The maximum sum of sizes of cross-intersecting families of subsets of a set

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Abstract

A set of sets is called a family. Two families $A$ and $B$ of sets are said to be cross-intersecting if each member of $A$ intersects each member of $B$. For any two integers $n$ and $k$ with $1 \leq k \leq n$, let $\binom{[n]}{\leq k}$ denote the family of subsets of $[n] = \{1, \ldots, n\}$ that have at most $k$ elements. We show that if $A$ is a non-empty subfamily of $\binom{[n]}{\leq r}$, $B$ is a non-empty subfamily of $\binom{[n]}{\leq s}$, $r \leq s$, and $A$ and $B$ are cross-intersecting, then

$$|A| + |B| \leq 1 + \sum_{i=1}^{s} \left( \binom{n}{i} - \binom{n-r}{i} \right),$$

and equality holds if $A = \{[r]\}$ and $B$ is the family of sets in $\binom{[n]}{\leq s}$ that intersect $[r]$.

1 Introduction

Unless stated otherwise, we shall use small letters such as $x$ to denote non-negative integers or elements of a set, capital letters such as $X$ to denote sets, and calligraphic letters such as $\mathcal{F}$ to denote families (that is, sets whose members are sets themselves). It
is to be assumed that arbitrary sets and families are finite. We call a set $A$ an $r$-element set if its size $|A|$ is $r$, that is, if it contains exactly $r$ elements.

The set $\{1, 2, \ldots\}$ of positive integers is denoted by $\mathbb{N}$. For $m \geq 0$ and $n \geq 0$, the set $\{i \in \mathbb{N}: m \leq i \leq n\}$ is denoted by $[m, n]$. We abbreviate $[1, n]$ to $[n]$. Note that $[0]$ is the empty set $\emptyset$, and $[n] = \{1, \ldots, n\}$ for $n \geq 1$. For a set $X$, the power set of $X$ (the set of subsets of $X$) is denoted by $2^X$. For any integer $r \geq 0$, the family of $r$-element subsets of $X$ is denoted by $\binom{X}{r}$, and the family of subsets of $X$ of size at most $r$ is denoted by $\binom{X}{\leq r}$. Thus, $\binom{X}{r} = \bigcup_{i=0}^{r} \binom{X}{i}$. If $x \in X$ and $\mathcal{F} \subseteq 2^X$, then we denote the family of sets in $\mathcal{F}$ which contain $x$ by $\mathcal{F}(x)$. We call $\mathcal{F}(x)$ a star of $\mathcal{F}$ if $\mathcal{F}(x) \neq \emptyset$.

We say that a set $A$ intersects a set $B$ if $A$ and $B$ have at least one common element (that is, if $A \cap B \neq \emptyset$). A family $\mathcal{A}$ is said to be intersecting if every two sets in $\mathcal{A}$ intersect. Note that the stars of a family $\mathcal{F}$ are the simplest intersecting subfamilies of $\mathcal{F}$. If $\mathcal{A}$ and $\mathcal{B}$ are families such that each set in $\mathcal{A}$ intersects each set in $\mathcal{B}$, then $\mathcal{A}$ and $\mathcal{B}$ are said to be cross-intersecting.

One of the most popular endeavours in extremal set theory is that of determining the size of a largest intersecting subfamily of a given family $\mathcal{F}$. This started in [10], which features the classical result, known as the Erdős-Ko-Rado (EKR) Theorem, that says that if $r \leq n/2$, then the size of a largest intersecting subfamily of $\binom{[n]}{r}$ is the size of $\binom{[n]}{r-1}$ of any star of $\binom{[n]}{r}$. There are many proofs of the EKR Theorem. Two of them are particularly short and beautiful: Katona’s [20], which introduced the elegant cycle method, and Daykin’s [8], which uses a fundamental result known as the Kruskal–Katona Theorem [21, 23] (see also [11, 22]). The EKR Theorem gave rise to some of the highlights in extremal set theory [1, 12, 17, 22, 28] and inspired many results, including generalizations (see, for example, [3, 27]), that establish how large a system of sets can be under certain intersection conditions; see [4, 9, 11, 13, 15, 18, 19].

A natural question to ask about intersecting subfamilies of a given family $\mathcal{F}$ is how large they can be. For cross-intersecting families, two natural parameters arise: the sum and the product of sizes of the families. The problem of maximizing the sum or the product of sizes of cross-$t$-intersecting subfamilies of a given family $\mathcal{F}$ has been attracting much attention. Many of the results to date are referenced in [5, 6, 7].

Hilton and Milner [17] proved that, for $1 \leq r \leq n/2$, if $\mathcal{A}$ and $\mathcal{B}$ are non-empty cross-intersecting subfamilies of $\binom{[n]}{r}$, then $|\mathcal{A}| + |\mathcal{B}| \leq \binom{n}{r} - \binom{n-r}{r} + 1$, and equality holds if $\mathcal{A} = \{[r]\}$ and $\mathcal{B} = \{B \in \binom{[n]}{r}: B \cap [r] = \emptyset\}$. To the best of the authors’ knowledge, this was the first result on the sizes of cross-intersecting families. Simpson [26] obtained a streamlined proof by means of the compression (also known as shifting) technique, which was introduced in the seminal EKR paper [10] and has proven to be a very useful tool in extremal set theory ([13] is a recommended survey on the properties and uses of compression operations). Frankl and Tokushige [16] instead used the Kruskal–Katona Theorem to establish the following stronger result: if $1 \leq r \leq s$, $n \geq r+s$, $\mathcal{A} \subseteq \binom{[n]}{r}$, $\mathcal{B} \subseteq \binom{[n]}{s}$, and $\mathcal{A}$ and $\mathcal{B}$ are cross-intersecting and non-empty, then $|\mathcal{A}| + |\mathcal{B}| \leq 1 + \binom{n}{s} - \binom{n-r}{s} - \binom{n-s}{s}$, and equality holds if $\mathcal{A} = \{[r]\}$ and $\mathcal{B} = \{B \in \binom{[n]}{s}: B \cap [r] = \emptyset\}$. The attainable upper bound on the maximum product of sizes for $1 \leq r \leq s \leq n/2$ was established in [24, 25].

In this paper, we solve the analogous maximum sum problem for the case where $\mathcal{A} \subseteq \binom{[n]}{\leq r}$ and $\mathcal{B} \subseteq \binom{[n]}{\leq s}$, using the compression technique. The following is our result, proved in the next section.
We now prove Theorem 1.

**Theorem 1** If \( n \geq 1 \), \( 1 \leq r \leq s \), \( A \subseteq \binom{[n]}{\leq r} \), \( B \subseteq \binom{[n]}{\leq s} \), and \( A \) and \( B \) are cross-intersecting and non-empty, then

\[
\left| A \right| + \left| B \right| \leq 1 + \sum_{i=1}^{s} \left( \binom{n}{i} - \binom{n-r}{i} \right),
\]

and equality holds if \( A = \{[r]\} \) and \( B = \{B \in \binom{[n]}{r}: B \cap [r] \neq \emptyset\} \).

The analogous problem for the product of sizes was solved in [2].

## 2 Proof of Theorem 1

We now prove Theorem 1.

For any \( i, j \in [n] \), let \( \delta_{i,j}: 2^{[n]} \rightarrow 2^{[n]} \) be defined by

\[
\delta_{i,j}(A) = \begin{cases} 
(A \setminus \{j\}) \cup \{i\} & \text{if } j \in A \text{ and } i \notin A; \\
A & \text{otherwise},
\end{cases}
\]

and let \( \Delta_{i,j}: 2^{[n]} \rightarrow 2^{[n]} \) be the compression operation defined by

\[
\Delta_{i,j}(A) = \{\delta_{i,j}(A): A \in \mathcal{A}\} \cup \{A \in \mathcal{A}: \delta_{i,j}(A) \in \mathcal{A}\}.
\]

Note that \( \Delta_{i,j} \) preserves the size of a family \( A \), that is,

\[
\left| \Delta_{i,j}(A) \right| = \left| A \right|.
\]

We will need this equality together with the following basic fact, which we prove for completeness.

**Lemma 1** If \( \mathcal{A} \) and \( \mathcal{B} \) are cross-intersecting subfamilies of \( 2^{[n]} \), then, for any \( i, j \in [n] \), \( \Delta_{i,j}(\mathcal{A}) \) and \( \Delta_{i,j}(\mathcal{B}) \) are cross-intersecting subfamilies of \( 2^{[n]} \).

**Proof.** Suppose \( A \in \Delta_{i,j}(\mathcal{A}) \) and \( B \in \Delta_{i,j}(\mathcal{B}) \). If \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \), then \( A \cap B \neq \emptyset \). Suppose \( A \notin \mathcal{A} \) or \( B \notin \mathcal{B} \). We may assume that \( A \notin \mathcal{A} \). Then, \( A = \delta_{i,j}(A') \neq A' \) for some \( A' \in \mathcal{A} \), so \( i \notin A' \), \( j \in A' \), \( i \in A \), and \( j \notin A \). Suppose \( A \cap B = \emptyset \). Then, \( i \notin B \), \( B \in \mathcal{B} \setminus \Delta_{i,j}(\mathcal{B}) \), and hence \( B, \delta_{i,j}(B) \in \mathcal{B} \). Thus, \( A' \cap B \neq \emptyset \) and \( A' \cap \delta_{i,j}(B) \neq \emptyset \). Since \( A \cap B = \emptyset \) and \( A' \cap B \neq \emptyset \), \( A' \cap B = \{j\} \). This yields \( A' \cap \delta_{i,j}(B) = \emptyset \), a contradiction. \( \square \)

If \( i < j \), then \( \Delta_{i,j} \) is called a left-compression. A family \( \mathcal{F} \subseteq 2^{[n]} \) is said to be compressed if \( \Delta_{i,j}(\mathcal{F}) = \mathcal{F} \) for every \( i, j \in [n] \) with \( i < j \) (that is, if \( \mathcal{F} \) is invariant under left-compressions). Thus, \( \mathcal{F} \) is compressed if and only if \( (\mathcal{F} \setminus \{j\}) \cup \{i\} \in \mathcal{F} \) for every \( i, j \in [n] \) and every \( F \in \mathcal{F} \) such that \( i < j \in F \) and \( i \notin F \).

A subfamily \( \mathcal{A} \) of \( 2^{[n]} \) that is not compressed can be transformed to a compressed subfamily of \( 2^{[n]} \) as follows. We choose one of the left-compressions that change \( \mathcal{A} \), and we apply it to \( \mathcal{A} \) to obtain a new subfamily of \( 2^{[n]} \). We keep on repeating this (always applying a left-compression to the last family obtained) until a family that is
invariant under each left-compression is obtained (such a point is indeed reached, because if \( \Delta_{ij}(\mathcal{F}) \neq \mathcal{F} \subseteq 2^n \) and \( i < j \), then \( 0 < \sum_{G \in \Delta_{i,j}(\mathcal{F})} \sum_{b \in G} b < \sum_{F \in \mathcal{F}} \sum_{a \in F} a \)).

If \( \mathcal{A}, \mathcal{B} \subseteq 2^n \) such that \( \mathcal{A} \) and \( \mathcal{B} \) are cross-intersecting, then, by Lemma [1] we can obtain \( \mathcal{A}^*, \mathcal{B}^* \subseteq 2^n \) such that \( \mathcal{A}^* \) and \( \mathcal{B}^* \) are compressed and cross-intersecting, \( |\mathcal{A}^*| = |\mathcal{A}| \), and \( |\mathcal{B}^*| = |\mathcal{B}| \). Indeed, similarly to the procedure above, if we can find a left-compression that changes at least one of \( \mathcal{A} \) and \( \mathcal{B} \), then we apply it to both \( \mathcal{A} \) and \( \mathcal{B} \), and we keep on repeating this (always performing this on the last two families obtained) until we obtain \( \mathcal{A}^*, \mathcal{B}^* \subseteq 2^n \) such that \( \mathcal{A}^* \) and \( \mathcal{B}^* \) are compressed.

**Proof of Theorem [1].** We prove the result by induction on \( n \). The result is trivial for \( n \leq 2 \). Consider \( n \geq 3 \).

Suppose \( r \geq n \). Then, \( s \geq n \) and \( \binom{n}{\leq s} = \binom{n}{s} = 2^n \). Since \( \mathcal{A} \) and \( \mathcal{B} \) are cross-intersecting, \( [n]\setminus A \notin B \) for each \( A \in \mathcal{A} \). Thus, \( B \subseteq 2^n \setminus \{[n]\setminus A: A \in \mathcal{A}\} \), and hence \( |B| \leq 2^n - |A| \). We have \( |A| + |B| \leq 2^n = 1 + |\{B \in 2^n: B \cap [n] \neq \emptyset\}| = 1 + |\{B \in (\frac{n}{s}) \setminus B \cap [r] \neq \emptyset\}| \).

Now suppose \( r < n \). If \( s > n \), then \( \binom{n}{\leq s} = 2^n = \binom{n}{\leq n} \). Thus, we may assume that \( s \leq n \). Since \( r < n \) and \( r \leq s \), we have

\[
 r < s \quad \text{or} \quad s < n. \tag{1}
\]

As explained above, we may assume that \( \mathcal{A} \) and \( \mathcal{B} \) are compressed.

Let \( \mathcal{A}_0 = \{A \in \mathcal{A}: n \notin A\} \), \( \mathcal{A}_1 = \{A \setminus \{n\} : n \in A \in \mathcal{A}\} \), \( \mathcal{B}_0 = \{B \in \mathcal{B} : n \notin B\} \), and \( \mathcal{B}_1 = \{B \setminus \{n\} : n \in B \in \mathcal{B}\} \). We have \( \mathcal{A}_0 \subseteq \binom{n-1}{s} \), \( \mathcal{A}_1 \subseteq \binom{n-1}{s} \), \( \mathcal{B}_0 \subseteq \binom{n-1}{s} \), and \( \mathcal{B}_1 \subseteq \binom{n-1}{s} \). Clearly, \( \mathcal{A}_0 \) and \( \mathcal{B}_0 \) are cross-intersecting. Since \( \mathcal{A} \) and \( \mathcal{B} \) are compressed, we clearly have that \( \mathcal{A}_0, \mathcal{A}_1, \mathcal{B}_0, \) and \( \mathcal{B}_1 \) are compressed. Thus, \( [r'] \in \mathcal{A}_0 \) for some \( r' \in [r] \), and if \( s < n \), then \( [s'] \in \mathcal{B}_0 \) for some \( s' \in [s] \).

Let \( \mathcal{C} = \{A \in \mathcal{A}_1 : A \cap B = \emptyset \text{ for some } B \in \mathcal{B}_1\} \). For each \( C \in \mathcal{C} \), let \( \bar{C} = [n-1] \setminus C \), \( C' = C \cup \{n\} \), and \( \bar{C}' = C \cup \{n\} \). Let \( \bar{C} = \{C : C \in \mathcal{C}\} \). For each \( C \in \mathcal{C} \), \( C' \in \mathcal{A} \) as \( C \in \mathcal{A}_1 \).

Suppose \( C \in \mathcal{C} \). Let \( \mathcal{D}_C = \{B \in \mathcal{B}_1 : B \cap C = \emptyset\} \). Suppose that there exists some \( B \in \mathcal{D}_C \) such that \( B \neq \bar{C} \). Then, \( B \subseteq [n-1] \setminus C \), and hence \( |n-1| \setminus (B \cup C) \neq \emptyset \). Let \( x \in [n-1] \setminus (B \cup C) \). Since \( B \in \mathcal{B}_1 \), \( B \cup \{n\} \in \mathcal{B} \). Let \( D = \delta_{x,n}(B \cup \{n\}) \). Since \( x \notin B \cup \{n\} \), \( D \in \mathcal{B} \). Since \( \mathcal{B} \) is compressed, \( D \in \mathcal{B} \). However, since \( x \notin C' \) and \( B \cap C = \emptyset \), we have \( C' \cap D = \emptyset \), which is a contradiction as \( \mathcal{A} \) and \( \mathcal{B} \) are cross-intersecting. Thus, \( \mathcal{D}_C \subseteq \{\bar{C}\} \). Since \( C \in \mathcal{C} \), we have \( \mathcal{D}_C \neq \emptyset \), so \( \mathcal{D}_C = \{\bar{C}\} \).

We have therefore shown that for any \( C \in \mathcal{C} \), \( \bar{C} \) is the unique set in \( \mathcal{B}_1 \) that does not intersect \( C \). Since \( C' \in \mathcal{A} \) and \( C' \cap \bar{C} = \emptyset \), the cross-intersection condition gives us \( \bar{C} \notin \mathcal{B}_0 \). Thus,

\[
 B_0 \cap \bar{C} = \emptyset. \tag{2}
\]

Let \( \mathcal{A}' = \mathcal{A}_1 \setminus \mathcal{C} \) and \( \mathcal{B}' = \mathcal{B}_0 \cup \bar{C} \). Clearly, \( \mathcal{A}' \subseteq \binom{n-1}{s} \), \( \mathcal{B}' \subseteq \binom{n-1}{s} \), \( \mathcal{A}_0 \) and \( \mathcal{B}_0 \) are cross-intersecting (as \( \bar{C} \subseteq \mathcal{B}_1 \)), and \( \mathcal{A}_1 \) and \( \mathcal{B}_1 \) are cross-intersecting.

**Claim 1** \( |\mathcal{A}_0| + |\mathcal{B}_0'| \leq 1 + |\{B \in \binom{n-1}{s} : B \cap [r] \neq \emptyset\}| \).

**Proof.** Since \( \mathcal{A}_0 \) is non-empty (as \( [r'] \in \mathcal{A}_0 \)), the claim follows by the induction hypothesis if \( \mathcal{B}_0' \) is non-empty too, and this is the case if \( s < n \) (as we then have \( [s'] \in \mathcal{B}_0' \)).
Suppose $s = n$ and $B'_0 = \emptyset$. By (1), $r \leq s - 1$. Since $s = n$, the sets in $\mathcal{A}_0$ intersect $[s - 1]$ (note that $\emptyset \notin \mathcal{A}$ as $B \neq \emptyset$), so $\mathcal{A}_0$ and $\{[s - 1]\}$ are cross-intersecting. By the induction hypothesis, $|\mathcal{A}_0| + |\{[s - 1]\}| \leq 1 + \{|B \in \binom{[n - 1]}{r} : B \cap [r] \neq \emptyset\}$, so $|\mathcal{A}_0| + |B'_0| = |\mathcal{A}_0| \leq \{|B \in \binom{[n - 1]}{r} : B \cap [r] \neq \emptyset\}$.

Claim 2 $|A'_1| + |B_1| \leq \{|B \in \binom{[n - 1]}{r} : B \cap [r] \neq \emptyset\}$.

Proof. The claim is immediate if $A'_1 = \emptyset$ and $B_1 = \emptyset$.

If $A'_1 \neq \emptyset$ and $B_1 \neq \emptyset$, then, by the induction hypothesis, $|A'_1| + |B_1| \leq 1 + \{|B \in \binom{[n - 1]}{r} : B \cap [r] \neq \emptyset\}$.

If $A'_1 = \emptyset$ and $B_1 \neq \emptyset$, then $|A'_1| + |B_1| = |B_1| \leq \{|B \in \binom{[n - 1]}{r} : B \cap [r] \neq \emptyset\}$ (as $[r] \in \mathcal{A}$), so $|A'_1| + |B_1| \leq \{|B \in \binom{[n - 1]}{r} : B \cap [r] \neq \emptyset\}$.

Finally, suppose that $A'_1 \neq \emptyset$ and $B_1 = \emptyset$.

Suppose $B = \{[n]\}$. Then, $s = n$. By the definition of $A'_1$, we have $\emptyset \notin A'_1$, so $A'_1$ and $\{[s - 1]\}$ are cross-intersecting (as $s = n$). By the induction hypothesis, $|A'_1| + |\{[s - 1]\}| \leq 1 + \{|B \in \binom{[n - 1]}{r} : B \cap [r - 1] \neq \emptyset\}$, so $|A'_1| + |B_1| = |A'_1| \leq \{|B \in \binom{[n - 1]}{r} : B \cap [r] \neq \emptyset\}$.

Now suppose $B \neq \{[n]\}$. Then, since $B$ is compressed and non-empty, $[s^*] \in B$ for some $s^* \in [s] \cap [n - 1]$. Thus, $A'_1$ and $\{[s^*]\}$ are cross-intersecting. If $s^* \leq s - 1$, then the claim follows as in the preceding paragraph. Suppose $s^* = s$. Let $\mathcal{E} = \{A \in A'_1 : 1 \in A\}$ and $\mathcal{E}' = A'_1 \setminus \mathcal{E}$. Then, $\mathcal{E}'$ is a subfamily of $\binom{\binom{[n - 1]}{r - 1}}{s^* - 1}$ and its sets intersect the $(s - 1)$-element set $[2, s]$. Let $\mathcal{F} = \{B \in \binom{[n - 1]}{r} : 1 \in B\}$ and $\mathcal{F}' = \{B \in \binom{[n - 1]}{r} : B \cap [2, r] \neq \emptyset\}$. Since $A'_1 \subseteq \binom{[n - 1]}{r} \subseteq \binom{[n - 1]}{s - 1}$, $|\mathcal{E}| \leq |\mathcal{F}|$. By the induction hypothesis, $|\mathcal{E}'| + |\{[2, s]\}| \leq |\mathcal{F}| + |\mathcal{F}'|$. We have $|A'_1| + |B_1| = |A'_1| = |\mathcal{E}| + |\mathcal{E}'| \leq |\mathcal{F}| + |\mathcal{F}'| = \{|B \in \binom{[n - 1]}{r} : B \cap [r] \neq \emptyset\}$.

We have

$$|A| + |B| = |A_0| + |A_1| + |B_0| + |B_1|$$

$$= (|A_0| + |B'_0|) + (|A'_1| + |B_1|) + |\mathcal{C}| - |\mathcal{C}|$$

(by 2)

$$= (|A_0| + |B'_0|) + (|A'_1| + |B_1|).$$

Therefore, by Claims 1 and 2, $|A| + |B| \leq 1 + \{|B \in \binom{[n - 1]}{r} : B \cap [r] \neq \emptyset\} + \{|B \in \binom{[n - 1]}{r} : B \cap [r] \neq \emptyset\} = 1 + \{|B \in \binom{[n - 1]}{r} : B \cap [r] \neq \emptyset\}$. □

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