A contribution of a $U(1)$-reducible connection to quantum invariants of links II: Links in rational homology spheres

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Abstract

We extend the definition of the $U(1)$-reducible connection contribution to the case of the Witten-Reshetikhin-Turaev invariant of a link in a rational homology sphere. We prove that, similarly to the case of a link in $S^3$, this contribution is a formal power series in powers of $q-1$ whose coefficients are rational functions of $q^\alpha$, their denominators being the powers of the Alexander-Conway polynomial. The coefficients of the polynomials in numerators are rational numbers, the bounds on their denominators are established with the help of the theorem proved by T. Ohtsuki in Appendix 2.

We derive a surgery formula for the $U(1)$-reducible connection contribution, which relates it to the similar contribution into the colored Jones polynomial of a surgery link in $S^3$.

Finally, we relate the $U(1)$-reducible connection contribution to a contribution of a $U(1)$-invariant stationary phase point to the Reshetikhin formula for the colored Jones polynomial in the appropriate semi-classical limit.

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1 Introduction

1.1 Motivation

In this paper we continue to study a new invariant of knots and links defined in [15] which we called a $U(1)$-reducible connection contribution to the colored Jones polynomial, or simply a $U(1)$-RC invariant. In [15] we studied the properties of this invariant for links in $S^3$. Our work was based on the $R$-matrix formula for the colored polynomial of a link.

Our task in this paper is twofold. First of all, we will show that the $U(1)$-RC invariant represents a particular stationary phase contribution to the integral in the Reshetikhin formula [9], [10], [11] for the colored Jones polynomial. This formula presents the expansion of the Jones polynomial in powers of log $q$ as an integral over the adjoint orbits of the Lie algebra $su(2)$. The orbits correspond to the $su(2)$ modules assigned to the link components and the integrand itself contains formal power series in log $q$ whose coefficients are $SU(2)$-invariant polynomial functions. If the Jones polynomial considered in the limit

$$q \to 1, \quad q^{\alpha_j} = \text{const},$$

where $\alpha_j$, $1 \leq j \leq L$ are the dimensions of the $su(2)$ modules assigned to $L$ components of a link, then the integral seems, at least formally, to be suitable for the stationary phase approximation. We will demonstrate that a contribution of a stationary point corresponding to a configuration were all the $su(2)$ integration variables belong to the same Cartan subalgebra, is well-defined and is equal to the $U(1)$-RC invariant.

The polynomials in the integrand of the Reshetikhin formula can be derived [1] from 3-valent diagrams (sometimes referred to as ‘chinese characters’) which come from the Kontsevich integral of a stringed link. Therefore, the relation between the Reshetikhin formula
and the $U(1)$-RC invariant will allow us [16] to formulate the latter solely in the language of 3-valent diagrams and Vassiliev invariants. However here we will use this new relation only in order to accomplish the second task of this paper - to extend the $U(1)$-RC invariant to links in rational homology spheres ($QHS$).

The Reshetikhin formula plays an important role in defining [12], [13] the trivial connection contribution to the Witten-Reshetikhin-Turaev (WRT) invariant of $QHS$ and, more generally, of links in $QHS$. We call this contribution the TCC invariant. In case of a link in $S^3$, the TCC invariant coincides with the colored Jones polynomial. The TCC invariant of a link in a general $QHS$ can be presented through an analog of the Reshetikhin formula. If a $QHS$ can be constructed by a surgery on a link $L^s \in S^3$, then the integrand of the Reshetikhin formula for the TCC invariant of a link $L$ in that $QHS$ can be expressed by a surgery formula in terms of the Reshetikhin integrand of the link $L \cup L^s \in S^3$. As a result, there exists a similar surgery formula which relates particular stationary phase contributions to both Reshetikhin integrals. We will use it in order to define the $U(1)$-RC invariant of a link $L$ in a $QHS$. The form and properties of this $U(1)$-RC invariant is very similar to those of the $U(1)$-RC invariant of links in $S^3$, except that the $U(1)$-RC invariant of links in $QHS$ is related to the TCC invariant of the links rather than to their WRT invariant, as in the case of $S^3$.

Since the Reshetikhin formula and the TCC invariant play a central role in our paper, we will have to review their properties prior to formulating our results.

1.2 Notations, definitions and basic properties

1.2.1 Multi-index and quantum notations

We adopt the same multi-index notations as in [15].

The quantum invariants of 3-manifolds and links depend on three parameters: $q$, $h$ and $K$ which are related

\[ q = \exp(2\pi i/K), \quad h = q - 1. \] (1.2)
We treat $q$, $h$, and $K$ as formal variables unless stated otherwise (the WRT invariant is defined for $K$ being a positive integer).

**Remark 1.1** Since

$$K^{-1} = \log(1 + h)/2\pi i = h/2\pi i + O(h^2) \quad \text{as } h \to 0,$$

then $\mathbb{C}[[h]] = \mathbb{C}[[K^{-1}]]$ and one can easily convert a power series in $h$ into a power series in $K^{-1}$ and back.

In this paper we will use a standard notation for a “quantum number”

$$[n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}.$$  \hfill (1.4)

### 1.2.2 Topological notations

First, we set the linking matrix notations. If two oriented knots $K_1, K_2$ in a 3-manifold $M$ represent trivial elements of $H_1(M; \mathbb{Q})$, then one can define their linking number $\text{lk}(K_1, K_2|_M)$.

Following the notations of C. Lescop [7], we denote the order of $K$ as an element of $H_1(M; \mathbb{Z})$ by $o(K)$ (for an $L$-component link $\mathcal{L}$ in a QHS $M$, $\mathcal{O} = (o_1, \ldots, o_L)$ denote the orders of its link components). Note that

$$\text{GCF}(o(K_1), o(K_2)) \text{lk}(K_1, K_2|_M) \in \mathbb{Z},$$  \hfill (1.5)

where GCF $(m, n)$ is the greatest common factor of $m$ and $n$.

We denote the entries of the linking matrix of an oriented link $\mathcal{L} \in M$ as

$$l_{ij}^{(\mathcal{L}|_M)} = \text{lk}(\mathcal{L}_i, \mathcal{L}_j|_M).$$  \hfill (1.6)

For a pair of oriented links $\mathcal{L}, \mathcal{L}' \in M$ we denote

$$l_{ij}^{(\mathcal{L}, \mathcal{L}'|_M)} = \text{lk}(\mathcal{L}_i, \mathcal{L}'_j|_M).$$  \hfill (1.7)
We use a shortcut notation for the quadratic form associated with the linking matrix of an oriented \(L\)-component link \(L\): 
\[
\text{lk}(L, M; x) = \sum_{1 \leq i \neq j \leq L} l_{ij}^{(L|M)} x_i x_j, 
\]
where \(x = (x_1, \ldots, x_L)\).

Let \(K\) be a knot in a manifold \(M\). A meridian of \(K\) is a simple cycle on the boundary of the tubular neighborhood of \(K\) which is contractible through that neighborhood. The meridian is uniquely defined. A parallel is a simple cycle on the boundary of the tubular neighborhood of \(K\) which has a unit intersection number with the meridian. A parallel is not unique (one can always add a meridian to it). A knot is called framed if a choice of a particular parallel has been made. We denote a framed knot as \(\hat{K}\). A link is framed if all of its components are framed. We denote such link as \(\hat{L}\).

Suppose that an oriented framed knot \(\hat{K}\) is a trivial element in \(H_1(M; \mathbb{Q})\). Then a linking number between \(K\) and its parallel is well-defined. We call this number a self-linking number of \(\hat{K}\). The self-linking numbers of the link components of \(\hat{L}\) form a diagonal part of the linking matrix \(l_{ij}^{(L|M)}\). We denote the signature of that matrix as \(\text{sign}(\hat{L}|M)\). For a pair of an unframed \(L\)-component link \(L\) and a framed \(L^s\)-component link \(\hat{L}^s\) we define the analog of (1.8) 
\[
\text{lk}(L \cup \hat{L}^s, M; x, y) = \text{lk}(L \cup L^s, M; x, y) + \sum_{j=1}^{L^s} l_{jj}^{(\hat{L}^s|M)} y_j^2, 
\]
where \(x = (x_1, \ldots, x_L)\), \(y = (y_1, \ldots, y_{L^s})\).

Given a quadratic form \(\text{lk}(\hat{L}^s, M; y)\), we denote its determinant as \(\det(\hat{L}^s|M)\) and its signature simply as \(\text{sign}(\hat{L}^s|M)\).

The quantum invariants of knots and links that we will consider in this paper, that is, the colored Jones polynomial, the WRT invariant and the contributions of various connections to those invariants, depend on the framing. However, this dependence is known to be quite simple. Namely, if we change the framing of a link component \(\hat{L}_j\) in such a way that its self-linking number increases by 1, then the corresponding invariants of \(\hat{L}\) are multiplied by
a factor $q^{(\alpha_j^2 - 1)/4}$, where $\alpha_j$ is a color assigned to $\mathcal{L}_j$. Therefore a quantum invariant of a link $\mathcal{L}$ in a QHS $M$ can be rendered framing-independent if we multiply it by an extra factor

$$q^{-\sum_{j=1}^{L} l_{ij}(\mathcal{L}|M)(\alpha_j^2 - 1)/4}. \quad (1.10)$$

Thus whenever we write a quantum invariant of an unframed knot or link, we assume that the factor (1.10) is included in the formula for that invariant.

If a knot $K$ is a trivial element of $H_1(M; \mathbb{Z})$ (that is, if $o(K) = 1$), then it can be endowed with a canonical framing for which the self-linking number is equal to zero.

Let $\hat{\mathcal{L}}^s$ be a framed $L^s$-component link in a 3-manifold $M$. A new 3-manifold $M'$ is constructed by Dehn’s surgery on $\hat{\mathcal{L}}^s$ in the following way: we cut out the tubular neighborhoods of the link components and then glue them back in such a way that the meridian on the boundary of the tubular neighborhood of each link component matches the parallel on the corresponding boundary of the link complement while the parallel on the boundary of the tubular neighborhood matches the meridian on the boundary of the complement with opposite orientation.

Suppose that the original manifold $M$ contained an oriented link $\mathcal{L}$. Then the linking numbers of the components of $\mathcal{L}$ in $M'$ are given by the formula

$$l_{ij}(\mathcal{L}|M') = l_{ij}(\mathcal{L}|M) - \sum_{1 \leq k,k' \leq L} l_{ik}(\mathcal{L},\mathcal{L}^s|M) l_{ik'}(\mathcal{L},\mathcal{L}^s|M)(l(\mathcal{L}^s|M))^{-1}kk', \quad (1.11)$$

where $(l(\mathcal{L}^s|M))^{-1}$ is the inverse of the matrix $l(\mathcal{L}^s|M)$.

We denote the order of $H_1(M, \mathbb{Z})$ as $h_1(M)$. For a pair of QHS related by a surgery on $\hat{\mathcal{L}}^s \subset M$

$$h_1(M') = |\det(\hat{\mathcal{L}}^s|M)| h_1(M). \quad (1.12)$$

If $M$ is a connected sum of two QHS, $M = M_1 \# M_2$, then

$$h_1(M) = h_1(M_1) h_1(M_2). \quad (1.13)$$
Let $\mathcal{L}$ and $\hat{\mathcal{L}}^s$ be a pair of oriented links in $S^3$, the second of them being framed. Suppose that Dehn’s surgery on $\hat{\mathcal{L}}^s$ produces a QHS $M$. Then the orders of the link components of $\mathcal{L}$ as elements of $H_1(M; \mathbb{Z})$ are

$$a_i = \text{LCM} \left( c_{ij}(\mathcal{L}, \hat{\mathcal{L}}^s) \middle| 1 \leq j \leq L^s \right)$$

(1.14)

where LCM $(m, n)$ denotes the least common multiple of $m$ and $n$, while

$$c_{ij}(\mathcal{L}, \hat{\mathcal{L}}^s) = \sum_{k=1}^{L^s} i_{ij}^{(\mathcal{L}, \hat{\mathcal{L}}^s)} (l(\mathcal{L}^s|S^3))^{-1}_{kj}.$$  

(1.15)

### 1.2.3 Alexander-Conway polynomial

In subsection 1.2.4 of [15] we set our notations and established some properties of the Alexander-Conway polynomial and Alexander-Conway function of knots and links in $S^3$. Here we have to extend these properties to knots and links in QHS.

Let $\mathcal{L}$ and $\hat{\mathcal{L}}^s$ be a pair of oriented links in $S^3$. Suppose that a QHS $M$ is constructed by Dehn’s surgery on $\hat{\mathcal{L}}^s$. Then we define the Alexander-Conway function of $\mathcal{L}$ in $M$ by a surgery formula

$$\nabla_A(\mathcal{L}, M; t) = (-1)^{L^s} \text{sign}(\hat{\mathcal{L}}^s|S^3) \frac{\nabla_A(\mathcal{L} \cup \hat{\mathcal{L}}^s, M; t)}{\{u^{(st)}(\mathcal{L})^{1/2} - (u^{(st)})^{-1/2}\}},$$

(1.16)

where $u^{(st)} = \prod_{i=1}^{L^s} t^{-c_{ij}(\mathcal{L}, \hat{\mathcal{L}}^s)}$,

and $c_{ij}(\mathcal{L}, \hat{\mathcal{L}}^s)$ are defined by eq. (1.15). The Alexander-Conway function has the following properties:

$$\nabla_A(\mathcal{L}, M; t^{-1}) = (-1)^{L^s} \nabla_A(\mathcal{L}, M; t)$$

(1.17)

and if we change an orientation of a component $\mathcal{L}_j$, then for the new oriented link $\mathcal{L}'$

$$\nabla_A(\mathcal{L}', M; t) = -\nabla_A(\mathcal{L}, M; t_1, \ldots, t_{j-1}, t_j^{-1}, \ldots, t_L).$$

(1.18)

The Alexander function satisfies the Torrence formula

$$\nabla_A(\mathcal{L}_0 \cup \mathcal{L}, M; 1, t) = \left( \prod_{j=1}^{L^s} t_j^{a_j^{(\mathcal{L}_0 \cup \mathcal{L}|M)}} - \prod_{j=1}^{L^s} t_j^{-a_j^{(\mathcal{L}_0 \cup \mathcal{L}|M)}} \right) \nabla_A(\mathcal{L}, M; t),$$

(1.19)
where $\mathcal{L}$ is an $L$-component link and $\mathcal{L}_0$ is a knot. If we take a connected sum of a QHS $M$ containing a link $\mathcal{L}$, with another QHS $M'$, then

$$\nabla_A(\mathcal{L}, M \# M'; t) = h_1(M') \nabla_A(\mathcal{L}, M; t).$$  \hspace{1cm} (1.20)

If $\mathcal{K}$ is a knot in $M$, then its Alexander-Conway polynomial is defined as

$$\Delta_A(\mathcal{K}, M; t) = (t^{1/2o(\mathcal{K})} - t^{-1/2o(\mathcal{K})})\nabla_A(\mathcal{K}, M; t).$$  \hspace{1cm} (1.21)

It satisfies the property

$$\Delta_A(\mathcal{K}, M; 1) = |H_1(M; \mathbb{Z})|/o(\mathcal{K}).$$  \hspace{1cm} (1.22)

Both Alexander-Conway polynomial of a knot and Alexander-Conway function of a link with at least 2 components are Laurent polynomials

$$\Delta_A(\mathcal{K}, M; t) \in \mathbb{Z}[t^{1/2o(\mathcal{K})}, t^{-1/2o(\mathcal{K})}],$$  \hspace{1cm} (1.23)

$$\nabla_A(\mathcal{L}, M; t) \in \mathbb{Z}[t^{1/2}, t^{-1/2}] \text{ if } L \geq 2.$$  \hspace{1cm} (1.24)

1.2.4 Lie algebra notations

In this paper we will be dealing with the Lie algebra $su(2)$, although it will be clear that most of our considerations can be easily extended to other simple Lie algebras. We denote the elements of $su(2)$ as $\vec{x}$. The Lie algebra $su(2)$ has a Killing form. We denote the corresponding scalar product in $su(2)$ as $\vec{x} \cdot \vec{y}$. A polynomial function $P(\vec{x}_1, \ldots, \vec{x}_n)$ on $su(2)$ is called $SU(2)$-invariant if it is invariant under the simultaneous adjoint action of $SU(2)$ on all of its variables.

1.2.5 The Reshetikhin Formula

Let $\mathcal{L}$ be an $L$-component link in $S^3$. We refer the readers to subsection 1.2.7 of [15] for the definition of the colored Jones polynomial of $\mathcal{L}$. P. Melvin and H. Morton considered an expansion of that polynomial in negative powers of

$$K = 2\pi i / \log q.$$  \hspace{1cm} (1.25)
They found that the coefficients of this expansion are polynomials of colors $\alpha$ assigned to the components of $L$

$$J_\alpha(L; q) = \sum_{m,n \geq 0} D_{m,n}(L) \alpha^{2m+1} K^{-n}. \quad (1.26)$$

Note that in view of Remark 1.1 the expansion (1.26) is equivalent to that of (1.46) of [15].

A few years ago N. Reshetikhin suggested [9], [10] to present the expansion (1.26) as an integral over adjoint orbits of the Lie algebra $su(2)$ which correspond to $su(2)$ modules of dimensions $\alpha$. We implemented this program in [11]. The only shortcoming of the arguments of [11] is that we based them on deriving the Melvin-Morton expansion (1.26) from a generic quantum Chern-Simons perturbation theory, which is not a rigorous mathematical tool yet. We easily corrected this in [13] by showing how to use Kontsevich integral as a foundation of the calculations which lead to the Reshetikhin formula.

**Theorem 1.2** For an $L$-component link $L \in S^3$ there exist homogeneous $SU(2)$-invariant polynomials $L_m(L; \alpha), L_{m,n}(L; \alpha)$ such that

$$\deg L_m(L; \alpha) = \deg L_{m,n}(L; \alpha) = m, \quad (1.27)$$

$$L_2(L; \alpha) = \sum_{1 \leq i \neq j \leq L} \frac{1}{i+j} \alpha_i \cdot \alpha_j, \quad (1.28)$$

$$L_m(L; \alpha \bar{n}) = \partial_{\alpha} L_m(L; \alpha) \bigg|_{\alpha = \alpha \bar{n}} = 0, \quad \text{for any } \bar{n} \in su(2) \text{ if } m \geq 3, \quad (1.29)$$

$$L_{0,0}(L; \alpha) = 1 \quad (1.30)$$

and the Melvin-Morton expansion (1.26) is equal to the following integral of a formal power series in $K^{-1}$ and $\alpha$

$$J_\alpha(L; q) = \sum_{m,n \geq 0} D_{m,n}(L) \alpha^{2m+1} K^{-n} \quad (1.31)$$

$$= (4\pi)^{-L} \int_{|\alpha|=\alpha} \left\{ \frac{d\alpha}{\alpha} \right\} \exp \left( \frac{1}{2} i\pi K^{1-m} \sum_{m \geq 2} L_m(L; \alpha) \right) \sum_{m,n \geq 0} L_{m,n}(L; \alpha) K^{-m-n}. \quad (1.31)$$

**Remark 1.3** The polynomials $L_m(L; \alpha)$ and $L_{m,n}(L; \alpha)$ are not invariants of $L$. In [11], [13] we showed how these polynomials may be derived from the Kontsevich integral of a stringed
link (in [1] we will translate these arguments into the language of Vassiliev invariants and 3-valent graphs). We may expect that all the polynomials $L_m(\mathcal{L}; \vec{\alpha})$ and $L_{m,n}(\mathcal{L}; \vec{\alpha})$ which satisfy the conditions of Theorem 1.2 come from the Kontsevich integral of a properly stringed link. In any case, from now on we will use in our formulas only the polynomials $L_m(\mathcal{L}; \vec{\alpha})$, $L_{m,n}(\mathcal{L}; \vec{\alpha})$ which come from the Kontsevich integral for a stringed link as described in [11] and [13].

We will modify the integral (1.31) in two ways. First, consider the Reshetikhin formula for a link $\mathcal{L} \cup \mathcal{L}'$ whose components carry the colors $\vec{\alpha}$ and $\vec{\beta}$. Let us expand the exponential into the powers of all monomials which depend on the variables $\vec{\beta}$ and then integrate over those variables. As a result, we will get a formula

$$J_{\vec{\alpha}, \vec{\beta}}(\mathcal{L}, \mathcal{L}'; q) = \sum_{m, m', n \geq 0} D_{m, m', n}(\mathcal{L} \cup \mathcal{L}'') \frac{\alpha^{2m+1} \beta^{2m'+1}}{2m+2m'+1} K^{-n}$$

$$= (4\pi)^{-L} \int_{|\vec{a}| = a} \left\{ \frac{d\vec{a}}{\alpha} \right\} \exp \left( \frac{1}{2} i\pi K^{1-m} \sum_{m \geq 2} L_m(\mathcal{L}; \vec{\alpha}) \right) \sum_{m,n\geq 0} L_{m,n}(\mathcal{L}, \mathcal{L}'; \vec{\alpha}, \vec{\beta}) K^{-m-n},$$

where $L_{m,n}(\mathcal{L}, \mathcal{L}'; \vec{\alpha}, \vec{\beta})$ are homogeneous polynomials of $\vec{\alpha}$ and $\vec{\beta}$, homogeneous in $\vec{\alpha}$

$$\deg_{\vec{\alpha}} L_{m,n}(\mathcal{L}, \mathcal{L}'; \vec{\alpha}, \vec{\beta}) = m, \quad L_{0,0}(\mathcal{L}, \mathcal{L}'; \vec{\alpha}, \vec{\beta}) = 1.$$ (1.33)

Now a substitution

$$\vec{\alpha} = Ka, \quad \vec{\alpha} = K\vec{\alpha}$$

turns eq.(1.32) into

$$J_{\vec{\alpha}, \vec{\beta}}(\mathcal{L}, \mathcal{L}'; q) = I_{\vec{\beta}}(\mathcal{L}, \mathcal{L}'; \vec{\alpha}/K; K),$$

$$I_{\beta}(\mathcal{L}, \mathcal{L}'; \vec{a}; K) = \left( \frac{K}{4\pi} \right)^L \int_{|\vec{a}| = a} \left\{ \frac{d\vec{a}}{\vec{a}} \right\} e^{\frac{i}{2} \pi KE(\mathcal{L}; \vec{\alpha})} P(\mathcal{L}, \mathcal{L}'; \vec{a}, \vec{\beta}; K),$$

where $I_{\beta}(\mathcal{L}, \mathcal{L}'; \vec{a}; K)$ is a formal power series in $\vec{a}$, $\vec{\beta}$ and $K$

$$I_{\beta}(\mathcal{L}, \mathcal{L}'; \vec{a}; K) = \sum_{m, m', n \geq 0} D_{m, m', n}(\mathcal{L} \cup \mathcal{L}') \frac{\alpha^{2m+1} \beta^{2m'+1}}{2m+2m'+1} K^{2m-n+1}.$$ (1.37)
and $\mathbf{E}(\mathcal{L}; \vec{a})$, $\mathbf{P}(\mathcal{L}, \mathcal{L}'; \vec{a}; \vec{\beta}; K)$ denote the formal power series

$$
\mathbf{E}(\mathcal{L}; \vec{a}) = \sum_{m \geq 2} L_m(\mathcal{L}; \vec{a}), \quad \mathbf{P}(\mathcal{L}, \mathcal{L}'; \vec{a}; \vec{\beta}; K) = \sum_{\frac{m, n \geq 0}{m \leq n}} L_{m,n}(\mathcal{L}, \mathcal{L}'; \vec{a}; \vec{\beta}) K^{-n}.
$$

(1.38)

Since for any $j$, $1 \leq j \leq L'$

$$
J_{\alpha, \beta}(\mathcal{L}, \mathcal{L}'(j); q) = J_{\alpha, \beta}(\mathcal{L}, \mathcal{L}'; q) \bigg|_{\beta_j = 1},
$$

(1.39)

then we may conclude from eq.(1.36) that

$$
L_{m,n}(\mathcal{L}, \mathcal{L}'(j); \vec{a}, \vec{\beta}(j)) = L_{m,n}(\mathcal{L}, \mathcal{L}'; \vec{a}; \vec{\beta}) \bigg|_{\beta_j = 1}.
$$

(1.40)

Indeed, in view of eq. (1.39), the polynomials $L_{m,n}(\mathcal{L}, \mathcal{L}'(j); \vec{a}, \vec{\beta}(j))$ as defined by eq. (1.40) would suite the formula (1.32) for $J_{\alpha, \beta}(\mathcal{L}, \mathcal{L}'(j); q)$. However, it follows easily from the calculations of [11] and [13] that we can make an even stronger statement: the stringing of $\mathcal{L}'(j)$ which yields the l.h.s. of eq.(1.40) is induced by the stringing of $\mathcal{L}'$ which yields the r.h.s. of that same equation (in fact, a more careful analysis would even show that the polynomials $L_{m,n}(\mathcal{L}, \mathcal{L}'; \vec{a}; \vec{\beta})$ do not depend on the stringing of $\mathcal{L}'$).

1.2.6 Formal power series and stationary phase approximation

In this paper we will perform a lot of calculations with formal power series. The basic operations with these series, such as addition, multiplication and term-by-term integration used in eq. (1.32), are well-known. However, we will need a potentially riskier procedure Here we will need another procedure - a stationary phase integration of an exponential of a formal power series. Therefore we have to explain which manipulations with formal power series we consider well defined.

Let $S(x; N) \in \mathbb{C}[[x]]$ be a formal power series in $x$ whose coefficients depend on an integer variable $N$

$$
S(x; N) = \sum_{n=0}^{\infty} s_n(N) x^n.
$$

(1.41)
**Definition 1.4** The series \((1.41)\) has a stable limit as \(N \to \infty\)

\[
\lim_{N \to \infty} S(x; N) = S(x), \quad S(x) = \sum_{n=0}^{\infty} s_n x^n,
\]

(1.42)

if for any \(m > 0\) there exists \(m' > 0\) such that if \(n < m\) and \(N > m'\), then \(s_n(N) = s_n\).

A map \(\mathcal{F} : \mathbb{C}[[x]] \to \mathbb{C}[[x]]\) is well-defined for us if for any \(S(x) \in \mathbb{C}[[x]]\)

\[
\mathcal{F}[S(x)] = \lim_{N \to \infty} \mathcal{F}[S(x|N)], \quad \text{where } S(x|N) = \sum_{n=0}^{N} s_n x^N.
\]

(1.43)

For two smooth functions \(f(x), g(x, K), \overline{x} = (x_1, \ldots, x_L)\) \(g(x, K)\) being analytical in the vicinity of \(K \to \infty\), consider an integral

\[
I(K) = K^{-L/2} \int e^{iKf(x)} g(x, K) \, dx.
\]

(1.44)

Suppose that \(x^{(st)}\) is a stationary point of \(f(x)\)

\[
\partial_{\overline{x}} f(x) \bigg|_{\overline{x} = x^{(st)}} = 0.
\]

(1.45)

Then

\[
I(x^{(st)})(K) = K^{-L/2} \int_{[x = x^{(st)}]} e^{iKf(x)} g(x, K) \, dx.
\]

(1.46)

denotes the contribution of the point \(x^{(st)}\) to the integral \(I(K)\) in the stationary phase approximation as \(K \to \infty\). \(I(x^{(st)})(K)\) is almost a formal power series in \(K^{-1}\)

\[
I(x^{(st)})(K) = e^{iKf(x^{(st)})} \sum_{n=0}^{\infty} I_n^{(x^{(st)})} K^{-n},
\]

(1.47)

whose coefficients are determined through a special procedure. We expand the following function in powers of \(x - x^{(st)}\):

\[
\exp iK \left( f(x) - f(x^{(st)}) - f_2(x^{(st)})(x) \right) g(x, K) = \sum_{m \geq 0} \sum_{n \geq \lceil |m|/3 \rceil} p_{m,n} \overline{x}^m K^{-n}
\]

(1.48)

where

\[
f_2(x^{(st)})(x) = \frac{1}{2} \sum_{i,j=1}^{L} \partial_{x_i} \partial_{x_j} f(x) \bigg|_{x = x^{(st)}} (x_i - x_i^{(st)})(x_j - x_j^{(st)}),
\]

(1.49)
so that

\[ I_n^{(x,n)} = K^{-L/2} \sum_{m \geq 0 \atop |m| \leq 6n} p_m \left| \frac{|m|}{n} \right| K \left| \frac{|m|}{n} \right| - n \int e^{iKf_2 \left( \left( x, x \right) \right) \left( x, m \right)} dx. \quad (1.50) \]

The gaussian integrals with polynomial prefactors can be calculated with the help of the following formula

\[ \int_{-\infty}^{+\infty} e^{iK A(x,x)} m d x = \left( i \pi \right)^{L/2} \left( \text{det } A \right)^{-1/2} \left( 2iK \right)^{-|m|} \frac{\partial^{|m|}}{\partial y^{|m|}} e^{iKA^{-1}(y,y)} \bigg|_{y=0}, \quad (1.51) \]

where \( A(x,x) \) is a non-degenerate quadratic form and \( A^{-1}(y,y) \) is the quadratic form with the inverse matrix.

1.2.7 The TCC invariant of links in rational homology spheres

First, let us recall the definition of the WRT invariant of links in 3-manifolds. For a link \( L \in S^3 \) and for a positive integer \( K \) we define

\[ Z_\omega(L, S^3; K) = J_\omega(L; q), \quad (1.52) \]

where \( q \) and \( K \) are related by eq. (1.2). If \( L \) is a framed link in a QHS \( M \), which can be constructed by Dehn’s surgery on a framed link \( \hat{L}^s \in S^3 \), then

\[ Z_\omega(L, M; K) = q^{\phi(n)}(L, \hat{L}^s; \omega) \sum_{0 < \gamma < K} q^{\frac{1}{2} \sum_{j=1}^{L^s} i \left( \hat{L}^s | S^3 \right) j \gamma} J_\omega^2(L \cup \hat{L}^s; q) \left\{ q^{2/2} - q^{-2/2} \right\}, \quad (1.53) \]

where \( q^{\phi(n)}(L, \hat{L}^s; \omega) \) is a factor which removes the dependence of \( Z_\omega(L, M; K) \) on the framing of \( M \) and \( L \)

\[ \phi(n)(L, \hat{L}^s; \alpha) = -\frac{3}{8} (K - 2) \text{sign} \left( \hat{L}^s | S^3 \right) - \frac{1}{4} \sum_{j=1}^{L^s} l^{(\hat{L}^s | S^3)}_{jj} - \frac{1}{4} \sum_{j=1}^{L} l^{(\hat{L}^s | M)}_{jj} \left( \alpha^2 - 1 \right) \quad (1.54) \]

and \( \hat{L} \) in the last sum is the link \( L \in M \) endowed with the framing induced by the 0-framing of \( L \in S^3 \) so that according to eq. (1.11)

\[ l^{(\hat{L}^s | M)}_{jj} = - \sum_{1 \leq k, k' \leq L} l^{(L, \hat{L}^s | S^3)}_{jk} l^{(L, \hat{L}^s | S^3)}_{jk'} (l^{(\hat{L}^s | S^3)}_{kk'})^{-1}, \quad (1.55) \]

The TCC invariant of a link \( L \) in a QHS \( M \) is defined by a surgery formula which is similar to (1.53).
Theorem 1.5 (see [13]) For an $L$-component link $L$ in a QHS $M$ there exists a unique set of invariants $D_{m,n}(L, M) \in \mathbb{C}$ such that the formal power series

$$Z^{(tr)}_{\alpha}(L, M; K) = \sum_{m,n \geq 0} D_{m,n}(L, M) \alpha^{2m+1} K^{-n}, \quad (1.56)$$

satisfies the following two conditions:

(1) If $M = S^3$, then

$$Z^{(tr)}_{\alpha}(L, S^3; K) = J_{\alpha}(L; q), \quad (1.57)$$

where $J_{\alpha}(L; q)$ is represented by its Melvin-Morton expansion (1.26).

(2) If $M$ is constructed by Dehn’s surgery on a framed link $\hat{L}^s \in S^3$, then

$$Z^{(tr)}_{\alpha}(L, M; K) = e^{(2\pi i / K)} \phi^{(tr)}(L, L^s; \alpha) \left( \frac{K}{2} \right)^{\frac{3}{2} L^s}$$

$$\times \int_{\vec{c} = 0} \left\{ d\vec{c} \left\{ \frac{\sin \pi |\vec{c}|}{\pi |\vec{c}|} \right\} \exp \frac{1}{2} i\pi K \left( \sum_{j=1}^{L^s} (c_j)^2 + E(L^s; \vec{c}) \right) \right\} P(L^s, L; \vec{c}; \alpha; K), \quad (1.58)$$

where $\int_{\vec{c} = 0}$ denotes a contribution of the point $\vec{c} = 0$ to the integral over $\vec{c}$ in the stationary phase approximation at $K \rightarrow \infty$.

The first of these invariants is

$$D_{0,0}(L, M) = \{ \alpha \} h_1^{-3/2}(M). \quad (1.59)$$

Since $E(L^s; \vec{c})$ and $P(L^s, L; \vec{c}; \alpha; K)$ are formal power series, then eq. (1.58) is just a short-cut for a more precise statement in the spirit of eq. (1.43)

$$Z^{(tr)}_{\alpha}(L, M; K) = \lim_{N \rightarrow \infty} e^{(2\pi i / K)} \phi^{(tr)}(L, L^s; \alpha) \left( \frac{K}{2} \right)^{\frac{3}{2} L^s}$$

$$\times \int_{\vec{c} = 0} \left\{ d\vec{c} \left\{ \frac{\sin \pi |\vec{c}|}{\pi |\vec{c}|} \right\} \exp \frac{1}{2} i\pi K \left( \sum_{j=1}^{L^s} (c_j)^2 + E(L^s; \vec{c}) \right) \right\} P(L^s, L; \vec{c}; \alpha; K|N). \quad (1.60)$$

Here we introduced truncated series

$$E(L; \vec{a} |N) = \sum_{2 \leq m \leq N} L_m(L; \vec{a}), \quad P(L, L'; \vec{a}; \beta; K|N) = \sum_{0 \leq m,n \leq N} L_{m,n}(L, L'; \vec{a}; \beta) K^{-n}. \quad (1.61)$$

Similarly to $J_{\alpha}(L; q)$, the TCC invariant $Z^{(tr)}_{\alpha}(L, M; K)$ has an integral representation.
Theorem 1.6 (see [13]) For an $L$-component link $\mathcal{L}$ and an $L'$-component link $\mathcal{L}'$ in a $\mathbb{Q}HS$ $M$ there exist $SU(2)$-invariant polynomials $L_m(\mathcal{L}, M; \vec{a})$, $L_{m,n}(\mathcal{L}, M; \vec{a})$ which satisfy the properties

$$\deg L_m(\mathcal{L}, M; \vec{a}) = \deg L_{m,n}(\mathcal{L}, M; \vec{a}; \beta) = m,$$  \hspace{1cm} (1.62)

$$L_2(\mathcal{L}, M; \vec{a}) = \sum_{1 \leq i < j \leq L} l_{ij}(\mathcal{L}; M) \vec{a}_i \cdot \vec{a}_j,$$  \hspace{1cm} (1.63)

$$L_m(\mathcal{L}, M; \vec{a}) = \partial_{\vec{a}} L_m(\mathcal{L}, M; \vec{a})|_{\vec{a}=\vec{a}_0} = 0, \text{ for any } \vec{a}_0 \in \mathfrak{su}(2) \text{ if } m \geq 3,$$  \hspace{1cm} (1.64)

$$L_{0,0}(\mathcal{L}, M; \vec{a}; \beta) = |H_1(M; \mathbb{Z})|^{-3/2} \{\beta\}$$  \hspace{1cm} (1.65)

and such that the formal power series

$$E(\mathcal{L}, M; \vec{a}) = \sum_{m \geq 2} L_m(\mathcal{L}, M; \vec{a}), \quad P(\mathcal{L}, \mathcal{L}', M; \vec{a}; \beta; K) = \sum_{m,n \geq 0} L_{m,n}(\mathcal{L}, \mathcal{L}', M; \vec{a}; \beta) K^{-\frac{3}{2}},$$  \hspace{1cm} (1.66)

combine into an integral representation

$$Z_{\vec{a}, \vec{b}}^{(t)}(\mathcal{L}, \mathcal{L}', M; K) = I_{\vec{b}}(\mathcal{L}, \mathcal{L}', M; \vec{a}/K; K),$$  \hspace{1cm} (1.67)

$$I_{\vec{b}}(\mathcal{L}, \mathcal{L}', M; \vec{a}; K) = \left(\frac{K}{4\pi}\right)^L \int_{[\vec{a}] = a} \left\{\frac{d\vec{a}}{2}\right\} e^{\frac{i}{4\pi KE(\mathcal{L}, M; \vec{a})}} P(\mathcal{L}, \mathcal{L}', M; \vec{a}; \beta; K).$$  \hspace{1cm} (1.68)

There are various ways to derive the series $E(\mathcal{L}, M; \vec{a})$, $P(\mathcal{L}, \mathcal{L}', M; \vec{a}; \beta; K)$ from the series $E(\mathcal{L} \cup \mathcal{L}^x; \vec{a}, \vec{c})$ and $P(\mathcal{L} \cup \mathcal{L}^x, \mathcal{L}', \vec{a}, \vec{c}; \beta; K)$. For our purposes here we will use one of the formulas of [13]:

$$e^{\frac{i}{4\pi KE(\mathcal{L}, M; \vec{a})}} P(\mathcal{L}, \mathcal{L}', M; \vec{a}; \beta; K) = e^{(2\pi i/K) \phi(t)}(\mathcal{L} \cup \mathcal{L}', \mathcal{L}^x; K[\vec{a}, \vec{b}]) \left(\frac{K}{2}\right)^{\frac{3}{2} L^x}$$

$$\times \int_{[\vec{c}] = \vec{c}(st)(\vec{a})} \{d\vec{c}\} \left\{\frac{\sin \pi |\vec{c}|}{\pi |\vec{c}|}\right\} \exp \frac{1}{2} i\pi K \left(\sum_{j=1}^{L^x} l_{jj}(\mathcal{L}^x; \vec{c}) \vec{c}_j^2 + E(\mathcal{L} \cup \mathcal{L}^x; \vec{a}, \vec{c})\right)$$

$$\times P(\mathcal{L} \cup \mathcal{L}^x, \mathcal{L}', \vec{a}, \vec{c}; \beta; K),$$  \hspace{1cm} (1.69)

where $\vec{c}(st)(\vec{a})$ is the stationary point of the exponential in the integral (1.69), which is a formal power series in $\vec{a}$

$$\vec{c}(st)(\vec{a}) = -\sum_{i=1}^{L} c_{ij}(\mathcal{L}, \mathcal{L}^x) \vec{a}_i + O(\vec{a}^2),$$  \hspace{1cm} (1.70)
Similarly to the case of $S^3$, the polynomials $L_m(\mathcal{L}, M; \vec{a})$, $L_{m,n}(\mathcal{L}, \mathcal{L}', M; \vec{a}; \beta)$ themselves are not invariants of $\mathcal{L}$. Equation (1.69) demonstrates that they depend on the stringing of the links $\mathcal{L}, \mathcal{L}', \mathcal{L}^a$ in $S^3$ (a more careful analysis [1] will show that they depend only on the stringing on $\mathcal{L}$ in $S^3$, but we will not use this fact here).

Applying eq. (1.40) to the r.h.s. of eq. (1.58) we find that for any $j$, $1 \leq j \leq L$

$$Z_{\mathcal{L}(j)}^{(tr)}(M; K) = Z_{\mathcal{L}}^{(tr)}(M; K)\big|_{\alpha_j=1}. \tag{1.71}$$

Applying eq. (1.40) to the r.h.s. of eq. (1.69) we conclude that eq. (1.40) holds also for links in $M$

$$L_{m,n}(\mathcal{L}, \mathcal{L}'(j), M; \vec{a}; \beta_{(j)}) = L_{m,n}(\mathcal{L}, \mathcal{L}', M; \vec{a}; \beta)\big|_{\beta_j=1}. \tag{1.72}$$

if the polynomials in both sides of this equation originate through the surgery formula (1.69) from the same stringing of $\mathcal{L}$ in $S^3$.

The multiplicativity of Reshetikhin’s integrand under an operation of disconnected sum of two links in $S^3$ leads to the following

**Proposition 1.7** Consider a connected sum of two $\mathbb{Q}HS$ $M\#M'$. If $\mathcal{L}, \mathcal{L}' \subset M$, then

$$E(\mathcal{L}, M\#M'; \vec{a}) = E(\mathcal{L}, M; \vec{a}), \tag{1.73}$$

$$P(\mathcal{L}, \mathcal{L}', M\#M'; \vec{a}; \beta; K) = Z^{(tr)}(M'; K) P(\mathcal{L}, \mathcal{L}', M; \vec{a}; \beta; K), \tag{1.74}$$

if we assume that the l.h.s. and r.h.s. of these equations originate from the same stringings of links in $S^3$ through the surgery formula (1.69).

### 1.3 Results

We prove three theorems which describe the $U(1)$-RC invariant of links in $\mathbb{Q}HS$ and its properties.
Theorem 1.8 (cf. Main Theorem of [15]) Let $\mathcal{L}$ be an oriented $L$-component link in a QHS $M$, whose Alexander-Conway function is not identically equal to zero

$$\nabla A(\mathcal{L}, M; t) \not\equiv 0, \quad t = (t_1, \ldots, t_L).$$  \hspace{1cm} (1.75)

Let $\mathcal{L}'$ be an $L'$-component link in $M$ with positive integers $\beta = (\beta_1, \ldots, \beta_{L'})$ assigned to its components. Then there exists a unique sequence of polynomial invariants

$$P_{\beta,n}(\mathcal{L}, \mathcal{L}', M; t) \in \mathbb{Z}[\frac{t}{2h_1(M)}], \quad n \geq 0,$$  \hspace{1cm} (1.76)

such that an expansion of $P_{\beta,n}(\mathcal{L}, \mathcal{L}', M; t)$ in powers of $(t - 1)$ has the form

$$P_{\beta,n}(\mathcal{L}, \mathcal{L}', M; t) = \sum_{m \geq 0} p_{m,n}(\mathcal{L}, \mathcal{L}', M; \beta) (t - 1)^m, \quad p_{m,n}(\mathcal{L}, \mathcal{L}', M; \beta) \in \mathbb{Q}[\beta]$$  \hspace{1cm} (1.77)

and the formal power series in $h$

$$\hat{Z}_{\beta}^{(r)}(\mathcal{L}, \mathcal{L}', M; t; h)$$

$$= \begin{cases} 
    h_1^{-3/2}(M) \sum_{n=0}^{\infty} P_{\beta,n}(\mathcal{L}') h^n & \text{if } L = 0, \\
    h^{-1} h_1^{-1/2}(M) \frac{1}{\nabla A(\mathcal{L}, M; t_1)} \sum_{n=0}^{\infty} \frac{P_{\beta,n}(\mathcal{L}, \mathcal{L}', M; t_1)}{\Delta^2_n(\mathcal{L}, M; t_1)} h^n & \text{if } L = 1, \\
    h^{-1} h_1^{-1/2}(M) \frac{1}{\nabla A(\mathcal{L}, M; t)} \sum_{n=0}^{\infty} \frac{P_{\beta,n}(\mathcal{L}, \mathcal{L}', M; t)}{\Delta^2_n(\mathcal{L}, M; t)} h^n & \text{if } L \geq 2
\end{cases} \hspace{1cm} (1.78)$$

satisfies the following properties:

(1) If the link $\mathcal{L}$ is empty (that is, if $L = 0$), then

$$\hat{Z}_{\beta}^{(r)}(\emptyset, \mathcal{L}', M; h) = Z^{(r)}_{\beta}(\mathcal{L}', M; K) \text{ in } \mathbb{Q}[\beta][[h]].$$  \hspace{1cm} (1.79)

(2) For any $j, 1 \leq j \leq L'$,

$$\hat{Z}_{\beta}^{(r)}(\mathcal{L}, \mathcal{L}', M; t; h) \bigg|_{\beta_j=1} = \hat{Z}_{\beta_{(j)}}^{(r)}(\mathcal{L}, \mathcal{L}_{(j)}', M; t; h).$$  \hspace{1cm} (1.80)
(3) Let \( \mathcal{L} \) be a non-empty link. Suppose that for a number \( j, 1 \leq j \leq L \), the link component \( \mathcal{L}_j \) is algebraically connected to \( \mathcal{L}_{(j)} \) and \( \nabla_A(\mathcal{L}_{(j)}; \mathcal{L}_j) \neq 0 \). Then

\[
\hat{Z}^{(r)}_{\beta_0}(\mathcal{L}_{(j)}, \mathcal{L}_j \cup \mathcal{L}'; \mathcal{L}_j; t; h) = \begin{cases} 
\sum_{\mu = \pm 1} \mu \left( \prod_{1 \leq i \leq L} t_{ij}^{(L|M)} \right)^{\mu \beta_0/2} \left( \frac{\hat{Z}^{(r)}(\mathcal{L}, \mathcal{L}', M; t; h)}{t_j = q^{\mu \beta_0}} \right) & \text{if } L \geq 2, \\
\hat{Z}^{(r)}(\mathcal{L}, \mathcal{L}'; q^{\beta_0}; h), & \text{if } L = 1.
\end{cases}
\] (1.81)

Thus defined, the polynomials \( P_{\beta;n}(\mathcal{L}, \mathcal{L}', M; t) \) have additional properties:

\[
P_{\beta;n}(\mathcal{L}, \mathcal{L}', M; t) \in \mathbb{Z}[t^{\pm 1/2}, \frac{1}{2h_1(M)}] \quad \text{if } \beta \text{ are odd,}
\] (1.82)

\[
P_{\beta;0}(\mathcal{L}, \mathcal{L}', M; t) = \begin{cases} 
\{ \beta \} & \text{if } L = 0, \\
\prod_{j=1}^{L'} \left( \frac{\Pi_{i=1}^{L} t_{ij}^{(L,M)}}{\Pi_{i=1}^{L} t_{ij}^{(L,M)}} \right)^{\beta_j/2} - \left( \frac{\Pi_{i=1}^{L} t_{ij}^{(L,M)}}{\Pi_{i=1}^{L} t_{ij}^{(L,M)}} \right)^{-\beta_j/2} & \text{if } L \geq 1,
\end{cases}
\] (1.83)

and if we reverse the orientation of a link component \( \mathcal{L}_j \), then for the new oriented link \( \bar{\mathcal{L}} \)

\[
P_{\beta;n}(\bar{\mathcal{L}}, \mathcal{L}', M; t) = P_{\beta;n}(\mathcal{L}, \mathcal{L}', M; t_1, \ldots, t_j^{-1}, \ldots, t_L).
\] (1.84)

The polynomials \( p_{\mu;n}(\mathcal{L}, \mathcal{L}', M; \beta) \) contain only odd powers of \( \beta_j 
\]

\[
p_{\mu;n}(\mathcal{L}, \mathcal{L}', M; \beta_1, -\beta_j, \ldots, \beta_L) = -p_{\mu;n}(\mathcal{L}, \mathcal{L}', M; \beta).
\] (1.86)

Finally, if we select an orientation for the components of \( \mathcal{L}' \), then the formal power series (1.78) can be rewritten as

\[
\hat{Z}^{(r)}_{\beta}(\mathcal{L}, \mathcal{L}', M; t; h) = \begin{cases} 
\frac{1}{2} h_1^{-3/2}(M) q^{\phi_{\beta}(\mathcal{L}, \mathcal{L}', M; \beta)} \sum_{n=0}^{\infty} P_{\beta;n}(\mathcal{L}') h^n & \text{if } L = 0, \\
\frac{1}{2} h_1^{-1/2}(M) q^{\phi_{\beta}(\mathcal{L}, \mathcal{L}', M; \beta)} \sum_{n=0}^{\infty} \frac{P_{\beta;n}(\mathcal{L}, \mathcal{L}', M; t_1)}{\nabla_A(\mathcal{L}, M; t_1)} h^n & \text{if } L = 1, \\
\frac{1}{2} h_1^{-1/2}(M) q^{\phi_{\beta}(\mathcal{L}, \mathcal{L}', M; \beta)} \sum_{n=0}^{\infty} \frac{P_{\beta;n}(\mathcal{L}, \mathcal{L}', M; t)}{\nabla_A(\mathcal{L}, M; t)} h^n & \text{if } L \geq 2,
\end{cases}
\] (1.87)

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where

$$\phi_{lk}(\mathcal{L}, \mathcal{L}', M; \beta) = 3\lambda_{\text{CW}}(M) + \frac{1}{2}\left(L + \sum_{1 \leq i < j \leq L} l_{ij}^{(\mathcal{L}|M)} + \sum_{1 \leq i < j \leq L'} l_{ij}^{(\mathcal{L}'|M)}(\beta_i - 1)(\beta_j - 1) - \sum_{j=1}^{L'} (\beta_j - 1)\right),$$

(1.88)

$$\Phi_{\beta}(\mathcal{L}, \mathcal{L}', M; t) = \prod_{i=1}^{L} t_i^{\frac{1}{2} \sum_{j=1}^{L'} l_{ij}^{(\mathcal{L}'|M)} (\beta_j - 1)},$$

(1.89)

while

$$P'_{\beta,m}(\mathcal{L}, \mathcal{L}', M; t) \in \mathbb{Z}[t^{\pm1/2}, 1/h_1(M)]$$

(1.90)

for any integer $\beta$ and

$$P'_{\beta,m}(\mathcal{L}, \mathcal{L}', M; t) = \sum_{m \geq 0} p'_{m,m}(\mathcal{L}, \mathcal{L}', M; \beta) (t - 1)^m, \quad p'_{m,m}(\mathcal{L}, \mathcal{L}', M; \beta) \in \mathbb{Q}[\beta].$$

(1.91)

**Remark 1.9** Obviously, if $M = S^3$, then this theorem is reduced to Main Theorem of [15] and

$$\hat{Z}^{(r)}(\mathcal{L}, \mathcal{L}', S^3; t; h) = \hat{J}^{(r)}(\mathcal{L}, \mathcal{L}'; t; h).$$

(1.92)

**Remark 1.10** It follows from eqs. (1.79) and (1.81) that for a knot $K$ and a link $\mathcal{L}'$ in a QHS $M$

$$Z^{(r)}_{\alpha,\beta}(K \cup \mathcal{L}', M; q^\alpha; h) = \hat{Z}^{(r)}_{\beta}(\mathcal{L}, \mathcal{L}', M; q^\alpha; h).$$

(1.93)

To formulate two other theorems, we substitute

$$t = e^{2\pi i\alpha}$$

(1.94)

into $\hat{Z}^{(r)}_{\beta}(\mathcal{L}, \mathcal{L}', M; t; h)$ and convert it into a formal power series in $K^{-1}$ (cf. Remark 1.1) which we call the $U(1)$-RC invariant of $\mathcal{L}$ and $M$

$$I^{(r)}_{\beta}(\mathcal{L}, \mathcal{L}', M; q; K) = e^{\frac{4\pi iK_{lk}(\mathcal{L}, M; q)}{2\pi i}} \hat{Z}^{(r)}_{\beta}(\mathcal{L}, \mathcal{L}', M; e^{2\pi i\alpha} h).$$

(1.95)
Theorem 1.11 Let $\mathcal{L}$ be a non-empty link in a $\mathbb{Q} \text{HS} M$ which satisfies (1.75). Suppose that $M$ is constructed by Dehn’s surgery on a link $\hat{\mathcal{L}}^s \in S^3$ such that neither of $L^s$ functions

$$c_j^{(st)}(a) = -\sum_{i=1}^{L} c_{ij}(\mathcal{L}, \hat{\mathcal{L}}^s) a_i,$$

where the coefficients $c_{ij}(\mathcal{L}, \hat{\mathcal{L}}^s)$ are defined by eq.(1.15), is identically equal to zero. Then

$$I_{\beta}^{(r)}(\mathcal{L}, \mathcal{L}', M; a; K) = e^{(2\pi i/K) \phi^{(tr)}(\mathcal{L} \cup \mathcal{L}', \mathcal{L}^s; \hat{\mathcal{L}}^s, K; \beta)} (2K)^{L^s/2}$$

$$\times \int_{\{d\zeta\} \{\sin(\pi \zeta)\}} \left\{ \frac{1}{2} i\pi K \sum_{j=1}^{L^s} l_{jj}^{(\hat{\mathcal{L}}^s|S^3)} c_j^2 \right\} I_{\beta}^{(r)}(\mathcal{L} \cup \mathcal{L}^s, \mathcal{L}', S^3; a, \zeta; K).$$

Note, that

$$\exp\left(\frac{1}{2} i\pi K \sum_{j=1}^{L^s} l_{jj}^{(\hat{\mathcal{L}}^s|S^3)} c_j^2 \right) I_{\beta}^{(r)}(\mathcal{L} \cup \mathcal{L}^s, \mathcal{L}', S^3; a, \zeta; K)$$

$$= e^{\frac{1}{2} i\pi K \text{lk}(\mathcal{L} \cup \mathcal{L}^s, S^3; a, \zeta)} \hat{Z}^{(r)}_{\beta}(\mathcal{L} \cup \mathcal{L}^s, \mathcal{L}', S^3; e^{2\pi i a}, e^{2\pi i \zeta}; \ell),$$

and $c^{(st)}(a)$ is the stationary point of the phase $\text{lk}(\mathcal{L} \cup \hat{\mathcal{L}}^s, S^3; a, \zeta)$.

Theorem 1.12 Let $\mathcal{L}$ be a link in a $\mathbb{Q} \text{HS} M$ which satisfies (1.75). Then

$$I_{\beta}^{(r)}(\mathcal{L}, \mathcal{L}', M; a; K) = \left(\frac{K}{4\pi}\right)^L \int_{\{d\alpha\} \{d\beta\}} \exp\left(\frac{1}{2} i\pi K \text{E}(\mathcal{L}, M; \alpha, \beta)\right) \text{P}(\mathcal{L}, \mathcal{L}', M; \alpha, \beta; K).$$

Here $\int_{\{d\alpha\} \{d\beta\}}$ is a contribution of the stationary phase points of the form $\alpha = a\bar{n}$, $\bar{n}$ being a unit vector in $su(2)$, to the integral (1.68).

Let us sketch the ideas of the proofs. We will prove the theorems in the reverse order. First of all, we show that for the links satisfying condition (1.75), the integral in the r.h.s. of eq.(1.99) is well-defined. This allows us to regard eqs.(1.99) and (1.95) as a definition of the power series $\hat{Z}^{(r)}_{\beta}(\mathcal{L}, \mathcal{L}', M; \ell; h)$. We prove that thus defined, $\hat{Z}^{(r)}_{\beta}(\mathcal{L}, \mathcal{L}', M; \ell; h)$ satisfies the properties (1.79)–(1.81). Then we use the uniqueness theorem of [15] to prove that $\hat{Z}^{(r)}_{\beta}(\mathcal{L}, \mathcal{L}', M; \ell; h)$ is the topological invariant of $\mathcal{L}, \mathcal{L}', M$ and that it coincides with the $U(1)$-RC invariant defined in [15] in the case when $M = S^3$. 

20
Next, we prove that the invariant $\hat{Z}_\beta^{(i)}(\mathcal{L}, \mathcal{L}', M; t; h)$ defined by eqs. (1.99) and (1.95) satisfies the surgery formula (1.97). The idea of the proof is to substitute the formula (1.69) into eq. (1.99). Then the stationary phase integral over $\vec{c}$ can be split into two integrals: a stationary phase integral over the directions of $\vec{c}$ which together with the integral over the directions of $\vec{a}$ yields $I_\beta^{(i)}(\mathcal{L} \cup \mathcal{L}^*, \mathcal{L}', S^3; a, c; K)$, and a stationary phase integral over the absolute values $|\vec{c}|$ which becomes the stationary phase integral over $\vec{c}$ in eq. (1.97).

Finally, since we have already established that $\hat{Z}_\beta^{(i)}(\mathcal{L}, \mathcal{L}', S^3; t; h)$ defined by eqs. (1.99) and (1.95) coincides with the $U(1)$-RC invariant of [15] and therefore admits the presentation (1.78), we use the surgery formula (1.97) to show that $\hat{Z}_\beta^{(i)}(\mathcal{L}, \mathcal{L}', M; t; h)$ also admits that presentation for a general $Q_{HS} M$.

2 The $U(1)$-RC contribution to the Reshetikhin integral

2.1 The definition of the $U(1)$-RC contribution

The definition of the $U(1)$-RC invariant of a link $\mathcal{L}$ depends on the number of its components $L$. If $L = 0$, that is, if $\mathcal{L}$ is an empty link, then we define

$$I_\beta^{(i)}(\emptyset, \mathcal{L}', M; K) = Z^{(ir)}_\beta(\mathcal{L}', M; K).$$

If $\mathcal{L}$ is non-empty, then we consider a truncated version of a general Reshetikhin formula (1.68)

$$I_\beta(\mathcal{L}, \mathcal{L}', M; a; K|N) = \left(\frac{K}{4\pi}\right)^L \int_{|\vec{a}| = a} \left\{ \frac{d\vec{a}}{\vec{a}} \right\} e^{\frac{i}{2}K E(\mathcal{L}, \mathcal{L}'; \vec{a} |N)} P(\mathcal{L}, \mathcal{L}', M; \vec{a}; \beta; K|N),$$

where $N$ is a positive integer. For all the unit vectors $\vec{n} \in su(2)$, the points

$$\vec{a} = a \vec{n}, \quad |\vec{n}| = 1$$

form a 2-dimensional sphere inside the integration space

$$|\vec{a}| = a.$$
Proposition 2.1 The points (2.3) are stationary points of $E(L, M; \vec{a}|N)$.

Proof. Since the condition (2.4) implies that the linear variations of $\vec{a}_j$ have to be orthogonal to $\vec{a}_j$, then the points (2.3) are stationary for $L_2(L, M; \vec{a})$ of (1.63). Conditions (1.64) show that these points are also stationary for all other polynomials $L_m(L, M; \vec{a})$.

We denote the stationary contribution of the sphere (2.3) to the integral (2.2) in the limit of $K \to \infty$ as

$$I_{\beta}^{(i)}(L, L', M; \vec{a}; K|N) = \left( \frac{K}{4\pi} \right)^L K a_1 \int_{|\vec{a}|=a} \left\{ \frac{d\vec{a}}{\vec{a}} \right\} e^{\frac{i}{2} \pi K E(L, M; \vec{a}|N)} P(L, L', M; \vec{a}; \beta; K|N).$$

Let us calculate this integral following the general procedures of subsection 1.2.6.

First of all, we have to reduce the sphere (2.3) to a single point. We can do this with the help of the $SU(2)$ symmetry of the integrand. Consider the integral (2.2). Suppose that we integrate over all the vectors except $\vec{a}_1$. The resulting expression is a function of $\vec{a}_1$, however, due to the $SU(2)$ symmetry, it does not depend on the direction of $\vec{a}_1$. Therefore the remaining integral over the directions of $\vec{a}_1$ is trivial and contributes only a factor $4\pi a_1$.

Thus we conclude that

$$I_{\beta}(L, L', M; \vec{a}; K|N) = \left( \frac{K}{4\pi} \right)^L K a_1 \int_{|\vec{a}|=a} \left\{ \frac{d\vec{a}}{\vec{a}} \right\} e^{\frac{i}{2} \pi K E(L, M; \vec{a}|N)} P(L, L', M; \vec{a}; \beta; K|N).$$

If $a_1 \neq 0$, then the stationary sphere (2.3) is reduced to the stationary point

$$\vec{a}_{(1)} = \vec{a}_{(1)} \vec{n}_s, \quad \vec{n}_s = \vec{a}_1/a_1$$

and

$$I_{\beta}^{(i)}(L, L', M; \vec{a}; K|N) = \left( \frac{K}{4\pi} \right)^L K a_1 \int_{\vec{a}_{(1)} = \vec{a}_1} \left\{ \frac{d\vec{a}_{(1)}}{\vec{a}_{(1)}} \right\} e^{\frac{i}{2} \pi K E(L, M; \vec{a}|N)} P(L, L', M; \vec{a}; \beta; K|N).$$

Remark 2.2 Fixing the direction of $\vec{a}_1$ in the integral (2.6) is similar to “breaking a strand” in the closure of a braid in [15].
Now we have to consider two separate cases. If $L = 1$, that is, if the link $\mathcal{L}$ is just a knot, then, in fact, there is no integration left in eq.(2.8). Therefore we define

$$I^{(r)}_2(\mathcal{L}, \mathcal{L}', M; a; K) = I_2(\mathcal{L}, \mathcal{L}', M; a; K) \quad \text{if } L = 1. \quad (2.9)$$

Furthermore, in this case $E(\mathcal{L}, M; \vec{a}) = 0$ because of eq.(1.64), and therefore

$$I_2(\mathcal{L}, \mathcal{L}', M; a; K) = Ka_1 \sum_{m, m_n \geq 0} D_{m, m_n} (\mathcal{L} \cup \mathcal{L}', M) a_2 \beta^{2m+1} K^{-n}. \quad (2.10)$$

(cf. eq.(1.56)).

If $L > 1$, then we proceed to expand the exponent and preexponential factor of the integrand in the vicinity of the stationary point. For a fixed unit vector $\vec{n}$ we introduce small vectors $\vec{x} = (x_1, \ldots, x_L)$ such that $\vec{x} \cdot \vec{n} = 0$ and

$$\vec{a}_j = a_j \left( \cos(|\vec{x}_j|) \vec{n} + \frac{\sin(|\vec{x}_j|)}{|\vec{x}_j|} \vec{x}_j \right) \quad \text{if } L = 1. \quad \text{if } L > 1. \quad (2.11)$$

Actually, it is convenient to identify a 2-dimensional space orthogonal to $\vec{n}$ with $\mathbb{C}$ (cf. the complex structure on the co-adjoint orbits), so that $\vec{x}$ become complex variables $\vec{x} \in \mathbb{C}$ and

$$\vec{x}_i \cdot \vec{x}_j = (x_i \bar{x}_j + \bar{x}_i x_j)/2. \quad (2.12)$$

The integration measure for the variables $\vec{x}$ is

$$\frac{d^2 \vec{a}_j}{a_j} = i \frac{a_j}{2} \frac{\sin(|\vec{x}_j|)}{|x_j|} \ dx_j d\bar{x}_j = \frac{i}{2} a_j \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \ (x_j \bar{x}_j)^n \right) d\bar{x}_j. \quad (2.13)$$

Of course, the $SU(2)$ symmetry reduction removes the integration over $x_1$.

Let us substitute the series (2.11) into $E(\mathcal{L}, M; \vec{a}_1 | N)$ and keep only the constant and quadratic terms in $\vec{x}, \bar{\vec{x}}$, while setting $x_1 = 0$. Only the term $L_2(\mathcal{L}, M; \vec{a})$ contributes to the constant term because of condition (1.64). Therefore

$$E(\mathcal{L}, M; \vec{a} | N) = \text{lk}(\mathcal{L}, M; a) + A^{(1)}(a; \vec{x}_1(1), \bar{x}_1(1) | N) + O(x_1^2 \bar{x}_1^2). \quad (2.14)$$
where \( A^{(1)}(\vec{a}; \vec{x}_1(1), \vec{x}(1)|N) \) is a bilinear form in \( \vec{x} \) and \( \vec{\bar{x}} \) whose coefficients are polynomials of \( \vec{a} \) (the index \( (1) \) indicates that we set \( x_1 = 0 \)). The terms quadratic in \( \vec{x} \) or \( \vec{\bar{x}} \) are absent due to the \( U(1) \) symmetry of the points \((2.3)\). This \( U(1) \) rotates the vectors around the axis of \( \vec{n} \) and therefore multiplies each \( x_j \) by a phase factor \( e^{i\phi} \) and each \( \bar{x}_j \) by a phase factor \( e^{-i\phi} \) (note that as a result, there are no linear terms in \( \vec{x}, \vec{\bar{x}} \) as we expected, since the points \((2.3)\) are stationary for the function \((2.14)\)).

Suppose that the form \( A^{(1)}(\vec{a}; \vec{x}_1(1), \vec{x}(1)|N) \) is non-degenerate. Then we can proceed with calculating the integral \((2.8)\) in the stationary phase approximation. We keep the constant and bilinear terms in the exponent, while expanding it in terms of higher order in \( \vec{x}, \vec{\bar{x}} \), so that

\[
e^{\frac{i}{2}i\pi KE(\mathcal{L}, M; \vec{a} | N)} \mathbf{P}(\mathcal{L}, \mathcal{L}', M; \vec{a}; \mathcal{L}; K | N) = e^{\frac{i}{2}i\pi K\mathrm{lk}(M, \mathcal{L}; \vec{a})} e^{\frac{i}{2}i\pi KA^{(1)}(\vec{a}; \vec{x}_1(1), \vec{x}(1)|N)} \sum_{m \geq 0} B_{m,n}(\vec{a}; \vec{x}_1(1), \vec{x}(1)|N) K^{-n},
\]

where \( B_{m,n}(\vec{a}; \vec{x}_1(1), \vec{x}(1)|N) \) are polynomials of \( \vec{x} \) and \( \vec{\bar{x}} \) of homogeneous degree \( m \), whose coefficients are polynomials of \( \vec{a} \). Then

\[
F_n(\vec{a}|N) = (i/2\pi)^{L-1} K \{ \vec{a} \} e^{\frac{i}{2}i\pi K\mathrm{lk}(M, \mathcal{L}; \vec{a})} \sum_{n=0}^{\infty} F_n(\vec{a}|N) K^{-n}.
\]

Here

\[
F_n(\vec{a}|N) = 4^L \sum_{0 \leq m \leq n} K^{L-m-1} \int_{-\infty}^{+\infty} \{ d\vec{x}_1(1) d\bar{\vec{x}}(1) \} e^{\frac{i}{2}i\pi KA^{(1)}(\vec{a}; \vec{x}_1(1), \bar{x}(1)|N)} B_{m,n-m}(\vec{a}; \vec{x}_1(1), \bar{x}(1)|N) e^{(2i/\pi K)A^{(1)}(\vec{a}; \vec{x}_1(1), \bar{x}(1)|N)} \bigg| \vec{x}_1(1) = 0, \bar{x}(1) = 0
\]

\[
= \frac{1}{\det A^{(1)}(\vec{a}|N)} \sum_{0 \leq m \leq n} K^m B_{m,n-m}(\vec{a}; \vec{x}_1(1), \bar{x}(1)|N) e^{(2i/\pi K)A^{(1)}(\vec{a}; \vec{x}_1(1), \bar{x}(1)|N)} \bigg| \vec{x}_1(1) = 0, \bar{x}(1) = 0
\]

\[
= \frac{S_n(\mathcal{L}, \mathcal{L}', M; \vec{a}; \beta|N)}{S_{d+1}^d(\mathcal{L}, M; \vec{a}|N)},
\]

where \( A^{(1)}(\vec{a}|N) \) is the matrix of the quadratic form \( A^{(1)}(\vec{a}; \vec{x}_1(1), \vec{x}(1)|N) \), \( \bar{A}^{(1)}(\vec{a}; \vec{y}_1(1), \bar{y}(1)|N) \) is the quadratic form whose matrix is the inverse of \( A^{(1)}(\vec{a}|N) \), \( S_n(\mathcal{L}, \mathcal{L}', M; \vec{a}; \beta|N) \) are polynomials in \( \vec{a} \) and we defined

\[
S_d(\mathcal{L}, M; \vec{a}|N) = \begin{cases} 
1 & \text{if } L = 1, \\
\det A^{(1)}(\vec{a}|N) & \text{if } L \geq 2.
\end{cases}
\]
Remark 2.3 We use the notation (2.18), because it follows from the $SU(2)$ invariance of $E(L, M; \vec{a}| N)$ that $det A(j; a| N)$ does not depend on $j$.

It follows from eq.(1.65) that

$$B_{0,0}(a; \vec{x}(1), \vec{x}(1)| N) = L_{0,0}(L, M; \vec{a}; \beta) + O(a) = h_1^{-3/2}(M) \{\beta\} + O(a), \quad (2.19)$$

and then, according to eq.(2.17),

$$S_0(L, L', M; \vec{a}; \vec{\beta}| N) = h_1^{-3/2}(M) \{\beta\} + O(a). \quad (2.20)$$

Proposition 2.4 If

$$S_d(L, M; \vec{a}| N) \neq 0, \quad (2.21)$$

then the stationary phase contribution (2.8) is well-defined and

$$I^{(r)}_{\beta}(L, L', M; \vec{a}; K| N) = \frac{i}{2\pi} L^{-1} K \{\alpha\} e^{\frac{i}{2} \text{Im}(M, \vec{a}; \beta)} \sum_{n=0}^{\infty} \frac{S_n(L, L', M; \vec{a}; \vec{\beta}| N)}{S_{d+1}(L, M; \vec{a}| N)} K^{-n}. \quad (2.22)$$

The polynomials $S_d(L, M; \vec{a}| N)$ and $S_n(L, L', M; \vec{a}; \vec{\beta}| N)$ have stable limits

$$\lim_{N \to \infty} S_d(L, M; \vec{a}| N) = S_d(L, M; \vec{a}), \quad \lim_{N \to \infty} S_n(L, L', M; \vec{a}; \vec{\beta}| N) = S_n(L, L', M; \vec{a}; \vec{\beta}),$$

$$S_d(L, M; \vec{a}) \in C[\alpha], \quad S_n(L, L', M; \vec{a}; \vec{\beta}) \in C[\beta][\alpha] \quad (2.23)$$

and

$$S_0(L, L', M; \vec{a}; \vec{\beta}) = h_1^{-3/2}(M) \{\beta\} + O(a). \quad (2.24)$$

Proof. Since we have just derived eq.(2.22), it remains only to establish the limits (2.23). They follow easily from the fact that a polynomial $L_m(L, M; \vec{a})$ contributes only to the terms of order $\vec{a}^m$, $|m| \geq m$ in $S_d(L, M; \vec{a}| N)$ and $S_n(L, L', M; \vec{a}; \vec{\beta}| N)$, while a polynomial $L_{m,n}(L, L', M; \vec{a}; \vec{\beta})$ contributes only to the terms of order $\vec{a}^m$, $|m'| \geq m$ in $S_n(L, L', M; \vec{a}; \vec{\beta})$, $n' \geq n$. \hfill \Box
**Definition 2.5** Let \( \mathcal{L} \) be a link in a QHS \( M \). If for a stringing of \( \mathcal{L} \), the corresponding determinant (2.18) is not identically equal to zero or if \( \mathcal{L} \) is an empty link, then the \( U(1) \)-RC contribution to the Reshetikhin integral (1.36), which we denote formally as

\[
I^{(r)}(r; \beta; L, L', M; a; K) = \left( \frac{K}{4\pi} \right)^L \int_{\|\vec{a}\| = a} \left\{ \frac{d\vec{a}}{\vec{a}} \right\} e^{\frac{i}{2\pi} K E(L, M; \vec{a})} \mathbf{P}(L, L', M; \vec{a}; \beta; K),
\]

is defined as

\[
I^{(r)}(\mathcal{L}, \mathcal{L}', M; a; \vec{a}; K) = \begin{cases} 
Z^{(tr)}(\mathcal{L}', M; K) & \text{if } L = 0, \\
I^{(r)}(\mathcal{L}, \mathcal{L}', M; a; \vec{a}; K) & \text{if } L = 1, \\
(i/2\pi)^{L-1} K \{\vec{a}\} e^{\frac{i}{2\pi} K \text{lk}(M, \mathcal{L}; a)} \sum_{n=0}^{\infty} \frac{S_n(L, \mathcal{L}', M; \vec{a}; \beta)}{S^{2n+1}_{\text{d}}(\mathcal{L}, M; \vec{a})} K^{-n} & \text{if } L \geq 2.
\end{cases}
\]

It follows from eq. (1.72), that

\[
I^{(r)}(\mathcal{L}, \mathcal{L}', M; a; \vec{a}; K) \bigg|_{\beta = 1} = I^{(r)}(\mathcal{L}, \mathcal{L}', M; a; \vec{a}; K),
\]

if both sides of this equation originated from the same stringing of links in \( S^3 \). Also it follows from Proposition 1.7 that

\[
I^{(r)}(\mathcal{L}, \mathcal{L}', M \# M'; a; \vec{a}; K) = Z^{(tr)}(M'; K) I^{(r)}(\mathcal{L}, \mathcal{L}', M; a; \vec{a}; K),
\]

where \( M' \) is another QHS.

### 2.2 Expansion of the \( U(1) \)-RC invariant

We are going to study the properties of the expansion of the formal power series (2.26) in powers of \( a \). First of all, we have to establish the relation between the determinants (2.18) of links and sublinks.

**Proposition 2.6** For a number \( j, 1 \leq j \leq L \), if the formal power series \( E(\mathcal{L}, M; \vec{a}) \) and \( E(\mathcal{L}_{(j)}, M; \vec{a}_{(j)}) \) for \( \mathcal{L} \) and its sublink \( \mathcal{L}_{(j)} \) come from the same stringing of the link \( \mathcal{L} \), then

\[
S_d(\mathcal{L}, M; \vec{a}) = -a_j S_d(\mathcal{L}_{(j)}, M; \vec{a}_{(j)}) \sum_{i \leq L \atop i \neq j} t^{(L|M)}_{ij} a_i + O(a_j^2).
\]

(2.29)
Proof. In view of Remark 2.3, we can assume that \( j \neq 1 \). Consider the matrix elements of the matrix \( A^{(1)}(a|N) \). Since the coefficients of the powers of \( \bar{e}_i \) in the expression (2.11) for \( \bar{a}_i \) are proportional to \( a_i \), then it is easy to see that \( A_{ik}^{(1)}(\mathcal{L}, M; a|N) \) is proportional to \( a_i a_k \) if \( i \neq k \) and to \( a_i \) if \( i = k \). Therefore we can introduce a matrix

\[
\bar{A}_{ik}^{(1)}(\mathcal{L}, M; a|N) = A_{ik}^{(1)}(\mathcal{L}, M; a|N)/a_k,
\]

whose coefficients are also polynomials of \( a \). Obviously,

\[
S_d(\mathcal{L}, M; a) = \{a_{(1)}\} \det \bar{A}^{(1)}(\mathcal{L}, M; a|N).
\]

We can expand \( \det \bar{A}^{(1)}(\mathcal{L}, M; a|N) \) in \( j \)-th row. Since all the coefficients \( \bar{A}_{jk}^{(1)}(\mathcal{L}, M; a|N) \), \( k \neq j \) are still proportional to \( a_j \), then

\[
\det \bar{A}^{(1)}(\mathcal{L}, M; a|N) = \bar{A}_{jj}^{(1)}(\mathcal{L}, M; a|N) \det \bar{A}^{(1)}(\mathcal{L}, M; a|N) + O(a_j)
\]

\[
= \bar{A}_{jj}^{(1)}(\mathcal{L}, M; a|N) \det \bar{A}^{(1)}(\mathcal{L}, M; a|N) + O(a_j),
\]

where \( \bar{A}^{(1)}(\mathcal{L}, M; a|N) \) is the \((j,j)\)-minor of \( \bar{A}^{(1)}(\mathcal{L}, M; a|N) \).

Since \( E(\mathcal{L}, M; \bar{a}) \) and \( E(\mathcal{L}_{(j)}, M; \bar{a}_{(j)}) \) come from the same stringing of \( \mathcal{L} \), then

\[
E(\mathcal{L}_{(j)}, M; \bar{a}_{(j)}) = E(\mathcal{L}, M; \bar{a})|_{a_j=0}
\]

and, as a result,

\[
\bar{A}^{(1)}(\mathcal{L}_{(j)}, M; a_{(j)}|N) = \bar{A}_{jj}^{(1)}(\mathcal{L}, M; a|N)|_{a_j=0}.
\]

Now consider the diagonal matrix element \( A_{jj}^{(1)}(\mathcal{L}, M; a|N) \). The part of it, which is linear in \( a_j \), comes from the terms in the polynomials \( L_m(\mathcal{L}, M; \bar{a}) \) which are linear in \( \bar{a}_j \). However, in view of eq.(1.64), only the polynomial \( L_2(\mathcal{L}, M; \bar{a}) \) gives a non-zero contribution, so that

\[
\bar{A}_{jj}^{(1)}(\mathcal{L}, M; a|N) = -\sum_{1 \leq i \leq L \atop i \neq j} l_{ij}^{(L|M)} a_i + O(a_j).
\]

Now we can conclude from eq.(2.29) written for both \( \mathcal{L} \) and \( \mathcal{L}_{(j)} \) as well as from eqs.(2.32), (2.34) and (2.35) that

\[
S_d(\mathcal{L}, M; a|N) = -a_j S_d(\mathcal{L}_{(j)}, M; a_{(j)}|N) \sum_{1 \leq i \leq L \atop i \neq j} l_{ij}^{(L|M)} a_i + O(a_j^2).
\]
Equation (2.29) follows if we take the stable limits of both sides of eq.(2.36) as $N \to \infty$. □

Let us establish a relation between the formal power series $I^{(i)}_\beta (\mathcal{L}, \mathcal{L}', M; \underline{a}; K)$ and similar series coming from the sublinks of $\mathcal{L}$.

**Proposition 2.7** Suppose that a link component $\mathcal{L}_j$ of an $L$-component link $\mathcal{L}$, $L > 1$, is algebraically linked to $\mathcal{L}_{(j)}$ (that is, there is a number $i$, $i \neq j$, such that $I^{(i)}_{ij} \neq 0$) and $S_d(\mathcal{L}_{(j)}, M; \underline{a}_{(j)}) \neq 0$. Then

1. $S_d(\mathcal{L}, M; \underline{a}) \neq 0$ and, moreover, the ratios $\{a\} S_n(\mathcal{L}, \mathcal{L}', M; \underline{a}; \beta)/S^{2n+1}_d(\mathcal{L}, M; \underline{a})$ are non-singular at $a_j = 0$;

2. if we expand $S_n(\mathcal{L}, \mathcal{L}', M; \underline{a}; \beta)/S^{2n+1}_d(\mathcal{L}, M; \underline{a})$ in powers of $a_j$ and substitute $a_j = a_j/K$, then

$$
\sum_{\mu=\pm 1} \mu^\mu I^{(i)}_{\beta}(\mathcal{L}, \mathcal{L}', M; \underline{a}; K) \bigg|_{a_j = \mu a_j/K} = I^{(i)}_{\alpha_j \beta}(\mathcal{L}_{(j)}, \mathcal{L}_j \cup \mathcal{L}', M; \underline{a}_{(j)}; K) 
$$

(2.37)

**Proof.** The fact that $S_d(\mathcal{L}, M; \underline{a}) \neq 0$ follows easily from eq.(2.29). Thus the stationary phase integral $I^{(i)}_{\beta}(\mathcal{L}, \mathcal{L}', M; \underline{a}; K|N)$ is well-defined for sufficiently large $N$. Let us study its expansion in powers of $a_j$. Instead of expanding the series (2.16) directly, we will work with the integral (2.8). Let us assume without the loss of generality that $j \neq 1$. We can split the integral (2.8) into two integrals

$$
I^{(i)}_{\beta}(\mathcal{L}, \mathcal{L}', M; \underline{a}; K|N) = \left( \frac{K}{4\pi} \right)^{L-2} \times K a_1 \int_{|\underline{a}_{(1)}| = a_{(1)}^{(1)}} \left\{ \frac{d\tilde{\alpha}^{(1)}(j)}{\tilde{\alpha}(1)} \right\} e^{\frac{1}{2} i \pi K E(\mathcal{L}_{(j)}, M; \underline{a}_{(j)}|N)} P^{(i)}_{(j)}(\mathcal{L}, \mathcal{L}', M; \tilde{a}(j); \beta; K|N),
$$

(2.38)

$$
P^{(i)}_{(j)}(\mathcal{L}, \mathcal{L}', M; \tilde{a}(j); \beta; K|N) = \frac{K}{4\pi} \int_{|\underline{a}| = a_j} \frac{d\tilde{a}_j}{a_j} e^{i \pi K \tilde{A}(\mathcal{L}_{(j)}; \underline{a}_{(j)})|N} \tilde{a}_j
$$

(2.39)

$$
\tilde{a}_j^{(st)}(\underline{a}_{(j)}) = \frac{\tilde{A}(\mathcal{L}_{(j)}; \underline{a}_{(j)}|N)}{\tilde{A}(\mathcal{L}_{(j)}; \underline{a}_{(j)}|N)} a_j.
$$

(2.40)
We derived this formula by splitting the exponent $E(\mathcal{L}, M; \vec{a}|N)$ of eq. (2.8) in 3 pieces: the terms which do not depend on $\vec{a}_j$ (that is, $E(\mathcal{L}(j), M; \vec{a}_j(j)|N)$), the terms which are linear in $\vec{a}_j$ (that is, $\vec{A}(\mathcal{L}(j); \vec{a}_j(j)|N) \cdot \vec{a}_j$) and the terms of higher order in $\vec{a}_j$. We expanded the exponential $e^{\frac{1}{\hbar} \pi K E(\mathcal{L}, M; \vec{a}|N)}$ in the third group of terms, combining this expansion with preexponential series $P(\mathcal{L}, \mathcal{L}', M; \vec{a}; \beta; K|N)$ and thus obtaining $P_{(j)}(\mathcal{L}, \mathcal{L}', M; \vec{a}; \beta; K|N)$:

$$P_{(j)}(\mathcal{L}, \mathcal{L}', M; \vec{a}; \beta; K|N) = \sum_{m \geq 0} P_{(j)}(\mathcal{L}, \mathcal{L}', M; \vec{a}; \beta|N) K^{-m}, \quad (2.41)$$

where $P_{(j)}(\mathcal{L}, \mathcal{L}', M; \vec{a}; \beta|N)$ are polynomials of $\vec{a}$ and $\beta$ of homogeneous degree $m$ in $\vec{a}_j$. Then we fixed the vectors $\vec{a}_j(j)$ near the stationary point (2.7) and performed the stationary phase integral (2.39) over $\vec{a}_j$. Note that

$$\vec{A}(\mathcal{L}(j); \vec{a}_j(j)|N) = \sum_{1 \leq i \leq L} \sum_{i \neq j} l_i^{(M)} \vec{a}_i + O(\vec{a}_j^2). \quad (2.42)$$

Since $\mathcal{L}_j$ is algebraically connected to $\mathcal{L}(j)$, then

$$\sum_{1 \leq i \leq L} \sum_{i \neq j} l_i^{(M)} \vec{a}_i \neq 0, \quad |\vec{A}(\mathcal{L}(j); \vec{a}_j(j)|N)| \neq 0, \quad (2.43)$$

and the stationary point (2.40) is well-defined. The integral (2.39) is calculated on a term-by-term basis with the help of the formula

$$\int_{|\vec{a}_j| = a_j} \frac{d\vec{a}_j}{a_j} e^{\frac{1}{\hbar} \pi K \vec{A}(\mathcal{L}(j); \vec{a}_j(j)|N) \cdot \vec{a}_j} P(\vec{a}_j) = -2\pi i \sum_{i \neq j} l_i^{(M)} \frac{e^{ia_j|\vec{b}|}}{|\vec{b}|} \bigg|_{\vec{b} = \pi K \vec{A}(\mathcal{L}(j); \vec{a}_j(j)|N)} \int_{|\vec{a}_j| = a_j} \frac{d\vec{a}_j}{a_j} e^{\frac{1}{\hbar} \pi K \vec{A}(\mathcal{L}(j); \vec{a}_j(j)|N) \cdot \vec{a}_j} P(\vec{a}_j) = -2\pi i \sum_{i \neq j} \frac{e^{ia_j|\vec{b}|} - e^{-ia_j|\vec{b}|}}{|\vec{b}|} \bigg|_{\vec{b} = \pi K \vec{A}(\mathcal{L}(j); \vec{a}_j(j)|N)} \quad (2.44)$$

which follows from a simple relation

$$\int_{|\vec{a}_j| = a_j} \frac{d\vec{a}_j}{a_j} e^{\frac{1}{\hbar} \pi K \vec{A}(\mathcal{L}(j); \vec{a}_j(j)|N) \cdot \vec{a}_j} P(\vec{a}_j) = -2\pi i \sum_{i \neq j} \frac{e^{ia_j|\vec{b}|} - e^{-ia_j|\vec{b}|}}{|\vec{b}|} \bigg|_{\vec{b} = \pi K \vec{A}(\mathcal{L}(j); \vec{a}_j(j)|N)} \quad (2.45)$$

where $P(\vec{a}_j)$ is a polynomial of $\vec{a}_j$. As a result,

$$P_{(j)}^{(1)}(\mathcal{L}, \mathcal{L}', M; \vec{a}_j(j); a_j; \beta; K|N) = e^{\frac{1}{\hbar} \pi K a_j|\vec{A}(\mathcal{L}(j); \vec{a}_j(j)|N)} \sum_{n \geq 0} \frac{P_{(j)}(k, m, n; \mathcal{L}, \mathcal{L}', M; \vec{a}_j(j); \beta|N)}{|\vec{A}(\mathcal{L}(j); \vec{a}_j(j)|N)|^{k+1}} a_j^m K^{-(n+1)}, \quad (2.46)$$
Proposition 2.7 follows from this by taking the stable limit of those ratios at scattering singularities. Thus the integration in (2.38) will yield the following expression in the preexponential factor (2.46). However, it follows from eq. (1.64) that performed in the same way as in (2.5), the only difference being the presence of denominators in the preexponential factor (2.46). However, it follows from eq. (1.64) that

\[ \tilde{A}(\mathcal{L}(j); \mathbf{a}(j)|N)|_{\mathbf{a}(j)=\mathbf{a}(j)\mathbf{n}} = \tilde{n} \sum_{1 \leq i \leq L} l^{(\mathcal{L}|M)} i \alpha_i, \]

and, as a result,

\[ |\tilde{A}(\mathcal{L}(j); \mathbf{a}(j)|N)|_{\mathbf{a}(j)=\mathbf{a}(j)\mathbf{n}} = \sum_{1 \leq i \leq L} l^{(\mathcal{L}|M)} i a_i \neq 0. \]

Therefore the denominators are non-zero in the vicinity of the stationary phase point \( \mathbf{a}(j) = \mathbf{a}(j)\mathbf{n} \), so we can expand them in powers of \( \mathbf{a}(1,j) \) after the substitution (2.11) without encountering singularities. Thus the integration in (2.38) will yield the following expression

\[ I^{(r)}_{\mathbf{a}}(\mathcal{L}, \mathcal{L}', M; \mathbf{a}; K|N) = K \{ \mathbf{a}(j) \} e^{\frac{i}{2} \pi K k(M, \mathcal{L}, \mathfrak{a})} \sum_{n \geq \frac{1}{2} k + 1 - \frac{1}{2} m} S_{(j) k, l, m, n}(\mathcal{L}, \mathcal{L}', M; \mathbf{a}(j); \beta|N) a_j^m \left( S_d(\mathcal{L}(j), M; \mathbf{a}(j)|N) \right)^{l+1} \left( \sum_{1 \leq i \leq L} l^{(\mathcal{L}|M)} i a_i \right)^{k+1} K^{-n}, \]

which represents the expansion of \( I^{(r)}_{\mathbf{a}}(\mathcal{L}, \mathcal{L}', M; \mathbf{a}; K|N) \) in powers of \( a_j \). Note that we did not touch \( a_j \) in the exponential \( e^{\frac{i}{2} \pi K k(M, \mathcal{L}, \mathfrak{a})} \). It is clear from eq. (2.50) that the ratios \( \{ \mathbf{a} \} S_n(\mathcal{L}, \mathcal{L}', M; \mathbf{a}; \beta|N)/S_d^{2n+1}(\mathcal{L}, M; \mathbf{a}|N) \) are non-singular at \( a_j = 0 \). The first claim of Proposition 2.7 follows from this by taking the stable limit of those ratios at \( N \to \infty \).

To prove the second claim of Proposition 2.7, we recall the relation between the integrands for \( I^{(r)}_{\mathbf{a}}(\mathcal{L}, \mathcal{L}', M; \mathbf{a}; K) \) and \( I^{(r)}_{\alpha_j, \beta}(\mathcal{L}(j); \mathcal{L}_j \cup \mathcal{L}', M; \mathbf{a}(j); K) \). Namely, the integrand for the stationary phase Reshetikhin integral of \( I^{(r)}_{\alpha_j, \beta}(\mathcal{L}(j); \mathcal{L}_j \cup \mathcal{L}', M; \mathbf{a}(j); K) \) can be obtained by
expanding the exponential of the integrand for $I^{(r)}_{\beta}(\mathcal{L}, \mathcal{L}', M; a; K)$ in terms which contain the powers of $\vec{a}_j$ and then by performing the integration over $\vec{a}_j$, $|\vec{a}_j| = \alpha_j/K$. This means that
\[
I^{(r)}_{\alpha_j, \beta}(\mathcal{L}(j), \mathcal{L} \cup \mathcal{L}', M; \vec{a}_{(j)}; K|N) = \tilde{I}^{(r)}_{\beta}(\mathcal{L}, \mathcal{L}', M; \vec{a}; K|N)|_{\vec{a}_j = \alpha_j/K}, \tag{2.51}
\]
\[
\tilde{I}^{(r)}_{\beta}(\mathcal{L}, \mathcal{L}', M; \vec{a}; K|N) = \left( \frac{K}{4\pi} \right)^{L-2} \times K a_1 \int_{|\vec{a}_{(1,j)}| = \alpha_{(1,j)}^{(q+1)}} \left\{ \frac{d\vec{a}_{(1,j)}}{a_{(1,j)}} \right\} e^{\frac{1}{2} \pi K E(\mathcal{L}(j), M; \vec{a}_{(j)}; K|N)} P_{(j)}(\mathcal{L}, \mathcal{L}', M; \vec{a}_{(j)}; a_j; \beta; K|N), \tag{2.52}
\]
\[
P_{(j)}(\mathcal{L}, \mathcal{L}', M; \vec{a}_{(j)}; a_j; \beta; K|N) = \frac{K}{4\pi} \int_{|\vec{a}_j| = a_j} \frac{d\vec{a}_j}{a_j} e^{i\pi K A(\mathcal{L}(j); \vec{a}_{(j)}; K|N). \vec{a}_j} \times \tilde{P}_{(j)}(\mathcal{L}, \mathcal{L}', M; \vec{a}; \beta; K|N). \tag{2.53}
\]
The latter integral can be calculated on a term-by-term basis with the help of eq.(2.45) (note that we do not have to expand $e^{i\pi K A(\mathcal{L}(j); \vec{a}_{(j)}; K|N). \vec{a}_j}$ in powers of $\vec{a}_j$ in the l.h.s. of this equation, because its r.h.s. is non-singular at $a_j = 0$). Comparing eqs.(2.44) and (2.45), we conclude that
\[
P_{(j)}(\mathcal{L}, \mathcal{L}', M; \vec{a}_{(j)}; a_j; \beta; K|N) = \sum_{\mu = \pm 1} \mu P_{(j)}(\mathcal{L}, \mathcal{L}', M; \vec{a}_{(j)}; \mu a_j; \beta; K|N). \tag{2.54}
\]
The similarity between eqs.(2.38) and (2.52) implies that
\[
\tilde{I}^{(r)}_{\beta}(\mathcal{L}, \mathcal{L}', M; a; K|N) = \sum_{\mu = \pm 1} \mu I^{(r)}_{\beta}(\mathcal{L}, \mathcal{L}', M; a_1, \ldots, a_j, \ldots, a_L; K|N) \tag{2.55}
\]
and eq.(2.37) follows by applying the stable limit $N \to \infty$ in view of eq.(2.51).

\section*{2.3 Uniqueness arguments}

We will use the uniqueness theorem of [15] in order to prove that the formal power series $I^{(r)}_{\beta}(\mathcal{L}, \mathcal{L}', M; a; K)$ as defined by eq.(2.26), is the invariant of the links $\mathcal{L}, \mathcal{L}'$, that is, it does not depend on their stringing. First of all, we prove that $I^{(r)}_{\beta}(\mathcal{L}, \mathcal{L}', M; a; K)$ is well-defined for algebraically connected links.
**Proposition 2.8** If $\mathcal{L}$ is algebraically connected, then for any stringing
\[ S_d(\mathcal{L}, M; \underline{a}) \neq 0. \]  

**Proof.** We prove this proposition by induction in the number of link components $L$. If $L = 1$, then, obviously, $S_d(\mathcal{L}, M; \underline{a}) = 1$. Suppose that eq. (2.56) holds for all $L - 1$-component algebraically connected links. Let $\mathcal{L}$ be an $L$-component algebraically connected link. Then it is easy to see that there exists a link component $\mathcal{L}_j$ such that $\mathcal{L}_j$ is also algebraically connected. According to the assumption of the induction, this means that $S_d(\mathcal{L}_j, M; \underline{a}_j) \neq 0$. Since $\mathcal{L}_j$ is algebraically connected to $\mathcal{L}_j$, then
\[ \sum_{1 \leq i \leq L \atop i \neq j} l_{ij}^{(L|M)} a_i \neq 0 \]  
and eq. (2.56) follows from eq. (2.36). \qed

Let $\mathcal{C}_L$ denote a set of stringings of a link $\mathcal{L}$ is a QHS $M$ such that for those stringings $S_d(\mathcal{L}, M; \underline{a}) \neq 0$. Let $\mathcal{L}_*$ denote the set of all links for which the set $\mathcal{C}_L$ is non-empty. For each stringing $c \in \mathcal{C}_L$ we cancel all common powers of $\underline{a}$ in the ratios
\[ \{a\} S_n(\mathcal{L}, \mathcal{L}', M; \underline{a}; \beta) / S^2_{d+1}(\mathcal{L}, M; \underline{a}) \]
and denote the result as $F_n^{(c)}(\mathcal{L}, \mathcal{L}', M; \underline{a}; \beta) / (G^{(c)}(\mathcal{L}; \underline{a}))^{2n+1}$. Finally, let $\epsilon = K^{-1}$. Now we are ready to apply the uniqueness theorem of [15].

**Proposition 2.9** The formal power series $I^{(r)}(\mathcal{L}, \mathcal{L}', M; \underline{a}; K)$ is a topological invariant of an oriented link $\mathcal{L}$ and a link $\mathcal{L}'$ in a QHS $M$, that is, $I^{(r)}(\mathcal{L}, \mathcal{L}', M; \underline{a}; K)$ does not depend on a stringing $c \in \mathcal{C}_L$ used in order to define the integrand of (2.25).

**Proof.** This result follows from Theorem 3.1(I), which is applicable in view of eq. (2.9) (coupled with eq. (1.67)), eq. (2.27) and Proposition 2.7. \qed

**Proposition 2.10** Let $\mathcal{L}$ be a link in $S^3$ such that
\[ S_d(\mathcal{L}, S^3; \underline{a}) \neq 0, \quad \nabla_A(\mathcal{L}, S^3; \underline{f}) \neq 0. \]  

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Then
\[ I_{\beta}(\mathcal{L}, \mathcal{L}', S^3; a; K) = \tilde{J}_{\beta}(r) \tilde{J}_{\beta}(\mathcal{L}, \mathcal{L}'; e^{2\pi i q}; h). \] (2.59)

where \( \tilde{J}_{\beta}(r) \) is defined by eq. (1.52,1).

Proof. This proposition follows from the relation
\[ I_{\beta}(\varnothing, \mathcal{L}', S^3; K) = \tilde{J}_{\beta}(\varnothing, \mathcal{L}'; h) = J_{\beta}(\mathcal{L}'; q) \] (2.60)
and Theorem 3.1(I).

Comparing the leading terms of the formal power series in both sides of eq. (2.59) we come to the following

**Corollary 2.11** If a link \( \mathcal{L} \) satisfies the conditions (2.58), then
\[ \{a\} \frac{S_0(\mathcal{L}, \mathcal{L}', S^3; a; \beta)}{S_d(\mathcal{L}, S^3; a)} = i^{-L}(2\pi)^{L-2} \frac{P_{\beta,0}(\mathcal{L}, \mathcal{L}', S^3; t)}{\nabla A(\mathcal{L}, S^3; t)}, \] (2.61)
where \( P_{\beta,0}(\mathcal{L}, \mathcal{L}', S; t) \) is given by eq. (1.83).

**Proposition 2.12** If \( \mathcal{L} \in \mathcal{L}_* \) then \( S_d(\mathcal{L}, M; a) \neq 0 \) for any stringing of \( \mathcal{L} \).

Proof. Let us add an extra component \( \mathcal{L}_0 \) to \( \mathcal{L} \) in such a way that
\[ l_{0j}^{(\mathcal{L}; M)} = 1 \quad \text{for all } j, 1 \leq j \leq L. \] (2.62)

Select a stringing of \( \mathcal{L}_0 \cup \mathcal{L} \) which is compatible with the stringing of \( \mathcal{L} \). Then, according to eq. (2.29),
\[ a_0^{-1}S_d(\mathcal{L}_0 \cup \mathcal{L}, M; a_0, a)|_{a_0=0} = -S_d(\mathcal{L}, M; a) \sum_{j=1}^{L} a_j \] (2.63)

This means that, as functions of \( a \),
\[ S_d(\mathcal{L}, M; a) \equiv 0 \quad \text{iff} \quad a_0^{-1}S_d(\mathcal{L}_0 \cup \mathcal{L}, M; a_0, a)|_{a_0=0} \equiv 0. \] (2.64)
On the other hand, in view of eq.(2.24),

\[
\frac{S_d(L_0 \cup L, M; a_0, a)}{a_0 S_0(L_0 \cup L, L', M; a_0, a; \beta)} \bigg|_{a_0=0} = 0 \iff a_0^{-1}S_d(L_0 \cup L, M; a_0, a) \bigg|_{a_0=0} \equiv 0. \tag{2.65}
\]

As a result,

\[
S_d(L, M; a) \equiv 0 \iff \frac{S_d(L_0 \cup L, M; a_0, a)}{a_0 S_0(L_0 \cup L, L', M; a_0, a; \beta)} \bigg|_{a_0=0} \equiv 0. \tag{2.66}
\]

Obviously, the link \(L_0 \cup L\) is algebraically connected, so in view of Proposition 2.8, it belongs to \(L_*\). Therefore the ratio \(S_d(L_0 \cup L, M; a_0, a)/(a_0 S_0(L_0 \cup L, L', M; a_0, a; \beta))\), as a part of \(I^{(r)}(L_0 \cup L, L', M; a_0, a; K)\), is the topological invariant of the link \(L_0 \cup L\), that is, it does not depend on its stringing. Then (2.66) proves Proposition 2.12. \(\square\)

**Proposition 2.13** For the links in \(S^3\), \(S_d(L, S^3; a) \not\equiv 0\) iff \(\nabla_A(L, S^3; t) \not\equiv 0\).

**Proof.** Throughout the proof we assume that \(M = S^3\) (we want to use \(M\), because later we will use the same proof in order to prove this proposition for any \(\mathbb{QHS} M\)). The proof is similar to the previous one. Let \(L\) be an \(L\)-component link in \(M\). We supplement it with an extra component \(L_0\) satisfying eq. (2.62) and thus come to (2.66). On the other hand, eq.(2.61) together with eqs.(1.83) and (2.24) imply that

\[
a_0^{-1}S_d(L_0 \cup L, M; a_0, a) \bigg|_{a_0=0} \equiv 0 \iff \nabla_A(L, S^3; t) \equiv 0. \tag{2.67}
\]

A combination of (2.66) and (2.67) proves the proposition. \(\square\)

A combination of Propositions 2.10 and 2.13 leads to the following

**Proposition 2.14** If a link \(L \subset S^3\) satisfies a condition \(\nabla_A(L, S^3; t) \not\equiv 0\), then the stationary phase integral (2.25) is well-defined and

\[
J^{(r)}(L, L', S^3; a; K) = \tilde{J}^{(r)}(L, L'; e^{2\pi ia}; h), \tag{2.68}
\]

where \(\tilde{J}^{(r)}(L, L'; t; h)\) is defined by eq. (1.52,I).
3 Surgery formula and rational expressions

3.1 A surgery formula for the $U(1)$-RC invariant

So far, we have defined the $U(1)$-RC invariant $I^{(r)}_{\beta}(\mathcal{L}, \mathcal{L}', M; \mathcal{g}, K)$ as a particular stationary phase contribution to the Reshetikhin integral and established its relation to the $U(1)$-RC invariant of links in $S^3$, as defined in [15]. Our next step is to prove a surgery formula which would express the $U(1)$-RC invariant of links in rational homology spheres in terms of the $U(1)$-RC invariant of links in $S^3$.

**Proposition 3.1** Let $\mathcal{L}$ be a link in a $\mathbb{Q}HS$ $M$ such that

$$S_d(\mathcal{L}, M; \mathcal{g}) \neq 0, \quad \nabla_A(\mathcal{L}, M; \mathcal{L}) \neq 0. \quad (3.1)$$

Suppose that $M$ can be constructed by Dehn’s surgery on a framed link $\hat{\mathcal{L}}^s \in S^3$ such that

$$\zeta^{(st)}(\mathcal{g}) \neq 0, \quad (3.2)$$

where the functions $\zeta^{(st)}(\mathcal{g})$ are defined by eq.(1.96). Then for any link $\mathcal{L}' \in M$, the $U(1)$-RC invariant of $\mathcal{L}, \mathcal{L}'$ is given by the surgery formula (1.97).

**Proof.** First of all, we note that

$$\nabla_A(\mathcal{L} \cup \mathcal{L}^s, S^3; e^{2\pi i \mathcal{g}}, e^{2\pi i \zeta^{(st)}(\mathcal{g})}) \neq 0, \quad (3.3)$$

because, according to eq.(1.16),

$$\nabla_A(\mathcal{L} \cup \mathcal{L}^s, S^3; e^{2\pi i \mathcal{g}}, e^{2\pi i \zeta^{(st)}(\mathcal{g})}) = \sign(\hat{\mathcal{L}}^s|S^3) \left\{ e^{i\pi \zeta^{(st)}(\mathcal{g})} - e^{-i\pi \zeta^{(st)}(\mathcal{g})} \right\} \nabla_A(\mathcal{L}, M; e^{2\pi i \mathcal{g}}). \quad (3.4)$$

Now let us substitute the truncated version of the stationary phase surgery formula (1.69) into the stationary phase integral (2.8) and combine both integrals into one

$$I^{(r)}_{\beta}(\mathcal{L}, \mathcal{L}', M; \mathcal{g}; K||N) \quad (3.5)$$

$$= \left( \frac{K}{4\pi} \right)^{L-1} \left( \frac{K}{2} \right)^{2L^s} K a_1 e^{(2\pi i / K) \phi^{(r)}(\mathcal{L} \cup \mathcal{L}', \hat{\mathcal{L}}^s; \mathcal{K} \mathcal{g}, \beta)}$$

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and the integral of the orbits to see that

Note that, generally speaking, over the radii \( c \) in the sense of eq.(2.23). We leave the details to the reader.

We can split the integral over \( \beta \) into the integral over the inner integral of (3.8) is well-defined. In fact, according to

\[
\int \left\{ \frac{d\bar{a}}{a} \right\} \int \left\{ \sin \frac{\pi |\bar{c}|}{|\bar{c}|} \right\}
\]

\[
\times \exp \frac{1}{2} i\pi K \left( \sum_{j=1}^{L^s} l^{(L^s)} c_j^2 + E(\mathcal{L} \cup \mathcal{L}^s; \bar{a}, \bar{c}, \beta; K|N) \right) P(\mathcal{L} \cup \mathcal{L}^s, \mathcal{L}'; \bar{a}, \bar{c}, \beta; K|N)
\]

\[
= \left( \frac{K}{\pi} \right)^L \frac{3^L s}{2^{L^s}} K a_1 e^{2\pi i/K, \delta^{(r)}(\mathcal{L} \cup \mathcal{L}^s, \beta; K)}
\]

\[
\times \int \left\{ \frac{d\bar{a}}{a} \right\} \int \left\{ \sin \frac{\pi |\bar{c}|}{|\bar{c}|} \right\}
\]

\[
\times \exp \frac{1}{2} i\pi K \left( \sum_{j=1}^{L^s} l^{(L^s)} c_j^2 + E(\mathcal{L} \cup \mathcal{L}^s; \bar{a}, \bar{c}, \beta; K|N) \right) P(\mathcal{L} \cup \mathcal{L}^s, \mathcal{L}'; \bar{a}, \bar{c}, \beta; K|N),
\]

where in view of eqs.(1.64)

\[
\bar{c}_{(s=)}(\bar{a}, \bar{c}) = - \sum_{i=1}^{L} c_i (\mathcal{L}, \mathcal{L}^s) a_i \bar{c}_i. \tag{3.6}
\]

Note that, generally speaking, \( I^{(r)}_{\beta}(\mathcal{L}, \mathcal{L}', M; \bar{a}; K||N) \neq I^{(r)}_{\beta}(\mathcal{L}, \mathcal{L}', M; \bar{a}; K||N) \), but it is easy to see that \( I^{(r)}_{\beta}(\mathcal{L}, \mathcal{L}', M; \bar{a}; K||N) \), as defined by eq. (3.5), is expressed in the same form as (2.22) and it has the same stable limit

\[
\lim_{N \to \infty} I^{(r)}_{\beta}(\mathcal{L}, \mathcal{L}', M; \bar{a}; K||N) = I^{(r)}_{\beta}(\mathcal{L}, \mathcal{L}', M; \bar{a}; K) \tag{3.7}
\]

in the sense of eq.(2.23). We leave the details to the reader.

We can split the integral over \( \bar{c} \) into the integral over the orbits \( |\bar{c}| = c \) and the integral over the radii \( c \) of the orbits

\[
I^{(r)}_{\beta}(\mathcal{L}, \mathcal{L}', M; \bar{a}; K||N) = e^{2\pi i/K, \delta^{(r)}(\mathcal{L} \cup \mathcal{L}', \beta; K')} (2K)^{L^s/2} \left( \frac{K}{\pi} \right)^{L^s-1} K a_1 \tag{3.8}
\]

\[
\times \int_{|\bar{c}|=c_{(s=)}(\bar{a})} \left\{ \sin(\pi |\bar{c}|) \right\} \exp \left( \frac{1}{2} i\pi K \sum_{j=1}^{L^s} l^{(L^s)} c_j^2 \right)
\]

\[
\times \int \left\{ \frac{d\bar{a}}{a} \right\} \int \left\{ \frac{d\bar{c}}{\bar{c}} \right\} e^{2\pi i/K, E(\mathcal{L} \cup \mathcal{L}^s, S^3; \bar{a}, \bar{c}, \beta; K||N)} P(\mathcal{L} \cup \mathcal{L}^s, \mathcal{L}', S^3; \bar{a}, \bar{c}, \beta; K|N).
\]

We can rearrange the integral in this way, because eqs.(3.2) and (3.3) guarantee in view of Proposition 2.13 that the inner integral of (3.8) is well-defined. In fact, according to
eq. (2.6) the inner integral (together with the appropriate normalization factor) yields

\[ I_{\frac{r}{2}}^{(l)}(\mathcal{L} \cup \mathcal{L}', S^3; \mathfrak{a}, \mathfrak{c}; K|N), \]

and so we come to the following surgery formula for the truncated invariants

\[ I_{\frac{r}{2}}^{(l)}(\mathcal{L}, \mathcal{L}', M; \mathfrak{a}; K||N) = e^{(2\pi i/K) \phi^{(l)}(\mathcal{L} \cup \mathcal{L}', L'; K, \beta)} (2K)^{L^*/2} \]

\[ \times \int_{c_{\mathfrak{a} \in \mathfrak{a}}} \{ \sin(\pi c) \} \exp \left( \frac{1}{2} i\pi K \sum_{j=1}^{L} j^{(l)}(L'|S^3) c_j^2 \right) I_{\frac{r}{2}}^{(l)}(\mathcal{L} \cup \mathcal{L}', \mathcal{L}', S^3; \mathfrak{a}, \mathfrak{c}; K|N). \]

We recover eq. (1.97) by taking the stable limits of both sides of this equation at \( N \to \infty. \)

Now we want to prove that two conditions (3.1) are equivalent to each other. This will bring us directly to Theorem 1.11. The proof requires a few steps.

**Proposition 3.2** Under the conditions of Proposition 3.1

\[ \{ \mathfrak{a} \} \frac{S_0(\mathcal{L}, \mathcal{L}', M; \mathfrak{a}; \beta)}{S_0(\mathcal{L}, \mathcal{M}; \mathfrak{a})} = i^{-L} (2\pi)^{L-2} h_1^{-1/2} (M) \frac{P_{\beta; 0}(\mathcal{L}, \mathcal{L}', M; e^{2\pi i\mathfrak{a}})}{\nabla_A(\mathcal{L}, \mathcal{M}; e^{2\pi i\mathfrak{a}})}, \]

where \( P_{\beta; 0}(\mathcal{L}, \mathcal{L}', M; \mathfrak{a}) \) is defined by eq. (1.83).

**Proof.** Compare the leading contribution to the stationary phase integral of eq. (1.97)

\[ I_{\frac{r}{2}}^{(l)}(\mathcal{L}, \mathcal{L}', M; \mathfrak{a}; K) = \frac{K}{2\pi i} \exp \left( -\frac{1}{2} i\pi K \left( \text{lk}(\mathcal{L} \cup \mathcal{L}', S^3; \mathfrak{a}, \mathfrak{c}(st)(\mathfrak{a})) - \sum_{j=1}^{L} j^{(l)}(L'|S^3) a_j^2 \right) \right) \]

\[ \times \left( \frac{\sin(\pi c^{(st)}(\mathfrak{a}))}{\nabla_A(\mathcal{L} \cup \mathcal{L}', S^3; e^{2\pi i\mathfrak{a}}, e^{2\pi i e(\text{st})(\mathfrak{a})})} e^{-\frac{3}{4} i\pi \text{sign}(\mathcal{L}'|S^3) (2K)^{L^*/2}} \times \int_{-\infty}^{+\infty} dc \sum_{j=1}^{L} j^{(l)}(L'|S^3) \right) \]

with eq. (2.26). Then relation (3.10) follows from the following equations:

\[ \text{lk}(\mathcal{L} \cup \mathcal{L}', S^3; \mathfrak{a}, \mathfrak{c}(st)(\mathfrak{a})) - \sum_{j=1}^{L} j^{(l)}(L'|M) a_j^2 = \text{lk}(\mathcal{L}, M; \mathfrak{a}) \]

(3.12)

(which follows from eq. (1.11)),

\[ \frac{\sin(\pi c^{(st)}(\mathfrak{a}))}{\nabla_A(\mathcal{L} \cup \mathcal{L}', S^3; e^{2\pi i\mathfrak{a}}, e^{2\pi i e(\text{st})(\mathfrak{a})})} = \frac{\text{sign}(\mathcal{L}'|S^3) (i/2)^{L^*}}{\nabla_A(\mathcal{L}, M; e^{2\pi i\mathfrak{a}})} \]

(3.13)
\[ \int_{-\infty}^{+\infty} dc \, e^{\frac{i}{2} \pi K \text{lk}(\hat{L}^s, S^3)} e^{-\frac{\pi}{2} \text{sign}(|\hat{L}^s|) \alpha} |\det(\hat{L}^s | M)|^{-\frac{1}{2}}, \]  
(3.14)

\[ e^{-\frac{\pi}{2} \text{sign}(|\hat{L}^s|) \alpha} \alpha = \text{sign}(\hat{L}^s | S^3), \]  
(3.15)

and
\[ |\det(\hat{L}^s | M)| = h_1(M) \]  
(3.16)

(which is a particular case of eq.(1.12)).

**Proposition 3.3** If eq. (3.10) holds for a QHS \( M \# M' \) such that \( \mathcal{L}, \mathcal{L}' \subset M \), then it also holds for \( M \).

**Proof.** This proposition follows easily from eq. (3.10) for \( M \# M' \) and from eq.(2.28), from (1.59) applied to \( Z^{(tr)}(M'; K) \) and from eqs.(1.20) and (1.13).

**Proposition 3.4** Let \( \mathcal{L} \subset M \) be a link satisfying the conditions (3.1). If there exists a knot \( \mathcal{L}_0 \subset M \) such that it is algebraically connected to \( \mathcal{L} \) and eq. (3.10) holds for a link \( \mathcal{L}_0 \cup \mathcal{L} \), then this equation also holds for the original link \( \mathcal{L} \).

**Proof.** Let us compare the expressions for both sides of eq.(3.10) in case of a link \( \mathcal{L} \) and in case of a link \( \mathcal{L}_0 \cup \mathcal{L} \) at \( a_0 = 0 \). First of all, a combination of eqs.(2.37) and (2.27) indicates that
\[ I_{\beta}^{(r)}(\mathcal{L}, \mathcal{L}', M; a; K) = \sum_{\mu = \pm 1} \mu I_{\beta}^{(s)}(\mathcal{L}_0 \cup \mathcal{L}, M; \mu/K, a; K). \]  
(3.17)

Comparing the leading terms on both sides of this equation we find that
\[ \{a\} \frac{S_0(\mathcal{L}, \mathcal{L}', M; a; \beta)}{S_a(\mathcal{L}, M; a)} = a_0 \{a\} \frac{S_0(\mathcal{L}_0 \cup \mathcal{L}, \mathcal{L}', M; a_0, a; \beta)}{S_a(\mathcal{L}_0 \cup \mathcal{L}, M; a_0, a)} \bigg|_{a_0 = 0} \]  
(3.18)

\[ \times \left( e^{i\pi \sum_{j=1}^{L} t_{ij}} e^{i\pi \sum_{j=1}^{L} t_{ij}^{(\mathcal{L}_0 \cup \mathcal{L}| M)}} a_j - e^{-i\pi \sum_{j=1}^{L} t_{ij}} e^{-i\pi \sum_{j=1}^{L} t_{ij}^{(\mathcal{L}_0 \cup \mathcal{L}| M)}} a_j \right). \]

On the other hand, since according to eq.(1.83),
\[ P_{\beta;0}(\mathcal{L}_0 \cup \mathcal{L}, \mathcal{L}', M; 1, t) = P_{\beta;0}(\mathcal{L}, \mathcal{L}', M; t), \]  
(3.19)

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and in view of eq.(1.19)

\[
P_{\beta_0}(\mathcal{L}, \mathcal{L}', M; e^{2\pi i a}) \frac{\nabla_A(\mathcal{L}, M; e^{2\pi i a})}{e^{2\pi i a}}
\]

(3.20)

\[
\begin{align*}
&= \left( e^{i\pi \sum_{j=1}^L l_{ij}^{(L_0 \cup L|M)|S^3}} - e^{-i\pi \sum_{j=1}^L l_{ij}^{(L_0 \cup L|M)|S^3}} \right) \frac{P_{\beta_0}(\mathcal{L}_0 \cup \mathcal{L}, \mathcal{L}', M; 1, e^{2\pi i a})}{\nabla_A(\mathcal{L}_0 \cup \mathcal{L}, M; 1, e^{2\pi i a})},
\end{align*}
\]

Equations (3.18) and (3.20) demonstrate that if we set \(a_0 = 0\) in eq. (3.10) written for the \(\mathcal{L}_0 \cup \mathcal{L}\), then we get the same equation for \(\mathcal{L}\).

\[\square\]

**Proposition 3.5** Equation (3.10) holds for any link \(\mathcal{L}\) in a \(\mathbb{Q}HS\) \(M\) which satisfies the conditions (3.1).

**Proof.** According to a slightly generalized theorem of [6], for any link \(\mathcal{L}\) in a \(\mathbb{Q}HS\) \(M\) there exists an algebraically split link \(\hat{\mathcal{L}}^s \subset S^3\) such that Dehn’s surgery on \(\hat{\mathcal{L}}^s\) produces a \(\mathbb{Q}HS\) \(M\# M'\) with \(\mathcal{L} \subset M\). Let us pick a knot \(\mathcal{L}_0 \subset S^3\) which has the following linking numbers with \(\mathcal{L}\) and \(\mathcal{L}_0\):

\[
l_{0i}^{(\mathcal{L}_0 \cup \mathcal{L}|S^3)} = \delta_{1i} + \sum_{j=1}^L l_{ij}^{(\mathcal{L}_0 \cup \mathcal{L}, \mathcal{L}|S^3)}, \quad l_{0j}^{(\mathcal{L}_0 \cup \mathcal{L}, \mathcal{L}^s|S^3)} = l_{jj}^{(\mathcal{L}^s|S^3)}. \quad (3.21)
\]

Then, according to eq. (1.11),

\[
l_{0i}^{(\mathcal{L}_0 \cup \mathcal{L}|M)} = \delta_{1i}. \quad (3.22)
\]

Thus \(\mathcal{L}_0\) is algebraically connected to \(\mathcal{L}\) in \(M\) and therefore the link \(\mathcal{L}_0 \cup \mathcal{L}\) satisfies the conditions (3.1). It also satisfies the condition (3.2), because according to eqs. (3.21), (1.96) and (1.15),

\[
c_j^{(st)}(a_0, a) = -a_0 + c_j^{(st)}(a), \quad e_j^{(st)}(a) = -\frac{1}{l_{jj}^{(\mathcal{L}^s|S^3)}} \sum_{i=1}^L l_{ij}^{(\mathcal{L}, \mathcal{L}^s|S^3)} a_i. \quad (3.23)
\]

Therefore, according to Proposition 3.2, eq. (3.10) holds for \(\mathcal{L}_0 \cup \mathcal{L}\) and \(M\# M'\). Then, according to Proposition 3.3, eq. (3.10) holds also for \(\mathcal{L}_0 \cup \mathcal{L}\) and \(M\), and therefore, according to Proposition (3.4), it holds for \(\mathcal{L}\) and \(M\).

\[\square\]

Now, repeating the same steps that led us from Corollary 2.11 to Proposition 2.13, we come to the following
Proposition 3.6 For a link \( L \) in a \( QHS \) \( M \), \( S_d(L, M; a) \neq 0 \) iff \( \nabla_A(L, M; t) \neq 0 \).

A combination of this proposition with Propositions 2.4 and 3.1 sums up to the following

**Proposition 3.7** If a link \( L \) in a \( QHS \) \( M \) satisfies the condition \( \nabla_A(L, M; t) \neq 0 \), then for any stringing of \( L \) the integral (2.25) is well-defined and it is a topological invariant of \( L \). If \( M \) can be constructed by Dehn’s surgery on a link \( \hat{L}^s \in S^3 \) which satisfies the condition (3.2), then the surgery formula (1.97) holds.

Now it remains to show that the ratios \( S_d(L, M; a)/S_n(L, L', M; a; \beta) \) of eq. (2.26) are rational functions of \( e^{2\pi ia} \), as suggested by eqs. (1.78) and (1.95).

### 3.2 Rational structure and integrality

#### 3.2.1 Surgery on algebraically split links

In this subsection we will complete the proof of our results by proving the following

**Proposition 3.8** Let \( L \) be an oriented link in a \( QHS \) \( M \) such that \( \nabla_A(L, M; t) \neq 0 \). Then for any link \( L' \subset M \) there exist the polynomials (1.90) such that they satisfy (1.91) and the formal power series \( \hat{Z}_\beta(r) (L, L', M; t; h) \) defined by eq. (1.87) is related to the modified \( U(1)-RC \) invariant

\[
\hat{I}_\beta^{(r)} (L, L', M; a; K) = e^{-\frac{1}{2}i\pi K \text{lk}(L, M; a)} I_\beta^{(r)} (L, L', M; a; K) \tag{3.24}
\]

as in eq. (1.95):

\[
\hat{Z}_\beta^{(r)} (L, L', M; e^{2\pi ia}; h) = \hat{I}_\beta^{(r)} (L, L', M; a; K). \tag{3.25}
\]

A proof of this proposition will require us to prove a few lemmas. We begin with a modification of Lemma 2.3 of [6] which was suggested by T. Ohtsuki.
Lemma 3.9 (H. Murakami, T. Ohtsuki) For a QHS $M$ and a prime number $K$ which does not divide $h_1(M)$, there exists an algebraically split link $\mathcal{L}^S \subset S^3$, such that Dehn’s surgery on $\mathcal{L}^S$ produces a QHS $M \# M_K$, where

$$M_K = L_{p_1,1} \# \cdots \# L_{p_N,1}.$$  

(3.26)

Here $L_{p_j,1}$ are lens spaces and the numbers $p = (p_1, \ldots, p_N)$ are not divisible by $K$.

We can use the proof of [6] if we observe that for even values of $h_1(M)$ the factors $E_0^k$ and $E_1^k$ do not appear in the decomposition of the linking pairing of $M$. \(\square\)

Now we can prove the following

Lemma 3.10 If Proposition 3.8 holds for any QHS which can be constructed by Dehn’s surgery on an algebraically split link in $S^3$, then it holds for any QHS.

Proof. Let $M$ be a QHS. According to our assumption, Proposition 3.8 holds for the QHS $M \# M_K$ described in the previous lemma. It also holds for a lens space $L_{p,1}$, because this manifold can be constructed by Dehn’s surgery on an unknot in $S^3$ with self-linking number $p$. Since $h_1(L_{p,1}) = |p|$, then according to eq. (1.87) (case of $L = 0$),

$$Z^{(tr)}(L_{p,1}; K) = |p|^{-3/2} q^{3\lambda_{\text{CW}}(L_{p,1})} \sum_{n=0}^{\infty} P'_n(L_{p,1}) h^n,$$

(3.27)

and

$$P'_n(L_{p,1}) \in \mathbb{Z}[1/p], \quad P'_0(L_{p,1}) = 1$$

and

$$\frac{1}{Z^{(tr)}(L_{p,1}; K)} = |p|^{-3/2} q^{-3\lambda_{\text{CW}}(L_{p,1})} \sum_{n=1}^{\infty} \tilde{P}_n(L_{p,1}) h^n, \quad \tilde{P}_n(L_{p,1}) \in \mathbb{Z}[1/p].$$

(3.28)

According to eq. (2.28),

$$\tilde{j}^{(tr)}_{\beta}(\mathcal{L}, \mathcal{L}', M; \mathfrak{q}; K) = \frac{\hat{Z}^{(tr)}_{\beta}(\mathcal{L}, \mathcal{L}', M \# M_K; e^{2\pi i \mathfrak{q}}; h)}{\prod_{j=1}^{N} Z^{(tr)}(L_{p_j,1}; K)}.$$ 

(3.29)

Since

$$h_1(M \# M_K) = h_1(M) \prod_{j=1}^{N} |p_j|, \quad \lambda_{\text{CW}}(M \# M_K) = \lambda_{\text{CW}}(M) + \sum_{j=1}^{N} \lambda_{\text{CW}}(L_{p_j,1}),$$

(3.30)
then it follows from eqs. (3.29) and (3.28) that $\hat{I}^r_L(\mathcal{L}, \mathcal{L}', M; q; K)$ can be presented in the 
(1.87) with $t = e^{2\pi i q}$ if we extend the ring (1.90) by $1/\{p\}$. Since $\{p\}$ is not divisible by $K$, 
then the intersection of all the extended rings for all prime $K$ which do not divide $h_1(M)$ is 
the original ring. This proves the lemma.

Another proof of Lemma 3.10 According to Corollary A2.2 of T. Ohtsuki’s Lemma A2.1 
of Appendix, for any QHS $M$ there exists an algebraically split link $\hat{L}^s \subset S^3$ such that a 
surgery on $\hat{L}^s$ produces a QHS $M \# M'$, where

$$M' = L_{p_1, q_1} \# \cdots \# L_{p_N, q_N},$$

(3.31)

and $1/\{p\} \in \mathbb{Z}[1/h_1(M)]$. A lens space $L_{p, q}$ can be constructed by a rational surgery on 
an unknot in $S^3$, therefore it satisfies Proposition 3.8 in view of [14] (note that for an empty 
link space eq. (2.1) sets $\hat{Z}^{(r)}(M; h) = Z^{(tr)}(M; K)$). We can also see this explicitly, because 
there is an analytic expression for $Z^{(tr)}(L_{p, q}; K)$ which can be found in [3] (see also [14] and 
references therein)

$$Z^{(tr)}(L_{p, q}; K) = \text{sign}(p) |p|^{-1/2} q^{3s(p, q)} q^{1/(2p)} - q^{-1/(2p)}$$

(3.32)

and the claim of the lemma follows from this expansion if we perform a division of formal 
power series in its r.h.s. in the same way as we did in eq. (3.29).

3.2.2 Integrality properties of the Alexander polynomial and its expansion

Before we proceed with the proof of Proposition 3.8, we have to strengthen the integrality 
properties (1.23) and (1.24). For a link $\mathcal{L}$ in a QHS $M$ we define

$$\nu_i(\mathcal{L}, M) = \sum_{1 \leq j \leq L \atop j \neq i} I^r_{ij}(\mathcal{L}|M) + 1.$$  

(3.34)
Suppose that $M$ can be constructed by a surgery on an algebraically split link $\hat{L}^s \subset S^3$. Then we define

$$\nu'(L, \hat{L}^s) = \sum_{j=1}^{L^s} \frac{l_{ij}(L, L^s|S^3)(p_j - l_{ij}(L^s|S^3))}{p_j},$$

(3.35)

where $p_j$ denote the self-linking numbers of the link components of $\hat{L}^s$

$$p_j = l_{ij}(\hat{L}^s|S^3).$$

(3.36)

We also assume that if $M = S^3$ (that is, if $\hat{L}^s$ is an empty link), then $\nu'(L, \varnothing) = 0$. Now we define

$$\Phi(L, \hat{L}^s; t) = t^{-\frac{1}{2}}(\nu(L, M) + \nu'(L, \hat{L}^s)).$$

(3.37)

The factor $\Phi(L, \hat{L}^s; t)$ absorbs the factors $1/2$ in the powers of $t$ and $\hat{t}$ in (1.23) and (1.24)

$$t^{1/2} \Phi(K, \hat{L}^s; t) \Delta_A(K, M; t) \in \mathbb{Z}[t^{1/o(K)}; t^{-1/o(K)}],$$

(3.38)

$$\Phi(L, \hat{L}^s; t) \nabla_A(L, M; t) \in \mathbb{Z}[t^{1/o}, t^{-1/o}] \quad \text{if } L \geq 2$$

(3.39)

If $M = S^3$, then these relations can be found in [17]. Otherwise they follow easily from eq.(1.16).

**Lemma 3.11** If $L$ is a link in a QHS $M$ is constructed by a surgery on an algebraically split link $\hat{L}^s \subset S^3$, then

$$\sum_{j=1}^{L^s} \frac{1}{2} \frac{p_j + 1}{p_j}, \frac{1}{2} \nu'(L, \hat{L}^s) \in \mathbb{Z}[1/h_1(M)].$$

(3.40)

**Proof.** Since $h_1(M) = \{p\}$, then $\mathbb{Z}[1/p_j] \subset \mathbb{Z}[1/h_1(M)]$ and it is enough to prove that

$$\frac{1}{2} \frac{p_j + 1}{p_j}, \frac{1}{2} \frac{l_{ij}(L, L^s|S^3)(p_j - l_{ij}(L^s|S^3))}{p_j} \in \mathbb{Z}[1/p_j].$$

(3.41)

Consider two cases. If $p_j$ is even, then $\mathbb{Z}[1/p_j] = \mathbb{Z}[1/2, 1/p_j]$ and (3.41) is obvious. If $p_j$ is odd, then $p_j + 1$ and $l_{ij}(L, L^s|S^3)(p_j - l_{ij}(L^s|S^3))$ are even and (3.41) follows from there. □
Lemma 3.12 Let $\mathcal{L}, \mathcal{L}'$ be a pair of links in a $\mathbb{Q}HS$ $M$ constructed by a surgery on an algebraically split link $\hat{L}^s \subset S^3$. Then we can expand the negative odd powers of the Alexander-Conway function around $u = u^*$ in the following way

\[
\frac{1}{\nabla_M^{2n+1}(\mathcal{L} \cup \mathcal{L}'; M; t, u)} = \prod_{j=1}^{L'} \left( \frac{u_j}{u_j^*} \right)^{\frac{1}{2}(\nu_j(\mathcal{L} \cup \mathcal{L}', M) + \nu'(\mathcal{L}', \hat{L}^s))} \sum_{m \geq 0} \left( \frac{w_{l,m,n}(t, u^*)}{\nabla_M^{2n+2l+1}(\mathcal{L} \cup \mathcal{L}'; M; t, u^*)} \right) (u/u^* - 1)^m,
\]

where $w_{l,m,n}(t, u^*) \in \mathbb{Z}[t^{\pm 1/\omega}, (u^*)^{\pm 1/\omega'}, 1/h_1(M)].$

Also if $K$ is a knot in $M,$ then

\[
\frac{1}{\Delta_M^{2n+1}(K, M; t)} = t^2 [\nu'(K, \hat{L}^s)] \sum_{m=0}^{\infty} w_m (t - 1)^m, \quad w_m \in \mathbb{Z}[1/h_1(M)].
\]

Proof. Let us first prove eq.(3.42). Consider the expansion

\[
\Phi(\mathcal{L} \cup \mathcal{L}', \hat{L}^s; t, u) \nabla_M(\mathcal{L} \cup \mathcal{L}', M; t, u)
\]

\[
= \Phi(\mathcal{L} \cup \mathcal{L}', \hat{L}^s; t, u^*) \nabla_M(\mathcal{L} \cup \mathcal{L}', M; t, u^*) + w(t, u^*, u/u^* - 1),
\]

where $w(t, u^*, s) \in \mathbb{Z}[t^{\pm 1/\omega}, (u^*)^{\pm 1/\omega'}, 1/h_1(M)][[s]], \quad w(t, s) = O(s).$

Then

\[
\frac{1}{\nabla_M^{2n+1}(\mathcal{L} \cup \mathcal{L}'; M; t, u)} = \frac{\Phi^{2n+1}(\mathcal{L} \cup \mathcal{L}', \hat{L}^s; t, u)}{\Phi(\mathcal{L} \cup \mathcal{L}', \hat{L}^s; t, u) \nabla_M(\mathcal{L} \cup \mathcal{L}', M; t, u)}^{2n+1}
\]

\[
= \frac{\Phi(\mathcal{L} \cup \mathcal{L}', \hat{L}^s; t, u)}{\Phi(\mathcal{L} \cup \mathcal{L}', \hat{L}^s; t, u^*)} \sum_{m \geq 0} \frac{w_{l,m,n}(t, u^*)}{\nabla_M^{2n+2l+1}(\mathcal{L} \cup \mathcal{L}'; M; t, u^*)} (u/u^* - 1)^m,
\]

where $w_{l,m,n}(t, u^*) \in \mathbb{Z}[t^{\pm 1/\omega}, (u^*)^{\pm 1/\omega'}, 1/h_1(M)].$

We kept the power in denominators odd by multiplying both denominator and numerator by an extra factor of $\Phi \nabla_M$ if needed. Also, we absorbed even powers of $\Phi$ in the polynomials $w$. Equation (3.42) follows easily from eq.(3.45).

The proof of eq. (3.43) is similar to that of eq. (3.42), except we have to use eq. (1.22) which determines the value of denominators. We leave the details to the reader. \qed

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3.2.3 Extra link component

Lemma 3.13 Let $M$ be a QHS which is constructed by a surgery on a link $\hat{L}^s \subset S^3$. Suppose that a link $L \subset M$, $L \geq 1$ satisfies the condition $\nabla_A(L, M; t) \not\equiv 0$. If we add a new component $L_0$ to $L$ in such a way that its linking numbers with the components of $L^s$ and $L$ in $S^3$ are given by eq. (3.21), then the claims of Proposition 3.8 hold for $L_0 \cup L$.

Proof. As we showed in the proof of Proposition 3.7, the $U(1)$-RC invariant $I_\beta^{(r)}(L_0 \cup L, L', M; a_0, a; K)$ can be expressed by the surgery formula (1.97) where the link $L$ is substituted by $L_0 \cup L$. On the other hand, Proposition 2.14 establishes that Proposition 3.8 holds for links in $S^3$, which allows us to present $\hat{I}_\beta^{(r)}(L_0 \cup L \cup L^s, L', M; a_0, a, c; K)$ in the form (1.87). As a result

$$\hat{I}_\beta^{(r)}(L_0 \cup L, L', M; a_0, a; K) = \sum_{n=0}^{\infty} F_n(a_0, a; q) h^{n-1}$$

(3.46)

where

$$F_n(a_0, a; q) = q^{\phi(L_0 \cup L \cup L', L; K_{a_0}, K_{a; s})} \Phi_{\beta}(L_0 \cup L \cup L^s, L', S^3; e^{2\pi i a_0}, e^{2\pi i a}, e^{2\pi i c})$$

(3.47)

$$\times (K/2)^{L^s/2} \int_{[z=\pm i]} (d\epsilon) \{ e^{\pi i \epsilon} - e^{-\pi i \epsilon} \} e^{\frac{i}{2} \pi K \mathbf{lk}(L_0 \cup L, M; a_0, a)} i^{-L^s}$$

Now we have to change the integration variable into

$$\bar{\epsilon} = \epsilon - \epsilon^{(st)}(\bar{a})$$

(3.48)

expand the preexponential factor in (3.47) in powers of $e^{2\pi i \bar{\epsilon}} - 1$ and integrate each term in this expansion individually. We use eq. (3.42) in which we substitute $L$ by $L_0 \cup L$, $L'$ by $L^s$, $L$ by $e^{2\pi i a_0}, e^{2\pi i a}$ and $u^*$ by $e^{2\pi i u^{(st)}(a_0, \bar{a})}$. Then the expansion of the preexponential factor in eq. (3.47) takes the form

$$\{ e^{\pi \epsilon} - e^{-\pi \epsilon} \} \Phi_{\beta}(L_0 \cup L \cup L^s, L', S^3; e^{2\pi i a_0}, e^{2\pi i a}, e^{2\pi i c})$$
$P_{2n}(L_0 \cup L \cup L^s, L', M; e^{2\pi i a_0}, e^{2\pi i \alpha}, e^{2\pi i \beta})$

\[
\frac{1}{\nabla A^{2n+1}(L_0 \cup L \cup L^s, M; e^{2\pi i a_0}, e^{2\pi i \alpha}, e^{2\pi i \beta})}
\]

\[
= \Phi_2(L_0 \cup L, L', M; e^{2\pi i a_0}, e^{2\pi i \alpha})e^{i\pi \sum_{j=1}^{L^s} \kappa_j x_j}
\]

\[
\sum_{0 \leq l \leq 2m+1} Q_{l,m,n}(e^{2\pi ia_0}, e^{2\pi i \alpha}, (e^{2\pi i \beta \kappa_j x_j} - 1)^m)
\]

\[
2n+2l+1,
\]

where

\[
\kappa_j = -\sum_{i=0}^{L} l_{ij}^{(L_0 \cup L \cup L^s)} + \sum_{i=1}^{L'} l_{ij}^{(L' \cup L^s)}(\beta_i - 1),
\]

\[
Q_{l,m,n}(t_0, L) \in \mathbb{Z}[t_0, t_0^{-1}, \frac{L}{L}, L^{-1}].
\]

and we used the fact that

\[
\Phi_2(L_0, L', S^3; t_0, L, u^{(st)}(t_0, L)) = \Phi_2(L_0 \cup L, L', M; t_0, L).
\]

The substitution (3.48) turns the exponent of eq. (3.47) into

\[
\text{lk}(L_0 \cup L \cup \hat{L}^s, S^3; a_0, a, c) = \text{lk}(\hat{L} \cup \hat{L}_0, M; a_0, a) + \sum_{j=1}^{L^s} p_j x_j^2
\]

where

\[
l_{ij}^{(\hat{L} \cup \hat{L}_0)} = -\sum_{j=1}^{L^s} \frac{l_{ij}^{(\hat{L}_0 \cup L \cup L^s)}2}{p_j}, \quad 0 \leq i \leq L
\]

and the numbers \( p \) denote the diagonal linking numbers of the algebraically split surgery link \( \hat{L}^s \) according to eq. (3.36). Now the gaussian integrals are easy to calculate: for any \( L^s \) integers \( m \)

\[
(K/2)^{L^s/2} \int_{-\infty}^{+\infty} \{dx\} \exp \frac{1}{2} \frac{i\pi}{2} \sum_{j=1}^{L^s} (Kp_j x_j^2 + 2(\kappa_j + 2m_j)x_j)
\]

\[
= e^{i\pi \text{sign}(\hat{L}^s \cup S^3)} | \{p_j\}|^{-1/2} q^{-\sum_{j=1}^{L^s} \frac{1}{p_j} (m_j^2 + m_j \kappa_j + \frac{1}{4}(\kappa_j)^2)}
\]

and therefore

\[
(K/2)^{L^s/2} \int_{-\infty}^{+\infty} \{dx\} e^{i\pi K \sum_{j=1}^{L^s} p_j x_j^2} e^{i\pi \sum_{j=1}^{L^s} \kappa_j x_j} \left\{ e^{2\pi i \beta} - 1 \right\}^m
\]

\[
= e^{i\pi \text{sign}(\hat{L}^s \cup S^3)} h_{1/2}(M) q^{-\sum_{j=1}^{L^s} \frac{(\kappa_j)^2}{p_j}} \sum_{n \geq \frac{1}{2|m|}} C_n h^n,
\]

\( C_n \in \mathbb{Z}[1/h(M)] \)
Assembling all phase factors from eqs. (3.47), (3.53) and (3.56), we can present their product as

\[
q^{\phi(t)}(L_0 \cup \mathcal{L}', \mathcal{L}'; e \phi_k(L_0 \cup \mathcal{L}, \mathcal{L}', S^3; \beta) \ e^{-\frac{1}{2}i\pi K \text{lk}(L_0 \cup \mathcal{L}, M; a_0, \underline{a})} (i - L_s \ e^{\frac{i}{2}i\pi K \text{lk}(L_0, \mathcal{L}_0, M; a_0, \underline{a}))} \]

\[
\times e^{\frac{i}{2}i\pi \text{sign}(\mathcal{L}^s|S^3) q \ \eta^3 \text{sign}(\mathcal{L}^s|S^3) \ \eta^3 \sum_{j=1}^{L_s} \left(p_j - \frac{2}{p_j}\right) - 3 \lambda_{\text{CW}}(M) \ \eta^3 \sum_{j=1}^{L_s} \frac{p_j + 1}{p_j} \times q^{\lambda_{\text{CW}}(M)}
\]

\[
= (-1)^{L_s} \text{sign}(\mathcal{L}^s|S^3) q^{\phi_k(L_0 \cup \mathcal{L}, \mathcal{L}', M; \beta)} = q^{\frac{3}{4} \text{sign}(\mathcal{L}^s|S^3) \ - \frac{1}{4} \sum_{j=1}^{L_s} \left(p_j - \frac{2}{p_j}\right) - 3 \lambda_{\text{CW}}(M)} \in \mathbb{Z}[1/h_1(M)].
\] (3.58)

Combining this with Lemma 3.11 we conclude that

\[
(1 + h)^{\frac{3}{4} \text{sign}(\mathcal{L}^s|S^3) \ - \frac{1}{4} \sum_{j=1}^{L_s} \left(p_j - \frac{2}{p_j}\right) - 3 \lambda_{\text{CW}}(M)} = (1 + h)^{\frac{1}{2} \sum_{j=1}^{L_s} \frac{p_j + 1}{p_j}} \times (1 + h)^{\frac{1}{2} \left(\sum_{j=1}^{L_s} \nu_j(L_0, \mathcal{L}^s) - \sum_{j=1}^{L_s} \nu_j(L_0, \mathcal{L}^s) (\beta_j - 1)\right)} \in \mathbb{Z}[1/h_1(M), \beta][h].
\] (3.59)

Expanding the expression (3.59) in powers of \(h\) as it appears in the product of phase factors (3.57), and combining this expansion with the integrals (3.47) calculated with the help of eqs. (3.49) and (3.56), we come to the following formula

\[
\tilde{f}_\beta(t) (L_0 \cup \mathcal{L}, \mathcal{L}', M; a_0, \underline{a}; K) = h^{-1} h_1^{-1/2}(M) q^{\phi_k(L_0 \cup \mathcal{L}, \mathcal{L}', M; \beta)} \Phi_\beta(L_0 \cup \mathcal{L}, \mathcal{L}', M; e^{2\pi i a_0}, e^{2\pi i \underline{a}})
\]

\[
\times \sum_{n=0}^{\infty} \ \left(\nabla \Phi_\beta(L_0 \cup \mathcal{L}, M; e^{2\pi i a_0}, e^{2\pi i \underline{a}}) \left\{e^{2\pi i \frac{1}{2}(a_0, \underline{a})} - 1\right\}\right)^{2n+1} h^n \quad \text{if } \beta \in \mathbb{Z}_+.
\] (3.60)

where

\[
Q_{\beta,n}(t_0, t) \in \mathbb{Z}[t_0^{+1}, t_0^{-1/2}, 1/h_1(M)] \quad \text{if } \beta \in \mathbb{Z}_+.
\] (3.61)

Thus the lemma is proved if we show that the factors \(e^{2\pi i \frac{1}{2}(a_0, \underline{a})} - 1\) can be canceled in the denominators of the r.h.s. of eq. (3.60).

Let us rewrite eq. (2.26) for the link \(L_0 \cup \mathcal{L}\) as a power series in \(h\), while substituting \(S_4(L_0 \cup \mathcal{L}, M; a_0, \underline{a})\) with its expression in terms of \(\nabla \Phi_\beta(L_0 \cup \mathcal{L}, M; e^{2\pi i a_0}, e^{2\pi i \underline{a}})\) which can be
derived from eq. (3.10)

\[
\tilde{I}^{(t)}_\beta(L_0 \cup \mathcal{L}, \mathcal{L}'; M; a_0, \underline{a}; K) = h^{-1} \sum_{n=0}^{\infty} \frac{S'_n(L_0 \cup \mathcal{L}, \mathcal{L}'; M; a_0, \underline{a}; \beta)}{(\{a\} S_0(L_0 \cup \mathcal{L}, \mathcal{L}'; M; a_0, \underline{a}; \beta) \nabla_A(L_0 \cup \mathcal{L}, M; e^{2\pi i a_0}, e^{2\pi i x})}^{2n+1} h^n,
\]

where

\[
S'_n(L_0 \cup \mathcal{L}, \mathcal{L}'; M; a_0, \underline{a}; \beta) \in \mathbb{Q}[\beta][[a_0, \underline{a}]].
\]

Note that we dropped the factor of \(a_0^{2n+1}\) in the denominators, because, according to claim 1 of Proposition 2.7, each term in the sum (3.63) is non-singular at \(a_0 = 0\), and therefore the powers of \(a_0\) can be canceled against the numerators.

Compare the series (3.60) and (3.62). Obviously, all \(L^s\) factors \(e^{2\pi i c^{(st)}(a_0, \underline{a})} - 1\) are divisible by the corresponding functions \(c^{(st)}(a_0, \underline{a})\), but according to eq. (2.24), \(S_0(L_0 \cup \mathcal{L}, \mathcal{L}', M; a_0, \underline{a}; \beta)\) is not divisible by either of \(c^{(st)}(a_0, \underline{a})\). Therefore \(Q_{\beta,n}(e^{2\pi i a_0}, e^{2\pi i x})\) is divisible by \(\{c^{(st)}(a_0, \underline{a})\}^{2n+1}\). Since the expressions

\[
\underline{u}^{(st)}(t_0, \underline{t}) - 1 = t_0^{-1} \underline{u}^{(st)}(\underline{t}) - 1
\]

do not factor over \(\mathbb{Z}[t_0^{1/2}, \underline{t}^{1/2}, 1/h_1(M)]\), then the divisibility of \(Q_{\beta,n}(e^{2\pi i a_0}, e^{2\pi i x})\) means that \(Q_{\beta,n}(t_0, \underline{t})\) is divisible by \(\{\underline{u}^{(st)}(t_0, \underline{t}) - 1\}^{2n+1}\):

\[
Q_{\beta,n}(t_0, \underline{t}) = \left\{\underline{u}^{(st)}(t_0, \underline{t}) - 1\right\}^{2n+1} P'_{\beta,n}(L_0 \cup \mathcal{L}, \mathcal{L}', M; t_0, \underline{t}),
\]

Substituting this expression into eq. (3.60) we get

\[
\tilde{I}^{(t)}_\beta(L_0 \cup \mathcal{L}, \mathcal{L}', M; a_0, \underline{a}; K) = h^{-1} h_1^{-1/2}(M) q^{\phi_{\underline{a}}(L_0 \cup \mathcal{L}, \mathcal{L}', M; \beta)} \Phi_\beta(L_0 \cup \mathcal{L}, \mathcal{L}', M; e^{2\pi i a_0}, e^{2\pi i x})
\times \sum_{n=0}^{\infty} \frac{P'_{\beta,n}(L_0 \cup \mathcal{L}, \mathcal{L}', M; e^{2\pi i a_0}, e^{2\pi i x})}{\nabla_A^{2n+1}(L_0 \cup \mathcal{L}, M; e^{2\pi i a_0}, e^{2\pi i x})},
\]

which is equivalent to eq. (1.87) for \(L_0 \cup \mathcal{L}\).
It remains to check eq. (1.91) for \( L_0 \cup L, L' \). We substitute the expansion (1.91) for the polynomials \( P^I_{2n}(L_0 \cup L \cup L', S^3; e^{2\pi i a}, e^{2\pi i b}, e^{2\pi i c}) \) in eq. (3.47) and integrate term by term. Then an easy power counting indicates that

\[
Q^I_{2n}(t_0, t) = \sum_{m_0, m \geq 0} q_{m_0, m}(\beta) (t_0 - 1)^{m_0} (t - 1)^m, \quad q_{m_0, m}(\beta) \in \mathbb{Q}[\beta].
\] (3.67)

According to eq. (3.65), this series is divisible by \( \{ u^{(st)}(t_0, t) - 1 \}^{2n+1} \). Thus performing this division at the level of formal power series in \( t_0 - 1, t - 1 \) we come to (1.91) for \( L_0 \cup L, L' \).

\[ \blacksquare \]

**Lemma 3.14** Proposition 3.8 holds for a link \( L \) in a QHS \( M \) if \( L \geq 2 \).

**Proof.** According to Lemma 3.10, we can assume that \( M \) is constructed by a surgery on an algebraically split link \( \hat{L} \subset S^3 \). Then, according to Lemma 3.13, Proposition 3.8 holds for the link \( L_0 \cup L \), where a knot \( L \) is described by eq. (3.21). Thus we start with eq. (3.66) and substitute it into eq. (3.17) which in view of eq. (3.22) can be rewritten for the invariants \( \hat{I} \) as

\[
\hat{I}^{(s)}_{\beta}(L, L', M; q^\mu, e^{2\pi i a}) = \sum_{\mu = \pm 1} C \sqrt{\mu} e^{i\pi \mu a} \hat{I}^{(s)}_{\beta}(L_0 \cup L, M; \mu/K, a/K). \] (3.68)

In view of eqs. (3.42) and (1.19),

\[
1 \over \nabla^{2n+1}(L_0 \cup L, M; q^\mu, e^{2\pi i a}) = \sum_{m \geq 0} \frac{w_{l,m,n}(e^{2\pi i a}) (q^\mu - 1)^m}{(n_{l,0})(e^{i\pi l a_1} - e^{-i\pi l a_1})^{2n+2l+1}},
\]

where we used a notation \( l_0 = l_{01}(L \cup L' | M) \). Although we know from eq. (3.22) that \( l_0 = 1 \), still we keep it explicitly in eq. (3.69), because we will use the same expression in the next lemma when that linking number has a different value.

It is easy to see that

\[
q^{\phi_{\kappa}(L_0 \cup L, L', M; \beta)} \Phi_{\beta}(L_0 \cup L, L', M; q^\mu, e^{2\pi i a}) = \frac{1}{q^{1+l_0 + (\mu - 1) \sum_{j=1}^L l_{j,0}}(L_0 \cup L, L' | M)(\beta_j - 1)} e^{-i\pi l_0 a_1} q^{\phi_{\kappa}(L, L', M; \beta)} \Phi_{\beta}(L, L', M; e^{2\pi i a}).
\] (3.70)
Since in view of the second relation of (3.40)

\[ q_{\mu_0}(L, L^*) + \frac{1}{2}(l_0 + 1) + \frac{1}{2} \sum_{j=1}^{L'} t_{0j} (C_{0j} L, L^{(M)}; \beta_j) \in \mathbb{Z}[1/h_1(M)][[h]], \]  

(3.71)

then we conclude from eqs. (3.66) and (3.68) that there exist polynomials

\[ Q_{\beta,n}(t) \in \mathbb{Z}[t^{\pm 1/2}, 1/h_1(M)] \]

such that

\[ \hat{I}^{(r)}(L, L', M; A; K) = h^{-1} h_1^{1/2}(M) q_{\phi_M(L, L', M; \beta)} \Phi_{\beta}(L, L', M; e^{2\pi i a}) \]

\[ \times \sum_{n=0}^{\infty} \frac{Q_{\beta,n}(e^{2\pi i a})}{\nabla_A(M, M; e^{2\pi i a})(e^{2\pi i a_2} - 1)^{2n+1} h^n} \]  

(3.72)

(here we used the fact that \( l_0 = 1 \)). Thus we have to prove that \( Q_{\beta,n}(t) \) is divisible by \((t_1 - 1)^{2n+1}\). To see this consider the same calculation that led us to eq. (3.72) but with a different knot \( L_0 \). Since \( L \geq 2 \), we can choose \( L_0 \) which satisfies conditions (3.21) with \( \delta_{2i} \) instead of \( \delta_{i1} \) in the expression for \( t_{0i}(L_0, L)^{S_3} \). As a result, we will get a similar formula for \( \hat{I}^{(r)}(L, L', M; A; K) \)

\[ \hat{I}^{(r)}(L, L', M; A; K) = h^{-1} h_1^{1/2}(M) q_{\phi_M(L, L', M; \beta)} \Phi_{\beta}(L, L', M; e^{2\pi i a}) \]

\[ \times \sum_{n=0}^{\infty} \frac{Q'_{\beta,n}(e^{2\pi i a})}{\nabla_A(M, M; e^{2\pi i a})(e^{2\pi i a_2} - 1)^{2n+1} h^n}, \]  

(3.73)

with some other polynomials \( Q'_{\beta,n}(t) \in \mathbb{Z}[t^{\pm 1/2}, 1/h_1(M)] \). Comparing eqs. (3.72) and (3.73) we find that

\[ (t_2 - 1)^{2n+1} Q_{\beta,n}(t) = (t_1 - 1)^{2n+1} Q'_{\beta,n}(t). \]  

(3.74)

Since the factors \((t_1 - 1)\) and \((t_2 - 1)\) are coprime, this means that \( Q_{\beta,n}(t) \) is indeed divisible by \((t_1 - 1)^{2n+1}\) so that

\[ Q_{\beta,n}(t) = (t_1 - 1)^{2n+1} P'_{\beta,n}(L, L', M; t), \quad P'_{\beta,n}(L, L', M; t) \in \mathbb{Z}[t^{\pm 1/2}, 1/h_1(M)], \]

(3.75)

and eq. (3.72) leads to eq. (1.87).
Relation (1.91) for \( L, L' \) follows easily from the same relation for \( L_0 \cup L, L' \). Indeed, if we substitute eq.(1.91) for \( L, L' \) into the r.h.s. of eq.(3.68), then it follows easily that

\[
Q_{\beta;n}(t) = \sum_{m \geq 0} q_{m;n}(\beta) (t - 1)^m, \quad q_{m;n}(\beta) \in \mathbb{Q}[\beta].
\]  

(3.76)

Now eq.(1.91) for \( L, L' \) emerges if we divide the formal power series (3.76) by \((t_1 - 1)^{2n+1}\).

\[ \Box \]

**Lemma 3.15** Proposition 3.8 holds for a link \( L \) in a \( QHS M \) if \( L = 1 \).

**Proof.** The proof is similar to that of the previous lemma. Let \( L \) be a one-component link in a \( QHS M \). We add to it an extra component \( L_0 \) which is the meridian of \( L \), so that

\[
l_{01}^{(L_0 \cup L|M)} = l_{01} = 1/o_1.
\]  

(3.77)

In view of Lemma 3.14, Proposition 3.8 holds for the link \( L_0 \cup L \). Therefore we repeat the same calculations that led us from eqs.(3.66) and (3.68) to eq.(3.72). This time however we can use eq.(1.21) for the Alexander-Conway polynomial of the knot \( L \). As a result, instead of eq.(3.72) we end up with the formula

\[
\hat{I}^{(r)}(L, L', M; a_1; K) = h^{-1} h_1^{-1/2}(M) q^{\phi_{\mu(L, L', M; \beta)}(L, L', M, e^{2\pi i a_1})} e^{-i\pi a_1/o_1}
\]

\[
\times \sum_{n=0}^{\infty} \frac{Q_{\beta;n}(e^{2\pi i a_1})}{\Delta_{\Lambda}^{2n+1}(L, M; e^{2\pi i a_1})} \Delta h^n,
\]  

(3.78)

\[ Q_{\beta;n}(t_1) \in \mathbb{Z}[t_1^{1/2}, 1/h_1(M)]. \]

Since \( L = 1 \), then according to eq.(2.26)

\[
\hat{I}^{(r)}(L, L', M; a_1; K) = \hat{I}^{(c)}(L, L', M; a_1; K) = I^{(c)}(L, L', M; a_1; K).
\]  

(3.79)

The latter invariant is given by a formal power series (2.10). Since this series is proportional to \( a_1 \), then

\[
\hat{I}^{(c)}(L, L', M; 0; K) = 0.
\]  

(3.80)
In view of eq. (1.22), this implies that

$$Q_{\beta n}(t_1) = (t_1^{1/\omega_1} - 1) P_{\beta n}'(\mathcal{L}, \mathcal{L}', M; t_1)$$  \hspace{1cm} (3.81)

with $P_{\beta n}'(\mathcal{L}, \mathcal{L}', M; t_1) \in \mathbb{Z}[t^{\pm 1/\omega_1}, 1/h_1(M)]$. Substituting this relation into eq. (3.78) we come to eq. (1.87).

Relation (1.91) for $\mathcal{L}$ follows easily from the same relation for $\mathcal{L}_0 \cup \mathcal{L}$ in the same way as in the previous lemma.

Lemma 3.16  Proposition 3.8 holds for an empty link $\mathcal{L}$, that is, if $L = 0$.

Proof. We place a knot $\mathcal{L}_0$ into $M$ in such a way that $\mathcal{L}_0$ represents a trivial element in $H_1(M; \mathbb{Z})$. According to the previous lemma, Proposition 3.8 holds for $\mathcal{L}_0$. On the other hand, according to eqs. (2.9), (1.67), (1.71) and (2.1)

$$\hat{I}_{\beta}^{(r)}(\emptyset, \mathcal{L}', M; K) = \hat{I}_{\beta}^{(r)}(\emptyset, \mathcal{L}', M; K) = \hat{I}_{\beta}^{(r)}(\mathcal{L}_0, \mathcal{L}', M; 1/K; K).$$  \hspace{1cm} (3.82)

Thus

$$\hat{I}_{\beta}^{(r)}(\emptyset, \mathcal{L}', M; K) = h^{-1} h_1^{-1/2}(M) q^{\phi_{\mathcal{L}_0}(\mathcal{L}_0) M^{\beta}} \Phi_{\beta}(\mathcal{L}_0, \mathcal{L}', M; q) (q^{1/2} - q^{-1/2}) \times \sum_{n=0}^{\infty} \frac{P_{\beta n}'(\mathcal{L}_0, \mathcal{L}', M; q)}{\Delta_{\Lambda}^{2n+1}(\mathcal{L}_0, M; q)} h^n. \hspace{1cm} (3.83)$$

Now we have to expand the instances of $q = 1 + h$ in the r.h.s. of this equation in powers of $h$. We do this calculation similarly to that of Lemma 3.14, except that we use eq. (3.43) instead of eq. (3.42). According to eq. (3.43),

$$\frac{1}{\Delta_{\Lambda}^{2n+1}(\mathcal{L}_0, M; q)} = q^{2v'(\mathcal{L}_0, \hat{\mathcal{L}}^s)} \sum_{m=0}^{\infty} w_m h^m, \quad w_m \in \mathbb{Z}[1/h_1(M)]. \hspace{1cm} (3.84)$$

Also it is easy to see that

$$h^{-1} (q^{1/2} - q^{-1/2}) q^{\phi_{\mathcal{L}_0}(\mathcal{L}_0) M^{\beta}} \Phi_{\beta}(\mathcal{L}_0, \mathcal{L}', M; q) = q^{\phi_{\emptyset}(\emptyset, \mathcal{L}', M; \beta)}. \hspace{1cm} (3.85)$$

Substituting eqs. (3.84) and (3.85) into eq. (3.83) we obtain an expression which is in the r.h.s. of eq. (1.87) for $L = 0$.

Three lemmas 3.14, 3.15 and 3.16 prove Proposition 3.8
3.3 Final proofs

Now we can prove the theorems of subsection 1.3.

Proof of Theorem 1.8. First of all, let us prove the existence of polynomials $P_{\beta n}(\mathcal{L}, \mathcal{L}', M; t)$. Consider the formal power series $\hat{Z}_{\beta}^{(r)}(\mathcal{L}, \mathcal{L}', M; t; h)$ as defined by eq. (3.25). Proposition 3.8 says that it can be presented in the form (1.87), (1.90). If we combine $\Phi_{\beta}(\mathcal{L}, \mathcal{L}', M; t)$ with the polynomials $P_{\beta n}(\mathcal{L}, \mathcal{L}', M; t)$, expand the factor $q^{(\mathcal{L}, \mathcal{L}', M; t)}$ in powers of $h = q - 1$ and write the resulting expression as a single power series in $h$, then, obviously, we come to eq. (1.78) with the polynomials $P_{\beta n}(\mathcal{L}, \mathcal{L}', M; t)$ satisfying (1.76) and (1.77). The property (1.82) follows from the fact that $\Phi_{\beta}(\mathcal{L}, \mathcal{L}', M; t) \in \mathbb{Z}[t, t^{-1}]$ if $\beta$ are odd.

In view of the definitions (3.25) and (3.24), eq. (1.79) follows from eq. (2.2), eq. (1.80) follows from eq. (2.27) while eq. (1.81) follows from eq. (2.37).

The uniqueness of the polynomials $P_{\beta n}(\mathcal{L}, \mathcal{L}', M; t)$ and their properties (1.84)–(1.86) follow from Theorem 3.1 of [15] in exactly the same way as in the case of $M = S^3$, which was considered in that paper.

The formula (1.83) follows from Propositions 3.5 and 3.6. \hfill \Box

Since we adopted eq. (3.25) as the definition of $\hat{Z}_{\beta}^{(r)}(\mathcal{L}, \mathcal{L}', M; t; h)$, then Theorem 1.12 is obvious and the claim of Theorem 1.11 is contained in Proposition 3.7. Finally, note that Theorem 1.11 can be extended to the case when a surgery is performed on a link in a general QHS, rather than in $S^3$.

Theorem 3.17 Let $\mathcal{L}$ be a non-empty link in a QHS $M'$ such that

$$\nabla_A(\mathcal{L}, M'; t) \neq 0. \quad (3.86)$$

Suppose that a QHS $M'$ is constructed by Dehn’s surgery on a link $\hat{\mathcal{L}}^s \in M$ such that neither of $\mathcal{L}^s$ functions

$$c_j^{(a)}(L) = -\sum_{i=1}^{L} c_{ij}(\mathcal{L}, \hat{\mathcal{L}}^s, M)a_i, \quad \text{where} \quad c_{ij}(\mathcal{L}, \hat{\mathcal{L}}^s, M) = \sum_{k=1}^{L^s} l_{ik}^{(\mathcal{L}, \mathcal{L}^s, M)}(l^{(\mathcal{L}^s, M)})_{kj}^{-1} \quad (3.87)$$
is identically equal to zero. Then

$$I_{\beta}^{(r)}(L, L', M', \underline{a}; K) = e^{(2\pi i/K) \phi^{(fr)}(L \cup L', \hat{L}^s, M; Ka, \beta)} (2K)^{L^s/2}$$

$$\times \int_{[c = c^{(st)}(\underline{a})]} \{d_c\} \{\sin(\pi_{\underline{c}})\} \exp \left( \frac{1}{2} i\pi K \sum_{j=1}^{L^s} l_{Lj}^{(\hat{L}^s|M)} c_j^2 \right) I_{\beta}^{(r)}(L \cup \hat{L}^s, L', M; \underline{a}, \underline{c}; K),$$

where the framing correction \(\phi^{(fr)}(L \cup L', \hat{L}^s, M; Ka, \beta)\) is defined by the formula similar to eq. (1.54)

$$\phi^{(fr)}(L, \hat{L}^s, M; \underline{a}) = -\frac{3}{8} (K - 2) \text{sign}(\hat{L}^s|M) - \frac{1}{4} \sum_{j=1}^{L^s} l_{Lj}^{(\hat{L}^s|M)} - \frac{1}{4} \sum_{j=1}^{L} l_{Lj}^{(\hat{L}^s|M)} (a_j^2 - 1).$$

Proof. A case of a surgery on a link in a QHS is similar to that of a surgery on a link in \(S^3\), because an obvious analog of the formula (1.69) holds if we substitute \(S^3\) by \(M\) and \(M\) by \(M'\). The proof of Proposition 2.14 also stays the same if we perform this substitution. \(\square\)

Appendix

A Construction of a rational homology sphere by a surgery on an algebraically split link

BY T. OHTSUKI

The aim of this note is to show Corollary A.2 below, which implies that any rational homology 3-sphere can be obtained from \(S^3\) by integral surgery along some algebraically split framed link after adding some lens spaces. For similar lemmas, see [8, 6].

Before showing the corollary we prepare notations of linking pairings. Let \(G\) be a finite Abelian group. A **linking pairing** on \(G\) is a non-singular symmetric bilinear map of \(G \times G\)

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to \( \mathbb{Q}/\mathbb{Z} \). For a non-singular symmetric integral \( n \times n \) matrix \( A \), we have an induced linking pairing \( \phi \) on \( \mathbb{Z}^n/\mathbb{A}\mathbb{Z}^n \) defined by \( \phi([v],[v']) = 'vA^{-1}v' \) for \( v,v' \in \mathbb{Z}^n \) whose images in \( \mathbb{Z}^n/\mathbb{A}\mathbb{Z}^n \) are denoted by \([v],[v']\); note that the right-hand side of this formula is well-defined in \( \mathbb{Q}/\mathbb{Z} \).

We denote this linking pairing by \( \iota(A) \). It is known [5, 2] that if two non-singular symmetric integral matrices \( A_1, A_2 \) give the same linking pairing \( \iota(A_1) \) and \( \iota(A_2) \) then there exists a unimodular integral matrix \( P \) such that

\[
{t}' P \cdot \left( A_1 \oplus (\pm 1) \oplus \cdots \oplus (\pm 1) \right) \cdot P = A_2 \oplus (\pm 1) \oplus \cdots \oplus (\pm 1).
\]

The set of linking pairing becomes an Abelian semigroup with respect to direct sum. Generators and relations of the semigroup are known [18, 4]. The generators in [18] are:

\[
[1/p^k], [d_p/p^k] \text{ on } \mathbb{Z}/p^k\mathbb{Z}
\]

for \( p \) odd primes, \( d_p \) a quadratic non-residue modulo \( p \)

\[
[1/2] \text{ on } \mathbb{Z}/2\mathbb{Z}, \quad [1/2^2], [-1/2^2] \text{ on } \mathbb{Z}/2^2\mathbb{Z}
\]

\[
[1/2^k], [-1/2^k], [3/2^k], [-3/2^k] \text{ on } \mathbb{Z}/2^k\mathbb{Z} \text{ for } k \geq 3
\]

\[
E_{0}^k \text{ on } \mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z} \text{ for } k \geq 1
\]

\[
E_{1}^k \text{ on } \mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z} \text{ for } k \geq 2
\]

where we denote by \([b/a] \) a linking pairing \( \phi \) on \( \mathbb{Z}/a\mathbb{Z} \) defined by \( \phi([v],[v']) = bvv'/a \) for \( v,v' \in \mathbb{Z} \) and we define \( E_{0}^k \) and \( E_{1}^k \) by

\[
E_{0}^k([v],[v']) = 'v \begin{pmatrix} 0 & 2^{-k} \\ 2^{-k} & 0 \end{pmatrix} v', \quad E_{1}^k([v],[v']) = 'v \begin{pmatrix} 2^{1-k} & 2^{-k} \\ 2^{-k} & 2^{1-k} \end{pmatrix} v'
\]

for \( v,v' \in \mathbb{Z} \oplus \mathbb{Z} \).

**Lemma A.1** Let \( \phi \) be any linking pairing on a finite Abelian group \( G \). Then there exists linking pairings \([b_1/a_1], [b_2/a_2], \cdots, [b_N/a_N] \) and integers \( n_1, n_2, \cdots, n_\nu \) such that

\[
\phi \oplus \bigoplus_{i=1}^{N} [b_i/a_i] = \iota(\bigoplus_{\xi=1}^{\nu} (n_\xi))
\]

and the order of \( G \) is divisible by each \( a_i \).
Proof. Since any linking pairing \( \phi \) is equal to a direct sum of some generators given above, it is sufficient to show the lemma when \( \phi \) is equal to each generator.

If \( \phi = [1/p^k] \), then we have \( \phi = \iota((p^k)) \).

If \( \phi = [d_p/p^k] \), then we have \( \phi \oplus [d_p/p^k] = 2[1/p^k] = \iota(2(p^k)) \) using the relation \( 2[d_p/p^k] = 2[1/p^k] \) in [18].

If \( \phi = [\pm 1/2^k] \), then we have \( \phi = \iota((\pm 2^k)) \).

If \( \phi = [\pm 3/2^k] \), then we have \( \phi \oplus [\pm 3/2^k] = 2[\mp 1/2^k] = \iota(2(\mp 2^k)) \) using a relation \( 2[\pm 3/2^k] = 2[\mp 1/2^k] \) in [4].

If \( \phi = E_{0}^k \) or \( E_{1}^k \), then we have
\[
E_{0}^k \oplus [3/2^k] = 3[1/2^k] = \iota(3(2^k))
\]
\[
E_{1}^k \oplus [-1/2^k] = [1/2^k] \oplus 2[-1/2^k] = \iota((2^k) \oplus 2(-2^k))
\]
using relations in [4], completing the proof.

Let \( L \) be a framed link in \( S^3 \), \( A \) the linking matrix of \( L \) and \( M \) the closed 3-manifold obtained by integral surgery along \( L \). Here we assume that \( M \) is a rational homology 3-sphere, which implies that the matrix \( A \) is non-singular. It is easily checked that \( H_1(M; \mathbb{Z}) \) is isomorphic to \( \mathbb{Z}^n/\mathbb{Z}^n \), the linking pairing on \( H_1(M; \mathbb{Z}) \) is equal to \( \iota(A) \) and the linking pairing on \( H_1(L(a, b); \mathbb{Z}) \) is equal to \( [b/a] \), where \( L(a, b) \) is the lens space of type \( (a, b) \).

Further, for a uni-modular integral matrix \( P \), a framed link whose linking matrix is equal to \( ^tP \cdot \left(A \oplus (\pm 1) \oplus \cdots \oplus (\pm 1)\right) \cdot P \) can be obtained from \( L \) by applying Kirby moves to \( L \); in particular the 3-manifold obtained by integral surgery along the new framed link is homeomorphic to \( M \). By Lemma A.1 putting \( \phi \) to be the linking pairing on \( H_1(M; \mathbb{Z}) \), we have

Corollary A.2 For any rational homology 3-sphere \( M \), there exist lens spaces of types \((a_1, b_1), (a_2, b_2), \cdots, (a_N, b_N)\) such that the connected sum of \( M \) and these lens spaces can be obtained by integral surgery along some algebraically split framed link and the order of \( H_1(M; \mathbb{Z}) \) is divisible by each \( a_i \).
Remark A.3 As for lens spaces in Corollary A.2, in fact, we need only lens spaces of types $(n, 1), (p^k, d_p)$ or $(2^k, \pm 3)$; we can see it by checking the proof of Lemma A.1.

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