The $Osp(8|4)$ singleton action
from the supermembrane$\ast\dagger$

Gianguido Dall’Agata$^1$, Davide Fabbri$^1$, Christophe Fraser$^2$,
Pietro Fré$^1$, Piet Termonia$^1$ and Mario Trigiante$^3$

$^1$ Dipartimento di Fisica Teorica, Università di Torino, via P. Giuria 1, I-10125 Torino,
Istituto Nazionale di Fisica Nucleare (INFN) - Sezione di Torino, Italy

$^2$ Dipartimento di Fisica Teorica, Università di Torino, via P. Giuria 1, I-10125 Torino and
Dipartimento di Fisica Politecnico di Torino, C.so Duca degli Abruzzi, 24, I-10129 Torino

$^3$ Department of Physics, University of Wales Swansea, Singleton Park,
Swansea SA2 8PP, United Kingdom

Abstract

Our goal is to study the supermembrane on an $AdS_4 \times M_7$ background, where
$M_7$ is a 7–dimensional Einstein manifold with $N$ Killing spinors. This is a direct
way to derive the $Osp(N|4)$ singleton field theory with all the additional properties
inherited from the geometry of the internal manifold. As a first example we consider
the maximally supersymmetric $Osp(8|4)$ singleton corresponding to the choice
$M_7 = S^7$. We find the explicit form of the action of the membrane coupled to this
background geometry and show its invariance under non–linearly realized super–
conformal transformations. To do this we introduce the supergroup generalization
of the solvable Lie algebra parametrization of non–compact coset spaces. We also
derive the action of quantum fluctuations around the classical configuration, show–
ing that this is precisely the singleton action. We find that the singleton is simply
realized as a free field theory living on flat Minkowski space.

$\ast$ Supported in part by EEC under TMR contract ERBFMRX-CT96-0045,
$\dagger$ Supported in part by EEC under TMR contract ERBFMRX-CT96-0012, in which M. Trigiante is
associated to Swansea University
1 Introduction

There has recently been renewed interest in compactified supergravity vacua of the form:

\[ M_D = \text{AdS}_{p+2} \times \mathcal{M}_{D-p-2} \]  

(1.1)

where \( D \) denotes the total dimension of space–time, \( \text{AdS}_{p+2} \equiv \frac{\text{SO}(2,p+1)}{\text{SO}(1,p+1)} \)

(1.2)
denotes an anti de Sitter space in \( p + 2 \) dimensions and \( \mathcal{M}_{D-p-2} \) is some choice of a compact Einstein manifold in the complementary \( D - p - 2 \) dimensions. This interest is due to a conjectured and partly proved holographic correspondence between the quantum dynamics of a conformal field theory (CFT) describing the infrared fixed point behaviour of a gauge theory (GT) in \( p + 1 \) dimensions and the classical tree–level dynamics of the Kaluza–Klein supergravity theory (KK) obtained by compactification and harmonic expansion on \( \mathcal{M}_{D-p-2} \). In this correspondence, originally proposed by Maldacena [1], GT is the effective field theory of a large number of \( D_p \)–branes, while the \( \text{AdS}_{p+2} \) metric describes the near horizon geometry of the corresponding classical brane solutions of \( D \)–dimensional supergravity. This correspondence was further generalised in [1] to the worldvolume conformal field theory of \( N \) coincident \( M \)–branes, which was conjectured to be dual to \( M \)–theory on the anti de Sitter background.

1.1 The algebraic basis of the holographic correspondence

The algebraic basis of Maldacena’s correspondence was an important observation recently stated in [2, 3], recalling the considerations made in [4, 5, 6, 7] on the famous membrane at the end of the world. Namely the anti de Sitter symmetry of the bulk theory is realised as conformal symmetry on the brane which is located at the boundary, hence holography.

To be explicit, consider an \( M_p \)–brane solution (where \( p = 2, 5 \)) of \( D = 11 \) M–theory, namely a metric

\[ ds_{11}^2 = \left( 1 + \frac{k}{r^{d}} \right)^{\frac{\bar{d}}{\bar{d}}} dx^I dx^J \eta_{IJ} + \left( 1 + \frac{k}{r^{d}} \right)^{\frac{\bar{d}}{\bar{d}}} dy^\hat{a} dy^\hat{b} \delta_{\hat{a}\hat{b}}. \]  

(1.3)

where

\[ d \equiv p + 1; \quad \bar{d} \equiv 11 - d - 2 \]  

(1.4)

are the world–volume dimensions of the \( p \)–brane and of its magnetic dual,

\[ r \equiv \sqrt{y^\hat{a} y^\hat{b} \delta_{\hat{a}\hat{b}}} \]  

(1.5)

is the radial distance from the brane in transverse space, \( I, J = 0, \ldots, d - 1 \) and \( \hat{a}, \hat{b} = d, \ldots, 10 \).

It has been known for some years [8, 9] that near the horizon \( (r \to 0) \) the exact metric (1.3) becomes approximated by the metric of the following 11–dimensional space:

\[ M_p^{\text{hor}} = \text{AdS}_{p+2} \times S^{9-p} \]  

(1.6)
that has
\[ T^\text{hor}_p = SO(2, p + 1) \times SO(10 - p) \] (1.7)
as isometry group.

It was observed \[^3\] that the Lie algebra of \( T^\text{hor}_p \) can be identified with the bosonic sector of a superalgebra \( SC_p \) admitting the interpretation of conformal superalgebra on the \( p \)-brane world–volume. The explicit identifications are
\[ T^\text{hor}_2 = SO(2, 3) \times SO(8) , \quad SC_2 = Osp(8|4), \]
\[ T^\text{hor}_5 = SO(2, 6) \times SO(5) , \quad SC_5 = Osp(2, 6|4), \] (1.8)
where \( Osp(8|4) \) is the real section of the complex orthosymplectic algebra \( Osp^c(8|4) \) having \( SO(8) \times Sp(4, \mathbb{R}) \) as bosonic sub-algebra, while \( Osp(2, 6|4) \) is the real section of the same complex superalgebra having \( SO(2, 6) \times (USp(4) \sim SO(5)) \) as bosonic sub-algebra.

In \[^3\] it was shown how to realize the transformations of \( SC_p \) as symmetries of the linearized \( p \)-brane world–volume action. In \[^1\] it was instead suggested that the non–linear Born Infeld effective action of the \( p \)-brane in the (1.6) background, is invariant under conformal like transformations that realize the group \( T^\text{hor}_p \). Since these transformations are similar but not identical to the standard conformal transformations, they have been named broken conformal transformations. Further developments in this direction appeared in \[^{10, 11}\].

These cases correspond to the choice \( M_{11-p-2} = S^{9-p} \) which, in the context of KK, yields the maximal number of preserved supersymmetries \[^4\]. However other choices of \( M_{11-p-2} \) are available. In \[^{12}\] it was shown that there is a one–to–one correspondence between Freund–Rubin compactifications of \( D = 11 \) supergravity \[^{14, 13, 16, 17, 18, 19}\] and \( Mp \)-brane solutions, in the sense that the Freund–Rubin solution on the manifold (1.1) with \( p = 2 \) or \( p = 5 \) is the near–horizon geometry of a suitable \( Mp \)-brane for each choice of \( M_{D-p-2} \). As KK vacuum the Freund–Rubin solution preserves \( N_M \) supersymmetries in \( AdS_{p+2} \) space where, by definition, \( N_M \) is the number of Killing spinors \( \eta_A \), defined as follows:
\[ \left[ D^M_m + e \Gamma_m \right] \eta_A = 0 \quad ; \quad A = 1, \ldots, N_M. \] (1.9)
In (1.9) \( D^M_m \) is the spinorial covariant derivative on \( \mathcal{M} \), \( \Gamma_m \) denotes the Dirac matrices in dimension \( \dim \mathcal{M} \) and \( e \) is related to the \( AdS \) radius. Hence for \( p = 2 \) the number of supercharges preserved near the horizon is \( 4N_G \), while in the bulk it is \( 1/2 \) of that number, namely \( 2N_G \).

\subsection*{1.2 \( G/H \)-branes}

Of particular interest are the \( G/H \) branes introduced in \[^{12}\] and already considered in \[^{20}\]. They correspond to the choice of a homogeneous coset manifold as internal space:
\[ \mathcal{M}_{D-p-2} = \frac{G}{H} \quad ; \quad \dim G - \dim H = D - p - 2 \] (1.10)
and are in one–to–one correspondence with the \( G/H \) Freund–Rubin compactifications of \( D=11 \) supergravity completely classified in \[^{18}\] and thoroughly studied in the eighties \[^{15}\].

\[^1\]In the case of the \( S^7 \)-compactification, the near horizon bulk theory is gauged \( N = 8 \) supergravity \[^{13}\]
The 7-dimensional coset manifolds for the $p=2$ case and the 4-dimensional coset manifolds for the $p=5$ case constitute a finite set and all the $N_{G/H}$ numbers are known (see [12] for a summary).

The case of the round and squashed seven spheres are the best known ($N_{G/H}=8$ and $N_{G/H}=1$) but in the eighties the Kaluza–Klein spectra have been systematically derived also for all the other solutions using the technique of harmonic expansions [22, 23]. The organization of these spectra in supermultiplets is known not only for the round $S^7$ [24] but also for the case of supersymmetric $M_{pqr}$ spaces

$$M^{pqr} \equiv \frac{SU(3) \times SU(2) \times U(1)}{SU(2) \times U(1) \times U(1)}$$

where $p, q, r \in \mathbb{Z}$ define the embedding of the $U(1)^2$ factor of $H$ in $G$. For $p=q=$ odd we have $N_{G/H}=2$, in all the other (non supersymmetric cases) we have $N_{G/H}=0$. The $N=2$ multiplet structure was obtained in [25].

Since much is known about these spaces, $G/H$–branes constitute an excellent laboratory where to make direct checks of the holographic correspondence:

$$\text{CFT on } \partial(AdS_{p+2}) \leftrightarrow \text{KK on } AdS_{p+2} \quad (1.11)$$

### 1.2.1 The qualitative difference between the round $S^7$ case and the lower supersymmetry cosets $G/H$

Let us now stress the qualitative difference between the case with maximal supersymmetry and the cases with lower supersymmetry. Recalling results that were obtained in the early eighties [22, 23], we know that, if the Freund Rubin coset manifold admits $N_{G/H}$ Killing spinors, then the structure of the isometry group $G$ is necessarily factorized in the following way:

$$G = G' \otimes SO\left(N_{G/H}\right)$$

where the $R$–symmetry factor $SO\left(N_{G/H}\right)$ can be combined with the isometry group $SO(2,3)$ of anti de Sitter space to produce the orthosymplectic algebra $Osp\left(N_{G/H}|4\right)$, while the factor $G'$ is the gauge–group of the vector multiplets. Correspondingly the three–dimensional world–volume action of the $CFT$ must have the following superconformal symmetry:

$$\mathcal{SC}_2^{G/H} = Osp\left(N_{G/H}|4\right) \times G' \quad (1.12)$$

where $G'$ is a flavour group. Here comes the essential qualitative difference between the maximal and lower supersymmetry cases. In the maximal case the harmonics on $G/H$ are labeled only by $R$–symmetry representations while in the lower susy case they depend both on $R$ labels and on representations of the gauge/flavour group $G'$. The structure of $Osp(8|4)$ supermultiplets determines completely their $R$–symmetry representation content so that the harmonic analysis becomes superfluous in this case. The eigenvalues of the internal laplacians which determine the Kaluza–Klein masses of the $Osp(8|4)$ graviton multiplets or, in the conformal reinterpretation of the theory, the conformal weights of the corresponding primary operators, are already fixed by supersymmetry and need not be calculated. In this sense the correspondence (1.11) is somewhat trivial in the maximal susy case: once the superconformal algebra $\mathcal{SC}_2$ has been identified with the super-isometry
group $Osp(8|4)$ the correspondence between conformal weights and Kaluza–Klein masses is simply guaranteed by representation theory of the superalgebra. On the other hand in the lower susy case the structure of the $Osp(N_{G/H}|4)$ supermultiplets fixes only their content in $SO\left(N_{G/H}\right)$ representations while the Kaluza–Klein masses, calculated through harmonic analysis depend also on $G'$ labels. In this case the holographic correspondence yields a definite prediction on the conformal weights that, as far as superconformal symmetry is concerned would be arbitrary. Explicit verification of these predictions would provide a much more stringent proof of the holographic correspondence and yield a deeper insight in its inner working. However in order to set up such a direct verification one has to solve a problem that was left open in Kaluza–Klein supergravity: the singleton problem.

1.3 The singleton problem in $G/H$ M2 branes

As it is well known both from [26, 27, 28, 29] and from the study of Kaluza–Klein supergravity in the eighties (for a review see [21]), apart from one exception, all the unitary irreducible representations of the $N$–extended anti de Sitter superalgebra correspond to supermultiplets of ordinary fields characterized by a mass and a spin and living in the bulk of anti de Sitter space. The massless representations are in one–to–one correspondence with the analogue massless multiplets of the $N$–extended Poincaré superalgebra and in addition there is a wealth of shortened massive multiplets that realize BPS saturated states of string theory or M–theory. These ordinary short and long multiplets appear in the Kaluza–Klein expansion of $D = 11$ supergravity around an anti de Sitter background. The exception is the lowest lying unitary irreducible representation of the $AdS$ superalgebra, the singleton, which does not admit a field theory realization in the bulk of $AdS$–space but which is the building block for all the other representations, in the sense that all supermultiplets can be obtained decomposing tensor products of the singleton supermultiplet.

In the eighties, the field theory realization of the singleton was considered by several authors [5, 6, 7, 30]. Following previous results on the non supersymmetric case [27, 29], it was realized that the singleton field theory lives on the boundary of $AdS$ space. It was also realized that the (super) anti de Sitter group acts as the (super) conformal group on its boundary, and thus on the singleton field theory. This fact led to the identification of the singleton theory with the world–volume conformal field theory on a brane placed at the boundary.

Recently, a deeper understanding of the singleton has been promoted by the holographic correspondence (1.11). The singleton field theory of $AdS_{p+2}$ lives in one lower dimension (i.e. $d = p+1$) since it is identified with the microscopic gauge field theory on the brane world–volume. The tensor product realization of the ordinary $AdS$–supermultiplets corresponds to the construction of composite operators in the world–volume theory playing the role of emission vertices for all KK states.

In the case of maximal supersymmetry there is little to discover from the group–theoretical view–point since, as already emphasized, the $Osp(8|4)$ singleton contains uniquely fixed representations of the $SO(8)$ $R$–symmetry group. Instead, in the lower susy case of $G/H$–branes there is a crucial group–theoretical information that needs to be extracted from dynamics. This is the representations of the flavour group $G'$ to which the singleton has to be assigned. Such an information, which is the prerequisite for any verification of the holographic correspondence (1.11), cannot be extracted from Kaluza–Klein
supergravity but can be provided only by the world–volume field theory.

It is for this reason that in the present paper we consider the derivation of the singleton field theory from the supermembrane action that couples consistently to any $D = 11$ supergravity background.

To avoid any confusion, we would like to stress here that we call singleton field theory the flat space limit of the free field theory of $[6, 7]$. We point out that, since we are going to find a theory living on a three–dimensional Minkowski space rather than on $S^2 \times S^1$, we have no scalar mass term which was instead required in $[6, 7]$ for conformal invariance. We will see that it can also be derived as the theory living on the solitonic brane of (1.3). However, we will also present the derivation of the interacting conformal field theory describing the dynamics of a single probe brane in the background of $N$ other coincident branes ($N$ large).

In $[18]$, another definition of the singleton field theory is given as the interacting non-abelian conformal field theory living on the boundary, and it is this theory which is dual to the bulk supergravity. Nonetheless, holography is also fundamental to the derivation of the free singleton field theory (hereafter simply referred to as the singleton field theory), as we will see later.

Furthermore, it is worth noting that though the free singleton field theory and the non-abelian conformal field theory are apparently unrelated, the former does provide information about the latter in the case of the $G/H$ branes, as it tells under which representation of $G'$ (1.12) the singletons transform (independently of whether they are free or interacting).

### 1.3.1 The singleton from the supermembrane

The route we follow is a priori conceptually simple. We consider the supermembrane action invariant with respect to $\kappa$–supersymmetry. It can be written in any background of the elfbein $E^a$, of the gravitino 1–form $\Psi^a$ and of the 3–form $A$ that are solutions of $D = 11$ supergravity. Specializing the background to be

$$AdS_4 \times \left( \frac{G}{H} \right) 7$$

we should obtain an interacting conformal field theory. Indeed after fixing reparametrization invariance which removes 3 of the 11 bosonic coordinates and after gauge fixing $\kappa$ supersymmetry which removes 16 of the 32 degrees of freedom we are left with 8 bosons and 8 fermions (on shell) which is the field content of the singleton field theory. The action must then be expanded around a classical solution, preserving the $AdS$ (i.e. superconformal) symmetry. This is the free field limit, yielding the singleton theory.

What is far from being trivial are the details along the route. There are three main questions one has to address in this programme:

1. The identification of the boundary on which the brane lives.
2. The choice of a suitable parametrization of anti de Sitter superspace.
3. How to expand the non–linear gauge–fixed action around a classical solution to obtain the unitary irreducible singleton representation.
Although our final goal is a description of $G/H$–branes, in the present paper we consider the solutions of the above problems in the case of maximal supersymmetry, namely for the supermembrane on the following background:

$$AdS_4 \times S^7.$$ 

As already emphasized, many aspects of the connection between the $M2$–brane and the singleton have been studied in the past [3, 4, 31]. However, an exact derivation of the singleton action from the supermembrane on this background has revealed to be problematic [3] and missed for a long time. Therefore what we do in this paper must be viewed both as a solution of so far unresolved questions and as a preparation to extract the $G'$ representation content of singleton field theories in the case of $G/H$–branes.

1.3.2 The organization of the paper

In section 2, we give our parametrization of the near–horizon geometry. In section 3 we present the rheonomic construction of the $\kappa$–supersymmetric supermembrane action in first order formalism.

Next, in section 4 we present the construction of the $\kappa$–fixed $M2$–brane action in the $AdS_4 \times S^7$ background comparing our result with the previously found partial results. We also elucidate the non-linear realization of the action of the superconformal $Osp(8|4)$ group on the fields and coordinates of the membrane inherited from the isometries of the background.

In section 5, we expand the fields in small fluctuations normal to the membrane, and thus we find the action of the singleton.

Finally, of the three appendices, A contains our conventions, including those of the normal coordinate expansion, B gives details about the conformal structure and the topology of the boundary, while C contains a detailed explanation of the solvable parametrization of $AdS_4$ as a coset space.

2 The near–horizon geometry

$M$–branes are classical solutions of eleven dimensional supergravity with the (1.3) metric. In particular, the $M2$–brane metric is obtained from (1.3) setting $d = 3, \tilde{d} = 6$:

$$ds^2 = \left(1 + \frac{k}{r^6}\right)^{-2/3} dx^I dx^J \eta_{IJ} + \left(1 + \frac{k}{r^6}\right)^{1/3} dy^a dy^b \delta_{ab},$$

(2.1)

where now $I, J = 0, 1, 2$ and $a, \hat{b} = 3, \ldots, 10$. The membrane is obviously located at $r = 0$.

It is now interesting to study the geometry near the brane. In the $r \to 0$ limit, the metric (2.1) becomes

$$ds^2 = \left(\frac{r}{R}\right)^4 dx^I dx^J \eta_{IJ} + \left(\frac{R}{r}\right)^2 dy^a dy^b \delta_{ab},$$

(2.2)

where $k = R^6$. Setting $\rho = \left(\frac{r}{R}\right)^2$ and using $dy^a dy^b \delta_{ab} = dr^2 + r^2 d\Omega_7^2$, where $d\Omega_7^2$ is the invariant metric on the sphere, the near–horizon metric can be seen explicitly to reduce
to $AdS_4 \times S^7$ in horospherical $\times$ hyperspherical coordinates [3]:

\[
d s^2 = \rho^2 \left( -dt^2 + dx^2 + dw^2 \right) + \frac{R^2}{4} \frac{1}{\rho^2} d\rho^2 + R^2 d\Omega_7^2.
\] (2.3)

It must be noted that the ratio of the $AdS$ radius to the $S^7$ radius is fixed to be $1/2$, and that after a redefinition of $\rho$ absorbing an $R/2$ factor, the metric has a $R^2/4$ factor in front which can be conformally scaled away retrieving the $AdS$ space in the solvable parametrization [12]:

\[
d \tilde{s}^2 = \rho^2 \left( -dt^2 + dx^2 + dw^2 \right) + \frac{1}{\rho^2} d\rho^2 + 4d\Omega_7^2.
\] (2.4)

An interesting point to note is that this approximated metric is in fact also an exact supergravity solution [14] and a stable quantum vacuum [35]. The metric near the brane is locally that of an $AdS$ space. The supergravity solution fixes the local metric, but there is still a certain arbitrariness in the choice of underlying global topology.

Anti de Sitter space $AdS_4$ can be represented as the hyperboloid given by the following algebraic locus in $\mathbb{R}^5$:

\[
Y_0^2 + Y_1^2 - Y_2^2 - Y_3^2 - Y_4^2 = 1.
\] (2.5)

We can partly parametrize this space with our coordinates $\rho, t, w, x$ as follows:

\[
\begin{align*}
Y_0 &= \rho t, \\
Y_1 &= \frac{1}{2} \left[ \rho + \frac{1}{\rho} + \rho \left( -t^2 + w^2 + x^2 \right) \right], \\
Y_2 &= \rho w, \\
Y_3 &= \frac{1}{2} \left[ \rho - \frac{1}{\rho} - \rho \left( -t^2 + w^2 + x^2 \right) \right], \\
Y_4 &= \rho x,
\end{align*}
\] (2.6)

with

\[
\begin{cases}
\rho \in [0, \infty[ \\
t, w, x \in ]-\infty, \infty[
\end{cases}
\]

where the coordinate range is a physical choice. Note that these coordinates do not parametrize the whole of the hyperboloid, but they are good coordinates for $AdS_4/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts as the inversion $Y \to -Y$.

Looking back at the metric (2.3), sections with $\rho$ fixed are locally isomorphic to flat Minkowski $M_3$. Furthermore, there exists an infinite number of classical solutions to the $D = 11$ brane-wave equations with $\rho = const, y^9 = const$ with Minkowski topology [3]. The $M2$-brane of (2.1) is just one of these membranes at $\rho$ fixed which has been pushed to $\rho \to 0$, that is part of the boundary of our space, but we should recover a proper CFT taking the $\rho \to \infty$ limit. Indeed it has to be noticed that this latter provides us with a theory on a conformally invariant support [10]. The first one instead yields a theory formulated at the horizon (which is not an invariant support), even if it realises the bulk invariances on the fields such that what we obtain is a conformally invariant theory.

For more details on its topology and its relation with the conformal boundary of $AdS$ we refer the reader to Appendix B.

\textsuperscript{2}Actually they cover only a part of $AdS_4/\mathbb{Z}_2$: the one for which $Y_1 + Y_3 \neq 0$. 

---
3 The supermembrane and $\kappa$–supersymmetry

It has been known for a long time that eleven dimensional supergravity can naturally be compactified on $AdS_4 \times S^7$, the simplest of the Freund–Rubin compactifications where the four–form field strength has non vanishing vacuum expectation value [14]. It is also known that this Freund–Rubin type solution can be seen as a consistent quantum vacuum of $D = 11$ membrane theory. Recently [35], it has been shown that the classical equations of motion of the effective theory of M–theory, namely $D = 11$ supergravity, evaluated on this background, cannot receive quantum corrections which are compatible with supersymmetry. This means that the $AdS_4 \times S^7$ vacuum is described by a fixed point where all the torsion, curvature and four form components are covariantly constant. $AdS_4 \times S^7$ is an exact solution of M–theory. We thus propose to find an explicit expression for the membrane world-volume theory in this background. This action should display super-conformal symmetry, which it inherits from the $AdS$ symmetry group of the background. This is so because the world-volume action is a generalized $\sigma$–model whose target space fields are the background fields. Then we build a three–dimensional interacting conformal field theory, the interactions describing the membrane dynamics.

To this effect we need to start from the $\kappa$ supersymmetric action of the $D = 11$ supermembrane. Although this latter has been derived long ago [36], we devote the next subsection to such a construction, because in the rheonomic first–order formalism the action of $\kappa$–supersymmetry becomes particularly simple and implementing its restriction to a specific background is very easy and clear. This formalism is equivalent to the geometric approach, from which it has been derived the action of the supermembrane on flat Minkowski space [37]. We point out that this approach has been used also to derive the actions for $Dp$–branes on a generic curved super background [38].

3.1 The first order “Polyakov” action of the supermembrane from rheonomy

The starting point for the formulation of the supermembrane action is the geometry of superspace and the rheonomic parametrization of the supergravity curvatures. These were obtained at the beginning of the eighties in [39]. The field content of $D = 11$ supergravity is given by the following set of exterior forms: the vielbein 1-form $E^a$ ($a = 0, 1, \ldots, 10$), the spin–connection 1-form $\omega^{ab}$, the gravitino fermionic 1–form $\Psi^a$ ($\alpha = 1, \ldots, 32$) and the 3–form $A$. The first three items in the above list constitute the dual description of the superPoincaré algebra in $D=11$. The last item, namely the 3–form $A$ extends it to a free–differential algebra which could be further enlarged by the addition of a 6–form $\tilde{A}$ whose field strength turns out to be the dual of that of $A$ upon implementation of the Bianchi identities [40]. The definition of the $D = 11$ curvatures is

$$R^a \equiv dE^a - \omega^{ab} \wedge E^c \eta_{bc} - \frac{1}{2} \bar{\Psi} \wedge \Pi^{ab} \Psi, \quad \rho \equiv d\Psi - \frac{1}{4} \omega^{ab} \wedge \Pi_{ab} \Psi, \quad (3.1)$$

$$R^{ab} \equiv d\omega^{ab} - \omega^{ac} \wedge \omega^{db} \eta_{cd}, \quad F[A] \equiv dA - \frac{1}{2} \bar{\Psi} \wedge \Pi^{ab} \Psi \wedge E_a \wedge E_b.$$
and the corresponding rheonomic solution of the superspace Bianchi identities is as follows:

\[ R^a = 0, \]
\[ F[A] = F_{a_1...a_4} E^{a_1} \wedge ... \wedge E^{a_4}, \]  \hfill (3.2)
\[ \rho = \rho_{ab} E^a \wedge E^b + \frac{i}{3} \left( \Pi_{abc} F_{abc} - \frac{1}{8} \Pi_{ab...24} F_{ab...24} \right) \Psi \wedge E^a, \]
\[ R^{ab} = as determined by second order formalism. \]

From these parametrizations one immediately obtains the supersymmetry transformations as superspace Lie derivatives \cite{21}:

\[ \delta E^a = i \bar{\epsilon} \Pi^a \Psi, \]
\[ \delta \Psi = D \epsilon - \frac{i}{3} \left( \Pi_{abc} F_{abc} - \frac{1}{8} \Pi_{ab...24} F_{ab...24} \right) \epsilon E^a, \]  \hfill (3.3)
\[ \delta A = -i \bar{\epsilon} \Pi^a \Psi \wedge E_a \wedge E_b \]

The basic idea to obtain the \( \kappa \)-supersymmetric action of the supermembrane is the following. We introduce a dreibein \( e^i \) \((i = 0, 1, 2)\) on the three–dimensional world–volume and we write the following first–order action functional

\[ S = \int \Pi^i E^a \wedge e^i \wedge e^j \epsilon_{ijk} + \alpha_1 \int \Pi^i \Pi^j \Pi^k e^i \wedge e^j \wedge e^k \frac{\epsilon_{ijk}}{3!} + \alpha_2 \int e^i \wedge e^j \wedge e^k \frac{\epsilon_{ijk}}{3!} + \alpha_3 q \int A \]  \hfill (3.4)

where \( q = \pm 1 \) is the “membrane charge”, while \( \alpha_1, \alpha_2, \alpha_3 \) are three real parameters to be determined by the following two conditions:

1. The variational equation in the 0–form \( \Pi^i \) must impose its identification with the projection of the target elfbein \( E^a \) onto the world–volume dreibein \( e^i \), namely:

\[ E^a = e^i \Pi^i. \]  \hfill (3.5)

2. The action should be invariant against \( \kappa \)–supersymmetry transformations. These are nothing else but ordinary supersymmetries of the background fields, as defined by eq. (3.3), with, however, a restricted supersymmetry parameter \( \epsilon \). The restriction corresponds to a world volume projection that halves the 32 components of the spinor parameter. Explicitly this is realized by setting:

\[ \epsilon = \frac{1}{2} \left( 1 - qi \bar{\Pi} \right) \kappa, \]
\[ \bar{\Pi} \equiv \frac{\epsilon_{ijk}}{3! \sqrt{-h}} \Pi_{ijk} = \frac{\epsilon_{ijk}}{3! \sqrt{-h}} \Pi_{ij} e^i \Pi_{jk} e^j \Pi_{ka} e^a, \]  \hfill (3.6)

where \( \kappa \) is a free 32–component spinor while the symbols \( h_{ij} \), \( h \) denote the world–volume metric and its determinant, respectively.

The invariance with respect to \( \kappa \) supersymmetry can be realized if \( \alpha_2, \alpha_3 \) have suitable values and if a suitable \( \kappa \)–variation of the world–volume vielbein is introduced. Before entering further details it is worth discussing how the \( \kappa \)–variation has to be conceived with respect to the world–volume fields. The action (3.4) defines a \( \sigma \)--model and the field configurations are embeddings

\[ \mathcal{WV}_3 \hookrightarrow \mathcal{SP}_{11|32} \]
of the bosonic three–dimensional world–volume $\mathcal{W}V_3$ into the $11 \oplus 32$–dimensional superspace $\mathcal{S}P_{11|32}$. Hence, given an explicit coordinatization of $\mathcal{S}P_{11|32}$, both the 11 bosonic coordinates $X^a$ and the 32 fermionic coordinates $\Theta^\alpha$ become fields depending on $\xi^0, \xi^1, \xi^2$, the three world–volume coordinates. The ordinary supersymmetry variations (3.3) are given by Lie derivatives and correspond to fermionic diffeomorphisms in superspace:

$$\begin{align*}
\delta X^a &= \epsilon^\alpha k^a_\alpha(X, \Theta) \\
\delta \Theta^\alpha &= \epsilon^\beta k^\alpha_\beta(X, \Theta)
\end{align*}$$

(3.7)

where $k^a_\alpha(X, \Theta), k^\alpha_\beta(X, \Theta)$ are, in general, functions of all the coordinates $X, \Theta$. Their explicit form depends on the choice of the coordinate frame. Replacing $\epsilon$ with its projected counterpart (3.6) what we said of ordinary supersymmetry holds true also for $\kappa$–supersymmetry. Hence the important point to be stressed is that the explicit form of $\kappa$–supersymmetries depends on the choice of the coordinate frame for superspace and can be either more or less involved. However, adopting the rheonomic point of view, the invariance of the action can be established in a way completely independent from such an explicit form.

The curvature definitions (3.1) and their rheonomic parametrizations (3.2) are based on a “mostly minus” flat metric $\eta_{ab} = \text{diag}(+, - , - , \ldots , - )$ and on gamma matrices $\Gamma^a$ generating the Clifford algebra $\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}$. In the context of $p$–brane solutions it is more convenient to use a “mostly plus” flat metric $\eta'_{ab} = \text{diag}(- , + , + , \ldots , + )$. A convenient formulation of eleven–dimensional supergravity using this metric is the superspace formulation introduced in [41], where the torsions and curvatures are defined as:

$$\begin{align*}
T^a &= dE^a + \omega^a_b E^b \\
T^\alpha &= dE^\alpha + \frac{1}{4} \omega^{ab}(\Gamma^a_\alpha^b E^b) \\
H &= dB \\
R^{ab} &= d\omega^{ab} + \omega^{a}_c \omega^{cb},
\end{align*}$$

(3.8)

on which one imposes the set of constraints

$$\begin{align*}
T^a &= E^a E^b \Gamma^a_\alpha^\beta E^\beta \\
T^\alpha &= \frac{1}{2} E^a E^b T^\alpha_\beta T^\beta_\alpha + E^\alpha E^\beta \left(8 \Gamma^{[\beta}_a \beta^\alpha H_{a[\beta]} + \Gamma^a_\alpha^\beta \beta^{\alpha} H^{[\beta]} \right) \\
H &= \frac{1}{4!} E^a_{\ldots E^a_{\ldots E^a}} H_{\xi_1 \xi_2 \xi_3} - \frac{1}{4!4!} E^a E^b E^\beta E^\alpha \Gamma^a_{\alpha \beta} E^c_{\ell d} H^{[\ell d]} \\
R_{\ell d} &= \frac{1}{2} E^a E^b R^{\ell d}_{\ell d} + 2 E^a E^\beta \left(4! \Gamma^{a}_\beta \beta^{a} H_{a b c d} + \Gamma^{a}_\beta \beta^{a} H^{[\ell d]} \right) + E^a E^\alpha \left(2 T^\alpha_\beta \beta^a \gamma_\alpha \gamma^a - T^\beta_\beta \beta^a \gamma_\alpha \gamma^a \right).
\end{align*}$$

(3.9)

Thus, now the $\kappa$–symmetry variations of the fields are

$$\begin{align*}
\delta_{\kappa} X^a &= 2 E^a \Gamma^a_\alpha^\beta \left[ \frac{1}{2} \beta (1 + q \Gamma) \right]^\beta, \\
\delta_{\kappa} B &= - \frac{2}{4!4!} E^b E^d E^\beta E^\alpha \Gamma^{a}_\beta \beta^{a} \left[ \frac{1}{2} \beta (1 + q \Gamma) \right]^\beta.
\end{align*}$$

(3.10)

$^3$For a detailed discussion of the notations and conventions we refer the reader to Appendix A.
\[ \delta_\kappa e^i = \frac{1}{(3-t)(t+1)} \left( 2h^{ij} + (t-1)\eta^{ik} \right) \left\{ 2 \left[ \frac{1}{2} (1 + q\bar{\Gamma})\kappa \right]^\alpha \Gamma_k^{\alpha \beta} E^\beta (\delta_j^k - \sqrt{-h} h^{kl} \eta_{ij}) \right. \\
- \left. \left[ \frac{1}{2} (1 + q\bar{\Gamma})\kappa \right]^{\alpha} \Gamma_{kl}^{\alpha \beta} E^\beta e^{klt} \left( \eta_{rj} + \frac{h_{rj}}{\sqrt{-h}} \right) \right\}, \]

where the operator
\[ \bar{\Gamma} \equiv \frac{\epsilon^{ijk}}{3!\sqrt{-h}} \Gamma_{ijk} = \frac{\epsilon^{ijk}}{3!\sqrt{-h}} \Pi_i^{\alpha} \Pi_j^{\beta} \Pi_k^{\gamma} \Gamma_{abc} \]

has the property \( \bar{\Gamma}^2 = 1 \),

\[ h_{ij} \equiv \Pi_i^{\alpha} \Pi_j^{\beta} \eta_{ab} \]

is the off-shell brane metric in flat indices and \( t \) its trace. It can be seen that the action (3.4) is invariant under \( \kappa \)-symmetry transformations if

\[ \alpha_1 = -1 ; \quad \alpha_2 = -1 ; \quad \alpha_3 = -4! \]

It can be understood that the language of superspace constraints is completely isomorphic to the rheonomic formalism. Indeed superspace constraints and rheonomic parametrizations are just different names for the same equations.

If one introduces the relations
\[ E^\alpha = \frac{1}{\sqrt{2}} \Psi^\alpha ; \quad B \equiv -\frac{1}{4!} A, \quad H = dB, \]

\[ T^\alpha = \frac{1}{\sqrt{2}} \rho^\alpha \]

\[ H_{\underline{a}_1 \cdots \underline{a}_4} = -\frac{1}{4!} F_{\underline{a}_1 \cdots \underline{a}_4}, \]

it can be checked that the rheonomic parametrizations (3.2) and the curvature definitions (3.1) exactly translate into (3.8) and (3.9). Furthermore (3.10) can be similarly translated into standard rheonomic formulae for the \( \kappa \)-symmetry variation of all the fields and the first two equations exactly reproduce the supersymmetry variations (3.3) of the background fields; the variation of the world–volume dreibein \( e^1 \) in (3.10) is the only novelty.

### 3.2 The bosonic equations of motion

A useful exercise to derive the existence of static membranes and later for the linearisation around such configurations is to vary the first order action (3.4) to obtain the equations of motion of the bosonic fields.

Set then the fermionic coordinates to zero, we parametrize the metric (2.3) with the following vielbeins

\[ E^i = \rho d\xi^i \delta^i_j \]

\[ E^* = \frac{1}{2} \frac{\rho}{R} \frac{d\rho}{\rho} \]

\[ E^{\hat{a}} = B^{\hat{a}}_{\underline{m}}(y)dy^{\underline{m}} \]

where \( B^{\hat{a}}_{\underline{m}} \) is a proper choice of vielbeins for the internal manifold \( G/H \), and

\[ B = E^i \wedge E^j \wedge E^k \epsilon_{kji} \]

\[ 3! \]
As already said, the variation with respect to $\Pi^a_i$ yields the embedding equation
\[ E^a = e^i \Pi^a_i. \]

Chosen to fix the world–volume frame in terms of the target space as
\[ e^i = E^i, \]
varying the action w.r.t. $e^i$ one gets the “stress–energy tensor” constraint:
\[ g_{\hat{m}\hat{n}}(y) \partial_1 y^\hat{m} \partial_1 y^\hat{n} = \frac{R^2}{4} \frac{1}{\rho^2} \partial_1 \rho \partial_1 \rho, \]
which is just the analogue in supermembrane theory of the Virasoro constraint of string theory.

This fixes the form of $\Pi^a_i$ in terms of $\rho$, $y^\hat{m}$ and $x^I$:
\[ \begin{align*}
\Pi^i_j &= \delta^i_j, \\
\Pi^i_\bullet &= \frac{R}{2} \frac{1}{\rho^2} \delta^i_1, \\
\Pi^a_i &= B_{\hat{m}a} \frac{1}{\rho} \partial_1 y^\hat{m} \delta^a_1 \end{align*} \]

All these equations are then useful to obtain the equations of motion of the bosonic fluctuations of the brane $\rho$ and $y^\hat{m}$. The variation w.r.t. $\rho$ yields:
\[ \delta S = \int \left\{ - \frac{R}{2} \Pi^i_\bullet \frac{1}{\rho^2} \partial_1 \rho d\xi^I \wedge e^j \wedge e_\epsilon \epsilon_{ijk} \right\} \delta \rho - \int d \left\{ \frac{R}{2} \Pi^i_\bullet \frac{1}{\rho^2} \delta^i_1 \hat{\epsilon} \wedge e_\epsilon \epsilon_{ijk} \right\} \delta \rho = 0 \]
and thus, substituting (3.16) and (3.18) into (3.19) we get the following non-linear equation for $\rho$
\[ \Box \rho - \frac{12}{R^2} (1 - q) \rho^3 = 0, \]
where $\Box \equiv \eta^{IJ} \partial_1 \partial_J$.

In the same way, from the variation with respect to $y^\hat{m}$ we obtain:
\[ \delta S = \int \left\{ \Pi^a_i \frac{\partial B_{\hat{m}a}}{\partial y^\hat{m}} \delta y^\hat{m} \wedge e^j \wedge e_\epsilon \epsilon_{ijk} \right\} \delta y^\hat{m} - \int d \left\{ \Pi^a_i \frac{\partial B_{\hat{m}a}}{\partial y^\hat{m}} \wedge e_\epsilon \epsilon_{ijk} \right\} \delta y^\hat{m} = 0 \]
and substituting again (3.16) and (3.18) into (3.21) we get the equations of motion for $y^\hat{m}$
\[ \Box \hat{y}^\hat{m} + \eta^{IJ} \Gamma^\hat{m}_{\hat{n}r} \partial_1 y^\hat{n} \partial_j y^\hat{r} + \eta^{IJ} \partial_1 y^\hat{m} \frac{\partial_1 \rho}{\rho} = 0, \]
where the $\Gamma$’s are the Christoffel symbols of the seven–manifold metric
\[ g_{\hat{m}\hat{n}}(y) = B_{\hat{m}a}^\hat{3}(y) B_{\hat{n}a}^\hat{3}(y) \eta_{\hat{a}\hat{b}}. \]

It is interesting now to notice that, with the choices (3.14), (3.15) and (3.16), these equations admit static solutions ($\rho = \text{const}$, $y^\hat{m} = \text{const}$) if and only if
\[ q = 1. \]
This fact selects the membrane action to be the one with $q = 1$ and yields the recipe to fix the world–volume diffeomorphisms in a way consistent with $\kappa$-symmetry gauge–fixing.
4 The supermembrane on the $AdS_4 \times S^7$ background

Having constructed the $\kappa$–supersymmetric action (3.4), in order to continue our programme we have to specialize it to the $AdS_4 \times S^7$ background. To achieve this point the coordinates of the $D = 11$ target superspace have to be split in the anti de Sitter ones and in that of the seven–sphere. After the splitting we need to fix a physical gauge such that eight of these bosonic coordinates and eight of the fermionic ones become fields on the brane world-volume.

To this end we have to find an explicit parametrization of the vielbeins as functions of these fields and to fix the three–dimensional diffeomorphisms and the $\kappa$–symmetry. Usually one has to deal with very complex objects because the 32 fermionic coordinates of the $D = 11$ space mix with the bosonic ones in complicated expressions. This kind of analysis, though straightforward in line of principle, is very difficult to perform in practice.

A nice way to overcome this obstacle is to use the Supersolvable parametrization of the vielbeins and the three–form field. This parametrization is perfectly equivalent to an a priori gauge–fixing of $\kappa$–symmetry and allows to half the fermionic coordinates (eight on the mass shell), simplifying the expressions one has to deal with.

As emphasized, the great technical advantage is that this fixes $\kappa$–symmetry a priori. It implies that one does not have to calculate first long and complex expressions which one later gauge fixes, but one works with compact formulae from the very start.

Let us then perform the $AdS_4 \times S^7$ spontaneous compactification of eleven–dimensional supergravity in a parametrization independent form and find the superconformal gauge–fixed action for the probe membrane on such a background.

4.1 The $AdS_4 \times S^7$ splitting

We already gave the constraints on the curvatures and torsions of the $D = 11$ space in (3.9). From these and the solution of the Bianchi identities, it follows the dynamics of the fields described by their equations of motion

\[
\Gamma^a_{\alpha\beta} T^{ab}_{\alpha\beta} = 0, \\
R_{ab} - \frac{1}{2} \eta_{ab} R = -288 \cdot 4! \left( H_{a\bar{c}i_1 \ldots i_4} H_{b\bar{c}j_1 \ldots j_4} - \frac{1}{8} \eta_{ab} H_{c_1 \ldots c_4} H_{d_1 \ldots d_4} H_{e_1 \ldots e_4} \right), \\
D_d H^d_{abc} = -\frac{1}{4} \epsilon_{abcd_1 \ldots d_4} \ldots i_1 \ldots i_4 H_{d_1 \ldots d_4} H_{e_1 \ldots e_4}.
\]

(4.1)

To find a consistent solution of these equations which parametrizes $AdS_4 \times S^7$ all the fields and quantities are to be split into four and seven dimensional ones and the antisymmetric tensor field $B$ has to satisfy the Freund–Rubin condition. Following [13] the $D = 11$ $\gamma^a$ matrices can be expressed in terms of the four–dimensional ones $\gamma^a$ and the seven–dimensional $\tau^a$ as follows

\[
\gamma^a = (1 \otimes \gamma^a, \tau^a \otimes \gamma^5)
\]

(4.2)

and the charge conjugation matrix can be expressed in terms of the four and seven dimensional ones:

\[
C = C_7 \otimes C_4 = 1 \otimes \gamma^0.
\]

(4.3)
This yields the splitting formulae for the eleven dimensional $\Gamma^{a}$ matrices. The bosonic
eleven–dimensional fields are relabeled as
\[ E^{a} = (E^{a}, E^{\hat{a}}), \quad \omega_{ab}^{\hat{b}} = (\omega_a^b, \omega_a^\hat{b}, \omega_a^{\hat{b}}), \quad B = B. \] (4.4)

The Freund Rubin solution of eleven dimensional supergravity can be obtained giving
an expectation value to the field–strength of the three–form field
\[ H_{a_1...a_4} = \frac{e}{4!} \epsilon_{a_1...a_4}, \] (4.5)
and imposing
\[ \omega_{a\hat{a}} = 0, \] (4.6)
\[ T_{ab}^{\hat{a}} = 0, \] (4.7)
which means the Lorentz connection factorizes and that the eleven–dimensional gravitino
has vanishing vev.

From this, and (4.1), the Einstein equations become that of an $AdS$ space and an
Einstein seven–manifold
\[ R_{ab} = 48e^2 \eta_{ab}, \] (4.8)
\[ R_{\hat{a}\hat{b}} = -24e^2 \delta_{\hat{a}\hat{b}}, \] (4.9)
of radius two times that of the anti de Sitter space. In particular we can choose any
homogeneous coset $G/H$. The case of the seven sphere we are considering in this paper,
corresponds to the choice that leads to a maximal number of preserved supersymmetries
in anti de Sitter space ($N=8$). This choice corresponds to the following Riemann tensors
\[ R_{ab}^{cd} = 32e^2 \delta_{ab}^{cd}, \] (4.10)
\[ R_{\hat{a}\hat{b}}^{\hat{c}\hat{d}} = -8e^2 \delta_{\hat{a}\hat{b}}^{\hat{c}\hat{d}}, \] (4.11)
Since the Gaussian curvature of any sphere is $K = 1/R^2$ and the curvature scalar is
proportional to $K$, we can now relate the vev of the antisymmetric tensor field to the
radii of the $AdS$ and $S^7$ spaces. This relation is given by
\[ e = \frac{1}{2R}. \] (4.12)

A natural way to split the eleven–dimensional fermions is simply as in standard di-
mensional reduction [44, 45] to write
\[ E^\alpha = E^{(A\alpha)} = \Psi_A^{\alpha}, \]
which are the fermions living on
\[ \mathcal{M}_{11} = \frac{OSp(8|4)}{SO(1,3) \times SO(7)}. \]

There is however a rather more elegant way to proceed, which is to write instead
\[ E^\alpha = E^{(\hat{\alpha} \alpha)} = \sum_{A=1}^{8} \eta_A^{\hat{\alpha}} \otimes \psi_A^{\alpha}, \] (4.13)
where $\psi_a^\alpha$ are Majorana $AdS_4$ fermions and $\eta_A$ are $c$-number $S^7$ Killing spinors, functions only of $\rho$ and $y^a$. This corresponds to the local decomposition

$$\mathcal{M}_{11} \approx \frac{OSp(8|4)}{SO(1,3) \times SO(8)} \times \frac{SO(8)}{SO(7)} = AdS_{(8|4)} \times S^7.$$ 

The elegance of this approach lies in the simplicity of the resulting super-vielbeins and it is indispensable for the generalisation to $G/H$ branes. To make (4.13) consistent with the Bianchi identities it is necessary to add fermion bilinears to the bosonic vielbein $B^a$ and connection $B^{\hat{a} b}$ of $S^7$.

$$E^{\hat{a}} = B^{\hat{a}}(y) - \frac{1}{8} \eta_A \tau^{\hat{a}} \eta_B A^{AB}(x, \theta), \quad (4.14)$$

$$\omega^{\hat{a} b} = B^{\hat{a} b}(y) + \frac{e}{4} \eta_A \tau^{\hat{a} b} \eta_B A^{AB}(x, \theta), \quad (4.15)$$

where $A_{AB}$ is the $SO(8)$ connection.

With the above choices the eleven dimensional constraints and Bianchi identities become relations on the four-dimensional quantities. The Ricci tensors for the $AdS_4$ space and for $S^7$ are then

$$R^{ab} = -16 e^2 E^a \land E^b + 2 e \psi_A \land \gamma_{cd} \psi_A e^{abc}d, \quad (4.16)$$

$$R^{\hat{a} b} = 4 e^2 E^{\hat{a}} \land E^b + 2 e \bar{\psi}_A \land \gamma^5 \psi_B \eta_A \tau^{\hat{a} b} \eta_B. \quad (4.17)$$

The torsions and gravitinos satisfy

$$D E^a = \bar{\psi}_A \land \gamma^a \psi_A, \quad (4.18)$$

$$D E^{\hat{a}} = \eta_A \tau^{\hat{a}} \eta_B \bar{\psi}_A \land \gamma^5 \psi_B, \quad (4.19)$$

$$\rho_A \equiv D \psi_A = -2 e E^a \land \gamma_a \gamma^5 \psi_A - e A_{AC} \psi_C, \quad (4.20)$$

the $SO(8)$ connection satisfies

$$d A_{AB} + e A_{AC} \land A_{CB} = 8 \bar{\psi}_A \land \gamma_5 \psi_B, \quad (4.21)$$

while the sphere spinors do indeed satisfy the Killing equation (cfr. equation (4.9))

$$D_{(Spb)} \bar{\eta}_A = e B^{\hat{a}} \tau_{\hat{a}} \eta_A, \quad (4.22)$$

where $D_{(Spb)} \equiv d + B(y)$. It is now easy to recognize that (4.16)–(4.21) are the Maurer Cartan equations of the $OSp(8|4)$ supergroup [44]. In fact, eqs. (4.16), (4.18), (4.20) and (4.21) can be rewritten as

$$d \omega^{ab} + \omega^{ac} \land \omega_c^b + 16 e^2 E^a \land E^b = 4 e \bar{\psi}_A \land \gamma^{ab} \gamma^5 \psi_A,$$

$$d E^a + \omega^a_c \land E^c = \bar{\psi}_A \land \gamma^a \psi_A, \quad (4.23)$$

$$d \psi_A + \frac{1}{4} \omega^{ab} \land \gamma_{ab} \psi_A + e A_{AB} \land \psi_B = -2 e E^a \land \gamma_a \gamma_5 \psi_A,$$

$$d A_{AB} + e A_{AC} \land A_{CB} = 8 \bar{\psi}_A \land \gamma_5 \psi_B,$$

which are the desired Maurer–Cartan equations of the $OSp(8|4)$ algebra given in standard form. Indeed an element of the $OSp(8|4)$ superalgebra can be defined as a graded matrix

$$\mu = \left( \begin{array}{cc} 1 \omega^{ab} \gamma_{ab} + 2 e E^a \gamma_5 \gamma_a & -8 e \gamma_5 \psi_B \\ \bar{\psi}_A & e A_{AB} \end{array} \right), \quad (4.24)$$
preserving the ortosymplectic matrix\(^4\):

\[
\Omega = \begin{pmatrix}
-C\gamma^5 & 0 \\
0 & 8\varepsilon
\end{pmatrix},
\]

(4.26)
i.e.

\[
\Omega \mu + \imath \mu \Omega = 0.
\]

(4.27)
The Maurer Cartan equations (4.23) can be retrieved by writing

\[
d\mu + \mu \wedge \mu = 0.
\]

Now we come to the main point in the construction of the solution. The curvature of the three–form \(B\) can be expressed in terms of AdS\(^4\) \(\times\) S\(^7\) quantities as follows:

\[
H = \frac{e}{4!} E^a \wedge \ldots \wedge E^d \epsilon_{d..a} + \frac{1}{4!} E^a \wedge E^b \wedge \psi_A \wedge \gamma_{ab} \psi_A + \frac{2}{4!} E^a \wedge E^b \wedge \eta_A \tau_b \eta_B \bar{\psi}_A \wedge \gamma_5 \gamma_a \psi_B + \frac{1}{4!} E^a \wedge E^b \wedge \eta_A \tau_{ab} \eta_B \bar{\psi}_A \wedge \psi_B,
\]

(4.28)
and, from the curvature definition \(dB = H\), it should be deduced the parametrization of the three–superform \(B\), but this can be done only if one uses a specific parametrization in terms of the coordinates. It is then useful to look at the supersolvable parametrization of our space.

4.2 The supersolvable algebra and the \(\kappa\)–symmetry gauge–fixing

The AdS superspace is defined as the following coset

\[
AdS^{(8|4)} = \frac{OSp(8|4)}{SO(1,3) \otimes SO(8)}
\]

and it is spanned by the four coordinates of the AdS\(^4\) manifold and by eight Majorana spinors (i.e. they have 32 real components) parametrizing the fermionic generators \(Q^A_\alpha\) of the superalgebra.

It has already been shown that the AdS manifold admits a suitable description in terms of a four dimensional solvable Lie algebra Solv \([12]\). The problem which we deal with is that of finding a supersolvable description of the superspace AdS\(^{(8|4)}\). It turns out that a solvable superalgebra \(SSolv\) containing Solv can be found inside \(OSp(8|4)\),

\[
H\mu + \mu^1 H = 0.
\]

(4.25)
This is the analogue of defining the bosonic anti de Sitter group \(SO(2,3)\) through the isomorphism \(SO(2,3) \sim Usp(2,2)/\mathbb{Z}_2\) where \(Usp(2,2) = Sp(4,\mathbb{C}) \cap SU(2,2)\). However, we can also define \(SO(2,3)\) through the alternative isomorphism \(SO(2,3) \sim Sp(4,\mathbb{R})/\mathbb{Z}_2\). In this case it just suffices to consider real symplectic matrices, removing the bosonic analogue of the second condition (4.23). Such a situation arises in the Majorana representation of gamma matrices. Here the spinorial representation which is symplectic is also real and naturally realizes the isomorphism \(SO(2,3) \sim Sp(4,\mathbb{R})/\mathbb{Z}_2\). Obviously all this carries over to the super Lie algebra case. Choosing the Majorana representation of gamma matrices it suffices to define the ortosymplectic group as the set of real graded matrices satisfying condition (4.27) and discard condition (4.23). This is what we do here with our chosen basis of gamma matrices.
Figure 1: The root diagram of $SO(2,3)$. The bosonic weights are represented by circles, and the fermionic weights by squares. The dilatation charge of horizontal planes in the diagram are on the left, while the worldvolume theory interpretation of the planes of generators are labelled to the right. The supersolvable algebra is the boxed subalgebra.

the only price one has to pay being a suitable projection of the fermionic generators $Q^A_\alpha \rightarrow Q'^A_\alpha = \mathcal{P}Q^A_\alpha$.

Just as the solvable description of $AdS_4$ allows to define the coordinates on the brane it will be seen that the supersolvable description of superspace yields the definition of the fermions living on the brane as the result of the equivalence of the projection operator $\mathcal{P}$ on the target fermionic coordinates, and the $\kappa$-symmetry projection operator.

It is perhaps worth pointing out that the supersolvable Lie algebra admits a very natural physical interpretation. The starting point is the decomposition of the $Osp(8|4)$ algebra of $AdS(8|4)$ isometries in terms of the superconformal algebra of the three-dimensional worldvolume theory of the brane.

We can decompose the $SO(2,3)$ Lie algebra of invariances of $AdS$ in terms of a 3 dimensional $SO(1,2)$ sub-algebra $\{L_\perp, L_{\pm}\}$, which is just the algebra of Lorentz rotations in the brane. Of the six remaining step operators of $SO(2,3)$, three are interpreted as worldvolume translations $\{\tau_\perp, \tau_{\pm}\}$, and the other three are the conformal boosts $\{\sigma_\perp, \sigma_{\pm}\}$. Of the two Cartan generators of $SO(2,3)$, one goes into $SO(1,2)$, while its orthogonal complement is the generator of dilatations $D$.

The fermions split as $4 \otimes 8_V = (2 \oplus 2) \otimes 8_V$. Physically each of the eight space-time supercharges, four-component $D = 4$ Majorana spinors, split into eight worldvolume supercharges, two-component $D = 3$ Majorana spinors, and eight matching worldvolume superconformal generators.

This decomposition is illustrated in the familiar “Union Jack” root diagram of $C_2$, which is the complexification of $SO(2,3)$ shown in figure (1). The fermionic supercharges form a square weight diagram within this figure, and the supertranslation algebra is then simply that the anticommutator of two fermions is given by vector addition of the corresponding weights in the diagram. The diagram can in fact be seen as a projection of the full $Osp(8|4)$ root diagram, since the $SO(8)$ roots lie on a perpendicular hyperplane, and so on this diagram they would be at the centre.

In figure (1) the decomposition into the algebra of the worldvolume superconformal algebra is simply the decomposition into horizontal planes. A natural way to characterize these planes is by their eigenvalues under $D$, i.e. by their dilatation charge. These charges are given by the numbers on the left of the figure. More formally, this charge provides a
$Z_5$ grading to $Osp(8|4)$, a preserved charge under the super Lie bracket. This is precisely the grading required to define the supersolvable Lie algebra of Appendix C, namely the solvable algebra of $Osp(8|4)$ is just the sub-algebra obtained by restricting $Osp(8|4)$ to negative grading complemented by $D$ itself. Its weight diagram is boxed in the diagram. It can also be seen that in our explicit representation of gamma matrices, this corresponds to restricting to triangular matrices. The generators of this supersolvable algebra are in fact a suitable choice of generators for the super coset space $AdS_{(8|4)}$. Their attraction comes from the fact that they are easily exponentiated since the power series expansion of the exponential only contains a finite number of terms, and so explicit expressions for the supervielbeins are easily obtained.

The choice of a gauge fixing condition is a subtle point. This condition must indeed be compatible with the classical solution of the brane–wave equations of motion chosen as the vacuum around which the perturbative theory is developed.

As it has been pointed out in [12], to have the static solutions needed to perform the correct expansion around the boundary configurations, the Grassmann coordinates one has to project away gauge–fixing the $\kappa$–symmetry are those parallel to the $\kappa$–symmetry projector. Since this same projector leaves the $Q$’s invariant (which are then recognised as the world–volume preserved supersymmetries) the proper choice to parametrize the super–$AdS$ space and obtain a $SSolv$ algebra is to use the generators $\{S_\pm, \sigma_\pm, \sigma_\perp, D\}$.

More details about the supersolvable algebra and the parametrization of the Super$AdS$ space can be found in the Appendix C.

We give here for the metric (2.3) the parametrizations of the vielbeins in terms of the four solvable coordinates $(\rho, t, w, x)$ and the eight four–dimensional fermions $(\theta^A_a)$:

\[
E^0 = -\rho dt - 2e\rho \tilde{\theta}^-_A \gamma^0 d\theta^-_A, \\
E^1 = \rho dw - 2e\rho \tilde{\theta}^-_A \gamma^1 d\theta^-_A, \\
E^2 = \frac{R}{2} \frac{1}{\rho} d\rho, \\
E^3 = \rho dx - 2e\rho \tilde{\theta}^-_A \gamma^3 d\theta^-_A,
\]  
(4.29)

and

\[
\psi^A = \sqrt{2e\rho} \begin{pmatrix} 0 \\ 0 \\ d\theta^A_1 \\ d\theta^A_2 \end{pmatrix},
\]
(4.30)

where $\theta^A = \frac{1-\gamma^5 \gamma^2}{2} \theta^A$ and $\tilde{\theta}^-_A = \theta^A \gamma^0$. It can also be found that the $SO(8)$ connection $A$, in this parametrization, is identically zero:

\[
A_{AB} = 0.
\]
(4.31)

To complete the parametrization of the target superspace one has to define also the vielbeins of the seven–sphere. Calling $y^{\hat{a}}$ the seven coordinates of the sphere, the parametrization we adopt is the following

\[
E^{\hat{a}} = \mathcal{B}^{\hat{a}} = -R \delta^{\hat{a}}_{\hat{m}} \frac{dy^{\hat{m}}}{1+y^2}.
\]
(4.32)
which is nothing else than the stereographic projection coordinate $s$.

The last thing to find is the parametrization of the Wess–Zumino term. This means solving $H = dB$ in this background. As we said in the last section this calculation can be done once obtained an explicit parametrization. So, with the parametrizations (4.29)–(4.32) given above, the $B$ field has the form

$$B = \frac{1}{4!} \cdot \frac{1}{4!} E^i \wedge E^j \wedge E^k \frac{\epsilon^{kji}}{3} - \frac{1}{2e} \frac{1}{4!} E^a \wedge \eta_A \tau^a \eta_B \bar{\psi}_A \wedge \psi_B. \quad (4.33)$$

We are now in position to obtain the complete action of the supermembrane on an $AdS_4 \times S^7$ background. The only missing point is the definition of the brane coordinates, i.e. the choice of a physical gauge.

### 4.3 The superconformal action in second order “Nambu-Goto” formalism

The $\kappa$-symmetry has already been fixed using the solvable parametrization. We fix the three–dimensional world–volume diffeomorphisms imposing the static gauge choice. As a consequence of what has been said in section (3.2), to obtain static solutions we have to identify

$$\xi^I \equiv (-t, w, x), \quad (4.34)$$

where $I = 0, 1, 2$, is the curved index of the brane. Once identified these coordinates, the form of the $\Pi$ fields can be deduced from the vielbein parametrizations. These can actually be projected along the basis of the brane cotangent space:

$$E^a = e^i \Pi^a_i = d\xi^I \Pi^a_I. \quad (4.35)$$

The brane metric can then be deduced from the embedding equations using the above parametrizations

$$h_{IJ}(\xi) = \Pi^{aI} \Pi^{bJ} \eta_{ab} = \frac{1}{16e^2} \rho \partial_I \rho \partial_J \rho + \frac{1}{4e^2} \frac{1}{(1 + y^2)^2} \partial_I y^\hat{a} \partial_J y_{\hat{a}} + \rho^2 \left( \eta_{IJ} - 4e^A \gamma^i \partial_I \theta^A_i \delta_{Ji} + 4e^{A\gamma^i} \partial_I \theta^A_i \partial_J \theta_B^i \right). \quad (4.36)$$

In second order formalism and with the embedded metric (4.36), the action of the membrane on the $AdS_4 \times S^7$ background is now

$$S = 2 \int \sqrt{-\det(h_{IJ})} d^3\xi + 4! \int B. \quad (4.37)$$

The expression for $B$ is given by

$$B = \frac{d^3\xi}{4!} \left[ \epsilon^{IJK} \rho^3 (\delta^i_j - 2e^{A\gamma^i} \partial_I \theta^A_i) (\delta^j_l - 2e^{A\gamma^j} \partial_J \theta^A_j) (\delta^k_m - 2e^{A\gamma^k} \partial_K \theta^A_k) \epsilon^{ijk} \right. + \left. \frac{1}{2e} \epsilon^{IJK} \partial_I y^\hat{a} \eta_A \tau_{\hat{a}} \eta_B \rho \partial_J \theta^B \right]. \quad (4.38)$$

It should be noticed that the field theory we have derived in this way has non trivial interactions and is highly non–linear.
Figure 2: The root diagram of $SO(2, 3)$, with simple roots $\epsilon_1$ and $\epsilon_2$. The Weyl reflection $W_{\epsilon_2}$ is shown in the picture.

An important feature displayed by this action is its invariance under conformal transformations. As already stressed in the introduction and before in this section, the super–$AdS$ group acts on the membrane as the group of superconformal transformations. This action is non–linearly realised $[2, 34]$ and this partly explains the complicated expression of (4.36) and (4.38).

We have completed here the programme started in $[2, 3, 33, 34]$, where the authors presented the theory restricted to only radial fluctuations or the free–field limit in the purely bosonic sector. In fact, keeping the $y$ and $\theta$ fixed and reducing only to the radial fluctuations the action takes the form

$$S = 2\int \rho^3 \left[ \sqrt{1 + \frac{R^2 \partial_I \rho \partial^I \rho}{\rho^4}} - 1 \right] d^3 \xi,$$

from which we can recover the action (15) presented in $[34]$, setting

$$\rho = \frac{\phi}{R^2},$$

where $w = \frac{1}{2}$ and $p = 2$.

If one properly identifies the radius $R$ as $[1]

$$R = l_p(2^5 \pi^2 N)^{1/6},$$

where $l_p$ is the eleven–dimensional Planck length and $N$ is the number of $M2$–branes, (4.37) can be interpreted as the action of one probe membrane in the background generated by the other $N – 1$ branes.

The explicit form of the non–linear realisation of the superconformal transformations on the world–volume is much simplified by the observation that although there are 6 superconformal generators, namely $\{D, \sigma_\perp, \sigma_\pm, S_\pm\}$ (see figure (2)), one needs to verify invariance only for two, namely dilatations and special conformal inversion. This is so because we can explicitly construct the operator that implements the Weyl reflection in the horizontal 0–grading plane (see figures (1) and (2)). Thus any superconformal generator in the lower part of the weight diagram can be constructed as a worldvolume Poincaré generator conjugated through the Weyl reflection $W_{\epsilon_2}$. Given that our worldvolume theory is Poincaré invariant, all that remains to be checked is dilatation invariance and invariance with respect to the finite group element $W_{\epsilon_2}$. We have:

$$W_{\epsilon_2} = \exp [\pi (E_{+\epsilon_2} - E_{-\epsilon_2})] = \exp [\pi (\tau_\perp - \sigma_\perp)].$$

(4.41)
Thus we only need the transformation induced by $\tau_\perp - \sigma_\perp \equiv K_3$ in order to be able to write any superconformal transformation. The action of the $Osp(8|4)$ generators on the coset space $AdS_{(8|4)}$ is simply given by the corresponding Killing vector. The transformation of the $\rho$ field, the coordinates and the spinor fields under dilatations is

$$
\begin{align*}
\delta \rho &= \rho, \\
\delta \xi^I &= -\xi^I, \\
\delta \theta^A_\alpha &= -\frac{1}{2}\theta^A_\alpha,
\end{align*}
$$

(4.42)

while their transformation under the Weyl rotation generator $K_3$ is

$$
\begin{align*}
\delta \rho &= \rho \xi^2, \\
\delta \xi^0 &= -\xi^0 \xi^2, \\
\delta \xi^1 &= -\xi^1 \xi^2, \\
\delta \xi^2 &= -\frac{1}{2} (1 + (\xi^2)^2) + \frac{1}{2} ((\xi^1)^2 - (\xi^0)^2) - \frac{e^2}{2} \theta^A_1 \theta^A_2 \theta^B_1 \theta^B_2, \\
\delta \theta^A_1 &= -\frac{1}{2} \xi^2 \theta^A_1, \\
\delta \theta^A_2 &= -\frac{1}{2} \xi^2 \theta^A_2.
\end{align*}
$$

(4.43)

An important thing to note here is that the transformation of the fields $f(\xi)$ dependent on the brane coordinates $\xi^I$ is their complete variation for the conformal transformations, i.e. $\delta f = f'(\xi') - f(\xi)$, in that (4.42)–(4.43) express the functional variation plus the $\xi$ dependent transformation of the fields. This observation leads then to the following identity for the variation of the derivatives of the fields

$$
\delta(\partial_I f(\xi)) = \partial_I \delta f - \partial_J f \partial_I \delta \xi^J,
$$

(4.44)

which is very useful for the verification of invariance of the action (4.37) under the transformations (4.42) and (4.43).

5 The singleton action from the supermembrane

Starting from the action (4.37) we can now recover the singleton field theory expanding the transverse coordinates around the classical solution of the brane–wave equations provided by

$$
\begin{align*}
\xi^I \equiv (-t, w, x), \quad \partial_I y^\dot{a} = 0, \quad \theta^A_\alpha = 0,
\end{align*}
$$

(5.1)

and

$$
\rho = \bar{\rho} = \text{const} : \quad \rho \to \infty \quad \text{or} \quad \rho \to 0.
$$

(5.2)

This means choosing a physical gauge, a point on the seven–sphere and then taking the limit to the boundary. We would like to stress here that the proper conformally invariant boundary of the $AdS$ space is given by the three–dimensional Minkowski space at $\rho \to \infty$ with some points at its infinity added. We will find the singleton field theory on such boundary. But it can be seen that the $AdS$ isometries act on the horizon such that it can be constructed a conformal invariant theory expanding around $\rho = 0$. Thus, we are going
to find that the singleton field theory describes the centre of mass degrees of freedom of the \( M2 \)-brane as a solution of the eleven-dimensional supergravity equations of motion.

We define the quantum expansion (small fluctuations) around the classical solution using the normal coordinate expansion \[46\]:

\[
X^M = x^M + \alpha' \phi^M. 
\] (5.3)

where \( x^M \) is the background value for \( X^M \) (the \( 11 + 32 \) superspace coordinates), \( \phi \) is the normal coordinate (quantum field) and \( \alpha' \) is related to the membrane tension (now \( M = (m, \mu) \)).

For the sake of simplicity we take the \( y^\hat{m} = 0 \) point on the sphere. Thus, the expansion formulae for the coordinates are

\[
\rho = \bar{\rho} + \alpha' \rho; \\
y^{\hat{m}} = \alpha' \rho; \\
\theta^A_{\hat{\alpha}} = \alpha' \rho. 
\] (5.4)

\( \bar{\rho} \) is different from zero, but constant.

Applying the normal coordinate expansion formulae as they are given in section A.1 to the supermembrane action, we can expand this as a power series in \( \alpha' \)

\[
\mathcal{L} = \sum_{n=0}^{\infty} \alpha'^{2(n-2)} \mathcal{L}_{(n)} 
\] (5.5)

The result in terms of the fluctuations \[5.4\] is that the vacuum membrane graph and the tadpole term are exactly zero:

\[
\mathcal{L}_{(0)} = 0 = \mathcal{L}_{(1)} 
\] (5.6)

and we should recover the singleton action from the order 1 term

\[
\mathcal{L}_{(2)} = \frac{1}{16 e^2 \bar{\rho}} \eta^{IJ} \partial_I \bar{\rho} \partial_J \bar{\rho} + \frac{\bar{\rho}}{4 e^2} \eta^{IJ} \partial_I \bar{y}^{\hat{m}} \partial_J \bar{y}^{\hat{m}} \delta_{\hat{m}\hat{m}} - 4e \bar{\rho}^3 \Theta^A \Theta^i \partial_I \Theta^A \delta^I. 
\] (5.7)

As it can be easily seen, taking the boundary limit, some terms in the action \[5.7\] vanish and some other diverge. This implies that, in order to keep it finite, we have to rescale these fluctuations with some power of \( \bar{\rho} \). This does not provide us the singleton action yet. Naively speaking in fact, one can see that all the other terms disappear as we go to the boundary. But we did not take into account the symmetry transformations. We will actually see that also these transformations diverge on the same limit and that the right rescaling of the fluctuations are the ones which let us retrieve the singleton action from \[5.7\].

Let us analyse then the symmetry of \[5.7\]. Since the action we obtained has \( \kappa \)-symmetry fixed, the supersymmetry transformations should preserve this gauge fixing. To this end one has to accompany any SUSY transformation with a \( \kappa \)-symmetry transformation such that one does not move the chosen configuration.

Following \[49\], we have fixed \( \kappa \)-symmetry by imposing

\[
(1 + \bar{\Gamma}) \theta = 0. 
\] (5.8)
Calling $\chi = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$, (5.8) can be read as $\theta_R = 0$.

In the $\gamma$–matrices basis chosen in the Appendix A, the generic value of $\bar{\Gamma}$ has the following block structure:

$$
\bar{\Gamma} = \begin{pmatrix}
-ACA^{-1} & A \\
(1 - C^2)A^{-1} & C
\end{pmatrix},
$$

(5.9)
enforced by the condition $\bar{\Gamma}^2 = 1$. For a SUSY plus $\kappa$–symmetry variation, the variation of the fermions is

$$
\delta \theta = \epsilon + \frac{(1 + \bar{\Gamma})}{2} \kappa,
$$

(5.10)
which, from (5.9), can be written as

$$
\begin{align*}
\delta \theta_L &= \epsilon_L + \frac{A}{2} \kappa_R, \\
\delta \theta_R &= \epsilon_R + \frac{1 + C}{2} \kappa_R.
\end{align*}
$$

(5.11)

To preserve the (5.8) gauge fixing, one has to impose $\delta \theta_R = 0$. Therefore, the compensating $\kappa$–symmetry transformations has parameter $\kappa_R = -2(1 + C)^{-1} \epsilon_R$. Thus, the complete SUSY transformation of the physical fermions is

$$
\delta \theta_L = \epsilon_L - A(1 + C)^{-1} \epsilon_R.
$$

(5.12)

Since we also fixed the world–volume diffeomorphisms imposing the static gauge (5.4), the total variation of $\theta_R$ as a field on the world–volume is

$$
\delta \theta^A = \epsilon^A - A(1 + C)^{-1} \epsilon^A - \partial_I \theta^A \delta_{e^+} x^I
$$

(5.13)

while the other field transformations are

$$
\begin{align*}
\delta \rho &= \delta \epsilon \rho - \partial_I \rho \delta_{e^+} x^I, \\
\delta y^a &= \delta \epsilon y^a - \partial_I y^a \delta_{e^+} x^I.
\end{align*}
$$

(5.14)

To derive the explicit form of these transformations we have then to find the value of $A$ and $C$ on our background, while to derive the transformation of the fluctuations one has also to expand the above equations and identify the terms with the same powers of $\alpha'$. As it is known, the classical configuration specified by (5.1) cannot preserve all the target space SUSY, but in the best case (like this) it preserves the half.

Our choice of the vacuum [31] imposes that in order to preserve SUSY

$$
\delta \psi = \delta \theta = 0.
$$

(5.15)

Since we projected the $\theta$ with the relation (5.8), condition (5.15) translates into the fact that the residual SUSY are those transformations parametrized by an $\epsilon$ which satisfies

$$
\bar{D} \epsilon = 0 \quad \text{and} \quad \frac{(1 - \bar{\Gamma})}{2} \epsilon = 0,
$$

(5.16)

where $\bar{D}$ is the supercovariant derivative and then implies that $\epsilon$ is a killing spinor. Thus we are left with the transformations (5.13) and (5.14) in which we set $\epsilon_L = 0$. 

23
We want the SUSY transformations on the world–volume. We have then to take the SUSY transformations (5.13)—(5.14) and make the expansion (5.4). It is straightforward to find that

$$C = +1,$$

$$A = \alpha' \frac{1}{2} \left[ \frac{\partial_I \bar{y}^a}{2e\bar{\rho}} \tau_a - \frac{\partial_I \bar{\rho}}{4e\bar{\rho}^2} \sigma^I \sigma^0 \right],$$

and, matching orders in $\alpha'$,

$$\delta \Theta^A_+ = \frac{1}{2} \left[ \frac{\partial_I \bar{y}^a}{2e\bar{\rho}} \eta^A \tau_\dot{a} \eta^B - \frac{\partial_I \bar{\rho}}{4e\bar{\rho}^2} \delta^{AB} \right] \sigma^I \sigma^0 \epsilon^B_+,$$

$$\delta \bar{\rho} = -8e^2 \bar{\rho}^2 \epsilon^A_+ \sigma^0 \Theta^A_-,$$

$$\delta \bar{y}^a = 4e^2 \bar{\rho} \eta^A \tau_\dot{a} \eta^B \epsilon^A_+ \sigma^0 \Theta^B_-,$$

where we have identified $i$ with $I$ since we are on flat space.

As we have already claimed, these transformations diverge as $\bar{\rho} \to 0$, but we can make them all finite if we rescale the fluctuations in the following way

$$\lambda = \bar{\rho}^2 \Theta^A_+, \quad \bar{P} = \frac{\bar{\rho}}{\sqrt{\bar{\rho}}}, \quad \bar{Y}^a = \sqrt{\bar{\rho}} \bar{y}^a.$$

In the (5.19) transformations there appear the killing spinors on the sphere $\eta^A$. These are functions of $y$ and, through these, of $\xi^I$. When fixing the gauge and taking the expansion (5.4), the leading term in $\alpha'$ for the $y$'s is zero and thus we can interpret $\eta^A \tau_\dot{a} \eta^B$ as a simple numerical matrix.

Finally, it can be seen that if we define

$$Y_{\dot{A}} \equiv \left\{ \bar{P}, \bar{Y} \right\}, \quad k \equiv \{-\delta^{AB}, \eta^A \tau_\dot{a} \eta^B\}$$

we can rewrite the singleton action as

$$\mathcal{L} = \eta^{IJ} \frac{1}{e^2} \partial_I Y_{\dot{A}} \partial_J Y_{\dot{A}} - 4e \bar{\lambda}^A \sigma^I \partial_I \lambda^A,$$

with supersymmetry transformations given by

$$\delta \lambda^A = \frac{1}{2e} k_{\dot{A}B} \partial_I Y_{\dot{A}} \sigma^I \sigma^0 \epsilon^B_+, \quad \delta Y_{\dot{A}} = 2e^2 k_{BC} \epsilon^B_+ \sigma^0 \lambda^C,$$

where we reinterpreted the indices as if we define

$$A \in \mathbf{8}_S, \quad \dot{A} \in \mathbf{8}_C, \quad \underline{A} \in \mathbf{8}_V,$$

and $k_{BC}$ is proportional to the triality matrices of $SO(8)$.

All the other superconformal transformations, under which (5.22) is invariant, can be retrieved applying the same method to the full transformations. We have verified that
(5.22) is indeed invariant under dilatations, supersymmetry and Weyl reflections and thus on the whole algebra.

Since the action and the supersymmetry transformations are independent from $\tilde{\rho}$, one should think that this action is superconformally invariant for any vev of $\rho$, but this is not the case. If $\rho$ has a finite vev value, different from zero, there are $\alpha'$ corrections to the action and the $K_3$ transformation is no more a symmetry of the theory since it depends explicitly on such vev $\tilde{\rho}$.

**Acknowledgements.** We are grateful to A. Lerda for many useful discussions and to G. Arcioni, F. Cordaro, L. Gualtieri for collaboration in the early stages of this work. We would also like to thank P. Pasti, D. Sorokin and M. Tonin for valuable comments and discussions about the proper choice of a consistent $\kappa$–symmetry gauge fixing.

**Appendix A: Notations and Conventions**

Latin letters are the indices of the bosonic coordinates, greek indices label the fermionic ones. Letters from the beginning of the alphabet are flat indices while middle alphabet letters label curved indices. Underlined indices refer to the eleven dimensional coordinates, $\underline{a}, \underline{m} = 0, \ldots, 10, \underline{\alpha}, \underline{\mu} = 1, \ldots 32$; normal indices span the AdS space $a, m = 0, \ldots 3, \alpha, \mu = 1, \ldots 4$ and the hatted indices label the seven–sphere $S^7$, $\hat{a}, \hat{m} = 1, \ldots 7, \hat{\alpha}, \hat{\mu} = 1, \ldots 8$. The membrane worldvolume is spanned by three bosonic coordinates labelled by $I = 0, 1, 2$ or $i = 0, 1, 2$, if curved or flat indices respectively. We use the mostly plus metric, i.e.

$$\eta_{\underline{a}\underline{b}} = \text{diag}\{- + + + + + + + + + +\}, \quad \eta_{ij} = \text{diag}\{- +\}. \quad (A.1)$$

A p-form $\phi_p$ is defined by

$$\phi_p = \frac{1}{p!} E_{a_1}^{a_p} \wedge \ldots \wedge E_{a_{p-1}}^{a_1} \phi_{\underline{a}_1 \ldots \underline{a}_p}; \quad (A.2)$$

the differential acts from the right

$$d(A_P B_q) = A_P d B_q + (-1)^p d A_P B_q \quad (A.3)$$

and the Levi-Civita tensor is defined as $\epsilon_{012} = +1$.

The **eleven–dimensional** gamma matrices $\gamma^a$ are elements of the Dirac algebra

$$\{\gamma^a, \gamma^\beta\} = 2 \eta^{ab}. \quad (A.4)$$

Through the charge conjugation matrix $C$ we define the matrices

$$\begin{align*}
(\Gamma^a)_{\underline{a}\underline{b}} & \equiv (\gamma^a)_{\underline{a}\underline{b}} \\
(\Gamma^a)_{\underline{a}\underline{b}} & \equiv C_{\underline{a}\underline{a}_1} (\gamma^2)_{\underline{a}_1\underline{b}} \\
(\Gamma^a)_{\underline{a}\underline{b}} & \equiv (\gamma^a)_{\underline{a}\underline{b}} C_{\underline{a}\underline{b}} \\
(\Gamma^a)_{\underline{a}\underline{b}} & \equiv C_{\underline{a}\underline{a}_1} (\gamma^2)_{\underline{a}_1\underline{b}} C_{\underline{a}\underline{b}}. \quad (A.5)
\end{align*}$$

25
Antisymmetrization $\Gamma^{a_1 \ldots a_n} \equiv \Gamma^{[a_1 \ldots a_n]}$ is understood with unit weight. The symmetric matrices are

$$\Gamma^a, \Gamma^{ab}, \Gamma^{ab \cdots c}, \Gamma^{a_1 \cdots a_n}, \Gamma^{a_1 \cdots a_2 \cdots a_0}.$$  \hspace{1cm} (A.6)

The cyclic identity in eleven dimensions reads

$$(\Gamma^{ab})_{\alpha \beta} (\Gamma^{c})_{\gamma \delta} = 0.$$  \hspace{1cm} (A.7)

The four–dimensional gamma matrices $\gamma^a$ satisfy the Dirac algebra

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}.$$  \hspace{1cm} (A.8)

The $\gamma^5$ is defined through the relation

$$\gamma^5 \equiv -\gamma^0 \gamma^1 \gamma^2 \gamma^3 = -\frac{1}{4!} \epsilon_{abcd} \gamma^a \gamma^b \gamma^c \gamma^d.$$  \hspace{1cm} (A.9)

Our (completely real) parametrization is given by

$$\gamma^0 = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} -\sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix},$$  \hspace{1cm} (A.10)

$$\gamma^2 = \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix},$$

and

$$\gamma^5 = \begin{pmatrix} 0 & i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix}.$$  \hspace{1cm} (A.11)

The charge conjugation matrix is $C = \gamma^0$ and

$$\gamma^0 \gamma^a \gamma^0 = \gamma^a \dagger = t \gamma^a.$$  \hspace{1cm} (A.12)

The seven–dimensional gamma matrices are $\tau^a$, and satisfy

$$\{\tau^a, \tau^b\} = -2\delta^{ab}.$$  \hspace{1cm} (A.13)

The killing spinors on the sphere are $\eta^A_\alpha$ (where $A = 1, \ldots, 8$ is the index of the $8_S$ of $SO(8)$), they are completely real (i.e. $\bar{\eta} = t\eta$) and satisfy the identity

$$\eta^A_\alpha \eta^B_\beta \eta^C_\delta \eta^D_\gamma \tau^a \tau^b \tau^c \tau^d = \frac{1}{2} \eta^A_\alpha \eta^B_\beta \eta^C_\delta \eta^D_\gamma \tau^a \tau^b \tau^c \tau^d - \frac{1}{4} \delta^{ab} \delta^{cd}.$$  \hspace{1cm} (A.14)

Some useful $\tau$ identities are:

$$\eta_A \tau^{\hat{a} \hat{b}} \eta_B \eta_C \eta_D \tau^{\hat{c} \hat{d}} \eta_A \eta^B \eta^C \eta^D = 4 \eta_A \tau^{\hat{a} \hat{b}} \eta_B \eta^C \eta^D \eta_A \eta^B \eta^C \eta^D - \frac{1}{8} \eta_A \eta_C \eta_D \eta_B \eta^A \eta^B \eta^C \eta^D.$$  \hspace{1cm} (A.15)

$$(\eta_A \delta_{BC}) \eta^A \eta^B = \frac{1}{16} \eta^a \eta^b \eta^c \eta^d \eta^e \eta^f \eta^g \eta^h,$$  \hspace{1cm} (A.16)

$$\eta_A \tau^{\hat{a}} \eta_B \eta_C \tau^{\hat{b}} \eta_D \tau^{\hat{c}} \eta^A \eta^B \eta^C \eta^D = \frac{1}{2} \eta_A \tau^{\hat{a}} \eta_B \eta_C \tau^{\hat{b}} \eta_D \eta^A \eta^B \eta^C \eta^D - 4 \eta_A \tau^{\hat{a}} \eta_B \eta^A \eta^B \eta^C \eta^D.$$  \hspace{1cm} (A.17)

The three–dimensional gamma matrices are

$$\hat{\sigma}^I \equiv \{i\sigma^2, \sigma^3, -\sigma^1\},$$  \hspace{1cm} (A.18)

and satisfy

$$t \hat{\sigma}^I = \hat{\sigma}^0 \hat{\sigma}^I \hat{\sigma}^0.$$  \hspace{1cm} (A.19)

We also write here two identities useful to verify the $\kappa$–symmetry of the action:

$$\bar{\Gamma} \Gamma_{ij} \epsilon^{ijk} = 2 \sqrt{-g} k^{jk} \Gamma_j,$$  \hspace{1cm} (A.20)

$$\bar{\Gamma} \Gamma_i = \frac{1}{2} \frac{\epsilon^{ijk}}{\sqrt{-g}} \Gamma_{jk} g_{il}.$$  \hspace{1cm} (A.21)
A.1: The membrane action and normal coordinates

To expand our membrane action around the classical solution (5.1) we make use of the normal coordinates [16, 47].

Though we do not enter the details, we give here some useful formulae as reference. We consider the normal coordinate expansion

\[ X^M = x^M + \alpha^I \phi^M (x) + \mathcal{O}(\alpha^3), \]  

where \( x^M \) is the classical value of \( X^M \) and \( \phi^M \) is the normal coordinate (quantum fluctuation). Its derivative with respect to the world-volume indices \( \xi^I \) is

\[ \partial_I X^M = \partial_I x^M + \alpha^I \nabla_I \phi^M - \frac{1}{3} \alpha^3 \partial_I x^S R^M_{\ P SQ} \phi^P \phi^Q + \mathcal{O}(\alpha^2). \]

From this one can derive

\[
\begin{align*}
g_{MN}(X^M) &= g_{MN}(x) - \frac{1}{3} \alpha^3 R_{M S I N J} \phi^S \phi^J + \mathcal{O}(\alpha^2), \\
B_{MNP}(X^M) &= B_{MNP}(x) + \alpha^2 \nabla^S B_{MNP}(x) \\
&\quad + \frac{1}{2} \alpha^3 \left( \nabla_{(S} \phi^{T J)} B_{MNP}(x) + R_{M S I N J}^{T} B_{N P T}(x) \right) \phi^S \phi^J + \mathcal{O}(\alpha^2),
\end{align*}
\]

and using these we find for the induced metric on the membrane world volume

\[
\begin{align*}
h_{IJ}(X) &= h_{IJ}(x) + 2 \alpha^2 \partial_I x^M \nabla_J \phi^N g_{MN} \\
&\quad + \alpha^3 \left( g_{MN} \nabla_I \phi^N \nabla_J \phi^N - R_{M P N Q} \partial_I x^M \partial_J x^N \phi^P \phi^Q \right) + \mathcal{O}(\alpha^2).
\end{align*}
\]

and for the inverse

\[
\begin{align*}
h^{IJ}(X) &= h^{IJ}(x) - 2 \alpha^2 h^{IK} h^{JL} \partial_{(K} x^M \nabla_{L)} \phi^N g_{MN} + \mathcal{O}(\alpha^3).
\end{align*}
\]

From these one can also recover the following expression for the determinant of the metric:

\[
\begin{align*}
\sqrt{-h}(X) &= \sqrt{-h(x)} \\
&\quad + \alpha^2 \sqrt{-h(x)} h^{IJ} g_{MN} \partial_I x^M \nabla_J \phi^N \\
&\quad + \frac{1}{2} \alpha^3 \sqrt{-h(x)} \left[ h^{IJ} \partial_{S I} \phi^S \partial_{J} \phi^S - h^{IJ} R_{M S I N J} \partial_I x^M \partial_J x^N \phi^S \phi^S \right. \\
&\quad + \left. \nabla_I \phi^S \nabla_J \phi^S g_{S I N J} \left( \partial_I x^M \partial_J x^N - \partial^J x^M \partial^I x^N - h^{IJ} \partial_K x^M \partial_K x^N \right) \right] \\
&\quad + \mathcal{O}(\alpha^2).
\end{align*}
\]

Filling these formulae in the action for the membrane,

\[
S = \frac{2}{\alpha^3} \int \sqrt{-h} d^3 \xi + \frac{4! 4!}{\alpha^3} \int B_{MNP}(X) \partial_I X^M \partial_J X^N \partial_K X^P \epsilon^{IJK} d^3 \xi,
\]

and expanding it in powers of \( \alpha' \),

\[
\mathcal{L} = \sum_{n=0}^{\infty} \alpha'^{\frac{n-2}{2}} \mathcal{L}_n,
\]

where

\[
\mathcal{L}_n = \sum_{\alpha=0}^{n} \binom{n}{\alpha} \mathcal{L}_{\alpha}^{(n-\alpha)}.
\]

27
we get

\[
\begin{align*}
\mathcal{L}_0 &= 2\sqrt{-h} + 4!\phi^2 R_{\phi \phi}^M \partial_I x^M \partial_J x^N \partial_K x^P \epsilon^{IJK} , \\
\mathcal{L}_1 &= 2\sqrt{-h} h^{IJ} g_{MN} \partial_I x^M \nabla_J \phi^N + 4!\phi^S H^{SMNP} \partial_I x^M \partial_J x^N \partial_K x^P \epsilon^{IJK} , \\
\mathcal{L}_2 &= \sqrt{-h} h^{IJ} g_{MN} \nabla_I \phi^M \nabla_J \phi^N - \sqrt{-h} h^{IJ} R_{MPNQ} \partial_I x^M \partial_J x^N \phi^P \phi^Q \\
&\quad + \sqrt{-h} \nabla_I \phi^P \nabla_J \phi^Q g_{MP} g_{NQ} (\partial^I x^M \partial^J x^N - \partial^I x^M \partial^J x^N - h^{IJ} \partial_K x^M \partial_K x^N) \\
&\quad + 4!\phi^S \nabla_I \phi^M H^{SMNP} \partial_I x^N \partial_K x^P \epsilon^{IJK} \\
&\quad + 4!\phi^2 R_{\phi \phi}^T Q_{[M} B_{NP]T} \phi^Q \phi^S \partial_I x^M \partial_J x^N \partial_K x^P \epsilon^{IJK} ,
\end{align*}
\]
(A.30)

which are the formulae used in the text to derive (5.6) and (5.7).

Appendix B: The boundary of the universe

As we have already said in section two, the four–manifold we consider as universe is \( U = \text{AdS}_4/\mathbb{Z}_2 \). It can be partly covered by the chart (B.4), diffeomorphic to the upper half \( \{ \rho > 0 \} \) of \( \mathbb{R}^4 \), equipped with the metric:

\[
ds^2 = \frac{d\rho^2}{\rho^2} + \rho^2 \eta_{IJ} dx^I dx^J , \quad \eta_{IJ} = \text{diag}(-1,1,1) ,
\]
(B.1)

We want now to discuss the concept of boundary of our universe. The first thing to remark is that, strictly speaking, a topological space has no boundary for its own topology (being at the same time an open and a closed set). To provide \( U \) with a boundary, first of all we have to immerse it in a “bigger” topological space, \( U' \supset U \), and then we have to determine the boundary \( \partial U \) with respect to the surrounding topology. In this sense, the choice of a boundary is somehow arbitrary, being arbitrary the choice of \( U' \).

The usual choice of boundary for an Anti–de Sitter space is made in the following way. First, we consider \( \text{AdS}_n \) as a hyperboloid in \( \mathbb{R}^{n+1} \) (as in section two). Secondly, we compactify \( \mathbb{R}^{n+1} \) with a hypersphere \( S^n \) given by the limiting points of the straight lines through the origin. Now we have a hyperboloid immersed in a new topological space, homeomorphic to the \((n+1)–\)dimensional disk, \( B^{n+1} \). Finally, we call \( \partial \text{AdS}_n \) the boundary of the hyperboloid as a subset of this \( B^{n+1} \). Its topology is the same of all the intersections \( \text{AdS}_n \cap S^n(R) \) between the hyperboloid of \( \mathbb{R}^{n+1} \) and the spheres of sufficiently large radius, \( R: S^1 \times \mathbb{S}^{n-2} \).

In our case, the actual space–time is the quotient \( \text{AdS}_4/\mathbb{Z}_2 \), so its boundary, even called “the end of the world”, has the topology \( (S^1 \times \mathbb{S}^2)/\mathbb{Z}_2 \), where the action of \( \mathbb{Z}_2 \) identifies points of the form \((\phi, p)\) and \((\phi + \pi, p')\), with \( p, p' \in \mathbb{S}^2 \) diametrically opposed points of the sphere.

The action of the isometry group of \( \text{AdS}_4/(\mathbb{Z}_2) \), \( SO(2,3) \), can be naturally extended to this boundary, which in turn can be provided with a metric chosen in a set of conformally equivalent ones. On \( (S^1 \times \mathbb{S}^2)/(\mathbb{Z}_2) \), \( SO(2,3) \) acts as the conformal group for this set of metrics. In this sense the “end of the world” constitutes the support for a conformal field theory.

Now, in “physical” coordinates \( \rho, t, w, x \), part of the boundary is given by the hyperplane \( \mathbb{R}^3(t,w,x) \) at \( \rho \rightarrow \infty \). This subspace inherits from the bulk metric (B.1) a set of
conformally minkowskian metrics, $ds^2 = \phi^2(-dt^2 + dw^2 + dx^2)$ which can be extended to the conformally invariant compactification of the minkowskian $\mathbb{R}^3$.

Finally, we want to remark that the properties of the "horizon", i.e. the site where the membrane lies, heavily depend on the choice of compactification. In "physical" coordinates $\rho, t, w, x$, the natural compactification of the space comprises the set $\rho \rightarrow 0$ (i.e. the set of limiting points of geodesics of the form $(t, w, x) = \text{const}, \rho \rightarrow 0$), which has the topology of $\mathbb{R}^3$. On the other hand, for the topology of the compactification previously described, all such geodesics converge to the same point of the boundary, i.e. the membrane shrinks to a single point.

**Appendix C: The supersolvable algebra**

Consider the superalgebra $Osp(N|4)$. Its bosonic subalgebra is $SO(2, 3) \times SO(N)$ and it is generated by the momenta $P_a$, the Lorentz generators $M_{ab}$ and the $SO(N)$ generators $T_{AB}$. In this Appendix $A, B = 1, \ldots, N$. The fermionic generators of the aforementioned algebra are $N$ Majorana spinors $Q^A_{\alpha}$ in 4 dimensions, where $C$ is the charge conjugation matrix in four dimensions that, in the representation of the Clifford algebra defined in Appendix A, coincides with $\gamma^0$.

The superspace is defined as the following quotient:

$$ AdS^{(N|4)} = \frac{Osp(N|4)}{SO(1, 3) \otimes SO(N)} $$

and it is spanned by the 4 coordinates of the $AdS_4$ manifold and by the $N$ Majorana spinors ($4N$ real components) parametrizing the generators $Q^A_{\alpha}$ of the superalgebra. It has been shown that the $AdS_4$ manifold admits a solvable description in terms of a 4 dimensional solvable Lie algebra $Solv$. The problem which will be dealt with in the present appendix is to find a supersolvable description of the superspace $AdS^{(N|4)}$, that is a decomposition of $Osp(N|4)$ of the following form:

$$ Osp(N|4) = (SO(1, 3) \otimes SO(N) \otimes \mathcal{Q}) \oplus SSolv, $$

where $\mathcal{Q}$ is a subset of the fermionic generators to be defined in the following. By supersolvable algebra we mean a superalgebra for which the $k^{th}$ Lie derivative (defined in terms of the supercommutator) vanishes for a finite $k$.

As pointed out in section 4, the only price which one has to pay in order to define a supersolvable algebra $SSolv$ as in eq. (C.2), is to perform a suitable projection of the fermionic generators:

$$ Q^A_+ = \mathcal{P}_- \cdot Q^A, $$

$$ Q^A_- = \mathcal{P}_+ \cdot Q^A, $$

$$ \mathcal{P}^2_\pm = \mathcal{P}_\pm; \mathcal{P}_+ \cdot \mathcal{P}_- = 0. $$

(differently from the notation used in section 4, in the present appendix the subscript "±" on fermionic generators denotes the action of the the projectors $\mathcal{P}_\pm$, i.e. generators lying in the upper or lower part of the diagram in Figure (1), unless the contrary is specified. Therefore referring to Figure (1), the fermionic generators in the upper part of the diagram are denoted by $Q^A_+$ while the generators $S^A$ in the lower part by $Q^A_-$. )
Indeed, as already discussed in section 4, the main idea underlying the construction rules of the supersolvable algebra generating $AdS^{(N|4)}$ as well as the solvable algebra generating $AdS$ is that of grading (figure (1)), i.e. the Cartan generator contained in the coset of $AdS_4$ defines a partition of the isometry generators into eigenspaces corresponding to positive, negative or null eigenvalues ($g_{(\pm 1)}$, $sg_{(\pm 1/2)}$, $sg_{(0)}$) and the structure of the solvable and supersolvable algebras ($Solv$ and $SSolv$) is the following:

\[ g = SO(2,3) \rightarrow g_{(-1)} \oplus g_{(0)} \oplus g_{(+1)}, \]
\[ Solv = \{\mathcal{C}\} \oplus g_{(-1)}, \]
\[ sg = Osp(N|4) \rightarrow g_{(-1)} \oplus sg_{(0)} \oplus g_{(+1)} \oplus sg_{(-1/2)} \oplus sg_{(1/2)}, \tag{C.4} \]
\[ sg_{(0)} = g_{(0)} \oplus SO(N), \]
\[ SSolv = \{\mathcal{C}\} \oplus g_{(-1)} \oplus sg_{(-1/2)}, \]

where $sg_{(\pm 1/2)}$ represents the grading induced by the Cartan generator on the fermionic isometries and the eigenspace $sg_{(+1/2)}$ not entering the construction of $SSolv$ is the space $\mathcal{Q} = \{Q^+_4\}$ in eq. (C.2) and generates the special conformal transformations. Moreover these generators on the chosen solution of the world volume theory, generate the local k-supersymmetry transformations ($(1 + \bar{\Gamma})/2 = \mathcal{P}_+$).

Let us now enter the details of the calculations. As far as the $SO(2,3)$ algebra is concerned let us use the following convention for the commutation relations between its generators $M_{IJ}$, $I, J = 0, 1, 2, 3, 5$:

\[ [M^{IJ}, M^{KL}] = -(\eta^{IL}M^{JK} + \eta^{IK}M^{JL} - \eta^{IK}M^{IL} - \eta^{IL}M^{JK}), \]
\[ M^{IJ} = -M^{JI}, \; \eta = \text{diag}\{-, +, +, +, -\}. \tag{C.5} \]

Let the momenta $P^a$ ($a, b, c = 0, \ldots, 3$) be defined as $P^a = M^{a5}$ and the Lorentz generator be $M^{ab}$.

The solvable algebra generating $AdS_4$ has the following form:

\[ Solv = \{\mathcal{C}, g_{(-1)}\}, \]
\[ \mathcal{C} = M^{25}; \; g_{(-1)} = \{T, W, X\}, \]
\[ T = M^{05} - M^{02} = M^{0-}, \]
\[ W = M^{15} - M^{12} = M^{1-}, \]
\[ X = M^{35} - M^{32} = M^{3-}, \tag{C.6} \]

where the value “-” for the index of the $SO(2,3)$ generators refers to the light–cone notation for the time–like direction 5 and the space–like direction 2.

Using the representation of the Clifford algebra defined in Appendix A, the spinorial representation of the momenta and of the Lorenz generators $\{P^a, M^{ab}\}$ consistent with the relations (C.3) is the following:

\[ M_{f}^{ab} = -\frac{1}{2} \gamma^{ab} = -\frac{1}{4} [\gamma^a, \gamma^b], \]
\[ P_f^a = \frac{1}{2} \gamma^5 \gamma^a. \tag{C.7} \]

With the adopted conventions, the anti–commutator of the supersymmetry generators is given by:

\[ \{Q^A_{\alpha}, Q^B_{\beta}\} = \frac{\delta^{AB}}{2} (\gamma^{ab} C \gamma^5)_{\alpha\beta} M_{ab} + \delta^{AB} (\gamma^5 \gamma^a C \gamma^5)_{\alpha\beta} P_a + i (C \gamma^5)_{\alpha\beta} T^{AB}, \tag{C.8} \]
From (C.8) it is clear that the presence of the $SO(N)$ generators $T^{AB}$ on the right hand side of the anticommutator between fermionic generators is an obstacle for the definition of a solvable superalgebra. As it will be shown that term disappears once the projection on $sg_{(-1/2)}$ is performed on the fermionic generators.

Now let us define the grading for the fermionic generators with respect to $C$. Since the adjoint action of $C$ on $Q^a$ is represented by the matrix $C^f = P^2_f = \gamma^5 \gamma^2 / 2$ acting on the fermions, the projector on the spaces $sg_{(\pm 1/2)}$ is given by:

$$P_\pm = \frac{1}{2} (\mathbb{1} \pm \gamma^5 \gamma^2),$$

$$sg_{(\pm 1/2)} = \{ Q_\pm^A \} = \{ P_\pm Q^A \}. \quad (C.9)$$

It is straightforward to verify that such a projection is compatible with the Majorana condition. The solvable superalgebra has then the following content:

$$SSolv = Solv \oplus sg_{(-1/2)} = \{ C, T, W, X \} \oplus \{ Q^A \}. \quad (C.10)$$

The fact that it closes follows from the rules (C.8) and from the grading of the generators: the anticommutator of two $Q^A$ has charge $-1$ with respect to $C$ and therefore is expressed only in terms of the $T, W, X$ generators and so on. Let us define the fermionic eigenmatrices $T^f_\pm, W^f_\pm, X^f_\pm$ of $C^f = 1/2 \gamma^5 \gamma^2 = P^2_f$:

$$[C^f, T^f_\pm] = \pm T^f_\pm, \quad [C^f, W^f_\pm] = \pm W^f_\pm, \quad [C^f, X^f_\pm] = \pm X^f_\pm,$$

$$T^f_\pm = P^0_f \pm M^{02}_f, \quad W^f_\pm = P^1_f \pm M^{12}_f, \quad X^f_\pm = P^3_f \pm M^{32}_f. \quad (C.11)$$

After some gamma-algebra one finds for the anticommutator of two $Q_-$ the following expression:

$$\{ Q_-, Q_- \} = -T^f_\pm C^5 T + W^f_\pm C^5 W + X^f_\pm C^5 X, \quad (C.12)$$

where we used the property: $P_- \{ T^f_\pm, W^f_\pm, X^f_\pm \} C = \{ T^f_\pm, W^f_\pm, X^f_\pm \} C$.

As far as the anti–commutation relations between the fermionic generators and the $AdS$ isometry generators are concerned, taking into account (C.11) they may be rewritten in the following way,

$$[T, Q^A] = T^f_\pm Q^A, \quad [W, Q^A] = W^f_\pm Q^A, \quad [X, Q^A] = X^f_\pm Q^A. \quad (C.13)$$

Projecting eqs. (C.13) on $sg_{(-1/2)}$, using the property $P_- \cdot T^f_\pm = P_- \cdot W^f_\pm = P_- \cdot X^f_\pm = 0$ (since the generators $\{ T^f_\pm, W^f_\pm, X^f_\pm \}$ shift the eigenvalue of $C^f$ by $+1$) one obtains:

$$[T, Q^A] = 0, \quad [W, Q^A] = 0, \quad [X, Q^A] = 0. \quad (C.14)$$

Defining for simplicity $Z^\mu = \{ T, W, X \}$; $Z^\mu_f = \{ T^f_\pm, W^f_\pm, X^f_\pm \}$, it is now possible to write the algebraic structure of $SSolv$:

$$[C, Z^\mu] = -Z^\mu, \quad (C.15)$$

$$[Z^\mu, Z^\nu] = 0,$$

$$[Z^\mu, Q^A_-] = 0,$$

$$[C, Q^A_-] = -\frac{Q^A_-}{2},$$

$$\{ Q^A_- , Q^B_- \} = \delta^{AB} \frac{h^\mu}{2} (Z^\mu_C \gamma^5) Z^\nu \quad h = diag\{-,+,+\}. \quad (C.16)$$
Using (A.10)–(A.11) conventions, the spinorial representation \( C^f \) of the Cartan generator has the form:

\[
C^f = \frac{1}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
\]  

(C.16)

and therefore the projectors are:

\[
P_+ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_- = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]  

(C.17)

In the spinorial representation of \( Solv \), besides \( C^f \) there are the maximal abelian generators \( T^f_- , W^f_- , X^f_- \) appearing in the coset representative through the following combinations:

\[
\sigma_\perp = \frac{1}{\sqrt{2}} X^f_- = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{2} \sigma^3 \\ 0 & 0 \end{pmatrix},
\]

\[
\sigma_+ = \frac{1}{2} (-T^f_+ + W^f_-) = \frac{1}{2} \begin{pmatrix} 0 & -1 + \sigma^1 \\ 0 & 0 \end{pmatrix},
\]

\[
\sigma_- = \frac{1}{2} (-T^f_- - W^f_-) = \frac{1}{2} \begin{pmatrix} 0 & -1 - \sigma^1 \\ 0 & 0 \end{pmatrix}.
\]  

(C.18)

where the “±” label on the operators \( \sigma \) refers to the light–cone index related to the coordinates \( t \) and \( w \).

Promoting all the bosonic matrices to graded ones preserving (4.26), the bosonic factor of the group representative is

\[
L_B = \text{Exp}(\sqrt{2}x\sigma_\perp + t(\sigma_+ + \sigma_-) + w(\sigma_+ - \sigma_-)) e^{aC^f},
\]  

(C.19)

and the coset representative of the superspace in the supersolvable parametrization has the form \( L = L_F L_B \) where

\[
L_F = \exp \left( \theta^A Q_1^A + \theta^A Q_2^A \right).
\]  

(C.20)

The left–invariant one–form is therefore given by

\[
\Omega = L^{-1} dL = \Omega_B + L^{-1}_B \Omega_F L_B.
\]  

(C.21)

From the left invariant form (C.21), we can finally obtain the vielbeins through the following projections:

\[
E^0 = \frac{1}{2} \text{Tr} \left( \gamma^5 \gamma^0 \Omega \right),
\]

\[
E^i = \frac{1}{2} \text{Tr} \left( \gamma^5 \gamma^i \Omega \right).
\]  

(C.22)

We have then parametrized our space with (2.4) metric. To go back to the horospherical coordinates (2.3), we have first to rescale all the vielbeins by a \( R/2 \) factor and then to reabsorb this factor in the \( \rho \) definition. The final parametrization is then given by (4.29)–(4.32) reported in section (4).
References

[1] J. Maldacena, preprint hep-th/9711200.

[2] P. Claus, R. Kallosh, and A. Van Proeyen, Nucl. Phys. B518, (1998) 117.

[3] P. Claus, R. Kallosh, J. Kumar, P. Townsend and A. Van Proeyen, preprint hep-th/9801206.

[4] M.J. Duff, Class. Quant. Grav. 5, (1988) 189;
M. Blencowe and M.J. Duff, Nucl. Phys. B310, (1988) 587.

[5] M.P. Blencowe and M.J. Duff, Phys. Lett. B203, (1988) 229.

[6] E. Bergshoeff, E. Sezgin and Y. Tanii, trieste preprint, IC/88/5, (1988).
E. Bergshoeff, A. Salam, E. Sezgin and Y. Tanii, Nucl. Phys. B305, (1988) 497.

[7] H. Nicolai, E. Sezgin and Y. Tanii, Nucl. Phys. B305 [FS23], (1988) 483.

[8] G.W. Gibbons and P.K. Townsend, Phys. Rev. Lett. 71, (1993) 3754.

[9] G. Gibbons, Nucl. Phys. B207, (1982) 337;
R. Kallosh and A. Peet, Phys. Rev. B46, (1992) 5223;
S. Ferrara, G. Gibbons and R. Kallosh, Nucl. Phys. B500, (1997) 75;
A. Chamseddine, S. Ferrara, G.W. Gibbons and R. Kallosh, Phys. Rev. D55, (1997) 3647.

[10] E. Witten, preprint hep-th/9802150.

[11] V. P. Nair and S. Randjbar-Daemi, preprint hep-th/9802187.

[12] L. Castellani, A. Ceresole, R. D’Auria, S. Ferrara, P. Frè and M. Trigiante, preprint hep-th/9803039.

[13] B. de Wit and H. Nicolai, Nucl. Phys. B208, (1982) 323;
B. Biran, F. Englert, B. de Wit and H. Nicolai, Phys. Lett. B124, (1983) 45;
B. de Wit, H. Nicolai and N. P. Warner, Nucl. Phys. B255, (1985) 29.

[14] P.G.O. Freund and M.A. Rubin, Phys. Lett B97, (1980) 233.

[15] M.J. Duff, B.E.W. Nilsson and C.N. Pope, Phys. Rep. 130, (1986) 1.

[16] F. Englert, Phys. Lett 119B, (1982) 339

[17] L. Castellani, R. D’Auria and P. Frè, Nucl. Phys. 239, (1984) 610.

[18] L. Castellani, L. J. Romans and N. P. Warner, Nucl. Phys. B241, (1984) 429.

[19] B. de Wit and H. Nicolai, Nucl. Phys. B231, (1984) 506;
B. de Wit and H. Nicolai, Nucl. Phys. B281, (1987) 211.

[20] M.J.Duff, H. Lü, C.N. Pope and E. Sezgin, Phys. Lett. B371, (1996) 206.
[21] L. Castellani, R. D’Auria and P. Fré, Supergravity and Superstring Theory: a geometric perspective. World Scientific 1990.

[22] R. D’Auria and P. Fré, *Ann. of Phys.* **157**, (1984) 1.

[23] R. D’Auria and P. Fré, *Ann. of Phys* **162**, (1985) 372.

[24] M. Günaydin and N. P. Warner, *Nucl. Phys.* **B272**, (1986) 99.

[25] A. Ceresole, P. Fré and H. Nicolai, *Class. Quant. Grav.* **2**, (1985) 133.

[26] P.A.M. Dirac, *J. Math. Phys.* **4**, (1963) 901.

[27] C. Fronsdal, *Phys. Rev.* **D26**, (1982) 1988.

[28] M. Flato and C. Fronsdal, *J. Math. Phys.* **22**, (1981) 1100.

[29] E. Angelopoulos, M. Flato, C. Fronsdal and D. Steinheimer, *Phys. Rev.* **D23**, (1981) 1278.

[30] E. Bergshoeff, A. Salam, E. Sezgin and Y. Tanii, *Phys. Lett.* **B205**, (1987) 237.

[31] E. Bergshoeff, M.J. Duff, C.N. Pope and E. Sezgin, *Phys. Lett.* **B199**, (1987) 69.

[32] E. Bergshoeff, M.J. Duff, C.N. Pope and E. Sezgin, *Phys. Lett.* **B224**, (1989) 71.

[33] R. Kallosh and A. Van Proeyen, preprint [hep-th/9804099].

[34] R. Kallosh, J. Kumar and A. Rajamaran, *Phys. Rev.* **D57**, (1998) 6452.

[35] R. Kallosh and A. Rajamaran, preprint [hep-th/9805041].

[36] E. Bergshoeff, E. Sezgin and P.K. Townsend, *Ann. of Phys.* **185**, (1988) 330.

[37] I. Bandos, D. Sorokin and D. V. Volkov, *Phys. Lett.* **B352**, (1995) 269.

[38] I. Bandos, D. Sorokin and M. Tonin, *Nucl. Phys.* **B497**, (1997) 275; I. Bandos, P. Pasti, D. Sorokin and M. Tonin, preprint [hep-th/9705064].

[39] R. D’Auria and P. Fré, *Nucl. Phys.* **B201**, (1982) 101.

[40] P. Fré, *Class. Quant.Grav.* **1**, (1984) L 81.

[41] A. Candiello and K. Lechner, *Nucl. Phys.* **B412**, (1994) 479.

[42] P. Pasti, D. Sorokin and M. Tonin, preprint [hep-th/9809213].

[43] E. Cremmer and B. Julia, *Nucl. Phys.* **B159**, (1979) 141.

[44] R. D’Auria and P. Fré, *Phys. Lett.* **B121**, (1983) 141.

[45] M. J. Duff, B. E. W. Nilsson and C. N. Pope, *Phys. Rep.* **130**, (1986) 1.

[46] L. Alvarez–Gaumé, D. Freedman and S. Mukhil, *Ann. of Phys.* **134**, (1981) 85.
[47] L. Eisenhart, “Riemannian Geometry”, Princeton Univ. Press, Princeton, N. J., 1965.

[48] S. Ferrara and A. Zaffaroni, preprint hep-th/9807090.

[49] R. Kallosh, Phys. Rev. D57, (1998) 1063 and preprint hep-th/9709202.