Creep via dynamical functional renormalization group

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We study a D-dimensional interface driven in a disordered medium. We derive finite temperature and velocity functional renormalization group (FRG) equations, valid in a $\epsilon = 4 - D$ expansion. These equations allow in principle for a complete study of the the velocity versus applied force characteristics. We focus here on the creep regime at finite temperature and small velocity. We show how our FRG approach gives the form of the $v - f$ characteristics in this regime, and in particular the creep exponent, obtained previously only through phenomenological scaling arguments.

Lines and surfaces can exhibit a remarkable variety of complex phenomena when they are driven by an external force over a disordered substrate. This situation is encountered in numerous physical systems such as growth phenomena [8], magnetic domain wall motion [9], fluid invasion of porous media [10], vortex systems [11], charge density waves (CDW) [12] or Wigner crystals [13].

In all these cases there is a competition between the elastic energy, which tends to keep the system straight, and the disorder. In addition to making the static interface rough, the disorder has also important consequences for the dynamics. At zero temperature $T = 0$, the interface remains pinned until a critical force $f_c$ is reached, whereas for large drive, it moves with a velocity proportional to the force, the disorder being averaged by the fast motion. Obtaining theoretically the velocity $v$ versus applied force $f$ characteristics is a very challenging problem since such a $v - f$ curve is directly measurable, and is in most cases one of the most important physical properties (e.g. the transport properties in vortex, CDW or Wigner crystal systems).

A very fruitful approach is to cast the depinning transition of an elastic system in the general framework of critical phenomena, with the velocity as an order parameter [14]. A functional renormalization group (FRG) treatment of the equation of motion [15] allows to obtain the various critical exponents characterizing the depinning transition.

At finite temperature, the situation is more complex since the interface can move by thermal activation below the zero temperature threshold force. This leads to a rounding of the depinning transition (see e.g. [16]) when $f \sim f_c$ and to a creep motion of the interface when $f \ll f_c$. Indeed it was realized that, due to the glassy nature of the static pinned interface, a moving interface would have to overcome divergent barriers as the applied force is reduced [3, 17] in contrast to much older theories that assumed that motion occurred only through finite barriers [18]. This lead to the proposal of a phenomenological theory of creep resulting in a highly non-linear $v - f$ characteristics of the form $v \propto \exp\left(-\left(U_c/T\right)f_c/f\right)^\mu$. Using scaling arguments, and assuming that relevant barriers for the dynamics scale with the same exponent as the energies of metastable static configurations, one obtains $\mu = (D - 2 + 2\zeta)/(2 - \zeta)$ where $D$ is the dimension of the interface and $\zeta$ the static roughening exponent of the interface. This remarkable formula relates dynamical properties to purely static quantities. The creep theory has been extremely successful in explaining various physical phenomena, in particular vortex systems [19], and the quantitative relations between the exponents has recently been verified by experiments on magnetic domain walls [20].

Unfortunately, up to now, the creep formula has remained phenomenological [21] and relies on a certain number of mostly unverified assumptions [22]. On the other hand, extending to finite temperatures and velocities the FRG techniques used at $T = 0$ has proved very challenging, although some interesting results could be obtained for the periodic systems driven within their internal space [23, 24]. We address this issue in the present paper. We give a generalization of the FRG equations valid for arbitrary temperature and velocity. These equations allow in principle for a complete study of the thermal effects on the $v - f$ characteristics. We show in this letter how they can be used to obtain the creep behavior and to derive the creep exponent from first principle (i.e. the equation of motion). We compare our findings with the phenomenological theory. A more refined study of the flow, a precise calculation of prefactors, and other
are left with the trivial friction law
\[ f = \eta v + F(r, vt + ur) + \zeta \]
where \( \eta \) is the friction coefficient and \( c \) is the elastic constant. For simplicity we assume here isotropic elasticity, but generalization to less symmetric elastic tensors can easily be done. Thermal fluctuations are described by the Langevin force \( \langle \zeta r, \zeta r' \rangle = 2\eta T \delta^D(r - r') \delta(t - t') \) (the brackets denote thermal averages). Disorder gives rise to a random force characterized by \( \langle \zeta \rangle = \xi \) the response at the interface is periodic, such that the trivial scaling of \( \xi \) is zero. We are then left with two free parameters (which can be chosen independent if needed): \( \zeta \), the roughening exponent of the interface relating longitudinal and transverse lengths \( u \sim r^{\zeta} \), and \( z \), the dynamical exponent relating space and time \( t \sim r^z \). Denoting \( S_D \) the surface of the sphere divided by \( (2\pi)^D \) and performing the rescalings, we obtain the following flow equations in \( \epsilon = 4 - D \) expansion

\[ \partial_t \Delta(u) = (\epsilon - 2\zeta)\Delta(u) + \zeta u \Delta'(u) + T \Delta''(u) \]
\[ + \int_{s > 0, s' > 0} e^{-s-s'} \Delta''(u) \Delta((s' - s)\lambda) - \Delta(u + (s' - s)\lambda)) \]
\[ - \Delta'(u - s\lambda) \Delta'(u + s\lambda) \]
\[ + \Delta'((s' + s)\lambda) \Delta'((u - s\lambda) - \Delta'(u + s\lambda)) \]  

These equations allow in principle for a complete description of the properties of the interface at finite temperatures and velocities.

For the deterministic \( T = 0 \) static \( v = 0 \) fixed point, the flow equations are greatly simplified since \( T_1 = 0 \) and \( \lambda_1 = 0 \). One recovers the usual formulas derived in the context of the depinning transition via replicas [8,23] or FRG [11]. The correlator \( \Delta \) flows to a fixed point \( \Delta^* \) which depends on the type of disorder. Exact solutions are known for random field [8] and periodic cases [26], and asymptotics for random bond. In

We consider a \( D \)-dimensional manifold without overhangs described by a height function \( u \), as shown in Figure, embedded in a space of dimension \( d \). If \( D = d \) this also describes periodic systems [24]. The system obeys the equation of motion

\[ (\eta \partial_t - c \nabla^2) u_{rt} = f - \eta v + F(r, vt + ur) + \zeta \]

The procedure is similar to the one used in [23], so we just give an outline here and leave technical details for [24]. We expand the weight \( e^{-\xi} \) up to second order in \( \Delta \), knowing that \( \xi = O(\Delta) \). We then integrate over the “fast” modes of the fields \( u \) and \( \hat{u} \) having Fourier components \( \Delta e^{\pm i q} \), where \( \Lambda e^{-\Lambda} < q < \Lambda \), where \( 1/\Lambda \) is a short distance cutoff. Reexponentiation gives an effective action similar to the original one, but with renormalized parameters, once we restore the original cutoff by rescaling the fields as \( u(r, t) = e^{-\xi} u^r c e^{\xi t} \) and \( \hat{u}(r, t) = e^{-\xi} \hat{u}^r c e^{\xi t} \).
all these cases, the fixed point function has a cusp ($\Delta''$ is discontinuous at the origin), which appears at a finite scale $l_c = \frac{1}{\epsilon} \log(1 + \frac{3}{\Delta''(0)})$ and which is responsible for the existence of the pinning force \[\xi.\] The scale at which the divergence of $\Delta''(0)$ occurs is the Larkin length, typical size of a segment of the interface wandering over the correlation length of the disorder. Its expression $\epsilon\lambda/\Lambda$ coincides with its standard expression, $R_c \simeq (\epsilon\lambda^2/\Lambda(0))^1/\epsilon$, when restoring the original $\lambda$, and denoting its range by $r_f$. In the following, we denote by $\alpha$ the length $-\Delta''(0^+)$, where $\alpha \simeq 0.4\epsilon^{2/3}(\int_{\Delta=0})^{1/3}$ for the random field case and $\alpha = \epsilon a/6$ for the periodic case where $a$ is the lattice spacing.

The flow equations \[\xi\] at finite but small velocity and zero temperature provide a natural derivation of the procedure used in \[\xi\] for calculating the critical force and the depinning exponents. Since up to the Larkin scale, disorder and displacements are small, the flow is trivial. After $l_c$, $\Delta$ has reached its fixed point $\Delta^*$. Provided that the velocity is small ($\lambda_l$ is smaller than the range $r_f$ of $\Delta^*$), our flow equations coincide with those used in \[\xi\] since we can replace the r.h.s. of the flow equations by the values at $0+$ of $\Delta^*$. The scale used to cut the flow in \[\xi\] appears here naturally when the length $\lambda_l$ is of the order of the range $r_f$ of $\Delta^*$. As in \[\xi\], we denote by $l_V$ the corresponding value of $l$. Note that $r_f$ can be quite different from the original range $r_f$ of the disorder. At larger scales, one crosses over for $l \sim l_V$ to a regime where the disorder acts as Langevin forces, and we are left with an effective Edwards Wilkinson equation of motion. We thus recover \[\xi\]

$$f_c = -\frac{c\Lambda^2\Delta''(0^+)}{-2 - \zeta}$$

This expression gives back, up to an $\epsilon$ coefficient, the dimensional estimate $f_c \simeq \frac{c\epsilon r_f}{\Lambda}$, since $R_c = \epsilon\lambda/\Lambda$ and $\alpha\epsilon^\zeta = 0.2\epsilon r_f$ in the random field case and $\alpha\epsilon^\zeta = \epsilon a/6$ in the periodic case.

Let us now analyse the finite temperature case. A finite temperature flows to zero with the free energy exponent $\epsilon = 2 - 2\zeta$. But the presence of the $T\Delta''(u)$ term in \[\xi\] is enough to smooth the behavior of the correlator near the origin, and it removes the divergence of $\Delta''(0)$. Indeed one expects $\lambda_l$ to remain analytic at all finite scales even if its limit $\Delta^*$ has a cusp. More precisely, during the flow, $-T\Delta''(u)$ increases and one may assume \[\xi\] that $-T\Delta''(0)$ converges, since the flow equation at the origin is $\partial_l\Delta_l(0) = (\epsilon - 2\zeta)\Delta_l(0) + T\Delta''(0)$. One can check that its limit is $\alpha^2$ (with $\alpha$ defined above). Using this property in the $T > 0$, $v = 0$ FRG equations for $\Delta$ and $T$ we obtain the precise way \[\xi\] the successive derivatives of $\Delta$ grow with $T$, and get

$$\frac{1}{T_l} \left( \Delta_l(0) - \Delta_l(\frac{T_l}{\alpha}x) \right) \to \sqrt{1 + x^2} - 1$$

The rounding close to the origin appears at a scale of the order of the temperature $T_l$. The form of $\Delta_l$ in this regime is given by $\Delta^*$ for $u \gg \xi_l = \frac{T_l}{\alpha}$ and by the rounding \[\xi\] for $u \ll \xi_l$, as shown on Figure \[\xi\].

FIG. 3. Generation of the cusp: two lengthscales appear. The width $\xi_l$ of the rounding, of order $T_l$, and the width $r_f^*$ of the fixed point function $\Delta^*$.

This property of the finite temperature equations enables us to investigate the creep regime. In addition to the Larkin scale $l_c$, two obvious lengthscales appear in the equations: the scale $l_T$ for which $\lambda_l = \xi_l$, the width of the rounding of $\Delta_l$, and as before, $l_V$ for which $\lambda_l$ is of the order of the range $r_f^*$ of $\Delta^*$, and above which disorder is washed out. Provided $f \ll f_c$, the velocity is arbitrarily small, and so is the initial value of $\lambda$. One has thus $l_c < l_T < l_V$. The equation for the temperature introduces in fact an additional lengthscale that we discuss later. Ignoring for the time being this additional lengthscales allows to distinguish two main regimes above the Larkin scale $l_c$. In the thermal regime $l_c < l < l_T$, we can replace the set of equations \[\xi\] by static ones obtained by replacing the r.h.s. occurences of $\lambda_l$ by zero, and not by $0+$ as for the depinning case. The quantity $-T_l\Delta''(0)$ becomes constant until $\lambda_l \sim \xi_l$. In this regime $\lambda$ grows rapidly. Physically this corresponds to a regime where barriers grow and the system can only move through thermal activation over those barriers.

Beyond $l_T$ one enters the depinning regime. The disorder is close to its fixed point and the other quantities now flow with the depinning flow equations, since the integrals over $s$ do no more see the behaviour at the origin but the completely different values of $\Delta^*$ at $0+$. Roughly speaking we can in this regime ignore the temperature, as far as motion is concerned. Physically this is because above the length $\epsilon\lambda/\Lambda \sim R_c(f_c/f)^{1/(2-\zeta)}$, that corresponds \[\xi\] to the criterion $\lambda_l \sim \xi_l$, the external force is dominant over the pinning force, and these lengthscales do not need thermal activation to move. For $l \gg l_T$, the temperature is not important for the motion. As for $T = 0$, we cut the flow at the scale $l_V$ where a perturbative expansion in disorder is well defined. We can use it to estimate the value of $\tilde{f}_l$. Since the renormalized temperature is very small it is enough to do this calculation.
as if $T_{1v} = 0$. Then it becomes obvious that $\tilde{f}_{iv}$ does not depend on the initial velocity, force or temperature.

Solving the flow equations for the three regimes yields a relation between initial quantities. Using $\kappa$ and replacing the initial value of $f$ by $f$ since in the creep regime $f \gg \eta v$, we get

$$\frac{\eta v}{f_{c}} = \frac{(2 - \zeta)SD}{(e^{1/\lambda})^{D-2}(\alpha e^{1/\lambda})^{2}} \frac{T}{c} \frac{f}{f_{c}}^{1+\mu} \exp \left( -\frac{U_{c}}{T} \left( \frac{f_{c}}{f} \right)^{\mu} - 1 \right)$$

(10)

where $\mu = (2 - \epsilon + 2\zeta)/(2 - \zeta)$ and a rough estimate of $U_{c}$ gives

$$U_{c} = \frac{c(\alpha e^{1/\lambda})^{2}(e^{1/\lambda})^{2-\epsilon}}{2(2 - \epsilon + 2\zeta)SD} = \frac{c(Cerf)^{2}R_{c}^{2-\epsilon}}{(2 - \epsilon + 2\zeta)SD}$$

(11)

where $C$ is a constant. We thus recover precisely the creep exponent $\mu$, obtained previously only via the phenomenological barrier arguments. Our derivation allows to obtain this value directly from the equation of motion without having to do any supplementary assumption. The fact that we do recover the simple scaling exponent $\mu$, is thus also a strong indication that the barriers important for the motion do scale with the free energy exponent (up to subleading corrections). Although our calculation is not expected to give, at that level, a precise estimate of the prefactors in the barriers $U_{c}$ it is conforming to also recover an expression similar to the one derived by scaling arguments, $U_{c} \sim \sigma^{2}R_{c}^{D-2}$, i.e. the barriers at the lengthscale of the Larkin length. As we showed above the FRG also identifies correctly the relevant length scales appearing in the simple scaling approach of the creep. This simple application of the FRG equations then gives a consistent picture for the creep.

The main interest of the present method is of course to allow in principle to go beyond the simple scaling. In this respect various questions remain. In particular the FRG equation for the temperature leads to an additional lengthscale at $\lambda_{T} \sim T_{1}^{3/2}$. Above this lengthscale the disorder starts renormalizing the temperature upwards. Although this effect has no impact on the creep exponent $\mu$ it can obviously lead to a modification of the barriers (or equivalently of the $1/T$ term in (10)). A more precise calculation of this amplitude, taking into account the full crossover around $l \sim l_{T}$ will be presented in [3]. Given the success of the finite temperature FRG for the creep regime, it is also tempting to apply these methods to the study of other regimes of the motion. In particular, close to the threshold at finite temperature, a scaling approach describing the motion as a succession of avalanches, and a complementary FRG approach based on the flow equations derived here, should help fixing the form of the force-velocity curve.