ON THE VANISHING OF COHOMOLOGY IN TRIANGULATED CATEGORIES

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Abstract. We study the vanishing of cohomology in a triangulated category, in particular vanishing gaps and symmetry.

1. Introduction

In this paper, we study the vanishing of cohomology in a triangulated category $T$. Given two objects $X$ and $Y$ of $T$, a very natural question arises when looking at their cohomology: can we detect the vanishing of $\text{Hom}_T(X, \Sigma^n Y)$ for large $n$ by looking at finite vanishing gaps? That is, is there a finite set $S$ of integers such that the implication

$$\text{Hom}_T(X, \Sigma^n Y) = 0 \text{ for } n \in S \implies \text{Hom}_T(X, \Sigma^n Y) = 0 \text{ for } n \gg 0$$

holds? Commutative local complete intersection rings provide examples where this is true. Namely, for such a ring $A$, the following was shown in [Jor] for a module $M$ of complexity $d$: if $N$ is an $A$-module, and there exists an integer $n > \dim A$ such that $\text{Ext}^i_A(M, N) = 0$ for $n \leq i \leq n + d$, then $\text{Ext}^i_A(M, N)$ vanishes for all $i > \dim A$. Over such a ring $A$, the complexity of an $A$-module is at most the codimension $c$ of $A$. Therefore, we can always detect vanishing of cohomology over $A$ by looking at gaps of length $c + 1$.

Another natural question is: does symmetry hold in the vanishing of cohomology in $T$? In other words, if $X$ and $Y$ are objects in $T$ such that $\text{Hom}_T(X, \Sigma^n Y)$ vanishes for $n \gg 0$, then does it necessarily follow that $\text{Hom}_T(Y, \Sigma^n X)$ also vanishes for $n \gg 0$? Again, commutative local complete intersection rings provide examples where this hold. Namely, it was shown in [AvB] that if $M$ and $N$ are modules over such a ring $A$, then the implication

$$\text{Ext}^i_A(M, N) = 0 \text{ for } i \gg 0 \implies \text{Ext}^i_A(N, M) = 0 \text{ for } i \gg 0$$

holds. Another class of rings where such symmetry holds are group algebras of finite groups, or, more generally, as we shall see, symmetric algebras with “finitely generated” cohomology.

The two questions raised are studied in Section 3 and Section 4, respectively. We obtain some affirmative answers when certain cohomology groups are finitely generated as a module over a ring acting centrally on our triangulated category, a concept we define in the following section. In particular, we show that Ext-symmetry holds for symmetric periodic algebras.

2. Preliminaries

Throughout this paper, we fix a triangulated category $T$ with a suspension functor $\Sigma$. Thus $T$ is an additive $\mathbb{Z}$-category together with a class of distinguished triangles satisfying Verdier’s axioms (cf. [Ver]).

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Recall that a thick subcategory of $\mathcal{T}$ is a full triangulated subcategory closed under direct summands. Now let $\mathcal{C}$ and $\mathcal{D}$ be subcategories of $\mathcal{T}$. We denote by $\text{thick}_\mathcal{T}(\mathcal{C})$ the full subcategory of $\mathcal{T}$ consisting of all the direct summands of finite direct sums of shifts of objects in $\mathcal{C}$. Furthermore, we denote by $\mathcal{C} \ast \mathcal{D}$ the full subcategory of $\mathcal{T}$ consisting of objects $M$ such that there exists a distinguished triangle
\[
C \to M \to D \to \Sigma C
\]
in $\mathcal{T}$, with $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Now for each $n \geq 2$, define inductively $\text{thick}_\mathcal{T}^n(\mathcal{C})$ to be $\text{thick}_\mathcal{T}^1(\text{thick}_\mathcal{T}^{n-1}(\mathcal{C}) \ast \text{thick}_\mathcal{T}^1(\mathcal{C}))$, and denote $\bigcup_{n=1}^{\infty} \text{thick}_\mathcal{T}^n(\mathcal{C})$ by $\text{thick}_\mathcal{T}(\mathcal{C})$. This is the smallest thick subcategory of $\mathcal{T}$ containing $\mathcal{C}$.

The aim of this paper is to study the vanishing of cohomology in triangulated categories satisfying a certain finite generation hypothesis. This finite generation hypothesis is expressed in terms of the graded center $Z^*(\mathcal{T})$ of our triangulated category $\mathcal{T}$. Recall therefore that for an integer $n \in \mathbb{Z}$, the degree $n$ component $Z^n(\mathcal{T})$ is the set of natural transformations $\text{Id} \xrightarrow{f} \Sigma^n$ satisfying $f_{\Sigma X} = (-1)^n f_X$ on the level of objects. For such a central element $f$ and objects $X, Y \in \mathcal{T}$, consider the graded group $\text{Hom}_\mathcal{T}^*(X, Y) = \oplus_{i \in \mathbb{Z}} \text{Hom}_\mathcal{T}(X, \Sigma^i Y)$. The element $f$ acts from the right on this graded group via the morphism $X \xrightarrow{f_X} \Sigma^n X$, and from the left via the morphism $Y \xrightarrow{f_Y} \Sigma^n Y$. Namely, given a morphism $g \in \text{Hom}_\mathcal{T}(X, \Sigma^n Y)$, the scalar product $fg$ is the composition $X \xrightarrow{f_X} \Sigma^n X \xrightarrow{\Sigma^n g} \Sigma^n \Sigma^{m+n} Y$, whereas $gf$ is the composition $X \xrightarrow{g} \Sigma^m Y \xrightarrow{f_{\Sigma^m Y}} \Sigma^m \Sigma^{m+n} Y$. However, since $\text{Id} \xrightarrow{f} \Sigma^n$ is a natural transformation, the diagram
\[
\begin{array}{ccc}
X & \xrightarrow{g} & \Sigma^n Y \\
| & & | \\
\Sigma^n X & \xrightarrow{f} & \Sigma^n \Sigma^{m+n} Y
\end{array}
\]
commutes, and so since $f_{\Sigma^m Y}$ equals $(-1)^{mn} \Sigma^m f_Y$ we see that $gf = (-1)^{mn} fg$.

This shows that $Z^*(\mathcal{T})$ acts graded-commutatively on $\text{Hom}_\mathcal{T}^*(X, Y)$ for all objects $X$ and $Y$ in $\mathcal{T}$.

Now let $R = \oplus_{i=0}^{\infty} R_i$ be a graded-commutative ring, that is, for homogeneous elements $r_1, r_2 \in R$ the equality $r_1 r_2 = (-1)^{|r_1||r_2|} r_2 r_1$ holds. We say that $R$ acts centrally on $\mathcal{T}$ if there exists a graded ring homomorphism $R \to Z^*(\mathcal{T})$.

If this is the case, then for every object $X \in \mathcal{T}$ there is a graded ring homomorphism $R \xrightarrow{\varphi_X} \text{Hom}_\mathcal{T}^*(X, X)$ with the following property: for all objects $Y \in \mathcal{T}$ the scalar actions from $R$ on $\text{Hom}_\mathcal{T}^*(X, Y)$ via $\varphi_X$ and $\varphi_Y$ are graded equivalent, i.e.
\[
\varphi_Y(r)f = (-1)^{|r||f|} f\varphi_X(r)
\]
for all homogeneous elements $r \in R$ and $f \in \text{Hom}_\mathcal{T}^*(X, Y)$. We say that the $R$-module $\text{Hom}_\mathcal{T}^*(X, Y)$ is eventually Noetherian, and write $\text{Hom}_\mathcal{T}^*(X, Y) \in \text{Noeth}^R$ if there exists an integer $n_0 \in \mathbb{Z}$ such that the $R$-module $\text{Hom}_\mathcal{T}^{\geq n_0}(X, Y)$ is Noetherian. Moreover, we say that $\text{Hom}_\mathcal{T}^*(X, Y)$ is eventually Noetherian of finite length, and write $\text{Hom}_\mathcal{T}^*(X, Y) \in \text{Noeth}^R$, if $\text{Hom}_\mathcal{T}^*(X, Y) \in \text{Noeth}^R$, and there exists an integer $n_0 \in \mathbb{Z}$ such that $\ell_{R_0}(\text{Hom}_\mathcal{T}(X, \Sigma^m Y)) < \infty$ for each $n \geq n_0$. Note that if $\text{Hom}_\mathcal{T}^*(X, Y)$ is eventually Noetherian (respectively, eventually Noetherian of finite length), then so is $\text{Hom}_\mathcal{T}^*(X', Y')$ for all objects $X' \in \text{thick}_\mathcal{T}(X)$ and $Y' \in \text{thick}_\mathcal{T}(Y)$. In particular, if our category $\mathcal{T}$ is classically finitely generated, that is, if there exists an object $G$ such that $\mathcal{T} = \text{thick}_\mathcal{T}(G)$, then $\text{Hom}_\mathcal{T}^*(X, Y) \in \text{Noeth}^R$ (respectively, $\text{Hom}_\mathcal{T}^*(X, Y) \in \text{Noeth}^R$) for all $X, Y \in \mathcal{T}$ if and only if $\text{Hom}_\mathcal{T}^*(G, G) \in \text{Noeth}^R$ (respectively, $\text{Hom}_\mathcal{T}^*(G, G) \in \text{Noeth}^R$).
Definition. Given objects $X$ and $Y$ of $T$, we define the complexity of the ordered pair $(X, Y)$ as

$$cx_T(X, Y) \overset{\text{def}}{=} \dim_{R^{ev}} \Hom_T^*(X, Y),$$

where $R^{ev}$ denotes the commutative graded subalgebra $\oplus_{i=1}^{\infty} R_{2i}$ of $R$. We define the complexity $cx_T X$ of the single object $X$ as $cx_T X \overset{\text{def}}{=} cx_T(X, X)$.

When studying vanishing of cohomology in $T$, we will only be dealing with objects $X, Y \in T$ with the property that the $R$-module $\Hom_T^*(X, Y)$ is eventually Noetherian of finite length. This motivates the choice of terminology. Namely, it follows from [BIKO, Proposition 2.6] that if $\Hom_T^*(X, Y)$ is eventually Noetherian of finite length, then the Krull dimension of the $R^{ev}$-module $\Hom_T^*(X, Y)$ equals the infimum of all non-negative integers $t$ with the following property: there exists a real number $a$ such that

$$\ell_{R^n}(\Hom_T(X, \Sigma^n Y)) \leq a n^{t-1}$$

for $n \gg 0$. A priori, the complexity of a pair is not finite. However, when $\Hom_T^*(X, Y)$ is eventually Noetherian of finite length, then the finiteness of $cx_T(X, Y)$ follows from the above together with [BIKO, Remark 2.1] and [AtM], Theorem 11.1.

It follows from the above alternative description of complexity that if $X$ and $Y$ are objects of $T$ with $\Hom_T^*(X, Y) \in \text{Noeth} R$, then $cx_T(X, Y) = 0$ if and only if $\Hom_T^*(X, Y)$ is eventually zero, that is, if $\Hom_T(X, \Sigma^n Y) = 0$ for $n \gg 0$. Now digress for a moment, and let $\Lambda$ be a ring. Then $\Lambda$ satisfies Auslander’s condition if for every finitely generated module $M$, there exists an integer $d_M$, depending only on $M$, satisfying the following: if $N$ is a finitely generated $\Lambda$-module and $\Ext^i_{\Lambda}(M, N) = 0$ for $n \gg 0$, then $\Ext^i_{\Lambda}(M, N) = 0$ for $n \geq d_M$. Motivated by this, we define a full subcategory $\mathcal{C}$ of $T$ to be a left Auslander subcategory if for every object $X \in \mathcal{C}$, there exists an integer $d_X$, depending only on $X$, such that the following holds: if $\Hom_T^*(X, Y)$ is eventually zero for some object $Y \in T$, then $\Hom_T(X, \Sigma^n Y) = 0$ for $n \geq d_X$. It is easy to see that this holds if and only if for all objects $X \in \mathcal{C}$ and $Y \in T$, the implication

$$\Hom_T(X, \Sigma^n Y) = 0 \text{ for } n \gg 0 \implies \Hom_T(X, \Sigma^n Y) = 0 \text{ for all } n \in \mathbb{Z}$$

holds. Dually, we can define right Auslander subcategories. Note that if an object $X \in T$ belongs to a left or right Auslander subcategory and $\Hom_T^*(X, X)$ is eventually zero, then $X = 0$.

3. Vanishing of cohomology

We start with the following result, the key ingredient in the main theorem. It shows that we can always reduce the complexity of an object whose endomorphism ring is eventually Noetherian of finite length. However, recall first the following notion. Let $R$ be a graded-commutative ring acting centrally on $T$, and let $X \in T$ be an object. Then, given a homogeneous element $r \in R$, we can complete the map $X \overset{\varphi_X(r)}{\longrightarrow} \Sigma^{[r]} X$ into a triangle

$$X \overset{\varphi_X(r)}{\longrightarrow} \Sigma^{[r]} X \rightarrow X/\sim \rightarrow \Sigma X.$$ 

The object $X/\sim$ is well defined up to isomorphism, and is called a Koszul object of $r$ on $X$.

Proposition 3.1. Let $R$ be a graded-commutative ring acting centrally on $T$, and let $X \in T$ be an object such that $\Hom_T^*(X, X) \in \text{Noeth} R$. Then if $cx_T X$ is
nonzero, there exists a homogeneous element \( r \in R \), of positive degree, whose Koszul object \( X/\langle r \rangle \) in the triangle

\[
X \xrightarrow{\varphi_X(r)} \Sigma^{|r|} X \to X/\langle r \rangle \to \Sigma X
\]

satisfies \( \text{cx}_T X/\langle r \rangle = \text{cx}_T X - 1 \).

Proof. Suppose \( \text{cx}_T X > 0 \). By [BIKO] Lemma 2.5, there exists an integer \( i_0 \) and a homogeneous element \( r \in R \), of positive degree, such that scalar multiplication

\[
\text{Hom}_T(X, \Sigma^i X) \xrightarrow{r} \text{Hom}_T(X, \Sigma^{|r|} X)
\]
is injective for \( i \geq i_0 \). Applying \( \text{Hom}_T(X, -) \) to the triangle

\[
X \xrightarrow{\varphi_X(r)} \Sigma^{|r|} X \to X/\langle r \rangle \to \Sigma X,
\]
we obtain a long exact sequence

\[
\cdots \to \text{Hom}_T(X, \Sigma^i X) \xrightarrow{\langle (−1)^r \rangle} \text{Hom}_T(X, \Sigma^{|r|} X) \to \text{Hom}_T(X, \Sigma^{|r|} X/\langle r \rangle) \to \cdots
\]
in cohomology. This long exact sequence induces a short exact sequence

\[
0 \to \text{Hom}_T^{\leq i_0}(X, X/\langle r \rangle) \xrightarrow{r} \text{Hom}_T^{\geq i_0 + |r|}(X, X) \to \text{Hom}_T^{\geq i_0}(X, X/\langle r \rangle) \to 0
\]
of eventually Noetherian \( R \)-modules of finite length, a sequence from which we deduce that \( \text{cx}_T(X, X/\langle r \rangle) = \text{cx}_T X - 1 \). From this exact sequence we also see that the element \( r \) annihilates \( \text{Hom}_T^{\geq i_0}(X, X/\langle r \rangle) \). Therefore, when applying \( \text{Hom}_T(-, X/\langle r \rangle) \) to our triangle, we obtain a short exact sequence

\[
0 \to \text{Hom}_T^{\geq i_0 + |r|}(X, X/\langle r \rangle) \to \text{Hom}_T^{\geq i_0 + |r| + 1}(X/\langle r \rangle, X/\langle r \rangle) \to \text{Hom}_T^{\geq i_0 + 1}(X, X/\langle r \rangle) \to 0
\]
of eventually Noetherian \( R \)-modules of finite length. This gives the inequality \( \text{cx}_T X/\langle r \rangle \leq \text{cx}_T(X, X/\langle r \rangle) \).

Since \( \text{Hom}_T(X, X/\langle r \rangle) \) is eventually Noetherian, there exists an integer \( n_0 \) such that the \( R \)-module \( \text{Hom}_T^{\geq n_0}(X, X/\langle r \rangle) \) is finitely generated. The \( R \)-scalar action factors through the ring homomorphism \( \varphi_X(X/\langle r \rangle) \), and therefore \( \text{Hom}_T^{\geq n_0}(X, X/\langle r \rangle) \) is also finitely generated as a module over \( \text{Hom}_T(X/\langle r \rangle, X/\langle r \rangle) \). Consequently, the rate of growth of the sequence \( \{\ell_{R_0}(\text{Hom}_T(X, \Sigma^n X/\langle r \rangle))\}_{n=0}^{\infty} \) is at most the rate of growth of \( \{\ell_{R_0}(\text{Hom}_T(X/\langle r \rangle, \Sigma^n X/\langle r \rangle))\}_{n=0}^{\infty} \), i.e. \( \text{cx}_T(X, X/\langle r \rangle) \leq \text{cx}_T X/\langle r \rangle \).

Now let \( X \) be an object of \( T \), and let \( r_1, \ldots, r_c \) be a sequence of homogeneous elements belonging to a graded-commutative ring \( R \) acting centrally on \( T \), with \( c \geq 2 \). Then we define the Koszul element \( X/(r_1, \ldots, r_c) \) inductively as

\[
X/(r_1, \ldots, r_c) \stackrel{\text{def}}{=} (X/(r_1, \ldots, r_{c-1}))/r_c.
\]

Suppose the \( R \)-module \( \text{Hom}_T^{\geq n_0}(X, X) \) is eventually Noetherian of finite length, and denote the complexity of \( X \) by \( c \). If \( c > 0 \), then the previous result guarantees the existence of a homogeneous element \( r_1 \in R \), of positive degree, such that \( \text{cx}_T X/\langle r_1 \rangle = c - 1 \). Since \( X/\langle r_1 \rangle \) belongs to thick \( T(X) \), the \( R \)-module \( \text{Hom}_T^{\geq n_0}(X/\langle r_1 \rangle, X/\langle r_1 \rangle) \) is also eventually Noetherian of finite length. Therefore, if \( c - 1 > 0 \), then we may use the above result again; there exists a homogeneous element \( r_2 \in R \), of positive degree, such that \( \text{cx}_T X/(r_1, r_2) = c - 2 \). Continuing like this, we obtain a sequence \( r_1, \ldots, r_c \) of homogeneous elements of \( R \), all of positive degree, together with triangles

\[
X \to \Sigma^{|r_1|} X \to X_1 \to \Sigma X
\]

\[
X_1 \to \Sigma^{|r_2|} X_1 \to X_2 \to \Sigma X_1
\]

\[
\vdots
\]

\[
X_{c-1} \to \Sigma^{|r_c|} X_{c-1} \to X_c \to \Sigma X_{c-1}
\]
in which $X_i = X//\langle r_1, \ldots, r_c \rangle$ and $\text{cx}_T X_i = c - i$ for $1 \leq i \leq c$. We say that the sequence $r_1, \ldots, r_c$ reduces the complexity of the object $X$. Note that such a sequence is not unique in general, and that the order matters. Note also that the sequence reduces the complexity of $\Sigma^n X$ for any $n \in \mathbb{Z}$, since $\text{cx}_T Y = \text{cx}_T \Sigma^n Y$ for all objects $Y$ in $T$.

We now prove our main result. It shows that, for two objects $X$ and $Y$, if $\text{Hom}_T^*(X, Y)$ contains a large enough “gap”, then $\text{Hom}_T^*(X, Y)$ is actually zero. The length of the gap depends on the sum of the degrees of a sequence reducing the complexity of $X$.

**Theorem 3.2.** Let $X$ and $Y$ be objects of $T$ with $\text{Hom}_T^*(X, X) \in \text{Noeth}^R$ for some graded-commutative ring $R$ acting centrally on $T$, and suppose $\text{thick}_T(X)$ is either a left or right Auslander subcategory of $T$. Let $r_1, \ldots, r_c$ be a sequence of positive degree homogeneous elements of $R$ reducing the complexity of $X$, where $c = \text{cx}_T X$. Then the following are equivalent:

(i) There exists an integer $n \in \mathbb{Z}$ such that $\text{Hom}_T(X, \Sigma^i Y) = 0$ for $n \leq i \leq n + |r_1| + \cdots + |r_c| - c$.

(ii) $\text{Hom}_T(X, \Sigma^i Y) = 0$ for all $i \in \mathbb{Z}$.

**Proof.** We argue by induction on $c$ that (i) implies (ii). If $c$ is zero, then $\text{Hom}_T^*(X, X)$ is eventually zero, and so $X = 0$ since $\text{thick}_T(X)$ is either a left or right Auslander subcategory of $T$. If $c > 0$, consider the triangle

$$X \to \Sigma^{|r_1|} X \to X//r_1 \to \Sigma X.$$

Applying $\text{Hom}_T(-, Y)$ to this triangle gives the long exact sequence

$$\cdots \to \text{Hom}_T(X, \Sigma^{i-1} Y) \to \text{Hom}_T(X//r_1, \Sigma^i Y) \to \text{Hom}_T(X, \Sigma^{i-|r_1|} Y) \to \cdots$$

in cohomology, from which we see that $\text{Hom}_T(X//r_1, \Sigma^i Y) = 0$ for

$$(n + |r_1|) \leq i \leq (n + |r_1|) + |r_2| + \cdots + |r_c| - (c - 1).$$

The complexity of $X//r_1$ is $c - 1$, and the sequence $r_2, \ldots, r_c$ is a complexity reducing sequence for this object. Therefore, by the induction hypothesis, we conclude that $\text{Hom}_T(X//r_1, \Sigma^i Y) = 0$ for all $i \in \mathbb{Z}$. The long exact sequence then shows that $\text{Hom}_T(X, \Sigma^i Y)$ is isomorphic to $\text{Hom}_T(X, \Sigma^{i+|r_1|} Y)$ for all integers $i$, and from (i) we then see that $\text{Hom}_T(X, \Sigma^i Y)$ must vanish for all $i \in \mathbb{Z}$. \(\square\)

If we interchange the objects $X$ and $Y$ in the theorem, then the corresponding result of course holds. We state this without proof.

**Theorem 3.3.** Let $X$ and $Y$ be objects of $T$ with $\text{Hom}_T^*(X, X) \in \text{Noeth}^R$ for some graded-commutative ring $R$ acting centrally on $T$, and suppose $\text{thick}_T(X)$ is either a left or right Auslander subcategory of $T$. Let $r_1, \ldots, r_c$ be a sequence of positive degree homogeneous elements of $R$ reducing the complexity of $X$, where $c = \text{cx}_T X$. Then the following are equivalent:

(i) There exists an integer $n \in \mathbb{Z}$ such that $\text{Hom}_T(Y, \Sigma^i X) = 0$ for $n \leq i \leq n + |r_1| + \cdots + |r_c| - c$.

(ii) $\text{Hom}_T(Y, \Sigma^i X) = 0$ for all $i \in \mathbb{Z}$.

We now use these results to study the vanishing of cohomology over Artin algebras. Let $\Lambda$ be a Noetherian ring, and denote the bounded derived category of finitely generated $\Lambda$-modules by $D^b(\Lambda)$. Furthermore, let $D^\text{perf}(\Lambda)$ be the thick subcategory of $D^b(\Lambda)$ generated by $\Lambda$; it consists of the perfect complexes, that is, objects isomorphic to bounded complexes of finitely generated projective $\Lambda$-modules. The *stable derived category* of $\Lambda$, denoted $D^b_{\text{st}}(\Lambda)$, is the Verdier quotient

$$D^b_{\text{st}}(\Lambda) \overset{\text{def}}{=} D^b(\Lambda)/D^\text{perf}(\Lambda).$$
This is a triangulated category whose suspension functor corresponds to that in $D^b(\Lambda)$. Moreover, by [Biko] Remark 5.1, the central action of a graded-commutative ring $R$ on $D^b(\Lambda)$ carries over to $D^b(\Lambda)$ via the ring homomorphism $Z^*(D^b(\Lambda)) \to Z^*(D^b_{\text{reg}}(\Lambda))$ induced by the natural quotient functor. Thus if $X$ and $Y$ are complexes in $D^b(\Lambda)$, then the natural map

$$\text{Hom}^i_{D^b(\Lambda)}(X, Y) \to \text{Hom}^i_{D^b(\Lambda)}(X, Y)$$

is an $R$-module homomorphism. If $\Lambda$ is also Gorenstein, that is, if the injective dimension of $\Lambda$ both as a left and as a right module over itself is finite, then by [Buc] Corollary 6.3.4] this homomorphism is eventually bijective. That is, if $\Lambda$ is a Noetherian Gorenstein ring, then the natural map

$$\text{Hom}_{D^b(\Lambda)}(X, \Sigma^n Y) \to \text{Hom}_{D^b(\Lambda)}(X, \Sigma^n Y)$$

is bijective for $n \gg 0$. Consequently, for any complexes $X$ and $Y$ in $D^b(\Lambda)$, if $\text{Hom}_{D^b(\Lambda)}(X, Y)$ is an eventually Noetherian $R$-module of finite length, then so is $\text{Hom}_{D^b(\Lambda)}(X, Y)$.

Suppose $\Lambda$ is an Artin algebra, that is, the center $Z(\Lambda)$ of $\Lambda$ is a commutative Artin ring over which $\Lambda$ is finitely generated as a module. Denote by $\text{mod} \Lambda$ the category of finitely generated left $\Lambda$-modules. If $\Lambda$ is Gorenstein, then denote by $\text{MCM}(\Lambda)$ the category of finitely generated maximal Cohen-Macaulay $\Lambda$-modules, i.e.

$$\text{MCM}(\Lambda) = \{ M \in \text{mod} \Lambda \mid \text{Ext}_i^\Lambda(M, \Lambda) = 0 \text{ for all } i > 0 \}.$$

It follows from general cotilting theory that this is a Frobenius exact category, in which the projective injective objects are the projective $\Lambda$-modules, and the injective envelopes are the left and right $\Lambda$-approximations. Therefore the stable category $\text{MCM}(\Lambda)$, which is obtained by factoring out all morphisms which factor through projective $\Lambda$-modules, is a triangulated category. Its shift functor is given by cokernels of left and right $\Lambda$-approximations, the inverse shift is the usual syzygy functor. It follows from work by Buchweitz, Happel and Rickard (cf. [Buc, Hap, Ric]) that $\text{MCM}(\Lambda)$ and the quotient category $D^b(\Lambda)/\text{perf}(\Lambda)$ are equivalent as triangulated categories. If $M$ and $N$ are maximal Cohen-Macaulay modules in $\text{mod} \Lambda$, then there is an isomorphism

$$\text{Ext}^i_{\Lambda}(M, N) \simeq \text{Hom}_{\text{MCM}(\Lambda)}(\Omega^i_{\Lambda}(M), N)$$

for every $n > 0$. We use this isomorphism to prove the following result. It shows that when a certain finiteness condition holds, then the thick subcategory in $\text{MCM}(\Lambda)$ generated by a module is a left and right Auslander subcategory.

**Proposition 3.4.** Let $\Lambda$ be an Artin Gorenstein algebra with Jacobson radical $\tau$, and let $M$ be a maximal Cohen-Macaulay module. If either $\text{Ext}^i_{\Lambda}(M, \Lambda/\tau)$ or $\text{Ext}^i_{\Lambda}(\Lambda/\tau, M)$ belongs to $\text{Noeth}^R$ for some graded-commutative ring $R$ acting centrally on $D^b(\Lambda)$, then $\text{thick}_{\text{MCM}(\Lambda)}(M)$ is a left and right Auslander subcategory of $\text{MCM}(\Lambda)$.

**Proof.** Suppose that $\text{Ext}^i_{\Lambda}(M, \Lambda/\tau) \in \text{Noeth}^R$. Let $X$ and $Y$ be maximal Cohen-Macaulay modules in $\text{mod} \Lambda$ with $X \in \text{thick}_{\text{MCM}(\Lambda)}(M)$, and suppose that $\text{Hom}^i_{\text{MCM}(\Lambda)}(X, Y)$ is eventually zero. We prove by induction on $\text{cx}_{\text{MCM}(\Lambda)} X$ that $\text{Hom}^i_{\text{MCM}(\Lambda)}(X, \Sigma^n Y) = 0$ for all $n \in \mathbb{Z}$.

Suppose $\text{cx}_{\text{MCM}(\Lambda)} X = 0$. The finiteness condition implies that the $R$-module $\text{Ext}^i_{\Lambda}(M, N)$ is eventually Noetherian of finite length for all $N \in \text{mod} \Lambda$, and so the same holds for $\text{Hom}^i_{\text{MCM}(\Lambda)}(X, X)$. Therefore $\text{Hom}^i_{\text{MCM}(\Lambda)}(X, X)$ is eventually zero, in particular $\text{Ext}^n_{\Lambda}(X, X) = 0$ for $n \gg 0$. Now consider the $R$-module $\text{Ext}^n_{\Lambda}(X, \Lambda/\tau)$. Since it belongs to $\text{Noeth}^R$, and the $R$-module structure factors through the ring
homomorphism $R \xrightarrow{\psi_X} \text{Ext}^i_A(X, X)$, we see that it must be eventually zero. The $A$-module $X$ therefore has finite projective dimension, and is isomorphic to the zero object in $\text{MCM}(A)$. Consequently $\text{Hom}_{\text{MCM}(A)}(X, \Sigma^n Y) = 0$ for all $n \in \mathbb{Z}$.

If $\text{cx}_{\text{MCM}(A)}(X) > 0$, then let $r \in R$ be a homogeneous element of positive degree such that $\text{cx}_{\text{MCM}(A)}(X/r) = \text{cx}_{\text{MCM}(A)}(X) - 1$. Since $\text{Hom}_{\text{MCM}(A)}(X, Y)$ is eventually zero, we see from the triangle

$$X \to \Sigma^{|r|}X \to X/r \to \Sigma X$$

that the same holds for $\text{Hom}_{\text{MCM}(A)}(X/r, Y)$. The Koszul object $X/r$ belongs to $\text{thick}_{\text{MCM}(A)}(M)$, hence by induction $\text{Hom}_{\text{MCM}(A)}(X/r, \Sigma^n Y) = 0$ for all $n \in \mathbb{Z}$.

From the triangle we obtain the isomorphism

$$\text{Hom}_{\text{MCM}(A)}(X, \Sigma^n Y) \simeq \text{Hom}_{\text{MCM}(A)}(X, \Sigma^{n+|r|} Y)$$

for all integers $n$, and this implies that $\text{Hom}_{\text{MCM}(A)}(X, \Sigma^n Y) = 0$ for all $n \in \mathbb{Z}$.

We have now proved that if $\text{Ext}^1_A(M, \Lambda/r) \in \text{Noeth}^R R$, then $\text{thick}_{\text{MCM}(A)}(M)$ is a left Auslander subcategory of $\text{MCM}(A)$. Virtually the same proof shows that $\text{thick}_{\text{MCM}(A)}(M)$ is also a right Auslander subcategory of $\text{MCM}(A)$. Moreover, an analogous proof shows that the same holds if $\text{Ext}^1_A(\Lambda/r, M) \in \text{Noeth}^R R$. \hfill $\Box$

Before proving the next result, we recall the following. Let $\Lambda$ be an Artin algebra with Jacobson radical $r$, and let $M \in \text{mod} \Lambda$ be a module with minimal projective and injective resolutions

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

and

$$0 \to M \to I^0 \to I^1 \to I^2 \to \cdots$$

respectively. Then the complexity and plextity of $M$, denoted $\text{cx}_\Lambda M$ and $\text{px}_\Lambda M$, respectively, are defined as

$$\text{cx}_\Lambda M \overset{\text{def}}{=} \inf \{ t \in \mathbb{N} \cup \{0\} \mid \exists a \in \mathbb{R} \text{ such that } \ell_{Z(\Lambda)}(P_n) \leq an^{t-1} \text{ for } n \gg 0\},$$

$$\text{px}_\Lambda M \overset{\text{def}}{=} \inf \{ t \in \mathbb{N} \cup \{0\} \mid \exists a \in \mathbb{R} \text{ such that } \ell_{Z(\Lambda)}(I^n) \leq an^{t-1} \text{ for } n \gg 0\},$$

where $Z(\Lambda)$ is the center of $\Lambda$. Now let $R$ be a graded-commutative ring acting centrally on $D^b(\Lambda)$. If $\text{Ext}^1_A(M, \Lambda/r) \in \text{Noeth}^R R$, then $\text{cx}_\Lambda M$ coincides with $\text{cx}_{D^b(\Lambda)} M$, and $\text{Ext}^1_A(\Lambda/r, M)$ also belongs to $\text{Noeth}^R R$. Therefore there exists a sequence $r_1, \ldots, r_c$ of homogeneous elements of $R$, all of positive degree, reducing the complexity of $M$ as an object in $D^b(\Lambda)$. Similarly, if $\text{Ext}^1_A(\Lambda/r, M)$ belongs to $\text{Noeth}^R R$, then $\text{px}_\Lambda M = \text{cx}_{D^b(\Lambda)} M$. In this case, there exists a homogeneous sequence in $R$ of length $\text{px}_\Lambda M$ reducing the complexity of $M$ as an object in $D^b(\Lambda)$. Using this and Proposition 3.3.4 we obtain the following vanishing results on cohomology over Gorenstein algebras. We prove only the first of these results; the proof of the other result is similar.

**Theorem 3.5.** Let $\Lambda$ be an Artin Gorenstein algebra with Jacobson radical $r$, and let $M \in \text{mod} \Lambda$ be a maximal Cohen-Macaulay module. Suppose $\text{Ext}^1_A(M, \Lambda/r) \in \text{Noeth}^R R$ for some graded-commutative ring $R$ acting centrally on $D^b(\Lambda)$, and let $r_1, \ldots, r_c$ be a sequence of positive degree homogeneous elements of $R$ reducing the complexity of $M$, where $c = \text{cx}_\Lambda M$. Then for any $N \in \text{mod} \Lambda$, the implications (i) $\iff$ (ii) and (iii) $\iff$ (iv) hold for the following statements:

(i) There exists a number $n > \text{id} \Lambda$ such that $\text{Ext}^i_A(M, N) = 0$ for $n \leq i \leq n + |r_1| + \cdots + |r_c| - c$.

(ii) $\text{Ext}^i_A(M, N) = 0$ for all $i > \text{id} \Lambda$.

(iii) There exists a number $n > \text{id} \Lambda$ such that $\text{Ext}^i_A(N, M) = 0$ for $n \leq i \leq n + |r_1| + \cdots + |r_c| - c$. 


(iv) \( \text{Ext}^i_A(N, M) = 0 \) for all \( i > \text{id} \Lambda \).

Proof. By [An1, Theorem 1.8], there exists an exact sequence

\[ 0 \to Q \to C \to N \to 0 \]

in \( \text{mod} \Lambda \), in which \( Q \) has finite projective dimension and \( C \) is maximal Cohen-Macaulay. Since \( Q \) also has finite injective dimension and \( \text{id} Q \) is at most \( \text{id} \Lambda \), there are isomorphisms \( \text{Ext}^i_A(M, N) \cong \text{Ext}^i_A(M, C) \) for \( i > \text{id} \Lambda \). Moreover, the module \( \Omega^i_A(N) \) is maximal Cohen-Macaulay, and \( \text{Ext}^i_A(N, M) \cong \text{Ext}^i_{\Lambda^{\text{id} \Lambda}}(\Omega^i_A(N), M) \) for \( i > \text{id} \Lambda \). We may therefore without loss of generality assume that \( N \) itself is maximal Cohen-Macaulay, and replace \( \text{id} \Lambda \) by 0 in the statements. The implications now follow from Theorem 3.2, Theorem 3.3, Proposition 3.4, and the fact that \( \text{Ext}^i_A(X, Y) \cong \text{Hom}_{\text{MCM}(\Lambda)}(X, \Sigma^i Y) \) when \( X \) and \( Y \) are maximal Cohen-Macaulay and \( i > 0 \).

\[ \square \]

Theorem 3.6. Let \( \Lambda \) be an Artin Gorenstein algebra with Jacobson radical \( \tau \), and let \( M \in \text{mod} \Lambda \) be a maximal Cohen-Macaulay module. Suppose \( \text{Ext}^i_A(\Lambda/\tau, M) \in \text{Noeth}^b R \) for some graded-commutative ring \( R \) acting centrally on \( D^b(\Lambda) \), and let \( r_1, \ldots, r_c \) be a sequence of positive degree homogeneous elements of \( R \) reducing the complexity of \( M \), where \( c = \text{px}_{\Lambda} M \). Then for any \( N \in \text{mod} \Lambda \), the implications (i) \( \iff \) (ii) and (iii) \( \iff \) (iv) hold for the following statements:

1. There exists a number \( n > \text{id} \Lambda \) such that \( \text{Ext}^i_A(M, N) = 0 \) for \( n \leq i \leq n + |r_1| + \cdots + |r_c| - c \).
2. \( \text{Ext}^i_A(M, N) = 0 \) for all \( i > \text{id} \Lambda \).
3. There exists a number \( n > \text{id} \Lambda \) such that \( \text{Ext}^i_A(N, M) = 0 \) for \( n \leq i \leq n + |r_1| + \cdots + |r_c| - c \).
4. \( \text{Ext}^i_A(N, M) = 0 \) for all \( i > \text{id} \Lambda \).

4. Symmetry

For a group algebra \( kG \) of a finite group \( G \) over a field \( k \), the group cohomology ring \( H^*(G, k) \) is graded-commutative and acts centrally on \( D^b(kG) \). Moreover, by a classical result of Evens and Venkov (cf. [Eva, Ve1, Ve2]), the cohomology ring is Noetherian, and \( \text{Ext}^*_{kG}(M, N) \) is a finitely generated \( H^*(G, k) \)-module for all \( M \) and \( N \) in \( \text{mod} kG \). Therefore the vanishing results from the previous section apply to group algebras.

Commutative local complete intersection rings also have finitely generated cohomology. For such a ring \( A \), it was shown in [Av2] that there exists a certain polynomial ring \( \hat{A}[x_1, \ldots, x_c] \) acting centrally on \( D^b(\hat{A}) \), where \( \hat{A} \) denotes the completion of \( A \) with respect to its maximal ideal. Again, for all finitely generated \( A \)-modules \( M \) and \( N \), the \( \hat{A}[x_1, \ldots, x_c] \)-module \( \text{Ext}^*_{\hat{A}}(\hat{M}, \hat{N}) \) is finitely generated. Consequently, vanishing results similar to those in the previous section also hold in this case.

A fascinating aspect of the vanishing of cohomology over both group algebras and commutative local complete intersections is symmetry. In [Av1B] it was shown that for finitely generated modules \( M \) and \( N \) over a commutative local complete intersection \( A \), the vanishing of \( \text{Ext}^i_A(M, N) \) for \( i \gg 0 \) implies the vanishing of \( \text{Ext}^i_A(N, M) \) for \( i \gg 0 \). The proof involves the theory of certain support varieties attached to each pair of \( A \)-modules. Denote by \( c \) the codimension of \( A \) and by \( K \) the algebraic closure of its residue field. A cone \( V^*_A(M, N) \) in \( K^c \) is associated to the ordered pair \( (M, N) \), with the following properties:

\[ V_A^*(M, N) = \{0\} \iff \text{Ext}^i_A(M, N) = 0 \text{ for } i \gg 0, \]

\[ V_A^*(M, N) = V_A^*(M, M) \cap V_A^*(N, N). \]
The symmetry in the vanishing of cohomology follows immediately from these properties. Similarly, the theory of support varieties for modules over group algebras of finite groups can be used to show that symmetry holds also for such algebras (cf. [Ben]).

We shall see in this section that in general there is no symmetry in the vanishing of cohomology over an Artin algebra, even when the algebra is selfinjective and has finitely generated cohomology in the sense of group algebras. But first, we study situations where symmetry holds. Let $k$ be a commutative Artin ring, and suppose $T$ is a Hom-finite triangulated $k$-category. In other words, for all objects $X, Y, Z \in T$ the group $\text{Hom}_T(X, Y)$ is a $k$-module of finite length, and composition

$$\text{Hom}_T(Y, Z) \times \text{Hom}_T(X, Y) \to \text{Hom}_T(X, Z)$$

is $k$-bilinear, where $D = \text{Hom}_k(-, k)$. A Serre functor on $T$ is a triangle equivalence $T \xrightarrow{S} T$, together with functorial isomorphisms

$$\text{Hom}_T(X, Y) \simeq D \text{Hom}_T(Y, SX)$$

of $k$-modules for all objects $X, Y \in T$. By [BoK], such a functor is unique if it exists. Following [Kel], for an integer $d \in \mathbb{Z}$, the category $T$ is said to be weakly $d$-Calabi-Yau if it admits a Serre functor which is isomorphic as a $k$-linear functor to $\Sigma^d$. If, in addition, this isomorphism is an isomorphism of triangle functors, then $T$ is $d$-Calabi-Yau. However, we will only be dealing with weakly $d$-Calabi-Yau categories. When $T$ is such a category, then for all objects $X, Y \in T$ there is an isomorphism

$$\text{Hom}_T(X, Y) \simeq D \text{Hom}_T(Y, \Sigma^d Y)$$

of $k$-modules. It follows immediately that if this holds, then $\text{Hom}_T(X, \Sigma^n Y) = 0$ for $n \gg 0$ if and only if $\text{Hom}_T(Y, \Sigma^n X) = 0$ for $n \ll 0$.

Now let $\Lambda$ be an Artin Gorenstein algebra. Following [Mon], we say that $\Lambda$ is stably symmetric if $\text{MCM}(\Lambda)$ is weakly $d$-Calabi-Yau for some integer $d \in \mathbb{Z}$. It was shown in that paper that if $\Lambda$ in addition satisfies Auslander’s condition, then symmetry holds in the vanishing of cohomology of $\Lambda$-modules. The following result shows that symmetry holds for modules with finitely generated cohomology.

**Theorem 4.1.** Let $\Lambda$ be a stably symmetric Artin Gorenstein algebra with Jacobson radical $t$. Let $M \in \text{mod} \Lambda$ be a module such that either $\text{Ext}_A^*(M, \Lambda/r)$ or $\text{Ext}_A^*(\Lambda/r, M)$ belongs to $\text{Noeth}^b R$ for some graded-commutative ring $R$ acting centrally on $D^b(\Lambda)$. Then for every $N \in \text{mod} \Lambda$, the following are equivalent:

(i) $\text{Ext}_A^i(M, N) = 0$ for $i \gg 0$.
(ii) $\text{Ext}_A^i(M, N) = 0$ for $i > \text{id} \Lambda$.
(iii) $\text{Ext}_A^i(N, M) = 0$ for $i \ll 0$.
(iv) $\text{Ext}_A^i(N, M) = 0$ for $i > \text{id} \Lambda$.

**Proof.** As in the proof of Theorem 3.5 there exists an exact sequence

$$0 \to Q_N \to C_N \to N \to 0$$

in $\text{mod} \Lambda$, in which $Q_N$ has finite projective (and injective) dimension, and $C_N$ is maximal Cohen-Macaulay. Thus there is an isomorphism $\text{Ext}_A^i(M, N) \simeq \text{Ext}_A^{i-\text{id} \Lambda}(\Omega^{\text{id} \Lambda}_A(M), C_N)$ for every $i > \text{id} \Lambda$. Moreover, since either $\text{Ext}_A^i(M, \Lambda/r)$ or $\text{Ext}_A^i(\Lambda/r, M)$ belongs to $\text{Noeth}^b R$, so do either $\text{Ext}_A^i(\Omega^{\text{id} \Lambda}_A(M), \Lambda/r)$ or $\text{Ext}_A^i(\Lambda/r, \Omega^{\text{id} \Lambda}_A(M))$. Therefore, as shown in the proof of Theorem 3.5 and by Theorem 4.6, the implication

$$\text{Ext}_A^i(\Omega^{\text{id} \Lambda}_A(M), C_N) = 0 \text{ for } i \gg 0 \Rightarrow \text{Ext}_A^i(\Omega^{\text{id} \Lambda}_A(M), C_N) = 0 \text{ for } i > 0$$
holds, showing that (i) implies (ii). To show that (iii) implies (iv), fix an exact sequence

$$0 \to Q_M \to C_M \to M \to 0$$

in mod \( \Lambda \), in which \( Q_M \) has finite projective (and injective) dimension, and \( C_M \) is maximal Cohen-Macaulay. There is an isomorphism \( \text{Ext}^i_\Lambda(N, M) \simeq \text{Ext}^i_\Lambda(N, C_M) \) for every \( i > \text{id} \Lambda \). Also, as above, since either \( \text{Ext}^i_\Lambda(M, \Lambda/\tau) \) or \( \text{Ext}^i_\Lambda(\Lambda/\tau, M) \) belongs to Noeth\( ^b \)\( R \), so does one of \( \text{Ext}^i_\Lambda(C_M, \Lambda/\tau) \) and \( \text{Ext}^i_\Lambda(\Lambda/\tau, C_M) \). Hence (iii) implies (iv) by Theorem 3.5 and Theorem 3.6.

By Theorem 4.1, the subcategory \( \text{thick}^{\text{MCM}}(\Lambda) \) of \( \text{MCM}(\Lambda) \) is a left and right Auslander subcategory. Moreover, by assumption \( \text{MCM}(\Lambda) \) is weakly \( d \)-Calabi-Yau for some integer \( d \in \mathbb{Z} \). Therefore the implications

$$\text{Ext}^i_\Lambda(M, N) = 0 \text{ for } i \gg 0 \iff \text{Ext}^i_\Lambda(C_M, N) = 0 \text{ for } i \gg 0$$

$$\iff \text{Hom}_{\text{MCM}(\Lambda)}(C_M, \Sigma^i N) = 0 \text{ for } i \gg 0$$

$$\iff \text{Hom}_{\text{MCM}(\Lambda)}(C_M, \Sigma^i C_M) = 0 \text{ for } i \in \mathbb{Z}$$

$$\iff \text{Hom}_{\text{MCM}(\Lambda)}(C_N, \Sigma^i C_M) = 0 \text{ for } i \in \mathbb{Z}$$

$$\iff \text{Ext}^i_\Lambda(C_N, C_M) = 0 \text{ for } i \gg 0$$

$$\iff \text{Ext}^i_\Lambda(N, M) = 0 \text{ for } i \gg 0$$

hold, and the proof is complete.

□

For an Artin algebra \( \Lambda \) with radical \( \tau \), if \( \text{Ext}^i_\Lambda(\Lambda/\tau, \Lambda/\tau) \in \text{Noeth}^b \)\( R \) for some graded-commutative ring \( R \) acting centrally on \( D^b(\Lambda) \), then \( \text{Ext}^i_\Lambda(M, N) \in \text{Noeth}^b \)\( R \) for all modules \( M, N \in \text{mod} \Lambda \). Moreover, if this holds, then \( \Lambda \) is automatically Gorenstein by [BIKO, Proposition 5.6]. Consequently, we obtain the following “global version” of Theorem 4.1.

**Theorem 4.2.** Let \( \Lambda \) be a stably symmetric Artin algebra with Jacobson radical \( \tau \), and suppose that \( \text{Ext}^i_\Lambda(\Lambda/\tau, \Lambda/\tau) \) belongs to Noeth\( ^b \)\( R \) for some graded-commutative ring \( R \) acting centrally on \( D^b(\Lambda) \). Then for all modules \( M, N \in \text{mod} \Lambda \), the following are equivalent:

(i) \( \text{Ext}^i_\Lambda(M, N) = 0 \text{ for } i \gg 0 \).

(ii) \( \text{Ext}^i_\Lambda(M, N) = 0 \text{ for } i > \text{id} \Lambda \).

(iii) \( \text{Ext}^i_\Lambda(N, M) = 0 \text{ for } i \gg 0 \).

(iv) \( \text{Ext}^i_\Lambda(N, M) = 0 \text{ for } i > \text{id} \Lambda \).

Next, we include a special case of this theorem. Recall that for a commutative Artin ring \( k \), an Artin \( k \)-algebra \( \Lambda \) is symmetric if there is an isomorphism \( \Lambda \simeq \text{Hom}_k(\Lambda, k) \) of \( \Lambda \)-bimodules. Such an algebra is necessarily selfinjective.

**Corollary 4.3.** Let \( \Lambda \) be a symmetric Artin algebra with Jacobson radical \( \tau \), and suppose that \( \text{Ext}^i_\Lambda(\Lambda/\tau, \Lambda/\tau) \) belongs to Noeth\( ^b \)\( R \) for some graded-commutative ring \( R \) acting centrally on \( D^b(\Lambda) \). Then for all modules \( M, N \in \text{mod} \Lambda \), the following are equivalent:

(i) \( \text{Ext}^i_\Lambda(M, N) = 0 \text{ for } i \gg 0 \).

(ii) \( \text{Ext}^i_\Lambda(M, N) = 0 \text{ for } i > 0 \).

(iii) \( \text{Ext}^i_\Lambda(N, M) = 0 \text{ for } i \gg 0 \).

(iv) \( \text{Ext}^i_\Lambda(N, M) = 0 \text{ for } i > 0 \).

**Proof.** By [Mor] Corollary 4.4], a symmetric Artin algebra is stably symmetric. □

We turn now to a particular class of algebras having finitely generated cohomology in the sense of Theorem 4.2 and Corollary 4.3. Details concerning the
following can be found in \([\text{SnS}]\) and \([\text{Sol}]\). Let \(k\) be a field and \(\Lambda\) a finite dimensional \(k\)-algebra, and denote the enveloping algebra \(\Lambda \otimes_k \Lambda^{op}\) of \(\Lambda\) by \(\Lambda^e\). For \(n \geq 0\), the \(n\)th Hochschild cohomology group of \(\Lambda\), denoted \(\text{HH}^n(\Lambda)\), is the vector space \(\text{Ext}_\Lambda^n(\Lambda, \Lambda)\). The graded vector space \(\text{HH}^*(\Lambda) = \text{Ext}_\Lambda^*(\Lambda, \Lambda)\) is a graded-commutative ring with Yoneda product, and for every \(M \in \text{mod}\Lambda\) the tensor product \(- \otimes_\Lambda M\) induces a homomorphism

\[
\text{HH}^*(\Lambda) \xrightarrow{\varphi_M} \text{Ext}_\Lambda^*(M, M)
\]

of graded \(k\)-algebras. If \(N \in \text{mod}\Lambda\) is another module and \(\eta \in \text{HH}^*(\Lambda)\) and \(\theta \in \text{Ext}_\Lambda^*(M, N)\) are homogeneous elements, then the relation \(\varphi_N(\eta) \circ \theta = (-1)^{|\eta||\theta|}\varphi_M(\eta)\) holds, where \(\circ\) denotes the Yoneda product. Therefore the Hochschild cohomology ring \(\text{HH}^*(\Lambda)\) acts centrally on \(D^b(\Lambda)\).

Suppose now in addition that \(\Lambda\) is indecomposable as an algebra, and that \(\Lambda/\tau \otimes_k \Lambda/\tau\) is semisimple (as happens for example when \(k\) is algebraically closed). Furthermore, suppose that \(\Lambda\) is a periodic algebra. That is, there exists a number \(p > 0\) such that \(\Lambda\) is isomorphic to \(\Omega^p_\Lambda(\Lambda)\) as a left \(\Lambda^e\)-module (i.e. as a bimodule). By \([\text{EhH}]\), \([\text{EHS}]\) and \([\text{ESn}]\), this happens for example when \(\Lambda\) is a selfinjective Nakayama algebra, a Möbius algebra or a preprojective algebra (see also \([\text{ESN}]\) and \([\text{Snes}]\) by \(\text{CSS}\) the condition implies that \(\Lambda\) is selfinjective. Letting \(Q_n\) denote the \(n\)th module in the minimal projective \(\Lambda^e\)-resolution of \(\Lambda\), we have an exact sequence

\[
0 \rightarrow \Lambda \rightarrow Q_{p-1} \rightarrow \cdots \rightarrow Q_0 \rightarrow \Lambda \rightarrow 0
\]

of bimodules, and we denote this by \(\mu\). This extension is an element of \(\text{HH}^p(\Lambda)\). If \(\theta\) is an element of \(\text{HH}^n(\Lambda)\) for some \(n > p\), then \(\theta = \bar{\theta}_i\mu^i\) for some \(i\) and a homogeneous element \(\bar{\theta}_i\) of degree not more than \(p\). Hence the Hochschild cohomology ring \(\text{HH}^*(\Lambda)\) is generated over \(\text{HH}^0(\Lambda)\) by the finite set of \(k\)-generators in \(\text{HH}^1(\Lambda), \ldots, \text{HH}^p(\Lambda)\), and therefore is Noetherian. If \(S\) is a simple non-projective \(\Lambda\)-module, then \(\mu \otimes_\Lambda S\) is the beginning of the minimal projective resolution of \(S\), since \(\Lambda/\tau \otimes_k \Lambda/\tau\) is semisimple. Therefore \(S\) must be periodic with period dividing \(p\). If \(N\) is any finitely generated \(\Lambda\)-module and \(\omega\) is an element of \(\text{Ext}^n_\Lambda(S, N)\) for some \(n > p\), then, as above, \(\omega = \bar{\omega}(\mu \otimes_\Lambda S)\) for some element \(\bar{\omega} \in \text{Ext}^m_\Lambda(S, N)\) with \(m \leq p\).

Therefore \(\text{Ext}^n_\Lambda(S, N)\) is finitely generated as a module over \(\text{HH}^*(\Lambda)\), and this shows that \(\text{Ext}^*_\Lambda(\Lambda/\tau, \Lambda/\tau)\) is a finitely generated \(\text{HH}^*(\Lambda)\)-module. The following result is therefore an application of Corollary \([\text{ESN}]\)

\[
\textbf{Theorem 4.4.} \text{ Let } k \text{ be a field, let } \Lambda \text{ be a symmetric periodic } k\text{-algebra with Jacobson radical } \tau, \text{ and suppose that } \Lambda/\tau \otimes_k \Lambda/\tau \text{ is semisimple. Then for all modules } M, N \in \text{mod}\Lambda, \text{ the following are equivalent:}

(i) \(\text{Ext}^*_\Lambda(M, N) = 0\) for \(i > 0\).
(ii) \(\text{Ext}^*_\Lambda(M, N) = 0\) for \(i > 0\).
(iii) \(\text{Ext}^*_\Lambda(N, M) = 0\) for \(i > 0\).
(iv) \(\text{Ext}^*_\Lambda(N, M) = 0\) for \(i > 0\).

We finish this paper with an example in which we look at selfinjective Nakayama algebras. As we have seen, these algebras are periodic and therefore have finitely generated cohomology. However, the example shows that unless the algebra is symmetric, symmetry does not necessarily hold in the vanishing of cohomology.
Example. Let $\Gamma$ be the circular quiver 

![Circular Quiver Diagram]

where $t \geq 2$ is an integer. Let $k$ be a field, denote by $k\Gamma$ the path algebra of $\Gamma$ over $k$, and let $J \subset k\Gamma$ be the ideal generated by the arrows. Fix an integer $n \geq 1$, let $\Lambda$ be the quotient algebra $k\Gamma/J^n$, and let $J\subset k\Gamma$ be the ideal generated by the arrows. Fix an integer $n \geq 1$, let $\Omega$ be the quotient algebra $k\Gamma/J^n$, and denote by $r$ the Jacobson radical of $\Lambda$. Then $\Lambda$ is a finite dimensional indecomposable selfinjective Nakayama algebra, and $\text{Ext}_\Lambda^*(\Lambda/\tau, \Lambda/\tau)$ is a finitely generated $\text{HH}^*(\Lambda)$-module (the ring structure of $\text{HH}^*(\Lambda)$ was studied and determined in [BLM] and [ErH]).

Write $n = qt + r$, where $0 \leq r < t$. Let $S_i$ be the simple module corresponding to the vertex $i$, and $P_i$ its projective cover. There is an exact sequence 

$$0 \to \Omega_\Lambda^2(S_i) \to P_{i+1(\text{mod } t)} \to P_i \to S_i \to 0,$$

and it is easy to see that $\Omega_\Lambda^2(S_i)$ is isomorphic to $S_{i+1+r(\text{mod } t)}$. Therefore the minimal projective resolution of $S_i$ is 

$$\cdots \to P_{i+3+2r} \to P_{i+2+2r} \to P_{i+1+r} \to P_{i+1} \to P_i \to S_i \to 0,$$

with $\Omega_\Lambda^2(S_i) = S_{i+j+jr}$ (all the indices are taken modulo $t$). A number of completely different situations may occur, depending on the values of the parameters $t$ and $r$. For example, if $r = 0$, then we see that all the simple modules appear infinitely many times as even syzygies in the minimal projective resolution of any simple module. Therefore, in this case, if $S$ and $S'$ are simple modules, then $\text{Ext}_\Lambda^n(S, S')$ is nonzero for infinitely many $n$.

Note that when $r = 0$, then $\Lambda$ is symmetric, and so by Theorem 4.4 symmetry holds in the vanishing of $\text{Ext}$. However, symmetry does not hold for all Nakayama algebras. For example, suppose $t \geq 3$ and $r = t - 1$. Then the exact sequences 

$$0 \to S_1 \to P_2 \to P_1 \to S_1 \to 0$$

$$0 \to S_2 \to P_3 \to P_2 \to S_2 \to 0$$

are the first parts of the minimal projective resolutions of $S_1$ and $S_2$, and therefore $\text{Ext}_\Lambda^n(S_1, S_2) \neq 0$ whenever $n$ is odd, whereas $\text{Ext}_\Lambda^n(S_2, S_1) = 0$ for all $n$. Thus in this situation there is no symmetry in the vanishing of $\text{Ext}$ over $\Lambda$.

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