Frame-like Lagrangians and presymplectic AKSZ-type sigma models

Konstantin Alkalaev\textsuperscript{a} and Maxim Grigoriev\textsuperscript{a,b}

\textsuperscript{a}I.E. Tamm Department of Theoretical Physics, P.N. Lebedev Physical Institute, Leninsky ave. 53, 119991 Moscow, Russia

\textsuperscript{b}Moscow Institute of Physics and Technology, Dolgoprudnyi, 141700 Moscow region, Russia

Abstract

We study supergeometric structures underlying frame-like Lagrangians. We show that for the theory in $n$ spacetime dimensions both the frame-like Lagrangian and its gauge symmetries are encoded in the target supermanifold equipped with the odd vector field, the closed 2-form of ghost degree $n - 1$, and the scalar potential of ghost degree $n$. These structures satisfy a set of compatibility conditions ensuring the gauge invariance of the theory. The Lagrangian and the gauge symmetries have the same structures as those of AKSZ sigma model so that frame-like formulation can be seen as its presymplectic generalization. In contrast to the conventional AKSZ model the generalization allows to describe systems with local degrees of freedom in terms of finite-dimensional target space. We argue that the proposed frame-like approach is directly related de Donder–Weyl polymomentum Hamiltonian formalism. Along with the standard field-theoretical examples like Einstein–Yang–Mills theory we consider free higher spin fields, multi-frame gravity, and parameterized systems. In particular, we propose the frame-like action for free totally symmetric massless fields that involves all higher spin connections on an equal footing.
1 Introduction

Manifly diffeomorphism-invariant formulations in terms of $p$-forms are extremely useful in the context of supergravity [1, 2] and higher spin gauge theories [3, 4, 5, 6, 7, 8, 9, 10] (for a review see [11]). One can usually assume the Lagrangian to be first order and to involve only the wedge product and de Rham differential. The structure of the respective action is remarkably similar to the standard extended Hamiltonian action while its gauge symmetries can be expressed through an associated BRST-type differential.

In this work we study the supergeometric structures underlying first order frame-like Lagrangians and propose a set of compatibility conditions for them which ensure the gauge invariance. More precisely, the basic object is the graded supermanifold, the target space $\mathcal{M}$, where the $p$-form fields take values so that the degree of a coordinate is a form degree of the respective field. The target space comes equipped with a presymplectic potential 1-form $\chi$, an odd vector field $Q$ of degree one, and a function $H_\mathcal{M}$ which is an analog of the usual Hamiltonian. This data is enough to formulate a first order Lagrangian which is by construction invariant under the diffeomorphisms and the extra gauge transformations generated by the odd vector field provided some natural compatibility conditions are satisfied.

In the special case where the presymplectic form is invertible the equations of motion take the form of a free differential algebra and the compatibility conditions require the odd vector field to be nilpotent. In this case the action can be reformulated as the so-called AKSZ sigma model [12] (for further developments and applications see, e.g., [13, 14, 15, 16, 17, 18, 19]) whose characteristic feature is that its Batalin-Vilkovisky (BV) structure is manifest already at the level of the classical action.

Although it is well-known that if the number of fields is finite and $n > 1$ the AKSZ sigma model is necessarily topological, the first order frame-like Lagrangians studied in this paper are deeply related to AKSZ sigma models and the Batalin-Vilkovisky approach. Namely, at the level of equations of motion any gauge theory can be represented as an AKSZ sigma model with infinite number of fields (the so-called parent formulation) by adding generalized auxiliary fields and (if necessary) parametrization [20].

The Lagrangian counterpart of the parent formulation is also known [21]. In this case however the interpretation of the resulting theory is more subtle. One either needs to truncate the resulting formulation or to carefully define the space of allowed field configurations [22]. It was shown that various frame-like actions can be systematically obtained by equivalent reduction of the parent formulation [21, 22]. Moreover, from this perspective all the structures of the reduced theory originate from the BV structure of the parent formulation. In particular, the canonical BV odd symplectic potential gives rise to the presymplectic form $\chi$, while the BRST differential determines the $Q$ odd vector field upon reduction.

First order frame-like action can be considered as a multidimensional generalization of the conventional extended Hamiltonian action. Moreover, in many cases
the above supermanifold, presymplectic potential and the generalized Hamiltonian
directly give rise to the polymomentum phase space, the canonical $n$-form, and the
Hamiltonian of the de Donder–Weyl polymomentum formulation of the theory (see,
e.g., [23, 24, 25]). Although straightforward for theories without gauge-invariance
or relatively simple models like Yang-Mills theory the identification is not so natural
as far as genuine diffeomorphism invariance like that of Einstein gravity or Chern-
Simons model comes into the game. In this case it seems that the target space
supermanifold itself plays a more fundamental role and is to be interpreted as a
proper version of the polymomentum phase space.

The paper is organized as follows. In Section 2 we discuss general properties
of frame-like Lagrangians, starting form the usual AKSZ construction in Section
2.1. In particular, we introduce basic structures of the AKSZ sigma model and
discuss gauge symmetries of its Lagrangian. In Section 2.2 we consider possible
generalizations of AKSZ sigma models. To this end, we introduce basic geometric
structures on the target manifold and propose a set of their compatibility condi-
tions that allow one to build a gauge invariant frame-like Lagrangian. In Section
the approach is illustrated by a number of field-theoretical examples. In par-
ticular, in Sections from 3.1 to 3.6 we reformulate various gravity models, including
their multi-frame generalizations and higher spin frame-like Lagrangians. In Sec-
tion 4 we extend our discussion to generic one-dimensional constrained Hamiltonian
systems, and the so-called parameterized systems. A relation of the presymplectic
AKSZ-type formulation to the de Donder–Weyl polymomentum formulation is es-	tablished in Section 4.3. Our notation and conventions, along with some basic facts
in supergeometry are collected in Appendix A.

2 First order frame-like Lagrangians

2.1 AKSZ sigma models

The AKSZ sigma model is defined in terms of two supermanifolds: the target space
$M$ and the source $X$. Target $M$ is equipped with odd nilpotent vector field $Q$
and the ghost degree $\text{gh}(\cdot)$ such that $\text{gh}(Q) = 1$. If coordinates on $M$ are $\Psi^A$, $A = 1, \ldots, \dim M$, then $Q = Q^A(\Psi) \partial_A$, where $\partial_A = \frac{\partial}{\partial \Psi^A}$ (see Appendix A).

Source $X$ is equipped with odd nilpotent vector field $d$ and a ghost degree also
denoted by $\text{gh}(\cdot)$, $\text{gh}(d) = 1$. Usually, $X$ is taken to be odd tangent bundle $T[1]X$
over a manifold $\mathcal{X}$, i.e. with inverse parity of the fibers. If $x^\mu$, $\mu = 1, \ldots, n = \dim X$
are coordinates on $X$ and $\theta^\mu$ with $\text{gh}(\theta^\mu) = 1$ on the fibers, then $d = \theta^\mu \frac{\partial}{\partial x^\mu}$ is the
de Rham differential.

This data is enough to define AKSZ model at the level of equations of motion.
The fields are ghost degree zero maps from $\mathcal{X}$ to $M$. To each target space
coordinate $\Psi^A$ of ghost degree $p$, $p \geq 0$ one associates a $p$-form field $\Psi^A(x, \theta) = \frac{1}{p!} \Psi^A_{\mu_1 \ldots \mu_p}(x) \theta^{\mu_1} \ldots \theta^{\mu_p}$, where $\text{gh}(\Psi^A_{\mu_1 \ldots \mu_p}(x)) = 0$, while coordinates with $p < 0$ do
not have associated fields. The equations of motion are

\[ d\Psi^A + Q^A(\Psi) = 0 \quad \text{gh}(\Psi^A) \geq 0, \quad Q^A(\Psi) = 0 \quad \text{gh}(\Psi^A) < 0. \quad (2.1) \]

Note that in the expressions of $Q^A(\Psi)$ fields $\Psi^A(x, \theta)$ associated with negative ghost degree coordinates should be put to zero. To avoid confusions one would better use different conventions for fields and target coordinates. However, in what follows we almost always assume that negative degree coordinates are absent. In this case, the second equation in (2.1) is missing and the equations of motion take the form of free differential algebra [26, 2, 1], also known as the unfolded formulation [27, 28].

The nontrivial property of AKSZ model is that its BV-BRST description is encoded in the same geometric structures. Fields of nonzero ghost degree can be introduced by simply taking a generic map from $X$ to $M$. In terms of components, one considers $\tilde{\Psi}^A(x, \theta) = \sum_{l=1}^{n} \frac{1}{l!} \overrightarrow{\Psi}^A_{\mu_1 \ldots \mu_l}(x) \theta^{\mu_1} \ldots \theta^{\mu_l}$, where $\text{gh}(\overrightarrow{\Psi}^A_{\mu_1 \ldots \mu_l}) = \text{gh}(\Psi^A) - l$, so that at $l = \text{gh}(\Psi^A)$ one finds ghost degree zero fields interpreted as fields of the original AKSZ model. Furthermore, the BRST differential on the BRST extended space of fields is determined in terms of $d$ and $Q$. Leaving aside technical details the BRST differential is determined by $s\tilde{\Psi}^A(x, \theta) = d\tilde{\Psi}^A(x, \theta) + Q^A[\tilde{\Psi}(x, \theta)]$. In particular, the equations of motion (2.1) are just $s\tilde{\Psi}^A(x, \theta) = 0$, where all fields of nonzero degree are put to zero.

In the same way the gauge symmetries are encoded in $s$ through $\delta \Psi^A = s\tilde{\Psi}^A$, where in the RHS one puts to zero all the components in nonzero degree except for degree one fields which are to be replaced by gauge parameters so that

\[ \delta_A \Psi^A = d\lambda^A - \lambda^C \partial_C Q^A, \quad (2.2) \]

where $\lambda^A$ is a gauge parameter associated to $\Psi^A$. More precisely, if $\text{gh}(\Psi^A) = p$ then $\lambda^A$ is a $(p - 1)$-form. Note that coordinates with $\text{gh}(\Psi^A) \leq 0$ do not give rise to gauge parameters.

To describe Lagrangian systems one assumes that $M$ is graded symplectic. More precisely, $M$ is additionally equipped with a nondegenerate 2-form $\sigma_{AB}$ of ghost degree $n - 1$ invariant with respect to odd vector field $Q$, i.e., $L_Q \sigma = 0$. Suppose now that $\chi_A$ is a symplectic potential and $L$ is a Hamiltonian of $Q$, so that $\sigma = d\chi$ and $dL = i_Q \sigma$, where $d$ denotes the de Rham differential on $M$ (see Appendix A). Then, one can define the action as

\[ S[\Psi] = \int_X \left( d\Psi^A \chi_A(\Psi) + L(\Psi) \right). \quad (2.3) \]

As before, replacing ghost degree zero fields $\Psi^A(x, \theta)$ with the complete multiplet $\tilde{\Psi}^A(x, \theta)$ in the expression for $S[\Psi]$ results in the Batalin–Vilkovisky master action $S[\tilde{\Psi}]$, while the symplectic structure gives rise to the antibracket on the space of all component fields entering $\tilde{\Psi}^A(x, \theta)$.

It is instructive to check gauge invariance of the above action. To begin with we can assume that $\sigma_{AB}$ is constant by virtue of the Darboux theorem. In this case, the
invariance of action (2.3) under (2.2) is straightforward. To perform this check in generic coordinates we recall the following simple observation: suppose that under the transformation $\delta \lambda$ the action transforms as $\delta \lambda S = M^{IJ} \partial S / \partial \Psi^J \partial \Psi^I$, where $M^{IJ}$ are some functions of fields and gauge parameters (here, for simplicity, we use condensed notation). In other words, the action is invariant modulo terms of order 2 in the equations of motion. Taking $M^{IJ}$ (graded) antisymmetric results in the so-called trivial gauge transformations which automatically preserve the action (see, e.g., [29]).

Although the transformation is not a symmetry of the action, it coincides with the symmetry transformation modulo terms vanishing on-shell. Indeed, adjusting the transformation as
$$
\delta' \lambda \Psi^I = \delta \lambda \Psi^I - M^{IJ} \partial S / \partial \Psi^J,
$$
(2.4)
one finds a genuine symmetry. This simple observation is useful in the sequel as it simplifies the description of gauge symmetries. It allows to only check the invariance modulo terms of second order in equations.

In general coordinates, the variation of action $S[\Psi]$ under (2.2) is given by
$$
\delta \lambda S[\Psi] = \int_X \left( - \lambda^C \partial_C K - (-)^{|A|} \mathcal{E}^A \lambda^C \partial_C T_A + \frac{1}{2} (-)^{|A|+|B|} \mathcal{E}^B \mathcal{E}^A \lambda^C \partial_C \sigma_{AB} \right),
$$
(2.6)
where
$$
\mathcal{E}^A = d\Psi^A + Q^A, \quad T_A = \partial_A L - Q^B \sigma_{BA}, \quad K = Q^A \partial_A L - \frac{1}{2} Q^A Q^B \sigma_{BA}.
$$
(2.7)

In the case of AKSZ model $K = 0$ and $T_A = 0$ so that the first two terms vanish identically, while the last term is precisely quadratic in the equations of motion $\mathcal{E}^A = 0$ (2.1). One concludes that modulo on-shell vanishing terms transformation (2.2) is a gauge symmetry of action $S[\Psi]$. Let us note that in AKSZ setting it is not difficult to find an explicit expression for genuine gauge symmetries. However, in the generalization we consider next the above indirect approach turns out to be useful. Note also that by construction AKSZ model is diffeomorphism invariant. Moreover, the diffeomorphisms seen as gauge transformations are just particular combinations of gauge symmetries (2.2).

### 2.2 Relaxing AKSZ conditions

Now we relax basic axioms of the AKSZ model discussed above. We restrict ourselves to finite-dimensional $M$ and try to find a generalization suitable for describing theories with local degrees of freedom. Recall that AKSZ with finite-dimensional $M$ is necessarily topological in spacetime dimension $n > 1$.

---

In deriving the expression for variation the following formula is useful
$$
\delta (d\Psi^A \chi_A) = \delta \Psi^B d\Psi^A \sigma_{AB} = (-1)^{|B|(n-1)} d\Psi^A \sigma_{AB} \delta \Psi^B.
$$
(2.5)
The main condition we give up here is the invertibility of the 2-form $\sigma$. Furthermore, we do not immediately assume that combinations $K$ and $T_A$ from (2.7) are vanishing. In addition, we admit that not all coordinates $\Psi^A$ with $\text{gh}(\Psi^A) > 0$ give rise to nontrivial gauge parameters.

The equations of motion following from (2.3) are given by

$$E_B \equiv d\Psi^A \sigma_{AB} + \partial_B L = 0.$$  
(2.8)

In the AKSZ case, these are equivalent to $E^A = 0$ that can be easily seen by introducing inverse matrix $(\sigma^{-1})^{AB}$.

The condition that transformations (2.2) preserve action (2.3) modulo terms quadratic in the equations of motion takes the form

$$\delta_\lambda S[\Psi] = \int_X \left( (-)^{|A|+|B|} E_A E_B \lambda^C R_{C}^{BA} + \text{total derivatives} \right),$$  
(2.9)

where $R_{C}^{AB}$ are some local functions of fields.

Using (2.8) and (2.9) along with the identity $E^A \sigma_{AB} = E_B - T_B$, and requiring the coefficients at 0th, 1st, and 2nd orders in $E^A$ to vanish one finds

$$\lambda^C \partial_C K + (-)^{|A|+|B|} T_A T_B \lambda^C R_{C}^{BA} = 0,$$

$$\frac{1}{2} \lambda^C \partial_C T_A + (-)^{|A|+|B|+|D|+1} \sigma_{AD} T_B \lambda^C R_{C}^{BD} = 0,$$

$$\frac{1}{2} \lambda^C \partial_C \sigma_{NM} + (-)^{|B|+|N|(|A|+|M|)+n+1} \sigma_{MA} \sigma_{NB} \lambda^C R_{C}^{BA} = 0.$$  
(2.10)

Obviously, the above expressions are not covariant under general coordinate transformations on $\mathcal{M}$ unless 2-form $\sigma$ nondegenerate. Note that even in this case a quantity $R_{C}^{AB}$ is not a tensor but rather a connection. For the system to have a clear geometrical interpretation one therefore should assume that $\mathcal{M}$ is to be equipped with additional structures and work in the special coordinate systems only. Leaving the study of associated geometry and most general axioms for future work we now formulate some minimal (perhaps too restrictive) set of axioms that guarantee that the gauge theory under consideration is consistent. We shall see that even this partial setting is sufficient for a variety of meaningful examples, see Section 3.

2.2.1 Basic structures and compatibility conditions

Without trying to be exhaustive we now assume $\mathcal{M}$ to have a structure of a trivial vector bundle $\mathcal{M} = \mathcal{M}_0 \times \mathcal{M}_1$ and refer to $\mathcal{M}_0$ and $\mathcal{M}_1$ as horizontal and vertical submanifolds, respectively. By analogy with the AKSZ case, let $\mathcal{M}$ be equipped with the following structures.

- Ghost degree $\text{gh}(\cdot)$. For simplicity we assume that $\text{gh}(\Psi^A) \geq 0$. However, in general it can be useful to allow for negative degrees as in AKSZ or BV setting.
• 1-form (pre-symplectic potential) $\chi = d\Psi^A \chi_A$ such that $\text{gh}(\chi_A) = n - 1 - \text{gh}(\Psi^A)$, where $n$ is the positive integer.

• Odd vector field $Q = Q^A(\Psi) \frac{\partial}{\partial \Psi_A}$ such that $\text{gh}(Q) = 1$ (note that $Q$ is not necessarily nilpotent).

• Function $L$ (potential or generalized Hamiltonian), $\text{gh}(L) = n$.

In addition, we introduce the differential subalgebra $I$ in the algebra of differential forms on $M$ determined by the decomposition $M_0 \times M_1$. Let $\phi^\alpha, v^i$ be adapted coordinates on $M$, i.e., $\phi^\alpha$ are coordinates on $M_0$ and $v^i$ are coordinates on the fibres $M_1$. Then, subalgebra $I$ is generated by $\phi^\alpha$, $d\phi^\alpha$ and $dv^i$.

Note that just like in the AKSZ setting the structures defined on $M$ depend on the positive integer $n$ to be identified with the space-time dimension. This is in contrast to the formulation at the level of equations of motion where the target space structures are not aware of which space-time manifold is involved.

The compatibility conditions read as

$$\sigma \in I,$$  \hspace{1cm} (2.11)  

$$\mathcal{T} = dL - i_Q \sigma \in I,$$  \hspace{1cm} (2.12)  

$$\mathcal{K} = QL - \frac{1}{2} \sigma(Q, Q) \in I,$$  \hspace{1cm} (2.13)  

where $\sigma(Q, Q) = i_Q i_Q \sigma$ and $\mathcal{T} = d \Psi^A \mathcal{T}_A$, cf. (2.7). These are precisely the conditions \[ (2.10) \] with $R^C_{AB}$ put to zero. Note that 1-form $\chi$ enters the above expressions only through 2-form $\sigma = d\chi$, while the second condition implies $\mathcal{L}_Q \sigma \in I$, where $\mathcal{L}_Q$ denotes Lie derivative along $Q$. Indeed, acting by $d$ on both sides of (2.12) one obtains $di_Q \sigma \in I$, and then uses formula (A.9). The second condition generalizes that of $Q$ being a Hamiltonian vector field with the Hamiltonian $L$. The third condition is much less trivial and is a generalization of the master equation for $L$ or the nilpotency condition for $Q$.

Provided that $\Psi^A$ are the adapted coordinates $\phi^\alpha, v^i$ on $M$, the component versions of (2.11) - (2.13) read as

$$\frac{\partial}{\partial v^i} \sigma_{AB} = 0,$$  \hspace{1cm} (2.14)  

$$\frac{\partial}{\partial v^i} \mathcal{T}_A = \frac{\partial}{\partial v^i} (\partial_B L - Q^A \sigma_{AB}) = 0,$$  

$$\frac{\partial}{\partial v^i} \mathcal{K} = \frac{\partial}{\partial v^i} (Q^A \partial_A L - \frac{1}{2} Q^B Q^A \sigma_{AB}) = 0.$$  

2.2.2 Gauge symmetries

By construction, action (2.3) is invariant under diffeomorphisms. In addition, consider the following transformations

$$\delta_\lambda \Psi^A = d\lambda^A - \lambda^C \partial_C Q^A,$$  \hspace{1cm} (2.15)
where $\lambda^A$ is a gauge parameter associated to the field $\Psi^A$ as described below

$$
\Psi^A = \begin{cases} 
\phi^\alpha & \lambda^\alpha = 0 \\
v^i, \text{gh}(v^i) = 0 & \lambda^i = 0 \\
v^i, \text{gh}(v^i) > 0 & \lambda^i \neq 0
\end{cases} = \lambda^A \tag{2.16}
$$

In other words, if $\Psi^A$ is a horizontal coordinate $\phi^\alpha$, then $\lambda^\alpha = 0$, while if $\Psi^A$ is a vertical coordinate $v^i$, \text{gh}(v^i) = p > 0$, then $\lambda^i$ is non-zero, $\text{gh}(\lambda^i) = p - 1$. If $\text{gh}(v^i) = 0$, then an associated gauge parameter is absent, $\lambda^i = 0$.

In terms of coordinates $\phi^\alpha$, $v^i$ transformations (2.15) take the form

$$
\delta v^i = d\lambda^i - \lambda^j \partial_j Q^i, \quad \delta \phi^\alpha = -\lambda^j \partial_j Q^\alpha, \tag{2.17}
$$

where $\partial_i = \frac{\partial}{\partial v^i}$ and $\partial_\alpha = \frac{\partial}{\partial \phi^\alpha}$. This explains the need to introduce the direct product structure $\mathcal{M} = \mathcal{M}_0 \times \mathcal{M}_1$: the vertical fields have associated differential gauge parameters, while horizontal fields have not. Finally, with this choice of $\lambda$ conditions (2.11)-(2.13) imply (2.10) (with vanishing $R^{AB}_C$) and the action (2.3) is gauge invariant.

Note that setting $R^{AB}_C = 0$ is possible only if one uses special coordinate systems (more precisely those induced by coordinates on the factors $\mathcal{M}_0$ and $\mathcal{M}_1$). A formulation in generic coordinates on $\mathcal{M}$ would require an introduction of nontrivial $R^{AB}_C$ along with a certain covariant differential on $\mathcal{M}$. These interesting and important issues will be considered elsewhere.

### 2.2.3 The stronger compatibility conditions

The compatibility conditions (2.11) - (2.13) are invariant under $L \rightarrow L + h$ with $h \in I$ such that $Qh \in I$. It could be convenient to decompose $L$ as $L^0 - H_M$ where the “minimal” $L^0$ satisfies (2.12), while $H_M$ is a generic ghost degree $n$ element from $I$ satisfying $QH_M \in I$.

In particular, suppose one can choose $\chi$ such that $\sigma = d\chi$ satisfies (2.11) and

$$
\mathcal{L}_{Q\chi} \in I . \tag{2.18}
$$

In this case using $L_Q = i_Q d - di_Q$ one finds $d(i_Q\chi) - i_Q\sigma \in I$ so that ”minimal” $L^0$ chosen as $L^0 = i_Q\chi$ satisfies (2.12), where $i_Q\chi = Q^A \chi_A$. The remaining freedom in $L$ is then described by the potential $H_M$, so that a general $L$ satisfying (2.12) can be represented as

$$
L = Q^A \chi_A - H_M, \quad H_M \in I \text{ and } QH_M \in I . \tag{2.19}
$$

Let us consider now condition (2.13). It is enough to check the condition for $L = L^0$. Condition (2.13) can be then written as $i_QdQ\chi - \frac{1}{2}i_Qi_Qd\chi \in I$. Using (A.7), (A.9), (A.11), along with the identity $i_QdQ\chi = \frac{1}{2} \mathcal{L}_{Qi_Q} \chi + \frac{1}{2} i_QdQ\chi$, one identically rewrites condition (2.13) as

$$
[\mathcal{L}_Q, i_Q] \chi \equiv i_Q^2 \chi \in I . \tag{2.20}
$$
It follows that for nilpotent $Q$ condition (2.13) is identically satisfied. To summarize: if one starts with nilpotent $Q$ and $\chi, H_M$ such that $\mathcal{L}_Q \chi \in I$, $H_M \in I$, $QH_M \in I$ then all the compatibility conditions are fulfilled. In all the examples we consider below the basic objects satisfy this more restrictive conditions.

Quite often condition (2.18) takes a strong form $\mathcal{L}_Q \chi = 0$ so that $dL = i_Q \sigma$. Then, condition (2.12) amounts to $QL = 0$ and $\sigma(Q, Q) = 0$.

3 Examples

3.1 The Einstein gravity

Let us consider the frame formulation of ordinary gravity. To apply the general construction developed in Section 2 one takes $\mathcal{M} = \Pi g$ with Poincaré algebra $g = \text{iso}(n - 1, 1)$

$$\begin{align*}
[L_{ab}, L_{cd}] &= \eta_{ac} L_{db} - \eta_{bc} L_{da} - \eta_{ad} L_{cb} + \eta_{bd} L_{ca} , \\
[P_a, L_{bc}] &= \eta_{ab} P_c - \eta_{ac} P_b , \\
[P_a, P_b] &= 0 ,
\end{align*}$$

(3.1)

where $\eta_{ab}$ is a flat canonical Minkowski metric, and indices run $a, b, \ldots = 0, \ldots, n - 1$. Coordinates on $\mathcal{M}$ are Grassmann odd $e^a$ and $\omega^{ab} = -\omega^{ba}$ so that an arbitrary element in $\mathcal{M}$ is parameterized as $\Psi = e^a P_a + \frac{1}{2} \omega^{ab} L_{ab}$. Coordinates $\omega^{ab}$ are vertical, while coordinates $e^a$ are horizontal. Odd vector field $Q$ on $\mathcal{M}$ is given by components

$$Q e^a = \omega^a e^c , \quad Q \omega^{ab} = \omega^a \omega^{cb} ,$$

(3.2)

so that $Q$ is the Chevalley-Eilenberg differential for the Poincaré algebra, $Q^2 = 0$.

As a presymplectic potential $\chi$ we take

$$\chi = de^a \chi_a + \frac{1}{2} d\omega^{ab} \chi_{ab} = \frac{1}{(n - 2)!} \epsilon_{abm_1 \ldots m_{n-2}} \omega^{ab} e^{m_1} \ldots e^{m_{n-2}} .$$

(3.3)

It is $Q$-invariant thanks to $o(n - 1, 1)$-invariance of the Levi-Civita tensor $\epsilon_{a_1 \ldots a_n}$ in $n$ dimensions.

The associated presymplectic 2-form $\sigma$ computed according to (A.3) reads as

$$\sigma = d\omega^{ab} de^c \mathcal{V}_{abc} .$$

(3.4)

Here and in what follows we use the following notation for the generalized volume forms

$$\mathcal{V}_{a_1 \ldots a_p} = \frac{1}{(n - p)!} \epsilon_{a_1 \ldots a_p c_1 \ldots c_{n-p}} e^{c_1} \ldots e^{c_{n-p}} , \quad p = 0, \ldots, n ,$$

(3.5)

that satisfy the identity

$$e^c \mathcal{V}_{a_1 \ldots a_p} = \mathcal{V}_{a_1 a_2 \ldots a_{p-1} c} \delta^c_{a_p} ,$$

(3.6)

where all indices $a_i$ are antisymmetrized with a unit weight.
Choosing potential \( L \) as

\[
L = Q^A \chi_A = \omega^a e^b \nabla_{ab},
\]

the corresponding action functional \((\ref{eq:3.7})\) takes the familiar form

\[
S_{GR}[\epsilon, \omega] = \int \left[ d\omega^{ab} + \omega^a e^b \right] V_{ab}.
\]

Here and in what follows the integral is taken over space-time manifold \( X \) unless otherwise specified. This is precisely the standard frame-like action of the Einstein gravity with zero cosmological constant. The expression in parenthesis is identified with the 2-form Lorentz curvature \( R^{ab} = d\omega^{ab} + \omega^a e^b \). Here, the ghost number zero component fields \( e^a = e^a_\mu dx^\mu, \omega^{ab} = \omega^{ab}_\mu dx^\mu \) enter \( e^a(x, \theta) \) and \( \omega^{ab}(x, \theta) \) as follows

\[
e^a = \theta^\mu e^a_\mu, \quad \omega^{ab} = \theta^\mu \omega^{ab}_\mu.
\]

One can explicitly check that all the compatibility conditions \((\ref{eq:2.14})\) are fulfilled. Using \((\ref{eq:3.7})\) one shows that \( L_Q \chi = 0 \) because the Levi-Civita tensor is Lorentz invariant, while both \( e^a \) and \( d\omega^{ab} \) are transformed by \( Q \) as a vector and bivector, respectively. Following the discussion of Section \(2.2.3\) one then easily proves the compatibility conditions in the form \((\ref{eq:2.11}) - (\ref{eq:2.13})\).

Action \((\ref{eq:3.8})\) can be non-trivially extended provided function \( L \) \((\ref{eq:3.7})\) is augmented as follows

\[
L \to L - H_M,
\]

where \( H_M \) is a new function such that \( H_M \in I, QH_M \in I \) and \( \text{gh}(H_M) = n \). Taking into account the ghost degree and the compatibility conditions the only nontrivial choice is to take \( H_M \) proportional to the volume \( n! \Lambda \epsilon_1 \ldots \epsilon_n \equiv \Lambda V \), where we generally trade prefactor \( \Lambda \) for the cosmological constant. The action with added term \((\ref{eq:3.10})\) is still invariant under local Lorentz symmetry transformations and therefore vertical and horizontal fields as well as vector field \( Q \) remain unchanged. Another way to obtain the action with \((\ref{eq:3.10})\) is to consider \( o(d-1,2) \)-covariant formalism for (anti-)de Sitter gravity, see Section \(3.5\).

### 3.1.1 Gauge symmetries.

Gauge transformations \((\ref{eq:2.17})\) with parameter \( \lambda^{ab}(x) \) are just conventional local Lorentz gauge transformations

\[
\delta_L \lambda^{ab} = -\chi^a \lambda^b + \chi^b \lambda^a,
\]

\[
\delta_L \omega^{ab} = \partial_\mu \lambda^{ab} - \lambda^c \omega^{ab}_\mu + \lambda^b \omega^{ca}_\mu + \lambda^a \omega^{bc}_\mu.
\]

The remaining gauge symmetries of the action \((\ref{eq:3.8})\) are precisely diffeomorphisms

\[
\delta_{\xi} \epsilon^a_\mu = \xi^\nu \partial_\nu \epsilon^a_\mu + \partial_\mu \xi^\nu \epsilon^a_\nu,
\]

\[
\delta_{\xi} \omega^{ab}_\mu = \xi^\nu \partial_\nu \omega^{ab}_\mu + \partial_\mu \xi^\nu \omega^{ab}_\nu.
\]

The point is that in contrast to Lorentz transformations frame-like gravity action \((\ref{eq:3.8})\) is not invariant with respect to transformations originating in translation.
subalgebra of the Poincaré algebra. The respective variation is proportional to
\[ \int R^a R^b \lambda^a e_{abcdm_1 m_2 \ldots m_d} \epsilon^{m_1 \ldots m_d} - 4 \epsilon^d \]
where \( \lambda^a \) is a gauge parameter associated with translations, i.e., with the frame field \( e^a \), while \( R^a \) and \( R^a \) are Lorentz curvature and the torsion. Obviously, in the linearized gravity the symmetry is restored.

At the nonlinear level one can still relate Poincaré translations to diffeomorphisms through
\[ \delta_{\xi}^{\text{transl}} e^a_\mu = \delta_{\xi}^{\text{diff}} e^a_\mu + \delta_{\lambda}^{\text{Lor}} e^a_\mu + R^a_{\mu \nu} \xi^\nu, \tag{3.13} \]
where diffeomorphism \( \xi^\mu \) and frame \( \xi^a \) vector parameters are related as \( \xi^\mu = e^a_\mu \xi^a \), and \( \lambda^{ab} = \omega^{abc} \xi^c \). Moreover, on the stationary surface diffeomorphisms can be expressed just in terms of translations and Lorentz rotations. Indeed, the equation of motion \( \delta S_{GR} / \delta \omega_{\mu}^{ab} = 0 \) implies that the torsion vanishes, \( R^a_{\mu \nu} = 0 \) so that on-shell one can identify Poincaré translations with diffeomorphisms. Furthermore, the vanishing torsion constraint expresses Lorentz connections via derivatives of the frame fields. It turns out that for the gravity the vertical and horizontal fields are respectively auxiliary and dynamical ones.

One concludes that the splitting between horizontal and vertical coordinates in the target space \( M \) naturally fits the gauge structure of the Poincaré gauge gravity. Vertical gauge transformations along with diffeomorphisms produce translational symmetry not seen within the geometrical setting of the theory.

### 3.2 Gravity + scalar field

To describe scalar field coupled to gravity we extend the superspace \( M \) of Section 3.1 by the horizontal coordinates \( \phi \), \( gh(\phi) = 0 \) and \( \pi^a \), \( gh(\pi^a) = 0 \). Additional components of the presymplectic potential (3.3) and odd vector field (3.2) are
\[ \chi(\partial / \partial \pi^a) = 0, \quad \chi(\partial / \partial \phi) = \pi^a \nu_a, \tag{3.14} \]
and
\[ Q \phi = 0, \quad Q \pi^a = \omega^a_{b} \pi^b. \tag{3.15} \]
Note that \( Q^2 = 0 \). By analogy with the pure gravity, condition \( L_Q \chi = 0 \) immediately follows from the invariance of the Levi-Civita tensor.

Consider function \( L = Q^A \chi_A - H_M \). As \( H_M \) we take scalar field covariant Hamiltonian (in the sense of de Donder–Weyl formalism) multiplied by the volume form, i.e.,
\[ H_M = (1/2 \pi^a \pi_a + m^2 \phi^2) \nu, \tag{3.16} \]
which is \( Q \)-invariant and depends on horizontal coordinates only. Coupling \( m^2 \) is the mass of a scalar. The entire action is familiar and reads as
\[ S[e, \omega, \phi, \pi] = S_{GR}[e, \omega] + \int [d\phi \pi^a \nu_a + (\pi^a \pi_a - m^2 \phi^2) \nu], \tag{3.17} \]
where the gravity action $S_{GR}[\epsilon, \omega]$ is given by (3.8). Using the identity (3.6) and eliminating $\pi^a$ one obtains

$$S[\epsilon, \omega, \phi] = S_{GR}[\epsilon, \omega] - \frac{1}{2} \int (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2) \mathcal{V}, \quad g^{\mu\nu} := e^a_\mu e^a_\nu.$$  \hspace{1cm} (3.18)

It is also instructive to write action (3.17) over the Minkowski background described by connections $\omega^{ab} = 0$ and $e^a = dx^a$. Then, action (3.17) takes the standard form

$$S[\phi, \pi] = \int d^n x (\pi^a \partial_a \phi + \frac{1}{2} \pi^a \pi_a - \frac{1}{2} m^2 \phi^2),$$  \hspace{1cm} (3.19)

which is the well-known 1st order massive scalar field action. In parallel, in the flat case the scalar field contribution to the presymplectic 1-form $\chi$ becomes

$$\frac{1}{(n-1)!} d\phi \pi^a dx^{b_1} \ldots dx^{b_{n-1}} \varepsilon_{a_{b_1} \ldots b_{n-1}}.$$  \hspace{1cm} (3.20)

It is a version of the canonical $n$-form of the covariant Hamiltonian formulation for the scalar field, see, e.g., [23].

### 3.3 Einstein-Yang-Mills theory

Consider Yang-Mills theory with the gauge algebra $g$, which is given by basis elements $[T^I, T^J] = C^{IJ}_K T^K$ and have the Killing form $\langle T^I, T^J \rangle = \delta^{IJ}$. To describe Yang-Mills theory minimally coupled to gravity we extend the superspace $M$ of the pure gravity of Section 3.1 with the following extra vertical coordinate $A^I$, $\text{gh}(A) = 1$ and extra horizontal coordinate $F^I_{ab} = -F^I_{ba}$, $\text{gh}(F^I_{ab}) = 0$.

The new components of the presymplectic potential (3.3) and odd vector field (3.2) are

$$\chi\left(\frac{\partial}{\partial A^I}\right) = F^I_{ab} \mathcal{V}_{ab}, \quad \chi\left(\frac{\partial}{\partial F^I_{ab}}\right) = 0,$$  \hspace{1cm} (3.21)

and

$$QA = \frac{1}{2} [A, A], \quad QF^I_{ab} = \omega^I_{a} F^c_{I} + \omega^I_{b} F^c_{I} + [A, F^I_{ab}].$$  \hspace{1cm} (3.22)

One can identify $Q$ as a Lie algebra cohomology differential for a direct sum of the Poincaré and the YM gauge algebra represented on $F^I_{ab}$. That $L_Q \chi = 0$ can be checked as follows. The YM contribution to $\chi$ can be written as

$$\chi_{YM} = \langle dA, F^I_{ab} \rangle \mathcal{V}_{ab}.$$  \hspace{1cm} (3.23)

Decomposing $Q$ as $Q_{GR} + Q_{YM}$ one finds that $L_{Q_{GR}} \chi_{YM} = 0$ in exactly the same way as before, while $L_{Q_{YM}} \chi_{YM} = 0$ follows from the invariance of the Killing form and the relation $L_{Q_{YM}} dA = [A, dA]$.

Function $L = \chi_A Q^A - H_M$ is given explicitly by

$$L = L_{GR} + \langle F^I_{ab}, [A, A] \rangle \mathcal{V}_{ab} - H_M.$$  \hspace{1cm} (3.24)
As $H_M$ we again take the de Donder–Weyl generalized Hamiltonian of the YM field multiplied by the volume form

$$H_M = -V \langle F^{ab}, F_{ab} \rangle,$$

(3.25)

which again depends on horizontal coordinates only and satisfies $QH_M = 0$.

The action takes the familiar form

$$S[e, \omega, A, F] = S_{GR}[e, \omega] + \int \langle dA + \frac{1}{2} [A, A], F^{ab} \rangle \mathcal{V}_{ab} + \int \langle F^{ab}, F_{ab} \rangle \mathcal{V},$$

where the gravity action $S_{GR}[e, \omega]$ is given by (3.8). Using the identity (3.6) and eliminating $F^{ab}_I$ one obtains

$$S[e, \omega, A] = S_{GR}[e, \omega] - \frac{1}{4} \int g^{\mu \rho} g^{\nu \sigma} \langle F^I_{\mu \nu}, F^I_{\rho \sigma} \rangle,$$

(3.27)

where $F^I_{\mu \nu} := \partial_{(\mu} A^I_{\nu)} - \partial_{\nu} A^I_{\mu} + [A_{(\mu}, A_{\nu)}]I$.

As before, in the flat background action (3.26) reduces to the well-known 1st order Yang-Mills action (actions of this type were originally proposed in [30])

$$S[A] = \int d^nx (F^{ab}_I (\partial_a A^I_b - \partial_b A^I_a + [A_a, A_b]^I) - F^{ab}_I F^{ab}_I).$$

(3.28)

### 3.4 Multi-frame theory

One can extend a target $M$ of Poincaré gravity by adding $\mathcal{N}$ copies of $\Pi g$ with $g$ Poincaré algebra (3.1) so that $M = \Pi (g \oplus g \oplus \ldots g)$. Grassmann odd variables $\omega^{ab}(i)$ and $e^a(i)$ where $i = 1, \ldots, \mathcal{N}$, are vertical and horizontal, respectively. Odd vector field generalizes (3.2) as

$$Q e^a(i) = \omega^a_c(i) e^c(i), \quad Q \omega^{ab}(i) = \omega^a_c(i) \omega^{cb}(i).$$

(3.29)

The presymplectic forms on $M$ are then sums of those on $\Pi g$, cf. (3.3), (3.4),

$$\chi = \sum_{i=1}^{\mathcal{N}} \chi(i), \quad \sigma = \sum_{i=1}^{\mathcal{N}} \sigma(i).$$

(3.30)

As $L$ we chose a direct generalization of that in the case of gravity:

$$L = \sum_{i=1}^{\mathcal{N}} Q^A \chi_A(i) + \sum_{i=1}^{\mathcal{N}} \frac{1}{n!} \Lambda(i) \epsilon_{m_1 \ldots m_n} e^{m_1}(i) \ldots e^{m_n}(i)$$

(3.31)

where $\Lambda(i)$ are the cosmological constants associated to each gravity sector.

To introduce multi-graviton interactions one mixes frames from different sectors adding terms like $\epsilon_a \ldots e^{a_1}(i_1) \ldots e^{a_n}(i_n)$ [31]. However, such terms break $\mathcal{N}$ local Lorentz symmetry groups (3.29) down to the diagonal subgroup. To restore $\mathcal{N} - 1$ Lorentz symmetries one introduce $\mathcal{N} - 1$ compensating fields.
To this end one further extends $\mathcal{M}_N$ to include new Grassmann odd coordinates $K^a_b(j)$, $j = 2, \ldots, N$ satisfying the matrix constraint $K^T \eta K = \eta$, so that the presymplectic 1-from (3.30) is unchanged, while odd vector field (3.29) acquires new components

$$QK^a_b(j) = \omega^a_c(1)K^c_b(j) - K^a_c(j)\omega^c_b(j), \quad j = 2, \ldots, N.$$  \hspace{1cm} (3.32)

Fields $K^a_b$ are horizontal. These are introduced in such a way that $Q$ acts on frames $\mathcal{e}^b(j) \equiv K^a_b(j)\mathcal{e}^b(j)$ as follows

$$Q[K^a_b(j)\mathcal{e}^b(j)] = \omega^a_b(1)[K^b_c(j)\mathcal{e}^c(j)],$$  \hspace{1cm} (3.33)

so that $\mathcal{e}^a$ can be used to build interaction cross-term supporting an overall local Lorentz invariance.

Leaving intact the presymplectic forms on $\mathcal{M}$ one modifies function $L$ as follows

$$L = Q^A\chi_A + \sum_{i_1, \ldots, i_N = 1}^{N} \beta^{i_1 \ldots i_N} \epsilon_{a_1 \ldots a_N} \left[ K^{a_1}_{b_1}(i_1)\mathcal{e}^{b_1}(i_1) \right] \cdot \left[ K^{a_N}_{b_N}(i_N)\mathcal{e}^{b_N}(i_N) \right],$$  \hspace{1cm} (3.34)

where $\beta^{i_1 \ldots i_N}$ are totally symmetric coupling constants, while $K^a_b(1) = \delta^a_b$. Then, the multi-frame action takes the form

$$S[e, \omega, K] = \int \left[ \sum_{i = 1}^{N} \left( d\omega^{ab}(i) + \omega^a_c(i)\omega^c_b(i) \right) V_{ab}(i) \right] + \sum_{i_1, \ldots, i_N = 1}^{N} \beta^{i_1 \ldots i_N} \epsilon_{a_1 \ldots a_N} \left[ K^{a_1}_{b_1}(i_1)\mathcal{e}^{b_1}(i_1) \right] \cdot \left[ K^{a_N}_{b_N}(i_N)\mathcal{e}^{b_N}(i_N) \right].$$  \hspace{1cm} (3.35)

By construction, the above action is invariant with respect to local Lorentz symmetry transformations read off from (3.29) and (3.32),

$$\delta^{\text{Lor}}_{\lambda}\omega^{ab}(i) = d\lambda^a(i) - \lambda^{ac}(i)\omega^c_b(i) + \lambda^b(i)\omega^a_c(i),$$  \hspace{1cm} (3.36)

$$\delta^{\text{Lor}}_{\lambda}\mathcal{e}^a(i) = -\lambda^b(i)\mathcal{e}^b(i), \quad \delta^{\text{Lor}}_{\lambda}K^a_b(j) = -\lambda^c(1)K^c_b(j) + \lambda^b(j)K^a_c,$$

where $i = 1, \ldots, N$ and $j = 2, \ldots, N$. Fields $K^a_b(j)$ are Stueckelberg fields that compensate broken local Lorentz symmetry. Using the gauge $K^a_b(j) = \delta^a_b$ one finds out that parameters $\lambda^{ab}(i)$ are set to satisfy matrix constraints

$$-\lambda(1)K(j) + K(j)\lambda(j) = 0, \quad j = 2, \ldots, N,$$  \hspace{1cm} (3.37)

for the gauge fixed $K(j) = \mathbb{I}_n$. It follows that $\lambda \equiv \lambda(1) = \lambda(2) = \ldots = \lambda(N)$, and the action retains a single local Lorentz symmetry with a common parameter $\lambda$ for all frame fields. The gauge fixed form of action (3.33) has been shown to describe consistent interactions of a single massless spin-2 field and $N - 1$ massive spin-2 fields $[31, 32]$.  

\footnote{This actions is similar to that proposed in [32], where the authors introduced $N$ Stueckelberg fields and operate with $N + 1$ unbroken local Lorentz invariances. Action (3.33) contains the minimum required number of $N - 1$ Stueckelberg fields needed to compensate $N - 1$ broken local Lorentz symmetries.}
3.5 First-order form of MMSW action for $\text{AdS}_n$ gravity

3.5.1 $n = 4$

Consider a superspace $M_0$ with Grassmann odd, vertical coordinates $\omega^{AB} = -\omega^{BA}$, $\text{gh}(\omega^{AB}) = 1$, and Grassmann even, horizontal coordinates $V^A$, $\text{gh}(V^A) = 0$ and $F^{AB} = -F^{BA}$, $\text{gh}(F^{AB}) = 2$. Take as $M \subset M_0$ the surface singled out by the following constraints

$$V^AV_A = -1, \quad F^{AB}V_B = 0.$$  (3.38)

Here and below indices $A, B, \ldots = 0, \ldots, n$ are raised and lowered by a flat canonical Minkowski metric $\eta_{AB} = (- - + \ldots)$. Odd vector field $Q$ with components

$$Q\omega^{AB} = \omega^A C\omega^{CB}, \quad QV^A = \omega^A B V^B, \quad QF^{AB} = \omega^A C F^{CB} + \omega^B C F^{AC}$$  (3.39)

can be identified with the Chevalley-Eilenberg differential of $o(n-1,2)$ algebra with coefficients in representation on vector $V$ and tensor $F$. Odd vector field $Q$ is tangent to $M$ (i.e., constraints (3.38) are invariant on the constraint surface).

Specializing to $n = 4$ we take a presymplectic potential $\chi$ in the form

$$\chi(\frac{\partial}{\partial V^A}) = \chi(\frac{\partial}{\partial F^{AB}}) = 0, \quad \chi(\frac{\partial}{\partial \omega^{AB}}) = \epsilon_{ABCDE} F^{CD} V^E,$$  (3.40)

while function $L$ is given by $L = Q^A \chi_A - H_M$, where

$$H_M = -\frac{1}{2} \epsilon_{ABCDE} F^{AB} F^{CD} V^E.$$  (3.41)

One can show that all the compatibility conditions (2.11)-(2.13) are satisfied due to $Q^2 = 0$ and the invariance of the $o(n-1,2)$ Levi-Civita tensor.

According to (2.3) we build the following action

$$S[\omega, V, F] = \int \epsilon_{ABCDE} (d\omega + \omega \omega)^{AB} F^{CD} V^E + \frac{1}{2} \epsilon_{ABCDE} F^{AB} F^{CD} V^E.$$  (3.42)

The above action can be shown to be dynamically equivalent to the standard MacDowell-Mansouri-Stelle-West (MMSW) action [33, 34], see formula (3.44) below. To this end, let us consider the Euler-Lagrange equation for $F^{AB}$ given by

$$\epsilon_{ABCDE} F^{CD} V^E + \epsilon_{ABCDE} (d\omega + \omega \omega)^{CD} V^E = 0.$$  (3.43)

Since this equation is algebraic with respect to $F^{AB}$, it is enough to solve it at a given point. By using the first equation in (3.38) one takes field $V^A$ in the form $V^A = \delta^A_{(n)}$, and finds that $F^{ab} = -(d\omega + \omega \omega)^{ab}$ where $a, b = 1, \ldots, n - 1$. Other components of $F$ vanish by virtue of the second condition in (3.38), and, moreover, they do not contribute to either of the terms in the action. One then concludes that $F^{AB}$ are auxiliary fields, and their elimination gives the MMSW gravity action

$$S[\omega, V] = -\frac{1}{2} \int \epsilon_{ABCDE} R^{AB} R^{CD} V^E, \quad R^{AB} := (d\omega + \omega \omega)^{AB}.$$  (3.44)
3.5.2 $n > 4$

In addition to the variables introduced in the case $n = 4$, let us also introduce extra horizontal variables $E^A$, $gh(E^A) = 1$ and $\pi_A$, $gh(\pi_A) = n - 1$ subjected to constraints

$$E^AV_A = 0, \quad \pi_AV^A = 0. \tag{3.45}$$

The nonvanishing components of presymplectic potential $\chi$ are

$$\chi\left(\frac{\partial}{\partial V^A}\right) = \pi_A, \quad \chi\left(\frac{\partial}{\partial \omega^{AB}}\right) = \frac{1}{(n-2)!} \epsilon_{ABCDB_1\ldots B_{n-4}E} F^{CD} E^{B_1} \ldots E^{B_{n-4}} V^E. \tag{3.46}$$

The odd vector field $Q$ determined by

$$Q\omega^{AB} = \omega^A_C \omega^{CB}, \quad QF^{AB} = \omega^A_C F^{CB} + \omega^B_C F^{AC}, \quad QV^A = \omega^A_B V^B, \quad QE^A = \omega^A_B E^B, \quad Q\pi_A = -\omega^B_A \pi_B, \tag{3.47}$$

is identified with the Chevalley-Eilenberg differential of $o(n - 1, 2)$ algebra with coefficients in representation on vectors $V$, $E$, $\pi$, and tensor $F$. One can check that the constraints (3.45) are $Q$-invariant while $L_Q \chi \in I$. More precisely,

$$L_Q \chi = -d\omega^A_B V^B \pi_A. \tag{3.48}$$

Note that in contrast to all the previous examples $L_Q \sigma \neq 0$.

Function $L$ is given by $L = Q^A \chi_A - H_M$, where

$$H_M = -\frac{1}{2(n-2)!} \epsilon_{ABCDB_1\ldots B_{n-4}E} F^{AB} F^{CD} E^{C_1} \ldots E^{C_{n-4}} V^E - E^A \pi_A, \tag{3.49}$$

so that the action takes the following form

$$S[\omega, V, F, E, \pi] = \int \left[ (dV^A + \omega^A_B V^B - E^A) \pi_A + \frac{1}{(n-2)!} \epsilon_{ABCDB_1\ldots B_{n-4}E} ((d\omega + \omega^C) F^{CD} + \frac{1}{2} F^{AB} F^{CD}) E^{C_1} \ldots E^{C_{n-4}} V^E \right]. \tag{3.50}$$

Fields $E_A$ and $\pi_A$ are clearly auxiliary. As before, the same is also true for $F^{AB}$. Eliminating the auxiliary fields by their own equations of motion one finally gets the standard action

$$S[\omega, V] = -\frac{1}{2(n-2)!} \int \epsilon_{ABCDB_1\ldots B_{n-4}E} R^{AB} R^{CD} E^{C_1} \ldots E^{C_{n-4}} V^E, \tag{3.51}$$

where $E^A := (dV + \omega V)^A$ and $R^{AB} := (d\omega + \omega)AB$.

3.6 Linearized frame-like actions

It is instructive to examine the perturbation theory for action (2.3) over background field values $\Psi^A_0$. The fluctuations over the background are defined as $\Psi^A = \Psi^A_0 + \ldots$.
Then, one finds that modulo additive constants and total derivative terms the quadratic action reads

\[
S_0[\Phi] = \int \left[ d\Phi^A \Phi^B \partial_B \chi_A(\Psi_0) + \frac{1}{2} \Phi^A \Phi^B \partial_B \partial_A L(\Psi_0) + \frac{1}{2} d\Psi_0^C \Phi^A \Phi^B \partial_B \partial_A \chi_C(\Psi_0) \right].
\]

(3.52)

The above action is again of the form (2.3). Adding total derivatives all the dependence on \(\chi_A\) can be expressed in terms of \(\sigma_{AB}\). More precisely, adding

\[
-\frac{1}{2} d(\Phi^A \Phi^B \partial_B \chi_A) = -\frac{1}{2} \left[ d\Phi^A \Phi^B (\partial_B \chi_A(\Psi_0) + (-)^{|A||B|} \partial_A \chi_B(\Psi_0)) \right. \\
+ (-1)^{|A|+|B|} \Phi^A \Phi^B d\Psi_0^C \partial_C \partial_B \chi_A(\Psi_0) \right]
\]

(3.53)

to (3.52) gives

\[
S_0[\Phi] = -\frac{1}{2} \int \left[ (-)^{|A|} d\Phi^A \Phi^B \sigma_{BA}(\Psi_0) - \right. \\
- \left. \Phi^A \Phi^B (\partial_B \partial_A L(\Psi_0) - (-)^{|C|} d\Psi_0^C \partial_B \sigma_{AC}(\Psi_0)) \right].
\]

(3.54)

The presymplectic 2-form \(\tilde{\sigma}_{AB} = \sigma_{AB}(\Psi_0)\) entering the expression for the linearized action does not depend on field variables. It may imply an extension of the compatibility conditions (2.14) in such a way that the quadratic approximation has more vertical fields than the original theory. Also, in deriving the quadratic action we do not assume the original action invariant under gauge symmetry transformations (2.15) or (2.17).

### 3.6.1 Linearized gravity

Let us consider the gravity action (3.8) linearized around Minkowski spacetime \(\mathbb{R}^{n-1,1}\) given by \(\omega_0^{ab} = 0\) and \(e_0^a = dx^a\). Let \(e^a\) and \(\omega^{ab}\) again denote dynamical fields. The linearized presymplectic 2-form is given by

\[
\tilde{\chi} = d\omega^{ab} e^c \tilde{V}_{abc}, \quad \tilde{\sigma} = d\omega^{ab} de^c \tilde{V}_{abc},
\]

(3.55)

where the basis forms \(\tilde{V} = \tilde{V}(\epsilon_0)\) (3.3) are built of the background frame field \(e_0^a\), while the linearized Poincaré algebra differential is determined by

\[
\hat{Q} e^a = \omega^a_c e_0^c, \quad \hat{Q} \omega^{ab} = 0.
\]

(3.56)

The linearized potential takes the form

\[
L = \omega^a_c \omega^{cb} \tilde{V}_{ab}.
\]

(3.57)

Finally, the linearized action (3.52) takes the form

\[
S_0[e, \omega] = \int \left[ d\omega^{ab} e^c \tilde{V}_{abc} + \omega^a_c \omega^{cb} \tilde{V}_{ab} \right].
\]

(3.58)
Adding a total derivative and using (3.54) along with \( de_0 = 0 \) satisfied by background it can be equivalently represented as

\[
S_0[e, \omega] = \int \left[ de^a - \frac{1}{2} \omega^a d e_0^d \right] \omega^{bc} V_{abc},
\]

(3.59)

which is manifestly invariant with respect to the gauge symmetry transformations \( \delta \lambda^a = d \lambda^a \) and \( \delta \omega^{ab} = 0 \), originating from local translations. One concludes that all fields can be treated now as vertical ones. Indeed, the extended compatibility conditions (2.14) are valid in this case because components of the presymplectic 2-form are field-independent.

Action (3.59) is directly generalized to higher spin fields which we consider next.

### 3.6.2 Frame-like action for spin-\( s \) massless fields

Consider a target superspace \( M \) with odd coordinates

\[
\omega^{a_1...a_{s-1},b_1...b_t}, \quad t = 0, ..., s - 1,
\]

(3.60)

which are irreducible Lorentz tensors satisfying tracelessness and Young symmetry conditions. Spin parameter \( s \) is some integer \( s = 1, 2, ... \). We assume that all coordinates are vertical ones. The odd vector field \( Q \) on \( M \) is given by components

\[
Q \omega^{a_1...a_{s-1},b_1...b_t} = \omega^{a_1...a_{s-1},b_1...b_t} e_0^e,
\]

(3.61)

where just like in the case of spin-2 field \( e_0 \) is an extra odd variable of ghost degree 1 interpreted as a target space parameter giving rise to background frame field. It is nilpotent, \( Q^2 = 0 \), and can be seen as a generalization of the linearized spin-2 differential (3.56).

In terms of field variables odd vector field \( Q \) induces the following gauge transformations

\[
\delta \lambda^a \omega^{a_1...a_{s-1},b_1...b_t} = d \lambda^a \omega^{a_1...a_{s-1},b_1...b_t} + \lambda^a \omega^{a_1...a_{s-1},b_1...b_t} e_0^e,
\]

(3.62)

where \( e_0 \) is the background frame field satisfying \( de_0 = 0 \) (more generally, to work in terms of generic frame one also introduces background Lorentz connection \( \omega_0^{ab} \) so that \( de_0 + \omega_0 e_0 = 0 \) and \( d \omega_0 + \omega \omega_0 = 0 \), i.e. the zero-curvature equations of the Poincaré algebra). Fields \( \omega^{a_1...a_{s-1},b_1...b_t} = d \omega^a \omega^{a_1...a_{s-1},b_1...b_t} \) are differential 1-forms on the spacetime manifold, while the gauge parameters \( \lambda^a \omega^{a_1...a_{s-1},b_1...b_t} \) are 0-forms.

Consider the following presymplectic 1-form and potential

\[
\chi = d \omega^{a_1...m_s...2} \omega^{m_1...m_{s-2}} b,c V_{abc},
\]

\[
L = \frac{1}{2} \omega^{a_1...m_s...k} \omega^{m_1...m_{s-2}} b,c e_0^k V_{abc},
\]

(3.63)

The associated 2-form reads as

\[
\sigma = d \omega^{a_1...m_s...2} d \omega^{m_1...m_{s-2}} b,c \bar{V}_{abc}.
\]

(3.64)
The presymplectic 2-form depends on the background frame field only that conforms with the compatibility condition \((2.11)\).

The above prerequisites are used to build an action functional according to \((2.3)\) as follows

\[
S_0[\omega] = \int \left[ d\omega^{am_1...m_{s-2}} - \frac{1}{2} \omega^{am_1...m_{s-2},k} \epsilon_{0k} \right] \omega^{b,c}_{m_1...m_{s-2}} V_{abc}, \tag{3.65}
\]

which is the frame-like action of massless spin-\(s\) fields on Minkowski spacetime \([3]\). This is a generalization of spin-2 case \((3.59)\).

It is important to note that \((3.65)\) does not depend on fields \(\omega^{a_1...a_{s-1},b_1...b_t}\) with \(t \geq 2\) (called extra fields). This is a new feature not present in other examples: the system contains fields that do not enter the action and hence should not be considered dynamical. We do not discuss here how this subtlety can be naturally handled in the formalism. \(\footnote{The extra fields do contribute to the higher spin action when considering interactions of FV-type [36, 35, 37, 10, 38].}\) In the next section we propose an alternative formulation where all the fields are at the equal footing. Furthermore, in Section 4.1 we describe generic mechanical systems from the frame-like perspective that naturally leads to the notion of extra fields as fields not entering a Lagrangian of a system but producing (a part of) its gauge symmetry invariance, see our comments below formula \((4.9)\).

It is claimed that the above action is the most general one containing only dimensionless coefficients. Generally, one may consider an action functional \((2.3)\) on \(M\) \((3.60)\) where all fields contribute and not only the two lowest rank ones identified with the generalized frame and Lorentz connections. However, using the dimensional analysis one finds out that other possible terms with extra fields in the searched-for action necessarily contain dimensionful coefficients. The extra field terms can either be set to zero directly (i.e., the corresponding coefficients), or be combined into total derivatives. Whence, the only scale invariant combination is given by \((3.65)\). The analogous reasoning applies in the case of \(AdS\) background, where the higher spin action of the type \((2.3)\) may have terms involving extra fields with overall dimensional coefficients proportional to inverse powers of the cosmological constant. One may require all such terms to combine into total derivatives giving the \(AdS\) higher spin action of the form \((3.65)\) with function \(L\) shifted as in \((3.10)\). This is in fact the extra field decoupling condition of Lopatin and Vasiliev \([5]\).

The above consideration of free higher-spin Lagrangians visualized as the presymplectic AKSZ-type models has many features in common with the search for Lagrangians within the unfolded formulation \([28, 10]\). In particular, it can be extended to other higher-spin systems like \([8, 33, 39, 40, 41]\) as well as to lower-spin supersymmetric models given within the unfolded formulation \([12]\). Also, it would be interesting to reconsider from this perspective the gauge symmetry structure of the frame-like Lagrangians for \(AdS\) mixed-symmetry massless fields proposed in \([7, 43, 44]\).

\(\footnote{Higher spin Lagrangians of \([3]\) are built as bilinear combinations of linearized gauge invariant curvatures so that these are manifestly gauge invariant. Modulo total derivative terms these can be shown to be of the form \((2.3)\).}\)
3.6.3 Extended frame-like action for spin-s massless fields

The extra fields \( \omega^{a_1...a_{s-1},b_1...b_t} \) with \( t \geq 2 \) do not enter the action (3.65). However, the gauge parameter \( \lambda^{a_1...a_{s-1},b_1b_2} \) associated to \( \omega^{a_1...a_{s-1},b_1b_2} \) is still needed to describe gauge invariance of the action. More precisely, the gauge transformations of \( \omega \) with \( t = 1 \) involves a gauge parameter \( \lambda_{b_1b_2} \) associated to the extra field \( \omega \) with \( t = 2 \), cf. (3.62). We now propose the extended action functional that contains fields associated to all coordinates of the target superspace \( \mathcal{M} \) (3.60) on equal footing.

It is convenient to represent elements of \( \mathcal{M} \) (3.60) as polynomials \( \omega(y,p) \) in auxiliary variables \( y^a \) and \( p_a \). The irreducibility conditions imposed on expansion coefficients of \( \omega(y,p) \) are encoded by

\[
\begin{align*}
  p^a \frac{\partial}{\partial p^a} \omega &= (s-1) \omega, \\
  p^a \frac{\partial}{\partial y^a} \omega &= 0, \\
  \frac{\partial}{\partial p^a} \frac{\partial}{\partial p_a} \omega &= 0, \\
  \frac{\partial}{\partial y^a} \frac{\partial}{\partial y_a} \omega &= 0.
\end{align*}
\]

Expanding \( \omega \) into homogeneous in \( y \) components one finds precisely \( s \) coordinates \( \omega_0, \ldots, \omega_{s-1} \) identified with higher spin connections (3.60).

In what follows we denote by \( \langle , \rangle \) the natural inner product on the space of polynomials in \( y, p \), i.e. one determined by \( \langle y^a, y^b \rangle = \eta^{ab} \) and \( \langle p^a, p^b \rangle = \eta^{ab} \). To describe the flat background one also introduces extra coordinates (they are external parameters from the point of view of \( \mathcal{M} \)) encoded in \( e_0 = e^a p_a \) and \( \omega_0 = \omega_0^a y^b p_a \). For simplicity put \( \omega_0 = 0 \) but the nontrivial \( \omega_0 \) can always be reinstated.

The space \( \mathcal{M} \) can be extended by extra coordinate \( R_{ab}(y,p) \) of ghost degree 0 and homogeneity \( s-1 \) in the auxiliary variables such that it is a rank 2 Lorentz tensor with the symmetry of \( o(d-1,1) \) rectangular two-row Young diagram. In addition, it is required to be traceless in the sector of \( y, p \) variables. The conditions can be summarized as

\[
( p_c \frac{\partial}{\partial p_c} - s + 1 ) R_{ab} = 0, \quad \frac{\partial}{\partial p_c} \frac{\partial}{\partial p^c} R_{ab} = 0, \quad p_c \frac{\partial}{\partial y_c} R_{ab} = 0, \quad \frac{\partial}{\partial p^[a} R_{bc]} = 0, \quad (3.67)
\]

where we only list the minimal set from which the remaining ones follow as consistency conditions. Note that for \( s > 2 \) it follows that coefficients of \( R_{ab}(y,p) \) are totally traceless in \( 2s \) indices while their tensor structure is precisely that of the spin \( s \) Weyl tensor. In the case of \( s = 2 \) the second condition in (3.67) is satisfied trivially so that the respective tensor has properties of the Riemann curvature.

The resulting supermanifold is equipped with the following odd nilpotent vector field \( Q \) determined by

\[
Q \omega = -e_0^a \frac{\partial}{\partial y^a} \omega + e_0^a c_0^b R_{ab}, \quad Q R_{ab} = 0. \quad (3.68)
\]

In terms of components

\[
Q \omega_i = -e_0^a \frac{\partial}{\partial y^a} \omega_{i+1} \quad i = 0, \ldots, s-2, \quad Q \omega_{s-1} = e_0^a c_0^b R_{ab}. \quad (3.69)
\]
We go further from $\mathcal{M}$ to $\hat{\mathcal{M}}$ which is a cotangent bundle $T^*[n-1]\mathcal{M}$ extended by $R_{ab}$. The coordinates on the fibres carry degree $n-2$ and are denoted by $\Lambda$. It is convenient to encode the fiber coordinates into $\Lambda(y,p)$ satisfying (3.66).

Let $\hat{\mathcal{M}}$ be equipped with the following 1-form and the potential (the potential is taken from [22])

$$\chi = \langle d\omega, \Lambda \rangle, \quad L = \chi(Q) - H_{\mathcal{M}}, \quad H_{\mathcal{M}} = -\frac{1}{2} \bar{V}^{ab} (\langle \omega_{1a}, \omega_{1b} \rangle - \langle p_a \omega_{1c}, p_b \omega_{1c} \rangle),$$  \hspace{1cm} (3.70)

where $\omega_{1a}$ is defined through $\omega_{1a}(y,p) = y^a \omega_{1a}(p)$, and the potential $H_{\mathcal{M}} = H_{\mathcal{M}}(e_0, \omega_1)$ is $Q$-invariant,

$$QH_{\mathcal{M}} = 0.$$  \hspace{1cm} (3.71)

The odd vector field $Q$ is extended to $\Lambda$-variables as follows

$$Q\Lambda = P[e_0^a y_a \Lambda] + q,$$  \hspace{1cm} (3.72)

where $P$ denotes the projector to the subspace (3.66) and $q$ is uniquely determined by

$$\langle d\omega, q \rangle = -d\omega_i \frac{\partial}{\partial \omega_i} H_{\mathcal{M}}.$$  \hspace{1cm} (3.73)

All the variables save for $\Lambda_{s-1}$ and $R_{ab}$ are vertical and hence have their associated gauge parameters. The introduced $Q, \chi, L$ satisfy the compatibility conditions (2.11) - (2.13). To see this let us restrict to $s > 2$ first (the spin 2 case is treated explicitly below). In the first step we set $H_{\mathcal{M}} = 0$ and find that the compatibility conditions are fulfilled provided the extra coordinate $R_{ab}(y,p)$ satisfies (3.67).

Turning on a non-vanishing $H_{\mathcal{M}}(e_0, \omega_1)$ the extra contributions in the first compatibility condition is

$$\langle q, d\omega \rangle = -dH_{\mathcal{M}}$$  \hspace{1cm} (3.74)

which is a definition of $q$ (3.73). And

$$\langle q, Q\omega \rangle = i_Q \langle q, d\omega \rangle = -i_Q dH_{\mathcal{M}} = -QH_{\mathcal{M}} = 0$$  \hspace{1cm} (3.75)

again thanks to the definition of $q$. In the last equality we used the $Q$-invariance property of $H_{\mathcal{M}}$ (3.71).

It follows that the frame-like action (2.3) determined by $Q, \chi, L$ is consistent. It is given explicitly by

$$S_0[\omega, \Lambda, R] = \int \sum_{i=0}^{s-1} \langle d\omega_i - e_0^a \frac{\partial}{\partial y^a} \omega_{i+1} + \delta_{i,s-1} e_0^a e_0^b R_{ab}, \Lambda_i \rangle$$

$$+ \frac{1}{2} \bar{V}^{ab} (\langle \omega_{1a}, \omega_{1b} \rangle - \langle p_a \omega_{1c}, p_b \omega_{1c} \rangle).$$  \hspace{1cm} (3.76)

Combinations of terms $d\omega_i - e_0^a \frac{\partial}{\partial y^a} \omega_{i+1} \equiv R_i$ are in fact linearized curvatures associated to fields $\omega_i$. Note that for $i = s-1$ curvature $R_i$ enters action (3.76) together with the extra coordinate $R_{ab}$. For $i \neq s-1$ one also represents $R_i$ as $R_i = (d+Q)\omega_i$.
so that the first terms in (3.76) can be written as \(\langle (d + Q)\omega_1, \Lambda_1 \rangle\). At the same time, the potential term \(H_M\) depends on fields \(\omega_1\) only.

Before showing that the above action indeed describes spin-\(s\) massless field let us discuss the interpretation of the various structures it involves. First of all, the odd vector field \(Q\) considered on \(M\) with coordinates \(\omega_i\) is directly related to the linear operator \(\sigma_- = e_0^a \frac{\partial}{\partial y^a}\) defined on the space of polynomials in auxiliary variables \(y, p, e_0\) seen as differential forms with values in (3.69). The operator \(\sigma_-\) and its cohomology are well-known within the unfolded description of higher spin fields [3, 11]. In the present consideration we prefer to work with \(Q\) rather than \(\sigma_-\) because it is naturally defined on generic function(s) of fields. In particular, potential \(H_M\) is naturally a \(Q\)-cocycle in the sense that it is \(Q\)-closed (3.71) and adding a \(Q\)-exact term leads to an equivalent action. Finally, the operator \(P(e^a y_a)\) entering (3.72) is conjugated to \(\sigma_-\) and is also known in the unfolded approach as \(\sigma_+\).

The extended frame-like action (3.76) is equivalent to the conventional frame-like action (3.63). To show this one observes that \(\Lambda_{s-1}\) and \(R_{ab}\) are auxiliary fields that can be expressed in terms of other fields using their own equations of motion. Indeed, using the gauge symmetry for \(\Lambda_{s-1}\) one can eliminate all the components of the image of \(P(e^a y_a)\), while the remaining components are precisely one to one with the components of \(R_{ab}\). The same applies to those components of \(\omega_i, i = 2, \ldots, s - 1\) and \(\Lambda_i, i = 1, \ldots, s - 2\) that cannot be eliminated using the gauge transformations for these fields. In particular, varying the Lagrange multipliers \(\Lambda_i, i = 1, \ldots, s - 2\) one arrives at the zero-curvature constraints \(R_i = 0, i = 1, \ldots, s - 2\). Finally, upon elimination one ends up with the reduced action depending on \(\omega_0, \omega_1\) and \(\Lambda_0\) and given by the (3.76) where all the other fields are put to zero. But this is precisely the action which was shown in [24] to produce frame-like action (3.63).

A crucial point in the above argument is that all the components of \(R_{ab}, \omega_i, i = 2, \ldots, s - 1\) and \(\Lambda_i, i = 1, \ldots, s - 1\) can be eliminated using the gauge symmetries and the equations of motion. This fact can be traced to the properties of the cohomology of \(\sigma_- = e_0^a \frac{\partial}{\partial y^a}\) entering the first term in (3.76). More precisely, 1-forms \(\omega\) can be identified with the linear in \(e_0^a\) elements of polynomials in \(e_0^a, p^a, y^i\) satisfying (3.63) and similarly for fields \(\Lambda_i\). One can actually show that all the components of \(\omega_i\) except \(\sigma_-\) cohomology and all the components of \(\Lambda_i\) except \((\sigma_-)^\dagger = \sigma_+\)-cohomology are generalized auxiliary (see, e.g., [24] for further details on generalized auxiliary fields) if one disregards \(H_M\) and \(R_{ab}\). Restricting to \(i > 1\) the only relevant \(\sigma_-\) cohomology class is the Weyl tensor in degree 2 and homogeneity \(s - 1\) in \(y\) along with its conjugate \(\sigma_+\) class in degree \(n - 2\). Because \(\omega_i\) are of degree 1 all \(\omega_i\) with \(i > 1\) are not in the cohomology and can be eliminated. In contrast, the component of \(\Lambda_{s-1}\) corresponding to \(\sigma_+\) cohomology can not because \(\Lambda\) is of degree \(n - 2\). However, in the expression for the action this component enters multiplied by \(R_{ab}(y, p)\) so that both the component and \(R_{ab}\) are auxiliary fields. In other words from this perspective \(R_{ab}\) plays a role of Lagrange multiplier needed to put to zero the unwanted component of \(\Lambda_{s-1}\). Alternatively, \(R_{ab}\) can be seen as an

\[^5\text{It is worth mentioning that implementation of zero-curvature constraints through the frame-like actions with Lagrange multipliers have been discussed in [27, 11].}\]
element eliminating (gluing) $Q$-cohomology in the space of linear in $\omega_i$ functions on $M$. These arguments can be seen as Lagrangian counterpart of the $\sigma_-$-cohomology method \cite{11} at the level of equations of motion.

Furthermore, one can generally allow new $R_{ab}$-type fields to enter the terms proportional to $\Lambda_i$ with $i < s - 1$. A particularly interesting option is to introduce $F_{ab}(y, p)$ in the term with $\Lambda_1$ (for consistency, one also needs to introduce its descendants in the terms proportional to $\Lambda_i$ with $i > 1$). This is the well-known trick to relax the Fronsdal equations because such $F_{ab}$ precisely corresponds to Fronsdal tensor. In its turn $F_{ab}$ is again related to $\sigma_-$-cohomology class known in the unfolded approach as “Einstein” cohomology.

Let us finally mention that the action (3.76) can be generalized to describe spin-$\frac{s}{2}$ massless fields on the (A)dS background. In this case one needs to allow nonvanishing background $\omega_0$ and add $\Lambda\sigma_+ = \Lambda \mathcal{P}(e^a_0 y_a)$ contribution to the first term, where $\Lambda$ is the cosmological constant. The odd vector field $Q$ acting on $\hat{M}$ will be augmented by $\Lambda\sigma_+$ and $\Lambda\sigma_-$ for coordinates $\omega(y, p)$ and $\Lambda(y, p)$, respectively.

**The spin $s = 2$ case.** Let us explicitly list all the structures in the case of spin 2. The coordinates on $\hat{M}$ are introduced according to

\[
\omega_0 = e^a p_a, \quad \omega_1 = \omega^a b^p p_a, \quad \Lambda_0 = \lambda_a p^a, \quad \Lambda_1 = \lambda_{ab} y^a p^b, \quad R_{ab} = R^{cd}_{ab} y_c p_d .
\]

Components of the odd vector field $Q$ are given by

\[
\begin{aligned}
Q e^a &= \omega^a b^p e^b, \\
Q \omega^{ab} &= e^c_0 e^d_0 R_{cd}^{ab}, \\
Q \lambda_a &= 0, \\
Q \lambda_{ab} &= \lambda_{[a} e^c_{b]} + \omega_{[a} e^c V_{b]} .
\end{aligned}
\]

(3.78)

The 1-form $\chi$ is

\[
\chi = \lambda_a d e^a + \frac{1}{2} \lambda_{ab} d \omega^{ab} ,
\]

(3.79)

and the potential $L$

\[
L = \chi(Q) + \frac{1}{2} \tilde{V}_{ab} \omega^a \omega^b .
\]

(3.80)

All the variables are vertical except for $\lambda_{ab}$ and $R$. In particular, there are gauge parameters $\xi^a, \xi^{ab}, \epsilon_a$ associated to respectively $e^a, \omega^{ab}, \lambda_a$. One can explicitly check that the compatibility conditions (2.11)-(2.13) are fulfilled. Namely,

\[
i_Q \sigma - dL = \lambda_{ab} \epsilon^c_0 e^d_0 R_{cd}^{ab}, \quad \sigma(Q, Q) = 0, \quad QL = 0 .
\]

(3.81)

In checking these relations we made use of the following consequences of Young symmetries of $R_{cd}^{ab}$.

\[
R_{cd}^{ab} e^c_0 e^d_0 \equiv 0, \quad \omega^a b R^{bc}_{ac} \equiv 0 .
\]

(3.82)

The linearized gravitational action takes the form

\[
S_0[e, \omega, \lambda, R] = \int \lambda_a (d e^a + \omega^a b^p e^b) + \lambda_{ab} (d \omega^{ab} + e^c_0 e^d_0 R_{cd}^{ab}) + \frac{1}{2} \tilde{V}_{ab} \omega^a \omega^b .
\]

(3.83)

The theory (3.83) can be made non-linear just by replacing the background fields $e_0$ and $\omega_0 = 0$ with dynamical fields $e$ and $\omega$. Note that one also needs to reinstate
the familiar $\omega^a, \omega^{cb}$ term in the expression for the curvatures in the second term. The resulting action will describe the standard Einstein gravitation theory action equivalent to (3.8).

4 (Polymomentum) Hamiltonian description

4.1 One-dimensional constrained Hamiltonian systems

It is well-known that equations of motion of generic one-dimensional Lagrangian system can be rewritten in a presymplectic Hamiltonian form, e.g. through hamiltonization. Moreover, the extended Hamiltonian action of a constrained system can be represented in the AKSZ form where the extended phase space of BFV-BRST formulation plays the role of the target space [14]. Whence, the system is frame-like one. The presymplectic representation of the Euler–Lagrange equations can be seen even without resorting to constrained Hamiltonian formalism. The easiest way to observe this for the Lagrangian $L(q, \dot{q})$ is to introduce auxiliary fields $v^i = \dot{q}^i$ and $p_i = \frac{\partial L(q, v)}{\partial \dot{v}^i}$ so that the system is equivalently represented as

$$S[p, q, v] = \int dt (p(\dot{q} - v) + L(q, v)).$$

(4.1)

Indeed, eliminating variables $p$ and $v$ by their own equations of motion gives back the starting point Lagrangian. At the same time, the equations of motion determined by $S$ have the presymplectic Hamiltonian form

$$\Psi^A \sigma_{AB} + \partial_B H = 0,$$

(4.2)

where $\Psi^A = (q^i, p_i, v^i)$, $H := p_i \dot{v}^i - L(q, v)$ and $\sigma := dp_i \wedge dq^i$. Using the system as a starting point of the presymplectic version [17, 18] of the Dirac–Bergmann algorithm one ends up with the reduced presymplectic system. This can equivalently be obtained using conventional Dirac–Bergmann approach: the reduced presymplectic system is just the constrained surface equipped with the pullbacks of the phase space symplectic form and the Hamiltonian.

We now take a different route and analyze what do general axioms of Section 2 tell us in the one-dimensional case. It turns out that constrained systems appear from a purely supergeometrical perspective without introducing the full-scale BFV-BRST formulation. Restricting to systems in $n = 1$ dimension we take $\mathcal{M}$ to be a presymplectic manifold with bosonic coordinates $\psi^i$ and fermionic $C^A$ of ghost degree 0 and 1 respectively. We assume ghost variables $C^A$ split into vertical $c^\alpha$ and horizontal $c^a$ while $\phi^i$ are also assumed horizontal. Taking into account the ghost degree, $\chi$ has the form $\chi = d\psi^i \chi_i(\psi)$. Similarly, $Q$ of degree 1 reads as $Q\psi^i = R^i_\alpha c^\alpha + R^i_\alpha c^a$ and $Qc^A = -\frac{1}{2} U^A_{BC} C^B C^C$ where $C^A = \{c^\alpha, c^a\}$. For simplicity we assume $R^i_\alpha$ to be of maximal rank and $Q$ nilpotent. In particular, $[R_A, R_B] = U^C_{AB} R_C$. A generic expression for $L$ is $L = c^\alpha T_\alpha(\psi) + c^a T_a(\psi)$. The compatibility
conditions (2.11)-(2.13) say
\[ \partial_i T_\alpha - \sigma_{ij} R^j_\alpha = 0, \quad L_{R_\alpha} \sigma = 0, \quad (4.3) \]
\[ \frac{1}{2} ( R_\alpha T_\beta - R_\beta T_\alpha ) = U^C_{\alpha \beta} T_C, \quad R_\alpha T_b = U^C_{ab} T_C. \quad (4.4) \]

A frame-like action determined by this data reads as
\[ S[\psi, c] = \int (d\psi^i \chi_i + L) = \int d\tau (\chi_i \dot{\psi}^i + c^\alpha T_\alpha + c^a T_a), \quad (4.5) \]
where \( c^\alpha, c^a \) are now 1-forms and are to be identified with Lagrange multipliers. The gauge transformations take the form:
\[ \delta_\lambda \psi^i = -R^i_\beta \lambda^\beta, \quad \delta c^\alpha = d\lambda^\alpha + \lambda^\beta U^a_{\beta \gamma} c^\gamma, \quad \delta c^a = \lambda^\beta U^a_{\beta \gamma} c^\gamma. \quad (4.6) \]

It is clear from the structure of the action and the gauge transformations that we are dealing with a generalization of a constrained Hamiltonian system where \( T_\alpha \) and \( T_a \) play the role analogous to the first and the second class constraints, respectively.

Under suitable regularity assumptions one can solve equations of motion \( T_a = 0 \) so that \( \mathcal{M} \) is replaced with the submanifold \( \mathcal{M}' \subset \mathcal{M} \) singled out by \( T_a = 0 \) and \( c^a = 0 \) and we take \( \psi^\mu \) to be the independent coordinates on \( \mathcal{M}' \). More precisely, taking \( T_a \) as a part of the coordinate system on \( \mathcal{M} \) one finds that \( T_a \) and \( c^a \) give rise to auxiliary fields. Their elimination results in the following action
\[ S[\psi, c] = \int d\psi^\mu \chi_\mu + c^\alpha T_\alpha. \quad (4.7) \]

The subtlety is that \( Q \) is in general not tangent to \( \mathcal{M}' \) (this happens if \( Qc^a \) and \( QT_a \) do not vanish when \( T_a = C^b = 0 \) so that its gauge symmetry can not be easily represented in terms of \( Q \). Assuming for simplicity \( Q \) and \( L \) are such that \( Q \) is tangent to \( \mathcal{M}' \) the expression for the gauge transformations take the usual form
\[ \delta_\lambda \psi^\mu = -R^\mu_\beta \lambda^\beta, \quad \delta c^\alpha = d\lambda^\alpha + \lambda^\beta U^a_{\beta \gamma} c^\gamma. \quad (4.8) \]

This system is nearly a usual first-class constrained Hamiltonian system. The only difference is that \( \sigma \) is not necessarily invertible. If one assumes \( \sigma_{\mu \nu} \) invertible one finds \( R^\mu_\alpha = \sigma^{\mu \nu} \partial_\nu T_\alpha \) so that \( R_\alpha = \{ T_\alpha, \cdot \} \), where \( \{ \cdot, \cdot \} \) is a Poisson bracket determined by \( \sigma \) on the space of \( \psi^\mu \)-variables so that indeed we are dealing with the conventional first-class constraint system.

If one stays in the symplectic framework an extra equivalent reduction is also possible. Namely, under the suitable regularity assumptions one can take \( T_\alpha \) as a part of the coordinate system. Just like in the previous step variables \( c^a \) and \( T_\alpha \) are auxiliary and can be eliminated. The action of the reduced system takes the form (here \( \psi^m \) are coordinates on \( \mathcal{M}'' \) singled out by \( T_\alpha = 0 \))
\[ S[\psi] = \int d\psi^m \chi_m, \quad (4.9) \]
while its gauge symmetries are determined by $Q$ restricted to the surface $\mathcal{M}'' \subset \mathcal{M}'$.

It is important to note that in contrast to the previous reductions this one, strictly speaking, takes us outside the class of systems described in Section 2.2. Indeed, although the 1-form field (Lagrange multiplier) associated to coordinate $e^{\alpha}$ has been eliminated using equations of motion the gauge invariance with parameter $\lambda^{\alpha}$ is still present. This is exactly the same situation as we met in the case of the frame-like Lagrangians for HS fields considered in Section 3.6.2: there are ghost coordinates on $\mathcal{M}$ that possess associated gauge parameters but do not possess associated fields (or, equivalently, the respective fields do not enter a given frame-like Lagrangian).

The following remarks are in order:

– the system has a vanishing Hamiltonian. To include the Hamiltonian one needs to treat one of the ghosts $e^{\alpha}$ (say, $e^{0}$) as a horizontal variable and the respective field as a background einbein field. A different way to introduce a genuine Hamiltonian is to reinterpret the system as a parameterized Hamiltonian system (for more details see the discussion in Section 4.2).

– The example is restricted to the case of irreducible systems. To describe reducible ones one allows for ghost variables of degree 2 (and higher) and lets $Q$ to encode reducibility relations.

– We have concentrated on geometry and gauge symmetries and have not discussed the dynamical implications (see, e.g., [49, 50] and references therein) of degenerate symplectic structures.

### 4.2 Parameterized systems

By construction frame-like Lagrangians describe diffeomorphism-invariant systems. In the above examples the non-diffeomorphism invariant theories are described by coupling them to the vacuum gravitational field. There is a different approach to describing such theories based on parametrization. It is well known (see e.g. [51, 52, 53]) that any theory can be made parametrization invariant by introducing extra fields and extra gauge transformations. More precisely, if $L[\phi, \partial_{\mu}\phi]$ is a Lagrangian of a system with fields $\phi^{i}$ let us consider a new system with fields $\phi^{i}, z^{a}$ depending on generic space time coordinates $x^{\mu}$. The parameterized action is given by

$$S[\phi, z] = \int d^{n-1}x \det(e)L[\phi, e^{\mu}_{a}\partial_{\mu}\phi], \quad e^{\mu}_{a} = \frac{\partial z^{a}}{\partial x^{\mu}}, \quad e^{\mu}_{a}e^{\nu}_{b} = \delta^{\mu}_{\nu}. \quad (4.10)$$

It is invariant under the following gauge transformation:

$$\delta_{\lambda}z^{a} = \lambda^{a}, \quad \delta\phi = \lambda^{a}e^{\mu}_{a}\partial_{\mu}\phi, \quad (4.11)$$

where $\lambda^{a}$ are components of a vector field parameterizing infinitesimal diffeomorphisms.

It turns out that parameterized systems can naturally be written in the frame-like form. Let us begin with the scalar field example. In addition to coordinates
$\phi, \pi^a$ introduced in Section 3.2 we need the following coordinates on $M$: $z^a, e^a, p_a, \varrho_a$ whose degrees are respectively 0, 1, $n - 1, n - 2$. The differential $Q$ is given by

$$Qz^a = -e^a, \quad Qe^a = 0, \quad Qp_a = 0, \quad Q\varrho_a = -p_a, \quad \Phi = Q\pi^a = 0.$$ (4.12) \hspace{1cm} (4.13)

The 1-form $\chi$ is given by

$$\chi = dz^a p_a + de^a \varrho_a + d\phi \pi^a \mathcal{V}_a.$$ (4.14)

Note that the last term coincides with the respective contribution in the 1-form of the scalar in a gravity background. Taking $H_M$ as in Section 3.2 results in the following action

$$S[z, p, e, \varrho, \pi, \phi] = \int \left[ (dz^a - e^a)p_a + de^a \varrho_a + d\phi \pi^a \mathcal{V}_a + \frac{1}{2}(\pi^b \pi_b - m^2 \phi^2) \right] \mathcal{V}.$$ (4.15)

Let us explicitly spell out its gauge symmetries (we only present nonvanishing transformations)

$$\delta z^a = \xi^a, \quad \delta \varrho_a = \epsilon_a, \quad (4.16)$$

where $\xi^a$ is the 0-form parameter associated with vertical coordinate $e^a$ while $\epsilon_a$ is the $n - 2$-form parameter associated to vertical coordinate $p_a$. Taking into account the equations of motion following from the above action (we only spell out those in the sector of $z, e, p, \varrho$-variables)

$$e^a - dz^a = 0, \quad de^a = 0, \quad p_a = U_a(e, \phi, d\phi, \pi, \varrho), \quad (4.17)$$

it is clear that variables $z^a, \varrho_a$ are Stueckelberg while $e^a, p_a$ are auxiliary so that all of them do not bring in new degrees of freedom. Indeed, these gauge symmetries are enough to put e.g. $z^a = x^a$ and $\varrho_a = 0$. The equations of motion then say $e^a = dz^a = dx^a$ and fix $p_a$ in terms of the remaining variables. Upon gauge fixation and elimination of auxiliary fields the action becomes just (3.19).

Let us stress the difference between the above parameterized formulation and the scalar field action in the flat gravity background described by $e^0_0, \omega^{ab}_0$. In the gravity case fields $e_0, \omega_0$ are to be treated as background fields. In contrast, in the above parameterized system all the fields enter the Lagrangian at the equal footing. These are equations of motion and gauge symmetries which make the additional fields non-dynamical.

More generally, if one is given with the frame-like action involving flat gravity background described by $e^0_0, \omega^{ab}_0$, one can systematically rewrite it in the parameterized form. Indeed, suppose that the frame-like description is provided by the action of the form

$$S[\Psi] = \int d^4 \Psi A^0_A(\Psi, e_0, \omega_0) + L(\Psi, e_0, \omega_0), \quad (4.18)$$

where $e_0$ and $\omega_0$ is the flat gravity background such that $d e_0 + \omega_0 e = 0, d \omega_0 + \omega_0 \omega_0 = 0$ and gauge symmetries are determined by $Q_0$. Introducing variables $z, e, p, \varrho$ in exactly the same way as in the scalar field example one takes

$$\chi = \chi^0 + dz^a p_a + de^a \varrho_a,$$ (4.19)
where in $\chi_0$ in the RHS one puts $\omega_0$ to zero and replaces $e_0$ with $e$. Taking $Q$ as in (4.12) in the sector of new variables and unchanged in the original sector one ends up with the parameterized description:

$$S[z,p,e,\rho,\Psi] = \int (dz^a - e^a)p_a + de^a\rho_a + d\Psi^A\chi_0^A(\Psi,e) + L(\Psi,e). \quad (4.20)$$

In this Lagrangian all fields can be treated as non-background. Of course in the gauge $z^a = x^a$ it is equivalent to the starting point system.

The following toy example illustrates that the above parametrization procedure reduces to the usual parameterized Hamiltonian description in the case of a Hamiltonian system. To see this let us take as $M$ a phase space of a Hamiltonian system with Hamiltonian $H(\psi)$ and symplectic potential $\chi = d\psi^a \chi_a(\psi)$ and extend it by odd degree 1 coordinate $e$ (einbein) so that the Hamiltonian action is $\int d\psi^a \chi_a - eH$. Following the above procedure and introducing variables $z,p,\rho$ the parameterized action reads as

$$S[\psi,e,z,p] = \int (d\psi^a \chi_a + dzp - e(H + p)), \quad (4.21)$$

(note that the term $de\rho$ is missing as $gh(\rho) = -1$ so that there are no dynamical fields associated to $\rho$) which is a standard Hamiltonian action of a usual parameterized Hamiltonian system (see, e.g., [29]).

It is instructive to write down an analog of (4.9) in this case. The constrained surface is parameterized by $\psi^a,t := z$ and the pullback of the 1-form is $d\psi^a \chi_a + H(\psi)dt$ so that the reduced action takes the well-known invariant form

$$S[\psi,t] = \int d\tau (d\psi^a \chi_a + Hdt) = \int \Phi^*(d\psi^a \chi_a + Hdt), \quad (4.22)$$

where $\Phi^*$ denotes the pullback induced by the map $\psi^a(\tau),t(\tau)$ from a time line to the constrained surface with coordinates $\psi^a,t$.

To complete the description of parameterized system let us mention that the structure of the Lagrangian and gauge transformations in the sector of $z,e,p,\rho$ variables explicitly coincides with that of the parent Lagrangians from [21, 22]. This is not a coincidence as frame-like formulation of a given (say in the metric-like formalism) system can be systematically derived from the parent formulation. Mention also that the above parameterized description does not contain $\omega_{ab}^\mu$ field. In fact this only applies to the minimal version. If for instance the starting point system is Lorentz invariant (this is of course always true if the system originates from that on a flat gravity background) one can systematically gauge this symmetry which results in $\omega_{ab}^\mu$ field present in the formulation. Analogous extension was discussed in the context of parent formulation in [22, 54].

### 4.3 Relation to the polymomentum phase space

Given a frame-like system with the target supermanifold $M$ equipped with $Q,\chi,L$ and the ghost degree let $M_P$ be the space of independent variables $x^\mu$ and dependent
ones $\Psi_{\mu_1 \ldots \mu_p A}(x)$. Since the formulation under consideration is the first-order, $M_P$ can be visualized as the multidimensional analog of the phase space. Moreover, as we are going to see $M_P$ can be merely identified with the polymomentum phase space of the de Donder–Weyl formalism. More precisely, we now show how the basic structures of the polymomentum approach arise from the frame-like formalism developed above. A systematic exposition of the polymomentum approach can be found in, e.g., [23, 25].

Space $M$ is equipped with 1-form $\chi$ of ghost degree $gh(\chi) = n - 1$. This form gives rise to an $n$-form on the phase space $M_P$. Indeed, substituting $\Psi^A$ in the expression for $\chi$ with $\Psi^A_{\mu_1 \ldots \mu_p A} dx^{\mu_1} \ldots dx^{\mu_p A}$ one arrives at $n$-form $\chi_P$ defined on $M_P$. This form can be identified with a version of the canonical $n$-form of the polymomentum approach, see, e.g., [23, 25].

$n$-form $\chi_P$ is in turn related to the canonical 1-form of the usual Hamiltonian formulation. Indeed, explicitly separating the space-time into the space $\Sigma$ with coordinates $x^i, i = 1, \ldots, n - 1$ and time $x^0$ and integrating $\chi_P$ over the space-like $(n - 1)$-dimensional hypersurface $\Sigma$ determined by $x^0 = \text{const}$ gives the following canonical 1-form

$$\int_{\Sigma} \delta \Psi^A \wedge \chi_A(\Psi).$$

(4.23)

It is the functional 1-form on the space of fields $\Psi^A_{\mu_1 \ldots \mu_p A}$ defined on $\Sigma$.

To see that (4.23) is the usual 1-form of the Hamiltonian formalism observe that frame-like action (2.3) takes the form

$$S[\Psi] = \int dx^0 \int_{\Sigma} dS \Psi^A \chi_A(\Psi) = \int dx^0 \int_{\Sigma} dS \Psi^A \chi_A(\Psi) + L(\Psi),$$

(4.24)

where $dS = dx^i \frac{\partial}{\partial x^i}$ is the spatial part of the de Rham differential.

As an illustration let us consider the scalar field and the Yang-Mills theory examples. For simplicity we take Minkowski spacetime as the background described in Cartesian coordinates $x^a$ by $e^a = dx^a$ and $\omega^{ab} = 0$.

In the case of the scalar field theory the respective component of the $n$-form $\chi_P$ is given by $\chi_P = \pi^a \nu_a$. Integrating over space $\Sigma$ gives the usual canonical 1-form

$$\int_{\Sigma} \pi^0 \delta \phi, \text{ and, as expected, } \pi^0 \text{ is the usual momenta.}$$

In the case of the Yang-Mills theory one obtains $\chi_P = \langle F^{ab}, dA_a \rangle \nu_b$. This can be identified with that of the canonical $n$-form of the covariant Hamiltonian formalism (see, e.g., [25]). After integrating over space $\Sigma$ one arrives at $\int_{\Sigma} \langle F^{a0}, \delta A_a \rangle$, so that again, as expected, $F^{a0}$ with $i = 1, \ldots, n - 1$ is the usual canonical momenta conjugated to the spatial components $A_i$ of the Yang-Mills field.

It follows from the above considerations that the structure of the polymomentum phase space as well as the usual phase space is completely determined by that of the supermanifold $M$. This suggests that $M$ is more fundamental object that can be used as a substitute of the polymomentum phase space. Note that from this perspective
frame-like actions considered in this work are multidimensional generalizations of the extended Hamiltonian actions for constrained systems in 1d.

Since we deal with a gauge theory the definition of the field space or phase space is ambiguous. More precisely, one can always add extra fields along with extra terms in the Lagrangian or the extra gauge transformations so that these extra variables do not bring in new degrees of freedom. Such variables are known as auxiliary and/or Stueckelberg fields. In particular, according to Section 4.2 one can always consider a parameterized version of the theory, where the spacetime coordinates are part of the target space $\mathbb{M}$ (as it is usually the case for the standard [23, 25] polymomentum description of theories not coupled to gravity).

### A Notation and conventions

Let $\mathcal{M}$ be a supermanifold with local coordinates $z^A$. Vector fields are left derivations of the algebra of smooth functions $\mathcal{C}_\mathcal{M}$ on $\mathcal{M}$, i.e., $V : \mathcal{C}_\mathcal{M} \to \mathcal{C}_\mathcal{M}$ is a linear map satisfying $V(fg) = (Vf)g + (-1)^{|V||f|}f(Vg)$, where $| \cdot | = 0, 1$ denotes a parity. Vector fields clearly form a left module over $\mathcal{C}_\mathcal{M}$.

In order to work with differential forms it is useful to consider the odd tangent bundle $\Pi T\mathcal{M}$ over $\mathcal{M}$. Differential forms are simply functions on $\Pi T\mathcal{M}$ which are polynomials in the fiber coordinates. By slight abuse of notation, the fiber coordinates are denoted by $dz^A$. The components of a $p$-form $\alpha$ are introduced as follows:

$$\alpha(z, dz) = \frac{1}{p!} dz^A \ldots dz^1 \alpha_{A_1 \ldots A_p}(z) \quad \text{(A.1)}$$

It follows that in our conventions the graded antisymmetry property of components is determined by $dz^A dz^B = (-1)^{|A||B|+|A|+1} dz^B dz^A$. In particular, for the components of a 2-form $\sigma$ one has

$$\sigma_{AB} = (-1)^{|A|+1(|B|+1)} \sigma_{BA} \quad \text{(A.2)}$$

De Rham differential is a natural vector field on $\Pi T\mathcal{M}$ given by

$$d = dz^A \frac{\partial}{\partial z^A}, \quad \text{(A.3)}$$

where $\frac{\partial}{\partial z^A} = \partial_A$ denotes a left derivative. For instance, for 0-forms $z^A$ and 1-form $\chi = dz^A \chi_A(z)$ one explicitly gets

$$dz^A = dz^A, \quad d(dz^B \chi_B) = \frac{(-1)^{|B|+1}}{2} dz^B dz^A (\partial_A \chi_B - (-1)^{|A||B|} \partial_B \chi_A). \quad \text{(A.4)}$$

In terms of components,

$$(d\chi)_{AB} = (-1)^{|B|+1} (\partial_A \chi_B - (-1)^{|A||B|} \partial_B \chi_A). \quad \text{(A.5)}$$

Let us also explicitly write down $d\sigma = 0$ for a 2-form $\sigma$:

$$\partial_A \sigma_{BC} (-1)^{|A||C|+|B|} + \text{cycle}(A, B, C) = 0. \quad \text{(A.6)}$$
The contraction of a vector field and a differential form is itself the following vector field on Π\(\mathcal{T}\)M:

\[ i_V = V^A \frac{\partial}{\partial (dz^A)} \]

In particular, the components can be expressed as follows

\[ \alpha_{A_1...A_p}(z) = i_{A_1} ... i_{A_p} \alpha(z, dz), \quad i_A := i_\frac{\partial}{\partial z^A}. \]

It is useful to define the Lie derivative as acting on forms through the Cartan formula

\[ \mathcal{L}_V = [i_V, d] = i_V d + (-1)^{|V|} di_V, \]

where \([\cdot, \cdot]\) denotes the graded commutator. \(\mathcal{L}_V\) is a vector field on Π\(\mathcal{T}\)M. Note also the following relations \([\mathcal{L}_V, d] = 0\). In particular, for functions one gets \(\mathcal{L}_V f = V^A \partial_A f\). The action on differential \(p\)-forms is easily found using the Leibnitz rule and the action on basic differentials:

\[ (-1)^{|V|} \mathcal{L}_V dz^A = di_V dz^A = d(V z^A) = dV^A = dz^B \partial_B V^A. \]

For instance, for 1-forms one gets

\[ (-1)^{|V|} \mathcal{L}_V \chi = dz^A \left( \partial_A V^B \chi_B + (-1)^{|A||V|} V^B \partial_B \chi_A \right). \]

Acknowledgments

We are indebted to I. Batalin for the collaboration at the early stage of this project. Useful discussions with G. Barnich, S. Lyakhovich, R. Metsaev, M. Vasiliev and A. Verbovetsky are gratefully acknowledged. The work of K.A. was supported by RFBR grant 11-01-00830. The work of M.G. was supported by RFBR grant 13-01-00386.

References

[1] R. D’Auria, P. Fre, P. K. Townsend, and P. van Nieuwenhuizen, “Invariance of actions, rheonomy and the new minimal N=1 supergravity in the group manifold approach,” Ann. Phys. 155 (1984) 423.

[2] R. D’Auria and P. Fre, “Geometric Supergravity in d = 11 and Its Hidden Supergroup,” Nucl. Phys. B201 (1982) 101–140.

[3] M. A. Vasiliev, “‘Gauge’ form of description of massless fields with arbitrary spin. (in russian),” Yad. Fiz. 32 (1980) 855–861.

[4] M. A. Vasiliev, “Equations of motion of interacting massless fields of all spins as a free differential algebra,” Phys. Lett. B209 (1988) 491–497.

[5] V. E. Lopatin and M. A. Vasiliev, “Free massless bosonic fields of arbitrary spin in d-dimensional de Sitter space,” Mod. Phys. Lett. A3 (1988) 257.

[6] Y. M. Zinoviev, “First order formalism for mixed symmetry tensor fields,” hep-th/0304067
[7] K. B. Alkalaev, O. V. Shaynkman, and M. A. Vasiliev, “On the frame-like formulation of mixed-symmetry massless fields in (A)dS(d),” *Nucl. Phys.* **B692** (2004) 363–393, [hep-th/0311164](https://arxiv.org/abs/hep-th/0311164).

[8] E. D. Skvortsov, “Frame-like Actions for Massless Mixed-Symmetry Fields in Minkowski space,” [0807.0903](https://arxiv.org/abs/0807.0903).

[9] N. Boulanger, N. Colombo, and P. Sundell, “A minimal BV action for Vasiliev’s four-dimensional higher spin gravity,” *JHEP* **1210** (2012) 043, [1205.3333](https://arxiv.org/abs/1205.3333).

[10] M. Vasiliev, “Cubic Vertices for Symmetric Higher-Spin Gauge Fields in (A)dSd,” *Nucl.Phys.* **B862** (2012) 341–408, [1108.5921](https://arxiv.org/abs/1108.5921).

[11] X. Bekaert, S. Cnockaert, C. Iazeolla, and M. A. Vasiliev, “Nonlinear higher spin theories in various dimensions,” [hep-th/0503128](https://arxiv.org/abs/hep-th/0503128).

[12] M. Alexandrov, M. Kontsevich, A. Schwartz, and O. Zaboronsky, “The Geometry of the master equation and topological quantum field theory,” *Int.J.Mod.Phys.* **A12** (1997) 1405–1430, [hep-th/9502010](https://arxiv.org/abs/hep-th/9502010).

[13] A. S. Cattaneo and G. Felder, “A path integral approach to the Kontsevich quantization formula,” *Commun. Math. Phys.* **212** (2000) 591–611, [math.qa/9902090](https://arxiv.org/abs/math.qa/9902090).

[14] M. A. Grigoriev and P. H. Damgaard, “Superfield BRST charge and the master action,” *Phys. Lett.* **B474** (2000) 323–330, [hep-th/9911092](https://arxiv.org/abs/hep-th/9911092).

[15] I. Batalin and R. Marnelius, “Superfield algorithms for topological field theories,” in “Multiple facets of quantization and supersymmetry”, M. Olshanetsky and A. Vainshtein, eds., pp. 233–251. World Scientific, 2002. [hep-th/0110140](https://arxiv.org/abs/hep-th/0110140).

[16] J.-S. Park, “Topological open p-branes,” [hep-th/0012141](https://arxiv.org/abs/hep-th/0012141).

[17] D. Roytenberg, “On the structure of graded symplectic supermanifolds and Courant algebroids,” [math/0203110](https://arxiv.org/abs/math/0203110).

[18] P. O. Kazinski, S. L. Lyakhovich, and A. A. Sharapov, “Lagrange structure and quantization,” *JHEP* **07** (2005) 076, [hep-th/0506093](https://arxiv.org/abs/hep-th/0506093).

[19] G. Barnich and M. Grigoriev, “A Poincare lemma for sigma models of AKSZ type,” *J.Geom.Phys.* **61** (2011) 663–674, [0905.0547](https://arxiv.org/abs/0905.0547).

[20] G. Barnich and M. Grigoriev, “First order parent formulation for generic gauge field theories,” *JHEP* **01** (2011) 122, [1009.0190](https://arxiv.org/abs/1009.0190).

[21] M. Grigoriev, “Parent formulation at the Lagrangian level,” *JHEP* **07** (2011) 061, [1012.1903](https://arxiv.org/abs/1012.1903).

[22] M. Grigoriev, “Parent formulations, frame-like Lagrangians, and generalized auxiliary fields,” *JHEP* **1212** (2012) 048, [1204.1793](https://arxiv.org/abs/1204.1793).

[23] M. J. Gotay, J. Isenberg, and J. E. Marsden, “Momentum maps and classical relativistic fields. I: Covariant field theory,” [physics/9801013](https://arxiv.org/abs/physics/9801013).

[24] I. V. Kanatchikov, “Canonical structure of classical field theory in the polymomentum phase space,” *Rept. Math. Phys.* **41** (1998) 49–90, [hep-th/9709229](https://arxiv.org/abs/hep-th/9709229).

[25] I. V. Kanatchikov, “Precanonical quantum gravity: Quantization without the space-time decomposition,” *Int. J. Theor. Phys.* **40** (2001) 1121–1149, [gr-qc/0012074](https://arxiv.org/abs/gr-qc/0012074).
[26] D. Sullivan, “Infinitesimal computations in topology,” *Inst. des Haut Etud. Sci. Pub. Math.* **47** (1977) 269.

[27] M. A. Vasiliev, “Consistent equations for interacting massless fields of all spins in the first order in curvatures,” *Annals Phys.* **190** (1989) 59–106.

[28] M. A. Vasiliev, “Actions, charges and off-shell fields in the unfolded dynamics approach,” *Int. J. Geom. Meth. Mod. Phys.* **3** (2006) 37–80, [hep-th/0504090](http://arxiv.org/abs/hep-th/0504090).

[29] M. Henneaux and C. Teitelboim, “Quantization of Gauge Systems,” Princeton, USA: Univ. Pr. (1992) 520 p.

[30] R. L. Arnowitt, S. Deser, and C. W. Misner, “The dynamics of general relativity,” in *Gravitation: an introduction to current research*, ch. 7, pp. 227–265. Wiley, 1962. [gr-qc/0405109](http://arxiv.org/abs/gr-qc/0405109).

[31] K. Hinterbichler and R. A. Rosen, “Interacting Spin-2 Fields,” *JHEP* **1207** (2012) 047, [1203.5783](http://arxiv.org/abs/1203.5783).

[32] S. Hasson, A. Schmidt-May, and M. von Strauss, “Metric Formulation of Ghost-Free Multivielbein Theory,” [1204.5202](http://arxiv.org/abs/1204.5202).

[33] S. W. MacDowell and F. Mansouri, “Unified geometric theory of gravity and supergravity,” *Phys. Rev. Lett.* **38** (1977) 739.

[34] K. S. Stelle and P. C. West, “Spontaneously broken De Sitter symmetry and the gravitational holonomy group,” *Phys. Rev. D* **21** (1980) 1466.

[35] M. A. Vasiliev, “Cubic interactions of bosonic higher spin gauge fields in AdS(5),” *Nucl. Phys.* **B616** (2001) 106–162, [hep-th/0106200](http://arxiv.org/abs/hep-th/0106200).

[36] E. Fradkin and M. A. Vasiliev, “Cubic Interaction in Extended Theories of Massless Higher Spin Fields,” *Nucl. Phys.* **B291** (1987) 141.

[37] K. Alkalaev and M. Vasiliev, “N=1 supersymmetric theory of higher spin gauge fields in AdS(5) at the cubic level,” *Nucl.Phys.* **B655** (2003) 57–92, [hep-th/0206068](http://arxiv.org/abs/hep-th/0206068).

[38] N. Boulanger, D. Ponomarev, and E. Skvortsov, “Non-abelian cubic vertices for higher-spin fields in anti-de Sitter space,” [1211.6979](http://arxiv.org/abs/1211.6979).

[39] Y. M. Zinoviev, “Toward frame-like gauge invariant formulation for massive mixed symmetry bosonic fields,” *Nucl. Phys.* **B812** (2009) 46–63, [0809.3287](http://arxiv.org/abs/0809.3287).

[40] D. S. Ponomarev and M. A. Vasiliev, “Frame-Like Action and Unfolded Formulation for Massive Higher-Spin Fields,” *Nucl. Phys.* **B839** (2010) 466–498, [1001.0062](http://arxiv.org/abs/1001.0062).

[41] R. Metsaev, “Extended Hamiltonian Action for Arbitrary Spin Fields in Flat And AdS Spaces,” *J.Phys.* **A46** (2013) 214021, [1112.0976](http://arxiv.org/abs/1112.0976).

[42] N. Misuna and M. Vasiliev, “Off-Shell Scalar Supermultiplet in the Unfolded Dynamics Approach,” [1301.2230](http://arxiv.org/abs/1301.2230).

[43] K. B. Alkalaev, “Two-column higher spin massless fields in AdS(d),” *Theor. Math. Phys.* **140** (2004) 1253–1263, [hep-th/0311212](http://arxiv.org/abs/hep-th/0311212).

[44] K. Alkalaev, O. Shaynkman, and M. Vasiliev, “Lagrangian formulation for free mixed-symmetry bosonic gauge fields in (A)dS(d),” *JHEP* **0508** (2005) 069, [hep-th/0501108](http://arxiv.org/abs/hep-th/0501108).

[45] M. A. Vasiliev, “Bosonic conformal higher–spin fields of any symmetry,” *Nucl. Phys.* **B829** (2010) 176–224, [0909.5226](http://arxiv.org/abs/0909.5226).
[46] D. M. Gitman and I. V. Tyutin, “Quantization of fields with constraints.”. Berlin, Germany: Springer (1990) 291 p. (Springer series in nuclear and particle physics).

[47] M. J. Gotay, J. M. Nester, and G. Hinds, “Presymplectic manifolds and the Dirac–Bergmann theory of constraints,” Journal of Mathematical Physics 19 (1978), no. 11, 2388–2399.

[48] M. Gotay and J. Nester, “Presymplectic Hamilton and Lagrange Systems, Gauge Transformations and the Dirac Theory of Constraints,” Lect. Notes Phys. 94 (1979) 272–279.

[49] K. Alkalaev and S. Lyakhovich, “On the consistency problem of interactions of (2+1) massive spinning particle,” Mod. Phys. Lett. A14 (1999) 2727–2737.

[50] J. Saavedra, R. Troncoso, and J. Zanelli, “Degenerate dynamical systems,” \textit{J. Math. Phys.} \textbf{42} (2001) 4383–4390, \texttt{hep-th/0011231}.

[51] R. L. Arnowitt, S. Deser, and C. W. Misner, “Dynamical Structure and Definition of Energy in General Relativity,” \textit{Phys. Rev.} \textbf{116} (1959) 1322–1330.

[52] K. Kuchar, “Dynamics of Tensor Fields in Hyperspace. 3.,” \textit{J. Math. Phys.} \textbf{17} (1976) 801–820.

[53] C. G. Torre, “Covariant phase space formulation of parametrized field theories,” \textit{J. Math. Phys.} \textbf{33} (1992) 3802–3812, \texttt{hep-th/9204055}.

[54] X. Bekaert and M. Grigoriev, “Notes on the ambient approach to boundary values of AdS gauge fields,” \textit{J. Phys.} \textbf{A46} (2013) 214008, \texttt{1207.3439}. 