AN ANALYTIC APPROACH TO A WEAK NON-ABELIAN KNESER-TYPE THEOREM

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ABSTRACT. We prove the following result due to Hamidoune using an analytic approach. Suppose that $A$ is a subset of a finite group $G$ with $|AA^{-1}| \leq (2 - \varepsilon)|A|$. Then there is a subgroup $H$ of $G$ and a set $X$ of size $O_\varepsilon(1)$ such that $A \subset XH$.

1. Introduction

In his blog (see also [Tao11]) Tao asked for a non-Abelian version of Kneser’s theorem and made a number of observations related to this as well as giving a conjectural form. This question was comprehensively answered by Hamidoune in [Ham10] using his isoperimetric method, but in these notes we shall describe a different, more analytic, approach. As it happens this is not a particularly efficient idea and in that sense these notes may be more of a curiosity than an essential contribution.

We remark also that Tao has written a second later blog entry comparing the two approaches (Hamidoune’s and the method here) and added a third qualitative explanation of the material below; it is certainly recommended if the reader is interested in this problem.

Suppose that $G$ is a (possibly non-Abelian) group and $A, B \subset G$. We define the product set of $A$ and $B$ to be $AB := \{ab : a \in A, b \in B\}$.

A coset of a subgroup in $G$ may be characterised as a non-empty set $H \subset G$ such that $|HH^{-1}| = |H|$. Our interest lies in what happens when we relax the condition to consider non-empty sets $A$ such that $|AA^{-1}| \leq K|A|$, where $K > 1$. When $G$ is Abelian sets of this form were studied by Freiman in his celebrated structure theory of set addition [Fre66, Fre73], and recently there has been considerable interest in extending this work to the non-Abelian setting.

As indicated, if $H$ is a coset of a subgroup then $|HH^{-1}| = |H| \leq K|H|$ for any $K > 1$. On the other hand if $H$ is such and $A \subset H$ has $|A| \geq |H|/K$ then $|AA^{-1}| \leq |H| \leq K|A|$. Since $AA^{-1} \subset HH^{-1}$ and $|HH^{-1}| = |H|$. It turns out that for $K$ sufficiently small this is the only way of constructing such sets $A$. In particular we have the following result of Freiman [Fre73].

**Proposition 1.1.** Suppose that $G$ is a group and $A \subset G$ has $|AA^{-1}| \leq K|A|$ for some $K < 1.5$. Then $A$ is contained in a (left) coset of a subgroup $H$ with $|H| \leq K|A|$.

It is instructive to see the proof of this since the definitions and tools will be useful later. In particular the proof motivates the introduction of convolution.
Suppose that \( f, g \in \ell^1(G) \). Then we define the convolution of \( f \) and \( g \) to be the function
\[
f \ast g(x) := \sum_{yz = x} f(y)g(z) \text{ for all } x \in G.
\]
The convolution is useful for two important reasons: the first, if \( A, B \subseteq G \) are finite then
\[
\text{supp } 1_A \ast 1_B = AB
\]
so that we can analyse the product set \( AB \) through the convolution \( 1_A \ast 1_B \). This is often rather easier to do than analysing \( 1_{AB} \) directly since the convolution is (typically) smoother.

The second reason convolution is important is that
\[
1_A \ast 1_B(x) = |A \cap xB^{-1}| \text{ for all } x \in G.
\]
To prove Proposition 1.1 we need both of these facts. If \( x \in A^{-1}A \) then \( x = a^{-1}a' \) for some \( a, a' \in A \) and so using the second fact we see that
\[
1_{A^{-1}} \ast 1_A(x) = |A^{-1} \cap (a^{-1}a'A^{-1})| = |aA^{-1} \cap a'A^{-1}|
\geq |aA^{-1}| + |a'A^{-1}| - |aA^{-1} \cup a'A^{-1}| \geq (2 - K)|A|;
\]
the first fact tells us that that if \( 1_{A^{-1}} \ast 1_A(x) \neq 0 \) then \( x \in A^{-1}A \) and so \( 1_{A^{-1}} \ast 1_A(x) \geq (2 - K)|A| \). Crucially this leads to a step in the values \( 1_{A^{-1}} \ast 1_A \) may take which will also be useful later.

Proof of Proposition 1.1 Suppose that \( x, y \in A^{-1}A \). By (1.1) there are more than \( |A|/2 \) pairs \( (a, a') \in A \times A \) such that \( x = a^{-1}a' \) and more than \( |A|/2 \) pairs \( (a'', a''' \in A \times A \) such that \( y = a''^{-1}a''' \). It follows that there must be two pairs \( (a, a') \) and \( (a'', a''') \) with \( a' = a'' \), and hence \( xy = a^{-1}a'a''^{-1}a''' = a^{-1}a''' \in A^{-1}A \). It follows that \( (A^{-1}A)^2 = A^{-1}A \), but \( A^{-1}A \) is also symmetric (and non-empty) and so \( A^{-1}A \) is a subgroup of \( G \). On the other hand for any \( a \in A \) we have \( a^{-1}A \subset A^{-1}A \) and so \( A \subset aA^{-1}A \) and the result is proved.

The restriction \( K < 1.5 \) in this argument is not simply an artefact of the proof: there is a qualitatively new structure which occurs at this threshold. Suppose that \( x \) is an element of order 4 and consider \( A := \{1_G, x\} \). Then \( |AA^{-1}| = 3 \) while \( |A| = 2 \) so that \( |AA^{-1}| \leq 1.5|A| \). On the other hand if \( H \) is a coset of a subgroup containing \( A \) then \( AA^{-1} \subset HH^{-1} \) and so \( HH^{-1} \) contains \( x \), an element of order 4. Thus the \( HH^{-1} \), which is a group, has size at least 4 and we conclude that the smallest coset containing \( A \) has size at least 4 which is bigger than \( 3 = 1.5|A| \).

While the set \( A \) above cannot be very efficiently contained in a subgroup, it can be very efficiently covered by a subgroup: the trivial subgroup. In light of this (and following Green and Ruzsa [GR06], but see also Tao [Tao10]) we say that a set \( A \) is \( K \)-covered (on the left) by a set \( B \) if there is a set \( X \) of at most \( K \) elements such that \( A \subset XB \). We shall then prove the following theorem.

Theorem 1.2 (Weak non-abelian Kneser). Suppose that \( G \) is a finite group and \( A \subseteq G \) has \( |AA^{-1}| \leq (2 - \varepsilon)|A| \). Then \( A \) is \( O_\varepsilon(1) \)-covered by a subgroup \( H \) of size at most \( O_\varepsilon(|A|) \).
Note that this result is not a characterisation in the way that Proposition 1.1 was. Additionally the example of a long arithmetic progression shows that one cannot hope to remove the $\varepsilon$ entirely without expanding the class of structure one wishes to cover by. This can be done, but is much harder than our work here; see [BGT11] for details.

Before discussing our approach we note that this result is a corollary of the work of Hamidoune. In [Ham10] he proved the following.

**Theorem 1.3** ([Ham10 Theorem 1]). Suppose that $G$ is a group and $A \subset G$ is finite. Then there is a subgroup $H$ of $G$ such that at least one of the following holds:

- (i) $A^{-1}HA = A^{-1}A$ and $|A^{-1}A| \geq 2|HA| - |H|$;
- (ii) $AHA^{-1} = AA^{-1}$ and $|AA^{-1}| \geq 2|AH| - |H|$.

Hamidoune’s theorem makes the connection with Kneser’s theorem [Kne53] clearer; we record it now for comparison. (A proof may also be found in [TV06, Theorem 5.5], which we mention for convenience as this is the standard text book for additive combinatorics.)

**Theorem 1.4** (Kneser’s theorem (symmetric version)). Suppose that $G$ is an Abelian group and $A \subset G$ is finite. Then there is a subgroup $H$ of $G$ such that

$$A - A + H = A - A \text{ and } |A - A| \geq 2|A + H| - |H|.$$

Theorem 1.2 follows from Hamidoune’s result.

**Proof of Theorem 1.2.** We apply Hamidoune’s theorem to get a subgroup $H$ such that $AHA^{-1} = AA^{-1}$ and $|AA^{-1}| \geq 2|AH| - |H|$.

Since $|AH| \geq |A|$ and $|AA^{-1}| \leq (2 - \varepsilon)|A|$ we conclude from the inequality that $|H| \geq \varepsilon|A|$. On the other hand if $A$ has non-empty intersection with $R$ left cosets of $H$, then $|AHA^{-1}| \geq R|H|$, whence $R \leq 2\varepsilon^{-1} - 1$. It follows that $A$ is $2\varepsilon^{-1}$-covered by $H$ and the result is proved. 

Note that the dependence on $\varepsilon$ here is sharp up to the multiplicative constant as can be seen by considering an arithmetic progression of length about $\varepsilon^{-1}$.

Our approach to Theorem 1.2 is based around the following idea much of which was also identified as important by Tao when he recorded the original question.

Roughly, we proceed by analysing $1_A^{-1} * 1_A$ which as a convolution is pretty smooth, but then we saw in (1.1) that for $K < 2$ there is a jump between when $1_A^{-1} * 1_A$ is zero and when it is non-zero. These two facts mean that $A^{-1}A$, the support of $1_A^{-1} * 1_A$ must be a ‘connected component’ in some sense which turns out to mean that it is a small union of cosets of a subgroup.

2. **Analytic proof of Theorem 1.2**

Some readers may wish to proceed assuming that $G$ is Abelian to get a sense of how the argument goes, although obviously in this setting the usual version of Kneser’s theorem is well-known and immediately yields Theorem 1.2 by the same argument we used to derive it from Hamidoune’s theorem in the general case.
We shall need two main results in our work. The first is a non-Abelian Bogolyubov-Ruzsa-type result (c.f. [Bog39, Ruz94]) from the paper [CST10] of Croot and Sisask. This provides us with a set which is an approximate group in the sense of [Tao08] and which is also correlated with our set $A$.

**Proposition 2.1.** Suppose that $G$ is a group, $A \subset G$ is a finite set with $|AA^{-1}| \leq K|A|$ and $k \in \mathbb{N}$ is a parameter. Then there is a symmetric neighbourhood of the identity, $X$, such that $|X| = \Omega_{K,k}(|A|)$ and

$$1_{A^{-1}} \ast 1_A \ast 1_{A^{-1}} \ast 1_A(x) \geq |A|^3/2K \text{ for all } x \in X^k.$$  

This is [CS10] Theorem 4.1 applied to the sets $A^{-1}$, $A$ and $A$, and using the fact that

$$\|1_{A^{-1}} \ast 1_A\|_{\ell^2(G)}^2 \geq \|1_A \ast 1_{A^{-1}}\|_{\ell^2(G)}^2$$

if $|AA^{-1}| \leq K|A|$. One should like to prove this by applying the Cauchy-Schwarz inequality but this requires that $|A^{-1}A| \leq K|A|$ which is not our hypothesis. However, in [Tao08 Lemma 4.3] Tao saw that

$$\|1_{A^{-1}} \ast 1_A\|_{\ell^2(G)}^2 = \|1_A \ast 1_{A^{-1}}\|_{\ell^2(G)}^2$$

since $\langle f \ast g, h \rangle = \langle g, f \ast h \rangle = \langle f, h \ast \tilde{g} \rangle$ for all functions $f, g, h \in \ell^1(G)$. (2.1) then follows from Cauchy-Schwarz on the right hand quantity and the hypothesis $|AA^{-1}| \leq K|A|$.

It may be worth noting that when $|AA^{-1}| < 2|A|$ we have $AA^{-1} = A^{-1}A$, and so the above switch is not necessary. Tao presented a proof of this fact in [Tao11], but it is more involved than the argument above so we have not recorded it here.

The paper [CST10] of Croot and Sisask is well worth reading and the proof of [CST10 Theorem 4.1] (and hence Proposition 2.1) is not long, although it is rather clever. One of the main points of their argument though is that they achieved good dependencies on $k$ and $K$, something we do not record as the second result we used is not blessed with such good dependencies.

We need a little notation: suppose that $G$ is a finite group and $X \subset G$. Then we write $\mathbb{P}_X$ for the uniform probability measure supported on $X$. Given a measure $\mu$ on $G$ we write $\tilde{\mu}$ for the measure assigning mass $\mu(x^{-1})$ to each $x \in G$, and similarly for functions. Finally, convolution of a function $f$ and a measure $\mu$ is defined point-wise by

$$f \ast \mu(x) = \int f(xz^{-1})d\mu(z) \text{ for all } x \in G.$$  

We can now state the result which is an easy corollary of [San11 Proposition 20.1]. (The result as stated in [San11] concerns functions in the Fourier-Eymard algebra but it is a short calculation (essentially in [San11 Lemma 6.1]) to show that $f \ast \tilde{f}$ has algebra norm bounded by its $L^\infty$-norm.)

**Proposition 2.2.** Suppose that $G$ is a finite group, $f \in \ell^2(G)$, $X$ is symmetric and $\mathbb{P}_G(X^4) \leq K\mathbb{P}_G(X)$ and $\nu \in (0,1]$ is a parameter. Then there are symmetric neighbourhoods of the identity $B' \subset B \subset X^4$ such that $\mathbb{P}_G(B') = \Omega_{K,\nu}(\mathbb{P}_G(X))$,

$$\sup_{x \in G} \|f \ast \tilde{f} \ast \mathbb{P}_B \ast \mathbb{P}_B - f \ast \tilde{f} \ast \mathbb{P}_B \ast \mathbb{P}_B(x)|_{L^\infty(\mathbb{P}_{x'B'})} \leq \nu\|f \ast \tilde{f}\|_{L^\infty(G)}$$
and
\[
\sup_{x \in G} \| f \ast \tilde{f} - f \ast \tilde{f} \ast B \ast \mathbb{P}_B \|_{L^2(P_{xB'})} \leq \nu \| f \ast \tilde{f} \|_{L^\infty(G)}.
\]

It is perhaps worth saying that very roughly this proposition makes quantitative the idea that if \( f \in L^2(G) \) then \( f \ast \tilde{f} \) is continuous. This in itself is a little involved as the appropriate quantitative notion of continuity is (necessarily) not in \( L^\infty \) but rather in a local \( L^2 \)-norm. The paper [GK09] was the first place to develop this idea in the Abelian context, and the above result is a non-Abelian extension localised to approximate groups.

**Proof of Theorem 1.2.** We apply Proposition 2.1 to the set \( A \) with \( k = 8 \) to get a symmetric neighbourhood of the identity, \( X \), with \( |X| = \Omega(|A|) \) such that
\[
1_{A^{-1}} \ast 1_A \ast 1_{A^{-1}} \ast 1_A(x) \geq |A|^3/4 \text{ for all } x \in X^k.
\]

First this tells us that
\[
|X^4||A|^3/4 \leq \sum_{x \in G} 1_{A^{-1}} \ast 1_A \ast 1_{A^{-1}} \ast 1_A(x) = |A|^4
\]
which combined with the lower bound on \( |X| \) gives \( |X^4| = O(|X|) \).

Now apply Proposition 2.2 to \( f = 1_{A^{-1}} \) with this set \( X \) and parameter \( \nu = \varepsilon/10 \). This tells us (on combining the two conclusions of the proposition using the triangle inequality) that
\[
\int |1_{A^{-1}} \ast 1_A(y) - 1_{A^{-1}} \ast 1_A \ast \widetilde{P}_B \ast \mathbb{P}_B(x)|^2 d\mathbb{P}_{xB'}(y) \leq 4\nu^2 |A|^2 \text{ for all } x \in G,
\]
since \( \|1_{A^{-1}} \ast 1_A\|_{L^\infty(G)} = |A| \). Now, suppose for a contradiction that there is some \( x \in G \) such that
\[
(2.2) \quad \varepsilon |A|/4 < 1_{A^{-1}} \ast 1_A \ast \widetilde{P}_B \ast \mathbb{P}_B(x) < 3\varepsilon |A|/4,
\]
in which case
\[
\int |1_{A^{-1}} \ast 1_A(y) - 1_{A^{-1}} \ast 1_A \ast \widetilde{P}_B \ast \mathbb{P}_B(x)|^2 d\mathbb{P}_{xB'}(y) > (\varepsilon/4)^2 |A|^2
\]
in light of (2.1). This contradicts our choice of \( \nu \) and hence there are no \( x \in G \) such that (2.2) holds.

On the other hand in light of the first conclusion in Proposition 2.2 for all \( x \in G \) we have
\[
|1_{A^{-1}} \ast 1_A \ast \widetilde{P}_B \ast \mathbb{P}_B(xy) - 1_{A^{-1}} \ast 1_A \ast \widetilde{P}_B \ast \mathbb{P}_B(x)| \leq \varepsilon |A|/10 \text{ for all } y \in B'.
\]

Thus by the triangle inequality and the fact that (2.2) does not hold we conclude that \( S := \{ x \in G : 1_{A^{-1}} \ast 1_A \ast \widetilde{P}_B \ast \mathbb{P}_B(x) > 3\varepsilon |A|/4 \} \) is invariant under right multiplication by elements of \( B' \), and hence by the group \( H \) generated by \( B' \). Now suppose, for a contradiction, that \( S \) is empty whence
\[
\varepsilon |A|^3/4 > \langle 1_{A^{-1}} \ast 1_A \ast \widetilde{P}_B \ast \mathbb{P}_B, 1_{A^{-1}} \ast 1_A \rangle
\]
\[
= \langle 1_{A^{-1}} \ast 1_A \ast 1_{A^{-1}} \ast 1_A, \widetilde{P}_B \ast \mathbb{P}_B \rangle.
\]
Of course, since $B \subset X^4$ we have that $\text{supp} \widehat{P_B } * P_B \subset X^{-4} X^4 = X^8$ and so
\[
\langle 1_{A^{-1}} * 1_A * 1_{A^{-1}} * 1_A, \widehat{P_B } * P_B \rangle \geq |A|^3 / 4.
\]
This leads to a contradiction if $\varepsilon$ is sufficiently small (which we may certainly assume) and so we conclude that $S$ is non-empty.

Since $S$ is non-empty (and $H$ right invariant) we note that there is some $z \in G$ such that
\[
3\varepsilon |A| / 4 \leq \langle P_z H, 1_{A^{-1}} * 1_A * \widehat{P_B } * P_B \rangle = \langle 1_A * P_z H, 1_A * \widehat{P_B } * P_B \rangle \leq \|1_A * P_z H\| \varepsilon(G) \|1_A * \widehat{P_B } * P_B \| \varepsilon(G) = \|1_A * P_z H\| \varepsilon(G) |A|;
\]
we conclude that there is some $x \in G$ for which $|A \cap xH| \geq 3\varepsilon |H| / 4$. Given this we first note that $|H| = O_{\varepsilon}(|A|)$; secondly, since $|H| \geq |B'| = \Omega_{\varepsilon}(|A|)$, we have $|x^{-1}A \cap H| = \Omega_{\varepsilon}(|A|)$.

Finally we decompose $G$ into left cosets of $H$, and suppose that there are $R$ cosets $yH$ with $|yH \cap A| > 0$. Then
\[
O(|A|) = |AA^{-1}| = |AA^{-1}x| \geq |A \cap (x^{-1}A \cap H)^{-1}| \geq R \Omega_{\varepsilon}(|A|).
\]
It follows that $R = O_{\varepsilon}(1)$ and hence $A$ is contained in $O_{\varepsilon}(|A|)$ left cosets of $H$ as required.

The bounds in Proposition 2.2 are very poor, but even in the Abelian setting they are at best exponential in $\nu^{-2}$. This can be seen by examining the Niveau sets of Ruzsa [Ruz91] (see also [GK09] and [Wol10]). This dependence means we necessarily get at best an exponential bound in $\varepsilon^{-2}$ in Theorem 1.2 whereas Hamidoune’s work is far better giving a linear bound.

To summarise what we have seen: our method is a more complicated way of getting a weaker result which cannot ever yield a result as strong as Hamidoune’s.

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