Complete ZX-calculi for the stabiliser fragment in odd prime dimensions

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We introduce a family of ZX-calculi which axiomatise the stabiliser fragment of quantum theory in odd prime dimensions. These calculi recover many of the nice features of the qubit ZX-calculus which were lost in previous proposals for higher-dimensional systems. We then prove that these calculi are complete, i.e. provide a set of rewrite rules which can be used to prove any equality of stabiliser quantum operations. Adding a discard construction, we obtain a calculus complete for mixed state stabiliser quantum mechanics in odd prime dimensions, and this furthermore gives a complete axiomatisation for the related diagrammatic language for affine co-isotropic relations.

The ZX-calculus is a powerful yet intuitive graphical language for reasoning about quantum computing, or, more generally, about operations on quantum systems [CK17; vdWet20]. It allows one to represent such quantum operations pictorially, and comes equipped with a set of rules which, in principle, make it possible to derive any equality between those pictures [Bac14; JPV17; NW17]. It has now had several applications in quantum information processing, from MBQC [DP10; Bac+21b], through quantum error correction codes [DL14; dBH17; Cha+18; GD18]. More recently, it has been used to obtain state-of-the-art optimisation techniques for quantum circuits [KvdW20; dBBW20; Dun+20] and faster classical simulation algorithms for general quantum computations [KvdWV22].

Despite its origins in categorical quantum mechanics and the diagrammatic language for finite-dimensional linear spaces [AC08; CD11; CK17] the literature on the ZX-calculus has been concerned almost exclusively with small-dimensional quantum systems, and even then mostly with the case of two-dimensional quantum systems, or qubits. The qubit ZX-calculus is remarkable in its simple treatment of stabiliser quantum mechanics, along with the fact that any diagram can be treated purely graph-theoretically, without concern to its overall layout, and without losing its quantum-mechanical interpretation. Those proposals that go beyond qubits lose many of these nice features, and are significantly more complicated than the qubit case [Ran14; WB14; BW15; Wan21b; Wan21a]. In particular, they eschew the prized “Only Connectivity Matters” (OCM) meta-rule, often cited as one of the key features in the qubit case. In these calculi, which can represent any linear map between the corresponding Hilbert spaces, it is also not necessarily obvious (at least, to us) how to pick out and work with the stabiliser fragment.

Stabiliser quantum mechanics is a simple yet particularly important fragment of quantum theory. While much less powerful than the full fragment—it can be efficiently clas-
sically simulated, even in odd prime dimensions [dBea12]—it has seen significant study [Ghe14; HNdR21] and forms the basis for a number of key methods in quantum information theory [Got99]. Operationally, it can be described as the fragment of quantum mechanics which is obtained if one allows only state preparation in the computational basis, and unitary operations from the Clifford groups [Got99]. In the qubit case, the stabiliser fragment of the ZX-calculus was proved complete in [Bac14] while ignoring global scalars, and extended to include scalars in [Bac15]. A simplified calculus based on these results but further reducing the set of axioms of the calculus was presented in [BPW17].

In this article, we present a simple ZX-calculus for stabiliser quantum mechanics in odd prime dimensions, and which recovers as many of the nice features of the qubit calculus as possible. In odd prime dimensions, stabiliser quantum mechanics can be given a particularly nice graphical presentation, owing to the group-theory underlying the corresponding Clifford groups [Nen02; App09; dBea12]. We then give this calculus a set of rewrite rules that is complete, i.e. rich enough to derive any equality of stabiliser quantum operations. In particular, it is a design priority to recover OCM, and to make explicit the stabiliser fragment and its group-theoretical underpinnings. Adding a discard construction [Car+19], we obtain a universal and complete calculus for mixed state stabiliser quantum mechanics in odd prime dimensions. By previous work [CK21], this gives a complete axiomatisation for the related diagrammatic language for affine co-isotropic relations, while still maintaining OCM.

Although we do not do so here, these calculi can naturally be extended to represent much larger fragments of quantum theory, up to the entire theory in odd prime dimensions [WB14]. However, finding a complete axiomatisation for such calculi will presumably be a much more complicated task, and we leave this for future work.

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1 Stabiliser quantum mechanics in odd prime dimensions

Throughout this paper, \( p \) denotes an arbitrary odd prime, and \( \mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} \) the ring of integers with arithmetic modulo \( p \). We also put \( \omega := e^{i \frac{2\pi}{p}} \), and let \( \mathbb{Z}_p^* \) be the group of units of \( \mathbb{Z}_p \). Since \( p \) is prime, \( \mathbb{Z}_p \) is a field and \( \mathbb{Z}_p^* = \mathbb{Z}_p \setminus \{0\} \) as a set. We also have need of the following definition:

\[
\chi_p(x) = \begin{cases} 
1 & \text{if there is no } y \in \mathbb{Z}_p \text{ s.t. } x = y^2; \\
0 & \text{otherwise}; 
\end{cases} 
\]  

which is just the characteristic function of the complement of the set of squares in \( \mathbb{Z}_p \).

The Hilbert space of a qupit [Got99; Wan+20] is \( \mathcal{H} = \text{span}\{|m\rangle \mid m \in \mathbb{Z}_p\} \cong \mathbb{C}^p \), and we write \( U(\mathcal{H}) \) the group of unitary operators acting on \( \mathcal{H} \). We have the following standard operators on \( \mathcal{H} \), also known as the clock and shift operators:

\[
Z |m\rangle := \omega^m |m\rangle \quad \text{and} \quad X |m\rangle := |m+1\rangle \quad \text{for any } m \in \mathbb{Z}_p.
\]  

In particular, note that \( ZX = \omega XZ \). We call any operator of the form \( \omega^k X^a Z^b \) for \( k, a, b \in \mathbb{Z}_p \) a Pauli operator. We say a Pauli operator is trivial if it is proportional to the
identity. The collection of all Pauli operators is denoted $\mathcal{P}$ and called the Pauli group. For $n \in \mathbb{N}^*$, the generalised Pauli group is $\mathcal{P}_n := \bigotimes_{k=1}^n \mathcal{P}$.

Of particular importance to us are the (generalised) Clifford groups. These are defined, for each $n \in \mathbb{N}^*$, as the normaliser of $\mathcal{P}_n$ in $U(\mathcal{H}^\otimes n)$: $C$ is a Clifford operator if for any $P \in \mathcal{P}_n$, $CPC^\dagger \in \mathcal{P}_n$. It is clear that every Pauli operator is Clifford, but there are non-Pauli Clifford operators. Some important examples are the Hadamard gate:

$$H |m\rangle = \frac{1}{\sqrt{d}} \sum_{n \in \mathbb{Z}_p} \omega^{mn} |n\rangle \quad \text{s.t.} \quad HXH^\dagger = Z \quad \text{and} \quad HZH^\dagger = X^{-1},$$

and the phase gate:

$$S |m\rangle = \omega^{2^{-1}m(m+1)} |m\rangle \quad \text{s.t.} \quad SXS^\dagger = \omega XZ \quad \text{and} \quad ZS^\dagger = Z.$$  

Yet another important example is the controlled-phase gate, which acts on $\mathcal{H} \otimes \mathcal{H}$,

$$E |m\rangle |n\rangle := \omega^{mn} |m\rangle |n\rangle.$$  

It is important to emphasise a key difference between the qupit and the qubit case: when $p \neq 2$, none of these operators are self-inverse. In fact, if $Q$ is a Pauli and $I$ the identity operator on $\mathcal{H}$, we have:

$$Q^p = I, \quad E^p = I \otimes I \quad \text{and} \quad H^4 = I.$$  

As a side note, equations (2), (3) and (4) imply that both $X$ and $Z$, and in fact every Pauli, have spectrum $\{\omega^k \mid k \in \mathbb{Z}_p\}$. As a result, we denote $|k : Q\rangle$ the eigenvector of a given Pauli $Q$ associated with eigenvalue $\omega^k$, and furthermore use the notation

$$|k_1, \ldots, k_n : Q\rangle = \bigotimes_{k=1}^n |k : Q\rangle.$$  

It follows from equation (2) that we can identify $|k : Z\rangle = |k\rangle$.

Now, for any $\alpha \in [0, 2\pi)$, the operator $e^{i\alpha}I$ is Clifford. However, we want to construct calculi with a finite axiomatisation. As a result, the diagrams in the calculus are countable and this makes it impossible for us to represent all such phases $e^{i\alpha}$. Unfortunately, finding a group of phases that behaves well diagrammatically is somewhat inconvenient, and involves some elementary number theory. For an odd prime $p$, we consider the group of phases given by:

- if $p \equiv 1 \mod 4$,
  $$\mathbb{P}_p := \left\{(-1)^s\omega^t \mid s, t \in \mathbb{Z} \right\};$$
- if $p \equiv 3 \mod 4$,
  $$\mathbb{P}_p := \left\{i^s\omega^t \mid s, t \in \mathbb{Z} \right\}.$$  

This of course covers all cases since $p$ is odd. Then, we restrict our attention to the reduced Clifford group,

$$\mathcal{C}_n = \{\lambda U \mid \lambda \in \mathbb{P}_p, U \text{ is Clifford and special unitary}\}.$$  

We call $\mathcal{C}_n^{\otimes n}$ the local Clifford group on $n$ qupits. It is clear from these examples that $\mathcal{C}_n$ is strictly larger than $\mathcal{C}_1^{\otimes n}$, but it turns out to not be that much larger:
Proposition 1 ([Cla06; Nen02]). The reduced Clifford group $C_n$ is generated by the gate-set \( \{H_j, S_j, E_{j,k} \mid j, k = 1, \ldots, n\} \).

Stabiliser quantum mechanics can be operationally described as the fragment of quantum mechanics in which the only operations allowed are initialisations and measurements in the eigenbases of Pauli operators, and unitary operations from the generalised Clifford groups. As before, we restrict our attention to the fragment of stabiliser quantum mechanics where only unitary operations from the reduced Clifford groups are allowed. Scalars are then taken from the monoid $G_p := \{0, \sqrt{p^r} \lambda \mid r \in \mathbb{Z}, \lambda \in \mathbb{F}_p\}$. Little is lost for the description of quantum algorithms, since we can always simplify by a global phase to make the Clifford generators special unitary. Thus, we can embed any stabiliser circuit into the calculus, and then calculate the relative phases of different branches of a computation without restriction.

Definition 2. The symmetric monoidal category $\text{Stab}_p$ has as objects $C^{pn}$ for each $n \in \mathbb{N}$, and morphisms generated by:

- $C \to C^p : \lambda \mapsto \lambda |0\rangle$;
- $C^{pn} \to C^{pn} : |\psi\rangle \mapsto U |\psi\rangle$ for any $U \in C_n$;
- $C^p \to C : |\psi\rangle \mapsto \langle 0|\psi\rangle$.

The monoidal product is given by the usual tensor product of linear spaces.

It is clear that $\text{Stab}_p$ is a subcategory of the category $\mathbf{FLin}$ of finite dimensional $\mathbb{C}$-linear spaces; it is also a PROP.

2 A ZX-calculus for odd prime dimensions

In this section, we present our family of ZX-calculi, with one for each odd prime. Relying on some of the group theoretical properties of the qudit Clifford groups, we can give a relatively simple presentation of the calculi, which avoids the need to explicitly consider rotations in $p$-dimensional space, significantly simplifying the presentation compared to previous work [Ran14]. These calculi are also constructed in order to satisfy the property of flexsymmetry, proposed in [Car21a; Car21b], and which allows one to recover the OCM meta-rule. OCM is an intuitively desirable feature for the design of a graphical language; anecdotally, it greatly simplifies the human manipulation of diagrams, including in the proofs of this paper. More formally, it means that the equational theory can be formalised in terms of double pushout rewriting over graphs rather than over hypergraphs as is necessary in the more general theory [Bon+22a; Bon+21; Bon+22b].

Another key concern is the issue of completeness, which we begin to address in this article. Outside of qubits [JPV17; NW17; Vil18], there has so far been a complete axiomatisation only for the stabiliser fragment in dimension $p = 3$ [Wan18]. We present an axiomatisation which is complete for the stabiliser fragment for any odd prime $p$, and leave the general case for future work.

2.1 Generators

For any odd prime $p$, consider the symmetric monoidal category $\mathbf{ZX}^{\text{Stab}}_p$ with objects $\mathbb{N}$ and morphisms (or, diagrams) generated by:

\[ \square : 1 \to 1 \quad \bigcirc : 1 \to 1 \quad \bigcirc : 1 \to 1 \]
where $x, y \in \mathbb{Z}_p$. We also introduce a generator $\star : 0 \to 0$ to simplify the calculus; it will correspond to a scalar whose representation in terms of the other generators depends non-trivially on the dimension $p$. Finally, the empty diagram $\emptyset : 0 \to 0$ is also a morphism in the language. Morphisms are composed by connecting output wires to input wires, and the monoidal product is given on objects by $n \otimes m = n + m$ and on morphisms by vertical juxtaposition of diagrams.

We extend this elementary notation with a first piece of syntactic sugar, which is standard for the ZX-calculus family: green spiders are defined inductively, for any $m, n \in \mathbb{N}$, by

\begin{align*}
\begin{array}{c}
m \vdash 1 : 1 \\
1 \vdash n : n
\end{array}
\end{align*}

(11)

It is clear that these diagrams have types $m + 1 \to 1$, $1 \to n + 1$ and $m \to n$ respectively. This construction can be justified by remarking that, under the equational theory to be presented in section 2.3, the unlabelled green fragment of the language forms a special commutative Frobenius algebra, which therefore admits a canonical form in terms of such “spiders” [CK17].

Red spiders are defined analogously, with a small twist:

\begin{align*}
\begin{array}{c}
\begin{array}{c}
m \vdash 1 : 1 \\
1 \vdash n : n
\end{array}
\end{array}
\end{align*}

(12)

The addition of these $1 \to 1$ red vertices (called antipodes) when compared to green spiders mimicks a construction of the ZH-calculus [BK19; Bac+21a], where they are sometimes dubbed “harvestmen”. They are crucial in order for the red spiders to be flexsymmetric in our equational theory and thus to recover the OCM meta-rule mentioned above.

We can then define labelled spiders by passing the label of a $0 \to 1$ diagram along an input wire:

\begin{align*}
\begin{array}{c}
\begin{array}{c}
m \vdash 1 : 1 \\
1 \vdash n : n
\end{array}
\end{array}
\end{align*}

and

\begin{align*}
\begin{array}{c}
\begin{array}{c}
m \vdash 1 : 1 \\
1 \vdash n : n
\end{array}
\end{array}
\end{align*}

(13)

In order to avoid clutter, we also use the following shorthand:

\begin{align*}
\begin{array}{c}
\begin{array}{c}
m \vdash 1 : 1 \\
1 \vdash n : n
\end{array}
\end{array}
\end{align*}

(14)

which we shall see represents the compositional inverse of $\begin{array}{c}
\begin{array}{c}
m \vdash 1 : 1 \\
1 \vdash n : n
\end{array}
\end{array}$.

2.2 Standard interpretation and universality

The standard interpretation of a $\mathcal{ZX}^{\text{Stab}}_p$-diagram is a symmetric monoidal functor $[-] : \mathcal{ZX}^{\text{Stab}}_p \to \mathcal{FLin}$ (the category of finite-dimensional $\mathbb{C}$-linear spaces). It is defined on objects as $[m] := \mathbb{C}^{p \times m}$, and on the generators of the morphisms as:

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
m \vdash 1 : 1 \\
1 \vdash n : n
\end{array}
\end{array}
\end{align*}
= \sum_{k \in \mathbb{Z}_p} \omega^{2^{-1}(xk+yk^2)} |k : Z|
\]

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
m \vdash 1 : 1 \\
1 \vdash n : n
\end{array}
\end{array}
\end{align*}
= \sum_{k \in \mathbb{Z}_p} \omega^{2^{-1}(xk+yk^2)} \langle k : Z |
By the functoriality of the standard interpretation, we then deduce that

$$\begin{align*}
\left[ \sum_{k \in \mathbb{Z}_p} |k : Z\rangle \langle k, k : Z| \right] &= \sum_{k \in \mathbb{Z}_p} |k, k : Z\rangle \langle k : Z| \\
\left[ \sum_{k \in \mathbb{Z}_p} |k, k : Z\rangle \langle k : Z| \right] &= \sum_{k \in \mathbb{Z}_p} |k, k : Z\rangle \langle k : Z| \\
\left[ \sum_{k \in \mathbb{Z}_p} \omega^{2^{-1}(xk+y^2)} \langle -k : X| \right] &= \sum_{k \in \mathbb{Z}_p} \omega^{2^{-1}(xk+y^2)} \langle k : X| \\
\left[ \sum_{k \in \mathbb{Z}_p} |k : Z\rangle \langle k : X| \right] &= \sum_{k \in \mathbb{Z}_p} |k : X\rangle \langle k, k : X| \\
\left[ \sum_{k \in \mathbb{Z}_p} |k : Z\rangle \langle k : Z| \right] &= \sum_{k \in \mathbb{Z}_p} |k : X\rangle \langle k, k : X| \\
\left[ \sum_{k \in \mathbb{Z}_p} \omega^{2^{-1}(xk+y^2)} | -k : X\rangle \langle k : Z| \right] &= \sum_{k \in \mathbb{Z}_p} \omega^{2^{-1}(xk+y^2)} | -k : X\rangle \langle k : Z| \\
\left[ \sum_{k, \ell \in \mathbb{Z}_p} |k, \ell : Z\rangle \langle \ell, k : Z| \right] &= \sum_{k \in \mathbb{Z}_p} \langle kk : Z| \\
\left[ * \right] &= -1
\end{align*}$$

By the functoriality of the standard interpretation, we then deduce that

$$\left[ \sum_{k \in \mathbb{Z}_p} \omega^{2^{-1}(xk+y^2)} |k : Z\rangle \langle k : Z| \right] = \sum_{k \in \mathbb{Z}_p} \omega^{2^{-1}(xk+y^2)} |k : Z\rangle \langle k : Z| \otimes^n, \quad (15)$$

and

$$\left[ \sum_{k \in \mathbb{Z}_p} \omega^{2^{-1}(xk+y^2)} | -k : X\rangle \langle k : Z| \right] = \sum_{k \in \mathbb{Z}_p} \omega^{2^{-1}(xk+y^2)} | -k : X\rangle \langle k : Z| \otimes^n. \quad (16)$$

There are a couple of peculiarities in this standard interpretation that must be remarked upon. Firstly, note that the red $1 \rightarrow 1$ spider is not the identity, but rather maps the label of any input $X$-eigenstate to its additive inverse. This feature is repeated on every red spider, and is another of the side-effects of imposing that the OCM rule must be sound for this interpretation.

Secondly, and much more subtly, we have made an unconventional choice for the interpretation of the label of spiders as a phase: there is an additional factor $2^{-1}$ which one might not expect. This might seem like an inconsequential choice which ought to be removed, but it has an important consequence for the equational theory. There is a specific rule in the axiomatisation, (GAUSS), which is sound for this choice of phases but not for the simpler one which omits the factor $2^{-1}$.

**Theorem 3** (Universality). The standard interpretation $[\cdot]$ is universal for the qupit stabiliser fragment, i.e. for any stabiliser operation $C : \mathbb{C}^m \rightarrow \mathbb{C}^m$ there is a diagram $D \in \mathsf{ZX}^	ext{Stab}_p$ such that $[D] = C$. Put formally, the co-restriction of $[\cdot]$ to $\text{Stab}_p$ is full.

### 2.3 Axiomatisation

We now begin to introduce rewrite rules with which to perform purely diagrammatic reasoning. By doing so we are in fact describing a PROP by generators and relations [BCR18], thus the swap is required to satisfy the following properties:

$$\begin{align*}
\begin{tikzpicture}
\end{tikzpicture}
&= \begin{tikzpicture}
\end{tikzpicture} \\
\begin{tikzpicture}
\end{tikzpicture}
&= \begin{tikzpicture}
\end{tikzpicture}
\end{align*}
\quad (17)$$
Figure 1: A presentation of the equational theory $\text{ZX}_{p}$, which is sound and complete for the stabiliser fragment. The equations hold for any $a, b, c, d \in \mathbb{Z}_p$ and $z \in \mathbb{Z}_p^*$. $\chi_p$ is the characteristic function of the complement of the set of squares in $\mathbb{Z}_p$, defined in equation (1).

Note that the last equation is required to hold for any diagram $D : n \to m$. This property states that our diagrams form a symmetric monoidal category. Furthermore, we want this category to be self-dual compact-closed, hence the cup and cap must satisfy:

$$\bigotimes = \bigodot$$

Furthermore, as long as the connectivity of the diagram remains the same, vertices can be freely moved around without changing the standard interpretation of the diagram. This is a consequence of the fact that we require our generators to be flexsymmetric, as shown in [Car21a; Car21b]. This amounts to imposing that all generators except the swap satisfy:

$$\sigma : n + m \to n + m$$

where $\sigma : n + m \to n + m$ stands for any permutation of the wires involving swap maps. We will consider all the previous rules as being purely structural and will not explicitly state their use. Using these rules, we can in fact deduce that both the green and red spiders (and their labelled varieties) are themselves flexsymmetric. This means that the language follows the OCM meta-rule, and we can formally treat any $\text{ZX}_{p}^{\text{Stab}}$-diagram as a graph.\(^1\)

\(^1\)Note that the graphs in question must be allowed to have loops and parallel edges, so are perhaps better called pseudographs or multigraphs.
whose vertices are the spiders, and whose edges are labelled by the $1 \rightarrow 1$ generators of the language.\footnote{There is a small ambiguity: $1 \rightarrow 1$ spiders can be treated as either edges or vertices. When considering diagrams, it matters little which, since any given graph is always to be understood as one of the many equivalent $\text{ZX}^\text{Stab}_p$-diagrams constructed formally out of the generators. Any computer implementation of the calculus will have to carefully resolve this ambiguity in its internal representation.}

Figure 1 presents the remainder of the equational theory $\text{ZX}_p$, which as we shall see axiomatises the stabiliser fragment of quantum mechanics in the qupit $\text{ZX}$-calculus. Firstly though, we must be sure that all of these rules are sound for the standard interpretation, i.e. it should not be possible to derive an equality of diagrams whose quantum mechanical interpretations are different.

**Theorem 4** (Soundness). The equational theory $\text{ZX}_p$ is sound for $\llbracket \_ \rrbracket$, i.e., for any $A, B \in \text{ZX}^\text{Stab}_p$, $\text{ZX}_p \vdash A = B$ implies $\llbracket A \rrbracket = \llbracket B \rrbracket$. Put formally, $\llbracket \_ \rrbracket$ factors through the projection $\text{ZX}^\text{Stab}_p \rightarrow \text{ZX}^\text{Stab}_p/\text{ZX}_p$.

This set of rewriting rule also turns out to be also complete:

**Theorem 5** (Completeness). The equational theory $\text{ZX}_p$ is complete for $\llbracket \_ \rrbracket$, i.e., for any $A, B \in \text{ZX}^\text{Stab}_p$, $\llbracket A \rrbracket = \llbracket B \rrbracket$ implies $\text{ZX}_p \vdash A = B$.

The proof of Theorem 5 will be the object of the following sections.

3 Some useful structures in $\text{ZX}^\text{Stab}_p$

The set of axioms in figure 1 is somewhat minimalistic. In this section we present how it will be manipulated in practice.

3.1 Elementary derivations

While we will see that the equations in figure 1 are enough to derive any equality of quantum operation in the stabiliser fragment, we describe here some rules which are derivable in $\text{ZX}_p$ and particularly useful. We start by the interaction between some $1 \rightarrow 1$ maps:

\begin{equation}
\begin{array}{c}
\begin{array}{c}
\text{red} \\
\text{yellow}
\end{array}
\end{array}
\end{equation}

\begin{equation}
\begin{array}{c}
\begin{array}{c}
\text{red} \\
\text{yellow}
\end{array}
\end{array}
\end{equation}

(20)

The following rules, very similar to the one of [BSZ17], will be central:

\begin{equation}
\begin{array}{c}
\begin{array}{c}
\text{red} \\
\text{yellow}
\end{array}
\end{array}
\end{equation}

\begin{equation}
\begin{array}{c}
\begin{array}{c}
\text{red} \\
\text{yellow}
\end{array}
\end{array}
\end{equation}

(21)
Here we see that the $1 \rightarrow 1$ red spider play the role of the antipode of Hopf algebra, we will then refer to it as the antipode. Finally, the pauli phases can easily move around diagrams:

\[
\begin{align*}
  x,0 & = \quad \quad \\
  x,0 & = \quad \quad \\
  x,y & = -x,y \\
  x,y & = -x,y \\
\end{align*}
\]

(22)

3.2 Meta-rules

From the equational theory follows more general patterns that we will often use as meta-rules.

**Changing colours** The qubit ZX-calculus admits a particularly elegant meta-rule: take any valid equation, and swap the colour of every spider while keeping everything else identical. Then the resulting equation is also derivable in the calculus. In fact this color swap transformation is equivalent to the functor mapping a diagram $D$ to $H^{\otimes m} D \circ H^{\otimes n}$. One might hope that such a rule would hold also in zx<p>. Unfortunately, the picture is a little more intricate and less pretty since now the analog of the Hadamard gate is not an involution. Using (Colour), one can see what happen while Hadamrd gates are going through a diagram:

\[
\begin{align*}
  a,b & = \quad \quad \\
  a,b & = \quad \quad \\
  a,b & = \quad \quad \\
\end{align*}
\]

(23)

This allows to formulate a variation of the color swap rules that holds for qudits as well:

**Proposition 6** (Colour change). If $A \in \text{ZX}_{p}^{\text{Stab}}[n,m]$ is a diagram, we can consider any spider which is not the antipode as vertices linked by the following four kind of edges:

let $S(A)$ be the $\text{ZX}_{p}^{\text{Stab}}$-diagram obtained from $A$ by :

1. swapping the colour of every vertex;
2. applying the following transformation on the edges:

\[
\begin{align*}
  \quad & \rightarrow \quad \\
  \quad & \rightarrow \quad \\
\end{align*}
\]

3. adding a minus to the Pauli part of every green spider.

Then $\text{zx}_p \vdash S(A) = (\quad \otimes m \circ A \circ (\quad \otimes n)$. 

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Application of this rule will simply be referred to as (COLOUR), since the colour change rule presented in figure 1 is a subcase. Note that, although $S$ is not functorial, it can be easily made into a functor, simply by post-composing with antipodes.

Note that the original colour change rule cannot hold for $\mathbf{ZX}_p^{\text{Stab}}$, since it does if and only if the antipode is trivial. To show this, it suffices to colour change the (G-ELIM) rule.

**Spider wars** Just like in the qubit case, there are rules for fusing spiders of the same colour, and splitting spiders of opposite colours. The fusion rules are pretty much the same, up to the use of antipodes for red spiders:

$$= \begin{array}{c}
\begin{array}{c}
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\end{array}
\end{array} \quad = \begin{array}{c}
\begin{array}{c}
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\end{array}
\end{array} 
\end{array} (24)$$

They are a straightforward consequence of (FUSION) and lemma 44 which eliminates self-loops. The splitting rules are more complicated, since unlike the qubit case, we now have two elementary rules for splitting: the Hopf identity (lemma 42) and (CHAR). Using a sequential application of these rules, it is straightforward to see that, for any $x, y \in \mathbb{N}$ such that $x \geq y$, we must have:

$$= \begin{array}{c}
\begin{array}{c}
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\end{array}
\end{array} \quad = \begin{array}{c}
\begin{array}{c}
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\end{array}
\end{array} 
\end{array} (25)$$

We dub this collection of meta-rules (SPIDER), along with deformations resulting from flexisymmetry.

### 3.3 Syntactic sugar for multi-edges

Before we move into the proof of completeness, we introduce some syntactic sugar to the language. Given equation (25), in $\mathbf{ZX}_p^{\text{Stab}}$, unlike the qubit case, spiders of opposite colours can be connected by more than one edge, and these multi-edges cannot be simplified. We therefore add some syntactic sugar to represent such multi-edges. These constructions add no expressiveness to the language, and are simply used to reduce the size of some recurring diagrams. They are shamelessly stolen from previous work \[ \text{[BSZ17; Zan18; CHP19; CP20]} \], and we use them to obtain a particularly nice representation of qupit graph states \[ \text{[Zho+03]} \]. Graph states play a central role in our proof of completeness, as they have in previous completeness results of the stabiliser fragments for dimensions 2 and 3 \[ \text{[Bac14; Dun+20; Wan18]} \]. In particular, these constructions permit a nice presentation of how graph states evolve under local Clifford operations.

Firstly, we extend $\mathbf{ZX}_p^{\text{Stab}}$ by *multipliers*, which are defined inductively by:

$$\begin{array}{c}
\begin{array}{c}
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\end{array}
\end{array} := \begin{array}{c}
\begin{array}{c}
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\end{array}
\end{array} \quad \text{and} \quad \begin{array}{c}
\begin{array}{c}
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\end{array}
\end{array} := \begin{array}{c}
\begin{array}{c}
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\ldots \ldots \ldots \ldots \\
\end{array}
\end{array} \end{array} (26)$$
We also define inverted multipliers, using the standard notation for graphical languages based on symmetric monoidal categories:

\[
\xrightarrow{\times} := \begin{tikzpicture}[baseline=0.5ex]
    \node (a) at (0,0) [draw, circle, scale=0.8] {};
    \node (b) at (1,0) [draw, circle, scale=0.8] {};
    \draw (a) to (b);
\end{tikzpicture}.
\]  

(27)

Now, it is straightforward using (CHAR) to see that, for any \( m \in \mathbb{N} \),

\[
\xrightarrow{m} = \xrightarrow{m \mod p},
\]  

(28)

and we can restrict the labels of multipliers to \( \mathbb{Z}_p \). Explicitly, then, for \( x \in \mathbb{Z}_p^* \),

\[
\xrightarrow{x^{-1}} = \begin{tikzpicture}[baseline=0.5ex]
    \node (a) at (0,0) [draw, circle, scale=0.8] {};
    \node (b) at (1,0) [draw, circle, scale=0.8] {};
    \draw (a) to (b);
\end{tikzpicture}.
\]  

(29)

**Proposition 7.** Multipliers satisfy the following equations under \( \times \mathbb{Z}_p \): for any \( x, y \in \mathbb{Z}_p \) and \( z \in \mathbb{Z}_p^* \),

\[
\begin{align*}
\xrightarrow{xy} &= \xrightarrow{yx} \quad & \xrightarrow{x^{-1}} &= \xrightarrow{x} \\
\xrightarrow{x} &= \xrightarrow{y} \quad & \xrightarrow{z^{-1}} &= \xrightarrow{z} \\
\xrightarrow{z} &= \xrightarrow{z^{-1}} \quad & \xrightarrow{x+y} &= \xrightarrow{x} \xrightarrow{y} \\
\xrightarrow{x} &= \xrightarrow{y} \quad & \xrightarrow{x} &= \xrightarrow{y} \\
\xrightarrow{x} &= \xrightarrow{y} \quad & \xrightarrow{z} &= \xrightarrow{z}
\end{align*}
\]  

(30)

which amounts to saying that the multipliers form a presentation of the field \( \mathbb{Z}_p \). They also satisfy the following useful copy and elimination identities:

\[
\begin{align*}
\xrightarrow{z} &= \xrightarrow{z} \\
\xrightarrow{a} &= \xrightarrow{a} \\
\xrightarrow{b} &= \xrightarrow{b}
\end{align*}
\]  

(31)

It is worth noting at this point, that multipliers are *not* flexsymmetric, thus OCM is technically lost and the diagrams cannot be treated as decorated graphs if multipliers are included. This is one reason, beyond simplicity, that multipliers are treated as syntactic sugar rather than as generators of the language. However, we can somewhat recover OCM if we are prepared to allow directed graphs with labelled edges instead.

Multipliers also satisfy the following equation:

\[
\xrightarrow{xy} = \xrightarrow{yx},
\]  

(32)

so we can also unambiguously extend the calculus with *H-boxes*:

\[
\xrightarrow{H} := \xrightarrow{\times}.
\]  

(33)

H-boxes exactly match the labelled Hadamard boxes introduced for the qutrit case [TM21] when \( p = 3 \). More explicitly, H-boxes amount to repeated Hadamard wires:

\[
\xrightarrow{\times} \xrightarrow{\times} \xrightarrow{\times} = \xrightarrow{\times} \xrightarrow{\times} \xrightarrow{\times}.
\]  

(34)

We have that for any \( x \in \mathbb{Z}_p \),

\[
\xrightarrow{1} = \xrightarrow{x^{-1}} \quad \text{and} \quad \xrightarrow{1} = \xrightarrow{x} = \xrightarrow{1}.
\]  

(35)

so that H-boxes are flexsymmetric, unlike multipliers.
Proposition 8. \( zp \) proves the following equations:

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
\node at (0,0) {z};
\node at (1,0) {p};
\end{tikzpicture}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
\node at (0,0) {x};
\node at (1,0) {y};
\end{tikzpicture}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
\node at (0,0) {x};
\node at (1,0) {x};
\end{tikzpicture}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
\node at (0,0) {0};
\node at (1,0) {1};
\end{tikzpicture}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
\node at (0,0) {x};
\node at (1,0) {y};
\end{tikzpicture}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
\node at (0,0) {0};
\node at (1,0) {1};
\end{tikzpicture}
\end{array}
\end{align*}
\]

It follows from this that the inverse Hadamard is just the \(-1\)-weighted Hadamard box:

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
\node at (0,0) {0};
\node at (1,0) {1};
\end{tikzpicture}
\end{array}
\end{align*}
\]

4 Representing Clifford states as graphs

Graph states are a now familiar tool of quantum information theory. Their properties are intricately linked with those of the Clifford group and more generally the stabiliser fragment. The original proof of completeness of the qubit stabiliser ZX-calculus also made extensive use of graph states. The qubit graph states have been generalised to the case of arbitrary (finite) dimension, and we present them here through the lens of \( Z_p \).

The idea is to associate a state in \( L^2(\mathbb{Z}_p) \otimes V \) to any graph on a set \( V \) of vertices. Unlike in the qubit case, it is more natural for qupits to consider \( \mathbb{Z}_p \)-edge-weighted graphs. We make the choice of identifying such a graph with its adjacency matrix \( G \in \mathbb{Z}_p^{V \times V} \). Then, the corresponding graph state is defined to be

\[
|G\rangle = \prod_{(u,v) \in V} E_{G_{uv}}^{G_{uv}} \bigotimes_{u \in V} |0 : X\rangle_u.
\]

(38)

In other words, to obtain the graph state, first initialise each qupit in the state \( |0 : X\rangle \). Then, for each edge in \( G \) with weight \( G_{uv} \), apply the entangling operation \( E_{G_{uv}}^{G_{uv}} \). Since \( E \) gates commute, they can be applied in any order.

As a simple example, the graph

\[
\begin{tikzpicture}
\draw (0,0) node {3};
\draw (1,0) node {3};
\draw (0,0) -- (1,0);
\end{tikzpicture}
\]

has adjacency matrix

\[
\begin{pmatrix}
0 & 3 & 1 \\
3 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix},
\]

(39)

and is associated with the graph state \( E_{1,2}E_{1,3}E_{2,3}(|0 : X\rangle \otimes |0 : X\rangle \otimes |0 : X\rangle) \).

4.1 Graph states in \( Z_p \)

The associated graph state is represented in \( Z_p \) as:

\[
\begin{tikzpicture}
\node at (0,0) {1};
\node at (1,0) {1};
\node at (0,1) {1};
\node at (1,1) {1};
\draw (0,0) -- (1,0); \draw (0,1) -- (1,1);
\end{tikzpicture}
\]

(40)

where the equality is obtained solely by fusing the green spiders. One immediately recognises the graph from equation (39) in the RHS of equation (40).
Here is a slightly more involved example:

![Graph State Diagram](image)

is associated to the state

\[
|G\rangle = \sum_{\text{valid state}} \text{state} \cdot |G\rangle
\]

where each spider is connected to an output wire.

Hopefully, it should be clear that from these examples that for any given graph \( G \in \mathbb{Z}^V_p \), one can obtain the \( ZX_{p}^{\text{Stab}} \)-diagram for the corresponding graph state by identifying each vertex with a green spider, and each edge with a correspondingly weight H-edge. More formally:

**Definition 9.** A \( ZX_{p}^{\text{Stab}} \)-diagram is a graph state diagram if

1. it contains only green spiders;
2. every spider is connected to a single output by a plain wire;
3. the spiders are connected only by H-edges.

Every graph state diagram is uniquely associated to a graph, and vice-versa. Since we are going to reason about such diagrams in generality and without referring to a specific graph, we use the following informal notation to represent the \( ZX_{p}^{\text{Stab}} \)-diagram associated to \( G \):

![Graph State Diagram](image)

In order to avoid having to track uninteresting scalars, we assume that the interpretation of this diagram is normalised. This amounts to including, along with the purely graph-theoretical part, a pair of scalars

\[
\left( \frac{1}{\sqrt{V}} \right) \otimes \left( \frac{1}{\sqrt{|E|}} \right),
\]

where \( V \) is the set of vertices in \( G \) and \( E \) the set of edges. These are simply the normalisation factors for states and entangling gates in \( G \) (see equations (70) and (71)). It should now be clear that

\[
|G\rangle = \left[ \frac{1}{\sqrt{V}} \right] \otimes \left[ \frac{1}{\sqrt{|E|}} \right] \cdot |G\rangle
\]

Note that this notation for graph states can be formalised using the “scalable” construction [CP20], but this would involve introducing rather more machinery than we really need.
4.2 Manipulating graph states

Having shown how to represent graph states within $\mathbf{ZX}^{\text{Stab}}$, we now give some natural rules for manipulating diagrams involving them under $\mathbf{zx}_{p}$.

Two graph states $|G\rangle$ and $|H\rangle$ are said to be local Clifford equivalent if there is a sequence of local Clifford operations that maps $|G\rangle$ to $|H\rangle$. It was shown in [BB07] that any local Clifford equivalence can be decomposed as a sequence of elementary local Clifford operations called local scaling and local complementation. These are particularly nice operations since their actions on a graph state $|G\rangle$ are straightforward to understand at the level of the graph $G$ defining the graph state.

Pauli stabilisers

It is well-known that graph states admit simple Pauli stabilisers, namely, for each $v \in V$,

$$X^{\gamma}_{v} \prod_{w \in N(v)} Z^{\gamma_{vw}}_{w} |G\rangle = |G\rangle.$$  \hspace{1cm} (45)

We can give a simple formulation of these rules within $\mathbf{ZX}^{\text{Stab}}_{p}$, and they can be derived in $\mathbf{zx}_{p}$:

**Proposition 10.** $\mathbf{zx}_{p}$ proves the Pauli stabiliser rules for graph states:

where the red spiders is connected to vertex $w$ in the graph $G$, and each neighbour $k$ of $w$ is connected to a green vertex with phase $(\gamma_{G_{kw}}, 0)$.

Local scaling

For any $\gamma \in \mathbb{Z}^{*}_{p}$, the $\gamma$-scaling about a vertex $w$ in a graph $G$ is given by:

$$(G^{\gamma}_{\circ \, w})_{uv} := \begin{cases} \gamma G_{uv} & \text{if } u = w \text{ or } v = w; \\ G_{uv} & \text{otherwise}. \end{cases}$$  \hspace{1cm} (46)

In other words, we apply a multiplicative scaling to all of the edges in the neighbourhood of $w$. For example:

$$\gamma_{a} \gamma_{b} \gamma_{\circ \, w} \rightarrow \gamma_{a} \gamma_{b} \gamma_{\circ \, w}.$$  \hspace{1cm} (47)

**Proposition 11.** Local scaling is derivable in $\mathbf{zx}_{p}$: for any graph $G \in \mathbb{Z}_{p}^{N \times N}$, $\gamma \in \mathbb{Z}_{p}$ and $w \in \{1, 2, \cdots, N\}$, $\mathbf{zx}_{p}$ proves that

$$G^{\gamma}_{\circ \, w} = G^{\gamma}_{\circ \, w},$$  \hspace{1cm} (48)

where in the LHS, the multiplier is connected to vertex $w$. 

---

14
Local complementation

For any $\gamma \in \mathbb{Z}_d^*$, the $\gamma$-weighted local $\mathbb{Z}_d$-complementation or $\gamma$-complementation about a vertex $w$ in a graph $G_{\mathbb{Z}_d^V \times V}$ is defined as:

$$(G \gamma \star w)_{uv} := \begin{cases} G_{uv} + \gamma G_{uw}G_{wv} & \text{if } u \neq v; \\ G_{uv} & \text{otherwise.} \end{cases}$$ \hfill (49)

This operation is somewhat harder than local scaling to get a good intuition for. It essentially operates on “cones” in $G$ with summit $w$. The simplest example is the following:

$$a \begin{array}{c} \gamma \star w \\ \rightarrow \end{array} b \gamma_{ab}. \hfill (50)$$

In a more complicated graph, local complementation about $w$ performs this simple operation for every such “cone” with summit $w$. For example,

$$a \begin{array}{c} \gamma \star w \\ \rightarrow \end{array} \begin{array}{c} 1 + \gamma a \\ \gamma_{ab} \end{array} \gamma_{b}. \hfill (51)$$

**Proposition 12.** Local complementation is derivable in $\text{zx}_p$: for any graph $G \in \mathbb{Z}_p^{N \times N}$, $\gamma \in \mathbb{Z}_p$ and $w \in \{1, 2, \cdots, N\}$, $\text{zx}_p$ proves that

$$0, -\gamma G^2_{1w} \cdots \gamma \star w = \begin{array}{c} \vdots \\ \vdots \\ 0, -\gamma G^2_{Nw} \end{array},$$

where in the LHS, the red phase is connected to vertex $w$, and each neighbour $v$ of $w$ is connected to a green phase $(0, -\gamma G^2_{vw})$.

Combining this with propositions 11 and 10, we get

**Corollary 13.** For any graph $G \in \mathbb{Z}_p^{N \times N}$, $w \in \{1, 2, \cdots, N\}$ and $x, y \in \mathbb{Z}_p$, $\text{zx}_p$ proves that

$$xG_{1w} + \gamma G^2_{1w} \cdots \begin{array}{c} x, y \\ \vdots \\ \vdots \\ xG_{Nw} + \gamma G^2_{Nw} \end{array} \gamma \star w = \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \end{array},$$

where in the LHS, the red phase is connected to vertex $w$, and each neighbour $v$ of $w$ is connected to a green phase with labels proportional to the edge weight $G_{vw}$.

5 Completeness

We now have all the necessary tools to show that our equational theory is complete. The structure of our proof is similar to the one used to show the completeness in the qubit
case [Bac14]. However, if the overall scheme is very similar, each step separately can involve different approaches more suited to the qupit situation. The plan is as follows. We identify a family of scalars, called elementary scalars, which correspond to those which appear when applying the rewrites of $\text{zx}_p$. We first show the completeness up to non-zero elementary scalars, which allows us to work with simpler diagrams without taking care of all the invertible scalars appearing along the way. Then, we show the completeness for elementary scalars independently, leading to a general proof of completeness. The proof of completeness up to non-zero elementary scalars, goes like this:

- Take two diagrams with the same interpretation.
- Put them in GS+LC form.
- Define the notion of rGS+LC form for a pair of diagrams in which some vertices are marked. Then show that in a rGS+LC pair if a vertex is marked on one side, it must also be marked on the other side, else the two diagrams cannot have the same interpretation.
- Show that two diagram forming an rGS+LC pair such that their marked vertices matches and having the same interpretation are equal modulo the equational theory.

5.1 Elementary scalars

The following is standard from categorical quantum mechanics:

**Lemma 14.** If $A, B \in \text{ZX}^\text{Stab}_p[0,0]$, then $A \otimes B = [A] \cdot [B] = [A \circ B]$, where $\cdot$ is the usual multiplication on $\mathbb{C}$ restricted to the monoid $\mathbb{G}$.

Now, as when we were defining the group of phases $P_p$, the set of normal forms for phases must depend on the prime $p$ in question:

**Definition 15.** An elementary scalar is a diagram $A \in \text{ZX}^\text{Stab}_p[0,0]$ which is a tensor product of diagrams from the collection $O_p \cup P \cup Q$: where

- if $p \equiv 1 \mod 4$,
  $$O_p = \left\{ \begin{array}{c}
\begin{array}{c}
\otimes \\
\ast
\end{array}
\end{array} \right\} , \quad (54)$$

- if $p \equiv 3 \mod 4$,
  $$O_p = \left\{ \begin{array}{c}
\begin{array}{c}
\circ, \ast, \ast, \circ
\end{array}
\end{array} \right\} , \quad (55)$$

  $$P = \left\{ \begin{array}{c}
\begin{array}{c}
\ast, 0
\end{array}
\end{array} \mid s \in \mathbb{Z}_p^* \right\} , \quad (56)$$

and

$$Q = \left\{ (\otimes^r) \otimes, \left( \begin{array}{c}
\otimes^r \\
\ast
\end{array} \right) \otimes^r \mid r \in \mathbb{Z} \right\} . \quad (57)$$

If $A, B \in \text{ZX}^\text{Stab}_p$, we say that $A$ and $B$ are equal up to an elementary scalar if there is an elementary scalar $C$ such that $A = B \otimes C$. In that case, we write $A \simeq B$.

Comparing with the definition of $\mathbb{G}$, the interpretation of the elements of $P$ correspond to powers of $\omega$, the elements of $Q$ to (possible negative) powers of $\sqrt{p}$, and the elements of $O_p$ to powers of $-1$ or $i$ depending on the value of $p$. This remark will naturally lead to the normal for for scalars in a few sections.

Now, as written, equality up to an elementary scalar might seem like a relation that is not symmetric and therefore not an equivalence relation.
Proposition 16. Every elementary scalar $C \in \mathsf{ZX}_p^\text{Stab}[0,0]$ has a multiplicative inverse, i.e. an elementary scalar $C^{-1} \in \mathsf{ZX}_p^\text{Stab}[0,0]$ such that
\[
C \otimes C^{-1} = C \circ C^{-1} = \mathbb{1}.
\] (58)

Proof. This is the content of lemmas 43, 45, 61, 63 and axioms (GAUSS) and (M-One).

In light of this fact, if $A \simeq B$, there is an elementary scalar $C$ such that $A = B \otimes C$, and then $B = B \otimes C \otimes C^{-1} = A \otimes C^{-1}$, so that $B \simeq A$.

Proposition 17. Every equation in $\mathsf{zx}_p$ can be loosened to equality up an elementary scalar by erasing every part of the LHS and RHS diagrams which is disconnected from the inputs and outputs.

Proof. This comes from a straightforward structural induction, and noting that every scalar diagram which appears in the axiomatisation (figure 1) can be straightforwardly rewritten under $\mathsf{zx}_p$ to an elementary scalar, and thus can be ignored.

Proposition 18. If $A \in \mathsf{ZX}_p^\text{Stab}[1,1]$ is a single-qupit Clifford diagram, then $\mathsf{zx}_p$ proves that
\[
\sigma, \tau \quad \sigma, 1 \\
\upsilon, \tau \quad 0, 1
\]
for some $s, t, u, v \in \mathbb{Z}_p$ and $w \in \mathbb{Z}_p^*$. Furthermore, this form is unique.

5.2 Relating stabiliser diagrams to graphs

In order to use the preceding tools in our completeness proof for the whole stabiliser fragment, we need to understand how an arbitrary stabiliser diagram can be related to a graph state diagram. Firstly, we rewrite any stabiliser diagram $D$ to a state using the Choi-Jamiolkowski isomorphism:

\[
\Rightarrow
\]

Any diagram $0 \rightarrow n$ obtained from the Choi-Jamiolkowski isomorphism can be rewritten to a graph state diagram, with single qupit Clifford operations acting on its output wires:

\[
\Rightarrow
\]

Definition 19 ([Bac14; Wan18]). A $\mathsf{GS+LC}$ diagram is a $\mathsf{ZX}_p^\text{Stab}$-diagram which consists of a graph state diagram with arbitrary single-qupit Clifford operations applied to each output. These associated Clifford operations are called vertex operators.
Proposition 20. Every stabiliser state $\mathbf{ZX}^{\text{Stab}}_p$-diagram can be rewritten, up to elementary scalars, to GS+LC form under $\mathbf{ZX}_p$.

In other words, $\mathbf{ZX}_p$ proves that, for any stabiliser $\mathbf{ZX}^{\text{Stab}}_p$-diagram $D : m \to n$, there is a graph $G$ on $m + n$ vertices and a set of vertex operators $(v_k)_{k=1}^{m+n}$ such that

$$D \simeq G$$

and we need only consider the question of whether $\mathbf{ZX}_p$ can prove the equality of two GS+LC diagrams.

5.3 Completeness modulo elementary scalars

Now, as a first step, we show that $\mathbf{ZX}_p$ can normalise any pair of diagrams with equal interpretations, up to elementary scalars. In particular, as was shown in the previous section, we can relax $\mathbf{ZX}_p$ to reason about equality up to elementary scalars by simply ignoring the scalar part of each rule, and make free use of the "scalarless" equational theory. We will take care of the resulting scalars in the next section.

This part of the completeness proof follows the general ideas of [Bac14]. The first step on the way to completeness is to note that, considering a diagram in GS+LC form, where the vertex operators have been normalised, we can obtain a yet more reduced diagram by absorbing as much as possible of the vertex operators into local scalings and local complementations. We then obtain the following form for the vertex operators:

Definition 21 ([Bac14]). A $\mathbf{ZX}^{\text{Stab}}_p$-diagram is in reduced GS+LC (or rGS+LC) form if it is in GS+LC form, and furthermore:

1. All vertex operators belong to the following set:

$$R = \left\{ s.t, \frac{s}{s.t} \mid s, t \in \mathbb{Z}_p \right\}.$$  \hspace{1cm}(62)

2. Two adjacent vertices do not have vertex operators that both include red spiders.

Proposition 22. If $D \in \mathbf{ZX}^{\text{Stab}}_p$ is a Clifford diagram, then there is a diagram $G \in \mathbf{ZX}^{\text{Stab}}_p$ in rGS+LC form such that $\mathbf{ZX}_p \vdash D \simeq G$.

Then, given two diagrams with equal interpretations, taking them both to rGS+LC makes the task of comparing the diagrams considerably easier. In particular, we can guarantee that the corresponding vertex operators in each diagram always have matching forms:

Definition 23 ([Bac14]). A pair of rGS+LC diagrams of the same type (i.e. whose graphs have the same vertex set $V$) is said to be simplified if there is no pair of vertices $q, p \in V$ such that $q$ has a red vertex operator in the first diagram but not the second, $q$ has a red vertex operator in the second diagram but not the first, and $q$ and $p$ are adjacent in at least one of the diagrams.
Proposition 24. Any pair $A, B$ of $rGS+LC$ diagrams of the same type (i.e. on the same vertex set) can be simplified.

For the sake of clarity, we shall say that the vertex operator (or equivalently, the vertex itself) is marked if it contains a red spider (i.e. it belongs to the right-hand form of definition 21). Then, two diagrams with the same interpretation can always be rewritten so that the marked vertices match:

Proposition 25. Let $C, D \in \mathbb{Z}_p^{\text{Stab}}$ be a simplified pair in $rGS+LC$ form, then $\llbracket C \rrbracket = \llbracket D \rrbracket$ only if the marked vertices in $C$ and $D$ are the same.

Finally, we have enough control over the pair of diagrams to finish the completeness proof:

Theorem 26. $\mathbb{Z}_p$ is complete for $J^{-K}$, i.e. if for any pair of diagrams $A, B \in \mathbb{Z}_p^{\text{Stab}}[0, n]$ with $n \neq 0$, $\llbracket A \rrbracket = \llbracket B \rrbracket$, then $\mathbb{Z}_p \vdash A \simeq B$.

5.4 Completeness of the scalar fragment

Finally, we are ready to leap-frog off of the previous section into the full completeness (including scalars). First, we need to find a normal form for diagrams which evaluate to 0. In fact, we need pick one normal form for each type $m \to n$:

Proposition 27. The “zero” scalar “destroys” diagrams: for any $m, n \in \mathbb{N}$ and $D \in \mathbb{Z}_p^{\text{Stab}}[m, n],

$$
\begin{array}{c}
 \circ 1.0 \\
 m : D \\
 n
\end{array} =
\begin{array}{c}
 \circ 1.0 \\
 m : \circ \\
 n
\end{array}
$$

We take the RHS diagram to be the “zero” diagram of type $m \to n$.

Now, we say that a scalar $C \in \mathbb{Z}_p^{\text{Stab}}[0, 0]$ is in normal form if it is either the zero diagram, or it belongs to the set

$$
\left\{ \begin{array}{c}
 \circ \\
 s \in \mathbb{Z}_p
\end{array} \right\} \otimes \left\{ \begin{array}{c}
 \circ \\
 r \in \mathbb{Z}
\end{array} \right\},
$$

if $p \equiv 1 \mod 4$, or to the set

$$
\left\{ \begin{array}{c}
 \circ \\
 s \in \mathbb{Z}_p
\end{array} \right\} \otimes \left\{ \begin{array}{c}
 \circ \\
 r \in \mathbb{Z}
\end{array} \right\},
$$

when $p \equiv 3 \mod 4$. It is straightforward to see, by evaluating $\llbracket - \rrbracket$ on each element, that the sets in equations (63) or (64) contain exactly one diagram for each scalar in $\mathbb{G}_p \setminus \{0\}$ (and the zero diagram $\circ 1.0$ corresponds to 0 $\in \mathbb{G}_p$). Comparing with the definition of $\mathbb{G}_p$ in equations (8)-(9), the first factor in the tensor product corresponds to powers of $-1$ or powers of the imaginary unit $i$; the second factor corresponds to powers of $\omega$; and the third factor to powers of $\sqrt{p}$.

Theorem 28. $\mathbb{Z}_p$ proves any scalar diagram equal to a scalar from either equations (63) or (64) (depending on the congruence of $p$ modulo 4), or to the zero scalar $\circ 1.0$.

The completeness for the whole stabiliser fragment follows immediately:
**Theorem 29.** The equational theory $zxp$ is complete for $\text{Stab}_p$, i.e. for any $ZX^{\text{Stab}}_p$-diagrams $A$ and $B$, if $\llbracket A \rrbracket = \llbracket B \rrbracket$, then $zxp \vdash A = B$. Put more formally, there is a commutative diagram

$$
\begin{array}{ccc}
ZX^{\text{Stab}}_p & \xrightarrow{p} & ZX^{\text{Stab}}_p/\text{zx}_p \\
\downarrow & & \downarrow \\
\text{Stab}_p & \rightarrow & \text{Stab}_p
\end{array}
$$

(65)

where $p$ is the projection $ZX^{\text{Stab}}_p \rightarrow ZX^{\text{Stab}}_p/\text{zx}_p$ and the vertical arrow is an isomorphism of categories.

6 Mixed states and relations

In this last section we use the work of [Car+19] to extend our completeness result to the mixed-state case. We then unravel the connection to the Lagrangian relation investigated in [CK21].

6.1 A complete graphical language for $\text{CPM}(\text{Stab}_p)$

We now extend our completeness result form $\text{Stab}_p$ to $\text{CPM}(\text{Stab}_p)$, the category of completely positive maps corresponding to mixed state stabiliser quantum mechanics, see [Sel07; CK17] for a formal definition. We will rely on the discard construction of [Car+19] to define a graphical language $(ZX^{\text{Stab}}_p)^\dagger$. It consists in adding to the equational theory one generator, the discard $\dagger : 1 \rightarrow 0$ and equations stating that this generator erases all isometries. In $\text{Stab}_p$, the isometries are generated by the following diagrams:

```
  0.1
  α.0.1.0
```

The equations to add are then:

```
  \[ = \]
  \[ = \]
  \[ = \]
  \[ = \]
  \[ = \]
  \[ = \]

\[ = \]

A new interpretation $\llbracket \cdot \rrbracket : ZX^{\text{Stab}}_p^\dagger \rightarrow \text{CPM}(\text{Stab}_p)$ is defined as $\llbracket \cdot \rrbracket : \rho \mapsto \text{Tr}(\rho)$ for the ground and for all $ZX^{\text{Stab}}_p$-diagram $D : n \rightarrow m$:

$$
\llbracket D \rrbracket^\dagger : \rho \mapsto \llbracket D \rrbracket^\dagger \rho \llbracket D \rrbracket.
$$

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Corollary 22 of [Car+19] provides a sufficient condition for the previous construction to extend to a universal and complete graphical language for \( \text{Stab}_p \) into a universal complete graphical language for \( \text{CPM}(\text{Stab}_p) \). This condition is for \( \text{Stab}_p \) to have enough isometries, meaning (we use here a little stronger condition than in [Car+19]) that for all maps \( f : A \rightarrow B \otimes X \) and \( g : A \rightarrow B \otimes Y \) such that:

\[
\begin{array}{ccc}
\text{f} & = & \text{g} \\
\delta & = & \delta
\end{array}
\]

there is an isometry \( V : X \rightarrow Y \) in \( \text{Stab}_p \) such that:

\[
\begin{array}{ccc}
\text{f} \circ \delta = & & \text{g} \circ \delta
\end{array}
\]

To prove this, we will use arguments similar to the proof that \( \text{Stab}_2 \) has enough isometry. Everything relies on the following lemma:

**Lemma 30.** Given any bipartition of the outputs of a \( \text{ZX}^{\text{Stab}}_p \)-diagram \( D : 0 \rightarrow n + m \). We can find unitaries \( A \) and \( B \) in \( \text{Stab}_p \) such that:

\[
\begin{align*}
D & = \begin{pmatrix}
|+ \rangle \otimes n \\
\vdots \\
\vdots \\
\vdots \\
\end{pmatrix} \\
A & = \begin{pmatrix}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{pmatrix} \\
B & = \begin{pmatrix}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{pmatrix}
\end{align*}
\]

Using this we can prove:

**Lemma 31.** \( \text{Stab}_p \) has enough isometries.

The proof is exactly the same as the qubit case, see the proof of Proposition 18 in [Car+19]. It then follows directly from [Car+19] that:

**Theorem 32.** \( (\text{ZX}^{\text{Stab}}_p)^+ \) is universal and complete for \( \text{CPM}(\text{Stab}_p) \).

### 6.2 Co-isotropic relations

It has been shown in [CK21; Com21; Com] that \( \text{CPM}(\text{Stab}_p) \) is equivalent to the category of affine co-isotropic relations up to scalars. More formally, we endow \( \mathbb{Z}^2_p \) with the symplectic form

\[
\omega \left( \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right) = ad - bc,
\]

and \( \mathbb{Z}^{2m}_p = \bigoplus_m \mathbb{Z}_p^2 \) with the direct sum symplectic form.

**Definition 33.** The symmetric monoidal category \( \text{AffColsoRel}_{\mathbb{Z}_p} \) has as objects \( \mathbb{N} \), and as morphisms, relations \( R : \mathbb{Z}^{2m}_p \rightarrow \mathbb{Z}^{2n}_p \) such that \( R \) viewed as a subset of \( \mathbb{Z}^m_p \times \mathbb{Z}^n_p \) is an affine co-isotropic subspace thereof.
Since [CK21] works in the scalarless ZX-calculus, we need to add one extra axiom, which suffices to eliminate all remaining (non-zero) scalars in $\text{Stab}_p$: we impose the rule (MOD) that $p = 1$. Diagrammatically, this amounts to quotienting $(ZX^\text{Stab}_p)^\dagger$ by the following rule:

$$\bullet \longrightarrow \begin{array}{c}
\end{array}$$

Then we can give an interpretation $[-]$ of $(ZX^\text{Stab}_p)^\dagger$ making it universal and complete for $\text{AffColsoRel}_{Z_p}$, and which is defined uniquely by the commutative diagram

$$
\begin{array}{ccc}
(ZX^\text{Stab}_p)^\dagger & \longrightarrow & \text{AffColsoRel}_{Z_p} \\
[-] & \downarrow & \downarrow \\
\text{CPM}(\text{Stab}_p) & \longrightarrow & \text{CPM}(\text{Stab}_p)/(\text{MOD})
\end{array}
$$

Explicitly, it is given by the identity on objects, $[m] = m$, and is defined on morphisms by: for $x, y \in Z_p$ and $z \in Z^*_p$

$$
[x, y] = \begin{cases}
\bigoplus_{k=1}^{m} \begin{bmatrix}
a_k \\
b_k
\end{bmatrix} + \bigoplus_{k=1}^{n} \begin{bmatrix}
-a_k \\
c_k
\end{bmatrix} & | a, b, c, k \in Z_p \text{ and } \sum_k b_k = \sum_k c_k \\
\end{cases}
$$

Note that all of these are actually affine Lagrangian relations. The only generator which has a co-isotropic but not Lagrangian semantics is the discard map:

$$[-\bullet] = \begin{cases}
(v, \bullet) & | v \in Z^2_p
\end{cases}
$$

As pointed out in [BCR18; BF18; CK21], the related category of affine Lagrangian relations over the field $\mathbb{R}[x, y]/(xy - 1)$ can be used to represent a fragment of electrical circuits. We expect that the axiomatisation of figure 1 can be adapted to that setting, but leave this for a future article.

### 6.3 Geometrical interpretation

The previous interpretation corresponds to a geometric intuition. All Clifford gates can be interpreted as affine transformations of the torus $Z_p \times Z_p$. This torus can be identified as a phase space, the position and momentum coordinates, $q$ and $p$ corresponding respectively to the first and second wire in the previous section semantics. In this section we will take as an example the case $p = 5$. 

---

22
The red triangle here will allow us to illustrate the geometrical action of stab.

The Pauli phase gates, acts as translations along the vertical or horizontal axis. The multiplication gates acts as a scaling. The Hadamard gate corresponds to a $\pi/2$-rotation and the antipode to a $\pi$-rotation. Finally, the pure Clifford gates corresponds to shears.

**Conclusion**

We have constructed a ZX-calculus which captures the stabiliser fragment in odd prime dimensions, whilst retaining many of the “nice” features of the qubit ZX-calculus. Of course, there are a few obvious questions that we leave for future work.

First amongst these is the question of whether a fully universal calculus can be obtained from the ideas we used here. The spiders we have used here labelled with elements $a, b \in \mathbb{Z}_p \times \mathbb{Z}_p$ and which can be interpreted as polynomials $x \mapsto ax + bx^2$ which parametrise the phases of the spider. Adding one additional term of degree 3 is already sufficient to obtain a universal calculus in prime dimensions (strictly) greater than 3 [HV12]. In fact, one might as well add all higher order of polynomials (mod $p$) since access to such higher degrees will hopefully prove useful in finding commutation relations for the resulting spiders.

Secondly, it remains to be seen how to formulate a universal ZX-calculus for non-prime dimensions, even for just the stabiliser fragment. For this, the methods in this article are clearly inadequate: for example local scaling is no longer an invertible operation and thus certainly not in the Clifford group.

Finally, the set of axioms we provide here is probably not minimal. It would be nice to see if a simplified version can be obtained, as was done in [BPW17] for the qubit case.
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A Some lemmas from elementary number theory

Proposition 34 (First supplement to Quadratic Reciprocity). $-1$ is a square mod $p$ if and only if $p$ is congruent to 1 mod 4.

Lemma 35. For any odd prime $p$ and $x, y \in \mathbb{Z}_p$, if neither $x$ nor $y$ is square, then $xy$ is a square.

Proof. The map $x \mapsto x^2$ is an endomorphism of $\mathbb{Z}_p$ with kernel $\{-1, 1\}$. In other words, the image of this map, the subgroup $Q_p$ of squares in $\mathbb{Z}_p^*$, must have index 2, and then $\mathbb{Z}_p^*/Q_p$ is a two element group. If neither $x$ nor $y$ is square, $xQ_p = yQ_p$ whence $(xy)Q_p = (xQ_p)(yQ_p) = (xQ_p)^2 = Q_p$, so that $xy \in Q_p$. □

Corollary 36. For any prime $p$ and $x \in \mathbb{Z}_p$, at least one of $-1$, $x$ or $-x$ is a square.
B Proof of universality

Proof of theorem 3. By proposition 1, in odd prime dimensions the generalised Clifford groups are generated by

\[ \begin{bmatrix} 1 & 1 \\ z & 1 \end{bmatrix} = S, \quad \begin{bmatrix} 2 \\ 0 \end{bmatrix} = H \quad \text{and} \quad \begin{bmatrix} 1 & 1 \\ z & 1 \end{bmatrix} = E. \] (70)

Since computational basis states and effects are easily represented as:

\[ \begin{bmatrix} 2x,0 \\ 0 \end{bmatrix} = |x\rangle \quad \text{and} \quad \begin{bmatrix} 2x,0 \\ 0 \end{bmatrix} = \langle x|, \] (71)

the standard interpretation is clearly universal for the stabiliser fragment by the functoriality of \([ - ]\).

C Proof of soundness

C.1 Proof of soundness

Proof of theorem 4. We prove that each of the equations in figure 1 is sound, and this extends to the entire equational theory by functoriality of \([ - ]\).

Firstly, straightforward calculation shows that, for any \(x, y \in \mathbb{Z}_p\),

\[ \begin{bmatrix} 2x,0 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{p}} \quad \text{and} \quad \begin{bmatrix} x,0 \\ 0 \end{bmatrix} = \sqrt{p} \omega^{2-2xz}, \quad \text{and} \quad \begin{bmatrix} x,y \\ 0 \end{bmatrix} = \sqrt{p}. \] (72)

We also have

\[ \begin{bmatrix} x,y \\ 0 \end{bmatrix} = \sum_{k \in \mathbb{Z}_p} \omega^{2-1(xz+yk^2)}, \] (73)

and using the harmonic series for the Kronecker delta, \([ x,0 ]\) = \(p \delta_{x,0}\).

(ZERO) and (ONE) then follow immediately from equation (72).

(FUSION) For any \(a, b, c, d \in \mathbb{Z}_p\),

\[ \begin{bmatrix} a,b \\ c,d \end{bmatrix} = \left( \sum_{k \in \mathbb{Z}_p} \omega^{2-1(ak+bk^2)} |k\rangle \otimes n (k) \otimes m \otimes \text{id}_{m'} - 1 \right) \] (74)

\[ \circ \left( \text{id}_{m-1} \otimes \sum_{\ell \in \mathbb{Z}_p} \omega^{2-1(c\ell+d\ell^2)} |\ell\rangle \otimes n' \otimes m' \right) \]

\[ = \sum_{k, \ell \in \mathbb{Z}_p} \omega^{2-1(ak+bk^2)} \omega^{2-1(c\ell+d\ell^2)} |k\rangle \otimes n \langle k| \otimes m-1 \langle k| \otimes n' \langle \ell| \otimes m' \] (75)

\[ = \sum_{k, \ell \in \mathbb{Z}_p} \delta_{k,\ell} \omega^{2-1(ak+bk^2)} \omega^{2-1(c\ell+d\ell^2)} |k\rangle \otimes n \langle k| \otimes m-1 \langle \ell| \otimes m' \] (76)
For any $a, b \in \mathbb{Z}_p$,

$$\left[ \begin{array}{c} \vdots \\ \; \; a, b \\ \; \; \vdots \\ \end{array} \right] = \left( \sum_{j \in \mathbb{Z}_p} |j : X| |j : Z| \otimes \cdots \otimes \sum_{j \in \mathbb{Z}_p} |j : X| |j : Z| \right) \circ \left( \sum_{\ell \in \mathbb{Z}_p} |\ell : Z| |\ell : -X| \otimes \cdots \otimes \sum_{\ell \in \mathbb{Z}_p} |\ell : Z| |\ell : -X| \right)$$

(79)

$$= \sum_{k \in \mathbb{Z}_p} \omega^{2^{-1}(ak+bk^2)} |k : X| |k : Z|$$

(80)

$$= \left[ \begin{array}{c} \vdots \\ \; \; a, b \\ \; \; \vdots \\ \end{array} \right]$$

(81)

(COLOUR) For any $a, b \in \mathbb{Z}_p$,

$$\left[ \begin{array}{c} \vdots \\ \; \; a, c, d \\ \; \; \vdots \\ \end{array} \right] = \sqrt{p} \omega^{-2^{-2}ac} \sum_{j, k, \ell \in \mathbb{Z}_p} \omega^{-2^{-1}a_j} \omega^{-2^{-1}(ck+dk^2)} \omega^{-2^{-1}a_\ell} |j \rangle |j \rangle | -k : X | |k : X| |\ell \rangle |\ell \rangle$$

(82)

$$= \sqrt{p} \omega^{-2^{-2}ac} \sum_{j, k, \ell \in \mathbb{Z}_p} \omega^{-2^{-1}a_j} \omega^{-2^{-1}(ck+dk^2)} \omega^{-2^{-1}a_\ell} \omega^{-2^{-1}a_j} |j \rangle |j \rangle | -k \rangle | -k \rangle$$

(83)

$$= \sqrt{p} \omega^{-2^{-2}ac} \sum_{j, k, \ell \in \mathbb{Z}_p} \omega^{-2^{-1}a_j} \omega^{-2^{-1}a_\ell} \omega^{-1} |j \rangle |j \rangle | -k \rangle | -k \rangle$$

(84)

$$= \sqrt{p} \omega^{-2^{-2}ac} \sum_{j, k, \ell \in \mathbb{Z}_p} \omega^{-2^{-1}a_j} \omega^{-2^{-1}a_\ell} \omega^{-1} |j \rangle |j \rangle | -k \rangle | -k \rangle$$

(85)

(SHEAR) For any $a, c, d \in \mathbb{Z}_p$,

$$\left[ \begin{array}{c} \vdots \\ \; \; a, 0 \\ \; \; \vdots \\ \end{array} \right] = \sum_{j, k, \ell \in \mathbb{Z}_p} \omega^{-2^{-3}a_j} \omega^{-2^{-3}(ck+dk^2)} \omega^{-2^{-3}a_\ell} |j \rangle |j \rangle | -k : X | |k : X| |\ell \rangle |\ell \rangle$$

(86)

$$= \sum_{j, k, \ell \in \mathbb{Z}_p} \omega^{-2^{-3}a_j} \omega^{-2^{-3}(ck+dk^2)} \omega^{-2^{-3}a_\ell} \omega^{-2^{-3}a_j} |j \rangle |j \rangle | -k \rangle | -k \rangle$$

(87)

$$= \sum_{j, k, \ell \in \mathbb{Z}_p} \omega^{-2^{-3}a_j} \omega^{-2^{-3}(ck+dk^2)} \omega^{-2^{-3}a_\ell} \omega^{-2^{-3}a_j} |j \rangle |j \rangle | -k \rangle | -k \rangle$$

(88)

$$= \sum_{j, k, \ell \in \mathbb{Z}_p} \omega^{-2^{-3}a_j} \omega^{-2^{-3}(ck+dk^2)} |j \rangle |j \rangle | -k : X | |k : X|$$

(89)
\[
\begin{align*}
\left[ \begin{array}{c}
(\cdot, \cdot)^{\otimes p} \\
\circ \quad \circ
\end{array} \right] &= \sqrt{p^p} \sum_{j,k \in \mathbb{Z}_p} | -k : X \rangle \langle k : X |^{\otimes p} | j \rangle^{\otimes p} \langle j | \\
&= \sum_{j,k \in \mathbb{Z}_p} \omega^{-pjk} | -k : X \rangle \langle j | \\
&= \sum_{j,k \in \mathbb{Z}_p} | -k : X \rangle \langle j | \\
&= \mathbb{1}
\end{align*}
\]

(Bigebra)
\[
\begin{align*}
\left[ \begin{array}{c}
\circ \quad \circ \\
\circ \quad \circ
\end{array} \right] &= p \sum_{i,j,k,\ell \in \mathbb{Z}_p} | -k : X \rangle | -\ell : X \rangle \langle k : X | i \rangle \langle \ell : X | j \rangle \langle k : X | j \rangle \langle \ell : X | i \rangle | i \rangle | j \rangle \\
&= \frac{1}{p} \sum_{i,j,k,\ell \in \mathbb{Z}_p} \omega^{-i+k} \omega^{-i+\ell} \omega^{-jk} \omega^{-j+\ell} | -k : X \rangle | -\ell : X \rangle | i \rangle \langle j | \\
&= \frac{1}{p^3} \sum_{i,j,k,\ell \in \mathbb{Z}_p} \omega^{i(o-k-\ell) + j(p-k-\ell) - km} \omega^{-n\ell} | m \rangle \langle o : X | p : X | \\
&= \frac{1}{p^3} \sum_{i,j,k,\ell \in \mathbb{Z}_p} \omega^{i(o-k-\ell) + j(p-k-\ell) - km} \omega^{-n\ell} | m \rangle \langle o : X | p : X | \\
&= \frac{1}{p^3} \sum_{i,j,k,\ell \in \mathbb{Z}_p} \omega^{i(o-k-\ell) + j(p-k-\ell) + \ell} \omega^{-km} \omega^{-n\ell} | m \rangle \langle o : X | p : X | \\
&= \frac{1}{p} \sum_{k,\ell,m \in \mathbb{Z}_p} \delta_{o,k} \delta_{p,k} + \ell \omega^{-km} \omega^{-n\ell} | m \rangle \langle o : X | p : X | \\
&= \frac{1}{p^3} \sum_{k,\ell,m \in \mathbb{Z}_p} \omega^{-km} \omega^{-n\ell} | m \rangle \langle k + \ell : X | k + \ell : X | \\
&= \frac{1}{\sqrt{p}} \sum_{k,\ell,m \in \mathbb{Z}_p} \omega^{k(n-m)} | m \rangle | n \rangle | -k - \ell : X \rangle \langle k + \ell : X | \langle k + \ell : X | \\
&= \sqrt{p} \sum_{k,\ell,m \in \mathbb{Z}_p} \delta_{m,n} | m \rangle | n \rangle | -k - \ell : X \rangle \langle \ell : X | \\
&= \sqrt{p} \sum_{m,\ell \in \mathbb{Z}_p} | m \rangle | m \rangle | -\ell : X \rangle \langle \ell : X |
\end{align*}
\]

(Copy)
\[
\begin{align*}
\left[ \begin{array}{c}
(\cdot, 0)^{\otimes p} \\
\circ \quad \circ
\end{array} \right] &= \sqrt{p} \sum_{j,k \in \mathbb{Z}_p} \omega^{2^{-1} x j} | k \rangle | -j : X \rangle | k \rangle | k \rangle
\end{align*}
\]
\[
= \sum_{j,k \in \mathbb{Z}_p} \omega^{2^{-1} j} \omega^{-j k} |k\rangle |k\rangle \\
= p \sum_{k \in \mathbb{Z}_p} \delta_{k,2^{-1} x} |k\rangle |k\rangle \\
= p \left| 2^{-1} x \right\rangle \left| 2^{-1} x \right\rangle \\
= \sum_{j \in \mathbb{Z}_p} \omega^{-2^{-1} j} |j : X\rangle \sum_{k \in \mathbb{Z}_p} \omega^{-2^{-1} x k} |k : X\rangle \\
= \sum_{j \in \mathbb{Z}_p} \omega^{-2^{-1} j} |j : X\rangle \sum_{k \in \mathbb{Z}_p} \omega^{2^{-1} x k} |-k : X\rangle \\
= \left[ \begin{array}{c}
0, 0 \\
0, z, 0 \\
z, z^{-1}, 0
\end{array} \right]
\]

(G-ELIM)
\[
\left[ \begin{array}{c}
0, 0 \\
0, z, 0 \\
z, z^{-1}, 0
\end{array} \right] = \sum_{k \in \mathbb{Z}_p} |k\rangle |k\rangle = \left[ \begin{array}{c}1 \\
1 \\
1
\end{array} \right]
\]

(R-ELIM)
\[
\left[ \begin{array}{c}
0, 0 \\
0, z, 0 \\
z, z^{-1}, 0
\end{array} \right] = \sum_{k \in \mathbb{Z}_p} |-k : X\rangle |k : X\rangle |-j : X\rangle |j : X\rangle = \sum_{k \in \mathbb{Z}_p} \delta_{k,-j} |-k : X\rangle |j : X\rangle \\
= \sum_{k \in \mathbb{Z}_p} |j : X\rangle |j : X\rangle = \left[ \begin{array}{c}1 \\
1 \\
1
\end{array} \right]
\]

(MULT) For any \( z \in \mathbb{Z}_p^* \),
\[
\left[ \begin{array}{c}
0, 0 \\
0, z, 0 \\
z, z^{-1}, 0
\end{array} \right] = \sum_{a,b,c,d \in \mathbb{Z}_p} \omega^{2^{-1} z - 1} a^2 \omega^{2^{-1} z b^2} \omega^{2^{-1} z - 1} c^2 \omega^{-ab} \omega^{-bc} \delta_{c,d} |a : Z\rangle |d : X\rangle \\
= \sum_{a,b,c \in \mathbb{Z}_p} \omega^{2^{-1} z - 1} a^2 \omega^{2^{-1} z b^2} \omega^{2^{-1} z - 1} c^2 \omega^{-ab} \omega^{-bc} |a : Z\rangle |c : X\rangle \\
= \sum_{a,b,c \in \mathbb{Z}_p} \omega^{2^{-1} z - 1} (a - zb + c)^2 \omega^{-z - 1} ac |a : Z\rangle |c : X\rangle \\
= \left( \sum_{b \in \mathbb{Z}_p} \omega^{2^{-1} z - 1} b^2 \right) \left( \sum_{a,c \in \mathbb{Z}_p} \omega^{-z - 1} ac |a : Z\rangle |c : X\rangle ight) \\
= \left( \sum_{b \in \mathbb{Z}_p} \omega^{2^{-1} z - 1} b^2 \right) \sqrt{p} \left( \sum_{a \in \mathbb{Z}_p} |a : Z\rangle \langle z^{-1} a : Z| \right) \\
= \left( \sum_{b \in \mathbb{Z}_p} \omega^{2^{-1} z - 1} b^2 \right) \sqrt{p} \left( \sum_{a \in \mathbb{Z}_p} |za : Z\rangle \langle a : Z| \right) \\
= \left( \sum_{b \in \mathbb{Z}_p} \omega^{2^{-1} z - 1} b^2 \right) \left( \sum_{a,c \in \mathbb{Z}_p} \omega^{-2ac} |c : X\rangle \langle a : Z| \right) \\
= \left( \sum_{b \in \mathbb{Z}_p} \omega^{2^{-1} z - 1} b^2 \right) \sqrt{p^2} \left( \sum_{a,c \in \mathbb{Z}_p} |c : X\rangle \langle c : X|^{\otimes z} |a : Z\rangle \langle a : Z|^{\otimes z} \right)
\]
\[(\text{GAUSS}) \text{ follows using equation (73) of [App09]:}\]
\[\begin{align*}
\left[ \begin{array}{c}
\ast \ast \\
\ast \\
\ast \\
\ast \\
\end{array} \right] &= \sum_{j \in \mathbb{Z}_p} \omega^{2^{-1}z_j^2} = -i^{\frac{p+3}{2}} \ell_p(z) = -i^{\frac{p+3}{2}} (-1)^{\chi_p(z)} = \left[ \begin{array}{c}
\ast \ast \\
\ast \\
\ast \\
\ast \\
\end{array} \right].
\end{align*}\]  
(123)

where we have used the fact that \(\ell_p(1) = 1\) (the Legendre symbol mod \(p\) of 1), and also that \(\ell_p(z) = (-1)^{\chi_p(z)}\) whenever \(z \neq 0\).

\[\begin{align*}
\left[ \begin{array}{c}
\ast \\
\ast \\
\ast \\
\ast \\
\end{array} \right] &= (-1)(-1) = 1 = \left[ \begin{array}{c}
\ast \\
\ast \\
\ast \\
\ast \\
\end{array} \right].
\end{align*}\]  
(124)

\[\begin{align*}
\left[ \begin{array}{c}
\ast \ast \ast \\
\ast \\
\ast \\
\ast \\
\end{array} \right] &= \sum_{j,k \in \mathbb{Z}_p} \omega^{2^{-1}(ak+bk^2)} \langle k : X | -z^j \rangle \langle j | \\
&= \sum_{j,k \in \mathbb{Z}_p} \omega^{2^{-1}(ak+bk^2)} \frac{\omega^{2^{-1}kz_j}}{\sqrt{p}} \langle j | \\
&= \frac{1}{\sqrt{p}} \sum_{j,k \in \mathbb{Z}_p} \omega^{2^{-1}(ak+bk^2)} \omega^{2^{-1}kz_j} \langle j | \\
&= \sum_{k \in \mathbb{Z}_p} \omega^{2^{-1}(ak+bk^2)} (-z : X) \\
&= \sum_{k \in \mathbb{Z}_p} \omega^{2^{-1}(-az^{-1}k+bz^{-2}k^2)} \langle k : X | \\
&= \left[ \begin{array}{c}
\ast \ast \ast \\
\ast \\
\ast \\
\ast \\
\end{array} \right].
\end{align*}\]  
(125-130)

\section*{D Proof of completeness}

\subsection*{D.1 Elementary derivations}

\textbf{Lemma 37.} Products of Hadamards are antipodes:

\[\] = \[\]

\textbf{Proof.}

\[\] = \[\] = \[\]

\textbf{Lemma 38.} Hadamards and antipodes commute:

\[\] = \[\]
**Lemma 39.** The inverse Hadamard is a product of Hadamards:

Proof.

Lemma 40. Units absorb antipodes:

Proof. The red equation is basically a subcase of \((M\text{-Elim})\):

and the green rule is obtained using \((\text{COLOUR})\):

**Lemma 41.** The Hadamards admit simple Euler decompositions:

Proof.

**Lemma 42.** The Hopf identity is derivable in \(zx_p\):
Proof.

Then, the second rule follows using the colour change meta-rule. □

**Lemma 43.** The following rules hold between the “elementary” scalars:

\[
\begin{array}{c}
\text{\textcolor{red}{\large \text{\textbullet}}} = \\
\text{\textcolor{green}{\large \text{\textbullet}}} = \\
\text{\textcolor{red}{\large \text{\textbullet}}} = \\
\text{\textcolor{green}{\large \text{\textbullet}}} = \\
\end{array}
\]

\[
\begin{array}{c}
\text{(Spider)} \\
\text{(Bigebra)} \\
\text{(Lem 40)} \\
\text{(Copy)}
\end{array}
\]

Proof.

\[
\begin{array}{c}
\text{\textcolor{red}{\large \text{\textbullet}}} = \\
\text{\textcolor{green}{\large \text{\textbullet}}} = \\
\text{\textcolor{red}{\large \text{\textbullet}}} = \\
\text{\textcolor{green}{\large \text{\textbullet}}} = \\
\end{array}
\]

\[
\begin{array}{c}
\text{(Spider)} \\
\text{(Lem 42)} \\
\text{(One)} \\
\text{(Lem 43)}
\end{array}
\]

(131)

**Lemma 44.** Self-loops on green spiders can be eliminated:

\[
\begin{array}{c}
\text{\textcolor{green}{\large \text{\textbullet}}} = \\
\text{\textcolor{red}{\large \text{\textbullet}}} = \\
\end{array}
\]

We include the colour-swapped version of this rule for completeness, even though it no longer includes a genuine self-loop.

Proof.

\[
\begin{array}{c}
\text{\textcolor{green}{\large \text{\textbullet}}} = \\
\text{\textcolor{red}{\large \text{\textbullet}}} = \\
\text{\textcolor{green}{\large \text{\textbullet}}} = \\
\text{\textcolor{red}{\large \text{\textbullet}}} = \\
\end{array}
\]

\[
\begin{array}{c}
\text{(R-Elim)} \\
\text{(Fusion)} \\
\text{(Lem 42)} \\
\text{(Lem 44)}
\end{array}
\]

(131)

The red version is obtained using (COLOUR). □

Proof.

\[
\begin{array}{c}
\text{\textcolor{green}{\large \text{\textbullet}}} = \\
\text{\textcolor{red}{\large \text{\textbullet}}} = \\
\text{\textcolor{green}{\large \text{\textbullet}}} = \\
\text{\textcolor{red}{\large \text{\textbullet}}} = \\
\end{array}
\]

\[
\begin{array}{c}
\text{(One)} \\
\text{(Lem 44)} \\
\text{(Copy)} \\
\text{(G-Elim)} \\
\text{(Lem 40)} \\
\text{(Fusion)}
\end{array}
\]

(132)

The rule \( \textcolor{green}{\Large \text{\textbullet}} = \textcolor{red}{\Large \text{\textbullet}} \) is immediate using (COLOUR). □

**Lemma 45.** The following rules hold between the “phase” scalars: for any \( a, b \in \mathbb{Z}_p \),

\[
\begin{array}{c}
\text{\textcolor{red}{\large \text{\textbullet}}} = \\
\text{\textcolor{green}{\large \text{\textbullet}}} = \\
\end{array}
\]

Proof.

\[
\begin{array}{c}
\text{\textcolor{red}{\large \text{\textbullet}}} = \\
\text{\textcolor{green}{\large \text{\textbullet}}} = \\
\text{\textcolor{red}{\large \text{\textbullet}}} = \\
\text{\textcolor{green}{\large \text{\textbullet}}} = \\
\end{array}
\]

\[
\begin{array}{c}
\text{(One)} \\
\text{(Lem 44)} \\
\text{(Copy)} \\
\text{(G-Elim)} \\
\text{(Lem 40)} \\
\text{(Fusion)}
\end{array}
\]

(132)

The rule \( \textcolor{red}{\Large \text{\textbullet}} \) is immediate using (COLOUR). □

**Lemma 46.** Green units absorb red rotations and vice-versa:

\[
\begin{array}{c}
\text{\textcolor{red}{\large \text{\textbullet}}} = \\
\text{\textcolor{green}{\large \text{\textbullet}}} = \\
\end{array}
\]

\[
\begin{array}{c}
\text{\textcolor{green}{\large \text{\textbullet}}} \quad \text{\textcolor{green}{\large \text{\textbullet}}} \\
\text{\textcolor{red}{\large \text{\textbullet}}} \quad \text{\textcolor{red}{\large \text{\textbullet}}}
\end{array}
\]

34
Lemma 47. The scalars in the bigebra law can be simplified to:

Proof.

Lemma 48. The green co-multiplication copies antipodes:

Proof.

Lemma 49. The bigebra law holds for arbitrary arities: for any $m, n \in \mathbb{N}$,

where in the diagram on the LHS, there are $m$ green and $n$ red spiders, and each green spider is connected to each red spider by a single wire.

Proof. The cases $m = 0$ or $n = 0$ correspond to the copy rules. The case $n = 1$ is the antipode copy rule (lemma 48) and the case $m = 1$ is trivial by the green identity rule. The case $n = 2, m = 2$ is lemma 47. The general case follows from a straightforward induction (which furthermore is analogous to the qubit case).

Lemma 50. The antipode can be rewritten as a multiplication:
Lemma 51. For any $x, y \in \mathbb{Z}_p$,

$\begin{align*}
x, y & = -x, y \\
x, y & = -x, y
\end{align*}$

Proof.

$\begin{align*}
x, y & = -x, y \\
x, y & = -x, y
\end{align*}$

The rule for red spiders is obtained from the green rule:

$\begin{align*}
x, y & = -x, y \\
x, y & = -x, y
\end{align*}$

Lemma 52. For any $x, y \in \mathbb{Z}_p$ and $m, n \in \mathbb{N}$,

$\begin{align*}
x, y & = m \cdot \hat{z}_n \\
x, y & = m \cdot \hat{z}_n \\
x, y & = m \cdot \hat{z}_n \\
x, y & = m \cdot \hat{z}_n
\end{align*}$

Proof.

$\begin{align*}
x, y & = m \cdot \hat{z}_n \\
x, y & = m \cdot \hat{z}_n \\
x, y & = m \cdot \hat{z}_n \\
x, y & = m \cdot \hat{z}_n
\end{align*}$

The rule for red spiders is obtained using (COLOUR) like in the proof of lemma 51.

Lemma 53. Green spiders copy red Pauli phases, and vice-versa: for any $x \in \mathbb{Z}_p$,

$\begin{align*}
x, 0 & = -x, 0 \\
x, 0 & = -x, 0
\end{align*}$
Proof.

The other rule is obtained using (COLOUR).

D.2 Multipliers

Lemma 54. Parallel multipliers sum: for any \( x, y \in \mathbb{Z}_p \):

\[
\begin{array}{c}
\text{Proof.} \\
\text{This is a straightforward consequence of (SPIDER).}
\end{array}
\]

Lemma 55. For any \( z \in \mathbb{Z}_p^* \),

\[
\begin{array}{c}
\text{Proof.} \\
The right rule is obtained using (COLOUR) as previously.
\end{array}
\]

Lemma 56. For any \( x, y \in \mathbb{N} \),

\[
\begin{array}{c}
\text{Proof.} \\
\end{array}
\]

Lemma 57. For any \( x \in \mathbb{Z}_p^* \),

\[
\begin{array}{c}
\end{array}
\]
Proof.

The second equality follows from the (Colour) meta-rule.

Lemmas 54-57 suffice to prove proposition 7 and 8, so we consider those rules proved from this point on, and adopt the multiplier notation.

Lemma 58. Spiders copy invertible multipliers: for any \( x \in \mathbb{Z}_p^* \),

\[
\begin{align*}
\text{Proof.} \\
(a) & \Rightarrow (\text{Def}) \Rightarrow (\text{Lem 48}) \Rightarrow (\text{Lem 49}) \Rightarrow (\text{R-Elim}) \\
(b) & \Rightarrow (\text{Colour}) \Rightarrow (\text{R-Elim})
\end{align*}
\]

Then (b) is obtained from (a) using already established properties of the multipliers:

\[
\begin{align*}
\text{Proof.} \\
(a) & \Rightarrow (\text{Def}) \Rightarrow (\text{Lem 48}) \Rightarrow (\text{Lem 49}) \Rightarrow (\text{R-Elim}) \\
(b) & \Rightarrow (\text{Colour}) \Rightarrow (\text{R-Elim})
\end{align*}
\]
Proof of proposition 11. This proposition now follows straightforwardly using lemma 58, the definition of H-boxes (equation (33)) and proposition 7.

Lemma 59. The action of multipliers on spiders is given by, for any \( x \in \mathbb{Z}_p^* \),

\[
\begin{align*}
\begin{array}{c}
\ast_1 \ast_2 \\
(\text{Lem 41})
\end{array} & \Rightarrow \\
\begin{array}{c}
\ast_1 \\
\text{(Lem 45)}
\end{array} \quad & \Rightarrow \\
\begin{array}{c}
\ast_2 \\
\text{(Lem 45)}
\end{array}
\end{align*}
\]

Proof. This follows straightforwardly using lemma 58 and (M-Elim).

Lemma 60. Any pure-Clifford states can be represented in both the red and green fragment: for any \( x \in \mathbb{Z}_p^* \),

\[
\begin{align*}
\begin{array}{c}
\ast_1 \\
(\text{Lem 42})
\end{array} & \Rightarrow \\
\begin{array}{c}
\ast_2 \\
\text{(Lem 45)}
\end{array} \quad & \Rightarrow \\
\begin{array}{c}
\ast_2 \\
\text{(Lem 45)}
\end{array}
\end{align*}
\]

Proof. Firstly, we prove the subcase \( x = 1 \) of (a):

\[
\begin{align*}
\begin{array}{c}
\ast_1 \\
(\text{Lem 60})
\end{array} & \Rightarrow \\
\begin{array}{c}
\ast_2 \\
\text{(Lem 60)}
\end{array} \quad & \Rightarrow \\
\begin{array}{c}
\ast_2 \\
\text{(Lem 60)}
\end{array}
\end{align*}
\]

Then the general case for any invertible \( x \) follows using lemma 59.

(b) follows once again using (COLOUR).

Lemma 61.

Proof. Since 1 is always a square:

\[
\begin{align*}
\begin{array}{c}
\ast_1 \\
(\text{Lem 60})
\end{array} & \Rightarrow \\
\begin{array}{c}
\ast_2 \\
(\text{Lem 60})
\end{array} \quad & \Rightarrow \\
\begin{array}{c}
\ast_2 \\
(\text{Lem 60})
\end{array}
\end{align*}
\]

Lemma 62. For any \( z \in \mathbb{Z}_p^* \),

\[
\begin{align*}
\begin{array}{c}
\ast_1 \\
(\text{Lem 60})
\end{array} & \Rightarrow \\
\begin{array}{c}
\ast_2 \\
(\text{Lem 60})
\end{array} \quad & \Rightarrow \\
\begin{array}{c}
\ast_2 \\
(\text{Lem 60})
\end{array}
\end{align*}
\]

Proof. If \( z \) is a square, then there is some \( \alpha \in \mathbb{Z}_p \) such that

\[
\begin{align*}
\begin{array}{c}
\ast_1 \\
(\text{Lem 60})
\end{array} & \Rightarrow \\
\begin{array}{c}
\ast_2 \\
(\text{Lem 60})
\end{array} \quad & \Rightarrow \\
\begin{array}{c}
\ast_2 \\
(\text{Lem 60})
\end{array}
\end{align*}
\]

If \( -z \) is a square, then there is again some \( \alpha \in \mathbb{Z}_p \) such that

\[
\begin{align*}
\begin{array}{c}
\ast_1 \\
(\text{Lem 60})
\end{array} & \Rightarrow \\
\begin{array}{c}
\ast_2 \\
(\text{Lem 60})
\end{array} \quad & \Rightarrow \\
\begin{array}{c}
\ast_2 \\
(\text{Lem 60})
\end{array}
\end{align*}
\]

Now, if neither \( z \) nor \( -z \) is a square, then by corollary 36, \( -1 \) must be a square. We then have

\[
\begin{align*}
\begin{array}{c}
\ast_1 \\
(\text{Lem 60})
\end{array} & \Rightarrow \\
\begin{array}{c}
\ast_2 \\
(\text{Lem 60})
\end{array} \quad & \Rightarrow \\
\begin{array}{c}
\ast_2 \\
(\text{Lem 60})
\end{array}
\end{align*}
\]
Lemma 63. For any $z \in \mathbb{Z}_p$,

Proof.

where we have freely used lemma 51 to commute antipodes and spiders throughout.

Lemma 64. If $p \equiv 3 \mod 4$,

Proof. If $p \equiv 3 \mod 4$, $-1$ is not a square, so that

Lemma 65. All the elementary “zero” diagrams are equal:

Proof.

Lemma 66. The “zero” diagram absorbs unlabelled elementary scalars:
Proof. First note that

\[
1,0 = (\text{Zero}) = 1,0
\]

so that

\[
1,0 = (\text{One}) = 1,0
\]

and

\[
1,0 = 1,0 (\text{One}) = 1,0
\]

Lemma 67. The zero diagram absorbs the phase scalars: for any \( x \in \mathbb{Z}_p \) and \( z \in \mathbb{Z}_p^* \),

\[
0, x = 0, z = 1,0
\]

Proof.

\[
0, x = 0, z = 1,0 \quad (\text{Zero}) = 1,0 \quad (\text{One})
\]

Lemma 68. Quadratic phases satisfy the following multiplication law: for any \( x, y \in \mathbb{Z}_p^* \),

\[
0, x \odot \chi_p(x) + \chi_p(y) + \chi_p(-1)
\]

Proof. This follows from applications of (GAUSS) and lemma 61.

Lemma 69. For any \( x, y \in \mathbb{Z}_p \),

\[
0, y = 1,0
\]

\[
x, 0 + y, 0 = 1,0
\]

Proof. This follows from applications of (GAUSS) and lemma 61.
Proof.

\[
\begin{align*}
&\quad x, 0 \quad 1.0 \quad \text{(Spider)} \\
&\quad y, 0 \quad 1.0 \\
&\quad \quad \quad = \\
&\quad \quad \quad \text{(Lem 42)} \\
&\quad \quad \quad = \\
&\quad \quad \quad \text{(Lem 42)} \\
&\quad \quad \quad \text{⊗}^2 \\
&\quad \quad \quad = \\
&\quad \quad \quad \text{(Spider)} \\
&\quad \quad \quad \text{(Lem 53)} \\
&\quad \quad \quad = \\
&\quad \quad \quad \text{(Lem 46)} \\
&\quad \quad \quad \text{⊗}^2 \\
&\quad \quad \quad = \\
&\quad \quad \quad \text{(Lem 42)} \\
&\quad \quad \quad \text{⊗}^2 \\
&\quad \quad \quad = \\
&\quad \quad \quad \text{(Lem 42)} \\
\end{align*}
\]

where we have freely used lemma 51 to commute antipodes and spiders throughout.

Lemma 70. The Euler decomposition of the Hadamards can be “improved” to:

\[
\begin{array}{ccc}
\text{0, 0} & \text{0, 1} & \text{1, 0} \\
\text{0, 0} & \text{0, 1} & \text{1, 0} \\
\end{array}
\]

Proof. This follows from applying lemma 60 to the decomposition of lemma 41, then using lemma 61 to simplify the scalar.

Lemma 71. Hadamard loops correspond to pure-Clifford operations: for any \(x \in \mathbb{Z}_p\) and \(z \in \mathbb{Z}_p^*\),

\[
\begin{array}{ccc}
\text{2} & \text{0, 2x} & \text{0, 0} \\
\text{z - 1} & \text{0, 0} & \text{0, 2x} \\
\end{array}
\]

Proof. The case \(x = 0\) is clear by decomposing the H-box. We begin by proving the case \(x = 1\):

\[
\begin{array}{ccc}
\text{0, 1} & \text{0, 1} & \text{0, 1} \\
\text{0, 1} & \text{0, 1} & \text{0, 1} \\
\text{0, 1} & \text{0, 1} & \text{0, 1} \\
\text{0, 1} & \text{0, 1} & \text{0, 1} \\
\text{0, 1} & \text{0, 1} & \text{0, 1} \\
\text{0, 1} & \text{0, 1} & \text{0, 1} \\
\end{array}
\]

The general case can be obtained by decomposing the weighted H-box into \(x\) H-loops using the sum rule from proposition 8. Then, under the assumption that the weight is invertible, the red version once again follows using (COLOUR) and the equations of proposition 8:

\[
\begin{array}{ccc}
\text{0, 0} & \text{0, 0} & \text{0, 0} \\
\text{0, 0} & \text{0, 0} & \text{0, 0} \\
\text{0, 0} & \text{0, 0} & \text{0, 0} \\
\text{0, 0} & \text{0, 0} & \text{0, 0} \\
\text{0, 0} & \text{0, 0} & \text{0, 0} \\
\text{0, 0} & \text{0, 0} & \text{0, 0} \\
\end{array}
\]

\]
Proof of proposition 10.

\begin{align*}
G & = (\text{Spider}) \\
G^{-1} & = (\text{Lem 46}) \\
(\text{Lem 53}) & = (\text{Spider}) \\
(\text{Lem 48}) & = (\text{Lem 46})
\end{align*}

Proof of proposition 11. This follows straightforwardly using proposition 7.

Lemma 72. For any $x, y \in \mathbb{Z}_p^*$,

\begin{align*}
0, x^2 & = 0, -1 \\
0, y^2 & = 0, 1
\end{align*}

Proof.

\begin{align*}
\text{(Lem 58)} & = \text{(Colour)} \\
\text{(Lem 60)-(b)} & = \text{(Bigebra)} \\
\text{(Lem 41)} & = \text{(Prop 8)} \\
\text{(Eq (33))} & = \text{(Bigebra)}
\end{align*}

and we can eliminate the $0, 1$ scalar using lemma 61.
Lemma 73. Let $\Sigma$ be a $\mathbb{Z}_p$-weighted star graph on $N \in \mathbb{N}$ vertices, i.e. it is a tree with $N - 1$ leaves. Order the vertices such that the first vertex is the only internal vertex, and it follows that all of the edges have weights $\Sigma_{1w}$ for $w$ ranging from 2 to $N$. Then,

$$0, \gamma \Sigma_1^2 = \ldots$$

where $\Sigma_{1w}$ is the graph obtained by adding an edge weighted in $\gamma \Sigma_{1m} \Sigma_{1n}$ between the wires connected to each pair of vertices $m, n \neq 1$.

Proof. The proof is by induction on the size $N$ of the tree. Assume the lemma is true for any tree of size $N - 1$. First, bend all the wires with green phases into inputs:

Then,

Then, we recognise the subtree with head 1 and leaves 3 to $N$, which is thus a tree of
size \( N - 1 \), to which we can apply the inductive hypothesis:

In the last step, we have pulled out the “head” of the \( N - 1 \) tree before applying lemma 49, since the \( \Sigma_{1w} \)-weighted Hadamards on each edge to the head are left unchanged by the local complementation.

Then, copy each \( \Sigma_{1w} \)-weighted Hadamard through the corresponding red spider on its right (which changes its colour and multiplies the weight by \(-1\)), and copy the multiplier and antipode through the green spiders below and to the left respectively. Fusing the resulting green spiders we get:

Bending the inputs back to outputs completes the proof.

Proof of proposition 12. The proof of this proposition is rather cumbersome to write in standard ZX\textsuperscript{Stab}. We give a sketch of the proof, and leave writing it out formally to future work, since it has a much clearer form in the scalable ZX\textsuperscript{Stab}-calculus.

The idea is to split out the subtree of \( G \) with head \( w \) and leaves each of the neighbours of \( w \). Then, one applies lemma 73, and adds the resulting edges into \( G \) using the additivity of parallel H-boxes (proposition 8).

D.3 Main completeness proofs

Proof of proposition 18. The single-qupit Clifford group is generated by the invertible generators under sequential composition. It therefore suffices to show that the composition of either of the above diagrams with such a generator can be rewritten to the claimed form. Now, every rewrite rule, except (ZERO) can be interpreted as an equality
up to an invertible scalar, if we simply ignore the parts of the equations disconnected from both the inputs and the outputs. Thus, we can freely use any of these rules in our proof of normalisation up to invertible scalars, and any equation derivable without (Zero).

(Multipliers) For the multipliers this is very straightforward: for any \( x \in \mathbb{Z}_d^* \),

\[
\begin{array}{ccc}
& x & \\
\downarrow & \downarrow & \\
& u, v & \\
\end{array}
= \begin{array}{ccc}
& x^{-1} & \\
\downarrow & \downarrow & \\
& u, v & \\
\end{array}
\]

(Hadamard) We can ignore the weight since this can be extracted as a multiplier and then the previous proof applies. Then, for the first normal form,

we need to split subcases based on the value of \( v \). If \( v = 0 \), then we have

\[
\begin{array}{ccc}
& x & \\
\downarrow & \downarrow & \\
& u, 0 & \\
\end{array}
= \begin{array}{ccc}
& x^{-1} & \\
\downarrow & \downarrow & \\
& u, 0 & \\
\end{array}
\]

\[
\approx \begin{array}{ccc}
& x^{-1} & \\
\downarrow & \downarrow & \\
& 0, t & \\
\end{array}
\]

\[
= \begin{array}{ccc}
& x^{-1} & \\
\downarrow & \downarrow & \\
& s + tu, 0 & \\
\end{array}
\]

\[
= \begin{array}{ccc}
& x^{-1} & \\
\downarrow & \downarrow & \\
& 0, t & \\
\end{array}
\]

\[
\approx \begin{array}{ccc}
& x^{-1} & \\
\downarrow & \downarrow & \\
& 0, 1 & \\
\end{array}
\]

\[
= \begin{array}{ccc}
& x^{-1} & \\
\downarrow & \downarrow & \\
& 0, 1 & \\
\end{array}
\]

where in the last step we have used (Shear) and (Spider) several times to commute the Paulis on the right back into the leftmost red and green spiders.
If $v \neq 0$, then,

The second normal form can be done in one go.

(Rotations) The red phase is once again very straightforward: let $x, y \in \mathbb{Z}_d$, then,

The green phases are more involved. Firstly, note that we can ignore the multiplier,
since we have:

\[ \begin{align*}
&\quad u,v \\
&= s,t \\
&\quad w \\
&\quad x,y
\end{align*} \]

and \( x, y \) are arbitrary. Then, the Pauli part can be straightforwardly taken care of using (Shear) to commute it through to the green spider on the right. Finally, we can view \( - \) as the \( y \)-fold composition of \( - \), so that we have reduced the proof to the normalisation of:

\[ \begin{align*}
&\quad 0,1 \quad u,v \\
&\quad \quad s,t
\end{align*} \]

On the right (in dashed lines) we recognise the composition of a Hadamard and a normal form, which we have already shown can be normalised. This done, we obtain the composition of a red spider and a normal form, which we have also already shown to be normalisable. Thus, we are done.

As for the second normal form, we have:

\[ \begin{align*}
&\quad 0,1 \quad s,1 \\
&= 0,1 \quad u,v \quad 0,1 \\
&\quad \quad s,1
\end{align*} \]

and we have again reduced to previously solved cases.

Unicity follows from the fact that none of these forms are equivalent, and there are therefore \( p^3(p^2 - 1) \) distinct forms, which matches the cardinality of the single qupit Clifford group.

Proof of proposition 20. Assume w.l.o.g. that the diagram \( D \) has type \( 0 \rightarrow n \) (i.e. we consider that the diagram has already been turned into a state via equation (60)), and furthermore that it contains no multipliers (equivalently, they have been unpacked into the sugarless calculus using equation (29)). Then, transform \( D \) as follows:

1. Use (Colour) to change every red spider into a green one, surrounded by Hadamards. The resulting diagram consists of only green spiders and Hadamards.

2. Use (G-ELIM) to add a green spider between any two subsequent Hadamards. The diagram now consists of only green spiders, connected either by plain edges or H-edges.

3. Use (SPIDER) to fuse any two spiders connected by plain edges, to eliminate any
loops, and use lemma 71 to eliminate any H-edge loops. The resulting diagram contains no loops, and furthermore spiders are connected only by H-edges.

4. Use the following rule from proposition 8:

\[ x \otimes y = x + y \]  \hspace{1cm} (133)

to fuse all H-edges between two spiders into a single weighted H-edge or no edge.

5. Use (G-Elim) and proposition 8 to obtain

\[ \ldots \otimes \ldots = \ldots \otimes \ldots \]  \hspace{1cm} (134)

which allows one to split any spider connected to more than one output into several green spiders, each connected to exactly one output.

The resulting diagram is nearly in GS+LC form, except it contains “internal” vertices which are not connected to an output. Diagrams in this form have been called graph-like in the literature. However, we can eliminate these internal vertices as follows. Let \( u \) be such a vertex, and let \( (a, b) \) be its phase. If \( u \) has no neighbours, then it is an elementary scalar diagram which we can ignore.

Otherwise, assume \( u \) has a neighbour \( v \) which is also connected to an output of the diagram. Then, we can eliminate the phase at \( u \) through the following manipulation. Throughout, we use local dilations (proposition 11) and local complementations (corollary 13) on the rest of the graph freely. We can ignore the green spiders introduced by corollary 13 since they are absorbed into the phases of the graph vertices, and track only the red spiders introduced which are the only “problematic” spiders that take us away from GS+LC form. Then, we can eliminate vertex \( u \) as follows:

where in the second step we have used the Euler decomposition of the Hadamard (lemma 41). The resulting green units are connected via a normal wire to a single other green spider, to which they can be fused using (SPIDER).

We still need to treat the case where \( u \) has only neighbours which are not connected to an output. In that case, letting \( v \) be any neighbour of \( u \), we can still eliminate \( u \) using a very similar argument and relying on the normal form for single-qubit operators. The
first step is analogous to the previous case:

\[
\begin{array}{c}
\text{Normalising the } C_1 \text{-diagram on } v \text{ using proposition 18, we obtain one of two forms which are essentially equivalent since we have}
\end{array}
\]

As a result, by the end of this procedure, we have eliminated \( u \) without reintroducing any new internal vertices, so we can repeat this process to eliminate each internal vertex in the diagram. The resulting diagram will then be in GS+LC form.

Proof of proposition 22. By proposition 20, every stabiliser diagram is equivalent to some GS+LC diagram, and furthermore, the vertex operators can be brought into the normal form of proposition 18. Using proposition 11, the multiplier part of this normal form can immediately be absorbed into a local scaling of the graph.

We now need to eliminate the leftmost red spider of the normal form. Consider the vertex operator acting on vertex \( u \) of the graph. If \( u \) has no neighbours, we can use lemma 46 to eliminate the red spider (up to a scalar in \( G \)). Otherwise, \( u \) has at least one neighbour. In this case, we can use lemma 13 to “copy” the red spider into a green spider on each neighbour of \( u \).

As in the qubit case, the set \( R \) is not stable under pre-composition by a green spider. However, by the proof of proposition 18, we see that, after normalisation, this operation applied to any element of \( R \) does not increase the number of red spiders. Thus, applying this procedure at most \( 2|V| \) times maps every vertex operator to one of the forms in \( R \).

In order to prove the second part of the simplification, assume that the preceding procedure has been applied to the diagram, so that all vertex operators are in \( R \). Let \( u, v \) be neighbouring vertices such that the vertex operators of both \( u \) and \( v \) contain a red spider. We are going to use a sequence of local complementations at \( u \) and \( v \) to eliminate the red spiders on both \( u, v \). Since we have just shown that the green spider these operations enact on neighbours can be corrected and the vertex operators brought back to \( R \), and that local complementations and scalings preserve the GS+LC form, we
ignore this action and consider only the part of the diagram connected to \( u \) and \( v \):

Then, normalising the vertex operators, we obtain

The red spiders can be eliminated using the previous strategy for mapping the vertex operators to \( R \), and so we have removed the pair of neighbouring red spiders. Repeating this process for each such pair renders a diagram in rGS+LC form.

**Proof of proposition 24.** Suppose this is not the case, i.e. there are vertices \( u, v \) such that \( u \) has a red vertex operator in \( A \) but not \( B \), \( v \) has a red vertex operator in \( B \) but not \( A \), and \( u, v \) are neighbours in \( A \). Then we perform a manipulation entirely analogous to the proof of proposition 22 in the diagram \( A \):

Then, after normalisation (proposition 18) and using proposition 22 to bring the diagram back to rGS+LC form, the vertex operator for \( u \) no longer contains a red vertex in \( A \) and the vertex operator for \( v \) contains a red vertex. Since we can do this for any pair of such red vertices in \( A \) and \( B \), it is clear that we can can simplify the pair of diagrams.

**Proof of theorem 28.** Let \( A \in ZX^\text{Stab}_p[0,0] \), then \( A \) consists of a tensor product of connected scalar diagrams. We first show that \( A \) can be rewritten to a tensor product of elementary scalars. If this is not already the case, pick some connected scalar diagram \( A' \) contained in \( A \). This diagram \( A' \) contains at least one red or green spider, from which we can extract a unit:

\[
A' = B \otimes \text{or} \quad A' = B \oplus
\]

where \( B \in ZX^\text{Stab}_p[0,1] \). By theorem 26 and the normal form for the single-qupit Clifford
group (proposition 18), we can rewrite $B$ to one of the forms

$$
\begin{align*}
\begin{array}{ccc}
B & \simeq & B \\
& = & = \\
\end{array}
\end{align*}
$$

(136)

since we know that these forms cover all single qupit Clifford states. Simplifying a bit, we therefore only need to consider the following forms:

$$
\begin{align*}
\begin{array}{ccc}
\bigcirc \bigcirc & \bigcirc \bigcirc \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{array}
\end{align*}
$$

(137)

The first of these is already an elementary scalar, and if $s$ or $t$ is 0, the second is also. If $s = 0$, the third diagram is also elementary. Thus, we can assume that $t \neq 0 \neq s$ and treat both middle diagrams in one go:

$$
\begin{align*}
\begin{array}{ccc}
\bigcirc \bigcirc & \bigcirc \bigcirc \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{array}
\end{align*}
$$

(138)

and both of these diagrams can be normalised using (M-Elim) and (Shear). Finally, the last scalar from equation (137) can be rewritten to:

$$
\begin{align*}
\begin{array}{ccc}
\bigcirc \bigcirc & \bigcirc \bigcirc \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{array}
\end{align*}
$$

(139)

We have shown that $A$ can be rewritten to a tensor product of elementary scalars. If this tensor product contains a zero diagram, then the whole diagram equal to the zero diagram by proposition 27. Otherwise, $A$ can further be rewritten to a scalar in normal form using the multiplication rules for elementary scalars, which are given by (M-One), lemmas 69 and 68, up to applying (Gauss) to decompose quadratic scalars.

Proof of theorem 30. First, $D$ can be put in GS-LC form. This allows to decompose $D$ in the following form:

$$
\begin{align*}
\begin{array}{ccc}
D & = & X \\
& = & Y \\
\end{array}
\end{align*}
$$

Where $X$ and $Y$ are compositions of E-gates constituting the edges within the same partition together with local Clifford gates, and $G$ is a bipartite graph-state gathering all vertices having a neighbor in a different partition and all corresponding edges. Thus,
we see that without loss of generality we can restrict to the case where $D$ is a bipartite graph-state.

Assuming that $D$ is a bipartite graph-state with partitions of size $n$ and $m$ with $n \leq m$ we show how to find $A$ and $B$ such that:

Let’s start with $n$ Bell’s pair. Plugging Generalized Hadamard boxes we get the following graph state:

Now, let’s say we want to add an edge with weight $w$ between the vertices $x$ and $y$ of this graph, to do so we first add an edge of weight 1 between $x'$ and $y$ and apply the right local complementation to $x'$ to obtain the desired edge. Finally we clean up all the unwanted edges inside the partition by using the convenient $E$-gates. The process is sketched in the following diagrams:

At the end, when all desired edges have been obtained, we still have to take care of the edges between $x$ and $x'$, but since we only kept vertices which are linked to the other partition in the final $D$, we know there is a neighbor $t$ that can be used to take care of this edge in the same way we used $x'$ before, first we add an edge between $t$ and $x'$, then we apply the needed local complementation on $t$ and finally we clean up the edges internal to the partitions.

This give up the wanted bipartite graph-state $D$ which conclude the proof.

Proof of 25. Let’s assume that there is a marked vertices $v$ in $C$ which is not marked in $D$. If $v$ is isolated in $C$ and not in $D$ then the two interpretations are obviously different. The same goes if $v$ is isolated in both diagram since then we can clearly differentiate the two corresponding states. So the only case remaining is when $v$ has neighbors in both diagrams.

We then apply to both diagram the same unitary, first a green Clifford map on the chosen vertex and appropriate gates for each edges in the neighborhood of $v$ in $C$. In $C$ we have:
And in $D$:

Here we compute up to local Cliffords on the wires. Thus, we see that in $C$, $v$ is now disconnected while it isn’t in $D$. Then $[D] \neq [C]$.

Proof of Theorem 26. We only show it for state by map-state duality.

First we rewrite $A$ and $B$ into a simplified pair of rGS-LC diagrams $A'$ and $B'$. So, the marked vertices are the same. It only remain to show that the graph state part with . But if we apply the gates corresponding to the edges in $A'$ to both diagrams, then we obtain a completely disconected graph in $A'$. And the two interpretation can only be equals if $B'$ is also now completely disconnected. It then follows that the edges in both diagrams were the same.

Finally, the only way for the interpretations to be the same is that all phases in the green vertices are equals, and then $A'$ can be rewritten in $B'$. So $\text{Stab}_p \vdash A = B$. 

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