SATISFACTION IS NOT ABSOLUTE

JOEL DAVID HAMKINS AND RUIZHI YANG

Abstract. We prove that the satisfaction relation $\mathcal{N} \models \varphi[\vec{a}]$ of first-order logic is not absolute between models of set theory having the structure $\mathcal{N}$ and the formulas $\varphi$ all in common. Two models of set theory can have the same natural numbers, for example, and the same standard model of arithmetic $\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle$, yet disagree on their theories of arithmetic truth; two models of set theory can have the same natural numbers and the same arithmetic truths, yet disagree on their truths-about-truth, at any desired level of the iterated truth-predicate hierarchy; two models of set theory can have the same natural numbers and the same reals, yet disagree on projective truth; two models of set theory can have the same $\langle H_{\omega_2}, \in \rangle$ or the same rank-initial segment $\langle V_\delta, \in \rangle$, yet disagree on which assertions are true in these structures.

On the basis of these mathematical results, we argue that a philosophical commitment to the determinateness of the theory of truth for a structure cannot be seen as a consequence solely of the determinateness of the structure in which that truth resides. The determinate nature of arithmetic truth, for example, is not a consequence of the determinate nature of the arithmetic structure $\mathbb{N} = \{0, 1, 2, \ldots \}$ itself, but rather, we argue, is an additional higher-order commitment requiring its own analysis and justification.

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1. Introduction

Many mathematicians and philosophers regard the natural numbers \(0, 1, 2, \ldots\), along with their usual arithmetic structure, as having a privileged mathematical existence, a Platonic realm in which assertions have definite, absolute truth values, independently of our ability to prove or discover them. Although there are some arithmetic assertions that we can neither prove nor refute—such as the consistency of the background theory in which we undertake our proofs—the view is that nevertheless there is a fact of the matter about whether any such arithmetic statement is true or false in the intended interpretation. The definite nature of arithmetic truth is often seen as a consequence of the definiteness of the structure of arithmetic \(\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle\) itself, for if the natural numbers exist in a clear and distinct totality in a way that is unambiguous and absolute, then (on this view) the first-order theory of truth residing in that structure—arithmetic truth—is similarly clear and distinct.

Feferman provides an instance of this perspective when he writes:

In my view, the conception [of the bare structure of the natural numbers] is completely clear, and thence all arithmetical statements are definite. [Fefer13, p.6–7] (emphasis original)

It is Feferman’s ‘thence’ to which we call attention, and many mathematicians and philosophers seem to share this perspective. Martin writes:

What I am suggesting is that the real reason for confidence in first-order completeness is our confidence in the full determinateness of the concept of the natural numbers. [Mar12, p. 13]

The truth of an arithmetic statement, to be sure, does seem to depend entirely on the structure \(\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle\), with all quantifiers restricted to \(\mathbb{N}\) and using only those arithmetic operations and relations, and so if that structure has a definite nature, then it would seem that the truth of the statement should be similarly definite.

Nevertheless, in this article we should like to tease apart these two ontological commitments, arguing that the definiteness of truth for a given mathematical structure, such as the natural numbers, the reals or higher-order structures such as \(H_{\omega_2}\) or \(V_\delta\), does not follow from the definite nature of the underlying structure in which that truth resides.
Rather, we argue that the commitment to a theory of truth for a structure is a higher-order ontological commitment, going strictly beyond the commitment to a definite nature for the underlying structure itself.

We shall make our argument by first proving, as a strictly mathematical matter, that different models of set theory can have a structure identically in common, even the natural numbers, yet disagree on the theory of truth for that structure.

- Models of set theory can have the same structure of arithmetic \( \langle \mathbb{N}, +, \cdot, 0, 1, < \rangle \), yet disagree on arithmetic truth.
- Models of set theory can have the same reals, yet disagree on projective truth.
- Models of set theory can have a transitive rank initial segment \( V_\delta \) in common, yet disagree about whether it is a model of ZFC.

In these cases and many others, the theory of a structure is not absolute between models of set theory having that structure identically in common. This is a stronger kind of non-absoluteness phenomenon than the usual observation, via the incompleteness theorem, that models of set theory can disagree on arithmetic truth, for here we have models of set theory, which disagree about arithmetic truth, yet agree on the structure in which that truth resides. Our mathematical claims will be made in sections 2 through 5. Afterwards, on the basis of these mathematical observations, we shall draw our philosophical conclusions in section 6.

Satisfaction is absolute! This slogan, heard by the first author in his graduate-student days—a fellow logic student would assert it with exaggerated double-entendre—was meant to evoke the idea that the satisfaction relation \( \mathcal{N} \models \varphi[\vec{a}] \) is absolute between all the various models of set theory able to express it. The set-theoretic universe \( V \), for example, has the same arithmetic truths as the constructible universe \( L \); and it doesn’t matter, when asking whether \( \mathcal{N} \models \varphi[\vec{a}] \), whether one determines the answer in the universe \( V \) or in a forcing extension \( V[G] \) of it. So the slogan is true to a very high degree, and in particular, it is true whenever the formula \( \varphi \) has standard-finite length in the metatheory (which is probably closer to what was actually meant by those asserting it), or between any two models of set theory, at least one of which has access to the satisfaction relation of the other (which covers such cases as \( L \subseteq V \) and \( V \subseteq V[G] \)), since any model of set theory that can see two satisfaction relations for a structure will see by induction on formulas that they must agree. Nevertheless, the main theorems of this article show that the slogan is not true in full generality, namely, satisfaction is not absolute, for there can be models of set theory with
a structure $N$ and sentence $\sigma$ in common, but which disagree about whether $N \models \sigma$.

Before proceeding further, we should like to remark on the folklore nature of some of the mathematical arguments and results contained in sections 2 through 5 of this article, including theorems 1 and 5 and their consequences, which are proved using only well-known classical methods. These arguments should be considered as a part of the mathematical folklore of the subject of models of arithmetic, a subject filled with many fascinating results about automorphisms of nonstandard models of arithmetic and of set theory and the images of non-definable sets in computably saturated models. For example, Schlipf [Sch78] proved many basic results about computably saturated models, including the case of $\langle M, X \rangle$ where $X$ is not definable in $M$, a case which figures in our theorems 1 and 5. For example, Schlipf proves that if $M \models ZF$ is resplendent, then there is a cofinal set of indiscernibles $I$ in the ordinals of $M$ such that for each $\alpha \in I$ we have $V^M_\alpha \prec M$ and $V^M_\alpha \cong M$; if $M$ is also countable and $V^M_\alpha \prec M$, then there are $2^\omega$ many distinct isomorphisms $V^M_\alpha \cong M$; and $M$ is isomorphic to some topless initial $N \prec M$, that is, for which $M$ has no supremum to the ordinals of $N$. Kossak and Kotlarski [KK88] identified circumstances under which a nondefinable subset $X$ of a countable model $M \models PA$ must have the maximum number of automorphic images in $M$, including the case where $X$ is an inductive satisfaction class. Schmerl subsequently proved that every undefinable class $X$ in such a model $M \models PA$ has continuum many automorphic images. In other work, Kossak and Kotlarski [KK92] proved that if $M$ is a model of $PA$ with a full inductive satisfaction class, then it has full inductive satisfaction classes $S_1$ and $S_2$ which disagree on a set of sentences that is coinitial with the standard cut. So the topic is well-developed and much is known. Concerning the specific results we prove in this article, experts in the area seem instinctively to want to prove them by means of resplendency and the other sophisticated contemporary ideas that frame the current understanding of the subject—showing the depth and power of those methods—and indeed one may prove the theorems via resplendency. Nevertheless, our arguments here show that elementary methods suffice.

2. Indefinite arithmetic truth

Let us begin with what may seem naively to be a surprising case, where we have two models of set theory with the same structure of arithmetic $\langle N, +, \cdot, 0, 1, < \rangle$, but different theories of arithmetic truth.
Theorem 1. Every consistent extension of ZFC has two models $M_1$ and $M_2$, which agree on the natural numbers and on the structure $(\mathbb{N}, +, \cdot, 0, 1, <)^{M_1} = (\mathbb{N}, +, \cdot, 0, 1, <)^{M_2}$, but which disagree their theories of arithmetic truth, in the sense that there is in $M_1$ and $M_2$ an arithmetic sentence $\sigma$, such that $M_1$ thinks $\sigma$ is true, but $M_2$ thinks it is false.

Thus, two models of set theory can agree on which natural numbers exist and agree on all the details of the standard model of arithmetic, yet disagree on which sentences are true in that model. The proof is elementary, but before giving the proof, we should like to place the theorem into the context of some classical results on arithmetic truth, particularly Krajewski’s work [Kra74, Kra76] on incompatible satisfaction classes, explained in theorem 2.

Inside every model of set theory $M \models \text{ZFC}$, we may extract a canonical model of arithmetic, the structure $(\mathbb{N}, +, \cdot, 0, 1, <)^{M}$, which we henceforth denote simply by $\mathbb{N}^M$, arising from what $M$ views as the standard model of arithmetic. Namely, $\mathbb{N}^M$ is the structure whose objects are the objects that $M$ thinks to be natural numbers and whose operations and relations agree with what $M$ thinks are the standard arithmetic operations and relations on the natural numbers. Let us define that a ZFC-standard model of arithmetic, or just a standard model of arithmetic (as opposed to the standard model of arithmetic), is a model of arithmetic that arises in this way as $\mathbb{N}^M$ for some model $M \models \text{ZFC}$. In other words, a standard model of arithmetic is one that is thought to be the standard model of arithmetic from the perspective of some model of ZFC. More generally, for any set theory $T$ we say that a model of arithmetic is a $T$-standard model of arithmetic, if it arises as $\mathbb{N}^M$ for some $M \models T$.

Every model of set theory $M \models \text{ZFC}$ has what it thinks is the true theory of arithmetic $\text{TA}^M$, the collection of $\sigma$ thought by $M$ to be (the Gödel code of) a sentence in the language of arithmetic, true in $\mathbb{N}^M$. (In order to simplify notation, we shall henceforth identify formulas with their Gödel codes.) The theory $\text{TA}^M$ is definable in $M$ by means of the recursive Tarskian definition of truth-in-a-structure, although it is not definable in $\mathbb{N}^M$, by Tarski’s theorem on the non-definability of...
truth. Note that when \( \mathbb{N}^M \) is nonstandard, this theory will include many nonstandard sentences \( \sigma \), which do not correspond to any actual assertion in the language of arithmetic from the perspective of the metatheory, but nevertheless, these sentences gain a meaningful truth value inside \( M \), where they appear to be standard, via the Tarski recursion as carried out inside \( M \). In this way, the ZFC-standard models of arithmetic can be equipped with a notion of truth, which obeys the recursive requirements of the Tarskian definition.

These relativized truth predicates are instances of the more general concept of a satisfaction class for a model of arithmetic (see [KS06] and [Kay91] for general background). Given a model of arithmetic \( \mathcal{N} = \langle \mathbb{N}, +, \cdot, 0, 1, \langle \rangle \rangle \), we say that a subclass \( \text{Tr} \subseteq \mathbb{N} \) is a truth predicate or a full satisfaction class for \( \mathcal{N} \)—we shall use the terms interchangeably—if every element of \( \text{Tr} \) is a sentence in the language of arithmetic, viewed by \( \mathcal{N} \), and such that \( \text{Tr} \) obeys the recursive Tarskian definition of truth:

1. (atomic) For each atomic sentence \( \sigma \) in \( \mathbb{N} \), we have \( \sigma \in \text{Tr} \) just in case \( \mathbb{N} \) thinks \( \sigma \) is true. (Note that the value of any closed term may be uniquely evaluated inside \( \mathcal{N} \) by an internal recursion, and so every model of arithmetic has a definable relation for determining the truth of atomic assertions.)
2. (conjunction) \( \sigma \land \tau \in \text{Tr} \) if and only if \( \sigma \in \text{Tr} \) and \( \tau \in \text{Tr} \).
3. (negation) \( \lnot \sigma \in \text{Tr} \) if and only if \( \sigma \notin \text{Tr} \).
4. (quantifiers) \( \exists x \varphi(x) \in \text{Tr} \) if and only if there is some \( n \in \mathbb{N} \) such that \( \varphi(\bar{n}) \in \text{Tr} \), where \( \bar{n} \) is the corresponding term \( 1 + \cdots + 1 \) as constructed in \( \mathcal{N} \).

Note that \( \text{Tr} \) is applied only to assertions in the language of arithmetic, not to assertions in the expanded language using the truth predicate itself (but we look at iterated truth predicates in section 3). A truth predicate \( \text{Tr} \) for a model \( \mathcal{N} = \langle \mathbb{N}, +, \cdot, 0, 1, \langle \rangle \rangle \) is inductive, if the expanded structure \( \langle \mathbb{N}, +, \cdot, 0, 1, \langle, \text{Tr} \rangle \rangle \) satisfies \( \text{PA(Tr)} \), the theory of \( \text{PA} \) in the language augmented with a predicate symbol for \( \text{Tr} \), so that mentions of \( \text{Tr} \) may appear in instances of the induction axiom. The theory of true arithmetic \( \text{TA}^M \) arising in any model of set theory \( M \) is easily seen to be an inductive truth predicate for the corresponding ZFC-standard model of arithmetic \( \mathbb{N}^M \) arising in that model of set theory, simply because ZFC proves that \( \mathbb{N} \) satisfies the second-order Peano axioms.

A principal case of Tarski’s theorem on the non-definability of truth is the fact that no model of arithmetic \( \mathcal{N} \) can have a truth predicate
that is definable in the language of arithmetic. This is simply because for every arithmetic formula \( \varphi(x) \) there is by the G"odel fixed-point lemma a sentence \( \sigma \) such that \( \text{PA} \vdash \sigma \iff \neg \varphi(\sigma) \), and so we would have either that \( \sigma \) is true in \( \mathcal{N} \) while \( \varphi(\sigma) \) fails, or that \( \sigma \) is false in \( \mathcal{N} \) while \( \varphi(\sigma) \) holds; either of these possibilities would mean that the collection of sentences satisfying \( \varphi \) could not satisfy the recursive Tarskian truth requirements, applied up to the logical complexity of \( \sigma \), which is a standard finite sentence.

Krajewski observed that a model of arithmetic can have different incompatible truth predicates, a fact we find illuminating for the context of this paper, and so we presently give an account of it. The argument is pleasantly classical, relying principally only on Beth’s implicit definability theorem and Tarski’s theorem on the non-definability of truth.

**Theorem 2** ([Kra74, Kra76]). There are models of arithmetic with different incompatible inductive truth predicates. Indeed, every model of arithmetic \( \mathcal{N}_0 \models \text{PA} \) that admits an (inductive) truth predicate has an elementary extension \( \mathcal{N} \) that admits several incompatible (inductive) truth predicates.

**Proof.** Let \( \mathcal{N}_0 = \langle \mathbb{N}_0, +, \cdot, 0, 1, \prec \rangle \) be any model of arithmetic that admits a truth predicate (for example, the standard model \( \mathbb{N}^M \) arising in any model of set theory \( M \)). Let \( T \) be the theory consisting of the elementary diagram \( \Delta(\mathcal{N}_0) \) of this model, in the language of arithmetic with constants for every element of \( \mathcal{N}_0 \), together with the assertion “\( \text{Tr} \) is a truth predicate,” which is expressible as a single assertion about \( \text{Tr} \), namely, the assertion that it satisfies the recursive Tarskian truth requirements. The theory \( T \) is consistent, because by assumption, \( \mathcal{N}_0 \) itself admits a truth predicate. Furthermore, any model of the theory \( T \) provides an elementary extension \( \mathcal{N} \) of \( \mathcal{N}_0 \), when reduced to the language of arithmetic, together with a truth predicate for \( \mathcal{N} \). Suppose toward contradiction that every elementary extension of \( \mathcal{N}_0 \) that admits a truth predicate has a unique such class. It follows that any two models of \( T \) with the same reduction to the language of the diagram of \( \mathcal{N}_0 \) must have the same interpretation for the predicate \( \text{Tr} \). Thus, the predicate \( \text{Tr} \) is implicitly definable in \( T \), in the sense of the Beth implicit definability theorem (see [CK90, thm 2.2.22]), and so by that theorem, the predicate \( \text{Tr} \) is explicitly definable in any model of \( T \) by a formula in the base language, the language of arithmetic. But this violates Tarski’s theorem on the non-definability of truth, which implies that no model of arithmetic can have a definable truth predicate. Thus, there must be a models of \( T \) with identical reductions \( \mathcal{N} \) to the
language of the diagram of $\mathcal{N}_0$, but different truth predicates on $\mathcal{N}$. In other words, $\mathcal{N}$ is an elementary extension of $\mathcal{N}_0$ having at least two different incompatible truth predicates. (And it is not difficult by similar reasoning to see that there must be an $\mathcal{N}$ having infinitely many distinct truth predicates.)

In the case where $\mathcal{N}_0$ admits an inductive full satisfaction class, we simply add $\text{PA}(\text{Tr})$ to the theory $\mathcal{T}$, with the result by the same reasoning that the elementary extension $\mathcal{N}$ will also have multiple inductive full satisfaction classes, as desired.

\[ \square \]

In fact, Krajewski [Kra76] proves that we may find elementary extensions $\mathcal{N}$ having at least any desired cardinal $\kappa$ many such full satisfaction classes.

We observe next the circumstances under which these various satisfaction classes can become the true theory of arithmetic inside a model of set theory. First, let’s note the circumstances under which a model of arithmetic is a ZFC-standard model of arithmetic. Let $\text{Th}(\mathbb{N})^{\text{ZFC}}$ be the set of sentences $\sigma$ in the language of arithmetic, such that $\text{ZFC} \vdash (\langle \mathbb{N}, +, \cdot, 0, 1, \langle \rangle \rangle \models \sigma)$. These are the arithmetic consequences of ZFC, the sentences that hold in every ZFC-standard model of arithmetic. If ZFC is consistent, then so is this theory, since it holds in the standard model of arithmetic of any model of ZFC. More generally, for any set theory $\mathcal{T}$ proving the existence of the standard model $\mathbb{N}$ and its theory $\text{Th}(\mathbb{N})$, we have the theory $\text{Th}(\mathbb{N})^\mathcal{T}$, consisting of the sentences $\sigma$ in the language of arithmetic that $\mathcal{T}$ proves to hold in the standard model.

Thanks to Roman Kossak (recently) and Ali Enayat (from some time ago) for discussions concerning the following proposition. The result appears in [Ena09] with further related analysis of the ZFC-standard models of arithmetic.

**Proposition 3.** The following are equivalent for any countable nonstandard model of arithmetic $\mathcal{N}$.

1. $\mathcal{N}$ is a ZFC-standard model of arithmetic. That is, $\mathcal{N} = \mathbb{N}^M$ for some $M \models \text{ZFC}$.

2. $\mathcal{N}$ is a computably saturated model of $\text{Th}(\mathbb{N})^{\text{ZFC}}$.

**Proof.** (1 $\rightarrow$ 2) Suppose that $\mathcal{N}$ is a countable nonstandard ZFC-standard model of arithmetic, arising as $\mathcal{N} = \mathbb{N}^M$ for some countable $M \models \text{ZFC}$. Clearly, $\mathcal{N}$ satisfies the theory $\text{Th}(\mathbb{N})^{\text{ZFC}}$. Note also that $\mathcal{N}$ admits an inductive satisfaction class, namely, the collection of arithmetic truths $\text{TA}^M$ as they are defined inside $M$. Any model of $\text{PA}$ with an inductive satisfaction class, we claim, is computably saturated. To see this, suppose that $p(x, \vec{n})$ is a computable type in the language of the
arithmetic with parameters $\vec{n} \in \mathcal{N}$ and that $p$ is finitely realized in $\mathcal{N}$. Since the type $p$ is computable, the model $\mathcal{N}$ computes its own version of the type $p^\mathcal{N}$, and this will agree with $p$ on all the standard-length formulas. Furthermore, the structure $\langle \mathcal{N}, \text{TA}^M \rangle$ can see that all the standard-finite initial segments of $p^\mathcal{N}$ are satisfiable according to the truth predicate $\text{TA}^M$, which by induction in the meta-theory agrees with actual truth on the standard-finite length formulas. Thus, by overspill (since we have induction for this predicate), it follows that some nonstandard length initial segment of $p^\mathcal{N}$ is satisfied in $\mathcal{N}$, and in particular $p$ itself is satisfied in $\mathcal{N}$. So $\mathcal{N}$ is a countable computably saturated model of $\text{Th}(\mathcal{N})^{\text{ZFC}}$, as desired.

(2 $\rightarrow$ 1) Conversely, suppose that $\mathcal{N}$ is a countable computably saturated model of $\text{Th}(\mathcal{N})^{\text{ZFC}}$. Consider the theory $T$ consisting of the ZFC axioms together with the assertions $\sigma^\mathcal{N}$ for every sentence $\sigma \in \text{Th}(\mathcal{N})$, which is consistent precisely because $\mathcal{N} \models \text{Th}(\mathcal{N})^{\text{ZFC}}$. By computable saturation, it follows that $T$ is coded in $\mathcal{N}$, since we may write down a computable type for this, and so there is a nonstandard finite theory $t \in \mathcal{N}$ whose standard part is exactly $T$. Since $\mathcal{N}$ agrees that any particular finite subtheory of $T$ is consistent, as each such instance is provable in $\text{Th}(\mathcal{N})^{\text{ZFC}}$, we may assume by cutting down to a nonstandard initial segment that $t$ is consistent in $\mathcal{N}$. Inside $\mathcal{N}$, build the canonical complete consistent Henkin theory $H$ extending $t$, and let $M \models H$ be the corresponding Henkin model. In particular, $M \models \text{ZFC}$, since this is a part of $t$, and so $\mathbb{N}^M$ is a ZFC-standard model of arithmetic and hence computably saturated. Note also that $\mathbb{N}^M$ has the same theory as $\mathcal{N}$, because this is part of $t$ and hence $H$. The structure $\mathcal{N}$ can construct an isomorphism from itself with an initial segment of $\mathbb{N}^M$, because for every $a \in \mathcal{N}$ it has a Henkin constant $\hat{a}$ witnessing $\hat{a} = 1 + \cdots + 1$ and it must be part of the theory that any $x < \hat{a}$ is some $\hat{b}$ for some $b < a$ in $\mathcal{N}$, since otherwise $\mathcal{N}$ would think $H$ is inconsistent. It follows that $\mathcal{N}$ and $\mathbb{N}^M$ have the same standard system. But any two countable computably saturated models of arithmetic with the same standard system are isomorphic, by the usual back-and-forth argument, and so $\mathcal{N} \cong \mathbb{N}^M$, showing that $\mathcal{N}$ is ZFC-standard, as desired. □

Thus, the countable nonstandard ZFC-standard models of arithmetic are precisely the countable computably saturated models of $\text{Th}(\mathbb{N})^{\text{ZFC}}$. One may similarly show that every uncountable model of $\text{Th}(\mathbb{N})^{\text{ZFC}}$ has an elementary extension to a ZFC-standard model of arithmetic, since the theory $\text{ZFC} + \{ \sigma^\mathcal{N} \mid \sigma \in \Delta(\mathcal{N}) \}$ is finitely consistent, where $\Delta(\mathcal{N})$
refers to the elementary diagram of $\mathcal{N}$ in the language with constants for every element of $\mathcal{N}$, and any model $M$ of this theory will have $\mathbb{N}^M$ as an elementary extension of $\mathcal{N}$, as desired. The equivalence stated in proposition 3 does not generalize to uncountable models, for there are uncountable computably saturated models of $\text{Th}(\mathbb{N})^\text{ZFC}$ that are $\omega_1$-like and rather classless, which means in particular that they admit no inductive truth predicates and therefore are not ZFC-standard.

We shall now extend proposition 3 to the case where the model carries a truth predicate. Let $\text{Th}(\mathbb{N}, \text{TA})^\text{ZFC}$ be the theory consisting of all sentences $\sigma$ for which ZFC $\vdash (\langle \mathbb{N}, +, \cdot, 0, 1, <, \text{TA} \rangle \models \sigma)$, where $\sigma$ is in the language of arithmetic augmented with a truth predicate and TA refers to the ZFC-definable set of true arithmetic assertions. If ZFC is consistent, then so is $\text{Th}(\mathbb{N}, \text{TA})^\text{ZFC}$, since it holds in the standard model of arithmetic, with the standard interpretation of TA, arising inside any model of ZFC. Note that $\text{Th}(\mathbb{N}, \text{TA})^\text{ZFC}$ includes the assertion that TA is an inductive truth predicate.

**Proposition 4.** The following are equivalent for any countable non-standard model of arithmetic $\mathcal{N}$ with a truth predicate $\text{Tr}$.

1. $\langle \mathcal{N}, \text{Tr} \rangle$ is a ZFC-standard model of arithmetic and arithmetic truth. That is, $\mathcal{N} = \mathbb{N}^M = \langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^M$ for some $M \models \text{ZFC}$ in which $\text{Tr} = \text{TA}^M$ is the theory of true arithmetic.
2. $\langle \mathcal{N}, \text{Tr} \rangle$ is a computably saturated model of $\text{Th}(\mathbb{N}, \text{TA})^\text{ZFC}$.

**Proof.** The proof of proposition 3 adapts to accommodate the expanded structure. If a countable nonstandard model $\langle \mathcal{N}, \text{Tr} \rangle$ arises as $\langle \mathbb{N}, \text{TA} \rangle^M$ for some $M \models \text{ZFC}$, then it admits an inductive truth predicate in the expanded language, and this implies that it is computably saturated just as above. Conversely, any countably computably saturated model of $\text{Th}(\mathbb{N}, \text{TA})^\text{ZFC}$ can build a model of the corresponding Henkin theory extending $\text{ZFC} + \{ \sigma^{(\mathbb{N}, \text{TA})} \mid \langle \mathcal{N}, \text{Tr} \rangle \models \sigma \}$. The corresponding Henkin model $M$ will have $\langle \mathcal{N}, \text{Tr} \rangle$ as an initial segment of $\langle \mathbb{N}^M, \text{TA} \rangle$, and so these two models have the same standard system, and since they also are elementarily equivalent and computably saturated, they are isomorphic by the back-and-forth construction. So $\langle \mathcal{N}, \text{Tr} \rangle$ is ZFC-standard. □

Let us now finally prove theorem 1. We shall give two proofs, one as a corollary to proposition 4 and another simpler direct proof.

**Proof of theorem 1.** (as corollary to proposition 4) By Beth’s theorem as in the proof of theorem 2, we may find two different truth predicates on the same model of arithmetic, with both $\langle \mathcal{N}, \text{Tr}_1 \rangle$ and $\langle \mathcal{N}, \text{Tr}_2 \rangle$ being computably saturated models of $\text{Th}(\mathbb{N}, \text{TA})^\text{ZFC}$. It follows by
proposition 4 that they arise as the standard model inside two different models of set theory, with \( \mathcal{N} = \mathbb{N}^{M_1} = \mathbb{N}^{M_2} \) and \( \text{Tr}_1 = \text{TA}^{M_1} \) and \( \text{Tr}_2 = \text{TA}^{M_2} \), establishing theorem 1. \( \square \)

We also give a simpler direct proof, as follows. (Thanks to W. Hugh Woodin for pointing out a simplification.)

Proof of theorem 1. (direct argument) Suppose that \( M_1 \) is any countable \( \omega \)-nonstandard model of set theory. It follows that \( M_1 \)'s version of the standard model of arithmetic \( \langle \mathbb{N}, +, \cdot, 0, 1, <, \text{TA} \rangle^{M_1} \), augmented with what \( M_1 \) thinks is the true theory of arithmetic, is countable and computably saturated. Since \( \text{TA}^{M_1} \) is not definable in the reduced structure \( \mathbb{N}^{M_1} = \langle \mathbb{N}, +, \cdot, 0, 1, <, \text{TA} \rangle^{M_1} \), it follows by saturation that there is an arithmetic sentence \( \sigma \in \text{TA}^{M_1} \) realizing the same 1-type in \( \mathbb{N}^{M_1} \) as another sentence \( \tau \) in \( \mathbb{N}^{M_1} \) with \( \tau \not\in \text{TA}^{M_1} \). To see this, consider the type \( p(s, t) \) containing all formulas \( \varphi(s) \leftrightarrow \varphi(t) \) for \( \varphi \) in the metatheory, plus the assertions \( s \in \text{TA} \) and \( t \not\in \text{TA} \). This is a computable type, and it is finitely realized in \( \langle \mathbb{N}, +, \cdot, 0, 1, <, \text{TA} \rangle^{M_1} \), precisely because otherwise we would be able to define \( \text{TA}^{M_1} \) in \( \mathbb{N}^{M_1} \). Thus, there are arithmetic sentences \( \sigma \) and \( \tau \) in \( M_1 \), whose Gödel codes realize the same 1-type in \( \mathbb{N}^{M_1} \), but such that \( M_1 \) thinks that \( \sigma \) is true and \( \tau \) is false. Since these objects have the same 1-type in \( \mathbb{N}^{M_1} \), it follows by the back-and-forth construction that there is an automorphism \( \pi : \mathbb{N}^{M_1} \rightarrow \mathbb{N}^{M_1} \) with \( \pi(\tau) = \sigma \). Let \( M_2 \) be a copy of \( M_1 \), witnessed by an isomorphism \( \pi^* : M_1 \rightarrow M_2 \) that extends \( \pi \), so that an element \( m \in \mathbb{N}^{M_1} \) sits inside \( M_1 \) the same way that \( \pi(m) \) sits inside \( M_2 \). Since \( \pi \) was an automorphism of \( \mathbb{N}^{M_1} \), the situation is therefore that \( M_1 \) and \( M_2 \) have exactly the same natural numbers \( \mathbb{N}^{M_1} = \mathbb{N}^{M_2} \). Yet, \( M_1 \) thinks that the arithmetic sentence \( \sigma \) is true in the natural numbers, while \( M_2 \) thinks \( \sigma \) is false, because \( M_1 \) thinks \( \tau \) is false, and so \( M_2 \) thinks \( \pi(\tau) \) is false, but \( \pi(\tau) = \sigma \). \( \square \)

3. Satisfacton is not absolute

In this section we aim to show that the non-absoluteness phenomenon is pervasive. For any sufficiently rich structure \( \mathcal{N} \) in any countable model of set theory \( M \), there are elementary extensions \( M_1 \) and \( M_2 \), which have the structure \( \mathcal{N}^{M_1} = \mathcal{N}^{M_2} \) in common, yet disagree about the satisfaction relation for this structure, in that \( M_1 \) thinks \( \mathcal{N} \models \sigma[\vec{a}] \) for some formula \( \sigma \) and parameters \( \vec{a} \), while \( M_2 \) thinks that \( \mathcal{N} \models \neg\sigma[\vec{a}] \) (see corollary 6). Even more generally, theorem 5 shows that for any non-definable class \( S \) in any structure \( \mathcal{N} \) in any countable model of set theory \( M \), there are elementary extensions \( M_1 \) and \( M_2 \) of \( M \),
which have the structure $N^{M_1} = N^{M_2}$ in common, yet disagree on
the interpretation of the class $S^{M_1} \neq S^{M_2}$. We provide several curious
instances as corollary applications.

**Theorem 5.** Suppose that $M$ is a countable model of set theory, $N$
is structure in $M$ in a finite language and $S \subseteq N$ is an additional
predicate, $S \in M$, that is not definable in $N$ with parameters. Then
there are elementary extensions $M \prec M_1$ and $M \prec M_2$, which agree
on the structure $N^{M_1} = N^{M_2}$, having all the same elements of it, the
same language and the same interpretations for the functions, relations
and constants in this language, yet disagree on the extension of the
additional predicate $S^{M_1} \neq S^{M_2}$, even though by elementarity these
predicates have all the properties in $M_1$ and $M_2$, respectively, that $S$
has in $M$.

**Proof.** Fix such a structure $N$ and non-definable class $S$ inside a count-
able model of set theory $M$. Fix a computable list of constant symbols
for the elements of $M$. Let $M_1$ be any countable computably saturated
elementary extension of $M$, in the language with constants for every
element of $M$. In particular, the structure $\langle N, S, m \rangle^{M_1}_{m \in N}$ is countable
and computably saturated (and this is all we actually require). Since $S$
is not definable from parameters, there are objects $s$ and $t$ in $N^{M_1}$ with
the same 1-type in $\langle N, m \rangle^{M_1}_{m \in N}$, yet $s \in S^{M_1}$ and $t \notin S^{M_1}$. Since this
latter structure is countable and computably saturated, it follows that
there is an automorphism $\pi : N \to N$ with $\pi(t) = s$ and $\pi(m) = m$ for
every $m \in N^M$. That is, $\pi$ is an automorphism of $N^{M_1}$ mapping $t$
to $s$, and respecting the copy of $N^M$ inside $N^{M_1}$. Let $M_2$ be a copy of $M_1$
containing $M$, witnessed by an isomorphism $\pi^* : M_1 \to M_2$ extending $\pi$
and fixing the elements of $M$. It follows that $N^{M_1} = N^{M_2}$ and $M_1$
thinks $s \in S$, but since $M_1$ thinks $t \notin S$, it follows that $M_2$ thinks
$\pi(t) \notin S$ and so $M_2$ thinks $s \notin S$, as desired. □

**Corollary 6.** Suppose that $M$ is a countable model of set theory, and
that $N$ is a sufficiently robust structure in $M$, in a finite language.
Then there are elementary extensions $M \prec M_1$ and $M \prec M_2$, which
agree on the natural numbers $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$ and on the structure $N^{M_1} = N^{M_2}$, having all the same elements of it, the same language and the
same interpretations for the functions, relations and constants in this
language, yet they disagree on what they each think is the standard
satisfaction relation $\mathcal{N} \models \sigma[\vec{a}]$ for this structure.

\[ M \prec M_1, M_2 \models \text{ZFC} \]
\[ \mathcal{N}^{M_1} = \mathcal{N}^{M_2}, \quad \mathcal{N}^{M_1} = \mathcal{N}^{M_2} \]

there are $\sigma$ and $\vec{a}$ for which

- $M_1$ believes $\mathcal{N} \models \sigma[\vec{a}]$
- $M_2$ believes $\mathcal{N} \models \neg \sigma[\vec{a}]$

**Proof.** By ‘sufficiently robust’ we mean that $\mathcal{N} = \langle N, \ldots \rangle$ interprets the standard model of arithmetic $\mathbb{N}$, so that it can handle the Gödel coding of formulas, and also that it is has a definable pairing function, so that it contains (Gödel codes for) all finite tuples of its elements. We are assuming that $\mathcal{N}$ is sufficiently robust from the perspective of the model $M$. It follows by Tarski’s theorem on the non-definability of truth that the satisfaction relation of $M$ for $\mathcal{N}$—that is, the relation $\text{Sat}(\varphi, \vec{a})$, which holds just in case $\mathcal{N} \models \varphi[\vec{a}]$ from the perspective of $M$—is not definable in $\mathcal{N}$. And so the current theorem is a consequence of theorem 5, using that relation. \qed

Many of the various examples of non-absoluteness that we have mentioned in this article can now be seen as instances of corollary 6, as in the following further corollary. In particular, statement (2) of corollary 7 is a strengthening of theorem 1, since we now get the models $M_1$ and $M_2$ as elementary extensions of any given countable model $M$.

**Corollary 7.** Every countable model of set theory $M$ has elementary extensions $M \prec M_1$ and $M \prec M_2$, respectively in each case, which...

1. agree on their natural numbers with successor $\langle \mathbb{N}, S \rangle^{M_1} = \langle \mathbb{N}, S \rangle^{M_2}$, as well as on natural-number addition and order $\langle \mathbb{N}, +, < \rangle^{M_1} = \langle \mathbb{N}, +, < \rangle^{M_2}$, but not on natural-number multiplication, so that $M_1$ thinks $a \cdot b = c$ for some particular natural numbers, but $M_2$ disagrees.

\[ M_1, M_2 \models \text{ZFC} \]
\[ \langle \mathbb{N}, +, < \rangle^{M_1} = \langle \mathbb{N}, +, < \rangle^{M_2} \]
\[ M_1 \text{ believes } \mathbb{N} \models a \cdot b = c \]
\[ M_2 \text{ believes } \mathbb{N} \models a \cdot b \neq c \]
(2) agree on their standard model of arithmetic \(\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^{M_1} = \langle N, +, \cdot, 0, 1, < \rangle^{M_2}\), but which disagree on their theories of arithmetic truth.

(3) agree on their natural numbers \(\mathbb{N}^{M_1} = \mathbb{N}^{M_2}\), their reals \(\mathbb{R}^{M_1} = \mathbb{R}^{M_2}\) and their hereditarily countable sets \(\langle \text{HC}, \in \rangle^{M_1} = \langle \text{HC}, \in \rangle^{M_2}\), but which disagree on their theories of projective truth.

(4) agree on the structure \(\langle H_{\omega_1}, \in \rangle^{M_1} = \langle H_{\omega_1}, \in \rangle^{M_2}\), but which disagree on truth in this structure.

(5) have a transitive rank-initial segment \(\langle V_{\delta}, \in \rangle^{M_1} = \langle V_{\delta}, \in \rangle^{M_2}\) in common, but which disagree on truth in this structure.

Proof. What we mean by “respectively” is that each case may be exhibited separately, using different pairs of extensions \(M_1\) and \(M_2\). Each statement in the theorem is an immediate consequences of corollary 6 or of theorem 5. Thanks to Roman Kossak for pointing out statement (1), which follows from theorem 5 because multiplication is not definable in Presburger arithmetic \(\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle\), as the latter is a decidable theory. For statement (2), we apply corollary 6 to the structure \(\mathcal{N} = \langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^{M}\). For statement (3), one should clarify exactly
what is meant by ‘projective truth,’ since it can be viewed variously as the full second-order theory of the standard model of arithmetic, using $P(N)^M$, or as the theory of the ordered real field $\langle \mathbb{R},\mathbb{Z},+,-,0,1,\rangle$ with a predicate for the integers, or as the theory of the set-theoretic structure $\langle V_{\omega+1},\in \rangle$, or of the structure of hereditarily countable sets $\text{HC} = \langle \text{HC},\in \rangle$. Nevertheless, these models are all interpretable in each other, and one can view projective truth as the satisfaction relation for any of them; and in each case, we make the conclusion of statement (3) by using that structure in corollary 6. Notice that one can similarly view arithmetic truth as residing in the structure of hereditarily finite sets $\langle \text{HF},\in \rangle$, which is mutually interpretable with the standard model of arithmetic via the Ackermann encoding, and in this case, statements (2), (3) and (4) can be seen as a progression, concerning the structures $H_\omega$, $H_{\omega_1}$ and $H_{\omega_2}$, a progression which continues, of course, to higher orders in set theory. Statement (5) is similarly an immediate consequence of corollary 6, using the structure $\mathcal{N} = \langle V_\delta,\in \rangle^M$. □

Let us explore a bit further how indefiniteness arises in the iterated truth-about-truth hierarchy. Beginning with the standard model of arithmetic $\mathbb{N}_0 = \langle \mathbb{N},+,-,0,1,\rangle$, we may define the standard truth predicate $\text{Tr}_0$ for assertions in the language of arithmetic, and consider the structure $\mathbb{N}_1 = \langle \mathbb{N},+,-,0,1,\text{Tr}_0 \rangle$, in the expanded language with a predicate for the truth of arithmetic assertions. In this expanded language, we may make assertions both about arithmetic and about arithmetic truth. Climbing atop this structure, let $\text{Tr}_1 = \text{Th}(\mathbb{N}_1)$ be its theory and form the next level of the iterated truths-about-truth hierarchy $\mathbb{N}_2 = \langle \mathbb{N},+,-,0,1,\text{Tr}_0,\text{Tr}_1 \rangle$ by appending this new truths-about-truth predicate. We may easily continue in this way, building the finite levels of the hierarchy, each new truth predicate telling us about the truth of arithmetic assertions involving only previous levels of the iterated truth hierarchy. There is a rich literature on various aspects of this iterated truth hierarchy, from Tarski [Tar83] to Kripke [Kri75] and many others, including the related development of the revision theory of truth [GB93]. Feferman [Fef91] treats the iteration of truth as an example where once one accepts certain statements about $\langle \mathbb{N},+,-,0,1,\text{Tr}_0,\ldots,\text{Tr}_n \rangle$, then one ought accept certain other statements about $\langle \mathbb{N},+,-,0,1,\text{Tr}_0,\ldots,\text{Tr}_n \rangle$.

**Corollary 8.** For every countable model of set theory $M$ and any natural number $n$, there are elementary extensions $M_1$ and $M_2$ of $M$, which have the same natural numbers $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$, the same iterated arithmetic truth predicates $(\text{Tr}_k)^{M_1} = (\text{Tr}_k)^{M_2}$ for $k < n$ and hence the same
iterated truth structure up to $n$,
$$\langle \mathbb{N}, +, 0, 1, <, \text{Tr}_0, \ldots, \text{Tr}_{n-1} \rangle^{M_1} = \langle \mathbb{N}, +, 0, 1, <, \text{Tr}_0, \ldots, \text{Tr}_{n-1} \rangle^{M_2},$$
but which disagree on the theory of this structure, and hence disagree on the next order of truth, $(\text{Tr}_n)^{M_1} \neq (\text{Tr}_n)^{M_2}$.

**Proof.** In other words, we will have $(\mathbb{N}_n)^{M_1} = (\mathbb{N}_n)^{M_2},$ yet $(\text{Tr}_n)^{M_1} \neq (\text{Tr}_n)^{M_2}$. This is an immediate consequence of corollary 6, using the model $\mathcal{N} = (\mathbb{N}_n)^M$. \hfill \square

In particular, even the cases $n = 0, 1$ or 2 are interesting. The case $n = 0$ amounts to theorem 1 and the case $n = 1$ shows that one can have models of set theory $M_1$ and $M_2$ which have the same standard model of arithmetic $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$ and the same arithmetic truth $\text{Tr}_0^{M_1} = \text{Tr}_0^{M_2},$ yet disagree in their theory of the theory of arithmetic truth $\text{Tr}_1^{M_1} \neq \text{Tr}_1^{M_2}$. Thus, even if one assumes a definite nature for the structure of arithmetic, and also for arithmetic truth, then still there is indefiniteness as to the nature of truths about arithmetic truth, and so on throughout the iterated truth-about-truth hierarchy; indefiniteness arises at any particular level.

The process of iterating the truth hierarchy of course continues transfinitely, for as long as we have some natural way of representing the ordinals inside $\mathbb{N}$, in order to undertake the Gödel coding of formulas in the expanded language and retain the truth predicates as subclasses of $\mathbb{N}$. If $\alpha$ is any computable ordinal, for example, then we have a representation of $\alpha$ inside $\mathbb{N}$ using a computable relation of order type $\alpha$, and we may develop a natural Gödel coding for ordinals up to $\alpha$ and formulas in the language $\{ +, 0, 1, <, \text{Tr}_\xi \}_{\xi < \alpha}$. If $\eta < \alpha$ and $\mathbb{N}_\eta = \langle \mathbb{N}, +, 0, 1, <, \text{Tr}_\xi \rangle_{\xi < \eta}$ is defined up to $\eta$, then we form $\mathbb{N}_{\eta+1}$ by adding the $\eta$th order truth predicate $\text{Tr}_\eta$ for assertions in the language of $\mathbb{N}_\eta$, which can make reference to the simpler truth predicates $\text{Tr}_\xi$ for $\xi < \eta$ using the Gödel coding established by the computability of $\alpha$. The higher levels of this truths-about-truth hierarchy provide truth predicates for assertions about lower-level truths-about-truth for arithmetic. We note that indefiniteness cannot arise at limit ordinal stages, since when $\lambda$ is a limit ordinal, then a sentence $\sigma$ is true at stage $\lambda$ just in case it is true at any stage after which all the truth predicates appearing in $\sigma$ have arisen. In other words, $\text{Tr}_\lambda = \bigcup_{\xi < \lambda} \text{Tr}_\xi$ for any limit ordinal $\lambda$. Nevertheless, we find it likely that there is version of corollary 8 revealing indefiniteness in the transfinite realm of the iterated truth hierarchy.

We conclude this section with some further applications of theorem 5 using non-definable predicates other than a satisfaction predicate.
Corollary 9.

1. Every countable model of set theory $M$ has elementary extensions $M \prec M_1$ and $M \prec M_2$, which agree on their standard model of arithmetic $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$ and have a computable linear order $\triangleleft$ on $\mathbb{N}$ in common, yet $M_1$ thinks $\langle \mathbb{N}, \triangleleft \rangle$ is a well-order and $M_2$ does not.

2. Similarly, every such $M$ has such $M_1$ and $M_2$, which agree on their standard model of arithmetic $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$, yet disagree on Kleene’s $\mathcal{O}$, with $\mathcal{O}^{M_1} \neq \mathcal{O}^{M_2}$.

3. Every countable model of set theory $M$ which thinks $0^\#$ exists has elementary extensions $M_1$ and $M_2$, which agree on the ordinals up to any desired uncountable cardinal $\kappa \in M$, on the constructible universe $L^{M_1}_{\kappa} = L^{M_2}_{\kappa}$ up to $\kappa$ and on the facts that $\kappa$ is an uncountable cardinal and $0^\#$ exists, yet disagree on which ordinals below $\kappa$ are the Silver indiscernibles. Similarly, we may ensure that they disagree on $0^\#$, so that $(0^\#)^{M_1} \neq (0^\#)^{M_2}$.

Proof. These are each consequences of theorem 5. For statement (1), consider the structure $\mathcal{N} = \mathbb{N}^M$, with the predicate $S = \text{WO}^M$, the set of indices of computably enumerable well-orderings on $\mathbb{N}$ in $M$. This is a $\Pi_1^1$-complete set of natural numbers, and hence not first-order definable in the structure $\mathcal{N}$ from the perspective of $M$. Thus, by theorem 5 we get models of set theory $M_1$ and $M_2$, elementarily extending $M$, which agree on $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$, yet disagree on $\text{WO}^{M_1} \neq \text{WO}^{M_2}$. So there is some c.e. relation $\triangleleft$ in common, yet $M_1$ (we may assume) thinks it is a well-order and $M_2$ does not. Since every c.e. order is isomorphic to a computable order, we may furthermore assume that $\triangleleft$ is computable, and the models will compute it with the same program, and both see that it is a linear order, as desired. Statement (2) follows as a consequence, since using the program computing $\triangleleft$, we may construct deviations between $\mathcal{O}^{M_1}$ and $\mathcal{O}^{M_2}$; or alternatively, statement (2) follows immediately from theorem 5 since $\mathcal{O}$ is not definable in $\mathbb{N}$. Statement (3) also follows from theorem 5 since the class of Silver indiscernibles below $\kappa$ is not definable in $\langle L_\kappa, \in \rangle$ in $M$, as from it we could define a truth predicate for that structure. So there must be extensions which agree on $\langle L_\kappa, \in \rangle^{M_1} = \langle L_\kappa, \in \rangle^{M_2}$ and on the fact that $\kappa$ is an uncountable cardinal and that $0^\#$ exists, yet disagree on the Silver indiscernibles below $\kappa$. Similarly, $0^\#$ also is not definable in $\langle L_\kappa, \in \rangle$, we may make them disagree on $0^\#$. □

We expect that the reader will be able to construct many further instances of the phenomenon. Here is another example we found striking.
Theorem 10. Every countable model of set theory $M$ has elementary extensions $M_1$ and $M_2$, which agree on the structure of their standard natural numbers $\langle \mathbb{N},+,\cdot,0,1,<\rangle^{M_1} = \langle \mathbb{N},+,\cdot,0,1,<\rangle^{M_2}$, and which have a set $A \subseteq \mathbb{N}$ in common, extensionally identical in $M_1$ and $M_2$, yet $M_1$ thinks $A$ is first-order definable in $\mathbb{N}$ and $M_2$ thinks it is not.

The proof relies on the following lemma, which was conjectured by the first author and asked on MathOverflow [Ham13b] specifically in connection with this application. The question was answered there by Andrew Marks, whose proof we adapt here.

Lemma 10.1 (Andrew Marks [Ham13b]). There is a subset $B \subseteq \mathbb{N} \times \mathbb{N}$, such that the set $\{ n \in \mathbb{N} \mid B_n \text{ is arithmetic} \}$ is not definable in the structure $\langle \mathbb{N},+,\cdot,0,1,<\rangle$, where $B_n = \{ (n,k) \in B \}$ denotes the $n^{\text{th}}$ section of $B$.

Proof. We identify $\mathbb{N} \times \mathbb{N}$ with $\mathbb{N}$ via Gödel pairing. We use $X'$ and $X^{(n)}$ to denote the Turing jump of $X$ and the $n^{\text{th}}$ Turing jump of $X$, respectively. Recall that a set $X \subseteq \mathbb{N}$ is $n$-generic if for every $\Sigma^0_n$ subset $S \subseteq \mathbb{N}^\omega$, there is an initial segment of $X$ that either is in $S$ or has no extension in $S$. A set is arithmetically generic if it is $n$-generic for every $n$. It is a standard fact that if $X$ is 1-generic, then $X' \equiv_T 0' \oplus X$, and if $X$ is $n$-generic and $Y$ is 1-generic relative to $X \oplus 0^{(n-1)}$, then $X \oplus Y$ is $n$-generic. Let $A = 0^{(\omega)} = \oplus_n 0^{(n)}$, which is Turing equivalent to the set of true sentences of first order arithmetic. We shall construct a set $B$ with the following features:

1. $\{ n \in \mathbb{N} \mid B_n \text{ is arithmetic} \} = A$. More specifically, if $n \in A$, then $B_n$ is $(n+1)$-generic and computable from $0^{(n+1)}$, and if $n \notin A$, then $B_n$ is arithmetically generic.
2. For each natural number $k$, the set $C_k = \oplus_{i \leq k} B_{m_i}$ is $(k+1)$-generic, where $m_i$ is the $i^{\text{th}}$ element of the set $\{ m \in \mathbb{N} \mid m \notin A \text{ or } k \leq m \}$.

Any set $B$ with these features, we claim, fulfills the lemma. To see this, we argue first by argue by induction that $B^{(n)} \equiv_T 0^{(n)} \oplus C_n$ for any natural number $n$. This is immediate for $n = 0$, since $C_0 = \oplus_m B_m$. If $B^{(n)} \equiv_T 0^{(n)} \oplus C_n$, then $B^{(n+1)} \equiv_T (0^{(n)} \oplus C_n) \equiv_T 0^{(n+1)} \oplus C_n$, since $C_n$ is $(n+1)$-generic and hence 1-generic relative to $0^{(n)}$, but $0^{(n+1)} \oplus C_n \equiv_T 0^{(n+1)} \oplus C_{n+1}$ because either $n \notin A$ and so $C_n = C_{n+1}$ or $n \in A$ so $C_{n+1} \equiv_T B_n \oplus C_n$, since $B_n \leq_T 0^{(n+1)}$. So we have established $B^{(n)} \equiv_T 0^{(n)} \oplus C_n$. Since $C_n$ is $(n+1)$-generic, it follows that $0^{(n)} \oplus C_n$ does not compute $0^{(n+1)}$, and so also $B^{(n)}$ does not compute $0^{(n+1)}$. In particular, $B^{(n)}$ does not compute $A$ for any $n$, and so $A$ is not arithmetically definable from $B$, as desired.
It remains to construct the set $B$ with features (1) and (2). We do so in stages, where after stage $n$ we will have completely specified $B_0, B_1, \ldots, B_n$ and finitely much additional information about $B$ on larger coordinates. To begin, let $B_0$ be any set satisfying the requirement of condition (1). We will ensure inductively that after each stage $n$, the set $C_{k,n} = B_{m_0} \oplus \ldots \oplus B_{m_j}$ is $(k+1)$-generic, where $k \leq n$ and $m_0, \ldots, m_j$ are the elements of \{ $m \in \mathbb{N}$ | $m \notin A$ or $k \leq m$ \} $\cap \{0, \ldots, n\}$. At stage $n > 0$, for each of the finitely many pairs $(i,k)$ with $i,k < n$, we let $S_{i,k}$ be the $i$th $\Sigma^0_{k+1}$ subset of $2^{<\omega}$, and if possible, we make a finite extension to our current approximation to $B$ so that the resulting approximation to $C_k$ extends an element of $S_{i,k}$, thereby ensuring this instance of (2). If there is no such extension, then since inductively $C_{k,n-1}$ is $(k+1)$-generic, there is already a finite part of our current approximation to $B$ that cannot be extended to extend an element of $S_{i,k}$, and this also ensures this instance of (2).

We complete stage $n$ by specifying $B_n$. If $n \notin A$, then we simply extend our current approximation to $B$ by ensuring that $B_n$ is arithmetically generic relative to $B_0 \oplus \ldots \oplus B_{n-1}$. This ensures this instance of (1) while maintaining our induction assumption that $C_{k,n}$ is $(k+1)$-generic for each $k < n$, since $C_{k,n-1}$ is $(k+1)$-generic and $B_n$ is $(k+1)$-generic relative to it; and similarly, $C_{n,n}$ is now $(n+1)$-generic. If $n \in A$, then we let $B_n$ be any $0^{(n+1)}$-computable $(n+1)$-generic set extending the finitely many bits of $B_n$ specified in the current approximation. For each $k < n$, let $j_0, \ldots, j_t$ be the elements of $A$ in the interval $[k,n)$. Since our new $B_n$ is $1$-generic relative to $0^{(n)}$, which can compute $B_{j_0} \oplus \ldots \oplus B_{j_t}$, it follows that that $B_{j_0} \oplus \ldots \oplus B_{j_t} \oplus B_n$ is $(k+1)$-generic, and so $C_{k,n}$ is $(k+1)$-generic, as the remaining elements in the finite join defining $C_{k,n}$ are mutually arithmetically generic with this; and since $C_{n,n}$ is $n+1$-generic, we maintain our induction assumption.

This completes the construction. We have fulfilled (1) explicitly by the choice of $B_n$, and we fulfilled (2) by systematically deciding all the required sets $S_{i,k}$.

\[ \square \]

Proof of theorem 10 Fix any countable model of set theory $M$. Apply lemma 10.1 inside $M$, to find a predicate $B \subseteq \mathbb{N} \times \mathbb{N}$ in $M$, such that the set $S = \{ n \in \mathbb{N} \mid B_n$ is arithmetic $\}$ is not definable in the structure $\mathcal{N} = \langle \mathbb{N}, +, \cdot, 0, 1, <, B \rangle$ from the perspective of $M$. It follows by theorem 5 applied to this structure that there are elementary extensions $M_1$ and $M_2$ of $M$, which agree on $\mathcal{N}^{M_1} = \mathcal{N}^{M_2}$ and in particular on $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$ and $B^{M_1} = B^{M_2}$, but not on $S^{M_1} \neq S^{M_2}$. In particular, there is some section $A = B_n$ that is arithmetic in $M_1$ (we may
assume), but not in $M_2$. But since the models agree on the predicate $B$, they agree on all the sections of $B$ and in particular have the set $A$ extensionally in common.

\[\square\]

4. INDEFINITENESS FOR SPECIFIC TYPES OF SENTENCES

Earlier in this article, we proved that the satisfaction relation $\mathcal{N} \models \sigma$ for a first-order structure $\mathcal{N}$ is not generally absolute between the various models of set theory containing that model and able to express this satisfaction relation. But the proofs of non-absoluteness did not generally reveal any specific nature for the sentences on which truth can differ in different models of set theory. We should now like to address this issue by presenting an alternative elementary proof of non-absoluteness, using reflection and compactness, which shows that the theory of a structure can vary on sentences whose specific nature we can identify.

Define that a cardinal $\delta$ is $\Sigma_n$-correct if $V_\delta \prec_{\Sigma_n} V$, and the reflection theorem shows that there is a proper class club $C^{(n)}$ of such cardinals. The cardinal $\delta$ is fully correct, if it is $\Sigma_n$-correct for every $n$. This latter notion is not expressible as a single assertion in the first-order language of ZFC, but one may express it as a scheme of assertions about $\delta$, in a language with a constant for $\delta$. Namely, let “$V_\delta \prec V$” denote the theory asserting of every formula $\varphi$ in the language of set theory, that $\forall x \in V_\delta [\varphi(x) \leftrightarrow \varphi(x)^{V_\delta}]$. Since every finite subtheory of this scheme is proved consistent in ZFC by the reflection theorem, it follows that the theory is finitely consistent and so, by the compactness theorem, ZFC $+$ $V_\delta \prec V$ is equiconsistent with ZFC. The reader may note that since $V_\delta \prec V$ asserts the elementarity separately for each formula, we may not deduce in $V$ that $V_\delta \models \text{ZFC}$, but rather only for each axiom of ZFC separately, that $V_\delta$ satisfies that axiom.

**Theorem 11.** Every countable model of set theory $M \models \text{ZFC}$ has elementary extensions $M_1$ and $M_2$, with a transitive rank-initial segment $\langle V_\delta, \in \rangle^{M_1} = \langle V_\delta, \in \rangle^{M_2}$ in common, such that $M_1$ thinks that the least natural number $n$ for which $V_\delta$ violates $\Sigma_n$-collection is even, but $M_2$ thinks it is odd.
Proof. Suppose that $M$ is a countable model of ZFC, and consider the theory:

$$T_1 = \Delta(M) + V_\delta \prec V + \{ m \in V_\delta \mid m \in M \}$$

$$+ \text{ the least } n \text{ such that } V_\delta \not\models \Sigma_n\text{-collection is even},$$

where $\Delta(M)$ is the elementary diagram of $M$, in the language of set theory having constants for every element of $M$. Note that the first three components of $T_1$ as it is described above are each infinite schemes, whereas the final assertion “the least $n\ldots$” which we take also to assert that there is such an $n$, is expressible as a single sentence about $\delta$ in the language of set theory, using the ZFC-definable satisfaction relation for $\langle V_\delta, \in \rangle$. We claim that this theory is consistent. Consider any finite subtheory $t \subseteq T_1$. We shall find a $\delta$ in $M$ such that $M$ with this $\delta$ will satisfy every assertion in $t$. Let $k$ be a sufficiently large odd number, so that every formula appearing in any part of $t$ has complexity at most $\Sigma_k$, and let $\delta$ be the next $\Sigma_k$-correct cardinal in $M$ above the largest rank of an element of $M$ whose constant appears in $t$. We claim now that $\langle M, \in^M, \delta, m \rangle_{m \in M} \models t$. First, it clearly satisfies all of $\Delta(M)$; and since $\delta$ is $\Sigma_k$-correct in $M$, we have $V^M_\delta \prec_{\Sigma_k} M$, and since also $\delta$ was large enough to be above any of the constants of $M$ appearing in $t$, we attain any instances from the first three components of the theory that are in the finite sub-theory $t$; since $\delta$ is $\Sigma_k$-correct in $M$, it follows that $V^M_\delta$ satisfies every instance of $\Sigma_k$-collection; but since $\delta$ is not a limit of $\Sigma_k$-correct cardinals (since it is the “next” one after a certain ordinal), it follows that $V^M_\delta$ does not satisfy $\Sigma_{k+1}$-collection, and so the least $n$ such that $V^M_\delta \not\models \Sigma_n$-collection is precisely $n = k + 1$, which is even. So $T_1$ is finitely consistent and thus consistent. Similarly, the theory

$$T_2 = \Delta(M) + V_\delta \prec V + \{ m \in V_\delta \mid m \in M \}$$

$$+ \text{ the least } n \text{ such that } V_\delta \not\models \Sigma_n\text{-collection is odd}$$

is also consistent.

As before, let $\langle M_1, M_2 \rangle$ be a computably saturated model pair, such that $M_1 \models T_1$ and $M_2 \models T_2$. It follows that $\langle V^M_\delta, V^{M_2}_\delta \rangle$ is a computably saturated model pair of elementary extensions of $\langle M, \in^M \rangle$, which are therefore elementarily equivalent in the language of set theory with constants for elements of $M$, and hence isomorphic by an isomorphism respecting those constants. So we may assume without loss of generality that $\langle M, \in^M \rangle \prec \langle V_\delta, \in \rangle^{M_1} = \langle V_\delta, \in \rangle^{M_2}$. Meanwhile, $M_1$ thinks that this $V_\delta$ violates $\Sigma_n$-collection first at an even $n$ and $M_2$ thinks it does so first for an odd $n$, since these assertions are part of the theories $T_1$ and $T_2$, respectively. \qed
An alternative version of theorem 11, with essentially the same proof, produces elementary extensions $M_1$ and $M_2$ of $M$ with a rank initial segment $V^M_\delta = V^{M_2}_\delta$ in common, but $M_1$ thinks the least $n$ for which $V_\delta$ is not $\Sigma_n$-correct is even, but $M_2$ thinks it is odd.

By looking not just at the parity of the least $n$ where $\Sigma_n$-collection (or $\Sigma_n$-correctness) fails, but rather, say, at the $k^{th}$ binary digit, we can easily make infinitely many different elementary extensions $M_1, M_2, \ldots$ of $M$, with the natural numbers $\mathbb{N}^{M_k}$ and $V^M_\delta$ all in common, but such that $M_k$ thinks that the least $n$ for which this $V_\delta$ violates $\Sigma_n$-collection is a number with exactly $k$ many prime factors. In particular, even though they have the same structure $\langle V_\delta, \in \rangle^M$, they each think specific incompatible things about the theory of this structure.

5. “Being a model of ZFC” is not absolute

In this section, we prove that the question of whether a given transitive rank initial segment $V_\delta$ of the universe is a model of ZFC is not absolute between models of set theory with that rank initial segment in common. Recall that a cardinal $\delta$ is worldly, if $V_\delta \models \text{ZFC}$.

**Theorem 12.** If $M$ is a countable model of set theory in which the worldly cardinals form a stationary proper class (it would suffice, for example, that $M \models \text{Ord is Mahlo}$), then there are elementary extensions $M_1$ and $M_2$, which have a transitive rank initial segment $\langle V_\delta, \in \rangle^{M_1} = \langle V_\delta, \in \rangle^{M_2}$ in common, such that $M_1$ thinks $V_\delta \models \text{ZFC}$ but $M_2$ thinks $V_\delta \not\models \text{ZFC}$. Moreover, such extensions can be found for which $\delta$ is fully correct in both $M_1$ and $M_2$, and furthermore in which $M \prec V^M_\delta = V^{M_2}_\delta$.

\[
\begin{array}{c}
M_1 \quad M_2 \\
\uparrow \quad \quad \quad \downarrow \\
V_\delta \quad V_\delta
\end{array}
\]

$M_1, M_2 \models \text{ZFC}$

$\langle V_\delta, \in \rangle^{M_1} = \langle V_\delta, \in \rangle^{M_2}$

$M_1$ believes $V_\delta \models \text{ZFC}$

$M_2$ believes $V_\delta \not\models \text{ZFC}$

**Proof.** Fix any countable model $M \models \text{ZFC}$, such that the worldly cardinals form a stationary proper class in $M$. That is, every definable proper class club $C \subseteq \text{Ord}$ in $M$ contains some $\delta$ that is worldly in $M$. Let $T_1$ be the theory consisting of the elementary diagram $\Delta(M)$ plus the scheme of assertions $V_\delta \prec V$, in the language with a new constant symbol for $\delta$, plus the assertions $a \in V_\delta$ for each constant symbol $a$ for an element $a \in M$, plus the assertion “$\delta$ is worldly,” which is to
say, the assertion that $V_\delta \models \text{ZFC}$. Suppose that $t$ is a finite subtheory of $T_1$, which therefore involves only finitely many instances from the $V_\delta \prec V$ scheme. Let $n$ be large enough so that all the formulas $\varphi$ in these instances have complexity at most $\Sigma_n$. Since the $\Sigma_n$-correct cardinals form a closed unbounded class and the worldly cardinals are stationary, there is a $\Sigma_n$-correct worldly cardinal $\delta$ in $M$. It follows that $M$ satisfies all the formulas in $t$ using this $\delta$, and so the theory $T_1$ is finitely consistent and hence consistent.

Let $T_2$ be the theory consisting of the elementary diagram $\Delta(M)$, the scheme $V_\delta \prec V$ plus the assertion “$\delta$ is not worldly.” This theory also is finitely consistent, since if $t \subseteq T_2$ is finite, then let $n$ be beyond the complexity of any formula appearing as an instance of $V_\delta \prec V$ in $t$, and let $\delta$ be a $\Sigma_n$ correct cardinal in $M$ that is not worldly (for example, we could let $\delta$ be the next $\Sigma_n$-correct cardinal after some ordinal; this can never be worldly since $V_\delta$ will not satisfy $\Sigma_n$-reflection). It follows that $M$ with this $\delta$ satisfies every assertion in $t$, showing that $T_2$ is finitely consistent and hence consistent.

Let $\langle M_1, M_2 \rangle$ be a computably saturated model pair, where $M_1 \models T_1$ and $M_2 \models T_2$. It follows that $\langle V_\delta^{M_1}, V_\delta^{M_2} \rangle$ is also a computably saturated model pair of models of set theory, and these both satisfy the elementary diagram of $M$. Consequently, they are isomorphic by an isomorphism that respects the interpretation of $M$ in them, and so by replacing with an isomorphic copy, we may assume that $\langle V_\delta, \in \rangle^{M_1} = \langle V_\delta, \in \rangle^{M_2}$. The theories $T_1$ and $T_2$ ensure that $V_\delta \prec V$ in both $M_1$ and $M_2$, that $M \prec V_\delta$, and furthermore, that $M_1 \models \delta$ is worldly and $M_2 \models \delta$ is not worldly, or in other words, $M_1 \models (V_\delta \models \text{ZFC})$, but $M_2 \models (V_\delta \not\models \text{ZFC})$, as desired. \[ \square \]

The hypothesis that the worldly cardinals form a stationary proper class is a consequence of the (strictly stronger) Lévy scheme, also known as $\text{Ord is Mahlo}$, asserting in effect that the inaccessible cardinals form a stationary proper class. This is in turn strictly weaker in consistency strength than the existence of a single Mahlo cardinal, since if $\kappa$ is Mahlo, then $V_\kappa \models \text{Ord is Mahlo}$. So these are all rather weak large cardinal hypotheses, in terms of the large cardinal hierarchy. Meanwhile, the conclusion already explicitly has large cardinal strength,
since $M_1 \models \delta$ is worldly. Furthermore, the “Moreover,...” part of the conclusion makes the hypothesis optimal, since if $M_1 \models \delta$ is worldly and fully correct, then the worldly cardinals of $M_1$ form a stationary proper class in $V_\delta^{M_1}$, as any definable class club there extends to a class club in $M_1$ containing $\delta$.

As an example to illustrate the range of non-absoluteness, consider the case where $M$ is a model of ZFC in which the worldly limits of Woodin cardinals are stationary. It follows from theorem 12 that there are elementary extensions $M_1$ and $M_2$ of $M$ with a common rank initial segment $\langle V_\delta, \in \rangle^{M_1} = \langle V_\delta, \in \rangle^{M_2}$, such that

$$M_1 \models \langle V_\delta \models \text{ZFC + there are a proper class of Woodin cardinals} \rangle,$$

but

$$M_2 \models \langle V_\delta \not\models \text{ZFC + there are a proper class of Woodin cardinals} \rangle.$$  

The proof shows that from the perspective of $M_2$, the common $V_\delta$ does not satisfy $\Sigma_n$-replacement for sufficiently large $n$, even though it satisfies full replacement from the perspective of $M_1$, which has all the same formulas.

We conclude this section with an observation that many have found curious. We learned this result from Brice Halimi [Hal], but the proof is classical, and the result may be folklore. If $\mathcal{M} = \langle M, \in^M \rangle$ is any model of set theory, and $\mathcal{M} \models \langle m, E \rangle$ is a first-order structure, then we may regard $\langle m, E \rangle$ as an actual first-order structure in $V$, with domain $\{ a \in M \mid \mathcal{M} \models a \in m \}$ and relation $a E b \leftrightarrow \mathcal{M} \models a E b$. Thus, we extract the existential content of the structure from $\mathcal{M}$ to form an actual structure in $V$.

**Theorem 13 ([Hal]).** Every model of ZFC has an element that is a model of ZFC. Specifically, if $\langle M, \in^M \rangle \models \text{ZFC}$, then there is an object $\langle m, E \rangle$ in $\mathcal{M}$, which when extracted as an actual structure in $V$, satisfies ZFC.

**Proof.** Consider first the case that $\mathcal{M}$ is $\omega$-nonstandard. By the reflection theorem, any particular finite fragment of ZFC is true in some $\langle V_\delta^M, \in^M \rangle$, and so by overspill there must be some $\delta$ in $\mathcal{M}$ for which $\mathcal{M}$ believes $\langle V_\delta, \in^M \rangle$ satisfies a nonstandard fragment of ZFC, and so in particular it will satisfy the actual ZFC, as desired. Meanwhile, if $\mathcal{M}$ is $\omega$-standard, then it must satisfy Con(ZFC) and so it can build the Henkin model of ZFC. So in any case, there is a model of ZFC inside $\mathcal{M}$. \qed
What surprises some is that the argument succeeds even when $\mathcal{M} \models \neg \text{Con}(\text{ZFC})$, for although $\mathcal{M}$ thinks there is no model of ZFC, nevertheless it has many actual models of ZFC, which it rejects because its $\omega$-nonstandard nature causes it to have a false understanding of what ZFC is. In the $\omega$-nonstandard case of theorem 13, the model $\langle V_\delta, \in \rangle^\mathcal{M}$ that is produced is actually a model of ZFC, although $\mathcal{M}$ thinks it fails to satisfy some nonstandard part of ZFC. Perhaps one way of highlighting the issue is to point out that the theorem asserts that every model $\mathcal{M}$ of ZFC has an element $\langle m, E \rangle$, such that for every axiom $\sigma$ of ZFC, the model $\mathcal{M}$ believes $\langle m, E \rangle \models \sigma$; this is different from $\mathcal{M}$ believing that $\langle m, E \rangle$ satisfies every axiom of ZFC, since when $\mathcal{M}$ is $\omega$-nonstandard it has additional axioms not present in the metatheory.

6. Conclusions

Let us now return to the philosophical issues with which we began this article. The main concern is whether the definiteness of arithmetic truth can be seen as a consequence of the definiteness of the underlying mathematical objects and structure, or to put it simply, whether definiteness-about-truth follows from definiteness-about-objects and definiteness-about-structure. Many mathematicians and philosophers seem to think that it does. We noted in the introduction, for example, that Feferman seems to support this when he writes:

In my view, the conception [of the bare order structure of the natural numbers $\mathbb{N}$, with its least element and the attendant operations of successor and predecessor] is completely clear, and thence all arithmetical statements are definite. [Fef13, p.6–7] (emphasis original)

Donald Martin, as we mentioned, makes a similar point in his article, presented in the same conference series:

The concept of the natural numbers is first-order complete: it determines truth values for all sentences of first-order arithmetic. That is, it implies each first-order sentence or its negation. (p. 3)

What I am suggesting is that the real reason for confidence in first-order completeness is our confidence in the full determinateness of the concept of the natural numbers. [Mar12, p. 13]

And others have made similar remarks. The idea, quite naturally, is that there can be no indeterminateness in our theory of arithmetic
truth, if the mathematical structure in which that truth resides is completely definite and determined. No blurriness or fuzziness will sneak into our judgement of arithmetic truths, if there is none in the underlying mathematical objects and structure. Thus, on this view, if there is any indeterminism or indefiniteness in mathematics, it was not born in the region between definite natural number objects and structure and definite arithmetic truth.

Our main thesis, in contrast, is that the definiteness of the theory of truth for a structure does not follow as a consequence of the definiteness the structure in which that truth resides. Even in the case of arithmetic truth and the standard model of arithmetic $\mathbb{N}$, we claim, it is a philosophical error to deduce that arithmetic truth is definite just on the basis that the natural numbers themselves and the natural number structure $\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle$ is definite. At bottom, our claim is that one does not get definiteness-of-truth for free from definiteness-of-objects and definiteness-of-structure, and that, rather, one must make a separate analysis and justification for this additional, higher-order claim about mathematical ontology.

The main part of our argument for this is a simple appeal to the mathematical results contained in the previous sections of this article, particularly theorems [H] and [F] and their corollaries, which describe situations in which it seems that we may have definiteness about objects and structure, but not about the truths residing in that structure. The point is that these situations, for all we know, could be the actual mathematical ontology of our set theoretic concepts, and thus we would be in error to deduce the definiteness of our theory of truth just from the definiteness of the mathematical structure in which that truth resides. Specifically, theorem [H] shows that there can be models of set theory $M_1$ and $M_2$, which agree completely on the objects, functions and relations of a particular mathematical structure, even on what they view as the standard model of arithmetic $\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle$, yet disagree on the theory of that structure. Imagine that we are actually living inside the world $M_1$, using the set theory provided by $M_1$ as our background set-theoretic ontology. Perhaps we are philosophically committed to the definite nature of the natural numbers $0, 1, 2, \ldots$, as we know and experience them inside $M_1$, finding them to have a clear and distinct existence. And what would it mean for us to assert that arithmetic truth is similarly definite, given our commitment to the definite nature of the natural numbers as we know them in $M_1$? Well, it would mean that in any other set-theoretic world having exactly the same natural numbers as we have in $M_1$, that the determination of arithmetic truth should work out exactly the same as it does for us in $M_1$. But this is a
claim that is refuted by the situation of theorem 1, for indeed there is another set-theoretic universe $M_2$, which has exactly the same natural numbers and natural number structure as we have in $M_1$, yet comes to different conclusions about which particular arithmetic sentences are true there. There is an arithmetic sentence $\sigma$ in $M_1$ and $M_2$ such that $M_1 \models \mathbb{N} \models \sigma$, but $M_2 \models \mathbb{N} \models \neg \sigma$. It seems that, living in $M_1$, we could have a clear and distinct concept of the natural numbers, yet meanwhile, there is another world out there, $M_2$, with the same natural numbers as we have and yet which has different arithmetic truths. Thus, it was a mistake for us to conclude the definiteness of arithmetic truth from our presumed definiteness of arithmetic objects and structure.

How exactly does this indefiniteness about truth arise? In order for it to happen, the two models $M_1$ and $M_2$ must have different second-order arithmetic. For example, neither model will be able to see the arithmetic truth predicates of the other, because both of these predicates obey the Tarskian truth conditions and one may easily prove by induction that any two predicates satisfying the Tarskian conditions must agree universally. So neither model can have access to the arithmetic truth predicate of the other, and in particular, the models cannot agree on the power set of the natural numbers; so they cannot agree on the real numbers. In this way, indefiniteness of truth is a manifestation of indefiniteness in the ontology of second-order objects. This is part of the reason why we hold that a commitment to a definite theory of truth for a structure is a higher-order ontological commitment that goes strictly beyond the commitment to a definite nature for the underlying structure itself. The assertion that there is definite arithmetic truth is a claim about the definiteness of certain second-order objects, asserting the unique existence of a truth predicate fulfilling the Tarski truth conditions. The existence and nature of such an object seems to be in another realm, the higher order realm of subsets of the structure, rather than directly realized inside the structure.

The claim of definiteness for arithmetic truth amounts in a certain sense to a claim that one’s meta-theoretic concept of natural number aligns with the natural number concept in the object theory. One may prove by induction in the meta-theory, after all, that the truth of any standard-finite arithmetic sentence (that is, finite with the respect to the meta-theoretic natural number concept) is invariant with respect to any change in the set-theoretic background having the same natural number objects and structure. This is simply another way to look upon the fact that any two truth predicates on a given model of arithmetic must agree on their judgements for standard-finite formulas. Perhaps
both Feferman and Martin, in their displayed quotations at the begin-
ing of this section, would agree that the perceived clarity of the
natural number concept leads them to have the same concept in both
the meta-theory as in the object theory. And surely if one can be
confident that one’s meta-theoretic natural number concept coincides
with the object-theoretic account, then one should expect definiteness
of arithmetic truth. But what we would desire is an account of how
this identity is supposed to work.

It would be natural to object to our argument using $M_1$ and $M_2$ on
the grounds that these models are $\omega$-nonstandard, and furthermore,
every instance of this kind of non-determinism will be with models
whose natural numbers are non-standard. Such an argument, however,
seems to us to beg the question, since the issue is whether definiteness
of truth follows from definiteness of objects and structure, not whether
it follows from definiteness of objects and structure and the knowledge
that those objects are ”standard” with respect to some other meta-
mathematical conception of standardness. One would want to know,
for example, about whether that conception of standardness is itself
definite, for it is easy enough to establish that the concept of whether
a given model of arithmetic is standard or not is not absolute between
all models of set theory. In any case, to augment the argument for
definite truth by making additional claims about the special nature
of the natural numbers, going beyond their mere definiteness, merely
serves to verify our main point, which is that indeed something more
needs to be said about it.

Another objection might come from a faction of inhabitants living in
the universe $M_1$, empirical realists with respect to mathematics, who
have direct mathematical experience of the mathematical objects of
$M_1$, but who after Kant reject transcendental knowledge of the objects
in $M_2$ as things-in-themselves outside of the ontology of their universe
$M_1$. Thus, they may reject the alternative arithmetic theory of truth
provided by $M_2$ as not providing an actual counterexample to the defi-
niteness of their theory of arithmetic truth, as the set-theoretic concepts
of $M_2$ lie inaccessibly beyond the framework of concepts available to
them in $M_1$. As a result, the nature and features of $M_2$ are pseudo-
problems for the people in $M_1$, and they will reject our argument as
not providing an actual counterexample to their claim of definite truth.
But do we have the burden of arguing with ostriches? The point is
that we have described how it could be that another universe has the
same natural numbers and natural number structure as exists in the
universe $M_1$, but different arithmetic truths, whether or not people
living only in $M_1$ can appreciate it, and of course such an alternative
universe will not be available inside \( M_1 \). A similar situation occurred with Kant’s views on the fundamental nature of geometry, when Kant’s promise was essentially destroyed with the rise of non-Euclidean geometry and developments in modern physics. Similar developments in set theory provide evidence of a pluralist nature for our fundamental set-theoretic conceptions, with the result that it is not so clear how we may determine in which mathematical universe we actually live. We take ourselves to have explained how it is possible that there can be indefiniteness in the theory of a structure that is definite, and so we place the burden on those who assert the definiteness of their theory of arithmetic truth to provide reasons that go beyond the definiteness of the natural number structure itself. For example, perhaps the inhabitants of \( M_1 \) will attempt to argue for the definiteness of their entire set-theoretic universe, arguing that it is the one true universe of all sets. If successful, this would certainly satisfy our objection.

Roman Kossak has objected to our argument by attempting to undercut the support provided for it by theorem 1. Although that theorem provides models of set theory \( M_1 \) and \( M_2 \) with the same natural number structures \( N^{M_1} = N^{M_2} \) and different truths \( M_1 \models N \models \sigma \) and \( M_2 \models N \models \neg \sigma \), Kossak argues that the meaning of the sentence \( \sigma \) has changed when moving from \( M_1 \) to \( M_2 \). Indeed, the proof shows that \( M_1 \) and \( M_2 \) are isomorphic by an isomorphism \( \pi \) mapping some other sentence \( \tau \) to \( \sigma \), and so in \( M_2 \), Kossak argues, the sentence \( \sigma \) carries the same meaning that \( \tau \) carried in \( M_1 \), rather than the meaning carried by \( \sigma \) itself in \( M_1 \). Similarly, in corollary 7 statement (4) and theorem 11 Kossak argues that the meaning of the ordinal \( \delta \) has changed in the move from \( M_1 \) to \( M_2 \).

This is an interesting and forceful objection, which if correct would undercut our main thesis. To reply, we argue that the meaning of any sentence \( \sigma \) is most naturally and best construed as precisely what that sentence itself expresses, and so the meaning of \( \sigma \) in \( M_1 \) is precisely the same as the meaning of \( \sigma \) in \( M_2 \), particularly in the light of the fact that also \( N^{M_1} = N^{M_2} \). In particular, all the fundamental syntactic features of \( \sigma \) are the same in \( M_1 \) as in \( M_2 \), including the quantifier complexity, the scopes of quantifiers and the construction tree witnessing that \( \sigma \) is a well-formed formula; these are all identical in \( M_1 \) as in \( M_2 \). Although we agree that the truth of an arithmetic sentence depends on the existence and nature of a second-order object—the arithmetic truth predicate—nevertheless the meaning of a sentence is more tightly connected with the syntactic nature of that sentence; the meaning is precisely whatever that sentence itself expresses. So it is not the meaning of \( \sigma \) that changed in the move from \( M_1 \) to \( M_2 \), we argue, but the
truth of \( \sigma \) that changed. There is of course an analogy between \( \sigma \) in \( M_2 \) and \( \tau \) in \( M_1 \), an analogy that is made precise and robust by the fact that they are automorphic images of each other. In the end, we find Kossak’s point to be not an objection to the argument, but rather an explanation of how it could come to be that arithmetic truth can vary without the objects and structure of arithmetic changing. That is, we take his observation to be an *explanation* of how the phenomenon can occur, rather than as a *refutation* of it. Similarly, to argue that \( \delta \) refers in \( M_2 \) to something other than \( \delta \) itself would seem to be a very hard task for Kossak. He replies to this by asking whether there is any meaning of \( \delta \) at all other than that given by the semantics of the ambient universe? We believe that there is, since \( \delta \) is a particular ordinal that both \( M_1 \) and \( M_2 \) have in common, and they agree on the cumulative hierarchy \( V_\delta \) up to stage \( \delta \), and one may inquire about the properties of this common object in \( M_1 \) and \( M_2 \). Our point is that because \( M_1 \) and \( M_2 \) agree on \( V_\delta \), they will be forced to have a certain level of agreement of their theories of truth for this structure, but this agreement is not necessarily universal and ultimately \( M_1 \) and \( M_2 \) can disagree on whether certain statements are true in that common structure.

In connection with this objection, Kossak considered whether there is any other method of producing examples where a given model of arithmetic has alternative truth predicates, and specifically, whether two models of set theory can have the same natural numbers, but non-isomorphic theories of arithmetic truth. That is, the question is whether one can have models for which

\[
\langle \mathbb{N}, +, \cdot, 0, 1, \langle \rangle M_1 = \langle \mathbb{N}, +, \cdot, 0, 1, \langle \rangle M_2, \text{ but}
\]

\[
\langle \mathbb{N}, +, \cdot, 0, 1, \langle, TA \rangle M_1 \not\sim \langle \mathbb{N}, +, \cdot, 0, 1, \langle, TA \rangle M_2.
\]

Jim Schmerl observed that indeed this situation occurs, arguing essentially as follows: the theory \( T = TA + \text{Th}(\mathbb{N}, TA)_{\text{ZFC}} \), in the language with a satisfaction predicate \( \text{Tr} \), is computable from \( 0^{(\omega)} \), but the theory of \( \langle \mathbb{N}, +, \cdot, 0, 1, \langle, TA \rangle \) has degree \( 0^{(\omega+\omega)} \), and so \( T \) is not complete in the expanded language. It follows that there are computably saturated models \( \langle \mathcal{N}, \text{Tr}_1 \rangle \) and \( \langle \mathcal{N}, \text{Tr}_2 \rangle \) of \( T \), which have the same reduct to the language of arithmetic, but which are not elementarily equivalent in the full language and hence also not isomorphic. But since these models are computably saturated models of \( \text{Th}(\mathbb{N}, TA)_{\text{ZFC}} \), it follows by proposition \( \Box \) that they are both \( \text{ZFC-standard} \), arising as the standard model of arithmetic and arithmetic truth \( \langle \mathbb{N}, TA \rangle_{M_1} \not\sim \langle \mathbb{N}, TA \rangle_{M_2} \) inside models of \( \text{ZFC} \), as desired. This kind of example therefore seems to address Kossak’s objection on the meaning of \( \sigma \), while still establishing
our main point, for in this case there is no automorphism providing one
with an alternative meaning for the sentence $\sigma$ on which the models
disagree.

Peter Koellner [Koe13] emphasizes that there is a hierarchy of po-
sitions to take on pluralism in mathematics, depending on where one
expects indeterminism to first arise in mathematics, if it does so at all.

But, in fact, many people are non-pluralists with regard to certain branches of mathematics and pluralists
with regard to others. For example, a very popular
view in the foundations of mathematics embraces non-
pluralism for first-order number theory while defending
pluralism for the higher reaches of set theory. Most peo-
ple would, for example, maintain that the Riemann Hypo-
thesis (which is equivalent to a $\Pi^0_1$-statement of arith-
metic) has a determinate truth-value and yet there are
many who maintain that CH (a statement of set theory,
indeed, of third-order arithmetic) does not have a deter-
minate truth-value. Feferman is an example of someone
who holds this position. So instead of two positions—
pluralism versus non-pluralism—we really have a hier-
archy of positions. [Koe13]

Summarizing the spectrum, on one end we have what might be de-
scribed as the hard-core set-theoretic Platonists, such as Isaacson [Isa08]
(and to a more qualified extent, Woodin and Maddy [Mad88a, Mad88b]),
who affirm the existence of the universe of all sets, in which set-theoretic
claims such as CH have a definite truth value that we might come to
know. Steel softens this position by allowing a restricted pluralism
of mutually interpretable theories, such as occurs in the interaction
of large cardinals, inner model theory and the theory of the axiom of
determinacy. Martin [Mar01] considers the case for determinism in
set theory, but declining a full endorsement, argues instead only that
there is at most one concept of set meeting his criteria, leaving ex-
plicitly open the possibility that in fact there is none. In the center
of the spectrum we are considering, Feferman [Fef99] is committed to
the Platonic existence of the natural numbers, while remaining circum-
spect about the definiteness of higher-order set-theoretic objects and
truth. He asserts that “the origin of Dedekind-Peano axioms is a clear
intuitive concept,” while the intuition for set theory “is a far cry from
what leads one to accept the Dedekind-Peano axioms,” and based on
the dichotomy between the ontological status of the structure of natu-
ral numbers and of the structure of higher types, he concludes that “the
Continuum Hypothesis is an inherently vague problem that no new axiom will settle in a convincingly definite way.” Meanwhile, Hamkins \cite{Ham, Ham13a, Ham11, Ham12, GH10} advances a more radical multiverse perspective that embraces pluralism even with respect to arithmetic truth. He criticizes the categoricity arguments for arithmetic definiteness—which are also used to justify higher-order definiteness, as in Isaacson \cite{Isa08}—as unsatisfactorily circular, in that they attempt to establish the definiteness of the natural number concept by appeal to second-order features involving a consideration of arbitrary subsets, which we would seem to have even less reason to think of as definite. More extreme positions, such as ultrafinitism, lie further along.

To conclude, perhaps Feferman and the others will reply to the argument we have given in this article by saying that their conception of the natural numbers \(0, 1, 2, \ldots\) and the natural number structure is so clear and distinct that they can use this overwhelming definiteness to see that arithmetic truth must also be definite. That is, perhaps they reply to our objection by saying that they did not base the definiteness of arithmetic truth solely on the fact that the natural number objects and structure are definite, but rather on something more, on the special nature of the definiteness of natural number objects and structure that they assert is manifest. For example, perhaps they would elaborate by explaining that arithmetic truth must be definite, because any indefiniteness would reveal a proper cut in the natural numbers: consider the natural numbers \(n\) for which \(\Sigma_n\) truth is definite and argue that this contains 0 and is closed under successors. Our response to such a reply, after first getting in the obvious objection about the indefinite nature of definiteness and whether it can be used in a mathematical induction in this way, would be to say that yes, indeed, this kind of further discussion and justification is exactly what we call for. Our main point, after all, is that definiteness-about-truth does not follow for free from definiteness-about-objects and that one must say something more about it.

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(J. D. Hamkins) MATHEMATICS, PHILOSOPHY, COMPUTER SCIENCE, THE GRADUATE CENTER OF THE CITY UNIVERSITY OF NEW YORK, 365 FIFTH AVENUE, NEW YORK, NY 10016 & MATHEMATICS, COLLEGE OF STATEN ISLAND OF CUNY, STATEN ISLAND, NY 10314

E-mail address: jhamkins@gc.cuny.edu

URL: http://jdh.hamkins.org

(R. Yang) SCHOOL OF PHILOSOPHY, FUDAN UNIVERSITY, 220 HANDAN ROAD, SHANGHAI, 200433 CHINA

E-mail address: yangruizhi@fudan.edu.cn