COTORSION PAIRS AND MODEL STRUCTURES ON MORITA RINGS

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Abstract. We study cotorsion pairs and abelian model structures on Morita rings Λ = (A B
A N B M A B) which are Artin algebras. Given cotorsion pairs (U, X) and (V, Y) in A-Mod
and B-Mod, respectively, one can construct four cotorsion pairs in Λ-Mod:

⊥(X, Y), ⊥(X, Y), ⊥(U, V), ⊥(U, V).

These cotorsion pairs have relations:

Δ(U, V) ⊆ (X, Y), ⊥(U, Y) ⊆ (X, Y).

An important feature is that they are not equal, in general. In fact, there even exists a Morita
algebra Λ, such that the four cotorsion pairs are pairwise different. The problem of identifica-
tions, i.e., when these inclusions are the same, are studied; the heredity and completeness of
these cotorsion pairs are investigated; and finally, various model structures on Λ-Mod are ob-
tained, by explicitly giving the corresponding Hovey triples and Quillen’s homotopy categories.
In particular, cofibrantly generated Hovey triples, and the Gillespie-Hovey triples induced by
compatible generalized projective (respectively, injective) cotorsion pairs, are explicitly con-
structed. All these Hovey triples obtained are pairwise different and “new” in some sense.
Some results are even new when M = 0 or N = 0.

Key words: Morita ring, cotorsion pair, model structure, Hovey triple, Quillen’s homotopy
category, Gorenstein-projective module, monomorphism category

1. Introduction

This paper is to study cotorsion pairs and abelian model structures on some Morita rings.

Morita rings Λ = (A B
A N B M A B), originated from equivalences of module categories ([M]), and
formulated in [Bas], are also called the rings of Morita contexts, and the formal matrix rings.
They are widely used in various aspects of mathematics; and for more information we refer to
[C], [G], [MR], [KT] and [GrP]. Throughout this paper, we assume that the considered Morita
rings are Artin algebras.

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Model structures, introduced by D. Quillen [Q1, Q2], provide common ideas and framework for many branches of mathematics. A triple \((\mathcal{C}, \mathcal{F}, \mathcal{W})\) of classes of objects of abelian category \(\mathcal{A}\) is a Hovey triple, if \(\mathcal{W}\) is thick and \((\mathcal{C} \cap \mathcal{W}, \mathcal{F})\) and \((\mathcal{C}, \mathcal{F} \cap \mathcal{W})\) are complete cotorsion pairs in \(\mathcal{A}\); and it is hereditary, if both the cotorsion pairs are hereditary. By M. Hovey [H2] (see also [BR]), abelian model structures on \(\mathcal{A}\) and the Hovey triples in \(\mathcal{A}\) are in one-to-one correspondence.

Of special interest are hereditary Hovey triples. In this case, \(\mathcal{C} \cap \mathcal{F}\) is a Frobenius category, \(\mathcal{C} \cap \mathcal{F} \cap \mathcal{W}\) is the class of projective-injective objects, and Quillen’s homotopy category is exactly the stable category \((\mathcal{C} \cap \mathcal{F})/(\mathcal{C} \cap \mathcal{F} \cap \mathcal{W})\). See [BR], [Bec], [Gil4].

J. Gillespie [Gil3] gives an approach to construct a hereditary Hovey triple \((^\perp \mathcal{Y}, \Theta^\perp, \mathcal{W})\), from two compatible complete hereditary cotorsion pairs \((\Theta, \Theta^\perp)\) and \((^\perp \mathcal{Y}, \mathcal{Y})\), where \(\mathcal{W}\) is given as in Theorem 2.9 and conversely, any hereditary Hovey triple in an abelian category \(\mathcal{A}\) is obtained in this way. This general construction \((^\perp \mathcal{Y}, \Theta^\perp, \mathcal{W})\) of hereditary Hovey triples will be called the Gillespie-Hovey triples. See Subsection 2.9 for details.

The module categories of Morita rings have been described ([G]), and cotorsion pairs and abelian model structures on the special case of triangular matrix rings (i.e., \(M = 0\)) have been studied ([ZPD]). Nevertheless, a general study of cotorsion pairs and abelian model structures on some Morita rings meets difficulties and induces a lot of new phenomena, even under the assumption of \(M \otimes_A N = 0 = N \otimes_B M\).

From cotorsion pairs \((\mathcal{U}, \mathcal{X})\) and \((\mathcal{V}, \mathcal{Y})\), respectively in \(A\text{-Mod}\) and \(B\text{-Mod}\), one can construct four kinds of cotorsion pairs in Morita rings. Quite different from the case of \(M = 0\) or \(N = 0\), the four cotorsion pairs are pairwise different, in general. The heredity, the problem of identifications, the completeness, and the specializations, of these cotorsion pairs are studied. It turns out that Morita rings are rich in producing cotorsion pairs. Even if one takes \((\mathcal{U}, \mathcal{X})\) and \((\mathcal{V}, \mathcal{Y})\) to be the projective or the injective cotorsion pair, what one gets in \(\Lambda\text{-Mod}\) are pairwise generally different and “new” cotorsion pairs.

Based on these, various model structures on \(\Lambda\text{-Mod}\) are obtained, by explicitly giving the Hovey triples and Quillen’s homotopy categories. In particular, cofibrantly generated Hovey triples, and the Gillespie-Hovey triples induced by compatible generalized projective (respectively, injective) cotorsion pairs, are explicitly constructed. All these Hovey triples obtained are pairwise different and “new” in some sense. Some results are new even for \(M = 0\) or \(N = 0\).

The paper is organized as follows.

1. Introduction
2. Preliminaries
3. (Hereditary) cotorsion pairs in Morita rings
4. Identifications
5. Completeness
6. Realizations
7. Abelian model structures on Morita rings

1.1. (Hereditary) cotorsion pairs in Morita rings. For a ring $R$, let $R$-Mod be the category of left $R$-modules. For a class $C$ of objects in abelian category $A$ and $X \in A$, by $\text{Ext}^1_A(X,C) = 0$ we mean $\text{Ext}^1_A(X,C) = 0$ for all $C \in C$. Let $\mathcal{C}^\perp$ be the full subcategory of objects $X$ with $\text{Ext}^1_A(X,C) = 0$. Similarly for $C^\perp$.

Given a class $\mathcal{X}$ of $A$-modules and a class $\mathcal{Y}$ of $B$-modules, three classes

$$((\mathcal{X}, \mathcal{Y}), \Delta(\mathcal{X}, \mathcal{Y}), \nabla(\mathcal{X}, \mathcal{Y}))$$

of modules over Morita ring $\Lambda$ are defined. See Subsection 3.1. In particular, one has

the monomorphism category $\text{Mon}(\Lambda) = \Delta(\text{A-Mod}, \text{B-Mod})$, and the epimorphism category $\text{Epi}(\Lambda) = \nabla(\text{A-Mod}, \text{B-Mod})$.

By $\text{Tor}^1_A(M, U) = 0$ we mean $\text{Tor}^1_A(M, U) = 0$ for all $U \in \mathcal{U}$. Constructions of (hereditary)
cotorsion pairs in $\Lambda$-Mod are given as follows.

Theorem 1.1. (Theorem 3.1) Let $\Lambda = (\mathcal{M}_A N_B)$ be a Morita ring with $\phi = 0 = \psi$, $(\mathcal{U}, \mathcal{X})$ and

$(\mathcal{V}, \mathcal{Y})$ cotorsion pairs in $\text{A-Mod}$ and $\text{B-Mod}$, respectively.

1. If $\text{Tor}^1_A(M, U) = 0 = \text{Tor}^1_B(N, V)$, then $((\mathcal{X}, \mathcal{Y}), (\mathcal{X}, \mathcal{Y}))$ is a cotorsion pair in $\Lambda$-Mod;

and it is hereditary if and only if so are $(\mathcal{U}, \mathcal{X})$ and $(\mathcal{V}, \mathcal{Y})$.

2. If $\text{Ext}^1_A(N, X) = 0 = \text{Ext}^1_B(M, Y)$, then $((\mathcal{X}, \mathcal{Y}), (\mathcal{X}, \mathcal{Y}))$ is a cotorsion pair in $\Lambda$-Mod;

and it is hereditary if and only if so are $(\mathcal{U}, \mathcal{X})$ and $(\mathcal{V}, \mathcal{Y})$.

Theorem 1.2. (Theorem 3.2) Let $\Lambda = (\mathcal{M}_A N_B)$ be a Morita ring with $M \otimes_A N = 0 = N \otimes_B M$, $(\mathcal{U}, \mathcal{X})$ and

$(\mathcal{V}, \mathcal{Y})$ cotorsion pairs in $\text{A-Mod}$ and $\text{B-Mod}$, respectively. Then

1. $(\Delta(\mathcal{U}, \mathcal{V}), \Delta(\mathcal{U}, \mathcal{V}))$ is a cotorsion pair in $\Lambda$-Mod; and if $M_A$ and $N_B$ are flat, then

it is hereditary if and only if so are $(\mathcal{U}, \mathcal{X})$ and $(\mathcal{V}, \mathcal{Y})$.

2. $(\Delta(\nabla(\mathcal{X}, \mathcal{Y}), \nabla(\mathcal{X}, \mathcal{Y}))$ is a cotorsion pair in $\Lambda$-Mod; and if $B_M$ and $AN$ are projective, then

it is hereditary if and only if so are $(\mathcal{U}, \mathcal{X})$ and $(\mathcal{V}, \mathcal{Y})$.

We stress that, the condition “$M \otimes_A N = 0 = N \otimes_B M$” in Theorem 1.2 can not be weakened as “$\phi = 0 = \psi$” in general, as Example 3.4 shows.

The four cotorsion pairs

$((\mathcal{X}, \mathcal{Y}), (\mathcal{X}, \mathcal{Y}))$, $(\Delta(\mathcal{U}, \mathcal{V}), \Delta(\mathcal{U}, \mathcal{V}))$, $((\mathcal{X}, \mathcal{Y}), (\mathcal{X}, \mathcal{Y}))$, $(\Delta(\nabla(\mathcal{X}, \mathcal{Y}), \nabla(\mathcal{X}, \mathcal{Y}))$

given in Theorems 1.1 and 1.2 have relations:

$$\Delta(\mathcal{U}, \mathcal{V}) \subseteq (\mathcal{X}, \mathcal{Y}); \quad \Delta(\nabla(\mathcal{X}, \mathcal{Y}) \subseteq (\mathcal{X}, \mathcal{Y})$$

See Theorem 3.5 for details. An important and interesting feature is that, they are not equal,
in general. In fact, there even exists an algebra $\Lambda$, such that the four cotorsion pairs above are
pairwise different. Such an example has been given in Example 4.3.
1.2. Identifications. If $M = 0$ or $N = 0$, Theorems 1.1 and 1.2 have been obtained by R. M. Zhu, Y. Y. Peng and N. Q. Ding [ZPD, 3.4, 3.6]. Moreover, they prove that (see [ZPD, 3.7])

$$\left(\frac{1}{\mathcal{U}}, \mathcal{V}\right) = \left(\Delta(\mathcal{U}, \mathcal{V}), \Delta(\mathcal{U}, \mathcal{V})^\perp\right)$$

and

$$\left(\left(\frac{1}{\mathcal{U}}\right), \left(\frac{1}{\mathcal{V}}\right)^\perp\right) = \left(\nabla(\mathcal{X}, \mathcal{Y}), \nabla(\mathcal{X}, \mathcal{Y})\right).$$

As pointed out above, in general, these are not true! We will study the problem of identifications, i.e., when the two equalities hold true. If they are equal, then one has the cotorsion pairs

$$\Delta(\mathcal{U}, \mathcal{V}), \left(\frac{1}{\mathcal{V}}\right)^\perp$$

both are explicitly given. Since $\frac{1}{\mathcal{U}}$ and $\Delta(\mathcal{U}, \mathcal{V})^\perp$ are usually difficult to determine, these identifications are of significance, in explicitly finding abelian model structures in Morita rings.

In the rest of this section, $A = (\begin{array}{cc} A & N \\ M & B \end{array})$ is a Morita ring which is an Artin algebra with $M \otimes_A N = 0 = N \otimes_B M$. We will not state this every time. For functors $T_A, H_A : \Lambda-Mod \rightarrow \Lambda-Mod$ and $T_B, H_B : B-Mod \rightarrow \Lambda-Mod$, we refer to Subsection 2.4. By $M \otimes_A U \subseteq \mathcal{Y}$ we mean $M \otimes_A U \in \mathcal{Y}$ for all $U \in \mathcal{U}$.

**Theorem 1.3.** (Theorem 14) Let $(\mathcal{U}, \mathcal{X})$ and $(\mathcal{V}, \mathcal{Y})$ be cotorsion pairs in $A$-Mod and in $B$-Mod, respectively.

1. Assume that $\text{Tor}_1^A(M, \mathcal{U}) = 0 = \text{Tor}_1^B(N, \mathcal{V})$. If $M \otimes_A U \subseteq \mathcal{Y}$ and $N \otimes_B V \subseteq \mathcal{X}$, then $\Delta(\mathcal{U}, \mathcal{V}) = \left(\frac{1}{\mathcal{V}}\right) = T_A(\mathcal{U}) \oplus T_B(\mathcal{V})$, and thus $T_A(\mathcal{U}) \oplus T_B(\mathcal{V}), \left(\frac{1}{\mathcal{U}}\right)^\perp$ is a cotorsion pair.

2. Assume that $\text{Ext}_B^1(M, \mathcal{Y}) = 0 = \text{Ext}_B^1(N, \mathcal{X})$. If $\text{Hom}_B(M, \mathcal{Y}) \subseteq \mathcal{U}$ and $\text{Hom}_B(N, \mathcal{X}) \subseteq \mathcal{V}$, then $\nabla(\mathcal{X}, \mathcal{Y}) = \left(\frac{1}{\mathcal{V}}\right)^\perp = H_A(\mathcal{X}) \oplus H_B(\mathcal{Y})$, and thus $\left(\frac{1}{\mathcal{U}}\right)^\perp, H_A(\mathcal{X}) \oplus H_B(\mathcal{Y})$ is a cotorsion pair.

Even if the two cotorsion pairs are not equal in general, there are possibilities that they can be equal for some special $A$, $B$, $M$ and $N$. The following result provide such important cases: cotorsion pairs

$$\left(\frac{1}{\mathcal{I}}, \mathcal{J}\right)$$

are not equal in general (cf. Example 1.3), but the following result claims that they can be the same in some special cases. For a ring $R$, let $R^\mathcal{P}$ (respectively, $R^\mathcal{I}$) be the full subcategory of $R$-Mod of projective (respectively, injective) modules, $R^\mathcal{P}^{\leq 1}$ (respectively, $R^\mathcal{I}^{\leq 1}$) the full subcategory of modules with projective (respectively, injective) dimension $\leq 1$.

**Theorem 1.4.** (Theorem 1.6) Assume that $A$ and $B$ are quasi-Frobenius rings, $AN$ and $BM$ are projective, and that $M_A$ and $N_B$ are flat. Then

1. $A$ is a Gorenstein ring with $\text{inj.dim}_A A \leq 1$. 

(2) \((\mathcal{I}_A, \mathcal{P}) \), \((\mathcal{I}_A, \mathcal{P}) = (\text{Mon}(\Lambda), \text{Mon}(\Lambda)^\perp)\); and it is exactly the Gorenstein-projective cotorsion pair \((\text{GP}(\Lambda), \Lambda^P \leq 1)\). So, it is complete and hereditary, and \(\text{GP}(\Lambda) = \text{Mon}(\Lambda) = \Lambda^\perp\). Therefore, \(\Lambda^P \leq 1\).

(2)′ \((\mathcal{P}_A, \mathcal{P}_A^\perp) = (\mathcal{I}_A, \text{Epi}(\Lambda)); and it is exactly the Gorenstein-injective cotorsion pair \((\Lambda^P \leq 1, \text{GI}(\Lambda))\). So, it is complete and hereditary, and \(\text{GI}(\Lambda) = \text{Epi}(\Lambda) = \Lambda^\perp\).

The conditions of Theorem 1.4 really and often occur. See Example 4.7. Note that \(\text{GP}(\Lambda) = \text{Mon}(\Lambda)\) is a new result, as an application of cotorsion theory and monomorphism category to Gorenstein-projective modules. See Remark 4.8.

1.3. Completeness. Completeness of a cotorsion pair is important, not only in the theory itself, but also in finding abelian model structures \([H2]\). See Theorem 2.6. In view of identifications, we only discuss the completeness of cotorsion pairs in Theorem 1.1.

If cotorsion pairs \((U, X)\) and \((V, Y)\) are cogenerated by sets \(S_1\) and \(S_2\), respectively, then the cotorsion pair \((\mathcal{I}_A, \mathcal{P})\) is cogenerated by the set \(S_1 \cup S_2\), and hence complete, by a well-known theorem \([ET2, \text{Theorem } 10]\) of P. C. Eklof and J. Trlifaj. See Proposition 5.1.

However, since the theorem of Eklof and Trlifaj has no dual version, in general, there is no information on the completeness of \((U, X)\). Also, it is more natural to start from the completeness of \((U, X)\). Thus, we need module-theoretical methods to the completeness of cotorsion pairs in Morita rings.

Take \((V, Y)\) to be an arbitrary complete cotorsion pair in \(B\)-Mod. Taking \((U, X) = (\mathcal{P}_A, \mathcal{I}_A)\) and applying Theorem 1.1(1), we have Theorem 1.5(1) below; taking \((U, X) = (\mathcal{P}_A, \mathcal{I}_A)\) and applying Theorem 1.1(2), we have Theorem 1.5(2) below.

Theorem 1.5. (Theorem 5.2) Assume that \(N_B\) is flat and \(B\) is projective. Let \((V, Y)\) be a complete cotorsion pair in \(B\)-Mod.

1. If \(M \otimes_A \mathcal{P} \subseteq Y\), then \((\mathcal{T}_A(\mathcal{P}) \oplus \mathcal{T}_B(V), (\mathcal{A}^\text{Mod})^\perp)\) is a complete cotorsion pair.

2. If \(\text{Hom}_A(N, \mathcal{I}) \subseteq Y\), then \((\mathcal{A}^\text{Mod}) \oplus \mathcal{H}_A(\mathcal{I}) \oplus \mathcal{H}_B(Y)\) is a complete cotorsion pair.

We stress that

1. If \(B\) is injective, then \(M \otimes_A \mathcal{P} \subseteq Y\) always holds;

2. If \(N_B\) is flat, then \(\text{Hom}_A(N, \mathcal{I}) \subseteq Y\) always holds.

Similarly, let \((U, X)\) be an arbitrary complete cotorsion pair in \(A\)-Mod. Taking \((V, Y) = (B\mathcal{P}, B\)-Mod\) and applying Theorem 1.1(1), we get Theorem 5.4(1); and taking \((V, Y) = (B\mathcal{P}, B\mathcal{I})\) and applying Theorem 1.1(2), we get Theorem 5.4(2).

1For some special cases, there are indeed dual versions. See e.g. \([ET1, \text{Theorem } 14(ii)]\).
1.4. Realizations. It turns out that Morita rings are rich in cotorsion pairs. In Theorem 1.1 (respectively, Theorem 1.2), even if one starts form the projective or the injective cotorsion pair in $A$-$Mod$ and $B$-$Mod$, what one gets in $\Lambda$-$Mod$ are already pairwise generally different (see Definition 4.1) and “new” cotorsion pairs. Here, by a “new” cotorsion pair we mean that it is generally different from the projective and the injective cotorsion pair, the Gorenstein-projective and the Gorenstein-injective cotorsion pair, and the flat cotorsion pair ([EJ, Lemma 7.1.4]). For details see Definition 6.2, Propositions 6.1, 6.3, 6.9 and 6.10.

1.5. Abelian model structures on Morita rings. A natural method of getting abelian model structures on Morita rings, is to see how abelian model structures on $A$-$Mod$ and $B$-$Mod$ can induce the ones on $\Lambda$-$Mod$.

A cofibrantly generated model category ([H1, 2.1.17]) enjoys nice properties. A Hovey triple $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ will be called cofibrantly generated, if both the cotorsion pairs $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are cogenerated by sets. For a Grothendieck category $\mathcal{A}$ with enough projective objects, a Hovey triple $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ in $\mathcal{A}$ is cofibrantly generated if and only if the corresponding model category is cofibrantly generated. See Proposition 2.7.

Theorem 1.6. (Theorem 7.1) Let $(\mathcal{U}', \mathcal{X}, \mathcal{W}_1)$ and $(\mathcal{V}', \mathcal{Y}, \mathcal{W}_2)$ be cofibrantly generated Hovey triples in $A$-$Mod$ and $B$-$Mod$, respectively.

1. Assume that $\text{Tor}_1^A(M, \mathcal{U}') = 0 = \text{Tor}_1^B(N, \mathcal{V}')$, $M \otimes_A \mathcal{U}' \subseteq \mathcal{Y} \cap \mathcal{W}_2$ and $N \otimes_B \mathcal{V}' \subseteq \mathcal{X} \cap \mathcal{W}_1$. Then

$$(T_A(\mathcal{U}') \oplus T_B(\mathcal{V}'), (\mathcal{X}, \mathcal{Y}), (\mathcal{W}_1, \mathcal{W}_2))$$

is a cofibrantly generated Hovey triple in $\Lambda$-$Mod$; and it is hereditary with $Ho(\Lambda) \cong Ho(A) \oplus Ho(B)$, if $(\mathcal{U}', \mathcal{X}, \mathcal{W}_1)$ and $(\mathcal{V}', \mathcal{Y}, \mathcal{W}_2)$ are hereditary.

2. Assume that $\text{Ext}_A^1(M, \mathcal{Y}) = 0 = \text{Ext}_B^1(N, \mathcal{X})$, $\text{Hom}_B(M, \mathcal{Y}) \subseteq \mathcal{U}' \cap \mathcal{W}_1$ and $\text{Hom}_A(N, \mathcal{X}) \subseteq \mathcal{V}' \cap \mathcal{W}_2$. Then

$$(\text{Hom}_A(\mathcal{X}) \oplus \text{Hom}_B(\mathcal{Y}), (\mathcal{W}_1, \mathcal{W}_2))$$

is a cofibrantly generated Hovey triple; and it is hereditary with $Ho(\Lambda) \cong Ho(A) \oplus Ho(B)$, if $(\mathcal{U}', \mathcal{X}, \mathcal{W}_1)$ and $(\mathcal{V}', \mathcal{Y}, \mathcal{W}_2)$ are hereditary.

For general Hovey triples (not assumed to be cofibrantly generated), we have

Theorem 1.7. (Theorem 7.3) Let $N_B$ be flat, $B$-$M$ projective, and $(\mathcal{V}', \mathcal{Y}, \mathcal{W})$ a Hovey triple in $B$-$Mod$.

1. If $M \otimes_A \mathcal{P} \subseteq \mathcal{Y} \cap \mathcal{W}$, then

$$(T_A(\mathcal{P}) \oplus T_B(\mathcal{V}'), (\text{A-Mod}), (\text{A-Mod}))$$

is a Hovey triple in $\Lambda$-$Mod$; and it is hereditary with $Ho(\Lambda) \cong Ho(B)$, if $(\mathcal{V}', \mathcal{Y}, \mathcal{W})$ is hereditary.

2. If $\text{Hom}_A(N, \mathcal{I}) \subseteq \mathcal{V}' \cap \mathcal{W}$, then

$$(\text{Hom}_A(\mathcal{I}) \oplus \text{Hom}_B(\mathcal{Y}), (\text{A-Mod}))$$

is a Hovey triple in $\Lambda$-$Mod$; and it is hereditary with $Ho(\Lambda) \cong Ho(B)$, if $(\mathcal{V}', \mathcal{Y}, \mathcal{W})$ is hereditary.
1.6. Generalized projective cotorsion pairs (gpcpts) and projective models. A complete cotorsion pair \((\mathcal{U}, \mathcal{X})\) in \(A\text{-Mod}\) is generalized projective, if \(\mathcal{U} \cap \mathcal{X} = A\mathcal{P}\) and \(\mathcal{X}\) is thick (see [Bec, 1.1.9]). A generalized projective cotorsion pair (or in short, a gpcpt) is always hereditary and not necessarily the projective cotorsion pair \((A\mathcal{P}, A\text{-Mod})\). Following [H2] and [Gil4], an abelian model structure is projective, if each object is fibrant, i.e., the corresponding Hovey triple is of the form \((\mathcal{U}, A\text{-Mod}, \mathcal{X})\). Note that gpcpts and projective models are in one-one correspondence, i.e., \((\mathcal{U}, \mathcal{X})\) is a gpcpt in \(A\text{-Mod}\) if and only if \((\mathcal{U}, A\text{-Mod}, \mathcal{X})\) is a Hovey triple. See Subsection 7.3.

Dually, one has the notion of a generalized injective cotorsion pair (or in short, gictp), and an injective model.

The following result deals with the important case of gpcpts (gictps) in Theorem 1.7, with stronger result, where the conditions “\(M \otimes_A \mathcal{P} \subseteq \mathcal{Y} \cap \mathcal{W}\)” and “\(\text{Hom}_A(N, A\mathcal{I}) \subseteq \mathcal{V}' \cap \mathcal{W}'\)” in Theorem 1.7 can be dropped in these cases.

**Theorem 1.8.** *(Theorem 7.9)* Assume that \(N_B\) is flat and \(_B M\) is projective.

1. Let \((\mathcal{V}, \mathcal{Y})\) and \((\mathcal{V}', \mathcal{Y}')\) be compatible gpcpts in \(B\text{-Mod}\), with Gillespie-Hovey triple \((\mathcal{V}', \mathcal{Y}, \mathcal{W})\). Then
   \[ (T_A(A\mathcal{P}) \oplus T_B(\mathcal{V}), \ (A\text{-Mod}_{\mathcal{V}})) \quad \text{and} \quad (T_A(A\mathcal{P}) \oplus T_B(\mathcal{V}'), \ (A\text{-Mod}_{\mathcal{V}'})) \]
   are compatible gpcpts in \(A\text{-Mod}\), with Gillespie-Hovey triple
   \[ (T_A(A\mathcal{P}) \oplus T_B(\mathcal{V}'), \ (A\text{-Mod}_{\mathcal{V}'}) , \ (A\text{-Mod}_{\mathcal{W}'}) ) \]
   and \(\text{Ho}(A) \cong (\mathcal{V}' \cap \mathcal{Y})/B\mathcal{P} \cong \text{Ho}(B)\).

2. Let \((\mathcal{V}, \mathcal{Y})\) and \((\mathcal{V}', \mathcal{Y}')\) be compatible gicpts, with Gillespie-Hovey triple \((\mathcal{V}', \mathcal{Y}, \mathcal{W})\). Then
   \[ (\text{Hom}(A\mathcal{I}) \oplus \text{H}_B(\mathcal{Y}), \ (A\text{-Mod}_{\mathcal{V}})) \quad \text{and} \quad (\text{Hom}(A\mathcal{I}) \oplus \text{H}_B(\mathcal{Y}'), \ (A\text{-Mod}_{\mathcal{V}'}) ) \]
   are compatible gicpts in \(A\text{-Mod}\), with Gillespie-Hovey triple
   \[ (\text{Hom}(A\mathcal{I}) \oplus \text{H}_B(\mathcal{Y}), \ (A\text{-Mod}_{\mathcal{V}}), \ (A\text{-Mod}_{\mathcal{W}}) ) \]
   and \(\text{Ho}(A) \cong (\mathcal{V}' \cap \mathcal{Y})/B\mathcal{I} \cong \text{Ho}(B)\).

A gpcpt \((\mathcal{V}, \mathcal{Y})\) in \(B\text{-Mod}\) gives compatible gpcpts \(B\mathcal{P}, B\text{-Mod}\) and \((\mathcal{V}, \mathcal{Y})\). A gictp \((\mathcal{V}, \mathcal{Y})\) in \(B\text{-Mod}\) gives compatible gicpts \((\mathcal{V}, \mathcal{Y})\) and \((B\text{-Mod}, B\mathcal{I})\). Thus, by Theorem 1.8 one has

**Corollary 1.9.** *(Corollaries 7.12, 7.13)* Suppose that \(N_B\) is flat and \(_B M\) is projective.
(1) Let \( (\mathcal{V}, \mathcal{Y}) \) be a gctp in \( \mathcal{B} \)-Mod. Then
\[
(T_\mathcal{A}(\mathcal{P}) \oplus T_\mathcal{B}(\mathcal{V}), \mathcal{A}-\text{Mod}, (A^{\text{Mod}}))
\]
is a hereditary Hovey triple, with \( \text{Ho}(\mathcal{A}) \cong \mathcal{V}/_{/\mathcal{B}} \mathcal{P} \).

In particular, if \( B \) is quasi-Frobenius, then \( (T_\mathcal{A}(\mathcal{P}) \oplus T_\mathcal{B}(\mathcal{B}\text{-Mod}), \mathcal{A}-\text{Mod}, (A^{\text{Mod}})) \) is a hereditary Hovey triple with \( \text{Ho}(\mathcal{A}) \cong \mathcal{B}\text{-Mod} \).

(2) Let \( (\mathcal{V}, \mathcal{Y}) \) be a gctp in \( \mathcal{B}\text{-Mod} \). Then
\[
(\mathcal{A}-\text{Mod}, H_\mathcal{A}(\mathcal{A}\mathcal{I}) \oplus H_\mathcal{B}(\mathcal{Y}), (A^{\text{Mod}}))
\]
is a hereditary Hovey triple, with \( \text{Ho}(\mathcal{A}) \cong \mathcal{Y}/_{/\mathcal{B}} \mathcal{I} \).

In particular, if \( B \) is quasi-Frobenius, then \( (\mathcal{A}-\text{Mod}, H_\mathcal{A}(\mathcal{A}\mathcal{I}) \oplus H_\mathcal{B}(\mathcal{B}\text{-Mod}), (A^{\text{Mod}})) \) is a hereditary Hovey triple with \( \text{Ho}(\mathcal{A}) \cong \mathcal{B}\text{-Mod} \).

Similarly, starting from compatible gctps (gictps) in \( \mathcal{A}-\text{Mod} \), one has the corresponding results. See Theorem 7.10, Corollaries 7.14 and 7.15.

We stress that all the abelian model structures obtained above are pairwise generally different and “new”. See Proposition 7.20 for details.

2. Preliminaries

2.1. Notations. For a ring \( R \), let \( R\text{-Mod} \) be the category of left \( R \)-modules, \( R\mathcal{P} \) (respectively, \( R\mathcal{I} \)) the full subcategory of \( R\text{-Mod} \) of projective (respectively, injective) modules; \( R\mathcal{P}^{<\infty} \) (respectively, \( R\mathcal{I}^{<\infty} \)) the full subcategory of \( R\text{-Mod} \) of finite projective (respectively, injective) dimension. Denote by \( \text{GP}(R) \) (respectively, \( \text{GI}(R) \)) the full subcategory of \( R\text{-Mod} \) of Gorenstein-projective (respectively, Gorenstein-injective) modules.

For a class \( \mathcal{C} \) of objects in abelian category \( \mathcal{A} \), let
\[
\begin{align*}
\perp \mathcal{C} & = \{ X \in \mathcal{A} \mid \text{Ext}_\mathcal{A}^1(X, \mathcal{C}) = 0 \}, \\
\perp \mathcal{C}^\perp & = \{ X \in \mathcal{A} \mid \text{Ext}_\mathcal{A}^1(\mathcal{C}, X) = 0 \}, \\
\mathcal{C}^\perp & = \{ X \in \mathcal{A} \mid \text{Ext}_\mathcal{A}^1(\mathcal{C}, X) = 0, \forall i \geq 1 \}, \\
\mathcal{C}^{\perp \perp} & = \{ X \in \mathcal{A} \mid \text{Ext}_\mathcal{A}^1(\mathcal{C}, X) = 0, \forall i \geq 1 \}.
\end{align*}
\]
For classes \( \mathcal{C} \) and \( \mathcal{D} \) of objects in \( \mathcal{A} \), by \( \text{Hom}_\mathcal{A}(\mathcal{C}, \mathcal{D}) = 0 \) we mean \( \text{Hom}_\mathcal{A}(\mathcal{C}, D) = 0 \) for all \( C \in \mathcal{C} \) and for all \( D \in \mathcal{D} \). Similarly for \( \text{Ext}_\mathcal{A}(\mathcal{C}, \mathcal{D}) = 0 \).

2.2. Morita rings. Let \( A \) and \( B \) be rings, \( B\mathcal{M}_A \) a \( B\)-\( A \)-bimodule, \( AN_B \) an \( A\)-\( B \)-bimodule, \( \phi : M \otimes_A N \rightarrow B \) a \( B \)-bimodule map, and \( \psi : N \otimes_B M \rightarrow A \) an \( A \)-bimodule map, such that
\[
m' \psi(n \otimes_B m) = \phi(n' \otimes_A n)m, \quad n' \phi(m \otimes_A n) = \psi(n' \otimes_B m)n, \quad \forall m, m' \in M, \quad \forall n, n' \in N. \quad (*)
\]

A Morita ring is \( \Lambda = \Lambda(\phi, \psi) := (A \mathcal{M}_B)_{AN_B} \), with componentwise addition, and multiplication
\[
\begin{pmatrix} a & n \\ m & b \end{pmatrix} \begin{pmatrix} a' & n' \\ m' & b' \end{pmatrix} = \begin{pmatrix} a_{m', n'} + \psi(n' \otimes_B m) \phi(n \otimes_A m') & a_{m', n'} + \psi(n' \otimes_B m) \phi(n \otimes_A m') \\ \phi(n' \otimes_A m') + bb' \end{pmatrix}.
\]
The assumptions \((*)\) guarantee the associativity of the multiplication (the converse is also true).

This construction is finally formulated in [Bas]. Throughout this paper, we will assume \( \phi = 0 = \psi \). This contains triangular matrix rings (i.e., \( M = 0 \) or \( N = 0 \)).
Throughout this paper, we assume that the considered Morita rings $A$ are Artin algebras, i.e., $A$ and $B$ are Artin $R$-algebras, where $R$ is a commutative artinian ring, $M$ and $N$ are finitely generated $R$-modules, such that $R$ acts centrally both on $M$ and $N$ (see [GrP, Proposition 2.2]).

2.3. Two expressions of modules over Morita rings. Let $\mathcal{M}(A)$ be the category with objects $(\frac{X}{Y}, f, g)$, where $X \in A\text{-Mod}$, $Y \in B\text{-Mod}$, $f \in \text{Hom}_B(M \otimes_A X, Y)$ and $g \in \text{Hom}_A(N \otimes_B Y, X)$, satisfy the conditions

$$g(n \otimes_B f(m \otimes_A x)) = \psi(n \otimes_B m)x, \quad f(m \otimes_A \phi(n \otimes_A y)) = \phi(m \otimes_A n)y, \quad \forall m \in M, n \in N, x \in X, y \in Y.$$  

The maps $f$ and $g$ are called the structure maps of $(\frac{X}{Y}, f, g)$.

For $\phi = 0 = \psi$, the conditions are just $g(1_N \otimes f) = f(1_M \otimes g)$.

A morphism in $\mathcal{M}(A)$ is $(\frac{X}{Y}, f, g) \rightarrow (\frac{X'}{Y'}, f', g')$, where $a : X \rightarrow X'$ and $b : Y \rightarrow Y'$ are respectively an $A$-map and a $B$-map, so that the following diagrams commute:

$$\begin{array}{ccc}
M \otimes_A X & \xrightarrow{1_M \otimes a} & M \otimes_A X' \\
\downarrow f & & \downarrow f' \\
Y & \xrightarrow{b} & Y'
\end{array} \quad \begin{array}{ccc}
N \otimes_B Y & \xrightarrow{1_N \otimes b} & N \otimes Y' \\
\downarrow g & & \downarrow g' \\
X & \xrightarrow{a} & X'
\end{array}$$

Let $\eta_{X,Y} : \text{Hom}_B(M \otimes_A X, Y) \cong \text{Hom}_A(X, \text{Hom}_B(M, Y))$ and $\eta'_{X,Y} : \text{Hom}_A(N \otimes_B Y, X) \cong \text{Hom}_B(Y, \text{Hom}_A(N, X))$ be the adjunction isomorphisms. For $f \in \text{Hom}_B(M \otimes_A X, Y)$ and $g \in \text{Hom}_A(N \otimes_B Y, X)$, put $\tilde{f} = \eta_{X,Y}(f)$ and $\tilde{g} = \eta'_{X,Y}(g)$. Thus

$$\tilde{f}(x) = "m \mapsto f(m \otimes_A x)", \quad \forall x \in X; \quad \tilde{g}(y) = "n \mapsto g(n \otimes_B y)", \quad \forall y \in Y.$$  

Using the bi-functorial property of the adjunction isomorphisms one knows that

$$fb = f'(1_M \otimes_A a) \quad \text{if and only if} \quad (M, b)\tilde{f} = \tilde{f}'a$$

and

$$ag = g'(1_N \otimes_B b) \quad \text{if and only if} \quad (N, a)\tilde{g} = \tilde{g}'b.$$  

Let $\mathcal{M}'(A)$ be the category with objects $(\frac{X}{Y}, \bar{f}, \bar{g})$, where $X \in A\text{-Mod}$, $Y \in B\text{-Mod}$, $\bar{f} \in \text{Hom}_A(X, \text{Hom}_B(M, Y))$ and $\bar{g} \in \text{Hom}_B(Y, \text{Hom}_A(N, X))$, such that the following diagrams commute:

$$\begin{array}{ccc}
X & \xrightarrow{(\psi, X)h_{A,X}} & \text{Hom}_A(N \otimes_B M, X) \\
\downarrow \bar{f} & \cong & \downarrow \eta_{M,X}' \\
\text{Hom}_B(M, Y) & \xrightarrow{(M, \bar{f})} & \text{Hom}_A(M, \text{Hom}_B(N, X))
\end{array} \quad \begin{array}{ccc}
Y & \xrightarrow{(\phi, Y)h_{B,Y}} & \text{Hom}_B(M \otimes_A N, Y) \\
\downarrow \bar{g} & \cong & \downarrow \eta_{N,Y} \\
\text{Hom}_A(N, X) & \xrightarrow{(N, \bar{f})} & \text{Hom}_A(N, \text{Hom}_B(M, Y))
\end{array}$$

where $h_{A,X} : X \rightarrow \text{Hom}_A(A, X)$ and $h_{B,Y} : Y \rightarrow \text{Hom}_B(B, Y)$ are the canonical isomorphisms. The maps $\tilde{f}$ and $\tilde{g}$ are also called the structure maps of $(\frac{X}{\bar{f}, \bar{g}})$.

For $\phi = 0 = \psi$, the conditions are just $(M, \bar{g})\tilde{f} = 0 = (N, \bar{f})\tilde{g}$. 
A morphism in $\mathcal{M}'(\Lambda)$ is $(\bar{\phi}, \bar{\psi}) : (\bar{\phi})_{g,\bar{g}} \rightarrow (\bar{\psi})_{f,\bar{g}}$, where $a : X \rightarrow X'$ and $b : Y \rightarrow Y'$ are respectively an $A$-map and a $B$-map, so that diagrams

\[
\begin{array}{ccc}
X & \xrightarrow{a} & X' \\
\downarrow f & & \downarrow \bar{f} \\
\text{Hom}_B(M, Y) & \xrightarrow{(M,b)} & \text{Hom}_B(M, Y')
\end{array}
\quad
\begin{array}{ccc}
Y & \xrightarrow{b} & Y' \\
\downarrow \bar{g} & & \downarrow g \\
\text{Hom}_A(N, X) & \xrightarrow{(N,a)} & \text{Hom}_A(N, X')
\end{array}
\]

commute. Then

\[
(\bar{\phi})_{f,\bar{g}} \mapsto (\bar{\phi})_{f,\bar{g}}, \quad "(\bar{\phi})_{f,\bar{g}} (\bar{\psi})_{f',\bar{g}'})" \mapsto "(\bar{\phi})_{f,\bar{g}} (\bar{\psi})_{f',\bar{g}'}"
\]
gives an isomorphism $\mathcal{M}(\Lambda) \cong \mathcal{M}'(\Lambda)$ of categories.

**Theorem 2.1.** (E. L. Green [G, 1.5]) *Let $\Lambda = (\frac{A}{M}, \frac{N}{B})$ be a Morita ring. Then $\Lambda\text{-Mod} \cong \mathcal{M}(\Lambda) \cong \mathcal{M}'(\Lambda)$ as categories.*

Throughout we will identify a $\Lambda$-module with $(\bar{\phi})_{f,\bar{g}}$. We will also use the expression $(\bar{\phi})_{f,\bar{g}}$, when it is more convenient. For convenience we will call $(\bar{\phi})_{f,\bar{g}}$ the second expression of a $\Lambda$-module. A sequence of $\Lambda$-maps

\[
\begin{array}{c}
\begin{array}{cc}
(X_1)_{f_1,\bar{g}_1} & \xrightarrow{(1)} (X_2)_{f_2,\bar{g}_2} \\
\downarrow & \downarrow \\
(X_3)_{f_3,\bar{g}_3} & \xrightarrow{(1)} (X_4)_{f_4,\bar{g}_4}
\end{array}
\end{array}
\]

is exact if and only if both the sequences $X_1 \xrightarrow{a_1} X_2 \xrightarrow{a_2} X_3$ and $Y_1 \xrightarrow{b_1} Y_2 \xrightarrow{b_2} Y_3$ are exact.

Also, in the second expressions of $\Lambda$-modules, a sequence of $\Lambda$-maps

\[
\begin{array}{c}
\begin{array}{cc}
(X_1)_{f_1,\bar{g}_1} & \xrightarrow{(1)} (X_2)_{f_2,\bar{g}_2} \\
\downarrow & \downarrow \\
(X_3)_{f_3,\bar{g}_3} & \xrightarrow{(1)} (X_4)_{f_4,\bar{g}_4}
\end{array}
\end{array}
\]

is exact if and only if both the sequences $X_1 \xrightarrow{a_1} X_2 \xrightarrow{a_2} X_3$ and $Y_1 \xrightarrow{b_1} Y_2 \xrightarrow{b_2} Y_3$ are exact.

2.4. Twelve functors and two recollements. Denote by $\Psi_X$ the composition $N \otimes_B M \otimes_A X \xrightarrow{\psi \otimes_X 1_X} A \otimes_A X \cong X$, and denote by $\Psi_Y$ the composition $M \otimes_A N \otimes_B Y \xrightarrow{1_M \otimes \bar{g}} B \otimes_B Y \cong Y$.

Let $\epsilon : M \otimes_A \text{Hom}_B(M, -) \rightarrow \text{Id}_{B\text{-Mod}}$ be the counit, and $\delta : \text{Id}_{A\text{-Mod}} \rightarrow \text{Hom}_B(M, M \otimes_A -)$ the unit, of the adjoint pair $(M \otimes_A -, \text{Hom}_B(M, -))$. Let $\epsilon' : N \otimes_B \text{Hom}_A(N, -) \rightarrow \text{Id}_{A\text{-Mod}}$ be the counit, and $\delta' : \text{Id}_{B\text{-Mod}} \rightarrow \text{Hom}_A(N, N \otimes_B -)$ the unit, of the adjoint pair $(N \otimes_B -, \text{Hom}_A(N, -))$.

Recall twelve functors involving $A$-Mod.

- $T_A : A\text{-Mod} \rightarrow A((\phi,\psi)^{-1})\text{-Mod}$, $X \mapsto (M \otimes_A X)_{1,M \otimes_A X, \psi X}$.
- $T_B : B\text{-Mod} \rightarrow A((\phi,\psi)^{-1})\text{-Mod}$, $Y \mapsto (N \otimes_B Y)_{\phi_Y, 1_N \otimes_B Y}$.

If $\phi = \psi = 0$, then $T_A X = (M \otimes_A X)_{1,0}$ and $T_B Y = (N \otimes_B Y)_{0,1}$.

- $U_A : A((\phi,\psi))\text{-Mod} \rightarrow A\text{-Mod}$, $(\bar{\phi})_{f,\bar{g}} \mapsto X$.
- $U_B : A((\phi,\psi))\text{-Mod} \rightarrow B\text{-Mod}$, $(\bar{\phi})_{f,\bar{g}} \mapsto Y$.
- $H_A : A\text{-Mod} \rightarrow A((\phi,\psi))\text{-Mod}$, $X \mapsto (\text{Hom}_A(N, X))_{\bar{\psi} X, \epsilon' X}$.
Note that $\overline{\Psi}_X = \text{Hom}_A(N, \Psi_X) \circ \delta'_{M@A^X}$; and $H_AX = (\text{Hom}_A(N,X))^\sim_{\Psi_X, 1}$ in the second expression.

- $\text{H}_B : B\text{-Mod} \rightarrow \Lambda_{(\phi, \psi)}\text{-Mod}$, $Y \mapsto (\text{Hom}_B(M,Y))_{\Psi_Y}$, $\text{H}_B Y = (\text{Hom}_B(M,Y))^{\sim}_{Y, \Psi_Y}$ in the second expression.

If $\phi = \psi = 0$, then $H_AX = (\text{Hom}_A(N,X))_{0, t_X}'$, $H_BY = (\text{Hom}_B(M,Y))_{Y, 0}'$; and it is convenient to use the second expression: $H_AX = (\text{Hom}_A(N,X))_{0, 1}'$, $H_BY = (\text{Hom}_B(M,Y))_{1, 0}'$.

- $\text{C}_A : \Lambda_{(\phi, \psi)}\text{-Mod} \rightarrow A\text{-Mod}$, $(X)_{f,g}' \mapsto \text{Coker} g$.
- $\text{C}_B : \Lambda_{(\phi, \psi)}\text{-Mod} \rightarrow B\text{-Mod}$, $(X)_{f,g}' \mapsto \text{Coker} f$.
- $\text{K}_A : \Lambda_{(\phi, \psi)}\text{-Mod} \rightarrow A\text{-Mod}$, $(X)_{f,g}' \mapsto \text{Ker} \tilde{f}$.
- $\text{K}_B : \Lambda_{(\phi, \psi)}\text{-Mod} \rightarrow B\text{-Mod}$, $(X)_{f,g}' \mapsto \text{Ker} \tilde{g}$.

If $\phi = \psi = 0$, then $(X)_{0, 0}'$ is a left $\Lambda$-module for any $A$-module $AX$, and $(\frac{0}{0})_{0, 0}'$ is a left $\Lambda$-module for any $B$-module $BY$. (In general, they are not left $\Lambda$-modules.) In this case, one has extra functors:

- $\text{Z}_A : A\text{-Mod} \rightarrow \Lambda_{(0,0)}\text{-Mod}$, $X \mapsto (X)_{0,0}'$.
- $\text{Z}_B : B\text{-Mod} \rightarrow \Lambda_{(0,0)}\text{-Mod}$, $Y \mapsto (\frac{0}{0})_{0,0}'$.

**Theorem 2.2.** ([GrP, 2.4]) *There are recollements of abelian categories (in the sense of [FP]):*

\[
\begin{array}{c}
A\text{-Mod} \xleftarrow{\text{C}_A} \Lambda_{(0,0)}\text{-Mod} \xrightarrow{\text{U}_B} \text{B-Mod} \\
\xleftarrow{\text{K}_A} \Lambda_{(0,0)}\text{-Mod} \xrightarrow{\text{H}_B} \text{B-Mod}
\end{array}
\]

and

\[
\begin{array}{c}
\text{B-Mod} \xleftarrow{\text{C}_B} \Lambda_{(0,0)}\text{-Mod} \xrightarrow{\text{U}_A} \text{A-Mod} \\
\xleftarrow{\text{K}_B} \Lambda_{(0,0)}\text{-Mod} \xrightarrow{\text{H}_A} \text{A-Mod}
\end{array}
\]

2.5. **Projective (injective) modules.** A left $\Lambda_{(\phi, \psi)}$-module $(L_1 L_2)_{f,g}$ is projective if and only if $(L_1 L_2)_{f,g} \cong T_AP \oplus T_BQ$ for some $P \in A\mathcal{P}$ and $Q \in B\mathcal{P}$; and it is injective if and only if $(L_1 L_2)_{f,g} \cong H_AI \oplus H_BJ$ for some $I \in A\mathcal{I}$ and $J \in B\mathcal{I}$.

Thus, if $\phi = 0 = \psi$, a left $\Lambda_{(0,0)}$-module $(L_1 L_2)_{f,g}$ is projective if and only if

\[
(L_1 L_2)_{f,g} \cong (L_1 P_{M@A^P})_{1,0} \oplus (N@B^Q)_{0,1} = (\frac{P@\text{Hom}_B(M,J)}{Q}(\text{Hom}_A(N,I)\oplus J))_{0, 0}'(\frac{0}{0}, \frac{0}{0})
\]

for some $P \in A\mathcal{P}$ and $Q \in B\mathcal{P}$; and it is injective if and only if

\[
(L_1 L_2)_{f,g} \cong (L_1 \text{Hom}_A(N,I))_{0, t_{L_1}} \oplus (\text{Hom}_B(M,J))_{t_{L_2}, 0} \cong (\frac{I@\text{Hom}_B(M,J)}{\text{Hom}_A(N,I)\oplus J})_{0, 0}'(\frac{0}{0}, \frac{0}{0}, \frac{0}{0})
\]
for some $I \in A$ and $J \in B$. Using the second expression of $\Lambda$-modules, a left $\Lambda$-module $(L_1, L_2)\sim f, g$ is injective if and only if
\[
\begin{pmatrix} I_{\text{Hom}_A(N, I)} & 0_{1,0} \\
0_{0,1} & 1_{0,0} \end{pmatrix} \sim \begin{pmatrix} I_{\text{Hom}_B(M, J)} & 0_{0,1} \\
0_{1,0} & 1_{0,0} \end{pmatrix}.
\]
See [GrP, 3.1].

2.6. Cotorsion Pairs. Let $\mathcal{A}$ be an abelian category. A pair $(C, F)$ of classes of objects of $\mathcal{A}$ is a cotorsion pair (see [S]), if $C = \perp F$ and $F = C^\perp$.

A cotorsion pair $(C, F)$ is complete, if for any object $X \in \mathcal{A}$, there are exact sequences
\[
0 \to F \to C \to X \to 0, \quad \text{and} \quad 0 \to X \to F' \to C' \to 0,
\]
with $C, C' \in C$, and $F, F' \in F$.

**Proposition 2.3.** ([EJ, 7.17]) Let $\mathcal{A}$ be an abelian category with enough projective objects and enough injective objects, and $(C, F)$ a cotorsion pair in $\mathcal{A}$. Then the following are equivalent:

(i) $(C, F)$ is complete;

(ii) For any object $X \in \mathcal{A}$, there is an exact sequence $0 \to F \to C \to X \to 0$ with $C \in C$ and $F \in F$;

(iii) For any object $X \in \mathcal{A}$, there is exact sequence $0 \to X \to F' \to C' \to 0$ with $C' \in C$ and $F' \in F$.

A cotorsion pair $(C, F)$ is cogenerated by a set $S$, if $F = S^\perp$. One should be careful with this terminology: in some reference, e.g., in [GT, p.99], it is also called “generated by”.

**Proposition 2.4.** Let $\mathcal{A}$ be a Grothendieck category with enough projective objects. Then any cotorsion pair in $\mathcal{A}$ cogenerated by a set is complete.

This result is given in [ET2, Theorem 10] for the module category of a ring, and has the generality by [SS] or [Bec, 1.2.2].

A cotorsion pair $(C, F)$ is hereditary, if $C$ is closed under the kernel of epimorphisms, and $F$ is closed under the cokernel of monomorphisms.

**Proposition 2.5.** ([GR, 1.2.10]) Let $\mathcal{A}$ be an abelian category with enough projective objects and enough injective objects, and $(C, F)$ a cotorsion pair in $\mathcal{A}$. Then the following is equivalent:

(i) $(C, F)$ is hereditary;

(ii) $C$ is closed under the kernel of epimorphisms;

(iii) $F$ is closed under the cokernel of monomorphisms;

(iv) $\text{Ext}_A^2(C, F) = 0$;

(v) $\text{Ext}_A^i(C, F) = 0$ for $i \geq 1$.

The proof of Proposition 2.5 really needs the assumption that abelian category $\mathcal{A}$ has enough projective objects and enough injective objects.
2.7. **Model structures.** A closed model structure on a category and a model category are introduced by D. Quillen [Q1] (see also [Q2]). A *closed model structure* on a category $\mathcal{M}$ is a triple $(\text{Cofib}(\mathcal{M}), \text{Fib}(\mathcal{M}), \text{Weq}(\mathcal{M}))$ of classes of morphisms, where the morphisms in the three classes are respectively called cofibrations, fibrations, and weak equivalences, satisfying (CM1) – (CM4):

1. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms in $\mathcal{M}$. If two of the morphisms $f$, $g$, $gf$ are weak equivalences, then so is the third.
2. If $f$ is a retract of $g$, and $g$ is a cofibration (fibration, weak equivalence), then so is $f$.
3. Given a commutative square

$$
\begin{array}{ccc}
A & \xrightarrow{a} & X \\
\downarrow{s} & & \downarrow{p} \\
B & \xrightarrow{b} & Y
\end{array}
$$

where $i \in \text{Cofib}(\mathcal{M})$ and $p \in \text{Fib}(\mathcal{M})$, if either $i \in \text{Weq}(\mathcal{M})$ or $p \in \text{Weq}(\mathcal{M})$, then there exists a morphism $s : B \to X$ such that $a = si$, $b = ps$.
4. Any morphism $f : X \to Y$ has a factorization $f = pi$ with $i \in \text{Cofib}(\mathcal{M}) \cap \text{Weq}(\mathcal{M})$ and $p \in \text{Fib}(\mathcal{M})$; and also $f = p'i'$ with $i' \in \text{Cofib}(\mathcal{M})$ and $p' \in \text{Fib}(\mathcal{M}) \cap \text{Weq}(\mathcal{M})$.

Following [H1] (also [Hir]), we will call a closed model structure just a *model structure*. A category is *bicomplete* if it has an arbitrary small limits and colimits. A *model category* is a bicomplete category equipped with a model structure ([H1, 1.1.4]).

For a model structure $(\text{Cofib}(\mathcal{M}), \text{Fib}(\mathcal{M}), \text{Weq}(\mathcal{M}))$ on category $\mathcal{M}$ with zero object, an object $X$ is *trivial* if $0 \to X$ is a weak equivalence, or, equivalently, $X \to 0$ is a weak equivalence. It is *cofibrant* if $0 \to X$ is a cofibration, and it is *fibrant* if $X \to 0$ is a fibration. For a model structure on category $\mathcal{M}$ with zero object ($\mathcal{M}$ is not necessarily a model category), Quillen’s homotopy category is the localization $\mathcal{M}[\text{Weq}(\mathcal{M})^{-1}]$, and is denoted by $\text{Ho}(\mathcal{M})$.

A model structure on an abelian category is an *abelian model structure*, provided that cofibrations are exactly monomorphisms with cofibrant cokernel, and that fibrations are exactly epimorphisms with fibrant kernel. This is equivalent to the original definition ([H2, 2.1, 4.2]), see also [Bec, 1.1.3]. An *abelian model category* is a bicomplete abelian category equipped with an abelian model structure.

2.8. **Hovey triples.** Let $\mathcal{A}$ be an abelian category. A triple $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ of classes of objects is a *Hovey triple* in $\mathcal{A}$ (see [H2]), if it satisfies the conditions:

1. The class $\mathcal{W}$ is *thick*, i.e., $\mathcal{W}$ is closed under direct summands, and if two out of three terms in a short exact sequence are in $\mathcal{W}$, then so is the third;
2. $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are complete cotorsion pairs.

**Theorem 2.6.** (Hovey correspondence) ([H2, Theorem 2.2]; also [BR, VIII 3.5, 3.6]) Let $\mathcal{A}$ be an abelian category. Then there is a one-to-one correspondence between abelian model structures
and Hovey triples in $\mathcal{A}$, given by

$$(\text{Cofib}(\mathcal{A}), \text{Fib}(\mathcal{A}), \text{Weq}(\mathcal{A})) \mapsto (\mathcal{C}, \mathcal{F}, \mathcal{W})$$

where $\mathcal{C} = \{\text{cofibrant objects}\}$, $\mathcal{F} = \{\text{fibrant objects}\}$, $\mathcal{W} = \{\text{trivial objects}\}$, with inverse

$$(\mathcal{C}, \mathcal{F}, \mathcal{W}) \mapsto (\text{Cofib}(\mathcal{A}), \text{Fib}(\mathcal{A}), \text{Weq}(\mathcal{A}))$$

where

- $\text{Cofib}(\mathcal{A}) = \{\text{monomorphisms with cokernel in } \mathcal{C}\}$
- $\text{Fib}(\mathcal{A}) = \{\text{epimorphisms with kernel in } \mathcal{F}\}$
- $\text{Weq}(\mathcal{A}) = \{\pi | i \text{ is monic, } \text{Coker } i \in \mathcal{C} \cap \mathcal{W}, p \text{ is epic, } \text{Ker } p \in \mathcal{F} \cap \mathcal{W}\}$.

We stress that, in Theorem 2.6, $\mathcal{A}$ is not necessarily to be bicomplete: although this is assumed in [H2, Theorem 2.2], however, the proof given there does not use the assumption “bicomplete”. (In fact, one can also read this from lines of [Gil2] and [Gil3].)

A cofibrantly generated model category has been introduced in [H1, 2.1.17]. Let $\mathcal{A}$ be a Grothendieck category with enough projective objects. A Hovey triple $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ in $\mathcal{A}$ will be called cofibrantly generated, if cotorsion pairs $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are cogenerated by sets. Note that a Grothendieck category is always bicomplete (see e.g. [KS, 8.3.27]).

**Proposition 2.7.** ([Bec, 1.2.7; 1.2.2]) Let $\mathcal{A}$ be a Grothendieck category with enough projective objects. Then a Hovey triple $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ in $\mathcal{A}$ is cofibrantly generated if and only if the corresponding abelian model category $\mathcal{A}$ is cofibrantly generated.

2.9. **Hereditary Hovey triples.** A Hovey triple $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ is hereditary, if both $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are hereditary cotorsion pairs. Hereditary Hovey triples enjoy the following pleasant property.

**Theorem 2.8.** ([Bec, 1.1.14]; [BR, VIII 4.2]; [Gil4, 4.3]) Let $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ be a hereditary Hovey triple in abelian category $\mathcal{A}$. Then $\mathcal{C} \cap \mathcal{F}$ is a Frobenius category (with the canonical exact structure), with $\mathcal{C} \cap \mathcal{F} \cap \mathcal{W}$ as the class of projective-injective objects. The composition $\mathcal{C} \cap \mathcal{F} \to \mathcal{A} \to \text{Ho}(\mathcal{A})$ induces a triangle equivalence $\text{Ho}(\mathcal{A}) \cong (\mathcal{C} \cap \mathcal{F})/(\mathcal{C} \cap \mathcal{F} \cap \mathcal{W})$, where $(\mathcal{C} \cap \mathcal{F})/(\mathcal{C} \cap \mathcal{F} \cap \mathcal{W})$ is the stable category of $\mathcal{C} \cap \mathcal{F}$ modulo $\mathcal{C} \cap \mathcal{F} \cap \mathcal{W}$.

Note that the definition of $\text{Ho}(\mathcal{A})$ does not need that $\mathcal{A}$ is bicomplete. By this result, hereditary Hovey triples $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ with $(\mathcal{C} \cap \mathcal{F}) \not\cong \mathcal{W}$ are of special interest.

Two cotorsion pairs $(\Theta, \Theta^\perp)$ and $(\perp \Upsilon, \Upsilon)$ are compatible (see [Gil3]), if $\text{Ext}^1_{\mathcal{A}}(\Theta, \Upsilon) = 0$ and $\Theta \cap \Theta^\perp = \perp \Upsilon \cap \Upsilon$. The compatibility depends on the order of two cotorsion pairs. This terminology of compatible is taken from [HJ].

J. Gillespie gives the following approach to construct all the hereditary Hovey triples.

**Theorem 2.9.** (Gillespie Theorem) ([Gil3, 1.1]) Let $\mathcal{A}$ be an abelian category, and $(\Theta, \Theta^\perp)$ and $(\perp \Upsilon, \Upsilon)$ complete hereditary cotorsion pairs in $\mathcal{A}$. If $(\Theta, \Theta^\perp)$ and $(\perp \Upsilon, \Upsilon)$ are compatible,
They will be called the monomorphism category. In particular, we put

$$W = \{W \in \mathcal{A} \mid \exists \text{ an exact sequence } 0 \to P \to F \to W \to 0 \text{ with } F \in \Theta, \ P \in \Upsilon\}$$

$$= \{W \in \mathcal{A} \mid \exists \text{ an exact sequence } 0 \to W \to P' \to F' \to 0 \text{ with } P' \in \Theta, \ F' \in \Upsilon\}.$$  

Conversely, any hereditary Hovey triple in $\mathcal{A}$ arises in this way.

For later applications, we will call the hereditary Hovey triple $(\perp, \Theta, \perp)$ in Theorem 2.9 the Gillespie-Hovey triple, induced by compatible complete hereditary cotorsion pairs $(\Theta, \Theta^\perp)$ and $(\perp, \Upsilon)$. Thus, the Gillespie-Hovey triples are exactly hereditary Hovey triples.

2.10. Gorenstein rings. A noetherian ring $R$ is a Iwanaga-Gorenstein ring, or a Gorenstein ring, if $\text{inj.d}im_R R < \infty$ and $\text{inj.d}im R_R < \infty$. In this case, it is well-known that

- $\text{inj.d}im_R R = \text{inj.d}im R_R$ and $R^{\mathcal{P} < \infty} = R^{I < \infty};$

- ([EJ, p. 211]) If $\text{inj.d}im R_R \leq n$, then $R^\mathcal{P} < \infty = R^\mathcal{P} \leq n = R^{I \leq n} = R^{I < \infty}$, where $R^\mathcal{P} \leq n$ (in $R^{I \leq n}$, respectively) is the full subcategory of $R\text{-Mod}$ consisting of modules $X$ with $\text{proj.d}im X \leq n$ (in $\text{inj.d}im X \leq n$, respectively).

- ([EJ, 11.5.3]) $\text{GP}(R) = \perp \geq 1 \ R^\mathcal{P} = \perp \geq 1 \ R^{\mathcal{P} < \infty}$, and $\text{GI}(R) = R^{I \geq 1} = (R^{I < \infty})^{\perp \geq 1};$

- ([H2, 8.6]) $(\text{GP}(R), \ R\text{-Mod}, \ R^{\mathcal{P} < \infty})$ and $(\text{GP}(R), \ R^{\mathcal{P} < \infty})$ are hereditary Hovey triples in $R\text{-Mod}$. In particular, $(\text{GP}(R), \ R^{\mathcal{P} < \infty})$ are complete hereditary cotorsion pairs.

3. (Hereditary) cotorsion pairs in Morita rings

3.1. Three classes of modules over a Morita ring. Let $\Lambda = (\frac{A}{B}, \frac{B}{A})$ be a Morita ring. For a class $\mathcal{X}$ of $A$-modules and a class $\mathcal{Y}$ of $B$-modules, define

$$\left(\frac{\mathcal{X}}{\mathcal{Y}}\right) = \{(\frac{X}{Y})_{f,g} \in \text{A-Mod} \mid X \in \mathcal{X}, \ Y \in \mathcal{Y}\};$$

$$\Delta(\mathcal{X}, \mathcal{Y}) = \{(\frac{L_1}{L_2})_{f,g} \in \text{A-Mod} \mid f : M \otimes_A L_1 \to L_2 \text{ and } g : N \otimes_B L_2 \to L_1$$

are monomorphisms, $\text{Coker} f \in \mathcal{Y}, \ \text{Coker} g \in \mathcal{X}\};$$

$$\nabla(\mathcal{X}, \mathcal{Y}) = \{(\frac{L_1}{L_2})_{f,g} \in \text{A-Mod} \mid \tilde{f} : L_1 \to \text{Hom}_B(M, L_2) \text{ and } \tilde{g} : L_2 \to \text{Hom}_A(N, L_1)$$

are epimorphisms, $\text{Ker} \tilde{f} \in \mathcal{X}, \ \text{Ker} \tilde{g} \in \mathcal{Y}\}.$

In particular, we put

$$\text{Mon}(\Lambda) : = \Delta(\text{A-Mod}, \ B\text{-Mod}) = \{(\frac{L_1}{L_2})_{f,g} \in \text{A-Mod} \mid f \text{ and } g \text{ are monomorphisms}\};$$

$$\text{Epi}(\Lambda) : = \nabla(\text{A-Mod}, \ B\text{-Mod}) = \{(\frac{L_1}{L_2})_{f,g} \in \text{A-Mod} \mid \tilde{f} \text{ and } \tilde{g} \text{ are epimorphisms}\}.$$

They will be called the monomorphism category and the epimorphism category of $\Lambda$, respectively.

It is clear that if $M \otimes_A N = 0 = N \otimes_B M$, then

$$\Delta(\mathcal{A}\mathcal{P}, \ \mathcal{B}\mathcal{P}) = \mathcal{A}\mathcal{P} \quad \text{and} \quad \nabla(\mathcal{A}I, \ \mathcal{B}I) = \mathcal{A}I.$$
3.2. Constructions on (hereditary) cotorsion pairs.

**Theorem 3.1.** Let $\Lambda = (\underline{A}, N, B)$ be a Morita ring with $\phi = 0 = \psi$. Let $(U, X)$ and $(V, Y)$ be cotorsion pairs in $A$-$Mod$ and $B$-$Mod$, respectively.

1. If $\text{Tor}_1^A(M, U) = 0 = \text{Tor}_1^B(N, V)$, then $(\Delta(U, V), (\Delta(U, V))^\perp)$ is a cotorsion pair in $\Lambda$-$Mod$; and moreover, it is hereditary if and only if so are $(U, X)$ and $(V, Y)$.

2. If $\text{Ext}_1^A(N, X) = 0 = \text{Ext}_1^B(M, Y)$, then $(\Delta(U, V), (\Delta(U, V))^\perp)$ is a cotorsion pair in $\Lambda$-$Mod$; and moreover, it is hereditary if and only if so are $(U, X)$ and $(V, Y)$.

**Theorem 3.2.** Let $\Lambda = (\underline{A}, N, B)$ be a Morita ring with $M \otimes_A N = 0 = N \otimes_B M$. Let $(U, X)$ and $(V, Y)$ be cotorsion pairs in $A$-$Mod$ and $B$-$Mod$, respectively. Then

1. $(\Delta(U, V), \Delta(U, V))^\perp)$ is a cotorsion pair in $\Lambda$-$Mod$.

Moreover, if $M_A$ and $N_B$ are flat, then $(\Delta(U, V), (\Delta(U, V))^\perp)$ is hereditary if and only if so are $(U, X)$ and $(V, Y)$.

2. $(\Delta(U, V), (\Delta(U, V))^\perp)$ is a cotorsion pair in $\Lambda$-$Mod$.

Moreover, if $BM$ and $AN$ are projective, then $(\Delta(U, V), (\Delta(U, V))^\perp)$ is hereditary if and only if so are $(U, X)$ and $(V, Y)$.

**Notation 3.3.** For convenience, we will call the cotorsion pairs in Theorem 3.1 the cotorsion pairs in Series I; and the ones in Theorem 3.2 the cotorsion pairs in Series II.

**Example 3.4.** The condition “$M \otimes_A N = 0 = N \otimes_B M$” in Theorem 3.2 can not be weakened as “$\phi = 0 = \psi$”, in general.

For example, taking $\Lambda = (\underline{A}, A, A)$ with $A \neq 0$ and $\phi = 0 = \psi$. Then for any class $U \subseteq A$-$Mod$ and any class $V \subseteq B$-$Mod$, one has $\Delta(U, V) = \{0\}$. In fact, if $(L_1, L_2)_{fg} \in \Delta(U, V)$, then $fg = 0 = gf$. However, $f : L_1 \to L_2$ and $g : L_2 \to L_1$ are monomorphisms. Thus $L_1 = 0 = L_2$.

But $\{0\}$ can not occur in any cotorsion pair (since $\Lambda \neq 0$).

We will compare the cotorsion pairs in Series I with the corresponding cotorsion pairs in Series II. Comparing cotorsion pair $(\Delta(U, V), (\Delta(U, V))^\perp)$ in Theorem 3.1 (1) with cotorsion pair $(\Delta(U, V), (\Delta(U, V))^\perp)$ in Theorem 3.2 (1), we get the assertion (1) below; comparing cotorsion pair $(\Delta(U, V), (\Delta(U, V))^\perp)$ in Theorem 3.1 (2) with cotorsion pair $(\Delta(U, V), (\Delta(U, V))^\perp)$ in Theorem 3.2 (2), we get the assertion (2) below.

**Theorem 3.5.** Let $\Lambda = (\underline{A}, N, B)$ be a Morita ring with $M \otimes_A N = 0 = N \otimes_B M$. Suppose that $(U, X)$ and $(V, Y)$ are cotorsion pairs in $A$-$Mod$ and $B$-$Mod$, respectively.

1. If $\text{Tor}_1^A(M, U) = 0 = \text{Tor}_1^B(N, V)$, then the cotorsion pairs $(\Delta(U, V), (\Delta(U, V))^\perp)$ in $\Lambda$-$Mod$ have a relation $\Delta(U, V)^\perp \subseteq (\Delta(U, V))^\perp$, or equivalently, $\Delta(U, V)^\perp \subseteq \Delta(U, V)$. 


(2) If $\text{Ext}^1_A(N, X) = 0 = \text{Ext}^1_B(M, Y)$, then the cotorsion pairs

$\left(\left(\frac{U}{V}, \frac{Y}{Z}\right), \left(\frac{U}{V}, \frac{Y}{Z}\right)^\perp\right)$

in $A\text{-Mod}$ have a relation $\perp \nabla(X, Y) \subseteq \left(\frac{U}{V}, \frac{Y}{Z}\right)$, or equivalently, $\left(\frac{U}{V}, \frac{Y}{Z}\right)^\perp \subseteq \nabla(X, Y)$.

**Remark 3.6.** If $M = 0$ or $N = 0$, then Theorems 3.1, 3.2 and 3.5 have been obtained by R. M. Zhu, Y. Y. Peng and N. Q. Ding [ZPD]. In particular, in that case one has

$\left(\left(\frac{U}{V}, \frac{Y}{Z}\right), \left(\frac{U}{V}, \frac{Y}{Z}\right)^\perp, \nabla(X, Y), \nabla(X, Y)^\perp\right)$

and

$\left(\left(\frac{U}{V}, \frac{Y}{Z}\right), \left(\frac{U}{V}, \frac{Y}{Z}\right)^\perp, \nabla(X, Y), \nabla(X, Y)^\perp\right)$.

See [ZPD, Proposition 3.7]. But, in general, they are not true! See Example 4.3.

### 3.3. Induced isomorphisms between $\text{Ext}^1$.

To prove Theorems 3.1 we need some preparations. In the following lemma, functors $F$ and $G$ are not required to be exact. This is important for applications.

**Lemma 3.7.** Let $R$ and $S$ be rings, $(F, G)$ an adjoint pair with $F : R\text{-Mod} \rightarrow S\text{-Mod}$.

1. For an $X \in R\text{-Mod}$, if $0 \rightarrow K \rightarrow P \rightarrow X \rightarrow 0$ is exact with $P$ projective, such that $0 \rightarrow FK \rightarrow FP \rightarrow FX \rightarrow 0$ is exact with $FP$ projective, then $\text{Ext}^1_R(FX, Y) \cong \text{Ext}^1_S(X, GY)$, $\forall Y \in S\text{-Mod}$.

2. For a $Y \in S\text{-Mod}$, if $0 \rightarrow Y \rightarrow I \rightarrow C \rightarrow 0$ is exact with $I$ injective, such that $0 \rightarrow GY \rightarrow GI \rightarrow GC \rightarrow 0$ is exact with $GI$ injective, then $\text{Ext}^1_S(FX, Y) \cong \text{Ext}^1_R(X, GY)$, $\forall X \in R\text{-Mod}$.

**Proof.** (1) Applying $\text{Hom}_R(-, GY)$ to $0 \rightarrow K \rightarrow P \rightarrow X \rightarrow 0$ and applying $\text{Hom}_S(-, Y)$ to $0 \rightarrow FK \rightarrow FP \rightarrow FX \rightarrow 0$, one gets a commutative diagram with exact rows

$$
\begin{array}{ccc}
\text{Hom}_R(P, \text{GY}) & \rightarrow & \text{Hom}_R(K, \text{GY}) \\
\text{Hom}_S(FP, Y) & \rightarrow & \text{Ext}^1_S(FX, Y) \\
\| & & \| \\
\text{Hom}_S(FK, Y) & \rightarrow & \text{Ext}^1_S(FX, Y) \rightarrow 0
\end{array}
$$

Then the assertion follows from the Five Lemma.

The assertion (2) is the dual of (1). □

**Lemma 3.8.** Let $\Lambda = \left(\frac{A}{B}\right)$ be a Morita ring with $\phi = \psi$, $X \in A\text{-Mod}$ and $Y \in B\text{-Mod}$. Then for any $L = \left(\frac{L_1}{L_2}\right)_{f,g} \in A\text{-Mod}$ one has

1. If $\text{Tor}^A_1(M, X) = 0$, then $\text{Ext}^1_A(T_A X, L) \cong \text{Ext}^1_A(X, U_A L)$.

2. If $\text{Tor}^B_1(N, Y) = 0$, then $\text{Ext}^1_A(T_B Y, L) \cong \text{Ext}^1_B(Y, U_B L)$.

3. If $\text{Ext}^1_A(N, X) = 0$, then $\text{Ext}^1_A(A X, L) \cong \text{Ext}^1_A(L, H_A X)$.

4. If $\text{Ext}^1_B(M, Y) = 0$, then $\text{Ext}^1_B(U_B L, Y) \cong \text{Ext}^1_A(L, H_B Y)$.
Proof. We only justify (1) and (3). The assertions (2) and (4) can be similarly proved.

(1) Take an exact sequence $0 \to K \to P \to X \to 0$ with $P$ projective. Since by assumption $\text{Tor}^A_1(M, X) = 0$, one has an exact sequence of $B$-modules

$$0 \to M \otimes_A K \to M \otimes_A P \to M \otimes_A X \to 0.$$

Applying $T_A$ (note that $T_A$ is not an exact functor), one gets an exact sequence of $\Lambda$-modules

$$0 \to \left( M \otimes_A K \right)_{1,0} \to \left( M \otimes_A P \right)_{1,0} \to \left( M \otimes_A X \right)_{1,0} \to 0$$

where $\left( M \otimes_A P \right)_{1,0}$ is a projective $\Lambda$-module. Consider adjoin pair $(T_A, U_A)$ between $\text{A-Mod}$ and $\text{A-Mod}$. Applying Lemma 3.7(1) to $X$, one gets $\text{Ext}^1_A(T_A X, L) \cong \text{Ext}^1_A(X, U_A L)$.

(3) Take an exact sequence $0 \to X \to I \to C \to 0$ with $I$ injective. Since by assumption $\text{Ext}^1_A(N, X) = 0$, one has an exact sequence of $B$-modules

$$0 \to \text{Hom}_A(N, X) \to \text{Hom}_A(N, I) \to \text{Hom}_A(N, C) \to 0.$$

Applying $H_A$ (note that $H_A$ is also not an exact functor) one gets an exact sequence of $\Lambda$-modules

$$0 \to \left( \text{Hom}_A(N, X) \right)_{0, \epsilon_X} \to \left( \text{Hom}_A(N, I) \right)_{0, \epsilon_I} \to \left( \text{Hom}_A(N, C) \right)_{0, \epsilon_C} \to 0$$

where $\left( \text{Hom}_A(N, I) \right)_{0, \epsilon_I}$ is an injective $\Lambda$-module. Consider adjoin pair $(U_A, H_A)$ between $\text{A-Mod}$ and $\text{A-Mod}$. Applying Lemma 3.7(2) to $X$, one gets $\text{Ext}^1_A(U_A L, X) \cong \text{Ext}^1_A(L, H_A X)$. \qed

Lemma 3.9. Let $\Lambda = (\underline{M_B^N})$ be a Morita ring with $\phi = \psi$, $X \subseteq \text{A-Mod}$, and $Y \subseteq \text{B-Mod}$.

1. If $\text{Tor}^A_1(M, X) = 0 = \text{Tor}^B_1(N, Y)$, then $\left( \frac{X}{Y} \right)^{\perp} = \text{Ext}_A^1(X, \perp) \cap \text{Ext}_B^1(Y, \perp)$.

2. If $\text{Ext}_A^1(N, X) = 0 = \text{Ext}_B^1(M, Y)$, then $\left( \frac{N}{M} \right)^{\perp} = \text{Ext}_A^1(N, \perp) \cap \text{Ext}_B^1(M, \perp)$.

Proof. (1) By definition $L = \left( \frac{L_1}{L_2} \right)_{f,g} \in \left( \frac{X}{Y} \right)^{\perp}$ if and only if $L_1 \in X^{\perp}$ and $L_2 \in Y^{\perp}$, or equivalently, $\text{Ext}_A^1(X, L_1) = 0 = \text{Ext}_B^1(Y, L_2)$. Since by assumption $\text{Tor}^B_1(M, X) = 0 = \text{Tor}^A_1(N, Y)$, it follows from Lemma 3.8(1) and (2) that $\text{Ext}_A^1(X, L_1) \cong \text{Ext}_B^1(T_A(X), L)$ and $\text{Ext}_B^1(Y, L_2) \cong \text{Ext}_A^1(T_B(Y), L)$. Thus, $L \in \left( \frac{L_1}{L_2} \right)_{f,g} \in \left( \frac{X}{Y} \right)^{\perp}$ if and only if $\text{Ext}_A^1(T_A(X), L) = 0 = \text{Ext}_B^1(T_B(Y), L)$

i.e., $L \in T_A(X)^{\perp} \cap T_B(Y)^{\perp}$.

(2) Similarly, $L = \left( \frac{L_1}{L_2} \right)_{f,g} \in \left( \frac{N}{M} \right)^{\perp}$ if and only if $\text{Ext}_A^1(L_1, \perp) = 0 = \text{Ext}_B^1(L_2, \perp) = 0$. Since $\text{Ext}_A^1(N, X) = 0$ and $\text{Ext}_B^1(M, Y) = 0$, by Lemma 3.8(3) and (4), $\text{Ext}_A^1(L_1, \perp) \cong \text{Ext}_B^1(L, H_A(X))$ and $\text{Ext}_B^1(L_2, \perp) \cong \text{Ext}_A^1(L, H_B(Y))$. Thus, $L \in \left( \frac{N}{M} \right)^{\perp}$ if and only if $L \in \text{Ext}^1_A(L, H_A(X)) \cap \text{Ext}^1_B(L, H_B(Y)).$ \qed

3.4. Proof of Theorem 3.1 (1) To prove that $\left( \frac{U}{V} \right)^{\perp}$ is a cotorsion pair, it suffices to show $\left( \frac{U}{V} \right)^{\perp} = \left( \frac{U}{V} \right)^{\perp}$.

In fact, since $(U, X)$ and $(V, Y)$ are cotorsion pairs, it follows that $\left( \frac{U}{V} \right)^{\perp} = \left( \frac{U}{V} \right)^{\perp}$. Since by assumption $\text{Tor}_A^1(M, U) = 0 = \text{Tor}_B^1(N, V)$, it follows from Lemma 3.7(1) that $\left( \frac{U}{V} \right)^{\perp} = \text{Ext}_A(U) \cap \text{Ext}_B(V)^{\perp} = \left( \text{Ext}_A(U) \cup \text{Ext}_B(V)^{\perp} \right)^{\perp}$. 

Thus

\[(\perp (\mathcal{X}^\perp))^{\perp} = (\perp \left(\perp (\mathcal{U}^\perp)\right)) = \perp[(\mathcal{T}_A(\mathcal{U}) \cup \mathcal{T}_B(\mathcal{V}))^\perp] = (\mathcal{T}_A(\mathcal{U}) \cup \mathcal{T}_B(\mathcal{V}))^{\perp} = (\mathcal{X}^\perp)\]

here one uses the fact \((\perp (\mathcal{S}^\perp))^{\perp} = \mathcal{S}^{\perp}\), for any class \(\mathcal{S}\) of modules.

If \((\mathcal{U}, \mathcal{X})\) and \((\mathcal{V}, \mathcal{Y})\) are hereditary, then \(\mathcal{X}\) and \(\mathcal{Y}\) are closed under taking the cokernels of monomorphisms. By the construction of \((\mathcal{X}^\perp)\), it is clear that \((\mathcal{X}^\perp)\) is also closed under taking the cokernels of monomorphisms, i.e., \((\perp (\mathcal{X}^\perp), (\mathcal{X}^\perp))\) is hereditary.

Conversely, let \((\perp (\mathcal{X}^\perp), (\mathcal{Y}^\perp))\) be hereditary. Using functors \(Z_A\) and \(Z_B\), one sees that \(\mathcal{X}\) and \(\mathcal{Y}\) are closed under taking the cokernels of monomorphisms, i.e., \((\mathcal{X}, (\mathcal{Y}^\perp))\) are hereditary.

(2) Similarly, it suffices to show \((\mathcal{U}^\perp) = (\perp (\mathcal{U}^\perp))\). In fact, by Lemma 3.9(2) one has

\[
(\mathcal{U}^\perp) = (\mathcal{X}^\perp) = \perp H_A(\mathcal{X}) \cap \perp H_B(\mathcal{Y}) = \perp(H_A(\mathcal{X}) \cup H_B(\mathcal{Y})).
\]

Thus

\[
\perp((\mathcal{U}^\perp)) = \perp\left(\mathcal{X}^\perp\right) = \perp\left[(\perp H_A(\mathcal{X}) \cup H_B(\mathcal{Y}))\right] = \perp(H_A(\mathcal{X}) \cup H_B(\mathcal{Y})) = (\mathcal{U}^\perp)
\]

here one uses the fact \((\perp (\mathcal{S}^\perp))^{\perp} = (\mathcal{S}^\perp)\), for any class \(\mathcal{S}\) of modules.

If \((\mathcal{U}, \mathcal{X})\) and \((\mathcal{V}, \mathcal{Y})\) are hereditary, then \(\mathcal{U}\) and \(\mathcal{V}\) are closed under taking the kernels of epimorphisms. By construction, \((\mathcal{U}^\perp)\) is also closed under taking the kernels of epimorphisms, i.e., \((\mathcal{U}^\perp), (\mathcal{U}^\perp))\) is hereditary. One can see the converse, by using functors \(Z_A\) and \(Z_B\).

**3.5. Induced isomorphisms between \(\text{Ext}^1\) (continued).**

**Lemma 3.10.** Let \(\Lambda = (\begin{array}{cc} A & N_B \\ M & B \end{array})\) be a Morita ring with \(\phi = 0 = \psi\), \(L = \left(\begin{array}{c}\ell_1 \\ \ell_2 \end{array}\right)_{f,g}\) a \(\Lambda\)-module.

1. If \(g\) is a monomorphism, then \(\text{Ext}_A^1(C_A L, X) \cong \text{Ext}_A^1(L, Z_A X), \forall X \in A\text{-Mod}.
2. If \(f\) is a monomorphism, then \(\text{Ext}_B^1(C_B L, Y) \cong \text{Ext}_B^1(L, Z_B Y), \forall Y \in B\text{-Mod}.
3. If \(\tilde{f}\) is an epimorphism, then \(\text{Ext}_A^1(Z_A X, L) \cong \text{Ext}_A^1(X, K_A L), \forall X \in A\text{-Mod}.
4. If \(\tilde{g}\) is an epimorphism, then \(\text{Ext}_B^1(Z_B Y, L) \cong \text{Ext}_B^1(Y, K_B L), \forall Y \in B\text{-Mod}.

**Proof.** We only prove (1) and (3). The assertions (2) and (4) can be similarly proved.

1. Taking an exact sequence

\[
0 \longrightarrow \left(\begin{array}{c} K_1 \\ K_2 \end{array}\right) \xrightarrow{\begin{array}{c} i_1 \\ i_2 \end{array}} \left(\begin{array}{c} p_1 \\ p_2 \end{array}\right) \xrightarrow{\begin{array}{c} p_1 \\ p_2 \end{array}} \left(\begin{array}{c} \ell_1 \\ \ell_2 \end{array}\right)_{f,g} \longrightarrow 0
\]
with \( \left( \frac{P_1}{P_2} \right)_{u,v} \) a projective module, one gets a commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
N \otimes_B K_2 & \overset{1 \otimes i_2}{\longrightarrow} & N \otimes_B P_2 & \overset{1 \otimes p_2}{\longrightarrow} & N \otimes_B L_2 & \longrightarrow & 0 \\
\downarrow t & & \downarrow v & & \downarrow g & & \\
0 & \longrightarrow & K_1 & \overset{i_1}{\longrightarrow} & P_1 & \overset{p_1}{\longrightarrow} & L_1 & \longrightarrow & 0
\end{array}
\]

Since \( g \) is a monomorphism, by Snake Lemma

\[
0 \longrightarrow \text{Coker } t \longrightarrow \text{Coker } v \longrightarrow \text{Coker } g \longrightarrow 0
\]

is exact. Since \( P = \left( \frac{P_1}{P_2} \right)_{u,v} \) is projective, \( \text{Coker } v = CA P \) is a projective \( A \)-module.

Consider adjoint pair \((CA, ZA)\) between \( A\text{-Mod} \) and \( A\text{-Mod} \). (Note that \( CA \) is not exact.) Applying Lemma \[3.7(1)\] to \( L = \left( \frac{L_1}{L_2} \right)_{f,g} \), one gets

\[
\text{Ext}_{A}^1(CAL, X) \cong \text{Ext}_{A}^1(L, ZAX), \quad \forall \ X \in A\text{-Mod}.
\]

(3) Similarly, taking an exact sequence of \( A\)-modules

\[
0 \longrightarrow \left( \frac{L_1}{L_2} \right)_{f,g} \longrightarrow \left( \frac{C_1}{C_2} \right)_{s,t} \longrightarrow 0
\]

with \( \left( \frac{L_1}{L_2} \right)_{u,v} \) an injective module, one gets a commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
M \otimes_A L_1 & \overset{1 \otimes \sigma_1}{\longrightarrow} & M \otimes_A I_1 & \overset{1 \otimes \pi_1}{\longrightarrow} & M \otimes_A C_1 & \longrightarrow & 0 \\
\downarrow f & & \downarrow u & & \downarrow s & & \\
0 & \longrightarrow & L_2 & \overset{\sigma_2}{\longrightarrow} & I_2 & \overset{\pi_2}{\longrightarrow} & C_2 & \longrightarrow & 0
\end{array}
\]

Using adjoint isomorphism, one gets a commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & L_1 & \overset{\sigma_1}{\longrightarrow} & I_1 & \overset{\pi_1}{\longrightarrow} & C_1 & \longrightarrow & 0 \\
\downarrow \tilde{f} & & \downarrow \tilde{u} & & \downarrow \tilde{s} & & \\
0 & \longrightarrow & \text{Hom}_B(M, L_2) & \overset{(M, \sigma_2)}{\longrightarrow} & \text{Hom}_B(M, I_2) & \overset{(M, \pi_2)}{\longrightarrow} & \text{Hom}_B(M, C_2)
\end{array}
\]

Since \( \tilde{f} \) is an epimorphism, by Snake Lemma that

\[
0 \longrightarrow \text{Ker } \tilde{f} \longrightarrow \text{Ker } \tilde{u} \longrightarrow \text{Ker } \tilde{s} \longrightarrow 0
\]

is exact. Since \( I = \left( \frac{I_1}{I_2} \right)_{u,v} \) is injective, \( \text{Ker } \tilde{u} = KA I \) is an injective \( A \)-module.

Consider adjoint pair \((ZA, KA)\) between \( A\text{-Mod} \) and \( A\text{-Mod} \). Applying Lemma \[3.7(2)\] to \( L = \left( \frac{L_1}{L_2} \right)_{f,g} \), one gets \( \text{Ext}_{A}^1(ZAX, L) \cong \text{Ext}_{A}^1(X, KA L), \quad \forall \ X \in A\text{-Mod} \). \( \square \)
3.6. **Key lemmas for Theorem 3.2** \[ \] The following lemma will play an important role in the proof of Theorem 3.2

**Lemma 3.11.** Let \( \Lambda = (A_N^M B) \) be a Morita ring with \( M \otimes_A N = 0 = N \otimes_B M \), \( X \subseteq A\)-Mod, and \( Y \subseteq B\)-Mod.

1. If \( X \supseteq \mathcal{I} \) and \( Y \supseteq \mathcal{I} \), then \( \Delta(\mathcal{I}^\perp, \mathcal{I}^\perp) = \mathcal{Z}_\Lambda(X) \cap \mathcal{Z}_\Lambda(Y) \).

2. If \( X \supseteq \mathcal{P} \) and \( Y \supseteq \mathcal{P} \), then \( \nabla(\mathcal{P}^\perp, \mathcal{P}^\perp) = \mathcal{Z}_\Lambda(X)^\perp \cap \mathcal{Z}_\Lambda(Y)^\perp \).

**Proof.** (1) Let \( L = (L_{\mathcal{I}}^1)_{f,g} \in \Delta(\mathcal{I}^\perp, \mathcal{I}^\perp) \). By definition \( f \) and \( g \) are monomorphisms, and \( \text{Coker} f \in \mathcal{I}^\perp \) and \( \text{Coker} g \in \mathcal{I}^\perp \). Since \( g \) is a monomorphism and \( \text{Ext}^1_\Lambda(C_A L, X) = \text{Ext}^1_\Lambda(\text{Coker} g, X) = 0 \), it follows from Lemma 3.10 that \( \text{Ext}^1_\Lambda(L, Z_A(X)) = 0 \), i.e., \( L \in \mathcal{Z}_\Lambda(X) \). Similarly, since \( f \) is a monomorphism and \( \text{Ext}^1_\Lambda(C_B L, Y) = \text{Ext}^1_\Lambda(\text{Coker} f, Y) = 0 \), by Lemma 3.11(2), \( \text{Ext}^1_\Lambda(L, Z_B(Y)) = 0 \), i.e., \( L \in \mathcal{Z}_\Lambda(X) \cap \mathcal{Z}_\Lambda(Y) \).

Conversely, let \( L = (L_{\mathcal{I}}^1)_{f,g} \in \mathcal{Z}_\Lambda(X)^\perp \cap \mathcal{Z}_\Lambda(Y)^\perp \), i.e., \( \text{Ext}^1_\Lambda(L, Z_A(X)) = 0 = \text{Ext}^1_\Lambda(L, Z_B(Y)) \).

**Claim 1:** \( \text{Hom}_\Lambda(g, X) : \text{Hom}_\Lambda(L_1, X) \rightarrow \text{Hom}_\Lambda(N \otimes_B L_2, X) \) is an epimorphism, for any module \( X \in \mathcal{X} \). In fact, for any \( A\)-map \( u : N \otimes_B L_2 \rightarrow X \), consider \( A\)-map \( g' = (\tilde{g}_0) : N \otimes_B L_2 \rightarrow X \otimes L_1 \) and the exact sequence of \( A\)-modules

\[ 0 \rightarrow X \rightarrow X \oplus L_1 \rightarrow L_1 \rightarrow 0. \]

Put \( f' = (0, f) : M \otimes_A (X \oplus L_1) \rightarrow L_2 \). Then \( (\frac{X \otimes L_1}{L_2})_{f',g'} \) is indeed a \( \Lambda \)-module. We stress that this is a place where one needs the assumption \( M \otimes_A N = 0 = N \otimes_B M \). Given any \( U \in A\)-Mod and \( V \in B\)-Mod, for arbitrary \( u \in \text{Hom}_\Lambda(M \otimes_A U, V) \) and \( v \in \text{Hom}_\Lambda(N \otimes_B V, U) \), \( (\frac{U}{V})_{u,v} \) is always a left \( \Lambda \)-module, since the conditions \( v(1_M \otimes u) = 0 \) and \( u(1_M \otimes v) = 0 \) automatically hold.

Then one can check that

\[ 0 \rightarrow (\tilde{X})_{1,0} \rightarrow (\frac{X \otimes L_1}{L_2})_{f',g'} \rightarrow (\frac{0}{1})_{L_2} \rightarrow 0 \]

is an exact sequence of \( \Lambda \)-modules. Since \( (\frac{X}{0})_{1,0} = Z_A X \in Z_A(X) \) and \( L \in Z_A(X)^\perp \cap Z_B(Y)^\perp \), this exact sequence splits. Thus there is a \( \Lambda \)-map

\[ (\frac{g}{\tilde{g}})_{L_2} \rightarrow (\frac{X \otimes L_1}{L_2})_{f',g'} \]

such that \( (\frac{0}{1})_{\frac{g}{\tilde{g}}} = \text{Id}_{L_1} \). So \( b = \text{Id}_{L_1} \) and \( \beta = \text{Id}_{L_2} \). Thus one gets a commutative diagram

\[ N \otimes_B L_2 \rightarrow N \otimes_B L_2 \]

and hence \( u = ag \). This proves Claim 1.

**Claim 2:** \( g \) is a monomorphism. In fact, embedding \( N \otimes_B L_2 \) into an injective \( A\)-module one has a monomorphism \( i : N \otimes_B L_2 \rightarrow I \). By assumption \( I \in \mathcal{X} \), hence \( \text{Hom}_\Lambda(g, I) \) :
Homₐ(L₁, I) → Homₐ(N ⊗ₐ L₂, I) is an epimorphism, by **Claim 1.** Hence there is an A-map v : L₁ → I such that vg = i. Thus, g is a monomorphism.

Similar as **Claim 1,** one has

**Claim 3:** Homₐ(f, Y) : Homₐ(L₂, Y) → Homₐ(M ⊗ₐ L₁, Y) is an epimorphism, for any module Y ∈ Y.

Similar as **Claim 2,** one has

**Claim 4:** f is a monomorphism.

We omit the similar proof of **Claim 3** and **Claim 4.**

Now, since g and f are monic, by Lemma 3.10(1) and (2) one has

$$\text{Ext}^1_1(\text{Coker } g, X) = \text{Ext}^1_1(C_A L, X) \cong \text{Ext}^1_1(L, Z_A(X)) = 0,$$

$$\text{Ext}^1_1(\text{Coker } f, Y) = \text{Ext}^1_1(C_B L, Y) \cong \text{Ext}^1_1(L, Z_B(Y)) = 0$$

By definition, $L = (L₁/L₂)_{f:g} \in \Delta(½X, ½Y)$. This completes the proof of (1).

(2) This can be similarly proved, however, it is difficult to say that it is the dual of (1), thus we include a justification. It will be much convenient to use the second expression of a Λ-module.

Let $L = (L₁/L₂)_{f:g} \in \nabla(X^⊥, Y^⊥)$, i.e., $f$ and $g$ are epimorphisms, and Ker $f \in X^⊥$ and Ker $g \in Y^⊥$. Since $f$ is an epimorphism and $\text{Ext}^1_1(X, K_A L) = \text{Ext}^1_1(X, Ker f) = 0$, by Lemma 3.10(3), $L \in Z_A(½X)$. Similarly, since $g$ is an epimorphism and $\text{Ext}^1_1(Y, K_B L) = \text{Ext}^1_1(Y, Ker g) = 0$, by Lemma 3.10(4), $L \in Z_B(½Y)$. Thus, $L \in Z_A(½X) \cap Z_B(½Y)$.

Conversely, let $L = (L₁/L₂)_{f:g} \in Z_A(½X) \cap Z_B(½Y)$.

**Claim 1:** Homₐ(Y, g̃) : Homₐ(Y, L₂) → Homₐ(Y, Homₐ(N, L₁)) is an epimorphism, for any module Y ∈ Y. In fact, ∀ u ∈ Homₐ(Y, Homₐ(N, L₁)), consider B-map $g̃' := (u, g̃) : Y ⊕ L₂ → Homₐ(N, L₁)$. Thus $g̃' ∈ Homₐ(N ⊗ₐ (Y ⊕ L₂), L₁)$. Put $f' = (g̃') : L₁ → Homₐ(M, Y) ⊕ Homₐ(g, M, L₂).$ Thus $f' ∈ Homₐ(M ⊗ₐ L₁, Y ⊕ L₂).$ Since $M ⊗ₐ N = 0 = N ⊗ₐ M$, $(Y ⊕ L₂)_{f', g̃'}$ is indeed a Λ-module.

Then one has the exact sequence

$$0 → (L₁/L₂)_{f,g̃} \xrightarrow{(0)} (L₁/Y ⊗ₐ L₂)_{f, g′} \xrightarrow{(0, 0)} (0) → 0.$$  

(We stress that it is much convenient to use the second expression of A-modules. Otherwise, say, it is not direct to see that $(0)$ is a Λ-map.)

Since $(0)_{0,0} = Z_B Y ∈ Z_B(½Y)$ and $L \in Z_A(½X) \cap Z_B(½Y)$, this exact sequence splits, i.e., there is a Λ-map

$$(a, b) : (L₁/L₂)_{f,g̃} \rightarrow (L₁/L₂)_{f,g̃}$$
such that \( (\alpha, b) \left( \frac{1}{(i)} \right) = \mathrm{Id}_L \). So \( \alpha = \mathrm{Id}_{L_1} \) and \( b = \mathrm{Id}_{L_2} \). This gives the commutative diagram

\[
\begin{array}{ccc}
Y \oplus L_2 & \xrightarrow{(u, 1)} & L_2 \\
\tilde{g} = (u, g) & \downarrow & \tilde{g} \\
\text{Hom}_A(N, L_1) & \xrightarrow{\alpha} & \text{Hom}_A(N, L_1)
\end{array}
\]

commutes. Hence \( u = \tilde{g}a \). This proves Claim 1.

**Claim 2:** \( \tilde{g} \) is an epimorphism. In fact, taking a \( B \)-epimorphism \( q : Q \to \text{Hom}_A(N, L_1) \) with \( Q \) projective. Then \( Q \in \mathcal{Y} \), hence \( \text{Hom}_B(Q, \tilde{g}) : \text{Hom}_B(Q, L_2) \to \text{Hom}_B(Q, \text{Hom}_A(N, L_1)) \) is an epimorphism. So there is a \( B \)-map \( v : Q \to L_2 \) with \( q = \tilde{g}v \). This proves Claim 2.

Similarly, \( \text{Hom}_A(X, \tilde{f}) : \text{Hom}_A(X, L_1) \to \text{Hom}_A(X, \text{Hom}_B(M, L_2)) \) is an epimorphism for any \( X \in \mathcal{X} \); and \( \tilde{f} \) is an epimorphism.

It follows from Lemma 3.10(3) and (4) that

\[
\text{Ext}^1_A(\mathcal{X}, \text{Ker} \tilde{f}) = \text{Ext}^1_A(\mathcal{X}, K_A L) \cong \text{Ext}^1_A(Z_A(\mathcal{X}), L) = 0
\]

and that

\[
\text{Ext}^1_B(\mathcal{Y}, \text{Ker} \tilde{g}) = \text{Ext}^1_B(\mathcal{Y}, K_B L) \cong \text{Ext}^1_A(Z_B(\mathcal{Y}), L) = 0.
\]

By definition, \( L = \left( \frac{L_1}{L_2} \right)_{f, \tilde{g}} \in \mathcal{N}(\mathcal{X}^\perp, \mathcal{Y}^\perp) \). This completes the proof.

**Lemma 3.12.** Let \( \Lambda = (\frac{A}{M} \frac{N}{B}) \) be a Morita ring with \( M \otimes_A N = 0 = N \otimes_B M \).

1. Assume that \( M_A \) and \( N_B \) are flat modules. Then \( \Delta(\mathcal{U}, \mathcal{V}) \) is closed under the kernels of epimorphisms if and only if \( \mathcal{U} \) and \( \mathcal{V} \) are closed under the kernels of epimorphisms.

2. Assume that \( BM \) and \( AN \) are projective. Then \( \nabla(\mathcal{X}, \mathcal{Y}) \) is closed under the cokernels of monomorphisms if and only if \( \mathcal{X} \) and \( \mathcal{Y} \) are closed under the cokernels of monomorphisms.

**Proof.** (1) Assume that \( \mathcal{U} \) and \( \mathcal{V} \) are closed under the kernels of epimorphisms. Let \( 0 \to (\frac{L_1}{L_2})_{f, g} \to (\frac{M_1}{M_2})_{u, v} \to (\frac{N_1}{N_2})_{s, t} \to 0 \) be an exact sequence with \( (\frac{M_1}{M_2})_{u, v}, (\frac{N_1}{N_2})_{s, t} \in \Delta(\mathcal{U}, \mathcal{V}) \). Thus \( u, v, s, t \) are monomorphisms, \( \text{Coker} u \in \mathcal{V}, \text{Coker} v \in \mathcal{U}, \text{Coker} s \in \mathcal{V}, \text{and Coker} t \in \mathcal{U} \).

Since \( M_A \) is flat, one has the commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \to & M \otimes_A L_1 & \xrightarrow{1 \otimes \alpha} & M \otimes_A M_1 & \to & M \otimes_A N_1 & \to & 0 \\
\downarrow f & & \downarrow s & & \downarrow s & & \downarrow s & & \downarrow s & & \downarrow s & & 0.
\end{array}
\]

Since \( 1 \otimes \alpha \) and \( u \) are monomorphisms, so is \( f \). By Snake Lemma and the assumption that \( \mathcal{V} \) is closed under the kernels of epimorphisms, one knows that \( \text{Coker} f \in \mathcal{V} \). Similarly, \( g \) is a monomorphism and \( \text{Coker} g \in \mathcal{U} \). By definition \( (\frac{L_1}{L_2})_{f, g} \in \Delta(\mathcal{U}, \mathcal{V}) \). This proves that \( \Delta(\mathcal{U}, \mathcal{V}) \) is closed under the kernels of epimorphisms.

Conversely, using functors \( T_A \) and \( T_B \), one sees that \( \mathcal{U} \) and \( \mathcal{V} \) are closed under the kernels of epimorphisms.
(2) Assume that \( \mathcal{X} \) and \( \mathcal{Y} \) are closed under the cokernels of monomorphisms. Let \( 0 \rightarrow (L_1)_{f,g} \rightarrow (M_1)_{u,v} \rightarrow (N_1)_{s,t} \rightarrow 0 \) be an exact sequence of \( \Lambda \)-modules with \( (L_1)_{f,g} \in \nabla(\mathcal{X}, \mathcal{Y}) \) and \( (M_1)_{u,v} \in \nabla(\mathcal{X}, \mathcal{Y}) \). Thus \( \tilde{f}, \tilde{g}, \tilde{u}, \tilde{v} \) are epimorphisms, \( \text{Ker} \tilde{f} \in \mathcal{X}, \text{Ker} \tilde{g} \in \mathcal{Y}, \text{Ker} \tilde{u} \in \mathcal{X}, \) and \( \text{Ker} \tilde{v} \in \mathcal{Y} \). Since \( BM \) is projective, one has the commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & L_1 & \longrightarrow & M_1 & \longrightarrow & N_1 & \longrightarrow & 0 \\
& & \downarrow{\tilde{f}} & & \downarrow{\tilde{g}} & & \downarrow{\tilde{s}} & & \\
0 & \longrightarrow & \text{Hom}_B(M, L_2) & \longrightarrow & \text{Hom}_B(M, M_2) & \longrightarrow & \text{Hom}_B(M, N_2) & \longrightarrow & 0.
\end{array}
\]

Since \( \tilde{u} \) and \( (M, \beta) \) are epimorphisms, so is \( \tilde{s} \). By Snake Lemma and the assumption that \( \mathcal{X} \) is closed under taking the cokernels of monomorphisms, one knows that \( \text{Ker} \tilde{s} \in \mathcal{X} \). Similarly, \( \tilde{f} \) is an epimorphism and \( \text{Ker} \tilde{f} \in \mathcal{Y} \). By definition \( (\frac{N_1}{N_2})_{s,t} \in \nabla(\mathcal{X}, \mathcal{Y}) \). This proves that \( \nabla(\mathcal{X}, \mathcal{Y}) \) is closed under the cokernels of monomorphisms.

Conversely, using functors \( H_A \) and \( H_B \), one sees that \( \mathcal{X} \) and \( \mathcal{Y} \) are closed under the cokernels of monomorphisms. \( \square \)

### 3.7. Proof of Theorem 3.2

(1) It suffices to prove \( \Delta(\mathcal{U}, \mathcal{V}) = \dagger(\Delta(\mathcal{U}, \mathcal{V})) \). In fact, \( \Delta(\mathcal{U}, \mathcal{V}) = \Delta(\dagger \mathcal{X}, \dagger \mathcal{Y}) \). Since \( \mathcal{X} \) contains all the injective \( \Lambda \)-modules and \( \mathcal{Y} \) contains all the injective \( B \)-modules, it follows from Lemma 3.11(1) that

\[
\Delta(\dagger \mathcal{X}, \dagger \mathcal{Y}) = \dagger(Z_A(\mathcal{X}) \cap Z_B(\mathcal{Y})) = \dagger(Z_A(\mathcal{X}) \cup Z_B(\mathcal{Y})).
\]

Thus

\[
\dagger(\Delta(\mathcal{U}, \mathcal{V})) = \dagger(\Delta(\dagger \mathcal{X}, \dagger \mathcal{Y})) = \dagger([\dagger(Z_A(\mathcal{X}) \cup Z_B(\mathcal{Y}))]) = \dagger(Z_A(\mathcal{U}) \cup Z_B(\mathcal{V})).
\]

By Lemma 3.12(1), \( \Delta(\mathcal{U}, \mathcal{V}) \) is closed under the kernels of epimorphisms if and only if \( \mathcal{U} \) and \( \mathcal{V} \) are closed under the kernels of epimorphisms. That is, \( (\Delta(\mathcal{U}, \mathcal{V}), \Delta(\mathcal{U}, \mathcal{V})) \) is hereditary if and only if \( (\mathcal{U}, \mathcal{X}) \) and \( (\mathcal{V}, \mathcal{Y}) \) are hereditary.

(2) Similarly, it suffices to show \( \nabla(\mathcal{X}, \mathcal{Y}) = (\dagger \nabla(\mathcal{X}, \mathcal{Y})) \). In fact, \( \nabla(\mathcal{X}, \mathcal{Y}) = \nabla(\mathcal{U}, \mathcal{V}) \). Since \( \mathcal{U} \) contains all the projective \( \Lambda \)-modules and \( \mathcal{V} \) contains all the projective \( B \)-modules, it follows from Lemma 3.11(2) that

\[
\nabla(\mathcal{U}, \mathcal{V}) = Z_A(\mathcal{U}) \cap Z_B(\mathcal{V}) = (Z_A(\mathcal{U}) \cup Z_B(\mathcal{V})) = \nabla(\mathcal{X}, \mathcal{Y}).
\]

Thus

\[
(\dagger \nabla(\mathcal{X}, \mathcal{Y})) = (\dagger \nabla(\mathcal{U}, \mathcal{V})) = (\dagger(Z_A(\mathcal{U}) \cup Z_B(\mathcal{V}))) = (Z_A(\mathcal{U}) \cup Z_B(\mathcal{V})) = \nabla(\mathcal{X}, \mathcal{Y}).
\]

By Lemma 3.12(2), \( \nabla(\mathcal{X}, \mathcal{Y}) \) is closed under the cokernels of monomorphisms if and only if \( \mathcal{X} \) and \( \mathcal{Y} \) are closed under the cokernels of monomorphisms. That is, \( (\dagger \nabla(\mathcal{X}, \mathcal{Y}), \nabla(\mathcal{X}, \mathcal{Y})) \) is hereditary if and only if \( (\mathcal{U}, \mathcal{X}) \) and \( (\mathcal{V}, \mathcal{Y}) \) are hereditary. \( \square \)
3.8. **Proof of Theorem 3.5**  
(1) By Theorem 3.4(1), one has cotorsion pair \((\perp_{\mathcal{X}}, \perp_{\mathcal{Y}}))\); and by Theorem 3.2(1), one has cotorsion pair \((\Delta(\mathcal{U}, \mathcal{V}), \Delta(\mathcal{U}, \mathcal{V})^\perp))\). We will prove \(\Delta(\mathcal{U}, \mathcal{V})^\perp \subseteq (\perp_{\mathcal{X}})^\perp\) and \((\perp_{\mathcal{X}})^\perp \subseteq (\perp_{\mathcal{Y}})^\perp\) by Lemma 3.11(1) one has
\[
\Delta(\mathcal{U}, \mathcal{V})^\perp = [\Delta(\perp_{\mathcal{X}}, \perp_{\mathcal{Y}})]^\perp = [\perp_{\mathcal{X}}]\mathcal{U}(\mathcal{X}) \cap \perp_{\mathcal{Y}}\mathcal{V}(\mathcal{Y})^\perp.
\]
Since by assumption \(\text{Tor}_1^A(M, \mathcal{U}) = 0 = \text{Tor}_1^B(N, \mathcal{V})\), it follows from Lemma 3.9(1) that
\[
(\perp_{\mathcal{X}}) = \left(\frac{u_A^+}{v_B^-}\right) = T_A(\mathcal{U})^\perp \cap T_B(\mathcal{V})^\perp = (T_A(\mathcal{U}) \cup T_B(\mathcal{V}))^\perp.
\]
Thus, to show \(\Delta(\mathcal{U}, \mathcal{V})^\perp \subseteq (\perp_{\mathcal{X}})^\perp\), it suffices to show
\[
T_A(\mathcal{U}) \cup T_B(\mathcal{V}) \subseteq \perp_{\mathcal{X}}\mathcal{U}(\mathcal{X}) \cap \perp_{\mathcal{Y}}\mathcal{V}(\mathcal{Y}).
\]
In fact, since \(N \otimes_B M = 0\), the structure map \(g = 0\) of any \(A\)-module in \(T_A(\mathcal{U})\) is a monomorphism, it follows from Lemma 3.10(1) that
\[
\text{Ext}_1^A(T_A(\mathcal{U}), \mathcal{Z}(\mathcal{X})) \cong \text{Ext}_1^A(C_A T_A(\mathcal{U}), \mathcal{X}) = \text{Ext}_1^A(\mathcal{U}, \mathcal{X}) = 0.
\]
By Lemma 3.10(2) one has
\[
\text{Ext}_1^A(T_A(\mathcal{U}), \mathcal{Z}(\mathcal{Y})) \cong \text{Ext}_1^A(C_B T_A(\mathcal{U}), \mathcal{Y}) = 0
\]
since \(C_B T_A = 0\). So \(T_A(\mathcal{U}) \subseteq \perp_{\mathcal{X}}\mathcal{U}(\mathcal{X}) \cap \perp_{\mathcal{Y}}\mathcal{V}(\mathcal{Y})\).

Similarly, by Lemma 3.10(1) one has
\[
\text{Ext}_1^A(T_B(\mathcal{V}), \mathcal{Z}(\mathcal{X})) \cong \text{Ext}_1^A(C_B T_B(\mathcal{V}), \mathcal{X}) = 0
\]
since \(C_B T_B = 0\). Since \(M \otimes_A N = 0\), the structure map \(f = 0\) of any \(A\)-module in \(T_B(\mathcal{V})\) is a monomorphism, it follows from Lemma 3.10(2) that
\[
\text{Ext}_1^A(T_B(\mathcal{V}), \mathcal{Z}(\mathcal{Y})) \cong \text{Ext}_1^A(C_B T_B(\mathcal{V}), \mathcal{Y}) = \text{Ext}_1^A(\mathcal{V}, \mathcal{Y}) = 0
\]
So \(T_B(\mathcal{V}) \subseteq \perp_{\mathcal{X}}\mathcal{U}(\mathcal{X}) \cap \perp_{\mathcal{Y}}\mathcal{V}(\mathcal{Y})\). This completes the proof of (1).

(2) Comparing cotorsion pair \((\perp_{\mathcal{X}}^\perp, (\perp_{\mathcal{Y}})^\perp)\) in Theorem 3.1(2) with \((\perp_{\mathcal{X}}^\perp, \perp_{\mathcal{Y}}^\perp)\) in Theorem 3.2(2), we will prove \(\perp_{\mathcal{X}}^\perp \subseteq (\perp_{\mathcal{Y}})^\perp\). This can be similarly done as (1). For convenience we include a brief justification. By Lemma 3.11(2) one has
\[
\perp_{\mathcal{X}}^\perp \mathcal{U}(\mathcal{X}) = \perp_{\mathcal{V}}^\perp(\mathcal{U})^\perp \subseteq (\perp_{\mathcal{Y}})^\perp = \perp_{\mathcal{Z}}(\mathcal{U})^\perp \cap \mathcal{Z}(\mathcal{V})^\perp.
\]
By Lemma 3.10(2), one has
\[
(\perp_{\mathcal{Y}})^\perp = \left(\perp_{\mathcal{X}}^\perp, \perp_{\mathcal{Y}}^\perp\right) = \perp_{\mathcal{H}}(\mathcal{X}) \cup \perp_{\mathcal{H}}(\mathcal{Y}).
\]
So, it suffices to show \(\mathcal{H}(\mathcal{X}) \cup \mathcal{H}(\mathcal{Y}) \subseteq \perp_{\mathcal{U}}(\mathcal{U})^\perp \cap \perp_{\mathcal{V}}(\mathcal{V})^\perp\).

In fact, since \(\text{Ext}_1^A(N, \mathcal{X}) = 0\), it follows from Lemma 3.3(3) that
\[
\text{Ext}_1^A(\mathcal{Z}(\mathcal{U}), \mathcal{H}(\mathcal{X})) \cong \text{Ext}_1^A(\mathcal{U} \mathcal{Z}(\mathcal{U}), \mathcal{X}) = \text{Ext}_1^A(\mathcal{U}, \mathcal{X}) = 0
\]
and
\[
\text{Ext}_1^A(\mathcal{Z}(\mathcal{V}), \mathcal{H}(\mathcal{X})) \cong \text{Ext}_1^A(\mathcal{U} \mathcal{Z}(\mathcal{V}), \mathcal{X}) = 0
\]
Thus \(\mathcal{H}(\mathcal{X}) \subseteq \perp_{\mathcal{U}}(\mathcal{U})^\perp \cap \perp_{\mathcal{V}}(\mathcal{V})^\perp\).
Since $\text{Ext}_B^1(M, \mathcal{Y}) = 0$, it follows from Lemma 3.8(4) that
$$\text{Ext}_A^1(Z_A(U), H_B(\mathcal{Y})) \cong \text{Ext}_A^1(U_BZ_A(U), \mathcal{Y}) = 0,$$
and
$$\text{Ext}_A^1(Z_B(V), H_B(\mathcal{Y})) \cong \text{Ext}_A^1(U_BZ_B(V), \mathcal{Y}) = \text{Ext}_A^1(V, \mathcal{Y}) = 0,$$
which show $H_B(\mathcal{Y}) \subseteq Z_A(U)^\perp \cap Z_B(V)^\perp$. This completes the proof. □

4. **Identifications**

We will prove that the four constructions of cotorsion pairs, given in Theorem 3.1 and Theorem 3.2, are pairwise generally different; and on the other hand, study the problem of identifications, i.e., we will show that, in many important cases, the cotorsion pairs in Series I coincide with the corresponding ones in Series II, and then we will get only two cotorsion pairs $(\Phi(U, V), (X, Y))$ and $(\Phi(U, V), (X, Y))$. Since the both cotorsion pairs are explicitly given, they can be used in finding Hovey triples, i.e., the abelian model structures on Morita rings.

4.1. **Generally different cotorsion pairs.** For use in Section 6, we introduce the following notion.

**Definition 4.1.** Let $\Omega$ be a class of Morita rings, $(X, Y)$ and $(X', Y')$ cotorsion pairs defined in $\Lambda$-$\text{Mod}$, for arbitrary Morita rings $\Lambda \in \Omega$. We say that $(X, Y)$ and $(X', Y')$ are generally different, provided that there exist $\Lambda \in \Omega$, such that $(X, Y) \neq (X', Y')$ in $\Lambda$-$\text{Mod}$.

**Example 4.2.** Generally different cotorsion pairs could be the same for some special Morita rings, as the following example shows.

Let $\Omega = \{\text{Morita ring } \Lambda = (A N M B) \mid M \otimes A = 0 = N \otimes B, M \text{ and } N \text{ are flat, } B \text{ and } A \text{ are projective}\}$. Then $(\Lambda P, \Lambda$-$\text{Mod})$ and $(\Lambda P, \Lambda$-$\text{Mod})^{\perp}$ are cotorsion pairs in $\Lambda$-$\text{Mod}$, $\forall \Lambda \in \Omega$. See Theorem 3.1.2.

If $M \neq 0$, then $(\Lambda P, \Lambda$-$\text{Mod})$ and $(\Lambda P, \Lambda$-$\text{Mod})^{\perp}$ are generally different cotorsion pairs. But they are the same for $\Lambda \in \Omega$ with $M = 0 = N$.

4.2. **The four cotorsion pairs are pairwise generally different.** By Theorems 3.1 and 3.2 the cotorsion pairs

$$((\Lambda P, \Lambda$-$\text{Mod})^{\perp}, (\Lambda P, \Lambda$-$\text{Mod})^{\perp}), ((\Lambda P, \Lambda$-$\text{Mod}), ((\Lambda P, \Lambda$-$\text{Mod}), \Lambda P,$$ and

$$((\Lambda P, \Lambda$-$\text{Mod})^{\perp}, (\Lambda P, \Lambda$-$\text{Mod})^{\perp}), ((\Lambda P, \Lambda$-$\text{Mod})^{\perp}, (\Lambda P, \Lambda$-$\text{Mod})^{\perp}),$$
are defined in $\Lambda$-$\text{Mod}$, $\forall \Lambda \in \Omega$, where

$\Omega = \{\Lambda = (A N M B) \mid M \otimes A N = 0 = N \otimes B M, M_A \text{ and } N_B \text{ are flat, } B M \text{ and } A N \text{ are projective}\}$. We will show that the four cotorsion pairs are pairwise generally different. For convenience, we will call the cotorsion pairs above the first, the second, the third, and the fourth cotorsion pairs.
Example 4.3. Let $A = B$ be the path algebra $k(1 \rightarrow 2)$, where $\text{char} \ k \neq 2$. Write the conjunction of paths from right to left. Thus $e_1 A e_2 = 0$ and $e_2 A e_1 \cong k$. Take $M = N = A e_2 \otimes_k e_1 A$. Then $M \otimes_A N = 0 = N \otimes_A M$. Let $\Lambda$ be the Morita ring $(\Lambda, \Lambda)$. Then $\Lambda \in \Omega$.

Note that $A M = A N$ is isomorphic to the simple projective left $A$-module $A e_2 = S_2$, and that $M_A = N_A$ is isomorphic to the simple projective right $A$-module $e_1 A$. Then $M \otimes_A A e_1 \cong A e_2 \otimes_k (e_1 A \otimes_A A e_1) \cong S_2$. To see the left $A$-module structure on $\text{Hom}_A(M, A e_1)$, note that $\text{Hom}_A(M, A e_1) \cong \text{Hom}_A(A e_2, A e_1) \cong e_2 A e_1 \cong k$ as $k$-spaces. For $f \in \text{Hom}_A(M, A e_1)$ given by $f(e_2 \otimes_k e_1) = e_1$, one has $e_1 f = f$. Thus $\text{Hom}_A(M, A e_1) \cong S_1$ as left $A$-modules. The Auslander-Reiten quiver of $A$ is

$$
\begin{array}{c}
\sigma \\
S_2 \\
\downarrow \\
A e_1 \\
\downarrow \\
S_1
\end{array}
$$

Take $(U, \mathcal{X}) = (A - \text{Mod}, A \mathcal{I}) = (V, \mathcal{Y})$. Note that $M \otimes_A U \not\cong \mathcal{X}$, $N \otimes_B V \not\cong \mathcal{X}$. Take $L = (A e_1)_{\sigma, \sigma}$. Then $L \in \text{Mon}(A) = \Delta(A - \text{Mod}, A - \text{Mod}) = \Delta(U, V)$ and $L \in (A e_1) = (\mathcal{Y})$.

Consider the exact sequence of $A$-modules

$$
0 \longrightarrow (A e_1)_{\sigma, \sigma} \xrightarrow{(0,1)} (A e_1 \oplus A e_1) \xrightarrow{(\sigma, \sigma)} (A e_1)_{\sigma, \sigma} \longrightarrow 0.
$$

This exact sequence does not split. In fact, if it splits, then there is a $A$-map $(a, b)_{(c, d)}$:

$$(A e_1 \oplus A e_1)_{(\sigma, \sigma)} \longrightarrow (A e_1)_{\sigma, \sigma}$$

such that $((a, b)_{(c, d)}) = (1)$, i.e., $a = 1 = c$. Since the following diagrams commute, $d + 1 = b$ and $b + 1 = d$, which is a contradiction, since $\text{char} \ k \neq 2$.

Thus $\text{Ext}_A^1(L, L) \neq 0$. This means $L \not\in \perp (A e_1)$. Since $L \in \text{Mon}(A)$, $\perp (A e_1)$ is not equal to the second one, i.e.,

$$
(\perp (\mathcal{Y}), (\mathcal{X})) = (\perp (\mathcal{I}), (\mathcal{I})) \neq (\mathcal{I}), (\mathcal{I}) = \Delta(U, V), \Delta(U, V).$

Since $(\mathcal{V}) = (A - \text{Mod}) = A - \text{Mod}$, it follows that $(\mathcal{V}) = (A - \text{Mod}, A \mathcal{I})$. Since $L \not\in \perp (\mathcal{I})$, the first cotorsion pair is not equal to the third one:

$$(\perp (\mathcal{Y}), (\mathcal{Y})) = (\perp (\mathcal{I}), (\mathcal{I})) \neq (\mathcal{I}, (\mathcal{I})).$$

Since $(A e_1) \not\in \text{Mon}(A) = \Delta(U, V)$, the second cotorsion pair is not equal to the third one:

$$
(\Delta(U, V), (\mathcal{I})) \neq (\mathcal{I}, (\mathcal{I})).$$
Thus \((\Delta(\mathcal{X}, \mathcal{Y}) = \nabla(\mathcal{X}, \mathcal{Y}) = \Lambda)\), i.e., the fourth cotorsion pair is exactly third cotorsion pair. Therefore, the first cotorsion pair is not equal to the fourth one, and the second cotorsion pair is not equal to the fourth one.

Finally, to see the third cotorsion pair is not equal to the fourth one, namely,
\[
((\mathcal{Y}), (\mathcal{Y})^\perp) \neq ((\mathcal{X}), (\mathcal{X})^\perp)
\]
we take \((\mathcal{U}, \mathcal{X}) = (\Lambda \mathcal{P}, \Lambda \mathcal{M}) = (\mathcal{V}, \mathcal{Y})\). Note that \(\text{Hom}_B(\mathcal{M}, \mathcal{Y}) \not\subseteq \mathcal{U}, \text{Hom}_A(\mathcal{N}, \mathcal{X}) \not\subseteq \mathcal{V}\).

Take \(L = (A_{\mathcal{P}})^{\perp} \mathcal{P}_{\mathcal{A}}\) as above. Then \(L \in (A\mathcal{P})^\perp = (\mathcal{Y})^\perp\). Since \(\mathcal{S}_A = \Lambda \rightarrow \text{Hom}_A(\mathcal{M}, \mathcal{A}) \cong S_1\) is exactly the epimorphism \(\pi : \mathcal{A} \mathcal{P} \rightarrow \mathcal{S}_1\), by definition \(L \in \text{Epi}(\Lambda) = \nabla(\mathcal{A}, \mathcal{M}) = \nabla(\mathcal{X}, \mathcal{Y})\). Since \(\text{Ext}_A^1(L, L) \neq 0\), \(L \not\in (\mathcal{A}\mathcal{P})^\perp\). Thus \(\nabla(\mathcal{X}, \mathcal{Y}) \neq (\mathcal{A}\mathcal{P})^\perp\), and hence the third cotorsion pair is not equal to the fourth one:
\[
((\mathcal{Y}), (\mathcal{Y})^\perp) = ((\mathcal{A}\mathcal{P})^\perp, (\mathcal{A}\mathcal{P})^\perp) \neq ((\mathcal{X}), (\mathcal{X})^\perp).
\]

All together, we have proved that the four cotorsion pairs are pairwise generally different. In fact, we have found an example \(\Lambda\), such that the four constructions of cotorsion pairs in \(\mathcal{A}\mathcal{M}\) are pairwise different.

4.3. Main results on identification.

**Theorem 4.4.** Let \(\Lambda = (\mathcal{A}\mathcal{N} \mathcal{M})\) be a Morita ring with \(\mathcal{A}\mathcal{B} \mathcal{N} = 0 = \mathcal{N} \mathcal{B}\mathcal{A}\mathcal{M}\), \((\mathcal{U}, \mathcal{X})\) and \((\mathcal{V}, \mathcal{Y})\) be cotorsion pairs in \(\mathcal{A}\mathcal{M}\) and in \(\mathcal{B}\mathcal{M}\), respectively.

1. Assume that \(\text{Tor}_A^1(\mathcal{M}, \mathcal{U}) = 0 = \text{Tor}_A^1(\mathcal{N}, \mathcal{V})\). If \(\mathcal{M} \mathcal{A} \mathcal{U} \subset \mathcal{Y}\) or \(\mathcal{N} \mathcal{B} \mathcal{V} \subset \mathcal{X}\), then
\[
(\Delta(\mathcal{U}, \mathcal{Y}), \Delta(\mathcal{U}, \mathcal{Y})^\perp) = ((\mathcal{Y}), (\mathcal{Y})^\perp).
\]
Thus \((\Delta(\mathcal{U}, \mathcal{Y}), (\mathcal{Y})^\perp)\) is a cotorsion pair in \(\mathcal{A}\mathcal{M}\).

Moreover, if \(\mathcal{M} \mathcal{A} \mathcal{U} \subset \mathcal{Y}\) and \(\mathcal{N} \mathcal{B} \mathcal{V} \subset \mathcal{X}\), then \(\Delta(\mathcal{U}, \mathcal{V}) = T_\mathcal{M}(\mathcal{U}) \oplus T_\mathcal{B}(\mathcal{V})\).

2. Assume that \(\text{Ext}_B^1(\mathcal{M}, \mathcal{Y}) = 0 = \text{Ext}_A^1(\mathcal{N}, \mathcal{X})\). If \(\text{Hom}_B(\mathcal{M}, \mathcal{Y}) \subset \mathcal{U}\) or \(\text{Hom}_A(\mathcal{N}, \mathcal{X}) \subset \mathcal{V}\), then
\[
(\nabla(\mathcal{X}, \mathcal{Y}), \nabla(\mathcal{X}, \mathcal{Y})) = ((\mathcal{Y}), (\mathcal{Y})^\perp).
\]
Thus \(((\mathcal{Y}), (\mathcal{Y}^\perp))\) is a cotorsion pair in \(\mathcal{A}\mathcal{M}\).

Moreover, if \(\text{Hom}_B(\mathcal{M}, \mathcal{Y}) \subset \mathcal{U}\) and \(\text{Hom}_A(\mathcal{N}, \mathcal{X}) \subset \mathcal{V}\), then \(\nabla(\mathcal{X}, \mathcal{Y}) = H_\mathcal{M}(\mathcal{X}) \oplus H_\mathcal{B}(\mathcal{Y})\).

4.4. Applications. In Theorem 4.3 taking one of \((\mathcal{U}, \mathcal{X})\) and \((\mathcal{V}, \mathcal{Y})\) being the projective cotorsion pair or the injective cotorsion pair, and another being an arbitrary cotorsion pair, one has

**Corollary 4.5.** Let \(\Lambda = (\mathcal{A}\mathcal{N} \mathcal{M})\) be a Morita ring with \(\mathcal{A}\mathcal{B} \mathcal{N} = 0 = \mathcal{N} \mathcal{B}\mathcal{A}\mathcal{M}\).

1. If \(\mathcal{N}\mathcal{B}\) is flat, then for any cotorsion pair \((\mathcal{V}, \mathcal{Y})\) in \(\mathcal{B}\mathcal{M}\) one has
\[
(\Delta(\mathcal{A}\mathcal{P}, \mathcal{V}), \Delta(\mathcal{A}\mathcal{P}, \mathcal{V})^\perp) = (\mathcal{A}\mathcal{M}\mathcal{Y}, (\mathcal{A}\mathcal{M}\mathcal{Y})^\perp).
\]
Thus \((\Delta(\mathcal{A}\mathcal{P}, \mathcal{V}), (\mathcal{A}\mathcal{M}\mathcal{Y}))\) is a cotorsion pair in \(\mathcal{A}\mathcal{M}\).
Moreover, if $M \otimes_A \mathcal{P} \subseteq \mathcal{Y}$ (e.g., this is the case if $B M$ is injective), then $\Delta(A \mathcal{P}, \mathcal{V}) = T_A(A \mathcal{P}) \oplus T_B(\mathcal{V})$.

(2) If $M_A$ is flat, then for any cotorsion pair $(\mathcal{U}, \mathcal{X})$ in $A$-Mod one has

$$(\Delta(\mathcal{U}, B \mathcal{P}), \Delta(\mathcal{U}, B \mathcal{P})^\perp) = (\, (\mathcal{X})_{B\text{-Mod}}^\perp, \, (\mathcal{X})_{B\text{-Mod}} \,) .$$

Thus $(\Delta(\mathcal{U}, B \mathcal{P}), \, (\mathcal{X})_{B\text{-Mod}})$ is a cotorsion pair in $A$-Mod.

Moreover, if $N \otimes_B \mathcal{P} \subseteq \mathcal{X}$ (e.g., this is the case if $A N$ is injective), then $\Delta(\mathcal{U}, B \mathcal{P}) = T_A(\mathcal{U}) \oplus T_B(B \mathcal{P})$.

(3) If $BM$ is projective, then for any cotorsion pair $(\mathcal{V}, \mathcal{Y})$ in $B$-Mod one has

$$(\, (\mathcal{V})_{A\text{-Mod}}^\perp, \, (\mathcal{V})_{A\text{-Mod}} \,) .$$

Thus $(\, (\mathcal{V})_{A\text{-Mod}}^\perp, \, (\mathcal{V})_{A\text{-Mod}} \,)$ is a cotorsion pair in $A$-Mod.

Moreover, if $\text{Hom}_A(N, A \mathcal{I}) \subseteq \mathcal{V}$ (e.g., this is the case if $B$ is quasi-Frobenius and $N_B$ is flat), then $\mathcal{V}(A \mathcal{I}, \mathcal{Y}) = H_A(A \mathcal{I}) \oplus H_B(\mathcal{Y})$.

(4) If $AN$ is projective, then for any cotorsion pair $(\mathcal{U}, \mathcal{X})$ in $A$-Mod one has

$$(\, (\mathcal{U})_{B\text{-Mod}}^\perp, \, (\mathcal{U})_{B\text{-Mod}} \,).$$

Thus $(\, (\mathcal{U})_{B\text{-Mod}}^\perp, \, (\mathcal{U})_{B\text{-Mod}} \,)$ is a cotorsion pair in $A$-Mod.

Moreover, if $\text{Hom}_B(M, B \mathcal{I}) \subseteq \mathcal{U}$ (e.g., this is the case if $A$ is quasi-Frobenius and $M_A$ is flat), then $\mathcal{V}(\mathcal{X}, B \mathcal{I}) = \text{Hom}_A(A \mathcal{X}) \oplus \text{Hom}_B(B \mathcal{I})$.

**Proof.** (1) Taking $(\mathcal{U}, \mathcal{X}) = (A \mathcal{P}, A$-Mod) in Theorem 4.4(1). Then $N \otimes_B \mathcal{V} \subseteq A$-Mod = $\mathcal{X}$.

By Theorem 4.4(1) one has $(\Delta(A \mathcal{P}, \mathcal{V}), \Delta(A \mathcal{P}, \mathcal{V})^\perp) = (\, (\mathcal{X})_{A\text{-Mod}}^\perp, \, (\mathcal{X})_{A\text{-Mod}} \,) .$

If $M \otimes_A \mathcal{P} \subseteq \mathcal{Y}$, i.e., $M \otimes_A \mathcal{U} = M \otimes_A \mathcal{P} \subseteq \mathcal{Y}$, then by Theorem 4.4(1), $\Delta(A \mathcal{P}, \mathcal{V}) = T_A(A \mathcal{P}) \oplus T_B(\mathcal{V})$.

Assume that $BM$ is injective. For any $P \in A \mathcal{P}$, as a left $B$-module, $M \otimes_A P$ is a direct summand of a direct sum of copies of $BM$. Since $BM$ is injective and $B$ is left noetherian (note that $A$ is assumed to be an Artin algebra, hence $B$ is an Artin algebra), $M \otimes_A P$ is an injective left $B$-module, and hence it is in $\mathcal{Y}$. Thus $M \otimes_A \mathcal{P} \subseteq \mathcal{Y}$, and hence $\Delta(A \mathcal{P}, \mathcal{V}) = T_A(A \mathcal{P}) \oplus T_B(\mathcal{V})$, by Theorem 4.4(1).

(2) Taking $(\mathcal{V}, \mathcal{X}) = (B \mathcal{P}, B$-Mod) in Theorem 4.4(1). Then $M \otimes_A \mathcal{U} \subseteq B$-Mod = $\mathcal{Y}$. By Theorem 4.4(1) one has $(\Delta(\mathcal{U}, B \mathcal{P}), \Delta(\mathcal{U}, B \mathcal{P})^\perp) = (\, (\mathcal{X})_{B\text{-Mod}}^\perp, \, (\mathcal{X})_{B\text{-Mod}} \,) .$

If $N \otimes_B \mathcal{P} \subseteq \mathcal{X}$, then by Theorem 4.4(1), $\Delta(\mathcal{U}, B \mathcal{P}) = T_A(\mathcal{U}) \oplus T_B(B \mathcal{P})$.

Assume that $A$ is $AN$ is injective. Then $N \otimes_B \mathcal{P} \subseteq A \mathcal{I} \subseteq \mathcal{X}$, and hence $\Delta(\mathcal{U}, B \mathcal{P}) = T_A(\mathcal{U}) \oplus T_B(B \mathcal{P})$.

(3) Taking $(\mathcal{U}, \mathcal{X}) = (A$-Mod, $A \mathcal{I})$ in Theorem 4.4(2). Then $\text{Hom}_B(M, \mathcal{Y}) \subseteq A$-Mod = $\mathcal{U}$. By Theorem 4.4(2) one has $(\, (\mathcal{V})_{A\text{-Mod}}^\perp, \, (\mathcal{V})_{A\text{-Mod}} \,).$
If \( \text{Hom}_A(N, \mathcal{I}) \subseteq \mathcal{V} \), then by Theorem 4.4(2), \( \nabla(\mathcal{I}, \mathcal{V}) = H_A(\mathcal{I}) \oplus H_B(\mathcal{V}) \).

Assume that \( B \) is quasi-Frobenius and \( N_B \) is flat. Then \( \text{Hom}_A(N, \mathcal{I}) \subseteq \mathcal{V} = B \mathcal{P} \subseteq \mathcal{V} \), and thus \( \nabla(\mathcal{I}, \mathcal{V}) = H_A(\mathcal{I}) \oplus H_B(\mathcal{V}) \), by Theorem 4.4(2).

(4) Taking \( (\mathcal{V}, \mathcal{Y}) = (B\text{-Mod}, B \mathcal{I}) \) in Theorem 4.4(2). Then \( \text{Hom}_A(N, \mathcal{X}) \subseteq B\text{-Mod} = \mathcal{V} \). By Theorem 4.4(2) one has \( \nabla(\mathcal{X}, B \mathcal{I}) = ((B\text{-Mod})^\perp, (B\text{-Mod})^\perp) \).

If \( \text{Hom}_B(M, B \mathcal{I}) \subseteq \mathcal{U} \), then by Theorem 4.4(2), \( \nabla(\mathcal{X}, B \mathcal{I}) = H_A(\mathcal{X}) \oplus H_B(B \mathcal{I}) \).

Assume that \( A \) is quasi-Frobenius and \( M_A \) is flat. Then \( \text{Hom}_B(M, B \mathcal{I}) \subseteq \mathcal{A} = \mathcal{A} \subseteq \mathcal{U} \), and hence \( \nabla(\mathcal{X}, B \mathcal{I}) = H_A(\mathcal{X}) \oplus H_B(B \mathcal{I}) \). \( \square \)

4.5. **Proof of Theorem 4.4** (1) By Theorem 3.5(1), one has cotorsion pairs

\[ (\{ Y \}, \{ Y \}) \]

with \( \nabla(\{ Y \}, \{ Y \}) \subseteq \Delta(\mathcal{A}, \mathcal{V}) \). To see that they are equal, it remains to prove \( \Delta(\mathcal{A}, \mathcal{V}) \subseteq \nabla(\{ Y \}, \{ Y \}) \).

Let \( L = (L_1 L_2) \in \Delta(\mathcal{A}, \mathcal{V}) \). By definition there are exact sequences

\[ 0 \rightarrow M \otimes_A L_1 \xrightarrow{f} L_2 \xrightarrow{p_1} \text{Coker} f \rightarrow 0 \quad \text{and} \quad 0 \rightarrow N \otimes_B L_2 \xrightarrow{g} L_1 \xrightarrow{p_2} \text{Coker} g \rightarrow 0 \]

with \( \text{Coker} f \in \mathcal{V} \) and \( \text{Coker} g \in \mathcal{U} \). Since \( N \otimes_B M = 0 = M \otimes_A N \), it follows that

\[ 1 \otimes p_1 : N \otimes_B L_2 \rightarrow N \otimes_B \text{Coker} f \quad \text{and} \quad 1 \otimes p_2 : M \otimes_A L_1 \rightarrow M \otimes_A \text{Coker} g \]

are isomorphisms.

**Case I**: Assume that \( M \otimes_A \mathcal{U} \subseteq \mathcal{V} \). Then

\[ M \otimes_A L_1 \cong M \otimes_A \text{Coker} g \in M \otimes_A \mathcal{U} \subseteq \mathcal{V} \]

Since \( (\mathcal{V}, \mathcal{Y}) \) is a cotorsion pair, the exact sequence

\[ 0 \rightarrow M \otimes_A L_1 \xrightarrow{f} L_2 \xrightarrow{p_1} \text{Coker} f \rightarrow 0 \]

splits. Thus there are \( B \)-maps \( f' : L_2 \rightarrow M \otimes_A L_1 \) and \( \sigma_1 : \text{Coker} f \rightarrow L_2 \) such that

\[ f' f = 1_{\otimes_A L_1}, \quad p_1 \sigma_1 = 1_{\text{Coker} f}, \quad f f' + \sigma_1 p_1 = 1_{L_2}, \quad f' \sigma_1 = 0. \]

Thus \( (f_1) : L_2 \rightarrow (M \otimes_A L_1) \oplus \text{Coker} f \) is a \( B \)-isomorphism, and

\[ L = (L_1 L_2) \in \Delta(\mathcal{A}, \mathcal{V}) \]

is a \( \Lambda \)-isomorphism, and

\[ 0 \rightarrow \left( N \otimes_B \text{Coker} f \right)_{0,1} \rightarrow \left( \begin{array}{l} L_1 \\ \text{Coker} f \end{array} \right)_{1,0} \rightarrow \left( \begin{array}{l} \text{Coker} g \\ M \otimes_A \text{Coker} g \end{array} \right)_{1,0} \rightarrow 0 \]

is an exact sequence of \( A \)-modules, i.e.,

\[ 0 \rightarrow T_B \text{Coker} f \rightarrow \left( \begin{array}{l} L_1 \\ \text{Coker} f \end{array} \right)_{1,0} \rightarrow T_A \text{Coker} g \rightarrow 0 \]

is exact.
Since Coker $f \in \mathcal{V}$ and $(\mathcal{V}, \mathcal{Y})$ is a cotorsion pair, by Lemma 3.3.2, $T_B \text{Coker } f \in (\mathcal{X})_{\ell}$. 

Since Coker $g \in \mathcal{U}$ and $(\mathcal{U}, \mathcal{X})$ is a cotorsion pair, by Lemma 3.3.1, $T_A \text{Coker } g \in (\mathcal{Y})_{\ell}$. Thus $L \cong \left( (M \otimes A L_1) \otimes_{\text{Coker } f} \right)_{(1,0)} \otimes (\mathcal{X})_{\ell} = (\mathcal{X})_{\ell}$.

**Case II:** Assume that $N \otimes B \mathcal{V} \subseteq \mathcal{X}$. This is similar to **Case I**. We include the main steps.

Since $N \otimes B L_2 \cong N \otimes B \text{Coker } f \in N \otimes B \mathcal{V} \subseteq \mathcal{X}$, the exact sequence

$$0 \rightarrow N \otimes B L_2 \xrightarrow{g} L_1 \xrightarrow{\mathcal{P}_2} \text{Coker } g \rightarrow 0$$

splits. Then $L = (L_2)^{1,0} \xrightarrow{f} (N \otimes B L_2)^{1,0} \rightarrow (N \otimes B L_2)^{1,0} \rightarrow (N \otimes B L_2)^{1,0} \rightarrow \ldots$ is an exact sequence of $\Lambda$-modules, i.e.,

$$0 \rightarrow T_A \text{Coker } g \rightarrow (N \otimes B L_2)^{1,0} \rightarrow T_B \text{Coker } f \rightarrow 0$$

is exact. By Lemma 3.3.1, $T_A \text{Coker } g \in (\mathcal{Y})_{\ell}$; and by Lemma 3.3.2, $T_B \text{Coker } f \in (\mathcal{Y})_{\ell}$. Thus $L \in (\mathcal{Y})_{\ell}$.

Finally, assume that $M \otimes A \mathcal{U} \subseteq \mathcal{Y}$ and $N \otimes B \mathcal{V} \subseteq \mathcal{X}$. Then from the proof above one sees that both $0 \rightarrow M \otimes A L_1 \xrightarrow{f} L_2 \xrightarrow{\mathcal{P}_1} \text{Coker } f \rightarrow 0$ and $0 \rightarrow N \otimes B L_2 \xrightarrow{g} L_1 \xrightarrow{\mathcal{P}_2} \text{Coker } g \rightarrow 0$ split, and

$$L \cong \left( (\mathcal{Y})^{1,0} \right)_{\ell} \cong \left( (\mathcal{Y})^{1,0} \right)_{\ell} \cong \left( (\mathcal{X})^{1,0} \right)_{\ell} = (\mathcal{X})_{\ell}$$

Thus $\Delta(\mathcal{U}, \mathcal{V}) \subseteq T_A(\mathcal{U}) \oplus T_B(\mathcal{V})$. The inclusion $T_A(\mathcal{U}) \oplus T_B(\mathcal{V}) \subseteq \Delta(\mathcal{U}, \mathcal{V})$ is clear. Thus shows $\Delta(\mathcal{U}, \mathcal{V}) = T_A(\mathcal{U}) \oplus T_B(\mathcal{V})$.

(2) By Theorem 3.3.2, one has cotorsion pairs

$$((\mathcal{Y})^{1,0}, (\mathcal{Y})^{1,0}), \quad (\Delta(\mathcal{X}, \mathcal{Y}), \Delta(\mathcal{X}, \mathcal{Y}))$$

with $(\mathcal{Y})^{1,0} \subseteq \Delta(\mathcal{X}, \mathcal{Y})$. To see that they are equal, it remains to prove $\Delta(\mathcal{X}, \mathcal{Y}) \subseteq (\mathcal{Y})^{1,0}$.

Here it is much more convenient to use the second expression of $\Lambda$-modules. Thus, let $L = \left( \frac{L_1}{L_2} \right)_{(1,0)} \otimes \left( \frac{\mathcal{X}}{\mathcal{Y}} \right)_{(1,0)} \otimes \left( \frac{\mathcal{X}}{\mathcal{Y}} \right)_{(1,0)}$, where $\tilde{f} \in \text{Hom}_A(X, \text{Hom}_B(M, Y))$ and $\tilde{g} \in \text{Hom}_B(Y, \text{Hom}_A(N, X))$. By definition there are exact sequences

$$0 \rightarrow \text{Ker } \tilde{f} \xrightarrow{i_1} L_1 \xrightarrow{\tilde{f}} \text{Hom}_B(M, L_2) \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \text{Ker } \tilde{g} \xrightarrow{i_2} L_2 \xrightarrow{\tilde{g}} \text{Hom}_A(N, L_1) \rightarrow 0$$

with $\text{Ker } \tilde{f} \subseteq \mathcal{X}$ and $\text{Ker } \tilde{g} \subseteq \mathcal{Y}$. Since $M \otimes A N = 0 = N \otimes B M$, it follows that

$$(N, i_1) : \text{Hom}_A(N, \text{Ker } \tilde{f}) \cong \text{Hom}_A(N, L_1) \quad \text{and} \quad (M, i_2) : \text{Hom}_B(M, \text{Ker } \tilde{g}) \cong \text{Hom}_B(M, L_2).$$

**Case I:** Assume that $\text{Hom}_B(M, \mathcal{Y}) \subseteq \mathcal{U}$. Then

$$\text{Hom}_B(M, L_2) \cong \text{Hom}_B(M, \text{Ker } \tilde{g}) \in \text{Hom}_B(M, \mathcal{Y}) \subseteq \mathcal{U}.$$
Since \((\mathcal{U}, \mathcal{X})\) is a cotorsion pair, the exact sequence
\[
0 \longrightarrow \text{Ker} \tilde{f} \xrightarrow{i_1} L_1 \xrightarrow{\tilde{f}} \text{Hom}_B(M, L_2) \longrightarrow 0
\]
splits. Thus there are \(A\)-maps \(\alpha : \text{Hom}_B(M, L_2) \longrightarrow L_1\) and \(\pi_1 : L_1 \longrightarrow \text{Ker} \tilde{f}\) such that
\[
\pi_1 i_1 = 1_{\text{Ker} \tilde{f}}, \quad \tilde{f} \alpha = 1_{\text{Hom}_B(M, L_2)}, \quad \alpha \tilde{f} + i_1 \pi_1 = 1_{L_1}, \quad \pi_1 \alpha = 0.
\]
Hence \(\left(\begin{smallmatrix} \pi_1 \\ i_1 \end{smallmatrix}\right) : L_1 \cong \text{Ker} \tilde{f} \oplus \text{Hom}_B(M, L_2)\) and
\[
\left(\begin{smallmatrix} \pi_1 \\ i_1 \end{smallmatrix}\right) : L = \left(\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right) \cong \left(\begin{smallmatrix} \text{Ker} \tilde{f} \oplus \text{Hom}_B(M, L_2) \\ L_2 \end{smallmatrix}\right) (0, 1), \left(\begin{smallmatrix} (N, \pi_1) \\ (N, \pi_1) \oplus \text{Hom}_B(M, L_2) \end{smallmatrix}\right) (0, 1) .
\]
Moreover,
\[
0 \to \left(\begin{smallmatrix} (M, \text{Ker} \tilde{g}) \\ \text{Ker} \tilde{g} \end{smallmatrix}\right) (0, 1) \xrightarrow{\left(\begin{smallmatrix} 1 \\ \text{Ker} \tilde{g} \end{smallmatrix}\right)} L_1 \xrightarrow{\text{Ker} \tilde{g} \oplus \text{Hom}_B(M, L_2)} (0, 1), \left(\begin{smallmatrix} (N, \pi_1) \\ (N, \pi_1) \oplus \text{Hom}_B(M, L_2) \end{smallmatrix}\right) \xrightarrow{\left(\begin{smallmatrix} 1 \\ \text{Ker} \tilde{g} \end{smallmatrix}\right)} \left(\begin{smallmatrix} (N, \pi_1) \\ (N, \pi_1) \oplus \text{Hom}_B(M, L_2) \end{smallmatrix}\right) (0, 1) \to 0
\]
is an exact sequence of \(A\)-modules, i.e.,
\[
0 \longrightarrow \text{H}_B \text{Ker} \tilde{g} \longrightarrow \left(\begin{smallmatrix} \text{Ker} \tilde{f} \oplus \text{Hom}_B(M, L_2) \\ L_2 \end{smallmatrix}\right) (0, 1), \left(\begin{smallmatrix} (N, \pi_1) \\ (N, \pi_1) \oplus \text{Hom}_B(M, L_2) \end{smallmatrix}\right) \longrightarrow \text{H}_A \text{Ker} \tilde{f} \longrightarrow 0
\]
is exact. (We stress that all the \(A\)-modules are in the second expression.)

Since \(\text{Ker} \tilde{g} \in \mathcal{Y}\) and \((\mathcal{Y}, \mathcal{Y})\) is a cotorsion pair, by Lemma 3.8(4), \(\text{H}_B \text{Ker} \tilde{g} \in (\mathcal{Y})^\perp\). Since \(\text{Ker} \tilde{f} \in \mathcal{X}\) and \((\mathcal{U}, \mathcal{X})\) is a cotorsion pair, by Lemma 3.8(3), \(\text{H}_A \text{Ker} \tilde{f} \in (\mathcal{X})^\perp\). Thus \(L \cong \left(\begin{smallmatrix} \text{Ker} \tilde{f} \oplus \text{Hom}_B(M, L_2) \\ L_2 \end{smallmatrix}\right) (0, 1), \left(\begin{smallmatrix} (N, \pi_1) \\ (N, \pi_1) \oplus \text{Hom}_B(M, L_2) \end{smallmatrix}\right) \in (\mathcal{Y})^\perp\).

**Case II:** Assume that \(\text{Hom}_A(N, \mathcal{X}) \subseteq \mathcal{V}\). This is similar to **Case I**. Since \(\text{Hom}_A(N, L_1) \cong \text{Hom}_A(N, \text{Ker} \tilde{f}) \in \text{Hom}_A(N, \mathcal{X}) \subseteq \mathcal{V}\), the exact sequence
\[
0 \longrightarrow \text{Ker} \tilde{g} \xrightarrow{i_2} L_2 \xrightarrow{\tilde{g}} \text{Hom}_A(N, L_1) \longrightarrow 0
\]
splits. Thus \(L = \left(\begin{smallmatrix} L_1 \\ \text{Ker} \tilde{g} \oplus \text{Hom}_A(N, L_1) \end{smallmatrix}\right) \left(\begin{smallmatrix} (M, \pi_1) \\ (L, \pi_1) \end{smallmatrix}\right) (0, 1)\) and
\[
0 \to \left(\begin{smallmatrix} \text{Ker} \tilde{f} \\ \text{Ker} \tilde{g} \oplus \text{Hom}_A(N, L_1) \end{smallmatrix}\right) (0, 1) \xrightarrow{\left(\begin{smallmatrix} 1 \\ \text{Ker} \tilde{g} \end{smallmatrix}\right)} L_1 \xrightarrow{\text{Ker} \tilde{g} \oplus \text{Hom}_A(N, L_1)} (0, 1), \left(\begin{smallmatrix} (M, \pi_1) \\ (N, \pi_1) \oplus \text{Hom}_B(M, L_2) \end{smallmatrix}\right) \xrightarrow{\left(\begin{smallmatrix} 1 \\ \text{Ker} \tilde{g} \end{smallmatrix}\right)} \left(\begin{smallmatrix} (M, \pi_1) \\ (M, \pi_1) \oplus \text{Hom}_B(M, L_2) \end{smallmatrix}\right) (0, 1) \to 0
\]
is exact, i.e.,
\[
0 \longrightarrow \text{H}_A \text{Ker} \tilde{f} \longrightarrow \left(\begin{smallmatrix} \text{Ker} \tilde{g} \oplus \text{Hom}_A(N, L_1) \\ \text{Ker} \tilde{g} \end{smallmatrix}\right) (0, 1) \longrightarrow \text{H}_B \text{Ker} \tilde{g} \longrightarrow 0
\]
is exact. By Lemma 3.8(3), \(\text{H}_A \text{Ker} \tilde{f} \in (\mathcal{Y})^\perp\); and by Lemma 3.8(4), \(\text{H}_B \text{Ker} \tilde{g} \in (\mathcal{Y})^\perp\). Thus \(L \in (\mathcal{Y})^\perp\).

Finally, assume that \(\text{Hom}_B(M, \mathcal{Y}) \subseteq \mathcal{U}\) and \(\text{Hom}_A(N, \mathcal{X}) \subseteq \mathcal{V}\). Then both \(0 \to \text{Ker} \tilde{f} \xrightarrow{i_1} L_1 \xrightarrow{\tilde{f}} \text{Hom}_B(M, L_2) \to 0\) and \(0 \to \text{Ker} \tilde{g} \xrightarrow{i_2} L_2 \xrightarrow{\tilde{g}} \text{Hom}_A(N, L_1) \to 0\) splits, and
\[
L = \left(\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right) \cong \left(\begin{smallmatrix} \text{Ker} \tilde{f} \oplus \text{Hom}_B(M, L_2) \\ \text{Ker} \tilde{g} \end{smallmatrix}\right) (0, 1) = \text{H}_A \text{Ker} \tilde{f} \oplus \text{H}_B \text{Ker} \tilde{g} \in \text{H}_A(\mathcal{X}) \oplus \text{H}_B(\mathcal{Y}).
\]
Conversely, it is clear that \(\text{H}_A(\mathcal{X}) \oplus \text{H}_B(\mathcal{Y}) \subseteq \nabla(\mathcal{X}, \mathcal{Y})\). Thus \(\nabla(\mathcal{X}, \mathcal{Y}) = \text{H}_A(\mathcal{X}) \oplus \text{H}_B(\mathcal{Y})\). □
4.6. Remark. In Theorem 3.5, taking one of \((U, X)\) and \((V, Y)\) being the projective cotorsion pair or the injective cotorsion pair, and another being an arbitrary cotorsion pair, we conclude as follows: where “=” follows from Corollary 4.5 and “\(\neq\)” follows from Example 4.3.

(1) If \((U, X) = (\mathcal{A}P, A\text{-Mod})\), and \((V, Y)\) is an arbitrary cotorsion pair in \(B\text{-Mod}\), then

\[
(\Delta(\mathcal{A}P, V), \Delta(\mathcal{A}P, V)^\perp) = (⊥(\mathcal{A}\text{-Mod})_Y, (\mathcal{A}\text{-Mod})_Y).
\]

But, in general \((\perp\nabla(\mathcal{A}\text{-Mod}, Y), \nabla(\mathcal{A}\text{-Mod}, Y))\) \(\neq\) \((⊥(\mathcal{A}P)_Y, (\mathcal{A}P)_Y)^\perp\).

(2) If \((U, X) = (A\text{-Mod}, \mathcal{A}I)\), and \((V, Y)\) an arbitrary cotorsion pair in \(B\text{-Mod}\), then in general \((\Delta(A\text{-Mod}, V), \Delta(A\text{-Mod}, V)^\perp) \neq \perp(\mathcal{A} \text{-Mod})_Y, (\mathcal{A} \text{-Mod})_Y\)). However, one has

\[
(\perp\nabla(\mathcal{A}I, Y), \nabla(\mathcal{A}I, Y)) = (\perp(\mathcal{A}\text{-Mod})_Y, (\mathcal{A}\text{-Mod})_Y)^\perp.
\]

(3) If \((U, X)\) is an arbitrary cotorsion pair in \(A\text{-Mod}\), and \((V, Y) = (B\mathcal{P}, B\text{-Mod})\), then

\[
(\Delta(U, B\mathcal{P}), \Delta(U, B\mathcal{P})^\perp) = \perp(\mathcal{X} B\text{-Mod})_Y, (\mathcal{X} B\text{-Mod})_Y).
\]

But, in general \((\perp\nabla(\mathcal{X}, B\text{-Mod}), \nabla(\mathcal{X}, B\text{-Mod}))\) \(\neq\) \((\perp_U B\mathcal{P}), (\perp_U B\mathcal{P})^\perp\).

(4) If \((U, X)\) is an arbitrary cotorsion pair in \(A\text{-Mod}\), and \((V, Y) = (B\text{-Mod}, B\mathcal{I})\), then in general \((\Delta(U, B\text{-Mod}), \Delta(U, B\text{-Mod})^\perp) \neq \perp(\mathcal{X} B\text{-Mod})_Y, (\mathcal{X} B\text{-Mod})_Y\)). However, one has

\[
(\perp\nabla(\mathcal{X}, B\mathcal{I}), \nabla(\mathcal{X}, B\mathcal{I})) = \perp(\mathcal{X} B\text{-Mod})_Y, (\mathcal{X} B\text{-Mod})_Y^\perp.
\]

The above information is listed in Table 1 below, where

\[
\mathcal{A} := A\text{-Mod}, \quad \mathcal{B} := B\text{-Mod}, \quad \text{proj. := projective.}
\]

Table 1: Cotorsion pairs in \(A\text{-Mod}\)
Theorem 4.6. Let \( (A, \Lambda N) \) be a Morita ring which is an Artin algebra with \( M \otimes_A N = 0 = N \otimes_B M \). Assume that \( A \) and \( B \) are quasi-Frobenius rings, that \( \Lambda N \) and \( B M \) are projective, and that \( MA \) and \( NB \) are flat. Then

1. \( A \) is a Gorenstein ring with \( \text{inj.dim}_A A \leq 1 \), and \( \Lambda P_{\leq \infty} = \Lambda P_{\leq 1} = (\Lambda P_A)^{\perp} = (\Lambda P_B)^{\perp} = \Lambda I_{\leq 1} = \Lambda I_{< \infty} \).

2. The cotorsion pair \((\perp A^T_B), (A^T_B)^{\perp}\) coincides with \((\text{Mon}(A), \text{Mon}(A)^{\perp})\); and it is exactly the Gorenstein-projective cotorsion pair \((\text{GP}(A), \Lambda P_{\leq 1})\). So, it is complete and hereditary, and \( \text{GP}(A) = \text{Mon}(A) = \perp A P, \text{Mon}(A)^{\perp} = \Lambda P_{\leq 1} \).
(2) The cotorsion pair \((\mathcal{P}^\Lambda, \mathcal{P}^\Lambda)\) coincides with \((\text{Epi}(\Lambda), \text{Epi}(\Lambda))\); and it is exactly the Gorenstein-injective cotorsion pair \((\mathcal{P}^{\Lambda^{\perp}}_\Lambda, \text{GI}(\Lambda))\). So, it is complete and hereditary, and
\[
\text{GI}(\Lambda) = \text{Epi}(\Lambda) = \Lambda I^+, \quad \text{Epi}(\Lambda) = \mathcal{P}^{\Lambda^{\perp}}_\Lambda.
\]

**Example 4.7.** (1) We give an example to justify the existence of the assumptions in Theorem 4.6. Let \(Q\) be the quiver
\[
\begin{array}{c}
1 \\
\downarrow \\
2 \\
\downarrow \\
3 \\
\vdots \\
n
\end{array}
\]
and \(A = kQ/J^h\), where \(J\) is the ideal of path algebra \(kQ\) generated by all the arrows, and \(2 \leq h \leq n\). Then \(A\) is a self-injective algebra, in particular, a quasi-Frobenius ring. Let \(e = e_i\), \(e' = e_j\), where \(1 \leq i < j \leq n\), satisfying \(j - i \geq h\). Then \(e' A e = e_j A e_i = 0\). Put \(M := A e \otimes_k e' A\). Then \(\Lambda\) is projective, and \(M \otimes_A M = (A e \otimes_k e' A) \otimes_A (A e \otimes_k e' A) = A e \otimes_k (e' A \otimes_A A e) \otimes_k e' A = 0\).

Take \(\Lambda = (\Lambda^\Lambda_M^A)\). Then \(\Lambda\) satisfies all the conditions in Theorem 4.6.

**Remark 4.8.** (1) Non-zero Morita rings \(\Lambda\) in Theorem 4.6 do not satisfy the sufficient condition for self-injective algebras in [GrP, Proposition 3.7]. In fact, \(\Lambda\) can not be quasi-Frobenius: otherwise \(\text{Mon}(\Lambda) = \text{GP}(\Lambda) = \Lambda\text{-Mod}\), which is absurd!

(2) Although Theorem 4.6 does not give new cotorsion pairs, in the sense that they are just the Gorenstein-projective (respectively, Gorenstein-injective) cotorsion pairs, however, \(\text{GP}(\Lambda) = \text{Mon}(\Lambda)\) is a new result. In the special case of triangular matrix rings, this is known, by [LiZ, Thm. 1.1], [XZ, Cor. 1.5], [Z2, Thm. 1.4], [LuoZ1, Thm. 4.1], [ECIT, Thm. 3.5]. For more relations between monomorphism categories and the Gorenstein-projective modules, we refer to [Z1], [GrP], [LuoZ2], [GaP], [ZX], [HLXZ].

4.8. Modules \((\mathcal{P}^\Lambda_{\text{nPB}})\) and \((\mathcal{I}^\Lambda_{\text{nPB}})\). To prove Theorem 4.6 we need the following fact, which is of independent interest.

**Lemma 4.9.** Let \(\Lambda = (\Lambda^\Lambda_M^N)\) be a Morita ring with \(M \otimes_A N = 0 = N \otimes_B M\).

(1) Assume that \(A\) and \(B\) are projective modules. Let \((\mathcal{P}^\Lambda_{\text{nPB}})_{f,g} \in (\mathcal{P}^\Lambda_{\text{nPB}})\). Then
\[
0 \rightarrow \left(\frac{N \otimes_B Q}{M \otimes_A P}\right)_{0,0} \xrightarrow{(\mathcal{P}^\Lambda_{\text{nPB}})_{1,0}} \left(\frac{N \otimes_B Q}{Q}\right)_{0,1} \oplus \left(\frac{N \otimes_B Q}{Q}\right)_{f,g} \rightarrow 0
\]
is a projective resolution of \((\mathcal{P}^\Lambda_{\text{nPB}})_{f,g}\). In particular, \(\text{proj.dim}(\mathcal{P}^\Lambda_{\text{nPB}})_{f,g} \leq 1\), \(\forall \ (\mathcal{P}^\Lambda_{\text{nPB}})_{f,g} \in (\mathcal{P}^\Lambda_{\text{nPB}})\).

(2) Assume that \(A\) and \(B\) are flat modules. Let \((\mathcal{I}^\Lambda_{\text{nPB}})_{f,g} \in (\mathcal{I}^\Lambda_{\text{nPB}})\). Then
\[
0 \rightarrow \left(\frac{I}{I}\right)_{f,g} \xrightarrow{(\mathcal{I}^\Lambda_{\text{nPB}})_{f,g}} \bigoplus_{i} \left(\frac{\text{Hom}_R(M,J)}{\text{Hom}_R(M,J)}\right)_{f,g} \rightarrow 0
\]
is an injective resolution of \((\mathcal{I}^\Lambda_{\text{nPB}})_{f,g}\). In particular, \(\text{inj.dim}(\mathcal{I}^\Lambda_{\text{nPB}})_{f,g} \leq 1\), \(\forall \ (\mathcal{I}^\Lambda_{\text{nPB}})_{f,g} \in (\mathcal{I}^\Lambda_{\text{nPB}})\).
Remark 4.10. The condition $M \otimes_A N = 0 = N \otimes_B M$ cannot be relaxed to $\phi = 0 = \psi$. Otherwise, for example in $(1)$, $\text{Ker} \left( \begin{array}{c} f' \\ g' \end{array} \right) = \left( \begin{array}{c} N \otimes_B Q \\ M \otimes_A P \end{array} \right) \rightarrow (1_{\gamma} \otimes_A g) - (1_{\gamma} \otimes_B f)$, which is no longer a projective left $\Lambda$-module. The similar remark for $(2)$.

Proof of Lemma 4.9

$(1)$ Thanks to the assumption $M \otimes_A N = 0 = N \otimes_B M$, the given maps are $\Lambda$-maps (otherwise $\left( \begin{array}{c} f' \\ g' \end{array} \right)$ is not necessarily a $\Lambda$-map in general, even if $\phi = 0 = \psi$). We omit the details. The given sequence of $\Lambda$-modules is exact, since

$$0 \rightarrow N \otimes_B Q \xrightarrow{(-\gamma)} P \oplus (N \otimes_B Q) \xrightarrow{(1,0)} P \rightarrow 0$$

and

$$0 \rightarrow M \otimes_A P \xrightarrow{(\gamma)} (M \otimes_A P) \oplus Q \xrightarrow{(f,1)} Q \rightarrow 0$$

are exact.

We claim that $\left( \begin{array}{c} N \otimes_B Q \\ M \otimes_A P \end{array} \right)_{\gamma,0}$ is a projective left $\Lambda$-module. In fact, since $A_N$ and $B_Q$ are projective, $N \otimes_B Q$ is a projective left $\Lambda$-module. Since $M \otimes_A N = 0$, it follows that $\left( \begin{array}{c} N \otimes_B Q \\ M \otimes_A P \end{array} \right)_{\gamma,0} = \left( \begin{array}{c} N \otimes_B Q \\ M \otimes_A N \otimes_B Q \end{array} \right)_{\gamma,0}$ is a projective left $\Lambda$-module. Similarly, $\left( \begin{array}{c} 0 \\ M \otimes_A P \end{array} \right)_{0,0}$ is a projective left $\Lambda$-module. Thus, $\left( \begin{array}{c} N \otimes_B Q \\ M \otimes_A P \end{array} \right)_{\gamma,0} = \left( \begin{array}{c} 0 \\ M \otimes_A P \end{array} \right)_{0,0} \oplus \left( \begin{array}{c} 0 \\ M \otimes_A P \end{array} \right)_{0,0}$ is a projective left $\Lambda$-module.

$(2)$ This can be similarly proved as $(1)$. Since $M \otimes_A N = 0 = M \otimes_A N$, the given sequence is an exact sequence of $\Lambda$-maps. Since $M_A$ is flat and $B_J$ is injective, $\text{Hom}_B(M,J)$ is an injective left $\Lambda$-module, and hence

$$\left( \text{Hom}_B(M,J) \right)_{0,0} = \left( \text{Hom}_B(M,J) \right)_{0,0}$$

is an injective left $\Lambda$-module. Similarly, $\left( \text{Hom}_A(N,I) \right)_{0,0}$ is an injective left $\Lambda$-module. Thus, $\left( \text{Hom}_A(N,I) \right)_{0,0} = \left( \text{Hom}_A(N,I) \right)_{0,0} \oplus \left( \text{Hom}_A(N,I) \right)_{0,0}$ is an injective left $\Lambda$-module.

Proof of Theorem 4.6

$(1)$ Since $A$ is quasi-Frobenius, $A_A \subseteq A_I$. Since $B$ is quasi-Frobenius and $B_M$ is projective, $B_M \subseteq B_I$. Since $M_A$ and $N_B$ are flat, it follows from Lemma 4.9(2) that $\text{inj.dim}_A(A_M,1,0) \leq 1$.

Similarly, since $A$ is quasi-Frobenius, $A_N \subseteq A_I$. Since $B$ is quasi-Frobenius, $B_B \subseteq B_I$. Since $M_A$ and $N_B$ are flat, $\text{inj.dim}_A(B_M,0,1) \leq 1$, by Lemma 4.9(2).

Thus, $\text{inj.dim}_A \Lambda \leq 1$. By the right module version of Lemma 4.9(2) one knows $\text{inj.dim}_A \Lambda \leq 1$. Thus $\Lambda$ is a Gorenstein ring.

Since $\Lambda$ is Gorenstein with $\text{inj.dim}_A \Lambda \leq 1$, it is well-known that $\Lambda \mathcal{P}^\infty = \Lambda \mathcal{P}^{\leq 1} = \Lambda \mathcal{T}^{\leq 1} = \Lambda \mathcal{T}^\infty$.

Since $A_N$ and $B_M$ are projective modules, it follows from Lemma 4.9(1) that $\left( \begin{array}{c} \Lambda \mathcal{P}^{\leq 1} \end{array} \right) \subseteq \Lambda \mathcal{P}^{\leq 1}$.

On the other hand, for any $\left( \begin{array}{c} X \\ Y \end{array} \right)_{f,g} \in \Lambda \mathcal{P}^{\leq 1}$, let $0 \rightarrow \left( \begin{array}{c} P_1 \\ P_2 \end{array} \right) \rightarrow \left( \begin{array}{c} P_1 \\ P_2 \end{array} \right) \rightarrow \left( \begin{array}{c} X \\ Y \end{array} \right)_{f,g} \rightarrow 0$ be a
projective resolution of $(\check{Y}_f, g)$. Then one has exact sequence $0 \to P_1 \to P_0 \to X \to 0$. Since $A \cdot N$ is projective, $P_1$ and $P_0$ are projective (cf. Subsection 2.5), and hence injective. Thus the exact sequence splits and hence $X$ is projective. Similarly, $Y$ is projective. This shows $(\check{Y}_f, g) \in (A_{\mathcal{P}})$. Hence $A \mathcal{P} \leq 1 = (A_{\mathcal{P}}) = (A_{\mathcal{P}}^\perp)$.

(2) By (1), $A$ is Gorenstein and $(A_{\mathcal{P}}^\perp) = \mathcal{A}_{\mathcal{P}}^{<\infty}$. Thus, $(\mathcal{A}_{\mathcal{P}}^\perp) = \mathcal{GP}(A)$ and $(\mathcal{A}_{\mathcal{P}}^\perp, T_{\mathcal{A}})$ is just the Gorenstein-projective cotorsion pair, so it is complete and hereditary.

By Theorem 3.35(1), $\text{Mon}(A)^\perp \subseteq (A_{\mathcal{P}}^\perp) = \mathcal{A}_{\mathcal{P}}^{<\infty}$. Thus, to see $(\text{Mon}(A), \text{Mon}(A)^\perp) = (\mathcal{GP}(A), \mathcal{A}_{\mathcal{P}}^{<\infty})$, it suffices to show $\text{Mon}(A) \subseteq \mathcal{GP}(A)$. Since $A$ is Gorenstein, $\mathcal{GP}(A) = \mathcal{A}_{\mathcal{P}}^{<\infty}$. See Subsection 2.9. While $\text{inj.dim}_A \Lambda \leq 1$, each projective $A$-module is of injective dimension $\leq 1$. It follows that $\mathcal{A}_{\mathcal{P}} = \mathcal{A}_{\mathcal{P}}^{\leq 1}$. Thus, it suffices to show $\text{Mon}(A) \subseteq \mathcal{A}_{\mathcal{P}}$, namely, it suffices to show

$$\text{Ext}_A^1(\text{Mon}(A), T_A(A_{\mathcal{P}} \oplus T_B(B_{\mathcal{P}})) = 0.$$  

This is indeed true. In fact, let $(\check{Y}_f, g) \in \text{Mon}(A)$. For any $P \in A_{\mathcal{P}}$, there is an exact sequence

$$0 \to (M_0_{\otimes A} P)_{0,0} \to T_A P = (M_0_{\otimes A} P)_{1,0} \to (p_0) \to 0.$$

By Lemma 3.10(2) one has

$$\text{Ext}_A^1((\check{Y}_f, g), (M_0_{\otimes A} P)_{0,0}) = \text{Ext}_B^1(\text{Coker} f, M \otimes_A P) = 0$$

since $M \otimes_A P$ is projective as a left $B$-module (and hence injective). By Lemma 3.10(1), one has

$$\text{Ext}_A^1((\check{Y}_f, g), (p_0)_{0,0}) = \text{Ext}_A^1(\text{Coker} g, P) = 0.$$

Thus $\text{Ext}_A^1((\check{Y}_f, g), T_A P) = 0$. This shows $\text{Ext}_A^1(\text{Mon}(A), T_A(A_{\mathcal{P}})) = 0$.

Similarly, $\text{Ext}_A^1(\text{Mon}(A), T_B(B_{\mathcal{P}})) = 0$. Thus $(\text{Mon}(A), \text{Mon}(A)^\perp) = (\mathcal{GP}(A), \mathcal{A}_{\mathcal{P}}^{<\infty})$, in particular, $\text{Mon}(A) = \mathcal{GP}(A) = \mathcal{A}_{\mathcal{P}}^{<\infty}$, $\text{Mon}(A)^\perp = \mathcal{A}_{\mathcal{P}}^{\leq 1}$.

The assertion $(2)'$ is the dual of (2).

\section{Completeness}

To study abelian model structures on Morita rings, a key step is to know the completeness of cotorsion pairs in Morita rings.

\subsection{Completeness via cogenerations by sets}

First, by [ET2, Theorem 10], one has

\begin{proposition}
Let $\Lambda = (\frac{A}{M} \mathcal{Y} \frac{B}{N})$ be a Morita ring with $\phi = 0 = \psi$, $(\mathcal{U}, \mathcal{X})$ and $(\mathcal{V}, \mathcal{Y})$ cotorsion pairs in $A$-Mod and $B$-Mod, cogenerated by sets $S_1$ and $S_2$, respectively.

(1) If $\text{Tor}_A^1(M, \mathcal{U}) = 0 = \text{Tor}_B^1(N, \mathcal{V})$, then cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is cogenerated by $T_A(S_1) \cup T_B(S_2)$, and hence complete.

(2) If $M \otimes_A N = N \otimes_B M$, then cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is cogenerated by $Z_A(S_1) \cup Z_B(S_2)$, and hence complete.
\end{proposition}
Proof. (1) By Theorem 3.1(1), \( (\frac{X}{Y}, \frac{Y}{Z}) \) is a cotorsion pair in \( \Lambda \)-Mod. By Lemma 3.9(1), \( (\frac{X}{Y}, \frac{Y}{Z}) = (\frac{S_1}{S_2}, \frac{S_2}{S_3}) = (T_A(S_1) \cup T_B(S_2)) \). Thus, \( (\frac{X}{Y}, \frac{Y}{Z}) \) is complete, by Proposition 3.4.

(2) Without loss of generality, one may assume that \( S_1 \supseteq \mathcal{A} \mathcal{P} \) and \( S_2 \supseteq \mathcal{B} \mathcal{P} \). Then by Lemma 3.11(2) one has \( \mathcal{V}(X, Y) = \mathcal{V}(S_1^2, S_2^2) = (Z_A(S_1) \cup Z_B(S_2)) \).

Proposition 3.1.5 gives some information on the completeness of the cotorsion pairs in Morita rings. However, since Proposition 2.4 has no dual versions in general, there are no results on the completeness of \((\frac{U}{V}, \frac{V}{W})\) and \((\Delta(U, V), \Delta(U, V)\); moreover, it is more natural to study the completeness of the cotorsion pairs given in Theorems 3.1 and 3.2 directly from the completeness of \((U, X)\) and \((V, Y)\), rather than requiring that they are cogenerated by sets. Thus, we need module-theoretical methods to the completeness of the cotorsion pairs in Morita rings.

Such a general investigation is difficult. We will deal with this question, by assuming that one of \((U, X)\) and \((V, Y)\) is arbitrary, and that another is the projective or injective cotorsion pair. In view of Section 4, we only consider cotorsion pairs in Theorem 3.1.

5.2. Main results on completeness. Take \((V, Y)\) to be an arbitrary complete cotorsion pair in \( B \)-Mod. For cotorsion pair \( (\frac{X}{Y}, \frac{Y}{Z}) \) in Theorem 3.1(1), taking \((U, X) = (\mathcal{A} \mathcal{P}, \Lambda \text{-Mod})\), we have assertion (1) below; for cotorsion pair \( (\frac{U}{V}, \frac{V}{W}) \) in Theorem 3.1(2), taking \((U, X) = (\Lambda \text{-Mod}, \Lambda \mathcal{I})\), we have assertion (2) below.

**Theorem 5.2.** Let \( \Lambda = (\frac{\mathcal{A} \mathcal{N}}{\mathcal{M} \mathcal{B}}) \) be a Morita ring with \( \phi = \psi = 0 \), and \((V, Y)\) a complete cotorsion pair in \( B \)-Mod. Suppose that \( N_B \) is flat and \( B M \) is projective.

1. If \( M \otimes_{\mathcal{A}} \mathcal{P} \subseteq \mathcal{Y} \), then \( (\frac{\mathcal{A} \text{-Mod} \mathcal{Y}}{\mathcal{Y}}) \) is a complete cotorsion pair in \( \Lambda \text{-Mod} \); and it is hereditary if \((V, Y)\) is hereditary.

Moreover, if \( M \otimes_{\mathcal{A}} N = 0 = N \otimes_{B} M \), then \( \frac{\mathcal{A} \text{-Mod} \mathcal{Y}}{\mathcal{Y}} = T_A(\mathcal{A} \mathcal{P}) \oplus T_B(\mathcal{Y}) \), and hence \( (T_A(\mathcal{A} \mathcal{P}) \oplus T_B(\mathcal{Y}), \frac{\mathcal{A} \text{-Mod} \mathcal{Y}}{\mathcal{Y}}) \) is a complete cotorsion pair; and it is hereditary if \((V, Y)\) is hereditary.

2. If \( \text{Hom}_A(N, \mathcal{I}) \subseteq \mathcal{V} \), then \( (\frac{\mathcal{A} \text{-Mod} \mathcal{Y}}{\mathcal{Y}}) \) is a complete cotorsion pair in \( \Lambda \text{-Mod} \); and it is hereditary if \((V, Y)\) is hereditary.

Moreover, if \( M \otimes_{\mathcal{A}} N = 0 = N \otimes_{B} M \), then \( \frac{\mathcal{A} \text{-Mod} \mathcal{Y}}{\mathcal{Y}} = \text{H}_A(\mathcal{I}) \oplus \text{H}_B(\mathcal{Y}) \), and hence \( (\frac{\mathcal{A} \text{-Mod} \mathcal{Y}}{\mathcal{Y}}, \text{H}_A(\mathcal{I}) \oplus \text{H}_B(\mathcal{Y})) \) is a complete cotorsion pair; and it is hereditary if \((V, Y)\) is hereditary.

**Remark 5.3.** (1) \( B M \) is injective, then \( M \otimes_{\mathcal{A}} \mathcal{P} \subseteq \mathcal{V} \) always holds.

(2) If \( B \) is quasi-Frobenius and \( N_B \) is flat, then \( \text{Hom}_A(N, \mathcal{I}) \subseteq \mathcal{V} \) always holds.

Take \((U, X)\) to be an arbitrary complete cotorsion pair in \( \Lambda \text{-Mod} \). For cotorsion pair \( (\frac{X}{Y}, \frac{Y}{Z}) \) in Theorem 3.1(1), taking \((V, Y) = (\mathcal{B} \mathcal{P}, B \text{-Mod})\), we have assertion (1) below; for cotorsion pair \( (\frac{U}{V}, \frac{V}{W}) \) in Theorem 3.1(2), taking \((V, Y) = (B \text{-Mod}, B \mathcal{I})\), we have assertion (2) below.
Theorem 5.4. Let $\Lambda = (\frac{A}{M} \frac{N}{P})$ be a Morita ring with $\phi = 0 = \psi$, and $(\mathcal{U}, \mathcal{X})$ a complete cotorsion pair in $\Lambda$-$\text{Mod}$. Suppose that $M_A$ is flat and $A_N$ is projective.

(1) If $N \otimes_B \mathcal{P} \subseteq \mathcal{X}$, then $(\mathcal{X} \otimes_{\text{B-Mod}} \mathcal{U})$, $(\mathcal{X} \otimes_{\text{B-Mod}} \mathcal{U}^\perp)$ is a complete cotorsion pair in $\Lambda$-$\text{Mod}$; and it is hereditary if $(\mathcal{U}, \mathcal{X})$ is hereditary.

Moreover, if $M \otimes_A N = 0 = N \otimes_B M$, then $(\mathcal{X} \otimes_{\text{B-Mod}} \mathcal{U}) = T_A(\mathcal{U}) \oplus T_B(\mathcal{B})$, and hence

$$(T_A(\mathcal{U}) \oplus T_B(\mathcal{B})), (\mathcal{X} \otimes_{\text{B-Mod}} \mathcal{U}^\perp)$$

is a complete cotorsion pair; and it is hereditary if $(\mathcal{U}, \mathcal{X})$ is hereditary.

(2) If $\text{Hom}_B(M, B\mathcal{T}) \subseteq \mathcal{U}$, then $(\mathcal{X} \otimes_{\text{B-Mod}} \mathcal{U})$, $(\mathcal{X} \otimes_{\text{B-Mod}} \mathcal{U}^\perp)$ is a complete cotorsion pair in $\Lambda$-$\text{Mod}$; and it is hereditary if $(\mathcal{U}, \mathcal{X})$ is hereditary.

Moreover, if $M \otimes_A N = 0 = N \otimes_B M$, then $(\mathcal{X} \otimes_{\text{B-Mod}} \mathcal{U})^\perp = \text{H}_A(\mathcal{X}) \oplus \text{H}_B(\mathcal{B} \mathcal{T})$, and hence

$$(\mathcal{X} \otimes_{\text{B-Mod}} \mathcal{U})$, $(\mathcal{X} \otimes_{\text{B-Mod}} \mathcal{U}^\perp)$$

is a complete cotorsion pair; and it is hereditary if $(\mathcal{U}, \mathcal{X})$ is hereditary.

Remark 5.5. (1) If $A_N$ is injective, then $N \otimes_B \mathcal{P} \subseteq \mathcal{X}$ always holds.

(2) If $A$ is quasi-Frobenius and $M_A$ is flat, then $\text{Hom}_B(M, B\mathcal{T}) \subseteq \mathcal{U}$ always holds.

5.3. Lemmas for Theorem [5.2] To prove Theorem 5.2(1), we need

Lemma 5.6. Let $\Lambda = (\frac{A}{M} \frac{N}{P})$ be a Morita ring with $\phi = 0 = \psi$. Suppose $B\mathcal{M}$ is projective. For a $\Lambda$-module $(\frac{L_1}{L_2})_{f,g}$, let $\pi : P \longrightarrow L_1$ be an epimorphism with $A\mathcal{P}$ projective, and $0 \longrightarrow Y\rightarrow V \rightarrow L_2 \rightarrow 0$ an exact sequence. Then there is an exact sequence of the form:

$$0 \rightarrow (\frac{K}{(M \otimes \mathcal{P}) \otimes V})_{\alpha, \beta} \rightarrow (\frac{P}{M \otimes \mathcal{P}})_{1,0} \oplus (\frac{N \otimes V}{V})_{0,1} \rightarrow (\frac{(f \otimes \pi)}{(g(1 \otimes \pi))})_{\pi} \rightarrow (\frac{(f_1)}{(L_2)})_{f, g} \rightarrow 0.$$

Proof. For convenience, rewrite the sequence as

$$0 \rightarrow (\frac{K}{(M \otimes \mathcal{P}) \otimes V})_{\alpha, \beta} \rightarrow (\frac{P}{(N \otimes \mathcal{P}) \otimes V})_{(1 \otimes \mathcal{P}) \otimes V} \rightarrow (\frac{(f_1)}{(f(1 \otimes \pi))})_{f, g} \rightarrow 0.$$

We claim that $(\frac{(\pi, g(1 \otimes \pi'))}{(f_1 \otimes \pi', \pi)})$ is a $\Lambda$-epimorphism. In fact, by $\phi = 0 = \psi$, $f(1 \otimes g) = 0 = g(1 \otimes f)$. Hence

$$f(1 \otimes g(1 \otimes \pi')) = 0 : M \otimes_A N \otimes_B V \longrightarrow L_2$$

and

$$g(1 \otimes f(1 \otimes \pi)) = 0 : N \otimes_B M \otimes P \longrightarrow L_1.$$
Thus, there is a $Y$-map. Clearly, it is an epimorphism.

It remains to see that $\text{Ker} \left( \frac{(\pi,g(1_N \otimes \pi'))}{(f(1_M \otimes \pi), \pi')} \right)$ is of the form $\left( (\pi,g(1_N \otimes \pi')) \right)_{\alpha,\beta}$.

In fact, as a $\Lambda$-module, $\text{Ker} \left( \frac{(\pi,g(1_N \otimes \pi'))}{(f(1_M \otimes \pi), \pi')} \right)$ is of the form $\left( \frac{K}{K'} \right)_{\alpha,\beta}$, where $K' = \text{Ker}(f(1_M \otimes \pi), \pi')$. Thus, it suffices to show $\text{Ker}(f(1_M \otimes \pi), \pi') \cong (\Lambda \otimes_A P) \otimes Y$.

Since $AP$ is projective, $M \otimes_A P$ is a direct summand of copies of $BM$, as a left $B$-module. While by assumption $BM$ is projective, it follows that $M \otimes_A P$ is a projective left $B$-module. Thus, there is a $B$-map $h$ such that the following diagram commutes:

$$
\begin{array}{c}
M \otimes_A P \xrightarrow{\text{1}_{M \otimes \pi}} M \otimes L_1 \xrightarrow{0}
\end{array}
$$

Then it is clear that

$$
0 \xrightarrow{} (M \otimes_A P) \oplus Y \xrightarrow{(1_h,0)} (M \otimes_A P) \oplus V \xrightarrow{(f(1_M \otimes \pi), \pi')} L_2 \xrightarrow{0}
$$
is exact. This completes the proof. \qed

To prove Theorem (2), we need the following lemma, in which it is more convenient to write a $\Lambda$-module in the second expression.

**Lemma 5.7.** Let $\Lambda = (\frac{A}{M} \otimes_B \frac{N}{B})$ be a Morita ring with $\phi = 0 = \psi$. Suppose $N_B$ is flat. For $\Lambda$-module $(\frac{L_2}{L_1})_{f,\tilde{g}}$, let $\sigma : L_1 \rightarrow I$ be a monomorphism with $AI$ injective, and $0 \rightarrow L_2 \xrightarrow{\sigma'} Y \xrightarrow{\pi} V \rightarrow 0$ an exact sequence. Then there is an exact sequence of the form:

$$
0 \rightarrow (L_1)_{f,\tilde{g}} \xrightarrow{((\alpha,\beta) \gamma)} (M,\sigma') \rightarrow \begin{pmatrix} I_{\text{Hom}_A(N,I)} & 0 \end{pmatrix}_{(0,1)} \oplus \begin{pmatrix} \text{Hom}_B(M,Y) \end{pmatrix}_{(1,0)} \rightarrow \begin{pmatrix} \text{Hom}_A(C_{N,I}Y) \end{pmatrix}_{\alpha,\beta} \rightarrow 0.
$$

**Proof.** Rewrite the sequence as

$$
0 \rightarrow (L_1)_{f,\tilde{g}} \xrightarrow{((\alpha,\beta) \gamma)} (M,\sigma') \rightarrow \begin{pmatrix} I_{\text{Hom}_B(M,Y)} \end{pmatrix}_{(0,1)} \oplus \begin{pmatrix} \text{Hom}_A(N,I) \end{pmatrix}_{(0,1)} \rightarrow \begin{pmatrix} \text{Hom}_A(C_{N,I}Y) \end{pmatrix}_{\alpha,\beta} \rightarrow 0.
$$
Since $\phi = 0 = \psi$, $(M, \tilde{g})\tilde{f} = 0 = (N, \tilde{f})\tilde{g}$, and hence $(M, (N, \sigma)\tilde{g})\tilde{f} = 0 = (N, (M, \sigma')\tilde{f})\tilde{g}$. Thus the following diagrams commute:

\[
\begin{array}{ccc}
\text{Hom}_B(M, L_2) & \xrightarrow{(N, \sigma)\tilde{g}} & \text{Hom}_A(N, L_1) \\
\text{Hom}(N, I) & \xrightarrow{(N, \sigma)\tilde{g}} & \text{Hom}(N, I)
\end{array}
\]

Therefore the map $\begin{pmatrix} (M, \sigma'\tilde{f}) \\ (N, \sigma'\tilde{g}) \end{pmatrix}$ is a $\Lambda$-map. Clearly, it is a monomorphism.

Write $\text{Coker} \begin{pmatrix} (M, \sigma'\tilde{f}) \\ (N, \sigma'\tilde{g}) \end{pmatrix}$ as $\begin{pmatrix} C' \\ \tilde{g} \end{pmatrix}$, then $C'$ is the cokernel of $B$-monomorphism $\begin{pmatrix} (N, \sigma'\tilde{g}) \\ \sigma' \end{pmatrix}$.

Since $N_B$ is flat and $A^I$ is injective, it follows that $\text{Hom}_A(N, I)$ is an injective left $B$-module. Thus there is a $B$-map $h$ such that the diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{L_2} & L_2 \\
\tilde{g} & \xrightarrow{h} & 0
\end{array}
\]

commutes. Therefore

\[
0 \longrightarrow L_2 \xrightarrow{(N, \sigma)\tilde{g}} \text{Hom}_A(N, I) \oplus Y \xrightarrow{\begin{pmatrix} 1 & -\tilde{h} \end{pmatrix}} \text{Hom}_A(N, I) \oplus V \longrightarrow 0
\]

is an exact sequence. It follows that $C' \cong \text{Hom}_A(N, I) \oplus V$. This completes the proof. $\square$

5.4. Proof of Theorem 5.2 (1) Since $N_B$ is flat, by Theorem [3.11], $(\mathcal{A}^\text{mod}, \mathcal{A}^\text{mod})$ is a cotorsion pair; and it is hereditary if $\mathcal{V}$ is hereditary.

Since $\mathcal{V}$ is complete, for any $\Lambda$-module $\begin{pmatrix} L_1 \\ L_2 \end{pmatrix}_{f,g}$, there is an exact sequence

\[
0 \longrightarrow Y \longrightarrow V \longrightarrow L_2 \longrightarrow 0
\]

with $V \in \mathcal{V}$ and $Y \in \mathcal{Y}$. Since $B M$ is projective, by Lemma 3.10 there is an exact sequence of $\Lambda$-modules of the form:

\[
0 \longrightarrow \begin{pmatrix} K \oplus P \end{pmatrix}_{\alpha, \beta} \longrightarrow \begin{pmatrix} P \oplus V \end{pmatrix}_{1, 0} \oplus \begin{pmatrix} N \oplus M \end{pmatrix}_{0, 1} \longrightarrow \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}_{f,g} \longrightarrow 0
\]
where \(AP\) is projective. Since by assumption \(M \otimes_A P \subseteq \mathcal{Y}, (M \otimes_A P) \oplus Y \in \mathcal{Y}\), and hence \(\left ( (M \otimes_A P) \oplus Y \right )_{\alpha,\beta} \in \left ( \text{A}^{-\text{Mod}} \right )_{\mathcal{Y}}\).

On the other hand, \(\left ( M \otimes_A P \right )_{1,0} = T_A P\) is a projective \(\Lambda\)-module, so it is in \(\left ( \text{A}^{-\text{Mod}} \right )_{\mathcal{Y}}\). Also, \(\left ( N \otimes_B V \right )_{0,1} = T_B V \in T_B (\mathcal{V})\). Since \(N_B\) is flat and \(\text{Ext}_B^2 (\mathcal{V}, \mathcal{Y}) = 0\), by Lemma 3.8(2), \(\left ( N \otimes_B V \right )_{0,1} = T_B V \in \left ( \text{A}^{-\text{Mod}} \right )_{\mathcal{Y}}\). This shows the completeness of \(\left ( \text{A}^{-\text{Mod}} \right )_{\mathcal{Y}}, \left ( \text{A}^{-\text{Mod}} \right )_{\mathcal{Y}}\).

Finally, if \(M \otimes_A N = 0 = N \otimes_B M\), then by Corollary 4.5(1) one has \(\left ( \text{A}^{-\text{Mod}} \right )_{\mathcal{Y}} = \Delta (A \mathcal{P}, \mathcal{V}) = T_A (A \mathcal{P}) \oplus T_B (\mathcal{V})\).

(2) Since \(BM\) is projective, by Theorem 5.12, \(\left ( \text{A}^{-\text{Mod}} \right )_{\mathcal{V}}, \left ( \text{A}^{-\text{Mod}} \right )_{\mathcal{V}}\) is a cotorsion pair; and it is hereditary if \((\mathcal{V}, \mathcal{Y})\) is hereditary.

Since \((\mathcal{V}, \mathcal{Y})\) is complete, for any \(\Lambda\)-module \(\left ( L_1 \right )_{f,g}\), there is an exact sequence \(0 \rightarrow L_2 \rightarrow Y \rightarrow V \rightarrow 0\) with \(Y \in \mathcal{Y}\) and \(V \in \mathcal{V}\). Since \(N_B\) is flat, by Lemma 5.7, there is an exact sequence of \(\Lambda\)-modules of the form:

\[
0 \rightarrow \left ( L_1 \right )_{f,g} \rightarrow \left ( \text{Hom}_A (N, I) \right )_{0,1} \oplus \left ( \text{Hom}_A (M, Y) \right )_{1,0} \rightarrow \left ( \text{Hom}_A (N, J) \oplus V \right )_{\alpha,\beta} \rightarrow 0
\]

where \(AI\) is injective. Since \(\left ( \text{Hom}_A (N, I) \right )_{0,1}\) is an injective \(\Lambda\)-module, it is in \(\left ( \text{A}^{-\text{Mod}} \right )_{\mathcal{Y}}\). Since \(BM\) is projective and \(\text{Ext}_B^2 (V, \mathcal{Y}) = 0\), it follows from Lemma 3.8(4) that \(\left ( \text{Hom}_A (M, Y) \right )_{1,0} = H_B Y \in \left ( \text{A}^{-\text{Mod}} \right )_{\mathcal{Y}}\).

Since by assumption \(\text{Hom}_A (N, I) \subseteq \mathcal{V}, \text{Hom}_A (N, I) \subseteq \mathcal{Y}\), and hence \(\left ( \text{Hom}_A (N, J) \oplus V \right )_{\alpha,\beta} \in \left ( \text{A}^{-\text{Mod}} \right )_{\mathcal{Y}}\). This proves the completeness of \(\left ( \text{A}^{-\text{Mod}} \right )_{\mathcal{Y}}, \left ( \text{A}^{-\text{Mod}} \right )_{\mathcal{Y}}\).

Finally, if \(M \otimes_A N = 0 = N \otimes_B M\), then by Corollary 4.5(3) one has \(\left ( \text{A}^{-\text{Mod}} \right )_{\mathcal{Y}} = \nabla (A I, \mathcal{Y}) = H_A (A I) \oplus H_B (\mathcal{Y})\).

5.5. Lemmas for Theorem 5.4. To see Theorem 5.4(1), we need

Lemma 5.8. Let \(\Lambda = \left ( \begin{smallmatrix} A & N \\ B & D \end{smallmatrix} \right )\) be a Morita ring with \(\phi = 0 = \psi\). Suppose \(AN\) is projective. For a \(\Lambda\)-module \(\left ( L_1 \right )_{f,g}\), let \(\pi : Q \rightarrow L_2\) be an epimorphism with \(B Q\) projective, and \(0 \rightarrow X \stackrel{\sigma}{\rightarrow} U \stackrel{\pi}{\rightarrow} L_1 \rightarrow 0\) an exact sequence. Then there is an exact sequence of the form:

\[
0 \rightarrow \left ( X \otimes (N \otimes Q) \right )_{\alpha,\beta} \rightarrow \left ( U \otimes (N \otimes U) \right )_{\alpha,\beta} = \left ( N \otimes Q \right )_{0,1} \rightarrow \left ( L_1 \right )_{f,g} \rightarrow 0.
\]

Proof. The proof is similar to Lemma 5.6. We include the points. Rewrite the sequence as:

\[
0 \rightarrow \left ( X \otimes (N \otimes U) \right )_{\alpha,\beta} \rightarrow \left ( U \otimes (N \otimes U) \right )_{\alpha,\beta} = \left ( N \otimes U \right )_{0,1} \rightarrow \left ( L_1 \right )_{f,g} \rightarrow 0.
\]
The map $\left( (\pi', g(1_N \otimes \pi)) \middle| (f(1_M \otimes \pi'), \pi) \right)$ is a $\Lambda$-epimorphism, since the diagrams commute:

$$
\begin{array}{c}
(M \otimes_A U) \oplus (M \otimes_A N \otimes_B Q) & \xrightarrow{(1 \otimes \pi', 1_M \otimes g(1_N \otimes \pi))} & M \otimes_A L_1 \\
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} & \xrightarrow{(f(1_M \otimes \pi'), \pi)} & L_2 \\
(M \otimes_A U) \oplus Q & \xrightarrow{\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}} & L_2
\end{array}
$$

$$
\begin{array}{c}
(M \otimes_B M \otimes_A U) \oplus (N \otimes_B Q) & \xrightarrow{(1_N \otimes f(1_M \otimes \pi'), 1_N \otimes \pi)} & N \otimes_B L_2 \\
\begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix} & \xrightarrow{(\pi', g(1_N \otimes \pi))} & L_1 \\
U \oplus (N \otimes_B Q) & \xrightarrow{\begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}} & L_1.
\end{array}
$$

It remains to prove $\operatorname{Ker}(\pi', g(1_N \otimes \pi)) \cong X \oplus (N \otimes_B Q)$. Since $A N$ is projective, $N \otimes_B Q$ is a projective left $A$-module. Thus, there is an $A$-map $h$ such that the diagram

$$
\begin{array}{c}
N \otimes_B Q & \xrightarrow{1_N \otimes \pi} & N \otimes L_2 & \rightarrow 0 \\
\downarrow h & & \downarrow g & \\
0 & \rightarrow & X & \xrightarrow{\alpha} U & \xrightarrow{\pi'} & L_1 & \rightarrow 0.
\end{array}
$$

commutes. Then

$$
\begin{array}{c}
0 & \rightarrow & X \oplus (N \otimes_B Q) & \xrightarrow{(\sigma, -h)} & U \oplus (N \otimes_B Q) & \xrightarrow{(\pi', g(1_N \otimes \pi))} & L_1 & \rightarrow 0
\end{array}
$$

is exact. This completes the proof. \qedhere

To prove Theorem 5.4 (2), we need the following lemma, in which the second expression of a $\Lambda$-module is more convenient.

**Lemma 5.9.** Let $\Lambda = \left( \frac{M}{N} \right)$ be a Morita ring with $\phi = 0 = \psi$. Suppose $M_A$ is flat. For $A$-module $\left( \frac{L_1}{L_2} \right)_{f, \tilde{g}}$, let $\sigma : L_2 \rightarrow J$ be a monomorphism with $BJ$ injective, and $0 \rightarrow L_1 \xrightarrow{\sigma'} X \xrightarrow{\pi} U \rightarrow 0$ an exact sequence. Then there is an exact sequence of the form:

$$
0 \rightarrow \left( \begin{bmatrix}
L_1 & L_2 \\
\frac{f}{f \tilde{g}}
\end{bmatrix}, \frac{(M, \sigma)\tilde{f}}{(N, \sigma')\tilde{g}} \right) \rightarrow \left( \begin{bmatrix}
X \\
\operatorname{Hom}_A(N,X)
\end{bmatrix} \oplus \left( \frac{\operatorname{Hom}_B(M,J)}{C} \right)_{\alpha, \beta} \right) \rightarrow 0.
$$

**Proof.** The proof is similar to Lemma 5.7. We include the points. First, as in the proof of Lemma 5.7, one can show that the map

$$
\begin{bmatrix}
(M, \sigma)\tilde{f} \\
(N, \sigma')\tilde{g}
\end{bmatrix} : \left( \begin{bmatrix}
L_1 & L_2 \\
\frac{f}{f \tilde{g}}
\end{bmatrix} \right) \rightarrow \left( \begin{bmatrix}
X \\
\operatorname{Hom}_A(N,X) \oplus J
\end{bmatrix} \oplus \left( \frac{\operatorname{Hom}_B(M,J)}{C} \right)_{\alpha, \beta}
\end{bmatrix}
$$

is a $\Lambda$-monomorphism. We omit the details.

Write $\operatorname{Coker} \left( \begin{bmatrix}
(M, \sigma)\tilde{f} \\
(N, \sigma')\tilde{g}
\end{bmatrix} \right) = \left( \frac{C'}{C} \right)_{\alpha, \beta}$. Then $C' \cong \operatorname{Coker} \left( \begin{bmatrix}
(M, \sigma)\tilde{f} \\
(N, \sigma')\tilde{g}
\end{bmatrix} \right)$. Since $M_A$ is flat and $BJ$ is injective, it follows that $\operatorname{Hom}_B(M,J)$ is an injective left $A$-module. Thus there is an $A$-map $h$
such that the diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & L_1 & \xrightarrow{\sigma'} & X & \xrightarrow{\pi} & U & \rightarrow & 0 \\
\downarrow j & & \downarrow h & & \downarrow & & & \\
0 & \rightarrow & \text{Hom}_B(M, L_2) & \xrightarrow{(M, \sigma')} & \text{Hom}_B(M, J) & & & \\
\end{array}
\]

commutes. Therefore

\[
\begin{array}{ccccccccc}
0 & \rightarrow & L_1 & \xrightarrow{(M, \sigma')f} & X \oplus \text{Hom}_B(M, J) & \xrightarrow{(\pi, 0)} & U \oplus \text{Hom}_B(M, J) & \rightarrow & 0 \\
\end{array}
\]

is exact, and hence \( C' \cong U \oplus \text{Hom}_B(M, J) \). \( \square \)

5.6. **Proof of Theorem 5.4** The proof is similar to Theorem 5.2.

(1) Since \( M_A \) is flat, by Theorem 3.11, \( (\perp_{B-\text{Mod}}, (\perp_{B-\text{Mod}})) \) is a cotorsion pair; and it is hereditary if \( (U, \mathcal{X}) \) is hereditary.

For any \( \Lambda \)-module \( (\frac{L_1}{L_2})_{f, \beta} \), since \( (U, \mathcal{X}) \) is complete, there is an exact sequence

\[
0 \rightarrow X \rightarrow U \rightarrow L_1 \rightarrow 0
\]

with \( U \in U, X \in \mathcal{X} \). Since \( A N \) is projective, by Lemma 5.8 there is an exact sequence of \( \Lambda \)-modules of the form:

\[
0 \rightarrow \left( \frac{X \oplus (N \otimes Q)}{K} \right)_{\alpha, \beta} \rightarrow \left( \frac{U}{M \otimes U} \right)_{1, 0} \oplus \left( \frac{N \otimes Q}{Q} \right)_{0, 1} \rightarrow \left( \frac{L_1}{L_2} \right)_{f, g} \rightarrow 0
\]

where \( BQ \) is projective. Since by assumption \( N \otimes B \mathcal{P} \subset \mathcal{X} \), it follows that \( X \oplus (N \otimes B Q) \in \mathcal{X} \), and hence \( \left( \frac{X \oplus (N \otimes Q)}{K} \right)_{\alpha, \beta} \in (\perp_{B-\text{Mod}}) \).

Since \( \left( \frac{N \otimes Q}{Q} \right)_{0, 1} \) is in \( \perp_{\Lambda \mathcal{P}} \), it is in \( \perp_{B-\text{Mod}} \). Since \( M_A \) is flat and \( \text{Ext}_A^1(U, \mathcal{X}) = 0 \), by Lemma 3.8(1), \( (\frac{U}{M \otimes U})_{1, 0} = T_A U \in \perp_{B-\text{Mod}} \). Thus, \( (\perp_{B-\text{Mod}}, (\perp_{B-\text{Mod}})) \) is complete.

Finally if \( M \otimes B N = 0 = N \otimes B M \), then by Corollary 4.5(2) one has \( \perp_{B-\text{Mod}} \) is complete.

(2) Since \( A N \) is projective, by Theorem 3.12, \( (\perp_{B-\text{Mod}}, (\perp_{B-\text{Mod}})) \) is a cotorsion pair; and it is hereditary if \( (U, \mathcal{X}) \) is hereditary.

For any \( \Lambda \)-module \( (\frac{X}{Y})_{f, \beta} \), since \( (U, \mathcal{X}) \) is complete, there is an exact sequence \( 0 \rightarrow L_1 \rightarrow X \rightarrow U \rightarrow 0 \) with \( X \in \mathcal{X}, U \in U \). Since \( M_A \) is flat, by Lemma 5.9 there is an exact sequence:

\[
0 \rightarrow \left( \frac{L_1}{L_2} \right)_{f, \beta} \rightarrow \left( \frac{X}{\text{Hom}_A(N, X)} \right)_{0, 1} \oplus \left( \frac{\text{Hom}_B(M, J)}{C} \right)_{0, 1} \rightarrow \left( \frac{U \oplus \text{Hom}_B(M, J)}{\alpha, \beta} \right) \rightarrow 0
\]

where \( B J \) is injective. Since \( \left( \frac{\text{Hom}_B(M, J)}{C} \right)_{0, 1} \) is an injective \( \Lambda \)-module, it is in \( \perp_{B-\text{Mod}} \). Since \( A N \) is projective and \( \text{Ext}_A^1(U, \mathcal{X}) = 0 \), by Lemma 3.8(3), \( \left( \frac{X}{\text{Hom}_A(N, X)} \right)_{0, 1} = H_A X \in \perp_{B-\text{Mod}} \).
Since by assumption $\text{Hom}_B(M, B\mathcal{I}) \subseteq \mathcal{U}$, $\text{Hom}_B(M, J) \in \mathcal{U}$, and hence $(U \oplus \text{Hom}_B(M, J))_{A, \mathfrak{I}, \mathfrak{J}} \in (B\text{-Mod})_{A, \mathfrak{I}, \mathfrak{J}}$. So $(B\text{-Mod})_{A, \mathfrak{I}, \mathfrak{J}}$ is complete.

Finally if $M \otimes_A N = 0 = N \otimes_B M$, then by Corollary 4.3.4 one has $(\oplus_{(B\text{-Mod})}) = \nabla(\mathcal{X}, B\mathcal{I}) = H_A(\mathcal{X}) \oplus H_B(B\mathcal{I})$.

5.7. **Remark.** Under the framework of one of $(\mathcal{U}, \mathcal{X})$ and $(\mathcal{V}, \mathcal{Y})$ being an arbitrary complete cotorsion pair, and another being the projective or the injective one, the careful reader will find that the completeness of the following cotorsion pairs

\[
\begin{align*}
(\mathcal{X}, \mathcal{Y}) &\text{ has special right}\ A\text{-approximation.} \\
(\mathcal{X}, \mathcal{Y}) &\text{ has special left}\ B\text{-approximation.}
\end{align*}
\]

have not been discussed (also they will be not used in constructing Hovey triples in Section 7). An interesting special cases of $((\alpha_{n, p})^\perp, (\alpha_{n, p})^\perp)$ and $(\perp \mathcal{X}, \perp \mathcal{Y})$ have been treated in Theorem 4.6.

5.8. **Triangular matrix rings.** For the case of $M = 0$ one has

**Proposition 5.10.** Let $\Lambda = (\frac{\mathcal{A}}{\mathcal{B}} \frac{\mathcal{N}}{\mathcal{I}})$ be an upper triangular matrix ring. Suppose that $(\mathcal{U}, \mathcal{X})$ and $(\mathcal{V}, \mathcal{Y})$ are complete cotorsion pairs in $A\text{-Mod}$ and $B\text{-Mod}$, respectively.

1. Assume that $\text{Tor}_A^1(N, \mathcal{V}) = 0$. If $N \otimes_B \mathcal{V} \subseteq \mathcal{X}$, then the cotorsion pair

\[
(\Delta(\mathcal{U}, \mathcal{V}), (\mathcal{X})) = (T_A(\mathcal{U}) \oplus T_B(\mathcal{V}), (\mathcal{X}))
\]

is complete.

2. Assume that $\text{Ext}_A^1(\mathcal{N}, \mathcal{X}) = 0$. If $\text{Hom}_A(\mathcal{N}, \mathcal{X}) \subseteq \mathcal{V}$, then the cotorsion pair

\[
((\mathcal{X}), \nabla(\mathcal{X}, \mathcal{Y})) = ((\mathcal{X}), H_A(\mathcal{X}) \oplus H_B(\mathcal{Y}))
\]

is complete.

For lower triangle matrix rings (i.e., $N = 0$) one has the similar results. We omit the details. For proof of Proposition 5.10 we need

**Lemma 5.11.** ([AA, 3.1]) Let $\mathcal{C}$ be an abelian category with enough projectives and injectives. Assume that $(A, B)$ be a hereditary cotorsion pair in $\mathcal{C}$, and $0 \rightarrow X \overset{f}{\rightarrow} Y \overset{g}{\rightarrow} Z \rightarrow 0$ be an exact sequence in $\mathcal{C}$.

1. Assume that $X$ and $Z$ have special right $A$-approximation, i.e., there are exact sequences:

\[
0 \rightarrow B_1 \rightarrow A_1 \rightarrow X \rightarrow 0, \quad 0 \rightarrow B_2 \rightarrow A_2 \rightarrow Z \rightarrow 0,
\]

with $A_i \in A$, $B_i \in B$, $i = 1, 2$. Then $Y$ has a special right $A$-approximation.

2. Assume that $X$ and $Z$ have special left $B$-approximation, i.e., there are exact sequences

\[
0 \rightarrow X \rightarrow B_1 \rightarrow A_1 \rightarrow 0, \quad 0 \rightarrow Z \rightarrow B_2 \rightarrow A_2 \rightarrow 0
\]

with $B_i \in B$, $A_i \in A$, $i = 1, 2$. Then $Y$ has a special left $B$-approximation.
Proof of Proposition 5.10. (1) By Theorem 4.4(1), \((\Delta(\mathcal{U}, \mathcal{V}), (\mathcal{X}, \mathcal{Y})) = (\mathcal{T}_A(\mathcal{U}) \oplus \mathcal{T}_B(\mathcal{V}), (\mathcal{X}, \mathcal{Y}))\) is a cotorsion pair. Let \((\overline{L}_1, \overline{L}_2)_g\) be a \(\Lambda\)-module. By the completeness of \((\mathcal{U}, \mathcal{X})\) and \((\mathcal{V}, \mathcal{Y})\), one has the exact sequences

\[
0 \rightarrow X \overset{\sigma_1}{\rightarrow} U \overset{\pi_1}{\rightarrow} L_1 \rightarrow 0, \quad 0 \rightarrow Y \overset{\sigma_2}{\rightarrow} V \overset{\pi_2}{\rightarrow} L_2 \rightarrow 0
\]

in \(A\)-Mod and \(B\)-Mod respectively, with \(U \in \mathcal{U}, \ X \in \mathcal{X}, \ V \in \mathcal{V}, \) and \(Y \in \mathcal{Y}\). Then

\[
0 \rightarrow (\overline{X}^0) \rightarrow (\overline{U}^0) \overset{(\overline{U}_0)_{\sigma_1}}{\rightarrow} (\overline{L}_1^0) \rightarrow 0
\]

is the special right \(\Delta(\mathcal{U}, \mathcal{V})\)-approximation of \((\overline{L}_1)_0\). Also, since \(N \otimes_B \mathcal{V} \subseteq \mathcal{X}\),

\[
0 \rightarrow (\overline{N} \otimes_B \mathcal{V}) \overset{1 \otimes_{\sigma_2}}{\rightarrow} (\overline{V}^1) \overset{(\overline{V}_1^0)_{\sigma_2}}{\rightarrow} (\overline{L}_2^0) \rightarrow 0
\]

is the special right \(\Delta(\mathcal{U}, \mathcal{V})\)-approximation of \((\overline{L}_2)_0\). Since

\[
0 \rightarrow (\overline{L}_1^0) \rightarrow (\overline{L}_1^1)_g \rightarrow (\overline{L}_2^0) \rightarrow 0
\]

is exact, it follows from Lemma 5.11(1) that \(\Lambda\)-module \((\overline{L}_1^1)_g\) has a special right \(\Delta(\mathcal{U}, \mathcal{V})\)-approximation

\[
0 \rightarrow (\overline{X}_s^0) \rightarrow (\overline{U}_h^1) \rightarrow (\overline{L}_1^1)_g \rightarrow 0
\]

with \((\overline{U}_h^1)_h \in \Delta(\mathcal{U}, \mathcal{V})\) and \((\overline{X}_s^0)_s \in (\mathcal{X}, \mathcal{Y})\). This proves the completeness.

The assertion (2) can be similarly proved. \(\square\)

Theorems 5.2, 5.4, and Proposition 5.10 are new, even when \(M = 0\) or \(N = 0\).

6. Realizations

In Table 1, taking \((\mathcal{U}, \mathcal{X})\) and \((\mathcal{V}, \mathcal{Y})\) to be the projective cotorsion pair or the injective cotorsion pair, we get Table 2 below. This section is to show that these cotorsion pairs in Table 2 are pairwise generally different and “new” in some sense. For details see Definitions 4.1 and 6.2, Propositions 6.1, 6.3, 6.9 and 6.10. All these results are new, even for \(M = 0\) or \(N = 0\). Thus, it turns out that Morita rings are rich in producing “new” cotorsion pairs.

6.1. Cotorsion pairs in Series I in Table 2 are pairwise generally different. To save the space, in Table 2 we use \(\mathcal{A} := A\)-Mod, \(\mathcal{B} := B\)-Mod, \(\text{proj.} :=\) projective, \(\mathcal{M} := \text{Mon}(\Lambda) = \Delta(A\text{-Mod}, B\text{-Mod})\) and \(\mathcal{E} := \text{Epi}(\Lambda) = \nabla(A\text{-Mod}, B\text{-Mod})\).

About Table 2: (i) It is clear that (see also Subsection 3.1)

\[
\Delta(\mathcal{A}\mathcal{P}, \mathcal{B}\mathcal{P}), \quad \Delta(\mathcal{A}\mathcal{P}, \mathcal{B}\mathcal{P})^\perp = (\mathcal{A}\mathcal{P}, \text{A-Mod}); \quad (\nabla(\mathcal{A}\mathcal{I}, \mathcal{B}\mathcal{I}), \nabla(\mathcal{A}\mathcal{I}, \mathcal{B}\mathcal{I})) = (\Lambda\text{-Mod}, \text{A}\mathcal{I}).
\]

(ii) The cotorsion pairs in columns 2 and 3 in Table 2 are cotorsion pairs in Series I, and the ones in columns 4 and 5 are the cotorsion pairs in Series II. See Notation 3.3.

Table 2: Cotorsion pairs in A-Mod
Hereditary cotorsion pairs in Series I

| Cotorsion pairs in Series II |
|-----------------------------|
| \( M \otimes_A N = 0 = N \otimes_B M \) |

| \((\mathcal{U}, \mathcal{V})\) | \((\mathcal{W}, \mathcal{Y})\) |
|-----------------------------|
| \(\text{Tor}_1(M, \mathcal{U}) = 0\) | \(\text{Tor}_1(N, \mathcal{V}) = 0\) |
| \(\text{Ext}^1(N, \mathcal{X}) = 0\) | \(\text{Ext}^1(M, \mathcal{W}) = 0\) |
| \((\mathcal{U}), (\mathcal{W})\) | \((\mathcal{V}), (\mathcal{Y})\) |
| \((\Delta(\mathcal{U}, \mathcal{V}), \Delta(\mathcal{W}, \mathcal{Y})\)) | \((\nabla(\mathcal{X}, \mathcal{Y}), \nabla(\mathcal{W}, \mathcal{Y}))\) |

- \(N = 0\) for \(A\) and \(B\).
- \(N = 0\) for \(A\) and \(B\).
- \(\nabla(X, Y)\) for \(X\) and \(Y\).

\[
\begin{array}{|c|c|c|}
\hline
\mathcal{P}, \mathcal{A} & (\mathcal{P}, \mathcal{B}) & (\mathcal{A}, \mathcal{T}) \\
\hline
\text{N.B flat:} & (\mathcal{A} \text{ proj.:}) & (\mathcal{M} \text{ flat:}) \\
\frac{1}{2} & \left(\frac{1}{2}\right) & \left(\frac{1}{2}\right) \\
\mathcal{A} \text{ proj.:} & (\mathcal{B} \text{ proj.:}) & (\mathcal{B} \text{ proj.:}) \\
\frac{1}{2} & \left(\frac{1}{2}\right) & \left(\frac{1}{2}\right) \\
\mathcal{A} \text{ proj.:} & (\mathcal{A} \text{ proj.:}) & (\mathcal{M} \text{ flat:}) \\
\frac{1}{2} & \left(\frac{1}{2}\right) & \left(\frac{1}{2}\right) \\
\mathcal{A} \text{ proj.:} & (\mathcal{M} \text{ flat:}) & (\mathcal{T} \text{ flat:}) \\
\frac{1}{2} & \left(\frac{1}{2}\right) & \left(\frac{1}{2}\right) \\
\end{array}
\]

From the proof of Proposition 6.1, we will see that, in the most cases, the eight hereditary cotorsion pairs in Series I in Table 2 are pairwise different.

**Proposition 6.1.** Let \(\Lambda = \left(\begin{array}{c}
A
\end{array}\right)\) be a Morita ring with \(\phi = 0 = \psi\). Then the eight hereditary cotorsion pairs in Series I in Table 2 are pairwise generally different, in the sense of Definition 4.1.

**Proof.** All together there are 28 situations.

**Step 1.** If \(A\) and \(B\) are not semisimple, then the cotorsion pairs in Series I in the same columns are pairwise different. This occupies 2\((\frac{1}{2})\) = 12 situations.

For example, since \(A\) is not semisimple, \(A\text{-Mod} \neq \mathcal{A}\mathcal{T}\). Thus \(A\text{-Mod} \neq \mathcal{A}\mathcal{T}\), and hence \((A\text{-Mod})\) \(\neq (\mathcal{A}\mathcal{T})\).

**Step 2.** The projective cotorsion pair \(A\text{-Mod}, \mathcal{T}\) is generally different from all other seven cotorsion pairs in Series I. This occupies 4 situations.

In fact, taking \(\mathcal{A}N = B = B\text{-Mod} \neq 0\), then

\[
\begin{align*}
\left(\begin{array}{c}
N \\
\end{array}\right)_{0,0} & \in \left(\begin{array}{c}
\mathcal{A} \text{-Mod} \\
\end{array}\right), \\
\left(\begin{array}{c}
\n \end{array}\right)_{0,0} & \in \left(\begin{array}{c}
\mathcal{A} \text{-Mod} \\
\end{array}\right), \\
\left(\begin{array}{c}
N \\
\end{array}\right)_{0,0} & \in \left(\begin{array}{c}
\mathcal{A} \text{-Mod} \\
\end{array}\right), \\
\left(\begin{array}{c}
\n \end{array}\right)_{0,0} & \in \Lambda\text{-Mod}
\end{align*}
\]

but \(\left(\begin{array}{c}
N \\
\end{array}\right)_{0,0} \notin \mathcal{A}\mathcal{P}\). Thus \(\mathcal{A}\mathcal{P}, \Lambda\text{-Mod}\) is generally different from

\[
\begin{align*}
\left(\begin{array}{c}
\mathcal{A} \text{-Mod} \\
\end{array}\right), \left(\begin{array}{c}
\mathcal{A} \text{-Mod} \\
\end{array}\right), \\
\left(\begin{array}{c}
\mathcal{A} \text{-Mod} \\
\end{array}\right), \left(\begin{array}{c}
\mathcal{A} \text{-Mod} \\
\end{array}\right), \\
\left(\begin{array}{c}
\mathcal{A} \text{-Mod} \\
\end{array}\right), \left(\begin{array}{c}
\mathcal{A} \text{-Mod} \\
\end{array}\right), \\
\left(\begin{array}{c}
\mathcal{A} \text{-Mod} \\
\end{array}\right), \left(\begin{array}{c}
\mathcal{A} \text{-Mod} \\
\end{array}\right)
\end{align*}
\]

(A-Mod, \(\Lambda\mathcal{T}\)).
Step 3. Similarly, the injective cotorsion pair \((A, \mod{A})\) is generally different from all other seven cotorsion pairs in Series I. This occupies 3 situations.

Step 4. For convenience, denote by \(R_{X, Y}\) the cotorsion pair where \((X, Y)\) is at the right hand side, i.e., \(R_{X, Y} = (\mathcal{A}X, \mathcal{A}Y)\). Similarly, \(L_{U, V} = (\mathcal{B}U, \mathcal{B}V)\).

Assume that \(A\) and \(B\) are not semisimple. Under some extra conditions we will show the following remaining 9 cases (listed in the order of comparing each cotorsion pair with the ones after):

\[
\begin{align*}
L_{A^P, b^P} & \neq R_{A, \mod{A}}; & L_{A^P, b^P} & \neq R_{A, \mod{A}}; & L_{A^P, b^P} & \neq R_{A, \mod{A}}; \\
R_{A, \mod{A}} & \neq L_{A, \mod{A}}; & R_{A, \mod{A}} & \neq L_{A, \mod{A}}; & L_{A^P, B-\mod{A}} & \neq R_{A, \mod{A}}; \\
L_{A^P, B-\mod{A}} & \neq R_{A, \mod{A}}; & L_{A, \mod{A}} & \neq L_{A, \mod{A}}; & L_{A^P, B-\mod{A}} & \neq R_{A, \mod{A}}; \\
L_{A^P, B-\mod{A}} & \neq R_{A, \mod{A}}; & R_{A, \mod{A}} & \neq L_{A, \mod{A}}; & L_{A^P, B-\mod{A}} & \neq R_{A, \mod{A}}; \\
L_{A^P, B-\mod{A}} & \neq R_{A, \mod{A}}; & R_{A, \mod{A}} & \neq L_{A, \mod{A}}; & L_{A^P, B-\mod{A}} & \neq R_{A, \mod{A}}. \\
\end{align*}
\]

To see the inequalities involving \(L_{A^P, b^P} = (\mathcal{A}P_{a^P}, \mathcal{A}P_{b^P})\), it suffices to show

\[
\begin{align*}
\mathcal{A}P_{a^P} & \neq \perp \mathcal{A}P_{a^P}, & \mathcal{A}P_{b^P} & \neq \perp \mathcal{A}P_{b^P}, & \mathcal{A}P_{a^P} & \neq \perp \mathcal{A}P_{b^P}.
\end{align*}
\]

Since \(B\) is not semisimple, there is a non-projective \(B\)-module \(Y\). By Lemma 3.8.2, \(T_B Y = (N \otimes_B Y)_{0,1} \subseteq \perp \mathcal{A}P_{a^P}\), but \((N \otimes_B Y)_{0,1} \notin \perp \mathcal{A}P_{b^P}\). This shows \(\mathcal{A}P_{a^P} \neq \perp \mathcal{A}P_{b^P}\).

Since \(A\) is not semisimple, there is a non-projective \(A\)-module \(X\). By Lemma 3.8.1, \(T_A X = (M \otimes_A X)_{1,0} \subseteq \perp \mathcal{A}P_{a^P}\) and \((M \otimes_A X)_{1,0} \notin \perp \mathcal{A}P_{b^P}\). This shows \(\mathcal{A}P_{a^P} \neq \perp \mathcal{A}P_{b^P}\).

For the next inequalities involving \(R_{A, \mod{A}} = (\perp \mathcal{A}P_{a^P}, \mathcal{A}P_{b^P})\), we need to find conditions such that

\[
\begin{align*}
\perp \mathcal{A}P_{a^P} & \neq \perp \mathcal{A}P_{b^P}, & \perp \mathcal{A}P_{a^P} & \neq \perp \mathcal{A}P_{b^P}.
\end{align*}
\]

Taking \(A = B = M = N \neq 0\) and a non-projective \(B\)-module \(Y\), then \(T_B Y = (N \otimes_B Y)_{0,1} = (Y)_{0,1} \not\subseteq \perp \mathcal{A}P_{a^P}\) by Lemma 3.8.2, but \((Y)_{0,1} \notin \perp \mathcal{A}P_{b^P}\) and \(\mathcal{A}P_{a^P} \neq \perp \mathcal{A}P_{b^P}\).

To see \(L_{A, \mod{A}} \neq R_{A, \mod{A}}; \quad L_{A^P, B-\mod{A}} \neq R_{A, \mod{A}}\), it suffices to show

\[
\begin{align*}
\mathcal{A}P_{a^P} & \neq \perp \mathcal{A}P_{a^P}, & \mathcal{A}P_{a^P} & \neq \perp \mathcal{A}P_{a^P}.
\end{align*}
\]

For a non-projective \(A\)-module \(X\), \(T_A X = (X \otimes_A M)_{1,0} \subseteq \perp \mathcal{A}P_{a^P}\) and \((X \otimes_A M)_{1,0} \notin \perp \mathcal{A}P_{a^P}\).

Finally, we show that \(L_{A, \mod{B}}\) is generally different from \(R_{A, \mod{A}}; \quad R_{A, \mod{A}}\). Taking \(A = B = M = N \neq 0\) and a non-projective \(A\)-module \(X\), it suffices to see

\[
\begin{align*}
\mathcal{A}P_{a^P} & \neq \perp \mathcal{A}P_{a^P}, & \mathcal{A}P_{a^P} & \neq \perp \mathcal{A}P_{a^P}.
\end{align*}
\]

In fact, by Lemma 3.8.2, \(T_A X = (M \otimes_A M)_{1,0} = (X \otimes X)_{1,0} \subseteq \perp \mathcal{A}P_{a^P}\) and \((X \otimes X)_{1,0} \notin \perp \mathcal{A}P_{a^P}\).

This completes the proof. \(\square\)
6.2. “New” cotorsion pairs in Series I in Table 2. Keep the notations \( R_A, y \) and \( L_{\mathcal{I}_A, y} \) as in the Step 4 of the proof of Proposition 6.1. Taking off the projective cotorsion pair and the injective one from Series I of Table 2, the remaining six hereditary cotorsion pairs

\[
R_{A, \mathcal{I}_A, \nu \mathcal{I}} = (\nu A^T_{A^T}, (A^T_{A^T})), \quad L_{A, \mathcal{P}_A, \nu \mathcal{P}} = ((A^P_{\mathcal{P}_A}), (A^P_{\mathcal{P}_A})^\perp),
\]

\[
R_{A, \text{Mod}, \nu \mathcal{I}} = (\nu (A^\text{Mod}_{A^\text{Mod}}), (A^\text{Mod}_{A^\text{Mod}})), \quad L_{A, \mathcal{P}, \text{Mod}} = ((A^P_{\mathcal{P}}), (A^P_{\mathcal{P}})^\perp),
\]

\[
R_{A, \text{Mod}, \nu \mathcal{I}} = (\nu (A^\text{Mod}_{B^\text{Mod}}), (A^\text{Mod}_{B^\text{Mod}})), \quad L_{A, \mathcal{P}, \text{Mod}} = ((A^P_{\mathcal{P}}), (A^P_{\mathcal{P}})^\perp)
\]

are “new”, in the following sense.

**Definition 6.2.** A cotorsion pair in \( \Lambda \text{-Mod} \) is said to be “new”, provided that it is generally different from all of the following cotorsion pairs:

- the projective cotorsion pair \((\Lambda^\text{P}, \Lambda^\text{Mod})\);
- the injective cotorsion pair \((\Lambda^\text{Mod}, \Lambda^\text{I})\);
- the Gorenstein-projective cotorsion pair \((\text{GP}(\Lambda), \Lambda^\text{P}<\infty)\), if \( \Lambda \) is a Gorenstein ring;
- the Gorenstein-injective cotorsion pair \((\Lambda^\text{P}<\infty, \text{GI}(\Lambda))\), if \( \Lambda \) is a Gorenstein ring;
- the flat cotorsion pair \((\Lambda^F, \Lambda^C)\), where \( \Lambda^F \) is the class of flat \( \Lambda \)-modules, and \( \Lambda^C \) is the class of cotorsion \( \Lambda \)-modules. See [FJ, Lemma 7.1.4].

**Proposition 6.3.** Let \( \Lambda = (\Lambda_M^N, \Lambda_B^N) \) be a Morita ring with \( \phi = 0 = \phi \). Then the following six cotorsion pairs

\[
R_{A, \mathcal{I}_A, \nu \mathcal{I}} \quad L_{A, \mathcal{P}_A, \nu \mathcal{P}} \quad R_{A, \text{Mod}, \nu \mathcal{I}} \quad L_{A, \mathcal{P}, \text{Mod}} \quad R_{A, \text{Mod}, \nu \mathcal{I}} \quad L_{A, \mathcal{P}, \text{Mod}}
\]

are “new”, in the sense of Definition 6.2.

To prove Proposition 6.3 we need some preparations.

**Lemma 6.4.** ([GaP, 4.15]) Let \( \Lambda = (\Lambda_M^N, \Lambda_B^N) \) be a Morita ring with \( N \otimes A N = 0 \). Assume that \( \Lambda N \) and \( N \Lambda \) are projective. If \( \Lambda \) is a Gorenstein ring, then so is \( \Lambda \).

**Lemma 6.5.** Let \( \Lambda = (\Lambda_M^N, \Lambda_B^N) \) be a Morita ring which is an Artin algebra and a Gorenstein ring, with \( \phi = 0 = \phi \). Then the cotorsion pairs \( R_{A, \mathcal{I}_A, \nu \mathcal{I}} \) and \( L_{A, \mathcal{P}_A, \nu \mathcal{P}} \) are generally different from the Gorenstein-projective cotorsion pair and the Gorenstein-injective cotorsion pair.

**Proof.** Take \( \Lambda \) to be the Morita ring \( \Lambda = (\Lambda_M^N, \Lambda_A^N) \), constructed in Example 4.3. Thus \( \Lambda \) is the path algebra \( k(1 \rightarrow 2) \) with \( \text{char} k \neq 2 \), \( N = A e_2 \otimes_k e_1 A \), and \( \text{char} k \neq 2 \). By Lemma 6.4 \( \Lambda \) is a Gorenstein algebra.

**Claim 1.** \( R_{A, \mathcal{I}_A, \nu \mathcal{I}} = (\nu (A^T_{A^T}), (A^T_{A^T})) \) is generally different from \((\text{GP}(\Lambda), \Lambda^\text{P}<\infty)\).

In fact, since \( N \otimes A S_2 = A e_2 \otimes_k (e_1 A e_2) = 0 \), \( (S_2)_{0,0} = T_{\Lambda} \), \( S_2 \) is a projective \( \Lambda \)-module, thus \( (S_2)_{0,0} \in \Lambda^\text{P}<\infty \), but \( (S_2)_{0,0} \not\in (A^T_{A^T}) \). Thus \( (A^T_{A^T}) \neq \Lambda^\text{P}<\infty \), and hence

\[
R_{A, \mathcal{I}_A, \nu \mathcal{I}} = (\nu (A^T_{A^T}), (A^T_{A^T})) \neq (\text{GP}(\Lambda), \Lambda^\text{P}<\infty).
\]

**Claim 2.** \( R_{A, \mathcal{I}_A, \nu \mathcal{I}} = (\nu (A^T_{A^T}), (A^T_{A^T})) \) is generally different from \((\Lambda^\text{P}<\infty, \text{GI}(\Lambda))\).
In fact, by Example 4.3 one knows \( L = (A_{e_1})_{\sigma, \sigma} \not\in (A_{I, P})^\perp \). The following \( \Lambda \)-projective resolution of \( \Lambda L \)

\[
0 \longrightarrow (S_2)_{0,0} \longrightarrow \left( \begin{smallmatrix} (c_1) \\ (\sigma_1) \end{smallmatrix} \right) \longrightarrow \left( \begin{smallmatrix} A_{e_1} \otimes S_2 \\ S_2 \otimes A_{e_1} \end{smallmatrix} \right) \left( \begin{smallmatrix} 1 & 0 \\ 0 & 0 \\ 1 \end{smallmatrix} \right) \longrightarrow \left( \begin{smallmatrix} A_{e_1} \\ A_{e_1} \end{smallmatrix} \right)_{\sigma, \sigma} \longrightarrow 0
\]

shows that \( \text{proj.dim}_\Lambda L = 1 \). So \( L \in \Lambda P^{<\infty} \), and hence \( (A_{I, P})^\perp \not\in (A_{P^{<\infty}}, \text{GI}(\Lambda)) \).

**Claim 3.** \( L_{A_P, B_P} = ((A_{P_{B_P}}), (A_{P_{B_P}})^\perp) \) is generally different from \((\text{GP}(\Lambda), \Lambda P^{<\infty}) \).

In fact, since \( \text{Hom}_\Lambda(N, S_1) = 0 \), \( (0, 0)_{0,0} = H_0 S_1 \) is an injective \( \Lambda \)-module, thus \((0, 0)_{0,0} \notin (A_{P_{B_P}})^\perp \). Thus \( (A_{P_{B_P}})^\perp \not\in (A_{P^{<\infty}}, \text{GI}(\Lambda)) \).

**Claim 4.** \( L_{A_P, B_P} = ((A_{P_{B_P}}), (A_{P_{B_P}})^\perp) \) is generally different from \((\Lambda P^{<\infty}, \text{GI}(\Lambda)) \).

In fact, since \( \text{Hom}_\Lambda(N, S_1) = 0 \), \( (\Lambda I)^{<\infty} = \Lambda P^{<\infty} \), but \( (0, 0)_{0,0} \notin (A_{P_{B_P}})^\perp \). Thus \( (A_{P_{B_P}})^\perp \not\in (A_{P^{<\infty}}, \text{GI}(\Lambda)) \).

This completes the proof. \( \square \)

**Lemma 6.6.** Let \( \Lambda = (A_{M_B}) \) be a Morita ring which is an Artin algebra and a Gorenstein ring, with \( \phi = 0 = \phi \). Then the cotorsion pairs

\[
R_{A-\text{Mod}}, \quad L_{\Lambda P, B-\text{Mod}}, \quad R_{\Lambda I, B-\text{Mod}}, \quad L_{A-\text{Mod}, B_P}\n\]

are generally different from the Gorenstein-projective cotorsion pair and the Gorenstein-injective cotorsion pair.

**Proof.** Choose quasi-Frobenius rings \( A \) and \( B \), bimodules \( BMA \) and \( ANB \), satisfying the following conditions (i), (ii), (iii), (iv):

(i) \( A \) and \( B \) are quasi-Frobenius and not semisimple;
(ii) \( AN \) and \( BM \) are non-zero projective modules, and \( MA \) and \( NB \) are flat;
(iii) \( \Lambda = (A_{M_B}) \) is a Morita ring with \( M \otimes_A N = 0 = N \otimes_B M \);
(iv) \( \Lambda \) is a noetherian ring.

By Remark 4.7, such \( \Lambda \)'s exist! By Theorem 4.10 \( \Lambda \) is a Gorenstein ring with \( \text{inj.dim} \Lambda \leq 1 \), the Gorenstein-projective cotorsion pair \((\text{GP}(\Lambda), \mathcal{P}^{\leq 1})\) is exactly \((A_{P_{B_P}}), (A_{P_{B_P}})^\perp)\), and the Gorenstein-injective cotorsion pair \((\Lambda P^{\leq 1}, \text{GI}(\Lambda))\) is exactly \((A_{P_{B_P}}), (A_{P_{B_P}})^\perp)\).

**Claim 1.** \( R_{A-\text{Mod}}, \quad L_{\Lambda I, B-\text{Mod}} \) and \( R_{\Lambda I, B-\text{Mod}} \) are generally different from the Gorenstein-projective cotorsion pair.
Since $A$ and $B$ are not semisimple, $A\Mod \neq A\I$ and $B\Mod \neq B\I$. Thus $\left( A^{\Mod}_{\I} \right) \neq \left( A^{\I}_{\I} \right)$ and $\left( B^{\Mod}_{\I} \right) \neq \left( B^{\I}_{\I} \right)$, and hence

$$R_{A\Mod, \_\I} = \left( \left( A^{\Mod}_{\I} \right), \left( A^{\Mod}_{\I} \right) \right) \neq \left( \left( A^{\I}_{\I} \right), \left( A^{\I}_{\I} \right) \right) = (GP(A), \_\P_{\leq 1})$$

and

$$R_{A\I, B\Mod} = \left( \left( A^{\I}_{\I} \right), \left( B^{\Mod}_{\I} \right) \right) \neq \left( \left( A^{\I}_{\I} \right), \left( A^{\I}_{\I} \right) \right) = (GP(A), \_\P_{\leq 1}).$$

**Claim 2.** $L_{A\P}, B\Mod$ and $L_{A\Mod, \_\P}$ are generally different from the Gorenstein-projective cotorsion pair.

Since $A$ is not semisimple, there is a non-projective $A$-module $X$. By Lemma [3.8](1), $T_{A}(X) = 0.1 \in \left( A^{\Mod}_{\I} \right)$, but $\left( M_{\I}X \right)_{0.1} \notin \left( A^{\P}_{\I} \right)$, which shows $\left( A^{\P}_{\I} \right) \neq \left( A^{\I}_{\I} \right)$. Hence

$$L_{A\P, B\Mod} = \left( \left( B^{\Mod}_{\I} \right), \left( B^{\Mod}_{\I} \right) \right) \neq \left( \left( A^{\I}_{\I} \right), \left( A^{\I}_{\I} \right) \right) = (GP(A), \_\P_{\leq 1}).$$

Similarly,$ L_{A\I, B\Mod} = \left( \left( A^{\I}_{\I} \right), \left( B^{\Mod}_{\I} \right) \right) \neq \left( \left( A^{\I}_{\I} \right), \left( A^{\I}_{\I} \right) \right) = (GP(A), \_\P_{\leq 1}).$

**Claim 3.** $R_{A\P, \_\P}$ and $R_{A\I, B\Mod}$ are generally different from the Gorenstein-projective cotorsion pair.

Since $B$ is not semisimple, there is a non-projective $B$-module $Y$. Then $T_{B}(Y) = \left( N_{\P_{\leq 1}} \right)_{0.1} \in \left( A^{\Mod}_{\I} \right)$, but $\left( M_{\I}Y \right)_{0.1} \notin \left( A^{\P}_{\I} \right)$, which shows $\left( A^{\P}_{\I} \right) \neq \left( A^{\I}_{\I} \right)$. Thus

$$R_{A\Mod, \_\I} = \left( \left( A^{\Mod}_{\I} \right), \left( A^{\Mod}_{\I} \right) \right) \neq \left( \left( A^{\I}_{\I} \right), \left( A^{\I}_{\I} \right) \right) = (\_\P_{\leq 1}, GI(A)).$$

Similarly,$ R_{A\I, B\Mod} = \left( \left( B^{\Mod}_{\I} \right), \left( B^{\Mod}_{\I} \right) \right) \neq \left( \left( A^{\I}_{\I} \right), \left( A^{\I}_{\I} \right) \right) = (\_\P_{\leq 1}, GI(A)).$

**Claim 4.** $L_{A\P}, B\Mod$ and $L_{A\Mod, \_\P}$ are generally different from the Gorenstein-injective cotorsion pair.

Since $B$ is not semisimple, $B\Mod \neq \_\P$. Thus $\left( B^{\Mod}_{\I} \right) \neq \left( B^{\I}_{\I} \right)$, and hence

$$L_{A\P, B\Mod} = \left( \left( B^{\Mod}_{\I} \right), \left( B^{\Mod}_{\I} \right) \right) \neq \left( \left( A^{\I}_{\I} \right), \left( A^{\I}_{\I} \right) \right) = (\_\P_{\leq 1}, GI(A)).$$

Similarly,$ L_{A\Mod, \_\P} = \left( \left( A^{\I}_{\I} \right), \left( B^{\Mod}_{\I} \right) \right) \neq \left( \left( A^{\I}_{\I} \right), \left( A^{\I}_{\I} \right) \right) = (\_\P_{\leq 1}, GI(A)).$

This completes the proof. \(\Box\)

We also need the following result due to P. A. Krylov and E. Yu. Yardykov [KY].

**Lemma 6.7.** ([KY, Corollary 2.5]) Let $L = (\frac{X}{Y})_{f,g}$ be a flat $A$-module. Then $\Coker g$ is a flat $A$-module and $\Coker f$ is a flat $B$-module.

**Proof of Proposition 6.3** By Proposition 6.1 these six cotorsion pairs are generally different from the projective cotorsion pair and the injective one; and they are generally different from the Gorenstein-projective cotorsion pair and the Gorenstein-injective one, by Lemmas 6.2 and 6.3. It remains to show that they are generally different from the flat cotorsion pair.

In fact, choose rings $A$ and $B$ such that they admit non flat modules (such a ring $A$ of course exists! For example, just take a finite-dimensional algebra $A$ which is not semisimple. Then $A$
has a finitely generated module $M$ which is not projective, and $M$ is not flat). Taking non-flat modules $AX$ and $BY$, by Lemma 5.7, all the following $\Lambda$-modules are not flat:

$$(X, 0), (0, Y), T_A X = (M \otimes_A X)_{1,0}, T_B Y = (N \otimes_B Y)_{0,1}.$$ 

However,

- For the cotorsion pair $R_{\mathcal{I}, \mathcal{J}}$, $\mathcal{I} = (\mathcal{A}^\top_{\mathcal{I}}) \supseteq M \otimes \mathcal{A} \mathcal{X}$, one has $T_A X = (M \otimes_A X)_{1,0} \notin \perp (M \otimes_A X)$, by Lemma 3.8.(1).

- For the cotorsion pair $R_{\mathcal{I}, \mathcal{J}}$, $\mathcal{I} = (\mathcal{A}^\top_{\mathcal{I}} \mathcal{L} \mathcal{M}) \supseteq M \otimes \mathcal{L} \mathcal{M}$, one has $T_B Y = (N \otimes_B Y)_{0,1} \notin \perp (N \otimes_B Y)$, by Lemma 3.8.(2).

- For the cotorsion pair $L_{\mathcal{P}, \mathcal{Q}}$, $\mathcal{P} = (\mathcal{A}^\top_{\mathcal{P}} \mathcal{B} \mathcal{M}) \supseteq M \otimes \mathcal{B} \mathcal{M}$, one has $T_A X = (X \otimes \mathcal{A} \mathcal{X})_{1,0} \notin \perp (X \otimes \mathcal{A} \mathcal{X})$, by Lemma 3.8.(1).

- For the cotorsion pair $L_{\mathcal{P}, \mathcal{Q}}$, $\mathcal{P} = (\mathcal{A}^\top_{\mathcal{P}} \mathcal{B} \mathcal{M}) \supseteq M \otimes \mathcal{B} \mathcal{M}$, one has $T_B Y = (Y \otimes \mathcal{B} \mathcal{M})_{0,1} \notin \perp (Y \otimes \mathcal{B} \mathcal{M})$, by Lemma 3.8.(1).

In conclusion, the five cotorsion pairs $R_{\mathcal{I}, \mathcal{J}}$, $R_{\mathcal{I}, \mathcal{J}}$, $L_{\mathcal{P}, \mathcal{Q}}$, $L_{\mathcal{P}, \mathcal{Q}}$, $L_{\mathcal{P}, \mathcal{Q}}$ are generally different from the flat cotorsion pair.

Finally, for the cotorsion pair $L_{\mathcal{P}, \mathcal{Q}}$, $\mathcal{P} = (\mathcal{A}^\top_{\mathcal{P}} \mathcal{B} \mathcal{M}) \supseteq M \otimes \mathcal{B} \mathcal{M}$, we take $\Lambda$ to be the Morita ring and $L = (\Lambda_{\mathcal{L} \mathcal{M}})_{\lambda, \sigma}^\top$, as given in Example 3.9. Then $L \notin (\Lambda^\top_{\mathcal{P}} \mathcal{B} \mathcal{M})$. But $L$ is not a flat $\Lambda$-module (otherwise, since $L$ is finitely generated, $L$ is projective, which is absurd).

This completes the proof. □

6.3. Cotorsion pairs in Series II in Table 2 are pairwise generally different. Also, in the most cases, the eight cotorsion pairs in Series II in Table 2 are pairwise different.

**Lemma 6.8.** Let $\Lambda = (\mathfrak{M}, \mathfrak{N})$ be a Morita ring with $M \otimes_A N = 0 = N \otimes_B M$, $(\mathcal{U}, \mathcal{X})$ and $(\mathcal{U'}, \mathcal{X'})$ cotorsion pairs in $\mathcal{A}$-$\mathcal{M}$, and $(\mathcal{V}, \mathcal{Y})$ and $(\mathcal{V'}, \mathcal{Y'})$ cotorsion pairs in $\mathcal{B}$-$\mathcal{M}$. Then

1. $\Delta(\mathcal{U}, \mathcal{V}) = \Delta(\mathcal{U'}, \mathcal{V'})$ if and only if $\mathcal{U} = \mathcal{U'}$ and $\mathcal{V} = \mathcal{V'}$.
2. $\nabla(\mathcal{X}, \mathcal{Y}) = \nabla(\mathcal{X'}, \mathcal{Y'})$ if and only if $\mathcal{X} = \mathcal{X'}$ and $\mathcal{Y} = \mathcal{Y'}$.

**Proof.** (1) This follows from the fact that $T_A U = (M \otimes_A U)_{1,0} \nsubseteq \Delta(\mathcal{U}, \mathcal{V})$, $\forall U \in \mathcal{U}$; $T_B V = (N \otimes_B V)_{0,1} \nsubseteq \Delta(\mathcal{U}, \mathcal{V})$, $\forall V \in \mathcal{V}$.

(2) This follows from the fact that $H_A X = (\text{Hom}_A(\mathcal{N}, \mathcal{X}))_{0,1} \notin \nabla(\mathcal{X}, \mathcal{Y})$ and $H_B Y = (\text{Hom}_B(\mathcal{Y}, \mathcal{Y}))_{1,0} \nsubseteq \nabla(\mathcal{X}, \mathcal{Y})$, $\forall Y \in \mathcal{Y}$, here we use the second expression of $\Lambda$-modules. □

**Proposition 6.9.** Let $\Lambda = (\mathfrak{M}, \mathfrak{N})$ be a Morita ring with $M \otimes_A N = 0 = N \otimes_B M$. Then the eight cotorsion pairs in Series II in Table 2 are pairwise generally different.

**Proof.** All together there are $\binom{8}{2} = 28$ situations.

**Step 1.** By Lemma 5.3, the cotorsion pairs in Series II in the same columns of Table 2 are pairwise different. This occupies $2 \binom{4}{2} = 12$ situations.
Step 2. \((A \mathcal{P}, A\text{-Mod})\) is generally different from all other cotorsion pairs in Series II in Table 2. This occupies 4 situations.

In fact, taking \(A\) to be a non semisimple ring and \(N\) a non injective \(A\)-module. Then \((N \downarrow 0), 0 \in A\text{-Mod}\). Since the map \(0 \longrightarrow \text{Hom}_A(N, N)\) is not epic, it follows that

\[(N \downarrow 0), 0 \notin \text{Epi}(A), (N \downarrow 0), 0 \notin \nabla(A\text{-Mod}, B\mathcal{I}), (N \downarrow 0), 0 \notin \nabla(A, B\text{-Mod}), (N \downarrow 0), 0 \notin A\mathcal{I}.

So \((A \mathcal{P}, A\text{-Mod})\) is generally different from \((\dagger, \text{Epi}(A), \text{Epi}(A)), (\dagger, \nabla(A\text{-Mod}, B\mathcal{I}), \nabla(A\text{-Mod}, B\mathcal{I})), (\dagger, \nabla(\mathcal{I}, B\text{-Mod}), \nabla(\mathcal{I}, B\text{-Mod})), \) and \((A\mathcal{P}, A\mathcal{I})\).

Step 3. Similarly, \((A\mathcal{P}, A\mathcal{I})\) is generally different from other cotorsion pairs in Series II. This occupies 3 situations.

Step 4. Assume that \(M \neq 0 \neq N\). It remains to show the following 9 cases:

\[
\begin{align*}
\text{Epi}(A) &\neq \Delta(A\mathcal{P}, B\text{-Mod})^\perp; & \text{Epi}(A) &\neq \Delta(A\text{-Mod}, B\mathcal{P})^\perp; & \text{Epi}(A) &\neq \text{Mon}(A)^\perp; \\
\Delta(A\mathcal{P}, B\text{-Mod})^\perp &\neq \nabla(A\text{-Mod}, B\mathcal{I}); & \Delta(A\mathcal{P}, B\text{-Mod})^\perp &\neq \nabla(A\mathcal{I}, B\text{-Mod}); \\
\nabla(A\text{-Mod}, B\mathcal{I}) &\neq \Delta(A\text{-Mod}, B\mathcal{P})^\perp; & \nabla(A\text{-Mod}, B\mathcal{I}) &\neq \text{Mon}(A)^\perp; \\
\Delta(A\text{-Mod}, B\mathcal{P})^\perp &\neq \nabla(A\mathcal{I}, B\text{-Mod}); & \nabla(A\mathcal{I}, B\text{-Mod}) &\neq \text{Mon}(A)^\perp.
\end{align*}
\]

First, we see the inequalities involving \(\text{Epi}(A) = \nabla(A\text{-Mod}, B\text{-Mod})\). Let \(A\mathcal{I}\) be the injective envelope of \(A\mathcal{N}\). By Lemma 3.10(1) one has

\[Z_A \mathcal{I} = (I \downarrow 0), 0 \in \Delta(A\mathcal{P}, B\text{-Mod})^\perp \cap \Delta(A\text{-Mod}, B\mathcal{P})^\perp \cap \Delta(A\text{-Mod}, B\text{-Mod})^\perp.
\]

But \(\tilde{g} : 0 \longrightarrow \text{Hom}_A(N, I)\) is not epic, so \((I \downarrow 0), 0 \notin \nabla(A\text{-Mod}, B\mathcal{I}) = \text{Epi}(A)\). This shows \(\text{Epi}(A) \neq \Delta(A\mathcal{P}, B\text{-Mod})^\perp, \text{Epi}(A) \neq \Delta(A\text{-Mod}, B\mathcal{P})^\perp\) and \(\text{Epi}(A) \neq \text{Mon}(A)^\perp\).

Next, we see the two inequalities involving \(\Delta(A\mathcal{P}, B\text{-Mod})^\perp\). By Lemma 3.10(1), \(Z_A \mathcal{N} = (N \downarrow 0), 0 \in \Delta(A\mathcal{P}, B\text{-Mod})^\perp\). But \(\tilde{g} : 0 \longrightarrow \text{Hom}_A(N, N) \neq 0\) is not epic, so \((N \downarrow 0), 0 \notin \nabla(A\text{-Mod}, B\mathcal{I})\). This shows the two inequalities.

Next, to see \(\nabla(A\text{-Mod}, B\mathcal{I}) \neq \Delta(A\text{-Mod}, B\mathcal{P})^\perp\) and \(\nabla(A\text{-Mod}, B\mathcal{I}) \neq \Delta(A\text{-Mod}, B\text{-Mod})^\perp\), Let \(B\mathcal{J}\) be the injective envelope of \(B\mathcal{M}\). By Lemma 3.10(2), \(Z_B \mathcal{J} = (\mathcal{J} \downarrow 0), 0 \in \Delta(A\text{-Mod}, B\mathcal{P})^\perp\) and \(\Delta(A\text{-Mod}, B\text{-Mod})^\perp\). But \(\tilde{f} : 0 \longrightarrow \text{Hom}_B(M, J)\) is not epic, so \((\mathcal{J} \downarrow 0), 0 \notin \nabla(A\text{-Mod}, B\mathcal{I})\).

Finally, to see the two inequalities involving \(\nabla(A\mathcal{I}, B\text{-Mod})\). Let \(A\mathcal{I}\) be the injective envelope of \(A\mathcal{N}\). By Lemma 3.10(1), \(Z_A \mathcal{I} = (I \downarrow 0), 0 \in \Delta(A\text{-Mod}, B\mathcal{P})^\perp\) and \(\Delta(A\text{-Mod}, B\text{-Mod})^\perp\). But \(\tilde{g} : 0 \longrightarrow \text{Hom}_A(N, I)\) is not epic, so \((I \downarrow 0), 0 \notin \nabla(A\mathcal{I}, B\text{-Mod})\). This shows \(\Delta(A\text{-Mod}, B\mathcal{P})^\perp \neq \nabla(A\mathcal{I}, B\text{-Mod})\) and \(\nabla(A\mathcal{I}, B\text{-Mod}) \neq \text{Mon}(A)^\perp\), \(\square\).

6.4. “New” cotorsion pairs in Series II in Table 2. In Series II of Table 2, taking off the projective cotorsion pair and the injective one, the remaining six cotorsion pairs are “new”.
Proposition 6.10. Let \( \Lambda = (\begin{array}{c} A \\ B \\ N \\ M \end{array}) \) be a Morita ring with \( M \otimes_A N = 0 = N \otimes_B M \). Then all the six cotorsion pairs

\[
(\text{Mon}(\Lambda), \text{Mon}(\Lambda)^\perp), \quad (\text{Epi}(\Lambda) \cap \text{GP}(\Lambda), \text{Mon}(\Lambda)),
\]

\[
(\Delta(A\mathcal{P}, B\text{-Mod}), \Delta(A\mathcal{P}, B\text{-Mod})^\perp), \quad (\nabla(A\text{-Mod}, B\mathcal{I}), \nabla(A\text{-Mod}, B\mathcal{I})^\perp)
\]

are “new”, in the sense of Definition 6.2.

To prove Proposition 6.10 we first show

Lemma 6.11. Let \( \Lambda = (\begin{array}{c} A \\ M \\ B \\ N \end{array}) \) be a Morita ring which is an Artin algebra and a Gorenstein ring, with \( M \otimes_A N = 0 = N \otimes_B M \). Then the cotorsion pairs \((\text{Mon}(\Lambda), \text{Mon}(\Lambda)^\perp)\) and \((\text{Epi}(\Lambda) \cap \text{GP}(\Lambda), \text{Mon}(\Lambda))\) are generally different from the Gorenstein-projective cotorsion pair and the Gorenstein-injective one.

Proof. Take \( \Lambda \) to be the Morita ring \( \Lambda = (\begin{array}{c} A \\ M \\ B \\ N \end{array}) \), constructed in Example 4.3. Thus \( \Lambda \) is the path algebra \( k(1 \rightarrow 2) \) with \( \text{char} \ k \neq 2 \), \( N = Ae_2 \otimes_k e_1 A \), and \( N \otimes_A N = 0 \). By Lemma 6.4, \( \Lambda \) is a Gorenstein algebra.

Claim 1. \((\text{Mon}(\Lambda), \text{Mon}(\Lambda)^\perp)\) is generally different from \((\text{GP}(\Lambda), \Lambda \mathcal{P}^{<\infty})\).

In fact, \( L = (\begin{array}{c} A \\ M \\ B \\ N \end{array}) \in \text{Mon}(\Lambda) \). By Claim 2 in the proof of Lemma 6.3, \( \text{proj.dim}_\Lambda L = 1 \). Thus \( L \) is not Gorenstein-projective (otherwise \( L \) is projective, which is absurd. Note that a Gorenstein-projective module of finite projective dimension is projective. See [EJ, 10.2.3]). So \( L \notin \text{GP}(\Lambda) \). Thus \( \text{Mon}(\Lambda) \neq \text{GP}(\Lambda) \), and hence

\[(\text{Mon}(\Lambda), \text{Mon}(\Lambda)^\perp) \neq (\text{GP}(\Lambda), \Lambda \mathcal{P}^{<\infty}).\]

Claim 2. \((\text{Mon}(\Lambda), \text{Mon}(\Lambda)^\perp)\) is generally different from \((\Lambda \mathcal{P}^{<\infty}, \text{GI}(\Lambda))\).

In fact, the following \( \Lambda \)-projective resolution of \( (\begin{array}{c} A \\ e_1 \\ 0 \\ 0 \end{array})_{0,0} \)

\[0 \rightarrow T_B S_2 = (\begin{array}{c} 0 \\ S_2 \\ 0 \\ 0 \end{array})_{0,0} \rightarrow \Lambda \rightarrow \Lambda \rightarrow 0 \]

shows that \( (\begin{array}{c} A \\ e_1 \\ 0 \\ 0 \end{array})_{0,0} \in \Lambda \mathcal{P}^{<\infty} \). Since \( N \otimes_A Ae_1 \cong S_2 \), \( (\begin{array}{c} A \\ e_1 \\ 0 \\ 0 \end{array})_{0,0} \notin \text{Mon}(\Lambda) \). Thus \( \text{Mon}(\Lambda) \neq (\Lambda \mathcal{P}^{<\infty}, \text{GI}(\Lambda)) \).

Claim 3. \((\text{Epi}(\Lambda), \text{Epi}(\Lambda))\) is generally different from \((\text{GP}(\Lambda), \Lambda \mathcal{P}^{<\infty})\).

In fact, by Claim 2, \( (\begin{array}{c} A \\ e_1 \\ 0 \\ 0 \end{array})_{0,0} \in \Lambda \mathcal{P}^{<\infty} \). Since \( \text{Hom}_\Lambda(N, Ae_1) \cong S_1 \neq 0 \) (cf. Example 4.3), \( (\begin{array}{c} A \\ e_1 \\ 0 \\ 0 \end{array})_{0,0} \notin \text{Epi}(\Lambda) \). Thus \( \text{Epi}(\Lambda) \neq (\Lambda \mathcal{P}^{<\infty}, \text{GI}(\Lambda)) \).

Claim 4. \((\text{Epi}(\Lambda), \text{Epi}(\Lambda))\) is generally different from \((\Lambda \mathcal{P}^{<\infty}, \text{GI}(\Lambda))\).
In fact, $L = \left(\frac{A}{\mathcal{A}}\right)_{\sigma,\sigma} \in \text{Epic}(\Lambda)$ and $\text{Ext}^1_{\Gamma}(L, L) \neq 0$ (cf. Example 4.3). So $L \notin \lambda \text{Epi}(\Lambda)$. However, $L \in \lambda \mathcal{P}^{<\infty}$. Thus $\lambda \text{Epi}(\Lambda) \neq \lambda \mathcal{P}^{<\infty}$, and hence $(\lambda \text{Epi}(\Lambda), \text{Epi}(\Lambda)) \neq (\lambda \mathcal{P}^{<\infty}, \text{GI}(\Lambda))$.

This completes the proof.

\textbf{Lemma 6.12.} Let $\Lambda = \left(\frac{A}{B}ight)$ be a Morita ring which is an Artin algebra and a Gorenstein ring, with $M \otimes_A N = 0 = N \otimes_B M$. Then the cotorsion pairs

$$(\Delta(A\mathcal{P}, B\text{-Mod}), \Delta(A\mathcal{P}, B\text{-Mod})^\perp), \quad (\lambda \nabla(A\text{-Mod}, B\mathcal{I}), \nabla(A\text{-Mod}, B\mathcal{I}))$$

are generally different from the Gorenstein-projective cotorsion pair and the Gorenstein-injective one.

\textbf{Proof.} Choose rings $A$ and $B$, bimodules $BMA$ and $ANB$, such that

(i) $A$ and $B$ are quasi-Frobenius and not semisimple;
(ii) $AN$ and $BM$ are non-zero projective modules, and $MA$ and $NB$ are flat;
(iii) $M \otimes_A N = 0 = N \otimes_B M$;
(iv) $\Lambda$ is an Artin algebra.

By Remark 4.7, such $\Lambda$’s always exist! By Theorem 4.6, $\Lambda$ is a Gorenstein ring with $\text{inj.dim}\Lambda \leq 1$, $(\text{GP}(\Lambda), \mathcal{P}^{\leq 1}) = (\text{Mon}(\Lambda), \text{Mon}(\Lambda)^\perp)$, and $(\lambda \mathcal{P}^{\leq 1}, \text{GI}(\Lambda)) = (\lambda \text{Epi}(\Lambda), \text{Epi}(\Lambda))$.

\textbf{Claim 1.} $(\Delta(A\mathcal{P}, B\text{-Mod}), \Delta(A\mathcal{P}, B\text{-Mod})^\perp)$ and $(\Delta(A\text{-Mod}, B\mathcal{P}), \Delta(A\text{-Mod}, B\mathcal{P})^\perp)$ are generally different from the Gorenstein-projective cotorsion pair.

In fact, since $A$ and $B$ are not semisimple, $A\text{-Mod} \neq A\mathcal{P}$ and $B\text{-Mod} \neq B\mathcal{P}$. By Lemma 5.8

$$(\Delta(A\mathcal{P}, B\text{-Mod}) \neq \Delta(A\text{-Mod}, B\text{-Mod}) \neq \text{Mon}(\Lambda), \text{and } \Delta(A\text{-Mod}, B\mathcal{P}) \neq \text{Mon}(\Lambda).$$

Thus

$$\Delta(A\mathcal{P}, B\text{-Mod}), \Delta(A\mathcal{P}, B\text{-Mod})^\perp \neq (\text{Mon}(\Lambda), \text{Mon}(\Lambda)^\perp) = (\text{GP}(\Lambda), \mathcal{P}^{\leq 1})$$

and

$$\Delta(A\text{-Mod}, B\mathcal{P}), \Delta(A\text{-Mod}, B\mathcal{P})^\perp \neq (\text{Mon}(\Lambda), \text{Mon}(\Lambda)^\perp) = (\text{GP}(\Lambda), \mathcal{P}^{\leq 1}).$$

\textbf{Claim 2.} $(\lambda \nabla(A\text{-Mod}, B\mathcal{I}), \nabla(A\text{-Mod}, B\mathcal{I}))$ and $(\lambda \nabla(A\mathcal{I}, B\text{-Mod}), \nabla(A\mathcal{I}, B\text{-Mod}))$ are generally different from the Gorenstein-projective cotorsion pair.

In fact, by Lemma 5.8(3), $Z\mathcal{A} = \left(\frac{A}{B}\right)_{0,0} \in \lambda \nabla(A\text{-Mod}, B\mathcal{I}) \cap \lambda \nabla(A\mathcal{I}, B\text{-Mod})$. But $f : M \otimes_A A \rightarrow 0$ is not monic, so $(\frac{A}{B})_{0,0} \notin \Delta(A\text{-Mod}, B\text{-Mod}) = \text{Mon}(\Lambda)$. This shows $\lambda \nabla(A\text{-Mod}, B\mathcal{I}) \neq \text{Mon}(\Lambda)$ and $\lambda \nabla(A\mathcal{I}, B\text{-Mod}) \neq \text{Mon}(\Lambda)$. Thus

$$\lambda \nabla(A\text{-Mod}, B\mathcal{I}), \lambda \nabla(A\text{-Mod}, B\mathcal{I}) \neq (\text{Mon}(\Lambda), \text{Mon}(\Lambda)^\perp) = (\text{GP}(\Lambda), \mathcal{P}^{\leq 1})$$

and

$$\lambda \nabla(A\mathcal{I}, B\text{-Mod}), \lambda \nabla(A\mathcal{I}, B\text{-Mod}) \neq (\text{Mon}(\Lambda), \text{Mon}(\Lambda)^\perp) = (\text{GP}(\Lambda), \mathcal{P}^{\leq 1}).$$

\textbf{Claim 3.} $(\Delta(A\mathcal{P}, B\text{-Mod}), \Delta(A\mathcal{P}, B\text{-Mod})^\perp)$ and $(\Delta(A\text{-Mod}, B\mathcal{P}), \Delta(A\text{-Mod}, B\mathcal{P})^\perp)$ are generally different from the Gorenstein-injective cotorsion pair.
In fact, let $I$ be the injective envelope of $AN$. By Lemma 5.10(1), $Z_A I = \left( \frac{t}{0} \right)_{0,0} \in \Delta_{(A \mathcal{P}, B \text{-Mod})} \cap \Delta(A \text{-Mod}, B \mathcal{P})$. But $\tilde{g} : 0 \to \text{Hom}_A(N, I)$ is not epic, so $\left( \frac{t}{0} \right)_{0,0} \notin \nabla(A \text{-Mod}, B \text{-Mod}) = \text{Epi}(A)$. This shows $\Delta_{(A \mathcal{P}, B \text{-Mod})} \neq \text{Epi}(A)$ and $\Delta(A \mathcal{P}, B \text{-Mod})^\perp \neq \text{Epi}(A)$. Thus

\[
\Delta_{(A \mathcal{P}, B \text{-Mod})}, \Delta(A \mathcal{P}, B \text{-Mod})^\perp \neq \left( \text{Epi}(A), \text{Epi}(A) \right) = (A \mathcal{P}^\perp, \text{GI}(A))
\]

and

\[
\Delta(A \mathcal{P}, B \text{-Mod}), \Delta(A \mathcal{P}, B \text{-Mod})^\perp \neq (A \mathcal{P}^\perp, \text{GI}(A)).
\]

**Claim 4.** $(A \nabla, B \mathcal{I}), (A \mathcal{I}, B \text{-Mod})$ and $(A \mathcal{I}, B \text{-Mod}), (\nabla(A \mathcal{I}, B \text{-Mod}), \nabla(A \mathcal{I}, B \text{-Mod}))$ are generally different from the flat cotorsion pair.

In fact, since $A$ and $B$ are not semisimple, $A \mathcal{I} \neq A \text{-Mod}$ and $B \mathcal{I} \neq B \text{-Mod}$. By Lemma 6.8, $\nabla(A \mathcal{I}, B \text{-Mod}) \neq \nabla(A \text{-Mod}, B \text{-Mod}) = \text{Epi}(A)$ and $\nabla(A \text{-Mod}, B \mathcal{I}) \neq \text{Epi}(A)$. Thus

\[
(\nabla(A \mathcal{I}, B \mathcal{I}), \nabla(A \mathcal{I}, B \text{-Mod})) \neq (A \mathcal{P}^\perp, \text{GI}(A))
\]

and

\[
(\nabla(A \text{-Mod}, B \mathcal{I}), \nabla(A \text{-Mod}, B \mathcal{I})) \neq (A \mathcal{P}^\perp, \text{GI}(A)).
\]

This completes the proof. \qed

**Proof of Proposition 6.10.** By Proposition 6.9, these six cotorsion pairs are generally different from the projective cotorsion pair and the injective one. By Lemmas 6.11 and 6.12, they are generally different from the Gorenstein-projective cotorsion pair and the Gorenstein-injective one. It remains to show that they are generally different from the flat cotorsion pair.

In fact, choose rings $A$ and $B$ such that they admit non flat modules (such a ring of course exists! See the proof of Proposition 6.3). Taking non flat modules $A X$ and $B Y$, by Lemma 6.7, all the following $A$-modules are not flat:

\[
( \tilde{Y} )_{0,0}, \left( \frac{0}{Y} \right)_{0,0}, T_A X = \left( \frac{X}{M \otimes A X} \right)_{1,0}, T_B Y = \left( \frac{N \otimes B Y}{Y} \right)_{0,1}.
\]

However,

- For the cotorsion pair $(\text{Mon}(A), \text{Mon}(A))$, one has $T_A X = \left( \frac{X}{M \otimes A X} \right)_{1,0} \in \text{Mon}(A)$.
- For the cotorsion pair $(\Delta(A \mathcal{P}, B \text{-Mod}), \Delta(A \mathcal{P}, B \text{-Mod})^\perp)$, one has $T_B Y = \left( \frac{N \otimes B Y}{Y} \right)_{0,1} \in \Delta(A \mathcal{P}, B \text{-Mod})$.
- For the cotorsion pair $(\nabla(A \text{-Mod}, B \mathcal{I}), \nabla(A \text{-Mod}, B \mathcal{I}))$, one has $\left( \frac{0}{Y} \right)_{0,0} \in \nabla(A \text{-Mod}, B \mathcal{I})$, by Lemma 6.10(4).
- For the cotorsion pair $(\Delta(A \mathcal{P}, B \mathcal{P}), \Delta(A \mathcal{P}, B \mathcal{P})^\perp)$, one has $T_A X = \left( \frac{X}{M \otimes A X} \right)_{1,0} \in \Delta(A \text{-Mod}, B \mathcal{P})$.
- For the cotorsion pair $(\nabla(A \mathcal{I}, B \text{-Mod}), \nabla(A \mathcal{I}, B \text{-Mod}))$, one has $\left( \frac{Y}{0} \right)_{0,0} \in \nabla(A \mathcal{I}, B \text{-Mod})$, by Lemma 6.10(3).

In conclusion, the five cotorsion pairs are different from the flat cotorsion pair.
Finally, to see $^\perp \mathrm{Epi}(\Lambda)$ is generally different from the flat cotorsion pair, choose a ring $A$ such that $A$ admits a flat (left) module which is not projective.

(For example, the ring $\mathbb{Z}$ of integers has a flat module $\mathbb{Z}/\mathbb{Q}$, but $\mathbb{Z}/\mathbb{Q}$ is not projective, or equivalently, $\mathbb{Z}/\mathbb{Q}$ is not free.)

Let $\Lambda = (A, 0, A) = A \times A$. Then $\mathrm{Epi}(\Lambda) = \Lambda$-Mod, and hence $^\perp \mathrm{Epi}(\Lambda) = \Lambda F$. By the choice of $A$, $^\perp \mathrm{Epi}(\Lambda) = \Lambda F$ is strictly contained in $\Lambda F$, the class of flat $\Lambda$-modules. It follows that $(^\perp \mathrm{Epi}(\Lambda), \mathrm{Epi}(\Lambda))$ is generally different from the flat cotorsion pair. \hfill \qed

7. Abelian model structures on Morita rings

Based on results in the previous sections, we will see how abelian model structures on $A$-Mod and $B$-Mod induce abelian model structures on Morita rings; and we will see that all these abelian model structures obtained on Morita rings are pairwise generally different, and they are generally different from the six well-known abelian model structures (cf. Proposition 7.20).

7.1. Cofibrantly generated Hovey triples in Morita rings. Let $R$ be a ring. Recall that a Hovey triple $(C, F, W)$ in $R$-Mod is cofibrantly generated, if both the cotorsion pairs $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are cogenerated by sets. If a model structure on $R$-Mod is clear in context, we write Quillen’s homotopy category simply as $\mathrm{Ho}(R)$.

**Theorem 7.1.** Let $\Lambda = (A, N, B)$ be a Morita ring with $M \otimes_A N = 0 = N \otimes_B M$, $(U', \mathcal{X}, W_1)$ and $(\mathcal{V}', \mathcal{Y}, W_2)$ cofibrantly generated Hovey triples in $A$-Mod and $B$-Mod, respectively.

1. Suppose that $\mathrm{Tor}_i^A(M, U') = 0 = \mathrm{Tor}_i^B(N, V')$, $M \otimes_A U' \subseteq \mathcal{X} \cap W_1$ and $N \otimes_B V' \subseteq \mathcal{X} \cap W_1$. Then

$$(T_A(U') \oplus T_B(V'), (X, Y), (\mathcal{W}_1, \mathcal{W}_2))$$

is a cofibrantly generated Hovey triple in $\Lambda$-Mod; and it is hereditary with $\mathrm{Ho}(\Lambda) \cong \mathrm{Ho}(A) \oplus \mathrm{Ho}(B)$, provided that $(U', \mathcal{X}, W_1)$ and $(\mathcal{V}', \mathcal{Y}, W_2)$ are hereditary.

2. Suppose that $\mathrm{Ext}_B^i(M, \mathcal{V}) = 0 = \mathrm{Ext}_A^i(N, \mathcal{X})$, $\mathrm{Hom}_B(M, \mathcal{V}) \subseteq U' \cap W_1$ and $\mathrm{Hom}_A(N, \mathcal{X}) \subseteq V' \cap W_2$. Then

$$(\mathcal{U}' \cap \mathcal{W}_1, \mathcal{X} \cap \mathcal{W}_2, \mathcal{V} \cap \mathcal{W}_2, \mathcal{Y} \cap \mathcal{W}_2)$$

is a cofibrantly generated Hovey triple, and it is hereditary with $\mathrm{Ho}(\Lambda) \cong \mathrm{Ho}(A) \oplus \mathrm{Ho}(B)$, provided that $(U', \mathcal{X}, W_1)$ and $(\mathcal{V}', \mathcal{Y}, W_2)$ are hereditary.

**Proof.** Put $U := U' \cap W_1$, $\mathcal{X} := \mathcal{X} \cap \mathcal{W}_1$, $\mathcal{V} := \mathcal{V} \cap \mathcal{W}_2$, $\mathcal{Y} := \mathcal{Y} \cap \mathcal{W}_2$.

Since $(U', \mathcal{X}, W_1)$ is a cofibrantly generated Hovey triple in $A$-Mod, $(U', \mathcal{X})$ and $(U', \mathcal{X}')$ are cotorsion pairs in $A$-Mod, cogenerated by, say, set $S_1$ and set $S_1'$, respectively. Similarly, $(\mathcal{V}, \mathcal{Y})$ and $(\mathcal{V}', \mathcal{Y}')$ are cotorsion pairs in $B$-Mod, cogenerated by, say, set $S_2$ and set $S_2'$, respectively.

1. Since $\mathrm{Tor}_i^A(M, U) \subseteq \mathrm{Tor}_i^A(M, U') = 0$ and $\mathrm{Tor}_i^B(N, V) \subseteq \mathrm{Tor}_i^B(N, V') = 0$, it follows from Theorem 7.2.1 that $(X, Y)$ is a cotorsion pair in $A$-Mod; and it is cogenerated by set $T_A(S_1) \oplus T_B(S_2)$, by Proposition 7.21.1.
Since \( M \otimes_A U \subseteq M \otimes_A U' \subseteq \mathcal{V} \) and \( N \otimes_B V \subseteq N \otimes_B V' \subseteq \mathcal{X} \), by Theorem \( \textup{[1.41]} \), \( (\mathbb{X}_\mathcal{Y}, \mathbb{Y}_\mathcal{X}) = T_A(U) \oplus T_B(V) \). Thus, \((T_A(U) \oplus T_B(V), (\mathbb{X}_\mathcal{Y})) \) is a cotorsion pair, cogenerated by set \( T_A(S_1) \oplus T_B(S_2) \).

Similarly, \((T_A(U') \oplus T_B(V'), (\mathbb{X}'_{\mathcal{Y}})) \) is a cotorsion pair, cogenerated by set \( T_A(S'_1) \oplus T_B(S'_2) \).

Since \( M \otimes_A U' \subseteq W_2 \) and \( N \otimes_B V' \subseteq W_1 \), one has

\[
(T_A(U') \oplus T_B(V')) \cap (W_{S_1}^{W_2}) = T_A(U' \cap W_1) \oplus T_B(V' \cap W_2) = T_A(U) \oplus T_B(V).
\]

Also, \( (\mathbb{X}_\mathcal{Y}) \cap (W_{S_1}^{W_2}) = (\mathbb{X}_\mathcal{Y}) \). Since \( W_1 \) and \( W_2 \) are thick, \( (W_{S_1}^{W_2}) \) is thick. Thus

\[
(T_A(U') \oplus T_B(V'), (\mathbb{X}_\mathcal{Y}), (W_{S_1}^{W_2}))
\]

is a cofibrantly generated Hovey triple.

If \((U', \mathcal{X}, W_1)\) and \((V', \mathcal{Y}, W_2)\) are hereditary Hovey triples, then so is \((T_A(U') \oplus T_B(V'), (\mathbb{X}_\mathcal{Y}), (W_{S_1}^{W_2}))\). Since \( M \otimes_A U' \subseteq \mathcal{V} \cap W_2 \) and \( N \otimes_B V' \subseteq \mathcal{X} \cap W_1 \), by Theorem \( \textup{[2.8]} \) one has

\[
\text{Ho}(A) \cong [(T_A(U') \oplus T_B(V')) \cap (\mathbb{X}_\mathcal{Y})]/((T_A(U') \oplus T_B(V')) \cap (\mathbb{X}_\mathcal{Y}))
\]

\[
\cong (T_A(U' \cap X) \oplus T_B(V' \cap Y))/((T_A(U \cap X) \oplus T_B(V \cap Y))
\]

\[
\cong [T_A(U' \cap X)/T_A(U \cap X)] \oplus [T_B(V' \cap Y)/T_B(V \cap Y)]
\]

\[
= \text{Ho}(A) \oplus \text{Ho}(B).
\]

(2) Since \( \text{Ext}_A^1(M, \mathcal{Y}) \subseteq \text{Ext}_A^1(M, \mathcal{X}) = 0 \) and \( \text{Ext}_B^1(N, \mathcal{X}) \subseteq \text{Ext}_B^1(N, \mathcal{Y}) = 0 \), by Theorem \( \textup{[3.11]} \), \( ((\mathbb{U}_{\mathcal{Y}}), \mathbb{U}_{\mathcal{Y}}, (\mathbb{U}_{\mathcal{Y}})) \) is a cotorsion pair in \( \Lambda\text{-Mod} \).

Since \( \text{Hom}_B(M, \mathcal{Y}) \subseteq \text{Hom}_B(M, \mathcal{Y}) \subseteq \mathcal{U}' \) and \( \text{Hom}_A(N, \mathcal{X})' \subseteq \text{Hom}_A(N, \mathcal{X}) \subseteq \mathcal{V}' \), by Theorems \( \textup{[1.4]} \) one has

\[
((\mathbb{U}_{\mathcal{Y}}), (\mathbb{U}_{\mathcal{Y}})_{\mathcal{Y}}) = (\mathcal{V}(\mathcal{X}'), (\mathcal{Y}'), (\mathcal{X}'))
\]

and \((\mathbb{U}_{\mathcal{Y}})_{\mathcal{Y}} = H_A(\mathcal{X}) \oplus H_B(\mathcal{Y}) \). By Proposition \( \textup{[5.11]} \), \( (\mathcal{V}(\mathcal{X}'), (\mathcal{Y}'), (\mathcal{X}')) \) is cogenerated by set \( Z_A(S'_1) \oplus Z_B(S'_2) \). Thus, \( ((\mathbb{U}_{\mathcal{Y}}), H_A(\mathcal{X}) \oplus H_B(\mathcal{Y})) \) is a cotorsion pair, cogenerated by set \( Z_A(S'_1) \oplus Z_B(S'_2) \).

Similarly, \( ((\mathbb{U}_{\mathcal{Y}}), H_A(\mathcal{X}) \oplus H_B(\mathcal{Y})) \) is a cotorsion pair, cogenerated by set \( Z_A(S_1) \oplus Z_B(S_2) \).

Note that \( (\mathbb{U}_{\mathcal{Y}}) \cap (W_{S_1}^{W_2}) = (\mathbb{U}_{\mathcal{Y}}) \cap (W_{S_1}^{W_2}) = (\mathbb{U}_{\mathcal{Y}}) \). Since \( \text{Hom}_A(N, \mathcal{X}) \subseteq W_2 \) and \( \text{Hom}_B(M, \mathcal{Y}) \subseteq W_1 \), one has

\[
(H_A(\mathcal{X}) \oplus H_B(\mathcal{Y})) \cap (W_{S_1}^{W_2}) = H_A(\mathcal{X} \cap W_1) \oplus H_B(\mathcal{Y} \cap W_2) = H_A(\mathcal{X}) \oplus H_B(\mathcal{Y})
\]

Since \( W_1 \) and \( W_2 \) are thick, \( (W_{S_1}^{W_2}) \) is thick. Thus

\[
((\mathbb{U}_{\mathcal{Y}}), H_A(\mathcal{X}) \oplus H_B(\mathcal{Y}), (W_{S_1}^{W_2}))
\]

is a cofibrantly generated Hovey triple.
If \((\mathcal{U}', \mathcal{X}, \mathcal{W}_1)\) and \((\mathcal{V}', \mathcal{Y}, \mathcal{W}_2)\) are hereditary, then \(((\mathcal{U}'_\mathcal{X}), \mathcal{H}_A(\mathcal{X}) \oplus \mathcal{H}_B(\mathcal{Y}), (\mathcal{W}_1^\mathcal{W}_2))\) is hereditary. Since \(\text{Hom}_A(\mathcal{N}, \mathcal{X}) \subseteq \mathcal{V}'\) and \(\text{Hom}_B(\mathcal{M}, \mathcal{Y}) \subseteq \mathcal{U}'\), by Theorem 7.3, one has

\[
\text{Ho}(\Lambda) \cong (((\mathcal{U}'_\mathcal{X}) \cap (\mathcal{H}_A(\mathcal{X}) \oplus \mathcal{H}_B(\mathcal{Y}))) / ((\mathcal{U}'_\mathcal{X}) \cap (\mathcal{H}_A(\mathcal{X}) \oplus \mathcal{H}_B(\mathcal{Y})))
\]

\[
\cong (\mathcal{H}_A(\mathcal{U}' \cap \mathcal{X}) \oplus \mathcal{H}_B(\mathcal{V}' \cap \mathcal{Y}) / ((\mathcal{H}_A(\mathcal{U}' \cap \mathcal{X}) \oplus \mathcal{H}_B(\mathcal{V}' \cap \mathcal{Y}))
\]

\[
\cong [\mathcal{H}_A(\mathcal{U}' \cap \mathcal{X}) / \mathcal{H}_A(\mathcal{U} \cap \mathcal{X})] \oplus [\mathcal{H}_B(\mathcal{V}' \cap \mathcal{Y}) / \mathcal{H}_B(\mathcal{V} \cap \mathcal{Y})]
\]

\[
\cong [(\mathcal{U}' \cap \mathcal{X}) / (\mathcal{U} \cap \mathcal{X})] \oplus [(\mathcal{V}' \cap \mathcal{Y}) / (\mathcal{V} \cap \mathcal{Y})]
\]

\[
= \text{Ho}(\Lambda) \oplus \text{Ho}(B).
\]

From Theorem 7.1 and its proof, one easily sees the following.

**Corollary 7.2.** Let \(\Lambda = (\mathcal{A} \mathcal{B} \mathcal{N})\) be a Morita ring with \(\mathcal{M} \mathcal{N} = 0 = \mathcal{N} \mathcal{B} \mathcal{M}\). Let \((\mathcal{U}, \mathcal{X})\) and \((\mathcal{U}', \mathcal{X}')\) be compatible hereditary cotorsion pairs in \(\Lambda\)-Mod, cogenerated by sets \(S_1\) and \(S_1'\), respectively, with Gillespie-Hovey triple \((\mathcal{U}', \mathcal{X}, \mathcal{W}_1)\). Let \((\mathcal{V}, \mathcal{Y})\) and \((\mathcal{V}', \mathcal{Y}')\) be compatible hereditary cotorsion pairs in \(\mathcal{B}\)-Mod, cogenerated by sets \(S_2\) and \(S_2'\), respectively, with Gillespie-Hovey triple \((\mathcal{V}', \mathcal{Y}, \mathcal{W}_2)\).

1. Assume that \(\text{Tor}_A^1(\mathcal{M}, \mathcal{U}') = 0 = \text{Tor}_B^1(\mathcal{N}, \mathcal{V}')\), \(\mathcal{M} \mathcal{N} \mathcal{U}' \subseteq \mathcal{Y}'\), \(\mathcal{N} \mathcal{B} \mathcal{V}' \subseteq \mathcal{X}'\). Then

\[
(\text{T}, (\mathcal{U}'_\mathcal{X}), (\mathcal{V}'_\mathcal{Y})) \text{ and } (\text{T}, (\mathcal{U}'_\mathcal{X}), (\mathcal{V}'_\mathcal{Y}))
\]

are compatible complete hereditary cotorsion pairs in \(\Lambda\)-Mod, with Gillespie-Hovey triple

\[
(\text{T}, (\mathcal{U}'_\mathcal{X}), (\mathcal{V}'_\mathcal{Y}), (\mathcal{W}_1^\mathcal{W}_2))
\]

and \(\text{Ho}(\Lambda) \cong \text{Ho}(\Lambda) \oplus \text{Ho}(B)\).

2. Assume that \(\text{Ext}_A^1(\mathcal{M}, \mathcal{V}) = 0 = \text{Ext}_B^1(\mathcal{N}, \mathcal{X})\), \(\text{Hom}_B(\mathcal{M}, \mathcal{Y}) \subseteq \mathcal{U}\) and \(\text{Hom}_A(\mathcal{N}, \mathcal{X}) \subseteq \mathcal{V}\). Then

\[
(\text{T}, (\mathcal{U}'_\mathcal{X}), (\mathcal{V}'_\mathcal{Y})) \text{ and } (\text{T}, (\mathcal{U}'_\mathcal{X}), (\mathcal{V}'_\mathcal{Y}))
\]

are compatible complete hereditary cotorsion pairs in \(\Lambda\)-Mod, with Gillespie-Hovey triple

\[
(\text{T}, (\mathcal{U}'_\mathcal{X}), (\mathcal{V}'_\mathcal{Y}), (\mathcal{W}_1^\mathcal{W}_2))
\]

and \(\text{Ho}(\Lambda) \cong \text{Ho}(\Lambda) \oplus \text{Ho}(B)\).

### 7.2. Hovey Triples in Morita Rings

We stress that, all the results in the rest of this section are not consequences of Theorem 7.1 or Corollary 7.2 since they need module-theoretical arguments on the completeness of cotorsion pairs in Morita rings, developed in Section 5. Thus, all these results are new even for \(\mathcal{M} = 0\) or \(\mathcal{N} = 0\).

**Theorem 7.3.** Let \(\Lambda = (\mathcal{A} \mathcal{M} \mathcal{N})\) be a Morita ring with \(\mathcal{M} \mathcal{N} = 0 = \mathcal{N} \mathcal{B} \mathcal{M}\). Let \((\mathcal{V}', \mathcal{Y}, \mathcal{W})\) be a Hovey triple in \(\mathcal{B}\)-Mod. Suppose that \(\mathcal{N} \mathcal{B}\) is flat and \(\mathcal{B} \mathcal{M}\) is projective.

1. If \(\mathcal{M} \mathcal{N} \mathcal{P} \subseteq \mathcal{Y} \cap \mathcal{W}\), then

\[
(\text{T}, (\mathcal{A} \mathcal{P}), (\mathcal{V}', (\mathcal{A} \mathcal{M} \mathcal{N} \mathcal{W}))).
\]

is a Hovey triple in \(\Lambda\)-Mod; and it is hereditary with \(\text{Ho}(\Lambda) \cong \text{Ho}(B)\), provided that \((\mathcal{V}', \mathcal{Y}, \mathcal{W})\) is hereditary.
(2) If \( \text{Hom}_A(N, A\mathcal{I}) \subseteq \mathcal{V}' \cap \mathcal{W} \), then
\[
(\mathcal{A}^{\text{Mod}}_{\mathcal{V}'}, \mathcal{H}_A(A\mathcal{I}) \oplus \mathcal{H}_B(\mathcal{Y}), \mathcal{A}^{\text{Mod}}_{\mathcal{W}'})
\]
is a Hovey triple in \( \text{A-Mod} \); and it is hereditary with \( \text{Ho}(A) \cong \text{Ho}(B) \), provided that \( (\mathcal{V}', \mathcal{Y}, \mathcal{W}) \) is hereditary.

**Proof.** Put \( \mathcal{V} := \mathcal{V}' \cap \mathcal{W} \), \( \mathcal{V}' := \mathcal{Y} \cap \mathcal{W} \). Since \( (\mathcal{V}', \mathcal{Y}, \mathcal{W}) \) is a Hovey triple in \( \text{B-Mod} \), \( (\mathcal{V}', \mathcal{Y}) \) and \( (\mathcal{V}', \mathcal{Y}') \) are complete cotorsion pairs in \( \text{B-Mod} \).

(1) Since \( M \otimes_A \mathcal{P} \subseteq \mathcal{Y} \), it follows from Theorem 5.2(1) that \( (T_A(A\mathcal{P}) \oplus T_B(\mathcal{V}'), (A_{\mathcal{V}'})^{\text{Mod}}) \) is a complete cotorsion pair in \( \text{A-Mod} \). Similarly, \( (T_A(A\mathcal{P}) \oplus T_B(\mathcal{V}'), (A_{\mathcal{Y}'})^{\text{Mod}}) \) is a complete cotorsion pair.

Since \( M \otimes_A \mathcal{P} \subseteq \mathcal{W} \), it follows that
\[
(T_A(A\mathcal{P}) \oplus T_B(\mathcal{V}')) \cap (A_{\mathcal{V}'})^{\text{Mod}} = T_A(A\mathcal{P}) \oplus T_B(\mathcal{V}' \cap \mathcal{W}) = T_A(A\mathcal{P}) \oplus T_B(\mathcal{V}).
\]
Clarity, \( (A_{\mathcal{V}'})^{\text{Mod}} \cap (A_{\mathcal{Y}'})^{\text{Mod}} = (A_{\mathcal{Y}'})^{\text{Mod}} \). Since \( \mathcal{W} \) is a thick class of \( \text{B-Mod} \), \( (A_{\mathcal{Y}'})^{\text{Mod}} \) is a thick class of \( \text{A-Mod} \). By definition \( (T_A(A\mathcal{P}) \oplus T_B(\mathcal{V}'), (A_{\mathcal{Y}'})^{\text{Mod}}) \) is a Hovey triple.

If \( (\mathcal{V}', \mathcal{Y}, \mathcal{W}) \) is a hereditary Hovey triple, then by Theorem 5.2(1), both \( (T_A(A\mathcal{P}) \oplus T_B(\mathcal{V}'), (A_{\mathcal{V}'})^{\text{Mod}}) \) and \( (T_A(A\mathcal{P}) \oplus T_B(\mathcal{V}'), (A_{\mathcal{Y}'})^{\text{Mod}}) \) are hereditary cotorsion pairs, and hence \( (T_A(A\mathcal{P}) \oplus T_B(\mathcal{V}'), (A_{\mathcal{Y}'})^{\text{Mod}}) \) is a hereditary Hovey triple. By Theorem 5.2 one has
\[
\text{Ho}(A) \cong ((T_A(A\mathcal{P}) \oplus T_B(\mathcal{V}')) \cap (A_{\mathcal{V}'})^{\text{Mod}})/((T_A(A\mathcal{P}) \oplus T_B(\mathcal{V}')) \cap (A_{\mathcal{Y}'})^{\text{Mod}})
\]
\[
\cong (T_A(A\mathcal{P}) \oplus T_B(\mathcal{V}' \cap \mathcal{Y}))/((T_A(A\mathcal{P}) \oplus T_B(\mathcal{V}' \cap \mathcal{Y}'))
\]
\[
\cong T_B(\mathcal{V}' \cap \mathcal{Y})/T_B(\mathcal{V}' \cap \mathcal{Y}')
\]
\[
\cong (\mathcal{V}' \cap \mathcal{Y})/(\mathcal{V}' \cap \mathcal{Y}') \cong \text{Ho}(B).
\]

(2) The proof is similar as (1). We include the main steps. Since \( \text{Hom}_A(N, A\mathcal{I}) \subseteq \mathcal{V} \), by Theorem 5.2(2), \( ((A_{\mathcal{V}'})^{\text{Mod}}, H_A(A\mathcal{I}) \oplus H_B(\mathcal{Y})) \) is a complete cotorsion pair. Similarly, \( ((A_{\mathcal{Y}'})^{\text{Mod}}, H_A(A\mathcal{I}) \oplus H_B(\mathcal{Y})) \) is a complete cotorsion pair.

Clearly \( (A_{\mathcal{V}'})^{\text{Mod}} \cap (A_{\mathcal{Y}'})^{\text{Mod}} = (A_{\mathcal{Y}'})^{\text{Mod}} \). Since \( \text{Hom}_A(N, A\mathcal{I}) \subseteq \mathcal{W} \), it follows that
\[
(H_A(A\mathcal{I}) \oplus H_B(\mathcal{Y})) \cap (A_{\mathcal{Y}'})^{\text{Mod}} = H_A(A\mathcal{I}) \oplus H_B(\mathcal{Y} \cap \mathcal{W}) = H_A(A\mathcal{I}) \oplus T_B(\mathcal{Y}').
\]

Also, \( (A_{\mathcal{Y}'})^{\text{Mod}} \) is a thick class of \( \text{A-Mod} \). By definition
\[
((A_{\mathcal{V}'})^{\text{Mod}}, H_A(A\mathcal{I}) \oplus H_B(\mathcal{Y})), (A_{\mathcal{Y}'})^{\text{Mod}})
\]
is a Hovey triple. Moreover, it is hereditary if \( (\mathcal{V}', \mathcal{Y}, \mathcal{W}) \) is hereditary. In this case, by Theorem 2.8 one has
\[
\text{Ho}(\text{A-Mod}) \cong ((H_A(A\mathcal{I}) \oplus H_B(\mathcal{Y})) \cap (A_{\mathcal{V}'})^{\text{Mod}})/(H_A(A\mathcal{I}) \oplus H_B(\mathcal{Y}))
\]
\[
\cong (H_A(A\mathcal{I}) \oplus H_B(\mathcal{Y} \cap \mathcal{W}))/((H_A(A\mathcal{I}) \oplus H_B(\mathcal{Y} \cap \mathcal{Y}'))
\]
\[
\cong H_B(\mathcal{V}' \cap \mathcal{Y})/H_B(\mathcal{V}' \cap \mathcal{Y}')
\]
\[
\cong (\mathcal{V}' \cap \mathcal{Y})/(\mathcal{V}' \cap \mathcal{Y}') \cong \text{Ho}(B).
\]
Corollary 7.4. Let $\Lambda = (\mathcal{A}, \mathcal{N})$ be a Morita ring with $M \otimes_A N = 0 = N \otimes_B M$, $(\mathcal{V}, \mathcal{Y})$ and $(\mathcal{V}', \mathcal{Y}')$ compatible complete hereditary cotorsion pairs in $\text{B-Mod}$, with Gillespie-Hovey triple $(\mathcal{V}', \mathcal{Y}, \mathcal{W})$. Suppose that $N_B$ is flat and $BM$ is projective.

1. If $M \otimes_A \mathcal{P} \subseteq \mathcal{V}'$, then
   $$(T_A(\mathcal{P}) \oplus T_B(\mathcal{V}'), (A_{\text{Mod}}^\mathcal{P})), (T_A(\mathcal{P}) \oplus T_B(\mathcal{V}'), (A_{\text{Mod}}^\mathcal{V}'))$$
   are compatible complete hereditary cotorsion pairs in $\text{A-Mod}$, with Gillespie-Hovey triple
   $$(T_A(\mathcal{P}) \oplus T_B(\mathcal{V}'), (A_{\text{Mod}}^\mathcal{V}'), (A_{\text{Mod}}^\mathcal{W}))$$
   and $\text{Ho}(\Lambda) \cong \text{Ho}(B)$.

2. If $\text{Hom}_A(N, \mathcal{I}) \subseteq \mathcal{V}$, then
   $$((A_{\text{Mod}}^\mathcal{V}), \text{H}_A(\mathcal{I}) \oplus \text{H}_B(\mathcal{Y})), ((A_{\text{Mod}}^\mathcal{V}), \text{H}_A(\mathcal{I}) \oplus \text{H}_B(\mathcal{Y}))$$
   are compatible complete hereditary cotorsion pairs in $\text{A-Mod}$, with Gillespie-Hovey triple
   $$((A_{\text{Mod}}^\mathcal{V}), \text{H}_A(\mathcal{I}) \oplus \text{H}_B(\mathcal{Y}), (A_{\text{Mod}}^\mathcal{W}))$$
   and $\text{Ho}(\Lambda) \cong \text{Ho}(B)$.

Similar as Theorem 7.3 starting from a Hovey triple in $\text{A-Mod}$ and using Theorem 5.4, we get

Theorem 7.5. Let $\Lambda = (\mathcal{A}, \mathcal{N})$ be a Morita ring with $M \otimes_A N = 0 = N \otimes_B M$. Let $(\mathcal{U}', \mathcal{X}, \mathcal{W})$ be a Hovey triple in $\text{A-Mod}$. Suppose that $\mathcal{M}_A$ is flat and $\mathcal{A}N$ is projective.

1. If $N \otimes_B \mathcal{P} \subseteq \mathcal{X} \cap \mathcal{W}$, then
   $$(T_A(\mathcal{U}') \oplus T_B(\mathcal{P}), (A_{\text{Mod}}^\mathcal{X}), (A_{\text{Mod}}^\mathcal{W}))$$
   is a Hovey triple; and it is hereditary with $\text{Ho}(\Lambda) \cong \text{Ho}(A)$, provided that $(\mathcal{U}', \mathcal{X}, \mathcal{W})$ is hereditary.

2. If $\text{Hom}_B(M, \mathcal{I}) \subseteq \mathcal{U}' \cap \mathcal{W}$, then
   $$((A_{\text{Mod}}^\mathcal{U}'), \text{H}_A(\mathcal{X}) \oplus \text{H}_B(\mathcal{I}), (A_{\text{Mod}}^\mathcal{W}))$$
   is a Hovey triple; and it is hereditary with $\text{Ho}(\Lambda) \cong \text{Ho}(A)$, provided that $(\mathcal{U}', \mathcal{X}, \mathcal{W})$ is hereditary.

Corollary 7.6. Let $\Lambda = (\mathcal{A}, \mathcal{N})$ be a Morita ring with $M \otimes_A N = 0 = N \otimes_B M$, $(\mathcal{U}, \mathcal{X})$ and $(\mathcal{U}', \mathcal{X}')$ compatible complete hereditary cotorsion pairs in $\text{A-Mod}$, with Gillespie-Hovey triple $(\mathcal{U}', \mathcal{X}, \mathcal{W})$. Suppose that $\mathcal{M}_A$ is flat and $\mathcal{A}N$ is projective.

1. If $N \otimes_B \mathcal{P} \subseteq \mathcal{X}'$, then
   $$(T_A(\mathcal{U}) \oplus T_B(\mathcal{P}), (A_{\text{Mod}}^\mathcal{X})), (T_A(\mathcal{U}) \oplus T_B(\mathcal{P}), (A_{\text{Mod}}^\mathcal{X}'))$$

□
are compatible complete hereditary cotorsion pairs in \( \Lambda\)-Mod, with Gillespie-Hovey triple
\[
(\mathcal{T}_A(\mathcal{U}) \oplus \mathcal{T}_B(\mathcal{P}), \mathcal{X}_{\text{B-Mod}}^\vee, \mathcal{W}_{\text{B-Mod}})
\]
and \( \text{Ho}(\Lambda) \cong \text{Ho}(A) \).

(2) If \( \text{Hom}_B(M, \mathcal{I}) \subseteq \mathcal{U} \), then
\[
(\mathcal{U}_{\text{B-Mod}}^\vee, \mathcal{H}_A(\mathcal{X}) \oplus \mathcal{H}_B(\mathcal{I})) \quad \text{and} \quad (\mathcal{U}_{\text{B-Mod}}^\vee, \mathcal{H}_A(\mathcal{X}^\vee) \oplus \mathcal{H}_B(\mathcal{I}))
\]
are compatible complete hereditary cotorsion pairs in \( \Lambda\)-Mod, with Gillespie-Hovey triple
\[
((\mathcal{U}_{\text{B-Mod}}^\vee), \mathcal{H}_A(\mathcal{X}) \oplus \mathcal{H}_B(\mathcal{I}), (\mathcal{W}_{\text{B-Mod}}))
\]
and \( \text{Ho}(\Lambda) \cong \text{Ho}(A) \).

7.3. Gillespie-Hovey triples in Morita rings, via generalized projective (injective) cotorsion pairs. The notion of generalized projective (injective) cotorsion pairs is essentially due to H. Becker [Bec].

Definition 7.7. (1) A complete cotorsion pair \((\mathcal{X}, \mathcal{Y})\) in an abelian category \(\mathcal{A}\) with enough projective objects is a generalized projective cotorsion pair, or in short, gictp, provided that
\[
\begin{align*}
(i) \quad \mathcal{X} \cap \mathcal{Y} = \mathcal{P}, \text{ where } \mathcal{P} \text{ is the class of projective objects of } \mathcal{A}; \\
(ii) \quad \text{the class } \mathcal{Y} \text{ is thick.}
\end{align*}
\]

(1') A complete cotorsion pair \((\mathcal{X}, \mathcal{Y})\) in an abelian category \(\mathcal{A}\) with enough injective objects is a generalized injective cotorsion pair, or in short, gictp, provided that
\[
\begin{align*}
(i') \quad \mathcal{X} \cap \mathcal{Y} = \mathcal{I}, \text{ where } \mathcal{I} \text{ is the class of injective objects of } \mathcal{A}; \\
(ii') \quad \text{the class } \mathcal{X} \text{ is thick.}
\end{align*}
\]

Example-Remark 7.8. A gictp (respectively, gictp) is not necessarily the projective (respectively, injective) cotorsion pair \((\mathcal{P}, \mathcal{A})\) (respectively, \((\mathcal{A}, \mathcal{I})\)).

(1) ([H2]) For a Gorenstein ring \(R\), the Gorenstein-projective cotorsion pair \((\text{GP}(R), R^{\leq \infty})\) is a gictp. Dually, \((R^{\leq \infty}, \text{GI}(R))\) is a gictp.

(2) Let \(\text{Ch}(R)\) be the complex category of modules over ring \(R\), \(\mathcal{E}\) the class of acyclic complexes, and \(\text{dgP}\) the class of dg projective complexes \(Q\) (see [Sp], [AF]), i.e., components of \(Q\) are projective and \(\text{Hom}^*(Q, \mathcal{E})\) is acyclic. By [EJX], \((\text{dgP}, \mathcal{E})\) is a cotorsion pair, and \(\text{dgP} \cap \mathcal{E}\) is exactly the class of projective objects of \(\text{Ch}(R)\). That is,
\[
\text{dgP} \cap \mathcal{E} = \bigoplus_{i \in \mathbb{Z}} P^i(P) \mid P \in R^{\leq \infty}
\]
where \(P^i(P) : \cdots \to 0 \to P \xrightarrow{i} P \to 0 \to \cdots\) is the complex with \(i\)-th and \((i + 1)\)-th component \(P\). By [Sp] (also [BN]), for any complex \(X\) there is an epimorphism \(Q \to X\) which is a quasi-isomorphism. Thus, \((\text{dgP}, \mathcal{E})\) is complete, and hence generalized projective. Dually, there is a gictp \((\mathcal{E}, \text{dgI})\). See [Gil1] for an important development of this work.

(3) Any gictp \((\mathcal{X}, \mathcal{Y})\) is hereditary, \(\mathcal{X}\) is a Frobenius category (with the canonical exact structure), and \(\mathcal{P}\) is the class of projective-injective objects.
(3') Any gictp $(X, \mathcal{Y})$ is hereditary, $\mathcal{Y}$ is a Frobenius category, and $\mathcal{I}$ is the class of projective-injective objects.

Taking gpctps or gctps in Corollary 7.4, we get a stronger and an improved result without extra conditions (i.e., the conditions “$M \otimes_A P \subseteq \mathcal{Y}$” and “$\text{Hom}_A(N, A\mathcal{I}) \subseteq \mathcal{Y}$” in Corollary 7.4 can be dropped). This is the reason we list it as a theorem.

**Theorem 7.9.** Let $\Lambda = (\frac{A}{M} N_B)$ be a Morita ring with $M \otimes_A N = 0 = N \otimes_B M$. Suppose that $N_B$ is flat and $bM$ is projective.

1. Let $(\mathcal{V}, \mathcal{Y})$ and $(\mathcal{V}', \mathcal{Y}')$ be compatible gpctps in $B\text{-}\text{Mod}$, with Gillespie-Hovey triple $(\mathcal{V}', \mathcal{Y}, \mathcal{W})$. Then
   $$\left( T_A(\mathcal{P}) \oplus T_B(\mathcal{V}), \left( \frac{A}{\mathcal{Y}} \right)^{\text{-}\text{Mod}} \right) \quad \text{and} \quad \left( T_A(\mathcal{P}) \oplus T_B(\mathcal{V}'), \left( \frac{A}{\mathcal{Y}'} \right)^{\text{-}\text{Mod}} \right)$$
   are compatible gpctps in $\Lambda\text{-}\text{Mod}$, with Gillespie-Hovey triple
   $$\left( T_A(\mathcal{P}) \oplus T_B(\mathcal{V}'), \left( \frac{A}{\mathcal{Y}'} \right)^{\text{-}\text{Mod}} \right)$$
   and $\text{Ho}(\Lambda) \cong (\mathcal{V}' \cap \mathcal{Y})/B\mathcal{P} \cong \text{Ho}(B)$.

2. Let $(\mathcal{V}, \mathcal{Y})$ and $(\mathcal{V}', \mathcal{Y}')$ be compatible gictp in $B\text{-}\text{Mod}$, with Gillespie-Hovey triple $(\mathcal{V}', \mathcal{Y}, \mathcal{W})$. Then
   $$\left( \left( \frac{A}{\mathcal{V}} \right)^{\text{-}\text{Mod}}, H_A(\mathcal{I}) \oplus H_B(\mathcal{Y}) \right) \quad \text{and} \quad \left( \left( \frac{A}{\mathcal{V}'} \right)^{\text{-}\text{Mod}}, H_A(\mathcal{I}) \oplus H_B(\mathcal{Y}') \right)$$
   are compatible gictp in $\Lambda\text{-}\text{Mod}$, with Gillespie-Hovey triple
   $$\left( \left( \frac{A}{\mathcal{V}'} \right)^{\text{-}\text{Mod}}, H_A(\mathcal{I}) \oplus H_B(\mathcal{Y}), \left( \frac{A}{\mathcal{W}} \right)^{\text{-}\text{Mod}} \right)$$
   and $\text{Ho}(\Lambda) \cong (\mathcal{V}' \cap \mathcal{Y})/B\mathcal{I} \cong \text{Ho}(B)$.

**Proof.** (1) Since $BM$ is projective, $M \otimes_A \mathcal{P} \subseteq B\mathcal{P}$. Since cotorsion pair $(\mathcal{V}', \mathcal{Y}')$ is generalized projective, $M \otimes_A \mathcal{P} \subseteq B\mathcal{P} = \mathcal{V}' \cap \mathcal{Y}' \subseteq \mathcal{Y}'$.

Thus, by Corollary 7.4(1),
   $$\left( T_A(\mathcal{P}) \oplus T_B(\mathcal{V}), \left( \frac{A}{\mathcal{Y}} \right)^{\text{-}\text{Mod}} \right) \quad \text{and} \quad \left( T_A(\mathcal{P}) \oplus T_B(\mathcal{V}'), \left( \frac{A}{\mathcal{Y}'} \right)^{\text{-}\text{Mod}} \right)$$
   are compatible complete hereditary cotorsion pairs in $\Lambda\text{-}\text{Mod}$, with Gillespie-Hovey triple
   $$\left( T_A(\mathcal{P}) \oplus T_B(\mathcal{V}'), \left( \frac{A}{\mathcal{Y}'} \right)^{\text{-}\text{Mod}} \right)$$
   and $\text{Ho}(\Lambda) \cong \text{Ho}(B) \cong (\mathcal{V}' \cap \mathcal{Y})/B\mathcal{P}$. Since
   $$\mathcal{A}\mathcal{P} = T_A(\mathcal{P}) \oplus T_B(\mathcal{B}\mathcal{P}) = \left\{ \left( \frac{M}{\mathcal{P} \otimes Q} \right) \mid P \in \mathcal{A}\mathcal{P}, \ Q \in \mathcal{B}\mathcal{P} \right\}$$
   and $M \otimes_A \mathcal{P} \subseteq \mathcal{Y}$, it follows that
   $$\left( T_A(\mathcal{P}) \oplus T_B(\mathcal{V}') \right) \cap \left( \frac{A}{\mathcal{Y}'} \right)^{\text{-}\text{Mod}} = T_A(\mathcal{P}) \oplus T_B(\mathcal{V} \cap \mathcal{Y}) = T_A(\mathcal{P}) \oplus T_B(\mathcal{B}\mathcal{P}) = \mathcal{A}\mathcal{P}.$$ Since $\mathcal{Y}$ is thick, $\left( \frac{A}{\mathcal{Y}} \right)^{\text{-}\text{Mod}}$ is thick. Thus, cotorsion pair $(T_A(\mathcal{P}) \oplus T_B(\mathcal{V}'), \left( \frac{A}{\mathcal{Y}'} \right)^{\text{-}\text{Mod}})$ is generalized projective. Similarly, $(T_A(\mathcal{P}) \oplus T_B(\mathcal{V}'), \left( \frac{A}{\mathcal{Y}'} \right)^{\text{-}\text{Mod}})$ is generalized projective.
(2) Since $N_{B}$ is flat, $\text{Hom}_{A}(N, \mathcal{A}) \subseteq B\mathcal{I}$. Since $(\mathcal{V}, \mathcal{Y})$ is generalized injective, $\text{Hom}_{A}(N, \mathcal{A}) \subseteq B\mathcal{I} = \mathcal{V} \cap \mathcal{V} \subseteq \mathcal{V}$.

Thus, by Corollary 7.4(2),

$$((\mathcal{U}_{B-\text{Mod}}), \text{Ho}(\mathcal{X}) \oplus H_{B}(B\mathcal{I})) \quad \text{and} \quad ((\mathcal{U}'_{B-\text{Mod}}), \text{Ho}(\mathcal{X}') \oplus H_{B}(B\mathcal{I}))$$

are compatible complete hereditary cotorsion pairs, with Gillespie-Hovey triple

$$((\mathcal{U}'_{B-\text{Mod}}), \text{Ho}(\mathcal{X}) \oplus H_{B}(B\mathcal{I}), (\mathcal{W}_{B-\text{Mod}}))$$

and $\text{Ho}(\Lambda) \cong \text{Ho}(B) \cong (\mathcal{U}' \cap \mathcal{X})/A\mathcal{I}$. Since

$$A\mathcal{I} = H_{A}(A\mathcal{I}) \oplus H_{B}(B\mathcal{I}) = \{(\text{Hom}_{A}(N, I)) \oplus \text{Hom}_{B}(M, J) \mid I \in A\mathcal{I}, J \in B\mathcal{I}\}$$

and $\text{Hom}_{A}(N, A\mathcal{I}) \subseteq \mathcal{V}$, it follows that

$$((A_{\text{Mod}}^{\mathcal{V}}) \cap (A\mathcal{I} \oplus H_{B}(Y)) = H_{A}(A\mathcal{I}) \oplus H_{B}(\mathcal{V} \cap \mathcal{Y}) = H_{A}(A\mathcal{I}) \oplus H_{B}(B\mathcal{I}) = A\mathcal{I}.$$ Since $\mathcal{V}$ is thick, $(A_{\text{Mod}}^{\mathcal{V}})$ is thick. Thus $((A_{\text{Mod}}^{\mathcal{V}}), \text{Ho}(A\mathcal{I}) \oplus H_{B}(\mathcal{Y}))$ is generalized injective. Similarly, $((A_{\text{Mod}}^{\mathcal{V}}), \text{Ho}(A\mathcal{I}) \oplus H_{B}(\mathcal{Y}''))$ is generalized injective. \hfill \Box

Similarly, taking gpcpts or gicpts in Corollary 7.6, we get a stronger and an improved result.

**Theorem 7.10.** Let $\Lambda = (A_{M})$ be a Morita ring with $M \otimes_{A} N = 0 = N \otimes_{B} M$. Suppose that $M_{A}$ is flat and $AN$ is projective.

(1) Let $(\mathcal{U}, \mathcal{X})$ and $(\mathcal{U}', \mathcal{X}')$ be compatible gpcpts in $A_{\text{Mod}}$, with Gillespie-Hovey triple $(\mathcal{U}', \mathcal{X}, \mathcal{W})$. Then

$$(T_{A}(\mathcal{U}) \oplus T_{B}(B\mathcal{P}), (\mathcal{X}_{B-\text{Mod}})) \quad \text{and} \quad (T_{A}(\mathcal{U}') \oplus T_{B}(B\mathcal{P}), (\mathcal{X}'_{B-\text{Mod}}))$$

are compatible gpcpts in $\Lambda_{\text{Mod}}$, with Gillespie-Hovey triple

$$(T_{A}(\mathcal{U}') \oplus T_{B}(B\mathcal{P}), (\mathcal{X}'_{B-\text{Mod}}), (\mathcal{W}_{B-\text{Mod}}))$$

and $\text{Ho}(\Lambda) \cong (\mathcal{U}' \cap \mathcal{X})/\mathcal{A}\mathcal{P} \cong \text{Ho}(A)$. \hfill \Box

(2) Let $(\mathcal{U}, \mathcal{X})$ and $(\mathcal{U}', \mathcal{X}')$ be compatible gicpts in $A_{\text{Mod}}$, with Gillespie-Hovey triple $(\mathcal{U}', \mathcal{X}, \mathcal{W})$. Then

$$(\mathcal{U}_{B-\text{Mod}}), \text{Ho}(\mathcal{X}) \oplus H_{B}(B\mathcal{I})) \quad \text{and} \quad ((\mathcal{U}'_{B-\text{Mod}}), \text{Ho}(\mathcal{X}') \oplus H_{B}(B\mathcal{I}))$$

are compatible gicpts in $\Lambda_{\text{Mod}}$, with Gillespie-Hovey triple

$$((\mathcal{U}'_{B-\text{Mod}}), \text{Ho}(\mathcal{X}) \oplus H_{B}(B\mathcal{I}), (\mathcal{W}_{B-\text{Mod}}))$$

and $\text{Ho}(\Lambda) \cong (\mathcal{U}' \cap \mathcal{X})/\mathcal{A}\mathcal{I} \cong \text{Ho}(A)$. \hfill \Box
7.4. Projective (Injective) models on Morita rings. An abelian model structure on (abelian) category $\mathcal{A}$ is projective (respectively, injective) if each object is fibrant (respectively, cofibrant), i.e., the Hovey triple is of the form $(\mathcal{X}, \mathcal{A}, \mathcal{Y})$ (respectively, $(\mathcal{A}, \mathcal{Y}, \mathcal{X})$). See [H2], [Gil2].

The following observation clarifies the relation between projective (respectively, injective) models and gpctps (respectively, gictps).

**Lemma 7.11.** ([Bec, 1.1.9]; [Gil3, 1.1]) Let $(\mathcal{X}, \mathcal{Y})$ be a complete cotorsion pair in abelian category $\mathcal{A}$ with enough projective objects and enough injective objects. Then

1. $(\mathcal{X}, \mathcal{A}, \mathcal{Y})$ is a (hereditary) Hovey triple if and only if $(\mathcal{X}, \mathcal{Y})$ is a generalized projective cotorsion pair.
2. $(\mathcal{A}, \mathcal{Y}, \mathcal{X})$ is a (hereditary) Hovey triple if and only if $(\mathcal{X}, \mathcal{Y})$ is a generalized injective cotorsion pair.

Any gpcp $(\mathcal{V}, \mathcal{Y})$ in $\mathcal{B}$-Mod gives compatible gpctps $(\mathcal{B}\mathcal{P}, \mathcal{B}$-Mod) and $(\mathcal{V}, \mathcal{Y})$. Any gictp $(\mathcal{V}, \mathcal{Y})$ in $\mathcal{B}$-Mod gives compatible gictps $(\mathcal{V}, \mathcal{Y})$ and $(\mathcal{B}$-Mod, $\mathcal{B}$I). Thus, by Theorem 7.9 one gets:

**Corollary 7.12.** Let $\Lambda = (\mathcal{A} \mathcal{N} \mathcal{M} \mathcal{B})$ be a Morita ring with $\mathcal{M} \otimes \mathcal{A} \mathcal{N} = 0 = \mathcal{N} \otimes \mathcal{B} \mathcal{M}$. Suppose that $\mathcal{N} \mathcal{B}$ is flat and $\mathcal{B} \mathcal{M}$ is projective.

1. Let $(\mathcal{V}, \mathcal{Y})$ be a gpcp in $\mathcal{B}$-Mod. Then

$$(\mathcal{T}(\mathcal{A}) \mathcal{P}) \oplus \mathcal{T}(\mathcal{V}), \mathcal{A}$-Mod, (\mathcal{A}$\mathcal{P}$))

is a hereditary Hovey triple, and $\text{Ho}(\Lambda) \cong \mathcal{V}/\mathcal{B}$\mathcal{P}$.

2. Let $(\mathcal{V}, \mathcal{Y})$ be a gicp in $\mathcal{B}$-Mod. Then

$$(\mathcal{A}$\mathcal{I}$, \mathcal{A}$\mathcal{I}$) \oplus \mathcal{H}(\mathcal{V}), (\mathcal{A}$\mathcal{P}$))

is a hereditary Hovey triple, and $\text{Ho}(\Lambda) \cong \mathcal{Y}/\mathcal{B}$\mathcal{I}$.

If $\mathcal{B}$ is quasi-Frobenius, then $(\mathcal{B}$-Mod, $\mathcal{B}$I) is a gpcp, and $(\mathcal{B}$\mathcal{P}$, \mathcal{B}$-Mod) is a gicp. By Corollary 7.12 one gets

**Corollary 7.13.** Let $\Lambda = (\mathcal{A} \mathcal{N} \mathcal{M} \mathcal{B})$ be a Morita ring with $\mathcal{M} \otimes \mathcal{A} \mathcal{N} = 0 = \mathcal{N} \otimes \mathcal{B} \mathcal{M}$. Suppose that $\mathcal{B}$ is quasi-Frobenius, $\mathcal{N} \mathcal{B}$ is flat and $\mathcal{B} \mathcal{M}$ is projective. Then

1. $(\mathcal{T}(\mathcal{A}) \mathcal{P}) \oplus \mathcal{T}(\mathcal{B}$-Mod), $\mathcal{A}$-Mod, (\mathcal{A}$\mathcal{P}$)) is a hereditary Hovey triple; and $\text{Ho}(\Lambda) \cong \mathcal{B}$-Mod.

2. $(\mathcal{A}$\mathcal{I}$, \mathcal{A}$\mathcal{I}$) \oplus \mathcal{H}(\mathcal{B}$-Mod), (\mathcal{A}$\mathcal{P}$)) is a hereditary Hovey triple; and $\text{Ho}(\Lambda) \cong \mathcal{B}$-Mod.

Similar as Corollary 7.12 by Theorem 7.10 one gets

**Corollary 7.14.** Let $\Lambda = (\mathcal{A} \mathcal{N} \mathcal{M} \mathcal{B})$ be a Morita ring with $\mathcal{M} \otimes \mathcal{A} \mathcal{N} = 0 = \mathcal{N} \otimes \mathcal{B} \mathcal{M}$. Suppose that $\mathcal{M} \mathcal{A}$ is flat and $\mathcal{A} \mathcal{N}$ is projective.
(1) Let \((\mathcal{U}, \mathcal{X})\) be a gctp in \(A\)-Mod. Then
\[
(T_A(\mathcal{U}) \oplus T_B(\mathcal{P}), \ A\text{-Mod}, \ \left(\frac{\mathcal{X}}{B\text{-Mod}}\right))
\]
is a hereditary Hovey triple, and \(\text{Ho}(\Lambda) \cong \mathcal{U}/A\mathcal{P}\).

(2) Let \((\mathcal{U}, \mathcal{X})\) be a gctp in \(A\)-Mod. Then
\[
(A\text{-Mod}, \ H_A(\mathcal{X}) \oplus H_B(\mathcal{I}), \ \left(\frac{\mathcal{U}}{B\text{-Mod}}\right))
\]
is a hereditary Hovey triple, and \(\text{Ho}(\Lambda) \cong \mathcal{X}/A\mathcal{I}\).

If \(A\) is quasi-Frobenius, then \((A\text{-Mod}, A\mathcal{I})\) is a gctp, and \((A\mathcal{P}, A\text{-Mod})\) is a gctp. By Corollary 7.15, one gets

**Corollary 7.15.** Let \(\Lambda = (\begin{smallmatrix} A & N \\ M & B \end{smallmatrix})\) be a Morita ring with \(M \otimes_A N = 0 = N \otimes_B M\). Suppose that \(A\) is quasi-Frobenius, \(M_A\) is flat and \(AN\) is projective. Then

1. \((T_A(A\text{-Mod}) \oplus T_B(\mathcal{P}), \ A\text{-Mod}, \ \left(\frac{\mathcal{X}}{B\text{-Mod}}\right))\) is a hereditary Hovey triple; and \(\text{Ho}(\Lambda) \cong A\text{-Mod}\).

2. \((A\text{-Mod}, \ H_A(A\text{-Mod}) \oplus H_B(\mathcal{I}), \ \left(\frac{\mathcal{X}}{B\text{-Mod}}\right))\) is a hereditary Hovey triple; and \(\text{Ho}(\Lambda) \cong A\text{-Mod}\).

### 7.5. Generally different Hovey triples.

**Fact 7.16.** Let \((\mathcal{C}, \mathcal{F}, \mathcal{W})\) and \((\mathcal{C}', \mathcal{F}', \mathcal{W}')\) be Hovey triples in abelian category \(\mathcal{A}\). If
\[
(\mathcal{C} \cap \mathcal{W}, \mathcal{F}) = (\mathcal{C}' \cap \mathcal{W}', \mathcal{F}'), \quad (\mathcal{C}, \mathcal{F} \cap \mathcal{W}) = (\mathcal{C}', \mathcal{F}' \cap \mathcal{W}')
\]
then \((\mathcal{C}, \mathcal{F}, \mathcal{W}) = (\mathcal{C}', \mathcal{F}', \mathcal{W}')\).

In fact, by Theorem 7.16, the corresponding two abelian model structures are the same. Thus \(\mathcal{W} = \mathcal{W}'\).

**Definition 7.17.** Let \(\Omega\) be a class of Morita rings, \((\mathcal{C}, \mathcal{F}, \mathcal{W})\) and \((\mathcal{C}', \mathcal{F}', \mathcal{W}')\) Hovey triples defined in \(A\text{-Mod}\), for arbitrary Morita rings \(\Lambda \in \Omega\). They are said to be generally different Hovey triples, provided that there is \(\Lambda \in \Omega\), such that \((\mathcal{C}, \mathcal{F}, \mathcal{W}) \neq (\mathcal{C}', \mathcal{F}', \mathcal{W}')\) in \(A\text{-Mod}\).

**Lemma 7.18.** Hovey triples \((\mathcal{C}, \mathcal{F}, \mathcal{W})\) and \((\mathcal{C}', \mathcal{F}', \mathcal{W}')\) in \(A\text{-Mod}\) are generally different if and only if \((\mathcal{C} \cap \mathcal{W}, \mathcal{F})\) and \((\mathcal{C}' \cap \mathcal{W}', \mathcal{F}')\) are generally different, or, \((\mathcal{C}, \mathcal{F} \cap \mathcal{W})\) and \((\mathcal{C}', \mathcal{F}' \cap \mathcal{W}')\) are generally different, as cotorsion pairs.

**Proof.** The “only if” part follows from Fact 7.16. Conversely, without loss of generality, we may assume that there are \(A, B, \ BMA\) and \(AN_B\), such that \(\mathcal{F} = \mathcal{F}'\) and \(\mathcal{C} \cap \mathcal{W} \neq \mathcal{C}' \cap \mathcal{W}'\). Then, either \(\mathcal{C} \neq \mathcal{C}'\), or \(\mathcal{W} \neq \mathcal{W}'\). Hence \((\mathcal{C}, \mathcal{F}, \mathcal{W}) \neq (\mathcal{C}', \mathcal{F}', \mathcal{W}')\) for the corresponding \(\Lambda\).

**Example 7.19.** Generally different Hovey triples could be the same in special cases.

For example, \((A\mathcal{P}, A\text{-Mod}, A\mathcal{I})\) and \((\left(\frac{A\mathcal{P}}{M\mathcal{P}}\right), \ \left(\frac{A\mathcal{P}}{M\mathcal{P}}\right)^\perp, A\text{-Mod}\)\) are Hovey triples. Since \((A\mathcal{P}, A\text{-Mod})\) and \((\left(\frac{A\mathcal{P}}{M\mathcal{P}}\right), \ \left(\frac{A\mathcal{P}}{M\mathcal{P}}\right)^\perp)\) are generally different (cf. Example 7.16), by Lemma 7.18, the two Hovey triples are generally different. But, if \(M = 0 = N\), then they are the same.
Proposition 7.20. (1) The two Hovey triples in Theorem 7.1 are generally different.
(2) The four Hovey triples in Theorems 7.3 and 7.5 are pairwise generally different.
(3) The four Hovey triples in Theorems 7.9 and 7.10 are pairwise generally different.
(4) The four Hovey triples in Corollaries 7.12 and 7.14 are pairwise generally different.
(5) The four Hovey triples in Corollaries 7.13 and 7.15 are pairwise generally different.
(6) All the Hovey triples in (1)-(5) are generally different from the following Hovey triples:
   • $(\Lambda \mathcal{P}, \Lambda \text{-Mod}, \Lambda \text{-Mod})$;
   • $(\Lambda \text{-Mod}, \Lambda \mathcal{I}, \Lambda \text{-Mod})$;
   • the Frobenius model $([\text{Gil}2]) : (\Lambda \text{-Mod}, \Lambda \text{-Mod}, \Lambda \mathcal{P})$ (if $\Lambda$ is quasi-Frobenius);
   • $(\text{GP}(\Lambda), \Lambda \text{-Mod}, \Lambda \mathcal{P}^{<\infty})$ (if $\Lambda$ is Gorenstein);
   • $(\Lambda \text{-Mod}, \text{GI}(\Lambda), \Lambda \mathcal{P}^{<\infty})$ (if $\Lambda$ is Gorenstein);
   • the flat-cotorsion Hovey triple $\text{(F}(\Lambda), \text{C}(\Lambda), \Lambda \text{-Mod})$ (see [BBE], [EJ, 7.4.3]).

Proof. (1) Let $k$ be a field. In Theorem 7.1 taking $\Lambda = (\begin{smallmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{smallmatrix})$ and $\mathcal{U}' = k \text{-Mod} = \mathcal{X}' = \mathcal{W}_1 = Y' = \mathcal{Y} = \mathcal{W}_2$, then all the conditions are satisfied. To see that $(\text{T}_{\Lambda}(\mathcal{U}') \oplus \text{T}_B(\mathcal{V}'), \text{\Lambda } \mathcal{P}^{<\infty})$ and $((\mathcal{U}', \mathcal{V}')) : (\mathcal{H}_A(\mathcal{X}) \oplus \mathcal{H}_B(\mathcal{Y}), (\mathcal{V}_1', \mathcal{V}_2'))$ are different Hovey triples, it suffices to see that cotorsion pairs $(\lambda \mathcal{P}, \Lambda \text{-Mod})$ and $(\Lambda \text{-Mod}, \lambda \mathcal{I})$ are different. This is clear since $\mathcal{A} \mathcal{P} \not\subseteq \Lambda \text{-Mod}.$

To show (2), (3), (4), (5), by the definition of generally different Hovey triples, it suffices to prove (5), since the Hovey triples in Corollaries 7.13 and 7.15 are respectively the special cases of the Hovey triples in Theorems 7.3 and 7.5 (or, in Theorems 7.9 and 7.10 or, in Corollaries 7.12 and 7.14). While for the four kinds of Hovey triples in Corollaries 7.13 and 7.15 one can easily see that they are pairwise generally different.

(6) It suffices to show that the four Hovey triples in Corollaries 7.13 and 7.15 are generally different from the six Hovey triples listed above. Then, all together there are 24 cases, and all these 24 cases are easy, except the following cases.

To see Hovey triple $(\text{T}_{\Lambda}(\mathcal{A} \mathcal{P}) \oplus \text{T}_B(\mathcal{B} \text{-Mod}), \Lambda \text{-Mod}, (\Lambda \text{-Mod}^{\mathcal{B}}))$ in Corollary 7.13(1) is generally different from $(\text{GP}(\Lambda), \Lambda \text{-Mod}, \Lambda \mathcal{P}^{<\infty})$ (if $\Lambda$ is Gorenstein), we take $\Lambda$ to be the Morita rings as in Theorem 7.6. Then $\Lambda \mathcal{P}^{<\infty} = (\mathcal{A} \mathcal{P}^{\mathcal{B}})$ if $\Lambda$ is not semisimple.

To see the Hovey triple $(\Lambda \text{-Mod}, \Lambda \mathcal{I}) \oplus \mathcal{H}_B(\mathcal{B} \text{-Mod}), (\Lambda \text{-Mod}^{\mathcal{B}}))$ in Corollary 7.13(2) is generally different from $(\Lambda \text{-Mod}, \text{GI}(\Lambda), \Lambda \mathcal{P}^{<\infty})$ (if $\Lambda$ is Gorenstein), we take $\Lambda$ to be the Morita rings as in Theorem 7.6. Then $\Lambda \mathcal{P}^{<\infty} = (\mathcal{A} \mathcal{P}^{\mathcal{B}})$ if $\Lambda$ is not semisimple.

To see the Hovey triple $(\text{T}_{\Lambda}(\Lambda \text{-Mod}) \oplus \text{T}_B(\mathcal{B} \mathcal{P}), \Lambda \text{-Mod}, \Lambda \mathcal{P}^{<\infty})$ (if $\Lambda$ is Gorenstein), we take $\Lambda$ to be the Morita rings as in Theorem 7.6. Then $\Lambda \mathcal{P}^{<\infty} = (\mathcal{A} \mathcal{P}^{\mathcal{B}})$ if $\Lambda$ is not semisimple.
To see the Hovey triple \((\text{A-Mod}, \mathbb{H}_{A}(\text{A-Mod}) \oplus \mathbb{H}_{B}(\text{B-T}))\) in Corollary 7.15(2) is generally different from \((\text{A-Mod}, \text{GI}(\Lambda), \Lambda^{P_{<\infty}}_{\text{B-Mod}})\) (if \(\Lambda\) is Gorenstein), we take \(\Lambda\) to be the Morita rings as in Theorem 4.6. Then \(\Lambda^{P_{<\infty}} = (\Lambda^{P_{<\infty}}_{\text{B-Mod}}) \neq (\Lambda^{P_{<\infty}})\) if \(B\) is not semisimple. □

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