Wavelets and renormalization group in quantum field theory problems

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Using continuous wavelet transform it is possible to construct a regularization procedure for scale-dependent quantum field theory models, which is complementary to functional renormalization group method in the sense that it sums up the fluctuations of larger scales in order to get the effective action at small observation scale (M.V.Altaisky, Phys. Rev. D 93(2016) 105043). The standard RG results for $\phi^4$ model are reproduced. The fixed points of the scale-dependent theory are studied in one loop approximation.

I. INTRODUCTION

Renormalization group (RG) has entered quantum field theory as a group of infinitesimal reparametrizations of the $S$ matrix emerging after the cancellation of ultraviolet divergences [1]. The RG method has become known in quantum electrodynamics (QED) since Gell-Mann and Low have shown the charge distribution surrounding a test charge in vacuum does not at small distances depend on a coupling constant, except for a scale factor, i.e., possesses a kind of self-similarity, that enables to express a "bare" charge at small scale using the measured value at large scale [2].

RG can be considered as a method of treating physical problems with a large number degrees of freedom, not taking all those at once, but treating them successively scale-by-scale [4, 5]. This resulted in an elegant theory of critical phenomena and was later generalized to many other stochastic systems [6].

Same idea of separating the fluctuations of different scales has been implemented, basically in experimental data processing, in a quite different way: using wavelets. This was first done in geophysics [7, 8], and then spread over all possible data, from face recognition and

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medical imaging to high energy physics and cosmology [9]. The only interference of the RG and the wavelet method seems to be the the lattice regularization in quantum field theory (QFT), which can be performed either by standard lattice methods, or by using the discrete wavelet basis [10–12]. The connections between these two seemingly different methods are still missing. The text below is an endeavor to fill this gap partially.

II. DIVERGENCES IN QUANTUM FIELD THEORY

The fundamental problem of quantum field theory is the problem of divergences of Feynman graphs. The infinities appearing in perturbation expansion of Feynman integrals are treated by different regularization methods, from maximal momentum cutoff and Pauli-Villars regularization, to $\epsilon$-expansion, and renormalization group methods, see e.g. [13] for a review. We restrict ourselves with a simple example of scalar $\phi^4$ field theory in $\mathbb{R}^d$, which, however, illustrates all main problems and approaches related to the problem of divergences in quantum field theory, see e.g. [13, 14].

Euclidean scalar field theory with $\phi^4$ interaction potential is determined by the generating functional

$$Z[J] = \mathcal{N} \int \exp \left( -S_E[\phi] + \int J(x)\phi(x) d^d x \right) \equiv \exp(W[J])$$

where

$$S_E[\phi] = \int d^d x \left[ \frac{1}{2} (\partial\phi)^2 + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 \right].$$

where $\mathcal{N}$ is a formal normalization constant. The connected Green functions, understood as statistical momenta of the field $\phi$ [15], are given by functional derivatives of the generating functional:

$$G^{(n)} \equiv \langle \phi(x_1) \ldots \phi(x_n) \rangle_c = \left. \frac{\delta^n \ln W[J]}{\delta J(x_1) \ldots \delta J(x_n)} \right|_{J=0}$$

The divergences of Feynman graphs in the perturbation expansion of the Green functions (2) with respect to the small coupling constant $\lambda$ emerge at coinciding arguments $x_i = x_k$. For instance, the bare two-point Green function

$$G_0^{(2)}(x - y) = \int \frac{d^d p}{(2\pi)^d} \frac{e^{-ip(x-y)}}{p^2 + m^2}$$

is divergent at $x = y$ for $d \geq 2$.

Since the Green functions in Euclidean quantum field theory have the probability meaning, it is quite obvious physically that neither of the joint probabilities of the measured
quantities can be infinite. The infinities seem to be caused by an inadequate choice of the functional space the fields belong to.

This standard approach to quantum field theory, based on $L^2(\mathbb{R}^d)$ fields disregards two important notes [16]:

1. To localize a particle in an interval $\Delta x$ the measuring device requests a momentum transfer of order $\Delta p \sim \hbar/\Delta x$. If the value of this momentum is too large we may get out of the applicability range of the initial model, in the sense that $\phi(x)$ at a fixed point $x$ has no experimentally verifiable meaning. What is meaningful, is the vacuum expectation of product of fields in certain region centered around $x$, the width of which $(\Delta x)$ is constrained by the experimental conditions of the measurement.

2. Even if the particle has been initially prepared on the interval $(x - \Delta x, x + \Delta x)$, the probability of registering it on this interval is generally less than unity: for the probability of registration depends on the strength of interaction and the ratio of typical scales of the measured particle and the measuring equipment. The maximum probability of registering an object of typical scale $\Delta x$ by the equipment with typical resolution $a$ is achieved when these two parameters are comparable. For this reason the probability of registering an electron by visual range photon scattering is much higher than by that of long radio-frequency waves. As mathematical generalization, we should say that if a measuring equipment with a given spatial resolution $a$ fails to register an object, prepared on spatial interval of width $\Delta x$ with certainty, then tuning the equipment to all possible resolutions $a'$ would lead to the registration. This certifies the fact of the existence of the object.

Most of the regularization methods applied to make the Green functions finite imply a certain type of self-similarity – the independence of physical observables on the scale transformation of an arbitrary parameter of the theory – the cutoff length or the normalization scale. Covariance with respect to scale transformations is expressed by renormalization group equation [13]. Its predecessor, the Kadanoff blocking procedure, averages the small-scale fluctuations up to a certain scale into a kind of effective interaction for a larger blocks, assuming the larger blocks interact with each other in the same way as their sub-blocks [17, 18]. However the theory based on the Fourier transform, such as quantum field theory is, have no explicit tools to regard the self-similarity in a fair way. An abstract harmonic
analysis based on some group $G$, wider than the group of translations $G : x \rightarrow x + b$, should be used to account for self-similarity. The simplest analysis of such type is based on the representations of affine group $G : x \rightarrow ax + b$ and is widely referred to as continuous wavelet transform.

III. CONTINUOUS WAVELET TRANSFORM IN QUANTUM FIELD THEORY

Continuous wavelet transform (CWT) is a generalization of the Fourier transform for the case when the scaling properties of the theory are important. Referring the reader to general reviews on wavelet transform [9, 19], and to the original papers devoted to the application of wavelet transform to quantum field theory [16, 20, 21], below we remind basic definitions of the wavelet formalism of quantum field theory.

Let $\mathcal{H}$ be a Hilbert space of states for a quantum field $|\phi\rangle$. Let $G$ be a locally compact Lie group acting transitively on $\mathcal{H}$, with $d\mu(\nu), \nu \in G$ being a left-invariant measure on $G$. Similarly to the Fourier representation $|\phi\rangle = \int |p\rangle dp \langle p|\phi\rangle$, any $|\phi\rangle \in \mathcal{H}$ can be decomposed with respect to a representation $U(\nu)$ of $G$ in $\mathcal{H}$ [22, 23]:

$$|\phi\rangle = \frac{1}{C_g} \int_G U(\nu)|g\rangle d\mu(\nu) \langle g|U^*(\nu)|\phi\rangle,$$

where $|g\rangle \in \mathcal{H}$ is referred to as an admissible vector, or a basic wavelet, satisfying the admissibility condition $C_g = \frac{1}{|g||} \int_G |\langle g|U(\nu)|g\rangle|^2 d\mu(\nu) < \infty$. The coefficients $\langle g|U^*(\nu)|\phi\rangle$ are referred to as wavelet coefficients. If the group $G$ is Abelian, the wavelet transform (4) coincides with the Fourier transform.

Next to the Abelian group is the group of the affine transformations of the Euclidean space $\mathbb{R}^d$:

$$G : x' = ax + b, x, b \in \mathbb{R}^d, a \in \mathbb{R}_+.$$

(For simplicity we assume the isotropic basic wavelet $g$ and drop rotation factor.) The unitary representation of the affine transform (5) with respect to the isotropic basic wavelet $g(x)$ can be written as follows:

$$U(a, b)g(x) = \frac{1}{a^d}g \left( \frac{x - b}{a} \right).$$

(In accordance to previous papers [16, 20] we use $L^1$ norm [19, 24] to keep the physical dimension of wavelet coefficients equal to the dimension of the original fields).
Wavelet coefficients of the Euclidean field $\phi(x)$ with respect to the basic wavelet $g(x)$ in $\mathbb{R}^d$ are
\[
\phi_a(b) = \int_{\mathbb{R}^d} \frac{1}{a^d} g\left(\frac{x - b}{a}\right) \phi(x) d^d x.
\] (7)

The function $\phi(x)$ can be reconstructed from its wavelet coefficients (7) using the formula (4):
\[
\phi(x) = \frac{1}{C_g} \int_{\mathbb{R}^d} \frac{1}{a^d} g\left(\frac{x - b}{a}\right) \phi_a(b) \frac{da d^d b}{a}.
\] (8)

The normalization constant is readily evaluated using Fourier transform: $C_g = \int_0^\infty |\tilde{g}(a)|^2 \frac{da}{a}$.

Substituting (8) into the field theory (1) we obtain the generating functional for the scale-dependent fields $\phi_a(x)$:
\[
Z_W[J_a] = N \int D\phi_a(x) \exp \left[ -\frac{1}{2} \int \phi_{a_1}(x_1) D(a_1, a_2, x_1 - x_2) \phi_{a_2}(x_2) \frac{da_1 d^d x_1}{a_1} \right.
\]
\[
\times \frac{da_2 d^d x_2}{a_2} - \int V_{a_1, \ldots, a_4} \phi_{a_1}(x_1) \cdots \phi_{a_4}(x_4) \frac{da_1 d^d x_1}{a_1} \frac{da_2 d^d x_2}{a_2}
\]
\[
\times \frac{da_3 d^d x_3}{a_3} \frac{da_4 d^d x_4}{a_4} + \int J_a(x) \phi_a(x) \frac{da d^d x}{a} \bigg],
\] (9)

with $D(a_1, a_2, x_1 - x_2)$ and $V_{a_1, \ldots, a_4}$ denoting the wavelet images of the inverse propagator and that of the interaction potential.

The Feynman diagram technique for the scale-dependent fields $\phi_a(x)$ is the same as for ordinary fields except for [16, 25]:

- each field $\tilde{\phi}(k)$ will be substituted by the scale component: $\tilde{\phi}(k) \rightarrow \tilde{\phi}_a(k) = \tilde{g}(ak)\tilde{\phi}(k)$.

- each integration in momentum variable is accompanied by corresponding scale integration:
\[
\frac{d^d k}{(2\pi)^d} \rightarrow \frac{d^d k}{(2\pi)^d} \frac{da}{a}.
\]

- each interaction vertex is substituted by its wavelet transform; for the $N$-th power interaction vertex this gives multiplication by factor $\prod_{i=1}^N \tilde{g}(a_i k_i)$.

According to these rules, the bare Green function in wavelet representation takes the form
\[
G_0^{(2)}(a_1, a_2, p) = \frac{\tilde{g}(a_1 p)\tilde{g}(-a_2 p)}{p^2 + m^2}.
\]

The finiteness of the loop integrals is provided by the following rule: there should be no scales $a_i$ in internal lines smaller than the minimal scale of all external lines [16]. Therefore
the integration in $a_i$ variables is performed from the minimal scale of all external lines up to the infinity. This corresponds to the assumption, that studying a system from outside one should not used functions with resolution better than the finest experimentally available scale. The integration over all scales will certainly drive us back to the known divergent theory.

IV. RENORMALIZATION GROUP FOR SCALE-DEPENDENT FIELDS

The only intersection between usual regularization methods of quantum field theory and the separation of different scales by means of wavelet transform up to very recently was the lattice regularization in wavelet basis [10–12, 21]. In [26] it was proposed to use CWT for regularization of quantum field theory along the lines the renormalization group is usually applied. To illustrate this let us rewrite the formalism of functional renormalization group for the effective averaging action, which accounts for the fluctuations with momentum $k$ integrating over fluctuations with momenta greater than $k$.

Let $\mathcal{F}_k$ be the space of functions the Fourier images of which are supported by $|p| \leq k$ domain. The effective action $S_k[\phi]$ is defined via

$$e^{-S_k[\phi]} = \int \mathcal{D}\chi P_k[\phi, \chi] e^{-S[\chi]},$$

where $P_k[\phi, \chi]$ is a projection of $\chi$ onto the space $\mathcal{F}_k$ [27]. If we know the action $S_k[\phi]$ we can coarse-grain to the next space $\mathcal{F}_{k-\Delta k}$ integrating over the functions $\tilde{\phi} \in \mathcal{D}_{k,\Delta k} = \mathcal{F}_k \setminus \mathcal{F}_{k-\Delta k}$, whose momenta are within the range $(k - \Delta k, k]$. The iteration of this procedure

$$e^{-S_{k-\Delta k}[\phi]} = \int \mathcal{D}[\tilde{\phi}] e^{-S_k[\phi+\tilde{\phi}]},$$

(10)

back to arbitrary small (IR) $k$ yields the scale decomposition

$$\cdots \subset \mathcal{F}_{k-2\Delta k} \subset \mathcal{F}_{k-\Delta k} \subset \mathcal{F}_k,$$

$$\mathcal{F}_k = \mathcal{D}_{k,\Delta k} \oplus \mathcal{D}_{k-\Delta k,2\Delta k} \oplus \mathcal{D}_{k-2\Delta k,3\Delta k} \oplus \cdots,$$

(11)

very similar to the Mallat sequence in wavelet analysis [28]. Doing so, one get an exact action $\Gamma[\phi] = \lim_{k \to 0} S_k[\phi]$. This is the essence of functional RG.

Working with wavelet transform we can do something complementary to functional renormalization group: we can sum up all fluctuations from infinitely large IR scale to a certain
finite scale of observation $A$ to obtain an effective action functional, which describes the physics at scale $A$. To work with 1PI diagrams we define the effective action via the Legendre transform of $W[J]$:

$$\Gamma[\phi_a] = -W_W[J_a] + \int J_a(x) \phi_a(x) \frac{da}{a} da$$

The functional derivatives of $\Gamma[\phi]$ are the renormalized vertex functions $\Gamma^{(n)}$.

Following [26] we consider $\Gamma^{(2)}$ and $\Gamma^{(4)}$ vertex functions for $\phi^4$ theory in $\mathbb{R}^d$ in one-loop level. The one loop contributions to the inverse propagator $\Gamma^{(2)}(A)$ and the vertex function $\Gamma^{(4)}(A)$ are given by diagrams shown in Fig. 1. After integration in scale arguments $d \ln a_i$ of the internal lines, the only difference between the wavelet-based theory and the standard one will be the presence of the squared cutoff functions $f^2(x)$ on each internal line, depending on the dimensionless momenta of the line $x = qA$. This gives

$$C_g^2 \frac{\Gamma^{(2)}(a_1, a_2, p)}{\bar{g}(a_1p)\bar{g}(-a_2p)} = p^2 + m^2 + \frac{\lambda}{2} T_g^d(\alpha),$$

where $d = 4$ is the dimension of Euclidean space, $\alpha = m \min(a_1, a_2)$ is the dimensionless scale of the tadpole diagram, for the inverse propagator; and similarly,

$$C_g^4 \frac{\Gamma^{(4)}(a_1p_1, a_2p_2, a_3p_3, a_4p_4)}{\bar{g}(a_1p_1)\bar{g}(a_2p_2)\bar{g}(a_3p_3)\bar{g}(a_4p_4)} = \lambda - \frac{3}{2} \lambda^2 X_g^d(A)$$

for the vertex function.

The values of the one-loop integrals

$$T_g^d(\alpha) = \frac{S_d m^{d-2}}{(2\pi)^d} \int_0^\infty f_\alpha(x) x^{d-1} dx,$$

$$X_g^d(A) = \int \frac{d^dq f_\alpha^2(qA)f_\alpha^2((q-s)A)}{(2\pi)^d [q^2 + m^2] [(q-s)^2 + m^2]}.$$
where \( s = p_1 + p_2, A = \min(a_1, a_2, a_3, a_4) \), depend on the wavelet cutoff function

\[
f_g(x) = \frac{1}{C_g} \int_x^\infty |\tilde{g}(a)|^2 \frac{da}{a}
\]

(16)

for the chosen wavelet \( g \).

The dependence of the effective coupling constant on the observation scale \( A \) can be obtained by taking the derivative with respect to the logarithm of observation scale \( \mu = -\ln A + \text{const} \). For the \( g_1 \) wavelet in \( d = 4 \) used in [26] this gives the flow equations

\[
\frac{\partial \lambda}{\partial \mu} = \frac{3\lambda^2 \alpha^2 \partial X_1^4}{16\pi^2} = \frac{3\lambda^2}{16\pi^2} \frac{2\alpha^2 + 1 - e^{2\alpha^2}}{\alpha^2} e^{-2\alpha^2},
\]

(17)

\[
\frac{1}{m^2} \frac{\partial m^2}{\partial \mu} = \frac{\lambda}{32\pi^2 \alpha^2} - \frac{\lambda}{16\pi^2} + \frac{\lambda}{16\pi^2} 2\alpha^2 e^{2\alpha^2} \text{Ei}_1(2\alpha^2),
\]

(18)

where \( \alpha = Am \) is the dimensionless scale, \( \text{Ei}_1(z) = \int_1^\infty \frac{e^{-xz}}{x} dx \) is the exponential integral of the first type. To find the scale dependence of the coupling constant \( \lambda = \lambda(\mu) \) explicitly. To do this we rewrite (17) as

\[
\frac{\partial \lambda}{\partial \mu} = \frac{3\lambda^2}{16\pi^2} B(\alpha^2), \quad B(x) = \left(2 + \frac{1}{x}\right) e^{-2x} - \frac{e^{-x}}{x}
\]

(19)

where \( x = \alpha^2 \) and \( d\mu = \frac{dx}{\alpha} \). The equation above can be solved now for the inverse coupling constant \( g = \frac{1}{x} \):

\[
dg = \frac{3}{32\pi^2} \left\{ \left(2 + \frac{1}{x}\right) e^{-2x} - \frac{e^{-x}}{x}\right\} \frac{dx}{x} \equiv \frac{3}{32\pi^2} dF(x), \quad F(x) = \frac{e^{-x}}{x} - \frac{e^{-2x}}{x} - \text{Ei}_1(x).
\]

Inverting the above equation we get the scale dependence of the coupling constant

\[
\lambda(x) = \lambda(x; x_1, \lambda_1) = \frac{1}{\lambda_1 + \frac{3}{16\pi^2} [F(x) - F(x_1)]}, \quad \lambda_1 \equiv \lambda(x_1)
\]

(20)

where \( \lambda_1 \equiv \lambda(x_1) \) is the boundary condition for the coupling constant \( \lambda \). The graphs of the ultraviolet behavior of \( \lambda(x) \) for different infrared boundary conditions \( \lambda_1 \) are shown in Fig. 2 below.

The zeros of the \( \beta \)-function \( \beta(\lambda, \mu) = \frac{3\lambda^2}{16\pi^2} B(x) \) except for the trivial case \( \lambda = 0 \) are determined by the equation

\[
B(x) = \left(2 + \frac{1}{x}\right) e^{-2x} - \frac{e^{-x}}{x} = 0
\]

(21)

The graph of the function \( B(x) \) is shown in Fig. 3.
The solutions of the equation $B(x) = 0$ are given by the equality $2x + 1 = e^x$, which can be satisfied for either $x = 0$ or

$$x_* = -\text{LambertW}(-1, -\frac{1}{2}e^{-1/2}) - \frac{1}{2} \approx 1.25643\ldots$$

Let there exist a fixed point value of the coupling constant $\lambda_* = \lambda(x_*)$, then, as it follows from the graph shown above:

- if $\lambda > \lambda_*$ to the left from $x_*$, the decrease of $\mu \to 0$ results in the decrease of $\lambda$
- if $\lambda < \lambda_*$ to the right from $x_*$, the increase of $\mu$ results in increase of $\lambda$

Therefore $\lambda_*$ is an IR stable fixed point.

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