On the $q$-Sumudu transform with two variables and some properties

Artan F. Alidema$^a$, Shkumbin V. Makolli$^b,\ast$

$^a$Department of Mathematics, Faculty of Mathematical and Natural Science, University of Prishtina, Mother Theresa p.n, 10000 Prishtina, Kosovo.

$^b$Department of Mathematics, Faculty of Mechanical Engineering, University of Prishtina, Agim Ramadani p.n, 1000 Pristina, Kosovo.

Abstract

In this paper we present some properties of double $q$-Sumudu transform in $q$-calculus by using the functions of two variables. Furthermore results on convergence, absolute convergence and convolution are discussed. At the end some examples are given to illustrate use of double $q$-Sumudu transform.

Keywords: Double $q$-Sumudu transform, convergence, convolution.

2020 MSC: 33D05, 33D60, 35A22, 42A85.

1. Introduction

The Sumudu transform was introduced by Watugala [24] and he applied it to the solution of ordinary differential equations. Asiru [7] and Belgacem [9] give the general and fundamental properties of the Sumudu transform. Watugala [25] has extended Sumudu transform to functions of two variables, and applied it to solving partial differential equations.

Tchuenche [22] applied the double Sumudu transform to an evolution equation of population dynamics. Kilicman and Gadain [19] gave the relations between double Laplace transform and double Sumudu transform and applied it to the solutions of non-homogenous wave equations. Debnath [11] presented general properties of the double Laplace transform, convolution and its properties. Convergence of double Sumudu transform was proved by Zulfiqar et al. [3].

In 1910, Jackson [17] presented a precise definition of so-called $q$-Jackson integral and developed $q$-calculus in a systematic way. It is well known that there are two types of $q$-Laplace transforms and they have been studied in detail by many authors ([1, 16, 20]).

The theory of $q$-analysis has been applied in many areas of mathematics, engineering, and physics, like in ordinary fractional calculus, optimal control problems, $q$-transform analysis and also in finding solutions of the $q$-difference and $q$-integral equations (see [2, 8, 18]).
Purohit and Kalla [20] evaluated the q-Laplace transforms of the q-Bessel functions, and they gave several useful cases of its application. Albayrak et al. [4, 5], have introduced q-analogue of Sumudu transform and investigated the fundamental properties of the q-Sumudu transform of certain q-polynomials. Brahim and Riah [10] introduced the q-analogue of the two dimensional Mellin transform, gave some properties and also proved the inversion formula of the q-two dimensional Mellin transform. Double q-Laplace transform was introduced by Sadjjang [21] as well as by Ganie [14]. The latter gave certain results on convergence, absolute convergence, convolution and its properties.

In this paper, we examine some properties of q-double Sumudu transform in q-calculus by using functions of two variables. We have shown some results regarding the convergence, absolute convergence and introduced convolution for q-double Sumudu transform.

The Sumudu transform of function \( f(x) \) is defined by Watugala in [24] as:

\[
S[f(x); s] = \frac{1}{s} \int_{0}^{\infty} e^{-\frac{s}{x}} f(x)dx, \quad s \in (-\tau_1, \tau_2),
\]

while \( f(x) \) is a function from the set of functions

\[
A = \{f(x)| \exists M, \tau_1, \tau_2 > 0, |f(x)| < Me^{\frac{M}{x}}, \text{ if } x \in (-1)^{1} \times [0, \infty)\}.
\]

The q-analogue of Sumudu transform by the q-Jackson integral is given by Albayrak et al. [6] as follows:

\[
F(s) = S_q[f(x); s] = \frac{1}{(1-q)s} \int_{0}^{\infty} e^{\frac{q}{x}} f(x)dx, \quad s \in (-\tau_1, \tau_2).
\]

Over the set of functions

\[
B = \{f(x)|\exists M, \tau_1, \tau_2 > 0, |f(x)| < Me^{\frac{M}{x}}, \text{ if } x \in (-1)^{1} \times [0, \infty)\}.
\]

Let \( f(x, y) \) be the function of two variables in the positive quadrant of Oxy plane then its double Sumudu transform is given by [22] as:

\[
S[f(x, t); (p, s)] = \frac{1}{ps} \int_{0}^{\infty} \int_{0}^{\infty} e^{(\frac{-1}{p} - \frac{1}{s})f(x, t)}dxdt.
\]

Double inverse Sumudu transform can be written as:

\[
S^{-1}(p, s) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\frac{s}{p}} dp \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} e^{\frac{p}{s}} f(p, s)ds,
\]

where \( f(p, s) \) is analytic functions for all \( p \) and \( s \) in the region defined by the inequalities Re (p) \( \geq \) \( \gamma \) and Re (s) \( \geq \delta \), while \( \gamma \) and \( \delta \) are real constants.

### 2. Auxiliary results

In this section we summarize the basic definitions and mathematical notations.

The q-factorials for \( q \in (0, 1) \) and \( a \in \mathbb{C} \) are defined as

\[
(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq)^k, \quad n = 1, 2, \ldots, \quad (a; q)_{\infty} = \lim_{n \to \infty} (a; q)_n = \prod_{k=0}^{n-1} (1 - aq)^k.
\]

Also we write \([a]_q = \frac{1-q^a}{1-q}, \quad [a]_q! = \frac{(q; a)_n}{(1-q)_n}, \quad n \in \mathbb{N}.
\]

The q-derivatives of a function are given by [18]

\[
(D_q f)(x) = \frac{(f(x) - f(qx))}{(1-q)x}, \quad \text{if} \quad x \neq 0, \quad (D_q f)(0) = f'(0).
\]
provided that $f'(0)$ exists. If $f$ is differentiable, then $(D_q f)(x)$ tends to $f'(x)$ as $q$ tends to 1. For $n \in \mathbb{N}$ we have $D_q^n = D_q \cdot (D_q^n)^1 = D_q^1$. The $q$-derivative of the product of two functions is defined as

$$D_q(f \cdot g)(x) = g(x)D_q f(x) + f(qx)D_q g(x).$$

The $q$-integrals from 0 to $a$ and from 0 to $\infty$ known as $q$-Jackson integrals are defined in [17]

$$\int_0^a f(x)d_q x = (1-q)a \sum_{n=-\infty}^\infty f(a^n)q^n, \quad \int_0^\infty f(x)d_q x = (1-q) \sum_{n=-\infty}^\infty f(q^n)q^n,$$

provided these sums converge absolutely. The integration by parts in terms of $q$-calculus is given by:

$$\int_a^b g(x)D_q f(x)d_q x = f(b)g(b) - f(a)g(a) - \int_a^b f(qx)D_q g(x)d_q x.$$ 

The $q$-analogues of the exponential functions are defined in [15, 18],

$$E^z_q = \sum_{n=0}^\infty q^n(n-1)/2 \frac{z^n}{[n]_q!} = (-1-qz)_\infty, \quad e^z_q = \sum_{n=0}^\infty \frac{z^n}{[n]_q!} = \frac{1}{(1-qz)_\infty}, |z| < \frac{1}{1-q}.$$ 

The $q$-exponentials are analogues of classical exponential functions and satisfy the relations $D_q e^z_q = e^z_q$, $D_q E^z_q = E^{qz}_q$, and $e^z_q E^q = e^{qz}_q e^1_q = 1$. Jackson also defined $q$-analogue of the gamma function $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx$, while many properties are found in [23, 26–28],

$$\Gamma_q(t) = \frac{(q; q)_\infty (1-q)^{1-t}, \quad t \neq 0, -1, -2, \ldots,}$$

that satisfies

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t), \quad \Gamma_q(1) = 1,$$

and $\lim_{q \to 1^-} \Gamma_q(t) = \Gamma(t), \quad \Re(t) > 0$. The $\Gamma_q$ function has the $q$-integral representation as

$$\Gamma_q(s) = \int_0^{1/(1-q)} t^{s-1}E_q^{-qt}d_q t = \int_0^{\infty/(1-q)} t^{s-1}E_q^{-qt}d_q t.$$ 

The $q$-integral representation of $\Gamma_q$ based on $q$-exponential function $e^z_q$ and $q$-integral representation of $q$-beta function are defined in [12] as: for all $s, t > 0$, we have

$$\Gamma_q(s) = K_q(s) \int_0^{\infty/(1-q)} t^{s-1}e^{-qt}d_q t$$

and $B_q(t, s) = K_q(t) \int_0^\infty x^{t-1} \frac{(-x^{1/s}q^s)}{(-q^s, -q^{-1/s}; q)_\infty}d_q x$, where in [10], $K_q(t) = \frac{(-q^{1-t}; q)_\infty}{(-q^t, -q^{1-t}; q)_\infty}$. If $\log(1-q) = \log(q) \in \mathbb{Z}$, we obtain

$$\Gamma_q(s) = K_q(s) \int_0^\infty t^{s-1}e^{-qt}d_q t = \int_0^\infty t^{s-1}E_q^{-qt}d_q t.$$ 

3. Two dimensional $q$-Sumudu transform

**Definition 3.1.** Let $q = (q_1, q_2) \in (0, 1)^2, (s, t) \in \mathbb{C}^2$ and let $f$ be a function of two variables $x$ and $y$ defined on $R_{q_1, q_2} \times \mathbb{R}_{q_2, t}$. Then the $q$-two dimensional Sumudu transform of $f$ is defined by the double integral is given as: ([13])

$$S_q(f)(s, t) = S_q[f(x, y)](s, t) = \frac{1}{(1-q)^2} \frac{1}{ps} \int_0^\infty \int_0^\infty e^\frac{1}{(1-q)^2} f(x, t)d_q x d_q t$$

(provided that this integral exists), where $R_{q_1, q_2} = \{ q^n, n \in \mathbb{Z} \}$ and

$$\{ e^{-x}_q = [1 - (1-q)x]^{-1/1-q}, \quad \text{for } 0 < x < \frac{1}{1-q}, \quad q < 1, \quad e^{-x}_q = [1 - (1-q)x]^{-1/1-q}, \quad \text{for } x \geq 0, \quad q > 1. \}$$

A. F. Alidema, Sh.V. Makolli, J. Math. Computer Sci., 25 (2022), 166–175 168
4. Convergence of q-double Sumudu transform

Theorem 4.1. Let $f(x,y)$ be a function of two variables continuous in $\mathbb{R}_{q_1,+} \times \mathbb{R}_{q_2,+}$ or continuous in the positive quadrant of $xy$ plane. If the integral

$$
\frac{1}{(1-q)q} \int_0^\infty \int_0^\infty e_q^{-\frac{q}{s}} f(x,t)dxdt
$$

converges at $s = s_0$ and $t = t_0$, then the integral in (4.1) converges for $s < s_0$, $p < p_0$.

We prove this Theorem by using following Lemma.

Lemma 4.2. If the integral

$$
\frac{1}{(1-q)^s} \int_0^\infty e_q^{-\frac{q}{s}} f(x,t)dt
$$

converges for $s = s_0$, then the integral converge for $s < s_0$.

Proof. Let us assume that, $\alpha(x,t) = \frac{1}{(1-q)^s} \int_0^\infty e_q^{-\frac{q}{s}} f(x,u)du$. It is obvious that $\alpha(x,0) = 0$ and $\lim_{t \to \infty} \alpha(x,t)$ exists, because integral $\frac{1}{(1-q)^s} \int_0^\infty e_q^{-\frac{q}{s}} f(x,t)dt$ converges at $s = s_0$.

Now let us denote

$$
\alpha_1(x,t) = \frac{1}{(1-q)^{s_0}} e_q^{-\frac{q}{s_0}} f(x,t).
$$

For $\zeta$ and $R$ such that $0 < \zeta < R$ we will consider

$$
\frac{1}{(1-q)^s} \int_0^\infty e_q^{-\frac{q}{s}} f(x,t)dt = \frac{1}{(1-q)^s} \int_0^\infty e_q^{-\frac{q}{s}} (1-q)e^{s_0} \alpha_1(x,t)dt
$$

It is obvious that

$$
\frac{1}{(1-q)^s} \int_0^\infty e_q^{-\frac{q}{s}} f(x,t)dt = \frac{1}{(1-q)^s} \int_0^\infty e_q^{-\frac{q}{s}} (1-q)e^{s_0} \alpha_1(x,t)dt.
$$

If we use partial integration we obtain

$$
= \frac{s_0}{s} \left[ e_q^{-\frac{q}{s_0}} \alpha(x,t) \right]_0^\infty + \frac{s_0 - s}{s} \int_0^\infty e_q^{-\frac{q}{s_0}} \alpha(x,t)dt
$$

Finally, if in the last expression we put $\zeta \to 0$, then due to $\lim_{\zeta \to 0} \alpha(x,\zeta) = \alpha(x,0) = 0$, we get

$$
\frac{1}{(1-q)^s} \int_0^\infty e_q^{-\frac{q}{s}} f(x,t)dt = \frac{s_0}{s} \left[ e_q^{-\frac{q}{s_0}} \alpha(x,t) \right]_0^\infty + \frac{s_0 - s}{s} \int_0^\infty e_q^{-\frac{q}{s_0}} \alpha(x,t)dt.
$$

Now if we let $R \to \infty$, then the first term on the right tends to zero when $s < s_0$, subsequently we get

$$
\frac{1}{(1-q)^s} \int_0^\infty e_q^{-\frac{q}{s}} f(x,t)dt = \frac{s_0}{s} \left[ e_q^{-\frac{q}{s_0}} \alpha(x,t) \right]_0^\infty + \frac{s_0 - s}{s} \int_0^\infty e_q^{-\frac{q}{s_0}} \alpha(x,t)dt.
$$

According to limit test for convergence of improper integrals we can prove that

$$
\frac{(s_0 - s)}{s^2} \int_0^\infty e_q^{-\frac{q}{s}} \alpha(x,t)dt
$$
converges if the following limit converges
\[
\lim_{t \to \infty} t^p e^{-\frac{1}{\alpha(x, qt)}} \alpha(x, qt), \text{ for } p > 1.
\]
It is obvious that for \( p = 2 \) and \( s < s_0 \) the above limit converges, considering the fact that \( \lim_{t \to \infty} \alpha(x, qt) \) converges. Therefore the following integral
\[
\frac{1}{(1-q)^s} \int_0^\infty e_{\frac{s}{q}}^{-\frac{1}{q}} f(x, t) \, dq \, dt
\]
converges for \( s < s_0 \).

**Lemma 4.3.** If the integral
\[
\frac{1}{(1-q)^s} \int_0^\infty e_{\frac{s}{q}}^{-\frac{1}{q}} f(x, t) \, dq \, dt
\]
converges for \( s \leq s_0 \) and the integral
\[
\frac{1}{(1-q)^p} \int_0^\infty e_{\frac{s}{q}}^{-\frac{1}{q}} f(x, s) \, dq \, dx
\]
converges for \( p = p_0 \), then integral (4.2) converges for \( p < p_0 \).

**Proof.** Same as above Lemma.

**Proof of Theorem 4.1.**
\[
\frac{1}{(1-q)^2} \frac{1}{p s} \int_0^\infty \int_0^\infty e_{p s}^{-\frac{1}{p}} f(x, t) \, dq \, dx \, dt = \frac{1}{(1-q)^p} \frac{1}{s} \int_0^\infty e_{p s}^{-\frac{1}{p}} \left\{ \int_0^\infty e_{(1-q)^s}^{-\frac{1}{q}} f(x, t) \, dq \right\} \, dq \, dx
\]
where \( h(x, t) = \frac{1}{(1-q)^s} \int_0^\infty e_{\frac{s}{q}}^{-\frac{1}{q}} f(x, t) \, dq \, dt \) and according to Lemma 4.3 it converges for \( s < s_0 \), while in accordance with Lemma 4.2 the following \( \frac{1}{(1-q)^p} \frac{1}{s} \int_0^\infty e_{\frac{s}{q}}^{-\frac{1}{q}} f(x, t) \, dq \, dx \) converges for \( p < p_0 \). Therefore, the initial integral, \( \frac{1}{(1-q)^2} \frac{1}{p s} \int_0^\infty \int_0^\infty e_{p s}^{-\frac{1}{p}} f(x, t) \, dq \, dx \, dt \) converges for \( s < s_0, p < p_0 \).

**Theorem 4.4 (Absolute convergence).** If integral
\[
\frac{1}{(1-q)^2} \frac{1}{p s} \int_0^\infty \int_0^\infty e_{p s}^{-\frac{1}{p}} f(x, t) \, dq \, dx \, dt
\]
converges absolute for \( s = s_0 \) and \( p = p_0 \), then it converges for \( s \leq s_0 \) and \( p \leq p_0 \).

**Proof.**
\[
e_{p s}^{-\frac{1}{p}} |f(x, t)| \leq e_{p_0 s_0}^{-\frac{1}{p_0}} |f(x, t)| \text{ for } |p \leq p_0 < +\infty, s \leq s_0 < +\infty,
\]
therefore
\[
\leq \frac{1}{(1-q)^2} \frac{1}{p s} \int_0^\infty \int_0^\infty e_{p s}^{-\frac{1}{p}} |f(x, t)| \, dq \, dx \, dt
\]
\[
\leq \frac{1}{(1-q)^2} \frac{1}{p s} \int_0^\infty \int_0^\infty e_{p_0 s_0}^{-\frac{1}{p_0}} |f(x, t)| \, dq \, dx \, dt
\]
\[
\leq \frac{1}{(1-q)^2} \frac{1}{p s} \int_0^\infty \int_0^\infty e_{p_0 s_0}^{-\frac{1}{p_0}} |f(x, t)| \, dq \, dx \, dt.
\]
The last integral converges based on the hypothesis, which means that the initial integral converges for \( p \leq p_0 \) and \( s \leq s_0 \).
Theorem 4.5. If \( f(x, y) \) is a periodic function of periods \( a \) and \( b \), \( f(x + a, y + b) = f(x, y) \), and if \( S_q[f(x, y)] \) exists, then

\[
S_q[f(x, y)](p, s) = \frac{1 - e^{-\frac{a}{p} - \frac{b}{s}}}{(1 - q)^2} \int_0^b \int_0^a e^{(-\frac{x}{p} - \frac{y}{s})} f(x, t) d_q x d_q t.
\]

Proof.

\[
S_q(f(x, y)) = \frac{1}{(1 - q)^2} \int_0^\infty \int_0^\infty e^{(-\frac{x}{p} - \frac{y}{s})} f(x, t) d_q x d_q t
\]

if we put \( x = u + a \) and \( t = v + b \), we get

\[
= \frac{1}{(1 - q)^2} \int_0^a \int_0^b e^{(-\frac{x}{p} - \frac{y}{s})} f(x, t) d_q x d_q t + \frac{1}{(1 - q)^2} \int_0^\infty \int_0^\infty e^{(-\frac{x}{p} - \frac{y}{s})} f(u, v) d_q u d_q v.
\]

Therefore

\[
S_q(f(x, t)) = \frac{1}{(1 - q)^2} \int_0^b \int_0^a e^{(-\frac{x}{p} - \frac{y}{s})} f(x, t) d_q x d_q t + e^{(-\frac{x}{p} - \frac{y}{s})} S_q f(x, t)
\]

\[
= [1 - e^{(-\frac{a}{p} - \frac{b}{s})}] S_q f(x, t) = \frac{1}{(1 - q)^2} \int_0^a \int_0^b e^{(-\frac{x}{p} - \frac{y}{s})} f(x, t) d_q x d_q t,
\]

\[
S_q f(x, t) = \frac{[1 - e^{(-\frac{a}{p} - \frac{b}{s})}] - 1}{(1 - q)^2} \int_0^b \int_0^a e^{(-\frac{x}{p} - \frac{y}{s})} f(x, t) d_q x d_q t.
\]

\( S_q \) is a double Sumudu transform of a periodic function.

\[\square\]

5. Double \( q \)-Sumudu convolution product

Definition 5.1. The convolution of \( f(x, y) \) and \( g(x, y) \) is defined as

\[
(f \ast g)(x, y) = \frac{1}{(1 - q)^2} \int_0^x \int_0^y f(\zeta, \mu) g(x - \zeta, y - \mu) d_q \zeta d_q \mu.
\]

Theorem 5.2 (Convolution Theorem). Let \( f_1(x, t) \) and \( f_2(x, t) \) be two functions having double \( q \)-double Sumudu transform. Then \( q \)-double Sumudu transform of the double convolution is given by:

\[
S_q[f_1(x, t) \ast f_2(x, t)](p, s) = S_q[f_1(x, t)]S_q[f_2(x, t)].
\]

Proof.

\[
S_q(f_1(x, t) \ast f_2(x, t))(p, s)
\]

\[
= \frac{1}{(1 - q)^4} \int_0^\infty \int_0^\infty \left\{ \int_0^t \int_0^t f_1(\zeta, \mu) f_2(x - \zeta, t - \mu) d_q \zeta d_q \mu \right\} d_q x d_q t
\]

\[
= \frac{1}{(1 - q)^4} \int_0^\infty \int_0^\infty \left\{ \int_0^t \int_0^t \left[ 1 + (q - 1) \left( \frac{x}{p} + \frac{t}{s} \right) \right]^{-\frac{1}{q-1}} f_1(\zeta, \mu) f_2(x - \zeta, t - \mu) d_q \zeta d_q \mu \right\} d_q x d_q t
\]

\[
= \frac{1}{(1 - q)^4} \int_0^\infty \int_0^\infty f_1(\zeta, \mu) \left\{ \int_0^t \int_0^t \left[ 1 + (q - 1) \left( \frac{x}{p} + \frac{t}{s} \right) \right]^{-\frac{1}{q-1}} f_2(x - \zeta, t - \mu) d_q \zeta d_q \mu \right\} d_q x d_q t.
\]

Let us replace \( x - \zeta = u \) and \( t - \mu = v \), and if we denote

\[
I = \int_0^x \int_0^t \left[ 1 + (q - 1) \left( \frac{x}{p} + \frac{t}{s} \right) \right]^{-\frac{1}{q-1}} f_2(x - \zeta, y - \mu) d_q \zeta d_q \mu
\]
in light of new variables, and if we consider upper bounds for \( x \) and \( t \), the integral \( I \) can also be written as following

\[
I = \left[ 1 + (q - 1) \left( \frac{\xi}{p} + \frac{\mu}{s} \right) \right]^{-\frac{1}{q-1}} \int_{-\infty}^0 \int_{-\infty}^0 \left[ 1 + (q - 1) \left( \frac{u}{p} + \frac{v}{s} \right) \right]^{-\frac{1}{q-1}} \times \left[ 1 + (q - 1) \left( \frac{u}{p} + \frac{v}{s} \right) \right]^{\frac{1}{q-1}} \left[ 1 + (q - 1) \left( \frac{u + \xi}{p} + \frac{v + \mu}{s} \right) \right]^{-\frac{1}{q-1}} f_2(u,v) d_\xi d_\mu d_q d_u d_v.
\]

Since both functions \( f_1(x,t) \) and \( f_2(x,t) \) are defined in the positive quadrant of the \( Oxy \) plane, it is obvious that

\[
I = \left[ 1 + (q - 1) \left( \frac{\xi}{p} + \frac{\mu}{s} \right) \right]^{-\frac{1}{q-1}} \int_{-\infty}^0 \int_{-\infty}^0 \left[ 1 + (q - 1) \left( \frac{u}{p} + \frac{v}{s} \right) \right]^{-\frac{1}{q-1}} \times \left[ 1 + (q - 1) \left( \frac{u}{p} + \frac{v}{s} \right) \right]^{\frac{1}{q-1}} \left[ 1 + (q - 1) \left( \frac{u + \xi}{p} + \frac{v + \mu}{s} \right) \right]^{-\frac{1}{q-1}} f_2(u,v) = f_2^*(u,v),
\]

from the last relation we can then express \( f_2(u,v) \) as

\[
f_2(u,v) = \left[ 1 + (q - 1) \left( \frac{\xi}{p} + \frac{\mu}{s} \right) \right]^{-\frac{1}{q-1}} \left[ 1 + (q - 1) \left( \frac{u}{p} + \frac{v}{s} \right) \right]^{\frac{1}{q-1}} \left[ 1 + (q - 1) \left( \frac{u + \xi}{p} + \frac{v + \mu}{s} \right) \right]^{-\frac{1}{q-1}} f_2^*(u,v),
\]

and after replacing it in the initial integral we will get:

\[
S_q\{f_1(x,t) \ast f_2(x,t)\} = \frac{1}{(1-q)^4p^2s^2} \int_{-\infty}^0 \int_{-\infty}^0 f_1(\zeta,\mu) \left[ 1 + (q - 1) \left( \frac{\xi}{p} + \frac{\mu}{s} \right) \right]^{-\frac{1}{q-1}} \times \left[ 1 + (q - 1) \left( \frac{u}{p} + \frac{v}{s} \right) \right]^{\frac{1}{q-1}} \left[ 1 + (q - 1) \left( \frac{u + \xi}{p} + \frac{v + \mu}{s} \right) \right]^{-\frac{1}{q-1}} f_2^*(u,v) d_\xi d_\mu d_q d_u d_v
\]

\[
= S_q\{f_1(\zeta,\mu)\} \left\{ \frac{1}{(1-q)^4p^2s^2} \int_{-\infty}^0 \int_{-\infty}^0 \left[ 1 + (q - 1) \left( \frac{u}{p} + \frac{v}{s} \right) \right]^{\frac{1}{q-1}} \left[ 1 + (q - 1) \left( \frac{u + \xi}{p} + \frac{v + \mu}{s} \right) \right]^{-\frac{1}{q-1}} f_2^*(u,v) d_\xi d_\mu d_q d_u d_v \right\}
\]

\[
= S_q\{f_1(\zeta,\mu)\} \cdot S_q\{f_2^*(u,v)\},
\]

\[
S_q\{f_1(x,t) \ast f_2(x,t)\}[p,s] = S_q\{f_1(x,t)\}S_q\{f_2(x,t)\}[p,s].
\]

\[
\square
\]

5.1. Properties of double \( q \)-Sumudu transform method

Some properties of \( q \)-double Sumudu transform are given as following.

a) Scaling: For a real number \( k \),

\[
S_q[kf(x,y)](p,s) = \frac{1}{(1-q)^2ps} \int_{-\infty}^0 \int_{-\infty}^0 ke_q(-\frac{x}{p} - \frac{y}{s}) f(x,y) d_q d_x d_y
\]

\[
= \frac{k}{(1-q)^2ps} \int_{-\infty}^0 \int_{-\infty}^0 e_q(-\frac{x}{p} - \frac{y}{s}) f(x,y) d_q d_x d_y kS_q[f(x,y)](p,s).
\]

b) Linearity:

\[
S_q[mf(x,y) + nS_qmf(x,y)](p,s) = mS_q[f(x,y)] + nS_q[f(x,y)](p,s),
\]
5.2. Examples

A. F. Alidema, Sh. V. Makolli, J. Math. Computer Sci., 25 (2022), 166–175

\[ S_q[mf(x, y) + nf(x, y)](p, s) = \frac{1}{(1-q)^2} \frac{1}{ps} \int_0^\infty \int_0^\infty mf(x, y) + nf(x, y) e_q^{\left( -\frac{x}{p} - \frac{y}{q} \right)} d_q x d_q y \]
\[ \times \frac{1}{(1-q)^2} \frac{1}{ps} \left\{ \int_0^\infty \int_0^\infty mf(x, y) e_q^{\left( -\frac{x}{p} - \frac{y}{q} \right)} d_q x d_q y \right\} \]
\[ + \frac{\infty}{\int_0^\infty} \int_0^\infty nf(x, y) e_q^{\left( -\frac{x}{p} - \frac{y}{q} \right)} d_q x d_q y \]
\[ = mS_q[f(x, y)](p, s) + nS_q[f(x, y)](p, s). \]

c): For \( a > 0, b > 0 \), and if we denote \( \overline{G}_q(p, s) = S_q[f(x, y)](p, s) \), we have:

\[ S_q[e^{\frac{\Delta_1}{p}} f(x, y)] = \frac{ab}{(a+p)(b+s)} \overline{G}_q \left( \frac{ap}{a+p} b + s \right) \]
\[ S_q[e^{\frac{\Delta_1}{p}} f(x, y)] = \frac{1}{(1-q)^2} \frac{1}{ps} \int_0^\infty \int_0^\infty e_q^{\left( -\frac{x}{p} - \frac{y}{q} \right)} f(x, y) d_q x d_q y \]
\[ = \frac{1}{(1-q)^2} \frac{1}{ps} \int_0^\infty \int_0^\infty e_q^{\left[ -\frac{\Delta_1}{p} \right]} e_q^{\left[ -\frac{\Delta_1}{q} \right]} f(x, y) d_q x d_q y \]
\[ = \frac{1}{(1-q)^2} \frac{1}{ps} \int_0^\infty \int_0^\infty e_q^{\left[ -\frac{\Delta_1}{p} \right]} e_q^{\left[ -\frac{\Delta_1}{q} \right]} f(x, y) d_q x d_q y \]
\[ = \frac{1}{(1-q)^2} \frac{1}{ps} \int_0^\infty \int_0^\infty e_q^{\left[ -\frac{\Delta_1}{p} \right]} e_q^{\left[ -\frac{\Delta_1}{q} \right]} f(x, y) d_q x d_q y \]
\[ = \frac{1}{(1-q)^2} \frac{1}{ps} \int_0^\infty \int_0^\infty \frac{ab}{(a+p)(b+s)} \overline{G}_q \left( \frac{ap}{a+p} b + s \right). \]

d): \( S_q[f(x)] = \frac{1}{(1-q)} \overline{G}_q(p) \), where \( \overline{G}_q(p) = \frac{1}{(1-q)} \frac{1}{p} \int_0^\infty e_q^{\left[ -\frac{x}{p} \right]} f(x) d_q x, \)

\[ S_q[f(x)] = \frac{1}{(1-q)^2} \frac{1}{ps} \int_0^\infty \int_0^\infty e_q^{\left[ -\frac{x}{p} - \frac{y}{q} \right]} f(x) d_q x d_q y \]
\[ = \frac{1}{(1-q)^2} \frac{1}{ps} \int_0^\infty \int_0^\infty e_q^{\left[ -\frac{x}{p} \right]} e_q^{\left[ -\frac{y}{q} \right]} f(x) d_q x d_q y \]
\[ = \frac{1}{(1-q)^2} \frac{1}{ps} \int_0^\infty \int_0^\infty e_q^{\left[ -\frac{\Delta_1}{p} \right]} f(x) d_q x d_q y \]
\[ = \frac{1}{(1-q)^2} \frac{1}{ps} \int_0^\infty \int_0^\infty e_q^{\left[ -\frac{\Delta_1}{p} \right]} f(x) d_q x d_q y \]
\[ = \frac{1}{(1-q)^2} \frac{1}{ps} \int_0^\infty \int_0^\infty e_q^{\left[ -\frac{\Delta_1}{p} \right]} f(x) d_q x d_q y \]
\[ = \frac{1}{(1-q)^2} \frac{1}{ps} \int_0^\infty \int_0^\infty e_q^{\left[ -\frac{\Delta_1}{p} \right]} f(x) d_q x d_q y \]
\[ = \frac{1}{(1-q)^2} \frac{1}{ps} \int_0^\infty \int_0^\infty e_q^{\left[ -\frac{\Delta_1}{p} \right]} f(x) d_q x d_q y \]
\[ = \frac{1}{(1-q)^2} \frac{1}{ps} \int_0^\infty \int_0^\infty \frac{ab}{(a+p)(b+s)} \overline{G}_q \left( \frac{ap}{a+p} b + s \right). \]

5.2. Examples

1. If \( f(x, y) = 1 \) for \( x > 0, y > 0 \), then for \( 1 < q < 2 \)

\[ S_q[1] = \frac{1}{(1-q)^2} \frac{1}{ps} \int_0^\infty \int_0^\infty e_q^{\left( -\frac{x}{p} - \frac{1}{q} \right)} d_q x d_q t \]
\[ = \frac{1}{(1-q)^2} \frac{1}{ps} \int_0^\infty e_q^{\left[ -\frac{1}{q} \right]} e_q^{\left[ -\frac{x}{p} \right]} d_q x d_q t \]
\[ = \frac{1}{(1-q)^2} \frac{1}{ps} \int_0^\infty e_q^{\left[ -\frac{1}{q} \right]} \left\{ \int_0^\infty \left[ 1 - (1-q) \frac{t}{s} \right]^{\frac{1}{q}} \right\} d_q x \]
\[ = \frac{1}{(1-q)^2} \frac{1}{ps} \int_0^\infty e_q^{\left[ -\frac{1}{q} \right]} \left\{ \int_0^\infty \left[ \frac{1}{1-q} \frac{t}{s} \right]^{\frac{1}{q}} \right\} d_q x \]
\[ = \frac{1}{(1-q)^2} \frac{1}{ps} \int_0^\infty e_q^{\left[ -\frac{1}{q} \right]} \left\{ \int_0^\infty \left[ \frac{1}{1-q} \frac{t}{s} \right]^{\frac{1}{q}} \right\} d_q x \]
\[
\begin{align*}
&= \frac{1}{(1-q)^2} \int_0^\infty e^{-\frac{x}{p}} d_q x \\
&= \frac{1}{(1-q)^2} \int_0^\infty \left[ 1 - (1-q) \frac{x}{p} \right]^{\frac{1}{1-q}} d_q x \\
&= \frac{1}{(1-q)^2} \int_0^\infty \left[ \frac{1}{1-q} \frac{2-q}{x-1} \right]^{\frac{2-q}{1-q}} d_q x \\
&= \frac{1}{(1-q)^2} \int_0^\infty \left[ \frac{2-q}{x-1} \right]^{\frac{2-q}{1-q}} d_q x.
\end{align*}
\]

2. If \( f(x, t) = \cos_q \left( \frac{x}{a} + \frac{t}{b} \right) \),

\[
S_q \left[ \cos_q \left( \frac{x}{a} + \frac{t}{b} \right) \right] = \frac{1}{(1-q)^2} \int_0^\infty \int_0^\infty \frac{e^{-\frac{x}{p} - \frac{t}{q}}}{2} e^{-\frac{i(x + \frac{t}{b})}{q}} d_q x d_q t.
\]

The last integral can be divided into two parts

\[
\begin{align*}
&= \frac{1}{(1-q)^2} \int_0^\infty \int_0^\infty \frac{e^{-\frac{x}{p} - \frac{t}{q}}}{2} e^{-\frac{i(x + \frac{t}{b})}{q}} d_q x d_q t + \frac{1}{(1-q)^2} \int_0^\infty \int_0^\infty \frac{e^{-\frac{x}{p} - \frac{t}{q}}}{2} e^{-\frac{i(x + \frac{t}{b})}{q}} d_q x d_q t.
\end{align*}
\]

Similarly as above we get

\[
\begin{align*}
&= \frac{1}{2} \left[ \frac{ba}{(1-q)^2(b-s)(a-pi)} + \frac{ba}{(1-q)^2(b+s)(a+pi)} \right] = \frac{ba(ba-sp)}{(1-q)^2(b^2+s^2)(a^2+p^2)}.
\end{align*}
\]

3. If \( f(x, t) = (xt)^n \),

\[
S_q[(xt)^n] = \frac{1}{(1-q)^2} \int_0^\infty \int_0^\infty \frac{e^{-\frac{x}{p} - \frac{t}{q}}}{2} (xt)^n d_q x d_q t = \frac{1}{(1-q)^2} \int_0^\infty \int_0^\infty \frac{e^{-\frac{x}{p} - \frac{t}{q}}}{2} x^n d_q x d_q t.
\]

If we substitute \( \frac{x}{p} = u \), we get

\[
\begin{align*}
&= \frac{1}{(1-q)^2} \int_0^\infty \int_0^\infty e^{-u(1-q)^n} u^n d_q u d_q t \\
&= \frac{1}{(1-q)^2} \int_0^\infty \int_0^\infty e^{-u(1-q)^n} u^n d_q u d_q t = \frac{1}{(1-q)^2} \left[ \frac{p^n}{s} \Gamma_q(n+1) \right] \int_0^\infty e^{-\frac{u}{q}} u^n d_q u.
\end{align*}
\]

In a similar manner if we substitute \( \frac{t}{s} = v \), we will get

\[
\begin{align*}
&= \frac{1}{(1-q)^2} \left[ \frac{p^n}{s} \Gamma_q(n+1) \right] \Gamma_q(n+1).
\end{align*}
\]

6. Conclusion

In this paper we have introduced some properties of double q-Sumudu transform and its convolution. Convergence and absolute convergence were also discussed, as well as the q-Sumudu transform of periodic functions. The results proved in this paper appear to be new and with certain applications to solving q-difference and q-integral equations.

References

[1] W. H. Abdi, On q-Laplace Transforms, Proc. Nat. Acad. Sci. India Sect. A, 29 (1960), 389–408.
[2] M. H. Abu Risha, M. H. Annaby, H. E. H. Ismail, Z. S. Mansour, Linear q-difference equations, Z. Anal. Anwend., 26 (2007), 481–494.
[3] Z. Ahmed, M. I. Idrrees, F. B. M. Belgacem, Z. Perven, On the convergence of double Sumudu transform, J. Nonlinear Sci. Appl., 13 (2019), 154–162.
[4] D. Albayrak, S. D. Purohit, F. Ucar, *On q-Sumudu transforms of certain q-polynomials*, Filomat, 27 (2013), 411–427.
[5] D. Albayrak, S. D. Purohit, F. Ucar, *On q-analogues of Sumudu transforms*, An. Stiint. Univ. “Ovidius” Constanța Ser. Mat., 21 (2013), 239–259.
[6] D. Albayrak, S. D. Purohit, F. Ucar, *Certain Inversion and Representation formulas for q-Sumudu Transforms*, Hacet. J. Math. Stat., 43 (2014), 699–713.
[7] M. A. Asiru, *Further Properties of the Sumudu Transform and its Applications*, Internat. J. Math. Ed. Sci. Tech., 33 (2002), 441–449.
[8] G. Bangerezako, *Variational calculus on q-nonuniform lattices*, J. Math. Anal. Appl., 306 (2005), 161–179.
[9] F. B. M. Belgacem, A. A. Karaballi, S. L. Kalla, *Analytical investigations of the Sumudu transform and applications to integral production equations*, Math. Probl. Eng., 2003 (2003), 103–118.
[10] K. Brahim, L. Riahi, *Two dimensional Mellin transform in quantum Calculus*, Acta Math. Sci. Ser. B (Engl. Ed.), 38 (2018), 546–560.
[11] L. Debnath, *The Double Laplace Transforms and Their Properties with Applications to Functional, Integral and Partial Differential Equations*, Int. J. Appl. Comput. Math., 2 (2016), 223–241.
[12] A. De Sole, V. G. Kac, *On integral representation of q-gamma and q-beta functions*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl., 16 (2005), 11–29.
[13] J. V. Ganie, A. Ahmad, R. Jain *Basic Analogue of Double Sumudu Transform and its Applicability in Population Dynamics*, Asian J. Math. Stat., 11 (2018), 12–17.
[14] J. V. Ganie, R. Jain, *On a system of q-Laplace transform of two variables with applications*, J. Comput. Appl. Math., 366 (2020), 12 pages.
[15] G. Gasper, M. Rahman, *Basic Hypergeometric Series*, Second ed., Cambridge University Press, Cambridge, (2004).
[16] W. Hahn, *Beitrage Zur Theorie der Heineschen Reihen, die 24 Integrale der hypergeometrischen q-Differenzengleichung, das q-Analog on der Laplace Transformation*, Math. Nachr., 2 (1949), 340–379.
[17] D. O. Jackson, T. Fukuda, O. Dunn, E. Majors, *On q-definite integrals*, Quart. J. Pure Appl. Math., 41 (1910), 193–203.
[18] V. Kac, P. Cheung, *Quantum Calculus*, Springer-Verlag, New York, (2002).
[19] A. Kilicman, H. E. Gadian, *On the application of Laplace and Sumudo transforms*, J. Franklin Inst., 347 (2010), 848–862.
[20] S. D. Purohit, S. L. Kalla ,*On q-Laplace transforms of the q-Bessel functions*, Fract. Calc. Appl. Anal., 10 (2007), 189–196.
[21] P. N. Sadjang, *On double q-Laplace transform and application*, arXiv, 2019 (2019), 27 pages.
[22] J. M. Tchuenche, N. S. Mbare, *An Application of the double Sumudu Transform*, J. Math. Anal. Appl. (Ruse), 1 (2007), 31–39.
[23] M. K. Wang, Y. M. Chu, *Refinements of transformation inequalities for zero-balanced hypergeometric functions*, Acta Math. Sci. Ser. B (Engl. Ed.), 37 (2017), 607–622.
[24] G. K. Watugala, *Sumudu transform: a new integral transform to solve differential equations and control engineering problems*, Internat. J. Math. Ed. Sci. Tech., 24 (1993), 35–43.
[25] G. K. Watugula, *The Sumudu transform for functions of two variables*, Math. Engrg. Indust., 8 (2002), 293–302.
[26] Z.-H. Yang, Y.-M. Chu, *Asymptotic formulas for gamma function with applications*, Appl. Math. Comput., 270 (2015), 665–680.
[27] Z.-H. Yang, W.-M. Qian, Y.-M. Chu, W. Zhang, *On rational bounds for the gamma function*, J. Inequal. Appl., 2017 (2017), 17 pages.
[28] Z.-H. Yang, W. Zhang, Y.-M. Chu, *Sharp Gautschi inequality for parameter 0 < p < 1 with applications*, Math. Inequal. Appl., 20 (2017), 1107–1120.
[29] T.-H. Zhao, Y.-M. Chu, H. Wang, *Logarithmically complete monotonicity properties relating to the gamma function*, Abstr. Appl. Anal., 2011 (2011), 13 pages.