A connection formula between the Ramanujan function and the $q$-Airy function

Takeshi MORITA‡

Abstract

We show a connection formula between two different $q$-Airy functions. One is called the Ramanujan function which appears in Ramanujan’s ”Lost notebook”. Another one is called the $q$-Airy function that obtained in the study of the second $q$-Painlevé equation. We use the $q$-Borel transformation and the $q$-Laplace transformation following C. Zhang to obtain the connection formula.

1 Introduction

In the study of the $q$-analysis, it is known that there exist several different $q$-special functions corresponding to a special function defined by a differential equation. For example, three types of $q$-Bessel functions are known. We denote them $J_{\nu}^{(1)}(x;q)$, $J_{\nu}^{(2)}(x;q)$ and $J_{\nu}^{(3)}(x;q)$ due to Ismail [6]. The first and the second one are called Jackson’s first and second $q$-Bessel function and the third one is called the Hahn-Exton $q$-Bessel function. Similarly, two types of $q$-Airy functions are known. We denote them $A_q(x)$ and $\text{Ai}_q(x)$. The first one is called the Ramanujan function and the second one is called the $q$-Airy function.

‡Graduate School of Information Science and Technology, Osaka University, 1-1 Machikaneyama-machi, Toyonaka, 560-0043, Japan. E-mail: t-morita@cr.math.sci.osaka-u.ac.jp
Three $q$-Bessel functions are given by

\[ J^{(1)}_{\nu}(x; q) := (q^{\nu+1}; q)_{\infty} \left( \frac{x}{2} \right)^{\nu} \sum_{n \geq 0} \frac{1}{(q^{\nu+1}; q)_{n}} \left( -\frac{x^{2}}{4} \right)^{n}, \]

\[ J^{(2)}_{\nu}(x; q) := (q^{\nu+1}; q)_{\infty} \left( \frac{x}{2} \right)^{\nu} \sum_{n \geq 0} \frac{q^{n^{2}}}{(q^{\nu+1}; q)_{n}} \left( -\frac{q^{\nu} x^{2}}{4} \right)^{n}, \]

\[ J^{(3)}_{\nu}(x; q) := (q^{\nu+1}; q)_{\infty} x^{\nu} \sum_{n \geq 0} \frac{q^{n(n+1)}}{(q^{\nu+1}; q)_{n}} (-x^{2})^{n}. \]

The Ramanujan function and the $q$-Airy function are defined by

\[ A_{q}(x) := \sum_{n \geq 0} \frac{q^{n^{2}}}{(q; q)_{n}} (-x)^{n}, \]

\[ \text{Ai}_{q}(x) := \sum_{n \geq 0} \frac{1}{(-q, q; q)_{n}} \left\{ (-1)^{n} q^{\frac{n(n-1)}{2}} \right\} (-x)^{n}. \]

Here, $(a; q)_{n}$ and $(a; q)_{\infty}$ are the $q$-Pochhammer symbol defined in section two. According to [1], the Ramanujan function $A_{q}(x)$ appears in the third identity on p.57 of Ramanujan’s “Lost notebook” [9] as follows (with $x$ replaced by $q^{\nu}$):

\[ A_{q}(-a) = \sum_{n \geq 0} \frac{a^{n} q^{n^{2}}}{(q; q)_{n}} = \prod_{n \geq 1} \left( 1 + \frac{a q^{2n-1}}{1 - q^{n} y_{1} - q^{2n} y_{2} - q^{3n} y_{3} - \cdots} \right), \]

where

\[ y_{1} = \frac{1}{(1 - q)\psi^{2}(q)}, \]
\[ y_{2} = 0, \]
\[ y_{3} = \frac{q + q^{3}}{(1 - q)(1 - q^{2})(1 - q^{3})\psi^{2}(q)} - \sum_{n \geq 0} \frac{(2n+1)q^{2n+1}}{1 - q^{2n+1}}, \]
\[ y_{4} = y_{1} y_{3}, \]
\[ \psi(q) = \sum_{n \geq 0} q^{\frac{n(n+1)}{2}} = \frac{(q^{2}; q^{2})_{\infty}}{(q; q^{2})_{\infty}}. \]
Ismail has pointed out that the Ramanujan function is a $q$-analogue of the Airy function [6]. We show more detail about these functions in section three.

It is known that there exist a relation for the $q$-Bessel functions and the $q$-Airy function. One is a relation between Jackson’s first and second $q$-Bessel function [4]:

$$J^{(2)}_{\nu}(x; q) = \left(-\frac{x^2}{4}; q\right)_{\infty} J^{(1)}_{\nu}(x; q).$$

Another is a relation between the Hahn-Exton $q$-Bessel function and the $q$-Airy function [8]:

$$J^{(3)}_{\nu}(x; q) = \left(-\frac{q}{q}; q\right)_{\infty} \frac{(q; q)}{\infty} x^{\nu} \text{Ai}_q(-qx^2),$$

where $q^{\nu} = -1$.

Other relations are not known. The main result in this paper is a relation between two $q$-Airy functions. It is not known any relation between the Ramanujan function and the $q$-Airy function, but these two functions are related by a connection formula and not by algebraic relation like (1) or (2).

Connection problems of linear $q$-difference equations between the origin and the infinity are studied by G. D. Birkhoff [2]. Watson gave a connection formula for the basic hypergeometric equation $2\varphi_1$ in 1910 [11]:

$$2\varphi_1 (a, b; c; q, x) = \frac{(b, c/a; q)_{\infty}(ax, q/a; q)_{\infty}}{(c, b/a; q)_{\infty}(x, q/x; q)_{\infty}} 2\varphi_1 (a, aq/c; aq/b; q, cq/abx)$$

$$+ \frac{(a, c/b; q)_{\infty}(bx, q/bx; q)_{\infty}}{(c, a/b; q)_{\infty}(x, q/x; q)_{\infty}} 2\varphi_1 (b, bq/c; bq/a; q, cq/abx).$$

Recently, C. Zhang has given some connection formulae of $q$-difference equations of the confluent type [12], [13] and [14]. Zhang gives a connection formula of Jackson’s $q$-Bessel function $J^{(1)}_{\nu}(x; q)$ [14]. In [14], Zhang introduced the $q$-Borel transformation and the $q$-Laplace transformation which are useful to study connection problems. In section four, we apply Zhang’s method to the $q$-Airy functions.

The connection formula of the $q$-Airy function gives a relation between the Ramanujan function and the $q$-Airy function as follows:
Theorem  For any $x \in \mathbb{C}^*$,

$$A_q^2 \left( -\frac{q^3}{x^2} \right) = \frac{1}{(q, -1; q)_\infty} \left\{ \theta \left( \frac{x}{q} \right) \operatorname{Ai}_q(-x) + \theta \left( -\frac{x}{q} \right) \operatorname{Ai}_q(x) \right\}.$$ 

Since our new relation shows an asymptotic behavior of the Ramanujan function near the infinity, it may be useful to study the Ramanujan function or similar type $q$-series.

### 2 Standard notations

In this section, we fix our notations. We assume that $q \in \mathbb{C}^*$ satisfies $0 < |q| < 1$. We define the $q$-Pochhammer symbol $(a; q)_n$.

**Definition 1** For any $n \in \mathbb{Z}_{\geq 0}$,

$$(a; q)_n = \begin{cases} 1, & n = 0, \\ (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), & n \geq 1, \end{cases}$$

and

$$(a; q)_\infty = \lim_{n \to \infty} (a; q)_n.$$ 

Moreover,

$$(a_1, a_2, \cdots, a_m; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty.$$ 

The $q$-difference operator $\sigma_q$ is given by $\sigma_q f(x) = f(qx)$. The basic hypergeometric series $r_{\varphi_s}$ is defined as follows.

**Definition 2** The basic hypergeometric series is given by

$$r_{\varphi_s}(a_1, \cdots, a_r; b_1, \cdots, b_s; q, x) := \sum_{n \geq 0} \frac{(a_1, \cdots, a_r; q)_n}{(b_1, \cdots, b_s; q)_n n!(q; q)_n} \left[ (-1)^n q^{\frac{n(n-1)}{2}} \right]^{1+s-r} x^n.$$ 

We define the theta function of Jacobi. We denote by $\theta_q(x)$ or more shortly $\theta(x)$. The theta function of Jacobi is given by following series;

**Definition 3** For any $x \in \mathbb{C}^*$,

$$\theta_q(x) = \theta(x) = \sum_{n \in \mathbb{Z}} q^{\frac{n(n-1)}{2}} x^n.$$ 

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The theta function has some important properties. The following lemma is called Jacobi’s triple product identity.

**Lemma 1** For any \( x \in \mathbb{C}^* \), we have

\[
\theta(x) = \left( q, -x, -\frac{q}{x}, q \right)_\infty.
\]

The theta function satisfies the following \( q \)-difference relation.

**Lemma 2** For any \( k \in \mathbb{Z} \), \( \theta(x) \) satisfies

\[
\theta(q^k x) = q^{-\frac{k(k-1)}{2}} x^{-k} \theta(x), \quad \forall x \in \mathbb{C}^*.
\]

From lemma 2, we remark that the function \( \theta(-\lambda x)/\theta(\lambda x), \forall \lambda \in \mathbb{C}^* \) satisfies a \( q \)-difference equation

\[
u(qx) = -u(x)
\]

which is also satisfied by the function \( u(x) = e^{\pi i \left( \frac{\log x}{\log q} \right)} \). From the definition, the theta function has the following inversion formula.

**Lemma 3** For any \( x \in \mathbb{C}^* \), one gets

\[
x \theta \left( \frac{1}{x} \right) = \theta(x).
\]

### 3 Two types of the \( q \)-analogue of the Airy function

There are two different \( q \)-analogue of the Airy function. One is called the Ramanujan function which appears in [9]. Ismail [6] pointed out that the Ramanujan function can be considered as a \( q \)-analogue of the Airy function. The other one is called the \( q \)-Airy function which is obtained by K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta and Y. Yamada [7]. In this section, we see the properties of these functions. We explain the reason why they are called \( q \)-analogue of the Airy function and we show \( q \)-difference equations which they satisfy.
3.1 The Ramanujan function $A_q(x)$

The Ramanujan function appears in Ramanujan’s "Lost notebook" [9]. Ismail has pointed out that the Ramanujan function can be considered as a $q$-analogue of the Airy function. The Ramanujan function is defined by the following convergent series:

$$A_q(x) := \sum_{n \geq 0} \frac{q^{n^2}}{(q; q)_n} (-x)^n = \varphi_1(-; 0; q, -qx).$$

In the theory of ordinary differential equations, the term Plancherel-Rotach asymptotics refers to asymptotics around the largest and smallest zeros. With $x = \sqrt{2n + 1} - 2\pi n \frac{t}{1}$ and for $t \in \mathbb{C}$, the Plancherel-Rotach asymptotic formula for Hermite polynomials $H_n(x)$ is

$$\lim_{n \to +\infty} e^{-\frac{x^2}{2}} \frac{1}{3! \pi^{-\frac{3}{2}} 2^{n+\frac{1}{2}} \sqrt{n!}} H_n(x) = \text{Ai}(t). \quad (3)$$

In [6], Ismail shows the $q$-analogue of (3):

**Proposition 1** One can get

$$\lim_{n \to \infty} q^{n^2} \frac{1}{n!} h_n(\sinh \xi_n|q) = A_q \left( \frac{1}{t^2} \right)$$

where $e^{\xi_n} = tq^{-\frac{n}{2}}$.

Here, $h_n(\cdot|q)$ is the $q$-Hermite polynomial. In this sense, we can deal with the Ramanujan function $A_q(x)$ as a $q$-analogue of the Airy function. The Ramanujan function satisfies the following $q$-difference equation:

$$(qx\sigma_q^2 - \sigma_q + 1) u(x) = 0. \quad (4)$$

**Remark 1** We remark that another solution of the equation (4) is given by

$$u(x) = \theta(x) \varphi_0(0, 0; -; q, -x).$$

Here,

$$\varphi_0(0, 0; -; q, -x) = \sum_{n \geq 0} \frac{1}{(q; q)_n} \left\{ (-1)^n q^{\frac{n(n-1)}{2}} \right\} (-x)^n$$

is a divergent series.
3.2 The \( q \)-Airy function \( \text{Ai}_q(x) \)

The \( q \)-Airy function is found by K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta and Y. Yamada [7], in their study of the \( q \)-Painlevé equations. This function is the special solution of the second \( q \)-Painlevé equations and given by the following series

\[
\text{Ai}_q(x) := \sum_{n \geq 0} \frac{1}{(-q, q; q)_n} \{(−1)^n q^{\frac{n(n−1)}{2}}\} (-x)^n = \varphi_1(0; −q; q, −x).
\]

T. Hamamoto, K. Kajiwara, N. S. Witte [5] proved following asymptotic expansions;

**Proposition 2** With \( q = e^{-\frac{2\pi i}{3}} \), \( x = −2ie^{-\frac{\pi i}{6}\delta^2} \) as \( \delta \to 0 \),

\[
\varphi_1(0; −q; q, −qx) = 2\pi^2\delta^2 e^{-\frac{\pi i}{3}ln^2\frac{\pi}{\delta^2} + \frac{\pi i}{6}ln^2\frac{\pi}{\delta^2}} \left[ \text{Ai} \left( se^{\frac{\pi i}{3}} \right) + O(\delta^2) \right],
\]

for \( s \) in any compact domain of \( \mathbb{C} \).

Here, \( \text{Ai}(\cdot) \) is the Airy function. From this proposition, we can regard the \( q \)-Airy function as a \( q \)-analogue of the Airy function.

We can easily check out that the \( q \)-Airy function satisfies the second order linear \( q \)-difference equation

\[
(\sigma_q^2 + x\sigma_q − 1) u(x) = 0.
\]

Another solution of the equation (5) is given by

\[
u(x) = e^{\pi i \left( \frac{\log x}{\log q} \right)} \varphi_1(0; −q; q, x) = e^{\pi i \left( \frac{\log x}{\log q} \right)} \text{Ai}_q(-x).
\]

3.3 Shearing transformations

We define a shearing transformation of a second order linear \( q \)-difference equation.

**Definition 4** For a \( q \)-difference equation

\[
a(x)u(q^2 x) + b(x)u(qx) + c(x)u(x) = 0,
\]

we define the shearing transformation as follows

\[
t^2 := x, \quad v(t) := u(t^2), \quad p := \sqrt{q}.
\]
The shearing transform of the equation (6) is given by
\[ a(t^2)v(p^2 t) + b(t^2)v(pt) + c(t^2)v(t) = 0. \]

By the shearing transformation, the equation
\[ (K \cdot x \sigma_q^2 - \sigma_q + 1) u(x) = 0 \]
is transformed to
\[ (K \cdot t^2 \sigma_p^2 - \sigma_p + 1) v(t) = 0, \]
where \( K \) is a fixed constant in \( \mathbb{C}^* \).

3.4 The \( q \)-Airy equation around the infinity

We consider the behavior of the equation (5) around the infinity. We set \( x = \frac{1}{t} \) and \( z(t) = u(\frac{1}{t}) \). Then \( z(t) \) satisfies
\[ \left( -\sigma_q^2 + \frac{1}{q^2 t} \sigma_q + 1 \right) z(t) = 0. \]

We set \( E(t) = \frac{1}{\theta(-q^2 t)} \) and \( f(t) = \sum_{n \geq 0} a_n t^n, \quad a_0 = 1 \). We assume that \( z(t) \) can be described as
\[ z(t) = E(t)f(t) = \frac{1}{\theta(-q^2 t)} \left( \sum_{n \geq 0} a_n t^n \right). \]

The function \( E(t) \) has the following property;

**Lemma 4** For any \( t \in \mathbb{C}^* \),
\[ \sigma_q E(t) = -q^2 t E(t), \quad \sigma_q^2 E(t) = q^5 t^2 E(t). \]

From this lemma, \( f(t) \) satisfies the following equation
\[ \left( -q^5 t^2 \sigma_q^2 - \sigma_q + 1 \right) f(t) = 0. \]

Since (8) is the same as (7) for \( K = -q^5 \), we obtain
\[ f(t) = \varphi_1(-; 0; q^2, q^5 t^2) = \Lambda_{q^2}(-q^3 t^2). \]
4 The $q$-Borel transformation, the $q$-Laplace transformation and the connection formula

In this section, we show a connection formula for $f(t)$. In order to obtain a connection formula, we need the $q$-Borel transformation and the $q$-Laplace transformation following Zhang [13].

4.1 The $q$-Borel transformation and the $q$-Laplace transformation

**Definition 5** For $f(t) = \sum_{n \geq 0} a_n t^n$, the $q$-Borel transformation is defined by

$$g(\tau) = (B_q f)(\tau) := \sum_{n \geq 0} a_n q^{-\frac{n(n-1)}{2}} \tau^n,$$

and the $q$-Laplace transformation is given by

$$(L_q g)(t) := \frac{1}{2\pi i} \int_{|\tau|=r} g(\tau) \theta \left( \frac{t}{\tau} \right) \frac{d\tau}{\tau}, \quad 0 < r < \frac{1}{|q^2|}.$$ 

The $q$-Borel transformation can be considered as a formal inverse of the $q$-Laplace transformation.

**Lemma 5** For any entire function $f$,

$$L_q \circ B_q f = f.$$ 

**Proof** We can prove this lemma calculating residues of the $q$-Laplace transformation around the origin.

The $q$-Borel transformation has following operational relation;

**Lemma 6** For any $l, m \in \mathbb{Z}_{\geq 0}$,

$$B_q(t^m \sigma_q^l) = q^{-\frac{m(m-1)}{2}} \tau^m \sigma_q^{l-m} B_q.$$
4.2 The connection formula of the $q$-Airy function

Applying the $q$-Borel transformation in $4.1$ to the equation (7) and using lemma $6$, we obtain the first order $q$-difference equation

$$g(q\tau) = (1 + q^2\tau)(1 - q^2\tau)g(\tau).$$

Since $g(0) = 1$, $g(\tau)$ is given by an infinite product

$$g(\tau) = \frac{1}{(-q^2\tau; q)_{\infty}(q^2\tau; q)_{\infty}}$$

which has single poles at

$$\{\tau; \tau = \pm q^{-2-k}, \quad \forall k \in \mathbb{Z}_{\geq 0}\}.$$ 

By Cauchy’s residue theorem, the $q$-Laplace transform of $g(\tau)$ is

$$f(t) = \frac{1}{2\pi i} \int_{|\tau|=r} g(\tau)\theta \left(\frac{t}{\tau}\right) \frac{d\tau}{\tau}$$

$$= -\sum_{k \geq 0} \text{Res} \left\{ g(\tau)\theta \left(\frac{t}{\tau}\right) \frac{1}{\tau}; \tau = -q^{-2-k} \right\}$$

$$-\sum_{k \geq 0} \text{Res} \left\{ g(\tau)\theta \left(\frac{t}{\tau}\right) \frac{1}{\tau}; \tau = q^{-2-k} \right\}$$

where $0 < r < r_0 := 1/|q^2|$. We can calculate the residue from lemma $7$ and lemma $2$.

**Lemma 7** For any $k \in \mathbb{N}$, $\lambda \in \mathbb{C}^*$, one can get;

1. $\text{Res} \left\{ \frac{1}{(\tau/\lambda; q)_{\infty}} \frac{1}{\tau}; \tau = \lambda q^{-k} \right\} = \frac{(-1)^{k+1}q^{k(k+1)/2}}{(q; q)_k(q; q)_{\infty}}$,

2. $\frac{1}{(\lambda q^{-k}; q)_{\infty}} = \frac{(-\lambda)^{-k}q^{k(k+1)/2}}{(\lambda; q)_{\infty} (q/\lambda; q)_k}, \quad \lambda \not\in q\mathbb{Z}$.

Summing up all of residues, we obtain

$$f(t) = \frac{\theta(q^2t)}{(q, -1; q)_{\infty}} \varphi_1 \left(0, -q; q, \frac{1}{t}\right) + \frac{\theta(-q^2t)}{(q, -1; q)_{\infty}} \varphi_1 \left(0, -q; q, -\frac{1}{t}\right).$$

Combining with lemma $3$, we get a connection formula for $z(t) = \mathcal{E}(t)f(t)$.

Finally, we acquire the following connection formula between the Ramanujan function and the $q$-Airy function.
**Theorem**  For any $x \in \mathbb{C}^*$,

$$A_q^2 \left( -\frac{q^3}{x^2} \right) = \frac{1}{(q, -1; q)_\infty} \left\{ \theta \left( \frac{x}{q} \right) Ai_q(-x) + \theta \left( -\frac{x}{q} \right) Ai_q(x) \right\}.$$  

Here, both $A_q(x)$ and $Ai_q(x)$ are defined by convergent series on whole of the complex plain. The connection formula above is valid for any $x \in \mathbb{C}^*$.

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