Einstein Manifolds As Yang-Mills Instantons

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ABSTRACT

It is well-known that Einstein gravity can be formulated as a gauge theory of Lorentz group where spin connections play a role of gauge fields and Riemann curvature tensors correspond to their field strengths. One can then pose an interesting question: What is the Einstein equations from the gauge theory point of view? Or equivalently, what is the gauge theory object corresponding to Einstein manifolds? We show that the Einstein equations in four dimensions are precisely self-duality equations in Yang-Mills gauge theory and so Einstein manifolds correspond to Yang-Mills instantons in $SO(4) = SU(2)_L \times SU(2)_R$ gauge theory. Specifically, we prove that any Einstein manifold with or without a cosmological constant always arises as the sum of $SU(2)_L$ instantons and $SU(2)_R$ anti-instantons. This result explains why an Einstein manifold must be stable because two kinds of instantons belong to different gauge groups, instantons in $SU(2)_L$ and anti-instantons in $SU(2)_R$, and so they cannot decay into a vacuum. We further illuminate the stability of Einstein manifolds by showing that they carry nontrivial topological invariants.

Keywords: Einstein manifold, Yang-Mills instanton, Self-duality

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1 Introduction

It seems that the essence of the method of physics is inseparably connected with the problem of interplay between local and global aspects of the world’s structure, as saliently exemplified in the index theorem of Dirac operators. Although Einstein field equations, being differential equations, are defined locally, they have to determine the structure of spacetime manifold on which they act, when a boundary condition for the differential equations is properly taken into account. Therefore, the local character of the Einstein equations would be intimately connected with the global topological structure of the underlying manifold [1]. The purpose of this letter is to explore how the topology of spacetime fabric is encoded into the local structure of Riemannian metrics using the gauge theory formulation of Euclidean gravity [2]. It turns out that the gauge theory formulation of gravity directly reveals the topological aspects of Einstein manifolds.

The physics on a curved spacetime becomes more transparent when expressed in a locally inertial frame and it is even indispensable when one want to couple spinors to gravity since spinors form a representation of $SO(4)$ rather than $GL(4, \mathbb{R})$. In this tetrad formalism, a Riemannian metric on spacetime manifold $M$ is replaced by a local basis for the tangent bundle $TM$, which is orthonormal tangent vectors $E_A (A = 1, \cdots, 4)$ on $M$. But, in any vector space, there is a freedom for the choice of basis and physical observables are independent of the arbitrary choice of a tetrad. As in any other gauge theory with local gauge invariance, to achieve local Lorentz invariance requires introducing a gauge field $\omega^A_B$ of the Lorentz group $SO(4)$. The gauge field of the local Lorentz group is called the spin connection. In the end, four-dimensional Einstein gravity can be formulated as a gauge theory of $SO(4)$ Lorentz group where spin connections play a role of gauge fields and Riemann curvature tensors correspond to their field strengths.

One can then pose an interesting question: What is the Einstein equations from the gauge theory point of view? Or equivalently, what is the gauge theory object corresponding to Einstein manifolds?

In order to answer to the above question, it will be important to notice the following mystic features [3, 4] existent only in the four dimensional space. Among the group of isometries of $d$-dimensional Euclidean space $\mathbb{R}^d$, the Lie group $SO(4)$ for $d \geq 3$ is the only non-simple Lorentz group and one can define a self-dual two-form only for $d = 4$. We will answer to the above question by noting such a plain fact that the Lorentz group $SO(4)$ is isomorphic to $SU(2)_L \times SU(2)_R$ and the Riemann curvature tensor $R^A_B = d\omega^A_B + \omega^A_C \wedge \omega^C_B$ is an $SO(4)$-valued two-form. One can thus apply two kinds of decomposition to spin connections and curvature tensors. The first decomposition is that the spin connections $\omega^A_B$ can be split into a pair of $SU(2)_L$ and $SU(2)_R$ gauge fields according to the splitting of the Lie algebra $SO(4) = SU(2)_L \oplus SU(2)_R$. (We will not be careful to discriminate a Lie group and its Lie algebra.) Accordingly the Riemann curvature tensor $R^A_B$ will also be decomposed into a pair of $SU(2)_L$ and $SU(2)_R$ curvature two-forms. The second decomposition is that, in four dimensions, the six-dimensional vector space $\Lambda^2T^*M$ of two-forms splits canonically into the sum of three-dimensional vector spaces of self-dual and anti-self-dual two forms,
i.e., $\Lambda^2 T^* M = \Lambda^2_+ \oplus \Lambda^2_-$. Therefore the Riemann curvature tensor $R^A_{\ B}$ will be split into a pair of self-dual and anti-self-dual two-forms. One can eventually combine these two decompositions.

Interestingly, the chiral splitting of $SO(4) = SU(2)_L \times SU(2)_R$ and the Hodge-$\ast$ decomposition $\Lambda^2 T^* M = \Lambda^2_+ \oplus \Lambda^2_-$ of two-forms are deeply correlated with each other due to the isomorphism between the Clifford algebra $\mathbb{C}l(4)$ in four-dimensions and the exterior algebra $\Lambda^* M = \bigoplus_{k=0}^4 \Lambda^k T^* M$ over a four-dimensional Riemannian manifold $M$. In particular, the Clifford map implies that the $SO(4)$ Lorentz generators $J^{AB} = \frac{1}{4}[\Gamma^A, \Gamma^B]$ in $\mathbb{C}l(4)$ have one-to-one correspondence with the space $\Lambda^2 T^* M$ of two-forms in $\Lambda^* M$. Since the spinor representation in even dimensions is reducible and its irreducible representations are defined by the chiral representations whose Lorentz generators are given by $J^{AB}_\pm = \frac{1}{2}(1 \pm \Gamma^5)J^{AB}$. Then the splitting of the Lie algebra $SO(4) = SU(2)_L \oplus SU(2)_R$ can be specified by the chiral generators $J^{AB}_\pm \in SU(2)_L$ and $J^{AB}_\pm \in SU(2)_R$ and the chiral splitting is precisely isomorphic to the decomposition $\Lambda^2 T^* M = \Lambda^2_+ \oplus \Lambda^2_-$ of two-forms on a four-manifold. It would be worthwhile to remark that these two decompositions actually occupy a central position in the Donaldson’s theory of four-manifolds.

In this paper we will systematically apply the gauge theory formulation of Einstein gravity to four-dimensional Riemannian manifolds and consolidate the chiral splitting of Lorentz group and the Hodge-$\ast$ decomposition of two-forms into the gauge theory formulation. A remarkable result, stated as a lemma in Section 2, comes out which sheds light on why the action of Einstein gravity is linear in curvature tensors in contrast to the quadratic action of Yang-Mills theory in spite of a close similarity to gauge theory and directly reveals the topological aspects of Einstein manifolds. We want to emphasize that our result is valid for general Einstein manifolds which are not addressed so far and thus completely generalizes the result for half-flat manifolds (so-called the gravitational instantons) which has been well-established as presented in a renowned review and a textbook. Moreover our result directs a new understanding to the Einstein equations.

The paper is organized as follows.

In Section 2, we apply the gauge theory formulation to four-dimensional Riemannian manifolds. We show that the Einstein equations in four dimensions are precisely self-duality equations in Yang-Mills gauge theory and so Einstein manifolds correspond to Yang-Mills instantons in $SO(4) = SU(2)_L \times SU(2)_R$ gauge theory. Specifically, we will prove a lemma to state that any Einstein manifold with or without a cosmological constant always arises as the sum of $SU(2)_L$ instantons and $SU(2)_R$ anti-instantons. This result explains why an Einstein manifold must be stable against small perturbations because two kinds of instantons belong to different gauge groups, instantons in $SU(2)_L$ and anti-instantons in $SU(2)_R$, and so they cannot decay into a vacuum.

In Section 3, we further illuminate the stability of Einstein manifolds by showing that they carry nontrivial topological invariants.

In Section 4, we will consider a coupling with gauge fields to understand how matter fields affect the structure of a vacuum Einstein manifold.

Finally, in Section 5, we discuss how our approach can be applied to get a new solution of Yang-
Mills instantons on a Riemannian manifold whenever an Einstein manifold is given [2]. Some generalizations to other dimensions, e.g., to three and five dimensions will be briefly addressed.

2 Einstein manifolds and Yang-Mills instantons

Four-dimensional Euclidean gravity can be formulated as a gauge theory using the language of $SO(4)$ gauge theory where the spin connections $\omega^{AB}$ are gauge fields with respect to $SO(4)$ rotations. We will follow Ref. [2] for the gauge theory formulation of Einstein gravity and also adopt the index notations in [2] except that we further distinguish the two kinds of Lie algebra indices with $a = 1, 2, 3$ and $\dot{a} = 1, 2, 3$ for $SU(2)_L$ and $SU(2)_R$, respectively, in $SO(4) = SU(2)_L \times SU(2)_R$ Lorentz group. In particular, the identities for the ’t Hooft symbols (Eqs. (3.13)-(3.19) in [2]) will be extensively used in this work.

If $M$ is an oriented four-manifold, the Hodge $\ast$-operation defines an automorphism of the vector space $\Lambda^2 T^* M$ of two-forms with the decomposition

$$\Lambda^2 T^* M = \Lambda^+_3 \oplus \Lambda^-_3$$  \hspace{1cm} (2.1)

where $\Lambda^+_3 \equiv P_\pm \Lambda^2 T^* M$ and $P_\pm = \frac{1}{2} (1 \pm \ast)$. The Hodge-$\ast$ decomposition (2.1) can harmoniously be incorporated with the Lie algebra isomorphism $SO(4) = SU(2)_L \oplus SU(2)_R$ according to the Clifford isomorphism $Cl(4) \cong \Lambda^* M$. In this respect, the ’t Hooft symbols $\eta^a_{AB}$ and $\eta^{\dot{a}}_{AB}$ take a superb mission consolidating the Hodge-$\ast$ decomposition (2.1) and the Lie algebra isomorphism $SO(4) = SU(2)_L \oplus SU(2)_R$, which intertwines the group structure carried by the Lie algebra indices $a = 1, 2, 3 \in SU(2)_L$ and $\dot{a} = 1, 2, 3 \in SU(2)_R$ with the spacetime structure of two-form indices $A, B$.

Since the spin connection $\omega_{AB} = \omega_{MAB} dx^M$ is a gauge field taking values in $SO(4)$ Lie algebra, first let us apply the Lie algebra decomposition $SO(4) = SU(2)_L \oplus SU(2)_R$ to it. This can explicitly be realized by considering the following splitting for spin connections [2]

$$\omega_{MAB} \equiv A^{(+a)}_{M} \eta^a_{AB} + A^{(-\dot{a})}_{M} \eta^{\dot{a}}_{AB}$$  \hspace{1cm} (2.2)

where $A^{(+a)} = A^{(+a)}_{M} dx^M$ and $A^{(-\dot{a})} = A^{(-\dot{a})}_{M} dx^M$ are $SU(2)_L$ and $SU(2)_R$ gauge fields, respectively. The Riemann curvature tensor $R_{AB} = \frac{1}{2} R_{MNAB} dx^M \wedge dx^N$ then takes a similar decomposition

$$R_{MAB} = F^{(+a)}_{MN} \eta^a_{AB} + F^{(-\dot{a})}_{MN} \eta^{\dot{a}}_{AB},$$  \hspace{1cm} (2.3)

where

$$F^{(\pm)}_{MN} = \partial_M A^{(\pm)}_N - \partial_N A^{(\pm)}_M + [A^{(\pm)}_M, A^{(\pm)}_N]$$  \hspace{1cm} (2.4)

are field strengths of $SU(2)_L$ and $SU(2)_R$ gauge fields in (2.2).

Now we will give an answer to the question raised before: What is the Einstein equations from the gauge theory point of view or what is the gauge theory object corresponding to Einstein manifolds?
Lemma. The Riemann curvature two-form \( R_{AB} = \frac{1}{2} R_{MNAB} dx^M \wedge dx^N \) is an \( SO(4) \)-valued field strength of the spin connections in (2.2) from the gauge theory point of view and thus can be decomposed into a pair of \( SU(2)_L \) and \( SU(2)_R \) field strengths. With the decomposition (2.3), the Einstein equation

\[
R_{AB} - \frac{1}{2} \delta_{AB} R + \delta_{AB} \Lambda = 0
\]

(2.5)

for a Riemannian manifold \( M \) is equivalent to the self-duality equation of Yang-Mills instantons

\[
F^{(\pm)} = \pm \frac{1}{2} \varepsilon_{AB}^{CD} F^{(\pm)}_{CD},
\]

(2.6)

where \( F^{(\pm)}_{AB} \eta^{a}_{AB} = F^{(\pm)}_{AB} \eta^{a}_{AB} = 2\Lambda \).

Proof. According to the Lie algebra splitting \( SO(4) = SU(2)_L \oplus SU(2)_R \), the Riemann curvature tensor \( R_{AB} = \frac{1}{2} R_{MNAB} dx^M \wedge dx^N \) in Eq. (2.3) has been decomposed into a pair of \( SU(2)_L \) and \( SU(2)_R \) field strengths defined by \( F^{(+)}_{MN} = \frac{1}{2} F^{(+)}_{MN} dx^M \wedge dx^N \) and \( F^{(-)}_{MN} = \frac{1}{2} F^{(-)}_{MN} dx^M \wedge dx^N \), respectively. Because \( F^{(\pm)} \) are curvature two-forms in gauge theory, we can apply the Hodge-* decomposition (2.1) to the \( SU(2) \) field strengths \( F^{(\pm)}_{AB} = E^M_{A} E^N_{B} F^{(\pm)}_{MN} \) as follows

\[
F^{(+)}_{AB} \equiv f^{ab}_{(+)} \eta^{a}_{AB} + f^{ab}_{(-)} \eta^{a}_{AB},
\]

(2.7)

\[
F^{(-)}_{AB} \equiv f^{ab}_{(+)} \eta^{a}_{AB} + f^{ab}_{(-)} \eta^{a}_{AB}.
\]

(2.8)

Using the above result, we get the following decomposition of the Riemann curvature tensor

\[
R_{ABCD} = f^{ab}_{(+)} \eta^{a}_{AB} \eta^{b}_{CD} + f^{ab}_{(-)} \eta^{a}_{AB} \eta^{b}_{CD} + f^{ab}_{(-)} \eta^{a}_{AB} \eta^{b}_{CD} + f^{ab}_{(-)} \eta^{a}_{AB} \eta^{b}_{CD}.
\]

(2.9)

Note that the curvature tensor has the symmetry property \( R_{ABCD} = R_{CDAB} \) from which one can get the following relations between coefficients in the expansion (2.9):

\[
f^{ab}_{(+)} = f^{ba}_{(+)}, \quad f^{ab}_{(-)} = f^{ba}_{(-)}, \quad f^{ab}_{(-)} = f^{ba}_{(-)}.
\]

(2.10)

The first Bianchi identity \( \varepsilon^{ACDE} R_{BCDE} = 0 \) further constrains the coefficients

\[
f^{ab}_{(+)} \delta^{ab} = f^{ab}_{(-)} \delta^{ab}.
\]

(2.11)

Hence the Riemann curvature tensor in (2.9) has 20 = (6 + 6 - 1) + 9 independent components, as is well-known [1].

The above results can be applied to the Ricci tensor \( R_{AB} = R_{ACBC} \) and the Ricci scalar \( R \equiv R_{AA} \) to yield

\[
R_{AB} = (f^{ab}_{(+)} \delta^{ab} + f^{ab}_{(-)} \delta^{ab}) \delta_{AB} + 2 f^{ab}_{(-)} \eta^{a}_{AC} \eta^{b}_{BC},
\]

(2.12)

\[
R = 4 (f^{ab}_{(+)} \delta^{ab} + f^{ab}_{(-)} \delta^{ab}),
\]

(2.13)
where a symmetric expression was taken in spite of the relation (2.11). After all, the Einstein tensor \( G_{AB} \equiv R_{AB} - \frac{1}{2} R \delta_{AB} \) has 10 independent components given by

\[
G_{AB} = 2 f^{a\dot{a}}_{(+)} \eta_{AC}^a \overline{\eta}_{BC}^\dot{a} - 2 f_{ab}^{\dot{a} \dot{b}} \delta^{ab} \delta_{AB}.
\] (2.14)

Note that the Einstein equation (2.5) can be recast as the form \( R_{AB} = \Lambda \delta_{AB} \) where \( \Lambda \) is a cosmological constant. Therefore the condition for an Einstein manifold can easily be read off from Eq. (2.12) and the result is given by

\[
f_{ab}^{\dot{+} \dot{+}} \delta^{ab} = f_{(--) \dot{+}}^{\dot{a} \dot{b}} = \frac{\Lambda}{2}, \quad f_{(+-)}^{\dot{a} \dot{a}} = 0.
\] (2.15)

Therefore, the curvature tensor for an Einstein manifold reduces to

\[
R_{ABCD} = f_{ab}^{\dot{+} \dot{+}} \eta_{AB}^a \eta_{CD}^b + f_{ab}^{\dot{+} \dot{-}} \eta_{AB}^a \overline{\eta}_{CD}^b + f_{ab}^{\dot{-} \dot{-}} \eta_{AB}^b \eta_{CD}^a (2.16)
\]

with the coefficients satisfying (2.15). Eq. (2.16) immediately shows that \( F_{AB}^{\pm} \) are the \( SU(2) \) field strengths obeying the self-duality equation in (2.6).

And one can verify that the converse is true too: If the Riemann curvature tensor is given by Eq. (2.16) and so satisfy the self-duality equations (2.6), the Einstein equation (2.5) is automatically satisfied with \( 2\Lambda = F_{AB}^{(+)} \eta_{AB}^{a} = F_{AB}^{(-)} \eta_{AB}^{\dot{a}} \). This completes the proof of the Lemma. \( \Box \)

Let us consider special classes of Einstein manifolds to illustrate how they easily arise from our general result. First, for gravitational instantons satisfying

\[
R_{EFAB} = \frac{1}{2} \varepsilon_{AB}^{CD} R_{EFCD},
\] (2.17)

we get the curvature tensor [2]

\[
R_{ABCD} = F_{AB}^{(+)} \eta_{CD}^a = f_{ab}^{\dot{+} \dot{+}} \eta_{AB}^a \eta_{CD}^b (2.18)
\]

with \( f_{ab}^{\dot{+} \dot{+}} \delta^{ab} = 0 \). Therefore the gravitational instanton is half-flat, i.e. \( F_{AB}^{(-)} \eta_{AB}^{a} = 0 \) and Ricci-flat, i.e. \( f_{ab}^{\dot{+} \dot{-}} \delta^{ab} = 0 \). Similarly, for gravitational anti-instantons satisfying

\[
R_{EFAB} = -\frac{1}{2} \varepsilon_{AB}^{CD} R_{EFCD},
\] (2.19)

the curvature tensor is given by

\[
R_{ABCD} = F_{AB}^{(-)} \eta_{CD}^{\dot{a}} = f_{ab}^{\dot{+} \dot{+}} \eta_{AB}^{\dot{a}} \eta_{CD}^b (2.20)
\]

with \( f_{ab}^{\dot{+} \dot{-}} \delta^{ab} = 0 \).

From the results (2.18) and (2.20), one can easily verify that gravitational instantons are \( SU(2) \) Yang-Mills instantons in the sense that they satisfy the self-duality equation (2.6). This result is not
new but has been well understood as presented in well-known reviews [3, 6]. Anyway it is interesting to notice that the solution of $F^{(\pm)}_{AB} = 0$ describes a Ricci-flat, Kähler manifold and so it is a Calabi-Yau 2-fold with $SU(2)$ holonomy. In other words, hyper-Kähler manifolds can be recast into the self-dual connections defined by the half-flat condition [7]. Indeed, one can easily show that self-dual connections satisfying the half-flat condition $F^{(+)}_{a AB} = 0$ admit the triple of Kähler forms defined by

$$J^a = \frac{1}{2} \eta_{AB} E^A \wedge E^B, \quad a = 1, 2, 3$$

(2.21)

which are all closed, i.e., $dJ^a = 0$. Similarly, it is easy to show that anti-self-dual connections satisfying the half-flat condition $F^{(-)}_{\dot{a} AB} = 0$ allow the triple of Kähler forms defined by

$$J^{\dot{a}} = \frac{1}{2} \bar{\eta}_{\dot{a} AB} E^A \wedge E^B, \quad \dot{a} = 1, 2, 3$$

(2.22)

which are also closed 2-forms, $dJ^{\dot{a}} = 0$.

For a Ricci-flat manifold obeying $R_{AB} = 0$, we get the condition from Eq. (2.12)

$$f_{ab}^{(+)} \delta^{ab} = f_{(-)}^{\dot{a}\dot{b}} \delta^{\dot{a}\dot{b}} = 0, \quad f_{ab}^{(+\dot{a})} = 0$$

(2.23)

and so the following decomposition

$$R_{ABCD} = f_{AB}^{(+)\dot{a}} \eta_{CD}^a + f_{AB}^{(-)\dot{a}} \bar{\eta}_{CD}^{\dot{a}}$$

(2.24)

with the traceless coefficients satisfying (2.23). Thus Eq. (2.24) is a particular case of the general result (2.16) with $\Lambda = 0$.

The decomposition (2.24) of Riemann curvature tensors for a Ricci-flat manifold is consistent with the double-dual condition

$$\varepsilon_{AB}^{A'B'} R_{AB'C'D'} = \theta_{AB}^{CD} \varepsilon^{C'D'}_{CD}$$

(2.25)

first introduced by Charap and Duff [8]. One can easily check that the curvature tensor in (2.24) obeys the double-dual condition (2.25) by using the self-duality relations for ’t Hooft symbols:

$$\eta_{AB}^a = \frac{1}{2} \varepsilon_{AB}^{CD} \eta_{CD}^a, \quad \bar{\eta}_{\dot{a} AB}^a = \frac{1}{2} \varepsilon_{AB}^{CD} \bar{\eta}_{\dot{a} CD}^a$$

(2.26)

It was noted in [8] that Ricci-flat spaces satisfy the condition (2.25) whose solution can be used to construct $SU(2)$ self-dual connections (Yang-Mills instantons) on a Ricci-flat manifold. For example, Euclidean Schwarzschild black-hole is a Ricci-flat manifold [9] and so the self-dual part of its spin connections can be implemented to find a Yang-Mills instanton on the black-hole geometry. However, our result (2.16) shows that not only a Ricci-flat manifold but also a general Einstein manifold obeys the double-dual condition (2.25) and the Einstein manifold can always be split into $SU(2)_L$ instantons
and $SU(2)_R$ anti-instantons. In the end, we have generalized the result in [8] to any Einstein manifold with or without a cosmological constant, which has not been stated anywhere so far.

One can draw a very interesting implication from the lemma we have proven. The $SU(2)$ field strengths in Eq. (2.3) are given by

$$F^{(\pm)} = dA^{(\pm)} + A^{(\pm)} \wedge A^{(\pm)}.$$  \hspace{1cm} (2.27)

The integrability condition, namely, the Bianchi identity, then reads as

$$D^{(\pm)}F^{(\pm)} \equiv dF^{(\pm)} + A^{(\pm)} \wedge F^{(\pm)} - F^{(\pm)} \wedge A^{(\pm)} = 0.$$  \hspace{1cm} (2.28)

Therefore the self-duality equation (2.6) immediately leads to the remarkable result that any Einstein manifold automatically satisfies the Yang-Mills equations of motion, i.e.,

$$D^{(\pm)} * F^{(\pm)} = \pm D^{(\pm)} F^{(\pm)} = 0 \quad \Leftrightarrow \quad D * F = D^{(\pm)} * F^{(\pm)} + D^{(-)} * F^{(-)} = 0$$  \hspace{1cm} (2.29)

where $*F$ means the Hodge $*$-operation on a two-form $F$. After all, our lemma sheds light on why the action of Einstein gravity is linear in curvature tensors contrary to the Yang-Mills action being quadratic in curvatures. If the action of Einstein gravity were quadratic in curvature tensors, four-manifolds obeying the equations of motion would not necessarily be given by $SU(2)$ Yang-Mills instantons and the four-manifold could be unstable in general as is well-known from gauge theory. Furthermore our lemma poses an intriguing issue about how to quantize an Einstein manifold, which will be discussed in the last section.

The “trace-free part” of the Riemann curvature tensor is called the Weyl tensor [1] defined by

$$W_{ABCD} = R_{ABCD} - \frac{1}{2}(\delta_{AC}R_{BD} - \delta_{AD}R_{BC} - \delta_{BC}R_{AD} + \delta_{BD}R_{AC}) + \frac{1}{6}(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC})R.$$  \hspace{1cm} (2.30)

The Weyl tensor shares all the symmetry structures of the curvature tensor and all its traces with the metric are zero. Therefore, one can introduce a similar decomposition for the Weyl tensor

$$W_{ABCD} \equiv g_{ab}^{(++)} \eta^{a}_{AB} \eta^{b}_{CD} + g_{(+-)}^{ab} \eta^{a}_{AB} \eta^{b}_{CD} + g_{(-+)}^{ab} \eta^{a}_{AB} \eta^{b}_{CD} + g_{(--)}^{ab} \eta^{a}_{AB} \eta^{b}_{CD}.$$  \hspace{1cm} (2.31)

The symmetry property of the coefficients in the expansion (2.31) is the same as Eq. (2.10) and the traceless condition, i.e. $W_{AB} \equiv W_{ACBC} = 0$, leads to the constraint for the coefficients:

$$g_{(++)}^{ab} = g_{(+-)}^{ab} = 0, \quad g_{(-+)}^{ab} = g_{(--)}^{ab} = 0.$$  \hspace{1cm} (2.32)

Hence the $SO(4)$-decomposition for the Weyl tensor is finally given by

$$W_{ABCD} = g_{(++)}^{ab} \eta^{a}_{AB} \eta^{b}_{CD} + g_{(--)}^{ab} \eta^{a}_{AB} \eta^{b}_{CD}.$$  \hspace{1cm} (2.33)

with the coefficients satisfying (2.32). One can see that the Weyl tensor has only $10 = 5 + 5$ independent components.
It is straightforward to determine the expansion coefficients $g_{(++)}^{ab} = \frac{1}{16} \eta_{AB}^{a} \eta_{CD}^{b} W_{ABCD}$ and $g_{(--)}^{\dot{a}\dot{b}} = \frac{1}{16} \eta_{AB}^{\dot{a}} \eta_{CD}^{\dot{b}} W_{ABCD}$ in Eq. (2.33) in terms of the coefficients in curvature tensors by substituting the results (2.9) and (2.12) into Eq. (2.30):

$$g_{(++)}^{ab} = f_{(++)}^{ab} - \frac{1}{3} \xi^{ab} f_{(++)}^{cd} \xi^{cd}, \quad g_{(--)}^{\dot{a}\dot{b}} = f_{(--)}^{\dot{a}\dot{b}} - \frac{1}{3} \xi^{\dot{a}\dot{b}} f_{(--)}^{\dot{c}\dot{d}} \xi^{\dot{c}\dot{d}}. \quad (2.34)$$

Then Eq. (2.33) can be written as follows

$$W_{ABCD} = f_{(++)}^{ab} \eta_{AB}^{a} \eta_{CD}^{b} + f_{(--)}^{\dot{a}\dot{b}} \eta_{AB}^{\dot{a}} \eta_{CD}^{\dot{b}} - \frac{1}{3} (f_{(++)}^{ab} \delta^{ab} + f_{(--)}^{\dot{a}\dot{b}} \delta^{\dot{a}\dot{b}}) (\delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC}). \quad (2.35)$$

Combining the results in Eqs. (2.9) and (2.35) gives us the well-known decomposition of the curvature tensor $R$ into irreducible components [10], schematically given by

$$R = \left( \begin{array}{ccc}
W^{+} + \frac{1}{12} s & B \\
B^T & W^{-} + \frac{1}{12} s \end{array} \right), \quad (2.36)$$

where $s$ is the scalar curvature, $B$ is the traceless Ricci tensor, and $W^{\pm}$ are the Weyl tensors.

One can consider the self-duality equation for Weyl tensors defined by $W_{EFAB} = \pm \frac{1}{2} \varepsilon_{EF}^{CD} W_{EFCD}$ [6]. An Einstein manifold is conformally self-dual if $g_{(--)}^{\dot{a}\dot{b}} = 0$ and conformally anti-self-dual if $g_{(++)}^{ab} = 0$ in Eq. (2.33). Note that the Weyl instanton (a conformally self-dual manifold) can also be regarded as a Yang-Mills instanton and $\mathbb{C}P^2$ is a well-known example [11].

But there is a subtle point for instantons with a non-zero cosmological constant. One can see from the condition (2.15) that $SU(2)$ field strengths in Eq. (2.16) do not decay to zero at an asymptotic region. It is not a problem for the case with $\Lambda > 0$, e.g. de Sitter space, because these spaces such as $S^4$ and $\mathbb{C}P^2$ are all compact [12]. So the corresponding Yang-Mills action can be finite even with the asymptotic condition (2.15). A trouble arises in the case with $\Lambda < 0$, e.g. anti-de Sitter space, because the gravitational action will diverge for noncompact geometries. To define a finite action for noncompact geometries, we may choose a reference background such that the physical (regularized) action of the reference background is defined to be zero as the ground state [13, 14] by subtracting an infinite contribution from the background solution. From the gauge theory point of view, this regularization can be realized [15] by expanding $SO(4)$ gauge fields $A_{M}^{(\pm)}$ around a classical background field $B_{M}^{(\pm)}$ (a.k.a., the background field method).

### 3 Topological invariants and stability of Einstein manifolds

Since Einstein manifolds carry a topological information in the form of Yang-Mills instantons as was shown above, it will be interesting to see how the topology of spacetime fabric is encoded into the local structure of gauge fields. In particular, the representation (2.16) provides us a powerful way to prove some inequalities about topological invariants for a compact Einstein manifold without...
The Euler characteristic $\chi(M)$ and the Hirzebruch signature $\tau(M)$ for a compact manifold $M$ are, respectively, given by \[6, 2\]

$$\chi(M) = \frac{1}{32\pi^2} \int_M \varepsilon^{ABCD} R_{AB} \wedge R_{CD}$$

\[3.1\]

$$\tau(M) = \frac{1}{24\pi^2} \int_M R_{AB} \wedge R_{AB}$$

\[3.2\]

It is obvious that $\chi(M) = 0$ only if $f^{ab}_{(++)} = f^{ab}_{(---)} = 0$, i.e., $M$ is flat. Furthermore, it is easy to get the Hitchin-Thorpe inequality \[3, 6\]

$$\chi(M) \pm 3\tau(M) = \frac{1}{\pi^2} \int_M d^4x \sqrt{g} \left( f^{ab}_{(++)} \right)^2 \geq 0$$

\[3.3\]

where the equality holds if and only if $f^{ab}_{(\pm\pm)} = 0$, i.e., $M$ is half-flat (a gravitational instanton).

For a noncompact Einstein manifold, the topological invariants have a complicated expression by including boundary terms \[6\]. The boundary terms introduce an intricate mixing of $SU(2)_L$ and $SU(2)_R$ gauge fields \[2\] whereas the bulk terms are completely separated into two sectors as was shown in Eqs. \[3.1\] and \[3.2\]. This mixing is triggered by the reduction of the Lorentz group on the boundary; $SU(2)_L \times SU(2)_R \rightarrow SU(2)_B$. For gravitational instantons where one of $SU(2)$’s decouples from the theory, the Euler number $\chi(M)$ has a nice interpretation in terms of the Chern-Simons form for an $SU(2)$ vector bundle on the boundary \[2\]. For general Einstein manifolds, we have not completely figured out the gauge theory formulation of boundary terms so far. Nevertheless, because we are using $SU(2)$ gauge fields as the basic variable, we believe the techniques which have developed for the corresponding Yang-Mills problem can be applied also to the gravitational case, which is under study \[15\].

Since an Einstein manifold carries nontrivial topological invariants, it explains why it is stable. Let us illustrate the deconstruction \[2, 16\] of Einstein manifolds with some examples; the Euclidean Schwarzschild metric \[9\] and the Fubini-Study metric on $CP^2$ \[11\]. More examples and their topological properties will be discussed in a companion paper \[15\]. The Euclidean Schwarzschild metric is not a gravitational instanton (not a half-flat manifold) though it is a Ricci-flat manifold \[9\]. The metric takes the form

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2).$$

\[3.4\]

It is easy to read off the nonvanishing coefficients in Eq. \[2.16\] from Eq. \(4.60\) in \[2\]:

$$f^{11}_{(++)} = \frac{m}{r^3}, \quad f^{22}_{(++)} = -\frac{m}{2r^3} = f^{33}_{(++)},$$

$$f^{11}_{(---)} = \frac{m}{r^3}, \quad f^{22}_{(---)} = -\frac{m}{2r^3} = f^{33}_{(---)}.$$
One can easily verify that the metric (3.4) is Ricci-flat, i.e., obeys the condition (2.23).

The result (3.5) plainly shows us that the Euclidean Schwarzschild solution (3.4) is the sum of an SU(2) instanton and an SU(2) anti-instanton. One can show [2] that the Euler number \( \chi(M) = 1 + 1 = 2 \) gets the equal contribution from the instanton and the anti-instanton where boundary terms identically vanish while the signature \( \tau(M) = 0 - 0 = 0 \) is zero for both sectors where the bulk contributions are precisely canceled by the \( \eta \)-function defined by a signature operator on the boundary [6]. Therefore, we check the lemma that a Ricci-flat four-manifold always arises as the sum of SU(2)\(_L\) instantons and SU(2)\(_R\) anti-instantons and so the Ricci-flat manifold should be stable at least perturbatively. This property is also true for an Einstein manifold as will be examined below.

The Fubini-Study metric on \( \mathbb{C}P^2 \) describes a compact Kähler and conformally self-dual manifold and is given by [11]

\[
ds^2 = \frac{r^2}{4(1 + \frac{\Lambda r^2}{6})} (\sigma_1^2 + \sigma_2^2) + \frac{r^2 \sigma_3^2}{(1 + \frac{\Lambda r^2}{6})^2} + dr^2
\]

where \( \sigma_i \) (\( i = 1, 2, 3 \)) are left-invariant 1-forms on the manifold of the group \( SU(2) \cong S^3 \) satisfying the exterior algebra \( d\sigma^i + \frac{1}{2} \epsilon^{ijk} \sigma^j \wedge \sigma^k = 0 \). It is straightforward to calculate the coefficients in Eq. (2.16) from the metric (3.6). The nonvanishing coefficients are given by

\[
f^{33}_{++} = \frac{\Lambda}{2}, \quad f^{11}_{--} = f^{22}_{--} = f^{33}_{--} = \frac{\Lambda}{6}.
\]

It is clear that the metric (3.6) is conformally self-dual, i.e., \( g^{ib}_{(-)} = 0 \) in Eq. (2.34). One can immediately see that the instantons and anti-instantons contribute a ratio of three to one to \( \chi(M) \) and \( \tau(M) \). Actually we get \( \chi(M) = \frac{3}{2} + \frac{3}{2} = 3 \) and \( \tau(M) = \frac{3}{2} - \frac{1}{2} = 1 \) [6].

Note that the Euler characteristic \( \chi(M) \) and the Hirzebruch signature \( \tau(M) \) are two topological invariants associated with the Atiyah-Patodi-Singer index theorem for an elliptic complex in four dimensions [3, 6] and thus they take integer values. Furthermore, as was shown in (3.1), the Euler characteristic \( \chi(M) \) for any Einstein manifold \( M \) takes a positive integer unless \( M \) is flat (which is true even for a noncompact manifold). This consideration enhances the reason why an Einstein manifold should be stable, at least, perturbatively. Suppose that \( M \) is an Einstein manifold such that it admits a metric \( g \) obeying (2.5). Given such a metric \( g \), one can continuously perturb it to a new metric \( g + \delta g \). But the metric perturbation \( g + \delta g \) cannot change the Euler characteristic \( \chi(M) \) because \( \chi(M) \) should not be changed by a continuous deformation. Hence the new metric \( g + \delta g \) has to describe the same Einstein manifold as before, which means that the continuous deformations \( \delta g \) correspond to zero modes or take values in the moduli space of Einstein metrics.

### 4 Einstein manifolds with a matter coupling

Our formalism can be fruitfully applied to the deformation theory of Einstein spaces. First of all, it will be interesting to see how the energy-momentum tensor \( T_{AB} \) of matter fields in the Einstein
equation
\[ G_{AB} + \Lambda \delta_{AB} = 8\pi GT_{AB} \]  \hspace{1cm} (4.1)
deforms the structure of an Einstein manifold described by Eq. (2.16). To be specific, consider the Einstein-Yang-Mills theory where the energy-momentum tensor of Yang-Mills gauge fields is given by
\[ T_{AB} = \frac{2}{g_{YM}^2} \text{Tr} \left( F_{AC}F_{BC} - \frac{1}{4} \delta_{AB} F_{CD}F^{CD} \right). \]  \hspace{1cm} (4.2)
Since the Yang-Mills field strength \( F_{AB} \) is a two-form taking values in the adjoint representation of gauge group \( G \), it can also be decomposed like (2.7) or (2.8) according to the Hodge-* decomposition (2.1)
\[ F_{AB} \equiv f^a(+) \eta^a_{AB} + f^\dot{a}(-) \bar{\eta}^\dot{a}_{AB}. \]  \hspace{1cm} (4.3)
It is then straightforward to calculate the energy-momentum tensor (4.2) which is given by
\[ T_{AB} = \frac{4}{g_{YM}^2} \text{Tr} \left( f^a(+) f^\dot{a}(-) \right) \eta^a_{AC} \bar{\eta}^\dot{a}_{BC}. \]  \hspace{1cm} (4.4)
Substituting Eqs. (2.14) and (4.4) into the equation (4.1) leads to the deformed relation instead of Eq. (2.15)
\[ f^{ab}(+) \delta^{ab} = f^{\dot{a}\dot{b}}(-) \delta^{\dot{a}\dot{b}} = \Lambda, \]
\[ f^{a\dot{a}}(+) = \frac{16\pi G}{g_{YM}^2} \text{Tr}(f^a(+) f^\dot{a}(-)). \]  \hspace{1cm} (4.5)
The Einstein equations written in the form (4.5) show us a crystal-clear picture how (non-)Abelian gauge fields deform the structure of the Einstein manifold. They introduce a mixing of \( SU(2)_L \) and \( SU(2)_R \) sectors without disturbing the conformal structure given by Eq. (2.35) and the instanton structure described by Eq. (2.16). This will not be the case for other fields such as scalar and Dirac fields, as was shown in [7].
An interesting but well-known point is that (anti-)self-dual Yang-Mills fields satisfying the following equation
\[ F_{AB} = \pm \frac{1}{2} \varepsilon_{ABCD} F_{CD} \]  \hspace{1cm} (4.6)
do not affect the Einstein structure of a manifold because \( f^\dot{a}(-) = 0 \) in Eq. (4.3) for Yang-Mills instantons or \( f^a(+) = 0 \) for Yang-Mills anti-instantons. This is, of course, due to the fact that the energy-momentum tensor (4.2) identically vanishes for Yang-Mills instantons obeying the self-duality equation (4.6) [2]. Therefore the Einstein structure is infinitely degenerate in the sense that one can add any number of Yang-Mills instantons without spoiling the Einstein condition of a four-manifold.
5 Discussion

It is a textbook statement \[1\] that gravity can be formulated as a gauge theory of local Lorentz symmetry. Nevertheless a thorough gauge theory formulation of gravity directly reveals the topological aspects of Einstein manifolds \[2\]. Indeed we were very pleased to find the proof of the Lemma we have given in Section 2. The lemma provides a completely new perspective about the Einstein equations and Einstein manifolds. In addition, it raises a sobering quantization issue of Einstein manifolds. The caveat is that an Einstein manifold consists of Yang-Mills instantons. The conventional perturbative path integral by the linearization of a metric, \( g_{MN} = \delta_{MN} + h_{MN} \), does not capture the nontrivial topology of a vacuum Einstein manifold. In general, a perturbative calculation around a generic background \( \bar{g}_{MN} \) will be involved with the instanton calculus because the vacuum manifold described by the metric \( \bar{g}_{MN} \) is a configuration of Yang-Mills instantons. Furthermore we expect that local fluctuations of spacetime geometry in quantum gravity, the so-called quantum foams, can accompany the quantum fluctuations of topology and change a global structure of spacetime fabric \[16\] and so it is necessary to quantize even the vacuum geometry itself near the Planck scale. Then the problem is how to quantize Einstein manifolds or equivalently Yang-Mills instantons. It may be imperative to go beyond the routine approach of quantum gravity.

Our result (2.6) can be applied to find an \( SU(2) \) Yang-Mills instanton on a general Einstein manifold which generalizes the result in \[8\] for Ricci-flat manifolds. Given an Einstein metric \( g \), one can calculate the spin connections and Riemann curvature tensors of the Einstein metric \( g \). Our lemma then says that the self-dual and anti-self-dual spin connections in Eq. (2.2) are automatically \( SU(2) \) (anti-)self-dual connections obeying Eq. (2.6) defined on the Einstein manifold whose metric is given by \( g \). It may be more transparent by rewriting Eq. (2.6) as the form \[2\]

\[
F_{MN}^{(\pm)} = \pm \frac{1}{2} \varepsilon^{RSPQ} g_{MR} g_{NS} F_{PQ}^{(\pm)}
\]  

(5.1)

where \( \sqrt{g} = \det E_M^A \) and \( \varepsilon^{RSPQ} \) is the metric independent Levi-Civita symbol with \( \varepsilon^{1234} = 1 \).

We will illustrate with the Fubini-Study metric (3.6) on \( \mathbb{C}P^2 \) that, whenever an Einstein metric is given, it is always possible to find an \( SU(2) \) Yang-Mills instanton on the Einstein manifold. Using the torsion free condition, \( T^A = dE^A + \omega^A_B \wedge E^B = 0 \), it is easy to get the spin connections and so \( SU(2) \) gauge fields in (2.2) for the metric (3.6):

\[
A^{(+)}_1 = A^{(+)}_2 = 0, \quad A^{(+)}_3 = -\frac{\Lambda r}{4} E^3, 
\]

(5.2)

\[
A^{(-)}_1 = -\frac{1}{2r} E^1, \quad A^{(-)}_2 = -\frac{1}{2r} E^2, \quad A^{(-)}_3 = -\frac{1}{2r} (1 + f) E^3,
\]

(5.3)

where

\[
E^1 = \frac{r}{2 \sqrt{f}} \sigma^1, \quad E^2 = \frac{r}{2 \sqrt{f}} \sigma^2, \quad E^3 = \frac{r}{2 f} \sigma^3
\]

(5.4)

with \( f(r) = 1 + \frac{\Lambda r^2}{6} \). It is straightforward to check that the self-dual gauge fields in (5.2) and the anti-self-dual gauge fields in (5.3) separately obey the self-duality equations in (5.1) (with + sign and
—sign, respectively), as was already verified in Eq. (3.7). Hence the Fubini-Study metric (3.6) can be used in this way to find $SU(2)$ Yang-Mills instantons on $\mathbb{C}P^2$.

It should be interesting to investigate a generalization of gauge theory formulation of Einstein gravity to other dimensions. Such a generalization to six dimensions was already considered using the Lie algebra isomorphism between $SO(6)$ Lorentz algebra and $SU(4)$ Lie algebra [17]. Of course, three and five dimensions can also be invited to the gauge theory formulation. In three dimensions, spin connections and Lorentz generators can be identified with $SU(2)$ gauge fields and Lie algebra generators, respectively, as $\omega_{AB} \equiv \varepsilon_{ABC} A^C$ and $J_{AB} \equiv \varepsilon^{ABC} T^C$ obeying the commutation relation $[T^A, T^B] = -\varepsilon^{ABC} T^C$. Then one can show that $R_{MNAB} \equiv \varepsilon_{ABC} F^C_{MN}$ where $F_{MN} = \partial_M A_N - \partial_N A_M + [A_M, A_N]$ with $A_M = A_A^M T^A$. After some work, it can be shown that an Einstein manifold satisfying the condition $R_{AB} = \Lambda \delta_{AB}$ corresponds to $F^C_{AB} = \frac{1}{2} \varepsilon_{ABC} \Lambda$. In particular, a Ricci-flat manifold with $\Lambda = 0$ is described by $SU(2)$ flat connections, i.e., $F_{AB} = 0$.

In five dimensions, the Lorentz group is $SO(5) = Sp(2)$ which is a simple Lie group. Therefore, five-dimensional gravity can be recast in the form of $Sp(2)$ Yang-Mills gauge theory. An interesting problem is to understand how $SU(2)$ Yang-Mills instantons and anti-instantons for four-dimensional Einstein manifolds can be embedded together into the simple group $SO(5) = Sp(2)$ and what is a corresponding gauge theory object for five-dimensional Einstein manifolds. In particular, if we consider a Kaluza-Klein compactification along the fifth direction, the five-dimensional metric will take the form

$$ds^2 = e^{\phi/\sqrt{3}} \left( g_{MN} dx^M dx^N + e^{-\sqrt{3} \phi} (dx^5 + A_M dx^M)^2 \right).$$

(5.5)

It is well-known that the resulting five-dimensional gravity reduces to Einstein-Maxwell-dilaton theory in four dimensions. It is then interesting to see how this Einstein-Maxwell-dilaton theory is embedded in $Sp(2)$ Yang-Mills gauge theory in five dimensions. Our gauge theory formulation here can be plainly applied to the four-dimensional gravity part coupling to $U(1)$ gauge fields and a dilaton. A detailed analysis for the gauge theory formulation of three- and five-dimensional gravity will be reported elsewhere.

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