finite unitary ring with minimal non-nilpotent group of units

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Abstract. Let $R$ be a finite unitary ring such that $R = R_0[R^*]$ where $R_0$ is the prime ring and $R^*$ is not a nilpotent group. We show that if all proper subgroups of $R^*$ are nilpotent groups, then the cardinal of $R$ is a power of prime number 2. In addition, if $(R/Jac(R))^*$ is not a $p$–group, then either $R \cong M_2(GF(2))$ or $R \cong M_2(GF(2)) \oplus A$ where $M_2(GF(2))$ is the ring of $2 \times 2$ matrices over the finite field $GF(2)$ and $A$ is a direct sum of finite field $GF(2)$.

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1 Introduction

The relations between rings and their groups of units is an interesting research subject. In [4], Groza has shown that if $R$ is a finite ring and at most one simple component of the semi-simple ring $R/Jac(R)$ is a field of order 2, then the group of units $R^*$ is a nilpotent group if and only if $R$ is a direct sum of two-sided ideals that are homomorphic images of group algebras of the form $SP$, where $S$ is a particular commutative finite ring, $P$ is

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a finite $p$-group, and $p$ is a prime number. More recently, Dolzan improves some results of Groza and describes the structure of an arbitrary finite ring with a nilpotent group of units (see [1]).

Let $X$ be a class of groups. We say that a group $G$ is a minimal non-$X$-group, if $G \notin X$, but all proper subgroups of $G$ belong to $X$. Minimal non-$X$-groups have been studied for various classes of groups $X$. For example, minimal non-abelian groups were studied by Miller and Moreno [5], while Schmidt [7], studied minimal non-nilpotent groups and he characterized such finite groups. The natural question is that what we can say about finite ring such that the group of units of ring is minimal non-$X$-group. In this paper, we study a finite ring $R$ with minimal non-nilpotent groups of the units and we prove that $|R| = 2^n$ for some positive integer $n$. More precisely, we prove the following theorem:

### Theorem 1

Let $R$ be a unitary ring of finite cardinality $2^m n$ and $m$ is an odd number such that $R^*$ is a minimal non-nilpotent group. Then $|R| = 2^n$. Also, if $(R/Jac(R))^*$ is not a $p-$group, then either $R \cong M_2(GF(2))$ or $R \cong M_2(GF(2)) \oplus A$ where $A$ is a direct sum of finite field of order two.

In this paper, $R$ denotes a ring with identity $1 \neq 0$, the Jacobson radical of $R$ by $Jac(R)$ and for an arbitrary finite set $X$, let $|X|$ denote the number of elements in $X$. We denote the group of units of $R$ by $R^*$, the order of element $x$ in $R^*$ by $o(x)$, and the group generated by $x$ by $\langle x \rangle$. The ring of $n \times n$ matrices over a ring $R$ is shown by $M_n(R)$, and the center of $R$, is the set of elements that commute with every element of $R$. The centralizer of the subset $X$ of $R$ is the set of all elements of $R$ which commute with every element of $X$ is denoted by $C_R(X)$. Also, for any pair $a, b \in R$, $[a, b] = ab - ba$ is the Lie product of $a$ and $b$ and $R_0[S]$ denotes the subring of $R$ which is generated over $R_0$, by $S \subseteq R$, where $R_0$ is the prime subring of $R$. The characteristic of $R$ is denoted by $CharR$ and $GF(p^m)$ is a finite field of order $p^m$ where $p$ is a prime number.
2 Results

We begin with the following useful lemma:

**Lemma 2.** Let $R$ be a unitary finite local ring with a nontrivial minimal ideal $I$, also let $\text{Jac}(R)$, the jacobson radical of $R$ be a commutative ideal, then we have $\text{ann}_R(I) \subseteq \text{Jac}(R)$.

**Proof.** By Theorem 2.4 from [3], there is an integer $m$ such that $\text{Jac}(R)^m = 0$. So, there exists a positive integer $n$, such that $I^n = 0$ and $I^{n-1} \neq 0$ where $2 \leq n \leq m$. Since $2n - 2 \geq n$, we see that $I^2 = (I^{n-1})^2 = 0$. Therefore $n = 2$. Let $u \in I$ and $h \in \text{Jac}(R)$. If $hu \neq 0$, then $RhuR = I = RuR$. Therefore $\sum_{\text{finite}} rhus = u$ where $r, s \in R$. Since $\text{Jac}(R)$ is a commutative ideal, we have

$$u = \sum_{\text{finite}} (rh)(us) = \sum_{\text{finite}} (us)(rh) = \sum_{\text{finite}} u(srh) = \sum_{\text{finite}} srhu.$$

Hence $((\sum_{\text{finite}} srh) - 1)u = 0$. Since $\sum_{\text{finite}} srh - 1 \in R^*$, we deduce that $u = 0$, which is a contradiction. So $hu = 0$ for all $h \in \text{Jac}(R)$, and hence $\text{Jac}(R) \subseteq \text{ann}_R(I)$. \(\square\)

**Remark 1.** Let $R = A \oplus B$ be a finite ring where $A$ and $B$ are two ideals in $R$. Then $R^* = A^* \oplus B^*$ and $1 = 1_A + 1_B$ where $1_A$ and $1_B$ are identity elements of $A$ and $B$ respectively. It is clear that $A^* + 1_B \leq R^*$ and $A^* + 1_B \cong A^*$.

**Lemma 3.** Let $R$ be a finite ring. If $|R|$ is an odd number, then $R = R_0[R^*]$.

**Proof.** By Lemma 1.1 from [4], the proof is clear. \(\square\)

Minimal non-nilpotent group are characterized by Schmidt as follow:
Theorem 4. [see (9.1.1) of [6]] Assume that every maximal subgroup of a finite group $G$ is nilpotent but $G$ itself is not nilpotent. Then:

(i) $G$ is soluble;

(ii) $|G| = p^m q^n$ where $p$ and $q$ are unequal prime numbers;

(iii) there is a unique Sylow $p$-subgroup $P$ and a Sylow $q$-subgroup $Q$ is cyclic. Hence $G = QP$ and $P \trianglelefteq G$.

Let $R$ be a finite ring such that $|R|$ is an odd number. In the following theorem, we show that $R^*$ is not a minimal non-abelian group.

Theorem 5. Let $R$ be a finite ring of order $m$, where $m$ is an odd number. If every proper subgroup of $R^*$ is an abelian group, then $R$ is a commutative ring.

Proof. Consider the finite ring $R$ which is minimal subject to this condition such that $R$ is not a commutative ring. Since every maximal subgroup of $R^*$ is abelian, we have $R^*$ is a minimal non-abelian group. By Lemma of [2], every ring with identity element of order $p$ or $p^2$ for prime number $p$, is a commutative ring. So we may assume that $|R| \notin \{p, p^2\}$. Let $S$ be a proper subring of $R$. It follows from Lemma [3] that $R = R_0[R^*]$, and hence $S^* \neq R^*$. By assumption $S^*$ is an abelian group and by Lemma [3] $S = S_0[S^*]$, is a commutative ring. So every proper subring of $R$ is a commutative ring. Let $|R| = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$ be the canonical decomposition of $|R|$ to the prime numbers $p_i$. Then we know that

$$R = R_1 \bigoplus R_2 \bigoplus \ldots \bigoplus R_k$$

where each ideal $R_i$ is of order $p_i^{\alpha_i}$. Let $H_i$ be a subgroup of $R^*$ such that $H_i \cong R_i^*$ for all $i$. If $k > 1$, then $H_i$ is an abelian subgroup of $R^*$. By minimality of $R$, we have $R_i$ is a commutative ring for all $i$ and then $R$ is a commutative ring, which is a contradiction. So suppose that $|R| = p^\beta$ where $p > 2$ is a prime number. We have two cases in respect of radical jacobson, either $Jac(R) = 0$ or $Jac(R) \neq 0$:

Case 1. Let $Jac(R) = 0$. By Wedderburn Structure Theorem $R \cong \bigoplus_{i=1}^t M_{n_i}(D_i)$ where $D_i$ is a finite field. If $t > 1$, then by minimality of $R$ and Remark [1] every $M_{n_i}(D_i)$
is commutative, and so $R$ is commutative, which is a contradiction. It follows that $t = 1$, and so $R \cong M_n(D)$, where $D$ is a finite field and $n$ is a positive integer. Since $R$ is not a commutative ring, we have $n > 1$. It follows from $(M_n(D))^* \cong R^*$ that $R^*$ is not a nilpotent group. Therefore $R^*$ is a minimal non-nilpotent group. But Sylow 2-subgroup of $R^*$ is neither cyclic nor normal, which is a contradiction by Theorem 3(iii).

Case 2. Let $Jac(R) \neq 0$. First suppose that $R^*$ is a nilpotent group. Since $-1 \in R^*$ and $o(-1) = 2$, we have $2 \mid |R^*|$. Since $Jac(R) \neq 0$, by Lemma 1.2 of [4], $1 + Jac(R)$ is a $p$–group. Let $P \in Syl_p(R^*)$ and $K$ be a subgroup of $R^*$ such that $R^* = PK$ and $P \cap K = 1$. By assumption $P$ and $K$ are abelian groups, and hence $R^*$ is an abelian group. By Lemma 3, $R = R_0[R^*]$ is commutative, which is a contradiction. Therefore $R^*$ is not a nilpotent group, and then $R^*$ is a minimal non-nilpotent group. By Theorem 4(i) $|R^*| = r^mq^n$ where $r, q$ are prime numbers. Since $2 \mid |R^*|$ and by Lemma (1.2) of [4], $1 + Jac(R)$ is a $p$–group, so we may assume that $r = p$ and $q = 2$. Since $(R/Jac(R))^* = R^* + Jac(R)/Jac(R)$, every proper subgroup of $(R/Jac(R))^*$ is an abelian group. By minimality of $R$, we have $R/Jac(R)$ is a commutative ring. Then $[R, R] \subseteq Jac(R)$. Let $P \in Syl_p(R^*)$ and $Q \in Syl_2(R^*)$. By Theorem 4(iii), either $P \triangleleft R^*$ or $Q \triangleleft R^*$. We claim that $P \triangleleft R^*$. Otherwise, by Theorem 4(iii), $P = \langle z \rangle$ is a cyclic subgroup of $R^*$ where $z \in P$. Since $1 + Jac(R) \subseteq P$, there is a positive integer $i$ such that $H = 1 + Jac(R) = \langle z^i \rangle$. Since $H \triangleleft R^*$ and $R^*$ is a non-nilpotent group, $H \neq P$. Since $HQ$ is an abelian subgroup of $R^*$ and $R = \langle z, Q \rangle$, we have $H \leq Z(R^*)$, and so $H \leq Z(R)$. Therefore, $Jac(R) \subseteq Z(R)$. Since $[R, R] \subseteq Jac(R)$, we have $uv - vu \in Jac(R)$ for all $u, v \in R^*$. It follows that $uvu^{-1}v^{-1} - 1 \in Jac(R) \subseteq Z(R)$, and so $uvu^{-1}v^{-1} \in Z(R)$ for all $u, v \in R^*$. Hence $R^*$ is a nilpotent group, which is a contradiction. Therefore $P \triangleleft R^*$, as claimed. By Theorem 4(iii), $Q = \langle x \rangle$ for some $x \in Q$. We claim that $R$ is a local ring. Otherwise, let $\{M_1, ..., M_k\}$ be the set of all maximal ideals of $R$, where $k > 1$. Since $R/Jac(R) = R/(M_1 \cap ... \cap M_k) \cong R/M_1 \times ... \times R/M_k$, we have $(R/Jac(R))^* \cong (R/M_1)^* \times ... \times (R/M_k)^*$. Let $\overline{Q} \in Syl_2((R/M_1)^* \times ... \times R/M_k)^*).$ Since $2 \mid |(R/M_1)^*|$ for all $1 \leq i \leq k$, we see that $\overline{Q}$ is not a cyclic group. But $(R/Jac(R))^* = R^* + Jac(R)/Jac(R) = PQ + Jac(R)/Jac(R)$, and then $Q$ is not a cyclic group, which is a contradiction. Hence $k = 1$, as claimed. Let $M = Jac(R)$. Since $1 + M$ is an abelian group, $M$ is a commutative ideal. Since $R/M$ is a finite field, $M$ is not a central ideal. So
there exits \( w \in M \) such that \( wx \neq xw \). By minimality of \( R \), we have \( R = R_0[w, x] \). Let \( I \) be a minimal ideal of \( R \). We follow the proof by separating it in two subcases, either \( Z(R) \cap I \neq 0 \) or \( Z(R) \cap I = 0 \):

**Subcase 1.** Let \( 0 \neq a \in Z(R) \cap I \). By Lemma 2, \( M \subseteq \text{ann}_R(I) = \{r \in R : rs = 0 \text{ for all } s \in I \} \). Since \( R/M \) is a finite field, we have \( M = \text{ann}_R(I) \). It follows from \( a \in Z(R) \) that \( I = Ra \) is two sided ideal. Since \( R/M \) is a finite field, we have \((R/M)^* = \langle x + M \rangle \) for some \( x \in R \) with \( \gcd(o(x + M), p) = 1 \). Let \( y \in R \setminus M \). Then \( y + M = x^i + M \) for some integer \( 0 \leq i \leq n - 1 \). Therefore \( y = x^i + s \) for some \( s \in M \). It follows that \( ya = x^ia + sa = x^ia \), and so \( I = \{0, xa, ..., x^n a\} \subseteq M \). Since \( xx^ia = x^iax, w(x^ia) = (x^ia)w \) and \( R = R_0[x, w] \), we have \( x^ia \in Z(R) \), and so \( I \subseteq Z(R) \).

By minimality of \( R \), we have \( R/I \) is a commutative ring, so \([R, R] \subseteq I \subseteq Z(R) \). Then \( uvu^{-1}v^{-1} - 1 \in I \subseteq Z(R) \) for all \( u, v \in R^* \). It follows that \( uvu^{-1}v^{-1} \in Z(R^*) \), and hence \( R^* \) is a nilpotent group, which is a contradiction.

**Subcase 2.** Let \( Z(R) \cap I = 0 \) and \( 0 \neq b \in I \). Since \( R = R_0[w, x] \) and \( M \) is commutative, we have \( bw = wb \), and so \( [b, x] \neq 0 \). Therefore \( R = R_0[b, x] \). We may assume that \( b = w \in I \). Let \( m_1, m_2 \in M \). Since \( M \) is a commutative ring and \( x^m_1, m_2x \in M \), we have

\[
(xm_1)m_2 = m_2(xm_1) = (m_2x)m_1 = m_1m_2x.
\]

Since \( R = R_0[x, w] \) and \( wm_1m_2 = m_1m_2w \), we conclude that \( M^2 \subseteq Z(R) \). If \( M^2 \neq 0 \), then by minimality of \( R \), \( R/M^2 \) is a commutative ring, so \( 0 \neq [R, R] \subseteq M^2 \cap I \). Since \( I \) is a minimal ideal and \( M^2 \) is an ideal, \( I \subseteq M^2 \subseteq Z(R) \), which is a contradiction. Hence \( M^2 = 0 \), and so by considering \( R \) as a local ring, for all \( s \in M \setminus \{0\} \), we have \( M \subseteq \text{ann}_R(s) \). We claim that \( I = M \), otherwise consider \( l \in M \setminus I \). Since \( R = R_0[x, w] \), we have \( l = \sum n_i x^i + c \) where \( c \in I \) and \( n_i \in R_0 \). Therefore \( l - c = \sum n_i x^i \in M \). Then \( \sum n_i x^i \in Z(R) \). If \( l - c \neq 0 \), then by minimality of \( R \), we have \( R/R(l-c) \) is a commutative ring. By similar argument was given in subcase 1, we have \( R(l-c) \subseteq Z(R) \). It follows from \( 0 \neq [R, R] \subseteq R(l-c) \cap I \) and \( I \) is a minimal ideal that \( I \subseteq R(l-c) \subseteq Z(R) \), which is a contradiction. Then \( l - c = 0 \), which is a contradiction. Therefore \( M = I \). Since \( R/M \) is a finite field, we have \( R = R^* \cup M \). Then \( |R^*| = |R| - |M| = o(x) |P| \). Let \( |R/M| = p^m \). Then \( |R| = p^m |M| \), and so \( |R| - |M| = (p^m - 1)|M| \), consequently, \( 1 + M = P \). Since \( P \langle x^2 \rangle \) is an abelian subgroup of \( R^* \) and \( R = R_0[w, x] \), we conclude that \( x^2 \in Z(R) \). If
\[ p \mid o(x+h) \text{ for some } h \in P, \text{ then } e = (x+h) \frac{o(x+h)}{p} \in P = 1 + M, \text{ and so } e - 1 \in M. \]

Since \( R = R_0[x+h,w] \) and \( M \) is commutative, \( e - 1 \in Z(R) \). Therefore \( M \cap Z(R) \neq 0 \), which is a contradiction. It follows that \( \gcd(o(x+h), p) = 1 \) for all \( h \in M \). By similar argument as above, \( (x+h)^2 \in Z(R) \) for all \( h \in M \). Then \( x^2 + xh + hx + h^2 \in Z(R) \). Since \( h^2 = 0 \) and \( x^2 \in Z(R) \), we have \( xh + hx \in Z(R) \cap M = 0 \). Therefore \( xh = -hx \) for all \( h \in M \).

Let \( 0 \neq h \in M \). Since \( (R/M)^* = \langle x + M \rangle \), we deduce that \( x + 1 + M = x^t + M \) for some integer \( t \). Then \( q = x^t - x - 1 \in M \). Since \( M \) is a commutative ideal, \( qw = wq \). It follows from \( qx = xq \) and \( R = R_0[x, w] \) that \( q = x^t - x - 1 \in M \cap Z(R) = 0 \). So \( x + 1 = x^t \).

Since \( x + 1 = x^t \not\in Z(R) \), we have \( t \) is an odd number. Then \( x^t h = (-1)^t h x^t = -hx^t \).

Therefore \( (x + 1)h = h(x + 1) \). But \( (x + 1)h = xh + h = -hx + h = -h(x + 1) \). Hence \( 2h = 0 \), and so \( h = 0 \) for all \( h \in M \), which is final our contradiction.

\[ \square \]

**Theorem 6.** Let \( R \) be a finite ring of order \( p^n \) where \( p \) is an odd prime number. If every proper subgroup of \( R^* \) is nilpotent, then \( R \) is a commutative ring.

**Proof.** Consider the finite ring \( R \) which is minimal with respect to these conditions, but it is not commutative. Then \( R^* \) is a minimal non-nilpotent group. By similar argument as case 1 of the previous theorem, we may assume that \( Jac(R) \neq 0 \). By Theorem 4 \( |R^*| = r^m q^n \) where \( r, q \) are prime numbers. By Lemma 1.2 from [4], \( 1 + Jac(R) \) is a \( p \)-group and then \( r = p \). Since \( -1 \in R^* \) and \( o(-1) = 2 \), we have \( q = 2 \). Let \( P \in Syl_p(R^*) \) and \( Q \in Syl_2(R^*) \). Let \( I \) be a minimal ideal of \( R \) that is contained in \( Jac(R) \). Then \( I^2 = 0 \), and hence \( I \) is commutative. By minimality of \( R \), we have \( R/I \) is commutative, so \( [R, R] \subseteq I \). We have two cases in respect of \( I \cap Z(R) \), either \( I \cap Z(R) \neq 0 \) or \( I \cap Z(R) = 0 \).

**Case 1.** Suppose that \( I \cap Z(R) \neq 0 \). Let \( 0 \neq c \in I \cap Z(R) \). By similar argument as the subcase 1 of case 2, in the above theorem, we have \( a I \subseteq Z(R) \), and then \( 1 + I \leq Z(R^*) \).

Since \( uu^{-1}v^{-1} - 1 \in I \), we have \( (R^*)' \leq Z(R^*) \), and so \( R^* \) is a nilpotent group, which is a contradiction.

**Case 2.** Let \( I \cap Z(R) = 0 \). First suppose that \( 1 + I \neq P \). Since \( 1 + I \triangleleft P \), there exists \( c \in I \setminus \{0\} \) such that \( 1 + c \in Z(P) \). We have \( (1 + I)Q \) is a proper nilpotent subgroup of \( R^* \). Then \( 1 + I \leq C_{R^*}(Q) \), and so \( c \in Z(R) \), which is a contradiction. Therefore \( 1 + I = P \) is an abelian subgroup of \( R^* \). By Theorem 4 either \( P \) is cyclic or \( Q \) is cyclic.
Since $1 + I = P < R^*$, by Theorem [4] $Q$ is cyclic. Since $P$ is an abelian group and $Q$ is cyclic, every proper subgroup of $R^*$ is an abelian group, which is a contradiction by Theorem [5].

If $2 \mid |R|$, then the above theorem is no longer valid. For example, let $R$ be the set of all $2 \times 2$ matrices over the finite field $GF(2)$. Then $R^* \cong S_3$, where $S_3$ is the symmetric group of order 6 and clearly, $S_3$ is a minimal non-abelian group. For simplicity, let $\Delta$ be the set of all rings $R$ in which either $R \cong M_2(GF(2))$ or $R \cong M_2(GF(2)) \oplus A$ where $A$ is a direct sum of finite field of order two.

**Remark 2.** Let $Sl(2, GF(2^m))$ be the kernel of the homomorphism $det(M_n(GF(2^m))) \to GF(2^m)^*$. We recall that when $m > 1$, then $Sl(2, GF(2^m))^* = ((Sl(2, GF(2^m)))^*)'$, and hence for $n > 1$ and $m > 1$, we have $(M_n(D))^*$ is not a minimal non-nilpotent group.

**Theorem 7.** Let $R$ be a unitary ring of finite cardinality $2^\beta$ such every proper subgroup of $R^*$ is a nilpotent group.

(a) If $Jac(R) = 0$, then $R \in \Delta$.

(b) If $Jac(R) \neq 0$, then $(R/Jac(R))^*$ is a cyclic $p$–group for some odd prime number $p$.

**Proof.** (a) We proceed the proof by induction on $\beta$. Since $R$ is a simple artinian ring, by the structure theorem of Artin-Wedderburn, we have $R \cong \bigoplus_{i=1}^t M_{n_i}(D_i)$, where every $D_i$ is a finite field. If $t = 1$, then $R \cong M_{n_1}(D)$, and clearly, $n_1 = 2$ and $D \cong GF(2)$, so $R \in \Delta$, and we are done. Let $t > 1$. First suppose that $n_i > 1$ for some $1 \leq i \leq t$. By Remark [2] if $M_{n_i}(D_i)$ is a minimal non-abelian group, consequently, $n_i = 2$ and $D_i = GF(2)$. If for some $j \neq i$, we have $n_j > 1$, then $R^*$ is not a minimal non-abelian, which is a contradiction. Therefore $M_{n_j}(D_j) \cong D_j$ for all $j \neq i$. Let $H \leq R^*$ such that $H \cong (M_{n_i}(GF(2)))^*$. If $|D_j^*| > 1$ for some $j \neq i$, then $R^* \neq H$, and so $H$ is a non-nilpotent proper subgroup of $R^*$, which is a contradiction. Consequently, $D_j \cong GF(2)$, and hence $R \in \Delta$.

(b) Suppose for a contradiction that $(R/Jac(R))^*$ is not a $p$–group. Let $\beta$ be the smallest positive integer number such that $R^*$ is a minimal non-nilpotent group and
(R/Jac(R))^* is a p-group. By Theorem 4, we have R^* = PQ where P \triangleleft G and Q is a cyclic Sylow subgroup. Let I \subseteq Jac(R) be a minimal ideal of R. It is easy to see that char(I) = 2 and I^2 = 0. Therefore 1 + I is an elementary abelian 2-group. Since (R/I)/(Jac(R/I)) \cong R/Jac(R), we have ((R/I)/(Jac(R/I)))^* is not a p-group, so by minimality of R, we have (R/I)^* is a nilpotent group. Let p > 2 be the prime number such that p \div |R^*|. Clearly, 2p \div |(R/Jac(R))^*|. Let \{M_1, ..., M_k\} be the set of all maximal ideals of R. We have

\[
R/Jac(R) \cong R/M_1 \oplus \ldots \oplus R/M_k.
\]

Since R/M_i is a simple ring, we have R/M_i \cong M_i(GF(2^{m_i})) for some positive integers n_i and m_i. If n_i > 1 for some i, then R/M_i \cong M_i(GF(2^{m_i})). Let x, y be two arbitrary elements of R^* such that xy \neq yx and gcd(o(x), o(y)) = 1. Since (R/I)^* is a nilpotent group, we have xy - yx \in I \subseteq Jac(R). Hence for all i, we have n_i = 1, and so R/M_i is a finite field. But gcd(|(R/M_i)^*|, 2) = gcd(2^{m_i} - 1, 2) = 1 for all i, so 2 \nmid |(R/Jac(R))^*|, which is a contradiction.

\[\square\]

Here we give an example for the statement of Theorem 7 part (b). Let GF(2)[x, y] be the free ring generated with two elements x and y over finite field GF(2). Let H be the ideal generated by \{x^2, y^3 + y + 1, xy - y^2x\}. Let R = GF(2)[x, y]/H, and let I be the ideal generated with x + H in R. Since xy - yx \notin H, we have R is a non-commutative ring. Let L be the ideal generated by t^3 + t + 1 in Z_2[t]. Since R/I \cong Z_2[t]/L and L is a maximal ideal, we have R/I is a finite field of order 8. Let (R/I)^* = \langle u + I\rangle. It is easy to check that I = \{0, x + H, ux + H, ..., u^7x + H\}. Clearly, 1 + I is an elementary abelian 2-group and R^* = (1 + I)(u) is a minimal non-nilpotent group.

Now we are ready to prove Theorem 1.

**Proof.** By Theorem 3 a \geq 1. Let |R| = 2^{a_1}p_2^{a_2}...p_k^{a_k} be the canonical decomposition of |R| to the prime numbers p_i. Then

\[
R = R_1 \oplus R_2 \oplus \cdots \oplus R_k,
\]

where each ideal R_i is of order p_i^{a_i}. By Theorem 4 we have R_2 \oplus \cdots \oplus R_k is a commutative
ring, and hence \((R_1)^*\) is minimal non-nilpotent group. Consequently, \(k = 1\). The rest of proof is clear by Theorem 7.

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