EXISTENCE-UNIQUENESS FOR NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS WITH DRIFT IN $\mathbb{R}^d$

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Abstract. In this article we consider a class of nonlinear integro-differential equations of the form

$$\inf_{\tau \in \mathcal{T}} \left\{ \int_{\mathbb{R}^d} \delta(u,x,y) \frac{k_r(x,y)}{|y|^{d+2s}} dy + b_r(x) \cdot \nabla u(x) + g_r(x) \right\} - \lambda^* = 0 \quad \text{in } \mathbb{R}^d,$$

where $0 < \lambda(2-2s) \leq k_r \leq \Lambda(2-2s), s \in \left(\frac{1}{2}, 1\right)$. The above equation appears in the study of ergodic control problems in $\mathbb{R}^d$ when the controlled dynamics is governed by pure-jump Lévy processes characterized by the kernels $k_r |y|^{-d-2s}$ and the drift $b_r$. Under a Foster-Lyapunov condition, we establish the existence of a unique solution pair $(u, \lambda^*)$ satisfying the above equation, provided we set $u(0) = 0$. Results are then extended to cover the HJB equations of mixed local-nonlocal type and this significantly improves the results in [4].

1. Introduction

Our chief goal in this article is to find a pair $(u, \lambda^*)$ that satisfies

$$\inf_{\tau \in \mathcal{T}} \left\{ \int_{\mathbb{R}^d} \delta(u,x,y) \frac{k_r(x,y)}{|y|^{d+2s}} dy + b_r(x) \cdot \nabla u(x) + g_r(x) \right\} - \lambda^* = 0 \quad \text{in } \mathbb{R}^d, \quad (1.1)$$

where $\delta(u,x,y) := u(x+y) + u(x-y) - 2u(x)$ and $\mathcal{T}$ is an indexing set. We impose the following assumptions on the kernel: Let $x \mapsto k_r(x,y)$ be continuous uniformly in $(y, \tau)$ and satisfies

$$k_r(x,y) = k_r(x,-y), \quad (2-2s)\lambda \leq k_r(x,y) \leq (2-2s)\Lambda \quad \forall x, y \in \mathbb{R}^d,$$

where $s \in \left(\frac{1}{2}, 1\right)$ and $0 < \lambda \leq \Lambda$. Let us introduce the following notations.

$$I_r[u](x) = \int_{\mathbb{R}^d} \delta(u,x,y) \frac{k_r(x,y)}{|y|^{d+2s}} dy; \quad \mathcal{L}_r[u](x) = I_r[u](x) + b_r(x) \cdot \nabla u(x). \quad (1.2)$$

Also, denote by $\omega_s(y) = \frac{1}{1+|y|^{d+2s}}$.

Remark 1.1. The symmetry property of the kernel $k_r$ is not used in this article, but we still make this assumption due to the following reason. In general, the nonlocal operator of Lévy type does not have a symmetrized form (cf. [1] Chapter 3) and this symmetrization (that is, the form involving $\delta(u,x,y)$) is possible when kernel $k_r$ is symmetric in the second variable. Therefore, keeping the assumption of symmetry makes the model physically relevant.

One of the main motivations to study (1.1) comes from the stochastic ergodic control problems where the random noise in the controlled dynamics corresponds to some $2s$-stable process. More precisely, suppose that the action set (or control set) $\mathcal{T}$ is a metric space and $\mathcal{U}$ denotes the collection of all stationary Markov controls, that is, collection of all Borel measurable functions $v: \mathbb{R}^d \to \mathcal{T}$.

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This class of functions plays a central role in the study of optimal control problems. Let us also assume that the martingale problem corresponding to the operator $\mathcal{L}^v$, defined by (see (1.2))

$$\mathcal{L}^v[f](x) = \mathcal{L}_v(x)[f](x),$$

is well-posed. In particular, for every $v \in \mathcal{U}$ there exists a family of probability measures $\{P_x \}_{x \in \mathbb{R}^d}$ on $\mathcal{D}([0, \infty), \mathbb{R}^d)$, the space of Cadlag functions on $[0, \infty)$ taking values in $\mathbb{R}^d$, such that $(P^{x}_v, X^v)$, where $X^v$ denotes the canonical coordinate process, solves the martingale problem. One albeit needs to impose certain regularity hypothesis on the kernels $k_\tau$ to guarantee well-posedness of the martingale problem, see for instance [17, 23]. Let $\mathbb{R}^d \times \mathcal{T} \ni (x, \tau) \rightarrow g_\tau(x)$ denotes the running cost and the goal is to minimize the ergodic cost criterion

$$J[v] := \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}^x_T \left[ \int_0^T g_v(X^v_t) \right],$$

over $\mathcal{U}$. We denote the optimal value by $\lambda^*$. It is then expected that the optimal value $\lambda^*$ would satisfy (1.1) (cf. [1, 3, 12, 21, 19]) and the measurable selectors of (1.1) would be the optimal controls in $\mathcal{U}$. Though the analogous problem for the local case (that is, $s = 1$) has been investigated extensively (see [3] and references therein), study of equation (1.1) remained open. Recently, ergodic control problem in $\mathbb{R}^d$ with dispersal type nonlocal kernel is considered by Brändle & Chasseigne in [13] whereas Barles et. al. [7] study the ergodic control problem for mixed integro-differential operators in periodic settings. In this article, we establish the existence and uniqueness of solution to (1.1) under a Foster-Lyapunov type condition.

We say a function $f : \mathbb{R}^d \to \mathbb{R}$ is inf-compact (or coercive) if for any $\kappa \in \mathbb{R}$ either $\{f \leq \kappa\}$ is empty or compact.

**Assumption 1.1.** We make following assumptions on the coefficients.

(A1) There exists $V \in C^2(\mathbb{R}^d)$, $V \geq 0$ and a function $0 \leq h \in C(\mathbb{R}^d)$, $V$ and $h$ are inf-compact, such that

$$\sup_{\tau \in \mathcal{T}} \mathcal{L}_\tau[V](x) \leq k_0 - h(x) \quad x \in \mathbb{R}^d, \quad (1.3)$$

for some $k_0 > 0$. This is called Foster-Lyapunov stability condition. The above condition also implies that $V \in L^1(\omega_s)$.

(A2) $\sup_{\tau \in \mathcal{T}} |g_\tau(x)| \leq h(x)$ and $\sup_{\tau \in \mathcal{T}} |g_\tau| \in \mathcal{O}(h)$, that is,

$$\lim_{|x| \to \infty} \frac{1}{1 + h(x)} \sup_{\tau \in \mathcal{T}} |g_\tau(x)| = 0.$$

(A3) For some $\mu \geq 0$ we have $V^{1+\mu} \in L^1(\omega_s)$ and

$$\sup_{\tau \in \mathcal{T}} \left[ \frac{|b_\tau|}{(1 + V(x))^{2s-1}\mu} + \frac{|g_\tau|}{(1 + V^{1+2\mu})} \right] \leq C, \quad (1.4)$$

$$\sup_{x,y} \frac{V(x + y)}{(1 + V(x))(1 + V(y))} + \limsup_{|x| \to \infty} \frac{1}{1 + V(x)} \sup_{|y-x| \leq 1} V(y) \leq C. \quad (1.5)$$

(A4) The map $x \rightarrow k_\tau(x, y)$ is uniformly continuous, uniformly in $\tau$ and $y$, that is,

$$|k_\tau(x_1, y) - k_\tau(x_2, y)| \leq g(|x_1 - x_2|) \quad \forall x_1, x_2 \in \mathbb{R}^d, \tau \in \mathcal{T}, y \in \mathbb{R}^d,$$

for some modulus of continuity $g$.

Note that (1.4) allows $b_\tau, g_\tau$ to be unbounded. Since $V$ is inf-compact, (1.4) holds for bounded $b_\tau, g_\tau$. This condition will be useful to find a growth-bound on the Hölder norm of solution $u$ in the unit ball $B_1(x)$. Condition (1.5) requires $V$ to have polynomial growth. For $C^{2s+\kappa}$ regularity of solutions we shall often impose the following condition on the coefficients.
(A5) For every compact set $K \subset \mathbb{R}^d$, there exists $\bar{\alpha} \in (0, 1)$ so that
$$\sup_{\tau \in \mathcal{T}} |b_\tau(x) - b_\tau(x')| + \sup_{\tau \in \mathcal{T}} |g_\tau(x) - g_\tau(x')| + \sup_{\tau \in \mathcal{T}} \sup_{y \in \mathbb{R}^d} |k_\tau(x, y) - k_\tau(x', y)| \leq C_K |x - x'|^{\bar{\alpha}}.$$

It can be easily seen that Assumption 1.1 holds for 2s-fractional Ornstein-Uhlenbeck type operators. Such operators are used by Fujita, Ishii & Loreti [20] to study large time behaviour of solutions to certain (local) Hamilton-Jacobi equation. Recently, Chasseigne, Ley & Nguyen [10] consider Ornstein-Uhlenbeck type drift with tempered 2s-stable nonlocal nonlocal version remained unsettled, mainly due to the unavailability of certain regularity estimates and appropriate Harnack type inequality for general nonlinear integro-differential operators. To understand the difficulty in proving Theorem 1.1, we recall the three main steps of the proof:

(a) Find solution $w_\alpha$ to the $\alpha$-discounted problem for $\alpha \in (0, 1)$;
(b) Establish the convergence of $\bar{w}_\alpha(x) := w_\alpha(x) - w_\alpha(0)$ to solution $u$ of (1.6);
(c) Show that $(u, \lambda^*)$ is unique.

For (a), we consider a more general class of discounted problem given by
$$\mathcal{L}w(x) = \inf_{\tau \in \mathcal{T}} (\mathcal{L}_\tau w + c_\tau(x)w(x) + g_\tau(x)) = 0 \quad \text{in} \; \mathbb{R}^d.$$ Fixing $c_\tau = -\alpha$ we obtain $\alpha$-discounted problem. It is possible to weaken Assumption 1.1 substantially to solve the above equation (see Assumption 2.1 in Section 2). In particular, we prove the following result.
Theorem 1.2. Suppose that $\{B1\} - \{B4\}$ hold. Then there exists $w \in C(\mathbb{R}^d) \cap \sigma(V)$, $V$ given by (2.2), satisfying
\[
\inf_{\tau \in T} \left( \mathcal{L}_\tau w + c_\tau w + g_\tau \right) = 0 \quad \text{in } \mathbb{R}^d.
\] (1.7)
In addition, if $\{B5\}$ holds, then $w \in C^{2s+}_{\text{loc}}(\mathbb{R}^d)$ and it is the unique solution in the class $\sigma(V)$.

If compared with the existing literature, the existence, uniqueness and the regularity results in Theorem 1.2 appear to be new. For instance, most of the existence results in the unbounded domains consider periodic setting, see Barles et. al. [4], Ciomaga, Ghilli & Topp [13]. Under the assumption that the coefficients are Lipschitz with sublinear growth, Jakobsen & Karlsen [22] establish comparison principle for bounded viscosity solutions to mixed Lévy-Itô type Isaacs equations. Biswas, Jakobsen & Karlsen [11] study the existence-uniqueness of viscosity solutions for parabolic integro-differential equations of Lévy-Itô type under the assumption that the Lévy kernel is having an exponentially decaying tail and the coefficients of (1.7) are Lipschitz. The existence-uniqueness result in Theorem 1.2 is obtained for a purely nonlocal equation that is not of Lévy-Itô type. Similar existence-uniqueness result for mixed integro-differential operator is established in Section 4. In a recent work, Meglioli & Punzo [24] consider linear equation involving $2s$-fractional Laplacian and $C^1$ drift and study uniqueness of the solutions. The methodology in [24] is of variational nature and cannot be adapted for operators of the form (1.7) (see Remark 2.2 for a detailed comparison). The regularity result plays a crucial role in obtaining the uniqueness. Two main ingredients to find $C^{2s+}$ regularity are $C^{1,\gamma}$ regularity of $w$ and the growth bounded on the Hölder norm of $w$ in $B_1(x)$ (see Lemma 2.5). We establish the $C^{1,\gamma}$ regularity for Isaacs type equations and without requiring the coefficients to be Hölder continuous (see Theorem 5.1 below for more detail). There is a large body of works dealing with the Hölder regularity of Isaacs type integro-differential equations. Many of these works are based on Ishii-Lions method and therefore, they require the coefficients to be Hölder continuous and $c_\tau$ to be negative, see for instance, [8, Corollary 4.1], [6, 16, 18, Theorem 3.1]. We do not require any such condition since our method is based on scaling argument and Liouville theorem used by Serra in [30, 31].

Since $c_\tau$ is negative (by $\{B1\}$) we can solve (1.7) in bounded domains, for instance, ball $B_n(0)$, with Dirichlet exterior condition. Then, applying the local Hölder estimate from [29] one can pass to the limit, as $n \to \infty$, to obtain a solution of (1.7), provided the solutions in the bounded domains are dominated by a fixed barrier function. This is required to pass the limit inside the nonlocal operator. We show that $V$ (see (2.4)) can be used as a barrier function. To establish uniqueness, we first show that $w \in C^{2s+}_{\text{loc}}(\mathbb{R}^d)$ and therefore, it is a classical solution to (1.7). Recall that $C^{2s+}$ estimate of $w$ requires global Hölder regularity of $w$ and local Hölder regularity of $\nabla w$ (cf. [30]). To attain this goal we first establish $C^{1,\gamma}$ estimate for $w$. This is done in Section 5. Once we have $C^{2s+}$ regularity (see Lemma 2.5) we can couple two solutions and use the barrier function $V$ to establish uniqueness.

Coming back to Theorem 1.1 we apply Theorem 1.2 to obtain solution $w_\alpha$ for the $\alpha$-discounted problem and define the normalized function $\bar{w}_\alpha(x) = w_\alpha(x) - w_\alpha(0)$. Note that
\[
\inf_{\tau \in T} \left( \mathcal{L}_\tau \bar{w}_\alpha + g_\tau \right) - \alpha \bar{w}_\alpha - \alpha w_\alpha(0) = 0 \quad \text{in } \mathbb{R}^d, \quad \bar{w}_\alpha(0) = 0.
\]
Thus to complete step (b) we only need to find a convergent subsequence of $\{\bar{w}_\alpha\}$, as $\alpha \to 0$, and pass the limit in the above equation. The equicontinuity of the family $\{\bar{w}_\alpha\}$ is generally obtained by employing a generalized Harnack’s type estimate (cf. [4, Theorem 3.3], [3, Lemma 3.6.3]). But, to the best of our knowledge, Harnack type estimate is not available for nonlinear integro-differential operators with gradient and zeroth order term. Therefore, we innovate a different method. We again use the Lyapunov function $V$ in (1.3) and show that $|\bar{w}_\alpha| \leq \kappa + V$ in $\mathbb{R}^d$ for some suitable constant $\kappa$ (see Lemma 3.1). Now applying the regularity result of [29] we can establish the equicontinuity of $\{\bar{w}_\alpha\}$. For the step (c), we again show that $u \in C^{2s+}_{\text{loc}}(\mathbb{R}^d)$ and then use the fact $u \in \sigma(V)$. It is worth
pointing out that equation (1.6) is not strictly monotone which makes the uniqueness tricky. There are only few works dealing with the uniqueness of non-monotone operators in bounded domains, see Caffarelli & Silvestre [1], Mou & Święch [27]. The $C^{2s+}$ regularity property is crucially used in the proof of uniqueness. Note that once we have $C^{1,\gamma}$ regularity, $C^{2s+}$ estimate follows from Serra [30] since (1.6) is of concave type. Similar approach does not work for Isaacs type equations.

Interestingly, the above approach can be generalized to study HJB equations for a larger family of integro-differential operators. In particular, we consider the operator

$$\mathcal{I}u(x) = \inf_{\tau \in T} \left[ \text{tr}(a_\tau(x)D^2u) + \tilde{I}_\tau[u] + b_\tau(x) \cdot \nabla u + g_\tau \right],$$

where $\tilde{I}_\tau$ is a general Lévy type nonlocal operator (see Section 4). We show that the above approach extends for these class of operators and we obtain the following result in Section 4.

**Theorem 1.3.** Suppose that Assumptions 1.1 and 2 hold. Then there exists a unique pair $(u, \lambda^*) \in \sigma(V) \times \mathbb{R}$ satisfying

$$\mathcal{I}u(x) - \lambda^* = 0 \quad \text{in } \mathbb{R}^d, \quad u(0) = 0. \quad (1.8)$$

Theorem 1.3 should be compared with Arapostathis et. al. [4] where similar problem has been considered for nonlocal kernels having compact support and finite measure. We conclude the introduction with an example satisfying Assumption 1.1. This example is inspired from [2].

**Example 1.1.** Consider a function $0 \leq V \in C^2(\mathbb{R}^d)$ satisfying $V(x) = |x|^{\gamma}$ for $|x| \geq 1$, and $\gamma \in (0, 2s)$. We observe that for any $|x| \leq 1$, $|V(x)| \leq C$ and therefore $V(x) \leq C + |x|^{\gamma}$ for all $x \in \mathbb{R}^d$. All the inequalities below are true up to a constant.

**Case-I** Let $x \in B_R$ and $R > 2$, then

$$\left| \int_{B_{R+1}} \delta(V, x, y) \frac{k_\tau(x, y)}{|y|^{d+2s}} \, dy \right| \leq \|D^2V\|_{L^\infty(B_{2R+1})} \int_{B_{R+1}} \frac{|y|^2}{|y|^{d+2s}} \, dy = \|D^2V\|_{L^\infty(B_{2R+1})} (R + 1)^{2-2s}$$

Now observe that for $|y| \geq R + 1$ and $|x| \leq R$, we have $|x \pm y| \geq 1$. Therefore

$$\left| \int_{B_{R+1}} \delta(V, x, y) \frac{k_\tau(x, y)}{|y|^{d+2s}} \, dy \right| \leq \int_{B_{R+1}} \frac{\gamma |x + y|^\gamma + |x - y|^\gamma - 2V(x)}{|y|^{d+2s}} \, dy$$

$$\lesssim \int_{B_{R+1}} \frac{2|x|^\gamma + 2(|x|^\gamma + |V(x)|)}{|y|^{d+2s}} \, dy$$

$$\lesssim (R + 1)^{\gamma - 2s} + (R^\gamma + \|V\|_{L^\infty(B_R)})^2 R^{-2s}.$$  

**Case-2** Let $x \in B_R$. Then

$$\left| \int_{B_{R+1}} \delta(V, x, y) \frac{k_\tau(x, y)}{|y|^{d+2s}} \, dy \right| \leq \|D^2V\|_{L^\infty(\mathbb{R}^d)} C_R. \quad (1.9)$$

Now to compute the integration on $B_{\frac{R}{2}}^c$, we define

$$J_\tau := \int_{\frac{R}{2} \leq |y| \leq \frac{|x|}{2}} \delta(V, x, y) \frac{k_\tau(x, y)}{|y|^{d+2s}} \, dy.$$

Note that $|y| \leq \frac{|x|}{2} \iff |x + y| \geq |x| - \frac{|x|}{2} \geq \frac{|x|}{2} \geq \frac{R}{2} \geq 1$. Therefore,

$$J_\tau = \int_{\frac{R}{2} \leq |y| \leq \frac{|x|}{2}} (|x + y|^\gamma + |x - y|^\gamma - 2|x|^\gamma) \frac{k_\tau(x, y)}{|y|^{d+2s}} \, dy$$

$$= |x|^{\gamma - 2s} \int_{\frac{R}{2|x|} \leq |z| \leq \frac{1}{2}} \left( \frac{x}{|x|} + z \right)^\gamma + \left( \frac{x}{|x|} - z \right)^\gamma - 2 \frac{k_\tau(x, |z|)}{|z|^{d+2s}} \, dz.$$
Now observe that for any $|z| < \frac{1}{2}$ we have $\frac{1}{|z|} \geq 1 - |z| \geq \frac{1}{2}$. Therefore we can apply Taylor’s formula to conclude

$$J_{\tau} \leq \tilde{C}|x|^{\gamma-2s} \int_{|z| < \frac{1}{2}} \frac{|z|^2}{|z|^{d+2s}} dz \leq C|x|^{\gamma-2s}. \quad (1.10)$$

Let us denote by $\tilde{\delta}(x, y) = |x + y|^{\gamma} + |x - y|^{\gamma} - 2|x|^{\gamma}$ and observe that

$$\int_{|x| \leq |y|} \delta(V, x, y) \frac{k_{\tau}(x, y)}{|y|^{d+2s}} dy = \int_{|x| \leq |y|} \left( \delta(V, x, y) - \tilde{\delta}(x, y) \right) \frac{k_{\tau}(x, y)}{|y|^{d+2s}} dy + \int_{|x| \leq |y|} \tilde{\delta}(x, y) \frac{k_{\tau}(x, y)}{|y|^{d+2s}} dy. \quad (1.11)$$

Furthermore, as for $|x| < 2|y|$ we have $||x + y|^{\gamma} - |x|^{\gamma}| < 9|y|^{\gamma}$, it can be easily seen that

$$\left| \int_{|x| \leq |y|} \tilde{\delta}(x, y) K_{\tau}(x, y) dy \right| \leq C \int_{|x| \leq |y|} \frac{|y|^\gamma}{|y|^{d+2s}} dy \leq C|x|^{\gamma-2s}. \quad (1.10)$$

Now to calculate the first integral on the rhs of (1.11), we observe that if both $|x + y|$ and $|x - y|$ are bigger than 1, then $\delta(V, x, y) - \tilde{\delta}(x, y) = 0$. Again, if $|x + y| \leq 1$ (or $|x - y| \leq 1$) then

$$\frac{|x|}{2} \leq |y| \leq |x| + |x + y| \leq |x| + 1 \leq \frac{3|x|}{2},$$

and hence,

$$\int_{\frac{|x|}{2} \leq |y|} \left| \delta(V, x, y) - \tilde{\delta}(x, y) \right| \frac{k_{\tau}(x, y)}{|y|^{d+2s}} dy \leq \int_{\frac{|x|}{2} \leq |y| \leq \frac{3|x|}{2}} \left| \delta(V, x, y) - \tilde{\delta}(x, y) \right| \frac{k_{\tau}(x, y)}{|y|^{d+2s}} dy \lesssim |x|^{\gamma-2s} \int_{\frac{|x|}{2} \leq |y| \leq \frac{3|x|}{2}} dy \leq C|x|^{\gamma-2s}. \quad (1.12)$$

Choose $\theta \geq 0, \gamma \in (s + \frac{1}{2}, 2s)$ satisfying

$$\theta + \gamma - 1 > 0, \quad \theta < (2s - \gamma)(2s - 1) < (\gamma - 1)(2s - 1).$$

Set $\mu = \frac{\theta}{\gamma(2s - 1)}$. Now suppose that $b_{\tau}(x) \cdot x \leq -|x|^\theta + 1$ outside a fixed compact set independent of $\tau$ and

$$\sup_{\tau \in T} |b_{\tau}| \leq C(1 + |x|^\theta), \quad \sup_{\tau \in T} |g_{\tau}| \leq C(1 + |x|^\frac{2s \theta}{2s - 1}),$$

then

$$\lim \sup_{|x| \to \infty} \sup_{\tau \in T} \frac{b_{\tau}(x) \cdot \nabla V(x)}{|x|^{\theta + \gamma - 1}} < 0. \quad (1.13)$$

Thus, fixing $R = 4$ in the above calculations, we get from (1.9), (1.10), (1.12) and (1.13) that

$$\sup_{\tau \in T} L_{\tau}[V](x) \leq k_0 + k_1|x|^\theta + \gamma - 1,$$

for some suitable constants $k_0, k_1$. Therefore to arrive at (1.3), we can take $h(x) = k_1|x|^\theta + \gamma - 1$. Moreover, since $\gamma(1 + \mu) < 2s$ we have $V^{1+\mu} \in L^1(\omega_s)$. By our choice of $\theta, \mu$ it is easily seen (1.4) holds and $V$ satisfies (1.5).

2. $\alpha$-DISCOUNTED HJB EQUATION WITH ROUGH KERNEL

In this section we prove the existence of a unique classical solution for the discounted problem. The results of this section will be proved under a weaker setting compared to Assumption 1.1. In this section we consider operators of the form

$$Iu(x) = \inf_{\tau \in T} \left( L_{\tau}u + c_{\tau}(x)u(x) + g_{\tau}(x) \right). \quad (2.1)$$

Note that $c_{\tau} = -\alpha$ corresponds to the $\alpha$-discounted problem.
Assumption 2.1. We impose following assumptions on the coefficients of the equation (2.1).

(B1) For some positive constant $c_0$ we have
\[
\sup_{\tau \in \mathcal{T}} c_\tau(x) \leq -c_0 \quad x \in \mathbb{R}^d.
\]

(B2) There exist an inf-compact $\mathcal{V} \in C^2(\mathbb{R}^d)$, $\mathcal{V} \geq 0$ and a positive inf-compact function $h \in C(\mathbb{R}^d)$, such that
\[
\sup_{\tau \in \mathcal{T}} (\mathcal{L}_\tau \mathcal{V}(x) + c_\tau(x) \mathcal{V}(x)) \leq k_0 1_{\mathcal{K}} - h(x), \quad x \in \mathbb{R}^d,
\]
for some $k_0 > 0$ and a compact set $\mathcal{K}$. In addition, \[ \lim_{|x| \to \infty} \frac{\sup_{\tau \in \mathcal{T}} |g_\tau(x)|}{h(x)} = 0. \]

(B3) For some $\mu \geq 0$ we have $\mathcal{V}^{1+\mu} \in L^1(\omega_x)$ and
\[
\sup_{\tau \in \mathcal{T}} \left[ \frac{|b_\tau|}{(1 + \mathcal{V})^{(2s-1)\mu}} + \frac{|g_\tau|}{(1 + \mathcal{V})^{1+2s\mu}} + \frac{|c_\tau|}{(1 + \mathcal{V})^{2s\mu}} \right] \leq C,
\]
\[
\sup_{x,y} \frac{\mathcal{V}(x+y)}{(1 + \mathcal{V}(x))(1 + \mathcal{V}(y))} + \frac{1}{\sup_{|x| \to \infty} \mathcal{V}(y)} \leq C.
\]

(B4) The maps $x \mapsto k_\tau(x,y), b_\tau(x), g_\tau(x), c_\tau(x)$ are locally uniformly continuous and locally bounded, uniformly in $\tau, y$.

(B5) For every compact set $K \subset \mathbb{R}^d$, there exists $\bar{\alpha} \in (0,1)$ so that
\[
\sup_{\tau \in \mathcal{T}} |b_\tau(x) - b_\tau(x')| + \sup_{\tau \in \mathcal{T}} |g_\tau(x) - g_\tau(x')| + |c_\tau(x) - c_\tau(x')| + \sup_{\tau \in \mathcal{T}} \sup_{y \in \mathbb{R}^d} |k_\tau(x,y) - k_\tau(x',y)|
\]
\[
\leq C_K |x - x'|^\bar{\alpha}.
\]

Assumptions (B1) and (B4) are standard whereas (B5) is generally imposed to obtain $C^{2s+p}$ regularity of the solutions. Let us now cite a class of examples that satisfy (2.2), (2.4) and (2.5).

Example 2.1. We recall the notations from Example 1.1. Suppose that
\[
\sup_{\tau \in \mathcal{T}} (b_\tau \cdot x) \leq C_0 |x|^\sigma + 1 \quad \text{where } \sigma \in [0,1].
\]
Choose $\gamma \in (0,2s)$ so that for $\sigma = 1$ we have $c_0 > \gamma C_0$ where $c_0$ is given by (B1). Let $0 \leq \mathcal{V} \in C^2(\mathbb{R}^d)$ be such that $\mathcal{V}(x) = |x|^\gamma$ for $|x| \geq 1$. Then the calculation in Example 1.1 reveals that
\[
\sup_{\tau \in \mathcal{T}} \int_{\mathbb{R}^d} \delta(V,x,y) \frac{k_\tau(x,y)}{|y|^{d+2s}} \leq C(1 + |x|^{\gamma - 2s}) \quad x \in B_1'/4.
\]
Therefore, if we set $h(x) = \kappa |x|^\gamma$ for $|x| \geq 1$, from (2.6) and (2.7) it is easily seen that (2.2) holds for suitable $\kappa$ and compact set $\mathcal{K}$. For (B3) to hold, we can choose $\mu \in [0, \frac{2}{\gamma} - 1)$ and restrict the family $\{b_\tau\}_{\tau \in \mathcal{T}}$ further to satisfy
\[
\sup_{\tau \in \mathcal{T}} |b_\tau(x)| \leq C(1 + |x|)^{\gamma(2s-1)}.
\]
It is quite possible for $b_\tau$ to have super-linear growth as we show in the example below.

Example 2.2. Let $\sigma, \theta$ are positive numbers satisfying
\[
\sigma - 1 < \theta < 4s^2, \quad \sigma < 2s(2s - 1).
\]
Consider a family of $\{b_\tau\}, \{c_\tau\}$ satisfying
\[
\sup_{\tau \in \mathcal{T}} |b_\tau(x)| \leq C(1 + |x|^{\sigma}), \quad \sup_{\tau \in \mathcal{T}} c_\tau(x) \leq -c_0(1 + |x|^\theta).
\]
Then we can choose \( \gamma \in (0, 2s) \) and \( \mu \in (0, \frac{2s}{\gamma} - 1) \) so that
\[
\sigma \leq \gamma \mu (2s - 1), \quad \text{and} \quad \theta \leq 2s \gamma \mu.
\]
Now let \( V \) satisfy \( V(x) = |x|^\gamma \) for \( |x| \geq 1 \). It can be easily checked that (2.2), (2.4) and (2.5) hold for \( h(x) \approx |x|^{\theta + \gamma} \).

By \( C^{\eta + \kappa}_0(\mathbb{R}^d) \) we denote the set of functions that are in \( C^{\eta + \kappa}(K) \) for every compact \( K \) and for some \( \kappa > 0 \), possibly depending on \( K \). More precisely, a function \( \ell \in C^{\eta + \kappa}_0(\mathbb{R}^d) \) if and only if for every compact set \( K \), there exists \( \kappa > 0 \) such that \( \ell \in C^{\eta + \kappa}(K) \). The remaining part of this section is devoted to the proof of Theorem 1.2. We first solve (1.7) in bounded domains (Theorem 2.3 in Section 5).

Without any loss of generality assume that \( \psi \in C^0(\Omega) \cap \mathfrak{g}(\Omega) \). Define
\[
t_0 = \inf \{ t > 0 : v + t > u \ \text{in} \ \mathbb{R}^d \}.
\]
It is evident that \( t_0 \leq 2 \sup_{\Omega}(u - v) \). Again, since \( \sup_{\Omega}(u - v) > 0 \), we must have \( t_0 > 0 \). Let \( \psi(x) = v(x) + t_0 \). From the definition we have \( \psi \geq u \). Since \( \psi - u \geq t_0 \) in \( \Omega^c \) and \( \Omega \) is bounded, it follows that \( \psi(x) = u(x) \) for some \( x \in \Omega \). Therefore, \( \psi \) is a valid test function at \( x \). Hence, from the definition of viscosity subsolution, we must have
\[
0 \leq \inf_{\tau \in T} \left( L_\tau (\psi(x) + c_\tau (x) \psi(x) + g_\tau (x)) \right) = \inf_{\tau \in T} \left( L_\tau v(x) + c_\tau (x) v(x) + g_\tau (x) \right) - c_0 t_0 \leq -c_0 t_0 < 0.
\]
But this is contradiction. Hence we must have \( v \geq u \) in \( \mathbb{R}^d \).

Next result is a \( C^{1, \gamma} \) regularity estimate. This is a special case of Theorem 5.1 which we prove in Section 5.

**Theorem 2.2.** Let \( u \) be a viscosity solution to
\[
\inf_{\tau \in T} \left[ L_\tau [u](x) + b_\tau (x) \cdot \nabla u(x) + g_\tau (x) \right] = 0 \quad \text{in} \ B_1.
\]
Let \( \sup_{\tau} \| b_\tau \|_{L^\infty(B_1)} \leq C \) and \( \gamma \in (0, 2s - 1) \). Suppose that for some modulus of continuity \( \rho \) we have
\[
|k_\tau (x_1, y) - k_\tau (x_2, y)| \leq \rho(|x_1 - x_2|) \quad \forall x_1, x_2 \in B_1, \ \tau \in T, \ y \in \mathbb{R}^d.
\]
Then we have
\[
\| u \|_{C^{1, \gamma}(B_1)} \leq C \left( \| u \|_{L^\infty(\mathbb{R}^d)} + \sup_{\tau \in T} \| g_\tau \|_{L^\infty(B_1)} \right).
\]
where the constant $C$ depends on $d, s, C_0, g, \lambda, \Lambda$.

Next we prove an existence result in the bounded domains.

**Theorem 2.3.** Let $\Omega$ be a bounded $C^1$ domain. Assume that (B1) and (B4) hold. Then there exists a viscosity solution $W$ satisfying

$$\inf_{\tau \in T} \left( \mathcal{L}_\tau W + c_\tau W + g_\tau \right) = 0 \quad \text{in } \Omega, \quad \text{and } W = 0 \quad \text{in } \Omega^c. \quad (2.8)$$

In addition, if (B5) holds, then $W \in C^{2s+\kappa}(\Omega) \cap C^\kappa(\mathbb{R}^d)$ for some $\kappa > 0$ and $W$ is the unique solution to (2.8).

**Proof.** Existence of a viscosity solution follows from [25, Corollary 5.7].

Next we show that $W \in C^\kappa(\mathbb{R}^d)$ for some $\kappa > 0$. Let $M = c_0^{-1} \sup \|g_\tau\|_{L^\infty(\Omega)}$. Using Lemma 2.1, we obtain $W \leq M$. Analogous argument also gives $W \geq -M$. Therefore, using the barrier function of [25, Lemma 5.10], it is standard to show that $|W| \leq CM \delta^\kappa(x)$, where $\delta(x) = \text{dist}(x, \Omega^c)$ (see for instance, [10, Theorem 2.6]). Again, by [29, Theorem 7.2], $W$ is Hölder continuous in the interior. It is now quite standard to show that $W \in C^\kappa(\mathbb{R}^d)$ for some $\kappa > 0$ (see again, the proof of [10, Theorem 2.6]). From Theorem 2.2 we also see that $W \in C^{1,\gamma}(\Omega_\sigma)$ for some $\gamma \in (0, 2s - 1)$ where

$$\Omega_\sigma = \{ x \in \Omega : \text{dist}(x, \Omega^c) > \sigma \}, \quad \sigma > 0.$$ 

Therefore, by (B5), the function

$$x \mapsto b_\tau(x) \cdot \nabla W(x) + c_\tau(x) W(x) + g_\tau(x)$$

is Hölder continuous in $\Omega_\sigma$, uniformly in $\tau$, for every $\sigma > 0$. Thus, using [30, Theorem 1.3], we obtain that $W \in C^{2s+\kappa}(\Omega_\sigma)$ for some $\kappa > 0$. Thus $W \in C^{2s+\kappa}(\Omega) \cap C^\kappa(\mathbb{R}^d)$. In particular, $W$ is a classical solution to (2.8).

To establish the uniqueness, let $\tilde{W}$ be a viscosity solution to (2.8). The preceding argument shows that $\tilde{W}$ is a classical solution. Hence, from the ellipticity property, it follows that

$$\sup_{\tau \in T} (\mathcal{L}_\tau (W - \tilde{W}) + c_\tau W - \tilde{W})) \geq 0 \quad \text{in } \Omega.$$ 

Since 0 is a supersolution to the above equation, applying Lemma 2.1 it follows that $W \leq \tilde{W}$ in $\mathbb{R}^d$. From the symmetry we also have $\tilde{W} \leq W$, giving us $W = \tilde{W}$. This completes the proof. \qed

**Remark 2.1.** In several places below we use the Hölder estimate from [29, Theorem 7.2]. Though the results of [29] are proved for nonlocal parabolic operators, we can still apply the results to our setting by treating the solutions as stationary functions of the time variable.

Our next step is to construct a solution for (2.8) in the whole space.

**Lemma 2.4.** Suppose that (B1), (B2) and (B4) hold. Let $W_n$ be a viscosity solution to (2.8) in the ball $\Omega = B_n$. Then there exists a subsequence $W_{n_k}$ such that $W_{n_k} \to w$, uniformly on compacts and $w \in C(\mathbb{R}^d) \cap L^1(\omega_s)$ satisfies

$$\inf_{\tau \in T} \left( \mathcal{L}_\tau w + c_\tau w + g_\tau \right) = 0 \quad \text{in } \mathbb{R}^d. \quad (2.9)$$

Furthermore, $|w(x)| \leq \frac{k_0}{c_0} + V(x)$ in $\mathbb{R}^d$.

**Proof.** Define $\tilde{V} = \frac{k_0}{c_0} + V$ where $k_0$ is given by (2.2). It then follows from (2.2) that

$$\sup_{\tau \in T} \left( \mathcal{L}_\tau \tilde{V} + c_\tau \tilde{V} + |g_\tau| \right) \leq \sup_{\tau \in T} (\mathcal{L}_\tau V + c_\tau V) + h - k_0 \leq 0 \quad \text{in } \mathbb{R}^d. \quad (2.10)$$

Let $W_n$ be a viscosity solution to (2.8) with $\Omega = B_n$, that is,

$$\inf_{\tau \in T} \left( \mathcal{L}_\tau W_n + c_\tau W_n + g_\tau \right) = 0 \quad \text{in } B_n, \quad \text{and } \quad W_n = 0 \quad \text{in } B_n^c. \quad (2.11)$$

Proof. Define $\tilde{V} = \frac{k_0}{c_0} + V$ where $k_0$ is given by (2.2). It then follows from (2.2) that

$$\sup_{\tau \in T} \left( \mathcal{L}_\tau \tilde{V} + c_\tau \tilde{V} + |g_\tau| \right) \leq \sup_{\tau \in T} (\mathcal{L}_\tau V + c_\tau V) + h - k_0 \leq 0 \quad \text{in } \mathbb{R}^d. \quad (2.10)$$

Let $W_n$ be a viscosity solution to (2.8) with $\Omega = B_n$, that is,

$$\inf_{\tau \in T} \left( \mathcal{L}_\tau W_n + c_\tau W_n + g_\tau \right) = 0 \quad \text{in } B_n, \quad \text{and } \quad W_n = 0 \quad \text{in } B_n^c. \quad (2.11)$$

Proof. Define $\tilde{V} = \frac{k_0}{c_0} + V$ where $k_0$ is given by (2.2). It then follows from (2.2) that

$$\sup_{\tau \in T} \left( \mathcal{L}_\tau \tilde{V} + c_\tau \tilde{V} + |g_\tau| \right) \leq \sup_{\tau \in T} (\mathcal{L}_\tau V + c_\tau V) + h - k_0 \leq 0 \quad \text{in } \mathbb{R}^d. \quad (2.10)$$

Let $W_n$ be a viscosity solution to (2.8) with $\Omega = B_n$, that is,

$$\inf_{\tau \in T} \left( \mathcal{L}_\tau W_n + c_\tau W_n + g_\tau \right) = 0 \quad \text{in } B_n, \quad \text{and } \quad W_n = 0 \quad \text{in } B_n^c. \quad (2.11)$$
Applying Lemma 2.1 on (2.10) and (2.11), we see that $|W_n| \leq \hat{V}$ in $\mathbb{R}^d$, for all $n$.

Next we show that for any compact set $\mathcal{K}, \{W_n\}_{n \geq n_0}$ is equicontinuous on $\mathcal{K}$, where $\mathcal{K} \subset B_{n_0}$. Let $R'$ be such that $\mathcal{K} \subset B_{R'}$. Without any loss of generality, we may assume that $R' + 4 < n_0$. Consider a smooth cut-off function $0 \leq \chi \leq 1$ such that $\chi = 1$ in the ball $B_{R'' + 2}$ and $\chi = 0$ outside the $B_{R'' + 3}$. Let $\tilde{\psi}_n = \chi W_n$ in $\mathbb{R}^d$. From (2.9) we see that

$$\inf_{\tau \in T} \left( \mathcal{L}_\tau \tilde{\psi}_n + c_\tau \tilde{\psi}_n + g_\tau \right) \leq 0 \quad \text{in } B_{R'' + 2}, \quad (2.12)$$

for all $n \geq n_0$. We claim that

$$\sup_{B_{R'' + 1}} \sup_{\tau \in T} |I_\tau[(1 - \chi)W_n]| \leq C. \quad (2.13)$$

Note that $|x + y| \geq R'' + 2$ and $|x| \leq R'' + 1$ imply that $|y| \geq 1$. Thus, for any $x \in B_{R'' + 1}$

$$\sup_{\tau \in T} |I_\tau[(1 - \chi)W_n]| \leq (2 - 2s) \int_{\mathbb{R}^d} |\delta((1 - \chi)W_n, x, y)| \frac{A}{|y|^{d+2s}} dy \leq 2(2 - 2s) \int_{|y| \geq 1} |W_n(x + y)| \frac{A}{|y|^{d+2s}} dy \leq 2(2 - 2s) \int_{|y| \geq 1} \hat{V}(x + y) \frac{A}{|y|^{d+2s}} dy \leq C_{R''} \int_{\mathbb{R}^d} \hat{V}(x + y) \frac{A}{1 + |x + y|^{d+2s}} dy \leq C.$$

This establishes (2.13). Also,

$$\sup_{\tau \in T} \sup_{B_{R'' + 1}} |c_\tau W_n| \leq C_{R''}(1 + \sup_{B_{R'' + 1}} V).$$

Thus, applying [29] Theorem 7.2 on (2.12), we obtain that for some $\alpha > 0$

$$\sup_{n \geq n_0} \|W_n\|_{C^0(B_{R''})} = \sup_{n \geq n_0} \left\| \tilde{\psi}_n \right\|_{C^0(B_{R''})} \leq C.$$

Hence, $\{W_n\}_{n \geq n_0}$ is equicontinuous on $\mathcal{K}$. Applying Arzelà-Ascoli theorem and a standard diagonalization argument we have $W_{n_k} \to w \in C(\mathbb{R}^d)$, uniformly on compact sets, along some subsequence $n_k \to \infty$. Since $|W_{n_k}| \leq \hat{V}$ for all $n_k$, we also get

$$|w| \leq \hat{V} \quad \text{and} \quad \lim_{n_k \to \infty} \|W_{n_k} - w\|_{L^1(\omega_k)} = 0. \quad (2.14)$$

Finally, applying the stability property of viscosity solutions [15 Lemma 5], we see that $w$ solves (2.9). In fact, the stability property can be seen as follows: Suppose $\varphi \in C^2(B_{2\kappa}(x))$ touches $w$ (strictly) from above at $x$ in $B_{2\kappa}(x)$, for some $\kappa > 0$. From the local uniform convergence above, we can find a sequence $(d_{n_k}, x_{n_k}) \in \mathbb{R} \times B_{n_k}(x)$ such that $x_{n_k} \to x$, $d_{n_k} \to 0$ and $\varphi + d_k$ would touch $W_{n_k}$ from above at the point $x_k$. Then letting

$$v_{n_k}(y) := \begin{cases} \varphi(y) + d_k & \text{for } y \in B_\kappa(x), \\ W_{n_k}(y) & \text{otherwise}, \end{cases}$$

we obtain from (2.11) and for large $n_k$ that

$$\inf_{\tau \in T} \left( \mathcal{L}_\tau v_{n_k}(x_{n_k}) + c_\tau v_{n_k}(x_{n_k}) + g_\tau(x_{n_k}) \right) \geq 0.$$

Using Assumption 2.1(B4) and (2.14), we can let $n_k \to \infty$ to get that

$$\inf_{\tau \in T} \left( \mathcal{L}_\tau v(x) + c_\tau v(x) + g_\tau(x) \right) \geq 0,$$
where \( v \) is defined in the same fashion replacing \( W_{n_k} \) by \( w \) and \( d_k \) by 0. Similarly, we can also check that \( w \) is a supersolution. This completes the proof. \( \square \)

Now we show that the solution \( w \) obtained in Lemma 2.4 belongs to the class \( \sigma(\mathcal{V}) \cap C^{2s+}_{\text{loc}}(\mathbb{R}^d) \).

**Lemma 2.5.** Let \( w \) be the solution to (2.9) obtained in Lemma 2.4. It holds that \( w \in \sigma(\mathcal{V}) \). In addition, if (BB), (LB) holds, then any solution \( v \) of (2.9) satisfying \( v \in \mathcal{O}(\mathcal{V}) \) is in \( C^{2s+}_{\text{loc}}(\mathbb{R}^d) \). In particular, \( w \) is a classical solution to (2.9).

**Proof.** First we show that the solution \( w \) obtained in Lemma 2.4 is in \( \sigma(\mathcal{V}) \). Recall that \( w = \lim_{n_k \to \infty} W_{n_k} \) where \( W_{n_k} \) solves (2.8) in the ball \( B_{n_k} \). Fix \( \varepsilon > 0 \). By (2.3) we have

\[
(\varepsilon h(x) - \sup_{\tau \in T} |g_r(x)|) \geq 0
\]

for all \( |x| \) large. Thus we can find a compact set \( \mathcal{K}_\varepsilon \ni \mathcal{K} \) so that

\[
\sup_{\tau \in T}(\mathcal{L}_\tau \varphi_\varepsilon + c_r \varphi_\varepsilon + |g_r|) \leq \sup_{\tau \in T}(\mathcal{L}_\tau \varphi_\varepsilon + c_r \varphi_\varepsilon + \varepsilon h) \leq 0 \quad \text{in } \mathcal{K}_\varepsilon,
\]

(2.15)

where \( \varphi_\varepsilon = \varepsilon \mathcal{V} \). Let \( \kappa = \sup_{n_k} \max_{\mathcal{K}_\varepsilon} |W_{n_k}| < \infty \). From (15) and (2.15) it follows that

\[
\sup_{\tau \in T}(\mathcal{L}_\tau (\kappa + \varphi_\varepsilon) + c_r (\kappa + \varphi_\varepsilon) + |g_r|) \leq 0 \quad \text{in } \mathcal{K}_\varepsilon.
\]

Applying Lemma 2.1 in the domain \( B_{n_k} \setminus \mathcal{K}_\varepsilon \) we obtain

\[
|W_{n_k}| \leq \kappa + \varphi_\varepsilon = \kappa + \varepsilon \mathcal{V} \quad \text{in } \mathbb{R}^d,
\]

for all \( n_k \) large. Hence, letting \( n_k \to \infty \), we get \( |w| \leq \kappa + \varepsilon \mathcal{V} \) in \( \mathbb{R}^d \) which in turn, implies

\[
\limsup_{|x| \to \infty} \frac{1}{1 + \mathcal{V}(x)} |w| \leq \varepsilon.
\]

From the arbitrariness of \( \varepsilon \) we have \( w \in \sigma(\mathcal{V}) \).

Now we prove the second part. Let \( v \) solve

\[
\inf_{\tau \in T}(\mathcal{L}_\tau v + c_r v + g_r) = 0 \quad \text{in } \mathbb{R}^d,
\]

(2.16)

and \( v \in \mathcal{O}(\mathcal{V}) \), that is, \( |v| \leq C(1 + \mathcal{V}) \) in \( \mathbb{R}^d \). Fix a point \( x_0 \in \mathbb{R}^d \) and let \( r = [1 + \mathcal{V}(x_0)]^{-1} \), where \( \mu \) is given by (13). Define \( v(x) = v(x_0 + rx) \). It then follows from (2.16) that

\[
\inf_{\tau \in T} \left( \tilde{I}_\tau [v] + r^{2s-1} b_r(x_0 + rx) \cdot \nabla v + r^2 c_r(x_0 + rx) v + r^2 g_r(x_0 + rx) \right) = 0 \quad \text{in } \mathbb{R}^d,
\]

(2.17)

where

\[
\tilde{I}_\tau [u](x) = \int_{\mathbb{R}^d} \delta(u, x, y) \frac{k_r(x_0 + rx, ry)}{|y|^{d+2s}} dy.
\]

Now consider a smooth cut-off function \( \xi : \mathbb{R}^d \to [0, 1] \) satisfying \( \xi = 1 \) in \( B_{\frac{1}{2}} \) and \( \xi = 0 \) in \( B_\frac{3}{2} \). We can re-write (2.17) as

\[
\inf_{\tau \in T} \left( \tilde{I}_\tau [\psi] + r^{2s-1} b_r(x_0 + rx) \cdot \nabla \psi + r^2 c_r(x_0 + rx) \psi + r^2 g_r(x_0 + rx) + \tilde{I}_\tau [(1 - \xi)v] \right) = 0 \quad \text{in } B_1,
\]

(2.18)

for \( \psi = \xi v \). Since \( |v| \leq C(1 + \mathcal{V}) \), from (2.3) we have

\[
\sup_{\tau \in T} \left( r^{2s-1} |b_r(x_0 + r\cdot)| + r^{2s} \frac{|g_r(x_0 + r\cdot)|}{1 + \mathcal{V}(x_0)} \right) \leq C, \quad \sup_{\tau \in T} \left( r^{2s} \frac{|c_r(x_0 + r\cdot)|}{1 + \mathcal{V}(x_0)} \right) \leq C,
\]

(2.19)
where $C$ is independent of $r$ and $x_0$. From (2.5) we also have $\|\psi\|_{L^\infty(\mathbb{R}^d)} \leq C(1 + \mathcal{V}(x_0))$. For $x \in B_1$ let us now compute

$$
\sup_{\tau \in T} |\tilde{T}_r[(1 - \xi)\psi]| \leq 2(2 - 2s) \int_{\mathbb{R}^d} |(1 - \xi(x + y))\psi(x + y)| \frac{1}{|y|^{d+2s}} dy
$$

$$
\leq 2(2 - 2s) \int_{|y|\geq 1/2} |\psi(x + y)| \frac{1}{|y|^{d+2s}} dy
$$

$$
\leq C \int_{\mathbb{R}^d} |\psi(x_0 + rx + ry)| \frac{1}{1 + |y|^{d+2s}} dy
$$

$$
\leq C \int_{\mathbb{R}^d} (1 + \mathcal{V}(x_0 + rx + ry)) \frac{1}{1 + |y|^{d+2s}} dy
$$

$$
\leq C \int_{|y|\leq r^{-1}} (1 + \mathcal{V}(x_0)) \frac{1}{1 + |y|^{d+2s}} dy + C \int_{|y|> r^{-1}} (1 + \mathcal{V}(x_0 + rx + ry)) \frac{1}{1 + |y|^{d+2s}} dy
$$

$$
\leq C(1 + \mathcal{V}(x_0)) + C \int_{|z|> 1} (1 + \mathcal{V}(x_0 + rz + z)) \frac{1}{1 + |z|^{d+2s}} dz
$$

$$
\leq C(1 + \mathcal{V}(x_0)) + C(1 + \mathcal{V}(x_0)) \int_{|z|> 1} \frac{1}{1 + |z|^{d+2s}} dz
$$

$$
\leq C(1 + \mathcal{V}(x_0)),
$$

(2.20)

where in the fifth and seventh line we use (2.5). Thus, using (2.15), (2.16), (2.20) and Theorem 2.2, we obtain

$$
\sup_{B_{1/2}} |\nabla \psi| \leq C(1 + \mathcal{V}(x_0)).
$$

Computing the derivative at $x = 0$ gives us

$$
|\nabla \psi(x_0)| \leq C(1 + \mathcal{V}(x_0))^{1+\mu}, \quad x_0 \in \mathbb{R}^d.
$$

(2.21)

This gives an estimate on the growth of $|\nabla \psi|$. Next we fix a point $x_0$ and let $\zeta(x) = \psi(x_0 + x)$. Let $\xi$ be the cut-off function as chosen above. Then, letting $\varphi = \xi \zeta$, we obtain

$$
\inf_{\tau \in T} \left( \tilde{T}_r[\varphi] + b_r(x_0 + x) \cdot \nabla \varphi + c_r(x_0 + x) \varphi + g_r(x_0 + x) + \tilde{T}_r[(1 - \xi)\zeta] \right) = 0 \quad \text{in } \mathbb{R}^d,
$$

(2.22)

where

$$
\tilde{T}_r[u](x) = \int_{\mathbb{R}^d} \delta(u, x, y) \frac{k_r(x_0 + x + ry)}{|y|^{d+2s}} dy.
$$

Let $x, x' \in B_1$. Using (155) we get

$$
|\tilde{T}_r[(1 - \xi)\zeta](x) - \tilde{T}_r[(1 - \xi)\zeta](x')|
$$

$$
\leq \int_{|y|\geq 1/2} |\delta((1 - \xi)\zeta, x, y)| \frac{|k_r(x_0 + x + y) - k_r(x_0 + x' + y)|}{|y|^{d+2s}} dy
$$

$$
+ 2(2 - 2s) \Lambda \int_{|y|\geq 1/2} |(1 - \xi(x + y))\zeta(x + y) - (1 - \xi(x' + y))\zeta(x' + y)| \frac{1}{|y|^{d+2s}} dy
$$

$$
\leq C|x - x'|^{\tilde{\alpha}} + C \int_{|y|\geq 1/2} |(\xi(x' + y) - \xi(x + y))\zeta(x + y)| \frac{1}{|y|^{d+2s}} dy
$$

$$
+ \int_{|y|\geq 1/2} |\zeta(x + y) - \zeta(x' + y)| \frac{1}{|y|^{d+2s}} dy
$$

$$
\leq C|x - x'|^{\tilde{\alpha}} + C|x - x'|(1 + \mathcal{V}(x_0)) + C|x - x'| \int_{|y|\geq 1/2} (1 + \mathcal{V}(x + y))^{1+\mu} \frac{1}{|y|^{d+2s}} dy
$$
This result should be compared with Meglioli & Punzo [24, Theorem 2.7]. Unlike [24], the above function \( c \) which is a contradiction. Therefore there exist an inf-compact \( C \).

The proof of Theorem 1.2 leads to the following Liouville type result. Suppose that

\[ \lambda - \lambda = 0 \]

for all \( \tau \in \mathcal{T} \), where in the third line we use [2.20] and [2.21], and in the last line we use the fact \( V^{1,\mu} = L^1(0,1) \). The above estimate gives Hölder continuity estimate of \( \hat{L} \) uniformly in \( \tau \in \mathcal{T} \). From [2.22] and Theorem 2.2 we first observe that \( v \in C^{1,\gamma}(B_1(x_0)) \) for some \( \gamma > 0 \). Therefore, using above estimates, we can apply [31, Theorem 1.3] to obtain that \( v \in C^{2s+k}(B_1(2x_0)) \) for some \( k > 0 \). Hence \( v \in C^{2s+k}(R^d) \), completing the proof.

Now we can complete the proof of Theorem 1.2.

**Proof of Theorem 1.2**: Existence and regularity follow from Lemma 2.3 and Lemma 2.5. So we only prove uniqueness. Let \( \tilde{w} \in \mathcal{O}(V) \) be a viscosity solution to (2.29). From Lemma 2.6 we see that \( \tilde{w} \in C^{2s+k}_\text{loc}(R^d) \). Hence \( w, \tilde{w} \) are classical solutions to (1.1). Now suppose, to the contrary, that \( w \neq \tilde{w} \) and, without any loss of generality, we let \( (\tilde{w} - w)_+ \geq 0 \). Letting \( v = \tilde{w} - w \) we obtain from (1.7) that

\[ \sup_{\tau \in \mathcal{T}} (\lambda_\tau v + c_\tau v) \geq 0 \quad \text{in } R^d. \tag{2.23} \]

If \( \sup_{R^d} (\tilde{w} - w)_+ \geq 0 \) is attained in \( R^d \), say at a point \( x_0 \), then we get from (2.23) that \( 0 > -c_\infty(\tilde{w} - w)_+ \geq 0 \), which is a contradiction. So we consider the situation where

\[ \sup_{B_m} v_+ \not\nearrow \sup_{R^d} v, \tag{2.24} \]

as \( m \to \infty \) and the sequence is strictly increasing. Define \( \mathcal{K} = \mathcal{K} \cup \{ V \leq 1 \} \) (see (2.2)) and let \( \kappa = \max_{\mathcal{K}} v_+ \). Letting \( \psi = v - \kappa \) we have from (2.23) that

\[ \sup_{\tau \in \mathcal{T}} (\lambda_\tau \psi + c_\tau \psi) \geq 0 \quad \text{in } R^d. \tag{2.25} \]

Using (2.24), we have \( \psi \geq 0 \) in \( \mathcal{K}^c \). Define

\[ t_0 = \inf\{ t > 0 : tV > \psi \quad \text{in } \mathcal{K}^c \}. \]

It is evident that \( t_0 \) is finite, and since \( \psi(z) > 0 \) for some \( z \in \mathcal{K}^c \), we also have \( t_0 > 0 \). We claim that \( t_0 \mathcal{V} \) must touch \( \psi \) from above. Suppose that, \( t_0 \mathcal{V} > \psi \) in \( \mathcal{K}^c \). Since \( \psi \in \mathcal{O}(V) \), there exists a compact set \( \mathcal{K}_1 \supseteq \mathcal{K} \) such that \( |\psi| < \frac{t_0}{2} \mathcal{V} \) in \( \mathcal{K}_1^c \). Again, on \( \partial \mathcal{K} \), we have \( \psi \leq 0 < t_0 \leq t_0 \mathcal{V} \). Therefore \( (t_0 \mathcal{V} - \psi) > 0 \) in \( \mathcal{K}_1 \setminus \mathcal{K} \). Thus we can find \( \varepsilon > 0 \) small enough so that \( (t_0 - \varepsilon) \mathcal{V} > \psi \) in \( \mathcal{K}^c \) which will contradict the definition of \( t_0 \). Hence \( t_0 \mathcal{V} \) must touch \( \psi \) at some point \( x_0 \in \mathcal{K}^c \). Since \( \psi \leq 0 \) in \( \mathcal{K} \), we also have \( t_0 \mathcal{V} \geq \psi \) in \( R^d \). Applying \( t_0 \mathcal{V} \) as a test function at \( x_0 \) in (2.25) and using (2.2) we get

\[ 0 \leq t_0 \sup_{\tau \in \mathcal{T}} (\lambda_\tau \mathcal{V}(x_0) + c_\tau \mathcal{V}(x_0)) \leq -t_0 h(x_0) < 0, \]

which is a contradiction. Therefore \( t_0 \) must be zero and \( (\tilde{w} - w)_+ = 0 \). This gives us uniqueness, completing the proof.

**Remark 2.2**: The proof of Theorem 1.2 leads to the following Liouville type result. Suppose that there exist an inf-compact \( C^2 \) function \( V \geq 0 \) and a compact set \( \mathcal{K} \) satisfying

\[ \sup_{\tau \in \mathcal{T}} (\lambda_\tau V(x) + c_\tau(x)V(x)) < 0 \quad x \in \mathcal{K}^c, \]

and \( c_\tau \leq -c_\infty < 0 \) for all \( \tau \in \mathcal{T} \). Then, if \( u \in \mathcal{O}(V) \) solves

\[ \inf_{\tau \in \mathcal{T}} \{ (\lambda_\tau u(x) + c_\tau(x)u(x)) = 0 \quad \text{in } R^d, \]

then \( u \equiv 0 \). The argument works for \( s \in (0,1) \) and \( u, b_\tau, c_\tau \) are only required to be continuous. This result should be compared with Meglioli & Punzo [24, Theorem 2.7]. Unlike [24], the above equation is nonlinear and requires no additional regularity on drift.
3. HJB FOR ERGODIC CONTROL PROBLEM

The main result of this section is the existence-uniqueness result for ergodic HJB problem, that is, to find a solution \((u, \lambda^*)\) satisfying
\[
\inf_{\tau \in T} \left( \mathcal{L}_\tau u + g_\tau \right) - \lambda^* = 0 \quad \text{in } \mathbb{R}^d.
\]
Letting \(c_\tau = -\alpha \in (-1, 0)\) in (B1), we see that \(V\) in Assumption 1.1 satisfies (B2)–(B3). Thus Theorem 1.2 is applicable under Assumption 1.1. Let \(w_\alpha\) be the unique solution to
\[
\inf_{\tau \in T} \left( \mathcal{L}_\tau w_\alpha + g_\tau \right) - \alpha w_\alpha = 0 \quad \text{in } \mathbb{R}^d,
\]
and \(w_\alpha \in \mathfrak{V}(V) \cap C^{2s+}\mathcal{C}_{\text{loc}}(\mathbb{R}^d)\). Define
\[
\bar{w}_\alpha(x) = w_\alpha(x) - w_\alpha(0).
\]
We claim that for some compact ball \(B\) we have
\[
|\bar{w}_\alpha(x)| \leq \max_B |\bar{w}_\alpha| + V(x) \quad x \in \mathbb{R}^d.
\]
Since \(w_\alpha \in \mathfrak{V}(V)\) and \(V\) is inf-compact, we have \(V \equiv w_\alpha\) inf-compact. Thus, there exists \(x_\alpha \in \mathbb{R}^d\) satisfying \((V - w_\alpha)(x_\alpha) = \inf_{\mathbb{R}^d}(V - w_\alpha)\) implying \(w_\alpha \leq V + w_\alpha(x_\alpha) - V(x_\alpha)\). From (3.2) we then have
\[
0 \leq \inf_{\tau \in T} \left( \mathcal{L}_\tau V(x_\alpha) + g_\tau(x_\alpha) \right) - \alpha w_\alpha(x_\alpha) \leq k_0 - (h(x_\alpha) - \inf_{\tau \in T} |g_\tau(x_\alpha)|) + \alpha V(0) - w_\alpha(0) - V(x_\alpha),
\]
by (B3). Since \(V\) is non-negative and \(\alpha w_\alpha \leq k_0 + \alpha V\) (by Lemma 2.3), the above inequality gives
\[
(h(x_\alpha) - \sup_{\tau \in T} |g_\tau(x_\alpha)|) \leq 2k_0 + 2\alpha V(0).
\]
Since \(\sup_{\tau \in T} |g_\tau| \in \mathfrak{V}(h)\) by (A2), there exists a compact ball \(B\), independent of \(\alpha \in (0, 1)\), so that \(x_\alpha \in B\). Hence
\[
\bar{w}_\alpha(x) \leq \bar{w}_\alpha(x) - V(x) + V(x) \leq \bar{w}_\alpha(x_\alpha) - V(x_\alpha) + V(x) \leq \max_B \bar{w}_\alpha + V(x).
\]
Again, since \((V + w_\alpha)\) is inf-compact, an argument similar to above gives that
\[
\bar{w}_\alpha(x) \geq \min_B \bar{w}_\alpha - V,
\]
for some compact ball \(B\). Combining the above estimates we get (3.3).

**Lemma 3.1.** It holds that \(\sup_{\alpha \in (0, 1)} \max_B |\bar{w}_\alpha| < \infty\).

**Proof.** Suppose, to the contrary, that
\[
\sup_{\alpha \in (0, 1)} \max_B |\bar{w}_\alpha| = \infty.
\]
Therefore, we can find a sequence \(\alpha_n \to \alpha_0 \in [0, 1]\) such that
\[
\max_B |\bar{w}_{\alpha_n}| \to \infty \quad \text{as } n \to \infty.
\]
Since \(|w_\alpha| \leq \frac{k_0}{\alpha} + V\), we must have \(\alpha_0 = 0\). Denote by \(\rho_n = \max_B |\bar{w}_{\alpha_n}|\). From (3.3) we observe that
\[
|\bar{w}_{\alpha_n}(x)| \leq \rho_n + V(x) \quad \text{in } \mathbb{R}^d.
\]
Letting \(\psi_n = \frac{1}{\rho_n} \bar{w}_{\alpha_n}\), we obtain from (3.2) that
\[
\inf_{\tau \in T} \left( \mathcal{L}_\tau \psi_n + \rho_n^{-1} g_\tau \right) - \alpha_n \psi_n - \frac{\alpha_n}{\rho_n} w_{\alpha_n}(0) = 0 \quad \text{in } \mathbb{R}^d.
\]
Following the arguments of Lemma 2.4 and using (3.4), it can be easily seen that for every compact 
$K$, $\{\psi_n\}_{n \geq 1}$ is in $C^k(K)$ for some $k > 0$, uniformly in $n$. Hence there exists a $\psi \in C(\mathbb{R}^d)$ such that 
$\psi_n \rightarrow \psi$ uniformly on compacts, along some subsequence. From (3.4) we also have $|\psi| \leq 1$ and $\max_B |\psi| = 1$. Using the stability property of viscosity solution in (3.5), we obtain 
$\psi$.

Next we prove the existence of a solution to the ergodic control problem.

**Theorem 3.2.** Suppose that (A1)–(A4) hold. Then for some sequence $\alpha_n \rightarrow 0$ we have 
$$
\lim_{n \rightarrow \infty} \tilde{w}_{\alpha_n} = u, \quad \alpha_n w_{\alpha_n}(0) = \lambda^*
$$
for some $(u, \lambda^*) \in C(\mathbb{R}^d) \times \mathbb{R}$, and

$$
\inf_{\tau \in T} (\mathcal{L}_\tau u + g_\tau) - \lambda^* = 0 \quad \text{in } \mathbb{R}^d.
$$
Moreover, if (A4) holds, then $u \in \mathcal{Q}(V) \cap C^{2s}_\text{loc}(\mathbb{R}^d)$.

**Proof.** From (3.2) we note that

$$
\inf_{\tau \in T} (\mathcal{L}_\tau \tilde{w}_\alpha + g_\tau) - \alpha \tilde{w}_\alpha - \alpha w_{\alpha}(0) = 0 \quad \text{in } \mathbb{R}^d.
$$

From Lemma 3.1 3.3 and the proof of Lemma 2.4 we see that the family $\{\tilde{w}_\alpha : \alpha \in (0,1)\}$ is locally equicontinuous. Also note that $\alpha|w_{\alpha}(0)| \leq \kappa + \alpha V(0)$. Hence we can find a sequence $\alpha_n \rightarrow 0$ such that

$$
\lim_{n \rightarrow \infty} \alpha_n w_{\alpha_n}(0) = \lambda^*
$$

$$
\lim_{n \rightarrow \infty} \tilde{w}_{\alpha_n} = u \quad \text{uniformly over compacts.}
$$

Using stability property of the viscosity solutions and passing the limit in (3.8) we obtain

$$
\inf_{\tau \in T} (\mathcal{L}_\tau u + g_\tau) - \lambda^* = 0 \quad \text{in } \mathbb{R}^d.
$$

This gives us (3.7). From (3.3) we also have $|u| \leq C + V$ in $\mathbb{R}^d$. Thus we can follow the proof of Lemma 2.5 to conclude that $u \in C^{2s}_\text{loc}(\mathbb{R}^d)$.

Thus we remain to show that $u \in \mathcal{Q}(V)$. Fix an $\varepsilon > 0$. Recall from Lemma 2.5 that $w_{\alpha_n} \in \mathcal{Q}(V)$. Thus, the functions $\varepsilon V - w_{\alpha}$ and $\varepsilon V + w_{\alpha}$ are inf-compact. Since $\sup_{\tau \in T} |g_\tau| \in \mathcal{Q}(h)$, we also have $\varepsilon h - \sup_{\tau \in T} |g_\tau|$ inf-compact. Thus the proof of (3.3) works with $V$ replaced by $\varepsilon V$ and we obtain a compact $B(\varepsilon)$ satisfying

$$
|\tilde{w}_{\alpha_n}(x)| \leq \max_{B(\varepsilon)} |\tilde{w}_{\alpha_n}| + \varepsilon V(x) \quad x \in \mathbb{R}^d.
$$

Letting $\alpha_n \rightarrow 0$ in the above equation we obtain

$$
|u(x)| \leq \max_{B(\varepsilon)} |u| + \varepsilon V(x) \quad x \in \mathbb{R}^d,
$$
which in turn, gives

$$
\limsup_{|x| \rightarrow \infty} \frac{|u(x)|}{1 + V(x)} \leq \varepsilon.
$$
From the arbitrariness of $\varepsilon$ it follows that $u \in \sigma(V)$, completing the proof. \hfill\square

Now we complete the proof of Theorem \ref{thm1}

**Proof of Theorem \ref{thm1}** Existence of $(u, \lambda)$ follows from Theorem \ref{thm3.2}. So we only prove uniqueness. Suppose that $(\hat{u}, \hat{\rho}) \in \sigma(V) \times \mathbb{R}$ is a solution to \eqref{eq1.0}. The proof of Lemma \ref{lem2.5} gives us that $u, \hat{u} \in C_{\text{loc}}^{2+}(\mathbb{R}^d)$, and therefore, both $u, \hat{u}$ are classical solutions.

Now suppose, to the contrary, that $\lambda^* \neq \hat{\rho}$. Assume, without loss of generality, that $\hat{\rho} < \lambda^*$ and define $\hat{u}_\epsilon := \hat{u} + \epsilon V$ for $\epsilon > 0$. Since $\hat{u}$ and $V$ are classical solution to \eqref{eq1.6} and \eqref{eq1.3}, respectively, we obtain
\[
\inf_{\tau \in T} [\mathcal{L}_\tau \hat{u}_\epsilon + g_\tau(x)] \leq \inf_{\tau \in T} (\mathcal{L}_\tau \hat{u} + g_\tau(x)) + \sup_{\tau \in T} \mathcal{L}_\tau (\epsilon V) \leq \hat{\rho} + \epsilon(k_0 - h) \leq \hat{\rho} + \epsilon k_0.
\]

We choose $\epsilon$ small enough such that $\hat{\rho} + \epsilon k_0 < \lambda^*$ which in turn, gives
\[
\inf_{\tau \in T} (\mathcal{L}_\tau \hat{u}_\epsilon + g_\tau(x)) < \lambda^*.
\]

Define $\phi_\epsilon = \epsilon V + \hat{u} - u$ and since $u, \hat{u} \in \sigma(V)$, observe that $\phi_\epsilon(x) \rightarrow +\infty$ as $|x| \rightarrow +\infty$. In other words $\phi_\epsilon$ is inf-compact, and so it attains minimum at a point in $\mathbb{R}^d$, say $y_\epsilon$. Denote by $\kappa_\epsilon = (\epsilon V + \hat{u} - u)(y_\epsilon)$ and note that $u + \kappa_\epsilon \leq \hat{u}_\epsilon$. From \eqref{eq1.0} and \eqref{eq3.10} we then obtain
\[
\lambda^* = \inf_{\tau \in T} (\mathcal{L}_\tau u(y_\epsilon) + g_\tau(y_\epsilon)) = \inf_{\tau \in T} (\mathcal{L}_\tau (u + \kappa_\epsilon)(y_\epsilon) + g_\tau(y_\epsilon)) \leq \inf_{\tau \in T} (\mathcal{L}_\tau \hat{u}_\epsilon(y_\epsilon) + g_\tau(y_\epsilon)) < \lambda^*.
\]

But this is a contradiction. Hence we must have $\hat{\rho} = \lambda^*$.

Next we show that $u = \hat{u}$. Denote by $v = \hat{u} - u$. Since $\hat{u}, u$ are classical solutions and $\hat{\rho} = \lambda^*$ we have
\[
\sup_{\tau \in T} \mathcal{L}_\tau v \geq 0 \quad \text{in } \mathbb{R}^d.
\]

Let $K = \{h \leq k_0\} \cup \{V \leq 1\}$. We claim that
\[
\sup_{\mathbb{R}^d} v = \max_K v.
\]

Denote by $\hat{\kappa} = \max_K v$ and $\hat{v} = v - \hat{\kappa}$. It is evident from \eqref{eq3.11} that
\[
\sup_{\tau \in T} \mathcal{L}_\tau \hat{v} \geq 0 \quad \text{in } \mathbb{R}^d.
\]

Let $t_0 = \inf\{t > 0 : \hat{v} < tV \text{ in } K^c\}$.

If $t_0 = 0$ we get the claim \eqref{eq3.12}. For $t_0 > 0$, the argument in Theorem \ref{thm1.2} gives that $t_0 V \geq \hat{v}$ in $\mathbb{R}^d$ and for some point $z \in K^c$, $t_0 V(z) = \hat{v}(z)$. Then
\[
0 \leq \sup_{\tau \in T} \mathcal{L}_\tau \hat{v}(z) \leq t_0 \sup_{\tau \in T} \mathcal{L}_\tau V(z) \leq t_0 (k_0 - h(z)) < 0.
\]

This is a contradiction. Hence we must have $t_0 = 0$, establishing \eqref{eq3.12}. Let $\tilde{z} \in K$ be such that $\sup_{\mathbb{R}^d} v = v(\tilde{z})$. Computing \eqref{eq3.11} at the point $z$ gives us
\[
0 \leq \sup_{\tau \in T} \mathcal{L}_\tau v(\tilde{z}) \leq \lambda \int_{\mathbb{R}^d} \delta(v, \tilde{z}, y) \frac{1}{|y|^{d+2s}} \, dy \leq 0.
\]

Hence $v = \text{constant}$ which gives us $u = \hat{u}$ since $u(0) = 0 = \hat{u}$. This completes the proof. \hfill \square
4. HJB EQUATION WITH MIXED LOCAL-NONLOCAL OPERATORS

In this section we generalize the results of Section 3 to a more general class of integro-differential operator involving both local and nonlocal terms. More precisely, we consider the following operator

\[ Iu(x) = \inf_{\tau \in T} \left[ \text{tr}(a_\tau(x)D^2u) + \tilde{I}_\tau[u](x) + b_\tau(x) \cdot \nabla u + g_\tau \right], \]

where \( T \) is an indexing set, \( a_\tau \) is positive definite matrix and

\[ \tilde{I}_\tau[u](x) = \int_{\mathbb{R}^d} (u(x+y) - u(x) - \nabla u(x) \cdot y 1_{B_1}(y))K_\tau(x,y) \, dy. \]

**Assumption 4.1.** We impose the following standard assumptions on the coefficients.

(i) \( \{a_\tau\}_{\tau \in T} \) is locally bounded, and

\[ \lambda |\xi|^2 \leq \xi a_\tau(x) \cdot \xi \leq \Lambda |\xi|^2 \quad \text{for all } x, \xi \in \mathbb{R}^d, \]

for some \( 0 < \lambda \leq \Lambda. \)

(ii) There exists a measurable \( K : \mathbb{R}^d \setminus \{0\} \to (0, \infty) \), locally bounded in \( \mathbb{R}^d \setminus \{0\} \), such that

\[ 0 \leq K_\tau(x,y) \leq K(y) \quad \text{for all } x \in \mathbb{R}^d, y \in \mathbb{R}^d \setminus \{0\}, \tau \in T, \]

and

\[ \int_{\mathbb{R}^d} \min\{|y|^2,1\}K(y) \, dy < \infty. \]

(iii) For every compact set \( K \), there exists \( \alpha (0, 1) \) such that

\[ \sup_{\tau \in T} \|a_\tau(x) - a_\tau(x')\| + \sup_{\tau \in T} \|b_\tau(x) - b_\tau(x')\| + \sup_{\tau \in T} \|g_\tau(x) - g_\tau(x')\| \leq C_K|x - x'|^\alpha, \]

and

\[ \sup_{\tau \in T} |K_\tau(x,y) - K_\tau(x',y)| \leq C_K|x - x'|^\alpha K(y) \quad y \in \mathbb{R}^d \setminus \{0\}, \]

for all \( x, x' \in K \).

Note that Assumption (4.1)(i) is the non-degeneracy condition and Assumption (4.1)(iii) is normally used to establish \( C^{2,\kappa} \) regularity. As before, we also need Foster-Lyapunov type condition which we state below. Denote by \( \tilde{K}(y) = 1_{B_1^c}(y)K(y) \).

**Assumption 4.2.** The following hold.

(i) There exist \( V \in C^2(\mathbb{R}^d), V \geq 0 \) and a function \( 0 \leq h \in C(\mathbb{R}^d) \), \( V \) and \( h \) are inf-compact, such that

\[ \sup_{\tau \in T} \left[ \text{tr}(a_\tau(x)D^2V) + \tilde{I}_\tau[V](x) + b_\tau(x) \cdot \nabla V \right] \leq k_0 - h(x) \quad x \in \mathbb{R}^d, \tag{4.1} \]

for some \( k_0 > 0 \).

(ii) \( \sup_{\tau \in T} |g_\tau(x)| \leq h(x) \) and \( \sup_{\tau \in T} |g_\tau| \in o(h) \).

(iii) For some \( \mu \geq 0 \) we have \( V^{1+\mu} \in L^1(\tilde{K}) \) and

\[ \sup_{\tau \in T} \frac{|g_\tau|}{(1 + V)^{1+2\mu}} \leq C, \tag{4.2} \]

\[ \sup_{x,y} \frac{V(x+y)}{(1 + V(x))(1 + V(y))} + \limsup_{|x| \to \infty} \frac{1}{1 + V(x)} \sup_{|y-x| \leq 1} V(y) \leq C. \tag{4.3} \]

Moreover, one of the following holds.
(a) There exists a kernel $\mathcal{J} : \mathbb{R}^d \setminus \{0\} \to (0, \infty)$, locally bounded in $\mathbb{R}^d \setminus \{0\}$, such that

$$r^{d+2}K(ry) \leq \mathcal{J}(y) \quad \text{for all} \quad r \in (0,1], \; y \in \mathbb{R}^d \setminus \{0\}, \quad \int_{\mathbb{R}^d} (|y|^2 \wedge 1) \mathcal{J}(y) \, dy < \infty, \quad (4.4)$$

and

$$\sup_{\tau \in \mathcal{T}} \frac{|b_{\tau}|}{(1+V)^{\mu}} \leq C, \quad (4.5)$$

where $\mu$ is same as above.

(b) $\sup_{\tau \in \mathcal{T}} \|b_{\tau}\|_{L^\infty(\mathbb{R}^d)} < \infty$.

In the spirit of Theorem 1.1, we can complete the proof of Theorem 1.3

**Proof of Theorem 1.3.** Since the proof is analogous to the proof of Theorem 1.1, we only provide the outline.

**α-discounted problem:** Applying [26, Theorem 5.7] we find a viscosity solution $W_n$ given by

$$\mathcal{I}W_n - \alpha W_n = 0 \quad \text{in} \quad B_n, \quad \text{and} \quad W_n = 0 \quad \text{in} \quad B^c_n,$$

where $\alpha \in (0,1)$. Now by Assumption 4.2 and the proof of Lemma 2.4 we get $|W_n| \leq \frac{k_0}{\alpha} + V$. Given $R' > 0$ we choose $n_0$ large enough so that $R' + 4 \geq n_0$. Then using the notations of Lemma 2.4 we have

$$\inf_{\tau \in \mathcal{T}} \left( \text{tr}(a_{\tau}(x)D^2\tilde{\psi}_n) + \tilde{I}_{\tau}[\tilde{\psi}_n] + b_{\tau} \cdot \tilde{\psi}_n + g_{\tau} + \tilde{I}_{\tau}[(1-\chi)W_n] \right) = 0 \quad \text{in} \quad B_{R'+2},$$

where $\tilde{\psi}_n = \chi W_n$. Using Assumption 4.1(ii) and (4.3) it can be easily checked that

$$\sup_{\tau \in B_{R'+1}} \sup_{\tau \in \mathcal{T}} |\tilde{I}_{\tau}[(1-\chi)W_n]| < \infty.$$

Thus, by [26, Theorem 4.2], $\{W_n\}_{n \geq n_0}$ is $\kappa$-Hölder continuous in $B_{R'}$, uniformly in $n$. Therefore, we can repeat the proof of Lemma 2.4 and find a subsequence $W_{n_k} \to w_\alpha \in \mathfrak{a}(V)$ such that

$$\mathcal{I}w_\alpha - \alpha w_\alpha = 0 \quad \text{in} \quad \mathbb{R}^d. \quad (4.6)$$

The unique solution of (1.8): As before, we denote by $w_\alpha(x) = u_\alpha(x) - w_\alpha(0)$. The argument of Lemma 3.1 goes through. Therefore, we can repeat the proof of Theorem 3.2 to obtain that, along some subsequence $\alpha_n \to 0$, we have $w_\alpha \to u$, $\alpha_n w_{\alpha_n}(0) \to \lambda^*$ and (passing limit in (4.6)

$$\mathcal{I}u - \lambda^* = 0 \quad \text{in} \quad \mathbb{R}^d, \quad u(0) = 0. \quad (4.7)$$

**Proof of Theorem 3.2** also gives us $u \in \mathfrak{a}(V)$. Next we argue that $u \in C_{\text{loc}}^{2+}(\mathbb{R}^d)$. To attain this goal we need an estimate analogous to (2.21). We use the notations of Lemma 2.5. Fix $x_0 \in \mathbb{R}^d$ and let $r = [1 + V(x_0)]^{-1'}, \; v(x) = u(x_0 + rx)$ and $\psi = \xi v$. Also, define

$$K_{\tau}^r = r^{d+2}K(x_0 + rx, ry), \quad \text{and} \quad \tilde{I}_{\tau}^r \xi(x) = \int_{\mathbb{R}^d} (\xi(x + y) - \xi(x) - \nabla \xi(x) \cdot y_1_{B_{\frac{1}{r}}}(y))K_{\tau}^r(x, y) \, dy.$$

It is easy to check from (4.7) that

$$\inf_{\tau \in \mathcal{T}} \left[ \text{tr}(a_{\tau}(x_0 + rx)D^2\psi) + \tilde{I}_{\tau}^r[\psi] + rb_{\tau}(x_0 + rx) \cdot \nabla \psi + r^2g_{\tau}(x_0 + rx) + \tilde{I}_{\tau}^r[(1-\xi)v] \right] - \lambda^* = 0 \quad \text{in} \quad \mathbb{R}^d. \quad (4.8)$$

Recall that $|v(x)| \leq C(1 + V(x_0 + rx))$. Therefore, for $x \in B_1$,

$$\sup_{\tau \in \mathcal{T}} |\tilde{I}_{\tau}^r[(1-\xi)v]| \leq Cr^{d+2} \int_{|y| \geq 1/2} (1 + V(x_0 + rx + ry))K(ry) \, dy$$

$$= Cr^2 \int_{|y| \geq 1/2} (1 + V(x_0 + rx + y))K(y) \, dy$$
\[ \leq Cr^2(1 + V(x_0)) \int_{|y| \leq 1} K(y) \, dy + C(1 + V(x_0)) \int_{\mathbb{R}^d} (1 + V(y)) \tilde{K}(y) \, dy \]

\[ \leq C(1 + V(x_0)) \left[ 1 + \int_{|y| \leq 1} |y|^2 K(y) \, dy + \|V\|_{L^1(K)} \right] \]

for some constant \( C \) independent of \( x_0, r \), where in the third line we use \((4.3)\). Under Assumption \(4.2(iii)(a)\), we can apply \([26, \text{Theorem 4.2}]\) on \((4.8)\) to obtain \( \eta \in (0, 1) \) satisfying

\[ |u(x_0) - u(x_0 + rx)| \leq C(1 + V(x_0))|x|^\eta \quad \text{for all } |x| \leq \frac{1}{2}, \quad (4.9) \]

and for all \( x_0 \in \mathbb{R}^d \). For Assumption \(4.2(iii)(b)\), we apply \([28, \text{Lemma 2.1}]\), to obtain \((4.9)\). Now consider \( |x| \leq 1 \). If \( |x| \leq r \), then \((4.9)\) gives

\[ |u(x_0) - u(x_0 + x)| \leq C(1 + V(x_0))|x|^\eta. \]

Now suppose, \( |x| > r \) and let \( e = \frac{x}{|x|}, k = \lfloor \frac{2|x|}{r} + \frac{1}{2} \rfloor \). Note that

\[ k \leq \frac{2|x|}{r} + \frac{1}{2} \leq k + 1 \Rightarrow \left| \frac{2|x|}{r} - k \right| \leq \frac{1}{2}. \]

Using \((4.3)\) and \((4.9)\) we have

\[ |u(x_0) - u(x_0 + x)| \leq \sum_{i=1}^{k} |u(x_0 + \frac{r}{2}(i - 1)e) - u(x_0 + \frac{r}{2}ie)| + |u(x_0 + \frac{r}{2}ke) - u(x_0 + x)| \]

\[ \leq C(1 + V(x_0)) \frac{k + 1}{2^\eta} \leq C_1(1 + V(x_0)) \frac{|x|}{r} \leq C_1(1 + V(x_0))^{1+\mu}|x|^\eta. \]

Combining the above estimate we get

\[ |u(x_0) - u(x_0 + x)| \leq C(1 + V(x_0))^{1+\mu}|x|^\eta \quad \text{for all } |x| \leq 1, \text{ and } x_0 \in \mathbb{R}^d. \]

Now using Assumption \(4.1(iii)\), \([28, \text{Theorem 5.3}]\) and the proof of Lemma \(2.5\) we see that \( u \in C_{\text{loc}}^2(\mathbb{R}^d) \). Hence \( u \) is a classical solution to \((4.7)\).

To complete the proof we only need to show uniqueness. Suppose that \((\hat{u}, \hat{\rho}) \in \mathfrak{o}(V) \times \mathbb{R} \) is a solution to \((4.7)\). From the above argument we have \( \hat{u} \in C_{\text{loc}}^2(\mathbb{R}^d) \). Therefore, the arguments in Theorem \(1.1\) gives that \( \hat{\rho} = \lambda^* \). The equality of \( u = \hat{u} \) can be established using a similar argument, provided we could show that for \( v = u - \hat{u} \), sup_{\mathbb{R}^d} v = v(\hat{z}) \) for some \( \hat{z} \in \mathbb{R}^d \), implies \( v = 0 \). We note from \((1.8)\) that

\[ \inf_{T \in T} \left[ \text{tr}(a_r(x)D^2(-v)) + T_r[-v](x) + b_r(x) \cdot \nabla(-v) \right] \leq \mathcal{I} \hat{u} - \mathcal{I} u = 0. \]

Letting \( \hat{v} = v(\hat{z}) - v \), we obtain from above that

\[ \inf_{T \in T} \left[ \text{tr}(a_r(x)D^2\hat{v}) + T_r[\hat{v}](x) + b_r(x) \cdot \nabla\hat{v} \right] \leq 0 \quad \text{in } \mathbb{R}^d. \]

Thus \( \hat{v} \) a non-negative super-solution. Fix any bounded domain \( \Omega \). Choose \( M \) large enough, so that \( \hat{v} < M \) in \( \Omega \). Defining \( v_M = \hat{v} \land M \) we get from above that

\[ \inf_{T \in T} \left[ \text{tr}(a_r(x)D^2v_M) + T_r[v_M](x) + b_r(x) \cdot \nabla v_M \right] \leq 0 \quad \text{in } \Omega, \]

and \( v_M(\hat{z}) = 0 \). From the weak Harnack principle \([26, \text{Theorem 3.12}]\), it then follows that \( v_M = 0 \) in \( \Omega \). Since \( \Omega, M \) are arbitrary, we have \( v = 0 \) in \( \mathbb{R}^d \). This completes the proof. \( \square \)
5. $C^{1,\gamma}$ REGULARITY

The main goal of this section is to establish $C^{1,\gamma}$ regularity of the viscosity solutions to the equation

$$
Au(x) := \inf_{\tau \in T} \sup_{t \in \mathcal{I}} \left[ I_{\tau t}(u(x) + b_{\tau t}(x) \cdot \nabla u(x) + g_{\tau t}(x)) \right] = 0,
$$

where $T, \mathcal{I}$ are some index sets and

$$
I_{\tau t}[u](x) = \int_{\mathbb{R}^d} \delta(u, x, y) \frac{k_{\tau t}(x, y)}{|y|^{d+2s}} dy.
$$

Note that (5.1) is more general than the one used in Theorem 2.2. For each $\tau \in T, t \in \mathcal{I}$, $k_{\tau t}$ is symmetric in $y$, that is, $k_{\tau t}(x, y) = k_{\tau t}(x, -y)$ and

$$(2 - 2s)\lambda \leq k_{\tau t}(x, y) \leq (2 - 2s)\Lambda \quad \text{for all } x, y, \quad 0 < \lambda \leq \Lambda.$$ 

By $\mathcal{L}_0(s)$ we denote the class of all kernels $k$ satisfying the above relation.

Let us define the Pucci extremal operators. The maximal operators, with respect to the class $\mathcal{L}_0(s)$, are defined as follows.

$$
M^+ u(x) = \sup_{I \in \mathcal{L}_0(s)} Iu(x) = (2 - 2s) \int_{\mathbb{R}^d} \frac{\Lambda \delta^+(u, x, y) - \Lambda \delta^-(u, x, y)}{|y|^{d+2s}},
$$

$$
M^- u(x) = \inf_{I \in \mathcal{L}_0(s)} Iu(x) = (2 - 2s) \int_{\mathbb{R}^d} \frac{\lambda \delta^+(u, x, y) - \lambda \delta^-(u, x, y)}{|y|^{d+2s}}.
$$

We impose the following assumptions on the coefficients.

(H1) $b_{\tau t}, g_{\tau t}$ are continuous and

$$
\sup_{\tau \in T, t \in \mathcal{I}} \|b_{\tau t}\|_{L^\infty} < \infty \quad \text{and} \quad \sup_{\tau \in T, t \in \mathcal{I}} \|g_{\tau t}\|_{L^\infty} < \infty.
$$

(H2) The map $x \mapsto k_{\tau t}(x, y)$ is uniformly continuous, uniformly in $\tau, t$ and $y$, that is,

$$(k_{\tau t}(x_1, y) - k_{\tau t}(x_2, y)) \leq \phi(|x_1 - x_2|) \quad \forall x_1, x_2 \in \mathbb{R}^d, \tau \in T, t \in \mathcal{I}, y \in \mathbb{R}^d,$$

where $\phi$ is the modulus of continuity.

As before, we denote by $\omega_s(y)$ the weight function $[1 + |y|^{d+2s}]^{-1}$. Let us define the following collection of test functions which will be used to define a norm on $I$: Let $M > 0, \Omega$ be a fixed domain and $x \in \Omega$. Let us consider the following set of functions:

$$
\mathcal{D}^x_M := \{ \phi \in L^1(\mathbb{R}^d, \omega) | \phi \in C^2(x), x \in \Omega, \|\phi\|_{L^1(\omega)} \leq M, \quad \text{and} \quad \|\phi - \phi(x) - (y - x) \cdot \nabla \phi(x)\| \leq M|y - x|^2, \forall y \in B_1(x) \},
$$

where $\omega$ is a weight function and $\phi \in C^2(x)$ means that there is a quadratic polynomial $q$ such that $\phi(y) = q(y) + o(|x - y|^2)$ for $y$ sufficiently close to $x$.

Let $\Omega = B_N$ be a ball of fixed radius $N$, now observe that if $\phi \in \mathcal{D}^x_M$, and $\omega_s(y) = (1 + |y|^{d+2s})^{-1}$ is a particular weight function, then

$$
\int_{B_1(x)} |\phi(y) - \phi(x) - (y - x) \cdot \nabla \phi(x)| dy \leq M \int_{B_1(x)} |x - y|^2 dy \leq M|B_1|
$$

$$
\Rightarrow \left| \int_{B_1(x)} \phi(y) dy - \int_{B_1(x)} \phi(x) dy - \int_{B_1(x)} (y - x) \cdot \nabla \phi(x) dy \right| \leq M|B_1|
$$

$$
\Rightarrow |\int_{B_1(x)} \phi(y) dy - \phi(x)|B_1| \leq M|B_1| \Rightarrow |\phi(x)|B_1| \leq M|B_1| + \int_{B_1(x)} \frac{|\phi(y)|(1 + |y|^{d+2s})}{1 + |y|^{d+2s}} dy
$$

$$
\Rightarrow |\phi(x)|B_1| \leq M|B_1| + M(1 + (1 + N)^{d+2s}) \Rightarrow |\phi(x)| \leq MC_{d,s,\Omega}.
$$

(5.2)
It is noteworthy that the right hand side of the last inequality does not depend on any particular choice of $\phi$ or $x$. In other words, we get the same bound as long as $x \in \Omega$ and $\phi \in \mathcal{D}_M^{x}$ for a fixed $M > 0$.

**Definition 5.1.** Let $\Omega$ be a domain and $\omega$ be a weight function, then for any non local operator $I$, the norm $\|I\|_{\omega}$ with respect to the weight function $\omega$ is defined as follows:

$$\|I\|_{\omega} = \sup_{x,M} \left\{ \frac{I[\phi](x)}{1 + M} \right\} \{\phi \in \mathcal{D}_M^{x} \text{, } x \in \Omega \text{, } M > 0\}.$$  \hspace{1cm} (5.3)

Our main result of this section is the following $C^{1,\gamma}$ regularity estimate of the solutions to (5.1).

**Theorem 5.1.** Let $s \in (1/2, 1)$. Also, \((H1)-(H2)\) hold in $B_1$ and $\sup_{r,t} \|b_{r,t}\|_{L^{\infty}(B_1)} \leq C_0$.

Then there exists a $\gamma \in (0, 2s - 1)$ such that for any bounded viscosity solution $u \in C(\mathbb{R}^d)$ to (5.1) in $B_1$ we have

$$\|u\|_{C^{1,\gamma}(B^{1/2}_1)} \leq C \left( \|u\|_{L^{\infty}(\mathbb{R}^d)} + \sup_{r,t} \|g_{r,t}\|_{L^{\infty}(B_1)} \right),$$

where the constant $C$ depends on $d, s, C_0, \rho, \lambda, \Lambda$.

We mainly follow the ideas in \cite{30} and \cite{31} for the proof of Theorem 5.1.

**Lemma 5.2.** Let $s > \frac{1}{2}$ and $I$ be an integro-differential operator, elliptic with respect to the class $\mathcal{L}_0(s)$, in particular an operator of the form $I_{\tau_r}$ in (5.1). Given $x_0 \in \mathbb{R}^d, r > 0, c > 0$, and $l(x) = a \cdot x + b$, define $\tilde{I}$ by

$$\tilde{I} \left( \frac{w(x_0 + r \cdot) - l(x_0 + r \cdot)}{c} \right) (x) = \frac{r^{2s}}{c} I(cw)(x_0 + rx)$$

Then $\tilde{I}$ is elliptic with respect to $\mathcal{L}_0(s)$ with the same ellipticity constants.

**Proof.** To see this, let us introduce the notation $\tau_r(x) = x_0 + r \cdot x$. Now from the definition of $\tilde{I}$ we observe that

$$\tilde{I}((w - l) \circ \tau_r)(x) = \frac{r^{2s}}{c} I(cw)(\tau_r(x))$$

$$\Rightarrow \tilde{I}(w)(x) = I(w \circ \tau_r^{-1} \circ \tau_r)(x) = \frac{r^{2s}}{c} I(c(w + l) \circ \tau_r^{-1})(\tau_r(x))$$

Now it is easy to see that

$$\tilde{I}(u)(x) - \tilde{I}(v)(x) \leq \frac{r^{2s}}{c} \left[ I(c(u + l) \circ \tau_r^{-1})(\tau_r(x)) - I(c(v + l) \circ \tau_r^{-1})(\tau_r(x)) \right]$$

$$\leq \frac{r^{2s}}{c} M^+(c(u - v) \circ \tau_r^{-1})(\tau_r(x)) = M^+(u - v)(x),$$

using the fact that the extremal operators $M^+$ and $M^-$ are translation invariant. Hence the proof.

Proof of the following result can be found in \cite{30} Lemma 4.3.

**Lemma 5.3.** Let $s > \frac{1}{2}, \beta \in (1, 2s)$ and define the for any $z \in \mathbb{R}^d$ and $r > 0$ the following affine function

$$l_{r,z}(x) = a^* \cdot (x - z) + b^*,$$
where
\[ a_i^* = \frac{\int_{B_r(z)} u(x)(x_i - z_i) \, dx}{\int_{B_r(z)} (x_i - z_i)^2 \, dx} \quad \text{and} \quad b^*(r, z) = \frac{\int_{B_r(z)} u(x) \, dx}{B_r(z)} \],

equivalently,
\[ (a^*, b^*) = \arg\min_{(a, b) \in \mathbb{R}^d \times \mathbb{R}} \int_{B_r(z)} (u(x) - a(x - z) + b)^2 \, dx. \]

If for some constant \( C_0 \) we have
\[ \sup_{r > 0} \sup_{z \in B_{\frac{1}{2}}} \tau^\beta \| u - I_{r, z} \|_{L^\infty(B_r(z))} \leq C_0, \]

then
\[ \| u \|_{C^\beta(B_{\frac{1}{2}})} \leq C(\| u \|_{L^\infty(\mathbb{R}^d)} + C_0), \]

where \( C \) depends on the exponent \( \beta \).

For our next result we need to introduce a class of scaled operators. For \( m \in \mathbb{N} \), let \( z_m \in B_{\frac{1}{2}} \) and
\[ \hat{I}^m[w](x) := \inf_{\tau \in \mathbb{T}_m} \sup_{r \in \mathbb{T}_m} \int_{\mathbb{R}^d} \delta(\phi, x, y) \frac{k_{r, \tau}(z_m + r_m x, r_m y)}{|y|^{d+2s}} \, dy, \tag{5.4} \]

where \( r_m > 0 \) and \( k_{r, \tau} \in L_0(s) \) for all \( \tau \in T_m, r \in T_m \). Now we recall the weak convergence of operators from [15]. A sequence of operators \( I_m \) is said to be weakly convergent to \( I \) (with respect to a weight function \( \omega \)), if for every \( \varepsilon > 0 \) small and test function \( \phi \) a point \( x_0 \in \Omega \), where \( \phi \) is a quadratic polynomial in \( B_{\varepsilon}(x_0) \) and \( \phi \in L^1(\omega) \), we have
\[ I_m[\phi](x) \to I[\phi](x) \quad \text{uniformly in } B_{\varepsilon/2}(x_0). \]

We need the following result on weak convergence.

**Lemma 5.4.** Let \( z_m \to z_0 \) and \( r_m \to 0 \) as \( m \to \infty \). Let \( \hat{I}^m \) be defined as above and the family \( \{k_{r, \tau} : \tau \in T_m, r \in T_m\} \) satisfy \( [H2] \) with the same modulus of continuity \( \delta \). Then, there exists a subsequence \( \hat{I}^{m_k} \) converges weakly to a translation invariant operator \( I_0 \) with \( I_0(0) = 0 \) where \( I_0 \) is elliptic with respect to the class \( L_0(s) \).

**Proof.** To prove the lemma, let us first define a translation invariant operator \( \hat{I}^m \) as follows
\[ \hat{I}^m(z_0)[\phi](x) = \inf_{\tau \in \mathbb{T}_m} \sup_{r \in \mathbb{T}_m} \int_{\mathbb{R}^d} \delta(\phi, x, y) \frac{k_{r, \tau}(z_0 + r_m x, r_m y)}{|y|^{d+2s}} \, dy. \]

We claim that for every bounded domain \( \Omega \) we have
\[ \| \hat{I}^m - \hat{I}^{m_k} \|_{L^\infty(\mathbb{R}^d)} \to 0, \tag{5.5} \]

where \( \| \cdot \|_{L^\infty(\mathbb{R}^d)} \) is given by [11]. To prove the convergence, consider a test function \( \phi \in D^M_x \), defined on \( \Omega \), and observe from [12] that
\[ |\hat{I}^m[\phi](x) - \hat{I}^{m_k}[\phi](x)| = \sup_{(r, \tau) \in T_m \times T_m} |\hat{I}_{r, \tau}[\phi](x) - \hat{I}^{m_k}_{r, \tau}[\phi](x)|. \]

Next we carefully calculate the second integral. Using [12] we see that
\[ \int_{B_1} \delta(\phi, x, y) \, dy \leq \int_{B_1} |\phi(x+y) + \phi(x-y) - 2\phi(x)| \, dy. \]
\[ = 2M \rho(|z_m + r_m x|) \int_{B_1} |y|^{d+2s} \, dy + \rho(z_m + r_m x) \int_{B_1} |\delta(\phi, x, y)| \, dy. \]
Thus, it follows from (5.3) that
\[
\left\| \tilde{I}^m - \hat{I}^m \right\|_{L^\Theta} \leq C_{s, \Omega} \sup_{x \in \Omega} \rho(z_m + r_m x - z - 0),
\]
which gives us (5.5). Again, by [15, Theorem 42], there exist a subsequence \( \tilde{I}^{m_k} \) converges weakly to some \( I_0 \). Combining with (5.5) it is easily seen that \( \tilde{I}^{m_k} \) converges weakly to \( I_0 \). It is also evident that \( I_0(0) = 0 \) and \( I_0 \) is elliptic with respect to the class \( \mathcal{L}_0(s) \) (cf. [31, Lemma 4.1]).

Now we can complete the proof of Theorem 5.1 with the help of Lemmas 5.2, 5.3 and 5.4.

**Proof of Theorem 5.1.** We prove the theorem by the method of contradiction. First we fix the choice of \( \gamma \). Let \( \gamma_0 \) be the Hölder exponent obtained with respect to the Pucci operators \( M^\pm \) in [14, Theorem 12.1] (see also [31, Theorem 2.1]). Fix \( \gamma \in (0, \min\{2s - 1, \gamma_0\}) \).

Now suppose that there exist \( I_k, u_k, b_k^r \) and \( \{g_{\tau r}\}_k \) satisfying
\[
\sup_{\tau, r} |b_k^r| \leq C_0 \quad \text{and} \quad \|u_k\|_{L^\infty(\mathbb{R}^d)} + \sup_{\tau, r} \left\| g_{\tau r}^k \right\|_{L^\infty(B_{1/2})} = 1,
\]
but \( \|u_k\|_{C^{\beta}(B_{1/2})} \xrightarrow{k \to \infty} +\infty \), where \( \beta = 1 + \gamma \).

In view of Lemma 5.3 there exist \( a^*(k, r, z) \) and \( b^*(k, r, z) \) such that
\[
\sup_{k, r > 0} \sup_{B_{1/2}} r^{-\beta} \|u_k - l_{k, r, z}\|_{L^\infty(B_r(z))} = +\infty,
\]
where
\[
(a^*(k, r, z), b^*(k, r, z)) = \arg\min_{(a, b) \in \mathbb{R}^d \times \mathbb{R}} \int_{B_r(z)} \left( u_k(x) - a \cdot (x - z) + b \right)^2 dx
\]
and \( l_{k, r, z}(x) = a^*(k, r, z) \cdot (x - z) + b^*(k, r, z) \).

Now for any \( r > 0 \), define \( \Theta \) as follows:
\[
\Theta(r) := \sup_k \sup_{r' \geq r} \sup_{z \in B_{1/2}} (r')^{-\beta} \|u_k - l_{k, r', z}\|_{L^\infty(B_{r'}(z))}.
\]

Since \( \|u_k\|_{L^\infty(\mathbb{R}^d)} \) is finite, we see that \( \Theta(r) < \infty \) for all \( r > 0 \), and therefore, (5.7) is well-defined. Furthermore, from (5.6) we observe that for any \( M > 0 \), however large, there exists \( \tilde{r} > 0, \tilde{k} \in \mathbb{N} \) and \( \tilde{z} \in B_{1/2} \) such that
\[
(\tilde{r})^{-\beta} \|u_{\tilde{k}} - l_{\tilde{k}, \tilde{r}, \tilde{z}}\|_{L^\infty(B_{\tilde{r}}(\tilde{z}))} > M.
\]

Therefore, since \( \Theta(r) \geq \Theta(\tilde{r}) > M \) for any \( 0 < r \leq \tilde{r} \), we get that \( \Theta(r) \uparrow \infty \) as \( r \downarrow 0 \). As \( \Theta(1/m) \uparrow +\infty \) with \( m \uparrow +\infty \), there exists \( r_m \geq \frac{1}{m}, k_m \in \mathbb{N} \) and \( z_m \in B_{1/2} \) such that
\[
\frac{1}{2} \Theta(r_m) \leq \frac{1}{2} \Theta(1/m) < (r_m)^{-\beta} \|u_{k_m} - l_{k_m, r_m, z_m}\|_{L^\infty(B_{r_m}(z_m))}.
\]
It is easily seen that \( r_m \) converges to zero. With this, let us define a new function \( v_m \) as follows
\[
v_m(x) := \left( \frac{u_{k_m} - l_{k_m, r_m, z_m}}{r_m^\beta \Theta(r_m)} \right) (z_m + r_m x).
\]
It is easy to see from the condition of minimality that
\[
\int_{B_1} v_m \, dx = 0; \quad \int_{B_1} v_m x_i \, dx = 0; \quad \text{and} \quad \|v_m\|_{L^\infty(B_1)} \geq 1/2.
\]
We next claim that
\[ ||v_m||_{L^\infty(B_{Rr})} \leq CR^\beta, \quad \forall R \geq 1. \] (5.10)

To prove the above growth bound, we first observe that for any \( z \in B_{\frac{1}{2}^r} \), \( k \in \mathbb{N} \) and \( r' \geq r > 0 \)
\[ ||u_k - l_{k,r',z}||_{L^\infty(B_{2r'}(z))} \leq (r')^\beta \Theta(r') \quad \text{and} \quad ||u_k - l_{k,2r',z}||_{L^\infty(B_{2r'}(z))} \leq (2r')^\beta \Theta(2r'). \]

Now by the monotonicity property of \( \Theta \), we conclude for any \( r' \geq r > 0 \)
\[ ||l_{k,2r',z} - l_{k,r',z}||_{L^\infty(B_r(z))} \leq (2r')^\beta \Theta(2r') + (r')^\beta \Theta(r') \leq (2^\beta + 1)(r')^\beta \Theta(r). \]

This implies that
\[ \sup_{k,z} |b^*(k, 2r', z) - b(k, r', z)| \leq C(\Theta(r))(r')^\beta \quad \text{and} \quad \sup_{k,z} |a^*(k, 2r', z) - a^*(k, r', z)| \leq C(\Theta(r))(r')^{\beta - 1}. \]

Now if we take \( R = 2^N, N \geq 1 \) and \( r_j = 2^j r \geq r, j = 1, \ldots, N \), in the above inequalities, then
\[
\frac{||a^*(k, Rr, z) - a^*(k, r, z)||}{r^{\beta - 1} \Theta(r)} \leq \sum_{j=1}^{N} \frac{||a^*(k, r_j, z) - a^*(k, r_{j-1}, z)||}{r^{\beta - 1} \Theta(r)} = \sum_{j=1}^{N} 2^{(\beta - 1)(j-1)} \frac{|a^*(k, r_j, z) - a^*(k, r_{j-1}, z)|}{2^{(\beta - 1)(j-1)} r^{\beta - 1} \Theta(r)} \leq C \sum_{j=1}^{N} 2^{(\beta - 1)(j-1)} \leq C \left( \frac{2^{\beta - 1} - 1}{2^{\beta - 1} - 1} \right) \leq C R^{\beta - 1}. \] (5.11)

Similarly, we can also prove that
\[
\frac{||b^*(k, Rr, z) - b^*(k, r, z)||}{r^{\beta} \Theta(r)} \leq C R^\beta. \] (5.12)

Now, observe that
\[ ||v_m||_{L^\infty(B_R)} \leq \frac{||u_{km} - l_{km,r_m,z_m}||_{L^\infty(B_{Rr_m}(z_m))}}{r_m^{\beta} \Theta(r_m)} + \frac{||l_{km,r_m,z_m} - l_{km,r_m,z_m}||_{L^\infty(B_{Rr_m}(z_m))}}{r_m^{\beta} \Theta(r_m)}. \]

Moreover, for any \( x \in B_{Rr_m}(z_m) \)
\[ |l_{km,r_m,z_m} - l_{km,r_m,z_m}| \leq |a^*(k_m, Rr_m, z_m) - a^*(k_m, r_m, z_m)| ||x - z_m|| + |b^*(k_m, Rr_m, z_m) - b^*(k_m, r_m, z_m)|| \leq Rr_m |a^*(k_m, Rr_m, z_m) - a^*(k_m, r_m, z_m)| + |b^*(k_m, Rr_m, z_m) - b^*(k_m, r_m, z_m)|. \]

By using (5.11) and (5.12) we have
\[
\frac{||l_{km,r_m,z_m} - l_{km,r_m,z_m}||_{L^\infty(B_{Rr_m}(z_m))}}{r_m^{\beta} \Theta(r_m)} \leq R \frac{|a^*(k_m, Rr_m, z_m) - a^*(k_m, r_m, z_m)|}{r_m^{\beta - 1} \Theta(r_m)} + \frac{|b^*(k_m, Rr_m, z_m) - b^*(k_m, r_m, z_m)|}{r_m^{\beta} \Theta(r_m)} \leq C(\beta) R^\beta + C(\beta) R^\beta. \] (5.13)

Ultimately, using the monotonicity property of \( \Theta \) and (5.13), we obtain
\[ ||v_m||_{L^\infty(B_R)} \leq (1 + 2C(\beta)) R^\beta. \] (5.14)
Therefore, as which in turn, implies that for small $r_m$ such that

$$\frac{\partial a_m^*}{\partial r_m} < 0$$

We first claim that

$$|a_m^*| \leq C(1 + \Theta(r_m)).$$

To see this, choose $l_m \in \mathbb{N}$ such that $2^{-l_m} \leq r_m < 2^{-(l_m-1)}$ and observe

$$|a^*(k_m, r_m, z_m) - a^*(k_m, 2^{(l_m-1)}r_m, z_m)| \leq \sum_{j=1}^{l_m} |a^*(k_m, 2^jr_m, z_m) - a^*(k_m, 2^{(j-1)}r_m, z_m)|$$

$$\leq \sum_{j=1}^{l_m} (2^{(j-1)}r_m)^{\beta-1} \left| \frac{a^*(k_m, 2^jr_m, z_m) - a^*(k_m, 2^{(j-1)}r_m, z_m)}{(2^{(j-1)}r_m)^{\beta-1}} \right|$$

$$\leq \Theta(r_m) \left( r_m 2^{(l_m-1)\beta} \right) \leq \Theta(r_m) \left( \frac{2\beta-1}{2^{\beta-1} - 1} \right).$$

Again, from the definition and monotonicity of $\Theta$

$$|u_{km}(x) - l_{km, 2^{(l_m-1)}r_m, z_m}(x)| \leq \Theta(2^{(l_m-1)}r_m)(2^{l_m-1}r_m)^{\beta} \leq \Theta(1/2)(2^{l_m-1}r_m)^{\beta} \forall x \in B_{2^{l_m-1}r_m}(z_m)$$

$$\implies |u_{km}(z_m) - b^*(k_m, 2^{(l_m-1)}r_m, z_m)| \leq \Theta(1/2) \left( 2^{l_m-1}r_m \right)^{\beta}, \text{ by substituting } x = z_m$$

which in turn, implies that for small $r_m$

$$|a^*(k_m, 2^{(l_m-1)}r_m, z_m)| \leq C\Theta(\frac{1}{2}).$$

So we have our desired result

$$|a^*(k_m, r_m, z_m)| \leq |a^*(k_m, r_m, z_m) - a^*(k_m, 2^{(l_m-1)}r_m, z_m)| + |a^*(k_m, 2^{(l_m-1)}r_m, z_m)| \leq C(\Theta(r_m) + 1).$$

Therefore, as $b_{r_m}, g_{r_m}$ are uniformly bounded, we have

$$r_m^{2\beta-\beta}\frac{\partial a_m^*}{\partial r_m} + \frac{g_{r_m}^*}{\Theta(r_m)} \rightarrow 0.$$  \hspace{1cm} (5.16)

Applying $\Theta$ Theorem 7.2 it then follows from (5.14)-(5.15), that the family $\{v_m : m \geq 1\}$ is locally Hölder continuous, uniformly in $m$. By the Arzela-Ascoli theorem we can extract a convergent subsequence of $v_m$. Let $v_m \rightarrow v \in C(\mathbb{R}^d)$ along some subsequence, as $m \rightarrow \infty$. It is also evident from (5.14) that

$$\|v\|_{L^\infty(\Omega)} \leq C(1 + R^3).$$

We claim that there exists a translation invariant operator $I_0$, elliptic with respect to $L_0(s)$, such that

$$I_0(v) = 0 \text{ in } \mathbb{R}^d.$$  \hspace{1cm} (5.17)

Once we have established (5.17), it then follows from $\Theta$ Theorem 3.1 that $v(x) = a \cdot x + b$. Passing the limit in the first two equations of (5.7) gives $a = 0, b = 0$ but it contradicts the third estimate in (5.9) that requires $\|v\|_{L^\infty(B_1)} \geq 1/2$. Hence we have a contradiction to (5.6).
Thus we remain to show \((5.17)\). Choosing a further subsequence, if required, we may assume that \(z_m \to z_0\) and \(v_m \to v\) uniformly on compacts. Recall the operator \(\tilde{I}^m\) from \((5.4)\). Now observe that for any bounded domain \(\Omega\) we have

\[
\tilde{I}^m v_m - r_m^{2s-1} K_0 |\nabla v_m| - r_m^{2s-\beta} |b^k_m \cdot a^m_k| + |g^m_k| \Theta(r_m) \leq 0 \quad \text{in} \quad \Omega,
\]

\[
\tilde{I}^m + r_m^{2s-1} K_0 |\nabla v_m| + r_m^{2s-\beta} |b^k_m \cdot a^m_k| + |g^m_k| \Theta(r_m) \leq 0 \quad \text{in} \quad \Omega.
\]

By Lemma \((5.4)\), \(\tilde{I}^{m_k}\) converges weakly to a translation invariant operator \(I_0\) which is elliptic with respect to \(L\). Thus \((5.17)\) follows from \((5.16)\) and \((5.18)\).

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