On the Problem of Reformulating Systems with Uncertain Dynamics as a Stochastic Differential Equation

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Abstract—We identify an issue in recent approaches to learning-based control that reformulate systems with uncertain dynamics using a stochastic differential equation. Specifically, we discuss the approximation that replaces a model with fixed but uncertain parameters (a source of epistemic uncertainty) with a model subject to external disturbances modeled as a Brownian motion (corresponding to aleatoric uncertainty).

I. PROBLEM FORMULATION AND ERROR IN LITERATURE

Consider a nonlinear system whose state at time $t \geq 0$ is $x(t) \in \mathbb{R}^n$, control inputs are $u(t) \in \mathbb{R}^m$, such that

$$\dot{x}(t) = f(x(t), u(t)), \quad t \in [0, T],$$

where $T > 0$, $x(0) = x_0 \in \mathbb{R}^n$ almost surely, i.e., $x(0)$ is known exactly, and $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is twice continuously differentiable.

In many applications, $f$ is not known exactly, and prior knowledge is necessary to safely control $x$. One such approach consists of assuming that $f$ lies in a known space of functions $\mathcal{H}$, and to impose a prior distribution in this space $P(\mathcal{H})$. For instance, by assuming that $f$ lies in a bounded reproducing kernel Hilbert space (RKHS), a common approach consists of imposing a Gaussian process prior on the uncertain dynamics $f \sim GP(m, k)$, where $m : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the mean function, and $k : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a symmetric positive definite covariance kernel function which uniquely defines $\mathcal{H}$.

An alternative consists of assuming that $f(x, u) = \phi(x, u)\theta$, where $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is known basis functions, and $\theta \in \mathbb{R}^p$ are unknown parameters. With this approach, one typically sets a prior distribution on $\theta$, e.g., a Gaussian $\theta \sim \mathcal{N}(\hat{\theta}, \Sigma_\theta)$, and updates this belief as additional data about the system is gathered.

Given these model assumptions and prior knowledge about $f$, safe learning-based control algorithms often consist of designing a control law $u$ satisfying different specifications, e.g., minimizing fuel consumption $\lVert u \rVert$, or satisfying constraints $x(t) \in X$ $\forall t \in [0, T]$, with $X$ a set encoding safety and physical constraints.

Next, we describe an issue with the mathematical formulation of the safe learning-based control problem that has appeared in recent research, slightly changing notations and assuming a finite-dimensional combination of features for clarity of exposition but without loss of generality. As in [5], consider the problem of safely controlling the uncertain system

$$\dot{x}(t) = \phi(x(t), u(t))\theta, \quad \theta \sim \mathcal{N}(\hat{\theta}, \Sigma_\theta),$$

where $t \in [0, T]$, $x(0) = x_0$, $\hat{\theta} \in \mathbb{R}^p$, and $\Sigma_\theta \in \mathbb{R}^{p \times p}$ is positive definite, with $\Sigma_\theta = B_\theta B_\theta^\top$ its Cholesky decomposition. Note that this formulation can be equivalently expressed in function space, where $f$ is drawn from a Gaussian process with mean function $m(x, u) = \phi(x, u)\theta$ and kernel $k(x, u, x', u') = \phi(x, u)\Sigma_\theta^{-1}\phi(x', u')^\top$. Their representation can be seen as a weight-space treatment of the GP approaches used in [3] and [4].

These works then proceed by introducing the Brownian motion $W(t)$, making the change of variable $\theta dt = \tilde{\theta} dt + B_\theta dW(t)$, and reformulating (2) as a stochastic differential equation (SDE)

$$dx(t) = \phi(x(t), u(t))\tilde{\theta} dt + \phi(x(t), u(t))B_\theta dW(t),$$

with $t \in [0, T]$ and $x(0) = x_0$. Unfortunately, (3) is not equivalent to (2). Indeed, the solution to (3) is a Markov process, whereas the solution to (2) is not. Intuitively, the increments of the Brownian motion $W(t)$ in (3) are independent, whereas in (2), $\theta$ is randomized only once, and the uncertainty in its realization is propagated along the entire trajectory. By making this change of variables for $\theta dt$, the temporal correlation between the trajectory $x(t)$ and the uncertain parameters $\theta$ is neglected. In the next section, we provide a few examples to illustrate the distinction between these two cases. The first demonstrates the heart of the issue on a simple autonomous system, whereas the second shows that analyzing the SDE reformulation (3) is insufficient to deduce the closed-loop stability of the system in (2).

II. COUNTER-EXAMPLES

A. Uncontrolled system

Consider the scalar continuous-time linear system

$$\dot{x}(t) = \theta, \quad \theta \sim \mathcal{N}(0, 1), \quad t \in [0, T],$$

where $T > 0$ and $x(0) = 0$ almost surely, i.e., $x(0)$ is known exactly. The solution to (4) satisfies $x(t) = \theta t$, i.e., each sample path is a linear (continuously differentiable) function of time $t \in [0, T]$. The marginal distribution of this stochastic process is Gaussian at any time $t \in [0, T]$, with $x(t) \sim \mathcal{N}(0, t^2)$. The increments of this process are not independent, since the increment $x(t_2) - x(t_1) = (t_2 - t_1)x(t_1)$ depends on $x(t_1) - x(0) = x(t_1)$ for any $t_2 > t_1 > 0$.

Using the change of variables described previously, one might consider substituting $dW(t)$ for $\theta dt$, where $W(t)$ is a standard Brownian motion, yielding the following SDE

$$dx(t) = dW(t), \quad x(0) = 0 \text{ (a.s.)} \quad t \in [0, T].$$

The solution of this SDE is a standard Brownian motion $x(t) = W(t)$ started at $W(0) = 0$. This stochastic process has different marginal distributions $x(t) \sim \mathcal{N}(0, t)$, has independent increments, and is not differentiable at any $t$ almost surely. We illustrate sample paths of these two different stochastic processes in Figure 1.

B. System with linear feedback

Starting from $x(t_0) = x_0 \in \mathbb{R}$, consider the controlled linear system

$$\dot{x}(t) = \theta x(t) + u(t) = (\theta + k)x(t), \quad \theta \sim \mathcal{N}(\hat{\theta}, 1), \quad t \in [0, T],$$

where $k \in \mathbb{R}$ is a feedback gain and $u(t) = kx(t)$ is the state-feedback control policy. Solutions to (6) take the form $x(t) = x_0e^{\theta t + k\int_{t_0}^t W(s) ds}$.

Choosing the gain $k = -\hat{\theta} + 1$ and simulating from $x_0 = 1$, one obtains the sample paths shown in Figure 2. We observe that some sampled trajectories are unstable, corresponding to samples of $\theta$ such that $\theta + k > 0$.

Note that the substitution $\theta dt = \tilde{\theta} dt + B_\theta dW(t)$ yields the SDE

$$dx(t) = (\theta + k)x(t)\tilde{\theta} dt + x(t) dW(t), \quad t \in [0, T].$$

The solution of this SDE is a geometric Brownian motion $x(t) = x_0e^{(\theta + k - \frac{1}{2}k)\int_{t_0}^t W(s) ds}$. Choosing the same control gain $k = -\hat{\theta} + 1$ and plotting sample paths in Figure 2 we observe that the system (7) is stochastically stable.
Fig. 1: Visualization of sample paths and confidence intervals for the open loop system given by (4) (green) and for the reformulation presented in (5) (red): the solutions of (4) and of (5) are distinct. The solid and dashed lines represent a handful of sample paths. The shaded regions represent the marginal 95% confidence intervals.

C. Discrete-time system

The observation we make in this note is well-known in the discrete-time problem setting. For example, starting from $x_0 \in \mathbb{R}$, the linear system with multiplicative uncertainty

$$x_{t+1} = \theta x_t, \quad \theta \sim \mathcal{N}(\bar{\theta}, 1), \quad t \in \mathbb{N},$$

(8)

is different from the system with additive disturbances

$$x_{t+1} = \theta x_t + w_k, \quad w_k \sim \mathcal{N}(0, 1), \quad t \in \mathbb{N},$$

(9)

where the disturbances $(w_k)_{k \in \mathbb{N}}$ are independent and identically distributed. We refer to [7, Chapter 4.7] for further discussions about this topic. We also refer to [8] for a recent analysis of systems of the form of (8) where the parameters $\theta$ are resampled at each time $t$.

III. IMPLICATIONS AND POSSIBLE SOLUTIONS

As (2) and (3) are generally not equivalent, the stability and constraint satisfaction guarantees derived for the SDE (3) in recent research [3]–[6] do not necessarily hold for the system (2). This could yield undesired behaviors when applying such algorithms, developed on an SDE formulation of dynamics (3), to safety-critical systems where uncertainty is better modeled by (2), i.e., dynamical systems with uncertain parameters that are not changing over time.

Although (3) is not equivalent to (2), it is interesting to ask whether (3) is a conservative reformulation of (2) for the purpose of safe control. For instance, given a safe set $\mathcal{X} \subset \mathbb{R}^n$, if one opts to encode safety constraints through joint chance constraints of the form

$$\mathbb{P}(x(t) \in \mathcal{X} \forall t \in [0, T]) \geq (1 - \delta),$$

(10)

where $\delta \in (0, 1)$ is a tolerable probability of failure, there may be settings where a controller $u$ satisfying (10) for the SDE (3) may provably satisfy (10) for the uncertain model (2). Indeed, as solutions of (3) may have unbounded total variation (as in the example presented above), which is not the case for solutions of (2), we make the conjecture that for long horizons $T$, a standard proportional-derivative-integral (PID) controller may better stabilize (3) than (2), and that similar properties hold for adaptive controllers.

Alternatively, approaches which bound the model error through the Bayesian posterior predictive variance [10] or confidence sets holding jointly over time [11], [12] exist. Given these probabilistic bounds, a policy can be synthesized yielding constraints satisfaction guarantees.

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