BILINEAR MATRIX INEQUALITIES AND POLYNOMIALS IN SEVERAL FREELY NONCOMMUTING VARIABLES

SRIRAM BALASUBRAMANIAN∗, NEHA HOTWANI†, AND SCOTT MCCULLOUGH

ABSTRACT. Matrix-valued polynomials in any finite number of freely noncommuting variables that enjoy certain canonical partial convexity properties are characterized, via an algebraic certificate, in terms of Linear Matrix Inequalities and Bilinear Matrix Inequalities.

1. INTRODUCTION

The main results of this article extend principal results of [HHLM08] on convex polynomials in freely noncommuting variables to the matrix-valued case and of [JKMMP21] on $xy$-convex polynomials to the matrix-valued setting in any finite number of freely noncommuting variables.

Fix a positive integer $g$. Given a positive integer $d$ and $d \times d$ matrices, $A_0, A_1, \ldots, A_g$, the expression

$$L_A(x) = A_0 - \sum_{j=1}^{g} A_j x_j$$

is a linear pencil, where $A = (A_0, A_1, \ldots, A_g)$. In the case the $A_j$ are hermitian the pencil is hermitian and, in this case, it is typically assumed that $A_0$ is positive definite. When $L_A$ is hermitian and $x \in \mathbb{R}^g$, the matrix $L_A(x)$ is hermitian and

$$L_A(x) \succeq 0$$

is a linear matrix inequality (LMI). Here $T \succeq 0$ indicates that the hermitian matrix $T$ is positive semidefinite. The (scalar) solution, or feasible, set of a hermitian pencil $L_A$,

$$\mathcal{D}_A[1] = \{ x \in \mathbb{R}^g : L_A(x) \succeq 0 \},$$

is a spectrahedron. Because $L_A$ is affine linear, it is evident that $\mathcal{D}_A[1]$ is convex. Spectrahedra figure prominently in numerous engineering applications. They are fundamental objects in semidefinite programming in convex optimization and in real algebraic geometry.

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Given $d \times d$ hermitian matrices $A_0, A_1, \ldots, A_g, B_1, \ldots, B_h, C_{pq}$, $1 \leq p \leq g$, $1 \leq q \leq h$, the expression
\[
L(x, y) = A_0 - \sum_{j=1}^{g} A_j x_j - \sum_{k=1}^{h} B_k y_k - \sum_{p, q=1}^{g, h} C_{pq} x_p y_q,
\]
is an $xy$-pencil. When all the coefficient matrices are hermitian, $L$ is a hermitian $xy$-pencil. For a hermitian $xy$-pencil, the inequality $L(x, y) \succeq 0$ is a Bilinear Matrix Inequality (BMI). Bilinear matrix inequalities appear in robust control. See for instance [KSVdS04, SGL94, vAB00] and the references therein and the MATLAB toolbox, https://set.kuleuven.be/optec/Software/bmisolver-a-matlab-package-for-solving-optimization-problems-with-bmi-constraints.

It is natural from multiple perspectives to consider the fully matricial analogs of LMIs and BMIs. For positive integers $n$, let $\mathbb{S}_n(\mathbb{C})$ denote the set of $n \times n$ hermitian matrices and let $\mathbb{S}_n(\mathbb{C}^g)$ denote the set of $g$-tuples from $\mathbb{S}_n(\mathbb{C})$. Given $X = (X_1, \ldots, X_g) \in \mathbb{S}_n(\mathbb{C}^g)$, let
\[
L_A(X) = A_0 \otimes I_n - \sum A_j \otimes X_j
\]
and let
\[
\mathcal{D}_A[n] = \{ X \in \mathbb{S}_n(\mathbb{C}^g) : L_A(X) \succeq 0 \}.
\]
The sequence $\mathcal{D}_A = (\mathcal{D}_A[n])_n$ is known as a free spectrahedron or LMI domain. While $\mathcal{D}_A[1]$ does not determine $A$, up to unitary equivalence, the free spectrahedra $\mathcal{D}_A$ does.

Free spectrahedra are matrix convex, meaning
\begin{enumerate}
  \item $\mathcal{D}_A$ is closed with respect to isometric compressions: if $X \in \mathcal{D}_A[n]$ and $V : \mathbb{C}^m \to \mathbb{C}^n$ is an isometry, then $V^* X V \in \mathcal{D}_A[m]$, where
    \[
    V^* X V = V^*(X_1, \ldots, X_g)V = (V^*X_1V, \ldots, V^*X_gV);
    \]
  \item $\mathcal{D}_A$ is closed under direct sums: if $X \in \mathcal{D}_A[n]$ and $Y \in \mathcal{D}_A[m]$, then $X \oplus Y \in \mathcal{D}_A[n+m]$, where
    \[
    (X \oplus Y)_j = \begin{pmatrix} X_j & 0 \\ 0 & Y_j \end{pmatrix}.
    \]
\end{enumerate}
In particular each $\mathcal{D}_A[n]$ is convex in the ordinary sense.

Free spectrahedra appear in the theories of completely positive maps and operator systems and spaces [Pa03, Pi03]. They appear in systems engineering problems governed by a signal flow diagram as explained in [dOH06, HMdOV09, CHSY]. They also produce tractable natural relaxations for optimizing over spectrahedra; e.g., the matrix cube problem [BtN02, DDSS17, HKMS17], which can be NP hard, but whose canonical free spectrahedral relaxation is a semidefinite program (SDP).

The fully matricial analog of BMIs is described below, after the introduction of polynomials in freely noncommuting variables.
1.1. Free Polynomials. The two types of partial convexity considered in this article are described in terms of free polynomials. Fix freely noncommuting variables $\chi_1, \ldots, \chi_k$. Given a word

\begin{equation}
(1.1) \quad w = \chi_{i_1} \cdots \chi_{i_\ell}
\end{equation}

in these variables and $T \in \mathbb{S}_n(\mathbb{C}^k)$, let

$$w(T) = T^w = T_{i_1} \cdots T_{i_\ell}.$$ 

Let $\mathcal{W}$ denote the collection of words in the variables $\chi$. A $d \times d$ matrix-valued free polynomial is an expression of the form

$$p(\chi) = \sum_{w \in \mathcal{W}} p_w w,$$

where the sum is finite and the $p_w \in M_d(\mathbb{C})$. The free polynomial $p$ is naturally evaluated at $T \in \mathbb{S}_n(\mathbb{C}^k)$ as

$$p(T) = \sum p_w \otimes T^w.$$ 

There is a natural involution $\ast$ on free polynomials that reverses the order of products in words so that, for $w$ in equation (1.1),

$$w^\ast = \chi_{i_\ell} \cdots \chi_{i_1};$$

and such that

$$p^\ast = \sum p_w^\ast w^\ast.$$ 

This involution is compatible with the adjoint operation on matrices,

$$p(T)^* = p^*(T).$$

A free polynomial $p$ is hermitian if $p^* = p$; equivalently, if $p(T)^* = p(T)$ for all $n$ and $T \in \mathbb{S}_n(\mathbb{C}^k)$.

From here on we often omit the adjectives matrix and free and simply refer to matrix-valued free polynomials as polynomials, particularly when there is no possibility of confusion.

Since the involution fixes the variables, $\chi_j^\ast = \chi_j$, we refer to $\chi_1, \ldots, \chi_k$ as hermitian variables. In Subsection 2.3, non-hermitian variables naturally appear.

1.2. Partial Convexity. Both types of partial convexity considered in this article involve partitioning freely noncommuting variables into two classes $x_1, \ldots, x_\mu$ and $y_1, \ldots, y_\mu$.\(^1\)

\[^1\text{For the results here, there is no loss in generality in assuming the number of $x$ and $y$ variables is the same.}\]
1.2.1. \( xy \)-convexity. Since matrix multiplication does not commute, we now update the definition of an \( xy \)-pencil as follows. (See [JKMMP21a].) A matrix-valued free polynomial of the form

\[
L(x, y) = A_0 - \sum_{j=1}^{\mu} A_j x_j - \sum_{k=1}^{\mu} B_k y_k - \sum_{p,q=1}^{\mu} C_{pq} x_p y_q - \sum_{p,q=1}^{\mu} D_{qp} y_q x_p,
\]

where \( A_j, B_k, C_{pq}, D_{qp} \) are all matrices of the same size, is an \( xy \)-pencil. The pencil \( L \) is naturally evaluated at a tuple \( (X, Y) \in \mathbb{S}_n(\mathbb{C}^n) \times \mathbb{S}_n(\mathbb{C}^n) \) as

\[
L(X, Y) = A_0 \otimes I_n - \sum_{j=1}^{\mu} A_j \otimes X_j - \sum_{k=1}^{\mu} B_k \otimes Y_k - \sum_{p,q=1}^{\mu} C_{pq} \otimes X_p Y_q - \sum_{p,q=1}^{\mu} D_{qp} \otimes Y_q X_p.
\]

When the \( A_j \) and \( B_k \) are hermitian and \( D_{qp} = C_{pq}^* \), the pencil \( L \) is a hermitian \( xy \)-pencil and \( L(X, Y) \succeq 0 \) is the matricial analog of a BMI. Assuming, as we usually do, \( A_0 \) is positive definite, writing \( \Sigma = (A_i, B_j, C_{ij}) \) and \( L_{\Sigma} = L \), let

\[
D_{\Sigma}[n] = \{(X, Y) \in \mathbb{S}_n^n \times \mathbb{S}_n^n : L_{\Sigma}(X, Y) \succeq 0 \}
\]

and let \( D_{\Sigma} \) denote the sequence \( (D_{\Sigma}[n])_n \). The set \( D_{\Sigma} \) is \( xy \)-convex, meaning \( D_{\Sigma} \) is

1. closed under direct sums; and
2. if \( (X, Y) \in D[n] \) and \( V : \mathbb{C}^m \rightarrow \mathbb{C}^n \) is an isometry such that \( V^*(X_i Y_j)V = V^* X_i VV^* Y_j V \), for all \( i, j \), then \( V^*(X, Y)V \in D_{\Sigma}[m] \).

A tuple \( ((X, Y), V) \) where \( (X, Y) \in \mathbb{S}_n(\mathbb{C}^n) \times \mathbb{S}_n(\mathbb{C}^n) \) and \( V : \mathbb{C}^m \rightarrow \mathbb{C}^n \) is an isometry such that \( V^*(X_i Y_j)V = V^* X_i VV^* Y_j V \), for all \( i, j \), is an \( xy \)-pair. A hermitian matrix-valued free polynomial \( p(x, y) \) is \( xy \)-convex if

\[
p(V^*(X, Y)V) \preceq (I_d \otimes V)^* p(X, Y) (I_d \otimes V)
\]

for all \( xy \)-pairs \( ((X, Y), V) \). It is nearly immediate that, if \( p \) is \( xy \)-convex, then the positivity set of \( -p \),

\[
D_{-p} = \{(X, Y) : p(X, Y) \preceq 0 \},
\]

is also \( xy \)-convex. Theorem 1.1 below provides an algebraic certificate characterizing \( xy \)-convex polynomials. When \( d = \mu = 1 \), it reduces to [JKMMP21, Theorem 1.4].

**Theorem 1.1.** Suppose \( p(x, y) \) is a hermitian \( d \times d \) matrix-valued polynomial. If \( p \) is \( xy \)-convex, then there exist a hermitian \( d \times d \) matrix-valued \( xy \)-pencil \( \lambda \), a positive integer \( N \) and an \( N \times d \) matrix-valued \( xy \)-pencil \( \Lambda \) such that

\[
p(x, y) = \lambda(x, y) + \Lambda(x, y)^* \Lambda(x, y).
\]

In particular, \( -p \) is the Schur complement of a Hermitian \( xy \)-pencil and \( D_{-p} \) is the feasible set of the BMI,

\[
\begin{pmatrix} I & \Lambda(x, y) \\ \Lambda(x, y)^* & -\lambda(x, y) \end{pmatrix} \succeq 0.
\]
The converse is easily seen to be true.

A proof of Theorem 1.1 is contained in the proof of Proposition 1.3 given in Section 3.

1.2.2. $a^2$-convexity. To maintain consistency with the literature, we now switch to freely noncommuting variables $a_1, \ldots, a_\mu$ and $x_1, \ldots, x_\mu$. A $d \times d$ matrix-valued hermitian polynomial $p(a,x)$ is **convex in** $x$ if for each positive integer $n$, each $A \in S_n(\mathbb{C}^\mu)$, each $X, Y \in S_n(\mathbb{C}^\mu)$ and each $0 < t < 1$, one has

$$p(A,tX + (1-t)Y) \preceq tp(A,X) + (1-t)p(A,Y).$$

A canonical example of a convex in $x$ polynomial is a hermitian **linear in** $x$ pencil; that is, a hermitian polynomial that is affine linear in $x$.

There is a fruitful alternate characterization of convexity in $x$. A tuple $((A,X),V)$ where $(A,X) \in S_n(\mathbb{C}^\mu) \times S_n(\mathbb{C}^\mu)$ and $V: \mathbb{C}^m \rightarrow \mathbb{C}^n$ is an isometry is an $a^2$-pair if $V^* A_i V = (V^* A_i V)^2$ for each $1 \leq i \leq \mu$. Equivalently $((A,X),V)$ is an $a^2$-pair if ran $V$ reduces $A$. As we will see in Proposition 2.1, a hermitian polynomial $p$ is convex in $x$, or $a^2$-convex, if and only if

$$p(V^*(A,X)V) \preceq (I_d \otimes V^*)p(A,X)(I_d \otimes V)$$

for all $a^2$-pairs $((A,X),V)$. Theorem 1.1 and Theorem 1.2 below – the latter of which is a matrix polynomial version of [HHLM08, Theorem 1.5] and [JKMMP21, Corollary 1.3] – are the main results of this article.

**Theorem 1.2.** Suppose $p(a,x)$ is a $d \times d$ matrix-valued hermitian polynomial. If $p(a,x)$ is convex in $x$, then there exist a $d \times d$ matrix-valued hermitian linear in $x$ pencil $L$, a positive integer $N$ and a $N \times d$ matrix-valued polynomial $\Lambda$ that is linear in $x$ such that

$$p(a,x) = L(a,x) + \Lambda(a,x)^* \Lambda(a,x).$$

In particular, $p$ has degree at most two in $x$ and $D_p$ is the feasible set of the affine linear in $x$ matrix inequality

$$\begin{pmatrix} I & \Lambda(a,x) \\ \Lambda(a,x)^* & -L(a,x) \end{pmatrix} \succeq 0.$$

The converse is evidently true.

A proof of Theorem 1.2 is given in Section 2. Proposition 1.3 below describes the relationship between $xy$-convexity and separate convexity in $x$ and $y$. It also extends [JKMMP21, Theorem 1.4] to both several $x$ and $y$ variables and matrix-valued polynomials.

**Proposition 1.3.** Let $p(x,y)$ be a $d \times d$ matrix-valued hermitian polynomial. The following statements are equivalent.

(i) $p$ is $xy$-convex
(ii) $p$ is convex in $x$ and $y$ separately.
(iii) \( p \) has the form given in equation (1.2).

In particular, \( p \) is \( xy \)-convex if and only if \( p \) is convex in \( x \) and \( y \) separately.

A proof of Proposition 1.3 is given in Section 3.

**Remark 1.4.** An example in the appendix of the arxiv version of [JKMMP21] shows that there is not a local version of Proposition 1.3. That is, as a local statement, separate convexity need not imply \( xy \)-convexity.

2. **Partially convex hermitian matrix-valued NC polynomials**

This section contains a proof of Theorem 1.2 and is organized as follows. Subsection 2.1 presents alternate formulations of \( a^2 \)-convexity. Needed versions of Amitsur’s no polynomial identities results are collected in Subsection 2.2. The border vector middle matrix representation for a type of Hessian for polynomials in \( a, x \) of degree two in \( x \) is reviewed in Subsection 2.3. The proof of Theorem 1.2 concludes in Subsection 2.4. Subsection 2.5 contains two corollaries that apply to \( xy \)-convex polynomials.

2.1. **Alternate formulations of convexity.** The proof of Theorem 1.2 makes use of the following characterization of \( a^2 \)-convex polynomials. It parallels [JKMMP21, Proposition 4.1] for \( xy \)-convex polynomials for \( \mu = 1 \) and, to some extent, appears as [JKMMP21, Proposition 1.5]. It also borrows liberally from the ideas in [PT-D+].

**Proposition 2.1.** For a \( d \times d \) matrix-valued hermitian polynomial \( p(a, x) \), the following statements are equivalent.

(i) The polynomial \( p \) is convex in \( x \);

(ii) If \( ((A, X), V) \) is an \( a^2 \)-pair, then

\[
(I_d \otimes V)^* p(A, X) (I_d \otimes V) \succeq p(V^*(A, X)V);
\]

(iii) For each tuple \((A, X) \in S_n(C^\mu) \times S_n(C^\mu)\), each positive integer \( m \) and all tuples \( \alpha, \delta \in S_m(C^\mu) \) and \( \beta \in M_{n,m}(C^\mu) \),

\[
(I_d \otimes W)^* p(R, S)(I_d \otimes W) \succeq p(W^*(R, S)W)
\]

where \( W^* = (I_n, 0) \in M_{n,n+m}(C) \),

\[
R = \left( \begin{array}{ccc} A_1 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & \alpha_{\mu} \end{array} \right) \in S_{n+m}(C^\mu)
\]

and

\[
S = \left( \begin{array}{ccc} X_1 & \beta_1 & \beta_1^* \\ \beta_1^* & \delta_1 & \beta_1^* \\ \vdots & \vdots & \vdots \\ \beta_{\mu}^* & \delta_{\mu} & \beta_{\mu}^* \end{array} \right) \in S_{n+m}(C^\mu).
\]
Proof. To prove (ii) implies (i), let \( A, X, Y \in S_n(\mathbb{C}^n) \) and \( t \in [0, 1] \) be given. Let
\[
\hat{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad \hat{X} = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}, \quad V = (\sqrt{t}I_n - \sqrt{1-t}I_n)^*.
\]
In particular, \(((\hat{A}, \hat{X}), V)\) is an \( a^2 \)-pair. Thus,
\[
p(A, tX + (1-t)Y) = p(V^*(\hat{A}, \hat{X})V) \leq (I_d \otimes V)^* p(\hat{A}, \hat{X}) (I_d \otimes V)
\]
\[
= (I_d \otimes V)^* \begin{pmatrix} p(A, X) & 0 \\ 0 & p(A, Y) \end{pmatrix} (I_d \otimes V)
\]
\[
= tp(A, X) + (1-t)p(A, Y),
\]
where the inequality is a consequence of the hypothesis. Hence \( p \) is convex in \( x \).

Now suppose item (iii) holds and let an \( a^2 \)-pair \(((A, X), V)\) be given. Since \( V : \mathbb{C}^m \to \mathbb{C}^n \) is an isometry whose range \( M \) reduces \( A_j \), the matrix representations of \( V, A_j \) and \( X_j \) with respect to the decomposition \( \mathbb{C}^n = M \oplus M^\perp \) take the forms
\[
\begin{pmatrix} I_M \nonumber \\ 0 \end{pmatrix}, \quad \begin{pmatrix} A_j |_M & 0 \\ 0 & A_j |_{M^\perp} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} P_M X_j P_M^* & Y_j \\ Y_j^* & P_{M^\perp} X_j P_{M^\perp} \end{pmatrix}
\]
respectively, where \( P_M \) denotes the orthogonal projection of \( \mathbb{C}^n \) onto \( M \). The conclusion of item (ii) now follows by identifying \( M \) with \( \mathbb{C}^m \) and observing that, under this identification, the operators \( V, A \) and \( X \) have the same form as \( W, R \) and \( S \) in the hypothesis. Hence item (iii) implies item (ii).

It remains to prove (i) implies (iii). To this end, let
\[
\hat{S} = \begin{pmatrix} X_1 & -\beta_1 \\ -\beta_1^* & \delta_1 \end{pmatrix}, \ldots, \begin{pmatrix} X_\mu & -\beta_\mu \\ -\beta_\mu^* & \delta_\mu \end{pmatrix}.
\]
By the convex in \( x \) hypothesis, it follows that
\[
(2.1) \quad \begin{pmatrix} p(A, X) & 0 \\ 0 & p(\alpha, \delta) \end{pmatrix} = p \left( R, \frac{1}{2} (S + \hat{S}) \right) \leq \frac{1}{2} (p(R, S) + p(R, \hat{S})).
\]
Multiplying the inequality of equation (2.1) by \((I_d \otimes W)^*\) on the left and \((I_d \otimes W)\) on the right gives
\[
p(A, X) = p(W^*(R, S)W) \leq \frac{1}{2} ((I_d \otimes W)^* [p(R, S) + p(R, \hat{S})] (I_d \otimes W).
\]
Thus, to complete the proof, it suffices to show.
\[
(2.2) \quad (I_d \otimes W)^* p(R, \hat{S}) (I_d \otimes W) = (I_d \otimes W)^* p(R, S) (I_d \otimes W).
\]
To this end, let
\[
U = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}
\]
and note \( (R, \widehat{S}) = U^*(R, S)U \). Consequently, \( p(R, \widehat{S}) = (I_d \otimes U^*)p(R, S)(I_d \otimes U) \), and equation (2.2) follows. \( \square \)

2.2. Faithful representations.

**Proposition 2.2.** Suppose \( p(a, x) \) is a hermitian polynomial. If \( p \) is convex in \( x \), then the degree of \( p \) in \( x \) is at most two.

**Proof.** Let \( d \) denote the size of \( p \). Thus \( p = \sum_w p_w w \) for some \( p_w \in M_d(\mathbb{C}) \). For \( \gamma \in \mathbb{C}^d \), define the polynomial \( p_\gamma \) by \( p_\gamma = \sum_w (\gamma^* p_w \gamma) w \). Since \( p \) is hermitian, it follows that \( p_\gamma \) is a hermitian polynomial with scalar coefficients. Also convexity of \( p \) in \( x \) implies the convexity of \( p_\gamma \) in \( x \). Hence, by [JKMMP21, Corollary 1.3],\(^2\) for each \( \gamma \in \mathbb{C}^d \), the degree of \( p_\gamma \) in \( x \) is at most two. Suppose the word \( w = w(a, x) \) is such that \( p_w \neq 0 \). Since the scalar field is \( \mathbb{C} \), it follows that there exists a \( \gamma \in \mathbb{C}^d \) such that \( \gamma^* p_w \gamma \neq 0 \). Since \( p_\gamma \) has degree at most two in \( x \), it follows that \( w(a, x) \) has degree at most two in \( x \). Hence \( p \) has degree at most two in \( x \). \( \square \)

The following lemma is a variant of the Amitsur-Levitski Theorem.

**Lemma 2.3.** If \( p(a) \) is a polynomial of degree at most \( m \geq 0 \) in the freely noncommuting variables \( a_1, \ldots, a_\mu \) and if there is an \( n \geq N(\mu, m) := \sum_{j=0}^{m} h^j \) and a nonempty open set \( \mathcal{U} \subseteq S_n(\mathbb{C}^\mu) \) such that \( p(U) = 0 \) for all \( U \in \mathcal{U} \), then \( p = 0 \).

**Proof.** Arguing by contradiction, suppose \( p \neq 0 \), but there is an \( n \geq N(\mu, m) \) and a nonempty open subset \( \mathcal{U} \subseteq S_n(\mathbb{C}^\mu) \) on which \( p \) vanishes. In this case, there is no loss of generality assuming the degree of \( p \) is \( m \). Since \( p \) vanishes on an open subset of \( S_n(\mathbb{C}^\mu) \), it vanishes on all of \( S_n(\mathbb{C}^\mu) \).

For the moment, assume \( n = N \). Let \( H \) denote the Hilbert space with orthonormal basis \( \mathcal{W} \), the words of length at most \( m \) in the variables \( a \). Hence \( \dim H = N \). Define linear maps \( S_j \), for \( j = 1, \ldots, g \), on \( H \) by \( S_j w = a_j w \), if \( w \in \mathcal{W} \) has length strictly less than \( m \), and \( S_j w = 0 \) if \( w \) has length \( m \). Observe that \( S_j^* w \) has length strictly less than the length of \( w \in \mathcal{W} \).

Let \( T_j = S_j + S_j^* \). Thus \( T = (T_1, \ldots, T_\mu) \in S_N(\mathbb{C}^\mu) \). A straightforward computation shows, when \( m \geq 1 \)

\[
p(T) \varnothing = \sum_{j=0}^{m-1} q_j + p_m,
\]

where \( q_j \) are homogeneous polynomials of degree \( j \) and \( p_m \) is the homogeneous of degree \( m \) part of \( p \). On the other hand, when \( m = 0 \),

\[
p(T) \varnothing = p_0 \varnothing.
\]

\(^2\)The same result, but with real scalars, appears as [HHLM08, Theorem 1.4]
Since the set \( \{ q_0, q_1, \ldots, q_{m-1}, p_m \} \subseteq H \) is linearly independent and, by assumption, \( p(T)\emptyset = 0 \), it follows that \( p_m = 0 \), contradicting the assumption that the degree of \( p \) is \( m \).

To complete the proof, if \( n > N \), then replace the tuple \( T \) by \( R = T \oplus 0 \) and \( \emptyset \) with \( \gamma = \emptyset \oplus 0 \), where the first 0 is the zero tuple in \( S_{n-N}(\mathbb{C}^\mu) \) and the second 0 is the zero vector in \( \mathbb{C}^{n-N} \), and observe that \( 0 = p(R)\gamma \) implies \( p(T)\emptyset = 0 \).

\[ \text{Lemma 2.4.} \quad \text{Let } q(a) = \sum w q_w w(a) \text{ be a } d \times d \text{ matrix (not necessarily hermitian) polynomial in the freely noncommuting variables } a_1, \ldots, a_\mu. \text{ If } q(A) = 0 \text{ for all } n \in \mathbb{N} \text{ and } A \in S_n(\mathbb{C}^\mu), \text{ then } q = 0. \]

\[ \text{Proof.} \quad \text{Since } q(a) \text{ is a } d \times d \text{ matrix polynomial, i.e. } q_w \in M_d(\mathbb{C}), \text{ it can be viewed as a } d \times d \text{ matrix } (q^{i,j}(a))^t_{i,j=1} \text{ of scalar polynomials. Suppose that } q \text{ is nonzero. Choose } i, j \text{ such that } q^{i,j}(a) \text{ is nonzero. Since } q^{i,j}(A) = 0 \text{ for all } n \in \mathbb{N} \text{ and } A \in S_n(\mathbb{C}^\mu), \text{ Lemma 2.3 implies } q^{i,j} \text{ is the zero polynomial, a contradiction.} \]

\[ \text{Proposition 2.5.} \quad \text{For each positive integer } \kappa \text{ and each } n \geq N = \sum_{j=0}^{\kappa} \mu^j \text{ there exist } A \in S_n(\mathbb{C}^\mu) \text{ and } v \in \mathbb{C}^n \text{ such that }\]

\[ \mathcal{M}_{A,v,\kappa} = \{ w(a)v : w(a) \text{ is a word with degree at most } \kappa \} \]

is linearly independent.

In particular, in the case \( \kappa = 1 \), there is an \( A \in S_{\mu+1}(\mathbb{C}^\mu) \) and a \( v \in \mathbb{C}^{\mu+1} \) such that \( \mathcal{M}_{A,v,1} \) is linearly independent.

\[ \text{Proof.} \quad \text{Fix } \kappa. \text{ Let } \mathcal{W}_\kappa \text{ denote the words in the (freely noncommuting) variables } a_1, \ldots, a_\mu \text{ of degree at most } \kappa. \text{ The cardinality of } \mathcal{W}_\kappa \text{ is } N = \sum_{j=0}^{\kappa} \mu^j. \text{ Given } c : \mathcal{W}_\kappa \to \mathbb{C}, \text{ let } c_w \text{ denote the value of } c \text{ at } w \in \mathcal{W}_\kappa. \text{ Let } \mathcal{C} \text{ denote the set of all functions } c : \mathcal{W}_\kappa \to \mathbb{C} \text{ such that } \sum_w |c_w|^2 = 1. \text{ Thus } \mathcal{C} \text{ is identified with the unit sphere in } \mathbb{C}^N \text{ and is thus compact.} \]

Given \( c \in \mathcal{C} \), let

\[ q_c(a) = \sum_{w \in \mathcal{W}_\kappa} c_w w. \]

Arguing by contradiction suppose, for each \( n \geq N \), for each \( C \in S_n(\mathbb{C}^\mu) \) and each \( \gamma \in \mathbb{C}^n \), there exists a \( c \in \mathcal{C} \) such that \( q_c(C)\gamma = 0 \). Given \( n \geq N \) and \( C \in S_n(\mathbb{C}^\mu) \) and \( \gamma \in \mathbb{C}^n \), let

\[ K_{C,\gamma} = \{ c \in \mathcal{C} : q_c(C)\gamma = 0 \}. \]

Thus \( K_{C,\gamma} \) is nonempty for all \( C \) and \( \gamma \). Likewise, since, for \( C \in S_n(\mathbb{C}^\mu) \) and \( \gamma \in \mathbb{C}^n \), the mapping

\[ C \ni c \mapsto q_c(C)\gamma \in \mathbb{C}^n \]

is continuous, the sets \( K_{C,\gamma} \) are compact. Given a positive integer \( M \), positive integers \( n_1, \ldots, n_M \geq N \), \( C^j \in S_{n_j}(\mathbb{C}^\mu) \) and \( \gamma_j \in \mathbb{C}^{n_j} \), for \( 1 \leq j \leq M \), observe that

\[ \bigcap_{j=1}^M K_{C^j,\gamma_j} = K_{\bigoplus C^j, \oplus \gamma_j} \neq \emptyset. \]
Hence \( \{ K_{C,\gamma} : C, \gamma \} \) has the finite intersection property. It follows that
\[
\cap_{C,\gamma} K_{C,\gamma} \neq \emptyset.
\]

Choosing any \( \tilde{c} \) in this intersection,
\[
q_{\tilde{c}}(C)\gamma = 0
\]
for all \( C \) and \( \gamma \). Consequently \( q_{\tilde{c}}(C) = 0 \) for all \( C \in \mathbb{S}_n(\mathbb{C}^n) \) and hence, by Lemma 2.3, \( q_{\tilde{c}} = 0 \).

Thus, \( \tilde{c}_w = 0 \) for all \( \gamma \), contradicting \( \tilde{c} \in C \). Hence, there exist \( C \) and \( \gamma \) such that \( \mathcal{M}_{C,\gamma} \) is linearly independent. Let \( \ell \) denote the size of \( C \); that is \( C \in \mathbb{S}_\ell(\mathbb{C}^n) \) and \( \mathcal{M}_{C,\gamma} \) is a subspace of \( \mathbb{C}^\ell \) of dimension \( N \). Let \( V \) denote the inclusion of \( \mathcal{M}_{C,\gamma} \) into \( \mathbb{C}^\ell \) and let \( B = V^*CV \). Since \( V^*A^\alpha V \gamma = A^\alpha \gamma \in \mathcal{M}_{C,\gamma} \) for words \( \alpha \) of length at most \( \kappa \), the set
\[
\{ w(B)\gamma : w \text{ is a word of length at most } \kappa \}
\]
is linearly independent.

Given \( m > N \), let \( A = B \oplus 0 \), where \( 0 \in \mathcal{M}_{m-N}(\mathbb{C}^n) \). Likewise let \( v = \gamma \oplus 0 \in \mathbb{C}^m = \mathbb{C}^N \oplus \mathbb{C}^{m-N} \) and note that \( \mathcal{M}_{A,v,C} \) is linearly independent. \( \square \)

2.3. The Border vector, middle matrix and non-hermitian variables. In this subsection, \( q(a, x) \) denotes a fixed polynomial that is homogeneous of degree two in \( x \) and \( d_a \) denote its degree in \( a \).

Enumerate the words in the variables \( a_1, \ldots, a_\mu \) of degree at most \( d_a \) as \( \{ m_1, \ldots, m_N \} \). In particular, \( N = \sum_{j=0}^{d_a} \mu^j \). For \( 1 \leq j, k \leq \mu \) and \( 1 \leq r, t \leq N \) there exist uniquely determined \( d \times d \) matrix-valued polynomials \( 3_{r,t}^{j,k}(a) \) such that
\[
q(a, x) = \sum_{j,k,r,t} (I_d \otimes m_{r}(a)^*x_j)3_{r,t}^{j,k}(a)(I_d \otimes x_k m_{t}(a)).
\]
(2.3)

In fact,
\[
(I_d \otimes m_{r}(a)^*x_j)3_{r,t}^{j,k}(a)(I_d \otimes x_k m_{t}(a)) = \sum_{s=1}^{N} \{ q_w, w : w = m_{r}(a)^*x_j m_{s}(a) x_k m_{t}(a) \}.
\]
(2.4)

Letting \( Z \) denote the block matrix indexed by \( ((j, r), (k, t)) \) with \( d \times d \) polynomial entries \( 3_{r,t}^{j,k}(a) \) and letting \( V(a)[x] \) the column vector with \( (k, t) \) entry \( I_d \otimes x_k m_{t}(a) \), equation (2.3) becomes,
\[
q(a, x) = V(a)[x]^* Z(a) V(a)[x].
\]
(2.5)

The polynomial \( V(a)[x] \) is the border vector and \( Z(a) \) is the middle matrix for \( q \). Equation (2.5) is the border vector-middle matrix representation of \( q \).

Before continuing, we pause to introduce non-hermitian freely noncommuting variables. Accordingly, let \( \chi_{1}, \ldots, \chi_{k}, z_1, \ldots, z_{\ell}, w_1, \ldots, w_{\ell} \) be freely noncommuting variables. Now let
* denote an involution on words in these variables that reverses the order of products and satisfies $\chi_j^* = \chi_j$ and $z_j^* = w_j$. Thus the $\chi$ variables are hermitian, but the $z, w$ variables are not. It is natural, and customary, to systematically use $z_j^*$ in place of $w_j$. A polynomial in this mix of variables is now a linear combination of words with matrix coefficients. A word in these variables evaluates at a tuple $(X, Z) \in S_n(\mathbb{C}^k) \times M_n(\mathbb{C}^l)$ in the natural way: replace $\chi_j$ with $X_j$ and similarly replace $z_j$ and $z_j^*$ with $Z_j$ and $Z_j^*$. The involution extends in the evident fashion to this mixed variable setting. Namely, the coefficient matrices are replaced by their adjoints and the involution is applied to the words. Finally, a polynomial is hermitian if $p^* = p$; equivalently $p(X, Z)^* = p^*(X, Z)$ for all tuples $(X, Z)$.

The definition of the border vector, as a polynomial, naturally extends to the case of non-hermitian $x$ variables. With this understanding, and given positive integers $m, n$, a tuple $B \in S_n(\mathbb{C}^r)$, a tuple $\beta \in M_{n,m}(\mathbb{C}^r)$ and tuple $\alpha \in S_m(\mathbb{C}^r)$,

$$
\sum_{j,k,r,t} (I_d \otimes m_Z(B)^\ast \beta_j) \mathcal{J}^{j,k}(\alpha) (I_d \otimes \beta_k^* m_t(B)) = V(B)[\beta^\ast]^* \mathcal{Z}(\alpha)V(B)[\beta^\ast].
$$

(2.6)

**Proposition 2.6.** If $W, R, S$ are given as in Proposition 2.1 item (iii), then

$$(I_d \otimes W)^\ast q(R, S) (I_d \otimes W) = q(A, X) + V(A)[\beta^\ast]^* \mathcal{Z}(\alpha)V(A)[\beta^\ast].$$

**Proof.** Suppose $w = \ell(a)x_j c(a)x_k r(a)$, where $\ell(a), c(a), r(a)$ are words. Compute

$$w(R, S) = \left ( \begin{array}{cc} \ell(A) & 0 \\ 0 & \ell(\alpha) \end{array} \right ) \left ( \begin{array}{cc} X_j & \beta_j \\ \beta_j^* & \delta_j \end{array} \right ) \left ( \begin{array}{cc} c(A) & 0 \\ 0 & c(\alpha) \end{array} \right ) \left ( \begin{array}{cc} X_k & \beta_k \\ \beta_k^* & \delta_k \end{array} \right ) \left ( \begin{array}{cc} r(A) & 0 \\ 0 & r(\alpha) \end{array} \right )$$

$$= \left ( \begin{array}{cc} \ell(A)X_jc(A)X_k r(A) + \ell(A)\beta_jc(\alpha)\beta_k^* r(A) & * \\ * & * \end{array} \right ).$$

Hence,

(2.7) $W^\ast w(R, S) W = \ell(A)X_jc(A)X_k r(A) + \ell(A)\beta_jc(\alpha)\beta_k^* r(A)$.

In particular, fixing $r, t, j, k$ and letting $Y = I_d \otimes W$, equations (2.4) and (2.7) give

$$Y^\ast (I_d \otimes S_j m_t(R))^\ast \mathcal{J}^{j,k}(R) (I_d \otimes S_k m_t(R)) Y$$

$$= Y^\ast \left ( \sum_{s=1}^N \{q_w \otimes w(R, S) : w = m_t(a)x_jm_s(a)x_k m_t(a) \} \right ) Y$$

(2.8)

$$= \sum_{s=1}^N \{q_w \otimes [w(A, X) + m_t(A)^\ast \beta_jm_s(\alpha)\beta_k^* m_t(A)] : w = m_t^a x_j m_s x_k m_t \}$$

$$(I_d \otimes X_j m_t(A))^\ast \mathcal{J}^{j,k}(A) (I_d \otimes X_k m_t(A))$$

$$+ (I_d \otimes \beta_j^* m_t(A))^\ast \mathcal{J}^{j,k}(\alpha) (I_d \otimes \beta_k^* m_t(A)).$$
Summing equation (2.8) over \( r, t, j, k \) and using equations (2.3), (2.5) and (2.6),

\[(I_d \otimes W)^* q(R, S)(I_d \otimes W) = q(A, X) + V(A)[\beta^*]^* Z(\alpha) V(A)[\beta^*].\]

\(\square\)

2.4. Proof of Theorem 1.2.

Proof of Theorem 1.2. Since \( p \) is convex in \( x \), Proposition 2.2 says its degree in \( x \) is at most two. Thus,

\[p(a, x) = L(a, x) + q(a, x),\]

where \( L(a, x) \) is affine linear in \( x \) and

\[q(a, x) = \sum_{w \in \Gamma} p_w w,\]

where \( \Gamma \) denotes words in the variables \( a, x \) that are homogeneous of degree two in \( x \). Since \( p \) is convex in \( x \), so is \( q \), and it suffices to prove that there exists an \( xy \)-pencil \( \Lambda \) such that \( q = \Lambda^* \Lambda \).

Let \( \kappa \) denote the degree of \( q \) in \( a \). By Proposition 2.5, there is an \( \ell \) such that for all \( n \geq \ell \) there exists an \( A \in S_n(\mathbb{C}^\mu) \) and a \( v \in \mathbb{C}^n \) such that

\[M_A, v, \kappa = \{ w(A)v : w \text{ is a word of length at most } \kappa \}\]

is linearly independent.

Fix \( n \geq \ell \) and choose \( C \in S_n(\mathbb{C}^\mu), v \in \mathbb{C}^n \) such that \( \mathcal{M}_{C, v} \) is linearly independent. For this \( C \) and a given \( H \in M_n(\mathbb{C}^\mu) \), the border vector evaluated at \((C, H^*)\) is

\[V(C)[H^*] = \bigoplus_{j=1}^\mu \begin{pmatrix} H_j^* m_1(C) \\ \vdots \\ H_j^* m_N(C) \end{pmatrix}.\]

By linear independence of \( \mathcal{M}_{C, v} \),

\[(2.9) \quad \{ V(C)[H^*]v : H \in M_n(\mathbb{C}^\mu) \} = \mathbb{C}^{\mu n N}.\]

Let \( \alpha \in S_n(\mathbb{C}^\mu) \) be given. To prove that \( Z(\alpha) \in M_d(\mathbb{C}) \otimes M_{\mu n N}(\mathbb{C}) \) is positive semidefinite, let \( z \in \mathbb{C}^d \otimes \mathbb{C}^{\mu n N} \) be given. There exist \( \gamma_1, \ldots, \gamma_d \in \mathbb{C}^d \) and \( u_1, \ldots, u_d \in \mathbb{C}^{\mu n N} \) such that \( z = \sum \gamma_a \otimes u_a \). By equation (2.9), for each \( 1 \leq a \leq d \), there exist \( H^a \in M_n(\mathbb{C}^\mu) \) such that \( u_a = V(C)[(H^a)^*]v \). Let \( \beta_j \) denote the \( d \times 1 \) block matrix with \((a, 1)\) entry \( H^a_j \). Thus \( \beta_j \in M_{dn, n}(\mathbb{C}) \) and \( \beta \in M_{dn, n}(\mathbb{C}^\mu) \). Let \( v_a = e_a \otimes v \in \mathbb{C}^d \otimes \mathbb{C}^n \), where \( \{ e_1, \ldots, e_d \} \) is the standard orthonormal basis for \( \mathbb{C}^d \).
Set $A = I_d \otimes C \in \mathbb{S}_{dn}(\mathbb{C}^n)$. Thus, $A$ is the direct sum of $C$ with itself $d$-times. Let $\Gamma = \sum_{b=1}^d \gamma_b \otimes v_b$ and compute

\[(I_d \otimes V(A)[\beta^*])\Gamma = \sum_{b=1}^d \gamma_b \otimes V(A)[\beta^*](e_b \otimes v)\]

\[= \sum_{b=1}^d \gamma_b \otimes V(C)[(H^b)^*]v = \sum_{b=1}^d \gamma_b \otimes u_b = z.\]  

(2.10)

Let $\delta \in \mathbb{S}_n(\mathbb{C}^n)$ be given and let $W, R, S$ have the form given in Proposition 2.1 item (iii). Since $q$ is convex in $x$, item (iii) of Proposition 2.1 implies

\[(I_d \otimes W)^*[q(R,S)](I_d \otimes W) \succeq q(A,X).\]  

(2.11)

Proposition 2.6 and equation (2.11) give

\[(I_d \otimes V(A)[\beta^*]) (I_d \otimes V(A)[\beta^*])^* \succeq 0.\]  

(2.12)

Combining equations (2.12) and (2.10) gives,

\[0 \leq (\mathcal{Z}(\alpha)(I_d \otimes V(A)[\beta^*])\Gamma, (I_d \otimes V(A)[\beta^*])\Gamma) = (\mathcal{Z}(\alpha)z, z)\]

and thus $\mathcal{Z}(\alpha) \succeq 0$.

At this point, it has been shown that there is an $\ell$ such that if $n \geq \ell$ and $\alpha \in \mathbb{S}_n(\mathbb{C}^n)$, then $\mathcal{Z}(\alpha) \succeq 0$. Hence, by a standard direct sum argument, $\mathcal{Z}(\alpha) \succeq 0$ for all $n$ and $\alpha \in \mathbb{S}_n(\mathbb{C}^n)$; that is $\mathcal{Z}$ is a positive polynomial. Hence $\mathcal{Z}$ factors [M] in the sense that there exists a (not necessarily square) matrix polynomial $F$ such that $\mathcal{Z}(\alpha) = F(\alpha)^*F(\alpha)$. Consequently,

\[q(a, x) = V(a)[x]^*\mathcal{Z}(\alpha)V(a)[x] = \Lambda(a, x)^*\Lambda(a, x),\]

where $\Lambda(a, x) = F(a)V(a)[x]$ is linear in $x$ and the proof is complete.  

\[\square\]

2.5. Biconvexity. This section concludes by collecting consequences of Theorem 1.2 for later use. Let $\mathcal{L}$ denote the set of words in $a, x$ of degree at most two in both $a$ and $x$, but excluding those of the forms $a_ja_i x_k x_m$ and $x_m x_k a_i a_j$.

Corollary 2.7. Suppose $p(a, x)$ is a hermitian $d \times d$ matrix polynomial. If $p$ is convex in $x$ and has degree at most two in $a$, then $p$ contains no words of the form $x_j x_k a_i a_m$ or $a_m a_k x_l x_j$; that is $p(a, x) \in M_d \otimes \text{span} \mathcal{L}$.

Proof. From Theorem 1.2,

\[p(a, x) = L(a, x) + \Lambda(a, x)^*\Lambda(a, x),\]

for matrix-valued polynomials $L$ and $\Lambda$, where $L$ is affine linear in $x$ and $\Lambda$ is linear in $x$. Since $p$ has degree at most two in $a$, it is immediate that $L(a, x)$ has degree at most two in $a$
and thus is a (matrix-valued) linear combination of elements of \( \mathcal{L} \). Let \( N \) denote the degree of \( \Lambda \) in \( a \) and, arguing by contradiction, suppose \( N \geq 2 \). Write
\[
\Lambda(a, x) = \sum_{u=0}^{N} \Lambda_u(a, x),
\]
where \( \Lambda_u(a, x) \) is homogeneous of degree \( u \) in \( a \). By assumption \( \Lambda_N(a, x) \neq 0 \). Hence, by Lemma 2.4, there exist \( A, X \) such that \( \Lambda_N(A, X) \neq 0 \). It follows that the matrix-valued polynomial of the single real variable \( t \),
\[
F(t) = \Lambda(tA, X) = \sum_{u=0}^{N} t^u \Lambda_u(A, X)
\]
has degree \( N \). Hence
\[
p(tA, X) = L(tA, X) + \Lambda(tA, X)^* \Lambda(tA, X) = L(tA, X) + F(t)^* F(t)
\]
has degree \( 2N \geq 4 \) in \( t \), contradicting the assumption that \( p \) has degree at most two in \( A \). We conclude that \( \Lambda(a, x) \) has degree at most one in both \( a \) and \( x \) and the proof is complete. □

**Corollary 2.8.** Suppose \( p(a, x) \) is a hermitian \( d \times d \) matrix polynomial. If \( p \) is convex in both \( a \) and \( x \) (separately), then \( p \) has degree at most two in both \( a \) and \( x \) and contains no words of the form \( x_j x_\ell \) or \( a_k a_m \); that is \( p(a, x) \in M_d \otimes \text{span} \mathcal{L} \).

**Proof.** If the hermitian polynomial \( p(a, x) \) is convex in both \( a \) and \( x \), then Theorem 1.2 holds with the roles of \( a \) and \( x \) interchanged. In particular, if \( p \) is convex in both \( a \) and \( x \), then \( p \) has degree at most two in both \( a \) and \( x \) and this result thus follows from Corollary 2.7. □

### 3. \( xy \)-Convex Hermitian Polynomials

Proposition 1.3 and Theorem 1.1 are proved in this section. The proof strategy is to show \( xy \)-convexity here is to proceed directly from Corollary 2.8. For notational consistency with [JKMMP21] we use \( x = (x_1, \ldots, x_\mu) \) and \( y = (y_1, \ldots, y_\mu) \) instead of \( a, x \) for the two classes of variables.

**Proposition 3.1 (Proposition 4.1, JKMMP21).** A triple \( ((X, Y), V) \) is an \( xy \)-pair if and only if, up to unitary equivalence, it has the block form
\[
(3.1) \quad X_j = \begin{pmatrix} X_{0j} & A_j & 0 \\ A_j^* & \ast & \ast \\ 0 & \ast & \ast \end{pmatrix}, \quad Y_k = \begin{pmatrix} Y_{0k} & 0 & C_k \\ 0 & \ast & \ast \\ C_k^* & \ast & \ast \end{pmatrix}, \quad V = (I \ 0 \ 0)^*,
\]

\( 1 \leq j, k \leq \mu \). Thus, a polynomial \( p(x, y) \in M_d(\mathbb{C}(x, y)) \) is \( xy \)-convex if and only if
\[
(I_d \otimes V)^* p(X, Y)(I_d \otimes V) - p(X_0, Y_0) \succeq 0
\]
for each \( xy \)-pair \( ((X, Y), V) \) of the form of equation (3.1).
Recall the definition of $\mathcal{L}$ from Subsection 2.5.

**Proof of Proposition 1.3.** To show that $p$ is convex in $x$ and $y$ separately, simply replace $X_1, X_2, Y$ in the proof [JKMMP21, Lemma 4.3] with $X^1, X^2, Y \in \mathbb{S}_n(\mathbb{C}^\mu)$. To prove item (ii) implies item (iii), let $\mathcal{W}_1$ denote the words of degree at most one in each of $x$ and $y$ separately, and let $\mathcal{W}_2$ denote the set of words that have degree at least two, but no more than two in each of $x, y$, but contains none of the words of the form $x_jx_\ell y_ky_m$ or $(x_jx_\ell y_ky_m)^*$, for $1 \leq j, k, \ell, m \leq \mu$.

Since $p(x, y)$ is convex in $x$ and $y$ separately, from Corollary 2.8, $p$ has the form,

$$p(x, y) = l(x, y) + q(x, y),$$

where

$$\ell(x, y) = \sum_{w \in \mathcal{W}_1} p_w w, \quad q(x, y) = \sum_{w \in \mathcal{W}_2} p_w w,$$

for some $p_w \in M_d(\mathbb{C})$.

Let $\mathcal{W}_{2,x}$ denote those words in $\mathcal{W}_2$ that have degree two in $x$. Define $\mathcal{W}_{2,y}$ similarly. A computation shows

$$\frac{1}{2} p_{x,x}(x, y)[x] = \frac{1}{2} q_{x,x}(x, y)[x] = \sum_{w \in \mathcal{W}_{2,x}} p_w w;$$

that is,

$$\frac{1}{2} p_{xx}(x, y)[x] = \frac{1}{2} q_{xx}(x, y)[x] = \sum_{j,k,\ell,m=1}^\mu [p_{x,x} x_j x_j x_\ell x_j x_\ell + p_{x,y} x_j x_j y_k x_\ell y_k + p_{y,y} x_j x_j y_k y_m x_\ell y_k + p_{y,y} y_k y_m x_\ell y_k y_m x_\ell + p_{y,y} x_j y_k y_m x_\ell y_k y_m x_\ell]$$

Similarly,

$$\frac{1}{2} p_{y,y}(x, y)[y] = \frac{1}{2} q_{y,y}(x, y)[y] = \sum_{w \in \mathcal{W}_{2,y}} p_w w$$

Since $p(x, y)$ is convex in $x$ and $y$ separately, the partial Hessian of $p$ with respect to $x$ as well as $y$ is positive. In particular,

$$p_{xx}(x, y)[x], \quad p_{yy}(x, y)[y] \succeq 0.$$  

Let $\mathcal{W}_{1,x}$ denote those words in $\mathcal{W}_1$ that have degree one in $x$. Define $\mathcal{W}_{1,y}$ similarly. By [M, Theorem 0.2], the positivity condition in equation (3.2) implies there exist an $N$ and $N \times d$ matrix-valued free polynomials $f(x, y)$ and $g(x, y)$ such that

$$p_{xx}(x, y)[x] = f(x, y)^* f(x, y) \quad p_{yy}(x, y)[y] = g(x, y)^* g(x, y),$$
where
\[ f(x, y) = \sum_{w \in W_{1,x}} f_w w = \sum_{j,k=1}^{\mu} f_{x_j} x_j + f_{y_k} y_k y_k + f_{y_k} x_j y_k \]
and similarly, \( g(x, y) = \sum_{w \in W_{1,y}} g_w w \).

Let \( W_{1,x,y} \) denote the words of degree one in both \( x \) and \( y \). Let \( x \) and \( y \) denote the column vectors
\[ x = (x_j)_{j=1}^{\mu}, \quad y = (y_j)_{j=1}^{\mu} \]
and let \( v \) denote the column vector
\[ v = (w)_{w \in W_{1,x,y}}. \]

Let
\[ W_x = \begin{pmatrix} x \end{pmatrix}, \quad W_y = \begin{pmatrix} y \end{pmatrix}. \]

Likewise, let \( F_0, F_1 \) and \( F \) denote the row vectors,
\[ F_0 = (f_{x_j})_{j=1}^{\mu}, \quad F_1 = (f_w)_{w \in W_{1,x,y}}, \quad F = (F_0 \ F_1) \]
and similarly
\[ G_0 = (g_{x_j})_{j=1}^{\mu}, \quad G_1 = (g_w)_{w \in W_{1,x,y}}, \quad G = (G_1 \ G_0). \]

Thus,
\[ f = F W_x, \quad g = G W_y \]
and
\[ W_x^* F^* F W_x = f^* f, \quad W_y^* G^* G W_y = g^* g. \]

Let
\[ P = \begin{pmatrix} p_{u,w} \end{pmatrix}_{w \in W_{1,x,y}} \]
and observe that
\[ F^* F = \begin{pmatrix} F_0^* F_0 & F_0^* F_1 \\ F_1^* F_0 & P \end{pmatrix}, \quad G^* G = \begin{pmatrix} P & G_0^* G_1 \\ G_1^* G_0 & G_0^* G_0 \end{pmatrix}. \]

Let
\[ M = \begin{pmatrix} F_0^* F_0 & F_0^* F_1 & 0 \\ F_1^* F_0 & P & G_1^* G_0 \\ 0 & G_0^* G_1 & G_0^* G_0 \end{pmatrix}, \quad W = \begin{pmatrix} x \\ v \\ y \end{pmatrix}, \]
and observe
\[ q(x, y) = W^* M W. \]

Since \( F^* F \) and \( G^* G \) are positive semidefinite, [T, Proposition 1] implies there is a \( d\mu \times d\mu \) matrix \( Q \) such that
\[ \hat{M} = M + \begin{pmatrix} 0 & 0 & Q \\ 0 & 0 & 0 \\ Q^* & 0 & 0 \end{pmatrix} = \begin{pmatrix} F_0^* F_0 & F_0^* F_1 & Q \\ F_1^* F_0 & P & G_1^* G_0 \\ Q^* & G_0^* G_1 & G_0^* G_0 \end{pmatrix} \succeq 0. \]
Letting $Q = (Q_{j,k})_{j,k=1}^\mu \in M_\mu \otimes M_d$, it follows that

$$p(x, y) = \lambda(x, y) + W^*\hat{\mathcal{M}}W,$$

where

$$\lambda(x, y) = l(x, y) - \left\{ \sum_{j,k=1}^{\mu} Q_{j,k}x_jy_k + Q_{j,k}^*y_kx_j \right\} = \sum_{j,k=1}^{\mu} \left[ p_{x,j}x_j + p_{y,j}y_j + (p_{x,j}y_k - Q_{j,k})x_jy_k + (p_{y,k}x_j - Q_{j,k}^*)y_kx_j \right].$$

Since $\hat{\mathcal{M}} \succeq 0$, there exists a matrix $R$ such that $\hat{\mathcal{M}} = R^*R$. Finally, letting $\Lambda(x, y) = RW^*$, it follows that $\Lambda(x, y)$ is a $d \times d$ matrix-valued $xy$-pencil and

$$p(x, y) = \lambda(x, y) + \Lambda(x, y)^*\Lambda(x, y).$$

To prove item (iii) implies item (i), let a triple $((X, Y), V)$ as in Proposition 3.1 be given and observe,

$$p(V^*(X, Y)V) = \lambda(V^*(X, Y)V) + \Lambda(V^*(X, Y)V)^*\Lambda(V^*(X, Y)V)$$

$$= (I_d \otimes V)^*\lambda(X, Y)(I_d \otimes V) + (I_d \otimes V)^*\Lambda(X, Y)^*(I_d \otimes V^*)\Lambda(X, Y)(I_d \otimes V)$$

$$\preceq (I_d \otimes V)^*\lambda(X, Y)(I_d \otimes V) + (I_d \otimes V)^*\Lambda(X, Y)^*\Lambda(X, Y)(I_d \otimes V)$$

$$= (I_d \otimes V)^*p(X, Y)(I_d \otimes V).$$

It follows from Proposition 3.1 that $p$ is $xy$-convex and the proof is complete. $\square$

It is well known that sums of squares representations can be certified with a semidefinite program (and hence are tractable). See for instance [KP10, KMP22, BS21, BKP03]. For simplicity, we treat the case $\mu = 1$. Thus, by Proposition 2.8,

$$p(x, y) = \sum p_{w,w},$$

where the sum is over words in $x, y$ of degree at most two in both $x$ and $y$, but excluding $x^2y^2$ and $y^2x^2$. Let

**Proposition 3.2.** The polynomial $p$ is $xy$-convex if and only if there is a positive semidefinite $Q = (Q_{j,k})_{j,k=1}^4$ such that

1. $Q_{j,k} = p_{w_jw_k}$ for $1 \leq j, k \leq 2$ and $3 \leq j, k \leq 4$;
2. $Q_{1,3} + Q_{3,1} = p_{w_1w_3}$; and
3. $Q_{2,4} + Q_{4,2} = p_{w_2w_4}$

where $w_1 = x, w_2 = y, w_3 = xy$ and $w_4 = yx$. 

**Sketch of proof.** Assuming such a $Q$ exists, factor $Q$ as $F^*F$ and write,

$$F = (F_1 \ F_2 \ F_3 \ F_4)$$
for some $N \times d$ matrices $F_j$. Let
\begin{equation}
\Lambda(x, y) = F_1x + F_2y + F_3yx + F_4xy.
\end{equation}
Thus $\Lambda(x, y)$ is an $xy$-pencil. A straightforward (but tedious) computation as in [JKMMP21] verifies,
\[ p(x, y) - \Lambda(x, y)^*\Lambda(x, y) = \lambda(x, y), \]
where $\lambda(x, y)$ has degree at most one in each of $x, y$.

Now suppose $p$ has the form
\[ p(x, y) = \lambda(x, y) + \Lambda(x, y)^*\Lambda(x, y), \]
where $\lambda(x, y)$ has degree at most one in each of $x, y$ and $\Lambda(x, y) = \Lambda_xx + \Lambda_yy + \Lambda_yx + \Lambda_yxy$.

In this case, let
\[ F = \begin{pmatrix} \Lambda_x & \Lambda_y & \Lambda_yx & \Lambda_yxy \end{pmatrix} \]
and check that $Q = F^*F \succeq 0$ has the desired properties. 

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Department of Mathematics, IIT Madras, Chennai - 600036, India.

Email address: bsriram@iitm.ac.in, bsriram80@yahoo.co.in

Department of Mathematics, IIT Madras, Chennai - 600036, India.

Email address: ma18d016@smail.iitm.ac.in

Scott McCullough, Department of Mathematics, University of Florida, Gainesville

Email address: sam@math.ufl.edu