Vanishing relaxation time dynamics of the Jordan Moore-Gibson-Thompson equation arising in nonlinear acoustics

Marcelo Bongarti, Sutthirut Charoenphon and Irena Lasiecka
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152 USA

Abstract
The (third-order in time) JMGT equation [15, 7] is a nonlinear (quasi-linear) Partial Differential Equation (PDE) model introduced to describe a nonlinear propagation of sound in an acoustic medium. The important feature is that the model avoids the infinite speed of propagation paradox associated with a classical second order in time equation referred to as Westervelt equation. Replacing Fourier’s law by Maxwell-Cattaneo’s law gives rise to the third order in time derivative scaled by a small parameter \( \tau \), the latter represents the thermal relaxation time parameter and is intrinsic to the medium where the dynamics occur. In this paper we provide an asymptotic analysis of the third order model when \( \tau \to 0 \). It is shown that the corresponding solutions converge in a strong topology of the phase space to a limit which is the solution of Westervelt equation. In addition, rate of convergence is provided for solutions displaying higher order regularity. This addresses an open question raised in [1], where a related JMGT equation has been studied and weak star convergence of the solutions when \( \tau \to 0 \) has been established. Thus, our main contribution is showing strong convergence on infinite time horizon, along with related rates of convergence valid on a finite time horizon. The key to unlocking the difficulty owns to a tight control and propagation of the “smallness” of the initial data in carrying the estimates at three different topological levels. The rate of convergence allows one then to estimate the relaxation time needed for the signal to reach the target. The interest in studying this type of problems is motivated by a large array of applications arising in engineering and medical sciences.

Keywords: Jordan-Moore-Gibson-Thompson equation; third-order evolutions; strong convergence of nonlinear flows; rate of convergence; uniform exponential decays; acoustic waves.

1 Introduction
Propagation of nonlinear waves in an acoustic environment has been a topic of great interest and activities. Broad range of physical applications including ultrasound technology, welding, lithotrips, thermotherapy, ultrasound cleaning, and sonochemistry [17, 8, 12, 29, 14] are just a few examples. In view of this, it is not surprising that mathematical models are of great interest and became a highly active field of research. We will be considering the Jordan-Moore-Gibson-Thompson (JMGT) equation, which, although simple, does display several mathematical intricacies of interest in PDE area and it is representative of the underlying physics. The JMGT equation written in the variable \( \psi \), which denotes velocity potential, can be recasted as

\[
\tau \psi_{ttt} + \psi_{tt} - c^2 \Delta \psi - b^\tau \Delta \psi_t = \frac{d}{dt}[k(\psi_t)^2]
\]

(1.1)
where $c$ denotes the speed of sound, $\delta > 0$ diffusivity of the sound ($b^r = \delta + \tau c^2$), $\tau > 0$ thermal relaxation time parameter and $k$ a parameter of nonlinearity, see [15, 16, 17] and references therein for derivation of the model. Rewriting equation in terms of the pressure $u \sim \rho_0 \psi_1$ ($\rho_0$= mass density) leads to

$$\tau u_{ttt} + (1 - 2ku)u_{tt} - c^2 \Delta u - b\Delta u_t = 2k(u_t)^2. \quad (1.2)$$

The presence of the constant $\tau$, which accounts for finite speed of propagation, removes the so called infinite speed of propagation paradox, see [12, 29, 6, 5]. In fact, when $\tau = 0$ the corresponding PDE becomes the Westervelt equation – which is of parabolic type – and with $\tau > 0$ the system is hyperbolic-like and its linear version represents a group. Since the parameter $\tau > 0$ is relatively small, it is essential to understand the effects of diminishing values of relaxation. This is a particularly delicate issue since the $\tau$-dynamics is governed by a generator which is singular as $\tau \rightarrow 0$. The goal of this paper is to consider the vanishing parameter $\tau \rightarrow 0$ and its consequences on the resulting evolution. Accordingly we will show that the Westervelt equation,

$$(1 - 2ku)u_{tt} - c^2 \Delta u - \delta \Delta u_t = 2k(u_t)^2 \quad (1.3)$$

is a strong limit of the JMGT equation, when the relaxation parameter vanishes. In this spirit we shall answer an open question raised in recent manuscript [1], where weak star convergence has been shown for a related model. In addition, the quantitative rate of convergence of the corresponding solutions will also be derived. As we shall see, the key in unlocking the difficulty is a good control of topological smallness of the initial data along with an appropriate calibration of the estimates at various topological levels.

A mathematical interest in third order equations stems also from the fact that an existence of semigroup for the linearization fails when the diffusivity $b = 0$ [13]. On the other hand, on physical grounds the parameter $\tau$ accounts for physically relevant finite speed of propagation of the waves. Thus, the analysis needs to reconcile "small" amplitude waves with the limit process. The main question to contend with is "how small is small ". This leads to a string of estimates with a minimal requirement imposed on the "smallness". The latter is the key in unlocking the difficulty with strong convergence.

### 1.1 The model and related literature

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ ($n = 2, 3$) with a $C^2$-boundary $\Gamma = \partial \Omega$ immersed in a resting medium. Let $A : \mathcal{D}(A) \subset L^2(\Omega) \rightarrow L^2(\Omega)$ be the unbounded, positive self-adjoint and densely defined operator defined by the action of the Dirichlet Laplacian, i.e., $A = -\Delta$ with $\mathcal{D}(A) = H^1_0(\Omega) \cap H^2(\Omega)$.

Let $\delta, c > 0$ denote diffusivity and speed of the sound, respectively; $k > 0$ denotes a nonlinear parameter and $b^r := \delta + \tau c^2$. Let $T > 0$ (could also be $T = \infty$) and the relaxation parameter $\tau \in \Lambda = (0, \tau_0]$ for some $0 < \tau_0 \in \mathbb{R}$ that will be taken as small as needed, but fixed. We then consider a family of third-order (in time) Jordan-Moore-Gibson-Thompson-Thompson (JMGT) equations

$$\tau u^r_{ttt} + (1 - 2k\tau^r)u^r_{tt} + c^2 A^r u + b^r A^r u_t = 2k(u_t^r)^2 \quad \text{on} \quad (0, T) \quad (1.4)$$

with the initial conditions: $u^r(0) = u_0$, $u^r_t(0) = u_1$, $u^r_{tt}(0) = u_2$. The wellposedness theory, [both local and global] for the model (1.4) has been well developed and known by now for each value of $\tau > 0$. This also includes regularity theory where for sufficiently smooth and compatible initial data one obtains smooth solutions [20, 17]. For reader’s convenience we shall recall some of the relevant results below. However, the focus of this paper is on the asymptotic analysis when $\tau \rightarrow 0$. 

2
Notation: Throughout this paper, \( L^2(\Omega) \) denotes the space of Lebesgue measurable functions whose squares are integrable and \( H^s(\Omega) \) denotes the \( L^2(\Omega) \)-based Sobolev space of order \( s \). We denote the inner product in \( L^2(\Omega) \) and by \( (u, v) = \int_{\Omega} uv d\Omega \) and the respective induced norm is denoted by \( \| \cdot \|_2 \). If less mentioned normed spaces appear, their norms will be indicated with a sub index, i.e., the norm of a space \( Y \) will be denoted by \( \| \cdot \|_Y \). Finally, we denote by \( \mathcal{L}(Y) \) the space of bounded linear operators from \( Y \) to itself equipped with the uniform norm. \( B(X) \) denotes a ball of radius \( r \) in a Banach space \( X \). The projection \( P : \mathbb{R}^3 \to \mathbb{R}^2 \) is defined as \( P(a, b, c) = (a, b) \). Various constants (generic) will be denoted by letters \( C, c, C_i \) – they may be different in different occurrences. \( C(s) \) denotes a continuous function for \( s \geq 0 \) and such that \( C(0) = 0 \).

We begin with collecting relevant results related to the wellposedness of solutions to (1.4) for each value of the parameter \( \tau > 0 \). Recalling that \( \mathcal{D}(A^{1/2}) = H^1_0(\Omega) \) we define \( \mathbb{H}_0, \mathbb{H}_1, \mathbb{H}_2 \) as follows

\[
\mathbb{H}_0 \equiv \mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2}) \times L^2(\Omega); \quad \mathbb{H}_1 \equiv \mathcal{D}(A) \times \mathcal{D}(A^{1/2}) \times L^2(\Omega); \quad \mathbb{H}_2 \equiv \mathcal{D}(A) \times \mathcal{D}(A) \times \mathcal{D}(A^{1/2}),
\]

with \( \mathcal{D}(A) \) and \( \mathcal{D}(A^{1/2}) \) being equipped with the graph norm and the product spaces \( \mathbb{H}_i \) \((i = 0, 1, 2) \) equipped with induced euclidean norms.

We rewrite the abstract equation (1.4) – in the variable \( U^\tau = (u^\tau, u_1^\tau, u_2^\tau)^T \) – as the first-order system

\[
\begin{align*}
U^\tau(t) &= A^\tau U^\tau(t) + \tau^{-1} F(U^\tau), \quad t > 0, \\
U^\tau(0) &= U_0 = (u_0, u_1, u_2)^T
\end{align*}
\] (1.5)

where \( A^\tau := M^\tau A^0 \) with \( A^0 \) and \( M^\tau \) given by

\[
A^0 := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c^2 A & -b^2 A & -1 \end{pmatrix}; \quad M^\tau := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \tau \end{pmatrix}; \quad F(U^\tau) := \begin{pmatrix} 0 \\ 0 \\ 2k[(u_1^\tau)^2 + u^\tau u_2^\tau] \end{pmatrix}.
\] (1.6)

Remark 1.1. Notice that the operator \( A^\tau \) becomes singular as \( \tau \to 0 \).

We shall also introduce spaces \( \mathbb{H}_i^\tau \equiv M^{1/2}_\tau \mathbb{H}_i \) for \( i = 0, 1, 2 \) with the topology \( \| U \|_{\mathbb{H}_i^\tau} = \| M^{1/2}_\tau U \|_{\mathbb{H}_i} \).

With the above setting, we are able to introduce the notion of solution for the nonlinear problem. Denote by \( \{T^\tau(t)\}_{t \geq 0} \) the \( C_0 \)–semigroup associated with the operator \( A^\tau \) for each value of \( \tau > 0 \).

This semigroup exists on each space \( \mathbb{H}_i^\tau, i = 0, 1, 2 \).

Definition 1.1 ([20]). We say that \( U^\tau(t) = (u^\tau(t), u_1^\tau(t), u_2^\tau(t))^T \) is a mild solution of (1.4) on \( [0, T] \) provided \( U^\tau \in C(0, T; \mathbb{H}_i) \) and \( U^\tau(t) \) satisfies the following integral equation

\[
U^\tau(t) = T^\tau(t)U(0) + \tau^{-1} \int_0^t T(t-s)F(U^\tau)ds
\]

The linear problem can be recasted by taking and \( F(U) = 0 \) and its wellposedness along with uniform exponential stability – on both \( \mathbb{H}_0 \) and \( \mathbb{H}_1 \) – was obtained in [26] under the condition (for stability) that \( \gamma^\tau := 1 - \tau c^2 b^{-1} > 0 \). Given a solution \( U^\tau \in \mathbb{H}_0^\tau \) its energy is defined via the \( u^\tau \)-dependent functional

\[
E^\tau(t) := E_0^\tau(t) + E_1^\tau(t), \text{ where } E_0^\tau(t) = \frac{\alpha}{2} \| u_1^\tau(t) \|_2^2 + \frac{c^2}{2} \| A^{1/2} u^\tau(t) \|_2^2 \] (1.7)

and, with \( z^\tau(t) := u_1^\tau(t) + \frac{c^2}{b} u^\tau(t) \in \mathcal{D}(A^{1/2}), E_1^\tau(t) \equiv \frac{b}{2} \| A^{1/2} z^\tau(t) \|_2^2 + \frac{\tau}{2} \| z_1^\tau(t) \|_2^2 + \frac{c^2 \gamma^\tau}{2b} \| u_1^\tau \|_2^2 \).
For $U^\tau$ in $H^1$, the corresponding energy is given by
\[ E^\tau(t) := E^\tau(t) + \|Au^\tau(t)\|_2^2, \]
(1.8)

We also introduce the energy associated with solutions in $H^2$: $E^\tau(t) := E^\tau(t) + \|Au^\tau(t)\|_2^2 + \tau\|A^{1/2}u^\tau_{tt}(t)\|_2^2$.

The following local and global wellposedness results are known [20]:

**Theorem 1.2 (local (in time) wellposedness in $H^1$, [20]).** Let $T > 0$ be arbitrary and $\gamma = 1 - \frac{c^2\tau}{b}$. There exists $\rho_{T^\tau}(\gamma) > 0$ such that if the initial data $U_0$ satisfies $E^\tau(0) \leq \rho_{T^\tau}(\gamma)$, then there exists a unique solution such that
\[ E^\tau(t) < C_{T^\tau}(E^\tau(0)) < \infty, \quad \text{for all } t \in [0, T) \]

The said solution depends continuously on the initial data in $H^1$ topology. An alternative statement holds for all $U_0 \in H^1$ and small time $T > 0$.

**Theorem 1.3 (global (in time) wellposedness, [20]).** Let $\gamma > 0$. Then, there exists $\rho_\gamma(\gamma) > 0$ such that if the initial data $U_0$ satisfies $E^\tau(0) \leq \rho_\gamma(\gamma)$, then there exists a unique solution such that
\[ E^\tau(t) \leq C_\gamma(E^\tau(0)), t > 0 \text{ and } E^\tau(t) \leq C_\gamma(\rho_\gamma)e^{-\omega t}, t > 0, \omega > 0 \]

The results above show, in particular, that there exists a nonlinear flow defined globally in time on a small ball (small radius) in $H^1$. In addition, the said solutions display additional regularity, provided the initial data are more regular as we make precise below.

**Theorem 1.4 (local (in time) wellposedness in $H^2$).** Let $T > 0$ be arbitrary and $\gamma = 1 - \frac{c^2\tau}{b}$. There exists $\rho = \rho_{T^\tau}(\gamma) > 0$, such that if the initial data $U_0 \in H^2$ and it satisfies $E^\tau(0) \leq \rho_{T^\tau}(\gamma)$, then there exists a unique solution $U(t)$ such that
\[ E^\tau(t) < C_{T^\tau}(E^\tau(0)) < \infty, \quad \text{for all } t \in [0, T) \]

The said solution depends continuously on the initial data in $H^2$ topology. In addition, for each $\tau > 0$, $u^\tau_{tt} \in C([0, T]; L^2(\Omega))$. An alternative statement holds for all $U_0 \in H^2$ and small time $T > 0$.

**Proof.** The result of Theorem 1.4 is proved by using a fixed-point argument with the techniques reminiscent to the proof of Theorem 1.2. For completeness, we provide the proof in the appendix. □

**Remark 1.2 (wellposedness with data small in lower topology).** Theorem 1.2 was stated exactly how it was stated in [20], however, as in Theorem 1.4, we can slightly modify the proof in [20] to obtain wellposedness and $H^1$-exponential stability asking for smallness of the data only in $H^0$. The strategy is the same as in the proof of Theorem 1.4, see Appendix A.

The problem of interest in this paper is an asymptotic analysis of solutions when the relaxation parameter $\tau$ tends to zero. It is expected that the limit solution should coincide (formally) with the solution of the Westervelt equation (Equation (1.4) with $\tau = 0$). In [2] asymptotic analysis has been performed for the linearized equation (1.4). It was shown there that the linear semigroups $U^\tau(t)$ converge strongly, in the topology of both $H^i$, ($i = 0, 1$) to the limit $U^0(0)$ – solution of strongly damped wave equation. Rates of convergence have been also shown for finite time horizon under additional regularity assumptions. The difficulty encountered lies in the fact that the operator $A^\tau$ is singular when $\tau \to 0$. As a consequence, more standard techniques based on Trotter Kato type
theorems [22] cannot be applied. However, the difficulty was handled by playing strategically two levels of the estimates with one allowing to derive the convergence rates.

The aim of the present work is to obtain related result valid for the nonlinear model (1.4). This entails to proving (1) Uniform (in $\tau$) exponential decay of nonlinear flows; (2) Strong Convergence of the flows (in an appropriate sense) with respect to vanishing relaxation parameter $\tau \geq 0$. As to the first task, this is routed – as expected – in a string of various estimates obtained via the use of multipliers. Careful analysis of “smallness” requirement for the initial data is critical in this step and also leads to new (more refined) results in the analysis of JMGT equations for a fixed $\tau > 0$. As to the second task, the main difficulty is topological compatibility of “smallness” of initial data (typical in quasilinear problems) required by the existence theory and the passage to the limit via density, where such “smallness” may not be retained. To unlock the difficulty, we were able to construct solutions with much less stringent assumptions on the said “smallness” which is just sufficient (it requires only the $\mathbb{H}_0$ topology) to carry the limit argument. This, in turn, entails to a new “look” at the energy estimates (Step 1) where such “reduced smallness” is traced and propagated.

This brings us to the very recent works [1] and [21] which study the JMGT–Kuznetsov model for acoustic velocity potential (whereas here we look for the JMGT model for acoustic pressure): $\tau \psi_{tt} + (1 - 2k\psi_t)\psi_t - c^2\Delta \psi - b\Delta \psi_t = 2\nabla \psi \nabla \psi_t$. For this model, the authors derive weak star convergence of solutions $\psi$ to the corresponding limit when $\tau = 0$. Because of an extra derivative in the equation, the topology of convergence is $\mathbb{H}_2$ with the associated “smallness” requirement imposed on the initial data. Remark 7.2 in [1] raises an open question whether this convergence can be improved to the strong one. Numerical results presented in [1] support the conjecture. The result of the present paper appears to confirm validity of this conjecture. Although the equation we study is not exactly the same – the topological considerations are related. A key feature to emphasize is that references [21] and [1] provide weak star convergence of solutions for the initial data which are small in $\mathcal{E}^r$ topology. In contrast, our approach provides not only strong convergence, but also the initial data are required to be small in a lowest topology $E^r$ and only bounded in $\mathcal{E}^r$. In addition, rates of convergence of classical solutions on every finite time interval are also established. On the other hand, the model studied in the present paper accounts for Westervelt nonlinearity rather than Kuznetsov. It is anticipated that the methodology developed in this paper would allow for an extension to Kuznetsov’s semilinear terms.

In closing the introduction, we list several other mathematical works dealing with various variants of JMGT model. Considerations related to regularity – including boundary regularity of linear models – have been studied in [4, 3]. MGT models under the effects of memory have been studied [10, 9]. The detailed analysis of regularity and decay on $\mathbb{R}^3$, by using Fourier analysis, has been conducted in [28].

1.2 Main Results

As anticipated in the introduction, our first goal is to obtain “good” stability estimates for the $\tau$ problem with the constants independent of $\tau$. However, the main challenge is to obtain these estimates with a minimal requirement of topological smallness imposed on the initial data. The latter challenge is stimulated by applicability of density argument where the size restrictions on the data becomes a predicament. The corresponding results, at two different topological levels, are formulated below.

**Theorem 1.5.** Let $U_0 \in \mathbb{H}_1$ and assume $\|U_0\|_{\mathbb{H}_1} \leq r_1$ for $r_1 > 0$. Then, there exists $\rho_1(r_1) > 0$ (independent of $\tau$) such that for $\|U_0\|_{\mathbb{H}_0} \leq \rho_1$, there exist $N_1 = N_1(r_1, \rho_1)$ and $\omega_1 > 0$ (independent of $\tau \in \Lambda = (0, \tau_0)$) such that

$$\mathcal{E}^r(t) \leq N_1(r_1, \rho_1)e^{-\omega_1 t},$$
for all \( t > 0 \) with \( U^\tau = (u^\tau, u^\tau_t, u^\tau_{tt}) \) a mild solution of (1.4) on \( \mathbb{H}_1 \).

**Theorem 1.6.** Let \( U_0 \in \mathbb{H}_2 \) and assume \( \|U_0\|_{\mathbb{H}_2} \leq r_2 \) for \( r_2 > 0 \). Then, there exists \( \rho_2(r_2) \) (independent of \( \tau \)) such that for any \( \|U_0\|_{\mathbb{H}_2} \leq \rho_2 \), there exists \( N_2 = N_2(r_2, \rho_2) \) and \( \omega_2 > 0 \) (independent of \( \tau \in \Lambda = (0, \tau_0) \)) such that

\[
\mathcal{E}^\tau(t) \leq N_2(r_2, \rho_2)e^{-\omega_2t},
\]

for all \( t > 0 \) with \( U^\tau = (u^\tau, u^\tau_t, u^\tau_{tt}) \) the solution of (1.4) on \( \mathbb{H}_2 \).

Theorems 1.5 and 1.6 allow the construction of a nonlinear flow corresponding to the limit \( \tau = 0 \). We denote this flow, as \( T^\tau(t) = U^\tau(t, U_0) \) and read: the nonlinear flow corresponding to the initial data \( U_0 \).

Our final result is to show that the Westervelt equation (see [18])

\[
(1 - 2ku^0_0)u^0_{tt} + c^2Au^0_0 + \delta Au^0_0 = 2k(u^0)^2,
\]

with the initial conditions \( u^0(0, \cdot) = u_0, \ u^0_t(0, \cdot) = u_1 \) is a limit of the JMGTE (equation (1.4)) when the thermal relaxation parameter vanishes (\( \tau \to 0 \)). Denoting \( U^0(t, V_0) \) the nonlinear flow for equation (1.9) corresponding to initial data \( V_0 \) and recalling \( P \) the projection on the first two coordinates (i.e. \( P(a, b, c) = (a, b) \), we have:

**Theorem 1.7.**

a) **[Rate of Convergence]** Let \( T > 0 \) and let \( U_0 \in \mathbb{H}_2 \) with \( \|U_0\|_{\mathbb{H}_2} \leq \rho \) for \( \rho = \min\{\rho_1, \rho_2\} \) (\( \rho_1, \rho_2 \) from Theorems 1.5 and 1.6 respectively). Then there exists a \( \tau \)-independent constant \( C_T \) such that

\[
\|P(U^\tau(t, U_0)) - U^0(t, PU_0)\|_{D(A) \times D(A^{1/2})} \leq C_T \|U_0\|^2_{\mathbb{H}_2}
\]

uniformly (in \( t \)) for \( t \in [0, T] \).

b) **[Strong Convergence]** Let \( U_0 \in \mathbb{H}_1 \) with \( \|U_0\|_{\mathbb{H}_2} \leq \rho \) for \( \rho \) as above. Then the following strong convergence

\[
\|P(U^\tau(t, U_0)) - U^0(t, PU_0)\|_{D(A) \times D(A^{1/2})} \to 0 \quad \text{as} \quad \tau \to 0
\]

holds uniformly on \([0, \infty)\).

The proofs of the theorems 1.5, 1.6, 1.7 are given in the subsequent sections. In what follows, we shall make few comments regarding the usage of various constants. (1) Within an identity or inequality we will use generic constants \( C_i, i = 1, 2, ..., \) the different indices are only to emphasize the difference of constants in the same expression. (2) The standard dependence on the fixed physical parameters \([b > 0, c > 0, \gamma^\tau > 0, k \in \mathbb{R}]\) may not be differentiated. However, critical dependence on \( \tau \) is always emphasized.

## 2 Uniform (in \( \tau \)) exponential decay \( \mathbb{H}_1^\tau \) — proof of Theorem 1.5

Let \( u^\tau \) be the solution for (1.4) whose existence is guaranteed in [20] for every \( \tau > 0 \). Fix \( r_1 > 0 \) such that \( \|U_0\|_{\mathbb{H}_1} \leq r_1 \). We shall show that there exists \( \rho_1(r_1) \) sufficiently small such that for \( \|U_0\|_{\mathbb{H}_1} \leq \rho_1 \) we have

\[
\mathcal{E}^\tau(t) + C_1(r_1, \rho_1) \int_0^t \mathcal{E}^\tau(\sigma)d\sigma \leq C_2(r_1)\mathcal{E}^\tau(0)
\]

with \( C_1, C_2 > 0, \rho_1 > 0 \) independent of \( \tau \), but possibly depending on \( r_1 \). Inequality in (2.1) provides the conclusion of Theorem 1.5 (see [24]). Therefore, our goal is to establish this inequality. The proof is accomplished through several steps organized as follows (with the details given later):
Step 1: With reference to the energies (1.7) and (1.8) we obtain the following $E_1^\tau$-energy identity

$$\frac{d}{dt}E_1^\tau(t) + \gamma^\tau \|u_{\tau t}\|^2_2 = (G(u^\tau), z^\tau),$$

where $G(u) := 2k(u^\tau u_{\tau t} + (u_{\tau})^2)$ and $z^\tau \equiv u_{\tau t}^\tau + \frac{c^2}{b} u^\tau$. These calculations are the same as in (see [20]).

Step 2: By Sobolev embeddings and interpolation inequalities we estimate the nonlinear terms to obtain that for every $\varepsilon > 0$ there exist constants $C_1(\varepsilon)$ and $C_2(\varepsilon)$ such that

$$E_1^\tau(t) + \int_0^t \left[ \gamma^\tau - (C_1(\varepsilon)\|u^\tau\|^2_{H^1_t}\|u^\tau\|^2_{H^2_t} + \varepsilon) \right] \|u_{\tau t}(\sigma)\|^2_2 d\sigma$$

$$\leq E_1^\tau(0) + \int_0^t \left[ C_2(\varepsilon)\|u^\tau\|_2\|u^\tau\|_{H^1_t} + \varepsilon \right] A^{1/2} u_{\tau t}(\sigma)\|_2^2 d\sigma,$$  \hspace{1cm} (2.2)

Step 3: Combination of Steps 1 and 2 along with another “energy recovery” estimate and calibration of the constants $\varepsilon > 0$ gives the first stabilizability inequality: there exist positive constants $C_i, (i = 1, 2, 3, 4)$ such that with $\gamma^* = \min\{\gamma^\tau, b, c^2\}$

$$E^\tau(t) + \int_0^t E^\tau(\sigma)|C_1\gamma^* - C_2[E^\tau(\sigma)]^{1/4} \|Au^\tau(\sigma)\|_2^{1/2} - C_3 E^\tau(\sigma)|d\sigma \leq C_4 E^\tau(0).$$

Step 4: Working further towards the estimate for the energy $E^\tau$ we obtain the existence of positive constants $\tilde{C}_i, (i = 1, 2, 3, 4)$ such that

$$E^\tau(t) + \int_0^t E^\tau(\sigma)|\tilde{C}_1\gamma^* - \tilde{C}_2[E^\tau(\sigma)]^{1/4} [E^\tau(\sigma)]^{1/4} - \tilde{C}_3 E^\tau(\sigma)|d\sigma \leq \tilde{C}_4 E^\tau(0).$$

Step 5: By considering $E^\tau(0) \leq r_1$ and $E^\tau(0) < \rho$ sufficiently small, we obtain – from modified Barrier’s Method – a priori global bounds. In fact, choosing $\rho$ such that $C_2\rho^{1/4}r_1^{1/4} - \tilde{C}_3\rho \leq \gamma^* C_1/2$ (with reference to the constants from Step 3) and $\tilde{C}_2\rho^{1/4}r_1^{1/4} - \tilde{C}_3\rho \leq \gamma^* \tilde{C}_1/2$ (with reference to the constants from Step 4) we have the existence of $C > 0$ such that the apriori bounds hold:

$$E^\tau(t) \leq C\rho \hspace{1cm} \text{and} \hspace{1cm} E^\tau(t) \leq C r_1,$$  \hspace{1cm} (2.3)

uniformly in $t > 0$ and $\tau \in (0, \tau_0]$. These a priori bounds imply the desired general inequality in (2.1) from where the uniform (in $\tau$) exponential decays follows via standard procedure as in, for example, [24, 23].

Next we shall focus on proving the outlined five steps above. Our analysis is focused on apriori bounds. The issue of existence and regularity of solutions [for a fixed $\tau$] has been dealt with in [20]-thus we can perform the estimates on smooth solutions with an eye on obtaining suitable a priori bounds which are uniform in the parameter $\tau$. .

Proof of Step 1: Rewriting (1.4) as

$$\tau u_{tt}^\tau + u_{tt}^\tau + c^2 Au^\tau + bAu_t^\tau = G(u^\tau),$$  \hspace{1cm} (2.4)

where $G(u^\tau) = 2ku^\tau u_{tt}^\tau + 2k(u_t^\tau)^2$, and recalling $E^\tau(t) = E_0^\tau(t) + E_1^\tau(t)$ where (as in (1.7)),

$$E_0^\tau(t) = \frac{1}{2} \|u^\tau_t\|^2_2 + \frac{c^2}{2} \|A^{1/2} u^\tau\|^2_2,$$
with $E_1^γ$ expanded as

$$E_1^γ(t) = \frac{τ^2}{2}∥u_t∥^2_2 + \frac{b}{2}∥A^{1/2}u_t^γ∥^2_2 + \frac{c^4}{2b}∥A^{1/2}u_t^γ∥^2_2 + c^2(Au_t^γ, u_t^γ) + \frac{τc^2}{b}(u_t^γ, u_t^γ) + \frac{c^2}{2b}∥u_t^γ∥^2_2, \quad (2.5)$$

one obtains, from the calculations identical to these in [20], the following::

**Lemma 2.1.**

$$\frac{d}{dt}E_1^γ(t) + γ^γ∥u_t^γ∥^2_2 = (G(u^γ), u_t^γ) + c^2b^{-1}u_t^γ, \quad (2.6)$$

**Proof of Step 2:** Sobolev Embedding $H^{3/4}(Ω) \hookrightarrow L^4(Ω)$ along with interpolation inequality implies that $∥f∥_{L^4} \leq ||f||_2^{1/4}||f||_H^{3/4}$, for all $f \in H^1(Ω)$. Moreover, we also have the Sobolev Embedding $H^2(Ω) \hookrightarrow L^∞(Ω)$ with the estimate: $∥u∥_{L^∞} \leq C∥u∥_{H^1}^{1/2}∥u∥_{H^2}^{1/2}$.

This gives:

$$∥u^γu_t^γ∥_2 \leq C∥u^γ∥_{L^∞}∥u_t^γ∥_2 \leq C∥u^γ∥_{H^1}^{1/2}∥u^γ∥_{H^2}∥u_t^γ∥_2^2 \quad (2.7)$$

and with a given $ε > 0$,

$$∥u^γu_t^γ, u_t^γ∥_2 \leq C_1(ε)∥u^γ∥_{H^1}∥u_t^γ∥_2^2 + εC_2∥A^{1/2}u_t^γ∥_2^2 \leq C_1(ε)∥u^γ∥_{H^1}∥u^γ∥_{H^2}∥u_t^γ∥_2^2 + εC_2∥A^{1/2}u_t^γ∥_2^2 \quad (2.8)$$

Since $∥f^2∥_2 = ||f||_{L^4}^4$ for all $f \in L^4(Ω)$ we have,

$$(∥u_t^γ∥_2^2, u_t^γ) \leq C_1(ε)∥u_t^γ∥_2∥u_t^γ∥_{H^1}∥A^{1/2}u_t^γ∥_2^2 + ε∥u_t^γ∥_2^2, \quad (2.9)$$

$$(∥u_t^γ∥_2^2, u_t^γ) \leq C_1(ε)∥u_t^γ∥_2∥u_t^γ∥_{H^1}∥A^{1/2}u_t^γ∥_2^2 + εC_2∥A^{1/2}u_t^γ∥_2^2. \quad (2.10)$$

Now, (2.7), (2.8), (2.9) and (2.10) together imply

$$(G(u^γ), z_t^γ) \leq (C_1(ε)∥u^γ∥_{H^1}∥u^γ∥_{H^2} + εC_2)∥u_t^γ∥_2^2 + (C_3(ε)∥u_t^γ∥_2∥u_t^γ∥_{H^1} + εC_4)∥A^{1/2}u_t^γ∥_2^2. \quad (2.11)$$

Inequalities (2.7) – (2.11) holds for all $t \in [0,T]$. After integration of (2.6) from 0 to $t \in [0,T]$ one obtains

$$E_1^γ(t) + \int_0^t \left[ γ^γ - (C_1(ε)∥u^γ∥_{H^1}∥u^γ∥_{H^2} + εC_2) \right]∥u_t^γ(σ)||_2^2dσ \leq E_1^γ(0) + \int_0^t \left[ C_3(ε)∥u_t^γ∥_2∥u_t^γ∥_{H^1} + εC_4 \right]∥A^{1/2}u_t^γ(σ)||_2^2dσ,$$

which reduces to (2.2) after noting $f \mapsto ||A^{1/2}f||_2$ defines a norm in $H_0^1(Ω)$ -thus completing Step 2. \[\square\]

**Proof of Step 3:** Here the goal is to obtain recovery of the integral of the total energy $E^γ(t) := E_0^γ(t) + E_1^γ(t)$. For this we need to construct $\frac{d}{dt}E_1^γ(t) + \frac{d}{dt}E_0^γ(t)$. From direct calculations – tedious but straightforward and reminiscent to [20] – we arrive at the identity

$$\frac{d}{dt}E^γ + (γ^γ - τC1)∥u_t^γ∥_2^2 + b∥A^{1/2}u_t^γ∥_2^2 = (G(u^γ), z_t^γ + u_t^γ) - τ\frac{d}{dt}(u_t^γ, u_t^γ). \quad (C_1 > 0) \quad (2.12)$$

Integrating inn time from 0 to $t$, an estimate for the last term on the right hand side of (2.12) yields:

$$-τ\int_0^t \frac{d}{dt}(u_t^γ(σ), u_t^γ(σ))dσ = -τ(u_t^γ(σ), u_t^γ(σ)) \bigg|_0^t \leq \frac{τ}{2} \left[ ||u_t^γ(t)||_2^2 + ||u_t^γ(t)||_2^2 + ||u_t^γ(0)||_2^2 + ||u_t^γ(0)||_2^2 \right]$$

8
\[
= \frac{\tau}{2} \|u^\tau_1(t)\|^2 + \frac{\tau\lambda}{c^2} \|u^\tau(t)\|^2 + \frac{\tau}{2} \|u^\tau_0(0)\|^2 + \frac{\tau\lambda}{c^2} \|u^\tau_0(0)\|^2 \leq \left(1 + \frac{\tau\lambda}{c^2}\right) (E^\tau(t) + E^\tau_0(0)),
\]
where we have used the expansion (2.5). Combining (2.13) with the identity (2.6) (to handle \(E^\tau(t)\) from (2.13)) and (2.12) to get (after integrating in time):

\[
E^\tau(t) + (\gamma^\tau - \tau C_1) \int_0^t \|u^\tau_1(\sigma)\|^2 d\sigma + b \int_0^t \|A^{1/2} u^\tau_1(\sigma)\|^2 d\sigma \leq C_2 E^\tau(0) + C_3 \int_0^t |G(u^\tau(\sigma)), \tau^\tau_1(\sigma) + z^\tau_1(\sigma)| d\sigma,
\]
(here \(\tau^\tau := u^\tau_1 + \left(\frac{c^2}{b} + 1\right) u^\tau\)) and after accounting for (2.11) we obtain the inequality

\[
E^\tau(t) + \int_0^t (\gamma^\tau - \tau C_1 - C_2(\varepsilon)\|u^\tau\|_{L^\infty} - \varepsilon C_3) \|u^\tau_0(\sigma)\|^2 d\sigma
\]
\[
+ \int_0^t (b - C_4(\varepsilon)\|u^\tau_1\|_{H^1} - \varepsilon C_3) \|A^{1/2} u^\tau_1(\sigma)\|^2 d\sigma \leq C_6 E^\tau(0).
\]

To reconstruct the integral \(\int_0^t \|A^{1/2} u^\tau(\sigma)\|^2 d\sigma\) we take the \(L^2\)-inner product of (2.4) with \(\lambda u^\tau\) (\(\lambda > 0\) will be chosen later) which gives

\[
\frac{\lambda b}{2} \|A^{1/2} u^\tau(t)\|^2 + \lambda c^2 \int_0^t \|A^{1/2} u^\tau(\sigma)\|^2 d\sigma = \frac{\lambda b}{2} \|A^{1/2} u^\tau(0)\|^2 + \lambda \int_0^t \|u^\tau_1(\sigma)\|^2 d\sigma
\]
\[
- \lambda \left[\tau(u^\tau_1, u^\tau) - \frac{\tau}{2} \|u^\tau_1\|^2 + (u^\tau_1, u^\tau)\right]\bigg|_0^t + \lambda \int_0^t (G(u^\tau(\sigma)), u^\tau(\sigma)) d\sigma.
\]

Denoting by \(I = \lambda \int_0^t \|u^\tau_1(\sigma)\|^2 d\sigma - \lambda \left[\tau(u^\tau_1, u^\tau) - \frac{\tau}{2} \|u^\tau_1\|^2 + (u^\tau_1, u^\tau)\right]\bigg|_0^t\) we estimate

\[
I \leq \lambda \left[\frac{\tau}{2} \|u^\tau_0(0)\|^2 + \frac{\tau + 1}{2} \|u^\tau(0)\|^2 + \frac{1}{2} \|u^\tau_0(0)\|^2\right] \leq \lambda C_1 \int_0^t \|A^{1/2} u^\tau_1(\sigma)\|^2 d\sigma + \lambda C_2 E^\tau_1(t) + E^\tau_1(0)
\]

which combined with (2.15) gives

\[
\frac{\lambda b}{2} \|A^{1/2} u^\tau(t)\|^2 + \lambda c^2 \int_0^t \|A^{1/2} u^\tau(\sigma)\|^2 d\sigma \leq \lambda C_1 [E^\tau(0) + E^\tau_1(t)]
\]
\[
+ \lambda C_2 \int_0^t \|A^{1/2} u^\tau_1(\sigma)\|^2 d\sigma + \lambda C_3 \int_0^t (G(u^\tau(\sigma)), u^\tau(\sigma)) d\sigma
\]

It remains to estimate the nonlinear term \((G(u^\tau), u^\tau)\). For this we have

\[
(G(u^\tau), u^\tau) = 2k(u^\tau u^\tau_t + (u^\tau_1)^2, u^\tau)
\]
\[
\leq \varepsilon C_1 \|u^\tau\|^2 + C_2(\varepsilon)\|u^\tau\|_{L^\infty} \|u^\tau_t\|^2 + C_3(\varepsilon)\|u^\tau_1\|_{H^1} \|u^\tau_t\|^2
\]
\[
\leq \varepsilon C_1 \|A^{1/2} u^\tau\|^2 + C_2(\varepsilon)\|u^\tau\|_{L^\infty} \|u^\tau_t\|^2 + C_3(\varepsilon)\|u^\tau_1\|_{H^1} \|A^{1/2} u^\tau_1\|^2,
\]
which holds for all \(t \in [0, T]\). Combining the result of Step 2 and (2.18) leads to

\[
\frac{\lambda b}{2} \|A^{1/2} u^\tau(t)\|^2 + \lambda (c^2 - \varepsilon C_1) \int_0^t \|A^{1/2} u^\tau(\sigma)\|^2 d\sigma \leq C_2 E^\tau(0)
\]
\[
+ \lambda \int_0^t (C_3(\varepsilon)\|u^\tau\|_{L^\infty} + C_4(\varepsilon)\|u^\tau\|_{L^2} + \varepsilon C_3) \|u^\tau_1(\sigma)\|^2 d\sigma
\]
Then, combining (2.14) with (2.19) and picking $0 < \lambda < \frac{b}{2}$, $\tau_0$ and $\varepsilon$ sufficiently small (and fixed) we obtain

\[ E^*(t) + \int_0^t (\gamma \tau - C_2 \|u^\tau\|_{L^\infty} \|u_{tt}^\tau\|_2) \, d\sigma + \int_0^t (C_3 b - C_4 \|u_t^\tau\|_2 \|H^1\|) A^{1/2} u_t^\tau(\sigma) \|2 \, d\sigma \\
+ C_5 c^2 \int_0^t \|A^{1/2} u^\tau(\sigma)\|_2^2 \, d\sigma \leq C_6 E^*(0). \tag{2.20} \]

Applying once more interpolation inequality $\|u^\tau\|_{L^\infty} \leq C \|u^\tau\|_{H^2}^{1/2} \|u^\tau\|_{H^1}^{1/2}$ leads to

\[ E^*(t) + \int_0^t E^*(\sigma) [\gamma \tau - C_2 \|A u^\tau(\sigma)\|_{L^3}^{1/2} |E^*(\sigma)|^{1/4} - C_3 E^*(\sigma)] \, d\sigma \leq C_4 E^*(0), \tag{2.21} \]

and here all the constants are positive and independent of $\tau$ and on time and $\gamma^* = \min\{\gamma, Cb, Cc^2\}$ where $C > 0$ is a suitable generic constant. This completes the proof of Step 3.

**Proof of Step 4:** Our objective in this step is to establish a stabilizability inequality for the energy $E^*$. Since $E^*(t) \approx E^*(t) + \|Au^\tau(t)\|_2^2$, it suffices to focus on the higher order term $\|Au^\tau(t)\|_2^2$. This quantity can be estimated using the multiplier $u^\tau \in L^2(\Omega)$. Taking the $L^2$-inner product of (2.4) with $\lambda u^\tau$ ($\lambda > 0$ to be later determined) we have,

\[ \lambda(G(u^\tau), Au^\tau) = \lambda \left\{ \frac{d}{dt} \left( \tau(u_{tt}^\tau, Au^\tau) - \frac{\tau}{2} \|A^{1/2} u_t^\tau\|^2 + (u_t^\tau, Au^\tau) \right) - \|A^{1/2} u_t^\tau\|^2_2 + \frac{b}{2} \frac{d}{dt} \|Au^\tau\|^2_2 + c^2 \|Au^\tau\|^2_2 \right\}, \]

then, integration in time from 0 to $t \in [0, T]$ gives

\[ \frac{\lambda b}{2} \|Au^\tau\|^2_2 + \lambda c^2 \int_0^t \|Au^\tau(\sigma)\|^2_2 \, d\sigma = \frac{\lambda b}{2} \|Au^\tau(0)\|^2_2 + \lambda \int_0^t \|A^{1/2} u_t^\tau(\sigma)\|^2_2 \, d\sigma \\
- \lambda \left[ \tau(u_{tt}^\tau, Au^\tau) - \frac{\tau}{2} \|A^{1/2} u_t^\tau\|^2_2 + (u_t^\tau, Au^\tau) \right]_0^t + \lambda \int_0^t (G(u^\tau(\sigma), Au^\tau(\sigma)) \, d\sigma. \tag{2.22} \]

Denoting $I = \left[ \tau(u_{tt}^\tau, Au^\tau) - \frac{\tau}{2} \|A^{1/2} u_t^\tau\|^2_2 + (u_t^\tau, Au^\tau) \right]_0^t$ we have

\[ I \leq \frac{\tau}{2} \|u_{tt}^\tau(t)\|^2_2 + \frac{\tau}{2} \|Au^\tau(t)\|^2_2 + \frac{\tau + 1}{2} \|A^{1/2} u_t^\tau(t)\|^2_2 + \frac{1}{2} \|A^{1/2} u^\tau(t)\|^2_2 \\
+ \frac{\tau}{2} \|u_t^\tau(0)\|^2_2 + \frac{\tau}{2} \|Au^\tau(0)\|^2_2 + \frac{\tau + 1}{2} \|A^{1/2} u_t^\tau(0)\|^2_2 + \frac{1}{2} \|A^{1/2} u^\tau(0)\|^2_2 \leq C_1 E^*(0) + \frac{\tau}{2} \|Au^\tau(t)\|^2_2 + C_2 E^*(t) \]

Then from (2.22) it follows that

\[ \frac{\lambda(b - \tau_0)}{2} \|Au^\tau(t)\|^2_2 + \lambda c^2 \int_0^t \|Au^\tau(\sigma)\|^2_2 \, d\sigma \leq C_1 (\lambda) E^*(0) + C_2 \lambda E^*(t) \\
+ \lambda C_3 \int_0^t \|A^{1/2} u_t^\tau(\sigma)\|^2_2 \, d\sigma + \lambda \int_0^t (G(u^\tau(\sigma), Au^\tau(\sigma)) \, d\sigma. \tag{2.23} \]

For the nonlinear part we have

\[ (G(u^\tau), Au^\tau) \leq \varepsilon C_1 \|Au^\tau\|^2_2 + C_2(\varepsilon) \|u^\tau\|^2_2 \|u_{tt}^\tau\|^2_2 + C_3(\varepsilon) \|u_t^\tau\|_2^2 \|u^\tau\|^3_2 \]

\[ \leq \varepsilon C_1 \|Au^\tau\|^2_2 + C_2(\varepsilon) \|u^\tau\|^2_2 \|u_{tt}^\tau\|^2_2 + C_3(\varepsilon) \|u_t^\tau\|_2^2 \|u^\tau\|^3_2 \|A^{1/2} u_t^\tau\|^2_2 \tag{2.24} \]
then a combination of (2.23), (2.24) and (2.20) gives
\[
E^\tau(t) + \frac{\lambda u}{4} \|Au^\tau(t)\|^2_2 + \lambda(c^2 - \varepsilon C_1) \int_0^t \|Au^\tau(\sigma)\|^2_2 d\sigma \\
+ \int_0^t (C_2 - (C_3 + \lambda C_4(\varepsilon))\|u^\tau\|_L^\infty)\|u_{tt}^\tau(\sigma)\|^2_2 d\sigma \\
+ \int_0^t (C_5 - (C_6 + \lambda C_7(\varepsilon))\|u_{tt}^\tau\|_2\|u_t^\tau\|_{H^1} - \lambda C_8))\|A^{1/2}u_t^\tau(\sigma)\|^2_2 d\sigma \\
+ C_9 \int_0^t \|A^{1/2}u_t^\tau(\sigma)\|^2_2 d\sigma \leq C_{10} E^\tau(0) + C_{11} \lambda E^\tau(t)
\] (2.25)
then, first fixing $\varepsilon > 0$ small and then fixing $\lambda << C_5/C_8$ it follows that
\[
E^\tau(t) + \int_0^t (\gamma^\tau - C_1\|u^\tau\|_L^\infty)\|u_{tt}^\tau(\sigma)\|^2_2 d\sigma + \int_0^t (C_2 - C_3\|u_{tt}^\tau\|_2\|u_t^\tau\|_{H^1})\|A^{1/2}u_t^\tau(\sigma)\|^2_2 d\sigma \\
+ C_4 \int_0^t \|A^{1/2}u_t^\tau(\sigma)\|^2_2 d\sigma + C_5 \int_0^t \|Au^\tau(\sigma)\|^2_2 \leq C_{10} E^\tau(0),
\] (2.26)
which reduces to
\[
E^\tau(t) + \int_0^t E^\tau(\sigma) [\gamma^* - C_1[E^\tau(\sigma)]^{1/4}[E^\tau(\sigma)]^{1/4} - C_2 E^\tau(\sigma)] d\sigma \leq C_3 E^\tau(0),
\] (2.27)
with all the constants being positive and independent of $\tau$ and on time. This completes the proof of Step 4.

**Remark 2.1.** Note that we also obtain the estimate
\[
E^\tau(t) + C_1(r_1, \rho_1) \int_0^t E^\tau(s) d\sigma \leq C_2(r_1, \rho_1) E^\tau(0)
\] (2.28)
for all solutions $u^\tau$ such that $E^\tau(0) \leq r_1$ and $E^\tau(t) \leq r_1(t)$ with $\rho_1$ sufficiently small. This is to say that under the smallness condition imposed on the lowest energy $E^\tau(0)$ -- which depends on the bound of higher energy $E^\tau(0)$ -- the lower energy of the solutions -- as a trajectory -- remains bounded by a multiple of $E^\tau(0)$ for all times. This fact will be used later several times.

### 3 Uniform (in $\tau$) exponential decay in $H^2_\sigma$ - the proof of Theorem 1.6

With an eye on higher topology of $H^2_\sigma$ space, we shall repeat the procedure of the previous section. We will now show that the solution for (1.4) is uniform (in $\tau$) exponentially stable in the topology of $H^2_\sigma$ -- recall that $H^2_\sigma = D(A) \times D(A) \times D(A^{1/2})$ -- under the smallness condition in $H^0_\sigma$ only. For $\tau > 0$, let $u^\tau$ be such solution and fix $r_2 > 0$ such that $\|U_0\|_{H^2} \leq r_2$ and let $\|U_0\|_{H^0} \leq \rho_2$, $\rho = \rho(r_2)$ sufficiently small will be determined in the course of the proof. Again, for the energy functional $E^\tau$ defined as $E^\tau(t) \approx E^\tau(t) + \|Au_t^\tau\|^2_2 + \tau\|A^{1/2}u_{tt}^\tau\|^2_2$, we seek to establish the general stabilizability inequality
\[
E^\tau(t) + C_0 \int_0^t \|A^{1/2}u_{tt}^\tau\|^2_2 d\sigma + C_1 \int_0^t E^\tau(\sigma) d\sigma \leq C_2 E^\tau(0),
\] (3.1)
with $C_0, C_1, C_2 > 0$ and independent of $\tau$ but they depend on $r_2$ and $\rho_2$. To accomplish this one needs to account for higher order terms $\|Au_t^\tau\|^2_2$ and $\|A^{1/2}u_{tt}^\tau\|$. The procedure is similar as in the previous case (Steps 1-5) of $H^1$ topology, so we shall concentrate only on the details which are different and require additional care. Recall that we are working with sufficiently smooth solutions guaranteed by the wellposedness and regularity theory, so our computations ahead can be rigorously justified.
Proof of (3.1): We take the $L^2$-inner product of (2.4) with $Au^\tau_t(t) \in L^2(\Omega)$ for all $t \in [0,T]$. Recalling that $G(u^\tau) = 2k(u^\tau u^\tau_t + (u^\tau)^2)$ we obtain,
\[
\frac{c^2}{2} \|Au^\tau\|^2_2 + b \int_0^t \|Au^\tau_t(\sigma)\|^2_2 d\sigma - \tau \int_0^t \|A^{1/2}u^\tau_t(\sigma)\|^2_2 d\sigma \\
= \frac{c^2}{2} \|Au^\tau(0)\|^2_2 - \left[\tau(u^\tau, Au^\tau_t) + \frac{1}{2} \|A^{1/2}u^\tau_t\|^2_2\right]_0^t + \int_0^t (G(u^\tau(\sigma)), Au^\tau_t(\sigma)) d\sigma. \tag{3.2}
\]

The next step is to take the $L^2$-inner product of (2.4) with $\lambda Au^\tau_t(t) \in L^2(\Omega)$ for all $t \in [0,T]$ (smooth solutions) where $\lambda > 0$ will be determined later. This leads to,
\[
\frac{\lambda \tau}{2} \|A^{1/2}u^\tau_t\|^2_2 + \frac{\lambda b}{2} \|Au^\tau_t\|^2_2 - \lambda c^2 \int_0^t \|Au^\tau_t(\sigma)\|^2_2 d\sigma + \lambda \int_0^t \|A^{1/2}u^\tau_t\|^2_2 d\sigma = \frac{\lambda \tau}{2} \|A^{1/2}u^\tau_t(0)\|^2_2 \\
+ \frac{\lambda b}{2} \|Au^\tau_t(0)\|^2_2 - \lambda c^2 \|(Au^\tau, Au^\tau_t)\|_0^t + \lambda \int_0^t (G(u^\tau(\sigma)), Au^\tau_t(\sigma)) d\sigma \tag{3.3}
\]
and then adding (3.2) with (3.3) we arrive at
\[
\frac{c^2}{2} \|Au^\tau\|^2_2 + \frac{\lambda \tau}{2} \|A^{1/2}u^\tau_t\|^2_2 + \frac{\lambda b}{2} \|Au^\tau_t\|^2_2 + (b - \lambda c^2) \int_0^t \|Au^\tau_t(\sigma)\|^2_2 d\sigma \\
+ (\lambda - \tau) \int_0^t \|A^{1/2}u^\tau_t(\sigma)\|^2_2 d\sigma = \frac{c^2}{2} \|Au^\tau(0)\|^2_2 + \frac{\lambda \tau}{2} \|A^{1/2}u^\tau_t(0)\|^2_2 + \frac{\lambda b}{2} \|Au^\tau_t(0)\|^2_2 \\
- \left[\tau(u^\tau, Au^\tau_t) + \frac{1}{2} \|A^{1/2}u^\tau_t\|^2_2 + \lambda c^2 \|(Au^\tau, Au^\tau_t)\|_0^t + \lambda \int_0^t (G(u^\tau(\sigma)), Au^\tau_t(\sigma)) d\sigma \right]. \tag{3.4}
\]

Now, letting $I = \left[\tau(u^\tau, Au^\tau_t) + \frac{1}{2} \|A^{1/2}u^\tau_t\|^2_2 + \lambda c^2 \|(Au^\tau, Au^\tau_t)\|_0^t + \lambda \int_0^t (G(u^\tau(\sigma)), Au^\tau_t(\sigma)) d\sigma \right]$ we estimate
\[
I \leq C_1(\lambda) \mathcal{E}_1(0) + \frac{\tau}{\lambda b} \|u^\tau_t(t)\|^2_2 + \frac{\tau + 1}{4} \|Au^\tau_t(t)\|^2_2 + \frac{\lambda c^2}{2} \|Au^\tau(t)\|^2_2 \\
\leq C_1(\lambda) \mathcal{E}_1(0) + C_2(\lambda) \mathcal{E}_1(0) + \frac{(\tau + 1)\lambda b}{4} \|Au^\tau(t)\|^2_2 \leq C_1(\lambda) \mathcal{E}_1(0) + \frac{(\tau + 1)\lambda b}{4} \|Au^\tau(t)\|^2_2. \tag{3.5}
\]

For the nonlinear terms we have: for all $\varepsilon > 0$ there exists constants $C_i(\varepsilon)$, $i = 1, 2, 3$ such that
\[
(G(u^\tau), Au^\tau_t) = (u^\tau u^\tau_t + (u^\tau)^2, Au^\tau_t) \\
\leq \varepsilon C_1 \|Au^\tau_t\|^2_2 + C_2(\varepsilon) \|u^\tau\|^4_2 \\|A^{1/2}u^\tau_t\|^2_2 + C_3(\varepsilon) \|u^\tau_t\|_2 \|u^\tau_t\|_{H^1} \|Au^\tau_t\|^2_2, \tag{3.6}
\]
and by Poincaré’s inequality, Sobolev’s embeddings, interpolation inequalities and product rule we obtain
\[
(G(u^\tau), Au^\tau_t) = (u^\tau u^\tau_t + (u^\tau_t)^2, Au^\tau_t) = \left(A^{1/2}(u^\tau u^\tau_t + (u^\tau_t)^2), A^{1/2}u^\tau_t\right) \leq \|\nabla(u^\tau u^\tau_t + (u^\tau_t)^2)\|_2 \|A^{1/2}u^\tau_t\|_2 \\
\leq C_1(\varepsilon) \|u^\tau_t\|^2_2 + \|A^{1/2}u^\tau_t\|^2_2 + (\varepsilon + \|u^\tau_t\|_{L^\infty}) \|A^{1/2}u^\tau_t\|^2_2 + C_2(\varepsilon) \|u^\tau_t\|^2_2 \|A^{1/2}u^\tau_t\|^2_2 \\
\leq C_1(\varepsilon) \|A^{1/2}u^\tau_t\|^2_2 \|A^{1/2}u^\tau_t\|^2_2 + (\varepsilon + \|u^\tau_t\|_{L^\infty}) \|A^{1/2}u^\tau_t\|^2_2 + C_2(\varepsilon) \|A^{1/2}u^\tau_t\|^2_2 \|Au^\tau_t\|^2_2 \\
\leq \left(\varepsilon + \|u^\tau_t\|_{L^\infty} + C_1(\varepsilon) \|A^{1/2}u^\tau_t\|^3_2 \|u^\tau_t\|^{3/2}_2 \right) \|A^{1/2}u^\tau_t\|^2_2 + C_2(\varepsilon) \|A^{1/2}u^\tau_t\|^2_2 \|Au^\tau_t\|^2_2. \tag{3.7}
\]

We then combine (3.4), (3.5), (3.6) and (3.7) we get
\[
\frac{c^2}{2} \|Au^\tau\|^2_2 + \frac{\lambda \tau}{2} \|A^{1/2}u^\tau_t\|^2_2 + \frac{\lambda (1 - \tau)}{4} \|Au^\tau_t\|^2_2 + \int_0^t (b - \lambda \varepsilon C_1 - \lambda C_2) \|Au^\tau_t(\sigma)\|^2_2 d\sigma \\
+ \int_0^t (\lambda - \tau - \varepsilon \lambda C_3) \|A^{1/2}u^\tau_t(\sigma)\|^2_2 d\sigma \leq C_4(\lambda) \mathcal{E}_1(0)
\]
and by making \( \tau_0, \lambda < \frac{b}{2C_1} \) and \( \varepsilon \) small enough, using the interpolation inequality, and taking into account inequality (2.20) we arrive at

\[
\mathcal{E}(t) + \int_0^t \frac{1}{8} b \| A^{1/2} u_{tt} (\sigma) \|^2 d\sigma + \gamma^* \int_0^t \mathcal{E}(\sigma) d\sigma + C_1 \int_0^t \| u_T \|^2_{H^2} \| u_T \|^2_{H^2} \| A^{1/2} u_{tt} (\sigma) \|^2 d\sigma + C_2 \int_0^t \| u_T \|^2_{H^2} \| u_T \|^2_{H^2} \| A^{1/2} u_{tt} (\sigma) \|^2 d\sigma \leq C_3 \mathcal{E}(0),
\]

which can be rewritten in terms of the energy functionals as:

\[
\mathcal{E}(t) + \int_0^t \| A^{1/2} u_{tt} (\sigma) \|^2 \left[ \frac{1}{8} b - [\mathcal{E}(\sigma)]^{1/4} [\mathcal{E}(\sigma)]^{1/4} \right] + \int_0^t \mathcal{E}(\sigma) (\gamma^* - C_1 [E^*(\sigma)]^{1/2} [\mathcal{E}(\sigma)]^{1/2}) d\sigma \leq C_2 \mathcal{E}(0),
\]

where \( \gamma^* = \min \{ \gamma^*, b \} \). Barrier’s Method applied to the last inequality asserts global boundedness of \( \mathcal{E}(\cdot) \) by a multiple of \( r_2 \), in a similar manner as in the previous section. This leads to the final estimate (3.1). The proof is complete. \( \square \)

**Remark 3.1.** Notice that the parameters responsible for \( H_1 \), stability estimates are \( \gamma^* > 0, b > 0, c^2 > 0 \).

### 4 Strong convergence and convergence rate - proof of Theorem 1.7

Our aim is to establish strong convergence of the flows \( U^\tau(t,U_0) \), when \( \tau \to 0 \) to a solution \( U^0(t,P U_0) \) of Westervelt equation (1.9) with initial data \( PU_0 \in P(\mathbb{H}_1) \). By strong we mean in the strong topology of the phase space \( \mathbb{H}_1 \). The argument will follow through the following two steps. In the first one we shall derive convergence rates — uniform convergence — for initial data in more regular space \( \mathbb{H}_2 \) and on the finite time horizon. In the second part we shall prove strong convergence in the phase space of dynamical system and for the initial data also taken from the phase space \( \mathbb{H}_1 \). In this context we wish to emphasize the following difficulty: standard argument used for this type of results is based on consistency, stability and density; However, in the case of quasilinear problems the initial data under consideration are required to be sufficiently small, which does not cooperate with the usual density argument. In order to deal with the issue a careful analysis of topological smallness is necessary. For this reasons the results on uniform exponential decays of solutions obtained in previous sections differentiate between the topology where the smallness and boundedness is required. The appropriate calibration of ”smallness” and regularity is critical for the argument.

To begin with let us denote \( x^\tau := u^\tau - u^0 \) where \( u^\tau \) is a solution of (1.4) and \( u^0 \) the solution of Westervelt equation (1.9). We will prove:

**Theorem 4.1.** For each arbitrary \( T > 0 \) and \( M > 0 \) there exists \( \rho_M > 0 \), sufficiently small, and an increasing continuous function \( K_T : \mathbb{R}^+ \to \mathbb{R}^+ \) such that the following inequality

\[
\| (x^\tau(t), x^\tau(t)) \|_{\mathbb{D}(A) \times \mathbb{D}(A^{1/2})}^2 = \| Ax^\tau(t) \|_2^2 + \| A^{1/2} x^\tau(t) \|_2 \leq \gamma K_T(\| U_0 \|_{\mathbb{H}_2}),
\]

is true for all initial data such that \( \| U_0 \|_{\mathbb{H}_2} \leq M \) and \( E^\tau(0) \leq \rho_M \), \( \tau \in \Lambda = (0, \tau_0] \) and all \( t \in [0,T] \).
Remark 4.1. Notice that part (a) of Theorem 1.7 – although written in the notation of the nonlinear flows – is equivalent to the first statement in Theorem (4.1).

Clearly, the regularity of $u^0$ is going to play the major role. The linear part of Westervelt equation generates an analytic semigroup and the system has “maximal regularity” \[11, 25\]. These are powerful tools in the study of the regularity of \((1.9)\). For instance, \[27\] provides a refined theory of wellposedness for \((1.9)\) within the $L^p$ framework. As to the Hilbert framework (relevant to this work) - from Theorem 1.1 in \[18\] we know that for the initial data $\|Au(0)\| + \|Au_t(0)\|$ sufficiently small one obtains unique global solution in $u^0 \in C^s([0, T], H^{2-s}(\Omega))$, $s = 0, 1, 2$. However, the above results will not be sufficient for our analysis. We will need a tighter control of the smallness and regularity – a result stated later in Theorem 4.2.

The proof of Theorem 4.1 will be accomplished through the following steps: We first prove that $x^\tau = u^\tau - u^0$ satisfy a (second order in time) PDE with forcing term dependent of $x$ and $u$ as well on their derivatives. Applying the multipliers $Ax^\tau_t$ and $x^\tau_{tt}$ reconstructs the terms on the left side of \((4.1)\). However, the right hand side becomes singular. Handling this singularity is the main component of the proof. We are faced with the following challenges: (1) Can the integral term $\tau \int_0^t \|u^\tau_{ttt}(\sigma)\|^2 d\sigma$ be controlled by the $H_2$ initial data which are small in $H_0$? (2) Can we work with the limit evolution $U^0(t)$ under less stringent regularity and smallness assumptions? Positive answers to these questions are given below. In fact, from the result of Theorem 1.4 we already know that for each $\tau > 0$, $u^\tau_{tt}(t)$ is in $L_2(\Omega)$. The issue is that of controlling it’s singularity when $\tau \to 0$. This is the content of the Lemma below.

**Lemma 4.1.** Let $U^\tau$ be a solution of \((1.4)\). Assume $\|U_0\|_{H_2} \leq M$ and $\|U_0\|_{H_0} < \rho_M$ with $\rho_M$ small. Then, there exists a continuous bounded function $C(s) > 0$, (independent of $\tau \in \Lambda$), such that

$$
\int_0^t \|u^\tau_{ttt}(\sigma)\|^2 d\sigma \leq \frac{1}{\tau} C(\|U_0\|_{H_2}^2).
$$

**Theorem 4.2.** Let $U^0 = (u^0_0, u^0_1)$ be a solution to the Westervelt equation with the initial conditions subject to the following assumption: For each $M > 0$ there exists $\rho_M$ small such that

$$
\frac{1}{2} \left[ \|u^0_0(0)\|^2 + \|A^{1/2}u^0_0(0)\|^2 \right] \leq \rho_M \quad \text{and} \quad \frac{1}{2} \left[ \|A^{1/2}u^0_1(0)\|^2 + \|Au^0(0)\|^2 \right] \leq M.
$$

Then, there exists constants $C(\rho_M, M)$ and $\omega_0 > 0$ such that

$$
\|Au^0(t)\|^2 + \|A^{1/2}u^0(0)\|^2 \leq C(M)e^{-\omega_0 t}, \|A^{1/2}u^0(t)\|^2 + \|u^0_0(t)\|^2 \leq \rho_M C(M)e^{-\omega_0 t}
$$

$$
\int_0^t \left[ \|u^0_0(s)\|^2 + \|Au^0(s)\|^2 + \|A^{1/2}u^0(s)\|^2 \right] ds \leq C(\rho, M), t > 0
$$

\[4.2\]

**Proof of Lemma 4.1:** Since $A^\tau$ generates a linear semigroup on each of the spaces $H_i$, $i = 0, 1, 2$, [2, 19, 26], linear semigroup theory allows to represent any solution of the non-homogenous problem $U^\tau_1 = A^\tau U^\tau(t) + F(t)$ with $F \in L^1(0, T, H_i)$, via the variation of parameters formula: $U^\tau(t) = e^{A^\tau t} U_0 + \int_0^t e^{-A^\tau (t-\sigma)} F(t-\sigma) d\sigma$. When $F \in W^{1,1}(0, T; H_i)$ and $A^\tau U_0 - F(0) \in H_i$, one also has $U^\tau(t) = e^{A^\tau t} (A^\tau U_0 - F(0)) + \int_0^t e^{A^\tau (t-\sigma)} F_t(\sigma) d\sigma$. We shall apply the formula with $F = \tau^{-1} [0, 0, 0, 2k ((u^\tau_0)^2 + u^\tau u^\tau_{ttt})]$ where $u^\tau$ is the solution of the nonlinear problem. By Theorem 1.4 we have that nonlinear solutions $U^\tau \in C([0, T]; H_2)$ for initial data also in $H_2$. For such solutions in $H_2$ one has that $F_t = 2k \tau^{-1} [0, 0, 3u^\tau u^\tau_0 + u^\tau u^\tau_{ttt}] \in L_\infty(H_0)$ and $[A^\tau U_0 - F(0)] \in H_0$. The latter can be deduced directly from the regularity of solutions in $H_2$ and Sobolev’s embeddings in the dimensions.
of $\Omega$ less or equal to three. This allows to consider the dynamics in the variable $V^r = U^r_t \in \mathbb{H}_0$ which leads to a familiar MGT equation in $\mathbb{H}_0$

$$\tau v^r_{ttt} + v^r_t + c^2 Au^r + bAv^r = G'(u^r) = 2k(3u^r_t u^r_{tt} + u^r T u^r_{ttt}) \in L_2(\Omega)$$

(4.3)

Repeating the calculations leading to the proof of (2.6), but applied to (4.3) we obtain

$$\gamma \int_0^t |v^r_t|^2\,d\sigma \leq |\int_0^t (G'(u^r), m(\sigma))\,d\sigma| + C(||A^{1/2} & 0\,v^r(0)||_2^2 + ||A^{1/2} v^r_t(0)||_2^2 + \tau ||v^r_{tt}(0)||_2^2)$$

(4.4)

where $m(t) := v^r_{tt} + c^2 b^{-1} v^r_t$. This leads to

$$\int_0^t \gamma ||u^r_{ttt}(\sigma)||_2^2 \,d\sigma \leq C[\gamma ||u^r_{ttt}(0)||_2^2 + \tau^{-1}||U_0||_{2m}^2 + \int_0^t (G'(u^r(\sigma)), m(\sigma))\,d\sigma].$$

(4.5)

It remains to estimate the nonlinear terms and $u^r_{ttt}(0)$. For the nonlinear terms we apply the estimates in Theorem 1.5 and Theorem 1.6. We thus obtain (for each $\varepsilon > 0$):

$$(2k)^{-1}(G'(u^r)), m) = 3(u^r_t u^r_{tt} + u^r_t u^r_{tt}, u^r_{tt}) + \frac{c^2}{2} (u^r_t u^r_{tt} + u^r_t u^r_{tt}, u^r_{tt})$$

$$\leq C_1(\varepsilon)||u^r_t||^2_4 + ||u^r||_L^4 + ||u^r_t||_2^4 + (\varepsilon + ||u^r||_L^\infty)||u^r_{ttt}||_2^2$$

$$\leq C_1(\varepsilon)||u^r_t||_2^2 + ||u^r_t||_4^4 + ||u^r_t||_2^4 + (\varepsilon + ||u^r||_H^2||u^r||_H^2)||u^r_{ttt}||_2^2$$

$$\leq C_1(\varepsilon)\mathbb{E}^{x}(0)|\mathbb{E}^{x}(0)|^{1/2} + (\varepsilon + |\mathbb{E}^{x}(\sigma)|)^{1/4}|\mathbb{E}^{x}(\sigma)|^{1/4})||u^r_{ttt}(\sigma)||_2^2, (4.6)$$

for all $\sigma > 0$. Returning back in (4.5) we have

$$\int_0^t \left(\gamma - \varepsilon - |\mathbb{E}^{x}(\sigma)|^{1/2}|\mathbb{E}^{x}(\sigma)|^{1/2}\right) ||u^r_{ttt}(\sigma)||_2^2 \,d\sigma \leq C_1(\varepsilon)\mathbb{E}^{x}(0)|\mathbb{E}^{x}(0)|^{1/2} + C\tau^{-1}||U_0||_{2m}^2 + C\tau||u^r_{ttt}(0)||_2^2.$$ (4.7)

Going back to the original equation, evaluating it at $t = 0$, gives

$$||u^r_{ttt}(0)||_2^2 \leq \tau^{-2}||A U_0||_2^2 + ||Au_1||_2^2 + ||u_2||_2^2 + ||u_0 u_2 + u^r_1||_2^2 \leq \tau^{-2}C(||U_0||_{2m}^2 + ||Au_0||_2^2 ||u_2||_2^2 + ||A^{1/2} u_1||_2^2$$

$$||u^r_{ttt}(0)||_2^2 \leq \tau^{-2}C(||U_0||_{2m}^2), (4.8)$$

and going back to (4.4), recalling positivity of $\gamma^r$, smallness of $\varepsilon$ and $\rho$ – note that the smallness of initial data propagates in time along the trajectories – and boundedness of $\mathbb{E}^{x} \mathbb{E}^{r}$, the assertion follows.

**Proof of Theorem 4.2.** We denote $2E_{u_0}(t) := ||u^r_0(t)||_2^2 + ||A^{1/2} u_0(0)||_2^2$, $2E_{u_1}(t) := ||u^r_1||_2^2 + ||A^{1/2} u^r_1(t)||_2^2 + ||A u_0(0)||_2^2$ and $2E_u(t) := ||A^{1/2} u^r_0(t)||_2^2 + ||Au(0)||_2^2$. From the assumption, then, we have, for each $M > 0$ the existence of a sufficiently small $\rho_M$ such that $E_u(0) \leq M$ and $E_{u_0}(0) \leq \rho_M$. The following calculations are patterned after [18] – however the result proved below has less regular initial data along with tighter control of the required smallness – a fact which is needed for the ultimate result. First, taking the $L^2$-inner product of (1.9) with $u^r_0(t) \in \mathcal{D}(A)$ we have

$$k(u^0, (u^0)^2) = \frac{d}{dt} \left[ \frac{1}{2} ||u^0||_2^2 + \frac{c^2}{2} ||A^{1/2} u^0||_2^2 + k(u^0, (u^0)^2) \right] + b||A^{1/2} u^0||_2^2,$$

(4.9)

integration in time then implies

$$\frac{1}{2} ||u^0||_2^2 + \frac{c^2}{2} ||A^{1/2} u^0||_2^2 + b \int_0^t ||A^{1/2} u^0(\sigma)||_2^2 \,d\sigma \leq C_1 E_u(0) - k(u^0, (u^0)^2) \bigg|_0^t - k \int_0^t (u^0(\sigma), (u^0(\sigma))^2)\,d\sigma.$$ (4.10)
and then the embeddings $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and $H^{1/2}(\Omega) \hookrightarrow L^3(\Omega)$ allows us to rewrite (4.10) as
\[
(C_1 - \|u^0\|_{L^\infty})\|u^0_t\|_2^2 + \frac{c^2}{2}\|A^{1/2}u^0_t\|_2^2 + \int_0^t \left[ b\|A^{1/2}u^0_\sigma - C_2\|u^0_\sigma\|_{H^{1/2}}^3 \right] d\sigma \\
\leq C_1 E_{u,0}(0) + k\|u^0(0)\|_{L^\infty}\|u^0_0(0)\|_2^2. \tag{4.11}
\]

Next, we take the $L^2$-inner product of (1.9) with $Au^0(t) \in L^2(\Omega)$ we have
\[
(G(u^0), Au^0) = \frac{d}{dt} \left[ (u^0_t, Au^0) + \frac{b}{2}\|A^{1/2}u^0_t\|_2^2 \right] - \|A^{1/2}u^0_t\|_2^2 + c^2\|Au^0\|_2^2, \tag{4.12}
\]
and with a combination of H"older’s Inequality and Sobolev embeddings we get, after integration by parts, that for every $\varepsilon > 0$,
\[
C_1\|Au^0\|_2^2 + (c^2 - \varepsilon) \int_0^t \|Au^0(\sigma)\|_2^2 d\sigma \leq C_2 E_{u,0}(0) + \int_0^t \|A^{1/2}u^0_\sigma\|_2^2 d\sigma \\
+ C_3\|u^0_t\|_2^2 + C_4(\varepsilon) \int_0^t \left[ \|u^0(\sigma)\|_{H^{1/2}}^2 + \|u^0_t(\sigma)\|_{H^{1/2}}^2 \right] d\sigma, \tag{4.13}
\]
where we have used the embeddings $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and $H^{3/4}(\Omega) \hookrightarrow L^2(\Omega)$. Finally, taking the $L^2$-inner product of (1.9) with $u^0_{tt}(t) \in L^2(\Omega)$ we have
\[
(G(u^0), u^0_{tt}) = \frac{d}{dt} \left[ c^2(u^0_t, Au^0_\tau) + \frac{b}{4}\|A^{1/2}u^0_t\|_2^2 \right] + \|u^0_\tau\|_2^2 - c^2\|A^{1/2}u^0_\tau\|_2^2, \tag{4.14}
\]
and then integrating from $0$ to $t$ in $[0,T]$ we have
\[
\int_0^t (1 - 2k\|Au^0(t)\|_{H^{1/2}}^2)\|A^{1/2}u^0(t)\|_{H^{1/2}}^2 \|u^0_t(\sigma)\|_2^2 d\sigma + C_1 \left[ \|A^{1/2}u^0_\tau(\sigma)\|_2^2 - \|u^0_\tau(\tau)\|_{H^{1/2}}^2 \right] \\
\leq C_2 E_{u,0}(0) + C_3 \left[ \int_0^t \|A^{1/2}u^0(\sigma)\|_2^2 d\sigma + \|A^{1/2}u^0_\tau\|_2^2 \right], \tag{4.15}
\]
where we have used the embedding $H^{1/2}(\Omega) \hookrightarrow L^3(\Omega)$. Using the inequalities $\|u\|_{L^\infty} \leq C\|Au\|_{H^{1/2}}^2$ and $\|u\|_{H^{1/2}} \leq \|u_0\|_{H^{1/2}}\|A^{1/2}u_0\|_{H^{1/2}}^2$ both dominated above by $E_{u,0}^{1/4}c_{u}^{1/4}$ and adding the expressions generated by each multiplier, the estimate (4.2) follows. The exponential decay follow from the given estimates in a standard manner via Barrier’s Method where we account for smallness of $E_u$ energy.

Now we are in position to prove Theorem 4.1.

**Proof of Theorem 4.1:** If $u^\tau$ is the solution of (1.4) and $u^0$ is the solution (1.9), then $x^\tau := u^\tau - u^0$ solves the following equation:
\[
x^\tau_{tt} + c^2Ax^\tau + bAx^\tau = -\tau u^\tau_{\tau\tau} + 2ku^\tau_x x^\tau + 2ku^0_x x^\tau + 2ku^\tau_{tt} x^\tau + 2ku^0_{tt} x^\tau \tag{4.16}
\]
with the zero initial conditions. Let $L(t)$ and $R(t)$ denote the left hand side and right hand side of (4.16). We start by taking the $L^2$-inner product of (4.16) with $Ax^\tau$ and $x^\tau_{tt}$, which – for the left hand side – give
\[
\int_0^t \int L(\sigma, Ax^\tau(\sigma))d\sigma = \frac{1}{2}\|A^{1/2}x^\tau\|_2^2 + \frac{c^2}{2}\|Ax^\tau\|_2^2 + b\int_0^t \|Ax^\tau(\sigma)\|_2^2 d\sigma \tag{4.17}
\]
\[
+ \int_0^t \int L(\sigma, x^\tau_{tt}(\sigma))d\sigma = \frac{b}{2}\|A^{1/2}x^\tau\|_2^2 + \int_0^t \|x^\tau_{tt}(\sigma)\|_2^2 d\sigma + c^2\int_0^t (Ax^\tau(\sigma), x^\tau_{tt}(\sigma))d\sigma \tag{4.18}
\]
and, for every $\varepsilon > 0$, the right hand sides give
\[
\int_0^t (R(\sigma), Ax_t^\tau(\sigma))d\sigma = \int_0^t (-\tau u_{ttt}^\tau + 2ku_t^\tau x_t^\tau + 2ku_t^0 x_t^\tau + 2ku_t^0 x_t^\tau, Ax_t^\tau)\ d\sigma
\]
\[
\leq \tau \int_0^t (u_{ttt}^\tau(\sigma), Ax_t^\tau(\sigma))d\sigma + \varepsilon \int_0^t \|Ax_t^\tau(\sigma)\|_2^2 d\sigma + C(\varepsilon) \int_0^t [\|u_t^\tau x_t^\tau\|_2^2 + \|u_t^\tau x_t^\tau\|_2^2 + \|u_t^0 x_t^\tau\|_2^2 + \|u_t^0 x_t^\tau\|_2^2] \ d\sigma
\]
(4.19)

and
\[
\int_0^t (R(\sigma), x_{ttt}^\tau(\sigma))d\sigma = \int_0^t (-\tau u_{ttt}^\tau + 2ku_t^\tau x_t^\tau + 2ku_t^0 x_t^\tau + 2ku_t^0 x_t^\tau, x_{ttt}^\tau)\ d\sigma
\]
\[
\leq \tau \int_0^t (u_{ttt}^\tau(\sigma), x_{ttt}^\tau(\sigma))d\sigma + \varepsilon \int_0^t \|x_{ttt}^\tau(\sigma)\|_2^2 d\sigma + C(\varepsilon) \int_0^t [\|u_t^\tau x_t^\tau\|_2^2 + \|u_t^\tau x_t^\tau\|_2^2 + \|u_t^0 x_t^\tau\|_2^2 + \|u_t^0 x_t^\tau\|_2^2] \ d\sigma.
\]
(4.20)

Moreover, notice that by Lemma 4.1
\[
\int_0^t (\tau u_{ttt}^\tau(\sigma), Ax_t^\tau(\sigma) + x_{ttt}^\tau(\sigma))d\sigma \leq \varepsilon \int_0^t \|Ax_t^\tau(\sigma)\|_2^2 + \|x_{ttt}^\tau(\sigma)\|_2^2 d\sigma + C_1(\varepsilon) \int_0^t \|u_{ttt}^\tau(\sigma)\|_2^2 d\sigma
\]
\[
\leq \varepsilon \int_0^t \|Ax_t^\tau(\sigma)\|_2^2 + \|x_{ttt}^\tau(\sigma)\|_2^2 d\sigma + \tau C_3(\varepsilon)(\|u_0^\tau\|_{L^2}^2),
\]
(4.21)

Then, adding the corresponding sides, equating left and right and taking $\varepsilon$ small we have
\[
E_{x\tau}(t) + C_1 \int_0^t [\|x_{ttt}^\tau(\sigma)\|_2^2 + \|Ax_t^\tau(\sigma)\|_2^2] d\sigma \leq C_2 \int_0^t \|Ax_t^\tau(\sigma)\|_2^2 d\sigma + \tau C_3(\|u_0^\tau\|_{L^2}^2)
\]
\[
+ C_4 \int_0^t [\|u_t^\tau x_t^\tau\|_2^2 + \|u_t^\tau x_t^\tau\|_2^2 + \|u_t^0 x_t^\tau\|_2^2 + \|u_t^0 x_t^\tau\|_2^2] d\sigma.
\]
(4.22)

We now estimate the four nonlinear terms. We have,
\[
\int_0^t \|u^\tau(x_{ttt}^\tau(\sigma))\|_2^2 d\sigma \leq \int_0^t \|u^\tau\|_{L^\infty}^2 \|x_{ttt}^\tau(\sigma)\|_2^2 d\sigma \leq C_1 \int_0^t \|A^{1/2} u^\tau\|_2 \|Au^\tau\|_2 \|x_{ttt}^\tau(\sigma)\|_2^2 d\sigma
\]
\[
\leq C_1[E^\tau(0)]^{1/2}[E^\tau(0)]^{1/2} \int_0^t \|x_{ttt}^\tau(\sigma)\|_2^2 d\sigma.
\]
where we have used a priori bounds for $u^\tau$ resulting from Theorem 1.5. The later controls small $H_0$ norms of solutions in terms of small initial data in that space (along with the bounded (not small) $H_1$ norms.) Similarly,
\[
\int_0^t \|u^\tau(\sigma)x^\tau(\sigma)\|_2^2 d\sigma \leq \int_0^t \|u^\tau\|_{L^\infty} \|x^\tau(\sigma)\|_2^2 d\sigma \leq C_1 \int_0^t \|A^{1/2}x^\tau(\sigma)\|_2 \|Ax^\tau(\sigma)\|_2 \|u_{ttt}^\tau\|_2^2 d\sigma
\]
\[
\leq \varepsilon \sup_{\sigma \in [0,T]} \|Ax^\tau(\sigma)\|_2^2 \left( \int_0^t \|u_{ttt}^\tau(\sigma)\|_2^2 d\sigma \right)^2 + C_2(\varepsilon) \sup_{\sigma \in [0,T]} \|A^{1/2}x^\tau(\sigma)\|_2^2,
\]
\[
\int_0^t \|u^\tau(\sigma)x_t^\tau(\sigma)\|_2^2 d\sigma \leq \int_0^t \|x_t^\tau\|_{L^\infty} \|u^\tau(\sigma)\|_2^2 d\sigma \leq C_1[E^\tau(0)]^{1/2}[E^\tau(0)]^{1/2} \int_0^t \|x_t^\tau(\sigma)\|_2^2 d\sigma,
\]
and finally
\[
\int_0^t \|u^\tau(\sigma)x_t^\tau(\sigma)\|_2^2 d\sigma \leq \int_0^t \|x_t^\tau\|_{L^\infty} \|u^\tau(\sigma)\|_2^2 d\sigma \leq C_1[E^\tau(0)]^{1/2}[E^\tau(0)]^{1/2} \int_0^t \|Ax_t^\tau(\sigma)\|_2^2 d\sigma.
\]
Now, taking $\varepsilon$ small and accounting, also, for smallness of $E^*(0)$ we rewrite (4.22) as

$$\|Ax^\tau(t)\|_2^2 + \|A^{1/2}x^\tau(t)\|_2^2 + C_1 \int_0^t [\|x^\tau_t(\sigma)\|_2^2 + \|Ax^\tau_t(\sigma)\|_2^2]d\sigma \leq \tau C_2(\|U_0\|_{H_2}^2) + C_3 \int_0^t E_{x^\tau}(\sigma)d\sigma,$$

(4.23)

where we have used the fact that

$$\|A^{1/2}x^\tau(t)\|_2^2 \leq 2 \int_0^t \|A^{1/2}x^\tau_t(\sigma)\|_2^2 d\sigma \leq C_1 \int_0^t [\|A^{1/2}x^\tau_t(\sigma)\|_2^2 + \|A^{1/2}x^\tau\|_2^2]d\sigma.$$

The final estimate then follows by Gronwall’s Inequality, that is,

$$\|Ax^\tau(t)\|_2^2 + \|A^{1/2}x^\tau(t)\|_2^2 + C_1 \int_0^t [\|x^\tau_t(\sigma)\|_2^2 + \|Ax^\tau_t(\sigma)\|_2^2]d\sigma \leq \tau e^{\tau t} C_2(\|U_0\|_{H_2}^2)$$

(4.24)

This completes the proof of the rate of convergence. We reiterate that these results hold under the assumption that the energy is $H_0$-small and $H_2$-finite.

**Remark 4.2.** Note that the proof of convergence rates holds on every fixed time interval. In order to obtain the convergence on the entire $\mathbb{R}^+$ we need to appeal to the uniform decay rates of the solutions of limiting and the limit problems.

**Strong convergence.** Now we move our attention to proving the strong convergence claimed in Part (b) of Theorem 1.7. We start by recalling that given $\varepsilon > 0$ and any $T > 0$, there exists $\tau \leq \tau_0(\varepsilon, T)$ we have

$$\|P(U^\tau(t, U_0)) - U^0(t, PU_0)\|_{D(A) \times D(A^{1/2})} \leq \varepsilon/3, \quad t \in [0, T]$$

(4.25)

for any initial data $U_0 \in H_2$ which are small in $H_0$. Our goal is to prove that the above inequality holds for all initial data $U_0, \in H_1$ which are small in $H_0$ and this is equivalent to showing

$$\|Ax^\tau(t)\|_2^2 + \|A^{1/2}x^\tau_t(t)\|_2^2 \leq \varepsilon \quad \text{for all} \quad \tau \in \Lambda, t > 0$$

(4.26)

**Remark 4.3.** Proving (4.26) is valid for the initial data in $H_1$ depends on the density of $H_2 \subset H_1$. It is essential here that the smallness of initial data is required only in $H_0$ -since otherwise smallness of the approximation in $H_2$ topology can not be guaranteed without having the smallness of the element approximated. This is the part where the fact that the analysis requires only $H_0$ smallness is critical.

The convergence stated below follows essentially from the proofs of Theorem 1.5 and Theorem 4.2. For reader’s convenience we will sketch the main ingredients of the argument.

**Lemma 4.2.** Let $U_0 \in H_1$ with $\|U_0\|_{H_0} \leq \rho$ ; and $\{U_0\}_{n \in \mathbb{N}} \subset H_2$ such that $\|U_n\|_{H_0} \leq C\rho$, with $\rho > 0$ sufficiently small, and $U_0^n \to U_0$ in $H_1$ as $n \to \infty$. Then, as $n \to \infty$ we have

a) $PU^\tau(t, U_0^n) \to PU^\tau(t, U_0)$ in $P^H_1 = D(A) \times D(A^{1/2})$, for every $\tau \in (0, \tau_0)$ and every $t \in [0, T], \quad T > 0$.

b) $U^0(t, PU_0^n) \to U^0(t, PU_0)$ in $P^H_1 = D(A) \times D(A^{1/2})$, for every $t \in [0, T], \quad T > 0$.

**Proof.** Let $u^\tau_n(t) := U^\tau(t, U_0^n), \quad u_0^\tau(t) := U^\tau(t, U_0)$ and $w^\tau_n := u_n^\tau - u_0^\tau$. Notice that $w^\tau_n$ satisfies (2.4) with $G(w^\tau_n)$ given by

$$G(w^\tau_n) = u_{n,t}^\tau w^\tau_n + u_0^\tau(w^\tau_{n,t}) + (u^\tau_n + u_{0,t}^\tau)w^\tau_n$$

(4.27)
which needs to be tested against the multipliers $w^\tau_n t_t, w^\tau_n u_t, u^\tau_n u_t$ and $A w^\tau_n u$. First notice that for all $\varepsilon > 0$ and $f \in L^2(\Omega)$ we have
\[
(G(w^\tau_n), f) \leq C(\varepsilon) \left( \| u^\tau_n t_t \|^2 + \| u^\tau_n u_t \|^2 + \| w^\tau_n u_t \|^2 + \| f \|^2 \right) + \varepsilon \| f \|^2
\]
\[
\leq C(\varepsilon) \left( \| U^\tau_n \|^2_{\mathbb{H}_0} + \| U^\tau_n \|^2_{\mathbb{H}_0} \right) \left( \| w^\tau_n u_t \|^2_{\mathbb{H}_1} + \| f \|^2 \right) \leq C(\varepsilon) \rho^2 \left( \| w^\tau_n \|^2_{\mathbb{H}_1} + \| f \|^2 \right),
\]
therefore, all the nonlinear terms can be estimated above by $(C(\varepsilon)\rho^2 + \varepsilon)(\| w^\tau_n \|^2_{\mathbb{H}_1} + \| f \|^2)$ with $f$ equal to one of $u_t u_t, Au u_t$ are absorbed by the dissipation due to positivity of $\gamma, b, c > 0$ – see steps 1 to 4 of the proof of Theorem 1.5 on Section 2) which can absorb the energy terms since we have smallness of $\rho$ and we can choose $\varepsilon$ small. Therefore, $\mathcal{E}^\tau[u^\tau_n]$ (energy in $\mathbb{H}_1$ with respect to $w^\tau_n$) is such that
\[
\mathcal{E}^\tau[u^\tau_n](t) + C_1 \int_0^t \mathcal{E}^\tau[w^\tau_n](\sigma)d\sigma \leq C_2 \| U^\tau_n \|^2_{\mathbb{H}_0} - U^\tau_n \|_{\mathbb{H}_1},
\]
which implies part a). Part b) follows by following the proof of Theorem 4.2 applied to equation (1.9).

To finalize the proof of Theorem 1.7 we need one more step based on the diagonal argument.

Let $U_0 \in \mathbb{H}_1, \| U_0 \|_{\mathbb{H}_0} \leq \rho$ and let $\{ U_0^n \}_{n \in \mathbb{N}} \subset \mathbb{H}_2$ such that $U_0^n \to U_0$ in $\mathbb{H}_1$ with $\mathbb{H}_0$-norm sufficiently small $\leq C\rho$. Combining (4.25) and Lemma 4.2 we obtain:
\[
\| PU^\tau(t, U_0^n) - U^\tau(t, PU_0^n) \|_{\mathcal{D}(A) \times \mathcal{D}(A^{1/2})} \leq \| PU^\tau(t, U_0^n) - PU^\tau(t, U_0^n) \|_{\mathcal{D}(A) \times \mathcal{D}(A^{1/2})} + \| U^\tau(t, PU_0^n) - U^\tau(t, PU_0^n) \|_{\mathcal{D}(A) \times \mathcal{D}(A^{1/2})}
\]
\[
\leq C_1 \int_0^t \mathcal{E}^\tau[w^\tau_n](\sigma)d\sigma \leq C_2 \| U^\tau_n \|^2_{\mathbb{H}_0} - U^\tau_n \|_{\mathbb{H}_1},
\]
The strong convergence in finite time is then proved.

For a given $\varepsilon > 0$ we select $N$ (Lemma 4.2) such that for all $t \in [0, T]$
\[
\| PU^\tau(t, U_0^n) - PU^\tau(t, U_0) \|_{\mathcal{D}(A) \times \mathcal{D}(A^{1/2})} \leq \frac{\varepsilon}{3}, \quad \tau \in \Lambda
\]
\[
\| PU^\tau(t, U_0^n) - PU^\tau(t, U_0) \|_{\mathcal{D}(A) \times \mathcal{D}(A^{1/2})} \leq \frac{\varepsilon}{3}.
\]
The continuity of $PU^\tau(t, U_0)$ is uniform in $\tau \in \Lambda$, hence
\[
\| PU^\tau(t, U_0) - U^\tau(t, PU_0) \|_{\mathcal{D}(A) \times \mathcal{D}(A^{1/2})} \leq \| PU^\tau(t, U_0) - PU^\tau(t, U_0^n) \|_{\mathcal{D}(A) \times \mathcal{D}(A^{1/2})}
\]
\[
\| PU^\tau(t, U_n) - U^\tau(t, PU^n N) \|_{\mathcal{D}(A) \times \mathcal{D}(A^{1/2})} + \| U^\tau(t, PU^n N) - U^\tau(t, PU_0^n) \|_{\mathcal{D}(A) \times \mathcal{D}(A^{1/2})} \leq \varepsilon + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\]
for all $t \in [0, T]$ and $\tau \in (0, \tau_0(\varepsilon, T)]$, where (4.25) were used in the middle term.

For the infinity time case we appeal for the uniform (in $\tau$) exponential stability in the following manner: we let $T_\varepsilon$ denote a time big enough such that
\[
\int_{T_\varepsilon}^{\infty} \rho^2(\sigma)d\sigma \leq C \int_{T_\varepsilon}^{\infty} e^{-\alpha \sigma} d\sigma \| U_0 \|_{\mathbb{H}_1} \leq \frac{\varepsilon}{2},
\]
which is possible since $2\mathcal{E}(t) \leq \| Au^\tau(t) \|^2 + \| Au^\tau(t) \|^2 + \| A^{1/2} \| \| u^\tau(t) \|^2 + \| A^{1/2} \| \| u^\tau(t) \|^2$ and therefore, uniform stability is inherited from the uniform stability asserted by Theorems 1.5 and 4.2. Finally, by taking $T = T_\varepsilon$ in our finite time proof, the result of Theorem 1.7 follows.

\section{Proof of Theorem 1.4}

As a starting point, we are going to use (see [2, 19, 20]), that given $f \in L^1(0, T; \mathcal{D}(A^{1/2}))$ and $\alpha > 0$, the linear problem
\[
\tau u_{ttt} + \alpha u_{tt} + c^2 Au + b Au_t = f,
\]
is wellposed – in the variable \( U = (u, u_t, u_{tt}) \) – and exponentially stable (with rates independent of \( \tau \) for \( \tau \) small) for initial data in \( \mathbb{H}_i \) \((i = 0, 1, 2)\). This is to say that, denoting by \( t \mapsto S(t) \) the semigroup generated by the evolution (A.1), there exist constants \( \omega_i, M_i > 0, \) \((i = 0, 1, 2)\) such that

\[
\|U(t)\|_{\mathbb{H}_i} = \|S(t)U_0\|_{\mathbb{H}_i} \leq M_i e^{-\omega_i t} \|U_0\|_{\mathbb{H}_i}.
\]  
(A.2)

Define \( X \) as the set

\[
X = \left\{ W = (w, w_t, w_{tt})^T \in C(0, T; \mathbb{H}_2) ; \sup_{t \in [0, T]} \|W(t)\|_{\mathbb{H}_2} < \infty \text{ and } \sup_{t \in [0, T]} \|W(t)\|_{\mathbb{H}_4} < \eta \right\}
\]

\((\eta > 0 \text{ will be taken to be sufficiently small later}) \) and equip it with the norm

\[
\|W\|_X^2 := \sup_{t \in [0, T]} \|W(t)\|_{\mathbb{H}_2}^2.
\]

Recalling the interpolation inequalities

\[
\|g\|_{L^\infty} \leq C \|g\|_{D(A^{1/2})}^{1/2} \|g\|_{D(A)}^{1/2}, \ g \in D(A)  \tag{A.3}
\]

and

\[
\|g\|_{L^4} \leq C \|g\|_{D(A)}^{1/4} \|g\|_{D(A^{1/2})}^{3/4}, \ g \in D(A^{1/2})  \tag{A.4}
\]

we observe that if \( W = (w, w_t, w_{tt}) \in X \), it follows that \( w, w_t \in D(A) \hookrightarrow L^{\infty}(\Omega) \) and \( w_{tt} \in D(A^{1/2}) \hookrightarrow L^4(\Omega) \) \((n = 2, 3, 4)\), for each \( t \in [0, T] \), and therefore \( f(w) := 2k(w_t^2 + w w_{tt}) \in C(0, T; D(A^{1/2})) \). This means that for each \( W \in X \), \( f(w) \) qualifies to be the right hand side of (A.1), and therefore it makes well defined the application \( \Upsilon \) that associates each \( W \in X \) to the solution \( (u, u_t, u_{tt})^T = U := \Upsilon(W) \in C(0, T; \mathbb{H}_2) \) for (A.1) with initial condition \( U_0 = (u(0), u_t(0), u_{tt}(0))^T \in \mathbb{H}_2 \). Moreover, the solution \( U \) is represented by the variation of parameters formula, i.e., for each \( t \in [0, T] \),

\[
U(t) = \Upsilon(W)(t) = S(t)U_0 + \int_0^t S(t - \sigma) (0, 0, \tau^{-1}f(w(t)))^T d\sigma. \tag{A.5}
\]

In addition, \( \Upsilon \) maps \( X \) into itself. In fact, for each \( t \in [0, T] \), uniform (in \( \tau \)) exponential stability implies that

\[
\|\Upsilon(W)(t)\|_{\mathbb{H}_2} \leq \|S(t)U_0\|_{\mathbb{H}_2} + \left\| \int_0^t (t - \sigma) F_\tau(W)(\sigma) d\sigma \right\|_{\mathbb{H}_2} \leq M_2 \|U_0\|_{\mathbb{H}_2} + \int_0^t M_2 e^{-\omega_2 (t - \sigma)} \|F_\tau(W)(\sigma)\|_{\mathbb{H}_2} d\sigma \leq M_2 \left( \|U_0\|_{\mathbb{H}_2} + \frac{C_\omega}{\tau} \sup_{t \in [0, T]} \|f(w(t))\|_{D(A^{1/2})} \right). \tag{A.6}
\]

and again for each \( t \in [0, T] \) – mostly omitted on the computations below – we estimate

\[
(2k)^{-1} \|f(w)\|_{D(A^{1/2})} \sim \|\nabla(w_t^2 + w w_{tt})\|_2 \leq 2\|w_t\|_{L^\infty} \|\nabla w_t\|_2 + \|w\|_{L^4} \|\nabla w_{tt}\|_2 + \|w_{tt}\|_{L^4} \|\nabla w\|_{L^4} \leq C \left[ \|w_t\|_{D(A^{1/2})}^{1/2} \|w_t\|_{D(A)}^{1/2} \|\nabla w_t\|_2 + \|w\|_{D(A^{1/2})}^{1/2} \|w\|_{D(A)}^{1/2} \|\nabla w_{tt}\|_2 \right]
\]

20
\[ + C \left( \| w_{tt} \|^2_{H^{1/2}} \right) \leq C \left( \sup_{t \in [0, T]} \| W(t) \|^2_{H^2} \right)^{1/2} \eta^{1/2}, \tag{A.7} \]

and then, back in (A.6) we conclude that
\[
\sup_{t \in [0, T]} \| Y(W)(t) \|_{H^5} \leq M_2 \left( \| U_0 \|_{H^6} + \frac{2kC_\omega}{\tau} \left( \sup_{t \in [0, T]} \| W(t) \|_{H^2} \right)^{1/2} \right) < \infty. \tag{A.8} \]

Similarly for \( H^7_0 \), for each \( t \in [0, T] \) we have have
\[
\| Y(W)(t) \|_{H^7} \leq M_1 \left( \| U_0 \|_{H^6} + \frac{C_\omega}{\tau} \sup_{t \in [0, T]} \| f(w)(t) \|_2 \right), \tag{A.9} \]

Moreover, for each \( t \in [0, T] \) – again mostly omitted – it holds that
\[
(2k)^{-1} \| f(w)(t) \|_2 = \| w_0^2 + w_{tt} \|_2 \\
\leq \| w_0^2 \|_{L^1} + \| f \|_{L^\infty} \| w_{tt} \|_2 \\
\leq C \left( \left( \sup_{t \in [0, T]} \| W(t) \|_{H^2} \right)^2 + \left( \sup_{t \in [0, T]} \| W(t) \|_{H^2} \right)^{1/2} \right)^{3/2} \\
\leq C \left\{ \eta^2 + \left[ \left( \sup_{t \in [0, T]} \| W(t) \|_{H^2} \right)^{1/2} \right] \right\},
\]

then, taking \( \eta = \eta(\tau) \) small enough so that
\[
\frac{2kC_\omega}{\tau} \left[ \eta^2 + \left( \sup_{t \in [0, T]} \| W(t) \|_{H^2} \right)^{1/2} \right] < \frac{\eta}{2M_1}
\]

and, as a consequence, \( \rho = \rho(\tau) \) small enough such that \( \rho < \frac{\eta}{2M_1} \), we can return to (A.9) to conclude that if \( \| U_0 \|_{H^6} < \rho \) we have
\[
\sup_{t \in [0, T]} \| Y(W)(t) \|_{H^6} \leq M_1 \left\{ \rho + \frac{2kC_\omega}{\tau} \left[ \eta^2 + \left( \sup_{t \in [0, T]} \| W(t) \|_{H^2} \right)^{1/2} \right] \right\} < \eta, \tag{A.10} \]

which completes the proof of invariance.

Next we prove that \( Y \) is a contraction if \( \eta \) is sufficiently small. For this, let \( W_1 = (w_1, w_{1t}, w_{1tt})^T, W_2 = (w_2, w_{2t}, w_{2tt})^T \in X \) and notice that,
\[
\| Y(W_1) - Y(W_2) \|_X = \sup_{t \in [0, T]} \| \int_0^t S(t - \sigma) \left[ F_r(W_1)(\sigma) - F_r(W_2)(\sigma) \right] d\sigma \|_{H^5} \\
\leq \frac{C_\omega}{\tau} \sup_{t \in [0, T]} \| f(w_1)(t) - f(w_2)(t) \|_{D(A^{1/2})}. \tag{A.11} \]

Next, observe that for each \( t \in [0, T] \) – mostly omitted on the computations below – we have
\[
(2k)^{-1} \| f(w_1)(t) - f(w_2)(t) \|_{D(A^{1/2})} = \| (w_{tt} + w_{tt}) (w_{tt} - w_{tt}) + (w_1 - w_2) w_{tt} + (w_1 - w_2) w_{tt} \|_{D(A^{1/2})}
\]

21
where

\[ Q_1(t) = \| (w_{1t} + w_{2t})(w_{1t} - w_{2t}) \|_{D(A^{1/2})}, \quad Q_2(t) = \| (w_1 - w_2)w_{1tt} \|_{D(A^{1/2})}, \quad Q_3(t) = \| (w_{1tt} - w_{2tt})w_2 \|_{D(A^{1/2})} \]

and then we estimate these three quantities (for each \( t \in [0, T] \)):

\[
Q_1(t) = \| (w_{1t} + w_{2t})(w_{1t} - w_{2t}) \|_{D(A^{1/2})} \\
\sim \| \nabla ((w_{1t} + w_{2t})(w_{1t} - w_{2t})) \|_2 \\
= \| (w_{1t} + w_{2t})\nabla (w_{1t} - w_{2t}) \|_2 + \| \nabla [(w_{1t} + w_{2t})](w_{1t} - w_{2t}) \|_2 \\
\leq \| w_{1t} + w_{2t} \|_{L^\infty} \| w_{1t} - w_{2t} \|_{D(A^{1/2})} + \| \nabla (w_{1t} + w_{2t}) \|_{L^\infty} \| w_{1t} - w_{2t} \|_{L^4} \\
\leq C \| w_{1t} + w_{2t} \|_{D(A^{1/2})} \| w_{1t} - w_{2t} \|_{D(A^{1/2})}^{1/2} + \| \nabla (w_{1t} + w_{2t}) \|_{D(A^{1/2})}^{3/4} \| w_{1t} - w_{2t} \|_{D(A^{1/2})}^{1/4} \\
\leq C \left[ \| (W_1 + W_2)(t) \|_{H^s} \right]^{1/2} \left[ \| (W_1 + W_2)(t) \|_{H^s} \right]^{1/2} \| (W_1 - W_2)(t) \|_{H^s} \\
+ C \left[ \| (W_1 + W_2)(t) \|_{H^s} \right]^{3/4} \| (W_1 - W_2)(t) \|_{H^s}^{3/4} \| (W_1 - W_2)(t) \|_{H^s} \\
\leq C \left\{ \eta^{1/2} \left[ \sup_{t \in [0, T]} \| (W_1 + W_2(t)) \|_{H^s} \right]^{1/2} + \eta^{3/4} \left[ \sup_{t \in [0, T]} \| (W_1 + W_2(t)) \|_{H^s} \right]^{3/4} \left[ \sup_{t \in [0, T]} \| (W_1 - W_2(t)) \|_{H^s} \right] \right\} \| W_1 - W_2 \|_X \\
\leq C \eta^{1/4} \| W_1 - W_2 \|_X,
\]

where \( C \) is a constant that does not depend on time.

\[
Q_2(t) = \| (w_1 - w_2)w_{1tt} \|_{D(A^{1/2})} \\
\sim \| w_{1tt} \nabla (w_1 - w_2) + (w_1 - w_2) \nabla w_{1tt} \|_2 \\
\leq C \| w_{1tt} \|_2^{1/2} \| w_{1tt} \|_{D(A^{1/2})}^{3/4} \| \nabla (w_1 - w_2) \|_2^{1/4} \| \nabla (w_1 - w_2) \|_{D(A^{1/2})}^{3/4} \\
+ C \| w_1 - w_2 \|_{D(A^{1/2})}^{1/2} \| w_1 - w_2 \|_{D(A)}^{1/2} \| \nabla w_{1tt} \|_2 \\
\leq \frac{C}{\tau} \left[ \| W_1(t) \|_{H^n} \right]^{1/4} \left[ \| W_2(t) \|_{H^n} \right]^{3/4} \| (W_1 - W_2)(t) \|_{H^s} + \frac{C}{\tau} \left[ \| W_1(t) \|_{H^n} \right]^{1/2} \| W_2(t) \|_{H^s} \\
+ \frac{C}{\tau} \left[ \| W_1(t) \|_{H^n} \right]^{1/2} \| (W_1 - W_2)(t) \|_{H^s} \| (W_1 - W_2)(t) \|_{H^s} \\
\leq \frac{C}{\tau} \left[ \sup_{t \in [0, T]} \| W_1(t) \|_{H^n} \right]^{1/4} \left[ \sup_{t \in [0, T]} \| W_2(t) \|_{H^n} \right]^{3/4} \left[ \sup_{t \in [0, T]} \| W_1(t) \|_{H^s} \right] \left[ \sup_{t \in [0, T]} \| W_2(t) \|_{H^s} \right] \\
+ \frac{C}{\tau} \left[ \sup_{t \in [0, T]} \| W_1(t) \|_{H^n} \right]^{1/2} \left[ \sup_{t \in [0, T]} \| W_2(t) \|_{H^n} \right] \left[ \sup_{t \in [0, T]} \| W_1(t) \|_{H^s} \right] \\
+ \frac{C}{\tau} \left[ \sup_{t \in [0, T]} \| W_1(t) \|_{H^n} \right]^{1/2} \left[ \sup_{t \in [0, T]} \| W_2(t) \|_{H^n} \right] \left[ \sup_{t \in [0, T]} \| W_1(t) \|_{H^s} \right] \\
\leq Q_1(t) + Q_2(t) + Q_3(t),
\]

(A.12)
\[ \leq \frac{C}{\tau} \left\{ \eta^{1/4} \left[ \sup_{t \in [0,T]} \|W_1(t)\|_{H^2_x} \right]^{3/4} + \eta^{1/2} \left[ \sup_{t \in [0,T]} \|(W_1 - W_2)(t)\|_{H^2_x} \right] \right\} \|W_1 - W_2\|_X \]

\[ \leq \frac{C}{\tau} \eta^{1/4} \|W_1 - W_2\|_X, \]

where \( C \) is a constant that does not depend on time.

\[ Q_3(t) = \| (w_{1tt} - w_{2tt})w_2 \|_{D(A^{1/2})} \]
\[ \sim \| w_2 \nabla (w_{1tt} - w_{2tt}) \|_2 + (w_{1tt} - w_{2tt}) \nabla w_2 \|_2 \]
\[ \leq C \| w_{1tt} - w_{2tt} \|_2^{1/4} \| w_{1tt} - w_{2tt} \|_{D(A^{1/2})}^{3/4} \| \nabla w_2 \|_2^{1/4} \| \nabla w_2 \|_{D(A^{1/2})}^{3/4} \]
\[ + C \| w_2 \|_{D(A^{1/2})} \| \nabla (w_{1tt} - w_{2tt}) \|_2 \]
\[ \leq \frac{C}{\tau} \left\{ \left[ \sup_{t \in [0,T]} \|W_1(t)\|_{H^2_x} \right]^{3/4} + \left[ \sup_{t \in [0,T]} \|(W_1 - W_2)(t)\|_{H^2_x} \right]^{1/2} \right\} \|W_1 - W_2\|_X \]

\[ \leq \frac{C}{\tau} \left\{ \eta^{1/4} \left[ \sup_{t \in [0,T]} \|W_1(t)\|_{H^2_x} \right]^{3/4} + \eta^{1/2} \left[ \sup_{t \in [0,T]} \|(W_1 + W_2)(t)\|_{H^2_x} \right]^{1/2} \right\} \leq \frac{C}{\tau} \eta^{1/4} \|W_1 - W_2\|_X, \]

where \( C \) is a constant that does not depend on time.

Therefore, back in (A.11) we have

\[ \| \Upsilon(W_1) - \Upsilon(W_2) \|_X \leq C_{\omega} \sup_{t \in [0,T]} \| f(w_1(t)) - f(w_2(t)) \|_{D(A^{1/2})} \]
\[ \leq \frac{2kC_{\omega}}{\tau} \sup_{t \in [0,T]} [Q_1(t) + Q_2(t) + Q_3(t)] \leq \frac{2kTC}{\tau^2} \eta^{1/4} \|W_1 - W_2\|_X. \quad (A.13) \]

which means that \( \Upsilon \) is a contraction as long as we take a – possibly smaller – \( \eta = \eta(\tau) \) such that \( \eta < \left( \frac{\tau^2}{2kC_{\omega}} \right)^{16} \). This completes the proof of local wellposedness.

References

[1]

[2] Bongarti, M., Charoenphon, S., and Lasiecka, I. Singular thermal relaxation limit for the Moore–Gibson–Thompson equation arising in propagation of acoustic waves. *Semigroups of Operators: Theory and Applications SOTA-2018* (2019), 147–182. Publisher: Springer.

[3] Bucci, F., and Eller, M. The Cauchy–Dirichlet problem for the Moore–Gibson–Thompson equation. *arXiv preprint arXiv:2004.11167* (2020).

[4] Bucci, F., and Pandolfi, L. On the regularity of solutions to the Moore–Gibson–Thompson equation: a perspective via wave equations with memory. *Journal of Evolution Equations 20*, 3 (2019), 1–31. Publisher: Springer.

[5] Cattaneo, C. Sulla conduzione del calore. *Atti Sem. Mat. Fis. Univ. Modena* 3 (1948), 83–101.

[6] Cattaneo, C. A form of heat–conduction equations which eliminates the parado of instantaneous propagation. *Comptes Rendus 247* (1958), 431.
[7] Christov, C., and Jordan, P. Heat conduction paradox involving second–sound propagation in moving media. Physical review letters 94, 15 (2005), 154301. Publisher: APS.

[8] Crighton, D. G. Model equations of nonlinear acoustics. Annual Review of Fluid Mechanics 11, 1 (1979), 11–33. Publisher: Annual Reviews 4139 El Camino Way, PO Box 10139, Palo Alto, CA 94303-0139, USA.

[9] Dell’Oro, F., Lasiecka, I., and Pata, V. The Moore–Gibson–Thompson equation with memory in the critical case. Journal of Differential Equations 261, 7 (2016), 4188–4222. Publisher: Elsevier.

[10] Dell’Oro, F., and Pata, V. On the Moore–Gibson–Thompson equation and its relation to linear viscoelasticity. Applied Mathematics and Optimization 261, 7 (2016), 4188–4222. Publisher: Elsevier.

[11] Denk, R., Hieber, M., and Pruss, J. R–boundedness, Fourrier multipliers and problems of elliptic and parabolic type. Memoires of American Mathematical Society 788 (2003). Publisher: AMS.

[12] Ekoue, F., d’Halloy, A. F., Gigon, D., Plantamp, G., and Zajdman, E. Maxwell–Cattaneo regularization of heat equation. World Academy of Science, Engineering and Technology 7 (2013), 05–23.

[13] Fattorini, H. O. The Cauchy Problem. Addison Wesley, 1983.

[14] Hamilton, M. F., Blackstock, D. T., and others. Nonlinear acoustics. Academic Press, 1997.

[15] Jordan, P. M. Nonlinear acoustic phenomena in viscous thermally relaxing fluids: Shock bifurcation and the emergence of diffusive solitons. The Journal of the Acoustical Society of America 124, 4 (2008), 2491–2491. Publisher: ASA.

[16] Jordan, P. M. Second-sound phenomena in inviscid, thermally relaxing gases. Discrete & Continuous Dynamical Systems-B 19, 7 (2014), 2189. Publisher: American Institute of Mathematical Sciences.

[17] Kaltenbacher, B. Mathematics of nonlinear acoustics. Evolution Equations and Control Theory 4, 4 (2015), 447–491.

[18] Kaltenbacher, B., and Lasiecka, I. Global existence and exponential decay rates for the Westervelt’s equation. Discrete and Continuous Dynamical Systems-Series S 2, 3 (2009), 503–525.

[19] Kaltenbacher, B., Lasiecka, I., and Marchand, R. Wellposedness and exponential decay rates for the Moore–Gibson–Thompson equation arising in high intensity ultrasound. Control and Cybernetics 40 (2011), 971–988.

[20] Kaltenbacher, B., Lasiecka, I., and Pospieszalska, M. K. Well-posedness and exponential decay of the energy in the nonlinear Jordan–Moore–Gibson–Thompson equation arising in high intensity ultrasound. Mathematical Models and Methods in Applied Sciences 22, 11 (2012), 1250035. Publisher: World Scientific.

[21] Kaltenbacher, B., and Nikolić, V. Vanishing relaxation time limit of the Jordan–Moore–Gibson–Thompson wave equation with Neumann and absorbing boundary conditions. Pure and Applied Functional Analysis 5 (2020), 1–26.

[22] Kato, T. Perturbation Theory for Linear Operators. Springer-Verlag Berlin Heidelberg, 1976.
[23] Lasiecka, I., and Ong, J. Global solvability and uniform decays of solutions to quasilinear hyperbolic equations with nonlinear boundary conditions. *Communications on PDE* 24, 11-12 (1999), 2069–2107. Publisher: Francis and Taylor.

[24] Lasiecka, I., Tataru, D., and others. Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping. *Differential and integral Equations* 6, 3 (1993), 507–533. Publisher: Khayyam Publishing, Inc.

[25] Lunardi, A. *Analytic Semigroups and Optimal regularity in parabolic problems*. Birkhäuser, 1995.

[26] Marchand, R., McDevitt, T., and Triggiani, R. An abstract semigroup approach to the third-order Moore–Gibson–Thompson partial differential equation arising in high-intensity ultrasound: structural decomposition, spectral analysis, exponential stability. *Mathematical Methods in the Applied Sciences* 35, 15 (2012), 1896–1929. Publisher: Wiley Online Library.

[27] Meyer, S., and Wilke, M. Optimal regularity and long-time behavior of solutions for the Westervelt equations. *Applied Mathematics and Optimization* 64 (2011), 257–271. Publisher: Springer Verlag.

[28] Pellicer, M., and Said-Houari, B. Wellposedness and Decay Rates for the Cauchy Problem of the Moore–Gibson–Thompson Equation Arising in High Intensity Ultrasound. *Applied Mathematics & Optimization* 80, 2 (Dec. 2017), 447–478. Publisher: Springer Science and Business Media LLC.

[29] Straughan, B. *Heat waves*. Springer Science & Business Media, 2011.