A Unified Convergence Analysis of First Order Convex Optimization Methods via Strong Lyapunov Functions

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Abstract
We present a unified convergence analysis for first order convex optimization methods using the concept of strong Lyapunov conditions. Combining this with suitable time scaling factors, we are able to handle both convex and strong convex cases, and establish contraction properties of Lyapunov functions for many existing ordinary differential equation models. Then we derive prevailing first order optimization algorithms, such as proximal gradient methods, heavy ball methods (also known as momentum methods), Nesterov accelerated gradient methods, and accelerated proximal gradient methods from numerical discretizations of corresponding dynamical systems. We also apply strong Lyapunov conditions to the discrete level and provide a more systematical analysis framework. Another contribution is a novel second order dynamical system called Hessian-driven Nesterov accelerated gradient flow which can be used to design and analyze accelerated first order methods for smooth and non-smooth convex optimizations.

Keywords: Unconstrained convex optimization, first order method, dynamical system, Lyapunov function, exponential decay, gradient flow, heavy ball system, asymptotic vanishing damping system, proximal gradient method, momentum method, Nesterov acceleration

1. Introduction
We consider first order methods for solving the unconstrained convex minimization problem

$$\min_{x \in V} f(x).$$

(1)

Above and throughout $V$ is a Hilbert space and $V^*$ is its dual space. First order optimization methods for solving (1) regains the popularity in the application of large scale machine learning (Bottou et al., 2018).

Denoted by $x^*$ a global minimizer of $f$ which is unique when $f$ is strictly convex. Instead of solving the Euler equation $\nabla f(x^*) = 0$, we consider continuous optimization methods which start from some ordinary differential equation (ODE)

$$x'(t) = G(x(t)), \quad t > 0.$$

(2)
Here in general, $\mathbf{x}$ is a vector-valued function of time variable $t$ and $\mathcal{G}$ is a vector field, which can be the negative gradient $-\nabla f$ or any reasonable alternate. We assume $\mathbf{x}^*$ (containing $\mathbf{x}^*$ as a component when $\mathbf{x}$ is a vector) is an equilibrium point of the autonomous dynamical system (2), i.e. $\mathcal{G}(\mathbf{x}^*) = 0$, and ideally this shall imply $\nabla f(\mathbf{x}^*) = 0$.

A simple example is $\mathcal{G} = -\nabla f$, with which the ODE (2) becomes the well known gradient flow $\mathbf{x}' = -\nabla f(\mathbf{x})$. For this standard model, the explicit (forward) Euler scheme leads to the gradient descent method for solving (1), and the implicit (backward) Euler scheme corresponds to the proximal point algorithm (Güler, 1991; Rockafellar, 1976). When extended to the composite case $f = h + g$ with smooth $h$ and non-smooth $g$, the semi-implicit discretization, also known as the forward-backward scheme, recovers the proximal gradient method (Parikh and Boyd, 2014).

Besides the gradient flow, many more (second order) dynamic systems, such as the heavy ball model (Polyak, 1964), the asymptotic vanishing damping (AVD) system (Su et al., 2016), the dynamic inertial of Newton system (Alvarez et al., 2002), and the ODE based variational method (Wibisono et al., 2016; Wilson et al., 2021) etc, have been developed to explain the acceleration mechanism and design new first order optimization methods as well. In Luo and Chen (2019), we have proposed the so-called Nesterov accelerated gradient flow and provided an explanation on the acceleration phenomena by using the so-called A-stability of ODE solvers. All the models mentioned here admit the unified first order form (2) with different ODE solvers.

The long time decay property of the continuous problem (2) is very important and gives insights on the rate of convergence of the corresponding optimization methods (Su et al., 2016). Appropriate discretizations of the above ODE systems will lead to accelerated first order methods such as the heavy ball method (Polyak, 1964), Nesterov’s accelerated gradient method (Nesterov, 1983), and the accelerated proximal gradient method (Beck and Teboulle, 2009; Tseng, 2008) etc. The analysis of discrete algorithms, however, is not a straightforward translation from the continuous level. A standard work flow is to design a Lyapunov function and establish the decay of that Lyapunov function; see Shi et al. (2018); Wilson et al. (2021); Siegel (2019) among many others. This procedure often involves tricky algebraic manipulation and tedious calculations. Indeed in Su et al. (2016, Section 6), the authors conclude that “a general theory mapping properties of ODEs into corresponding properties for discrete updates would be a welcome advance.”

In this paper we will propose such a theory using a new concept: strong Lyapunov condition. Recall that, in order to study the stability of an equilibrium of a dynamical system, e.g. (2), Lyapunov introduced the so-called Lyapunov function $\mathcal{L}(\mathbf{x})$ (see Khalil, 2001), which is nonnegative and satisfies $\mathcal{L} (\mathbf{x}^*) = 0$ and the Lyapunov condition:

$$- \nabla \mathcal{L}(\mathbf{x}) \cdot \mathcal{G}(\mathbf{x}) \text{ is locally positive near the equilibrium point } \mathbf{x}^*. \quad (3)$$

Then the (local) decay property of $\mathcal{L}(\mathbf{x}(t))$ along the trajectory $\mathbf{x}(t)$ of the autonomous system (2) can be derived immediately

$$\frac{d}{dt} \mathcal{L}(\mathbf{x}(t)) = \nabla \mathcal{L}(\mathbf{x}) \cdot \mathbf{x}'(t) = \nabla \mathcal{L}(\mathbf{x}) \cdot \mathcal{G}(\mathbf{x}) < 0.$$ 

Therefore $\mathcal{L}(\mathbf{x}(t)) \to 0$ as $t \to \infty$ from which we may conclude $x(t) \to x^*$ or $f(x(t)) \to f(x^*)$ as $t \to \infty$. However, this can only imply the convergence not the rate of convergence, i.e., how fast $\mathcal{L}(\mathbf{x}(t))$ approaches to zero.
To establish the convergence rate of $L(x(t))$, we introduce the following strong Lyapunov condition: $L(x)$ is a Lyapunov function and there exist constant $q \geq 1$, strictly positive function $c(x)$ and function $p(x)$ such that

$$-\nabla L(x) \cdot G(x) \geq c(x)L^q(x) + p^2(x)$$

holds true near $x^*$. From this, we can easily derive the exponential decay $L(x(t)) = O(e^{-ct})$ for $q = 1$ and the algebraic decay $L(x(t)) = O(t^{1/(1-q)})$ for $q > 1$. We emphasize that the condition (4) is not only restricted to the strongly convex case. It can be established for convex case; see (56) for the gradient flow and (101) for the AVD system.

We apply our framework to design and analyze first-order optimization methods, especially the accelerated gradient methods, for smooth and non-smooth convex optimization problems. We believe our unified convergence analysis is more transparent and systematic. Specifically, once the dynamical system (2) is discretized in time by

$$x_{k+1} - x_k = \alpha_k \tilde{G}(x_k, x_{k+1}),$$

where $\tilde{G}(x_k, x_{k+1})$ is an approximation of $G(x_{k+1})$, a sequence of points $\{x_k\}$ is produced. Given some strong Lyapunov function $L(x)$ that possess fast decay in the continuous level, a discrete Lyapunov function $L_k = L(x_k)$ appear naturally. Due to the discretization error, the discrete dynamic system (5) may not be faithful to the continuous one (2). Whence, it is nontrivial to say that the scheme (5) preserves the decay property in the discrete level. Fortunately, the strong Lyapunov condition (4) works for $L_k$ and we will use it to guide the designing and analysis of optimization algorithms. A paradigm of our analysis is summarized in the following three steps.

- First expand the difference

$$L(x_{k+1}) - L(x_k) = (\nabla L(x_{k+1}), x_{k+1} - x_k) - R_1,$$

where $R_1 \geq 0$ is the Bregman divergence of $L$. The negative remainder $-R_1$ is introduced due to the convexity of $L$ which can be built-in when designing $L$.

- Then compare the right hand side of the discretization (5) with $\alpha_k \tilde{G}(x_{k+1})$ and obtain

$$L(x_{k+1}) - L(x_k) \leq \alpha_k (\nabla L(x_{k+1}), \tilde{G}(x_{k+1})) - R_1 + R_2,$$

where the positive term $R_2$ comes from the lagging of discretization, i.e., $\tilde{G}(x_k, x_{k+1}) - G(x_{k+1})$, which is generally nonzero for using partial information from $x_k$.

- Finally applying strong Lyapunov property (4) at $x_{k+1}$ to (6) will bring more negative term $-p^2(x_{k+1})$, which together with $-R_1$ cancels the lagging effect $R_2$ and thus implies

$$L_{k+1} - L_k \leq -\alpha_k L^q_{k+1},$$

from which linear or sub-linear decay rate of the sequence $\{L_k\}$ can be derived.
Here, we mention a most related work Wilson et al. (2021). They derived dynamical models from the Bregman–Lagrangian and showed an equivalence between the technique of estimate sequences devised by Nesterov (2013) and a family of Lyapunov functions in both continuous and discrete time. Note that their attentions were only paid to smooth objectives and they treated convex case and strongly convex case separately. In this work, however, we handle both convex and strongly convex cases simultaneously by introducing a time scaling factor and unify the verification of the contraction of Lyapunov function via the tool of strong Lyapunov condition which is also generalizable to non-smooth cases.

The rest of this paper is outlined as follows. Section 2 is responsible for a brief review of preliminary inequalities involving convex functions, and Section 3 introduces the strong Lyapunov condition and also provides some key estimates. As a revisit of the gradient descent method and the proximal point algorithm, Sections 4 and 5 shall apply our Lyapunov framework to the gradient flow and the scaled gradient flow, respectively. After that, Sections 6 and 8 focus on some typical second-order dynamical systems and give the corresponding convergence rate analysis via strong Lyapunov functions. Finally, Section 9 ends this paper with some concluding remarks.

2. Bounds on Convex Functions

This section gives a quick review of several bounds on convex functions. Throughout, we consider both smooth convex functions over the entire space $V$ and extended-value function $f : V \to \mathbb{R} \cup \{+\infty\}$. For the latter, the effective domain of $f$ is denoted by $\text{dom } f := \{x \in V : f(x) < \infty\}$.

2.1 Convex functions

A continuous function $f$ is called convex if

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y) \quad \forall \, x, y \in \text{dom } f,$$

for all $\alpha \in [0, 1]$, and it is called strictly convex if the above inequality holds strictly

$$f(\alpha x + (1 - \alpha)y) < \alpha f(x) + (1 - \alpha)f(y) \quad \forall \, x, y \in \text{dom } f \text{ and } x \neq y,$$

for all $\alpha \in (0, 1)$. A convex function is called $\mu$-strongly convex with parameter $\mu > 0$ if

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y) - \frac{\mu}{2} \alpha(1 - \alpha)\|x - y\|^2 \quad \forall \, x, y \in \text{dom } f,$$

for all $\alpha \in [0, 1]$.

The function $f$ is coercive if $f(x) \to \infty$ when $\|x\| \to \infty$. When $f$ is $\mu$-strongly convex with $\mu > 0$, then it is not hard to see $f$ is coercive. But convexity itself cannot imply the coercivity, e.g. $f(x) = e^{-x}$. The following results are classical, and proofs, which are skipped for the sake of brevity, can be found in Ekeland and Témam (1987, Proposition 1.2) or Ciarlet (1989, Theorem. 8.2.2).

**Theorem 2.1** If $f$ is convex and coercive, then the problem (1) admits at least one solution $x^* \in V$, which is unique if we assume further $f$ is strictly convex.
2.2 Convex function classes

Let \( C^1 \) consist of all continuous differentiable functions on \( V \). Denote by \( C^{1,1}_L \) the set of all \( C^1 \) functions, the gradient of which is Lipschitz continuous with constant \( 0 < L < \infty \):

\[
\| \nabla f(x) - \nabla f(y) \|_* \leq L \| x - y \| \quad \forall x, y \in V,
\]

where, for \( g \in V^* \), the dual norm is

\[
\| g \|_* := \sup_{v \in V} \frac{\langle g, v \rangle}{\| v \|} = \sup_{v \in V \setminus \{0\}} \frac{\langle g, v \rangle}{\| v \|}.
\]

We now introduce several function classes of convex functions. For \( \mu > 0 \), we use \( S^0_\mu \) to denote the set of all \( \mu \)-strongly convex functions, and \( S^0_0 \) for convex functions, where the superscript 0 indicates the function is only continuous and may not be differentiable. Also, any \( f \in S^0_\mu \) is assumed to be closed and proper (\( \text{dom} f \neq \emptyset \)). Moreover, for all \( \mu \geq 0 \) we set \( S^1_\mu := S^0_\mu \cap C^1 \). For constants \( 0 \leq \mu \leq L < \infty \), we introduce the function class

\[
S^{1,1}_{\mu,L} := \{ f \in S^1_1 : \mu \| x - y \|^2 \leq \langle \nabla f(x) - \nabla f(y), x - y \rangle \leq L \| x - y \|^2 \forall x, y \in V \}.
\]

Set \( S^{1,1}_{\mu,L} = S^1_{\mu,L} \cap C^{1,1}_L \). It can be shown that \( S^{1,1}_{\mu,L} = S^1_{\mu,L} \); see Lessard et al. (2016).

2.3 Bregman divergence and various bounds

For \( f \in C^1 \), define

\[
D_f(y, x) := f(y) - f_1(y; x) = f(y) - f(x) - \langle \nabla f(x), y - x \rangle,
\]

where \( f_1(y; x) := f(x) + \langle \nabla f(x), y - x \rangle \) is its linear Taylor expansion at \( x \). If \( f \) is convex, then for fixed \( x \in V \), \( D_f(\cdot, x) \) is also convex and thus \( D_f(y, x) \geq 0 \). When \( f \) is strictly convex, \( D_f(y, x) = 0 \) iff \( x = y \), and \( D_f(y, x) \) is called the Bregman divergence associated with \( f \), which is in general not symmetric.

We then introduce its symmetrization, the symmetrized Bregman divergence,

\[
2M_{\nabla f}(x, y) := D_f(y, x) + D_f(x, y) = \langle \nabla f(x) - \nabla f(y), x - y \rangle.
\]

By the fundamental theorem of calculus

\[
D_f(y, x) = \left\langle \int_0^1 \nabla f(x + \xi(y - x)) - \nabla f(x) \, d\xi, y - x \right\rangle = \int_0^1 2M_{\nabla f}(x_\xi, x) \frac{d\xi}{\xi}, \quad x_\xi := x + \xi(y - x).
\]

Based on (12), we can shift the bound for \( D_f(y, x) \) to \( M_{\nabla f}(x, y) \) and vice versa. The following bounds can be found in Nesterov (2013, Chapter 2).

**Lemma 2.1** For \( f \in C^{1,1}_L \), we have the upper bound

\[
\max\{D_f(y, x), M_{\nabla f}(x, y)\} \leq \frac{L}{2} \| x - y \|^2.
\]
For \( f \in S_{\mu}^1 \) with \( \mu \geq 0 \), we have the lower bound
\[
\min\{D_f(y, x), M_{\nabla f}(x, y)\} \geq \frac{\mu}{2}\|x - y\|^2.
\]
(14)
For \( f \in S_{0, L}^1 \), we have the lower bound
\[
\min\{D_f(y, x), M_{\nabla f}(x, y)\} \geq \frac{1}{2L}\|\nabla f(y) - \nabla f(x)\|^2_*.
\]
(15)
For \( f \in S_{\mu}^1 \) with \( \mu > 0 \), we have the upper bound
\[
\max\{D_f(y, x), M_{\nabla f}(x, y)\} \leq \frac{1}{2\mu}\|\nabla f(y) - \nabla f(x)\|^2_*.
\]
(16)
All the above inequalities hold for all \( x, y \in \text{dom} \ f \).

2.4 Bounds involving a global minimum

We list specific examples of inequalities when one variable is \( x^* \) for which \( \nabla f(x^*) = 0 \). Then \( D_f(x, x^*) = f(x) - f(x^*) \) is the so-called optimality gap and \( 2M_{\nabla f}(x, x^*) = \langle \nabla f(x), x - x^* \rangle \).

Corollary 2.1 For \( f \in S_{0, L}^1 \), we have
\[
\frac{1}{2L}\|\nabla f(x)\|^2_2 \leq f(x) - f(x^*) \leq \frac{L}{2}\|x - x^*\|^2,
\]
(17)
\[
\frac{1}{L}\|\nabla f(x)\|^2_2 \leq \langle \nabla f(x), x - x^* \rangle \leq L\|x - x^*\|^2.
\]
(18)
For \( f \in S_{\mu}^1 \) with \( \mu > 0 \), we have
\[
\frac{\mu}{2}\|x - x^*\|^2 \leq f(x) - f(x^*) \leq \frac{1}{2\mu}\|\nabla f(x)\|^2_*,
\]
(19)
\[
\mu\|x - x^*\|^2 \leq \langle \nabla f(x), x - x^* \rangle \leq \frac{1}{\mu}\|\nabla f(x)\|^2_*,
\]
(20)
\[
\langle \nabla f(x), x - x^* \rangle \geq f(x) - f(x^*) + \frac{\mu}{2}\|x - x^*\|^2.
\]
(21)
For \( f \in S_{\mu, L}^1 \) with \( \mu \geq 0 \), we have
\[
\langle \nabla f(x), x - x^* \rangle \geq \frac{\mu L}{\mu + L}\|x - x^*\|^2 + \frac{1}{\mu + L}\|\nabla f(x)\|^2_*.
\]
(22)
All the above inequalities hold for all \( x, y \in \text{dom} \ f \).

Inequalities (17)-(20) are direct consequence of Lemma 2.1 and (21) is the definition of \( \mu \)-convex. The refined lower bound (22) of \( M_{\nabla f} \) can be found in Nesterov (2013, Theorem 2.1.12).

To the end, we extend (19) and (21) to the nonsmooth case. Recall that the sub-gradient \( \partial f \) of a proper and convex function \( f \) is a set-valued function and can be defined as follows
\[
\partial f(x) := \{ p \in V^* : f(y) - f(x) \geq \langle p, y - x \rangle \quad \forall y \in V \}.
\]
(23)
Any \( p \in \partial f(x) \) will be also called a sub-gradient of \( f \) at \( x \).
Corollary 2.2  For \( f \in S^0_\mu \) with \( \mu > 0 \), we have
\[
\frac{\mu}{2} \|x - x^*\|^2 \leq f(x) - f(x^*) \leq \frac{1}{2\mu} \|p\|^2,
\]
\[
\mu \|x - x^*\|^2 \leq \langle p, x - x^* \rangle \leq \frac{1}{\mu} \|p\|^2,
\]
\[
\langle p, x - x^* \rangle \geq f(x) - f(x^*) + \frac{\mu}{2} \|x - x^*\|^2,
\]
where \( p \in \partial f(x) \) and \( x \in \text{dom} f \).

3. Strong Lyapunov Functions

In this section we consider the autonomous dynamical system
\[
x'(t) = \mathcal{G}(x(t)), \quad t > 0,
\]
where \( x : \mathbb{R}_+ \to \mathcal{V} \) and \( \mathcal{G} : \mathcal{V} \to \mathcal{V}^* \) is a vector field. Here the Hilbert space \( \mathcal{V} \) may not be the space \( \mathcal{V} \) for the original optimization (1). We mainly consider smooth \( \mathcal{G} \), with which the well-posedness of (24) is usually evident by standard ODE theory, under mild condition on \( \mathcal{G} \) (Lipschitz continuity). Let \( x^* \) be an equilibrium point of the dynamic system (24), i.e. \( \mathcal{G}(x^*) = 0 \). We are interested in the convergence of the trajectory \( x(t) \) to \( x^* \) as \( t \to \infty \).

3.1 Strong Lyapunov condition and decay property

Originally the Lyapunov function is constructed to study the stability of an equilibrium point. To obtain the convergence rate, we need a stronger condition than merely \(-\nabla \mathcal{L}(x) \cdot \mathcal{G}(x)\) is locally positive. If there exist a compact subset \( \mathcal{W} \subseteq \mathcal{V} \), a positive function \( c(x) > 0, \forall x \in \mathcal{W} \), a constant \( q \geq 1 \), and a function \( p(x) : \mathcal{V} \to \mathbb{R} \) such that \( \mathcal{L}(x) \geq 0 \)
\[
-\nabla \mathcal{L}(x) \cdot \mathcal{G}(x) \geq c(x) \mathcal{L}^q(x) + p^2(x) \quad \forall x \in \mathcal{W}.
\]
then we call \( \mathcal{L} \) a locally (\( \mathcal{W} \subset \mathcal{V} \)) or globally (\( \mathcal{W} = \mathcal{V} \)) strong Lyapunov function. We use \( \mathcal{A}(c, q, p, \mathcal{W}) \) to denote the strong Lyapunov condition (25) and simplify it as \( \mathcal{A}(c, q, p) \) when \( \mathcal{W} = \mathcal{V} \).

Theorem 3.1  Assume that \( \mathcal{L}(x) \) satisfies \( \mathcal{A}(c, q, p, \mathcal{W}) \). If the trajectory \( x(t) \) to (24) satisfies that \( \{x(t) : t \geq t_0\} \subset \mathcal{W} \) for some \( t_0 \geq 0 \), then for all \( t \geq t_0 \),
\[
\mathcal{L}(x(t)) \leq \begin{cases} 
\mathcal{L}(x(t_0)) \exp \left( -\int_{t_0}^t c(x(s)) \, ds \right) & \text{if } q = 1, \\
\left( (q - 1) \int_{t_0}^t c(x(s)) \, ds + \mathcal{L}(x(t_0))^{1-q} \right)^{1/(1-q)} & \text{if } q > 1.
\end{cases}
\]

Proof  By the assumption \( \mathcal{A}(c, q, p, \mathcal{W}) \), for all \( t \geq t_0 \),
\[
\frac{d}{dt} \mathcal{L}(x(t)) = \nabla \mathcal{L}(x(t)) \cdot x'(t) = \nabla \mathcal{L}(x(t)) \cdot \mathcal{G}(x(t)) 
\leq - c(x(t)) \mathcal{L}^q(x(t)) - p^2(x(t))
\leq - c(x(t)) \mathcal{L}^q(x(t)).
\]
The case $q = 1$ is trivial from (27). Assume $q > 1$. Then we have
\[
\frac{d}{dt} \mathcal{L}^{1-q} = (1 - q) \frac{\mathcal{L}'(t)}{\mathcal{L}^q} \geq c(x(t))(q - 1),
\]
and it follows that
\[
\mathcal{L}^{1-q} - \mathcal{L}(0)^{1-q} \geq (q - 1) \int_{t_0}^t c(x(s)) ds, \quad t \geq t_0,
\]
which proves (26).

3.2 Generalization to non-smooth convex optimization

Generally, the field $\mathcal{G}$ can be a set-value mapping, which may occur when $f$ is convex but non-smooth, which yields the differential inclusion
\[
x'(t) \in \mathcal{G}(x(t)), \quad t > 0.
\]

To emphasize the dependence of sub-gradient $\partial f$, we modify the notation $\mathcal{G}(x)$ to $\mathcal{G}(x, \partial f(x))$ and use $\mathcal{G}(x, d(x))$ for one particular direction $d \in \partial f(x)$. Then (28) can be also written as $x' = \mathcal{G}(x, d(x))$.

Similarly a Lyapunov function $\mathcal{L}(x)$ may not be smooth and $\partial \mathcal{L}(x, \partial f)$ is used to emphasize the dependence of sub-gradient of $f$. For one particular direction $d \in \partial f(x)$, $\partial \mathcal{L}(x, d)$ is a single-valued vector function.

The strong Lyapunov condition can be generalized to the non-smooth case as follows. We call $\mathcal{L} : \mathcal{V} \to \mathbb{R}^+$ a locally Lyapunov function of the flow field $\mathcal{G}$ if $\mathcal{L}(x^*) = 0$ and there exist a nonnegative function $c(x) \geq 0$, a constant $q \geq 1$, a compact subset $\mathcal{W} \subset \mathcal{V}$, a function $p(x) : \mathcal{V} \to \mathbb{R}$, and $d(x) \in \partial f(x)$ such that $\mathcal{L}(x) \geq 0$ for all $x \in \mathcal{W}$ and
\[
- \partial \mathcal{L}(x, d) \cdot \mathcal{G}(x, d) \geq c(x) \mathcal{L}^q(x) + p^2(x), \quad \forall x \in \mathcal{W}.
\]

If $c(x) > 0$, for all $x \in \mathcal{W}$, then we call $\mathcal{L}$ locally ($\mathcal{W} \subset \mathcal{V}$) or globally ($\mathcal{W} = \mathcal{V}$) strong Lyapunov function. We still use $\mathcal{A}(c, q, p, \mathcal{W})$ to denote the strong Lyapunov condition (25) and use $\mathcal{A}(c, q, p)$ when $\mathcal{W} = \mathcal{V}$.

Note that when verifying the strong Lyapunov property (29), for non-smooth functions, we only need to find one sub-gradient in $\partial f$.

3.3 Difference of Lyapunov functions

We then move to the discrete case. The following identities are obvious by the definition of $D_{\mathcal{L}}(\cdot, \cdot)$. When $\mathcal{L}$ is convex, various bounds on $D_{\mathcal{L}}(\cdot, \cdot)$ can be used to bound the difference of Lyapunov functions.

Lemma 3.1 Assume $\mathcal{L}$ is differentiable. Then for any two points $x_k, x_{k+1} \in \mathcal{V}$
\[
\mathcal{L}(x_{k+1}) - \mathcal{L}(x_k) = \begin{cases} 
\langle \nabla \mathcal{L}(x_k), x_{k+1} - x_k \rangle + D_{\mathcal{L}}(x_{k+1}, x_k), \\
\langle \nabla \mathcal{L}(x_{k+1}), x_{k+1} - x_k \rangle - D_{\mathcal{L}}(x_k, x_{k+1}).
\end{cases}
\]
The two points \( x_k \) and \( x_{k+1} \) will be connected by some numerical discretization of (24). For example, for the implicit Euler method, \( x_{k+1} - x_k = \alpha \mathcal{G}(x_{k+1}) \). Then the strong Lyapunov property can be applied to \( \langle \nabla \mathcal{L}(x_{k+1}), \mathcal{G}(x_{k+1}) \rangle \) which will bring more negative terms on the upper bound of \( \mathcal{L}(x_{k+1}) - \mathcal{L}(x_k) \) and convergence can be further derived.

On the other hand, if we use the explicit Euler method \( x_{k+1} - x_k = \alpha \mathcal{G}(x_k) \), the vector field is evaluated at the current point \( x_k \), there will be a positive term \( D_{\mathcal{L}}(x_{k+1}, x_k) \approx \alpha^2 \|x_{k+1} - x_k\|^2 \) on the upper bound. We then use the strong Lyapunov function at \( x_k \) to bring negative terms which scales like \( \mathcal{O}(\alpha) \). Then choosing step size \( \alpha \) small enough, we can cancel the positive \( \mathcal{O}(\alpha^2) \) term.

By Corollary 2.2, for \( f \in S^0_{\mu, L} \), we can use the definition of the convexity to control the difference: for any \( d_{k+1} \in \partial f(x_{k+1}) \)

\[
f(x_{k+1}) - f(x_k) \leq \langle d_{k+1}, x_{k+1} - x_k \rangle - \frac{\mu}{2} \|x_{k+1} - x_k\|^2. \tag{31}
\]

Besides the gradient at two end points \( \{x_k, x_{k+1}\} \), we may also use another intermediate point.

**Lemma 3.2** For \( f \in S^1_{\mu, L} \) and arbitrary three points \( \{x_k, y, x_{k+1}\} \), we have

\[
f(x_{k+1}) - f(x_k) \leq \langle \nabla f(y), x_{k+1} - x_k \rangle + \frac{L}{2} \|x_{k+1} - y\|^2
\]

\[
- \max \left\{ \frac{\mu}{2} \|y - x_k\|^2, \frac{1}{2L} \|\nabla f(y) - \nabla f(x_k)\|^2 \right\}.
\]

**Proof** We split the difference \( f(x_{k+1}) - f(x_k) = f(x_{k+1}) - f(y) + f(y) - f(x_k) \). For the first term, we use (13)

\[
f(x_{k+1}) - f(y) \leq \langle \nabla f(y), x_{k+1} - y \rangle + \frac{L}{2} \|x_{k+1} - y\|^2
\]

and for the second term, we use either (14) or (15)

\[
f(y) - f(x_k) \leq \langle \nabla f(y), y - x_k \rangle - \max \left\{ \frac{\mu}{2} \|y - x_k\|^2, \frac{1}{2L} \|\nabla f(y) - \nabla f(x_k)\|^2 \right\}.
\]

Summing these two inequalities completes the proof of this lemma.

\[\square\]

### 3.4 Decay rate of discrete Lyapunov functions

Start from an initial guess \( x_0 \), for \( k = 0, 1, \ldots \), a generic one step method for (24) can be written as \( x_{k+1} = E(\alpha_k, x_k) \), where \( \alpha_k \) is the time step size. A discrete Lyapunov sequence is naturally defined as \( \{\mathcal{L}_k = \mathcal{L}(x_k), k = 0, 1, 2, \ldots \} \).

Although the strong Lyapunov property ensures the decay of \( \mathcal{L}(x(t)) \) in the continuous level, it is nontrivial to design a numerical scheme that preserves the decay property in the discrete level, i.e., the decay of the sequence \( \{\mathcal{L}_k\} \).

To establish the convergence rate, the key is to have a discrete version of Theorem 3.1 which will yield the convergence (or boundness) of \( \{\mathcal{L}_k\} \), the discrete analogue of \( \mathcal{L}(x(t)) \). We present the following decay rate of positive sequences satisfying certain inequalities.
Theorem 3.2 Let \( \{A_k : k \geq 0\} \) be a positive sequence.

1. If
   \[
   A_{k+1} - A_k \leq -\alpha_k A_k - p_k^2, \quad k \geq 0,
   \]
   holds for some nonnegative sequence \( \{\alpha_k : k \geq 0\} \subset [0, 1) \), then
   \[
   A_k \leq \rho_k A_0 \quad \text{and} \quad \sum_{i=0}^{\infty} \frac{p_i^2}{\rho_i} \leq C A_0,
   \]  \hfill (32)
   where
   \[
   \rho_0 = 1, \quad \rho_k = \prod_{i=0}^{k-1} (1 - \alpha_i) \in (0, 1], \quad k \geq 1.
   \]

2. If
   \[
   A_{k+1} - A_k \leq -\alpha_k A_{k+1} - p_k^2, \quad k \geq 0,
   \]
   holds for some nonnegative sequence \( \{\alpha_k : k \geq 0\} \), then \( (32) \) holds true with
   \[
   \rho_0 = 1, \quad \rho_k = \prod_{i=0}^{k-1} \frac{1}{1 + \alpha_i} \in (0, 1], \quad k \geq 1.
   \]

3. If
   \[
   A_{k+1} - A_k \leq -\alpha A_k^2, \quad k \geq 0,
   \]  \hfill (33)
   holds for some \( \alpha > 0 \), then
   \[
   A_k \leq \frac{A_0}{1 + \alpha A_0 k},
   \]  \hfill (34)

4. If
   \[
   A_{k+1} - A_k \leq -\alpha A_{k+1}^2, \quad k \geq 0,
   \]  \hfill (35)
   holds for some \( \alpha > 0 \), then
   \[
   A_k \leq (1 + \delta) \frac{A_0}{1 + \alpha A_0 k}, \quad \text{with} \quad \delta = \frac{\alpha A_0}{1 + \alpha A_0}.
   \]  \hfill (36)

Proof The cases (1) and (2) are straightforward and thus skipped. We now consider case (3). Inequality (33) implies \( A_{k+1} \leq A_k \). Consider the reciprocal of the sequence

\[
\frac{1}{A_{k+1}} - \frac{1}{A_k} = \frac{A_k - A_{k+1}}{A_k A_{k+1}} \geq \alpha \frac{A_k}{A_{k+1}} \geq \alpha.
\]

Then sum to get the estimate \( (34) \).

For a sequence satisfying \( (35) \), we still have \( A_{k+1} \leq A_k \) and

\[
\frac{1}{A_{k+1}} - \frac{1}{A_k} = \frac{A_k - A_{k+1}}{A_k A_{k+1}} \geq \alpha \frac{A_{k+1}}{A_k}.
\]  \hfill (37)
Obviously (36) holds for \(k = 0\). Assume it holds for \(k \geq 1\). If \(A_{k+1} \geq A_k/(1 + \delta)\), then (37) implies
\[
\frac{1}{A_{k+1}} \geq \frac{1}{A_k} + \frac{\alpha}{1 + \delta} \geq \frac{1 + \alpha A_0(k + 1)}{A_0(1 + \delta)}.
\]
Namely (36) holds for \(k + 1\). Otherwise \(A_{k+1} < A_k/(1 + \delta)\), then
\[
A_{k+1} \leq \frac{1 + \alpha A_0}{1 + \alpha A_0 k} \leq (1 + \delta) \frac{A_0}{1 + \alpha A_0 (k + 1)}.
\]
The last inequality can be easily verified by the definition of \(\delta\).

The convergence rates \(\rho_k\) given in Theorem 3.2 depend on the step size \(\{\alpha_k : k \geq 0\}\). Within the allowed range of \(\alpha_k\), the larger is the step size, the better is the decay rate.

### 3.5 Decay rate of parameters

For most accelerated optimization methods, there is a sequence of parameter \(\{\gamma_k\}\) defined by
\[
\gamma_{k+1} - \gamma_k = \alpha_k (\mu - \gamma_{k+1}),
\]
which is an implicit Euler discretization of ODE \(\gamma' = \mu - \gamma\). The step size \(\alpha_k\) is determined by the parameters \(L\) and/or \(\mu\). Let
\[
\rho_0 = 1, \quad \rho_k = \frac{1}{\prod_{i=0}^{k-1} (1 + \alpha_i)}, \quad k \geq 1.
\]
By definition (39), \(\{\rho_k\}\) satisfies the relation
\[
\rho_{k+1} - \rho_k = -\alpha_k \rho_{k+1}, \quad \rho_{k+1} = \frac{1}{1 + \alpha_k} \rho_k,
\]
which implies \(\rho_k\) is monotone decreasing. The formula (38) of \(\gamma_k\) yields
\[
\frac{1}{1 + \alpha_k} = \frac{\gamma_{k+1}}{\gamma_k + \mu \alpha_k} \leq \frac{\gamma_{k+1}}{\gamma_k},
\]
and it follows that
\[
\rho_k \leq \frac{\gamma_k}{\gamma_0}. \tag{40}
\]
Namely the decay rate of \(\gamma_k\) will give an upper bound of the convergence rate \(\rho_k\). We will present estimates when \(\alpha_k\) and \(\gamma_k\) are related. We first present an identity on \(\gamma_k\) in terms of the ratio \(\alpha_k/\gamma_k\).

**Lemma 3.3** Given \(\mu \geq 0\), \(\gamma_0 > 0\) and some positive real sequence \(\{\alpha_k\}_{k=0}^\infty\), define \(\{\gamma_k\}_{k=0}^\infty\) by (38). Then \(\gamma_k > 0\) and we have
\[
\gamma_k = \frac{\gamma_0 \prod_{i=0}^{k-1} (1 + t_i \mu)}{1 + \gamma_0 \left[ \prod_{i=0}^{k-1} (1 + t_i \mu) - 1 \right] / \mu}, \tag{41}
\]
where \( t_k = \alpha_k/\gamma_k \) and for \( \mu = 0 \) we made the convention

\[
\frac{1}{\mu} \left[ \prod_{i=0}^{k-1} (1 + t_i \mu) - 1 \right] := \sum_{i=0}^{k-1} t_i,
\]

(42)

which is compatible with the right hand side as \( \mu \to 0 \).

**Proof** Consider the difference

\[
\frac{1}{\gamma_{k+1}} - \frac{1}{\gamma_k} = \frac{\gamma_k - \gamma_{k+1}}{\gamma_k \gamma_{k+1}} = \frac{\alpha_k (\gamma_{k+1} - \mu)}{\gamma_{k+1} \gamma_k} = \frac{t_k (\gamma_{k+1} - \mu)}{\gamma_{k+1}},
\]

which implies the recursion, for \( \mu > 0 \),

\[
\frac{1}{\gamma_{k+1}} - \frac{1}{\mu} = \frac{1}{1 + t_k \mu} \left( \frac{1}{\gamma_k} - \frac{1}{\mu} \right).
\]

(43)

Starting from \( \gamma_0 > 0 \) and then solving (43), we get the formula (41).

The identity also implies if \( \gamma_0 > \mu \), then \( \{\gamma_k\} \) is decreasing and converges to \( \mu \) from above. If \( \gamma_0 < \mu \), then \( \{\gamma_k\} \) is increasing and converges to \( \mu \) from below. And if \( \gamma_0 = \mu \), then \( \gamma_k = \mu \) for all \( k \geq 1 \). For all cases, we have

\[
\min\{\gamma_0, \mu\} \leq \lambda_k \leq \max\{\gamma_0, \mu\}, \quad k = 0, 1, 2, \ldots.
\]

We then consider the ratio \( \alpha_k^2/\gamma_k \) is bounded below, which leads to accelerated rate. For simplicity, we present the results for the case \( \gamma_0 = rL \geq \mu \). Refined analysis involving optimized \( \gamma_0 \) can be found in Nesterov (2018, Lemma 2.2.4).

**Lemma 3.4** Given \( L \geq \mu \geq 0 \), \( \gamma_0 = rL \geq \mu \), define \( (\alpha_k, \gamma_k) \) by

\[
\begin{align*}
\gamma_{k+1} &= \gamma_k + \alpha_k (\mu - \gamma_{k+1}), \\
L\alpha_k^2 &= \gamma_k (1 + B\alpha_k), \quad \alpha_k > 0,
\end{align*}
\]

(44)

where \( B \geq 0 \). Then \( \gamma_k > 0 \) and we have the following bound of \( \rho_k \).

- If \( B = 0 \), then

\[
\rho_k \leq \min \left\{ \left( \frac{\sqrt{r+1} + 1}{\sqrt{r+1} + 1 + \sqrt{r}k} \right)^2, \left( 1 + \sqrt{\frac{\mu}{L}} \right)^{-k} \right\}.
\]

(45)

- If \( B \geq 1/2 \), then

\[
\rho_k \leq \min \left\{ \left( \frac{2}{2 + \sqrt{r}k} \right)^2, \left( 1 + \sqrt{\frac{\mu}{L}} \right)^{-k} \right\}.
\]

(46)
Proof Consider the difference of $1/\sqrt{\rho_k}$ and use (40), we get
\[
\frac{1}{\sqrt{\rho_{k+1}}} - \frac{1}{\sqrt{\rho_k}} = \frac{\rho_k - \rho_{k+1}}{\sqrt{\rho_k \rho_{k+1}}(\sqrt{\rho_k} + \sqrt{\rho_{k+1}})} = \frac{\alpha_k}{\sqrt{\rho_k}(1 + \sqrt{1 + \alpha_k})} = \sqrt{r} \frac{\sqrt{1 + B \alpha_k}}{1 + \sqrt{1 + \alpha_k}}.
\]
If $B = 0$, then $L \alpha_k^2 = \gamma_k \leq \max\{\gamma_0, \mu\} = r L$. Therefore, $\alpha_k \leq \sqrt{r}$ and thus
\[
\frac{1}{\sqrt{\rho_{k+1}}} - \frac{1}{\sqrt{\rho_k}} = \frac{1}{1 + \sqrt{1 + \alpha_k}} \geq \frac{1}{\sqrt{r + 1 + 1}}.
\]
Therefore, we obtain
\[
\rho_k \leq \left( \frac{\sqrt{r + 1 + 1}}{\sqrt{1 + 1 + \sqrt{r} k}} \right)^2. \tag{47}
\]
If $B \geq 1/2$, then consider the function
\[
\chi(t) := \frac{\sqrt{1 + B t}}{1 + \sqrt{1 + t}}, \quad t \geq 0.
\]
An elementary calculation proves that $\chi(t) \geq \chi(0) = 1/2$ for all $t \geq 0$. Therefore, we have
\[
\frac{1}{\sqrt{\rho_{k+1}}} - \frac{1}{\sqrt{\rho_k}} \geq \frac{1}{2} \sqrt{r},
\]
which implies
\[
\rho_k \leq \left( \frac{2}{2 + \sqrt{r} k} \right)^2. \tag{48}
\]
Note that both of the sublinear rates (47) and (48) hold for all $\mu \geq 0$.

If $\mu > 0$, then it is evident that
\[
\alpha_k^2 = \frac{\gamma_k}{L} (1 + B \alpha_k) \geq \frac{\gamma_k}{L} \geq \frac{\mu}{L}, \tag{49}
\]
so we have that
\[
\rho_k \leq \left( 1 + \frac{\mu}{L} \right)^{-k}.
\]
This finishes the proof of this lemma.

4. Gradient Flow and Euler Methods

In this section we will study the gradient flow. The implicit Euler method is equivalent to the proximal method and the explicit Euler method to gradient descent methods. Convergence analyses are derived from the strong Lyapunov property for various Lyapunov functions.
4.1 Gradient flow

The simplest dynamic system is the gradient flow
\[ x'(t) = -\nabla f(x(t)), \]
with initial condition \( x(0) = x_0 \). Namely \( \mathcal{G}(x) = -\nabla f(x) \). A natural Lyapunov function is the so-called optimality gap
\[ \mathcal{L}(x) = f(x) - f(x^*). \]

4.1.1 Strongly convex case

To get the convergence rate, we verify the strong Lyapunov property. When \( f \in S^1_{\mu^*} \) with \( \mu > 0 \), by (19), we get the global strong Lyapunov property \( A(2\mu, 1, 0) \) or \( A(\mu, 1, p) \) with \( p^2 = \|\nabla f(x)\|^2/2 \). Consequently, by Theorem 3.1, this yields the exponential decay \( O(e^{-t}) \) of the optimality gap \( f(x(t)) - f(x^*) \) along the trajectory of the gradient flow (50).

We have more candidates for strong Lyapunov functions besides the optimality gap (51). For example, we can use the squared distance \( \mathcal{L}(x) = \frac{1}{2}\|x - x^*\|^2 \). If \( f \in S^1_{\mu,L} \) with \( 0 < \mu \leq L < \infty \), then using (22) implies that
\[
-\nabla \mathcal{L}(x) \cdot \mathcal{G}(x) = \langle \nabla f(x), x - x^* \rangle \geq \frac{\mu L}{L + \mu} \|x - x^*\|^2 + \frac{1}{L + \mu} \|\nabla f(x)\|^2
= \frac{2\mu L}{L + \mu} \mathcal{L}(x) + \frac{1}{L + \mu} \|\nabla f(x)\|^2.
\]

Hence \( \mathcal{L} \) satisfies \( A(2\mu L/(L + \mu), 1, p) \) with \( p^2(x) = \|\nabla f(x)\|^2/(L + \mu) \). The extra positive term \( p^2 \) is useful for the analysis of the gradient descent method.

Another choice is the combination of the previous two:
\[ \mathcal{L}(x) = f(x) - f(x^*) + \frac{\mu}{2} \|x - x^*\|^2. \]

If \( f \in S^1_{\mu^*} \) with \( \mu > 0 \), then by (21), we have
\[ \langle \nabla f(x), x - x^* \rangle \geq f(x) - f(x^*) + \frac{\mu}{2} \|x - x^*\|^2 = \mathcal{L}(x), \]
and this bound verifies the strong Lyapunov property \( A(\mu, 1, p) \) since
\[ -\nabla \mathcal{L}(x) \cdot \mathcal{G}(x) = \|\nabla f(x)\|^2 + \mu(\nabla f(x), x - x^*) \geq \mu \mathcal{L}(x) + \|\nabla f(x)\|^2. \]

The above extra positive term \( p^2 = \|\nabla f(x)\|^2 \) is also useful for the analysis of the gradient descent method.

4.1.2 Convex case

When \( \mu = 0 \), the previous strong Lyapunov properties are degenerated. Hence, coercivity of \( f \) is needed.

Define the sub-level set of \( f \) on a given constant value \( c \) as
\[ S_c(f) = \{x : f(x) \leq c\}. \]
As \( f \) is convex, so is \( S_c(f) \). The set \( \text{argmin} f \) where \( f \) attains its minimum value \( f_{\text{min}} \) can be written as \( S_{f_{\text{min}}}(f) \).
**Lemma 4.1** Let $f$ be convex and coercive. For a given finite value $f_0$, there exists a constant $R_0$ such that

$$\max_{x^* \in \arg\min f} \max_{x \in S_{f_0}} \|x - x^*\| \leq R_0. \quad (54)$$

**Proof** When $f$ is coercive, $S_{f_0}$ is bounded. Otherwise we can find a sequence $\{x_n\}$ with $f(x_n) \leq f_0$, but $\|x_n\| > n$, for $n = 1, 2, \ldots$, contradicting the coercivity of $f$. As $\arg\min f \subseteq S_{f_0}$, it is also bounded and (54) follows.

For $L(x) = f(x) - f(x^*)$, where $x^*$ is an arbitrary but fixed point in the minimum set $\arg\min f$, as $-\nabla L(x) \cdot \mathcal{G}(x) = \|\nabla f(x)\|^2_* \geq 0$, we conclude that the trajectory of gradient flow $x(t)$ satisfies $x(t) \in S_{f_0}(f)$ with $f_0 = f(x_0)$. Furthermore, assuming coercivity and using the convexity, we have

$$f(x) - f(x^*) \leq \langle \nabla f(x), x - x^* \rangle \leq R_0 \|\nabla f(x)\|_* \quad \forall x \in S_{f_0}(f). \quad (55)$$

Then it is straightforward to show

$$-\nabla L(x) \cdot \mathcal{G}(x) = \|\nabla f(x)\|^2_* \geq \frac{1}{R_0^2} L^2(x) \quad \forall x \in S_{f_0}(f). \quad (56)$$

Hence, the strong Lyapunov property holds with $q = 2$ and $W = S_{f_0}(f)$. Applying Theorem 3.1 implies the sublinear rate $O(1/t)$ of the optimality gap $f(x(t)) - f(x^*)$ along the trajectory of the gradient flow provided the function is coercive and convex.

**4.2 Proximal point algorithm**

Consider the implicit Euler method for the gradient flow

$$x_{k+1} = x_k - \alpha_k \nabla f(x_{k+1}), \quad (57)$$

which can be written as

$$x_{k+1} = \text{prox}_{\alpha_k f}(x_k) := \arg\min_x \left\{ f(x) + \frac{1}{2\alpha_k} \|x - x_k\|^2 \right\}, \quad (58)$$

and is known as the proximal point algorithm (PPA).

**Theorem 4.1** Assume $f \in S^1_\mu$ with $\mu > 0$. The sequence $\{x_k\}$ generated by (58) satisfies

$$\mathcal{L}_{k+1} \leq \frac{1}{1 + \mu \alpha_k} \mathcal{L}_k,$$

for all $\alpha_k > 0$, where

$$\mathcal{L}_k = \mathcal{L}(x_k) = f(x_k) - f(x^*) + \frac{\mu}{2} \|x_k - x^*\|^2. \quad (59)$$

**Proof** We first use the convexity of $\mathcal{L}$, then the discretization, and last the strong Lyapunov property (53) to get

$$\mathcal{L}_{k+1} - \mathcal{L}_k \leq \langle \nabla \mathcal{L}(x_{k+1}), x_{k+1} - x_k \rangle = \alpha_k \langle \nabla \mathcal{L}(x_{k+1}), \mathcal{G}(x_{k+1}) \rangle \leq -\mu \alpha_k \mathcal{L}_{k+1}.$$ 

The linear convergence rate then follows. \[\blacksquare\]
4.3 Gradient descent method

Next we present analysis for the explicit Euler method for the gradient flow, which is exactly
the gradient descent method

\[ x_{k+1} = x_k - \alpha_k \nabla f(x_k). \] (60)

**Theorem 4.2** Assume \( f \in S_{\mu,L}^{1,1} \) with \( 0 < \mu \leq L < \infty \). Let \( \{ x_k \} \) be the sequence generated by (60). Then for \( \alpha_k \leq 2/(L + \mu) \), we have

\[ \mathcal{L}_{k+1} \leq (1 - \mu \alpha_k) \mathcal{L}_k, \]

where \( \mathcal{L}_k \) is define by (59). The optimal value \( \alpha_k = 2/(L + \mu) \) gives

\[ \mathcal{L}_{k+1} \leq \frac{L - \mu}{L + \mu} \mathcal{L}_k, \]

and the quasi-optimal value \( \alpha_k = 1/L \) gives

\[ \mathcal{L}_{k+1} \leq (1 - \mu/L) \mathcal{L}_k. \]

**Proof** As \( f \in S_{\mu,L}^{1} \subset S_{\mu}^{1} \), we have verified the strong Lyapunov property \( A(c,1,p) \) (cf. (53)). Note that \( \mathcal{L} \in S_{2\mu,L+\mu}^{1} \). Using the identity (30), the upper bound of \( D_{\mathcal{L}} \), and the strong Lyapunov condition at \( x_k \), we have

\[ \mathcal{L}_{k+1} - \mathcal{L}_k \leq (\nabla \mathcal{L}(x_k), x_{k+1} - x_k) + \frac{L + \mu}{2} \| x_{k+1} - x_k \|^2 \\
= -\alpha_k (\nabla \mathcal{L}(x_k), \nabla f(x_k)) + \frac{L + \mu}{2} \| x_{k+1} - x_k \|^2 \\
\leq -\mu \alpha_k \mathcal{L}_k - \alpha_k \| \nabla f(x_k) \|^2 + \frac{L + \mu}{2} \alpha_k^2 \| \nabla f(x_k) \|^2. \]

Then for \( \alpha_k \leq 2/(L + \mu) \), we will have \( \mathcal{L}_{k+1} - \mathcal{L}_k \leq -\mu \alpha_k \mathcal{L}_k \) and the linear convergence follows. \( \square \)

One can also choose

\[ \mathcal{L}(x) = f(x) - f(x^*) \quad \text{or} \quad \mathcal{L}(x) = \frac{1}{2} \| x - x^* \|^2, \]

and prove the linear convergence of the gradient descent methods using the strong Lyapunov property. The details is left to the interested readers.

4.4 Proximal gradient method

In some applications, the non-smooth convex function \( f \) has the composite structure \( f = h + g \), where \( h \in S_{\mu,L}^{1} \) is smooth and \( g \) is convex but nonsmooth. We may also assume that \( h \in S_{0,L}^{1,1} \) and \( g \in S_{\mu}^{0} \) as we can split \( h + g \) as \( (h(x) - \frac{\mu}{2} \| x \|^2) + (g(x) + \frac{\mu}{2} \| x \|^2) \). One important example is the least absolute shrinkage and selection operator (LASSO) problem (Tibshirani, 1996)

\[ \min_x f(x) := \frac{1}{2} \| Ax - b \|^2 + \rho \| x \|_1, \]
which is also known as basis pursuit in the signal processing context (Chen et al., 1999). Application and generalization of LASSO to a variety of problems can be found in Tibshirani (1996, Table 1).

We start from the proximal gradient (PG) method which is also known as operator splitting or forward-backward splitting (Parikh and Boyd, 2014). Namely we use an explicit scheme for smooth part $h$ and an implicit scheme for non-smooth part $g$:

$$\frac{x_{k+1} - x_k}{\alpha_k} \in -\nabla h(x_k) - \partial g(x_{k+1}),$$

which is an implicit-explicit method for the generalized gradient flow

$$x' \in \mathcal{G}(x, \partial f(x)) := -\partial f(x) = -\nabla h(x) - \partial g(x).$$

It can be written using the proximal operator

$$x_{k+1} = \text{prox}_{\alpha_k g}(x_k - \alpha_k \nabla h(x_k)),$$

and summarized as the following algorithm.

**Algorithm 1** PG method for minimizing $f = h + g$, $h \in \mathcal{S}_\mu^1$ with $0 \leq \mu \leq L < \infty$

**Input:** $x_0 \in V$.

1. for $k = 0, 1, \ldots$ do
2. Choose $\alpha_k \in (0, 2/L)$.
3. Compute $y_k = x_k - \alpha_k \nabla h(x_k)$.
4. Update $x_{k+1} = \text{prox}_{\alpha_k g}(y_k)$.
5. end for

We consider the Lyapunov function (51), i.e., $\mathcal{L}(x) = f(x) - f(x*)$. Let $d(x) \in \partial f(x)$ be an arbitrary direction in the sub-gradient. Assume $f \in \mathcal{S}_\mu^0$ with $\mu > 0$. Thanks to Corollary 2.2, we have

$$-\partial \mathcal{L}(x, d(x)) \cdot \mathcal{G}(x, d(x)) = \|d(x)\|^2_* \geq \mu \mathcal{L}(x) + \frac{1}{2} \|d(x)\|^2_*,$$

which means $\mathcal{L}$ satisfies strong Lyapunov property $\mathcal{A}(\mu, 1, p)$ with $p = \|d(x)\|^2_*/2$. When $\mu = 0$, assuming additionally $f$ is coercive, then

$$-\partial \mathcal{L}(x, d(x)) \cdot \mathcal{G}(x, d(x)) \geq \frac{1}{2 R_0^2} \mathcal{L}^2(x) + \frac{1}{2} \|d(x)\|^2_*,$$

where $R_0$ is defined in (54). That is when $\mu = 0$, $\mathcal{L}$ satisfies strong Lyapunov property $\mathcal{A}(1/(2R^2), 2, p, \mathcal{S}_{f_0})$ with $p = \|d(x)\|^2_*/2$ for arbitrary direction in the sub-gradient $d(x) \in \partial f(x)$.

**Lemma 4.2** Assume $f = h + g$ where $h \in \mathcal{S}_\mu^1$ with $0 \leq \mu \leq L < \infty$. Let $\{(x_k, y_k)\}$ be the sequence generated by Algorithm 1. Let

$$q_{k+1} = (y_k - x_{k+1})/\alpha_k \in \partial g(x_{k+1}),$$

$$d_{k+1} = \nabla h(x_{k+1}) + q_{k+1} \in \partial f(x_{k+1}),$$

$$d_{k+\frac{1}{2}} = \nabla h(x_k) + q_{k+1} = (x_k - x_{k+1})/\alpha_k.$$
Then for $0 \leq \alpha_k \leq 2/L$, 
\[
    f(x_{k+1}) - f(x_k) \leq \alpha_k \left( \frac{L\alpha_k}{2} - 1 \right) \min \left\{ \|d_{k+1}\|_*^2, \|d_{k+\frac{1}{2}}\|_*^2 \right\}.
\]

**Proof** Applying (30) and (15) to $h$ and using the convexity of $g$, we have the bound 
\[
    f(x_{k+1}) - f(x_k) = h(x_{k+1}) - h(x_k) + g(x_{k+1}) - g(x_k)
\]
\[
    \leq \langle d_{k+1}, x_{k+1} - x_k \rangle - \frac{1}{2L} \|\nabla h(x_{k+1}) - \nabla h(x_k)\|_*^2 - \frac{\mu}{2} \|x_{k+1} - x_k\|^2.
\]

We use the discretization (61) to replace $x_{k+1} - x_k = -\alpha_k d_{k+\frac{1}{2}}$ and get 
\[
    \langle d_{k+1}, x_{k+1} - x_k \rangle = -\alpha_k \langle d_{k+1}, d_{k+\frac{1}{2}} \rangle = -\alpha_k \|d_{k+1}\|_*^2 + \alpha_k \langle d_{k+1}, d_{k+1} - d_{k+\frac{1}{2}} \rangle.
\]

Note that $d_{k+1} - d_{k+\frac{1}{2}} = \nabla h(x+1) - \nabla h(x_k)$. We can control the cross term as 
\[
    \alpha_k \left| \langle d_{k+1}, d_{k+1} - d_{k+\frac{1}{2}} \rangle \right| \leq \frac{1}{2L} \|\nabla h(x_{k+1}) - \nabla h(x_k)\|_*^2 + \frac{L\alpha_k^2}{2} \|d_{k+1}\|_*^2.
\]

Adding them together, we get the desired inequality with bound $\|d_{k+1}\|_*$. We can switch to $d_{k+\frac{1}{2}}$ in a similar fashion and obtain a slightly better inequality 
\[
    f(x_{k+1}) - f(x_k) \leq \alpha_k \left( \frac{(L - \mu)\alpha_k}{2} - 1 \right) \|d_{k+\frac{1}{2}}\|_*^2,
\]

and the range of the step size can be enlarged to $0 < \alpha_k \leq 2/(L - \mu)$. □

The vector $d_{k+\frac{1}{2}}$ is the so-called gradient mapping (see Nesterov, 2013; Luo and Chen, 2019). The gradient $d_{k+1} \in \partial f(x_{k+1})$ is useful in the analysis while $d_{k+\frac{1}{2}}$ is practical in the implementation.

**Theorem 4.3** Assume $f = h + g$ where $h \in S^1_{\mu, L}$ with $0 \leq \mu \leq L < \infty$. When $\mu = 0$, assume further that $f$ is coercive. Let the sequence $\{x_k\}$ be generated by Algorithm 1 with fixed step size $\alpha_k = 1/L$. Then we have 
\[
    \mathcal{L}_{k+1} - \mathcal{L}_k \leq \begin{cases} 
        -\frac{\mu}{L} \mathcal{L}_{k+1} & \text{if } \mu > 0, \\
        -\frac{1}{2R^2_0} \mathcal{L}_{k+1}^2 & \text{if } \mu = 0,
    \end{cases}
\]

where $\mathcal{L}_k = f(x_k) - f(x^*)$. Consequently, for all $k \geq 0$, it holds that 
\[
    \mathcal{L}_k \leq \begin{cases} 
        \mathcal{L}_0 (1 + \mu/L)^{-k} & \text{if } \mu > 0, \\
        \left( 1 + \frac{\mathcal{L}_0}{1 + 2R^2_0 \mathcal{L}_0} \right) \frac{\mathcal{L}_0}{2R^2_0 + \mathcal{L}_0 k} & \text{if } \mu = 0.
    \end{cases}
\]
**Proof** First of all, we have the relation $L_{k+1} - L_k = f(x_{k+1}) - f(x_k)$ and, by Lemma 4.2, the choice $\alpha_k = 1/L$ implies

$$L_{k+1} - L_k \leq -\frac{1}{2L} \|d_{k+1}\|^2_*.$$

The strong Lyapunov property at $x_{k+1}$ reads as

$$\begin{cases} \frac{1}{2\mu} \|d_{k+1}\|^2_* \geq f(x_{k+1}) - f(x^*) & \mu > 0, \\ R_0 \|d_{k+1}\|_* \geq f(x_{k+1}) - f(x^*) & \mu = 0, \end{cases}$$

which can be proved similarly to (63) and (64).

For $\mu > 0$, we then have

$$L_{k+1} - L_k \leq -\frac{1}{2L} \|d_{k+1}\|^2_* \leq -\frac{\mu}{L} L_{k+1}.$$

The desired result (66) then follows.

When $\mu = 0$, we first conclude $L_{k+1} - L_k \leq 0$ for all $k \geq 0$. That is $f(x_k) \leq f(x_0)$ for all $k \geq 0$ so that we can use bound (54) and the strong Lyapunov property implies

$$L_{k+1} - L_k \leq -\frac{1}{2L} \|d_{k+1}\|^2_* \leq -\frac{1}{2L} R_0^2 L_{k+1},$$

which proves (66) for $\mu = 0$. The sub-linear rate (67) follows from Theorem 3.2. \[\square\]

5. Gradient Flows with Time Scaling

In this section we shall introduce a rescaled gradient flow for $f \in S^1_\mu$ and deal with the strongly convex case $\mu > 0$ and convex case $\mu = 0$ in a unified way.

5.1 Scaled gradient flows

Consider the following first-order ODE system

$$\begin{cases} x' = -\nabla f(x)/\gamma, \\ \gamma' = \mu - \gamma, \end{cases} \quad (68)$$

with arbitrary initial value $x(0) = x_0$ and $\gamma(0) = \gamma_0 > 0$. Let $x = (x, \gamma)$ and write (68) as

$$x'(t) = G(x(t)).$$

We introduce a Lyapunov function

$$L(x) := f(x) - f(x^*) + \frac{\gamma}{2} \|x - x^*\|^2.$$

\[\text{(69)}\]
Let us verify the strong Lyapunov property as follows

\[-\nabla L(x) \cdot G(x) = \langle \nabla f(x), x - x^* \rangle + \frac{\gamma - \mu}{2} \| x - x^* \|^2 + \frac{1}{\gamma} \| \nabla f(x) \|^2 \]

\[
\geq f(x) - f(x^*) + \frac{\gamma}{2} \| x - x^* \|^2 + \frac{1}{\gamma} \| \nabla f(x) \|^2 \\
= L(x) + \frac{1}{\gamma} \| \nabla f(x) \|^2 \tag{70}.
\]

Hence $L$ is a strong Lyapunov function of $G$ with $q = 1$, $c(x) = 1$ and $p^2(x) = \| \nabla f(x) \|^2 / \gamma$.

By Theorem 3.1, we have

\[ L(x(t)) \leq e^{-t} L(0), \quad t \geq 0. \tag{71} \]

Note that even for $\mu = 0$, we can still achieve the exponential decay, i.e., the linear convergence rate rather than the sub-linear rate.

In the continuous level, exponential decay can also be viewed as nothing but a time rescaling. Introducing the parameter $\gamma$ governed by the equation $\gamma' = \mu - \gamma$ brings the effect of rescaling and allows us to handle $\mu \geq 0$ in a unified way.

### 5.2 Scaled proximal point algorithm

Convergence analysis of the implicit Euler methods for smooth or non-smooth convex functions are almost identical. Therefore in the following we present the non-smooth case only, i.e., $f \in S_\mu^0$ with $\mu \geq 0$.

Given any time step size $\alpha_k > 0$, the implicit Euler method reads as

\[
\begin{cases}
\frac{x_{k+1} - x_k}{\alpha_k} & \in G^\gamma(x_{k+1}, \gamma_k, \partial f(x_{k+1})) := -\frac{1}{\gamma_k} \partial f(x_{k+1}) \\
\frac{\gamma_{k+1} - \gamma_k}{\alpha_k} & = G^\gamma(x_k, \gamma_{k+1}) := \mu - \gamma_{k+1}.
\end{cases} \tag{72}
\]

Denoted by $t_k = \alpha_k / \gamma_k$, the update of $x_{k+1}$ in (72) can be written using the proximal operator

\[ x_{k+1} = \text{prox}_{t_k f}(x_k) := \arg\min_x \left\{ f(x) + \frac{1}{2t_k} \| x - x_k \|^2 \right\}. \tag{73} \]

We still use the Lyapunov function (69) and the strong Lyapunov condition: for any $d(x) \in \partial f(x)$

\[- \partial L(x, d(x)) \cdot G(x, d(x)) \geq L(x) + \frac{1}{\gamma} \| d(x) \|^2 \tag{74}.\]

can be verified similarly to (70).

To study the change of discrete Lyapunov function

\[ L_k = f(x_k) - f(x^*) + \frac{\gamma_k}{2} \| x_k - x^* \|^2, \]

we shall move along the path $(x_k, \gamma_k) \rightarrow (x_{k+1}, \gamma_k) \rightarrow (x_{k+1}, \gamma_{k+1})$. To be able to use the strong Lyapunov property, we will try to evaluate the vector field $G$ at $(x_{k+1}, \gamma_{k+1})$. In (73), only the step size $t_k = \alpha_k / \gamma_k$ enters the algorithm and $(\alpha_k, \gamma_k)$ can be solved in terms of $t_k$. Therefore in (75) we estimate the rate by $t_k$. 20
Theorem 5.1 Assume $f$ is in $S_{\mu}^0$ with $\mu \geq 0$. Then for (72) with any $\alpha_k > 0$, we have

$$L_{k+1} \leq \frac{1}{1 + \alpha_k} L_k.$$  

Consequently for any $\gamma_0 > 0$,

$$L_k \leq \frac{L_0}{1 + \gamma_0 \left[ \prod_{i=0}^{k-1} (1 + t_i \mu) - 1 \right] / \mu},$$  \hspace{1cm} (75)

where $t_k = \alpha_k / \gamma_k$ is the rescaled step size and for $\mu = 0$ we used the convention (42).

Proof First of all, the direction $d_{k+1} = \frac{x_k - x_{k+1}}{t_k} \in \partial f(x_{k+1})$. We split the difference as

$$L_{k+1} - L_k = L(x_{k+1}, \gamma_k) - L(x_k, \gamma_k) + L(x_{k+1}, \gamma_{k+1}) - L(x_{k+1}, \gamma_k) := I_1 + I_2.$$  

As $L$ is linear in $\gamma$ and $G^\gamma$ is independent of $(x, \gamma)$

$$I_2 = \langle \nabla \gamma L(x_{k+1}), \gamma_{k+1} - \gamma_k \rangle = \alpha_k \langle \nabla \gamma L(x_{k+1}), G^\gamma(x_{k+1}, d_{k+1}) \rangle.$$  

As for fixed $\gamma > 0$, the function $L(\cdot, \gamma) \in S_{\gamma+\mu}^0$, we obtain that

$$I_1 \leq \langle \partial_\gamma L(d_{k+1}, \gamma_k), x_{k+1} - x_k \rangle - \frac{\mu + \gamma_k}{2} \|x_{k+1} - x_k\|^2$$

$$= \alpha_k \langle \partial_\gamma L(d_{k+1}, \gamma_k), G^\gamma(d_{k+1}, \gamma_{k+1}) \rangle - \frac{\mu + \gamma_k}{2} \|x_{k+1} - x_k\|^2$$

$$+ \alpha_k \left( \frac{1}{\gamma_{k+1}} - \frac{1}{\gamma_k} \right) \|d_{k+1}\|^2_*$$

$$\leq \alpha_k \langle \partial_\gamma L(d_{k+1}, \gamma_k), G^\gamma(d_{k+1}, \gamma_{k+1}) \rangle + \frac{\alpha_k}{\gamma_{k+1}} \|d_{k+1}\|^2_*.$$  

Adding all together and using the strong Lyapunov condition (74), we get

$$L_{k+1} - L_k \leq \alpha_k \langle \partial L(d_{k+1}, \gamma_{k+1}), G(d_{k+1}, \gamma_{k+1}) \rangle + \frac{\alpha_k}{\gamma_{k+1}} \|d_{k+1}\|^2_* \leq - \alpha_k L_{k+1}.$$  

By recursion, we have

$$L_k \leq \frac{L_0}{\prod_{i=0}^{k-1} (1 + \alpha_i)} = L_0 \prod_{i=0}^{k-1} \frac{\gamma_{i+1}}{(1 + t_i \mu) \gamma_i} = \frac{L_0}{\prod_{i=0}^{k-1} (1 + t_i \mu) \gamma_0},$$

and (75) follows from the identity (41) on $\gamma_k$. \hfill \blacksquare

For explicit methods, sub-linear rate is expected for $\mu = 0$ (see Theorem 4.3). The implicit method, however, retains the linear rate uniformly for all $\mu \geq 0$. The larger is step size $t_k$, the better is the convergence rate. On the other hand, uniform bound $\alpha_k \geq \alpha > 0$ implies the exponential increasing of $t_k$ and the proximal operator is harder to evaluate. In the limiting case $t_k = \infty$, it goes back to the original optimization problem.
6. Heavy Ball Flow and Momentum Methods

The well-known heavy ball (HB) flow system (Polyak, 1964) reads as follows

\[ x'' + \gamma x' + \beta \nabla f(x) = 0, \quad t > 0, \numbered{76} \]

where \( \gamma, \beta > 0 \) are constant parameters and initial conditions are given by \( x(0) = x_0, \ x'(0) = x_1 \). Discretization of (76) leads to the so-called heavy ball method (a.k.a the momentum method). In this section we shall study the momentum method using the strong Lyapunov functions for the strongly convex case \( f \in S^{1,\mu,L} \) with \( 0 < \mu \leq L < \infty \).

6.1 Literature review

In the early 1960s, Polyak (1964) considered the HB model (76) together with its discretization

\[ x_{k+1} = x_k - s_k \nabla f(x_k) + \alpha_k (x_k - x_{k-1}), \numbered{77} \]

which covers a large class of iterative solvers for solving linear algebraic systems. Applying the asymptotic bound between the matrix norm and the spectral radius, Polyak (1964, Theorem 9) established the local convergence result for (76) and (77) (with stronger smoothness condition \( f \in C^2 \)) via spectral analysis and obtained the minimum spectral radius

\[ \rho^* = \rho(\alpha^*, s^*) = \frac{\sqrt{L} - \sqrt{\mu}}{\sqrt{L} + \sqrt{\mu}}, \]

where

\[ \alpha^* = \left( \frac{\sqrt{L} - \sqrt{\mu}}{\sqrt{L} + \sqrt{\mu}} \right)^2, \quad s^* = \frac{4}{(\sqrt{L} + \sqrt{\mu})^2}. \]

However, it was shown in Lessard et al. (2016) that the HB method with parameters optimized for linear ODEs does not guarantee the global convergence for general nonlinear objective function, which justifies the limitation of spectral analysis and the need of Lyapunov analysis.

Recently, Ghadimi et al. (2015) established the first global ergodic convergence rate

\[ f(\bar{x}_k) - f(x^*) \leq \begin{cases} \frac{C}{k+1}, & \mu = 0, \\ C \left( 1 - \frac{\mu}{L} \right)^k, & \mu > 0, \end{cases} \numbered{78} \]

for the HB method (77), where \( \bar{x}_k = \frac{1}{k+1} \sum_{i=0}^{k} x_i \) denotes the Cesaéro average. Later on, Sun et al. (2018) considered \( \beta = 1, \ \gamma > 0 \) and the Lyapunov function

\[ \mathcal{L}(x) := f(x) - f(x^*) + \frac{1}{2} \| v \|^2, \numbered{79} \]

which has a nice physical meaning. Indeed, if we treat \( x(t) \) as the trajectory of a particle and understand \( f(x) - f(x^*) \) as the potential energy, then \( v = x' \) is the velocity and \( \frac{1}{2} \| v \|^2 \) is the kinetic energy. Therefore \( \mathcal{L} \) defined in (79) is the total energy consisting of summation of the potential energy and the kinetic energy.
Hence, instead of reducing the potential energy only in the gradient flow, the HB flow (76) reduces the total energy, and an easy calculation yields the global Lyapunov property
\(- \nabla L(x) \cdot G(x) = \gamma \|v\|^2\). Unfortunately, we only have the boundedness \(L(x(t)) \leq L(x_0), 0 < t < \infty\), due to the absence of strong Lyapunov property. To get the global convergence rate \(O(1/t)\), further assumptions (such as coercivity of \(f\)) are needed which may not be easy to verify. Moreover, Sun et al. (2018) improved the ergodic rate (78) to non-ergodic sense.

Another choice \(\gamma = 2\sqrt{\mu}, \beta = 1\) has been considered in Siegel (2019); Wilson et al. (2021). This is nothing but a rescaling of the time variable \(t = \sqrt{\mu}\tau\) for (80) and thus linear convergence \(O(e^{-\sqrt{\mu}/L})\) can be established under the new time variable \(\tau > 0\). In Siegel (2019); Wilson et al. (2021), several methods for the HB system (80) with provable accelerated convergence rate \((1 - \sqrt{\mu/L})^k\) have been presented. We also note that Shi et al. (2018) considered \(\gamma = 2\sqrt{\mu}\) and \(\beta = 1 + \sqrt{\mu/s}\) with \(s > 0\) and require \(f \in \mathcal{S}_{\mu,L}^1 \cap C^2\). They introduced another Lyapunov function
\[ L(x) = f(x(t)) - f(x^*) + \frac{1}{4\beta} \|x'(t)\|^2 + \frac{\gamma^2}{4\beta} \|x(t) + \frac{x'(t)}{\gamma} - x^*\|^2, \]
and obtained the exponential decay
\[ L(x(t)) \leq e^{-\sqrt{\mu}/4}L(0). \]
Besides, they also established the linear rate \(O((1 - \mu/L)^k)\) for the corresponding discretization. As we shall show below, our analysis based on strong Lyapunov conditions is simpler.

6.2 HB model with suitable parameters

In our recent work (Luo and Chen, 2019), we considered \(\gamma = 2\) and \(\beta = 1/\mu\). In this case, (76) is equivalent to a first-order system
\[
\begin{align*}
x' &= v - x, \\
v' &= x - v - \nabla f(x)/\mu.
\end{align*}
\]
Let \(x = (x,v)\) and let the above right hand side be \(\mathcal{G}(x)\). The HB system (80) can be abbreviated as \(x'(t) = \mathcal{G}(x(t))\). We choose the Lyapunov function
\[ L(x) := f(x) - f(x^*) + \frac{\mu}{2} \|v - x^*\|^2. \]
Besides, we present the following identity which is trivial but very useful for our analysis in both the continuous and discrete level
\[ 2(u - v, v - w) = \|u - w\|^2 - \|u - v\|^2 - \|v - w\|^2 \quad \forall u, v, w \in V. \]

We now verify the strong Lyapunov property of (81) as follows. A direct computation leads to
\[
- \nabla L(x) \cdot \mathcal{G}(x) = \langle \nabla f(x), x - x^* \rangle - \mu(x - v, v - x^*)
= \langle \nabla f(x), x - x^* \rangle - \frac{\mu}{2} \left( \|x - x^*\|^2 - \|x - v\|^2 - \|v - x^*\|^2 \right).
\]
Recalling (21) and our assumption that \( f \in S_{\mu,L}^1 \) with \( 0 < \mu \leq L < \infty \), this implies

\[
- \nabla \mathcal{L}(x) \cdot \mathcal{G}(x) \geq f(x) - f(x^*) + \frac{\mu}{2} \|v - x^*\|^2 + \frac{\mu}{2} \|x - v\|^2 = \mathcal{L}(x) + \frac{\mu}{2} \|x - v\|^2. \tag{83}
\]

Consequently \( \mathcal{L} \) is a strong Lyapunov function with \( q = 1 \), \( c(x) = 1 \) and \( p^2(x) = \mu \|x - v\|^2 / 2 \). By Theorem 3.1, it follows that

\[
\mathcal{L}(x(t)) \leq e^{-t} \mathcal{L}(0), \quad t \geq 0.
\]

### 6.3 A semi-implicit discretization

Given \( \alpha_k > 0 \), we consider a semi-implicit scheme for solving the HB system (80):

\[
\begin{aligned}
    x_{k+1} &= x_k + \alpha_k (v_k - x_{k-1}), \\
    v_{k+1} &= v_k + \alpha_k (x_{k+1} - v_k) - \frac{\alpha_k}{\mu} \nabla f(x_{k+1}),
\end{aligned} \tag{84}
\]

which is a Gauss-Seidel type iteration. We first treat \( v \) as known as \( v_k \) and solve the first equation to get \( x_{k+1} \) and then with known \( x_{k+1} \) to solve the second equation to update \( v_{k+1} \). A discrete analogue to (81) is

\[
\mathcal{L}_k := \mathcal{L}(x_k) := f(x_k) - f(x^*) + \frac{\mu}{2} \|v_k - x^*\|^2, \tag{85}
\]

where \( x_k = (x_k, v_k) \).

**Lemma 6.1** Assume \( f \in S_{\mu,L}^1 \) with \( 0 < \mu \leq L < \infty \). Then for the scheme (84) with any \( \alpha_k > 0 \), we have

\[
\mathcal{L}_{k+1} - \mathcal{L}_k \leq -\alpha_k \mathcal{L}_{k+1} + \frac{\alpha_k^2}{2\mu} \|
abla f(x_{k+1})\|^2. \tag{86}
\]

**Proof** We split the difference along the path \((x_k, v_k)\) to \((x_{k+1}, v_{k+1})\) and then to \((x_{k+1}, v_{k+1})\):

\[
\mathcal{L}_{k+1} - \mathcal{L}_k = \mathcal{L}(x_{k+1}, v_k) - \mathcal{L}(x_k, v_k) + \mathcal{L}(x_{k+1}, v_{k+1}) - \mathcal{L}(x_{k+1}, v_k) := I_1 + I_2.
\]

Again the idea is to express the difference in terms of \( \langle \nabla \mathcal{L}(x_{k+1}), \mathcal{G}(x_{k+1}) \rangle \) and then use the strong Lyapunov property.

For item \( I_2 \), we use the fact \( \mathcal{L}(x_{k+1}, \cdot) \) is \( \mu \)-convex to get

\[
I_2 \leq \langle \nabla v \mathcal{L}(x_{k+1}, v_{k+1}), v_{k+1} - v_k \rangle - \frac{\mu}{2} \|v_{k+1} - v_k\|^2
= \alpha_k \langle \nabla v \mathcal{L}(x_{k+1}), \mathcal{G}^v(x_{k+1}) \rangle - \frac{\mu}{2} \|v_{k+1} - v_k\|^2.
\]

We now estimate \( I_1 \) as follows

\[
I_1 = f(x_{k+1}) - f(x_k) = \langle \nabla f(x_{k+1}), x_{k+1} - x_k \rangle - Df(x_k, x_{k+1})
\leq \langle \nabla f(x_{k+1}), x_{k+1} - x_k \rangle - \frac{\mu}{2} \|x_{k+1} - x_k\|^2
= \langle \nabla x \mathcal{L}(x_{k+1}), x_{k+1} - x_k \rangle - \frac{\mu}{2} \|x_{k+1} - x_k\|^2.
\]
In the last step, we can switch from point \((x_{k+1}, v_k)\) to \((x_{k+1}, v_{k+1})\) because \(\nabla_x L = \nabla f(x)\) is independent of \(v\).

Then we use the discretization (84) to replace \(x_{k+1} - x_k\) and compare with the flow evaluated at \(x_{k+1} = (x_{k+1}, v_{k+1})\):

\[
\langle \nabla_x L(x_{k+1}), x_{k+1} - x_k \rangle = \alpha_k \langle \nabla_x L(x_{k+1}), G(x_{k+1}) \rangle + \alpha_k \langle \nabla f(x_{k+1}), v_k - v_{k+1} \rangle.
\]

Observing the bound for \(I_2\), we use Cauchy–Schwarz inequality to bound the second term by that

\[
\alpha_k \|\nabla f(x_{k+1})\| \|v_k - v_{k+1}\| \leq \frac{\alpha_k^2}{2\mu} \|\nabla f(x_{k+1})\|^2 + \frac{\mu}{2} \|v_k - v_{k+1}\|^2.
\] (87)

Adding all together, we get

\[
L_{k+1} - L_k \leq \alpha_k \langle \nabla L(x_{k+1}), G(x_{k+1}) \rangle + \frac{\alpha_k^2}{2\mu} \|\nabla f(x_{k+1})\|^2.
\]

Then applying the strong Lyapunov property \(A(1, 1, p)\) at \(x_{k+1}\), cf. (83), yields that

\[
L_{k+1} - L_k \leq -\alpha_k L_{k+1} - \frac{\mu}{2} \|x_{k+1} - x_k\|^2 - \frac{\mu \alpha_k}{2} \|x_{k+1} - v_{k+1}\|^2 + \frac{\alpha_k^2}{2\mu} \|\nabla f(x_{k+1})\|^2. \tag{88}
\]

This completes the proof with extra negative terms.

Although there are additional negative terms in (88), they cannot be used to cancel the positive term involving \(\|\nabla f(x_{k+1})\|^2\) as they are not directly related.

### 6.4 Momentum methods

To cancel the positive term in the right hand side of (86), we add one extra gradient descent step to (84):

\[
\begin{align*}
y_k &= x_k + \alpha_k (v_k - y_k), \\
v_{k+1} &= v_k + \alpha_k (y_k - v_{k+1}) - \frac{\alpha_k}{\mu} \nabla f(y_k), \\
x_{k+1} &= y_k - \frac{1}{L} \nabla f(y_k).
\end{align*} \tag{89}
\]

Here we use \(y_k\) for the intermediate approximation and \(x_{k+1}\) is an extra gradient descent step obtained from \(y_k\). Note that \(\nabla f(y_k)\) appears twice in each iteration but it can be evaluated only once.

Following the proof of the previous section, we are able to establish the contraction of the Lyapunov function (85) for the modified scheme (89), which gives a momentum method and will be summarized later in Algorithm 2.

**Theorem 6.1** Assume \(f \in \mathcal{S}_{\mu, L}^1\) with \(0 < \mu \leq L < \infty\). If \(\alpha_k\) satisfies \(L \alpha_k^2 \leq \mu (1 + \alpha_k)\), then for (89) we have the contraction property

\[
L_{k+1} \leq \frac{1}{1 + \alpha_k} L_k. \tag{90}
\]
In particular, choosing
\[ \alpha_k = \sqrt{\frac{\mu}{L}}, \quad \text{or} \quad \alpha_k = \frac{\mu + \sqrt{\mu^2 + 4L\mu}}{2L}, \]
we have the accelerated linear convergence rate
\[ \mathcal{L}_{k+1} \leq \frac{1}{1 + \sqrt{\mu/L}} \mathcal{L}_k. \]

**Proof** In Lemma 6.1, we have already proved that
\[ (1 + \alpha_k) \mathcal{L}(y_k, v_{k+1}) - \mathcal{L}(x_k, v_k) \leq \frac{\alpha_k^2}{2\mu} \|\nabla f(y_k)\|_*^2. \tag{91} \]
Thanks to (13) and the extra gradient descent step in (89), we have the sufficient decay
\[ \mathcal{L}(x_{k+1}, v_{k+1}) - \mathcal{L}(y_k, v_{k+1}) = f(x_{k+1}) - f(y_k) \leq -\frac{1}{2L} \|\nabla f(y_k)\|_*^2. \tag{92} \]
Multiplying $1 + \alpha_k$ to (92) and adding to (91), we get
\[ (1 + \alpha_k) \mathcal{L}_{k+1} - \mathcal{L}_k \leq \left( \frac{\alpha_k^2}{2\mu} - \frac{1 + \alpha_k}{2L} \right) \|\nabla f(y_k)\|_*^2 \leq 0, \]
as $L\alpha_k^2 \leq \mu(1 + \alpha_k)$. This implies (90). The rest part is obvious and therefore we conclude
the proof of this theorem. \hfill \blacksquare

We now present a simplified version in the following algorithm which drops the sequence \( \{v_k\} \) from (89). Verification of the equivalence is straightforward.

**Algorithm 2** Momentum method for minimizing $f \in \mathcal{S}_{\mu,L}^1$ with $0 < \mu \leq L < \infty$

**Input:** $x_0, y_0 \in V$.

1: for $k = 0, 1, \ldots$ do
2: Update $x_{k+1} = y_k - \frac{1}{L} \nabla f(y_k)$.
3: Update $y_{k+1} = \begin{cases} \frac{\alpha y_k}{1 + \alpha} - \frac{x_k}{(1 + \alpha)^2} + \frac{2 + \alpha}{(1 + \alpha)^2} x_{k+1}, & \text{if} \ \alpha = \sqrt{\frac{\mu}{L}}, \\ \frac{\alpha^2 y_k}{(1 + \alpha)^2} - \frac{x_k}{(1 + \alpha)^2} + \frac{2x_{k+1}}{1 + \alpha}, & \text{if} \ \alpha = \frac{\mu + \sqrt{\mu^2 + 4L\mu}}{2L}. \end{cases}$
4: end for

7. Asymptotic Vanishing Damping System

In this section we study a second order ODE model, the so-called asymptotic vanishing damping (AVD) system (Su et al., 2016):
\[ x'' + \frac{r}{t} x' + \nabla f(x) = 0, \quad t \geq t_1 > 0, \tag{93} \]
where $r > 0, f \in \mathcal{S}_{0,L}^1$ is smooth convex and initial conditions are $x(t_1) = x_0, x'(t_1) = x_1$.\]
7.1 Existing works

The AVD model (93) was firstly derived and analyzed in Su et al. (2016) for the case $r \geq 3$ then further studied in Aujol and Dossal (2017); Attouch et al. (2019) for $r > 0$. The positive constant $r$ is very crucial for both the designing of Lyapunov function and the convergence rate analysis.

For $r \geq 3$, Su et al. (2016, Theorem 5) proved that

$$f(x(t)) - f(x^*) \leq \frac{(r - 1)^2}{2t^2} \|x_0 - x^*\|^2,$$  \hspace{1cm} (94)

by using the Lyapunov function

$$\mathcal{L}(t) := f(x(t)) - f(x^*) + \frac{(r - 1)^2}{2t^2} \left\| x + \frac{t}{r - 1} x' - x^* \right\|^2.$$  \hspace{1cm} (95)

Additionally, if $f$ is strongly convex, then they also obtained a faster decay rate (Su et al., 2016, Theorem 8)

$$f(x(t)) - f(x^*) \leq Ct^{-2r/3},$$

by the Lyapunov function

$$\mathcal{L}(t) := f(x(t)) - f(x^*) + \frac{2r^2}{9t^2} \left\| x + \frac{3t}{2r} x' - x^* \right\|^2.$$  \hspace{1cm} (96)

Later on, Aujol and Dossal (2017) introduced a Lyapunov function

$$\mathcal{L}(t) := f(x(t)) - f(x^*) + \frac{\lambda^2}{2t^2} \left\| x + \frac{t}{\lambda} x' - x^* \right\|^2 + \frac{\xi}{2t^2} \|x(t) - x^*\|^2,$$  \hspace{1cm} (96)

with $\lambda = 2\beta \min\{1, r/(2\beta + 1)\}$ and $\xi = \lambda |\lambda + 1 - r|$, and also generalized (94) to

$$f(x(t)) - f(x^*) \leq \begin{cases} Ct^{-2}, & \text{if } r \geq 2\beta + 1, \\ Ct^{-2r/(2\beta+1)}, & \text{if } 0 < r < 2\beta + 1, \end{cases}$$  \hspace{1cm} (97)

where $(f - f(x^*))^\beta$ is convex with $\beta > 0$. Around the same time, Attouch et al. (2019) obtained the estimate (97) for $\beta = 1$, with the corresponding Lyapunov function (96) taking $\beta = 1$. More importantly, they considered numerical discretizations for (93) and proved the sublinear convergence rate

$$f(x_k) - f(x^*) \leq \begin{cases} Ck^{-2}, & \text{if } r \geq 3, \\ Ck^{-2r/3}, & \text{if } 0 < r < 3, \end{cases}$$  \hspace{1cm} (98)

which matches the convergence rate (97) with $\beta = 1$ for the continuous level.
7.2 Strong Lyapunov condition

Here, we only focus on the case \( r = 3 \). As discussed in Luo and Chen (2019), the AVD model (93) with \( r = 3 \) is equivalent to NAG flow (see (115) in Section 8) with suitable time scaling. We shall apply our strong Lyapunov condition to establish the decay rates of continuous problem and its numerical discretizations.

Let \( v = x + tx' / 2 \) and introduce an auxiliary function \( \gamma = 4t^{-2} \). Then the AVD model (93) with \( r = 3 \) can be rewritten as

\[
\begin{align*}
&x' = \sqrt{\gamma} (v - x), \\
v' = - \nabla f(x) / \sqrt{\gamma}, \\
&\gamma' = - \gamma^{3/2},
\end{align*}
\]

and the Lyapunov function (95) reads equivalently as follows

\[
L(x) := f(x(t)) - f(x^*) + \frac{\gamma(t)}{2} \|v(t) - x^*\|^2,
\]

where \( x = (x, v, \gamma) \). Let us write the right hand side of (99) as \( G(x) \). It follows that

\[
- \nabla L(x) \cdot G(x) = \sqrt{\gamma} \langle \nabla f(x), x - x^* \rangle + \frac{\gamma^{3/2}}{2} \|v - x^*\|^2 \geq \sqrt{\gamma} L(x).
\]

Therefore \( L \) is a strong Lyapunov function of (93) with \( c(x) = \sqrt{\gamma}, q(x) = 1 \) and \( p(x) = 0 \).

By Theorem 3.1, we obtain the decay rate \( L(t) = O(t^{-2}) \), which coincides with (94).

7.3 Gauss-Seidel iteration with extra gradient step

Given any time step size \( \alpha_k > 0 \), we consider the following semi-implicit scheme for (99):

\[
\begin{align*}
\frac{x_{k+1} - x_k}{\alpha_k} &= \sqrt{\gamma_k} (v_k - x_{k+1}), \\
\frac{v_{k+1} - v_k}{\alpha_k} &= - \nabla f(x_{k+1}) / \sqrt{\gamma_k}, \\
\frac{\gamma_{k+1} - \gamma_k}{\alpha_k} &= - \sqrt{\gamma_k \gamma_{k+1}},
\end{align*}
\]

which is a Gauss-Seidel type discretization. Mimicking (100), for \( x_k = (x_k, v_k, \gamma_k) \), we introduce the discrete Lyapunov function

\[
L_k := L(x_k) = f(x_k) - f(x^*) + \frac{\gamma_k}{2} \|v_k - x^*\|^2.
\]

A one iteration analysis is given below.

**Lemma 7.1** If \( f \in S_{0, L}^1 \), then for the semi-implicit scheme (102a) with any step size \( \alpha_k > 0 \), we have

\[
L_{k+1} - L_k \leq - \alpha_k \sqrt{\gamma_k} L_{k+1} - \frac{\gamma_k}{2} \|v_{k+1} - v_k\|^2 + \alpha_k \sqrt{\gamma_k} \langle \nabla f(x_{k+1}), v_k - v_{k+1} \rangle,
\]

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which implies
\[
\mathcal{L}_{k+1} - \mathcal{L}_k \leq -\alpha_k \sqrt{\gamma_k} \mathcal{L}_{k+1} + \frac{\alpha_k^2}{2} \|\nabla f(x_{k+1})\|^2.
\]  

**Proof**  Let us calculate the difference \( \mathcal{L}_{k+1} - \mathcal{L}_k = I_1 + I_2 + I_3 \) where
\[
\begin{align*}
I_1 &:= \mathcal{L}(x_{k+1}, v_k, \gamma_k) - \mathcal{L}(x_k, v_k, \gamma_k), \\
I_2 &:= \mathcal{L}(x_{k+1}, v_{k+1}, \gamma_k) - \mathcal{L}(x_{k+1}, v_k, \gamma_k), \\
I_3 &:= \mathcal{L}(x_{k+1}, v_{k+1}, \gamma_k) - \mathcal{L}(x_{k+1}, v_{k+1}, \gamma_k).
\end{align*}
\]

Set \( \tau_k = \sqrt{\gamma_k/\gamma_{k+1}} \). Below, we shall estimate the above three terms one by one.

It is evident that
\[
I_3 = \langle \nabla_\gamma \mathcal{L}(x_{k+1}), \gamma_{k+1} - \gamma_k \rangle = \alpha_k \tau_k \langle \nabla_\gamma \mathcal{L}(x_{k+1}), G^\gamma(x_{k+1}) \rangle.
\]

For item \( I_2 \), we use the fact \( \mathcal{L}(x_{k+1}, \cdot, \gamma_k) \) is \( \gamma_k \)-convex to get
\[
I_2 \leq \langle \nabla v \mathcal{L}(x_{k+1}, v_{k+1}, \gamma_k), v_{k+1} - v_k \rangle - \frac{\gamma_k}{2} \|v_{k+1} - v_k\|^2
\]
\[
= -\langle \sqrt{\gamma_k} (v_{k+1} - x^*), \nabla f(x_{k+1}) \rangle - \frac{\gamma_k}{2} \|v_{k+1} - v_k\|^2
\]
\[
= -\tau_k \langle \sqrt{\gamma_{k+1}} (v_{k+1} - x^*), \nabla f(x_{k+1}) \rangle - \frac{\gamma_k}{2} \|v_{k+1} - v_k\|^2,
\]
and in view of (102b), we have
\[
I_2 \leq \alpha_k \tau_k \langle \nabla v \mathcal{L}(x_{k+1}), G^v(x_{k+1}) \rangle - \frac{\gamma_k}{2} \|v_{k+1} - v_k\|^2.
\]

We then estimate \( I_1 \) as follows
\[
I_1 = f(x_{k+1}) - f(x_k) \leq \langle \nabla f(x_{k+1}), x_{k+1} - x_k \rangle = \langle \nabla_x \mathcal{L}(x_{k+1}), x_{k+1} - x_k \rangle.
\]

In the last step, we have switched from point \((x_{k+1}, v_k, \gamma_k)\) to \((x_{k+1}, v_{k+1}, \gamma_{k+1})\) because \( \nabla_x \mathcal{L} = \nabla f(x) \) is independent of \((v, \gamma)\). Then we use the discretization (102a) to replace \( x_{k+1} - x_k \) and compare with the flow evaluated at \( x_{k+1} = (x_{k+1}, v_{k+1}, \gamma_{k+1}) \):
\[
\langle \nabla_x \mathcal{L}(x_{k+1}), x_{k+1} - x_k \rangle = \alpha_k \tau_k \langle \nabla_x \mathcal{L}(x_{k+1}), G^x(x_{k+1}) \rangle + \alpha_k \sqrt{\gamma_k} \langle \nabla f(x_{k+1}), v_k - v_{k+1} \rangle.
\]

Whence, adding all together and applying the strong Lyapunov condition \( \mathcal{A}(\sqrt{\gamma}, 1, 0) \) at \( x_{k+1} \) (cf. (101)) yield that
\[
\mathcal{L}_{k+1} - \mathcal{L}_k \leq -\alpha_k \sqrt{\gamma_k} \mathcal{L}_{k+1} - \frac{\gamma_k}{2} \|v_{k+1} - v_k\|^2 + \alpha_k \sqrt{\gamma_k} \langle \nabla f(x_{k+1}), v_k - v_{k+1} \rangle.
\]

This proves (104). Besides, applying Cauchy–Schwarz inequality gives (105) and completes the proof of this lemma. ■

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Obviously, one cannot obtain contraction result of $\mathcal{L}_k$ from Lemma 7.1. To cancel the positive term in (105), we then modify (102a) by adding one gradient descent step:

$$\begin{align*}
\frac{y_k - x_k}{\alpha_k} &= \sqrt{\gamma_k} (v_k - y_k), \\
\frac{v_{k+1} - v_k}{\alpha_k} &= -\nabla f(y_k)/\sqrt{\gamma_k}, \\
\frac{x_{k+1} - y_k}{\alpha_k} &= -\frac{1}{L} \nabla f(y_k), \\
\frac{\gamma_{k+1} - \gamma_k}{\alpha_k} &= -\sqrt{\gamma_k \gamma_{k+1}}.
\end{align*}$$

(108a, 108b, 108c, 108d)

Thanks to Lemma 7.1, we have

$$\tilde{\mathcal{L}}_{k+1} - \mathcal{L}_k \leq -\alpha_k \sqrt{\gamma_k} \tilde{\mathcal{L}}_{k+1} + \frac{\alpha_k^2}{2} \|\nabla f(y_k)\|^2_*,$$

where

$$\tilde{\mathcal{L}}_{k+1} := f(y_k) - f(x^*) + \frac{\gamma_{k+1}}{2} \|v_{k+1} - x^*\|^2.$$

(109)

Moreover, by (13) and the gradient descent step of $x_{k+1}$ in (108a), we see that

$$\mathcal{L}_{k+1} - \tilde{\mathcal{L}}_{k+1} = f(x_{k+1}) - f(y_k) \leq -\frac{1}{2L} \|\nabla f(y_k)\|^2_*.$$

(110)

This promises the contraction property

$$\mathcal{L}_{k+1} - \mathcal{L}_k \leq -\alpha_k \sqrt{\gamma_k} \mathcal{L}_{k+1},$$

(111)

provided that $L\alpha_k^2 \leq 1 + \alpha_k \sqrt{\gamma_k}$.

Before the convergence analysis, let us simplify (108a). If $L\alpha_k^2 = 1 + \alpha_k \sqrt{\gamma_k}$, then by (108a) and (108b), we have

$$v_{k+1} = x_{k+1} + \frac{x_{k+1} - x_k}{\alpha_k \sqrt{\gamma_k}}.$$

Plugging this into (108a) and using (108d) imply that

$$y_{k+1} = x_{k+1} + \frac{x_{k+1} - x_k}{L\alpha_k^2 \sqrt{L\alpha_k+1}}.$$

Thus the sequences $\{v_k\}$ and $\{\gamma_k\}$ can totally be dropped.

**Theorem 7.1** For (108a), we have

$$\mathcal{L}_{k+1} - \mathcal{L}_k \leq -\alpha_k \sqrt{\gamma_k} \mathcal{L}_{k+1}.$$

(112)

This implies that

$$\mathcal{L}_k \leq \mathcal{L}_0 \times \prod_{i=0}^{k-1} \frac{1}{1 + \alpha_i \sqrt{\gamma_i}} = \frac{\gamma_k}{\gamma_0} \mathcal{L}_0,$$

where the rate of convergence is given by, with $r = \gamma_0/L$,

$$\frac{\gamma_k}{\gamma_0} \leq \left(1 + \frac{\sqrt{r}}{2 + \sqrt{r}}\right)^2 \left(\frac{2}{2 + \sqrt{r}}\right)^2.$$

(113)
**Proof** By the above discussion, the contraction \((112)\) is evident since \(L\alpha_k^2 = 1 + \alpha_k \sqrt{\gamma_k}\). According to the equation of \(\{\gamma_k\}\) in \((108d)\), it is clear that 

\[
\frac{\gamma_k}{\gamma_0} = \prod_{i=0}^{k-1} \frac{1}{1 + \alpha_i \sqrt{\gamma_i}}.
\]

It remains to prove the decay rate of \(\gamma_k\). We have \(\gamma_k \geq \gamma_{k+1} \) and thus 

\[
\sqrt{\gamma_{k+1}} - \sqrt{\gamma_k} = \frac{\gamma_{k+1} - \gamma_k}{\sqrt{\gamma_{k+1}} + \sqrt{\gamma_k}} = -\frac{\alpha_k \sqrt{\gamma_k} \gamma_{k+1}}{\sqrt{\gamma_{k+1}} + \sqrt{\gamma_k}} \leq -\frac{\alpha_k}{2} \gamma_{k+1}.
\]

As \(\alpha_k = \sqrt{1 + \alpha_k \sqrt{\gamma_k}}/\sqrt{L} \geq 1/\sqrt{L}\), we have 

\[
\sqrt{\gamma_{k+1}} - \sqrt{\gamma_k} \leq -\frac{1}{2\sqrt{L}} \gamma_{k+1}.
\]

Applying Theorem 3.2 (4) to the sequence \(\{\sqrt{\gamma_k}\}\), we get the decay estimate \((113)\) and finish the proof of this theorem.

### 7.4 Gauss-Seidel iteration with extrapolation

Instead of \((108a)\), let us consider another modified scheme

\[
\begin{align*}
\frac{y_k - x_k}{\alpha_k} &= \sqrt{\gamma_k}(v_k - y_k), \quad (114a) \\
\frac{v_{k+1} - v_k}{\alpha_k} &= -\nabla f(y_k)/\sqrt{\gamma_k}, \quad (114b) \\
\frac{x_{k+1} - x_k}{\alpha_k} &= \sqrt{\gamma_k}(v_{k+1} - x_{k+1}), \quad (114c) \\
\frac{\gamma_{k+1} - \gamma_k}{\alpha_k} &= -\sqrt{\gamma_k} \gamma_{k+1}, \quad (114d)
\end{align*}
\]

where we used an extrapolation step \((114c)\) to update \(x_{k+1}\). This is different from the gradient descent step in \((108a)\). By Lemma 7.1, we have

\[
\hat{L}_{k+1} - L_k \leq -\alpha_k \sqrt{\gamma_k} \hat{L}_{k+1} - \frac{\gamma_k}{2} \|v_{k+1} - v_k\|^2 + \alpha_k \sqrt{\gamma_k} \langle \nabla f(y_k), v_k - v_{k+1} \rangle,
\]

where \(\hat{L}_{k+1}\) is defined by \((109)\). In addition, \((110)\) becomes

\[
L_{k+1} - \hat{L}_{k+1} = f(x_{k+1}) - f(y_k) \leq \langle \nabla f(y_k), x_{k+1} - y_k \rangle + \frac{L}{2} \|x_{k+1} - y_k\|^2.
\]

Combining \((114a)\) with \((114c)\) gives the relation

\[
(1 + \alpha_k \sqrt{\gamma_k})(x_{k+1} - y_k) = \alpha_k \sqrt{\gamma_k}(v_{k+1} - v_k),
\]

which implies that

\[
L_{k+1} - \hat{L}_{k+1} \leq \frac{\alpha_k \sqrt{\gamma_k}}{1 + \alpha_k \sqrt{\gamma_k}} \langle \nabla f(y_k), v_{k+1} - v_k \rangle + \frac{L\alpha_k^2 \gamma_k}{(1 + \alpha_k \sqrt{\gamma_k})^2} \|v_{k+1} - v_k\|^2.
\]
Therefore, if \( L\alpha_k^2 \leq 1 + \alpha_k \sqrt{\gamma_k} \), then the contraction (111) follows immediately.

Moreover, if \( L\alpha_k^2 = 1 + \alpha_k \sqrt{\gamma_k} \), then we claim that (114a) coincides with (108a). It is sufficient to verify that (114c) is identical to (108c). Indeed, inserting (108a) and (108b) into (108c) gives

\[
x_{k+1} = \frac{x_k + \alpha_k \sqrt{\gamma_k} v_{k+1}}{1 + \alpha_k \sqrt{\gamma_k}} = \frac{x_k + \alpha_k \sqrt{\gamma_k} (v_k - \alpha_k \nabla f(y_k) / \sqrt{\gamma_k})}{1 + \alpha_k \sqrt{\gamma_k}}
\]

\[
= \frac{x_k + \alpha_k \sqrt{\gamma_k} v_k}{1 + \alpha_k \sqrt{\gamma_k}} - \frac{\alpha_k^2 \nabla f(y_k)}{1 + \alpha_k \sqrt{\gamma_k}} = y_k - \frac{1}{L} \nabla f(y_k).
\]

For other choice that violates the relation \( L\alpha_k^2 = 1 + \alpha_k \sqrt{\gamma_k} \), we cannot obtain the equivalence. For simplicity, we will not consider general choices here.

### 8. A Family of Nesterov Accelerated Gradient Methods

The last two sections treat the HB model (76) and the AVD model (93) for strongly convex case \((\mu > 0)\) and convex case \((\mu = 0)\), respectively. Apart from this, we have not considered accelerated methods for the composite case \(f = h + g\).

In this section, we shall propose a novel second order dynamical system called the **Hessian-driven Nesterov accelerated gradient (HNAG)** flow that involves a built-in time scaling and unifies the analysis for \(\mu \geq 0\). We will design several accelerated first order optimization methods based on numerical discretizations of our HNAG flow system. Moreover, we extend this model to the composite setting and propose two accelerated proximal gradient methods. As before, the convergence analysis will be established via the strong Lyapunov condition.

#### 8.1 Nesterov accelerated gradient flow

In our recent work (Luo and Chen, 2019), for \(f \in S^1_{\mu} \) with \(\mu \geq 0\), we have introduced a new ODE model

\[
\gamma x'' + (\gamma + \mu) x' + \nabla f(x) = 0, \quad \gamma' = \mu - \gamma,
\]

with initial conditions \(x(0) = x_0, x'(0) = x_1\) and \(\gamma(0) = \gamma_0 > 0\). For algorithmic designing and convergence analysis, we prefer the alternative formulation as an ODE system

\[
\begin{cases}
x' = v - x, \\
v' = \frac{\mu}{\gamma}(x - v) - \frac{1}{\gamma} \nabla f(x), \\
\gamma' = \mu - \gamma.
\end{cases}
\]

An appropriate numerical discretization of (116) recovers exactly Nesterov’s optimal method constructed from estimate sequence (Nesterov, 2013, Chapter 2). Hence, we call (115) and (116) **Nesterov accelerated gradient (NAG)** flows. Exponential decay of the Lyapunov function (100) has been established and it was also proved that Gauss-Seidel iteration with one extra gradient descent step lead to a variant of Nesterov accelerated gradient method; see Luo and Chen (2019).
Motivated by the dynamical inertial Newton (DIN) system proposed by Alvarez et al. (2002) and Hessian-driven damping models Attouch et al. (2012, 2020), we further propose a new second order dynamical system, which is called the Hessian-driven Nesterov accelerated gradient (HNAG) flow and reads as follows

$$\gamma x'' + (\gamma + \mu) x' + \beta \gamma \nabla^2 f(x)x' + (1 + \mu \beta + \gamma \beta') \nabla f(x) = 0, \quad (117)$$

where $\beta > 0$ is any positive smooth (continuous differentiable) function on $[0, \infty)$ and $\gamma$ is the same time scaling factor as that in (115).

Obviously, the HNAG flow model (117) requires stronger smoothness $f \in \mathcal{C}^2 \cap \mathcal{S}_\mu^1$ than NAG flow (115). Therefore direct discretization based on (117) is restrictive and might be expensive due to the existence of the Hessian matrix. Fortunately, as observed in (Alvarez et al., 2002), if we write (117) as the first-order system

$$\begin{cases} x' = v - x - \beta \nabla f(x), \\
v' = \frac{\mu}{\gamma} (x - v) - \frac{1}{\gamma} \nabla f(x), \\
\gamma' = \mu - \gamma, \quad (118)\end{cases}$$

no $\nabla^2 f(x)$ is needed. The formulation (118) can be also thought of as a modified model of our previous NAG flow (116) by adding one more damping term $-\beta \nabla f(x)$ to the system. In the next, we will see this minor modification brings faster decay of the gradient. Under standard assumption $f \in \mathcal{S}^1_{\mu,L}$ with $0 \leq \mu \leq L < \infty$, existence and uniqueness of classical solution $(x, v) \in \mathcal{C}^1 \times \mathcal{C}^1$ to (118) can be easily concluded from conventional theory of ODE.

For $x = (x, v, \gamma)$, we still use the Lyapunov function $\mathcal{L}(x) := f(x) - f(x^*) + \frac{\gamma}{2} \|v - x^*\|^2$, and denote by $\mathcal{G}(x)$ the right hand side of (118), which then becomes $x' = \mathcal{G}(x)$. Observing the identity (82) and the $\mu$-convexity of $f$ (cf. (21)), a direct computation gives

$$-\nabla \mathcal{L}(x) \cdot \mathcal{G}(x) = -\mu \langle x - v, v - x^* \rangle + \langle \nabla f(x), x - x^* \rangle + \beta \|\nabla f(x)\|_*^2 + \frac{\gamma - \mu}{2} \|v - x^*\|^2$$

$$\geq \mathcal{L}(x) + \beta \|\nabla f(x)\|_*^2 + \frac{\mu}{2} \|x - v\|^2. \quad (119)$$

Hence $\mathcal{L}$ is a strong Lyapunov function of (118) and satisfies $A(q, c, p)$ with $q = 1$, $c(x) = 1$, and $p^2(x) = \beta \|\nabla f(x)\|_*^2 + \frac{\gamma}{2} \|x - v\|^2$. Invoking Theorem 3.1, one can prove the exponential decay

$$\mathcal{L}(x(t)) + \int_0^t e^{s-t} \beta(s) \|\nabla f(x(s))\|_*^2 \, ds \leq e^{-t} \mathcal{L}(x(0)), \quad t \geq 0. \quad (120)$$

Thanks to the built-in scaling factor $\gamma$, this holds true for $\mu \geq 0$ in a unified and simpler way. Additionally, as one may see from (120), the extra gradient norm square term $\beta \|\nabla f(x)\|_*^2$ in (119) brings faster decay of the gradient.
8.2 Nesterov accelerated gradient method

Let us apply the Gauss-Seidel type discretization to (118) and obtain
\[
\begin{aligned}
\frac{x_{k+1} - x_k}{\alpha_k} &= v_k - x_{k+1} - \beta_k \nabla f(x_k), \\
\frac{v_{k+1} - v_k}{\alpha_k} &= \frac{\mu}{\gamma_k} (x_{k+1} - v_{k+1}) - \frac{1}{\gamma_k} \nabla f(x_{k+1}), \\
\frac{\gamma_{k+1} - \gamma_k}{\alpha_k} &= \mu - \gamma_{k+1},
\end{aligned}
\] (121)

where $\alpha_k > 0$ is the time step size. Given the current iterate $x_k = (x_k, v_k, \gamma_k)$, one compute $x_{k+1}$ and $v_{k+1}$ successively from the first and the second equations and then update the parameter $\gamma_{k+1}$ by the last one. We have three parameters $(\alpha_k, \beta_k, \gamma_k)$ in (121) and will set
\[
\alpha_k \beta_k = 1/L, \quad L \alpha^2_k = \gamma_k(2 + \alpha_k), \quad \alpha_k > 0. \quad (122)
\]

Although there are two gradient evaluations in the $k$-th iteration of (121), the second one $\nabla f(x_{k+1})$ can be reused in the $k+1$-th iteration for updating $x_{k+2}$. Moreover, introduce an extra variable
\[
y_k = x_k - \frac{1}{L} \nabla f(x_k),
\] (123)

we can obtain an equivalent form of (121) which requires only one gradient evaluation in each iteration and has been summarized in Algorithm 3.

**Algorithm 3** NAG method for minimizing $f \in S^{1}_{\mu,L}$ with $0 \leq \mu \leq L < \infty$

**Input:** $\gamma_0 > 0, x_0, v_0 \in V$

1: Initialization $y_0 = x_0 - \frac{1}{L} \nabla f(x_0)$.

2: for $k = 0, 1, \ldots$ do

3: Compute $\alpha_k = (\gamma_k + \sqrt{\gamma_k^2 + 8L \gamma_k})/(2L)$.

4: Update $x_{k+1} = (y_k + \alpha_k v_k)/(1 + \alpha_k)$.

5: Compute $y_{k+1} = x_{k+1} - \frac{1}{L} \nabla f(x_{k+1})$.

6: Update $v_{k+1} = \frac{\gamma_k v_k + \mu \alpha_k x_{k+1}}{\gamma_k + \mu \alpha_k} + \frac{L \alpha_k}{\gamma_k + \mu \alpha_k} (y_{k+1} - x_{k+1})$.

7: Update $\gamma_{k+1} = (\mu \alpha_k + \gamma_k)/(1 + \alpha_k)$.

8: end for

**Lemma 8.1** For Algorithm 3, we have
\[
\mathcal{L}_{k+1} - \frac{1}{2L} \| \nabla f(x_{k+1}) \|^2_2 \leq \frac{1}{1 + \alpha_k} \left( \mathcal{L}_k - \frac{1}{2L} \| \nabla f(x_k) \|^2_2 \right) \quad \forall k \geq 0, \quad (124)
\]

where $\mathcal{L}_k = \mathcal{L}(x_k) = f(x_k) - f(x^*) + \frac{\alpha_k}{2} \| v_k - x^* \|^2$.

**Proof** Following the proof of Lemma 7.1, we have the difference $\mathcal{L}_{k+1} - \mathcal{L}_k = I_1 + I_2 + I_3$, where $I_1, I_2$ and $I_3$ are defined in (106a). Below, we shall estimate these three terms one by one.
As $L$ is linear in terms of $\gamma$, we see

$$I_3 = \langle \nabla_x L(x_{k+1}), \gamma_{k+1} - \gamma_k \rangle = \alpha_k \langle \nabla_x L(x_{k+1}), G^\gamma(x_{k+1}) \rangle. \quad (125)$$

For the second item $I_2$, we use the fact $L(x_{k+1}, \cdot, \gamma_k)$ is $\gamma_k$-convex to get

$$I_2 \leq \langle \nabla_v L(x_{k+1}, v_{k+1}, \gamma_k), v_{k+1} - v_k \rangle - \frac{\gamma_k}{2} \|v_{k+1} - v_k\|^2$$

$$= \alpha_k \langle \nabla_v L(x_{k+1}), G^v(x_{k+1}) \rangle - \frac{\gamma_k}{2} \|v_{k+1} - v_k\|^2. \quad (126)$$

In the last step, as the parameter $\gamma$ is canceled in the product $\langle \nabla_v L(x), G^v(x) \rangle$, we can switch the variable $(x_{k+1}, \gamma_k)$ to $(x_{k+1}, \gamma_{k+1})$.

We now focus on the first one $I_1$:

$$I_1 \leq \langle \nabla_x L(x_{k+1}), x_{k+1} - x_k \rangle - \frac{1}{2\ell} \|\nabla f(x_{k+1}) - \nabla f(x_k)\|_*^2.$$ 

In the first term, we can switch $(x_{k+1}, v_k, \gamma_k)$ to $x_{k+1}$ because $\nabla_x L = \nabla f(x)$ is independent of $(v, \gamma)$. Then we use the discretization (121) to replace $x_{k+1} - x_k$ and compare with the flow evaluated at $x_{k+1}$:

$$\langle \nabla_x L(x_{k+1}), x_{k+1} - x_k \rangle = \alpha_k \langle \nabla_x L(x_{k+1}), G^x(x_{k+1}) \rangle$$

$$+ \alpha_k \beta_k \langle \nabla f(x_{k+1}), \nabla f(x_{k+1}) - \nabla f(x_k) \rangle$$

$$+ \alpha_k \langle \nabla f(x_{k+1}), v_k - v_{k+1} \rangle.$$ 

Observing the bound (126) for $I_2$, we use Cauchy–Schwarz inequality to bound the last term as follows

$$\alpha_k \|\nabla f(x_{k+1})\|_* \|v_k - v_{k+1}\| \leq \frac{\alpha_k^2}{2\gamma_k} \|\nabla f(x_{k+1})\|_*^2 + \frac{\gamma_k}{2} \|v_k - v_{k+1}\|^2.$$

We use the identity (82) for the cross term

$$\alpha_k \beta_k \langle \nabla f(x_{k+1}), \nabla f(x_{k+1}) - \nabla f(x_k) \rangle$$

$$= - \frac{\alpha_k \beta_k}{2} \|\nabla f(x_k)\|_*^2 + \frac{\alpha_k \beta_k}{2} \|\nabla f(x_{k+1})\|_*^2 + \frac{\alpha_k \beta_k}{2} \|\nabla f(x_{k+1}) - \nabla f(x_k)\|_*^2.$$

Adding all together and applying strong Lyapunov property $\mathcal{A}(1,1,p^2)$ with $p^2 = \beta \|\nabla f(x)\|_*^2 + \frac{\beta}{2} \|x - v\|^2$ at $x_{k+1}$ (but with $\beta_k$ not $\beta_{k+1}$) yields that

$$L_{k+1} - L_k \leq - \alpha_k L_{k+1} - \frac{\alpha_k \beta_k}{2} \|\nabla f(x_k)\|_*^2$$

$$+ \frac{1}{2} \left( \frac{\alpha_k^2}{\gamma_k} - \alpha_k \beta_k \right) \|\nabla f(x_{k+1})\|_*^2$$

$$+ \frac{1}{2} \left( \alpha_k \beta_k - \frac{1}{L} \right) \|\nabla f(x_{k+1}) - \nabla f(x_k)\|^2.$$

By our choice of parameters (122):

$$\alpha_k \beta_k - \frac{1}{L} = 0, \quad \frac{\alpha_k^2}{\gamma_k} - \alpha_k \beta_k = (1 + \alpha_k) \frac{1}{L},$$

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and consequently,
\[ \mathcal{L}_{k+1} - \mathcal{L}_k \leq -\alpha_k \mathcal{L}_{k+1} - \frac{1}{2L} \| \nabla f(x_k) \|^2 + (1 + \alpha_k) \frac{1}{2L} \| \nabla f(x_{k+1}) \|^2. \]
Rearranging the above inequality gives the desired estimate (124).

For the extra variable \( y_k \) defined by (123), we have by (13) that
\[ f(y_k) \leq f(x_k) - \frac{1}{2L} \| \nabla f(x_k) \|_2^2, \]
which implies
\[ \mathcal{L}_k - \frac{1}{2L} \| \nabla f(x_k) \|^2 \geq f(y_k) - f(x^*) + \frac{\gamma_k}{2} \| v_k - x^* \|^2 = \mathcal{L}(y_k, v_k, \gamma_k) \geq 0. \]
Therefore it is easy to conclude from (124) that
\[ \mathcal{L}(y_k, v_k, \gamma_k) \leq \mathcal{L}_k - \frac{1}{2L} \| \nabla f(x_k) \|^2 \leq \rho_k \left( \mathcal{L}_0 - \frac{1}{2L} \| \nabla f(x_0) \|^2 \right), \]
where \( \rho_k \) is defined by (39).

According to the above discussion, we conclude the following result. As we see, due to the slightly larger step size, in the inequality (49), the constant 1 is increased to 2 and thus the rate (127) is slightly better.

**Theorem 8.1** For Algorithm 3 with \( \gamma_0 = rL \geq \mu \), we have
\[ f(y_k) - f(x^*) + \frac{\gamma_k}{2} \| v_k - x^* \|^2 \leq \rho_k \mathcal{L}_0 \quad \forall k \geq 0, \]
where \( \rho_k \) is defined by (39) and satisfies the estimate
\[ \rho_k \leq \min \left\{ \left( \frac{\sqrt{2}}{\sqrt{2} + \sqrt{r} k} \right)^2, \left( 1 + \sqrt{\frac{2 \mu}{L}} \right)^{-k} \right\}. \] (127)

### 8.3 Accelerated proximal gradient methods

We now move to the nonsmooth case. Let \( f = h + g \) and assume that \( f \in \mathcal{S}_L^0 \) with \( \mu \geq 0 \), \( h \in \mathcal{S}_L^1 \) is the smooth part and the non-smooth part \( g \) is convex and lower semicontinuous.

In this setting, the HNAG flow (118) becomes
\[ \begin{cases} 
  x' \in v - x - \beta \partial f(x), \\
  v' \in \frac{\mu}{\gamma} (x - v) - \frac{1}{\gamma} \partial f(x), \\
  \gamma' = \mu - \gamma.
\end{cases} \] (128)

For \( x = (x, v, \gamma) \), let the right hand side of (128) be \( \mathcal{G}(x) \) and we write \( \mathcal{G}(x, d) \) if \( \partial f(x) \) is replaced by some \( d \in \partial f(x) \). We still use the Lyapunov function (100). Similar with (119), one can easily verify the strong Lyapunov property: for any \( d \in \partial f(x) \),
\[ -\partial \mathcal{L}(x, d) \cdot \mathcal{G}(x, d) \geq \mathcal{L}(x) + \beta \| d \|_*^2 + \frac{\mu}{2} \| x - v \|^2. \] (129)
Yet, unlike the smooth case (118), it is generally hard to obtain classical $C^1$ solution $(x, v)$ of (128) since the subdifferential $\partial f(x)$ is a set-valued mapping and discontinuity may occur in $x'$ and $v'$. Also it is nontrivial to establish the corresponding nonsmooth version of the exponential decay (120). Here we skip further discussion and investigation on these topics but restrict ourselves to algorithm analysis based on the strong Lyapunov condition (129).

To utilize the separable structure of $f = h + g$, given the previous iterate $(x_k, v_k)$, we first find $x_{k+1}$ such that

$$\frac{x_{k+1} - x_k}{\alpha_k} \in v_k - x_{k+1} - \beta_k \nabla h(x_k) - \beta_k \partial g(x_{k+1}), \quad (130)$$

where the operator splitting, also known as forward-backward method, is used. Let $y_k = x_k - \alpha_k \beta_k \nabla h(x_k)$, then we can write $x_{k+1} = \prox_{s_k g}(w_k)$ where

$$w_k = \frac{y_k + \alpha_k v_k}{1 + \alpha_k}, \quad s_k = \frac{\alpha_k \beta_k}{1 + \alpha_k}.$$

Note that we also have

$$q_{k+1} := \frac{w_k - x_{k+1}}{s_k} \in \partial g(x_{k+1}), \quad (131)$$

which can be reused to discretize the second equation of (128).

In summary, we obtain a semi-implicit scheme for (128):

$$\begin{cases}
\frac{x_{k+1} - x_k}{\alpha_k} = v_k - x_{k+1} - \beta_k (\nabla h(x_k) + q_{k+1}), \\
\frac{v_{k+1} - v_k}{\alpha_k} = \mu \frac{1}{\gamma_k} (x_{k+1} - v_{k+1}) - \frac{1}{\gamma_k} (\nabla h(x_{k+1}) + q_{k+1}), \\
\frac{\gamma_k + \gamma_k}{\alpha_k} = \mu - \gamma_{k+1},
\end{cases} \quad (132)$$

where $q_{k+1}$ is defined by (131). We chose the parameters $\alpha_k$ and $\beta_k$ by the rule

$$\alpha_k = \sqrt{\frac{\gamma_k}{L}}, \quad \alpha_k \beta_k = \frac{1}{L}, \quad (133)$$

and simplify (132) to obtain the following algorithm which is named by accelerated proximal gradient (APG) method.

---

**Algorithm 4** APG method for minimizing $f = h + g$, $h \in S_{\mu,L}^1$ with $0 \leq \mu \leq L < \infty$

**Input:** $\gamma_0 > 0$, $x_0$, $v_0 \in V$.
1. Initialization $y_0 = x_0 - \frac{1}{L} \nabla h(x_0)$.
2. for $k = 0, 1, \ldots$ do
   3. Compute $w_k = \frac{y_k + \alpha_k v_k}{1 + \alpha_k}$ and $s_k = \frac{1}{L(1 + \alpha_k)}$.
   4. Update $x_{k+1} = \prox_{s_k g}(w_k)$.
   5. Compute $y_{k+1} = x_{k+1} - \frac{1}{L} \nabla h(x_{k+1})$.
   6. Update $v_{k+1} = x_{k+1} + (y_{k+1} - y_k)/(\alpha_k + \mu/L)$.
   7. Update $\alpha_{k+1} = \sqrt{(\alpha_k^2 + \alpha_k \mu/L)/(1 + \alpha_k)}$.
3. end for

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Theorem 8.2 For Algorithm 4, we have the contraction property

\[ \mathcal{L}_{k+1} \leq \frac{\mathcal{L}_k}{1 + \alpha_k} - \frac{\| \nabla h(x_k) + q_{k+1} \|^2}{1 + \alpha_k} \quad \forall k \geq 0, \]  

(134)

where \( \mathcal{L}_k = \mathcal{L}(x_k) = f(x_k) - f(x^*) + \frac{\gamma_0}{2} \| v_k - x^* \|^2 \). When \( \gamma_0 = rL \geq \mu \), it holds that

\[ \mathcal{L}_k \leq \mathcal{L}_0 \times \min \left\{ \left( \frac{\sqrt{r} + 1 + \sqrt{r}k}{\sqrt{r} + 1 + \sqrt{r}k} \right)^2, \left( 1 + \sqrt{\frac{\mu}{L}} \right)^{-k} \right\}. \]  

(135)

Proof Following the proof of Lemma 8.1, we have the difference \( \mathcal{L}_{k+1} - \mathcal{L}_k = I_1 + I_2 + I_3 \), where \( I_1, I_2 \) and \( I_3 \) are defined in (106a).

Clearly, the estimates (126) and (125) for \( I_2 \) and \( I_3 \) keep unchanged here:

\[ I_3 = \alpha_k \langle \nabla \gamma \mathcal{L}(x_{k+1}), \mathcal{G}^\gamma(x_{k+1}) \rangle, \]

\[ I_2 \leq \alpha_k \langle \nabla v \mathcal{L}(x_{k+1}), \mathcal{G}^v(x_{k+1}, d_{k+1}) \rangle - \frac{\gamma_k}{2} \| v_{k+1} - v_k \|^2, \]

where \( d_{k+1} := \nabla h(x_{k+1}) + q_{k+1} \in \partial f(x_{k+1}) \) and \( q_{k+1} \) is defined by (131). Observing that

\[ I_1 = \mathcal{L}(x_{k+1}, v_k, \gamma_k) - \mathcal{L}(x_k, v_k, \gamma_k) = g(x_{k+1}) - g(x_k) + h(x_{k+1}) - h(x_k), \]

we use the fact \( q_{k+1} \in \partial g(x_{k+1}) \) and (15) to estimate \( I_1 \) as follows

\[ I_1 \leq \langle d_{k+1}, x_{k+1} - x_k \rangle - \frac{1}{2L} \| \nabla h(x_{k+1}) - \nabla h(x_k) \|^2. \]  

(136)

We use the discretization (132) to replace \( x_{k+1} - x_k \) and compare with the flow evaluated at \( x_{k+1} = (x_{k+1}, v_{k+1}, \gamma_{k+1}) \):

\[ \langle d_{k+1}, x_{k+1} - x_k \rangle = \alpha_k \langle d_{k+1}, \mathcal{G}^\gamma(x_{k+1}, d_{k+1}) \rangle \]

\[ + \alpha_k \beta_k \langle d_{k+1}, \nabla h(x_{k+1}) - \nabla h(x_k) \rangle \]

\[ + \alpha_k \langle d_{k+1}, v_k - v_{k+1} \rangle. \]

Thanks to the negative term in the bound of \( I_2 \), the last term is bounded by

\[ \alpha_k \| d_{k+1} \|^2 \| v_k - v_{k+1} \| \leq \frac{\alpha_k^2}{2\gamma_k} \| d_{k+1} \|^2 + \frac{\gamma_k}{2} \| v_k - v_{k+1} \|^2. \]

The cross term is expanded by combination of squares (cf. (82)),

\[ \frac{1}{L} \langle d_{k+1}, \nabla h(x_{k+1}) - \nabla h(x_k) \rangle = -\frac{1}{2L} \| \nabla h(x_k) + q_{k+1} \|^2 + \frac{1}{2L} \| d_{k+1} \|^2 \]

\[ + \frac{1}{2L} \| \nabla h(x_{k+1}) - \nabla h(x_k) \|^2. \]

We now get the estimate for \( I_1 \) as follows

\[ I_1 \leq \alpha_k \langle d_{k+1}, \mathcal{G}^\gamma(x_{k+1}, d_{k+1}) \rangle + \frac{\gamma_k}{2} \| v_k - v_{k+1} \|^2 + \frac{1}{2L} \| d_{k+1} \|^2 \]

\[ - \frac{1}{2L} \| \nabla h(x_k) + q_{k+1} \|^2. \]  

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Putting all together and using strong Lyapunov property (129) imply that
\[
L_{k+1} - L_k \leq \alpha_k (\nabla L(x_{k+1}, d_{k+1}), G(x_{k+1}))
\]
\[
+ \left( \frac{1}{2L} + \frac{\alpha_k^2}{2\gamma_k} \right) \|d_{k+1}\|_s^2 - \frac{1}{2L} \|\nabla h(x_k) + q_{k+1}\|_s^2
\]
\[
\leq - \alpha_k L_{k+1} + \left( \frac{1}{2L} + \frac{\alpha_k^2}{2\gamma_k} - \frac{1}{L} \right) \|d_{k+1}\|_s^2 - \frac{1}{2L} \|\nabla h(x_k) + q_{k+1}\|_s^2
\]
\[
= - \alpha_k L_{k+1} - \frac{1}{2L} \|\nabla h(x_k) + q_{k+1}\|_s^2.
\]
This proves (134). The final decay rate comes from (45) and (133).

If we choose
\[
\alpha_k = \sqrt{\frac{\gamma_k}{4L}}, \quad \beta_k = \frac{1}{2L\alpha_k},
\]
then we have the identity
\[
\frac{\alpha_k^2 \beta_k^2 L}{2} + \frac{\alpha_k^2}{2\gamma_k} - \alpha_k \beta_k = - \frac{\alpha_k \beta_k}{2} = - \frac{1}{4L}.
\]
This will keep the negative term \(- \|d_{k+1}\|_s^2\) and implies the faster convergence of gradient; see the theorem below. As the proof is a simple modification of previous one, we only present the result below.

**Theorem 8.3** If \(f = h + g\) where \(h \in S_{\mu,L}^1\) with \(0 \leq \mu \leq L < \infty\), then for the splitting scheme (132) with parameter setting (137), we have the contraction property
\[
L_{k+1} - L_k \leq -\alpha_k L_{k+1} - \frac{1}{4L} \|d_{k+1}\|_s^2,
\]
and it follows that
\[
L_k + \frac{1}{4L} \sum_{i=0}^{k-1} \rho_k \|d_{i+1}\|_s^2 \leq \rho_k L_0.
\]
Above \(\rho_k\) is defined by (39) and satisfies the estimate, when \(\gamma_0 = rL \geq \mu\),
\[
\rho_k \leq \min \left\{ \left( \frac{2 + \sqrt{r + 4} + \sqrt{r^2 + 4}}{2 + \sqrt{r + 4} + \sqrt{r^2 + 4} + \sqrt{r^2 + 4}} \right)^2, \left( 1 + \sqrt{\frac{\mu}{4L}} \right)^{-k} \right\}.
\]
The bound of the sub-gradient yields that
\[
\min_{0 \leq i \leq k} \|d_{i+1}\|_s \leq \frac{4L\rho_k}{\sum_{i=0}^{k} \rho_i},
\]
and asymptotically, we have a faster decay rate of gradient \(\|d_{i+1}\|_s^2 = o(2L\rho_k\rho_0)\).
8.4 Another accelerated proximal gradient method

We now apply the operator splitting to the first equation of (128) by considering \( x' = v - x \) first and \( x' \in -\beta(\nabla h(x) + \partial g(x)) \) next. The later can be further split via the proximal-gradient method. That is

\[
\begin{aligned}
\frac{y_k - x_k}{\alpha_k} &= v_k - y_k, \\
x_{k+1} &= \text{prox}_{\alpha_k \beta_k g}(y_k - \alpha_k \beta_k \nabla h(y_k)).
\end{aligned}
\]

Letting

\[
d_f(y_k) := \frac{y_k - x_{k+1}}{\alpha_k \beta_k} \in \partial g(x_{k+1}) + \nabla h(y_k),
\]

we have the following ‘middle’ point discretization of HNAG flow (128):

\[
\begin{aligned}
\frac{x_{k+1} - x_k}{\alpha_k} &= v_k - y_k - \beta_k d_f(y_k), \\
v_{k+1} - v_k &= \frac{\mu}{\gamma_k} (y_k - v_{k+1}) - \frac{1}{\gamma_k} d_f(y_k), \\
\gamma_{k+1} - \gamma_k &= \mu - \gamma_{k+1}.
\end{aligned}
\]

The point \( y_k \) is an intermediate point of \( x_k \) and \( x_{k+1} \), and in the vector field \( G(x, v, \gamma) \) the first variable will be evaluated at \( y_k \). We note that \( d_f(y_k) \) is nothing but the gradient mapping used in Luo and Chen (2019); see also \( d_{k+1/2} \) defined in Lemma 4.2.

We use the step size

\[
L \alpha_k^2 = \gamma_k (1 + \alpha_k), \quad \alpha_k \beta_k = 1/L,
\]

and summarize (143) in Algorithm 5. Note that \( \alpha_k \) is slightly larger than that in (133).

**Algorithm 5** New APG method for minimizing \( f = h + g, \ h \in \mathcal{S}_{\mu,L}^1 \) with \( 0 \leq \mu \leq L < \infty \)

**Input:** \( \gamma_0 > 0, \ x_0, \ v_0 \in V \) and \( s = 1/L \).

1: for \( k = 0, 1, \ldots \) do

2: Compute \( \alpha_k = \left( \gamma_k + \sqrt{\gamma_k^2 + 4L \gamma_k} \right) / (2L) \).

3: Compute \( y_k = \frac{x_k + \alpha_k v_k}{1 + \alpha_k} \).

4: Update \( x_{k+1} = \text{prox}_{sg}(y_k - s \nabla h(y_k)) \).

5: Update \( v_{k+1} = \frac{\gamma_k v_k + \mu \alpha_k y_k}{\gamma_k + \mu \alpha_k} + \frac{\gamma_k (1 + \alpha_k) x_{k+1} - y_k}{\alpha_k} \).

6: Update \( \gamma_{k+1} = \frac{\gamma_k + \mu \alpha_k}{1 + \alpha_k} \).

7: end for

We will establish the convergence analysis via the strong Lyapunov condition. A key tool is the following estimate at \( y \), which allows us to modify (129) for later use.
Lemma 8.2 Assume $f = h + g$ and $h \in S_{\mu,L}$ with $0 \leq \mu \leq L < \infty$. Let $x = \text{prox}_{sg}(y - s\nabla h(y))$ with $s = 1/L$ and $d_f(y) = (y - x)/s$. We have the following inequality

$$
\langle d_f(y), y - x^* \rangle \geq f(x) - f(x^*) + \frac{\mu}{2}\|y - x^*\|^2 + \frac{1}{2L}\|d_f(y)\|_*^2.
$$

(145)

Proof Let $q(x) = d_f(y) - \nabla h(y)$. Then by definition $q(x) \in \partial g(x)$ and thus

$$
g(x) - g(x^*) \leq \langle q(x), x - x^* \rangle.
$$

For $h \in S_{\mu,L}$, we use Lemma 3.2 to conclude

$$
h(x) - h(x^*) \leq \langle \nabla h(y), x - x^* \rangle - \frac{\mu}{2}\|y - x^*\|^2 + \frac{L}{2}\|y - x\|^2.
$$

Adding the above two estimates together yields that

$$
f(x) - f(x^*) \leq \langle d_f(y), x - x^* \rangle - \frac{\mu}{2}\|y - x^*\|^2 + \frac{L}{2}\|y - x\|^2.
$$

Now split $\langle d_f(y), x - x^* \rangle = \langle d_f(y), y - x^* \rangle + \langle d_f(y), x - y \rangle$ and use the fact $x - y = -d_f(y)/L$ to get the desired estimate (145).

Theorem 8.4 For Algorithm 5, we have

$$
L_{k+1} \leq \frac{L_k}{1 + \alpha_k} \quad \forall k \in \mathbb{N},
$$

(146)

and this implies, when $\gamma_0 = r L \geq \mu$,

$$
L_k \leq L_0 \times \min\left\{ \left( \frac{2}{2 + \sqrt{r} k} \right)^2, \left( 1 + \sqrt{\frac{\mu}{L}} \right)^{-k} \right\}.
$$

(147)

Proof Using the refined convexity lower bound (145), the strong Lyapunov property at $y_k$ reads as

$$
- \partial \mathcal{L} (d_f(y_k), v_{k+1}, \gamma_{k+1}) \cdot \mathcal{G} (d_f(y_k), v_{k+1}, \gamma_{k+1})
\geq \mathcal{L} (x_{k+1}, v_{k+1}, \gamma_{k+1}) + \left( \frac{1}{2L} + \beta_k \right) \|d_f(y_k)\|_*^2,
$$

(148)

which can proved analogously to (129).

The estimate of $L_{k+1} - L_k = I_1 + I_2 + I_3$ is almost in line with that of Theorem 8.2, where $I_1$, $I_2$ and $I_3$ are defined in (106a). The difference comes from the first item $I_1$. Recall (142) and let $q(x_{k+1}) := d_f(y_k) - \nabla h(y_k) \in \partial g(x_{k+1})$. By convexity of $g$, it follows that

$$
g(x_{k+1}) - g(x_k) \leq \langle q(x_{k+1}), x_{k+1} - x_k \rangle,
$$

and we use Lemma 3.2 to conclude

$$
h(x_{k+1}) - h(x_k) \leq \langle \nabla h(y_k), x_{k+1} - x_k \rangle + \frac{L}{2}\|x_{k+1} - y_k\|^2.
$$
By \((142)\), we see
\[ x_{k+1} - y_k = -\frac{1}{L} d_f(y_k), \]
and a routine calculation yields the bound
\[ \mathcal{L}_{k+1} - \mathcal{L}_k \leq \alpha_k \partial \mathcal{L}(d_f(y_k), v_{k+1}, \gamma_{k+1}) \cdot \mathcal{G}(d_f(y_k), v_{k+1}, \gamma_{k+1}) + \left( \frac{1}{2L} + \frac{\alpha_k^2}{2\gamma_k} \right) \|d_f(y_k)\|_2^2. \]
Applying the strong Lyapunov property \((148)\) and the relation \((144)\), we get
\[ \mathcal{L}_{k+1} - \mathcal{L}_k \leq -\alpha_k \mathcal{L}_{k+1} + \left( \frac{\alpha_k^2}{2\gamma_k} - \frac{\alpha_k}{2L} - \frac{1}{2L} \right) \|d_f(y_k)\|_2^2 = -\alpha_k \mathcal{L}_{k+1}. \]
This proves \((146)\). To the end, recalling \((46)\) and \((144)\) proves \((147)\).

9. Concluding Remarks

By using the tool of Lyapunov function and introducing the concept of strong Lyapunov condition, we present a unified self-contained framework for first-order optimization methods including gradient descent method, proximal point algorithm, proximal gradient method, heavy ball (momentum) method, and Nesterov accelerated gradient method.

However, we notice that a systematical way to find an appropriate Lyapunov function satisfying the strong Lyapunov condition is not presented in this work. When \(\nabla f\) is linear, it is possible to use control theory to design one; see Lessard et al. (2016) for more details. On the other hand, we have not considered non-Euclidean setting that involves the Bregman divergence (or preconditioning effect), which is related to the mirror descent models (Wibisono et al., 2016; Krichene et al., 2015).

Different from existing works using Lyapunov analysis and involving complicated algebraic calculations, the strong Lyapunov condition can be verified much more systematically by inequalities of convex functions. Besides, by suitable time scaling factor, we can handle the convex case and strong convex case in a unified way. Furthermore, the strong Lyapunov condition can be easily used in the discretization to establish the convergence of the algorithms. This together with continuous dynamical system renders effective tools for designing and analysis of convex optimization algorithms.

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