GEVREY REGULARITY AND EXISTENCE OF NAVIER-STOKES- NERNST- PLANCK- POISSON SYSTEM IN CRITICAL BESOV SPACES

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(Communicated by Alain Miranville)

Abstract. The paper deals with the Cauchy problem of Navier-Stokes-Nernst-Planck-Poisson system (NSNPP). First of all, based on so-called Gevrey regularity estimates, which is motivated by the works of Foias and Temam [J. Funct. Anal., 87 (1989), 359-369], we prove that the solutions are analytic in a Gevrey class of functions. As a consequence of Gevrey estimates, we particularly obtain higher-order derivatives of solutions in Besov and Lebesgue spaces. Finally, we prove that there exists a positive constant C such that if the initial data \((u_0, n_0, c_0) = (u^h_0, u^3_0, n_0, c_0)\) satisfies

\[
\|(n_0, c_0, u^h_0)\|_{B^{-2+3/q}_q \times B^{-2+3/q}_q \times B^{-1+3/p}_p} + \|u^3_0\|_{B^{-1+3/p}_p} \|u^3_0\|_{B^{-1+3/p}_p}^{1-\alpha} \leq 1/C
\]

for \(p, q, \alpha\) with \(1 < p < q \leq 2p < \infty\), then global existence of solutions with large initial vertical velocity component is established.

1. Introduction and main results. In this paper, we study a dissipative system of partial differential equations modeling the flow of electrohydrodynamics. This system (NSNPP) consists of the Navier-Stokes equations with a source term coupled with the Nernst-Planck-Poisson equations for electronic charges:

\[
\begin{align*}
\dot{u} + u \cdot \nabla u - \Delta u + \nabla p &= \Delta \phi \nabla \phi, \quad \text{in } (0, \infty) \times \mathbb{R}^3, \\
\nabla \cdot u &= 0, \quad \text{in } (0, \infty) \times \mathbb{R}^3, \\
\dot{n} + u \cdot \nabla n - \Delta n &= -\nabla \cdot (n \nabla \phi), \quad \text{in } (0, \infty) \times \mathbb{R}^3, \\
\dot{c} + u \cdot \nabla c - \Delta c &= \nabla \cdot (c \nabla \phi), \quad \text{in } (0, \infty) \times \mathbb{R}^3, \\
\Delta \phi &= n - c, \quad \text{in } (0, \infty) \times \mathbb{R}^3, \\
(u, n, p)|_{t=0} &= (u_0, n_0, p_0), \quad \text{in } \mathbb{R}^3.
\end{align*}
\]

Here \(u(t, x)\) is a vector in \(\mathbb{R}^3\), \(P(t, x), n(t, x), c(t, x)\) and \(\phi(t, x)\) are scalars. The (NSNPP) describes the dynamic coupling between incompressible flows and diffuse...
charge systems and finds application in biology, chemistry and pharmacology. See [30, 31, 33, 34] for more details.

The above system (1) has been studied by several authors. Schmuck [36] and Ryham [35] obtained the global existence of weak solutions with Neumann and Dirichlet boundary conditions respectively. Li [32] studied the quasineutral limit in periodic domain. When \( \Omega = \mathbb{R}^3 \), Joseph [31] established the existence of a unique smooth local solution for smooth initial data by making use of Kato’s semigroup ideas. Zhao et al. [12, 13, 40, 41, 42] studied the local and global well-posedness in the critical Lebesgue spaces, modulation spaces, Triebel-Lizorkin spaces and Besov spaces.

In particular, if \( n = c = 0 \), then (1) becomes the problem related to the classical Navier-Stokes equations

\[
\begin{aligned}
\left\{ \begin{array}{l}
\dot{u} + u \cdot \nabla u - \Delta u + \nabla p = 0, \\
\nabla \cdot u = 0, \\
u |_{t=0} = u_0,
\end{array} \right. & \quad \text{in } (0, \infty) \times \mathbb{R}^3, \\
\nabla \cdot u = 0, & \quad \text{in } (0, \infty) \times \mathbb{R}^3, \\
u |_{t=0} = u_0, & \quad \text{in } \mathbb{R}^3,
\end{aligned}
\]

which has been widely studied during the past eighty more years. Leray [22] proved the global existence of weak solutions of (2), but the uniqueness and regularity of weak solutions are remaining the big open problems. It has been proved that the Cauchy problem (2) is globally well-posed for small initial data in a series of function spaces including particularly the following critical spaces: \( \dot{H}^{\frac{1}{2}}, L^3, \dot{B}_p^1 \dot{B}_p^{1+\frac{2}{p}} (3 < p < \infty), \) BMO\(^{-1} \), see Fujita and Kato [14], Kato [19], Kozono and Yamazaki [8], Koch and Tataru [20]; Xiao [29, 38] proved this property in the space \( Q_\infty^{-1}(\mathbb{R}^3) \).

Biswas [4] introduced \( \mathbb{V}_{\theta,p} \) and homogeneous Besov type spaces \( \mathbb{B}_p^{-\delta} \) and then established Gevrey regularity of a class of dissipative equations in \( \mathbb{B}_p^{-\delta} \) and \( \mathbb{V}_{\theta,p} \). Biswas and Swanson [5] studied Gevrey regularity of Navier-Stokes equations with space-periodic boundary conditions in \( \dot{F}_p^{-1+N/N/p} \). Bae on [2] studied the Gevrey regularity of Lei-Lin solutions [21] of Navier-Stokes equations in \( \dot{F}_p^{-1} \). Recently, Bae, Biswas and Tadmor [3] obtained analyticity of Navier-Stokes equations in critical Besov spaces \( \dot{B}_p^{-1+N/N/p} \).

Let \( S \) be the Schwartz class of rapidly decreasing functions, \( S' \) be the space of tempered distributions, \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) denote Fourier and inverse Fourier transforms of \( L^1 \) functions, respectively, which are defined by

\[
\mathcal{F} f = \hat{f}(\xi) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{-ix \cdot \xi} f(x) dx, \quad \mathcal{F}^{-1} f = \check{f}(x) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{ix \cdot \xi} \hat{f}(\xi) d\xi.
\]

More generally, Fourier transform of any \( f \in S' \), given by \( (\mathcal{F} f, g) = (f, \mathcal{F} g) \) for any \( g \in S \). Let \( \mathcal{C} \) be the annulus \( \{ \xi \in \mathbb{R}^3 : \frac{3}{4} \leq |\xi| \leq \frac{5}{4} \} \) and \( \mathcal{D}(\Omega) \) be a space of smooth compactly supported functions on the domain \( \Omega \). There exist radial functions \( \chi \) and \( \varphi \), valued in the interval \( [0, 1] \), belonging respectively to \( \mathcal{D}(B(0, \frac{2}{3})) \) and \( \mathcal{D}(\mathcal{C}) \), and such that

\[
\forall \xi \in \mathbb{R}^3, \quad \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j} \xi) = 1, \quad \forall \xi \in \mathbb{R}^3 \setminus \{0\}, \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) = 1,
\]

\[
|j - j'| \geq 2 \Rightarrow \operatorname{supp} \varphi(2^{-j} \xi) \cap \operatorname{supp} \varphi(2^{-j'} \xi) = \emptyset,
\]

\[
j \geq 1 \Rightarrow \operatorname{supp} \chi(\xi) \cap \operatorname{supp} \varphi(2^{-j} \xi) = \emptyset.
\]

Define the set \( \tilde{\mathcal{C}} = B(0, \frac{2}{3}) + \mathcal{C} \). Then we have

\[
|j - j'| \geq 5 \Rightarrow 2^j \tilde{\mathcal{C}} \cap 2^j \mathcal{C} = \emptyset.
\]
Furthermore, we have
\[ \forall \xi \in \mathbb{R}^3, \quad \frac{1}{2} \leq \chi^2(\xi) + \sum_{j \geq 0} \varphi^2(2^{-j}\xi) \leq 1, \quad \forall \xi \in \mathbb{R}^3, \quad \frac{1}{2} \leq \sum_{j \in \mathbb{Z}} \varphi^2(2^{-j}\xi) \leq 1. \]

From now on, we write \( h = F^{-1}\varphi \) and \( \hat{h} = F^{-1}\chi \). The homogeneous dyadic blocks \( \Delta_j \) and \( S_j \) are defined by
\[ \Delta_j u = \varphi(2^{-j}D)u = 2^{3j} \int_{\mathbb{R}^3} h(2^j y) u(x - y) dy \quad \text{if} \quad j \in \mathbb{Z}, \]
\[ S_j u = \chi(2^{-j}D)u = 2^{3j} \int_{\mathbb{R}^3} \hat{h}(2^j y) u(x - y) dy \quad \text{if} \quad j \in \mathbb{Z}. \]

Here \( D = (D_1, D_2, D_3) \) and \( D_j = i^{-1}D_j(i^2 = -1) \). The set \( \{\Delta_j, S_j\}_{j \in \mathbb{Z}} \) is called the Littlewood-Paley decomposition. Formally, \( \Delta_j = S_j - S_{j-1} \) is a frequency projection to the annulus \( \{\xi : 2^j \leq |\xi| < 2^{j+1}\} \), and \( S_j = \sup_{j' \leq j-1} \Delta_{j'} \) is a frequency projection to the ball \( \{\xi : |\xi| \leq 2^j\} \). We denote by \( S^r_k \) the space of tempered distributions \( f \) such that \( \lim_{j \to \infty} S_j f = 0 \) in \( S' \). Recall that for \( s \in \mathbb{R} \) and \((p, r) \in [1, \infty] \times [1, \infty] \), the homogeneous Besov spaces \([17]\) are denoted by \( \dot{B}^s_{p, r}(\mathbb{R}^3) \)
\[ \dot{B}^s_{p, r}(\mathbb{R}^3) = \left\{ f \in S'_h : \|f\|_{B^s_{p, r}(\mathbb{R}^3)} < \infty \right\}, \]
where
\[ \|f\|_{B^s_{p, r}(\mathbb{R}^3)} = \begin{cases} \left( \sum_{k \in \mathbb{Z}} \left(2^{ks}\|\Delta_k f\|_p\right)^r \right)^{\frac{1}{r}} , & \text{if} \quad 1 \leq p \leq \infty, 1 \leq r < \infty, s \in \mathbb{R}, \\ \sup_{k \in \mathbb{Z}} 2^{ks}\|\Delta_k f\|_p , & \text{if} \quad 1 \leq p \leq \infty, r = \infty, s \in \mathbb{R}. \end{cases} \]

It is well-known that if either \( s < 3/p \) or \( s = 3/p \) and \( r = 1 \), then \( \dot{B}^s_{p, r}(\mathbb{R}^3) \) is a Banach space.

Let us now recall the definition of the Chemin-Lerner space \( \mathfrak{L}^p(0, T; \dot{B}^s_{p, r}(\mathbb{R}^3)) \): for \( 0 < T \leq \infty, s \in \mathbb{R} \) and \( 1 \leq p, r, \rho \leq \infty \) (with the usual convention if \( r = \infty \) or \( \rho = \infty \)). The Chemin-Lerner space is defined by
\[ \mathfrak{L}^p(0, T; \dot{B}^s_{p, r}(\mathbb{R}^3)) := \left\{ f \in S'(((0, T), S^r_k)) : \|f\|_{\mathfrak{L}^p(0, T; \dot{B}^s_{p, r}(\mathbb{R}^3))} < \infty \right\}, \]
where
\[ \|f\|_{\mathfrak{L}^p(0, T; \dot{B}^s_{p, r}(\mathbb{R}^3))} = \left( \sum_{k \in \mathbb{Z}} \left(2^{ks}\|\Delta_k f\|_{L^p((0, T); L^r)}\right)^r \right)^{\frac{1}{r}}. \]

We define the usual space \( L^p(0, T; \dot{B}^s_{p, r}(\mathbb{R}^3)) \) associated with the norm
\[ \|f\|_{L^p(0, T; \dot{B}^s_{p, r}(\mathbb{R}^3))} = \left( \int_0^T \left( \sum_{k \in \mathbb{Z}} \left(2^{ks}\|\Delta_k f\|_{L^r}\right)^r \right)^{\frac{1}{r}} \right)^{1/p} \]

By Minkowski’s inequality, it is readily to verify that
\[
\begin{align*}
\|f\|_{\mathfrak{L}^p(0, T; \dot{B}^s_{p, r}(\mathbb{R}^3))} & \leq \|f\|_{L^p(0, T; \dot{B}^s_{p, r}(\mathbb{R}^3))}, & \text{if} \quad \rho \leq r, \\
\|f\|_{L^p(0, T; \dot{B}^s_{p, r}(\mathbb{R}^3))} & \leq \|f\|_{\mathfrak{L}^p(0, T; \dot{B}^s_{p, r}(\mathbb{R}^3))}, & \text{if} \quad r \leq \rho.
\end{align*}
\]
A constant $\tilde{C}$ exists which satisfies the following properties: if $s_1$ and $s_2$ are real numbers such that $s_1 < s_2$ and $\theta \in (0,1)$, then we have, for any $(p,r,\rho,\rho_1,\rho_2) \in [1,\infty]^5$ and any $1/\rho = \theta/\rho_1 + (1-\theta)/\rho_2$, 
\[
\begin{cases}
\|f\|_{\dot{B}^{s_1}_p, \tau_{s_1} + (1-\theta)s_2} \leq \tilde{C}\|f\|_{\dot{B}^{\theta s_2}_p, \tau_{s_2}} \|f\|_{\dot{B}_{p, \tau_{s_1}}^{1-\theta}} \\
\|f\|_{L^p(0;\dot{B}^{s_1}_p, \tau_{s_1} + (1-\theta)s_2)} \leq \tilde{C}\|f\|_{L^p(0;\dot{B}^{\theta s_2}_p, \tau_{s_2})} \|f\|_{L^{s_2}(0;\dot{B}_{p, \tau_{s_1}}^{1-\theta})}, \\
\|f\|_{L^p(0;\dot{B}^{s_1}_p, \tau_{s_1} + (1-\theta)s_2)} \leq \tilde{C}\|f\|_{L^p(0;\dot{B}^{\theta s_2}_p, \tau_{s_2})} \|f\|_{L^{s_2}(0;\dot{B}_{p, \tau_{s_1}}^{1-\theta})}.
\end{cases}
\]

(4)

Because $\dot{B}^{3}_{p,1}$ is embedded in $L^\infty$, we deduce that whenever $1 \leq p \leq \infty$, the product of two functions in $\dot{B}^{3}_{p,1}$ is also in $\dot{B}^{3}_{p,1}$ and such that for some constant $\tilde{C} > 0$,
\[
\|uv\|_{\dot{B}^{3}_{p,1}} \leq \tilde{C}\|u\|_{\dot{B}^{3}_{p,1}} \|v\|_{\dot{B}^{3}_{p,1}}.
\]

(5)
The homogeneous paraproduct of $v$ and $u$ is defined by 
\[
T_u v := \sum_j S_{j-1} u \Delta_j v.
\]
The homogeneous remainder of $v$ and $u$ is defined by 
\[
R(u,v) := \sum_{|k-j| \leq 1} \Delta_k u \Delta_j v := \sum_k \Delta_k u \tilde{\Delta}_k v \quad \text{and} \quad \tilde{\Delta}_k := \Delta_{k-1} + \Delta_k + \Delta_{k+1}.
\]

We have the following Bony's decomposition 
\[
wv = T_u v + R(u,v) + T_v u.
\]

(6)

For any operator $T : \dot{B}^{s}_p, \tau \to \dot{B}^{s}_p, \tau$, we set $\|u\|_{T \dot{B}^{s}_p, \tau} := \|Tu\|_{\dot{B}^{s}_p, \tau}$. Let $\Lambda$ be the Fourier multiplier whose symbol is given by $|\xi|_1 = \sum_{i=1}^3 |\xi_i|$ and $\lambda = 0.1$. We emphasize that here $\Lambda \equiv \Lambda_1$ is quantified by the $\ell^1$ norm rather than the usual $L^2$ norm associated with $\Lambda_2 := (\Delta)^{1/2}$. $e^{\sqrt{\lambda} T} A$ is a Fourier multiplier operator whose symbol is given by $e^{\sqrt{\lambda} |\xi|_1}$. A function $f$ is said to be Gevrey regular if 
\[
\|e^{\sqrt{\phi} T} f\|_{\dot{B}^{s}_p, \tau} < +\infty,
\]
for some $s \in \mathbb{R}$ and $1 \leq p, \tau \leq \infty$. We mention that the finiteness of the corresponding Gevrey norm implies that the functions are analytic.

Let $E_0 := \dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3) \times \dot{B}^{-2+3/q}_{q,1}(\mathbb{R}^3) \times \dot{B}^{-2+3/q}_{q,1}(\mathbb{R}^3)$. We introduce a vector space $\Theta_T := \mathbb{V}_T \times \mathbb{V}_T \times \mathbb{V}_T$ and with the usual product norm $\|(u,v,w)\|_{\Theta_T} := \|u\|_{\mathbb{V}_T} + \|v\|_{\mathbb{V}_T} + \|w\|_{\mathbb{V}_T}$, 
\[
\begin{align*}
\mathbb{X}_T^\lambda &:= \left\{u \in \mathbb{X}_T : \nabla \cdot u = 0, u \in \mathcal{A}^1(0,T; \dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3)) \cap \mathcal{L}^\infty(0,T; \dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3)) \right\}, \\
\mathbb{V}_T^\lambda &:= \left\{u \in \mathcal{A}^1(0,T; \dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3)) \cap \mathcal{L}^\infty(0,T; \dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3)) \right\},
\end{align*}
\]
and 
\[
\|u\|_{\mathbb{X}_T^\lambda} := \|u\|_{L^\infty(0,T; \dot{B}^{-1+3/p}_{p,1})} + \|u\|_{L^1(0,T; \dot{B}^{-1+3/p}_{p,1})}, \quad \|u\|_{\mathbb{V}_T^\lambda} := \|u\|_{L^\infty(0,T; \dot{B}^{-1+3/p}_{p,1})} + \|u\|_{L^1(0,T; \dot{B}^{-1+3/p}_{p,1})}.
\]

Meanwhile, let 
\[
\Theta_T^\lambda = C([0,T]; \dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3)) \times C([0,T]; \dot{B}^{-2+3/q}_{q,1}(\mathbb{R}^3)) \times C([0,T]; \dot{B}^{-2+3/q}_{q,1}(\mathbb{R}^3)).
\]
For simplicity, when \( T = \infty, \lambda = 1 \), we denote \( \mathcal{X}_\infty^1 := \mathcal{X}, Y_\infty^1 := \mathcal{Y}, \Theta_\infty^1 := \Theta \) and \( \Theta_\infty^C := \Theta^C \).

Inspired by the seminal work of Foias-Temam in [15], and also motivated by papers [1, 2, 3, 4, 5] concerning the Gevrey regularity of solutions, we obtain the Gevrey class regularity for the system (1) in the context of critical Besov spaces.

**Theorem 1.1.** Let \( 1 < p < q \leq 2p < \infty, \frac{1}{p} + \frac{1}{q} > \frac{1}{3}, 1 < q < 6, \frac{1}{p} - \frac{1}{q} \leq \frac{1}{3}, \nabla \cdot u_0 = 0 \) and \( (u_0, n_0, c_0) \in E_0 \). Then the system (1) admits a unique local solution on \([0, T)\) such that \( (u, n, c) \in \Theta_\infty^1 \cap \Theta_\infty^C \). Furthermore, there exists \( \epsilon > 0 \), such that if \( (u_0, n_0, c_0) \) satisfies \( \| (u_0, n_0, c_0) \|_{E_0} \leq \epsilon \), then the solution is global.

As a consequence of working with Gevrey norms, we obtain higher-order derivatives of solutions in Besov and Lebesgue spaces.

**Corollary 1.** The solution \((u, n, c)\) in Theorem 1.1 enables us to establish the following estimates on the high-order derivatives in Besov and Lebesgue spaces, that is, for all \( 0 < t < T \), there exist positive constants \( C_1, C_2, C_3 \) and \( C \) such that

(i) If \( m > 0 \), then

\[
\| (D^m u, D^m n, D^m c) \|_{\dot{B}^{-1+3/p}_{p,1} \times \dot{B}^{-2+3/q}_{q,1} \times \dot{B}^{-2+3/q}_{q,1}} \lesssim C m^m t^{-m}.
\]

(ii) If \( k_1 > -1 + 3/p \) and \( 1 < p \leq 2 \), then

\[
\| D^{k_1} u(t) \|_{L^p} \lesssim C_1^{k_1+1-3/p} (k_1 + 1 - 3/p)^{k_1+1-3/p} t^{-k_1+1-3/p}.
\]

(iii) If \( k_2 > -2 + 3/q \) and \( 1 < q \leq \frac{3}{2} \), then

\[
\| D^{k_1} n(t) \|_{L^q} \lesssim C_2^{k_2+2-3/q} (k_2 + 2 - 3/q)^{k_2+2-3/q} t^{-k_2+2-3/q}.
\]

(iv) If \( k_2 > -2 + 3/q \) and \( 1 < q \leq \frac{3}{2} \), then

\[
\| D^{k_1} c(t) \|_{L^q} \lesssim C_2^{k_2+2-3/q} (k_2 + 2 - 3/q)^{k_2+2-3/q} t^{-k_2+2-3/q}.
\]

It is worth mentioning that for any \( t \in (0, T) \) in Theorem 1.1, we obtain the solution

\[
(u(t), n(t), c(t)) \in C^\infty(\mathbb{R}^3) \times C^\infty(\mathbb{R}^3) \times C^\infty(\mathbb{R}^3).
\]

Motivated by [39] concerning the global well-posedness of 3D incompressible Navier-Stokes equations with the third component of the initial velocity field being large, applying the local existence mentioned in Theorem 1.1 (in this case \( \lambda = 0 \)) as well as the standard continuity argument, the system (1) has a unique global solution. Our main result reads as follows:

**Theorem 1.2.** Let \( 1 < p < q \leq 2p < \infty, \frac{1}{p} + \frac{1}{q} > \frac{1}{3}, 1 < q < 6, \frac{1}{p} - \frac{1}{q} \leq \frac{1}{3} \). Let \( (u_0, n_0, c_0) \in E_0 \) with \( \nabla \cdot u_0 = 0 \) and \( u_0 = (u_0^1, u_0^2, u_0^3) = (u_0^1, u_0^2, u_0^3) \). There exists a positive constant \( C \) such that if the initial data \((u_0, n_0, c_0)\) satisfies

\[
C(\| (n_0, c_0) \|_{\dot{B}^{-2+3/q}_{q,1} \times \dot{B}^{-2+3/q}_{q,1} \times \dot{B}^{-2+3/q}_{q,1}} + \| u_0^1 \|_{\dot{B}^{-1+3/p}_{p,1}} + \| u_0^3 \|_{\dot{B}^{-1+3/p}_{p,1}} \| u_0^3 \|_{\dot{B}^{-1+3/p}_{p,1}} \| u_0^3 \|_{\dot{B}^{-1+3/p}_{p,1}} \leq 1.
\]

Then the system (1) admits a unique global-in-time solution such that \((u, n, c) \in \Theta \cap \Theta^C\).

We mention that our results do not impose any smallness conditions on the third component \( u_0^3 \) of the initial velocity field. It improves the recent result of [42], where the exponent form of the initial smallness condition is replaced by a polynomial form.
Notations. Throughout the paper, $\tilde{c}$ and $\tilde{C}$ stand for harmless constants. Let $A$ and $B$ be two operators, we denote $[A; B] := AB - BA$. For $X$ a Banach space and $I$ an interval of $\mathbb{R}$, we denote by $C(I; X)$ in the set of continuous functions on $I$ with value in $X$. $(d_j)_{j \in \mathbb{Z}}$ will be a generic element of $\ell^1(\mathbb{Z})$ such that $d_j \geq 0$ and $\sum_{j \in \mathbb{Z}} d_j = 1$.

2. Preliminaries.

Lemma 2.2 ([17]). Let $0 < r < R, C := \{ \xi \in \mathbb{R}^3 : r \leq |\xi| \leq R \}$ be an annulus and $B := B(0, R) = \{ \xi \in \mathbb{R}^3 : 0 \leq |\xi| \leq R \}$ be a ball. A positive constant $\tilde{C}$ exists such that for any nonnegative integer $k$, any couple $(p, q)$ in $[1, \infty]^2$ with $q \geq p \geq 1$, and any function $u$ of $L^p(\mathbb{R}^3)$, we have

$$\text{Supp } \hat{u} \subseteq \lambda B \Rightarrow \| D^k u \|_{L^q(\mathbb{R}^3)} \leq \tilde{C} \| \theta^p u \|_{L^q(\mathbb{R}^3)} \leq \tilde{C} \| \theta^p u \|_{L^q(\mathbb{R}^3)} \leq \tilde{C} \| u \|_{L^p(\mathbb{R}^3)}.$$
Lemma 2.4. Let $1/r_1 + 1/r_2 = 1$ and $r_1 \geq 2$. Let $1 < p < q < \infty$ and $1/p + 1/q > \frac{1}{3}$. Assume that $u \in X_T$ and $n \in Y_T$. Then we have
\[
\|u \cdot \nabla n\|_{L_p^1(\mathbb{R}^n, \nabla(\lambda \nabla n)_{-2+3/q})}^2 \\
\lesssim \|u\|_{L_p^1(\mathbb{R}^n, \nabla(\lambda \nabla n)_{-2+3/p})}^2 \|u\|_{L_p^1(\mathbb{R}^n, \nabla(\lambda \nabla n)_{-2+3/q})}^2 \|n\|_{L_p^1(\mathbb{R}^n, \nabla(\lambda \nabla n)_{-2+3/q})}^2 \\
+ \|u\|_{L_p^1(\mathbb{R}^n, \nabla(\lambda \nabla n)_{-2+3/p})}^2 \|u\|_{L_p^1(\mathbb{R}^n, \nabla(\lambda \nabla n)_{-2+3/q})}^2 \|n\|_{L_p^1(\mathbb{R}^n, \nabla(\lambda \nabla n)_{-2+3/q})}^2 \\
+ \|u\|_{L_p^1(\mathbb{R}^n, \nabla(\lambda \nabla n)_{-2+3/p})}^2 \|n\|_{L_p^1(\mathbb{R}^n, \nabla(\lambda \nabla n)_{-2+3/q})}^2 \|\frac{1}{q} \cdot n\|_{L_p^1(\mathbb{R}^n, \nabla(\lambda \nabla n)_{-2+3/q})}^2.
\]  
(11)

Proof. Let $(N, U) := (\lambda \nabla n, (\lambda \nabla n)_u)$. Thanks to Bony’s paraproduct decomposition (6), we have
\[
e^{\lambda \nabla n} \Delta_j(u \nabla n) = e^{\lambda \nabla n} \Delta_j(T_{e^{-\lambda \nabla n} U} \nabla e^{-\lambda \nabla n} N) \\
+ R(e^{-\lambda \nabla n} U, \nabla e^{-\lambda \nabla n} N) + T_{e^{-\lambda \nabla n} N} e^{-\lambda \nabla n} U.
\]  
(12)

Choosing $1 \leq r_1, r_2 \leq \infty$, such that $\frac{1}{r_2} + \frac{1}{r_1} = 1$, note that $r_1 \geq 2$, applying (4) and Lemmas 2.1 and 2.2 give rise to
\[
\|\Delta_j e^{\lambda \nabla n}(T_{e^{-\lambda \nabla n} U} \nabla e^{-\lambda \nabla n} N)\|_{L_p^1(L^p)} \\
\lesssim \sum_{|j-j'| \leq 4} 2^{j'} \sum_{k \leq j'-2} 2^{\frac{3k}{q}} \|\Delta_k \nabla \|_{L_p^1(L^p)} \|\Delta_j \nabla N\|_{L_p^1(L^p)} \\
\lesssim \sum_{|j-j'| \leq 4} 2^{j}(3+\frac{3}{q}) \sum_{k \leq j'-2} 2^{(1-\frac{3}{q})k} \|\Delta_k \nabla \|_{L_p^1(L^p)} \|\Delta_j \nabla N\|_{L_p^1(L^p)} \\
\lesssim 2^{j}(2-\frac{3}{q}) \sum_{|j-j'| \leq 4} \|\Delta_j \nabla \|_{L_p^1(L^p)} \|\Delta_j \nabla N\|_{L_p^1(L^p)}
\]  
(13)

Thanks to $1 < p < q < \infty$ and $r_1 \geq 2$, Lemmas 2.1 and 2.2 and (4), we arrive at
\[
\|\Delta_j e^{\lambda \nabla n}(T_{e^{-\lambda \nabla n} N} e^{-\lambda \nabla n} U)\|_{L_p^1(L^p)} \\
\lesssim 2^{3j} \frac{1}{q} \sum_{|j-j'| \leq 4} \|\Delta_j \nabla \|_{L_p^1(L^p)} \|\Delta_j \nabla N\|_{L_p^1(L^p)} \\
\lesssim 2^{3j} \frac{1}{q} \sum_{|j-j'| \leq 4} 2^{j}(1-\frac{3}{p}-\frac{2}{r_2}) \sum_{k \leq j'-2} 2^{k(3-2/r_1)} \|\Delta_k \nabla \|_{L_p^1(L^p)} \|\Delta_j \nabla N\|_{L_p^1(L^p)}
\]  
(14)

In the case $\frac{1}{p} + \frac{1}{q} > 1$, we can find $1 < q' \leq \infty$ such that $\frac{1}{q} + \frac{1}{q'} = 1$, applying Lemmas 2.1 and 2.2 and (4), we infer that for some fixed constant $N_0$,
\[
\|\Delta_j e^{\lambda \nabla n} R(e^{-\lambda \nabla n} U, \nabla e^{-\lambda \nabla n} N)\|_{L_p^1(L^p)} \\
\lesssim 2^{3j(1-1/q)} \sum_{j' \geq j-N_0} 2^{j'} 2^{j'(3/p-3/q')} \|\Delta_j \nabla \|_{L_p^1(L^p)} \|\Delta_j \nabla N\|_{L_p^1(L^p)} \\
\lesssim 2^{j(3/p)} \sum_{j' \geq j-N_0} 2^{j'} 2^{j'(3/p-3/q')} \|\Delta_j \nabla \|_{L_p^1(L^p)} \|\Delta_j \nabla N\|_{L_p^1(L^p)}
\]  
(15)
If \( \frac{1}{p} + \frac{1}{q} \leq 1 \), Lemmas 2.1 and 2.2 together with \( \frac{1}{q} + \frac{1}{p} > \frac{1}{4} \) and (4) ensure that

\[
\| \Delta_j e^{\lambda \sqrt{\Delta} A} R(e^{-\lambda \sqrt{\Delta} A} U, \nabla e^{-\lambda \sqrt{\Delta} A} N) \|_{L^p_t(L^q)} \\
\lesssim 2^{j(1+3/p)} \sum_{j' \geq j-N_0} \| \Delta_j' U \|_{L^p_t(L^q)} \| \tilde{\Delta}_{j'} N \|_{L^q_t(L^\infty)} \\
\lesssim 2^{j(1+3/p)} \sum_{j' \geq j-N_0} 2^{j(1-3/p-3/q)} d_{j'} \| U \|_{L^p_t(B_{p,1}^{-1+3/p+2/r_2})} \\
\times 2^{j(-2+2/r_1+3/q)} \| \tilde{\Delta}_{j'} N \|_{L^q_t(L^q)} \\
\lesssim 2^{j(2-3/q)} d_{j'} \| U \|_{L^p_t(B_{p,1}^{-1+3/p})} \| N \|_{L^q_t(B_{q,1}^{-2+3/q})} \| N \|_{L^q_t(B_{q,1}^{3/q})}.
\]

Combining the above estimates (13)-(16) and taking summation for \( j \in \mathbb{Z} \), we arrive at (11). We thus conclude the proof of Lemma 2.4. \( \square \)

**Lemma 2.5.** Assume that \( n \in \mathbb{V}_T \) and \( c \in \mathbb{V}_T \). Let \( 1 < q < 6, r_1 \geq 2 \) and \( 1/r_1 + 1/r_2 = 1 \). \( \Delta \phi = n - c \), we thus have

\[
\| \nabla \cdot (n \nabla \phi) \|_{L^1_t(B_{q,1}^{2+3/q})} \\
\lesssim \| n \|_{L^p_t(B_{q,1}^{2+3/q})} \| n \|_{L^p_t(B_{q,1}^{3/q})} \| (n, c) \|_{L^p_t(B_{q,1}^{2+3/q})} \| (n, c) \|_{L^p_t(B_{q,1}^{2+3/q})} \\
+ \| n \|_{L^p_t(B_{q,1}^{2+3/q})} \| n \|_{L^p_t(B_{q,1}^{3/q})} \| (n, c) \|_{L^p_t(B_{q,1}^{2+3/q})} \| (n, c) \|_{L^p_t(B_{q,1}^{2+3/q})}.
\]

**Proof.** Let \((N, \Phi) := (e^{\lambda \sqrt{\Delta} A} n, e^{\lambda \sqrt{\Delta} A} \phi)\). Thanks to Bony’s paraproduct decomposition (6), we have

\[
e^{\lambda \sqrt{\Delta}} \Delta_j (n \nabla \phi) = e^{\lambda \sqrt{\Delta}} \Delta_j (T_{e^{-\lambda \sqrt{\Delta} A} N} \nabla e^{-\lambda \sqrt{\Delta} A} \phi) \\
+ R(e^{-\lambda \sqrt{\Delta} A} N, \nabla e^{-\lambda \sqrt{\Delta} A} \phi) + T_{\nabla e^{-\lambda \sqrt{\Delta} A} e^{-\lambda \sqrt{\Delta} A} N},
\]

Taking advantage of Lemmas 2.1 and 2.2 and (4), note that \( r_1 \geq 2 \), we thus have

\[
\| \Delta_j e^{\lambda \sqrt{\Delta} A} (T_{e^{-\lambda \sqrt{\Delta} A} N} \nabla e^{-\lambda \sqrt{\Delta} A} \phi) \|_{L^p_t(L^q)} \\
\lesssim \sum_{|j-j'| \leq 4, k \leq j'-2} 2^{j'k} \| \Delta_k N \|_{L^p_t(L^q)} \| \Delta_j' \phi \|_{L^q_t(L^\infty)} \\
\lesssim \sum_{|j-j'| \leq 4, k \leq j'-2} 2^{j'(1-2/r_2+3/q)} d_{j'} \| N \|_{L^p_t(B_{q,1}^{2+3/q})} \| N \|_{L^p_t(B_{q,1}^{2+3/q})} \| N \|_{L^p_t(B_{q,1}^{2+3/q})} \| N \|_{L^p_t(B_{q,1}^{2+3/q})} \\
\lesssim 2^{j(1-3/q)} d_{j'} \| N \|_{L^p_t(B_{q,1}^{2+3/q})} \| N \|_{L^p_t(B_{q,1}^{2+3/q})} \| N \|_{L^p_t(B_{q,1}^{2+3/q})} \| N \|_{L^p_t(B_{q,1}^{2+3/q})} \| N \|_{L^p_t(B_{q,1}^{2+3/q})}.
\]
It follows from Lemmas 2.1 and 2.2 , $r_1 \geq 2$ and (4) that
\[
\| \Delta J e^{\lambda \nabla T A} (T e^{-\lambda \nabla T A} N e^{-\lambda \nabla T A} \nabla \Phi) \|_{L^r_\lambda(L^q)} \\
\lesssim \sum_{|j-j'| \leq 4} \sum_{k \leq j'-2} 2^{k+\frac{3}{q}} \| \Delta_k \Phi \|_{L^q_\lambda(L^q)} \| \Delta_j \n \|_{L^2_\lambda(L^q)} \\
\lesssim \sum_{|j-j'| \leq 4} 2^{j'(-3/2+2/\nu)} 2^{k(1-2/r_1)} \times \| N \|_{L^2_\lambda(B_{q,1}^{2+3/q+2/\nu})} 2^{k(1/2+\frac{3}{q})} \| \Delta_k \Phi \|_{L^q_\lambda(L^q)} \\
\lesssim 2^{j(1-3/q)} d_j \| N \|_{L^2_\lambda(B_{q,1}^{2+3/q})} \| \Phi \|_{L^2_\lambda(B_{q,1}^{2+3/q})} \| \Phi \|_{L^2_\lambda(B_{q,1}^{2+3/q})}.
\]

In the case of $2 \leq q < 6$, let $\frac{1}{q} + \frac{1}{q} = 1$, applying Lemmas 2.1 and 2.2 and (4), for some constant $N_0$, we see that
\[
\| \Delta J e^{\lambda \nabla T A} R(e^{-\lambda \nabla T A} N, \nabla e^{-\lambda \nabla T A} \Phi) \|_{L^r_\lambda(L^q)} \\
\lesssim 2^{3j/q} \sum_{j' \geq j-N_0} \| \Delta_j \Phi \|_{L^q_\lambda(L^q)} \| \Delta_j \n \|_{L^2_\lambda(L^q)} \\
\lesssim 2^{3j/q} \sum_{j' \geq j-N_0} 2^{j'(1-6/q)} \| N \|_{L^2_\lambda(B_{q,1}^{2+3/q})} \| \Phi \|_{L^2_\lambda(B_{q,1}^{2+3/q})} \\
\lesssim 2^{3j(1-3/q)} d_j \| N \|_{L^2_\lambda(B_{q,1}^{2+3/q})} \| \Phi \|_{L^2_\lambda(B_{q,1}^{2+3/q})} \| \Phi \|_{L^2_\lambda(B_{q,1}^{2+3/q})}.
\]

In the case of $1 \leq q < 2$, there exists $2 < \lambda_0 \leq \infty$ such that $1/q + 1/\lambda_0 = 1$, from Lemmas 2.1 and 2.2 and (4), we see that
\[
\| \Delta J e^{\lambda \nabla T A} R(e^{-\lambda \nabla T A} N, \nabla e^{-\lambda \nabla T A} \Phi) \|_{L^r_\lambda(L^q)} \\
\lesssim 2^{3j(1-1/q)} \sum_{j' \geq j-N_0} \| \Delta_j \Phi \|_{L^q_\lambda(L^q)} \| \Delta_j \n \|_{L^2_\lambda(L^\lambda_0)} \\
\lesssim 2^{3j(1-1/q)} \sum_{j' \geq j-N_0} 2^{j'(1/q-1/\lambda_0)} \| N \|_{L^2_\lambda(B_{q,1}^{2+3/q})} \| \Phi \|_{L^2_\lambda(B_{q,1}^{2+3/q})} \| \Phi \|_{L^2_\lambda(B_{q,1}^{2+3/q})}.
\]

Combining the above estimates (19)-(22) shows that
\[
\| \nabla \cdot (\nabla \Phi) \|_{L^2_\lambda(e^{\lambda \nabla T A} B_{q,1}^{2+3/q})} \\
\lesssim \| \n \|_{L^2_\lambda(B_{q,1}^{2+3/q})} \| \n \|_{L^2_\lambda(B_{q,1}^{2+3/q})} \| \Phi \|_{L^2_\lambda(B_{q,1}^{2+3/q})} \\
+ \| \n \|_{L^2_\lambda(B_{q,1}^{2+3/q})} \| \n \|_{L^2_\lambda(B_{q,1}^{2+3/q})} \| \Phi \|_{L^2_\lambda(B_{q,1}^{2+3/q})} \\
\lesssim \| \n \|_{L^2_\lambda(B_{q,1}^{2+3/q})} \| \n \|_{L^2_\lambda(B_{q,1}^{2+3/q})} \| \n \|_{L^2_\lambda(B_{q,1}^{2+3/q})} \| \n \|_{L^2_\lambda(B_{q,1}^{2+3/q})} \| \Phi \|_{L^2_\lambda(B_{q,1}^{2+3/q})} \\
+ \| \n \|_{L^2_\lambda(B_{q,1}^{2+3/q})} \| \n \|_{L^2_\lambda(B_{q,1}^{2+3/q})} \| \n \|_{L^2_\lambda(B_{q,1}^{2+3/q})} \\
\lesssim \| \n \|_{L^2_\lambda(B_{q,1}^{2+3/q})} \| \n \|_{L^2_\lambda(B_{q,1}^{2+3/q})} \| \n \|_{L^2_\lambda(B_{q,1}^{2+3/q})} \| \n \|_{L^2_\lambda(B_{q,1}^{2+3/q})} \| \n \|_{L^2_\lambda(B_{q,1}^{2+3/q})}.
\]

Since $\Delta \Phi = n - c$, we have (17) from Lemma 2.1 and (23). This completes the proof of Lemma 2.5.
Lemma 2.6. Let \(1/r_1 + 1/r_2 = 1, r_1 \geq \max\{2, \frac{2pq}{pq + 3p - 3q}\}\). Let \(1 < p < q \leq 2p, \frac{1}{p} - \frac{1}{q} \leq \frac{1}{3}\). Assume that \(f, g \in \mathcal{Y}_T\), then we have

\[
\|f \nabla (-\Delta)^{-1} g + g \nabla (-\Delta)^{-1} f\|_{L^{2}_{T}(e^{\lambda \sqrt{T} A} \dot{B}_{q,1}^{0})} \\
\lesssim \|f\|_{L^{2}_{T}(e^{\lambda \sqrt{T} A} \dot{B}_{q,1}^{-2+3/q+2/r_1})} \|g\|_{L^{2}_{T}(e^{\lambda \sqrt{T} A} \dot{B}_{q,1}^{-2+3/q+2/r_2})} \\
+ \|f\|_{L^{2}_{T}(e^{\lambda \sqrt{T} A} \dot{B}_{q,1}^{-2+3/q+2/r_2})} \|g\|_{L^{2}_{T}(e^{\lambda \sqrt{T} A} \dot{B}_{q,1}^{-2+3/q+2/r_1})} \\
\lesssim \|f\|_{L^{2}_{T}(e^{\lambda \sqrt{T} A} \dot{B}_{q,1}^{-2+3/q+3})} \|g\|_{L^{2}_{T}(e^{\lambda \sqrt{T} A} \dot{B}_{q,1}^{-2+3/q+3})} \\
+ \|f\|_{L^{2}_{T}(e^{\lambda \sqrt{T} A} \dot{B}_{q,1}^{-2+3/q+3})} \|g\|_{L^{2}_{T}(e^{\lambda \sqrt{T} A} \dot{B}_{q,1}^{-2+3/q+3})} \\
(24)
\]

Proof. Let \((F, G) := (e^{\lambda \sqrt{T} A} f, e^{\lambda \sqrt{T} A} g)\). Thanks to Bony’s paraproduct decomposition (6), we have

\[
\Delta_j e^{\lambda \sqrt{T} A} \left(f \nabla (-\Delta)^{-1} g + g \nabla (-\Delta)^{-1} f\right) \\
= \sum_{|j'| - j| \leq 4} e^{\lambda \sqrt{T} A} \left(\Delta_j \Delta_{j'} e^{-\lambda \sqrt{T} A} F \nabla (-\Delta)^{-1} S_j e^{-\lambda \sqrt{T} A} G \\
+ \Delta_j S_{j'} e^{-\lambda \sqrt{T} A} F \nabla (-\Delta)^{-1} \Delta_{j'} e^{-\lambda \sqrt{T} A} G \right) \sum_{|j'-j| \leq 4} \left(\Delta_j \Delta_{j'} e^{-\lambda \sqrt{T} A} G \nabla (-\Delta)^{-1} S_j e^{-\lambda \sqrt{T} A} F \\
+ \Delta_j S_{j'} e^{-\lambda \sqrt{T} A} G \nabla (-\Delta)^{-1} \Delta_{j'} e^{-\lambda \sqrt{T} A} F \right) \\
+ \sum_{|j'| - j| \leq 4} \sum_{|k' - j'| \leq 1} \left(\Delta_j \Delta_{j'} e^{-\lambda \sqrt{T} A} F \Delta_k \nabla (-\Delta)^{-1} g \\
+ \Delta_j \Delta_{j'} e^{-\lambda \sqrt{T} A} G \Delta_k \nabla (-\Delta)^{-1} e^{-\lambda \sqrt{T} A} F \right) \\
:= I_1 + I_2 + I_3.
\]

For \(I_1\), choosing \(1 \leq r_1, r_2 \leq \infty\), such that \(\frac{1}{r_2} + \frac{1}{r_1} = 1\), note that \(r_1 \geq 2\), applying (4) and Lemmas 2.1 and 2.2 gives rise to

\[
\|I_1\|_{L^{2}_{T}(L^{p})} \leq \sum_{|j'-j| \leq 4} \|\Delta_j' F\|_{L^{2}_{T}(L^{p})} \sum_{k \leq j' - 2} 2^{k(1 + \frac{3}{4} - \frac{1}{r_2} - \frac{3}{p})} 2^{k(-2 + \frac{3}{4} + \frac{3}{p})} \|\Delta_k G\|_{L^{2}_{T}(L^{q})} \\
+ \sum_{|j'-j| \leq 4} 2^{-j'} \|\Delta_j' G\|_{L^{2}_{T}(L^{p})} \sum_{k \leq j' - 2} 2^{k(2 + \frac{3}{4} - \frac{3}{p} - \frac{1}{r_2})} 2^{k(-2 + \frac{3}{4} + \frac{3}{p})} \|\Delta_k F\|_{L^{2}_{T}(L^{q})} \\
\leq \sum_{|j'-j| \leq 4} \|\Delta_j' F\|_{L^{2}_{T}(L^{p})} 2^{j'(1 + \frac{3}{4} - \frac{1}{r_2} - \frac{3}{p})} \|G\|_{L^{2}_{T}(B_{q,1}^{2+3/q+2/r_1})} \\
+ \sum_{|j'-j| \leq 4} 2^{-j'} 2^{j'(2 + \frac{3}{4} - \frac{3}{p} - \frac{1}{r_2})} \|\Delta_j' G\|_{L^{2}_{T}(L^{p})} \|F\|_{L^{2}_{T}(B_{q,1}^{2+3/q+2/r_1})} \\
\lesssim 2^{j'(1 - \frac{3}{2})} d_{j'} \|f\|_{L^{2}_{T}(e^{\lambda \sqrt{T} A} \dot{B}_{q,1}^{-2+3/q+3})} \|g\|_{L^{2}_{T}(e^{\lambda \sqrt{T} A} \dot{B}_{q,1}^{-2+3/q+3})} \\
+ \|f\|_{L^{2}_{T}(e^{\lambda \sqrt{T} A} \dot{B}_{q,1}^{-2+3/q+3})} \|g\|_{L^{2}_{T}(e^{\lambda \sqrt{T} A} \dot{B}_{q,1}^{-2+3/q+3})}.
\]

(26)
For the estimate of the second term $I_2$, using an argument similar to the one used in $I_1$, we obtain immediately

$$
\|I_2\|_{L^2_p(L^p)} \lesssim 2^{(1 - \frac{1}{p})} d_j \|f\|_{\Sigma^2_p(e^{\lambda \psi_T}B_{q,1}^{-2+3/q+2/r_1})} \|g\|_{\Sigma^2_p(e^{\lambda \psi_T}B_{q,1}^{-2+3/q+2/r_1})} \tag{27}
$$

where

$$
\begin{align*}
I_{31} &:= \sum_{k \geq j - 2} \sum_{|k'| \leq k} \Delta_j e^{\lambda \psi_T} \Delta_k \left( (-\Delta)^{-1} \Delta_k e^{-\lambda \psi_T} F \right) \left( \partial_m (-\Delta)^{-1} \Delta_k e^{-\lambda \psi_T} G \right), \\
I_{32} &:= \sum_{k \geq j - 2} \sum_{|k'| \leq k} 2 \Delta_j e^{\lambda \psi_T} \nabla \cdot \left( (-\Delta)^{-1} \Delta_k e^{-\lambda \psi_T} F \right) \left( \partial_m \nabla (-\Delta)^{-1} \Delta_k e^{-\lambda \psi_T} G \right), \\
I_{33} &:= \sum_{k \geq j - 2} \sum_{|k'| \leq k} \Delta_j e^{\lambda \psi_T} \partial_m \left( (-\Delta)^{-1} \Delta_k e^{-\lambda \psi_T} F \right) \left( \Delta_k e^{-\lambda \psi_T} G \right). \tag{28}
\end{align*}
$$

Moreover, since $I_{32}$ can be treated similarly to $I_{33}$, we treat $I_{31}$ and $I_{33}$ only. We first consider $I_{31}$,

$$
\begin{align*}
||I_{31}\|_{L^2_p(L^p)} &\leq 2^{2j} \sum_{k \geq j - 2} \sum_{|k'| \leq k} \left( (-\Delta)^{-1} \Delta_k F \right) \left( \partial_m (-\Delta)^{-1} \Delta_k G \right) \|L^2_p(L^p)\| \\
&\leq 2^{2j} \sum_{k \geq j - 2} \sum_{|k'| \leq k} 2^{k(-2 + \frac{q}{p} - \frac{3}{2})} \|\Delta_k F\|_{L^2_p(L^p)} 2^{-k'} \|\Delta_k G\|_{L^2_p(L^p)} \\
&\leq 2^{2j} \sum_{k \geq j - 2} 2^{k(-2 + \frac{q}{p} - \frac{3}{2})} 2^{k(-2 + \frac{q}{p} + \frac{2}{r_1})} \|\Delta_k F\|_{L^2_p(L^p)} 2^{k(-2 + \frac{q}{p} + \frac{2}{r_1})} \|\Delta_k G\|_{L^2_p(L^p)} \\
&\lesssim 2^{(1 - \frac{1}{p})} d_j \|f\|_{\Sigma^2_p(e^{\lambda \psi_T}B_{q,1}^{-2+3/q+2/r_1})} \|g\|_{\Sigma^2_p(e^{\lambda \psi_T}B_{q,1}^{-2+3/q+2/r_1})} \quad (29)
\end{align*}
$$

By Lemmas 2.1 and 2.2, we have

$$
\begin{align*}
||I_{33}\|_{L^2_p(L^p)} &\leq 2^{2j} \sum_{k \geq j - 2} \sum_{|k'| \leq k} \left( (-\Delta)^{-1} \Delta_k F \right) \left( \Delta_k G \right) \|L^2_p(L^p)\| \\
&\leq 2^{2j} \sum_{k \geq j - 2} \sum_{|k'| \leq k} 2^{k(-2 + \frac{q}{p} - \frac{3}{2})} \|\Delta_k F\|_{L^2_p(L^p)} \|\Delta_k G\|_{L^2_p(L^p)} \\
&\leq 2^{2j} \sum_{k \geq j - 2} 2^{-k' \frac{3}{2}} 2^{k(-2 + \frac{q}{p} + \frac{2}{r_1})} \|\Delta_k F\|_{L^2_p(L^p)} 2^{k(-2 + \frac{q}{p} + \frac{2}{r_1})} \|\Delta_k G\|_{L^2_p(L^p)} \\
&\lesssim 2^{(1 - \frac{1}{p})} d_j \|f\|_{\Sigma^2_p(e^{\lambda \psi_T}B_{q,1}^{-2+3/q+2/r_1})} \|g\|_{\Sigma^2_p(e^{\lambda \psi_T}B_{q,1}^{-2+3/q+2/r_1})} \quad (30)
\end{align*}
$$

Summing over $m = 1, 2, 3$, we obtain the estimate of $I_3$,

$$
\|I_3\|_{L^2_p(L^p)} \lesssim 2^{(1 - \frac{1}{p})} d_j \|f\|_{\Sigma^2_p(e^{\lambda \psi_T}B_{q,1}^{-2+3/q+2/r_1})} \|g\|_{\Sigma^2_p(e^{\lambda \psi_T}B_{q,1}^{-2+3/q+2/r_1})}. \tag{31}
$$

Thus we obtain Lemma 2.6 by (26), (27) and (31). \(\square\)

**Lemma 2.7.** Let $1 < p < \infty$. Assume that $u \in \mathcal{X}_T$ and $\nabla \cdot u = 0$, we obtain

$$
\|u \cdot \nabla u\|_{L^2_p(e^{\lambda \psi_T}B_{p,1}^{-1+3/p}(\mathbb{R}^3))} \lesssim \|u\|_{\mathcal{X}_T} \|u\|_{\mathcal{X}_T}. \tag{32}
$$
Proof. By interpolation (4), the algebra property (5) and Lemma 2.2, the proof of Lemma 2.7 is a similar argument as the proof of Lemmas 2.4-2.6. Here, we omit the details.

3. Proof of Theorem 1.1.

Proposition 1 ([10]). Let \((\chi, \| \cdot \|)\) be a Banach space, \(B : \chi \times \chi \to \chi\) a bilinear operator with norm \(K\) and \(L : \chi \to \chi\) a continuous operator with norm \(M < 1\). Let \(y \in \chi\) satisfy \(4K\|y\|_\chi < (1 - M)^2\). Then the equation \(u = y + L(u) + B(u, u)\) has a unique solution in the ball \(B(0, \frac{1 - M}{2K})\).

Proposition 2. Let \(u_L = e^{t\Delta}u_0, n_L = e^{t\Delta}n_0\) and \(c_L = e^{t\Delta}c_0\). \((u, n, c)\) is a mild solution of system (1) on \([0, T] \times \mathbb{R}^3\) with initial data \((u_0, n_0, c_0)\) if and only if \((u, n, c) = (u_L + \bar{u}, n_L + \bar{n}, c_L + \bar{c})\) with

\[
\bar{u} = -\int_0^t e^{(t-s)\Delta}P(u_L \cdot \nabla u_L - \Delta \phi_L \nabla \phi_L)(\cdot, s)ds
- \int_0^t e^{(t-s)\Delta}P(u_L \cdot \nabla \bar{u} + \bar{u} \cdot \nabla u_L + \Delta \phi_L \nabla \bar{\phi} + \Delta \bar{\phi} \nabla \phi_L)(\cdot, s)ds
- \int_0^t e^{(t-s)\Delta}P(\bar{u} \cdot \nabla \bar{u} + \bar{\phi} \nabla \bar{\phi}')(\cdot, s)ds
:= y_1 + L_1((\bar{u}, \bar{n}, \bar{c})) + B_1((\bar{u}, \bar{n}, \bar{c})),
\]

\[
\bar{n} = -\int_0^t e^{(t-s)\Delta}(u_L \cdot \nabla n_L + \nabla \cdot (n_L \cdot \nabla \phi_L))(\cdot, s)ds
- \int_0^t e^{(t-s)\Delta}(\bar{u} \cdot \nabla \bar{n} + \bar{n} \cdot \nabla u_L + \nabla \cdot (n_L \cdot \nabla \bar{\phi} + \Delta \bar{\phi})
+ \nabla \cdot (\bar{n} \cdot \nabla \phi_L))(\cdot, s)ds
\]

\[
:= y_2 + L_2((\bar{u}, \bar{n}, \bar{c})), B_2((\bar{u}, \bar{n}, \bar{c})), (\bar{u}, \bar{n}, \bar{c}),
\]

\[
\bar{c} = -\int_0^t e^{(t-s)\Delta}(u_L \cdot \nabla c_L
- \nabla \cdot (c_L \cdot \nabla \phi_L))(\cdot, s)ds
\]

\[
- \int_0^t e^{(t-s)\Delta}(u_L \cdot \nabla \bar{c} + \bar{u} \cdot \nabla c_L - \nabla \cdot (c_L \cdot \nabla \bar{\phi})
- \nabla \cdot (\bar{c} \cdot \nabla \phi_L))(\cdot, s)ds
\]

\[
:= y_3 + L_3((\bar{u}, \bar{n}, \bar{c})), B_3((\bar{u}, \bar{n}, \bar{c})), (\bar{u}, \bar{n}, \bar{c}),
\]

\[
(\bar{u}, \bar{n}, \bar{c})|_{t=0} = (0, 0, 0), \quad \Delta \phi_L = n_L - c_L, \quad \Delta \bar{\phi} = \bar{n} - \bar{c}.
\]

(33)

Proof. The proof of Proposition 2 is easy, so we skip it here.

Applying Lemma 2.3, we have

\[
\|(u_L, n_L, c_L)\|_{L^\infty([0, T]; e^{2\Delta} \hat{B}^{1+3/p}_{p,1}) \times L^\infty([0, T]; e^{2\Delta} \hat{B}^{-2+3/q}_{q,1}) \times L^\infty([0, T]; e^{2\Delta} \hat{B}^{-2+3/q}_{q,1})}
\leq C\|(u_0, n_0, c_0)\|_{\hat{B}^{1+3/p}_{p,1} \times \hat{B}^{-2+3/q}_{q,1} \times \hat{B}^{-2+3/q}_{q,1}}
\]
and
\[
\lim_{T \to 0} \|(u_L, n_L, c_L)\|_{L^1([0,T];e^{\lambda \sqrt{T} \Lambda} \dot{B}^{1+3/p}_{p,1}) \times L^1([0,T];e^{\lambda \sqrt{T} \Lambda} \dot{B}^{3/q}_{q,1}) \times L^1([0,T];e^{\lambda \sqrt{T} \Lambda} \dot{B}^{3/q}_{q,1})} = 0,
\]
thus, from above let \( \eta > 0 \) be a sufficiently small constant and we can define
\[
\xi_1 = \sup \left\{ T_1 > 0 : \| u_L \|_{L^1([0,T];e^{\lambda \sqrt{T} \Lambda} \dot{B}^{1+3/p}_{p,1}) \leq \eta} \right\},
\]
\[
\xi_2 = \sup \left\{ T_1 > 0 : \| n_L \|_{L^1([0,T];e^{\lambda \sqrt{T} \Lambda} \dot{B}^{3/q}_{q,1})} \leq \eta \right\},
\]
\[
\xi_3 = \sup \left\{ T_1 > 0 : \| c_L \|_{L^1([0,T];e^{\lambda \sqrt{T} \Lambda} \dot{B}^{3/q}_{q,1})} \leq \eta \right\}.
\]
We can choose \( \xi = \min \{ \xi_1, \xi_2, \xi_3 \} \) and we can take \( \eta \) sufficiently small and \( T \leq \xi \).

Let
\[
F_T = L^1([0,T];e^{\lambda \sqrt{T} \Lambda} \dot{B}^{1+3/p}_{p,1}) \times L^1([0,T];e^{\lambda \sqrt{T} \Lambda} \dot{B}^{3/q}_{q,1}) \times L^1([0,T];e^{\lambda \sqrt{T} \Lambda} \dot{B}^{3/q}_{q,1}).
\]

Therefore, by Lemmas 2.4-2.7, we conclude that
\[
\| B \|_\Theta_T = \|(B_1, B_2, B_3)((\bar{u}, \bar{n}, \bar{c}), (\bar{u}, \bar{n}, \bar{c}))\|_\Theta_T \]
\[
\leq \| (\bar{u}_L \cdot \nabla \bar{u} + \bar{u} \cdot \nabla u_L + \Delta \phi_L \nabla \bar{\phi}, \bar{u} \cdot \nabla \bar{n} + \nabla \cdot (\bar{n} \cdot \nabla \bar{\phi}, \bar{u} \cdot \nabla \bar{c} - \nabla \cdot (\bar{c} \cdot \nabla \bar{\phi}))\|_{F_T}
\]
\[
\leq K\|(\bar{u}, \bar{n}, \bar{c})\|_\Theta^2_T.
\]
On the other hand, and noting that \( 1/r_1 + 1/r_2 = 1 \), we can take
\[
\max \{2, \frac{2pq}{pq + 3p - 3q} \} \leq r_1 < \infty.
\]
Using Lemmas 2.4-2.7, we deduce that
\[
\| L \|_\Theta_T = \|(L_1, L_2, L_3)(\bar{u}, \bar{n}, \bar{c})\|_\Theta_T \]
\[
\leq \| (u_L \cdot \nabla \bar{u} + \bar{u} \cdot \nabla u_L + \Delta \phi_L \nabla \bar{\phi}, u_L \cdot \nabla \bar{n} + \nabla \cdot (n_L \cdot \nabla \phi), u_L \cdot \nabla \bar{c} - \nabla \cdot (c_L \cdot \nabla \phi))\|_{F_T}
\]
\[
\leq \tilde{C}\|(\bar{u}, \bar{n}, \bar{c})\|_\Theta_T \left( \eta^{1/r_1} + \eta^{1/r_2} \right) \left( \| u_0 \|_{\dot{B}^{1+3/p}_{p,1}}^{1-\frac{1}{r_2}} + \| u_0 \|_{\dot{B}^{3/q}_{q,1}}^{1-\frac{1}{r_2}} + \| u_0 \|_{\dot{B}^{3/q}_{q,1}}^{1-\frac{1}{r_2}} \right)
\]
\[
+ \| n_0 \|_{\dot{B}^{1+3/p}_{p,1}}^{1-\frac{1}{r_2}} + \| c_0 \|_{\dot{B}^{3/q}_{q,1}}^{1-\frac{1}{r_2}} \right)
\]
\[
:= M_1\|(\bar{u}, \bar{n}, \bar{c})\|_\Theta_T,
\]
where we can choose \( M_1 < 1 \), if \( \eta, T \) small enough and \( \max \{2, \frac{2pq}{pq + 3p - 3q} \} \leq r_1 < \infty \).

Thanks to Lemmas 2.4-2.7, we get
\[
\| y \|_\Theta_T = \|(y_1, y_2, y_3)\|_\Theta_T \]
\[
\leq \| (u_L \cdot \nabla u_L + \Delta \phi_L \nabla \phi_L, u_L \cdot \nabla n_L + \nabla \cdot (n_L \cdot \nabla \phi_L), u_L \cdot \nabla c_L - \nabla \cdot (c_L \cdot \nabla \phi_L))\|_{F_T}
\]
\[
\leq \| u_L \|_{L^2([0,T];e^{\lambda \sqrt{T} \Lambda} \dot{B}^{1+3/p}_{p,1})} \| u_L \|_{L^2([0,T];e^{\lambda \sqrt{T} \Lambda} \dot{B}^{3/q}_{q,1})}
\]
\[
+ \| n_L \|_{L^2([0,T];e^{\lambda \sqrt{T} \Lambda} \dot{B}^{3/q}_{q,1})} \| n_L \|_{L^2([0,T];e^{\lambda \sqrt{T} \Lambda} \dot{B}^{3/q}_{q,1})}
\]
\[
+ \| c_L \|_{L^2([0,T];e^{\lambda \sqrt{T} \Lambda} \dot{B}^{3/q}_{q,1})} \| c_L \|_{L^2([0,T];e^{\lambda \sqrt{T} \Lambda} \dot{B}^{3/q}_{q,1})}
\]
\[
+ \| n_L \|_{L^2([0,T];e^{\lambda \sqrt{T} \Lambda} \dot{B}^{3/q}_{q,1})} \| n_L \|_{L^2([0,T];e^{\lambda \sqrt{T} \Lambda} \dot{B}^{3/q}_{q,1})}
\]
\[
\leq \tilde{C} \eta \|(u_0, n_0, c_0)\|_{E_0}.
\]
Proof of local Gevrey regularity of Theorem 1.1. We can take \( \eta \) and \( T \) small enough such that \( \dot{C} \eta \| (u_0, n_0, c_0) \|_{E_0} \leq \frac{(1 - M)^2}{4K} \) for \( i = 1 \) or \( i = 2 \), applying Proposition 1, we thus get that there exists a positive time \( T > 0 \) such that the system (33) has a unique solution \((\tilde{u}, \tilde{n}, \tilde{c})\) on \([0, T]\). Therefore, applying Proposition 2 yields that the local Gevrey regularity of solution to the system (1).

4. Proof of Corollary 1: Decay of Besov norms and Lebesgue norms.

Theorem 1.1 tells us that the solution is locally in the Gevrey regular, that is, the energy bound \( \| (u, n, c) \|_{\Theta} < \infty \) for \( \lambda = 1 \). Specifically, we can show that a solution \((u, n, c)\) satisfies

\[
\sup_{t > 0} \left\| \left( e^{\sqrt{t} \Lambda} u, e^{\sqrt{t} \Lambda} n, e^{\sqrt{t} \Lambda} c \right) \right\|_{\dot{B}^{-1-3/p, 2+3/q}_p \times \dot{B}^{-2+3/q}_q} < +\infty. \tag{40}
\]

As we have showed the solution \((u, n, c)\) of system (1) in Gevrey classes for which the estimate (40) holds, thanks to (3), (40) and using the same argument as Corollary 3.3 in [1], for \( m > 0 \), there exists positive constant \( C \) such that

\[
\| (D^m u, D^m n, D^m c) \|_{\dot{B}^{-1-3/p, 2+3/q}_p \times \dot{B}^{-2+3/q}_q} = \| (D^m e^{\sqrt{t} \Lambda} u, D^m e^{\sqrt{t} \Lambda} n, D^m e^{\sqrt{t} \Lambda} c) \|_{\dot{B}^{-1-3/p, 2+3/q}_p \times \dot{B}^{-2+3/q}_q} \\
\leq C m m t^{-\frac{3}{2}} \left\| \left( e^{\sqrt{t} \Lambda} u, e^{\sqrt{t} \Lambda} n, e^{\sqrt{t} \Lambda} c \right) \right\|_{L_t^\infty \dot{B}^{-1-3/p, 2+3/q}_p \times L_t^\infty \dot{B}^{-2+3/q}_q} \\
\leq C m m t^{-\frac{3}{2}} \| e^{\sqrt{t} \Lambda} u, e^{\sqrt{t} \Lambda} n, e^{\sqrt{t} \Lambda} c \|_{L_t^\infty \dot{B}^{-1-3/p, 2+3/q}_p \times L_t^\infty \dot{B}^{-2+3/q}_q} \\
\lesssim C m m t^{-\frac{3}{2}}. \tag{41}
\]

By using the relation between homogeneous Besov spaces and homogeneous Triebel-Lizorkin spaces \( \dot{F}^s_{p_0, p} \) (the definition of Triebel-Lizorkin spaces, see [9]), note that \( \ell^{p_0} \rightarrow \ell^2 \) for \( p_0 \leq 2 \) and \( \dot{F}^s_{p_0, 2} = \dot{W}^{s, p_0} := (-\Delta)^{-s/2} L^{p_0} \), one thus has

\[
\dot{B}^s_{p_0, 1} \hookrightarrow \dot{F}^s_{p_0, p_0} \hookrightarrow \dot{F}^s_{p_0, p} \hookrightarrow \dot{F}^s_{p_0, 2} = \dot{W}^{s, p_0}. \tag{42}
\]

Applying (42) and (40), along the same line with (41), for \( k_1 \geq -1 + 3/p \) and \( 1 < p \leq 2 \), there exists a positive constant \( C_1 \) so that

\[
\| D^{k_1} u \|_{L^p} = \| D^{k_1+1-3/p} e^{\sqrt{t} \Lambda} D^{-1+3/p} e^{\sqrt{t} \Lambda} u \|_{L^p} \\
\leq C_1^1 + 1 - 3/p \| D^{k_1+1-3/p} e^{\sqrt{t} \Lambda} u \|_{L^p} \\
= C_1^1 + 1 - 3/p \| D^{k_1+1-3/p} e^{\sqrt{t} \Lambda} u \|_{L^p} \\
\leq C t^{1 + 1 / 2} \| e^{\sqrt{t} \Lambda} u \|_{\dot{B}^{-1-3/p}_p} \\
\lesssim C_1^1 + 1 - 3/p \| e^{\sqrt{t} \Lambda} u \|_{\dot{B}^{-1-3/p}_p}. \tag{43}
\]

There exist positive constants \( C_2 \) and \( C_3 \), for \( k_2 \geq 2 + 3/q, 1 < q \leq \frac{3}{2} \), whence a similar argument as that in (43) gives rise to

\[
\| D^{k_2} u \|_{L^q} \lesssim C_2^{k_2 + 2 - 3/q} (k_2 + 2 - 3/q) k_2 + 2 - 3/q, \\
\| D^{k_2} u \|_{L^q} \lesssim C_2^{k_2 + 2 - 3/q} (k_2 + 2 - 3/q) k_2 + 2 - 3/q. \tag{44}
\]

We complete the proof of Corollary 1.
5. Proof of Theorem 1.2: Global existence with large vertical velocity component. The goal of this section is to present the proof of the global existence of system (1) with large vertical velocity component. Let \( u = (u^1, u^2, u^3) = (u^h, u^c) \) and \( \text{div}_h u^h = \partial_1 u^1 + \partial_2 u^2 \).

Lemma 5.1 ([39]). Let \( 1 < p < 1 \) and \( \alpha \in (0, \infty) \). One obtains
\[
\| \Delta_j (u^3 u^h) \|_{L^1_T(L^p)} \leq d_j 2^{-3j/p} (\| u^h \|_{L^\infty(B_{p,1}^{1+3/p})} \| u^h \|^{1/p}_{L^1(B_{p,1}^{1+3/p})}) \times \| u^3 \|^{1-1/p}_{L^1(B_{p,1}^{1+3/p})} + \| u^h \|^{1+1/p}_{L^1(B_{p,1}^{1+3/p})} \| u^3 \|^{1-1/p}_{L^1(B_{p,1}^{1+3/p})}. \tag{45}
\]

Lemma 5.2 ([39]). Assume that \( 1 < p < 6 \), we then have
\[
\| \Delta_j (u^3 \text{div}_h u^h) \|_{L^1_T(L^p)} \quad \leq \quad d_j 2^{j(1-3/p)} (\| u^h \| \| u^h \|^{1+1/p}_{L^1(B_{p,1}^{1+3/p})} \| u^3 \|^{1-1/p}_{L^1(B_{p,1}^{1+3/p})} + \| u^3 \|^{1+1/p}_{L^1(B_{p,1}^{1+3/p})} \| u^3 \|^{1-1/p}_{L^1(B_{p,1}^{1+3/p})}). \tag{46}
\]

wherelse
\[
\alpha = \begin{cases} 1/p, & 1 < p < 5, \\ \varepsilon, & 5 \leq p < 6. \end{cases}
\]

for \( 0 < \varepsilon < \frac{6}{p} - 1 \).

- The estimate of the pressure \( P \).

Taking div to the first equation of system (1) yields that
\[
-\Delta P = \text{div}_h \text{div}_h (u^h \otimes u^h) + 2 \partial_3 \text{div}_h (u^3 u^h) + \partial_3^2 (u^3) + \text{div}(n - c)\nabla (-\Delta)^{-1} (n - c) \tag{47}
\]

By virtue of \( \nabla \cdot u = 0 \), it is clear that
\[
\nabla P = \nabla (-\Delta)^{-1} [\text{div}_h \text{div}_h (u^h \otimes u^h) + 2 \partial_3 \text{div}_h (u^3 u^h) + 2 \partial_3 (u^3 \text{div}_h u^h) + \text{div}(n - c)\nabla (-\Delta)^{-1} (n - c)]. \tag{48}
\]

Proposition 3. Let \( 1 < p < q \leq 2p \leq 12, \frac{1}{p} - \frac{1}{q} \leq \frac{1}{5} \). Let \( (u, n, c) \in \Theta_T \) with \( \nabla \cdot u = 0 \). Then (48) have unique solutions \( \nabla P \in L^q_T(B_{p,1}^{-1+3/p}) \) so that for \( t \in [0, T] \), there holds
\[
\| \nabla P \|_{L^q_T(B_{p,1}^{-1+3/p})} \leq C \| W(u^h, u^3) + \tilde{C} \| (n, c) \| L^q_T(B_{p,1}^{-1+3/p}) \| (n, c) \| L^q_T(B_{p,1}^{-1+3/p}). \tag{49}
\]

Proof. Applying the operator \( \Delta_j \) to (48), taking the \( L^1_T(L^p) \) norm, using Lemmas 2.6, 2.7, 5.1 and 5.2 yields that
\[
\| \Delta_j (\nabla P) \|_{L^1_T(L^p)} \quad \leq \quad 2^j \| \Delta_j (u^h \otimes u^h) \|_{L^1_T(L^p)} + 2^j \| \Delta_j (u^3 u^h) \|_{L^1_T(L^p)} + 2^j \| \Delta_j (u^3 \text{div}_h u^h) \|_{L^1_T(L^p)}
\]
\[
+ \| \Delta_j ((n - c)\nabla (-\Delta)^{-1} (n - c)) \|_{L^1_T(L^p)} \quad \leq \quad C 2^{j(1-3/p)} \| W(u^h, u^3) + C d_j 2^{j(1-3/p)} \| (n, c) \| L^q_T(B_{p,1}^{-1+3/p}) \| (n, c) \| L^q_T(B_{p,1}^{-1+3/p}), \tag{50}
\]

where
\[
W(u^h, u^3) := \| u^h \|_{L^\infty(B_{p,1}^{1+3/p})} \| u^h \|_{L^\infty(B_{p,1}^{1+3/p})} + \| u^h \|_{L^\infty(B_{p,1}^{1+3/p})} \| u^3 \|^{1-1/p}_{L^1(B_{p,1}^{1+3/p})} + \| u^h \|^{1/p}_{L^1(B_{p,1}^{1+3/p})} \| u^3 \|^{1-1/p}_{L^1(B_{p,1}^{1+3/p})} + \| u^3 \|^{1+1/p}_{L^1(B_{p,1}^{1+3/p})} \| u^3 \|^{1+1/p}_{L^1(B_{p,1}^{1+3/p})} + \| u^3 \|^{1+1/p}_{L^1(B_{p,1}^{1+3/p})} \| u^3 \|^{1+1/p}_{L^1(B_{p,1}^{1+3/p})} \tag{51}
\]
from which, the term (49) follows readily. Finally, these complete the proof of Proposition 3.

- The estimate of $u^h$ and $u^3$.

The first equation of (1) implies that

$$\partial_t u^h - \Delta u^h = -u \cdot \nabla u^h - \nabla_h p - (n - c)\nabla_h (-\Delta)^{-1}(n - c)$$

(52)

and

$$\partial_t u^3 - \Delta u^3 = -u \cdot \nabla u^3 - \partial_3 p - (n - c)\partial_3 (-\Delta)^{-1}(n - c).$$

(53)

**Proposition 4.** Let $1 < p < q \leq 2p < \infty$, $\frac{1}{p} + \frac{1}{q} > \frac{1}{2}$, $1 < q < 6$, $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{3}$. Let $(u, n, c) \in \Theta_T$ with $\nabla \cdot u = 0$. The equations (52) and (53) satisfy

$$\|u^h\|_{L^\infty(\bar{B}_{p,1}^{-1+3/p})} + \bar{c}\|u^h\|_{L^2(\bar{B}_{p,1}^{1+3/p})} \leq \|u_0^h\|_{L^p(\bar{B}_{p,1}^{-1+3/p})} + \bar{C}\mathcal{W}(u^h, u^3) + \bar{C}\|(n, c)\|_{L^\infty(\bar{B}_{q,2}^{-2+3/q})}||(n, c)||_{L^2(\bar{B}_{q,1}^{3/q})}$$

(54)

and

$$\|u^3\|_{L^\infty(\bar{B}_{p,1}^{-1+3/p})} + \bar{c}\|u^3\|_{L^2(\bar{B}_{p,1}^{1+3/p})} \leq \|u_0^3\|_{L^p(\bar{B}_{p,1}^{-1+3/p})} + \bar{C}\|u^h\|_{L^2(\bar{B}_{p,1}^{3/p})}||u^3||_{L^2(\bar{B}_{p,1}^{3/p})} + \bar{C}\mathcal{W}(u^h, u^3) + \bar{C}\|(n, c)\|_{L^\infty(\bar{B}_{q,2}^{-2+3/q})}||(n, c)||_{L^2(\bar{B}_{q,1}^{3/q})}.$$  

(55)

**Proof.** We first prove (54), applying the operator $\Delta_j$ to (52). Denote $T_{\varepsilon}(x) := \sqrt{x^2 + \varepsilon^2}$, taking $L^2$ inner product of the resulting equation with

$$(T_{\varepsilon}(\Delta_j u^h))^{p-1}T'_{\varepsilon}(\Delta_j u^h)$$

as that in [11], then let $\varepsilon \to 0$, an integration by parts yields

$$\frac{1}{p} \frac{d}{dt} \|\Delta_j u^h\|^p_{L^p} - \int_{\mathbb{R}^3} \Delta_j u^h \cdot |\Delta_j u^h|^{p-2}\Delta_j u^h dx = -\int_{\mathbb{R}^3} \Delta_j (u \cdot \nabla u^h + \nabla_h p + (n - c)\nabla_h (-\Delta)^{-1}(n - c))|\Delta_j u^h|^{p-2}\Delta_j u^h dx.$$  

(56)

Thanks to [11, 27], there exists a positive constant $\bar{c}_p$ fulfilling

$$-\int_{\mathbb{R}^3} \Delta_j u^h \cdot |\Delta_j u^h|^{p-2}\Delta_j u^h dx \geq \bar{c}_p 2^{2j}||\Delta_j u^h||_{L^p}^p.$$  

(57)

From (56) and (57), we thus infer the inequality

$$\frac{d}{dt} \|\Delta_j u^h\|_{L^p} + \bar{c}_p 2^{2j}||\Delta_j u^h||_{L^p} \lesssim ||\Delta_j (u \cdot \nabla u^h)||_{L^p} + ||\Delta_j \nabla_h p||_{L^p} + ||\Delta_j (n - c)\nabla_h (-\Delta)^{-1}(n - c)||_{L^p}.$$  

(58)
which after time integrating over \([0, t]\), applying Lemmas 2.6, 2.7, 5.1, 5.2 and Proposition 3 yields that

\[
\|\Delta_j u^h\|_{L^p_t(L^p)} + C2^{2j} \|\Delta_j u^h\|_{L^1_t(L^p)} \\
\lesssim \|\Delta_j u^h\|_{L^p} + 2^j \|\Delta_j (u^h \otimes u^h)\|_{L^1_t(L^p)} + 2^j \|\Delta_j (u^3 u^h)\|_{L^p} \\
+ \|\Delta_j \nabla_h p\|_{L^1_t(L^p)} + \|\Delta_j (n - c) \nabla_h (-\Delta)^{-1}(n - c)\|_{L^1_t(L^p)} \\
\lesssim \|\Delta_j u^h\|_{L^p} + C2^{j} 2^{-3/p} d_j \|\Delta_j (u^h)\|_{L^p(B_{p,1}^{1+3/p})} \|u^h\|_{\Sigma_F(B_{p,1}^{-1+3/p})} \\
+ \|u^h\|_{\Sigma_F(B_{p,1}^{1+3/p})} \|u^3\|_{L^p(B_{p,1}^{1+3/p})} \|u^h\|_{L^1_t(L^p)}^{1/p} \\
+ C2^{2j} 2^{-3/p} d_j \|\nabla u^h\|_{L^p(B_{p,1}^{1+3/p})}^{1/p} \\
+ C2^{j} 2^{(1-3/p)} d_j \|\nabla u^h, u^3\|_{L^p(B_{p,1}^{1+3/p})} \|u^3\|_{L^p(B_{p,1}^{1+3/p})} \|u^h\|_{L^1_t(L^p)}^{1/p} \\
\lesssim \|\Delta_j u^h\|_{L^p} + C2^{j} 2^{(1-3/p)} d_j \|\Delta_j (u^h)\|_{L^p(B_{p,1}^{1+3/p})} \|u^h\|_{\Sigma_F(B_{p,1}^{1+3/p})} \\
+ C2^{j} 2^{(1-3/p)} d_j \|\nabla u^h, u^3\|_{L^p(B_{p,1}^{1+3/p})} \|u^3\|_{L^p(B_{p,1}^{1+3/p})} \|u^h\|_{L^1_t(L^p)}^{1/p},
\]

from which, we obtain the desired estimate (54).

Finally, as the proof of (59), let us apply \(\Delta_j\) to (53), we need to make some modifications as that [11], by virtue of Lemmas 2.6, 5.1 and Proposition 3, we have

\[
\|\Delta_j u^3\|_{L^p_t(L^p)} + C2^{2j} \|\Delta_j u^3\|_{L^1_t(L^p)} \\
\lesssim \|\Delta_j u^3\|_{L^p} + \|\Delta_j (u^3 u^h)\|_{L^1_t(L^p)} + 2^j \|\Delta_j (u^3 u^h)\|_{L^p} \\
+ \|\Delta_j \nabla_h p\|_{L^1_t(L^p)} + \|\Delta_j (n - c) \nabla_h (-\Delta)^{-1}(n - c)\|_{L^1_t(L^p)} \\
\lesssim \|\Delta_j u^3\|_{L^p} + d_j 2^{(1-3/p)} \|u^h\|_{L^p(B_{p,1}^{1+3/p})} \|u^3\|_{L^p(B_{p,1}^{1+3/p})} \\
+ C2^{2(j-3/p)} d_j \|\nabla u^3\|_{L^p(B_{p,1}^{1+3/p})} \|u^3\|_{L^p(B_{p,1}^{1+3/p})} \|u^h\|_{L^1_t(L^p)}^{1/p},
\]

Multiplying (60) by \(2^{j-3/p}\) and summing up over \(j\) yield (78). Therefore, we complete the proof of Proposition 4.

**The estimate of \(n\).**

Let consider the second equation of (1), we have

\[
\partial_t n - \Delta n = -u \cdot \nabla n - \nabla \cdot (n \nabla \phi).
\]

(61)

**Proposition 5.** Let \(1 < p < q \leq 2p < \infty, \frac{1}{p} + \frac{1}{q} > \frac{1}{3}, 1 < q < 6, \frac{1}{p} - \frac{1}{q} \leq \frac{1}{3} \). Let \((u, n, c) \in \Theta_T\) with \(\nabla \cdot u = 0\). Then (61) leads to

\[
\|n\|_{\Sigma_F(B_{p,1}^{2+3/q})} + C\|n\|_{L^p_t(B_{p,1}^{2+3/q})} \leq \|n_0\|_{B_{p,1}^{2+3/q}} + \|u^h\|_{L^1_t(L^p)} \\
+ \|u^3\|_{L^p(B_{p,1}^{1+3/p})} \|u^h\|_{L^p(B_{p,1}^{1+3/p})} \|n\|_{\Sigma_F(B_{p,1}^{2+3/q})} \\
+ C\|n\|_{\Sigma_F(B_{p,1}^{2+3/q})} \|n, c\|_{L^p(B_{p,1}^{3/q})} \|n, c\|_{L^p(B_{p,1}^{2+3/q})}.
\]

(62)
Proof. Applying the operator \( \Delta_j \) to (61), as the same estimates (56), we obtain
\[
\frac{1}{q} \int_{\mathbb{R}^3} |\Delta_j n|^q L^q dx - \int_{\mathbb{R}^3} \Delta \Delta_j n \cdot |\Delta_j n|^{q-2} \Delta_j ndx = - \int_{\mathbb{R}^3} \Delta_j (u \cdot \nabla n + \nabla (n \nabla \phi)) |\Delta_j n|^{q-2} \Delta_j ndx.
\] (63)

Thanks to [11, 27], there exists a positive constant \( \tilde{c} \) so that
\[
- \int_{\mathbb{R}^3} \Delta \Delta_j n \cdot |\Delta_j n|^{q-2} \Delta_j ndx \geq \tilde{c} 2^{2j} \|\Delta_j n\|_{L^q}^q.
\] (64)

It is clear that, formally, we have the following homogeneous paraproduct decomposition
\[
u \nabla n = \mathcal{T} u \nabla n + R(u, \nabla n) + T_{\nabla u}.
\] (65)

Besides, because \( \nabla \cdot u = 0 \), by using the standard argument, one obtains
\[
\int_{\mathbb{R}^3} \Delta_j (\mathcal{T} u \nabla n) |\Delta_j n|^{q-2} \Delta_j ndx = \sum_{|j-j'| \leq 5} \int_{\mathbb{R}^3} [\Delta_j; S_{j'-1} u] \Delta_j \nabla n |\Delta_j n|^{q-2} \Delta_j ndx 
+ \sum_{|j-j'| \leq 5} \int_{\mathbb{R}^3} (S_{j'-1} u - S_{j-1} u) \Delta_j \nabla n |\Delta_j n|^{q-2} \Delta_j ndx.
\] (66)

Therefore, substituting (64), (65) and (66) into (63), using an argument for the \( L^q \) energy estimate as that in [11], we reach at
\[
\|\Delta_j n\|_{L^\infty_\partial (L^s)} + \tilde{c} 2^{2j} \int_0^t \|\Delta_j n\|_{L^s} d\tau
\leq \|\Delta_j n_0\|_{L^s} + \tilde{C} \sum_{|j-j'| \leq 5} \|\Delta_j; S_{j'-1} u\|_{L^1_\partial (L^s)} |\Delta_j \nabla n|_{L^1_\partial (L^s)} 
+ C \sum_{|j-j'| \leq 5} \|\Delta_j (S_{j'-1} u - S_{j-1} u)\|_{L^1_\partial (L^s)} |\Delta_j \nabla n|_{L^1_\partial (L^s)} 
+ \tilde{C} \|\Delta_j (R(u, \nabla n))\|_{L^1_\partial (L^s)} + \tilde{C} \|\Delta_j \nabla \cdot (n \nabla \phi)\|_{L^1_\partial (L^s)}.
\] (67)

Applying the classical estimate on commutator [10] and Lemma 2.1 yields
\[
\sum_{|j-j'| \leq 5} \|\Delta_j; S_{j'-1} u\|_{L^1_\partial (L^s)} |\Delta_j \nabla n|_{L^1_\partial (L^s)}
\leq \sum_{|j-j'| \leq 5} \sum_{k \leq j'-2} 2^{k(1+3/p)} \|\Delta_k u^h\|_{L^1_\partial (L^p)} |\Delta_j n|_{L^\infty_{\partial} (L^q)} 
+ \sum_{|j-j'| \leq 5} \sum_{k \leq j'-2} 2^{k} \|\Delta_k u^3(\tau)\|_{L^1_\partial (L^\infty)} |\Delta_j n(\tau)|_{L^\infty_{\partial} (L^q)}
\leq \sum_{|j-j'| \leq 5} d_{j'} 2^{j'(-2-3/q)} \|u^h\|_{L^1_\partial (B_{p,1}^{1+3/p})} \|n\|_{L^\infty_{\partial} (B_{q,1}^{2+3/q})}
+ \sum_{|j-j'| \leq 5} \|u^3\|_{L^1_\partial (B_{p,1}^{1+3/p})} \|u^h\|_{L^1_\partial (B_{p,1}^{1+3/p})} \|\Delta_j n|_{L^\infty_{\partial} (L^q)}
\leq d_{j'} 2^{j'(-2-3/q)} (\|u^h\|_{L^1_\partial (B_{p,1}^{1+3/p})} + \|u^3\|_{L^1_\partial (B_{p,1}^{1+3/p})} \|u^h\|_{L^1_\partial (B_{p,1}^{1+3/p})} \|n\|_{L^\infty_{\partial} (B_{q,1}^{2+3/q})}),
\] (68)
Now, using the same type of computations as in (68), we get
\[
\sum_{|j-j'|\leq 5} \| (S_{j'-1} u - S_{j-1} u) \Delta_j \Delta_{j'} \nabla n \|_{L^1_t(L^\infty)} \\
\leq \sum_{|j-j'|\leq 5} \| (S_{j'-1} \nabla u^h - S_{j-1} \nabla u^h) \|_{L^1_t(L^\infty)} \| \Delta_{j'} n \|_{L^\infty_t(L^\infty)} \\
+ \sum_{|j-j'|\leq 5} \| S_{j'-1} \nabla u^3(\tau) - S_{j-1} \nabla u^3(\tau) \|_{L^1_t(L^\infty)} \| \Delta_{j'} n(\tau) \|_{L^\infty_t(L^\infty)} \\
\leq d_3 2^{i(2-3/q)} (\| u^h \|_{L^1_t(B^{1+3/p}_{p,1})} + \| u^3 \|_{L^1_t(B^{1+3/p}_{p,1})}^{1-1/p} \| u^h \|_{L^1_t(B^{1+3/p}_{p,1})}^{1/p}) \| n \|_{L^\infty_t(B^{-2+3/q}_{q,1})},
\]
(69)

Note that \(1 < q < 6\), we get that
\[
\| \Delta_j R(u, \nabla n) \|_{L^1_t(L^\infty)} \\
\leq \sum_{j' \geq j-N_0} \| \Delta_j u^h \|_{L^1_t(L^\infty)} \| S_{j'-2} \nabla h n \|_{L^\infty_t(L^\infty)} \\
+ \sum_{j' = j-N_0} \| \Delta_j u^3 \|_{L^1_t(L^\infty)} \| S_{j'-2} \partial_3 n \|_{L^\infty_t(L^\infty)} \\
\leq \sum_{j' \geq j-N_0} \sum_{j' \leq j'} (2^{j''(1+3/q)} \| \Delta_{j''} n \|_{L^\infty_t(L^p)} 2^{j''(3/p-3/q)} \| \Delta_{j''} u^h \|_{L^1_t(L^p)}) \\
+ \| \Delta_j u^3 \|_{L^1_t(L^\infty)} 2^{j''(1+3/q)} \| \Delta_{j''} n \|_{L^\infty_t(L^p)} \\
\leq d_3 2^{i(2-3/q)} (\| u^h \|_{L^1_t(B^{1+3/p}_{p,1})} + \| u^3 \|_{L^1_t(B^{1+3/p}_{p,1})}^{1-1/p} \| u^h \|_{L^1_t(B^{1+3/p}_{p,1})}^{1/p}) \| n \|_{L^\infty_t(B^{-2+3/q}_{q,1})},
\]
(70)

So plugging the above inequalities (68)-(70) into (67) and keeping in mind Lemma 2.5, we conclude
\[
\| \Delta_j n \|_{L^\infty_t(L^\infty)} + \| \Delta_j \nabla u^h \|_{L^\infty_t(L^\infty)} \leq \| \Delta_j n_0 \|_{L^\infty} + d_3 2^{i(2-3/q)} (\| u^h \|_{L^1_t(B^{1+3/p}_{p,1})} + \| u^3 \|_{L^1_t(B^{1+3/p}_{p,1})}^{1-1/p} \| u^h \|_{L^1_t(B^{1+3/p}_{p,1})}^{1/p}) \| n \|_{L^\infty_t(B^{-2+3/q}_{q,1})} \\
+ d_3 2^{i(2-3/q)} (\| n \|_{L^\infty_t(B^{2+3/q}_{q,1})} \| n, c \|_{L^1_t(B^{3/q}_{q,1})} + \| n \|_{L^1_t(B^{3/q}_{q,1})} \| (n, c) \|_{L^\infty_t(B^{-2+3/q}_{q,1})}).
\]
(71)

Then multiplying both sides by \(2^i \) and then taking the \(L^1(\mathbb{Z})\)-norm, we obtain Proposition 5.

- **The estimate of \(c\).**

Thanks to the second equation of (1), we have
\[
\partial_t c - \Delta c = -u \cdot \nabla c + \nabla \cdot (c \nabla \phi).
\]
(72)

**Proposition 6.** Let \(1 < p < q < 2p < \infty, \frac{1}{p} + \frac{1}{q} > \frac{1}{2}, 1 < q < \frac{6}{3}, \frac{1}{p} - \frac{1}{q} = \frac{1}{2}. \) Let \((u, n, c) \in \Theta_T \) with \(\nabla u = 0\). Then the equation (72) leads to
\[
\| c \|_{L^\infty_t(B^{2+3/q}_{q,1})} + \| c \|_{L^1_t(B^{3/q}_{q,1})} \leq \| c_0 \|_{B^{-2+3/q}_{q,1}} \\
+ (\| u^h \|_{L^1_t(B^{1+3/p}_{p,1})} + \| u^3 \|_{L^1_t(B^{1+3/p}_{p,1})}^{1-1/p} \| u^h \|_{L^1_t(B^{1+3/p}_{p,1})}^{1/p}) \| c \|_{L^\infty_t(B^{2+3/q}_{q,1})} \\
+ C(\| c \|_{L^\infty_t(B^{2+3/q}_{q,1})} \| (n, c) \|_{L^1_t(B^{3/q}_{q,1})} + \| c \|_{L^1_t(B^{3/q}_{q,1})} \| (n, c) \|_{L^\infty_t(B^{2+3/q}_{q,1})}).
\]
(73)

**Proof.** Similarly to the Proposition 5, we can prove Proposition 6 easily. □
Proof of Theorem 1.2. Let \( T^* \) be a maximal time of existence introduced in Theorem 1.1. Hence, to prove Theorem 1.2, we only need to prove that \( T^* = \infty \)
with \( (u, n, c) \in \Theta \cap \Theta^c \) provided that there holds (7).

Let \( \eta, \vartheta \) be positive constants, which will be determined later on, we define

\[
\mathcal{T} := \{t \in [0, T^*): \|u^h\|^2_\mathcal{L}^p(\dot{B}^{-1+3/p}_{p,1}) + \tilde{c}\|u^h\|^2_\mathcal{L}^1(\dot{B}^{1+3/p}_{p,1}) \leq 4\|u_0^h\|^2_\mathcal{L}^p(\dot{B}^{-1+3/p}_{p,1}) + \eta := \mathfrak{A}_0, \\
\|u^3\|^2_\mathcal{L}^p(\dot{B}^{-1+3/p}_{p,1}) + \tilde{c}\|u^3\|^2_\mathcal{L}^1(\dot{B}^{1+3/p}_{p,1}) \leq 4\|u_0^3\|^2_\mathcal{L}^p(\dot{B}^{-1+3/p}_{p,1}) + 1 := \mathfrak{B}_0, \\
\|(n, c)\|_{\mathcal{L}^p(\dot{B}^{-2+3/q}_{q,1}) \times \mathcal{L}^q(\dot{B}^{-2+3/q}_{q,1})} + \tilde{c}||(n, c)||_{\mathcal{L}^1(\dot{B}^{3/q}_{q,1}) \times \mathcal{L}^1(\dot{B}^{3/q}_{q,1})} \leq 4\|(n_0, 0)\|_{\dot{B}^{-2+3/q}_{q,1} \times \dot{B}^{-2+3/q}_{q,1}} := \mathfrak{C} \}.
\] (74)

Applying Propositions 5 and 6, we obtain

\[
\|(n, c)\|_{\mathcal{L}^\infty(\dot{B}^{-1+3/q}_{q,1}) \times \mathcal{L}^\infty(\dot{B}^{-2+3/q}_{q,1})} + \tilde{c}||(n, c)||_{\mathcal{L}^1(\dot{B}^{3/q}_{q,1}) \times \mathcal{L}^1(\dot{B}^{3/q}_{q,1})} \\
\leq \|(n_0, 0)\|_{\dot{B}^{-2+3/q}_{q,1} \times \dot{B}^{-2+3/q}_{q,1}} + \tilde{C}||(u^h)\|_{\mathcal{L}^1(\dot{B}^{1+3/p}_{p,1})} + \|(u^3)\|_{\mathcal{L}^1(\dot{B}^{1+3/p}_{p,1})} \\
\times \|(n, c)\|_{\mathcal{L}^\infty(\dot{B}^{-2+3/q}_{q,1}) \times \mathcal{L}^\infty(\dot{B}^{-2+3/q}_{q,1})} + \tilde{C}||(n, c)||_{\mathcal{L}^\infty(\dot{B}^{-2+3/q}_{q,1})}||(n, c)||_{\mathcal{L}^1(\dot{B}^{3/q}_{q,1})} \\
\leq \|(n_0, 0)\|_{\dot{B}^{-2+3/q}_{q,1} \times \dot{B}^{-2+3/q}_{q,1}} + \tilde{C}||(\mathfrak{A}_0 + \mathfrak{A}_0^1/p, \mathfrak{B}_0^{1-1/p})||(n, c)||_{\mathcal{L}^\infty(\dot{B}^{-2+3/q}_{q,1})} \\
+ \tilde{C}||(n, c)||_{\mathcal{L}^\infty(\dot{B}^{-2+3/q}_{q,1}) \times \mathcal{L}^\infty(\dot{B}^{-2+3/q}_{q,1})}.
\] (75)

Using the Cauchy-Schwartz inequality \( ab \leq \frac{\alpha}{2}a^2 + \frac{\beta}{2}b^2 \) for \( \varepsilon = 2\sqrt{c} \) yields

\[
\mathcal{W}(u^h, u^3) := \|u^h\|_{\mathcal{L}^1(\dot{B}^{1+3/p}_{p,1})}^2 + \|u^3\|_{\mathcal{L}^1(\dot{B}^{1+3/p}_{p,1})}^2 + \|u^h\|^2_\mathcal{L}^p(\dot{B}^{-1+3/p}_{p,1}) + \|u^3\|^2_\mathcal{L}^p(\dot{B}^{-1+3/p}_{p,1}) \\
+ \|u^h\|_{\mathcal{L}^{1+1/p}(\dot{B}^{1+3/p}_{p,1})}^2 + \|u^3\|_{\mathcal{L}^{1+1/p}(\dot{B}^{1+3/p}_{p,1})}^2 \\
\leq (\tilde{C}\mathfrak{A}_0 + \tilde{C}\mathfrak{A}_0^1/p\mathfrak{B}_0^{1-1/p}) (\tilde{c}\|u^h\|_{\mathcal{L}^1(\dot{B}^{1+3/p}_{p,1})} + \|u^h\|_{\mathcal{L}^\infty(\dot{B}^{-1+3/p}_{p,1})}) \\
+ \tilde{C}(\tilde{c}\|u^h\|_{\mathcal{L}^1(\dot{B}^{1+3/p}_{p,1})} + \|u^h\|_{\mathcal{L}^\infty(\dot{B}^{-1+3/p}_{p,1})})^{1+1/p} \\
\times (\tilde{c}\|u^3\|_{\mathcal{L}^1(\dot{B}^{1+3/p}_{p,1})} + \|u^3\|_{\mathcal{L}^\infty(\dot{B}^{-1+3/p}_{p,1})})^{1+1/p} \\
+ \tilde{C}(\tilde{c}\|u^h\|_{\mathcal{L}^1(\dot{B}^{1+3/p}_{p,1})} + \|u^h\|_{\mathcal{L}^\infty(\dot{B}^{-1+3/p}_{p,1})})^{1+1/p} \\
\times (\tilde{c}\|u^3\|_{\mathcal{L}^1(\dot{B}^{1+3/p}_{p,1})} + \|u^3\|_{\mathcal{L}^\infty(\dot{B}^{-1+3/p}_{p,1})})^{1-\alpha}.
\] (76)

Thanks to (76) and Proposition 4, we have

\[
\|u^h\|^2_\mathcal{L}^p(\dot{B}^{-1+3/p}_{p,1}) + \tilde{c}\|u^h\|^2_\mathcal{L}^1(\dot{B}^{1+3/p}_{p,1}) \\
\leq \|u^h\|_{\dot{B}^{-1+3/p}_{p,1}} + \tilde{C}\mathcal{W}(u^h, u^3) + \tilde{C}||(n, c)||_{\mathcal{L}^\infty(\dot{B}^{-2+3/q}_{q,1})}||(n, c)||_{\mathcal{L}^1(\dot{B}^{3/q}_{q,1})} \\
\leq \|u^h\|_{\dot{B}^{-1+3/p}_{p,1}} + \left(\frac{\tilde{C}}{c}\mathfrak{A}_0 + \frac{\tilde{C}}{c}\mathfrak{A}_0^1/p\mathfrak{B}_0^{1-1/p} + \frac{\tilde{C}}{c}\mathfrak{A}_0\mathfrak{B}_0^1\mathfrak{B}_0^{-1}\right) (\tilde{c}\|u^h\|_{\mathcal{L}^1(\dot{B}^{1+3/p}_{p,1})}) \\
+ \|u^h\|^2_\mathcal{L}^p(\dot{B}^{-1+3/p}_{p,1}) + \frac{\tilde{C}}{c}\mathfrak{A}_0^2.
\] (77)
We deduce from (76) and Proposition 4 that
\[
\|u^3\|_{\mathcal{L}_c^\infty(\hat{B}_{p,1}^{-1+3/p})} + \tilde{c}\|u^3\|_{\mathcal{L}_c^1(\hat{B}_{p,1}^{1+3/p})}
\leq \|u_0^3\|_{\mathcal{B}_{p,1}^{-1+3/p}} + \tilde{C}\|u^h\|_{\mathcal{L}_c^2(\mathcal{B}_{p,1})}\|u^3\|_{\mathcal{L}_c^2(\hat{B}_{p,1}^{3/p})} + \tilde{C}\mathcal{W}(u^h, u^3)
\]
\[+ \tilde{C}\|(n, c)\|_{\mathcal{L}_c^2(\hat{B}_{q,1}^{-2+3/q})}\|(n, c)\|_{\mathcal{L}_c^2(\hat{B}_{q,1}^{3/q})} \tag{78}\]
\[
\leq \|u_0^3\|_{\mathcal{B}_{p,1}^{-1+3/p}} + \frac{\tilde{C}}{c}\|\tilde{c}\|_{\mathcal{L}_c^1(\hat{B}_{p,1}^{1+3/p})} + \|u^3\|_{\mathcal{L}_c^\infty(\hat{B}_{p,1}^{-1+3/p})} + \frac{\tilde{C}}{c}\mathcal{C}_2 + \left(\frac{\tilde{C}}{c}\mathcal{A}_0 + \frac{\tilde{C}}{c}\mathcal{A}_0^{1+1/p}\mathcal{A}_0^{-1/p} + \frac{\tilde{C}}{c}\tilde{A}_0^{1+\alpha}\mathcal{A}_0^{-1-\alpha}\right). \tag{79}\]

Thanks to (74), taking \(C > 0\) large enough in (7) follows that
\[
\frac{\tilde{C}}{c}(\mathcal{A}_0 + \mathcal{A}_0^{1+1/p}\mathcal{A}_0^{-1/p}) + \tilde{C}\mathcal{C}_3 \leq 1/2, \quad \eta = \frac{4\tilde{C}}{c}\mathcal{C}_2,
\]
which show that
\[
\|(n, c)\|_{\mathcal{L}_c^\infty(\hat{B}_{q,1}^{-2+3/q}\times\hat{B}_{q,1}^{3/q})} + \left(\frac{\tilde{C}}{c}\mathcal{A}_0 + \frac{\tilde{C}}{c}\mathcal{A}_0^{1+1/p}\mathcal{A}_0^{-1/p} + \frac{\tilde{C}}{c}\tilde{A}_0^{1+\alpha}\mathcal{A}_0^{-1-\alpha}\right) \leq \frac{1}{4}. \tag{80}\]

We thus obtain
\[
\|u^h\|_{\mathcal{L}_c^\infty(\hat{B}_{p,1}^{-1+3/p})} + \tilde{c}\|u^h\|_{\mathcal{L}_c^1(\hat{B}_{p,1}^{1+3/p})} \leq 2\|u_0^h\|_{\mathcal{B}_{p,1}^{-1+3/p}} + \frac{\eta}{2},
\]
\[
\|u^3\|_{\mathcal{L}_c^\infty(\hat{B}_{p,1}^{1+3/p})} + \tilde{c}\|u^3\|_{\mathcal{L}_c^1(\hat{B}_{p,1}^{1+3/p})} \leq 2\|u_0^3\|_{\mathcal{B}_{p,1}^{1+3/p}} + \frac{\eta}{2}, \tag{81}\]
\[
\|(n, c)\|_{\mathcal{L}_c^\infty(\hat{B}_{q,1}^{-2+3/q}\times\hat{B}_{q,1}^{3/q})} + \tilde{c}\|(n, c)\|_{\mathcal{L}_c^1(\hat{B}_{q,1}^{3/q})} \leq 2\|(n_0, c_0)\|_{\hat{B}_{q,1}^{-2+3/q}\times\hat{B}_{q,1}^{3/q}}.
\]

Thus (81) contradicts with the definition (74), in view of the standard continuity argument, we can conclude that \(T = T^*\). Therefore, we complete the proof of Theorem 1.2.

Acknowledgements. The authors are grateful to the referees for their careful reading and valuable comments and suggestions that improved the presentation of the paper.
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Received August 2016; revised March 2017.

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