ON DISCRETENESS OF SUBGROUPS IN RANK ONE

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Abstract. We prove discreteness criteria for Zariski dense subgroups of Sp(n, 1). Discreteness criteria for Zariski-dense subgroups of SU(n, 1) also follow from this.

We further revisit the Jørgensen type inequality of Cao and Parker in Q. J. Math., 62(3):523–543, 2011 which gives a necessary condition for discreteness of a two generator non-elementary subgroup of Sp(n, 1) where one of the generators is loxodromic. We show that the inequality of Cao and Parker is strict for such discrete subgroups.

1. Introduction

Let \( \mathbb{H} \) be the division ring of Hamilton’s quaternions. Let \( \mathbb{H}^n_F \) denote the \( n \)-dimensional hyperbolic space over \( F = \mathbb{H} \) or \( \mathbb{C} \). Let Sp\((n, 1)\), resp. SU\((n, 1)\), be the linear group that acts on \( \mathbb{H}^n_{\mathbb{H}} \), resp. \( \mathbb{H}^n_{\mathbb{C}} \), by isometries. It is a standard fact that the only rank one symmetric spaces of non-compact type with linear isometric group actions are the real, complex and quaternionic hyperbolic spaces. It is an old problem to obtain discreteness criteria in the respective isometry groups. Discreteness problem in the Möbius group, i.e. in the orientation-preserving isometry group of the real hyperbolic space, has seen extensive investigations in the literature, e.g. \([AH90]\), \([Bea95]\), \([Mar89]\), \([WLC05]\), \([LW09]\). In this paper we investigate discreteness criteria for subgroups of Sp\((n, 1)\) and SU\((n, 1)\). We shall mostly work in the group Sp\((n, 1)\). However, the methods restrict to the complex numbers and give us analogous results for SU\((n, 1)\). In the following we summarize the main results.

Recall that an element \( g \in \text{Sp}(n, 1) \) is elliptic if it has a fixed point on \( \mathbb{H}^n_{\mathbb{H}} \). It is parabolic, resp. loxodromic (or hyperbolic), if it has a unique fixed point, resp. exactly two fixed points on the boundary \( \partial \mathbb{H}^n_{\mathbb{H}} \). A parabolic element having all eigenvalues 1 is called a Heisenberg translation. It is well-known, see \([CG74]\), that an elliptic or loxodromic isometry is conjugate to a diagonal element in Sp\((n, 1)\). One may associate certain conjugacy invariants to elliptic and loxodromic elements as follows.

A loxodromic element in Sp\((n, 1)\) is conjugate to a matrix of the form

\[
g = \text{diag}(\lambda_1, \lambda^{-1}_1, \lambda_3, \ldots, \lambda_{n+1}),
\]

where \(|\lambda_1| > 1\), and \(|\lambda_i| = 1\) for \( i = 3, \ldots, n + 1 \). Cao and Parker defined the following conjugacy invariant in \([CP11]\):

\[
\delta_{cp}(g) = \max\{|\lambda_i - 1| : i = 3, \ldots, n + 1\},
\]

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\[ M_g = 2\delta_{cp}(g) + |\lambda_1 - 1| + |\bar{\lambda}_1^{-1} - 1|. \]

An eigenvalue \( \lambda \) of a matrix in \( \text{Sp}(n,1) \) is called negative-type or positive-type according as the Hermitian length of the corresponding eigenvector is negative or positive. An elliptic element in \( \text{Sp}(n,1) \) is conjugate to a matrix of the form

\[(1.2) \quad g = \text{diag}(\lambda_1, \ldots, \lambda_{n+1}),\]

where for all \( i \), \( |\lambda_i| = 1 \), and we choose the underlying Hermitian form so that \( \lambda_1 \) is a negative-type eigenvalue and all others are positive-type eigenvalues. Define

\[(1.3) \quad \delta(g) = \max\{ |\lambda_1 - 1| + |\lambda_i - 1| : i = 2, \ldots, n + 1 \}.\]

Clearly, \( \delta(g) \) is an invariant of the conjugacy class of the elliptic element \( g \).

Let \( T_{s,\zeta} \) be a Heisenberg translation in \( \text{Sp}(n,1) \). We may assume (cf. [CG74, p. 70]) that up to conjugacy,

\[(1.4) \quad T_{s,\zeta} = \begin{pmatrix} 1 & 0 & 0 \\ s & 1 & \zeta^* \\ \zeta & 0 & 1 \end{pmatrix},\]

where \( \text{Re}(s) = \frac{1}{2} |\zeta|^2 \).

1.1. **Discreteness using a test map.** A subgroup \( G \) of \( \text{Sp}(n,1) \) is called Zariski dense if it does not fix a point on \( \mathbb{H}^n \cup \partial \mathbb{H}^n \), and neither it preserves a totally geodesic subspace of \( \mathbb{H}^n \). We call an elliptic element regular if it has mutually distinct classes of eigenvalues. We prove the following.

**Theorem 1.1.** Let \( G \) be Zariski dense in \( \text{Sp}(n,1) \).

1. Let \( g \in \text{Sp}(n,1) \) be a regular elliptic element such that \( g \) does not have a negative type eigenvalue 1 or \(-1\), and \( \delta(g) < 1 \). If \( \langle g, h \rangle \) is discrete for every loxodromic element \( h \in G \), then \( G \) is discrete.
2. Let \( g \in \text{Sp}(n,1) \) be a loxodromic element such that \( M_g < 1 \). If \( \langle g, h \rangle \) is discrete for every loxodromic element \( h \in G \), then \( G \) is discrete.
3. Let \( g \in \text{Sp}(n,1) \) be a Heisenberg translation such that \( |\zeta| < \frac{1}{2} \). If \( \langle g, h \rangle \) is discrete for every loxodromic element \( h \in G \), then \( G \) is discrete.

Restricting everything to the complex numbers, as a by-product of the proof of the above theorem, the following holds.

**Corollary 1.2.** Let \( G \) be Zariski dense in \( \text{SU}(n,1) \).

1. Let \( g \in \text{SU}(n,1) \) be a regular elliptic element such that \( g \) does not have a negative type eigenvalue 1 or \(-1\), and \( \delta(g) < 1 \). If \( \langle g, h \rangle \) is discrete for every loxodromic element \( h \in G \), then \( G \) is discrete.
2. Let \( g \in \text{SU}(n,1) \) be loxodromic element such that \( M_g < 1 \). If \( \langle g, h \rangle \) is discrete for every loxodromic element \( h \in G \), then \( G \) is discrete.
3. Let \( g \in \text{SU}(n,1) \) be a Heisenberg translation such that \( |\zeta| < \frac{1}{2} \). If \( \langle g, h \rangle \) is discrete for every loxodromic element \( h \in G \), then \( G \) is discrete.

We also have the following result which follows using similar arguments as in the proof of Theorem 1.1.

**Corollary 1.3.** Let \( G \) be Zariski dense in \( \text{Sp}(n,1) \), resp. \( \text{SU}(n,1) \).
(1) Let \( g \in \text{Sp}(n,1) \), resp. \( \text{SU}(n,1) \), be a regular elliptic element such that \( \delta(g) < 1 \). If \( \langle g, h \rangle \) is discrete and non-elementary for every regular elliptic element \( h \in G \), then \( G \) is discrete.

(2) Let \( g \in \text{Sp}(n,1) \), resp. \( \text{SU}(n,1) \), be loxodromic element such that \( M_g < 1 \). If \( \langle g, h \rangle \) is discrete for every regular elliptic \( h \in G \), then \( G \) is discrete.

(3) Let \( g \in \text{Sp}(n,1) \), resp. \( \text{SU}(n,1) \), be a Heisenberg translation such that \( |\zeta| < \frac{1}{2} \). If \( \langle g, h \rangle \) is discrete for every regular elliptic \( h \in G \), then \( G \) is discrete.

We obtain the above results by using certain versions of the Jørgensen inequality in \( \text{Sp}(n,1) \). One of these inequalities was obtained by Cao and Parker in \cite{CP11}, see Theorem CP below, and the others are straight-forward adaptations of the inequalities in \cite{CT10} and \cite{HP96} respectively.

Note that the above results show that the discreteness of \( G \) is determined by the two generator subgroups \( \langle g, h \rangle \), where \( h \in G \), but the generator \( g \) is fixed and need not be an element from \( G \). After fixing such a generator, loxodromic, resp. regular elliptic, elements from \( G \) are enough to determine the discreteness. We note here that discreteness of Möbius subgroups using such a ‘test map’ \( g \) has been studied in \cite{Che04}, \cite{Yan09}, \cite{Cao12}, \cite{YZ14}, \cite{GMS18}. Our results may be thought of as counterparts of some of these works in the complex and quaternionic hyperbolic geometries.

A version of Corollary 1.2 has been obtained by Qin and Jiang in \cite{QJ12}. Qin and Jiang applied an inequality of Friedland and Hersonsky from \cite{FH93} that is valid for subgroups of automorphisms of a normed linear algebra over complex numbers. We have applied a different set of generalized Jørgensen inequalities as mentioned above, and accordingly, the above corollary gives some quantitative bounds for the test maps. These bounds may be useful for algorithmic purposes to test discreteness of a subgroup. In the discreteness criteria of Qin and Jiang, only regular elliptic elements of finite order were considered as test maps. In Corollary 1.2 we have no restriction on the order of the regular elliptic test map.

1.2. Extremality of the Cao-Parker inequality. We further revisit the Jørgensen inequality of Cao and Parker in \cite{CP11}. Recall that the quaternionic cross ratio of four distinct points \( z_1, z_2, z_3, z_4 \) on \( \partial H^n_\mathbb{H} \) is defined as:

\[
[z_1, z_2, z_3, z_4] = \langle z_3, z_1 \rangle \langle z_3, z_2 \rangle^{-1} \langle z_4, z_2 \rangle \langle z_4, z_1 \rangle^{-1},
\]

where \( z_i \) denote the lift to \( \mathbb{H}^{n+1} \) of a point \( z_i \) on \( \partial H^n_\mathbb{H} \). Now, Cao and Parker’s theorem may be stated as follows.

**Theorem CP.** (Cao and Parker) \cite{CP11} Let \( g \) and \( h \) be elements of \( \text{Sp}(n,1) \) such that \( g \) is loxodromic element with fixed points \( u, v \in \partial H^n_\mathbb{H} \), and \( M_g < 1 \). If \( \langle g, h \rangle \) is non-elementary and discrete, then

\[
|h(u), u, v, h(v)|^\frac{1}{2} |h(u), v, u, h(v)|^\frac{1}{2} \geq 1 - \frac{1}{M^2_g}.
\]

The above result also holds for \( \text{SU}(n,1) \) and stronger than earlier results in similar direction. We investigate what happens if the above inequality is an equality. We prove the following.

**Theorem 1.4.** Let \( g, h \) in \( \text{Sp}(n,1) \) be as in the above theorem. Then equality can not hold in (1.5).
Combining Theorem CP and Theorem 1.3 we now have the following analogue of the Jørgensen’s inequality for non-elementary two-generator subgroups of $\text{Sp}(n, 1)$, where one of the generators is loxodromic.

**Theorem 1.5.** Suppose $g$ and $h$ are elements of $\text{Sp}(n, 1)$ such that $g$ is a loxodromic element with fixed points $u, v \in \partial \mathbb{H}_n$ and $M_g < 1$. If

$$|\langle h(u), u, v, h(v) \rangle|^{\frac{1}{2}} |\langle h(u), v, u, h(v) \rangle|^{\frac{1}{2}} \leq \frac{1 - M_g}{M_g^2},$$

then the subgroup $\langle g, h \rangle$ is either elementary or not discrete.

In [WJC13], Wang, Jiang and Cao have obtained a generalized version of the Jørgensen inequality for two generator subgroups of $\text{SL}(2, \mathbb{C})$ where one of the generators is loxodromic. Wang, Jiang and Cao embed $\text{SL}(2, \mathbb{C})$ in $\text{Sp}(1, 1)$ and applied Theorem CP to obtain their result. Using local analysis over complex numbers, in [GMT19] it has been shown that the inequality of Wang, Jiang and Cao is strict. This also follows as an immediate corollary to Theorem 1.4.

**Structure of the paper.** In section 2 basic notions and preliminary results are noted. In section 3 we give the proof of Theorem 1.1 and Corollary 1.3. We prove Theorem 1.4 in section 4.

### 2. Preliminaries

**2.1. The quaternionic hyperbolic space.** We begin with some background material on quaternionic hyperbolic geometry. Much of this can be found in [CG74] [KP03].

Let $\mathbb{H}^{n,1}$ be the right vector space over $\mathbb{H}$ of quaternionic dimension $(n+1)$ (so real dimension $4n+4$) equipped with the quaternionic Hermitian form for $z = (z_0, \ldots, z_n), w = (w_0, \ldots, w_n)$,

$$\langle z, w \rangle = -(\bar{z}_0 w_1 + \bar{z}_1 w_0) + \sum_{i=2}^{n} \bar{z}_i w_i.$$

Thus the Hermitian form is defined by the matrix

$$J_2 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & I \end{pmatrix}.$$

Equivalently, one may also use the Hermitian form given by the following matrix wherever convenient.

$$J_1 = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix}.$$

Following Section 2 of [CG74], let

$$V_0 = \{ z \in \mathbb{H}^{n,1} - \{0\} : \langle z, z \rangle = 0 \} , \quad V_- = \{ z \in \mathbb{H}^{n,1} : \langle z, z \rangle < 0 \}.$$ 

It is obvious that $V_0$ and $V_-$ are invariant under $\text{Sp}(n, 1)$. We define an equivalence relation $\sim$ on $\mathbb{H}^{n,1}$ by $z \sim w$ if and only if there exists a non-zero quaternion $\lambda$ so that $w = z\lambda$. Let $[z]$ denote the equivalence class of $z$. Let $\mathbb{P} : \mathbb{H}^{n,1} - \{0\} \rightarrow \mathbb{H} \mathbb{P}^n$ be the right projection map given by $\mathbb{P} : z \mapsto [z]$. The $n$ dimensional quaternionic hyperbolic space is defined to be $\mathbb{H}_n^\mathbb{P} = \mathbb{P}(V_-)$ with boundary $\partial \mathbb{H}_n^\mathbb{P} = \mathbb{P}(V_0)$. 
In the model using $J_2$, there are two distinct points $0$ and $\infty$ on $\partial H^n$. For $z_1 \neq 0$, the projection map $\mathbb{P}$ is given by

$$\mathbb{P}(z_1, z_2, \ldots, z_{n+1}) = (z_2z_1^{-1}, \ldots, z_{n+1}z_1^{-1}),$$

and accordingly we choose boundary points

(2.1) \hspace{1cm} \mathbb{P}(0, 1, \ldots, 0, 0)^t = 0.

(2.2) \hspace{1cm} \mathbb{P}(1, 0, \ldots, 0, 0)^t = \infty.

In the model using $J_1$, we mark $\mathbb{P}(1, 0, \ldots, 0, 0)^t$ as the origin $0 = (0, 0, \ldots, 0)^t$ of the quaternionic hyperbolic ball.

Now consider the Lie group $Sp(n, 1) = \{ A \in GL(n + 1, H) : A^* J_2 A = J_2 \}$. An element $g \in Sp(n, 1)$ acts on $H^n = H^n \cup \partial H^n$ as $g(z) = \mathbb{P}g\mathbb{P}^{-1}(z)$. Thus the isometry group of $H^n$ is given by $PSp(n, 1) = Sp(n, 1)/\{I, -I\}$.

For a matrix (or a vector) $T$ over $H$, let $T^* = \overline{T}^t$. Let $A$ be an element in $Sp(n, 1)$. Then one can choose $A$ to be of the following form.

(2.3) \hspace{1cm} A = \begin{pmatrix} a & b & \gamma^* \\ c & d & \delta^* \\ \alpha & \beta & U \end{pmatrix},

where $a, b, c, d$ are scalars, $\gamma, \delta, \alpha, \beta$ are column matrices in $H^{n-1}$ and $U$ is an element in $M(n-1, H)$. Then, it is easy to compute that

$$A^{-1} = \begin{pmatrix} \bar{d} & \bar{b} & -\beta^* \\ \bar{c} & \bar{a} & -\alpha^* \\ -\delta & -\gamma & U^* \end{pmatrix}.$$ 

**Convention.** In this paper, to describe conjugacy types of non-elliptic elements in $Sp(n, 1)$ we shall always use the Hermitian form $J_2$, whence for elliptic elements always $J_1$ will be used.

2.2. **Cross Ratios.** We note the following lemma concerning cross ratios.

**Lemma 2.1.** [CP11] Let $0, \infty \in \partial H^n$ stand for the $(0, 1, \ldots, 0)^t$ and $(1, 0, \ldots, 0)^t \in H^{n,1}$ under the projection map $\mathbb{P}$, respectively and let $h \in PSp(n, 1)$ be given by (2.3). Then

$$|h(0), 0, \infty, h(\infty)| = |bc|,$$

$$|h(\infty), \infty, 0, h(0)| = |ad|,$$

$$|\infty, 0, h(\infty), h(0)| = \frac{|bc|}{|ad|}.$$

2.3. **Useful Results.** The following theorem from [CG74] will be used later.

**Theorem 2.2.** [CG74] Let $G$ be a Zariski dense subgroup of $Sp(n, 1)$. Then $G$ is either discrete or dense in $Sp(n, 1)$.

Recall that a subgroup $G$ of $Sp(n, 1)$ is called elementary if it has a finite orbit in $H^n \cup \partial H^n$. If all of its orbits are infinite then $G$ is non-elementary. A subgroup $G$ is non-elementary if it contains two non-elliptic elements with distinct fixed points.
2.4. Generalized Jørgensen Inequalities. In [HP96, Appendix], Hersonsky and Paulin proved a version of Shimizu’s lemma for subgroups in SU(n, 1). The following quaternionic version of [HP96, Proposition A.1] is a straightforward adaption of the proof of Hersonsky and Paulin.

**Theorem 2.3.** Suppose $T_{s, \zeta}$ be an Heisenberg translation in Sp(n, 1) and $A$ be an element in Sp(n, 1) of the form (2.3). Set
\[
(2.4) \quad t = \text{Sup}\{|b|, |\beta|, |\gamma|, |U - I|\}, \quad M = |s| + 2|\zeta|.
\]
If
\[
(2.5) \quad Mt + 2|\zeta| < 1,
\]
then the group generated by $A$ and $T_{s, \zeta}$ is either non-discrete or fixes $\infty$.

For two generator subgroups of Sp(n, 1) with an elliptic generator, Cao and Tan have defined the following invariant in [CT10]:
\[
\delta_{ct}(g) = \max\{|\lambda_i - \lambda_1|^2 : i = 2, \ldots, n + 1\}.
\]
This quantity, though a conjugacy invariant, does not resemble the conjugacy invariant $\delta_{cp}(g)$ of Cao and Parker. In order to have a consistent family of invariants for two generator subgroups with a semisimple generator, we define the conjugacy invariant $\delta(g)$ in (1.3), and observe that the following version of Jørgensen inequality can be obtained by mimicking the proof of Cao and Tan of [CT10, Theorem 1.1]. We refer to [CT10] for the proof. We remark that the following result also holds for SU(n, 1).

**Theorem 2.4.** Let $g$ and $h$ be elements of Sp(n, 1). Suppose that $g$ is a regular elliptic element with fixed point $q$. If
\[
(2.6) \quad \cosh\left(\frac{\rho(q, h(q))}{2}\right) |\delta(g)| < 1,
\]
then the group $\langle g, h \rangle$ generated by $g$ and $h$ is either elementary or non-discrete.

Let $g$ be a regular elliptic. We shall use the Hermitian form $J_1$ in this case. Since $\cosh(\frac{\rho(q, h(q))}{2}) |\delta(g)|$ is invariant under conjugation, we may assume that $g$ is of the following form having fixed point $q = (0, \ldots, 0)^t \in \mathbb{H}_n^m$:
\[
g = \begin{pmatrix}
\lambda_1 & 0 \\
0 & L
\end{pmatrix},
\]
where $L = \text{diag}(\lambda_2, \ldots, \lambda_{n+1})$. Let
\[
h = (a_{i,j})_{i,j=1,\ldots,n+1} = \begin{pmatrix}
a_{1,1} & \beta \\
\alpha & A
\end{pmatrix}.
\]
Then a simple computation shows that
\[
\cosh\left(\frac{\rho(q, h(q))}{2}\right) = |a_{1,1}|.
\]
So (2.6) is the following inequality under the hypothesis of the theorem:
\[
|a_{1,1}| |\delta(g)| < 1.
\]
3. Proof of Theorem 1.1

If possible suppose $G$ is not discrete. Then $G$ must be dense in $\text{Sp}(n, 1)$ by Theorem 2.2. Note that the set $\mathcal{L}$ of loxodromic elements in $\text{Sp}(n, 1)$ forms an open subset of $\text{Sp}(n, 1)$. Let $\text{Fix}(g)$ denote the fixed point set of $g$ on $\partial \mathbb{H}^n_{\overline{\mathbb{H}}}$. Let $\mathcal{L} - F_g$ be the subgroup of $\text{Sp}(n, 1)$ that stabilizes $\text{Fix}(g)$. The subgroup $\mathcal{L} - F_g$ is closed in $\text{Sp}(n, 1)$. Hence $\mathcal{L} - F_g$ is still an open subset in $\text{Sp}(n, 1)$.

(1) Let $g$ be a regular elliptic. We can assume that $q = 0$ is a fixed point of $g$ on $\mathbb{H}^n_{\overline{\mathbb{H}}}$. Since, $G$ is dense in $\text{Sp}(n, 1)$, there is a sequence of loxodromic element $\{h_m\}$ in $\mathcal{L} \cap G$ such that $h_m \rightarrow I$. Let

$$h_m = \left( \begin{array}{cc} a_{1,1}^{(m)} & \beta^{(m)}
\alpha^{(m)} & A^{(m)}
\end{array} \right).$$

Since $q = 0$ is a fixed point of $g$, for $h = h_m$ the left hand side of (2.6) becomes $|a_{1,1}^{(m)}| \delta(g)$. Note that $\langle g, h_m \rangle$ is discrete by our assumption. Suppose, for large $m$, $\langle g, h_m \rangle$ is elementary. If $g$ and $h_m$ have a common fixed point, then the fixed point $u$ must belong to $\partial \mathbb{H}^n_{\overline{\mathbb{H}}}$, and $g$ will fix the midpoint of the quaternionic line joining $u$ and $q$. Since a regular elliptic element has a unique fixed point, this contradicts the regularity of $g$. If $g$ keeps the quaternionic line $L$ joining $q$ with the fixed points of $h_m$ invariant, then $g|_L$ would be an involution, and hence must have a negative type eigenvalue 1 or $-1$, again a contradiction. So, $\langle g, h_m \rangle$ must be discrete and non-elementary.

Since, $\langle g, h_m \rangle$ is discrete and non-elementary, by Theorem 2.4

$$|a_{1,1}^{(m)}| \delta(g) \geq 1.$$ 

But $a_{1,1}^{(m)} \rightarrow 1$ and $\delta(g) < 1$. This is a contradiction.

(2) Let $g$ be loxodromic. Assume without loss of generality

$$g(0) = 0, \quad g(\infty) = \infty.$$ 

Since $g \in \overline{G}$, there exists a sequence $\{h_n\}$ of loxodromic elements in $(\mathcal{L} - F_g) \cap G$ such that $h_n \rightarrow g$. Thus, $h_n, g$ does not have a common fixed point, and $\langle h_n, g \rangle$ is non-elementary for each $n$. Let

$$h_n = \left( \begin{array}{ccc}
a_n & b_n & \gamma_n^*
\gamma_n & \delta_n^* & \alpha_n
\end{array} \right).$$

By Theorem CP,

$$|a_n b_n|^{1/2} |b_n c_n|^{1/2} \geq 1 - M_g M_g^2.$$ 

But $b_n c_n \rightarrow 0$ as $n \rightarrow \infty$, hence

$$\frac{1 - M_g M_g^2}{M_g^2} \leq 0,$$

which is a contradiction since $M_g < 1$.

(3) Let $g$ be a Heisenberg translation. Since, $g \in \overline{G}$, there exist a sequence of loxodromic elements $\{h_n\} \in (\mathcal{L} - F_g) \cap G$ such that

$$h_n \rightarrow g.$$
Since \( \langle g, h_n \rangle \) is discrete and non-elementary, by Theorem 2.3
\[
M t_n + 2|\zeta| \geq 1.
\]
But \( t_n \to 0 \) as \( n \to \infty \). Thus for large \( n \), \( |\zeta| \geq \frac{1}{2} \). This is a contradiction as \( |\zeta| < \frac{1}{2} \) is given.

This proves the theorem.

3.1. **Proof of Corollary 1.3** Noting that the set of regular elliptic elements in \( \text{Sp}(n, 1) \) forms an open subset, using arguments similar to the above we have Corollary 1.3

4. **Proof of Theorem 1.4**

**Lemma 4.1.** Let \( g, h \) in \( \text{Sp}(n, 1) \) be as in Theorem CP. Consider the sequence \( h_0 = h, h_{k+1} = h_k g h_k^{-1} \). If equality holds in (1.5), then for all \( k \geq 1 \),
\[
|\langle h_k(u), u, h_k(v) \rangle|^{\frac{1}{2}} = \frac{1}{M_g}, \quad |\langle h_k(u), v, u, h_k(v) \rangle|^{\frac{1}{2}} = \frac{1 - M_g}{M_g}.
\]

**Proof.** Up to conjugacy, let \( g = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & L \end{pmatrix} \), where \( L = \text{diag}(\lambda_1, \ldots, \lambda_{n-1}) \), \( |\lambda| > 1 \), \( |\lambda_i| = 1 \).

Following similar approach as in [CP11], consider the sequence \( h_0 = h, h_{k+1} = h_k g h_k^{-1} \). Let
\[
h_k = \begin{pmatrix} a_k & b_k & \gamma_k^* \\ c_k & d_k & \delta_k^* \\ \alpha_k & \beta_k & U_k \end{pmatrix}.
\]

Then
\[
h_{k+1} = \begin{pmatrix} a_{k+1} & b_{k+1} & \gamma_{k+1}^* \\ c_{k+1} & d_{k+1} & \delta_{k+1}^* \\ \alpha_{k+1} & \beta_{k+1} & U_{k+1} \end{pmatrix}
= \begin{pmatrix} a_k & b_k & \gamma_k^* \\ c_k & d_k & \delta_k^* \\ \alpha_k & \beta_k & U_k \end{pmatrix} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & L \end{pmatrix} \begin{pmatrix} \tilde{a}_k & \tilde{b}_k & -\beta_k^* \\ \tilde{c}_k & \tilde{a}_k & -\alpha_k^* \\ -\delta_k & -\gamma_k & U_k \end{pmatrix}.
\]

Then we have
\[
a_{k+1} = a_k \lambda \tilde{b}_k + b_k \lambda^{-1} \tilde{c}_k - \gamma_k^* L \delta_k,
b_{k+1} = a_k \lambda \tilde{b}_k + b_k \lambda^{-1} \tilde{a}_k - \gamma_k^* L \gamma_k,
c_{k+1} = c_k \lambda \tilde{d}_k + d_k \lambda^{-1} \tilde{c}_k - \delta_k^* L \delta_k,
d_{k+1} = c_k \lambda \tilde{d}_k + d_k \lambda^{-1} \tilde{a}_k - \delta_k^* L \gamma_k.
\]

The above equations, along with the ones obtained from \( gg^{-1} = I \), give the following.
\[
|b_{k+1} c_{k+1}|^{\frac{1}{2}} \leq M_g |a_k d_k|^\frac{1}{2} |b_k c_k|^\frac{1}{2}.
\]

This yields,
\[
|b c_1|^\frac{1}{2} \leq \frac{1}{M_g}.
\]
That is,

\[(4.2) \quad M_g (1 + |b_1 c_1|^\frac{1}{2}) \leq 1.\]

It follows from Corollary 1.2 (4) of [CP11] that the above inequality can not be strict, otherwise the group \( \langle g, h \rangle \) would be either elementary or non-discrete. So, we must have

\[(4.3) \quad M_g (1 + |b_1 c_1|^\frac{1}{2}) = 1.\]

From [CP11 Lemma 2.3 (34)] we note that

\[|a_1 d_1| \leq |b_1 c_1|^\frac{1}{2} + 1.\]

Now observe that

\[
\left( \frac{1 - M_g}{M_g^2} \right)^2 \leq |a_1 b_1 c_1 d_1|
\]

\[
\leq |a_1 d_1||b_1 c_1|
\]

\[
\leq |a_1 d_1| M_g^2 |b_0 c_0||a_0 d_0|
\]

\[
\leq M_g^2 |a_1 d_1| \left( \frac{1 - M_g}{M_g^2} \right)^2
\]

\[
\leq M_g^2 (1 + |b_1 c_1|^\frac{1}{2})^2 \left( \frac{1 - M_g}{M_g^2} \right)^2
\]

\[
\leq \left( \frac{1 - M_g}{M_g^2} \right)^2.
\]

Thus,

\[|a_1 d_1|^\frac{1}{2} = \frac{1}{M_g}.\]

Now the lemma follows by induction. \( \Box \)

4.1. **Proof of Theorem 1.4** If possible, suppose the equality in (1.5) holds. From the proof of the above theorem, we have

\[
M_g^2 (1 + |b_1 c_1|^\frac{1}{2}) = 1.
\]

Hence

\[
M_g^2 (1 + |b_1 c_1|) \leq M_g^2 (1 + |b_1 c_1|^\frac{1}{2})^2 = 1.
\]

If \( M_g^2 (1 + |b_1 c_1|) < 1 \), then as in [CP11 Section 3], \( \langle g, h \rangle \) is either elementary or non-discrete, which is a contradiction. So we must have

\[(4.4) \quad M_g^2 (1 + |b_1 c_1|) = 1.\]

Now from (4.3) and (4.4) we have the following equality:

\[
\frac{1 - M_g^2}{M_g^2} = \left( \frac{1 - M_g}{M_g} \right)^2.
\]

This implies, \( M_g = 1 \), which is again a contradiction. This completes the proof.
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