On tails of symmetric and totally asymmetric α-stable distributions

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Abstract

We estimate up to universal constants tails of symmetric and totally asymmetric α-stable distributions in terms of functions of the parameters of these distributions.

1 Introduction

Stable distributions, which constitute important family of distributions used in stochastic modeling, have usually no closed formulas for densities and distribution functions. Partial solutions to this problem are series and asymptotic expansions, see [4] and [1, Chapt. XVII, Sect. 6]. There exists also rich literature on numerical calculation of stable densities and distribution functions, see [2] and references therein. In this article we will be interested in ‘qualitative’ behavior of tails of symmetric and totally asymmetric α-stable distributions. More precisely, we will be interested in the description of these tails in terms of functions of the parameters of a distribution up to universal constants. Let X be an α-stable random variable. The asymptotic behavior of \( P(X > t) \) as \( t \to +\infty \) is well known [3, Property 1.2.15] but the value of the tail \( P(X > t) \) for moderate values of \( t \) seems to be not well investigated even in the case of symmetric α-stable distribution.

To obtain our estimates we will use completely elementary methods - Chebyshev and Paley-Zygmund inequalities.

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2 Estimates of tails of symmetric and totally asymmetric $\alpha$-stable random variables, $\alpha \in (0, 1)$

2.1 Tails of totally asymmetric $\alpha$-stable random variables, $\alpha \in (0, 1)$

First we will work with a strictly asymmetric $\alpha$-stable random variable (r.v. in short) $X$, $\alpha \in (0, 1)$, whose characteristic function is given by

$$\mathbb{E} \exp(itX) = \exp \left( \int_0^\infty e^{itx} - 1 \frac{dx}{x^{\alpha+1}} \right). \quad (1)$$

Any strictly $\alpha$-stable random variable has the same distribution as some multiple of $X$.

The main tool of our analysis will be the decomposition of $X$ into the sum of two independent random variables $X_k$ and $X_k$, where $k > 0$, and whose characteristic functions are given respectively by

$$\mathbb{E} \exp(itX_k) = \exp \left( \int_0^k e^{itx} - 1 \frac{dx}{x^{\alpha+1}} \right), \quad \mathbb{E} \exp(itX_k) = \exp \left( \int_k^\infty e^{itx} - 1 \frac{dx}{x^{\alpha+1}} \right).$$

$X_k$ is positive, has moments of any order and we have the following identities

$$\mathbb{E}X_k = \int_0^k x \frac{dx}{x^{\alpha+1}} = \frac{k^{1-\alpha}}{1-\alpha},$$

$$\mathbb{E}(X_k - \mathbb{E}X_k)^2 = \int_0^k x^2 \frac{dx}{x^{\alpha+1}} = \frac{k^{2-\alpha}}{2-\alpha}.$$ 

Next, we observe that $X^k$ has compound Poisson distribution and may be represented as the sum $X^k = \sum_{n=1}^{N^k} X_n^k$, where $N^k$ has Poisson distribution with the expectation $\int_0^\infty 1 \frac{dx}{x^{\alpha+1}} = 1/(\alpha k)$ and $X_n^k$, $n = 1, 2, \ldots$, are i.i.d. r.v.s, independent from $N^k$, with the distribution

$$\mathbb{P}(X_n^k \geq x) = \begin{cases} 1/2 & \text{if } x \leq k, \\ \left(\frac{x}{k}\right)^\alpha & \text{if } x > k. \end{cases} \quad (2)$$

We have the following result

**Theorem 1** Let $X$ be totally asymmetric $\alpha$-stable random variable with $\alpha \in (0, 1)$, whose characteristic function is given by (1). One has the following estimates of tails of $X$:

- if $\alpha \in (0, 1/2]$ and $x \geq k_\alpha := (1/\alpha)^{1/\alpha}$ (which is equivalent with $ax^\alpha \geq 1$) then
  $$\mathbb{P}(X \geq x) \geq \frac{1}{e} \frac{1}{ax^\alpha}, \quad (2)$$
\[ P(X \geq 3x) \leq P(X \geq \frac{k^{1-\alpha}}{1-\alpha} + 2x) = P(X \geq E_{k_\alpha} + 2x) \]
\[ \leq \left( \frac{\alpha}{2-\alpha} \left( \frac{k_\alpha}{x} \right)^{2-\alpha} + \frac{1}{e} \sum_{n=1}^{\infty} \frac{n^\alpha + 1}{n!} \right) \frac{1}{\alpha x^\alpha} \]
\[ \leq 1.71 \frac{1}{\alpha x^\alpha}; \quad (3) \]

- if \( \alpha \in (1/2, 1) \) and \( x \geq 4 \) then
\[ P \left( X \geq \frac{1}{2} \frac{1}{1-\alpha} + x \right) \geq \frac{1}{12e} \frac{1}{\alpha x^\alpha} \geq \frac{1}{12e} \frac{1}{x^\alpha}, \quad (4) \]
\[ P \left( X \geq \frac{2}{1-\alpha} + 2x \right) \leq P \left( X \geq \frac{k^{1-\alpha}}{1-\alpha} + 2x \right) = P \left( X \geq E_{k_\alpha} + 2x \right) \]
\[ \leq \left( \frac{\alpha}{2-\alpha} \left( \frac{k_\alpha}{x} \right)^{2-\alpha} + \frac{1}{e} \sum_{n=1}^{\infty} \frac{n^\alpha + 1}{n!} \right) \frac{1}{\alpha x^\alpha} \]
\[ \leq 3 \frac{1}{\alpha x^\alpha} \leq 6 \frac{1}{x^\alpha}. \quad (5) \]

**Proof.** Let \( \alpha \in (0, 1/2] \). The lower bound (2) follows from the following easy calculation. Since \( P \left( X_{k_\alpha} > 0 \right) = 1 \) for \( x \geq k_\alpha \) we have
\[ P(X \geq x) = P(X_{k_\alpha} + X^{k_\alpha} \geq x) \geq P(X^{k_\alpha} \geq x) \]
\[ \geq P(N^{k_\alpha} = 1) P(X^{k_\alpha} \geq x) \]
\[ = \frac{1}{e} \left( \frac{k_\alpha}{x} \right)^{\alpha} = \frac{1}{e} \frac{1}{\alpha x^\alpha}. \]

The upper bound (3) for the tail of \( X \) follows from the estimates: for \( x \geq k_\alpha \) and \( \alpha \in (0, 1/2) \) one has
\[ E_{X_{k_\alpha}} = \frac{k^{1-\alpha}}{1-\alpha} = \frac{\alpha}{1-\alpha} k_\alpha \leq k_\alpha \leq x \]
and
\[ P(X \geq 3x) \leq P \left( X \geq \frac{k^{1-\alpha}}{1-\alpha} + 2x \right) = P \left( X_{k_\alpha} + X^{k_\alpha} \geq E_{k_\alpha} + 2x \right) \]
\[ \leq P(X_{k_\alpha} - E_{k_\alpha} \geq x) + P(X^{k_\alpha} \geq x). \quad (6) \]

From the Chebyshev inequality
\[ P(X_{k_\alpha} - E_{k_\alpha} \geq x) \leq \frac{E(X_{k_\alpha} - E_{k_\alpha})^2}{x^2} \]
\[ = \frac{\alpha}{2-\alpha} \left( \frac{k_\alpha}{x} \right)^{2-\alpha} \frac{1}{\alpha x^\alpha}. \quad (7) \]

Next, to bound \( P(X^{k_\alpha} \geq x) \) let us notice that for \( x > 0 \) and \( n = 1, 2, \ldots \) we have
\[ \left\{ X^{k_\alpha} \geq x \text{ and } N^{k_\alpha} = n \right\} \subset \left\{ X_{i}^{k_\alpha} \geq \frac{x}{n} \text{ for some } i = 1, 2, \ldots, n \text{ and } N^{k_\alpha} = n \right\} \]
From the Paley-Zygmund inequality and the obvious estimate

thus

From (6)-(8) we get

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and since \( x \geq k_\alpha \) and \( \alpha \leq 1/2 \) we obtain

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From the Paley-Zygmund inequality and the obvious estimate \( k_\alpha > 1 \), for \( \alpha \in (1/2, 1) \) we get

From this and the estimate \( \mathbb{P}(X_{k_\alpha} \geq x) \geq (1/e)(k_\alpha/x)\alpha \) (valid for any \( \alpha \in (0, 1) \) not only for \( \alpha \in (0, 1/2) \) we obtain (3)). To bound \( \mathbb{P}(X_{k_\alpha} \geq \frac{2}{1-\alpha} + 2x) \)

let us notice that for \( \alpha > 1/2 \),

\[
\mathbb{E}X_{k_\alpha} = \frac{k_\alpha^{1-\alpha}}{1-\alpha} = \frac{(1/\alpha)^{(1-\alpha)/\alpha}}{1-\alpha} \leq \frac{1/\alpha}{1-\alpha} \leq \frac{2}{1-\alpha},
\]
hence
\[
P \left( X \geq \frac{2}{1 - \alpha} + 2x \right) \leq P \left( X \geq \frac{k_1}{1 - \alpha} + 2x \right) = P \left( X \geq E X k_1 + 2x \right)
\]
and using the same reasoning as for \( \alpha \in (0, 1/2) \) we obtain
\[
P \left( X \geq \frac{2}{1 - \alpha} + 2x \right) \leq \left( \frac{\alpha}{2 - \alpha} \left( \frac{k_1}{x} \right)^{2 - \alpha} + \frac{1}{e} \sum_{n=1}^{\infty} \frac{n^{\alpha + 1}}{n!} \right) \frac{1}{\alpha x^\alpha}.
\]
Now to obtain \([3]\) it is enough to notice that
\[
\frac{\alpha}{2 - \alpha} \left( \frac{k_1}{x} \right)^{2 - \alpha} + \frac{1}{e} \sum_{n=1}^{\infty} \frac{n^{\alpha + 1}}{n!} \leq 1 + \frac{1}{e} \sum_{n=1}^{\infty} \frac{n^{1+1}}{n!} = 3
\]
and \(1/\alpha \leq 2\).

2.2 Tails of symmetric \( \alpha \)-stable random variables, \( \alpha \in (0, 1) \)

Now let \( \tilde{X} \) be a strictly symmetric \( \alpha \)-stable random variable with \( \alpha \in (0, 1) \), whose characteristic function is given by
\[
E \exp (it \tilde{X}) = \exp \left( \int_{-\infty}^{\infty} e^{itx} - 1 \frac{dx}{x^{\alpha+1}} \right).
\]
As a consequence of Theorem 1 we have

**Corollary 2** If \( \tilde{X} \) is a symmetric \( \alpha \)-stable random variable with \( \alpha \in (0, 1/2] \), whose characteristic function is given by \([2]\) and \( x \geq \tilde{k}_\alpha := (2/\alpha)^{1/\alpha} \) then one has the following estimates of tails of \( \tilde{X} \):

\[
P \left( \tilde{X} \geq \frac{1}{2} x \right) \geq \frac{1}{1 - e^{1/\alpha}} x^{\alpha}; \quad (10)
\]

\[
P \left( \tilde{X} \geq 3x \right) \leq 1.71 \frac{1}{\alpha x^\alpha}. \quad (11)
\]

**Proof.** From the form of the characteristic function of \( \tilde{X} \) we have that \( X = X_1 - X_2 \) where \( X_1 \) and \( X_2 \) have the same distribution as \( X \). If \( \alpha \in (0, 1/2] \) and \( x \geq \tilde{k}_\alpha \) then \( \alpha x^\alpha \geq 2 \), \( x/2 \geq \tilde{k}_\alpha \) and the lower bound \([10]\) follows from estimates \([2]\) and \([3]\):

\[
P \left( \tilde{X} \geq \frac{1}{2} x \right) \geq P (X_1 \geq x) P (X_2 \leq \frac{1}{2} x) \geq \frac{1}{e} \frac{1}{\alpha x^\alpha} \left( 1 - 1.71 \frac{1}{\alpha x^\alpha} \right) \geq \frac{1}{e} \frac{1}{\alpha x^\alpha} \left( 1 - \frac{1.71}{2} \right) \geq \frac{1}{1 - e^{1/\alpha}} x^{\alpha}.
\]

The upper bound is obvious, since \( P (X_2 \geq 0) = 1 \).
Let now $\alpha \in (1/2, 1)$. To obtain estimates of the tails of $\tilde{X}$ we again, similarly as for $X$, split $\tilde{X}$ into the sum $\tilde{X} = \tilde{X}_k + \tilde{X}_k^\alpha$, where $k > 0$ and $\tilde{X}_k = \sum_{i=1}^{\tilde{N}_k} \tilde{X}_i$, where $\tilde{N}_k$ has Poisson distribution with the expectation $2 \int_{1/\alpha}^{\infty} \frac{dx}{x^{\alpha+1}} = 2/(\alpha k^n)$ and $\tilde{X}_i\in [1,2,\ldots]$, are symmetric i.i.d. r.v.s, independent from $\tilde{N}_k$, with the distribution

$$P(\tilde{X}_i \geq x) = P(\tilde{X}_i \leq -x) = \begin{cases} \frac{1}{2} & \text{if } x = k, \\ \frac{1}{2} \left(\frac{1}{2}\right)^\alpha & \text{if } x > k. \end{cases}$$

**Proposition 3** If $\tilde{X}$ is a symmetric $\alpha$-stable random variable with $\alpha \in (1/2, 1)$, whose characteristic function is given by (7) and $x \geq 16$ then one has the following estimates of tails of $\tilde{X}$:

$$P(\tilde{X} \geq x) \geq \frac{1}{2e \cdot x^\alpha}, \quad P(\tilde{X} \leq 2x) \leq \frac{45}{x^\alpha}.$$

**Proof.** For $\alpha \in (1/2, 1)$ we define $\tilde{k}_\alpha := (2/\alpha)^{1/\alpha} \in (2, 16)$. Now, for $x \geq 16$ the lower bound follows from the estimates

$$P(\tilde{X} \geq x) \geq P(\tilde{X}_{\tilde{k}_\alpha} \geq 0) P(\tilde{X}_{\tilde{k}_\alpha} \geq x) \geq \frac{1}{2} P(\tilde{N}_{\tilde{k}_\alpha} = 1) P(\tilde{X}_{\tilde{k}_\alpha} \geq x) = \frac{1}{2} \frac{1}{\tilde{k}_\alpha^{\alpha/2}} \frac{1}{x} \geq \frac{1}{2e \cdot x^\alpha}.$$

To obtain the upper bound we again use the Chebyshev inequality. For $x > 0$ we estimate

$$P\left(\left|\tilde{X}_{\tilde{k}_\alpha}\right| \geq x\right) \leq \frac{E \tilde{X}_{\tilde{k}_\alpha}^2}{x^2} = \frac{2\tilde{k}_\alpha^{2-\alpha}}{2-\alpha} \leq \frac{256}{3} \frac{1}{x^2},$$

where we used the fact that the function $\alpha \mapsto 2\tilde{k}_\alpha^{2-\alpha}/(2-\alpha)$ is decreasing on $[1/2, 1]$. Next, to bound $P(\tilde{X}_{\tilde{k}_\alpha} \geq x)$ we proceed similarly as in the proof of Theorem 1. We notice that for $x > 0$ we have

$$\left\{\tilde{X}_{\tilde{k}_\alpha} \geq x \text{ and } \tilde{N}_{\tilde{k}_\alpha} = n\right\} \subset \left\{\tilde{X}_{\tilde{k}_\alpha} \geq \frac{x}{n} \text{ for some } i = 1, 2, \ldots, n \text{ and } \tilde{N}_{\tilde{k}_\alpha} = n\right\}$$
and
\[ P \left( \tilde{X}_{k_{\alpha}} \geq \frac{x}{n} \text{ for some } i = 1, \ldots, n \text{ and } N_{k_{\alpha}} = n \right) \]
\[ \leq P \left( N_{k_{\alpha}} = n \right) \sum_{i=1}^{n} P \left( \tilde{X}_{k_{\alpha}} \geq \frac{x}{n} \right) \]
\[ \leq \frac{1}{e} \frac{n}{n!} \frac{1}{2} \left( \frac{k_{\alpha}}{x/n} \right)^{\alpha} \leq \frac{1}{2e} \frac{n}{n!} \frac{2^{\alpha} n^{\alpha}}{x^\alpha} \]
thus
\[ P \left( \tilde{X}_{k_{\alpha}} \geq x \right) = \sum_{n=1}^{\infty} P \left( \tilde{X}_{k_{\alpha}} \geq x \text{ and } N_{k_{\alpha}} = n \right) \]
\[ \leq \sum_{n=1}^{\infty} \frac{1}{2e} \frac{n^{\alpha+1}}{n!} \frac{1}{x^\alpha} . \quad (13) \]

From \( (12) \) and \( (13) \) for \( x \geq 1 \) we finally get
\[ P \left( \tilde{X} \geq 2x \right) \leq \frac{256}{6} \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{1}{2e} \frac{n}{n!} \frac{2^{\alpha} n^{\alpha}}{x^\alpha} \]
\[ \leq \left( 42.7 + \frac{1}{e} \sum_{n=1}^{\infty} \frac{n^2}{n!} \right) \frac{1}{x^\alpha} \leq \frac{45}{x^\alpha}. \]

\[ \blacksquare \]

3 Estimates of tails of symmetric and totally asymmetric \( \alpha \)-stable random variables, \( \alpha \in (1, 2) \)

In this section we will deal with symmetric and totally asymmetric \( \alpha \)-stable random variables, \( \alpha \in (1, 2) \).

3.1 Tails of totally asymmetric \( \alpha \)-stable random variables, \( \alpha \in (1, 2) \)

First we will consider totally asymmetric r.v. with the characteristic function
\[ E \exp (itX) = \exp \left( \int_{0}^{+\infty} e^{itx} - 1 - itx \frac{dx}{x^{\alpha+1}} \right) . \quad (14) \]

Opposite to the previous case \( \alpha \in (0, 1) \), the support of the distribution of a r.v. \( X \) with the characteristic function given by \( (14) \) is the whole real line. To estimate \( P \left( \tilde{X} \geq t \right) \) or \( P \left( \tilde{X} \leq t \right) \) similarly as in the case \( \alpha \in (0, 1) \) we split \( X \) into the sum \( X = X_{1} + X^{1} \), where
\[ E \exp (itX_{1}) = \exp \left( \int_{0}^{1} e^{itx} - 1 - itx \frac{dx}{x^{\alpha+1}} \right) . \quad (15) \]
\[ \mathbb{E}\exp(itX^1) = \exp \left( \int_1^{+\infty} e^{itx} - 1 - itx \frac{dx}{x^{\alpha+1}} \right). \]

It is easy to calculate
\[ \int_1^{+\infty} e^{itx} - 1 - itx \frac{dx}{x^{\alpha+1}} = \int_1^{+\infty} e^{itx} - 1 \frac{dx}{x^{\alpha+1}} - \frac{1}{\alpha - 1}i \]

thus \( X^1 \) has the compound Poisson distribution with the expected number of summands \( \int_1^{+\infty} \frac{dx}{x^{\alpha+1}} = 1/\alpha \) minus \( 1/(\alpha - 1) \). More precisely,
\[ X^1 = \sum_{n=1}^{N} Z_n - \frac{1}{\alpha - 1}, \]

where \( N \) has Poisson distribution with the parameter (expectation) \( (1/\alpha) \), and \( Z_n, n = 1, 2, \ldots \) are i.i.d. r. vs., independent from \( N \) and such that
\[ \mathbb{P}(Z_n \geq z) = \alpha \int_z^{+\infty} \frac{dx}{x^{\alpha+1}} = z^{-\alpha} \text{ for } z \geq 1. \]

Lemma 4 For \( z \geq 1 \) one has the following lower bound
\[ \mathbb{P}(X^1 \geq z - \frac{1}{\alpha - 1}) \geq \mathbb{P}(N = 1) \mathbb{P}(Z_1 \geq z) = e^{-1/\alpha} \frac{1}{\alpha} \frac{1}{z^\alpha} \quad (16) \]

and the following upper bound
\[ \mathbb{P}(X^1 \geq z - \frac{1}{\alpha - 1}) \leq \left( e^{-1/\alpha} \sum_{k=1}^{+\infty} \frac{1}{\alpha k!} \right) \frac{1}{z^\alpha} \leq \frac{5}{\sqrt[4]{e}} \frac{1}{z^\alpha}. \quad (17) \]

Proof. To estimate that \( X^1 \geq z - 1/(\alpha - 1) \), where \( z \geq 1 \), we notice that
\[ \mathbb{P}(X^1 \geq z - \frac{1}{\alpha - 1}) \geq \mathbb{P}(N = 1) \mathbb{P}(Z_1 \geq z) = e^{-1/\alpha} \frac{1}{\alpha} \frac{1}{z^\alpha}. \]

and \( e^{-1/\alpha}/\alpha \geq 1/(2\sqrt{e}) \), since, by simple calculus, the function \( \alpha \mapsto e^{-1/\alpha}/\alpha \) is decreasing on the interval \([1, 2]\).

On the other hand, whenever \( \sum_{i=1}^{N} Z_i \geq z \) and \( N = k \) we have that at least for one \( i = 1, 2, \ldots, k \), \( Z_i \geq z/k \) which occurs with probability no greater than \( \sum_{i=1}^{k} \mathbb{P}(Z_i \geq z/k) \), thus
\[ \mathbb{P}(X^1 \geq z - \frac{1}{\alpha - 1}) \leq \\
\leq \sum_{k=1}^{+\infty} \mathbb{P}(N = k) \left( \sum_{i=1}^{k} \mathbb{P}(Z_i \geq z/k) \right) \\
\leq e^{-1/\alpha} \sum_{k=1}^{+\infty} \frac{1}{\alpha k!} \left( \frac{z}{k} \right)^{-\alpha} \\
= \left( e^{-1/\alpha} \sum_{k=1}^{+\infty} \frac{k^{\alpha+1}}{\alpha k!} \right) \frac{1}{z^\alpha} \\
\leq \left( \frac{1}{\sqrt[4]{e}} \sum_{k=1}^{+\infty} \frac{k^3}{k!} \right) \frac{1}{z^\alpha} = \frac{5}{\sqrt[4]{e}} \frac{1}{z^\alpha}. \]
Now we proceed to analyse $X_1$ which is much more delicate task. First we will prove

**Lemma 5** For $0 \leq y \leq 2/(2 - \alpha)$ one has

$$P(X_1 \geq y) \leq e^{4e^2/3}e^{-\frac{1}{2}(2-\alpha)y^2},$$
$$P(X_1 \leq -y) \leq e^{4/3}e^{-\frac{1}{2}(2-\alpha)y^2}.$$  

**Proof.** We calculate

$$E\exp(tX_1) = \exp\left(\int_0^1 e^{tx} - 1 - tx \frac{dx}{x^{\alpha+1}}\right)$$
$$= \exp\left(\frac{1}{2} \frac{t^2}{2 - \alpha} + \int_0^1 e^{tx} - 1 - tx - \frac{1}{2}t^2x^2 \frac{dx}{x^{\alpha+1}}\right)$$

and since for $tx \geq 0$, $e^{tx} - 1 - tx - \frac{1}{2}t^2x^2 \leq \frac{e^t}{3}t^3x^3$, for $t \geq 0$ we estimate

$$E\exp(tX_1) \leq \exp\left(\frac{1}{2} \frac{t^2}{2 - \alpha} + \frac{e^t}{6} \int_0^1 x^3 \frac{dx}{x^{\alpha+1}}\right)$$
$$= \exp\left(\frac{1}{2} \frac{t^2}{2 - \alpha} + \frac{e^t}{6} \frac{t^3}{3 - \alpha}\right). \quad (18)$$

Now, for $0 \leq y \leq 2/(2 - \alpha)$, taking $t_y = (2 - \alpha)y$ we get $t_y \leq 2$ and by Chebyshev’s inequality we get

$$P(X_1 \geq y) \leq E\exp(t_yX_1)e^{-t_yy} \leq \exp\left(-\frac{1}{2} (2 - \alpha) y^2 + e^2 \frac{8}{6}\right).$$

Similarly, since for $tx \leq 0$, $\left|e^{tx} - 1 - tx - \frac{1}{2}t^2x^2 \right| \leq \frac{1}{6} \left|t^3x^3\right|$, for $t \leq 0$ we have

$$E\exp(tX_1) \leq \exp\left(\frac{1}{2} \frac{t^2}{2 - \alpha} + \frac{1}{6} \frac{t^3}{3 - \alpha}\right).$$

Again, for $0 \leq y \leq 2/(2 - \alpha)$, taking $t_y = -(2 - \alpha)y$ by Chebyshev’s inequality we get

$$P(X_1 \leq -y) \leq E\exp(-t_yX_1)e^{t_yy} \leq \exp\left(-\frac{1}{2} (2 - \alpha) y^2 + \frac{8}{6}\right).$$

**Remark 6** If we restrict to $0 \leq y \leq \kappa/(2 - \alpha)$ ($\kappa > 0$) we get the following estimates:

$$P(X_1 \geq y) \leq e^{\kappa^3/6}e^{-\frac{1}{2}(2-\alpha)y^2},$$
$$P(X_1 \leq -y) \leq e^{\kappa^3/6}e^{-\frac{1}{2}(2-\alpha)y^2}.$$
Now we will prove similar lower bounds. We will use the Paley-Zygmund inequality. We have

**Lemma 7** For \( y \in \left[ \frac{2}{\sqrt{2} - \alpha}, \frac{1}{(2 - \alpha)} \right] \) one has

\[
P \left( X_1 \geq \frac{1}{4} y \right) \geq 2 \cdot 10^{-5} e^{-\left(2 - \alpha\right) y^2}
\]

(19)

while for \( y \in \left[ -\frac{2}{2 - \alpha}, -\frac{2}{\sqrt{2} - \alpha} \right] \) one has

\[
P \left( X_1 \leq -\frac{1}{24} y \right) \geq 4 \cdot 10^{-2} e^{-\left(2 - \alpha\right) y^2}.
\]

(20)

**Proof.** Since for \( tx \geq 0, e^{tx} - 1 - tx - \frac{1}{2} t^2 x^2 \geq 0 \), for \( t \geq 0 \) we estimate

\[
\mathbb{E} \exp (tX_1) \geq \exp \left( \frac{t^2}{2 - \alpha} \right).
\]

(21)

Next, notice that for \( y \geq \frac{2}{\sqrt{2} - \alpha} \) we have \( \frac{1}{2} y - \frac{1}{2 - \alpha} y \geq \frac{1}{2} y - \frac{2 - \alpha}{4} y = \frac{1}{4} y \)

and for \( t_y = (2 - \alpha) y \) and \( \lambda = 1/e \) by (21) we have

\[
\frac{1}{t_y} \ln \left( \lambda \mathbb{E} \exp (t_y X_1) \right) \geq \frac{1}{t_y} \ln \left( \lambda \exp \left( \frac{t_y^2}{2 - \alpha} \right) \right)
\]

\[
= \frac{1}{2} \frac{t_y}{2 - \alpha} + \frac{\ln \lambda}{t_y} = \frac{1}{2} y - \frac{1}{2} - \frac{1}{2 - \alpha} y
\]

\[
\geq \frac{1}{4} y.
\]

This, together with the Paley-Zygmund inequality, (21) and (18) (notice that for \( y \leq \frac{1}{(2 - \alpha)} \), \( t_y \leq 1 \)) yields (19):

\[
P \left( X_1 \geq \frac{1}{4} y \right) \geq P \left( X_1 \geq \frac{1}{t_y} \ln \left( \lambda \mathbb{E} \exp (t_y X_1) \right) \right)
\]

\[
= P \left( \exp (t_y X_1) \geq \lambda \mathbb{E} \exp (t_y X_1) \right)
\]

\[
\geq \left( 1 - \frac{1}{e} \right) \frac{\left( \mathbb{E} \exp (t_y X_1) \right)^2}{\mathbb{E} \exp (2t_y X_1)}
\]

\[
\geq \left( 1 - \frac{1}{e} \right)^2 \exp \left( \frac{t_y^2}{2 - \alpha} \right)
\]

\[
= e^{-e^{-2t^2/6 + 2\ln(e - 1) - 2 - (2 - \alpha)(y^2)}} \geq 2 \cdot 10^{-5} e^{-\left(2 - \alpha\right) y^2}.
\]

For negative tails we use estimate \( e^{tx} - 1 - tx - \frac{1}{2} t^2 x^2 \geq \frac{1}{6} t^3 x^3 \) for \( tx \leq 0 \), which for \( t \leq 0 \) yields

\[
\mathbb{E} \exp (tX_1) \geq \exp \left( \frac{1}{2} \frac{t^2}{2 - \alpha} - \frac{1}{6} |t^3| \right).
\]

(22)
Next, notice that for $2/\sqrt{2 - \alpha} \leq y \leq 2/(2 - \alpha)$ we have $1/y \leq (2 - \alpha)/y$ and $(2 - \alpha)^2 y^2 \leq 4 \leq 2y$ so

$$\frac{1}{2} y - \ln \left( \lambda^2 \exp \left( t_y X_1 \right) \right) \geq \frac{1}{2} y - \frac{1}{6} (2 - \alpha)^2 y^2 \geq \frac{1}{2} y - \frac{1/2}{2 - \alpha} \frac{2 - \alpha}{4} y - \frac{1}{3} y = \frac{1}{24} y.$$ 

From this for $t_y = - (2 - \alpha) y$ and $\lambda = 1/\sqrt{e}$, by (22) we have

$$\frac{1}{|t_y|} \ln \left( \lambda^2 \exp \left( t_y X_1 \right) \right) \geq \frac{1}{|t_y|} \ln \left( \lambda \exp \left( \frac{t_y^2}{2} \frac{2 - \alpha}{6} - \frac{1}{6} |t_y|^2 \right) \right) = \frac{1}{2} \frac{|t_y|}{2 - \alpha} + \ln \left( \frac{1}{|t_y|} \right) \frac{1}{6} |t_y|^2 = \frac{1}{2} y - \frac{1/2}{2 - \alpha} \frac{1}{y} - \frac{1}{6} (2 - \alpha)^2 y^2 \geq \frac{1}{24} y.$$ 

This, together with the Paley-Zygmund inequality (and since for $tx \leq 0, e^{tx} \leq 1 - tx - \frac{t^2}{2} x^2 \leq 0$) yields

$$\mathbb{P} \left( X_1 \leq - \frac{1}{24} y \right) \geq \mathbb{P} \left( \frac{1}{|t_y|} \ln \left( \lambda^2 \exp \left( t_y X_1 \right) \right) \geq \mathbb{P} \left( \exp \left( t_y X_1 \right) \geq \lambda \exp \left( t_y X_1 \right) \right) \right. \geq \left( 1 - \frac{1}{\sqrt{e}} \right)^2 \frac{\left( \mathbb{E} \exp \left( t_y X_1 \right) \right)^2}{\mathbb{E} \exp \left( 2t_y X_1 \right)} \geq \left( 1 - \frac{1}{\sqrt{e}} \right)^2 \frac{\exp \left( \frac{3}{2 - \alpha} \right)}{\exp \left( \frac{9}{2 - \alpha} \right)} = e^{2 \ln(\sqrt{e} - 1) - 7/3} e^{-(2 - \alpha) y^2} \geq 4 \cdot 10^{-2} e^{-(2 - \alpha) y^2}.$$ 

To complete the picture we will now estimate $\mathbb{P} \left( X_1 \leq - y \right)$ in the case $y > 2/(2 - \alpha)$.

**Lemma 8** For $y \geq 2/(2 - \alpha)$ one has

$$\mathbb{P} \left( X_1 \leq - y \right) \leq \exp \left( - \left( \frac{1}{2} y + \frac{1}{\alpha - 1} \right)^{\alpha/(\alpha - 1)} / \left( \frac{1}{2 - \alpha} + \frac{1}{\alpha - 1} \right)^{1/(\alpha - 1)} \right)$$

and

$$\mathbb{P} \left( X_1 \leq - \left( \frac{1}{e} - \frac{1}{4} \right) y \right) \geq \left( 1 - \frac{1}{e^2} \right)^2 \exp \left( - \left( \frac{4}{\sqrt{2}} \frac{1}{e} y + \frac{1}{\alpha - 1} \right)^{\alpha/(\alpha - 1)} / \left( \frac{1}{2 - \alpha} + \frac{1}{\alpha - 1} \right)^{1/(\alpha - 1)} \right).$$

**Proof.** For $t < -1$ first we split

$$\int_{-1}^{1} e^{tx} - 1 - tx \frac{dx}{x^{\alpha + 1}} = \int_{-1}^{1/|t|} e^{tx} - 1 - tx \frac{dx}{x^{\alpha + 1}} + \int_{1/|t|}^{1} e^{tx} - 1 - tx \frac{dx}{x^{\alpha + 1}}.$$
For $t < -1$ and $0 \leq x \leq 1/|t|$ we calculate $e^{tx} - 1 - tx \leq \frac{1}{2}t^2 x^2 \leq t^2 x^2$ and we get

$$\int_0^{1/|t|} e^{tx} - 1 - tx \frac{dx}{x^{\alpha+1}} \leq \int_0^{1/|t|} t^2 x^2 \frac{dx}{x^{\alpha+1}} = \frac{1}{2 - \alpha} |t|^\alpha.$$  

Next, for $x > 1/|t|$ we bound $e^{tx} - 1 - tx \leq -tx = |t|x$ and get

$$\int_{1/|t|}^1 e^{tx} - 1 - tx \frac{dx}{x^{\alpha+1}} \leq |t| \int_{1/|t|}^1 x \frac{dx}{x^{\alpha+1}} = \frac{1}{\alpha - 1} (|t|^\alpha - |t|).$$  

Finally, we arrive at

$$\int_0^1 e^{tx} - 1 - tx \frac{dx}{x^{\alpha+1}} \leq \left( \frac{1}{2 - \alpha} + \frac{1}{\alpha - 1} \right) |t|^\alpha - \frac{1}{\alpha - 1} |t|$$

which yields that for $t < -1$

$$E \exp (tX_1) \leq \exp \left( \left( \frac{1}{2 - \alpha} + \frac{1}{\alpha - 1} \right) |t|^\alpha - \frac{1}{\alpha - 1} |t| \right).$$

Let $y > 2/ (2 - \alpha)$ and $t_y < -1$ be such that

$$\alpha \left( \frac{1}{2 - \alpha} + \frac{1}{\alpha - 1} \right) |t_y|^\alpha - 1 = y + \frac{1}{\alpha - 1}. \quad (23)$$  

We estimate

$$P(X_1 < -y) \leq E \exp (t_y X) e^{t_y y}$$

$$\leq \exp \left( \left( \frac{1}{2 - \alpha} + \frac{1}{\alpha - 1} \right) |t_y|^\alpha - \frac{1}{\alpha - 1} |t_y| - y |t_y| \right)$$

$$= \exp \left( - (\alpha - 1) \left( \frac{1}{2 - \alpha} + \frac{1}{\alpha - 1} \right) |t_y|^\alpha \right)$$

$$= \exp \left( - \left( \left( y + \frac{1}{\alpha - 1} \right) (\alpha - 1)^{(\alpha - 1)/\alpha} / \alpha \right)^{\alpha/(\alpha - 1)} / \left( \frac{1}{2 - \alpha} + \frac{1}{\alpha - 1} \right)^{1/(\alpha - 1)} \right)$$

$$\leq \exp \left( - \left( \frac{1}{2} \left( y + \frac{1}{\alpha - 1} \right) \right)^{\alpha/(\alpha - 1)} / \left( \frac{1}{2 - \alpha} + \frac{1}{\alpha - 1} \right)^{1/(\alpha - 1)} \right),$$

where we used the estimate $\inf_{t \in (1, 2)} (\alpha - 1)^{(\alpha - 1)/\alpha} / \alpha = 1/2$.

On the other hand for $t < -1$ and $0 \leq x \leq 1/|t|$ we have $e^{tx} - 1 - tx \geq \frac{1}{e} t^2 x^2$ and we get

$$\int_0^{1/|t|} e^{tx} - 1 - tx \frac{dx}{x^{\alpha+1}} \geq \int_0^{1/|t|} \frac{t^2 x^2}{e} \frac{dx}{x^{\alpha+1}} = \frac{1}{e} \frac{1}{2 - \alpha} |t|^\alpha.$$

Similarly, for $x > 1/|t|$ we bound $e^{tx} - 1 - tx \geq -\frac{1}{e} t x = \frac{1}{e} |t|x$. This for $y \leq -2/ (2 - \alpha)$ and $t_y$ defined by (23) yields

$$\int_{1/|t|}^1 e^{tx} - 1 - tx \frac{dx}{x^{\alpha+1}} \geq \frac{1}{e} |t| \int_{1/|t|}^1 x \frac{dx}{x^{\alpha+1}} = \frac{1}{e} \frac{1}{\alpha - 1} (|t|^\alpha - |t|).$$
Finally we arrive at the estimate
\[
\mathbb{E} \exp(tX_1) \geq \exp \left( \frac{1}{e} \left( \frac{1}{2 - \alpha} + \frac{1}{\alpha - 1} \right) |t|^\alpha - \frac{1}{e} \frac{1}{\alpha - 1} |t| \right)
\]
which for \( \tilde{t}_y > 1 \) satisfying
\[
\left( \frac{1}{2 - \alpha} + \frac{1}{\alpha - 1} \right) |\tilde{t}_y|^{\alpha - 1} = \frac{1}{\alpha - 1} + y
\]
\[(24)\]
yields
\[
\left( \frac{1}{2 - \alpha} + \frac{1}{\alpha - 1} \right) |\tilde{t}_y|^\alpha - \frac{1}{\alpha - 1} |\tilde{t}_y| = |\tilde{t}_y| y
\]
and for \( \lambda = 1/e^2 \) we get
\[
\frac{1}{|\tilde{t}_y|} \ln \left( \lambda \mathbb{E} \exp (\tilde{t}_y X) \right) \geq \frac{1}{|\tilde{t}_y|} \left( \ln (\lambda) + \frac{1}{e} \left( \frac{1}{2 - \alpha} + \frac{1}{\alpha - 1} \right) |\tilde{t}_y|^{\alpha - 1} - \frac{1}{e} \frac{1}{\alpha - 1} |\tilde{t}_y| \right)
\]
\[
= \frac{1}{|\tilde{t}_y|} \left( \ln (\lambda) + \frac{1}{e} |\tilde{t}_y| - y \right) = \frac{1}{e} y - \frac{1}{2 \tilde{t}_y}.
\]
\[(25)\]
To estimate \( 1/|\tilde{t}_y| \) let us notice that from (24) for \( y \geq 2/(2 - \alpha) \geq 2 \) we have
\[
|\tilde{t}_y| \geq |\tilde{t}_y|^{\alpha - 1} \geq \frac{1}{2 - \alpha} + \frac{y}{\alpha - 1} \geq 1 \geq \frac{2}{y}
\]
which together with (25) yields
\[
\frac{1}{|\tilde{t}_y|} \ln \left( \lambda \mathbb{E} \exp (\tilde{t}_y X) \right) \geq \frac{1}{e} y - \frac{1}{2 |\tilde{t}_y|} \geq \left( \frac{1}{e} - \frac{1}{4} \right) y.
\]
Finally, using the Paley-Zygmund inequality we arrive at
\[
P \left( X_1 \leq - \left( \frac{1}{e} - \frac{1}{4} \right) y \right)
\]
\[
\geq P \left( X_1 \leq - \frac{1}{|\tilde{t}_y|} \ln \left( \lambda \mathbb{E} \exp (\tilde{t}_y X_1) \right) \right)
\]
\[
= P \left( \exp (\tilde{t}_y X) \geq \lambda \mathbb{E} \exp (\tilde{t}_y X_1) \right)
\]
\[
\geq \left( 1 - \frac{1}{e^2} \right)^2 \frac{\mathbb{E} \exp (\tilde{t}_y X_1)^2}{\exp (2t_y X_1)}
\]
\[
\geq \left( 1 - \frac{1}{e^2} \right)^2 \exp \left( \frac{2}{e} \left( \frac{1}{2 - \alpha} + \frac{1}{\alpha - 1} \right) |\tilde{t}_y|^{\alpha - 1} - \frac{1}{e} \frac{1}{\alpha - 1} |\tilde{t}_y| \right)
\]
\[
\exp \left( \frac{1}{2 - \alpha} + \frac{1}{\alpha - 1} \right) |\tilde{t}_y|^{\alpha - 1} \left( \frac{1}{2 - \alpha} + \frac{1}{\alpha - 1} \right) |\tilde{t}_y|^{\alpha - 1}
\]
\[
\geq \left( 1 - \frac{1}{e^2} \right)^2 \exp \left( - \left( 4 - \frac{2}{e} \right) \left( \frac{1}{2 - \alpha} + \frac{1}{\alpha - 1} \right) |\tilde{t}_y|^{\alpha} \right)
\]
\[
= \left( 1 - \frac{1}{e^2} \right)^2 \exp \left( - \left( 4 - \frac{2}{e} \right) \left( y + \frac{1}{\alpha - 1} \right)^{\alpha/(\alpha - 1)} / \left( \frac{1}{2 - \alpha} + \frac{1}{\alpha - 1} \right)^{1/(\alpha - 1)} \right)
\]
\[
\geq \left( 1 - \frac{1}{e^2} \right)^2 \exp \left( - \left( 4 - \frac{2}{e} \right) \left( y + \frac{1}{\alpha - 1} \right)^{\alpha/(\alpha - 1)} / \left( \frac{1}{2 - \alpha} + \frac{1}{\alpha - 1} \right)^{1/(\alpha - 1)} \right).
\]
As an easy consequence of Lemmas 4, 5 and 7 we have the following theorem.

**Theorem 9** Let $X$ be a strictly asymmetric $\alpha$-stable random variable, $\alpha \in (1, 2)$, with the characteristic function \[ \text{[14]} \]. For any $y \in \left[ \frac{2}{\sqrt{2} - \alpha}, 1/(2 - \alpha) \right]$ one has the following estimates

\[
\mathbb{P} \left( X \geq 2y - \frac{1}{\alpha - 1} \right) \leq \frac{5}{\sqrt{e}} \frac{1}{y^\alpha} + e^{4e^2/3} e^{-\frac{1}{2}(2-\alpha)y^2}, \tag{26}
\]

\[
\mathbb{P} \left( X \geq \frac{1}{4} y - \frac{1}{\alpha - 1} \right) \geq \frac{e^{-4}}{2\sqrt{e}} 10^{-5} \frac{1}{y^\alpha} + \frac{1}{2\sqrt{e}} 10^{-5} e^{-(2-\alpha)y^2}; \tag{27}
\]

while for $y \geq 1/(2 - \alpha)$ one has

\[
\mathbb{P} \left( X \geq 2y - \frac{1}{\alpha - 1} \right) \leq \frac{5}{\sqrt{e}} \frac{1}{y^\alpha} + e^{4e^2/3} e^{-\frac{1}{2}(2-\alpha)y^2}, \tag{28}
\]

\[
\mathbb{P} \left( X \geq y - \frac{1}{\alpha - 1} \right) \geq \frac{e^{-4}}{\sqrt{e}} 10^{-5} \frac{1}{y^\alpha}. \tag{29}
\]

**Proof.** To prove (26) we estimate

\[
\mathbb{P} \left( X \geq 2y - \frac{1}{\alpha - 1} \right) \leq \mathbb{P} \left( X^1 \geq y - \frac{1}{\alpha - 1} \right) + \mathbb{P} \left( X \geq y \right)
\]

and then use (17) and the upper bound for $\mathbb{P} \left( X \geq y \right)$ from Lemma 5.

To prove (27) for $y \geq 1$ we write

\[
\mathbb{P} \left( X \geq y - \frac{1}{\alpha - 1} \right) \geq \mathbb{P} \left( X^1 \geq y - \frac{1}{\alpha - 1} \right) \mathbb{P} \left( X \geq \frac{1}{2\sqrt{2} - \alpha} \right)
\]

and then use (19) and (14) to obtain

\[
\mathbb{P} \left( X \geq \frac{1}{4} y - \frac{1}{\alpha - 1} \right) \geq \mathbb{P} \left( X \geq y - \frac{1}{\alpha - 1} \right) \geq \frac{1}{2\sqrt{e}} \frac{1}{y^\alpha} 2 \cdot 10^{-5} e^{-4}. \tag{30}
\]

Next, for $y \in \left[ \frac{2}{\sqrt{2} - \alpha}, 2/(2 - \alpha) \right]$ we also have

\[
\mathbb{P} \left( X \geq \frac{1}{4} y - \frac{1}{\alpha - 1} \right) \geq \mathbb{P} \left( X^1 \geq 1 - \frac{1}{\alpha - 1} \right) \mathbb{P} \left( X \geq \frac{1}{4} y \right)
\]

which, together with (19) and (14) gives

\[
\mathbb{P} \left( X \geq \frac{1}{4} y - \frac{1}{\alpha - 1} \right) \geq \frac{1}{2\sqrt{e}} \frac{3}{2 \cdot 10^{-5} e^{-(2-\alpha)y^2}}. \tag{31}
\]

Summing corresponding sides of estimates (30) and (31) we get (27).

To prove (28), using (15) we simply calculate that

\[
\mathbb{E} X_1 = 0, \quad \mathbb{E} X_1^2 = \int_0^1 x^2 \frac{dx}{x^{\alpha+1}} = \frac{1}{2 - \alpha}
\]

and

\[
\mathbb{E} X_1^4 = 3(\mathbb{E} X_1^2)^2 + \int_0^1 x^4 \frac{dx}{x^{\alpha+1}} = \frac{3}{(2 - \alpha)^2} + \frac{1}{4 - \alpha}.
\]
From this we easily get for any $y > 0$ the estimate
\[
P(X_1 \geq y) \leq \frac{\mathbb{E}X_1^4}{y^4} \leq \frac{3}{(2-\alpha)^2} + \frac{1}{(4-\alpha)y^4}
\]
and since for $y \geq 1/(2-\alpha)$
\[
\frac{1}{(4-\alpha)y^4} \leq \frac{1}{(2-\alpha)^2y^4} \leq \frac{1}{y^2} \leq \frac{1}{y^\alpha},
\]
using also (17), we obtain (28):
\[
P(X \geq 2y - \frac{1}{\alpha - 1}) \leq P(X_1 \geq y) + P(X^1 \geq y - \frac{1}{\alpha - 1})
\leq \frac{3}{y^\alpha} + \frac{1}{y^\alpha} + 5 \sqrt{e} \leq \frac{8}{y^\alpha}.
\]
The proof of (29) is exactly the same as the proof of (30).

Remark 10 Let us notice that from (27) it follows that for $y = 2\sqrt{2 - \alpha}$ the probability $P(X \geq 1/(2-\alpha))$ is of order $O(1)$.

Remark 11 For $\delta \in (0, 1/e)$ the equation $\delta \cdot y = \ln y$ has exactly two solutions $1 < y_1 < e < y_2$, and the larger one satisfies
\[
\frac{1}{\delta} \ln \frac{1}{\delta} < y_2 < \frac{2}{\delta} \ln \frac{1}{\delta}.
\]
From this we get that for $\alpha \approx 2$, the term containing $1/y^\alpha$ in (24) and (27) starts to dominate the term containing $\exp \left( -\kappa(2-\alpha)y^2 \right)$, $\kappa \in \{1/2, 1\}$, already for
\[
y = O \left( \sqrt{\frac{1}{2 - \alpha} \ln \frac{1}{2 - \alpha}} \right).
\]

Finally, to complete the picture, we analyse the decay of left tails of $X$. We have

Theorem 12 Let $X$ be a strictly asymmetric $\alpha$-stable random variable, $\alpha \in (1, 2)$, with the characteristic function (14). For any $y \in [2/\sqrt{2 - \alpha}, 2/(2 - \alpha)]$ one has the following estimates
\[
P(X \leq -y - \frac{1}{\alpha - 1}) \leq e^{4/3} e^{-\frac{1}{2}(2-\alpha)y^2},
\]
while for $y \geq 2/(2 - \alpha)$ one has
\[
P(X \leq -y - \frac{1}{\alpha - 1}) \leq e^{-\frac{1}{24} y - \frac{1}{\alpha - 1}} \geq \frac{4}{e} \cdot 10^{-2} e^{-(2-\alpha)y^2};
\]
and
\[
P(X \leq -y - \frac{1}{\alpha - 1}) \leq \exp \left( - \left( \frac{1}{2} \left( y + \frac{1}{\alpha - 1} \right) \right)^{\alpha/(\alpha - 1)} \right) / \left( \frac{1}{2 - \alpha} + \frac{1}{\alpha - 1} \right)^{1/(\alpha - 1)}
\]
and
\[ P \left( X \leq - \left( \frac{1}{e} - \frac{1}{4} \right) y - \frac{1}{\alpha - 1} \right) \geq \frac{1}{4} \exp \left( - \left( \sqrt{1 - \frac{2}{\alpha}} \left( y + \frac{1}{\alpha - 1} \right) - \frac{1}{2 - \alpha} + \frac{1}{\alpha - 1} \right)^{1/(\alpha - 1)} \right). \] (37)

**Proof.** Estimate (34) follows from Lemma 5 and the fact that \( X^1 \geq -1/(\alpha - 1) \). Estimate (35) follows from Lemma 7 and the fact that \( P (X^1 = -1/(\alpha - 1)) = e^{-1/\alpha} \geq 1/e \).

Similarly, estimate (36) follows from Lemma 8 and the fact that \( X^1 \geq -1/(\alpha - 1) \) while estimate (37) follows from the estimate \( P (X^1 = -1/(\alpha - 1)) = e^{-1/\alpha} \geq 1/e \), Lemma 8 and the inequality \((1 - 1/e^2)^2 /e \geq 1/4\).

\[ \Box \]

### 3.2 Tails of symmetric \( \alpha \)-stable random variables, \( \alpha \in (1, 2) \)

Now we will consider totally asymmetric r.v. with the characteristic function
\[ E \exp (itX) = \exp \left( \int_{-\infty}^{+\infty} e^{itx} - 1 - itx \frac{dx}{x^{\alpha+1}} \right). \] (38)

Naturally, \( X \) is the difference of two independent, identically distributed r.v. vs \( \bar{X} \) and \( X, X = \bar{X} - \bar{X} \) where
\[ E \exp (it\bar{X}) = E \exp \left( it\bar{X} \right) = \exp \left( \int_{0}^{+\infty} e^{itx} - 1 - itx \frac{dx}{x^{\alpha+1}} \right). \]

Further, we may write
\[ X = X_1 + \bar{X} - \bar{X}_1 - \bar{X}_1, \]
where \( \bar{X}_1, \bar{X}_1, \bar{X}_1, \bar{X}_1 \) are independent r.v. vs such that
\[ E \exp (it\bar{X}_1) = E \exp \left( it\bar{X}_1 \right) = \exp \left( \int_{0}^{+\infty} e^{itx} - 1 - itx \frac{dx}{x^{\alpha+1}} \right) \]
and \( \bar{X}_1, \bar{X}_1 \) have shifted compound Poisson distribution, \( \bar{X}_1 = \sum_{n=1}^{N} \bar{Z}_n - 1/(\alpha - 1), \bar{X}_1 = \sum_{n=1}^{N} \bar{Z}_n - 1/(\alpha - 1), \) where \( N, \hat{N} \) have Poisson distribution with the parameter (expectation) \((1/\alpha), \) and \( \bar{Z}_n, \bar{Z}_n, n = 1, 2, \ldots \) are i.i.d. r.v. vs., independent from \( \hat{N}, \tilde{N} \) and such that
\[ P (\tilde{Z}_n \geq z) = P (\bar{Z}_n \geq z) = \alpha \int_{z}^{+\infty} \frac{1}{x^{\alpha+1}} \frac{dx}{x} - z^{-\alpha} \text{ for } z \geq 1. \]

We have

**Theorem 13** Let \( X \) be a symmetric \( \alpha \)-stable random variable, \( \alpha \in (1, 2), \) with the characteristic function (38). For any \( y \in [2/\sqrt{2} - \alpha, 1/(2 - \alpha)] \) one has the following estimate
\[ P (X \geq 3y) \leq \frac{5}{\sqrt{e}} y^{\alpha} + 2 \cdot 104 e^{-\frac{1}{2} (2 - \alpha)^2 y^2}, \] (39)
and for any $y$ such that $\sqrt{1400/(2-\alpha)} \leq y \leq 1/(2-\alpha)$ one has
\[
P\left( X \geq \frac{1}{8} y \right) \geq \frac{e^{-d}}{\sqrt{e}} 10^{-6} \frac{1}{y^{\alpha}} + \frac{1}{\sqrt{e}} 10^{-6} e^{-(2-\alpha)y^2} ;
\]  
(40)
while for $y \geq 1/(2-\alpha)$ one has
\[
P\left( X \geq 3y \right) \leq \frac{12}{y^{\alpha}}
\]  
(41)
and for $y$ such that $y \geq \max(32,1/(2-\alpha))$
\[
P\left( X \geq y \right) \geq \frac{e^{-5}}{2\sqrt{e}} 10^{-5} \frac{1}{y^{\alpha}} .
\]  
(42)

**Proof.** To prove (39) we use estimates (26) and (34):
\[
P\left( X \geq 3y \right) \leq \frac{5}{\sqrt{e} y^{\alpha}} + \frac{e^{4e^2/3} e^{-\frac{1}{2}(2-\alpha)y^2} + e^{4/3} e^{-\frac{1}{2}(2-\alpha)y^2}}{\sqrt{e} y^{\alpha}}
\]
\[
\leq \frac{5}{\sqrt{e} y^{\alpha}} + \frac{e^{4/3} e^{-\frac{1}{2}(2-\alpha)y^2} + e^{4/3} e^{-\frac{1}{2}(2-\alpha)y^2}}{\sqrt{e} y^{\alpha}}
\]
\[
\leq \frac{5}{\sqrt{e} y^{\alpha}} + 2 \cdot 10^3 e^{-\frac{1}{2}(2-\alpha)y^2}.
\]

To prove (40) we use (27) and Lemma 5 and for $y$ such that $\sqrt{1400/(2-\alpha)} \leq y \leq 1/(2-\alpha)$ we get:
\[
P\left( X \geq \frac{1}{8} y \right) \geq \frac{1}{\sqrt{e} y^{\alpha}} + \frac{e^{-4}}{2\sqrt{e}} 10^{-5} \frac{1}{y^{\alpha}} + \frac{1}{2\sqrt{e}} 10^{-5} e^{-(2-\alpha)y^2}
\]
\[
\geq \left( \frac{e^{-4}}{2\sqrt{e}} 10^{-5} \frac{1}{y^{\alpha}} + \frac{1}{2\sqrt{e}} 10^{-5} e^{-(2-\alpha)y^2} \right) \left( 1 - e^{-4\alpha/3} e^{-\frac{1}{2}(2-\alpha)(y/8)^2} \right) \frac{1}{e}
\]
\[
\geq \frac{1}{\sqrt{e}} \left( 1 - \frac{1}{e} \right) \left( \frac{e^{-4}}{2\sqrt{e}} 10^{-5} \frac{1}{y^{\alpha}} + \frac{1}{2\sqrt{e}} 10^{-5} e^{-(2-\alpha)y^2} \right)
\]
\[
\geq \frac{e^{-4}}{2\sqrt{e}} 10^{-5} \frac{1}{y^{\alpha}} + \frac{1}{\sqrt{e}} 10^{-6} e^{-(2-\alpha)y^2},
\]
where we have used the estimate $4e^2/3 - \frac{1}{2}(2-\alpha)(y/8)^2 \leq -1$ valid for $y \geq \sqrt{1400/(2-\alpha)}$ and $(1-1/e)/e \geq 0.2$.

To prove (41) we use estimates (25) and (32):
\[
P\left( X \geq 3y \right) \leq \frac{8}{y^{\alpha}} + \frac{3}{(2-\alpha)^2 y^4} + \frac{1}{(4-\alpha)y^4} \leq \frac{12}{y^{\alpha}},
\]
where we have also used (33).
Finally, to prove \(42\) we use \(29\) and \(28\), and estimate

\[
P\left( X \geq \frac{1}{2} y \right) \geq P\left( \bar{X} + \tilde{X} \geq y - \frac{1}{\alpha - 1} \right) P\left( \bar{X} + \tilde{X} \leq \frac{1}{2} y - \frac{1}{\alpha - 1} \right) \\
\geq \frac{e^{-4}}{\sqrt{e}} 10^{-5} \frac{1}{y^{\alpha}} \left( 1 - P\left( \tilde{X} + \bar{X} > \frac{1}{2} y - \frac{1}{\alpha - 1} \right) \right) \\
\geq \frac{e^{-4}}{\sqrt{e}} 10^{-5} \frac{1}{y^{\alpha}} \left( 1 - \frac{8}{(y/2)^{\alpha}} \right) \frac{1}{e} \\
\geq \frac{e^{-5}}{2\sqrt{e}} 10^{-5} \frac{1}{y^{\alpha}}
\]

where we have used the estimate \(8/(y/2)^{\alpha} \leq 16/y \leq 1/2\) for \(y \geq 32\).

\[\square\]

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