ON SOME UNBOUNDED DOMAINS FOR A MAXIMUM PRINCIPLE

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ABSTRACT. In this paper, we study some characterizations of unbounded domains. Among these, so-called G-domain is introduced by Cabre for the Aleksandrov-Bakelman-Pucci maximum principle of second order linear elliptic operator in a non-divergence form. This domain is generalized to wG-domain by Vitolo for the maximum principle of an unbounded domain, which contains G-domain. We study the properties of these domains and compare some other characterizations. We prove that sA-domain is wG-domain, but using the Cantor set, we are able to construct a example which is wG-domain but not sA-domain.

1. INTRODUCTION

We consider the second order elliptic operator in the following non-divergence form

\[(1.1) \quad Lu(x) = a_{ij}(x)D_{ij}u(x) + b_i(x)D_iu(x) + c(x)u\]

in a given domain \(\Omega\) in \(\mathbb{R}^n\), where \(D_i = \frac{\partial}{\partial x_i}\), \(D_{ij} = D_iD_j\). The operator is called a uniformly elliptic if, for some positive constants \(\lambda, \Lambda\),

\[(1.2) \quad \lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \forall \xi \in \Omega, \quad c(x) \leq 0.\]

For the elliptic operator, there is a well known property called maximum principles. For the bounded domain, it can be written as follows:

**Theorem 1.1.** Let \(Lu \geq 0\) for some bounded domain \(\Omega\), then

\[\sup_{\Omega} u \leq \sup_{\partial\Omega} u.\]
For an unbounded domain, one can consider the following simple example:

\[ \Delta u = 0 \text{ in } \mathbb{R}^n_+, \quad u = 0 \text{ on } \partial \mathbb{R}^n_+. \]

Here, \( \Delta \) is the Laplace operator, \( \mathbb{R}^n_+ \) is an upper half plane. For the Dirichlet value problem, we have infinitely many solutions of the form \( u(x) = u(x_1, x_2, \ldots, x_n) := kx_n \) for any \( k \in \mathbb{R} \).

Thus, unlike bounded domains, the maximum principle is not easy to obtain for the unbounded domains, hence there are recent publications regarding the subject. For example, one may refer to [1, 2, 3, 5] and references therein.

**Definition 1.1** ([1]). We say that the maximum principle holds for the operator \( L \) in \( \Omega \) if

\[
\begin{align*}
(1.3) \quad & Lw \geq 0 \quad \text{in } \Omega, \\
(1.4) \quad & \limsup_{x \to \partial \Omega} w(x) \leq 0
\end{align*}
\]

imply \( w \leq 0 \) in \( \Omega \).

Using an improved classical Alexandrov-Bakelman-Pucci maximum principle, Cabre [2] obtained the maximum principle above for the following type of domains, which will be denoted by G-domain hereafter.

**Definition 1.2** ([2]). We say that \( \Omega \) satisfies a condition \( G \) if there exist positive constants \( \sigma < 1, \tau < 1 \) and \( R_0 \) such that

\[
(1.5) \quad \forall x \in \Omega \quad \exists B_{R_x} \quad \text{s.t.} \quad |B_{R_x} \setminus \Omega_{x, \tau}| \geq \sigma |B_{R_x}|,
\]

where \( B_{R_x} \) is a ball containing \( x \) of radius \( R_x \leq R_0 \) and \( \Omega_{x, \tau} \) is the component of \( \Omega \cap B_{R_x/\tau} \) to which \( x \) belongs.

As noted by Vitolo [5], the G-domain contains connected open sets with finite measure, infinite cylinders, and strips. The explanations are presented in the next section.

By Cafagna and Vitolo [3], G-domain was generalized to wG-domain, and they obtained the maximum principle.

**Definition 1.3** ([3]). We say that \( \Omega \) satisfies a condition \( wG \) if there exist positive constants \( \sigma < 1 \) and \( \tau < 1 \) such that

\[
(1.6) \quad \forall x \in \Omega \quad \exists B_{R_x} \quad \text{s.t.} \quad |B_{R_x} \setminus \Omega_{x, \tau}| \geq \sigma |B_{R_x}|,
\]
where $B_{R_x}$ is a ball containing $x$ of radius $R_x$ and $\Omega_{x, \tau}$ is the component of $\Omega \cap B_{R_x/\tau}$ to which $x$ belongs.

Observe that in the definition, we do not impose any restriction on the boundedness of radius $R$ unlike G-domain. It is immediate to see that G-domain is wG-domain. In the next section, we present examples of wG-domain, which is not G-domain.

The following A-domain appear in the book by O. A. Ladyzhenskaya and N. N. Uraltseva [4].

**Definition 1.4.** A domain $\Omega$ is called **A-domain** if there exists a constant $\sigma > 0$ and $R > 0$, such that for each $y \in \partial \Omega$ and $r \in (0, R)$, the Lebesgue measure

$$|B_r(y) \setminus \Omega| \geq \sigma |B_r|,$$

where $B_r(y)$ is the ball of radius $r > 0$, centered at $y$.

Similar to wG-domain, we may also consider the following sA-domain. But in this case, the condition is stronger than A condition unlike G condition.

**Definition 1.5.** A domain $\Omega$ is called **sA-domain** if there exists a constant $\sigma > 0$, such that for each $y \in \partial \Omega$ and $r > 0$, the Lebesgue measure

$$|B_r(y) \setminus \Omega| \geq \sigma |B_r|,$$

where $B_r(y)$ is the ball of radius $r > 0$, centered at $y$.

There are examples which are sA-domain, but not A-domain, which is also presented in the next section.

So far, we introduce 4 types of condition, G, wG, A, sA conditions. By its definition, it is rather easy to tell the inclusion of between G and wG, A and sA. In the paper, we show that the sA condition imply the wG condition, but the converse does not hold.

**Theorem 1.2.** Any sA domain is wG domain, but the converse does not hold.

It is proved in Theorem 2.2 and Theorem 2.3. The main idea for a counter example is to use Cantor set for its construction, such that, we are able to construct locally A-domain, but not sA-domain for big $r$ for any $\sigma$.

2. **Main Results**

In this section, we prove main results of the paper, and some known and unknown
but simple related facts. Firstly, we enlist some known examples of G-domain.

**Example 2.1.** Any connected open set $\Omega$ with finite measure is a G-domain. Namely $\Omega$ does satisfy Definition 1.2. Let the Lebesgue measure of $\Omega$, $|\Omega| = m$, choose sufficiently large $R$ such that $|B_R| \geq 2m$. Then $\Omega$ satisfies Definition with $\sigma = \frac{1}{2}$, for any $\tau > 1$, $R_0 = R$. Note that for any $x \in \Omega$, there exists $B_{R_x} = B_R(x)$,

$$|B_{R_x} \setminus \Omega_{x,\tau}| \geq |B_R| - |\Omega| \geq \frac{1}{2}|B_R|.$$ 

**Example 2.2.** Any infinite cylinder and strips are G-domain. Let $\Omega = \{x \in \mathbb{R}^n \mid |x'| \leq r, r > 0, x = (x', x_n)\}$. Then for each $x \in \Omega$, there exists $B_{R_x} = B_{2r}(x)$ such that, for any $\tau > 1$,

$$|B_{R_x} \setminus \Omega_{x,\tau}| \geq \frac{1}{4^n}|B_{2r}|$$

for some $y \in \mathbb{R}^n \setminus \Omega$. Domains of strips case is similar.

**Example 2.3.** A checked domain is also G-domain. Let

$$\Omega_1 := \{x \in \mathbb{R} \mid x \in (2i, 2i + 1) \text{ for some integer } i\}, \quad \Omega := \Omega_1 \times \Omega_1.$$ 

Note that for any $x \in \Omega$, $B_{10}(x)$ contains a unit square in $\mathbb{R}^n \setminus \Omega$. Similarly, one can prove n-dimensional case.

The G-domain (Definition 1.2) is generalized to wG-domain (Definition 1.3). The following examples show that the converse does not hold.

**Example 2.4.** Any open connected cone whose closure is different from the whole space is wG-domain, but not G-domain. For example we consider 2-dimensional case. Let $\Omega := \{x \in \mathbb{R}^2 \mid x_2 > x_1 \wedge x_2 > -x_1, x = (x_1, x_2)\}$. For any $x \in \Omega$, choose $B_{2|x|}(0)$. With this ball, it is easy to that it satisfies the definition 1.3. But, note that for any positive $y$, the point $(0, \sqrt{2y})$ in $\Omega$ has a distance of $y$ to its boundary. Thus at least $B_{y/2}$ is needed to touch outside of $\Omega$ containing the point. This means that one can not impose the boundedness of $R$ in the definition 1.2. Thus in all $\Omega$ is not G-domain.

**Example 2.5.** Let

$$\Omega_1 := \{x \in \mathbb{R} \mid x \in (2^i, 2^{i+1}) \text{ for some natural number } i\}, \quad \Omega := \Omega_1 \times \Omega_1.$$ 

Similar to the previous example, $\Omega$ is wG-domain, but not G-domain.

As discussed in the introduction, there are examples which are sA-domain, but not A-domain.
Theorem 2.1. There exist a domain which does satisfies A condition, but not sA condition.

Proof. Consider the following domain in $\mathbb{R}^2$

$$\Omega := \{ x \in \mathbb{R}^2 \mid x_2 < x_1^2, x = (x_1, x_2) \}.$$ 

It is immediate to see that $\Omega$ is A-domain considering the unit ball centering on its boundary. Considering the unit ball centered at the origin,

$$|B_r(0) \setminus \Omega| = 2 \int_0^1 \sqrt{1+\sqrt{1+r^2}} \sqrt{r^2 - x^2 - x^2} \, dx \leq 2 \cdot r \cdot \sqrt{r}.$$ 

Thus,

$$\frac{|B_r(0) \setminus \Omega|}{|B_r|} \to 0 \text{ as } r \to \infty.$$ 

□

The next theorem implies that the sA condition imply the wG condition.

Theorem 2.2. Any domain $\Omega$ which satisfies Definition 1.5 does satisfy Definition 1.3.

Proof. Let $\Omega$ be a sA-domain, $x$ be an arbitrary point in $\Omega$, $d(x)$ be a distance of $x$ to $\partial\Omega$, and $|x - y| = d(x)$ for some $y \in \partial\Omega$. Choose $R = 2d(x)$, then $x \in B_{2d(x)}(y)$ and

$$|B_{2d(x)}(y) \setminus \Omega_{x,\tau}| \geq |B_{2d(x)}(y) \setminus \Omega| \geq \sigma |B_{2d(x)}|$$

by Definition 1.5. Thus in all, $\Omega$ is wG domain with the same $\sigma$ and for any $\tau < 1$. □

For the next, we will present an example which is wG-domain, but not sA-domain. Thus, the converse of the previous theorem does not hold.

First consider the domain in $\mathbb{R}^2$ using the Cantor set. The Cantor set $C$ is defined by

$$C = [0, 1] \setminus \cup_{m=1}^{\infty} \cup_{k=0}^{3^{m-1}-1} \left( \frac{3k+1}{3m}, \frac{3k+2}{3m} \right).$$

For each positive integer $m$, let

$$D_m := \cup_{k=0}^{3^{m-1}-1} \left( \frac{3k+1}{3m}, \frac{3k+2}{3m} \right), \quad D := \cup_{m=1}^{\infty} D_m.$$ 

Note that $|C| = 0$, $|D| = 1$. Now we define a set operations as follows: for any set $S$, we set $-S = \{-x \mid x \in S\}$, $m + S = S + m := \{m + x \mid x \in S\}$.

We define an open set in $\mathbb{R}^2$. 
\[\Omega_1^+ := \bigcup_{m=1}^{\infty} ((m - 1) + D_m) \times (-\infty, +\infty), \quad \Omega_1^- := -\Omega_1^+, \quad \Omega_1 := \Omega_1^+ \cup \Omega_1^-.
\]

Note that \(\Omega_1\) satisfies Definition 1.3, but is not connected. To connect these components, we add the following sets: for any \(m \in \mathbb{N}\),
\[B_m := ([m - 1, m) \cup (-m, 1 - m]) \times \left(-\frac{1}{2 \cdot 3^m}, \frac{1}{2 \cdot 3^m}\right).
\]
Now we take \(\Omega_1'\) to be \(\bigcup_{m=1}^{\infty} B_m \cup \Omega_1\). Then \(\Omega_1'\) is wG-domain. For \(\Omega_1'\), it is not easy to check that it satisfies Definition 1.5 due to the irregularity of the domain. It is difficult to estimate \(|B_r(0) \setminus \Omega_1'|\).

We will modify the above idea to obtain the following theorem.

**Theorem 2.3.** There is wG-domain, which is not sA-domain.

**Proof.** First recall that the Cantor set \(C\) is defined by
\[C = [0, 1] \setminus \bigcup_{m=1}^{\infty} \cup_{k=0}^{3^m-1} \left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right).
\]
For each positive integer \(m\), let
\[D_m := \bigcup_{k=0}^{3^{m-1}-1} \left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right), \quad D := \bigcup_{m=1}^{\infty} D_m.
\]
Note that \(|C| = 0, |D| = 1\).

Now we define a set operations as follows: for any set \(S\), we set \(m+S = \{m+x \mid x \in S\}\). Also we define
\[E := \bigcup_{m=1}^{\infty} ((m - 1) + D_m).
\]
Let
\[\Omega := \{x \in \mathbb{R}^n \mid |x| \in E\} \cup \{x \in \mathbb{R}^n \mid x = (x_1, x'), |x_1| \in [m-1, m), |x'| \leq \frac{1}{3^m} \text{ for some } m \in \mathbb{N}\}.
\]
It is easy to see that \(\Omega\) is connected since \(x' = 0\) is contained in \(\Omega\). Observe that
\[
(2.2) \quad \frac{|\Omega \cap B_r|}{|B_r|} \nearrow 1 \quad \text{as } r \nearrow \infty.
\]
For any \(x \in \Omega\) and \(|x| \leq m\), then \(x \in B_{\frac{1}{3^m}}(y)\) for some \(y \in \partial \Omega\), and \(B_{\frac{1}{3^m}}(z) \subset \mathbb{R}^n \setminus \Omega, B_{\frac{1}{3^m}}(z) \subset B_{\frac{1}{3^m}}(y)\). This is due to the fact that if \(|x| \leq m\), then \(x\) belongs to a locally connected component of width \(\frac{1}{3^m}\). Thus in all, \(\Omega\) is wG-domain.

But \(0 \in \Omega\) and \(B_r(0)\) is a disjoint union of \(B_r \setminus \Omega\) and \(B_r \cap \Omega\), we have that
\[
\frac{|B_r(0) \setminus \Omega|}{|B_r|} \searrow 0 \quad \text{as } r \nearrow \infty
\]
due to (2.2). Thus \(\Omega\) is not sA-domain. \(\Box\)
Remark 2.6. In the definition of sA-domain, one may replace $\Omega$ by $\Omega_{x,\tau}$ as in the definition of $G$ or $wG$-domain. But the above example in the proof still works as a counterexample.

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