Neutrino Mass, Muon Anomalous Magnetic Moment, and Lepton Flavor Nonconservation

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Abstract

If the generating mechanism for neutrino mass is to account for both the newly observed muon anomalous magnetic moment as well as the present experimental bounds on lepton flavor nonconservation, then the neutrino mass matrix should be almost degenerate and the underlying physics be observable at future colliders. We illustrate this assertion in two specific examples, and show that $\Gamma(\mu \to e\gamma)/m_\mu^5$ and $\Gamma(\tau \to e\gamma)/m_\tau^5$ are in the ratio $(\Delta m^2)^2_{\text{sol}}/2$, and $\Gamma(\tau \to \mu\gamma)/m_\tau^5$ are in the ratio $(\Delta m^2)^2_{\text{sol}}/2$, respectively, where the $\Delta m^2$ parameters are those of solar and atmospheric neutrino oscillations and bimaximal mixing has been assumed.
Any mechanism for generating a mass matrix for the three neutrinos $\nu_e$, $\nu_\mu$, and $\nu_\tau$ will have side effects \cite{1}, among which are lepton-flavor violating processes such as $\mu \to e \gamma$, $\tau \to \mu \gamma$, and $\mu - e$ conversion in nuclei, as well as an extra contribution to the muon anomalous magnetic moment \cite{2}. If the scale of this new physics is very high, as in the simplest models of neutrino mass \cite{3, 4}, then these side effects are suppressed by the high scale and are totally negligible phenomenologically. However, if this scale is of order 1 TeV or less, as in two recent proposals \cite{5, 6}, then the exciting possibility exists for all of these effects to be visible in present and future laboratory experiments.

In view of the newly announced measurement \cite{2} of the muon anomalous magnetic moment:

$$a_{\mu}^{\exp} = \frac{g_\mu - 2}{2} = 116592020(160) \times 10^{-11},$$  
(1)

which differs from the standard-model (SM) prediction \cite{7} by 2.6$\sigma$:

$$\Delta a_{\mu} = a_{\mu}^{\exp} - a_{\mu}^{SM} = 426 \pm 165 \times 10^{-11},$$  
(2)

a relatively large positive new contribution to $a_{\mu}$ is needed, hinting thus at possible new physics just above the electroweak scale. One may be tempted to believe that it is due to some new physics which has not appeared anywhere else before. On the other hand, a much better established hint of new physics already exists, i.e. neutrino mass from neutrino oscillations, so it is important to ask the question: Are they related?

In this paper we assume that the generating mechanism for neutrino mass is responsible for at least a significant part of the deviation shown in Eq. (2). We show that unless the neutrino mass matrix is almost degenerate, i.e. with 3 nearly equal mass eigenvalues, the $a_{\mu}$ measurement is in conflict with the $\tau \to \mu \gamma$ rate. This is because of the nearly maximal $\nu_\mu - \nu_\tau$ mixing for atmospheric neutrino oscillations \cite{8}, as explained below. We study two examples, one of which will be shown to be completely consistent with all other flavor-nonconserving
processes as well. We predict the relative decay rates of $\mu \rightarrow e\gamma$, $\tau \rightarrow e\gamma$, and $\tau \rightarrow \mu\gamma$ in terms of neutrino oscillation data, and show that these processes constrain the common neutrino mass scale and the solar neutrino oscillation solution in a very interesting range. In addition, the underlying new physics should be observable at future collider experiments.

Consider the following mass eigenstates of the 3 active neutrinos:

$$\nu_1 = \cos \theta \nu_e - \frac{\sin \theta}{\sqrt{2}} (\nu_\mu - \nu_\tau),$$  \hspace{1cm} (3)

$$\nu_2 = \sin \theta \nu_e + \frac{\cos \theta}{\sqrt{2}} (\nu_\mu - \nu_\tau),$$  \hspace{1cm} (4)

$$\nu_3 = \frac{1}{\sqrt{2}} (\nu_\mu + \nu_\tau),$$  \hspace{1cm} (5)

with masses $m_1 \leq m_2 \leq m_3$ respectively. This choice is dictated by the present knowledge of neutrino data regarding atmospheric \cite{atmospheric} and solar \cite{solar} neutrino oscillations. Specifically, $\nu_\mu - \nu_\tau$ mixing is assumed to be maximal to explain the atmospheric data (we comment on the effect of small allowed deviations from this assumption later), and $\nu_e$ mixes with the other two neutrinos with angle $\theta$ to account for the solar data. The $3 \times 3$ Majorana neutrino mass matrix in the $(\nu_e, \nu_\mu, \nu_\tau)$ basis is then given by

$$M_\nu = \begin{bmatrix}
    c^2 m_1 + s^2 m_2 & sc(m_2 - m_1)/\sqrt{2} & sc(m_1 - m_2)/\sqrt{2} \\
    sc(m_2 - m_1)/\sqrt{2} & (s^2 m_1 + c^2 m_2 + m_3)/2 & (-s^2 m_1 - c^2 m_2 + m_3)/2 \\
    sc(m_1 - m_2)/\sqrt{2} & (-s^2 m_1 - c^2 m_2 + m_3)/2 & (s^2 m_1 + c^2 m_2 + m_3)/2
\end{bmatrix},$$  \hspace{1cm} (6)

where $s \equiv \sin \theta$ and $c \equiv \cos \theta$. For $\theta = \pi/4$, it is known as bimaximal mixing.

In the Higgs triplet model \cite{Higgs} with $\xi \sim (3, 1)$ under the standard $SU(2)_L \times U(1)_Y$ gauge group, we have the interaction

$$f_{ij}[\xi^0 \nu_i \nu_j + \xi^+ (\nu_i l_j + l_i \nu_j)]/\sqrt{2} + \xi^{++} l_i l_j + h.c.$$  \hspace{1cm} (7)

which gives $(M_\nu)_{ij} = 2f_{ij}\langle \xi^0 \rangle$, and establishes a one-to-one correspondence between the neutrino mass matrix and the interaction terms. The smallness of $M_\nu$ follows from the smallness of $\langle \xi^0 \rangle$ \cite{Higgs}, while the couplings $f_{ij}$ can be large and the triplet mass $m_\xi$ can be
the order of the electroweak scale. Therefore, it follows from Eq. (7) that the muon $g - 2$ contribution is proportional to $f_{\mu e}^2 + f_{\mu\mu}^2 + f_{\mu\tau}^2$, whereas the $\tau \rightarrow \mu \gamma$ amplitude is proportional to $f_{\tau e} f_{\mu e} + f_{\tau \mu} f_{\mu \mu} + f_{\tau \tau} f_{\tau \mu}$. The former is proportional to $(m_3^2 + c^2 m_2^2 + s^2 m_1^2)/2$ and the latter to $(m_3^2 - c^2 m_2^2 - s^2 m_1^2)/2$. This means that a suppression of the $\tau \rightarrow \mu \gamma$ rate (relative to the muon $g - 2$) is possible only if $m_1 \simeq m_2 \simeq m_3$, i.e. a nearly degenerate neutrino mass matrix.

In the leptonic Higgs doublet model \[6\], $\mathcal{M}_\nu$ comes from the terms

$$\frac{1}{2} M_i \nu_{IR}^2 + h_{ij} \bar{N}_{IR} (\nu_j \eta^0 - l_j L \eta^+) + h.c.,$$

where $\eta \sim (2, 1/2)$ and carries lepton number $L = -1$, while the singlet fermions $N_R$ have $L = 0$. We assume now that all the heavy $N_R$‘s are equal in mass. Hence Eqs. (3) to (6) imply

$$h_{ij} = \begin{bmatrix} 2ch_1 & -\sqrt{2}sh_1 & \sqrt{2}sh_1 \\ 2sh_2 & \sqrt{2}ch_2 & -\sqrt{2}ch_2 \\ 0 & \sqrt{2}h_3 & \sqrt{2}h_3 \end{bmatrix},$$

with $m_i = 4h_i^2 (\eta^0)^2/M$. Again, $m_i$ is small because $<\eta^0>$ is small \[6\], allowing thus $h_i$ to be large and $M$ the order of the electroweak scale. In this case, the muon $g - 2$ contribution is proportional to $(m_3 + c^2 m_2 + s^2 m_1)/2$ and the $\tau \rightarrow \mu \gamma$ amplitude to $(m_3 - c^2 m_2 - s^2 m_1)/2$, again suppressing the latter relative to the former in the limit of degenerate neutrino masses.

In both of the above models, there are large contributions to $\Delta a_\mu$ as well as $l_i \rightarrow l_j \gamma$ coming from the interactions of Eqs. (7) and (8), as shown in Fig. 1. In the triplet model,

$$\Delta a_\mu = \sum_l \frac{10}{3} \frac{f_{\mu l}^2}{(4\pi)^2} \frac{m_\mu^2}{m_\xi^2}.$$  \hspace{1cm} (10)$$

In the limit of a degenerate neutrino mass matrix, i.e. $m_1 = m_2 = m_3 = 2f<\xi^0>$, this implies

$$m_\xi < 1174\sqrt{\alpha f} \text{ GeV},$$  \hspace{1cm} (11)$$

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where $\alpha_f = f^2/4\pi$ and the 90% confidence-level limit $\Delta a_\mu > 215 \times 10^{-11}$ has been used [4].

In the doublet model,

$$\Delta a_\mu = \sum_i \frac{h_i^2}{(4\pi)^2} \frac{m_i^2}{m_\mu^2} F_2(s_{N_i}),$$

where $s_{N_i} \equiv m_{N_i}^2/m_\eta^2$ and

$$F_2(x) = \frac{1 - 6x + 3x^2 + 2x^3 - 6x^2 \ln x}{6(1 - x)^4}.$$  

(13)

Assuming $s_{N_i} = 1$ [which gives $F_2(1) = 1/12$] and using Eq. (9) with all $h$’s equal, we then obtain

$$m_\eta < 371 \sqrt{\alpha_h} \text{ GeV},$$

(14)

where $\alpha_h = h^2/4\pi$. Comparison of Eq. (11) and Eq. (14) implies that masses below 1 TeV are expected in either model.

The $l_i \to l_j\gamma$ rate divided by the $l_i \to l_j\nu_i\bar{\nu}_j$ rate is given by

$$R(l_i \to l_j\gamma) = \frac{96\pi^3\alpha}{G_F^2 m_i^4} \left( |f_{M1}|^2 + |f_{E1}|^2 \right),$$

(15)

where $\alpha \simeq 1/137$ and $G_F$ is the Fermi constant. In the doublet model, the magnetic and electric dipole moment form factors are given by

$$f_{M1} = f_{E1} = \sum_k \frac{h_k l_k h_{kl} m_i^2}{4(4\pi)^2} \frac{m_i^2}{m_\eta^2} F_2(1).$$

(16)
For $\tau \to \mu\gamma$,
\[
\sum_k h_{k\tau}h_{k\mu} = 2(h_3^2 - c^2h_2^2 - s^2h_1^2) \simeq 2(h_3^2 - h_2^2) \simeq \frac{h_2^2}{m_\nu^2} (\Delta m^2)_{atm},
\]
where $m_\nu$ is the common mass of the 3 neutrinos. Hence the $\tau \to \mu\gamma$ branching fraction is given by
\[
B(\tau \to \mu\gamma) = B(\tau \to \mu\nu\bar{\nu}) \frac{\pi \alpha}{192 G_F^2} \left( \frac{\alpha_h}{m_\eta^2} \right)^2 \frac{(\Delta m^2)^2_{atm}}{m_\nu^4}.
\]

Suppose we do not have neutrino mass degeneracy, but rather a hierarchical neutrino mass matrix, then $(\Delta m^2)_{atm}/m_\nu^2$ would be equal to one, and using Eq. (14), we would obtain $B(\tau \to \mu\gamma) > 8.0 \times 10^{-6}$, well above the experimental upper limit of $1.1 \times 10^{-6}$. Note that this result, while presented for a specific model, is actually very general. If $\nu_3 = c\nu_\mu + s\nu_\tau$, there would be a suppression factor of $s^2/c^2$, but this is not available because atmospheric neutrino data require nearly maximal $\nu_\mu - \nu_\tau$ mixing.

Similarly, the $\mu \to e\gamma$ and $\tau \to e\gamma$ branching fractions are given by
\[
B(\mu \to e\gamma) = \frac{\pi \alpha}{192 G_F^2} \left( \frac{\alpha_h}{m_\eta^2} \right)^2 \frac{2s^2c^2}{m_\nu^4} (\Delta m^2)_{sol}^2,
\]
\[
B(\tau \to e\gamma) = B(\tau \to e\nu\bar{\nu}) B(\mu \to e\gamma).
\]
Hence we have the interesting relationship
\[
\frac{\Gamma(\mu \to e\gamma)}{m_\mu^5} : \frac{\Gamma(\tau \to e\gamma)}{m_\tau^5} : \frac{\Gamma(\tau \to \mu\gamma)}{m_\tau^5} = 2s^2c^2(\Delta m^2)_{sol}^2 : 2s^2c^2(\Delta m^2)_{sol}^2 : (\Delta m^2)_{atm}^2.
\]

The $\mu - e$ conversion ratio $R_{\mu e}$ in nuclei is given by
\[
R_{\mu e} = \frac{8\alpha^5 m_\mu^5 Z_{eff}^4 Z F_p(p_e)^2}{\Gamma_{capt} q^4} \left[ |f_{E0} + f_{M1}|^2 + |f_{E1} + f_{M0}|^2 \right],
\]
where $q^2 \simeq -m_\mu^2$ and for $^{13}$Al, $Z_{eff} = 11.62$, $F_p = 0.66$, and $\Gamma_{capt} = 7.1 \times 10^5 s^{-1} \|[1], [2]\]$. The charge-radius form factors are given by
\[
f_{E0} = -f_{M0} = \sum_i \frac{h_i h_{ie} m_\mu^2}{2(4\pi)^2 m_\eta^2} F_1(s_{Ni}),
\]
where

\[ F_1(x) = \frac{2 - 9x + 18x^2 - 11x^3 + 6x^3 \ln x}{36(1 - x)^4}, \]  

(24)

with \( F_1(1) = 1/24 \). In Fig. 2, using

\[ (\Delta m^2)_{atm} = 3 \times 10^{-3} \text{ eV}^2, \]  

(25)

and assuming the large-angle matter-enhanced solution of solar neutrino oscillations with

\[ (\Delta m^2)_{sol} = 3 \times 10^{-5} \text{ eV}^2, \]  

(26)

we plot \( B(\tau \to \mu \gamma) \), \( B(\mu \to e \gamma) \), and \( R_{\mu e} \) as functions of \( m_\nu \) for \( s^2 = c^2 = 1/2 \) and \( \alpha_h/m_\eta^2 = (371 \text{ GeV})^{-2} \). Hence these lines should be considered as lower bounds in the case of bimaximal mixing for neutrino oscillations.

We note that at \( m_\nu = 0.2 \text{ eV} \), \( B(\mu \to e \gamma) \) is at its present upper limit \([12]\) of \( 1.2 \times 10^{-11} \). If \( m_\nu > 0.2 \text{ eV} \) is desired, then the constraint from the nonobservation of neutrinoless double beta decay \([13]\) requires the \( m_{ee} \) element of Eq. (6) to be less than 0.2 eV. This is easily achieved by making \( m_1 < 0 \) but keeping \( m_{2,3} > 0 \), without affecting any of our results presented so far. However, we must then choose the large-angle mixing solution of solar neutrino oscillations, implying the observation of \( \mu \to e \gamma \) and \( \mu-e \) conversion in the planned experiments with the sensitivities down to \( 2 \times 10^{-14} \) \([14]\) and \( 2 \times 10^{-17} \) \([15]\) respectively. From Fig. 2 we see that an order-of-magnitude improvement of the present \( \tau \to \mu \gamma \) bound will also test this specific prediction. Thus \( B(\tau \to \mu \gamma) \), neutrinoless double beta decay, \( B(\mu \to e \gamma) \), and \( \mu-e \) conversion are all complementary to one another in probing the connection between \( m_\nu \) and \( \Delta a_\mu \).

However, the neutrino mixings need not be exactly bimaximal. Indeed, the mixing element \( |V_{e3}| \) is constrained to be small but may still be nonzero. Obviously the rate \( B(\tau \to \mu \gamma) \) is completely independent of this parameter and our conclusion that neutrinos must be degenerate in mass to explain the observed \( \Delta a_\mu \) remains unchanged. However, \( B(\mu \to e \gamma) \),
Figure 2: Lower bounds on $B(\tau \rightarrow \mu \gamma)$, $B(\mu \rightarrow e \gamma)$, and $R_{\mu e}$ from the measurement of $a_\mu$ in the leptonic Higgs doublet model, assuming bimaximal mixing of degenerate neutrinos.

and $R_{\mu e}$ receive additional contributions proportional to $|V_{e3}|^2(\Delta m^2)_{\text{atm}}^{2/3}$. For example, if $|V_{e3}| \sim 0.1$ one needs $m_\nu \sim 1$ eV to satisfy the present experimental bounds. Therefore, no fine tuning in the parameters of Fig. 2 is needed to comply with data if $|V_{e3}| \neq 0$. Nevertheless, the planned $\mu \rightarrow e \gamma$ experiments offer sensitive probes of the small mixing angle $|V_{e3}|$ in this scenario.

In the triplet model, the relevant form factors are calculated in Ref. [10]. We have again the relationship given by Eq. (21), but the corresponding $R_{\mu e}$ is not suppressed as in the
doublet model. The reason is that the form factors $f_{E0,M0}$ are now functions of $m_{\ell_i}^2/m_\xi^2$ which are different for different charged leptons, unlike $m_{N_i}^2/m_\eta^2$ which are the same for all $N$'s. As a result, $R_{\mu e}$ is of order $10^{-6}$ independent of $m_\nu$, which is definitely ruled out by experiment. In addition, the $\mu \to eee$ branching fraction [which occurs at tree level, but is suppressed by $(\Delta m^2)^2_{sol}$] also exceeds the present experimental bound for $m_\nu < 2.7$ eV if $s^2 = c^2 = 1/2$, again assuming the large-angle matter-enhanced solution of solar neutrino oscillations. Thus the triplet model cannot explain $\Delta a_\mu$ even if neutrino masses are degenerate. It is of course still perfectly viable as a model of neutrino masses [5], but it will have no significant contribution to the muon $g-2$.

Since the $g-2$ announcement [2], there have been many papers [17] dealing with its possible explanation. Ours is the only one relating it to another existing hint of new physics, i.e. neutrino mass from neutrino oscillations. A glance at Fig. 2 shows that $m_\nu = 0.2$ eV is a very interesting number. It is the present upper limit of a Majorana neutrino mass from neutrinoless double beta decay; it also corresponds to the present upper limits of $B(\mu \to e\gamma)$ and $\mu - e$ conversion in nuclei. Planned experiments on all three fronts are in progress and will test our proposed connection between $m_\nu$ and $\Delta a_\mu$. They will also probe the possibly nonzero neutrino mixing angle $V_{e3}$. In addition, the $\tau \to \mu \gamma$ branching fraction is just an order-of-magnitude away, and Eq. (14) implies that the leptonic Higgs doublet $(\eta^+, \eta^0)$ as well as the fermion singlets $N_{iR}$ are not far away from being discovered in future colliders as proposed in Ref. [6]. A neutrino mass of 0.2 eV is also very relevant in cosmology [18] and astrophysics [19].

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Erratum

Equation (12) is missing a minus sign on the right-hand side. This renders our model as it stands unsuitable for explaining the positive $\Delta a_\mu$ observed. On the other hand, if we extend our model to include supersymmetry as recently proposed [1], then a positive $\Delta a_\mu$ from the exchange of the supersymmetric particles $\tilde{N}_i$ and $\tilde{\eta}$ can be obtained. Equation (12) becomes correct with $m_\eta$ replaced by $m_{\tilde{\eta}}$ and $s_{N_i} \equiv m_{\tilde{N}_i}^2/m_{\tilde{\eta}}^2$. With this replacement in the rest of our paper, our results remain unchanged.

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