Regularity Criteria for the Kuramoto–Sivashinsky Equation in Dimensions Two and Three

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Abstract
We propose and prove several regularity criteria for the 2D and 3D Kuramoto–Sivashinsky equation, in both its scalar and vector forms. In particular, we examine integrability criteria for the regularity of solutions in terms of the scalar solution \( \phi \), the vector solution \( u \triangleq \nabla \phi \), as well as the divergence \( \text{div}(u) = \Delta \phi \), and each component of \( u \) and \( \nabla u \). We also investigate these criteria computationally in the 2D case, and we include snapshots of solutions for several quantities of interest that arise in energy estimates.

Keywords Kuramoto–Sivashinsky · Navier–Stokes equations · Regularity · Global well-posedness

Mathematics Subject Classification 35A01 · 35K25 · 35K51 · 35K58 · 35B65 · 35B10 · 65M70

1 Introduction
For those with an interest in nonlinear PDEs, the two-dimensional (2D) Kuramoto–Sivashinsky equation (KSE) is full of tantalizing possibilities. There is such strong diffusion in the equation that the nonlinear term seems incapable of overpowering it,
and yet, it is precisely the structure of the nonlinear term that has so far thwarted all efforts at a general proof of global well-posedness. It is sometimes thought that this is due to the backward diffusion term in the equations, but this term serves mainly as the source of energy and instability, fueling a kind of self-perpetuating chaos. Instead, the major difficulty is due to the fact that the nonlinear term does not vanish in any known energy estimates, similar to the vorticity stretching term in the Navier–Stokes equations (NSE), where an analogous barrier arises. This paper aims to quantify these difficulties by proving several integrability-type component-based regularity criteria for well-posedness, and also to shed light on well-posedness issues via numerical simulations.

For those with a more practical eye, the KSE enjoys a wealth of applications. Originally proposed in the late 1970s by Kuramoto, Sivashinsky, and Tsuzuki in investigations of crystal growth (Kuramoto and Tsuzuki 1975, 1976) and flame-front instabilities (Sivashinsky 1977) (see also Sivashinsky 1980), it has since made many appearances, such as in studies of inclined planes (Sivashinsky and Michelson 1980), and has even been shown to be a general feature of certain unstable behaviors (Misbah and Valance 1994). Although we consider both 2D and 3D cases, we note that, at least in certain situations such as flame-front propagation, the 2D case is more relevant, as it describes the evolving 2D front of a 3D flame. However, the 3D case is of interest in terms of making analogies with the 3D NSE.

For the $N \geq 2$ case, short-time existence of smooth solutions (specifically, Gevrey class regularity) was proved in Biswas and Swanson (2007). The triviality of steady states was studied in Cao and Titi (2006). Variations of the 2D KSE have been studied in, e.g., Ambrose and Mazzucato (2021), Guo Boling (1993), Coti-Zelati et al. (2021), Feng and Mazzucato (2020), Ioakim and Smyrlis (2016), Larios and Yamazaki (2020), Tomlin et al. (2018), as well as variations on its boundary conditions (Galaktionov et al. 2008; Larios and Titi 2016; Pokhozhaev 2008). The question of the global well-posedness of KSE for $N \geq 2$ in the periodic case or $\mathbb{R}^N$ is still open in general; however, in dimensions $N = 2$ and 3 for the case of radially symmetric initial data in an annular domain, global well-posedness was proved in Bellout et al. (2003), assuming homogeneous Neumann boundary conditions. It was shown in Ambrose and Mazzucato (2018) (see also Ambrose and Mazzucato 2021; Coti-Zelati et al. 2021; Feng and Mazzucato 2020) that, in the case where there are no linearly growing modes, global existence holds for sufficiently small initial data in a certain function space based on the Wiener algebra. We also mention (Sell and Taboada 1992), which studied global existence and attractors in 2D thin domains (see also Guo Boling 1993; Benachour et al. 2014).

The two major difficulties in the $N \geq 2$ case are (i) the fact that the nonlinear term does not vanish in energy estimates (since the solution is not divergence-free), and hence no $L^p$ norm is conserved, and (ii) the fact that, due to the 4th-order derivatives, no maximum principle is known to hold. However, we note an interesting recent result in Ibdah (2021) that proves global regularity (without any smallness condition) for a modified version of the Michelson–Sivashinsky equation. While lower-order than KSE, this equation shares many similarities with KSE in that it also does not conserve $L^p$ norms, has no known maximum principle, and has no divergence-free condition.
In contrast to the 2D KSE, the 1D KSE is a seemingly limitless playground, where one has essentially everything one could want. Well-posedness is straightforward (Nicolaenko and Scheurer 1984; Tadmor 1986), the large-time dynamics are chaotic (unlike in the case of the 1D Burgers’ equation, where the large-time dynamics are trivial) but finite-dimensional (Collet et al. 1993b; Nicolaenko et al. 1985; Otto 2009), and much work on quantities of interest, such as the existence of the attractor and estimates on its dimension, have seen excellent progress in recent decades, see, e.g., Collet et al. (1993a), Collet et al. (1993b), Constantin et al. (1989a), Constantin et al. (1989b), Foias et al. (1985a), Foias et al. (1985b), Foias et al. (1989), Goluskin and Fantuzzi (2019), Goodman (1994), Grujić (2000), Hyman and Nicolaenko (1986), Il’yashenko (1992), Nicolaenko et al. (1986), Robinson (2001), Tadmor (1986), Temam (1997), Kostianko et al. (2018) and the references therein.

Faced with an obstacle in proving global well-posedness in case $N = 2$ or 3, a natural strategy is to investigate its criterion following analogous works for the 3D NSE such as (2.3) from Escauriaza et al. (2003), Kiselev and Ladyzhenskaya (1957), Prodi (1959), Serrin (1962). In this endeavor, we first obtained criterion that seemed to be natural extensions from those of the NSE (see (2.11) and Remark 2.3). To our surprise, subsequently we were able to improve such results significantly (see Theorem 3.1 and Remark 3.2). This motivated us to pursue another direction of research that has caught much attention in the past few decades, specifically component reduction of such classical regularity criterion (see (2.12)). Despite a large amount of work dedicated to such results on various systems including the NSE, magnetohydrodynamics (MHD) system, and surface quasi-geostrophic (SQG) equations, divergence-free property of velocity field was crucial in all of their results; consequently, we are not aware of any example of a PDE that does not involve divergence-free velocity field and yet admit component reduction results. Unexpectedly, we were able to obtain such results by making use of special structure of the KSE, which seems to be a unique property that is absent in the NSE or Burgers’ equation (see Remark 3.5).

Now let us write $\partial_t \triangleq \frac{\partial}{\partial t}$ and introduce several equations of our main concern. First, the $ND$ KSE in vector form is given by

$$\partial_t u + (u \cdot \nabla)u + \lambda \Delta u + \Delta^2 u = 0, \quad (1.1)$$

with appropriate boundary conditions (here taken to be periodic) and initial data $u^{in}$. Here, $\lambda > 0$ is a constant. We note that one often considers the KSE with $\lambda = 1$ and a domain with size determined by a parameter $L > 0$, such as $[-\frac{L}{2}, \frac{L}{2}]^N$. We choose to work with a fixed domain $\mathbb{T}^N \triangleq [-\pi, \pi]^N$ and a general parameter $\lambda$; these formulations are equivalent under the rescaling $u'(x, t) \mapsto (L/2\pi)^{-3} u(\frac{x}{L/2\pi}, \frac{t}{(L/2\pi)^d})$ after setting $\lambda = (L/2\pi)^2$.

The scalar form of (1.1) is formally given by

$$\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + \lambda \Delta \phi + \Delta^2 \phi = 0, \quad (1.2)$$

By setting $u = \nabla \phi$ in (1.2), one formally obtains (1.1). Our analytical results throughout this manuscript go through independently of the fact that $u = \nabla \phi$; i.e., we will
obtain results on (1.1) and (1.2) separately and independently of each other. For comparison, let us recall the NSE:

\[ \partial_t u + (u \cdot \nabla) u + \nabla \Pi = \Delta u, \quad \nabla \cdot u = 0, \quad (1.3) \]

where \( \Pi \) represents the pressure. Due to the condition \( \nabla \cdot u = 0 \), an \( L^2(\mathbb{T}^N) \)-inner product of \( (u \cdot \nabla) u \) with \( u \) vanishes for the NSE, which is crucial in energy estimates. However, this does not happen for the KSE, nor for the ND Burgers’ equation

\[ \partial_t u + (u \cdot \nabla) u = \Delta u, \quad (1.4) \]

with \( N \geq 2 \), although Burgers’ equation enjoys a maximum principle allowing for a proof of global well-posedness; see, e.g., Solonnikov et al. (1968); Pooley and Robinson (2016) for details. Currently, no such maximum principle is known to hold for the KSE, and hence, global well-posedness for arbitrary smooth initial data remains an open problem. The case \( \Pi \equiv 0 \) and \( \nabla \cdot u = 0 \) condition being dropped reduces the NSE to Burgers’ equation.

**Remark 1.1** A major interest of ours is the Navier–Stokes Eq. (1.3), where, not counting the pressure, the nonlinearity is (at least notationally) the same as in Eq. (1.1). Hence, we aim to prove as much as possible without using the assumption that \( u = \nabla \phi \).

However, all of our results can be translated to results about Eq. (1.2) by thinking of \( u \) as merely a notation for \( \nabla \phi \); for instance, \( u_1 = \partial_1 \phi, \nabla \cdot u = \Delta u \), and so on.

However, note that if one does not assume that \( u = \nabla \phi \), then the vector and scalar forms of the KSE may not be equivalent. Indeed, suppose that \( \phi \) is a given smooth solution to (1.2) with initial data \( \phi^{in} \). Automatically, \( u \triangleq \nabla \phi \) is a solution of (1.1) with initial data \( u^{in} \triangleq \nabla \phi^{in} \). However, given a solution \( u \) to the vector form (1.1), it may not be possible to find a solution \( \phi \) to the scalar form such that \( u = \nabla \phi \), since \( u \) might not be guaranteed to be a pure gradient; for instance, when the initial data \( u_0 \) for the vector form is not a pure gradient.

Indeed, denoting the Helmholtz–Hodge decomposition of a solution \( u \) to (1.1) by \( u = \nabla q + v \) where \( \nabla \cdot v = 0 \) (we denote the Leray–Helmholtz projection \( P_\sigma u = v \)), a straightforward calculation shows, at least formally, that

\[ \frac{1}{2} \frac{d}{dt} \|v\|^2_{L^2} + \|\Delta v\|^2_{L^2} = \lambda \|\nabla v\|^2_{L^2} - 2(\nabla q \cdot S, v), \quad (1.5) \]

where \( S \triangleq \frac{1}{2} (\nabla v - (\nabla v)^T) \) is the anti-symmetric part of the gradient of \( v \). If \( (\nabla q \cdot S, v) \neq 0 \), there is potential for the gradient part of \( u \) to “push” the solution off of the gradient manifold. Moreover, even if one could show analytically that \( P_\sigma u = 0 \) is preserved by (1.1), in computations, small errors could be amplified due to the destabilizing term in (1.1), and the non-gradient part of the solution could grow rapidly. To test this computationally, we simulated the vector Eq. (1.1) starting from a pure-gradient initial condition. The results are shown in Fig. 1, where we observe that the deviation from the gradient manifold grows exponentially fast in time (see Fig. 1).
The growth of the $L^2$-norm of the divergence-free part $v = P_\sigma u$ of a vector solution $u$ of (1.1) ($\lambda = 8.1$; i.e., 24 unstable modes), with initial data $u^{in} = \nabla \phi^{in}$, where $\phi^{in}$ is given by (4.1). Since $P_\sigma u^{in} \equiv 0$, one expects that $P_\sigma u(t) \approx 0$ for $t > 0$, but this is not what is observed computationally.

Therefore, we choose to study both (1.1) and (1.2) independently, although if one assumes that $u = \nabla \phi$, then all the results we obtain for (1.1), specifically Theorems 3.1, 3.6, 3.10, 3.12, 3.13, and 3.16, immediately imply equivalent results for $\phi$ that solves (1.2). Informally, we give a comprehensive list our results of regularity criteria here for clarity:

1. in terms of $u$ in 2D and 3D (Theorem 3.1);
2. in terms of $u_1$ in 2D (Theorem 3.6);
3. in terms of $u_1, u_2$ in 3D (Theorem 3.10);
4. in terms of $\nabla u$ in 2D and 3D (Theorem 3.12);
5. in terms of $\nabla \cdot u$ in 2D and 3D (Theorem 3.13);
6. in terms of $\partial_2 u_2$ in 2D (Theorem 3.16);
7. in terms of $\phi$ in 2D and 3D (Theorem 3.18);
8. in terms of $\partial_{12} \phi$ in 2D (Theorem 3.20).

Remark 1.2 We note that the proofs of the results on individual components are non-standard, and do not follow as in, e.g., the Navier–Stokes case; a major difference being the loss of the divergence-free condition, which is what makes the multi-dimensional KSE difficult in the first place.

This paper is organized as follows. In Sect. 2, we lay out notation and mention some preliminary results. In Sect. 3, we state and prove our main results. In Sect. 4, we present and discuss our computational results.
2 Preliminaries

For simplicity, we write \( \partial_j \triangleq \frac{\partial}{\partial x_j} \) for \( j \in \{1, \ldots, N\} \) and \( \int F \triangleq \int_{\mathbb{T}^N} F(x)dx; \) for further brevity we may write \( f_x \triangleq \partial_x f, \) etc., when \( f \) is a scalar field. We write \( A \lesssim a, b B \) when there exists a constant \( C = C(a, b) \geq 0 \) such that \( A \leq CB. \) We recall that we may write \( f(x) = \sum_{k \in \mathbb{Z}^N} \hat{f}(k)e^{ik \cdot x}, \) and the inhomogeneous and homogeneous Sobolev norms

\[
\|f\|_{H^s} = \left( \sum_{k \in \mathbb{Z}^N} (1 + |k|^{2s})|\hat{f}(k)|^2 \right)^{\frac{1}{2}},
\|f\|_{\dot{H}^s} = \left( \sum_{k \in \mathbb{Z}^N} |k|^{2s}|\hat{f}(k)|^2 \right)^{\frac{1}{2}},
\]

respectively. We denote by \( \langle \cdot, \cdot \rangle \) the standard \( L^2(\mathbb{T}^N) \)-inner product, and by \( \langle \cdot, \cdot \rangle \) the \( H^{-1}-H^1 \) duality paring.

Let us formally write down the definition of a strong solution to the KSE equation (1.1).

**Definition 2.1** We call \( u \) a strong solution to the KSE (1.1) over a time interval \([0, T]\) if for any \( \psi \in C^\infty(\mathbb{T}^N), \)

\[
\langle \partial_t u, \psi \rangle + (u \cdot \nabla u, \psi) + (\Delta u, \Delta \psi) = \lambda(\nabla u, \nabla \psi)
\]

(2.1)

for almost all \( t \in [0, T], \) and

\[
u \in L^\infty([0, T]; H^1(\mathbb{T}^N)), u \in L^2([0, T]; H^3(\mathbb{T}^N)), \]

(2.2a)

\[
u \in C([0, T]; H^s(\mathbb{T}^N)) \quad \forall \; s \in [0, 1), \partial_t u \in L^2([0, T]; H^{-1}(\mathbb{T}^N)).
\]

(2.2b)

and moreover, the initial data \( u_0 \in H^1(\mathbb{T}^N) \) is satisfied in the sense that \( (u(t), \psi) \to (u_0, \psi) \) as \( t \downarrow 0. \)

Standard arguments show (as in, e.g., Larios and Yamazaki 2020) that the KSE is locally well-posed in \( H^1(\mathbb{T}^N) \) for both \( N \in \{2, 3\}: \)

**Theorem 2.2** In case \( N \in \{2, 3\}, \) given any initial data \( u^{in} \in H^1(\mathbb{T}^N), \) there exists an interval \([0, T]) \) where \( T = T(u^{in}) > 0 \) over which the KSE (1.1) has a unique strong solution starting from \( u^{in}. \)

While the proof is standard and follows from the works within Larios and Yamazaki (2020), because we could not locate its proof in the literature, especially the case \( N = 3, \) for completeness we sketch its outline in the “Appendix.” On the other hand, by Bellout et al. (2003, Theorem 1), we know that (1.2) is locally well-posed in \( L^2(\mathbb{T}^N) \) for both \( N \in \{2, 3\}. \)
Let us recall two prominent regularity criteria for the 3D NSE (1.3):

\[ u \in L^r_T L^p_x \text{ where } \frac{3}{p} + \frac{2}{r} \leq 1, \ p \in [3, \infty], (2.3a) \]

\[ \nabla u \in L^r_T L^p_x \text{ where } \frac{3}{p} + \frac{2}{r} \leq 2, \ p \in [\frac{9}{4}, \infty] (2.3b) \]

due to da Veiga (1995), Iscauriaza and Seregin (2003), Prodi (1959), Serrin (1962).

Let us follow the proof of (2.3a) on NSE (1.3) to examine what type of criteria we may expect for the ND KSE. Taking \( L^2(\mathbb{T}^N) \)-inner products on (1.1) with \( -\Delta u \) gives us

\[ \frac{1}{2} \frac{d}{dt} \| \nabla u \|^2_{L^2} + \| \Delta \nabla u \|^2_{L^2} = \int (u \cdot \nabla) u \cdot \Delta u + \lambda \| \Delta u \|^2_{L^2}. \]  

(2.4)

Let us already mention that in the case of the KSE (1.1), we need to additionally estimate \( L^2(\mathbb{T}^N) \)-norm so that the sum with (2.4) gives an \( H^1(\mathbb{T}^N) \)-estimate because the solution to the KSE (1.1) does not preserve mean-zero property in sharp contrast to the NSE as \( \int (u \cdot \nabla) u \neq 0 \) (this is certainly not a problem if we assume \( u = \nabla \phi \)). As we will see in (3.4) and (3.5), this will not give us much trouble, and because we only want to illustrate heuristics here, let us omit these details. Now, for clarity, we consider \( p \in (N, \infty) \) first, and the case \( p = \infty \) afterward. We estimate by Hölder’s inequality

\[ \int (u \cdot \nabla) u \cdot \Delta u \leq \| u \|^p_{L^p} \| \nabla u \|^p_{L^2} \| \Delta u \|^2_{L^2}. \]  

(2.5)

We can continue by Gagliardo–Nirenberg inequality of

\[ \| \nabla f \|_{L^{\frac{2p}{p-N}}} \lesssim \| f \|_{L^{\frac{2p}{p}}}^{\frac{p-N}{2p}} \| f \|_{H^2}^{\frac{p+N}{2p}} \]  

(2.6)

for \( p \in \{ (N, \infty) \text{ if } N = 2, \ [N, \infty) \text{ if } N = 3 \} \)

to bound by

\[ \int (u \cdot \nabla) u \cdot \Delta u \lesssim \| u \|^p_{L^p} \| \nabla u \|^p_{L^2} \| \nabla \nabla u \|_{L^2} \| \nabla u \|_{L^2}^{\frac{p-N}{2p}} \| \nabla u \|_{H^2}^{\frac{p+N}{2p}} \]

\[ \leq \frac{1}{4} \| \Delta \nabla u \|^2_{L^2} + C \left( \| u \|^p_{L^p} \| \nabla u \|_{L^2}^{\frac{4p-N}{2p}} + \| u \|^p_{L^p} \right) \| \nabla u \|^2_{L^2} \]  

(2.7)

by Young’s inequality. This estimate is not valid for the case \( N = p = 2 \) because it is false that \( \| \Delta u \|_{L^\infty} \) can be bounded by a constant multiple of \( \| \Delta u \|_{H^1} \) when \( N = 2 \). In case \( p = +\infty \), we easily use the interpolation of
\[ \|f\|_{H^2}^2 \leq \left( \int |\xi|^{3} |\hat{f}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} \left( \int |\xi|^{3} |\hat{f}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} = \|f\|_{H^3} \|f\|_{H^1} \quad (2.8) \]

and estimate

\[ \int (u \cdot \nabla)u \cdot \Delta u \leq \|u\|_{L^\infty} \|\nabla u\|_{L^2} \|\Delta u\|_{L^2} \]

\[ \leq \frac{1}{4} \|\Delta \nabla u\|_{L^2}^2 + C \|u\|_{L^\infty}^4 \|\nabla u\|_{L^2}^2 \quad (2.9) \]

by Young’s inequality. Using (2.8), we can also estimate the linear term

\[ \lambda \|\Delta u\|_{L^2}^2 \leq \lambda \|\Delta \nabla u\|_{L^2} \|\nabla u\|_{L^2} \leq \frac{1}{4} \|\Delta \nabla u\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2. \quad (2.10) \]

Applying (2.7)–(2.10) to (2.4) leads to

\[ \frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \frac{1}{2} \|\Delta \nabla u\|_{L^2}^2 \lesssim \|\nabla u\|_{L^2}^2 (\|u\|_{L^{\frac{4^p}{3p-N}}}^{\frac{4^p}{3p-N}} + \|u\|_{L^p} + 1) \]

with \( \frac{4^p}{3p-N} = \frac{4}{3} \) in case \( p = \infty \). This leads to a regularity criteria of

\[ u \in L^r_T L^p_x \quad \text{where} \quad \left\{ \begin{array}{ll}
p \in (N, \infty), r \in \left[ \frac{4}{3}, 2 \right] & \text{if } N = 2, \\
p \in [N, \infty), r \in \left[ \frac{4}{3}, 2 \right] & \text{if } N = 3, \\
\end{array} \right. \]

satisfy

\[ \frac{N}{p} + \frac{2}{r} = \frac{3}{2} + \frac{N}{2p}. \quad (2.11) \]

**Remark 2.3** If we take \( N = 2 \) and then \( p > 2 \) arbitrarily close to 2 in (2.11), then we see that the right hand side of (2.11) is almost 2, very reminiscent of (2.3b) in terms of \( \nabla u \) for the 3D NSE (1.3). In fact, (2.11) may be heuristically explained via scaling argument as follows. Informally assuming that \( \lambda = 0 \) in (1.1), we realize that if \( u(t, x) \) is its solution, then so is \( u_{\beta}(t, x) \triangleq \beta^3 u(\beta^4 t, \beta x) \) and \( \|u_{\beta}\|_{L^r_T L^p_x} = \|u\|_{L^r_{\beta^4 t} L^p_{\beta x}} \) if and only if \( \frac{N}{p} + \frac{2}{r} = \frac{3}{2} + \frac{N}{2p} \). We are indebted to one of the referees for pointing this out.

As we emphasized after (2.7), we cannot obtain the case \( N = p = 2 \) due to the fact that \( H^1(\mathbb{T}^2) \hookrightarrow L^\infty(\mathbb{T}^2) \) is false in general. For the NSE, there are typically div-curl lemma or Brezis-Wainger type inequality arguments to work around this issue and improve \( p > N \) to \( p = N \). Unfortunately, both methods faced technical difficulty for the KSE. In fact, we will actually see in Theorem 3.1 that it is possible to improve (2.11) significantly by rather an unconventional approach that seems unique to the KSE.

Very recently, the research direction of component reduction from the criteria in (2.3) for the 3D NSE has received much attention. While we acknowledge that there have been many extensions and improvements, let us list the following examples as a
reference: starting from an initial data that is sufficiently smooth, there exists a unique solution to the 3D NSE for all $t > 0$ if for any $j, k \in \{1, 2, 3\}$,

$$u_j \in L^r_T L^p_x \text{ where } \frac{3}{p} + \frac{2}{r} \leq \frac{5}{8}, \quad p \in \left(\frac{24}{5}, \infty\right] \quad (2.12a)$$

$$\partial_j u_j \in L^r_T L^p_x \text{ where } \frac{3}{p} + \frac{2}{r} < \frac{3(p + 2)}{4p}, \quad p > 2, \quad r \in [1, \infty), \quad (2.12b)$$

$$\partial_k u_j \in L^r_T L^p_x \text{ for } k \neq j \text{ where } \frac{3}{p} + \frac{2}{r} < \frac{p + 3}{2p}, \quad p > 3, \quad r \in [1, \infty), \quad (2.12c)$$

due to Cao and Titi (2011), Kukavica and Ziane (2006) (see also Cao and Titi 2008; Kukavica and Ziane 2007; Zhou and Pokorny 2010). We point out that while the dependence on $u = (u_1, u_2, u_3)$ or $\nabla u = (\partial_j u_k)_{1 \leq j, k \leq 3}$ was remarkably reduced down respectively to $u_j, \partial_j u_j$, or $\partial_k u_j$, the integrability condition of 1 and 2 in (2.3) deteriorated respectively to $\frac{5}{8}, \frac{3(p+2)}{4p}$, or $\frac{p+3}{2p}$ in (2.12). This is the price one must pay for reducing components; to the best of our knowledge, despite much effort by many mathematicians, it remains unknown if we can improve $\frac{5}{8}$ up to 1 or $\frac{3(p+2)}{4p}$, $\frac{p+3}{2p}$ up to 2, as in (2.3). All the proofs of such component reduction results rely crucially on the divergence-free property of $u$ in (1.3), and some of such results have been extended to other systems of PDEs that involve divergence-free velocity field (e.g., Cao and Wu 2010; Yamazaki 2016 on MHD system; Yamazaki 2013 on SQG equations). In short, the proof of (2.12a) relies on the fact that upon $\frac{\| (\partial_1 u, \partial_2 u) \|^2_{L^2}}{\| \nabla \|_{L^2}}$-estimate of the 3D NSE (1.3), one can separate $u_3$ within the nonlinear terms as follows:

$$\int (u \cdot \nabla) u \cdot (\partial_1^2 + \partial_2^2) u = -\sum_{i,j=1}^{3} \sum_{k=1}^{2} \int \partial_k u_i \partial_1 u_j \partial_2 u_j \lesssim \int |u_3| \| \nabla u \| \| \nabla (\partial_1, \partial_2) u \|.$$

For example, we can estimate by summing the terms when $(i, j, k) = (1, 1, 2)$ and $(i, j, k) = (2, 1, 2)$ to bound

$$\int \partial_2 u_1 \partial_1 u_1 \partial_2 u_1 + \partial_2 u_2 \partial_2 u_1 \partial_2 u_1 = -\int (\partial_2 u_1)^2 \partial_3 u_3 \lesssim \int |u_3| \| \nabla u \| \| \nabla (\partial_1, \partial_2) u \|$$

where the divergence-free property played an important role. Despite the large number of papers devoted to this direction of research, we are not aware of any example of a PDE that admitted such component reduction of regularity criteria without a divergence-free property. Remarkably, we have been able to prove such results for the KSE (see Theorems 3.6, 3.10, 3.16, and 3.20).
3 Regularity Criteria for Nd KSE and Their Proofs

The following computations hold on $\mathbb{R}^N$ or $\mathbb{T}^N$ for $N \in \{2, 3\}$, but we choose to focus on the latter, and also elaborate more in the case $N = 2$ as our main concern. Results in Sects. 3.1–3.4 focus on (1.1) independently of (1.2) while Sect. 3.5 focuses on (1.2) independently of (1.1).

3.1 Result on $u$

Let us continue the $\dot{H}^1(\mathbb{T}^N)$-estimate from (2.4). For the sake of generality, let us also aim to attain a criterion in $W^m_{x,p}$ for $m \in [0, 1)$ so that taking $m = 0$ reduces to the standard criterion in $L^p_T$. We proceed as follows:

$$\int (u \cdot \nabla) u \cdot \Delta u \leq \|u\|_{L^{\frac{Np}{N-mp}}} \|\Delta u\|_{L^{\frac{2Np}{Np+mp-N}}}$$

(3.1)

by Hölder’s inequality for $p$ in an appropriate range to be specified subsequently. Now we use two Gagliardo–Nirenberg inequalities:

$$\|f\|_{L^{\frac{2Np}{Np+mp-N}}} \lesssim \|f\|_{L^2}^{\frac{(4+m)p-N}{4p}} \|f\|_{H^2}^{\frac{N-mp}{4p}}$$

(3.2a)

for any $N \in \{2, 3\}$, $p \in \begin{cases} \left[1, \frac{N}{m}\right] & \text{if } m \in (0, 1), \\ \left[1, \infty\right] & \text{if } m = 0, \end{cases}$

$$\|\nabla f\|_{L^{\frac{2Np}{Np+mp-N}}} \lesssim \|f\|_{L^2}^{\frac{(2+m)p-N}{4p}} \|f\|_{H^2}^{\frac{(2-m)p+N}{4p}}$$

(3.2b)

$$\begin{cases} p \in \left[1, \infty\right] & \text{if } N = 2, m \in (0, 1), \\ p \in \left[\frac{3}{m+2}, \infty\right] & \text{if } N = 3, m \in (0, 1), \\ p \in \left(1, \infty\right] & \text{if } N = 2, m = 0, \\ p \in \left[\frac{3}{2}, \infty\right) & \text{if } N = 3, m = 0. \end{cases}$$

Now if $m \in (0, 1)$, then we apply Sobolev embedding $W^{m,p}(\mathbb{T}^N) \hookrightarrow L^{Np/mp}(\mathbb{T}^N)$ assuming $p < \frac{N}{m}$ as part of hypothesis in case $m \in (0, 1)$ and see that

$$\int (u \cdot \nabla) u \cdot \Delta u \lesssim \|u\|_{W^{m,p}} \|\nabla u\|_{L^2}^{\frac{(m+3)p-N}{2p}} \|\nabla u\|_{H^2}^{\frac{(1-m)p+N}{2p}}$$

$$\leq \frac{1}{8} \|\nabla u\|_{L^2}^2 + C(\|u\|_{W^{m,p}}^{\frac{4p}{(3+mp)p-N}} + 1) \|\nabla u\|_{L^2}^2$$

(3.3)

by Young’s inequality. As mentioned, we also need an $L^2(\mathbb{T}^N)$ estimate; thus, taking $L^2(\mathbb{T}^N)$-inner products on (1.1) with $u$ we obtain

$$\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 = \lambda \|\nabla u\|_{L^2}^2 - \int (u \cdot \nabla) u \cdot u$$

(3.4)
where we estimate
\[
\int (u \cdot \nabla) u \cdot u \\
\leq \|u\|_{L^{N-m/p}}^{Np} \|u\|_{L^{Np+m/p-N}}^{2Np} \|\nabla u\|_{L^{Np+m/p-N}}^{2Np} \\
\lesssim \|u\|_{W^{m,p}} \|u\|_{L^2}^{(m+3)p-N} \|\nabla u\|_{L^2}^{(1-m)p+N} \\
\lesssim \|u\|_{W^{m,p}} \left(\|u\|_{L^2}^{(m+3)p-N} \|\nabla u\|_{L^2}^{4p} \|\Delta \nabla u\|_{L^2}^{4p} + \|u\|_{L^2}^2\right) \\
\leq \frac{1}{8} \|\Delta \nabla u\|_{L^2}^2 + C \left(1 + \|u\|_{W^{m,p}} + \|u\|_{W^{m,p}}^{(3+m)p-N}\right) \|u\|_{H^1}^2.
\] (3.5)

by Hölder’s inequality, Sobolev embedding $W^{m,p}(\mathbb{T}^N) \hookrightarrow L^{N-m/p}(\mathbb{T}^N)$ assuming $p < \frac{N}{m}$ in case $m \in (0, 1)$ as part of hypothesis, (3.2a) and (3.2b), and Young’s inequality. Applying (2.10) and (3.3) to (2.4), and (3.5) to (3.4), we showed the following criteria:

**Theorem 3.1** Suppose $u \in L^r_T W^{m,p}_x$ where $m \in [0, 1)$, and

\[
\begin{cases}
  p \in [1, \frac{N}{m}], r \in \left[\frac{4}{3}, \frac{4}{1+m}\right] & \text{if } N = 2, m \in (0, 1), \\
  p \in \left[\frac{N}{m+2}, \frac{N}{m}\right], r \in \left[\frac{4}{3}, 4\right] & \text{if } N = 3, m \in (0, 1),
\end{cases}
\] (3.6)

and

\[
\begin{cases}
  p \in (\frac{N}{2}, \infty], r \in \left[\frac{4}{3}, 4\right] & \text{if } N = 2, m = 0, \\
  p \in (\frac{N}{2}, \infty], r \in \left[\frac{4}{3}, 4\right] & \text{if } N = 3, m = 0,
\end{cases}
\] (3.7)

satisfy

\[
\frac{N}{p} + \frac{2}{r} = \frac{3 + m}{2} + \frac{N}{2p}.
\] (3.8)

Then, the ND KSE is globally well-posed in $H^1(\mathbb{T}^N)$.

**Remark 3.2** In comparison with Remark 2.3, because we improved the range of $p$ in (2.11) to (3.7) in Theorem 3.1, the right hand side of (3.8) with $m = 0$ can achieve $\frac{3}{2} + \frac{N}{2p} > 2$ for $p \in (\frac{N}{2}, N)$; in fact, taking $p \approx \frac{N}{2}$ gives $\frac{3}{2} + \frac{N}{2p}$ to be almost $\frac{3}{2}$. We recall from Remark 2.3 that $\frac{N}{p} + \frac{2}{r} = \frac{3}{2} + \frac{N}{2p}$ can be explained heuristically via scaling argument. We also emphasize that this new approach in the proof of Theorem 3.1 is impossible for the NSE. Indeed, if we choose $\|\Delta u\|_{L^{Np+m/p-N}}^{2Np}$ in the first Hölder’s inequality of (3.1), then this term already cannot be handled by the diffusion in the NSE, specifically $\|\Delta u\|_{L^2}^2$ in $H^1$-estimate. It is the fourth-order diffusion in the KSE that is allowing us to proceed with this new approach.
3.2 Result on $u_1$

Next, we obtain a criterion on only one component $u_1$ in the 2D case while on $(u_1, u_2)$ in the 3D case.

**Proposition 3.3** Suppose $u_1 \in L^p_T W^{m,p}_x$ where $m \in [0, 1)$ and

$$
\begin{aligned}
    & p \in [1, \frac{2}{m}), r \in (\frac{4}{3}, \frac{4}{1+m}] & \text{if } m \in (0, 1), \\
    & p \in (1, \infty), r \in [\frac{4}{3}, 4) & \text{if } m = 0,
\end{aligned}
$$

satisfy

$$
\frac{2}{p} + \frac{2}{r} = \frac{1}{p} + \frac{3 + m}{2}.
$$

Then, $u_2 \in L^\infty_T L^2_\lambda \cap L^2_T H^2_\lambda$.

**Proof** Taking $L^2(T^2)$-inner products with $u_2$ on the second component of (1.1) gives

$$
\frac{1}{2} \frac{d}{dt} \|u_2\|^2_{L^2} + \|\Delta u_2\|^2_{L^2} = -\int (u \cdot \nabla)u_2 u_2 + \lambda \|\nabla u_2\|^2_{L^2} \tag{3.9}
$$

where

$$
\lambda \|\nabla u_2\|^2_{L^2} \leq \lambda \|u_2\|_{L^2} \|\Delta u_2\|_{L^2} \leq \frac{1}{4} \|\Delta u_2\|^2_{L^2} + C \|u_2\|^2_{L^2}, \tag{3.10}
$$

and

$$
-\int (u \cdot \nabla)u_2 u_2 = -\int u_1(\partial_1 u_2)u_2 - \int u_2 \partial_2 u_2 u_2
\begin{align*}
&= -\int u_1(\partial_1 u_2)u_2 - \int \frac{1}{3} \partial_2 (u_2)^3 \\
&\leq \|u_1\|_{L^{2-p} m-p} \|u_2\|_{L^{2p-mp=2}} \|\nabla u_2\|_{L^{2p-mp=2}} \\
&\leq \|u_1\|_{W^{m,p}} \|u_2\|_{L^2} \left( \|u_1\|_{W^{m,p}}^{\frac{(3-m)p-2}{2p}} + \|u_1\|_{W^{m,p}} \right) \|u_2\|_{L^2} \tag{3.11}
\end{align*}
$$

by Hölder’s inequality, Sobolev embedding $W^{m,p}(T^2) \hookrightarrow L^{\frac{2p}{3-m}}(T^2)$ if $m \in (0, 1)$ so that $p < \frac{2}{m}$ by hypothesis, (3.2a) and (3.2b), and Young’s inequality. Applying (3.10) and (3.11) to (3.9) implies the desired result. □

**Proposition 3.4** Suppose $u_1 \in L^p_T W^{m,p}_x$ where $m \in [0, 1)$ and

$$
\begin{aligned}
    & p \in [1, \frac{2}{m}), r \in (\frac{4}{3}, \frac{4}{1+m}] & \text{if } m \in (0, 1), \\
    & p \in (1, \infty), r \in [\frac{4}{3}, 4) & \text{if } m = 0,
\end{aligned}
$$

by hypothesis, (3.2a) and (3.2b), and Young’s inequality. Applying (3.10) and (3.11) to (3.9) implies the desired result.
satisfy

\[ \frac{2}{p} + \frac{2}{r} = \frac{1}{p} + \frac{3 + m}{2}. \]

Then, \( u_1 \in L_T^\infty L_x^2 \cap L_T^2 H_x^2 \).

**Proof** We take \( L^2(\mathbb{T}^2) \)-inner products on the first component of (1.1) with \( u_1 \) and compute

\[
\frac{1}{2} \frac{d}{dt} \|u_1\|_{L^2}^2 + \|\Delta u_1\|_{L^2}^2 = -\int (u \cdot \nabla) u_1 u_1 + \lambda \|\nabla u_1\|_{L^2}^2 \tag{3.12}
\]

where \( \lambda \|\nabla u_1\|_{L^2}^2 \leq \frac{1}{4} \|\Delta u_1\|_{L^2}^2 + C \|u_1\|_{L^2}^2 \) similarly to (3.10), while

\[
- \int (u \cdot \nabla) u_1 u_1 \leq \|u_2\|_{L^\infty} \|\nabla u_1\|_{L^2} \|u_1\|_{L^2}
\]

\[
\lesssim \|u_2\|_{H^2} \|u_1\|_{L^2} \|u_1\|_{L^2} \|\Delta u_1\|_{L^2} \leq \frac{1}{4} \|\nabla u_1\|_{L^2} + C \|u_2\|_{H^2} \|u_1\|_{L^2}^2 \tag{3.13}
\]

by Sobolev embedding \( H^2(\mathbb{T}^2) \hookrightarrow L^\infty(\mathbb{T}^2) \), (3.10) and Young’s inequality. This implies that \( u_1 \in L_T^\infty L_x^2 \cap L_T^2 H_x^2 \) because we know \( \|u_2(t)\|_{H^2} \in L_T^2 \) due to Proposition 3.3.

**Remark 3.5** Interestingly, the previous proof of Proposition 3.4 would not work for the 2D Burgers’ equation if we neglect the fact that its solution has the maximum principle in the following way. Starting from \( u_1 \in L_T^\infty L_x^p \cap L_T^r H_x^2 \) for some \( p \) and \( r \), we can still deduce that \( u_2 \in L_T^\infty L_x^2 \cap L_T^2 H_x^1 \) analogously to Proposition 3.3. However, this does not seem to be sufficient to lead to \( u_1 \in L_T^\infty L_x^2 \cap L_T^2 H_x^1 \) because in the estimate of

\[
- \int u_2 \partial_2 u_1 u_1
\]

within (3.13), the diffusion for the Burgers’ equation only gives \( \|\nabla u_1\|_{L^2} \) so that we have no choice but to bound this by

\[
\|u_2\|_{L^\infty} \|\partial_2 u_1\|_{L^2} \|u_1\|_{L^2}.
\]

However, while \( \|\nabla u_2\|_{L^2} \in L_T^2 \), this estimate will not go through immediately since \( H^1(\mathbb{T}^2) \not\hookrightarrow L^\infty(\mathbb{T}^2) \). Thus, Proposition 3.4 really is a special feature of the KSE due to the fourth-order diffusion.

By Propositions 3.3 and 3.4, we have \( u \in L_T^\infty L_x^2 \) and thus we deduce that the 2D KSE is globally well-posed due to Theorem 3.1 as follows:
Theorem 3.6 Suppose \( u_1 \in L_T^r W_x^{m,p} \) where \( m \in [0, 1) \) and
\[
\begin{cases}
  p \in [1, \frac{2}{m}), r \in \left(\frac{4}{3}, \frac{4}{1+m}\right) & \text{if } m \in (0, 1), \\
  p \in (1, \infty], r \in \left[\frac{4}{3}, 4\right) & \text{if } m = 0,
\end{cases}
\]
satisfy
\[
\frac{2}{p} + \frac{2}{r} = \frac{1}{p} + \frac{3 + m}{2}.
\]
Then, the 2D KSE is globally well-posed in \( H^1(\mathbb{T}^2) \).

Remark 3.7 As mentioned, Theorem 3.6 is very interesting because all previous component reduction results relied on divergence-free condition (NSE, MHD system, and SQG equations).

Another remarkable feature of Theorem 3.6 is that we obtained component reduction at no cost in terms of \( \frac{3}{2} + \frac{1}{p} \) (cf. “1” in (2.3a) and “\( \frac{5}{8} \)” in (2.12a)). As mentioned, reducing the Serrin criteria (2.3a) to \( u_3 \) while retaining \( \frac{3}{p} + \frac{2}{r} = 1 \) is a well-known open problem that has caught much attention in the literature.

We are able to extend our result to the 3D case as follows:

Proposition 3.8 Suppose \( u_1, u_2 \in L_T^r W_x^{m,p} \) where \( m \in [0, 1) \) and
\[
\begin{cases}
  p \in \left[\frac{3}{m+2}, \frac{3}{m}\right), r \in \left(\frac{4}{3}, 4\right] & \text{if } m \in (0, 1), \\
  p \in \left[\frac{3}{2}, \infty\right], r \in \left[\frac{4}{3}, 4\right] & \text{if } m = 0,
\end{cases}
\]
satisfy
\[
\frac{3}{p} + \frac{2}{r} = \frac{3}{2p} + \frac{3 + m}{2}.
\]
Then, \( u_3 \in L_T^\infty L_x^2 \cap L_T^2 H_x^2 \).

Proof We consider the third component of (1.1), take \( L_T^2(\mathbb{T}^3) \)-inner products with \( u_3 \) to obtain
\[
\frac{1}{2} \frac{d}{dt} \|u_3\|_{L_x^2}^2 + \|\Delta u_3\|_{L_x^2}^2 = -\int (u \cdot \nabla)u_3u_3 + \lambda \|\nabla u_3\|_{L_x^2}^2
\]
(3.14)
where \( \lambda \|\nabla u_3\|_{L_x^2}^2 \leq \frac{1}{4} \|\Delta u_3\|_{L_x^2}^2 + C \|u_3\|_{L_x^2}^2 \) similarly to (3.10), while
\[
\int (u \cdot \nabla)u_3u_3 = -\int u_1 \partial_1 u_3 + u_2 \partial_2 u_3 u_3
\]
Proposition 3.9 Suppose \( u \in L^{3 \frac{3}{m-2p}}(\mathbb{T}^3) \) applies to (3.14), Grönwall’s inequality, and Young’s inequality. We apply these estimates to (3.16) and because

\[
\lesssim (\|u_1\|_{L^{\frac{3p}{3-mp}}} + \|u_2\|_{L^{\frac{3p}{3-mp}}}) \|u_3\|_{L^{\frac{6p}{3+pmp-3}}} \|\nabla u_3\|_{L^{\frac{6p}{3+pmp-3}}}
\lesssim (\|u_1\|_{W^{m,p}} + \|u_2\|_{W^{m,p}}) \|u_3\|_{L^{2}} \|\frac{(3+m)p-3}{2p} \|u_3\|_{H^2}^{\frac{2p}{2}}
\leq \frac{1}{4} \|\Delta u_3\|^2_{L^2} + C \left( \sum_{k=1}^{2} \|u_k\|_{W^{m,p}}^{\frac{4p}{3+m(p-3)}} + \|u_k\|_{W^{m,p}} \right) \|u_3\|^2_{L^2} \tag{3.15}
\]

by Hölder’s inequality, Sobolev embedding \( L^{3 \frac{3}{m-2p}}(\mathbb{T}^3) \leftrightarrow W^{m,p}(\mathbb{T}^3) \) if \( m > 0 \), (3.2a) and (3.2b), and Young’s inequality. Applying these estimates to (3.14), Grönwall’s inequality completes the proof. \( \square \)

Proposition 3.9 Suppose \( u_1, u_2 \in L^r_T W^{m,p}_x \) where \( m \in [0, 1) \) and

\[
\begin{align*}
    p &\in [\frac{3}{m+2}, \frac{3}{m}], & r &\in \left(\frac{4}{3}, 4\right) & \text{if } m \in (0, 1), \\
    p &\in \left[\frac{3}{2}, \infty\right], & r &\in \left[\frac{4}{3}, 4\right] & \text{if } m = 0,
\end{align*}
\]

satisfy

\[
\frac{3}{p} + \frac{2}{r} = \frac{3}{2p} + \frac{3+m}{2}.
\]

Then, \( u_1, u_2 \in L^\infty_T L^3 \cap L^2_T H^2_L \).

Proof We fix \( j \in \{1, 2\} \) arbitrarily and take \( L^2(\mathbb{T}^3) \)-inner product on the \( j \)th component of (1.1) with \( u_j \) to obtain

\[
\frac{1}{2} \frac{d}{dt} \|u_j\|^2_{L^2} + \|\Delta u_j\|^2_{L^2} = -\int (u \cdot \nabla)u_j u_j + \lambda \|\nabla u_j\|^2_{L^2} \tag{3.16}
\]

where \( \lambda \|\nabla u_3\|^2_{L^2} \leq \frac{1}{4} \|\Delta u_3\|^2_{L^2} + C \|u_3\|^2_{L^2} \) similarly to (3.10), while

\[
-\int (u \cdot \nabla)u_j u_j = -\sum_{k=1}^{3} \int u_k \partial_k u_j u_j
\]

where \( \int u_j \partial_j u_j u_j = 0 \) so that for \( k \in \{1, 2\} \setminus \{j\}, \)

\[
-\int (u \cdot \nabla)u_j u_j = -\int u_k \partial_k u_j u_j + u_3 \partial_3 u_j u_j
\lesssim (\|u_k\|_{L^3} + \|u_3\|_{L^3}) \|\nabla u_j\|_{L^6} \|u_j\|_{L^2}
\leq \frac{1}{4} \|\Delta u_j\|^2_{L^2} + C \left(1 + \|u_k\|^2_{L^3} + \|u_3\|^3_{L^3} \|u_3\|^2_{H^2} \right) \|u_j\|^2_{L^2} \tag{3.17}
\]

by Hölder’s inequality, Sobolev embedding \( H^1(\mathbb{T}^3) \leftrightarrow L^6(\mathbb{T}^3) \), Gagliardo–Nirenberg inequality, and Young’s inequality. We apply these estimates to (3.16) and because
\[ \int_0^T \| u_k \|_{L^3}^2 \lesssim \int_0^T \| u_k \|_{W^{m, \frac{3}{m+t}}}^2 \, dt < \infty, \]
\[ \int_0^T \| u_3 \|_{L^2}^\frac{3}{2} \| u_3 \|_{H^2}^\frac{1}{2} \, dt \leq \| u_3 \|_{L^\infty L^2} \int_0^T \| u_3 \|_{H^2}^\frac{1}{2} \, dt < \infty \]
due to Sobolev embedding \( W^{m, \frac{3}{m+t}} (\mathbb{T}^3) \hookrightarrow L^3(\mathbb{T}^3) \) if \( m \in (0, 1) \), hypothesis, and Proposition 3.8, Grönwall’s inequality completes the proof. \( \square \)

Due to Propositions 3.8 and 3.9, we see that \( u \in L^\infty L^2_x \) and thus Theorem 3.1 again leads to the following theorem.

**Theorem 3.10** Suppose \( u_1, u_2 \in L^r_T W^{m-p}_x \) where \( m \in [0, 1) \) and

\[
\begin{align*}
  p & \in \left[ \frac{3}{m+2}, \frac{3}{m} \right), r \in (\frac{4}{3}, 4) & \text{if } m \in (0, 1), \\
  p & \in \left[ \frac{3}{2}, \infty \right], r \in \left[ \frac{4}{3}, 4 \right] & \text{if } m = 0,
\end{align*}
\]
satisfy

\[
\frac{3}{p} + \frac{2}{r} = \frac{3}{2p} + \frac{3+m}{2}.
\]

Then the 3D KSE is globally well-posed in \( H^1(\mathbb{T}^3) \).

**Remark 3.11** We leave it as an interesting problem to attain a regularity criterion for the 3D KSE in terms of one component instead of two components as we did in Theorem 3.10.

### 3.3 Result on \( \nabla u \)

We aim to obtain a criterion in terms of \( \| \nabla u \|_{L^p} \), motivated the case of the NSE in (2.3b). We can start from (2.4) and rely on (2.10). For the nonlinear term we can write

\[
\int (u \cdot \nabla) u \cdot \Delta u = -\sum_{i,j,k} \int \left( \partial_k u_i \partial_i u_j \partial_k u_j - \frac{1}{2} \partial_i u_i (\partial_k u_j)^2 \right) \lesssim \int |\nabla u|^3. \tag{3.18}
\]

Now let us point out again that if we follow the standard approach of the NSE, then we would compute for \( p \in [2, \infty] \),

\[
C \int |\nabla u|^3 \lesssim \| \nabla u \|_{L^p} \| \nabla u \|_{L^2} \| \nabla u \|_{L^\frac{2p}{p-2}}^2 \lesssim \| \nabla u \|_{L^p} \| \nabla u \|_{L^2} \| \nabla u \|_{L^\frac{2p-N}{2p}}^2 \| \nabla u \|_{H^2}^{\frac{N}{2p}} \leq \frac{1}{4} \| \Delta u \|_{L^2}^2 + C (\| \nabla u \|_{L^p}^{4p-N} + \| \nabla u \|_{L^p}) \| \nabla u \|_{L^2}^2
\]
by Hölder’s, Gagliardo–Nirenberg, and Young’s inequalities. This only leads to a criterion of \( \nabla u \in L_T^r L_x^p \) where \( \frac{N}{p} + \frac{2}{r} = 2 + \frac{N}{2p} \), \( p \in [2, \infty) \), \( r \in [1, \frac{8}{8-N}] \). In fact, we approach this estimate differently from the NSE as follows in both cases \( N \in \{2, 3\} \):

\[
C \int |\nabla u|^3 \lesssim \|\nabla u\|_{L^p} \|\nabla u\|_{L^2}^{2p} \lesssim \|\nabla u\|_{L^p} \|\nabla u\|_{H^2}^{\frac{N}{2p}} \\
\leq \frac{1}{4} \|\Delta u\|_{L^2}^2 + C(\|\nabla u\|_{L^p}^{\frac{4p-N}{2p}} + \|\nabla u\|_{L^p}) \|\nabla u\|_{L^2}^2 \tag{3.19}
\]

by Hölder’s inequality, (3.2a) and Young’s inequality. We also continue from (3.4) and estimate the nonlinear term as

\[
-\int (u \cdot \nabla) u \cdot u \leq \|\nabla u\|_{L^p} \|u\|_{L^2}^{2p} \lesssim \|\nabla u\|_{L^p} \|u\|_{L^2}^{\frac{4p-N}{2p}} \|u\|_{H^2}^{\frac{N}{2p}} \\
\leq \frac{1}{8} \|\Delta u\|_{L^2}^2 + C(\|\nabla u\|_{L^p}^{\frac{4p-N}{2p}} + \|\nabla u\|_{L^p}) \|u\|_{L^2}^2 \tag{3.20}
\]

by Hölder’s inequality (3.2a) and Young’s inequality. Considering (3.18), (3.19), and (3.20) in (2.4) and (3.4) gives us the following result.

**Theorem 3.12** Suppose \( \nabla u \in L_T^r L_x^p \) where \( p \in [1, \infty) \), \( r \in [1, \frac{4}{4-N}] \) satisfy

\[
\frac{N}{p} + \frac{2}{r} = 2 + \frac{N}{2p}.
\]

Then, the ND KSE is globally well-posed in \( H^1(\mathbb{T}^N) \).

Taking \( p = 1 \) turns \( 2 + \frac{N}{2p} \) to \( 2 + \frac{N}{2} \). We emphasize again that \( 2 + \frac{N}{2} \) is remarkably larger than 2 in (2.3b).

### 3.4 Results on \( \nabla \cdot u \), and \( \partial_2 u_2 \)

Next, we aim to obtain a result on divergence \( \nabla \cdot u \), which is an improvement of \( \nabla u \), and in fact, even \( \partial_2 u_2 \) at the optimal level.

If we decide to work on the \( H^1(\mathbb{T}^N) \)-estimate, then we can look at (2.4) again

\[
\frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 = \int (u \cdot \nabla) u \cdot \Delta u + \lambda \|\Delta u\|_{L^2}^2.
\]

Now if we integrate by parts, we can get

\[
\int (u \cdot \nabla) u \cdot \Delta u = -\sum_{i,j,k=1}^N \int \left( (\partial_i u_i) u_j \partial_k^2 u_j + u_i u_j \partial_i \partial_k^2 u_j \right)
\]
or
\[
\int (u \cdot \nabla) u \cdot \Delta u = - \sum_{i,j,k=1}^N \int \left( \partial_k u_i \partial_i u_j \partial_k u_j - \frac{1}{2} \partial_i u_i |\partial_k u_j|^2 \right).
\]

Although we have separated the divergence \( \nabla \cdot u \) in \( (\partial_i u_i) u_j \partial_k^2 u_j \) or \( \frac{1}{2} \partial_i u_i |\partial_k u_j|^2 \), this will require regularity criteria of an additional term besides the divergence due to \( u_i u_j \partial_k^2 u_j \) or \( \partial_k u_i \partial_i u_j \partial_k u_j \).

In fact, thanks to Theorem 3.1 we can work in \( L^2(T^N) \)-norm and avoid this issue. Indeed, we can restart from (3.4) where we can estimate \( \lambda \| \nabla u \|_{L^2}^2 \leq \frac{1}{4} \| \Delta u \|_{L^2}^2 + C \| u \|_{L^2}^2 \) identically to (3.10), while

\[
- \int (u \cdot \nabla) u \cdot u = - \int (u \cdot \nabla) \frac{1}{2} |u|^2 = \frac{1}{2} \int (\nabla \cdot u) |u|^2
\]

(note that this term vanishes for the NSE but not for the ND KSE when \( N > 1 \)). Now if we estimate

\[
\int (\nabla \cdot u) |u|^2 \leq \| \nabla \cdot u \|_{L^\infty} \| u \|_{L^2}^2,
\]

we can get a criterion in terms of \( \int_0^T \| \nabla \cdot u \|_{L^\infty} d\tau \) analogously to the 3D Euler equations (Beale et al. 1984). But to make full use of the fourth-order diffusion, we compute similarly to (3.19)

\[
\frac{1}{2} \int (\nabla \cdot u) |u|^2 \leq \frac{1}{2} \| \nabla \cdot u \|_{L^p} \| u \|_{L^p}^2 \lesssim \| \nabla \cdot u \|_{L^p} \| u \|_{L^2} \frac{4p-N}{2p} \| u \|_{H^2}^\frac{N}{2p} \leq \frac{1}{4} \| \Delta u \|_{L^2} + C(\| \nabla \cdot u \|_{L^p}^{\frac{4p-N}{2p}} + \| \nabla \cdot u \|_{L^p}) \| u \|_{L^2}^2 (3.21)
\]

by Hölder’s inequality, (3.2a), and Young’s inequality. Because Theorem 3.1 implies that \( L^\infty_T L^2_x \)-bound leads immediately to the global well-posedness of the ND KSE, we obtain the following result.

**Theorem 3.13** Suppose \( \nabla \cdot u \in L^r_T L^p_x \) where \( p \in [1, \infty], r \in [1, \frac{4}{4-N}] \) satisfy

\[
\frac{N}{p} + \frac{2}{r} = 2 + \frac{N}{2p}.
\]

Then, the ND KSE is globally well-posed in \( H^1(T^N) \).

By a similar trick to the proof of Theorem 3.6, we can also obtain the following result in the 2D case:

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Proposition 3.14 Suppose $\partial^2 u_2 \in L_T^r L_x^p$ where $p \in [1, \infty], r \in [1, 2]$ satisfy

$$\frac{2}{p} + \frac{2}{r} = 2 + \frac{1}{p}.$$ 

Then, $u_1 \in L_T^\infty L_x^2 \cap L_T^2 H_x^2$.

Proof This is almost identical to the previous computation for the criteria of $\nabla \cdot u$. We restart from (3.12) where $\lambda \|\nabla u_1\|_{L_2}^2$ may be handled again identically to (3.10) while similarly to (3.21)

$$-\int (u \cdot \nabla) u_1 u_1 = \frac{1}{2} \int \partial^2 u_2 (u_1)^2 \leq \frac{1}{2} \|\partial^2 u_2\|_{L^p} \|u_1\|_{L^2}^2 \lesssim \|\partial^2 u_2\|_{L^p} \|u_1\|_{L^2}^2 \|u_1\|_{H^2}^2 \leq \frac{1}{4} \|\Delta u_1\|_{L_2}^2 + C(\|\partial^2 u_2\|_{L^p} \|u_1\|_{L^2}^2). \quad (3.22)$$

Now applying (3.22) to (3.12) completes the proof of Proposition 3.14. \qed

We can raise this regularity immediately as follows:

Proposition 3.15 Suppose $\partial^2 u_2 \in L_T^r L_x^p$ where $p \in [1, \infty], r \in [1, 2]$ satisfy

$$\frac{2}{p} + \frac{2}{r} = 2 + \frac{1}{p}.$$ 

Then, $u_2 \in L_T^\infty L_x^2 \cap L_T^2 H_x^2$.

Proof We restart from (3.9) again where $\lambda \|\nabla u_2\|_{L_2}^2$ may be handled identically to (3.10) while

$$-\int (u \cdot \nabla) u_2 u_2 = -\int u_1 \partial_1 u_2 u_2 \leq \|u_1\|_{L^\infty} \|\partial_1 u_2\|_{L_2} \|u_2\|_{L_2} \leq (\|\nabla u_2\|_{L_2}^2 + \|u_1\|_{L^\infty}^2 \|u_2\|_{L_2}^2) \lesssim (\|u_2\|_{L^2}^2 \|\Delta u_2\|_{L_2} + \|u_1\|_{H^2}^2 \|u_2\|_{L_2}^2) \leq \frac{1}{4} \|\Delta u_2\|_{L_2}^2 + C(1 + \|u_1\|_{H^2}^2)\|u_2\|_{L_2}^2. \quad (3.23)$$

by the Gagliardo–Nirenberg inequality as in (3.10), the Sobolev embedding $H^2(\mathbb{T}^2) \hookrightarrow L^\infty(\mathbb{T}^2)$, and Young’s inequality. Therefore, applying (3.23) to (3.9) leads to

$$\frac{1}{2} \frac{d}{dt} \|u_2\|_{L_2}^2 + \|\Delta u_2\|_{L_2}^2 \leq \frac{1}{2} \|\Delta u_2\|_{L_2}^2 + C(1 + \|u_1\|_{H^2}^2)\|u_2\|_{L_2}^2.$$ 

Now applying Proposition 3.14 completes the proof of Proposition 3.15. \qed
By Propositions 3.14 and 3.15, we deduce that \( u \in L_T^\infty L^2_x \) and thus by Theorem 3.1 we see that the 2D KSE is globally well-posed in \( H^1(\mathbb{T}^2) \) as follows:

**Theorem 3.16** Suppose \( \partial_2 u_2 \in L_T^p L^p_x \) where \( p \in [1, \infty] \), \( r \in [1, 2] \) satisfy

\[
\frac{2}{p} + \frac{2}{r} = 2 + \frac{1}{p}.
\]

Then, 2D KSE (1.1) is globally well-posed in \( H^1(\mathbb{T}^2) \).

**Remark 3.17** We emphasize again that Theorem 3.16 is surprising due to the KSE lacking divergence-free property and that \( 2 + \frac{1}{p} \) in Theorem 3.16 is same as the condition in Theorem 3.12; in comparison, \( \frac{3(2+p)}{4p} \) for \( p > 2 \) in (2.12b) is much smaller than “2” in (2.3b).

Lastly, we aim for \( \partial_2 u_1 \). It is well known that in general, such a criterion is expected to be more difficult than \( \partial_2 u_2 \) where the components of \( \nabla \) and \( u \) match; indeed, \( \frac{p+3}{2p} < \frac{3(p+2)}{4p} \) in (2.12b) and (2.12c). Now if we return again to (3.4) where we can rely on the same estimates in (3.10) to handle the term \( \lambda \|\nabla u\|_{L^2}^2 \), then we are only faced with a nonlinear term of

\[
\int (u \cdot \nabla)u \cdot u = -\int ((u \cdot \nabla)u_1u_1 + (u \cdot \nabla)u_2u_2).
\]

The first term allows us to separate \( \partial_2 u_1 \) as follows:

\[
-\int (u \cdot \nabla)u_1u_1 = -\int (u_1 \partial_1 u_1u_1 + u_2 \partial_2 u_1u_1) = -\int u_2 \partial_2 u_1u_1; \quad (3.24)
\]

however, for the second term we are faced with

\[
\int (u \cdot \nabla)u_2u_2 = -\int (u_1 \partial_1 u_2u_2 + u_2 \partial_2 u_2u_2) = -\int u_1 \partial_1 u_2u_2.
\]

We seem to face a significant difficulty here. We can integrate by parts to deduce \( \frac{1}{2} \int \partial_1 u_1u_2u_2 \), but \( \partial_2 u_1 \) is nowhere to be found. One idea would be to rely on some analogue of anisotropic Sobolev inequality that has proven to be useful in the theory of fluid PDEs such as the NSE and the MHD system (e.g., Cao and Wu 2011, Lemma 1.1). However, those results are typically all on the whole space such as \( \mathbb{R}^2 \) because they rely on the decay at infinity; i.e., while

\[
f(x) = \int_{-\infty}^{x} \partial_z f(z)dz,
\]

an analogous attempt in \( \mathbb{T}^2 \) gives

\[
f(x) = \int_{-\pi}^{x} \partial_z f(z)dz + f(-\pi).
\]
Remarkably, there is a trick that seems to be special to the KSE here. Let us make our argument precise by working on the equation of \( \phi \) rather than \( u \) and attain a criterion in terms of \( \partial_{12}\phi \), informally \( \partial_1 u_2 \) as desired (see Theorem 3.20).

### 3.5 Results on \( \phi \) and \( \partial_{12}\phi \)

We emphasize that the following two results focus on (1.2), independently of (1.1). Let us start with a criterion in terms of \( \phi \).

**Theorem 3.18** Suppose \( \phi \in L^r_T W^m_p \) where \( m \in (0, 1) \), and

\[
\begin{align*}
    p &\in [1, \frac{N}{m}], \quad r \in (2, \frac{4}{m}] \quad \text{if } N = 2, \ m \in (0, 1), \\
    p &\in \left( \frac{N}{m+2}, \frac{N}{m} \right], \quad r \in (2, \infty) \quad \text{if } N = 3, \ m \in (0, 1), \\
    p &\in (\frac{N}{2}, \infty], \quad r \in [2, \infty) \quad \text{if } N \in \{2, 3\}, \ m = 0.
\end{align*}
\]

satisfy

\[
\frac{N}{p} + \frac{2}{r} = \frac{2 + m}{2} + \frac{N}{2p}.
\]

Then, the ND KSE (1.2) is globally well-posed in \( L^2(\mathbb{T}^N) \).

**Remark 3.19** Very recently, Feng and Mazzucato (2020) showed a blowup criterion of \( \phi \in L^\infty_T L^2_x \) for 2D KSE. If \( N = 2 \), we only require \( \frac{2}{p} + \frac{2}{r} \leq 1 + \frac{1}{p} \), allowing cases such as \( L^\infty_T L^p_x \) for any \( p > 1 \) and \( L^4_T L^2_x \).

**Proof** We take \( L^2(\mathbb{T}^N) \)-inner products on (1.2) with \( \phi \) to deduce

\[
\frac{1}{2} \frac{d}{dt} \| \phi \|_{L^2}^2 + \| \Delta \phi \|_{L^2}^2 = -\lambda \int \Delta \phi \phi - \frac{1}{2} \int |\nabla \phi|^2 \phi. \tag{3.25}
\]

Now

\[
-\int \Delta \phi \phi \leq \| \Delta \phi \|_{L^2} \| \phi \|_{L^2} \leq \frac{1}{4} \| \Delta \phi \|_{L^2}^2 + C \| \phi \|_{L^2}^2 \tag{3.26}
\]

by Young’s inequality, while

\[
-\frac{1}{2} \int |\nabla \phi|^2 \phi \leq \frac{1}{2} \| \phi \|_{L^{\frac{Np}{m+mp}}} \| \nabla \phi \|_{L^{\frac{2Np}{Np+mp}}}^2 \\
\lesssim \| \phi \|_{W^{m,p}} \| \phi \|_{L^2} \| \phi \|_{H^{\frac{(m+2)p-N}{2p}}} \| \phi \|_{W^{m,p}} \lesssim \frac{N}{4} \| \Delta \phi \|_{L^2}^2 + C \left( \| \phi \|_{W^{m,p}}^{\frac{4p}{(2+m)p-N}} + \| \phi \|_{W^{m,p}} \right) \| \phi \|_{L^2}^2 \tag{3.27}
\]

by Hölder’s inequality, Sobolev embedding \( W^{m,p}(\mathbb{T}^N) \hookrightarrow L^{\frac{Np}{N-mp}}(\mathbb{T}^N) \) in case \( m \in (0, 1) \) so that \( p < \frac{N}{m} \) by hypothesis, (3.2b), and Young’s inequality. By applying (3.26)
and (3.27) to (3.25), we deduce that $\phi \in L^\infty_T L^2_x \cap L^2_T H^2_x$ and thus the desired result. \hfill \Box

In the following theorem, we assume $N = 2$ and aim for a result analogous to a criterion of (1.1) in terms of $\partial_1 u_2$ as promised.

**Theorem 3.20** Suppose $\partial_{12} \phi \in L^r_T L^p_x$ where $p \in [1, \infty], \ r \in [1, 2]$ satisfy

$$\frac{2}{p} + \frac{2}{r} = 2 + \frac{1}{p}.$$ 

Then, 2D KSE (1.2) is globally well-posed in $L^2(T^2)$.

**Proof** Let us consider the Eq. (1.2), take $L^2(T^2)$-inner products with $-\Delta \phi$ so that

$$\frac{1}{2} \frac{d}{dt} \| \nabla \phi \|^2_{L^2} + \| \Delta \nabla \phi \|^2_{L^2}$$

$$= \lambda \| \Delta \phi \|^2_{L^2} - \frac{1}{2} \int \partial_1 ((\partial_1 \phi)^2 + (\partial_2 \phi)^2) \partial_1 \phi - \frac{1}{2} \int \partial_2 ((\partial_1 \phi)^2 + (\partial_2 \phi)^2) \partial_2 \phi$$

$$\triangleq \lambda \| \Delta \phi \|^2_{L^2} + I + II.$$ (3.28)

It is straightforward to estimate

$$\lambda \| \Delta \phi \|^2_{L^2} \leq \frac{1}{8} \| \Delta \nabla \phi \|^2_{L^2} + C \| \nabla \phi \|^2_{L^2}$$ (3.29)

by Young’s inequality. Next, we estimate

$$I = - \int \frac{1}{3} \partial_1 (\partial_1 \phi)^3 + \partial_2 \phi \partial_{12} \phi \partial_1 \phi$$ (3.30)

$$\leq \| \partial_2 \phi \|_{L^{\frac{2p}{p-1}}} \| \partial_{12} \phi \|_{L^p} \| \partial_1 \phi \|_{L^{\frac{2p}{p-1}}}$$

$$\lesssim \| \partial_2 \phi \|_{L^2}^{\frac{2p-1}{2p}} \| \partial_2 \phi \|_{H^2}^{\frac{1}{2p}} \| \partial_{12} \phi \|_{L^p} \| \partial_1 \phi \|_{L^{\frac{2p}{p-1}}} \| \partial_1 \phi \|_{H^2}^{\frac{1}{2p}}$$

$$\leq \frac{1}{8} (\| \partial_1 \Delta \phi \|^2_{L^2} + \| \partial_2 \Delta \phi \|^2_{L^2})$$

$$+ C (\| \partial_{12} \phi \|_{L^p} + \| \partial_2 \phi \|_{L^{\frac{4p}{4p-1}}} + \| \partial_{12} \phi \|_{L_p^{\frac{2p}{p-1}}} + \| \partial_2 \phi \|_{L_p^{\frac{2p}{p-1}}}) \| \nabla \phi \|^2_{L^2}$$

by Hölder’s inequality, (3.2a) and Young’s inequality. Next, we get a break here as it turns out that we can apply identical estimates in $I$ to $II$ because

$$II = - \frac{1}{2} \int \partial_2 ((\partial_1 \phi)^2 + (\partial_2 \phi)^2) \partial_2 \phi$$

$$= - \int [(\partial_1 \phi)(\partial_1 \phi) + (\partial_2 \phi)(\partial_2 \phi)] \partial_2 \phi = - \int \partial_1 \phi \partial_{12} \phi \partial_2 \phi.$$ (3.31)
By applying (3.29)–(3.31) to (3.28), we obtain
\[ \phi \in L^\infty_T \dot{H}^1_x \cap L^2_T \dot{H}^3_x. \]
It is not difficult to ensure the lower regularity at \( L^\infty_T L^2_x \) from such high regularity by going back to (3.25). Indeed, we can compute from (3.25)
\[
\frac{1}{2} \frac{d}{dt} \| \phi \|_{L^2}^2 + \| \Delta \phi \|_{L^2}^2 \leq \lambda \| \nabla \phi \|_{L^2}^2 \| \nabla \phi \|_{L^\infty} + \frac{1}{2} \| \phi \|_{L^2} \| \nabla \phi \|_{L^2} \| \nabla \phi \|_{L^\infty} \leq \| \phi \|_{L^2} \| \Delta \phi \|_{L^2} + \| \phi \|_{L^2} \| \nabla \phi \|_{L^2} (\| \nabla \phi \|_{L^2} + \| \Delta \nabla \phi \|_{L^2})
\]
(3.32)
where we used the same estimate in (3.10) and the Sobolev embedding \( H^2(\mathbb{T}^2) \hookrightarrow L^\infty(\mathbb{T}^2) \). This estimate (3.32) immediately implies
\[
\frac{d}{dt} \| \phi \|_{L^2} \leq \| \nabla \phi \|_{L^2} + \| \Delta \nabla \phi \|_{L^2} + \| \nabla \phi \|_{L^2} (\| \nabla \phi \|_{L^2} + \| \Delta \nabla \phi \|_{L^2})
\]
(3.33)
due to Young’s inequality. Integrating (3.33) over \([0, t]\) now and relying on the fact that \( \phi \in L^\infty_T \dot{H}^1_x \cap L^2_T \dot{H}^3_x \) complete the proof of Theorem 3.20.

4 Computational Results

In this section, we present some computational results. In particular, we simulate solutions to the 2D KSE. All simulations are done for the scalar form of the system (1.2), (except for the simulation used to generate Fig. 1, which directly simulated (1.1)), with the slight modification of subtracting the spatial average at each time step, as is common in simulations of the KSE (see, e.g., Kalogirou et al. 2015). Note that this modification formally preserves the differential form of the KSE, system (1.1). In this section, we assume that \( u = \nabla \phi \); that is, here, \( u \) is essentially a notation for \( \nabla \phi \).

The existing literature on computational studies of the 2D KSE is fairly limited, but we refer to Kalogirou et al. (2015) where an exhaustive computational study of the KSE was carried out in various rectangular domains. Various modifications to the 2D KSE have been investigated computationally in several works; see, e.g., Tomlin et al. (2018); Larios and Yamazaki (2020).

Our simulations were computed via MATLAB (R2020a) on the periodic square domain \([−\pi, \pi)^2\), using a uniform rectangular grid \(256 \times 256\). Derivatives were computed using spectral methods (also called pseudo-spectral methods), specifically using MATLAB’s \texttt{fftn} and \texttt{ifftn} implementations of the ND discrete Fourier transform, making sure to respect the 2/3’s dealiasing law for quadratic nonlinearities, as well as the vanishing of the Nyquist frequency for only odd-order derivatives. Time stepping was carried out using a 4th-order ETD-RK4 method (see, e.g., Kassam and Trefethen 2005; Kennedy and Carpenter 2003) to handle the linear terms implicitly, while the nonlinear term was computed explicitly and in physical space. In particular, a complex contour with 64 nodes (technically 32 nodes plus a symmetry condition) was used to handle the removable singularities in the ETD-RK4 coefficients, as proposed in Kassam and Trefethen (2005). We used a uniform time-step of \( \Delta t = 2.1227 \times 10^{-5} \), well below the advective CFL given by \( \Delta t < \Delta x/(\frac{1}{2} \| \nabla \phi \|_{L^\infty((0,T,L^\infty(\mathbb{T}^2))}) \approx 4.4 \times 10^{-5} \).
Fig. 2 The spectrum at time steps $t = 0.0, t = 0.001, t = 0.002, \ldots, 10.0$ (log–log scale), starting in gray and getting darker with increasing time. Highlighted are the initial spectrum (cyan plus), the final ($t = 10.0$) spectrum (red asterisk), and the average spectrum (green circle), (averaged over $9 \leq t \leq 10$) (Color figure online)

All simulations presented here were well-resolved at each time step, in the sense that the magnitude of the energy spectrum of $\phi$ was below machine precision ($\epsilon = 2.2204 \times 10^{-16}$ in MATLAB) for all wave numbers above the $2/3$’s dealiasing cutoff (i.e., for all $k \in \mathbb{Z}^2$ with $|k| \geq 256/3$). This can be seen in the plots of the spectrum at each time step in Fig. 2. We used roughly the largest $\lambda$ possible at this resolution ($256^2$) while maintaining well-resolvedness; namely $\lambda = 29.1$ (found experimentally), meaning that 96 modes were unstable (not counting the zero mode, which was set to zero); that is, $\# \{k \in \mathbb{Z}^2 \setminus \{0\} : |k|^2 \triangleq k_1^2 + k_2^2 < 29.1\} = 96$. However, we observed in tests that the behavior of the solution presented here was qualitatively similar for a wide range of $\lambda$ values.

For concreteness, we use the initial data $\phi^{in}$ studied in Kalogirou et al. (2015); namely

$$\phi^{in}(x, y) = \sin(x + y) + \sin(x) + \sin(y).$$

(4.1)

However, we also tested several other choices of initial data, including data based on randomly generated Fourier amplitudes, and found qualitatively similar results to those presented here. Thus, for simplicity of presentation, we only include the results from simulations with initial data (4.1).

We computed the $L^p$ norms for $1 \leq p \leq \infty$ in physical space (using Riemann sums for $p < \infty$), with the modification of dividing by the area of the domain; that
Fig. 3 $L^p$ norms of $\phi$ (left), $\phi_x$ (middle) and $\phi_y$ (right). Note that $u_1 = \phi_x$, and $u_2 = \phi_y$ ($p = 1, \frac{3}{2}, 2, 3, 4, 5, 6, \infty$)

is, in this section only, unless otherwise noted, we denote

$$\|f\|_{L^p} \triangleq \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(x, y)|^p \, dx \, dy. \quad (4.2)$$

for $p < \infty$, so that $\|f\|_{L^p} \leq \|f\|_{L^\infty} \triangleq \text{ess sup}_{(x,y) \in \mathbb{T}^2} |f(x, y)|$. This is done only for aesthetic purposes so that it is easier to see the plots on the same figure, see Figs. 3 and 4. From these graphics, it appears that all the criteria proven in the present work hold. Indeed, it appears that $\phi, \phi_x, \phi_y, u_1 \triangleq \phi_x, u_2 \triangleq \phi_y, \phi_{xx}, \phi_{xy} \equiv \phi_{yx}, \phi_{yy}$, and $\text{div}(u) = \Delta \phi$ are all in $L^\infty([0, T], L^\infty)$ for all $T > 0$, at least for the initial data and $\lambda$-values we tested. Thus, in these tests, we do not see evidence for ill-posedness.

In the context of the 2D NSE, it is common to look for regions in, e.g., the energy–enstrophy plane or enstrophy–palenstrophy plane, etc., where the attractor lies (see, e.g., Cao et al. 2021; Dascaliuc et al. 2005, 2007, 2008, 2010; Emami and Bowman 2018). In the same spirit, we consider analogous planes combining $\|\phi\|_{L^2}^2$ (“Energy”), $\|\nabla \phi\|_{L^2}^2$ (“Enstrophy”), and $\|\Delta \phi\|_{L^2}^2$ (“Palenstrophy”), where we have borrowed the names for these quantities from the NSE setting even though they have different physical meanings in the context of the KSE. In Fig. 5, these plots are displayed. Here, we see that the large-time dynamics are constrained to a relatively small region of these planes with a roughly elliptical shape.

We also display some quantities that occur in standard energy estimates. For example, in taking the inner product with $\phi$ in (1.2) as in (3.25), the nonlinear term $|\nabla \phi|^2/2$ (bottom left of Fig. 7) yields the term $\frac{1}{2} \int_{\Omega} \phi |\nabla \phi|^2 \, dx$ (Fig. 6). If this term happened to be positive, or even if it had a positive integral over arbitrary time intervals of the form $[0, T], T > 0$ this would be sufficient to obtain a bound on the $L^2$ norm on finite time intervals, and hence a proof of global well-posedness of the KSE. Our numerical tests in Fig. 6 (left) show that this term is typically positive (at least, for the simulations we performed), but can intermittently become negative for small windows of time. In Fig. 7 (bottom middle), the integrand of this term, $\phi |\nabla \phi|^2$ can also be seen at time $t = 10.0$. Here, we see that the integrand is mostly positive in space, but has regions where it is strongly negative. A deeper understanding of these negative regions may be important to understanding the well-posedness of the KSE. Note also that, strictly speaking, the positivity of $(\phi, |\nabla \phi|^2) + \|\Delta \phi\|_{L^2}^2$ would be sufficient to bound the $L^2$
Fig. 4 $L^p$ norms of $\phi_{xx}$ (top left), $\phi_{xy}$ (top right) and $\phi_{yy}$ (bottom left), and $\Delta \phi$ (bottom right). Note that $\text{div}(u) = \Delta \phi$ ($p = 1, \frac{3}{2}, 2, 3, 4, 5, 6, \infty$)

Fig. 5 Plots of combinations of “Energy” $\|\psi\|_L^2$, “Enstrophy” $\|\nabla \psi\|^2_L$ and “Palenstrophy” $\|\Delta \psi\|_L^2$ for times $0 \leq t \leq 3$. At $t = 0$, the solutions are close to the origin, and move roughly northeast until reaching a region in the northeast where they tend to remain, moving about chaotically

norm, as from (3.25), one has

\[
\frac{1}{2} \frac{d}{dt} \|\phi\|^2_L + \frac{1}{2} (|\nabla \phi|^2, \phi) + \|\Delta \phi\|^2_L = -\lambda (\Delta \phi, \phi) \leq \frac{1}{2} \|\Delta \phi\|^2_L + \frac{\lambda^2}{2} \|\phi\|^2_L.
\]

so that

\[
\frac{d}{dt} \|\phi\|^2_L + (|\nabla \phi|^2, \phi) + \|\Delta \phi\|^2_L \leq \lambda^2 \|\phi\|^2_L.
\]
Fig. 6  Quantities of interest in energy estimates: $\frac{1}{2}(\phi, |\nabla \phi|^2)$ (top left), $(\phi, |\nabla \phi|^2) + \|\Delta \phi\|_{L^2}^2$ (top right), $(u \cdot \nabla u, u) = -\frac{1}{2}(\Delta \phi, |\nabla \phi|^2)$ (bottom left), and $2(u \cdot \nabla u, u) + \|\Delta u\|_{L^2}^2$ (bottom right). While the quantities in the left column have intermittent negativity, in our simulations, the quantities in the right column were positive for all $t \geq 0$ (only $0 \leq t \leq 1$ displayed for visibility), but similar behavior was observed out to $t = 10$. Norms were not normalized as in (4.2).

A nearly identical calculation yields

$$\frac{d}{dt} \|u\|_{L^2}^2 + 2(u \cdot \nabla u, u) + \|\Delta u\|_{L^2}^2 \leq \lambda^2 \|u\|_{L^2}^2.$$

Thus, in Fig. 6 we also display $(\phi, |\nabla \phi|^2) + \|\Delta \phi\|_{L^2}^2$ and $2(u \cdot \nabla u, u) + \|\Delta u\|_{L^2}^2$. In the figure, we see that the terms coming quantities coming from the nonlinear terms are mostly positive, but occasionally become negative. However, after adding the norm of the Laplacian, the quantity remains positive for all time. Of course, if one could prove that such positivity holds for general initial data, one could prove global well-posedness.
Fig. 7 Solution $\phi$ to (1.2) at time $t = 10.0$, and various quantities derived from $\phi$. Here, $\Delta = \partial_{xx} + \partial_{yy}$. Note that in terms of $u \equiv \nabla \phi$, one has $\text{div}(u) = \Delta \phi$.

Similar observations can be made at the level of the $u$ Eq. (1.1). In particular, regarding the cubic term in (3.4), we notice

$$
(u \cdot \nabla u, u) = \frac{1}{2} \int_{\Omega} u \cdot \nabla |u|^2 \, dx = -\frac{1}{2} \int_{\Omega} (\nabla \cdot u)|u|^2 \, dx = -\frac{1}{2} \int_{\Omega} \Delta \phi |\nabla \phi|^2 \, dx.
$$

(4.3)

Note that, according to (3.4), positivity of this term would immediately imply $u \in L^\infty(0, T; L^2)$, and hence global existence for the KSE. Thus, we also display $\Delta \phi |\nabla \phi|^2 / 2$ in Fig. 7 (bottom right), where we make the perhaps unsurprising observation that integrand in (4.3) is not of a single sign; however, it is surprising that integrand evidently has positive values that are far greater than the magnitude of the negative values. Moreover, the regions where the integrand is negative seem to be concentrated in small, relatively thin pockets, while the positive regions dominate the domain. Indeed, in simulations negative values of $(u \cdot \nabla u, u)$ were observed only for short intervals of time (no larger than roughly $[t, t + 0.003]$), but after a short time
(roughly $t > 0.08$), $(u \cdot \nabla u, u)$ remained positive. This phenomenon was observed repeatedly in simulations, indicating that one possible route to proving global well-posedness of (1.1) might be to show that the negative part of $\Delta \phi |\nabla \phi|^2 / 2$ remain sufficiently small in comparison with its positive part.

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5 Appendix

Here, we leave details concerning the local well-posedness of the KSE (1.1) in $H^1(\mathbb{T}^N)$ for $N \in \{2, 3\}$, formally stated in Theorem 2.2. The proof follows the argument in Larios and Yamazaki (2020), which in turn follows Majda and Bertozzi (2002); we only sketch the main steps. We consider a Galerkin approximation with $P_n$ being the projection onto the Fourier modes of order up to $n \in \mathbb{N} \cup \{0\}$:

$$P_n u(x) \triangleq \sum_{|k| \leq n} \hat{u}(k) e^{ix \cdot k}.$$

We consider the following Galerkin approximation system

$$\begin{align*}
\partial_t u^n + P_n ((u^n \cdot \nabla) u^n) &= -\lambda \Delta u^n - \Delta^2 u^n, \\
0 &= P_n u^\text{in}(.).
\end{align*}$$

(5.1a)\hspace{1cm} (5.1b)

Relying on Majda and Bertozzi (2002, Theorem 3.1), we can deduce the following statement:

**Proposition 5.1** Given initial data $u^\text{in} \in H^1(\mathbb{T}^N)$, there exists $T = T(\|u^\text{in}\|_{H^1}) > 0$ such that (5.1a)–(5.1b) has a solution

$$u^n \in L^\infty([0, T]; H^1(\mathbb{T}^N)) \cap L^2([0, T]; H^3(\mathbb{T}^N));$$

(5.2)

additionally, $\partial_t u^n \in L^2([0, T]; H^{-1}(\mathbb{T}^N))$. Moreover, these bounds are independent of $n$. Finally, if $T^*$ is the maximal existence time and $T^* < \infty$, then $\lim \sup_{T \to T^*} \|u^n(t)\|_{H^1} = +\infty$.

**Proof** By Majda and Bertozzi (2002, Theorem 3.1), it can be immediately shown that given $u^\text{in} \in H^1(\mathbb{T}^N)$, there exists a unique solution $u^n \in C^1([0, T_n), H^1(\mathbb{T}^N) \cap O^n)$.
for some $T_n > 0$ and $O^M \triangleq \{ f \in H^1(\mathbb{T}^N) : \| f \|_{H^1} \leq M \}$. Now we can take $L^2(\mathbb{T}^N)$ inner products on (5.1a) with $u$ and then with $-\Delta u^n$, sum the resulting equations and prove that there exists a constant $c \geq 0$ such that

$$\frac{d}{dt} \| u^n \|_{H^1} \leq c(1 + \| u^n \|_{H^1})^3$$

We can fix such a constant $c > 0$ and deduce that $H^1(\mathbb{T}^N)$-norm does not blow up for all $t < T^* \triangleq \frac{T_n}{2}$ for all $n \in \mathbb{N}$ and $u^n$ has the regularity of (5.2). This regularity leads to $\int_0^T \| \partial_t u^n \|_{H^{-1}}^2 \, d\tau \lesssim 1$ by (1.1) and a standard argument (see Majda and Bertozzi 2002, Cor. 3.2) completes the proof of Proposition 5.1.

From the regularity of the solution to the Galerkin approximation due to Proposition 5.1, we can deduce the following convergence results by standard compactness lemma (e.g., Simon 1987, Theorem 5; Simon 1990, Lemma 4): weak* in $L^\infty(0, T; H^1(\mathbb{T}^N))$; weak in $L^2(0, T; H^3(\mathbb{T}^N))$; strong in $L^2(0, T; H^s(\mathbb{T}^N))$ for $s \in [2, 3)$ and $C([0, T]; H^s(\mathbb{T}^N))$ for $s \in [0, 1)$. Thereafter, verifying that the limiting solution indeed solves the KSE and proving its uniqueness is standard. We refer to, e.g., Larios and Yamazaki (2020) for details.

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