Bayesian Analysis of the Chaplygin Gas and Cosmological Constant Models using the SNe Ia Data

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Abstract

The type Ia supernovae observational data are used to estimate the parameters of a cosmological model with cold dark matter and the Chaplygin gas. This exotic gas, which is characterized by a negative pressure varying with the inverse of density, represents in this model the dark energy responsible for the acceleration of the Universe. The Chaplygin gas model depends essentially on four parameters: the Hubble constant, the velocity of the sound of the Chaplygin gas, the curvature of the Universe and the fraction density of the Chaplygin gas and the cold dark matter. The Bayesian parameter estimation yields $H_0 = 62.1^{+3.3}_{-3.4}$ km/Mpc.s, $\Omega_{m0} = -0.84^{+1.51}_{-1.23}$, $\Omega_{c0} = 0.0^{+0.82}_{-0.0}$, $\Omega_{k0} = 1.40^{+1.16}_{-1.16}$, $\tilde{A} = c^2_{s} = 0.93^{+0.07}_{-0.21}$, $t_0 = 14.2^{+2.8}_{-1.3}$ Gyr and $q_0 = -0.98^{+1.02}_{-0.62}$. These and other results indicate that a Universe completely dominated by the Chaplygin gas is favoured, what reinforces the idea that the Chaplygin gas may unify the description for dark matter and dark energy, at least as the type Ia supernovae data are concerned. A closed and accelerating Universe is also favoured. The Bayesian statistics indicates that the Chaplygin gas model is more likely than the standard cosmological constant ($\Lambda$CDM) model at 55.3% confidence level when an integration on all free parameters is performed. Assuming the spatially flat curvature, this percentage mounts to 65.3%. On the other hand, if the density of dark matter is fixed at zero value, the Chaplygin gas model becomes more preferred than the $\Lambda$CDM model at 91.8% confidence level. Finally, the hypothesis of flat Universe and baryonic matter ($\Omega_b = 0.04$) implies a Chaplygin gas model preferred over the $\Lambda$CDM at a confidence level of 99.4%.

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1 Introduction

The combined data from the measurements of the spectrum of anisotropies of the cosmic microwave background radiation and from the observations of high redshift type Ia supernovae indicate that the matter content of the Universe today may be very probably described by cold dark matter and dark energy, in a proportion such that $\Omega_{dm} \approx 0.3$ and $\Omega_{de} \approx 0.7$. These dark components of the matter content of the Universe manifest themselves only through their gravitational effects. At same time, a fraction of the dark components agglomerates at small scales (cold dark matter) while the other
fraction seems to be a smooth component (dark energy). The dark energy must exhibit negative pressure, since it would be the responsible for the present acceleration of the Universe as indicated by the type Ia supernovae observations, while the cold matter must have zero (or almost zero) pressure, in order that it can gravitationally collapse at small scales.

The nature of these mysterious matter components of the Universe is object of many debates. The cold dark matter may be, for example, axions which result from the symmetry breaking process of Grand Unified Theories in the very early Universe. But, since Grand Unified Theories, and their supersymmetrical versions, remain a theoretical proposal, the nature of cold dark matter is yet an open issue.

A cosmological constant is, in principle, the most natural candidate to describe the dark energy. It contributes with an homogeneous, constant energy density, its fluctuation being strictly zero. However, if the origin of the cosmological constant is the vacuum energy, there is a discrepancy of about 120 orders of magnitude between its theoretical value and the observed value of dark energy [4]. This situation can be ameliorate, but not solved, if supersymmetry is taken into account. Another candidate to represent dark energy is quintessence, which considers a self-interacting scalar field, which interpolates a radiative era and a vacuum dominated era [5, 6, 7]. But the quintessence program suffers from fine tuning of microphysical parameters [8].

Recently, an alternative to both the cosmological constant and to quintessence to describe dark energy has been proposed: the Chaplygin gas [9, 10, 11, 12]. The Chaplygin gas is characterized by the equation of state

\[ p = -\frac{A}{\rho}, \]  

where \( A \) is a constant. Hence, the pressure is negative while the sound velocity is positive, avoiding instability problems at small scales [13]. The Chaplygin gas has been firstly conceived in studies of adiabatic fluids [14], but recently it has been identified an interesting connection with string theories [15, 16, 17]. Some extensions of the Chaplygin gas have been proposed [12] through the equation of state

\[ p = -\frac{A}{\rho^{1+\alpha}}. \]  

However, the connection with string theories is lost unless \( \alpha = 1 \), and for this reason we will only consider in what follows the equation of state [11].

Considering a relativistic fluid with the equation of state [11], the equation for the energy-momentum conservation relations leads, in the case of an homogeneous and isotropic Universe, to the following relation between the fluid density and the scale factor \( a \):

\[ \rho = \sqrt{\frac{A}{a^6} + \frac{B}{a^5}}, \]  

where \( B \) is an integration constant. This relation shows that initially the Chaplygin gas behaves as a pressureless fluid, acquiring later a behaviour similar to a cosmological constant. So, it interpolates a decelerated phase of expansion to an accelerated one, in way close to that of the quintessence program.

In this work, we will constrain the parameters associated with the Chaplygin gas model (CGM) using the type Ia supernovae (SNe Ia) observational data. Specifically, we will consider a model where the dynamics of the Universe is driven in principle by pressureless matter and by the Chaplygin gas. The luminosity distance for the configuration where, in general, Chaplygin gas and dark matter are present, is evaluated from which the relation between the magnitude and the redshift \( z \) is established. The observational data are then considered, and they are fitted using four free parameters: the ratio of the density fraction, with respect to the critical density \( \rho_c \), of the pressureless matter and of the Chaplygin gas, \( \Omega_m0 \) and \( \Omega_c0 \) respectively, the sound velocity of the Chaplygin gas \( A \), in terms of the velocity of light, the curvature parameter \( \Omega_k0 \) and the Hubble parameter \( H_0 \). All these parameters are evaluated today.

The fact that we consider a model containing pressureless matter and the Chaplygin gas, implies that in principle we ignore the possibility that the Chaplygin gas could unify the description of dark energy and dark matter [11, 12]. This unification is suggested by the fact that, at perturbative level, the Chaplygin
gas does not agglomerate at very large scales while it may agglomerate at small scales. The possible unification of both dark components of the matter content of the Universe through the Chaplygin gas has increased the interest on this new possible exotic fluid. A recent analysis of type Ia supernovae data correlated with the CGM used extensively this idea, employing the generalized equation of state in a flat Universe and excluding, \textit{ab initio}, the possibility to have a dark matter component. In this work, the authors tried to restrict the possible values of the parameter $\alpha$, but their results indicate a yet large range of possible values for this parameter. In the Ref. \cite{19}, the authors studied, using type Ia supernovae data, essentially two scenarios: a flat Universe with the generalized Chaplygin gas and dark matter; a non-flat Universe with the ordinary Chaplygin gas and baryonic matter (with a fixed $\Omega_{b} = 0.04$). For the last case, the most favoured configuration was obtained for a closed Universe with $\Omega_{b} = 0.78$, using a $\chi^2$ statistic. As it will be seen later, this result is consistent with the more general analysis we will perform here.

The unification program using the Chaplygin gas has been recently criticized \cite{20, 21}. In our opinion this question remains an open issue, for the following reasons. In Ref. \cite{20}, the authors consider a model containing only the Chaplygin gas, and find a spectrum for the clustering of matter that is not in agreement with the 2dFGRS observations, unless the parameter $\alpha$ in \cite{2} is very small, $\alpha < 10^{-5}$. However, we must not forget that there are baryons in the Universe. Even the quantity of baryons is small, it presences claims at least for a two fluid model, whose behaviour is in general quite different of a single fluid model \cite{22}. Moreover, we may rise the question of what is really observed in the matter power spectrum. In Ref. \cite{21}, the authors are more stringent by stating that, when the large scale structure data are crossed with the CMB data, the likelihood in the configuration space leads to a range of value for the $\alpha$ parameter that is very near zero and $\Lambda \sim 1$, i.e., essentially a cosmological constant model. However, the CMB analysis suffers from a high degeneracy in the space of the parameters, and the crossing of these data with those coming from type Ia supernovae must bring again the Chaplygin gas to scene. Here, we open the possibility to have dark matter present, not excluding at the same time the possibility to have a pure CGM.

Our goal here, in contrast with the Refs. \cite{18, 19}, is to perform an extensive and comprehensive analysis of the problem of fitting the type Ia supernovae data using the ordinary Chaplygin gas given by \cite{1}. This restriction on the parameter $\alpha$ keeps us in contact with a string motivation for the Chaplygin gas. At the same time, our intention is to leave space for the presence of other fluids besides the Chaplygin gas as well as a spatial curvature term, constraining these parameters (together with the Chaplygin gas sound velocity and the Hubble parameter) only through the confrontation with the supernovae type Ia data. In this sense, the present work is more general than the previous ones, excepting for the restriction on the equation of state, restriction dictated by the theoretical motivation for the Chaplygin gas. A detailed description of the Bayesian statistics method of analysis is presented, as well as the meaning of the observational limits on the different free parameters of the model obtained through this method. A comparison with the cosmological constant model ($\Lambda$CDM) is exhibited, which is one of the main purpose of this work.

It will be shown, in particular, that the CGM, in what concerns the supernovae type Ia data, favours strongly, at 2$\sigma$, a closed Universe ($\Omega_{k0} = 0.84^{+1.54}_{-1.29}$), peaked in the zero value for the cold dark matter density ($\Omega_{m0} = 0.0^{+0.82}_{-0.0}$) and a sound velocity $c_{s}^2 = 0.93^{+0.07}_{-0.21}$ near, but not equal, to the value that corresponds to the cosmological constant. The present age of the Universe is $t_{0} = 14.2^{+2.8}_{-1.3}$ Gy in the CGM, a bit smaller than that in the $\Lambda$CDM model ($t_{0} = 15.4^{+3.4}_{-1.9}$ Gy), but still compatible with other astronomical measurements \cite{23}. The deceleration parameter is highly negative: $q_{0} = -0.98^{+1.92}_{-0.62}$. The CGM is always prefered with respect to the $\Lambda$CDM model, with a confidence level of 55.3% when all free parameters are considered. This confidence level is considerably higher if the curvature or the dark matter density is fixed to zero. An almost flat Universe is predicted if dark matter is absent, $\Omega_{k0} = 0.17^{+0.83}_{-1.58}(\Omega_{m0} = 0)$. Later, we will make a brief comparison between the results obtained with the recent data coming from the anisotropy of the cosmic microwave background (including the data coming from the WMAP) \cite{24, 25}, 2dFGRS measurements \cite{26} and Lyman $\alpha$ forest \cite{27}, even if a proper combined analysis is outside the scope of the present paper.

In next section, the Chaplygin gas model is described. In section 3, the Bayesian probabilistic analysis is described. In section 4, the cosmological parameters are estimated, and in section 5 our conclusions
are presented.

2 The Chaplygin gas model and the least-squares fitting

We now proceed by constructing the cosmological model based on the Chaplygin gas, which we will test against the type Ia supernovae observations. We must specify the observable cosmological parameters in the model: we will deal with the Hubble parameter $H_0$, the curvature parameter $\Omega_{k0}$, the dark matter density parameter $\Omega_{m0}$, the Chaplygin gas density parameter $\Omega_{c0}$ and the Chaplygin gas sound velocity $c_s^2$. All these parameters are evaluated today.

The sound velocity of the Chaplygin gas today, in units of $c$, is given by

$$c_s^2 = \bar{A} = \frac{A}{\rho_{c0}}.$$  \hspace{1cm} (4)

The ratios of the density fractions with respect to the current critical density are related by $\Omega_{k0} + \Omega_{m0} + \Omega_{c0} = 1$. For the Chaplygin gas case we take $\Omega_{m0} \geq 0$ and $\Omega_{c0} \geq 0$, so $\Omega_{k0} \leq 1$. When the limit $\bar{A} \to 1$ is taken ($c = 1$ hereafter), the cosmological constant model is obtained from the Chaplygin model, so $\Omega_{c0}$ is relabeled as $\Omega_\Lambda$ (the ratio of the vacuum energy density with respect to the current critical density), and the only range restriction is $\Omega_{m0} \geq 0$. It will be verified that the best-fitting and the Bayesian inference suggest $\Omega_{m0} \approx 0$, this result becomes quite interesting if we take into account some recent considerations about a unification of cold dark matter and dark energy in Chaplygin gas models [12]. Moreover, closed models will be favoured.

The equation governing the evolution of our model is

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \left(\frac{\rho_{m0}}{a^3} + \sqrt{\frac{A}{a^6}} + \Omega_{k0}\right).$$  \hspace{1cm} (5)

It can be rewritten as

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left(\frac{\Omega_{m0}}{a^3} + \Omega_{c0} \sqrt{\frac{\bar{A}}{a^6}} + \frac{\Omega_{k0}}{a^2}\right).$$  \hspace{1cm} (6)

where $H_0$ is the Hubble parameter today, $\Omega_i = \frac{\rho_i}{\rho_c}$ ($\rho_i$ denoting a matter component and $\rho_c$ the critical density), $\Omega_{k0} = \frac{3kc^2}{8\pi G}$ and the scale factor was normalized to unity today, $a_0 = 1$.

The luminosity distance is obtained by standard procedures [28], using the equation for the light trajectory in the above specified background, and its definition,

$$D_L = \left(\frac{1}{4\pi}\frac{L}{l}\right)^{1/2}$$  \hspace{1cm} (7)

where $L$ is the absolute luminosity of the source, and $l$ is the luminosity measured by the observer. This expression can be rewritten as

$$D_L = (1 + z)r,$$  \hspace{1cm} (8)

$r$ being the co-moving distance of the source. Taking into account the definitions of absolute and apparent magnitudes in terms of the luminosity $L$ and $l$, $M$ and $m$ respectively, we finally obtain the relation valid for the three types of spatial section:

$$m - M = 5 \log \left\{ \frac{(1 + z) c}{H_0 \sqrt{\left|\Omega_{k0}\right|}} \times f \left[ \sqrt{\left|\Omega_{k0}\right|} \times \right. \\ \times \int_0^z \sqrt{\frac{\Omega_{m0}(1 + z')^3 + \Omega_{c0}\sqrt{A + (1 - A)(1 + z')^6} + \Omega_{k0}(1 + z')^2}} dz' \left. \right] \right\}$$  \hspace{1cm} (9)

where $f(x) = \sinh(x)$ for $\Omega_{k0} > 0$ (open Universe with $k < 0$), $f(x) = \sin(x)$ for $\Omega_{k0} < 0$ (closed Universe with $k > 0$) and $f(x) = x$ for $\Omega_{k0} = 0$ (flat Universe with $k = 0$).

4
Following the same lines, we can obtain the expression of the age of the Universe today:

\[ t_0 = \int_0^\infty \frac{dz'}{H_0(1 + z') \left[ \Omega_{m0}(1 + z')^3 + \Omega_{\cd}(A + (1 - A)(1 + z')^6 + \Omega_{\Lambda 0}(1 + z')^2) \right]^2} , \]  

(11)

which will be another parameter evaluated for the different models.

The deceleration parameter \( q \) is defined as \( q = -1 - \ddot{H}/H^2 \), and its value calculated today, using Eq. (6), reads

\[ q_0 = \frac{\Omega_{m0} + \Omega_{\cd}(1 - 3\bar{A})}{2} , \]  

(12)

so depending on the values of the three parameters above, an accelerating Universe \((q_0 > 0)\) or decelerating Universe \((q_0 < 0)\) is obtained. When \( \bar{A} = 1 \) the well-known result for the cosmological constant model is recovered, \( q_0 = (\Omega_{m0}/2) - \Omega_{\cd} \).

Another useful result is the calculation whether the Universe will expand eternally or not from the present time. A simple method used here searches for the roots of \( a(t) \) in the r.h.s. of Eq. (6); if there is one that is a real value and greater than \( a_0 = 1 \), then the Universe is not always expanding in the future.

We proceed by fitting the SNe Ia data using the model described above. Essentially, we compute the quantity distance moduli,

\[ \mu_0 = 5 \log \left( \frac{D_L}{Mpc} \right) + 25 \]  

(13)

and compare the same distance moduli as obtained from observations. The quality of the fitting is characterized by the \( \chi^2 \) parameter of the least-squares statistic, as defined in Ref. [2],

\[ \chi^2 = \sum_i \frac{(\mu_{0,i} - \mu^t_{0,i})^2}{\sigma^2_{\mu_{0,i}} + \sigma^2_{mz,i}} . \]  

(14)

In this expression, \( \mu_{0,i} \) is the measured value, \( \mu^t_{0,i} \) is the value calculated through the model described above, \( \sigma^2_{\mu_{0,i}} \) is the measurement error, \( \sigma^2_{mz,i} \) is the dispersion in the distance modulus due to the dispersion in galaxy redshift due to peculiar velocities. This quantity we will taken as

\[ \sigma^2_{mz} = \frac{\partial \log D_L}{\partial z} \sigma_z . \]  

(15)

where, following Ref. [2], \( \sigma_z = 200 \text{ km/s} \).

In Table 1 the data concerning 26 type Ia supernovae with the error bar are displayed. These are essentially the type Ia supernovae employed in the first works on the problem of the acceleration of the Universe [2]. We restrict ourselves to this sample (samples with up to about one hundred type Ia supernovae are now available), since it contains some of the better studied SNe Ia. It must be stressed that since one of the goals of this work is to compare competitive models, the choice of the sample is not so essential, provided it is not too small neither contains doubtful data.

There are two methods to determine the relationship between the shape of SNe Ia light curve and its peak luminosity: MLCS (Multicolor Light Curve Shapes) [32] and a template fitting method (\( \Delta m_{15}(B) \)) [33]. For both methods, the fit determines the light curve parameters and their uncertainties. The set of 26 SNe Ia used in this article are within the expected statistical uncertainties range of the two methods. Specifically, we use the parameters obtained by the template fitting method in our analyses.

In the tables 2 and 3 we evaluate, in fact, \( \chi^2_\nu \), the estimated errors for degree of freedom, i.e., \( \chi^2 \) divided by 26, the number of type Ia supernovae chosen in this article. The values of \( \chi^2_\nu \) for the best-fitting are indeed small, between 0.75 and 0.78, if compared to other SNe Ia analyses [2], due to our choice of SNe Ia that avoided those with large observational errors. In previous works [30, 31], the restricted case of spatially flat Universe was analysed by means of best-fitting parameter estimation.

The Table 2 lists the best-fitting parameters of the cosmological constant model (the limit \( \bar{A} \rightarrow 1 \) of the Chaplygin gas model). When the spatial section is open or flat, the best-fitting favours a flat Universe with \( H_0 = 61.8 \text{ km/Mpc.s} \), \( \Omega_{m0}/\Omega_\Lambda = 0.24/0.76 \) and \( t_0 = 16.5 \text{ Gy} \), approximately the same
Table 1: The SNe Ia data of the 26 supernovae used in this article, obtained by the template fitting method ($\Delta m_{15}(B)$).

| SNe Ia | z     | $\mu_0(\sigma_{\mu_0})$ | SNe Ia | z     | $\mu_0(\sigma_{\mu_0})$ |
|--------|-------|-------------------------|--------|-------|-------------------------|
| 1992al | 0.014 | 34.13 (0.14)            | 1992bp | 0.080 | 37.96 (0.15)            |
| 1992bo | 0.018 | 34.88 (0.21)            | 1992br | 0.087 | 38.09 (0.36)            |
| 1992bc | 0.020 | 34.77 (0.15)            | 1992aq | 0.111 | 38.33 (0.23)            |
| 1992ag | 0.026 | 35.53 (0.20)            | 1996J  | 0.30  | 40.99 (0.25)            |
| 1992P  | 0.026 | 35.59 (0.16)            | 1996K  | 0.38  | 42.21 (0.18)            |
| 1992bg | 0.035 | 36.49 (0.21)            | 1996E  | 0.43  | 42.03 (0.22)            |
| 1992bl | 0.043 | 36.53 (0.20)            | 1996U  | 0.43  | 42.34 (0.17)            |
| 1992bh | 0.045 | 36.87 (0.17)            | 1997cl | 0.44  | 42.26 (0.16)            |
| 1990af | 0.050 | 36.67 (0.25)            | 1995K  | 0.48  | 42.49 (0.17)            |
| 1993ag | 0.050 | 37.11 (0.19)            | 1997cj | 0.50  | 42.70 (0.16)            |
| 1993O  | 0.052 | 37.31 (0.14)            | 1996I  | 0.57  | 42.83 (0.21)            |
| 1992bs | 0.064 | 37.63 (0.18)            | 1996H  | 0.62  | 43.01 (0.15)            |
| 1992ae | 0.075 | 37.77 (0.19)            | 1997ck | 0.97  | 44.30 (0.19)            |

Table 2: The best-fitting parameters, i.e., when $\chi^2_\nu$ is minimum, for each type of space section of the cosmological constant model. $H_0$ is given in km/Mpc.s and $t_0$ in Gy.

| Parameters | Cosmological constant model with $k \leq 0$ | Cosmological constant model with $k > 0$ |
|------------|---------------------------------------------|------------------------------------------|
| $\chi^2_\nu$ | 0.7743                                      | 0.7539                                   |
| $H_0$      | 61.8                                        | 62.4                                     |
| $\Omega_k$ | 0.00                                        | -0.80                                    |
| $\Omega_m$ | 0.24                                        | 0.57                                     |
| $\Omega_\Lambda$ | 0.76                                      | 1.23                                     |
| $t_0$      | 16.5                                        | 15.7                                     |
| $q_0$      | -0.64                                       | -0.95                                    |

result estimated by other research groups [2, 3]. But a closed Universe gives a better fitting of the parameters, since $\chi^2_\nu$ is smaller, suggesting a different Universe, with positive curvature and younger. It is clear that small amounts of $\chi^2_\nu$ lead to totally different parameters, i.e., the parameter estimation is highly dependent on $\chi^2_\nu$.

The best-fitting parameters for the Chaplygin gas model are given by table 3. Slightly smaller values of $\chi^2_\nu$ favour the Chaplygin gas model over the cosmological constant model, independent of the spatial
Table 3: The best-fitting parameters, i.e., when $\chi^2$ is minimum, for each type of space section of the Chaplygin gas model. $H_0$ is given in km/Mpc.s, $\bar{A}$ in units of $c$ and $t_0$ in Gy.

section type. The best-of-all fitting suggests a closed Universe dominated by the Chaplygin gas, featuring a positive curvature, small value of $\Omega_m$, and the smallest age. In the case of open or flat spatial section, the best-fit gives a limit case of a flat Universe without cold dark matter content, just filled by the energy density of the Chaplygin gas.

Nevertheless, the $\chi^2$ best-fitting analysis has many limitations; the following questions cannot be answered. How much worse is one parameter set compared to other one, for example in table 2 or 3? What is the likelihood of a closed Universe using the Chaplygin gas model (table 3)? And the likelihood of the age of Universe to be in the range 13 to 15 Gy?
as explained in Ref. 2:

\[
p(\mu_0 \mid H_0, \Omega_{m0}, \Omega_\epsilon, \bar{A}) = \prod_i \frac{1}{\sqrt{2\pi(\sigma^2_{\mu_0,i} + \sigma^2_{mz,i})}} \exp\left[-\frac{(\mu_{0,i}^c - \mu_{0,i})^2}{2(\sigma^2_{\mu_0,i} + \sigma^2_{mz,i})}\right],
\]

which can be written using the \(\chi^2\) calculated previously, Eq. (14), because the product of exponentials is the exponential of the sum present in \(\chi^2\):

\[
p(\mu_0 \mid H_0, \Omega_{m0}, \Omega_\epsilon, \bar{A}) \propto \exp\left(-\frac{\chi^2}{2}\right).
\]

Finally, by using the relation (17), the probability of the parameters \((H_0, \Omega_{m0}, \Omega_\epsilon, \bar{A})\) conditional on the set of distance modulus \(\mu_0\) can be written in a normalized form:

\[
p(H_0, \Omega_{m0}, \Omega_\epsilon, \bar{A} \mid \mu_0) = \frac{\exp\left(-\frac{\chi^2}{2}\right)}{\int_{-\infty}^{\infty} dH_0 \int_{-\infty}^{\infty} d\Omega_{m0} \int_{-\infty}^{\infty} d\Omega_\epsilon \int_{-\infty}^{\infty} \exp\left(-\frac{\chi^2}{2}\right) d\bar{A}},
\]

where the integrals are performed on the allowed region of the parameter space.

Obviously, as the least-squares error \(\chi^2\) statistic is minimum, the probability is maximum. The values of the parameters \((H_0, \Omega_{m0}, \Omega_\epsilon, \bar{A})\) of this maximum are the most likely parameter values, when thought simultaneously.

But when we ask for the best value of one parameter, the \(\chi^2\) analysis is not robust. For example, the \(\chi^2\) minimum can be located in a narrow region with small values of \(\chi^2\) whereas another larger region can also have small values of \(\chi^2\), and this information is not taken into account to determine one of the parameters \((H_0, \Omega_{m0}, \Omega_\epsilon, \bar{A})\) for the \(\chi^2\) minimum.

With the Bayesian probability (20), or likelihood, or PDF, defined in the four-dimensional parameter space, we can construct a probability for one parameter by marginalizing, i.e., integrating with relation to the other parameters, the so-called marginal probability. For example, the marginal probability for \(H_0\) is defined as

\[
p(H_0 \mid \mu_0) = \int_{-\infty}^{\infty} d\Omega_{m0} \int_{-\infty}^{\infty} d\Omega_\epsilon \int_{-\infty}^{\infty} p(H_0, \Omega_{m0}, \Omega_\epsilon, \bar{A} \mid \mu_0) d\bar{A},
\]

where again the integration region is restricted to the parameter space with physical meaning. The peak of this PDF provides the maximum likelihood estimate of the parameter \(H_0\), which is usually different from the value of \(H_0\) for \(\chi^2\) minimum, because generally, \(p(H_0, \Omega_{m0}, \Omega_\epsilon, \bar{A} \mid \mu_0)\) is not a four-dimensional Gaussian PDF (or some distribution alike). This maximum likelihood estimate of \(H_0\) is a more robust estimate because the rest of the parameter space is the integrate region for each value of \(H_0\), so a larger region with high PDF values (but not the maximum) is not discarded.

The availability of a PDF for a parameter also allows us to calculate the likelihood of arbitrary hypothesis, for example: \(H_0\) greater than some value, \(H_0\) between some range, etc. They are performed by calculating the CDF (cumulative distribution function), i.e., the integral of the PDF over the specified region. The inverse problem also appears: for a given likelihood value, calculate the region in the parameter space which has a CDF equal to some likelihood value.

A well-known example is the credible or confidence region of 1\(\sigma\), 2\(\sigma\) and 3\(\sigma\) likelihoods. To define this type of region, we use the property that Gaussian PDF function, \(\sigma^{-1}(2\pi)^{-1/2}\exp(-x^2/2\sigma^2)\), when integrated over the region \(-1\sigma < x < 1\sigma\), gives a probability of approx. 68.27\%; with the region \(-2\sigma < x < 2\sigma\) the obtained likelihood is approx. 95.45\%; and the range \(-3\sigma < x < 3\sigma\) yields a CDF of approx. 99.73\%. So, \(\sigma\) values mean mean probabilities.

For an arbitrary PDF function \(p(x)\), the calculation of the 2\(\sigma\) (for example) credible or confidence region in one dimension is usually defined as follows: obtain the PDF level between zero and the peak PDF (when \(x = x_0\)) which have the intersection with \(p(x)\), \(x_+\) and \(x_-\), so that \(\int_{x_-}^{x_+} p(x)dx \simeq 95.45\). The estimation of \(x\) using 2\(\sigma\) credible region is described as \((x_0)^{x_+ - x_0}_{x_- - x_0}\), meaning that the PDF is peaked at
x_0 and the CDF of the region \( x_- < x < x_+ \) is equal to 2\( \sigma \) (95.45\%). See, for example, the thin line PDF in figure 1 with its 1\( \sigma \) and 2\( \sigma \) credible regions, PDF levels and intersections. In \( n \) dimensions, the intersection produces a \( n \)-dimensional region that becomes the integrate region, for an example in two dimensions, figure 3 shows the 1\( \sigma \), 2\( \sigma \) and 3\( \sigma \) credible regions.

The marginal joint probability (PDF) as function of two parameters, for example \( p(\Omega_m, \Omega_c) \), is an integral in the two dimensional parameter space of \((H_0, \bar{A})\),

\[
p(\Omega_m, \Omega_c | \mu_0) = \int_{-\infty}^{\infty} H_0 \int_{-\infty}^{\infty} p(H_0, \Omega_m, \Omega_c, \bar{A} | \mu_0) d\bar{A}.
\] (22)

Analogously, the marginalization method can be used to obtain, for example, \( p(H_0, \Omega_m, \Omega_c | \mu_0) \) from \( p(H_0, \Omega_m, \Omega_c, \bar{A} | \mu_0) \) by means of one integral over the \( \bar{A} \) parameter space (or marginalizing over \( \bar{A} \)).

Any quantity depending on the parameters can also have a probability for it. For example, in the case of \( t_0 \), the dynamical age of Universe, the PDF is obtained from:

\[
p(t_0 | \mu_0) = \int_{-\infty}^{\infty} dH_0 \int_{-\infty}^{\infty} d\Omega_m0 \int_{-\infty}^{\infty} d\Omega_c0 \int_{-\infty}^{\infty} p(H_0, \Omega_m0, \Omega_c0, \bar{A} | t_0, \mu_0) d\bar{A},
\] (23)

where the integrand gives the likelihood of the parameters which give a certain age \( t_0 \), so the four-dimensional integral is a sum of the likelihood of all possible parameters which produces a Universe with age \( t_0 \). To avoid the computation of a four-dimensional integral for each value \( t_0 \), it is better to calculate the age of Universe today for the four-dimensional parameter space and store the cumulative probabilities for each value of \( t_0 \).

### 4 Analyses of the estimated parameters

We have performed some long calculations using the Bayesian approach to obtain the parameter estimations and answers for some hypothesis. In the Chaplygin gas case, the five parameters, \((H_0, \Omega_k0, \Omega_m0, \Omega_c0, \bar{A})\), the age of the Universe \( t_0 \) and the deceleration parameter \( q_0 \) were estimated with a central value and a 2\( \sigma \) (95.45\%) credible region. Each independent parameter estimate used a marginal likelihood of the type of Eq. (21), where three-dimensional integrals are computed for each value of the parameter (and in the integrand, \( \chi^2 \) needs the calculation of about one hundred numerical integrals, or four times the number of supernovae).

An ideal calculation to compute the \( n \)-dimensional integrals would include an infinite number of samples of a parameter space with infinite volume, but in practical estimations we chose a finite region of the parameter space (such that outside it the probabilities are almost null) and a finite number of samples. Considering \( n \) samples for each parameter dimension, the estimation of one parameter needs \( n^4 \) computations of \( \chi^2 \) (as there are four independent parameters), and by proper marginalization the PDF of each parameter is calculated. For this article, \( n \approx 30 \), and the number of computations of \( \chi^2 \) is at least \( n^4 \approx 10^6 \) just for the Chaplygin gas model (some calculations are repeated because it is not worth storing \( 10^6 \) results due to computational memory constraints), or \( 100n^4 \approx 10^8 \) numerical integrals of the type in Eq. (10). We also calculate \( n^4 \approx 10^6 \) times the age of Universe today to discard parameter space regions which are not physical \( (t_0 \) is not real, etc) and to estimate \( t_0 \). Likewise, the \( q_0 \) estimate demands \( n^4 \) calculations of Eq. (12) for \( q_0 \). The probability of eternally expanding Universe consists of \( n^4 \) calculations of the roots of Eq. (9). The other cases listed in tables 4 and 5 demands \( n^3 \) or \( n^2 \) calculations, instead of \( n^4 \).

Tables 4 and 5 summarizes the Bayesian parameter estimation results for seven different cases. As expected, the central values of the Bayesian parameter estimation and the best-fitting parameters of tables 2 or 3 are different. In most cases, the likelihood functions (PDF) of the parameters have a shape similar to a Gaussian function, but with some asymmetry, consequently they are not shown here because the central value and credible region description are enough to realize the PDF behaviour.
Table 4: The estimated parameters for the cosmological constant model, using the Bayesian analysis to obtain the peak of the marginal probability and the credible region for each parameter. $H_0$ is given in km/Mpc.s, $\dot{A}$ in units of c and $t_0$ in Gyr.

| Parameters | Cosmological constant model | Cosmological constant model with $k = 0$ | Cosmological constant model with $\Omega_{m0} = 0$ |
|------------|-----------------------------|------------------------------------------|-----------------------------------------------|
| $H_0$      | $62.2 \pm 3.1$              | $62.2 \pm 3.1$                           | $61.4 \pm 3.0$                                |
| $\Omega_{k0}$ | $-0.80^{+1.45}_{-1.34}$   | $0$                                      | $0.55^{+0.50}_{-0.37}$                        |
| $\Omega_{m0} > 0$ | $0.58^{+0.56}_{-0.58}$ | $0.24^{+0.21}_{-0.16}$                   | $0$                                           |
| $\Omega_{A}$      | $1.21^{+0.81}_{-0.91}$   | $0.76^{+0.16}_{-0.21}$                   | $0.45^{+0.37}_{-0.50}$                        |
| $t_0$        | $15.4^{+3.4}_{-1.9}$       | $15.5^{+3.3}_{-1.9}$                     | $19.3^{+5.3}_{-3.1}$                          |
| $q_0$        | $-0.99^{+0.75}_{-0.52}$   | $-0.75^{+0.44}_{-0.14}$                  | $-0.47^{+0.50}_{-0.37}$                       |
| $p(\Omega_{k0} < 0)$ | $81.6\%(1.33\sigma)$ | $-$                                      | $0.03\%$                                      |
| $p(\Omega_{A} > 0)$ | $99.8\%(3.03\sigma)$ | $99.999993\%(4.86\sigma)$               | $95.7\%(2.02\sigma)$                         |
| $p(q_0 < 0)$     | $99.6\%(2.86\sigma)$   | $99.97\%(3.64\sigma)$                   | $96.1\%(2.06\sigma)$                         |
| $p(\dot{a} > 0)$   | $93.7\%(1.86\sigma)$   | $99.1\%(2.61\sigma)$                    | $37.8\%(0.49\sigma)$                         |

Table 5: The estimated parameters for the Chaplygin gas model, using the Bayesian analysis to obtain the peak of the marginal probability and the credible region for each parameter. $H_0$ is given in km/Mpc.s, $\dot{A}$ in units of c and $t_0$ in Gyr.

| Parameters | Chaplygin gas model with $k = 0$ | CGM with $\Omega_{m0} = 0$ | CGM with $k = 0$ and $\Omega_{m0} = 0.04$ |
|------------|---------------------------------|---------------------------|-------------------------------------------|
| $H_0$      | $62.1^{+3.3}_{-3.4}$            | $61.4 \pm 2.8$            | $61.9 \pm 2.8$                            | $61.8 \pm 2.8$ |
| $\Omega_{k0}$ | $-0.84^{+1.51}_{-1.23}$   | $0$                                      | $0.17^{+0.83}_{-1.58}$                     | $0$ |
| $\Omega_{m0} > 0$ | $0.00^{+0.82}_{-0.82}$  | $0.00^{+0.35}_{-0.00}$       | $0$                                       | $0.04$ |
| $\Omega_{A} > 0$   | $1.40^{+1.15}_{-1.36}$   | $1.0^{+0.00}_{-0.35}$       | $0.83^{+1.59}_{-0.83}$                     | $0.96$ |
| $0 \leq \dot{A} \leq 1$ | $0.93^{+0.07}_{-0.21}$ | $0.93^{+0.07}_{-0.20}$       | $0.78^{+0.22}_{-0.19}$                     | $0.87^{+0.13}_{-0.18}$ |
| $t_0$        | $14.2^{+2.8}_{-1.5}$          | $13.1^{+0.5}_{-1.0}$        | $14.5^{+2.9}_{-1.7}$                       | $14.8^{+2.4}_{-1.5}$ |
| $q_0$        | $-0.98^{+1.02}_{-0.62}$       | $-0.65^{+0.32}_{-0.27}$     | $-0.56^{+0.61}_{-0.26}$                    | $-0.83^{+0.28}_{-0.16}$ |
| $p(\Omega_{k0} < 0)$ | $84.0\%(1.41\sigma)$ | $-$                                      | $55.7\%(0.77\sigma)$                     | $-$ |
| $p(q_0 < 0)$     | $97.5\%(2.24\sigma)$   | $99.96\%(3.55\sigma)$       | $95.2\%(1.98\sigma)$                      | $99.997\%(4.17\sigma)$ |
| $p(\dot{a} > 0)$   | $93.9\%(1.87\sigma)$   | $99.7\%(2.99\sigma)$        | $63.8\%(0.91\sigma)$                      | $100\%$ |
4.1 The Hubble parameter $H_0$

The parameter estimation of $H_0$ is almost the same for all cases, around 62 km/Mpc.s with a narrow credible region of ±3 km/Mpc.s. This estimation is compatible with other SNe Ia analyses [2, 3]. Hence, the choice of Chaplygin gas model or $\Lambda CDM$ does not have any significant consequence on the $H_0$ estimates we have made.

4.2 The curvature density parameter $\Omega_{k0}$

Likewise, the curvature density parameter $\Omega_{k0}$ estimation is almost the same, $-0.84^{+1.51}_{-1.23}$ and $-0.80^{+1.45}_{-1.34}$, assuming the Chaplygin gas model or $\Lambda CDM$, respectively, and in both cases a closed Universe ($\Omega_{k0} < 0$) is preferred at 84.0% (1.41σ) and 81.6% (1.33σ) confidence levels, also respectively, in agreement with the conclusions of many other previous works on the subject [2, 3]. In the $CGM$, a spatially flat Universe is ruled out at 67.5% (0.98σ) confidence level, i.e., the PDF of $\Omega_{k0} = 0$ is smaller than the PDF of $-1.42 < \Omega_{k0} < 0$ and this region has a CDF of 67.5% (0.98σ). The $\Lambda CDM$ model shows a similar behaviour, a spatially flat Universe is ruled out at 63.3% (0.90σ) level, i.e., the PDF of $\Omega_{k0} = 0$ is smaller than the PDF of $-1.46 < \Omega_{k0} < 0$.

But assuming the hypothesis that $\Omega_m = 0$, describing a Universe empty of cold dark matter and totally dominated by the Chaplygin gas, we have quite different estimates. For the $CGM$, the Bayesian analysis gives $\Omega_{k0} = 0.17^{+0.83}_{-1.58}$ with a closed Universe favoured at 55.7% (0.77σ), and a flat Universe is strongly preferred at 83.3% (1.38σ) confidence level, i.e., the PDF of $\Omega_{k0} = 0$ is smaller than the PDF of $0 < \Omega_{k0} < 0.30$ and this region has a small CDF of 16.7%.

The same hypothesis, $\Omega_m = 0$, applied to the $\Lambda CDM$ model, leads to the estimation $\Omega_{k0} = 0.55^{+0.50}_{-0.37}$, so now an open Universe is quite favourable at 99.97% (3.60σ) confidence level. A spatially flat Universe is ruled out at 99.92% (3.36σ) confidence level, i.e., the PDF of $\Omega_{k0} = 0$ is smaller than the PDF of $0 < \Omega_{k0} < 1.43$.

Therefore, the $\Omega_m = 0$ case clearly discriminates the Chaplygin gas and the $\Lambda CDM$ models, as their estimates are quite different. This discrimination is still enhanced if the additional hypothesis of spatially flat Universe is assumed (to be compatible with inflationary models of the primordial Universe, or some CMB estimations, for example), when the $CGM$ agrees at a high confidence level while $\Lambda CDM$ is ruled out.

4.3 The cold dark matter density parameter $\Omega_m$

But the cold dark matter density parameter $\Omega_m$ depends quite on the cosmological models. Figure 11 clearly shows that the PDF for the Chaplygin gas and the $\Lambda CDM$ models behave differently, $\Omega_m = 0.00^{+0.82}_{-0.00}$ and $\Omega_m = 0.58^{+0.56}_{-0.58}$, respectively. So the Chaplygin gas model favours small values of $\Omega_m$ at a high level of confidence, for example, $\Omega_m < 0.39$ at 1σ (68.27%) confidence level, while $\Lambda CDM$ estimates $\Omega_m < 0.39$ with a probability of 31.1% (0.40σ), or, $\Omega_m = 0.58^{+0.30}_{-0.28}$ at 1σ level.

While the most favoured value of $\Omega_m$ is zero for the $CGM$, the cosmological constant model rules out $\Omega_m = 0$ at 89.5% (1.62σ) confidence level, i.e., the PDF of $\Omega_m = 0$ (equal to 0.56) is smaller than the PDF of $0 < \Omega_m < 0.99$ which has a CDF of 89.5%.

On the hypothesis that the Universe is spatially flat, the $CGM$ and $\Lambda CDM$ have narrower confidence regions, see tables 2 and 3. It is worth noting that the $CGM$ continues to indicate a $\Omega_m$ peaked at zero, but the $\Lambda CDM$ model rules out $\Omega_m = 0$ completely, at 99.94% (3.44σ) confidence level (when $0 < \Omega_m < 0.63$ the PDF is greater than the PDF of $\Omega_m = 0$).

Based on the SNe Ia data, we can conclude that, assuming the Chaplygin gas model, the estimated values of $\Omega_m$ are decreased (with respect to $\Lambda CDM$), or if independent estimations of $\Omega_m$ suggest low values then the Chaplygin gas model is favoured over the $\Lambda CDM$ model.

4.4 The Chaplygin gas density parameter $\Omega_c$

The Chaplygin gas density parameter $\Omega_c$ estimation is quite spread, i.e., it has a large credible region, $\Omega_c = 1.40^{+1.15}_{-1.16}$. Only for the particular case of a spatially flat Universe ($k = 0$) the estimation of $\Omega_c$ is
Figure 1: The PDF of $\Omega m_0$ for the Chaplygin gas model (bold lines) and the cosmological constant model (thin lines). The solid lines are the PDF, the $2\sigma$ (95.45%) credible regions are given by dashed lines and the $1\sigma$ (68.27%) regions are delimited by dotted lines. The Chaplygin gas case has maximum probability 2.55 for $\Omega m_0 = 0.0$, the $2\sigma$ credible region reads $0 \leq \Omega m_0 < 0.82$ with probability level 0.26, and the $1\sigma$ credible region is given by $0 \leq \Omega m_0 < 0.39$ with PDF level 1.10. The $\Lambda CDM$ has probability peak of 1.11 when $\Omega m_0 = 0.58$, the $2\sigma$ credible region reads $0 \leq \Omega m_0 < 1.13$ with probability level 0.31, and the $1\sigma$ credible region is given by $0.20 \leq \Omega m_0 < 0.88$ with PDF level 0.80. This graphics clearly shows that small values of $\Omega m_0$ are more preferred by the Chaplygin gas model. For the $\Lambda CDM$ model obtained from the Chaplygin gas model as $\bar{A} = 1$, the dark energy density $\Omega_\Lambda$ is positive at a $3.3\sigma$ (99.8%) confidence level, and the estimation $\Omega_\Lambda = 1.21 \pm 0.81$ becomes more peaked for a spatially flat Universe, $\Omega_\Lambda = 0.76 \pm 0.16$. See table 2 for more estimates.

4.5 $\bar{A}$, the sound velocity of the Chaplygin gas

The analysis of the parameter $\bar{A}$ is very conclusive, because $\Lambda CDM$ is a special case of the Chaplygin gas model when $\bar{A} = 1$, therefore we can compare the probabilities of each cosmological model. Indeed, figure 2 shows that the best estimation reads $\bar{A} = 0.93 \pm 0.07$, with peak probability level 4.30, while the $\Lambda CDM$ limit ($\bar{A} = 1$) has a slightly lower likelihood of 4.12. More specifically, the Chaplygin gas model is the more preferred model over the range $0.87 \leq \bar{A} < 1$ because the PDF is greater than 4.12, and this region represents 55.3% (0.76σ) of the total probability. So, even if there is no additional hypothesis about the curvature parameter $\Omega_k$ or the cold dark matter density parameter $\Omega m_0$, the Bayesian analysis explicitly estimates that the Chaplygin gas model is more likely than $\Lambda CDM$ at 0.76σ (55.3%) confidence level, although the peak likelihood is just 4% greater than $\Lambda CDM$ likelihood.

Under the hypothesis that $\Omega m_0 = 0$, so the Chaplygin gas is the only content of the Universe, figure 2 gives $\bar{A} = 0.78 \pm 0.22$ and indicates that the Chaplygin gas model has a peak likelihood of 4.86, quite greater than the $\Lambda CDM$ model probability level of 0.87 (for $\bar{A} = 1$), and on a broader region $0.67 \leq \bar{A} < 1$ the PDF is greater than 0.87. The parameter $\bar{A}$ is inside this region with likelihood of 91.8% (1.74σ), i.e., the Chaplygin gas model is preferred over the $\Lambda CDM$ model at 1.74σ (91.8%) confidence level.

Assuming now the hypothesis of a flat Universe, $\Omega k_0 = 0$, we obtain $\bar{A} = 0.93 \pm 0.07$ with a peak
The likelihood of 5.12, greater than the $\Lambda CDM$ model probability level of 4.18 (for $\bar{A} = 1$). The region $0.87 \leq \bar{A} < 1$ has a PDF greater than 4.18, which has a CDF of 63.5% (0.91$\sigma$), i.e., the Chaplygin gas model is preferred over $\Lambda CDM$ at 63.5% (0.91$\sigma$) confidence level.

In the framework of the cosmological model estimations based on SNe Ia, the baryonic matter has the same behaviour of the cold dark matter, so we can assume a typical value of 0.04 for the baryonic density parameter by setting $\Omega_{m0} = 0.04$. Still imposing the hypothesis of a flat Universe, the last column of table 3 shows that $\bar{A} = 0.87^{+0.13}_{-0.18}$ with a maximum PDF of 5.56 and the region $0.59 \leq \bar{A} < 1$ having greater PDF than the PDF of $\Lambda CDM$ limit ($\bar{A} = 1$), 0.13. The CDF of this region means that the CGM is preferred over $\Lambda CDM$ at 99.4% (2.72$\sigma$) confidence level, under the hypothesis $\Omega_{k0} = 0$ and $\Omega_{m0} = 0.04$. Not shown in the $\Lambda CDM$ model parameter estimates read, $H_0 = 64.4^{+2.2}_{-2.1} \text{ km/Mpc.s}$ and $t_0 = 24.2^{+0.8}_{-0.6} \text{ Gy}$, with an age of the Universe excessively higher than those obtained through other astronomical estimations.

Combining both hypothesis, $\Omega_{k0} = 0$ and $\Omega_{m0} = 0$, the parameter estimation is quite physically acceptable for the CGM. Not show explicitly in table 3 we have obtained $H_0 = 61.9^{+2.8}_{-2.8} \text{ km/Mpc.s}$, $\bar{A} = 0.89^{+0.15}_{-0.18}$, $t_0 = 14.7^{+2.1}_{-1.5} \text{ Gy}$ and $q_0 = -0.81^{+0.32}_{-0.13}$ with $p(q_0 < 0) = 99.996% (4.12\sigma)$. The Chaplygin gas model is strongly favoured over the $\Lambda CDM$ model at 99.93% (3.38$\sigma$) confidence level, as the PDF of $A = 0.85$ is equal to 5.53 and the likelihood of $A = 1$ is 0.02, so a large region $0.47 \leq \bar{A} < 1$ has PDF levels greater than 0.02. Moreover, the $\Lambda CDM$ model produces totally unphysical estimates:

![Graph showing the PDF of $\bar{A}$ for the Chaplygin gas model (bold lines) and the same with $\Omega_{m0} = 0$ (thin lines). The solid lines are the PDF, the 2$\sigma$ (95.45%) credible regions are given by dashed lines and the 1$\sigma$ (68.27%) regions are delimited by dotted lines. The non-restricted case has maximum probability 4.30 for $\bar{A} = 0.93$, the 2$\sigma$ credible region reads $0.72 \leq \bar{A} < 1$ with probability level 0.85, and the 1$\sigma$ credible region is given by $0.84 \leq \bar{A} < 1$ with PDF level 3.77. The probability of the $\Lambda CDM$ obtained when $\bar{A} = 1$ is 4.12, so the Chaplygin gas model is more likely (has greater PDF) in the range $0.87 \leq \bar{A} < 1$, which has a CDF of 55.3% (0.76$\sigma$). The $\Omega_{m0} = 0$ case has probability peak of 4.86 when $\bar{A} = 0.78$, the 2$\sigma$ credible region reads $0.59 \leq \bar{A} < 1$ with probability level 0.25, the 1$\sigma$ credible region is given by $0.71 \leq \bar{A} < 0.89$ with PDF level 3.39, and the $\Lambda CDM$ has a PDF level of only 0.87 which suggests that the Chaplygin gas model is strongly preferred if there is no cold dark matter ($\Omega_{m0} = 0$) content in the Universe, because the range $0.67 \leq \bar{A} < 1$ has a CDF of 91.8% (1.74$\sigma$) such that the PDF level is greater than 0.87.](image-url)
\[ t_0 = 368^{+13}_{-12} \times 10^3 \text{Gy}. \]

### 4.6 The present age \( t_0 \) of the Universe

Looking at the dynamical age of the Universe today, \( t_0 \), given in tables 4 and 5 we verify that the Chaplygin gas model estimates lower values of \( t_0 \) as well as narrower credible regions \( (t_0 = 14.2^{+2.5}_{-1.8}) \), when compared to the estimations of the \( \Lambda \text{CDM} \) model \( (t_0 = 15.4^{+3.4}_{-1.9}) \). By means of independent estimations of \( t_0 \) from different observations (globular agglomerates, etc), the Chaplygin gas model would be more preferred if, for example, these estimations give \( t_0 \) between 13Gy to 14Gy.

If we impose some constraints, \( k = 0 \) or \( \Omega_m = 0 \), we still have lower estimates for \( t_0 \) of the \( \text{CGM} \) than the \( \Lambda \text{CDM} \) model estimations. The latter are strongly sensible to the value of \( \Omega_m \): for example, if \( k = 0 \), we obtain \( t_0 = 24.2^{+0.8}_{-0.9} \) Gy for \( \Omega_m = 0.04 \) and totally unphysical value of \( t_0 = 368^{+13}_{-12} \times 10^3 \) Gy for \( \Omega_m = 0 \). Quite different, the \( \text{CGM} \) produces \( t_0 \) values with a weak dependence on \( \Omega_{k0} \) and \( \Omega_m \).

### 4.7 The deceleration parameter \( q_0 \) of the Universe

All the estimations based on the Chaplygin gas model or the cosmological constant model suggest an accelerating Universe today, i.e., \( q_0 < 0 \), with a high confidence level (at least more than 95%).

### 4.8 The eternally expanding Universe

The probability of having a eternally expanding Universe, given here by demanding \( \dot{a} > 0 \) during the future evolution of the Universe, is estimated to be almost one (at least more than 99%), excepting in the \( \Omega_m = 0 \) case for both the \( \text{CGM} \) and \( \Lambda \text{CDM} \) when \( p(\dot{a} > 0) \) is far from unity, see tables 4 and 5.

### 4.9 Two-dimensional analysis in the \((\Omega_m, \Omega_\Lambda)\) and \((\Omega_m, \Omega_\sigma)\) parameter space

Finally, it is useful to analyze the two-dimensional joint PDF, for example, in the parameter space of \((\Omega_m, \Omega_\Lambda)\), where the PDF \( p(\Omega_m, \Omega_\Lambda \mid \mu_0) \) is given by the integral of \( p(H_0, \Omega_m, \Omega_\Lambda \mid \mu_0) \) over the \( H_0 \) space. The well-known credible regions for the \( \Lambda \text{CDM} \) model are reproduced here, see figure 3. It is worth noting that the probability peak gives \((\Omega_m, \Omega_\Lambda) = (0.60, 1.27)\) which are different from the central values listed in table 4 \( \Omega_m = 0.58 \) and \( \Omega_\Lambda = 1.21 \), due to the integration over the \( \Omega_\Lambda \) space to obtain \( p(\Omega_m \mid \mu_0) \) from \( p(\Omega_m, \Omega_\Lambda \mid \mu_0) \), and analogously for \( p(\Omega_\Lambda \mid \mu_0) \). The interpretation is the following : \((\Omega_m, \Omega_\Lambda) = (0.60, 1.27)\) means that these values are the most likely simultaneously values of \((\Omega_m, \Omega_\Lambda)\) independent of the parameter \( H_0 \); while \( \Omega_m = 0.58 \) is the most likely value of \( \Omega_m \) independent of the other parameters \((H_0, \Omega_\Lambda)\), and \( \Omega_\Lambda = 1.21 \) is the most likely value independent of the other parameters \((H_0, \Omega_m)\). Figure 4 shows \( p(\Omega_m \mid \mu_0) \) which is obtained by integrating \( p(\Omega_m, \Omega_\Lambda \mid \mu_0) \) over the \( \Omega_\Lambda \) space.

The contour curves are defined such that the enclosed regions have a cumulative probability equal to \( 1\sigma \) (68.27\%), \( 2\sigma \) (95.45\%) and \( 3\sigma \) (99.73\%) credible levels. For example, this means simply that \( \Omega_m \) and \( \Omega_\Lambda \) have values simultaneously inside the \( 2\sigma \) (95.45\%) region with a likelihood of 95.45\%. The line \( \Omega_m + \Omega_\Lambda = 1 \) represents a spatially flat Universe \((k = 0, \Omega_{k0} = 0)\), the parameter space above corresponds to a closed Universe \((k > 0, \Omega_{k0} < 0)\), and below, to an open Universe \((k < 0, \Omega_{k0} > 0)\). As the closed Universe region clearly dominates and presents high PDF values, a closed Universe is estimated at 1.33\( \sigma \) (81.6\%) confidence level.

For the case of the Chaplygin gas model, the PDF \( p(\Omega_m, \Omega_\sigma \mid \mu_0) \) is a two-dimensional integral given by Eq. (22), and figure 4 displays its behaviour with credible regions quite different from the \( \Lambda \text{CDM} \). The PDF peak is now located at \((\Omega_m, \Omega_\sigma) = (0.0, 0.85)\), contrasting with the central values \( \Omega_m = 0.00 \) and \( \Omega_\sigma = 1.40 \) of table 5 due to the Bayesian integrations. Of course, the integral of \( p(\Omega_m, \Omega_\sigma \mid \mu_0) \) over the \( \Omega_\sigma \) space yields \( p(\Omega_m \mid \mu_0) \), shown in figure 4. Once more, a spatially closed Universe is strongly favoured at 1.41\( \sigma \) (84.0\%) confidence level, despite the PDF peak being inside the spatially open Universe region.

These two graphics also emphasize the totally different likelihoods for \((\Omega_{k0}, \Omega_m) = (0, 0.04)\) in the \( \text{CGM} \) and \( \Lambda \text{CDM} \) model. This point corresponds to \((\Omega_{m0}, \Omega_\Lambda) = (0.04, 0.96)\) in figure 3 for \( \Lambda \text{CDM} \),
with a PDF level of 0.02, well below the PDF peak 2.07, so a large region of the \((\Omega_m, \Omega_{\Lambda})\) parameter space has greater PDF than 0.02 yielding a CDF of 99.6% (2.91\sigma), i.e., the simultaneously hypothesis of flat Universe and \(\Omega_m = 0.04\) (representing a typical baryonic density parameter of 0.04) is ruled out at 99.6% (2.91\sigma) assuming the \(\Lambda CD M\). The limit case of \((\Omega_k, \Omega_m) = (0, 0)\) gives a confidence level of 99.93% (3.39\sigma).

Figure 3 clearly shows that the point \((\Omega_m, \Omega_{\Lambda}) = (0.04, 0.96)\) is near the maxima of PDF, their PDF levels are 1.41 and 1.47, respectively. Therefore, the region with smaller PDF than 1.41 is almost the whole of the parameter space, and its CDF is equal to 98.6%, i.e., for the \(CD M\) the simultaneously hypothesis of flat Universe and \(\Omega_m = 0.04\) is favoured at 98.6% (2.45\sigma) confidence level. In the same
way, the case of $(\Omega_k, \Omega_m) = (0, 0)$ is favoured at a confidence level of 99.3% (2.69σ).

## 5 Conclusion

In the present paper constraints on a Chaplygin gas model using type Ia supernovae data were settled out using a Bayesian statistics. The model contains as matter content the Chaplygin gas and a pressureless fluid, whose density parameters are $\Omega_c$ and $\Omega_m$ respectively. The curvature term $\Omega_k$, the sound velocity of the Chaplygin gas $\bar{A}$ and the Hubble parameter $H_0$ are also free parameters. At the 2σ level, the
results indicate $H_0 = 62.1^{+3.3}_{-3.4}$ km/Mpc.s, $\Omega_{k0} = -0.84^{+1.51}_{-1.23}$, $\Omega_{m0} = 0.0^{+0.82}_{-0.0}$, $\Omega_{c0} = 1.40^{+1.15}_{-1.16}$. This results must be compared with those obtained replacing the Chaplygin gas by the cosmological constant ($\bar{\Lambda}$ model). The $\Lambda$CDM model is preferred with respect to $\Lambda$CDM at a competitive level with respect to $\Lambda$CDM model at 99.4% confidence level. In general, the predicted value for the sound velocity of the Chaplygin gas is close to the cosmological constant value ($\bar{\Lambda} = 1$), but after integrating on the various parameters the $\Lambda$CDM case becomes quite disfavoured.

Other important differences between the CGM and $\Lambda$CDM concern the pressureless matter parameter and the age of the Universe. For the former, the CGM favours a zero value for this cold dark matter component, in agreement with the idea that the Chaplygin gas may unify dark matter and dark energy, and in contrast with $\Lambda$CDM where a non-negligible fraction of the matter in the Universe must appear under the form of dark matter.

A very remarkable discrimination between the CGM and $\Lambda$CDM occurs when the pressureless matter parameter represents the baryonic matter, $\Omega_{b0} = 0.04$ and the Universe is spatially flat. This case is favoured at 98.6% of confidence level for the CGM, while for the $\Lambda$CDM model it is excluded with 99.6% of confidence level. This result renders the CGM quite attractive in view of the predictions of almost all primordial inflationary scenarios, which lead to a flat Universe, and also in view of the unification program for dark energy and dark matter through the Chaplygin gas.

One of the most important conclusions of this work is that the predicted value for the dark matter parameter $\Omega_{m0}$ is peaked in the zero value. This reinforces the idea that the Chaplygin gas may unify dark matter and dark energy as its behaviour in terms of the scale factor suggests. This unification program has been criticized because, besides some other reasons, the matter power spectrum in a pure CGM exhibits oscillations that are not observed in the recent 2dFGRS data, and the phase space of possible configuration is highly concentrated around the cosmological constant value. This seems to be a strong argument against the unified model. However, we must remark that the authors of Ref. 20 employ an one fluid model. Even if the CGM may unify dark matter and dark energy, baryons exists anyway, even if in a small fraction ($\Omega_{b0} \sim 0.04$). This may seems to be irrelevant, but the behaviour of a two fluid models is generally very different from an one fluid model (see, for example, Ref. 22), and this point deserves, in our opinion, a deeper analysis. In what concerns the results of Ref. 20, we observe that the crossing of all observational data, including type Ia supernovae, may put the Chaplygin gas again at a competitive level with respect to $\Lambda$CDM. Our results indicate clearly that the Chaplygin gas is preferred with respect to $\Lambda$CDM if only type Ia supernovae data is taken into account.

The CGM predicts a Universe younger than $\Lambda$CDM, but still in agreement with other astronomical data, in particular with the age of globular clusters. Both models favour a closed Universe, with a very small difference for the value of the curvature parameter. On the other hand, the value of the Hubble parameter is essentially the same in both models, as well as the deceleration parameter $q_0$, for which the data indicate a highly negative value, near $-1$. The value of the deceleration parameter becomes less negative for a flat Universe and when $CDM$ is absent.

In Ref. 48, statistics of gravitational lenses where used to constrain the proportion of dark matter and Chaplygin gas in a flat Universe, and the authors found $\Omega_{m0} \sim 0.2$. We remark also that our results are consistent with the particular case described in Ref. 19.

These results may be compared with those recently obtained from the WMAP observatory for the spectrum of the anisotropy of the cosmic microwave background radiation. Using also the 2dFGRS and Lyman-$\alpha$ forest data, and fixing a $\Lambda$CDM model, it has been obtained that $\Omega_{m0}h^2 = 0.133 \pm 0.006$, $h = 0.72 \pm 0.03$, where $H_0 = 100h$Mpc/km.s, and $\Omega_{k0} = 0.02 \pm 0.02$. We notice that the dark matter component is much smaller than the value deduced from the supernovae data for $\Lambda$CDM ($\Omega_{m0} = 0.58^{+0.06}_{-0.08}$), and at same time the Hubble parameter is greater. The smaller uncertainty on the dark matter component is natural in this case, since the CMB spectrum gives a better estimation on the
total matter in the Universe through the position of the first acoustic peak, and it is verified that an almost flat Universe is favoured by the data, for the $\Lambda$CDM model, in contrast with what the supernovae data indicate for the same theoretical framework.

For the CGM there is a large uncertainty also in the estimation of the curvature parameter if only supernovae data are used. But, its quite remarkable that, if the dark matter density is fixed to zero in this model the type Ia supernovae data indicates an almost flat Universe: $\Omega_{k0} = 0.17^{+0.83}_{-1.58}$. It must be stressed that a proper comparison between the WMAP results with our analysis for the Chaplygin gas requires that the WMAP data must be analysed using the CGM, what is one of the natural extension of the present work.

We hope that, in the future, more SNe Ia data (from the SNAP project [39, 40], etc) with small observational errors will impose stringent constraints on the parameter estimation for the CGM and the $\Lambda$CDM model, so the parameter credible regions become narrow enough to rule out one of these cosmological models. For example, the estimation of the parameter $\bar{A}$ could favour one of the models with high confidence level ($>2\sigma$), the estimation of $\Omega_{m0}$ and $t_0$ could be incompatible with other independent and well accepted estimations therefore excluding some cosmological models, etc.

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