The transition constant
for arithmetic hyperbolic reflection groups

V. V. Nikulin

Abstract. Using the results and methods of our papers [1], [2], we show that the degree of the ground field of an arithmetic hyperbolic reflection group does not exceed 25 in dimensions \( n \geq 6 \), and 44 in dimensions 3, 4, 5. This significantly improves our estimates obtained in [2]–[4]. We also use recent results in [5] and [6] to reduce the last bound to 35. We also review and correct the results of [1], §1.

Keywords: group generated by reflections, arithmetic group, hyperbolic space, number field, field of definition, quadratic form.

§1. Introduction

The transition constant was introduced in 1981 in our paper [1] and denoted by \( N(14) \). It is equal to the maximal degree of the ground fields of V-arithmetic connected edge graphs of minimality 14 with four vertices. It was shown in [1] that the number of such fields is finite, \( N(14) \) is a finite effective constant, and the degrees of the ground fields of arithmetic hyperbolic reflection groups in dimensions \( n \geq 10 \) (of the hyperbolic space) do not exceed \( N(14) \).

In [2] we showed that the degrees of the ground fields of arithmetic hyperbolic reflection groups in dimensions \( n \geq 6 \) do not exceed the maximum of \( N(14) \) and 11 (here 11 is the upper bound for the degrees of the ground fields of plane hyperbolic reflection groups with quadrangular fundamental polygon of minimality 14). In [7] we showed that the degrees of the ground fields of arithmetic hyperbolic reflection groups do not exceed the maximum of \( N(14) \) and the degrees of the ground fields of arithmetic hyperbolic reflection groups in dimensions \( n = 2, 3 \). In general, the transition constant \( N(14) \) is fundamental since if the degree of the ground field of an arithmetic hyperbolic reflection group is greater than \( N(14) \), then this field coincides with the ground field of some special plane reflection group (see [1]–[4], [7]).

In [2] we claimed that \( N(14) \leq 56 \). Here we use similar but more complicated arguments to show that \( N(14) \leq 25 \).
As another application of our methods and results (see §5), we show that the degrees of the ground fields of arithmetic hyperbolic reflection groups do not exceed 25 in dimensions \( n \geq 6 \), and 44 in dimensions \( n = 3, 4, 5 \). These bounds are much better than those in [2]–[4]. We also obtain a proof of finiteness in dimension \( n = 3 \) that is different from Agol’s original proof in [8].

Using the results of [8], Belolipetsky [5] obtained the bound 35 in dimension \( n = 3 \). Maclachlan [6] obtained the bound 11 in dimension \( n = 2 \). Using these bounds and our results in [7], we can improve the bound 44 to 35 in dimensions \( n = 4, 5 \).

We hope that our results and methods will be important for the further classification of arithmetic hyperbolic reflection groups (there is an example in Theorem 5.6).

In §6 we survey and correct those results of [1], §1 that are important for our methods (see also Remark 5.1).

The author is grateful to Professor C. Maclachlan for his letter and a preprint of [6].

A preliminary version of this paper was first published as a preprint [9].

\section*{2. Survey of some basic facts about hyperbolic fundamental polyhedra}

In this section we recall some basic definitions and results on fundamental chambers (always for discrete reflection groups) in hyperbolic spaces and their Gram matrices (see [1], [10]–[12]).

We work with Klein’s model of the hyperbolic space \( \mathcal{L} \) associated with a hyperbolic form \( \Phi \) of signature \((1, n)\) over the field \( \mathbb{R} \) of real numbers, where \( n = \dim \mathcal{L} \). Let \( V = \{ x \in \Phi \mid x^2 > 0 \} \) be the cone determined by \( \Phi \), and let \( V^+ \) be one of the two halves of this cone. Then \( \mathcal{L} = \mathcal{L}(\Phi) = V^+/\mathbb{R}^+ \) is the set of rays in \( V^+ \). We write \([x]\) for the element of \( \mathcal{L} \) determined by the ray \( \mathbb{R}^+ x, x \in V^+ \), where \( \mathbb{R}^+ \) is the set of positive real numbers. The hyperbolic distance is given by the formula

\[
\text{ch}(\rho([x],[y])) = \frac{x \cdot y}{\sqrt{x^2 y^2}}, \quad [x],[y] \in \mathcal{L}.
\]

Then the curvature of \( \mathcal{L} \) is equal to \(-1\).

Every half-space \( \mathcal{H}^+ \) in \( \mathcal{L} \) determines (and is in turn determined by) an orthogonal element \( e \in \Phi \) with square \( e^2 = -2 \):

\[
\mathcal{H}^+ = \mathcal{H}^+_e = \{[x] \in \mathcal{L} \mid x \cdot e \geq 0 \}.
\]

This half-space is bounded by the hyperplane

\[
\mathcal{H} = \mathcal{H}_e = \{[x] \in \mathcal{L} \mid x \cdot e = 0 \}
\]

orthogonal to \( e \). If half-spaces \( \mathcal{H}^+_e \) and \( \mathcal{H}^+_f \) with \( e_1^2 = e_2^2 = -2 \) have a common non-empty open subset in \( \mathcal{L} \), then \( \mathcal{H}^+_e \cap \mathcal{H}^+_f \) is an angle of magnitude \( \phi \), where \( 2 \cos \phi = e_1 \cdot e_2 \) if \(-2 < e_1 \cdot e_2 \leq 2 \), and the distance between the hyperplanes \( \mathcal{H}_{e_1} \) and \( \mathcal{H}_{e_2} \) is equal to \( \rho \), where \( 2 \text{ch} \rho = e_1 \cdot e_2 \) if \( e_1 \cdot e_2 > 2 \).
A convex polyhedron $\mathcal{M}$ in $\mathcal{L}$ is an intersection of finitely many half-spaces $\mathcal{H}_e^+$, $e \in P(\mathcal{M})$, where $P(\mathcal{M})$ is the set of all vectors with square $-2$ that are orthogonal to the faces (of codimension 1) of $\mathcal{M}$ and directed outward. The matrix

$$A = (a_{ij}) = (e_i \cdot e_j), \quad e_i, e_j \in P(\mathcal{M}),$$

(2.1)

is called the Gram matrix $\Gamma(\mathcal{M}) = \Gamma(P(\mathcal{M}))$ of $\mathcal{M}$. It uniquely determines $\mathcal{M}$ up to motions of $\mathcal{L}$. If $\mathcal{M}$ is sufficiently general, then $P(\mathcal{M})$ generates the form

$$\Phi = \sum_{e_i, e_j \in P(\mathcal{M})} a_{ij}X_iY_j \mod \ker,$$

(2.2)

and the set $P(\mathcal{M})$ is naturally identified with a subset of $\Phi$ and determines $\mathcal{M}$.

The polyhedron $\mathcal{M}$ is a fundamental polyhedron (or a fundamental chamber) for a discrete reflection group $W$ in $\mathcal{L}$ if and only if for all $i \neq j$ we have $a_{ij} \geq 0$ and $a_{ij} = 2\cos \frac{\pi}{m_{ij}}$, where $m_{ij} \geq 2$ is an integer, when $a_{ij} < 2$. Symmetric real matrices $A$ satisfying these conditions and having all diagonal elements equal to $-2$ are said to be fundamental (and then the set $P(\mathcal{M})$ formally corresponds to the set of indices of $A$). As usual, we identify fundamental matrices with the fundamental graphs $\Gamma$. Their vertices correspond to elements of $P(\mathcal{M})$. Two vertices $e_i \neq e_j \in P(\mathcal{M})$ are connected by a thin edge of integer weight $m_{ij} \geq 3$ if $0 < a_{ij} = 2\cos \frac{\pi}{m_{ij}} < 2$, by a thick edge if $a_{ij} = 2$, and by a dotted edge of weight $a_{ij}$ if $a_{ij} > 2$. In particular, two vertices $e_i$ and $e_j$ are not joined by an edge if and only if $e_i \cdot e_j = a_{ij} = 2\cos \frac{\pi}{2} = 0$ (equivalently, $e_i$ and $e_j$ are perpendicular or orthogonal). Examples of such graphs are given in Figs. 1–8 below.

Given a real $t > 0$, we say that the fundamental matrix $A = (a_{ij})$ (and the corresponding fundamental polyhedron $\mathcal{M}$) has minimality $t$ if $a_{ij} < t$ for all $a_{ij}$. Here we follow [1] and [12]. Minimality $t = 14$ is especially important in the present paper.

It is known that the fundamental domains of arithmetic hyperbolic groups must have finite volume. Suppose that this holds for the fundamental polyhedron $\mathcal{M}$ of some discrete hyperbolic reflection group. Vinberg [10] proved that $\mathcal{M}$ is the fundamental polyhedron of an arithmetic reflection group $W$ in $\mathcal{L}$ if and only if all the cyclic products

$$b_{i_1...i_m} = a_{i_1i_2}a_{i_2i_3}...a_{i_{m-1}i_m}a_{i_mi_1}$$

(2.3)

are algebraic integers, the field $\mathbb{K} = \mathbb{Q}(\{a_{ij}\})$ is totally real and the form (2.2) is negative definite for every embedding $\mathbb{K} \to \mathbb{R}$ not equal to the identity on the ground field $\mathbb{K} = \mathbb{Q}(\{b_{i_1...i_m}\})$ generated by the cyclic products (2.3).

A fundamental real matrix $A = (a_{ij})$, $a_{ij} = e_i \cdot e_j$, $e_i, e_j \in P(\mathcal{M})$ (or the corresponding graph) with a hyperbolic form $\Phi$ in (2.2) is said to be $V$-arithmetic if it satisfies these conditions of Vinberg (here we do not require that the corresponding polyhedron $\mathcal{M}$ has finite volume). It is well known (and easily provable; see the arguments in § 4) that a subset $P \subset P(\mathcal{M})$ also determines a $V$-arithmetic matrix $(e_i \cdot e_j)$, $e_i, e_j \in P$, with the same ground field $\mathbb{K}$ if $P$ is hyperbolic, that is, the form (2.2) corresponding to $P$ is hyperbolic.
§ 3. V-arithmetic edge polyhedra

A fundamental chamber $\mathcal{M}$ (and the corresponding Gram matrix $A$ and its graph) is called an edge chamber (matrix or graph) if all the hyperplanes $\mathcal{H}_e$, $e \in P(\mathcal{M})$, contain one of the two distinct vertices $v_1$, $v_2$ of a one-dimensional edge $v_1v_2$ of $\mathcal{M}$. In what follows we always assume that the vertices $v_1$ and $v_2$ are finite. Such edge chambers are said to be finite. Suppose that $\dim \mathcal{L} = n$. Then $P(\mathcal{M})$ consists of $n + 1$ elements: $e_1, e_2$ and $n - 1$ elements of $P(\mathcal{M}) - \{e_1, e_2\}$. Here $P(\mathcal{M}) - \{e_1, e_2\}$ corresponds to the hyperplanes containing the edge $v_1v_2$ of $\mathcal{M}$, $e_1$ to those containing $v_1$ but not $v_2$, and $e_2$ to those containing $v_2$ but not $v_1$. Then the set $P(\mathcal{M})$ is hyperbolic (it has a hyperbolic Gram matrix), but the subsets $P(\mathcal{M}) - \{e_1\}$ and $P(\mathcal{M}) - \{e_2\}$ are negative definite (they have a negative definite Gram matrix) and determine Coxeter graphs. Only the element $u = e_1 \cdot e_2$ of the Gram matrix of $\mathcal{M}$ can be greater than 2. Thus $\mathcal{M}$ has minimality $t > 2$ if and only if $u = e_1 \cdot e_2 < t$.

By the considerations above, the Gram graph $\Gamma(P(\mathcal{M}))$ of an edge chamber $\mathcal{M}$ has exactly one hyperbolic connected component $P(\mathcal{M})^{\text{hyp}}$ (containing $e_1$ and $e_2$) and several negative definite connected components. Clearly, the Gram matrix $\Gamma(P(\mathcal{M})^{\text{hyp}})$ corresponds to an edge chamber of dimension $\#P(\mathcal{M})^{\text{hyp}} - 1$. If $\mathcal{M}$ is V-arithmetic, then $\mathcal{M}$ and the hyperbolic connected component $\Gamma(P(\mathcal{M})^{\text{hyp}})$ have the same ground field $\mathbb{K}$.

**Theorem 3.1** ([1], Theorem 2.3.1). Given any $t > 0$, there is an effective constant $N(t)$ such that for every V-arithmetic edge chamber of minimality $t$ with ground field $\mathbb{K}$ of degree more than $N(t)$ over $\mathbb{Q}$, the number of vertices in the hyperbolic connected component of the Gram graph is less than 4.

The proof of this theorem in [1] (and in [12]) also shows that the set of all possible ground fields $\mathbb{K}$ of the hyperbolic connected components with at least four vertices for V-arithmetic edge chambers of minimality $t$ is finite. Moreover, the set of Gram graphs $\Gamma(P(\mathcal{M})^{\text{hyp}})$ of minimality $t$ with at least four vertices fixed is finite. Using this and following [2], we shall give a simpler definition of the constants $N(t), t > 0$, and $N(14)$.

3.1. Arithmetic Lannér graphs with at least four vertices. We recall that Lannér graphs are the Gram graphs of bounded fundamental hyperbolic simplices. They are characterized as those hyperbolic fundamental graphs all of whose proper subgraphs are Coxeter graphs. They were classified by Lannér [13]. Fig. 1 shows all arithmetic Lannér graphs with at least four vertices (only one Lannér graph with at least 4 vertices is not arithmetic). As usual, we replace thin edges of weight $k$ by $(k-2)$-edges for small $k$. There are only three ground fields of Lannér graphs with at least four vertices (see [14]):

$$\mathcal{FL}^4 = \{\mathbb{Q}, \mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{5})\}.$$  \hspace{1cm} (3.1)

3.2. Arithmetic triangle graphs. Triangle graphs are the Gram graphs of bounded fundamental triangles in the hyperbolic plane (we do not consider unbounded triangles). Equivalently, these are the Lannér graphs with three vertices. The triangle graphs are shown in Fig. 2, where $k, l, m \geq 2$ and $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} < 1$. 


Arithmetic triangles were listed in [15]. All bounded arithmetic triangles are given by the following triples \((k, l, m)\) (assuming that \(k \leq l \leq m\)):

\[
(2, 3, 7 - 12), (2, 3, 14), (2, 3, 16), (2, 3, 18), (2, 3, 24), (2, 3, 30), (2, 4, 5 - 8), (2, 4, 10), (2, 4, 12), (2, 4, 18), (2, 5, 5), (2, 5, 6), (2, 5, 8), (2, 5, 10), (2, 5, 20), (2, 5, 30), (2, 6, 6), (2, 6, 8), (2, 6, 12), (2, 7, 7), (2, 7, 14), (2, 8, 8), (2, 8, 16), (2, 9, 18), (2, 10, 10), (2, 12, 12), (2, 12, 24), (2, 15, 30), (2, 18, 18), (3, 3, 4 - 9), (3, 3, 12), (3, 3, 15), (3, 4, 4), (3, 4, 6), (3, 4, 12), (3, 5, 5), (3, 6, 6), (3, 6, 18), (3, 8, 8), (3, 8, 24), (3, 10, 30), (3, 12, 12), (4, 4, 4 - 6), (4, 4, 9), (4, 5, 5), (4, 6, 6), (4, 8, 8), (4, 16, 16), (5, 5, 5), (5, 5, 10), (5, 5, 15), (5, 10, 10), (6, 6, 6), (6, 12, 12), (6, 24, 24), (7, 7, 7), (8, 8, 8), (9, 9, 9), (9, 18, 18), (12, 12, 12), (15, 15, 15).
\]

Their ground fields were found in [16]. They are the fields

\[
\mathcal{FT} = \{\mathbb{Q}\} \cup \{\mathbb{Q}(\sqrt{a}) \mid a = 2, 3, 5, 6\} \cup \{\mathbb{Q}(\sqrt{2}, \sqrt{3}), \mathbb{Q}(\sqrt{2}, \sqrt{5})\} \cup \left\{\mathbb{Q}\left(\cos \frac{2\pi}{b}\right) \mid b = 7, 9, 11, 15, 16, 20\right\}.
\]
3.3. V-arithmetic finite edge graphs with four vertices for $2 < u < 14$.

Using the classification of Coxeter graphs, it is easy to draw all possible finite edge graphs $\Gamma^{(4)}$ with four vertices and $u = e_1 \cdot e_2 > 2$. They correspond to all three-dimensional finite fundamental edge polyhedra with connected Gram graph and $u > 2$. They are shown in Fig. 3 and give five types of graphs $\Gamma^{(4)} = \Gamma^{(4)}_i$, $1 \leq i \leq 5$. One can list all possible integer parameters $s, k, r, p \geq 2$ for such graphs since $\Gamma^{(4)} - \{e_1\}$ and $\Gamma^{(4)} - \{e_2\}$ are Coxeter graphs. These lists will be given in §4.

**Definition 3.1.** Suppose that $1 \leq i \leq 5$ and $t > 0$. We write $\Gamma^{(4)}_i(t)$ for the set of all V-arithmetic finite edge graphs $\Gamma^{(4)}_i$ with four vertices and of minimality $t$, that is, with $2 < u < t$. The set of ground fields of such graphs is denoted by $\mathcal{F}\Gamma^{(4)}_i(t)$.

The V-arithmetic graphs $\Gamma^{(4)}_i$ with $2 < u < t$ are particular cases of the graphs of V-arithmetic hyperbolic edge polyhedra whose hyperbolic connected component has four vertices and minimality $t$. Thus, by Theorem 3.1, the degrees (over $\mathbb{Q}$) of the ground fields in $\mathcal{F}\Gamma^{(4)}_i(t)$ are bounded by the effective constant $N(t)$. It follows that the sets $\Gamma^{(4)}_i(t)$ and $\mathcal{F}\Gamma^{(4)}_i(t)$ of V-arithmetic graphs and fields are also finite.

Conversely, Theorem 3.1 follows from the finiteness of the sets of fields because of the following simple assertion.

**Proposition 3.1.** The ground field of any V-arithmetic hyperbolic edge chamber of minimality $t > 0$ whose Gram graph has a hyperbolic connected component with at least four vertices belongs to one of the finite sets of fields $\mathcal{F}\mathcal{L}^4$, $\mathcal{F}\mathcal{T}$ or $\mathcal{F}\Gamma^{(4)}_i(t)$, $1 \leq i \leq 5$. In particular, Theorem 3.1 is equivalent to the finiteness of the sets of fields $\mathcal{F}\Gamma^{(4)}_i(t)$, $1 \leq i \leq 5$.

**Proof.** See [2].

The degrees of the fields in $\mathcal{F}\mathcal{L}^4$ and $\mathcal{F}\mathcal{T}$ are bounded above by 2 and 5 respectively.

Minimality $t = 14$ is especially important in the theory of arithmetic hyperbolic reflection groups. Using the methods of proof of Theorem 3.1 in [1], we prove the following explicit upper bounds improving those in [2] (for example, in [2] we only claimed that $N(14) \leq 56$).
Theorem 3.2. The degrees of the fields in $\mathcal{F}\Gamma_1^{(4)}(14)$ and $\mathcal{F}\Gamma_2^{(4)}(14)$ are bounded by (less than or equal to) 23. The degrees of the fields in $\mathcal{F}\Gamma_3^{(4)}(14)$, $\mathcal{F}\Gamma_4^{(4)}(14)$, $\mathcal{F}\Gamma_5^{(4)}(14)$ are bounded by 25. Thus one can take $N(14) = 25$ (or $N(14) \leq 25$) for the constant $N(14)$ in Theorem 3.1.

Theorem 3.2 will be proved in the next section.

§ 4. Ground fields of $V$-arithmetic connected finite edge graphs of minimality 14 with four vertices

In this section we obtain explicit upper bounds for the degrees of the fields in the finite sets $\mathcal{F}\Gamma_i^{(4)}(14)$, $1 \leq i \leq 5$ (see Definition 3.1) and prove Theorem 3.2. We also obtain further important information about these sets of fields.

4.1. Some results on hyperbolic numbers. As in the proof of Theorem 3.1 in [1], we use the following results from [1], § 1 (they are reviewed and corrected in § 6 below).

Theorem 4.1 ([1], Theorem 1.2.1). Suppose that $\mathbb{F}$ is a totally real algebraic number field and to every embedding $\sigma : \mathbb{F} \to \mathbb{R}$ there corresponds an interval $[a_\sigma, b_\sigma]$ in $\mathbb{R}$ with

$$\prod_\sigma \frac{b_\sigma - a_\sigma}{4} < 1.$$

We also fix a positive integer $m$ and intervals $[s_1, t_1], \ldots, [s_m, t_m]$ in $\mathbb{R}$. Then there is a constant $N(s_1, t_1)$ such that if $\alpha$ is a totally real algebraic integer and the
following inequalities hold for all embeddings $\tau : \mathbb{F}(\alpha) \to \mathbb{R}$:

$$s_i \leq \tau(\alpha) \leq t_i, \quad \tau = \tau_1, \ldots, \tau_m,$$

$$\alpha_{\tau|\mathbb{F}} \leq \tau(\alpha) \leq \beta_{\tau|\mathbb{F}}, \quad \tau \neq \tau_1, \ldots, \tau_m,$$

then

$$[\mathbb{F}(\alpha) : \mathbb{F}] \leq N(s_i, t_i).$$

**Theorem 4.2** ([1], Theorem 1.2.2). Under the hypotheses of Theorem 4.1 one can choose $N(s_i, t_i)$ in the form $N(s_i, t_i) = N$, where $N$ is the least positive integer solution of the inequality

$$N \ln \frac{1}{R} - M \ln (2N + 2) - \ln B \geq \ln S. \quad (4.1)$$

Here

$$M = [\mathbb{F} : \mathbb{Q}], \quad B = \sqrt{\text{discr} \mathbb{F}}, \quad \text{(4.2)}$$

$$R = \sqrt{\prod_{\sigma} b_\sigma - a_\sigma}, \quad S = \prod_{i=1}^{m} \frac{2r_i}{b_\sigma - a_\sigma}, \quad \text{(4.3)}$$

where

$$\sigma_i = \tau_i | \mathbb{F}, \quad r_i = \max\{|t_i - a_\sigma_i|, |b_\sigma - s_i|\}. \quad \text{(4.4)}$$

We note that proofs of Theorems 4.1, 4.2 use a variant of Fekete’s theorem (1923) on the existence of non-zero polynomials over $\mathbb{Z}$ of bounded degree with small deviation from zero on appropriate intervals (see [1], Theorem 1.1.1, as well as the survey and some corrections in § 6, Theorems 6.1, 6.2).

Unfortunately, Theorems 4.1, 4.2 usually give poor bounds for the degree. They should mainly be regarded as existence theorems showing that the degree is bounded and giving an explicit upper bound for that degree. In this paper we use appropriate explicit polynomials to get better bounds for the degree in similar cases. This is the main difference between the present paper and previous work of the author on this subject.

We shall use the following assertion. Similar arguments occur in the proofs of Theorems 4.1, 4.2 (see § 6).

**Lemma 4.1.** Suppose that $\mathbb{F}$ is a totally real algebraic number field and to every embedding $\sigma : \mathbb{F} \to \mathbb{R}$ there corresponds an interval $[a_\sigma, b_\sigma]$ in $\mathbb{R}$. We also fix a positive integer $m$ and intervals $[s_1, t_1], \ldots, [s_m, t_m]$ in $\mathbb{R}$. Let $P(x)$ be a non-zero polynomial over the ring of integers of $\mathbb{F}$. We put

$$\delta(\sigma) = \max_{x \in [a_\sigma, b_\sigma]} |P^{\sigma}(x)|, \quad \sigma : \mathbb{F} \to \mathbb{R},$$

and $a_i = \max_{x \in [s_i, t_i]} |P^{\sigma_i}(x)|$, where $\sigma_i = \tau_i|\mathbb{F}$ (see below). Suppose that $\prod_{\sigma} \delta(\sigma) < 1$. Let $\alpha$ be a totally real algebraic integer such that the following inequalities hold for all embeddings $\tau : \mathbb{F}(\alpha) \to \mathbb{R}$:

$$s_i \leq \tau(\alpha) \leq t_i, \quad \tau = \tau_1, \ldots, \tau_m,$$

$$\alpha_{\tau|\mathbb{F}} \leq \tau(\alpha) \leq \beta_{\tau|\mathbb{F}}, \quad \tau \neq \tau_1, \ldots, \tau_m.$$

Then we have the following bound for the degree $[\mathbb{F}(\alpha) : \mathbb{F}]$:

$$1 \leq [\mathbb{F}(\alpha) : \mathbb{F}] \leq \frac{\ln \prod_{i} a_i - \ln \prod_{i} \delta(\sigma_i)}{-\ln \prod_{\sigma} \delta(\sigma)}$$

if $P(\alpha) \neq 0$. We note that $\alpha$ does not exist if the right-hand side of the last estimate is less than 1.

**Proof.** Since $P(\alpha) \neq 0$, we get

$$1 \leq |N_{\mathbb{F}(\alpha)/\mathbb{Q}}(P(\alpha))| = \left| \prod_{\tau} \tau(P(\alpha)) \right| = \prod_{\tau} |P^\tau(\tau(\alpha))|$$

$$= \prod_{\tau \neq \tau_i} |P^\tau(\tau(\alpha))| \prod_{i=1}^{m} |P^{\tau_i}(\tau_i(\alpha))| \leq \prod_{\tau \neq \tau_i} \max_{[a_r, b_r, [\bar{a}_r, \bar{b}_r, \bar{[a]}]]} |P^\tau(x)| \prod_{i=1}^{m} \max_{[s_i, t_i]} |P^{\sigma_i}(x)|$$

$$\leq \prod_{\tau} \max_{[a_r, b_r, [\bar{a}_r, \bar{b}_r, \bar{[a]}]]} |P^\tau(x)| \prod_{i=1}^{m} \max_{[s_i, t_i]} |P^{\sigma_i}(x)| \prod_{i=1}^{m} a_i \prod_{i=1}^{m} \delta(\sigma_i)^{[\mathbb{F}(\alpha) : \mathbb{F}]}$$

Taking the logarithms of the left- and right-hand sides of the last chain of formulae and using the inequality $\prod_{\sigma} \delta(\sigma) < 1$, we get the desired assertion.

The proof of Lemma 4.1 is similar to the proofs of Theorems 4.1, 4.2 (see §6), which use Fekete polynomials.

### 4.2. Ground fields of some V-arithmetic connected finite edge graphs with three vertices and given minimality.

Although we are mainly interested in connected edge graphs with four vertices, some connected graphs with three vertices are also important.

All connected finite edge graphs with three vertices are shown in Fig. 4: these are $\Gamma_1^{(3)}$, where $s, k \geq 3$, and $\Gamma_2^{(3)}$, where $d \geq 3$. We denote their ground fields for minimality $t > 2$ by $\mathcal{F}_{\Gamma_1^{(3)}}(t)$ and $\mathcal{F}_{\Gamma_2^{(3)}}(t)$. This means that $2 < \sigma^{(+)}(u) < t$. 

![Figure 4: Connected finite edge graphs with three vertices](image-url)
4.2.1. Ground fields of $\Gamma_2^{(3)}(t)$. We first consider the graphs $\Gamma_2^{(3)}(t)$. The corresponding Gram matrix is equal to

$$
\begin{pmatrix}
-2 & u & 0 \\
u & -2 & 2\cos\frac{\pi}{d} \\
0 & 2\cos\frac{\pi}{d} & -2
\end{pmatrix},
$$

(4.5)

where $d \geq 3$ is an integer and $u$ is a totally real algebraic integer. The ground field is $K = \mathbb{Q}(u^2, \cos^2\frac{\pi}{d})$. The determinant $d(u)$ of the Gram matrix is given by

$$
d(u) = u^2 - 4\sin^2\frac{\pi}{d}.
$$

It follows that the graph $\Gamma_2^{(3)}$ is V-arithmetic of minimality $t > 2$ if and only if for $\alpha = u^2$ we have

$$
0 < \sigma(\alpha) < \sigma(4\sin^2\frac{\pi}{d})
$$

(4.6)

for all $\sigma : K \to \mathbb{R}$ different from $\sigma^+$ (the identity) and

$$
4 < \sigma^+(\alpha) < t^2.
$$

(4.7)

Hence $\mathbb{F}_d = \mathbb{Q}(\cos^2\frac{\pi}{d}) \subset K = \mathbb{Q}(\alpha)$. We are especially interested in the cases $t = 14$ and $t = 16$. Let us estimate the degree $n = [K : \mathbb{Q}]$.

Suppose that $d = 3$. Then $\alpha$ satisfies the inequalities

$$
0 < \sigma(\alpha) < 3, \quad 4 < \sigma^+(\alpha) < t^2.
$$

Consider the polynomial

$$
P(x) = x^3(x - 1)^4(x - 2)^4(x - 3)^3(x^2 - 3x + 1)^3
$$

(4.8)

of degree 20 with integer coefficients. The maximum of $|P(x)|$ on the interval $[0, 3]$ is equal to $\delta = \frac{884736}{9765625} = 0.0905969664$. Lemma 4.1 yields the bound

$$
[K : \mathbb{Q}] \leq 1 + \frac{\ln P(t^2)}{-\ln \delta}.
$$

For $t = 14$ we get $[K : \mathbb{Q}] \leq 44$, and for $t = 16$ we get $[K : \mathbb{Q}] \leq 47$. (Theorems 4.1, 4.2 give poorer upper bounds: 76 for $t = 14$ and 78 for $t = 16$.)

Remark 4.1. Our assertions above concerning the polynomial $P(x)$ in one variable $x$ are easily checked up to the required accuracy using a computer. Polynomials are very easy functions for a computer.

We used a GP/PARI Calculator, version 2.2.13. It enables one to find the derivative of a polynomial (using the command ‘deriv’) and the roots of a polynomial (using the command ‘polroots’). Therefore all the assertions above on $P(x)$ are easily checked using standard calculations. The same can be done for all the polynomials in one variable encountered below which all have ‘reasonable’ degrees and ‘reasonable’ coefficients.
Suppose that $d = 4$. Then $\alpha$ satisfies the inequalities
\[ 0 < \sigma(\alpha) < 2, \quad 4 < \sigma^{(+)}(\alpha) < t^2. \]
We consider the polynomial
\[ P(x) = x(x - 1)^2(x - 2) \]  
of degree 4 with integer coefficients. The maximum of $|P(x)|$ on the interval $[0, 2]$ is equal to $\delta = \frac{1}{4}$. Lemma 4.1 yields that
\[ [K : \mathbb{Q}] \leq 1 + \frac{\ln P(t^2)}{-\ln \delta}. \]
For $t = 14$ and $t = 16$ we get $[K : \mathbb{Q}] \leq 16$. (Theorems 4.1, 4.2 yield only the bound 31.)

Suppose that $d = 5$. Consider the polynomial
\[ P(x) = x(x - 1)\left(x + 1 - 4 \sin^2 \frac{\pi}{5}\right)\left(x - 4 \sin^2 \frac{\pi}{5}\right) \]  
of degree 4 over the ring of integers of $\mathbb{F}_5$. The maximum of $|P(x)|$ on the interval $[0, 4 \sin^2 \frac{\pi}{5}]$ is equal to $\delta_1 = 0.04559\ldots$. For the conjugate polynomial
\[ P^\sigma(x) = x(x - 1)\left(x + 1 - 4 \sin^2 \frac{2\pi}{5}\right)\left(x - 4 \sin^2 \frac{2\pi}{5}\right), \]
the maximum of $|P^\sigma(x)|$ on the conjugate interval $[0, 4 \sin^2 \frac{2\pi}{5}]$ is equal to $\delta_2 = 2.1419\ldots$. Clearly, $P(\alpha)$ is not equal to zero. Lemma 4.1 yields that
\[ [K : \mathbb{F}_5] \leq \frac{\ln P(t^2) - \ln \delta_1}{-\ln (\delta_1 \delta_2)}. \]
For $t = 14$ and $t = 16$ we respectively get $[K : \mathbb{F}_5] \leq 10$ and $[K : \mathbb{Q}] \leq 20$. (Theorems 4.1, 4.2 only give $[K : \mathbb{F}_5] \leq 27$ for $t = 14$ and $[K : \mathbb{F}_5] \leq 28$ for $t = 16$.)

Suppose that $d \geq 6$. Consider the polynomial
\[ P(x) = x\left(x - 4 \sin^2 \frac{\pi}{d}\right) \]  
over the ring of integers of $\mathbb{F}_d$. For a given $\sigma : \mathbb{F}_d \to \mathbb{R}$ the maximum of $|P^\sigma(x)|$ on the interval $[0, \sigma(4 \sin^2 \frac{\pi}{d})]$ is obviously equal to $4\sigma(\sin^4 \frac{\pi}{d})$. Clearly, $P(\alpha) \neq 0$. Then we have (actually repeating the proof of Lemma 4.1 in this case)
\[ 1 \leq |N_{K/Q}(P(\alpha))| < N_{\mathbb{F}_d/Q}\left(4 \sin^4 \frac{\pi}{d}\right)^{[K:\mathbb{F}_d]}|P(t^2)| = \left(\frac{\gamma(d)^2}{4[\mathbb{F}_d : \mathbb{Q}]^2}\right)^{[K:\mathbb{F}_d]}|P(t^2)|. \]
where, for $d \geq 3$,
\[ N_{\mathbb{F}_d/Q}\left(4 \sin^2 \frac{\pi}{d}\right) = \gamma(d) = \begin{cases} p & \text{if } d = p^t > 2 \text{ for a prime } p, \\ 1 & \text{otherwise} \end{cases} \]  
and $[\mathbb{F}_d : \mathbb{Q}] = \frac{\varphi(d)}{2}$, where $\varphi(d)$ is the Euler totient function.
Putting $m = \left[ \mathbb{K} : F_d \right]$, we obtain the inequality

$$m(\varphi(d) \ln 2 - 2 \ln \gamma(d)) < \ln t^2 + \ln \left( t^2 - 4 \sin^2 \frac{\pi}{d} \right) - \ln \left( 4 \sin^4 \frac{\pi}{d} \right).$$

We easily see that $\varphi(d) \ln 2 - 2 \ln \gamma(d) > 0$ for $d \geq 6$. Hence,

$$1 \leq m < \frac{\ln t^2 + \ln \left( t^2 - 4 \sin^2 \frac{\pi}{d} \right) - \ln \left( 4 \sin^4 \frac{\pi}{d} \right)}{\varphi(d) \ln 2 - 2 \ln \gamma(d)}.$$  \hspace{1cm} (4.13)

Accordingly, we have $\left[ \mathbb{K} : \mathbb{Q} \right] \leq \frac{m}{2} \varphi(d)$, where $m$ satisfies (4.13).

We now use the trivial bounds $\varphi(d) \geq \sqrt{d-2}$ and $\gamma(d) \leq d$. Then (4.13) gives that only the following values of $d \geq 6$ are possible for $t = 14$, with corresponding bounds for $\left[ \mathbb{K} : \mathbb{Q} \right]$.

- $d = 6$, then $[\mathbb{K} : \mathbb{Q}] \leq 8$; $d = 7$, then $[\mathbb{K} : \mathbb{Q}] \leq 138$
- $d = 8$, then $[\mathbb{K} : \mathbb{Q}] \leq 18$; $d = 9$, then $[\mathbb{K} : \mathbb{Q}] \leq 18$
- $d = 10$, then $[\mathbb{K} : \mathbb{Q}] \leq 10$; $d = 11$, then $[\mathbb{K} : \mathbb{Q}] \leq 30$
- $d = 12$, then $[\mathbb{K} : \mathbb{Q}] \leq 10$; $d = 13$, then $[\mathbb{K} : \mathbb{Q}] \leq 24$
- $d = 14$, then $[\mathbb{K} : \mathbb{Q}] \leq 9$; $d = 15$, then $[\mathbb{K} : \mathbb{Q}] \leq 8$
- $d = 16$, then $[\mathbb{K} : \mathbb{Q}] \leq 8$; $d = 17$, then $[\mathbb{K} : \mathbb{Q}] \leq 16$
- $d = 18$, then $[\mathbb{K} : \mathbb{Q}] \leq 9$; $d = 19$, then $[\mathbb{K} : \mathbb{Q}] \leq 18$
- $d = 20$, then $[\mathbb{K} : \mathbb{Q}] \leq 8$; $d = 21$, then $[\mathbb{K} : \mathbb{Q}] \leq 12$
- $d = 22$, then $[\mathbb{K} : \mathbb{Q}] \leq 10$; $d = 23$, then $\mathbb{K} = F_{23}$
- $d = 24$, then $[\mathbb{K} : \mathbb{Q}] \leq 12$; $d = 25$, then $\mathbb{K} = F_{25}$
- $d = 26$, then $[\mathbb{K} : \mathbb{Q}] \leq 12$; $d = 27$, then $\mathbb{K} = F_{27}$
- $d = 28$, then $[\mathbb{K} : \mathbb{Q}] \leq 12$; $d = 29$, then $\mathbb{K} = F_{29}$
- $d = 30$, then $[\mathbb{K} : \mathbb{Q}] \leq 12$; $d = 31$, then $\mathbb{K} = F_{31}$
- $d = 32$, then $\mathbb{K} = F_{32}$; $d = 33$, then $\mathbb{K} = F_{33}$
- $d = 34$, then $\mathbb{K} = F_{34}$; $d = 35$, then $\mathbb{K} = F_{35}$
- $d = 36$, then $[\mathbb{K} : \mathbb{Q}] \leq 12$; $d = 37$, then $\mathbb{K} = F_{37}$
- $d = 38$, then $\mathbb{K} = F_{38}$; $d = 39$, then $\mathbb{K} = F_{39}$
- $d = 40$, then $\mathbb{K} = F_{40}$; $d = 42$, then $[\mathbb{K} : \mathbb{Q}] \leq 12$
- $d = 44$, then $\mathbb{K} = F_{44}$; $d = 45$, then $\mathbb{K} = F_{45}$
- $d = 46$, then $\mathbb{K} = F_{46}$; $d = 48$, then $\mathbb{K} = F_{48}$
- $d = 50$, then $\mathbb{K} = F_{50}$; $d = 52$, then $\mathbb{K} = F_{52}$
- $d = 54$, then $\mathbb{K} = F_{54}$; $d = 56$, then $\mathbb{K} = F_{56}$
- $d = 58$, then $\mathbb{K} = F_{58}$; $d = 60$, then $\mathbb{K} = F_{60}$
- $d = 62$, then $\mathbb{K} = F_{62}$; $d = 66$, then $\mathbb{K} = F_{66}$
- $d = 70$, then $\mathbb{K} = F_{70}$; $d = 72$, then $\mathbb{K} = F_{72}$
- $d = 78$, then $\mathbb{K} = F_{78}$; $d = 80$, then $\mathbb{K} = F_{80}$
For \( t = 16 \) only the following values of \( d \geq 6 \) are possible, with corresponding bounds for \([K : \mathbb{Q}]\).

\[
\begin{align*}
&d = 84, \text{ then } K = F_{84}; & d = 90, \text{ then } K = F_{90}; \\
&d = 96, \text{ then } K = F_{96}; & d = 102, \text{ then } K = F_{102}; \\
&d = 120, \text{ then } K = F_{120}.
\end{align*}
\]

\[
\begin{align*}
&d = 6, \text{ then } [K : \mathbb{Q}] \leq 8; & d = 7, \text{ then } [K : \mathbb{Q}] \leq 144; \\
&d = 8, \text{ then } [K : \mathbb{Q}] \leq 18; & d = 9, \text{ then } [K : \mathbb{Q}] \leq 21; \\
&d = 10, \text{ then } [K : \mathbb{Q}] \leq 10; & d = 11, \text{ then } [K : \mathbb{Q}] \leq 30; \\
&d = 12, \text{ then } [K : \mathbb{Q}] \leq 10; & d = 13, \text{ then } [K : \mathbb{Q}] \leq 24; \\
&d = 14, \text{ then } [K : \mathbb{Q}] \leq 9; & d = 15, \text{ then } [K : \mathbb{Q}] \leq 8; \\
&d = 16, \text{ then } [K : \mathbb{Q}] \leq 12; & d = 17, \text{ then } [K : \mathbb{Q}] \leq 24; \\
&d = 18, \text{ then } [K : \mathbb{Q}] \leq 12; & d = 19, \text{ then } [K : \mathbb{Q}] \leq 18; \\
&d = 20, \text{ then } [K : \mathbb{Q}] \leq 12; & d = 21, \text{ then } [K : \mathbb{Q}] \leq 12; \\
&d = 22, \text{ then } [K : \mathbb{Q}] \leq 10; & d = 23, \text{ then } K = F_{23}; \\
&d = 24, \text{ then } [K : \mathbb{Q}] \leq 12; & d = 25, \text{ then } K = F_{25}; \\
&d = 26, \text{ then } [K : \mathbb{Q}] \leq 12; & d = 27, \text{ then } K = F_{27}; \\
&d = 28, \text{ then } [K : \mathbb{Q}] \leq 12; & d = 29, \text{ then } K = F_{29}; \\
&d = 30, \text{ then } [K : \mathbb{Q}] \leq 12; & d = 31, \text{ then } K = F_{31}; \\
&d = 32, \text{ then } K = F_{32}; & d = 33, \text{ then } K = F_{33}; \\
&d = 34, \text{ then } K = F_{34}; & d = 35, \text{ then } K = F_{35}; \\
&d = 36, \text{ then } [K : \mathbb{Q}] \leq 12; & d = 37, \text{ then } K = F_{37}; \\
&d = 38, \text{ then } K = F_{38}; & d = 39, \text{ then } K = F_{39}; \\
&d = 40, \text{ then } K = F_{40}; & d = 42, \text{ then } [K : \mathbb{Q}] \leq 12; \\
&d = 44, \text{ then } K = F_{44}; & d = 45, \text{ then } K = F_{45}; \\
&d = 46, \text{ then } K = F_{46}; & d = 48, \text{ then } K = F_{48}; \\
&d = 50, \text{ then } K = F_{50}; & d = 52, \text{ then } K = F_{52}; \\
&d = 54, \text{ then } K = F_{54}; & d = 56, \text{ then } K = F_{56}; \\
&d = 58, \text{ then } K = F_{58}; & d = 60, \text{ then } K = F_{60}; \\
&d = 62, \text{ then } K = F_{62}; & d = 64, \text{ then } K = F_{64}; \\
&d = 66, \text{ then } K = F_{66}; & d = 68, \text{ then } K = F_{68}; \\
&d = 70, \text{ then } K = F_{70}; & d = 72, \text{ then } K = F_{72}; \\
&d = 78, \text{ then } K = F_{78}; & d = 80, \text{ then } K = F_{80}; \\
&d = 84, \text{ then } K = F_{84}; & d = 90, \text{ then } K = F_{90}; \\
&d = 96, \text{ then } K = F_{96}; & d = 102, \text{ then } K = F_{102}; \\
&d = 120, \text{ then } K = F_{120}. \\
\]
To improve these results, we apply polynomials similar to the one used for $d = 5$. For $d \geq 5$ we take the polynomial
\[
P(x) = x(x-1)\left(x + 1 - 4\sin^2 \frac{\pi}{d}\right)\left(x - 4\sin^2 \frac{\pi}{d}\right)
\] (4.14)
over the ring of integers of $\mathbb{F}_d$. Given an embedding $\sigma: \mathbb{F}_d \rightarrow \mathbb{R}$, we denote the maximum value of $|P^\sigma(x)|$ on the interval $[0, \sigma(4\sin^2 \frac{\pi}{d})]$ by $\delta(\sigma)$ (to find it, we calculate the zeros of the derivative on this interval). Note that all the $\frac{\varphi(d)}{2}$ embeddings $\sigma$ are given by $\sigma(4\sin^2 \frac{\pi}{d}) = 4\sin^2 \frac{kr}{d}$, where $1 \leq k \leq \frac{d}{2}$ and $(k, d) = 1$, and $\sigma(+) \equiv k = 1$.

Clearly, $P(\alpha) \neq 0$. Lemma 4.1 yields that
\[
1 \leq m \leq \frac{\ln P^{\sigma(+)}(t^2) - \ln \delta(\sigma)}{-\ln (\prod_{\sigma} \delta(\sigma))}
\] (4.15)
if $\prod_{\sigma} \delta(\sigma) < 1$. This usually gives a better bound for $m$ than (4.13), but requires more complicated calculations. We calculate (4.15) taking $t = 14$, $t = 16$ and $d$ as in the lists above. For some values of $d$ this gives better upper bounds for $m$. The final results are stated in the following theorems.

**Theorem 4.3.** For $d \geq 3$ all V-arithmetic finite connected edge graphs $\Gamma_2^{(3)}(14)$ (see Fig. 4) of minimality 14 (equivalently, all totally real algebraic integers $\alpha = u^2$ over $\mathbb{F}_d = \mathbb{Q}(\cos \frac{\pi}{d})$ satisfying conditions (4.6) and (4.7) for $t = 14$) are possible only for the values of $d$ listed below. We also have the following bounds for the degree $m = [K : \mathbb{F}_d]$ of the ground field $K = \mathbb{Q}(\alpha) \supset \mathbb{F}_d$.

- $d = 3$, $m \leq 44$; $d = 4$, $m \leq 16$; $d = 5$, $m \leq 10$;
- $d = 6$, $m \leq 8$; $d = 7$, $m \leq 9$; $d = 8$, $m \leq 8$;
- $d = 9$, $m \leq 5$; $d = 10$, $m \leq 5$; $d = 11$, $m \leq 4$;
- $d = 12$, $m \leq 5$; $d = 13$, $m \leq 3$; $d = 14$, $m \leq 3$;
- $d = 15$, $m \leq 2$; $d = 16$, $m \leq 3$; $d = 17$, $m \leq 2$;
- $d = 18$, $m \leq 3$; $d = 20$, $m \leq 2$; $d = 21$, $m \leq 2$;
- $d = 22$, $m \leq 2$; $d = 24$, $m \leq 3$; $d = 26$, $m \leq 2$;
- $d = 28$, $m \leq 2$; $d = 30$, $m \leq 3$; $d = 36$, $m \leq 2$;
- $d = 42$, $m \leq 2$.

For the remaining $d = 19, 23, 25, 27, 29, 31–35, 38–40, 44–46, 48, 50, 52, 54, 56, 58, 60, 66, 70, 72, 78, 84, 90, 102, 120$ we have $m = 1$ and $K = \mathbb{F}_d$. In particular, $[K : \mathbb{Q}] \leq 44$ for $d = 3$; $[K : \mathbb{Q}] \leq 27$ for $d = 7$; $[K : \mathbb{Q}] \leq 20$ for $d = 5, 11$; $[K : \mathbb{Q}] \leq 18$ for $d = 13$; $[K : \mathbb{Q}] \leq 16$ for the other values of $d$.

**Theorem 4.4.** For $d \geq 3$ all V-arithmetic finite connected edge graphs $\Gamma_2^{(3)}(16)$ (see Fig. 4) of minimality 16 (equivalently, totally real algebraic integers $\alpha = u^2$ over $\mathbb{F}_d = \mathbb{Q}(\cos \frac{\pi}{d})$ satisfying conditions (4.6) and (4.7) for $t = 16$) are possible
only for the values of \( d \) listed below. We also have the following bounds for the degree \( m = [K : F_d] \) of the ground field \( K = \mathbb{Q}(\alpha) \supset F_d \).

\[
\begin{align*}
d = 3, & \quad m \leq 47; \quad d = 4, & \quad m \leq 16; \quad d = 5, & \quad m \leq 10; \\
d = 6, & \quad m \leq 8; \quad d = 7, & \quad m \leq 10; \quad d = 8, & \quad m \leq 9; \\
d = 9, & \quad m \leq 5; \quad d = 10, & \quad m \leq 5; \quad d = 11, & \quad m \leq 4; \\
d = 12, & \quad m \leq 5; \quad d = 13, & \quad m \leq 3; \quad d = 14, & \quad m \leq 3; \\
d = 15, & \quad m \leq 2; \quad d = 16, & \quad m \leq 3; \quad d = 17, & \quad m \leq 2; \\
d = 18, & \quad m \leq 4; \quad d = 19, & \quad m \leq 2; \quad d = 20, & \quad m \leq 3; \\
d = 21, & \quad m \leq 2; \quad d = 22, & \quad m \leq 2; \quad d = 24, & \quad m \leq 3; \\
d = 26, & \quad m \leq 2; \quad d = 28, & \quad m \leq 2; \quad d = 30, & \quad m \leq 3; \\
d = 36, & \quad m \leq 2; \quad d = 42, & \quad m \leq 2. \\
\end{align*}
\]

For the remaining \( d = 23, 25, 27, 29, 31–35, 38–40, 44–46, 48, 50, 52, 54, 56, 58, 60, 62, 64, 66, 70, 72, 78, 80, 84, 90, 96, 102, 120 \) we have \( m = 1 \) and \( K = F_d \).

In particular, \( [K : \mathbb{Q}] \leq 47 \) for \( d = 3 \); \( [K : \mathbb{Q}] \leq 30 \) for \( d = 7 \); \( [K : \mathbb{Q}] \leq 20 \) for \( d = 5, 11 \); \( [K : \mathbb{Q}] \leq 18 \) for \( d = 8, 13, 19 \); \( [K : \mathbb{Q}] \leq 16 \) for the other values of \( d \).

Theorem 4.3 has the following consequence.

**Theorem 4.5.** The set of fields \( \mathcal{F}_{12}^{(3)}(t) \) is finite and their degrees over \( \mathbb{Q} \) do not exceed 44 (see Theorem 4.3 for more details of these fields for various values of \( d \)).

4.2.2. **Ground fields of some of the graphs \( \Gamma_{12}^{(3)}(t) \).** Here we consider the ground fields in \( \mathcal{F}_{12}^{(3)}(t) \) for the V-arithmetic graphs \( \Gamma_{12}^{(3)}(t) \) with parameters \( 3 \leq s \leq k \leq 5 \) (see Fig. 4). The corresponding Gram matrix is equal to

\[
\begin{pmatrix}
-2 & u & 2 \cos \frac{\pi}{s} \\
u & -2 & 2 \cos \frac{\pi}{k} \\
2 \cos \frac{\pi}{s} & 2 \cos \frac{\pi}{k} & -2
\end{pmatrix},
\]

where \( s, k \geq 3 \) are integers and \( u \) is a totally real algebraic integer. The ground field is

\[
K = \mathbb{Q}\left(u^2, \cos^2 \frac{\pi}{s}, \cos^2 \frac{\pi}{k}, u \cos \frac{\pi}{s} \cos \frac{\pi}{k}\right).
\]

The determinant \( d(u) \) of the Gram matrix (4.16) is given by

\[
\frac{d(u)}{2} = u^2 + 4u \cos \frac{\pi}{s} \cos \frac{\pi}{k} + 4\cos^2 \frac{\pi}{s} + 4\cos^2 \frac{\pi}{k} - 4.
\]

It follows that \( \Gamma_{12}^{(3)} \) is V-arithmetic of minimality \( t > 2 \) if and only if

\[
0 < -\tilde{\sigma}\left(2 \cos \frac{\pi}{s} \cos \frac{\pi}{k}\right) - \sqrt{\sigma\left(4\sin^2 \frac{\pi}{s} \sin^2 \frac{\pi}{k}\right)} < \tilde{\sigma}(u) < -\tilde{\sigma}\left(2 \cos \frac{\pi}{s} \cos \frac{\pi}{k}\right) + \sqrt{\sigma\left(4\sin^2 \frac{\pi}{s} \sin^2 \frac{\pi}{k}\right)}
\]

(4.17)
for all $\sigma : \mathbb{K} \to \mathbb{R}$ different from $\sigma^{(+)}$, and
\[
4 < \sigma^{(+)}(u^2) < t^2,
\]
(4.18)
where $\tilde{\sigma}$ extends $\sigma : \mathbb{K} = \mathbb{Q}(u^2) \to \mathbb{R}$ to $\mathbb{Q}(u) = \mathbb{K}(\cos \frac{\pi}{s} \cos \frac{\pi}{k})$.

**Case $s = k = 3$.** In this case $\mathbb{K} = \mathbb{Q}(u)$ and
\[
-2 < \sigma(u) < 1, \quad 2 < \sigma^{(+)}(u) < t.
\]
Consider the polynomial
\[
P(x) = (x + 2)^3(x + 1)^4x^4(x - 1)^3(x^2 + x - 1)^3
\]
(4.19)
with integer coefficients. It is analogous to (4.8). We get
\[
[\mathbb{K} : \mathbb{Q}] \leq 1 + \frac{\ln P(t)}{-\ln \delta},
\]
where $\delta = 0.0905969664$. For $t = 14$ we get $[\mathbb{K} : \mathbb{Q}] \leq 23$. (Theorems 4.1, 4.2 give only the bound 57.)

**Case $s = 3, k = 4$.** In this case $\mathbb{Q}(\sqrt{2}) \subset \mathbb{K}(u)$ and $\mathbb{K} = \mathbb{Q}(u^2)$. We have $\sqrt{2} \in \mathbb{K}$ if and only if $u \in \mathbb{K}$. In this case,
\[
d(u) = u^2 + \sqrt{2}u - 1.
\]
We have
\[
\tilde{\sigma}\left(-\frac{1}{\sqrt{2}}\right) - \sqrt{\frac{3}{2}} < \tilde{\sigma}(u) < \tilde{\sigma}\left(-\frac{1}{\sqrt{2}}\right) + \sqrt{\frac{3}{2}}, \quad 4 < \sigma^{(+)}(u^2) < t^2.
\]
Consider the polynomial
\[
P(x) = x^4(x + \sqrt{2})^4(x^2 + \sqrt{2}x - 1)
\]
(4.20)
over the ring of integers of $\mathbb{Q}(\sqrt{2})$. The maximum value of $|P(x)|$ on the interval $[-1/\sqrt{2} - \sqrt{3}/2, -1/\sqrt{2} + \sqrt{3}/2]$ is equal to $\delta_1 = 0.09375\ldots$. The maximum of the conjugate polynomial $P^g(x) = x^4(x - \sqrt{2})^4(x^2 - \sqrt{2}x - 1)$ on the interval $[1/\sqrt{2} - \sqrt{3}/2, 1/\sqrt{2} + \sqrt{3}/2]$ is equal to $\delta_2 = 0.09375\ldots$. Lemma 4.1 yields that
\[
[\mathbb{K} : \mathbb{Q}(\sqrt{2})] < \frac{\ln P(t) - \ln \delta_1}{-\ln(\delta_1\delta_2)}
\]
if $\sqrt{2} \in \mathbb{K}$, and
\[
[\mathbb{K}(u) : \mathbb{Q}(\sqrt{2})] < \frac{\ln (P(t)P^g(-t)) - \ln (\delta_1\delta_2)}{-\ln(\delta_1\delta_2)}
\]
if $\sqrt{2} \notin \mathbb{K}$.

For $t = 14$ we get $[\mathbb{K} : \mathbb{Q}] \leq 12$ in both cases.

**Case $s = k = 4$.** Then $u \in \mathbb{K}$ and $\mathbb{K} = \mathbb{Q}(u)$. We have
\[
-2 < \sigma(u) < 0, \quad 2 < \sigma^{(+)}(u) < 14.
\]
Consider the polynomial
\[ P(x) = (x + 2)(x + 1)^2x \] (4.21)
with integer coefficients. It is analogous to (4.9). By Lemma 4.1,
\[ [K : \mathbb{Q}] \leq 1 + \frac{\ln P(t)}{-\ln 4}. \]
For \( t = 14 \) we get \([K : \mathbb{Q}] \leq 8.\)

Case \( s = 3, k = 5. \) Then \( u \in K \) and \( \mathbb{F}_5 = \mathbb{Q}(\cos^2 \frac{\pi}{5}) \subset K = \mathbb{Q}(u). \) Hence,
\[ \frac{d(u)}{2} = u^2 + 2u \cos \frac{\pi}{5} + 4 \cos^2 \frac{\pi}{5} - 3. \]
We have
\[ -\sigma \left( \cos \frac{\pi}{5} \right) - \sqrt{3} \sigma \left( \sin \frac{\pi}{5} \right) < \sigma(u) < -\sigma \left( \cos \frac{\pi}{5} \right) + \sqrt{3} \sigma \left( \sin \frac{\pi}{5} \right), \] (4.22)
\[ 2 < \sigma^{(+)}(u) < 14. \]

Consider the polynomial
\[ P(x) = x(x + 1) \left( x + 3 - 4 \sin^2 \frac{\pi}{5} \right) \left( x + 2 - 4 \sin^2 \frac{\pi}{5} \right) \times \left( x^2 + 2x \cos \frac{\pi}{5} + 4 \cos^2 \frac{\pi}{5} - 3 \right) \] (4.23)
over the ring of integers of \( \mathbb{F}_5. \) The maximum \( \delta(\sigma) \) of \(|P^\sigma(x)|\) on the corresponding intervals (4.22) is equal to \( \delta_1 = 0.0690098 \ldots \) if \( \sigma(\sin \frac{\pi}{5}) = \sin \frac{\pi}{5} \) and to \( \delta_2 = 1.2383316 \ldots \) if \( \sigma(\sin \frac{\pi}{5}) = \sin \frac{3\pi}{5}. \) By Lemma 4.1,
\[ [K : \mathbb{F}_5] < \frac{\ln P(t) - \ln \delta_1}{-\ln(\delta_1 \delta_2)}. \]
For \( t = 14 \) we get \([K : \mathbb{F}_5] \leq 7 \) and \([K : \mathbb{Q}] \leq 14.\)

Case \( s = 4, k = 5 \) (the most difficult case). Here \( \mathbb{F}_5 = \mathbb{Q}(\cos^2 \frac{\pi}{5}) \subset K = \mathbb{Q}(u^2) \) and \( u \in K \) if and only if \( \sqrt{2} \in K. \) We have
\[ \frac{d(u)}{2} = u^2 + 2\sqrt{2} u \cos \frac{\pi}{5} + 4 \cos^2 \frac{\pi}{5} - 2 = u^2 + \left( \frac{\sqrt{2}}{2} + \frac{\sqrt{10}}{2} \right) u + \left( -\frac{1}{2} + \frac{\sqrt{5}}{2} \right), \]
\[ a(\tilde{\sigma}) = -\tilde{\sigma} \left( \sqrt{2} \cos \frac{\pi}{5} \right) - \sqrt{\sigma \left( 2 \sin^2 \frac{\pi}{5} \right)} \]
\[ < \tilde{\sigma}(u) < -\tilde{\sigma} \left( \sqrt{2} \cos \frac{\pi}{5} \right) + \sqrt{\sigma \left( 2 \sin^2 \frac{\pi}{5} \right)} = b(\tilde{\sigma}) \] (4.24)
for all \( \sigma : K \to \mathbb{R} \) different from \( \sigma^{(+)}, \) and
\[ 4 < \sigma^{(+)}(u^2) < t^2. \] (4.25)
Consider the following polynomial of degree 8:

\[ P(x) = \left( x^2 + \left( \frac{\sqrt{2}}{2} + \frac{\sqrt{10}}{2} \right) x - \frac{1}{2} + \frac{\sqrt{5}}{2} \right) \times \left( 4x^3 + (3\sqrt{2} + 3\sqrt{10})x^2 + (6 + 4\sqrt{5})x + \frac{5\sqrt{2}}{2} + \frac{10}{2} \right) \times \left( 4x^3 + (3\sqrt{2} + 3\sqrt{10})x^2 + (7 + 3\sqrt{5})x + \frac{3\sqrt{2}}{2} + \frac{10}{2} \right) \]  

(4.26)

ger over the ring of integers of \( \mathbb{F}_{4,5} = \mathbb{Q}(\sqrt{2}, \cos^2 \frac{\pi}{5}) = \mathbb{Q}(\sqrt{2}, \sqrt{5}) \).

All the roots of the polynomials \( P(\tilde{\sigma})(x) \) belong to the corresponding intervals \([a(\tilde{\sigma}), b(\tilde{\sigma})]\). The maxima \( \delta(\tilde{\sigma}) \) of \(|P(\tilde{\sigma})(x)|\) on the corresponding intervals (4.24) are as follows. On the interval

\[ [a(\tilde{\sigma}), b(\tilde{\sigma})] = [-1.97537668 \ldots, -0.31286893 \ldots] \]

the maximum of \(|P(\tilde{\sigma})(x)|\) is equal to \( \delta_1 = 0.045593135 \ldots \) if \( \tilde{\sigma}(\sqrt{2}) = \sqrt{2} \) and \( \tilde{\sigma}(\sqrt{5}) = \sqrt{5} \). On the interval

\[ [a(\tilde{\sigma}), b(\tilde{\sigma})] = [0.31286893 \ldots, 1.97537668 \ldots] \]

it is equal to \( \delta_2 = 0.045593135 \ldots \) if \( \tilde{\sigma}(\sqrt{2}) = -\sqrt{2} \) and \( \tilde{\sigma}(\sqrt{5}) = \sqrt{5} \). On the interval

\[ [a(\tilde{\sigma}), b(\tilde{\sigma})] = [-0.90798099 \ldots, 1.78201304 \ldots] \]

it is equal to \( \delta_3 = 2.14190686 \ldots \) if \( \tilde{\sigma}(\sqrt{2}) = \sqrt{2} \) and \( \tilde{\sigma}(\sqrt{5}) = -\sqrt{5} \). On the interval

\[ [a(\tilde{\sigma}), b(\tilde{\sigma})] = [-1.78201304 \ldots, 0.90798099 \ldots] \]

it is equal to \( \delta_4 = 2.14190686 \ldots \) if \( \tilde{\sigma}(\sqrt{2}) = -\sqrt{2} \) and \( \tilde{\sigma}(\sqrt{5}) = -\sqrt{5} \).

By Lemma 4.1,

\[ [\mathbb{K} : \mathbb{F}_{4,5}] \leq \frac{\ln P(t) - \ln \delta_1}{-\ln(\delta_1 \delta_2 \delta_3 \delta_4)} \]

if \( \sqrt{2} \in \mathbb{K} \), and

\[ [\mathbb{K}(u) : \mathbb{F}_{4,5}] \leq \frac{\ln (P(t)P'(t)) - \ln(\delta_1 \delta_2)}{-\ln(\delta_1 \delta_2 \delta_3 \delta_4)} \]

if \( \sqrt{2} \notin \mathbb{K} \), where \( \tau(\sqrt{2}) = -\sqrt{2} \) and \( \tau(\sqrt{5}) = \sqrt{5} \).

For \( t = 14 \), in the first case we obtain \([\mathbb{K} : \mathbb{F}_{4,5}] \leq 5 \) and \([\mathbb{K} : \mathbb{Q}] \leq 20 \), and in the second we obtain \([\mathbb{K}(u) : \mathbb{F}_{4,5}] \leq 11 \) and \([\mathbb{K}(u) : \mathbb{Q}] \leq 44 \). Then \([\mathbb{K} : \mathbb{Q}] \leq 22 \). Thus \([\mathbb{K} : \mathbb{Q}] \leq 22 \) in both cases. (Theorems 4.1, 4.2 give only the bound \([\mathbb{K} : \mathbb{Q}] \leq 72 \).)

Case \( s = k = 5 \). Then \( \mathbb{F}_5 = \mathbb{Q}(\cos^2 \frac{\pi}{5}) \subset \mathbb{K} = \mathbb{Q}(u) \) and

\[ -2 < \sigma(u) < \sigma \left( 4 \sin^2 \frac{\pi}{5} \right) - 2, \quad (4.27) \]

\[ 2 < \sigma^{(+)}(u) < 14. \]
Consider the polynomial
\[ P(x) = (x + 2)(x + 1) \left( x + 3 - 4 \sin^2 \frac{\pi}{5} \right) \left( x + 2 - 4 \sin^2 \frac{\pi}{5} \right). \] (4.28)

It is analogous to (4.10). The maximum \( \delta(\sigma) \) of \(|P^\sigma(x)|\) on the corresponding interval (4.27) is equal to \( \delta_1 = 0.0455931 \ldots \) if \( \sigma(\sin^2 \frac{\pi}{5}) = \sin^2 \frac{\pi}{5} \), and to \( \delta_2 = 2.1419068 \ldots \) if \( \sigma(\sin^2 \frac{\pi}{5}) = \sin^2 \frac{2\pi}{5} \).

By Lemma 4.1,
\[ [\mathbb{K} : \mathbb{F}_5] \leq \frac{\ln P(t) - \ln \delta_1}{-\ln(\delta_1\delta_2)}. \]

For \( t = 14 \) we get \([\mathbb{K} : \mathbb{F}_5] \leq 6\) and \([\mathbb{K} : \mathbb{Q}] \leq 12\).

As a result we get the following assertion.

**Theorem 4.6.** We have the following upper bounds for the degrees of the ground fields \( \mathbb{K} \in \mathcal{F}\Gamma_1^{(3)}(14) \), that is, for \( V \)-arithmetic edge graphs \( \Gamma_1^{(3)} \) of minimality 14 (for \( 2 < u < 14 \)) with parameters \( 3 \leq s \leq k \leq 5 \) (see Fig. 4):

(i) \([\mathbb{K} : \mathbb{Q}] \leq 23 \) for \( s = k = 3 \);
(ii) \([\mathbb{K} : \mathbb{Q}] \leq 12 \) for \( s = 3, k = 4 \);
(iii) \([\mathbb{K} : \mathbb{Q}] \leq 8 \) for \( s = k = 4 \);
(iv) \( \mathbb{F}_5 = \mathbb{Q}(\sqrt{5}) \subset \mathbb{K} \) and \([\mathbb{K} : \mathbb{F}_5] \leq 7 \) for \( s = 3, k = 5 \);
(v) for \( s = 4, k = 5 \) we have \( \mathbb{F}_{4,5} = \mathbb{Q}(\sqrt{2}, \sqrt{5}) \subset \mathbb{K} \), \([\mathbb{K} : \mathbb{F}_{4,5}] \leq 5 \) if \( \sqrt{2} \in \mathbb{K} \), and \( \mathbb{F}_5 = \mathbb{Q}(\sqrt{5}) \subset \mathbb{K} \), \([\mathbb{K} : \mathbb{F}_5] \leq 11 \) if \( \sqrt{2} \notin \mathbb{K} \);
(vi) \( \mathbb{F}_5 = \mathbb{Q}(\sqrt{5}) \subset \mathbb{K} \) and \([\mathbb{K} : \mathbb{F}_5] \leq 6 \) for \( s = k = 5 \).

In particular, \([\mathbb{K} : \mathbb{Q}] \leq 23 \) for all fields \( \mathbb{K} \in \mathcal{F}\Gamma_1^{(3)}(14) \) with parameters \( 3 \leq s \leq k \leq 5 \).

**4.3. Fields in \( \mathcal{F}\Gamma_1^{(4)}(14) \).** For the graph \( \Gamma_1^{(4)}(14) \) (Fig. 5) we assume that \( s, k, r, p \geq 3 \). The subgraphs \( \Gamma_1^{(4)} - \{e_1\} \) and \( \Gamma_1^{(4)} - \{e_2\} \) must be Coxeter graphs. It follows that we need consider only the following cases (up to obvious symmetries): either \( s = k = 3 \) and \( 5 \geq r \geq p \geq 3 \), or \( s = p = 3 \) and \( 5 \geq r \geq k \geq 4 \); the totally real algebraic integer \( u \) satisfies the inequalities \( 2 < u < 14 \).

![Figure 5. The graph \( \Gamma_1^{(4)} \)](image-url)
Since every graph $\Gamma_1^{(4)}(14)$ with parameters $s, k, r, p$ contains subgraphs $\Gamma_1^{(3)}(14)$ with parameters $s, k$ and $r, p$, we see that every field $\mathbb{K} \in \mathcal{F}_1^{(4)}(14)$ with parameters $s, k, r, p$ belongs to $\mathcal{F}_1^{(3)}(14)$ with parameters $s, k$, and to $\mathcal{F}_1^{(3)}(14)$ with parameters $p, r$. Thus the degree of the field $\mathbb{K} \in \mathcal{F}_1^{(4)}(14)$ with parameters $s, k, r, p$ does not exceed the minimum of the maxima of the degrees of fields in $\mathcal{F}_1^{(3)}(14)$ with parameters $s, k$ and the maxima of the degrees of fields in $\mathcal{F}_1^{(3)}(14)$ with parameters $p, r$. Therefore we obtain the following result from Theorem 4.6.

**Proposition 4.1.** We have the following upper bounds for the degrees of the ground fields $\mathbb{K} \in \mathcal{F}_1^{(4)}(14)$, that is, for $\mathbb{V}$-arithmetic edge graphs $\Gamma_1^{(4)}$ of minimality 14 (for $2 < u < 14$), depending on the values of the parameters $s, k, p, r$ (see Fig. 5):

(i) $[\mathbb{K} : \mathbb{Q}] \leq 23$ for $s = k = p = r = 3$;

(ii) $[\mathbb{K} : \mathbb{Q}] \leq 12$ for $s = k = 3, r = 4, p = 3$;

(iii) $\mathbb{F}_5 = \mathbb{Q}(\sqrt[5]{5}) \subset \mathbb{K}$ and $[\mathbb{K} : \mathbb{F}_5] \leq 7$ for $s = k = 3, r = 5, p = 3$;

(iv) $[\mathbb{K} : \mathbb{Q}] \leq 8$ for $s = k = 3, r = p = 4$;

(v) $\mathbb{F}_{4,5} = \mathbb{Q}(\sqrt[4]{2}, \sqrt[5]{5}) \subset \mathbb{K}$ and $[\mathbb{K} : \mathbb{F}_{4,5}] \leq 5$ for $s = k = 3, r = 5, p = 4$;

(vi) $\mathbb{F}_5 = \mathbb{Q}(\sqrt[5]{5}) \subset \mathbb{K}$ and $[\mathbb{K} : \mathbb{F}_5] \leq 6$ for $s = k = 3, r = p = 5$;

(vii) $[\mathbb{K} : \mathbb{Q}] \leq 12$ for $s = p = 3, r = k = 4$;

(viii) $\mathbb{F}_{4,5} = \mathbb{Q}(\sqrt[4]{2}, \sqrt[5]{5}) \subset \mathbb{K}$ and $[\mathbb{K} : \mathbb{F}_{4,5}] \leq 3$ for $s = p = 3, r = 5, k = 4$;

(ix) $\mathbb{F}_5 = \mathbb{Q}(\sqrt[5]{5}) \subset \mathbb{K}$ and $[\mathbb{K} : \mathbb{F}_5] \leq 6$ for $s = p = 3, r = k = 5$.

In particular, $[\mathbb{K} : \mathbb{Q}] \leq 23$ for all fields $\mathbb{K} \in \mathcal{F}_1^{(4)}(14)$.

One can significantly improve Proposition 4.1 by considering the graphs $\Gamma_1^{(4)}$ directly. We hope to do this in future publications.

**4.4 Fields in $\mathcal{F}_2^{(4)}(14)$.** For the graph $\Gamma_2^{(4)}(14)$ (Fig. 6) we assume that $s, k, p \geq 3$ are integers, $u$ is a totally real algebraic integer and $2 < u < 14$. Moreover, there are only the following possibilities: $3 \leq s \leq k \leq 5, p = 3; s = k = 3, p = 4, 5$.

![Figure 6. The graph $\Gamma_2^{(4)}$](image)

Since every graph $\Gamma_2^{(4)}(14)$ with parameters $s, k, p$ contains a subgraph $\Gamma_1^{(3)}(14)$ with parameters $s, k$, we see that every field $\mathbb{K} \in \mathcal{F}_2^{(4)}(14)$ with parameters $s, k, p$ belongs to $\mathcal{F}_1^{(3)}(14)$ with parameters $s, k$. Thus the degrees of the fields $\mathbb{K} \in \mathcal{F}_2^{(4)}(14)$ with parameters $s, k, p$ do not exceed the degrees of the fields in $\mathcal{F}_1^{(3)}(14)$ with parameters $s, k$. Using Theorem 4.6, we get the following result.
Proposition 4.2. We have the following upper bounds for the degrees of ground fields $K \in \mathcal{F}_2^{(4)}(14)$, that is, for $V$-arithmetic edge graphs $\Gamma_2^{(4)}$ of minimality 14 (for $2 < u < 14$), depending on the values of the parameters $s, k, p$ (see Fig. 6):

(i) $[K : \mathbb{Q}] \leq 23$ for $s = k = 3, p = 3, 4, 5$;
(ii) $[K : \mathbb{Q}] \leq 12$ for $s = 3, k = 4, p = 3$;
(iii) $[K : \mathbb{Q}] \leq 8$ for $s = k = 4, p = 3$;
(iv) $F_5 = \mathbb{Q}(\sqrt{5}) \subset K$ and $[K : F_5] \leq 7$ for $s = 3, k = 5, p = 3$;
(v) for $s = 4, k = 5, p = 3$ we have $F_{4, 5} = \mathbb{Q}(\sqrt{2}, \sqrt{5}) \subset K$, $[K : F_{4, 5}] \leq 5$ if $\sqrt{2} \not\in K$, and $F_5 = \mathbb{Q}(\sqrt{5}) \subset K$, $[K : F_5] \leq 11$ if $\sqrt{2} \not\in K$;
(vi) $F_5 = \mathbb{Q}(\sqrt{5}) \subset K$ and $[K : F_5] \leq 6$ for $s = k = 5, p = 3$.

In particular, $[K : \mathbb{Q}] \leq 23$ for all $K \in \mathcal{F}_2^{(4)}(14)$.

One can significantly improve Proposition 4.2 by considering the graphs $\Gamma_2^{(4)}$ directly. We hope to do this in future publications.

4.5. Fields in $\mathcal{F}_3^{(4)}(14)$. For the graph $\Gamma_3^{(4)}(14)$ (Fig. 7) we assume that $s \geq 2, k, r \geq 3$ are integers, $u$ is a totally real algebraic integer and $2 < u < 14$. Moreover, there are only the following possibilities: $s = 2, k = 3, r = 3, 4, 5; s = 2, k = 4, 5, r = 3; 3 \leq s \leq k \leq 5, r = 3; s = k = 3, r = 4, 5.$

The ground field $K = \mathbb{Q}(u^2)$ contains the cyclic products

$$\cos^2 \frac{\pi}{s}, \cos^2 \frac{\pi}{k}, \cos^2 \frac{\pi}{r}, u^2, u \cos \frac{\pi}{s} \cos \frac{\pi}{k} \cos \frac{\pi}{r}.$$ 

The determinant $d(u)$ of the Gram matrix is given by

$$-\frac{d(u)}{4} = u^2 \sin^2 \frac{\pi}{r} + u \cdot 2 \cos \frac{\pi}{s} \cos \frac{\pi}{k} \cos \frac{\pi}{r} + 4 \cos^2 \frac{\pi}{r} - 4 \sin^2 \frac{\pi}{s} \sin^2 \frac{\pi}{k}.$$ (4.29)

Let

$$D = 4 \cos^2 \frac{\pi}{s} \cos^2 \frac{\pi}{k} \cos^2 \frac{\pi}{r} + 16 \sin^2 \frac{\pi}{s} \sin^2 \frac{\pi}{k} \sin^2 \frac{\pi}{r} - 16 \sin^2 \frac{\pi}{r} \cos^2 \frac{\pi}{r}$$

be the discriminant of the quadratic polynomial (4.29) in $u$. A graph $\Gamma_3^{(4)}(14)$ is $V$-arithmetic if and only if, for $\tau : K(u) \to \mathbb{R}$ different from $\sigma^{(+)}$ on $K$, we have

$$-2\tilde{\tau}(\cos \frac{\pi}{s} \cos \frac{\pi}{k} \cos \frac{\pi}{r}) - \sqrt{\tau(D)} < \tau(u) < -2\tilde{\tau}(\cos \frac{\pi}{s} \cos \frac{\pi}{k} \cos \frac{\pi}{r}) + \sqrt{\tau(D)},$$

(4.30)

where $\tilde{\tau}$ extends $\tau$.

Since every graph $\Gamma_3^{(4)}(14)$ with parameters $s, k, r$ contains subgraphs $\Gamma_2^{(3)}(14)$ with parameter $s$ if $s \geq 3$, and with parameter $k, 3 \leq k \leq 5$, we see that every field $K \in \mathcal{F}_3^{(4)}(14)$ with parameters $s, k, r$ belongs to $\mathcal{F}_2^{(3)}(14)$ with parameter $s$ if $s \geq 3$, and to $\mathcal{F}_2^{(3)}(14)$ with parameter $k, 3 \leq k \leq 5$. Thus the degrees of the fields $K \in \mathcal{F}_3^{(4)}(14)$ with parameters $s, k, r$ do not exceed the degrees of the fields in $\mathcal{F}_2^{(3)}(14)$ with parameter $s$ if $s \geq 3$, and the degrees of the fields in $\mathcal{F}_2^{(3)}(14)$ with parameter $k, 3 \leq k \leq 5$. Using Theorem 4.3, we get the following result.
Proposition 4.3. We have the following upper bounds for the degrees of ground fields $K \in \mathcal{F} \Gamma^{(4)}_3(14)$, that is, for $V$-arithmetic edge graphs $\Gamma^{(4)}_3$ of minimality 14 (for $2 < u < 14$), depending on the values of the parameters $s, k, r$ (see Fig. 7):

(i) $[K : \mathbb{Q}] \leq 44$ for $s = 2, k = 3, r = 3, 4, 5$;
(ii) $[K : \mathbb{Q}] \leq 16$ for $s = 2, k = 4, r = 3$;
(iii) $F_5 = \mathbb{Q}(\sqrt{5}) \subset K$ and $[K : F_5] \leq 10$ for $s = 2, k = 5, r = 3$;
(iv) $[K : \mathbb{Q}] \leq 44$ for $s = k = 3, r = 3, 4, 5$;
(v) $[K : \mathbb{Q}] \leq 16$ for $s = 3, k = 4, r = 3$;
(vi) $F_5 = \mathbb{Q}(\sqrt{5}) \subset K$ and $[K : F_5] \leq 10$ for $s = 3, k = 5, r = 3$;
(vii) $[K : \mathbb{Q}] \leq 16$ for $s = k = 4, r = 3$;
(viii) $F_5 = \mathbb{Q}(\sqrt{5}) \subset K$ and $[K : F_5] \leq 10$ for $s = 4, k = 5, r = 3$;
(ix) $F_5 = \mathbb{Q}(\sqrt{5}) \subset K$ and $[K : F_5] \leq 10$ for $s = k = 5, r = 3$.

We now improve the poor bounds for $s = 2, k = 3$ and $s = k = 3$ in Proposition 4.3.

Suppose that $s = 2$. Then, for $\alpha = u^2$ and $\sigma : K \rightarrow \mathbb{R}$, we get

$$0 < \sigma(\alpha) < \sigma\left(\frac{D}{4\sin^4 \frac{r}{2}}\right)$$

if $\sigma \neq \sigma^{(+)}$.

For $s = 2, k = r = 3$ we get

$$0 < \sigma(\alpha) < \frac{8}{3}, \quad 4 < \sigma^{(+)}(\alpha) < 14^2.$$ 

Consider the polynomial

$$P(x) = x(x-1)^2(x-2)^2(x^2-3x+1)^2.$$  \hspace{1cm} \text{(4.31)}

The maximum of $|P(x)|$ on the interval $[0, \frac{4}{3}]$ is equal to $\delta = 0.148148 \ldots$. By Lemma 4.1,

$$[K : \mathbb{Q}] \leq 1 + \frac{\ln P(14^2)}{-\ln \delta} \leq 25.9.$$ 

Thus $[K : \mathbb{Q}] \leq 25$. 

\begin{figure}[h]
\centering
\includegraphics{figure7.png}
\caption{The graph $\Gamma^{(4)}_3$}
\end{figure}
For \( s = 2, k = 3, r = 4 \) we get
\[
0 < \sigma(\alpha) < 2, \quad 4 < \sigma(+)\alpha < 14^2.
\]

Consider the polynomial
\[
P(x) = x(x - 1)^2(x - 2).
\] (4.32)
The maximum of \(|P(x)|\) on the interval \([0, 2]\) is equal to \(\delta = \frac{1}{4}\). By Lemma 4.1,
\[
[K : \mathbb{Q}] \leq 1 + \frac{\ln P(14^2)}{-\ln \delta} < 16.3.
\]
Thus \([K : \mathbb{Q}] \leq 16.

For \( s = 2, k = 3, r = 5 \) we get
\[
0 < \sigma(\alpha) < \frac{16}{5} \sigma\left(\sin^2 \frac{2\pi}{5}\right), \quad 4 < \sigma(+)\alpha < 14^2.
\]

Consider the polynomial
\[
P(x) = x(x - 1)\left(x + 1 - 4 \sin^2 \frac{\pi}{5}\right)
\] (4.33)
over the ring of integers of the field \(\mathbb{F}_5 = \mathbb{Q}(\cos^2 \frac{\pi}{5})\). The maximum of \(|P(x)|\) on the interval \([0, 4 - \frac{16}{5} \sin^2 \frac{2\pi}{5}]\) is equal to \(\delta_1 = 0.0844582\ldots\), and the maximum of \(|P^g(x)|\) on the interval \([0, 4 - \frac{16}{5} \sin^2 \frac{\pi}{5}]\) with non-identity \(g : \mathbb{F}_5 \to \mathbb{R}\) is equal to \(\delta_2 = 1.51554175\ldots\). By Lemma 4.1,
\[
[K : \mathbb{F}_5] \leq \frac{\ln P(14^2) - \ln \delta_1}{-\ln (\delta_1 \delta_2)} \leq 8.91.
\]
Thus \([K : \mathbb{F}_5] \leq 8.

Suppose that \(s = k = r = 3\). Then
\[
0 < \sigma(u^2) < x_1^2 = 2.15612498\ldots, \quad 4 < \sigma(+)u^2 < 14^2,
\]
where \(x_1\) is the left root of the polynomial \(3x^2 + x - 5\) defined in (4.29). Consider the polynomial
\[
P(x) = x(x - 1)(x - 2).
\] (4.34)
The maximum of \(|P(x)|\) on the interval \([0, 2.15612498\ldots]\) is equal to \(\delta = 0.38918054\ldots\). By Lemma 4.1,
\[
[K : \mathbb{Q}] \leq 1 + \frac{\ln P(14^2)}{-\ln \delta} \leq 17.8.
\]
Thus \([K : \mathbb{Q}] \leq 17.

Suppose that \(s = k = 3, r = 4\). Then
\[
0 < \sigma(u^2) < x_1^2 = 1.30901699\ldots, \quad 4 < \sigma(+)u^2 < 14^2,
\]
where \(x_1\) is the left root of the polynomial \(2x^2 + \sqrt{2}x - 1\) defined in (4.29). Consider the polynomial

\[
P(x) = x(x - 1)^2.
\]

(4.35)
The maximum of \(|P(x)|\) on the interval \([0, 1.30901699]\ldots\) is equal to \(\delta = 0.14814\ldots\). By Lemma 4.1,

\[
[K : \mathbb{Q}] \leq 1 + \frac{\ln P(14^2)}{-\ln \delta} \leq 9.3.
\]

Thus \([K : \mathbb{Q}] \leq 9\).

Suppose that \(s = k = 3, r = 5\). Then \(F_5 = \mathbb{Q}(\cos \frac{\pi}{5}) \subset K = \mathbb{Q}(u)\). In this case, \(\sigma(D) < 0\) if \(\sigma(\sin^2 \frac{\pi}{5}) = \sin^2 \frac{\pi}{5}\). Hence we have \(K = F_5\).

We obtain the following result.

**Theorem 4.7.** We have the following upper bounds for the degrees of ground fields \(K \in \mathcal{F}\Gamma_3^{(4)}(14)\), that is, for \(V\)-arithmetic edge graphs \(\Gamma_3^{(4)}\) of minimality 14 (for \(2 < u < 14\)), depending on the values of the parameters \(s, k, r\) (see Fig. 7):

(i) \([K : \mathbb{Q}] \leq 25\) for \(s = 2, k = r = 3\);
(ii) \([K : \mathbb{Q}] \leq 16\) for \(s = 2, k = 3, r = 4\);
(iii) \(F_5 = \mathbb{Q}(\sqrt{5}) \subset K\) and \([K : F_5] \leq 8\) for \(s = 2, k = 3, r = 5\);
(iv) \([K : \mathbb{Q}] \leq 16\) for \(s = 2, k = 4, r = 3\);
(v) \(F_5 = \mathbb{Q}(\sqrt{5}) \subset K\) and \([K : F_5] \leq 10\) for \(s = 2, k = 5, r = 3\);
(vi) \([K : \mathbb{Q}] \leq 17\) for \(s = k = r = 3\);
(vii) \([K : \mathbb{Q}] \leq 9\) for \(s = k = 3, r = 4\);
(viii) \(K = F_5 = \mathbb{Q}(\sqrt{5})\) for \(s = k = 3, r = 5\);
(ix) \([K : \mathbb{Q}] \leq 16\) for \(s = 3, k = 4, r = 3\);
(x) \(F_5 = \mathbb{Q}(\sqrt{5}) \subset K\) and \([K : F_5] \leq 10\) for \(s = 3, k = 5, r = 3\);
(xi) \([K : \mathbb{Q}] \leq 16\) for \(s = k = 4, r = 3\);
(xii) \(F_5 = \mathbb{Q}(\sqrt{5}) \subset K\) and \([K : F_5] \leq 8\) for \(s = 4, k = 5, r = 3\);
(xiii) \(F_5 = \mathbb{Q}(\sqrt{5}) \subset K\) and \([K : F_5] \leq 10\) for \(s = k = 5, r = 3\).

In particular, \([K : \mathbb{Q}] \leq 25\) for all \(K \in \mathcal{F}\Gamma_3^{(4)}(14)\).

One can significantly improve Theorem 4.7 by considering the graphs \(\Gamma_3^{(4)}\) directly in all cases. We hope to do this in future publications.

**4.6. Fields in \(\mathcal{F}\Gamma_5^{(4)}(14)\).** For the graph \(\Gamma_5^{(4)}(14)\) (Fig. 8) we assume that \(s, k\) are integers, \(3 \leq s \leq k\), and \(u\) is a totally real algebraic integer, \(2 < u < 14\).

![Figure 8. The graph \(\Gamma_5^{(4)}\)](image)

The ground field \(K = \mathbb{Q}(u^2)\) contains the cyclic products

\[
\cos^2 \frac{\pi}{k}, \cos^2 \frac{\pi}{s}, \quad u^2.
\]
Therefore \( F_{s,k} = \mathbb{Q}(\cos^2 \frac{\pi}{s}, \cos^2 \frac{\pi}{k}) \subset \mathbb{K} = \mathbb{Q}(u^2) \). The determinant \( d(u) \) of the Gram matrix is given by

\[
-\frac{d(u)}{4} = u^2 - 4 \sin^2 \frac{\pi}{k} \sin^2 \frac{\pi}{s}.
\]

A graph \( \Gamma_5^{(4)} \) is V-arithmetic if and only if, for \( \alpha = u^2 \), we have

\[
0 < \sigma(\alpha) < \sigma \left( 4 \sin^2 \frac{\pi}{k} \sin^2 \frac{\pi}{s} \right)
\]

for all \( \sigma: \mathbb{K} \to \mathbb{R} \) different from \( \sigma^{(+)} \), and

\[
4 < \sigma^{(+)}(\alpha) < 14^2.
\]

Every graph \( \Gamma_5^{(4)}(14) \) with parameters \( s, k \) contains subgraphs \( \Gamma_2^{(3)}(14) \) with parameters \( d = s \) and \( d = k \). Thus every field \( \mathbb{K} \in \mathcal{F} \Gamma_5^{(4)}(14) \) with parameters \( s, k \), \( 3 \leq s \leq k \), belongs to \( \mathcal{F} \Gamma_2^{(3)}(14) \) with parameter \( s \) and \( \mathcal{F} \Gamma_2^{(3)}(14) \) with parameter \( k \). Theorem 4.3 yields that \( [\mathbb{K} : \mathbb{Q}] \leq 20 \) if one of the numbers \( s, k \) does not belong to \( \{3, 7\} \). Assume that both numbers \( s, k \) belong to \( \{3, 7\} \).

Suppose that \( s = k = 3 \). Then \( 0 < \sigma(\alpha) < \frac{9}{4} \). Consider the polynomial

\[
P(x) = x(x - 1)^2(x - 2)^2(x^2 - 3x + 1).
\]

The maximum of \( |P(x)| \) on the interval \( [0, \frac{9}{4}] \) is equal to \( \delta = 0.21570956 \ldots \) By Lemma 4.1,

\[
[K : \mathbb{Q}] \leq 1 + \frac{\ln P(14^2)}{-\ln \delta} \leq 25.06
\]

and \( [K : \mathbb{Q}] \leq 25 \).

Suppose that \( s = 3, k = 7 \). In this case we have

\[
0 < \sigma(\alpha) < 3 \sigma \left( \sin^2 \frac{\pi}{7} \right)
\]

for \( \sigma: \mathbb{F}_7 = \mathbb{Q}(\cos^2 \frac{\pi}{7}) \to \mathbb{R} \). Consider the polynomial

\[
P(x) = x^2(x - 1)^2(x - 2)(x^2 - 3x + 1) \left( x + 1 - 4 \sin^2 \frac{\pi}{7} \right) \left( x - 4 \sin^2 \frac{\pi}{7} \right)
\]

over the ring of integers of the field \( \mathbb{F}_7 \). It may be regarded as an appropriate combination of the polynomials (4.8) for \( d = 3 \) and (4.14) for \( d = 7 \). For \( \sigma_k: \mathbb{F}_7 \to \mathbb{R} \) with \( \sigma_k(\sin^2 \frac{\pi}{7}) = \sin^2 \frac{k\pi}{7} \), \( k = 1, 2, 3 \), we obtain that the maxima \( \delta_k = \delta(\sigma_k) \), \( k = 1, 2, 3 \), of \( |P^{\sigma_k}(x)| \) on the intervals \( [0, 3\sigma_k(\sin^2 \frac{\pi}{7})] \) are respectively equal to \( \delta_1 = 0.0050709048 \ldots \), \( \delta_2 = 0.110711589 \ldots \), \( \delta_3 = 1.23817314 \ldots \). By Lemma 4.1,

\[
[K : \mathbb{F}_7] \leq \frac{\ln P(14^2) - \ln \delta_1}{-\ln(\delta_1 \delta_2 \delta_3)} \leq 7.3.
\]

Thus \( [K : \mathbb{F}_7] \leq 7 \) and \( [K : \mathbb{Q}] \leq 21 \).
Suppose that $s = k = 7$. Then $0 < \sigma(\alpha) < \sigma(4\sin^4\frac{\pi}{7})$. Consider the polynomial

$$P(x) = x(x - 1) \left( x + 1 - 4\sin^2\frac{\pi}{7} \right) \left( x - 4\sin^2\frac{\pi}{7} \right), \quad (4.39)$$

which is just $(4.14)$ with $d = 7$. In the notation above for $\sigma_k$, $k = 1, 2, 3$, the maxima of $|P^{\sigma_k}(x)|$ on the intervals $[0, \sigma_k(4\sin^4\frac{\pi}{7})]$ are respectively equal to $\delta_1 = 0.0289099468\ldots$, $\delta_2 = 0.52203650\ldots$, $\delta_3 = 2.93338845\ldots$. As above, $[K : F_7] \leq 7$ and $[K : \mathbb{Q}] \leq 21$.

We now get the final result.

**Theorem 4.8.** We have the following upper bounds for the degrees of ground fields $K \in \mathcal{F}\Gamma_5^{(4)}(14)$, that is, for $V$-arithmetic edge graphs $\Gamma_5^{(4)}$ of minimality 14 (for $2 < u < 14$), depending on the values of the parameters $s, k$ (see Fig. 8):

(i) $[K : \mathbb{Q}] \leq 25$ for $s = k = 3$;
(ii) $[K : \mathbb{Q}] \leq 21$ for $s = 3, k = 7$;
(iii) $[K : \mathbb{Q}] \leq 21$ for $s = k = 7$;
(iv) $[K : \mathbb{Q}] \leq 20$ for the other $s, k$, $3 \leq s \leq k$.

In particular, $[K : \mathbb{Q}] \leq 25$ for all $K \in \mathcal{F}\Gamma_5^{(4)}(14)$.

This theorem can be significantly improved by further use of the results and methods of our papers [2]–[4]. We hope to do this in future publications.

**4.7. Fields in $\mathcal{F}\Gamma_4^{(4)}(14)$.** For the graph $\Gamma_4^{(4)}(14)$ (Fig. 9) we assume that $k \geq 2$, $s, r \geq 3$ are integers, $u$ is a totally real algebraic integer and $2 < u < 14$. Moreover, there are only the following possibilities: $s = 3, r = 3, 4, 5$; $s = 4, 5, r = 3$.

![Figure 9. The graph $\Gamma_4^{(4)}$](image)

The ground field $K = \mathbb{Q}(u^2)$ contains the cyclic products

$$\cos^2\frac{\pi}{s}, \quad \cos^2\frac{\pi}{r}, \quad \cos^2\frac{\pi}{k}, \quad u^2, \quad u \cos\frac{\pi}{s} \cos\frac{\pi}{k}.$$ 

This case may be regarded as the specialization of the graph $\Gamma_1^{(4)}$ with $p = 2$. The determinant $d(u)$ of the Gram matrix is given by

$$\frac{-d(u)}{4} = u^2 + 4u \cos\frac{\pi}{s} \cos\frac{\pi}{k} + 4 \cos^2\frac{\pi}{s} - 4 \sin^2\frac{\pi}{k} \sin^2\frac{\pi}{r}.$$
We obtain
\[ \sigma \left( u^2 + 4u \cos \frac{\pi}{s} \cos \frac{\pi}{k} + 4 \cos^2 \frac{\pi}{s} - 4 \sin^2 \frac{\pi}{k} \sin^2 \frac{\pi}{r} \right) < 0 \quad (4.40) \]
for \( \sigma \neq \sigma^{(+)} \) and
\[ 4 < \sigma^{(+)}(u^2) < 14^2. \quad (4.41) \]

Suppose that \( k = 2 \). Then we have
\[ 0 < \sigma(u^2) < 4 - \sigma \left( 4 \cos^2 \frac{\pi}{s} \right) - \sigma \left( 4 \cos^2 \frac{\pi}{r} \right). \]

If \( s = r = 3 \), we get \( 0 < \sigma(u^2) < 2 \). Using the polynomial \((4.9)\) as above, we obtain that \([K : Q] \leq 16\).

If \( (s, r) = (3, 4) \) or \((4, 3)\), then we have \( 0 < \sigma(u^2) < 1 \). Using the polynomial \( P(x) = x(x - 1) \), we obtain that \([K : Q] \leq 8\).

If \( (s, r) = (3, 5) \) or \((5, 3)\), then we have \( 0 < \sigma(u^2) < \sigma \left( 4 \sin^2 \frac{\pi}{5} \right) - 1 \). Using the polynomial \( P(x) = x(x - 1)(x + 1 - 4 \sin^2 \frac{\pi}{5}) \), we obtain that \([K : F_5] \leq 5\) and \([K : Q] \leq 10\).

Suppose that \( k \geq 3 \). Then
\[ \tilde{u} = u^2 + 4u \cos \frac{\pi}{s} \cos \frac{\pi}{k} + 4 \cos^2 \frac{\pi}{s} \quad (4.42) \]
is a totally positive algebraic integer (since the minimum of this quadratic polynomial in \( u \) is equal to \( 4 \cos^2 \frac{\pi}{s} \sin^2 \frac{\pi}{k} \)) belonging to \( K \), and \( \Gamma_k^{(4)} \) is V-arithmetic if and only if
\[ 0 < \sigma \left( 4 \cos^2 \frac{\pi}{s} \sin^2 \frac{\pi}{k} \right) \leq \sigma(\tilde{u}) < \sigma \left( 4 \sin^2 \frac{\pi}{r} \sin^2 \frac{\pi}{k} \right) < 4 \quad (4.43) \]
for all \( \sigma : K \to \mathbb{R} \) different from the (identity) \( \sigma^{(+)} \). For \( \sigma^{(+)} \) we have
\[ 4 < 4 + 8 \cos \frac{\pi}{s} \cos \frac{\pi}{k} + 4 \cos^2 \frac{\pi}{s} < \sigma^{(+)}(\tilde{u}) < 14^2 + 56 \cos \frac{\pi}{s} \cos \frac{\pi}{k} + 4 \cos^2 \frac{\pi}{s} < 16^2. \quad (4.44) \]

It follows that \( K = Q(\tilde{u}) \).

In what follows we use only the inequalities
\[ 0 < \sigma(\tilde{u}) < \sigma \left( 4 \sin^2 \frac{\pi}{r} \sin^2 \frac{\pi}{k} \right) < 4, \quad 4 < \sigma^{(+)}(\tilde{u}) < 16^2. \]

This case is analogous to that of \( \Gamma_5^{(4)}(14) \) above, where we replace the upper bound \( 14^2 \) by \( 16^2 \) (or use \((4.44)\) if necessary) and put \( s = \min(r, k) \). One uses Theorem 4.4 instead of Theorem 4.3. The analogous bad cases are as follows.

When \( r = k = 3 \) we have \([K : Q] \leq 25\) (using the quantity \( 14^2 + 56 \cos \frac{\pi}{s} \cos \frac{\pi}{k} + 4 \cos^2 \frac{\pi}{s} \) instead of \( 14^2 \)).

When \( r = 3, k = 7 \) we have \([K : Q] \leq 21\).
We finally obtain the following result.

**Theorem 4.9.** We have the following upper bounds for the degrees of ground fields $\mathbb{K} \in \mathcal{F}_4^{(4)}(14)$, that is, for $V$-arithmetic edge graphs $\Gamma_4^{(4)}$ of minimality 14 (for $2 < u < 14$), depending on the values of the parameters $k$, $s$, $r$ (see Fig. 9):

(i) $[\mathbb{K} : \mathbb{Q}] \leq 16$ for $k = 2$, $s = r = 3$;
(ii) $[\mathbb{K} : \mathbb{Q}] \leq 8$ for $k = 2$, $(s, r) = (3, 4), (4, 3)$;
(iii) $[\mathbb{K} : \mathbb{Q}] \leq 10$ for $k = 2$, $(s, r) = (3, 5), (5, 3)$;
(iv) $[\mathbb{K} : \mathbb{Q}] \leq 25$ for $r = k = 3$;
(v) $[\mathbb{K} : \mathbb{Q}] \leq 21$ for $r = 3$, $k = 7$;
(vi) $[\mathbb{K} : \mathbb{Q}] \leq 20$ for the other values of $k$, $s$, $r$.

*In particular, $[\mathbb{K} : \mathbb{Q}] \leq 25$ for all $\mathbb{K} \in \mathcal{F}_4^{(4)}(14)$.*

This theorem can be significantly improved by further use of the results and methods of our papers [2]–[4]. We hope to do this in future publications.

The proof of Theorem 3.2 follows from what has been obtained above.

§ 5. Applications

Using [1]–[4], [7], [12], [17], [18], we get the following applications to arithmetic hyperbolic reflection groups.

**Theorem 5.1.** In dimensions $n \geq 10$ (of the hyperbolic space) the ground field of any arithmetic hyperbolic reflection group belongs to one of the finite sets of fields $\mathcal{F}L^4$, $\mathcal{F}T$ or $\mathcal{F}\Gamma_i^{(4)}(14)$, $1 \leq i \leq 5$. In particular, its degree does not exceed 25.

*Proof.* We showed in [1], [12] (for more details see [2]) that such a field belongs to $\mathcal{F}L^4$, $\mathcal{F}T$ or $\mathcal{F}\Gamma_i^{(4)}(14)$, $1 \leq i \leq 5$. Then its degree does not exceed 25 by Theorem 3.2 (see the proof of the following theorem).

**Theorem 5.2.** In dimensions $n \geq 6$, the ground field of an arithmetic hyperbolic reflection group belongs to one of the finite sets of fields $\mathcal{F}L^4$, $\mathcal{F}T$, $\mathcal{F}\Gamma_i^{(4)}(14)$, $1 \leq i \leq 5$, or $\mathcal{F}\Gamma_{2,4}(14)$. In particular, its degree does not exceed 25.

*Proof.* This is similar to the proof of Theorem 9 in [2], which says the same but with the bound 56.

As shown in [2], such a field belongs to $\mathcal{F}L^4$, $\mathcal{F}T$, $\mathcal{F}\Gamma_i^{(4)}(14)$, $1 \leq i \leq 5$, or $\mathcal{F}\Gamma_{2,4}(14)$. It is well known that the degree of any field in $\mathcal{F}L^4$ does not exceed 2 (see § 3.1). By [15] (see also [16], [19], [20]), the degrees of fields in $\mathcal{F}T$ do not exceed 5 (see § 3.2). We showed in [2] that the degrees of fields in $\mathcal{F}\Gamma_{2,4}(14)$ (the set of ground fields of arithmetic hyperbolic quadrangles of minimality 14) do not exceed 11. By Theorem 3.2, the degrees of fields in $\mathcal{F}\Gamma_i^{(4)}(14)$, $1 \leq i \leq 5$, do not exceed 25. The theorem is proved.

**Theorem 5.3.** In dimensions $n = 4, 5$, the ground field of an arithmetic hyperbolic reflection group belongs to one of the finite sets of fields $\mathcal{F}\Gamma_i^{(6)}(14)$, $\mathcal{F}\Gamma_i^{(6)}(14)$, $\mathcal{F}\Gamma_i^{(6)}(14)$, $\mathcal{F}\Gamma_i^{(7)}(14)$, $\mathcal{F}\Gamma_i^{(7)}(14)$ or $\mathcal{F}\Gamma_{2,5}(14)$ unless it belongs to one of the sets $\mathcal{F}L^4$, $\mathcal{F}T$, $\mathcal{F}\Gamma_i^{(4)}(14)$, $1 \leq i \leq 5$, and $\mathcal{F}\Gamma_{2,4}(14)$ in Theorem 5.2. In particular, its degree does not exceed 44.
Proof. This is similar to the proof of Theorem 2.6 in [3], which says the same but with the bound 138.

As shown in [3], such a field belongs to one of the sets $\mathcal{F} \Gamma_1^{(6)}(14)$, $\mathcal{F} \Gamma_2^{(6)}(14)$, $\mathcal{F} \Gamma_1^{(7)}(14)$, $\mathcal{F} \Gamma_2^{(7)}(14)$ or $\mathcal{F} \Gamma_{2,5}(14)$, and the degrees of the fields in $\mathcal{F} \Gamma_{2,5}(14)$ (the set of ground fields of arithmetic pentagons of minimality 14) do not exceed 12.

Suppose that the field belongs to one of the sets $\mathcal{F} \Gamma_1^{(6)}(14)$, $\mathcal{F} \Gamma_2^{(6)}(14)$, $\mathcal{F} \Gamma_3^{(6)}(14)$, $\mathcal{F} \Gamma_1^{(7)}(14)$ or $\mathcal{F} \Gamma_2^{(7)}(14)$. All the diagrams $\Gamma_1^{(6)}(14)$, $\Gamma_2^{(6)}(14)$, $\Gamma_3^{(6)}(14)$, $\Gamma_1^{(7)}(14)$ and $\Gamma_2^{(7)}(14)$ contain a subdiagram $\Gamma_2^{(3)}(14)$. Then the degree of the field does not exceed 44 by Theorem 4.5. The theorem is proved.

Theorem 5.4. In dimension $n = 3$, the ground field of an arithmetic hyperbolic reflection group belongs to one of the finite sets of fields $\mathcal{F} \Gamma_6^{(4)}(14)$, $\mathcal{F} \Gamma_1^{(5)}(14)$, $\mathcal{F} \Gamma_4^{(6)}(14)$ or $\mathcal{F} \Gamma_2(14)$ unless it belongs to one of the sets of fields $\mathcal{F} \Gamma_1^{(6)}(14)$, $\mathcal{F} \Gamma_2^{(6)}(14)$, $\mathcal{F} \Gamma_3^{(6)}(14)$, $\mathcal{F} \Gamma_1^{(7)}(14)$, $\mathcal{F} \Gamma_2^{(7)}(14)$, $\mathcal{F} \Gamma_{2,5}(14)$ and $\mathcal{F} \Gamma_{2,4}(14)$ in Theorems 5.2, 5.3. In particular, its degree does not exceed 44.

Proof. This is similar to the proof of Theorem 3.8 in [4], which says the same but with the bound 909.

As shown in [4], such a field belongs to one of the sets $\mathcal{F} \Gamma_6^{(4)}(14)$, $\mathcal{F} \Gamma_1^{(5)}(14)$, $\mathcal{F} \Gamma_4^{(6)}(14)$, $\mathcal{F} \Gamma_2(14)$.

The set $\mathcal{F} \Gamma_2(14)$ consists of the ground fields of arithmetic hyperbolic reflection groups acting on a hyperbolic plane with fundamental polygon of minimality 14. It was shown in [2] (using the results of [21]) that the degree of the ground field of an arithmetic hyperbolic reflection group on a hyperbolic plane does not exceed 44.

All the diagrams $\Gamma_6^{(4)}(14)$, $\Gamma_1^{(5)}(14)$, $\Gamma_4^{(6)}(14)$ contain a subdiagram $\Gamma_2^{(3)}(14)$. Then the degree of the field does not exceed 44 by Theorem 4.5. The theorem is proved.

We remark that Theorem 5.4 also gives a proof of the finiteness of the number of arithmetic reflection groups in dimension 3 different from the proof in [8].

It was shown in [5] (using the results of [8]) that the degrees of the ground fields of arithmetic hyperbolic reflection groups in dimension 3 do not exceed 35. It was also shown in [6] that the degrees of the ground fields of arithmetic hyperbolic reflection groups in dimension 2 do not exceed 11.

The results of [5]–[7] enable us to improve the upper bound 44 in Theorem 5.3 for dimensions 4, 5.

Theorem 5.5. In dimensions $n = 4, 5$, the ground field of an arithmetic hyperbolic reflection group belongs to one of the sets $\mathcal{F} \Gamma_i^{(4)}(14)$, $\mathcal{F} \Gamma_i^{(5)}(14)$, $\mathcal{F} \Gamma_i^{(6)}(14)$, $\mathcal{F} \Gamma_i^{(7)}(14)$, $\mathcal{F} \Gamma_{2,5}(14)$ and $\mathcal{F} \Gamma_{2,4}(14)$ in Theorems 5.2, 5.3. In particular, its degree does not exceed 35.

Proof. The first assertion is proved in [7].

The degrees of fields in $\mathcal{F} \Gamma_i^{(4)}$, $\mathcal{F} \Gamma_i^{(5)}$ do not exceed 5 (see §§3.1, 3.2). By Theorem 3.2, the degrees of fields in $\mathcal{F} \Gamma_i^{(4)}(14)$, $1 \leq i \leq 5$, do not exceed 25.
By [5], the degrees of fields in dimension \( n = 3 \) do not exceed 35. By [6], the degrees of fields in dimension \( n = 2 \) do not exceed 11. The theorem is proved.

Theorems 4.5, 4.3 may be regarded as a step towards the classification of arithmetic hyperbolic reflection groups over an arbitrary field.

**Theorem 5.6.** Given an arithmetic hyperbolic reflection group, assume that a narrow face of minimality 14 (which exists by [12]) of the fundamental chamber contains an edge such that the hyperbolic connected component of the graph of this edge has a subgraph \( \Gamma_2^{(3)} \) with parameter \( d \) (see Fig. 4). Then \( d \leq 120 \). If \( d \geq 44 \), then the ground field of the reflection group is \( F_d = \mathbb{Q}(\cos^2 \frac{\pi}{d}) \). (See Theorem 4.3 for more precise results.)

**Remark 5.1.** In our paper [2] we used Theorems 4.1, 4.2 in some rare exceptional cases. But an elementary arithmetical inaccuracy led to a slightly incorrect condition instead of the correct condition (4.1). This gave us better bounds than they really are. Thus we actually proved in [2] that \( N(14) \leq 120 \) instead of the claimed bound \( N(14) \leq 56 \). (In the present paper we have shown that \( N(14) \leq 25 \) and thus corrected the inaccuracy.) The same applies to our papers [3] for the dimensions \( n = 4, 5 \) and [4] for \( n = 3 \). But the upper bounds for the degree \( N \) claimed in these papers are already sufficiently high: \( N \leq 138 \) in [3] and \( N \leq 909 \) in [4]. Therefore our use of the slightly incorrect condition instead of (4.1) did not influence the final bounds in those papers. (In the present paper we have shown that \( N \leq 44 \) in dimensions \( n = 3, 4, 5 \).)

### § 6. Appendix: Hyperbolic numbers

Here we review and adjust the results of [1], § 1 (see Theorems 1.1.1, 1.2.2 in [1], which are analogous to Theorems 6.2, 6.4 of the present paper).

#### 6.1. Fekete’s theorem.

The following important theorem, to which this subsection is devoted, was obtained by Fekete [22]. Although he considered the case of \( \mathbb{Q} \) (as far as we know), his method of proof immediately extends to totally real algebraic number fields.

**Theorem 6.1** (Fekete). Suppose that \( F \) is a totally real algebraic number field and to every embedding \( \sigma : F \to \mathbb{R} \) there corresponds an interval \([a_\sigma, b_\sigma]\) in \( \mathbb{R} \) and a real number \( \lambda_\sigma > 0 \). Suppose that \( \prod_\sigma \lambda_\sigma = 1 \). Then for every non-negative integer \( n \) there is a non-zero polynomial \( P_n(T) \in \mathcal{O}[T] \) of degree at most \( n \) over the ring \( \mathcal{O} \) of integers of \( F \) such that the following inequality holds for each \( \sigma \):

\[
\max_{x \in [a_\sigma, b_\sigma]} |P_\sigma^n(x)| \leq \lambda_\sigma |\text{discr } F|^{1/(2[F:Q])} 2^{n/(n+1)} (n+1) \left( \prod_{\sigma'} \frac{b_{\sigma'} - a_{\sigma'}}{4} \right)^{n/(2[F:Q])}.
\]

**Proof.** Put \( N = [F : \mathbb{Q}] \) and let \( \gamma_1, \ldots, \gamma_N \) be a basis of \( \mathcal{O} \) over \( \mathbb{Z} \). Consider a non-zero polynomial

\[
P_n(T) = \sum_{i=0}^n \sum_{j=1}^N \alpha_{ij} \gamma_j T^i \in \mathcal{O}[T]
\]
of degree at most \( n \), where \( \alpha_{ij} \in \mathbb{Z} \) are not all zero. For every \( \sigma : \mathbb{F} \to \mathbb{R} \) we consider the real functions \( P_n^\sigma(x) \) on the interval \( [a_\sigma, b_\sigma] \).

We make the change of variables

\[
x = x(z) = \frac{b_\sigma + a_\sigma}{2} + \frac{b_\sigma - a_\sigma}{2} \cos z.
\]

If \( z \) runs through the interval \([0, \pi]\), then \( x \) runs through \([a_\sigma, b_\sigma]\). We put \( Q_n^\sigma(z) = P_n^\sigma(x(z)) \). Since \( Q_n^\sigma(z) \) is an even trigonometric polynomial, we have

\[
Q_n^\sigma(z) = \sum_{k=0}^n A_{k\sigma} \cos(kz), \quad (6.2)
\]

where

\[
A_{k\sigma} = \frac{1}{\pi} \int_{-\pi}^{\pi} P_n^\sigma \left( \frac{b_\sigma + a_\sigma}{2} + \frac{b_\sigma - a_\sigma}{2} \cos z \right) \cos(kz) \, dz
\]

\[
= \sum_{i=0}^n \sum_{j=1}^N \left( \frac{1}{\pi} \int_{-\pi}^{\pi} \gamma_j^\sigma \left( \frac{b_\sigma + a_\sigma}{2} + \frac{b_\sigma - a_\sigma}{2} \cos z \right)^i \cos(kz) \, dz \right) \alpha_{ij}
\]

for \( k \geq 1 \), and

\[
A_{0\sigma} = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_n^\sigma \left( \frac{b_\sigma + a_\sigma}{2} + \frac{b_\sigma - a_\sigma}{2} \cos z \right) \, dz
\]

\[
= \sum_{i=0}^n \sum_{j=1}^N \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \gamma_j^\sigma \left( \frac{b_\sigma + a_\sigma}{2} + \frac{b_\sigma - a_\sigma}{2} \cos z \right)^i \, dz \right) \alpha_{ij}.
\]

Thus

\[
A_{k\sigma} = \sum_{i=0}^n \sum_{j=1}^N c_{k\sigma ij} \alpha_{ij} \quad (6.3)
\]

are linear functions of \( \alpha_{ij} \), where

\[
c_{k\sigma ij} = \gamma_j^\sigma \left( \frac{b_\sigma + a_\sigma}{2} + \frac{b_\sigma - a_\sigma}{2} \right)^i \cos(kz) \, dz
\]

for \( k \geq 1 \), and

\[
c_{0\sigma ij} = \gamma_j^\sigma \left( \frac{b_\sigma + a_\sigma}{2} + \frac{b_\sigma - a_\sigma}{2} \cos z \right)^i \, dz.
\]

We observe from these formulae that \( c_{k\sigma ij} = 0 \) if \( i < k \), and

\[
c_{k\sigma kj} = \gamma_j^\sigma \cdot 2 \left( \frac{b_\sigma - a_\sigma}{4} \right)^k \quad \text{if} \quad k \geq 1
\]

\((c_{0\sigma ij} = \gamma_j^\sigma)\). Hence, by ordering the subscripts \( k\sigma \) and \( ij \) lexicographically, we see that the matrix of the linear forms (6.3) is upper block-triangular with the
above $N \times N$ matrices $(c_{0\sigma 0j})$ and $(c_{k\sigma kj})$, $1 \leq k \leq n$, on the diagonal. Hence the determinant of the matrix (6.3) takes the form

$$\Delta = \det(\gamma^\sigma_{j})^{n+1} \cdot 2^{Nn} \left( \prod_{\sigma} \frac{b_{\sigma} - a_{\sigma}}{4} \right)^{n(n+1)/2}.$$  

Since $\prod_{\sigma} \lambda_{\sigma}^{n+1} = (\prod_{\sigma} \lambda_{\sigma})^{n+1} = 1$, it follows from Minkowski’s theorem on linear forms (see, for example, [23], [24]) there are $\alpha_{ij} \in \mathbb{Z}$, not all equal to zero, such that $|A_k\sigma| \leq \lambda_{\sigma} |\Delta|^{1/(N(n+1))}$. Hence, by (6.2),

$$\max_{z} |Q^\sigma_{n}(z)| \leq \lambda_{\sigma}(n + 1)|\Delta|^{1/(N(n+1))}.$$

Since $\det(\gamma^\sigma_{j})^{2} = \text{discr} \mathbb{F}$, the theorem is proved.

Taking $\lambda_{j} = 1$, we get a particular case to be used below.

**Theorem 6.2** (Fekete). Suppose that $\mathbb{F}$ is a totally real algebraic number field and to every embedding $\sigma: \mathbb{F} \to \mathbb{R}$ there corresponds an interval $[a_{\sigma}, b_{\sigma}]$ in $\mathbb{R}$. Then for every non-negative integer $n$ there is a non-zero polynomial $P_{n}(T) \in \mathbb{O}[T]$ of degree at most $n$ over the ring $\mathbb{O}$ of integers of $\mathbb{F}$ such that the following inequality holds for each $\sigma$:

$$\max_{x \in [a_{\sigma}, b_{\sigma}]} |P^\sigma_{n}(x)| \leq |\text{discr} \mathbb{F}|^{1/(2[\mathbb{F}:\mathbb{Q}])} \cdot 2^{n/(n+1)}(n+1) \left( \prod_{\sigma'} \frac{b_{\sigma'} - a_{\sigma'}}{4} \right)^{n/(2[\mathbb{F}:\mathbb{Q}])}. \quad (6.4)$$

6.2. Hyperbolic numbers. The totally real algebraic numbers $\{\alpha\}$ to be considered here are very similar to the Pisot–Vijayaraghavan numbers [24], although the latter are not totally real.

**Theorem 6.3.** Suppose that $\mathbb{F}$ is a totally real algebraic number field and to every embedding $\sigma: \mathbb{F} \to \mathbb{R}$ there corresponds an interval $[a_{\sigma}, b_{\sigma}]$ in $\mathbb{R}$ with

$$\prod_{\sigma} \frac{b_{\sigma} - a_{\sigma}}{4} < 1.$$  

We also fix a positive integer $m$ and intervals $[s_1, t_1], \ldots, [s_m, t_m]$ in $\mathbb{R}$. Then there is a constant $N(s_i, t_i)$ such that if $\alpha$ is a totally real algebraic integer and the following inequalities hold for the embeddings $\tau: \mathbb{F}(\alpha) \to \mathbb{R}$:

$$s_i \leq \tau(\alpha) \leq t_i, \quad \tau = \tau_1, \ldots, \tau_m;$$

$$a_{\tau[\mathbb{P}]} \leq \tau(\alpha) \leq b_{\tau[\mathbb{P}]}, \quad \tau \neq \tau_1, \ldots, \tau_m;$$

then

$$[\mathbb{F}(\alpha) : \mathbb{F}] \leq N(s_i, t_i).$$

**Theorem 6.4.** Under the hypotheses of Theorem 6.3 one can choose $N(s_i, t_i)$ in the form $N(s_i, t_i) = N(S)$, where $N(S)$ is the smallest positive integer solution $n$ of the inequality

$$n \ln \frac{1}{R} - M \ln(2n + 2) - \ln B \geq \ln S. \quad (6.5)$$
Here

\[ M = [F : \mathbb{Q}], \quad R = \sqrt{\prod_{\sigma} \frac{b_{\sigma} - a_{\sigma}}{4}}, \quad \text{(6.6)} \]

\[ B = \sqrt{|\text{discr} F|}, \quad S = \prod_{i=1}^{n} (2er_{i}(b_{\sigma_{i}} - a_{\sigma_{i}})^{-1}), \quad \text{(6.7)} \]

where \( \sigma_{i} = \tau_{i} | F \), \( r_{i} = \max \{|t_{i} - a_{\sigma_{i}}|, |b_{\sigma_{i}} - s_{i}|\} \). We asymptotically have

\[ N(s_{i}, t_{i}) \sim \frac{\ln S}{\ln R}. \]

**Proofs of Theorems 6.3, 6.4.** We shall use the following assertion.

**Lemma 6.1.** Suppose that \( Q_{n}(T) \in \mathbb{R}[T] \) is a non-zero polynomial over \( \mathbb{R} \) of degree at most \( n > 0 \), \( a < b \) and \( M_{0} = \max_{[a,b]} \{|Q_{n}(x)|\} \). Then, for \( x \geq b \),

\[ |Q_{n}(x)| \leq \frac{M_{0}(x-a)^{n}n^{n}}{(b-a)^{n}n!} < \frac{M_{0}(x-a)^{n}e^{n}}{(b-a)^{n}\sqrt{2\pi n}} < \frac{M_{0}(x-a)^{n}e^{n}}{(b-a)^{n}}. \]

**Proof.** Let \( \alpha_{0} < \alpha_{1} < \cdots < \alpha_{n} \). By the Lagrange interpolation formula,

\[ Q_{n}(x) = \sum_{i=0}^{n} Q_{n}(\alpha_{i})F_{i}(x), \]

where

\[ F_{i}(x) = \frac{(x - \alpha_{0})(x - \alpha_{1})\cdots(x - \alpha_{i-1})(x - \alpha_{i+1})\cdots(x - \alpha_{n})}{(\alpha_{i} - \alpha_{0})(\alpha_{i} - \alpha_{1})\cdots(\alpha_{i} - \alpha_{i-1})(\alpha_{i} - \alpha_{i+1})\cdots(\alpha_{i} - \alpha_{n})}. \]

Taking \( \alpha_{i} = a + \frac{i(b-a)}{n} \), \( 0 \leq i \leq n \), we obtain for \( x \geq b \) that

\[ |Q_{n}(x)| \leq \frac{M_{0}(x-a)^{n}}{(b-a)^{n}n!} \sum_{i=0}^{n} \frac{1}{i!(n-i)!} = \frac{M_{0}(x-a)^{n}2^{n}}{(b-a)^{n}}. \]

By Stirling’s formula, \( n! = \sqrt{2\pi n} \left( \frac{n}{e} \right)^{n}, \) where \( 0 < \lambda_{n} < \frac{1}{12n} \). Hence \( \frac{n^{n}}{n!} < \frac{e^{n}}{\sqrt{2\pi n}} < e^{n} \). This proves the lemma.

We return to proving the theorems. For any given \( n \) we consider the polynomial \( P_{n}(T) \in \mathbb{Q}[T] \) whose existence is ensured by Theorem 6.2. Putting \( N = [F(\alpha) : F] \) and \( M = [F : \mathbb{Q}] \), we use Theorem 6.2 and Lemma 6.1 to conclude that

\[ \left| \prod_{\tau} \tau(P_{n}(\alpha)) \right| = \prod_{\tau} |P_{n}^{\tau}(\tau(\alpha))| = \prod_{\tau \neq \tau_{1}} P_{n}^{\tau}(\tau(\alpha)) \prod_{i=1}^{m} |P_{n}^{\tau_{i}}(\tau_{i}(\alpha))| \]

\[ \leq \prod_{\tau \neq \tau_{1}} \max_{[a_{\tau}, b_{\tau}]} \left| P_{n}^{\tau}(x) \right| \prod_{i=1}^{m} \max_{[s_{i}, t_{i}]} \left| P_{n}^{\tau_{i}}(x) \right| \]

\[ \leq (|\text{discr} F|^{1/(2M)} \cdot 2(n+1)R^{n/M}) N^{M} \prod_{i=1}^{m} \frac{r_{i}^{n}e^{n}}{\left( \frac{b_{\sigma_{i}} - a_{\sigma_{i}}}{2} \right)^{n}} \]

\[ = R^{nN} B^{N} S^{n}(2n + 2)^{MN}. \quad \text{(6.8)} \]
Since $R < 1$, there is a sufficiently large $n_0$ such that

$$R^{n_0} B(2n_0 + 2)^M \leq \frac{1}{S}. \tag{6.9}$$

If $N > n_0$, we have

$$R^{n_0 N} B^N S^{n_0} (2n_0 + 2)^{MN} \leq S^{n_0 - N} < 1$$

since $S > 1$. Using this and the chain of inequalities (6.8), we get

$$\left| \prod \tau(\alpha) \right| < 1.$$

But

$$\prod \tau(\alpha) = N_{F(\alpha)/Q}(P_{n_0}(\alpha)) \in \mathbb{Z},$$

whence $P_{n_0}(\alpha) = 0$. Therefore $N \leq n_0$, a contradiction.

We have thus proved that $N \leq n_0$, where $n_0$ is a positive integer solution of inequality (6.9). Clearly, (6.9) is equivalent to the inequality

$$n_0 \ln \frac{1}{R} - M \ln (2n_0 + 2) - \ln B \geq \ln S.$$

This completes the proof of the theorem.

**Bibliography**

[1] V. V. Nikulin, “On the classification of arithmetic groups generated by reflections in Lobachevsky spaces”, *Izv. Akad. Nauk SSSR Ser. Mat.* 45:1 (1981), 113–142; English transl., *Math. USSR-Izv.* 18:1 (1982), 99–123.

[2] V. V. Nikulin, “On ground fields of arithmetic hyperbolic reflection groups”, *Groups and symmetries* (Montreal, Canada 2007), CRM Proc. Lecture Notes, vol. 47, Amer. Math. Soc., Providence, RI 2009, pp. 299–326.

[3] V. V. Nikulin, “On ground fields of arithmetic hyperbolic reflection groups. II”, *Mosc. Math. J.* 8:4 (2008), 789–812.

[4] V. V. Nikulin, “On ground fields of arithmetic hyperbolic reflection groups. III”, *J. Lond. Math. Soc.* (2) 79:3 (2009), 738–756.

[5] M. Belolipetsky, “On fields of definition of arithmetic Kleinian reflection groups”, *Proc. Amer. Math. Soc.* 137:3 (2009), 1035–1038.

[6] C. Maclachlan, “Bounds for discrete hyperbolic arithmetic reflection groups in dimension 2”, *Bull. Lond. Math. Soc.* 43:1 (2011), 111–123.

[7] V. V. Nikulin, “Finiteness of the number of arithmetic groups generated by reflections in Lobachevsky spaces”, *Izv. Ross. Akad. Nauk Ser. Mat.* 71:1 (2007), 55–60; English transl., *Izv. Math.* 71:1 (2007), 53–56.

[8] I. Agol, “Finiteness of arithmetic Kleinian reflection groups”, *Proceedings of the international congress of mathematicians*, vol. II (Madrid, Spain 2006), Eur. Math. Soc., Zürich 2006, pp. 951–960.

[9] V. V. Nikulin, *The transition constant for arithmetic hyperbolic reflection groups*, arXiv:0910.5217.
The transition constant for hyperbolic groups

1005

E. B. Vinberg, “Discrete groups generated by reflections in Lobachevskii spaces”, Mat. Sb. 72 (114):3 (1967), 471–488; English transl., Math. USSR-Sb. 1:3 (1967), 429–444.

E. B. Vinberg, “Hyperbolic reflection groups”, Uspekhi Mat. Nauk 40:1 (1985), 29–66; English transl., Russian Math. Surveys 40:1 (1985), 31–75.

V. V. Nikulin, “On arithmetic groups generated by reflections in Lobachevsky spaces”, Izv. Akad. Nauk SSSR Ser. Mat. 44:3 (1980), 637–669; English transl., Math. USSR-Izv. 16:3 (1981), 573–601.

F. Lannér, “On complexes with transitive groups of automorphisms”, Comm. Sém., Math. Univ. Lund 11 (1950), 1–71.

E. B. Vinberg, “The non-existence of crystallographic groups of reflections in Lobachevskij spaces of large dimension”, Trudy Moskov. Mat. Obshch., vol. 47, Moscow Univ. Press, Moscow 1984, pp. 68–102; English transl., Trans. Moscow Math. Soc., 1985, 75–112.

K. Takeuchi, “Arithmetic triangle groups”, J. Math. Soc. Japan 29:1 (1977), 91–106.

K. Takeuchi, “Commensurability classes of arithmetic triangle groups”, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 24:1 (1977), 201–212.

V. V. Nikulin, “Discrete reflection groups in Lobachevsky spaces and algebraic surfaces”, Proceedings of the International Congress of Mathematicians, vol. 1 (Berkeley, CA 1986), Amer. Math. Soc., Providence, RI 1987, pp. 654–671.

V. V. Nikulin, The remark on arithmetic groups generated by reflections in Lobachevsky spaces, Preprint MPI/91 Max-Planck-Institut für Mathematik, Bonn 1991.

K. Takeuchi, “A characterization of arithmetic Fuchsian groups”, J. Math. Soc. Japan 27:4 (1975), 600–612.

K. Takeuchi, “Arithmetic Fuchsian groups with signature (1, e)”, J. Math. Soc. Japan 35:3 (1983), 381–407.

D. D. Long, C. Maclachlan, and A. W. Reid, “Arithmetic Fuchsian groups of genus zero”, Pure Appl. Math. Q. 2:2 (2006), 569–599.

M. Fekete, “Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten”, Math. Z. 17:1 (1923), 228–249.

Z. I. Borevich and I. R. Shafarevich, Number theory, 3rd ed., Nauka, Moscow 1985; English transl. of 1st ed., Academic Press, New York–London 1966.

J. W. S. Cassels, An introduction to Diophantine approximation, Cambridge Univ. Press, New York 1957; Russian transl., Inostr. Lit., Moscow 1961.

V. V. Nikulin

Department of Pure Mathematics,
The University of Liverpool
Steklov Mathematical Institute, RAS
E-mail: vnikulin@liv.ac.uk, vnikulin@list.ru

Received 4/AUG/10
Translated by THE AUTHOR