Tamagawa Number
Conjecture for zeta Values

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Abstract

Spencer Bloch and the author formulated a general conjecture (Tamagawa number conjecture) on the relation between values of zeta functions of motives and arithmetic groups associated to motives. We discuss this conjecture, and describe some application of the philosophy of the conjecture to the study of elliptic curves.

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Mysterious relations between zeta functions and various arithmetic groups have been important subjects in number theory.

(0.0) zeta functions ↔ arithmetic groups.

A classical result on such relation is the class number formula discovered in 19th century, which relates zeta functions of number field to ideal class groups and unit groups. As indicated in (0.1)–(0.3) below, the formula of Grothendieck expressing the zeta functions of varieties over finite fields by etale cohomology groups, Iwasawa main conjecture proved by Mazur-Wiles, and Birch and Swinnerton-Dyer conjectures for abelian varieties over number fields, considered in 20th century, also have the form (0.0).

(0.1) Formula of Grothendieck.

zeta functions ↔ etale cohomology groups.

(0.2) Iwasawa main conjecture.

zeta functions ↔ ideal class groups, unit groups.

(0.3) Birch Swinnerton-Dyer conjectures (see 4).

zeta functions ↔ groups of rational points, Tate-Shafarevich groups.

Here in (0.2), “zeta elements” mean cyclotomic units which are units in cyclotomic fields and closely related to zeta functions. Roughly speaking, the relations

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(often conjectural) say that the order of zero or pole of the zeta function at an integer point is equal to the rank of the related finitely generated arithmetic abelian group (Tate, the conjecture (0.3), Beilinson, Bloch, ...) and the value of the zeta function at an integer point is related to the order of the related arithmetic finite group.

In [BK], Bloch and the author formulated a general conjecture on (0.0) (Tamagawa number conjecture for motives). Further generalizations of Tamagawa number conjecture by Fontaine, Perrin-Riou, and the author [FP], [Pe1] [Ka1], [Ka2] have the form

(0.4) zeta functions (= Euler products, analytic)
↔ zeta elements (= Euler systems, arithmetic)
↔ arithmetic groups.

Here the first ↔ means that zeta functions enter the arithmetic world transforming themselves into zeta elements, and the second ↔ means that zeta elements generate “determinants” of certain etale cohomology groups.

The aim of this paper is to discuss (0.4) in an expository style. We review (0.1) in §1, and then in §2, we describe the generalized Tamagawa number conjecture (0.4), the relation with (0.2), and an application of the philosophy (0.4) to (0.3).

In this paper, we fix a prime number $p$. For a commutative ring $R$, let $Q(R)$ be the total quotient ring of $R$ obtained from $R$ by inverting all non-zerodivisors.

1. Grothendieck formula and zeta elements

Let $X$ be a scheme of finite type over a finite field $F_q$. We assume $p$ is different from char($F_q$).

In this §1, we first review the formula (1.1.2) of Grothendieck representing zeta functions of $p$-adic sheaves on $X$ by etale cohomology. We then show that those zeta functions are recovered from $p$-adic zeta elements (1.3.5).

1.1. Zeta functions and etale cohomology groups in positive characteristic case. The Hasse zeta function $\zeta(X, s) = \prod_{x \in |X|} (1 - \sharp \kappa(x)^{-s})^{-1}$, where $|X|$ denotes the set of all closed points of $x$ and $\kappa(x)$ denotes the residue field of $x$, has the form $\zeta(X, s) = \zeta(X/F_q, q^{-s})$ where

$$\zeta(X/F_q, u) = \prod_{x \in |X|} (1 - u^{\deg(x)})^{-1}, \quad \deg(x) = [\kappa(x) : F_q]. \quad (1.1.1)$$

A part of Weil conjectures was that $\zeta(X/F_q, u)$ is a rational function in $u$, and it was proved by Dwork and then slightly later by Grothendieck. The proof of Grothendieck gives a presentation of $\zeta(X/F_q, u)$ by using etale cohomologyy. More generally, for a finite extension $L$ of $Q_p$ and for a constructible $L$-sheaf $\mathcal{F}$ on $X$, Grothendieck proved that the $L$-function $L(X/F_q, \mathcal{F}, u)$ has the presentation

$$L(X/F_q, \mathcal{F}, u) = \prod_m \det_L(1 - \varphi_q u ; H^m_{et,c}(X \otimes_{F_q} \overline{F}_q, \mathcal{F}))^{-1} \quad (1.1.2)$$
where $H^m_{et,c}$ is the etale cohomology with compact supports and $\varphi_q$ is the action of the $q$-th power morphism on $X$.

In the case $L = \mathbb{Q}_q = \mathcal{F}$, $\zeta(X/\mathbb{F}_q, u) = L(X/\mathbb{F}_q, \mathcal{F}, u)$.

1.2. $p$-adic zeta elements in positive characteristic case. Determinants appear in the theory of zeta functions as above, rather often. The regulator of a number field, which appears in the class number formula, is a determinant. Such relation with determinant is well expressed by the notion of “determinant module”.

If $R$ is a field, for an $R$-module $V$ of dimension $r$, $\det_R(V)$ means the 1-dimensional $R$-module $\Lambda(V)$ for a bounded complex $C$ of $R$-modules whose cohomologies $H^m(C)$ are finite dimensional, $\det_R(C)$ means $\otimes_{m\in \mathbb{Z}} \{ \det_R(H^m(C)) \}^{(-1)^m}$.

This definition is generalized to the definition of an invertible $R$-module $\det_R(C)$ associated to a perfect complex $C$ of $R$-modules for a commutative ring $R$ (see [KM]). $\det^{-1}_R(C)$ means the inverse of the invertible module $\det_R(C)$.

By a pro-$p$ ring, we mean a topological ring which is an inverse limit of finite rings whose orders are powers of $p$. Let $\Lambda$ be a commutative pro-$p$ ring. By a ctf $\Lambda$-complex on $X$, we mean a complex of $\Lambda$-sheaves on $X$ for the etale topology with constructible cohomology sheaves and with perfect stalks. For a ctf $\Lambda$-complex $\mathcal{F}$ on $X$, $R\Gamma_{et,c}(X, \mathcal{F})$ ($\_c$ means with compact supports) is a perfect complex over $\Lambda$.

For a commutative pro-$p$ ring $\Lambda$ and for a ctf $\Lambda$-complex $\mathcal{F}$ on $X$, we define the $p$-adic zeta element $\zeta(X, \mathcal{F}, \Lambda)$ which is a $\Lambda$-basis of $\det^{-1}_\Lambda R\Gamma_{et,c}(X, \mathcal{F})$. Consider the distinguished triangle

$$R\Gamma_{et,c}(X, \mathcal{F}) \to R\Gamma_{et,c}(X \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \mathcal{F}) \to (1 - \varphi) R\Gamma_{et,c}(X \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \mathcal{F}).$$

Since $\det$ is multiplicative for distinguished triangles, (1.2.1) induces an isomorphism

$$\det^{-1}_\Lambda R\Gamma_{et,c}(X, \mathcal{F}) \cong \det^{-1}_\Lambda R\Gamma_{et,c}(X \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \mathcal{F}) \otimes_{\Lambda} \det_{\Lambda} R\Gamma_{et,c}(X \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \mathcal{F}) \cong \Lambda.$$  

We define $\zeta(X, \mathcal{F}, \Lambda)$ to be the image of $1 \in \Lambda$ in $\det^{-1}_\Lambda R\Gamma_{et,c}(X, \mathcal{F})$ under (1.2.2).

It is a $\Lambda$-basis of the invertible $\Lambda$-module $\det^{-1}_\Lambda R\Gamma_{et,c}(X, \mathcal{F})$.

1.3. Zeta functions and $p$-adic zeta elements in positive characteristic case. Let $L$ be a finite extension of $\mathbb{Q}_p$, let $O_L$ be the valuation ring of $L$, and let $\mathcal{F}$ be a constructible $O_L$-sheaf on $X$. We show that the zeta function $L(X/\mathbb{F}_q, \mathcal{F}_L, u)$ of the $L$-sheaf $\mathcal{F}_L = \mathcal{F} \otimes_{O_L} L$ is recovered from a certain $p$-adic zeta element as in (1.3.5) below. Let

$$\Lambda = O_L[[\text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)]] = \lim_n O_L[\text{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)].$$

Let $s(\Lambda)$ be the $\Lambda$-module $\Lambda$ which is regarded as a sheaf on the etale site of $X$ via the natural action of $\text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$. Then

$$H^m_{et,c}(X, \mathcal{F} \otimes_{O_L} s(\Lambda)) \cong \lim_n H^m_{et,c}(X \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}, \mathcal{F}).$$
where the transition maps of the inverse system are the trace maps. From this, we can deduce that $H^m_{\text{et},c}(X, \mathcal{F} \otimes_{O_L} s(\Lambda))$ is a finitely generated $O_L$-module for any $m$. Hence we have $Q(\Lambda) \otimes_{A} R\Gamma_{\text{et},c}(X, \mathcal{F} \otimes_{O_L} s(\Lambda)) = 0$ and this gives an identification canonical isomorphism

$$Q(\Lambda) \otimes_{\Lambda} \det_{\Lambda}^{-1} R\Gamma_{\text{et},c}(X, \mathcal{F} \otimes_{O_L} t(\Lambda)) = Q(\Lambda).$$

(1.3.3)

Note

$$Q(\Lambda) = Q(\lim_{\leftarrow} O_L[1/u]/(u^n - 1)) \supset Q(O_L[u]) = L(u).$$

(1.3.4)

By a formal argument, we can prove the following (1.3.5) (1.3.6) which show

zeta function = zeta element, zeta value = zeta element, respectively.

$$L(X/F_q, \mathcal{F}_L, u) = \zeta(X, \mathcal{F} \otimes_{O_L} s(\Lambda), \Lambda) \text{ in } Q(\Lambda).$$

(1.3.5)

If $H^m_{\text{et},c}(X, \mathcal{F}_L) = 0$ for any $m$, $L(X/F_q, \mathcal{F}_L, u)$ has no zero or pole at $u = 1$, and

$$L(X/F_q, \mathcal{F}_L, 1) = \zeta(X, \mathcal{F}, O_L) \text{ in } L.$$  

(1.3.6)

2. Tamagawa number conjecture

In 2.1, we describe the generalized version of Tamagawa number conjecture. In 2.2 (resp. 2.3), we consider $p$-adic zeta elements associated to 1 (resp. 2) dimensional $p$-adic representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and their relations to (0.2) (resp. (0.3)).

2.1. The conjecture. Let $X$ be a scheme of finite type over $\mathbb{Z}[1/p]$. For a complex of sheaves $\mathcal{F}$ on $X$ for the etale topology, we define the compact support version $R\Gamma_{\text{et},c}(X, \mathcal{F})$ of $R\Gamma_{\text{et}}(X, \mathcal{F})$ as the mapping fiber of

$$R\Gamma_{\text{et}}(\mathbb{Z}[1/p], Rf_!\mathcal{F}) \to R\Gamma_{\text{et}}(R, Rf_!\mathcal{F}) \oplus R\Gamma_{\text{et}}(\mathbb{Q}_p, Rf_!\mathcal{F}).$$

where $f : X \to \text{Spec}(\mathbb{Z}[1/p])$.

It can be shown that for a commutative pro-$p$ ring $\Lambda$ and for a ctf $\Lambda$-complex $\mathcal{F}$ on $X$, $R\Gamma_{\text{et},c}(X, \mathcal{F})$ is perfect.

The following is a generalized version of the Tamagawa number conjecture [BK] (see [FP], [Pe_1], [Ka_1], [Ka_2]). In [BK], the idea of Tamagawa number of motives was important, but it does not appear explicitly in this version.

Conjecture. To any triple $(X, \Lambda, \mathcal{F})$ consisting of a scheme $X$ of finite type over $\mathbb{Z}[1/p]$, a commutative pro-$p$ ring $\Lambda$, and a ctf $\Lambda$-complex on $X$, we can associate a $\Lambda$-basis $\zeta(X, \mathcal{F}, \Lambda)$ of

$$\Delta(X, \mathcal{F}, \Lambda) = \det_{\Lambda}^{-1} R\Gamma_{\text{et},c}(X, \mathcal{F}),$$

where
which we call the $p$-adic zeta element associated to $\mathcal{F}$, satisfying the following conditions (2.1.1)-(2.1.5).

(2.1.1) If $X$ is a scheme over a finite field $\mathbb{F}_q$, $\zeta(X, \mathcal{F}, \Lambda)$ coincides with the element defined in §3.2.

(2.1.2) (rough form) If $\mathcal{F}$ is the $p$-adic realization of a motive $M$, $\zeta(X, \mathcal{F}, \Lambda)$ recovers the complex value $\lim_{s \to \infty} s^{-e} L(M, s)$ where $L(M, s)$ is the zeta function of $M$ and $e$ is the order of $L(M, s)$ at $s = 0$.

(2.1.3) If $\Lambda'$ is a pro-$p$ ring and $\Lambda \to \Lambda'$ is a continuous homomorphism, $\zeta(X, \mathcal{F} \otimes_{\Lambda} \Lambda', \Lambda')$ coincides with the image of $\zeta(X, \mathcal{F}, \Lambda)$ under $\Delta(X, \mathcal{F} \otimes_{\Lambda} \Lambda', \Lambda') \cong \Delta(X, \mathcal{F}) \otimes_{\Lambda} \Lambda'$.

(2.1.4) For a distinguished triangle $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ with common $X$ and $\Lambda$, we have

$$\zeta(X, \mathcal{F}, \Lambda) = \zeta(X, \mathcal{F}', \Lambda) \otimes \zeta(X, \mathcal{F}'', \Lambda) \quad \text{in} \quad \Delta(X, \mathcal{F}, \Lambda) = \Delta(X, \mathcal{F}', \Lambda) \otimes_{\Lambda} \Delta(X, \mathcal{F}'', \Lambda).$$

(2.1.5) If $Y$ is a scheme of finite type over $\mathbb{Z}[\frac{1}{p}]$ and $f : X \to Y$ is a separated morphism,

$$\zeta(Y, Rf_* \mathcal{F}, \Lambda) = \zeta(X, \mathcal{F}, \Lambda) \quad \text{in} \quad \Delta(Y, Rf_* \mathcal{F}, \Lambda) = \Delta(X, \mathcal{F}, \Lambda).$$

By this (2.1.5), the constructions of $p$-adic zeta elements are reduced to the case $X = \text{Spec}(\mathbb{Z}[\frac{1}{p}])$. How to formulate the part (4.1.2) of this conjecture is reduced to the case of motives over $\mathbb{Q}$ by (2.1.5) and $L(M, s) = L(Rf!(M), s)$ (by philosophy of motives), where $f : X \to \text{Spec}(\mathbb{Z}[\frac{1}{p}])$.

The conditions (2.1.3)-(2.1.5) are formal properties which are analogous to formal properties of zeta functions. The conditions (2.1.1) and (2.1.3)-(2.1.5) can be interpreted as

(2.1.6) The system $(X, \Lambda, \mathcal{F}) \mapsto \zeta(X, \mathcal{F}, \Lambda)$ is an “Euler system”.

In fact, let $L$ be a finite extension of $\mathbb{Q}_p$, $S$ a finite set of prime numbers containing $p$, and let $T$ be a free $O_L$-module of finite rank endowed with a continuous $O_L$-linear action of $\text{Gal}(\mathbb{Q}/\mathbb{Q})$ which is unramified outside $S$. For $m \geq 1$, let $R_m = O_L(\text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}))$ and let

$$z_m = \zeta_{R_m}(\mathbb{Z}[\frac{1}{p}], j_m \cdot (T \otimes_{O_L} \sigma(R_m)), R_m) \in \det^{-1} R_{et,c}(\mathbb{Z}[\zeta_m, \frac{1}{mS}], T).$$

Then the conditions (4.1.1) and (4.1.3)-(4.1.5) tell that when $m$ varies, the $p$-adic zeta elements $z_m$ form a system satisfying the conditions of Euler systems formulated by Kolyvagin [Ko].

We illustrate the relation (2.1.2) with zeta functions.

Let $M$ be a motive over $\mathbb{Q}$, that is, a direct summand of the motive $H^m(X)(r)$ for a proper smooth scheme $X$ over $\mathbb{Q}$ and for $r \in \mathbb{Z}$, and assume that $M$ is endowed with an action of a number field $K$. Then the zeta function $L(M, s)$ lives in $\mathbb{C}$, and the $p$-adic zeta element lives in the world of $p$-adic etale cohomology. Since these two worlds are too much different in nature, $L(M, s)$ and the $p$-adic zeta element are not simply related.
However in the middle of $\mathbf{C}$ and the $p$-adic world,

(a) there is a 1 dimensional $K$-vector space $\Delta_K(M)$ constructed by the Betti realization and the de Rham realization of $M$, and $K$-groups (or motivic cohomology groups) associated to $M$.

Let $\infty$ be an Archimedean place of $K$. Then

(b) there is an isomorphism

$$
\Delta_K(M) \otimes_K K_{\infty} \cong K_{\infty}
$$

cast by Hodge theory and $K$-theory.

Let $w$ be a place of $K$ lying over $p$, let $M_w$ be the representation of $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ over $K_w$ associated to $M$, and let $T$ be a $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-stable $O_{K_w}$-lattice in $M_w$. Then

(c) there is an isomorphism

$$
\Delta_K(M) \otimes_K K_w \cong \text{det}_{O_{K_w}}^{-1} R\Gamma_{et,c}(\mathbf{Z}_{p}^{1}, j_*M_w)
$$

where $j : \text{Spec}(\mathbf{Q}) \to \text{Spec}(\mathbf{Z}_{p}^{1})$, constructed by $p$-adic Hodge theory and $K$-theory.

See [FP] how to construct (a)-(c) (constructions require some conjectures).

The part (2.1.2) of the conjecture is:

(d) there exists a $K$-basis $\zeta(M)$ of $\Delta_{K}(M)$ (called the rational zeta element associated to $M$), which is sent to $\lim_{s \to 0} s^{-e} L(M, s)$ under the isomorphism (b) where $e$ is the order of $L(M, s)$ at $s = 0$, and to $\zeta(\mathbf{Z}_{p}^{1}, j_*T, O_{K_w})$ in $\text{det}_{O_{K_w}}^{-1} R\Gamma_{et,c}(\mathbf{Z}_{p}^{1}, j_*M_w)$ under the isomorphism (c).

The existence of $\zeta(M)$ having the relation with $\lim_{s \to 0} s^{-e} L(M, s)$ was conjectured by Beilinson [Be].

How zeta functions and $p$-adic zeta elements are related is illustrated in the following diagram.

The left upper arrow with a question mark shows the conjecture that the map

$$\{\text{motives}\} \to \{\text{zeta functions}\}$$

factor through automorphic representations, which
is a subject of non-abelian class field theory (Langlands correspondences). As the other question marks indicate, we do not know how to construct zeta elements in general, at present.

2.2. \( p \)-adic zeta elements for 1 dimensional Galois representations.

Let \( \Lambda \) be a commutative pro-\( p \) ring, and assume we are given a continuous homomorphism

\[
\rho : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_n(\Lambda)
\]

which is unramified outside a finite set \( S \) of prime numbers \( S \) containing \( p \). Let \( \mathcal{F} = \Lambda^{\otimes n} \) on which \( \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \) acts via \( \rho \), regarded as a sheaf on \( \text{Spec}(\mathbb{Z}[\frac{1}{S}]) \) for the etale topology. We consider how to construct the \( p \)-adic zeta element \( \zeta(\mathbb{Z}[\frac{1}{S}], \mathcal{F}, \Lambda) \).

In the case \( n = 1 \), we can use the “universal objects” as follows. Such \( \rho \) comes from the canonical homomorphism

\[ \rho_{\text{univ}} : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_1(\Lambda_{\text{univ}}) \] where \( \Lambda_{\text{univ}} = \mathbb{Z}_p[[\text{Gal}(\mathbb{Q}(\zeta_{Np^\infty})/\mathbb{Q})]] \)

for some \( N \geq 1 \) whose set of prime divisors coincide with \( S \) and for some continuous ring homomorphism \( \Lambda_{\text{univ}} \to \Lambda \). We have \( \mathcal{F} \cong \mathcal{F}_{\text{univ}} \otimes_{\Lambda_{\text{univ}}} \Lambda \). Hence \( \zeta(\mathbb{Z}[\frac{1}{S}], \mathcal{F}_{\text{univ}}, \Lambda_{\text{univ}}) \) should be defined to be the image of \( \zeta(\mathbb{Z}[\frac{1}{S}], \mathcal{F}_{\text{univ}}, \Lambda_{\text{univ}}) \). As is explained in [Ka2] Ch. I, 3.3, \( \zeta(\mathbb{Z}[\frac{1}{S}], \mathcal{F}_{\text{univ}}, \Lambda_{\text{univ}}) \) is the pair of the \( p \)-adic Riemann zeta function and a system of cyclotomic units. Iwasawa main conjecture is regarded as the statement that this pair is a \( \Lambda_{\text{univ}} \)-basis of \( \Delta(\mathbb{Z}[\frac{1}{S}], \mathcal{F}_{\text{univ}}, \Lambda_{\text{univ}}) \).

2.3. \( p \)-adic zeta elements for 2 dimensional Galois representations.

Now consider the case \( n = 2 \). The works of Hida, Wiles, and other people suggest that the universal objects \( \Lambda_{\text{univ}} \) and \( \mathcal{F}_{\text{univ}} \) for 2 dimensional Galois representations in which the determinant of the action of the complex conjugation is -1, are given by

\[
\Lambda_{\text{univ}} = \lim_{\leftarrow n} \text{p-adic Hecke algebras of weight 2 and of level } Np^n,
\]

\[
\mathcal{F}_{\text{univ}} = \lim_{\leftarrow n} H^1 \text{ of modular curves of level } Np^n.
\]

Beilinson [Be] discovered rational zeta elements in \( K_2 \) of modular curves, and the images of these elements in the etale cohomology under the Chern class maps become \( p \)-adic zeta elements, and the inverse limit of these \( p \)-adic zeta elements should be \( \zeta(\mathbb{Z}[\frac{1}{S}], \mathcal{F}_{\text{univ}}, \Lambda_{\text{univ}}) \) at least conjecturally. By using this plan, the author obtained \( p \)-adic zeta elements for motives associated to eigen cusp forms of weight \( \geq 2 \), from Beilinson elements. Here it is not yet proved that these \( p \)-adic zeta elements are actually basis of \( \Delta \), but it can be proved that they have the desired relations with values \( L(E, \chi, 1) \) and \( L(f, \chi, r) \) \((1 \leq r \leq k - 1)\) for elliptic curves over \( \mathbb{Q} \) (which are modular by [Wi], [BCDT]) and for eigen cusp forms of weight \( k \geq 2 \), and for Dirichlet characters \( \chi \). Beilinson elements are related in the Archimedean world to \( \lim_{s \to 0} s^{-1} L(E, \chi, s) \) for elliptic curves \( E \) over \( \mathbb{Q} \), but not related to \( L(E, \chi, 1) \). However since they become universal (at least conjecturally) in the inverse limit in
the $p$-adic world, we can obtain from them $p$-adic zeta elements related to $L(E, \chi, 1)$. Using these elements and applying the method of Euler systems [Ko], [Pe2], [Ru2], [Ka3], we can obtain the following results ([Ka4]).

**Theorem.** Let $E$ be an elliptic curve over $\mathbb{Q}$, let $N \geq 1$, and let $\chi : \Gal(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \cong (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}$ be a homomorphism. If $L(E, \chi, 1) \neq 0$, the $\chi$-part of $E(\mathbb{Q}(\zeta_N))$ and the $\chi$ part of the Tate-Shafarevich group of $E$ over $\mathbb{Q}(\zeta_N)$ are finite.

The $p$-adic $L$-function $L_p(E)$ of $E$ is constructed from the values $L(E, \chi, 1)$.

**Theorem.** Let $E$ be an elliptic curve over $\mathbb{Q}$ which is of good reduction at $p$.

1. $\text{rank}(E(\mathbb{Q})) \leq \text{ord}_s L_p(E)$.

2. Assume $E$ is ordinary at $p$. Let $\Lambda = \mathbb{Z}_p[[\Gal(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})]]$. Then the $p$-primary Selmer group of $E$ over $\mathbb{Q}(\zeta_{p^\infty})$ is $\Lambda$-cotorsion and its characteristic polynomial divides $p^n L_p(E)$ for some $n$.

This result was proved by Rubin in the case of elliptic curves with complex multiplication ([Ru1]).

As described above, we can obtain $p$-adic zeta elements of motives associated to eigen cusp forms of weight $\geq 2$. For such modular forms, we can prove the analogous statement as the above (2).

Mazur and Greenberg conjectured that the characteristic polynomial of the above $p$-primary Selmer group and the $p$-adic $L$-function divide each other.

**References**

[BK] Bloch, S. and Kato, K., Tamagawa numbers of motives and $L$-functions, in The Grothendieck Festschrift, 1, Progress in Math., 86, Birkhäuser (1990), 333–400.

[BCDT] Breuil, C., Conrad, B, Diamond, F., Taylor, R., On the modularity of elliptic curves over $\mathbb{Q}$: wild 3-adic exercises, J. Amer. Math. Soc., 14 (2001), 834–939.

[FP] Fontaine, J. -M., and Perrin-Riou, B., Autour des conjectures de Bloch et Kato, cohomologie Galoisienne et valeurs de fonctions $L$, Proc. Symp. Pure Math. 55, Amer. Math. Soc., (1994), 599–706.

[Ka1] Kato, K., Iwasawa theory and $p$-adic Hodge theory, Kodai Math. J., 16 (1993), 1–31.

[Ka2] Kato, K., Lectures on the approach to Iwasawa theory for Hasse-Weil $L$-functions via $B_{dR}$, I, Arithmetic algebraic geometry (Trento, 1991), 50–163, Lecture Notes in Math., 1553, Springer, Berlin (1993).

[Ka3] Kato, K., Euler systems, Iwasawa theory, and Selmer groups, Kodai Math. J., 22 (1999), 313–372.

[Ka4] Kato, K., $p$-adic Hodge theory and values of zeta functions of modular forms, preprint.
[KM] Knudsen, F., and Mumford, D., The projectivity of the moduli space of stable curves I, *Math. Scand.*, 39, 1 (1976), 19–55.

[Ko] Kolyvagin, V. A., Euler systems, in The Grothendieck Festchrift, 2, Birkhöuser (1990), 435–483.

[Pe₁] Perrin-Riou, B., Fonction L p-adiques des représentations p-adiques, *Astérisque* 229 (1995).

[Pe₂] Perrin-Riou, B., Systemes d’Euler p-adiques et théorie d’Iwaswa, *Ann. Inst. Fourier*, 48 (1998), 1231–1307.

[Ru₁] Rubin, K., The “main conjecture” of Iwasawa theory for imaginary quadratic fields, *Inventiones math.*, 103 (1991), 25–68.

[Ru₂] Rubin, K., Euler systems, Hermann Weyl Lectures, *Annals of Math. Studies*, 147, Princeton Univ. Press (2000).

[Wi] Wiles, A., Modular elliptic curves and Fermat’s last theorem, *Ann. of Math.*, 141 (1995), 443–551.