1. Introduction

The large-N limit of Gauge Theory was proposed in 1974 by 't Hooft [1] for quantum chromodynamics (QCD). The dimensionality of the gauge group SU($N_c$) was used as a parameter, considering $N_c$ as a large number and performing an expansion in $1/N_c$. The motivation was an expansion in the inverse number of field-components $N$ in statistical mechanics where it is known as the $1/N$-expansion, and is a standard method for non-perturbative investigations.

The expansion of QCD in $1/N_c$ rearranges diagrams of perturbation theory in a way which is consistent with a string picture of strong interaction, whose phenomenological consequences agree with experiment. The accuracy of the leading-order term, which is often called large-N QCD or multicolor QCD, is expected to be of the order of the ratios of meson widths to their masses, i.e. about 10–15%.
While QCD is simplified in the large-$N_c$ limit, it is still not yet solved. Generically, it is a problem of infinite matrices, rather than of infinite vectors as in the theory of second-order phase transitions in statistical mechanics.

We start these Lectures by showing how the $1/N$-expansion works for the $O(N)$-vector models, and describing some applications to the four-Fermi interaction and the nonlinear sigma model. Then we concentrate on the Yang–Mills theory at large $N_c$.

The methods described in these Lectures are developed mostly in the seventies and the beginning of the eighties. They are used over and over again when discussing a relation between Gauge Theory and String Theory. The conformal invariance, described in Section 2 for the four-Fermi interaction, is used in the modern approaches based on the AdS/CFT correspondence [2] (discussed in the lectures by L. Thorlacius at this School). The notion of the large-$N$ limit, described in Section 3, is widely used, in particular, in Matrix Theory [3] (discussed in the lectures by W. Taylor at this School). The loop equation, described in Section 4, has been recently applied [4] for studying the AdS/CFT correspondence. The large-$N$ reduced models, described in Section 5, are applied to a nonperturbative formulation of superstring [5] and very recently to noncommutative gauge theory [6].

The content of my fifth lecture at this School, which was devoted to an application of the large-$N$ methods to Matrix Theory at finite temperature, is not included in the text. It is mostly contained in Ref. [7].

2. $O(N)$ Vector Models

The simplest models, which become solvable in the limit of a large number of field components, deal with a field which has $N$ components forming an $O(N)$ vector in an internal symmetry space. A model of this kind was first considered by Stanley [8] in statistical mechanics and is known as the spherical model. The extension to quantum field theory was done by Wilson [9] both for the four-Fermi and $\varphi^4$ theories.

In the framework of perturbation theory, the four-Fermi interaction is renormalizable only in $d = 2$ dimensions and is non-renormalizable for $d > 2$. The $1/N$-expansion resums perturbation-theory diagrams after which the four-Fermi interaction becomes renormalizable to each order in $1/N$ for $2 \leq d < 4$. An analogous expansion exists for the nonlinear $O(N)$ sigma model.

The $1/N$ expansion of the vector models is associated with a resummation of Feynman diagrams. A very simple class of diagrams — the bubble graphs — survives to the leading order in $1/N$. This is why the large-$N$
limit of the vector models is solvable. Alternatively, the large-$N$ solution is nothing but a saddle-point solution in the path-integral approach. The existence of the saddle point is due to the fact that $N$ is large. This is to be distinguished from a perturbation-theory saddle point which is due to the fact that the coupling constant is small. Taking into account fluctuations around the saddle-point results in the $1/N$-expansion of the vector models.

We begin this Section with a description of the $1/N$-expansion of the $N$-component four-Fermi theory analyzing the bubble graphs. Then we introduce functional methods and construct the $1/N$-expansion of the $O(N)$-symmetric nonlinear sigma model. At the end we discuss the factorization in the $O(N)$ vector models at large $N$.

2.1. FOUR-FERMI INTERACTION

The action of the $O(N)$-symmetric four-Fermi interaction in a $d$-dimensional Euclidean space is defined by

$$S[\bar{\psi},\psi] = \int d^d x \left( \bar{\psi} \hat{\partial} \psi + m \bar{\psi} \psi - \frac{G}{2} (\bar{\psi} \psi)^2 \right).$$

(1)

Here $\hat{\partial} = \gamma_\mu \partial_\mu$ and $\psi = (\psi_1, \ldots, \psi_N)$ is a spinor field which forms an $N$-component vector in an internal symmetry space so that

$$\bar{\psi} \psi = \sum_{i=1}^N \bar{\psi}_i \psi_i.$$  

(2)

In $d = 2$ this model was studied in the large-$N$ limit in Ref. [10] and is often called the Gross–Neveu model.

The dimension of the four-Fermi coupling constant $G$ is $\text{dim}[G] = m^{2-d}$. For this reason, the perturbation theory for the four-Fermi interaction is renormalizable in $d = 2$ but is non-renormalizable for $d > 2$ (and, in particular, in $d = 4$). This is why the old Fermi theory of weak interactions was replaced by the modern electroweak theory, where the interaction is mediated by the $W^\pm$ and $Z$ bosons.

The action (1) can be equivalently rewritten as

$$S[\bar{\psi},\psi,\chi] = \int d^d x \left( \bar{\psi} \hat{\partial} \psi + m \bar{\psi} \psi - \chi \bar{\psi} \psi + \frac{\chi^2}{2G} \right),$$

(3)

where $\chi$ is an auxiliary field. The two forms of the action, (1) and (3), are equivalent due to the equation of motion

$$\chi = G \bar{\psi} \psi.$$  

(4)
which can be derived by varying the action (3) with respect to $\chi$.

In the path-integral quantization, where the partition function is defined by

$$Z = \int D\chi D\bar{\psi} D\psi e^{-S[\bar{\psi}, \psi, \chi]}$$

(5)

with $S[\bar{\psi}, \psi, \chi]$ given by Eq. (3), the action (1) appears after performing the Gaussian integral over $\chi$. Therefore, one alternatively gets

$$Z = \int D\bar{\psi} D\psi e^{-S[\bar{\psi}, \psi]}$$

(6)

with $S[\bar{\psi}, \psi]$ given by Eq. (1).

The perturbative expansion of the $O(N)$-symmetric four-Fermi theory can be conveniently represented using the formulation (3) via the auxiliary field $\chi$. Then the diagrams are of the type of those in Yukawa theory, and resemble the ones for QED with $\psi$ and $\bar{\psi}$ being an analog of the electron-positron field and $\chi$ being an analog of the photon field. However, the auxiliary field $\chi(x)$ does not propagate, since it follows from the action (3) that

$$D_0 (x - y) \equiv \langle \chi(x) \chi(y) \rangle_{\text{Gauss}} = G \delta^{(d)}(x - y)$$

(7)

or

$$D_0 (p) \equiv \langle \chi(-p) \chi(p) \rangle_{\text{Gauss}} = G$$

(8)

in momentum space.

It is convenient to represent the four-Fermi vertex as the sum of two terms

$$\begin{array}{c}
\begin{array}{c}
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\end{array}
\end{array}
\quad = \quad
\begin{array}{c}
\begin{array}{c}
\uparrow \\
\uparrow \\
\uparrow \\
\uparrow \\
\end{array}
\end{array}
\quad - \quad
\begin{array}{c}
\begin{array}{c}
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\end{array}
\end{array}$$

(9)

where the empty space inside the vertex is associated with the propagator (7) (or (8) in momentum space). The relative minus sign makes the vertex antisymmetric in both incoming and outgoing fermions as is prescribed by the Fermi statistics.

The diagrams that contribute to second order in $G$ for the four-Fermi vertex are depicted, in these notations, in Figure 1. The O($N$) indices propagate through the solid lines so that the closed line in the diagram in Figure 1b corresponds to the sum over the O($N$) indices which results in a factor of $N$. Analogous one-loop diagrams for the propagator of the $\bar{\psi}$-field are depicted in Figure 2.

**Remark on the one-loop Gell-Mann–Low function of four-Fermi theory**

Evaluating the diagrams in Figure 1 which are logarithmically divergent in $d = 2$, and noting that the diagrams in Figure 2 do not contribute to
Figure 1. Diagrams of the second order of perturbation theory for the four-Fermi vertex. The diagram b) involves the sum over the O(N) indices.

Figure 2. One-loop diagrams for the propagator of the ψ-field. The diagram b) involves the sum over the O(N) indices.

the wave-function renormalization of the ψ-field, which emerges to the next order in G, one gets for the one-loop Gell-Mann–Low function

\[ B(G) = -\frac{(N - 1) G^2}{2\pi}. \]  

(10)

The four-Fermi theory in 2 dimensions is asymptotically free as was first noted by Anselm [11] and rediscovered in Ref. [10].

The vanishing of the one-loop Gell-Mann–Low function in the Gross–Neveu model for \( N = 1 \) is related to the same phenomenon in the Thirring model. The latter model is associated with the vector-like interaction \((\bar{\psi} \gamma_\mu \psi)^2\) of one species of fermions with \(\gamma_\mu\) being the \(\gamma\)-matrices in 2 dimensions. Since a bispinor has in \(d = 2\) only two components \(\psi_1\) and \(\psi_2\),
both the vector-like and the scalar-like interaction (1) for $N = 1$ reduce to $\bar{\psi}_1 \psi_1 \bar{\psi}_2 \psi_2$ since the square of a Grassmann variable vanishes. Therefore, these two models coincide. For the Thirring model, the vanishing of the Gell-Mann–Low function for any $G$ was shown by Johnson [12] to all loops.

2.2. BUBBLE GRAPHS AS ZEROTH ORDER IN $1/N$

The perturbation-theory expansion of the O($N$)-symmetric four-Fermi theory contains, in particular, the diagrams of the type depicted in Figure 3 which are called bubble graphs. Since each bubble has a factor of $N$, the contribution of the $n$-bubble graph is $\propto G^{n+1} N^n$ which is of the order of

$$G^{n+1} N^n \sim G$$

as $N \to \infty$ since

$$G \sim \frac{1}{N}.$$  

Therefore, all the bubble graphs are essential to the leading order in $1/N$.

Let us denote

$$\cdots = G + \ldots + G^2 \bigcirc + G^{n+1} \bigcirc \cdots \bigcirc + \ldots$$

(13)

In fact the wavy line is nothing but the propagator $D$ of the $\chi$ field with the bubble corrections included. The first term $G$ on the RHS of Eq. (13) is nothing but the free propagator (8).

Summing the geometric series of the fermion-loop chains on the RHS of Eq. (13), one gets analytically\(^1\)

$$D^{-1}(p) = \frac{1}{G} - N \int \frac{d^d k}{(2\pi)^d} \frac{sp(\hat{k} + i m)(\hat{k} + \hat{p} + i m)}{(k^2 + m^2)((k + p)^2 + m^2)}.$$  

(14)

\(^1\)Recall that the free Euclidean fermionic propagator is given by $S_0(p) = (i\hat{p} + m)^{-1}$ due to Eqs. (3), (5) and the additional minus sign is associated with the fermion loop.
This determines the exact propagator of the $\chi$ field at large $N$. It is $O(N^{-1})$ since the coupling $G$ is included in the definition of the propagator.

The idea is now to change the order of summation of diagrams of perturbation theory using $1/N$ rather than $G$ as the expansion parameter. Therefore, the zeroth-order propagator of the expansion in $1/N$ is defined as the sum over the bubble graphs (13) which is given by Eq. (14). Some of the diagrams of the new expansion for the four-Fermi vertex are depicted in Figure 4. The first diagram is proportional to $G$ while the second and third ones are proportional to $G^2$ or $G^3$, respectively, and therefore are of order $O(N^{-1})$ or $O(N^{-2})$ with respect to the first diagram. The perturbation theory is thus rearranged as the $1/N$-expansion.

The general structure of the $1/N$-expansion is the same for all vector models, say, for the $N$-component nonlinear sigma model which is considered in Subsection 2.4.

The main advantage of the expansion in $1/N$ for the four-Fermi interaction, over the perturbation theory, is that it is renormalizable in $d < 4$ while the perturbation-theory expansion in $G$ is renormalizable only in $d = 2$. Moreover, the $1/N$-expansion of the four-Fermi theory in $2 < d < 4$ demonstrates [9] an existence of an ultraviolet-stable fixed point, i.e. a nontrivial zero of the Gell-Mann–Low function.

In order to show that the $1/N$-expansion of the four-Fermi theory is renormalizable in $2 \leq d < 4$, let us analyze indices of the diagrams of the $1/N$-expansion. First of all, we shall get rid of an ultraviolet divergence of the integral over the $d$-momentum $k$ in Eq. (14). The divergent part of the integral is proportional to $\Lambda^{d-2}$ (logarithmically divergent in $d = 2$) with $\Lambda$ being an ultraviolet cutoff. It can be canceled by choosing

$$G = \frac{g^2}{N} \Lambda^{2-d}, \quad (15)$$

Figure 4. Some diagrams of the $1/N$-expansion for the $O(N)$ four-Fermi theory. The wavy line represents the (infinite) sum of the bubble graphs (13).
where $g^2$ is a proper dimensionless constant which is not necessarily positive since the four-Fermi theory is stable with either sign of $G$. The power of $\Lambda$ in Eq. (15) is consistent with the dimension of $G$. This prescription works for $2 < d < 4$ where there is only one divergent term while another divergence $\propto p^2 \ln \Lambda$ emerges additionally in $d = 4$. This is why the consideration is not applicable in $d = 4$.

The propagator $D(p)$ is therefore finite, and behaves at large momenta $|p| \gg m$ as

$$D(p) \propto \frac{1}{|p|^{d-2}}. \quad (16)$$

The standard power-counting arguments then show that the only divergent diagrams appear in the propagators of the $\psi$ and $\chi$ fields, and in the $\psi$-$\chi$-$\psi$ three-vertex. These divergencies can be removed by a renormalization of the coupling $g$, mass, and wave functions of $\psi$ and $\chi$.

This completes a demonstration of renormalizability of the $1/N$-expansion for the four-Fermi interaction in $2 \leq d < 4$. For more detail, see Ref. [13].

2.3. SCALE AND CONFORMAL INVARIANCE OF FOUR-FERMI THEORY

The coefficient in Eq. (15) can easily be calculated in $d = 3$. To evaluate the divergent part of the integral in Eq. (14), we put $p = 0$ and $m = 0$. Remembering that the $\gamma$-matrices are $2 \times 2$ matrices in $d = 3$, we get

$$\int_{\Lambda} d^3 k \ sp \hat{k} \hat{k} = 2 \int_{\Lambda} d^3 k \ \frac{1}{k^2} = \frac{1}{\pi^2} \int_{\Lambda} d|k| = \frac{\Lambda}{\pi^2}. \quad (17)$$

Note that the integral is linearly divergent in $d = 3$ and $\Lambda$ is the cutoff for the integration over $|k|$. This divergence can be canceled by choosing $G$ according to Eq. (15) with $g$ equal to

$$g_\ast = \pi. \quad (18)$$

To calculate in $d = 3$ the coefficient of proportionality in Eq. (16), let us choose $G = \pi^2/\Lambda$ as is prescribed by Eqs. (15), (18) and put in Eq. (14) $m = 0$ since we are interested in the asymptotics $|p| \gg m$. Then the RHS of Eq. (14) can be rearranged as

$$D^{-1}(p) = -2N \int \frac{d^3 k}{(2\pi)^3} \left[ \frac{k^2 + kp}{k^2(k + p)^2} - \frac{1}{k^2} \right] = \frac{N|p|}{8}. \quad (19)$$

The integral is obviously convergent.

Equation (19) (or (16) in $d$ dimensions) is remarkable since it shows that the scale dimension of the field $\chi$ changes its value from $l_\chi = d/2$ in
 Perturbation theory to $l_\psi = 1$ in the zeroth order of the $1/N$ expansion (remember that the momentum-space propagator of a field with the scale dimension $l$ is proportional to $|p|^{2l-d}$). This appearance of scale invariance in the $1/N$-expansion of the four-Fermi theory at $2 < d < 4$ was first pointed out by Wilson [9] and implies that the Gell-Mann–Low function $B(g)$ has a zero at $g = g^*$ which is given in $d = 3$ by Eq. (18).

The (logarithmic) anomalous dimensions of the fields $\psi$, $\chi$, and of the $\bar{\psi}\chi\psi$ three-vertex in $d = 3$ to order $1/N$ can be found as follows. The $1/N$-correction to the propagator of the $\psi$-field is given by the diagram depicted in Figure 5a). Since we are interested in an ultraviolet behavior, we can put again $m = 0$. Analytically, we have

$$S^{-1}(p) = i\hat{p} + \frac{8i}{N} \int_{\Lambda} \frac{d^3k}{(2\pi)^3} \frac{\hat{k} + \hat{p}}{|k|(k + p)^2}. \quad (20)$$

The (logarithmically) divergent contribution emerges from the domain of integration $|k| \gg |p|$ so we can expand the integrand in $p$. The $p$-independent term vanishes after integration over the directions of $k$ so that we get

$$S^{-1}(p) = i\hat{p} \left[ 1 + \frac{8}{N} \int_{\Lambda} \frac{d^3k}{(2\pi)^3} \frac{1}{|k|^3} \right] = i\hat{p} \left[ 1 + \frac{2}{3\pi^2N} \ln \frac{\Lambda^2}{p^2} + \text{finite} \right]. \quad (21)$$

The diagram, which gives a non-vanishing contribution to the three-vertex in order $1/N$, is depicted in Figure 5b. It reads analytically

$$\Gamma(p_1, p_2) = 1 + \frac{8}{N} \int_{\Lambda} \frac{d^3k}{(2\pi)^3} \frac{(\hat{k} + \hat{p}_1)(\hat{k} + \hat{p}_2)}{|k|(k + p_1)^2(k + p_2)^2}, \quad (22)$$

where $p_1$ and $p_2$ the incoming and outgoing fermion momenta, respectively. The logarithmic domain is $|k| \gg |p|$ with $|p|_{\text{max}}$ being the largest of $|p_1|$ \[ \text{a)} \quad \text{b)} \]

Figure 5. Diagrams for the $1/N$-correction to the $\psi$-field propagator (a) and the three-vertex (b).
\[ \Gamma(p_1, p_2) = 1 - \frac{2}{\pi^2 N} \ln \frac{\Lambda^2}{p_{\text{max}}^2} + \text{finite}. \] (23)

An analogous calculation of the $1/N$ correction for the field $\chi$ is a bit more complicated since involves three two-loop diagrams (see Ref. [14]). The resulting expression for $D^{-1}(p)$ reads

\[ (N D(p))^{-1} = \frac{\Lambda}{g^2} + \left[ \frac{-\Lambda}{\pi^2} + \frac{|p|}{8} \right] + \frac{1}{\pi^2 N} \left[ 2\Lambda - |p| \left( \frac{2}{3} \ln \frac{\Lambda^2}{p^2} + \text{finite} \right) \right]. \] (24)

The linear divergence is canceled to order $1/N$ providing $g$ is equal to

\[ g_* = \pi \left( 1 + \frac{1}{N} \right), \] (25)

which determines $g_*$ to order $1/N$. After this $D^{-1}(p)$ takes the form

\[ D^{-1}(p) = \frac{N |p|}{8} \left( 1 - \frac{16}{3\pi^2 N} \ln \frac{\Lambda^2}{p^2} \right). \] (26)

To make all three expressions (21), (23), and (26) finite, we need logarithmic renormalizations of the wave functions of $\psi$- and $\chi$-fields and of the vertex $\Gamma$. This can be achieved by multiplying them by the renormalization constants

\[ Z_i(\Lambda) = 1 - \gamma_i \ln \frac{\Lambda^2}{\mu^2} \] (27)

where $\mu$ stands for a reference mass scale and $\gamma_i$ are anomalous dimensions. The index $i$ stands for $\psi$, $\chi$, or $v$ for the $\psi$- and $\chi$-propagators or the three-vertex $\Gamma$, respectively. We have, therefore, calculated

\[ \gamma_\psi = \frac{2}{3\pi^2 N}, \]
\[ \gamma_v = -\frac{2}{\pi^2 N}, \]
\[ \gamma_\chi = -\frac{16}{3\pi^2 N}. \] (28)

to order $1/N$. Due to Eq. (4) $\gamma_\chi$ coincides with the anomalous dimension of the composite fields $\tilde{\psi}\psi$

\[ \gamma_{\tilde{\psi}\psi} = \gamma_\chi. \] (29)

Note, that

\[ Z_\psi^2 Z_v^{-2} Z_\chi = 1. \] (30)
This implies that the effective charge is not renormalized and is given by Eq. (25). Thus, the nontrivial zero of the Gell-Mann–Low function persists to order $1/N$ (and, in fact, to all orders of the $1/N$-expansion).

If $g$ is chosen exactly at the critical point $g^*$, then the renormalization-group equations

$$\frac{\mu d\ln \Gamma_i}{d\mu} = \gamma_i \left(g^2\right),$$

where $\Gamma_i$ stands generically either for vertices or for inverse propagators, possess the scale invariant solutions

$$\Gamma_i \propto \mu^{\gamma_i(g^2)}.$$ (32)

For the four-Fermi theory in $d = 3$, Eq. (32) yields

$$S(p) = \frac{1}{ip} \left(\frac{p^2}{\mu^2}\right)^{\gamma_\phi},$$ (33)

$$D(p) = \frac{8}{N|p|} \left(\frac{p^2}{\mu^2}\right)^{\gamma_\chi},$$ (34)

$$\Gamma(p_1, p_2) = \left(\frac{\mu^2}{p_1^2}\right)^{\gamma_v} f \left(\frac{p_2^2}{p_1^2}, \frac{p_1 p_2}{p_1^2}\right),$$ (35)

where $f$ is an arbitrary function of the dimensionless ratios which is not determined by scale invariance. The indices here obey the relation

$$\gamma_v = \gamma_\phi + \frac{1}{2} \gamma_\chi$$ (36)

which guarantees that Eq. (30), implied by scale invariance, is satisfied.

The indices $\gamma_i$ are given to order $1/N$ by Eqs. (28). When expanded in $1/N$, Eqs. (33) and (34) obviously reproduces Eqs. (21) and (26). Therefore, one gets the exponentiation of the logarithms which emerge in the $1/N$-expansion. The calculation of the next terms of the $1/N$-expansion for the indices $\gamma_i$ is contained in Ref. [15].

Scale invariance implies, in a renormalizable quantum field theory, more general conformal invariance as is first pointed out in Refs. [16, 17]. The conformal group in a $d$-dimensional space-time has $(d+1)(d+2)/2$ parameters as is illustrated by Table 1. More about the conformal group can be found in the lecture by Jackiw [18].

A heuristic proof [16] of the fact that scale invariance implies conformal invariance is based on the explicit form of the conformal current $K^\alpha_\mu$, which is associated with the special conformal transformation, via the energy-momentum tensor:

$$K^\alpha_\mu = \left(2x_\nu x^\alpha - x^2 \delta^\alpha_\nu\right) \theta_{\mu\nu}.$$ (37)
Differentiating, we get
\[ \partial_\mu K_\mu^\alpha = 2x^\alpha \theta_{\mu \mu}, \]
which is proportional to divergence of the dilatation current. Therefore, both the dilatation and conformal currents vanish simultaneously when \( \theta_{\mu \nu} \) is traceless which is provided, in turn, by the vanishing of the Gell-Mann–Low function.

Conformal invariance completely fixes three-vertices as was first shown by Polyakov [19] for scalar theories. The proper formula for the four-Fermi theory reads [20]
\[ \Gamma(p_1, p_2) = \mu^{2\gamma_\psi} \frac{\Gamma(d/2)\Gamma(d/2 - \gamma_\psi)}{\Gamma(\gamma_\psi)} \times \int \frac{d^dk}{\pi^{d/2}} \frac{\hat{k} + \hat{p}_1}{(\hat{k} + p_1)^2} \frac{\hat{k} + \hat{p}_2}{(\hat{k} + p_2)^2} \frac{1}{|k|^{d-2+2\gamma_\psi-\gamma_\chi/2}}, \]
where the coefficient in the form of the ratio of the \( \Gamma \)-functions is prescribed by the normalization (33) and (34) and the indices are related by Eq. (36) but can be arbitrary otherwise.\(^2\)

\(^2\)The only restriction \( \gamma_\psi \geq 0 \) is imposed by the Källén–Lehmann representation of the propagator while there is no such restriction on \( \gamma_\chi \) since it is a composite field.
Equation (39), which results from conformal invariance, unambiguously fixes the function $f$ in Eq. (35). In contrast to infinite-dimensional conformal symmetry in $d = 2$, the conformal group in $d > 2$ is less restrictive. It fixes only the tree-point vertex while, say, the four-point vertex remains an unknown function of two variables.

The integral on the RHS of Eq. (39) looks in $d = 3$ very much like that in Eq. (22) and can easily be calculated to the leading order in $1/N$ when only the region of integration over large momenta with $|k| \gtrsim |p|_{\text{max}} \equiv \max\{|p_1|, |p_2|\}$ is essential to this accuracy.

Let us first note that the coefficient in front of the integral is $\propto \gamma_v \sim 1/N$, so that one has to peak up the term $\sim 1/\gamma_v$ in the integral for the vertex to be of order 1. This term comes from the region of integration with $|k| \gtrsim |p|_{\text{max}}$. Recalling that $|p_1 - p_2| \lesssim |p|_{\text{max}}$ in Euclidean space, one gets

$$
\int \frac{d^3 k}{2\pi} \frac{\hat{k} + \hat{p}_1}{[(k + p_1)^2]^{1+\gamma_x/2} \, [k + p_2)^2]^{1+\gamma_x/2} \, |k|^{1+2 \gamma_x/2} |k|^1} = \frac{1}{\gamma_v (p^2_{\text{max}})^{\gamma_v}},
$$

where Eq. (36) has been used and

$$
\Gamma(p_1, p_2) = \left( \frac{\mu^2}{p^2_{\text{max}}} \right)^{\gamma_v}.
$$

While the integral in Eq. (40) is divergent in the ultraviolet for $\gamma_v < 0$, this divergence disappears after the renormalization.

Equation (23) is reproduced by Eq. (40) when expanding in $1/N$. This dependence of the three-vertex solely on the largest momentum is typical for logarithmic theories in the ultraviolet region where one can put, say, $p_1 = 0$ without changing the integral with logarithmic accuracy. This is valid if the integral is fast convergent in infrared regions which is our case.

**Remark on broken scale invariance**

Scale (and conformal) invariance at a fixed point $g = g^*$ holds only for large momenta $|p| \gg m$. For smaller values of momenta, scale invariance is broken by masses. In fact, any dimensional parameter $\mu$ breaks scale invariance. If the bare coupling $g$ is chosen in the vicinity of $g^*$, then scale invariance holds even in the massless case only for $|p| \gg \mu$ where $g(p)$ approaches $g^*$ while it is broken if $|p| \lesssim \mu$. 
2.4. NONLINEAR SIGMA MODEL

The nonlinear $O(N)$ sigma model\(^3\) in 2 Euclidean dimensions is defined by the partition function

\[
Z = \int D\vec{n} \delta \left( \vec{n}^2 - \frac{1}{g^2} \right) e^{-\frac{1}{4} \int d^2x (\partial_\mu \vec{n})^2} \tag{42}
\]

where $\vec{n} = (n_1, \ldots, n_N)$ is an $O(N)$ vector. While the action in Eq. (42) is pure Gaussian, the model is not free due to the constraint

\[
\vec{n}^2(x) = \frac{1}{g^2}, \tag{43}
\]

which is imposed on the $\vec{n}$ field via the (functional) delta-function.

The sigma model in $d = 2$ is sometimes considered as a toy model for QCD since it possesses:

1) asymptotic freedom [21];
2) instantons for $N = 3$ [22].

The action in Eq. (42) is $\sim N$ as $N \to \infty$ but the entropy, \textit{i.e.} a contribution from the measure of integration, is also $\sim N$ so that a straightforward saddle point is not applicable.

To overcome this difficulty, we introduce an auxiliary field $u(x)$, which is $\sim 1$ as $N \to \infty$, and rewrite the partition function (42) as

\[
Z \propto \int \uparrow Du(x) \int D\vec{n}(x) e^{-\frac{1}{2} \int d^2x \left[ (\partial_\mu \vec{n})^2 - u \left( \vec{n}^2 - \frac{1}{g^2} \right) \right]}, \tag{44}
\]

where the contour of integration over $u(x)$ is parallel to imaginary axis.

Doing the Gaussian integration over $\vec{n}$, we get

\[
Z \propto \int \uparrow Du(x) e^{-\frac{N}{2} \text{Sp} \ln \left( -\partial_\mu^2 + u(x) \right) + \frac{1}{g^2} \int d^2x u(x)}. \tag{45}
\]

The first term in the exponent is nothing but the sum of one-loop diagrams in 2 dimensions

\[
\frac{N}{2} \text{Sp} \ln \left( -\partial_\mu^2 + u(x) \right) = \sum_n \frac{1}{n} \tag{46}
\]

\(^3\)The name comes from elementary particle physics where a nonlinear sigma model in 4 dimensions is used as an effective Lagrangian for describing low-energy scattering of the Goldstone $\pi$-mesons.
where the auxiliary field $u$ is denoted again by the wavy line.

Now the path integral over $u(x)$ in Eq. (45) is a typical saddle-point one: the action $\sim N$ while the entropy $\sim 1$ since only one integration over $u$ is left. The saddle-point equation for the nonlinear sigma model

$$\frac{1}{g^2} - NG(x,x; u_{sp}) = 0$$

while $G$ is defined by

$$G(x,y; u) = \left\langle y \left\| \frac{1}{-\partial_{\mu}^2 + u} \right\| x \right\rangle.$$ (48)

The coupling $g^2$ in Eq. (47) is $\sim 1/N$ as is prescribed by the constraint (43) which involves a sum over $N$ terms on the LHS. This guarantees that a solution to Eq. (47) exists. Next orders of the $1/N$-expansion for the 2-dimensional sigma model can be constructed analogously to the previous Subsections.

The $1/N$-expansion of the 2-dimensional nonlinear sigma model has many advantages over perturbation theory, which is usually constructed solving explicitly the constraint (43), say, choosing

$$n_N = \frac{1}{g} \left[ 1 - g^2 \sum_{a=1}^{N-1} n_a^2 \right]$$

and expanding the square root in $g^2$. Only $N-1$ dynamical degrees of freedom are left so that the $O(N)$-symmetry is broken in perturbation theory down to $O(N-1)$. The particles in perturbation theory are massless (like Goldstone bosons) and it suffers from infrared divergencies.

On the contrary, the solution to Eq. (47) has the form

$$u_{sp} = m_R^2 \equiv \Lambda^2 e^{-\frac{4\pi}{N}}.$$ (50)

where $\Lambda$ is an ultraviolet cutoff. Therefore, all $N$ particles acquire the same mass $m_R$ in the $1/N$-expansion so that the $O(N)$ symmetry is restored. This appearance of mass is due to dimensional transmutation which says in this case that the parameter $m_R$ rather than the renormalized coupling constant $g_R^2$ is observable. The emergence of the mass cures the infrared problem.

To show that (50) is a solution to Eq. (47), let us look for a translationally invariant solution $u_{sp}(x) \equiv m_R^2$. Then Eq. (47) in the momentum space reads

$$\frac{1}{g^2} = N \int_0^\Lambda \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 + m_R^2} = N \int_0^\Lambda \frac{dk^2}{k^2 + m_R^2} = N \ln \frac{\Lambda^2}{m_R^2}.$$ (51)
The exponentiation results in Eq. (50).

Equation (51) relates the bare coupling $g^2$ and the cutoff $\Lambda$ and allows us to calculate the Gell-Mann–Low function yielding

$$B(g^2) \equiv \frac{\Lambda dg^2}{d\Lambda} = -\frac{Ng^4}{2\pi}.$$  \hspace{1cm} (52)

The analogous one-loop perturbation-theory formula for any $N$ reads [21]

$$B(g^2) = -\frac{(N-2)g^4}{2\pi}.$$  \hspace{1cm} (53)

Thus, the sigma-model is asymptotically free in 2-dimensions for $N > 2$ which is the origin of the dimensional transmutation. There is no asymptotic freedom for $N = 2$ since $O(2)$ is Abelian.

2.5. LARGE-$N$ FACTORIZATION IN VECTOR MODELS

The fact that a path integral has a saddle point at large $N$ implies a very important feature of large-$N$ theories — the factorization. It is a general property of the large-$N$ limit and holds not only for the $O(N)$ vector models. However, it is useful to illustrate it by these solvable examples.

The factorization at large $N$ holds for averages of singlet operators, for example

$$\langle u(x_1) \cdots u(x_k) \rangle \equiv Z^{-1} \int D\phi e^{-\frac{N}{2} \text{Sp} [-\partial^2 + u] + \frac{1}{2g^2} \int d^2x u(\phi) \cdot \cdots \cdot u(\phi)}$$  \hspace{1cm} (54)

in the 2-dimensional sigma model.

Since the path integral has a saddle point at some $u(x) = u_{sp}(x)$ which is, in fact, $x$-independent due to translational invariance, we get to the leading order in $1/N$:

$$\langle u(x_1) \cdots u(x_k) \rangle = u_{sp}(x_1) \cdots u_{sp}(x_k) + O\left(\frac{1}{N}\right),$$  \hspace{1cm} (55)

which can be written in the factorized form

$$\langle u(x_1) \cdots u(x_k) \rangle = \langle u(x_1) \rangle \cdots \langle u(x_k) \rangle + O\left(\frac{1}{N}\right).$$  \hspace{1cm} (56)

Therefore, $u$ becomes “classical” as $N \to \infty$ in the sense of the $1/N$-expansion. This is an analog of the WKB-expansion in $\hbar = 1/N$. “Quantum” corrections are suppressed as $1/N$.

We shall return to discussing the large-$N$ factorization in the next Section when considering the large-$N$ limit of QCD.
3. Large-N QCD

The method of the $1/N$-expansion can be applied to QCD. This was done by ‘t Hooft [1] using the inverse number of colors for the gauge group $SU(N_c)$ as an expansion parameter.

For a $SU(N_c)$ gauge theory without virtual quark loops, the expansion goes in $1/N_c^2$ and rearranges diagrams of perturbation theory according to their topology. The leading order in $1/N_c^2$ is given by planar diagrams, which have a topology of a sphere, while the expansion in $1/N_c^2$ plays the role of a topological expansion. This reminds an expansion in the string coupling constant in string models of the strong interaction, which also has a topological character.

Virtual quark loops can be easily incorporated in the $1/N_c$-expansion. One distinguishes between the ‘t Hooft limit when the number of quark flavors $N_f$ is fixed as $N_c \to \infty$ and the Veneziano limit [23] when the ratio $N_f/N_c$ is fixed as $N_c \to \infty$. Virtual quark loops are suppressed in the ‘t Hooft limit as $1/N_c$ and lead in the Veneziano limit to the same topological expansion as dual-resonance models of strong interaction.

The simplification of QCD in the large-$N_c$ limit is due to the fact that the number of planar graphs grows with the number of vertices only exponentially rather than factorially as do the total number of graphs. Correlators of gauge invariant operators factorize in the large-$N_c$ limit which looks like the leading-order term of a “semiclassical” WKB-expansion in $1/N_c$.

We begin this Section with a description of the double-line representation of diagrams of QCD perturbation theory and rearrange it as the topological expansion in $1/N_c$. Then we discuss some properties of the $1/N_c$-expansion for a generic matrix-valued field.

3.1. INDEX OR RIBBON GRAPHS

In order to describe the $1/N_c$-expansion of the Yang–Mills theory, it is convenient to represent the gauge field by a Hermitean $N \times N$ matrix

$$A^{ij}_\mu(x) = g \sum_a A^a_\mu(x)[t^a]^{ij}. \quad (57)$$

Here $[t^a]^{ij}$ are the generators of the gauge group ($a = 1, \ldots, N^2_c - 1$ for $SU(N_c)$) with the normalization

$$\text{tr} \ t^a t^b = \frac{1}{2} \delta^{ab}. \quad (58)$$

The (Euclidean) action reads

$$S[A] = \int d^dx \frac{1}{2g^2} \text{tr} \ F^2_{\mu\nu}, \quad (59)$$
where

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \]  

(60)

is the non-Abelian field strength and \( g \) is the coupling constant.

The propagator of the matrix field \( A_{ij}^a(x) \) has the form

\[
\left\langle A_{ij}^a(x) A_{kl}^b(y) \right\rangle_{\text{Gauss}} = \frac{1}{2} \left( \delta^{il} \delta^{kj} - \frac{1}{N_c} \delta^{ij} \delta^{kl} \right) D_{\mu\nu}(x-y),
\]  

(61)

where we have assumed, as usual, a gauge fixing to define the propagator in perturbation theory. For instance, one has

\[
D_{\mu\nu}(x-y) = \frac{g^2}{4\pi^2} \frac{\delta_{\mu\nu}}{(x-y)^2}
\]  

(62)

in the Feynman gauge.

Equation (61) can be immediately derived from the standard formula

\[
\left\langle A_{\mu}^a(x) A_{\nu}^b(y) \right\rangle_{\text{Gauss}} = \delta^{ab} D_{\mu\nu}(x-y)
\]  

(63)

multiplying by the generators of the \( SU(N_c) \) gauge group according to the definition (57) and using the completeness condition

\[
\sum_{a=1}^{N_c^2-1} (t^a)^ij (t^a)^kl = \frac{1}{2} \left( \delta^{il} \delta^{kj} - \frac{1}{N_c} \delta^{ij} \delta^{kl} \right) \text{ for } SU(N_c),
\]  

(64)

where the factor of 1/2 is due to the normalization (58) of the generators.

We concentrate in this Section only on the structure of diagrams in the index space, \( i.e. \) the space of the indices associated with the \( SU(N_c) \) group. We shall not consider, in most cases, space-time structures of diagrams which are prescribed by Feynman’s rules.

Omitting at large \( N_c \) the second term in parentheses on the RHS of Eq. (61), we depict the propagator by the double line

\[
\left\langle A_{ij}^a(x) A_{kl}^b(y) \right\rangle_{\text{Gauss}} \propto g^2 \delta^{il} \delta^{kj} = \begin{array}{c} i \end{array} \begin{array}{c} j \end{array} \begin{array}{c} l \end{array} \begin{array}{c} k \end{array}.
\]  

(65)

Each line represents the Kronecker delta-symbol and has orientation which is indicated by arrows. This notation is obviously consistent with the space-time structure of the propagator which describes a propagation from \( x \) to \( y \).

The arrows are due to the fact that the matrix \( A_{ij}^a \) is Hermitian and its off-diagonal components are complex conjugate. The independent fields are, say, the complex fields \( A_{ij}^i \) for \( i > j \) and the diagonal real fields \( A_{ii}^i \). The arrow represents the direction of the propagation of the indices of the
complex field \( A^{ij}_\mu \) for \( i > j \) while the complex-conjugate one, \( A^{ij}_\mu = (A^{ji}_\mu)^* \), propagates in the opposite direction. For the real fields \( A^{ii}_\mu \), the arrows are not essential.

The double-line notation appears generically in all models describing matrix fields in contrast to vector (in internal symmetry space) fields whose propagators are depicted by single lines as in the previous Section.

The three-gluon vertex, which is generated by the action (59), is depicted in the double-line notations as

\[
\begin{array}{c}
\gamma_{\mu_1\mu_2\mu_3} (p_1, p_2, p_3) \\
= \delta_{\mu_1\mu_2} (p_1 - p_2)_{\mu_3} + \delta_{\mu_2\mu_3} (p_2 - p_3)_{\mu_1} + \delta_{\mu_1\mu_3} (p_3 - p_1)_{\mu_2},
\end{array}
\]

is antisymmetric as well. We consider all three gluons as incoming so that their momenta obey \( p_1 + p_2 + p_3 = 0 \). The full vertex is symmetric as is prescribed by Bose statistics.

The color structure in Eq. (66) can alternatively be obtained by multiplying the standard vertex

\[
\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3} (p_1, p_2, p_3) = f^{a_1a_2a_3} \gamma_{\mu_1\mu_2\mu_3} (p_1, p_2, p_3)
\]

by \((f^{a_1})^{ij_1j_2} (f^{a_2})^{j_2j_3} (f^{a_3})^{j_3j_4}\) with \( f^{abc} \) being the structure constants of the SU(\( N_c \)) group, and using the formula

\[
f^{a_1a_2a_3} (f^{a_1})^{ij_1j_2} (f^{a_2})^{j_2j_3} (f^{a_3})^{j_3j_4} = \frac{1}{2} \left( \delta^{i_1j_3} \delta^{i_2j_1} \delta^{i_3j_2} - \delta^{i_1j_2} \delta^{i_3j_1} \delta^{i_2j_3} \right)
\]

which is a consequence of the completeness condition (64).

The four-gluon vertex involves six terms — each of them is depicted by a cross — which differ by interchanging of the color indices. We depict the color structure of the four-gluon vertex for simplicity in the case when \( i_1 = j_2 = i \), \( i_2 = j_3 = j \), \( i_3 = j_4 = k \), \( i_4 = j_1 = l \) but \( i, j, k, l \) take on
different values. Then only the following term is left

\[
\begin{align*}
\sum_{i} \sum_{k} \sum_{j} \propto g^{-2}
\end{align*}
\]

(70)

and there are no deltas on the RHS since the color structure is fixed. In other words, we pick up only one color structure by equaling indices pairwise.

The diagrams of perturbation theory can now be completely rewritten in the double-line notation [1]. The simplest one which describes the one-loop correction to the gluon propagator is depicted in Figure 6.\(^4\) This diagram involves two three-gluon vertices and a sum over the \(N_c\) indices which is associated with the closed index line. Therefore, the relative contribution of this diagram is \(\sim g^2 N_c\). In order for the large-\(N_c\) limit to be nontrivial, the bare coupling constant \(g^2\) should satisfy

\[
g^2 \sim \frac{1}{N_c}.
\]

(71)

This dependence on \(N_c\) is similar to Eqs. (12) and (47) for the vector models and is prescribed by the asymptotic-freedom formula

\[
g^2 = \frac{24\pi^2}{11N_c \ln \left(\Lambda / \Lambda_{QCD}\right)}
\]

(72)

of the pure SU\((N_c)\) gauge theory.

Thus, the relative contribution of the diagram of Figure 6 is of order

\[
\text{Figure 6} \sim g^2 N_c \sim 1
\]

(73)

in the large-\(N_c\) limit.

\(^4\)Here and in the most figures below the arrows of the index lines are omitted for simplicity.
LARGE-N GAUGE THEORIES

The double lines of the diagram in Figure 6 can be viewed as bounding a piece of a plane. Therefore, these lines represent a two-dimensional object rather than a one-dimensional one as the single lines do in vector models. These double-line graphs are often called in mathematics the ribbon graphs or fatgraphs. We shall see below their connection with Riemann surfaces.

Remark on the \( U(N_c) \) gauge group

As is said above, the second term in the parentheses on the RHS of Eq. (64) can be omitted at large \( N_c \). Such a completeness condition emerges for the \( U(N_c) \) group whose generators \( T^A \) \((A = 1, \ldots, N_c^2)\) are

\[
T^A = \left(t^a, \frac{1}{\sqrt{2N_c}}\right), \quad \text{tr} T^A T^B = \frac{1}{2} \delta^{AB}.
\]

They obey the completeness condition

\[
\sum_{A=1}^{N_c^2} (T^A)^{ij} (T^A)^{kl} = \frac{1}{2} \delta^{il} \delta^{kj} \quad \text{for} \quad U(N_c).
\]

The point is that elements of both the \( SU(N_c) \) group and the \( U(N_c) \) group can be represented in the form \( U = \exp iB \), where \( B \) is a general Hermitean matrix for \( U(N_c) \) and a traceless Hermitean matrix for \( SU(N_c) \).

Therefore, the double-line representation of perturbation-theory diagrams which is described in this Section holds, strictly speaking, only for the \( U(N_c) \) gauge group. However, the large-\( N_c \) limit of both the \( U(N_c) \) group and the \( SU(N_c) \) group is the same.

3.2. PLANAR AND NON-PLANAR GRAPHS

The double-line representation of perturbation theory diagrams in the index space is very convenient to estimate their orders in \( 1/N_c \). Each gluon propagator contributes a factor of \( g^2 \) Each three- or four-gluon vertex contributes a factor of \( g^{-2} \). Each closed index line contributes a factor of \( N_c \). The order of \( g \) in \( 1/N_c \) is given by Eq. (71).

Let us consider a typical diagram for the gluon propagator depicted in Figure 7. It has eight three-gluon vertices and four closed index lines which coincides with the number of loops. Therefore, the relative order of this diagram in \( 1/N_c \) is

\[
\text{Figure 7} \sim \left(g^2 N_c\right)^4 \sim 1.
\]

The diagrams of the type in Figure 7, which can be drawn on a sheet of a paper without crossing any lines, are called the planar diagrams.
such diagrams, an adding of a loop inevitably results in adding of two three-gluon (or one four-gluon) vertex. A planar diagram with $n_2$ loops has $n_2$ closed index lines. It is of order

$$n_2\text{-loop planar diagram } \sim \left( g^2 N_c \right)^{n_2} \sim 1, \quad (77)$$

so that all planar diagrams survive in the large-$N_c$ limit.

Let us now consider a non-planar diagram of the type depicted in Figure 8. This diagram is a three-loop one and has six three-gluon vertices. The crossing of the two lines in the middle does not correspond to a four-gluon vertex and is merely due to the fact that the diagram cannot be drawn on a sheet of a paper without crossing the lines. The diagram has only one closed index line. The relative order of this diagram in $1/N_c$ is

$$\text{Figure 8 } \sim g^6 N_c \sim \frac{1}{N_c^2}. \quad (78)$$

It is therefore suppressed at large $N_c$ by $1/N_c^2$.

The non-planar diagram in Figure 8 can be drawn without line-crossing on a surface with one handle which is usually called in mathematics a torus or the surface of genus one. A plane is then equivalent to a sphere and has genus zero. Adding a handle to a surface produces a \textit{hole} according to mathematical terminology. A general Riemann surface with $h$ holes has genus $h$. 

Figure 7. Double-line representation of a four-loop diagram for the gluon propagator. The sum over the $N_c$ indices is associated with each of the four closed index lines whose number is equal to the number of loops. The contribution of this diagram is $\sim g^8 N_c^4 \sim 1$.

Figure 8. Double-line representation of a three-loop non-planar diagram for the gluon propagator. The diagram has six three-gluon vertices but only one closed index line (while three loops!). The order of this diagram is $\sim g^6 N_c \sim 1/N_c^2$. 
The above evaluations of the order of the diagrams in Figures 6–8 can now be described by the unique formula

\[ \text{genus-} \ell \text{ diagram } \sim \left( \frac{1}{N_c^2} \right)^{\text{genus}}. \]  

Thus, the expansion in \(1/N_c\) rearranges perturbation-theory diagrams according to their topology \([1]\). For this reason, it is referred to as the topological expansion or the genus expansion. The general proof of Eq. (79) for an arbitrary diagram is given in Subsection 3.3.

Only planar diagrams, which are associated with genus zero, survive in the large-\(N_c\) limit. This class of diagrams is an analog of the bubble graphs in the vector models. However, the problem of summing the planar graphs is much more complicated than that of summing the bubble graphs. Nevertheless, it is simpler than the problem of summing all the graphs, since the number of the planar graphs with \(n_0\) vertices grows at large \(n_0\) exponentially \([24, 25]\)

\[ \#_p(n_0) \equiv \# \text{ of planar graphs } \sim \text{const}^{n_0}, \]  

while the number of all the graphs grows with \(n_0\) factorially. There is no dependence in Eq. (80) on the number of external lines of a planar graph which is assumed to be much less than \(n_0\).

It is instructive to see the difference between the planar diagrams and, for instance, the ladder diagrams which describe \(e^+e^-\) elastic scattering in QED. Let the ladder has \(n\) rungs. Then there are \(n!\) ladder diagrams, but only one of them is planar. This simple example shows why the number of planar graphs is much smaller than the number of all graphs, most of which are non-planar.

We shall discuss in these Lectures what is known about solving the problem of summing the planar graphs.

Equation (79) holds, strictly speaking, only for the relative order while the contribution of tree diagrams to a connected \(n\)-point Green’s function is \(\sim (g^2)^{n-1}\) which is its natural order in \(1/N_c\). In order to make contributions of all planar diagrams to be of the same order \(\sim 1\) in the large-\(N_c\) limit, independently of the number of external lines, it is convenient to contract the Kronecker deltas associated with external lines.

Let us do this in a cyclic order as is depicted in Figure 9 for a generic connected diagram with three external gluon lines. The extra deltas which are added to contract the color indices are depicted by the single lines. They can be viewed as a boundary of the given diagram. The actual size of the boundary is not essential — it can be shrunk to a point. Then a bounded piece of a plane will be topologically equivalent to a sphere with a puncture.
I shall prefer to draw planar diagrams in a plane with an extended boundary (boundaries) rather than in a sphere with a puncture (punctures).

It is clear from the graphic representation that the diagram in Figure 9 is associated with the trace over the color indices of the three-point Green’s function

\[ G_{\mu_1\mu_2\mu_3}^{(3)}(x_1, x_2, x_3) \equiv \frac{1}{N_c} \left( \text{tr} \left[ A_{\mu_1}(x_1) A_{\mu_2}(x_2) A_{\mu_3}(x_3) \right] \right). \]  

(81)

We have introduced here the factor $1/N_c$ to make $G_3$ of $O(1)$ in the large-$N_c$ limit. Therefore, the contribution of the diagram in Figure 9 having one boundary should be divided by $N_c$.

The extension of Eq. (81) to multi-point Green’s functions is obvious:

\[ G_{\mu_1\cdots\mu_n}^{(n)}(x_1, \ldots, x_n) \equiv \frac{1}{N_c} \left( \text{tr} \left[ A_{\mu_1}(x_1) \cdots A_{\mu_n}(x_n) \right] \right). \]  

(82)

The factor $1/N_c$, which normalizes the trace, provides the natural normalization

\[ G^{(0)} = 1 \]  

(83)

of the averages.

Though the two terms in the index-space representation (66) of the three-gluon vertex look very similar, their fate in the topological expansion is quite different. When the color indices are contracted anti-clockwise, the first term leads to the planar contributions to $G^{(3)}$, the simplest of which is
Figure 10. Planar a) and non-planar b) contributions of the two color structures in Eq. (66) for three-gluon vertex to $G^{(3)}$ in the lowest order of perturbation theory.

depicted in Figure 10a. The anti-clockwise contraction of the color indices in the second term leads to a non-planar graph in Figure 10b which can be drawn without crossing of lines only on a torus. Therefore, the two color structures of the three-gluon vertex contribute to different orders of the topological expansion. The same is true for the four-gluon vertex.

Remark on oriented Riemann surfaces

Each line of an index graph of the type depicted in Figure 9 is oriented. This orientation continues along a closed index line while the pairs of index lines of each double line have opposite orientations. The overall orientation of the lines is prescribed by the orientation of the external boundary which we choose to be, say, anti-clockwise like in Figure 10a. Since the lines are oriented, the faces of the Riemann surface associated with a given graph are oriented too — all in the same way — anti-clockwise. Vice versa, such an orientation of the Riemann surfaces unambiguously fixes the orientation of all the index lines. This is the reason why we omit the arrows associated with the orientation of the index lines: their directions are obvious.

Remark on cyclic-ordered Green’s functions

The cyclic-ordered Green’s functions (82) naturally arise in the expansion of the trace of the path-ordered non-Abelian phase factor for a closed contour.
One gets

\[
\left\langle \frac{1}{N_c} \text{tr} P e^{i \int \! \! \! \! \int \! d^d x A_\mu (x)} \right\rangle = \sum_{n=0}^{\infty} i^n \int_{\Gamma} d^d x_1 \frac{d^d x_2}{x_1} \cdots \frac{d^d x_{n-1}}{x_1} \int_{x_1} d^d x_n G^{(n)}_{\mu_1 \cdots \mu_n} (x_1, \ldots, x_n) \, .
\]

The reason is because the ordering along a closed path implies the cyclic-ordering in the index space.

Remark on generating functionals for planar graphs

By connected or disconnected planar graphs we mean, respectively, the graphs which were connected or disconnected before the contraction of the color indices as is illustrated by Figure 11. The graph in Figure 11a is connected planar while the graph in Figure 11b is disconnected planar.

The usual exponential relation between the generating functionals \( W [J] \) and \( Z [J] \) for connected graphs and all graphs, does not hold for the planar graphs. The reason is that an exponentiation of such a connected planar diagram for the cyclic-ordered Green’s functions (82) can give disconnected non-planar ones.

The generating functionals for all and connected planar graphs can be constructed [26] by means of introducing non-commutative sources \( j_\mu (x) \). “Non-commutative” means that there is no way to transform \( j_{\mu_1} (x_1) j_{\mu_2} (x_2) \) into \( j_{\mu_2} (x_2) j_{\mu_1} (x_1) \). This non-commutativity of the sources reflects the cyclic-ordered structure of the Green’s functions (82) which possess only cyclic symmetry.

Using the short-hand notations

\[
\mathbf{j} \circ A \equiv \sum_\mu \int d^d x \, \mathbf{j}_\mu (x) A_\mu (x) ,
\]

Figure 11. Example of a) connected and b) disconnected planar graphs.
Figure 12. Graphic derivation of Eq. (89): $Z[j]$ is denoted by an empty box, $W[j]$ is denoted by a shadow box, $j$ is denoted by a filled circle. By picking a leftmost external line of a planar graph, we end up with a connected planar graph, whose remaining external lines are somewhere to the right interspersed by disconnected planar graphs. It is evident that $jZ[j]$ plays the role of a new source for the connected planar graph. If we instead pick up the rightmost external line, we get the inverse order $Z[j]j$, which results in Eq. (90).

where the symbol $\circ$ includes the sum over the d-vector (or whatever available) indices except for the color ones, we write down the definitions of the generating functionals for all planar and connected planar graphs, respectively, as

$$Z[j] \equiv \sum_{n=0}^{\infty} \left\langle \frac{1}{N_c} \text{tr} \left( j \circ A \right)^n \right\rangle$$ (86)

and

$$W[j] \equiv \sum_{n=0}^{\infty} \left\langle \frac{1}{N_c} \text{tr} \left( j \circ A \right)^n \right\rangle_{\text{conn}}.$$ (87)

The planar contribution to the Green’s functions (82) and their connected counterparts can be obtained, respectively, from the generating functionals $Z[j]$ and $W[j]$ applying the non-commutative derivative which is defined by

$$\frac{\delta}{\delta j_\mu(x)} j_\nu(y) f(j) = \delta_{\mu\nu} \delta^{(d)}(x-y) f(j),$$ (88)

where $f$ is an arbitrary function of $j$’s. In other words the derivative picks up only the leftmost variable.

The relation which replaces the usual one for planar graphs is

$$Z[j] = W[jZ[j]],$$ (89)

while the cyclic symmetry says

$$W[jZ[j]] = W[Z[j]j].$$ (90)

A graphic derivation of Eqs. (89) and (90) is given in Figure 12. In other words, given $W[j]$, one should construct an inverse function as the solution to the equation

$$j_\mu(x) = J_\mu(x)W[j],$$ (91)
after which Eq. (89) says
\[ Z[j] = W[J]. \] (92)

More about this approach to the generating functionals for planar graphs can be found in Ref. [27].

An iterative solution to Eq. (91) for the Gaussian case can be easily found. In the Gaussian case, only \( G^{(2)} \) is nonvanishing which yields
\[ W[j] = 1 - g^2 j \circ D \circ j, \] (93)

where the propagator \( D \) is given by Eq. (62). Using Eq. (89), we get explicitly
\[ Z[j] = 1 - g^2 \int d^4x d^4y D_{\mu\nu}(x - y) j_{\mu}(x) Z[j] j_{\nu}(y) Z[j]. \] (94)

While this equation for \( Z[j] \) is quadratic, its solution can be written only as a continued fraction due to the non-commutative nature of the variables. In order to find it, we rewrite Eq. (94) as
\[ Z[j] = \frac{1}{1 + g^2 \int d^4x d^4y D_{\mu\nu}(x - y) j_{\mu}(x) Z[j] j_{\nu}(y) Z[j]}, \] (95)

whose iterative solution reads [26]
\[ Z[j] = \frac{1}{1 + g^2 j \circ D \circ j} \frac{1}{1 + g^2 j \circ D \circ j} \frac{1}{1 + g^2 j \circ D \circ j} \cdots. \] (96)

3.3. TOPOLOGICAL EXPANSION AND QUARK LOOPS

It is easy to incorporate quarks in the topological expansion. A quark field belongs to the fundamental representation of the gauge group \( SU(N_c) \) and its propagator is represented by a single line
\[ \langle \bar{\psi}_i \psi_j \rangle \propto \delta_{ij} = i \rightarrow j. \] (97)

The arrow indicates, as usual, the direction of propagation of a (complex) field \( \psi \). We shall omit these arrows for simplicity.

The diagram for the gluon propagator which involves one quark loop is depicted in Figure 13a. It has two three gluon vertices and no closed index
Figure 13. Diagrams for the gluon propagator with a quark loop which is represented by the single lines. The diagram a) involves one quark loop and has no closed index lines so that its order is $\sim g^2 \sim 1/N_c$. The diagram b) involves three loops one of which is a quark loop. Its relative order is $\sim g^6 N_c^2 \sim 1/N_c$.

The lines so that its order in $1/N_c$ is

$$\text{Figure 13a} \sim g^2 \sim \frac{1}{N_c}. \quad (98)$$

Analogously, the relative order of a more complicated tree-loop diagram in Figure 13b, which involves one quark loop and two closed index lines, is

$$\text{Figure 13b} \sim g^6 N_c^2 \sim \frac{1}{N_c}. \quad (99)$$

It is evident from this consideration that quark loops are not accompanied by closed index lines. One should add a closed index line for each quark loop in order for a given diagram with $L$ quark loops to have the same double-line representation as for pure gluon diagrams. Therefore, given Eq. (79), diagrams with $L$ quark loops are suppressed at large $N_c$ by

$$L \text{ quark loops} \sim \left( \frac{1}{N_c} \right)^{L+2\text{-genus}}. \quad (100)$$

The single-line representation of the quark loops is similar to the one of the external boundary in Figure 9. Moreover, such a diagram emerges when one calculates perturbative gluon corrections to the vacuum expectation value of the quark operator

$$O_{\Gamma} = \frac{1}{N_c} \bar{\psi} \Gamma \psi, \quad (101)$$

where $\Gamma$ stands for one of the combinations of the gamma-matrices:

$$\Gamma = I, \gamma_5, \gamma_\mu, i\gamma_\mu \gamma_5, \Sigma_{\mu\nu} = \frac{1}{2i} [\gamma_\mu, \gamma_\nu], \ldots. \quad (102)$$
The factor of $1/N_c$ is introduced in (101) to make it $\mathcal{O}(1)$ in the large-$N_c$ limit. Therefore, the external boundary can be viewed as a single line associated with valence quarks. The difference between virtual quark loops and external boundaries is that each of the latter comes along with the factor of $1/N_c$ due to the definitions (82) and (101).

In order to prove Eqs. (79) and its quark counterpart (100), let us consider a generic diagram in the index space which has $n_0^{(3)}$ three-point vertices (either three-gluon or quark-gluon ones), $n_0^{(4)}$ four-gluon vertices, $n_1$ propagators (either gluon or quark ones), $n_2$ closed index lines, $L$ virtual quark loops and $B$ external boundaries. Its order in $1/N_c$ is

$$
\frac{1}{N_c^B} (g^2)^{n_1-n_0^{(3)}-n_0^{(4)}} N_c^{n_2-n_1+n_0-B} \sim N_c^{n_2-n_1+n_0-B} 
$$

(103)

where the total number of vertices $n_0 = n_0^{(3)} + n_0^{(4)}$ is introduced. The extra factor of $1/N_c^B$ is due to the extra normalization factor of $1/N_c$ in operators associated with external boundaries.

The exponent on the RHS of Eq. (103) can be expressed via the Euler characteristics $\chi$ of a given graph of genus $h$. Let us first mention that a proper Riemann surface, which is associated with a given graph, is open and has $B+L$ boundaries (represented by single lines). This surface can be closed by attaching a cap to each boundary. The single lines then become double lines together with the lines of the boundary of each cap. We have already considered this procedure when deducing Eq. (100) from Eq. (79).

The number of faces for a closed Riemann surface constructed in such a manner is $n_2 + L + B$, while the number of edges and vertices are $n_1$ and $n_0$, respectively. Euler’s theorem says that

$$
\chi = 2 - 2h = n_2 + L + B - n_1 + n_0.
$$

(104)

Therefore the RHS of Eq. (103) can be rewritten as

$$
N_c^{n_2-n_1+n_0-B} = N_c^{2-2h-L-2B}.
$$

(105)

We have thus proven that the order in $1/N_c$ of a generic graph does not depend on its order in the coupling constant and is completely expressed via the genus $h$ and the number of virtual quark loops $L$ and external boundaries $B$ by

$$
\text{generic graph} \sim \left( \frac{1}{N_c} \right)^{2h+L+2(B-1)}.
$$

(106)

For $B = 1$, we recover Eqs. (79) and (100).
Remark on the order of gauge action
We see from Eq. (82) that the natural variables for the large-$N_c$ limit are the matrices $A_\mu$ which include the factor of $g$ (see Eq. (57)). In these variables, the action (59) is $\mathcal{O}(N_c^2)$ at large $N_c$, since $g^2 \sim 1/N_c$ and the trace is $\sim N_c$.

This result can be anticipated from the free theory because the kinetic part of the action involves the sum over $N_c^2 - 1$ free gluons. Therefore, the non-Abelian field strength (60) is $\sim 1$ for $g^2 \sim 1/N_c$.

The fact that the action is $\mathcal{O}(N_c^2)$ in the large-$N_c$ limit is a generic property of the models describing matrix fields. It will be crucial for developing saddle-point approaches at large $N_c$ which are considered below.

Remark on phenomenology of multicolor QCD
While $N_c = 3$ in the real world, there are phenomenological indications that $1/N_c$ may be considered as a small parameter. We have already mentioned some of them in the text — the simplest one is that the ratio of the $\rho$-meson width to its mass, which is $\sim 1/N_c$, is small. Considering $1/N_c$ as a small parameter immediately leads to qualitative phenomenological consequences which are preserved by the planar diagrams associated with multicolor QCD, but are violated by the non-planar diagrams.

The most important consequence is the relation of the $1/N_c$-expansion to the topological expansion in the dual-resonance model of hadrons. Vast properties of hadrons are explained by the dual-resonance model. A very clear physical picture behind this model is that hadrons are excitations of a string with quarks at the ends.

I shall briefly list some consequences of multicolor QCD:

1) The “naive” quark model of hadrons emerges at $N_c = \infty$. Hadrons are built out of (valence or constituent) quark and antiquark $q\bar{q}$, while exotic states like $qq\bar{q}\bar{q}$ do not appear.

2) The partial width of decay of the $\phi$-meson, which is built out of $s\bar{s}$ (the strange quark and antiquark), into $K^+K^-$ is $\sim 1/N_c$, while that into $\pi^+\pi^-\pi^0$ is $\sim 1/N_c^2$. This explains Zweig’s rule. The masses of the $\rho$- and $\omega$-mesons are degenerate at $N_c = \infty$.

3) The coupling constant of meson-meson interaction is small at large $N_c$.

4) The widths of glueballs are $\sim 1/N_c^2$, i.e. they should be even narrower than mesons built out of quarks. The glueballs do not interact or mix with mesons at $N_c = \infty$.

All these hadron properties (except the last one) approximately agree with experiment, and were well-known even before 1974 when multicolor QCD was introduced. Glueballs are not yet detected experimentally (maybe because of their property listed in the item 4).
3.4. LARGE-$N_C$ FACTORIZATION

The vacuum expectation values of several colorless or white operators, which are singlets with respect to the gauge group, factorize in the large-$N_c$ limit of QCD (or other matrix models). This property is similar to that already discussed in Subsection 2.5 for the vector models.

The simplest gauge-invariant operator in a pure SU($N_c$) gauge theory is the square of the non-Abelian field strength:

$$O(x) = \frac{1}{N_c} \text{tr} F_{\mu\nu}^2(x).$$

(107)

The normalizing factor is the same as in Eqs. (81), (82), which provides the natural normalization

$$\left\langle \frac{1}{N_c} \text{tr} F_{\mu\nu}^2(x) \right\rangle = \left\langle \frac{g^2}{2N_c} F_{\mu\nu}^a(x) F_{\mu\nu}^a(x) \right\rangle \sim 1. \quad (108)$$

The contribution of all planar graphs to the average on the LHS of Eq. (108) is of order 1 in accord with the general formula (106) for $B = 1$.

In order to verify the factorization in the large-$N_c$ limit, let us consider the index space diagrams for the average of the product of two colorless operators $O(x_1)$ and $O(x_2)$ given by (107). It involves a factorized part when gluons are emitted and absorbed by the same operators. The contribution of the factorized part is of order 1 as above.

Alternatively, the connected correlator of the two operators is associated with the general formula (106) for two boundaries $B = 2$. Its contribution is suppressed by $1/N_c^2$ in the large-$N_c$ limit. For this correlator, at least one gluon line is emitted and absorbed by different operators $O(x_1)$ and $O(x_2)$. Notice, that these graphs themselves are planar, while the suppression comes from the number of boundaries.

This example illustrates the general property that only (planar) diagrams with gluon lines emitted and absorbed by the same operators survive as $N_c \to \infty$. Since correlations between the colorless operators $O(x_1)$ and $O(x_2)$ are of order $1/N_c^2$, the factorization property holds as $N_c \to \infty$:

$$\left\langle \frac{1}{N_c} \text{tr} F^2(x_1) \frac{1}{N_c} \text{tr} F^2(x_2) \right\rangle = \left\langle \frac{1}{N_c} \text{tr} F^2(x_1) \right\rangle \left\langle \frac{1}{N_c} \text{tr} F^2(x_2) \right\rangle + O\left(\frac{1}{N_c^2}\right). \quad (109)$$

For a general set of gauge-invariant operators $O_1, \ldots, O_n$, the factorization property can be represented by

$$\langle O_1 \cdots O_n \rangle = \langle O_1 \rangle \cdots \langle O_n \rangle + O\left(\frac{1}{N_c^2}\right). \quad (110)$$
This is analogous to Eq. (56) for the vector models.

The factorization in large-$N_c$ QCD was first discovered by A.A. Migdal in the late seventies. An important observation that the factorization implies a semiclassical nature of the large-$N_c$ limit of QCD was done by Witten [28]. We shall discuss this in the next two Subsections.

The factorization property also holds for gauge-invariant operators constructed from quarks like in Eq. (101) as a consequence of Eq. (106).

**Remark on factorization beyond perturbation theory**

The large-$N_c$ factorization (110) has been shown above to all orders of perturbation theory. It can be also verified at all orders of the strong coupling expansion in the SU($N_c$) lattice gauge theory. A non-perturbative proof of the factorization will be given in the next Section by using quantum equations of motion (the loop equations).

### 3.5. THE MASTER FIELD

The large-$N_c$ factorization in QCD assumes that gauge-invariant objects behave as $c$-numbers, rather than as operators. Likewise the vector models, this suggests that the path integral is dominated by a saddle point.

We already saw in Subsection 2.5 that the factorization in the vector models does not mean that the fundamental field itself, for instance $\vec{n}$ in the sigma-model, becomes “classical”. It is the case, instead, for a singlet composite field.

We are now going to apply a similar idea to the Yang–Mills theory whose partition function reads

$$Z = \int D A^a_\mu e^{-S}.$$  \hspace{1cm} (111)

The action, $\sim N_c^2$, is large as $N_c \to \infty$, but the “entropy” is also $\sim N_c^2$ due to the $N_c^2 - 1$ integrations over $A^a_\mu$:

$$D A^a_\mu \sim e^{N_c^2}.$$ \hspace{1cm} (112)

Consequently, the saddle-point equation of the large-$N_c$ Yang–Mills theory is *not* the classical one.

The idea is to rewrite the path integral over $A_\mu$ for the Yang–Mills theory as that over a colorless composite field $\Phi [A]$, likewise it was done in Subsection 2.4 for the sigma-model. The expected new path-integral representation of the partition function (111) would be something like

$$Z \propto \int D \Phi \frac{1}{\delta^2 S[\Phi]} e^{-N_c^2 S[\Phi]}.$$  \hspace{1cm} (113)
The Jacobian
\[
\frac{\partial \Phi[A]}{\partial A_\mu^a} \equiv e^{-N_c^2 J[\Phi]} \tag{114}
\]
in Eq. (113) is related to the old entropy factor, so that \( J[\Phi] \sim 1 \) in the large-\( N_c \) limit.

The original partition function (111) can be then rewritten as
\[
Z \propto \int D\Phi e^{N_c^2 J[\Phi] - N_c^2 S[\Phi]}, \tag{115}
\]
where \( S[\Phi] \) represents the Yang–Mills action in the new variables. The new “entropy” factor \( D\Phi \) is \( O(1) \) because the variable \( \Phi[A] \) is a color singlet.

The large parameter \( N_c \) enters Eq. (115) only in the exponent. Therefore, the saddle-point equation can be immediately written:
\[
\frac{\delta S}{\delta \Phi} = \frac{\delta J}{\delta \Phi}. \tag{116}
\]
Remembering that \( \Phi \) is a functional of \( A_\mu^a \): \( \Phi \equiv \Phi[A] \), we rewrite the saddle-point equation (116) as
\[
\frac{\delta S}{\delta A_\mu^a} = (\nabla_\mu F_{\mu\nu})^a = \frac{\delta J}{\delta A_\mu^a}. \tag{117}
\]
It differs from the classical Yang–Mills equation by the term on the RHS coming from the Jacobian (114).

Given \( J[\Phi] \) which depends on the precise form of the variable \( \Phi[A] \), Eq. (117) has a solution
\[
A_\mu(x) = A_\mu^{cl}(x). \tag{118}
\]
Let us first assume that there exists only one solution to Eq. (117). Then the path integral is saturated by a single configuration (118), so that the vacuum expectation values of gauge-invariant operators are given by their values at this configuration:
\[
\langle O \rangle = O \left( A_\mu^{cl}(x) \right). \tag{119}
\]
The factorization property (110) will obviously be satisfied.

An existence of such a classical field configuration in multicolor QCD was conjectured by Witten [28]. It was discussed in the lectures by Coleman [29] who called it the master field. Equation (117) which determines the master field is often referred to as the master-field equation.

A subtle point with the master field is that a solution to Eq. (117) is determined only up to a gauge transformation. To preserve gauge invariance, it is more reasonable to speak about the whole gauge orbit as a solution of
Eq. (117). However, this will not change Eq. (119) since the operator $O$ is gauge invariant.

The conjecture about an existence of the master field has surprisingly rich consequences. Since vacuum expectation values are Poincaré invariant, the RHS of Eq. (119) does. This implies that $A_{\mu}^{\text{cl}}(x)$ must itself be Poincaré invariant up to a gauge transformation: a change of $A_{\mu}^{\text{cl}}(x)$ under translations or rotations can be compensated by a gauge transformation. Moreover, there must exist a gauge in which $A_{\mu}^{\text{cl}}(x)$ is space-time independent: $A_{\mu}^{\text{cl}}(x) = A_{\mu}^{\text{cl}}(0)$. In this gauge, rotations must be equivalent to a global gauge transformation, so that $A_{\mu}^{\text{cl}}(0)$ transforms as a Lorentz vector.

In fact, the idea about such a master field in multicolor QCD may be incorrect as was pointed out by Haan [30]. The conjecture about an existence of only one solution to the master-field equation (117) seems to be too strong. If several solutions exists, one needs an additional averaging over these solutions. This is a very delicate matter, since this additional averaging must still preserve the factorization property. One might better think about this situation as if $A_{\mu}^{\text{cl}}(0)$ would be an operator in some Hilbert space rather than a $c$-valued function. Such an operator-valued master field is sometimes called the master field in the \textit{weak} sense, while the above conjecture about a single classical configuration of the gauge field, which saturates the path integral, is called the master field in the \textit{strong} sense.

The concept of the master field is rather vague until a precise form of the composite field $\Phi[A]$, and consequently the Jacobian $\Phi[A]$ that enters Eq. (117), is not defined. However, what is important is that the master field (in the weak sense) is space-time independent. This looks like a simplification of the problem of solving large-$N_c$ QCD. A Hilbert space, in which the operator $A_{\mu}^{\text{cl}}(0)$ acts, should be specified by $\Phi[A]$. We shall consider in the next Subsection a realization of these ideas for the case of $\Phi[A]$ given by the trace of the non-Abelian phase factor for closed contours.

\textit{Remark on non-commutative probability theory}

An adequate mathematical language for describing the master field in multicolor QCD (and, generically, in matrix models at large $N_c$) was found by I. Singer in 1994. It is based on the concept of free random variables of non-commutative probability theory, introduced by Voiculescu [31]. How to describe the master field in this language and some other applications of non-commutative free random variables to the problems of planar quantum field theory are discussed in Refs. [32, 33].
3.6. 1/N_C AS SEMICLASSICAL EXPANSION

A natural candidate for the composite operator Φ [A] from the previous Subsection is given by the trace of the non-Abelian phase factor for closed contours — the Wilson loop. It is labeled by the loop C in the same sense as the field A_μ (x) is labeled by the point x, so we shall use the notation

\[ \Phi (C) \equiv \Phi [A] = \frac{1}{N_c} \text{tr} P e^{i \oint_C d\sigma^\mu A_\mu (x)}. \] (120)

Nobody up to now managed to reformulate QCD at finite N_c in terms of Φ (C) in the language of path integral. This is due to the fact that self-intersecting loops are not independent (they are related by the so-called Mandelstam relations [34]), and the Jacobian is huge. The reformulation was done [35] in the language of Schwinger–Dyson or loop equations which will be described in the next Section.

Schwinger–Dyson equations are a convenient way of performing the semiclassical expansion, which is an alternative to the path integral. Let us illustrate an idea how to do this by an example of the ϕ^3 theory. The RHS of the Schwinger–Dyson equations is proportional to the Planck’s constant \( \bar{h} \). In the semiclassical limit \( \bar{h} \to 0 \), we get

\[ \left( -\partial^2 + m^2 \right) \langle \varphi (x_1) \ldots \varphi (x_n) \rangle + \frac{\lambda}{2} \langle \varphi^2 (x_1) \ldots \varphi (x_n) \rangle = 0, \] (121)

whose solution is of the factorized form

\[ \langle \varphi (x_1) \ldots \varphi (x_n) \rangle = \langle \varphi (x_1) \rangle \ldots \langle \varphi (x_n) \rangle + \mathcal{O} (\bar{h}) \] (122)

provided that

\[ \langle \varphi (x) \rangle \equiv \varphi_{cl} (x) \] (123)

obeys

\[ \left( -\partial^2 + m^2 \right) \varphi_{cl} (x) + \frac{\lambda}{2} \varphi_{cl}^2 (x) = 0. \] (124)

Equation (124) is nothing but the classical equation of motion for the \( \varphi^3 \) theory, which specifies extrema of the action entering the path integral. Thus, we have reproduced, using the Schwinger–Dyson equations, the well-known fact that the path integral is dominated by a classical solution as \( \bar{h} \to 0 \). It is also clear how to perform the semiclassical expansion in \( \bar{h} \) in the language of the Schwinger–Dyson equations: one should solve them by iterations.

The reformulation of multicolor QCD in terms of the loop functionals Φ (C) is, in a sense, a realization of the idea of the master field in the weak sense, when the master field acts as an operator in the space of loops.
Remark on the large-$N_c$ limit as statistical averaging

There is yet another, pure statistical, explanation why the large-$N_c$ limit is a “semiclassical” limit for the collective variables $\Phi(C)$. The matrix $U_{ij}[C_{xx}]$, that describes the parallel transport along a closed contour $C_{xx}$, can be reduced by the gauge transformation to

$$U[C_{xx}] = \Omega[C_{xx}] \text{diag} \left( e^{ig\alpha_1(C)}, \ldots, e^{ig\alpha_{N_c}(C)} \right) \Omega^+[C_{xx}].$$

Then $\Phi(C)$ reads

$$\Phi(C) = \frac{1}{N_c} \sum_{j=1}^{N_c} e^{ig\alpha_j(C)}.$$  

The phases $\alpha_j(C)$ are gauge invariant and normalized so that $\alpha_j(C) \sim 1$ as $N_c \to \infty$. For simplicity we omit below all the indices (including space ones) except color.

The commutator of $\Phi$’s can be estimated using the representation (126). Since $[\alpha_i, \alpha_j] \propto \delta_{ij}$, one gets

$$[\Phi(C), \Phi(C')] \sim g^2 \frac{1}{N_c} \sim \frac{1}{N_c^2}$$

in the limit (71), i.e. the commutator can be neglected as $N_c \to \infty$, and the field $\Phi(C)$ becomes classical.

Note that the commutator (127) is of order $1/N_c^2$. One factor $1/N_c$ is because of $g$ in the definition (126) of $\Phi(C)$, while the other has a deep reason. Let us image the summation over $j$ in Eq. (126) as some statistical averaging. It is well-known in statistics that such averages weakly fluctuate as $N_c \to \infty$, so that the dispersion is of order $1/N_c$. It is the factor which emerges in the commutator (127).

We see that the factorization is valid only for the gauge-invariant quantities which involve the averaging over the color indices, like that in Eq. (126). There is no reason to expect factorization for gauge invariants which do not involve this averaging, for instance for the phases $\alpha_j(C)$. Moreover, their commutator is $\sim 1$, so that $\alpha_j(C)$’s strongly fluctuate even at $N_c = \infty$. An explicit example of such strongly fluctuating gauge-invariant quantities was first constructed in Ref. [30].

The résumé to this Remark is that the factorization is due to the additional statistical averaging in the large-$N_c$ limit. There is no reason to assume an existence of the master field in the strong sense in order to explain the factorization.
4. QCD in Loop Space

QCD can be entirely reformulated in terms of the colorless composite field $\Phi(C)$ — the trace of the Wilson loop for closed contours. This fact involves two main steps:

i) All the observables are expressed via $\Phi(C)$.

ii) Dynamics is entirely reformulated in terms of $\Phi(C)$.

This approach is especially useful in the large-$N_c$ limit where everything is expressed via the vacuum expectation value of $\Phi(C)$ — the Wilson loop average. Observables are given by summing the Wilson loop average over paths with the same weight as in free theory. The Wilson loop average obeys itself a close functional equation — the loop equation.

We begin this Section with presenting the formulas which relate observables to the Wilson loops. Then we translate quantum equation of motion of Yang–Mills theory into loop space. We derive the closed equation for the Wilson loop average as $N_c \to \infty$ and discuss its various properties, including a non-perturbative regularization. Finally, we briefly comment on what is known about solutions of the loop equation.

4.1. OBSERVABLES IN TERMS OF WILSON LOOPS

All observables in QCD can be expressed via the Wilson loops $\Phi(C)$ defined by Eq. (120). This property was first advocated by Wilson [36] on a lattice. Calculation of QCD observables can be divided in two steps:

1) Calculation of the Wilson loop averages for arbitrary contours.

2) Summation of the Wilson loop averages over the contours with some weight depending on a given observable.

At finite $N_c$, observables are expressed via the $n$-loop averages

$$W_n(C_1, \ldots, C_n) = \langle \Phi(C_1) \cdots \Phi(C_n) \rangle,$$

which are analogous to the $n$-point Green functions for $\varphi^3$ theory. The appropriate formulas for the continuum theory can be found in Ref. [37].

Great simplifications occur in these formulas at $N_c = \infty$, when all observables are expressed only via the one-loop average

$$W(C) = \langle \Phi(C) \rangle \equiv \left\langle \frac{1}{N_c} \text{tr} \left( P e^{i \oint_C \! dx^\mu A_\mu} \right) \right\rangle.$$

This is associated with the quenched approximation.

For example, the average of the product of two colorless quark vector currents (101) is given at large $N_c$ by

$$\langle \bar{\psi} \gamma_\mu \psi(x_1) \bar{\psi} \gamma_\nu \psi(x_2) \rangle = \sum_{C \ni x_1, x_2} J_{\mu\nu}(C) \langle \Phi(C) \rangle,$$
Figure 14. Contours in the sum over paths representing observables: a) in Eq. (130) and b) in Eq. (131). The contour a) passes two nailed points $x_1$ and $x_2$. The contour b) passes three nailed points $x_1$, $x_2$, and $x_3$.

where the sum runs over contours $C$ passing through the points $x_1$ and $x_2$ as is depicted in Figure 14a. An analogous formula for the (connected) correlators of three quark scalar currents reads

$$\langle \bar{\psi}(x_1) \psi(x_2) \bar{\psi}(x_3) \rangle_{\text{conn}} = \sum_{C \ni x_1, x_2, x_3} J(C) \langle \Phi(C) \rangle,$$

where the sum runs over contours $C$ passing through the three points $x_1$, $x_2$, and $x_3$ as is depicted in Figure 14b. A general (connected) correlator of $n$ quark currents is given by a similar formula with $C$ passing through $n$ points $x_1, \ldots, x_n$ (some of them may coincide).

The weights $J_{\mu\nu}(C)$ in Eq. (130) and $J(C)$ in Eq. (131) are completely determined by free theory. If quarks were scalars rather than spinors, then we would get

$$J(C) = e^{-\frac{1}{2}m^2 r - \frac{1}{2} \int_0^\tau dt \, \dot{z}_\mu(t)} = e^{-m L(C)} \quad \boxed{\text{scalar quarks}},$$

where $L(C)$ is the length of the (closed) contour $C$. For spinor quarks, an additional disentangling of the gamma-matrices is needed (ee, e.g., Ref. [38]).

Remark on renormalization of Wilson loops

Perturbation theory for $W(C)$ can be obtained by expanding the path-ordered exponential in the definition (129) in $g$ (see Eq. (84)) and averaging over the gluon field $A_\mu$. Because of ultraviolet divergencies, we need a (gauge invariant) regularization. After such a regularization introduced, the Wilson loop average for a smooth contour $C$ of the type in Figure 15a reads

$$W(C) = e^{-g^2(N_v^2 - 1) L(C) \frac{a}{\alpha_s N_c}} W_{\text{ren}}(C),$$

where $a$ is the cutoff, $L(C)$ is the length of $C$, and $W_{\text{ren}}(C)$ is finite when expressed via the renormalized charge $g_R$. The exponential factor is due to
the renormalization of the mass of a heavy test quark. This factor does not emerge in the dimensional regularization where $d = 4 - \varepsilon$. The multiplicative renormalization of the smooth Wilson loop was shown in Refs. [39, 40, 41].

If the contour $C$ has a cusp (or cusps) but no self-intersections as is illustrated by Figure 15b, then $W(C)$ is still multiplicatively renormalizable [42]:

$$W(C) = Z(\gamma) W_{\text{ren}}(C),$$

while the (divergent) factor $Z(\gamma)$ depends on the cusp angle (or angles) $\gamma$ (or $\gamma'$s) and $W_{\text{ren}}(C)$ is finite when expressed via the renormalized charge $g_R$.

4.2. SCHWINGER–DYSON EQUATIONS FOR WILSON LOOP

Dynamics of (quantum) Yang–Mills theory is described by the quantum equation of motion

$$\nabla^a F_{\mu}^b(x) \overset{\text{w.s.}}{=} \hbar \frac{\delta}{\delta A_\mu^a(x)},$$

which is understood in the weak sense, i.e. for the averages

$$\left\langle \nabla^a F_{\mu}^b(x) \right\rangle = \hbar \left\langle \frac{\delta}{\delta A_\mu^a(x)} Q[A] \right\rangle.$$  (136)

The standard set of Schwinger–Dyson equations of Yang–Mills theory emerges when the functional $Q[A]$ is chosen in the form of the product of $A$'s as in Eq. (82).

Strictly speaking, the last statement is incorrect, since we have not added, in Eqs. (135) and (136), contributions coming from the variation of gauge-fixing and ghost terms in the Yang–Mills action. However, these two contributions are mutually cancelled for gauge-invariant functionals $Q[A]$. We shall deal below only with such gauge-invariant functionals (the Wilson loops). This is why we have not considered the contribution of the gauge-fixing and ghost terms.
It is also convenient to use the matrix notation (57), when Eq. (135) for the Wilson loop takes on the form

\[
\left\langle \frac{1}{N_c} \text{tr} P \nabla_\mu F_{\mu\nu}(x) e^{i \oint_C d\xi^\mu A_\mu} \right\rangle = \left\langle \frac{g^2}{2N_c} \text{tr} \frac{\delta}{\delta A_\nu(x)} P e^{i \oint_C d\xi^\mu A_\mu} \right\rangle,
\]

where we have restored the units with \(\hbar = 1\).

The variational derivative on the RHS can be calculated by virtue of the formula

\[
\frac{\delta A^{ij}_\mu(y)}{\delta A^kl_\nu(x)} = \delta_{\mu\nu} \delta^{(d)}(x-y) \left( \delta^{ij} \delta^{kl} - \frac{1}{N_c} \delta^{ij} \delta^{kl} \right)
\]

which is a consequence of

\[
\frac{\delta A^a_\mu(y)}{\delta A^b_\nu(x)} = \delta_{\mu\nu} \delta^{(d)}(x-y) \delta^{ab}.
\]

The second term in the parentheses in Eq. (138) — same as in Eq. (64) — is because \(A_\mu\) is a matrix from the adjoint representation of SU\((N_c)\).

By using Eq. (138), we get for the variational derivative on RHS of Eq. (137):

\[
\text{tr} \frac{\delta}{\delta A_\nu(x)} P e^{i \oint_C d\xi^\mu A_\mu} = i \oint_C dy_\nu \delta^{(d)}(x-y) \times
\]

\[
\left[ \frac{1}{N_c} \text{tr} P e^{i \oint_{C_{yx}} d\xi^\mu A_\mu} \frac{1}{N_c} \text{tr} P e^{i \oint_{C_{xy}} d\xi^\mu A_\mu} - \frac{1}{N_c^3} \text{tr} P e^{i \oint_{C} d\xi^\mu A_\mu} \right].
\]

The contours \(C_{yx}\) and \(C_{xy}\), which are depicted in Figure 16, are the parts of the loop \(C\): from \(x\) to \(y\) and from \(y\) to \(x\), respectively. They are always closed due to the presence of the delta-function. It implies that \(x\) and \(y\) should be the same points of space but not necessarily of the contour \(i.e.\) they may be associated with different values of the parameter \(\sigma\).
We finally rewrite Eq. (137) as
\[
\left\langle \frac{1}{N_c} \text{tr} \mathcal{P} \nabla_\mu F_{\mu\nu} (x) \ e^{i \oint_C d\xi A_\mu} \right\rangle = \ i \lambda \oint_C dy_\nu \delta^{(d)} (x - y) \left[ \langle \Phi (C_{yx}) \Phi (C_{xy}) \rangle - \frac{1}{N^2} \langle \Phi (C) \rangle \right]
\] (141)

where we have introduced
\[
\lambda = \frac{g^2 N_c}{2}.
\] (142)

Notice that the RHS of Eq. (141) is completely represented via the (closed) Wilson loops.

4.3. PATH AND AREA DERIVATIVES

As we already mentioned, the RHS of Eq. (141) is completely represented via the (closed) Wilson loops. It is crucial for the loop-space formulation of QCD that the LHS of Eq. (141) can also be represented in loop space as some operator applied to the Wilson loop. To do this we need to develop a differential calculus in loop space.

Loop space consists of arbitrary continuous closed loops, $C$. They can be described in a parametric form by the functions $x_\mu (\sigma) \in L_2^5$ where $\sigma_0 \leq \sigma \leq \sigma_f$ and $\mu = 1, \ldots, d$, which take on values in a $d$-dimensional Euclidean space. The functions $x_\mu (\sigma)$ can be discontinuous, generally speaking, for an arbitrary choice of the parameter $\sigma$. The continuity of the loop $C$ implies a continuous dependence on parameters of the type of proper length.

The functions $x_\mu (\sigma) \in L_2$ which are associated with the elements of loop space obey the following restrictions:

i) The points $\sigma = \sigma_0$ and $\sigma = \sigma_f$ are identified: $x_\mu (\sigma_0) = x_\mu (\sigma_f)$ — the loops are closed.

ii) The functions $x_\mu (\sigma)$ and $A_{\mu\nu} x_\nu (\sigma) + \alpha_\mu$, with $A_{\mu\nu}$ and $\alpha_\mu$ independent of $\sigma$, represent the same element of the loop space — rotational and translational invariance.

iii) The functions $x_\mu (\sigma)$ and $x_\mu (\sigma')$ with $\sigma' = f (\sigma), \ f' (\sigma) \geq 0$ describe the same loop — reparametrization invariance.

An example of functionals which are defined on the elements of loop space is the Wilson loop average (129) or, more generally, the $n$-loop average (128).

The differential calculus in loop space is built out of the path and area derivatives.

\footnote{Let us remind that $L_2$ stands for the Hilbert space of functions $x_\mu (\sigma)$ whose square is integrable over the Lebesgue measure: \( \int_{\sigma_0}^{\sigma_f} \ d\sigma x_\mu^2 (\sigma) < \infty. \)}
The area derivative of a functional $\mathcal{F}(C)$ is defined by the difference

$$\frac{\delta \mathcal{F}(C)}{\delta \sigma_{\mu\nu}(x)} \equiv \frac{1}{|\delta \sigma_{\mu\nu}|} \left[ \mathcal{F} \left( \begin{array}{c} x \ \mu \\ x \ \nu \end{array} \right) - \mathcal{F} \left( \begin{array}{c} x \ \mu \\ x \end{array} \right) \right]$$

(143)

where an infinitesimal loop $\delta C_{\mu\nu}(x)$ is attached to a given loop at the point $x$ in the $\mu\nu$-plane and $|\delta \sigma_{\mu\nu}|$ stands for the area enclosed by the $\delta C_{\mu\nu}(x)$. For a rectangular loop $\delta C_{\mu\nu}(x)$, one gets $\delta \sigma_{\mu\nu} = dx_{\mu} \wedge dx_{\nu}$, where the symbol $\wedge$ implies antisymmetrization.

Analogously, the path derivative is defined by

$$\partial_\mu \mathcal{F}(C_{xx}) \equiv \frac{1}{|\delta x_{\mu}|} \left[ \mathcal{F} \left( \begin{array}{c} x \ \mu \\ x \end{array} \right) - \mathcal{F} \left( \begin{array}{c} x \ \mu \\ x \end{array} \right) \right]$$

(144)

where $\delta x_{\mu}$ is an infinitesimal path along which the point $x$ is shifted from the loop and $|\delta x_{\mu}|$ stands for the length of the $\delta x_{\mu}$.

These two differential operations are well-defined for so-called functionals of the Stokes type which satisfy the backtracking condition — they do not change when an appendix passing back and forth is added to the loop at some point $x$:

$$\mathcal{F} \left( \begin{array}{c} x \ \mu \\ x \end{array} \right) = \mathcal{F} \left( \begin{array}{c} x \ \mu \\ x \end{array} \right).$$

(145)

This condition is equivalent to the Bianchi identity of Yang–Mills theory and is obviously satisfied by the Wilson loop (129) due to the properties of the non-Abelian phase factor. Such functionals are known in mathematics as Chen integrals.

A simple example of the Stokes functional is the area of the minimal surface, $A_{\text{min}}(C)$. It obviously satisfies Eq. (145). Otherwise, the length $L(C)$ of the loop $C$ is not a Stokes functional, since the lengths of contours on the LHS and RHS of Eq. (145) are different.

For the Stokes functionals, the variation on the RHS of Eq. (143) is proportional to the area enclosed by the infinitesimally small loop $\delta C_{\mu\nu}(x)$ and does not depend on its shape. Analogously, the variation on the RHS of Eq. (144) is proportional to the length of the infinitesimal path $\delta x_{\mu}$ and does not depend on its shape.
If \( x \) is a regular point (like any point of the contour for the functional (129)), the RHS of Eq. (144) vanishes due to the backtracking condition (145). In order for the result to be nonvanishing, the point \( x \) should be a marked (or irregular) point. A simple example of the functional with a marked point \( x \) is

\[
\Phi^a[C_{xx}] = \frac{1}{N_c} \text{tr} \left( t^a \mathbf{P} e^{i \int_{C_{xx}} d\xi^\mu A_\mu(\xi)} \right)
\]  

(146)

with the SU\((N_c)\) generator \( t^a \) inserted in the path-ordered product at the point \( x \).

The area derivative of the Wilson loop is given by the Mandelstam formula

\[
\frac{\delta}{\delta \sigma_{\mu\nu}(x)} \frac{1}{N_c} \text{tr} \mathbf{P} e^{i \int_{C_{xx}} d\xi^\mu A_\mu} = \frac{i}{N_c} \text{tr} \mathbf{P} F_{\mu\nu}(x) e^{i \int_{C_{xx}} d\xi^\mu A_\mu}.
\]

(147)

In order to prove it, it is convenient to choose \( \delta C_{\mu\nu}(x) \) to be a rectangle in the \( \mu\nu \)-plane and straightforwardly use the definition (143). The sense of Eq. (147) is very simple: \( F_{\mu\nu} \) is a curvature associated with the connection \( A_\mu \).

The functional on the RHS of Eq. (147) has a marked point \( x \), and is of the type in Eq. (146). When the path derivative acts on such a functional according to the definition (144), the result reads

\[
\frac{1}{N_c} \text{tr} \mathbf{P} B(x) e^{i \int_{C_{xx}} d\xi^\mu A_\mu} = \frac{1}{N_c} \text{tr} \mathbf{P} \nabla_\mu B(x) e^{i \int_{C_{xx}} d\xi^\mu A_\mu},
\]

(148)

where

\[
\nabla_\mu B = \partial_\mu B - i [A_\mu, B]
\]

(149)

is the covariant derivative in the adjoint representation.

Combining Eqs. (147) and (148), we finally represent the expression on the LHS of Eq. (137) (or Eq. (141)) as

\[
\frac{1}{N_c} \text{tr} \mathbf{P} \nabla_\mu F_{\mu\nu}(x) e^{i \int_{C_{xx}} d\xi^\mu A_\mu} = \frac{1}{N_c} \frac{\delta}{\delta \sigma_{\mu\nu}(x)} \frac{i}{N_c} \text{tr} \mathbf{P} e^{i \int_{C_{xx}} d\xi^\mu A_\mu},
\]

(150)

i.e. via the action of the path and area derivatives on the Wilson loop. It is therefore rewritten in loop space.

A résumé of the results of this subsection is presented in Table 2 as a vocabulary for translation of Yang–Mills theory from the language of ordinary space in the language of loop space.
Remark on Bianchi identity for Stokes functionals
The backtracking relation (145) can be equivalently represented as

\[ \epsilon_{\mu\nu\lambda\rho} \partial_\mu F_{\nu\lambda}(x) \frac{\delta}{\delta \sigma_{\mu\nu}(x)} \Phi(C) = 0, \]  

(151)

by choosing the appendix in Eq. (145) to be an infinitesimal straight line in the \( \rho \)-direction and geometrically applying the Stokes theorem. Using Eqs. (147) and (148), Eq. (151) can in turn be rewritten as

\[ \epsilon_{\mu\nu\lambda\rho} \frac{1}{N_c} \text{tr} \mathbf{P} \nabla_\mu F_{\nu\lambda}(x) e^{i \int_c A_\mu} = 0. \]  

(152)

Therefore, Eq. (151) represents the Bianchi identity in loop space.

Remark on relation to variational derivative
The standard variational derivative, \( \delta / \delta x_\mu(\sigma) \), can be expressed via the path and area derivatives by the formula

\[ \frac{\delta}{\delta x_\mu(\sigma)} = \dot{x}_\nu(\sigma) \frac{\delta}{\delta \sigma_{\mu\nu}(x(\sigma))} + \sum_{i=1}^{m} \partial_\mu^{x_i} \delta(\sigma - \sigma_i), \]  

(153)
where the sum on the RHS is present for the case of a functional having \( m \) marked (irregular) points \( x_i \equiv x(\sigma_i) \). A simplest example of the functional with \( m \) marked points is just a function of \( m \) variables \( x_1, \ldots, x_m \).

By using Eq. (153), the path derivative can be calculated as the limiting procedure

\[
\partial^{x(\sigma)}_\mu = \int_{\sigma - 0}^{\sigma + 0} d\sigma' \frac{\delta}{\delta x(\sigma')}.
\]

The result is obviously nonvanishing only when \( \partial^{x(\sigma)}_\mu \) is applied to a functional with \( x(\sigma) \) being a marked point.

It is nontrivial that the area derivative can also be expressed via the variational derivative [40]:

\[
\frac{\delta}{\delta \sigma_{\mu\nu}(x(\sigma))} = \int_{\sigma - 0}^{\sigma + 0} d\sigma' (\sigma' - \sigma) \frac{\delta}{\delta x(\sigma')} \frac{\delta}{\delta x(\sigma)}. \tag{155}
\]

The point is that the six-component quantity, \( \delta/\delta \sigma_{\mu\nu}(x(\sigma)) \), is expressed via the four-component one, \( \delta/\delta x(\sigma) \), which is possible because the components of \( \delta/\delta \sigma_{\mu\nu}(x(\sigma)) \) are dependent due to the loop-space Bianchi identity (151).

### 4.4. LOOP EQUATIONS

By virtue of Eq. (150), Eq. (141) can be represented completely in loop space:

\[
\partial^x_\mu \frac{\delta}{\delta \sigma_{\mu\nu}(x)} \left\langle \Phi(C) \right\rangle = \lambda \int_C dy_\nu \delta^{(d)}(x - y) \left\langle \left[ \Phi(C_{yx}) \Phi(C_{xy}) - \frac{1}{N_c^2} \Phi(C) \right] \right\rangle, \quad \tag{156}
\]

or, using the definitions (128) and (129) of the loop averages, as

\[
\partial^x_\mu \frac{\delta}{\delta \sigma_{\mu\nu}(x)} W(C) = \lambda \int_C dy_\nu \delta^{(d)}(x - y) \left[ W_2(C_{yx}, C_{xy}) - \frac{1}{N_c^2} W(C) \right]. \quad \tag{157}
\]

This equation is not closed. Having started from \( W(C) \), we obtain another quantity, \( W_2(C_1, C_2) \), so that Eq. (157) connects the one-loop average with a two-loop one. This is similar to the case of the (quantum) \( \phi^3 \)-theory, whose Schwinger–Dyson equations connect the \( n \)-point Green functions with different \( n \). We shall derive this complete set of equations for the \( n \)-loop averages in this Subsection later on.
However, the two-loop average factorizes in the large-$N_c$ limit:

$$W_2 (C_1, C_2) = W (C_1) W (C_2) + O \left( \frac{1}{N_c^2} \right),$$  \hspace{1cm} (158)

as was discussed in Subsection 3.4. Keeping the constant $\lambda$ (defined by Eq. (142)) fixed in the large-$N_c$ limit as is prescribed by Eq. (71), we get [35]

$$\frac{\partial^x}{\delta \sigma_{\mu \nu} (x)} W (C) = \lambda \oint_{C} dy_\nu \delta^{(d)}(x - y) W (C_{yx}) W (C_{xy})$$  \hspace{1cm} (159)

as $N_c \to \infty$.

Equation (159) is a closed equation for the Wilson loop average in the large-$N_c$ limit. It is referred to as the loop equation.

To find $W(C)$, Eq. (159) should be solved in the class of Stokes functionals with the initial condition

$$W(0) = 1$$  \hspace{1cm} (160)

for loops which are shrunk to points.

The factorization (158) can itself be derived from the chain of loop equations. Proceeding as before, we get

$$\frac{1}{\lambda} \frac{\partial^x}{\delta \sigma_{\mu \nu} (x)} W_n (C_1, \ldots, C_n)$$

$$= \oint_{C_1} dy_\nu \delta^{(d)}(x - y) \left[ W_{n+1} (C_{xy}, C_{yx}, \ldots, C_n) - \frac{1}{N_c^2} W_n (C_1, \ldots, C_n) \right]$$

$$+ \sum_{j \geq 2} \frac{1}{N_c^2} \oint_{C_j} dy_\nu \delta^{(d)}(x - y) \left[ W_{n-1} (C_1 C_j, \ldots, C_{j-1}, C_j, \ldots, C_n) - W_n (C_1, \ldots, C_n) \right].$$  \hspace{1cm} (161)

Here $x$ belongs to $C_1$; $C_1 C_j$ stands for the joining of $C_1$ and $C_j$; $C_j$ means that $C_j$ is omitted.

Equation (161) looks like the Schwinger–Dyson equation for the $\phi^3$-theory. Moreover, the number of colors $N_c$ enters Eq. (161) simply as a scalar factor $N_c^{-2}$, likewise Plank’s constant $\hbar$ enters in the $\phi^3$-theory. It is the major advantage of the use of loop space. What is said in Subsection 3.6 about the “semiclassical” nature of the $1/N_c$-expansion of QCD is explicitly realized in Eq. (161). Its expansion in $1/N_c$ is straightforward.

At $N_c = \infty$, Eq. (161) is simplified to

$$\frac{\partial^x}{\delta \sigma_{\mu \nu} (x)} W_n (C_1, \ldots, C_n) = \lambda \oint_{C_1} dy_\nu \delta^{(d)}(x - y) W_{n+1} (C_{yx}, C_{xy}, \ldots, C_n).$$  \hspace{1cm} (162)
This equation possesses a factorized solution

\[ W_n (C_1, \ldots, C_n) = W (C_1) \cdots W (C_n) + O \left( \frac{1}{N_c^2} \right) \]  

provided \( W (C) \) obeys Eq. (159) which plays the role of a “classical” equation in the large-\( N_c \) limit. Thus, we have given a non-perturbative proof of the large-\( N_c \) factorization of the Wilson loops.

4.5. RELATION TO PLANAR DIAGRAMS

The perturbation-theory expansion of the Wilson loop average can be calculated from Eq. (84) which we represent in the form

\[
W (C) = 1 + \sum_{n=2}^{\infty} i^n \oint_{C} dx_{1}^{\mu_1} \oint_{C} dx_{2}^{\mu_2} \cdots \oint_{C} dx_{n}^{\mu_n} \\
\times \theta_c (1, 2, \ldots, n) G^{(n)}_{\mu_1 \mu_2 \cdots \mu_n} (x_1, x_2, \ldots, x_n),
\]  

where \( \theta_c (1, 2, \ldots, n) \) orders the points \( x_1, \ldots, x_n \) along contour in the cyclic order and \( G^{(n)}_{\mu_1 \cdots \mu_n} \) is given by Eq. (82). This \( \theta \)-function has the meaning of the propagator of a test heavy particle which lives in the contour \( C \).

We assume, for definitiveness, the dimensional regularization throughout this Subsection to make all the integrals well-defined.

Each term on the RHS of Eq. (164) can be conveniently represented by the diagram in Figure 17, where the integration over the contour \( C \) is associated with each point \( x_i \) lying in the contour \( C \).

These diagrams are analogous to those discussed in Subsection 3.2 with one external boundary — the Wilson loop in the given case. In the large-\( N_c \)
Figure 18. Planar diagrams for $W(C)$: a) of order $\lambda$ with gluon propagator, and of order $\lambda^2$ b) with two noninteracting gluons and c) with the three-gluon vertex. Diagrams of order $\lambda^2$ with one-loop insertions to gluon propagator are not drawn.

limit, only planar diagrams survive. Some of them, which are of the lowest order in $\lambda$, are depicted in Figure 18.

The large-$N_c$ loop equation (159) describes the sum of the planar diagrams. Its iterative solution in $\lambda$ reproduces the set of planar diagrams for $W(C)$ provided the initial condition (160) and some boundary conditions for asymptotically large contours are imposed.

Equation (164) can be viewed as an ansatz for $W(C)$ with some unknown functions $G^{(n)}_{\mu_1...\mu_n}(x_1,...,x_n)$ to be determined by the substitution into the loop equation. To preserve symmetry properties of $W(C)$, the functions $G^{(n)}$ must be symmetric under a cyclic permutation of the points $1,...,n$ and depend only on $x_i - x_j$ (translational invariance). A main advantage of this ansatz is that it automatically corresponds to a Stokes functional, due to the properties of vector integrals, and the initial condition (160) is satisfied.

The action of the area and path derivatives on the ansatz (164) is easily calculable. For instance, the area derivative reads

$$\frac{\delta W(C)}{\delta \sigma_{\mu\nu}(z)} = \sum_{n=1}^{\infty} i^n \oint_{C} dx_1^{\mu_1} \cdots \oint_{C} dx_n^{\mu_n} \theta_c(1,2,...,n) \times \left[ \left( \partial_\mu^2 \delta_{\nu\alpha} - \partial_\nu^2 \delta_{\mu\alpha} \right) G^{(n+1)}_{\alpha\mu_1...\mu_n}(z,x_1,...,x_n) 
+ \left( \delta_{\mu\beta} \delta_{\nu\alpha} - \delta_{\mu\alpha} \delta_{\nu\beta} \right) G^{(n+2)}_{\alpha\beta\mu_1...\mu_n}(z,z,x_1,...,x_n) \right].$$

The analogy with the Mandelstam formula (147) is obvious.

More about solving the loop equation by the ansatz (164) can be found in Refs. [37, 43, 44].
4.6. LOOP-SPACE LAPLACIAN AND REGULARIZATION

The loop equation (159) is not yet entirely formulated in loop space. It is a $d$-vector equation whose both sides depend explicitly on the point $x$ which does not belong to loop space. The fact that we have a $d$-vector equation for a scalar quantity means, in particular, that Eq. (159) is overspecified.

A practical difficulty in solving Eq. (159) is that the area and path derivatives, $\delta/\delta \sigma_{\mu\nu}(x)$ and $\partial_\mu$, which enter the LHS are complicated, generally speaking, non-commutative operators. They are intimately related to the Yang–Mills perturbation theory where they correspond to the non-Abelian field strength $F_{\mu\nu}$ and the covariant derivative $\nabla_\mu$. However, it is not easy to apply these operators to a generic functional $W(C)$ which is defined on elements of loop space.

A much more convenient form of the loop equation can be obtained by integrating both sides of Eq. (159) over $dx_\nu$ along the same contour $C$, which yields

$$\oint_C dx_\nu \frac{\delta}{\delta \sigma_{\mu\nu}(x)} W(C) = \lambda \oint_C dx_\mu \oint_C dy_\mu \delta^{(d)}(x-y) W(C_{yx}) W(C_{xy}).$$  \hspace{1cm} (166)

Now both the operator on the LHS and the functional on the RHS are scalars without labeled points and are well-defined in loop space. The operator on the LHS of Eq. (166) can be interpreted as an infinitesimal variation of elements of loop space.

Equations (159) and (166) are completely equivalent. A proof of equivalence of scalar Eq. (166) and original $d$-vector Eq. (159) is based on the important property of Eq. (159) whose both sides are identically annihilated by the operator $\partial_\mu$. It is a consequence of the identity

$$\nabla_\mu \nabla_\nu F_{\mu\nu} = -\frac{i}{2} [F_{\mu\nu}, F_{\mu\nu}] = 0$$ \hspace{1cm} (167)

in the ordinary space. Due to this property, the vanishing of the contour integral of some vector is equivalent to vanishing of the vector itself, so that Eq. (159) can in turn be deduced from Eq. (166).

Equation (166) is associated with the so-called second-order Schwinger–Dyson equation

$$\int d^d x \nabla_\mu F_{\mu\nu}^a (x) \frac{\delta}{\delta A_\nu^a(x)} \overset{w.s.}{=} \hbar \int d^d x d^d y \delta^{(d)}(x-y) \frac{\delta}{\delta A_\nu^a(y)} \frac{\delta}{\delta A_\nu^a(x)}$$ \hspace{1cm} (168)

in the same sense as Eq. (159) is associated with Eq. (135). It is called “second order” since the RHS involves two variational derivatives with respect to $A_\nu$. 
The operator on the LHS of Eq. (166) is a well-defined object in loop space. When applied to regular functionals which do not have marked points, it can be represented, using Eqs. (154) and (155), in an equivalent form

\[ \Delta \equiv \oint_{\Sigma} \delta \mu(x) \frac{\partial}{\partial \sigma} \left( \int_{\sigma_0}^{\sigma} \frac{d\sigma'}{\delta x_\mu(\sigma')} \right) . \]  

(169)

As was first pointed out by Gervais and Neveu [45], this operator is nothing but a functional extension of the Laplace operator, which is known in mathematics as the Levy operator.\(^6\) Equation (166) can be represented in turn as an (inhomogeneous) functional Laplace equation

\[ \Delta W(C) = \lambda \oint_{\Sigma} \delta \mu(x) \oint_{\Sigma} \delta \mu(y) \delta^{(d)}(x - y) W(C_{yx}) W(C_{xy}) . \]  

(170)

We shall refer to this equation as the loop-space Laplace equation.

The form (170) of the loop equation is convenient for a non-perturbative ultraviolet regularization.

The idea is to start from the regularized version of Eq. (168), replacing the delta-function on the RHS by the kernel of the regularizing operator:

\[ \delta^{ab} \delta^{(d)}(x - y) \xrightarrow{\text{Reg.}} \left\langle \frac{y \left| R^{ab} \right| x}{x \left| R^{ab} \right| y} \right\rangle = \delta^{ab} \delta^{(d)}(x - y) \]  

(171)

with

\[ R^{ab} = \left( e^{a^2 \nabla^2/2} \right)^{ab} , \]  

(172)

where \( \nabla_\mu \) is the covariant derivative in the adjoint representation. The regularized version of Eq. (168) is

\[ \int d^d x \nabla_\mu F_{\mu\nu}^a (x) \frac{\delta}{\delta A_\nu^a (x)} \xrightarrow{\text{w.s.}} h \int d^d x d^d y \left\langle \frac{y \left| R^{ab} \right| x}{x \left| R^{ab} \right| y} \right\rangle \frac{\delta}{\delta A_\nu^b (y)} \frac{\delta}{\delta A_\mu^a (x)} . \]  

(173)

To translate Eq. (173) in loop space, we use the path-integral representation

\[ \left\langle \frac{y \left| R^{ab} \right| x}{x \left| R^{ab} \right| y} \right\rangle = \int Dr(t) e^{-\frac{1}{4} \int_0^{\tau} \partial_t^2 s^2(t)} 2 \text{tr} \left[ t^a U(r_{yx}) t^b U(r_{xy}) \right] \]  

(174)

with

\[ U(r_{yx}) = P e^{i \int_x r A_\mu^a (r)} , \]  

(175)

\(^6\)See the book by Levy [46] and a review [47].
Figure 19. Contours $C_{yx} r_{xy}$ and $C_{xy} r_{yx}$ which enter the RHS's of Eqs. (176) and (177). where the integration is over regulator paths $r_\mu(t)$ from $x$ to $y$ whose typical length is $\sim a$.

Calculating the variational derivatives on the RHS of Eq. (173), using Eq. (174) and the completeness condition (64), we get as $N \to \infty$:

$$\int d^d x d^d y \left< y \left| R^{ab} \right| x \right> \frac{\delta}{\delta A^b_\nu(y)} \frac{\delta}{\delta A^a_\mu(x)} \Phi(C) = \lambda \oint_C dx_\mu \oint_C dy_\mu \times \int_{r(0) = x}^{r(a^2) = y} Dr(t) e^{-\frac{1}{2} \int_0^{a^2} dt \dot{r}^2(t)} \Phi(C_{yx} r_{xy}) \Phi(C_{xy} r_{yx}), \quad (176)$$

where the contours $C_{yx} r_{xy}$ and $C_{xy} r_{yx}$ are depicted in Figure 19. Averaging over the gauge field and using the large-$N_c$ factorization, we arrive at the regularized loop-space Laplace equation [48]

$$\Delta W(C) = \lambda \oint_C dx_\mu \oint_C dy_\mu \int_{r(0) = x}^{r(a^2) = y} Dr(t) e^{-\frac{1}{2} \int_0^{a^2} dt \dot{r}^2(t)} W(C_{yx} r_{xy}) W(C_{xy} r_{yx}) \quad (177)$$

which manifestly recovers Eq. (170) when $a \to 0$.

The constructed regularization is non-perturbative while perturbatively reproduces regularized Feynman diagrams. An advantage of this regularization of the loop equation is that the contours $C_{yx} r_{xy}$ and $C_{xy} r_{yx}$ on the RHS of Eq. (177) both are closed and do not have marked points if $C$ does not have. Therefore, Eq. (177) is written entirely in loop space.

**Remark on functional Laplacian**

It is worth noting that the representation of the functional Laplacian on the RHS of Eq. (169), which involves the standard variational derivatives, is defined for a wider class of functionals than Stokes functionals. It is easier to deal with the whole operator $\Delta$, rather than separately with the area and path derivatives.
The functional Laplacian is parametric invariant and possesses a number of remarkable properties. While a finite-dimensional Laplacian is an operator of the second order, the functional Laplacian is that of the first order and satisfies the Leibnitz rule

\[ \Delta (UV) = \Delta (U) V + U \Delta (V). \] (178)

The functional Laplacian can be approximated [49] in loop space by a (second-order) partial differential operator in such a way to preserve these properties in the continuum limit. This loop-space Laplacian can be inverted to determine a Green function \( G(C, C') \) in the form of a sum over surfaces \( S_{C,C'} \) connecting two loops which is analogous to the sum-over-path representation of the Green function of the ordinary Laplacian. The standard perturbation theory can then be recovered by iterating Eq. (170) (or its regularized version (177)) in \( \lambda \) with the Green function of the loop-space Laplacian.

### 4.7. SURVEY OF NON-PERTURBATIVE SOLUTIONS

While the loop equations were proposed long ago, not much is known about their non-perturbative solutions. We briefly list some of the results.

It was shown in Ref. [50] that area law

\[ W(C) \equiv \langle \Phi(C) \rangle \propto e^{-K \cdot A_{\text{min}}(C)} \] (179)

satisfies the large-\( N_c \) loop equation for asymptotically large \( C \). However, a self-consistency equation for \( K \), which should relate it to the bare charge and the cutoff, was not investigated. In order to do this, one needs more detailed information about the behavior of \( W(C) \) for intermediate loops.

The free bosonic Nambu–Goto string which is defined as a sum over surfaces spanned by \( C \)

\[ W(C) = \sum_{S: \partial S = C} e^{-K \cdot A(S)}, \] (180)

with the action being the area \( A(S) \) of the surface \( S \), is not a solution for intermediate loops. Consequently, QCD does not reduce to this kind of string, as was originally expected in Refs. [51, 52, 53, 54, 55]. Roughly speaking, the ansatz (180) is not consistent with the factorized structure on the RHS of Eq. (159).

Nevertheless, it was shown that if a free string satisfies Eq. (159), then the same interacting string satisfies the loop equations for finite \( N_c \). Here “free string” means, as usual in string theory, that only surfaces of genus
zero are present in the sum over surfaces, while surfaces or higher genera are associated with a string interaction. The coupling constant of this interaction is \( O(N_c^{-2}) \).

A formal solution of Eq. (159) for all loops was found by Migdal [56] in the form of a fermionic string

\[
W(C) = \sum_{S: \partial S = C} \int D\psi e^{-\int d^2 \xi [\bar{\psi}\sigma_k \partial_k \psi + \bar{\psi}\psi m^4 \sqrt{g}]},
\]

(181)

where the world sheet of the string is parametrized by the coordinates \( \xi_1 \) and \( \xi_2 \) for which the 2-dimensional metric is conformal, i.e. diagonal. The field \( \psi(\xi) \) describes 2-dimensional elementary fermions (elves) living in the surface \( S \), and \( m \) stands for their mass. Elves were introduced to provide factorization which now holds due to some remarkable properties of 2-dimensional fermions. For large loops, the internal fermionic structure becomes frozen, so that the empty string behavior (179) is recovered. For small loops, the elves are necessary for asymptotic freedom. However, it is unclear whether or not the string solution (181) is practically useful for a study of multicolor QCD, since the methods of dealing with the string theory in four dimensions are not yet developed.

A very interesting solution of the large-\( N_c \) loop equation on a lattice was found by Eguchi and Kawai [57]. They showed that the SU(\( N_c \)) gauge theory on an infinite lattice reduces at \( N_c = \infty \) to the model on a hypercube. The equivalence is possible only at \( N_c = \infty \), when the space-time dependence is absorbed by the internal symmetry group. More about this large-\( N_c \) reduction will be said in the next Section.

4.8. WILSON LOOPS IN QCD

Two-dimensional QCD is popular since the paper by \( 't \) Hooft [58] as a simplified model of QCD.

One can always choose the axial gauge \( A_1 = 0 \), so that the commutator in the non-Abelian field strength (60) vanishes in two dimensions. Therefore, there is no gluon self-interaction in this gauge and the theory looks, at the first glance, like the Abelian one.

The Wilson loop average in QCD can be straightforwardly calculated via the expansion (164) where only disconnected (free) parts of the correlators \( G^{(n)} \) for even \( n \) should be left, since there is no interaction. Only the planar structure of color indices contributes at \( N_c = \infty \). Diagrammatically, the diagrams of the type depicted in Figure 18a and Figure 18b are relevant for contours without self-intersections, while that in Figure 18c should be omitted in two dimensions.
The color structure of the relevant planar diagrams can be reduced by using the completeness condition (64) at large $N_c$. We have

$$W(C) = 1 + \sum_{k} (-\lambda)^k \oint_{C} dx_1^{\mu_1} \oint_{C} dx_2^{\nu_1} \cdots \oint_{C} dx_{2k-1}^{\mu_k} \oint_{C} dx_{2k}^{\nu_k}$$

$$\times \theta_c(1,2,\ldots,2k) D_{\mu_1\nu_1}(x_1-x_2) \cdots D_{\mu_k\nu_k}(x_{2k-1}-x_{2k}),$$

where the points $x_1, \ldots, x_{2k}$ are still cyclic ordered along the contour. We can exponentiate the RHS of Eq. (182) to get finally

$$W(C) = e^{-\frac{1}{2} \oint_{C} dx \oint_{C} dy D_{\mu\nu}(x-y)}. \quad (183)$$

This is the same formula as in the Abelian case if $\lambda$ stands for $e^2$.

The propagator $D_{\mu\nu}(x,y)$ is, strictly speaking, the one in the gauge $A_1 = 0$ which reads

$$D_{\mu\nu}(x-y) = \frac{1}{2} \delta_{\mu 2} \delta_{\nu 2} |x_1-y_1| \delta^{(1)}(x_2-y_2). \quad (184)$$

However, the contour integral on the RHS of Eq. (183) is gauge invariant, and we can simply choose instead

$$D_{\mu\nu}(x-y) = \delta_{\mu\nu} \frac{1}{4\pi} \ln \frac{\ell^2}{(x-y)^2}, \quad (185)$$

where $\ell$ is an arbitrary parameter of the dimension of length. Nothing depends on it because the contour integral of a constant vanishes.

The contour integral in the exponent on the RHS of Eq. (183) can be graphically represented as is depicted in Figure 20, where $x_2 = y_2$ due to the delta-function in Eq. (184) and the bold line represents $|x_1-y_1|$. This gives

$$\oint_{C} dx^{\mu} \oint_{C} dy^{\nu} D_{\mu\nu}(x-y) = A(C) \quad (186)$$
where \( A(C) \) is the area enclosed by the contour \( C \). We get finally

\[
W(C) = e^{-\frac{1}{2}A(C)}
\]

(187)

for the contours without self-intersections.

Therefore, area law holds in two dimensions both in the non-Abelian and Abelian cases. This is, roughly speaking, because of the form of the two-dimensional propagator (185) which falls down with the distance only logarithmically in the Feynman gauge.

The difference between the Abelian and non-Abelian cases shows up for the contours with self-intersections.

We first note that the simple formula (186) does not hold for contours with arbitrary self-intersections.

The simplest contours with one self-intersection are depicted in Figure 21. There is nothing special about the contour in Figure 21a. Equation (186) still holds in this case with \( A(C) \) being the total area \( A(C) = A_1 + A_2 \).

The Wilson loop average for the contour in Figure 21a coincides both for the Abelian and non-Abelian cases and equals

\[
W(C) = e^{-\frac{1}{2}(A_1+A_2)}.
\]

(188)

This is nothing but the exponential of the total area.

For the contour in Figure 21b, we get

\[
\oint_C dx^\mu \oint_C dy^\nu D_{\mu\nu}(x-y) = A_1 + 4A_2.
\]

(189)

This is easy to understand in the axial gauge where the ends of the propagator line can lie both on the exterior and interior loops, or one end at the
exterior loop and the other end on the interior loop. These cases are illustrated by Figure 22. The contributions of the diagrams in Figure 22a,b,c,d are $A_1 + A_2$, $A_2$, $A_2$, and $A_2$, respectively. The result given by Eq. (189) is obtained by summing over all four diagrams.

For the contour in Figure 21b, the Wilson loop average is

$$W (C) = e^{-\frac{\lambda}{2}(A_1 + 4A_2)}$$

in the Abelian case and

$$W (C) = (1 - \lambda A_2) e^{-\frac{\lambda}{2}(A_1 + 2A_2)}$$

in the non-Abelian case at $N_c = \infty$. They coincide only to the order $\lambda$ as they should. The difference to the next orders is because only the diagrams with one propagator line connecting the interior and exterior loops are planar and, therefore, contribute in the non-Abelian case. Otherwise, the diagram is non-planar and vanishes as $N_c \to \infty$. Notice, that the exponential of the total area $A (C) = A_1 + 2A_2$ of the surface with the folding, which is enclosed by the contour $C$, appears in the exponent for the non-Abelian case. The additional pre-exponential factor could be associated with an entropy of foldings of the surface.

The Wilson loop averages (188) and (191) in QCD$_2$ at large $N_c$ as well as the ones for contours with arbitrary self-intersections, which have a generic form

$$W (C) = P (A_1, \ldots, A_n) e^{-\frac{\lambda}{2} \text{Area}}$$

where $P$ is a polynomial of the areas of individual windows and $\text{Area}$ is the total area of the surface with foldings, were first calculated in Ref. [59] by solving the two-dimensional loop equation and in Ref. [60] by applying the non-Abelian Stokes’ theorem.
Remark on the string representation
A nice property of QCD at large \( N_c \) is that the exponential of the area enclosed by the contour \( C \) emerges\(^7\) for the Wilson loop average \( W(C) \). This is as it should for the Nambu–Goto string (180). However, the additional pre-exponential factors (like that in Eq. (191)) are very difficult to interpret in the stringy language. They may become negative for large loops which is impossible for a bosonic string. This explicitly demonstrates in \( d = 2 \) the statement of the previous subsection that the Nambu–Goto string is not a solution of the large-\( N_c \) loop equation.

5. Large-\( N \) Reduction

The large-\( N_c \) reduction was first discovered by Eguchi and Kawai [57] who showed that the Wilson lattice gauge theory on a \( d \)-dimensional hypercubic lattice is equivalent at \( N_c = \infty \) to the one on a hypercube with periodic boundary conditions. This construction is based on an extra \((Z_{N_c})^d\) symmetry which the reduced theory possesses to each order of the strong coupling expansion.

Soon after it was recognized that a phase transition occurs in the reduced model with decreasing the coupling constant, so that this symmetry is broken in the weak coupling regime. To cure the construction at weak coupling, the quenching prescription was proposed by Bhanot, Heller and Neuberger [61] and elaborated by many authors. An elegant alternative reduction procedure based on twisting prescription was advocated by Gonzalez-Arroyo and Okawa [62]. Each of these prescriptions results in the reduced model which is fully equivalent to multicolor QCD, both on the lattice and in the continuum.

While the reduced models look as a great simplification, since the space-time is reduced to a point, they still involve an integration over \( d \) infinite matrices which is in fact a continual path integral. It is not clear at the moment whether or not this is a real simplification of the original theory which can make it solvable. Nevertheless, the reduced models are useful and elegant representations of the original theory at large \( N_c \).

We shall start this Section by a simplest example of a pure matrix scalar theory. The quenched reduced model for this case was proposed by Parisi [63] on the lattice end elaborated by Gross and Kitazawa [64] in the continuum, while the twisted reduced model was advocated by Eguchi and Nakayama [65]. Then we concentrate on the Eguchi–Kawai reduction of Yang–Mills theory.

\(^7\)This is not true, as is already discussed, in the Abelian case for contours with self-intersections.
5.1. REDUCTION OF SCALAR FIELD

Let us begin with the simplest example of a pure matrix scalar theory on a lattice whose partition function is defined by the path integral

\[ Z = \int \prod_x \prod_{i \geq j} d\varphi_{x}^{ij} e^{\sum_x N_c \text{tr} \left( -V[\varphi_x] + \sum_\mu \varphi_x \varphi_{x+a\hat{\mu}} \right)} \quad (193) \]

Here \( \varphi_x \) is a \( N_c \times N_c \) Hermitian matrix field with \( x \) running over sites of a hypercubic lattice and \( V[\varphi] \) is some interaction potential, say

\[ V[\varphi] = \frac{M}{2} \varphi^2 + \frac{\lambda_3}{3} \varphi^3 + \frac{\lambda_4}{4} \varphi^4. \quad (194) \]

The prescription of the large-\( N_c \) reduction is formulated as follows. We substitute

\[ \varphi_{x} \rightarrow S_x \Phi S_x^\dagger \quad (195) \]

where

\[ [S_x]^{kj} = e^{ip^k_x \delta S x^\mu} = \text{diag} \left( e^{i p^1_x}, \ldots, e^{i p^{N_c}_x} \right) \quad (196) \]

is a diagonal unitary matrix which eats the coordinate dependence, so that \( \Phi \) does not depend on \( x \).

The averaging of a functional \( F[\varphi_x] \) which is defined with the same weight as in Eq. (193),

\[ \langle F[\varphi_x] \rangle \equiv \frac{1}{Z} \int \prod_x d\varphi_x e^{\sum_x N_c \text{tr} \left( -V[\varphi_x] + \sum_\mu \varphi(x) \varphi(x+a\hat{\mu}) \right)} F[\varphi_x], \quad (197) \]

can be calculated at \( N_c = \infty \) by

\[ \langle F[\varphi_x] \rangle \rightarrow a^{N_c d} \int \prod_{-\pi}^{\pi} \prod_{i=1}^{N_c} \frac{dp^\mu_i}{2\pi} \langle F[S_x \Phi S_x^\dagger] \rangle_{\text{Reduced}} \quad (198) \]

where the average on the RHS is calculated [63] for the quenched reduced model whose averages are defined by

\[ \langle F[\Phi] \rangle_{\text{Reduced}} \equiv \frac{1}{Z_{\text{Reduced}}} \int \prod_i d\Phi_{ij} e^{-N_c \text{tr} V[\Phi] + N_c \sum_{ij} |\Phi_{ij}|^2 \sum_\mu \cos \left( (p^\mu_i - p^\mu_j) a \right)} F[\Phi]. \quad (199) \]

The partition function of the reduced model reads

\[ Z_{\text{Reduced}} = \int \prod_{i \geq j} d\Phi_{ij} e^{-N_c \text{tr} V[\Phi] + N_c \sum_{ij} |\Phi_{ij}|^2 \sum_\mu \cos \left( (p^\mu_i - p^\mu_j) a \right)} \quad (200) \]
which can be deduced, modulo the volume factor, from the partition function (193) by the substitution (195).

Notice that the integration over the momenta $p_i^\mu$ on the RHS of Eq. (198) is taken after the calculation of averages in the reduced model. Such variables are usually called quenched in statistical mechanics which clarifies the terminology.

Since $N_c \to \infty$ it is not necessary to integrate over the quenched momenta in Eq. (198). The integral should be recovered if $p_i^\mu$’s would be uniformly distributed in a $d$-dimensional hypercube. Moreover, a similar property holds for the matrix integral over $\Phi$ as well, which can be substituted by its value at the saddle point configuration $\Phi_s$:

$$\left\langle F[\varphi_{x1}] \right\rangle \to F[S_x \Phi_s S_x^\dagger],$$

where the momenta $p_i^\mu$ are uniformly distributed in the hypercube. Therefore, this saddle point configuration plays the role of a master field in the sense of Subsection 3.5.

In order to show how Eq. (198) works, let us demonstrate how the planar diagrams of perturbation theory for the scalar matrix theory (193) are recovered in the quenched reduced model.

The quenched reduced model (200) is of the general type discussed in Section 3. The propagator is given by

$$\left\langle \Phi_{ij} \Phi_{kl} \right\rangle_{\text{Gauss}} = \frac{1}{N_c} G(p_i - p_j) \delta_{il} \delta_{kj}$$

with

$$G(p_i - p_j) = \frac{1}{M - \sum_{\mu} \cos \left( (p_i^\mu - p_j^\mu) a \right)}.$$

It is convenient to associate the momenta $p_i$ and $p_j$ in Eq. (203) with each of the two index lines representing the propagator and carrying, respectively, indices $i$ and $j$. Remember, that these lines are oriented for a Hermitean matrix $\Phi$ and their orientation can be naturally associated with the direction of the flow of the momentum. The total momentum carried by the double line is $p_i - p_j$.

The simplest diagram which represents the correction of the second order in $\lambda_3$ to the propagator is depicted Figure 23. The momenta $p_i$ and $p_j$ flows along the index lines $i$ and $j$ while the momentum $p_k$ circulates along the index line $k$. The contribution of the diagram in Figure 23 reads

$$\frac{\lambda_3^2}{N_c^2} G(p_i - p_j)^2 \sum_k G(p_i - p_k) G(p_k - p_j),$$
Figure 23. Simplest planar diagram of the second order in $\lambda_3$ for the propagator in the quenched reduced model (200). The momentum $p_i$ flows along the index line $i$. The momentum $p_i - p_j$ is associated with the double line $ij$.

where the summation over the index $k$ is just a standard one over indices forming a closed loop.

In order to show that the quenched-model result (204) reproduces the correction to the propagator in the original theory on an infinite lattice, we pass to the variables of the total momenta flowing along the double lines:

$$p_i - p_j = p; \quad p_k - p_j = q; \quad p_i - p_k = p - q,$$

(205)

which is obviously consistent with the momentum conservation at each of the two vertices of the diagram in Figure 23. Since $p_k$'s are uniformly distributed in the hypercube, the summation over $k$ can be substituted as $N_c \to \infty$ by the integral

$$\frac{1}{N_c} \sum_k f(p_k) \Rightarrow a^d \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^d q}{(2\pi)^d} f(q).$$

(206)

The prescription (198) then gives the correct expression

$$a^d \frac{\lambda_3^2}{N_c} G(p)^2 \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^d q}{(2\pi)^d} G(q) G(p-q)$$

(207)

for the second-order contribution of the perturbation theory for the propagator on the lattice.

It is now clear how a generic planar diagram is recovered by the reduced model. We first represent the diagram by the double lines and associate the momentum $p_i^\mu$ with an index line carrying the index $i$. Then we write down the expression for the diagram in the reduced model with the propagator (203). Passing to the momenta flowing along the double lines, similar to Eq. (205), we get an expression which coincides with the integrand of the Feynman diagram for the theory on the whole lattice. It is crucial that such
a change of variables can always be done for a planar diagram consistently with the momentum conservation at each vertex. The last step is that the summation over indices of closed index lines reproduces the integration over momenta associated with each of the loops according to Eq. (206). It is assumed that the number of loops is much less than \( N_c \) which is always true for a given diagram since \( N_c \) is infinite.

We thus have shown how planar diagrams of the lattice theory defined by the partition function (193) are recovered by the reduced model (200). The lattice was needed only as a regularization to make all integrals well-defined and was not crucial in the consideration. This construction can be formulated directly for the continuum theory \([64, 66]\) where the propagator turns into

\[
G(p_i - p_j) = \frac{1}{(p_i - p_j)^2 + m^2} \tag{208}
\]

and a Lorentz-invariant regularization can be achieved by choosing \( p^2 < \Lambda^2 \).

**Remark on the twisted reduced model**

An alternative reduction procedure is based on the twisting prescription \([62]\). We again perform the unitary transformation (195) with the matrices \( S_x \) being expressed via a set of \( d \) (unitary) \( N_c \times N_c \) matrices \( \Gamma_\mu \) by

\[
S_x = \Gamma_{x_1/a} \Gamma_{x_2/a} \Gamma_{x_3/a} \Gamma_{x_4/a} \tag{209}
\]

where the coordinates of the (lattice) vector \( x_\mu \) are measured in the lattice units. The matrices \( \Gamma_\mu \) are explicitly constructed in Ref. \([62]\) and commute by

\[
\Gamma_\mu \Gamma_\nu = Z_{\mu\nu} \Gamma_\nu \Gamma_\mu \tag{210}
\]

with \( Z_{\mu\nu} = Z_{\nu\mu}^\dagger \) being elements of \( Z_{N_c} \).

For the twisting reduction prescription, Eq. (198) is valid providing the average on the RHS is calculated for the **twisted reduced model** which is defined by the partition function \([65]\)

\[
Z_{TRM} = \int d\Phi e^{-N_c \text{tr} V[\Phi] + N_c \sum_\mu \text{tr} \Gamma_\mu \Phi \Gamma_\mu^\dagger \Phi} . \tag{211}
\]

We can change the order of \( \Gamma \)'s in Eq. (209) defining a more general path-dependent factor

\[
S_x = \text{P} \prod_{l \in C_{x\infty}} \Gamma_\mu \tag{212}
\]

The path-ordered product in this formula runs over all links \( l = (z, \mu) \) forming a path \( C_{x\infty} \) from infinity to the point \( x \).
Due to Eq. (210), changing the form of the path multiplies \( S_x \) by the Abelian factor

\[
Z(C) = \prod_{\Box \in S; \partial S = C} Z_{\mu \nu}(\Box)
\]  

(213)

where \((\mu, \nu)\) is the orientation of the plaquette \( \Box \). The product runs over any surface spanned by the closed loop \( C \) which is obtained by passing the original path forward and the new path backward. Due to the Bianchi identity

\[
\prod_{\Box \in \text{cube}} Z_{\mu \nu}(\Box) = 1
\]  

(214)

where the product goes over six plaquettes forming a 3-dimensional cube on the lattice, the product on the RHS of Eq. (213) does not depend on the form of the surface \( S \) and is a functional of the loop \( C \).

It is now easy to see that under this change of the path we get

\[
[S_x]_{ij} [S_x^\dagger]_{kl} \rightarrow |Z(C)|^2 [S_x]_{ij} [S_x^\dagger]_{kl}
\]  

(215)

and the path-dependence is canceled because \( |Z(C)|^2 = 1 \). This is a general property which holds for the twisting reduction prescription of any even \((i.e.\) invariant under the center \( Z_{N_c} \)) representation of \( SU(N_c) \).

5.2. REDUCTION OF YANG–MILLS FIELD

The statement of the Eguchi–Kawai reduction of the Yang–Mills field says that the theory on a \( d \)-dimensional space-time is equivalent at \( N_c = \infty \) to the reduced model which is nothing but its reduction to a point. The action of the reduced model is given by

\[
S_{\text{EK}} = \frac{1}{2g^2\Lambda^d} \tr [A_\mu, A_\nu]^2,
\]  

(216)

where \( A_\mu \) are \( d \) space-independent matrices and \( \Lambda \) is a dimensionful parameter.

A naive statement of the Eguchi–Kawai reduction is that the averages coincide in both theories, for example,

\[
\left\langle \frac{1}{N_c} \tr P e^{i \oint d \xi^\mu A_\mu(\xi)} \right\rangle_{d-\text{dim}} = \left\langle \frac{1}{N_c} \tr P e^{i \oint d \xi^\mu A_\mu} \right\rangle_{\text{EK}}
\]  

(217)

where the LHS is calculated with the action (59) and the RHS is calculated with the reduced action (216). Strictly speaking, this naive statement is valid only in \( d = 2 \) or supersymmetric case for the reason which will be explained in a moment.
The precise equivalence is valid only if the average of open Wilson loops vanish in the reduced model:

$$\left\langle \frac{1}{N_c} \text{tr} \mathbf{P} e^{i \int_{C_{yx}} d\xi^\mu A_\mu} \right\rangle_{\text{EK}} = 0,$$

(218)

as it does in the $d$-dimensional theory due to the local gauge invariance under which

$$\left( \mathbf{P} e^{i \int_{C_{yx}} d\xi^\mu A_\mu} \right)_{ij} \rightarrow \left( \Omega^\dagger(y) \mathbf{P} e^{i \int_{C_{yx}} d\xi^\mu A_\mu} \Omega(x) \right)_{ij}.$$

(219)

The point is that this gauge invariance transforms in the reduced model into (global) rotation of the reduced field by constant matrices $\Omega$:

$$A_\mu \rightarrow \Omega^\dagger A_\mu \Omega.$$

(220)

which does not guarantee such vanishing in the reduced model.

There exists, however, a symmetry of the reduced action (216) under the shift of $A_\mu$ by a unit matrix$^8$:

$$A_\mu^{ij} \rightarrow A_\mu^{ij} + a_\mu \delta^{ij},$$

(221)

which is often called the $\mathbb{R}^d$-symmetry. Under the transformation (221), we get

$$\left( \mathbf{P} e^{i \int_{C_{yx}} d\xi^\mu A_\mu} \right)_{ij} \rightarrow e^{i(y^\mu - x^\mu)a_\mu} \left( \mathbf{P} e^{i \int_{C_{yx}} d\xi^\mu A_\mu} \right)_{ij}$$

(222)

which guarantees, if the symmetry is not broken, the vanishing of the open Wilson loops

$$W_{\text{EK}}(C_{yx}) \equiv \left\langle \frac{1}{N_c} \text{tr} \mathbf{P} e^{i \int_{C_{yx}} d\xi^\mu A_\mu} \right\rangle_{\text{EK}} = 0$$

(223)

in the reduced model.

The equivalence of the two theories can then be shown using the loop equation which reads for the reduced model

$$\partial_x^\mu \frac{\delta}{\delta A_\mu(x)} W_{\text{EK}}(C) = \left\langle \frac{1}{N_c} \text{tr} \mathbf{P} [A_\mu, [A_\mu, A_\nu]] e^{i \int_{C_{xx}} d\xi^\mu A_\mu} \right\rangle_{\text{EK}}$$

$$= \lambda \Lambda^d \left\langle \frac{1}{N_c} \text{tr} \frac{\partial}{\partial A_\nu} e^{i \int_{C_{xx}} d\xi^\mu A_\mu} \right\rangle_{\text{EK}}$$

$$= \lambda \Lambda^d \oint_{\mathcal{C}} dy_\nu W_{\text{EK}}(C_{yx}) W_{\text{EK}}(C_{xy}).$$

(224)

$^8$This symmetry is rigorously defined on a lattice where it is associated with a direction-dependent $\mathbb{Z}_{N_c}$ transformation.
The RHS is pretty much similar to the one in Eq. (159) while $\delta^{(d)}(x-y)$ is missing.

This delta function can be recovered if the $R^d$ symmetry is not broken since

$$W_{EK}(C_{yx}) \sim \frac{\delta^{(d)}(x-y)}{\delta^{(d)}(0)} W_{EK}(C_{yx}) \quad (225)$$

due to Eq. (223) for the open loops.

This is not a rigorous argument since a regularization is needed. What actually happens is the following. If we smear the delta function introducing

$$\delta^{(d)}_{\Lambda}(x) = \left( \frac{\Lambda}{\sqrt{2\pi}} \right)^d e^{-x^2 \Lambda^2 / 2} , \quad (226)$$

then

$$\frac{1}{\delta^{(d)}_{\Lambda}(0)} \left( \delta^{(d)}_{\Lambda}(0) \right)^2 \propto \Lambda^d e^{-x^2 \Lambda^2} \rightarrow \delta^{(d)}(x) , \quad (227)$$

reproducing the delta function.

5.3. $R^d$-SYMMETRY IN PERTURBATION THEORY

Since $N_c$ is infinite, the $R^d$-symmetry can be broken spontaneously. The point is that the large-$N_c$ limit plays the role of a statistical averaging, as is mentioned already in Subsection 3.6, and phase transitions are possible for infinite number of degrees of freedom. This phenomenon occurs in perturbation theory of the reduced model for $d \geq 3$.

The perturbation theory can be constructed expanding the fields around solutions of the classical equation

$$[A_\mu, [A_\mu, A_\nu]] = 0 . \quad (228)$$

Any diagonal matrix

$$A^{cl}_\mu \equiv p_\mu = \text{diag} \left\{ p^{(1)}_\mu, \ldots, p^{(N_c)}_\mu \right\} \quad (229)$$

is a solution to Eq. (228).

The perturbation theory of the reduced model can be constructed expanding around the classical solution (229):

$$A_\mu = A^{cl}_\mu + g A^q_\mu , \quad (230)$$

where $A^q_\mu$ is off-diagonal.

Substituting (230) into the action (216), we get

$$S_{EK} = \text{tr} \left\{ \frac{1}{2} [p_\mu, A^q_\nu]^2 - \frac{1}{2} [p_\mu, A^q_\mu]^2 \right\} + \text{higher orders} . \quad (231)$$
To fix the gauge symmetry (220), it is convenient to add

$$S_{g.f.} = \text{tr} \left\{ \frac{1}{2} [p_{\mu}, A_{\mu}^q]_2^2 + [p_{\mu}, b][p_{\mu}, c] \right\}, \quad (232)$$

where $b$ and $c$ are ghosts.

The sum of (231) and (232) gives

$$S_2 = \text{tr} \left\{ \frac{1}{2} [p_{\mu}, A_{\mu}^q]_2^2 + [p_{\mu}, b][p_{\mu}, c] \right\}$$

up to quadratic order in $A_{\mu}^q$.

Doing the Gaussian integral over $A_{\mu}^q$, we get at the one-loop level:

$$\int dp_{\mu} A_{\mu}^q e^{-S_2} \ldots = \int \prod_{k=1}^N dp_{\mu}^{(k)} \prod_{i<j} \left[ (p_{\mu}^{(i)} - p_{\mu}^{(j)})^2 \right]^{1-d/2} \ldots , \quad (234)$$

where the integration over $p_{\mu}$ accounts for equivalent classical solutions.

For $d = 1$ the product on the RHS of Eq. (234) reproduces the Vandermonde determinant. For $d = 2$ it vanishes and does not affect dynamics. For $d \geq 3$ the measure is singular and the eigenvalues collapse. This leads us to a spontaneous breakdown of the $R_d$ in perturbation theory.

The equivalence between the $N_c = \infty$ Yang–Mills theory on a whole space and the reduced model can be provided [61] introducing a quenching prescription similar to the one described in Subsection 5.1. Then no collapse of eigenvalues happens and $d$-dimensional planar graphs are reproduced by the reduced model. More about the quenching prescription in Yang–Mills theory can be found in the reviews [44, 67] and cited there original papers.

Remark on supersymmetric case
In a supersymmetric gauge theory, there is an extra contribution from fermions to the exponent on the RHS of Eq. (234). Since the integration over fermions results in the extra factor $[(p_{\mu}^{(i)} - p_{\mu}^{(j)})^2]^{\text{tr} I/2}$, this yields finally the exponent $1 - d/2 + \text{tr} I/2$. It vanishes in $d = 4$ for either Majorana or Weyl fermions and in $d = 10$ for the Majorana–Weyl fermions. Therefore, the $R_d$-symmetry is not broken and no quenching is needed in the supersymmetric case [68, 5].

5.4. TWISTED REDUCED MODEL
The continuum version of the twisted reduced model can be constructed [69] by substituting $A_\mu \rightarrow A_\mu - \gamma_\mu$ into the action (216), where the matrices $\gamma_\mu$ obey the commutation relation

$$[\gamma_\mu, \gamma_\nu] = B_{\mu\nu} I , \quad (235)$$
where $B_{\mu\nu}$ is an antisymmetric tensor and $d$ is even. This is possible only for infinite Hermitean matrices (operators). An example of such matrices is $x$ and $p$ operators in quantum mechanics. Eq. (235) is a continuum version of Eq. (210).

The Wilson loop averages in the twisted reduced model are defined by

$$ W_{\text{TEK}}(C_{yx}) = \left\langle \frac{1}{N_c} \text{tr} P e^{-i \int_{C_{yx}} d\gamma^\mu \gamma_\mu} \frac{1}{N_c} \text{tr} P e^{i \int_{C_{yx}} d\gamma^\mu A_\mu} \right\rangle_{\text{TEK}}. $$

They vanish for open loops which is provided by the vanishing of the trace of the path-ordered exponential of $\gamma_\mu$ in this definition. For closed loops this factor does not vanish and is needed to provide the equivalence with $d$-dimensional Yang–Mills perturbation theory, since the classical extrema of the twisted reduced model are $A_\mu^d = \gamma_\mu$ and the perturbation theory is constructed expanding around this classical solution.

The proof of the equivalence can be done using the loop equation quite similarly to that of Subsection 5.2 for the Eguchi–Kawai model with an unbroken $R^d$ symmetry.

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