Finite dimensional representations of rational Cherednik algebras

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Abstract

A complete classification and character formulas for finite-dimensional irreducible representations of the rational Cherednik algebra of type $A$ are given. Less complete results for other types are obtained. Links to the geometry of affine flag manifolds and Hilbert schemes are discussed.

1 Main results

1.1 Preliminaries

Fix a finite Coxeter group $W$ in a complex vector space $\mathfrak{h}$. Thus, $\mathfrak{h}$ is the complexification of a real Euclidean vector space and $W$ is generated by a finite set $S \subset W$ of reflections $s \in S$ with respect to certain hyperplanes $\{H_s\}_{s \in S}$ in that Euclidean space.

For each $s \in S$, we choose a nonzero linear function $\alpha_s \in \mathfrak{h}^*$ which vanishes on $H_s$ (the positive root corresponding to $s$), and let $\alpha_s^\vee = 2(\alpha_s, -)/(\alpha_s, \alpha_s) \in \mathfrak{h}$ be the corresponding coroot. The group $W$ acts naturally on the set $S$ by conjugation.

Put $\ell := \dim \mathfrak{h}$. Let $\wedge^i \mathfrak{h}$, $i = 0, 1, \ldots, \ell$, denote the $i$-th wedge power of the tautological reflection representation. Thus, $\wedge^0 \mathfrak{h} = \text{triv}$ is the trivial 1-dimensional representation of $W$, and $\wedge^i \mathfrak{h} = \text{sign}$ is the sign representation. It is known that if $\mathfrak{h}$ is an irreducible $W$-module (as we will assume), then the representations $\wedge^i \mathfrak{h}$ are irreducible for all $i$.

According to [EG], for each $W$-invariant function $c : S \to \mathbb{C}$, $c \mapsto c_s$, one defines a rational Cherednik algebra $H_c = H_c(W)$ to be an associative algebra generated by the vector spaces $\mathfrak{h}$, $\mathfrak{h}^*$, and the set $W$, with defining relations (cf. formula [EG, (1.15)] for $t = 1$) given by

$$
\begin{align*}
    w \cdot x \cdot w^{-1} &= w(x) , & w \cdot y \cdot w^{-1} &= w(y) , & \forall y \in \mathfrak{h} , & x \in \mathfrak{h}^* , & w \in W \\
    x_1 \cdot x_2 &= x_2 \cdot x_1 , & y_1 \cdot y_2 &= y_2 \cdot y_1 , & \forall y_1, y_2 \in \mathfrak{h} , & x_1, x_2 \in \mathfrak{h}^* \\
    y \cdot x - x \cdot y &= \langle y, x \rangle - \sum_{s \in S} c_s \langle y, \alpha_s \rangle \langle \alpha_s^\vee, x \rangle \cdot s , & \forall y \in \mathfrak{h} , & x \in \mathfrak{h}^* .
\end{align*}
$$

Thus, the elements $x \in \mathfrak{h}^*$ generate a subalgebra $\mathbb{C}[\mathfrak{h}] \subset H_c$ of polynomial functions on $\mathfrak{h}$, the elements $y \in \mathfrak{h}$ generate a subalgebra $\mathbb{C}[\mathfrak{h}^*] \subset H_c$, and the elements $w \in W$ span a copy of the group algebra $\mathbb{C}W$ sitting naturally inside $H_c$.

Given a finite dimensional $W$-module $\tau$, extend it to a module $\tilde{\tau}$ over $\mathbb{C}[\mathfrak{h}^*] \# W$, the crossed product algebra, by letting $\mathbb{C}[\mathfrak{h}^*]$ act via the evaluation map $P \mapsto P(0)$. Following [DO] and [BEG], define a standard $H_c$-module $M(\tau)$ to be the induced module $M(\tau) := H_c \otimes_{\mathbb{C}[\mathfrak{h}^*] \# W} \tilde{\tau}$. This module is known, by [DO], to have a unique simple quotient, to be denoted $L(\tau)$. Furthermore, in [OR] an abelian category $\mathcal{O}(H_c)$ (which was denoted by $\mathcal{O}_0(H_c)$ in [BEG]) of ‘bounded below’ $H_c$-modules has been introduced, similar to one
defined by Bernstein-Gelfand-Gelfand in the case of semisimple Lie algebras. By definition, the objects of \( \mathcal{O}(H_c) \) are finitely-generated \( H_c \)-modules \( M \) such that the \( \mathfrak{h} \)-action on \( M \) is locally-nilpotent. The modules \( \{L(\tau)\}_{\tau \in \text{Irrep}(W)} \) were shown to be precisely the simple objects of the category \( \mathcal{O}(H_c) \).

In this paper we will be interested in classification and characters of \textit{finite-dimensional} \( H_c \)-modules. They clearly belong to the category \( \mathcal{O}(H_c) \), but exist only for a certain special set of parameters ‘\( c \)’.

\textbf{Remark.} Let \( \varepsilon : W \to \mathbb{C}^\times \) be a group character. Then there exists an isomorphism \( H_c \cong H_{c,\varepsilon} \) acting identically on \( \mathfrak{h} \) and \( \mathfrak{h}^* \), and sending each \( w \in W \) to \( \varepsilon(w) \cdot w \), so the representation categories of \( H_c \) and \( H_{c,\varepsilon} \) are equivalent. Under this equivalence, representations with lowest weight \( \tau \) go to representations with lowest weight \( \varepsilon \otimes \tau \). That explains symmetry between our results for \( H_c \) and for \( H_{c,\varepsilon} \), at various places below. ◊

\subsection{1.2 Type A.}

We start with the case \( W = S_n \), the Symmetric group, and \( \mathfrak{h} = \mathbb{C}^{n-1} \), so that the parameter ‘\( c \)’ runs over the line \( \mathbb{C} \).

The following result provides a complete classification of irreducible finite dimensional representations of the algebra \( H_c = H_c(S_n) \).

\textbf{Theorem 1.2 (Classification of finite dimensional representations in type \( A_{n-1} \)).}

(i) The only values of \( c \in \mathbb{C} \) for which nonzero finite dimensional representations of \( H_c \) exist are the rational numbers of the form: \( \pm r/n \), where \( r \in \mathbb{N} \) with \( (r,n) = 1 \).

(ii) Let \( c = \pm r/n \) with \( r \in \mathbb{N} \), \( (r,n) = 1 \). Then the only irreducible finite dimensional representation of \( H_c \) is \( L(\text{triv}) \), in the ‘+’ case, resp. \( L(\text{sign}) \), in the ‘−’ case.

\textbf{Remark.} Some special cases of this Theorem can be found in [CO], [Dz], and [Go]. Also, it was pointed out by Cherednik [Ch2 p.65] that Theorem 1.2 can be deduced from the results of [Ch2] (see Section 7.3 below for more details). ◊

\subsection{1.3 Geometric realization}

Let \( \mathbb{P}^k \) denote the complex projective space of dimension \( k \), and let \( \mathbb{C}_{pk} \) denote the constant sheaf on \( \mathbb{P}^k \) (extended by zero, whenever viewed as a sheaf on \( \mathbb{P}^l \supset \mathbb{P}^k \)). Let \( \mathcal{P}_{erv}(\mathbb{P}^{n-1}) \) be the abelian category of perverse sheaves on \( \mathbb{P}^{n-1} \) which are constructible with respect to the standard stratification \( \mathbb{P}^{n-1} = \text{pt} \cup \mathbb{C}^1 \cup \mathbb{C}^2 \cup \ldots \cup \mathbb{C}^{n-1} \). The perverse sheaves \( \mathbb{C}_{pk}[k] \), \( k = 0,1,\ldots,n-1 \), are exactly the simple objects of the category \( \mathcal{P}_{erv}(\mathbb{P}^{n-1}) \).

\textbf{Theorem 1.3.} Let \( W = S_n \), and \( c = r/n \) with \( (r,n) = 1 \), \( r \in \mathbb{N} \). Then

(i) The block of the category \( \mathcal{O}(H_c) \) containing the (only) finite dimensional representation is equivalent to the abelian category \( \mathcal{P}_{erv}(\mathbb{P}^{n-1}) \). The equivalence sends the perverse sheaf \( \mathbb{C}_{pk}[k] \) to the \( H_c \)-module \( L(\wedge^k \mathfrak{h}) \), \( \forall k = 0,1,\ldots,n-1 \).

(ii) The class of \( L(\wedge^k \mathfrak{h}) \) in the Grothendieck group of the category \( \mathcal{O}(H_c) \) is given by the formula \( [L(\wedge^k \mathfrak{h})] = \sum_{j=k}^n (-1)^{j-k} \cdot [M(\wedge^j \mathfrak{h})] \), for any \( k = 0,1,\ldots,n-1 \).
(iii) Any block in \( \mathcal{O}(H_c) \) other than the one considered in (i) is semisimple.

**Remark.** If \( c = -r/n \), the category \( \mathcal{O}(H_c) \) has a similar structure, since we have an isomorphism \( H_c \simeq H_{-c} \).  

### 1.4 Other types

We are going to generalize Theorem 1.2 to other Coxeter groups \( W \). Given a group homomorphism \( \varepsilon : W \to \mathbb{C}^\times \), introduce the following

**Notation:**

\[
 r := \frac{2}{\ell} \sum_{s \in S} c_s \cdot \varepsilon(s).
\]

For example, if \( c \) and \( \varepsilon \) are constant functions on \( S \), then \( c = \varepsilon \cdot r/h \), where we put \( h := 2 \cdot |S|/\ell \). It is well known, see e.g. [Hu, 3.18], that the number \( h \) thus defined is equal to the Coxeter number of \( W \).

**Theorem 1.4.** Assume that \( c, \varepsilon \) are constant on \( S \), and we are in one of the following situations.

1. \( r = k \cdot h + 1 \), where \( k \) is a nonnegative integer;
2. \( W = W(B_n) \), \( r = 2k + 1 \), where \( k \) is a nonnegative integer, such that \( (r, n) = 1 \).
3. \( W = W(D_n) \), \( r = 2k + 1 \), where \( k \) is a nonnegative integer, such that \( (r, n-1) = 1 \).

Then the only finite dimensional irreducible \( H_c \)-module is \( L(\varepsilon) \).

The proof of (a generalization of) this theorem is given in Section 6. For analogues of Theorem 1.4 for the double-affine Hecke algebra see [Ch2, §8].

**Remark.** For types other than type \( A \) there may exist more than one irreducible finite dimensional representation of \( H_c \) for a given \( c \), and they may have lowest weights which are not 1-dimensional. As we will see below, it happens for example, in type \( D_4 \). This seems to be an '\( L \)-packet' phenomenon, cf. §7.1.

Let \( \{x_i\} \) and \( \{y_i\} \) be dual bases of \( \mathfrak{h} \) and \( \mathfrak{h}^* \), respectively. Following [BEG], put \( h := \frac{1}{2} \left( \sum_{i=1}^{\ell} x_i y_i + y_i x_i \right) \in H_c \). This element commutes with \( W \) in \( H_c \). It is known that the \( h \)-action on any \( V \in \mathcal{O}(H_c) \) is locally finite, moreover, for each \( a \in \mathbb{C} \), the generalized \( h \)-eigenspace \( V_a \) corresponding to the eigenvalue \( a \) is finite dimensional (and, of course, \( W \)-stable). These (generalized) eigenspaces \( V_a \) will be referred to as weight spaces of \( V \), so \( V = \bigoplus_a V_a \), where the sum ranges (by [BEG]) over a finite union of (one-sided) arithmetic progressions with step one each.

For a finite dimensional \( W \)-module \( E \) and \( w \in W \), write \( \text{Tr}(w, E) \) for the trace of \( w \)-action in \( E \). Now, given \( V \in \mathcal{O}(H_c) \) with weight decomposition \( V = \bigoplus_a V_a \), and \( w \in W \), define the character of \( V \) as a formal infinite series in (complex) powers of \( t \) given by:

\[
 \text{Tr}(w, V_a) := \sum_a \text{Tr}(w, V_a) \cdot t^a.
\]

For example, the character of the standard module \( M(\tau) \) is equal to

\[
 \text{Tr}_{M(\tau)}(w \cdot t^h) = \frac{t^{\kappa(c, \tau)} \cdot \text{Tr}(w, \tau)}{\det(1 - t \cdot w)}, \quad \text{where} \quad \kappa(c, \tau) = \frac{\ell}{2} - \sum_{s \in S} c_s \cdot s|_{\tau}. \tag{1.5}
\]
Here and elsewhere, ‘det’ denotes the determinant of the reflection representation, and
\(\sum_{s \in S} c_s s|_\tau\) denotes the scalar by which the indicated central element of \(CW\) acts in
the simple \(W\)-module \(\tau\). Formula (1.5) follows from the natural identification \(M(\tau) \simeq \mathbb{C}[h] \otimes \tau\),
and the fact that the lowest weight of \(M(\tau)\) equals \(\kappa(c, \tau)\), see e.g. [BEG].

**Theorem 1.6 (Character formula).** Under the assumptions of either Theorem 1.2 or
Theorem 1.4 the representation \(L(\varepsilon)\) has the character:
\[
\text{Tr}_{L(\varepsilon)}(w \cdot t^h) = \varepsilon(w) \cdot t^{-(r-1)t/2} \frac{\det(1 - t^r \cdot w)}{\det(1 - t \cdot w)}, \quad \forall w \in W.
\]
In particular, \(\dim L(\varepsilon) = r^\ell\).

For the rest of the subsection assume that \(W\) is a Weyl group of a simple Lie algebra. Write \(Q\) for the root lattice in \(h^*\). The group \(W\) acts naturally on \(Q\). This induces, for any
integer \(r > 0\), an \(W\)-action on \(Q/r \cdot Q\), making the \(\mathbb{C}\)-vector space \(\mathbb{C}[Q/r \cdot Q]\),
of \(\mathbb{C}\)-valued functions on \(Q/r \cdot Q\), a \(W\)-module. The following corollary of the character formula above is a generalization of [BEG, Conjecture 5.10].

**Proposition 1.7.** In the situation of either Theorem 1.2 or Theorem 1.4, and \(c > 0\), one
has: \(L(\varepsilon)|_W \simeq \mathbb{C}[Q/r \cdot Q]\).

**Proof.** E. Sommers showed in [So] (using a case by case argument) that if \((r, h) = 1\) then, for any \(w \in W\), one has \(\text{Tr}(w, \mathbb{C}[Q/r \cdot Q]) = r^{\text{fix}(w)}\), where \(\text{fix}(w)\) is the dimension
of the space of invariants of the \(w\)-action in \(h\). On the other hand, we calculate
\[
\text{Tr}(w, L(\varepsilon)) = \varepsilon(w) \cdot t^{-(r-1)t/2} \frac{\det(1 - t^r \cdot w)}{\det(1 - t \cdot w)} = r^{\text{fix}(w)}.
\]
Hence, the \(W\)-modules \(L(\varepsilon)\) and \(\mathbb{C}[Q/r \cdot Q]\) have equal characters. The result follows.

Given a \(W\)-module \(E\) and \(\tau \in \text{Irrep}(W)\), let \(\mathcal{E} := (\tau \otimes E)^W\) denote the \(\tau\)-isotypic
component of \(E\).

**Corollary 1.9.** In the situation of either Theorem 1.2 or Theorem 1.4, and \(c > 0\), we have:
\[
\dim(\mathcal{L}(\varepsilon)) = \frac{1}{|W|} \cdot \sum_{w \in W} \text{Tr}(w, \tau) \cdot r^{\text{fix}(w)}.
\]

**Proof.** Use (1.8) and the general formula: \(\dim(\mathcal{E}) = \frac{1}{|W|} \cdot \sum_{w \in W} \text{Tr}(w, \tau) \cdot \text{Tr}(w, E)\).
1.5 Representations of the spherical subalgebra

An analogue of Theorem 1.2 also holds for finite dimensional representations of the spherical subalgebra \( eH_0e \subset H_c \), where \( e = \frac{1}{|W|} \sum_{w \in W} w \), see [EG], [BEG]. To formulate it, note that, for any \( H_c \)-module \( V \), the space \( e \cdot V \) has a natural \( eH_0e \)-module structure, and the element \( h \) preserves the subspace \( e \cdot V \subset V \). Also, write \( d_i, i = 1, \ldots, \ell \), for the degrees of the basic \( W \)-invariant polynomials on \( h \).

Theorem 1.10 (Character formula for \( eH_0e \)). (i) In type \( A_{n-1} \), the only positive values of \( c \) for which nonzero finite dimensional representations of \( eH_0e \) exist are \( r/n \), where \( r = 1, 2, \ldots \), with \( (r,n) = 1 \).

(ii) Under the assumptions of either Theorem 1.2 or Theorem 1.4, \( e \cdot L(\text{triv}) \) is the only finite-dimensional simple \( eH_0e \)-module, if \( c > 0 \).

(iii) We have

\[
\text{Tr}_{e \cdot L(\text{triv})}(t^h) = t^{-(r-1)/2} \prod_{i=1}^{\ell} \frac{(1 - t^{d_i+r-1})}{(1 - t^{d_i})}.
\]

In particular, we have: \( \text{dim}(e \cdot L(\text{triv})) = \prod_i \frac{d_i + r - 1}{d_i} = \frac{1}{r} \left( \frac{r+n-1}{n} \right) \) in type \( A_{n-1} \).

Remark. Finite dimensional representations of \( eH_0e \) for \( c < -1 \) have the same structure as those for \( c > 0 \), since the algebra \( eH_0e \) is isomorphic to \( eH_{c-1}e \), see e.g. [BEG2]). For \( W = S_n \) and \( -1 \leq c \leq 0 \), there are no finite dimensional representations. Indeed, if \( V \) were a nonzero finite dimensional representation of \( eH_0e \) then \( V' = H_c e \otimes_{eH_0e} V \) would have been a nonzero finite dimensional representation of \( H_c \) such that \( e \cdot V' \neq 0 \). On the other hand, from Theorem 1.2 it follows that such representation does not exist: \( e \cdot L(\text{sign}) = 0 \) in this case. ♦

We postpone the proof of parts (i),(ii) of Theorem 1.10 until later sections. However, assuming that the character formula for \( L(\text{triv}) \) is already known, part (iii) of the Theorem is an immediate corollary of the following general lemma from the theory of complex reflection groups.

Lemma 1.11 (Solomon [S]). Let \( W \) be a complex reflection group acting in a complex vector space \( \mathfrak{h} \) of dimension \( \ell \). Then one has an equality of rational functions in \( t, y \):

\[
\frac{1}{|W|} \sum_{w \in W} \frac{\det(1 + y \cdot w)}{\det(1 - t \cdot w)} = \prod_{i=1}^{\ell} \frac{1 + y \cdot t^{d_i-1}}{1 - t^{d_i}}.
\]

Proof of Lemma. The LHS is the two-variable Poincaré series of the space \( \Omega^\bullet(\mathfrak{h})^W \) of invariant polynomial differential forms on \( \mathfrak{h} \) (powers of \( y \) count the rank of the form and powers of \( t \) the homogeneity degrees of the polynomial coefficients). However, as was shown in [S], \( \Omega^\bullet(\mathfrak{h})^W \) is a free supercommutative algebra generated by basic invariants
Therefore, it generates a proper

\[ \text{H} \] cannot contain the whole

\[ \text{N} \]

The result of Solomon may be seen as claiming that the projection \( \tau \)-module is a singular vector, belonging to the

\[ \text{L} \text{triv} \text{H} \text{c} \text{m} \text{o} \text{d} \text{u} \text{l} \text{e} \text{r} \]

1.7 The Gorenstein property of \( \text{L} \text{triv} \).

The action of the commutative subalgebra \( \mathbb{C}[\mathfrak{h}] \subset \text{H} \text{c} \) on the lowest weight vector in the simple module \( \text{L} \text{triv} \) clearly generates the whole space \( \text{L} \text{triv} \). Therefore one has \( \text{L} \text{triv} \subset \mathbb{C}[\mathfrak{h}]/I \), where \( I \) is an ideal in \( \mathbb{C}[\mathfrak{h}] \). Thus, \( \text{L} \text{triv} \) becomes a positively graded commutative algebra.

**Proposition 1.13.** If \( \dim \text{L} \text{triv} < \infty \), then \( \text{L} \text{triv} = \mathbb{C}[\mathfrak{h}]/I \) is a Gorenstein algebra.

Proof. Observe first that the module \( \text{L} \text{triv} \) has a canonical (Jantzen type) contravariant nondegenerate bilinear form \( J(-,-) \), see [DO], [Dz]. To make this form *invariant* rather than *contravariant*, we use the \( \text{SL}_2(\mathbb{C}) \)-action on any finite dimensional \( \text{H} \text{c} \text{m} \text{o} \text{d} \text{u} \text{l} \text{e} \text{r} \) that has been constructed in [BEG, Remark after Proposition 3.8]. In particular, let \( F \)

\[ p_1, ..., p_\ell \]

and their differentials \( dp_1, ..., dp_\ell \) As the RHS of the equality in the lemma is exactly the Poincaré series of such free algebra, the Lemma follows.

To obtain Theorem 1.1(iii), it suffices to use the character formula of Theorem 1.6, note that \( \text{Tr}_{\epsilon \cdot \text{L} \text{triv}}(t^\mathfrak{h}) = \frac{1}{|W|} \sum_{w \in W} \text{Tr}_{\epsilon \cdot \text{L} \text{triv}}(w \cdot t^\mathfrak{h}) \), and to substitute \( y = -t^\mathfrak{r} \) in Lemma 1.11.

**Remark.** Compatibility of the dimension formula for \( \epsilon \cdot \text{L} \text{triv} \) arising from Theorem 1.1(iii) with the one given, in the special case \( \tau = \text{triv} \), by Corollary 1.9 is equivalent, modulo the standard identity \( \prod_i d_i = |W| \), to the Shephard-Todd formula: \( \sum_{w \in W} r^{\text{fix}(w)} = \prod_{i=1}^\ell (r + d_i - 1) \). 

1.6 Absence of self-extensions

**Proposition 1.12.** (i) For any simple object \( V \in \mathcal{O}(\text{H} \text{c}) \) one has: \( \text{Ext}^1_{\mathcal{O}(\text{H} \text{c})}(V, V) = 0 \). In particular,

(ii) In all the situations considered in this section, any finite-dimensional \( \text{H} \text{c} \text{-m} \text{o} \text{d} \text{u} \text{l} \text{e} \text{r} \) is a multiple of a (unique) simple \( \text{H} \text{c} \text{-m} \text{o} \text{d} \text{u} \text{l} \text{e} \text{r} \).

Proof. Let \( \tau \in \text{Irrep}(W) \) be the lowest weight of \( V \). Let \( N \) be an extension of \( V \) by \( V \), and \( N_0 \) the lowest nonzero generalized eigenspace of \( \mathfrak{h} \) in \( N \). Any nonzero vector \( v \in N_0 \) is a singular vector, belonging to the \( W \)-module \( \tau \), since the \( W \)-action is semisimple. Therefore, it generates a proper \( \text{H} \text{c} \text{-s} \text{u} \text{b} \text{m} \text{o} \text{d} \text{u} \text{l} \text{e} \text{r} \) in \( N \) (because it is a quotient of \( M(\tau) \), it cannot contain the whole \( N_0 \)), which has to be \( V \). This shows that \( N = V \oplus V \). 

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\[ \text{Recall that } \mathfrak{h}/W := \text{Spec} \mathbb{C}[\mathfrak{h}]/W \text{ is an affine space with coordinates } p_1, ..., p_\ell, \text{ by Chevalley theorem. The result of \( J(-,-) \) may be seen as claiming that the projection } \pi: \mathfrak{h} \to \mathfrak{h}/W \text{ induces an isomorphism } \pi^* : \Omega^* (\mathfrak{h}/W) \to \Omega^* (\mathfrak{h}/W) \text{. Here is a sketch of proof of this result different from the one given in [BEG].} \]

To prove the claim it suffices to verify that \( \pi^* \) is an isomorphism on the complement of a codimension 2 subset in \( \mathfrak{h}/W \). Further, the claim is clear over the generic point of \( \mathfrak{h}/W \). Hence, one is reduced to proving the claim at the generic point of a root hyperplane. The latter case is essentially the one-dimensional situation that can be easily verified directly.

\[ \text{[BEG, Remark after Proposition 3.8]. In particular, let } F \]

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\[ \text{For a geometric interpretation of the Proposition cf. } \S \text{7.1.} \]
be the ‘Fourier transform’ endomorphism of $L(\text{triv})$ corresponding to the action of the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2(\mathbb{C})$. Since the Fourier transform interchanges $\mathfrak{h}$ and $\mathfrak{h}^*$, the bilinear form: $\Phi : (v_1, v_2) \mapsto J(v_1, Fv_2)$ on $L(\text{triv})$ is compatible with the algebra structure on $L(\text{triv}) = \mathbb{C}[\mathfrak{h}]/I$, i.e. for any $x \in \mathfrak{h}^*$, we have $\Phi(xv_1, v_2) = \Phi(v_1, xv_2)$. The $SL_2$-representation theory now shows that the top degree homogeneous component of the graded algebra $\mathbb{C}[\mathfrak{h}]/I$ is 1-dimensional. Moreover, the form $\Phi$ being nondegenerate, it provides $L(\text{triv})$ with a Frobenius algebra structure induced by a linear function on $\mathbb{C}[\mathfrak{h}]/I$ that factors through the top degree homogeneous component. Thus, the scheme $\text{Spec}(L(\text{triv}))$ is Gorenstein (of dimension zero).

1.8 Structure of the paper

The paper is organized as follows. In §2, we introduce an important tool of this paper – the BGG resolutions of irreducible finite dimensional representations. In §3, using the Knizhnik-Zamolodchikov functor $\mathcal{F}$ and the representation theory of Iwahori-Hecke algebras at roots of unity [DJ1,DJ2], we prove the existence of the BGG resolution in the type A case. This will play a crucial step in the proof of Theorem 1.2, Theorem 1.3, and Character Theorem 1.6 (for type A) given in §5. Section 4 contains some general results about category $\mathcal{O} (\mathcal{H}_c)$, and also a proof of Theorem 1.4(i) and Theorem 1.10(i)-(ii). Classification Theorem 1.2 is proved in §5, using the trace techniques developed in [BEG, §5]. In §6 we prove the results on finite dimensional representations of rational Cherednik algebras for types B and D. This section contains a proof of Theorem 1.4(ii)-(iii). Finally, in §7 we discuss connections with other topics, such as the affine flag variety, and the geometry of the Hilbert scheme of the affine plane.

Remark. After the bulk of this paper had been written, we received a preliminary version of a very interesting preprint by I. Gordon [Go], containing results essentially equivalent to our Theorems 1.4(i), and 1.6(i)-(ii). (for the case $c = 1 + \frac{1}{h}$). The main ideas of [Go] are very similar to ours. However, reading [Go] has allowed us to improve a number of the results of the present paper.

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2 The BGG resolution

2.1 Special case: the Coxeter number

One dimensional representations of the algebra $\mathcal{H}_c(W)$ are easily described by the following

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3This functor was originally introduced by Opdam, and exploited in [BEG]. For a more detailed study of the Knizhnik-Zamolodchikov functor the reader is referred to [GGOR].
well known elementary result.

**Proposition 2.1.** (i) The assignment

\[ w \mapsto \text{Id}, \quad x \mapsto 0, \quad y \mapsto 0, \quad (\forall w \in W, x \in \mathfrak{h}^*, y \in \mathfrak{h}) \]

can be extended to a 1-dimensional \( H_c \)-module if and only if the equation \( 2 \sum_{s \in S} c_s = \ell \) holds.

(ii) If \( c \) is a constant function, then the above assignment can be extended to a 1-dimensional \( H_c \)-module if and only if \( c = 1/h \).

**Proof.** The assignment above sends the last commutation relation in (1.1) to

\[ 0 = y \cdot x - x \cdot y = \langle y, x \rangle \cdot \text{Id} - \sum_{s \in S} c_s \langle y, \alpha_s \rangle \langle \alpha_s^\vee, x \rangle \cdot \text{Id}. \]

But for any conjugacy class \( C \) of reflections, one has

\[ \langle y, x \rangle = \frac{\ell}{2|C|} \sum_{s \in C} \langle y, \alpha_s \rangle \langle \alpha_s^\vee, x \rangle, \quad \forall y \in \mathfrak{h}, x \in \mathfrak{h}^* \]

(the two sides are clearly proportional, and the proportionality coefficient is found by substituting base elements \( x = x_i, y = y_i \), and summing over \( i = 1, \ldots, \ell \)). Thus we see that the assignment of the Proposition gives rise to a 1-dimensional \( H_c \)-module if and only if we have \( 2 \sum_{s \in S} c_s = \ell \).

Part (ii) is now immediate from the identity \( 2|S| = h \cdot \ell \), see [Hu, 3.18] (in the Weyl group case it goes back to Kostant, [Ko]). \( \Box \)

**Remark.** More generally, let \( H_c = H_c(V, \Gamma, \omega) \) be a symplectic reflection algebra associated to a symplectic vector space \( (V, \omega) \) and a finite group \( \Gamma \subset Sp(V, \omega) \), see [EG]. Let \( S \subset \Gamma \) denote the set of symplectic reflections. Then, a similar computation shows that trivial 1-dimensional \( H_c \)-modules exist if and only if the following linear equation on \( c \) holds: \( \sum_{s \in S} c_s \cdot \omega_s = \omega \) (see [EG] for the notation \( \omega_s \)). \( \Diamond \)

### 2.2 BGG resolutions of finite dimensional modules

Since the 1-dimensional \( H_c \)-module considered in Proposition 2.1 restricts to the trivial representation \( \text{triv} \) of the group \( W \), it is isomorphic to \( L(\text{triv}) \). We have the following Cherednik algebra analogue of the Bernstein-Gelfand-Gelfand resolution of the trivial representation of a semisimple Lie algebra (it also appears in [Go]).

**Proposition 2.2.** Let \( H_c \) be the rational Cherednik algebra associated to \( W \), and let \( 2 \sum_{s \in S} c_s = \ell \). Then there exists a resolution of \( L(\text{triv}) \) by standard modules, of the form

\[ 0 \to M(\wedge^2 \mathfrak{h}^*) \to \ldots \to M(\wedge^\ell \mathfrak{h}^*) \to M(\mathfrak{h}^*) \to M(\text{triv}) \to L(\text{triv}) \to 0. \]

Moreover, when restricted to the polynomial subalgebra \( \mathbb{C}[\mathfrak{h}] \subset H_c \), the resolution above becomes the standard Koszul complex:

\[ 0 \to \mathbb{C}[\mathfrak{h}] \otimes \wedge^2 \mathfrak{h}^* \to \ldots \to \mathbb{C}[\mathfrak{h}] \otimes \wedge^\ell \mathfrak{h}^* \to \mathbb{C}[\mathfrak{h}] \otimes \mathfrak{h}^* \to \mathbb{C}[\mathfrak{h}] \to \mathbb{C} \to 0. \]
We build the required resolution explicitly by induction. The first nontrivial
arrow is the augmentation $M(\text{triv}) = \mathbb{C}[h] \rightarrow L(\text{triv}) = \mathbb{C}$. Its kernel is the space of
polynomials vanishing at 0, which is an $H_c$-submodule, generated (already over $\mathbb{C}[h]$) by
linear polynomials, which are singular vectors in this module. Thus, we have $H_c$-module
morphisms: $M(h^*) \rightarrow M(\text{triv}) \rightarrow L(\text{triv}) = \mathbb{C}$. Now, we claim that the kernel of the first
map is a submodule of $M(h^*)$, which is generated (already over $\mathbb{C}[h]$) by $\wedge^2 h^*$ sitting in
degree 1 of $M(h^*)$. This is clear because, from commutative algebra point of view, we
have just the first two terms of the Koszul resolution of $\mathbb{C}$. Thus, we obtain $H_c$-module
morphisms:

$$
0 \rightarrow M(\wedge^2 h^*) \rightarrow \ldots \rightarrow M(\wedge^2 h^*) \rightarrow M(h^*) \rightarrow M(\text{triv}) \rightarrow L(\text{triv}) \rightarrow 0.
$$

The proof of Theorem 2.3 is given in Section 3. Theorem 2.3 will be used in the proof
of Theorem 1.2.

Similarly, outside of type $A$, we have the following (weaker) generalization of Proposition 2.2.

Theorem 2.4 (BGG-resolution). Let $W$ be a finite Weyl group and ‘$c$’ be a constant
of the form $\frac{1}{h} + k$, where $k = 0, 1, 2, \ldots$. Then $L(\text{triv})$ admits a resolution of the form

$$
0 \rightarrow M(\wedge^2 h^*) \rightarrow \ldots \rightarrow M(\wedge^2 h^*) \rightarrow M(h^*) \rightarrow M(\text{triv}) \rightarrow L(\text{triv}) \rightarrow 0.
$$

For $k = 1$ this theorem is contained in [Go]. The proof of Theorem 2.4 is given in
Section 4. It will be used for the proof of Theorem 1.4(i).

3 The Hecke algebra and the proof of Theorem 2.3

3.1 The Hecke algebra and Specht modules

Recall that to a finite Coxeter group $(W, S)$, and a $W$-invariant function $q : S \rightarrow \mathbb{C}^\ast$ one
associates the corresponding Hecke algebra $H_q = H_q(W)$. In this subsection we will recall
from [DJ1, DJ2] some known facts about the representation theory of $H_q$ for root system
of type $A$ which will be used below.

Remark. We adopt the normalisation in which the quadratic relations for the generators
$\{T_i = T_{s_i}\}_{i=1,\ldots,\ell}$ of our Hecke algebra $H_q = H_q(W)$ read: $(T_i - 1)(T_i + q) = 0$. In
[DJ1, DJ2], the quadratic relations for $T_i$ are $(T_i + 1)(T_i - q) = 0$, so the $T_i$ of [DJ1]
are different from ours by a factor $q$, and our $q$ corresponds to $q^{-1}$ of [DJ1, DJ2].
Let \( e \) be the order of \( q \) (equal to infinity if \( q \) is not a root of 1). Recall [DJ1] that for every partition \( \lambda \) of \( n \), we have the Specht module \( S^\lambda = S^\lambda(q) \subset \mathcal{H}_q \) over \( \mathcal{H}_q \), which is flat with respect to \( q \), viewed as a parameter. We also have its quotient \( D^\lambda \), which is nonzero if and only if \( \lambda \) is \( e \)-regular (i.e. multiplicities of all parts of the partition \( \lambda \) are smaller than \( e \)), and is irreducible when nonzero; this gives the full list of irreducible \( \mathcal{H}_q \)-modules without repetitions. Moreover, by [DJ1, 7.6], the module \( D^\lambda \) occurs in \( S^\lambda \) with multiplicity 1.

Recall also from [DJ2] that \( S^\lambda \) and \( S^\mu \) are in the same block if and only if \( \lambda \) and \( \mu \) have the same \( e \)-core (see [JK] for definition of core). In particular, if \( \lambda \) is an \( e \)-core itself (i.e., has no hook of length \( e \); in this case it is also \( e \)-regular), then \( S^\lambda \) lies in a block separate from other \( S^\mu \) and, furthermore, \( D^\lambda \neq 0 \). Hence, if \( \lambda \) is an \( e \)-core, then the only irreducible module in the block of \( S^\lambda \) is \( D^\lambda \), and consequently \( S^\lambda \) is simple.

### 3.2 The Hecke algebra at the primitive \( n \)-th root of 1

The simplest nontrivial special case \( e = n \) (i.e., \( q \) is a primitive \( n \)-th root of unity) will be especially important for us. In this case, \( n \)-core diagrams are all but single-hook diagrams, i.e. diagrams \( \lambda^{\ast} \) corresponding to representations \( \Lambda^\ast \mathfrak{h} \) of \( S_n \), \( i = 0, ..., n - 1 \). The \( n \)-core of \( \lambda^{\ast} \) is the empty diagram. Thus all Specht modules \( S^\lambda^{\ast} \) belong to the same block, while other Specht modules are all irreducible.

Moreover, it is seen from [DJ1] that if \( \lambda \neq \lambda^{\ast}, \forall i \), then \( S^\lambda \) is projective. Therefore, the block of \( S^\lambda \) consists of multiples of \( S^\lambda \). Thus, we have:

\[
\mathcal{H}_q = (\bigoplus_{\lambda \text{ is an } e\text{-core}} \text{End}_c S^\lambda) \oplus A,
\]

where \( A \) is an indecomposable, non-semisimple algebra, corresponding to the only non-semisimple block of the representation category of \( \mathcal{H}_q \).

Further, we need a more precise description of the Specht modules for \( \lambda = \lambda^{\ast} \), which can be inferred from [DJ1, DJ2]. First of all, \( D^{\lambda^{\ast}} \) is nonzero if and only if \( i \neq n - 1 \).

**Proposition 3.1.** The Specht module \( S^{\lambda^{\ast}} \) has composition factors \( D^{\lambda^{\ast}}, D^{\lambda^{\ast} - 1} \) (only one of them if the other is zero or not defined).

(Here and elsewhere, each simple composition factor is written as many times as it occurs in the composition series.)

### 3.3 Knizhnik-Zamolodchikov functor on standard modules

Abusing notation, we will write \( M(\lambda) \) for the standard module over \( \mathcal{H}_c \) whose lowest weight is the representation of \( S_n \) corresponding to a partition \( \lambda \).

Let \( c \) be any complex number, and \( q = e^{2\pi ic} \). Recall ([OR], see also [BEG]) that for any \( V \in \mathcal{O}(\mathcal{H}_c) \), its localization to \( \mathfrak{h}^{\ast \ast} := \mathfrak{h} \setminus (\cup_{s \in S} H_s) \) (= set of regular elements), is a vector bundle \( V|_{\mathfrak{h}^{\ast \ast}} \) with flat connection, i.e., a local system. Assigning to \( V \) the monodromy of that local system gives the “Knizhnik-Zamolodchikov functor” \( \text{KZ} : \mathcal{O}(\mathcal{H}_c) \to \text{Rep}(\mathcal{H}_q) \), see [GGOR] for more details. Put \( \tilde{S}^\lambda := \text{KZ}(M(\lambda)) \).
Lemma 3.2. The $H_q$-modules $\tilde{S}^\lambda$ and $S^\lambda$ have the same composition factors.

Proof. We begin with the following standard result (see e.g. [CG, Lemma 2.3.4]).

Claim: Let $A$ be a flat, finite rank algebra over $\mathbb{C}[\![t]\!]$. Let $M, N$ be two $\mathbb{C}[\![t]\!]$-flat, finite rank $A$-modules. Suppose that $M \otimes_{\mathbb{C}[\![t]\!]} \mathbb{C}(\!(t)\!)$ and $N \otimes_{\mathbb{C}[\![t]\!]} \mathbb{C}(\!(t)\!)$ are isomorphic as $A \otimes_{\mathbb{C}[\![t]\!]} \mathbb{C}(\!(t)\!)$-modules. Then $M/tM$ and $N/tN$ have the same composition factors as $A/tA$-modules. □

Now, in our particular situation, we have two flat (holomorphic) families of modules over $H_q$ (where $q = e^{2\pi ic}$), namely $S^\lambda(c)$ and $\tilde{S}^\lambda(c)$. They are obviously isomorphic at $c = 0$, and hence for regular $c$, by a standard deformation argument. Now, the Claim above implies that, for special value of $c$, the modules $S^\lambda$ and $\tilde{S}^\lambda$ have the same composition factors. Lemma 3.2 is proved. □

Corollary 3.3. For $c = r/n$, where $(r, n) = 1$, $r \in \mathbb{Z}$, we have:

1) If $\lambda \neq \lambda^i$, $\forall i$, then $\tilde{S}^\lambda$ is isomorphic to the Specht module $S^\lambda$, and in particular is irreducible.

2) If $\lambda = \lambda^i$, then $\tilde{S}^\lambda$ belongs to the non-semisimple block and has composition factors $D^{\lambda^i}$ and $D^{\lambda^{i-1}}$ (only one of them if the other is zero).

Proof. This follows from Lemma 3.2 and Proposition 3.1, since $q = e^{2\pi ic}$ is a primitive $n$-th root of unity. □

3.4 Construction of the BGG resolution

Throughout this subsection let $c = r/n$, where $(r, n) = 1$, $r \in \mathbb{N}$.

Proposition 3.4. (i) If $\lambda \neq \lambda^i$, $\forall i$, then $M(\lambda)$ is irreducible.

(ii) If $\lambda \neq \mu$ are Young diagrams, and $\text{Hom}(M(\lambda), M(\mu)) \neq 0$. Then $\lambda = \lambda^i$ and $\mu = \lambda^{i-1}$.

Proof. According to a result by Opdam-Rouquier presented in ([BEG, Lemma 2.10]), the Knizhnik-Zamolodchikov functor is faithful when restricted to standard modules. Therefore, statement (i) follows from the fact, if $\lambda \neq \lambda^i$, $\forall i$, then $\tilde{S}^\lambda$ does not belong to the block of $\tilde{S}^\mu$, for $\mu \neq \lambda$, which is a consequence of Corollary 3.3.

To prove statement (ii), we observe that again by Corollary 3.3, $\lambda = \lambda^i$ and $\mu = \lambda^{i-1}$ for some $i$ and $j$. If $j > i$, the statement is clear since one knows, see e.g. formula (4.3) of §4 below, that: $\kappa(c, \lambda^i) > \kappa(c, \lambda^j)$. On the other hand, if $j < i - 1$, then any nonzero homomorphism from $M(\lambda^i)$ to $M(\lambda^j)$ would give rise, through the Knizhnik-Zamolodchikov functor and [BEG, Lemma 2.10], to a nonzero homomorphism from $\tilde{S}^{\lambda^i}$ to $\tilde{S}^{\lambda^j}$. But such a homomorphism does not exist, since these modules have no composition factors in common. □

Proposition 3.5. The space $\text{Hom}(M(\lambda^i), M(\lambda^{i-1}))$ is one-dimensional.
Proof. The modules $\tilde{S}^{\lambda^i}$ and $\tilde{S}^{\lambda^{i-1}}$ have one composition factor in common, so we see that $\text{Hom}(M(\lambda^i), M(\lambda^{i-1}))$ is at most one dimensional. Thus we are left to show that this space is nonzero.

By Deligne’s criterion, the connection defined by Dunkl operators has regular singularities. Therefore, by the Riemann-Hilbert correspondence, the $(\mathcal{H}_c)|_{\mathfrak{h}^\text{reg}}$-module $M(\lambda^i)|_{\mathfrak{h}^\text{reg}}$ (where $"\cdot"|_{\mathfrak{h}^\text{reg}}$ denotes localization to the Zariski open affine set of regular points in $\mathfrak{h}$) is reducible for $j = 1, \ldots, n - 2$, since so is, $\tilde{S}^{\lambda^i}$, its image under KZ. Hence, $M(\lambda^j)$ itself is reducible (it has a nonzero proper submodule $M(\lambda^j) \cap N$, where $N$ is a nonzero proper submodule of $M(\lambda^j)|_{\mathfrak{h}^\text{reg}}$). This implies that another standard module maps to $M(\lambda^j)$ nontrivially. This yields the result for $i > 1$.

It remains to prove the statement for $i = 1$ (and in particular to show that $M(\text{triv})$ is reducible). To do this, observe that the validity of the statement for $i > 1$ implies that, for any $i \geq 1$, the module $\tilde{S}^{\lambda^i}$ is included in an exact sequence

$$0 \to D_{\lambda^i} \to \tilde{S}_{\lambda^i} \to D_{\lambda^{i-1}} \to 0$$

(to afford maps $\tilde{S}^{\lambda^i} \to \tilde{S}^{\lambda^{i-1}}$ coming from $M(\lambda^i) \to M(\lambda^{i-1})$, $i > 1$). Hence, we have a surjective map $\tilde{S}^{\lambda^i} \to \tilde{S}^{\lambda^0}$. By the Riemann-Hilbert correspondence, this map gives rise to a nonzero map $f : M(\lambda^i)|_{\mathfrak{h}^\text{reg}} \to M(\lambda^0)|_{\mathfrak{h}^\text{reg}}$. Let $N \subset M(\lambda^i)|_{\mathfrak{h}^\text{reg}}$ be defined by $N = f^{-1}(M(\lambda^0))$. This is a nonzero $\mathcal{H}_c$-module. Let $P = N \cap M(\lambda^i)$. Then $P$ is a nonzero submodule of $M(\lambda^i)$ which maps nontrivially to $M(\lambda^0)$. This means that for some $\tau \in \text{Irrep}(\mathfrak{h}_c)$ we have a diagram of nonzero maps of $\mathcal{H}_c$-modules: $M(\lambda^i) \leftarrow M(\tau) \to M(\lambda^0)$. But it follows from the above that the only choice for $\tau$ is $\tau = \lambda^1$. Thus, we have a nonzero map $M(\lambda^1) \to M(\lambda^0)$. We are done. \qed

Propositions 3.4 and 3.5 imply the following corollary (we will often identify $\mathfrak{h}^* \simeq \mathfrak{h}$).

Corollary 3.6. For $c = r/n > 0$, $(r, n) = 1$, $r \in \mathbb{N}$, there exists a (unique up to scaling) complex of $\mathcal{H}_c$-modules

$$0 \to M(\lambda^{n-1}) \to \ldots \to M(h) \to M(\text{triv}) \to L(\text{triv}) \to 0. \quad (3.7)$$

3.5 Exactness of the BGG resolution

The following proposition implies Theorem 2.3.

Proposition 3.8. The complex $(3.7)$ is exact.

To prove the Proposition, we introduce the notion of thick object of $\mathcal{O}(\mathcal{H}_c)$. We say that $V \in \mathcal{O}(\mathcal{H}_c)$ is thick if $V|_{\mathfrak{h}^\text{reg}} \neq 0$. Otherwise we say that $V$ is thin. In other words, $V$ is thick if and only if its Gelfand-Kirillov dimension equals $n - 1$, and thin if and only if it is $< n - 1$.

Lemma 3.9. If $c = r/n > 0$, $(r, n) = 1$, then $L(\tau)$ is thin if and only if $\tau = \text{triv}$. 

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This lemma implies in particular that all $L(\tau)$ except $L(\text{triv})$ are infinite dimensional.

**Proof.** First of all, $L(\text{triv})$ is thin. Indeed, we have an exact sequence $M(\lambda^1) \to M(\lambda^0) \to V \to 0$, and $L(\text{triv})$ is a quotient of $V$. Applying to this sequence the Knizhnik-Zamolodchikov functor and using the fact that this functor is exact, we get an exact sequence $\tilde{S}^\lambda \to \tilde{S}^{\lambda^0} \to KZ(V) \to 0$. Thus, $KZ(V) = 0$, so $V$ and hence $L(\text{triv})$ is thin.

Now, we claim that all other $L(\tau)$ are thick. Indeed, by the Riemann-Hilbert correspondence, $KZ(L(\tau))$ is either zero or irreducible, and the set of all irreducibles obtained this way has to be the full set of irreducibles $D^\lambda$ of $\mathcal{H}_q$. But the number of irreducibles in our case is $p(n) - 1$, where $p(n)$ is the number of partitions of $n$. Hence, for all $\tau \neq \text{triv}$, the module $KZ(L(\tau))$ has to be nonzero. Thus, $L(\tau)$ is thick, for any $\tau \neq \text{triv}$. □

**Corollary 3.10.** For any $i = 0, 1, \ldots, n - 2$, the module $M(\lambda^i)$ is included in an exact sequence

$$0 \to L(\lambda^{i+1}) \to M(\lambda^i) \to L(\lambda^i) \to 0.$$ 

**Proof.** Let $J(\lambda^i)$ be the maximal proper submodule of $M(\lambda^i)$. By exactness of the functor $KZ$ we get $KZ(J(\lambda^i)) = D^{\lambda^{i+1}}$. Since any $L(\tau)$ for $\tau \neq \text{triv}$ is thick, by Lemma 3.9 we see that $J(\lambda^i)$ must have a composition factor $L(\lambda^{i+1})$ with possibly other composition factors all isomorphic to $L(\text{triv})$. But $L(\text{triv})$ cannot occur in $J(\lambda^i)$ since $\kappa(c, \text{triv})$, see (4.3), is smaller than the lowest eigenvalue of $\mathbf{h}$ in $J(\lambda^i)$. Thus, $J(\lambda^i) \simeq L(\lambda^{i+1})$. □

Now, Proposition 3.8 easily follows from Corollary 3.10.

### 4 Results on category $\mathcal{O}(\mathcal{H}_c)$

#### 4.1 $p$-function and characters

Let $(W, S)$ be a Coxeter group, $C \subset S$ a conjugacy class of reflections, and $\tau \in \text{Irrep}(W)$. For $s \in S$, the equation $s^2 = 1$ implies $\tau(s)^2 = \text{Id}$, hence the linear map $\tau(s)$ has eigenvalues $\pm 1$. For any $s \in C$, let $q$ be the number of $(+1)$-eigenvalues, and $p$ the number of $(-1)$-eigenvalues of $\tau(s)$. Then, $q + p = \dim \tau$, and also $\text{Tr}(s, \tau) = q - p$. Now, the central element $\sum_{s \in C} s$ acts in $\tau$ as a scalar, say $p(C, \tau)$. We compute:

$$p(C, \tau) \cdot \dim \tau = \text{Tr}(\sum_{s \in C} s, \tau) = |C| \cdot \text{Tr}(s, \tau) = |C| \cdot (q - p) = |C| \cdot (\dim \tau - 2p).$$

On the other hand, the scalar $p(C, \tau)$ is well-known to be an algebraic integer, for any conjugacy class $C$ in a finite group and for any irreducible module $\tau$. Thus, we conclude

$$p(C, \tau) = \frac{|C| \cdot (q - p)}{\dim \tau} = |C| - \frac{2|C|}{\dim \tau} \cdot p \quad \text{is an integer}. \quad (4.1)$$

Write $p(\tau) := \sum_{C \subset S} p(C, \tau)$ for the scalar by which the central element $\sum_{s \in S} s$ acts in $\tau$. From (4.1) we deduce that $p(\tau)$ is an integer, moreover, we have:

$$-|S| = \text{p(sign)} < p(\tau) < p(\text{triv}) = |S|, \quad \forall \tau \in \text{Irrep}(W), \tau \neq \text{triv, sign}. \quad (4.2)$$
In the special case $\tau = \wedge^i h$ that will play a role below, one easily calculates using \((\ref{4.1})\):

$$p(\wedge^i h) = |S| - h \cdot i, \quad \text{hence} \quad \kappa(c, \wedge^i h) = \ell/2 - c \cdot p(\wedge^i h) = \ell/2 + c \cdot h(i - \ell/2).$$ \hfill (4.3)

(This follows also from Proposition \ref{2.2} because the degrees in the consecutive terms of the Koszul resolution go with step 1.)

Next, let $K(\mathcal{O}(H_c))$ be the Grothendieck group of the category $\mathcal{O}(H_c)$. Recall that classes $\{[M(\tau)] \in K(\mathcal{O}(H_c)), \tau \in \text{Irrep}(W)\}$ form a $\mathbb{Z}$-basis in $K(\mathcal{O}(H_c))$. Thus, for any object $V$ there are certain (uniquely determined) integers $a_{\tau} \in \mathbb{Z}$ such that in $K(\mathcal{O}(H_c))$ we have: $[V] = \sum_{\tau \in \text{Irrep}(W)} a_{\tau} \cdot [M(\tau)]$. Now, assuming ‘c’ is a constant function on $S$, from formula \((\ref{4.3})\) for the character of standard modules we get

$$\text{Tr}_V(w \cdot \ell^h) = \frac{t^{\ell/2}}{\det(1 - t \cdot w)} \cdot \sum_{\tau \in \text{Irrep}(W)} a_{\tau} \cdot \text{Tr}(w, \tau) \cdot t^{-c \cdot p(\tau)}. \hfill (4.4)$$

In particular, we find

$$\text{Tr}_V(\ell^h) = \frac{t^{\ell/2}}{(1 - t)^{\ell}} \cdot \sum_{\tau \in \text{Irrep}(W)} a_{\tau} \cdot \dim \tau \cdot t^{-c \cdot p(\tau)}. \hfill (4.5)$$

4.2 Proof of the character Theorem \ref{1.6} and Theorem \ref{1.4}(uniqueness)

Without loss of generality, we may (and will) restrict our attention to the case $\varepsilon = 1$, i.e. $c > 0$. The assumptions of Theorem \ref{1.4} imply that $(r, h) = 1$ in all the cases at hand. Hence, $q = e^{2\pi ic}$ is a primitive root of unity of order $h$. Hence, by \cite[Lemma 4.3]{Go}, $\mathcal{O}(H_c) = \mathcal{O}' \oplus \mathcal{O}''$, where $\mathcal{O}'$ is generated by the $L(\wedge^i h), i = 0, \ldots, \ell - 1$, and $\mathcal{O}''$ is generated by the other simples in $\mathcal{O}(H_c)$. Thus, the simple and standard modules in $\mathcal{O}''$ coincide, hence no simple module in $\mathcal{O}''$ can be finite dimensional.

Now, fix an irreducible finite dimensional $H_c$-module $L$.

**Proof of the character Theorem \ref{1.6}.** Our argument is similar in spirit to H. Weyl’s proof of his character formula for a compact Lie group.

First of all, as we have seen above, $L$ must be an object of $\mathcal{O}'$. Therefore in the Grothendieck group of the category $\mathcal{O}'$ we can write $[L] = \sum a_i \cdot [M(\wedge^i h)]$, for some integers $a_i$. Hence, applying a version of formula \((\ref{4.3})\) in our specific situation and using \((\ref{4.3})\), we find: $\text{Tr}_L(\ell^h) = t^{-\ell(c \cdot h - 1)/2} \cdot Q(t^{c \cdot h})/(1 - t)^{\ell}$, where $Q(z) = \sum_{i=0}^{\ell} z^i \cdot \binom{\ell}{i} \cdot a_i$.

The polynomial $Q \in \mathbb{C}[z]$ has degree $\ell$, and the ratio $Q(t^{c \cdot h})/(1 - t)^{\ell}$ cannot have a pole at $t = 1$ since it is equal to dim $L$, which is finite. We deduce that $Q(z) = m \cdot (1 - z)^{\ell}$, for some constant $m \in \mathbb{C}$. It follows that, for any $i$ we have: $a_i = m \cdot (-1)^i$. Thus we find: $[L] = m \cdot \sum_{i=0}^{\ell} (-1)^i \cdot [M(\wedge^i h)]$. This implies, since $a_i \in \mathbb{Z}$, that the constant $m$ must be an integer. But if the class $[L] \in K(\mathcal{O}')$ of a nonzero object $L \in \mathcal{O}'$ is divisible by an integer $m$, then the object $L$ cannot be simple unless $m = \pm 1$. Hence the irreducibility of $L$ forces $m = \pm 1$. Therefore, for the character of $L$ we get $\text{Tr}_L(\ell^h) = \pm t^{-\ell(c \cdot h - 1)/2} \cdot (1 - t)^{\ell}/(1 - t)^{\ell}$.

Letting $t \to 1$ here we must get the dimension of $L$, which is positive. Therefore, one must
have the ‘+’-sign in the formula. Thus, \([L] = \sum_{i=0}^{\ell} (-1)^i \cdot [M(\wedge^i \mathfrak{g})]\). This proves that the character of \(L\) is given by the formula of Theorem 1.4.

It remains to show that \(L \simeq L(\text{triv})\). Suppose \(L \simeq L(\wedge^j \mathfrak{g})\), for some \(j > 0\). Then, since \(\kappa(\wedge^j \mathfrak{g}) > \kappa(\text{triv})\) by (4.3), the power expansion: \(\text{Tr}_L(t^h) = \sum (\dim L_n) \cdot t^a\) cannot contain \(t^\kappa(\text{triv})\) with a non-zero coefficient. But on the other hand, it is clear e.g. from formula (4.3) that \(t^\kappa(\text{triv})\) does appear in the power expansion of \(\sum_{i=0}^{\ell} (-1)^i \cdot \text{Tr}_{M(\wedge^i \mathfrak{g})}(t^h)\) with a non-zero coefficient. The contradiction forces \(j = 0\), and \(L \simeq L(\text{triv})\).

This completes the proof of Theorem 1.4 and at the same time yields the uniqueness claim in Theorem 1.4. \(\square\)

4.3 Morita functors

Let \(\varepsilon : W \to \mathbb{C}^\times\) be a character of \(W\), and \(1_{\varepsilon}\) the characteristic function of the subset \(\{s \in S \mid \varepsilon(s) = -1\}\). We write \(e_\varepsilon = \frac{1}{|W|} \sum_{w \in W} \varepsilon(w) \cdot w\) for the central idempotent in \(\mathbb{C}W\) corresponding to \(\varepsilon\).

**Proposition 4.6.** There exists a filtered algebra isomorphism \(\phi_\varepsilon : e\mathbb{H} e \xrightarrow{\sim} e_\varepsilon \mathbb{H} e_\varepsilon + 1_{\varepsilon} e_\varepsilon\).

**Proof.** Let \(A_1(c)\) denote the first algebra, and \(A_2(c)\) the second one. Let \(\mathcal{D}(\mathfrak{h}^{\text{reg}})\) denote the algebra of differential operators on the regular part of \(\mathfrak{h}\). The algebras \(A_1(c), A_2(c)\), each carry a filtration given by \(\deg(\mathfrak{h}^*) = \deg(w) = 0\), \(\deg(\mathfrak{h}) = 1\), and a grading given by \(\deg(\mathfrak{h}^*) = 1\), \(\deg(\mathfrak{h}) = -1\), \(\deg(w) = 0\). Similarly, \(\mathcal{D}(\mathfrak{h}^{\text{reg}})\) has a filtration by order of differential operators, and grading by their homogeneity degree.

As explained in [EG], [BEG] we have the spherical homomorphism \(\theta_c : A_1(c) \to \mathcal{D}(\mathfrak{h}^{\text{reg}})\), and \(\varepsilon\)-antispherical’ homomorphism \(\theta^{-}_c : A_2(c) \to \mathcal{D}(\mathfrak{h}^{\text{reg}})\).

It is shown in [BEG] that for generic \(c\) (i.e. all but countably many), the images of these two homomorphisms are the same. Namely, for an orthonormal basis \(y_i\) of \(\mathfrak{h}\), the elements \(e(\sum y_i^2) e_1(\sum y_i^2) e \in A_1(c)\) and \(e_c(\sum y_i^2) e \in A_2(c)\), go to the same operator, the so-called Calogero-Moser operator \(L_c\) (see [EG]); the same holds for the elements \(eP(x) e\) and \(e\cdot P(x) e\), where \(P\) is a polynomial on \(\mathfrak{h}\). On the other hand, \(A_1(c)\) is generated by \(e(\sum y_i^2) e\) and \(e \cdot \mathbb{C}[\mathfrak{h}] \cdot e\), and similarly for \(A_2(c)\). This proves: \(\text{Image}(\theta_c) = \text{Image}(\theta^{-}_c)\), for generic ‘\(c\)’. We are going to show that the equality holds, in effect, for any ‘\(c\)’.

To this end, recall that the parameter ‘\(c\)’ varies over an affine space \(\mathbb{C}^d\). It is known, see [EG], that for any \(c \in \mathbb{C}^d\), both \(\theta_c\) and \(\theta^{-}_c\) are injective filtration preserving morphisms, such that the associated graded maps are also injective. Moreover, the dimensions of associated graded components of \(\text{Image}(\theta_c)\), resp. of \(\text{Image}(\theta^{-}_c)\), are independent of ‘\(c\)’.

Thus, as ‘\(c\)’ runs over \(\mathbb{C}^d\), one may treat the spaces \(\text{Image}(\theta_c)\), resp. \(\text{Image}(\theta^{-}_c)\), as the fibers of a filtered subbundle \(\text{Image}(\theta^+)\), resp. \(\text{Image}(\theta^-)\), in a trivial vector bundle on \(\mathbb{C}^d\) with fiber \(\mathcal{D}(\mathfrak{h}^{\text{reg}})\), an infinite dimensional filtered vector space.

By an argument two paragraphs above we know that, on a dense subset \(U \subset \mathbb{C}^d\), the two vector subbundles coincide: \(\text{Image}(\theta^+)_{|U} = \text{Image}(\theta^-)_{|U}\). It follows that they coincide everywhere, i.e., \(\text{Image}(\theta_c) = \text{Image}(\theta^{-}_c)\), for any \(c \in \mathbb{C}^d\). Thus the filtered isomorphism claimed by the Proposition may be given by \(\phi_c := (\theta^{-}_c)^{-1} \circ \theta_c\). \(\square\)
Let $C$ be the set of those $c$ for which $H_c e V = V$ for any $V \in \mathcal{O}(H_c)$. This condition is equivalent to saying that the functors $F_c : \mathcal{O}(H_c) \to \mathcal{O}(eH_c e)$ and $G_c : \mathcal{O}(eH_c e) \to \mathcal{O}(H_c)$ given by $F_c(V) = eV$, $G_c(Y) = H_c e \otimes eH_c e Y$ are mutually inverse equivalences of categories. It is easy to see that $c \in C$ if and only if $H_c e \cdot V = V$ for all irreducible objects of $\mathcal{O}(H_c)$.

The isomorphism $\phi_c$ of Proposition 4.6 induces an equivalence $\Phi_c : \mathcal{O}(eH_c e) \to \mathcal{O}(e_h H_{c+1} e_c)$.

**Lemma 4.7.** If $c$ and $-c - 1$ belong to $C$ then the shift functor $S_{c,\varepsilon} : \mathcal{O}(H_c) \to \mathcal{O}(H_{c+1})$, given by: $V \mapsto S_{c,\varepsilon}(V) = H_{c+1} e_\varepsilon \otimes e_h H_{c+1} e_\varepsilon \Phi_c(eV)$ is an equivalence of categories.

**Proof.** The functor $S_{c,\varepsilon}$ is a composition of three functors: $S_{c,\varepsilon} = G^{-1}_c \circ \Phi_c \circ F_c$, where $G^{-1}_c(Y) := H_{c+1} e_\varepsilon \otimes e_h H_{c+1} e_\varepsilon Y$. The functor $F_c$ is an equivalence since $c \in C$, and the functor $\Phi_c$ is an equivalence by definition. To show that $G^{-1}_c : \mathcal{O}(e_h H_{c+1} e_\varepsilon) \to \mathcal{O}(H_{c+1})$ is an equivalence, recall that there exists an isomorphism $H_c \simeq H_{c-1}$ which acts trivially on $\mathfrak{h}$ and $\mathfrak{h}^*$ and maps any reflection $s \in S$ to $-s$. Applying this isomorphism and using that $-c - 1 \in C$, we get the result. \qed

**Proposition 4.8.** Let $c$ be a constant and $\varepsilon$ be the sign character. The shift functor $S_{c,\varepsilon} : \mathcal{O}(H_c) \to \mathcal{O}(H_{c+1})$ is an equivalence of categories in either of the following two cases

(i) If $|c|$ is sufficiently large;

(ii) (see \cite{Go}) If $c = \frac{1}{h} + k$, where $k = 0, 1, 2, \ldots$.

**Proof.** To handle case (i), let $|c|$ be large enough. Then for any $\sigma, \tau \in \text{Irrep}(S_n)$, either $\kappa(c, \sigma) = \kappa(c, \tau)$ or $|\kappa(c, \sigma) - \kappa(c, \tau)| = c \cdot |p(\tau) - p(\sigma)|$ is large. This shows that for any $\tau$, the $\mathfrak{h}$-weight spaces of $M(\tau)$ and $L(\tau)$ coincide up to a large degree. Thus, $L(\tau)$ contains an invariant, and hence is generated by its invariants, as desired.

To prove (ii), we will use \cite[Lemma 4.3]{Go}, which says in particular that $M(\tau)$ is irreducible unless $\tau = \wedge^i \mathfrak{h}$ (this Lemma is stated in \cite{Go} for Weyl groups, but remains true for a general Coxeter group, see \cite[Remark 6.9]{BGK}). By this lemma, it suffices to show that if $c > 0$ then $L(\wedge^i \mathfrak{h})$ is generated by its invariants, and furthermore if $c > 1$ then $L(\wedge^i \mathfrak{h})$ is also generated by its anti-invariants. In other words, we must show that $e L(\wedge^i \mathfrak{h}) \neq 0$ for $c > 0$, and (using the isomorphism $e H_c e \simeq e H_{c-1} e$, see \cite{BEG2}) that $e L(\wedge^i \mathfrak{h}) \neq 0$ for $c < -1$.

Using Lemma 1.11 and the formula for $\kappa(c, \wedge^i \mathfrak{h})$ given in \cite{13}, we obtain: $\text{Tr}_{e h(l \mathfrak{h})}(l^h) = t^{(1-c \cdot h)1/2} \cdot \prod_{i=1}^{\ell} (1 - t^{m_i+1})^{-1} \cdot \sum t^{c h(l \mathfrak{h}) + m_j + \ldots + m_i}$, where $m_i = d_i - 1$ denote the exponents of $W$. For $c > 0$, we see that the powers of $t$ in the first nonvanishing term in the formal series expansion of the function above are increasing with $i$. Therefore, these functions are linearly independent. Hence the result. For $c < -1$, the lowest terms are decreasing (note that $m_j < h$, $\forall j$). Hence, they are also independent. (This argument is due to I. Gordon, \cite[Lemmas 4.4, 4.5]{Go}). \qed
5 Classification of finite-dimensional modules in type A

5.1 The trace formula

In this subsection, we consider the case $W = S_n$. Let $\text{Tr} : H_c \to H_c/[H_c,H_c]$ be the tautological projection. It follows from [BEG, Theorem 1.8(i)], that the space $H_c/[H_c,H_c]$ is 1-dimensional for all $c$ except possibly countably many. Also, as has been shown in §5 of [BEG], for all ‘$c$’ but at most countably many, one has $\text{Tr}(1_{H_c}) \neq 0$. Hence, there exists at most a countable set $T \subset \mathbb{C}$ such that one has: for any fixed noncommutative polynomial $P$ in the generators $w \in S_n$, $\{x_i\}$, and $\{y_i\}$ (dual bases in $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively), there exists a function $f_P : \mathbb{C} \times T \to \mathbb{C}$ such that $\text{Tr}(P) = f_P(c) \cdot \text{Tr}(1_{H_c}) \pmod{[H_c,H_c]}$, $\forall c \in \mathbb{C} \setminus T$.

**Lemma 5.1.** The function $f_P$ is rational. Moreover, for all but finitely many values of $c$, one has $\text{Tr}(P - f_P(c) \cdot 1_{H_c}) = 0$.

**Proof.** Let $F^*_c H_c$ be the standard increasing filtration of $H_c$ such that $\mathbb{C} S_n$ has filtration degree zero, and any element in $\mathfrak{h}$, $\mathfrak{h}^*$ has filtration degree one, see [BEG]. Then for each $N = 1, 2, 3, \ldots$, the set $Z_N \subset \mathbb{C}$ of all the ‘$c$’ for which $P \in \mathbb{C} + [F_N H_c,F_N H_c]$ is semialgebraic. Therefore, either the set $Z_N$ is finite or its complement is finite. If all sets $Z_N$ are finite then for all ‘$c$’ but countably many, $P \notin \mathbb{C} + [H_c,H_c]$, which contradicts to the fact that $P \in \mathbb{C} + [H_c,H_c]$ for $c \in \mathbb{C} \setminus T$. Thus, for some $N$ the complement to $Z_N$ is finite. Moreover, since the set of those ‘$c$’ for which $1 \in [F_N H_c,F_N H_c]$ is finite, we see that for all but finitely many $c$, we have $P \in \mathbb{C} \oplus [F_N H_c,F_N H_c]$. For such ‘$c$’, there exists a unique complex number $f_P(c) \in \mathbb{C}$ (depending on ‘$c$’ as a rational function, of course) such that $P - f_P(c) \cdot 1_{H_c} \in [H_c,H_c]$. The lemma is proved. \hfill $\Box$

**Proposition 5.2.** Fix $s \in W$. For all $c \in \mathbb{C}$ but countably many, in $\mathbb{C}((z))$ we have:

$$\text{Tr}(s e^{z h}) = \frac{g_{s,c}(z)}{(nc)^{n-1}} \cdot \text{Tr}(1), \quad \text{where} \quad g_{s,c}(z) = e^{(1-nc)(n-1)z} \cdot \frac{\det(1 - e^{ncz} s)}{\det(1 - e^{z} s)}. \quad (5.3)$$

Moreover, for any $N = 1, 2, \ldots$, there exists a finite subset $Y_N \subset \mathbb{C}$ such that equation (5.3) holds modulo $z^N$ for all $c \notin Y_N$.

**Proof.** Let $P = s \cdot h^{k}/k!$. By Lemma 5.1, there exists a rational function $f_P$ and a finite set $X_P$ such that $\text{Tr}(P) = f_P(c) \cdot \text{Tr}(1)$, $c \notin X_P$. In particular, we have infinitely many $c = r/n$, $r > 0$, $(r,n) = 1$, which do not belong to $X_P$. For such $r$, by Theorem 1.8, $f_P(c)$ equals the $k$-th term of the series $g_{s,c}(z)/(nc)^{n-1}$ (indeed, it suffices to evaluate the trace in the corresponding finite dimensional representation). Since $f_P$ is a rational function, this equality holds identically. This implies the Theorem. \hfill $\Box$

**Remark.** Recall the notation $\text{fix}(s)$ for the dimension of the fixed point set of an element $s \in S_n$ in $\mathfrak{h}$. A computation similar to (1.8) yields $g_{s,c}(0) = (n \cdot c)^{\text{fix}(s)}. \Diamond$
5.2 Proof of Theorem 1.2

It is sufficient to prove the result for \( c > 0 \), since there is an isomorphism \( H_c \simeq H_{-c} \).

According to [BEG, §3], the set of those \( c > 0 \) for which there are finite dimensional representations of \( H_c \) is of the form \( S + \mathbb{Z}_+ \), where \( S \) is a subset of \( \{1/n, ..., 1 - 1/n\} \). Fix \( r_0/n \in S \). Then \( H_c \) has finite dimensional representations for all \( c \) of the form \( c = k + (r_0/n) \), \( k = 0, 1, 2, ... \).

Take \( k' \) to be sufficiently large and let \( V \) be a finite dimensional representation of \( H_{k + (r_0/n)} \). It is clear that \( V \) belongs to the category \( \mathcal{O}(H_c) \). Hence, formula (5.4) yields:

\[
\text{Tr}_V(s \cdot t^h) = t^{\frac{n-1}{2}} \cdot \frac{Q_s(t^c)}{\det(1 - t \cdot s)} , \quad \text{where} \quad Q_s(t) = \sum_{\tau \in \text{irrep}(S_n)} a_\tau \cdot \text{Tr}(s, \tau) \cdot t^{-p(\tau)}. \tag{5.6}
\]

Further, for each integer \( j \geq 0 \), we consider the finite dimensional representation \( V_j := S_{c+j-1} \cdot ... \cdot S_c(V) \) of \( H_{c+j} \). Since \( c \) is large, for all \( i \geq 0 \), the functors \( S_{c+i} \) are equivalences of categories, due to Proposition 4.8. Hence, the character of \( V_j \) is given by the same linear combination of characters of standard modules, that is given by the formula

\[
\text{Tr}_{V_j}(s \cdot e^{z \hbar}) = e^{\frac{n-1}{2}} \cdot \frac{Q_s(e^{(c+j)z})}{\det(1 - e^{z \cdot s})}, \quad \forall s \in S_n, \ j = 0, 1, 2, \ldots . \tag{5.4}
\]

On the other hand, by Proposition 5.2, for any \( N \), we have

\[
\text{Tr}_{V_j}(s \cdot e^{z \hbar}) = \dim(V_j) \cdot \frac{g_{s,c+j}(z)}{(n \cdot (c + j))^{n-1}} (\text{mod } z^N), \tag{5.5}
\]

for all but finitely many \( j \). We note that \( g_{s,c+j}(0) = (c + j)^{\text{fix}(s)} \), by Remark at the end of §5.1. Observe further that the RHS of formula (5.4) has a pole at \( z = 0 \) of order \( \text{fix}(s) \). Thus, comparing formulas (5.4) and (5.5), we deduce

\[
e^{\frac{n-1}{2}} \cdot \frac{Q_s(e^{(c+j)z})}{\det(1 - e^{z \cdot s})} = K \cdot g_{s,c+j}(z) (\text{mod } z^N) \quad \text{holds for all but finitely many } j,
\]

where \( K \) is a constant independent of \( z, j \), and of \( s \in S_n \). Now, since \( Q_s \) is a polynomial of fixed degree, and the integer \( N \) is arbitrarily large, this implies that \( Q_s(t) = K \cdot t^{-n(n-1)/2}/\det(1 - t^n \cdot s) \). We conclude that for all \( c = r/n \), \( r = r_0 + kn \), one has

\[
\text{Tr}_V(s \cdot t^h) = K \cdot t^{(1-r)(n-1)/2} \cdot \frac{\det(1 - t^r \cdot s)}{\det(1 - t \cdot s)}, \quad \forall s \in S_n. \tag{5.6}
\]

Next, we claim that \( (r, n) = 1 \). To show this, we prove that if \( (r, n) = d > 1 \), then the function on the right hand side of (5.6) has poles in \( \mathbb{C}^* \). Indeed, let \( s \in S_n \) be the cycle of order \( n \). Then we get

\[
\text{Tr}_V(s \cdot t^h) = K \cdot t^{(1-r)(n-1)/2} \cdot \frac{(1 - t^r)(1 - t)}{(1 - t^r)(1 - t^n)}. \tag{5.7}
\]
This function has a pole at \( t = e^{2\pi i/d} \), which proves statement (i) of the theorem.

Statement (ii) was proved in the course of the proof of Theorem 2.3 in §3. Namely, it was shown there that for all \( \tau \neq \text{triv} \) the irreducible representation \( L(\tau) \) is infinite dimensional.

5.3 Proof of Theorem 1.3

For each \( i = 0, 1, \ldots, n - 1 \) set \( L_i = L(\wedge^i \mathfrak{h}) \) and \( M_i = M(\wedge^i \mathfrak{h}) \). Let \( P_i \) denote the indecomposable projective cover of \( L_i \) in the category \( \mathcal{O}(\mathfrak{h}_c) \), which exists by [Gu]. Then the BGG-reciprocity proved in [Gu], shows that each \( P_i \) has a two-step standard flag with submodule \( M_{i+1} \subset P_i \) and such that \( P_i/M_{i+1} \cong M_i \) (where \( M_0 := 0 \)). Put \( P = \oplus_{i=0}^{n-1} P_i \).

By general principles, the block of the finite-dimensional representation in \( \mathcal{O}(\mathfrak{h}_c) \) is equivalent to the category of finite-dimensional representations of the associative algebra \( A = \text{Hom}_{\mathcal{O}(\mathfrak{h}_c)}(P, P) \), viewed as a bigraded algebra \( A = \oplus_{i,j \in \{0, \ldots, n-1\}} A_{ij} \), where \( A_{ij} = \text{Hom}_{\mathcal{O}(\mathfrak{h}_c)}(P_i, P_j) \). Furthermore, from the analysis of morphisms in the category of \( \mathcal{H}_c \)-modules, that we have carried out earlier in this section, one deduces the following

**Claim.** The algebra \( A \) is isomorphic to \( \mathbb{C}Q/J \), a quotient of the Path algebra of the following quiver \( Q \):

\[
\begin{array}{cccccccc}
\bullet & \cdots & \bullet & \bullet \\
0 & g_0 & 1 & g_1 & 2 & \ldots & g_{n-3} & n-2 & g_{n-2} & n-1 & \bullet \\
& f_0 & f_1 & f_2 & f_{n-3} & f_{n-2} & f_n & g_0 & & & \\
\end{array}
\]

by the two-sided ideal \( J \subset \mathbb{C}Q \) generated by relations:

\[
g_0 \circ f_0 = 0, \quad f_{i+1} \circ f_i = 0, \quad g_i \circ g_{i+1} = 0, \quad f_i \circ g_i = g_{i+1} \circ f_{i+1}, \quad \forall i = 0, \ldots, n-3.
\]

On the other hand, it is well-known that the category \( \mathcal{Perv}(\mathbb{P}^{n-1}) \) is also equivalent to the category of finite-dimensional representations of the algebra \( \mathbb{C}Q/J \).

This proves Theorem 1.3(i). Part (ii) of this Theorem follows from Corollary 3.10, and part (iii) from the fact that \( M(\tau) \) is simple unless \( \tau = \wedge^i \mathfrak{h} \) for some \( i \).

5.4 Proof of Theorems 2.4, 1.4(i), and Theorem 1.10(i)-(ii)

Proposition 4.8 and the results of §2 implies Theorem 2.4. Indeed, it is easy to see that the shift functors for \( c = k + \frac{1}{h} \) map standard modules to standard modules. Therefore, applying the shift functors \( k \) times to the BGG resolution at \( c = 1/h \), we get the BGG resolution for \( c = k + \frac{1}{h} \). This also implies Theorem 1.4(i), since a representation having the BGG resolution is necessarily finite dimensional. Finally, at this point we obtain Theorem 1.10(i)(ii), since we have shown that for positive \( c = r/n \), \( (r, n) = 1 \), the functor \( F_c : \mathcal{O}(\mathfrak{h}_c) \rightarrow \mathcal{O}(\mathfrak{eH}_c) \), \( V \mapsto \mathfrak{e} \cdot V \) is an equivalence of categories.
6 Results for types B = C, and D

6.1 Type B

Let $W$ be the Weyl group of type $B_n$, viewed as a subgroup in $GL(C^n)$. Write $\{e_i\}_{i=1,...,n}$ for the standard basis in $C^n$. Then $c = (c_1, c_2)$, where $c_1$ corresponds to roots of the form $\pm e_i \pm e_j$, and $c_2$ corresponds to roots of the form $\pm e_i$. The Coxeter number of $W$ is $h = 2n$.

Theorem 6.1. (i) Let $k$ be a nonnegative integer. Then for any $c$ such that $c_1(n-1)+c_2 = (2k+1)/2$, there exists a lowest weight finite dimensional representation $N(\text{triv})$ of $H_c$ with lowest weight $\text{triv}$ and character given by

$$\text{Tr}|_{N(\text{triv})}(s \cdot t^h) = t^{-kn} \cdot \frac{\det(1 - t^{2k+1}, s)}{\det(1 - t \cdot s)}.$$ (6.2)

(ii) For all but countably many solutions $c$ of $c_1(n-1)+c_2 = (2k+1)/2$, the $H_c$-module $N(\text{triv})$ is irreducible, i.e., isomorphic to $L(\text{triv})$.

6.2 Proof of Theorem 6.1

Let $u$ be a variable, and $c_k(u) = (u, \frac{2k+1}{2} - (n-1)u) \in C[u]_{\geq 2}$. Then $H_{c_k(u)}$ is a (flat) algebra over $C[u]$. It follows from Proposition 2.1 that there exists a unique 1-dimensional $H_{c_0(u)}$-module $V_0$ that restricts to the trivial representation of $W$. We have an isomorphism $V_0 = C[u]$ of $C[u]$-modules.

We use the Shift functor $S_{c_0, \varepsilon}$, where the character $\varepsilon : W \to C^\times$ is given by $\varepsilon(s_{e_i} - e_j) = 1$, $\varepsilon(s_{e_i}) = -1$. Define an $H_{c_k(u)}$-module $V_k := S_{c+(k-1), \varepsilon} \circ \cdots \circ S_{c, \varepsilon}(V_0)$.

Proposition 6.3. $V_k$ is a coherent $C[u]$-module, whose generic fiber is isomorphic to $L(\text{triv}, c)$, and has character (6.3).

Proof. The fact that $V_k$ is coherent is clear, since $H_{c, e_\varepsilon}$ is finite as a right $e_\varepsilon H_{c} e_\varepsilon$-module. We will prove the remaining claims by induction in $k$. For $k = 0$, the statement is clear, so assume it is known for $k = m$ and let us prove it for $k = m + 1$.

First of all, $V_{m+1}$ is generically nonzero, since $eV_m$ is generically nonzero (by the character formula for $V_m$).

On the other hand, let us look at the fiber of $V_{m+1}$ at $u = 0$. In this case the conclusion of both parts of Theorem 6.1 is obvious, since $H_c = H_{0, c_2} = CS_n \times H_{c_2}(A_1)^{\otimes n}$. Thus, $V_{m+1}|_{u=0} = L(\text{triv}, (0, c_2))$ and its character is given by formula (6.2).

This implies that $V_{m+1}$ is torsion free at $u = 0$. Indeed, since $V_{m+1}$ is generically nonzero, its torsion at $u = 0$ is a representation of $H_{0, c_2}$ of smaller dimension than that of $L(\text{triv}, (0, c_2))$, which means (by Theorem 6.1 for $c_1 = 0$) that this torsion must be zero.

Thus, the fiber of $V_{m+1}$ at generic point $u$ has the character equal to that of $L(\text{triv}, c_{m+1}(0))$. It follows that $L(\text{triv}, c_{m+1}(u))$ is contained as a constituent in this fiber, and hence has the dimension at most that of $L(\text{triv}, c_{m+1}(0))$. On the other hand, the
module $L(\text{triv},c_{m+1}(u))$ is the quotient of the standard module $M(\text{triv},c_{m+1}(u))$ by the radical of Jantzen-type contravariant form in $M(\text{triv},c_{m}(u))$. Therefore, the dimension of $L(\text{triv},c_{m+1}(u))$ for generic $u$ is at least the dimension of $L(\text{triv},c_{m+1}(0))$. Hence, these dimensions are equal, and in particular the generic fiber is irreducible.

To summarize, we have shown that for generic $u$, the fiber at $u$ is an irreducible representation with the character of $L(\text{triv},c_{m+1}(0))$. Thus the induction step has been established, and the Proposition is proved. □

Thus, we have shown that $L(\text{triv},c_k(u))$ is finite dimensional for all values of $u$, and generically has the character (6.2). Let $K$ be the kernel of the contravariant form in $M(\text{triv},c_k(u))$, considered as a $\mathbb{C}[u]$-bilinear form on a free $\mathbb{C}[u]$-module $M(\text{triv},c_k(u))$ (thus, $K$ is itself a free $\mathbb{C}[u]$-module). Let $N(\text{triv},c_k(u)) = M(\text{triv},c_k(u))/K$. This is a $\mathbb{C}[u]$-free lowest weight $\mathcal{H}_{c_k(u)}$-module, with character (6.2). Taking the fibers of this module, we get the representations whose existence is claimed in Theorem 6.1(i).

Statement (ii) of the Theorem now follows immediately from the $u = 0$ case by a deformation argument. Theorem 6.1 is proved.

Remark. In fact, for each $k$ the set of points $(c_1,c_2)$ where $N(\text{triv}) \neq L(\text{triv})$ is finite and consists of rational points. The latter follows from a result proved in [DJO], [DO], saying that the determinant of the contravariant form on weight subspaces of $M(\text{triv})$ is a product of linear functions of $c$ with rational coefficients. ◊

Proposition 6.4. If $c = (2k+1)/h$, where $(2k+1,h) = 1$, then $N(\text{triv})$ is irreducible.

Proof. The character formula (4.3) implies that there exist integers $a_\tau$ such that

$$\text{Tr}_{L(\text{triv})}(t^h) = t^{n/2} \cdot Q(t)/(1-t)^n \quad \text{where} \quad Q(t) = \sum_{\tau \in \text{Irrep}(W)} a_\tau \cdot \dim \tau \cdot t^{-(2k+1)p(\tau)/h},$$

The value of this function at $t = 1$, that is the dimension of $L(\text{triv})$, is given by the expression $\frac{(-1)^n}{n!} \cdot (2k+1)^n Q_{t=1} = \sum_{\tau \in \text{Irrep}(W)} a_\tau \cdot (\dim \tau) \cdot [(2k+1) \cdot p(\tau)/h]^n$. Since $p(\tau)$ is an integer, the RHS is clearly divisible by $(2k+1)^n$, the dimension of $N(\text{triv})$. Thus, $L(\text{triv}) = N(\text{triv})$, and we are done. □

In particular, Theorem 1.4(ii) is proved.

Remark. The same argument as in the proof of Proposition 6.4 can be used to prove the following more general fact. Suppose that $c = (c_1,c_2)$ is a solution of the equation $(n-1) \cdot c_1 + c_2 = (2k+1)/2$, such that $c_1,c_2$ are rational numbers whose numerators are divisible by $2k+1$ (when the numbers are written as irreducible fractions). Then $N(\text{triv}) = L(\text{triv})$. ◊

Remark. Similar arguments to the ones used in this subsection can be employed to study finite dimensional representations of $\mathcal{H}_c(G_2)$, using that if $c = (0,c_2)$ then $\mathcal{H}_c(G_2) = \mathbb{C}[\mathbb{Z}/2\mathbb{Z}] \rtimes \mathcal{H}_{c_2}(A_2)$. Specifically, for any solution $c = (c_1,c_2)$ of the equation $3c_1 + 3c_2 = 3k+1$, where $k$ is a nonnegative integer, one can show the existence of a finite dimensional lowest weight representation $N(\text{triv})$ with character given by the formula of Theorem 6.4. This is the case in particular if $c$ is the constant equal to $k/2 + 1/6$. Note that for even
k this follows (since \(h = 6\)) from Theorem 6.4(i). Furthermore, in the latter case one has \(N(\triv) = L(\triv)\). ◊

6.3 Type \(D_n\)

Now let \(W\) be of type \(D_n\). In this case the Coxeter number is \(h = 2(n - 1)\).

**Theorem 6.5.** (i) If \(c = (2k + 1)/h\) where \(k\) is a nonnegative integer, then there exists a finite dimensional lowest weight module \(N(\triv)\) over \(H_c\) with character given by (6.2).

(ii) If \((2k + 1, h) = 1\) then \(N(\triv)\) is irreducible (i.e. is isomorphic to \(L(\triv)\)).

**Proof.** Recall that \(H_{c,0}(B_n) = \mathbb{C}[\mathbb{Z}/2] \times H_c(D_n)\). By Theorem 6.1, for \(c = r/h\), there is a lowest weight representation \(N(\triv)\) of \(H_{c,0}(B_n)\), which has character given by (6.2). Restricting this representation to \(H_c(D_n)\), we obtain a required representation of \(H_c(D_n)\).

The proof of its irreducibility for \((2k + 1, n) = 1\) is analogous to the proof of Proposition 6.4. The theorem is proved. \(\square\)

In particular, Theorem 6.4(iii) is proved.

6.4 An example

An example of the algebra \(H_c(D_4)\) shows, as we will now see, that in the simplest non-relatively prime case \(c = 1/2\) the module \(N(\triv)\), constructed in the Theorem above, is not irreducible, and actually contains a submodule whose lowest weight is the reflection representation tensored with a character.

In more detail, we start with the algebra \(H_c(B_4)\) with parameter \(c = (c_1, c_2)\) such that \(6c_1 + 2c_2 = 3\). Then by Theorem 6.1, \(N(\triv)\) is an 81-dimensional lowest weight representation of \(H_c(B_4)\), with character \((t^{-1} + 1 + t)^4\). Generically, this representation is irreducible. Moreover, for \(c_1 = 0\) hence generically, we have an \(H_c\)-module map \(\psi : M(\mathfrak{g}) \rightarrow M(\triv)\), and \(N(\triv) = M(\triv)\setminus\text{Image}(\psi)\), where \(\text{Image}(\psi)\) is generated by a copy of \(\mathfrak{g}\) sitting in degree 3 in \(M(\triv) = \mathbb{C}[\mathfrak{g}]\). There are two copies of \(\mathfrak{g}\) in \(M(\triv) = \mathbb{C}[\mathfrak{g}]\), obtained by taking the partial derivatives of the two invariant polynomials in degree 4. There is a copy of \(\mathfrak{g} \subset \mathbb{C}[\mathfrak{g}] = M(\triv)\) consisting of singular vectors, i.e. those annihilated by the elements of \(\mathfrak{g} \subset H_c\) acting in \(\mathbb{C}[\mathfrak{g}]\) via the Dunkl operators. This copy is spanned by partial derivatives of some polynomial \(Q\) of degree 4, invariant under \(W = W(B_4)\). Such a polynomial has the form \(a \cdot (\sum_{i=1}^3 x_i^2)^2 - \sum_{i=1}^3 x_i^4\). An explicit calculation with Dunkl operators shows that \(a = c_1\). So, generically \(L(\triv)\) is the local ring of the isolated critical point of \(Q\) at the origin.

However, sometimes the origin is not an isolated critical point, in which case the local ring is not finite dimensional, and hence cannot be isomorphic to \(L(\triv)\). This happens, for instance, if \(c_1 = 1/p\), \(p = 1, ..., 4\); in particular, for \(c_1 = 1/2\) (in which case \(c_2 = 0\), so we are essentially in the \(D_4\) case). In this case, we have additional polynomials in degree 3, killed by Dunkl operators. An explicit calculation shows that this additional space of singular vectors in \(\mathbb{C}[\mathfrak{g}] = M(\triv)\) is isomorphic, as a representation of \(W\), to \(\mathfrak{g} \otimes \varepsilon\) where
ε is the character such that ε(s_e_i−e_j) = 1 and ε(s_e_i) = −1. More specifically, a basis of this space is formed by the polynomials f_i = \prod_{j \neq i} x_j. These polynomials are definitely nonzero in N(triv), which shows that N(triv) ≠ L(triv). In fact, it is easy to see that we have an exact sequence

\[ 0 \to L(\mathfrak{h} \otimes \varepsilon) \to N(\text{triv}) \to L(\text{triv}) \to 0. \]

7 Connections to other topics

7.1 Relation to the affine flag variety.

Cherednik introduced an associative \( \mathbb{C} \)-algebra depending on two complex parameters \( q, t \in \mathbb{C}^* \), called the double affine Hecke algebra, see e.g. [Ch1]. The double affine Hecke algebra \( \mathcal{H}_{q,t} \) has generators: \( T_i, X_i^{±1}, Y_i^{±1}, i = 1, \ldots, \ell \), with certain defining relations, analogous to those in the affine Hecke algebra (the latter is a subalgebra in \( \mathcal{H}_{q,t} \) generated by the \( T_i \)'s and \( X_i^{±1} \)'s). Cherednik showed that the elements \( \{ T_w \cdot X_i^{a} \cdot Y_j^{b} \mid w \in W; i, j = 1, \ldots, \ell; a, b \in \mathbb{Z} \} \), form a Poincaré-Birkhoff-Witt type \( \mathbb{C} \)-basis in \( \mathcal{H}_{q,t} \).

The rational Cherednik algebra \( \mathcal{H}_c \) may be thought of, see [EG], as the following two-step degeneration of the double affine Hecke algebra:

\[ \mathcal{H}_{q,t} = \mathcal{H}_{q,t}^{\text{elliptic}} \rightsquigarrow \mathcal{H}_c^{\text{trigon}} \rightsquigarrow \mathcal{H}_c^{\text{rational}} = \mathcal{H}_c. \]

Otherwise put, the algebra \( \mathcal{H}_{q,t} \) can be regarded as a formal \( \mathbb{C}[[\varepsilon]] \)-deformation of \( \mathcal{H}_c \) upon the substitution: \( t = e^{\varepsilon^2 c}, X_i = e^{\varepsilon x_i}, Y_i = e^{\varepsilon y_i} \). In [Ch2, p.65], it is observed that this deformation is in effect trivial (as had been conjectured by the second author).

Before our paper was written, Cherednik announced [Ch2, p.64] a classification of finite dimensional representations of the double affine Hecke algebra \( \mathcal{H}_{q,t} \) of type A. As was pointed out in [Ch2], this allows one to classify finite dimensional representations of \( \mathcal{H}_c(S_n) \) (i.e. our Theorem 1.2), and obtain the dimension formula for them using the corresponding results for the double affine Hecke algebra.

Let us explain, for example, why the trigonometric Cherednik algebra \( \mathcal{H}_c^{\text{trigon}} \) is trivial as a formal deformation of the rational Cherednik algebra \( \mathcal{H}_c \) (the argument is due to Cherednik). Recall that the (formal) trigonometric Cherednik algebra \( \hat{\mathcal{H}}^{\text{trigon}}(\varepsilon) \), can be realized, via the Dunkl representation, as a complete topological \( \mathbb{C}[[\varepsilon]] \)-algebra (topologically generated by \( W, \mathbb{C}[\mathfrak{h}] \), and trigonometric Dunkl operators

\[ T_y^{\text{trigon}}(\varepsilon) = \partial_y - \sum_{\alpha \in R_+} c_\alpha \cdot \frac{\varepsilon \cdot (\alpha, y)}{1 - e^{-\varepsilon \alpha}} (1 - s_\alpha) + \frac{\varepsilon}{2} \sum_{\alpha \in R_+} c_\alpha \cdot (\alpha, y), \quad y \in \mathfrak{h} \]

(\( \varepsilon \) is a deformation parameter). Since the function \( \frac{1}{\varepsilon} - \frac{\varepsilon}{(1 - e^{-\varepsilon \alpha})} \) is regular at \( \varepsilon = 0 \), we find

\[ T_y^{\text{trigon}}(0) = \partial_y - \sum_{\alpha \in R_+} c_\alpha \cdot \frac{\alpha \cdot (\alpha, y)}{\alpha} (1 - s_\alpha) + \varepsilon \cdot F = T_y^{\text{rational}} + \varepsilon \cdot F, \]

where \( T_y^{\text{rational}} \) is the rational Dunkl operator and \( F \in \mathbb{C} W \ltimes \mathbb{C}[\mathfrak{h}][[\varepsilon]] \). It follows readily that there is a topological \( \mathbb{C}[[\varepsilon]] \)-algebra isomorphism: \( \mathcal{H}_c^{\text{trigon}}(\varepsilon) \simeq \mathbb{C}[[\varepsilon]] \hat{\otimes} \mathcal{H}_c \). We see
that, if for some $c$, the algebra $H_c$, has representation in a $d$-dimensional $\mathbb{C}$-vector space ($d < \infty$), then the $\mathbb{C}[[\epsilon]]$-algebra $H^{\text{trig}}(\epsilon) \simeq \mathbb{C}[[\epsilon]] \hat{\otimes} H_c$ has a $\mathbb{C}[[\epsilon]]$-linear representation in $\mathbb{C}[[\epsilon]]^d$. Furthermore, we observe that all the operators in $\mathbb{C}[[\epsilon]]^d$ corresponding to the generators $x_i$, $i = 1, \ldots, \ell$, of $H_c$ are necessarily nilpotent, i.e. the infinite series involved are actually polynomials in $\epsilon$. Hence, it makes sense to specialize the formal variable $\epsilon$ to the complex number: $\epsilon = 1$, to obtain a well-defined action in $\mathbb{C}^d$ of the original $\mathbb{C}$-algebra $H^{\text{trig}}_c$. This way, one proves

**Proposition 7.1 (Cherednik).** There is a natural equivalence of the categories of finite dimensional $H_c$- and $H^{\text{trig}}_c$-modules that preserves the $\mathbb{C}$-dimension of the modules.  

Motivated by the well-known geometric interpretation of the affine Hecke algebra (see [CG, ch. 7] for a review), E. Vasserot [Va] has produced a similar construction for the algebra $H_{\hat{\mathfrak{g}}_l}$. In more detail, let $R$ be the root system of a complex semisimple Lie algebra $\mathfrak{g}$, let $\mathfrak{g} = \mathfrak{g}(\hat{\mathfrak{g}})$ be the corresponding loop Lie algebra, and $G$ the corresponding Kac-Moody group. One has the Springer resolution $\tilde{\mathfrak{g}} \to \mathfrak{g}$ and the Steinberg variety $Z = \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$. All the (infinite dimensional) varieties $\mathfrak{g}, \tilde{\mathfrak{g}}$, and $Z$ acquire a natural $G$-action by conjugation, and a natural $C^*$-action by dilations, thus become $G \times C^*$-varieties. Vasserot [Va] succeeded in defining an equivariant $K$-group $K^{G \times C^*}(Z)$ with a convolution-type (non-commutative) algebra structure, and proved (roughly speaking) an algebra isomorphism $H \simeq K^{G \times C^*}(Z)$, where $H$ is the double affine Hecke algebra associated to the dual root system $R^\vee$.

It is very likely that the argument in [Va] can be used to prove a similar algebra isomorphism $H^{\text{trig}} \simeq H^{G \times C^*}(Z)$, where $H^{G \times C^*}(\cdot)$ stands for $G \times C^*$-equivariant Borel-Moore homology. We note at this point that no analogous interpretation of the rational Cherednik algebra $H^{\text{rational}}_c = H_c$ is available at present.

Using the isomorphism $H \simeq K^{G \times C^*}(Z)$, Vasserot obtained a complete classification of simple ‘bounded below’ $H$-modules in geometric terms. A similar classification of simple ‘bounded below’ $H^{\text{trig}}$-modules, based on the isomorphism $H^{\text{trig}} \simeq H^{G \times C^*}(Z)$ should be possible. Although classifications of that type do not allow, in general, to distinguish finite dimensional representations among all ‘bounded below’ representations, they do allow to construct some of them. In this manner, some simple finite dimensional $H$-modules have been constructed in [Va, §9.3]. It seems certain that their trigonometric analogues exist as well. By Proposition 7.1, finite dimensional $H^{\text{trig}}_c$-modules arising in that way may also be seen as $H_c$-modules. A comparison with [Va, §9.3] shows that, in type $A$, the geometric construction of loc. cit. produces in effect all simple finite dimensional $H_c$-modules. In other types, this is definitely not true. The complexity of the situation for types $B, D$, observed both in the present paper and in [Va], suggests that some kind of ‘$L$-packet phenomenon’ resulting from the failure of $G$-orbits to be simply-connected is involved.

Let $B$ denote the flag variety for $G$, the so-called affine flag manifold. Recall that simple $H$-modules have been realized in [Va] as subquotients of $H^*(B_\mathfrak{c}_c)$, the cohomology of certain fixed point sets $B_\mathfrak{c}_c$ in $B$. In the case of a nil-elliptic element $x \in \mathfrak{g}$ the corresponding set $B_\mathfrak{c}_c$ turns out to be a (finite dimensional) smooth projective variety, see [KL]; moreover, the total cohomology $H^*(B_\mathfrak{c}_c)$ turns out to be a simple $H$-module. A trigonometric analogue
of this should provide a realization of finite dimensional simple $H_{C}^{\text{trigon}}$-modules, hence of $H_{c}$-modules, as the total cohomology groups of appropriate smooth projective subvarieties in $B$. The Poincaré duality for such varieties would explain the Gorenstein property in Proposition 1.13.

Let $x \in g$ be a nil-elliptic element. We note that the computation of the Poincaré polynomials for varieties $B_{x}^{c}$ (carried out in [LS] in type A and in [So] in general) produces exactly the same answer as our formula in Theorem 1.6. In particular, for the Euler characteristic, the computation in [LS], [So] gives: $\chi(B_{x}^{c}) = r^{\lambda}$, where the integer $r$ is related to $x$ as explained e.g. in [Va, §8]. This agrees with our Proposition 1.7.

Replacing the affine flag manifold by a loop Grassmannian $Gr$, one constructs similarly the ‘spherical’ Steinberg variety $Z_{\text{spher}}^{s}$. One expects to have algebra isomorphisms $eH_{e} \cong K^{G \times C}(-Z_{\text{spher}}^{s})$, and $eH_{e}^{\text{trigon}} \cong H^{G \times C}(-Z_{\text{spher}}^{s})$. This should give rise to a geometric construction of simple ‘bounded below’ $eH_{e}$-modules and $eH_{e}^{\text{trigon}}$-modules, respectively. Our expectations are supported by the computations of the Euler characteristic of certain fixed point varieties $Gr_{x}^{s}$ carried out in [So]. The result of Sommers reads: $\chi(Gr_{x}^{s}) = \prod_{i=1}^{d} \frac{d_{i}+r-1}{d_{i}}$, which is according to our Theorem 1.10 exactly the dimension of a simple finite dimensional $eH_{e}$-module.

### 7.2 Relation to the geometry of Hilbert scheme

Let $\text{Hilb}^{n}(\mathbb{C}^{2})$ be the Hilbert scheme of $n$-points in $\mathbb{C}^{2}$, that is, a scheme parametrizing all codimension $n$ ideals in the polynomial ring $\mathbb{C}[x, y]$ in two variables. It is known that $\text{Hilb}^{n}(\mathbb{C}^{2})$ is a smooth connected algebraic variety of dimension $2n$. There is a natural (ample) determinant line bundle $L$ on $\text{Hilb}^{n}(\mathbb{C}^{2})$, the $n$-th wedge power of the tautological bundle of rank $n$. The natural action on the plane $\mathbb{C}^{2}$ of the torus $T = \mathbb{C}^{*} \times \mathbb{C}^{*}$, by diagonal matrices, lifts to a $T$-action on $\text{Hilb}^{n}(\mathbb{C}^{2})$, making $L$ a $T$-equivariant line bundle.

There is a natural Hilbert-Chow morphism $\pi : \text{Hilb}^{n}(\mathbb{C}^{2}) \to \text{Sym}^{n}(\mathbb{C}^{2}) = \mathbb{C}^{2n}/S_{n}$. This morphism is proper, and one puts $\text{Hilb}_{0}^{n}(\mathbb{C}^{2}) := \pi^{-1}(0)$, the zero-fiber of $\pi$. The scheme $\text{Hilb}_{0}^{n}(\mathbb{C}^{2})$ parametrizes all codimension $n$ ideals in $\mathbb{C}[x, y]$ set-theoretically concentrated at the origin $0 \in \mathbb{C}^{2}$; alternatively, $\text{Hilb}_{0}^{n}(\mathbb{C}^{2})$ parametrizes codimension $n$ ideals in the formal power series ring $\mathbb{C}[[x, y]]$. It is known that $\text{Hilb}_{0}^{n}(\mathbb{C}^{2})$ is a reduced and irreducible $T$-stable subscheme in $\text{Hilb}^{n}(\mathbb{C}^{2})$ of dimension $(n - 1)$.

Giving an algebraic $T$-action on a vector space $V$ is equivalent to giving a bigrading $V = \oplus_{i,j \in \mathbb{Z}} V_{ij}$. If all the bigraded components are finite-dimensional, we may introduce the formal character $\chi_{T}(V) := \sum_{i,j \in \mathbb{Z}} q^{i} \cdot t^{j} \cdot \dim V_{ij} \in \mathbb{C}[[q, q^{-1}, t, t^{-1}]]$.

The results of M. Haiman [Ha2],[Ha3] imply the following 2-variable character formula

$$
\chi_{T}(H^{0}(\text{Hilb}^{n}_{0}(\mathbb{C}^{2}), L^{\oplus k})) = C_{n}^{(k)}(q, t) , \quad \forall k = 0, 1, 2, \ldots,
$$

where the $C_{n}^{(k)}(q, t)$ are $(q, t)$-analogues of Catalan numbers introduced in [Ha2, (1.10)]. In particular, the dimension of the cohomology space is given by the usual Catalan number:

$$
\dim H^{0}(\text{Hilb}^{n}_{0}(\mathbb{C}^{2}), L^{\oplus k}) = \frac{1}{n \cdot k + 1} \binom{n(k + 1)}{n} , \quad \forall k = 0, 1, 2, \ldots.
$$

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Now, given $c \in \mathbb{C}$, write $L_c(\text{triv}) = L(\text{triv})$ (to keep track of the parameter ‘$c$’) for the simple $H_e(S_n)$-module corresponding to the trivial $S_n$-representation. We see that the RHS of formula (7.3) equals, by Theorem [4.7], the dimension of the simple finite dimensional $eH_e$-module:

$$\dim H^0(\operatorname{Hilb}_n^0(\mathbb{C}^2), \mathcal{L}^{\otimes k}) = \dim \left( e \cdot L_c(\text{triv}) \right) \quad \text{for} \quad c = \frac{1}{n} + k, \ k = 0, 1, 2, \ldots . \quad (7.4)$$

We will now provide a heuristic ‘explanation’ of equality (7.4).

Let $h = \mathbb{C}^n$. Write $\mathbb{C}[h \oplus h]^W$ for the space of diagonal $W$-invariants. Let $m = \mathbb{C}[h \oplus h]^W$ denote the augmentation ideal in $\mathbb{C}[h \oplus h]^W$ formed by all diagonal $W$-invariants without constant term, and $C_o := \mathbb{C}[h \oplus h]^W/m$, the corresponding 1-dimensional module.

The Hilbert-Chow morphism gives a canonical isomorphism $H^0(\operatorname{Hilb}_n^0(\mathbb{C}^2), O_{\operatorname{Hilb}_n^0(\mathbb{C}^2)}) = \mathbb{C}[h \oplus h]^W$. Thus, the space $H^0(\operatorname{Hilb}_n^0(\mathbb{C}^2), \mathcal{L}^{\otimes k})$ acquires a natural $\mathbb{C}[h \oplus h]^W$-module structure. Furthermore, thanks to [14, (25)], we have:

$$H^0(\operatorname{Hilb}_n^0(\mathbb{C}^2), \mathcal{L}^{\otimes k}) = H^0(\operatorname{Hilb}_n^0(\mathbb{C}^2), \mathcal{L}^{\otimes k})/m \cdot H^0(\operatorname{Hilb}_n^0(\mathbb{C}^2), \mathcal{L}^{\otimes k}).$$

Therefore, the space on the LHS of (7.4) may be rewritten as follows

$$H^0(\operatorname{Hilb}_n^0(\mathbb{C}^2), \mathcal{L}^{\otimes k}) \simeq H^0\left( \operatorname{Hilb}_n^0(\mathbb{C}^2), \mathcal{L} \otimes_C \prod_{i=1}^{n-1} O_{\operatorname{Hilb}_n^0(\mathbb{C}^2)} \mathcal{L} \right) \otimes_{\mathbb{C}[h \oplus h]^W} C_o. \quad (7.5)$$

We now turn to the RHS of (7.4). Given $c = \frac{1}{n} + k$, introduce the shorthand notation $H^{(k)} = H^{h/k}$. By the definition of Shift functor; see Lemma [4.7] and Proposition [4.8(ii)], for $c = \frac{1}{n} + k$ where $k = 0, 1, \ldots , n$, we have

$$L_c(\text{triv}) = S_{k-1} h^{-1} \circ \cdots \circ S_{1+k-1} h^{-1}(\mathbb{C}) = H^{(k)} e - \bigotimes_{eH^{(k-1)} e} eH^{(k-1)} e - \bigotimes_{eH^{(k-2)} e} eH^{(k-2)} e \cdots \bigotimes_{eH^{(1)} e} eH^{(1)} e - \bigotimes_{eH^{(0)} e} eC. \quad (7.6)$$

where we have repeatedly exploited the algebra isomorphism: $e_- H^{(k+1)} e_- \simeq eH^{(k)} e$, see Proposition [4.6]. Similarly, one has:

$$e \cdot L_c(\text{triv}) = \left( eH^{(k)} e_- \bigotimes_{eH^{(k-1)} e} eH^{(k-1)} e_- \bigotimes_{eH^{(k-2)} e} eH^{(k-2)} e_- \cdots \bigotimes_{eH^{(1)} e} eH^{(1)} e_- \right) \bigotimes_{eH^{(0)} e} eC. \quad (7.7)$$

Next, recall that the algebra $H_e$ has a canonical increasing filtration such that the generators $x \in h^*, y \in h$ are assigned filtration degree 1, and elements of the Weyl group $W$ are assigned filtration degree 0. The filtration on $H_e$ induces natural increasing filtrations on other objects. For the corresponding associated graded objects one has:

$$\operatorname{gr} H_e \simeq \mathbb{C}[h \oplus h]^W, \ \operatorname{gr}(eH_e e) \simeq \mathbb{C}[h \oplus h]^W \simeq H^0(\operatorname{Hilb}_n^0(\mathbb{C}^2), O_{\operatorname{Hilb}_n^0(\mathbb{C}^2)}) \text{ and}$$

$$\operatorname{gr}(eH_e e_-) \simeq \mathbb{C}[h \oplus h]^\text{sign} \simeq H^0(\operatorname{Hilb}_n^0(\mathbb{C}^2), \mathcal{L}). \quad (7.8)$$

The last two isomorphisms in (7.8) suggest to think of (7.7) as a ‘quantum’ analogue of (7.3).
To make this analogy more precise, observe first that the canonical filtration on \( H_c \) is stable under the adjoint action \( \text{ad}\ h : u \mapsto [h, u] = h \cdot u - u \cdot h \), hence induces an \( \text{ad}\ h \)-action on the associated graded algebra. It is immediate to check that transporting this action under the first isomorphism in (7.8) we get: \( \text{ad}\ h(x) = x, \forall x \in h^* \), and \( \text{ad}\ h(y) = -y, \forall y \in h \). Thus, \( \text{gr} H_c \) may be viewed as a bi-graded algebra \( \text{gr} H_c = \oplus_{p,q \in \mathbb{Z}} \mathbb{H}^{p,q} \), where the space \( \mathbb{H}^{p,q} \) is defined to have total degree \( p + q \) and such that \( \text{ad}\ h |_{\mathbb{H}^{p,q}} = (p-q) \cdot \text{Id}_{\mathbb{H}^{p,q}} \).

Then, all the isomorphisms in (7.8) become bigraded isomorphisms, where the bidegrees of the tensor product in the RHS of (7.6). We introduce an increasing filtration on \( \text{gr} H_c \) to be the induced filtration

\[
F_\bullet(L_c(\text{triv})) := \sum_{\{a_1, \ldots, a_k \geq 0 \mid a_1 + \ldots + a_k \leq a\}} F_{a_k} (H^{(k)}e_{-}) \otimes_c F_{a_{k-1}} (eH^{(k-1)}e_{-}) \otimes_c \ldots \otimes_c F_{a_1} (eH^{(1)}e_{-}) \otimes_c \mathbb{C}
\]

where, for any \( a \in \mathbb{Z} \), the corresponding \( a \)-th term of the filtration on the RHS above is defined as the image of the \( \mathbb{C} \)-vector space

\[
\mathbb{H}^{(k)}e_{-} \otimes_c \mathbb{H}^{(k-1)}e_{-} \otimes_c \ldots \otimes_c \mathbb{H}^{(1)}e_{-} \otimes_c \mathbb{C}
\]

under the canonical projection from a tensor product over \( \mathbb{C} \) to the corresponding tensor product over the algebras \( e\mathbb{H}^{(l)}e \).

The filtration on \( L_c(\text{triv}) \) thus defined is clearly \( h \)-stable. Therefore, we obtain a well-defined \( h \)-action on the associated graded space. As we have done above in the case of the algebra \( H_c \) itself, this allows to view \( \text{gr} L_c(\text{triv}) \) as a bi-graded space

\[
\text{gr} L_c(\text{triv}) = \oplus_{p,q \in \mathbb{Z}} L_{p,q} \quad \text{where} \quad h |_{L_{p,q}} = (p-q) \cdot \text{Id}_{L_{p,q}},
\]

and such that the original grading corresponds to the total grading, that is, we have:

\[
\text{gr}^a L(\text{triv}) = \oplus_{\{(p,q) \mid p+q = a\}} L_{p,q}.
\]

Similar considerations and definitions apply to the \( e\mathbb{H}_c e \)-module \( e \cdot L(\text{triv}) \). In particular, we define an increasing filtration on \( e \cdot L(\text{triv}) \) using the isomorphism of (7.7), and view \( \text{gr}(e \cdot L(\text{triv})) \) as a bi-graded space, that is as a \( T \)-module. Thus, we may define the bigraded character of \( \text{gr}(e \cdot L(\text{triv})) \), a two-variable formal series \( \chi_{e \cdot L(\text{triv})}(q,t) \in \mathbb{C}[q,q^{-1},t,t^{-1}] \).

The following is a considerable refinement of equation (7.4).

**Conjecture 7.10.** Let \( c = \frac{1}{n} + k \). Then there is a canonical \( T \)-module isomorphism

\[
\text{gr}(e \cdot L_c(\text{triv})) \simeq H^0(\text{Hilb}_0^n(\mathbb{C}^2), \mathcal{L}^{\otimes k}) , \quad \text{for any} \quad k = 0,1,\ldots
\]

In particular, \( \chi_{e \cdot L(\text{triv})}(q,t) = C_n^{(k)}(q,t) \), is the \((q,t)\) Catalan number.

\[\text{We remark that the lowest weight line } \text{triv} \subset L_c(\text{triv}) \text{ does not usually belong to the first (the smallest) term of the filtration on } L_c(\text{triv}).\]
To formulate an analogue of Conjecture 7.10 for the $H_c$-module $L_c(triv)$, we need to recall (see [Ha3]) that there is an ‘unusual’ tautological rank $n!$ vector bundle $\mathcal{R}$ on $\text{Hilb}^n(C^2)$ whose fibers afford the regular representation of the group $W = S_n$.

**Conjecture 7.11.** Let $c = \frac{1}{n} + k + 1$. Then there is a canonical $W$-equivariant $\mathbb{T}$-module isomorphism

$$\text{gr}(L_c(triv)) \simeq \text{sign} \otimes H^0(\text{Hilb}^n(C^2), \mathcal{R} \otimes \mathcal{L}^{\otimes k}), \quad \text{for any } k = 0, 1, \ldots .$$

Case $k = 0$ of Conjecture 7.11 is true; it amounts (modulo [Ha3]) to the main result of Gordon [Go] on diagonal harmonics. Conjecture 7.10 is easy for $k = 0$; Case $k = 1$ of Conjecture 7.10 follows from Conjecture 7.11 for $k = 0$.

**Remark.** If true, Conjecture 7.11 would provide a formula, see [Ha3, (106)], for the bigraded multiplicities $[\text{gr}(L_c(triv)) : \tau]$, $\tau \in \text{irrep}(S_n)$, in terms involving among other things the $(q,t)$-Kostka polynomials, whose positivity has been conjectured by Macdonald and proved by Haiman.

**Remark.** The geometric structures considered in §6.1 and §6.2 should be related to each other by a kind of ‘Langlands duality’ for rational Cherednik algebras, similar somewhat to the existing Langlands duality for affine Hecke algebras, cf. e.g. [CG, Introduction]. These matters are not understood at the moment.

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*Haiman denotes this vector bundle by ‘$P$’, in honor of Procesi.*

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