The Willmore flow of Hopf-tori in the 3-sphere

Ruben Jakob

Abstract. In this article, the author investigates flow lines of the classical Willmore flow, which start to move in a smooth parametrization of a Hopf-torus in $S^3$. We prove that any such flow line of the Willmore flow exists globally, in particular does not develop any singularities, and subconverges to some smooth Willmore-Hopf-torus in every $C^m$-norm. Moreover, if in addition the Willmore energy of the initial immersion $F_0$ is required to be smaller than or equal to the threshold $8\pi^2\sqrt{2}$, then the unique flow line of the Willmore flow, starting to move in $F_0$, converges fully to a conformally transformed Clifford torus in every $C^m$-norm, up to time dependent, smooth reparametrizations. Key instruments for the proofs are the equivariance of the Hopf-fibration $\pi : S^3 \to S^2$ w.r.t. the effect of the $L^2$-gradient of the Willmore energy applied to smooth Hopf-tori in $S^3$ and to smooth closed regular curves in $S^2$, a particular version of the Lojasiewicz–Simon gradient inequality, and a well-known classification and description of smooth, arc-length parametrized solutions of the Euler–Lagrange equation of the elastic energy functional in terms of Jacobi elliptic functions and elliptic integrals, dating back to the 80s.

1. Introduction

In this article, the author investigates the long-term behavior of the classical Willmore flow:

$$\partial_t f_t = -\frac{1}{2} \left( \Delta_{f_t} \mathbf{H}_{f_t} + Q(A_{f_t}^0)(\mathbf{H}_{f_t}) \right) \equiv -\nabla_{L^2} \mathcal{W}(f_t),$$  

(1)

moving families of smooth immersions $f_t$ of a compact smooth torus $\Sigma$ into the standard 3-sphere $S^3$. It is well known that the $L^2$-gradient flow of the Willmore functional (2) has a unique smooth short-time solution, starting to move in any prescribed smooth initial immersion $F_0 : \Sigma \to M$, for any Riemannian target manifold $M$ of dimension $\geq 3$. But there are only few results addressing the long-term behavior and full convergence of this flow. Inspired by Kuwert’s and Schätzle’s optimal convergence result on the classical Willmore flow, moving spherical immersions into $\mathbb{R}^3$—see [15], Theorem 5.2—and also by the recent article [8], treating the Willmore flow of tori of revolution in $\mathbb{R}^3$, we are going to prove a global existence—a subconvergence—and a full convergence result in Theorem 1 for flow lines of the Willmore flow in $S^3$, starting to move in parametrizations of arbitrary smooth Hopf-tori in $S^3$. More precisely, we

Mathematics Subject Classification: 53C42, 53E40, 35R01, 58J35, 11Z05

Keywords: Willmore flow, Willmore functional, Hopf-tori, Elastic energy.
are going to employ classical ideas and results due to Singer and Langer [17, 18] about the classification and quantitative analysis of elastic curves in simply connected space forms \( M \) and some modern improvements and clarifications in [21, 23]—treating the case in which \( M \) is the hyperbolic plane—in order to accurately estimate the size of the gap between the Willmore energy of the Clifford torus and of any further Willmore-Hopf-torus in \( S^3 \) within the framework of Elliptic Integrals and Elliptic Functions. Such an estimate plays a key role in our ambitious pursuit of an optimal statement about “full, smooth convergence” of flow lines of the Willmore flow in \( S^3 \) to the Clifford torus—at least up to conformal equivalence; see our Theorem 1. The basic notion of this article is the “Willmore energy of a closed surface”:

\[ W(f) := \int_\Sigma K^M_f + \frac{1}{4} |H_f|^2 \, d\mu_f, \quad (2) \]

which is well defined for \( C^2 \)-immersions \( f : \Sigma \to M \) mapping any closed smooth orientable surface \( \Sigma \) into an arbitrary smooth Riemannian manifold \( M \), where \( K^M_f(x) \) denotes the sectional curvature of \( M \) w.r.t. the “immersed tangent plane” \( Df_x(T_x\Sigma) \) in \( T_{f(x)}M \). In those cases being relevant in this article, we will only have \( K_f \equiv 0 \) for \( M = \mathbb{R}^n \) or \( K_f \equiv 1 \) for \( M = S^n, n \geq 3 \).

Now, given an immersion \( f : \Sigma \to S^3 \), we endow the torus \( \Sigma \) with the pullback \( f^*g_{\text{euc}} \) of the Euclidean metric on \( S^3 \), i.e., with coefficients \( g_{ij} := \langle \partial_i f, \partial_j f \rangle_{\mathbb{R}^4} \), and we let \( A_f \) denote the second fundamental form of the immersion \( f \), defined on pairs of tangent vector fields \( X, Y \) on \( \Sigma \) by

\[ A_f(X, Y) \equiv A_{f,S^3}(X, Y) :=
\]

\[ = D_X(D_Y(f)) - P^{\text{Tan}(f)}(D_X(D_Y(f))) \equiv (D_X(D_Y(f)))^{\perp_f} \quad (3) \]

where \( D_X(V) \big|_X \) denotes the projection of the derivative of a vector field \( V : \Sigma \to \mathbb{R}^4 \) in direction of the tangent vector field \( X \) into the respective fiber \( T_{f(x)}S^3 \) of \( TS^3 \), \( P^{\text{Tan}(f)} : \bigcup_{x \in \Sigma} \{x\} \times T_{f(x)}S^3 \to \bigcup_{x \in \Sigma} \{x\} \times T_{f(x)}f(\Sigma) =: \text{Tan}(f) \) denotes the bundle morphism which projects the entire tangent space \( T_{f(x)}S^3 \) orthogonally onto its subspace \( T_{f(x)}f(\Sigma) \), the tangent space of the immersion \( f \) in \( f(x) \), for every \( x \in \Sigma \), and where \( ^{\perp_f} \) abbreviates the bundle morphism \( \text{Id}_{T_{f(x)}S^3} - P^{\text{Tan}(f)} \). Furthermore, \( A^0_f \) denotes the trace-free part of \( A_f \), i.e.,

\[ A^0_f(X, Y) := A_{f,S^3}(X, Y) - \frac{1}{2} g(X, Y) H_{f,S^3} \]

and \( H_{f,S^3} := \text{Trace}(A_{f,S^3}) \equiv g^{ij}_f A_{f,S^3}(\partial_i, \partial_j) \) (“Einstein’s summation convention”) denotes the mean curvature vector of \( f \), which we shall always abbreviate by \( H_f \), if there cannot arise any confusion with the mean curvature vector \( H_{f,\mathbb{R}^4} \) of \( f \) interpreted as an immersion into \( \mathbb{R}^4 \). Finally, \( Q(A^0_f) \) operates on sections \( \phi \) into the normal bundle of \( f \), by assigning \( Q(A^0_f)(\phi) := A^0_f(e_i, e_j)(A^0_f(e_i, e_j), \phi) \), which is by definition again a section of the normal bundle of \( f \) within \( TS^3 \). We know from [29, Section
2], respectively, from Proposition 3.1 in [24] that the first variation of the Willmore functional $\delta W(f, \phi)$ in a smooth immersion $f : \Sigma \rightarrow S^3$, in direction of a smooth section $\phi$ of the normal bundle of $f$ within $TS^3$, is given by:

$$\delta W(f, \phi) = \frac{1}{2} \int_{\Sigma} \langle \Delta_{\perp} f H f + Q(A^0_f)(H f), \phi \rangle \, d\mu_f =: \int_{\Sigma} \langle \nabla_{L^2} W(f), \phi \rangle \, d\mu_f,$$

which justifies the second equality sign in (1).

In contrast to Kuwert’s and Schätzle’s convergence theorem in [15] and in contrast to Theorem 1.3 in the new article [8], treating the Willmore flow of tori of revolution in $\mathbb{R}^3$, we do not necessarily have to start the Willmore flow below the prominent $8\pi$-energy threshold, because our approach does not rely on the Li–Yau inequality [20]—relating the pointwise 2-dimensional Hausdorff-density of the image of an immersion into some $\mathbb{R}^n$ to its Willmore energy. Instead, we will exploit a certain type of “dimension reduction”—worked out in Sect. 3 below—by means of a careful analysis of the Hopf-fibration, we will prove in Sect. 4 the absence of singularities along the considered flow lines of the Willmore flow in $S^3$, and we will compute the critical levels of the elastic energy (15) as precisely as possible in our appendix, Section 5, in order to obtain in Sect. 4 the surprisingly strong “energy threshold” $8\pi^2/\sqrt{2}$ for full, smooth convergence of the considered flow lines to the Clifford torus in $S^3$, up to Möbius-transformations of $S^3$, which is the final result of Theorem 1 below.

Remark 1. We should point out here, at the end of the introduction, that the entire investigation of this article was motivated by the examination of another variant of the Willmore flow, namely of the evolution equation

$$\partial_t f_t = -\frac{1}{2} |A^0_{f_t}|^{-4} \left( \Delta_{\perp} f_t H f_t + Q(A^0_{f_t})(H f_t) \right) \equiv - |A^0_{f_t}|^{-4} \nabla_{L^2} W(f_t),$$

for differentiable families of $C^4$-immersions $f_t : \Sigma \rightarrow M$, with $M = \mathbb{R}^n$ or $M = S^n$. As already pointed out in the author’s article [11], the “umbilic-free condition” $|A^0_{f_t}|^2 > 0$ on $\Sigma$, implies $\chi(\Sigma) = 0$ for the Euler characteristic of the surface $\Sigma$, which forces the flow (5) to be only well defined on families of sufficiently smooth umbilic-free tori, immersed into $\mathbb{R}^n$ or $S^n$. In [11], the author has proved short-time existence and uniqueness of flow lines of this flow, starting to move in a umbilic-free immersion of some fixed smooth compact torus $\Sigma$ into $\mathbb{R}^n$ or $S^n$, for $n \geq 3$. The big advantage of this flow—compared to the classical Willmore flow—is its conformal invariance, which means that any family $\{f_t\}_{t \in [0, T]}$ of $C^4$-immersions $f_t : \Sigma \rightarrow S^n$ without any umbilic points, i.e., with $|A^0_{f_t}|^2 > 0$ on $\Sigma \forall t \in [0, T]$, solves flow equation (5) on $\Sigma \times [0, T]$, if and only if its composition $\Phi(f_t)$ with an arbitrary applicable Möbius-transformation $\Phi$ of $S^n$ solves the same flow equation on $\Sigma \times [0, T]$ again. This property allows us to project flow lines of this flow with original target manifold $\mathbb{R}^3$ into $S^3$, which is a compact Lie-group, diffeomorphic to $SU(2)$, the unit sphere within the division algebra $\mathbf{H}$ of quaternions and which can be fibered by great circles by means of the Hopf-fibration $\pi : S^3 \rightarrow S^2$. These rather elementary algebraic
and topological properties of the 3-sphere finally turned out to be very useful, in order to investigate the classical Willmore flow in the 3-sphere, and the method of this article can actually be interpreted as a straightforward continuation of Pinkall’s paper [25] about concrete conformal embeddings of flat tori into $S^3$ and about Hopf-tori which are “Willmore” but not “minimal” in $S^3$. Unfortunately, flow lines of the conformally invariant Willmore flow (5) might in general develop singularities, even if they start moving in smooth parametrizations of Hopf-tori with Willmore energies below $8\pi$. The author has already started to address this apparently difficult problem in his preprint [12], which relies on the preparatory results of Sects. 2 and 3 in this article.

\section{Main results and main tools}

The basic idea of this article is the use of the Hopf-fibration

$$S^1 \hookrightarrow S^3 \xrightarrow{\pi} S^2$$

and its equivariance w.r.t. the first variation of the Willmore energy along Hopf-tori in $S^3$ and closed curves in $S^2$; see formula (32). In order to work with the most effective formulation of the Hopf-fibration, we shall consider $S^3$ as the subset of the four-dimensional $\mathbb{R}$-vector space $H$ of quaternions, whose elements have length 1, i.e., $S^3 := \{ q \in H \mid \bar{q} \cdot q = 1 \}$. We shall use the usual notation for the generators of the division algebra $H$, i.e., $1, i, j, k$. We therefore decompose every quaternion in the way $q = q_1 + i q_2 + j q_3 + k q_4$, for unique “coordinates” $q_1, q_2, q_3, q_4 \in \mathbb{R}$, and especially the conjugate of any quaternion $q$ can be written as $\bar{q} = q_1 - i q_2 - j q_3 - k q_4$. Moreover, we identify

$$S^2 = \{ q \in \text{span}\{1, j, k\} \mid \bar{q} \cdot q = 1 \} = S^3 \cap \text{span}\{1, j, k\},$$

and we shall use the particular involution $q \mapsto \bar{q}$ of $H$, which fixes the generators $1, j$ and $k$, but sends $i$ to $-i$. Following Section 2 of [25], we employ this involution to write the Hopf-fibration in the elegant way

$$\pi : H \longrightarrow H, \quad q \mapsto \bar{q} \cdot q.$$  \hfill (6)

We gather its most important properties in the following elementary lemma.

\textbf{Lemma 1.} \begin{enumerate}
\item $\pi(S^3) = S^3 \cap \text{Span}\{1, j, k\} = S^2$.
\item $\pi(e^{i\phi}q) = \pi(q)$, $\forall \phi \in \mathbb{R}$ and $\forall q \in S^3$.
\item The group $S^3$ acts isometrically on $S^3$ by right (and left) multiplication, and it acts isometrically on $S^2$, which means that every $r \in S^3$ induces the rotation

$$q \mapsto \bar{r} \cdot q \cdot r, \quad \text{for} \quad q \in S^2.$$ \hfill (7)

Moreover, there holds

$$\pi(q \cdot r) = \bar{r} \cdot \pi(q) \cdot r \quad \forall q, r \in S^3,$$ \hfill (8)
\end{enumerate}
which means that right multiplication on $S^3$ translates equivariantly via the Hopf-fibration to rotation in $S^2$.

(4) The differential of $\pi$ in any $q \in H$, applied to some $v \in H$, reads

$$D\pi_q(v) = \tilde{v} \cdot q + \tilde{q} \cdot v.$$  \hfill (9)

Proof. Assertions (1)–(3) follow immediately from the definition in (6) and can also be found in Section 2 of [25]. In order to prove formula (9), we choose a smooth curve $[t \mapsto \gamma(t)]$, satisfying $\gamma(0) = q$ and $\gamma'(0) = v$ and compute by means of the linearity of the differential of the involution $I : q \mapsto \tilde{q}$:

$$D\pi_q(v) = \partial_t \pi(\gamma(t))|_{t=0} = \partial_t (\tilde{\gamma}(t) \gamma(t))|_{t=0}$$

$$= DI(\gamma(0)) \cdot \gamma'(0) + I(\gamma(0)) \cdot \gamma'(0) = I(v) \cdot q + \tilde{q} \cdot v.$$  \hfill \Box

By means of the Hopf-fibration, we introduce Hopf-tori in the following definition. See also Section 2 of [25] for further explanations:

**Definition 1.**  (1) Let $\gamma : [a, b] \longrightarrow S^2$ be a regular, smooth and closed curve in $S^2$, and let $\eta : [a, b] \rightarrow S^3$ be a smooth lift of $\gamma$ w.r.t. $\pi$ into $S^3$, i.e., a smooth map from $[a, b]$ into $S^3$ satisfying $\pi \circ \eta = \gamma$. We define

$$X(s, \varphi) := e^{i\varphi} \cdot \eta(s), \quad \forall (s, \varphi) \in [a, b] \times [0, 2\pi].$$  \hfill (10)

and note that $(\pi \circ X)(s, \varphi) = \gamma(s), \forall (s, \varphi) \in [a, b] \times [0, 2\pi]$.

(2) We call this map $X$ the “Hopf-torus-immersion” and its image, respectively $\pi^{-1}(\text{trace}(\gamma))$ the “smooth Hopf-torus” in $S^3$ w.r.t. the smooth curve $\gamma$.

Preparing ourselves for precise computations which involve Hopf-tori and lifts w.r.t. the Hopf-fibration $\pi$, we need the following lemma, already anticipating the elementary result of Lemma 3 in the appendix.

**Lemma 2.** Let $\gamma : R/LZ \longrightarrow S^2$ regularly parametrize a smooth closed curve in $S^2$. Moreover, let $\eta : [0, L] \rightarrow S^3$ be a smooth lift of $\gamma$ w.r.t. $\pi$, having constant speed $|\eta'| \equiv 1$ and intersecting the fibers of $\pi$ perpendicularly, see here Lemma 3 below. Then, there holds:

(1) $\eta' = u \eta$ for some function $u : [0, L] \longrightarrow \text{span}[j, k]$ satisfying $|u(s)| \equiv 1$.

(2) $\partial_\varphi X(s, \varphi) = ie^{i\varphi} \eta(s)$ and thus $\mathfrak{H}(\eta'(s) ie^{i\varphi} \eta(s)) \equiv 0$ and $|\partial_\varphi X(s, \varphi)| \equiv 1$.

(3) $\partial_s X(s, \varphi) = e^{i\varphi} u(s) \eta(s)$, and thus also $|\partial_s X(s, \varphi)| \equiv 1$ and

$$\mathfrak{H}(\partial_s X(s, \varphi) \cdot \partial_\varphi X(s, \varphi)) \equiv 0, \quad \forall (s, \varphi) \in [0, L] \times [0, 2\pi].$$  \hfill (11)

(4) $\gamma'(s) = 2 \tilde{\eta}(s) u(s) \eta(s), \forall s \in [0, L]$. Every horizontal smooth lift $\eta$ of $\gamma$ w.r.t. $\pi$ intersects each fiber of $\pi$ exactly $m \geq 1$ times on $[0, L]$ with constant speed 1, if and only if $\gamma$ performs $m$ loops through its trace on $[0, L]$ with constant speed 2. In this case, the length of trace($\gamma$) is $2L/m$.  

Proof. (1) From $|\eta|^2 \equiv 1$ and $|\eta'|^2 \equiv 1$ on $[0, L]$ we infer that $\eta'(s)$ is a unit vector, which is perpendicular to $\eta(s)$, for every $s \in [0, L]$. Therefore, we can obtain the vector $\eta'(s)$ by means of a rotation of the vector $\eta'(s)$ about a right angle within $S^3$. In $\mathbf{H}$ this can be achieved by means of left multiplication with a unit-length quaternion $u(s)$, being contained in $\text{Span}[i, j, k]$. Hence, since we assume that $\eta$ is a horizontal lift of $\gamma$ w.r.t. $\pi$, we infer from the second statement of Lemma 1 that $u(s) \in \text{Span}[j, k]$, for every $s \in [0, L]$.

(2) The assertions of the second part of the lemma follow immediately from formula (10) and from the assumption on the lift $\eta$ to be horizontal w.r.t. $\pi$.

(3) We compute:

$$
\frac{\partial}{\partial s} X(s, \varphi) \cdot \overline{\varphi} X(s, \varphi) = \overline{\eta}(s) \overline{u}(s) e^{-i\varphi} i e^{i\varphi} \eta(s) = \overline{\eta}(s) \overline{u}(s) i \eta(s) = \overline{\eta}(s) i \eta(s).
$$

Using the formula $\Re(i \overline{\eta}(s) i e^{i\varphi} \eta(s)) \equiv 0$ from the second part of this lemma, we obtain especially $\Re(i \overline{\eta}(s) i \eta(s)) \equiv 0$, which proves formula (11).

(4) Using formula (9) and the chain rule, we compute:

$$
\gamma'(s) = (\pi \circ \eta)'(s) = D_\pi \eta(s)(\eta'(s)) = (I(\eta'(s)) \eta(s)) + (I(\eta(s)) \eta'(s)) = (\eta'(s) \overline{u}(s) \eta(s)) + (\eta(s) u(s) \eta(s)) = 2\eta(s) u(s) \eta(s),
$$

(12)

because $u(s) \in \text{Span}[j, k] \forall s \in \mathbf{R}/L\mathbf{Z}$, and therefore $\overline{u} \equiv u$. In particular, there holds $|\gamma'| \equiv 2$ on $\mathbf{R}/L\mathbf{Z}$. In fact, this statement is equivalent to $|\eta'| \equiv 1$ on $\mathbf{R}/L\mathbf{Z}$ on account of formula (12) combined with the uniqueness of horizontal smooth lifts w.r.t. $\pi$, as stated and proved in Lemma 3 below.

Remark 2. We can conclude from point (4) of Lemma 2 that every horizontal smooth lift $\eta$ of a closed regular smooth curve $\gamma : [0, L] \rightarrow S^2$ w.r.t. $\pi$ covers the path $\gamma$ with exactly half the speed of $\gamma$. This fact will not cause any technical problems in the sequel. It only leads to a multiplication of the “elastic energy” $\frac{1}{2} \int_{S^1} 1 + |\kappa_\gamma|^2 \, d\mu_\gamma$ by the factor 2. The algebraic background of this phenomenon is the short exact sequence of groups

$$
1 \rightarrow \mathbf{Z}/2 \rightarrow \text{Spin}(3) \xrightarrow{\mathcal{U}} \text{SO}(3) \rightarrow 1
$$

where $\mathcal{U}$ is defined by

$$
\text{Spin}(3) \ni [q \mapsto q \cdot r] \mapsto [q \mapsto \bar{r} q r] \in \text{SO}(3), \quad \text{for every } r \in S^3,
$$

(13)

and satisfies $\pi(q \cdot r) = \mathcal{U}(r)(\pi(q))$, for every $q, r \in S^3$, on account of formula (8). The above exact sequence only rephrases the topological fact that $\mathcal{U}$ is the universal covering of $\text{SO}(3)$.

In order to estimate, “how large” the subset of all Hopf-tori within the set of all smoothly immersed tori in $S^3$ actually is, we shall follow Section 2 of [25] and introduce “abstract Hopf-tori.”
**Definition 2.** Let \( \gamma : [0, L/2] \to \mathbb{S}^2 \) be a path with constant speed 2 which traverses a simple closed smooth curve in \( \mathbb{S}^2 \) of length \( L > 0 \) and encloses the area \( A \) of the domain on \( \mathbb{S}^2 \), “lying on the left hand side” when performing one loop through the trace of \( \gamma \). We assign to \( \gamma \) the lattice \( \Gamma_\gamma \) which is generated by the vectors \((2\pi, 0)\) and \((A/2, L/2)\), and we call the quotient \( M_\gamma := \mathbb{C}/\Gamma_\gamma \) the “abstract Hopf-torus” corresponding to trace(\( \gamma \)).

**Remark 3.** Pinkall proved in Proposition 1 of [25] that for every simple closed smooth curve \( \gamma \) in \( \mathbb{S}^2 \) its preimage \( \pi^{-1}(\text{trace}(\gamma)) \) can be isometrically mapped onto its corresponding “abstract Hopf-torus” \( M_\gamma := \mathbb{C}/\Gamma_\gamma \), as defined in Definition 2. Taking the uniformization theorem into account, Pinkall derived from this result in Section 3 of [25], that every conformal class of a compact Riemann surface of genus one can be realized by a Hopf-torus, i.e., by the preimage \( \pi^{-1}(\text{trace}(\gamma)) \) of some appropriate smooth, simple and closed curve \( \gamma \), as introduced in Definition 1. Therefore, the subset of Hopf-tori is sufficiently large, such that the main results of this article can be considered to be significant.

In order to rule out inappropriate parametrizations of Hopf-tori along the flow lines of the Willmore flow, we shall introduce “topologically simple” maps between tori in the following definition.

**Definition 3.** Let \( \Sigma_1 \) and \( \Sigma_2 \) be two compact tori, i.e., two compact sets which are homeomorphic to the standard torus \( \mathbb{S}^1 \times \mathbb{S}^1 \). We term a continuous map \( F : \Sigma_1 \to \Sigma_2 \) simple, if it has mapping degree \( \pm 1 \), i.e., if the induced map
\[
(F_* : H_2(\Sigma_1, \mathbb{Z}) \xrightarrow{\cong} H_2(\Sigma_2, \mathbb{Z}))
\]
is an isomorphism between these two singular homology groups in degree 2.

**Remark 4.** (1) We should point out here—especially regarding the proof of the third part of Theorem 1—that if \( \Sigma_1 \) is a smooth compact manifold of genus 1, i.e., a “smooth compact torus” without self-intersections, and if \( F : \Sigma_1 \to \mathbb{S}^n \) is a smooth immersion of \( \Sigma_1 \) into \( \mathbb{S}^n \), \( n \geq 3 \), such that \( F \) maps \( \Sigma_1 \) simply onto its image \( \Sigma_2 := F(\Sigma_1) \), then \( F \) is a smooth diffeomorphism between \( \Sigma_1 \) and \( \Sigma_2 \), if and only if \( \Sigma_2 \) is a smooth compact manifold of genus 1, as well. This follows immediately from our Definition 3, Propositions 4.5 and 4.7 in Chapter VIII of [9] and the remark following Proposition 4.7 in Chapter VIII of [9].

(2) If \( \gamma : \mathbb{R}/2\pi \mathbb{Z} \to \mathbb{S}^2 \) is a smooth, regular closed path, which traverses its trace exactly once with speed 2, and if \( \eta : [0, L] \to \pi^{-1}(\text{trace}(\gamma)) \) is a horizontal smooth lift of \( \gamma \) w.r.t. \( \pi \), as discussed in Lemma 3 below, then the standard parametrization \( X : [0, L] \times [0, 2\pi] \to \pi^{-1}(\text{trace}(\gamma)) \) of the Hopf-torus \( \pi^{-1}(\text{trace}(\gamma)) \) in \( \mathbb{S}^3 \), given by \( X(s, \varphi) := e^{i\varphi} \eta(s) \), has the effect of a typical simple parametrization, namely to cover \( \pi^{-1}(\text{trace}(\gamma)) \) “essentially once.” Precisely this means that \( X^{-1}(\{z\}) \) consists of exactly one element for \( \mathcal{H}^2 \)-almost every \( z \in \pi^{-1}(\text{trace}(\gamma)) \).
Now we can state the main results of this article:

**Theorem 1.** [Global existence and Subconvergence] Let $\gamma_0 : S^1 \to S^2$ be a smooth, closed and regular path in $S^2$. Let $F_0 : \Sigma \to S^3$ be an arbitrary smooth immersion, which maps a compact smooth torus $\Sigma$ simply onto the Hopf-torus $\pi^{-1}(\text{trace}(\gamma_0))$ in the sense of Definition 3. Then, the following statements hold:

(I) There is a unique smooth global solution $\{P(t, 0, F_0)\}_{t \geq 0}$ of the Willmore flow (5) on $[0, \infty) \times \Sigma$, starting in $F_0$ at time $t = 0$. The immersions $P(t, 0, F_0)$ remain umbilic-free and map $\Sigma$ simply onto Hopf-tori $\forall t \geq 0$.

(II) This global solution $\{P(t, 0, F_0)\}$ of Eq. (1) subconverges in every $C^m$-norm—up to reparametrizations—to smooth Willmore-Hopf-tori in $S^3$. More precisely we have: For every sequence $t_j \to \infty$, there are a subsequence of times $t_{jk} \to \infty$, a sequence of diffeomorphisms $\varphi_k : \Sigma \to \Sigma$ and a smooth immersion $\hat{F} : \Sigma \to S^3$ mapping $\Sigma$ simply onto some smoothly immersed Willmore-Hopf-torus in $S^3$, such that there holds

$$P(t_{jk}, 0, F_0) \circ \varphi_k \to \hat{F} \quad \text{in } C^m(\Sigma, \mathbb{R}^4),$$

as $k \to \infty$, $\forall m \in \mathbb{N}_0$.

(III) If the initial immersion $F_0 : \Sigma \to S^3$ maps the smooth torus $\Sigma$ simply onto a Hopf-torus in $S^3$ with Willmore energy $\mathcal{W}(F_0) \leq \frac{8\pi^2}{\sqrt{2}}$, then there is a smooth family of smooth diffeomorphisms $\Psi_t : \Sigma \to \Sigma$, such that the reparametrization $\{P(t, 0, F_0) \circ \Psi_t\}$ of the flow line $\{P(t, 0, F_0)\}$, starting in $F_0$ at time $t = 0$, consists of smooth diffeomorphisms between the smooth torus $\Sigma$ and their images in $S^3$ for sufficiently large $t$, and the family $\{P(t, 0, F_0) \circ \Psi_t\}$ converges fully in $C^m(\Sigma, \mathbb{R}^4)$, for each $m \in \mathbb{N}_0$, to some smooth diffeomorphism $F^*$ between $\Sigma$ and a conformally transformed Clifford torus in $S^3$.

□

Remark 5. In this paper, “convergence in $C^m(M, \mathbb{R}^4)$” will always refer to the classical meaning of $C^m$-convergence of immersions $f_j : M \to \mathbb{R}^4$ being defined on a compact smooth surface $M$ without boundary, as formulated for instance in Definition C.10 of Appendix C in [8], although in Breuning’s general compactness theorem, Theorem 1.3 in [2], the conclusion has to be stated in terms of a weaker notion of “locally smooth convergence.” Therefore, the reader might want to compare the classical notion of convergence in the Banach space $C^m(M, \mathbb{R}^4)$ with two geometrically motivated, but weaker types of convergence, which have been formulated in Definitions 2.1 and C.7 of [8]—two different types of convergence which essentially coincide in case $M$ is compact and closed on account of Remark C.8 in [8]. Finally, Proposition C.9 in [8] clarifies the functional-analytic meaning of this latter type of convergence and allows us to work throughout the paper in terms of classical terminology.

For the proof of the $C^m$-subconvergence (14) in Part II of Theorem 1, we will need the following special version of Breuning’s general compactness theorem in [2] for
proper smooth immersions of closed manifolds into $\mathbb{R}^n$. Breuning’s Theorem 1.3 in [2] is actually a generalization of Langer’s compactness theorem in [16], and it appeared already in [14, Theorem 4.2], without proof.

**Proposition 1.** [2, Theorems 1.1 and 1.3] Let $F_j : \mathcal{M} \longrightarrow \mathbb{R}^4$ be a sequence of smooth immersions, where $\mathcal{M}$ is a compact surface without boundary. If there is a ball $B_R(0) \subset \mathbb{R}^4$ with $F_j(\mathcal{M}) \subset B_R(0)$, for every $j \in \mathbb{N}$, and if there are constants $C_1 > 0$ and $C_2(m) > 0$ for each $m \in \mathbb{N}_0$, such that

$$A(F_j) := \int_{\mathcal{M}} d\mu_{F_j} \leq C_1 \text{ and } \| (\nabla_{\perp} F_j)^m (A_{F_j}) \|_{L^\infty(\mathcal{M})} \leq C_2(m)$$

holds for every $j \in \mathbb{N}$ and each $m \in \mathbb{N}_0$, then there exists a smooth immersion $\hat{F} : \mathcal{M} \longrightarrow \mathbb{R}^4$, some subsequence $\{F_{jk}\}$ and some sequence of smooth diffeomorphisms $\varphi_k : \mathcal{M} \xrightarrow{\sim} \mathcal{M}$, such that $F_{jk} \circ \varphi_k \longrightarrow \hat{F}$ converge in $C^m(\mathcal{M}, \mathbb{R}^4)$, as $k \rightarrow \infty$, for every $m \in \mathbb{N}_0$, in the sense of Proposition C.9 in [8].

Moreover, for the proof of the full convergence of the auxiliary flow (51) to smooth parametrizations of great circles in $S^2$, we will need the “Lojasiewicz–Simon gradient inequality” for the elastic energy functional $E$ applied to immersions $F : \Sigma \longrightarrow \mathbb{R}^4$ of a torus $\Sigma$ into $\mathbb{R}^4$, which is proved in Theorem 3.1 in [4]: For any critical immersion $F^* : \Sigma \longrightarrow \mathbb{R}^4$ of the Willmore functional $\tilde{W}(F) := \frac{1}{4} \int_{\Sigma} |\tilde{H}_F, \mathbb{R}^4|^2 d\mu_F$ there exist constants $\theta \in (0, \frac{1}{2}]$, $c \geq 0$ and $\sigma > 0$, only depending on $F^*$, such that for every closed curve $\gamma \in C^4_{\text{reg}}(\mathbb{S}^1, \mathbb{S}^2)$ satisfying $\| \gamma - \gamma^* \|_{C^4(\mathbb{S}^1, \mathbb{R}^3)} \leq \sigma$ there holds:

$$|E(\gamma) - E(\gamma^*)|^{1-\theta} \leq c \left( \int_{\mathbb{S}^1} |\nabla L^2 E(\gamma)|^2 d\mu_{\gamma} \right)^{1/2}.$$  

**Proof.** First of all, we recall the Lojasiewicz–Simon gradient inequality for the Willmore functional $\mathcal{W}$ applied to immersions $F : \Sigma \longrightarrow \mathbb{R}^4$ of a torus $\Sigma$ into $\mathbb{R}^4$, which is proved in Theorem 3.1 in [4]: For any critical immersion $F^* : \Sigma \longrightarrow \mathbb{R}^4$ of the Willmore functional $\tilde{W}(F) := \frac{1}{4} \int_{\Sigma} |\tilde{H}_F, \mathbb{R}^4|^2 d\mu_F$ there exist constants $\theta \in (0, \frac{1}{2}]$, $c \geq 0$ and $\sigma > 0$, such that for every immersion $F \in C^4(\Sigma, \mathbb{R}^4)$ satisfying $\| F - F^* \|_{C^4(\Sigma, \mathbb{R}^4)} \leq \sigma$ there holds:

$$|\tilde{W}(F) - \tilde{W}(F^*)|^{1-\theta} \leq c \left( \int_{\Sigma} |\nabla L^2 \tilde{W}(F)|^2 d\mu_F \right)^{1/2}.$$  

(16)
Now we recall from [24], formula (2.6), that the $L^2$-gradient of the functional $\tilde{\mathcal{W}}$ coincides with the $L^2$-gradient of the functional $\mathcal{W}$ in $C^4$-immersions $F: \Sigma \rightarrow S^3 \subset R^4$, i.e., such immersions satisfy:

$$\triangle_F^{1/2}(H_{F,R^4}) + Q(A^0_{F,R^4})(H_{F,R^4}) = \triangle_F^{1/2}(H_{F,S^3}) + Q(A^0_{F,S^3})(H_{F,S^3}), \quad (17)$$

although their second fundamental tensors and mean curvature vectors do not coincide, namely there holds:

$$A_{F,R^4} = A_{F,S^3} - F g_F, \quad H_{F,R^4} = H_{F,S^3} - 2F, \quad (18)$$

for $C^2$-immersions $F: \Sigma \rightarrow S^3 \subset R^4$ by formulae (2.2) and (2.3) in [24]. Hence, we have due to $\langle \hat{H}_{F,S^3}, F \rangle_R^4 \equiv 0$ and $|F|^2 \equiv 1: |H_{F,R^4}|^2 = |\hat{H}_{F,S^3}|^2 + 4$ for $C^2$-immersions $F: \Sigma \rightarrow S^3 \subset R^4$, and we consequently obtain together with formulae (16) and (17):

$$|\mathcal{W}(F) - \mathcal{W}(F^*)|^{1-\theta} = |\tilde{\mathcal{W}}(F) - \tilde{\mathcal{W}}(F^*)|^{1-\theta} \leq c \left( \int_\Sigma |\nabla L^2 \tilde{\mathcal{W}}(F)|^2 d\mu_F \right)^{1/2} = c \left( \int_\Sigma |\nabla L^2 \mathcal{W}(F)|^2 d\mu_F \right)^{1/2}, \quad (19)$$

for two $C^4$-immersions $F, F^*: \Sigma \rightarrow S^3$ satisfying $\| F - F^* \|_{C^4(\Sigma,R^4)} \leq \sigma$, provided $F^*$ is Willmore in $S^3$, i.e., satisfies the equation

$$\triangle_{F^*}^{1/2}(H_{F^*,S^3}) + Q(A^0_{F^*,S^3})(H_{F^*,S^3}) \equiv 0 \quad \text{on } \Sigma,$$

and provided the number $\sigma > 0$ is sufficiently small. Now we infer from formula (32) that the Willmore-Hopf-tori in $S^3$ correspond exactly to the elastic curves in $S^2$, i.e., to the critical points of $E$ in $C^4_{\text{reg}}(S^1, S^2)$. Moreover, we note that the Hopf-fibration $\pi: S^3 \rightarrow S^2$ is a smooth submersion, and it can therefore be locally transformed into an orthogonal projection from $R^3$ onto $R^2$. Hence, fixing some elastic curve $\gamma^*: S^1 \rightarrow S^2$, we infer from estimate (19), combined with formulae (33) and (34), the existence of some $\sigma > 0$, such that for any closed curve $\gamma \in C^4_{\text{reg}}(S^1, S^2)$ with $\| \gamma - \gamma^* \|_{C^4(S^1,R^3)} \leq \sigma$ there holds:

$$\pi^{1-\theta}|E(\gamma) - E(\gamma^*)|^{1-\theta} = |\mathcal{W}(F_{\gamma}) - \mathcal{W}(F_{\gamma^*})|^{1-\theta} \leq c \left( \int_\Sigma |\nabla L^2 \mathcal{W}(F_{\gamma})|^2 d\mu_{F_{\gamma}} \right)^{1/2} = c \sqrt{\pi} \left( \int_{S^1} |\nabla L^2 \mathcal{E}(\gamma)|^2 d\mu_{\gamma} \right)^{1/2},$$

where $F_{\gamma}$ and $F_{\gamma^*}$ are appropriately chosen, simple $C^4$-parametrizations of the Hopf-tori $\pi^{-1}(\text{trace}(\gamma))$ and $\pi^{-1}(\text{trace}(\gamma^*))$ in $S^3$, corresponding to the closed $C^4$-curves $\gamma$ and $\gamma^*$.

3. The link between the Hopf-fibration and the first variation of the Willmore functional

The central result of this section is formula (32), the “Hopf-Willmore-identity,” which shows that the differential of the Hopf-fibration takes the first variation of the
Willmore functional evaluated in an immersed Hopf-torus in $S^3$ into the first variation of the elastic energy functional, evaluated in the corresponding, projected curve in $S^2$. Expressed in a more algebraic language: the Hopf-fibration and its differential transform the pair $(V, \nabla l_z) V)$ naturally from its effect on smooth immersed Hopf-tori in $S^3$ to its effect on smooth closed regular curves in $S^2$. Consequently, the Hopf-fibration transforms flow lines of the Willmore flow in $S^3$ into flow lines of the classical “elastic energy flow” (51) in $S^2$. To start out, we recall some basic differential geometric terms in the following definition. As in the introduction, we endow the unit 3-sphere with the Euclidean scalar product of $\mathbb{R}^4$, i.e., we set $g_{S^3} := (\cdot, \cdot)_{\mathbb{R}^4}$.

**Definition 4.**

1. For any fixed C$^2$—immersion $G : \Sigma \rightarrow S^3$ and any smooth chart $\psi$ of an arbitrary coordinate neighborhood $\Sigma'$ of a fixed smooth compact torus $\Sigma$, we will denote the resulting partial derivatives on $\Sigma'$ by $\partial_i$, $i = 1, 2$, the coefficients $g_{ij} := (\partial_i G, \partial_j G)$ of the first fundamental form of $G$ w.r.t. $\psi$ and the associated Christoffel-symbols $(\Gamma_G)_{kl}^m := g^{mj} (\partial_k G, \partial_j G)$ of $(\Sigma', G^*(g_{S^3}))$.

2. For any vector field $V \in C^2(\Sigma, \mathbb{R}^4)$, we define the first covariant derivatives $\nabla^G_i (V) \equiv \nabla^G_i (V)$, $i = 1, 2$, w.r.t. $G$ as the projections of the usual partial derivatives $\partial_i (V)(x)$ of $V : \Sigma \rightarrow \mathbb{R}^4$ into the respective tangent spaces $T_G(x)S^3$ of the 3-sphere, $\forall x \in \Sigma$, and the second covariant derivatives by

$$\nabla^G_{kl}(V) \equiv \nabla^G_k \nabla^G_l (V) := \nabla^G_k (\nabla^G_l (V)) - (\Gamma_G)_{kl}^m \nabla^G_m (V).$$

Moreover, we define the projections of its first derivatives into the normal bundle of the immersed surface $G(\Sigma)$ within $TS^3$ by

$$\nabla^G_i (V) \equiv (\nabla^G_i (V))^\perp := \nabla^G_i (V) - P_{\text{Tan}(G)} (\nabla^G_i (V))$$

and the “normal second covariant derivatives” of $V$ w.r.t. $G$ by

$$\nabla^G_k \nabla^G_i (V) := \nabla^G_k (\nabla^G_i (V) \perp) - (\Gamma_G)_{kl}^m \nabla^G_m (V).$$

3. For a smooth regular curve $\gamma : [a, b] \rightarrow S^2$ and a smooth tangent vector field $W$ on $S^2$ along $\gamma$, we will denote by $\nabla_{\partial_t \gamma} (W)$ the classical covariant derivative of the vector field $W$ w.r.t. the tangent vector field $\partial_t \gamma$ along $\gamma$—being projected into $TS^2$—moreover by $\nabla_{\nu} (W)$ the covariant derivative of the tangent vector field $W$ along $\gamma$ w.r.t. the unit tangent vector field $\overrightarrow{\gamma}(t)$ along $\gamma$, and by $\nabla^\perp (W)$ the orthogonal projection of the tangent vector field $\nabla_{\nu} (W)$ into the normal bundle of the curve $\gamma$ within $TS^2$.

4. For any fixed closed regular curve $\gamma \in C^\infty_{\text{reg}}(S^1, S^2)$, we denote by $\Gamma(\gamma^*TS^2)$ the $\mathbb{R}$-vector space of smooth sections of the pullback bundle $\gamma^*TS^2$, i.e., of all vector fields $\eta \in C^\infty(S^1, \mathbb{R}^3)$ meeting the additional condition:

$$\eta(x) \in T_\gamma(x)S^2 \text{ in every } x \in S^1.$$
Finally, we define the subspace $\Gamma^+(\gamma^*TS^2)$ of the vector space $\Gamma(\gamma^*TS^2)$ by the requirement, to consist of all those sections $\eta \in \Gamma(\gamma^*TS^2)$, which are normal along the prescribed path $\gamma$, i.e., which additionally satisfy $\langle \eta(x), \partial_x \gamma(x) \rangle_{\mathbb{R}^3} = 0$ in every $x \in S^1$.

Preparing ourselves for the fundamental Proposition 4 below, we firstly need the following elementary proposition, which results from straightforward computations on the basis of Section 2.1 in [6].

**Proposition 3.** The $L^2$-gradient of the elastic energy $E(\gamma) := \int_{S^1} 1 + |\vec{\kappa}_\gamma|^2 \, d\mu_\gamma$, with $\vec{\kappa}_\gamma$ as in (24) below, evaluated in an arbitrary closed curve $\gamma \in C^\infty_{reg}(S^1, S^2)$, reads exactly:

$$\nabla_{L^2} E(\gamma)(x) = 2 \left( \frac{\nabla_{\gamma}}{|\gamma'|} \right)^2 (\vec{\kappa}_\gamma)(x) + |\vec{\kappa}_\gamma|^2 (x) \vec{\kappa}_\gamma(x) + \vec{\kappa}_\gamma(x), \quad \text{for } x \in S^1. \quad (21)$$

Using the abbreviation $\partial_\gamma := \frac{\partial_x \gamma}{|\partial_x \gamma|}$ for the partial derivative of $\gamma$ normalized by arc-length, the leading term on the right hand side of Eq. (21) reads:

$$\left( \frac{\nabla_{\gamma}}{|\gamma'|} \right)^2 (\vec{\kappa}_\gamma)(x) = \left( \nabla_{\partial_\gamma} \right)^2 \left( (\partial_{ss} \gamma)(x) - \langle \gamma(x), \partial_{ss} \gamma(x) \rangle \gamma(x) \right)$$

$$= (\partial_\gamma)^4 (\gamma)(x) - ((\partial_\gamma)^4 (\gamma)(x), \gamma(x)) \gamma(x)$$

$$-((\partial_\gamma)^4 (\gamma)(x), \partial_\gamma \gamma(x)) \partial_\gamma \gamma(x) + |(\nabla_{\partial_\gamma} (\gamma)(x)|^2 \partial_{ss} \gamma(x), \quad (22)$$

for $x \in S^1$. The fourth normalized derivative $(\partial_\gamma)^4 (\gamma) \equiv \left( \frac{\partial_x}{|\partial_x \gamma|} \right)^4 (\gamma)$ is nonlinear w.r.t. $\gamma$, and at least its leading term can be computed in terms of ordinary partial derivatives of $\gamma$:

$$\begin{align*}
(\partial_\gamma)^4 (\gamma) &= \frac{(\partial_\gamma)^4 (\gamma)}{|\partial_\gamma \gamma|^4} - \frac{1}{|\partial_\gamma \gamma|^4} \left( (\partial_\gamma)^4 (\gamma), \frac{\partial_x \gamma}{|\partial_x \gamma|} \right) \frac{\partial_x \gamma}{|\partial_x \gamma|} \\
&+ C((\partial_\gamma)^2 (\gamma), \partial_\gamma \gamma) \cdot (\partial_\gamma)^3 (\gamma) \\
&\text{+ rational expressions which only involve } (\partial_\gamma)^2 (\gamma) \text{ and } \partial_\gamma \gamma, \quad (23)
\end{align*}$$

where $C : \mathbb{R}^6 \rightarrow \text{Mat}_{3,3}(\mathbb{R})$ is a $\text{Mat}_{3,3}(\mathbb{R})$-valued function, whose components are rational functions in $(y_1, \ldots, y_6) \in \mathbb{R}^6$. \hfill \qed

Now we are ready to collect some basic differential geometric formulae in Proposition 4 for an arbitrary simple immersion $F$ mapping a compact torus $\Sigma$ onto the Hopf-torus $\pi^{-1}(\text{trace}(\gamma))$ corresponding to some closed smooth regular curve $\gamma : S^1 \rightarrow S^2$; see here Definition 3. We shall understand below in detail that the following proposition paves the path to the decisive Proposition 5 and that therefore Proposition 4 constitutes the technical foundation for the proofs of our main results in Theorem 1.
Proposition 4. Let \( F : \Sigma \rightarrow S^3 \) be an immersion which maps the compact torus \( \Sigma \) simply onto some Hopf-torus in \( S^3 \), and let \( \gamma : S^1 \rightarrow S^2 \) be a smooth regular parametrization of the closed curve \( \pi \circ F \). Let moreover

\[
\vec{\kappa}_\gamma := -\frac{1}{|\gamma'|^2} (\gamma' \cdot v_\gamma') \cdot v_\gamma
\]

be the curvature vector along the curve \( \gamma \), for a unit normal field \( v_\gamma \) along the trace of \( \gamma \), and \( \kappa_\gamma := \langle \vec{\kappa}_\gamma, v_\gamma \rangle_{\mathbb{R}^3} \) the signed curvature along \( \gamma \). Then, there is some \( \varepsilon = \varepsilon(F, \gamma) > 0 \), such that for an arbitrarily fixed point \( s^* \in S^1 \) the following differential geometric formulae hold for the immersion \( F \):

\[
A_F(\eta_F(s)) = N_F(\eta_F(s)) \begin{pmatrix} 2\kappa_\gamma(s) & 1 \\ 1 & 0 \end{pmatrix}
\]

where \( \eta_F : S^1 \cap B_\varepsilon(s^*) \rightarrow \Sigma \) denotes an arbitrary horizontal smooth lift of \( \gamma \mid_{S^1 \cap B_\varepsilon(s^*)} \) w.r.t. the fibration \( \pi \circ F \), as introduced in Lemma 3 below, and \( N_F \) denotes a fixed unit normal field along the immersion \( F \). This implies

\[
H_F(\eta_F(s)) = \text{trace} A_F(\eta_F(s)) = 2\kappa_\gamma(s) N_F(\eta_F(s))
\]

\forall s \in S^1 \cap B_\varepsilon(s^*), for the mean curvature vector of \( F \) and also

\[
A_F^0(\eta_F(s)) = N_F(\eta_F(s)) \begin{pmatrix} \kappa_\gamma(s) & 1 \\ 1 & -\kappa_\gamma(s) \end{pmatrix}
\]

and consequently \(|A_F^0|^2(\eta_F(s)) = 2(\kappa_\gamma(s)^2 + 1) \), and also

\[
Q(A_F^0)(H_F)(\eta_F(s)) = 4 (\kappa_\gamma^3(s) + \kappa_\gamma(s)) N_F(\eta_F(s)),
\]

\[
\Delta_F^\perp(H_F)(\eta_F(s)) = 4 \left( \nabla_{\gamma'}|\gamma'| \right)^2 (\kappa_\gamma(s)) N_F(\eta_F(s)),
\]

and finally for the traced sum of all covariant derivatives of \( A_F \) of order \( k \in \mathbb{N} \):

\[
|\nabla^\perp F^k (A_F)(\eta_F(s))|^2 = 2^{2+2k} \left| \nabla_{\gamma'}|\gamma'|^k (\vec{\kappa}_\gamma)(s) \right|^2
\]

\forall s \in S^1 \cap B_\varepsilon(s^*). In particular, we derive

\[
\nabla_{L^2} W(F)(\eta_F(s)) = 2 \left( \nabla_{\gamma'}|\gamma'| \right)^2 (\kappa_\gamma(s) + \kappa_\gamma^3(s) + \kappa_\gamma(s)) N_F(\eta_F(s)),
\]

and the “Hopf-Willmore-identity”:

\[
D\pi_F(\eta_F(s)) \left( \nabla_{L^2} W(F)(\eta_F(s)) \right) = 4 \left( \nabla_{\gamma'}|\gamma'| \right)^2 (\vec{\kappa}_\gamma) + |\vec{\kappa}_\gamma|^2 \vec{\kappa}_\gamma + \vec{\kappa}_\gamma(s)
\]

\equiv 4 \nabla_{L^2} E(\gamma)(s)
\]
\[ \forall s \in S^1 \cap B_e(s^*), \text{ where there holds } \pi \circ F \circ \eta_F = \gamma \text{ on } S^1 \cap B_e(s^*); \text{ see Lemma 3.} \]

Finally, we will prove that

\[ \mathcal{W}(F) \equiv \int_{\Sigma} 1 + \frac{1}{4} |H_F|^2 \, d\mu_F = \pi \int_{S^1} 1 + |\kappa_\gamma|^2 \, d\mu_\gamma = \pi \mathcal{E}(\gamma), \]  

and

\[ \int_{\Sigma} |\nabla_{L^2} \mathcal{W}(F)|^2 \, d\mu_F = 4\pi \int_{S^1} |\nabla_{L^2} \mathcal{E}(\gamma)|^2 \, d\mu_\gamma. \]  

Proof. Let \( 2L \) denote the length of the curve image \((\pi \circ F) \subset S^2\). Without loss of generality, we may require \( \gamma \) to perform only one loop through its trace. Hence, we can parametrize the trace of \( \gamma \) on \( R/LZ \) in such a way that it has constant speed 2 on \( R/LZ \). We therefore assume during the entire proof of this proposition that \( \gamma \) is defined on \( R/LZ \), satisfies \( |\gamma'| \equiv 2 \) on \( R/LZ \) and that \( s^* \) is a fixed point in \([0, L]\). We infer from Lemma 3 the existence of horizontal smooth lifts \( \eta : [0, L] \to \pi^{-1}(\text{trace}(\gamma)) \) of \( \gamma \) w.r.t. \( \pi \), and we infer from point (4) of Lemma 2 that every such lift \( \eta \) must have constant speed \(|\gamma'| \equiv 1 \) on \([0, L]\) and intersects each fiber of \( \pi \) exactly once. Hence, also statements (1)–(3) of Lemma 2 can be applied in the sequel. In order to simplify our computations, we choose the explicit parametrization \( X(s, \varphi) := e^{i\varphi} \eta(s) \), for \((s, \varphi) \in [0, L] \times [0, 2\pi]\), from formula (10) of \( \pi^{-1}(\text{trace}(\gamma)) \), which gives us the opportunity to apply many ideas arising in Sections 2 and 3 of [25]. Firstly, we infer from Lemma 2 that the vector \( N_X(s, \varphi) := u(s) e^{-i\varphi} \eta(s) \) is orthogonal to \( \partial_s X(s, \varphi) \) and \( \partial_\varphi X(s, \varphi) \) within \( T_{X(s, \varphi)} S^3 \) and therefore a unit normal field along \( \pi^{-1}(\text{trace}(\gamma)) \).\(^1\)

Now, we apply formula (9) with \( q := X(s, \varphi) \in \pi^{-1}(\text{trace}(\gamma)) \) and \( v := N_X(s, \varphi) \) and compute similarly to (12):

\[ D\pi_{X(s, \varphi)}(N_X(s, \varphi)) = I(N_X(s, \varphi)) \cdot X(s, \varphi) + I(X(s, \varphi)) \cdot N_X(s, \varphi) \]
\[ = I(iu(s) e^{-i\varphi} \eta(s)) \cdot (e^{i\varphi} \eta(s)) + I(e^{i\varphi} \eta(s)) \cdot (iu(s) e^{-i\varphi} \eta(s)) \]
\[ = -\tilde{\eta}(s) e^{i\varphi} \tilde{u}(s) i e^{i\varphi} \eta(s) + \tilde{\eta}(s) e^{-i\varphi} iu(s) e^{-i\varphi} \eta(s) \]
\[ = -\tilde{\eta}(s) \tilde{u}(s) i \eta(s) + \tilde{\eta}(s) iu(s) \eta(s) = 2\tilde{\eta}(s) i u(s) \eta(s), \]  

(35)

where we used that \( iu(s) = -u(s) i \) and \( \tilde{u}(s) = u(s) \). Moreover, we compute:

\[ \tilde{\eta}(s) i u(s) \eta(s) \tilde{u}(s) u(s) \eta(s) = \tilde{\eta}(s) i \tilde{u}(s) \tilde{\eta}(s) u(s) \eta(s) = \tilde{\eta}(s) i \eta(s), \]

(36)

and we know that \( \Re(\tilde{\eta}i \eta) \equiv 0 \). Combining this with the fact that \( \gamma'(s) = 2\tilde{\eta}(s) u(s) \eta(s) \) is tangential at the curve \( \gamma = \pi \circ \eta \) in its point \( \gamma(s) \), formula (36) shows that there holds \( \tilde{\eta} i u \in \Gamma^1(\gamma^* TS^2) \), being here even a normal section along \( \gamma \) of constant length one. Hence, (35) shows that the differential \( D\pi \) of the Hopf-fibration maps the one-dimensional normal bundle of the immersion \( X \) within \( TS^3 \) isomorphically onto

\(^1\)See here also formula (20) in [25].
the normal bundle of the curve $\gamma = \pi \circ \eta$ within $TS^2$. Moreover, we verify by means of formula (12) that
\[
D\pi_X(s,\varphi)(\partial_s X(s,\varphi)) = I(\partial_s X(s,\varphi)) \cdot X(s,\varphi) + I(X(s,\varphi)) \cdot \partial_s X(s,\varphi)
\]
\[
= I(e^{i\varphi} u(s) \eta(s)) \cdot (e^{i\varphi} \eta(s)) + I(e^{i\varphi} \eta(s)) \cdot (e^{i\varphi} u(s) \eta(s))
\]
\[
= \tilde{\eta}(s) \tilde{u}(s) e^{-i\varphi} e^{i\varphi} \eta(s) + \tilde{\eta}(s) e^{-i\varphi} e^{i\varphi} u(s) \eta(s) = 2\tilde{\eta}(s) u(s) \eta(s) = \gamma'(s)
\]
(37)

and
\[
D\pi_X(s,\varphi)(\partial_\varphi X(s,\varphi)) = I(\partial_\varphi X(s,\varphi)) \cdot X(s,\varphi) + I(X(s,\varphi)) \cdot \partial_\varphi X(s,\varphi)
\]
\[
= I(i e^{i\varphi} \eta(s)) \cdot (e^{i\varphi} \eta(s)) + I(e^{i\varphi} \eta(s)) \cdot (i e^{i\varphi} \eta(s))
\]
\[
= -\tilde{\eta}(s) i e^{-i\varphi} e^{i\varphi} \eta(s) + \tilde{\eta}(s) e^{-i\varphi} i e^{i\varphi} \eta(s) \equiv 0.
\]
(38)

Formulae (37) and (38) show that $D\pi$ maps the tangent bundle of the immersion $X$ in $TS^3$ onto the tangent bundle of the closed curve $\gamma$ in $TS^2$ with a necessarily one-dimensional kernel. Again using Lemma 2, $i u(s) = -u(s) i$ and $u^2 \equiv -1$, we obtain as in formula (21) of [25]:
\[
\partial_s X(s,\varphi) = -2\kappa_\gamma(s) \partial_s X(s,\varphi) - \partial_\varphi X(s,\varphi),
\]
\[
\partial_\varphi X(s,\varphi) = -i u(s) i e^{-i\varphi} \eta(s) = -u(s) e^{-i\varphi} \eta(s) = -\partial_s X(s,\varphi),
\]
(39)

where we have introduced the curvature function $\kappa_\gamma(s)$ of the curve $\gamma = \pi \circ \eta$ in the point $s \in [0, L]$ by the relation:
\[
u'(s) = 2i \kappa_\gamma(s) u(s),
\]
(40)

as in formula (22) of [25]. Since $u$ has no zeroes, equation (40) defines a unique smooth function on $[0, L]$, and one can easily verify, using the formula $(\tilde{\eta} i u \eta)' = \tilde{\eta} \tilde{u} i u \eta + \tilde{\eta} i u' \eta + \tilde{\eta} i u^2 \eta$ and the fact that $u^2 \equiv -1$, that this function $\kappa_\gamma$ exactly coincides with the signed curvature function of the curve $\gamma$, as defined below definition (24). Hence using (36), we can write $\tilde{k}_\gamma = \kappa_\gamma \tilde{\eta} i u \eta$. Moreover, on account of Lemma 2 we know that $[\partial_s X, \partial_\varphi X]$ is an orthonormal frame along $\pi^{-1}(\text{trace}(\gamma))$, implying that $(g_X)_{ij} = \delta_{ij}$, and thus (39) yields the coefficients of the 2nd fundamental form $A_X$ w.r.t. the unit normal $N_X$:
\[
A_X(s,\varphi) = N_X(s,\varphi) \begin{pmatrix} 2\kappa_\gamma(s) & 1 \\ 1 & 0 \end{pmatrix}.
\]
(41)

This yields
\[
H_X(s,\varphi) \equiv \text{trace} A_X(s,\varphi) = 2 \kappa_\gamma(s) N_X(s,\varphi)
\]
(42)

for the mean curvature vector of $X$ and also
\[
A_X^0(s,\varphi) = N_X(s,\varphi) \begin{pmatrix} \kappa_\gamma(s) & 1 \\ 1 & -\kappa_\gamma(s) \end{pmatrix},
\]
(43)
in particular
\[ |A_X^0|^2(s, \varphi) = 2(\kappa_\gamma(s)^2 + 1), \]

and \( \mathcal{Q}(A_X^0)(H_X)(s, \varphi) = (A_X^0)_{ij}((A_X^0)^{ij}) = 4(\kappa_\gamma^3 + \kappa_\gamma)(s) N_X(s, \varphi) \).

Recalling now that \( N_X(s, \varphi) = i u(s) e^{-i \varphi} \eta(s) \) and formula (42), we have:
\[
\nabla_s^\perp H_X(s, \varphi) = \left( 2\kappa_\gamma(s) i u(s) e^{-i \varphi} \eta(s) + 2\kappa_\gamma(s) i u'(s) e^{-i \varphi} \eta(s) + 2\kappa_\gamma(s) i u(s) e^{-i \varphi} u(s) \eta(s) \right)^{\perp X} = 2\kappa_\gamma(s) N_X(s, \varphi). \tag{45}
\]

In order to achieve this result, one has to derive from \( |u|^2 = 1, |\eta|^2 = 1, \eta^\prime = u \eta \) and \( u(s) \in \text{Span}\{j, k\}, \forall s \in [0, L] \):
\[
0 = \ddot{u}' u + \ddot{u} u' = \ddot{u}' u + \dddot{u} \dot{u} = 2\Re(\ddot{u}' u) = 2\Re(\dddot{u} \dot{u})
\]
and thus \( (\ddot{u} u') = -i \det(u, u') \), and then compute:
\[
\Re(N_X(s, \varphi) i u'(s) e^{-i \varphi} \eta(s)) = \Re(\dddot{u}(s) e^{i \varphi} \eta(s))
\]
\[
= -2\Re(\dddot{u}(s)) \det(u(s), u'(s)) e^{-i \varphi} \eta(s) = -\det(u(s), u'(s)) \Re(\dddot{u}(s)) \Re(\eta(s)) = 0.
\]

Also using that \( \Re(\dddot{u} \dot{u}) = 0 \), and then:
\[
\Re(N_X(s, \varphi) i u(s) e^{-i \varphi} \eta(s)) = \Re(\dddot{u}(s) u(s)) \Re(\eta(s)) = 0.
\]

for \( (s, \varphi) \in [0, L] \times [0, 2\pi] \). Moreover, combining formulae (39) and (42) we obtain:
\[
\nabla_\varphi^\perp H_X(s, \varphi) = 2\kappa_\gamma(s) \left( \partial_\varphi N_X(s, \varphi) \right)^{\perp X} = 2\kappa_\gamma(s) \left( \partial_\varphi X(s, \varphi) \right)^{\perp X} = 0. \tag{46}
\]

This immediately implies also
\[
\nabla_\varphi^\perp \nabla_\varphi^\perp H_X(s, \varphi) = 0 \quad \text{and} \quad \nabla_\varphi^\perp \nabla_s^\perp H_X(s, \varphi) = 0 = \nabla_s^\perp \nabla_\varphi^\perp H_X(s, \varphi)
\]
for \( (s, \varphi) \in [0, L] \times [0, 2\pi] \). Finally, we derive from (45) by analogy:
\[
\nabla_s^\perp N_X(s, \varphi) = \nabla^\perp X(s, \varphi) = 2\kappa_\gamma(s) N_X(s, \varphi)
\]
for \( (s, \varphi) \in [0, L] \times [0, 2\pi] \), and therefore also:
\[
(\nabla_s^\perp)^k X(s, \varphi) = N_X(s, \varphi) \left( \begin{array}{cc} 2\kappa_\gamma^{(k)}(s) & 0 \\ 0 & 0 \end{array} \right) \tag{47}
\]
for \( k \in \mathbb{N} \), which immediately yields \( |(\nabla^\perp)^k(A_X)(s, \varphi)|^2 = 4(\kappa_\gamma^{(k)}(s))^2 \) on \( [0, L] \times [0, 2\pi] \). Together with \((g_X)^{ij} = \delta_{ij} = (g_X)^{ij} \) on \([0, L] \times [0, 2\pi]\), we instantly arrive at the expression for the normal Laplacian of \( H_X \):
\[
\triangle_X^\perp (H_X)(s, \varphi) = \nabla_s^\perp \nabla_s^\perp H_X(s, \varphi) = 2\kappa_\gamma''(s) N_X(s, \varphi), \tag{48}
\]
see here also Section 2 in [11]. Combining formulae (44) and (48), we obtain:

\[ \nabla_{L^2} W(X)(s, \varphi) = \left( \kappa''_\gamma(s) + 2 \kappa^3_\gamma(s) + 2 \kappa_\gamma(s) \right) N_X(s, \varphi) \]  

(49)

for \( (s, \varphi) \in [0, L] \times [0, 2\pi] \). Now, the curvature vector \( \vec{\kappa}_\gamma \) and its covariant derivatives \( \left( \nabla_{\gamma'} \right)^k (\kappa_\gamma) \) w.r.t. the unit tangent vector field \( \gamma' \) along \( \gamma \) is invariant w.r.t. smooth reparametrization of \( \gamma \), unlike the usual derivatives \( k_\gamma^{(k)} \) of the signed curvature \( k_\gamma \) w.r.t. the original parameter \( s \in [0, L] \). Since the path \( \gamma = \pi \circ \eta \) was assumed to have constant speed 2, we thus obtain here:

\[ \left( \nabla_{\gamma'} \right)^k (\vec{\kappa}_\gamma)(s) = \frac{1}{2k} k_\gamma^{(k)}(s) \eta(s) i u(s) \eta(s) = \frac{1}{2k} k_\gamma^{(k)}(s) v_\gamma(s), \]  

(50)

for every \( k \in \mathbb{N} \) and \( \forall s \in [0, L] \). Moreover, taking Lemma 3 into account, we know that for every horizontal smooth lift \( \gamma \) w.r.t. \( \pi \) and for every simple parametrization \( F : \Sigma \rightarrow \pi^{-1} \text{trace}(\gamma) \), there is a horizontal smooth lift \( \eta_F : \mathbb{R} / L \mathbb{Z} \cap B_\epsilon (s^*) \rightarrow \Sigma \) of \( \gamma |_{\mathbb{R} / L \mathbb{Z} \cap B_\epsilon (s^*)} \) w.r.t. \( \pi \circ F \), i.e., \( \eta_F \) satisfies \( \eta = F \circ \eta_F \) on \( \mathbb{R} / L \mathbb{Z} \cap B_\epsilon (s^*) \).

Hence, taking also into account the invariance of traced tensors w.r.t. smooth local coordinate transformations and also the invariance of the curvature vector \( \vec{\kappa}_\gamma \) w.r.t. smooth reparametrizations of the path \( \gamma \), we can immediately derive from formulae (41)–(50) the assertions (25)–(31). Moreover, we note that formula (35) describes a geometric property of the pair \( (\pi, D\pi) \), which is thus independent of the chosen parametrization of the Hopf-torus \( \pi^{-1}(\text{trace}(\gamma)) \). Hence, a combination of formulae (31), (35) and (50) immediately yields assertion (32).

Finally, since the standard parametrization \( X : [0, L] \times [0, 2\pi] \rightarrow \pi^{-1}(\text{trace}(\gamma)) \) in formula (10) covers \( \pi^{-1}(\text{trace}(\gamma)) \) only once, as pointed out in Remark 4, since \( F \) has mapping degree \( \pm 1 \), and since \( \gamma \) has constant speed 2 on \( \mathbb{R} / L \mathbb{Z} \), we obtain from the classical area-formula, formula (42) and Lemma 2:

\[ \mathcal{W}(F) = | \deg(F) | \mathcal{W}(X) = \int_{0}^{2\pi} \int_{0}^{L} 1 + \frac{1}{4} |2\kappa_\gamma(s)|^2 \, ds \, d\varphi \]

\[ = 2\pi \int_{0}^{L} 1 + |\kappa_\gamma|^2 \, ds = \pi \int_{\mathbb{R} / L \mathbb{Z}} 1 + |\kappa_\gamma|^2 \, d\mu_\mathcal{E} \equiv \pi \, \mathcal{E}(\gamma). \]

Similarly, we can infer from formulae (49) and (50) that there holds:

\[ \int_{\Sigma} |\nabla_{L^2} \mathcal{W}(F)|^2 \, d\mu_F = | \deg(F) | \int_{0}^{2\pi} \int_{0}^{L} |\nabla_{L^2} \mathcal{W}(X)|^2(s, \varphi) \, ds \, d\varphi \]

\[ = 2\pi \int_{0}^{L} |\kappa''_\gamma(s) + 2\kappa^3_\gamma(s) + 2\kappa_\gamma(s)|^2 \, ds \]

\[ = 4\pi \int_{\mathbb{R} / L \mathbb{Z}} \left| 2 \left( \nabla_{\gamma'} \right)^{\perp} (\vec{\kappa}_\gamma) + |\vec{\kappa}_\gamma|^2 \vec{\kappa}_\gamma + \vec{\kappa}_\gamma \right|^2 \, d\mu_\mathcal{E}, \]

which is just the asserted formula (34) on account of the invariance of the differential operator \( \nabla_{L^2} \mathcal{E}(\gamma) \) w.r.t. reparametrization of the curve \( \gamma \). \( \square \)
Proposition 5. Let \( T > 0 \) be arbitrarily chosen, and let \( \gamma_t : S^1 \to S^2 \) be a smooth family of closed smooth regular curves, for \( t \in [0, T] \). Moreover, let \( F_t : \Sigma \to S^3 \) be an arbitrary smooth family of smooth immersions, which map some compact smooth torus \( \Sigma \) simply onto the Hopf-tori \( \pi^{-1}(\text{trace}(\gamma_t)) \subset S^3 \), for every \( t \in [0, T] \). Then, the following statement holds:

The family of immersions \( \{F_t\} \) moves according to the Willmore flow Eq. (1) on \([0, T] \times \Sigma\)—up to smooth, time-dependent reparametrizations \( \Phi_t \) with \( \Phi_0 = \text{Id}_\Sigma \)—if and only if there is a smooth family \( \sigma_t : S^1 \to S^1 \) of reparametrizations with \( \sigma_0 = \text{Id}_S \), such that the family \( \{\gamma_t \circ \sigma_t\} \) satisfies the “elastic energy evolution equation”

\[
\partial_t \tilde{\gamma}_t = - \left( 2 \left( \nabla_{\tilde{\gamma}_t} \right)^2 (\kappa_{\tilde{\gamma}_t}) + |\kappa_{\tilde{\gamma}_t}|^2 \kappa_{\tilde{\gamma}_t} + \kappa_{\tilde{\gamma}_t} \right) = - \nabla_{L^2} \mathcal{E}(\gamma_t)
\]

on \([0, T] \times S^1\), where \( \nabla_{L^2} \mathcal{E} \) denotes the \( L^2 \)-gradient of \( \mathcal{E}(\gamma) = \int_{S^1} 1 + |\kappa_\gamma|^2 \, d\mu_\gamma \).

Proof. The easier direction of the assertion follows immediately from the “Hopf-Willmore-identity” (32) and the ordinary chain rule. For, suppose a smooth family \( F_t : \Sigma \to \pi^{-1}(\text{trace}(\gamma_t)) \) of smooth immersions, which parametrize \( \pi^{-1}(\text{trace}(\gamma_t)) \) simply, solves the Willmore equation (5) on \([0, T] \times \Sigma \). In this case, we apply Lemma 3 and choose some smooth family of horizontal smooth lifts \( \eta_{F_t} : S^1 \cap B_{\epsilon_t}(s^*) \to \Sigma \) of \( \gamma_t|_{S^1 \cap B_{\epsilon_t}(s^*)} \), for some arbitrary \( s^* \in S^1 \) and some \( \epsilon_t > 0 \), i.e., such that \( \gamma_t = \pi \circ F_t \circ \eta_{F_t} \) on \( \bigcup_{t \in [0, T]} \{t\} \times (S^1 \cap B_{\epsilon_t}(s^*)) \), and conclude by means of formula (32):

\[
\partial_t \gamma_t(s) = \partial_t (\pi \circ F_t \circ \eta_{F_t})(s) = D\pi_{F_t(\eta_{F_t}(s))}(\partial_t (F_t \circ \eta_{F_t})(s))
\]

\[
= D\pi_{F_t(\eta_{F_t}(s))}(\partial_t F_t)(\eta_{F_t}(s)) + D\pi_{F_t(\eta_{F_t}(s))}(D\eta_{F_t}(s)(F_t))(\partial_t (\eta_{F_t})(s))
\]

\[
= D\pi_{F_t(\eta_{F_t}(s))}(D\eta_{F_t}(s)(F_t)(\partial_t (\eta_{F_t})(s)))
\]

\[
= - \nabla_{L^2} \mathcal{E}(\gamma_t)(s) + D\pi_{F_t(\eta_{F_t}(s))}(D\eta_{F_t}(s)(F_t)(\partial_t (\eta_{F_t})(s)))
\]

on \( \bigcup_{t \in [0, T]} \{t\} \times (S^1 \cap B_{\epsilon_t}(s^*)) \). Now, the vector \( \partial_t (\eta_{F_t})(s) \) is contained in the tangent space \( T_{\eta_{F_t}(s)} \Sigma \) touching \( \Sigma \) at the point \( \eta_{F_t}(s) \), for every \( s \in S^1 \cap B_{\epsilon_t}(s^*) \) and every \( t \in [0, T] \). Then, \( D\eta_{F_t}(s)(F_t)(\partial_t (\eta_{F_t})(s)) \) is a tangent vector of \( \pi^{-1}(\text{trace}(\gamma_t)) \) at its point \( F_t(\eta_{F_t}(s)) \), and formula (38) shows that \( D\pi_{F_t(\eta_{F_t}(s))}(D\eta_{F_t}(s)(F_t)(\partial_t (\eta_{F_t})(s))) \) is a tangent vector of \( \gamma_t \) in its point \( (\pi \circ F_t \circ \eta_{F_t})(s) = \gamma_t(s) \) on \( \bigcup_{t \in [0, T]} \{t\} \times (S^1 \cap B_{\epsilon_t}(s^*)) \). Since the vector \( \nabla_{L^2} \mathcal{E}(\gamma_t)(s) \) is contained in the normal space of the curve \( \gamma_t \) within \( TS^2 \) in its point \( \gamma_t(s) \) and since \( s^* \in S^1 \) was arbitrarily chosen, we conclude that the family \( \{\gamma_t\} \) solves the equation

\[
\left( \partial_t \gamma_t(s) \right)_{\perp} = - \nabla_{L^2} \mathcal{E}(\gamma_t)(s)
\]
using the fact that the smooth family \( \{ \gamma_t \} \) solves equation (52) on \([0, T] \times S^1\) and that it consists of smooth, closed and regular curves only, we can follow the lines of the author’s article [11, p. 1177], solving a certain ODE in order to construct an appropriate smooth family of diffeomorphisms \( \sigma_t : S^1 \to S^1 \), for \( t \in [0, T] \), satisfying \( \sigma_0 = \text{Id}_{S^1} \), such that the composition \( \gamma_t \circ \sigma_t \) solves equation (51) on \([0, T] \times S^1\), just as asserted. Vice versa, if there holds

\[
\partial_t \gamma_t(s) = -\nabla_{L^2} \mathcal{E}(\gamma_t)(s), \tag{53}
\]

\( \forall (t, s) \in [0, T] \times S^1 \), for some family of smooth closed regular curves \( \gamma_t : S^1 \to S^2 \), \( t \in [0, T] \), then—again using Lemma 3—we first choose some family of horizontal smooth lifts \( \eta_{F_t} : S^1 \cap B_{\varepsilon_t}(s^*) \longrightarrow \Sigma \) of \( \gamma_t \mid S^1 \cap B_{\varepsilon_t}(s^*) \) w.r.t. \( \pi \circ F_t \) on \([0, T] \) for an arbitrarily given smooth family of simple parametrizations \( F_t : \Sigma \longrightarrow \pi^{-1}(\text{trace}(\gamma_t)) \), and then we compute by means of \( \gamma_t = \pi \circ F_t \circ \eta_{F_t} \) on \( S^1 \cap B_{\varepsilon_t}(s^*) \), formula (32) and the chain rule:

\[
\begin{align*}
D\pi_{F_t(\eta_{F_t}(s))} \cdot \left( -\nabla_{L^2} \mathcal{W}(F_t)(\eta_{F_t}(s)) \right) \\
= -\nabla_{L^2} \mathcal{E}(\gamma_t)(s) = \partial_t(\gamma_t)(s) = \partial_t(\pi \circ F_t \circ \eta_{F_t})(s) \\
= D\pi_{F_t(\eta_{F_t}(s))} \cdot (\partial_t(\gamma_t)(F_t(\eta_{F_t}(s)))) + D\pi_{F_t(\eta_{F_t}(s))} \cdot \left( D\eta_{F_t}(s)(F_t) \cdot (\partial_t(\eta_{F_t})(s)) \right)
\end{align*}
\]  

(54)

on \( \bigcup_{t \in [0, T]} \{ t \} \times (S^1 \cap B_{\varepsilon_t}(s^*)) \) and for every family of horizontal smooth lifts \( \eta_{F_t} \) of \( \gamma_t \mid S^1 \cap B_{\varepsilon_t}(s^*) \) w.r.t. \( \pi \circ F_t \). Now, we fix some point \( x \in \Sigma \) arbitrarily. For every \( t \in [0, T] \), there is at least one point \( s_x(t) \in S^1 \) satisfying

\[
\gamma_t(s_x(t)) = (\pi \circ F_t)(x). \tag{55}
\]

Since every path \( \gamma_t \) is required to be regular, i.e., \( \gamma_t'(s) \neq 0 \) \( \forall s \in S^1 \) in every \( t \in [0, T] \), and since \( \{ \gamma_t \} \) and \( \{ F_t \} \) are required to be smooth families of smooth maps, the implicit function theorem guarantees that the solutions \( s_x(t) \) to equation (55) can be chosen in such a way, that \( [t \mapsto s_x(t)] \) is a smooth function mapping \([0, T] \) into \( S^1 \). On account of (55) and Lemma 3, we can choose the horizontal smooth lift \( \eta_{F_t} \) for every \( t \in [0, T] \) in such a way that exactly \( \eta_{F_t}(s_x(t)) = x \) holds. Inserting this into formula (54), we obtain

\[
D\pi_{F_t(x)} \cdot \left( -\nabla_{L^2} \mathcal{W}(F_t)(x) \right) = D\pi_{F_t(x)} \cdot \left( \partial_F F_t(x) \right) + D\pi_{F_t(x)} \cdot \left( D_x(F_t)(\partial_t(\eta_{F_t})(s_x(t))) \right), \tag{56}
\]

for every fixed \( x \in \Sigma \) and for \( t \in [0, T] \). Now, as we have computed in formulae (35)–(38), the Hopf-differential \( D\pi_{F_t} \) maps the one-dimensional normal bundle of the immersion \( F_t \) in \( TS^3 \) isomorphically onto the normal bundle of the regular curve \( \gamma_t \) within \( TS^2 \) and the tangent bundle of the immersion \( F_t \) onto the tangent bundle of \( \gamma_t \) in \( TS^2 \), which implies in particular that \( D\pi_{F_t(\eta_{F_t}(s))} \cdot \left( D\eta_{F_t}(s)(F_t)(\partial_t(\eta_{F_t})(s)) \right) \) is a tangent vector of the curve \( \pi \circ F_t \circ \eta_{F_t} = \gamma_t \) in its point \( \gamma_t(s) \). Hence, since
\( \nabla_{L^2} \mathcal{W}(F_t) \) is a section of the normal bundle of the immersion \( F_t \), a comparison of the normal components in (56) particularly yields the following modified flow equation:

\[
\left( \partial_t F_t \right)^{F_t}_{\perp} (x) = - \nabla_{L^2} \mathcal{W}(F_t)(x)
\]  
(57)

\( \forall (t, x) \in [0, T] \times \Sigma \), where \( (V_t)^{F_t}_{\perp} \) denotes the projections of vector fields \( V_t \in \Gamma(F_t^* TS^3) \) along the immersion \( F_t \) onto the subspace \( \Gamma^{\perp}(F_t^* TS^3) \) of smooth sections of the normal bundle of \( F_t \), for every \( t \in [0, T] \). Now we can follow the lines of the author’s article [11], p. 1177, solving a certain system of ODE’s, in order to infer from the facts that the smooth family \( \{ F_t \} \) solves Eq. (57) on \( [0, T] \times \Sigma \) and that it consists of smooth immersions only, that one can construct an appropriate smooth family of smooth diffeomorphisms \( \Phi_t : \Sigma \rightarrow \Sigma \), for \( t \in [0, T] \), satisfying \( \Phi_0 = \text{Id}_\Sigma \), such that the composition \( F_t \circ \Phi_t \) solves the Willmore flow equation (1) on \( [0, T] \times \Sigma \), which proves also the second direction of the assertion.

\[ \square \]

4. Proof of Theorem 1

Part I Short-time existence and uniqueness of the Willmore flow (1) is classical. In particular, for any smooth closed regular curve \( \gamma_0 : S^1 \rightarrow S^2 \) and for any smooth parametrization \( F_0 : \Sigma \rightarrow S^3 \) of the corresponding smooth Hopf-torus \( \pi^{-1}(\text{trace}(\gamma_0)) \) there is a unique smooth short-time solution \( \mathcal{P}(\cdot, 0, F_0) \) of evolution equation (1) on \( [0, \varepsilon] \times \Sigma \) with \( \mathcal{P}(0, 0, F_0) = F_0 \) on \( \Sigma \). Hence, there has to exist a unique maximal time \( T_{\text{max}} \in (\varepsilon, \infty] \), such that this short-time solution of equation (1) extends from \( [0, \varepsilon] \times \Sigma \) to a unique smooth solution of the same evolution equation on \( [0, T_{\text{max}}] \times \Sigma \). Moreover, we know from Theorem 1.1 in [6] that there is a unique global, smooth solution \( \{ P(t, 0, \gamma_0) \}_{t \geq 0} \) to equation (51), satisfying the Cauchy problem

\[
\partial_t \tilde{\gamma}_t = - \nabla_{L^2} \mathcal{E}(\tilde{\gamma}_t), \quad \text{with} \quad \tilde{\gamma}_0 = \gamma_0 \quad \text{on} \quad S^1.
\]  
(58)

Now, on account of the requirement that \( F_0 \) is a simple map from \( \Sigma \) onto \( \pi^{-1}(\text{trace}(\gamma_0)) \) and recalling the uniqueness of the Willmore flow, we obtain from Proposition 5 that the short-time solution \( \{ \mathcal{P}(t, 0, F_0) \}_{t \in [0, \varepsilon]} \) of the Willmore flow (1), starting in \( F_0 \), consists of simple smooth maps from \( \Sigma \) onto \( \pi^{-1}(\text{trace}(P(t, 0, \gamma_0))) \), for every \( t \in [0, \varepsilon] \). Hence, the unique short-time flow line \( \{ P(t, 0, F_0) \}_{t \in [0, \varepsilon]} \) of the Willmore flow corresponds to the unique solution \( \{ P(\cdot, 0, \gamma_0) \} \) of the Cauchy problem (58), restricted to \( [0, \varepsilon] \times S^1 \), via the Hopf-fibration \( \pi \), in the precise sense, that there is a family of horizontal smooth lifts \( \eta_{\mathcal{P}(t, 0, F_0)} : S^1 \cap B_{\varepsilon_t}(\pi^*) \rightarrow \Sigma \) of \( P(t, 0, \gamma_0) \) w.r.t. \( \pi \circ \mathcal{P}(t, 0, F_0) \), satisfying

\[
P(t, 0, \gamma_0) = \pi \circ \mathcal{P}(t, 0, F_0) \circ \eta_{\mathcal{P}(t, 0, F_0)}
\]  
(59)

on \( \bigcup_{t \in [0, \varepsilon]} \{ t \times (S^1 \cap B_{\varepsilon_t}(\pi^*)) \} \), for every fixed \( \pi^* \in S^1 \); see Lemma 3 for the existence of such horizontal smooth lifts. Now, since we know from Theorem 1.1 in [6], that
the maximal solution \( \{ P(t, 0, \gamma_0) \}_{t \geq 0} \) of the initial value problem (58) is global and smooth, we immediately derive from Proposition 5, from formula (59), and from the uniqueness of flow lines of the Willmore flow, that there has to hold “\( T_{\text{max}} = \infty \),” that formula (59) holds on \( \bigcup_{t \in [0, \infty)} \{ t \} \times \left( S^1 \cap B_{\epsilon_i}(s^i) \right) \) and that the resulting global smooth solution \( \{ P(t, 0, F_0) \}_{t \geq 0} \) of the Willmore flow equation is unique. Moreover, using formula (27) and the fact that every immersion \( P(t, 0, F_0) \) has to be a smooth and simple parametrization of the Hopf-torus \( \pi^{-1}(\text{trace}(P(t, 0, \gamma_0))) \)—because identity (59) continues to hold for any \( t \geq 0 \)—we see that the flow line \( \{ P(t, 0, F_0) \}_{t \geq 0} \) of the Willmore flow consists of umbilic-free immersions only, just as asserted in Part I of this theorem.

**Part II** Now we start to prove “subconvergence” of the Willmore flow, as \( t \to \infty \). First of all, given the global flow line \( \{ F_t \}_{t \geq 0} := \{ P(t, 0, F_0) \}_{t \geq 0} \) of the Willmore flow (1) starting in some prescribed immersion \( F_0 \), which maps the torus \( \Sigma \) simply onto some prescribed Hopf-torus \( \pi^{-1}(\text{trace}(\gamma_0)) \), we infer from the proof of Part I of this theorem that there is a unique and global flow line \( \{ \gamma_t \}_{t \geq 0} := \{ P(t, 0, \gamma_0) \}_{t \geq 0} \) of evolution equation (51), starting in the prescribed curve \( \gamma_0 \), which corresponds to the global flow line \( \{ F_t \} \) of equation (1) in the precise sense of identity (59), holding for \( t \geq 0 \). Now we note that

\[
\frac{d}{dt} E(\gamma_t) = - \int_{S^1} \left( 2 \left( \nabla_{\gamma_t}^\perp / |\gamma_t| \right)^2 (\kappa_{\gamma_t})^2 + |\kappa_{\gamma_t}|^2 \kappa_{\gamma_t} + \kappa_{\gamma_t} \right)^2 d\mu_{\gamma_t} \leq 0,
\]

for any \( t \geq 0 \), i.e., that the function \( t \mapsto E(P(t, 0, \gamma_0)) \) is not increasing on \([0, \infty)\).

In particular, there exists some constant \( K(\gamma_0) > 0 \), such that both

\[
\text{length}(P(t, 0, \gamma_0)) \leq K(\gamma_0) \quad \text{and} \quad \int_{S^1} |\kappa_{P(t, 0, \gamma_0)}|^2 d\mu_{P(t, 0, \gamma_0)} \leq K(\gamma_0)
\]

(60) hold for every \( t \in [0, \infty) \). Now, applying the elementary inequality

\[
\left( \int_{S^1} |\kappa_{\gamma}| d\mu_{\gamma} \right)^2 \geq 4\pi^2 - \text{length}(\gamma)^2
\]

holding for every closed smooth regular path \( \gamma : S^1 \to S^2 \), see [28, Proposition 1], one can easily derive the lower bound

\[
\text{length}(\gamma) \geq \min \left\{ \pi, \frac{3\pi^2}{E(\gamma)} \right\},
\]

(61)

see Lemma 2.9 in [6], for any closed smooth regular path \( \gamma : S^1 \to S^2 \). Combining estimate (61) with statement 60, we see that the lengths of the curves \( P(t, 0, \gamma_0) \) can be uniformly bounded from below, as well:

\[
\text{length}(\text{trace}(P(t, 0, \gamma_0))) \geq \min \left\{ \pi, \frac{3\pi^2}{E(\gamma_0)} \right\} =: l(\gamma_0), \quad \forall t \in [0, \infty),
\]

(62)

which particularly rules out extinction of flow lines of the flow (51) at any time \( t \in [0, \infty) \). Now, using both (60) and (62), one can argue as in Section 4.2 in [6],
using also Lemma 2.6 in [6], or also as in Steps 1-6 of the proof of Theorem 1.1 of [5] that there hold uniform curvature estimates for the flowing curves $\gamma_t = P(t, 0, \gamma_0)$, namely:

$$\left\| \left( \nabla_{\gamma_t} \right)^m_{\gamma_t} \right\|_{L^\infty(S^1)} \leq C(\gamma_0, m), \quad \text{for } t \in [0, \infty),$$

for some constant $C(\gamma, m) > 0$, for each $m \in \mathbb{N}_0$. From this fact and from the compactness of $S^2$, one can easily conclude that for every sequence $t_k \not\to \infty$ there exists some subsequence $\{t_{k_j}\}$ and some regular smooth closed curve $\gamma_\infty : S^1 \to S^2$ such that reparametrizations via $\tilde{\gamma}_{t_{k_j}}$ of $\gamma_{t_{k_j}}$ to arc-length converge:

$$\tilde{\gamma}_{t_{k_j}} \to \gamma_\infty \neq \{\text{point}\} \quad \text{in } C^m(S^1, \mathbb{R}^2), \quad \forall m \in \mathbb{N}_0,$$

as $j \to \infty$. For ease of exposition, we rename the subsequence $\{t_{k_j}\}$ into $\{t_j\}$ again. We note here that trace$(\gamma_\infty)$ cannot be degenerated, i.e., a point on $S^2$, because of the lower bound (62) for the lengths of the traces of curves $\gamma_t = P(t, 0, \gamma)$, for $t \in [0, \infty)$. Moreover, we can derive from line (64) that $A(F_{t_j}) := \int_{S^3} d\mu_{F_{t_j}} \leq \text{const.} \quad \forall j \in \mathbb{N}$, and from formula (30) and estimate (63) we obtain the uniform curvature bounds for the immersions $F_t = \mathcal{P}(t, 0, F_0)$:

$$\left\| (\nabla_{F_t}^\perp)^m_{F_t} (A_{F_t}, S^3) \right\|_{L^\infty(S)} \leq \text{Const}(F_0, m), \quad \text{for } t \in [0, \infty),$$

for each $m \in \mathbb{N}_0$, where we indicated that we can only derive from this argument estimates for the covariant derivatives $(\nabla_{F_t}^\perp)^m_{F_t} (A_{F_t}, S^3)$ in the normal bundle of $F_t$ within $TS^3$. Now, arguing as in the proof of Lemma 2.1 in [24], one can prove by induction that for any smooth section $\Phi$ of the normal bundle of some smooth immersion $F : \Sigma \to S^3 \subset \mathbb{R}^4$ within $TS^3$ there holds:

$$(\nabla_{F_t}^\perp)^m_{F_t} (A_{F_t}, S^3) = (\nabla_{F_t}^\perp)^m_{F_t} (A_{F_t}, S^3) \circ \cdots \circ (\nabla_{F_t}^\perp)^m_{F_t} (A_{F_t}, S^3) \circ \Phi \quad \text{for } n \geq 0,$$

for any order $m \in \mathbb{N}$ and any $i_j \in \{1, 2\}$. Moreover, we recall from formula (18) that $A_{F_t} = A_{F_t}^3 F g_{F_t}$. Since the first covariant derivative $\nabla_{F_t}^\perp (F)$ of the immersion $F$ is a section of the tangent bundle of $F$, its projection $\nabla_{F_t}^\perp (F)$ into the normal bundle of $F$ within $\mathbb{R}^4$ vanishes identically on $\Sigma$, see also here the proof of Lemma 2.1 in [24]. Hence, every further derivative of $\nabla_{F_t}^\perp (F)$ vanishes as well. Combining this insight with the fact that $\nabla{F_t}^\perp (g_{F_t}) = 0$—by definition of $\nabla_{F_t}^\perp$—and with formulae (18), (65) and (66)—here applied to $\Phi = A_{F_t}^3$—we arrive at stronger uniform curvature bounds for $F_t = \mathcal{P}(t, 0, F_0)$:

$$\left\| (\nabla_{F_t}^\perp)^m_{F_t} (A_{F_t}, \mathbb{R}^4) \right\|_{L^\infty(S)} = \left\| (\nabla_{F_t}^\perp)^m_{F_t} (A_{F_t}, S^3) \right\|_{L^\infty(S)} \leq \text{Const}(F_0, m), \quad \text{for } t \in [0, \infty),$$

for each $m \in \mathbb{N}$. Finally, we have again by estimate (65):

$$\left| A_{F_t, \mathbb{R}^4}(x) \right|^2 = \left| A_{F_t, S^3}(x) - F_t(x) - g_{F_t}(x) \right|^2$$

$$\equiv g_{F_t}^{ij}(x) g_{F_t}^{jk}(x) ((A_{F_t, S^3} - F_t g_{F_t})_{ij}(x), (A_{F_t, S^3} - F_t g_{F_t})_{jk}(x)) \mathbb{R}^4$$

$$= \left| A_{F_t, S^3}(x) \right|^2(x) + 2 \leq \text{Const}(F_0, 0)^2 + 2, \quad \forall x \in \Sigma,$
for every \( t \in [0, \infty) \). Hence, we may apply here Proposition 1 to the family of immersions \( F_t = \mathcal{P}(t, 0, F_0) : \Sigma \to S^3 \) and obtain the existence of a subsequence \( \{F_{t_{jk}}\} \) and of \( C^\infty \)-diffeomorphisms \( \varphi_k : \Sigma \to \Sigma \), such that

\[
F_{t_{jk}} \circ \varphi_k \to \tilde{F} \quad \text{converge in } C^m(\Sigma, \mathbb{R}^4), \quad \text{as } k \to \infty,
\]

(69)

for each \( m \in \mathbb{N}_0 \), in the sense of Definition C.10 or also Proposition C.9 in [8], where \( \tilde{F} \) is again a \( C^\infty \)-smooth immersion of \( \Sigma \) into \( S^3 \), satisfying the bounds (65), (67) and (68). Combining convergence (69) with convergence (64) and with identity (59), we see that there holds:

\[
\pi^{-1}(\text{trace}(\gamma_\infty)) \leftarrow \pi^{-1}(\text{trace}(\gamma_{t_{jk}})) = \text{image}(F_{t_{jk}}) \]

\[
= \text{image}(F_{t_{jk}} \circ \varphi_k) \to \text{image}(\tilde{F}),
\]

(70)

in Hausdorff-distance, as \( k \to \infty \). Equation (70) particularly shows that there holds \( \text{image}(\tilde{F}) = \pi^{-1}(\text{trace}(\gamma_\infty)) \), and since we know already that \( \gamma_\infty \) is a non-degenerate, regular closed curve in \( S^2 \), this means that the limit immersion \( \tilde{F} \) maps the torus \( \Sigma \) onto a non-degenerate Hopf-torus in \( S^3 \), namely \( \pi^{-1}(\text{trace}(\gamma_\infty)) \), just as asserted in line (14). Obviously, the immersion \( \tilde{F} \) is a simple map from \( \Sigma \) onto \( \pi^{-1}(\text{trace}(\gamma_\infty)) \) in the sense of Definition 3, because it is the uniform limit of the simple parametrizations \( F_{t_{jk}} \circ \varphi_k \), taking the proved convergence (14), respectively (69), into account.

Finally, we prove that \( \tilde{F} \) is actually “Willmore.” To this end, we firstly use formula 60, in order to integrate the derivative of the function \( t \mapsto \mathcal{E}(\gamma_t) \) over the interval \([0, T)\), for any fixed \( T > 0 \):

\[
\limsup_{T \to \infty} \int_0^T \int_{S^1} |\nabla_{L^2} \mathcal{E}(\gamma_t)|^2 \, d\mu_{\gamma_t} \, dt = \limsup_{T \to \infty} (\mathcal{E}(\gamma_0) - \mathcal{E}(\gamma_T)) \leq \mathcal{E}(\gamma_0) < \infty.
\]

(71)

Therefore, the limit \( \int_0^\infty \int_{S^1} |\nabla_{L^2} \mathcal{E}(\gamma_t)|^2 \, d\mu_{\gamma_t} \, dt \) exists and is finite. We can now apply Lemma 3.1 in [5], respectively, Lemma 2.5 in [6], in order to derive from the uniform curvature bounds in (63), as well:

\[
\| \nabla_t \circ \left( \frac{\nabla_{\gamma_t}^m(\kappa_{\gamma_t})}{|\nabla_{\gamma_t}|} \right) \|_{L^\infty(S^1)} \leq C^*(\gamma_0, m), \quad \text{for } t \in [0, \infty),
\]

(72)

along the flow (51), for some positive constant \( C^*(\gamma_0, m) \), for each \( m \in \mathbb{N}_0 \). Combining estimates (72) again with estimates (60) and (63), we infer via the usual chain- and product/quotient rule:

\[
\left| \frac{d}{dt} \left( \int_{S^1} |\nabla_{L^2} \mathcal{E}(\gamma_t)|^2 \, d\mu_{\gamma_t} \right) \right| \leq C(\gamma_0), \quad \text{for } t \in [0, \infty),
\]

for some constant \( C(F_0) > 0 \). Together with (71), we achieve full convergence:

\[
\int_{S^1} |\nabla_{L^2} \mathcal{E}(\gamma_t)|^2 \, d\mu_{\gamma_t} \to 0 \quad \text{as } t \to \infty.
\]
In particular, there has to be some subsequence \( \{t_{ji}\} \) of the particular sequence \( t_j \to \infty \) of convergence (64), such that

\[
|\nabla L^2 \mathcal{E}(\gamma_{t_{ji}})|^2 \to 0 \quad \text{in } \mathcal{H}^1-\text{almost every point of } S^1
\]
as \( l \to \infty \). Hence, together with the smooth convergence in (64) we conclude:

\[
0 \leftarrow \nabla L^2 \mathcal{E}(\gamma_{t_{ji}}) \to \nabla L^2 \mathcal{E}(\gamma_\infty) \quad \text{in } \mathcal{H}^1-\text{almost every point of } S^1.
\]

Therefore, the limit curve \( \gamma_\infty \) is a smooth elastic curve, i.e., satisfies: \( \nabla L^2 \mathcal{E}(\gamma_\infty) \equiv 0 \) on \( S^1 \). Hence, the “Hopf-Willmore-formula” (32) and formula (35) immediately imply that every smooth and simple parametrization \( F^* : \Sigma \to S^3 \) of \( \pi^{-1}(\text{trace}(\gamma_\infty)) \) satisfies \( \nabla L^2 \mathcal{W}(F^*) \equiv 0 \) on \( \Sigma \), and especially the smooth limit immersion \( \tilde{F} \) in convergence (69) turns out to be “Willmore,” which finishes the proof of Part II of the theorem.

**Part III** Instead of directly proving the assertion of this part of the theorem about flow lines of the Willmore flow (1), which start to move with Willmore energy smaller than or equal to \( 8 \pi \), we will at first prove the full convergence of flow lines \( P(\cdot, 0, \gamma_0) \) of the simpler flow (51), moving closed smooth curves in \( C^\infty_{\text{reg}}(S^1, S^2) \), which start in some fixed curve \( \gamma_0 \in C^\infty_{\text{reg}}(S^1, S^2) \) whose trace is \( (\pi \circ F_0)(\Sigma) \) in \( S^2 \) and whose elastic energy \( \mathcal{E}(\gamma_0) \) is smaller than or equal to \( \frac{8\pi}{\sqrt{2}} \), on account of formula (33). We recall from line 60 that the function \( [t \mapsto \mathcal{E}(P(t, 0, \gamma_0))] \) is not increasing on \([0, \infty)\), for every initial curve \( \gamma_0 \in C^\infty_{\text{reg}}(S^1, S^2) \). Therefore, our initial condition \( \mathcal{E}(\gamma_0) \leq \frac{8\pi}{\sqrt{2}} \)” implies that \( \lim_{t \to \infty} \mathcal{E}(P(t, 0, \gamma_0)) \) exists and satisfies: \( \lim_{t \to \infty} \mathcal{E}(P(t, 0, \gamma_0)) \in \left[ 2\pi, \frac{8\pi}{\sqrt{2}} \right] \). Moreover, from Part II we recall the existence of some sequence \( \{\gamma_{t_j}\} := \{P(t_j, 0, \gamma_0)\} \), with \( t_j \nrightarrow \infty \), and of some sequence of smooth diffeomorphisms \( \psi_j : S^1 \to S^1 \), such that the reparametrizations \( \gamma_{t_j} \circ \psi_j \) have constant speed on \( S^1 \) and converge in every \( C^m(S^1, \mathbb{R}^3) \)-norm to some elastic curve \( \gamma_\infty \), again being parametrized with constant speed. We obtain therefore:

\[
2\pi \leq \mathcal{E}(\gamma_\infty) = \lim_{j \to \infty} \mathcal{E}(\gamma_j \circ \psi_j) = \lim_{t \to \infty} \mathcal{E}(P(t, 0, \gamma_0)) \leq \frac{8\pi}{\sqrt{2}}. \quad (73)
\]

**Case 1:** \( \mathcal{E}(\gamma_\infty) > 2\pi \). In this case, formula (73) contradicts Proposition 6 below, stating that there are no critical values of \( \mathcal{E} \) in the interval \( \left[ 2\pi, \frac{8\pi}{\sqrt{2}} \right] \).

**Case 2:** \( \mathcal{E}(\gamma_\infty) = 2\pi \), i.e., in this second case the limit curve \( \gamma_\infty \) is a smooth and regular parametrization of some great circle in \( S^2 \) of constant speed 1.

We are going to prove the assertion of Part III of this theorem in this remaining second case in the following three steps.

**Step 1:** As on p. 2187 in [7] we firstly assume that the function \( [t \mapsto \mathcal{E}(P(t, 0, \gamma_0))] \) was not strictly monotonically decreasing for \( t \in [0, \infty) \). In this case, there was some finite time \( t^* \geq 0 \), such that \( \partial_t(\mathcal{E}(P(t, 0, \gamma_0))) \big|_{t=t^*} = 0 \). Then, we would have:

\[
0 = -\int_{S^1} |\nabla L^2 \mathcal{E}(P(t, 0, \gamma_0))|^2 \, d\mu_{P(t,0,\gamma_0)} \big|_{t=t^*} \quad \text{on account of equation (51)},
\]

implying that the path \( P(t^*, 0, \gamma_0) \) would parametrize an elastic curve with elastic energy...
\[ \mathcal{E}(P(t^*, 0, \gamma_0)) \in [2\pi, \frac{8\pi}{\sqrt{2}}], \text{hence exactly with elastic energy } 2\pi \text{ on account of Proposition 6. Again using the weak monotonicity of the function } [t \mapsto \mathcal{E}(P(t, 0, \gamma_0))], \text{this would imply that } \mathcal{E}(P(t, 0, \gamma_0)) = 2\pi \text{ for every } t \geq t^*, \text{ and therefore again on account of Eq. (51); } 0 = \partial_t \mathcal{E}(P(t, 0, \gamma_0))) = -\int_{\mathbb{S}^1} |\nabla_{L^2} \mathcal{E}(P(t, 0, \gamma_0))|^2 \, d\mu_{P(t, 0, \gamma_0)}, \text{ for } t \geq t^*. \text{Combining this again with evolution Eq. (51), we see that the flow line } \{P(t, 0, \gamma_0)\}_{t \geq 0} \text{ would not move at all for } t \geq t^* \text{ and thus would have got stuck in a smooth parametrization } \gamma^* \text{ of some great circle in } \mathbb{S}^2. \text{In particular we would obtain here:}

\[ P(t, 0, \gamma_0) \longrightarrow \gamma^* \text{ in } C^m(\mathbb{S}^1, \mathbb{R}^3), \text{ as } t \to \infty, \text{ for every } m \in \mathbb{N}_0. \tag{74} \]

**Step 2:** We shall suppose throughout in steps 2 and 3 of the proof of this third part of the theorem that the function \([t \mapsto \mathcal{E}(P(t, 0, \gamma_0))]\) is strictly monotonically decreasing on \([0, \infty)\). Combining this assumption with estimates (60), (62) and (63) and with our version of the Lojasiewicz–Simon-inequality, Proposition 2, we will be able, to exactly copy the reasoning of Section 4.4 in [26], in order to arrive at the full smooth convergence—below in (80)—of the constant speed-reparametrization \(\tilde{P}(\cdot, 0, \gamma_0)\) of the global smooth flow line \(P(\cdot, 0, \gamma_0)\) of evolution equation (51), instead of working out the much more general and thus more complicated technique of Lemma 4.1 in Section 4 of [4], respectively, of Theorem 1.2 in Section 5 of [7]. The gist of Rupp’s and Spener’s argument in [26] consists of the pragmatic and new idea, to simply work with the entire reparametrized flow line \(\tilde{P}(\cdot, 0, \gamma_0)\)—and not only with a convergent subsequence of it yielding some smooth and stationary limit curve, as we did above following, e.g., Section 5 of [7]—and to realize that this reparametrization can and should be written down explicitly, allowing for further precise investigation.

In order to translate Definition 4.9 of [26] to our situation, we should identify \(\mathbb{S}^1\) with \([-\pi, \pi]/(-\pi \sim \pi)\) via the map \(s \mapsto x = \frac{\log(s)}{i}\), where “log” denotes the principal branch of the natural logarithm, and we should rather consider the curves \(x \mapsto f_t(x) := P(t, 0, \gamma_0)(\exp(ix)), \text{ for } x \in [-\pi, \pi], \text{ instead of } s \mapsto P(t, 0, \gamma_0)(s), \text{ for } s \in \mathbb{S}^1, \text{ for each fixed time } t \in [0, \infty).\) Working with this slightly changed notation, we are able to define exactly as in Definition 4.9 of [26]:

\[ \check{f}_t(x) := f_t(\psi_t(x)), \text{ for } (x, t) \in [-\pi, \pi] \times [0, \infty), \tag{75} \]

where \(\psi_t : [-\pi, \pi] \overset{\sim}{\longrightarrow} [-\pi, \pi]\) is the inverse of the smooth diffeomorphism

\[ \varphi_t(y) := \frac{2\pi}{\text{length}(P(t, 0, \gamma_0))} \int_{-\pi}^y |\partial_x f_t(z)| \, dz, \text{ for } y \in [-\pi, \pi], \]

defined here for each fixed \(t \in [0, \infty).\) The reparametrization in (75) automatically yields the smooth reparametrization \(\tilde{P}(t, 0, \gamma_0)(s) := \check{f}_t(\frac{\log(s)}{i}), \text{ for } s \in \mathbb{S}^1, \text{ of the original flow line } P(\cdot, 0, \gamma_0).\) Now, Lemma 4.10 of [26] and estimate (60) yield the important comparison:

\[ \| \check{f}_t \|_{L^2([-\pi, \pi], \mathcal{L}^1)} \leq \sqrt{4\pi} \sqrt{\frac{1}{\ell(y_0)} + 4 \mathcal{E}(f_0)} \| f_t \|_{L^2([-\pi, \pi], \mu_{f_{i,t}})}, \tag{76} \]
for each fixed $t \in [0, \infty)$, whose proof can be adopted here from [26] without any alteration, taking also formula (2.14) in [6] into account. For the convenience of the reader, we shall now sketch the proof of Theorem 1.2 in [26], in order to arrive at the desired convergence (80): First of all, we apply Proposition 2 to some small $C^4$-neighborhood of our smooth elastic limit curve $\gamma_\infty$, stating that there are constants $\theta \in (0, \frac{1}{2}], c \geq 0$ and $\sigma > 0$, only depending on $\gamma_\infty$, such that for every curve $\gamma \in C^4_{reg}(S^1, S^2)$ satisfying $\| \gamma - \gamma_\infty \|_{C^4(S^1, \mathbb{R}^3)} \leq \sigma$ there holds:

$$|E(\gamma) - E(\gamma_\infty)|^{1-\theta} \leq c \left( \int_{S^1} |\nabla L_2 E(\gamma)|^2 \, d\mu_\gamma \right)^{1/2}. \quad (77)$$

Moreover, we recall here from the beginning of the proof of this third part of the theorem that there is an increasing sequence of times $\{t_j\}$, such that our reparametrized curves $\tilde{P}(t_j, 0, \gamma_0)$ converge in every $C^m$-norm to $\gamma_\infty$, as $j \to \infty$. We can therefore choose $j_0$ that large, such that the suprema

$$s_j := \sup \{ s \geq t_j | \| \tilde{P}(t, 0, \gamma_0) - \gamma_\infty \|_{C^4(S^1, \mathbb{R}^3)} < \sigma \text{ for every } t \in [t_j, s] \}$$

are well defined and satisfy $s_j > t_j$ for each $j > j_0$, where $\sigma$ denotes again the constant from inequality (77). We finally recall that here the function $[t \mapsto E(\tilde{f}_t)] = E(P(t, 0, \gamma_0))$ is strictly monotonically decreasing and converges to $E(\gamma_\infty)$ as $t \to \infty$, on account of statement (73). We can therefore also introduce the smooth, strictly monotonically decreasing and positive function $G(t) := (E(\tilde{f}_t) - E(\gamma_\infty))^\theta$, for $t \in [0, \infty)$, with exponent $\theta$ from inequality (77), and we compute by means of the chain rule, evolution equation (51) and the invariance of the elastic energy w.r.t. smooth reparametrization:

$$-\frac{d}{dt} G(t) = -\frac{d}{dt} \left( (E(\tilde{f}_t) - E(f_\infty))^\theta \right) = -\frac{d}{dt} \left( (E(P(t, 0, \gamma_0)) - E(\gamma_\infty))^\theta \right)$$

$$= \theta (E(P(t, 0, \gamma_0)) - E(\gamma_\infty))^\theta - 1 \int_{S^1} |\nabla L_2 E(P(t, 0, \gamma_0))|^2 \, d\mu_{P(t, 0, \gamma_0)}$$

$$= \theta (E(\tilde{P}(t, 0, \gamma_0)) - E(\gamma_\infty))^\theta - 1 \| \nabla L_2 E(P(t, 0, \gamma_0)) \|_{L^2(\mu_{P(t, 0, \gamma_0)})}$$

$$\geq \frac{\theta}{c} \| \partial_t P(t, 0, \gamma_0) \|_{L^2(S^1, \mu_{P(t, 0, \gamma_0)})}, \text{ for } t \in [t_j, s_j] \text{ and each } j > j_0,$$

where we could apply the Lojasiewicz–Simon gradient inequality (77) in the last line to the reparametrized curves $\tilde{P}(t, 0, \gamma_0)$, being sufficiently close to the great circle $\gamma_\infty$ in $C^4(S^1, \mathbb{R}^3)$ for $t \in [t_j, s_j]$, by definition of the suprema $s_j$. Combining the above inequality $-\frac{d}{dt} G(t) \geq \frac{\theta}{c} \| \partial_t P(t, 0, \gamma_0) \|_{L^2(S^1, \mu_{P(t, 0, \gamma_0)})}$ with inequality (76),
we achieve as in formula (4.9) of [26] the decisive estimate:

$$- \frac{d}{dt} G(t) \geq C(\theta, c, \mathcal{E}(\gamma_0), l(\gamma_0)) \| \partial_t \tilde{P}(t, 0, \gamma_0) \|_{L^2(S^1, H^1)}, \quad \text{for } t \in [t_j, s_j),$$

(78)
for each $j > j_0$. Integration of inequality (78) from $t_j$ to any $T \in (t_j, s_j)$ yields:

$$\| \tilde{P}(T, 0, \gamma_0) - \tilde{P}(t_j, 0, \gamma_0) \|_{L^2(S^1, H^1)} \leq \int_{t_j}^{T} \| \partial_t \tilde{P}(t, 0, \gamma_0) \|_{L^2(S^1, H^1)} \, dt \leq \frac{1}{C(\theta, c, \mathcal{E}(\gamma_0), l(\gamma_0))} G(t_j),$$

(79)
whose right hand side converges to 0 as $j \to \infty$ because of statement (73). Now, as in the proof of Theorem 1.2 in [26] we can combine statement (79) with the general subconvergence of the elastic energy flow (51) from Section 4.2 in [6], which we had already used in the proof of the second part of this theorem, and with the fact that exactly the reparametrized curves $\tilde{P}(t_j, 0, \gamma_0)$ converge smoothly to the great circle $\gamma_\infty$, in order to prove that some of the suprema $s_j$, say $s_J$ for some large $J > j_0$, cannot be finite. Hence, by definition of $s_J$ we can conclude hereby that $\| \tilde{P}(t, 0, \gamma_0) - \gamma_\infty \|_{C^4(S^1)} < \sigma$ holds for every $t \geq t_J$, implying that also inequality (78) holds for every $t \geq t_J$. Herewith we can finally infer that the function $[t \mapsto \| \partial_t \tilde{P}(t, 0, \gamma_0) \|_{L^2(S^1, H^1)}]$ is of class $L^1([0, \infty), \mathbb{R})$, and thus the reparametrized flow line $\{\tilde{P}(t, 0, \gamma_0)\}_{t \geq 0}$ converges fully in $L^2(S^1, H^1)$ as $t \to \infty$, and its $L^2$-limit has to be the great circle $\gamma_\infty$, which we had started to work with at the beginning of the proof of Part III of the theorem. Now, combining this full $L^2$-convergence with the uniform curvature estimates (63)—which hold for any smooth reparametrization of the original flow line $\{P(t, 0, \gamma_0)\}_{t \geq 0}$ of flow (51)—we finally infer that:

$$\tilde{P}(t, 0, \gamma_0) \longrightarrow \gamma_\infty \quad \text{in } C^m(S^1, \mathbb{R}^3), \quad \text{as } t \to \infty, \quad \text{for every } m \in \mathbb{N}_0.$$  

(80)

Hence, just as in conclusion (74) of Step I of the considered second case within the proof of Part III of this theorem, we have arrived in (80) at the full and smooth convergence of our flow line $\{P(t, 0, \gamma_0)\}_{t \geq 0}$ of flow (51) - up to the smooth reparametrization in (75) - to a smooth parametrization $\gamma_\infty$ of some great circle in $S^2$.

**Step 3:** On account of the proof of Part I of Theorem 1, we know that the unique flow line $\{\mathcal{P}(\cdot, 0, F_0)\}$ of the Willmore flow (5) corresponds to the global flow line $\{P(t, 0, \gamma_0)\}_{t \geq 0}$ of flow (51) via the Hopf-fibration in the precise sense of formula (59). Hence, every immersion $\mathcal{P}(t, 0, F_0) : \Sigma \longrightarrow S^3$ yields a smooth and simple map from the torus $\Sigma$ onto the Hopf-torus $\pi^{-1}(\text{trace}(P(t, 0, \gamma_0)))$, for $t \geq 0$. Moreover, combining the convergence in line (80) with the strict monotonicty of the function $[t \mapsto \mathcal{E}(P(t, 0, \gamma_0))]$ and with formula (33), we can infer from the Li–Yau inequality [20] that the immersions $\mathcal{P}(t, 0, F_0)$ are smooth diffeomorphisms between $\Sigma$ and their images $\pi^{-1}(\text{trace}(P(t, 0, \gamma_0)))$, for sufficiently large $t \geq t_0 >> 1$, showing in particular that the Hopf-tori $\pi^{-1}(\text{trace}(P(t, 0, \gamma_0)))$ are smooth compact manifolds of
genus one, for \( t \geq t_0 \gg 1 \). Furthermore, since we know that \( \mathcal{E}(\gamma_\infty) = 2\pi \) and thus \( \mathcal{W}(\pi^{-1}(\text{trace}(\gamma_\infty))) = 2\pi^2 \) by formula (33), we can conclude from Theorem A in [22] that the limit Hopf-torus \( \pi^{-1}(\text{trace}(\gamma_\infty)) \) is a conformal image \( M \left( \frac{1}{\sqrt{2}} (\mathbb{S}^1 \times \mathbb{S}^1) \right) \) of the Clifford torus, for some suitable \( M \in \text{Mob}(\mathbb{S}^3) \). Moreover, by Lemma 3 there are smooth horizontal lifts \( \eta_t: \mathbb{S}^1 \setminus \{y^*_t\} \to \mathbb{S}^3 \), for arbitrarily chosen points \( y^*_t \in \mathbb{S}^1 \), of the curves \( \tilde{P}(t, 0, \gamma_0) \) from line (80) w.r.t. the Hopf-fibration \( \pi \), and by Lemma 2 and Remark 4 the maps \( Y_t: (\mathbb{S}^1 \setminus \{y^*_t\}) \times [0, 2\pi] \to \mathbb{S}^3 \) explicitly given by

\[
X_t(s, \varphi) := e^{i\varphi} \eta_t(s), \quad \text{for } t \geq t_0, \tag{81}
\]

are isometric parametrizations of the Hopf-tori \( \pi^{-1}(\text{trace}(P(t, 0, \gamma_0))) \), covering \( \pi^{-1}(\text{trace}(P(t, 0, \gamma_0))) \) exactly once up to a subset of vanishing \( H^2 \)-measure, for every fixed \( t \geq t_0 \). Now, choosing a smooth finite atlas for the torus \( \Sigma \) and a subordinate smooth partition of unity, one can use restrictions of the parametrizations \( X_t \) in (81) to appropriate open subsets of \( \mathbb{S}^1 \times [0, 2\pi] \), in order to construct smooth immersions \( Y_t: \Sigma \to \mathbb{S}^3 \), which map the torus \( \Sigma \) simply onto \( \pi^{-1}(\text{trace}(P(t, 0, \gamma_0))) \), for every \( t \geq t_0 \). In particular, by Definition 3 the induced maps in singular homology

\[
(Y_t)_{*2}: H_2(\Sigma, \mathbb{Z}) \xrightarrow{\cong} H_2(\pi^{-1}(\text{trace}(P(t, 0, \gamma_0))), \mathbb{Z}) \tag{82}
\]

are isomorphisms, for every fixed \( t \geq t_0 \). Since we know already that the Hopf-tori \( \pi^{-1}(\text{trace}(P(t, 0, \gamma_0))) \) and also \( \Sigma \) are smooth compact manifolds of genus one, and that \( Y_t \) are immersions of \( \Sigma \) into \( \mathbb{S}^3 \), for every \( t \geq t_0 \), we can conclude from statement (82) and Remark 4 that the maps \( Y_t \) are actually smooth diffeomorphisms between \( \Sigma \) and \( \pi^{-1}(\text{trace}(P(t, 0, \gamma_0))) \), for every \( t \geq t_0 \). Now, recalling (75) we set \( \tilde{\gamma}_t := \tilde{P}(t, 0, \gamma_0) \), for \( t \geq 0 \), we fix some \( y^* \in \mathbb{S}^1 \), we choose some point \( s^* \in \mathbb{S}^1 \setminus \{y^*\} \) arbitrarily, and we also recall from the first part of Lemma 3 that for every fixed \( t^* \geq t_0 \) and every fixed \( s^* \in \mathbb{S}^1 \setminus \{y^*\} \), \( Y_t^*: \mathbb{S}^1 \setminus \{y^*\} \to \pi^{-1}(\text{trace}(\tilde{\gamma}_t)) \) is a unique horizontal, smooth lift \( \eta_t(s^*, q^*): \mathbb{S}^1 \setminus \{y^*\} \to \pi^{-1}(\text{trace}(\tilde{\gamma}_t)) \) w.r.t. the Hopf-fibration \( \pi \), satisfying \( \eta_t(s^*, q^*)(s^*) = q^* \), and that this horizontal lift was obtained via the unique smooth flow generated by the initial value problem (89), here with generating vector field \( V_{\tilde{\gamma}} := V_{\tilde{\gamma}_t} \). Now, since the flow line \( \{\tilde{\gamma}_t\}_{t \geq t_0} \) is a smooth family of smooth closed regular paths, also the corresponding generating vector fields \( [(q, t) \mapsto V_{\tilde{\gamma}_t}(q)] \)—to be substituted into line (89)—are smooth sections of \( T\mathbb{S}^2 \), which depend smoothly on the time \( t \) as well. Interpreting the time \( t \) as an additional real parameter of the generating vector field \( V_{\tilde{\gamma}_t} \), Theorem 1.5.3 in [13] guarantees us that the unique solution of initial value problem (89) with smooth right hand side \( [(q, t) \mapsto V_{\tilde{\gamma}_t}(q)] \) depends smoothly on the initial value \( q^* \) and also on the additional parameter \( t \geq t_0 \). We can therefore construct a smooth family of horizontal lifts \( \eta_t = \eta_t(s^*, q^*): \mathbb{S}^1 \setminus \{y^*\} \to \mathbb{S}^3 \) of \( \tilde{\gamma}_t|_{\mathbb{S}^1 \setminus \{y^*\}} \) w.r.t. \( \pi \) with \( \eta_t(s^*, q^*)(s^*) = q_t \) in such a way that \( q_t \to q_\infty, \) as \( t \to t_\infty \), for some point \( q_\infty \in \pi^{-1}(\gamma_\infty(s^*)) \), making here also use of the full convergence of \( \{\tilde{\gamma}_t\} \) to the great circle-parametrization \( \gamma_\infty \) in (80). By successive differentiation -
w.r.t. $\tau \in \mathbb{R}$—of equation (89) for any fixed $t \geq t_0$, which reads here:

$$\frac{d\eta_t^{(s^*,q_t)}}{ds}(s) = V_{\gamma_t}(\eta_t^{(s^*,q_t)}(s)), \quad \text{for every } s \in S^1 \setminus \{y^*\},$$

and again using convergence (80) successively for every $m \in \mathbb{N}_0$, one infers inductively that any sequence $t_j \not\to \infty$ possesses a certain subsequence $\{t_{jk}\}$, such that

$$\eta_{t_{jk}} \equiv \eta_{t_{jk}}^{(s^*,q_{t_{jk}})} \longrightarrow \eta_{\infty} \quad \text{in } C^m_{\text{loc}}(S^1 \setminus \{y^*\}, \mathbb{R}^4), \quad \text{for each } m \in \mathbb{N}_0, \quad (83)$$

as $k \to \infty$, for some smooth limit function $\eta_{\infty} : S^1 \setminus \{y^*\} \longrightarrow \mathbb{S}^3$ satisfying $\eta_{\infty}(s^*) = q_{\infty}$. The limit function $\eta_{\infty}$ has to be a smooth horizontal lift of the smooth limit curve $\gamma_{\infty}|_{S^1 \setminus \{y^*\}}$ w.r.t. $\tau$, because of:

$$\pi \circ \eta_{\infty} \equiv \pi \circ \eta_{t_{jk}} = \tilde{\gamma}_{t_{jk}} \longrightarrow \gamma_{\infty} \quad \text{in } C^m_{\text{loc}}(S^1 \setminus \{y^*\}, \mathbb{R}^3), \quad \text{as } k \to \infty,$$

for each $m \in \mathbb{N}_0$, using here both (80) and (83). Now, the first part of Lemma 3 guarantees us that the horizontal lift $\eta_{\infty}$ of $\gamma_{\infty}|_{S^1 \setminus \{y^*\}}$ is uniquely determined by the additional equation $\eta_{\infty}(s^*) = q_{\infty} \in \pi^{-1}(\gamma_{\infty}(s^*))$. Combining this again with convergence (83), thus the principal of subsequences guarantees us that there holds actually:

$$\eta_t \equiv \eta_t^{(s^*,q_t)} \longrightarrow \eta_{\infty} \quad \text{in } C^m_{\text{loc}}(S^1 \setminus \{y^*\}, \mathbb{R}^4), \quad \text{for each } m \in \mathbb{N}_0, \quad (84)$$

as $t \to \infty$, where $y^* \in S^1$ had been fixed arbitrarily. Since the parametrizations $Y_t : \Sigma \longrightarrow \pi^{-1}(\text{trace}(P(t,0,\gamma_0))) = \pi^{-1}(\text{trace}(\tilde{\gamma}_t))$ had been defined via a fixed smooth atlas of $\Sigma$ and by means of restrictions of the immersions $X_t$ in (81) to appropriate open subsets of $S^1 \times [0, 2\pi]$, the full smooth convergence of $\{\eta_t\}$ in (84) implies that:

$$Y_t \longrightarrow F^* \quad \text{converge in } C^m(\Sigma, \mathbb{R}^4), \quad \text{as } t \to \infty, \quad (85)$$

for each $m \in \mathbb{N}_0$, in the sense of Definition C.10 in [8]. Here, the limit map $F^*$ can be constructed exactly as the immersions $Y_t$, but using now the above horizontal lift $\eta_{\infty}$ of the great circle-parametrization $\gamma_{\infty}$ instead of the lifts $\eta_t$ in formula (81). As the lift $\eta_{\infty}$ is horizontal, also the limit map $F^*$ turns out to be an immersion of $\Sigma$ into $\mathbb{S}^3$. Moreover, we note that the map $F^*$ maps $\Sigma$ onto $\pi^{-1}(\text{trace}(\gamma_{\infty})) = M\left(\frac{1}{\sqrt{2}}(S^1 \times S^1)\right)$, because convergences (80) and (85) imply:

$$M\left(\frac{1}{\sqrt{2}}(S^1 \times S^1)\right) \leftarrow \pi^{-1}(\text{trace}(P(t,0,\gamma_0))) = \text{image}(Y_t) \longrightarrow \text{image}(F^*)$$

in Hausdorff-distance, as $t \to \infty$. Since the parametrizations $Y_t$ are simple maps, we infer from convergence (85) and directly from Definition 3 that the limit map $F^*$ in (85) is actually a simple map from $\Sigma$ onto $M\left(\frac{1}{\sqrt{2}}(S^1 \times S^1)\right)$. Now, combining this with
the fact that both $\Sigma$ and $M\left(\frac{1}{\sqrt{2}}(S^1 \times S^1)\right)$ are smooth compact manifolds of genus one, and that $F^*$ is an immersion of $\Sigma$ into $S^3$, we can argue—just as above—by means of Remark 4 that $F^*$ is a smooth diffeomorphism between the smooth tori $\Sigma$ and $M\left(\frac{1}{\sqrt{2}}(S^1 \times S^1)\right)$. Moreover, recalling that both the immersions $P(t,0, F_0)$ and $Y_t$ are smooth diffeomorphisms between $\Sigma$ and their images $\pi^{-1}(\text{trace}(P(t,0,\gamma_0)))$, in every $t \geq t_0$—for $t_0$ sufficiently large—we conclude that there is a unique diffeomorphism $\Theta : \Sigma \xrightarrow{\cong} \Sigma$ such that $P(t_0,0, F_0) = Y_{t_0} \circ \Theta^{-1}$ holds on $\Sigma$, at time $t = t_0$. Now, since both families $\{P(t,0, F_0)\}_{t \geq t_0}$ and $\{Y_t \circ \Theta^{-1}\}_{t \geq t_0}$ parametrize the Hopf-tori $\pi^{-1}(\text{trace}(P(t,0,\gamma_0)))$, for $t \geq t_0$, Proposition 5 yields a smooth family of smooth diffeomorphisms $\Xi_t : \Sigma \longrightarrow \Sigma$ satisfying

$$P(t,0, F_0) = Y_t \circ \Theta^{-1} \circ (\Xi_t)^{-1}, \quad \text{for every } t \geq t_0.$$

Hence, we infer from statement (85) the full, smooth convergence

$$P(t,0, F_0) \circ (\Xi_t \circ \Theta) = Y_t \longrightarrow F^* \text{ in } C^m(\Sigma, R^4), \quad \text{as } t \to \infty,$$

for each $m \in N_0$, in the sense of Definition C.10 in [8], where $F^*$ is the desired smooth diffeomorphism between $\Sigma$ and some conformally transformed Clifford torus, namely $M\left(\frac{1}{\sqrt{2}}(S^1 \times S^1)\right)$. □

Acknowledgements

The author would like to thank Professor Itai Shafrir and Professor Yehuda Pinchover and especially the kind referee on account of their helpful suggestions. The author was funded by the Ministry of Absorption of the State of Israel in the years 2019–2022.

Data availability Data sharing is not applicable to this article, because datasets were neither generated nor analyzed during the current study.

Declarations

Conflict of interest The author declares that there are neither financial nor non-financial interests related to this article.

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5 Appendix

In this appendix, we firstly prove existence of local, horizontal and smooth lifts of some arbitrary smooth, closed path \( \gamma : S^1 \to S^2 \) w.r.t. fibrations of the type \( \pi \circ F \) for “simple” parametrizations \( F : \Sigma \to \pi^{-1}(\text{trace}(\gamma)) \subset S^3 \), in the sense of Definition 3. Since \( \gamma \) should be allowed to have self-intersections in view of our Theorem 1, we cannot blindly apply the general theory of smooth fiber bundles over smooth base manifolds. Instead, we have to construct here such local lifts elementarily, using Lemmata 1 and 2 and the theory of “ODEs.”

**Lemma 3.** Let \( \gamma : S^1 \to S^2 \) be a smooth, closed and regular path in \( S^2 \), and let \( F : \Sigma \to S^3 \) be a smooth immersion, which maps a smooth compact torus \( \Sigma \) simply and smoothly onto the Hopf-torus \( \pi^{-1}(\text{trace}(\gamma)) \subset S^3 \).

1. For every fixed \( s^* \in S^1 \) and \( q^* \in \pi^{-1}(\gamma(s^*)) \subset S^3 \), there is a unique horizontal, smooth lift \( \eta(s^*,q^*) : \operatorname{dom}(\eta(s^*,q^*)) \to \pi^{-1}(\text{trace}(\gamma)) \), defined on a non-empty, open interval \( \operatorname{dom}(\eta(s^*,q^*)) \subset S^1 \), of \( \gamma : S^1 \to S^2 \) w.r.t. the Hopf-fibration \( \pi \), such that \( \operatorname{dom}(\eta(s^*,q^*)) \) contains the point \( s^* \) and such that \( \eta(s^*,q^*) \) attains the value \( q^* \) in \( s^* \); i.e., \( \eta(s^*,q^*) \) is a smooth path in the torus \( \pi^{-1}(\text{trace}(\gamma)) \), which intersects the fibers of \( \pi \) perpendicularly and satisfies:

\[
(\pi \circ \eta(s^*,q^*))(s) = \gamma(s) \quad \forall s \in \operatorname{dom}(\eta(s^*,q^*)) \quad \text{and} \quad \eta(s^*,q^*)(s^*) = q^*, \tag{86}
\]

and there is only one such function \( \eta(s^*,q^*) \) mapping the open interval \( \operatorname{dom}(\eta(s^*,q^*)) \subset S^1 \) into \( \pi^{-1}(\text{trace}(\gamma)) \).

2. There is some \( \epsilon = \epsilon(F, \gamma) > 0 \), such that for every fixed \( s^* \in S^1 \) and every \( x^* \in (\pi \circ F)^{-1}(\gamma(s^*)) \subset S^2 \), there is a horizontal smooth lift \( \eta_F(s^*,x^*) \) of \( \gamma\big|_{S^1 \cap B_\epsilon(s^*)} \) w.r.t. the fibration \( \pi \circ F : \Sigma \to \text{trace}(\gamma) \subset S^2 \), attaining the value \( x^* \) in \( s^* \), i.e., \( \eta_F(s^*,x^*) \) is a smooth path in the torus \( \Sigma \) which intersects the fibers of \( \pi \circ F \) perpendicularly and satisfies:

\[
(\pi \circ F \circ \eta_F(s^*,x^*))(s) = \gamma(s) \quad \forall s \in S^1 \cap B_\epsilon(s^*) \quad \text{and} \quad \eta_F(s^*,x^*)(s^*) = x^*. \tag{87}
\]

In particular, for the above \( \epsilon = \epsilon(F, \gamma) > 0 \) the function \( \eta_F : S^1 \to S^2 \) maps the set \( \mathcal{L}(\gamma\big|_{S^1 \cap B_\epsilon(s^*)}, \pi \circ F) \) of horizontal smooth lifts of \( \gamma\big|_{S^1 \cap B_\epsilon(s^*)} \) w.r.t. \( \pi \circ F \) surjectively onto the set \( \mathcal{L}(\gamma\big|_{S^1 \cap B_\epsilon(s^*)}, \pi) \) of horizontal smooth lifts of \( \gamma\big|_{S^1 \cap B_\epsilon(s^*)} \) w.r.t. \( \pi \).

**Proof.** Without loss of generality, we may assume here that \( \gamma \) performs only one loop through its closed trace, and that it is defined on \( \mathbb{R}/L\mathbb{Z} \) with \( |\gamma|_{S^2} = 2 \), where \( 2L \) is the length of the trace of \( \gamma \). We consider the unique smooth vector field \( V_\gamma \in \Gamma(T(\pi^{-1}(\text{trace}(\gamma)))) \subset \Gamma(TS^3) \), which intersects the fibers of the Hopf-fibration perpendicularly, satisfies \( |V_\gamma|_{S^3} \equiv 1 \) throughout on \( \pi^{-1}(\text{trace}(\gamma)) \) and

\[
D \pi q^*(V_\gamma(q^*)) = c^* \gamma'(0), \quad \text{for some} \quad q^* \in \pi^{-1}(\gamma(0)) \quad \text{and some} \quad c^* > 0. \tag{88}
\]
Now, the unique flow $\Psi$ on $\pi^{-1}(\text{trace}(\gamma))$ generated by the initial value problem
\[
\frac{dv}{d\tau}(\tau) = V_\gamma(v(\tau)) \quad \text{and} \quad v(0) = q,
\] (89)
for any fixed point $q \in \pi^{-1}(\text{trace}(\gamma))$, exists eternally, because the Hopf-torus $\pi^{-1}(\text{trace}(\gamma))$ is a compact subset of $S^3$, which can locally be parametrized by smooth charts, similarly to a smooth compact 2-manifold without any boundary points. We can readily infer from the requirements on $V_\gamma$, that the flow lines of the resulting eternal flow $\Psi : \mathbb{R} \times \pi^{-1}(\text{trace}(\gamma)) \longrightarrow \pi^{-1}(\text{trace}(\gamma))$ intersect the fibers of $\pi$ perpendicularly, have constant speed 1 and are mapped by $\pi$ onto $\text{trace}(\gamma)$. Now we choose some arbitrary initial point $q_0 \in \pi^{-1}(\text{trace}(\gamma))$, and we derive from the properties
\[
|\Psi_\tau(q_0)|_{g_3} \equiv 1 \quad \text{and} \quad |\frac{d}{d\tau}\Psi_\tau(q_0)|_{g_3} = |V_\gamma(\Psi_\tau(q_0))|_{g_3} \equiv 1
\]
as in Lemma 2 that there is some unit vector field $\tau \mapsto u(\tau) = u_{\gamma,q_0}(\tau) \in \text{Span}\{j, k\} \subset H$, satisfying $V_\gamma(\Psi_\tau(q_0)) = u(\tau) \cdot \Psi_\tau(q_0)$ in $H$, for every $\tau \in \mathbb{R}$, and we can consequently compute exactly as in formula (12):
\[
\frac{d}{d\tau}(\tau \circ \Psi_\tau(q_0)) = (I(V_\gamma(\Psi_\tau(q_0)))) \cdot \Psi_\tau(q_0)) + (I(\Psi_\tau(q_0)) \cdot V_\gamma(\Psi_\tau(q_0)))
\]
\[
= 2 I(\Psi_\tau(q_0)) \cdot u(\tau) \cdot \Psi_\tau(q_0) \equiv 2 I(\Psi_\tau(q_0)) \cdot V_\gamma(\Psi_\tau(q_0)),
\] (90)
for every $\tau \in \mathbb{R}$. Now, since $\pi$ maps the Hopf-torus $\pi^{-1}(\text{trace}(\gamma))$ onto the trace of $\gamma$, we know a-priori that the trace of the path $[\tau \mapsto \tau \circ \Psi_\tau(q_0)]$ is contained in the trace of the curve $\gamma$. Moreover, Eq. (90) shows us:
\[
|\frac{d}{d\tau}(\tau \circ \Psi_\tau(q_0))| \equiv 2 \equiv |\gamma'(\tau)|, \quad \forall \tau \in \mathbb{R},
\] (91)
and for every initial point $q_0 \in \pi^{-1}(\text{trace}(\gamma))$. Hence, taking especially $q_0 \in \pi^{-1}(\gamma(0))$ we have $\tau \circ \Psi_0(q_0) = \gamma(0)$, and therefore assumption (88) and Eq. (91) prove that there has to hold:
\[
\tau \circ \Psi_\tau(q_0) = \gamma(\tau), \quad \forall \tau \in \mathbb{R},
\] (92)
and for every $q_0 \in \pi^{-1}(\gamma(0))$. Moreover, combining identity (92) with the group property $\Psi_{\tau_1+\tau_2} = \Psi_{\tau_1} \circ \Psi_{\tau_2}$” of the flow $\Psi$, it follows that for any initial point $q_0 \in \pi^{-1}(\text{trace}(\gamma))$—not only for $q_0 \in \pi^{-1}(\gamma(0))$—the path $[\tau \mapsto \tau \circ \Psi_\tau(q_0)]$ parametrizes the trace of $\gamma$ exactly once, as $\tau$ increases from 0 to $L$, which means that any such flow line $\{\Psi_\tau(q_0)\}_{\tau \in [0,L]}$ intersects each fiber of $\pi$ over the trace of $\gamma$ exactly once. Now, for any $s^* \in (0, L)$ and $q^* \in \pi^{-1}(\gamma(s^*))$ we consider the unique flow line of $\Psi$, which starts moving in the point $\Psi_{s^*}^{-1}(q^*) = \Psi_{-s^*}(q^*)$ at time $\tau = 0$. Applying now statement (92) to the initial point $q_0 := \Psi_{s^*}^{-1}(q^*) \in \pi^{-1}(\gamma(0))$ we can easily infer, that the function
\[
\eta^{s^*, q^*}(\tau) := \Psi(\tau, \Psi_{s^*}^{-1}(q^*)) \quad \text{for} \quad \tau \in (0, L)
\]
satisfies the first property in (86), where \( \text{dom}(\eta(s^*, q^*)) \subset S^1 \) can be any open and connected subset \( \neq S^1 \), which contains the prescribed point \( s^* \) —here being identified with some open subinterval of \( (\mathbb{R}/L\mathbb{Z}) \setminus \{0\} \)—and we also verify that

\[
\eta(s^*, q^*)(s^*) = \Psi(s^*, \Psi_{s^*}^{-1}(q^*)) = \Psi_{s^*}^{-1}(q^*) = q^* ,
\]

which is just the second property in (86). Uniqueness of such a local, horizontal smooth lift of \( \gamma \) follows easily from the above construction.

2) First of all, since \( F : \Sigma \longrightarrow S^3 \) is required to be an immersion whose image is \( \pi^{-1}(\text{trace}(\gamma)) \) and since \( \Sigma \) is compact, there is some \( \delta = \delta(F) > 0 \) such that for an arbitrarily fixed \( q \in \pi^{-1}(\text{trace}(\gamma)) \) the preimage \( F^{-1}(B_\delta(q)) \) consists of finitely many disjoint open subsets of \( \Sigma \), which are mapped diffeomorphically onto their images in \( B_\delta(q) \cap \pi^{-1}(\text{trace}(\gamma)) \) via \( F \). Moreover, from the first part of the lemma we infer the existence of horizontal smooth lifts \( \eta : \text{dom}(\eta) \rightarrow \pi^{-1}(\text{trace}(\gamma)) \) of \( \gamma \vert \text{dom}(\eta) \) w.r.t. \( \pi \). Now, we infer from Lemma 2 and from the compactness of \( S^1 \) that there is some \( C = C(\gamma) > 0 \) such that

\[
|\eta'(s)|^2(s) = \frac{|\gamma'|^2(s)}{4} \leq C , \quad \forall s \in \text{dom}(\eta) ,
\]

holds for every horizontal lift \( \eta \) of \( \gamma \vert \text{dom}(\eta) \) w.r.t. \( \pi \). Hence, there is some \( \epsilon = \epsilon(F, \gamma) > 0 \), such that for every \( \tilde{s} \in S^1 \) there is some \( \tilde{q} \in S^3 \) such that \( \text{trace}(\eta\vert_{S^1\cap B_\epsilon(\tilde{s})}) \subset B_\delta(\tilde{q}) \). Now, using the fact that \( F \) maps each connected component of \( F^{-1}(B_\delta(\tilde{q})) \) diffeomorphically onto its image in \( B_\delta(\tilde{q}) \cap \pi^{-1}(\text{trace}(\gamma)) \), we obtain the existence of at least one smooth map \( \eta_F : S^1 \cap B_\epsilon(\tilde{s}) \longrightarrow F^{-1}(B_\delta(\tilde{q})) \) satisfying \( F \circ \eta_F = \eta \) on \( S^1 \cap B_\epsilon(\tilde{s}) \), and thus also \( \pi \circ F \circ \eta_F = \gamma \) on \( S^1 \cap B_\epsilon(\tilde{s}) \). Again in combination with the first part of the lemma, we finally infer from this construction that for every \( s^* \in S^1 \) and every \( x^* \in (\pi \circ F)^{-1}(\gamma(s^*)) \) there is a smooth map \( \eta_F(s^*, x^*) : S^1 \cap B_\epsilon(s^*) \rightarrow \Sigma \), which possesses the two desired properties in (87). Moreover, it immediately follows from the horizontal property of every constructed lift \( \eta \) w.r.t. \( \pi \) in the first part of this lemma that every constructed lift \( \eta_F \) of \( \gamma \) w.r.t. \( \pi \circ F \) intersects the fibers of \( \pi \circ F \) perpendicularly w.r.t. the induced pullback metric \( F^*(g_{\text{euc}}) \). Hence, the last assertion of the lemma now turns out to be evident. \( \square \)

**Remark 6.** It is important to understand that horizontal lifts w.r.t. \( \pi \) of closed curves in \( S^2 \) would in general not close up in \( S^3 \). As explained in [25, p. 381], a horizontal lift \( \eta \) w.r.t. \( \pi \) of a simple, closed path \( \gamma : S^1 \longrightarrow S^2 \), which performs \( k \geq 2 \) loops and encloses the area \( A \) on \( S^2 \), closes up, if and only if there holds the relation \( A = \frac{4\pi}{k} \). Consider, for example, the standard parametrization \( p(\varphi) := [\cos(2\varphi) + j \sin(2\varphi)] \) of a great circle in \( S^2 \). Its preimage w.r.t. \( \pi \) is the Clifford torus, and one can easily infer at first from Remark 3, that the Clifford torus is conformally equivalent to the special parallelogram \( D \) in \( \mathbb{C} \) with vertices \((0,0),(2\pi,0), (\pi, \pi) \) and \((3\pi, \pi) \), and then secondly check that any horizontal lift \( \eta \) of \( p \) corresponds to a certain diagonal in \( D \), which indeed closes up exactly when the parameter \( \varphi \) reaches the value \( 2\pi \).
Finally, we are going to employ the complete integrability of the Euler–Lagrange equation of the elastic energy $E$—see the right hand side of formula (21)—and the theory of elliptic integrals and Jacobi elliptic functions, in order to precisely compute the critical values of the elastic energy $E$, particularly in order to exclude critical values of $E$ in the surprisingly large interval $(2\pi, \frac{8\pi}{\sqrt{2}})$.

**Proposition 6.** (1) Up to isometries of $S^2$, there are only countably many different smooth closed curves $\gamma : S^1 \rightarrow S^2$, parametrized with constant speed, which are critical points of the elastic energy $E$, called “closed elastic curves on $S^2.”$ Vice versa, for each pair of positive integers $(m, n)$ with $\gcd(m, n) = 1$ and $\frac{m}{n} \in (0, 2 - \sqrt{2})$ there is—up to isometric equivalence—a unique arc-length parametrized elastic curve $\gamma_{(m,n)}$ in $S^2$, which closes up after $n$ periods and traverses some fixed great circle exactly $m$ times.

(2) There are no critical values of the elastic energy $E$ in the interval $(2\pi, \frac{8\pi}{\sqrt{2}})$.

Proof of the first part of Proposition 6: As explained in [17], every smooth closed stationary curve $\gamma : [a, b]/(a \sim b) \rightarrow S^2$ of the elastic energy $E$ satisfies the differential equation

$$2 \left( \nabla_{\gamma'} \right)^2 (\kappa_{\gamma'}) + |\kappa_{\gamma'}|^2 \kappa_{\gamma} + \tilde{\kappa}_{\gamma} \equiv \nabla_{L^2} E(\gamma) \equiv 0, \quad \text{on } [a, b]. \tag{93}$$

Now, Langer and Singer have pointed out in [17], that equation (93) holds for a non-geodesic closed curve $\gamma$, if and only if the signed curvature $\kappa_{\gamma}$ of its parametrization with speed $|\gamma'| \equiv 1$ satisfies the ordinary differential equation

$$\left( \frac{d\kappa}{ds} \right)^2 = -\frac{1}{4} \kappa^4 - \frac{1}{2} \kappa^2 + A, \quad \text{on } \mathbb{R}, \tag{94}$$

for some integration constant $A \in \mathbb{R}$, depending on the respective solution $\gamma$ of (93). Moreover, for such non-geodesic solutions $\gamma$ of (93), i.e., having non-constant curvature $\kappa_{\gamma} \neq 0$, Eq. (94) is equivalent to the KdV-type-equation

$$\left( \frac{du}{ds} \right)^2 = -u^3 - 2u^2 + 4Au, \quad \text{on } \mathbb{R}, \tag{95}$$

to be satisfied by the square $\kappa_{\gamma}^2$ of the curvature of the arc-length parametrized solution $\gamma$, which is simply obtained by pointwise multiplication of Eq. (94) with $4\kappa_{\gamma}^2$. One can easily verify that for every non-geodesic closed solution $\gamma$ of (93), the polynomial $P(x) := x^3 + 2x^2 - 4Ax$, occurring on the right hand side of Eq. (95), must have three different real roots $-\alpha_1 < 0 = \alpha_2 < \alpha_3$ satisfying the algebraic relations

$$\alpha_1 - \alpha_3 = 2 \quad \text{and} \quad \alpha_1 \alpha_3 = 4A. \tag{96}$$
Now, as explained in Section 2 of [17] the solutions \( u \) of Eq. (95) are exactly given by the Jacobi elliptic functions\(^2\) of the particular type\(^3\):

\[
\alpha(s) = \alpha_3 \operatorname{cn}^2(r \cdot s; p), \quad \forall s \in \mathbb{R},
\]

(97)

with

\[
\begin{align*}
    r &:= \frac{1}{2} \sqrt{\alpha_1 + \alpha_3} = \frac{1}{\sqrt{2}} \sqrt{\alpha_3 + 1} \quad \text{and} \quad p := \sqrt{\frac{\alpha_3}{\alpha_3 + \alpha_1}} = \frac{1}{\sqrt{2}} \sqrt{\alpha_3 + 1}.
\end{align*}
\]

(98)

Hence, the modulus \( p \) occurring in formula (97) has to be contained in the open interval \((0, \sqrt{2})\), and combining formulae (96) and (98) one can express the roots \( \alpha_3 \) and \( \alpha_1 \) of the polynomial \( P(x) := x^3 + 2 x^2 - 4 A x \), occurring in equation (95), in terms of \( p \):

\[
\alpha_3 = \frac{2 p^2}{1 - 2 p^2} \quad \text{and} \quad \alpha_1 = \frac{2 p^2}{1 - 2 p^2} + 2 = \frac{2 - 2 p^2}{1 - 2 p^2}.
\]

Moreover, combining these formulae again with formula (96) the modulus \( p \) in formula (97) automatically yields the integration constant \( A \) appearing in equations (94) and (95), and also the frequency \( r \) in (97):

\[
A = \frac{1}{4} \alpha_1 \alpha_3 = \frac{p^2 - p^4}{(1 - 2 p^2)^2} > 0, \quad r = \frac{1}{\sqrt{2 - 4 p^2}} \in \left(\frac{1}{\sqrt{2}}, \infty\right).
\]

(99)

Formula (97) particularly implies that every arc-length parametrized solution \( \gamma \) of equation (93) performs a periodic path on \( S^2 \) with

\[
\text{one period of } \gamma = 4 \frac{K(p)}{r} = 4 \sqrt{2 - 4 p^2} K(p)
\]

(100)

on account of formula (2.2.5) in [19] and formula (99), where

\[
K(p) := \int_0^{\pi/2} \frac{1}{\sqrt{1 - p^2 \sin^2(\varphi)}} \, d\varphi
\]

(101)

denotes the complete elliptic integral of the first kind with parameter \( p \in [0, \frac{1}{\sqrt{2}}] \); see [19], Sections 3.1 and 3.8, and [3], pp. 8–17. Another important consequence of equation (97) is that every arc-length parametrized solution \( \gamma \) of (93) possesses a well-defined wavelength \( \Lambda(\gamma) \in \mathbb{R} \) whose quotient \( \frac{\Lambda}{2 \pi} \) has to be rational, because \( \gamma \) is supposed to be a closed curve on \([0, L]\), with \( L := \text{length}(\gamma) \).\(^4\) Hence, for every non-geodesic, closed and arc-length parametrized solution \( \gamma \) of equation (93) there

---

\(^2\)See, e.g., Chapter 2 in [19] or pp. 18–32 in [3] for an introduction to this subject.

\(^3\)See here the precise computation of the “annual” shift of the perihelion of a relativistic planetary orbit in Section 5.5 of [19] or also Appendix A.1 of [23], in order to obtain a clean derivation of formulae (97) and (98).

\(^4\)See here also Sections 2, 3 and 5 in [17] and Section 4 in [18] for further motivation, to attach an essentially unique “wavelength” \( \Lambda \) to any closed solution \( \gamma \) of equation (93).
is a unique pair \((m, n) \in \mathbb{N} \times \mathbb{N}\) of positive integers with \(\gcd(m, n) = 1\), such that \(\Lambda = \frac{m}{n} 2\pi\), which means geometrically that this particular path \(\gamma\) closes up after \(n\) periods, respectively, “lobes”—whose common lengths are given by formula \((100)\)—and traverses some fixed great circle in \(S^2\) \(m\) times, while its arc-length parameter \(s\) runs from 0 to \(L\).\(^5\) Hence, up to isometric equivalence we are able to count non-geodesic, closed and arc-length parametrized elasticae on \(S^2\) systematically, which has already proved the first assertion of the first part of Proposition 6.

Now, in order to prove the entire classification of closed elastic curves on \(S^2\), as asserted in the first part of Proposition 6, we have to understand vice versa, for which quotients \(\frac{m}{n}\) of coprime, positive integers there are actually closed elastic curves “\(\gamma(m,n)\),” possessing the aforementioned two geometric properties, according to the respective pairs \((m, n)\). To this end, we employ the second quantitative ingredient of the proof of Proposition 6, namely the following formula \((102)\), taken directly from Section 4 of [18], p. 148, which expresses the wavelength \(\Lambda\) of a non-geodesic, closed, arc-length parametrized solution \(\gamma\) of equation \((93)\) as a function of the above parameter \(p \in (0, \frac{1}{\sqrt{2}})\):

\[
\Lambda(\gamma(p)) = 2\pi \varepsilon \Lambda_0(\psi(p), p) - 2(3 - 4p^2) \frac{1}{\sqrt{1 - (1 - p^2) \sin^2(\psi(p)) \sin(\psi(p))}} K(p) \quad (102)
\]

where \(\psi(p) := \arcsin\left(\sqrt{\frac{8}{3}} \frac{1-2p^2}{1-4p^2}\right)\). \(\varepsilon := \frac{4p^2-1}{|4p^2-1|}\). \(\Lambda_0\) denotes the Heuman-Lambda function—see [3], pp. 35–37 and pp. 344–349—and \(K(p)\) had been introduced in \((101)\). By means of the computations in Section 3.8 in [19], one can verify that the first derivative w.r.t. \(p\) of the function in \((102)\) reads \(^6\):

\[
\frac{d\Lambda(\gamma(p))}{dp} = \frac{2\sqrt{8} \left(1 - p^2\right) K(p) - E(p)\right)}{p \left(1 - p^2\right) \sqrt{1 - 2p^2(3 - 4p^2) \sqrt{1 - p^2 \sin^2(\psi(p))}}} \quad (103)
\]

for \(p \in (0, \frac{1}{\sqrt{2}})\), where \(E(p) := \int_0^{\pi/2} \sqrt{1 - p^2 \sin^2(\varphi)} \, d\varphi\) denotes the complete elliptic integral of the second kind. Since we have \((1 - p^2) K(p) - E(p) < 0\) for \(p \in \left(0, \frac{1}{\sqrt{2}}\right)\), and furthermore \(\sin(\psi(0)) = \frac{\sqrt{8}}{3}, \Lambda_0(\arcsin\left(\frac{\sqrt{8}}{3}\right), 0) = \frac{\sqrt{8}}{3}, \sin(\psi\left(\frac{1}{2}\right)) = 1\) and \(\Lambda_0\left(\frac{\pi}{2}, \frac{1}{2}\right) = 1\), formula \((103)\) shows us that the function \(p \mapsto \Lambda(\gamma(p))\) in \((102)\) decreases strictly monotonically from its initial value \(\Lambda(0) = -2\pi \cdot \frac{\sqrt{8}}{3} - 2\frac{\sqrt{2}}{3} \pi = -\sqrt{2}(2\pi)\) to its minimal value \(\min_{p \in [0, \frac{1}{\sqrt{2}}]} \Lambda(p) = \lim_{p \nearrow \frac{1}{2}} \Lambda(p) = -2\pi - 2K\left(\frac{1}{2}\right) \approx -9.65469\) as \(p\) increases from 0 to \(\frac{1}{2}\), then jumps at the point \(p = \frac{1}{2}\) from its minimal value \(\min_{p \nearrow \frac{1}{2}} \Lambda(p)\) to its maximal value \(\max_{p \in [0, \frac{1}{\sqrt{2}}]} \Lambda(p) = \lim_{p \searrow \frac{1}{2}} \Lambda(p) = 2\pi - 2K\left(\frac{1}{2}\right) \approx 2.91169\), and finally decreases strictly monotonically to its minimal value \(\min_{p \leq \frac{1}{2}} \Lambda(p)\).

\(^5\) Compare here also with Section 5.4 in [21] and with Section 3 in [23] for closed elasticae in the hyperbolic plane.

\(^6\) Compare here also with Proposition 4.1 in [18] for further information.

\(^7\) Note that the function \([p \mapsto \varepsilon(p)]\) jumps at \(p = \frac{1}{2}\) from \(-1\) to \(1\) and that \(\Lambda_0\left(\frac{\pi}{2}, \frac{1}{2}\right) = 1\).
from this maximum to its final value $\Lambda\left(\frac{1}{\sqrt{2}}\right) = 2 \pi \Lambda_0\left(0, \frac{1}{\sqrt{2}}\right) = 0$. Since we are allowed to shift the value of the wavelength $\Lambda(p)$ in the first interval $[-2\pi - 2K\left(\frac{1}{2}\right), -\sqrt{2}(2\pi)]$ about the height of the jump of $\Lambda(p)$, i.e., about $4\pi$, to the right in $\mathbb{R}$ into the interval $[2\pi - 2K\left(\frac{1}{2}\right), 2\pi (2 - \sqrt{2})]$, we conclude that for each coprime pair $(m,n) \in \mathbb{N} \times \mathbb{N}$ satisfying

$$\frac{m}{n} \in \left(0, 1 - \frac{K\left(\frac{1}{2}\right)}{\pi}\right) \cup \left[1 - \frac{K\left(\frac{1}{2}\right)}{\pi}, 2 - \sqrt{2}\right) = (0, 2 - \sqrt{2}) \quad (104)$$

there is a “unique” arc-length parametrized elastic curve $\gamma(m,n)$, which closes up after $n$ periods and traverses some fixed great circle $C$ in $S^2$ exactly $m$ times - “unique” only up to the action of all those isometries of $S^2$, which leave the great circle $C$ invariant, just as claimed in the first part of the proposition. □

Proof of the second part of Proposition 6: Relying on the proof of the first part of Proposition 6, we attempt to prove its second part in a “straightforward manner,” which means that we fix a pair of coprime integers $(m,n) \in \mathbb{N} \times \mathbb{N}$ satisfying condition (104) and that we simply combine formulae (97) and (100) with the definition of the elastic energy in (15), in order to compute both length and elastic energy of the unique solution $\gamma(m,n)$ of equation (93) directly. This is actually possible, because the given data “$(m,n)$” determine the value of the wavelength $\Lambda = \frac{m}{n} 2\pi$ and thus also a unique value of the parameter $p = p(\gamma(m,n))$ in (98), inverting the strictly monotonic wavelength function $p \mapsto \Lambda(\gamma(p))$ in (102). Hence, also recalling that we only work with arc-length parametrized elastic curves $\gamma(m,n)$, we can firstly compute by means of formula (100), abbreviating here $p = p(\gamma(m,n))$ $^8$:

$$\text{length}(\gamma(m,n)) = n \text{ periods of } \gamma(m,n) = 4n \frac{K(p)}{r} = 4n \sqrt{2 - 4p^2} K(p), \quad (105)$$

and together with formulae (15) and (97) and with formulae (3.15) and (3.27) in [19], we obtain furthermore:

$$\mathcal{E}(\gamma(m,n)) = \int_0^{4n} \frac{K(p)}{r} \left[1 + \kappa_{\gamma(m,n)}^2(s)\right] ds = \int_0^{4n} \frac{K(p)}{r} \left[1 + \alpha_3 \text{cn}^2(r \cdot s; p)\right] ds \quad (106)$$

$$= 4n \sqrt{2 - 4p^2} K(p) + \frac{2p^2}{1 - 2p^2} \int_0^{4n} \frac{K(p)}{r} \text{cn}^2(u; p) du$$

$$= 4n \sqrt{2 - 4p^2} K(p) + \frac{2p^2}{1 - 2p^2} 4n \sqrt{2 - 4p^2} \frac{1}{p^2} \left[E(p) - (1 - p^2)K(p)\right]$$

$$= 4n \sqrt{2 - 4p^2} K(p) \left(1 + \frac{2p^2}{1 - 2p^2}\right) + \frac{16n}{\sqrt{2 - 4p^2}} (E(p) - K(p))$$

$$= \frac{8n}{\sqrt{2 - 4p^2}} (2E(p) - K(p)), \quad (106)$$

$^8$Compare here also with Proposition 9 in [21] and with Propositions 3.3 and 3.4 in [23].
a formula which has essentially already appeared in the proof of Corollary 6.4 in [23] for closed wavelike elasticae in the hyperbolic plane, being similar to the slightly simpler formula (46) in [21] for the elastic energy of closed orbitlike elasticae in the hyperbolic plane. Now, formula (106) is not only useful for numerical purposes, i.e., in order to compute elastic energies $\mathcal{E}(\gamma_{(m,n)})$ effectively, but it also meets the aim of the second part of Proposition 6, to rigorously determine a “rather accurate” lower bound for all possible elastic energies $\mathcal{E}(\gamma_{(m,n)})$ of non-geodesic elastic curves. The key observation in this situation is that the function $f(p) := \frac{1}{\sqrt{1 - 2p^2}} \left( 2E(p) - K(p) \right)$—appearing on the right hand side of formula (106)—is actually strictly monotonically increasing on the entire open interval $\left( 0, \frac{1}{\sqrt{2}} \right)$. In order to prove this, we firstly compute by means of formulae (3.8.7) and (3.8.12) in [19]:

$$\frac{d}{dp} \left( 2E(p) - K(p) \right) = \frac{2}{p} \left( \frac{E(p) - K(p)}{p} - \frac{E(p) - (1 - p^2) K(p)}{p (1 - p^2)} \right)$$

$$= \frac{1 - 2p^2}{p (1 - p^2)} E(p) - \frac{1}{p} K(p) < \frac{1}{p} (E(p) - K(p)) < 0,$$

(107)

for $p \in \left( 0, \frac{1}{\sqrt{2}} \right)$, showing first of all that the function $p \mapsto 2E(p) - K(p)$ decreases monotonically from $\frac{2}{\sqrt{2}}$ in $p = 0$ to $2E\left( \frac{1}{\sqrt{2}} \right) - K\left( \frac{1}{\sqrt{2}} \right) > 0$ in $p = \frac{1}{\sqrt{2}}$, and we can continue by deriving the function $f$ w.r.t. $p$:

$$\frac{df}{dp} = \frac{1}{\sqrt{1 - 2p^2}} \left( \frac{1 - 2p^2}{p (1 - p^2)} E(p) - \frac{1}{p} K(p) \right) + \frac{2p}{(1 - 2p^2)^{3/2}} (2E(p) - K(p))$$

$$= \frac{\sqrt{1 - 2p^2}}{p (1 - p^2)} E(p) - \frac{1}{p \sqrt{1 - 2p^2}} K(p) + \frac{4p}{(1 - 2p^2)^{3/2}} E(p) - \frac{2p}{(1 - 2p^2)^{3/2}} K(p)$$

$$= \frac{1}{p (1 - p^2) (1 - 2p^2)^{3/2}} E(p) - \frac{1}{p (1 - p^2) (1 - 2p^2)^{3/2}} K(p)$$

Moreover, we see that the function $g(p) := \frac{1}{1 - p^2} E(p) - K(p)$ satisfies $g(0) = 0$, and that by formulae (3.8.7) and (3.8.12) in [19] its derivative is:

$$\frac{dg}{dp} = \frac{E(p) - K(p)}{(1 - p^2) p} + \frac{2p}{(1 - p^2)^2} E(p) - \frac{E(p) - (1 - p^2) K(p)}{p (1 - p^2)}$$

$$= \frac{2p}{(1 - p^2)^2} E(p) - \frac{p}{1 - p^2} K(p) > \frac{2p}{1 - p^2} E(p) - \frac{p}{1 - p^2} K(p)$$

for every $p \in \left( 0, \frac{1}{\sqrt{2}} \right)$, using that here $(1 - p^2) \in \left( \frac{1}{2}, 1 \right)$ and $E(p) > 0$, and furthermore that by (107) $\min_{p \in \left[ 0, \frac{1}{\sqrt{2}} \right]} \left( 2E(p) - K(p) \right) = 2E\left( \frac{1}{\sqrt{2}} \right) - K\left( \frac{1}{\sqrt{2}} \right) > 0$. Hence,

\[\text{Compare here also with Lemma 4.1 in [23] and with p. 19 in [17].}\]
we can infer that \( g(p) > 0 \) for \( p \in \left(0, \frac{1}{\sqrt{2}}\right) \) and that therefore \( f \) increases strictly monotonically from \( f(0) = \frac{\pi}{2} \) to \( \infty \), as \( p \) runs from 0 to \( \frac{1}{\sqrt{2}} \). Hence, without even guessing for which pair \((m, n)\) of coprime integers—respectively, for which value of the modulus \( p = p(\gamma(m, n)) \in \left(0, \frac{1}{\sqrt{2}}\right) \)—the elastic energy \( \mathcal{E}(\gamma(m, n)) \) in (106) might attain its minimal value among all non-geodesic elasticae on \( S^2 \), we can roughly, but rigorously estimate by means of formula (106) and \( \inf_{p \in \left(0, \frac{1}{\sqrt{2}}\right)} f(p) = f(0) = \frac{\pi}{2} \):

\[
\mathcal{E}(\gamma(m, n)) > \frac{16}{\sqrt{2}} f(0) = \frac{8\pi}{\sqrt{2}} \approx 17,71532
\]

for coprime pairs \((m, n) \in \mathbb{N} \times \mathbb{N}\) satisfying condition (104), where we have also used the fact that we must have “\( n \geq 2 \)” on account of condition (104) that \( p(\gamma(m, n)) \in \left(0, \frac{1}{\sqrt{2}}\right) \) and \( f \) increases strictly monotonically from \( f(0) \) to \( f(p(\gamma(m, n))) \), for any fixed coprime pair \((m, n)\) satisfying condition (104).

**Remark 7.** In order to assess the quality of both formula (106) and estimate (108), we mention here a totally different method yielding a precise formula for the elastic energy of any non-geodesic elastic curve \( \gamma(m, n) \) on \( S^2 \) and also a rigorous lower bound for all those elastic energies, which is \( 16 \sqrt{\frac{\pi}{3}} \).\(^{10}\) The starting point of this second method is the perfect match between equation (2.2) in [10] with equation (94), simply with \( v = -A, G = 1 \) and \( \mu = -\frac{1}{2} \) in equation (2.2) of [10]. Therefore, the formulae in Lemma 1 and Remark 2 of [10] yield here the coefficients \( a_2 = \frac{1}{48} - \frac{A}{4} \) and \( a_3 = \frac{1}{1728} + \frac{A}{48} \) of a concrete polynomial equation \( y^2 = 4x^3 - a_2x - a_3 \), whose set of solutions \([x : y : 1] \in \mathbb{C}P^2\) yields a particular elliptic curve \( E(m, n) \subset \mathbb{C}P^2 \), which turns out to carry all the relevant information about the elastic curve \( \gamma(m, n) \), for any fixed pair \((m, n)\) of positive, coprime integers with \( \frac{m}{n} \in (0, 2 - \sqrt{2}) \). One can easily infer from the above concrete formulae for the coefficients \( a_2, a_3 \) that the discriminant \( D(F) := a_2^3 - 27a_3^2 \) of the polynomial \( F(x) = 4x^3 - a_2x - a_3 \) vanishes, if and only if the integration constant \( A \) satisfies \( A = 0 \) or \( A = -\frac{1}{4} \). However, formula (99) rules out these two possibilities. Hence, for any \( p = p(\gamma(m, n)) \in \left(0, \frac{1}{\sqrt{2}}\right) \) the corresponding polynomial \( F(x) = 4x^3 - a_2x - a_3 \) has a non-vanishing—here actually a negative—discriminant \( D(F) \). Hence, by the uniformization theorem—see, e.g., Theorem 2.9 in [1]—there exists a unique lattice \( \Omega \equiv \Omega_{(m, n)} := \omega_1 \mathbb{Z} \oplus \omega_2 \mathbb{Z} \in \mathbb{C}, \) with \( \Im \left(\frac{\omega_2}{\omega_1}\right) > 0, \) such that the corresponding Weierstrass-\( \wp \)-function \( \wp(z, \Omega) := \frac{1}{z^2} + \sum_{\omega \in \Omega \setminus \{0\}} \frac{1}{(\omega - z)^2} - \frac{1}{\omega^2} \) solves the complex ordinary differential equation

\[(\wp'(z))^2 = 4(\wp(z))^3 - a_2 \wp(z) - a_3 \equiv F(\wp(z)), \quad \forall z \in \mathbb{C}/\Omega, \quad (109)\]

and therefore “parametrizes” the elliptic curve \( E(m, n) \).\(^{11}\) Now, Eq. (109) and Lemmata 1 and 2 in [10] reveal the surprising relation between the elliptic curve \( E(m, n) \) and our

\(^{10}\)Obviously, the number \( 16 \sqrt{\frac{\pi}{3}} \) is smaller than our threshold \( \frac{8\pi}{\sqrt{2}} \) from formula (108).

\(^{11}\)See here [27, pp. 165–170], and formula (3.4) in [10].
elastic curve $\gamma(m,n)$, because they yield the equation $\wp(x + x_0) = -i\frac{\kappa'(x)}{4} - \frac{\kappa^2(x)}{8} - \frac{1}{24}$, for every $x \in \mathbb{R}$, where $x_0 \in \mathbb{C} \setminus \left(\frac{1}{2}\Omega \oplus \mathbb{R}\right)$ is some suitably chosen point and $\kappa$ the signed curvature function of $\gamma(m,n)$, as in Eq. (94). In combination with equations (95) and (97) this implies that the Weierstrass-$\wp$-function $\wp(z, \Omega)$ has a real primitive period, say $\omega_1$, namely the real primitive period of the Jacobi elliptic function $cn(r \cdot; p)$, precisely: $\omega_1(\wp) = 4\sqrt{2} - 4 p^2 K(p)$ by formula (100). This identity converts the formula “$E(\gamma(m,n)) = 8 \cdot n \cdot \Re(\eta(\omega_1(\wp), \Omega(m,n))) + \frac{2}{\pi} \cdot n \cdot \omega_1(\wp)$” from Theorem 5 in [10] into a computational tool—being comparable to formula (106), but much less effective—which can be used as in (108), in order to prove the lower bound $16\sqrt{\frac{\pi}{3}}$ for the elastic energies of all non-geodesic elasticae $\gamma(m,n)$ on $S^2$. □

We should also check the accuracy of the second statement of our Proposition 6 by means of a comparison between several numerical values of energies—first table below—and lengths—second table—of elasticae $\gamma(m,n)$ and our threshold “$\frac{8\pi}{\sqrt{2}}$” from formula (108). To this end, we apply formulae (105) and (106) for coprime integers $1 \leq m \leq 4$ and $1 \leq n \leq 7$, satisfying condition (104). The curious reader can compare these values with the corresponding ones for closed, orbitlike elasticae in the hyperbolic plane, collected in Table 1 of [21].

| $m$/$n$ | $m = 1$ | $m = 2$ | $m = 3$ | $m = 4$ |
|------|--------|--------|--------|--------|
| $n = 1$ | – | – | – | – |
| $n = 2$ | 19.17 | – | – | – |
| $n = 3$ | 38.38 | – | – | – |
| $n = 4$ | 62.88 | – | – | – |
| $n = 5$ | 96.62 | 55.01 | – | – |
| $n = 6$ | 134.95 | – | – | – |
| $n = 7$ | 192.23 | 98.87 | 74.97 | 62.89 |

| $m$/$n$ | $m = 1$ | $m = 2$ | $m = 3$ | $m = 4$ |
|------|--------|--------|--------|--------|
| $n = 1$ | – | – | – | – |
| $n = 2$ | 14.68 | – | – | – |
| $n = 3$ | 13.68 | – | – | – |
| $n = 4$ | 13.98 | – | – | – |
| $n = 5$ | 13.77 | 28.51 | – | – |
| $n = 6$ | 13.99 | – | – | – |
| $n = 7$ | 13.15 | 27.95 | 41.63 | 60.22 |
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Ruben Jakob
Mathematics Department
Israel Institute of Technology
3200003 Haifa
Israel
E-mail: rubenj@technion.ac.il

Accepted: 26 September 2023