quarter of a great circle.

**Remark 2.** The groups $S^3$ of unit quaternions is isomorphic to $SU(2)$. The standard isomorphism maps $a+bi+ cj+dk \in S^3$ to $\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$.

Thus, the Hamilton model describes also $SU(2)$.

**Remark 3.** A similar model on the circle $S^1$ describes $Pin^{-}(2)$.

### 7. Two stories about one two-fold covering

**7.1. The covering.** The group $SO(n)$ with $n \geq 3$ is known to have a unique two-fold covering. The covering space is known as $Spin(n)$, it is also a Lie group. For $n = 3$, the latter group is known to coincide with the group $S^3$ of unit quaternions. There are many descriptions of the covering $S^3 \rightarrow SO(3)$. The Hamilton model of $S^3$ coupled with biflippers of rotations on $S^2$ provides probably the simplest description.

**7.2. A biflipper view on the covering.** The description of rotations of $S^2$ via biflipper given above in Section 4 can be converted obviously into a model for $SO(3)$ similar to the Hamilton’s vector-arcs model for unit quaternions, see section 6.

In this model, elements of $SO(3)$ are interpreted as classes of equivalent 0-dimensional biflippers on $S^2$. Recall that a 0-dimensional biflipper on $S^2$ is two pairs of antipodal points connected with an arc-arrow,
and two such biflippers are equivalent iff they lie on the same great circle and can be obtained from each other by a rotation of the great circle.

There is an obvious two-fold covering, in which the base is this model of $SO(3)$ and the total space is the Hamilton model for the group of unit quaternions. Namely, each end point of an vector-arc is completed by its antipode. The preimage of a biflipper consists of four vector-arcs, but they split into two pairs of equivalent vector-arcs.

This is a group homomorphism, since in both models the multiplication is defined by the same head to tail rule.

7.3. The action of unit quaternions in the 3-space. A traditional description of the same covering looks as follows.

The group $S^3$ of unit quaternions acts in $\mathbb{H}$ by formula $\rho_q(p) = qpq^{-1} = qpq^*$.

This action commutes with the conjugation $p \mapsto p^*$. Indeed, $\rho_q(p^*) = qp^*q^* = ((q^*)^*(p^*)^*)^* = (qpq^*)^* = (\rho_q(p))^*$.

Therefore the action of $S^3$ in $\mathbb{H}$ preserves all the structures defined by the conjugation: the norm and the decomposition into scalar and vector parts. Indeed,

$$\rho_q(p_v) = \rho_q\left(\frac{p - p^*}{2}\right) = \frac{\rho_q(p) - \rho_q(p^*)}{2} = \frac{\rho_q(p) - \rho_q(p)^*}{2} = (\rho_q(p))_v,$$

$$\rho_q(p_s) = \rho_q\left(\frac{p + p^*}{2}\right) = \frac{\rho_q(p) + \rho_q(p^*)}{2} = \frac{\rho_q(p) + \rho_q(p)^*}{2} = (\rho_q(p))_s,$$

$$|\rho_q(p)| = \sqrt{\rho_q(p)(\rho_q(p))^*} = \sqrt{\rho_q(p)\rho_q(p^*)} = \sqrt{(qpq^*)(qpq^*)} = \sqrt{qppq^*} = |p|.$$  

In particular, the space $\mathbb{R}^3$ of vector quaternions is invariant, and $S^3$ acts on $\mathbb{R}^3$ by isometries. The action is a homomorphism $S^3 \to O(3)$. Since $S^3$ is connected, the image of this homomorphism lies in $SO(3)$.

**Theorem 6.** A unit vector quaternion $v$ acts in $\mathbb{R}^3$ as the symmetry about the line generated by $v$.

**Proof.** The statement that we are going to prove admits the following reformulation: for the linear operator $\mathbb{R}^3 \to \mathbb{R}^3 : u \mapsto vuv^*$, the vector
v is mapped to itself and each unit vector u orthogonal to v is an eigenvector with eigenvalue $-1$.

Let us verify the first statement. Since v is a unit vector, $vv^* = |v|^2 = 1$. Therefore $vv^* = v$.

Now let us verify the second statement. Since u is a unit vector orthogonal to v, $vu = v \times u - v \cdot u = v \times u$. Therefore, $vv^* = -uvv = -(v \times u - v \cdot u)v = -(v \times u)v$. Vector $v \times u$ is orthogonal to v. Therefore $-(v \times u)v = -(v \times u) \times v + (v \times u) \cdot v = -(v \times u) \times v = -u$. The latter equality holds true, because $(a \times b) \times c = a \cdot c - b \cdot c$ for any orthogonal unit vectors $a$, $b$ (e.g., $(i \times j) \times k = k \times i = j$).

**Theorem 7 (Euler-Rodrigues-Hamilton).** For any unit quaternion $q \in S^3$ the map $\mathbb{R}^3 \to \mathbb{R}^3 : p \mapsto qpq^*$ is the rotation of $\mathbb{R}^3$ about the axis generated by a unit vector $u$ by the angle $\theta$, where $\theta$ and $u \in \mathbb{R}^3$ are such that $q = \cos \frac{\theta}{2} + u \sin \frac{\theta}{2}$.

**Proof.** By Theorem 5 any unit quaternion $q$ can be presented as a product of unit vectors $v$ and $w$. In this proof it will be more convenient to use a modification of this presentation, the fraction presentation $q = v^{-1}w = -vw$ discussed above.

By Theorem 6 a unit vector acts as a symmetry about the line generated by this vector. Thus, $\rho_q$ is the composition of the symmetries $\rho_{-v} u$ and $\rho_w$. We know that the composition of symmetries about lines is a rotation by the angle the half of the angle between the lines. On the other hand, $q_s = (v(-w)) = -v \cdot (-w) = v \cdot w = \cos \alpha$, where $\alpha$ is the angle between the vectors $v$ and $w$. Thus $q_s = \cos \frac{\theta}{2}$, where $\theta$ is the rotation angle.

The vector part $q_v$ of the product of two unit vectors $v$ and $-w$ is collinear to $v \times (-w)$. The cross product of vectors is perpendicular to the vectors. On the other hand, we know that composition of symmetries about lines is a rotation about the axis perpendicular to the lines. Thus the vector $q_v$ is collinear to the axis of the rotation $\rho_q$. The length $|q_v|$ is $|\sin \frac{\theta}{2}|$, because $|q| = 1$ and $q_s = \cos \frac{\theta}{2}$. Therefore $q_v = u \sin \frac{\theta}{2}$ for some unit vector $u$ collinear to the axis of rotation.

The quaternion $q$ can be written down as $a + bi + cj + dk$. It is defined by the rotation up to multiplication by $-1$. Its components $a, b, c, d$ are called the *Euler parameters* for this rotation. They are calculated as follows: $a = \cos \frac{\theta}{2}$, $b = u_x \sin \frac{\theta}{2}$, $c = u_y \sin \frac{\theta}{2}$, $d = u_z \sin \frac{\theta}{2}$, where $u_x, u_y$ and $u_z$ are coordinates of the unit vector $u$ directed along the rotation axis.

Given a 0-dimensional biflipper $\overrightarrow{AB}$ defining a rotation, the Euler parameters of the rotations can be recovered just by choosing one point
from $A$ and $B$, say $v \in A$ and $w \in B$ (recall that $A$ and $B$ are pairs of antipodal points on $S^2$), and quaternion multiplication of the representative: $q = vw$. The components of $q$ are the Euler parameters of the rotation.

8. On the hyperbolic plane

8.1. Well-known facts. We will use the Poincaré model, in which the hyperbolic plane is represented by an open unit disk in $\mathbb{R}^2$. In this model, a line is either a diameter of the disk or an arc cut on the disk by a circle orthogonal to the boundary circle of the disk. The boundary circle of the disk is the absolute. (The absolute is not contained in the hyperbolic plane, it is rather something like a horizon.)

On the hyperbolic plane, two lines may intersect in a single point, or be disjoint. In the latter case, their closures on the absolute may be disjoint, and then the lines are said to be ultra-parallel, or have a common point, and then the lines are said to be parallel.

Two ultra-parallel lines have a unique common perpendicular. Conversely, two lines perpendicular to the same line are ultra-parallel. The set of lines perpendicular to the same line is called a hyperbolic pencil of lines. The line which is their common perpendicular is called the axis of the pencil.

The set of all lines passing through the same point, is called an elliptic pencil of lines. Their common points is called the center of the pencil.

The set of all lines whose closures share the same point on the absolute is called a parabolic pencil of lines. Their common point on the absolute is called the center of the pencil.

Observe that lines of a pencil of any type fill the whole hyperbolic plane, and only one line of the pencil passes through any point except the center of an elliptic pencil.

There are two kinds of flippers: lines and points. In the Poincaré model, the flip in a line, which is a diameter of the disk, is the restriction of the usual Euclidean symmetry with respect to the line of the diameter; the flip in a line, which is cut by a circle $C$, is a restriction of the plane inversion with respect to $C$. A flip in a point is a composition of flips in any two lines passing through this point and orthogonal to each other. Any line and any point is a flipper.
8.2. Biflippers. Proper biflippers can be formed of two points, a point and a line, which may be disjoint or the point may belong to the line, and a pair of lines. In the Poincaré model of the hyperbolic plane proper biflippers look as follows.

\[
\begin{array}{cccc}
\text{rotation} & \text{parallel motion} & \text{translation} & \text{glide reflections} & \text{reflections} \\
\end{array}
\]

A biflipper, which encodes a rotation or parallel motion, consists of lines. Except rotations and parallel motion, each isometry admit biflippers that include a mirror of dimensions 0, and one can use the construction of Theorem 1 in order to change the dimensions of both flippers simultaneously and get an equivalent biflipper.

Biflippers can be varied in their equivalence classes also in a continuous manner according to Theorem 2.

**Theorem 8.** For any isometry \( S \) of the hyperbolic plane, there is a pencil \( P(S) \) of lines, which is invariant under any isometry from the centralizer \( C(S) \) of \( S \). Any one-dimensional flipper from any biflipper, which encodes \( S \), belongs to \( P(S) \). Moreover, \( C(S) \) acts transitively in the set of lines belonging to \( P(S) \): for any \( L, M \in P(S) \) there exists \( T \in C(S) \) such that \( T(L) = M \). In particular, for any \( L \in P(S) \) there are biflippers \( \overrightarrow{AL} \) and \( \overrightarrow{LB} \), which encode \( S \).

**Proof.** If \( S \) is a translation, then it has a unique invariant line. It is called the axis of \( S \). In this case, \( P(S) \) is the hyperbolic pencil of all lines perpendicular to the invariant line.

If \( S \) is parallel motion, then the induced map of the absolute has a unique fixed point. Then \( P(S) \) is the parabolic pencil of all lines whose closure contains this point.

If \( S \) is a rotation, then it has a fixed point, and \( P(S) \) is the elliptic pencil of all lines passing through this point.

If \( S \) is a glide reflection, then (as in the case of translation) it has a unique invariant line called the axis of \( S \) and \( P(S) \) is the corresponding hyperbolic pencil of lines.

If \( S \) is a reflection in a line, then \( P(S) \) is the hyperbolic pencil of all lines perpendicular to this line.

8.3. Head to tail methods. Let \( S \) and \( T \) be isometries of the hyperbolic plane. In this section we describe how to find a biflipper for \( S \circ T \), given biflippers of \( S \) and \( T \).
If \( S \) and \( T \) are rotations, then we act as in the case of two rotations in \( \mathbb{R}^2 \). Connect the fixed points of \( S \) and \( T \) by a line \( L \). It belongs to both \( \mathcal{P}(S) \) and \( \mathcal{P}(T) \). By Theorem 8 there exist biflippers \( \overrightarrow{AL} \) and \( \overrightarrow{LB} \), which encode \( T \) and \( S \), respectively. The biflipper \( \overrightarrow{AB} \) encodes \( S \circ T \).

If \( S \) or \( T \) is a rotation and the other of them is not, then we act as in the case of rotation and translation. Let, say, \( S \) be a rotation. Choose a line from the \( \mathcal{P}(T) \), which passes through the center of \( S \) and hence belong to \( \mathcal{P}(S) \). Then do the same as above.

Any other situation can be easily reduced to one of these two. Choose any biflippers \( \overrightarrow{AB} \) and \( \overrightarrow{CD} \) representing \( T \) and \( S \). If \( A \) is not a line, change both \( A \) and \( B \) to make \( A \) a line.

Consider the biflipper \( \overrightarrow{BC} \) and the corresponding isometry \( U = F_C \circ F_B \). Lines of the pencil \( \mathcal{P}(U) \) cover the whole plane. Choose a point \( Q \in A \) and a line \( L \in \mathcal{P}(U) \) passing through \( Q \). By Theorem 8, there is a biflipper \( \overrightarrow{LM} \) which encodes \( U \). So,

\[
S \circ T = F_D \circ F_C \circ F_B \circ F_A = F_D \circ U \circ F_A = F_D \circ F_M \circ F_L \circ F_A
\]

Since \( L \) and \( A \) intersect, either \( L = A \) and then \( S \circ T = F_D \circ F_M \) (and we are done), or \( F_L \circ F_A \) is a rotation and we are in the situation considered above.

However, in most cases this reduction to rotations is not needed and there is a direct simpler construction. Consider those cases.

Let \( S \) and \( T \) be parallel motion. Then the pencils \( \mathcal{P}(S) \) and \( \mathcal{P}(T) \) intersect (a common line is connecting the points on the absolute which are the centers of these pencils), and this line can be made the head of one biflipper and the tail of the other one.

Let both \( S \) and \( T \) have invariant lines \( L \) and \( M \) (i.e., each of \( S \) and \( T \) is either a translation, or a glide reflection, or a reflection) and \( L \), \( M \) be non parallel lines distinct from each other. Then either \( L \) and \( M \) intersect, or they are ultra parallel. In the latter case the pencils \( \mathcal{P}(S) \) and \( \mathcal{P}(M) \) have a common line and it can be made the head of a biflipper encoding \( T \) and tail of the biflipper encoding \( S \). In the former case, an intersection point of \( L \) and \( M \) can be made the head of a biflipper encoding \( T \) and tail of the biflipper encoding \( S \).

A hyperbolic pencil of lines orthogonal to line \( L \) and parabolic pencil of lines with central point \( Q \) on the absolute which does not belong to (the closure of) \( L \) have a common line. This allows to achieve head to tail biflippers in the cases when one of the isometries is a parallel motion and the other one has invariant line not containing the center of the parabolic pencil of the other one.
Thus, the only situations, in which there is no biflippers for $S$ and $T$ in head to tail position, are when either both $S$ and $T$ have invariant lines and these lines are parallel, or one of the isometries is parallel motion the other has an invariant line and this line is parallel to the lines of the pencil of lines of the parallel motion. These are degenerate situations and for them one has to make a preliminary replacement of $S$ and $T$ which would not affect $S \circ T$. The preliminary replacement can be done so that $T$ would be replaced by a rotation, as it was described above.

9. Back to Euclidean spaces

9.1. On line. Any isometry of a line is either the reflection in a point or a translation. A translation is encoded by a biflipper made of two points. A reflection in a point can be presented by a biflipper made of the point and the whole line, no matter in which order.

9.2. Multiplication of biflippers. For any isometries $S : \mathbb{R}^k \to \mathbb{R}^k$ and $T : \mathbb{R}^l \to \mathbb{R}^l$, the direct product $S \times T : \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^k \times \mathbb{R}^l : (x, y) \mapsto (S(x), T(y))$ is an isometry of $\mathbb{R}^{k+l} = \mathbb{R}^k \times \mathbb{R}^l$. If $A \subset \mathbb{R}^k$ and $A' \subset \mathbb{R}^l$ are affine subspaces, then $A \times A'$ is an affine subspace of $\mathbb{R}^k \times \mathbb{R}^l = \mathbb{R}^{k+l}$ and $F_A \times F_{A'} = F_{A \times A'}$.

Theorem 9. For any biflippers $\overrightarrow{AB}$ and $\overrightarrow{CD}$, which encode isometries $S : \mathbb{R}^k \to \mathbb{R}^k$ and $T : \mathbb{R}^l \to \mathbb{R}^l$, the isometry $S \times T : \mathbb{R}^{k+l} \to \mathbb{R}^{K+l}$ is encoded by $\overrightarrow{(A \times C)(B \times D)}$.

Proof. Notice that if $F, F' : \mathbb{R}^k \to \mathbb{R}^k$ commute and $G, G' : \mathbb{R}^l \to \mathbb{R}^l$ commute, then $F \times G$ commutes with $F' \times G'$. Therefore

$$F_{B \times D} \circ F_{A \times C} = (F_B \times \text{id}) \circ (\text{id} \times F_D) \circ (F_A \times \text{id}) \circ (\text{id} \times F_C)$$

$$= (F_B \times \text{id}) \circ (F_A \times \text{id}) \circ (\text{id} \times F_D) \circ (\text{id} \times F_C)$$

$$= ((F_B \circ F_A) \times \text{id}) \circ (\text{id} \times (F_D \circ F_C))$$

$$= (F_B \circ F_A) \times (F_D \circ F_C) = S \times T.$$ 

Corollary. Let $A, B$ be affine subspaces of $\mathbb{R}^k$, and $C$ be an affine subspace of $\mathbb{R}^l$. Then the biflipper $\overrightarrow{(A \times C)(B \times C)}$ encodes the direct
product of the isometry of $\mathbb{R}^k$ encoded by $\overrightarrow{AB}$ by the identity of $\mathbb{R}^l$ (no matter what $C$ is).

Observe, that all isometries of the plane, except rotations by angles $0 < \varphi < \pi$, can be obtained as products of isometries of the line. Their biflippers can be obtained as products as well.

9.3. In Euclidean $n$-space. As follows from the well-known classification of isometries of $\mathbb{R}^n$ (see, e.g., M. Berger [1]), any isometry of $\mathbb{R}^n$ is isometric to a direct product of isometries in factors of dimension at most two. Therefore, any isometry of $\mathbb{R}^n$ can be encoded by a biflipper.

This statement can be obtained also as a corollary of much more general results by Djoković [2]. He proved, in particular, that any isometry of a non-degenerate inner product space over any field can be presented as composition of two involutions-automorphisms.

From this it follows that any isometry of an affine space with a non-degenerate bilinear form can be presented as a composition of two flips.

This implies that in the classical simply connected complete spaces of constant curvature, isometries can be presented as compositions of two flips (i.e. by biflippers).

One can ask if there exist simple and useful descriptions for biflippers of the composition of isometries presented by biflippers.

10. In Euclidean 3-space

Biflippers in the 3-space look as follows:
Here really all possible mutual positions of two proper distinct affine subspaces of $\mathbb{R}^3$ are presented. We take into account if one of the subspaces is contained in the other or not, if the subspaces are parallel, perpendicular or form a generic angle, if the lines are skew.

10.1. **Decompositions into products.** Each of these biflippers can be obtained as products of biflippers in plane and line. Therefore one can use Theorem 9 for recognizing the isometries encoded by the biflippers. Moreover, the biflippers for translations, rotations, reflections, glide reflections and two of four biflippers for symmetries about a line satisfy the assumptions of Corollary of Theorem 9 and encode a product of a plane isometry by the identity map of the line. The reader can easily verify this.

**Central symmetries.** Biflippers which encode the central symmetries are products of biflippers encoding central symmetries of plane and line. For instance:

\[
\begin{array}{c}
\begin{array}{c}
\text{plane} \\
\text{point}
\end{array}
\end{array}
\quad = 
\begin{array}{c}
\begin{array}{c}
\text{plane} \\
\text{point}
\end{array}
\end{array}
\quad \times 
\begin{array}{c}
\begin{array}{c}
\text{plane} \\
\text{point}
\end{array}
\end{array}
\]

**Symmetries about a line.** Two biflippers which encode 1-flips and are formed by a plane and a point are products of biflippers which encode reflections of plane and line. For instance:

\[
\begin{array}{c}
\begin{array}{c}
\text{plane} \\
\text{point}
\end{array}
\end{array}
\quad = 
\begin{array}{c}
\begin{array}{c}
\text{plane} \\
\text{point}
\end{array}
\end{array}
\quad \times 
\begin{array}{c}
\begin{array}{c}
\text{plane} \\
\text{point}
\end{array}
\end{array}
\]

The two biflippers which encode symmetries about a line of the 3-space are degenerations of two biflippers for glide symmetries about a line. On our picture of all 3D biflippers, the biflippers of glide symmetries about a line are right under the corresponding biflippers for symmetries about a line.

**Glide symmetries about a line.** These biflippers for glide symmetries about a line are direct products of biflippers for a glide reflection of plane and a reflection of line.

\[
\begin{array}{c}
\begin{array}{c}
\text{plane} \\
\text{point}
\end{array}
\end{array}
\quad = 
\begin{array}{c}
\begin{array}{c}
\text{plane} \\
\text{point}
\end{array}
\end{array}
\quad \times 
\begin{array}{c}
\begin{array}{c}
\text{plane} \\
\text{point}
\end{array}
\end{array}
\]
Screw motions. A screw motion is a direct product of a plane rotation by a translation of a line. By Theorem 9, a biflipper for a screw motion can be obtained as a product of biflippers for rotation and translation:

Rotary reflections. A rotary reflection is a direct product of a plane rotation by a reflection of line in a point. The biflippers for rotary reflections drawn above can be obtained (by Theorem 9) as products of biflippers for rotation and reflection. For instance:

10.2. Generic isometries. Isometries of the last two types, namely, screw motions and rotary reflections, are generic in the following sense. The space of all isometries of $\mathbb{R}^3$ is a Lie group of dimension 6. It consists of two connected components. The connected component, which contains the identity, consists of isometries preserving orientation; the other component consists of isometries reversing orientation. Screw motions constitute an open dense subset in the set of orientation preserving isometries and rotary reflections constitute an open dense subset in the set of orientation reversing isometries.

For any isometry of $\mathbb{R}^3$, there exists a biflipper, which can be obtained as a limit of biflippers of screw motions or rotary reflections. There may be also biflippers that are not such limits.

For example, a biflipper of a rotation, which is formed by two intersecting lines, can be presented as a limit of biflippers formed by skew lines, that is biflippers of screw motions. A biflipper of a rotation, which is formed by two planes, is not a limit of biflippers of screw motions.

In the study of biflippers in $\mathbb{R}^3$, we will restrict ourselves to the case of generic isometries. We leave to the reader extensions of the results to more special isometries. At least, for biflippers that are limits of biflippers of screw motions, it can be easily done by passing to the limit.

10.3. Screw motions. As we saw above, an ordered pair of skew lines is a biflipper for a screw motion. Recall that a screw motion is the composition of a translation with a rotation about a line parallel to the direction of the translation. For the composition of flips in skew lines, the
translation acts along the line, which is the common perpendicular to the skew lines, by the distance equal twice the distance between the skew lines and a rotation about the same common perpendicular by the angle which is twice the angle between the lines.

**Theorem 10.** Two biflippers in the Euclidean 3-space that consist of two skew lines are equivalent iff the common perpendicular to the skew lines in one of the biflippers coincides with the common perpendicular to the skew lines in the other biflipper and the biflippers can be obtained from each other by a translation along the common perpendicular followed by a rotation about it.

**Proof.** Let $\overrightarrow{AB}$ is a biflipper formed of skew lines $A$ and $B$ with common perpendicular $L$. Translations along $L$ and rotations about $L$ generate a commutative group. The screw motions encoded by $\overrightarrow{AB}$ belongs to this group. Therefore, by Theorem 2 translations along $L$ and rotations about $L$ map $\overrightarrow{AB}$ to equivalent biflippers.

By the way, this centralizer group for the screw motion can be obtained also from the decomposition of the screw motion into a direct product of a plane rotation and a line translation: the centralizer of a plane rotation is the group of all plane rotations with the same center, the centralizer of a line translation is the group of all line translations, the centralizer of their direct product is the product of the centralizers.

Let $\overrightarrow{CD}$ be a biflipper equivalent to $\overrightarrow{AB}$. By the classification of biflippers in the 3-space made above, $C$ and $D$ are skew lines. Only one line is invariant under a screw motion, and the common perpendicular of two skew lines is invariant under the reflections in the lines, and hence under their composition. Thus, $C$ and $D$ have the same common perpendicular as $A$ and $B$. By an appropriate composition $T$ of a translation along this line and rotation along it, we can map $A$ to $C$. Then $\overrightarrow{CT(B)}$ is equivalent to $\overrightarrow{AB}$ and hence to $\overrightarrow{CD}$. So, $F_{T(B)} \circ F_C = F_D \circ F_C$. By multiplying this equality by $F_C$ from the right, we get $F_{T(B)} = F_D$ and hence $T(B) = D$. □

**10.4. Head to tail rule for screw motions.** Take biflippers $\overrightarrow{AB}$ and $\overrightarrow{CD}$ for the screw motions, and in each of them extend the arrow connecting the flippers to the axes of the screw motions (the common perpendicular $X$ for lines $A$ and $B$ and the common perpendicular $Y$ for $C$ and $D$). Find the common perpendicular $Z$ to $X$ and $Y$. By gliding the biflippers along their axes and rotating about the axes, make the head flipper of the first biflipper and the tail flipper of the second biflipper coinciding with $Z$. Find the common perpendicular
to the tail flipper of the first biflipper and head flipper of the second biflipper. Connect these flippers with an arrow along this common perpendicular. Erase the old arrows and their common perpendicular.

10.5. Rotary reflections. As we saw above, an ordered pair formed of transversal line and plane is a biflipper for a rotary reflection. Recall that a rotary reflection is a composition of a reflection in a plane with a rotation about a line which is perpendicular to the plane. The rotation and reflection commute, therefore the order does not matter.

\[ F_B \circ F_A = (F_B \circ F_C) \circ (F_C \circ F_A) = F_B \circ F_C \circ F_D \]

rotary reflection = rotation \circ reflection in plane

As we saw above, a rotary reflection is a direct product of a planar rotation and reflection of a line. The centralizer group of a rotary reflection is the product of the centralizers of the factors. Therefore the centralizer of a rotary reflection is the direct product of the group of plane rotations about the fixed point of the rotary reflection and the reflection of the 3-space in the plane of rotations.

A rotary reflection has a fixed point and invariant plane and line orthogonal to each other and intersecting in the fixed point. Flippers in a biflipper, which encodes a rotary reflection are a line and a plane, the plane contains the invariant line and the line is contained in the invariant plane and passes through the fixed point.

Using these one can prove that two biflippers of rotary reflections are equivalent iff they can be obtained from each other by a rotation about the invariant line and reflection in the plane flipper of the biflipper with a simultaneous change of the flippers’ ordering.

10.6. No head to tail for rotary reflections. If two rotary reflections are such that their invariant lines are skew and the common perpendicular to the lines does not pass through the fixed points of both rotary reflections, then there are no biflippers of the rotary reflections with a common flipper. Thus, there is no head to tail method that works for composing any two rotary reflections in the 3-space.
10.7. **Screw motions help.** However, one can easily calculate composition $S \circ T$ of rotary reflections using a head to tail rule for composing two screw motions. For this, first choose biflippers $\overrightarrow{AB}$ and $\overrightarrow{CD}$ for $T$ and $S$, respectively, such that $B$ and $C$ are planes. Then consider the biflipper $\overrightarrow{BC}$. Since it consists of planes, it encodes either translation (if $B \parallel C$) or rotation (otherwise). Take any plane $E$ orthogonal to both $B$ and $C$. By Theorem 1, the biflipper $\overrightarrow{B'C'}$ formed of the lines $B' = B \cap E$ and $C' = C \cap E$ is equivalent to $\overrightarrow{BC}$. Thus $S \circ T = F_A \circ F_B \circ F_C \circ F_D$, and we deal with the composition of screw motions or their degenerations, and can apply the technique of Section 10.3.

10.8. **No head to tail for composition of rotary reflection and screw motion.** Similarly, if a rotary reflection and screw motion are such that their invariant lines are skew and the common perpendicular to the lines does not pass through the fixed point of the rotary reflection, then the rortoreflection and screw motion have no biflippers with a common mirror, and there is no head to tail rule that would work for composing them.

10.9. **Screw motions help again.** However, the composition can be easily calculated. For this one can use again a decomposition of a rotary reflection into a composition of a screw motion and reflection.

Say, if $T$ is a rotary reflection, and $S$ is a screw motion, then choose biflippers $\overrightarrow{AB}$ and $\overrightarrow{CD}$ for $T$ and $S$, respectively, such that $A$ is a plane. Let $E$ be any line on $A$ and $F$ be a plane containing $E$ and perpendicular to $A$. Then $F_A = F_E \circ F_F$ and $S \circ T = F_D \circ F_C \circ F_B \circ F_A = F_D \circ F_C \circ F_B \circ F_E \circ F_F$. For $F_D \circ F_C \circ F_B \circ F_E$ one can apply a head to tail composition method and replace it by $F_X \circ F_Y$. Now we have to reduce the composition $F_X \circ F_Y \circ F_F$, where $X$ and $Y$ are lines and $F$ is a plane. Denote the common perpendicular of $X$ and $Y$ by $Z$.

If $F$ is not parallel to $Z$, then, by a translation along $Z$, we move the point $Z \cap Y$ to $F$ and, by a rotation about $Z$ place $Y$ on $F$. As $Y \subset F$, the composition $F_Y \circ F_F$ is a reflection in a plane $W$ containing $Y$ and perpendicular to $F$. Hence $S \circ T = F_X \circ F_W$ and we are done.

If $F$ is parallel to $Z$, then by a rotation about $Z$ we make $Y$ perpendicular to $F$, and observe that, as soon as this is done, $F_Y \circ F_F = F_{Y \cap F}$. Hence, $S \circ T = F_X \circ F_Y \circ F_F = F_X \circ F_{Y \cap F}$. 

11. In hyperbolic 3-space

11.1. Well-known facts. We will use the Poincaré model, in which the hyperbolic 3-space is represented by an open unit ball $H^3$ in $\mathbb{R}^3$. The boundary 2-sphere $\partial H^3$ of $H^3$ is called the absolute.

In this model, a plane is either an open unit disk, which is cut on $H^3$ by a plane passing through the origin, or a surface, which cut on $H^3$ by a 2-sphere orthogonal to $\partial H^3$. In any case the boundary of a plane in $\mathbb{R}^3$ is a circle which lies on $\partial H^3$. It is called the absolute of this plane. A line in $H^3$ is an arc which can be presented as the intersection of two different planes. A line is either a diameter of the ball $H^3$, or an arc cut on $H^3$ by a circle orthogonal to $\partial H^3$.

Two planes may intersect in a line or be disjoint. In the latter case, their absolutes may be disjoint and then the planes are said to be ultra-parallel or be tangent to each other and then the planes are said to be parallel. Two ultra-parallel planes have a unique common perpendicular line. Conversely, two planes perpendicular to the same line are ultra-parallel.

There are three kinds of flippers: planes, lines and points. Any plane, line or point is a flipper. In the Poincaré model, the flip in a plane, which passes through the origin of $\mathbb{R}^3$ is the restriction of the usual Euclidean symmetry of $\mathbb{R}^3$ with respect to the 2-subspace, which cut the plane on $H^3$; the flip in a plane, which is cut by a 2-sphere $S$, is a restriction of the inversion of $\mathbb{R}^3$ with respect to $S$. A flip in a line is a composition of flips in any two planes passing through the line and orthogonal to each other. Similarly, the flip in a point is the composition of flips in any three planes passing through this point and orthogonal to each other.

11.2. Biflippers. We restrict ourselves to generic biflippers leaving to the reader to compile a complete list of biflippers in the hyperbolic 3-space. As in other situations that we considered above, we just list ordered pairs of flippers and figure out what the compositions of the corresponding flips are. See [3], chapter IV.
Exercise. Find an explicit descriptions for equivalence of the biflippers and head to tail rules. Advice: revisit sections 3 and 10.

12. Full Möbius group.

An isometry of the hyperbolic 3-space $H^3$ acts on the absolute $\partial H^3$. Recall that the absolute $\partial H^3$ is a 2-sphere. In the Poincaré model, $\partial H^3$ is the unit sphere $S^2 = \{ x \in \mathbb{R}^3 \mid |x| = 1 \}$ in $\mathbb{R}^3$. Isometries of $H^3$ induce on it transformations which preserve angles, but do not necessarily preserve any metric or orientation. They form a group, which is called the full Möbius group. As an abstract group, it is isomorphic to the group of all isometries of the hyperbolic 3-space. The transition from an isometry of $H^3$ to the corresponding transformation of $\partial H^3$ is an isomorphism between the groups.

The Möbius group is a subgroup of the full Möbius group formed by orientation preserving transformations. A general trend to discriminate orientation reversing maps and non-orientable manifolds in the 2-dimensional case is rationalized by an attention to complex structures. Elements of the Möbius group are identified with fractional linear transformations with complex coefficients, the 2-sphere is identified with the complex projective line $\mathbb{CP}^1$ and the Möbius group with the group of its complex projective automorphisms.

In our context, orientation reversing elements of the full Möbius group are not any worse than orientation preserving ones. The group of isometries of $H^3$ includes orientation reversing isometries. The group is generated by reflections in planes. Respectively, the full Möbius group is generated by involutions, which fixed point sets are circles (the absolutes of the planes). If the fixed point set is a great circle, then the involution of $S^2$ is the restriction of the reflection of $\mathbb{R}^3$ in the plane, which cut the circle on the sphere. If the fixed point set is not a great circle, it is cut on $S^2$ by a sphere $S$ orthogonal to $S^2$, and the involution is the restriction to $S^2$ of the inversion of $\mathbb{R}^3$ in $S$.

As was mentioned in the beginning of the paper, the notions of flip, flipper and biflipper are naturally extended to this setup. There are two kinds of flippers: circles that are cut on $S^2$ by planes and pairs of points. The flips corresponding to circles are the involutions which
were described above as the generators of the Möbius group. The flips corresponding to pairs of points are compositions of two commuting flips of the first kind.

In the Möbius group there are many involutions without fixed points. They are induced by flips on $H^3$ with one point flippers. These involutions of $S^2$ share the fixed point set in $S^2$ (the empty), hence they are not determined by their fixed point sets and are not qualified to be flips in $S^2$.

The stereographic projection $st : S^2 \setminus pt \to \mathbb{R}^2$ is a conformal isomorphism, and we can use it for pictures. Here is the list of plane images of the biflippers. The first row is filled by biflippers of Möbius transformations. The biflippers of the second row correspond to orientation reversing transformations.

Conclusion. Presentations of isometries as compositions of two flips allow one to find geometric algorithms for calculating compositions. In the two-dimensional classical homogeneous spaces this works perfectly. The two-dimensional geometry is the most interesting for the pedagogical applications.

As the dimension grows the algorithms become more complicated. In the three-dimensional Euclidean space the algorithms are still easy, but for some pairs of isometries, flip-flop decompositions of them such that the same flipper appears in both decompositions do not exist.

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