ISOPERIMETRIC INEQUALITY ON ASYMPTOTICALLY FLAT MANIFOLDS WITH NONNEGATIVE SCALAR CURVATURE

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Abstract. In this note, we consider the isoperimetric inequality on asymptotically flat manifolds with nonnegative scalar curvature, and improve it by using Hawking mass (see Theorem 1.2). We also obtain a rigidity result when equality holds for the classical isoperimetric inequality on an asymptotically flat manifold with nonnegative scalar curvature (see Theorem 1.3).

1. Introduction

The isoperimetric inequality and isoperimetric surfaces have a very long history and many important applications in mathematics, for instance, see [1], [2] etc, among other things, Huisken observed ADM mass of an asymptotically flat manifold (see Definition 1.1 below) appeared in the expansion of isoperimetric ratio when the volume is large enough, see [6] and [3] (for the case of coordinates sphere, see [4]). Inspired by these facts, it is natural to ask if there is any relationship between the isoperimetric inequality and quasi-local mass for any fixed enclosed volume. In this short note, we are able to use Hawking mass to improve the isoperimetric inequality in some cases. In order to present our result, we need some notations.

Definition 1.1. A complete and connected three manifold \((M^3, g)\) is said to be asymptotically flat (AF) (with one end) if there is a positive constant \(C > 0\) and compact subset \(K\) such that \(M \setminus K\) is diffeomorphic to \(\mathbb{R}^3 \setminus B_R(0)\) for some \(R > 0\) and in the standard coordinates in \(\mathbb{R}^3\), the metric \(g\) satisfies:

\[
g_{ij} = \delta_{ij} + \sigma_{ij}
\]

with

\[
|\sigma_{ij}| + r|\partial \sigma_{ij}| + r^2|\partial^2 \sigma_{ij}| = Cr^{-1},
\]

where \(r\) and \(\partial\) denote the Euclidean distance and standard derivative operator on \(\mathbb{R}^3\) respectively, and \(M \setminus K\) is called end of \(M\).
Our main idea is to use the weak solution of inverse mean curvature \((4)\) in an asymptotically flat manifold \((M^3, g)\). Actually, for any \(x \in M\), it was proved that there is a weak solution \((G_t)_{t>\infty}\) of \((4)\) with initial condition \(x\) in \([5]\). One important property for this weak solution is that for each \(t \in \mathbb{R}\), \((G_t)\) has the least boundary area among all domains containing it, i.e. \((G_t)\) is a minimizing hull in \((M^3, g)\). Another interesting property is that the Hawking mass of \(K_t = \partial G_t\) which is defined as the following

\[
m_H(t) = \frac{(\text{Area}(K_t))^\frac{1}{2}}{(16\pi)^\frac{3}{2}} (16\pi - \int_{K_t} H^2),
\]

is nondecreasing on \(t\); here, \(H\) is the mean curvature of \(K_t(x) = \partial G_t\) with respect to outward unit normal vector. By using this quantity, we are able to estimate the area of \(K_t\) in terms of the volume of \(G_t\), see \((15)\) below; hence, we obtain Theorem 1.2. To do that, we need to parametrize \(t\) by \(v\), which is the volume of \(G_t\), and it turns out that this function \(t(v)\) is Lipschitz; for details, see Lemma 3.4 below. Let \(m(v) \triangleq m_H(t(v))\), \(B(v) \triangleq \text{Area}(K_t(v))\), and

\[
A(v) \triangleq \inf \{\mathcal{H}^2(\partial^* \Omega) : \Omega \subset M \text{ is a Borel set with finite perimeter, and } \mathcal{L}^3(\Omega) = v\}.
\]

Then our theorem can be stated as

**Theorem 1.2.** Suppose \((M^3, g)\) is a simply connected and asymptotically flat (AF) manifold with nonnegative scalar curvature if \(M\) has a single end and admits no minimal \(S^2\), then for any \(v > 0\), any \(x \in M\),

\[
A(v) \leq (36\pi)^\frac{1}{3} \left( \int_0^v (1 - (16\pi)^{\frac{1}{2}} B^{-\frac{1}{2}}(v)m(v))^{\frac{1}{2}} \right)^{\frac{3}{2}}.
\]

When scalar curvature of \(M\) is nonnegative, and \(M\) satisfies some topology conditions, then \(m(v) \geq 0\); we see that in this case \(A(v) \leq (36\pi)^\frac{1}{2} v^{\frac{3}{2}}\). Comparing this with the Euclidean case in which \(m(v) = 0\), we observe the following heuristic phenomenon: to enclose the same volume, isoperimetric surfaces in a manifold with bigger mass have smaller area. We believe such a phenomenon can also be observed in the case asymptotically hyperbolic manifolds, and we will discuss this problem in a future paper. With these facts in mind, it is natural to ask what happens if there is \(v_0 > 0\) with \(A(v_0) = (36\pi)^\frac{1}{2} v_0^{\frac{3}{2}}\)? Our next theorem give an answer to this question. Namely,
**Theorem 1.3.** Suppose $(M^3, g)$ is a simply connected and asymptotically flat (AF) manifold with nonnegative scalar curvature and admits no minimal $S^2$ and $M$ has a single end, then there is $v_0 > 0$ with

$$A(v_0) = (36\pi)^{\frac{1}{3}}v_0^{\frac{8}{3}}$$

if and only if $(M^3, g)$ is isometric to $\mathbb{R}^3$.

**Remark 1.4.** We wonder assumption of nonexistence minimal $S^2$ on $(M^3, g)$ is necessary in above two theorems. However, we use this to handle the difficult which is from jump of inverse mean curvature flow.

The basic outline of the paper is as follows. In Section 2, we introduce some notation and basic facts of weak solutions of inverse mean curvature flow from [5]; in Section 3, we prove the main results.

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## 2. Preliminary

In this section, we introduce some notations and present some facts that will be needed in the proof of Theorem 1.2 and Theorem 1.3, most of them are from [5]. As in [5], a classical solution of the inverse mean curvature flow(IMCF) in $(M^3, g)$ is a smooth family of embedded hypersurfaces $N_t = F(N, t)$ satisfying the following evolution equation

$$\frac{\partial F}{\partial t} = H^{-1}\nu, \quad 0 \leq t \leq T$$

where $H$ is the mean curvature of $N_t$ at $F(x, t)$ with respect to outward unit vector $\nu$ for any $x \in N$. Generally, the evolution equation [5] has no classical solution, in order to overcome this difficult, the level set arguments was established in [5], i.e. these evolving surfaces were given as the level-sets of a scalar function $u$ via $N_t = \partial \{x \in M : u(x) < t \}$, and $u$ satisfying the degenerate elliptic equation

$$\text{div}_M\left(\frac{\nabla u}{|\nabla u|}\right) = |\nabla u|$$

where the left hand side describes the mean curvature of level-sets and the right hand side yields the inverse speed.

By the definition of AF manifolds, for any $x \in M \setminus K$, we may consider $x$ as $(x^1, x^2, x^3)$, which is the the standard coordinates in $\mathbb{R}^3$. It was observed in [5] that $v(x) = C \log |x|$ is a weak subsolution of [5] on $M \setminus K$ (please see the precise definition of weak subsolution of [5] in P.365 in [5]), here
\[ |x| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}. \] With this weak subsolution one is able to prove the existence of the weak solution of (5) on \( M \) with any nonempty precompact smooth open set \( E_0 \) as initial condition (See weak Theorem 3.1 in [5]). The key idea is to use the following approximate equation which is called as elliptic regularization:

\[
\begin{cases}
E^\epsilon u^\epsilon \triangleq \text{div}(\frac{\nabla u^\epsilon}{\sqrt{|\nabla u^\epsilon|^2 + \epsilon^2}}) - \sqrt{|\nabla u^\epsilon|^2 + \epsilon^2} = 0, & \text{in } \Omega_L \\
u^\epsilon = 0, & \text{on } \partial E_0 \\
u^\epsilon = L - 2, & \text{on } \partial F_L
\end{cases}
\]

Here and in the sequel, \( F_L \triangleq \{v < L\} \), for any large \( L > 0 \), and \( \Omega_L \triangleq F_L \setminus \bar{E}_0 \), let \( W^\epsilon(x, z) \triangleq u^\epsilon(x) - \varepsilon z \) be a function on \( \Omega_L \times \mathbb{R} \), then we have

\[
\text{div}(\frac{\nabla W^\epsilon}{|\nabla W^\epsilon|}) = |\nabla W^\epsilon|,
\]

or equivalently, the level set \( N_t^\epsilon \triangleq \{(x, z) \in \Omega_L \times \mathbb{R} : W^\epsilon(x, z) = t\} \) is a slice of the inverse mean curvature flow in the domain \( \Omega_L \times \mathbb{R} \) for any \( t > 0 \), and actually it is the classical solution to (4). Due to the Approximate Existence Lemma 3.5 in [5], we know that (6) admits a classical solution.

Also, we have the following compactness lemma and its proof can be found in [5](P.398)

**Lemma 2.1.** Let \((M^3, g)\) be an AF manifold, and \(E_0\) be a precompact set of \( M\) with smooth boundary, then there are subsequences \( \epsilon_i \to 0, L_i \to \infty, N_i^\epsilon = N_i^\epsilon \) such that

\[
N_i^\epsilon \to \tilde{N}_t = N_t \times \mathbb{R}, \text{ locally in } C^1, \quad a.e \ t \geq 0
\]

where \( N_t = \partial E_t \) and \((E_t)_{t>0}\) is the unique weak solution of (4) with \( E_0 \) as the initial condition.

3. Proof of the main theorems

Let \( B_\mu(x) \) be any geodesic ball with radius \( \mu > 0 \) and center \( x \) in \((M, g)\), and let \( E_0 = B_\mu(x) \), we consider the following boundary problem

\[
\begin{cases}
E^\epsilon u^\epsilon \triangleq \text{div}(\frac{\nabla u^\epsilon}{\sqrt{|\nabla u^\epsilon|^2 + \epsilon^2}}) - \sqrt{|\nabla u^\epsilon|^2 + \epsilon^2} = 0, & \text{in } \Omega_L \\
u^\epsilon = 0, & \text{on } \partial E_0 \\
u^\epsilon = L - 2, & \text{on } \partial F_L
\end{cases}
\]

then due to Lemma 2.1 we know there are subsequences \( \epsilon_i \to 0, L_i \to \infty, N_i^\epsilon = N_i^\epsilon \) such that

\[
N_i^\epsilon \to \tilde{N}_t = N_t \times \mathbb{R}, \text{ locally in } C^1, \quad a.e \ t \geq 0
\]
where $N_t = \partial E_t$ and $(E_t)_{t>0}$ is the unique weak solution of (11) with the initial condition $E_0 = B_\mu(x)$. Also, by proof of Lemma 8.1 in [5], we know that, by some translations on $t$, $(E_t)$ converges locally in $C^1$ to the unique weak solution of (11) with $\{x\}$ as the initial condition, which is denoted by $(G_t)_{t>-\infty}$, and together with the Regularity Theorem 1.3 in [5], we see that $N_t = \partial E_t$ and $K_t = \partial G_t$ are $C^{1,\alpha}$- hypersurface of $(M, g)$, for some $0 < \alpha \leq \frac{1}{2}$.

For simplicity, as in the proof of Lemma 8.1 in [5], for each $\mu > 0$, we may take a suitable transformation on $t$, so that the weak solution $(E_t)$ for initial value problem (11) is defined on $[-T(\mu), \infty)$, here $T(\mu) \to \infty$ as $\mu$ approaches to zero, and $(E_t)_{-T(\mu) \leq t < \infty}$ locally converges to $(G_t)_{-\infty < t < \infty}$ which is the weak solution of (11) with single point $\{x\}$ as the initial condition. Let

$$V_t \triangleq \text{Vol}(\{(x, z) \in \Omega_L \times \mathbb{R} : W^\epsilon(x, z) < t, \ |z| \leq \frac{1}{2}\}).$$

Note that level set of $W^\epsilon$ is a classical solution to (11), we see that $V_t$ is a smooth function of $t$, and further more, we have

**Lemma 3.1.** Let $\chi_{\{|z| \leq \frac{1}{2}\}}(x, z)$ be the characteristic function of the domain $\mathbb{D} = \{(x, z) \in \Omega_L \times \mathbb{R} : |z| \leq \frac{1}{2}\}$, then

$$\frac{dV_t}{dt} = \int_{N^\epsilon_t} H^{-1}_{\epsilon} \chi_{\{|z| \leq \frac{1}{2}\}}(x, z) dS > 0,$$

here and in the sequel $H_{\epsilon}$ denotes the mean curvature of $N^\epsilon_t$ in $\mathbb{D}$ with respect to unit normal direction $\frac{\nabla W^\epsilon}{|\nabla W^\epsilon|}$.

**Proof.** Due to the Co-area formula, we see that

$$V_t = \int_{\mathbb{D}} \chi_{\{|z| \leq \frac{1}{2}\}}(x, z) \chi_{\{W^\epsilon < t\}}(x, z) dv$$

$$= \int_{-\infty}^{\infty} \int_{\{W^\epsilon = \sigma\}} \frac{\chi_{\{|z| \leq \frac{1}{2}\}}(x, z) \chi_{\{W^\epsilon < t\}}(x, z)}{|\nabla W^\epsilon|} dSd\sigma$$

$$= \int_{-\infty}^{t} \int_{\{W^\epsilon = \sigma\}} \frac{\chi_{\{|z| \leq \frac{1}{2}\}}(x, z)}{|\nabla W^\epsilon|} dSd\sigma$$

which implies

$$\frac{dV_t}{dt} = \int_{N^\epsilon_t} H^{-1}_{\epsilon} \chi_{\{|z| \leq \frac{1}{2}\}}(x, z) dS > 0,$$

Thus, we finish to prove the lemma. \hfill \Box

A direct conclusion of Lemma 3.1 is the following
Corollary 3.2. Let $W^\epsilon$ be a classical solution to \([4]\) on $\mathbb{D}$, $v = \text{Vol}(\{(x, z) \in \Omega_L \times \mathbb{R} : W^\epsilon(x, z) < t, |z| \leq \frac{1}{2}\})$, then $t$ is a smooth function of $v$, i.e. $t = t(v)$ and

$$
\frac{dt}{dv} = \left( \int_{N_t^\epsilon} H^{-1}_\epsilon \chi_{\{|z| \leq \frac{1}{2}\}}(x, z)dS \right)^{-1}
= \left( \int_{N_t^\epsilon \cap \{|z| \leq \frac{1}{2}\}} H^{-1}_\epsilon dS \right)^{-1}.
$$

Let $(G_t)_{t>\infty}$ be the weak solution of \([4]\), we have

Lemma 3.3. For any $v > 0$ either there is time $t$ with $\text{Vol}(G_t) = v$ or $v$ is a jump volume for \([4]\), i.e. time $t_1 > -\infty$ with

$$
\text{Vol}(G_{t_1}) < v \leq \text{Vol}(G_{t_1}^+),
$$

here $G_{t_1}^+$ is the strictly minimizing hull for $G_{t_1}$.

Proof. Let

$$
t_0 = \inf\{t \in \mathbb{R} : \text{Vol}(G_t) \geq v\},
$$

and

$$
\tau_0 = \sup\{t \in \mathbb{R} : \text{Vol}(G_t) \leq v\},
$$

then $t_0 \geq \tau_0$. By \([5]\), we know that $K_t = \partial G_t$ converges to $K_{t_0}^+$ in $C^1$ local sense when $t$ decreases to $t_0$ and $K_t$ converges to $K_{\tau_0}$ in $C^1$ local sense when $t$ increases to $\tau_0$, hence $\text{Vol}(G_{t_0}^+) \geq v \geq \text{Vol}(G_{\tau_0})$. If $t_0 > \tau_0$, we find it contradicts to the definition of $t_0$ or $\tau_0$, which implies $t_0 = \tau_0$, hence, either $v$ satisfies $\text{Vol}(G_{t_0}) = v$ or $\text{Vol}(G_{t_0}) < v \leq \text{Vol}(G_{t_0}^+)$, therefore, Lemma 3.3 is proved. □

Next lemma is on the relation between $t$ and volume of $(G_t)_{t>\infty}$ which is the weak solution of \([4]\).

Lemma 3.4. For any $v > 0$, let

$$
t(v) = \begin{cases} 
t, & \text{Vol}(G_t) = v \\
t, & \text{Vol}(G_t) < v \leq \text{Vol}(G_t^+),
\end{cases}
$$

then $t$ is a Lipschitz function and

$$
\frac{dt}{dv} \leq \left( \int_{K_t} H^2 \right)^{\frac{3}{2}} \cdot (\text{Area}(K_t))^{-\frac{3}{2}},
$$

here $K_t = \partial G_t$. 

Proof. For any fixed $v > 0$, let $t^i(v) = t^i$ with $v = Vol(\{(x, z) \in \Omega : W^i(x, z) < t^i, |z| \leq \frac{1}{2}\})$, then by Lemma 2.1, we see that $t^i(v)$ converges to $t(v)$ (here, without loss of generality, we assume the initial condition $B_\mu(x)$ shrink to $x$ when $i$ approaches to infinity). Next, according to Corollary 3.2
\[
\frac{dt^i}{dv} = \left( \int_{N^i_t \cap \{|z| \leq \frac{1}{2}\}} H^{-1}_i dS \right)^{-1}
\leq \left( \int_{N^i_t \cap \{|z| \leq \frac{1}{2}\}} H^2_i dS \right)^{\frac{1}{2}} \left( \frac{Area(N^i_t \cap \{|z| \leq \frac{1}{2}\})}{\frac{1}{2}} \right)^{-\frac{3}{2}}
\]
Hence, for any $v_1 \geq v_2$, we have
\[
t^i(v_1) - t^i(v_2) \leq \int_{v_2}^{v_1} \left( \int_{N^i_t \cap \{|z| \leq \frac{1}{2}\}} H^2_i dS \right)^{\frac{1}{2}} \left( \frac{Area(N^i_t \cap \{|z| \leq \frac{1}{2}\})}{\frac{1}{2}} \right)^{-\frac{3}{2}} dv
\]
According to (5.6) in [5], we see that for any $T > -T(\mu)$, for all $t \in [-T(\mu), T]$
\[
\int_{N^i_t \cap \{|z| \leq \frac{1}{2}\}} H^2_i dS \leq C(T),
\]
here $C(T)$ is a constant depends only on $T$, and also by (5.12) in [5], we see that for a.e. $t > -T(\mu)$, we have
\[
\int_{N^i_t \cap \{|z| \leq \frac{1}{2}\}} H^2_i dS \rightarrow \int_{\bar{N}^i_t \cap \{|z| \leq \frac{1}{2}\}} H^2 dS
\]
then let $i \rightarrow \infty$, and by bounded convergence theorem, we see that
\[
t(v_1) - t(v_2) \leq \int_{v_2}^{v_1} \left( \int_{\bar{N}^i_t \cap \{|z| \leq \frac{1}{2}\}} H^2 dS \right)^{\frac{1}{2}} \left( \frac{Area(\bar{N}^i_t \cap \{|z| \leq \frac{1}{2}\})}{\frac{1}{2}} \right)^{-\frac{3}{2}} dv
\]
\[
= \int_{v_2}^{v_1} \left( \int_{K_t} H^2 dS \right)^{\frac{1}{2}} \cdot \left( \frac{Area(K_t)}{\frac{1}{2}} \right)^{-\frac{3}{2}} dv
\]
and finish to prove the Lemma.

In order to prove Theorem 1.2 and Theorem 1.3, we need nondecreasing of $A(v)$, namely,

Lemma 3.5. Let $(M^3, g)$ be an AF manifold with nonnegative scalar curvature and admits no minimal $S^2$, then $A(v)$ is nondecreasing.

In order to prove above lemma, we need to construct a compact manifold from the AF manifold $M$, more precisely, note that $(M^3, g)$ is AF, hence we may take a large compact domain $\Omega \subset M$ so that $M \setminus \Omega$ is diffeomorphic to $\mathbb{R}^3 \setminus B_{R+4}$, hence, for simplicity, we just assume $\Omega \setminus K$ is diffeomorphic to $B_{R+4} \setminus B_{\frac{R}{2}}$, here $K$ is a compact domain of $M$. On the other hand, we
observe that the standard sphere with radius \( \frac{1}{2} \) can be expressed as \( \mathbb{S}^2(\lambda) = (\mathbb{R}^3, g_S = \frac{dx_1^2 + dx_2^2 + dx_3^2}{(1 + \lambda^2 |x|^2)^2}) \). Let

\[
\bar{g} = \begin{cases} 
  g, & \text{inside } \mathbb{B}_{R+3} \\
  \eta g + (1 - \eta) g_S, & \text{on } \mathbb{B}_{R+4} \setminus \mathbb{B}_{R+3} \\
  g_S, & \text{outside } \mathbb{B}_{R+4}
\end{cases}
\]

here \( \eta \) is a smooth function with \( \eta = 1 \) in \( \mathbb{B}_{R+3} \) and vanishes outside \( \mathbb{B}_{R+4} \).

Thus \((M^3, \bar{g})\) can be regarded as a compact manifold which is denoted by \( \bar{M} \).

We also need the following lemma from [7, Lemma 1].

**Lemma 3.6.** [Meeks-Yau] Let \( \iota \) be the infinimum of the injectivity radius of points in \( \{ x \in \bar{M} \mid d(x, S_{\frac{R}{2}}) > 4 \} \). Let \( K > 0 \) be the upper bound of the curvature of \( \bar{M} \) outside \( \mathbb{B}_{\frac{R}{2}} \). Let \( S_{\frac{R}{2}} \) be the coordinate sphere with radius \( \frac{R}{2} \), suppose \( N \) is a minimal surface and suppose \( x \in N \) is a point satisfying \( d(x, S_{\frac{R}{2}}) \geq \frac{4}{2} \), then

\[
|N \cap B_x(r)| \geq 2\pi K^{-2} \int_0^r \tau^{-1} (\sin K \tau)^2 d\tau
\]

where \( r = \min\{\frac{4}{2}, \iota\} \).

**Proof of Lemma 3.6** Suppose \( A(v) \) is not nondecreasing, then there is \( v_1 < v_2 \) with \( A(v_1) > A(v_2) \). By geometry measure theory, there is a compact domain \( \Omega_0 \subset \bar{M} \) with smooth boundary \( \Sigma_0 \) so that

\[
\text{Area}(\Sigma_0) = \inf \{ \text{Area}(\partial \Omega) : \Omega \subset \bar{M}, \text{Vol}_g(\Omega) \geq v_1 \},
\]

We claim that \( \text{Vol}(\Omega_0) > v_1 \) provided \( R \) large enough, therefore, \( \Sigma_0 \) is a stable minimal surface in \( \bar{M} \). In fact, suppose \( \text{Vol}(\Omega_0) = v_1 \), for any \( \epsilon > 0 \), we assume there is a compact domain \( \mathbb{D}_2 \subset M \) with \( \text{Vol}(\mathbb{D}_2) = v_2 \) and \( \text{Area}(\partial \mathbb{D}_2) < A(v_2) + \epsilon \), and without loss of generality, we assume \( \mathbb{D}_2 \) is contained in \( \Omega \), then we have

\[
\text{Area}(\Sigma_0) \leq \text{Area}(\partial \mathbb{D}_2) < A(v_2) + \epsilon < A(v_1),
\]

which implies \( \Omega_0 \) cannot contained in \( \Omega \) completely.

If \( \Omega_0 \) is contained the domain outside \( \mathbb{B}_{R+4} \), then by solution of isoperimetric problem on the standard sphere, we see that when \( R \) and \( \lambda \) becomes large the diameter of \( \Omega_0 \) in \( \bar{M} \) is uniform bounded, however, for any fixed \( R \), take \( \lambda \) large enough, we see that the metric \( \bar{g} \) restricted on \( \Omega_0 \) is almost Euclidean, then by a translation in \( \mathbb{R}^3 \), we may find a domain \( \Omega_1 \) which is contained in \( \mathbb{B}_R \setminus \mathbb{B}_{\frac{R}{2}} \subset \Omega \) and is isometric to \( \Omega_0 \) in \( \mathbb{R}^3 \), hence, the volume and area of the
boundary of \( \Omega_1 \) is very close to these of \( \Omega_0 \) with respect to metric \( \bar{g} \) provided \( R \) and \( \lambda \) is large enough, by a small perturbation on \( \Omega_1 \) if necessary, we may assume \( \text{Vol}(\Omega_1) = \text{Vol}(\Omega_0) \), and \( A(v_1) \leq \text{Area}(\partial \Omega_1) \leq \text{Area}(\Sigma_0) + \epsilon \) with respect to metric \( \bar{g} \), which is contradiction to (14), provided \( \epsilon \) is small enough.

For the remain case, by the co-area formula, we see that we may find a coordinate sphere \( S_\rho \) with \( \text{Area}(S_\rho \cap \Omega_0) < \epsilon \), and \( R + 4 \leq \rho \leq 2R \). By the solution of classical isoperimetric problem on the standard sphere, we may get a domain in \( \Omega \) which is still denoted by \( \Omega_2 \) with \( \text{Vol}(\Omega_2) = v_1 \), we again get \( A(v_1) \leq \text{Area}(\partial \Omega_1) \leq \text{Area}(\Sigma_0) + 2\epsilon \) with respect to metric \( \bar{g} \), which is contradiction to (14), provided \( \epsilon \) is small enough. Therefore, \( \text{Vol}(\Omega_0) > v_1 \), and hence, as we claimed before \( \Sigma_0 \) is a stable minimal surface in \( \bar{M} \).

Finally, we want to prove the minimal surface \( \Sigma_0 \) is contained in \( \bar{B}_{R+1} \) when \( R \) is large enough, hence it is in \( \Omega \). Actually, for any \( x \in \Sigma_0 \setminus \bar{B}_{R+1} \), note that \( (M, g) \) is AF, we may assume \( \iota > \frac{R}{2} \) and \( K \leq CR^{-3} \) outside \( \bar{B}_{R+1} \), by (13), we that

\[
\text{Area}(\Sigma_0) \geq CR^2,
\]

here \( C \) is a uniform constant, however, by (14) we see that when \( R \) is large enough, it is a contradiction. Thus, \( \Sigma_0 \) is contained in \( \Omega \), in particular, it is a stable minimal surface in \( (M^3, g) \), then by proof of Lemma 4.1 in [5], we see that \( (M^3, g) \) contains a minimal \( S^2 \), thus we finish to prove the Lemma.

\[ \square \]

Now, we can prove Theorem 1.2 and Theorem 1.3.

**Proof of Theorem 1.2 and Theorem 1.3** By a direct computation and Lemma 3.4, we see that

\[
\frac{dB}{dv} \leq B^{-\frac{1}{2}} \left( \int_{K_t} H^2 \right)^{\frac{1}{2}}.
\]

By the definition of Hawking mass of \( K_{i(v)} \), we see that

\[
\int_{K_t} H^2 = 16\pi - (16\pi)^{\frac{3}{2}} B^{-\frac{1}{2}} m(v),
\]

combine with above inequality, we obtain

...
\[ B(v) \leq (36\pi)^{\frac{1}{3}} \left( \int_0^v (1 - (16\pi)^{\frac{1}{2}} B^{-\frac{1}{2}}(v)m(v))^{\frac{1}{3}} \right)^{\frac{2}{3}}. \]

If \( v \) is not a jump volume, then there is a \( G_t \) with \( Vol(G_t) = v \), hence in this case, we have

\[ A(v) \leq Area(K_t) = B(v) \leq (36\pi)^{\frac{1}{3}} \left( \int_0^v (1 - (16\pi)^{\frac{1}{2}} B^{-\frac{1}{2}}(v)m(v))^{\frac{1}{3}} \right)^{\frac{2}{3}}; \]

otherwise, there is \( G_\tau \) with \( v_1 = Vol(G_\tau) < v \leq Vol(G_\tau^+) = v_2 \), hence \( t(v) = \tau \), and thus \( B(v) = B(v_1) \).

\[ A(v) \leq A(v_2) \leq Area(K_\tau^+) = Area(K_\tau) = B(v_1) = B(v), \]

here we have used Lemma 3.5 in the first inequality. Thus, we finish to Theorem 1.2.

Suppose there is \( v_0 > 0 \) with \( A(v_0) = (36\pi)^{\frac{1}{3}} v_0^{\frac{2}{3}} \). We claim that in this case \( v_0 \) is not a jump volume. Suppose not, then we may find \( v_1 < v_0 \leq v_2 \), with \( Vol(G_{t_1}) = v_1 \) and \( Vol(G_{t_1}^+) = v_2 \), by nondecreasing of \( A(v) \), we see that \( A(v_1) \leq A(v_0) \leq A(v_2) \), however,

\[ A(v_1) \leq Area(K_{t_1}) \leq (36\pi)^{\frac{1}{3}} v_1^{\frac{2}{3}}, \]

\[ A(v_0) = (36\pi)^{\frac{1}{3}} v_0^{\frac{2}{3}}, \]

\[ A(v_2) \leq Area(K_{t_1}^+) = Area(K_{t_1}) \]

Combine these inequalities together, we see \( v_0 \leq v_1 \), which is a contradiction, hence \( v_0 \) is not a jump volume.

Suppose there is non flat point \( x \), we consider the weak solution of (4) with initial condition \( x \), then by Lemma 8.1 in [5], \( m(v) > 0 \), for \( v > 0 \), together with (15), we that there is \( t > -\infty \) with \( Vol(G_t) = v_0 \), hence,

\[ A(v_0) \leq B(v_0) < (36\pi)^{\frac{1}{3}} v_0^{\frac{2}{3}}, \]

which is a contradiction. Thus we finish to prove Theorem 1.3. \( \square \)

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