Global existence and large time behavior of weak solutions to the two-phase flow

Ya-Ting Wang *a,b and Ling-Yun Shou †a,b

*School of Mathematical Sciences, Capital Normal University, Beijing 100048, P.R. China
†Academy for Multidisciplinary Studies, Capital Normal University, Beijing 100048, P.R. China

Abstract

In this paper, we consider a two-phase flow model consisting of the compressible Navier-Stokes equations with degenerate viscosity coupled with the compressible Navier-Stokes equations with constant viscosities via a drag force, which can be derived from Chapman-Enskog expansion for the compressible Navier-Stokes-Vlasov-Fokker-Planck system. For general initial data, we establish the global existence of weak solutions with finite energy to the initial value problem in the three-dimensional periodic domain, and prove the convergence of global weak solutions to its equilibrium state as the time tends to infinity.

Key words: Two-phase flow, Compressible Navier-Stokes equations, Weak solutions, Global existence, Large time behavior

1 Introduction

The two-phase flow models can simulate a variety of physical phenomena describing the mixture of two different flows with appropriate interactions, and play an important role in many applied scientific areas, such as nuclear, chemical-process, petroleum, cryogenic, bio-medical, oil-and-gas, microtechnology, and so on [2, 15, 21, 24, 51]. In the present paper, we consider the initial value problem (IVP) for the
following two-phase flow model in the periodic domain $T^3 := \mathbb{R}^3/\mathbb{Z}^3$:

\[
\begin{aligned}
&n_t + \text{div}(nv) = 0, \\
&(nv)_t + \text{div}(nv \otimes v) + \nabla n = -\kappa n(v - u) + \eta \text{div}(n\mathbb{D}(v)), \\
&\rho_t + \text{div}(\rho u) = 0, \\
&(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P(\rho) = \kappa n(v - u) + \mu \Delta u + (\mu + \lambda) \nabla \text{div} u,
\end{aligned}
\]

(1.1)

with the initial data

\[
(n, nv, \rho, \rho u)(x,0) = (n_0, m_0, \rho_0, \tilde{m}_0)(x), \quad x \in T^3,
\]

(1.2)

where $n = n(x, t) \geq 0$ and $v = v(x, t) \in \mathbb{R}^3$ denote the density and velocity of compressible Navier-Stokes equations $(1.1)_1-(1.1)_2$ with degenerate viscosity, and $\rho = \rho(x, t) \geq 0$ and $u = u(x, t) \in \mathbb{R}^3$ stand for the density and velocity of compressible Navier-Stokes equations $(1.1)_3-(1.1)_4$ with constant viscosities, $\kappa n(v - u)$ is the drag force term. $\mathbb{D}(v) := \frac{\nabla v \otimes (\nabla v)^T}{2}$ is the deformation tensor. The coefficients $\kappa$, $\eta$, $\mu$ and $\lambda$ are constants satisfying

\[
\kappa > 0, \quad \eta > 0, \quad \mu > 0, \quad 2\mu + \lambda > 0.
\]

The pressure $P(\rho)$ takes the form

\[
P(\rho) = A \rho^\gamma,
\]

with $\gamma > \frac{3}{2}$ the adiabatic exponent and $A > 0$ a constant.

The two-phase flow model (1.1) can be derived from the compressible Navier-Stokes-Vlasov-Fokker-Planck equations with a local alignment force as follows

\[
\begin{aligned}
f_t + \xi \cdot \nabla_x f &= \text{div}_\xi((\xi - u)f) + \frac{1}{\eta} \text{div}_\xi((\xi - v) + \nabla_x f), \\
\rho_t + \text{div}_x(\rho u) &= 0, \\
(\rho u)_t + \text{div}_x(\rho u \otimes u) + \nabla_x P(\rho) &= \kappa n(v - u) + \mu \Delta_x u + (\mu + \lambda) \nabla_x \text{div}_x u,
\end{aligned}
\]

(1.3)

where $f$ is the distribution function associated with the particles, and $n$ and $nv$ are the macroscopical density and momentum defined by

\[
n(x, t) := \int_{\mathbb{R}^3} f(x, \xi, t)d\xi, \quad nv(x, t) := \int_{\mathbb{R}^3} \xi f(x, \xi, t)d\xi.
\]

The fluid-particle system (1.3) arises in modelling the sedimentation of suspensions, sprays, combustion \cite{22, 41, 47, 48}, etc. Recently, Li-Wang-Wang \cite{31} applied the Chapman-Enskog expansion for the fluid-particle model (1.3) around the local Maxwellian

\[
\frac{n(x, t)}{\sqrt{(2\pi)^3}} e^{-\frac{||\xi - (x, t)||^2}{2}}.
\]
to obtain the two-phase flow equations
\[
\begin{aligned}
& n_t + \text{div}_x(nv) = 0, \\
& (nv)_t + \text{div}_x(nv \otimes v) + \nabla_x n = -\kappa n(v - u) + \eta \text{div}_x(n\mathbb{D}_x(v)) - \eta \int_{\mathbb{R}^3} \xi \otimes \xi \cdot \nabla_x \Pi d\xi, \\
& \rho_t + \text{div}(\rho u) = 0, \\
& (\rho u)_t + \text{div}(\rho u \otimes u) + \nabla_x P(\rho) = \kappa n(v - u) + \mu \Delta_x u + (\mu + \lambda)\nabla_x \text{div}_x u,
\end{aligned}
\]  
(1.4)

where the term $\Pi$ is governed by the microscopic part.

In the limiting case $\eta \to 0$, the system (1.4) converges to the coupled Euler-Navier-Stokes two-phase flow model
\[
\begin{aligned}
& n_t + \text{div}(nv) = 0, \\
& (nv)_t + \text{div}(nv \otimes v) + \nabla n = -\kappa n(v - u), \\
& \rho_t + \text{div}(\rho u) = 0, \\
& (\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P(\rho) = \kappa n(v - u) + \mu \Delta u + (\mu + \lambda)\nabla \text{div}_x u, \quad x \in \mathbb{T}^3, \ t > 0.
\end{aligned}
\]  
(1.5)

When the microscopic effect in (1.4) is considered to be suitably small, we can derive the two-phase flow model (1.1) as a suitable approximate system of both (1.3) and (1.5). To our best knowledge, there are few results available on mathematical analysis for the two-phase flow model (1.1). In [32, 33], the authors showed the existence and nonlinear stability of stationary solutions to the inflow/outflow problem in a half line.

The coupled Euler-Navier-Stokes two-phase flow model (1.5) has been derived rigorously in [13] as a fluid-dynamical limit of the compressible Navier-Stokes-Vlasov-Fokker-Planck equations (1.3). Global well-posedness and time-decay rates of strong solutions to (1.5) for initial data close to a constant equilibrium state have been investigated both in Sobolev spaces [12, 26, 49] and in critical Besov spaces [30]. In addition, there are many important works on other related two-phase flow models; refer to [6–8, 39, 46] and references therein.

The two-phase flow system (1.1) for $\kappa = 0$ can reduce to the compressible Navier-Stokes equations
\[
\begin{aligned}
& \rho_t + \text{div}(\rho u) = 0, \\
& (\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P(\rho) = \text{div}(\mu(\rho)\nabla u + \lambda(\rho) \nabla u). 
\end{aligned}
\]  
(1.6)

We would like to mention some important progress made recently on the existence of global weak solutions with large aptitude for the compressible Navier-Stokes equations (1.6). In the case that both $\mu(\rho)$ and $\lambda(\rho)$ are constants, Lions first established the global existence of weak solutions to the $d$-dimensional ($d \geq 2$) compressible Navier-Stokes equations (1.6) for general initial data, where the pressure satisfies the $\gamma$-law $P(\rho) = \rho^\gamma$ with $\gamma \geq \frac{d\mu}{d+2}$ ($d = 2, 3$) and $\gamma > \frac{d}{2}$ ($d \geq 4$). The range for the adiabatic constant has been relaxed to any $\gamma > \frac{d}{2}$ by Feireisl-Novotný-Petzeltová [16], any $\gamma > 1$ by Jiang-Zhang [25] if the
initial data is spherically symmetric, and \( \gamma = 1 \) by Plotnikov-Weigant [43] in the two-dimensional case. Indeed, Hu [23] studied the concentration phenomenon of the convective term \( \rho u \otimes u \) for \( \gamma \in [1, \frac{4}{3}] \). In addition, Feireisl [17] showed that such weak solutions asymptotically converge to its equilibrium state as the time grows up, and then Feireisl [17] proved the global existence of weak solutions to (1.6) for general pressure laws allowed to be non-monotone on a compact set. Bresch-Jabin [9] developed new compactness estimates of the density and obtained global weak solutions to (1.6) for thermodynamically unstable pressure laws and anisotropic viscosity coefficients.

When \( \mu(\rho) \) and \( \lambda(\rho) \) depend on the density which are degenerate at vacuum, such viscous compressible equations appear in the description of shallow water [20, 38] or geophysical flows [35, 42], and can be derived from the fluid-dynamical approximation to the Boltzmann equation [29]. Bresch et. al. [3-5] showed that for \( \mu(\rho) \) and \( \lambda(\rho) \) verifying \( \lambda(\rho) = \rho \mu'(\rho) - \mu(\rho) \), the system (1.6) had a new entropy inequality providing higher regularity of the density, which was applied to prove the global existence of weak solutions for compressible Navier-Stokes equations with force terms [3, 4]. Mellet-Vasseur [37] investigated the weak stability of global solutions to (1.6) by deriving a \( L^\infty(0,T;L\log L) \)-type estimate of the velocity. Li-Xin [28] and Vasseur-Yu [45] used two different ways to construct the approximate sequence and established the global existence of weak solutions to (1.6) independently. Bresch-Vasseur-Yu [10] obtained global weak solutions for more general viscous stress tensors.

In this paper, we aim to establish the existence and large time behavior of global weak solutions with finite energy to the IVP (1.1)-(1.2) for general initial data.

First, we give the definition of global weak solutions to the IVP (1.1)-(1.2) below.

**Definition 1.1.** \((n, nv, \rho, \rho u)\) with \( n \geq 0 \) and \( \rho \geq 0 \) is said to be a global weak solution to the IVP (1.1)-(1.2) if for any time \( T > 0 \), the following properties hold:

- **Integrability conditions.**
  
  \[
  \begin{align*}
  n & \in L^\infty(0,T;L^1(\mathbb{T}^3)), & \nabla \sqrt{n} & \in L^\infty(0,T;L^2(\mathbb{T}^3)), \\
  \rho & \in L^\infty(0,T;L^\gamma(\mathbb{T}^3)), & \rho^{\frac{\gamma}{\gamma-1}} & \in L^1(0,T;L^1(\mathbb{T}^3)), \\
  \sqrt{nv} & \in L^\infty(0,T;L^2(\mathbb{T}^3)), & \sqrt{\rho u} & \in L^\infty(0,T;L^2(\mathbb{T}^3)), \\
  u & \in L^2(0,T;H^1(\mathbb{T}^3)), & \sqrt{nv} - \sqrt{nu} & \in L^2(0,T;L^2(\mathbb{T}^3)).
  \end{align*}
  \]

- **Equations.**

  For any test function \( \phi \in D(\mathbb{T}^3 \times [0,T]) \), the equations (1.1)\(_1\), (1.1)\(_2\) and (1.1)\(_4\) are satisfied in the
following sense:

\[
\begin{align*}
\int_{\mathbb{T}^3} n_0 \phi(0) dx + \int_0^T \int_{\mathbb{T}^3} (n \phi_t + n v \cdot \nabla \phi) dx dt = 0, \\
\int_{\mathbb{T}^3} m_0^i \phi(0) dx + \int_0^T \int_{\mathbb{T}^3} [n v^i \phi_t + \sqrt{n} v^i \sqrt{n} \cdot \nabla \phi - \partial_t n \phi] dx \\
= \int_0^T \int_{\mathbb{T}^3} [k n u^i \phi - k n u^i \phi - \eta n v^i \Delta \phi - \eta n v^i \nabla \phi \partial_i \phi] dx dt, \quad i = 1, 2, 3, \\
\int_{\mathbb{T}^3} \tilde{m}_0^i \phi(0) dx + \int_0^T \int_{\mathbb{T}^3} [\mu n u^i \phi_t + \mu n u^i u \cdot \nabla \phi + A \rho^\gamma \partial_i \phi] dx \\
= \int_0^T \int_{\mathbb{T}^3} [k n u^i \phi - k n v^i \phi + \mu n u^i \nabla \phi + (\mu + \lambda) \text{div} u \partial_i \phi] dx dt, \quad i = 1, 2, 3,
\end{align*}
\]

and the equation (1.1) holds in the sense of renormalized solutions, i.e., for \( b \in C^1(\mathbb{R}) \) satisfies \( b'(z) = 0 \) with \( z \in \mathbb{R} \) large enough,

\[\int_{\mathbb{T}^3} b(\rho_0) \phi(0) dx + \int_0^T \int_{\mathbb{T}^3} [b(\rho) \phi_t + b(\rho) u \cdot \nabla \phi - (\rho b'(\rho) - b(\rho)) \text{div} u \phi] dx dt = 0. \tag{1.10}\]

In addition, it holds for any test function \( \varphi \in \mathcal{D}(\mathbb{T}^3) \) that

\[
\begin{align*}
\lim_{t \to 0} \int_{\mathbb{T}^3} n \varphi dx = & \int_{\mathbb{T}^3} n_0 \varphi dx, \quad \lim_{t \to 0} \int_{\mathbb{T}^3} n v \varphi dx = \int_{\mathbb{T}^3} m_0 \varphi dx, \\
\lim_{t \to 0} \int_{\mathbb{T}^3} \rho \varphi dx = & \int_{\mathbb{T}^3} \rho_0 \varphi dx, \quad \lim_{t \to 0} \int_{\mathbb{T}^3} \rho u \varphi dx = \int_{\mathbb{T}^3} \tilde{m}_0 \varphi dx.
\end{align*}
\]  

- **Energy inequality.**

For a.e. \( t \in (0, T) \), there holds

\[
\begin{align*}
\int_{\mathbb{T}^3} \frac{1}{2} n |v|^2 + n \log n - n + 1 + \frac{1}{2} \rho |u|^2 + \frac{A \rho^\gamma}{\gamma - 1} dx \\
+ \int_0^t \int_{\mathbb{T}^3} (k n |v - u|^2 + \mu |\nabla u|^2 + (\mu + \lambda) (\text{div} u)^2) dx dt \\
\leq \int_{\mathbb{T}^3} \left( \frac{1}{2} \frac{|m_0|^2}{n_0} + n_0 \log n_0 - n_0 + 1 + \frac{1}{2} \frac{|	ilde{m}_0|^2}{\rho_0} + \frac{A \rho_0^\gamma}{\gamma - 1} \right) dx.
\end{align*}
\]  

Then, we have the global existence of weak solutions to the IVP (1.1)-(1.2) as follows.

**Theorem 1.1.** Assume that the initial data \((n_0, m_0, \rho_0, \tilde{m}_0)\) satisfies

\[
\begin{align*}
0 \leq n_0 & \in L^1(\mathbb{T}^3), \quad n_0 \log n_0 \in L^1(\mathbb{T}^3), \quad \nabla \sqrt{n_0} \in L^2(\mathbb{T}^3), \\
m_0 = 0 \text{ in } \{x \in \mathbb{T}^3 \mid n_0(x) = 0\}, \quad \frac{|m_0|^2}{n_0} \in L^1(\mathbb{T}^3), \quad \frac{m_0^{2+n_0}}{n_0^{1+n_0}} \in L^1(\mathbb{T}^3), \\
0 \leq \rho_0 & \in L^1(\mathbb{T}^3) \cap L^\gamma(\mathbb{T}^3), \\
\tilde{m}_0 = 0 \text{ in } \{x \in \mathbb{T}^3 \mid \rho_0(x) = 0\}, \quad \frac{|	ilde{m}_0|^2}{\rho_0} \in L^1(\mathbb{T}^3),
\end{align*}
\]
with \( \eta_0 > 0 \) any small constant. Then the IVP (1.1)-(1.2) admits a global weak solution \((n, nv, \rho, pu)\) in the sense of Definition 1.1. Moreover, the solution \((n, nv, \rho, pu)\) satisfies for any time \( T > 0 \) that

\[
\begin{align*}
\int_{\Omega^3} n \, dx &= \int_{\Omega^3} n_0 \, dx, \\
\int_{\Omega^3} \rho \, dx &= \int_{\Omega^3} \rho_0 \, dx, \quad t \in (0, T), \\
\int_{\Omega^3} (nv + pu) \, dx &= \int_{\Omega^3} (m_0 + \tilde{m}_0) \, dx, \quad t \in (0, T),
\end{align*}
\]

(1.14)

and

\[
\begin{align*}
\text{ess sup}_{t \in [0, T]} \int_{\Omega^3} |\nabla \sqrt{n}|^2 \, dx + \int_0^T \int_{\Omega^3} |\nabla \sqrt{n}|^2 \, dx \, dt &\leq C, \\
\text{ess sup}_{t \in [0, T]} \int_{\Omega^3} n(1 + |v|^2) \log (1 + |v|^2) \, dx &\leq C_T,
\end{align*}
\]

(1.15)

where \( C > 0 \) is a constant independent of the time \( T > 0 \), and \( C_T > 0 \) is a constant dependent of the time \( T > 0 \).

**Remark 1.1.** By similar arguments, Theorem 1.1 can be extended to the two-dimensional case for any adiabatic constant \( \gamma > 1 \).

Next, we study the large time behavior of global weak solutions to the IVP (1.1)-(1.2).

**Theorem 1.2.** Let the assumptions (1.13) be satisfied, and \((n, nv, \rho, pu)\) be the global weak solution to the IVP (1.1)-(1.2) given by Theorem 1.1. Then \((n, v, \rho, u)\) converges to its equilibrium state \((n_c, u_c, \rho_c, u_c)\) in the sense

\[
\lim_{t \to \infty} \int_{\Omega^3} (n|v - u_c|^2 + |n - n_c|^p + \rho|u - u_c|^2 + |\rho - \rho_c|^\gamma) \, dx = 0, \quad p \in [1, 3),
\]

(1.16)

where the constants \( n_c, \rho_c \) and \( u_c \) are denoted by

\[
n_c := \int_{\Omega^3} n_0 \, dx, \quad \rho_c := \int_{\Omega^3} \rho_0 \, dx, \quad u_c := \frac{\int_{\Omega^3} (m_0 + \tilde{m}_0) \, dx}{\int_{\Omega^3} (n_0 + \rho_0) \, dx}.
\]

(1.17)

**Remark 1.2.** Theorem 1.2 implies that the velocities of the compressible Navier-Stokes equations (1.1)_1-(1.1)_2 with degenerate viscosity and the compressible Navier-Stokes equations (1.1)_3-(1.1)_4 with constant viscosities are aligned when the time tends to infinity.

The rest part of this paper is arranged as follows. In Section 2, we state the global well-posedness of strong solutions to the approximate system. Section 3 is devoted to the a-priori estimates of approximate solutions. In Section 4, we vanish the artificial viscosities and show the convergence of approximate sequence to a global weak solution for (1.1) with an artificial pressure term. In Section 5, we vanish the artificial pressure and prove Theorem 1.1 concerning global existence of weak solutions to the original IVP (1.1)-(1.2). Theorem 1.2 on the large time behavior of global weak solutions will be shown in Section 6.
2 Approximate sequence

Inspired by Li-Xin [28], for \( \varepsilon, \delta \in (0, 1) \) and \( \gamma_0 > \max\{\gamma + 4\} \), we are ready to solve the following approximate problem:

\[
\begin{aligned}
\begin{cases}
  n_t + \text{div}(nv) = \varepsilon \sqrt{n} \Delta \sqrt{n} + \varepsilon \sqrt{n} \text{div}((\nabla \sqrt{n})^2 \nabla \sqrt{n}) + \varepsilon n^{-12}, \\
n(v_t + v \cdot \nabla v) + \nabla n + \varepsilon |v|^3 v + \varepsilon n^{-12} v \\
  = -\kappa n(v - u) + \eta \text{div}(n \nabla (v)) + \sqrt{\varepsilon} \text{div}(n \nabla v) + \varepsilon \sqrt{n} |\nabla \sqrt{n}|^2 \nabla \sqrt{n} \cdot \nabla v, \\
\rho_t + \text{div}(\rho u) = \varepsilon \Delta \rho, \\
\rho (u_t + u \cdot \nabla u) + \nabla (A \rho^\gamma \delta + \delta \rho^\gamma) + \varepsilon |u|^2 \rho \\
  = \kappa n(v - u) + \mu \Delta u + (\mu + \lambda) \nabla \text{div} u + \varepsilon \nabla u \cdot \nabla \rho, \\
(n, v, \rho, u) (x, 0) = (n_{0, \delta}, v_{0, \delta}, \rho_{0, \delta}, u_{0, \delta})(x), \quad x \in \mathbb{T}^3, \quad t > 0,
\end{cases}
\end{aligned}
\]

The regularized initial data \((n_{0, \delta}, v_{0, \delta}, \rho_{0, \delta}, u_{0, \delta})\) is constructed by

\[
\begin{aligned}
\begin{cases}
  n_{0, \delta} := (\sqrt{n_0} \ast j_\delta)^2 + \varepsilon \text{div}, \\
v_{0, \delta} := \frac{(n_0 - \frac{1 + m_0}{2 + m_0} m_0) \ast j_\delta}{n_0 + m_0}, \\
\rho_{0, \delta} := \rho_0 \ast j_\delta + \delta \geq \delta_0, \\
u_{0, \delta} := \frac{\sqrt{n_0} \ast j_\delta}{\sqrt{\rho_0 \ast j_\delta}},
\end{cases}
\end{aligned}
\]

where \(j_\delta\) is the smooth function satisfying

\[
\begin{aligned}
\begin{cases}
  \|j_\delta\|_{L^1} = 1, \quad 0 \leq j_\delta \leq \delta^{- \frac{m_0}{2 + m_0}}, \\
  |\nabla j_\delta| \leq C \delta^{- \frac{m_0}{2 + m_0}}, \\
  j_\delta \ast f \rightarrow f \quad \text{in} \quad L^p(\mathbb{T}^3), \quad \text{as} \quad \delta \rightarrow 0, \quad \forall f \in L^p(\mathbb{T}^3), \quad p \in [1, \infty),
\end{cases}
\end{aligned}
\]

with some constant \(C > 0\) independent of \(\delta\).

It is easy to verify as \(\delta \rightarrow 0\) that

\[
\begin{aligned}
\begin{cases}
  \rho_{0, \delta} \rightarrow \rho_0, \quad \text{in} \quad L^\gamma(\mathbb{T}^3), \\
  \rho_{0, \delta}|u_{0, \delta}|^2 \rightarrow \frac{|m_0|^2}{\rho_0} \quad \text{in} \quad L^1(\mathbb{T}^3), \\
n_{0, \delta} \rightarrow n_0 \quad \text{in} \quad L^1(\mathbb{T}^3), \\
\nabla \sqrt{n_{0, \delta}} \rightarrow \nabla \sqrt{n_0} \quad \text{in} \quad L^2(\mathbb{T}^3), \\
n_{0, \delta}|v_{0, \delta}|^{2 + m_0} \rightarrow \frac{|m_0|^{2 + m_0}}{n_0^{1 + m_0}} \quad \text{in} \quad L^1(\mathbb{T}^3), \\
n_{0, \delta}|v_{0, \delta}|^2 = \frac{m_0^{2 + m_0}}{n_0^{1 + m_0}} [(n_0 - \frac{1 + m_0}{2 + m_0} m_0) \ast j_\delta]^2 \\
  \rightarrow n_0 \frac{m_0^{2 + m_0}}{n_0^{1 + m_0}} \quad \text{in} \quad L^1(\mathbb{T}^3), \\
n_{0, \delta} \rightarrow \frac{|m_0|^2}{n_0} \quad \text{in} \quad L^1(\mathbb{T}^3).
\end{cases}
\end{aligned}
\]
Denote
\[ E_{0,\delta} := \int_{T^3} \left( \frac{1}{2} |v_{0,\delta}|^2 + n_{0,\delta} \log n_{0,\delta} - n_{0,\delta} + 1 + \frac{1}{2} \rho_{0,\delta} |u_{0,\delta}|^2 + \frac{A\rho_{0,\delta}}{\gamma - 1} \right) \]
\[ + \frac{\delta \rho_{0,\delta}}{\rho_0 - 1} + \varepsilon_{0,\delta} n^{-12} \right) dx. \]
(2.5)

By (2.3) and (2.4), it follows that
\[ E_{0,\delta} \leq C, \]
\[ \lim_{\delta \to 0} E_{0,\delta} = 0, \]
\[ \int_{T^3} n_{0,\delta} (1 + |v_{0,\delta}|^2) \log (1 + |v_{0,\delta}|^2) dx \leq C, \]
(2.6)
with \( C > 0 \) a constant independent of \( \delta \).

We have the global well-posedness of strong solutions to the approximate problem (2.1)-(2.6).

**Proposition 2.1.** Let \( \delta \in (0, 1) \), and the assumptions of Theorem 1.1 hold. There for suitably small \( \varepsilon \in (0, \frac{1}{4}) \), the IVP (2.1) has a unique strong solution \( (n_\varepsilon, v_\varepsilon, \rho_\varepsilon, u_\varepsilon) \) satisfying for any \( T > 0 \) that

\[
\begin{align*}
\inf_{(x,t) \in T^3 \times [0,T]} n_\varepsilon(x,t) &> 0, \\
\inf_{(x,t) \in T^3 \times [0,T]} \rho_\varepsilon(x,t) &> 0, \\
n_\varepsilon, v_\varepsilon, \rho_\varepsilon, u_\varepsilon &\in C([0,T]; H^2(T^3)) \cap L^2(0,T; H^3(T^3)), \\
(n_\varepsilon)_t, (v_\varepsilon)_t, (\rho_\varepsilon)_t, (u_\varepsilon)_t &\in L^2(0,T; H^1(T^3)).
\end{align*}
\]

The local well-posedness of the strong solution to the IVP (2.1) for \( \varepsilon, \delta \in (0, 1) \) can be proved in a standard way based on linearization techniques and fixed point arguments. We omit the proof here for brevity, and the reader can refer to for example [36]. By virtue of the a-priori estimates established in Section 3 below, we are able to to extend the local approximate sequence \( (n_\varepsilon, v_\varepsilon, \rho_\varepsilon, u_\varepsilon) \) to a global one and prove Proposition 2.1.

**Remark 2.1.** The approximate framework of (2.1)_1-(2.1)_2 can be applied to the construction of approximate solutions for the three-dimensional isentropic compressible Navier-Stokes equations with constant viscosities.

### 3 The a-priori estimates

In this section we derive the uniformly a-priori estimates of approximate sequence given by Proposition 2.1. These estimates shall capture the basic energy, the Bresch-Desjardins type entropy, the Mellet-Vasseur type estimate, and the higher integrability of the density \( \rho \), which are applied to obtain the upper and lower bounds of the densities by virtue of De Giorgi iteration, and then the higher-order estimates can be shown by standard regularity estimates for nonlinear parabolic equations.

First of all, we have the basic energy estimates.
Lemma 3.1. For any \( \varepsilon \in (0, \frac{1}{4}) \), \( \delta \in (0, 1) \) and given time \( T > 0 \), let \( (n, v, \rho, u) \) be any strong solution to the IVP (2.1) for \( t \in (0, T] \). Then, under the assumptions of Theorem 1.1, we have \( n, \rho \geq 0 \) and

\[
\begin{align*}
&\sup_{t \in [0,T]} \int_{\mathbb{T}^3} (n|v|^2 + n + \rho|u|^2 + \rho^\gamma + \delta \rho^{\gamma_0} + \varepsilon n^{-12})dx \\
&+ \int_0^T \int_{\mathbb{T}^3} (\eta n|\nabla(v)|^2 + \kappa n|v-u|^2 + \mu |\nabla u|^2 + (\mu + \lambda)(\text{div} \, u)^2)dxdt \\
&+ \int_0^T \int_{\mathbb{T}^3} (\varepsilon n|\nabla v|^2 + \varepsilon n|v|^5 + \varepsilon (1 + |v|^2)(|\nabla \sqrt{n}|^2 + |\nabla \sqrt{n}||^4) + \varepsilon n^{-12}|v|^2 + \varepsilon^2 n^{-25})dxdt \\
&+ \int_0^T \int_{\mathbb{T}^3} \varepsilon (|u|^4 + |\nabla \rho|^2 + \rho^\gamma - 2 + \delta \rho^{\gamma_0 - 2})dxdt \leq C_T,
\end{align*}
\]  

(3.1)

where \( C_T > 0 \) is a constant independent of \( \varepsilon \) and \( \delta \).

Proof. First, according to the maximum principle for the parabolic equations (2.1) and (2.1)3, both \( n \) and \( \rho \) are nonnegative. Then one obtains after integrating (2.1) and (2.1)3 over \( \mathbb{T}^3 \) that

\[
\frac{d}{dt} \int_{\mathbb{T}^3} n dx + \varepsilon \int_{\mathbb{T}^3} (|\nabla \sqrt{n}|^2 + |\nabla \sqrt{n}||^4)dx = \varepsilon \int_{\mathbb{T}^3} n^{-12}dx,
\]  

(3.2)

and

\[
\frac{d}{dt} \int_{\mathbb{T}^3} \rho dx + \varepsilon \int_{\mathbb{T}^3} |\nabla \rho|^2dx = 0.
\]  

(3.3)

By (2.1)3-(2.1)4, we show

\[
\frac{d}{dt} \int_{\mathbb{T}^3} \left( \frac{1}{2} \rho |u|^2 + A \frac{\rho^{\gamma}}{\gamma - 1} + \frac{\delta \rho^{\gamma_0}}{\gamma_0 - 1} \right) dx - \int_{\mathbb{T}^3} (\kappa n(v-u) \cdot u + \mu |\nabla u|^2 + (\mu + \lambda)(\text{div} \, u)^2)dx \\
+ \int_{\mathbb{T}^3} (\varepsilon (\rho^{\gamma-2} + \delta \gamma_0 \rho^{\gamma_0 - 2}) |\nabla \rho|^2 + \varepsilon |u|^4)dx = 0.
\]  

(3.4)

Meanwhile, we take the \( L^2 \)-inner of (2.1)2 with \( v \) to obtain

\[
\begin{align*}
\frac{d}{dt} \int_{\mathbb{T}^3} \frac{1}{2} n|v|^2dx &+ \int_{\mathbb{T}^3} \kappa n(v-u) \cdot v dx + \int_{\mathbb{T}^3} \eta n|\nabla(v)|^2dx \\
&+ \int_{\mathbb{T}^3} (\varepsilon n|v|^5 + \sqrt{\varepsilon} n|\nabla v|^2 + \frac{\varepsilon}{2} n^{-12}|v|^2 + \frac{\varepsilon}{2} |\nabla \sqrt{n}||^4|v|^2)dx \\
&\leq \int_{\mathbb{T}^3} n \text{div} \, v dx + \varepsilon \int_{\mathbb{T}^3} n|\nabla v|^2dx + \frac{\varepsilon}{4} \int_{\mathbb{T}^3} |\nabla \sqrt{n}||^2|v|^2dx.
\end{align*}
\]  

(3.5)

One deduces after adding (3.4)-(3.5) together that

\[
\begin{align*}
\frac{d}{dt} \int_{\mathbb{T}^3} \left( \frac{1}{2} n|v|^2 + \frac{1}{2} \rho |u|^2 + A \frac{\rho^{\gamma}}{\gamma - 1} + \frac{\delta \rho^{\gamma_0}}{\gamma_0 - 1} \right) dx &+ \int_{\mathbb{T}^3} (\kappa n|v-u|^2 + \eta n|\nabla(v)|^2 + \mu |\nabla u|^2 + (\mu + \lambda)(\text{div} \, u)^2)dx \\
&+ \int_{\mathbb{T}^3} (\varepsilon n|v|^5 + (\sqrt{\varepsilon} - \varepsilon) n|\nabla v|^2 + \frac{\varepsilon}{2} n^{-12}|v|^2 + \frac{\varepsilon}{2} |\nabla \sqrt{n}||^4|v|^2)dx \\
&+ \int_{\mathbb{T}^3} (\varepsilon |u|^4 + (\gamma \rho^{\gamma-2} + \delta \gamma_0 \rho^{\gamma_0 - 2}) |\nabla \rho|^2)dx \leq \int_{\mathbb{T}^3} n \text{div} \, v dx.
\end{align*}
\]  

(3.6)
We would like to mention that if we estimate the usual term \( n \log n \) to cancel the first term on the right-hand side of (3.5), then the diffusion terms in (2.1) will cause an additional difficulty since \( 1 + \log n \) may not be positive. To overcome this difficulty, we have by Young’s inequality and the fact \( |\nabla v|^2 \leq 3|\nabla(v)|^2 \) that
\[
\int_{T^n} n \div v dx \leq \int_{T^n} n \chi(n) \div v dx + \frac{n}{4} \int_{T^n} n|\nabla v|^2 dx + \frac{C}{\eta},
\] (3.7)
where \( \chi(s) \geq 0 \) is a smooth function on \( \mathbb{R}_+ \) satisfying \( \chi(s) = 0 \) for \( s \in [0, 1] \), \( \chi(s) = 1 \) for \( s \geq \epsilon \) and \( 3\chi(s) + 2s\chi'(s) \geq 0 \) for \( s \geq 0 \). Denoting \( \Pi(n) = \int_0^n s^{-1}\chi(s)ds \), we get from (2.1) and \( \Pi'(n) + 2n\Pi''(n) = \int_0^n s^{-1}\chi(s)ds + 3\chi(n) + 2n\chi'(n) \geq 0 \) that
\[
\frac{d}{dt} \int_{T^n} \Pi(n) dx - \int_{T^n} (\Pi'(n) + 2n\Pi''(n))|\nabla\sqrt{n}|^2(1 + |\nabla\sqrt{n}|^2) dx
= -\int_{T^n} n\chi(n) \div v dx + \epsilon \int_{T^n} \Pi'(n)n^{-12} dx
\leq -\int_{T^n} n\chi(n) \div v dx + \frac{\epsilon^2}{26} \int_{T^n} n^{-25} dx + C,
\] (3.8)
where one has used \( |\Pi'(s)| \leq C(1 + s) \) and the simple fact
\[
n^{-q} \leq C(n^{-25} + 1), \quad q \in (0, 25).
\] (3.9)

In addition, according to (2.1) and \( |\nabla v|^2 \leq 3|\nabla(v)|^2 \), we also get
\[
\frac{d}{dt} \int_{T^n} \frac{\epsilon}{78} n^{-12} dx + \int_{T^n} \left( \frac{50\epsilon^2}{13} n^{-13}|\nabla\sqrt{n}|^2(1 + |\nabla\sqrt{n}|^2) + \frac{2\epsilon^2}{13} n^{-25} \right) dx
= \frac{\epsilon}{6} \int_{T^n} n^{-12} \div v dx \leq \frac{\epsilon^2}{13} \int_{T^n} n^{-25} dx + \frac{1}{2} \int_{T^n} n|\nabla v|^2 dx.
\] (3.10)

The combination of (3.2)-(3.10) and the Grönwall inequality leads to (3.1).

\begin{remark}
We are able to recover the energy inequality uniformly in time after taking the limit as \( \epsilon \to 0 \), see the proof of Proposition 4.1.
\end{remark}

Next, in order to obtain the uniform spatial derivative estimate of \( \sqrt{n} \), we show a Bresch-Desjardins type entropy estimates.

\begin{lemma}
For any \( \epsilon \in (0, \frac{1}{4}) \), \( \delta \in (0, 1) \) and given time \( T > 0 \), let \( (n, v, \rho, u) \) be any strong solution to the IVP (2.1) for \( t \in (0, T] \). Then, under the assumptions of Theorem 1.1, it holds
\[
\sup_{t \in [0, T]} \int_{T^n} (|\nabla\sqrt{n}|^2 + \epsilon|\nabla\sqrt{n}|^4) dx + \int_0^T \int_{T^n} (n|\mathcal{K}(v)|^2 + n|\nabla v|^2 + |\nabla\sqrt{n}|^2) dx dt
+ \epsilon \int_0^T \int_{T^n} (|\nabla^2\sqrt{n}|^2 + |\nabla\sqrt{n}|^2|\nabla^2\sqrt{n}|^2) dx dt + \epsilon^2 \int_0^T \int_{T^n} |\nabla\sqrt{n}|^4 |\nabla^2\sqrt{n}|^2 dx dt \leq C_T,
\] (3.11)
where \( \mathcal{K}(v) := \frac{\nabla v - (\nabla v)^T}{2} \) denotes the antisymmetric part of the gradient, and \( C_T > 0 \) is a constant independent of \( \epsilon \) and \( \delta \).
\end{lemma}
Proof. Let $c_0 > 0$ be a constant to be chosen later. We denote the effective velocity

$$w := v + c_0 \nabla \log n$$

to rewrite the mass equation (2.1)$_1$ by

$$n_t + \text{div}(nw) = c_0 \Delta n + G,$$

and the momentum equation (2.1)$_2$ by

$$(nw)_t + \text{div}(nw \otimes w) + \nabla n - c_0 \Delta (nw) + \varepsilon n|v|^3 v + \varepsilon n^{-12} v$$

$$= (\eta + \sqrt{\varepsilon} - 2c_0) \text{div}(n\nabla w) + \sqrt{\varepsilon} \text{div}(n\mathcal{K}(v)) - (c_0^2 + (\eta + \sqrt{\varepsilon} - 2c_0)c_0) \text{div}(n\nabla^2 \log n)$$

$$+ \nabla G + Gv + \varepsilon n|\nabla \sqrt{n}|^2 \nabla \sqrt{n} \cdot \nabla,$$

where one has used the following equalities:

$$\begin{cases}
(n\nabla \log n)_t = -\nabla \text{div}(nv) + \nabla G, \\
\text{div}(nv \otimes \nabla \log n + n\nabla \log n \otimes v) = \Delta (nv) - 2 \text{div}(n\nabla v) + \nabla \text{div}(nv), \\
\text{div}(n\nabla \log n \otimes \nabla \log n) = \Delta (n\nabla \log n) - \text{div}(n\nabla^2 \log n).
\end{cases}$$

Thus, choosing

$$c_0 = \eta + \sqrt{\varepsilon},$$

we derive the following reformulated equations:

$$\begin{cases}
n_t + \text{div}(nw) = c_0 \Delta n + G, \\
n(w_t + w \cdot \nabla w) + \nabla n + \kappa n(v - u) + c_0 w \Delta n - c_0 \Delta (nw) + \varepsilon n|v|^3 v + \varepsilon n^{-12} v \quad (3.12)
\end{cases}$$

Thence we obtain after taking the $L^2(\mathbb{T}^3)$-inner product of (3.12)$_2$ with $w$ and making use of (3.12)$_1$ that

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{T}^3} \frac{1}{2} |w|^2 dx &+ \int_{\mathbb{T}^3} (c_0 + \sqrt{\varepsilon})|n(\mathcal{K}(v))|^2 + 2c_0|\nabla \sqrt{n}|^2 + \nabla n \cdot v + \kappa n(v - u)v dx \\
&= -c_0 \int_{\mathbb{T}^3} (\kappa n(v - u) + \varepsilon n|v|^3 v) \cdot \nabla \log n dx - \int_{\mathbb{T}^3} G \text{div} v dx - c_0 \int_{\mathbb{T}^3} G \Delta \log n dx \quad (3.13)
\end{aligned}$$

$$+ \int_{\mathbb{T}^3} G \left(\frac{1}{2} |w|^2 - c_0 \nabla \log n \cdot w\right) dx + \varepsilon \int_{\mathbb{T}^3} \nabla n|\nabla \sqrt{n}|^2 \nabla \sqrt{n} \cdot \nabla v \cdot w dx - \varepsilon \int_{\mathbb{T}^3} n^{-12} v \cdot w dx.$$

We estimate the terms on the right-hand side of (3.13) as follows. First, we can show after using Hölder’s
and Young’s inequalities that

\[-c_0 \int_{\mathbb{T}^3} (\kappa n(v - u) + \varepsilon n|v|^3) \cdot \nabla \log n dx\]
\[\leq \kappa c_0 \left( \int_{\mathbb{T}^3} n|v - u|^2 dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{T}^3} |\nabla \sqrt{n}|^2 dx \right)^{\frac{1}{2}} + 2c_0 \varepsilon \left( \int_{\mathbb{T}^3} n|v|^5 dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{T}^3} |\nabla \sqrt{n}|^4 |v|^2 dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{T}^3} n^{-4} dx \right)^{\frac{1}{2}} \]
\[\leq c_0 \int_{\mathbb{T}^3} |\nabla \sqrt{n}|^2 dx + \varepsilon \int_{\mathbb{T}^3} n|v|^2 dx + \frac{\varepsilon}{2} \int_{\mathbb{T}^3} |\nabla \sqrt{n}|^4 |v|^2 dx + C \int_{\mathbb{T}^3} n|v - u|^2 dx + C\varepsilon \int_{\mathbb{T}^3} n^{-25} dx + C\varepsilon. \]

Similarly, one has

\[-\int_{\mathbb{T}^3} G \text{ div } u dx\]
\[= -\varepsilon \int_{\mathbb{T}^3} \sqrt{n}(\Delta \sqrt{n} + \text{div}(|\nabla \sqrt{n}|^2 \nabla \sqrt{n})) \text{ div } u dx + \varepsilon \int_{\mathbb{T}^3} n^{-12} \text{ div } u dx \]
\[\leq \varepsilon^2 \int_{\mathbb{T}^3} (\Delta \sqrt{n} + \text{div}(|\nabla \sqrt{n}|^2 \nabla \sqrt{n}))^2 dx + C \int_{\mathbb{T}^3} n|\text{div } u|^2 dx + C\varepsilon^2 \int_{\mathbb{T}^3} n^{-25} dx. \]

And as the property

\[\int_{\mathbb{T}^3} \text{div}(|\nabla \sqrt{n}|^2 \nabla \sqrt{n}) \Delta \sqrt{n} dx = -\int_{\mathbb{T}^3} |\nabla \sqrt{n}|^2 \nabla \sqrt{n} \cdot \nabla \Delta \sqrt{n} dx\]
\[= \int_{\mathbb{T}^3} |\nabla \sqrt{n}|^4 |\nabla \sqrt{n}|^2 dx + \frac{1}{2} \int_{\mathbb{T}^3} |\nabla |\nabla \sqrt{n}|^2|^2 dx, \]

the fourth term on the right-hand side of (3.13) can be decomposed by

\[-c_0 \int_{\mathbb{T}^3} G \Delta \log n dx\]
\[= -2c_0 \varepsilon \int_{\mathbb{T}^3} \sqrt{n}(\Delta \sqrt{n} + \text{div}(|\nabla \sqrt{n}|^2 \nabla \sqrt{n}))(n^{-\frac{1}{2}} \Delta \sqrt{n} - n^{-1}|\nabla \sqrt{n}|^2) dx\]
\[= -\varepsilon c_0 \int_{\mathbb{T}^3} n^{-12} \Delta \log n dx \]
\[= -\varepsilon c_0 \int_{\mathbb{T}^3} (2|\Delta \sqrt{n}|^2 + 2|\nabla \sqrt{n}|^2 |\nabla \sqrt{n}|^2 + |\nabla |\nabla \sqrt{n}|^2|^2 + 48n^{-13}|\nabla \sqrt{n}|^2) dx\]
\[+ 2c_0 \varepsilon \int_{\mathbb{T}^3} (\Delta \sqrt{n} + \text{div}(|\nabla \sqrt{n}|^2 \nabla \sqrt{n}))n^{-\frac{1}{2}} |\nabla \sqrt{n}|^2 dx. \]

Note that the Sobolev inequality implies that

\[\left\{ \begin{array}{ll}
||\nabla \sqrt{n}||_{L^6} \leq C||\nabla^2 \sqrt{n}||_{L^2} \leq C||\Delta \sqrt{n}||_{L^2} \\
||\nabla \sqrt{n}||_{L^4} \leq ||\nabla \sqrt{n}||_{L^2} ||\nabla \sqrt{n}||_{L^4} \leq C||\nabla \sqrt{n}||_{L^2} ||\nabla \sqrt{n}||_{L^4} \leq C||\nabla \sqrt{n}||_{L^2} ||\nabla \sqrt{n}||_{L^4} + ||\nabla \sqrt{n}||_{L^2}
\end{array} \right.\]
which together with (3.9) gives rise to

\[
\varepsilon \int_{\Omega^3} (\Delta \sqrt{n} + \text{div}(|\nabla \sqrt{n}|^2 \nabla \sqrt{n})) n^{-\frac{1}{2}} |\nabla \sqrt{n}|^2 \, dx \\
\leq \varepsilon \|\Delta \sqrt{n}\|_{L^2} \|\nabla \sqrt{n}\|_{L^4} n^{-\frac{1}{2}} \|L^4} \|\nabla \sqrt{n}\|_{L^8} \|\nabla \sqrt{n}\|_{L^2} n^{-\frac{1}{2}} \|L^2}
\leq \frac{C_0 \varepsilon}{4} \int_{\Omega^3} (2|\Delta \sqrt{n}|^2 + 2|\nabla \sqrt{n}|^2 |\nabla \sqrt{n}|^2 + |\nabla \sqrt{n}|^2| v |^2 + 48n^{-13}|\nabla \sqrt{n}|^2) \, dx
\]

(3.17)

Moreover, by (3.9), the equation of \(\sqrt{n}\) reads

\[
2(\sqrt{n})_t - \varepsilon \Delta \sqrt{n} - \varepsilon \text{div}(|\nabla \sqrt{n}|^2 \nabla \sqrt{n}) = -2v \cdot \nabla \sqrt{n} - \sqrt{n} \text{div} v + \varepsilon \sqrt{n}^{-25}
\]

(3.21)
One multiplies (3.21) by $\varepsilon(\Delta \sqrt{n} + \text{div}(\nabla \sqrt{n}^2 \sqrt{n}))$ and integrates the resulted equation by parts to get

$$
\frac{d}{dt} \int_{T_3} \left( \varepsilon |\nabla \sqrt{n}|^2 + \frac{\varepsilon}{2} |\nabla \sqrt{n}|^4 \right) dx + \varepsilon^2 \int_{T_3} (\Delta \sqrt{n} + \text{div}(\nabla \sqrt{n}^2 \sqrt{n}))^2 dx 
+ 25\varepsilon^2 \int_{T_3} n^{-13}(|\nabla \sqrt{n}|^2 + |\nabla \sqrt{n}|^4) dx 
= \varepsilon \int_{T_3} (\Delta \sqrt{n} + \text{div}(\nabla \sqrt{n}^2 \sqrt{n})) \nabla \sqrt{n} \text{div} \nu dx 
+ 2\varepsilon \int_{T_3} (\Delta \sqrt{n} + \text{div}(\nabla \sqrt{n}^2 \sqrt{n})) \nu \cdot \nabla \sqrt{n} dx 
\leq \frac{\varepsilon^2}{8} \int_{T_3} (\Delta \sqrt{n} + \text{div}(\nabla \sqrt{n}^2 \sqrt{n}))^2 dx 
+ \varepsilon C \int_{T_3} |\nabla \sqrt{n}|^2 \nu(1 + |\nabla \sqrt{n}|^2) dx,
$$

which together with (3.20) and the fact

$$
\int_{T_3} (\Delta \sqrt{n} + \text{div}(\nabla \sqrt{n}^2 \sqrt{n}))^2 dx 
\geq \frac{1}{2} \int_{T_3} (\text{div}(\nabla \sqrt{n}^2 \sqrt{n}))^2 dx - 2 \int_{T_3} (\Delta \sqrt{n})^2 dx 
= \frac{1}{2} \int_{T_3} (|\nabla \sqrt{n}|^4 |\nabla^2 \sqrt{n}|^2 + (\nabla \sqrt{n} \cdot \nabla |\nabla \sqrt{n}|^2)^2 + |\nabla \sqrt{n}|^2 |\nabla |\nabla \sqrt{n}|^2|^2) dx - 2 \int_{T_3} (\Delta \sqrt{n})^2 dx 
$$

leads to

$$
\frac{d}{dt} \int_{T_3} \left( \frac{1}{2} |n|w|^2 + \varepsilon |\nabla \sqrt{n}|^2 + \frac{\varepsilon}{2} |\nabla \sqrt{n}|^4 \right) dx 
+ \int_{T_3} ((\eta + 2\sqrt{n} |A(v)|^2 + c_0 |\nabla \sqrt{n}|^2 + \nabla n \cdot \nu + kn(\nu - u)\nu) dx 
+ \frac{\varepsilon}{4} (c_0 - 2\varepsilon) \int_{T_3} (|\Delta \sqrt{n}|^2 + |\nabla \sqrt{n}|^2 |\nabla^2 \sqrt{n}|^2 + \frac{1}{2} |\nabla |\nabla \sqrt{n}|^2|^2 + 24n^{-13}|\nabla n|^2) dx 
+ \frac{\varepsilon^2}{4} \int_{T_3} ((|\nabla \sqrt{n}|^4 |\nabla^2 \sqrt{n}|^2 + (\nabla \sqrt{n} \cdot \nabla |\nabla \sqrt{n}|^2)^2 + |\nabla \sqrt{n}|^2 |\nabla |\nabla \sqrt{n}|^2|^2) dx 
+ 25\varepsilon^2 \int_{T_3} n^{-13}(|\nabla \sqrt{n}|^2 + |\nabla \sqrt{n}|^4) dx 
\leq C \int_{T_3} (n|\text{div} v|^2 + n|\nu - u|^2) dx 
+ \varepsilon \int_{T_3} (n|v|^3 + |\nabla \sqrt{n}|^4 + |\nabla \sqrt{n}|^2 + |v|^2 |\nabla \sqrt{n}|^2 (1 + |\nabla \sqrt{n}|^2) + n^{-25}) dx + C\varepsilon.
$$

The combination of (3.1), (3.4), (3.7)-(3.8) and (3.22) gives rise to (3.11). The proof of Lemma 3.2 is completed. 

**Remark 3.2.** We will show that the Bresch-Desjardins type estimates do not depend on time after the limit process as $\varepsilon \to 0$, see the proof of Proposition 4.1.

The following Mellet-Vasseur type estimate will be used to prove the strong convergence of $\sqrt{n}\nu$ in $L^2(0,T;L^2(\mathbb{T}^3))$. 

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Lemma 3.3. For any \( \varepsilon \in (0, \frac{1}{4}) \), \( \delta \in (0,1) \) and given time \( T > 0 \), let \( (n,v,\rho,u) \) be any strong solution to the IVP (2.1) for \( t \in (0,T] \). Then, under the assumptions of Theorem 1.1, it holds
\[
\sup_{t \in [0,T]} \int_{T^3} n(1 + |v|^2) \log(1 + |v|^2) dx \leq C_T, \tag{3.23}
\]
where \( C_T > 0 \) is a constant independent of \( \varepsilon \) and \( \delta \).

Proof. Multiplying (2.1) by \((1 + \log(1 + |v|^2))v\), we obtain after integration by parts that
\[
\frac{d}{dt} \int_{T^3} \frac{n(1 + |v|^2)}{2} \log(1 + |v|^2) dx + \int_{T^3} n \log(1 + |v|^2)(\eta \|D(v)\|^2 + \sqrt{\varepsilon} |\nabla v|^2) dx + \int_{T^3} \varepsilon n|v|^5(1 + \log(1 + |v|^2)) dx + \frac{\varepsilon}{2} \int_{T^3} (1 + |v|^2) \log(1 + |v|^2) |\nabla \sqrt{n}|^4 dx \\
+ \kappa \int_{T^3} n|v|^2(1 + \log(1 + |v|^2)) dx
= \frac{\varepsilon}{2} \int_{T^3} (1 + |v|^2) \sqrt{\eta \epsilon} \Delta \sqrt{n} \log(1 + |v|^2) dx + \frac{\varepsilon}{2} \int_{T^3} n^{-12}(1 + v^2) \log(1 + |v|^2) dx - \varepsilon \int_{T^3} n^{-12}|v|^2(1 + \log(1 + |v|^2)) dx + \int_{T^3} v \cdot \nabla n(1 + \log(1 + |v|^2)) dx \\
+ \kappa \int_{T^3} n u \cdot v(1 + \log(1 + |v|^2)) dx. \tag{3.24}
\]

The first term on right-hand side of (3.24) can be estimated as
\[
\frac{\varepsilon}{2} \int_{T^3} (1 + |v|^2) \sqrt{\eta \epsilon} \Delta \sqrt{n} \log(1 + |v|^2) dx \\
\leq -\frac{\varepsilon}{8} \int_{T^3} |\nabla \sqrt{n}|^2(1 + |v|^2) \log(1 + |v|^2) dx + \frac{\varepsilon}{2} \int_{T^3} n \log(1 + |v|^2) |\nabla v|^2 dx + C \varepsilon \int_{T^3} |\nabla \sqrt{n}|^2 |v|^2 dx + C \varepsilon \int_{T^3} n |\nabla v|^2 dx. \tag{3.25}
\]

In addition, one has
\[
\frac{\varepsilon}{2} \int_{T^3} n^{-12}(1 + |v|^2) \log(1 + |v|^2) dx - \varepsilon \int_{T^3} n^{-12}|v|^2(1 + \log(1 + |v|^2)) dx \leq C \varepsilon \int_{T^3} n^{-25} dx + C \varepsilon,
\]
and
\[
\int_{T^3} v \cdot \nabla n(1 + \log(1 + |v|^2)) dx \leq C \int_{T^3} n |v|^2 dx + C \int_{T^3} n |\nabla v|^2.
\]

For the last term on the right-hand side of (3.24), we make use of (3.1), (3.11) and the fact \( \log(1 + |v|^2) \leq |v|^\frac{5}{4} \) that
\[
\kappa \int_{T^3} n u \cdot v(1 + \log(1 + |v|^2)) dx \\
\leq C \|u\|_{L^\infty} \|\sqrt{n}v\|_{L^2}^2 + \|n^{\frac{5}{4}}\|_{L^\infty} \|n^{\frac{5}{4}}v^{\frac{5}{4}}\|_{L^2} \tag{3.26}
\]
\[
\leq C \|\nabla u\|_{L^2} \|\sqrt{n}v\|_{H^1} \|\sqrt{n}v\|_{L^2} + \|\sqrt{n}\|_{H^1} \|\sqrt{n}v\|_{L^2} \leq C \int_{T^3} |\nabla v|^2 dx + C.
\]

Inserting the above estimates (3.25)-(3.26) into (3.24) and applying the Grönwall inequality, we gain (3.23). The proof of Lemma 3.3 is completed. \( \square \)
Then, we establish the uniform $L^{γ+1}(0, T; L^{γ+1}(T^3))$ estimate of $ρ$ so as to show the strong convergence of the pressure.

**Lemma 3.4.** For any $ε ∈ (0, {1 \over 2})$, $δ ∈ (0, 1)$ and given time $T > 0$, let $(u, v, ρ, u)$ be any strong solution to the IVP (2.1) for $t ∈ (0, T]$. Then, under the assumptions of Theorem 1.1, we have

$$∫_0^T ∫_{T^3} (ρ^{γ+1} + δρ^{γ0+1})dxdt ≤ C_{δ,T},$$

(3.27)

where $C_{δ,T} > 0$ is a constant independent of $ε$.

**Proof.** Denote the operator $(-Δ)^{-1} : W^{k-2,p}(T^d) → W^{k,p}(T^d)$ for $k ∈ R$ and $p ∈ (1, ∞)$ by

$$(-Δ)^{-1} f = g, \quad g \text{ is the solution of the problem } -Δ g = f, \quad ∫_{T^3} f = 0.$$ (3.28)

One obtains after applying $(-Δ)^{-1} \text{ div}$ to (2.1) that

$$Aρ^{γ} + δρ^{γ0} - (2μ + λ) \text{ div } u
= (-Δ)^{-1} \text{ div}(ρu) + (-Δ)^{-1} \text{ div}(ρu ⊗ u) + κ(-Δ)^{-1} \text{ div}(n(u - v))$$

$$+ ε(-Δ)^{-1} \text{ div}(∇ρ ⊗ u) + ε(-Δ)^{-1} \text{ div}(|u|^8 u).$$

(3.29)

Then we multiply (3.29) by $ρ$ and integrate the resulted equality over $T^3 × (0, T)$ to gain

$$∫_0^T ∫_{T^3} (Aρ^{γ+1} + δρ^{γ0+1})dxdt$$

$$= ∫_{T^3} ρ(-Δ)^{-1} \text{ div}(ρu)|_{t=0}^t$$

$$+ ∫_0^T ∫_{T^3} ρ(-Δ)^{-1} \text{ div}(ρu ⊗ u) - u∇(-Δ)^{-1} \text{ div}(ρu)]dxdt$$

$$+ κ ∫_0^T ∫_{T^3} ρ(-Δ)^{-1} \text{ div}(n(u - v))dxdt$$

$$+ ε ∫_0^T ∫_{T^3} (∇ρ · ∇(-Δ)^{-1} \text{ div}(ρu) + ρ(-Δ)^{-1} \text{ div}(∇ρ ⊗ u) + ρ(-Δ)^{-1} \text{ div}(|u|^8 u))dxdt.$$ (3.30)

It can be verified by (3.1), (3.11), the Sobolev inequality and $L^p(T^3)$ ($p ∈ (1, ∞)$) boundedness of the operator $(-Δ)^{-1}∂_i∂_j$ for $1 ≤ i, j ≤ 3$ that

$$κ ∫_0^T ∫_{T^3} ρ(-Δ)^{-1} \text{ div}(n(u - v))dxdt$$

$$≤ ||ρ||_{L^2(0,T;L^2)} ||n(u - v)||_{L^2(0,T;L^p(T^3))}$$

$$≤ CT^{{1 \over 2}} ||ρ||_{L^∞(0,T;L^∞(T^3))} ||∇n||_{L^∞(0,T;L^2(T^3))} ||n(u - v)||_{L^2(0,T;L^2(T^3))} ≤ C_{δ,T}.$$ (3.31)

Similarly, due to $γ_0 > 4$, one has

$$ε ∫_0^T ∫_{T^3} (∇ρ · ∇(-Δ)^{-1} \text{ div}(ρu) + ρ(-Δ)^{-1} \text{ div}(∇ρ ⊗ u) + ρ(-Δ)^{-1} \text{ div}(|u|^8 u))dxdt$$

$$≤ ε||∇ρ||_{L^2(0,T;L^2)} ||ρu||_{L^2(0,T;L^2)} + Cε||ρ||_{L^∞(0,T;L^∞(T^3))} ||∇ρ||_{L^2(0,T;L^2)} ||u||_{L^2(0,T;L^6(T^3))}$$

$$+ Cε||ρ||_{L^∞(0,T;L^{12}(T^3))} ||u^9||_{L^1(0,T;L^{12}(T^3))} ≤ C_{δ,T}.$$ (3.32)
Since the other terms on the right-hand side of (3.30) can be controlled in a standard way as [18, 35], we omit the detail.

Inspired by [28], we take the advantage of Lemma 3.1-3.4 and the De Giorgi iteration to have the upper and lower bounds of two densities.

Lemma 3.5. For any $\varepsilon \in (0, \frac{1}{4})$, $\delta \in (0,1)$ and given time $T > 0$, let $(n, v, \rho, u)$ be any strong solution to the IVP (2.1) for $t \in (0, T)$. Then, under the assumptions of Theorem 1.1, it holds

$$0 < \frac{1}{C_{\varepsilon,T}} \leq n(x, t) \leq C_{\varepsilon,T}, \quad 0 < \frac{1}{C_{\varepsilon,T}} \leq \rho(x, t) \leq C_{\varepsilon,T}, \quad (x, t) \in \mathbb{T}^3 \times (0, T),$$

(3.31)

where $C_{\varepsilon,T} > 1$ is a constant.

Proof. First, it holds by (3.1), (3.11) and the Sobolev inequality that

$$\sup_{(x,t) \in \mathbb{T}^3 \times (0, T)} n(x, t) \leq C \sup_{t \in [0, T]} \|\sqrt{n(t)}\|_{L^2}^2 \leq C_{\varepsilon,T}.$$

(3.32)

Next, the lower bound of $n$ can be shown by a De Giorgi-type procedure. It is easy to verify that $\vartheta := n^{-\frac{1}{2}}$ satisfies

$$2\vartheta_t + 2v \cdot \nabla \vartheta - \vartheta \div v + \varepsilon \vartheta^{27} + 2\varepsilon \vartheta^{-1} |\nabla \vartheta|^2 + 2\varepsilon \vartheta^{-5} |\nabla \vartheta|^4 = \varepsilon \Delta \vartheta + \varepsilon \div (\vartheta^{-4} |\vartheta|^2 \nabla \vartheta).$$

(3.33)

Multiplying (3.33) by $(\vartheta - k)_+$ with $k \geq \|n_{0,\vartheta}^{-\frac{1}{2}}\|_{L^\infty(\mathbb{T}^3)}$ and integrating the resulted equation over $\mathbb{T}^3 \times (0, t)$, we have

$$\sup_{t \in [0,T]} \int_{\mathbb{T}^3} |(\vartheta - k)_+|^2 dx + \varepsilon \int_0^T \int_{\mathbb{T}^3} (|\nabla (\vartheta - k)_+|^2 + n^2 |\nabla (\vartheta - k)_+|^4) dx$$

$$\leq 3 \int_0^T \int_{\mathbb{T}^3} (\vartheta - k)_+ |v| |\nabla \vartheta| dx dt + \int_0^T \int_{\mathbb{T}^3} \vartheta |v| |\nabla (\vartheta - k)_+| dx dt$$

$$\leq 4 \int_0^T \int_{\mathbb{T}^3} n^{-\frac{1}{2}} |v| |\nabla (\vartheta - k)_+| dx dt$$

$$\leq \frac{\varepsilon}{2} \int_0^T \int_{\mathbb{T}^3} n^2 |\nabla (\vartheta - k)_+|^4 dx dt + C \int_0^T \int_{\mathbb{T}^3} n^{-\frac{1}{2}} |v| \|\vartheta_{\geq k}\|_{L^\infty} dx dt,$$

(3.34)

where one has used $(\vartheta - k)_+|_{t=0} = 0$ and $(\vartheta - k)_+ \leq \vartheta$. It follows from (3.1) that

$$\int_0^T \int_{\mathbb{T}^3} n^{-\frac{1}{2}} |v| \|\vartheta_{\geq k}\|_{L^\infty} dx dt$$

$$\leq (\int_0^T \int_{\mathbb{T}^3} n|v|^5 dx dt)^{\frac{1}{5}} \left( \int_0^T \int_{\mathbb{T}^3} n^{-\frac{4}{5}} \|\vartheta_{\geq k}\|_{L^\infty} dx \right)^{\frac{4}{5}}$$

$$\leq T^{\frac{1}{5}} \left( \int_0^T \int_{\mathbb{T}^3} n|v|^5 dx dt \right)^{\frac{1}{5}} \left( \int_0^T \int_{\mathbb{T}^3} n^{-25} dx \right)^{\frac{1}{5}} \|\vartheta_{\geq k}\|_{L^\infty} \|\vartheta\|_{L^\infty} \|\vartheta\|_{L^\infty} \leq C_{\varepsilon,T} |\vartheta|_{L^\infty} |\vartheta|_{L^\infty} \|\vartheta_{\geq k}\|_{L^\infty}.$$

This together with (3.34) yields

$$\sup_{t \in [0,T]} \int_{\mathbb{T}^3} (\vartheta - k)_+^2 dx + \varepsilon \int_0^T \int_{\mathbb{T}^3} (|\nabla (\vartheta - k)_+|^2 + n^2 |\nabla (\vartheta - k)_+|^4) dx dt \leq C_{\varepsilon,T} |\vartheta|_{L^\infty} |\vartheta|_{L^\infty} \|\vartheta_{\geq k}\|_{L^\infty}.$$
On the other hand, using the Gagliardo-Nirenberg inequality, we deduce for any $h > k$ that

$$|\vartheta > h| \leq (h - k)^{-\frac{q}{p}} \|\vartheta - k\|_p \|\varpartial - k\|^\frac{q}{1 - p} = C(h - k)^{-\frac{q}{p}} \left( \sup_{t \in [0,T]} \int_{T^3} |(\vartheta - k)|^2 \, dx \right)^{\frac{q}{2}} \left( \int_0^T \int_{T^3} |\nabla(\vartheta - k)|^2 \, dx \, dt \right)^{\frac{1}{2}}. \tag{3.36}$$

Combining (3.35)-(3.36) together, we get

$$|\vartheta > h| \leq C\varepsilon (h - k)^{-\frac{q}{p}} |\vartheta > k|^{\frac{q}{2}, \frac{k}{2}}, \quad h > k.$$ 

Thus, by virtue of the De Giorgi lemma (see, e.g. [50, Lemma 4.1.1]), there is a constant $C\varepsilon > 0$ such that

$$\vartheta(x, t) \leq C\varepsilon, \quad (x, t) \in T^3 \times (0, T),$$

the lower bound of $n$ follows.

Furthermore, we turn to show the upper bound of $\rho$. By the Sobolev inequality, one obtains after multiplying (2.1)$_3$ by $p \rho^{p - 1}$ for $p \in [2, \infty)$ that

$$\frac{d}{dt} \int_{T^3} \rho^p \, dx + \varepsilon p(\rho - 1) \int_{T^3} \rho^{p - 2} |\nabla \rho|^2 \, dx$$

$$\leq p(\rho - 1) ||\nabla||^2 \|\rho\|^{2 - \frac{q}{p} + \varepsilon} L^2(\|\nabla\|^2)$$

$$\leq p(\rho - 1) ||\nabla||^2 \|\rho\|^{2 - \frac{q}{p} + \varepsilon} L^2(\|\nabla\|^2 + ||\nabla||^2)$$

$$\leq \varepsilon p(\rho - 1) \int_{T^3} \rho^{p - 2} |\nabla \rho|^2 \, dx + C\varepsilon(1 + ||\nabla||^{10}) \int_{T^3} \rho^p \, dx,$$

from which and (3.1) we deduce

$$\sup_{t \in [0,T]} \int_{T^3} \rho^p \, dx \leq C_p, \quad p \in [2, \infty), \tag{3.37}$$

where $C_p$ is a constant dependent of $\varepsilon, T$ and $p$. Hence we take the $L^2(T^3)$ inner product of (2.1)$_3$ with $(\rho - \ell)_+$ for $\ell \geq \|\rho_0, \ell\|_{L^{\infty}}$ to have

$$\sup_{t \in [0,T]} \int_{T^3} |(\rho - \ell)_+|^2 \, dx + \varepsilon \int_0^T \int_{T^3} |\nabla(\rho - \ell)_+|^2 \, dx \, dt$$

$$\leq \int_0^T \int_{T^3} \rho u \cdot \nabla(\rho - \ell)_+ \, dx \, dt$$

$$\leq \frac{1}{\varepsilon} \int_0^T \int_{T^3} \mathbb{I}_{\varrho > \ell} \rho^2 |\nabla|^2 \, dx \, dt + \frac{\varepsilon}{4} \int_{T^3} |\nabla(\rho - \ell)_+|^2 \, dx \, dt. \tag{3.38}$$

Due to (3.1) and (3.37), the first term on the right-hand side of (3.38) can be estimated by

$$\int_0^T \int_{T^3} \mathbb{I}_{\varrho > \ell} \rho^5 \, dx \, dt \leq \left( \int_0^T \int_{T^3} \rho^5 \, dx \, dt \right)^{\frac{1}{5}} \left( \int_0^T \int_{T^3} \mathbb{I}_{\varrho > \ell} \rho^2 \, dx \, dt \right)^{\frac{1}{2}} \left( \int_0^T \int_{T^3} \rho^{10} \, dx \, dt \right)^{\frac{1}{10}}$$

$$\leq C_{\varepsilon, T} |\rho |^{\frac{1}{5}, \frac{2}{5}, \frac{1}{10}}.$$
This implies for any \( m > \ell \) that
\[
|\rho > m| \leq (m - \ell)^{-\frac{m}{2}} \|(\rho - \ell)_+\|_{L^\infty(0,T;L^\infty)}^\frac{m}{2} \int_0^T \int_{T^3} |\nabla(\rho - \ell)_+|^2 dxdt)^\frac{1}{2}.
\]
\[
\leq T^\frac{2}{3} (m - \ell)^{-\frac{m}{2}} \left( \sup_{t \in [0,T]} \int_{T^3} (\rho - \ell)^2 dx \right)^\frac{2}{3} \left( \int_0^T \int_{T^3} |\nabla(\rho - \ell)_+|^2 dxdt \right)^\frac{1}{3}
\]
\[
\leq C_{\varepsilon,T}(m - \ell)^{-\frac{m}{2}} |\rho > \ell|^{\frac{m}{2}}.
\]
Thus, the De Giorgi lemma implies the upper bound of \( \rho \).

Finally, it is easy to verify that \( \rho^{-1} \) satisfies
\[
(\rho^{-1})_t - \text{div}(\rho^{-1}u) + 2u \cdot \nabla \rho^{-1} + 2\varepsilon \rho |\nabla \rho^{-1}|^2 = \varepsilon \Delta \rho^{-1}.
\]  
(3.39)

Note that the negative term \( 2\varepsilon \rho |\nabla \rho^{-1}|^2 \) in (3.39) don’t influence the energy estimates, and therefore we are able to estimate the \( L^\infty(0,T;L^p(T^3)) \) of \( \rho^{-1} \) for any \( p \in [2,\infty) \) and employ a similar De Giorgi-type procedure to derive the lower bound of \( \rho \). The proof of Lemma 3.5 is completed.

\[\square\]

**Lemma 3.6.** For any \( \varepsilon \in (0,\frac{1}{2}) \), \( \delta \in (0,1) \) and given time \( T > 0 \), let \( (n,v,\rho,u) \) be any strong solution to the IVP (2.1) for \( t \in (0,T) \). Then, under the assumptions of Theorem 1.1, it holds
\[
\sup_{t \in [0,T]} \|(n,v,\rho,u)(t)\|_{H^2} + \int_0^T \left( \|(n,v,\rho,u)(t)\|_{H^3}^2 + \|(n_1,\nu,\rho_1,u_1)(t)\|_{H^1}^2 \right) dt \leq C_{\varepsilon,T},
\]
(3.40)
where \( C_{\varepsilon,T} > 0 \) is a constant.

**Proof.** First, it is easy to verify that
\[
2(\sqrt{n})_t - \varepsilon \text{div}(1 + |\nabla \sqrt{n}|^2)\nabla \sqrt{n} = - \text{div}(\sqrt{n}v + \nabla W) - \frac{1}{|T^3|} \int_{T^3} (v \cdot \nabla \sqrt{n} - \varepsilon \sqrt{n}^{-25}) dx,
\]  
(3.41)
where \( W(\cdot,t) \) for \( t > 0 \) is the unique solution of the following elliptic problem:
\[
\Delta W = v \cdot \nabla \sqrt{n} - \varepsilon \sqrt{n}^{-25} - \frac{1}{|T^3|} \int_{T^3} (v \cdot \nabla \sqrt{n} - \varepsilon \sqrt{n}^{-25}) dx, \quad \int_{T^3} W dx = 0.
\]  
(3.42)
Applying (3.1), (3.11), (3.31) and \( L^p \)-estimates of the elliptic equation (3.42)(cf. [50]) that
\[
\|\nabla W(t)\|_{L^p} \leq \|\nabla^2 W(t)\|_{L^\frac{3p}{2p-3p}} \leq C_p \|v(t)\|_{L^p} \|\nabla \sqrt{n}(t)\|_{L^1} \|\nabla \sqrt{n}(t)\|_{L^2}^{\frac{1}{2}} + C_p \|v(t)\|_{L^p}, \quad t > 0, \quad p \in (1,\infty).
\]  
(3.43)
Setting
\[
W^* := \sqrt{n} + \frac{1}{2|T^3|} \int_0^T \int_{T^3} (v \cdot \nabla W^* - \varepsilon \sqrt{n}^{-25}) dxdt,
\]
we rewrite (3.41) by
\[
2W^*_t - \varepsilon \text{div}(|\nabla W^*|^2|\nabla W^*|) = \text{div}(\varepsilon \nabla W^* - \sqrt{n}v - \nabla W).
\]  
(3.44)
One derives from (3.1), (3.11), (3.31), (3.43) and $L^p$-estimates of the parabolic equation (3.44) (cf. [1]) that
\[
\int_0^T \|\nabla \sqrt{n}(t)\|^2_{L^p} dt = \int_0^T \|\nabla W^*(t)\|^2_{L^p} dt \\
\leq C_p + C_p(\int_0^T \|v(t)\|^p_{L^p} dt)^2 + \frac{1}{2} \int_0^T \|\nabla \sqrt{n}(t)\|^2_{L^p} dt, \quad p \in (1, \infty),
\]
which implies
\[
\int_0^T \|\nabla \sqrt{n}(t)\|^2_{L^p} dt \leq C_p + C_p(\int_0^T \|v(t)\|^p_{L^p} dt)^2, \quad p \in (1, \infty). \tag{3.45}
\]

Next, we show the high-order estimates of $v$. Note that the equation (2.1) can be rewrite as
\[
v_t - \left(\frac{n}{2} + \sqrt{\epsilon} \right) \Delta v - \frac{n}{2} \nabla \div v + \epsilon |v|^3 v + \kappa v \\
= -v \cdot \nabla v - n^{-1} \nabla n + \kappa u - \epsilon n^{-3} v + \epsilon n^{-\frac{1}{2}} |\nabla \sqrt{n}|^2 \nabla \sqrt{n} \cdot \nabla v. \tag{3.46}
\]

Multiplying (3.46) by $-2\Delta v$ and integrating the resulted equation over $T^3$, we get by (3.1) and (3.31) that
\[
\frac{d}{dt} \int_{T^3} |v|^2 dx + \int_{T^3} \left((\eta + 2\sqrt{\epsilon})|\Delta v|^2 + \eta |\nabla \div v|^2 + 2\varepsilon |v|^3 |\nabla v|^2 + 2\varepsilon |v|^3 |\nabla v|^2 + 2\kappa |\nabla v|^2\right) dx \\
\leq \|\Delta v\|_{L^2}^2 \|v\|_{L^5}^2 \|\nabla v\|_{L^2}^2 + \|\nabla n\|_{L^2}^2 \|v\|_{L^2}^2 + \|u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\nabla \sqrt{n}^3\|_{L^3} \|\nabla v\|_{L^2}^2 \|\Delta v\|_{L^2}^2 \\
\leq \frac{\eta}{2} \|\Delta v\|_{L^2}^2 + C(\|v\|_{L^5}^2 + \|\nabla \sqrt{n}^3\|_{L^3}^2) \|\nabla v\|_{L^2}^2 + C_{\varepsilon, T}, \tag{3.47}
\]
which together with (3.45), (3.47), the Grönwall inequality and the Sobolev inequality gives rise to
\[
\sup_{t \in [0,T]} \|\nabla v(t)\|_{L^2}^2 + \|\nabla^2 v\|_{L^2(0,T;L^2)}^2 + \|v\|_{L^1(0,T;L^\infty)}^2 + \|\nabla v\|_{L^2(0,T;L^\infty)}^2 \\
+ \|\sqrt{n}\|_{L^2(0,T;L^\infty)}^2 \leq C_{\varepsilon,T}. \tag{3.48}
\]
The Sobolev inequality (cf. [27]), (3.45) and (3.48) thus imply that
\[
\|v\|_{L^q(0,T;L^p)} + \|v\|_{L^q(0,T;L^p)} + \|\sqrt{n}\|_{L^q(0,T;L^p)} \leq C_{\varepsilon,T}, \quad q \in [2, \infty), \tag{3.49}
\]
and therefore it holds by (3.49), $L^p$ estimates for the parabolic equations (3.44) and (3.46) and the Sobolev inequality (cf. [27]) that
\[
\|v\|_{L^\infty(0,T;L^\infty)} + \|(\nabla^2 \sqrt{n}, \nabla^2 v)\|_{L^2(0,T;L^2)} \\
\leq C_{\varepsilon,T}(1 + \|v\|_{L^2(0,T;L^2)}^2) \|\nabla v\|_{L^2(0,T;L^2)} + \|\sqrt{n}\|_{L^2(0,T;L^2)} \tag{3.50}
\]
With the help of (3.1), (3.11), (3.31) and (3.48)-(3.50), we employ $L^p$ estimates for the parabolic equations (3.44) and (3.46) again to gain
\[
\sup_{t \in [0,T]} \|(n, v)(t)\|_{H^2} + \|(n, v)\|_{L^2(0,T;H^3)} + \|(n_t, v_t)\|_{L^2(0,T;H^1)} \\
\leq C_{\varepsilon,T}(1 + \|v\|_{L^2(0,T;H^1)} + \|\sqrt{n}\|_{L^2(0,T;H^1)} + \|\sqrt{n}\|_{L^2(0,T;H^1)} \|v\|_{L^2(0,T;H^1)} \leq C_{\varepsilon,T},
\]

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where we have used the fact
\[
\| \sqrt{n} \|_{L^2(0,T;H^1)}^3 \| \nabla v \|_{L^2(0,T;L^2(T^3))}^2 + \| \sqrt{n} \|_{L^2(0,T;L^2(T^3))} \leq \| \sqrt{n} \|_{L^2(0,T;L^2(T^3))}^3 \| \nabla v \|_{L^2(0,T;L^2(T^3))}^2 \| \nabla v \|_{L^2(0,T;L^2(T^3))}^2.
\]
By similar computations, one can show the expected high-order estimates of \((\rho, u)\). The proof of Lemma 3.6 is completed. \(\square\)

4 Vanishing artificial viscosities

In this section, for \(\delta \in (0, 1)\) and \(\gamma_0 > \max\{\gamma + 4\}\), we turn to consider the following IVP:

\[
\begin{cases}
n_t + \text{div}(nv) = 0, \\
(nv)_t + \text{div}(nv \otimes v) + \nabla n = -\kappa n(v - u) + \eta \text{div}(n\mathbb{D}(v)), \\
\rho_t + \text{div}(\rho u) = 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla (A\rho^\gamma + \delta\rho^\gamma_0) = \kappa n(v - u) + \mu \Delta u + (\mu + \lambda)\text{div} u, \\
(n, v, \rho, u)(x, 0) = (n_0, v_0, \rho_0, u_0)(x), \quad x \in T^3,
\end{cases}
\]

where \((n_0, v_0, \rho_0, u_0)\) is given by (2.2).

We have the global existence of weak solutions to the IVP (4.1) below.

**Proposition 4.1.** Let \(\delta \in (0, 1)\), and the assumptions of Theorem 1.1 hold. Then there exists a global weak solution \((n, nv, \rho, pu)\) to the IVP (4.1) with \(n, \rho \geq 0\) satisfying the following properties for any \(T > 0\):

- **The conservation laws of momentum and mass hold:**
  \[
  \int_{T^3} n dx = \int_{T^3} n_0 dx, \quad \int_{T^3} \rho dx = \int_{T^3} \rho_0 dx, \quad t \in (0, T),
  \]
  \[
  \int_{T^3} (nv + pu) dx = \int_{T^3} (n_0 v_0 + \rho_0 u_0) dx, \quad t \in (0, T).
  \]

- **The energy inequality holds:**
  \[
  \begin{align*}
  \sup_{t \in [0,T]} \int_{T^3} & \left( \frac{1}{2} u^2 + n \log n - n + 1 + \frac{1}{2} \rho |u|^2 + \frac{A \rho^\gamma}{\gamma - 1} + \frac{\delta \rho^\gamma_0}{\gamma_0 - 1} \right) dx \\
  + & \int_{0}^{t} \int_{T^3} (\kappa n|v - u|^2 + \mu |u|^2 + (\mu + \lambda)(\text{div} u)^2) dx d\tau \leq E_{0, \delta},
  \end{align*}
  \]
  where \(E_{0, \delta}\) is given by (2.5).
• Bresch-Desjardins type entropy inequality holds:

There exists a constant \( C > 0 \) uniformly in \( \delta \) and \( T \) such that

\[
\sup_{t \in [0, T]} \int_{\mathbb{T}^3} |\nabla \sqrt{n}|^2 dx + \int_0^T \int_{\mathbb{T}^3} |\nabla \sqrt{n}|^2 dx dt \leq C, \tag{4.5}
\]

• The Mellet-Vasseur type estimate follows:

There exists a constant \( C_T > 0 \) independent of \( \delta \) such that

\[
\text{ess sup}_{t \in [0, T]} \int_{\mathbb{T}^3} n(1 + |v|^2) \log (1 + |v|^2) dx \leq C_T. \tag{4.6}
\]

**Proof of Proposition 4.1.** For \( \varepsilon \in (0, \frac{1}{2} \gamma_0) \), \( \delta \in (0, 1) \) and \( \gamma_0 > \max\{\gamma + 4\} \), let \((n_\varepsilon, v_\varepsilon, \rho_\varepsilon, u_\varepsilon)\) be the global strong solution to the approximate problem (2.1)-(2.6) given by Proposition 2.1. For any given time \( T > 0 \), it follows from the a-priori estimates established in Lemmas 3.1-3.4 that there is a limit \((n, m, \rho, u)\) such that as \( \varepsilon \to 0 \), we have

\[
\begin{align*}
&n_\varepsilon \rightharpoonup n \quad \text{in} \quad L^\infty(0, T; L^1(\mathbb{T}^3)), \\
n_\varepsilon v_\varepsilon \rightharpoonup m \quad \text{in} \quad L^\infty(0, T; L^1(\mathbb{T}^3)), \\
\rho_\varepsilon \to \rho \quad \text{in} \quad L^\infty(0, T; L^{\gamma_0}(\mathbb{T}^3)), \\
u_\varepsilon \to u \quad \text{in} \quad L^2(0, T; H^1(\mathbb{T}^3)).
\end{align*}
\tag{4.7}
\]

One deduces by (3.1) and (3.11) that

\[
\sup_{\varepsilon \in (0, \frac{1}{2})} \|\nabla n_\varepsilon\|_{L^\infty(0, T; L^2)} \leq 2 \sup_{\varepsilon \in (0, \frac{1}{2})} \left( \|\nabla \sqrt{n_\varepsilon}\|_{L^\infty(0, T; L^2)} \|\nabla \sqrt{n_\varepsilon}\|_{L^\infty(0, T; L^2)} \right) < \infty,
\]

and

\[
\sup_{\varepsilon \in (0, \frac{1}{2})} \|(n_\varepsilon)\varepsilon\|_{L^2(0, T; W^{-1,1})} \leq C \sup_{\varepsilon \in (0, \frac{1}{2})} \left( \|n_\varepsilon v_\varepsilon\|_{L^2(0, T; H^1)} + \|\nabla n_\varepsilon \Delta \nabla n_\varepsilon\|_{L^2(0, T; L^1)} + \|\nabla \sqrt{n_\varepsilon} \Delta \nabla n_\varepsilon\|_{L^2(0, T; L^1)} \right) \\
+ \varepsilon \|n_\varepsilon^{-12}\|_{L^2(0, T; L^1)} \leq C \sup_{\varepsilon \in (0, \frac{1}{2})} \left( \|\nabla n_\varepsilon\|_{L^\infty(0, T; L^2)} \|\nabla n_\varepsilon\|_{L^\infty(0, T; L^2)} + \|\nabla \sqrt{n_\varepsilon} \|_{L^\infty(0, T; L^2)} \|\Delta \nabla \sqrt{n_\varepsilon}\|_{L^2(0, T; L^1)} \right) \\
+ \varepsilon \|\nabla \sqrt{n_\varepsilon}\|_{L^\infty(0, T; H^1)} \|\nabla \sqrt{n_\varepsilon}\|_{L^\infty(0, T; L^6)} + \|n_\varepsilon^{-25}\|_{L^1(0, T; L^1)} \leq C(1 + \varepsilon^{\frac{1}{4}}) < \infty,
\]

where in the last inequality we have used the fact

\[
\varepsilon^{\frac{1}{4}} \int_0^T \|\nabla \sqrt{n_\varepsilon}\|_{L^6}^t dt \leq \int_0^T \|\nabla \sqrt{n_\varepsilon}\|_{L^2}^t \left( \varepsilon \|\nabla \sqrt{n_\varepsilon}\|_{L^2}^t \right)^{\frac{1}{4}} \left( \varepsilon \|\nabla \sqrt{n_\varepsilon}\|_{L^2}^t \right)^{\frac{9}{12}} dt \leq C \varepsilon \int_0^T \|\nabla \sqrt{n_\varepsilon}\|_{L^2}^t dt \leq C.
\]
Thus, one employs the Aubin-Lions lemma and $W^{1,\frac{3}{2}}(\mathbb{T}^3) \hookrightarrow L^p(\mathbb{T}^3)$ ($1 \leq p < 3$) to derive (up to a sequence) that

\[
\begin{align*}
 n_\varepsilon &\rightarrow n \quad \text{in} \quad C([0, T]; L^p(\mathbb{T}^3)), \quad 1 \leq p < 3, \\
 \sqrt{n_\varepsilon} &\rightharpoonup \sqrt{n} \quad \text{in} \quad L^\infty(0, T; H^1(\mathbb{T}^3)), \\
 \sqrt{n_\varepsilon} &\rightarrow \sqrt{n} \quad \text{in} \quad C([0, T]; L^p(\mathbb{T}^3)), \quad 1 \leq p < 6. \\
\end{align*}
\tag{4.8}
\]

Due to (3.1) and (3.11), it is easy to show

\[
\sup_{\varepsilon \in (0, \frac{1}{4})} \left( \|\nabla (n_\varepsilon v_\varepsilon)\|_{L^2(0, T; L^1)} + \|(n_\varepsilon v_\varepsilon)\|_{L^1(0, T; H^{-6})} \right) < \infty,
\]

for some suitably large constant $s_0 > 0$. Therefore, we apply Aubin-Lions lemma and Lemma C.1 in [34] to gain

\[
\begin{align*}
 n_\varepsilon v_\varepsilon &\rightarrow n v \quad \text{in} \quad L^2(0, T; L^p(\mathbb{T}^3)), \quad 1 \leq p < \frac{3}{2}, \\
 n_\varepsilon v_\varepsilon &\rightarrow n v \quad \text{in} \quad C([0, T]; L^\frac{2}{3}(\mathbb{T}^3)), \\
\end{align*}
\tag{4.9}
\]

where $C([0, T]; X_{\text{weak}})$ denote the space of continuous functions on $[0, T]$ with values in $X$ equipped with the weak topology, and $v$ is given by

\[
v := \begin{cases} 
\frac{m}{n}, & \text{if } n > 0, \\
0, & \text{if } n = 0.
\end{cases}
\]

Similarly, one gets as $\varepsilon \to 0$ that

\[
\begin{align*}
 \rho_\varepsilon &\rightarrow \rho \quad \text{in} \quad C([0, T]; H^{-1}(\mathbb{T}^3)), \\
 \rho_\varepsilon &\rightarrow \rho \quad \text{in} \quad C([0, T]; L^\gamma_{\text{weak}}(\mathbb{T}^3)), \\
 \rho_\varepsilon u_\varepsilon &\rightarrow \rho u \quad \text{in} \quad C([0, T]; L^{\frac{2}{1+\gamma}}_{\text{weak}}(\mathbb{T}^3)), \\
 \rho_\varepsilon u_\varepsilon &\rightarrow \rho u \quad \text{in} \quad C([0, T]; H^{-1}(\mathbb{T}^3)), \\
\end{align*}
\tag{4.10}
\]

By (??)-(4.10), it also holds

\[
\begin{align*}
 \rho_\varepsilon u_\varepsilon \otimes u_\varepsilon &\rightarrow \rho u \otimes u \quad \text{in} \quad \mathcal{D}'(\mathbb{T}^3 \times (0, T)), \\
 n_\varepsilon v_\varepsilon - n_\varepsilon u_\varepsilon &\rightarrow n v - n u \quad \text{in} \quad \mathcal{D}'(\mathbb{T}^3 \times (0, T)). \\
\end{align*}
\tag{4.11}
\]

In addition, as $\varepsilon \to 0$, it is easy to verify that

\[
\begin{align*}
 \varepsilon \sqrt{n_\varepsilon} \Delta \sqrt{n_\varepsilon} + \varepsilon \sqrt{n} \text{div}(\nabla \sqrt{n_\varepsilon})^2 \nabla \sqrt{n_\varepsilon} + \varepsilon n^{-1}_\varepsilon 2 &\rightarrow 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{T}^3 \times (0, T)), \\
 \sqrt{n} \text{div}(n_\varepsilon \nabla v_\varepsilon) + \varepsilon \sqrt{n_\varepsilon} \text{div}(\nabla \sqrt{n_\varepsilon})^2 \nabla \sqrt{n_\varepsilon} v_\varepsilon &\rightarrow 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{T}^3 \times (0, T)), \\
 \varepsilon \nabla \varepsilon \nabla \varepsilon - \varepsilon n_\varepsilon |v_\varepsilon|^3 v_\varepsilon &\rightarrow 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{T}^3 \times (0, T)), \\
 \varepsilon \Delta n_\varepsilon &\rightarrow 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{T}^3 \times (0, T)), \\
 -\varepsilon \nabla u_\varepsilon \cdot \nabla \rho_\varepsilon - \varepsilon |u_\varepsilon|^8 u_\varepsilon &\rightarrow 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{T}^3 \times (0, T)).
\end{align*}
\tag{4.12}
\]
Next, we gain the strong convergence of $\sqrt{n_\varepsilon}v_\varepsilon$. For any constant $L > 0$, write

$$
\int_0^T \int_{\mathbb{T}^3} |\sqrt{n_\varepsilon}v_\varepsilon - \sqrt{n}v|^2 dx dt
\leq \int_0^T \int_{\mathbb{T}^3} |\sqrt{n_\varepsilon}v_\varepsilon 1_{|v_\varepsilon| \leq L} - \sqrt{n}v 1_{|v| \leq L}|^2 dx dt
+ 2 \int_0^T \int_{\mathbb{T}^3} |\sqrt{n_\varepsilon}v_\varepsilon 1_{|v_\varepsilon| > L}|^2 dx dt + 2 \int_0^T \int_{\mathbb{T}^3} |\sqrt{n}v 1_{|v| > L}|^2 dx dt.
\tag{4.13}
$$

We need to show that the first term on the right-hand side of (4.13) tends to 0 as $\varepsilon \to 0$. Let $\zeta > 0$, and $Q_\zeta := \{(x, t) \in \mathbb{T}^3 \times (0, T) \mid n(x, t) \geq \zeta > 0\}$. According to (4.8) and the Egorov theorem, for any $\zeta' > 0$, there is a sufficiently small constant $\varepsilon_1 \in (0, \varepsilon_0)$ and a set $\tilde{Q}_{\zeta'} \subset Q_\zeta$ such that for any $\varepsilon \in (0, \varepsilon_1)$, it holds

$$
|Q_\zeta / Q_{\zeta'}| \leq \zeta', \quad n(x, t) \geq \frac{\zeta}{2} > 0, \quad (x, t) \in \tilde{Q}_{\zeta'}.
\tag{4.14}
$$

Thus, there follows by (4.8), (4.9) and (4.14) that

$$
\sqrt{n_\varepsilon}v_\varepsilon \to \sqrt{n}v \quad \text{a.e. in} \quad \tilde{Q}_{\zeta'},
$$

from which and the dominated convergence theorem we infer

$$
\lim_{\varepsilon \to 0} \int_{\tilde{Q}_{\zeta'}} |\sqrt{n_\varepsilon}v_\varepsilon 1_{|v_\varepsilon| \leq L} - \sqrt{n}v 1_{|v| \leq L}|^2 dx dt = 0.
\tag{4.15}
$$

Combining (4.14)-(4.15) together, we have for any fixed $L, \zeta, \zeta' > 0$ that

$$
\int_0^T \int_{\mathbb{T}^3} |\sqrt{n_\varepsilon}v_\varepsilon 1_{|v_\varepsilon| \leq L} - \sqrt{n}v 1_{|v| \leq L}|^2 dx dt
\leq \int_{\tilde{Q}_{\zeta'}} |\sqrt{n_\varepsilon}v_\varepsilon 1_{|v_\varepsilon| \leq L} - \sqrt{n}v 1_{|v| \leq L}|^2 dx dt + \int_{Q_\zeta / Q_{\zeta'}} |\sqrt{n_\varepsilon}v_\varepsilon 1_{|v_\varepsilon| \leq L} - \sqrt{n}v 1_{|v| \leq L}|^2 dx dt
+ \int_{\mathbb{T}^3 \times (0, T) / Q_\zeta} \left( |(\sqrt{n_\varepsilon} - \sqrt{n})v_\varepsilon 1_{|v_\varepsilon| \leq L} + \sqrt{n}(v 1_{|v| \leq L} - v_\varepsilon 1_{|v_\varepsilon| \leq L}) \right)^2 dx dt
\leq \int_{\tilde{Q}_{\zeta'}} |\sqrt{n_\varepsilon}v_\varepsilon 1_{|v_\varepsilon| \leq L} - \sqrt{n}v 1_{|v| \leq L}|^2 dx dt + CL \int_0^T \int_{\mathbb{T}^3} |\sqrt{n_\varepsilon} - \sqrt{n}| dx dt + CL(\sqrt{\zeta'} + \sqrt{\zeta}),
$$

which together with (??), (4.15) and the arbitrariness of $\zeta', \zeta$ leads to

$$
\lim_{\varepsilon \to 0} \int_0^T \int_{\mathbb{T}^3} |\sqrt{n_\varepsilon}v_\varepsilon 1_{|v_\varepsilon| \leq L} - \sqrt{n}v 1_{|v| \leq L}|^2 dx dt = 0, \quad L > 0.
\tag{4.16}
$$

We estimate the last term on the right-hand side of (4.13). For a.e. $(x, t) \in \{(x, t) \in \mathbb{T}^3 \times (0, T) \mid n(x, t) > 0\}$, there follows by (4.8), (4.9) and the fact $n_\varepsilon > 0$ that

$$
n|v|^2 \log (1 + |v|^2) = \frac{|m|^2 \log (n^2 + |m|^2) - 2|m|^2 \log n}{n} \varepsilon = \lim_{\varepsilon \to 0} \frac{|n_\varepsilon v_\varepsilon|^2 \log (n_\varepsilon^2 + |n_\varepsilon v_\varepsilon|^2) - 2|n_\varepsilon v_\varepsilon|^2 \log n_\varepsilon}{n_\varepsilon} \varepsilon = \lim_{\varepsilon \to 0} n_\varepsilon |v_\varepsilon|^2 \log (1 + |v_\varepsilon|^2).
\tag{4.17}
$$

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It clearly holds on \( \{ (x,t) \in T^3 \times (0,T) \mid n(x,t) = 0 \} \) that

\[
n|v|^2 \log (1 + |v|^2) = 0 \leq n|v|^2 \log (1 + |v|^2). \tag{4.18}
\]

By (4.17)-(4.18) and the Fatou lemma, we prove for a.e. \( t \in (0,T) \) that

\[
\int_{T^3} n(1 + |v|^2) \log (1 + |v|^2) dx \leq \liminf_{\varepsilon \to 0} \int_{T^3} n_\varepsilon(1 + |v_\varepsilon|^2) \log (1 + |v_\varepsilon|^2) dx \leq C_T. \tag{4.19}
\]

Due to (4.19), the last term on the right-hand side of (4.13) can be controlled by

\[
\int_0^T \int_{T^3} |\sqrt{n\nu} v|_{|v| \geq L}^2 dx dt \leq \frac{\int_0^T \int_{T^3} n|v|^2 \log (1 + |v|^2) dx}{\log (1 + L^2)} \leq \frac{C_T}{\log (1 + L^2)}. \tag{4.20}
\]

Similarly, one gets

\[
\int_0^T \int_{T^3} |\sqrt{n_\varepsilon \nu} v_\varepsilon|_{|v_\varepsilon| \geq L}^2 dx dt \leq \frac{C_T}{\log (1 + L^2)}. \tag{4.21}
\]

Since the constant \( L > 0 \) can be arbitrarily large, we substitute the estimates (4.20)-(4.21) into (4.13) and then make use of (4.16) to obtain

\[
\sqrt{n_\varepsilon v_\varepsilon} \to \sqrt{n} v \quad \text{in} \quad L^2(0,T; L^2(T^3)), \tag{4.22}
\]

which leads to

\[
n_\varepsilon v_\varepsilon \otimes v_\varepsilon \to \sqrt{n} v \otimes \sqrt{n} v \quad \text{in} \quad L^1(0,T; L^1(T^3)). \tag{4.23}
\]

Then, we aim to prove that the limit \( (n, \nu, \rho, \rho u) \) satisfies the conservation laws of mass and momentum. One can take advantage of (4.10), (4.12) and the fact \( 1 \in L^\infty(T^3) \) to deduce

\[
\lim_{\varepsilon \to 0} \int_{T^3} \rho_\varepsilon dx = \int_{T^3} \rho_0 dx, \quad t \in (0,T). \tag{4.24}
\]

Similarly, one can derive

\[
\lim_{\varepsilon \to 0} \int_{T^3} n_\varepsilon dx = \int_{T^3} n_0 dx, \quad t \in (0,T), \tag{4.25}
\]

\[
\int_{T^3} \nu dx = \int_{T^3} n_0 \nu_0 dx + \int_0^t \int_{T^3} \kappa(nu - n\nu) dx d\tau, \quad t \in (0,T), \tag{4.26}
\]

\[
\int_{T^3} \rho u dx = \int_{T^3} \rho_0 \nu_0 dx + \int_0^t \int_{T^3} \kappa(nu - nu) dx dt, \quad t \in (0,T). \tag{4.27}
\]

By (4.23)-(4.27), (4.3) follows.

Then, we claim as \( \varepsilon \to 0 \) that

\[
\rho_\varepsilon \to \rho \quad \text{a.e. in} \quad T^3 \times (0,T), \tag{4.28}
\]

which together with (3.27) and the Egorov theorem implies

\[
A\rho_\varepsilon^\gamma + \delta \rho_\varepsilon^\gamma_0 \to A\rho^\gamma + \delta \rho^\gamma_0 \quad \text{in} \quad L^1(0,T; L^1(T^3)). \tag{4.29}
\]
By virtue of (4.7)-(4.29), one can conclude that the limit \( (n, nv, \rho, \rho u) \) indeed satisfies the equations (4.1) in the sense of distributions.

To prove (4.28), we need the following property of the effect viscous flux, which can be shown by repeating same arguments as used in [16,35].

**Lemma 4.1.** There holds
\[
\lim_{\varepsilon \to 0} \int_0^T \int_{T^3} \left( A\rho_\varepsilon \gamma + \delta \rho_\varepsilon^{\gamma_0} - (2\mu + \lambda) \text{div} u_\varepsilon \right) \rho_\varepsilon dx dt = \int_0^T \int_{T^3} \left( A\overline{p_\varepsilon} + \delta \overline{p_\varepsilon^{\gamma_0}} - (2\mu + \lambda) \text{div} u \right) \rho dx dt.
\]
(4.30)

where \( f \) denotes the weak limit of \( f_\varepsilon \) as \( \varepsilon \to 0 \).

**Proof of the strong convergence of of the density \( \rho_\varepsilon \).** We are going to show (4.28). First, from \( u \in L^2(0,T;H^1(T^3)) \), \( \rho \in L^2(0,T;L^2(T^3)) \) and the arguments of renormalized solutions (cf. [14,34]) \((\rho, u)\) satisfies
\[
(\rho \log \rho)_t + \text{div}(\rho \log \rho) + \rho \text{div} u = 0 \quad \text{in } \mathcal{D}(T^3 \times (0,T)).
\]
(4.31)

One deduces from (4.31) that
\[
\int_0^t \int_{T^3} \rho \text{div} u dx dt = \int_{T^3} \rho_0 \log \rho_0 dx - \int_{T^3} \rho \log \rho dx, \quad t \in (0,T).
\]
(4.32)

Next, multiplying (2.1) by \( 1 + \log \rho_\varepsilon \), we have
\[
\int_0^t \int_{T^3} \rho_\varepsilon \text{div} u_\varepsilon dx dt \leq \int_{T^3} \rho_{0,\delta} \log \rho_{0,\delta} dx - \int_{T^3} \rho_\varepsilon \log \rho_\varepsilon dx, \quad t \in (0,T).
\]
(4.33)

It follows by (4.30) and (4.32)-(4.33) that
\[
\lim_{\varepsilon \to 0} \int_{T^3} (\rho_\varepsilon \log \rho_\varepsilon - \rho \log \rho) dx \\
\leq \lim_{\varepsilon \to 0} \int_0^T \int_{T^3} (\rho \text{div} u - \rho_\varepsilon \text{div} u_\varepsilon) dx dt \\
= \lim_{\varepsilon \to 0} \int_0^T \int_{T^3} (\rho(A\gamma + \delta \gamma_0) - \rho_\varepsilon(A\gamma + \delta \gamma_0) dx dt \leq 0,
\]
where we have used the monotonicity of \( A\gamma + \delta \gamma_0 \). Since \( \rho \to \rho \log \rho \) is strictly convex, we end up with (4.28).

**Proof of the uniform-in-time basic energy inequality and Bresch-Desjardins estimate.** By (2.1), (4.8) and (4.22), we show as \( \varepsilon \to 0 \) that
\[
n_t + \text{div}(nv) = 0.
\]
(4.34)
Since it follows
\[ \int_{T^3} (n_\varepsilon \log n_\varepsilon - n_\varepsilon + 1) dx \]
\[ = \int_0^t \int_{T^3} \log n_\varepsilon (\varepsilon \sqrt{n_\varepsilon} \Delta \sqrt{n_\varepsilon} + \varepsilon \sqrt{n_\varepsilon} \div (|\nabla \sqrt{n_\varepsilon}|^2 \nabla \sqrt{n_\varepsilon} + \varepsilon n_\varepsilon^{-12})) dx dr \]  
\[ + 2 \int_0^t \int_{T^3} \sqrt{n_\varepsilon} \nabla \sqrt{n_\varepsilon} v_\varepsilon dx dr, \quad 0 < t < T, \]  
we have after taking the limit as \( \varepsilon \to 0 \) in (4.35) and employing (4.8) and (4.22) that
\[ \int_{T^3} (n \log n - n + 1) dx = 2 \int_0^t \int_{T^3} \sqrt{n} v \cdot \nabla \sqrt{n} dx dr, \quad 0 < t < T. \]  
(4.36)

Meanwhile, we make use of the lower semi-continuity of weak limits in (3.6) to get
\[ \text{ess sup}_{t \in [0,T]} \int_{T^3} \left( \frac{1}{2} n |v|^2 + \frac{1}{2} \rho |u|^2 + \frac{A \rho^\gamma}{\gamma - 1} + \frac{\delta \rho^{\gamma_0}}{\gamma_0 - 1} \right) dx \]
\[ + \int_0^t \int_{T^3} (\kappa n |v - u|^2 + \mu |\nabla u|^2 + (\mu + \lambda) (\div u)^2) dx dr \]
\[ \leq \text{ess sup}_{t \in [0,T]} \int_{T^3} \left( \frac{1}{2} n_\varepsilon |v_\varepsilon|^2 + \frac{1}{2} \rho_\varepsilon |u_\varepsilon|^2 + \frac{A \rho_\varepsilon^\gamma}{\gamma - 1} + \frac{\delta \rho_\varepsilon^{\gamma_0}}{\gamma_0 - 1} \right) dx \]
\[ + \liminf_{\varepsilon \to 0} \varepsilon \int_0^t \int_{T^3} (n_\varepsilon |v_\varepsilon - u_\varepsilon|^2 + \mu |\nabla u_\varepsilon|^2 + (\mu + \lambda) (\div u_\varepsilon)^2) dx dr \]
\[ + \liminf_{\varepsilon \to 0} \varepsilon \int_0^t \int_{T^3} (n_\varepsilon |v_\varepsilon|^5 + (1 + |v_\varepsilon|^2) |\nabla \sqrt{n_\varepsilon}|^2 (1 + |\nabla \sqrt{n_\varepsilon}|^2) + n_\varepsilon^{-25}) dx dr \]
\[ \leq E_{0,\delta} - 2 \int_0^t \int_{T^3} \sqrt{n} v \cdot \nabla \sqrt{n} dx dr, \]  
(4.37)

where \( E_{0,\delta} \) is given by (2.5). Adding (4.36) and (4.37) together, we show (4.4).

Similarly, we prove the uniform-in-time Bresch-Desjardins estimate. By (3.4) and (3.22), it holds as \( \varepsilon \to 0 \) that
\[ \text{ess sup}_{t \in [0,T]} \int_{T^3} \left( \frac{1}{2} n |v| + c_0 \nabla \log n \right|^2 + \frac{1}{2} \rho |u|^2 + \frac{A \rho^\gamma}{\gamma - 1} + \frac{\delta \rho^{\gamma_0}}{\gamma_0 - 1} \right) dx \]
\[ + \int_0^t \int_{T^3} (c_0 |\nabla \sqrt{n}|^2 + \kappa n |v - u|^2 + \mu |\nabla u|^2 + (\mu + \lambda) (\div u)^2) dx dr \]
\[ \leq C + C \liminf_{\varepsilon \to 0} \int_{T^3} (n_\varepsilon (\div v_\varepsilon)^2 + n_\varepsilon |v_\varepsilon - u_\varepsilon|^2) dx - 2 \int_0^t \int_{T^3} \sqrt{n} v \cdot \nabla \sqrt{n} dx dr \]
\[ + \liminf_{\varepsilon \to 0} \varepsilon \int_0^t \int_{T^3} (n_\varepsilon |v_\varepsilon|^5 + |\nabla \sqrt{n_\varepsilon}|^4 + |\nabla \sqrt{n_\varepsilon}|^2 + |v_\varepsilon|^2 |\nabla \sqrt{n_\varepsilon}|^2 (1 + |\nabla \sqrt{n_\varepsilon}|^2) + n_\varepsilon^{-25}) dx. \]
(4.38)

The combination of (3.9), (4.36) and (4.38) gives rise to (4.5).

5 Vanishing artificial pressure

In this section, we are ready to show that the sequence of weak solutions to the IVP (4.1) converges to a expected weak solution to the original IVP (1.1) as \( \delta \to 0 \). Since the \( L^\infty(0,T; L^\gamma(\mathbb{T}^3)) \)-bound of the
density \( \rho_\delta \) is not enough to get the convergence of the pressure, we need the higher integrability estimates, which can be proved similarly as in Lemma 3.4 and the arguments in [35, Page 174].

**Lemma 5.1.** Let \( T > 0 \), and \((n, nv, \rho u, u)\) be a weak solution to the IVP (4.1) given by Proposition 4.1 for \( t \in (0, T] \). Then, under the assumptions of Theorem 1.1, we have

\[
\int_0^T \int_{\mathbb{T}^3} (\rho^{\frac{\gamma}{2\gamma}} + \delta \rho^{\gamma_0 + \frac{\gamma}{2\gamma}}) dx dt \leq C,
\]

(5.1)

where \( C > 0 \) is a constant independent of \( \delta \).

**Proof of Theorem 1.1 on the global existence of weak solutions.** Let \( \delta \in (0, 1) \), and \((n_\delta, n_\delta v_\delta, \rho_\delta, \rho_\delta u_\delta)\) be a global weak solution to the IVP (2.1) given by Proposition 4.1. For any \( T > 0 \), with the help of the uniform estimates (4.2)-(4.5) and (5.1), there exists a limit \((n, m, \rho, \tilde{m})\) such that as \( \delta \to 0 \), it holds

\[
\begin{cases}
\delta \int_0^T \int_{\mathbb{T}^3} \rho_\delta^{\gamma_0} dx dt \to 0, \\
\sqrt{n_\delta} \xrightarrow{a} \sqrt{n} \quad \text{in} \quad L^{\infty}(0, T; H^1(\mathbb{T}^3)), \\
n_\delta v_\delta \xrightarrow{a} m \quad \text{in} \quad L^\infty(0, T; L^6(\mathbb{T}^3)), \\
\rho_\delta \to \rho \quad \text{in} \quad L^{\frac{\gamma_0}{\gamma}}(0, T; L^{\frac{2\gamma}{\gamma_0} - 1}(\mathbb{T}^3)), \\
u_\delta \to u \quad \text{in} \quad L^2(0, T; H^1(\mathbb{T}^3)).
\end{cases}
\]

(5.2)

By using the Aubin-Lions lemma as well as the uniform estimates (4.2)-(4.6), we can denote

\[
v := \left\{ \begin{array}{ll}
\frac{\sqrt{n}}{m}, & \text{if } n > 0, \\
0, & \text{if } n = 0,
\end{array} \right. 
\]

and have as \( \delta \to 0 \) (up to a sequence) that

\[
\begin{cases}
\sqrt{n_\delta} \to \sqrt{n} \quad \text{in} \quad C([0, T]; L^p(\mathbb{T}^3)), \quad p \in [1, 6), \\
n_\delta \to n \quad \text{in} \quad C([0, T]; L^p(\mathbb{T}^3)), \quad p \in [1, 3), \\
n_\delta v_\delta \to nv \quad \text{in} \quad L^2(0, T; L^p(\mathbb{T}^3)), \quad p \in [1, \frac{3}{2}), \\
\sqrt{n_\delta v_\delta} \to \sqrt{nv} \quad \text{in} \quad L^2(0, T; L^2(\mathbb{T}^3)), \\
\rho_\delta \to \rho \quad \text{in} \quad C([0, T]; H^{-1}(\mathbb{T}^3)), \\
\rho_\delta \to \rho \quad \text{in} \quad C([0, T]; L^{\gamma}很棒(\mathbb{T}^3)), \\
\rho_\delta u_\delta \to \rho u \quad \text{in} \quad C([0, T]; H^{-1}(\mathbb{T}^3)), \\
\rho_\delta u_\delta \to \rho u \quad \text{in} \quad C([0, T]; L^{\frac{2\gamma}{\gamma_0}}很棒(\mathbb{T}^3)).
\end{cases}
\]

(5.3)

Since the proof of (5.3) is quite similar to the arguments in [37], we omit the details for brevity.
Then, we aim to prove
\[ \rho_{\delta} \to \rho \quad \text{a.e. in } \mathbb{T}^3 \times (0, T). \]  
(5.4)

If (5.4) holds, then it follows
\[ \rho_{\delta}^\gamma \to \rho^\gamma \quad \text{in } L^1(0, T; L^1(\mathbb{T}^3)). \]  
(5.5)

Arguing similarly as the proof of Proposition 4.1, one can conclude from (2.4)-(2.6), (4.2)-(4.6) and (5.2)-(5.5) that the limit \((n, nv, \rho, \rho u)\) indeed is a global weak solution to the IVP (1.1)-(1.2) in the sense of Definition 1.1 satisfying (1.15).

To show the strong converge of \(\rho_{\delta}\), we consider the following cut-off function (cf. [16]):

\[ T_k(s) := \begin{cases} s, & \text{if } 0 \leq s \leq k, \\ \text{smooth and concave,} & \text{if } k \leq s \leq 3k, \\ 2k, & \text{if } s \geq 3k. \end{cases} \]  
(5.6)

As in [16,35], we can prove the following property of effect viscous flux:

**Lemma 5.2.** For any \(k > 0\), we have
\[
\lim_{\delta \to 0} \int_0^T \int_{\mathbb{T}^3} \left( A \rho_{\delta}^\gamma + \delta \rho_{\delta}^\gamma - (2\mu + \lambda) \operatorname{div} u_{\delta} \right) T_k(\rho_{\delta}) \, dx \, dt
= \int_0^T \int_{\mathbb{T}^3} \left( A \rho^\gamma + \delta \rho^\gamma - (2\mu + \lambda) \operatorname{div} u \right) \bar{T}_k(\rho) \, dx \, dt,
\]  
(5.7)

where \(\bar{T}\) denotes the weak limit of \(f_{\epsilon}\) as \(\delta \to 0\).

**Lemma 5.3.** For any \(k > 0\), there exists a constant \(C\) independent of \(\delta\) and \(k\) such that
\[
\limsup_{\delta \to 0} \int_0^T \int_{\mathbb{T}^3} |T_k(\rho_{\delta}) - T_k(\rho)|^{\gamma+1} \, dx \, dt \leq C.
\]  
(5.8)

**Proof.** From the facts that \(\rho^\gamma\) is convex and \(T_k(\rho)\) is concave, we have by (5.7) that
\[
\limsup_{\delta \to 0} \int_0^T \int_{\mathbb{T}^3} |T_k(\rho_{\delta}) - T_k(\rho)|^{\gamma+1} \, dx \, dt
\leq \lim_{\delta \to 0} \int_0^T \int_{\mathbb{T}^3} (\rho_{\delta}^\gamma - \rho^\gamma)(T_k(\rho_{\delta}) - T_k(\rho)) \, dx \, dt + \lim_{\delta \to 0} \int_0^T \int_{\mathbb{T}^3} (\rho^\gamma - \rho^\gamma)(T_k(\rho) - \bar{T}_k(\rho)) \, dx \, dt
= \lim_{\delta \to 0} \int_0^T \int_{\mathbb{T}^3} (\rho_{\delta}^2 T_k(\rho_{\delta}) - \rho^\gamma T_k(\rho)) \, dx \, dt,
= \frac{(2\mu + \lambda)}{A} \limsup_{\delta \to 0} \int_0^T \int_{\mathbb{T}^3} |\operatorname{div} u_{\delta} T_k(\rho_{\delta}) - \operatorname{div} u \bar{T}_k(\rho)| \, dx \, dt
\leq C \sup \| \operatorname{div} u_{\delta} \|_{L^2(0; T; L^2)} \lim_{\delta \to 0} (\|T_k(\rho_{\delta}) - T_k(\rho)\|_{L^2(0; T; L^2)} + \|T_k(\rho) - \bar{T}_k(\rho)\|_{L^2(0; T; L^2)}).
\]

The above estimate as well as (3.1) and (5.1) yields (5.8).
Proof of strong convergence of the density $\rho_\delta$. Introduce a series of functions

$$L_k(z) = \begin{cases} 
  z \log z, & 0 \leq z \leq k, \\
  z \log k + z \int_k^{\infty} \frac{T_k(s)}{s} ds, & z \geq k. 
\end{cases} \quad (5.9)$$

According to [35], both $L_k(\rho)$ and $L_k(\rho_\delta)$ satisfies the renormalized properties:

$$L_k(\rho) + \text{div}(L_k(\rho)u) + T_k(\rho) \text{ div } u = 0, \quad \text{in } \mathcal{D}'((0,T) \times \mathbb{T}^3), \quad (5.10)$$

$$L_k(\rho_\delta) + \text{div}(L_k(\rho_\delta)u_\delta) + T_k(\rho_\delta) \text{ div } u_\delta = 0, \quad \text{in } \mathcal{D}'((0,T) \times \mathbb{T}^3). \quad (5.11)$$

To prove $\overline{L_k(\rho)} = L_k(\rho)$, we deduces from (5.10)-(5.11) that

$$\int_{\mathbb{T}^3} (\overline{L_k(\rho)} - L_k(\rho)) dx = \lim_{\delta \to 0} \int_0^T \int_{\mathbb{T}^3} (T_k(\rho) \text{ div } u - T_k(\rho_\delta) \text{ div } u_\delta) dx dt, \quad (5.12)$$

By (5.7) and (5.12), it holds as $k \to \infty$ that

$$\int_{\mathbb{T}^3} (\overline{L_k(\rho)} - L_k(\rho)) dx$$

$$= \int_0^T \int_{\mathbb{T}^3} T_k(\rho) \text{ div } u + \frac{1}{2\mu + \lambda} \lim_{\delta \to 0} \int_0^T \int_{\mathbb{T}^3} (A \rho_\delta^2 - (2\mu + \lambda) \text{ div } u_\delta) T_k(\rho_\delta) dx dt$$

$$- \frac{A}{2\mu + \lambda} \lim_{\delta \to 0} \int_0^T \int_{\mathbb{T}^3} \rho_\delta^2 T_k(\rho_\delta) dx dt$$

$$\leq \int_0^T \int_{\{\rho \geq k\}} |T_k(\rho) - \overline{T_k(\rho)}| \text{ div } u dx dt + \int_0^T \int_{\{\rho \leq k\}} |T_k(\rho) - \overline{T_k(\rho)}| \text{ div } u dx dt$$

$$\leq C \| u \|_{L^2(\{\rho \geq k\})} + C k^{-\gamma - 1} \to 0.$$ 

Furthermore, we have as $k \to \infty$ that

$$\| L_k(\rho) - \rho \log \rho \|_{L^1(0,T;L^1(\mathbb{T}^3))} \leq C \int_0^T \int_{\{\rho \geq k\}} |\rho \log \rho| dx dt \leq C k^{-\gamma + \frac{3}{2}} \int_0^T \int_{\{\rho \geq k\}} \rho^\gamma dx dt \to 0, \quad (5.14)$$

and

$$\| L_k(\rho_\delta) - \rho_\delta \log \rho_\delta \|_{L^1(0,T;L^1(\mathbb{T}^3))} \leq C k^{-\gamma + \frac{3}{2}} \int_0^T \int_{\{\rho \geq k\}} \rho_\delta^\gamma dx dt \to 0. \quad (5.15)$$

With the aid of (5.14) and the lower semi-continuity of weak limits, it further holds

$$\| L_k(\rho) - \rho \log \rho \|_{L^1(0,T;L^1(\mathbb{T}^3))} \leq \liminf_{\delta \to 0} \| L_k(\rho_\delta) - \rho_\delta \log \rho_\delta \|_{L^1((0,T) \times \mathbb{T}^3)} \to 0, \quad \text{uniformly in } \delta. \quad (5.16)$$

By (5.13)-(5.16), we conclude that

$$\int_{\mathbb{T}^3} (\rho \log \rho - \rho \log \rho) dx \leq 0, \quad \text{a.e. } \quad t \in (0,T) \quad (5.17)$$

This combined with $\overline{\rho \log \rho} \geq \rho \log \rho$ leads to (5.4). The proof of Theorem 1.1 is completed.
6 Large time behavior

In this section, we are ready to study the large time behavior of global weak solutions to the IVP (1.1)-(1.2).

Lemma 6.1. Let the assumptions (1.13) be satisfied, and \( (n, n v, \rho, \rho u) \) be the global weak solution to the IVP (1.1)-(1.2) given by Theorem 1.1. Then there holds for a.e. \( 0 \leq s < t < \infty \) (including \( s = 0 \)) that

\[
\tilde{E}(t) + \int_s^t \int_{\Omega_3} (\kappa n|v - m_1|^2 + \mu |\nabla u|^2 + (\mu + \lambda)(\text{div} \, u)^2) \, dx \, d\tau \leq \tilde{E}(s),
\]

(6.1)

where the modified energy \( \tilde{E}(t) \) is defined by

\[
\tilde{E}(t) := \int_{\Omega_3} \left( \frac{1}{2} n|v - m_1(t)|^2 + n \log n - n + 1 + \frac{1}{2} \rho|u - m_2(t)|^2 + \frac{A \rho^\gamma}{\gamma - 1} \right) dx
\]

(6.2)

and \( C_{n_0, \rho_0} \) and \( m_i(t), i = 1, 2, \) are given by

\[
C_{n_0, \rho_0} := \int_{\Omega_3} n_0 dx \int_{\Omega_3} \rho_0 dx, \quad m_1(t) := \frac{\int_{\Omega_3} n v dx}{\int_{\Omega_3} n dx}, \quad m_2(t) := \frac{\int_{\Omega_3} \rho u dx}{\int_{\Omega_3} \rho dx}. \tag{6.3}
\]

Proof. One deduces from (1.14)_2, (2.4), (4.26)-(4.27) and (5.3) that

\[
m_1(t) = \frac{1}{\int_{\Omega_3} n dx} \left( \int_{\Omega_3} m_0 dx + \int_0^t \int_{\Omega_3} \kappa (n u - n v) dx \, d\tau \right), \quad t > 0,
\]

(6.4)

\[
m_2(t) = \frac{1}{\int_{\Omega_3} \rho dx} \left( \int_{\Omega_3} \rho_0 dx + \int_0^t \int_{\Omega_3} \kappa (n v - n u) dx \, d\tau \right), \quad t > 0,
\]

(6.5)

which together with (1.14)_1 yields

\[
\int_{\Omega_3} \left( \frac{1}{2} n|v - m_1|^2 + \frac{1}{2} \rho|u - m_2|^2 \right) dx
\]

\[
= \int_{\Omega_3} \left( \frac{1}{2} n|v|^2 + \frac{1}{2} \rho|u|^2 \right) dx - \int_{\Omega_3} \left( \frac{1}{2} m_1(t) \right)^2 \int_{\Omega_3} n dx - \int_{\Omega_3} \left( \frac{1}{2} m_2(t) \right)^2 \int_{\Omega_3} \rho dx
\]

\[
= \int_{\Omega_3} \left( \frac{1}{2} n|v|^2 + \frac{1}{2} \rho|u|^2 \right) dx - \frac{1}{2} \left( \int_{\Omega_3} m_0 dx \right)^2 + \left( \int_{\Omega_3} \rho_0 dx \right)^2
\]

\[
- \left( \int_{\Omega_3} n_0 dx \int_{\Omega_3} \rho_0 dx \int_0^t \int_{\Omega_3} \kappa (n u - n v) dx \, d\tau \right)
\]

\[
- \int_{\Omega_3} \left( n_0 + \rho_0 \right) dx \int_{\Omega_3} \rho_0 dx \left( \int_0^t \int_{\Omega_3} \kappa (n v - n u) dx \, d\tau \right)^2.
\]

(6.6)

And we make use of (1.14) and (6.4)-(6.5) again to obtain

\[
\frac{1}{2} C_{n_0, \rho_0} |(m_1 - m_2)(t)|^2
\]

\[
= \frac{\int_{\Omega_3} n_0 dx \int_{\Omega_3} \rho_0 dx}{\int_{\Omega_3} (n_0 + \rho_0) dx} \left( \left| \int_{\Omega_3} m_0 dx \right|^2 + \left( \frac{\int_{\Omega_3} \rho_0 dx}{\int_{\Omega_3} n_0 dx} \right)^2 \int_0^t \int_{\Omega_3} \kappa (n u - n v) dx \, d\tau \right)^2
\]

\[
= \frac{1}{2} C_{n_0, \rho_0} |(m_1 - m_2)(0)|^2 + \left( \int_{\Omega_3} m_0 dx \int_{\Omega_3} \rho_0 dx \int_0^t \int_{\Omega_3} \kappa (n u - n v) dx \, d\tau \right)^2
\]

(6.7)

\[
+ \frac{\int_{\Omega_3} (n_0 + \rho_0) dx}{\int_{\Omega_3} n_0 dx \int_{\Omega_3} \rho_0 dx} \left( \int_0^t \int_{\Omega_3} \kappa (n v - n u) dx \, d\tau \right)^2.
\]

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By (1.12) and (6.6)-(6.7), (6.1) holds for \( s = 0 \).

In addition, owing to (6.1) for \( s = 0 \), we deduce for any nonnegative test function \( \psi = \psi(t) \in \mathcal{D}((0, T)) \) that
\[
- \int_0^\infty \psi \tilde{E}(t)dt + \int_0^\infty \psi \int_{\mathbb{T}^3} (\kappa|v-u|^2 + \mu|\nabla u|^2 + (\mu + \lambda)(\text{div} u)^2) dx dt \leq 0, \tag{6.8}
\]
Let \( \psi_{\varepsilon} = \psi_{\varepsilon}(t) \in \mathcal{D}(0, \infty) \) for \( \varepsilon \in (0, 1) \) be the Friedrichs mollifier. Then letting \( \psi(t) = \psi_{\varepsilon}(t - \cdot) \) in (6.8), we deduce for any \( 0 < s \leq \tau \leq t < \infty \) that
\[
\frac{d}{dt} \bar{E} * \psi_{\varepsilon}(\tau) + \left( \int_{\mathbb{T}^3} (\kappa|v-u|^2 + \mu|\nabla u|^2 + (\mu + \lambda)(\text{div} u)^2) dx \right) * \psi_{\varepsilon}(\tau) \leq 0. \tag{6.9}
\]
Integrating (6.9) over \([s, t]\) and taking the limit as \( \varepsilon \to 0 \), we prove (6.1) for a.e. \( 0 < s < t < \infty \).

**Proof of Theorem 1.2 on the large time behavior of weak solutions.** Let the assumptions of Theorem 1.1 hold, and \((n, nv, \rho, u)\) be a global weak solution to the IVP (1.1)-(1.2) given by Theorem 1.1. For any \( s \geq 0 \) and \( t \in (0, 1) \), denote the sequence
\[
(n_s, n_s v_s, \rho_s, u_s)(x, t) := (n, nv, \rho, u)(x, t + s).
\]
By (1.12), (1.14), (1.15)\(_1\) and (6.1), we get the uniform bounds
\[
\begin{align*}
\sup_{s \geq 0} \left( \|\sqrt{n_s}\|_{L^\infty(0, 1; H^1)} + \|\nabla \sqrt{n_s}\|_{L^2(0, 1; L^2)} + \|\nabla n_s\|_{L^2(0, 1; L^2)} \right) < \infty, \\
\sup_{s \geq 0} \left( \|\sqrt{n_s} v_s\|_{L^\infty(0, 1; L^2)} + \|\rho_s\|_{L^\infty(0, 1; L^\gamma)} + \|\sqrt{\rho_s} u_s\|_{L^\infty(0, 1; L^2)} \right) < \infty,
\end{align*} \tag{6.10}
\]
and the strong convergences
\[
\begin{align*}
\lim_{s \to \infty} \left( \|\nabla n_s\|_{L^2(0, 1; L^2)} + \|n_s\|_{L^\infty(0, 1; L^l)} + \|\sqrt{n_s}(v_s - u_s)\|_{L^2(0, 1; L^2)} + \|\nabla u_s\|_{L^2(0, 1; L^2)} \right) &= 0, \\
\lim_{s \to \infty} \left\| \frac{\int_{\mathbb{T}^3} n_s v_s dx}{\int_{\mathbb{T}^3} n_s dx} - \frac{\int_{\mathbb{T}^3} \rho_s u_s dx}{\int_{\mathbb{T}^3} \rho_s dx} \right\|_{L^2(0, 1)} &= 0. \tag{6.11}
\end{align*}
\]
Owing to (6.10), (1.14), (6.11)\(_1\) and the Sobolev inequality, it holds as \( s \to \infty \) that
\[
\begin{align*}
\|n_s - n_c\|_{L^2(0, T; L^2)} &\to 0, \quad n_c := \int_{\mathbb{T}^3} n_0 dx, \\
\|u_s - \frac{\int_{\mathbb{T}^3} \rho_s u_s dx}{\int_{\mathbb{T}^3} \rho_s dx}\|_{L^2(0, 1; L^6)} &\leq (1 + \sup_{t \in (0, 1)} \|\rho(t)\|_{L^1}) \|u_s - \int_{\mathbb{T}^3} u_s dx\|_{L^2(0, 1; L^6)} \to 0, \tag{6.12} \\
\|\sqrt{n_s}(v_s - u_s)\|_{L^2(0, 1; L^2)} &\to 0, \\
\|\sqrt{n_s}(v_s - u_s)\|_{L^2(0, 1; L^2)} + \sup_{t \in (0, 1)} \|\sqrt{n_s}(t)\|_{L^\infty} u - \frac{\int_{\mathbb{T}^3} \rho_s u_s dx}{\int_{\mathbb{T}^3} \rho_s dx}\|_{L^2(0, 1; L^\infty)} &\to 0.
\end{align*}
\]

Since \((n_s)_t = -\text{div}(\sqrt{n_s^3} u_s)\) is uniformly bounded in \(L^\infty(0, 1; W^{-1/2}(T^3))\), by virtue of (6.10), (6.12), and the Aubin-Lions lemma, there is a subsequence of \(n_s\) (still denoted by the same index) such that as \(s \to \infty\), we have

\[
n_s \to n_c \quad \text{in} \quad C([0, T]; L^q(T^3)), \quad q \in [1, 3).
\]

(6.13)

Then, by similar arguments as used in (5.1), we are able to show

\[
\int_0^1 \int_{T^3} \rho_n^{3/5} \, dx \, dt \leq C,
\]

for a constant \(C > 0\) independent of \(s\), so there exist two functions \(\rho \in L^{5/3-1}(0, 1; L^{5/3-1}(T^3))\) and \(\rho\gamma \in L^{5/3-1}(0, 1; L^{5/3-1}(T^3))\) such that up to a sequence if necessary, it holds as \(s \to \infty\) that

\[
\begin{aligned}
\rho_s &\to \rho \quad \text{in} \quad L^{5/3-1}(0, 1; L^{5/3-1}(T^3)), \\
\rho_s^{\gamma} &\to \rho\gamma \quad \text{in} \quad L^{5/3-1}(0, 1; L^{5/3-1}(T^3)).
\end{aligned}
\]

Set

\[
G(P) = P^\alpha, \quad P \in \mathbb{R}_+, \quad 0 < \alpha < \frac{5}{3} - \frac{1}{\gamma},
\]

so that by (6.14), as \(s \to \infty\), there exists two limits \(\overline{G(\rho)} \in L^{p_1}(0, 1; L^{p_1}(T^3))\) and \(\overline{G(\rho\gamma)} \rho\gamma \in L^{p_2}(0, 1; L^{p_2}(T^3))\) with \(\frac{1}{p_1} + \frac{1}{p_2} < 1\) satisfying

\[
\begin{aligned}
G(\rho_s^{\gamma}) &\to \overline{G(\rho)} \quad \text{in} \quad L^{p_1}(0, 1; L^{p_1}(T^3)), \\
G(\rho_s^{\gamma}) \rho_s^{\gamma} &\to \overline{G(\rho)} \rho\gamma \quad \text{in} \quad L^{p_2}(0, 1; L^{p_2}(T^3)).
\end{aligned}
\]

As in [19], we apply the dive-curl lemma to get the strong convergence of \(\rho_s\). Consider the two vector functions

\[
(G(\rho_s^{\gamma}), 0, 0, 0), \quad (\rho_s^{\gamma}, 0, 0, 0).
\]

By the arguments of renormalized solutions of (1.10), it holds

\[
G(\rho_s^{\gamma})_t = -\text{div}(G(\rho_s^{\gamma}) u_s) - (\gamma \alpha - 1)G(\rho_s^{\gamma}) \text{ div } u_s \quad \text{in} \quad \mathcal{D}'(T^3 \times (0, 1)),
\]

which implies that

\[
\text{div}_{t,x} \left( G(\rho_s^{\gamma}), 0, 0, 0 \right) = G(\rho_s^{\gamma})_t
\]

is strongly compact in \(W^{-1,q_1}(T^3 \times (0, 1))\) for some \(q_1 > 1\). It is easy to check that

\[
\text{curl}_{t,x} (\rho_s^{\gamma}, 0, 0, 0) = \left( \begin{array}{cccc}
0 & -\partial_{x_1} \rho_s^{\gamma} & -\partial_{x_2} \rho_s^{\gamma} & -\partial_{x_3} \rho_s^{\gamma} \\
\partial_{x_1} \rho_s^{\gamma} & 0 & 0 & 0 \\
\partial_{x_2} \rho_s^{\gamma} & 0 & 0 & 0 \\
\partial_{x_3} \rho_s^{\gamma} & 0 & 0 & 0
\end{array} \right)
\]

is strong compact in \(W^{-1,q_2}(T^3 \times (0, 1))\) for some \(q_2 > 1\). Therefore, employing the \(L^p\) type dive-curl lemma (cf. [52]), we have

\[
\overline{G(\rho\gamma)} \rho\gamma = \overline{G(\rho\gamma)} \rho\gamma.
\]
Since \( G(P) \) is strictly monotone with respect to the variable \( P \), we deduce
\[
G(\rho^c) = G(\rho^\gamma), \tag{6.15}
\]
By (6.15) and the convexity of \( L^p_+ \), it holds
\[
\rho_s^\gamma \to \rho^\gamma \quad \text{in} \quad L^1(0, 1; L^1(\mathbb{T}^3)),
\]
which together (1.14) and (6.14) gives rise to
\[
\rho_s \to \rho_c := \int_{\mathbb{T}^3} \rho_0 \, dx \quad \text{in} \quad L^p(0, 1; L^p(\mathbb{T}^3)), \quad 1 \leq p < \frac{5}{3} \gamma - 1. \tag{6.16}
\]
Since \( (\rho_s)_t \) is uniform bounded in \( L^\infty(0, 1; L^{\frac{2\gamma}{\gamma - 1}}(\mathbb{T}^3)) \), we further have
\[
\rho_s \to \rho_c \quad \text{in} \quad C([0, 1]; L^\gamma_{weak}(\mathbb{T}^3)). \tag{6.17}
\]
Finally, due to (6.1), the energy \( \bar{E}(t) \) defined by (6.2) is non-increasing and bounded from below, and therefore \( \bar{E}(t) \) converge to a constant \( \bar{E}_\infty \geq 0 \) as \( t \to \infty \):
\[
\bar{E}_\infty := \limsup_{t \to \infty} \bar{E}(t).
\]
By (6.16)-(6.17) and the Fatou lemma, there is a sequence \( s_k \to \infty \) such that
\[
\bar{E}_\infty = \lim_{s_k \to \infty} \int_{s_k}^{s_{k+1}} \bar{E}(\tau) \, d\tau \\
= \lim_{s_k \to \infty} \int_0^1 \int_{\mathbb{T}^3} \left( \frac{1}{2} n_s |v_s - \frac{\int_{\mathbb{T}^3} n_s v_s \, dx}{\int_{\mathbb{T}^3} n_s \, dx} |^2 + n_s \log n_s - n_s + 1 \\
+ \frac{1}{2} \rho_s |u_s - \frac{\int_{\mathbb{T}^3} \rho_s u_s \, dx}{\int_{\mathbb{T}^3} \rho_s \, dx} |^2 + \frac{A \rho_s^\gamma}{\gamma - 1} \right) \, dx \, d\tau \\
+ \frac{\int_{\mathbb{T}^3} n_0 \, dx}{\int_{\mathbb{T}^3} \rho_0 \, dx} \lim_{s_k \to \infty} \int_0^1 \int_{\mathbb{T}^3} \left( n_s v_s \, dx - \frac{\int_{\mathbb{T}^3} \rho_s \, dx}{\int_{\mathbb{T}^3} \rho_s \, dx} \right)^2 \, d\tau \\
= \int_{\mathbb{T}^3} \left( n_c \log n_c - n_c + 1 + \frac{A \rho_c^\gamma}{\gamma - 1} \right) \, dx \\
\leq \liminf_{t \to \infty} \int_{\mathbb{T}^3} \left( n \log n - n + 1 + \frac{A \rho_t^\gamma}{\gamma - 1} \right) \, dx \\
\leq \limsup_{t \to \infty} \bar{E}(t) = \bar{E}_\infty,
\]
so we have
\[
\lim_{t \to \infty} \int_{\mathbb{T}^3} \left( n \log n - n + 1 + \frac{A \rho_t^\gamma}{\gamma - 1} \right) \, dx = \int_{\mathbb{T}^3} \left( n_c \log n_c - n_c + 1 + \frac{A \rho_c^\gamma}{\gamma - 1} \right) \, dx = \bar{E}_\infty. \tag{6.18}
\]
which together with (6.13) gives rise to
\[
\lim_{t \to \infty} \int_{\mathbb{T}^3} |\rho - \rho_c|^\gamma \, dx = 0. \tag{6.19}
\]
With the aid of (6.18)-(6.19), it further holds
\[
\begin{cases}
\lim_{t \to \infty} \int_{\mathbb{T}^3} (n |v - \frac{\int_{\mathbb{T}^3} n v \, dx}{\int_{\mathbb{T}^3} n \, dx}|^2 + \rho |u - \frac{\int_{\mathbb{T}^3} \rho u \, dx}{\int_{\mathbb{T}^3} \rho \, dx}|^2) \, dx = 0,
\lim_{t \to \infty} \left| \frac{\int_{\mathbb{T}^3} n v \, dx}{\int_{\mathbb{T}^3} n \, dx} - \frac{\int_{\mathbb{T}^3} \rho u \, dx}{\int_{\mathbb{T}^3} \rho \, dx} \right| = 0.
\end{cases} \tag{6.20}
\]
Making use of the conservation laws (1.14), we substitute the equality
\[ \int_{T_3} \rho u dx = \frac{1}{\int_{T_3} \rho dx} \left( \int_{T_3} (m_0 + \tilde{m}_0) dx - \int_{T_3} n_0 dx \int_{T_3} n dx \right) \]
into (6.20) to derive
\[ \lim_{t \to \infty} \left( \| \int_{T_3} n v dx \|_{L^2(0,1)} - \| u_c \|_{L^2(0,1)} + \| \int_{T_3} \rho u dx \|_{L^2(0,1)} - \| u_c \|_{L^2(0,1)} \right) = 0, \quad u_c := \frac{\int_{T_3} (m_0 + \tilde{m}_0) dx}{\int_{T_3} (n_0 + \rho_0) dx}. \]
This together with (6.20) yields
\[ \lim_{t \to \infty} \int_{T_3} (n|v - u_c|^2 + \rho|u - u_c|^2) dx = 0. \quad (6.21) \]
The combination of (6.13), (6.19) and (6.21) gives rise to (1.16). The proof of Theorem 1.2 is completed.

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