Two new classes of compactly supported radial basis functions for approximation of discrete and continuous data

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Radial basis functions (RBFs), first proposed for interpolation of scattered data, have gotten the scientific community interest in the past two decades, with applications ranging from interpolation with the dual reciprocity approach of the boundary element method to mesh-free finite element applications. The use of compactly supported RBFs (CSRBFs) has spread due to their localization properties. However, the mathematical derivation of continuous polynomial functions for different continuity requirements can be quite cumbersome, and although a few classes of functions have already been proposed, there is still room for nonpolynomial trial functions. In this paper, two new classes of CSRBFs are presented for which the continuity requirements are guaranteed. To justify the novelty claim, a view of RBF literature is conducted. The first function class proposed consists of a combination of different inverse polynomial functions, similar to inverse quadric functions, and is thus called inverse class. The second class is called the rational class, in which the functions are obtained as a ratio of two polynomial functions. The proposed functions are used on an approximation software, which takes advantage of their simplicity. The results demonstrate the accuracy and convergence of the proposed functions when compared to some referenced CSRBFs.

KEYWORDS
approximation, compactly supported radial basis functions, least squares methods, nonpolynomial radial basis functions

1 | INTRODUCTION

Numerical approximation procedures, which use radial basis functions (RBFs), are quite common, especially for approximation of scattered data sets. With the onset of compactly supported RBFs (CSRBFs), this usage became ever more promising, since the compactness of the support radius guarantees localization of the properties, which, in turn, assures sparse problem matrices. This advantage, since they just depend on a single variable (radius) and unless the radial basis is not compactly supported, affects the results only on a neighborhood (support) of the given center, surpasses several other approximation applications. For the past two decades, the applications have sprung and research has been

Abbreviations: CSRBF, compactly supported radial basis function; RBF, radial basis function.
promising on the use of polynomial RBFs on compact support. The mathematical derivation of such RBFs is, though, quite elaborate, as can be seen in the works of Wendland, Hubbert, and Zhu.

Two novel classes of CSRBFs are presented here, which can be used both for approximation of unknown functions and for solution of differential equations, though, here, only the former is demonstrated. These two new classes are created using very simple construction rules that guarantee their continuity. A few of these functions are then tested against a few of the well-accepted literature functions.

A thorough review of CSRBF literature is conducted to justify the novelty claim, and then the development of the novel classes is presented. This development requires only elementary calculus and is one of the main advantages of these functions.

To test the new classes, a traditional testing function is attempted and solved through a discrete least-squares procedure. The results are compared.

Finally, the proposed functions are established as two new classes, and some of the following research challenges on the application of such CSRBFs are outlined.

2 | A REVIEW OF RBFs

The RBFs can be used for several different applications throughout the engineering and physical sciences. In this section, a comprehensive review is conducted, which can help interested researchers to get acquainted both with the traditional RBFs (with infinite support) and with compactly supported RBFs. Although the number of shown functions is large, it was decided that all the functions studied should be presented for completeness.

If one considers \( \mathbb{R}^+ = \{ x \in \mathbb{R}, x \geq 0 \} \) to be the set of nonnegative numbers and \( f : \mathbb{R}^+ \to \mathbb{R} \) to be a continuous function such that \( f(0) \geq 0 \), the RBF \( \mathbb{R} \) will be the function such that (see the work of Goldberg et al)

\[
f(||P - Q||),
\]

where \((P, Q) \in \mathbb{R}^n\) and \( ||.|| \) denotes the euclidean distance between \( P \) and \( Q \), i.e.,

\[
r = ||P - Q|| = [(P - Q) \cdot (P - Q)]^{1/2}.
\]

If one chooses \( m \) points \( Q_j \), for \( j = 1, m \), \( m \) real coefficients \( a_j \) can be found such that the linear combination

\[
g(P) = \sum_{j=1}^{m} a_j f(||P - Q_j||)
\]

interpolates the given function. This function \( g \) can also be called an RBF, though the terminology linear combination of approximating RBFs is favored.

Classical RBFs, which return nonzero values for \( r < \infty \), that is, with infinite support, are as follows: Duchon's thin plate splines,

\[
f(r) = r^2 \log(r).
\]

Hardy's multiquadrics function,

\[
f(r) = (c^2 + r^2)^{1/2},
\]

inverse multiquadrics,

\[
f(r) = (c^2 + r^2)^{-1/2},
\]

Gaussian,

\[
f(r) = e^{-r^2},
\]

quadrics,

\[
f(r) = 1 + r^2,
\]

and inverse quadrics,

\[
f(r) = \frac{1}{1 + r^2}.
\]
Due to the infinite support of the classical RBFs, any interpolation routine would behave as a field problem, without localization characteristics, returning full matrices for inversion. There are, of course, several numerical techniques to deal with issues of interpolation and approximation with RBFs, but these are usually rather expensive. These problems sparked the proposition of functions with local domain. The proposed function should vanish for \( r \) greater than a given measure (from now on called support or support radius). This function is called a CSRBF, several of which can be found on the literature. The first CSRBFs to be mentioned are those proposed by Buhmann, ie,

\[
f(r) = \begin{cases} 
\frac{1}{3} + r^2 - \frac{4}{3} r^3 + 2r^2 \log(r) & \text{for } 0 \leq r \leq 1, \\
0 & \text{otherwise},
\end{cases}
\]

and

\[
f(r) = \begin{cases} 
\frac{1}{15} + \frac{19}{6} r^2 - \frac{16}{3} r^3 + 3r^4 - \frac{16}{15} r^5 + \frac{1}{6} r^6 + 2r^2 \log(r) & \text{for } 0 \leq r \leq 1, \\
0 & \text{otherwise}.
\end{cases}
\]

All CSRBFs present the characteristics of both equations earlier (10 and 11), in which the behavior is different if the radius is smaller or greater than the support radius (in this case unity). It is easy to scale the functions such that the unity radius can represent accurately the behavior of any function within a given class. All CSRBFs described further will be evaluated only for \( r \leq 1 \), vanishing for \( r > 1 \).

Wendland proposed polynomial CSRBFs of minimal degree, ie,

\[
\text{We}_{C^0,PD_3}(r) = (1-r),
\]

\[
\text{We}_{C^1,PD_3}(r) = (1-r)^3(3r + 1),
\]

\[
\text{We}_{C^2,PD_3}(r) = (1-r)^5(8r^2 + 5r + 1),
\]

\[
\text{We}_{C^3,PD_3}(r) = (1-r)^2,
\]

\[
\text{We}_{C^4,PD_3}(r) = (1-r)^4(4r + 1),
\]

\[
\text{We}_{C^5,PD_3}(r) = (1-r)^6(35r^2 + 18r + 3)
\]

\[
\text{We}_{C^7,PD_3}(r) = (1-r)^8(32r^3 + 25r^2 + 8r + 1),
\]

\[
\text{We}_{C^9,PD_3}(r) = (1-r)^3,
\]

\[
\text{We}_{C^{10},PD_3}(r) = (1-r)^5(5r + 1),
\]

\[
\text{We}_{C^{11},PD_3}(r) = (1-r)^5(16r^2 + 7r + 1).
\]

The previous functions were named according to their continuity and positive definiteness, as proven in other works. Thus, function \( \text{We}_{C^0,PD_3} \) of Equation (12) has \( C^0 \) continuity (continuous with discontinuous derivatives) and is positive definite on \( \mathbb{R}^1 \), and function \( \text{We}_{C^{11},PD_3} \) (Equation (17)) is continuous and possesses continuous derivatives up to fourth degree and is positive definite on a three-dimensional space.

The paper written by Wong et al also includes higher-order Wendland functions, ie,

\[
\text{We}_{C^{10},PD_3}(r) = (1-r)^{10}(5 + 50r + 210r^2 + 450r^3 + 429r^4),
\]

\[
\text{We}_{C^{11},PD_3}(r) = (1-r)^{12}(9 + 108r + 566r^2 + 1644r^3 + 2697r^4 + 2048r^5).
\]

Wendland, on his doctoral dissertation, also proposed some “thin plate” like functions, ie,

\[
\text{We}_{C^{TP},PD_3}^T(r) = (1-r)^4,
\]

\[
\text{We}_{C^{10},PD_3}^{TP}(r) = 1 + 12r^2 - 16r^3 + 3r^4 + 12r^2 \log(r),
\]

\[
\text{We}_{C^{11},PD_3}^{TP}(r) = 1 - 18r^2 + 8r^3 + 9r^4 + 24r^3 \log(r)
\]
There are two possible $C^2$ continuous solutions for thin plate like functions that are positive definite on $\mathbb{R}^3$.

Wu, who first introduced the minimal degree polynomial compactly supported radial basis functions, shortly before Wendland, proposed different polynomials,\textsuperscript{16,17} ie,

$$W_u^{TP}_{C^n PD_1} (r) = (1 - r)^5,$$  \hfill (27)

$$W_u^{TP}_{C^n PD_3} (r) = 3 + 80r^2 - 120r^3 + 45r^4 - 8r^5 + 60r^2 \log(r),$$  \hfill (28)

$$W_u^{TP}_{C^n PD_4} (r) = 1 - 30r^2 - 10r^3 + 45r^4 - 6r^5 - 60r^3 \log(r),$$  \hfill (29)

$$W_u^{TP}_{C^n PD_5} (r) = 1 - 20r^2 + 80r^3 - 45r^4 - 16r^5 + 60r^4 \log(r),$$  \hfill (30)

$$W_u^{TP}_{C^n PD_6} (r) = (1 - r)^5,$$  \hfill (31)

$$W_u^{TP}_{C^n PD_8} (r) = 2 + 95r^2 - 160r^3 + 90r^4 - 32r^5 + 5r^6 + 60r^2 \log(r),$$  \hfill (32)

$$W_u^{TP}_{C^n PD_9} (r) = 1 - 45r^2 - 60r^3 + 135r^4 - 36r^5 + 5r^6 - 120r^3 \log(r).$$  \hfill (33)

Some CSRBFs used in meshless finite element methods and on the smooth particle hydrodynamics method\textsuperscript{18} can also be mentioned, such as the exponential (still $f(r) = 0$ for $r > 1$), ie,

$$f(r) = e^{-\alpha r}.$$  \hfill (47)

The cubic spline

$$f(r) = \begin{cases} \frac{3}{2}r - 4r^2 + 4r^3, & \text{for } r \leq \frac{1}{2}, \\ \frac{4}{3} - 4r + 4r^3 - \frac{4}{3}r^3 & \text{for } \frac{1}{2} < r \leq 1, \end{cases} $$  \hfill (48)

or the quadric spline

$$f(r) = 1 - 6r^2 + 8r^3 - 3r^4.$$  \hfill (49)
If one expands from polynomial functions only, there are also the so called Euclid’s hat functions,\(^3\),\(^{19}\) which can be obtained as in the references. Some of these functions are shown at Equations 50 to 54, ie,

\[
\psi_1(r) = 1 - r, \quad (50)
\]

\[
\psi_2(r) = \frac{2}{\pi} \left( \arccos(r) - r\sqrt{1 - r^2} \right), \quad (51)
\]

\[
\psi_3(r) = 1 - \frac{1}{4\pi} \left( (1 + 4\pi)r - r^3 \right), \quad (52)
\]

\[
\psi_4(r) = \frac{2}{\pi} \arccos(r) - \frac{1}{2\pi} r(5 - r^2)\sqrt{1 - r^2}, \quad (53)
\]

\[
\psi_5(r) = 1 - \frac{1}{8\pi^2} \left( (3 + 2\pi + 8\pi^2)r - (3 + 2\pi)r^3 \right). \quad (54)
\]

Note that all \(\psi_s\) are strictly positive definite on \(\mathbb{R}^s\), but all are only continuous without derivative continuity (\(C^0\)).

Quite a few functions were also developed as covariance functions for application on geodesics. Some of them will be mentioned, such as those due to Gneiting.\(^{20}\) Fasshauer\(^3\) described those functions, ie,

\[
\tau_s(l)(r) = (1 - r)^l \left( 1 + lr - \frac{(l + 1)(l + 2 + s)}{s} r^2 \right), \quad (55)
\]

which are \(C^2\) continuous and strictly positive definite and radial on \(\mathbb{R}^s\) provided \(l \geq (s + 5)/2\).

Gaspari and Cohn\(^{21,22}\) also proposed CSRBFs to approximate the first three autoregressive covariance functions: Gaussian and the second- and third-order Markov models. The actual description of such functions is quite elaborate and can also be found in the work of Moreaux.\(^{23}\) It is interesting to notice that the compactly supported fifth-order piecewise correlation function obtained by Gaspari and Cohn presents an inverse term

\[
f(r, a_k, a_l, c) = \begin{cases} 
  g_1(r/c)n_kn_l, & 0 \leq r \leq c/2; \\
  g_2(r/c)n_kn_l, & c/2 \leq r \leq c; \\
  g_3(r/c)n_kn_l, & c \leq r \leq 3c/2; \\
  g_4(r/c)n_kn_l, & 3c/2 \leq r \leq 2c; \\
  0, & 2c \leq r;
\end{cases} \quad (56)
\]

where \(c, a_k,\) and \(a_l\) are length parameters (\(c\) is half the support radius), \(n_k = (2 + 6a_k + 44a_k^2)^{-1}\), and the functions \(g_i\) are\(^{22}\)

\[
g_i(z) = \frac{d_i}{z} + \sum_{j=0}^{5} b_{ij}z^j; \quad (57)
\]

where \(b_{ij}\) and \(d_i\) are parameters dependent on \(a_k\) and \(a_l\). The \(d_1\) parameter must be null to avoid a singularity at the origin \((d_1 = 0)\). The other constants are

\[
\begin{align*}
  b_{10} &= 2 + 44a_k a_l + 3a_k + 3a_l; \\
  b_{11} &= 0; \\
  b_{12} &= -40(1 + 8a_k a_l - a_k - a_l)/3; \\
  b_{13} &= 10(1 + 8a_k a_l - 2a_k - 2a_l); \\
  b_{14} &= 16(1 + 2a_k a_l - a_k - a_l); \\
  b_{15} &= 16(-3 - 7a_k a_l + 4a_k + 4a_l)/3;
\end{align*} \quad (58)
\]
\[ b_{20} = \frac{(16 + 204a_k a_l - 35a_k - 35a_l)}{2}; \]
\[ b_{21} = 5(-4 - 36a_k a_l + 13a_k + 13a_l); \]
\[ b_{22} = 20(2 + 20a_k a_l - 11a_k - 11a_l)/3; \]
\[ b_{23} = 10; \]
\[ b_{24} = 8(-2 - 8a_k a_l + 5a_k + 5a_l); \]
\[ b_{25} = 16(1 + 5a_k a_l - 3a_k - 3a_l)/3; \]
\[ b_{30} = -\frac{(244a_k a_l - 189a_k - 189a_l)}{2}; \]
\[ b_{31} = 5(44a_k a_l - 27a_k - 27a_l); \]
\[ b_{32} = -20(20a_k a_l - 9a_k - 9a_l)/3; \]
\[ b_{33} = -20(2a_k a_l - a_k - a_l); \]
\[ b_{34} = 8(8a_k a_l - 3a_k - 3a_l); \]
\[ b_{35} = 16(-3a_k a_l + a_k + a_l)/3; \]
\[ b_{40} = 256a_k a_l; \]
\[ b_{41} = -320a_k a_l; \]
\[ b_{42} = 320a_k a_l/3; \]
\[ b_{43} = 40a_k a_l; \]
\[ b_{44} = -32a_k a_l; \]
\[ b_{45} = 16a_k a_l/3; \]
\[ b_{50} = 256a_k a_l; \]
\[ b_{51} = -320a_k a_l; \]
\[ b_{52} = 320a_k a_l/3; \]
\[ b_{53} = 40a_k a_l; \]
\[ b_{54} = -32a_k a_l; \]
\[ b_{55} = 16a_k a_l/3; \]
\[ d_2 = (-8 - 84a_k a_l + 29a_k + 29a_l)/12; \]
\[ d_3 = (460a_k a_l - 243a_k - 243a_l)/12; \]
\[ d_4 = -128a_k a_l/3. \]

From all the literature reviewed, the only other rational function mentioned was the nonuniform radial rational B-spline, proposed by Schaback,\(^{19}(p12)\) i.e.,

\[
\begin{align*}
 f(r) &= \begin{cases} 
 (85 - 720x^3 + 3528x^4 - 3780x^5 - 960x^6 + 2160x^7 + 192x^8)/85 & \text{for } 0 \leq r \leq 1/2, \\
 4(x - 1)^6(16x^4 + 96x^3 + 156x^2 + 56x - 9)/(85x) & \text{for } 1/2 \leq r \leq 1, \\
 0 & \text{otherwise.}
\end{cases}
\end{align*}
\]

Other functions can be found in the literature that are radial, positive definite, and vanish for a distance greater than a support radius, but these functions presented here are, by far, the most cited among them. As mentioned on the previous paragraphs, none of them was assembled as an inverse polynomial function or as a polynomials ratio. This sparked our interest in investigating the properties of such functions.

\section{Construction}

The continuous experience with numerical methods for different applications, as well as an interest in differential geometry, sparked my interest in attempting different functions that could effectively approximate (and, sometimes, interpolate) a given data set. Polynomial functions had already been exploited on depth, and both Wendland\(^4,5\) and Wu\(^17\) proposed minimal degree polynomial functions. Not much existed on polynomial ratios, or on inverse polynomials, though they can be very effective for approximation procedures.\(^24\) One of the greatest advantages of the proposed approach is the computational easiness of the functions, which requires only basic calculus. It is surprising that this venue has not been explored so far.
3.1 Inverse functions

Inverse functions are composed by a linear combination of inverse polynomials. To facilitate the reference, the “order” of the function will be the order of the highest dividing polynomial. In this section, different combinations will be attempted to assure continuity of the RBFs and their derivatives at the center and at the support radius (hereupon fixed as one, for ease of writing, without loss of generality). Although the computation is very easy and because of this, it will be shown in detail.

The inverse functions will always be constructed from the general formula, ie,

\[ I_N(r) = \sum_{i=0}^{N} c_i \frac{1}{1 + r^i}. \]  

(64)

3.1.1 First inverse function

The first inverse function to be proposed is

\[ I_1(r) = \frac{c_1}{1 + r} - \frac{c_0}{2}. \]  

(65)

To assure that the proposed function can serve well in the representation of continuous functions at \( r = 1 \), the function will have to obey the following conditions: \( F(0) = 1 \), and \( F(1) = 0 \),

\[
\begin{align*}
    c_1 + \frac{c_0}{2} &= 1, \\
    \frac{c_1}{2} + \frac{c_0}{2} &= 0.
\end{align*}
\]  

(66)

As a result, the coefficients are found to be \( c_0 = -2 \) and \( c_1 = 2 \). Rewriting Equation (65), we have

\[ I_1(r) = \frac{2}{1 + r} - 1. \]  

(67)

Function \( I_1 \) is presented in Figure 1 to show its continuity. The derivative, though, would not be continuous either at \( r = 0 \) or at \( r = 1 \).

The second inverse function presents itself differently, since the continuity of the derivative could either be enforced at the center or at the support radius.

3.1.2 Second inverse function

The second function proposed as inverse function is

\[ I_2(r) = \frac{c_2}{1 + r^2} + \frac{c_1}{1 + r} + \frac{c_0}{2}. \]  

(68)

Again, to assure that the proposed function can serve well in the representation of continuous functions at \( r = 1 \), the function will have to obey the following conditions: \( F(0) = 1 \), and \( F(1) = 0 \). For this case, however, there are two possibilities regarding the derivative, ie,

\[ \frac{dI_2}{dr} \bigg|_{r=0} = 0, \]  

(69)

or

\[ \frac{dI_2}{dr} \bigg|_{r=1} = 0. \]  

(70)
First, the derivative of the $I_2$ function is obtained, ie,
\[
\frac{dI_2}{dr} = -\frac{2c_2 r}{(1 + r^2)^2} - \frac{c_1}{(1 + r^2)^2}.
\] (71)

On the first condition (Equation 69), the following system of equations is obtained:
\[
\begin{align*}
c_2 + c_1 + \frac{c_0}{2} &= 1, \\
\frac{c_0}{2} + c_1 + \frac{c_2}{2} &= 0, \\
c_1 &= 0.
\end{align*}
\] (72)

The result of which is
\[
\begin{align*}
c_0 &= -2, \\
c_1 &= 0, \\
c_2 &= 2.
\end{align*}
\] (73)

The expression of $I_{2C}(r)$ (index $C$ indicates that continuity of the derivative is assured at the center of the radial function) is
\[
I_{2C}(r) = \frac{2}{1 + r^2} - 1.
\] (74)

Function $I_{2C}$ is shown in Figure 2, along with its derivative.

For the hypothesis of null derivative at $r = 1$ (Equation 70), the following system of equations is obtained:
\[
\begin{align*}
c_2 + c_1 + \frac{c_0}{2} &= 1, \\
\frac{c_0}{2} + c_1 + \frac{c_2}{2} &= 0, \\
-\frac{c_2}{2} - \frac{c_1}{4} &= 0.
\end{align*}
\] (75)

The result of which is
\[
\begin{align*}
c_0 &= -2, \\
c_1 &= 4, \\
c_2 &= -2.
\end{align*}
\] (76)

The expression of $I_{2S}(r)$ ($S$ indicates continuity of the derivative at the support radius) is
\[
I_{2S}(r) = -\frac{2}{1 + r^2} + \frac{4}{1 + r} - 1.
\] (77)

Function $I_{2S}$ is shown in Figure 3, along with its derivative.
3.1.3  Third inverse function

The third proposed inverse function, which will be one of our test functions, is

\[ I_3(r) = \frac{c_3}{1 + r^3} + \frac{c_2}{1 + r^2} + \frac{c_1}{1 + r} + \frac{c_0}{2}. \]  

(78)

Once more, to assure that the proposed function can serve well in the representation of continuous functions at \( r = 1 \), the function will have to obey the following conditions: \( I_3(0) = 1 \), \( I_3(1) = 0 \), and two extra conditions regarding the first derivative, which should vanish for both \( r = 0 \) and \( r = 1 \).

The first derivative of \( I_3 \) is

\[ \frac{dI_3}{dr} = -\frac{3c_1 r^2}{(1 + r^3)^2} - \frac{2c_2 r}{(1 + r^2)^2} - \frac{c_1}{(1 + r)^2}. \]  

(79)

The system of equations obtained by substituting the conditions at the proposed function is

\[
\begin{align*}
    c_3 + c_2 + c_1 + \frac{c_0}{2} &= 1, \\
    \frac{c_2}{2} + \frac{c_1}{2} + \frac{c_0}{2} &= 0, \\
    c_1 &= 0, \\
    -3c_3 - 2c_2 - \frac{c_1}{4} &= 0,
\end{align*}
\]  

(80)

which results \( c_0 = -2 \), \( c_1 = 0 \), \( c_2 = 6 \), \( c_3 = -4 \).

The final expression of \( I_3(r) \) is

\[ I_3(r) = -\frac{4}{1 + r^3} + \frac{6}{1 + r^2} - 1. \]  

(81)

Function \( I_3 \) is shown in Figure 4, along with its first derivative.

3.1.4  Fourth inverse function

The next inverse function proposed is

\[ I_4(r) = \frac{c_4}{1 + r^4} + \frac{c_3}{1 + r^3} + \frac{c_2}{1 + r^2} + \frac{c_1}{1 + r} + \frac{c_0}{2}. \]  

(82)

As before, to assure that the proposed function can serve well in the representation of continuous functions, the function will have to obey five conditions: \( I_4(0) = 1 \), \( I_4(1) = 0 \), \( \left. \frac{dI_4}{dr} \right|_{r=0} = 0 \), \( \left. \frac{dI_4}{dr} \right|_{r=1} = 0 \), and another condition that will have to be established to obtain the coefficients. This fifth condition, with respect to the second derivative of \( I_4 \), can either be

\[ \left. \frac{d^2I_4}{dr^2} \right|_{r=0} = 0, \]  

(83)

or

\[ \left. \frac{d^2I_4}{dr^2} \right|_{r=1} = 0. \]  

(84)

The derivatives of \( I_4 \) can be obtained, ie,

\[ \frac{dI_4}{dr} = -\frac{4c_4 r^3}{(1 + r^4)^2} - \frac{3c_3 r^2}{(1 + r^3)^2} - \frac{2c_2 r}{(1 + r^2)^2} - \frac{c_1}{(1 + r)^2}. \]  

(85)
When the curvature of the function vanishes at the support radius \( r = 1 \), the obtained equation is equivalent to the one already enforced for the first derivative. Therefore, there is no need to impose this condition to the function, or to add superscript \( C \) or \( S \).

For the first condition of null curvature at the origin \( r = 0 \), the system of equations obtained is

\[
\begin{align*}
\sum_{i=0}^{N} c_i r^i &= 1, \\
\frac{\sum_{i=0}^{N} c_i}{2} + \frac{c_1}{2} + c_2 + c_3 + c_4 &= 0, \\
-c_4 - \frac{3c_3}{4} - \frac{2c_2}{4} - c_1 &= 0, \\
-2c_2 + 2c_1 &= 0,
\end{align*}
\]

and the result of the system: \( c_0 = -2, c_1 = 0, c_2 = 0, c_3 = 8, c_4 = -6 \).

The final expression for \( I_4(r) \) is

\[
I_4(r) = -\frac{6}{1 + r^4} + \frac{8}{1 + r^3} - 1.
\]

Function \( I_4 \) is shown at Figure 5, along with its first and second derivatives. This function will also be tested against some of the well-known CSRBFS.

### 3.1.5 General inverse function formula

In this section, we will describe the procedure for deriving higher-order inverse functions.

A general formula for inverse functions can be obtained from the general description of the function, ie,

\[
I_N(r) = \sum_{i=0}^{N} \frac{c_i}{1 + r^i}.
\]

From this definition, the derivatives up to order \( M \) of the inverse functions can be obtained, ie,

\[
\begin{align*}
\frac{dI_N}{dr} &= \sum_{i=1}^{N} c_i \frac{-i r^{i-1}}{(1 + r^i)^2}, \\
\frac{d^2 I_N}{dr^2} &= \sum_{i=1}^{N} - ic_i \frac{(i-1)(r^{i-2})(1 + r^i)^2 - 2(r^{i-1})(1 + r^i)ir^{i-1}}{(1 + r^i)^4}, \\
\frac{d^3 I_N}{dr^3} &= \sum_{i=1}^{N} - ic_i \frac{(i-1)(r^{i-2})(1 + r^i) - 2(r^{i-1})ir^{i-1}}{(1 + r^i)^3}, \\
\frac{d^4 I_N}{dr^4} &= \sum_{i=1}^{N} - ic_i \frac{(i-1)r^{i-2} + (i-1)r^{2i-2} - 2ir^{2i-2}}{(1 + r^i)^3},
\end{align*}
\]
\[
\frac{d^2 I_N}{dr^2} = \sum_{i=1}^{N} -ic_i \frac{(i-1)r^{i-2} - (1 + i)r^{2i-2}}{(1 + r^i)^3},
\]
\[(94)\]

or

\[
\frac{d^3 I_N}{dr^3} = \sum_{i=1}^{N} -ic_i \frac{(i-1)(i-2)r^{i-3} - (1 + i)(2i - 2)r^{2i-3}(1 + r^i)^3}{(1 + r^i)^3} - 3((i-1)r^{i-2} - (1 + i)r^{2i-2})i r^{i-1} \]
\[(95)\]

\[
\frac{d^3 I_N}{dr^3} = \sum_{i=1}^{N} -ic_i \frac{(i-1)(i-2)r^{i-3} - (1 + i)(2i - 2)r^{2i-3}(1 + r^i)}{(1 + r^i)^3} - 3((i-1)r^{i-2} - (1 + i)r^{2i-2})i r^{i-1} \]
\[(96)\]

\[
\frac{d^3 I_N}{dr^3} = \sum_{i=1}^{N} -ic_i \frac{(i-1)(i-2)r^{i-3} - (1 + i)(2i - 2)r^{2i-3}(1 + r^i)}{(1 + r^i)^3} - 3((i-1)r^{i-2} - (1 + i)r^{2i-2})i r^{i-1} \]
\[(97)\]

\[
\frac{d^3 I_N}{dr^3} = \sum_{i=1}^{N} -ic_i \frac{(i-1)(i-2)r^{i-3} - (1 + i)(2i - 2)r^{2i-3} + (i-1)(i-2)r^{2i-3} - (1 + i)(2i - 2)r^{3i-3} - 3i(i-1)r^{2i-3} + 3i(1 + i)r^{3i-3}}{(1 + r^i)^3} \]
\[(98)\]

\[
\frac{d^3 I_N}{dr^3} = \sum_{i=1}^{N} -ic_i \frac{(i-1)(i-2)r^{i-3} - (1 + i)(2i - 2) - (i - 1)(i - 2) + 3i(i - 1))r^{2i-3} + ((1 + i)(2i - 2) - 3i(1 + i))r^{3i-3}}{(1 + r^i)^3} \]
\[(99)\]

\[
\frac{d^3 I_N}{dr^3} = \sum_{i=1}^{N} -ic_i \frac{(i-1)(i-2)r^{i-3} + (2i - 2 - 4i^2 + 4i + i^2 - i + 2 - 3i^2 + 3i)r^{2i-3} + 2i - 2 - 4i^2 + 4i - 3i + 6(2)r^{3i-3}}{(1 + r^i)^3} \]
\[(100)\]

\[
\frac{d^3 I_N}{dr^3} = \sum_{i=1}^{N} -ic_i \frac{(i-1)(i-2)r^{i-3} + (6i - 6i^2)r^{2i-3} - (1 - 2i)(i + 2)r^{3i-3}}{(1 + r^i)^3} \]
\[(101)\]

or

\[
\frac{d^3 I_N}{dr^3} = \sum_{i=1}^{N} -ic_i \frac{r^{i-3} \frac{1}{(1 + r^i)^4} [((i-1)(i-2) - 6i(i-1) - (1 - 2i)(i + 2)r^2]}{(1 + r^i)^3} \]
\[(102)\]

For the fourth derivative, one gets

\[
\frac{d^4 I_N}{dr^4} = \sum_{i=1}^{N} -ic_i \frac{1}{(1 + r^i)^5} \left[ ((i-1)(i-2)(i-3)r^{i-4} - 6i(i-1)(2i - 3)r^{2i-4} - (1 - 2i)(i + 2)(3i - 3)r^{3i-4})(1 + r^i)^4 \right. \\
\left. - 4((i-1)(i-2)r^{i-3} - 6i(i-1)r^{2i-3} - (1 - 2i)(i + 2)r^{3i-3})(1 + r^i)^3]r^{i-1} \right] \]
\[(103)\]

\[
\frac{d^4 I_N}{dr^4} = \sum_{i=1}^{N} -ic_i \frac{1}{(1 + r^i)^5} \left[ ((i-1)(i-2)(i-3)r^{i-4} - 6i(i-1)(2i - 3)r^{2i-4} - (1 - 2i)(i + 2)(3i - 3)r^{3i-4})(1 + r^i)^4 \right. \\
\left. - 4((i-1)(i-2)r^{i-3} - 6i(i-1)r^{2i-3} - (1 - 2i)(i + 2)r^{3i-3})r^{i-1} \right] \]
\[(104)\]

\[
\frac{d^4 I_N}{dr^4} = \sum_{i=1}^{N} -ic_i \frac{1}{(1 + r^i)^5} \left[ ((i-1)(i-2)(i-3)r^{i-4} - 6i(i-1)(2i - 3)r^{2i-4} - 6i(i-1)(2i - 3)r^{2i-4} - 6i(i-1)(2i - 3)r^{2i-4} \right. \\
\left. - 6i(i-1)(2i - 3)r^{3i-4} - (1 - 2i)(i + 2)(3i - 3)r^{3i-4} - (1 - 2i)(i + 2)(3i - 3)r^{4i-4} \right. \\
\left. - 4i(i-1)(i-2)r^{2i-4} + 24i^2(i-1)r^{3i-4} + 4i(1 - 2i)(i + 2)r^{4i-4} \right] \]
\[(105)\]
\[
\frac{d^4I_N}{dr^4} = \sum_{i=1}^{N} -ic_i \frac{1}{(1 + r^i)^5} [(i-1)(i-2)(i-3)r^{i-4} + (i-1)(i-2)(i-3) - 6i(i-1)(2i-3) \\
- 4i(i-1)(i-2))r^{2i-4} + (-6i(i-1)(2i-3) - (1-2i)(i+2)(3i-3) \\
+ 24i^2(i-1))r^{3i-4} + (-1-2i)(i+2)(3i-3) + 4i(1-2i)(i+2))r^{4i-4}],
\]
\[
\frac{d^4I_N}{dr^4} = \sum_{i=1}^{N} -ic_i \frac{1}{(1 + r^i)^5} [(i-1)(i-2)(i-3)r^{i-4} + (i-1)((i^2 - 6 + 6 + 12i^2 + 18i - 4i^2 + 8i)r^{2i-4} \\
+ (1-2i)(i+2)(i+3)r^{4i-4}],
\]
\[
\frac{d^4I_N}{dr^4} = \sum_{i=1}^{N} -ic_i \frac{1}{(1 + r^i)^5} [(i-1)(i-2)(i-3)r^{i-4} + (i-1)(6 + 21i - 15i^2)r^{2i-4} \\
+ (i-1)(-6 + 27i + 18i^2)r^{3i-4} + (1-2i)(i+2)(i+3)r^{4i-4}],
\]

Once the derivatives, here, only up to fourth order, are obtained, the system of equations for the coefficients can be assembled and solved. It is already known that the application of the inverse function and its derivatives at the center, \( r = 0 \), and at the support, \( r = 1 \), constructs a system of equations for the determination of the coefficients. The first six equations are built attributing unit value to the function at \( r = 0 \) and null value to the function at \( r = 1 \) and to its first and second derivatives at both \( r = 0 \) and \( r = 1 \).

The first line is known to be
\[
I_N(0) = \frac{c_0}{2} + \sum_{i=1}^{N} c_i = 1,
\]
and the second line
\[
I_N(1) = \frac{c_0}{2} + \sum_{i=1}^{N} c_i = 0,
\]
both obtained from the conditions applied to the function at the center and at the support radius.

From the linear combination of these two equations (Equation (109) and Equation (110)), it will always follow that \( c_0 = -2 \).

Applying now the null derivative condition for the center (Equation (111)) and the support radius (Equation (112)),
\[
\frac{dl_N}{dr} \bigg|_{r=0} = -c_1 = 0,
\]
and
\[
\frac{dl_N}{dr} \bigg|_{r=1} = \sum_{i=0}^{N} -\frac{i}{4}c_i = 0.
\]

It is noticeable that, in order to insure that the first derivative vanishes at \( r = 0 \), the first coefficient must vanish: \( c_1 = 0 \).

For the vanishing condition of the second derivatives at \( r = 0 \) (Equation (113)) and at \( r = 1 \) (Equation (114)),
\[
\frac{d^2l_N}{dr^2} \bigg|_{r=0} = 2c_1 - 2c_2 = 0,
\]
and
\[
\frac{d^2l_N}{dr^2} \bigg|_{r=1} = \sum_{i=0}^{N} \frac{2i}{8}c_i = 0.
\]

It can be noticed that the vanishing second derivative condition at \( r = 1 \) (Equation (114)) is equivalent to the condition due to Equation (114) (null first derivative at \( r = 1 \)). Furthermore, according to Equation (113), \( c_2 = 0 \).
**TABLE 1** Inverse functions and continuity

| Function | Continuity | Observation |
|----------|------------|-------------|
| $I_1(r) = \frac{2}{1+r} - 1$ | $C^0$ | Center |
| $I_2^*(r) = \frac{2}{1+r^2} - 1$ | $C^0$ | Center |
| $I_3^*(r) = -\frac{2}{1+r^2} + \frac{8}{1+r} - 1$ | $C^0$ | Support |
| $I_4(r) = -\frac{4}{1+r^2} + \frac{10}{1+r} - 1$ | $C^1$ | |
| $I_5^*(r) = -\frac{6}{1+r^2} + \frac{1}{1+r} - 1$ | $C^2$ | Center |
| $I_6(r) = -\frac{8}{1+r^2} + \frac{15}{1+r^2} - 1$ | $C^2$ | Support |
| $I_7^* = \frac{55}{3(1-r^2)} - \frac{140}{3(1-r^2)} + \frac{91}{11(1-r^2)} - 1$ | $C^3$ | |
| $I_8^* = \frac{26}{3(1-r^2)} + \frac{22}{1+r^2} - 1$ | $C^4$ | Center |
| $I_9^* = -\frac{182}{3(1-r^2)} + \frac{91}{11(1-r^2)} + \frac{408}{31(1-r^2)} - 1$ | $C^5$ | Support |
| $I_{10}^* = -\frac{56}{1+r} + \frac{210}{1+r^2} - \frac{114}{1+r^2} - 1$ | $C^6$ | |

Center - The next higher derivative is continuous at the center ($r = 0$). Support - The next higher derivative is continuous at the support radius ($r = 1$).

From the conditions on the higher-order derivatives,

$$\frac{d^3 I_N}{dr^3} \bigg|_{r=0} = -6c_1 - 6c_3 = 0,$$

(115)

$$\frac{d^3 I_N}{dr^3} \bigg|_{r=1} = \sum_{i=0}^{N} i(i-2)(i+2) \frac{c_i}{8} = 0,$$

(116)

$$\frac{d^4 I_N}{dr^4} \bigg|_{r=0} = 24c_1 + 24c_2 - 24c_4 = 0,$$

(117)

$$\frac{d^4 I_N}{dr^4} \bigg|_{r=1} = \sum_{i=0}^{N} -\frac{3(i+1)(i-1)-1}{4}c_i = 0,$$

(118)

$$\frac{d^5 I_N}{dr^5} \bigg|_{r=0} = -120c_1 - 120c_5 = 0,$$

(119)

and

$$\frac{d^5 I_N}{dr^5} \bigg|_{r=1} = \sum_{i=0}^{N} -\frac{(2i-2)(i+2) + 5*(i-4)(i+4)i}{8} c_i = 0.$$

(120)

From these equations for the coefficients, it was possible to obtain the formulae for $I_6, I_7, I_8^*, I_9^*, I_{10}^*$, and $I_{10}$. They will be shown in Table 1 in Section 3.3.

On the next section, another class of functions is proposed.

### 3.2 Rational functions

A rational function as proposed here is obtained by dividing two polynomials, the denominator being of the same order or one order higher than the numerator. Next, the procedure for obtaining the coefficients shall be detailed, ie,

$$R_{M/M} = \frac{\sum_{i=0}^{M} n_i r^i}{\sum_{j=0}^{M} d_j r^j},$$

(121)

and

$$R_{M/M+1} = \frac{\sum_{i=0}^{M} n_i r^i}{\sum_{j=0}^{M+1} d_j r^j}.$$  

(122)

The first of these functions, for $M = 1$, will be a continuous function and will turn out to be equal to $I_1$ (Equation (67)).
3.2.1 | Rational function of type 1/1

Defining the test function such as in Equation (123), assembled as a ratio between two first-order polynomials, ie,

\[
R_{1/1}(r) = \frac{n_0 + n_1 r}{d_0 + d_1 r},
\]

which can be evaluated at points \( r = 0 \) and \( r = 1 \),

\[
R_{1/1}(0) = \frac{n_0}{d_0}, \quad (124)
\]

\[
R_{1/1}(1) = \frac{n_0 + n_1}{d_0 + d_1}, \quad (125)
\]

Calculating now the derivative of function \( R_{1/1}(r) \), ie,

\[
\frac{dR_{1/1}(r)}{dr} = \frac{n_1(d_0 + d_1 r) - (n_0 + n_1 r)d_1}{(d_0 + d_1 r)^2} = \frac{n_1 d_0 - n_0 d_1}{(d_0 + d_1 r)^2}, \quad (126)
\]

and evaluating this derivative at points \( r = 0 \) and \( r = 1 \),

\[
\left. \frac{dR_{1/1}}{dr} \right|_{r=0} = \frac{n_1 d_0 - n_0 d_1}{d_0^2}, \quad (127)
\]

\[
\left. \frac{dR_{1/1}}{dr} \right|_{r=1} = \frac{n_1(d_0 + d_1) - (n_0 + n_1) d_1}{(d_0 + d_1)^2}, \quad (128)
\]

both conditions can be tested.

Attempting to ensure continuity of the derivative at the center, ie,

\[
\begin{align*}
R_{1/1}(0) &= 1, \\
R_{1/1}(1) &= 0, \\
\left. \frac{dR_{1/1}}{dr} \right|_{r=0} &= 0,
\end{align*}
\]

the following nonlinear system of equations is obtained:

\[
\begin{align*}
\frac{n_0}{d_0} &= 1, \\
\frac{n_0 + n_1}{d_0 + d_1} &= 0, \\
\frac{n_1 d_0 - n_0 d_1}{d_0^2} &= 0,
\end{align*}
\]

which leads to an impossibility, since, from the first equation, \( d_0 \) must be equal to \( n_0 \). From the second, it can be inferred that \( n_1 = -n_0 \) and that \( d_0 + d_1 \neq 0 \). When this third rule is applied, an incoherence arises, since the only “possible” solution is \( d_1 = n_1 = -d_0 \), and that contradicts the second equation. Therefore, it is not possible to ensure first derivative continuity at the center and must be attempted at the support radius.

Attempting, thus, this condition

\[
\begin{align*}
R_{1/1}(0) &= 1, \\
R_{1/1}(1) &= 0, \\
\left. \frac{dR_{1/1}}{dr} \right|_{r=1} &= 0,
\end{align*}
\]

the attained system of equations would be

\[
\begin{align*}
\frac{n_0}{d_0} &= 1, \\
\frac{n_0 + n_1}{d_0 + d_1} &= 0, \\
\frac{n_1(d_0 + d_1) - (n_0 + n_1) d_1}{(d_0 + d_1)^2} &= 0.
\end{align*}
\]
Once more the data generates an inconsistency. It can only be stated that \( d_0 = n_0,\ n_1 = -n_0,\) and that \( d_1 \neq n_1.\)

The previous analysis shows that the choice of \( d_1\) does not affect the continuity properties of the function, but the continuity of the first derivative cannot be warranted.

One of the possible options for \( R_{1/1}\) is function \( I_1,\) as stated previously, obtained for \( d_1 = 1.\) This function can be seen at Figure 6. As with function \( I_1,\) Figure 6 shows only the function, since the derivative would not be continuous either at the center or at the support radius, ie,

\[
R_{1/1}(r) = I_1 = \frac{2}{1 + r} - 1 = \frac{2 - 1 - r}{1 + r},
\]

(133)

\[
R_{1/1}(r) = \frac{1 - r}{1 + r}.
\]

(134)

### 3.2.2 Rational function of type 1/2

The second rational function can be postulated as a ratio between a first- and a second-order polynomials, ie,

\[
R_{1/2}(r) = \frac{n_0 + n_1 r}{d_0 + d_1 r + d_2 r^2}.
\]

(135)

This function can be evaluated at the center, \( r = 0,\) and at the support, \( r = 1,\) ie,

\[
R_{1/2}(0) = \frac{n_0}{d_0},
\]

(136)

\[
R_{1/2}(1) = \frac{n_0 + n_1}{d_0 + d_1 + d_2}.
\]

(137)

Obtaining now the derivative \( R_{1/2}(r)\)

\[
\frac{dR_{1/2}(r)}{dr} = \frac{n_1 (d_0 + d_1 r + d_2 r^2) - (n_0 + n_1 r)(d_1 + 2 d_2 r)}{(d_0 + d_1 r + d_2 r^2)^2}
\]

\[
= \frac{n_1 d_0 + n_1 d_1 r + n_1 d_2 r^2 - n_0 d_1 r - 2 n_0 d_2 r - n_1 d_1 r - 2 n_1 d_2 r^2}{(d_0 + d_1 r + d_2 r^2)^2},
\]

(138)

\[
\frac{dR_{1/2}(r)}{dr} = \frac{n_1 d_0 - n_0 d_1 - 2 n_0 d_2 r - n_1 d_2 r^2}{(d_0 + d_1 r + d_2 r^2)^2},
\]

(139)

and evaluating this derivative at points \( r = 0\) and \( r = 1,\)

\[
\left. \frac{dR_{1/2}}{dr} \right|_{r=0} = \frac{n_1 d_0 - n_0 d_1}{d_0^2},
\]

(140)

\[
\left. \frac{dR_{1/2}}{dr} \right|_{r=1} = \frac{n_1 d_0 - n_0 d_1 - 2 n_0 d_2 - n_1 d_2}{(d_0 + d_1 + d_2)^2}.
\]

(141)

The continuity condition for the derivative at the support radius, at \( r = 1,\) cannot be enforced, since it would generate an inconsistency: from the continuity of the function at the support radius, it is known that \( n_1 = -n_0,\) and the first derivative condition would require that

\[
\left. \frac{dR_{1/2}}{dr} \right|_{r=1} = \frac{-n_0 d_0 - n_0 d_1 - 2 n_0 d_2 + n_0 d_2}{(d_0 + d_1 + d_2)^2} = 0,
\]

(142)
which means a vanishing numerator and would be inconsistent for the numerator to vanish.

It can, though, be demanded that the derivative at the center vanishes, at \( r = 0 \), which would lead to

\[
\begin{align*}
R_{1/2}(0) & = 1, \\
R_{1/2}(1) & = 0, \\
dR_{1/2} / dr \bigg|_{r=0} & = 0,
\end{align*}
\]  

and

\[
\begin{align*}
\frac{n_0}{d_0} & = 1, \\
\frac{n_0 d_0 + n_1 d_1}{(d_0 + d_1 r)^2} & = 0, \\
\frac{n_0 n_1 - n_0 d_2}{(d_0 + d_1 r)^2} & = 0.
\end{align*}
\]  

It is, thus, obtained that \( n_0 = d_0 \) and \( d_1 = n_1 = -d_0 \), for the continuity of the first derivative at the origin. The difference from the \( R_{1/1} \) condition is that, for the \( R_{1/2} \) function, there is a new \( d_2 \) coefficient. Another continuity requirement should be sought to determine this coefficient.

Computing the second derivative,

\[
\frac{d^2 R_{1/2}(r)}{dr^2} = \frac{1}{(d_0 + d_1 r + d_2 r^2)^3} \left[ (-2 n_0 d_2 - 2 n_1 d_2 r)(d_0 + d_1 r + d_2 r^2)^2 \\
- (n_1 d_0 - n_0 d_1 - 2 n_0 d_2 r - n_1 d_2 r^2)2(d_0 + d_1 r + d_2 r^2)(d_1 + 2d_2 r) \right],
\]  

\[
\frac{d^2 R_{1/2}(r)}{dr^2} = \frac{1}{(d_0 + d_1 r + d_2 r^2)^3} \left[ (-2 n_0 d_2 - 2 n_1 d_2 r)(d_0 + d_1 r + d_2 r^2) \\
- (n_1 d_0 - n_0 d_1 - 2 n_0 d_2 r - n_1 d_2 r^2)2(d_1 + 2d_2 r) \right].
\]  

If the solution is sought at \( r = 0 \), the only possible solution would be \( d_2 = 0 \), which would undermine the second degree polynomial requirement of function \( R_{1/2} \). The second derivative continuity requirement must, thus, be for \( r = 1 \), giving \( d_1 + 2d_2 = 0 \), or \( d_2 = -\frac{d_1}{2} = \frac{n_0}{2} \).

The function becomes

\[
R_{1/2} = \frac{d_0 - n_0 r}{d_0 - n_0 r + \frac{n_0}{2} r^2},
\]  

or

\[
R_{1/2} = \frac{2 - 2r}{2 - 2r + r^2}. 
\]  

The \( R_{1/2} \) function is shown in Figure 7. The derivatives are also shown to illustrate their continuity requirements.
If the continuity condition for the first derivative were to be enforced at the support, it would again lead to an inconsistency

\[
\begin{align*}
R_{1/2}(0) &= 1, \\
R_{1/2}(1) &= 0, \\
\left. \frac{dR_{1/2}}{dr} \right|_{r=1} &= 0,
\end{align*}
\]

(151)

giving

\[
\begin{align*}
\frac{n_0}{d_0+n_1} &= 1, \\
\frac{n_0}{d_0+d_1+2d_2} &= 0, \\
\frac{n_0d_0-n_0d_1-2n_0d_2-n_1d_2}{(d_0+d_1+2d_2)^2} &= 0.
\end{align*}
\]

(152)

From the first equation follows that \(d_0 = n_0\). The second equation requires that \(n_1 = -n_0\) and that \(d_0 + d_1 + 2d_2 \neq 0\). From the third,

\[
\begin{align*}
n_1d_0 - n_0d_1 - 2n_0d_2 - n_1d_2 &= 0, \\
-n_0n_0 - n_0d_1 - 2n_0d_2 + n_0d_2 &= 0, \\
n_0 + d_1 + d_2 &= 0,
\end{align*}
\]

(153)\hspace{1cm} (154)\hspace{1cm} (155)

which, once again, leads to an impossibility.

It is worth mentioning that the continuity consideration for the second derivative at \(r = 1\), for a function that already does not possess continuous first derivative, is unnecessary. Any other polynomial obtained varying \(d_2\) would serve equally as a denominator and assure first derivative continuity at the center. Continuity conditions should not be enforced, thus, for higher-order derivatives, after a discontinuity arises at any lesser-order derivative.

### 3.2.3 Rational function of type 2/2

Defining the test function such as in Equation 156, assembled as a ratio between two second-order polynomials,

\[
R_{2/2}(r) = \frac{n_0 + n_1r + n_2r^2}{d_0 + d_1r + d_2r^2}.
\]

(156)

The \(R_{2/2}\) function can then be evaluated at the center and the support radius, ie,

\[
R_{2/2}(0) = \frac{n_0}{d_0}
\]

(157)

and

\[
R_{2/2}(1) = \frac{n_0 + n_1 + n_2}{d_0 + d_1 + d_2}.
\]

(158)

Obtaining the derivative of \(R_{2/2}(r)\),

\[
\frac{dR_{2/2}(r)}{dr} = \frac{(d_0 + d_1r + d_2r^2)(n_1 + n_2r) - (n_0 + n_1r + n_2r^2)(d_1 + d_2r)}{(d_0 + d_1r + d_2r^2)^2},
\]

(159)

and evaluating this derivative at points \(r = 0\) and \(r = 1\),

\[
\left. \frac{dR_{2/2}}{dr} \right|_{r=0} = \frac{d_0n_1 - n_0d_1}{d_0^2},
\]

(160)

\[
\left. \frac{dR_{2/2}}{dr} \right|_{r=1} = \frac{(d_0 + d_1 + d_2)(n_1 + 2n_2) - (n_0 + n_1 + n_2)(d_1 + 2d_2)}{(d_0 + d_1 + d_2)^2}.
\]

(161)
Obtaining, then, the second derivative of the sought function

\[ \frac{d^2R_{2/2}(r)}{dr^2} = \frac{1}{(d_0 + d_1 r + d_2 r^2)^4} \left[ (d_0 + d_1 r + d_2 r^2)^2 \right] \left[ 2(d_0 + d_1 r + d_2 r^2)n_2 
+ (d_1 + 2d_2)(n_1 + 2n_2 r) - (2(n_0 + n_1 r + n_2 r^2)d_2
+ (n_1 + 2n_2)(d_1 + 2d_2 r)) - ((d_0 + d_1 r + d_2 r^2)(n_1 + 2n_2 r) - (n_0 + n_1 r + n_2 r^2)(d_1 + 2d_2 r)] \right) \]

which may be written, simplifying the denominator term, as follows:

\[ \frac{d^2R_{2/2}(r)}{dr^2} = \frac{1}{(d_0 + d_1 r + d_2 r^2)^3} \left[ (d_0 + d_1 r + d_2 r^2) \right] \left[ 2(d_0 + d_1 r + d_2 r^2)n_2 
+ (d_1 + 2d_2)(n_1 + 2n_2 r) - (2(n_0 + n_1 r + n_2 r^2)d_2
+ (n_1 + 2n_2)(d_1 + 2d_2 r)) - ((d_0 + d_1 r + d_2 r^2)(n_1 + 2n_2 r) - (n_0 + n_1 r + n_2 r^2)(d_1 + 2d_2 r)] \right) \]

This second derivative can thus be evaluated at points \( r = 0 \) and \( r = 1 \), ie,

\[ \frac{d^2R_{2/2}}{dr^2} \bigg|_{r=0} = \frac{1}{d_0} \left[ d_0 \left( 2d_0 n_2 + d_1 n_1 - (2n_0 d_2 + n_1 d_1) \right) \right] - (d_0 n_1 - n_0 d_1) \left( 2d_1 \right), \]

and

\[ \frac{d^2R_{2/2}}{dr^2} \bigg|_{r=1} = \frac{1}{(d_0 + d_1 + d_2)^3} \left[ (d_0 + d_1 + d_2) \right] \left[ 2(d_0 + d_1 + d_2)n_2 
+ (d_1 + 2d_2)(n_1 + 2n_2) - (2(n_0 + n_1 + n_2)d_2
+ (n_1 + 2n_2)(d_1 + 2d_2)) - ((d_0 + d_1 + d_2)(n_1 + 2n_2) - (n_0 + n_1 + n_2)(d_1 + 2d_2)] \right) \]

To find a compactly supported radial basis function, the coefficients for both the numerator and the denominator must be found to ensure the compactness of the support. This is accomplished stating that the function must evaluate to unity at \( r = 0 \) and zero for \( r = 1 \). To find the coefficients that cannot be obtained by these identities, the continuity of the derivatives should be enforced. It is important to notice that, since the function is rational, one of the coefficients can remain independent and does not have to be defined, ie,

\[ \begin{cases} 
R_{2/2}(0) = 1, \\
R_{2/2}(1) = 0, \\
\frac{dR_{2/2}}{dr} \bigg|_{r=0} = 0, \\
\frac{dR_{2/2}}{dr} \bigg|_{r=1} = 0. 
\end{cases} \]

The nonlinear system of equations becomes:

\[ \begin{cases} 
\frac{d_0}{d_0} = 1, \\
\frac{n_0 + n_1 + n_2}{d_0 + d_1 + d_2} = 0, \\
\frac{d_0 n_1 - n_0 d_1}{d_0} = 0, \\
\frac{(d_0 + d_1 + d_2)(n_1 + 2n_2) - (n_0 + n_1 + n_2)(d_1 + 2d_2)}{(d_0 + d_1 + d_2)^2} = 0. 
\end{cases} \]
The first condition states that
\[ \frac{n_0}{d_0} = 1, \]  
(168)

which gives
\[ n_0 = d_0. \]  
(169)

The second states that
\[ \frac{n_0 + n_1 + n_2}{d_0 + d_1 + d_2} = 0, \]  
(170)
establishing that
\[ n_0 + n_1 + n_2 = 0, \]  
(171)
and
\[ d_0 + d_1 + d_2 \neq 0. \]  
(172)

From the first derivative continuity at \( r = 0 \), third condition
\[ \frac{d_0 n_1 - n_0 d_1}{d_0^2} = 0, \]  
(173)
which gives
\[ n_1 = d_1, \]  
(174)
assured, obviously, that
\[ d_0 \neq 0. \]  
(175)

The aforementioned requirement is obvious since, if \( d_0 = n_0 \), only first- and second-order terms would exist both in the numerator and the denominator, and they could be simplified, turning into a lesser type rational function.

The fourth condition, first derivative at \( r = 1 \), is
\[ \frac{(d_0 + d_1 + d_2)(n_1 + 2n_2) - (n_0 + n_1 + n_2)(d_1 + 2d_2)}{(d_0 + d_1 + d_2)^2} = 0, \]  
(176)
which, with the already established condition of the nonvanishing denominator, becomes
\[ (d_0 + d_1 + d_2)(n_1 + 2n_2) - (n_0 + n_1 + n_2)(d_1 + 2d_2) = 0. \]  
(177)

The aforementioned expression can be analyzed term by term, with attention to the factors of each term. The first factor of the first term, \( (d_0 + d_1 + d_2) \), cannot vanish, and the first factor of the second term, \( (n_0 + n_1 + n_2) \), certainly vanishes (from condition 171). Thus,
\[ n_1 = -2n_2. \]  
(178)

Taking the aforementioned equation, Equation 178, and Equation 171, it follows that
\[ n_0 - 2n_2 + n_2 = 0, \]  
(179)
\[ n_0 = n_2. \]  
(180)

Therefore, \( n_0 = d_0, n_1 = -2d_0, n_2 = d_0, d_1 = -2d_0, \) and \( d_2 \) is indeterminate.

If the second derivative continuity condition is tested at \( r = 0 \), an inconsistency arises, ie,
\[ \left. \frac{d^2 R_{1/2}}{dr^2} \right|_{r=0} = \frac{1}{d_0^3} \left[ d_0 \left( 2d_0 n_2 + d_1 n_1 - (2n_0 d_2 + n_1 d_1) \right) - (d_0 n_1 - n_0 d_1) \right] 2d_1 = 0, \]  
(181)
\[ d_0 \left( 2d_0 n_2 + d_1 n_1 - (2n_0 d_2 + n_1 d_1) \right) - (d_0 n_1 - n_0 d_1) \right] 2d_1 = 0, \]  
(182)
and, as \( n_0 = d_0 \) and \( n_1 = d_1 \), the second term vanishes, giving
\[ d_0 \left( 2d_0 n_2 - 2n_0 d_2 \right) = 0, \]  
(183)
or

\[ d_2 = n_2, \]  

(184)

which is inconsistent because the same function would appear on the numerator and the denominator, simplifying into a constant unit function.

Testing, thus, the second derivative at \( r = 1, \)

\[
\frac{1}{(d_0 + d_1 + d_2)^3} \left[ (d_0 + d_1 + d_2)(2(d_0 + d_1 + d_2)n_2 + (d_1 + 2d_2)(n_1 + 2n_2) - (2(n_0 + n_1 + n_2)d_2
\]

\[ + (n_1 + 2n_2)(d_1 + 2d_2)) - ((d_0 + d_1 + d_2)(n_1 + 2n_2) - (n_0 + n_1 + n_2)(d_1 + 2d_2))2(d_1 + 2d_2) \] = 0,

(185)

considering that \( n_1 + 2n_2 = 0 \) and \( n_0 + n_1 + n_2 = 0, \)

\[
\frac{1}{(d_2 - d_0)^3} \left[ (d_2 - d_0)(2(d_2 - d_0)d_0 + 0 - (0 + 0)) - (0 - 0)2(2d_2 - 2d_0) \right] = 0,
\]

(186)

\[
\frac{2d_0}{d_2 - d_0} = 0,
\]

(187)

which adds nothing to the already established solution.

The option of testing also the continuity of third derivatives would not add precision to the proposed functions, or to the determination of the parameters. Computing, nonetheless, the third derivative,

\[
\frac{d^3 R_{2/2}(r)}{dr^3} = \frac{1}{(d_0 + d_1r + d_2r)^3} \left[ (d_0 + d_1r + d_2r)^4 \right] \left[ (d_0 + d_1r + d_2r)^2 \right]
\]

\[
+ (d_1 + 2d_2r) (d_0 + d_1r + d_2r)^2 n_2
\]

\[
+ \left( d_1 + 2d_2r \right)(n_1 + 2n_2r) - \left( 2(n_0 + n_1 + n_2r^2) \right) d_2
\]

\[
+ \left( n_1 + 2n_2r \right)(d_1 + 2d_2r))
\]

\[
- (4(\left( d_0 + d_1r + d_2r^2 \right)(n_1 + 2n_2r)
\]

\[
- (n_0 + n_1 + n_2r^2)(d_1 + 2d_2r)) d_2
\]

\[
+ 2 \left( d_0 + d_1r + d_2r^2 \right) n_2
\]

\[
+ (d_1 + 2d_2r)(n_1 + 2n_2r)
\]

\[
- 2\left( n_0 + n_1 + n_2r^2 \right) d_2
\]

\[
+ (n_1 + 2n_2r)(d_1 + 2d_2r))(d_1 + 2d_2r))
\]

\[
- 3\left( (d_0 + d_1r + d_2r^2)(d_0 + d_1r + 2d_2r^2) n_2
\]

\[
+ (d_1 + 2d_2r)(n_1 + 2n_2r) - (2(n_0 + n_1 + n_2r^2) d_2
\]

\[
+ (n_1 + 2n_2r)(d_1 + 2d_2r))
\]

\[
- 2\left( (d_0 + d_1r + d_2r^2 \right)(n_1 + 2n_2r)
\]

\[
- (n_0 + n_1 + n_2r^2)(d_1 + 2d_2r))
\]

\[
(d_1 + 2d_2r))(d_1 + 2d_2r) \right].
\]

(188)

Testing for \( \frac{d^3 R_{2/2}(r)}{dr^3} \bigg|_{r=0} = 0, \)

\[
\frac{d^3 R_{2/2}(r)}{dr^3} \bigg|_{r=0} = \frac{1}{d_0^3} \left[ d_0 (d_0 (4d_1n_2 + 2d_2n_1 - (4n_1d_2 + 2n_2d_1)) + d_1(2d_0n_2 + d_1n_1 - (2n_0d_2 + n_1d_1))
\]

\[
- (4(d_0n_1 - n_0d_1) d_2 + 2(2d_0n_2 + d_1n_1 - (2n_0d_2 + n_1d_1)) d_1))
\]

\[
- 3(d_0 (2d_0n_2 + d_1n_1 - (2n_0d_2 + n_1d_1)) - 2(d_0n_1 - n_0d_1) d_1) d_1 \right] = 0,
\]

(189)
which can be analyzed, eliminating terms

\[
\begin{align*}
    d_0 (4d_1 n_2 + 2d_2 n_1 - (4n_1 d_2 + 2n_2 d_1)) + d_1 (2d_0 n_2 - 2n_0 d_2) - 2 (2d_0 n_2 - 2n_0 d_2) d_1 - 3 (2d_0 n_2 - 2n_0 d_2) d_1 &= 0, \\
    d_0 (2d_1 n_2 - 2d_2 n_1) - 4d_1 (2d_0 n_2 - 2n_0 d_2) &= 0, \\
    2d_0 d_1 n_2 - 2d_0 d_2 n_1 - 8d_0 d_1 n_2 + 8n_0 d_1 d_2 &= 0, \\
    -6d_0 d_1 n_2 + 6d_0 d_1 d_2 &= 0,
\end{align*}
\]

or \(d_2 = n_2\), inconsistent by the motives explained previously. Testing for \[\frac{d^3 R_{2/2}(r)}{dr^3}\] at \(r = 1\),

\[
\begin{align*}
    \left. \frac{d^3 R_{2/2}(r)}{dr^3} \right|_{r=1} &= \frac{1}{(d_0 + d_1 + d_2)^4} \left[ (d_0 + d_1 + d_2) ((d_0 + d_1 + d_2) (4 (d_1 + 2d_2) n_2 + 2d_2 (n_1 + 2n_2) \\
    - (4 (n_1 + 2n_2) d_2 + 2n_2 (d_1 + 2d_2)) \\
    + (d_1 + 2d_2) (2 (d_0 + d_1 + d_2) n_2 + (d_1 + 2d_2) n_1 + 2n_2) \\
    (n_1 + 2n_2) - (2 (n_0 + n_1 + n_2) d_2 + (n_1 + 2n_2) (d_1 + 2d_2)) \\
    (d_1 + 2d_2)) - (4 ((d_0 + d_1 + d_2) \\
    (n_1 + 2n_2) - (n_0 + n_1 + n_2) (d_1 + 2d_2)) d_2 + 2 (2 (d_0 + d_1 + d_2) n_2 + (d_1 + 2d_2) (n_1 + 2n_2) \\
    - (2 (n_0 + n_1 + n_2) d_2 + (n_1 + 2n_2) (d_1 + 2d_2)) \\
    (d_1 + 2d_2)) - 3 ((d_0 + d_1 + d_2) \\
    (2 (d_0 + d_1 + d_2) n_2 + (d_1 + 2d_2) (n_1 + 2n_2) \\
    - (2 (n_0 + n_1 + n_2) d_2 + (n_1 + 2n_2) (d_1 + 2d_2)) \\
    - 2 ((d_0 + d_1 + d_2) (n_1 + 2n_2) - (n_0 + n_1 + n_2) \\
    (d_1 + 2d_2)) (d_1 + 2d_2)) (d_1 + 2d_2) \right] = 0. 
\end{align*}
\]

It can be obtained that, after extensive manipulation,

\[
\begin{align*}
    \frac{1}{(d_0 + d_1 + d_2)^4} \left[ 6(d_0 + d_1 + d_2)^2 n_2 (d_1 + 2d_2) \right] = 0,
\end{align*}
\]

which leads to the same inconsistent result previously obtained: \(d_1 = -2d_2\), which is the same as \(d_2 = n_2\).

It will be adopted the value \(d_2 = 2\) for the \(R_{2/2}\) function (Figure 8),

\[
R_{2/2}(r) = \frac{1 - 2r + r^2}{1 - 2r + 2r^2}. 
\]
3.2.4 | Rational function of type 2/3

The next rational function can be postulated as a ratio between a second- and a third-order polynomials,

\[ R_{2/3}(r) = \frac{n_0 + n_1 r + n_2 r^2}{d_0 + d_1 r + d_2 r^2 + d_3 r^3}. \]  

(197)

The function parameters can be obtained from the conditions on the function and the derivatives at \( r = 0 \) and at \( r = 1 \). For \( r = 1 \),

\[ R_{2/3}(1) = 0 \Rightarrow n_0 + n_1 + n_2 = 0; \]  

(198)

\[ R'_{2/3}(1) = 0 \Rightarrow n_1 + 2n_2 = 0; \]  

(199)

and for \( r = 0 \),

\[ R_{2/3}(0) = 1 \Rightarrow n_0 = d_0; \]  

(200)

\[ R'_{2/3}(0) = 0 \Rightarrow n_1 = d_1; \]  

(201)

\[ R''_{2/3}(0) = 0 \Rightarrow n_2 = d_2. \]  

(202)

Solving the nonlinear system of equations, it is obtained, from Equations 198 (in light of Equation 200) and 198, \( n_2 = d_0 \) and \( n_1 = -2d_0 \). Thus, it results that, from Equation 200, \( d_1 = -2d_0 \), and, from Equation 201, \( d_2 = d_0 \).

Therefore,

\[ R_{2/3}(r) = \frac{1 - 2r + r^2}{1 - 2r + r^2 + r^3}. \]  

(203)

where \( d_3 \) was arbitrarily set to unit (\( d_3 = 1 \)).

Continuity of the second derivative cannot be guaranteed at \( r = 1 \). Moreover, continuity of third the derivative either at the center or at the support radius cannot be guaranteed.

Figure 9 presents this function and its first and second derivatives.

3.2.5 | Rational function of type 3/3

Defining the rational function of type 3/3, constructed by the ratio of two cubic polynomials,

\[ R_{3/3}(r) = \frac{n_0 + n_1 r + n_2 r^2 + n_3 r^3}{d_0 + d_1 r + d_2 r^2 + d_3 r^3}. \]  

(204)

The conditions for the function at \( r = 0 \) and at \( r = 1 \) are obtained after the derivatives. For \( r = 1 \),

\[ R_{3/3}(1) = 0 \Rightarrow n_0 + n_1 + n_2 + n_3 = 0; \]  

(205)

\[ R'_{3/3}(1) = 0 \Rightarrow n_1 + 2n_2 + 3n_3 = 0; \]  

(206)

\[ R''_{3/3}(1) = 0 \Rightarrow 2n_2 + 6n_3 = 0; \]  

(207)

and for \( r = 0 \),

\[ R_{3/3}(0) = 1 \Rightarrow n_0 = d_0; \]  

(208)

\[ R'_{3/3}(0) = 0 \Rightarrow n_1 = d_1; \]  

(209)

\[ R''_{3/3}(0) = 0 \Rightarrow n_2 = d_2. \]  

(210)
Solving the aforementioned system of equations, it is obtained, from Equation 205 (using Equation 208), Equation 205, and Equation 206, \( n_3 = -d_0, n_2 = 3d_0, \) and \( n_1 = -3d_0. \) Thus, it results that, from Equation 208, \( d_1 = -3d_0, \) and, from Equation 209, \( d_2 = 3d_0. \)

Therefore,

\[
R_{3/3}(r) = \frac{1 - 3r + 3r^2 - r^3}{1 - 3r + 3r^2 + r^3},
\]

(211)

where \( d_3 \) was arbitrarily set to unity (\( d_3 = 1 \)).

The continuity of third derivatives cannot be guaranteed either at 0 or at 1. Figure 10 presents this function and its derivatives.

### 3.2.6 Rational function of type 3/4

Defining the rational function of type 3/4, constructed by dividing a cubic polynomial by a fourth-order polynomial,

\[
R_{3/4}(r) = \frac{n_0 + n_1 r + n_2 r^2 + n_3 r^3}{d_0 + d_1 r + d_2 r^2 + d_3 r^3 + d_4 r^4}
\]

(212)

The conditions for the function at \( r = 0 \) and at \( r = 1 \) are obtained after the derivatives.

For \( r = 1,\)

\[
R_{3/4}(1) = 0 \Rightarrow n_0 + n_1 + n_2 + n_3 = 0;
\]

(213)

\[
R'_{3/4}(1) = 0 \Rightarrow n_1 + 2n_2 + 3n_3 = 0;
\]

(214)

\[
R''_{3/4}(1) = 0 \Rightarrow 2n_2 + 6n_3 = 0;
\]

(215)

and for \( r = 0,\)

\[
R_{3/4}(0) = 1 \Rightarrow n_0 = d_0;
\]

(216)

\[
R'_{3/4}(0) = 0 \Rightarrow n_1 = d_1;
\]

(217)

\[
R''_{3/4}(0) = 0 \Rightarrow n_2 = d_2;
\]

(218)

\[
R'''_{3/4}(0) = 0 \Rightarrow n_3 = d_3.
\]

(219)

Solving the aforementioned system of equations, it is obtained, from Equation 213 (using Equation 216), Equation 213, and Equation 214, \( n_3 = -d_0, n_2 = 3d_0, \) and \( n_1 = -3d_0. \) Thus, it results that, from Equation 216, \( d_1 = -3d_0, \) from Equation 217, \( d_2 = 3d_0, \) and, from Equation 218, \( d_3 = -d_0.\)

Therefore,

\[
R_{3/4}(r) = \frac{1 - 3r + 3r^2 - r^3}{1 - 3r + 3r^2 + r^3},
\]

(220)

where \( d_4 \) was arbitrarily set to unity (\( d_4 = 1 \)).

The continuity of the third derivative cannot be guaranteed at 1, or of fourth derivative for either the center or the support radius. Figure 11 presents this function and its derivatives.

Higher-order functions can be obtained in much the same way, but this procedure will not be shown here. Functions \( R_{4/4}, R_{5/4}, \) and \( R_{5/5} \) were also obtained, though, and will be shown next (Table 2).
### TABLE 2  Rational functions and continuity

| Function         | Continuity | Observation |
|------------------|------------|-------------|
| \( R_{1/2}(r) = \frac{1-r}{1+r} \) | \( C^2 \) | Center |
| \( R_{1/2}(r) = \frac{1-r^2}{2-r^2} \) | \( C^0 \) | Center |
| \( R_{1/2}(r) = \frac{1-r^2}{2-r^2} \) | \( C^2 \) | Center |
| \( R_{2/3}(r) = \frac{1-r^3}{1-r^3} \) | \( C^2 \) | Center |
| \( R_{3/4}(r) = \frac{1-r^4}{1-r^4} \) | \( C^2 \) | Center |
| \( R_{4/5}(r) = \frac{1-r^5}{1-r^5} \) | \( C^2 \) | Center |

Center - The next higher derivative is continuous at the center \((r = 0)\).

### 3.3 Summary of proposed functions

To summarize the proposed function classes, Tables 1 and 2 are presented. In these tables, the continuity properties are given for each function.

### 4 NUMERICAL RESULTS

In order to investigate the behavior of these novel classes of proposed functions, the approximation of Franke’s function is attempted (Figure 12). This function, proposed by Richard Franke in his research report alongside other functions, has gained notoriety in the approximation community by its mathematical characteristics.

The approximation domain will be limited to \( 0 \leq x, y \leq 1 \), ie,

\[
f(x) = 0.75e^{-\left(\frac{(0x-2)^2+2y-2^2}{4}\right)} + 0.75e^{-\left(\frac{(0x-2)^2-2y-2^2}{4}+0.5\right)} + 0.5e^{-\left(\frac{(0x-2)^2-2y-2^2}{4}+0.2\right)} - 0.2e^{-(9x-4)^2-(9y-7)^2}.\]  

\[(221)\]

### FIGURE 12  Representation of Franke’s function
The approximation problem is modeled using clouds of squarely distributed function center points, using 9, 16, 36, 121, 256, and 441 functions (for 3, 4, 6, 11, 16, and 21 function centers along each border of the square domain, squarely distributed inside). A discrete least squares procedure, in which the error computed at a discrete set of points is used to approximate the sought function, is performed using 2601 squarely distributed evaluation points (51 points on each face). After the calculation is performed for each approximating function, the normalized root mean squared error is estimated as the root mean residue of the algebraic equation (i.e., the difference between the computed value and the function value), summed over all the evaluation points and normalized by the amplitude of the function.

The results for the Wendland class of functions are shown in Table 3. These results are presented in Figure 13. The residue error values are plotted against the number of approximating functions using logarithmic scales in an attempt to capture the convergence rates of the chosen method for each of the tested functions.

The results for the Wu class of functions are shown in Table 4. Figure 14 shows these results.

| Number | 9    | 16   | 36   | 121  | 256  | 441  |
|--------|------|------|------|------|------|------|
| \(W_{e_{c_r^2-PD_1}}\) | 0.112965 | 0.045241 | 0.030487 | 0.005877 | 0.005603 | 0.005465 |
| \(W_{e_{c_r^2-PD_2}}\) | 0.097005 | 0.045059 | 0.009278 | 0.002749 | 0.002995 | 0.002821 |
| \(W_{e_{c_r^2-PD_3}}\) | 0.092619 | 0.050196 | 0.008352 | 0.003868 | 0.003986 | 0.003950 |
| \(W_{e_{c_r^2-PD_4}}\) | 0.105475 | 0.073850 | 0.010344 | 0.003961 | 0.003865 | 0.003815 |
| \(W_{e_{c_r^2-PD_5}}\) | 0.106367 | 0.047833 | 0.012505 | 0.003959 | 0.003215 | 0.003550 |
| \(W_{e_{c_r^2-PD_6}}\) | 0.109309 | 0.063335 | 0.012450 | 0.006160 | 0.005872 | 0.005811 |
| \(W_{e_{c_r^2-PD_7}}\) | 0.105475 | 0.073850 | 0.010344 | 0.003961 | 0.003865 | 0.003815 |
| \(W_{e_{c_r^2-PD_8}}\) | 0.106367 | 0.047833 | 0.012505 | 0.003959 | 0.003215 | 0.003550 |
| \(W_{e_{c_r^2-PD_9}}\) | 0.109309 | 0.063335 | 0.012450 | 0.006160 | 0.005872 | 0.005811 |
| \(W_{e_{c_r^2-PD_{10}}}\) | 0.110355 | 0.068139 | 0.018520 | 0.014271 | 0.012545 | 0.013142 |

| Number | 9    | 16   | 36   | 121  | 256  | 441  |
|--------|------|------|------|------|------|------|
| \(W_{e_{c_r^2-PD_1}}\) | 0.114166 | 0.066342 | 0.020108 | 0.014931 | 0.012815 | 0.013317 |
| \(W_{e_{c_r^3-PD_1}}\) | 0.103099 | 0.063335 | 0.012450 | 0.006160 | 0.005872 | 0.005811 |
| \(W_{e_{c_r^3-PD_2}}\) | 0.103099 | 0.063335 | 0.012450 | 0.006160 | 0.005872 | 0.005811 |
| \(W_{e_{c_r^3-PD_3}}\) | 0.103099 | 0.063335 | 0.012450 | 0.006160 | 0.005872 | 0.005811 |
| \(W_{e_{c_r^3-PD_4}}\) | 0.103099 | 0.063335 | 0.012450 | 0.006160 | 0.005872 | 0.005811 |
| \(W_{e_{c_r^3-PD_5}}\) | 0.103099 | 0.063335 | 0.012450 | 0.006160 | 0.005872 | 0.005811 |
| \(W_{e_{c_r^3-PD_6}}\) | 0.103099 | 0.063335 | 0.012450 | 0.006160 | 0.005872 | 0.005811 |
| \(W_{e_{c_r^3-PD_7}}\) | 0.103099 | 0.063335 | 0.012450 | 0.006160 | 0.005872 | 0.005811 |
| \(W_{e_{c_r^3-PD_8}}\) | 0.103099 | 0.063335 | 0.012450 | 0.006160 | 0.005872 | 0.005811 |
| \(W_{e_{c_r^3-PD_9}}\) | 0.103099 | 0.063335 | 0.012450 | 0.006160 | 0.005872 | 0.005811 |
| \(W_{e_{c_r^3-PD_{10}}}\) | 0.103099 | 0.063335 | 0.012450 | 0.006160 | 0.005872 | 0.005811 |
The results for the proposed inverse functions are shown in Table 5.
The results for the rational functions are shown in Table 6.

Each set of results for the proposed functions is plotted, respectively, at Figures 15 and 16.

As can be observed, the inverse functions perform as well as both Wendland or Wu functions, and the rational functions also perform similarly. Rational functions present a higher computed error and do not seem to converge at the same rate of the others. However, the rational function results for root mean square residue error using 9 functions or 16 functions are comparable, if not better than the results with other classes. They seem to be adequate for rough estimates.

The convergence rate is calculated by linear regression for each of the tested functions assuming a negative power law for the error function,

$$\epsilon(N_{RBF}) = A(N_{RBF})^{-B},$$

where $A$ is a mathematical parameter corresponding to a possible solution with one approximating function, $B$ is the convergence rate, $\epsilon$ is the normalized root mean squared residue error, and $N_{RBF}$ is the number of RBFs.
The calculated parameters for the given data are shown in Tables 7 to 10. On the given tables, the values for the parameters $A$ and $B$ and their respective asymptotic standard errors $\sigma_A$ and $\sigma_B$ are shown.

For Wendland, Wu, and inverse functions, the first parameter evaluates between one and two and a half for most of the tested functions. The asymptotic standard error associated to this parameter was high in some of the analyses, but this accounts for the lack of more evaluations. Some of the rational functions behaved in a similar way as the lower-order polynomial ratios, whereas the higher-order polynomial ratios behaved differently. The accuracy appeared good for the initial values. With respect to the convergence rate $B$, it is observed that the values for the proposed functions are not quite distant from those for well-accepted CSRBFs and that, with a few exceptions, the standard deviation is acceptable. Particularly, the inverse function class presents an average $B$ value of 1.15, whereas Wendland functions average $B$ value
TABLE 8  Convergence parameters for Wu functions

| Functions | $A$ | $\sigma_A$ | $B$ | $\sigma_B$ |
|-----------|-----|------------|-----|------------|
| WuC$_{0\cap PD_1}$ | 1.48531 | $\pm 0.6512$ | 1.18572 | $\pm 0.1845$ |
| WuC$_{0\cap PD_1}$ | 2.31583 | $\pm 0.5389$ | 1.44127 | $\pm 0.1002$ |
| WuC$_{0\cap PD_1}$ | 1.58911 | $\pm 0.5980$ | 1.28688 | $\pm 0.1600$ |
| WuC$_{0\cap PD_1}$ | 1.27493 | $\pm 0.5463$ | 1.17298 | $\pm 0.1801$ |
| WuC$_{0\cap PD_1}$ | 2.44285 | $\pm 0.4272$ | 1.42567 | $\pm 0.07523$ |
| WuC$_{0\cap PD_1}$ | 1.86561 | $\pm 0.6373$ | 1.34955 | $\pm 0.1461$ |
| WuC$_{0\cap PD_1}$ | 1.39919 | $\pm 0.5655$ | 1.22078 | $\pm 0.1707$ |
| WuC$_{0\cap PD_1}$ | 1.74837 | $\pm 0.5980$ | 1.17298 | $\pm 0.1600$ |
| WuC$_{0\cap PD_1}$ | 1.26220 | $\pm 0.5506$ | 1.15775 | $\pm 0.1830$ |
| WuC$_{0\cap PD_1}$ | 1.20239 | $\pm 0.4609$ | 1.10051 | $\pm 0.1597$ |
| WuC$_{0\cap PD_1}$ | 1.34561 | $\pm 0.5470$ | 1.18222 | $\pm 0.1710$ |

TABLE 9  Convergence parameters for inverse functions

| Functions | $A$ | $\sigma_A$ | $B$ | $\sigma_B$ |
|-----------|-----|------------|-----|------------|
| I$_1$ | 2.00992 | $\pm 0.4911$ | 1.31306 | $\pm 0.1041$ |
| I$_2$ | 1.65109 | $\pm 0.5400$ | 1.25661 | $\pm 0.1387$ |
| I$_3$ | 0.482108 | $\pm 0.1582$ | 0.618397 | $\pm 0.1218$ |
| I$_4$ | 1.83753 | $\pm 0.6326$ | 1.34670 | $\pm 0.1472$ |
| I$_5$ | 1.71046 | $\pm 0.4792$ | 1.33147 | $\pm 0.1196$ |
| I$_6$ | 1.76165 | $\pm 0.6986$ | 1.33370 | $\pm 0.1693$ |
| I$_7$ | 1.12377 | $\pm 0.3489$ | 1.16899 | $\pm 0.1304$ |
| I$_8$ | 0.974848 | $\pm 0.3204$ | 1.11380 | $\pm 0.1372$ |
| I$_9$ | 1.14979 | $\pm 0.4641$ | 1.16153 | $\pm 0.1694$ |
| I$_{10}$ | 1.41038 | $\pm 0.447$ | 1.17827 | $\pm 0.1333$ |
| I$_{11}$ | 0.73588 | $\pm 0.2823$ | 0.997944 | $\pm 0.1574$ |
| I$_{12}$ | 0.997117 | $\pm 0.4006$ | 1.08842 | $\pm 0.1671$ |
| I$_{13}$ | 1.27593 | $\pm 0.4098$ | 1.12993 | $\pm 0.1633$ |

TABLE 10  Convergence parameters for rational functions

| Functions | $A$ | $\sigma_A$ | $B$ | $\sigma_B$ |
|-----------|-----|------------|-----|------------|
| R$_{1/1}$ | 2.00992 | $\pm 0.4911$ | 1.31306 | $\pm 0.1041$ |
| R$_{1/2}$ | 1.19093 | $\pm 0.1780$ | 1.04848 | $\pm 0.06181$ |
| R$_{1/2}$ | 2.03042 | $\pm 0.6584$ | 1.40419 | $\pm 0.1393$ |
| R$_{1/3}$ | 1.51190 | $\pm 0.5705$ | 1.27271 | $\pm 0.1602$ |
| R$_{1/3}$ | 0.658524 | $\pm 0.1930$ | 0.958627 | $\pm 0.1195$ |
| R$_{1/4}$ | 0.522879 | $\pm 0.2018$ | 0.843464 | $\pm 0.1538$ |
| R$_{1/4}$ | 0.271619 | $\pm 0.08378$ | 0.602867 | $\pm 0.1137$ |
| R$_{1/5}$ | 0.237852 | $\pm 0.07287$ | 0.524382 | $\pm 0.1086$ |
| R$_{1/5}$ | 0.265056 | $\pm 0.1039$ | 0.578286 | $\pm 0.1429$ |

is 1.12, and Wu functions have $B_{av} = 1.23$. The higher-order rational functions present this convergence rate lower than the others, as was already observed on the plots (Figure 16).

5 | FINAL REMARKS

The proposed sets of functions, especially the inverse functions, when compared to two well-accepted CSRBF classes, present similar convergence parameters. For the rational class, the convergence rate is lower, especially for higher-order functions. These results confirm the initial hypothesis that the proposed classes of CSRBFs can be used to correctly approximate mathematical functions: not only the estimated error for most of the analyses performed was similar but also the convergence rate also returns comparable values. Rational functions seem to be accurate for rough estimates using a small number of approximating functions, whereas inverse functions can be used on convergence analyses.
These results confirm that the proposed functions perform well for approximation problems by a discrete least squares evaluation, as long as the number of approximation functions is sufficient (for a squarely distributed function center cloud) and the number of least squares discrete evaluation points is enough to capture the peculiarities of the specific point cloud.

The results also demand more work on the mathematical properties of the proposed classes of functions, such as the complete definition of their positive definiteness dimensionality, for this is an essential property toward a completeness of solution and mathematical convergence proof. The ease and simplicity of derivation, though, help in this direction.

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CONFLICT OF INTEREST
The author declares no potential conflict of interests.

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