Analytical finite-Lamor-radius and finite-orbit-width model for the LIGKA code and its application to KGAM and shear Alfvén physics

Ph. Lauber, Z. Lu

MPI für Plasmaphysik, Boltzmannstr. 2, D-85748 Garching, Germany
E-mail: philipp.lauber@ipp.mpg.de

Abstract. In this paper, a finite Larmor radius (FLR) and finite orbit width (FOW) model within the LIGKA [1] framework is derived and its implementation in LIGKA is verified against analytical theory. The model is based on an expansion in $k_{\perp} \rho_i$ and thus valid for low and intermediate mode numbers. Assuming Maxwellian distribution functions and considering circulating ions only, analytical expressions for the non-adiabatic response in the quasi-neutrality (QN) and the gyrokinetic moment equation (GKM) are derived. It is shown how these expressions, also valid for $n \neq 0$, can be connected to results in the literature for $n = 0$. Verification tests with analytical literature [2] are carried out for mode frequency, damping and radial mode propagation. A set of parameters is chosen that is based on recent experiments at ASDEX Upgrade [3] featuring strong EGAM (energetic particle induced geodesic acoustic mode) and BAE (beta-induced Alfvén eigenmode) activity. For finite $n$, the model can be used to determine the local radiative damping consistently, which is needed for the fast evaluation of linear AE (Alfvén eigenmode) stability boundaries that are typically required by energetic particle transport models.

1. Introduction

On the way to reliable, fast and automated tools for determining the linear stability boundaries for various classes of Alfvénic modes [4–6], it is desirable to employ a hierarchy of models that allows one to balance accuracy and computational effort in a controlled way. This requires a well understood relation between the physics models of different complexity. In addition, for non-linear hybrid [7, 8] and fully non-linear models [9–12], a comprehensive linear gyrokinetic model is useful not only for benchmarks, but also for choosing relevant scenarios and qualify its sensitivity with respect to linear stability.

This work intends to pave the way for a low-cost, semi-analytical model embedded within the LIGKA/HAGIS framework. Whereas the global LIGKA solver takes into account FLR and FOW effects in a non-perturbative way already, the local model [13, 14] did not account for these important corrections, that are known to be crucial for KGAM physics (kinetic GAM) [2, 15, 16], radiative damping of Alfvén eigenmodes (AEs) and the drive due to energetic particles.

In this paper, the derivation of the model is presented. The motivation for presenting this detailed derivation is the intention, to be able to replace analytical expressions by numerical integrals at any stage, e.g. given by a particle following code like HAGIS [7, 17] for the unperturbed orbit integrals, or by numerical integrals over velocity space for non-Maxwellian distribution functions, or both. In this way, one can relate deviations from the analytical dispersion relations (local solution of GK system) or global linear mode properties (global solver) to geometry and/or distribution function effects. In fact, both local and global solvers can work with both analytical or numerically evaluated input coefficients. Thus, also
the effects of multiple species, fast ions, electrons and passing/trapped particles can be analysed in a straightforward way. The FOW expressions for the various coefficients that are derived in the following, are similar to the ones in ref. [18] that is based on a ballooning formulation. The main difference to ref. [18] and [19] is the asymmetry with respect to diamagnetic effects [13, 20] and a more general expansion (Taylor expansion) with respect to FOW effects. This expansion limits the applicability of the model to cases where \( k_r \theta_i \leq 1 \). In principle, this assumption has to be checked by global calculations case by case, however for typical AE problems the upper limit for toroidal mode numbers is of order \( n \approx 40 - 60 \).

2. Development of the model

2.1. Quasi-neutrality: terms \( (r - r_0) \)

The basic equations for the gyrokinetic LIGKA model[1, 13, 21, 22] are briefly summarised in appendix 8.1. The expression for the non-adiabatic part of the perturbed distribution function \( h \) contains the integration over the unperturbed orbits (dashed quantities in formula 1).

\[
\dot{h} = i e \sum_m \int_{-\infty}^{t_f} dt' e^{i[n(\varphi'-\varphi)-m(\theta'-\theta)-\omega(t'-t)]} e^{-im\theta} \]  
\[
\frac{\partial F_0}{\partial E} [\omega - \omega_s] J_0(k_{\perp} \theta_i) \left[ \phi_m(r') - (1 - \frac{\omega_d(r', \theta')}{\omega}) \psi_m(r') \right] \]  

(1)

Here, \( m \) represents the poloidal mode number, \( \varphi \) and \( \theta \) are the toroidal and poloidal angle, \( \omega \) the mode frequency, \( \omega_s \) the diamagnetic frequency, \( \omega_e = \frac{eB}{m} \), \( \tau = \frac{T_e}{T_i} \), and \( F_0 \) the equilibrium distribution function, \( E \) the energy coordinate and \( J_0 \) the Bessel function with the argument \( k_{\perp} \theta_i \). \( \phi \) and \( \psi \) are the perturbed electrostatic and electromagnetic super-potential \( \omega A_{\parallel} = (\nabla \psi)_{\parallel} \) that need to be evaluated along the particle trajectory. These potentials are expanded in Taylor series around \( r \) [21] whereas the equilibrium quantities’ variation is neglected, i.e. \( \phi(r') \approx \phi(r) + (r' - r) \phi'(r) + \frac{1}{2} (r' - r)^2 \phi''(r) \) where \( r' = r(t') \) and \( r = r(t) \). This methodology is similar to ref. [21], but there, the geodesic component of the drift velocity was omitted and most coefficients were expressed in terms of numerical integrals which means that no connection to analytical GAM, kinetic GAM (KGAM) and BAE physics was possible. In order to ensure consistent ordering, also the operator \( J^0_0 \) is expanded to second order (see next section). Using the periodicity of the radial excursion:

\[
r(t') - r(t) = \left[ r(t') - r_0 \right] - \left[ r(t) - r_0 \right] = \sum_{l=-l_{\text{max}}}^{l_{\text{max}}} r_l \left[ e^{i l \omega_l (t'-t)} e^{i l \omega_l (t-t_0)} - e^{i l \omega_l (t-t_0)} \right]
\]

with \( r_l = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} dt' [r(t') - r_0] e^{-i l \omega_l t'} \), \( r_0 \) the average radial position of a circulating particle, \( t_0 \) the timepoint when the particle passes through the reference position \( r_0 \) and assuming that for circulating particles \( v_{\parallel, \perp} \) is independent from \( \theta \) and thus \( \hat{\omega} = v_{\parallel} / (qR) \), one can write:

\[
r(\theta) = \int_0^t v_{\text{dr}} \cdot \nabla r dt = - \int \frac{v_{\parallel}^2 + v_{\perp}^2}{2 \omega_l \Omega_e R} \sin \theta' d\theta' = r_0 + \sigma_0 \frac{t^2 + s^2/2}{\omega_l} \cos \theta = r_0 + \sigma_0 \hat{v}_d \cos \theta
\]

where \( \sigma_0 = \pm 1 \) indicates co and counter-passing particles, \( \omega_l = v_{\parallel} / (qR), \omega_l = \sqrt{2T/m}, v_{\parallel} = v_{\parallel} / v_{\text{th}}, \) \( s = v_{\parallel} / v_{\text{th}}, v_d = v_{\parallel} / (\Omega_e R), \Omega_e = eB / m \) and \( \hat{v}_d = v_d(v_{\parallel}^2 + v_{\perp}^2/2) / v_{\text{th}} \). The drift velocity \( v_d \) is part of the operator \( \nabla \phi / i = [i \hat{v}_d \sin \theta / i \hat{v}_d + \hat{v}_d \cos \theta / (ir) \hat{v}_d + \omega_D] \phi \), where \( \omega_D \) is the toroidal precession frequency that is assumed to be much smaller than the transit frequency and the curvature drifts. Thus, with \( \varrho = \frac{v_{\parallel}}{\hat{v}_d} = v_d / \omega_l \) and \( t = t - t_0 \) one finally obtains:

\[
r(t') - r(t) = \sum_{l=-l_{\text{max}}}^{l_{\text{max}}} r_{l, \sigma} \left[ e^{i l \omega_l (t'-t)} e^{i l \omega_l t} - e^{i l \omega_l t} \right] \quad \text{with} \quad r_{l, \sigma} = \sigma_0 \varrho (t + \frac{s^2}{2t})
\]  

(2)
Now, the integration of $\bar{h}$ over $t$ and $\theta$ (projected into matrix form by applying the projection operator $\int d\theta e^{i\Theta}$) can be carried out, leading to the following expression for the propagator coefficient $\bar{h}$:

$$\bar{h} = - \sum_p \sum_m \sum_k \sum_l \sum_\sigma \left[ \frac{a_{m,k,\sigma} K_{m,p,k,l,\sigma} r_{l,\sigma}^g}{\omega - \omega_D - (H\sigma S_m + k + l)\omega_t} - \frac{a_{m,k,\sigma} K_{m,p,k,l,\sigma} r_{l,\sigma}^g}{\omega - \omega_D - (H\sigma S_m + k)\omega_t} \right]$$

Here, the definitions $a_{m,k,\sigma} = \frac{1}{\pi^2} \int_{-\tau_l/2}^{\tau_l/2} dt e^{-\sqrt{\Theta^2 - (\sigma S_m + k)\omega_t^2}}$ with $S_m = \nu(q_0) - m$, $\tau_l = 2\pi/\omega_t$ and $K_{m,p,k,l,\sigma} = \frac{1}{2\pi^2} \int_{-\pi}^{\pi} d\theta e^{-\sqrt{\Theta^2 - (\sigma S_m + k + l)\omega t(\theta)}}$ with $S_p = \nu(q_0) - p$ are used. The minus sign arises from the time integration. The index $j$ runs from $j = 0, 1, 2$ depending on the order of the Taylor expansion $t^j$. If this order is odd, i.e. $j = 1$, and if we assume symmetric distribution function with respect to the pitch angle (i.e index $\sigma$) we can use that $\tau_{\sigma = +1} = -\tau_{\sigma = -1}$, $r_{\sigma = +1} = r_{\sigma = -1}$. The symmetry relations $a_{m,k,1} = a_{m,-k-1}$ and $K_{m,p,k,l,1} = K_{m,-p,-k,l,-1}$ hold and thus the indices $k \to -k$ and $l \to -l$ can be relabelled, and the summation over $\sigma$, i.e. $\frac{1}{(\omega - \omega_D) - C_1^2} = \frac{1}{(\omega - \omega_D) + C_2^2}$ can be carried out. Using the abbreviation $x = (\omega - \omega_D)/(k||_{k,l}v \Omega_k)$ with $k||_{k,l} = (S_m + k + l)/(qR)$ we arrive for $j = 1$ at:

$$\bar{h} = - \sum_p \sum_m \sum_k \sum_l \left[ \frac{2a_{m,k} K_{m,p,k,l} r_{l,1}(S_m + k + l)\omega_t}{(\omega - \omega_D)^2} - \frac{2a_{m,k} K_{m,p,k,l} r_{l,1}(S_m + k)\omega_t}{(\omega - \omega_D)^2} \right]$$

where $\tilde{N}(x_{k|l}) = \frac{x_{k|l} t^2 + x^2/2}{x_{k|l} t^2}$ and $\tilde{N}(x_k) = \frac{x_k t^2 + x^2/2}{x_k t^2}$. The non-adiabatic density response $\int d\mathbf{v} \cdot (\phi_m(r_0) - \psi_m(r_0))' = -\gamma_{\nu}(r)\tilde{N}(x)(\phi_m(r_0) - \psi_m(r_0))'$ can be calculated by assuming a Maxwellian distribution function, i.e. $F_0 = \frac{n(r)}{\sqrt{2\pi} v^3} \cdot e^{-s^2/2}$ and $\partial F_0/\partial E = -F_0/T(r)$ via integrating over velocity space:

$$N(x) \equiv \frac{-2\pi \cdot 2^{3/2}}{\pi x^3/2} \int_0^\infty dt \int ds \frac{x(t^2 + \frac{\gamma^2}{2} e^{-s^2} e^{-t^2}}{t(x^2 - t^2)} [1 - \tilde{\omega}_s] =$$

$$= (1 - \frac{\tilde{\omega}_s}{\omega})(x + (x^2 + \frac{1}{2} Z(x)) - \tilde{\omega}_s x + x^3 + (x^4 + \frac{1}{4} Z(x)]$$

where $\tilde{\omega}_s \equiv \frac{\omega_s}{\omega} = \frac{\omega_s}{\gamma}(1 + \eta \gamma + \eta \gamma^2 - \frac{\gamma}{2})$ and $\tilde{\omega}_s \equiv \frac{\nu_{km}}{\nu_{km}} \frac{1}{\nu_{km}} \frac{\nu_{km}}{\nu_{km}}$. Two minus signs arise and thus cancel: one from $\partial F_0/\partial E$ and the other from the definition of the plasma dispersion function $Z(x)$: $\int_0^\infty dt \frac{2x e^{-t^2}}{(x^2 - t^2)} = -\sqrt{\pi} Z(x)$. Finally, after using $a_{km} = \delta_{km}$, $K_{m,p,k,l,1} = \delta_{m,-k-l}$ and summing over $k$ and $l$, we obtain:

$$\sum_{m,p} -\frac{en(r)\theta}{T_e(r)} \tilde{N}(x)(\phi_m(r_0) - \psi_m(r_0))' =$$

$$\sum_{m,p} -\frac{en(r)\theta}{T_e(r)} \left[ \delta_{p,m-1} N_{1 - N_0} + \delta_{p,m+1} N_{N_0 - N_0} \right] (\phi_m(r_0) - \psi_m(r_0))'$$

$$= \sum_{m,p} -\frac{en(r)\theta}{T_e(r)} \left[ \delta_{p,m-1} N_{1 - N_0} + \delta_{p,m+1} N_{1} \right] (\phi_m(r_0) - \psi_m(r_0))'$$

where $N_1 = N(x_1) = N(\tilde{\omega}_s x|x|)$, $N_{-1} = N(x_{-1}) = N(\tilde{\omega}_s x|x|)$ and $N_0 = N(x_0) = N(\tilde{\omega}_s x|x|)$. 

3
The last term of eqn (1) proportional to \((\omega_d/\omega)\psi_m(r)\) is calculated in the same way. Due to the extra powers of \(v_d \sim t^2 + s^2/2\) the integral equivalent to \(N(x)\), called \(F(x)\) reads:

\[
\frac{\omega_d^\pm}{2\omega} F(x) = -\frac{2\pi}{\sqrt{3}} \frac{\omega_d^\pm}{2\omega} \int_0^\infty dt \int ds \ s \ t(x^2 + \frac{s^2}{2}) e^{-s^2} e^{-t^2} \left[1 - \frac{\omega_d}{\omega}\right] = \\
= \frac{\omega_d^\pm}{2\omega} \left[ \left(1 - \frac{\omega_d}{\omega}\right) \left(\frac{3}{2} x^4 + (x^4 + x^2 + \frac{1}{2}) Z(x)\right)\right] \\
+ \eta \frac{\omega_d}{\omega} \left[ x^5 + x^3 + 2x + Z(x) [x^6 + \frac{1}{2} x^4 + x^2 + \frac{3}{4}] \right]
\]

(3)

The sideband coupling is different than above due to the \(\theta\) dependence of \(\omega_d\); instead of \(a_{k,m,\sigma}\) we have to use \(a_{k,m,\sigma}^G\) \(\frac{\sqrt{\omega_d^\pm}}{\sqrt{\omega_d^\pm}} \frac{\omega_d^\pm}{2\omega} \left[ \left(1 - \frac{\omega_d}{\omega}\right) \left(\frac{3}{2} x^4 + (x^4 + x^2 + \frac{1}{2}) Z(x)\right)\right] + \eta \frac{\omega_d}{\omega} \left[ x^5 + x^3 + 2x + Z(x) [x^6 + \frac{1}{2} x^4 + x^2 + \frac{3}{4}] \right]

For the first time we see second order coupling terms appearing, as expected when including finite orbits width effects.

2.2. Quasi-neutrality: terms \(\sim (r - r_0)^2\)

The other terms of eqn (1) proportional to \((r - r_0)^2\) equation are calculated similarly. The details of the calculation differ slightly, especially when using symmetry properties to integrate over co- and counter passing particles. The details are given in the appendix.

Using these results, the propagator term \(\sim (r - r_0)^2(\phi - \psi)''\) becomes:

\[
\sum_{m,p} \frac{-2n(r) \partial}{T_i(r)} \hat{M}(x)(\phi - \psi)'' = \sum_{m,p} \frac{-2n(r) \partial}{T_i(r)} \left[ \delta_{m,0} \left(\frac{\omega_d^+}{2\omega} \tau (F_2 - F_1) + \frac{\omega_d^-}{2\omega} \tau (F_1 - F_0)\right)\right] \psi_m(r_0)''
\]

(4)

The explicit expression for \(M\) (that is very similar to \(F\)) is given in the appendix.

Finally, the last term of eqn (1) proportional to \((\omega_d/\omega)(r - r_0)\psi_m''(r)\) is calculated in the same way. Due to the extra powers of \(v_d \sim t^2 + s^2/2\) \(t^2 + s^2/2\) the integral equivalent to \(N(x)\) and \(M(x)\), called \(I(x)\) reads:

\[
\sum_{m,p} \frac{-2n(r) \partial}{T_i(r)} \hat{I}(x)(\phi - \psi)'' = \sum_{m,p} \frac{-2n(r) \partial}{T_i(r)} \left[ \delta_{m,0} \left(\frac{2\omega_d^+}{2\omega} (I_1 - I_3) (I_1 - I_0) + \frac{2\omega_d^-}{2\omega} (I_1 - I_2) (I_1 - I_0)\right) \right] \psi''
\]

Here, terms proportional to \(\delta_{m,0}\) have been omitted since they will not contribute to the final dispersion relation.
2.3. FLR Effects

So far, FOW effects for the QN equation have been derived. In order to add FLR effects consistently, we have to include the second order expansion of the operator \( J_{0}^{\perp}(k_{\perp} \vartheta_{i}) \approx 1 - k_{\perp}^{2} \vartheta_{i}^{2} / 2 \). It should be noted that these FLR corrections do not contain any poloidal angle \( \theta \) dependence and thus are fundamentally different from the FOW effects, that depend via \( \nu_{v \varphi} \) explicitly on \( \theta \). FLR effects simply modify the expressions for the coefficients \( D, N \) and \( H \) as derived in [13, 14, 23]: due to the expansion in \( k_{\perp} \vartheta_{i} \), we have to add a term with the extra factor \( \frac{v_{th}^{2}}{v_{th}^{2} + k_{\perp}^{2}} = \frac{v_{th}^{2}}{v_{th}^{2}} \hat{\vartheta}^{2} \) to the velocity space integrals. The zero-orbit width (ZOW) expression for \( D \) is \( D^{m}(x) = (1 - \omega_{m}^{2}) xZ(x) - \frac{\omega_{m}^{2}}{\omega_{m}} \eta(x^{2} + xZ(x)(x^{2} - \frac{1}{2})) \). The FLR-modified \( D \) coefficient is:

\[
D_{\text{FLR}}^{m} = \frac{2 \cdot 2 \pi}{\sqrt{b/2} v_{th}^{3}} \int_{0}^{\infty} v_{th} dt \int_{0}^{\infty} ds s v_{th}^{2} \frac{x^{2} s^{2} e^{-s^{2} e^{-t^{2}}}}{(x^{2} - t^{2})} \left[ 1 - \tilde{\omega}_{s} \right]
\]

and thus for \( D_{\text{FLR}}^{m} \) we get (just sign of \( 1/2 \) flipped):

\[
D_{\text{FLR}}^{m}(x) = (1 - \omega_{m}^{2}) xZ(x) - \frac{\omega_{m}^{2}}{\omega_{m}} \eta(x^{2} + xZ(x)(x^{2} + \frac{1}{2}))
\]

If we add this FLR correction to the N terms, we obtain:

\[
N_{\text{FLR}}^{m}(x) = (1 - \omega_{m}^{2}) x^{2} + xZ(x)(x^{2} + 1) - \frac{\omega_{m}^{2}}{\omega_{m}} \eta(x^{4} + 2x^{2} + xZ(x)(x^{4} + \frac{3}{2}x^{2} + \frac{3}{2})).
\]

in comparison to the ZOW expression \( N^{m}(x) = (1 - \omega_{m}^{2}) [x^{2} + xZ(x)(x^{2} + \frac{1}{2})] - \frac{\omega_{m}^{2}}{\omega_{m}} \eta[x^{2}(x^{2} + \frac{1}{2}) + xZ(x)(\frac{1}{2} + x^{2})] \). As shown in [14], \( D \) is a diagonal term only, and thus we define

\[
\hat{D} = \hat{D}^{m}(x_{m}) = D^{m}(x_{m}) - \hat{\vartheta}^{2} D_{\text{FLR}}^{m}(x_{m})
\]

whereas

\[
\hat{N} = \hat{N}^{m}(x) = \delta_{p,m-1} \frac{\omega_{d}^{+}}{2 \omega} (N^{m}(x_{m+1}) - \hat{\vartheta}^{2} N_{\text{FLR}}^{m}(x_{m+1})) + \delta_{p,m+1} \frac{\omega_{d}^{+}}{2 \omega} \left[ N^{m}(x_{m-1}) - \hat{\vartheta}^{2} N_{\text{FLR}}^{m}(x_{m-1}) \right]
\]

For the GKM we also need:

\[
\hat{N}^{G} = \hat{N}^{Gm}(x) = \delta_{p,m-1} \frac{\omega_{d}^{+}}{2 \omega} (N^{Gm}(x_{m}) - \hat{\vartheta}^{2} N_{\text{FLR}}^{Gm}(x_{m})) + \delta_{p,m+1} \frac{\omega_{d}^{+}}{2 \omega} \left[ N^{Gm}(x_{m}) - \hat{\vartheta}^{2} N_{\text{FLR}}^{Gm}(x_{m}) \right]
\]

and finally, for \( H_{\text{FLR}} \) we get:

\[
H_{\text{FLR}}^{m}(x) = (1 - \omega_{m}^{2}) \left( x^{4} + \frac{5}{2} x^{2} + (x^{4} + 2x^{2} + \frac{3}{2}) xZ(x) \right) - \omega_{m}^{2} \eta(x^{6} + 3x^{4} + \frac{13}{2} x^{2} + xZ(x)(x^{6} + \frac{5}{2} x^{4} + \frac{9}{2} x^{2} + \frac{15}{4}))
\]

in comparison to the ZOW expression: \( H^{m}(x) = (1 - \omega_{m}^{2}) (xZ(x)(\frac{1}{2} + x^{2} + x^{4}) + \frac{3x^{2}}{2} + x^{4}) - \eta \frac{\omega_{m}^{2}}{\omega_{m}} [xZ(x)(\frac{3}{4} + x^{2} + \frac{x^{4}}{2} + x^{6}) + 2x^{2} + x^{4} + x^{6}] \) and thus

\[
\hat{H} = \delta_{m} \frac{\omega_{d}^{+} \omega_{d}^{-}}{4 \omega^{2}} (H^{m}(x_{m-1}) + H^{m}(x_{m+1})) \quad \text{with} \quad \hat{H}^{m}(x_{m}) = H^{m}(x_{m}) - \hat{\vartheta}^{2} H_{\text{FLR}}^{m}(x_{m})
\]
Note that in the GAM limit a factor of 2 arises when assuming symmetry with respect to $x_{m-1}$ and $x_{m+1}$ and such in the asymptotic limit $x \gg 1$ in $\tilde{H} = \delta_{m,p} \left( \frac{7}{4} + 23/(8x_{m+1}^2) + \ldots \right)$. In the final dispersion relation, another factor of 2 appears when writing the pre-factors of the integrals in terms of $\omega_{th}^2 = 2T/m$ and $v_A$. These FLR terms correspond to the expressions called $V, W, U, T$ in refs.[18, 19]. However, here the expressions are written in a more general form, i.e. they are valid for finite toroidal mode numbers and take into account the arising asymmetry with respect to $\omega_m$ which can be of importance for intermediate poloidal mode numbers.

2.4. QN summary

Finally we can collect all FLR/FOW corrections and sum over ions and electrons. For the QN equation the matrix structure is the following:

$$
\begin{pmatrix}
\hat{D} + \hat{\rho}^2 M_d & \hat{\rho} N_{-1} & \hat{\rho}^2 M_{-2} \\
\hat{\rho} N_{1} & \hat{D} + \hat{\rho}^2 M_d & \hat{\rho} N_{-1} \\
\hat{\rho}^2 M_2 & \hat{\rho} N_{1} & \hat{D} + \hat{\rho}^2 M_d
\end{pmatrix}
\begin{pmatrix}
(\phi - \psi)_{m-1} \\
(\phi - \psi)_m \\
(\phi - \psi)_{m+1}
\end{pmatrix}
= 
\begin{pmatrix}
\hat{\rho} F_d & \hat{\rho} N_{-1} & \hat{\rho}^2 I_{-1} \\
\hat{\rho} F_d & \hat{\rho} N_{1} & \hat{\rho}^2 I_{-1} \\
\hat{\rho} F_d & \hat{\rho} N_{1} & \hat{\rho}^2 I_{-1}
\end{pmatrix}
\begin{pmatrix}
\psi_{m-1} \\
\psi_m \\
\psi_{m+1}
\end{pmatrix}
$$

with $\hat{D} = [1 + D_e] + \tau [1 + \hat{D}]$, where electron FLR effects have been omitted, $\tau = T_e/T_i$ and the relation $\omega_{de,m} = -\tau \omega_{di,m}$ was used. For brevity only the central three poloidal harmonics (i.e. the main poloidal harmonic and the first two sidebands) have been visualised, however, for the final dispersion relation (see below) five harmonics, i.e. 2 upper and 2 lower sidebands have to be used.

2.5. GKM summary

The procedure how to treat the gyrokinetic moment equation and how to combine with the QN equation in order to derive the final dispersion relation, is described in detail in ref. [14] and will not repeated here. We just focus on the modifications of the coefficients of the term $\int d^3v \, c_{n} v_d \cdot \nabla J_0 \hat{h}$ and the term $\frac{3\omega_{th}^2}{4v_A^2} \nabla \cdot \frac{\partial \delta_b}{\partial \nabla}$ that arises from eqn. 11 in ref.[14]. Thus, the procedure for all terms is similar to the QN section with the difference that the system now is one order higher in $v_d$. The summary of the coefficients is given in the appendix. The non-adiabatic part of GKM equations with FLR/FOW corrections (corresponding to the ZOW version in eqn. 19, ref. [14] reads then:

$$
\begin{pmatrix}
\hat{N}_1^G + \hat{\rho}^2 Q^G_1 & \hat{N}_1^G + \hat{\rho}^2 Q^G_1 & \hat{N}_1^G + \hat{\rho}^2 Q^G_1 \\
\hat{N}_1^G + \hat{\rho}^2 Q^G_1 & \hat{N}_1^G + \hat{\rho}^2 Q^G_1 & \hat{N}_1^G + \hat{\rho}^2 Q^G_1 \\
\hat{N}_1^G + \hat{\rho}^2 Q^G_1 & \hat{N}_1^G + \hat{\rho}^2 Q^G_1 & \hat{N}_1^G + \hat{\rho}^2 Q^G_1
\end{pmatrix}
\begin{pmatrix}
(\phi - \psi)_{m-1} \\
(\phi - \psi)_m \\
(\phi - \psi)_{m+1}
\end{pmatrix}
+ 
\begin{pmatrix}
\hat{H} + \hat{\rho}^2 L^G_1 & \hat{H} + \hat{\rho}^2 L^G_1 & \hat{H} + \hat{\rho}^2 L^G_1 \\
\hat{H} + \hat{\rho}^2 L^G_1 & \hat{H} + \hat{\rho}^2 L^G_1 & \hat{H} + \hat{\rho}^2 L^G_1 \\
\hat{H} + \hat{\rho}^2 L^G_1 & \hat{H} + \hat{\rho}^2 L^G_1 & \hat{H} + \hat{\rho}^2 L^G_1
\end{pmatrix}
\begin{pmatrix}
\psi_{m-1} \\
\psi_m \\
\psi_{m+1}
\end{pmatrix}
\equiv RHS
$$

In order to complete the system, the left hand side of the GKM equation $\nabla_\perp \frac{\omega^2}{v_A^2} \nabla_\perp \phi \rightarrow \omega^2/v_A^2 k_A^2 \phi$, $b \cdot \nabla (\nabla \times (\nabla \psi)) | b \cdot b \rightarrow -k_A^2 \frac{\omega^2}{v_A^2} \phi$ and on the right hand side the polarisation term $\frac{3\omega^2}{8v_A^2} k_A^4 \phi \rightarrow \frac{3\omega^2}{4R_\phi^2} k_A^4 \phi$ and diamagnetic correction term $b \times (\frac{\omega^2}{2v_A^2}) \cdot \nabla \nabla_\perp \phi \rightarrow \omega^2 \phi/v_A^2 k_A^2 \phi$ have to be added.
The terms $\sim \phi$ can be combined and when $\phi$ is replaced in favour of $\psi$ by using QN including lowest order polarisation, the adiabatic part of the dispersion relation reads:

$$k_1^2 \left[ \hat{\omega}^2 (1 - \hat{\omega}_* / \hat{\omega}) (1 - (\tau / 2 + 3/8) k_1^2 q_1^2 \psi - k_1^2 \rho R_0^2 \psi) \right] = RHS$$

which is the usual kinetic Alfvén wave dispersion relation with diamagnetic corrections on the left hand side where $\hat{\omega} = \omega / \omega_A = \omega R_0 / v_A$. When solved numerically, no simplifications with respect to the operators acting on background quantities are made, as it is done above for illustrative purposes.

### 3. General dispersion relation

At this point, the system can be solved numerically (e.g. Nyqvist contour method, as described in ref. [14]), either locally by choosing $k_1 \cdot q_1$ as a parameter or globally by solving the 4-th order differential equation. The results of the comparison of the numerical solutions and the analytical limits in the literature will be discussed below.

In order to make contact to existing analytical literature we solve the QN for $\phi - \psi$ and substitute into the GKM. Collecting all second order radial derivatives, neglecting FOW effects and expanding in $\hat{\phi}$ leads to the expression:

$$\hat{\omega}^2 - \hat{k}_1^2 \rho R_0^2 = \left( \frac{3}{8} + \tau / 2 \right) \hat{\omega}_*^2 q_1^2 k_1^2 + \frac{\omega^2 \mu_0 \varepsilon n_i R_0^2}{T_i} \left[ - (H_{F,0}^0 + H_{F,L,1}^0) + \frac{N_{F,1}^0 N_{G,1}^{-1}}{D_{-1}} + \frac{N_{F,0}^0 N_{G,1}^{-1}}{D_{1}} \right]$$

where the pre-factor is $\omega^2 \mu_0 \varepsilon n_i R_0^2 / T_i = 2 \omega^2 \rho \hat{\omega}_*^2 / v_1^2$ and when combined with $\frac{\omega^2 \rho \hat{\omega}_*^2}{4 \hat{v}_1}$, $\left( \frac{\omega^2 \rho}{4 \hat{v}_1} \right)$ or $\left( \frac{\omega^2 \rho}{2 \hat{v}_1} \right)$ leads to $v_{th,i}^2 / (2 v_1^2)$. Note, that the upper index is necessary to describe the poloidal mode number to be used for the $\omega^{G}_n$ terms, here given relative to the central poloidal mode number.

The FOW corrections to order $O(\hat{\phi}^2)$ are the following ones:

$$\rho^2 \left[ \begin{array}{c} L_{G,0}^0 \frac{F_{-2}^0 F_{G,0}^0}{D_{-2}} + \frac{F_{G,0}^0 F_{-2}^0}{D_{-2}} N_{-1}^0 \frac{G_{-1,1}^1 Q_{-1}^1}{D_{-1}} + \frac{G_{-1}^1 Q_{-1,1}^1}{D_{-1}} + \frac{G_{-1}^1 Q_{-1}^1}{D_{-1}} + \frac{F_{G,0}^0 F_{-2}^0}{D_{-2}} + \frac{F_{G,0}^0 F_{-2}^0}{D_{-2}} \end{array} \right]$$

One immediately can see that in case of symmetry with respect to the poloidal sidebands - such as for $(n = 0, m = 0)$, or high $(m, n)$ - the expression can be greatly simplified, as shown in the next section.
4. GAM limit
In order to make contact to existing GAM literature [2, 15, 16, 18, 24] and for code verification, the \( n = 0 \) limit is discussed now in more detail. In case of \( n = 0 \), keeping symmetrically \( m = -2, \ldots, 2 \), the dispersion relation - keeping in a first step only FLR effects - simplifies significantly since positive and negative indices are equivalent:

\[
\hat{\omega}^2 = \left( \frac{3}{8} + \frac{\tau}{2} \right) \omega^2 \hat{q}^2 k_1^2 + \frac{v_{th}^2}{2 v_A^2} \left( -2 H_1^0 + \frac{2 N_{i0}^0 N_{G,1}^{G,1}}{D_1} \right)
\]

Using the large argument expansion for the plasma dispersion functions, i.e. \( x = \omega / \omega_i \gg 1 \) reduces the expression to \( \hat{\omega}^2 = \left( \frac{3}{8} + \tau \right) \hat{\rho}^2 \omega^2 + \beta (7/4 + \tau + (\beta / q^2 \hat{\omega}^2)(23/8 + 2\tau + \tau^2 + \beta^2)/2 - \hat{\rho}^2[13/4 + 3\tau + \tau^2]) \) where \( \beta = v_{th}^2 / v_A^2 \) and \( \omega_1^2 / \omega^2 = \beta/(q^2 \hat{\omega}^2) \).

Now we add the FOW corrections: the leading FOW term \( L_0^{G,0} \) can be reduced to \( L^G = 4(L_2^G - L_0^G) = 4(\frac{490}{16e^2} - \frac{249}{16e^2}) \) and when using that \( x_2 = \omega / (2\omega_i) = x_1/2 \), \( L^G = \frac{747\omega_i^2}{16e^2} \). Similarly, the other FOW terms reduce to \( (x^2)^2 \left( \frac{491}{16} \tau + \frac{35}{8} \tau^2 + \tau^3 \right) \). All together, the dispersion relation can be written as:

\[
\hat{\omega}^2 = \left( \frac{3}{4} + \tau \right) \hat{\rho}^2 \omega^2 + \beta \left( \frac{7}{4} + \tau + (\beta / q^2 \hat{\omega}^2)(23/8 + 2\tau + \tau^2 + \beta^2)/2 - \hat{\rho}^2[13/4 + 3\tau + \tau^2] \right)
\]

that is identical to the result given in [2], noting that \( \hat{\rho}^2 = v_{th}^2 k_1^2 / 2\Omega_i^2 \) and \( \beta / \omega^2 = q^2 \omega_i^2 / \omega^2 \). The normalisation to the Alfvén frequency in \( \hat{\omega} \) illustrates the connection between GAM and BAE physics for \( k_1 = 0 \) [18, 23], but it can be straightforwardly changed to the more appropriate normalisation \( \omega / \omega_i \) for GAM physics.

5. Verification of numerical results
The full system of equation, consisting of the QN and GKM equations including the FOW/FLR matrices described in sections 2.4 and 2.5 is compared to the dispersion relation, eqn. 7. Since both sign and magnitude of FLR and FOW effects differ, we compare them separately. In the figures they are called ‘only pol.’ (for including only the polariisation term) and ‘pol.+ FLR’ and ‘all’, i.e. polarisation, FLR and FOW together. A simplified experimental ASDEX Upgrade case, called the ‘NLED base case’ is chosen [3], that at the given time point exhibits strong EGAM activity. Here, we analyse this case for \( n = 0 \) in the absence of energetic particles and with a circular equilibrium. The main parameters are the following ones: \( B_0 = 2.2 T, R_0 = 1.67 m, q_0 = 2.4 \) with slightly reversed shear and \( q_{min} = 2.284 \) at \( q_s = 0.5 \) where \( q_s \) is the square root of normalised poloidal flux. The other parameters are \( T_0 = T_0 = 2.47 kE \) and \( n_e = n_D = 1.72 \cdot 10^{19} m^{-3} \). The kinetic profiles and their polynomial fits are given in ref. [3]. In the figures 1 and 2, the level of agreement is shown for two different radial positions of the same case: in fig.1, \( q = 2.33 \) at \( q_s = 0.3 \) and in fig.2 \( q = 3.33 \) at \( q_s = 0.9 \). Dimensionless parameters are given in the figure captions. In both cases the agreement for small \( k_r q_i \) values is satisfactory, whereas for larger values, i.e. shorter wave lengths, differences arise. These deviations are due the fact that eqn. (7) relies on an expansion of the plasma dispersion functions and that in eqn.(7) all \( \hat{\omega} \)’s on the right hand side of the equation have been replaced by the corresponding value for \( k_r q_i = 0 \), whereas the numerical solution does not make these simplifications. Note, that for \( q_s = 0.9 \) the agreement for \( k_r q_i \leq 0.15 \) is almost perfect (fig.2), as expected for large \( \omega / \omega_i \). In fig. 3, the imaginary parts, i.e. the damping rate is plotted for the same two cases described above. Again, agreement with analytical theory is found (formula 9 in ref.[2]) for small \( k_r q_i \). Interestingly,
Figure 1. Numerical solutions of the dispersion relation and comparison to analytical expressions as given by eqn. 7: ‘only pol.’ refers to including only the polarisation term $3/4 + \tau$ in eqn. 7, whereas ‘pol. + FLR’ refers to additionally including all FLR corrections. ‘All’ refers to polarisation terms, FLR and FOW effects. Here, $q = 2.33$ at $\rho_s = 0.3$ (the peak of the experimentally measured EGAM density perturbation). Other dimensionless parameters are $v_{th,i}^2/v_A^2 = 0.003$ and $\omega/\omega_t = 4.11$.

Figure 2. The same comparison as described in fig.1, but at a different radial location: $q = 3.33$ at $\rho_s = 0.9$. Other dimensionless parameters are $v_{th,i}^2/v_A^2 = 0.00072$ and $\omega/\omega_t = 5.75$.

The damping rates for large $k_r \rho_i$ start to decrease, which seems to be a consequence of the upshift of the real mode frequency: since the real frequency dominates the argument of the exponential factor, $\gamma \sim (k_r \rho_i)^2 e^{-(\omega/\omega_t)^2/4}$, the increase of the pre-factor $k_r \rho_i$ is overcompensated by the exponential factor depending on the real frequency shift, as plotted in figs.1 and 2.

Finally, for the same case, the $n = 2$ solution at the BAE accumulation point is compared to the KGAM ($n = 0$) solution (BAE/GAM degeneracy [18, 23]). Its rational surface is $q = 2.5$ at $\rho_s = 0.74$ and thus the central poloidal harmonic is $m = 5$. Two sidebands, i.e. $m = 3 - 7$ are included in the results presented in fig. 4. For both $n = 0$ and $n = 2$ diamagnetic effects were switched on and compared to the $\omega_s = 0$ solutions. In the KGAM case, the $\omega_s$ effects are not visible since $\omega_s^{m=1}(1 + \eta) = 0.0022\omega_A$ is quite small for this discharge at $\rho_s = 0.74$. However, for $n = 2$ and $\omega_s^{m=5}(1 + \eta) = 0.011\omega_A$ the usual
Figure 3. The absolute value of the damping rates of the the cases described in figs.1 and 2 with full FLR/FOW effects compared to the analytical formulae given in [2].

Figure 4. The frequency/damping rate dependence on $k_r \varrho_i$ (color code) for $n = 2$ at the rational surface $q = 2.5$ in comparison to $n = 0$ with and without diamagnetic effects (i.e. $\omega_* = 0$). The poloidal mode number spectrum is chosen around the central harmonic $m = 5 \pm 2$.

downshift [13] is found together with reduced damping due to the destabilising effect of $\omega_*$ in the BAE branch [13, 23]. Also, the KGAM and BAE solutions for $\omega_* = 0$ agree, apart from a minor difference that is due to the slight shift of the radial grid-point from the rational surface.

6. Conclusions
In this paper, a generalised FOW/FLR model for LIGKA has been presented, together with some successful numerical verification tests. Due to its one-to-one correspondence with expressions given by numerical orbit integration and phase space integration, straightforward comparisons can be carried out in order to understand the differences due to geometry (e.g. elongation effects in EGAMs), trapped particles or anisotropic distribution functions. Also the assumption that the ions are treated in the ‘fast circulating particle’ approximation, can be dropped. This typically leads to a reduction of the GAM frequency and to an increased damping rate, leading to the conclusion that the ‘fast circulation ion’
approximation gives a lower limit for the local KGAM damping rate. This will be discussed in detail in further publications.

Based on this framework, the analytical model can be extended using existing literature (e.g. elongation effects [25], trapped particle effects [26] or anisotropic distribution functions [27, 28]) and related at every level to fully numerical evaluations of corresponding quantities (e.g. \( d_{k,m}, G_{k,m,l}, K_{k,m,l} \) or \( D, N, H, \ldots \)). This procedure is envisaged to lead to a fast local model that incorporates consistently the main crucial physics elements for KGAM/EGAM and AE physics. Also, the same coefficients \( D, N, H, \ldots \) can be used as input to the global solver, allowing one to qualify the validity of local calculations. Other applications of this methodology with respect to symmetry breaking and related transport are discussed in ref. [29].

7. Acknowledgements

Discussions with F. Zonca, A. Bierwage, N. Markoski, I. Chavdarovski, T. Hayward-Schneider, F. Palermo and B. D. Scott are acknowledged. Support by the Eurofusion Enabling Research projects ‘NAT’ and ‘NLED’ and National Natural Science Foundation of China under Grant No. 11605186 are appreciated by Z. X. Lu. This work has been carried out within the framework of the EUROfusion Consortium and has received funding from the Euratom research and training programme 2014-2018 under grant agreement No 633053. The views and opinions expressed herein do not necessarily reflect those of the European Commission.

8. Appendix

8.1. System of equations: QN and GKM

The system of equations, as derived in [21] and further developed in [1, 13, 14] consists of the quasi-neutrality equation (QN)

\[
0 = \sum_a e_a \int \, d^2v \left\{ J_0 f \right\}_a + \nabla \cdot \frac{m_i m_i \nabla \phi}{B^2} \tag{8}
\]

and the linearised GK momentum equation (GKM):

\[
- \frac{\partial}{\partial t} \left[ \nabla \cdot \frac{1}{v_A^2} \nabla \phi \right] + B \cdot \nabla \left( \frac{\nabla \times (\nabla \times \left( \frac{\nabla \psi}{i \omega} \right))_B}{B} \right) + \left( b \times \nabla \left( \frac{\nabla \psi}{i \omega} \right) \right)_B \cdot \nabla \mu_0 \frac{\partial \psi}{\partial t} \tag{9}
\]

By using \( f = h + H_1 \frac{\partial F_0}{\partial E} - \frac{e}{i \omega} \left( \nabla \mu_0 \right)_B \cdot \left( b \times \nabla \right) \), the adiabatic response is split off and thus the non-adiabatic part remains to be integrated (eqn. 1) which is the main subject of this paper.

8.2. Calculation of terms \( (r - r_0)^2 \) in the QN

The radial excursion term - now squared - is again expanded in Fourier harmonics:

\[
[r(t') - r(t)]^2 = \frac{1}{2} \sum_{l_1=-l_{1,\text{max}}}^{l_{1,\text{max}}} \sum_{l_2=-l_{2,\text{max}}}^{l_{2,\text{max}}} \left[ e^{il_1 \omega_1 (t-t')} e^{il_2 \omega_1 (t-t_0)} - e^{il_1 \omega_1 (t-t_0)} \right] \left[ e^{il_2 \omega_1 (t-t')} e^{il_2 \omega_1 (t-t_0)} - e^{il_2 \omega_1 (t-t_0)} \right]
\]
The time integrals are for the terms $\sim a_{m,k,\sigma}$ and $K_{m,p,k,l}$, i.e. terms without drifts in eqn. (1) are:

$$\frac{ie}{2\pi} \int_0^{2\pi} d\theta e^{i\theta} \int_{-\infty}^{t} dt e^{-i[\omega - \omega_D - (H\sigma S_m + k)\omega_1](t - t')} \sum_{l_2} \hat{a}_{k,m} e^{-i[S_m \theta - (H\sigma S_m + k)\omega_1]l_2} e^{i k \omega_1(t' - t)}$$

$$\cdot \sum_{l_2} \hat{r}_{l_2,\sigma} \left[e^{i l_2 \omega_1(t' - t)} - e^{i l_2 \omega_1 t'}\right]$$

$$= - \frac{e}{2} \sum_{p \neq m} \sum_{k} \sum_{l_1} \sum_{l_2} \sum_{\sigma} \left[ \frac{\hat{a}_{k,m} K_{m,p,k,l_1, l_2} r_{l_1,\sigma} r_{l_2,\sigma}}{\omega - \omega_D - (H\sigma S_m + k + l_1 + l_2)\omega_1} - \frac{\hat{a}_{k,m} K_{m,p,k,l_1, l_2} r_{l_1,\sigma} r_{l_2,\sigma}}{\omega - \omega_D - (H\sigma S_m + k + l_1)\omega_1} \right]$$

$$= - \frac{e}{2} \sum_{p \neq m} \sum_{k} \sum_{l_1} \sum_{l_2} \sum_{\sigma} \hat{a}_{k,m} K_{m,p,k,l_1, l_2} \theta^2 \left[ \frac{2(\omega - \omega_D)}{(\omega - \omega_D)^2 - [(S_m + k + l_1)\omega_1]^2} \right]$$

$$\cdot \left[ \frac{x_{m,k,l_1,l_2}^2}{x_{m,k,l_1,l_2}^2 - t^2} - \frac{x_{m,k,l_1}^2}{x_{m,k,l_1}^2 - t^2} - \frac{x_{m,k,l_2}^2}{x_{m,k,l_2}^2 - t^2} + \frac{x_{m,k}^2}{x_{m,k}^2 - t^2} \right]$$

$$= - \frac{e}{\omega - \omega_D} \sum_{p \neq m} \sum_{k} \sum_{l_1} \sum_{l_2} \sum_{\sigma} \hat{a}_{k,m} K_{m,p,k,l_1, l_2} \theta^2 \left[ f_{k+l_1+l_2} - f_{k+l_1} - f_{k+l_2} + f_k \right]$$

Summing over $k, l_1, l_2$ with $K_{m,p,k,l_1,l_2+1} = \delta_{m,(k+l_1+l_2)}$ gives:

$$- \frac{e}{\omega - \omega_D} \sum_{p \neq m} \sum_{\sigma} \theta^2 \left[ 2\delta_{p,m} [(f_0 - f_1) + (f_0 - f_1)] + \delta_{p,m+2} [(f_2 - f_1) + (f_0 - f_1)] \right]$$

$$+ \delta_{p,m-2} [(f_2 - f_1) + (f_0 - f_1)]$$

where two terms $f_x - f_y$ are combined in the following way:

$$\frac{x^2}{x^2 - t^2} - \frac{y^2}{y^2 - t^2} = \frac{t^2(y^2 - x^2)}{(x^2 - t^2)(y^2 - t^2)}$$

(10)

in order to avoid the explicit cancellation of the divergent term $\int_0^\infty dt \frac{x^2}{x^2 - t^2}$ due to $\frac{\omega^2}{\omega_1^2}$. The velocity space integrals are then:

$$M_x - M_y = \frac{-2\pi}{\pi^{3/2}} \int dt \int ds \frac{(y^2 - x^2)t^2(t^2 + x^2)^2e^{-x^2 + t^2}}{t^2(x^2 - t^2)(y^2 - t^2)} \left[ 1 - \omega_1 \right] =$$

$$= \frac{1}{2x} \left[ 1 - \frac{\omega_1}{\omega} \right] \left[ \frac{3}{2} x + x^3 + (x^4 + x^2 + \frac{1}{2}) Z(x) \right]$$

12
Next, terms of order \( \omega^2 (r - r_0)^2 \) have the following form:

\[
\omega^2 \int dt \int ds \frac{(y^2 - x^2)e^{-s^2}e^{-t^2}(t^2 + 1/2s^2)^3}{\pi^{3/2}} \left[ 1 - \hat{\omega}_s \right] =
\]

\[
\delta(t - t_0) \left( \frac{\omega^2}{2\omega} \int ds \frac{(y^2 - x^2) e^{-s^2} e^{-t^2} (t^2 + 1/2s^2)^3}{\pi^{3/2}} \right) [1 - \hat{\omega}_s] =
\]

\[
\left( \frac{\omega^2}{2\omega} \right) \left( \frac{1}{2\pi} \right) \left( \frac{x^5 + 2x^3 + 3x + (x^6 + 3/2x^4 + 3/2x^2 + 3/4)Z(x) \right) - \delta(t - t_0) \left( \frac{\omega^2}{2\omega} \right) \left( \frac{1}{2\pi} \right) \left( \frac{x^7 + 3/2x^5 + 7/2x^3 + 27/4x + Z(x) \right) \right].
\]

These expressions are identical with \( F(x) \) and \( G(x) \) as given in [18]. As above for \( \mathcal{F} \), the \( 1/t^2 \) has been avoided by using eqn. (10).

8.3. Summary of expressions for non-adiabatic FOW contributions in the GKM:

Terms of order \( \omega \) in the GKM proportional to \( \phi - \psi \) are considered:

\[
\sum_{m,p} \omega^2 \mu e^2 n_i R_0^G \frac{\hat{\omega}}{T_i} \mathcal{F}^G(\phi - \psi)' = \sum_{m,p} \frac{\omega^2 \Omega^2 \nu^2}{R_0^G \nu^2 c_i h_i} \left[ \delta_{p,m-2} \frac{\omega^2}{2\omega} \left( F_1 - F_0 \right) + \delta_{p,m+2} \frac{\omega^2}{2\omega} \left( F_1 - F_0 \right) \right]
\]

where velocity space integrals \( \mathcal{F} \), assuming a Maxwellian distribution, are identical to eqn.3. Note, however, that the indices (i.e. sideband information) of \( \mathcal{F} \) and \( \mathcal{F}^G \) are different due to different combinations of \( a, a^G, K \) and \( K^G \).

Next, terms of order \( \omega (r - r_0)^2 \) in the GKM proportional to \( \phi - \psi \) are considered:

\[
\sum_{m,p} \omega^2 \mu e^2 n_i R_0^G \frac{\hat{\omega}}{T_i} \hat{Q}^G(x_{(2,1,0)})(\phi - \psi)'' = \sum_{m,p} \frac{2\omega^2 \Omega^2 \nu^2}{R_0^G \nu^2 c_i h_i} \left[ \delta_{p,m+1} \frac{\omega^2}{2\omega} \left( Q_2^G - 2Q_1^G + Q_0^G \right) + \frac{2\omega^2}{2\omega} \left[ 2Q_0^G - Q_{-1}^G - Q_1^G \right] \right]
\]

where the velocity space integration gives:

\[
Q_2^G - Q_0^G = \frac{-2\pi \hat{\omega}_s}{\pi^{3/2}} \int dt \int ds \frac{(y^2 - x^2)e^{-s^2}e^{-t^2}(t^2 + 1/2s^2)^3}{\pi^{3/2}} [1 - \hat{\omega}_s] =
\]
It turns out that only the diagonal term contributes to the dispersion relation \( \text{eqn. 5} \). The next, terms of order \( (\frac{\omega}{\omega_0})^2(r - r_0) \) in the GKM proportional to \( \psi \) are considered:

\[
\sum_{m,p} \delta \mathcal{L}^G \psi' = \sum_{m,p} \left[ \delta_{p,m-1} \left( \frac{\omega^+}{2\omega} \right)^2 \left( I_0^G - I_1^G \right) + \frac{\omega^+ \omega^-}{(2\omega)^2} \left( \frac{I_2^G - I_1^G + I_0^G - I_1^G}{2} \right) + \right]
\]

\[
\delta_{p,m+1} \left( \frac{\omega^-}{2\omega} \right)^2 \left( I_0^G - I_1^G \right) + \frac{\omega^- \omega^-}{(2\omega)^2} \left( \frac{I_2^G - I_1^G + I_0^G - I_1^G}{2} \right) \right] \psi'
\]

\[
\text{with}
\]

\[
I_G = \frac{-2\pi \cdot 2}{\pi s/2} \left( \frac{\omega^+}{2\omega} \right)^2 \int dt \int ds s \frac{x^e^{-s}e^{-t^2}}{s^2} \left( t^2 + \frac{1}{2} s^2 \right) [1 - \omega^+] = \left( \frac{\omega^-}{2\omega} \right)^2 \left[ \left( 1 - \frac{\omega}{\omega} \right) \right]
\]

\[
\left[ \frac{1}{2} \omega^+ \omega^- \left( \frac{2\omega}{2\omega} \right)^2 \left( I_0^G - I_1^G \right) + \frac{\omega^- \omega^-}{(2\omega)^2} \left( \frac{I_2^G - I_1^G + I_0^G - I_1^G}{2} \right) \right] \psi'
\]

Finally, the highest order terms, \( (\frac{\omega}{\omega_0})^2(r - r_0)^2 \) in the GKM proportional to \( \psi \) are given:

\[
\sum_{p,m} \delta \mathcal{L}^G \psi'' = \delta_{p,m} \left( \frac{\omega^+}{2\omega} \right)^2 \left[ L_0^G - 2L_1^G \right] + \left( \frac{\omega^-}{2\omega} \right)^2 \left[ L_1^G - 2L_0^G + L_1^G \right] + 2 \frac{\omega^+ \omega^-}{(2\omega)^2} \left[ 2L_1^G + 2L_1^G - 2L_1^G - 2L_1^G + 2L_1^G \right] + \delta_{p,m=\pm 2} \ldots = \delta_{p,m} \delta \mathcal{L}^G \psi''
\]

It turns out that only the diagonal term contributes to the dispersion relation eqn. 5. The \( \pm 2 \) sidebands \( \mathcal{L}^G_{\pm 2} \) are omitted here.

\[
F_x^G - F_y^G = \frac{-2\pi \cdot 2}{\pi s/2} \left( \frac{\omega^+}{2\omega} \right)^2 \int dt \int ds s \frac{(y^2 - x^2)e^{-s}e^{-t^2}}{(s^2 - t^2)(y^2 - t^2)} \left( t^2 + \frac{1}{2} s^2 \right) [1 - \omega^+] = \left( \frac{\omega^-}{2\omega} \right)^2 \left[ \left( 1 - \frac{\omega}{\omega} \right) \right]
\]

\[
\left[ \left( \frac{1}{2} \omega^+ \omega^- \left( \frac{2\omega}{2\omega} \right)^2 \left( I_0^G - I_1^G \right) + \frac{\omega^- \omega^-}{(2\omega)^2} \left( \frac{I_2^G - I_1^G + I_0^G - I_1^G}{2} \right) \right) \psi'
\]

\[
\text{with}
\]

\[
\text{finally, the highest order terms, } (\frac{\omega}{\omega_0})^2(r - r_0)^2 \text{ in the GKM proportional to } \psi \text{ are given:}
\]

\[
\sum_{p,m} \delta \mathcal{L}^G \psi'' = \delta_{p,m} \left( \frac{\omega^+}{2\omega} \right)^2 \left[ L_0^G - 2L_1^G \right] + \left( \frac{\omega^-}{2\omega} \right)^2 \left[ L_1^G - 2L_0^G + L_1^G \right] + 2 \frac{\omega^+ \omega^-}{(2\omega)^2} \left[ 2L_1^G + 2L_1^G - 2L_1^G - 2L_1^G + 2L_1^G \right] + \delta_{p,m=\pm 2} \ldots = \delta_{p,m} \delta \mathcal{L}^G \psi''
\]

It turns out that only the diagonal term contributes to the dispersion relation eqn. 5. The \( \pm 2 \) sidebands \( \mathcal{L}^G_{\pm 2} \) are omitted here.
References

[1] Lauber P, Günter S, Könies A and Pinches S D 2007 Journal Of Computational Physics 226 447–465
[2] Zonca, F and Chen, L 2008 EPL 83 35001
[3] Lauber P 2015 The NLED Base Case URL http://www2.ipp.mpg.de/~pwl/NLED_AUG/data.html
[4] Hayward-Schneider T, Lu Z X, Wang X, Lauber P and Bottino A 2018 presented at the THEORY OF FUSION PLASMAS JOINT VARENNA - LAUSANNE INTERNATIONAL WORKSHOP, Villa Monastero, Varenna, Italy, August 27 - 31, 2018, to be submitted to PPCF 2018
[5] Figueiredo A, Rodrigues P, Borba D, Coelho R, Fazendeiro L, Ferreira J, Loureiro N, Nabais F, Pinches S, Polevoi A and Sharapov S 2016 Nuclear Fusion 56 076007
[6] Waltz R and Bass E 2013 13th IAEA Technical Meeting on Energetic Particles in Magnetic Confinement Systems (1720 September 2013)
[7] Pinches S D, Appel L C, Candy J, Sharapov S E, Berk H L, Borba D, Breizman B N, Hender T C, Hopcraft K I, Huysmans G T A and Kerner W 1998 Computer Physics Communications 111 133–149
[8] White R B and Chance M S 1984 Physics of Fluids 27 2466
[9] Bottino A, Vernay T, Scott B, Hatzky R, Jolliet S, McMillan B, Tran T and Villard L 2011 Plasma Phys. Controlled Fusion 53 124027
[10] Bottino A and Sonnendrücker E 2015 J. Plasma Phys. 81 435810501
[11] Lin Z, Hahm T S, Lee W W, Tang W M and White R B 1998 Science 281 1835–1837
[12] Candy J and Waltz R E 2003 Phys. Rev. Lett. 91(4) 045001
[13] Lauber P, Brüdgam M, Curran D, Igochine V, Sassenberg K, Günter S, Maraschek M, Garcia-Munoz M, Hicks N and the ASDEX Upgrade Team 2009 Plasma Physics and Controlled Fusion 51 124009
[14] Lauber P 2013 Physics Reports 533 33 – 68
[15] Sugama H and Watanabe T H 2006 Journal of Plasma Physics 72(06) 825–828
[16] Sugama H and Watanabe T H 2008 Journal of Plasma Physics 74(01) 139–140
[17] Pinches S 1996 Ph.D. Thesis, The University of Nottingham
[18] Zonca F, Chen L, Santoro R A and Dong J Q 1998 Plasma Physics and Controlled Fusion 40 2009
[19] Lu Z, Wang X, Lauber P and Zonca F 2018 Nuclear Fusion 58 082021
[20] Zonca F, Chen L, Botrugno A, Buratti P, Cardinali A, Cesarino R and Ridolfini V 2009 Nuclear Fusion 49 085009
[21] Qin H, Tang W M and Rewoldt G 1999 Physics of Plasmas 6 2544–2562
[22] Bierwage A and Lauber P 2017 Nuclear Fusion 57 116063
[23] Zonca F, Chen L and Santoro R 1996 Plasma Physics and Controlled Fusion 38 2011–2028
[24] Qin Z, Chen L and Zonca F 2009 Plasma Physics and Controlled Fusion 51 012001
[25] Gao Z, Peng L, Wang P, Dong J and Sanuki H 2009 Nuclear Fusion 49 045014
[26] Chavdarovski I and Zonca F 2009 Plasma Physics and Controlled Fusion 51 115001
[27] Xie H S 2013 Physics of Plasmas 20 092125
[28] Girardo J B, Zarzoso D, Dumont R, Garbet X, Sarazin Y and Sharapov S 2014 Physics of Plasmas 21 092507
[29] Lu Z X, Wang X, Lauber P, Fable E, Bottino A, Hornsby W, Zonca F and Angioni C 2018 presented at the THEORY OF FUSION PLASMAS JOINT VARENNA - LAUSANNE INTERNATIONAL WORKSHOP, Villa Monastero, Varenna, Italy, August 27 - 31, 2018, to be submitted to PPCF 2018