SPECTRUM IS RATIONAL IN DIMENSION ONE

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Abstract. A bounded measurable set $\Omega \subset \mathbb{R}^d$ is called a spectral set if it admits some exponential orthonormal basis \( \{ e^{2\pi i \langle \lambda, x \rangle} : \lambda \in \Lambda \} \) for $L^2(\Omega)$. In this paper, we show that in dimension one $d = 1$, any spectrum $\Lambda$ with $0 \in \Lambda$ of a spectral set $\Omega$ with Lebesgue measure normalized to 1 must be rational. Combining previous results that spectrum must be periodic, the Fuglede’s conjecture on $\mathbb{R}^1$ is now equivalent to the corresponding conjecture on all cyclic groups $\mathbb{Z}_n$.

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1. Introduction

Let $\Omega$ be a bounded measurable set on $\mathbb{R}^d$ with positive Lebesgue measure. We say that $\Omega$ is a spectral set if we can find a countable set $\Lambda$ such that $E(\Lambda) = \{ e^{2\pi i \langle \lambda, x \rangle} : \lambda \in \Lambda \}$ forms an orthonormal basis for $L^2(\Omega)$. If such $\Lambda$ exists, we call $\Lambda$ a spectrum of $\Omega$ and $(\Omega, \Lambda)$ is referred as a spectral pair. The study of spectral sets was initiated by Fuglede [5], who proposed a famous conjecture, which is now known as the spectral set conjecture:

Fuglede’s Conjecture (1974). $\Omega$ is a spectral set on $\mathbb{R}^d$ if and only if $\Omega$ is a translational tile on $\mathbb{R}^d$.

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By *translational tile* we mean there exists a set $\mathcal{J}$ (called a *tiling set*) such that
\[
\mathbb{R}^d = \bigcup_{t \in \mathcal{J}} (\Omega + t), \text{ and } |(\Omega + t) \cap (\Omega + t')| = 0
\]
for all $t \neq t' \in \mathcal{J}$, where $| \cdot |$ denotes the Lebesgue measure. Fuglede proved that this conjecture holds if the spectrum or the tiling set is assumed to be a lattice. He also illustrated that triangle and circle cannot be spectral and they are of course not translational tile either. The conjecture remains open for 30 years until Tao [24] gave the first counterexample on $\mathbb{R}^d$ for $d \geq 5$, and the counterexample was later modified to show the conjecture is false in both directions for $d \geq 3$ [12, 13]. The conjecture is still open for $\mathbb{R}^1$ and $\mathbb{R}^2$. Despite the counterexamples, it is still widely believed that conjecture itself is true under some natural extra assumptions. Laba [15] proved that the conjecture is true if $\Omega$ is a union of two intervals. A recent progress by Lev and Matolcsi [19] showed that the conjecture is true in all dimensions if $\Omega$ is assumed to be convex.

There are also strong evidences that Fuglede’s conjecture may be true in dimension one since tiling set and spectrum exhibits a rather rigid structure. Without loss of generality, we normalize $\Omega$ to have Lebesgue measure 1. Lagarias and Wang [16] showed that any tiling set of a translational tile must be periodic (see Section 3 for the definition) and contained in the set of rational numbers. In the other direction, Iosevich and Kolountzakis [8] proved that any spectrum must be periodic. It has been conjectured for quite some time that any spectrum for a spectral set $\Omega$ in $\mathbb{R}^1$ with $|\Omega| = 1$ must be rational as well. A partial result of this conjecture was proved by Bose and Madan [1] for very special $\Omega$’s.

Our main result of this paper is to settle the rationality conjecture of the spectrum.

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}$ be a bounded spectral set with spectrum $\Lambda$. Assume that $|\Omega| = 1$ and $0 \in \Lambda$. Then $\Lambda$ is rational.

The proof of this theorem is first motivated by the weak tiling property of spectral sets observed recently by Lev and Matolcsi [19]. The weak tiling property turns the rationality problem into a weighted tiling problem (Section 3). Our aim is to show that the weights are periodic (Section 4).

With obvious modification, we can also define spectral sets and translational tiles on any locally compact abelian groups and study its Fuglede’s conjecture. In [2 Theorem 1.3], assuming that the result of our Theorem 1.1 holds, Dutkay and Lai proved that Fuglede’s conjecture on $\mathbb{R}^1$ and all cyclic groups are equivalent [2 Theorem 1.3]. Consequently and more precisely, we now have

**Theorem 1.2.** (1) All bounded tiles on $\mathbb{R}^1$ are spectral sets if and only if for all integers $n \geq 1$, all tiles in $\mathbb{Z}_n$ are spectral sets in $\mathbb{Z}_n$.

(2) All bounded spectral sets on $\mathbb{R}^1$ are translational tiles if and only if for all integers $n \geq 1$, all spectral sets in $\mathbb{Z}_n$ are tiles in $\mathbb{Z}_n$.

We remark that Fuglede’s conjecture on finite cyclic group has received significant attention in recent years and notable progress has been made. As of today the Fuglede’s conjecture has been shown to hold for the cyclic groups $\mathbb{Z}_{p^a}$, $\mathbb{Z}_{p^aq}$, $\mathbb{Z}_{pqr}$ and $\mathbb{Z}_{p^aq^2}$ [4, 14].
where \( p, q, r \) are distinct primes. We should also mention that Fuglede’s conjecture is also verified for the groups \( \mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_{p^2} \times \mathbb{Z}_p \) and the \( p \)-adic group \( \mathbb{Q}_p \).

We organize the paper as follows: In Section 2, we will mention some preliminary notations and introduce the weak tiling result by Lev and Matolcsi. In Section 3, we will turn our rationality problem into the weighted tiling problem. The proof of Theorem 1.1 and the weighted tiling problem will be given in Section 4.

2. Preliminaries

2.1. Notations. We denote by \( B(x, r) \) the Euclidean ball with center \( x \) and radius \( r \). In this paper, all measures are locally finite and Borel which may be positive or complex and it will be stated explicitly. A complex measure \( \mu \) is called translation-bounded if there exists a constant \( C > 0 \) and \( r > 0 \) such that

\[
\sup_{x \in \mathbb{R}^d} |\mu|(B(0, r) + x) \leq C,
\]

where \( |\mu| \) denotes the variation measure of \( \mu \). We denote by \( \delta_\lambda \) the Dirac measure at the point \( \lambda \) and \( \delta_\Lambda = \sum_{\lambda \in \Lambda} \delta_\lambda \) for a countable set \( \Lambda \).

Let \( \alpha \) be a tempered distribution and \( \varphi \) be a Schwartz function. The action of \( \alpha \) on the Schwartz space is denoted by \( \langle \alpha, \varphi \rangle \). The Fourier transform of a tempered distribution \( \alpha \) is defined to be the tempered distribution \( \hat{\alpha} \) such that

\[
\langle \hat{\alpha}, \varphi \rangle = \langle \alpha, \hat{\varphi} \rangle.
\]

All translation-bounded measures are tempered distributions. The support of a tempered distribution \( \alpha \) is the smallest closed set \( K \) such that if the support of a Schwartz function \( \varphi \) is inside \( K^c \), the complement of \( K \), then \( \langle \alpha, \varphi \rangle = 0 \). We denote it by \( \text{supp}(\alpha) \).

The characteristic function of a set \( \Omega \) is defined to be \( 1_\Omega \). The Fourier transform of \( f \in L^1(\mathbb{R}^1) \) is defined to be

\[
\hat{f}(\xi) = \int f(x) e^{-2\pi i (\xi, x)} \, dx.
\]

Furthermore, given a finite set \( \mathcal{A} \), the Fourier transform of \( \delta_\mathcal{A} \) is defined to be

\[
\hat{\delta}_\mathcal{A}(\xi) = \sum_{a \in \mathcal{A}} e^{-2\pi i (a, \xi)}.
\]

We also define

\[
m_\mathcal{A}(\xi) = \frac{1}{|\mathcal{A}|} \hat{\delta}_\mathcal{A}(\xi)
\]

\(|\mathcal{A}|\) means its cardinality) which people sometimes refer it as the mask function of \( \mathcal{A} \).

A countable set \( \Lambda \) is uniformly discrete if

\[
\inf_{\lambda \neq \lambda' \in \Lambda} |\lambda - \lambda'| > 0.
\]

If \((\Omega, \Lambda)\) is a spectral pair with \( \Omega \) a bounded set, then mutual orthogonality implies that for all \( \lambda \neq \lambda' \in \Lambda \), we have that

\[
\widehat{1_\Omega}(\lambda - \lambda') = 0.
\]
But $\hat{1}_\Omega$ is a continuous function with $\hat{1}_\Omega(0) = |\Omega| > 0$, $\Lambda$ must be uniformly discrete. Due to the uniform discreteness, $\delta_{\Lambda}$ must be a translation-bounded measure. Moreover, a classical result of Landau [17] shows that the density of $\Lambda$ is equal to $|\Omega|$, i.e.

\[
(2.2) \quad \lim_{R \to \infty} \frac{|\Lambda \cap B(x,R)|}{R^d} = |\Omega|
\]

uniformly in $x \in \mathbb{R}^d$.

2.2. Weak Tiling. Most of the results presented in this subsection are taken from [19].

Lev and Matolcsi observed that all spectral sets admits a weak tiling with the tiling measure can be obtained by the vague limit of the auto-correlation measures of spectra. More precisely, they prove

Theorem 2.1 ([19]). Let $(\Omega, \Lambda)$ be a spectral pair. Then there exists a positive measure $\gamma$ with the following properties.

1. $\gamma$ is a positive translation-bounded measure.
2. $\gamma$ is supported in $\{ \xi \in \mathbb{R}^d : \hat{1}_\Omega(\xi) = 0 \} \cup \{0\}$.
3. $\gamma = \delta_0$ in a neighborhood of the origin.
4. $\hat{\gamma}$ is also a positive and translation-bounded measure.
5. $\hat{\gamma} = m(\Omega)\delta_0$ on the open set $\Delta(\Omega) = \{ t \in \mathbb{R}^d : m(\Omega \cap (\Omega + t)) > 0 \}$.

Moreover, $\gamma = \hat{\nu}$ where $\nu$ is obtained by taking the vague limit from either of the following measures

\[
\frac{1}{|\Lambda_n|}\delta_\Lambda * \delta_{-\Lambda_n}, \text{ or } \frac{1}{|\Lambda_n|}\delta_{\Lambda_n} * \delta_{-\Lambda_n}
\]

where $\Lambda_n = \Lambda \cap B(0,r_n)$ and $|\Lambda_n|$ denotes its cardinality. In this case, we also have

$$f \ast \gamma = 1,$$

where $f = |\Omega|^{-2} |\hat{\Omega}|^2$.

We note that the way of the construction of $\gamma$ is implicit in the proof of the theorem and we will use it. Using Theorem 2.1 the following result was obtained in their paper.

Theorem 2.2 ([19]). Let $(\Omega, \Lambda)$ be a spectral pair. Then there exists a positive measure $\mu$ such that

$$1_\Omega \ast \mu = 1_{\Omega^c}$$

The following lemma illustrates how one may compute the limiting measure $\gamma$ in some special cases, particularly on $\mathbb{R}^1$.

Lemma 2.3. (1) Suppose that $\Lambda = L$ is a lattice. Then $\gamma = \delta_L$ and $\hat{\gamma} = |\Omega|\delta_{L^\perp}$, where $L^\perp$ is the dual lattice of $L$.

(2) Suppose that $\Lambda = p\mathbb{Z} + \mathcal{A}$, where $\mathcal{A} \subset [0,p)$ Then

$$\gamma = \frac{p}{|\mathcal{A}|}\delta_{p\mathbb{Z}} \ast (\delta_\mathcal{A} * \delta_{-\mathcal{A}}) \text{ and } \hat{\gamma} = \frac{1}{p|\mathcal{A}|}|\hat{\mathcal{A}}|^2 \cdot \delta_{p\mathbb{Z}}.$$
Proof. For (1), since $\Lambda - \lambda = \Lambda$ for all $\lambda \in \Lambda$. The measure $\frac{1}{|\Lambda_n|} \delta_\Lambda * \delta_{-\Lambda_n}$ is just $\delta_\Lambda$. Hence, the vague limit is $\delta_\Lambda$ as desired. For (2), we let $\Lambda_n = (\mathbb{Z} \cap (-np, np)) + \mathcal{A}$

$$\nu_n = \frac{1}{(2n + 1)(|\mathcal{A}|)} \delta_\Lambda * \delta_{-\Lambda_n} = \frac{1}{(2n + 1)(|\mathcal{A}|)} \delta_{p\mathbb{Z}} * \delta_{\mathcal{A}} * \delta_{p\mathbb{Z} \cap (-np, np)} * \delta_{-\mathcal{A}}.$$

Using (1) and rearranging the convolutions, in the vague limit, we have

$$\gamma = \frac{p}{|\mathcal{A}|} \delta_{p\mathbb{Z}} * (\delta_{\mathcal{A}} * \delta_{-\mathcal{A}}).$$

This shows

$$\hat{\gamma} = \frac{1}{p|\mathcal{A}|} |\hat{\delta}_{\mathcal{A}}|^2 \cdot \delta_{\frac{1}{p}\mathbb{Z}}.$$

□

In general, $\hat{\gamma}$ may not be a discrete measure in higher dimension as illustrated in [19].

3. Periodicity of spectra on $\mathbb{R}^1$

3.1. Periodicity of Spectra. We say that a uniformly discrete set $\Lambda$ is periodic with a period $p$ if

$$\Lambda + p = \Lambda.$$ 

This is equivalent to saying that $\Lambda$ is a finite union of translates of the lattice $p\mathbb{Z}$. i.e. $\Lambda = p\mathbb{Z} + \mathcal{A}$ for some finite set $\mathcal{A} \subset [0, p)$. Let us first recall that any spectra of $\Omega$ must be periodic due to a result by Iosevich and Kolountzakis [3].

Theorem 3.1 ([3]). Suppose that $\Omega \subset \mathbb{R}^1$ is a bounded measurable set of measure 1 and it is a spectral set with spectrum $\Lambda$. Then $\Lambda$ is periodic with a positive integer period $p$. Moreover, $\Omega$ is a mult-tile by the translate $p^{-1}\mathbb{Z}$. i.e.

$$\sum_{n \in \mathbb{Z}} 1_\Omega \left( x - \frac{n}{p} \right) = p.$$ 

From this theorem, we can write

$$\Lambda = p\mathbb{Z} + \mathcal{A}$$ 

for some integers $p \geq 1$ and $\mathcal{A} \subset [0, p)$ with $0 \in \mathcal{A}$ (since by a translation, we can assume $0 \in \Lambda$). Using the result of weak tiling, we have the following lemma.

Lemma 3.2. Suppose that $\Omega$ is a bounded measurable set of measure 1 and it is a spectral set with spectrum $\Lambda$. Write

$$\Lambda = p\mathbb{Z} + \mathcal{A}$$ 

with $0 \in \mathcal{A}$. Then we have

$$1_\Omega * \mu = 1_{\Omega \mathcal{C}}.$$ 

where $\mu = |m_\mathcal{A}|^2 \cdot \delta_{\frac{1}{p}\mathbb{Z}\setminus\{0\}}$ and $m_\mathcal{A}$ is the mask function of $\mathcal{A}$ defined in [2, 4].
Proof. Using Lemma 2.3, we know that \( \hat{\gamma} = \frac{1}{|\mathcal{A}|} \delta_{\mathbb{Z}} \cdot |\hat{\mathcal{A}}|^2 \). As a spectrum, the density of \( \Lambda \) must equal to \( |\Omega| = 1 \) by (2.2). However, \( \Lambda \) also has density \( |\mathcal{A}| / p \). Therefore, \( p = |\mathcal{A}| \).

We have then
\[
\hat{\gamma} = \frac{1}{|\mathcal{A}|} \delta_{\mathbb{Z}} \cdot |\hat{\mathcal{A}}|^2 = \frac{1}{|\mathcal{A}|} \delta_{\mathbb{Z}} \cdot |m\mathcal{A}|^2 \cdot \delta_{\mathbb{Z}}
\]
and \( \mu \) is obtained directly by removing the origin. \( \square \)

By including the origin, we can rewrite the above lemma as the following equation
\[
1 = \sum_{n \in \mathbb{Z}} v_n 1_{\Omega} \left( x - \frac{n}{p} \right), \quad v_n = |m\mathcal{A}(n/p)|^2.
\]

We note that \( v_0 = 1 \) which we will see it will be very important later on. We say that a weights sequence \( \{v_n\}_{n \in \mathbb{Z}} \) is periodic if there exists \( M \geq 1 \) such that
\[
v_n = v_m \text{ if } n \equiv m \pmod{M}.
\]

**Lemma 3.3.** Let \( v_n = |m\mathcal{A}(n/p)|^2 \) and assume \( 0 \in \mathcal{A} \). Then the following are equivalent.

1. \( \mathcal{A} \subset \mathbb{Q} \).
2. \( \{v_n\}_{n \in \mathbb{Z}} \) is periodic.
3. There exists an \( M \neq 0 \) such that \( v_M = 1 \).

**Proof.** (1) \( \Rightarrow \) (2). Suppose that \( \mathcal{A} \subset \mathbb{Q} \). Then we can write \( \mathcal{A} = \frac{1}{q} \{a_1, \ldots, a_p\} \), where all \( a_j \) are integers. In this case, For all \( n \in \mathbb{Z} \), we let \( q = Qp \) and \( n = qk + r \) with \( 0 \leq r \leq q - 1 \), we have
\[
v_n = \frac{1}{|\mathcal{A}|} \left| \hat{\mathcal{A}} \left( \frac{n}{p} \right) \right|^2 = \frac{1}{|\mathcal{A}|} \left| \hat{\mathcal{A}} \left( \frac{r}{p} \right) \right|^2
\]
This shows \( \{v_n\}_{n \in \mathbb{Z}} \) is periodic.

(2) \( \Rightarrow \) (3) is direct since \( v_0 = 1 \) and \( \{v_n\}_{n \in \mathbb{Z}} \) is periodic with period \( M \), so we must have \( v_M = 1 \).

(3) \( \Rightarrow \) (1) Suppose that there exists \( M \geq 1 \) such that \( v_0 = v_M = 1 \). We have that
\[
\left| \frac{1}{|\mathcal{A}|} \hat{\mathcal{A}} \left( \frac{q}{p} \right) \right|^2 = 1.
\]
But \( \left| \hat{\mathcal{A}} \left( \frac{q}{p} \right) \right| \leq |\mathcal{A}| \) and the equality holds if and only if all exponentials \( e^{2\pi i a q / p} = 1 \) with \( a \in \mathcal{A} \). Hence, \( aq/p \) is an integer and thus \( a \) is rational. \( \square \)

### 3.2. Weighted Tiling Problem.

Due to the above lemma, the main question that will be sufficient to solve the rationality problem is the following:

**Weighted tiling problem:** Given weights \( v_n \geq 0 \) with \( v_0 = 1 \) such that
\[
\sum_{n \in \mathbb{Z}} v_n 1_{\Omega}(x - \frac{n}{p}) = 1 \text{ a.e.}
\]

where \( \Omega \) is a spectral set, can we prove that the weights sequence \( \{v_n\} \) is periodic?
Proposition 3.4. Under the assumption of the weighted tiling problem, there exists an $S \subset \mathbb{Z}$ with $|S| = p$ such that

$$\sum_{s \in S} v_{n-s} = 1, \forall n \in \mathbb{Z}.$$ 

Proof. By Theorem 3.1, we know that a spectral set $\Omega$ must be a multi-tile by the translations of $\frac{1}{p}\mathbb{Z}$. Hence,

$$\sum_{n \in \mathbb{Z}} 1_{\Omega} \left( x - \frac{n}{p} \right) = p.$$ 

We now apply a multi-tile decomposition. For each $x \in [0,1/p)$ we let

$$J_x = \left\{ k \in \mathbb{Z} : x + \frac{k}{p} \in \Omega \right\}.$$ 

As $\Omega$ is a multatile of level $p$, $|J_x| = p$ for almost all $x$. Let $S \subset \mathbb{Z}$ be a finite set such that $|S| = p$, we define

$$\Omega_S = \left\{ x \in \left[ 0, \frac{1}{p} \right) : J_x = S \right\}.$$ 

We now collect all those $S$ such that $|\Omega_S| > 0$ and denote it by $\mathcal{S}$. As $\Omega$ is bounded, there are only finitely many elements in $\mathcal{S}$. Then $\Omega$ admits the following decomposition.

$$\Omega = \bigcup_{S \in \mathcal{S}} \left( \Omega_S + \frac{1}{p}S \right), \bigcup_{S \in \mathcal{S}} \Omega_S = \left[ 0, \frac{1}{p} \right).$$ 

Moreover, $\Omega_S$ are disjoint. We now apply the weighted tiling assumption:

$$1 = \sum_{n \in \mathbb{Z}} v_n \sum_{S \in \mathcal{S}} \sum_{s \in S} 1_{\Omega_S} \left( x - \frac{n-s}{p} \right)$$

$$= \sum_{S \in \mathcal{S}} \sum_{s \in S} v_{n-s} \sum_{n \in \mathbb{Z}} 1_{\Omega_S} \left( x - \frac{n}{p} \right)$$

$$= \sum_{n \in \mathbb{Z}} \sum_{S \in \mathcal{S}} \left( \sum_{s \in S} v_{n-s} \right) 1_{\Omega_S} \left( x - \frac{n}{p} \right).$$

As $\Omega_S$ are disjoint and forms a partition of $[0,1/p)$, we have

$$\sum_{n \in \mathbb{Z}} \sum_{S \in \mathcal{S}} 1_{\Omega_S} \left( x - \frac{n}{p} \right) = 1.$$ 

This forces

$$\sum_{s \in S} v_{n-s} = 1$$

for all $n \in \mathbb{Z}$ and for all $S \in \mathcal{S}$. The proof is now complete.

Proposition 3.5. Suppose that the weights sequence $\{v_n\}_{n \in \mathbb{Z}}$ has the property that there exists an $S \subset \mathbb{Z}$ with $|S| = p$ such that

$$\sum_{s \in S} v_{n-s} = 1, \forall n \in \mathbb{Z}.$$
Let $I_S = [0,1) + S$. We have
\[
\sum_{n \in \mathbb{Z}} v_n 1_{I_S}(x - n) = 1.
\]

Proof. The left hand side above is equal to
\[
\sum_{n \in \mathbb{Z}} v_n 1_{[0,1) + S}(x - n) = \sum_{s \in S} \sum_{n \in \mathbb{Z}} v_n 1_{[0,1)}(x - n - s)
= \sum_{n \in \mathbb{Z}} \left( \sum_{s \in S} v_{n-s} \right) 1_{[0,1)}(x - n)
= \sum_{n \in \mathbb{Z}} 1_{[0,1)}(x - n) = 1.
\]

This completes the proof. \hfill \Box

The above proposition indicates that it suffices to solve the periodic tiling problem when $\Omega$ is a finite union of intervals.

3.3. Fourier Analytic Criterion. We now further study the weighted tiling problem using Fourier analysis. Fourier techniques have been used frequently for the study of translational tiling, see e.g. \cite{Jog}. Results in this subsection are mostly known and we collect all necessary results here for our summary.

**Proposition 3.6.** Let $\mu = \sum_{n \in \mathbb{Z}} v_n \delta_n$, where $0 \leq v_n \leq 1$ and let $f \in L^1(\mathbb{R}^1)$. Suppose that $f * \mu = 1$. Then we have
\[
supp(\hat{\mu}) \subset \{ \xi \in \mathbb{R}^1 : \hat{f}(\xi) = 0 \} \cup \{0\}.
\]

Proof. Since $0 \leq v_n \leq 1$, the proof can be modified directly from the proof of \cite{Jog} Theorem 4.1 and we thus omit the details. \hfill \Box

We notice that in the weighted tiling problem, $f = 1_{\Omega}$ and $\hat{f}$ is an analytic function. Thus the zero set of $\hat{f}$ is uniformly discrete and hence $\hat{\mu}$ is supported in a uniformly discrete set. The following result can be found in \cite{Hof} Lemma 4.5.

**Proposition 3.7.** Let $\mu$ be a positive translation-bounded measure on $\mathbb{R}^d$. Suppose that for some $r > 0$ and some $\tau \in \mathbb{R}^d$,
\[
supp(\hat{\mu}) \cap B_r(\tau) = \{\tau\},
\]
where $B_r(\tau)$ is the ball of radius $r$ centered at $\tau$. Then, there exists $a_\tau \in \mathbb{C}$ such that
\[
\hat{\mu} = a_\tau \delta_\tau \text{ on } B_r(\tau).
\]

For $f = 1_{\Omega}$ we already stated that the zero set of $\hat{f}$ is uniformly discrete because $\hat{f}$ is an analytic function, and therefore $\hat{\mu}$ is supported in a uniformly discrete set by Proposition 3.6. Using Proposition 3.7, we know that on each point $\tau$ in the support of $\hat{\mu}$, it is a Dirac mass of some complex weight. Since a distribution is completely determined by its localization via partitions of unity (see \cite{Ste} Section 6.19), we obtain the following corollary.
Corollary 3.8. Suppose that $1_{\Omega} * \mu = 1$ for some bounded measurable set $\Omega \subset \mathbb{R}^1$ and $\mu$ is translation-bounded and positive. Then $\hat{\mu}$ is a complex measure whose support is uniformly discrete.

4. Rationality of the spectra

4.1. Proof of the Main Theorem. We now prove our main theorem in this subsection. We will first need the following lemma.

Lemma 4.1. Let $\{\theta_1, \ldots, \theta_J\}$ be a finite set of positive irrational numbers in $(0, 1)$ such that $\theta_i - \theta_j \notin \mathbb{Q}$ for all $i \neq j$. Suppose that for some $a_j, b_j \in \mathbb{R}$, we have

\[(4.1) \quad a_0 + \sum_{j=1}^J (a_j \cos(2\pi \theta_j n) + b_j \sin(2\pi \theta_j n)) = 0\]

for all positive integer $n$. Then all $a_j = b_j = 0$.

Proof. Let $V$ be the $\mathbb{Q}$-span of $\{\theta_1, \ldots, \theta_J\}$. Without loss of generality, we let $\theta_1, \ldots, \theta_\ell$ be the basis of $V$, so that for all $j = \ell + 1, \ldots, J$,

$$\theta_j = \sum_{k=1}^\ell c_{jk} \theta_k$$

where $c_{jk} \in \mathbb{Q}$. Let also $M$ be the least common multiples (l.c.m.) of the denominators of all $c_{jk}$. We will use $\{x\}$ to denote the fractional part of $x$ and $\lfloor x \rfloor$ denote the largest integers smaller than or equal to $x$, so that $x = \lfloor x \rfloor + \{x\}$. Since by assumption $\{\theta_1, \ldots, \theta_\ell\}$ are rationally independent, by the Kronecker theorem (see e.g. [7, Lemma 3.13]), for all $(x_1, \ldots, x_\ell) \in [0, 1)^\ell$, we can find $n \in \mathbb{N}$ so that

$$\left| x_i - \{n \theta_i\} \right| < \epsilon$$

for all $i = 1, \ldots, \ell$. In other words, the set of all tuples $(\{n \theta_1\}, \ldots, \{n \theta_\ell\})$ is dense in $[0, 1)^\ell$.

Putting $Mn$ into $(4.1)$, it becomes

\begin{align*}
& a_0 + \sum_{j=1}^\ell (a_j \cos(2\pi M \theta_j n) + b_j \sin(2\pi M \theta_j n)) + \\
& \sum_{j=\ell+1}^J \left( a_j \cos(2\pi \left( \sum_{k=1}^\ell M c_{jk} \theta_k n \right) + b_j \sin(2\pi \left( \sum_{k=1}^\ell M c_{jk} \theta_k n \right)) \right) = 0.
\end{align*}

From the density result of the Kronecker theorem, we can pass limit and obtain

\[(4.2) \quad a_0 + \sum_{j=1}^\ell (a_j \cos(2\pi M x_j) + b_j \sin(2\pi M x_j)) + \\
\sum_{j=\ell+1}^J \left( a_j \cos(2\pi \left( \sum_{k=1}^\ell M c_{jk} x_k \right) + b_j \sin(2\pi \left( \sum_{k=1}^\ell M c_{jk} x_k \right)) \right) = 0.

Let $e_1, \ldots, e_\ell$ be the standard basis for $\mathbb{R}^\ell$. For $j = \ell + 1, \ldots, J$, let also

$$e_j = (c_{j1}, \ldots, c_{j\ell}),$$
so that $Me_j \in \mathbb{Z}^\ell$. Then (4.2) is rewritten as

$$a_0 + \sum_{j=1}^{J}(a_j \cos(2\pi \langle Me_j, x \rangle) + b_j \sin(2\pi \langle Me_j, x \rangle)) = 0$$

for all $x \in [0,1)^\ell$. Note that the vectors $Me_1, ..., Me_J$ are all distinct non-zero integer vectors since $\theta_i - \theta_j \notin \mathbb{Q}$. By the linear independence of sine and cosine functions of integer frequencies in the space of continuous functions on $[0,1)^\ell$, we must have all $a_j = b_j = 0$. This completes the proof. \(\square\)

**Theorem 4.2.** Let $\Omega$ be a bounded spectral set with $|\Omega| = 1$ and $\Lambda$ be a spectrum with $0 \in \Lambda$. Then $\Lambda$ is rational.

**Proof.** We know that we can write $\Lambda = p\mathbb{Z} + A$ and $v_n = \left| \frac{1}{|A|} \mathcal{A}(\frac{a}{p}) \right|^2 \geq 0$ and $v_0 = 1$ satisfies

$$\sum_{n \in \mathbb{Z}} v_n 1_{\Omega}(x - \frac{n}{p}) = 1.$$ 

Using Proposition 3.5, we obtain that for some finite set $S$,

$$\sum_{n \in \mathbb{Z}} v_n 1_{S}(x - n) = 1.$$ 

Let

$$\mu = \sum_{n \in \mathbb{Z}} v_n \delta_n.$$ 

We have that

$$1_{S} * \mu = 1.$$ 

Therefore, by Proposition 3.6

$$\text{supp}(\hat{\mu}) \subset \{ \xi \in \mathbb{R} : \hat{1_{S}}(\xi) = 0 \} \cup \{0\}.$$ 

Note that $\hat{1_{S}}(\xi) = \hat{\delta}(\xi) 1_{[0,1)}(\xi)$. Define

$$\Theta = \{ \xi \in [0,1) : \hat{1_{S}}(\xi) = 0 \} = \{\theta_1, ..., \theta_N\}.$$ 

As $S \subset \mathbb{Z}$, we have that $\hat{1_{S}}$ is an integer periodic function and we thus have the support of $\hat{\mu}$ is contained in the union of lattice:

(4.3) $$\{ \xi \in \mathbb{R} : \hat{1_{S}}(\xi) = 0 \} \cup \{0\} = \{0, \theta_1, ..., \theta_N\} + \mathbb{Z}.$$ 

By Corollary 3.8 $\hat{\mu}$ is a complex measure supported on $\{0, \theta_1, ..., \theta_N\} + \mathbb{Z}$. Let also $\theta_0 = 0$. We can write

$$\hat{\mu} = \sum_{n \in \mathbb{Z}} \sum_{j=0}^{N} w_{n,\theta_j} \delta_{n+\theta_j}.$$ 

**Claim:** For all $k \in \mathbb{Z}$ and for all $j = 0, 1, ..., N$, we have $w_{k,\theta_j} = w_{0,\theta_j}$. 

Proof of claim: Let $\psi$ be a Schwartz function that is supported in a small neighborhood of the origin so that $\psi(0) = 1$ and $\psi(x) = 0$ for all $x \in \{0, \theta_1, ..., \theta_N\} + \mathbb{Z}$ other than 0. Let $\varphi_k(x) = \psi(x - k - \theta_j)$. Then $\varphi_k$ is supported around $k + \theta_j$. Therefore,

$$w_{k,\theta_j} = \langle \hat{\mu}, \varphi_k \rangle = \langle \mu, e^{2\pi i (k+\theta_j)x} \rangle = \sum_{n \in \mathbb{Z}} v_n e^{2\pi i \theta_j n} \hat{\psi}_k(n) = \langle \hat{\mu}, \varphi_0 \rangle = w_{0,\theta_j}.$$

From the claim, $\hat{\mu}$ is constant on each of the coset. We can write

$$\hat{\mu} = \sum_{j=0}^{N} w_{\theta_j} \delta_{z+\theta_j} = \left( \sum_{j=0}^{N} w_{\theta_j} \delta_{\theta_j} \right) * \delta_z$$

and $w_{\theta_j} \in \mathbb{C}$. Taking inverse Fourier transform, we have

$$\mu = \sum_{n \in \mathbb{Z}} \left( \sum_{j=0}^{N} w_{\theta_j} e^{-2\pi i \theta_j n} \right) \cdot \delta_n.$$

This means that

$$v_n = \sum_{j=0}^{N} w_{\theta_j} e^{-2\pi i \theta_j n}$$

are positive real numbers. We now decompose $\Theta \cup \{0\}$ into equivalent class of rational numbers

$$\Theta_0 \cup \Theta_1 \cup ... \cup \Theta_J,$$

so that each $\theta_j \in \Theta_j$ can be written as $\theta_j = \tilde{\theta}_j + r_{\theta_j}$, where $r_{\theta_j} \in \mathbb{Q}$, $\tilde{\theta}_i - \tilde{\theta}_j \not\in \mathbb{Q}$ for $i \neq j$ and $\tilde{\theta}_0 = 0$ (so that $\Theta_0 \subset \mathbb{Q}$). Hence,

$$v_n = \sum_{j=0}^{J} \sum_{\theta \in \Theta_j} w_{\theta} e^{-2\pi i (\tilde{\theta}_j + r_{\theta_j}) n}$$

Let $M$ be the least common multiple (l.c.m.) of the denominators of all $r_{\theta_j}$, we have that

$$v_{Mn} = \sum_{\theta \in \Theta_0} w_{\theta} + \sum_{j=1}^{J} \left( \sum_{\theta \in \Theta_j} w_{\theta} \right) e^{-2\pi i \tilde{\theta}_j M n}$$

(note that $\tilde{\theta}_0 = 0$). Denote by $W_j = \sum_{\theta \in \Theta_j} w_{\theta}$ for $j = 0, 1, ..., J$ and we thus have

$$v_{Mn} = W_0 + \sum_{j=1}^{J} W_j e^{-2\pi i \tilde{\theta}_j M n}.$$

We further write $W_j = X_j + i Y_j$ where $X_j, Y_j$ are the real and imaginary part of $W_j$. As $v_{Mn}$ is real and all $\theta_j$ are in $(0, 1)$, this shows that the imaginary part of $v_{Mn}$,

$$Y_0 + \sum_{j=1}^{J} \left( Y_j \cos(2\pi \tilde{\theta}_j M n) - X_j \sin(2\pi \tilde{\theta}_j M n) \right) = 0$$

for all $n \in \mathbb{N}$. By Lemma 4.11, we have

$$X_j = Y_j = 0$$
for all $j = 1, \ldots, N$ and $Y_0 = 0$. Hence,

$$v_{Mn} = X_0 = W_0 = \sum_{\theta \in \Theta_0} w_{\theta}$$

and thus it is real. But $W_0 = v_0 = 1$, we have shown that $v_M = 1$ and $M \neq 0$. This shows that $A$ must be rational by Lemma 3.3.

**4.2. Solution to the Weighted Tiling Problem.** The information of the spectral sets is used with Proposition 3.5 to obtain that $\text{supp}(\hat{\mu})$ is contained in a finite union of translates of a lattice as in (4.3). We can then obtain the representation (4.4) of the weights $v_n$. In fact, the following proposition proved in Wang and Zhou [25] establishes the support is a finite union of translates of a lattices without assuming any information about spectral sets.

**Proposition 4.3.** Let $\Lambda$ be a uniformly discrete set in $\mathbb{R}$ and $\eta = \sum_{\beta \in \Lambda} v_\beta \delta_\beta$ where \{v_\beta\} are bounded. Assume that $\text{supp}(\hat{\eta}) \subseteq \mathbb{Z}$. Then

(A) There exist $\theta_1, \ldots, \theta_m \in [0, 1)$ such that $\Lambda = \bigcup_{j=1}^m (\theta_j + \mathbb{Z})$.

(B) There exist $c_1, \ldots, c_m$ such that $v_\beta = c_j$ for all $\beta \in \theta_j + \mathbb{Z}$. Thus

$$\eta = \sum_{j=1}^m c_j \sum_{k \in \mathbb{Z}} \delta_{\theta_j + k}.$$ 

In other words, $\eta$ is periodic.

(C) $\hat{\eta} = \sum_{n \in \mathbb{Z}} \left( \sum_{j=1}^m c_j e^{2\pi i \theta_j n} \right) \delta_n$.

Using this proposition, we can actually solve the weighted tiling problem without assuming that $f = \mathbf{1}_\Omega$ with $\Omega$ a bounded spectral set.

**Theorem 4.4.** Let $f \in L^1(\mathbb{R})$ be such that $\hat{f}$ is analytic and let $v_n \geq 0$ be a set of weights with $v_0 = 1$. Suppose that

$$\sum_{n \in \mathbb{Z}} v_n f(x - n) = 1.$$ 

Then $v_n$ must be periodic.

**Proof.** Let $\mu = \sum_{n \in \mathbb{Z}} v_n \delta_n$. Then we have $f * \mu = 1$. Then $\text{supp}(\hat{\mu})$ is contained in a uniformly discrete set, namely the zero set of $\hat{f}$ union with the origin, and $\hat{\mu}$ is a complex measure. Using Proposition 4.3 with $\eta = \hat{\mu}$, the support of $\hat{\mu}$ must be a finite union of translates of a lattice. Proposition 4.3(C) implies that (4.4) similarly holds. Hence, following the rest of the argument in the same way as Theorem 4.2, we obtain $v_n$ is periodic with period $M$.

A stronger result of Lev and Olevskii [18], using quasicrystals, also shows that if $\mu$ and $\hat{\mu}$ are all uniformly discrete complex measures on $\mathbb{R}^1$, then both $\mu$ and $\hat{\mu}$ will have supports contained in a finite union of translates of a lattice. This result will also suffice to prove Theorem 4.4. In the end, we illustrate (4.4) through a simple example.

**Example 4.5** Let $\Omega = [0, 1/2] \cup [1, 3/2]$. Then it admits a spectrum $2\mathbb{Z} + A$ where $A = \{0, 1/2\}$. Hence, $v_n = m_A(n/2) = \cos^2(\pi n/4)$. Note that in this case $S = \{0, 2\}$. 

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Hence, the zero set of $I_{kS}$ is precisely $\{0, 1/4, 3/4\} + \mathbb{Z}$. Our representation can be obtained as follows:

$$\cos^2 \frac{\pi n}{4} = \frac{1}{2} + \frac{1}{4} e^{2\pi i n/4} + \frac{1}{4} e^{-2\pi i n/4} = \frac{1}{2} + \frac{1}{4} e^{2\pi i n/4} + \frac{1}{4} e^{2\pi i 3n/4}$$

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