A TRACE FORMULA FOR VARIETIES OVER A DISCRETELY VALUED FIELD

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Abstract. We study the motivic Serre invariant of a smoothly bounded algebraic or rigid variety $X$ over a complete discretely valued field $K$ with perfect residue field $k$. If $K$ has characteristic zero, we extend the definition to arbitrary $K$-varieties using Bittner’s presentation of the Grothendieck ring and a process of Néron smoothening of pairs of varieties.

The motivic Serre invariant can be considered as a measure for the set of unramified points on $X$. Under certain tameness conditions, it admits a cohomological interpretation by means of a trace formula. In the curve case, we use T. Saito's geometric criterion for cohomological tameness to obtain more detailed results. We discuss some applications to Weil-Châtelet groups, Chow motives, and the structure of the Grothendieck ring of varieties.

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1. Introduction

Let $R$ be a complete discrete valuation ring, with quotient field $K$ and perfect residue field $k$. If $X$ is a smoothly bounded rigid $K$-variety (e.g. smooth, quasi-compact and separated), then one can associate to $X$ its so-called motivic Serre invariant $S(X)$. If $X/R$ is a formal weak Néron model for $X$, then $S(X)$ is the class of the special fiber $X_s$ in the quotient of the Grothendieck ring of $k$-varieties modulo the ideal generated by the class of the torus $\mathbb{G}_{m,k}$. Of course, one has to show that this value only depends on $X$ and not on the choice of a weak Néron model. If $X$ is smooth and quasi-compact, this was proven in [32] using the theory of motivic integration on formal $R$-schemes [43], and the general case can be deduced from this result.

By definition, the generic fiber $X_\eta$ of $X$ is an open rigid subvariety of $X$ which contains all the unramified points on $X$. Since $X$ is smooth over $R$, its special fiber is a good measure for the set of unramified points on $X$. Therefore, one can consider the motivic Serre invariant $S(X)$ as a measure for the set of unramified points on $X$. It is natural to ask if this invariant admits a cohomological interpretation in terms of the Galois action on the étale cohomology of $X$. This is indeed the case: under certain finiteness and tameness conditions on $X$, a trace formula expresses the Euler characteristic of $S(X)$ in terms of the trace of a monodromy operator on the tame $\ell$-adic cohomology of $X$ [35, 6.4].

The main themes of the present article are the following:

1. study of the error term in the trace formula in the non-tame case,
2. generalization of the definition of the motivic Serre invariant to arbitrary algebraic $K$-varieties, if $K$ has characteristic zero,
3. realization morphisms and structure of the Grothendieck ring of varieties.

These themes are tightly interwoven.
In Section 2 we recall the definition of the Grothendieck ring $K_0(Var_k)$ of varieties over an arbitrary field $k$, and its localization $M_k$ w.r.t. the class $L$ of the affine line. This ring is still poorly understood, and one of the main problems is to decide when two $k$-varieties $X$ and $Y$ define the same class in $K_0(Var_k)$ or $M_k$. To this end, it is important to have some “computable” realization morphisms from $K_0(Var_k)$ to more concrete rings. If $k$ has characteristic zero, the main tools are the theorems of Bittner (Theorem 2.3) and Larsen and Lunts (Theorem 2.4). Both use resolution of singularities and weak factorization, which explains the restriction on the characteristic. These theorems imply the existence of some fundamental realization morphisms: stably birational classes (Theorem 2.4), Albanese (Corollary 2.5), Chow motives (Theorem 2.8). Larsen and Lunts’ Theorem gives a beautiful description of the Grothendieck ring modulo the ideal generated by the class of the affine line, but it tells nothing about the localized Grothendieck ring $M_k$ (the same holds for the Albanese realization).

In positive characteristic, we are considerably less equipped. In Proposition 2.9 we formulate the classical technique of “spreading out” on the level of Grothendieck rings. This tool allows to reduce questions about $K_0(Var_k)$ and $M_k$ to a finitely generated base ring. As an application, we define the Poincaré polynomial for arbitrary separated morphisms of finite type of schemes (Section 2.5 and Section 8). Roughly speaking, it is the only constructible invariant which is compatible with base change and gives the correct result over a finite field (viz. the polynomial whose coefficients are given by the virtual Betti numbers, which are defined in terms of the weight filtration on $\ell$-adic cohomology). We refer to Theorem 8.12 for the exact statement. The Poincaré polynomial defines ring morphisms $K_0(Var_k) \to \mathbb{Z}[T]$ and $M_k \to \mathbb{Z}[T,T^{-1}]$ for an arbitrary field $k$, which provide a new way to distinguish elements in these Grothendieck rings (Proposition 2.11).

Next, we turn our attention to the motivic Serre invariant. Let $R$ be a complete discrete valuation ring, with quotient field $K$ and perfect residue field $k$. If $X$ is a smooth and proper $K$-variety, then the associated rigid $K$-variety $X^{an}$ is separated, smooth and quasi-compact, so $S(X) := S(X^{an}) \in K_0(Var_k)/(L-1)$ is well-defined. Our main result in this setting (Theorem 5.4) states that if $K$ has characteristic zero, this invariant can be uniquely extended to an additive and multiplicative invariant on the category of $K$-varieties, i.e. a ring morphism $S : M_K \to K_0(Var_k)/(L-1)$. This ring morphism is interesting for two reasons: it defines the motivic Serre invariant for arbitrary $K$-varieties, and it provides a new realization of $M_K$ which is computable in significant cases (see e.g. Theorem 7.5). We use it in Proposition 7.9 to show that the realization morphisms to (effective and non-effective) Chow motives are not injective.

The existence of the morphism $S$ can be deduced from Bittner’s theorem once we understand how the motivic Serre invariant behaves w.r.t. the blow-up relations. To this end, we extend Néron’s smoothening process to pairs of smooth $K$-varieties in Section 3 (Theorem 3.9). This result implies the existence of weak Néron models for bounded and smooth pairs (Definition 3.14). Theorem 3.9 is proved by using the detailed information on the centers of the blow-ups in the classical smoothening process [13, 3.5.2] and Greenberg’s Theorem [22].

In Section 4 we study and compare different notions of boundedness for rigid and algebraic varieties. In particular, we show that, if $K$ has characteristic zero, a smooth $K$-variety $X$ is bounded iff it has a compactification without unramified
points at infinity; then this holds for every smooth compactification (Proposition 4.6). This result is used to prove that the motivic Serre invariant of a $K$-variety without unramified points is zero (Proposition 5.8).

Section 6 deals with the trace formula in the tame case. We assume that the residue field $k$ of $R$ is algebraically closed, and for each $d > 0$ prime to the characteristic exponent $p$ of $k$, we denote by $K(d)$ the unique degree $d$ extension of $K$ inside a fixed algebraic closure. If $X$ is a tame smooth and proper $K$-variety (in the sense of Definition 6.1) and $d > 0$ is an integer prime to $p$, then the trace formula (Proposition 6.3) states that the Euler characteristic of the motivic Serre invariant of the $K(d)$-variety $X \times_K K(d)$ equals the trace of a generator of the tame monodromy group $G(K^t/K(d))$ on the tame $\ell$-adic cohomology of $X$. If $K$ has characteristic zero, then by formal arguments, this result extends to any $K$-variety whose isomorphism class belongs to the subring of $K_0(Var_K)$ generated by the classes of tame smooth and proper varieties (Theorem 6.4). In particular, if $k$ has characteristic zero, the trace formula holds for any $K$-variety. This yields a sufficient and necessary cohomological condition for the existence of a rational point (Corollary 6.6).

Without tameness conditions, the trace formula no longer holds. We take a closer look at the case of curves in Section 7. A computation on the nearby cycles yields an expression for the error term in the trace formula in terms of the geometry of a regular model with normal crossings (Theorem 7.3). Intriguingly, this expression appears to be related to Saito’s geometric criterion for cohomological tameness [11], and this relation shows that the trace formula holds for cohomologically tame curves of genus $\neq 1$ (Theorem 7.4) and cohomologically tame elliptic curves (Theorem 7.5).

In the case of elliptic curves we can make explicit computations of the motivic Serre invariant and the error term using the Kodaira-Néron reduction table.

The case of genus 1 curves without rational point brings some surprises. Even if the wild ramification acts trivially, the trace formula can fail (Proposition 7.7), and more fundamentally, the motivic Serre invariant does not admit any general cohomological (nor even motivic) interpretation (at least if we work with rational coefficients). The cause is the fact that there are cohomologically tame elliptic curves $E$ over $K$ with non-trivial Weil-Châtelet group whose motivic Serre invariant has non-zero Euler characteristic. If $X$ is a non-trivial $E$-torsor then the Chow motives of $E$ and $X$ are isomorphic (and hence their $\ell$-adic cohomology spaces are isomorphic as Galois modules), but their motivic Serre invariants have distinct Euler characteristics ($S(X) = 0$ since $X$ has no $K$-rational point). Therefore, the trace formula can not hold for both $X$ and $E$ (over finite fields this situation cannot occur since the Weil-Châtelet group of an elliptic curve over a finite field is zero). Reversing the arguments, we can use the validity of the trace formula in certain cases to recover triviality results about the Weil-Châtelet group (Proposition 7.8).

Finally, we use the local version of Saito’s criterion to prove a trace formula for the analytic Milnor fiber (Theorem 7.11).

**Notation.** We denote by $(Sch)$ the category of schemes. For any scheme $S$, we denote by $(Sch/S)$ the category of schemes over $S$. If $S = \text{Spec } A$ is affine, we write also $(Sch/A)$ instead of $(Sch/S)$. If $x$ is a point on $S$, we will denote by $k(x)$ its residue field. We write $S^o$ for the set of closed points on $S$. We denote by 

$$\cdot_{\text{red}} : (Sch/S) \rightarrow (Sch/S) : X \mapsto X_{\text{red}}$$
the functor mapping a $S$-scheme $X$ to its maximal reduced closed subscheme $X_{\text{red}}$.

For any scheme $S$, a $S$-variety is a reduced separated $S$-scheme of finite type. For any separated $S$-scheme of finite type $X$, we denote by $\text{Sm}(X)$ the open subscheme of $X$ consisting of the points where $X$ is smooth over $S$. If we want to make the base scheme $S$ explicit, we’ll write $\text{Sm}(X/S)$ instead of $\text{Sm}(X)$.

For any field $F$, we denote by $F^a$ an algebraic closure, and by $F^s$ the separable closure of $F$ in $F^a$. Starting from Section 3, $R$ denotes a discrete valuation ring, with quotient field $K$ and residue field $k$. The maximal ideal of $R$ will be denoted by $\mathfrak{m}$. We fix a separable closure $K^s$ of $K$, and we denote by $R^{sh}$ the strict henselization of $R$ in $K^s$, and by $K^{sh} \subset K^s$ its quotient field. We denote by $k^s$ the residue field of $R^{sh}$. The field $k^s$ is a separable closure of $k$. Moreover, we denote by $K^t$ the tame closure of $K$ inside $K^s$. We fix a prime $\ell$ invertible in $k$. Additional assumptions will be indicated at the beginning of each section.

If $R$ is complete, and we fix a value $0 < \theta < 1$, then there exists a unique absolute value $|\cdot|$ on $K^s$ such that $|a| = \theta^{v(a)}$ for each $a$ in $K^*$, where $v$ denotes the discrete valuation on $K^*$. This absolute value makes $K$ into a non-archimedean field.

We’ll consider the generic fiber functor
\[(\cdot)_K : (\text{Sch}/R) \to (\text{Sch}/K) : X \mapsto X_K = X \times_R K\]
as well as the special fiber functor
\[(\cdot)_s : (\text{Sch}/R) \to (\text{Sch}/k) : X \mapsto X_K = X \times_R k\]

For any scheme or rigid variety $X$ and any prime $\ell$, we put
\[H(X, \mathbb{Q}_\ell) = \bigoplus_{i \geq 0} H^i(X, \mathbb{Q}_\ell)\]
where $H^i(X, \mathbb{Q}_\ell)$ is the $i$-th $\ell$-adic cohomology space, and we view $H(X, \mathbb{Q}_\ell)$ as a $\mathbb{Z}$-graded vector space. Similar notation applies for cohomology with compact supports. If $H = \bigoplus_{i \in \mathbb{Z}} H^i$ is a finite dimensional graded vector space over some field $F$, and $M$ is a graded endomorphism of $H$, then we put
\[\text{Trace}(M \mid H) = \sum_{i \in \mathbb{Z}} (-1)^i \text{Trace}(M \mid H^i)\]

2. The Grothendieck ring of varieties.

2.1. Definition and realization morphisms.

**Definition 2.1 (Grothendieck ring).** Let $S$ be any Noetherian scheme. We define the Grothendieck group of $S$-varieties $K_0(\text{Var}_S)$ as the abelian group with

- **generators:** isomorphism classes $[X/S]$ of separated $S$-schemes of finite type $X$
- **relations:** if $Y \to X$ is a closed immersion, then $[X/S] = [(X - Y)/S] + [Y/S]$ (“scissor relations”).

The product $[X/S] \cdot [Y/S] = [(X \times_S Y)/S]$ defines a ring structure on $K_0(\text{Var}_S)$, and we call this ring the Grothendieck ring of $S$-varieties.

Moreover, we put $L_S = [\mathbb{A}^1_S/S]$ and $M_S = K_0(\text{Var}_S)[L_S^{-1}]$.

A morphism of Noetherian schemes $T \to S$ induces base change morphisms of rings $K_0(\text{Var}_S) \to K_0(\text{Var}_T)$ and $M_S \to M_T$. Moreover, a separated morphism of finite type $S \to U$ induces forgetful morphisms of abelian groups $K_0(\text{Var}_S) \to K_0(\text{Var}_U)$ and $M_S \to M_U$ (the definition of the latter requires some care): if $X$...
is a separated $S$-scheme of finite type, then $[X/S]_{S_i}$ is mapped to $[X/U]_{L_i}$, for any $i \in \mathbb{Z}$.

The following properties follow immediately from the definition: for any separated $S$-scheme of finite type $X$, the natural closed immersion $X_{red} \to X$ gives rise to the equality $[X/S] = [X_{red}/S]$ in $K_0(Var_S)$. Likewise, the base change morphisms $K_0(Var_S) \to K_0(Var_{S_{red}})$ and $\mathcal{M}_S \to \mathcal{M}_{S_{red}}$ are ring isomorphisms.

If the base scheme $S$ is clear from the context, we write $[X]$ and $\mathbb{L}$ instead of $[X/S]$ and $\mathbb{L}_S$. If $S$ is affine, say $S = \text{Spec} A$, then we write $K_0(Var_A)$ and $\mathcal{M}_A$ instead of $K_0(Var_S)$ and $\mathcal{M}_S$.

Even if $k$ is a field of characteristic zero, the Grothendieck ring $K_0(Var_k)$ is not very well understood. Poonen showed that $K_0(Var_k)$ is not a domain. It is not known if the natural morphism $K_0(Var_k) \to \mathcal{M}_k$ is injective (i.e. if $L$ is a zero divisor in $K_0(Var_k)$). We refer to [31] for some intriguing questions and results.

Now let $k$ be any field. By its definition, $K_0(Var_k)$ is the universal additive and multiplicative invariant for the category $Var_k$ of $k$-varieties: any invariant of $k$-varieties with values in a ring $A$ which is additive w.r.t. closed immersions and multiplicative w.r.t. the product of $k$-varieties, defines a ring morphism $K_0(Var_k) \to A$.

Here are some well-known examples we will need:

1. **Counting rational points:** if $k$ is a finite field, then there exists a unique ring morphism

   $\sharp : K_0(Var_k) \to \mathbb{Z}$

   which maps $[X]$ to $\sharp X(k)$ (the number of $k$-rational points) for each separated $k$-scheme of finite type $X$. It localizes to a ring morphism $\sharp : \mathcal{M}_k \to \mathbb{Q}$.

2. **Etale realization:** let $k$ be any field, and denote by $G_k$ the absolute Galois group $G(k^s/k)$. We fix a prime $\ell$ invertible in $k$, and we denote by $\text{Rep}_{G_k}(\mathbb{Q}_\ell)$ the abelian tensor category of $\ell$-adic representations of $G_k$ (i.e. finite dimensional $\mathbb{Q}_\ell$-vector spaces with a continuous left action of $G_k$). The tensor structure on $\text{Rep}_{G_k}(\mathbb{Q}_\ell)$ defines a ring structure on the Grothendieck group $K_0(\text{Rep}_{G_k}(\mathbb{Q}_\ell))$. As pointed out in [34], there exists a unique ring morphism

   $\acute{\text{e}}t : K_0(Var_k) \to K_0(\text{Rep}_{G_k}(\mathbb{Q}_\ell))$

   such that

   $$\acute{\text{e}}t([X]) = \sum_{i \geq 0} (-1)^i [H^i(X \times_k k^s, \mathbb{Q}_\ell)]$$

   for each separated $k$-scheme of finite type $X$. It localizes to a ring morphism $\acute{\text{e}}t : \mathcal{M}_k \to K_0(\text{Rep}_{G_k}(\mathbb{Q}_\ell))$ (since $\acute{\text{e}}t(\mathbb{L}) = [\mathbb{Q}_\ell (-1)]$ is invertible in $K_0(\text{Rep}_{G_k}(\mathbb{Q}_\ell))$).

3. **The $\ell$-adic Euler characteristic:** if $k$ is any field and $\ell$ is a prime invertible in $k$, then there exists a unique ring morphism

   $\chi_{\text{top}} : \mathcal{M}_k \to \mathbb{Z}$

   such that

   $$\chi_{\text{top}}([X]) = \sum_{i \geq 0} (-1)^i \dim H^i(X \times_k k^s, \mathbb{Q}_\ell)$$

   for each separated $k$-scheme of finite type $X$. It can also be obtained by composing the etale realization $\acute{\text{e}}t$ with the forgetful morphism

   $$K_0(\text{Rep}_{G_k}(\mathbb{Q}_\ell)) \to K_0(\mathbb{Q}_\ell) = \mathbb{Z}$$

   The morphism $\chi_{\text{top}}$ is independent of $\ell$ (this is well known: if $k$ has characteristic zero it follows from comparison with singular cohomology; if $k$ is finite from
the cohomological interpretation of the zeta function; if \( k \) is any field of positive characteristic by spreading out and reduction to a finite base field).

(4) The Hodge-Deligne realization: assume \( k = \mathbb{C} \), and define the Hodge-Deligne polynomial \( HD(X; u, v) \) of a separated \( \mathbb{C} \)-scheme of finite type \( X \) by

\[
HD(X; u, v) = \sum_{k \geq 0} (-1)^k h^{p,q}(H^k_{\text{ét}}(X(\mathbb{C}), \mathbb{C})) u^p v^q
\]

where \( h^{p,q}(H^k_{\text{ét}}(X, \mathbb{C})) \) denotes the dimension of the \((p,q)\)-component of Deligne’s mixed Hodge structure on \( H^k_{\text{ét}}(X, \mathbb{C}) \). Then \( HD(\cdot; u, v) \) is additive and multiplicative, so there exists a unique ring morphism

\[
HD : K_0(\text{Var}_\mathbb{C}) \to \mathbb{Z}[u, v]
\]

mapping \([X]\) to \( HD(X; u, v)\) for any separated \( \mathbb{C} \)-scheme of finite type \( X \). It localizes to a ring morphism \( HD : M_\mathbb{C} \to \mathbb{Z}[u, v, u^{-1}, v^{-1}] \).

The definition of \( HD \) generalizes to an arbitrary base field \( k \) of characteristic zero, either by invoking the Lefschetz principle (the Hodge numbers are algebraic invariants) or by using Bittner’s presentation of the Grothendieck ring (Theorem 2.3).

As a general rule, whenever \( \mu \) is a group morphism from \( K_0(\text{Var}_k) \) or \( M_\mathbb{C} \) to some abelian group \( A \), we write \( \mu([X]) \) instead of \( \mu([X]) \) for any separated \( k \)-scheme of finite type \( X \).

2.2. Bittner’s presentation and the theorem of Larsen and Lunts. Let \( k \) be any field.

**Definition 2.2.** We denote by \( K_0^{(bl)}(\text{Var}_k) \) the abelian group given by

- generators: isomorphism classes \([X]_{bl}\) of smooth, projective \( k \)-varieties \( X \)
- relations: \([\emptyset]_{bl} = 0\), and if \( Y \) is a closed subvariety of \( X \), smooth over \( k \), \( X' \to X \) is the blow-up of \( X \) with center \( Y \), and \( E = X' \times_X Y \) is the exceptional divisor, then \([X']_{bl} - [E]_{bl} = [X]_{bl} - [Y]_{bl}\) ("blow-up relations").

The product \([X]_{bl} \cdot [Y]_{bl} = [X \times_k Y]_{bl}\) defines a ring structure on \( K_0^{(bl)}(\text{Var}_k) \).

The ring \( K_0^{(bl)}(\text{Var}_k)' \) is defined in the same way, replacing "projective" by "proper".

Note that the product is well-defined, since blow-ups commute with flat base change. It follows immediately from the definition that there exist unique ring morphisms

\[
\alpha : K_0^{(bl)}(\text{Var}_k) \to K_0(\text{Var}_k)
\]

\[
\alpha' : K_0^{(bl)}(\text{Var}_k)' \to K_0(\text{Var}_k)
\]

mapping \([X]_{bl}\) to \([X]\) for any smooth, projective (resp. proper) \( k \)-variety \( X \).

**Theorem 2.3** (Bittner [10], Thm. 3.1). If \( k \) has characteristic zero, then the natural ring morphisms

\[
\alpha : K_0^{(bl)}(\text{Var}_k) \to K_0(\text{Var}_k)
\]

\[
\alpha' : K_0^{(bl)}(\text{Var}_k)' \to K_0(\text{Var}_k)
\]

are isomorphisms.
It follows easily from Hironaka’s resolution of singularities that $\alpha$ and $\alpha'$ are surjective. Using Weak Factorization \[3\], Bittner also proved injectivity.

Recall that two smooth, projective, connected $k$-varieties $X, Y$ are called stably birational if there exist integers $m, n \geq 0$ such that $X \times_k \mathbb{P}^m_k$ and $Y \times_k \mathbb{P}^n_k$ are birational. This defines an equivalence relation on the set of smooth, projective, connected $k$-varieties. We denote by $SB$ the set of equivalence classes and by $\mathbb{Z}[SB]$ the free abelian group on $SB$.

**Theorem 2.4** (Stably birational realization \[29\]). If $k$ has characteristic zero, then there exists a unique isomorphism of abelian groups

$$\Phi_{SB} : K_0(Var_k) / \mathbb{L} K_0(Var_k) \to \mathbb{Z}[SB]$$

mapping a smooth, projective, connected $k$-variety to its equivalence class in $SB$.

As explained in \[29, 2.4+7\] the existence of $\Phi_{SB}$ follows immediately from Theorem 2.3, and the fact that it is an isomorphism follows easily from Weak Factorization \[3\]. In \[29\] it was assumed that $k$ is algebraically closed, but this is not necessary \[28, p. 28\].

**Corollary 2.5** (Albanese realization \[38\]). Assume that $k$ has characteristic zero, denote by $AV_k$ the monoid of isomorphism classes of abelian varieties over $k$, and by $\mathbb{Z}[AV_k]$ the associated monoid ring. There exists a unique ring morphism

$$\text{Alb} : K_0(Var_k) \to \mathbb{Z}[AV_k]$$

which sends the class $[X]$ of a smooth, projective, connected $k$-variety $X$ to the isomorphism class of its Albanese $\text{Alb}(X)$ in $\mathbb{Z}[AV_k]$.

In particular, if $A, B$ are abelian varieties over $k$, then $[A] = [B]$ in $K_0(Var_k)$ iff $A \cong B$.

**Proof.** The Albanese is invariant under stably birational equivalence. \qed

Note that $\text{Alb}() = 0$, so that $\text{Alb}$ does not localize to a realization of $\mathcal{M}_k$.

### 2.3. Specialization to Chow motives.

Let $k$ be any field, denote by $Mot_k^{eff}$ the category of effective Chow motives over $k$ with rational coefficients, and by $Mot_k$ the category of Chow motives with rational coefficients. The natural functor

$$Mot_k^{eff} \to Mot_k$$

is additive and compatible with the tensor product, so it yields a natural ring morphism

$$\rho : K_0(Mot_k^{eff}) \to K_0(Mot_k)$$

which induces an isomorphism

$$K_0(Mot_k^{eff})([\mathbb{L}_{mot}]^{-1}) \cong K_0(Mot_k)$$

where $\mathbb{L}_{mot}$ denotes the Lefschetz motive. I do not know if $\rho$ is injective (i.e. if $[\mathbb{L}_{mot}]$ is not a zero divisor in $K_0(Mot_k^{eff})$). Using the fact that $Mot_k^{eff} \to Mot_k$ is fully faithful, it is easily seen that $\rho$ is injective if one assumes the following conjecture:

**Conjecture 2.6** (Goettsche \[20\], Conj. 2.5). If $M$ and $N$ are objects of $Mot_k^{eff}$, then $[M] = [N]$ in $K_0(Mot_k^{eff})$ iff $M$ and $N$ are isomorphic.
Proposition 2.7. Assume that Conjecture 2.6 holds. If $M$ and $N$ are objects of $\text{Mot}_k$, then $[M] = [N]$ in $K_0(\text{Mot}_k)$ iff $M \cong N$ in $\text{Mot}_k$. Moreover, $\rho$ is injective.

Proof. If $[M] = [N]$ in $K_0(\text{Mot}_k)$, then there exists an object $P$ in $\text{Mot}_k$ such that $M \oplus P \cong N \oplus P$. For $i > 0$, $P \otimes \mathbb{L}_\text{mot}^i$, $M \otimes \mathbb{L}_\text{mot}^i$ and $N \otimes \mathbb{L}_\text{mot}^i$ are effective, and then

\[(M \otimes \mathbb{L}_\text{mot}^i) \oplus (P \otimes \mathbb{L}_\text{mot}^i) \cong (N \otimes \mathbb{L}_\text{mot}^i) \oplus (P \otimes \mathbb{L}_\text{mot}^i)\]

in $\text{Mot}_k^{\text{eff}}$. Hence, $[M \otimes \mathbb{L}_\text{mot}^i] = [N \otimes \mathbb{L}_\text{mot}^i]$ in $K_0(\text{Mot}_k^{\text{eff}})$ and by Conjecture 2.6 this implies that $M \otimes \mathbb{L}_\text{mot}^i$ and $N \otimes \mathbb{L}_\text{mot}^i$ are isomorphic in $\text{Mot}_k^{\text{eff}}$. Tensoring with $\mathbb{L}_\text{mot}^{-i}$ shows that $M \cong N$ in $\text{Mot}_k$.

This easily implies the injectivity of $\rho$: any element $\alpha$ of $K_0(\text{Mot}_k^{\text{eff}})$ can be written as $[M] - [N]$ with $M$ and $N$ objects of $\text{Mot}_k^{\text{eff}}$, and $\rho(\alpha) = 0$ means that $M$ and $N$ have the same class in $K_0(\text{Mot}_k)$. Hence, $M$ and $N$ are isomorphic in $\text{Mot}_k$, and therefore also in $\text{Mot}_k^{\text{eff}}$, so $\alpha = 0$. □

It was shown in [20] that Conjecture 2.5 follows from the Beilinson-Murre Conjecture [26, Conj. 2.1+5.1+Thm. 5.2]

We denote, for each smooth and projective variety $X$ over $k$, by $M(X)$ the motive $(X, \text{id})$ associated to $X$ in $\text{Mot}_k^{\text{eff}}$. With slight abuse of notation, we'll use the same notation for its image $(X, \text{id}, 0)$ in $\text{Mot}_k$.

Theorem 2.8 (Gillet-Soulé [19], Guillen-Navarro Aznar [25], Bittner [10]). Assume that $k$ has characteristic zero. There exist unique ring morphisms

\[
\chi^{\text{eff}} : K_0(\text{Var}_k) \to K_0(\text{Mot}_k^{\text{eff}})
\]

\[
\chi : \mathcal{M}_k \to K_0(\text{Mot}_k)
\]

such that, for any smooth and projective $k$-variety $X$, $\chi^{\text{eff}}(X)$ (resp. $\chi(X)$) is the class of $M(X)$ in $K_0(\text{Mot}_k^{\text{eff}})$ (resp. $K_0(\text{Mot}_k)$).

The question about the existence of such a morphism $\chi^{\text{eff}}$ was raised already by Grothendieck in a letter to Serre [10] letter of 16/8/1964; he also asked how far the morphism $\chi^{\text{eff}}$ is from being bijective. It is known that $\chi^{\text{eff}}$ is not injective: isogenous abelian varieties have isomorphic Chow motives with rational coefficients, while, if $k$ has characteristic zero, the classes of two abelian varieties in $K_0(\text{Var}_k)$ coincide iff the varieties are isomorphic, because of the existence of the Albanese realization (Corollary 2.5).

However, this example does not answer the following question: Is $\chi$ injective? It is not known if $\mathbb{L}$ is a zero divisor in $K_0(\text{Var}_k)$, and the Albanese realization $\text{Alb}$ maps $\mathbb{L}$ to zero, so it is not clear if two non-isomorphic abelian varieties have distinct classes in $\mathcal{M}_k$.

We will show in Proposition 7.3 that, for an appropriate base field $k$ of characteristic zero, $\chi$ is non-injective. I do not know if $\chi$ and $\chi^{\text{eff}}$ are surjective.

Remark. Theorem 2.8 still holds if we replace rational coefficients by integer coefficients [19 Thm. 4]. By Theorem 2.8 we only have to check that Chow motives satisfy the blow-up relations. For rational coefficients, this was proven in [25, 5.1], but the same proof holds for $\mathbb{Z}$-coefficients (see [4, 0.1.3] for a computation of the Chow groups). If we work with $\mathbb{Z}$-coefficients, I do not know if $\chi$ and $\chi^{\text{eff}}$ are injective. □
2.4. Spreading out. Let $k$ be any field. We denote by $\mathcal{A}_k$ the set of finitely generated sub-$\mathbb{Z}$-algebras of $k$, ordered by inclusion. Then $k$ is the limit of the direct system $\mathcal{A}_k$ in the category of rings. If $X$ is a $k$-variety, and $A$ is an object in $\mathcal{A}_k$, then a $A$-model for $X$ is an $A$-variety $X_A$ endowed with an isomorphism $X \cong X_A \times_A k$. An $A$-model for a morphism of $k$-varieties $f : X \to Y$ is a morphism of $A$-varieties $f_A : X_A \to Y_A$ such that $X_A$ and $Y_A$ are $A$-models for $X$, resp. $Y$, and such that $f$ coincides with the morphism $X_A \times_A k \to Y_A \times_A k$ obtained from $f_A$ by base change (modulo the identifications $X_A \times_A k \cong X$ and $Y_A \times_A k \cong Y$).

For any pair of objects $A, A'$ in $\mathcal{A}_k$ with $A \subset A'$, we consider the natural base change morphisms
\[
\phi_A^{A'} : K_0(\text{Var}_A) \to K_0(\text{Var}_{A'})
\]
\[
\psi_A^{A'} : \mathcal{M}_A \to \mathcal{M}_{A'}
\]
as well as
\[
\phi_A^k : K_0(\text{Var}_A) \to K_0(\text{Var}_k)
\]
\[
\psi_A^k : \mathcal{M}_A \to \mathcal{M}_k
\]
We obtain direct systems of rings $(K_0(\text{Var}_A), \phi_A^k)$ and $(\mathcal{M}_A, \psi_A^k)$ indexed by the directed set $\mathcal{A}_k$, and the morphisms $\phi_A^k$ and $\psi_A^k$ induce morphisms
\[
\phi : \lim_{A \in \mathcal{A}_k} K_0(\text{Var}_A) \to K_0(\text{Var}_k)
\]
\[
\psi : \lim_{A \in \mathcal{A}_k} \mathcal{M}_A \to \mathcal{M}_k
\]
The classical technique of “spreading out” can be formulated on the level of Grothendieck rings in the following way.

**Proposition 2.9** (Spreading out). The natural ring morphisms
\[
\phi : \lim_{A \in \mathcal{A}_k} K_0(\text{Var}_A) \to K_0(\text{Var}_k)
\]
\[
\psi : \lim_{A \in \mathcal{A}_k} \mathcal{M}_A \to \mathcal{M}_k
\]
are isomorphisms.

**Proof.** Surjectivity follows from the fact that for any $k$-variety $X$, there exist an object $A$ in $\mathcal{A}_k$ and a $A$-model $X_A$ for $X$, by [23 8.8.2]. Injectivity follows from the following facts:

- if $A$ is an object of $\mathcal{A}_k$, and $U_A$ and $V_A$ are $A$-varieties, then the canonical map
\[
\lim_{A \subset A' \in \mathcal{A}_k} \text{Hom}_{A'}(U_A \times_A A', V_A \times_A A') \to \text{Hom}_k(U_A \times_k k, V_A \times_k k)
\]
is a bijection [23 8.8.2]. Moreover, if $f_A : U_A \to V_A$ is a morphism of $A$-varieties such that the induced morphism $f_A : U_A \times_A k \to V_A \times_A k$ is a closed (resp. open) immersion, then there exists an object $A'$ in $\mathcal{A}_k$ with $A \subset A'$ such that the natural morphism $f_A' : U_A \times_A A' \to V_A \times_A A'$ is a closed (resp. open) immersion [23 8.10.5].

Proposition 2.9 provides a convenient way to construct additive and multiplicative invariants of $k$-varieties. We will give an illustration in Section 2.5 (see also the Appendix).
2.5. The Poincaré polynomial. Let $k$ be any field. It is, in general, a non-trivial problem to decide whether the classes of two $k$-varieties $X, Y$ in $K_0(\text{Var}_k)$ are distinct. To this aim, it is important to know some “computable” realization morphisms on $K_0(\text{Var}_k)$. If $k$ has characteristic zero, we’ve encountered many of these in the preceding sections, but in positive characteristic, we’re considerably less equipped. For $k = \mathbb{C}$ or $k$ a finite field, one can define the virtual Betti numbers $\beta_i(X)$ and the Poincaré polynomial $P(X; T)$ of a $k$-variety $X$ using Deligne’s theory of weights [17] [18]. By spreading out, these invariants can be generalized to arbitrary base fields. These definitions seem to be known to experts, but since we could not find a reference for their construction and main properties, we found it worthwhile to include the arguments in the Appendix. We summarize in the following theorem the facts we’ll need in the remainder of this article.

**Theorem 2.10.** Let $k$ be any field. For any separated $k$-scheme of finite type $X$, its Poincaré polynomial $P(X; T) \in \mathbb{Z}[T]$ has degree $d$, with $d$ the dimension of $X$. The coefficient of $T^d$ in $P(X; T)$ equals the number of irreducible components of dimension $d$ of $X \times_k k^s$. The value $P(X; 1)$ is equal to the Euler characteristic $\chi_{\text{top}}(X)$. If $X$ is smooth and proper over $k$, then

$$P(X; T) = \sum_{i \geq 0} (-1)^i b_i(X) T^i$$

with $b_i(X) = \dim H^i(X \times_k k^s, \mathbb{Q}_\ell)$ for any prime $\ell$ invertible in $k$.

There exists a unique ring morphism $P : K_0(\text{Var}_k) \to \mathbb{Z}[T]$ mapping $[X]$ to $P(X; T)$ for any separated $k$-scheme of finite type $X$. The morphism $P$ maps $\mathbb{L}$ to $T^2$ and localizes to a ring morphism $P : \mathcal{M}_k \to \mathbb{Z}[T; T^{-1}]$.

**Proof.** See Appendix (Section 8), in particular Propositions 8.6, 8.7 and 8.10. □

The existence and properties of the Poincaré polynomial yield the following useful criterion to distinguish elements of the localized Grothendieck ring.

**Corollary 2.11.** Let $k$ be any field, and let $X$ and $Y$ be separated $k$-schemes of finite type such that $[X] = [Y]$ in $\mathcal{M}_k$. Then $X$ and $Y$ have the same dimension $d$, and $X \times_k k^s$ and $Y \times_k k^s$ have the same number of irreducible components of dimension $d$. In particular, if $X$ is non-empty, then $[X] \neq 0$ in $\mathcal{M}_k$. If $X$ and $Y$ are proper and smooth over $k$, then they have the same $\ell$-adic Betti numbers (for $\ell$ invertible in $k$).

The first part of Corollary 2.11 (concerning the dimension and the geometric number of irreducible components of maximal dimension) was proven in [31] 4.7 by a different method (their proof was formulated for $K_0(\text{Var}_k)$ but holds also for $\mathcal{M}_k$).

2.6. Zero divisors. In [38], Poonen has shown that $K_0(\text{Var}_k)$ is not a domain if $k$ is a field of characteristic zero. Other examples of zero-divisors were constructed by Kollár [28] Ex.6] and by Liu and Sebag [31] 5.11. To my best understanding, these proofs don’t say anything about $\mathcal{M}_k$. The authors construct elements $\alpha$ and $\beta$ in $K_0(\text{Var}_k)$ such that $\alpha \cdot \beta = 0$, and to show that neither $\alpha$ nor $\beta$ are zero, they use the stably birational realization $\Phi_{SB}$ (Theorem 2.3) or the Albanese realization (Corollary 2.5). However, each of these realization morphisms maps $\mathbb{L}$ to 0, so they do not allow to conclude that $\alpha$ and $\beta$ are non-zero in $\mathcal{M}_k$. 
To my knowledge, the only case where it has been shown that $\mathcal{M}_k$ is not a domain, is the case where $k$ is not separably closed \([40, 3.5]\) (the result is stated there for $K_0(Var_k)$ but works also for $\mathcal{M}_k$; it generalizes \([34, \text{Thm. 25}]\)). We’ll give a new proof of this result, which does not use $\ell$-adic cohomology. We refer to Proposition 7.10 for another example of a zero-divisor in $\mathcal{M}_k$ (for appropriate $k$).

**Proposition 2.12.** If $k$ is any field which is not separably closed, then $K_0(Var_k)$ and $\mathcal{M}_k$ are not domains.

**Proof.** Choose a non-trivial finite Galois extension $k'$ of $k$, and put $d = [k' : k]$. Then $([\text{Spec } k'] - d) \cdot [\text{Spec } k'] = 0$ in $K_0(Var_k)$. We’ll prove that $[\text{Spec } k'] \neq 0$ and $[\text{Spec } k'] \neq d$ in $\mathcal{M}_k$. By Proposition 2.9 it is enough to show that $[X] \neq 0, d$ in $\mathcal{M}_A$ for every object $A$ in $\mathcal{O}_k$ and every $A$-model $X$ of $\text{Spec } k'$. If $y$ is a closed point on $X$ then $k(y)$ is finite, and applying the point counting morphism $\sharp : M_y \to \mathbb{Q}$ (Section 2.1) we see that $[X \times_{\text{Spec } A} y] \neq 0$ in $\mathcal{M}_y$. This implies $[X] \neq 0$ in $\mathcal{M}_A$. It remains to show that $[X] \neq d$ in $\mathcal{M}_A$.

Localizing $A$ we may assume that $X$ is irreducible. The function field $k(X)$ is a field extension of degree $d$ of the quotient field $k(A)$ of $A$, since $k(X) \otimes_{k(A)} k \cong k'$. Base changing to an object $A'$ in $\mathcal{O}_k$ with $A \subset A'$ we may assume that every automorphism of $k'$ over $k$ is induced by an automorphism of $X$ over $A$ \([23, 8.8.2]\). Localizing $A$ again we may suppose that $X$ is a Galois cover of $\text{Spec } A$.

By Chebotarev’s density theorem for $\text{Spec } A$ (see \([44]\)), there exists a closed point $x$ on $\text{Spec } A$ which does not split completely in the Galois cover $X$. It suffices to show that $[X \times_A k(x)] \neq d$ in $\mathcal{M}_x$. This can be seen by applying the point counting morphism $\sharp : \mathcal{M}_x \to \mathbb{Q}$. \qed

I do not know whether $\mathcal{M}_k$ is a domain if $k$ is separably closed, or if $K_0(Var_k)$ is a domain if $k$ is separably closed and has characteristic $p > 0$. If $k'$ is a purely inseparable finite field extension of $k$, I do not know if $[\text{Spec } k'] \neq [\text{Spec } k]$ in $K_0(Var_k)$.

3. **Néron smoothening of pairs**

3.1. **Pairs of varieties.** Let $S$ be any scheme. A pair of $S$-varieties $(X, A)$ consists of a $S$-variety $X$ and a closed subvariety $A$ of $X$. We say that the pair $(X, A)$ is proper, smooth, . . . if this holds for both $X$ and $A$. A morphism of pairs of $S$-varieties $f : (Y, B) \to (X, A)$ is a morphism of $S$-varieties $f : Y \to X$ such that $f(B) \subset A$. Since $B$ is reduced, this implies that the restriction of $f$ to $B$ factors through a morphism of $S$-varieties $f : B \to A$. We embed the category of $S$-varieties in the category of pairs by $X \mapsto (X, \emptyset)$.

We denote by $R$ a discrete valuation ring, with quotient field $K$ and residue field $k$. The maximal ideal of $R$ will be denoted by $\mathfrak{m}$. We fix a separable closure $K^s$ of $K$, and we denote by $R^{sh}$ the strict henselization of $R$ in $K^s$, and by $K^{sh} \subset K^s$ its quotient field. We denote by $k^s$ the residue field of $R^{sh}$. The field $k^s$ is a separable closure of $k$.

If $(X, A)$ is a pair of $R$-varieties, then their generic fibers $(X_K, A_K)$ form a pair of $K$-varieties. We say that the pair $(X, A)$ is generically smooth if $(X_K, A_K)$ is a smooth pair of $K$-varieties.

We recall two properties which we’ll frequently use: if $Y$ is a smooth $k$-variety, then $Y(k^s)$ is schematically dense in $Y$ \([13, 2.2.13]\) and if $X$ is a smooth $R$-variety, then the natural reduction map $X(R^{sh}) \to X_s(k^s)$ is surjective \([13, 2.3.5]\).
3.2. Néron smoothening.

**Definition 3.1** (Néron smoothening). If \( X \) is a generically smooth \( R \)-variety, then a Néron smoothening of \( X \) is a morphism of \( R \)-varieties \( h : Y \to X \) with the following properties:

- \( Y \) is smooth over \( R \)
- \( h_K : Y_K \to X_K \) is an isomorphism.
- \( h \) satisfies the following “weak valuative criterion”: the natural map \( \phi : Y(R^h) \to X(R^h) \) is bijective.

Note that injectivity of \( \phi \) follows already from the fact that \( h \) is separated. Any generically smooth \( R \)-variety \( X \) admits a Néron smoothening, by [13] 3.5.2.

**Definition 3.2** (Admissible ideal sheaf). If \( Y \) is any \( R \)-variety, an ideal sheaf \( \mathcal{I} \) on \( Y \) is called admissible if it contains an element of the maximal ideal \( \mathcal{M} \) of \( R \).

**Lemma 3.3.** Let \( Y \) be any \( R \)-variety, let \( \mathcal{I} \) be an admissible locally principal ideal sheaf on \( Y \), and denote by \( Z \) the closed subscheme of \( Y \) defined by \( \mathcal{I} \). If

\[
\{a \in Y(R^h) \mid a_\ast \in Z(k^a)\} = \emptyset
\]

then \( Sm(Y) \subseteq Y - Z \).

**Proof.** We may as well assume that \( Y \) is connected and smooth. Then \( Z_{\text{red}} \) either is empty or coincides with the special fiber \( Y_s \). But \( Y_s(k^a) \) is dense in \( Y \), and any point in \( Y_s(k^a) \) lifts to a section in \( Y(R^h) \), so \( Z \) is empty. \( \square \)

**Proposition 3.4.** Let \( X \) be a generically smooth \( R \)-variety. If \( Z \to X \) is a smoothening*, then the induced morphism \( Sm(Z) \to X \) is a Néron smoothening in the sense of Definition 3.1. Conversely, if \( h : Y \to X \) is a Néron smoothening, then there exists a smoothening* \( g : Z \to X \) such that \( Y \) and \( Sm(Z) \) are isomorphic as \( X \)-schemes.

**Proof.** It is obvious from the definition that \( Sm(Z) \to X \) is a Néron smoothening if \( Z \to X \) is a smoothening*. Conversely, let \( h : Y \to X \) be a smoothening*. By Nagata’s embedding theorem, there exist a proper morphism \( \overline{h} : \overline{Y} \to X \) and a dense open immersion \( j : Y \to \overline{Y} \) such that \( h = \overline{h} \circ j \). Since \( h_K \) is an isomorphism, \( \overline{h}_K \) and \( j_K \) are isomorphisms, and since \( Y \subset Sm(\overline{Y}) \) and \( h \) is a Néron smoothening, \( \overline{h} \) is a smoothening*.

If \( k \) is perfect, this implies automatically that \( j \) is an isomorphism onto \( Sm(\overline{Y}) \) (because any point in \( Sm(\overline{Y})_{s}(k^a) \) lifts to a section in \( \overline{Y}(R^h) \), which has to be contained in \( Y(R^h) \) since \( Y \to X \) is a Néron smoothening).

If \( k \) is not perfect, this needs not be true (take \( X = \overline{Y} = A^1_k \) and \( Y = X - \{x\} \) with \( x \) any closed point on \( X_s \) whose residue field is inseparable over \( k \)) so we have to modify \( \overline{Y} \). Let \( U \) be the complement of \( Y_s \) in \( \overline{Y}_s \), with its reduced closed subscheme structure, and denote by \( \mathcal{I} \) its defining ideal sheaf. Let \( b : Z \to \overline{Y} \) be the blow-up with center \( U \). Since \( b \) is an isomorphism over \( Y \),

\[
\overline{h} \circ b : Z \to X
\]
is again a smoothening.

We put
\[ V := Z \times_{Y} U \]
This is a closed subscheme of \( Z \), defined by the invertible sheaf \( \mathcal{I} \mathcal{O}_{Z} \). Since \( Y \to X \) is a Néron smoothening,
\[
\{ a \in Z(R^{sh}) \mid a \in V(k^{\ast}) \} = \emptyset
\]
so Lemma 3.3 implies that \( V \) is disjoint from \( Sm(Z) \). On the other hand, \( b : Z \setminus V \to Y \) is an isomorphism and \( Y \) is smooth, so we conclude that \( Sm(Z) = Z \setminus V \), and \( Y \) and \( Sm(Z) \) are isomorphic as \( X \)-schemes.

**Definition 3.5** (Néron smoothening of pairs). Let \( (X, A) \) be a generically smooth pair of \( R \)-varieties. A Néron smoothening of \( (X, A) \) is a morphism of pairs of \( R \)-varieties \( h : (Y, B) \to (X, A) \) such that \( h : Y \to X \) is a Néron smoothening of \( X \) and \( h : B \to A \) is a Néron smoothening of \( A \).

This definition implies in particular that \( B \) is the schematic closure of \( h_{K}^{-1}(A_{K}) \subset Y_{K} \) in \( Y \). Note that \( h : (Y, B) \to (X, A) \) is a Néron smoothening as soon as \( h : Y \to X \) is a Néron smoothening, \( B \) is smooth over \( R \), and \( h_{K} : B_{K} \to A_{K} \) is an isomorphism: then a section \( a \) in \( A(R^{sh}) \) lifts uniquely to a section \( a' \) in \( Y(R^{sh}) \), which is automatically included in \( B(R^{sh}) \) since \( a'_{K} \in B_{K}(K^{sh}) \) and \( B \) is closed in \( Y \).

**Definition 3.6** (Strict transform and admissible blow-up). Let \( (X, A) \) be a pair of \( R \)-varieties. If \( h : Y \to X \) is a morphism of \( R \)-varieties such that \( h_{K} \) is an isomorphism, then the strict transform of \( A \) in \( Y \) is the schematic closure of \( h_{K}^{-1}(A_{K}) \) in \( Y \).

If \( \mathcal{I} \) is an admissible ideal sheaf on \( X \), we define the blow-up of \( (X, A) \) at the center \( \mathcal{I} \) as the morphism of pairs of \( R \)-varieties
\[ h : (Y, B) \to (X, A) \]
where \( h : Y \to X \) is the blow-up of \( X \) at \( \mathcal{I} \), and \( B \) is the strict transform of \( A \) in \( Y \). We call such a morphism \( h \) an admissible blow-up of \( (X, A) \).

We denote for any \( R \)-variety \( Z \) by \( Z^{\flat} \) the maximal \( R \)-flat closed subscheme of \( Z \), i.e. the closed subscheme of \( Z \) defined by the \( \mathfrak{M} \)-torsion ideal. Then the admissible blow-up of \( (X, A) \) at the ideal \( \mathfrak{M} \mathcal{O}_{X} \) is the natural morphism \( (X^{\flat}, A^{\flat}) \to (X, A) \).

In general, if \( (Y, B) \to (X, Z) \) is any admissible blow-up, then \( B \) is flat over \( R \). Moreover, the natural maps \( Y(R^{sh}) \to X(R^{sh}) \) and \( B(R^{sh}) \to A(R^{sh}) \) are bijections, by the valuative criterion for properness; so we can identify any subset \( E \) of \( X(R^{sh}) \) (resp. \( A(R^{sh}) \)) with its inverse image in \( Y(R^{sh}) \) (resp. \( B(R^{sh}) \)).

The main result of this section is the following.

**Theorem 3.7.** Let \( (X, A) \) be a generically smooth pair of \( R \)-varieties. There exists a composition \( h : (Y, B) \to (X, A) \) of admissible blow-ups, such that \( Sm(B) = Sm(Y) \cap B \) and such that the restriction of \( h \) to \( (Sm(Y), Sm(B)) \to (X, A) \) is a Néron smoothening of \( (X, A) \).

The proof of Theorem 3.7 follows after Proposition 3.13. First, we need some preliminary results.
Lemma 3.8. Let $Z$ be a $R$-variety, and let $a$ be a section in $Z(R^{sh})$. Let $C$ be a closed subscheme of $Z_s$, and assume that $Z$ is smooth over $R$ at $a_s \in Z_s(k^s)$, and that $C$ is smooth over $k$ at $a_s$. Denote by $Z' \to Z$ the blow-up with center $C$, and by $a'$ the unique lifting of $a$ to $Z'(R^{sh})$. Then $Z'$ is smooth over $R$ at $a'_s \in Z'_s(k^s)$.

Proof. Since blowing up commutes with flat base change, we may assume that $Z$ is smooth over $R$, and that $C$ is smooth over $k$. Denote by $D \to Z$ the dilatation with center $C$ (see [13, §3.2.1]); then $D$ is an open subscheme of $Z'$ in a natural way. By the universal property of the dilatation [13, 3.2.1], $a'$ factors through $D$, so $a'_s \in D_s(k^s)$. But $D$ is smooth over $R$ by [13, 3.2.3].

Proposition 3.9. Let $(X, A)$ be a pair of generically smooth $R$-varieties, and assume that the natural map $Sm(X)(R^{sh}) \to X(R^{sh})$ is bijective. There exists a composition $h: (Y, B) \to (X, A)$ of admissible blow-ups such that $Sm(B) \to B$ and $Sm(Y) \to Y$ are Néron smoothenings.

Proof. We may assume that $A$ is flat over $R$. By [13, 3.4.2], there exists a composition

$$A' = A_r \xrightarrow{h_{r-1}} \ldots \xrightarrow{h_1} A_1 \xrightarrow{h_0} A_0 = A$$

such that $h_i$ is the blow-up at a closed subscheme $C_i$ of the special fiber $(A_i)_s$, for $i = 0, \ldots, r-1$, and such that the natural map $Sm(A')(R^{sh}) \to A'(R^{sh}) = A(R^{sh})$ is bijective. Moreover, we may assume that each center $C_i$ is $E$-permissible (in the sense of [13, p.71]) with $E = A(R^{sh})$. This implies in particular that the $k$-smooth locus $U_i$ of $C_i$ is open and dense in $C_i$, and that none of the $k^s$-valued points of $C_i - U_i$ lift to a section in $A_i(R^{sh})$.

Now let $X' \to X$ be the composition

$$X' = X_r \xrightarrow{g_{r-1}} \ldots \xrightarrow{g_1} X_1 \xrightarrow{g_0} X_0 = X$$

with $g_i$ the blow-up of $X_i$ at $C_i$, for $i = 0, \ldots, r-1$. Then $A_i$ is canonically isomorphic to the strict transform of $A$ in $X_i$, for each $i$, and these isomorphisms identify the restriction of $g_i$ to $A_i$ with the morphism $h_i$. In particular, $A'$ is canonically isomorphic to the strict transform $A$ in $X'$.

If $a$ is any section in $A'(R^{sh})$, then $X'$ is smooth over $R$ at $a_s \in X'_s(k^s)$, by Lemma 3.8. This implies that $Sm(A') \subset Sm(X')$, because every point in $Sm(A')_s(k^s)$ lifts to a section in $Sm(A')_s(R^{sh})$ and $Sm(A')_s(k^s)$ is schematically dense in $Sm(A')_s$. Again applying [13, 3.4.2], we can find a composition $f: Y \to X'$ of admissible blow-ups such that $f$ is an isomorphism over $Sm(X')$, and such that $Sm(Y)(R^{sh}) = Y(R^{sh})$. If we denote by $B$ the strict transform of $A'$ in $Y$, then the map $B \to A'$ is an isomorphism over $Sm(A')$, and in particular, $Sm(B)(R^{sh}) = B(R^{sh})$.

The result in Proposition 3.9 is not yet strong enough to produce a Néron smoothening of the pair $(X, A)$, since it does not guarantee that $Sm(B)$ is a closed subscheme of $Sm(Y)$. For this purpose, we introduce a new invariant.

Definition 3.10. Let $(X, A)$ be a pair of $R$-varieties, and denote by $\mathcal{I}_A$ the defining ideal sheaf of $A$ on $X$. If $a$ is a section in $X(R^{sh})$, and $x$ is the image of $a_s$ in $X$, then we define the contact of $a$ and $A$ by

$$c_A(a) = \{ \min v(a^*f) \mid f \in (\mathcal{I}_A)_x \} \in \mathbb{N} \cup \{ \infty \}$$

where $v$ denotes the discrete valuation on $R^{sh}$.
Note that \( c_A(a) = 0 \) iff \( x \notin A \), and \( c_A(a) = \infty \) iff \( a \in A(R^{sh}) \).

**Lemma 3.11.** Assume that \( R \) is excellent. Let \((X, A)\) be a generically smooth pair of \( R \)-varieties, and let \( C \) be a closed subscheme of \( A_s \). Put

\[
E_C = \{ a \in X(R^{sh}) \mid a_s \in C(k^s) \}
\]

and assume that \( E_C \cap A(R^{sh}) = \emptyset \). Then there exists a value \( c > 0 \) such that \( c_A(a) \leq c \) for every \( a \in E_C \).

**Proof.** By \([21, 5.6]\) \( R^{sh} \) is excellent, so we may assume that \( R = R^{sh} \), and that there exists a closed immersion \( X \to \mathbb{A}_R^n \) for some \( n > 0 \). Let \( F_1, \ldots, F_t \) be a system of generators of the defining ideal of \( A \) in \( \mathbb{A}_R^n \), and assume that \( c_A \) is unbounded on \( E_C \). Then in particular, for any \( \nu > 0 \), there exists a point \( x \in \mathbb{A}_R^n(R) = R^n \) such that \( x_s \in C(k) \) and \( F_i(x) \equiv 0 \mod t^\nu \) for all \( i \). Since \( R \) is excellent, it follows from Greenberg’s Theorem \([22, \text{Thm. 1}]\) that there exists a section \( a \in E_C \) which is contained in \( A(R) \); so we arrive at a contradiction. \( \square \)

**Lemma 3.12.** Let \((X, A)\) be a pair of \( R \)-varieties, let \( a \) be a section of \( X(R^{sh}) \) which is not contained in \( A(R^{sh}) \), and let \( C \) be a closed subscheme of \( X_s \). Denote by \((X', A')\) \( \to (X, A) \) the admissible blow-up with center \( C \). Then \( c_{A'}(a) \leq c_A(a) \).

If, moreover, \( C \) is a closed subscheme of \( A_s \) and \( a_s \in C(k^s) \), then \( c_{A'}(a) < c_A(a) \).

**Proof.** We may assume that \( X \) is affine. We choose a uniformizer \( \pi \) in \( R \). Let \( f \) be an element of the defining ideal of \( A \) in \( X \) such that \( c_A(a) = v(a^*f) \). Since the pull-back of \( f \) to \( X' \) vanishes on \( A' \), we see immediately that \( c_{A'}(a) \leq c_A(a) \).

Now assume that \( a_s \in C(k^s) \) and that \( C \) is a closed subscheme of \( A_s \), and denote by \( I_C \) the defining ideal of \( C \) in \( X \). If we denote by \( D \to X \) the dilatation of \( X \) with center \( C \), then \( D \) is an open subscheme of \( X' \) and \( a \) is contained in \( D(R^{sh}) \subset X'(R^{sh}) \). Moreover, since \( f \) vanishes on \( C \) and \( \pi \) generates \( I_C \mathcal{O}_D \), there exists an element \( f' \) in \( \mathcal{O}_{X'}(D) \) such that \( f' = \pi f \). Then \( f' \) vanishes on \( A'_K \) because \( f \) vanishes on \( A_K \), and as \( A' \) is the schematic closure of \( A'_K \) in \( A_s \), we see that \( f' \) vanishes on \( A' \). Moreover, \( v(a^*f') = v(a^*f) - 1 \) so \( c_{A'}(a) < c_A(a) \). \( \square \)

**Proposition 3.13.** Let \((X, A)\) be a pair of generically smooth \( R \)-varieties, and assume that the natural morphisms \( Sm(X) \to X \) and \( Sm(A) \to A \) are Néron smoothenings. There exist a composition \((Y, B) \to (X, A)\) of admissible blow-ups with centers contained in \( (\text{the strict transform of}) \ A \), such that \( Sm(B) = Sm(Y) \cap B \), and such that \( Sm(Y) \to Y \) and \( Sm(B) \to B \) are Néron smoothenings.

**Proof.** Denote by \( D \) the complement of \( Sm(X) \cap Sm(A) \) in \( Sm(A) \) (with its reduced closed subscheme structure), and denote by \( \overline{D} \) its schematic closure in \( A_s \). Since every point of \( Sm(A)_s(k^s) \) lifts to a section of \( A(R^{sh}) \), we see that \( Sm(A)_s(k^s) \subset Sm(X)_s(k^s) \), so \( D(k^s) = \emptyset \). Denote by \( C \) the complement of \( Sm(X) \cap Sm(A) \) in \( Sm(X) \cap A \) (with its reduced closed subscheme structure), and by \( \overline{C} \) its schematic closure in \( A_s \), and put

\[
E_{\overline{C}} = \{ a \in X(R^{sh}) \mid a_s \in \overline{C}(k^s) \}
\]

Denote by \( S \) the completion of \( R^{sh} \). The morphism \( Sm(A) \times_R S \to A \times_R S \) is a Néron smoothening by \([13, 3.6.6]\), so \( Sm(A)(S) = A(S) \). Since \( \overline{C} \) is disjoint from \( Sm(A) \), we have

\[
\{ b \in A(S) \mid b_s \in \overline{C}(k^s) \} = \emptyset
\]
Since $S$ is excellent, we can apply Lemma 3.11 and we see that
\[ M(X, A) := \max \{ c_A(a) \mid a \in E_{\overline{\tau}} \} \]
is well-defined and finite (we put $\max \emptyset = 0$). We will argue by induction on $M(X, A)$.

**Induction basis:** assume $M(X, A) = 0$. This is only possible if $E_{\overline{\tau}} = \emptyset$, since for every section $a \in E_{\overline{\tau}}$, $a_s$ belongs to $A_s(k^s)$, so $c_A(a) > 0$. Moreover, since any point of $Sm(X)_s(k^s)$ lifts to a section in $Sm(X)(R^{sh})$, $E_{\overline{\tau}} = \emptyset$ implies $C(k^s) = \emptyset$.

If $k$ is perfect, then we get $C = \emptyset$, so $Sm(X) \cap Sm(A) = Sm(X) \cap A$. Also, in this case $D(k^s) = \emptyset$ implies that $D = \emptyset$ and $Sm(A) \subset Sm(X)$, so we are done.

If $k$ is not perfect, we consider the admissible blow-up
\[ h : (Y, B) \to (X, A) \]
with center $\overline{C} \cup \overline{D}$. Then Lemma 3.13 shows that $h^{-1}(\overline{C} \cup \overline{D})$ is disjoint from $Sm(Y)$ and $Sm(B)$, so $h$ induces isomorphisms
\[ Sm(Y) \cong Sm(X) - (\overline{C} \cup \overline{D}) = Sm(X) - C \]
\[ Sm(B) \cong Sm(A) - (\overline{C} \cup \overline{D}) = Sm(A) - D = Sm(A) \cap Sm(X) \]
Hence, $Sm(B) = Sm(Y) \cap B$, and $Sm(Y) \to Y$ and $Sm(B) \to B$ are Néron smoothings.

**Induction step:** assume $M := M(X, A) > 0$, and suppose that Proposition 3.15 holds for all pairs as in the statement with $M(\cdot, \cdot) < M$. Let $h_1 : (X_1, A_1) \to (X, A)$ be the admissible blow-up with center $\overline{C}$. By [13, 3.4.2] there exists a composition of admissible blow-ups $h_2 : (X_2, A_2) \to (X_1, A_1)$ such that $Sm(X_2) \to X_2$ is a Néron smoothening. Applying Proposition 3.9 we may suppose that $Sm(A_2) \to A_2$ is also a Néron smoothening.

Denote by $C_2$ the complement of $Sm(X_2) \cap Sm(A_2)$ in $Sm(X_2) \cap A_2$ (with its reduced closed subscheme structure), and by $\overline{C}_2$ its schematic closure in $(A_2)_s$. We put
\[ E_{\overline{\tau}_2} = \{ a \in X_2(R^{sh}) \mid a_s \in \overline{C}_2(k^s) \} \]
Since $\overline{C}_2 \subset (h_1 \circ h_2)^{-1}(\overline{C})$, Lemma 3.12 implies that $c_{A_2}(a) < M$ for each element $a$ of $E_{\overline{\tau}_2}$, so $M(X_2, A_2) < M$ and we may conclude by the induction hypothesis. \(\square\)

**Proof of Theorem 3.7.** By [13, 3.4.2], there exists a composition of admissible blow-ups $(X', A') \to (X, A)$ such that the natural map $Sm(X')(R^{sh}) \to X'(R^{sh})$ is a bijection. By Proposition 3.9 we can find a composition $h : (X'', A'') \to (X', A')$ of admissible blow-ups such that the maps $Sm(A'') \to A''$ and $Sm(X'') \to X''$ are Néron smoothings. Finally, we apply Proposition 3.13 to the pair $(X'', A'')$. \(\square\)

**Definition 3.14** (Weak Néron models of pairs). *If $(X_K, A_K)$ is a smooth pair of $K$-varieties, then a weak Néron model for $(X_K, A_K)$ is a smooth pair of $R$-varieties $(Y, B)$ endowed with an isomorphism of pairs of $K$-varieties $f : (Y_K, B_K) \to (X_K, A_K)$ such that the natural map $Y(R^{sh}) \to Y_K(K^{sh})$ is a bijection.*

Note that $B(R^{sh}) \to B_K(K^{sh})$ will automatically be a bijection: any section $a$ in $Y(R^{sh})$ with $a_K \in B_K(K^{sh})$ belongs to $B(R^{sh})$, since $B$ is closed in $Y$.

If $V_K$ is a smooth $K$-variety, then a smooth $R$-variety $W$ endowed with an isomorphism of $K$-varieties $g : W_K \to V_K$ is a weak Néron model for $V_K$ (in the sense of [13]) iff $(W, \emptyset)$ is a weak Néron model for $(V_K, \emptyset)$ w.r.t. the map $g$. 


Moreover, \((Y, B, f)\) is a weak Néron model for \((X, A)\) iff \((Y, f)\) is a weak Néron model for \(X\) and \((B, f|_B)\) is a weak Néron model for \(A\).

The following proposition gives a necessary and sufficient condition for the existence of a weak Néron model.

**Proposition 3.15.** A smooth pair of \(K\)-varieties \((X, A)\) admits a weak Néron model, iff \(X_K(K^\mathrm{sh})\) is bounded in \(X_K\) (in the sense of \([13, 1.1.2]\)).

**Proof.** By \([13, 3.5.7]\), \(X_K(K^\mathrm{sh})\) is bounded in \(X_K\) iff there exists a \(R\)-variety \(X'\) whose generic fiber is isomorphic to \(X_K\) and such that the natural map \(X'(R^\mathrm{sh}) \to X_K'(K^\mathrm{sh})\) is a bijection, so boundedness is obviously a necessary condition for the existence of a weak Néron model.

Let us prove that it is also sufficient. We fix an isomorphism between \(X_K'\) and \(X_K\). If we denote by \(A'\) the scheme-theoretic closure of \(A_K\) in \(X'\), then the generic fiber of \((X', A')\) is isomorphic to the pair \((X_K, A_K)\).

By Theorem 3.7 there exists a Néron smoothening \(g : (Y, B) \to (X', A')\). The pair \((Y, B)\) endowed with the isomorphism \(g_K : (Y_K, B_K) \to (X_K', A_K) \cong (X_K, A_K)\) is a weak Néron model of \((X_K, A_K)\). □

**Corollary 3.16.** Any smooth and proper pair of \(K\)-varieties admits a weak Néron model.

**Proof.** For any proper \(K\)-variety \(X_K\), \(X_K(K^\mathrm{sh})\) is bounded in \(X_K\) by \([13, 1.1.6]\). □

We’ll take a closer look at this boundedness condition in the next section.

### 4. Bounded varieties and weak Néron models

We keep the notations of Section 3 and we assume moreover that \(R\) is complete.

**Definition 4.1** (Bounded and smoothly bounded varieties). Let \(L\) be a discretely valued field, and let \(X\) be a \(L\)-variety. We say that \(X\) is bounded if \(X(L^\mathrm{sh})\) is bounded in \(X\) (in the sense of \([13, 1.1.2]\)). We say that \(X\) is smoothly bounded if \(X\) is bounded and the natural map \(\text{Sm}(X)(L^\mathrm{sh}) \to X(L^\mathrm{sh})\) is a bijection.

**Remark.** If \(\text{Sm}(X)\) is bounded and the natural map \(\text{Sm}(X)(L^\mathrm{sh}) \to X(L^\mathrm{sh})\) is a bijection, then \(X\) is smoothly bounded by \([13, 1.1.4]\). The converse holds if the ring of integers of \(L\) is excellent \([13, 1.1.9]\). □

**Definition 4.2** (Bounded and smoothly bounded rigid varieties). We say that a rigid \(K\)-variety \(X\) is bounded if it is separated and there exists a quasi-compact open subvariety \(V\) of \(X\) such that the natural map \(V(K') \to X(K')\) is a bijection for each finite unramified extension \(K'\) of \(K\). If, moreover, \(V\) is smooth, then we call \(X\) smoothly bounded.

If \(X\) is a rigid \(K\)-variety, then strictly speaking, the set \(X(K^a)\) is not defined since \(K^a\) is not an affinoid \(K\)-algebra. Therefore, we put \(X(K^a) = \bigcup_{K'/K} X(K')\) where \(K'\) runs through the finite extensions of \(K\) inside the fixed algebraic closure \(K^a\). The set \(X(K^\mathrm{sh})\) is defined similarly. If \(Y\) is a \(K\)-variety, then the analytification map \(Y^\text{an} \to Y\) induces natural bijections \(Y(K^a) = Y^\text{an}(K^a)\) and \(Y(K^\mathrm{sh}) = Y^\text{an}(K^\mathrm{sh})\).

The definition of a bounded rigid variety appeared earlier in \([13, 1.2]\) and \([36, 5.6]\). The following proposition compares it to Definition 4.1 for algebraic varieties.
Proposition 4.3. Let $X$ be an algebraic variety over $K$, and denote by $X^{an}$ its analytification.

(a) If $E$ is a subset of $X(K^a)$, then $E$ is bounded in $X$ iff there exists a quasi-compact open subvariety $V$ of $X^{an}$ such that $E$ is contained in $V(K^a)$.

(b) In particular, $X^{an}$ is bounded iff $X$ is bounded, and $X^{an}$ is smoothly bounded iff $X$ is smoothly bounded.

Proof. If $E$ is bounded in $X$, the existence of a subvariety $V$ as in the statement follows easily from the definition [13, 1.1.2]. So suppose conversely that $V$ is a quasi-compact open subvariety of $X^{an}$ such that $E$ is contained in $V(K^a)$.

Choose a finite cover of $X$ by affine open subschemes $U_1, \ldots, U_r$. It is clear from the definition [13, 1.1.2] and the Maximum Modulus Principle [11, 6.2.1.4] that, for any affine $K$-variety $U$, a subset $F$ of $U(K^a)$ is bounded in $U$ iff there exists a quasi-compact open subvariety $W$ of $U^{an}$ such that $E$ is contained in $W(K^a)$. Therefore, it suffices to construct, for each index $i \in \{1, \ldots, r\}$, a quasi-compact open subvariety $V_i$ of $(U_i)^{an}$ such that $V(K^a) \subset \bigcup V_i(K^a)$.

Now $\{(U_1)^{an}, \ldots, (U_r)^{an}\}$ is an admissible open cover of $X^{an}$ [11, 0.3.3], and 

is an admissible open cover of $V$. Since $V$ is quasi-compact, this cover can be refined by a finite affinoid cover $W = \{W_1, \ldots, W_q\}$. If we define $V_i$ as the union of those members $W_j$ of the cover $W$ which are contained in $(U_i)^{an}$, for $i = 1, \ldots, r$, then $V_i$ is a quasi-compact open subvariety of $(U_i)^{an}$ and $\bigcup V_i(K^a) = V(K^a)$. This concludes the proof of (a).

Applying this result to $E = X(K^a)$ we see that $X^{an}$ is bounded iff $X$ is bounded. Since, moreover, $X$ is smooth at a closed point $x$ iff $X^{an}$ is smooth at $x$ [16, 5.2.1], we see that $X^{an}$ is smoothly bounded iff $X$ is smoothly bounded.

Corollary 4.4. Let $S$ be a discrete valuation ring, with quotient field $L$ and residue field $k$, and let $R$ be its completion. Let $X$ be a $L$-variety, and assume either that $X$ is smooth or that $S$ is excellent. Then $(X \times_L K)^{an}$ is bounded iff $X$ is bounded, and $(X \times_L K)^{an}$ is smoothly bounded iff $X$ is smoothly bounded.

Proof. We fix an embedding of $L^*$ in $K^s$. We know from Proposition 4.3 that $(X \times_L K)^{an}$ is bounded iff $(X \times_L K)(K^s)$ is bounded in $X \times_L K$. This is also equivalent to the property that $X(K^s)$ is bounded in $X$ [13, 1.1.5], which implies that $X(L^s)$ is bounded in $X$.

Assume, conversely, that $X$ is bounded. We have to show that $X(K^s)$ is bounded in $X$. If we denote by $K'$ the closure of $L^s$ inside the completion of $K^s$, then $K^s$ is a subfield of $K'$. By [13, 1.1.5], $X(L^s)$ (viewed as a subset of $X(K')$) is bounded in $X \times_L K'$. Since $X$ is smooth or $S$ (and hence $S^s$ [21, 5.6]) is excellent, we can apply [19, 3.6.10] and we see that $X(L^s)$ is dense in $X(K')$ (w.r.t. the topology induced by the valuation on $L$). It is clear from the definition [13, 1.1.2] that this implies that $X(K')$ is bounded in $X$. Therefore, $X(K^s)$ is bounded in $X$.

Now assume that $S$ is excellent. By [21, 17.7.2], $Sm(X \times_L K)$ is canonically isomorphic to $Sm(X) \times_L K$. Combining this with Proposition 4.3 we see that $X$ is smoothly bounded if $(X \times_L K)^{an}$ is smoothly bounded. Conversely, if $X$ is smoothly bounded, then $Sm(X)$ is bounded since $S$ is excellent, so $(Sm(X) \times_L K)^{an}$ is bounded by the first part of Corollary 4.3. Hence, to show that $(X \times_L K)^{an}$ is
smoothly bounded, it suffices to show that the natural map

\[ Sm((X \times_L K)^{an})(K^{sh}) \to (X \times_L K)^{an}(K^{sh}) \]

is a bijection. By [16, 5.2.1], the source of this map is canonically isomorphic to

\[ (Sm(X \times_L K))^{an}(K^{sh}) \]

so it is enough to show that

\[ Sm(X \times_L K)(K^{sh}) \to (X \times_L K)(K^{sh}) \]

is a bijection. This follows from [13, 3.6.10] by the same arguments as above. \( \square \)

Lemma 4.5. If \( f : Y \to X \) is a proper morphism of separated rigid \( K \)-varieties, then \( Y \) is bounded if \( X \) is bounded. The same is true if \( f : Y \to X \) is a proper morphism of \( K \)-varieties.

Proof. Since \( K \) is a discretely valued field, the analytification of a proper morphism of \( K \)-varieties is a proper map of separated rigid \( K \)-varieties by the concluding remarks in [16, \S 5.2], so we only have to prove the result in the rigid analytic setting. There it follows from the fact that the inverse image of a quasi-compact open subvariety under a proper morphism is again quasi-compact. \( \square \)

Let \( X \) be a variety over an arbitrary field \( F \). A compactification of \( X \) is a dense open immersion \( X \to \overline{X} \) of \( X \) into a proper \( F \)-variety \( \overline{X} \). Such a compactification always exists by Nagata’s embedding theorem. We denote by \( \partial \overline{X} \) the complement of \( X \) in \( \overline{X} \) (with its reduced closed subscheme structure). If \( X \) is smooth and \( F \) has characteristic zero, then \( X \) admits a smooth compactification by Hironaka’s resolution of singularities.

Proposition 4.6. Let \( L \) be a discretely valued field, and let \( X \) be a smooth \( L \)-variety. We assume that \( X \) admits a smooth compactification. The following properties are equivalent:

1. \( X \) is bounded
2. there exists a compactification \( \overline{X} \) of \( X \) such that \( \partial \overline{X}(L^{sh}) = \emptyset \)
3. for every smooth compactification \( \overline{X} \) of \( X \), \( \partial \overline{X}(L^{sh}) = \emptyset \)

So, in characteristic zero, boundedness means that there are “no unramified points at infinity”.

Proof. The implication (2) \( \Rightarrow \) (1) was shown in [13, 1.1.10] (only assuming that the ring of integers of \( L \) is excellent) and (3) \( \Rightarrow \) (2) follows from our assumption. So let us prove (1) \( \Rightarrow \) (3). By Proposition 4.4 we may assume that \( L \) is complete. We denote its ring of integers by \( R \) and its residue field by \( k \). Let \( \overline{X} \) be any smooth compactification of \( X \). Let \( \mathfrak{X} \) be a weak Néron model for \( \overline{X}^{an} \). By boundedness of \( X \) and [12, 4.4] we may assume that there exists an open formal subscheme \( \mathfrak{Y} \) of \( \mathfrak{X} \) such that \( \mathfrak{Y}(L) \) is contained in \( X^{an} \), and such that \( \mathfrak{Y}(L') = X^{an}(L') \) for each finite unramified extension \( L'/L \).

Suppose that any closed point \( x \) on \( \mathfrak{X}_s \) whose residue field is separable over \( k \) is contained in \( \mathfrak{Y}_s \). Then

\[ \overline{X}(L^{sh}) = \mathfrak{X}_s(L^{sh}) = \mathfrak{Y}_s(L^{sh}) = X(L^{sh}) \]

and the lemma is proven. Hence, we may assume that there exists a closed point \( x \) in the complement of \( \mathfrak{Y}_s \) in \( \mathfrak{X}_s \) whose residue field is separable over \( k \). Passing to a finite unramified extension of \( R \), we may suppose that \( x \in \mathfrak{X}_s(k) \).
The tube $|x|$ of $x$ in $\mathfrak{X}$ is an open rigid subvariety of $\mathfrak{X}_\eta$ (see [9] 1.1.2), and hence of $X^{an}$. Since $\mathfrak{X}$ is formally smooth over $R$, the map

$$\mathfrak{X}(R/\mathfrak{M}^{n+1}) \to \mathfrak{X}(R/\mathfrak{M}^n)$$

is surjective for each $n \geq 0$, and by completeness of $R$, $x$ lifts to a section in $\mathfrak{X}(R)$. Hence, there is an isomorphism of $R$-algebras

$$\mathcal{O}_{\mathfrak{X},x} \cong R[[y_1, \ldots, y_d]]$$

with $d = \dim(X)$, by [13] 3.1.2. Moreover, by [9] 0.2.7, $|x|$ is canonically isomorphic to the generic fiber of the special formal $R$-scheme $\text{Spf} \mathcal{O}_{\mathfrak{X},x}$, which is the open unit polydisc $\mathbb{B}_L^d$ of dimension $d$ over $L$. Moreover, by our assumptions, all $L$-valued points of $|x|$ are contained in $(|x| \cap (\overline{\partial X})^{an})(L)$. However, since this implies that $\mathbb{B}_L^d \subset (\overline{\partial X})^{an}$, while on the other hand $\dim(\overline{\partial X}) < d$, we arrive at a contradiction.

\[\square\]

**Remark.** If $L$ is a henselian discretely valued field and $X$ an irreducible $L$-variety with a smooth $L$-rational point, then $X(L)$ is dense in $X$. This is well-known and can be proven in an elementary way; it can also be deduced from the existence of weak Néron models using an argument similar to the one in the proof of (1) $\Rightarrow$ (3).

\[\square\]

**Definition 4.7** (Weak Néron model of a rigid variety [14], Def. 1.3). Let $X$ be a separated rigid $K$-variety. A weak Néron model for $X$ is a smooth separated formal $R$-scheme $\mathfrak{X}$, topologically of finite type, endowed with an open immersion $h : \mathfrak{X}_\eta \to X$, such that $h$ induces a bijection $\mathfrak{X}_\eta(K') \to X(K')$ for each finite unramified extension $K'/K$.

**Proposition 4.8.** A separated rigid $K$-variety $X$ admits a weak Néron model iff $X$ is smoothly bounded.

**Proof.** This condition is obviously necessary. It is also sufficient: observe that, if $V$ is a smooth quasi-compact open subvariety of $X$ with $V(K^{sh}) = X(K^{sh})$, a weak Néron model for $V$ is also a weak Néron model for $X$, and apply [14] 3.3].

We establish some elementary properties of weak Néron models which we’ll need in the following section.

**Lemma 4.9.** Let $S$ be a discrete valuation ring, with maximal ideal $\mathfrak{M}$ and quotient field $L$, and let $R$ be its completion. Let $X$ be a smooth and bounded $L$-variety, and let $Y$ be a smooth $S$-variety endowed with an isomorphism $f : Y_L \to X$ such that $(Y,f)$ is a weak Néron model for $X$. We put $Y_K = Y_L \times_L K$. Denote by $\mathfrak{Y} \to \text{Spf} R$ the formal $\mathfrak{M}$-adic completion of $Y \to \text{Spec} S$ and by $h$ the composition

$$\mathfrak{Y} \longrightarrow (Y_K)^{an} \xrightarrow{f^{an}} (X \times_L K)^{an}$$

where the first arrow is the canonical open immersion [9] 0.3.5). Then $(\mathfrak{Y}, h)$ is a weak Néron model for $(X \times_L K)^{an}$.

**Proof.** We only have to show that the canonical open immersion $\mathfrak{Y}_\eta \to (Y_K)^{an}$ induces a bijection $\mathfrak{Y}_\eta(K') \to (Y_K)^{an}(K')$, for any finite unramified extension $K'/K$. By definition of the analytification functor $(\cdot)^{an}$ (see e.g. [9] 0.3.3), there is a natural map of locally ringed sites $(Y_K)^{an} \to Y_K$ which induces a canonical bijection $Y_K(K') = (Y_K)^{an}(K')$. Moreover, since $Y$ is a weak Néron model for $Y_L$,
it follows from [13, 3.6.7] that $Y \times_S R$ is a weak Néron model for $Y_K$, so the natural map $Y(R') \to Y_L(K')$ is a bijection, with $R'$ the normalization of $R$ in $K'$. Hence, the result follows from the canonical bijections $\mathfrak{M}_Y(K') = \mathfrak{M}(R') = Y(R')$. □

**Lemma 4.10.** If $X, Y$ are smoothly bounded rigid varieties over $K$, and if $(X, f)$ and $(Y, g)$ are weak Néron models of $X$, resp. $Y$, then

$$(X \times R \mathfrak{M}, f \times_K g : X_N \times_K \mathfrak{M}_Y \to X \times_K Y)$$

is a weak Néron model for $X \times_K Y$.

**Proof.** Since smoothness is preserved under base-change, and the composition of two smooth morphisms is again smooth, we see that $X \times R \mathfrak{M}$ is a smooth $\text{stft}$ formal $R$-scheme. Note also that the fibered product commutes with taking generic fibers [12, 4.6], so that the generic fiber of $X \times R \mathfrak{M}$ is canonically isomorphic to $X_N \times \mathfrak{M}_Y$. As a fiber product of two open immersions, the morphism $f \times_K g$ is again an open immersion. It follows immediately from the universal property of the fiber product that $(X \times R \mathfrak{M}, f \times_K g)$ is a weak Néron model for $X \times_K Y$. □

5. Motivic Serre invariants for algebraic varieties

In this section, we assume that $R$ is complete, and that the residue field $k$ of $R$ is perfect.

**Definition 5.1** (Motivic Serre invariant). Let $X$ be a smoothly bounded rigid $K$-variety, and let $(X, h)$ be a weak Néron model for $X$. We define the motivic Serre invariant $S(X)$ of $X$ by

$$S(X) = [X_s] \in K_0(Var_k)/(\mathbb{L} - 1)$$

This invariant only depends on $X$, and not on the choice of a weak Néron model.

If $Y$ is a smoothly bounded $K$-variety, then the associated rigid $K$-variety $Y_{an}$ is smoothly bounded by Proposition 4.3, so $S(Y_{an})$ is well-defined, and we put

$$S(Y) = S(Y_{an}) \in K_0(Var_k)/(\mathbb{L} - 1)$$

The fact that $S(X)$ only depends on $X$, and not on the choice of a weak Néron model was proven in [32, 4.5.3] for $X$ smooth and quasi-compact, using the theory of motivic integration on formal schemes, and in [36, 5.11] for $X$ smooth and bounded. The proof of [36, 5.11] also applies to the case where $X$ is smoothly bounded. Note that $S(X) = 0$ if $X(K^{sh}) = \emptyset$, and more generally, $S(X) = S(X')$ if $X$ is a bounded open rigid subvariety of $X$ such that $X(K^{sh}) = X'(K^{sh})$.

**Lemma 5.2.** Let $L$ be a discretely valued field, with perfect residue field $k$, and let $K$ be its completion. If $X$ is a smooth and bounded $L$-variety, and $(Y, f)$ is a weak Néron model for $X$, then

$$S(X \times_L K) = [Y_s] \in K_0(Var_k)/(\mathbb{L} - 1)$$

In particular, this value only depends on $X \times_L K$ and not on the chosen weak Néron model.

**Proof.** This follows immediately from Lemma 4.9. □
Lemma 5.3. Let $X$ be a smooth and bounded $K$-variety, and $A$ a closed subvariety of $X$, smooth over $K$. Denote by $h : X' \to X$ the blow-up of $X$ at $A$, and by $E$ the exceptional divisor $h^{-1}(A)$. Then

$$S(X') - S(E) = S(X) - S(A)$$

in $K_0(\text{Var}_K)/((\mathbb{L} - 1))$.

Proof. Let $((Y, B), f)$ be a weak Néron model for $(X, A)$. By Lemma 5.2, $S(X) = [Y_s]$ and $S(A) = [B_s]$ in $K_0(\text{Var}_K)/((\mathbb{L} - 1))$. Denote by $h : Y' \to Y$ the blow-up of $Y$ at $B$, and by $F = h^{-1}(B)$ the exceptional divisor. Since $B$ is smooth over $R$, $Y'$ and $F$ are also smooth over $R$. Moreover, since blowing up commutes with flat base change, the isomorphism

$$f : (Y_K, B_K) \to (X, A)$$

induces an isomorphism

$$f' : (Y'_K, F_K) \to (X', E)$$

We’ll show that $((Y', F), f')$ is a weak Néron model for $(X', E)$. We only have to prove that any $K^{sh}$-valued point $x$ on $Y'_K$ extends to a section in $Y'((R^{sh})$. Since $(Y, B)$ is a weak Néron model for $(X, A)$, the point $h(x) \in Y_K(K^{sh})$ extends to a section $a$ in $Y((R^{sh})$. But $h$ is proper, so $x$ itself extends to a section in $Y'((R^{sh})$.

This implies that $S(X') = [Y'_s]$ and $S(E) = [F_s]$, and since $h$ restricts to an isomorphism $Y' - F \to Y - B$, we have $[Y'_s] - [F_s] = [Y_s] - [B_s]$ in $K_0(\text{Var}_K)$, so the result follows. \qed

Theorem 5.4. Assume that $K$ has characteristic zero. There exists a unique ring morphism

$$S : \mathcal{M}_K \to K_0(\text{Var}_K)/((\mathbb{L} - 1))$$

such that $S([X]) = S(X)$ for any smooth and proper $K$-variety $X$. It satisfies $S([X]) = S(X)$ for any smoothly bounded $K$-variety $X$, and $S(\mathbb{L} - 1) = 0$.

Proof. By Theorem 2.8 and Lemma 5.3, there exists a unique morphism of abelian groups

$$S : K_0(\text{Var}_K) \to K_0(\text{Var}_K)/((\mathbb{L} - 1))$$

such that $S([X]) = S(X)$ for any smooth and proper $K$-variety $X$. By Lemma 4.10 and the fact that the analytification functor $(\cdot)^{an}$ commutes with fiber products, $S$ is a morphism of rings. We have

$$S(\mathbb{L}) = S(\mathbb{P}^1_K) - S(\text{Spec } K) = \mathbb{L} - 1$$

in $K_0(\text{Var}_K)/((\mathbb{L} - 1))$, so $S$ localizes to a ring morphism on $\mathcal{M}_K$ and $S(\mathbb{L} - 1) = 0$. It remains to show that $S([X]) = S(X)$ if $X$ is smoothly bounded. We proceed by induction on the dimension of $X$.

If $X$ has dimension 0, then $X$ is proper and smooth over $K$, so $S([X]) = S(X)$ by definition. Suppose that $\dim(X) > 0$. Since $K$ has characteristic zero and $X$ is reduced, the $K$-smooth locus $Sm(X)$ of $X$ is open and dense in $X$. But $X$ is smoothly bounded, so $(X - Sm(X))(K^{sh})$ is empty (and in particular, $X - Sm(X)$ is smoothly bounded). Since

$$\dim(X - Sm(X)) < \dim(X)$$

we know that $S([X - Sm(X)]) = S(X - Sm(X)) = 0$ by the induction hypothesis. By additivity, $S([X]) = S([Sm(X)])$, so we may assume that $X$ is smooth over $K$. 

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We embed $X$ as a dense open subscheme in a smooth proper $K$-variety $\overline{X}$, and we denote the boundary $\overline{X} - X$ by $\partial \overline{X}$. Since $X$ is bounded, we know that $\partial \overline{X}(K^{sh}) = \emptyset$, by Proposition 4.6. Again by our induction hypothesis, this implies that $S(\partial \overline{X}) = 0$, so

$$S([X]) = S([\overline{X}]) = S(\overline{X}) = S(X)$$

as required. \hfill \Box

**Definition 5.5** (Motivic Serre invariant of an algebraic variety). Assume that $K$ has characteristic zero. For any separated $K$-scheme of finite type $X$, we define the motivic Serre invariant $S(X)$ of $X$ as the image of $[X]$ under the morphism

$$S : K_0(Var_K) \to K_0(Var_K)/(L - 1)$$

**Example 5.6.** If $X$ is the cusp $\text{Spec} K[x, y]/(x^2 - y^3)$, then $X^{an}$ is not bounded, so $X^{an}$ does not admit a weak Néron model (and neither does $X$). However, we can break up $X$ into the disjoint union of the origin $O$ and its complement

$$Y = \text{Spec} K[x, x^{-1}, y, y^{-1}]/(x^2 - y^3)$$

Since $Y$ is isomorphic to the torus $G_m,K$, we get $S(Y) = 1$ in $K_0(Var_K)/(L - 1)$. Alternatively, we can use the fact that the normalization of $Y$ is isomorphic to $\mathbb{A}^1_K$ and that the inverse image of the singular point in this normalization consists of a unique $K$-point.

**Example 5.7.** Let $X$ be a rational projective curve with $\delta$ nodes, and no other singularities, and suppose that all the nodes and their tangent directions are rational over $K$. Then the normalization $\overline{X}$ is isomorphic to $\mathbb{P}^1_K$, and over each node of $X$ lie exactly 2 points of $\overline{X}$, which are $K$-rational. Hence,

$$S(X) - \delta = S(\overline{X}) - 2\delta$$

whence $S(X) = 2 - \delta$ in $K_0(Var_K)/(L - 1)$.

**Lemma 5.8.** Assume that $K$ has characteristic zero, and let $X$ be a variety over $K$. If $X(K^{sh}) = \emptyset$, then $S(X) = 0$, and if $k = k^s$ and $X(K)$ is finite, then $S(X) = \sharp X(K)$.

**Proof.** By additivity, it suffices to prove the result when $X(K^{sh}) = \emptyset$. Then $X$ is smoothly bounded, so $S(X) = S(X^{an})$. But $S(X^{an}) = 0$ since the empty formal scheme is a weak Néron model for $X^{an}$. \hfill \Box

6. The trace formula

In this section, we assume that $R$ is complete and $k$ algebraically closed, and we fix a prime number $\ell$ invertible in $k$. For each integer $d > 0$ prime to the characteristic exponent $p$ of $k$, we denote by $K(d)$ the unique extension of degree $d$ of $K$ in a fixed separable closure $K^s$. We denote by $K^t$ the tame closure of $K$ in $K^s$.

For any pro-finite group $H$, we denote by $\text{Rep}_H(\mathbb{Q}_\ell)$ the abelian tensor category of $\ell$-adic representations of $H$ (i.e. finite dimensional $\mathbb{Q}_\ell$-vector spaces endowed with a continuous left action of $H$) and by $K_0(\text{Rep}_H(\mathbb{Q}_\ell))$ its Grothendieck ring. For each element $h$ of $H$, we consider the unique ring morphism

$$Tr_h : K_0(\text{Rep}_H(\mathbb{Q}_\ell)) \to \mathbb{Q}_\ell$$
mapping $[M]$ to $\text{Trace}(h|\ M)$ for any $\ell$-adic representation $M$ of $H$.

Denote by $G_K$ the monodromy group $G(K^s/K)$. Consider the étale realization morphism

$$\text{ét}: K_0(\text{Var}_K) \to K_0(\text{Rep}_{G_K}(\mathbb{Q}_\ell))$$

from Section 2.1

If we denote by $G'_K$ the tame monodromy group $G(K'/K)$, then there is a natural surjective morphism $G_K \to G'_K$ whose kernel is the wild inertia group $P$. This morphism induces a canonical morphism of rings

$$K_0(\text{Rep}_{G'_K}(\mathbb{Q}_\ell)) \to K_0(\text{Rep}_{G_K}(\mathbb{Q}_\ell))$$

Since $P$ is a pro-$p$-group and $\ell$ is prime to $p$, the functor

$$(\cdot)^P: \text{Rep}_{G_K}(\mathbb{Q}_\ell) \to \text{Rep}_{G'_K}(\mathbb{Q}_\ell)$$

is exact, so it defines a morphism of abelian groups

$$(\cdot)^P: K_0(\text{Rep}_{G_K}(\mathbb{Q}_\ell)) \to K_0(\text{Rep}_{G'_K}(\mathbb{Q}_\ell))$$

which is left inverse to

$$K_0(\text{Rep}_{G'_K}(\mathbb{Q}_\ell)) \to K_0(\text{Rep}_{G_K}(\mathbb{Q}_\ell))$$

Hence, the latter morphism is injective, and we may identify $K_0(\text{Rep}_{G'_K}(\mathbb{Q}_\ell))$ with its image in $K_0(\text{Rep}_{G_K}(\mathbb{Q}_\ell))$. Then an element $\alpha$ of $K_0(\text{Rep}_{G_K}(\mathbb{Q}_\ell))$ belongs to $K_0(\text{Rep}_{G'_K}(\mathbb{Q}_\ell))$ iff $(\alpha)^P = \alpha$.

**Definition 6.1 (Tame varieties).** If $X$ is a smooth and proper $K$-variety, then we say that $X$ is tame if there exists a regular proper $R$-variety $Y$ such that $Y_s$ is a tame strict normal crossings divisor (i.e. the multiplicity of each component is tame) and such that $Y_K$ is isomorphic to $X$. Such a model $Y$ will be called a tame $R$-model for $X$.

The tame Grothendieck ring of varieties over $K$ is the subring $K'_0(\text{Var}_K)$ of $K_0(\text{Var}_K)$ generated by the isomorphism classes $[X]$ of tame smooth proper $K$-varieties $X$.

Of course, if $k$ has characteristic zero, then any smooth and proper $K$-variety is tame, and $K'_0(\text{Var}_K) = K_0(\text{Var}_K)$.

**Lemma 6.2.** The image of the étale realization morphism

$$\text{ét}: K'_0(\text{Var}_K) \to K_0(\text{Rep}_{G_K}(\mathbb{Q}_\ell))$$

is contained in $K_0(\text{Rep}_{G'_K}(\mathbb{Q}_\ell))$. If $X$ is a $K$-variety such that $[X]$ belongs to $K'_0(\text{Var}_K)$, then

$$\text{ét}(X) = \sum_{i \geq 0} (-1)^i [H^i_c(X \times_K K^s, \mathbb{Q}_\ell)]$$

in $K_0(\text{Rep}_{G'_K}(\mathbb{Q}_\ell))$.

**Proof.** If $X$ is a tame, smooth and proper $K$-variety, and $Y$ is a tame $R$-model for $X$, then by [39, 2.23], the $\ell$-adic nearby cycles complex $R\psi_\eta(\mathbb{Q}_\ell)$ of $Y$ is tame, i.e. $P$ acts trivially on $R\psi_\eta(\mathbb{Q}_\ell)$ for each $i \geq 0$. By the spectral sequence [1] 1.2.2.3 this implies that $P$ acts trivially on $H^i(X \times_K K^s, \mathbb{Q}_\ell)$, for each $i \geq 0$. Since the isomorphism classes of tame smooth and proper $K$-varieties $X$ generate the subring $K'_0(\text{Var}_K)$ of $K_0(\text{Var}_K)$, we see that the image of étale realization morphism

$$\text{ét}: K'_0(\text{Var}_K) \to K_0(\text{Rep}_{G_K}(\mathbb{Q}_\ell))$$
Corollary 6.6. If \( \text{Proof.} \) The "if" part follows from Lemma 5.8 and Corollary 6.5. For the converse implication we can take for vectors \( U \) a rational point iff there exists a subvariety \( \text{Corollary 6.5.} \) this follows from Proposition 6.3 and Lemma 6.2. \( \square \)

The following elementary example was given in \([37, \S 2]\). Since \( \ell \) is invertible in \( \hat{k} \), and \( P \) is a pro-\( p \)-group, there is a canonical isomorphism

\[ H^i_c(X \times_K K^t, \mathbb{Q}_\ell) \cong H^i_c(X \times_K K^s, \mathbb{Q}_\ell)^P \]

for any \( K \)-variety \( X \) and each \( i \geq 0 \). Hence,

\[ \hat{\epsilon}(X)^P = \sum_{i \geq 0} (-1)^i[H^i_c(X \times_K K^t, \mathbb{Q}_\ell)] \]

in \( K_0(\text{Rep}_{G^t_K}(\mathbb{Q}_\ell)) \). If \( [X] \) belongs to \( K_0^0(\text{Var}_K) \) then \( \hat{\epsilon}(X)^P = \hat{\epsilon}(X) \) by the first part of the proof, and we are done. \( \square \)

Proposition 6.3 (Trace formula for tame varieties). Let \( \varphi \) be a topological generator of the tame monodromy group \( G^t_K \). If \( X \) is a tame smooth and proper \( K \)-variety, then

\[ \chi_{\text{top}}(S(X \times_K K(d))) = \text{Trace}(\varphi^d \mid H(X \times_K K^t, \mathbb{Q}_\ell)) \]

for each integer \( d > 0 \) prime to \( p \).

Proof. This follows immediately from the trace formula in \([37, 5.4]\) and the comparison theorem for étale cohomology \([9, 7.5.4]\). See also \([37, 5.4]\) for an explicit expression in terms of a tame \( R \)-model of \( X \). \( \square \)

Theorem 6.4 (Trace formula). Assume that \( K \) has characteristic zero. If \( d > 0 \) is an integer prime to \( p \) and \( \varphi \) is a topological generator of the tame Galois group \( G^t_K \) of \( K(d) \), then the following diagram of ring morphisms commutes:

\[
\begin{array}{cccccc}
K_0^0(\text{Var}_K) & \xrightarrow{\hat{\epsilon}} & K_0(\text{Var}_K(d)) & \xrightarrow{S} & K_0(\text{Var}_K)/(\mathbb{L} - 1) & \\
\downarrow & & \downarrow & & \downarrow & \\
K_0(\text{Rep}_{G^t_K}(\mathbb{Q}_\ell)) & \xrightarrow{\text{Tr}_{\varphi^d}} & \mathbb{Q}_\ell &
\end{array}
\]

(the upper left horizontal morphism is the natural base change morphism). In particular, for any \( K \)-variety \( X \) such that \( [X] \) belongs to \( K_0^0(\text{Var}_K) \), we have

\[ \chi_{\text{top}}(S(X \times_K K(d))) = \text{Trace}(\varphi^d \mid H_c(X \times_K K^t, \mathbb{Q}_\ell)) \]

Proof. Since the classes \( [X] \) of tame smooth proper \( K \)-varieties generate \( K_0^0(\text{Var}_K) \), this follows from Proposition 6.3 and Lemma 6.2. \( \square \)

Corollary 6.5. If \( k \) has characteristic zero, then for any \( K \)-variety \( X \),

\[ \chi_{\text{top}}(S(X)) = \text{Trace}(\varphi \mid H_c(X \times_K K^s, \mathbb{Q}_\ell)) \]

Corollary 6.6. If \( k \) has characteristic zero, and if \( X \) is a \( K \)-variety, then \( X \) has a rational point iff there exists a subvariety \( U \) of \( X \) such that

\[ \text{Trace}(\varphi \mid H(U \times_K K^s, \mathbb{Q}_\ell)) \neq 0 \]

Proof. The “if” part follows from Lemma 5.8 and Corollary 6.5. For the converse implication we can take for \( U \) a rational point on \( X \). \( \square \)

There are examples of (non-tame) smooth and proper \( K \)-varieties \( X \) such that

\[ \chi_{\text{top}}(S(X)) \neq \text{Trace}(\varphi \mid H_c(X \times_K K^t, \mathbb{Q}_\ell)) \]

The following elementary example was given in \([37, \S 5]\): let \( R \) be the ring of Witt vectors \( W(F_p) \) over the algebraic closure of a finite field \( F_p \) of characteristic \( p \), and
put $X = \text{Spec } K[T]/(T^p - p)$. Then $X$ is smooth and proper over $K$, and since $X(K) = \emptyset$, we have $S(X) = 0$. On the other hand, $H^i(X \times_K K^t, \mathbb{Q}_\ell) = 0$ for $i > 0$, and $H^0(X \times_K K^t, \mathbb{Q}_\ell)$ is isomorphic to $\mathbb{Q}_\ell$ with the trivial $G'_K$-action, so that

$$\text{Trace}(\varphi|H(X \times_K K^t, \mathbb{Q}_\ell)) = 1$$

Of course, it would be very interesting to obtain a cohomological interpretation of $\chi_{\text{top}}(S(X))$ in terms of $\ell\text{t}(X)$ if $X$ is not tame, already in the case where $X$ is smooth and proper over $K$. We will see below that this is not always possible (Proposition 7.1).

**Definition 6.7** (Error term). Let $\varphi$ be a topological generator of the tame Galois group $G'_K$. If $X$ is any smooth and proper $K$-variety, we put

$$e(X) = \text{Trace}(\varphi|H(X \times_K K^t, \mathbb{Q}_\ell)) - \chi_{\text{top}}(S(X))$$

We say that the trace formula holds for $X$ iff $e(X) = 0$.

In particular, by Corollary 6.5 the trace formula holds for any $K$-variety $X$ if $k$ has characteristic zero.

7. Trace formula for curves

In this section, we assume that $R$ is complete and $k$ is algebraically closed. We fix a prime $\ell$ invertible in $k$. We denote by $\varphi$ a topological generator of the tame Galois group $G(K^t/K)$, and by $P \subset G(K^t/K)$ the wild inertia group.

**Definition 7.1** (Cohomological tameness). If $X$ is a $K$-variety, we say that $X$ is cohomologically tame if $P$ acts trivially on $H^i_c(X \times_K K^t, \mathbb{Q}_\ell)$ for each $i \geq 0$.

If $X$ is a tame smooth proper $K$-variety, then $X$ is cohomologically tame (cf. proof of Lemma 6.2). We will study the validity of the trace formula for smooth, proper, geometrically connected curves over $K$, and we will see that there are remarkable connections with T. Saito’s criterion for cohomological tameness [41].

7.1. A general result for curves. If $Y$ is a regular $R$-variety and $Y_s$ is a normal crossings divisor, we denote the irreducible components of $(Y_s)_{\text{red}}$ by $E_i$, $i \in I$, and we denote by $N_i$ the multiplicity of $E_i$ in $Y_s$. We write $Y_s = \sum_{i \in I} N_i E_i$ as usual. For each $i \in I$, we put

$$E_i^o = E_i \setminus (\cup_{j \neq i} E_j)$$

and we denote by $\text{Sm}(E_i^o)$ its $k$-smooth locus. If $Y_s$ has strict normal crossings, then $\text{Sm}(E_i^o) = E_i^o$.

**Definition 7.2** (Wild locus). Let $Y$ be a regular $R$-variety such that $Y_s$ is a normal crossings divisor. If $k$ has characteristic $p > 0$, then we define the wild locus $W_Y$ of $Y$ as the disjoint union of the subvarieties $\text{Sm}(E_i^o)$ of $Y$ with $N_i = p^{e_i}$ for some $e_i > 0$. If $k$ has characteristic zero, we put $W_Y = \emptyset$.

**Theorem 7.3.** Let $X$ be a smooth and proper curve over $K$, and let $Y$ be a regular $R$-model for $X$ such that $Y_s$ has strict normal crossings. Then

$$e(X) = \chi_{\text{top}}(W_Y)$$

so the trace formula holds for $X$ iff $\chi_{\text{top}}(W_Y) = 0$. 
Proof. If \( y \) is a closed point of \( Y_s \), then the computation of the tame nearby cycles in [II 1.3.3] shows that

\[
\text{Trace}(\varphi | R\psi^t_\eta(Q_\ell)_y) = \begin{cases} 
0 & \text{if } y \in Sm(Y_s) \cup W_Y \\
1 & \text{else.} 
\end{cases}
\]

Moreover, by [II 3.3], the complex \( R\psi^t_\eta(Q_\ell) \) is tamely constructible (in the sense of [35]), so [35 6.3] applies and

\[
\text{Trace}(\varphi | H(X \times_K K^t, \mathbb{Q}_\ell)) = \text{Trace}(\varphi | H(Y_s, R\psi^t_\eta(Q_\ell))) = \chi_{\text{top}}(Sm(Y_s)) + \chi_{\text{top}}(W_Y)
\]

Since \( Y \) is regular, \( Sm(Y) \) is a weak Néron model for \( X \) (cf. remark following [13 3.1.2]) so \( \chi_{\text{top}}(S(X)) = \chi_{\text{top}}(Sm(Y_s)) \). \( \square \)

7.2. Curves of genus \( \neq 1 \).

**Theorem 7.4.** Let \( X \) be a proper smooth geometrically connected curve over \( K \) of genus \( g \neq 1 \), and assume that \( X \) is cohomologically tame. Then the trace formula holds for \( X \).

**Proof.** In view of Corollary [6.3] we may suppose that \( k \) has characteristic \( p > 0 \). Let \( Y \) be a relatively minimal regular \( R \)-model with normal crossings of \( X \) (\( RMN \)-model in the terminology of [II 3.1.1]), with \( Y_s = \sum_{i \in I} E_i \). Then by Saito's criterion [II 3.11], the fact that \( X \) is cohomologically tame implies that \( E_i^p \) is smooth and \( \chi_{\text{top}}(E_i^p) = 0 \) if \( p \) divides \( N_i \), so \( \chi_{\text{top}}(W_Y) = 0 \) and we may conclude by Theorem [6.3]. \( \square \)

7.3. Elliptic curves.

**Theorem 7.5.** Let \( X \) be an elliptic curve over \( K \).

- \( X \) has multiplicative reduction iff \( S(X) = 0 \)
- \( X \) has additive reduction iff \( S(X) \in \{1, 2, 3, 4\} \). In this case, \( S(X) = n \), with \( n \) the number of connected components of the special fiber of the Néron minimal model of \( X \). More precisely:
  - \( S(X) = 1 \) iff \( X \) is of type \( II \) or \( II^* \);
  - \( S(X) = 2 \) iff \( S(X) \) is of type \( III \) or \( III^* \);
  - \( S(X) = 3 \) iff \( X \) is of type \( IV \) or \( IV^* \);
  - \( S(X) = 4 \) iff \( X \) is of type \( I_{1, \nu}^* \), \( \nu \geq 0 \).
- \( X \) has good reduction \( \overline{X} \) iff \( S(X) \notin \{0, 1, 2, 3, 4\} \), and in this case, \( S(X) = \lceil \overline{X} \rceil \)

In particular, \( \chi_{\text{top}}(S(X)) = 0 \) iff \( X \) has semi-stable reduction. Moreover, the trace formula holds for \( X \) iff we're in one of the following situations:

- \( X \) is cohomologically tame,
- \( p = 2 \) and \( X \) is of type \( III \) or \( III^* \).

**Proof.** By definition, \( S(X) = [\mathcal{A}_s] \in K_0(Var_k)/(L-1) \) where \( \mathcal{A} \) is the Néron model of \( X \). It follows immediately that \( S(X) = 0 \) if \( X \) has multiplicative reduction, \( S(X) = \lceil \overline{X} \rceil \) if \( X \) has good reduction \( \overline{X} \) and \( S(X) = n \) if \( X \) has additive reduction, with \( n \) the number of connected components of \( \mathcal{A}_s \). The values for \( n \) can be read from the Kodaira-Néron reduction table (see e.g. [46 IV.9]). We only have to check that \( \lceil \overline{X} \rceil \notin \{0, 1, 2, 3, 4\} \subset K_0(Var_k)/(L-1) \) if \( X \) has good reduction \( \overline{X} \).
However, for any elliptic curve $E$ over $k$, its Poincaré polynomial $P(E; T)$ is equal to $1 + 2T + T^2$ which is not congruent to any integer modulo $P(L - 1; T) = T^2 - 1$.

By Theorem 7.3, Saito’s criterion [41 3.11] and direct computation on the reduction table, we see that the trace formula holds for $X$ if we’re in one of the two cases described in the statement (for a more precise analysis, see below). □

Let us investigate the cases where $X$ is not cohomologically tame. By Saito’s criterion [41 3.11] this happens exactly in the following situations:

(1) $k$ has characteristic 2, and $X$ has type $II$, $II^*$, $III$, $III^*$, or $I_\nu$, $\nu \geq 0$.

(2) $k$ has characteristic 3, and $X$ has type $II$, $II^*$, $IV$ or $IV^*$.

Using the expression for $e(X)$ in Theorem 7.3 we can read the following values from the reduction table:

(1) Suppose that $k$ has characteristic 2. If $X$ has type $II$ or $II^*$, then $e(X) = 1$.
   If $X$ has type $III$ or $III^*$, then $e(X) = 0$ and the trace formula holds.
   If $X$ has type $I_\nu$, $\nu \geq 0$, then $e(X) = -2$.

(2) Suppose that $k$ has characteristic 3. If $X$ has type $II$ or $II^*$ then $e(X) = 1$.
   If $X$ has type $IV$ or $IV^*$ then $e(X) = -1$.

Remark. It seems reasonable to expect that the trace formula holds for all cohomologically tame abelian varieties $A$. If $k$ has characteristic zero, this follows from Corollary [43]. If $k$ has positive characteristic, the trace formula holds if $A$ does not have purely additive reduction, and also if $A$ is the Jacobian of a curve. Details and further results will appear in a forthcoming paper. □

7.4. Curves of genus 1 without rational point. Finally, we discuss the case of curves of genus 1 without rational point. Let $X$ be a smooth, proper, geometrically connected $K$-curve of genus 1. Then its Jacobian $Jac(X)$ is an elliptic curve. If we denote by $m(X)$ the order of the torsor $X$ in the group $H^1(K, Jac(X))$, then the reduction type of $X$ is equal to $m(X)$ times the reduction type of $Jac(X)$, by [30 6.6] (i.e. the multiplicities of the components of the reduction are multiplied by $m(X)$).

Theorem 7.6. Let $X$ be a smooth, proper, geometrically connected $K$-curve of genus 1, and assume that $X(K)$ is empty. Then $S(X) = 0$, and

$$e(X) = \chi_{top}(S(Jac(X))) + e(Jac(X))$$

The trace formula holds for $X$ iff

1. $k$ has characteristic 0, or
2. $k$ has characteristic $p > 0$ and $Jac(X)$ has semi-stable reduction.

Proof. The fact that $X(K)$ is empty implies that $S(X) = 0$, since the empty scheme is a weak Néron model for $X$. Moreover, there exists a canonical $G_K^t$-equivariant isomorphism

$$H(X \times_K K^t, \mathbb{Q}_\ell) \cong H(Jac(X) \times_K K^t, \mathbb{Q}_\ell)$$

so that

$$Trace(\varphi | H(X \times_K K^t, \mathbb{Q}_\ell)) = Trace(\varphi | H(Jac(X) \times_K K^t, \mathbb{Q}_\ell))$$

and

$$e(X) = \chi_{top}(S(Jac(X))) + e(Jac(X))$$
Hence, the trace formula holds for $X$ iff

$$\chi_{\text{top}}(S(\text{Jac}(X))) + e(\text{Jac}(X)) = 0$$

We know from Corollary 6.5 that the trace formula holds for $X$ if $k$ has characteristic zero, so assume that $k$ has characteristic $p > 0$. The computations in Section 7.5 show that

$$\chi_{\text{top}}(S(\text{Jac}(X))) + e(\text{Jac}(X)) = 0$$

iff $\text{Jac}(X)$ has semi-stable reduction. □

Proposition 7.7. If $k$ has characteristic $p > 0$, then there exists a smooth, proper, geometrically connected curve $X$ over $K$ of genus 1 such that $X$ is cohomologically tame and such that the trace formula does not hold for $X$.

Proof. Choose a cohomologically tame elliptic curve $E$ over $K$ such that $E$ has additive reduction. This is possible for any value of $p$, by Saito’s criterion [41, 3.11]. Since $k$ is algebraically closed and $K$ is complete, we have $H^1(K, E) \neq 0$ (as noted in [30, 6.7] this follows from the results in [3] in the mixed characteristic case, and from those in [8] in the equicharacteristic case). Any non-zero element in $H^1(K, E)$ corresponds to a smooth, proper, geometrically connected curve $X$ over $K$ of genus 1 without rational point, whose Jacobian is isomorphic to $E$. By the existence of a $G_K$-equivariant isomorphism

$$H(X \times_K K^*, \mathbb{Q}_\ell) \cong H(E \times_K K^*, \mathbb{Q}_\ell)$$

we know that $X$ is cohomologically tame. Since the trace formula holds for $E$, by Theorem 7.5 we see that the trace formula holds for $X$ iff

$$\chi_{\text{top}}(S(X)) = \chi_{\text{top}}(S(E))$$

However, the left hand side vanishes, while the right hand side is non-zero by Theorem 7.6. □

The example shows that $\chi_{\text{top}}(S(X))$ can, in general, not be computed from the étale realization $\hat{\text{et}}(X)$ (nor even from the Chow motive with rational coefficients $M(X)$ of $X$ since $X$ and $\text{Jac}(X)$ have the same étale realization (and isomorphic Chow motives [42, 3.3])). We will see below (proof of Proposition 7.9) that, even if $k$ has characteristic zero, $S(X)$ can in general not be computed from $M(X)$ (even though $\chi_{\text{top}}(S(X))$ can be computed from $\hat{\text{et}}(X)$ by the trace formula in Corollary 9.5).

Over a finite field $\mathbb{F}_q$, the situation of Proposition 7.7 does not occur: every smooth, proper, geometrically connected curve $X$ of genus 1 over $\mathbb{F}_q$ admits a rational point, since $H^1(\mathbb{F}_q, E) = 0$ for every elliptic curve $E$ over $\mathbb{F}_q$. This result can be interpreted as a consequence of Grothendieck’s trace formula: if $X$ is a $E$-torsor then $\hat{\text{et}}(X) = \hat{\text{et}}(E)$, so since $E$ has a rational point the same holds for $X$.

Playing with these ideas, we recover the following classical result.

Proposition 7.8. Let $E$ be an elliptic curve over $K$ with additive reduction.

1. If $k$ has characteristic zero, then $H^1(K, E) = 0$.
2. If $k$ has characteristic $p > 0$, then $H^1(K, E)$ is a $p$-group.

Proof. 1. We know that the trace formula holds if $k$ has characteristic zero, by Corollary 6.5. Since for any $E$-torsor $X$, $\hat{\text{et}}(X) = \hat{\text{et}}(E)$, and $\chi_{\text{top}}(S(E)) \neq 0$, we conclude that $S(X) \neq 0$, so $X$ has a rational point.
2. Assume that $H^1(K, E)$ contains an element whose order $m$ is not a power of $p$. It corresponds to a smooth, proper, geometrically connected curve $X$ of genus 1, with $Jac(X) \cong E$. Since the reduction type of $X$ is equal to $m$ times the reduction type of $E$, we see that the trace formula holds for $X$, by Theorem 7.3 since the wild locus of the minimal regular model with normal crossings is empty. This contradicts Theorem 7.6. (For a more direct proof: $Tr(\varphi | H(X \times_K K^t, \mathbb{Q}_t)) = 0$ by the computation in the proof of Theorem 7.3; a similar computation shows that $Tr(\varphi | H(E \times_K K^t, \mathbb{Q}_t)) \neq 0$, which is a contradiction.)

**Proposition 7.9.** Assume that $K$ has characteristic zero. The natural ring morphisms

$$\chi^{eff} : K_0(Var_K) \to K_0(Mot_K^{eff})$$

$$\chi : \mathcal{M}_K \to K_0(Mot_K)$$

from Section 2.8 are both non-injective.

**Proof.** Let $E$ be an elliptic curve over $K$ with good reduction, and let $X$ be a non-trivial $E$-torsor. Such a torsor $X$ exists since $H^1(K, E) \neq 0$ by [45]. We have $\chi^{eff}(E) = \chi^{eff}(X)$ by [12, 3.3], but $S(E) \neq 0$ by Theorem 7.11 while $S(X) = 0$ since $X$ has no rational point. The ring morphism

$$S : \mathcal{M}_K \to K_0(Var_k)/(L - 1)$$

from Theorem 5.4 maps $[X]$ and $[E]$ to $S(X)$, resp. $S(E)$, so $[E] \neq [X]$ in $\mathcal{M}_K$. □

**Proposition 7.10.** Assume that $K$ has characteristic zero. If $A$ is an abelian variety over $K$ with good reduction, then $[A]$ is a zero divisor in $K_0(Var_K)$, in $\mathcal{M}_K$, and in $K_0(Var_K)/(L - 1)$.

**Proof.** let $X$ be a non-trivial $A$-torsor. Such a torsor $X$ exists since $H^1(K, A) \neq 0$ by [45]. Then $S(X) = 0$ and $S(A) = [A] \in K_0(Var_k)/(L - 1)$, with $A$ the reduction of $A$. Since the Poincaré polynomial $P(A; T)$ is not divisible by $P(L - 1) = T^2 - 1$ we see that $S(A) \neq 0$ and therefore $[A] \neq [X]$ in $K_0(Var_K)/(L - 1)$. However, $A \times_K A$ and $X \times_K A$ are isomorphic over $K$ and hence $([A] - [X]) \cdot [A] = 0$ in $K_0(Var_K)$. □

It is shown in [31, 5.11] that, more generally, $[A]$ is a zero-divisor in $K_0(Var_k)$ if $k$ is a field of characteristic zero and $A$ is an abelian variety over $k$ such that $H^1(k, A) \neq 0$, but their proof doesn’t extend to $\mathcal{M}_k$.

7.5. **The local case.** Following [11], we can also state a local variant of Theorem 7.4. The category of special formal $R$-schemes is defined as in [35, 2.2]. The generic fiber $\mathfrak{X}_\eta$ of a special formal $R$-scheme $\mathfrak{X}$ is a bounded rigid $K$-variety, by [30, 5.8]. We denote by $\mathfrak{X}_0$ the reduction of $\mathfrak{X}$, i.e. the closed subscheme defined by the largest ideal of definition on $\mathfrak{X}$.

**Theorem 7.11** (Local case). Let $X$ be a normal flat $R$-variety of pure relative dimension 1, and let $x$ be a closed point of $X$ such that $X - \{x\}$ is smooth over $R$. Denote by $\mathfrak{F}_x$ the generic fiber of the special formal $R$-scheme $\text{Spf} \hat{O}_{X,x}$. If $P$ acts trivially on $H^1(\mathfrak{F}_x \hat{\otimes}_R \hat{K}^s, \mathbb{Q}_t)$, then

$$\chi_{\text{top}}(S(\mathfrak{F}_x)) = \text{Trace}(\varphi | H(\mathfrak{F}_x \hat{\otimes}_R \hat{K}^t, \mathbb{Q}_t)) = \text{Trace}(\varphi | R\psi_\eta^*(\mathbb{Q}_t)_x)$$
Note that $S(\mathcal{F}_x)$ is well-defined since $\mathcal{F}_x$ is a bounded and smooth rigid $K$-variety. Following the terminology in [37, 39], we call $\mathcal{F}_x$ the analytic Milnor fiber of $X$ at $x$.

Proof. The second equality follows from [7, 3.5]. By [41, 4.12] there exists a proper morphism $h : Y \to X$ of $R$-varieties such that $h_K : Y_K \to X_K$ is an isomorphism, $Y$ is regular, $Y_s = \sum_{i \in I} N_i E_i$ is a strict normal crossings divisor, and

$$\chi_{top}(W_Y \cap h^{-1}(x)) = 0$$

If we denote by $\mathfrak{f}$ the formal completion of $Y$ along $h^{-1}(x)$, then $h$ induces an isomorphism $\mathfrak{f} \cong \mathcal{F}_x$ because $h$ is proper.

Since $Y$ is regular, it follows from [13, 3.1.2] that $Sm(\mathfrak{f}) \to \mathfrak{f}$ is a special Néron smoothening (in the sense of [35, 4.11]) and we see from [35, 4.14] that

$$\chi_{top}(S(\mathcal{F}_x)) = \chi_{top}(Sm(\mathfrak{f})) = \sum_{N_i = 1} \chi_{top}(E_i \cap h^{-1}(x))$$

Moreover, there is a canonical $G(K^t/K)$-equivariant isomorphism

$$H(\mathfrak{f} \times_K K^t, \mathcal{O}_F) \cong H(h^{-1}(x), R\psi_Y(\mathcal{O}_F)|_{h^{-1}(x)})$$

by the comparison results in [7, 3.5]. Now the arguments in the proof of Theorem 7.3 show that

$$\chi_{top}(S(\mathcal{F}_x)) = \text{Trace}(\varphi | H(\mathcal{F}_x \times_K K^t, \mathcal{O}_F))$$

\hfill \Box

Theorem 7.12. Let $X$ be a flat, proper, normal $R$-variety of pure relative dimension 1 such that $X - Sm(X)$ is a finite set of points, and such that the $\ell$-adic nearby cycles of $X$ are tame. Then the trace formula holds for $X_K$.

Proof. By [41, 4.12] there exists a proper morphism $h : Y \to X$ of $R$-varieties such that $h_K : Y_K \to X_K$ is an isomorphism, $Y$ is regular, $Y_s = \sum_{i \in I} N_i E_i$ is a strict normal crossings divisor, and $\chi_{top}(W_Y) = 0$. Now the result follows from Theorem 7.3. \hfill \Box

8. Appendix: the Poincaré polynomial

Let $k$ be any field. It is, in general, a non-trivial problem to decide whether the classes of two $k$-varieties $X$, $Y$ in $K_0(Var_k)$ are distinct. (Larsen and Lunts formulated the following question in [29]: does $[X] = [Y]$ in $K_0(Var_k)$ imply that $X$ and $Y$ are piecewise isomorphic? See [31] for results in this direction.)

To distinguish elements in $K_0(Var_k)$, it is important to know some “computable” realization morphisms on $K_0(Var_k)$. If $k$ has characteristic zero, we’ve encountered many of these in the preceding sections, but in positive characteristic, we’re less equipped. In this section, we’ll show how the so-called Poincaré polynomial can be defined over arbitrary base fields by means of a standard spreading out technique.

We recall the following notation: for any field $k$, any prime $\ell$ invertible in $k$, and any separated $k$-scheme of finite type $X$, we denote by $b_i(X)$ the $i$-th $\ell$-adic Betti number of $X$:

$$b_i(X) = \dim H^i(X \times_k k^\times, \mathcal{O}_F)$$

It is known that this value is independent of $\ell$ in the following cases:

- $k$ has characteristic zero (by comparison with singular cohomology)
\* \( k \) has characteristic \( p > 0 \) and \( X \) is smooth and proper over \( k \) (if \( k \) is finite this follows from the cohomological expression for the zeta function and purity of weight [27, p. 27]; the general case follows by spreading out to reduce to a finite base field).

To be precise, \( b_i(X) \) not only depends on the scheme \( X \) but also on the base field \( k \); if we want to make this explicit, we write \( b_i(f) \) instead of \( b_i(X) \), with \( f : X \to \text{Spec } k \) the structural morphism.

8.1. Characteristic zero. If \( k \) is a field of characteristic zero, there exists a unique ring morphism \( P : K_0(\text{Var}_k) \to \mathbb{Z}[T] \) mapping the class \([X]\) of a smooth and proper \( k\)-variety to the polynomial

\[
P(X; T) = \sum_{i \geq 0} (-1)^i b_i(X) T^i
\]

Uniqueness and existence follow from Theorem 6.6. The morphism \( P \) can also be obtained by composing the Hodge-Deligne realization \( HD \) with the ring morphism

\[
\mathbb{Z}[u, v] \to \mathbb{Z}[T] : a(u, v) \mapsto a(T, T)
\]

For any element \( \alpha \) of \( K_0(\text{Var}_k) \), we call \( P(\alpha) \) the Poincaré polynomial of \( \alpha \); for any separated \( k\)-scheme of finite type \( Y \), we put \( P(Y; T) = P([Y]) \) and we call this element of \( \mathbb{Z}[T] \) the Poincaré polynomial of \( Y \). Then \( P(Y; T) = HD(Y; T, T) \), and \( P(Y; 1) = HD(Y; 1, 1) \) is the Euler characteristic \( \chi_{\text{top}}(Y) \) of \( Y \).

If we write

\[
P(Y; T) = \sum_{i \geq 0} (-1)^i \beta_i(Y) T^i
\]

then \( \beta_i(Y) \) is known as the \( i \)-th virtual Betti number of \( Y \). If \( Y \) is proper and smooth, then \( \beta_i(Y) = b_i(Y) \). Note that, in general, \( \beta_i(Y) \) can be negative: for instance,

\[
P(\mathbb{G}_m, k; T) = P(\mathbb{P}^1_k; T) - P(\{0\}; T) - P(\{\infty\}; T) = T^2 - 1
\]

For \( k = \mathbb{C} \),

\[
\beta_i(Y) = \sum_{j \geq 0} (-1)^{i+j} \dim Gr_i^W H_j^i(Y(\mathbb{C}), \mathbb{Q})
\]

The invariants \( P(Y; T) \) and \( \beta_i(Y) \) not only depend on the scheme \( Y \) but also on the base field \( k \); if we want to make this explicit, we’ll write \( P(f; T) \) and \( \beta_i(f) \), with \( f : Y \to \text{Spec } k \) the structural morphism.

8.2. Finite base field. We can also define a Poincaré polynomial for a finite base field \( k \), using Deligne’s theory of weights. Denote by \( q \) the cardinality of \( k \). Recall that, for any integer \( w \geq 0 \), a Weil number of weight \( w \) (w.r.t. \( q \)) is an algebraic integer \( \alpha \) such that \( |i(\alpha)| = q^{w/2} \) for each embedding \( i : \mathbb{Q}(\alpha) \to \mathbb{C} \). A fundamental result by Deligne [18, 3.3.4] says the following: if \( X \) is a separated \( k\)-scheme of finite type, and \( \ell \) a prime invertible in \( k \), then for any integer \( i \geq 0 \), each eigenvalue \( \alpha \) of the geometric Frobenius on \( H^i_c(X \times_k k^s, \mathbb{Q}_l) \) is a Weil number, and its weight \( w(\alpha) \) is contained in \( \{0, \ldots, i\} \). Moreover, if \( X \) is proper and smooth over \( k \), then \( w(\alpha) = i \) ("purity of weight" [18, 3.3.5]).

**Definition 8.1.** Assume that \( k \) is finite. For any separated \( k\)-scheme of finite type \( X \) and each pair of integers \( i, j \geq 0 \), we define \( \beta_{ij}^\ell(X) \) as the number of weight \( i \) eigenvalues (counted with multiplicities) of the geometric Frobenius on \( H^j_c(X \times_k k^s, \mathbb{Q}_l) \).
We put \( \beta_i(X) = \sum_{j \geq 0} (-1)^{i+j} \beta^j_i(X) \), and we call this integer the \( i \)-th virtual Betti number of \( X \). We define the Poincaré polynomial \( P(X; T) \) of \( X \) by

\[
P(X; T) = \sum_{i \geq 0} (-1)^i \beta_i(X) T^i
\]

The virtual Betti numbers \( \beta_i(X) \), and hence the Poincaré polynomial \( P(X; T) \), are independent of \( \ell \) : as noted in \([27, p. 28 (2b)]\), \((-1)^{i+1} \beta^j_i(X) \) is the degree of the “weight \( i \) part” of the zeta function of \( X \) (beware that Katz’ definition of virtual Betti number differs from ours by a factor \((-1)^i\)). By purity of weight, \( b_i(X) = \beta^j_i(X) = \beta_i(X) \) if \( X \) is proper and smooth over \( k \). The invariants \( P(X; T) \) and \( \beta^j_i(X) \) not only depend on the scheme \( X \), but also on the base field \( k \). If we want to make the base field explicit, we’ll write \( P(f; T) \) and \( \beta_i(f) \) instead, with \( f : X \to \text{Spec } k \) the structural morphism.

**Lemma 8.2** (Additivity and multiplicativity). Assume that \( k \) is finite. There exists a unique ring morphism

\[
P : K_0(\text{Var}_k) \to \mathbb{Z}[T]
\]

which maps the class \([X]\) of any \( k \)-variety \( X \) to the Poincaré polynomial \( P(X; T) \).

**Proof.** Uniqueness is obvious. Well-definedness and additivity follow immediately from the excision long exact sequence; multiplicativity from the Künneth formula. Alternatively, \( P(X; T) \) can be computed from the étale realization \( \ell \hat{\text{et}}(X) \).

As noted above, the Poincaré polynomial still has the property

\[
P(X; T) = \sum_{i \geq 0} (-1)^i b_i(X) T^i
\]

for any proper and smooth \( k \)-variety \( X \), by purity of weight; however, it is not clear if this property uniquely defines the morphism \( P : K_0(\text{Var}_k) \to \mathbb{Z}[T] \) (unless we assume the existence of resolution of singularities for \( k \)-varieties).

### 8.3. Base field of characteristic \( p > 0 \)

Let \( X \) be a Noetherian scheme, and let \( B \) be a set. We denote by \( X^\circ \) the set of closed points of \( X \). We say that a function \( a : X^\circ \to B \) is constructible, if there exists a stratification \( \mathcal{S} \) of \( X \) into locally closed subsets, such that \( a \) is constant on \( S \cap X^\circ \) for each member \( S \) of \( \mathcal{S} \). Likewise, we say that a function \( b : X \to B \) is constructible if there exists a stratification \( \mathcal{T} \) of \( X \) into locally closed subsets, such that \( b \) is constant on \( T \) for each member \( T \) of \( \mathcal{T} \). We denote by \( \mathcal{C}(X, B) \), resp. \( \mathcal{C}(X^\circ, B) \), the ring of constructible functions on \( X \), resp. \( X^\circ \), with values in \( B \).

If \( X \) is a Jacobson scheme (e.g. of finite type over a field, or over \( \mathbb{Z} \)) then any constructible function \( a : X^\circ \to B \) extends uniquely to a constructible function \( a : X \to B \).

**Proposition 8.3.** Let \( k \) be a finite field. For any separated \( k \)-scheme of finite type \( X \), there exists a unique ring morphism

\[
P : K_0(\text{Var}_X) \to \mathcal{C}(X, \mathbb{Z}[T])
\]

such that \( P([Y])(x) = P(f_x; T) \) for every separated morphism of finite type \( f : Y \to X \) and every closed point \( x \) of \( X \). Here \( f_x : Y \times_X x \to \text{Spec } k(x) \) is the morphism obtained from \( f \) by base change.
If $g : X' \to X$ is a morphism of separated $k$-schemes of finite type, then the diagram
\[
\begin{array}{ccc}
K_0(\text{Var}_X) & \longrightarrow & K_0(\text{Var}_{X'}) \\
\downarrow P & & \downarrow P \\
C(X, \mathbb{Z}[T]) & \stackrel{(\cdot)g}{\longrightarrow} & C(X', \mathbb{Z}[T])
\end{array}
\]
commutes (the horizontal arrows are the natural base change morphisms).

**Proof.** Uniqueness of $P$ is obvious, since an element of $C(X, \mathbb{Z}[T])$ is determined by its values on $X^o$. To prove its existence, first note that the function $x \mapsto P(f_x; T)$ is constructible on $X^o$ since the sheaves $R^if_!(\mathbb{Q}_\ell)$ are mixed [18, 3.3.1]. Hence, this function extends uniquely to a function $P(f; T)$ in $C(X, \mathbb{Z}[T])$. The invariant $P(\cdot; T)$ satisfies the scissors relations in $K_0(\text{Var}_X)$: since the property of being a closed (resp. open) immersion is stable under base change, we can reduce to the case where $X$ is a point, which was proven in Lemma 8.2.

Commutativity of the base change diagram is also immediately reduced to the case where $X$ and $X'$ are points; this case is clear from the definition of the virtual Betti numbers. □

**Corollary 8.4.** Let $k$ be any field of characteristic $p > 0$. Using the notation in Section 2.4, there exists a unique ring morphism
\[
P : K_0(\text{Var}_k) \to \mathbb{Z}[T]
\]
such that, for any object $A$ of $\omega_k$ and any separated $A$-scheme of finite type $f : X \to \text{Spec } A$,
\[
(P \circ \phi)([X]) = P(f; T)(\eta) \in \mathbb{Z}[T]
\]
where $\eta$ is the generic point of $\text{Spec } A$.

**Proof.** This follows from Proposition 2.9 and Proposition 8.3. □

**Definition 8.5.** For any field $k$ of characteristic $p > 0$ and any separated $k$-scheme of finite type $X$, we define the Poincaré polynomial $P(X; T)$ of $X$ as the image of $[X]$ under the ring morphism
\[
P : K_0(\text{Var}_k) \to \mathbb{Z}[T]
\]
Writing
\[
P(X; T) = \sum_{i \geq 0} (-1)^i \beta_i(X) T^i
\]
we call $\beta_i(X) \in \mathbb{Z}$ the $i$-th virtual Betti number of $X$.

If we want to make the base field explicit, we write $P(f; T)$ and $\beta_i(f)$, with $f : X \to \text{Spec } k$ the structural morphism.

Note that the definition of $P(X; T)$ and $\beta_i(X)$ does not require the choice of a prime $\ell$ (since the definition over finite fields is independent of $\ell$).

**Remark.** We should point out that, if $k$ is finitely generated, the Poincaré polynomial can also be realized as the composition of the realization
\[
\mu_k : K_0(\text{Var}_k) \to K_0(\text{Rep}_{G_k, \mathbb{Q}_\ell})[T] : [X] \mapsto \sum_{i \geq 0} \left( \sum_{j \geq 0} (-1)^j [Gr^W_i H^j_\ell(X \times_k k^s, \mathbb{Q}_\ell)] \right) T^i
\]
from [34] with the forgetful ring morphism

\[ K_0(\text{Rep}_{G_k} \mathbb{Q}_\ell)[T] \to K_0(\mathbb{Q}_\ell)[T] \cong \mathbb{Z}[T] \]

\[ \square \]

### 8.4. Arbitrary base field

**Proposition 8.6.** For any field \( k \) and any separated \( k \)-scheme of finite type \( X \), \( P(X; 1) = \chi_{\text{top}}(X) \).

**Proof.** If \( k \) has characteristic zero, this follows from the fact that the equality holds for smooth and proper \( k \)-varieties, since their isomorphism classes generate \( K_0(\text{Var}_k) \) by Hironaka’s resolution of singularities. If \( k \) is finite, it follows immediately from the definition. If \( k \) is any field of characteristic \( p > 0 \), it follows from the finite field case and the fact that for any object \( A \in \mathcal{A}_k \) and any separated morphism of finite type \( f : X \to \text{Spec} A \), the function

\[ \text{Spec} A \to \mathbb{Z} : x \mapsto \chi_{\text{top}}(X \times_{\text{Spec} A} x) \]

is constructible, by constructibility of the sheaves \( R^1 f_!(\mathbb{Q}_\ell) \) and proper base change [33 VI(3.2)]. \[ \square \]

**Proposition 8.7.** Let \( k \) be any field, and \( X \) a separated \( k \)-scheme of finite type, of dimension \( n \). Then the Poincaré polynomial \( P(X; T) \) has degree \( 2n \), and the coefficient \( \beta_{2n}(X) \) of \( T^{2n} \) is equal to the number of irreducible components of dimension \( n \) of \( X \times_k k^s \).

**Proof.** We may assume that \( X \) is reduced. Passing to a finite separable extension of \( k \), we may assume that the irreducible components of \( X \) are geometrically irreducible. Now we proceed by induction on \( n \). If \( n = 0 \), then the statement is clear, so assume that we have proven the result for varieties of dimension \( < n \) over any field. Then taking away closed subvarieties from \( X \) of dimension \( < n \) does not change the value of \( \beta_i(X) \) for \( i \geq 2n \), so we may as well assume that the connected components of \( X \) are geometrically irreducible. By additivity, it suffices to consider the case where \( X \) itself is geometrically irreducible.

First, assume that \( k \) has characteristic zero. We may suppose that \( k \) is algebraically closed. The class \([X]\) of \( X \) in \( K_0(\text{Var}_k) \) can be written as the class \([Y]\) of a smooth, proper, irreducible \( k \)-variety \( Y \) plus a \( \mathbb{Z} \)-linear combination of classes \([Z_i]\) of \( k \)-varieties \( Z_i \) of dimension \( < n \), by Hironaka’s resolution of singularities. Hence, by the induction hypothesis, the Poincaré polynomial of \( X \) has degree at most \( 2n \), and \( \beta_{2n}(X) = b_{2n}(Y) = 1 \).

Now assume that \( k \) has characteristic \( p > 0 \). There exist an object \( A \in \mathcal{A}_k \) and a model \( X' \) for \( X \) over \( A \); by [23 9.7.7], we may assume that \( X \times_{\text{Spec} A} x \) is geometrically irreducible for each closed point \( x \) of \( \text{Spec} A \). By definition of the Poincaré polynomial, we may suppose that \( k \) is finite. Then the Poincaré polynomial of \( \chi \) has degree at most \( 2n \); and \( G_{2n}^W H_c^j(X \times_k k^s, \mathbb{Q}_\ell) \) vanishes for \( j \neq 2n \), because \( R^i f_!(\mathbb{Q}_\ell) \) is mixed of weight \( \leq i \) by [13 3.3.1]. Hence,

\[ \beta_{2n}(X) = \dim G_{2n}^W H_c^{2n}(X \times_k k^s, \mathbb{Q}_\ell) \]

Moreover, by [33 VI(11.3)] there exists a Galois-equivariant isomorphism

\[ H_c^{2n}(X \times_k k^s, \mathbb{Q}_\ell) \to \mathbb{Q}_\ell(-n) \]

so \( H_c^{2n}(X \times_k k^s, \mathbb{Q}_\ell) \) has pure weight \( 2n \) and \( \beta_{2n}(X) = 1 \). \[ \square \]
8.5. Arbitrary base scheme.

**Definition 8.8.** For any separated morphism of finite type \( f : Y \to X \) in \((\text{Sch})\), we denote by \( P(f; T) \) the function

\[
P(f; T) : X \to \mathbb{Z}[T] : x \mapsto P(f_x; T)
\]

where \( f_x : Y \times_X x \to \text{Spec} k(x) \) is the morphism obtained by base change. We call \( P(f; T) \) the Poincaré polynomial of \( f \). Writing \( P(f; T) \) as

\[
\sum_{i \geq 0} (-1)^i \beta_i(f) T^i
\]

we call the function \( \beta_i(f) : X \to \mathbb{Z} \) the \( i \)-th virtual Betti number of \( f \).

**Lemma 8.9** (Base Change). Let \( g : X' \to X \) be a morphism of schemes, and let \( f : Y \to X \) be a separated morphism of finite type. If we denote by \( f' : Y \times_X X' \to X \) the morphism obtained from \( f \) by base change, then \( P(f'; T) = P(f; T) \circ g \).

**Proof.** It suffices to consider the case where \( X = \text{Spec} k \) and \( X' = \text{Spec} k' \) with \( k \subset k' \) fields. If \( k \) has characteristic zero, the result follows from the fact that the \( \ell \)-adic Betti numbers are invariant under extension of the base field \([33, \text{VI}(4.3)]\).

If \( k \) has characteristic \( p > 0 \), it suffices to note that the diagram

\[
\begin{align*}
K_0(\text{Var}_A) & \xrightarrow{\phi_A^k} K_0(\text{Var}_k) \\
\phi_A^{k'} & \downarrow \\
K_0(\text{Var}_{k'}) & \xrightarrow{P} \mathbb{Z}[T]
\end{align*}
\]

commutes for each object \( A \) of \( \mathfrak{A}_k \) (both paths from \( K_0(\text{Var}_A) \) to \( \mathbb{Z}[T] \) coincide with the morphism \( P(\cdot)(\eta) \) with \( \eta \) the generic point of \( \text{Spec} A \)). \( \square \)

**Proposition 8.10.** Let \( X \) be a locally Noetherian scheme, and let \( f : Y \to X \) be a smooth and proper morphism. Then \( P(f; T) \) is locally constant, and for any point \( x \) of \( X \) and any integer \( i \geq 0 \), \( \beta_i(f)(x) = b_i(f_x) \) where \( f_x : Y \times_X x \to x \) is the morphism obtained from \( f \) by base change.

**Proof.** By definition, \( \beta_i(f)(x) = \beta_i(f_x) \) for each \( i \geq 0 \). If \( k(x) \) has characteristic zero, then \( \beta_i(f_x) = b_i(f_x) \) by definition; if \( k(x) \) is finite, the same holds by purity of weight. If \( k(x) \) has characteristic \( p > 0 \), we can always find an object \( A \) of \( \mathfrak{A}_{k(x)} \) and a smooth and proper \( A \)-model \( h : Z \to \text{Spec} A \) for \( f_x \) by \([23, \text{8.10.5}]\) and \([24, \text{17.7.8}]\). By definition, \( P(f_x; T) = P(h; T)(\eta) \) where \( \eta \) is the generic point of \( \text{Spec} A \).

For any point \( y \) of \( \text{Spec} A \), we denote by \( h_y : Y \times_A k(y) \to \text{Spec} k(y) \) the morphism obtained by base change. If \( y \) is closed, then \( k(y) \) is finite, and since \( h \) is smooth and proper, \( \beta_i(h)(y) = b_i(h_y) \). However, both sides of the equality are constructible as functions in \( y \in \text{Spec} A \); for the left hand side this follows from \( \text{Proposition 8.3} \) and for the right hand side by applying proper base change to the lisse sheaf \( R^i h_*(\mathbb{Q}_{\ell}) \) for any prime \( \ell \) invertible in \( k(x) \) \([33, \text{VI}(2.3+4.2)]\). Hence, \( \beta_i(f_x) = \beta_i(h)(\eta) = b_i(h_\eta) = b_i(f_x) \)

(the last equality follows from invariance of \( \ell \)-adic Betti numbers under extension of the base field \([33, \text{VI}(4.3)]\)).

Finally, the fact that \( P(f; T) \) is locally constant follows from the fact that the function \( x \mapsto b_i(f_x) \) is locally constant on \( X \) : we may assume that there exists
a prime $\ell$ invertible on $X$, and we apply proper base change to the lisse sheaf $R^i f_*(\mathbb{Q}_\ell)$.

Proposition 8.11 (Constructibility). Let $X$ be a Noetherian scheme. For any separated morphism of finite type $f : Y \to X$, the map

$$P(f; T) : X \to \mathbb{Z}[T]$$

is constructible.

Proof. By Noetherian induction, it suffices to find a non-empty open subscheme $U$ of $X$ such that $P(f; T)$ if constant on $U$, so we may assume that $X$ is integral and affine, say $X = \text{Spec } B$, and that there exists a prime $\ell$ invertible on $X$. By the canonical isomorphism $(Y_{\text{red}} \times_X X)_{\text{red}} \cong (Y \times_X X)_{\text{red}}$ for any point $x$ of $X$, we may suppose that $Y$ is reduced.

By [23, 8.8.2] there exists a finitely generated sub-$\mathbb{Z}[1/\ell]$-algebra $C$ of $B$, and a reduced separated $C$-scheme of finite type $Y'$, such that $Y$ is isomorphic to $Y' \times_C B$ over $B$. By Lemma 8.9, we may assume that $B = C$ and $Y = Y'$. Then the sheaves $R^i f_i^*(\mathbb{Q}_\ell)$ are mixed [13, 3.3.1], so there exists a non-empty open subset $U$ of $X$ such that $P(f_{\eta}; T) = P(f_{\eta}; T)$ for any pair of closed points $x, y$ on $X$. By definition of the Poincaré polynomial, this implies that $P(f_{\eta}; T) = P(f_{\eta}; T)$ for any pair of points $x, y$ of $U$ which lie over a closed point of $\text{Spec } \mathbb{Z}[1/\ell]$.

Hence, we may assume that the generic point $\eta$ of $X$ lies over the generic point of $\text{Spec } \mathbb{Z}[1/\ell]$. We proceed by induction on the dimension $n$ of $Y_{\eta} = Y \times_X \eta$.

If $Y_{\eta}$ is empty, then there exists an open neighbourhood $V$ of $\eta$ in $X$ such that the fibers of $f$ over $V$ are empty [23, 9.2.6], hence $P(f; T) = 0$ on $V$. So assume that $n \geq 0$ and that the result has been proven for morphisms for which the dimension of the generic fiber is $< n$. Let $\overline{f} : \overline{Y} \to X$ be a compactification of the morphism $f$ (i.e. $\overline{f}$ is proper and there exists a dense open immersion $j : Y \to \overline{Y}$ with $f = \overline{f} \circ j$). Denote by $\partial \overline{Y}$ the complement of $Y$ in $\overline{Y}$ (with its reduced closed subscheme structure). Then $\partial \overline{Y} \times_X \eta$ has dimension $< n$, so by the induction hypothesis and additivity of the Poincaré polynomial, we may as well assume that $Y = \overline{Y}$, i.e. that $f$ is proper.

Since $k(\eta)$ has characteristic zero, and $Y_{\eta}$ is reduced, there exists a proper birational morphism of $k(\eta)$-varieties $h' : Z' \to Y_{\eta}$ such that $Z'$ is proper and smooth over $k(\eta)$. Shrinking $X$, we may suppose that $h$ is obtained by base change from a proper birational morphism of $X$-varieties $h : Z \to Y$ with $Z$ smooth and proper over $X$, by [23, 8.8.2+9.6.1] and [24, 17.7.11]. Then we can find open subschemes $U$ and $V$ of $Z$, resp. $Y$, such that $h$ restricts to an isomorphism $U \cong V$, and such that $(Z - U) \times_X \eta$ and $(Y - V) \times_X \eta$ have dimension $< n$. By additivity and our induction hypothesis, it suffices to prove the result for the proper and smooth morphism $Z \to X$; this case was settled in Proposition 8.10.

Theorem 8.12. The Poincaré polynomial $P(\cdot ; T)$ is the unique invariant which associates to every separated morphism of finite type $f : Y \to X$ in $(\text{Sch})$ a function $P(f; T) : X \to \mathbb{Z}[T]$ with the following properties:

1. Constructibility: If $X$ is Noetherian, then $P(f; T)$ is constructible.
2. Base change: If $g : X' \to X$ is a morphism in $(\text{Sch})$ and $f' : Y \times_X X' \to X'$ is the morphism obtained by base change, then

$$P(f'; T) = P(f; T) \circ g$$
(3) If $X = \text{Spec } k$ with $k$ a finite field, then $P(f; T)$ is the Poincaré polynomial defined in Section 8.2.

If $X$ is a Noetherian scheme, then there exists a unique ring morphism

$$P(\cdot; T) : K_0(\text{Var}_X) \to C(X, \mathbb{Z}[T])$$

such that $P(Y; T) = P(f; T)$ for any separated $X$-scheme of finite type $f : Y \to X$.

If $g : X' \to X$ is a morphism of Noetherian schemes, then the square

$$
\begin{array}{ccc}
K_0(\text{Var}_X) & \to & K_0(\text{Var}_{X'}) \\
\downarrow P(\cdot; T) & & \downarrow P(\cdot; T) \\
C(X, \mathbb{Z}[T]) & \to & C(X', \mathbb{Z}[T]) \\
\end{array}
$$

commutes (the horizontal arrows are the natural base change morphisms).

Proof. We proved in Lemma 8.9 and Proposition 8.11 that the Poincaré polynomial satisfies (1) and (2), and (3) is clear by definition.

Let us show that such an invariant is unique. By (2), $P(\cdot; T)$ is uniquely determined by its values on morphisms $f : Y \to X$ with $X = \text{Spec } k$ and $k$ a field. If $f : X \to \text{Spec } k$ a separated morphism of finite type, we can find a finitely generated sub-$\mathbb{Z}$-algebra $C$ of $k$ and a separated morphism of finite type $h : Z \to \text{Spec } C$ such that $X$ is $k$-isomorphic to $Z \times_C k$, by [23, 8.8.2]. Then

$$P(f; T) = P(h; T)(\eta)$$

with $\eta$ the generic point of $\text{Spec } C$, by (2). The function $P(h; T)$ is a constructible function, by (1), so it is uniquely determined by its values on the closed points of $\text{Spec } C$, which have finite residue field. Hence, $P(h; T)$ is uniquely determined, by (2) and (3).

It only remains to show that $P(\cdot, T)$ satisfies the scissor relations in $K_0(\text{Var}_X)$, if $X$ is a Noetherian scheme. Since the property of being a closed (resp. open) immersion is stable under base change, we can reduce to the case where $X$ is a point; this case is clear from Lemma 8.2 and the definition of the Poincaré polynomial. □

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