Existence of a calibrated regime switching local volatility model

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Abstract
By Gyöngy’s theorem, a local and stochastic volatility model is calibrated to the market prices of all European call options with positive maturities and strikes if its local volatility (LV) function is equal to the ratio of the Dupire LV function over the root conditional mean square of the stochastic volatility factor given the spot value. This leads to a stochastic differential equation (SDE) nonlinear in the sense of McKean. Particle methods based on a kernel approximation of the conditional expectation, as presented in Guyon and Henry-Labordère [Risk Magazine, 25, 92–97], provide an efficient calibration procedure even if some calibration errors may appear when the range of the stochastic volatility factor is very large. But so far, no global existence result is available for the SDE nonlinear in the sense of McKean. When the stochastic volatility factor is a jump process taking finitely many values and with jump intensities depending on the spot level, we prove existence of a solution to the associated Fokker–Planck equation under the condition that the range of the squared stochastic volatility factor is not too large. We then deduce existence to the calibrated model by extending the results in Figalli [Journal of Functional Analysis, 254(1), 109–153].

KEYWORDS
calibration, diffusions nonlinear in the sense of McKean, Dupire’s local volatility, Fokker–Planck systems, local and stochastic volatility models
Calibration to the market prices of European call options is a major concern in mathematical finance. According to Breeden and Litzenberger (1978), the knowledge of the market prices of those options for a continuum of strikes and maturities is equivalent to the knowledge of the marginal distributions of the underlying asset under the pricing measure. Thus, to be consistent with the market prices, a calibrated model must have marginal distributions that coincide with those given by the market. More specifically, in this paper, we address the question of existence of a special class of calibrated local and stochastic volatility (LSV) models.

LSV models, introduced by Lipton (2002) and Piterbarg (2007), can be interpreted as an extension of Dupire’s local volatility (LV) model, described in Dupire (1994), to get more consistency with real markets. A typical LSV model has the dynamics

\[ dS_t = rS_t dt + f(Y_t)\sigma(t, S_t)S_t dW_t, \]

for the stock under the risk-neutral probability measure, where \((Y_t)_{t \geq 0}\) is a stochastic process that may be correlated with \((S_t)_{t \geq 0}\), \(r\) is the risk-free rate, \(\sigma\) is a deterministic function, and \(W\) is an one-dimensional Brownian motion. Meanwhile, according to Dupire (1994), given the European call option prices \(C(t, K)\) for all positive maturities \(t\) and strikes \(K\), the process \(S^D_t\) following the dynamics

\[ dS^D_t = rS^D_t dt + \sigma_{Dup}(t, S^D_t)S^D_t dW_t, \]

where \(\sigma_{Dup}(t, K) := \sqrt{\frac{2\delta C(t, K) + rK\delta K C(t, K)}{K^2 \delta^2 K C(t, K)}}\) is the Dupire LV function, is calibrated to the European option prices, which means that for all positive maturities \(t\) and strikes \(K\), the process \(S^D_t\) has the same distribution as \(S^D_t\). This leads to the following stochastic differential equation (SDE):

\[ dS_t = rS_t dt + \frac{f(Y_t)}{\sqrt{E[f^2(Y_t)|S_t]}}\sigma_{Dup}(t, S_t)S_t dW_t, \]

which is nonlinear in the sense of McKean, because the diffusion term depends on the law of \((S_t, Y_t)\) through the conditional expectation in the denominator.

Getting existence and uniqueness to this kind of SDE is still an open problem. Nevertheless some local (in time) existence results to the restriction of the associated Fokker–Planck equation to a compact spatial domain have been found by Abergel and Tachet (2010) using a perturbative approach, when the first-order derivative of the stochastic volatility factor \(f'\) is small enough and in the case where \(Y\) is solution to an autonomous SDE. From a numerical point of view, calibration can be achieved through the resolution of the associated Fokker–Planck PDE according to Ren, Madan, and Qian (2007). Moreover, particle methods based on a kernel approximation of the conditional expectation, as presented by Guyon and Henry-Labordère (2012), provide an efficient calibration procedure even if some calibration errors may appear when the range of the stochastic volatility process \((f(Y_t))_{t \geq 0}\) is very large.

Recently, advances have been made for analogous models. Alfonsi, Labart, and Lelong (2016) proved existence and uniqueness of stochastic local intensity models calibrated to collateralized debt obligation (CDO) tranche prices, where the discrete feature of the loss process makes the conditional expectation simpler to handle. Moreover, Guennoun and Henry-Labordère (2016) showed that in an...
LV model enhanced with jumps, the particle method applied to a well-defined nonlinear McKean SDE with a regularized volatility function gives call prices that converge to the market prices as the regularization parameter goes to 0, for all strikes and maturities.

In this paper, we prove existence for a special class of calibrated LSV models. Let $d \geq 2$, $\mathcal{Y} := \{1, \ldots, d\}$ and $f : \mathcal{Y} \to \mathbb{R}_+^*$. The process $(S_t)_{t \geq 0}$ describing the underlying asset follows the dynamics

$$dS_t = rS_t dt + \frac{f(Y_t)}{\sqrt{E[f^2(Y_t)|S_t]}} \sigma_{Dup}(t, S_t) S_t dW_t,$$

and the process $(Y_t)_{t \geq 0}$ is a jump process, taking values in $\mathcal{Y}$. Moreover, it follows the dynamics

$$\forall j \in \{1, \ldots, d\} \setminus \{Y_t\}, \quad \mathbb{P}(Y_{t+dt} = j|F_t) = q_{Y,j}(\log(S_t)) dt + O((dt)^2),$$

where $(F_t)_{t \geq 0}$ is the filtration generated by the process $(S,Y)$, for $1 \leq i \neq j \leq d$, the functions $q_{ij} : \mathbb{R} \to \mathbb{R}_+$ are the intensities of the jumps and depend on the level of the spot. We assume that those intensities are uniformly bounded, that $r$ is constant, and that the initial condition $(S_0, Y_0)$ and the Brownian motion $W$ are independent. We call the process $(S_t, Y_t)_{t \geq 0}$ a calibrated regime switching local volatility (RSLV) model. Let us consider the log-price $X := \log S$ of the asset. We define $\sigma_{Dup}(t,x) := \sigma_{Dup}(t, e^x)$, for $t \geq 0$ and $x \in \mathbb{R}$, and denote by $\mu$ the distribution of $(X_0, Y_0)$, which is a probability measure on $\mathbb{R} \times \mathcal{Y}$. The process $X$ follows the dynamics

$$dX_t = \left( r - \frac{1}{2} \frac{f^2(Y_t)}{E[f^2(Y_t)|X_t]} \bar{\sigma}_{Dup}(t, X_t) \right) dt + \frac{f(Y_t)}{\sqrt{E[f^2(Y_t)|X_t]}} \bar{\sigma}_{Dup}(t, X_t) dW_t,$$

\begin{equation}
(X_0, Y_0) \sim \mu.
\end{equation}

In the same way, we define the process $X^D := \log(S^D)$ following the dynamics

$$dX_t^D = \left( r - \frac{1}{2} \bar{\sigma}_{Dup}^2(t, X_t^D) \right) dt + \bar{\sigma}_{Dup}^2(t, X_t^D) dW_t,$$

\begin{equation}
X_0^D \sim \mu_{X_0},
\end{equation}

where $\mu_{X_0}$ is the distribution of $X_0$. Let us set $\lambda_i := f^2(i) > 0$ for $i \in \{1, \ldots, d\}$, and for $t > 0$, denote by $p_i(t,x)$ the conditional density of $X_t$ given $\{Y_i = i\}$ multiplied by $\mathbb{P}(Y_i = i)$. It means that, for any measurable and nonnegative function $\phi$,

$$\mathbb{E} \left[ \phi(X_t) 1_{\{Y_t = i\}} \right] = \int_{\mathbb{R}} \phi(x) p_i(t,x) dx.$$

The Fokker–Planck partial differential system (PDS) associated to SDE (2) and (3) writes, for $1 \leq i \leq d$,

$$\partial_t p_i = -\partial_x \left( \left[ r - \frac{1}{2} \frac{\lambda_i \sum_{l=1}^d p_l \bar{\sigma}_{Dup}^2}{\sum_{l=1}^d \lambda_l p_l} \right] p_i \right) + \frac{1}{2} \partial_{xx} \left( \frac{\lambda_i \sum_{l=1}^d p_l \bar{\sigma}_{Dup}^2}{\sum_{l=1}^d \lambda_l p_l} \right) + \sum_{j=1}^d q_{ij} p_j \text{ in } (0,T] \times \mathbb{R},$$

\begin{equation}
(6)
\end{equation}
where the constant $T > 0$ is the finite time horizon and $q_{ij} = -\sum_{j \neq i} q_{ij}$. Moreover, for any continuous and bounded function $\varphi : \mathbb{R} \to \mathbb{R}$,

$$
\lim_{t \to 0} \int_{\mathbb{R}} \varphi(x)p_i(t,x)dx = \int_{\mathbb{R}} \varphi(x)\mu_i(dx), \tag{7}
$$

where $\mu_i = \mu(\cdot, \{i\})$. We will call (RS) the PDS (6) and (7).

Up to a reordering, we can assume that without loss of generality that $f$ is increasing and therefore $\lambda_1 \leq \cdots \leq \lambda_d$. Let us remark that if $\lambda_1 = \lambda_d$, then each $p_i$ is a solution to the Fokker–Planck PDE associated with Dupire’s diffusion, with initial condition $\mu_i$.

We introduce the following condition on the family $(\lambda_i)_{1 \leq i \leq d}$, under which we will obtain existence of a solution to SDE (2) and (3). We denote by $(e_1, \ldots, e_d)$ the canonical basis of $\mathbb{R}^d$ and for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we define $x^\perp = \{y = (y_1, \ldots, y_d) \in \mathbb{R}^d, \sum_{i=1}^d x_i y_i = 0 \}$.

**Condition (C).** There exists a symmetric positive definite $d \times d$ matrix $\Gamma$ such that for $1 \leq k \leq d$, the $d \times d$ matrix $\Gamma^{(k)}$ with coefficients

$$
\Gamma^{(k)}_{ij} = \frac{\lambda_i + \lambda_j}{2} \left( \Gamma_{ij} + \Gamma_{kk} - \Gamma_{ik} - \Gamma_{jk} \right), \quad 1 \leq i, j \leq d, \tag{9}
$$

is positive definite on the space $e_k^\perp$.

Under Condition (C), which ensures a coercivity property that enables us to establish energy estimates on the vector $p := (p_1, \ldots, p_d)$, existence of a solution to SDE (2) and (3) can be proved in three steps:

1. When $\mu_i$, $1 \leq i \leq d$ have square integrable densities on $\mathbb{R}$, we define a variational formulation to (RS), called $V_{L^2}(\mu)$, and we apply Galerkin’s method to show that $V_{L^2}(\mu)$ has a solution.

2. For $\mu$ a probability measure on $\mathbb{R} \times \mathcal{Y}$, we define a weaker variational formulation called $V(\mu)$. We take advantage of the fact that if $p$ is a solution to $V(\mu)$ then $\sum_{i=1}^d p_i$ is solution to Dupire’s PDE (8) with initial condition $\mu_{X_0}$. This enables to get control of the explosion rate of the $L^2$ norm of $p_i$, as $t \to 0$, for $1 \leq i \leq d$, and we extend the results obtained in Step 1 to show existence of a solution to $V(\mu)$.

3. Finally, we extend the results in Figalli (2008), which give equivalence between the existence of a solution to a Fokker–Planck system and the existence of a solution to the associated martingale problem with time marginals given by that solution to the Fokker–Planck system, and we show that the existence result in Step 2 implies existence of a weak solution to SDE (2) and (3).

The paper is organized as follows. In Section 2, we discuss the role played by Condition (C), we give sufficient and simple conditions under which (C) is satisfied and we state our existence result on calibrated RSLV models. Sections 3 and 4 are dedicated to the proofs of the existence results from Section 2. Beforehand, let us introduce some additional notation.
Notation

- For an interval $I \subset \mathbb{R}$, we denote by $L^2(I)$ the space of measurable real-valued functions defined on $I$, which are square integrable for the Lebesgue measure. For $k \geq 1$, and $u, v \in (L^2(I))^k$, we use the notation

$$(u, v)_k = \int_I \sum_{i=1}^k u_i(x)v_i(x)dx, \ |u|_k = (u, u)_k^{\frac{1}{2}},$$

and we define $L(I) := (L^2(I))^d$. We also define $L := L(\mathbb{R})$ and we denote by $L'$ its dual space.

- For $m \geq 1$, we denote by $H^m(I)$ the Sobolev space of real-valued functions on $I$ that are square integrable together with all their distribution derivatives up to the order $m$. We define the space $H(I) := (H^1(I))^d$, endowed with the usual scalar product and norm

$$\langle u, v \rangle_d = \int_I \sum_{i=1}^d (u_i(x)v_i(x) + \partial_x u_i(x)\partial_x v_i(x))dx, \ |u|_d = \langle u, u \rangle_d^{\frac{1}{2}},$$

and we define $H := H(\mathbb{R})$. We denote by $H'$ its dual space, and by $\langle \cdot, \cdot \rangle$ the duality product between $H$ and $H'$.

- For $1 \leq p \leq \infty$, we denote by $W^{1,p}(\mathbb{R})$ the Sobolev space of functions belonging to $L^p(\mathbb{R})$, and that have a first-order derivative in the sense of distributions that also belongs to $L^p(\mathbb{R})$.

- For $n \geq 1$, $\mathcal{O}$ an open subset of $\mathbb{R}^n$ and $0 \leq k \leq \infty$, we denote by $C^k(\mathcal{O})$ the set of functions $\mathcal{O} \to \mathbb{R}$ that are continuous and have continuous derivatives up to the order $k$, by $C^k_c(\mathcal{O})$ the set of functions in $C^k(\mathcal{O})$ that have compact support on $\mathcal{O}$, and by $C^k_b(\mathcal{O})$ the set of functions in $C^k(\mathcal{O})$ that are uniformly bounded on $\mathcal{O}$ together with their $p$th-order derivatives, for $p \leq k$.

- For $n \geq 1$, we denote by $\mathcal{M}_n(\mathbb{R})$ the set of $n \times n$ matrices with real-valued coefficients. We denote by $I_n \in \mathcal{M}_n(\mathbb{R})$ the identity matrix, and by $J_n \in \mathcal{M}_n(\mathbb{R})$ the matrix where all the coefficients are equal to $1$. We denote $\mathcal{M}_n(\mathbb{R})$ for a $n \times p$ matrix, we denote its transpose by $A^*$ and we define $||A||_\infty := \max\{|A_{ij}|, 1 \leq i \leq n, 1 \leq j \leq p\}$.

- For $y \in \mathbb{R}$, we denote its positive part by $y^+ := \max(0, y)$ and its negative part by $y^- := \min(0, y)$. For $k \geq 2$ and $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$, we denote its positive part by $x^+ := (x_1^+, \ldots, x_k^+)$, and its negative part by $x^- := (x_1^-, \ldots, x_k^-)$.

- We denote by $\mathcal{P}(\mathbb{R})$ (resp., $\mathcal{P}(\mathbb{R} \times \mathcal{Y})$) the set of probability measures on $\mathbb{R}$ (resp., $\mathbb{R} \times \mathcal{Y}$).

- We define $D := (\mathbb{R}^+)^d \setminus \{(0, \ldots, 0)\}$.

- For notational simplicity, for a function $g$ defined on $\mathbb{R}^2$ and $t \in \mathbb{R}$, we may sometimes use the notation $g(t) := g(t, \cdot)$. 


2 | MAIN RESULTS

In this section, we give the main results concerning SDE (2) and (3). We assume that the jump intensities of the stochastic volatility factor $Y$ are bounded and that the European call prices given by the market have sufficient regularity, so that the following assumptions hold.

**Assumption (B).** The function $\tilde{\sigma}_{Du,p}$ belongs to the space $L^\infty([0, T], W^{1, \infty}(\mathbb{R}))$ and there exists a constant $\tilde{\sigma} > 0$ such that almost everywhere (a.e.) on $[0, T] \times \mathbb{R}$, $\sigma \leq \tilde{\sigma}_{Du,p}$. Moreover, there exists a constant $\sqrt{q} > 0$ such that a.e. on $[0, T] \times \mathbb{R}$, $q_{ij} \leq \sqrt{q}$ for $1 \leq i, j \leq d$.

**Assumption (H).** The function $\tilde{\sigma}_{Du,p}$ is continuous and there exist two constants $H_0 > 0$ and $\chi \in (0, 1)$ such that

$$\forall s, t \in [0, T], \forall x \in \mathbb{R}, \quad |\tilde{\sigma}_{Du,p}(s, x) - \tilde{\sigma}_{Du,p}(t, x)| \leq H_0|t - s|^{\chi}.$$  

We introduce a variational formulation to give sense to the PDS (RS). Let us assume that $\rho := (p_1, \ldots, p_d)$ is a classical solution to the PDS (RS), such that $\rho$ takes values in $D$ and for $t \in (0, T)$, $\rho(t, \cdot) \in H$. For $\nu := (\nu_1, \ldots, \nu_d) \in H$, and $i \in \{1, \ldots, d\}$, we multiply (6) by $\nu_i$ and integrate over $\mathbb{R}$. Through integration by parts, we obtain that for $i \in \{1, \ldots, d\}$, the following equality holds in the classical sense and also in the sense of distributions on $(0, T)$,

$$\frac{d}{dt} \int_{\mathbb{R}} \nu_i p_i dx = \int_{\mathbb{R}} \partial_\nu \lambda_i \left( r - \frac{1}{2} \frac{\lambda_i \sum_{l=1}^{d} p_l}{\sum_{l=1}^{d} \lambda_l p_l} \tilde{\sigma}_{Du,p}^2 \right) p_i dx$$

$$- \frac{1}{2} \int_{\mathbb{R}} \partial_\nu \lambda_i \left( \frac{\lambda_i \sum_{l=1}^{d} p_l}{\sum_{l=1}^{d} \lambda_l p_l} \right) \tilde{\sigma}_{Du,p}^2 p_i dx + \int_{\mathbb{R}} \nu_i \sum_{j=1}^{d} q_{ij} p_j dx. \quad (10)$$

The term $\partial_\nu \left( \frac{\lambda_i \sum_{l=1}^{d} p_l}{\sum_{l=1}^{d} \lambda_l p_l} \tilde{\sigma}_{Du,p}^2 \right)$ rewrites

$$\partial_\nu \left( \frac{\lambda_i \sum_{l=1}^{d} p_l}{\sum_{l=1}^{d} \lambda_l p_l} \tilde{\sigma}_{Du,p}^2 \right) = \frac{2 \lambda_i \sum_{l=1}^{d} p_l}{\sum_{l=1}^{d} \lambda_l p_l} \tilde{\sigma}_{Du,p} \partial_\nu \tilde{\sigma}_{Du,p} + \tilde{\sigma}_{Du,p}^2 \partial_\nu \left( \frac{\lambda_i \sum_{l=1}^{d} p_l}{\sum_{l=1}^{d} \lambda_l p_l} \right).$$

Let us define the function $M : D \to M_d(\mathbb{R})$ where for $\rho \in D$, $M(\rho)$ is the matrix with coefficients

$$1 \leq i, j \leq d, \quad M_{ij}(\rho) = \frac{(\lambda_i \sum_{l=1}^{d} \lambda_l p_l) (\lambda_i - \lambda_j) p_i}{\left( \sum_{l=1}^{d} \lambda_l p_l \right)^2} 1_{\{i=j\}} + \frac{\lambda_i p_i (\lambda_i - \lambda_j) p_i}{\left( \sum_{l=1}^{d} \lambda_l p_l \right)^2} 1_{\{i \neq j\}}$$

and the function $A : D \to M_d(\mathbb{R})$ by $A = \frac{1}{2} (I_d + M)$. Then it is easy to check the equality

$$\forall 1 \leq i \leq d, \quad \frac{1}{2} \partial_\nu \left( \frac{\lambda_i \sum_{l=1}^{d} p_l}{\sum_{l=1}^{d} \lambda_l p_l} \right) = (A(\rho) \partial_\nu p)_i.$$  

Summing Equation (10) over the index $1 \leq i \leq d$, the following equality holds in the sense of distributions:
\[ \forall v \in H, \frac{d}{dt}(v, p)_d - r(\partial_x v, p)_d + \left( \partial_x v, \frac{1}{2} R(p) \sigma_{\text{Dup}}(\sigma_{\text{Dup}} + 2 \partial_x \sigma_{\text{Dup}}) \Lambda p \right)_d + \left( \partial_x v, \sigma_{\text{Dup}}^2 A(p) \partial_x p \right)_d = (Q, v)_d, \] (11)

where we denote by \( \Lambda \) the diagonal \( d \times d \) matrix with coefficients \((\lambda_i)_{1 \leq i \leq d}\), we define for \( \rho \in D \), \( R(\rho) := \sum_{i=1}^d \rho_i \), for \( x \in \mathbb{R}, Q(x) = (a_{ij}(x))_{1 \leq i, j \leq d} \). We will say that a vector \( p \) satisfies the variational formulation \( V(\mu) \) if \( p \in L^2_{\text{loc}}((0, T]; H) \cap L^\infty_{\text{loc}}((0, T]; L) \), \( p \) takes values in \( D \) a.e. on \((0, T) \times \mathbb{R} \), \( p \) solves Equation (11) in the sense of distributions on \((0, T) \), and finally,

\[ p(t, \cdot) \xrightarrow{t \to 0^+} p_0 := (\mu_1, \ldots, \mu_d), \]

which means that \( \forall v \in H \), \( \lim_{t \to 0^+}(p(t), v)_d = \sum_{i=1}^d \int_{\mathbb{R}} v_i d\mu_i \). Let us remark that as \( \sum_{i=1}^d q_{ji} = 0 \) for \( 1 \leq j \leq d \), it is easy to check that if \( p \) solves \( V(\mu) \), then \( \sum_{i=1}^d p_i \) solves the variational formulation associated to Dupire’s PDE, that we call \( LV(\mu_{x_0}) \), where for \( v \in \mathcal{P}(\mathbb{R}) \), a solution \( u \) to \( LV(v) \) satisfies \( u \in L^2_{\text{loc}}((0, T]; H^1(\mathbb{R})) \cap L^\infty_{\text{loc}}((0, T]; L^2(\mathbb{R})) \), \( u \geq 0 \), \( \forall v \in H \), \( H^1(\mathbb{R}), \frac{d}{dt}(v, u) - r(\partial_x v, u) + \left( \partial_x v, \frac{1}{2} \sigma_{\text{Dup}}(\sigma_{\text{Dup}} + 2 \partial_x \sigma_{\text{Dup}}) u \right)_1 + \frac{1}{2} \left( \partial_x v, \sigma_{\text{Dup}}^2 \partial_x u \right)_1 = 0, \]

in the sense of distributions on \((0, T) \) and \( \forall v \in H \) weakly-*.

### 2.1 About condition (C)

Let us first discuss the role played by Condition (C) and give sufficient conditions on the family \((\lambda_i)_{1 \leq i \leq d}\), that is, on the range of the squared stochastic factor \( f^2(Y) \), for (C) to be satisfied. We introduce the notion of uniform coercivity.

**Definition 2.1.** Given a domain \( D \subset \mathbb{R}^d \), a function \( G : D \to \mathcal{M}_d(\mathbb{R}) \) is uniformly coercive on \( D \) with a coefficient \( c > 0 \)

\[ \forall \rho \in D, \forall \xi \in \mathbb{R}^d, \xi^* G(\rho) \xi \geq c \xi^* \xi. \]

First, let us note that the function \( A \) is bounded.

**Lemma 2.2.** Any matrix \( B \) in the image \( A(D) \) of \( D \) by \( A \) satisfies \( ||B||_\infty \leq \frac{1}{2} (1 + \frac{\lambda_d}{\lambda_1}). \)

**Proof.** For \( \rho \in D, A(\rho) = \frac{1}{2}(I_d + M(\rho)). \) It is sufficient to check that \( ||M(\rho)||_\infty \leq \frac{\lambda_d}{\lambda_1}. \) As for \( 1 \leq l \leq d, \lambda_l > 0, \rho_l \geq 0 \) and \( \sum_{i=1}^d \lambda_i \rho_l > 0 \), we obtain the result as for \( 1 \leq i \neq j \leq d, \)

\[ \frac{-\lambda_d}{\lambda_1} \leq -\lambda_j \frac{\lambda_i \rho_l}{(\sum_i \lambda_i \rho_l)} \frac{\sum_{i \neq j} \lambda_i \rho_l}{\sum_i \lambda_i \rho_l} \leq M_{ij}(\rho) \leq \frac{\lambda_i \rho_l}{(\sum_i \lambda_i \rho_l)} \frac{\sum_{i \neq j} \lambda_i \rho_l}{\sum_i \lambda_i \rho_l} \leq 1 \leq \frac{\lambda_d}{\lambda_1}, \]

\[ \frac{-\lambda_d}{\lambda_1} \leq -1 \leq \frac{\sum_{i \neq l} \lambda_l \rho_l \sum_{i \neq l} (-\lambda_l) \rho_l}{(\sum_i \lambda_i \rho_l)^2} \leq M_{ili}(\rho) \leq \frac{\sum_{i \neq l} \lambda_l \rho_l}{\sum_i \lambda_i \rho_l} \frac{\sum_{i \neq l} \lambda_l \rho_l}{\sum_i \lambda_i \rho_l} \leq \frac{\lambda_d}{\lambda_1}. \]

\( \square \)
The role of Condition (C) is to ensure the existence of a matrix $\Pi \in S_{d+}^+(\mathbb{R})$ such that $\Pi A$ is uniformly coercive on $D$. The introduction of the matrix $\Pi$ is somehow reminiscent of the boundedness-by-entropy technique introduced by Jüngel (2015) and used to establish existence of global weak solutions to cross-diffusions, as in Desvillettes, Lepoutre, and Moussa (2014). Proposition 2.3 is proved in Appendix B.

**Proposition 2.3.** Condition (C) is equivalent to the existence of a matrix $\Pi \in S_{d+}^+(\mathbb{R})$ such that the function $\Pi A$ is uniformly coercive on $\mathcal{D}$ with a coefficient $\kappa \in \left(0, \frac{l_{\max}(\Pi)}{2}\right)$.

Making Condition (C) explicit does not seem to be an easy task in general, but we give here simpler criteria that are all proved in Appendix B. By a slight abuse of notation, we will say that a matrix $\Gamma$ satisfies (C) if $\Gamma \in S_{d+}^+(\mathbb{R})$, and for $1 \leq k \leq d$, the matrix $\Gamma^{(k)}$ defined by (9) is positive definite on $e_k^\perp$.

- For $d = 2$, Condition (C) is always satisfied, for the choice $\Gamma = I_2$.
- For $d = 3$, let us define $r_1 = \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_3} \geq 2$, $r_2 = \frac{\lambda_1}{\lambda_3} + \frac{\lambda_3}{\lambda_2} \geq 2$, $r_3 = \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} \geq 2$. Condition (C) is satisfied if and only if $\frac{1}{\sqrt{(r_1-2)(r_2-2)}} + \frac{1}{\sqrt{(r_2-2)(r_3-2)}} + \frac{1}{\sqrt{(r_1-2)(r_3-2)}} > \frac{1}{4}$, with the convention that $\frac{1}{0} = +\infty$.
- For $d \geq 4$, we give in Appendix B a numerical procedure to check if there exists a diagonal matrix that satisfies Condition (C). Moreover, if $\frac{\lambda_d}{\lambda_1} < \left(\frac{d+1}{d-1}\right)^2$, then Condition (C) is satisfied for the choice $\Gamma = I_d$.

As $\frac{d+1}{d-1}$ decreases to 1 when $d \to \infty$, the previous sufficient condition is more and more restrictive as $d$ grows and obtaining existence for stochastic volatility factors taking infinitely many values seems out of reach. This seems consistent with the hypotheses used in the framework of Abergel and Tachet (2010), where the authors assume that the first-order derivative of the stochastic volatility factor is small enough and thus control the range of $f$. We now state our existence results.

### 2.2 Existence to $V(\mu)$ and to SDE (2) and (3)

**Theorem 2.4.** Under Condition (C), Assumptions (B) and (H), there exists a solution $p \in C((0, T], L)$ to $V(\mu)$, such that $\sum_{i=1}^{d} p_i$ is the unique solution to $LV(\mu X_0)$.

**Theorem 2.5.** Under Condition (C), Assumptions (B) and (H), there exists a weak solution $X$ to SDE (2) and (3). Moreover, we have that for $t \in [0, T]$, $X_t^{(d)} = X_t^D$.

We prove Theorem 2.4 in Section 3, and Theorem 2.5 in Section 4.

### 3 PROOF OF THEOREM 2.4

We first establish preliminary existence and uniqueness results to the variational formulation $LV$.

#### 3.1 Existence and uniqueness for $LV$

All the proofs of this subsection are gathered in Appendix C.
Lemma 3.1. Under Assumption (B), for $v \in \mathcal{P}(\mathbb{R})$, the solutions of $LV(v)$ are continuous and positive on $(0, T] \times \mathbb{R}$.

In addition, if $\tilde{\sigma}_{D_{up}} \in L^\infty([0, T], W^{1,\infty}(\mathbb{R}))$, and $\tilde{\sigma}_{D_{up}}$ are continuous, then $\tilde{\sigma}_{D_{up}}$ and $\tilde{\sigma}_{D_{up}}^2$ have the Lipschitz property in the space variable, uniformly in time, so that existence and pathwise uniqueness hold for (4) and (5). Moreover, Assumptions (H) and (B) imply that there exists $\tilde{H}_0 > 0$ such that

$$\forall s, t \in [0, T], \forall x, y \in \mathbb{R}, |\tilde{\sigma}_{D_{up}}(s, x) - \tilde{\sigma}_{D_{up}}(t, y)| \leq \tilde{H}_0|t - s|^2 + |x - y|.$$ 

Then for $v \in \mathcal{P}(\mathbb{R})$, those conditions are sufficient to obtain uniqueness to $LV(v)$ and Aronson-like upper bounds on the solutions of $LV(v)$.

Proposition 3.2. Under Assumptions (B) and (H), there exists a unique solution $u$ to $LV(\mu_{X_0})$ and for $t \in [0, T]$, $X_t^D$ has the density $u(t, \cdot)$, Moreover, there exists a finite constant $\zeta$, independent from $\mu_{X_0}$, such that for a.e. $t \in (0, T)$, $||u(t)||^2_{L^2} \leq \frac{\zeta}{\sqrt{t}}$.

3.2 Existence of a solution to PDS (RS) when $\mu_i, 1 \leq i \leq d$ have $L^2(\mathbb{R})$ densities

In this section, for $1 \leq i \leq d$, the measure $\mu_i$ is assumed to have a square integrable density denoted by $p_{0,i}$ and we define $p_0 := (p_{0,1}, \ldots, p_{0,d}) \in L$. We will say that a function $p$ satisfies the stronger variational formulation $V_{L^2}(\mu)$ if $p \in L^2([0, T]; H) \cap L^\infty([0, T]; L)$, $p$ takes values in $D$ a.e. in $(0, T) \times \mathbb{R}$, satisfies Equation (11) in the sense of distributions on $(0, T)$, and $p(0, \cdot) = p_0$.

If we only have $p \in L^2([0, T]; H) \cap L^\infty([0, T]; L)$ then the initial condition $p(0, \cdot) = p_0$ does not make sense. We will show that if $p$ moreover solves Equation (11), then $p$ is a.e. equal to a continuous function from $[0, T]$ to $L$. We consider in what follows this continuous representative and therefore the initial condition makes sense. We now give an existence result to $V_{L^2}(\mu)$.

Theorem 3.3. Under Condition (C), Assumptions (B) and (H), $V_{L^2}(\mu)$ has a solution $p \in C([0, T], L)$ such that $\sum_{i=1}^d p_i$ is equal to the unique solution to $LV(\mu_{X_0})$ a.e. in $[0, T] \times \mathbb{R}$.

To prove Theorem 3.3, we use Galerkin’s procedure, as in Temam (1979, III, 1.3). Standard PDE techniques such as Galerkin’s method can also be found in the Lions and Magenes (1972) reference book. As $H$ is a separable Hilbert space, there exists a sequence $(w_k)_{k \geq 1} = ((w_{k1}, \ldots, w_{kd}))_{k \geq 1}$ of linearly independent elements, which is total in $H$. It is not at all obvious to preserve the condition that $p$ takes values in $D$ a.e. on $[0, T] \times \mathbb{R}$ at the discrete level. That is why, for $\epsilon > 0$, we introduce for $\rho \in (\mathbb{R}_+)^d$ the approximation $M_{\epsilon, j}(\rho)$ of $M$ defined on $(\mathbb{R}_+)^d$ by

$$M_{\epsilon, j}(\rho) = \frac{\sum_{i \neq j} \lambda_i \rho_i \sum_{i \neq j} (\lambda_i - \lambda_j) \rho_j}{e^2 \sqrt{\sum_{i} \lambda_i \rho_i ^2}} 1_{\{i \neq j\}}, \quad 1 \leq i, j \leq d. \quad (12)$$

We introduce the approximation $A_{\epsilon}$ of $A$ defined on $(\mathbb{R}_+)^d$ by $A_{\epsilon} = \frac{1}{2}(I_d + M_{\epsilon})$, and we define, for $\rho \in (\mathbb{R}_+)^d$, $\tilde{R}_{\epsilon}(\rho) = \frac{\sum_{i=1}^d \rho_i}{\epsilon \sqrt{\sum_{i=1}^d \lambda_i \rho_i}}$. We first introduce a variational formulation $V_{\epsilon}(\mu)$ and we will say that a function $p_{\epsilon}$ is a solution to $V_{\epsilon}(\mu)$ if $p_{\epsilon} \in L^2([0, T]; H) \cap L^\infty([0, T]; L)$,

$$\forall \nu \in H, \quad (Q\nu + r\partial_\nu \nu, p_{\epsilon})_d = \frac{d}{dt}(\nu, p_{\epsilon})_d + \left(\partial_\nu \nu, \tilde{\sigma}_{D_{up}}^2 A_{\epsilon}(p_{\epsilon}^+) \partial_\nu p_{\epsilon}\right)_d + \left(\partial_\nu \nu, \frac{1}{2} R_{\epsilon}(p_{\epsilon}^+) \tilde{\sigma}_{D_{up}} + 2 \partial_\nu \tilde{\sigma}_{D_{up}} \Lambda p_{\epsilon}\right)_d, \quad (13)$$

where $Q \in L^\infty(\mathbb{R} \times \mathcal{D}(\mathbb{R}^d), \mathbb{R}^d)$ is a bounded function, and $\Lambda$, $\partial_\nu$, $\tilde{\sigma}_{D_{up}}$, $R_{\epsilon}$, $\tilde{\sigma}_{D_{up}}$, and $\tilde{\sigma}_{D_{up}}$ are defined as in the main text.
in the sense of distributions on \((0, T)\) and \(p_e(0, \cdot) = p_0\). As we will also show that any solution \(p_e\) to \(V_e(\mu)\) is a.e. equal to a continuous function \([0, T] \to L\), we will consider this continuous representative and the initial condition \(p_e(0, \cdot) = p_0\) makes sense. To take advantage of the fact that under Condition (C), there exists \(\Pi \in S^+_d(\mathbb{R})\) and \(\kappa > 0\) such that \(\Pi A\) is uniformly coercive on \(D\) with the coefficient \(\kappa\), we remark that as \(v \in H \to \Pi v \in H\) is a bijection, the previous equation is equivalent to

\[
\forall v \in H, \quad (Q\Pi v + r\Pi \partial_x v, p_e)_d = \frac{d}{dt}(v, \Pi p_e)_d + \left(\partial_x v, \tilde{\sigma}_{\text{Dup}}\Pi_\epsilon \left(p_e^+\right) \partial_x p_e\right)_d + \left(\partial_x v, \frac{1}{2} R_e \left(p_e^+\right) \tilde{\sigma}_{\text{Dup}}(2\tilde{\sigma}_{\text{Dup}} + 2\partial_x \tilde{\sigma}_{\text{Dup}})\Pi \Lambda p_e\right)_d.
\]

To apply Galerkin’s procedure, we introduce, for \(\epsilon > 0\) and \(m \geq 1\), the approximate variational formulation \(V_{\epsilon,m}(\mu)\), where any solution \((g_{\epsilon,1}^m, \ldots, g_{\epsilon,m}^m) \in (C^0([0, T], \mathbb{R}))^m\) is such that the function \(t \in [0, T] \to p_{\epsilon,m}^m(t) = \sum_{j=1}^m g_{\epsilon,j}^m(t) w_j\) satisfies, for \(1 \leq i \leq m\), \(p_{\epsilon,m}^m(0) = p_0^m\), where \(p_0^m\) is the orthogonal projection of \(p_0\) in \(L\) on the space spanned by \((w_j)_{1 \leq j \leq m}\), and

\[
(Q\Pi w_i + r\Pi \partial_x w_i, p_{\epsilon,m}^m(t))_d = \frac{d}{dt}(w_i, \Pi p_{\epsilon,m}^m(t))_d + \left(\partial_x w_i, \tilde{\sigma}_{\text{Dup}}\Pi_\epsilon \left(p_{\epsilon,m}^+(t)\right) \partial_x p_{\epsilon,m}^m(t)\right)_d + \frac{1}{2} \left(\partial_x w_i, R_e \left(p_{\epsilon,m}^+(t)\right) \tilde{\sigma}_{\text{Dup}}(2\tilde{\sigma}_{\text{Dup}} + 2\partial_x \tilde{\sigma}_{\text{Dup}})\Pi \Lambda p_{\epsilon,m}^m(t)\right)_d.
\]

By a slight abuse of notation, we may identify \((g_{\epsilon,1}^m, \ldots, g_{\epsilon,m}^m)\) with \(p_{\epsilon,m}^m\) and say that \(p_{\epsilon,m}^m\) is a solution to \(V_{\epsilon,m}(\mu)\) if \((g_{\epsilon,1}^m, \ldots, g_{\epsilon,m}^m)\) solves \(V_{\epsilon,m}(\mu)\). We denote by \(W^m \in S^+_m(\mathbb{R})\) the nonsingular Gram matrix of the linearly independent family \((\sqrt{\Pi} w_i)_{1 \leq i \leq m}\), with coefficients \(W_{ij}^m = (w_i, \Pi w_j)_d, 1 \leq i, j \leq m\). For \(z \in \mathbb{R}^m\) and \(t \geq 0\), we define \(K_{\epsilon,1}(t, z)\) the matrix with coefficients

\[
K_{\epsilon,1,i,j}(t, z) = -\frac{1}{2} \left(\partial_x w_i, R_e \left(\sum_{k=1}^m z_k w_k\right)\right) \tilde{\sigma}_{\text{Dup}}(t)(2\tilde{\sigma}_{\text{Dup}} + 2\partial_x \tilde{\sigma}_{\text{Dup}})\Pi \Lambda w_j, 1 \leq i, j \leq m,
\]

where \(K_{\epsilon,2}(t, z)\) the matrix with coefficients

\[
K_{\epsilon,2,i,j}(t, z) = -\left(\partial_x w_i, \tilde{\sigma}_{\text{Dup}}(t)\Pi_\epsilon \left(\sum_{k=1}^m z_k w_k\right)\right) \partial_x w_j, 1 \leq i, j \leq m,
\]

and \(K_{\epsilon,3}\) is the constant matrix where for \(1 \leq i, j \leq m\), \(K_{\epsilon,3,i,j} = (Q\Pi w_i + r\Pi \partial_x w_i, w_j)_d\). We then define \(F_{\epsilon,k}(t, z) := (W^m)^{-1} K_{\epsilon,k}(t, z)\), for \(k = 1, 2\), and \(F_{\epsilon,3}(z) := (W^m)^{-1} K_{\epsilon,3}\). Finally we define for \(z \in \mathbb{R}^m\), \(G_{\epsilon}(t, z) := F_{\epsilon,1}(t, z) + F_{\epsilon,2}(t, z) + F_{\epsilon,3}(z)\). Solving \(V_{\epsilon,m}(\mu)\) then boils down to solving the ODE with unknown \(g_{\epsilon}^m\),

\[
(g_{\epsilon}^m)'(t) = G_{\epsilon}^m(t, g_{\epsilon}^m(t)),
\]

\[
g_{\epsilon}^m(0) = g_{\epsilon,0}^m,
\]

where the vector \(g_{\epsilon,0}^m\) is the expression of \(p_0^m\) on the basis \((w_i)_{1 \leq i \leq m}\). To prove existence and uniqueness to \(V_{\epsilon,m}(\mu)\), \(m \geq 1, \epsilon > 0\), we check that the function \(G_{\epsilon}^m\) is locally Lipschitz and that the functions \(K_{\epsilon,k}\), \(k = 1, 2\) are bounded. The proof of Lemma 3.4 is postponed to Appendix A.

**Lemma 3.4.** Under Assumption (B), for \(m \geq 1\), the function \(G_{\epsilon}^m\) is locally Lipschitz in \(z\), uniformly in \(t \in [0, T]\).
To obtain that the functions $K_{\epsilon, k}^m$, $1 \leq k \leq 2$ are bounded, we easily check that the following inequalities hold.

**Lemma 3.5.** The functions $R$ and $R_\epsilon$, $\epsilon > 0$, satisfy $\| R \|_{L^\infty(D)} \leq \frac{1}{\lambda_1} \| R_\epsilon \|_{L^\infty((\mathbb{R}^e)^d)} \leq \frac{1}{\lambda_1}$.

**Lemma 3.6.** For $\epsilon > 0$ and $B \in A_\epsilon((\mathbb{R}_+)^d)$, $\| B \|_{\infty} \leq \frac{1}{2} \left( 1 + \frac{\lambda_d}{\lambda_1} \right)$.

**Proof.** For $\epsilon > 0$, we have that $A_\epsilon(0) = \frac{1}{2} I_d$, so $\| A_\epsilon(0) \|_{\infty} = \frac{1}{2}$. For $\rho \in D$ and $\epsilon > 0$, it is clear that $\| M_\epsilon(\rho) \|_{\infty} \leq \frac{\lambda_d}{\lambda_1}$, where the inequality of the r.h.s. comes from the proof of Lemma 2.2, and this concludes the proof.

Under Assumption (B), let us define $\tilde{\sigma}$ a positive constant such that a.e. on $[0, T] \times \mathbb{R}$, $\tilde{\sigma}_{\text{Dup}} \leq \tilde{\sigma}$.

**Lemma 3.7.** Under Assumption (B), for $m \geq 1$, the functions $K_{\epsilon, 1}^m$ and $K_{\epsilon, 2}^m$ are uniformly bounded on $[0, T] \times \mathbb{R}$.

**Proof.** Using Assumption (B), Lemmas 3.5 and 3.6, for $t \in [0, T]$, $x \in \mathbb{R}$, $\rho \in D$, we have
\[
\left| \frac{1}{2} R_\epsilon(\rho) \tilde{\sigma}_{\text{Dup}}(t, x) (\tilde{\sigma}_{\text{Dup}}(t, x) + 2 \partial_\rho \tilde{\sigma}_{\text{Dup}}(t, x)) \right| \leq \frac{1}{2 \lambda_1} \tilde{\sigma}(\tilde{\sigma} + 2 \| \partial_\rho \tilde{\sigma}_{\text{Dup}} \|_{\infty}),
\]
\[
\| A_\epsilon(\rho) \tilde{\sigma}_{\text{Dup}}^2(t, x) \|_{\infty} \leq \frac{1}{2} \left( 1 + \frac{\lambda_d}{\lambda_1} \right) \tilde{\sigma}^2.
\]
This is sufficient to show that $K_{\epsilon, 1}^m$ and $K_{\epsilon, 2}^m$ are uniformly bounded, as the functions $(\omega_{ik} \partial_{x_i} w_j)_{1 \leq i, j \leq m, 1 \leq k \leq d}$ and $(\partial_{x_i} w_{ik} \partial_{x_i} w_j)_{1 \leq i, j \leq m, 1 \leq k \leq d}$ belong to $L^1(\mathbb{R})$.

**Lemma 3.8.** Under Assumption (B), $V_{\epsilon}^m(\mu)$ has a unique solution.

**Proof.** By Lemma 3.4 and Caratheodory’s theorem (see, e.g., Hale, 2009, Theorems 5.2, 5.3), there exists a unique maximal absolutely continuous solution $g_{\epsilon}^m$ on an interval $[0, T^*)$, for some $T^* > 0$. We show that $T^* = \infty$, so that $g_{\epsilon}^m$ is defined on $[0, T]$ and is the unique solution to $V_{\epsilon}^m(\mu)$. By Lemma 3.7, the function $H : (t, z) \to (W(m))^{-1} K_{\epsilon, 1}^m(t, z) + (W(m))^{-1} K_{\epsilon, 2}^m(t, z)$ is uniformly bounded on $\mathbb{R}_+ \times \mathbb{R}^m$, we can denote by $\gamma > 0$ an uniform upper bound of the coefficients of $H$ and we obtain that for $1 \leq i \leq m$,
\[
\left| \left( g_{\epsilon, i}^m \right)'(t) \right| = \left| \sum_{j=1}^m H(t, g_{\epsilon}^m(t))_{ij} g_{\epsilon, j}^m(t) \right| \leq \gamma \sum_{j=1}^m g_{\epsilon, j}^m(t).
\]
Summing over the index $i \in \{ 1, \ldots, m \}$, we have that for $t \in [0, T^*)$,
\[
\sum_{i=1}^m g_{\epsilon, i}^m(t) \leq \sum_{i=1}^m g_{\epsilon, i}^m(0) + \int_0^t \sum_{i=1}^m \left| g_{\epsilon, i}^m \right|(s) \| ds \leq \sum_{i=1}^m g_{\epsilon, i}^m(0) + m \gamma \int_0^t \sum_{i=1}^m g_{\epsilon, i}^m(s) \| ds.
\]
If $T^*$ was finite, then the function $t \to \sum_{i=1}^m |g_{\epsilon, i}^m(t)|$ would explode as $t \to T^*$. By Gronwall’s lemma applied on Inequality (15), we obtain that $T^* = \infty$, and this concludes the proof.

Before showing the existence of a subsequence of $(p_{\epsilon}^m)_{m \geq 1}$ converging to a solution to $V_\epsilon(\mu)$, where for $m \geq 1$, $p_{\epsilon}^m = \sum_{j=1}^m g_{\epsilon, j}^m w_j$, we check that $\Pi A_\epsilon$ is uniformly coercive on $D$, uniformly in $\epsilon > 0$.

**Lemma 3.9.** If $\Pi A_\epsilon$ is uniformly coercive on $D$ with coefficient $\kappa \in \left( 0, \frac{L_{\text{lip}}(1)}{\lambda_1} \right)$, then for $\epsilon > 0$, $\Pi A_\epsilon$ is uniformly coercive on $(\mathbb{R}_+)^d$ with coefficient $\kappa$. 
Proof. For \( \epsilon > 0 \), \( A_{c}(0) = \frac{1}{2} l_d \) so \( \forall \xi \in \mathbb{R}^d, \xi^* \Pi A_{c}(0) \xi = \frac{1}{2} \xi^* \Pi \xi \geq \frac{l_{\text{min}}(\Pi)}{2} \xi^* \xi \geq \kappa \xi^* \xi \). For \( \rho \in D \), if \( \epsilon \leq \sum \lambda_i \rho_i \), then \( A_{c}(\rho) = A(\rho) \) and by hypothesis \( \forall \xi \in \mathbb{R}^d, \xi^* \Pi A_{c}(\rho) \xi \geq \kappa \xi^* \xi \). If \( \epsilon > \sum \lambda_i \rho_i \), then for \( 1 \leq i, j \leq d \), \( M_{e,ij}(\rho) = \left( \frac{1}{\epsilon} \sum \lambda_i \rho_i \right)^2 M_{ij}(\rho) \), with \( \left( \frac{1}{\epsilon} \sum \lambda_i \rho_i \right)^2 < 1 \). If \( \xi^* \Pi M_c(\rho) \xi \leq 0 \), then \( \xi^* \Pi M_c(\rho) \xi \geq \xi^* \Pi M_c(\rho) \xi \geq \kappa \xi^* \xi \). If \( \xi^* \Pi M_c(\rho) \xi > 0 \), then \( \xi^* \Pi M_c(\rho) \xi \geq 0 \) and \( \xi^* \Pi A_c(\rho) \xi \geq \frac{1}{2} \xi^* \xi \geq \kappa \xi^* \xi \), so \( \Pi A_c \) is uniformly coercive on \( (\mathbb{R}^d)^2 \) with coefficient \( \kappa \). \( \Box \)

We now compute energy estimates on the solution \( p_c^m \) to \( V_c^m(\mu) \), for \( m \geq 1 \) and \( \epsilon > 0 \). Replacing \( w_i \) by \( p_c^m \) in (14), we have

\[
\frac{1}{2} \frac{d}{dt} \left[ \sqrt{\Pi p_c^m} \right]_d^2 - (Q \Pi p_c^m, p_c^m)_d = -\left( \partial_x p_c^m, \Sigma^2 \partial_t \Pi A_c \left( \left( p_c^m \right)^+ \right) \partial_x p_c^m \right)_d \\
- \left( \partial_x p_c^m, \left( \frac{1}{2} R_{\epsilon} \left( \left( p_c^m \right)^+ \right) \Sigma \partial_t \Pi \Lambda - \partial_t \Pi \right) p_c^m \right)_d.
\]

For \( \eta > 0 \), by Young’s inequality, there exists a finite constant \( C \), not depending on \( \epsilon \), such that

\[
\left| \left( \partial_x p_c^m, \left( \frac{1}{2} R_{\epsilon} \left( \left( p_c^m \right)^+ \right) \Sigma \partial_t \Pi \Lambda - \partial_t \Pi \right) p_c^m \right)_d \right| \leq C \left( \eta \left| \partial_x p_c^m \right|_d^2 + \frac{1}{4\eta} \left| p_c^m \right|_d^2 \right),
\]

as \( R_{\epsilon} \) (see Lemma 3.5) and \( \Sigma \partial_t \Pi \Lambda - \partial_t \Pi \) are bounded. Under Condition (C), by Lemma 3.9, we have that for \( \epsilon > 0 \), \( \Pi A_c \) is uniformly coercive with coefficient \( \kappa \). Moreover, as \( \Sigma \partial_t \Pi \Lambda - \partial_t \Pi \) is bounded from below by \( \sigma > 0 \), we have that

\[
\left( \partial_x p_c^m, \Sigma \partial_t \Pi A_c \left( \left( p_c^m \right)^+ \right) \partial_x p_c^m \right)_d \geq \kappa \sigma^2 \left| \partial_x p_c^m \right|_d^2.
\]

We then choose \( \eta = \frac{\kappa}{2C} \sigma^2 \), so that \( C \eta = \frac{\kappa}{2} \sigma^2 \). Moreover, for \( b := d^2 \| \Pi \|_\infty \) we have that \( \forall \xi \in \mathbb{R}^d, |\xi^* \Pi \xi| \leq b \xi^* \xi \). Gathering the previous inequalities we have that \( \frac{1}{2} \frac{d}{dt} \left| \sqrt{\Pi p_c^m} \right|_d^2 - \left( b + \frac{C^2}{2\kappa \sigma^2} \right) \left| p_c^m \right|_d^2 \leq \left( \frac{\kappa}{2 \sigma^2} \right) \left| \partial_x p_c^m \right|_d^2 \leq 0 \). As \( |p_c^m|_d^2 \leq \frac{1}{l_{\text{min}}(\Pi)} \left| \sqrt{\Pi p_c^m} \right|_d^2 \), we also have

\[
\frac{1}{2} \frac{d}{dt} \left| \sqrt{\Pi p_c^m} \right|_d^2 - \frac{1}{l_{\text{min}}(\Pi)} \left( b + \frac{C^2}{2\kappa \sigma^2} \right) \left| \sqrt{\Pi p_c^m} \right|_d^2 \leq \left( \frac{\kappa}{2 \sigma^2} \right) \left| \partial_x p_c^m \right|_d^2 \leq 0.
\]

Integrating Inequality (17), and using the fact that \( l_{\text{min}}(\Pi)|p_c^m|_d^2 \leq |\sqrt{\Pi p_c^m}|_d^2 \leq l_{\text{max}}(\Pi)|p_c^m|_d^2 \), we obtain the following lemma.

Lemma 3.10. The following energy estimates hold, for \( m \geq 1 \) and \( \epsilon > 0 \),

\[
\sup_{t \in [0,T]} |p_c^m(t)|_d^2 \leq \frac{l_{\text{max}}(\Pi)}{l_{\text{min}}(\Pi)} \left( b + \frac{C^2}{2\kappa \sigma^2} \right) T |p_0|_d^2,
\]

\[
\int_0^T |\partial_x p_c^m(t)|_d^2 dt \leq \frac{l_{\text{max}}(\Pi)}{\kappa \sigma^2} \left( b + \frac{C^2}{2\kappa \sigma^2} \right)^2 T |p_0|_d^2,
\]

\[
\int_0^T |p_c^m|^2 dt \leq \left( T \frac{l_{\text{max}}(\Pi)}{l_{\text{min}}(\Pi)} + \frac{l_{\text{max}}(\Pi)}{\kappa \sigma^2} \right) \left( b + \frac{C^2}{2\kappa \sigma^2} \right)^2 T |p_0|_d^2.
\]
Now we can prove existence to \( V_\epsilon(\mu) \), \( \epsilon > 0 \).

**Proposition 3.11.** There exists a solution \( p_\epsilon \) to \( V_\epsilon(\mu) \) for \( \epsilon > 0 \).

**Proof.** By Lemma 3.10, the family \( (p_\epsilon^m)_{m \geq 0} \) remains in a bounded set of \( L^2([0, T]; H) \cap L^\infty([0, T]; L) \), so there exists an element \( p_\epsilon \in L^2([0, T]; H) \cap L^\infty([0, T]; L) \) and a subsequence, for notational simplicity also called \( (p_\epsilon^m)_{m \geq 1} \), that has the following convergence:

\[
\begin{align*}
p_\epsilon^m & \to p_\epsilon \text{ in } L^2([0, T], H) \text{ weakly,} \\
p_\epsilon^m & \to p_\epsilon \text{ in } L^\infty([0, T], L) \text{ weakly-}\ast.
\end{align*}
\]

We now show that there exists a subsequence of \( (p_\epsilon^m)_{m \geq 1} \) that converges a.e. on \((0, T) \times \mathbb{R}\) to \( p_\epsilon \). Equality (14) rewrites for \( 1 \leq j \leq m \),

\[
\frac{d}{dt}(w_j, \Pi p_\epsilon^m)_d + (w_j, Gp_\epsilon^m) = 0,
\]

where for \( \hat{p} \in H, G\hat{p} \in H' \), and

\[
\forall v \in H, \langle v, G\hat{p} \rangle = -r(\partial_x v, \Pi\hat{p})_d + \left( \partial_x v, \left( \frac{1}{2} R_\epsilon(\hat{p}^+) \hat{\sigma}_{\text{Dup}} (\hat{\sigma}_{\text{Dup}} + 2\partial_x \hat{\sigma}_{\text{Dup}}) \right) \Pi \hat{\sigma} \right)_d,
\]

\[
+ \left( \partial_x v, \hat{\sigma}_{\text{Dup}}^2 \Pi A_\epsilon (\hat{p}^+) \partial_x \hat{p} \right)_d - (Q\Pi v, \hat{p})_d.
\]

Let us note that we have the inequality

\[
||G_\epsilon \hat{p}||_{H'} \leq d||\Pi||_{\infty} \left( r + \lambda \frac{d}{\lambda_1} ||\sigma||_{\infty} + \frac{d}{2} \left( 1 + \lambda \frac{d}{\lambda_1} \right) ||\sigma||_{\infty}^2 + d \bar{q} \right) ||\hat{p}||_d.
\]

(21)

By Lemma 3.10, the family \( (p_\epsilon^m)_{m \geq 1} \) is bounded in \( L^2([0, T], H) \) and through Equality (21), the family \( (G_\epsilon p_\epsilon^m)_{m \geq 1} \) is bounded in \( L^2([0, T], H') \), and the computations in Temam (1979, (iii), p. 285) give that for any bounded open subset \( \mathcal{O} \subset \mathbb{R} \), modulo the extraction of a subsequence, \( p_\epsilon^m|_{\mathcal{O}} \to p_\epsilon|_{\mathcal{O}} \) in \( L^2([0, T], L(\mathcal{O})) \) strongly and a.e. on \([0, T] \times \mathcal{O}\). We define for \( n \geq 1 \), \( \mathcal{O}_n = (-n, n) \), so that \( \mathcal{O}_n \subset \mathcal{O}_{n+1} \), and \( \bigcup_{n \geq 1} \mathcal{O}_n = \mathbb{R} \). By diagonal extraction, we get from \( (p_\epsilon^m)_{m \geq 1} \) a subsequence, also called \( (p_\epsilon^m)_{m \geq 1} \), such that

\[
p_\epsilon^m \to p_\epsilon \text{ a.e. on } [0, T] \times \mathbb{R}.
\]

We now show that \( p_\epsilon \) is a solution to the variational formulation \( V_\epsilon(\mu) \). For \( j \geq 1 \), \( \psi \in C^1([0, T], \mathbb{R}) \), with \( \psi(T) = 0 \), and \( m \geq j \), we have by integration by parts,

\[
\begin{align*}
\int_0^T \psi(t) & \left( Q\Pi w_j + r\Pi \partial_x w_j, p_\epsilon^m(t) \right)_d dt \\
= & -\int_0^T \psi'(t)(w_j, \Pi p_\epsilon^m(t))_d dt - \psi(0)(w_j, \Pi p_\epsilon^m(0))_d \\
+ & \int_0^T \psi(t) \left( \partial_x w_j, \hat{\sigma}_{\text{Dup}}^2 \Pi A_\epsilon \left( (p_\epsilon^m)^+(t) \right) \partial_x p_\epsilon^m(t) \right)_d dt \\
+ & \frac{1}{2} \int_0^T \psi(t) \left( \partial_x w_j, R_\epsilon \left( (p_\epsilon^m)^+(t) \right) \hat{\sigma}_{\text{Dup}} (\hat{\sigma}_{\text{Dup}} + 2\partial_x \hat{\sigma}_{\text{Dup}}) \Pi L p_\epsilon^m(t) \right)_d dt.
\end{align*}
\]
As \( p_0^m \) converges strongly to \( p_0 \) in \( L \), and \( p_e^m \) converges to \( p_e \) weakly in \( L^2([0, T]; L) \), the following convergences hold,

\[
- \int_0^T \psi'(t)(w_j, \Pi p_0^m(t))_d dt - \psi(0)(w_j, \Pi p_0^m(0))_d \to - \int_0^T \psi'(t)(w_j, \Pi p_e(t))_d dt - \psi(0)(w_j, \Pi p_0)_d,
\]

\[
\int_0^T \psi(t)(Q\Pi w_j + r\Pi \partial_x w_j, p_0^m(t))_d dt \to \int_0^T \psi(t)(Q\Pi w_j + r\Pi \partial_x w_j, p_e(t))_d dt.
\]

We deal with the remaining terms in Equation (22). First, we have that

\[
\left| \int_0^T \psi(t) \left( \partial_x w_j, \tilde{\sigma}_{Dup}^2(t) \Pi A_e \left( p_e^m(t) \right) \partial_x p_e(t) \right)_d dt \right|
\]

\[
- \int_0^T \psi(t) \left( \partial_x w_j, \tilde{\sigma}_{Dup}^2(t) \Pi A_e \left( (p_e^m)^+ (t) \right) \partial_x p_e^m(t) \right)_d dt
\]

\[
\leq \int_0^T \psi(t) \left( \partial_x w_j, \tilde{\sigma}_{Dup}^2(t) \Pi A_e \left( (p_e^m)^+ (t) \right) \left( \partial_x p_e(t) - \partial_x p_e^m(t) \right) \right)_d dt
\]

\[
+ \int_0^T \psi(t) \left( \partial_x w_j, \tilde{\sigma}_{Dup}^2(t) \Pi \left( A_e \left( p_e^m(t) \right) - A_e \left( (p_e^m)^+ (t) \right) \right) \partial_x p_e^m(t) \right)_d dt \right|.
\]  \hspace{1cm} (23)

The first term in the r.h.s. of Inequality (23) converges to 0 as \( m \to \infty \), as \( \partial_x p_e^m \) converges weakly to \( \partial_x p_e \) in \( L^2([0, T], L) \), the function \( A_e \) is uniformly bounded by Lemma 3.6 and \( \tilde{\sigma}_{Dup} \) is also bounded. For the second term, by Cauchy–Schwarz inequality and Inequality (19) we have that

\[
\left| \int_0^T \left( \psi(t) \tilde{\sigma}_{Dup}^2(t) \left( A_e^+ \left( (p_e^m)^+ (t) \right) - A_e^+ \left( (p_e^m)^+ (t) \right) \right) \right) \Pi \partial_x w_j, \partial_x p_e^m(t) \right)_d dt \right|^2
\]

\[
\leq \frac{l_{\max}(\Pi)}{2\sigma^2} e^{-\frac{2l_{\min}(\Pi)}{b+e^2}} T |p_0|_d \int_0^T \left| \psi(t) \tilde{\sigma}_{Dup}^2(t) \left( A_e^+ \left( (p_e^m)^+ (t) \right) - A_e^+ \left( (p_e^m)^+ (t) \right) \right) \right| \Pi \partial_x w_j \right|_d dt,
\]

and the r.h.s. goes to 0 as \( m \to \infty \), as \( A_e \) is continuous and uniformly bounded on \( (\mathbb{R}^d)^+ \) and \( p_e^m \to p_e \) a.e. on \([0, T] \times \mathbb{R} \), so we conclude that

\[
\int_0^T \psi(t)(\partial_x w_j, \tilde{\sigma}_{Dup}^2(t) \Pi A_e \left( (p_e^m)^+ (t) \right) \partial_x p_e^m(t))_d dt
\]

\[
\to \int_0^T \psi(t)(\partial_x w_j, \tilde{\sigma}_{Dup}^2(t) \Pi A_e \left( (p_e^m)^+ (t) \right) \partial_x p_e(t))_d dt. \]  \hspace{1cm} (24)

Likewise, as the function \( R_e \) is continuous and bounded on \( (\mathbb{R}^d)^+ \), and as \( p_e^m \to p_e \) a.e. on \( (0, T] \times \mathbb{R} \), we have that \( R_e(p_e^m)^+ \to R_e(p_e)^+ \) a.e. on \([0, T] \times \mathbb{R} \), and using similar arguments, we also obtain that

\[
\int_0^T \psi(t)(\partial_x w_j, R_e \left( (p_e^m)^+ (t) \right) \tilde{\sigma}_{Dup} \left( \tilde{\sigma}_{Dup} + 2\partial_x \tilde{\sigma}_{Dup} \right) \Pi A p_e^m(t))_d dt
\]

\[
\to \int_0^T \psi(t)(\partial_x w_j, R_e \left( (p_e)^+ (t) \right) \tilde{\sigma}_{Dup} \left( \tilde{\sigma}_{Dup} + 2\partial_x \tilde{\sigma}_{Dup} \right) \Pi A p_e(t))_d dt.
\]
As the family \((w_j)_{j \geq 1}\) is total in \(H\), and the function \(v \to \Pi v\) is a bijection from \(H\) to \(H\), we conclude that

\[
\forall v \in H, \int_0^T \psi(t)(Qv + r \partial_x v, p_v(t)) dt = -\int_0^T \psi'(t)(v, p_v(t)) dt - \psi(0)(v, p_0)
\]

\[
+ \int_0^T \psi(t) \left( \partial_x v, A_v \left( (p_v)^+(t) \right)^2 \right) dt + \frac{1}{2} \int_0^T \psi(t) \left( \partial_x v, R_v \left( (p_v)^+(t) \right) \right) dt \tag{25}
\]

If in addition, \(\psi \in C_\infty([0, T], \mathbb{R})\), we obtain that \(p_v\) satisfies Equation (13) in the sense of distributions on \((0, T)\). Then by Inequality (21) and Temam (1979, III, Lemma 1.2), \(p_v\) has representatives in \(C([0, T], L)\). Moreover, as \((v, p_v)_d, (Qv + r \partial_x v, p_v)_d, (\partial_x v, \sigma_{\text{Dup}}^2 A_v (p_v)^+ \partial_x p_v)_d, (\partial_x v, \frac{1}{2} R_v (p_v)^+ \sigma_{\text{Dup}}(\sigma_{\text{Dup}} + \partial_x \sigma_{\text{Dup}}) \Lambda p_v)_d\), all belong to \(L^2((0, T), \mathbb{R})\), we have that the function \((v, p_v)_d\) belongs to \(H^1((0, T), \mathbb{R})\), so thanks to Brezis (1983, Corollary 8.10), Equation (25) also holds by replacing \(p_0\) by \(p_v\). Then if we choose \(\psi\) such that \(\psi(0) \not= 0\), we obtain by comparing the two versions of Equation (25), that \(\forall v \in H, (v, p_v(0) - p_0)_d = 0\), and this concludes the proof for the existence of a solution to \(V_\epsilon(\mu)\).

\[
\text{Proposition 3.12. The solutions to } V_\epsilon(\mu) \text{ are nonnegative.}
\]

\[
\text{Proof. Let } p_v \text{ be a solution to } V_\epsilon(\mu). \text{ According to Temam (1979, Lemma 1.2, p. 261), one can take } p_v \text{ as a test function in (13) and obtain}
\]

\[
\frac{1}{2} \frac{d}{dt} [p_v]^2 + \left( \partial_x p_v^+, \frac{1}{2} R_v (p_v)^+ \sigma_{\text{Dup}}(\sigma_{\text{Dup}} + \partial_x \sigma_{\text{Dup}}) \Lambda - r I_d \right) p_v \right)_d = (Q p_v^- p_v)_d.
\]

For \(f \in H^1(\mathbb{R}), \partial_x f^- = 1_{\{f < 0\}} \partial_x f\), so we have \(\forall i \in \{1, \ldots, d\}, p_v^+ \partial_x p_v^- = 0, \partial_x p_v^- \partial_x p_v^i = (\partial_x p_v^i)^2\). As a consequence, by (12) p.8 we obtain that \(\partial_x p_v^- \sum_{j \neq i} M_{e,i,j} (p_v^+)^2 \partial_x p_v^j = 0\), and

\[
(\partial_x p_v^-, A_v (p_v^+ \partial_x p_v^i)_d = \int_{\mathbb{R}} \sum_{i=1}^d A_{e,i} (p_v^+)^2 (\partial_x p_v^-)^2 dx.
\]

Let us show that for \(1 \leq i \leq d\),

\[
A_{e,i} (p_v^+)^2 (\partial_x p_v^-)^2 \geq \frac{\lambda_1}{2 \lambda_d} (\partial_x p_v^-)^2 \geq 0. \tag{26}
\]

For \(1 \leq i \leq d\), \(2 A_{e,i} (p_v^+)^2 = 1 + \frac{(\sum_{j \neq i} \lambda_j (p_v^+)^2)(\sum_{j \neq i} (p_v^+)^2)}{e^2(\sum_{j=1}^d \lambda_j (p_v^+)^2)^2} - \frac{(\sum_{j \neq i} \lambda_j (p_v^+)^2)^2}{e^2(\sum_{j=1}^d \lambda_j (p_v^+)^2)^2} \). We distinguish the cases \(p_v^+ > 0\) and \(p_v^+ = 0\). In the first case, \(\partial_x p_v^- = 1_{\{p_v < 0\}} \partial_x p_v^+ = 0\), so (26) is true. In the second case, let us define \(\beta := (\sum_{j \neq i} \lambda_j (p_v^+)^2) = (\sum_{j=1}^d \lambda_1 (p_v^+)^2), \) and note that \((\lambda_1 \sum_{j \neq i} (p_v^+)^2) \geq \frac{\lambda_1}{\lambda_d} \beta\). Thus we obtain

\[
2 A_{e,i} (p_v^+) \geq 1 + \left( \frac{\lambda_1}{\lambda_d} - 1 \right) \frac{\beta^2}{e^2 \sqrt{\beta^2}} \geq \frac{\lambda_1}{\lambda_d},
\]
as \(0 \leq \frac{\beta^2}{e^{2\sqrt{p}}} \leq 1\), and (26) is also true. As a consequence, we have that \((\partial_x p_e^-, \bar{\sigma}^2_{\text{Dup}} A_e(p_e) \partial_x p_e)_d \geq \sigma^2 \frac{\lambda_1}{2 \lambda_d} |\partial_x p_e^-|^2_d\). By Young’s inequality, for \(\eta > 0\),

\[
\left( \partial_x p_e^-, \left( \frac{1}{2} R_e(p_e^+) \bar{\sigma}_{\text{Dup}}(\bar{\sigma}_{\text{Dup}} + 2 \partial_x \bar{\sigma}_{\text{Dup}}) \Lambda - r I_d \right) p_e \right)_d \geq -K \left( \eta |\partial_x p_e^-|^2_d + \frac{1}{4\eta} |p_e^-|^2_d \right),
\]

where \(K = \frac{1}{2} \lambda_1 \frac{\lambda_1}{2 \lambda_d} \bar{\sigma}(\sigma + 2|\partial_x \bar{\sigma}_{\text{Dup}}|_{\infty}) + r\). We set \(\eta = \frac{\sigma^2}{4 K \lambda_d}\), so that

\[
\frac{1}{2} \frac{d}{dt} |p_e^-|^2_d - (Q p_e^-, p_e^-)_d = \left( \frac{K^2 \lambda_d}{\sigma^2} \right) |p_e^-|^2_d \leq (Q p_e^-, p_e^+)_d - \frac{\sigma^2}{4 \lambda_d} |\partial_x p_e^-|^2_d.
\]

(27)

We also check that \((Q p_e^-, p_e^+)_d \leq 0\). Indeed, as for \(i \in \{1, \ldots, d\}\), \(p_{e,i}^- p_{e,i}^+ = 0\) and \(q_{ij} \geq 0\) for \(j \neq i\),

\[
(Q p_e^-, p_e^+)_d = \int_{\mathbb{R}} \sum_{i=1}^d p_{e,i}^- \left( \sum_{j=1}^d q_{ij} p_{e,j}^- \right) dx = \int_{\mathbb{R}} \sum_{i \neq j} q_{ij} p_{e,j}^- p_{e,i}^+ dx \leq 0,
\]

so the r.h.s. of (27) is nonpositive. Moreover, \((Q p_e^-, p_e^-)_d \leq d \bar{q}|p_e^-|^2_d\), so we have the inequality

\[
\frac{1}{2} \frac{d}{dt} |p_e^-|^2_d - \left( \frac{d \bar{q} + \frac{K^2 \lambda_d}{\sigma^2} \lambda_1}{\lambda_1} \right) |p_e^-|^2_d \leq 0.
\]

We thus obtain that the function \(t \to \exp \left( -2 \left( \frac{d \bar{q} + \frac{K^2 \lambda_d}{\sigma^2} \lambda_1}{\lambda_1} \right) t \right) |p_e^-(t)|^2_d\) is nonincreasing. As \(p_e^-(0) = 0\), we can conclude that \(p_e^- \equiv 0\).

We now check that if \(p\) is the limit of a sequence \((p_k)_{k \geq 1}\), where \(p_k\) solves \(V_{\epsilon_k}(\mu)\) for \(k \geq 1\), and \((\epsilon_k)_{k \geq 1}\) a sequence decreasing to 0 as \(k \to \infty\), then \(p\) takes values in \(D\).

**Lemma 3.13.** Let \((p_k)_{k \geq 1}\) be a sequence such that for \(k \geq 1\), \(p_k\) is a solution to \(V_{\epsilon_k}(\mu)\), where \(\epsilon_k \to 0\). If \((p_k)_{k \geq 1}\) converges to \(\bar{p}\) in the sense

\[
p_k \to \bar{p} \text{ in } L^2([0, T]; H) \text{ weakly},
\]

\[
p_k \to \bar{p} \text{ in } L^\infty([0, T]; L) \text{ weakly-*},
\]

\[
p_k \to \bar{p} \text{ a.e. on } (0, T) \times \mathbb{R},
\]

then under Assumptions (B) and (H), \(\sum_{i=1}^d \bar{p}_i\) is the unique solution to \(L V(\mu_{X_0})\) and \(\bar{p}\) takes values in \(D\).

**Proof.** For \(k \geq 1\), we define \(u_k := \sum_{i=1}^d p_{k,i}\), and \(u := \sum_{i=1}^d \bar{p}_i\). Then \(u_k\) satisfies, for \(\psi \in C_c^\infty((0, T), \mathbb{R})\) and \(v \in H^1(\mathbb{R})\),

\[
0 = -\int_0^T \psi'(t)(v, u_k)_1 dt - \int_0^T \psi(t) r(\partial_x v, u_k)_1 dt + \int_0^T \psi(t) \frac{1}{2} \left( \partial_x v, \bar{\sigma}_{\text{Dup}}^2 \partial_x u_k \right)_1 dt + \int_0^T \psi(t) \left( \partial_x v, \frac{1}{2} \bar{\sigma}_{\text{Dup}}(\bar{\sigma}_{\text{Dup}} + 2 (\partial_x \bar{\sigma}_{\text{Dup}})) \right)_1 dt,
\]

\[
+ \int_0^T \psi(t) \left( \partial_x v, \frac{1}{2} \bar{\sigma}_{\text{Dup}}(\bar{\sigma}_{\text{Dup}} + 2 (\partial_x \bar{\sigma}_{\text{Dup}})) \right)_1 \frac{\sum_{i=1}^d \lambda_i p_{k,i}}{\epsilon_k \vee \left( \sum_{i=1}^d \lambda_i p_{k,i} \right)} dt,
\]

\[
0 = -\int_0^T \psi'(t)(v, u)_1 dt - \int_0^T \psi(t) r(\partial_x v, u)_1 dt + \int_0^T \psi(t) \frac{1}{2} \left( \partial_x v, \bar{\sigma}_{\text{Dup}}^2 \partial_x u \right)_1 dt + \int_0^T \psi(t) \left( \partial_x v, \frac{1}{2} \bar{\sigma}_{\text{Dup}}(\bar{\sigma}_{\text{Dup}} + 2 (\partial_x \bar{\sigma}_{\text{Dup}})) \right)_1 dt,
\]

\[
+ \int_0^T \psi(t) \left( \partial_x v, \frac{1}{2} \bar{\sigma}_{\text{Dup}}(\bar{\sigma}_{\text{Dup}} + 2 (\partial_x \bar{\sigma}_{\text{Dup}})) \right)_1 \frac{\sum_{i=1}^d \lambda_i p_{k,i}}{\epsilon_k \vee \left( \sum_{i=1}^d \lambda_i p_{k,i} \right)} dt,
\]
as for \( i \in \{1, \ldots, d\} \), \( \sum_{j=1}^d a_{ij} = 0 \) and \( \sum_{j=1}^d A_{e,ij} = \frac{1}{2} \). As \( p_k \to \tilde{p} \) in \( L^2([0,T]; H) \) weakly, the terms on the r.h.s. of the first line converge to \(- \int_0^T \psi'(t)(v,u)_1 dt - \int_0^T \psi(t)\sigma(u)_1 dt + \int_0^T \psi(t)\frac{1}{2}(\partial_x v, \tilde{\sigma}^2_{D_u p} \partial_x u)_1 dt \). It is sufficient to show that the term on the r.h.s. of the second line converges to \( \int_0^T \psi(t)(\partial_x v, \frac{1}{2}\tilde{\sigma}_{D_u p}(\tilde{\sigma}_{D_u p} + 2(\partial_x \tilde{\sigma}_{D_u p}))u)_1 dt \). For a.e. \((t,x) \in [0,T] \times \mathbb{R}, \) we have that \( u_k(t,x) \to u(t,x) \). Let us remark that by Proposition 3.12, for \( k \geq 1 \), and \( 1 \leq i \leq d, \) \( p_{ki} \geq 0 \), so \( \tilde{p} \) and \( u \) are nonnegative. If \( u(t,x) > 0 \), then \( \sum_{i=1}^d \frac{\lambda_{ip_{ki}}}{\epsilon_i \sqrt{\sum_{j=1}^d \lambda_{jp_{ki}}}}(t,x) \to 1 \). If \( u(t,x) = 0 \), \( \sum_{i=1}^d \frac{\lambda_{ip_{ki}}}{\epsilon_i \sqrt{\sum_{j=1}^d \lambda_{jp_{ki}}}} u_k(t,x) \to 0 \) as \( 0 \leq \sum_{i=1}^d \frac{\lambda_{ip_{ki}}}{\epsilon_i \sqrt{\sum_{j=1}^d \lambda_{jp_{ki}}}} \leq 1 \). Let us use the following decomposition, \( u_k = 1_{\{u>0\}} u_k + 1_{\{u=0\}} u_k \). We first study the limit, as \( k \to \infty \), of

\[
I_1(k) := \int_0^T \psi(t) \left( \frac{1}{2} \tilde{\sigma}_{D_u p}(\tilde{\sigma}_{D_u p} + 2(\partial_x \tilde{\sigma}_{D_u p})) \right) \frac{\sum_{i=1}^d \lambda_{ip_{ki}}}{\epsilon_i \sqrt{\sum_{j=1}^d \lambda_{jp_{ki}}}} 1_{\{u>0\}} \partial_x v, u_k \right) dt.
\]

The function \( u_k \) converges weakly to \( u \) in \( L^2([0,T], L^2(\mathbb{R})) \). Moreover, the function

\[
\frac{1}{2} \tilde{\sigma}_{D_u p}(\tilde{\sigma}_{D_u p} + 2(\partial_x \tilde{\sigma}_{D_u p})) \frac{\sum_{i=1}^d \lambda_{ip_{ki}}}{\epsilon_i \sqrt{\sum_{j=1}^d \lambda_{jp_{ki}}}} 1_{\{u>0\}} \partial_x v
\]

converges strongly to \( \frac{1}{2} \tilde{\sigma}_{D_u p}(\tilde{\sigma}_{D_u p} + 2(\partial_x \tilde{\sigma}_{D_u p})) 1_{\{u>0\}} \partial_x v, \) in \( L^2([0,T], L^2(\mathbb{R})) \). Indeed, the convergence is a.e. on \([0,T] \times \mathbb{R}, \)

\[
\left| \frac{1}{2} \tilde{\sigma}_{D_u p}(\tilde{\sigma}_{D_u p} + 2(\partial_x \tilde{\sigma}_{D_u p})) \frac{\sum_{i=1}^d \lambda_{ip_{ki}}}{\epsilon_i \sqrt{\sum_{j=1}^d \lambda_{jp_{ki}}}} 1_{\{u>0\}} \partial_x v \right| \leq \frac{1}{2} \tilde{\sigma}(\tilde{\sigma} + 2||\partial_x \tilde{\sigma}||_{\infty})||\partial_x v|| \in L^2([0,T], L^2(\mathbb{R}))
\]

and we conclude by dominated convergence. Therefore, \( I_1(k) \) converges as \( k \to \infty, \) to

\[
\int_0^T \psi(t) \left( \frac{1}{2} \tilde{\sigma}_{D_u p}(\tilde{\sigma}_{D_u p} + 2(\partial_x \tilde{\sigma}_{D_u p})) \partial_x v, 1_{\{u>0\}} u \right)_1 dt = \int_0^T \psi(t) \left( \frac{1}{2} \tilde{\sigma}_{D_u p}(\tilde{\sigma}_{D_u p} + 2(\partial_x \tilde{\sigma}_{D_u p})) \partial_x v, u \right)_1 dt,
\]

because \( u \geq 0 \). We then study the term

\[
I_2(k) := \int_0^T \psi(t) \left( \frac{1}{2} \tilde{\sigma}_{D_u p}(\tilde{\sigma}_{D_u p} + 2(\partial_x \tilde{\sigma}_{D_u p})) \frac{\sum_{i=1}^d \lambda_{ip_{ki}}}{\epsilon_i \sqrt{\sum_{j=1}^d \lambda_{jp_{ki}}}} 1_{\{u=0\}} \partial_x v, u_k \right) dt.
\]

We note that \( |I_2(k)| \leq \frac{1}{2} \tilde{\sigma}(\tilde{\sigma} + 2||\partial_x \tilde{\sigma}||_{\infty}) \int_0^T |\psi(t)| ||\partial_x v||_1 ||u_k||_1 \) and the r.h.s. converges to 0 as \( k \to \infty, \) as \( u_k \to u \) weakly in \( L^2([0,T], L) \). With the same arguments as the proof of Proposition 3.11, \( u \in C([0,T], L^1(\mathbb{R})) \) and \( u = \mu_{X_0} \) so that \( u \) solves \( LV(\mu_{X_0}), \) which by Proposition 3.2, has a unique solution. Moreover, by Lemma 3.1, \( u > 0 \) a.e. on \([0,T] \times \mathbb{R}. \) Finally, as \( \tilde{p} \) is nonnegative, \( \tilde{p} \) takes values in \( D \) a.e. on \([0,T] \times \mathbb{R}. \)

We can now conclude the proof of Theorem 3.3. Using the same computations as those to obtain Lemma 3.10, it is easy to check that the family \((p_k)_{k \geq 0}, \) where \( p_k \) is a solution to \( V_{\epsilon}(\mu), \) satisfies energy estimates uniform in \( \epsilon. \) Using (21), we also obtain that the family \((\frac{dp}{dt})_{k \geq 0} \) is bounded in \( L^2([0,T], H^\prime). \)
Similar to the proof of Proposition 3.11, there exists a subsequence \((p_{e_k})_{k \geq 1}\) converging to a function \(p \geq 0\), weakly in \(L^2([0, T], H)\) and a.e. on \([0, T] \times \mathbb{R}\), as \(e_k \to 0\). For notational simplicity, we redefine for \(k \geq 1\), \(p_k := p_{e_k}\). To show that for \(v \in H\) and \(\psi \in C^1([0, T], \mathbb{R})\) with \(\psi(T) = 0\),

\[
\int_0^T \left( \psi(t) \partial_x v, A_{e_k}(p_k) \sigma_{Dup}^2 \partial_x p_k \right) dt \to \int_0^T \left( \psi(t) \partial_x v, A(p) \sigma_{Dup}^2 \partial_x p \right) dt,
\]

using the same computation as in Inequality (23), it is sufficient to check that \(A_{e_k}(p_k) \to A(p)\) a.e in \([0, T] \times \mathbb{R}\). By Lemma 3.13, for a.e. \((t, x) \in [0, T] \times \mathbb{R}\), \(p_k(t, x) \to p(t, x) > 0\), so we have that for \(k\) big enough, \(A_{e_k}(p_k(t, x)) = A(p_k(t, x))\) and the convergence holds true. Using similar arguments, we also obtain that

\[
\int_0^T \left( \psi(t) \partial_x v, \frac{1}{2} R_{e_k}(p_k) \sigma_{Dup} \left( \sigma_{Dup} + \partial_x \sigma_{Dup} \right) \Lambda p_k \right) dt
\]

\[\to \int_0^T \left( \psi(t) \partial_x v, \frac{1}{2} R(p) \sigma_{Dup} \left( \sigma_{Dup} + \partial_x \sigma_{Dup} \right) \Lambda p \right) dt.
\]

As \(p_k \to p\) weakly in \(L^2([0, T], L)\), we finally obtain the equality

\[
\int_0^T \psi(t)(Qv + r \partial_x v, p(t)) dt = -\int_0^T (\psi'(t)v, p)_d dt - (v, p_0)_d \psi(0) + \int_0^T (\psi(t) \partial_x v, A(p) \sigma_{Dup}^2 \partial_x p) dt
\]

\[+ \int_0^T \left( \psi(t) \partial_x v, \frac{1}{2} R(p) \sigma_{Dup} \left( \sigma_{Dup} + \partial_x \sigma_{Dup} \right) \Lambda p \right) dt,
\]

and we conclude the proof with the same arguments as in the proof of Proposition 3.11 to obtain existence of a solution to \(V_{L^2}(\mu)\) and the fact that \(p\) is a.e. equal to a function that belongs to \(C([0, T], L)\). By Lemma 3.13, \(\sum_{i=1}^d p_i\) solves \(LV(\mu_{X_0})\), which has a unique solution by Proposition 3.2 and this concludes the proof.

### 3.3 Existence of a solution to \(V(\mu)\)

We first introduce a lemma that will be useful later in the proof.

**Lemma 3.14.** Let \(\gamma \geq 0\) and let \(\phi\) be a nonnegative measurable function, s.t. \(\forall t \in (0, T]\), \(\int_t^\infty \phi^2(s) ds \leq \frac{\gamma}{\sqrt{t}}\). Then for \(t \in (0, T]\), \(\int_0^t \phi(s) ds \leq \frac{\sqrt{\gamma t^\frac{3}{2}}}{2^{\frac{3}{2}} - 1}\).

**Proof.** By monotone convergence and the Cauchy–Schwarz inequality, we obtain for \(t \in (0, T]\),

\[
\int_0^t \phi(s) ds = \sum_{k=0}^{\infty} \int_{t^2^{-k}}^{t^2^{-k+1}} \phi(s) ds \leq \sum_{k=0}^{\infty} \sqrt{t^2^{-k+1}} \int_{t^2^{-k+1}}^{t^2^{-k}} \phi^2(s) ds
\]

\[\leq \sum_{k=0}^{\infty} \sqrt{t^2^{-k+1}} \int_{t^2^{-k+1}}^{t} \phi^2(s) ds \leq \sqrt{t} \sum_{k=0}^{\infty} 2^{-k^{\frac{1}{2}}} \sqrt{\frac{\gamma}{2^{-k^\frac{1}{2}}} \sqrt{t}} = \frac{\sqrt{\gamma t^\frac{3}{2}}}{2^{\frac{3}{2}} - 1}.
\]

\(\square\)

We now prove Theorem 2.4. For \(\sigma > 0\), we denote by \(h_\sigma\) the density of a centered normal law with variance \(\sigma^2\). Now let \((\sigma_k)_{k \geq 0}\) be a sequence decreasing to 0 as \(k \to \infty\). In order to rely on Theorem 3.3,
for $1 \leq i \leq d$, we approximate $\mu_i$ by a sequence of measures $(\mu_{i,\sigma_k})_{k \geq 1}$, weakly converging to $\mu_i$, and that have densities in $L^2(\mathbb{R})$. To do so, we apply a convolution product on $\mu_i$ by setting $\mu_{i,\sigma_k} := \mu_i \ast h_{\sigma_k}$. The measure $\mu_{i,\sigma_k}$ is absolutely continuous with respect to the Lebesgue measure, with the density

$$
\mu_{i,\sigma_k}(x) := \frac{1}{2\pi \sigma_k^2} \int_\mathbb{R} e^{-\frac{(x-y)^2}{2\sigma_k^2}} d\mu_i(y).
$$

With Jensen’s inequality and Fubini’s theorem, we check that $\mu_{i,\sigma_k} \in L^2(\mathbb{R})$. Indeed,

$$
\int_{\mathbb{R}} \mu_{i,\sigma_k}^2(x) dx = \int_{\mathbb{R}} \left( \frac{1}{2\pi \sigma_k^2} \int_\mathbb{R} e^{-\frac{(x-y)^2}{2\sigma_k^2}} d\mu_i(y) \right)^2 dx \leq \frac{1}{2\pi \sigma_k^2} \int_\mathbb{R} \int_\mathbb{R} e^{-\frac{(x-y)^2}{2\sigma_k^2}} dx d\mu_i(y) = \frac{1}{2\pi \sigma_k^2}.
$$

For $k \geq 1$, let us define $p_{0,\sigma_k} := (\mu_{1,\sigma_k}, \ldots, \mu_{d,\sigma_k}) \in L$ and let us denote by $p_k$ a solution to the variational formulation $V_{L^2}(p_{0,\sigma_k})$, which exists as an application of Theorem 3.3. We compute energy estimates on $p_k$. As $p_{k,i}$ is nonnegative for $1 \leq i \leq d$ and $\sum_{i=1}^d p_{k,i}$ is the solution to $LV(\sum_{i=1}^d p_{0,i}) \ast h_{\sigma_k}$, we have by Proposition 3.2 that for a.e. $t \in (0, T)$, $|p_k(t)|_{d}^2 \leq \| \sum_{i=1}^d p_{k,i}(t) \|_{L^2}^2 \leq \frac{\zeta}{\sqrt{t}}$, so that

$$
\int_0^T |p_k(t)|_{d}^2 dt \leq 2 \zeta \sqrt{T}.
$$

Using $\Pi p_k$ as a test function in (11) and the fact that $\Pi A$ is uniformly coercive on $D$ with coefficient $\kappa$, (17) still holds when we replace $p_{0}^m$ by $p_e$ and we obtain after integration over time that

$$
\forall t \in (0, T), \int_0^T \left| \partial_s p_k(s) \right|_{d}^2 ds \leq \frac{l_{\max}(\Pi)}{\kappa \sigma^2} \frac{\zeta}{\sqrt{t}} e^{\frac{2}{\kappa \sigma^2}} \left( b + \frac{c^2}{2 \kappa \sigma^2} \right)^{\frac{1}{2}}.
$$

(28)

Let us remark that the estimates $\int_0^T |p_k(t)|_{d}^2 dt$ and $\int_0^T \left| \partial_s p_k(t) \right|_{d}^2 ds$, for $t \in (0, T)$ have bounds that are independent from the choice of $\sigma_k$. Then for a sequence $(s_n)_{n \geq 1}$ with values in $(0, T)$ and decreasing to 0 as $n \to \infty$, there exists a function $p$ defined a.e. on $(0, T) \times \mathbb{R}$ such that for each $n \geq 1$, there exists a limit $p$ and a subsequence called again $p_k$ converging weakly to $p$ in $L^2((0, T), L)$ and weakly-* in $L^1([s_n, T], L)$. Similar to the proof of Proposition 3.11, we can also suppose, modulo the extraction of a subsequence that $p_k \to p$ a.e. on $[s_n, T] \times \mathbb{R}$ as $k \to \infty$. By diagonal extraction, we obtain a subsequence, called again $(p_k)_{k \geq 0}$ such that

$$
p_k \to p \text{ in } L^2((0, T]; L) \text{ weakly},
p_k \to p \text{ in } L^2_{loc}((0, T]; H) \text{ weakly},
p_k \to p \text{ in } L^\infty_{loc}((0, T]; L) \text{ weakly-*},
p_k \to p \text{ a.e. in } (0, T] \times \mathbb{R}.
$$

Here we show that $p$ solves $V(\mu)$ under the assumption that $p$ takes values in $D$ a.e. on $(0, T) \times \mathbb{R}$. For $\psi \in C^1([0, T], \mathbb{R})$, such that $\psi(T) = 0$ and $\psi \in H$, as $p_k \to p$ weakly in $L^2((0, T), L)$, $\int_0^T (\psi'(s)v, p_k(s)) ds \to \int_0^T (\psi'(s)v, p(s)) ds$ and $\int_0^T \psi(t)(Qv, p_k) dt \to \int_0^T \psi(t)(Qv, p) dt$. As the sequence $(p_k)_{k \geq 0}$ converges to $p$ weakly in $L^2((0, T), L)$, $R$ is bounded and continuous on $D$, and $R(p_k) \to R(p)$ a.e. on $(0, T) \times \mathbb{R}$, we have that
\begin{align*}
\int_0^T \psi(t)(\partial_x v, R(p(t))\bar{\sigma}_{Dup}(\bar{\sigma}_{Dup} + \partial_x \bar{\sigma}_{Dup})\Lambda p(t))_d t \\
\quad \overset{k \to \infty}{\to} \frac{1}{2} \int_0^T \psi(t)(\partial_x v, R(p(t))\bar{\sigma}_{Dup}(\bar{\sigma}_{Dup} + \partial_x \bar{\sigma}_{Dup})\Lambda p(t))_d t.
\end{align*}

To check that $\int_0^T (\psi(s)\partial_x v, \bar{\sigma}_{Dup}^2 A(p_k)\partial_x p_k)_d s \overset{k \to \infty}{\to} \int_0^T (\psi(s)\partial_x v, \bar{\sigma}_{Dup}^2 A(p)\partial_x p)_d s$, we use the decomposition

\begin{equation}
\left| \int_0^T (\psi(s)\partial_x v, \bar{\sigma}_{Dup}^2 (A(p_k)\partial_x p_k - A(p)\partial_x p))_d s \right| \\
\leq \left| \int_0^T (\psi(s)\partial_x v, \bar{\sigma}_{Dup}^2 (A(p_k)\partial_x p_k - A(p)\partial_x p))_d s \right| \\
+ \left| \int_0^T (\psi(s)\partial_x v, \bar{\sigma}_{Dup}^2 A(p_k)\partial_x p_k)_d s \right| + \left| \int_0^T (\psi(s)\partial_x v, \bar{\sigma}_{Dup}^2 A(p)\partial_x p)_d s \right|.
\end{equation}

The first term in the r.h.s. of Inequality (29) goes to zero, with same arguments as those used to obtain the convergence in (24). Moreover, we have that for $P \in (p_k)_{k \geq 1} \cup \{p\}$, by Lemma 2.2,

\begin{equation*}
\left| \int_0^t \psi(s)(\partial_x v, \bar{\sigma}_{Dup}^2 A(P)\partial_x P)_d s \right| \leq \frac{d}{2} \left( 1 + \frac{\lambda_d}{\lambda_1} \right) \bar{\sigma}^2 ||\psi||_{\infty} ||\partial_x v||_d \int_0^t |\partial_x P|_d d s.
\end{equation*}

Let us define $Y := \frac{1}{21/4-1} \left( \frac{l_{\max}(\Omega) \gamma^2}{\kappa^2} e^{\min(||\mu||_1(\kappa + c^2 \kappa^2))} \right)^{1/2}$. By Lemma 3.14 and (28), we have that for $t \in (0, T)$ and $k \geq 1$, $\int_0^t |\partial_x P(s)|_d d s \leq Y t^{1/4}$, so the two last terms of the r.h.s. of Inequality (29), can be made arbitrarily small, uniformly in $k$, for $t$ small enough. This is enough to obtain that the term in the l.h.s. of Inequality (29) converges to 0 as $k \to \infty$. Finally, we also have that $(v, p_0, \sigma_k)_d \to \sum_{i=1}^d \int_{\mathbb{R}} v_i(x)\mu_i(dx)$ as $k \to \infty$. Gathering all the convergence results we get that

\begin{align*}
\int_0^T \psi(t)(Qv + r\partial_x v, p(t))_d t = & - \int_0^T (\psi'(t)v, p)_d t - \psi(0) \sum_{i=1}^d \int_{\mathbb{R}} v_i(x)\mu_i(dx) \\
& + \int_0^T (\psi(t)\partial_x v, A(p)\bar{\sigma}_{Dup}^2 \partial_x p)_d t \\
& + \int_0^T (\psi(t)\partial_x v, \frac{1}{2} R(p)\bar{\sigma}_{Dup}(\bar{\sigma}_{Dup} + \partial_x \bar{\sigma}_{Dup})\Lambda p)_d t.
\end{align*}

and this is enough to obtain (11) in the sense of distributions on $(0, T)$. As it is easy to check that $s \to (v, p(s))_d$ belongs to $H^1((t, T))$ for $t \in (0, T)$, we also have the following integration by parts formula:

\begin{align*}
\int_t^T \psi(t)(Qv + r\partial_x v, p(t))_d t = & - \int_t^T (\psi'(t)v, p)_d t - (v, p(t))_d \psi(t) + \int_t^T (\psi(t)\partial_x v, A(p)\bar{\sigma}_{Dup}^2 \partial_x p)_d t \\
& + \int_t^T (\psi(t)\partial_x v, \frac{1}{2} R(p)\bar{\sigma}_{Dup}(\bar{\sigma}_{Dup} + \partial_x \bar{\sigma}_{Dup})\Lambda p)_d t.
\end{align*}
Letting $t \to 0$, all the integrals in the previous equality converge as the functions $s \to |p(s)|_d$, $s \to |\partial_x p|_d$ belong to $L^1((0, T], \mathbb{R})$, and the functions $\psi, \psi', R, A, \tilde{\sigma}$ are bounded. Comparing with Equality (30), we obtain that

$$\lim_{t \to 0} (\psi, \mu(t))_d \geq \sum_{i=1}^d \int_{\mathbb{R}} v_i(x) \mu_i(dx).$$

Moreover, with the same arguments as at the end of the proof of Proposition 3.11, we have that $\mu \in C((0, T], L)$, and this concludes the proof of Theorem 2.4.

We now show that $\mu$ takes values in $D$ a.e. on $(0, T] \times \mathbb{R}$. To do so, as $\mu$ is the limit of the nonnegative sequence $(p_k)_{k \geq 1}$, $p \geq 0$, so it is sufficient to show that $u := \sum_{i=1}^d p_i$ is a solution to $LV(\mu_{x_0})$ and conclude by Lemma 3.1. For $\psi \in C^1([0, T], \mathbb{R})$, and $\mu \in H^1(\mathbb{R})$, as $u_k := \sum_{i=1}^d p_{k,i} \to u$ in $L^2((0, T], L^2(\mathbb{R}))$ weakly, we have that $\int_0^T \psi'(t)(v, u_{\mu_k})_1 dt \to \int_0^T \psi'(t)(v, u)_1 dt$, and

$$\int_0^T \psi(t)(\partial_x v, (-r + \frac{1}{2} \tilde{\sigma}^2 + \tilde{\sigma} \tilde{\sigma}_D) u_k)_1 dt \to \int_0^T \psi(t)(\partial_x v, (-r + \frac{1}{2} \tilde{\sigma}^2 + \tilde{\sigma} \tilde{\sigma}_D) u)_1 dt.$$

To prove that $\int_0^T \psi(t)(\partial_x v, \tilde{\sigma}^2 \partial_x u_k) dt \to \int_0^T \psi(t)(\partial_x v, \tilde{\sigma}^2 \partial_x u) dt$, we use the decomposition

$$\left| \int_0^T \psi(s)(\partial_x v, \tilde{\sigma}^2 \partial_x u_k - \partial_x u)_1 ds \right| \leq \int_0^T \psi(s)(\partial_x v, \tilde{\sigma}^2 \partial_x u_k - \partial_x u)_1 ds + \int_0^t \psi(s)(\partial_x v, \tilde{\sigma}^2 \partial_x u_k)_1 ds + \int_0^t \psi(s)(\partial_x v, \tilde{\sigma}^2 \partial_x u)_1 ds$$

and the reasoning is the same as for the convergence in (29), as we have that

$$\forall t \in (0, T), \int_0^t |\partial_x u_k| ds = \int_0^t \left| \sum_{i=1}^d \partial_x p_{k,i}(s) \right| ds \leq \sqrt{d} \int_0^t |\partial_x p_k(s)|_d ds \leq \sqrt{d} Ty_1^{1/4}.$$ (32)

The initial condition is treated in the same way as for $p$. This is sufficient to prove that $u$ solves $LV(\mu_{x_0})$ and assert that $\mu$ takes values in $D$ by Lemma 3.1.

4 | PROOF OF THEOREM 2.5

To prove Theorem 2.5, we use Theorem 4.2, which is a generalization of Figalli (2008, Theorem 2.6) to make a link between the existence of a solution to a Fokker–Planck system and the existence of a solution to the corresponding martingale problem. Theorem 4.2 is proved in Appendix D. There exist several generalizations of Figalli (2008, Theorem 2.6), among whom we can mention Fournier and Hauray (2016), where the coefficients of the generator are no longer bounded but have linear
growth, and Zhang (2013), where the author deals with a partial integro differential equation with a Lévy generator.

### 4.1 | Generalization of Figalli (2008, Theorem 2.6)

Given bounded functions \((b_i)_{1 \leq i \leq d}, (a_i)_{1 \leq i \leq d}, (q_{ij})_{1 \leq i, j \leq d}\) defined on \([0, T] \times \mathbb{R}\) and a finite measure \(\mu_0\) on \(\mathbb{R} \times \mathcal{Y}\), we study the following PDS, where for \(1 \leq i \leq d\),

\[
\partial_t \mu_i + \partial_x (b_i \mu_i) - \frac{1}{2} \partial_{xx}^2 (a_i \mu_i) - \sum_{j=1}^{d} q_{ij} \mu_j = 0 \text{ in } (0, T) \times \mathbb{R}
\]

\[
\mu_i(0) = \mu_0(\cdot, \{i\}).
\]

**Definition 4.1.** A family of vectors of Borel measures \((\mu_1(t, \cdot), \ldots, \mu_d(t, \cdot))_{t \in [0, T]}\) is a solution to the PDS (33) and (34) if \(B := \sup_{t \in [0, T]} \max_{1 \leq i \leq d} \mu_i(t, \mathbb{R}) < \infty\), for any function \(\phi\) defined on \(\mathbb{R} \times \mathcal{Y}\) such that \(\forall i \in \{1, \ldots, d\}, \phi(\cdot, i) \in C^\infty_c(\mathbb{R})\),

\[
\frac{d}{dt} \int_{\mathbb{R}} \sum_{i=1}^{d} \phi(x, i) \mu_i(t, dx) = \sum_{i=1}^{d} \int_{\mathbb{R}} \left( b_i(t, x) \partial_x \phi(x, i) + \frac{1}{2} a_i(t, x) \partial_{xx}^2 \phi(x, i) \right) \mu_i(t, dx) + \sum_{i=1}^{d} \sum_{j=1}^{d} \left( \int_{\mathbb{R}} q_{ij}(t, x) \phi(x, i) \mu_j(t, dx) \right),
\]

in the distributional sense on \((0, T)\), and for \(1 \leq i \leq d\) and \(\psi \in C^2_b(\mathbb{R})\), the function \(t \to \int_{\mathbb{R}} \psi(x) \mu_i(t, dx)\) is continuous on \((0, T)\) and converges to \(\int_{\mathbb{R}} \psi(x) \mu_0(dx, \{i\})\) as \(t \to 0\).

We suppose that all the coefficients \(b_i, a_i, q_{ij}, 1 \leq i, j \leq d\), are uniformly bounded on \([0, T] \times \mathbb{R}\), that the coefficients \((a_i)_{1 \leq i \leq d}\) are nonnegative, that the coefficients \((q_{ij})_{1 \leq i, j \leq d}\) are nonnegative functions for \(i \neq j\), and \(q_{ii} = -\sum_{j \neq i} q_{ij}\). We introduce the SDE

\[
dX_t = b_Y(t, X_t) dt + \sqrt{a_Y(t, X_t)} dW_t,
\]

where \(Y_t\) is a stochastic process with values in \(\mathcal{Y}\), and that satisfies for \(j \neq Y_t\),

\[
\mathbb{P}(Y_{t+dt} = j| (X_s, Y_s)_{0 \leq s \leq t}) = q_{Y,j}(t, X_t) dt + O((dt)^2).
\]

We define \(E = \{(X, Y), X \in C([0, T], \mathbb{R}), Y \text{ càdlàg with values in } \mathcal{Y}\}\) endowed with the Skorokhod topology. For a probability measure \(m\) on \(\mathbb{R} \times \mathcal{Y}\), a probability measure \(\nu\) on \(E\) is a martingale solution to SDE (36) with initial condition \(m\) if under the probability \(\nu\), the canonical process \((X, Y)\) on \(E\) satisfies \((X_0, Y_0) \sim m\) and for any function \(\phi\) defined on \(\mathbb{R} \times \mathcal{Y}\) s.t. \(\forall i \in \{1, \ldots, d\}, \phi(\cdot, i) \in C^2_b(\mathbb{R})\), the process

\[
\phi(X_t, Y_t) - \phi(X_0, Y_0) - \int_0^t \left( \frac{1}{2} a_Y(s, X_s) \partial_{xx}^2 \phi(X_s, Y_s) + b_Y(s, X_s) \partial_x \phi(X_s, Y_s) + \sum_{i=1}^{d} q_{Y,i}(s, X_s) \phi(X_s, i) \right) ds
\]
is a \( v \)-martingale. Now we can state the generalization of Figalli (2008, Theorem 2.6), which will be used in the following section.

**Theorem 4.2.** Let \( (\mu_1(t, \cdot), \ldots, \mu_d(t, \cdot)) \in (0, T) \) be a solution to the PDS (33) and (34) where the initial condition \( \mu_0 \) is a probability measure on \( \mathbb{R} \times \mathcal{Y} \). Then SDE (36) with initial distribution \( \mu_0 \) has a martingale solution \( v \) which satisfies the following representation formula:

\[
\forall \psi \in C_c^\infty(\mathbb{R}), \quad \int_{\mathbb{R}} \psi(x)\mu_i(t, dx) = \int_E \psi(X_i)1_{\{Y_i = 1\}}dv(X, Y), \quad 1 \leq i \leq d.
\]

### 4.2 | Proof of Theorem 2.5

Let \( p \) be a solution to \( V(\mu) \) with the properties stated in Theorem 2.4. To show that \( (p_1(t, x)dx, \ldots, p_d(t, x)dx) \in (0, T) \) satisfies the variational formulation in the sense of distributions (35), we check that for \( 1 \leq i \leq d, \) \( \partial_x^2 \frac{\sum_{k=1}^d p_k}{\sum_{k=1}^d \lambda_k p_k} \lambda_i p_i \in H^1(\mathbb{R}) \), and \( \frac{1}{2} \partial_x^2 \left( \partial_x^2 \frac{\sum_{k=1}^d p_k}{\sum_{k=1}^d \lambda_k p_k} \lambda_i p_i \right) = \partial_x^2 \left( A(p)\partial_x p_i \right) + \frac{1}{2} \partial_x^2 \frac{\sum_{k=1}^d p_k}{\sum_{k=1}^d \lambda_k p_k} \lambda_i p_i \partial_x^2 \partial_x^2 \partial_x \lambda_j p_i \). As \( \partial_x^2 \in L^\infty([0, T], W^{1,\infty}(\mathbb{R})) \), it is sufficient to check that \( \frac{\sum_{k=1}^d p_k}{\sum_{k=1}^d \lambda_k p_k} \lambda_i p_i \in H^1(\mathbb{R}) \) and that \( \frac{1}{2} \partial_x^2 \left( \frac{\sum_{k=1}^d p_k}{\sum_{k=1}^d \lambda_k p_k} \lambda_i p_i \right) = (A(p)\partial_x p_i)(s, \cdot) \).

**Lemma 4.3.** Let \( p \) be a solution to \( V(\mu) \). Then, for \( 1 \leq i \leq d \) and a.e. \( s \in (0, T), \) \( \frac{\sum_{k=1}^d p_k}{\sum_{k=1}^d \lambda_k p_k} \lambda_i p_i (s, \cdot) \in H^1(\mathbb{R}) \) and \( \frac{1}{2} \partial_x^2 \left( \frac{\sum_{k=1}^d p_k}{\sum_{k=1}^d \lambda_k p_k} \lambda_i p_i \right)(s, \cdot) = (A(p)\partial_x p_i)(s, \cdot) \).

**Proof.** It is sufficient to show that for \( 1 \leq i, j \leq d \) and a.e. \( s \in (0, T), \) \( \frac{\sum_{k=1}^d p_k}{\sum_{k=1}^d \lambda_k p_k} \lambda_i p_i (s, \cdot) \in H^1(\mathbb{R}) \) and

\[
\partial_x \left( \frac{p_i p_j}{\sum_k \lambda_k p_k} \right)(s, \cdot) = \left( \frac{p_i \partial_x p_j + p_j \partial_x p_i}{\sum_k \lambda_k p_k} - \frac{p_i p_j}{\sum_k \lambda_k p_k} \right) \sum_k \lambda_k \partial_x p_k (s, \cdot),
\]

as we conclude by linearity. For a.e. \( s \in (0, T), \) \( p(s, \cdot) \in H \) and therefore \( \frac{p_i p_j}{\sum_k \lambda_k p_k} (s, \cdot) \) and

\[
\left( \frac{p_i \partial_x p_j + p_j \partial_x p_i}{\sum_k \lambda_k p_k} - \frac{p_i p_j}{\sum_k \lambda_k p_k} \right) \sum_k \lambda_k \partial_x p_k (s, \cdot)
\]

belong to \( L^2(\mathbb{R}) \), as \( \forall i \in \{1, \ldots, d\}, \) \( \frac{p_i}{\sum_k \lambda_k p_k} \in [0; \frac{1}{\lambda_1}], \) a.e. on \( (0, T) \times \mathbb{R} \). It is sufficient to show that for a compact set \( K \subset \mathbb{R}, \phi \in C_c^\infty(\mathbb{R}) \) with support included in \( K \), and a.e. \( s \in (0, T), \)

\[
\int_K \partial_x \phi(x) \left( \frac{p_i p_j}{\sum_k \lambda_k p_k} \right)(s, x)dx = -\int_K \phi(x) \left( \frac{p_i \partial_x p_j + p_j \partial_x p_i}{\sum_k \lambda_k p_k} - \frac{p_i p_j}{\sum_k \lambda_k p_k} \right) \sum_k \lambda_k \partial_x p_k (s, x)dx.
\]

Let \( (p_n)_{n \geq 1} \) be a regularizing sequence, where \( p_n \in C_c^\infty(\mathbb{R}) \), with a support included in \( (-\frac{1}{n}, \frac{1}{n}) \), \( \int_{\mathbb{R}} p_n = 1 \), and \( p_n \geq 0 \) for \( n \geq 1 \). We define \( p_{n,i}(s, \cdot) := p_n \ast p_i(s, \cdot) \). As for a.e \( (s, x) \in (0, T) \times \mathbb{R}, \sum_k p_k(s, x) > 0 \) by Lemma 3.1, and \( \lambda_i > 0 \) for \( 1 \leq i \leq d \), we have that that for \( n \geq 1, x \in \mathbb{R} \) and a.e \( s \in (0, T), \) \( \sum_{i=1}^d \lambda_i p_n,i(s, x) > 0 \). Then for a.e. \( s \in (0, T), \) we have the equality
\[
\int_K \partial_x \phi(x) \left( \frac{p_{n,i}p_{n,j}}{\sum_k \lambda_k p_{n,k}} \right) (s, x) dx \\
= - \int_K \phi(x) \left( \frac{p_{n,i} \partial_x p_{n,j} + p_{n,j} \partial_x p_{n,i}}{\sum_k \lambda_k p_{n,k}} - \frac{p_{n,i}p_{n,j}}{(\sum_k \lambda_k p_{n,k})^2} \sum_k \lambda_k \partial_x p_{n,k} \right) (s, x) dx.
\]

Moreover, for a.e. \( s \in (0, T) \), \((p_{n,i}(s, \cdot))_{n \geq 0}, i \in \{1, \ldots, d\} \) are sequences of nonnegative functions in \( C^\infty(\mathbb{R}) \cap H^1(\mathbb{R}) \) such that for \( 1 \leq i \leq d \), the following strong convergences hold, by Brezis (1983, Theorem 4.22):

\[
p_{n,i}(s, \cdot) \to p_i(s, \cdot) \text{ in } L^2(\mathbb{R}),
\]

\[
\partial_x p_{n,i}(s, \cdot) \to \partial_x p_i(s, \cdot) \text{ in } L^2(\mathbb{R}).
\]

For a.e. \( s \in (0, T) \), modulo the extraction of a subsequence (which can depend on \( s \)), we can assume that for \( 1 \leq l \leq d \) and a.e. \( x \in K \), \( p_{n,l}(s, x) \to p_l(s, x) \). By Lemma 3.1, as \( \sum_{i=1}^d p_i(s, x) > 0 \) for a.e. \( x \in \mathbb{R} \), we have \( \frac{p_{n,i}}{\sum_k \lambda_k p_{n,k}}(s, x) \to \frac{p_i}{\sum_k \lambda_k p_k} (s, x) \) for a.e. \( x \in K \). Then,

\[
\left\| \left( \frac{p_{n,i}p_{n,j}}{\sum_k \lambda_k p_{n,k}} - \frac{p_i p_j}{\sum_k \lambda_k p_k} \right) (s, \cdot) \right\|_{L^2(K)} \leq \frac{1}{\lambda_1} \left\| \left( p_{n,j} - p_j \right) (s, \cdot) \right\|_{L^2(K)}
\]

\[
+ \left\| \left( p_j \left( \frac{p_{n,i}}{\sum_k \lambda_k p_{n,k}} - \frac{p_i}{\sum_k \lambda_k p_k} \right) \right) (s, \cdot) \right\|_{L^2(K)}.
\]

The first term of the r.h.s. converges to 0 as \( n \to \infty \), as \( p_{n,j}(s, \cdot) \to p_j(s, \cdot) \) in \( L^2(\mathbb{R}) \), and the second term also converges to 0 by dominated convergence as \( \frac{p_{n,i}}{\sum_k \lambda_k p_{n,k}}(s, \cdot) \to \frac{p_i}{\sum_k \lambda_k p_k}(s, \cdot) \) a.e. on \( K \), and \( \forall i \in \{1, \ldots, d\}, \forall n \geq 1, \frac{p_{n,i}}{\sum_k \lambda_k p_{n,k}}(s, x) \in [0, 1] \). This ensures that

\[
\int_K \partial_x \phi(x) \left( \frac{p_{n,i}p_{n,j}}{\sum_k \lambda_k p_{n,k}} \right) (s, x) dx \to \int_K \partial_x \phi(x) \left( \frac{p_i p_j}{\sum_k \lambda_k p_k} \right) (s, x) dx,
\]

for a.e. \( s \in (0, T) \). With similar arguments, letting \( n \to \infty \) in the r.h.s. of (37), we have the convergence of the r.h.s. term of (37) to

\[
- \int_K \phi(x) \left( \frac{p_i \partial_x p_j + p_j \partial_x p_i}{\sum_k \lambda_k p_k} - \frac{p_i p_j}{(\sum_k \lambda_k p_k)^2} \sum_k \lambda_k \partial_x p_k \right) (s, x) dx,
\]

for a.e. \( s \in (0, T) \) and this concludes the proof. \( \square \)
Then by Theorem 4.2, as \(\lambda_i = f^2(i)\) for \(1 \leq i \leq d\), there exists a measure \(\nu\) under which \((X_0, Y_0) \sim \mu\)
and for any function \(\phi\) defined on \(\mathbb{R} \times \mathcal{Y}\) s.t. \(\forall i \in \{1, \ldots, d\}, \phi(\cdot, i) \in C^2_b(\mathbb{R})\), the function

\[
\phi(X_t, Y_t) - \phi(X_0, Y_0) - \int_0^t \frac{1}{2} \hat{\sigma}_{Dup}(s, X_s)f^2(Y_s) \sum_{k=1}^d \frac{p_k}{\hat{\lambda}_k p_k} (s, X_s) \partial_{XX}^2 \phi(X_s, Y_s) ds
\]

\[
- \int_0^t \left( r - \frac{1}{2} \hat{\sigma}_{Dup}(s, X_s)f^2(Y_s) \sum_{k=1}^d \frac{p_k}{\hat{\lambda}_k p_k} (s, X_s) \right) \partial_s \phi(X_s, Y_s) - \sum_{l=1}^d q_l(X_s) \phi(X_s, l) ds,
\]

is a \(\nu\)-martingale. Moreover, for \(g : \mathbb{R} \to \mathbb{R}\) a continuous bounded function and \(h : \mathcal{Y} \to \mathbb{R}\), we have that

\[
\mathbb{E}[h^2(Y_s)g(X_s)] = \sum_{i=1}^d \mathbb{E}[h^2(Y_s)1_{\{Y_s=y_i\}}g(X_s)] = \int_{\mathbb{R}} g(x) \sum_{i=1}^d h^2(y_i)p_i(t, x) dx.
\]

Taking \(h \equiv 1\) and \(h \equiv f\), we check that the time marginals of \(X\) are given by \(\sum_{i=1}^d p_i\) and that

\[
\mathbb{E}[f^2(Y_s)|X_s] = \sum_{i=1}^d \frac{\hat{\lambda}_k p_k}{\sum_{k=1}^d p_k} (s, X_s).\]

Therefore \(\nu\) is a solution to the martingale problem associated to SDE (2), and we obtain existence of a weak solution to SDE (2) by Kurtz (2011, Theorem 2.3). This solution has the same time marginals as the solution to SDE (4).

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APPENDIX A: PROOF OF LEMMA 3.4

Proof. Let $m \geq 1$ and $\varepsilon > 0$. As $F^m_{e,3}$ is obviously globally Lipschitz in $z$ uniformly in $t$, it is sufficient to show that the functions $K^m_{e,1}$ and $K^m_{e,2}$ are locally Lipschitz in $z$ uniformly in $t$ to prove Lemma 3.4. We only give the proof for $K^m_{e,2}$, as the arguments are similar for $K^m_{e,1}$. For $\Theta \in L^1(\mathbb{R})$, $1 \leq i, j \leq d$, let us define

$$g^\Theta_{ij}(t, z) := \int_{\mathbb{R}} \sigma^2_{Dup}(t, x)\Theta(x)M_{e,ij}\left(\sum_{k=1}^{m} z_k w_k(x)\right)^{+} dx, (t, z) \in [0, T] \times \mathbb{R}^m.$$ 

Let us show that $g^\Theta_{ij}$ is locally Lipschitz in $z$ uniformly in $t$ and conclude by linearity. Indeed, for $(t, z) \in [0, T] \times \mathbb{R}^m$, the coefficients of $K^m_{e,2}(t, z)$ are linear combinations of functions $g^\Theta_{ij}(t, z)$ and $\int_{\mathbb{R}} \sigma^2_{Dup}(t, x)\Theta(x)dx$, where $1 \leq i, j \leq d$ and $\Theta \in \{\partial \partial_x w_{ac} \partial_x w_{bd}, 1 \leq a, b \leq m, 1 \leq i, j \leq d\}$, which is a bounded subset of $L^1(\mathbb{R})$.

It is easy to check that for $1 \leq i, j \leq d$, the function $\rho \in (\mathbb{R}_+)^d \rightarrow M_{e,ij}(\rho)$ is locally Lipschitz, and so is the function $\rho \in \mathbb{R}^d \rightarrow M_{e,ij}(\rho^+)$, as the positive part function is 1-Lipschitz. Now, let $C$
be a compact subset of $\mathbb{R}^m$, $z, \tilde{z} \in C$, and for $x \in \mathbb{R}$, $\rho(x) = \sum_{k=1}^{m} z_k w_k(x)$, and $\tilde{\rho} = \sum_{k=1}^{m} \tilde{z}_k w_k(x)$. We define $\Gamma(\cdot)$ as the observation that if $\rho = \sum_{k=1}^{m} \tilde{z}_k w_k(x)$ achieves the coercivity property. The intuition behind the choice of $\Gamma$ is $\sum_{k=1}^{m} \tilde{z}_k w_k(x)$, which has nonzero coefficients only on the diagonal and in the $k$th row, and $\tilde{\rho}$, $\tilde{w}_k \in L^\infty(\mathbb{R})$ and the function $(x, z) \in \mathbb{R} \times C \rightarrow \sum_{k=1}^{m} z_k w_k(x) \in \mathbb{R}^d$ takes values for a.e. $x \in \mathbb{R}$ in a bounded subset of $\mathbb{R}^d$, so there exists an uniform bound $K < \infty$ s.t. $\forall z, \tilde{z} \in C$,

$$
\left\| M_{e,i} \left( \left( \sum_{k=1}^{m} z_k w_k(x) \right)^+ \right) - M_{e,i} \left( \left( \sum_{k=1}^{m} \tilde{z}_k w_k(x) \right)^+ \right) \right\|_\infty \leq K \|\tilde{z} - z\|_\infty,
$$

so that after integration, $|g_{ij}(z) - g_{ij}(\tilde{z})| \leq K \sigma^2 \|\Theta\|_{L^1} \|z - \tilde{z}\|_\infty$, and this concludes the proof. □

**APPENDIX B: ABOUT CONDITION (C)**

In Subsection B.1, we prove Proposition 2.3. In Subsection B.2, we give a necessary and sufficient condition for a diagonal matrix to satisfy Condition (C). We also give a numerical procedure to check if there exists a diagonal matrix that satisfies (C). Then, in Subsection B.3, we focus on the case $d = 3$, and give a simple necessary and sufficient condition for (C) to be satisfied. When $d = 3$, (C) is satisfied if and only if it is satisfied by a diagonal matrix. When $d \geq 4$, we do not know if this property still holds.

**B.1 Proof of Proposition 2.3**

We consider matrices $\Pi$ with the form $J_d + \epsilon \Gamma$, where $\Gamma \in S_d^{++}(\mathbb{R})$ and $\epsilon > 0$, and we show that if $\Gamma$ satisfies (C), then for $\epsilon$ small enough, $(J_d + \epsilon \Gamma)A$ achieves the coercivity property. The intuition behind the choice of $J_d$ is the observation that if $(p_1, \ldots, p_d)$ is a solution to $V(\mu)$ then $\sum_{i=1}^{d} p_i$ is a solution to Dupire’s PDE. This also translates into the algebraic property that $J_d M = 0$. We define $1 := (1, \ldots, 1) \in \mathbb{R}^d$.

**Proposition B.1.** If $\Gamma \in S_d^{++}(\mathbb{R})$ satisfies Condition (C), then there exists $z > 0$ such that

$$
\forall x \in 1^1, \forall \rho \in D, x^* \Gamma A(\rho) x \geq z x^* x. \quad (B.1)
$$

Moreover, for $\epsilon > 0$ small enough, the function $(J_d + \epsilon \Gamma)A$ is uniformly coercive on $D$.

*Proof.* For $\rho \in D$, we introduce the notation $\bar{\lambda}(\rho) := \sum_{i} \lambda_i \sum_{j} \frac{\rho_j}{\rho_i}$ and remark that $\bar{\lambda}(D) = [\lambda_1, \lambda_d]$. The matrix $A(\rho)$ rewrites as the convex combination $A(\rho) = \sum_{i=1}^{d} w_i(\rho) A_k(\rho)$ with the weight $w_i(\rho) = \frac{\lambda_i}{\sum_{j} \lambda_j \rho_j}$ of the matrix $A_k(\rho)$ which has nonzero coefficients only on the diagonal and the $k$th row, and is defined by

$$
(A_k(\rho))_{ij} = \frac{1}{2} \left( 1_{\{i=j\}} \frac{\lambda_i}{\bar{\lambda}(\rho)} + 1_{\{i=k\}} \left( 1 - \frac{\lambda_j}{\bar{\lambda}(\rho)} \right) \right), \quad 1 \leq i, j \leq d.
$$

We prove that for $1 \leq k \leq d$, there exists $z_k > 0$ such that

$$
\forall x \in 1^1, \forall \rho \in D, 2 x^* \rho \Gamma A_k(\rho) x \geq z_k x^* x, \quad (B.2)
$$

and we can set $z = \min_{1 \leq k \leq d} \frac{z_k}{2 \lambda_d} > 0$ to obtain (B.1). As we have the equality

$$
2 \bar{\lambda}(\rho) \Gamma A_k(\rho) = \left( \lambda_j (\Gamma_{ij} - \Gamma_{ik}) \right)_{1 \leq i, j \leq d} + \left( \bar{\lambda}(\rho) \Gamma_{ik} \right)_{1 \leq i, j \leq d},
$$
we have that for \( x \in \mathbb{I}^1 \), as \( \sum_{i=1}^{d} x_i = 0 \) and \( x_k = -\sum_{i \neq k} x_i \),

\[
2x^* \overline{\lambda}(\rho) \Gamma A_k(\rho)x = \sum_{i \neq k, j \neq k} \lambda_j (\Gamma_{ij} - \Gamma_{ik} - \Gamma_{kj} + \Gamma_{kk}) x_i x_j
\]

\[
= \sum_{i \neq k, j \neq k} \frac{\lambda_i + \lambda_j}{2} (\Gamma_{ij} - \Gamma_{ik} - \Gamma_{kj} + \Gamma_{kk}) x_i x_j.
\]  

(B.3)

As \( \Gamma^{(k)} \) is positive definite on \( e_k^\perp \), there exists \( \varepsilon_k > 0 \) such that \( \forall x \in \mathbb{I}^1, 2x^* \overline{\lambda}(\rho) \Gamma A_k(\rho)x \geq \varepsilon_k \sum_{i \neq k} x_i^2 \). By Cauchy–Schwarz inequality, \( x_i^2 = (\sum_{i \neq k} x_i)^2 \leq (d-1) \sum_{i \neq k} x_i^2 \), so \( \sum_{i \neq k} x_i^2 \geq \frac{1}{d} x^* x \) and we can set \( z_k = \frac{\varepsilon_k}{d} \) to satisfy (B.2).

Now, we show that for \( \varepsilon > 0 \) small enough, the function \( (J_d + e\Gamma)A \) is uniformly coercive on \( D \). For \( \rho \in D \), as \( A(\rho) = \frac{1}{2}(I_d + M(\rho)) \) and for \( 1 \leq j \leq d \), it is easy to check that \( \sum_{i=1}^{d} M_{ij}(\rho) = 0 \), we have that \( J_d M = 0 \) and \( (J_d + e\Gamma)A(\rho) = \frac{1}{2}J_d + e\Gamma A(\rho) \). For \( x \in \mathbb{R}^d \), we decompose \( x = u + v \) where \( u \in \mathbb{R}^1 \) and \( v \in \mathbb{I}^1 \). As \( J_d v = 0 \), we have

\[
x^* \left( \frac{1}{2}J_d + e\Gamma A(\rho) \right) x = u^* \left( \frac{1}{2}J_d + e\Gamma A(\rho) \right) u + e v^* \Gamma A(\rho) v + e u^* \Gamma A(\rho) u + e v^* \Gamma A(\rho) u.
\]

We will use Young’s inequality: \( \forall \eta > 0, \forall a, b \in \mathbb{R}, ab \leq \eta a^2 + \frac{1}{4\eta} b^2 \). For \( 1 \leq i, j \leq d \), by Young’s inequality and Lemma 2.2, \( \langle \Gamma A(\rho) \rangle_{ij} = \sum_{k=1}^{d} \Gamma_{ik} A(\rho)_{kj} \leq \frac{d}{2} ||\Gamma||_\infty (1 + \frac{\lambda_d}{\lambda_1}) \), so for \( a, b \in \mathbb{R} \), and \( \eta > 0 \), \( \langle \Gamma A(\rho) \rangle_{ij} ab \leq -\frac{d}{2} ||\Gamma||_\infty (1 + \frac{\lambda_d}{\lambda_1}) (\eta a^2 + \frac{1}{4\eta} b^2) \). Then for \( \eta > 0 \),

\[
(v^* \Gamma A(\rho) u) \wedge (u^* \Gamma A(\rho) v) \geq -\frac{d^2}{2} ||\Gamma||_\infty (1 + \frac{\lambda_d}{\lambda_1}) (\eta u^* u + \frac{1}{4\eta} u^* u).
\]

Moreover, we have that \( u^* \Gamma A(\rho) u \geq -\frac{d^2}{2} ||\Gamma||_\infty (1 + \frac{\lambda_d}{\lambda_1}) u^* u \). Using (B.1) and \( J_d u = du \), we get that for \( \eta > 0 \),

\[
x^* \left( \frac{1}{2}J_d + e\Gamma A(\rho) \right) x \geq \left( \frac{d}{2} - e \frac{d^2}{2} ||\Gamma||_\infty (1 + \frac{\lambda_d}{\lambda_1}) \right) u^* u
\]

\[
+ e \left( z - d^2 ||\Gamma||_\infty (1 + \frac{\lambda_d}{\lambda_1}) \eta \right) v^* v.
\]

For \( 0 < e < \left( d ||\Gamma||_\infty (1 + \frac{\lambda_d}{\lambda_1}) \right)^{-1} \), we check that

\[
\eta_1 := \left( \frac{2}{e d ||\Gamma||_\infty (1 + \frac{\lambda_d}{\lambda_1})} - 2 \right)^{-1} < \frac{z}{d^2 ||\Gamma||_\infty (1 + \frac{\lambda_d}{\lambda_1})} =: \eta_2,
\]

and with the choice \( \eta \in (\eta_1, \eta_2) \), we see that the function \( (J_d + e\Gamma)A \) is uniformly coercive on \( D \).
We can now prove Proposition 2.3. If a matrix $\Gamma \in S^+_d(\mathbb{R})$ satisfies (C), then by Proposition B.1, for $\epsilon > 0$ small enough, the function $(J_d + \epsilon \Gamma)A$ is uniformly coercive on $D$. Moreover, $(J_d + \epsilon \Gamma) \in S^+_d(\mathbb{R})$ as $J_d \in S^+_d(\mathbb{R})$.

Conversely, if $\Pi \in S^+_d(\mathbb{R})$ is such that the function $\Pi A$ is uniformly coercive on $D$ with a coefficient $c > 0$, with the same computation as for Equality (B.3), we have that for $x \in \mathbb{1}^\perp$, $1 \leq k \leq d$, and as $e_k \in D$, we have that

$$
\sum_{i \neq k, j \neq k} \frac{\lambda_i + \lambda_j}{2} (\Pi_{ij} - \Pi_{ik} - \Pi_{jk} + \Pi_{kk}) x_i x_j
$$

$$
= 2x^* \bar{\lambda}(e_k) \Pi A_k(e_k) x = 2x^* \bar{\lambda}(e_k) \Pi A(e_k)x \geq 2\lambda_1 c \sum_{i=1}^d x_i^2,
$$

so $\Pi$ satisfies (C). Finally, it is obvious that if $\Pi A$ is uniformly coercive, then $\Pi A$ is uniformly coercive with a coefficient $c \leq \frac{I_{min}(\Pi)}{2}$.

**B.2 The diagonal case**

For $k \geq 1$, $\delta := (\delta_1, \ldots, \delta_k) \in \mathbb{R}^k$, let us denote by $Diag(\delta) \in \mathcal{M}_k(\mathbb{R})$ the diagonal matrix with coefficients $\delta_1, \ldots, \delta_k$.

**Proposition B.2.** For $d \geq 2$ and $\alpha := (\alpha_1, \ldots, \alpha_d) \in (\mathbb{R}^+)^d$, $Diag(\alpha)$ satisfies Condition (C) if and only if

$$
\forall 1 \leq k \leq d, \quad \frac{2}{\alpha_k} + \sum_{i \neq k} \frac{1}{\alpha_i} > \sqrt{\sum_{i \neq k} \frac{\lambda_i}{\alpha_i} \sum_{j \neq k} \frac{1}{\lambda_j \alpha_i}}.
$$

(B.4)

**Proof.** For $1 \leq k \leq d$, the symmetric matrix $D^{(k)}$ with coefficients

$$
D^{(k)}_{ij} = \frac{\lambda_i + \lambda_j}{2} (\alpha_{k1_{i=j}} + \alpha_k - \alpha_k 1_{i=k} - \alpha_k 1_{j=k})
$$

for $1 \leq i, j \leq d$, is positive definite on $e_k^\perp$ if and only if the matrix $\bar{D}^{(k)}$ defined as $D^{(k)}$ with its $k$th row and $k$th column removed, is positive definite on $\mathbb{R}^{d-1}$. Here we only show how to deal with the case $k = d$, but the same arguments can be used for the indices $1 \leq k \leq d - 1$. The matrix $\bar{D}^{(d)}$ has coefficients

$$
\bar{D}^{(d)}_{ij} = \frac{\lambda_i + \lambda_j}{2} (\alpha_{i1_{i=j}} + \alpha_d),
$$

for $1 \leq i, j \leq d - 1$. We define $\Delta := Diag(\sqrt{\frac{\lambda_i}{\alpha_i}})_{1 \leq i \leq d-1}$. The matrix $\bar{D}^{(d)}$ rewrites

$$
\bar{D}^{(d)} = \Delta \Delta + \frac{\alpha_d}{2} (\lambda_i + \lambda_j)_{1 \leq i, j \leq d-1} = \Delta \left( I_{d-1} + \frac{\alpha_d}{2} \left( \frac{\lambda_i + \lambda_j}{\sqrt{\lambda_i \alpha_i \sqrt{\lambda_j \alpha_j}}} \right)_{1 \leq i, j \leq d-1} \right) \Delta
$$

$$
= \Delta \left( I_{d-1} + \frac{\alpha_d}{2} (ab + ba^*) \right) \Delta,
$$
where \( a = \left( \sqrt{\frac{1}{\alpha_i}} \right)_{1 \leq i \leq d-1} \) and \( b = \left( \frac{1}{\sqrt{\lambda_i\alpha_i}} \right)_{1 \leq i \leq d-1} \). The matrix \( \tilde{D}^{(d)} \) is positive definite if and only if the matrix \( (I_{d-1} + \frac{ab^*}{2}(ab^* + ba^*)) \), has positive eigenvalues. The columns of the matrix \( ab^* + ba^* \) are linear combinations of \( a \) and \( b \). If \( a \) and \( b \) are not colinear (resp., colinear), then the matrix \( ab^* + ba^* \) has eigenvalues 0 with multiplicity \( d-2 \) and \( \sum_{i=1}^{d-1} a_i b_i - \sqrt{\sum_{i=1}^{d-1} a_i^2 \sqrt{\sum_{i=1}^{d-1} b_i^2}} < 0 \) for the eigenvector \( a - \frac{\sqrt{\sum_{i=1}^{d-1} a_i^2}}{\sqrt{\sum_{i=1}^{d-1} b_i^2}} b \) (resp., 0 with multiplicity \( d-1 \)), and \( \sum_{i=1}^{d-1} a_i b_i + \sqrt{\sum_{i=1}^{d-1} a_i^2 \sqrt{\sum_{i=1}^{d-1} b_i^2}} > 0 \) for the eigenvector \( a + \frac{\sqrt{\sum_{i=1}^{d-1} a_i^2}}{\sqrt{\sum_{i=1}^{d-1} b_i^2}} b \). Thus, \( \tilde{D}^{(d)} \) is definite positive if and only if

\[
1 + \frac{\alpha_d}{2} \left( \sum_{i \neq d} \frac{1}{\alpha_i} - \sqrt{\sum_{i \neq d} \frac{\lambda_i}{\alpha_i} \sum_{i \neq d} \frac{1}{\lambda_i\alpha_i}} \right) > 0,
\]

which is equivalent to (B.4) for \( k = d \). Using the same arguments on \( \tilde{D}^{(k)} \) for \( 1 \leq k \leq d-1 \), we obtain (B.4).

The choice \( \alpha = (1, \ldots, 1) \in \mathbb{R}^d \) in Inequality (B.4) gives a sufficient condition for the identity matrix \( I_d \) to satisfy Condition (C).

**Corollary B.3.** If the condition

\[
\max_{1 \leq k \leq d} \sqrt{\sum_{i \neq k} \lambda_i \sum_{i \neq k} \frac{1}{\lambda_i}} < d + 1 \tag{B.5}
\]

is satisfied then Condition (C) is satisfied for the choice \( \Gamma = I_d \). In particular, if \( \frac{\lambda_d}{\lambda_1} < \left( \frac{d+1}{d-1} \right)^2 \), then (C) is satisfied for the choice \( \Gamma = I_d \).

Moreover, for \( d = 2 \), Inequality (B.5) is always satisfied, as for \( k = 1, 2 \), \( \sqrt{\lambda_k \frac{1}{\lambda_k}} = 1 < 3 \).

**Corollary B.4.** If \( d = 2 \), then Condition (C) is satisfied for the choice \( \Gamma = I_2 \).

From Inequality (B.4), we deduce a method to check numerically whether there exists a diagonal matrix that satisfies Condition (C). We suppose that the values of \( \lambda_1, \ldots, \lambda_d \) are not equal, otherwise by Corollary B.3, \( I_d \) satisfies Condition (C). For \( z = (z_1, \ldots, z_d) \in (\mathbb{R}^2)^d \), let us denote by \( C(z) = \{ x \in \mathbb{R}^2 | \exists \mu_1, \ldots, \mu_d > 0, \sum_{i=1}^{d} \mu_i = 1, x = \sum_{i=1}^{d} \mu_i z_i \} \), the strict convex envelope of \( z \). Let \( \alpha := (\alpha_1, \ldots, \alpha_d) \in (\mathbb{R}^+)^d \), such that \( \text{Diag}(\alpha) \) satisfies Condition (C). By Proposition B.2, (B.4) holds and rewrites

\[
\left( \frac{1}{\alpha_k} + \sum_{i=1}^{d} \frac{1}{\alpha_i} \right)^2 > \left( \sum_{i=1}^{d} \frac{\lambda_i}{\alpha_i} - \frac{\lambda_k}{\alpha_k} \right) \left( \sum_{i=1}^{d} \frac{1}{\lambda_i\alpha_i} - \frac{1}{\lambda_k\alpha_k} \right), \quad 1 \leq k \leq d. \tag{B.6}
\]
If we define \( \tilde{\lambda} = \sum_{i=1}^{d} \lambda_i \sum_{k=1}^{\frac{1}{\lambda_i}} \in C((\lambda_i)_{1\leq i \leq d}) \), then Inequality (B.6) writes
\[
1 + \frac{\frac{1}{\alpha_k}}{\sum_{i=1}^{d} \frac{1}{\lambda_i}} \left( 2 + \frac{1}{\lambda_k} \tilde{\lambda} + \lambda_k \tilde{\lambda}^{-1} \right) > \tilde{\lambda} \tilde{\lambda}^{-1}, \quad 1 \leq k \leq d. \tag{B.7}
\]

We deduce that there exists a diagonal matrix that satisfies Condition (C) if and only if there exists \( (x, y) \in C((\lambda_1, \frac{1}{\lambda_1}), \ldots, (\lambda_d, \frac{1}{\lambda_d})) \) and a probability distribution \((p_1, \ldots, p_d) \in (\mathbb{R}_+)^d\) such that
\[
\sum_{i=1}^{d} \lambda_i p_i = x, \quad \sum_{i=1}^{d} \frac{1}{\lambda_i} p_i = y, \quad \forall k \in \{1, \ldots, d\}, \quad p_k > (xy - 1) \left( 2 + \frac{x}{\lambda_k} + \lambda_k y \right)^{-1}. \tag{B.8}
\]

For \((x, y)\) in the closure of \( C((\lambda_1, \frac{1}{\lambda_1}), \ldots, (\lambda_d, \frac{1}{\lambda_d})) \), let us define \( M_0(x, y) = (xy - 1) \sum_{i=1}^{d} (2 + \frac{x}{\lambda_i} + \lambda_i y)^{-1} \), \( M_{-1}(x, y) = (xy - 1) \sum_{i=1}^{d} (2 + \frac{x}{\lambda_i} + \lambda_i y)^{-1} \), and \( M_1(x, y) = (xy - 1) \sum_{i=1}^{d} \lambda_i (2 + \frac{x}{\lambda_i} + \lambda_i y)^{-1} \). If moreover \( M_0(x, y) < 1 \), we also define \( X(x, y) = \frac{x-M_1(x,y)}{1-M_0(x,y)} \), \( Y(x, y) = \frac{y-M_{-1}(x,y)}{1-M_0(x,y)} \).

**Proposition B.5.** If \( \lambda_1, \ldots, \lambda_d \) are not all equal, there exists a diagonal matrix that satisfies Condition (C) if and only if there exists \((x, y) \in C((\lambda_1, \frac{1}{\lambda_1}), \ldots, (\lambda_d, \frac{1}{\lambda_d}))\) such that
\[
M_0(x, y) < 1 \quad \text{and} \quad (X(x, y), Y(x, y)) \in C\left( \left( \lambda_i, \frac{1}{\lambda_i} \right)_{1 \leq i \leq d} \right). \tag{B.9}
\]

**Proof.** If a diagonal matrix \( \text{Diag}(\alpha) \), where \( \alpha := (\alpha_1, \ldots, \alpha_d) \in (\mathbb{R}_+)^d \), satisfies Condition (C), then (B.7) holds and it is easy to check that if we set \( p_i = \frac{1}{\alpha_i} \) for \( 1 \leq i \leq d \), \( x = \sum_{i=1}^{d} \lambda_i p_i \) and \( y = \sum_{i=1}^{d} \frac{p_i}{\lambda_i} \), the conditions in (B.8) hold so the conditions in (B.9) are satisfied, as
\[
\left( X(x, y), Y(x, y) \right) = \frac{1}{1 - M_0(x, y)} \sum_{k=1}^{d} \left( p_k - (xy - 1) \left( 2 + \frac{x}{\lambda_k} + \lambda_k y \right)^{-1} \right) \left( \lambda_k \frac{1}{\lambda_k} \right).
\]

Conversely, if \((x, y) \in C((\lambda_1, \frac{1}{\lambda_1})_{1 \leq i \leq d})\) and satisfies (B.9), then there exists a probability distribution \((q_1, \ldots, q_d) \in (\mathbb{R}_+)^d\) such that \( X(x, y) = \sum_{i=1}^{d} \lambda_i q_i \) and \( Y(x, y) = \sum_{i=1}^{d} \frac{1}{\lambda_i} q_i \) and it is easy to check that if we set \( p_i = (1 - M_0(x, y)) q_i + (xy - 1) (2 + \frac{x}{\lambda_i} + \lambda_i y)^{-1} \) for \( 1 \leq i \leq d \) then the conditions in (B.8) are satisfied.

Given a discretization parameter \( n \), the numerical procedure that follows builds a grid \( \mathcal{G} \) that consists in \((d - 1)(n - 1)^2\) points of \( C((\lambda_1, \frac{1}{\lambda_1}), \ldots, (\lambda_d, \frac{1}{\lambda_d})) \) and returns whether there exists a point \((x, y) \in \mathcal{G}\) satisfying (B.9). The border of the convex set \( C((\lambda_1, \frac{1}{\lambda_1}), \ldots, (\lambda_d, \frac{1}{\lambda_d})) \) has a simple shape. Indeed, it is a polygon with vertices \((\lambda_i, \frac{1}{\lambda_i})_{1 \leq i \leq d}\) and edges \(\{(\lambda_i, \frac{1}{\lambda_i}), (\lambda_{i+1}, \frac{1}{\lambda_{i+1}})\}_{1 \leq i \leq d}\), where we define \((\lambda_{d+1}, \frac{1}{\lambda_{d+1}}) = (\lambda_1, \frac{1}{\lambda_1})\).
The advantage of such a numerical procedure is that it operates in a bounded convex of $\mathbb{R}^2$ with a simple shape, instead of a bounded convex of $\mathbb{R}^d$, as Inequality (B.4) would suggest. For $d = 5$, the procedure gives that in the example $\lambda_1 = 1.188$, $\lambda_2 = 1.192$, $\lambda_3 = 5.205$, $\lambda_4 = 8.738$, and $\lambda_5 = 9.310$, Condition (C) is satisfied, although Inequality (B.5) is not (Figure B.1).

In the case $d \geq 3$, another sufficient condition to obtain (B.9) can be obtained from the local behavior of the function $(x, y) \mapsto (X(x, y), Y(x, y))$ for $(x, y) \in C((\lambda_i, \frac{1}{\lambda_i})_{1 \leq i \leq d})$, around the points $(\lambda_i, \frac{1}{\lambda_i})$. For $2 \leq i \leq d - 1$, let $B_i$ be the ball centered in $(\lambda_i, \frac{1}{\lambda_i})$ with radius $r < \min_{2 \leq i \leq d} \left\{ |\lambda_i - \lambda_{i-1}| \right\} \wedge \min_{2 \leq i \leq d - 1} \left\{ \frac{|\lambda_{i+1} \lambda_i - \lambda_i \lambda_{i-1}|}{\sqrt{1 + \lambda_i^2 \lambda_{i-1}^2}} \right\}$, for the Euclidean norm. For $(x, y) \in C((\lambda_i, \frac{1}{\lambda_i})_{1 \leq i \leq d}) \cap B_i$ and $z = (x - \lambda_i, y - \frac{1}{\lambda_i})$, $v_1 = \frac{1}{\sqrt{1 + \lambda_i^2 \lambda_{i-1}^2}}(1, \lambda_i \lambda_{i-1})$, $v_2 = \frac{1}{\sqrt{1 + \lambda_i^2 \lambda_{i+1}^2}}(1, \lambda_i \lambda_{i+1})$, we have that $z^*z < r^2$, $z^*v_1 > 0$ and $z^*v_2 > 0$. Similarly, the condition $(X(x, y), Y(x, y)) \in C((\lambda_i, \frac{1}{\lambda_i})_{1 \leq i \leq d}) \cap B_i$ rewrites $z^*z' < r^2$, $z^*v_1' > 0$ and $z^*v_2' > 0$, where

```plaintext
1: for i = 1, ..., d - 1 do
2:   for k1 = 1, ..., n - 1 do
3:     x = \lambda_i + (\lambda_{i+1} - \lambda_i) \frac{k1}{n}
4:     y_{\text{min}} = \frac{1}{\lambda_i} \frac{n-k1}{n} + \frac{1}{\lambda_{i+1}} \frac{k1}{n}
5:     y_{\text{max}} = \frac{1}{\lambda_i} - \frac{x-\lambda_i}{\lambda_i \lambda_d}
6:   for k2 = 1, ..., n - 1 do
7:     y = y_{\text{min}} + (y_{\text{max}} - y_{\text{min}}) \frac{k2}{n}
8:     M_0 = (xy - 1) \sum_{k=1}^d \left(2 + \frac{x_i}{\lambda_i} + \lambda_i y\right)^{-1}
9:     if M_0 < 1 then
10:    M_1 = (xy - 1) \sum_{k=1}^d \lambda_i \left(2 + \frac{x_i}{\lambda_i} + \lambda_i y\right)^{-1}
11:    X = \frac{x-M_1}{1-M_0}
12:    if \lambda_1 < X < \lambda_d then
13:       j = \sum_{i=1}^d 1_{x > \lambda_i}
14:       z_{\text{min}} = \frac{1}{\lambda_j} - \frac{X-\lambda_i}{\lambda_j \lambda_{i+1}}
15:       z_{\text{max}} = \frac{1}{\lambda_1} - \frac{X-\lambda_i}{\lambda_1 \lambda_d}
16:       M_{-1} = (xy - 1) \sum_{k=1}^d \frac{1}{\lambda_i} \left(2 + \frac{x_i}{\lambda_i} + \lambda_i y\right)^{-1}
17:       Y = \frac{y-M_{-1}}{1-M_0}
18:       if z_{\text{min}} < Y < z_{\text{max}} then
19:          return TRUE
20: end if
21: end if
22: end if
23: end for
24: end for
25: end for
26: return FALSE
```
because for $2 \leq i \leq d - 1$, $(\lambda_i, \frac{1}{\lambda_i})$ is a fixed point for the function $(X, Y)$. As $M_0(\lambda_i, \frac{1}{\lambda_i}) = 0$ and the functions $M_0$, $X$, and $Y$ are smooth on $C((\lambda_i, \frac{1}{\lambda_i})_{i \leq d}) \cap B$, we have that for $r$ small enough, $M_0(x, y) < 1$ and $z' = \Gamma z + \mathcal{O}(z^2)$, where $\Gamma$ is the matrix $(\nabla X, \nabla Y)$ built with the gradients of $X$ and $Y$. By scaling, it is then sufficient to find a vector $z$ such that $z^* v_1 = 1$, $z^* v_2 > 0$, $z^* \Gamma^* v_1 > 0$, $z^* \Gamma^* v_2 > 0$ in order to have (B.9) satisfied. Using the orthonormal basis $(u_1, \overline{v}_1)$, where $\overline{v}_1 := \frac{1}{\sqrt{1+\gamma^2_i}}(-\lambda_i \lambda_{i-1}, 1)$, we rewrite $z = z_a u_1 + z_{\beta} \overline{v}_1$, $u_2 = u_2 a u_1 + u_2 \beta \overline{v}_1, G^* v_1 = g_1 a u_1 + g_1 \beta \overline{v}_1, G^* v_2 = g_2 a u_1 + g_2 \beta \overline{v}_1$. Let us remark that with that choice of basis, we have that $u_2 > 0$. The previous conditions simplify into $z_a = 1, z_{\beta} > -u_2 a / u_2 \beta, g_1 a z_{\beta} > -g_1 a, g_2 a z_{\beta} > -g_2 a$. It is sufficient to discuss according to the signs of $g_1 a$ and $g_2 a$. If $g_1 a > 0$ and $g_2 a > 0$ then (B.9) is satisfied as it is sufficient to choose $z_{\beta} > \max(-u_2 a / u_2 \beta, -g_1 a / g_2 a, -g_2 a / g_2 a)$. If $g_1 a > 0$ and $g_2 a < 0$ then (B.9) is satisfied if $(-u_2 a / u_2 \beta) \lor (-g_1 a / g_2 a) < -g_2 a / g_2 a$. If $g_1 a < 0$ and $g_2 a > 0$ then (B.9) is satisfied if $(-u_2 a / u_2 \beta) \lor (-g_2 a / g_2 a) < -g_1 a / g_1 a$. The cases $g_1 a = 0$ or $g_2 a = 0$ are also easy to handle and we can deduce a simple procedure that gives a sufficient condition ensuring the existence of a diagonal matrix satisfying condition (C).

**B.3 The case $d = 3$**

In the following, we study the case $d = 3$. We recall that

$$r_1 = \frac{\lambda_3}{\lambda_2} + \frac{\lambda_2}{\lambda_3} \geq 2, \quad r_2 = \frac{\lambda_3}{\lambda_1} + \frac{\lambda_1}{\lambda_3} \geq 2, \quad r_3 = \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} \geq 2.$$
Let us first explicit the link between the values of $r_1, r_2, r_3$.

**Lemma B.6.** The values of $r_1, r_2, r_3$ are linked by

$$r_3 \in \left\{ \frac{1}{2} \left( r_1 r_2 - \sqrt{\left( r_1^2 - 4 \right) \left( r_2^2 - 4 \right)} \right), \frac{1}{2} \left( r_1 r_2 + \sqrt{\left( r_1^2 - 4 \right) \left( r_2^2 - 4 \right)} \right) \right\}.$$  

**Proof.** As $r_1 = \frac{\lambda_2}{\lambda_3} + \frac{\lambda_3}{\lambda_1} \geq 2$ and $r_2 = \frac{\lambda_1}{\lambda_3} + \frac{\lambda_3}{\lambda_2} \geq 2$, we have that $\frac{\lambda_1}{\lambda_3}, \frac{\lambda_2}{\lambda_1} \in \{ \frac{1}{2} (r_1 - \sqrt{r_1^2 - 4}), \frac{1}{2} (r_1 + \sqrt{r_1^2 - 4}) \}$ and $\frac{\lambda_1}{\lambda_3}, \frac{\lambda_2}{\lambda_2} \in \{ \frac{1}{2} (r_2 - \sqrt{r_2^2 - 4}), \frac{1}{2} (r_2 + \sqrt{r_2^2 - 4}) \}$. When computing $r_3 = \frac{\lambda_1}{\lambda_3} + \frac{\lambda_2}{\lambda_1} + \frac{\lambda_3}{\lambda_2}$, we obtain the result, as the coefficients of the terms $r_1 \sqrt{r_2^2 - 4}$ and $r_2 \sqrt{r_1^2 - 4}$ vanish.  

We now give the main result concerning the case $d = 3$.

**Proposition B.7.** There is equivalence between

(i) The inequality

$$\frac{1}{\sqrt{(r_1 - 2)(r_2 - 2)}} + \frac{1}{\sqrt{(r_2 - 2)(r_3 - 2)}} + \frac{1}{\sqrt{(r_1 - 2)(r_3 - 2)}} > \frac{1}{4} \tag{B.10}$$

holds, with the convention $\frac{1}{0} = +\infty$;

(ii) Condition (C) is satisfied by a diagonal matrix;

(iii) Condition (C) is satisfied.

We now prove Proposition B.7. We first show that (i) $\Rightarrow$ (ii).

**Lemma B.8.** (a) If $\min(r_1, r_2, r_3) = 2$, then there exists a diagonal matrix that satisfies Condition (C), that is, (ii) holds.

(b) If $\min(r_1, r_2, r_3) > 2$ and Inequality (B.10) holds, then (ii) holds.

**Proof.** We prove the first assertion. Let $\alpha_1, \alpha_2, \alpha_3 > 0$ and $p_i = \frac{\alpha_i}{\alpha_1 + \alpha_2 + \alpha_3}, i = 1, 2, 3$. Inequality (B.6) rewrites, for $k = 1$,

$$1 + 2p_1 + p_1^2 > \left( \frac{\lambda_2}{\lambda_3} + \frac{\lambda_3}{\lambda_2} \right) p_2 p_3 + p_2^2 + p_3^2.$$  

Writing $p_1^2 = (1 - p_2 - p_3)^2$, and then using the fact that $p_1 + p_2 + p_3 = 1$, we obtain that

$$4 > p_2 p_3 \frac{(r_1 - 2)}{p_1}. \tag{B.11}$$

With similar computations for $k = 2, 3$ we moreover obtain

$$4 > p_1 p_3 \frac{(r_2 - 2)}{p_2} \quad \text{and} \quad 4 > p_1 p_2 \frac{(r_3 - 2)}{p_3}. \tag{B.12}$$

To prove Lemma B.8, it is sufficient to exhibit a probability distribution $(p_1, p_2, p_3) \in (\mathbb{R}_+^*)^3$ such that Inequalities (B.11) and (B.12) are satisfied. In the case where $r_1 = r_2 = 2$, we have that $\lambda_1 = \lambda_2 = \lambda_3$, so $r_3 = 2$, and the choice $p_1 = p_2 = p_3 = \frac{1}{3}$ satisfies (B.11) and (B.12). In the case where $r_1 = 2, r_2 > 2$
and \( r_3 > 2 \), we have that it is sufficient to choose \( p_1 \in (0, \min(1, \frac{4}{r_2-2}, \frac{4}{r_3-2})) \) and \( p_2 = p_3 = \frac{1-p_1}{2} \) to satisfy (B.11) and (B.12).

To prove the second assertion, using the same computations as previously, it is easy to check that if Inequality (B.10) is satisfied then (B.11) and (B.12) are satisfied for the choice \( p_i = \frac{\sqrt{r_i-2} - \frac{1}{\sqrt{r_i-2} + \sqrt{r_i-2} + \sqrt{r_i-2}}}{\sqrt{r_1-2} + \sqrt{r_2-2} + \sqrt{r_3-2}} > 0 \), for \( i = 1, 2, 3 \), so that the matrix \( \text{Diag}(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3}) \) satisfies Condition (C). \( \square \)

As the relation \((ii) \Rightarrow (iii)\) is trivial, we have obtained \((i) \Rightarrow (ii) \Rightarrow (iii)\). As when \( \min(r_1, r_2, r_3) = 2 \), (B.10) holds, then to prove \((iii) \Rightarrow (i)\), it is sufficient to show that in the case \( \min(r_1, r_2, r_3) > 2 \), if Condition (C) is satisfied then (B.10) holds. Let us remark that we can assume without loss of generality that \( r_1 \leq r_2 \leq r_3 \), so in what follows we suppose that

\[
2 < r_1 \leq r_2 \leq r_3.
\]

The next lemma deals with the case \( \frac{r_3-2}{r_3+2} \geq \frac{r_1-2}{r_1+2} + \frac{r_2-2}{r_2+2} \).

**Lemma B.9.** Let us assume that \( \min(r_1, r_2, r_3) > 2 \). Then Inequality (B.10) is equivalent to

\[
\left\{ \sqrt{(r_1 - 2)(r_2 - 2)} \leq 4 \right\} \text{ or } \left\{ \sqrt{(r_1 - 2)(r_2 - 2)} > 4 \text{ and } r_3 < 16 \left( \frac{r_1 - 2 + \sqrt{r_1 - 2}}{\sqrt{(r_1 - 2)(r_2 - 2)} - 4} \right)^2 + 2 \right\}. \tag{B.13}
\]

In particular, if moreover \( \frac{r_3-2}{r_3+2} \geq \frac{r_1-2}{r_1+2} + \frac{r_2-2}{r_2+2} \), then Inequality (B.10) holds.

**Proof.** Under the assumption that \( \min(r_1, r_2, r_3) > 2 \), Inequality (B.10) rewrites \( \frac{1}{\sqrt{r_3-2}}(\frac{1}{\sqrt{r_1-2}} + \frac{1}{\sqrt{r_2-2}}) > \frac{1}{4} - \frac{1}{\sqrt{(r_1-2)(r_2-2)}} \), so it is equivalent to (B.13). The term \( \frac{r_1-2}{r_1+2} + \frac{r_2-2}{r_2+2} \) rewrites

\[
\frac{r_1 - 2}{r_1 + 2} + \frac{r_2 - 2}{r_2 + 2} = 2\sqrt{(r_1 - 2)(r_2 - 2)} \left( \sqrt{(r_1 - 2)(r_2 - 2) - 4} \right) + 4 \left( \sqrt{r_1 - 2} + \sqrt{r_2 - 2} \right)^2.
\]

Hence if \( \sqrt{(r_1 - 2)(r_2 - 2)} > 4 \), then \( \frac{r_1-2}{r_1+2} + \frac{r_2-2}{r_2+2} > 1 \). Thus, as \( 1 > \frac{r_3-2}{r_3+2} \), if

\[
\frac{r_3 - 2}{r_3 + 2} \geq \frac{r_1 - 2}{r_1 + 2} + \frac{r_2 - 2}{r_2 + 2},
\]

then \( \sqrt{(r_1 - 2)(r_2 - 2)} \leq 4 \) and Inequality (B.10) holds. \( \square \)

Let us now suppose that Condition (C) is satisfied by a matrix \( \Gamma \in S_{3}^{++}(\mathbb{R}) \). We consider, for \( k = 1, 2, 3 \), the matrices \( \Gamma^{(k)} \) with coefficients

\[
\Gamma_{ij}^{(k)} = \frac{\lambda_i + \lambda_j}{2} (\Gamma_{ij} + \Gamma_{kk} - \Gamma_{ik} - \Gamma_{jk}), \quad 1 \leq i, j \leq 3.
\]
We define
\[ v_1 := \Gamma_{22} + \Gamma_{33} - 2\Gamma_{23}, \quad v_2 := \Gamma_{11} + \Gamma_{33} - 2\Gamma_{13}, \quad v_3 := \Gamma_{11} + \Gamma_{22} - 2\Gamma_{12}. \]

The matrix \( \Gamma^{(3)} \) rewrites
\[
\Gamma^{(3)} = \begin{pmatrix}
\frac{\lambda_1 v_2}{2} & \frac{\lambda_1 + \lambda_2}{2} \times \frac{v_1 + v_2 - v_3}{2} & 0 \\
\frac{\mu_2 v_1}{2} & \frac{\lambda_2 v_1}{2} & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

We deduce that \( \Gamma^{(3)} \) is positive definite on \( e_{\frac{1}{3}} \) if and only if
\[ v_1 > 0, \quad v_2 > 0 \quad \text{(B.14)} \]
and the determinant of the matrix \( (\Gamma^{(3)})_{1\leq i,j\leq 2} \) is positive, which rewrites \( 16v_1v_2 > (\sqrt{\lambda_1} + \sqrt{\lambda_2})^2(v_1 + v_2 - v_3)^2 = (r_3 + 2)(v_1 + v_2 - v_3)^2 \) and therefore
\[ -4 \frac{r_2 - 2}{r_2 + 2} v_1v_2 > v_1^2 + v_2^2 + v_3^2 - 2(v_1v_2 + v_2v_3 + v_1v_3), \quad \text{(B.15)} \]
With similar computations for \( \Gamma^{(1)} \) and \( \Gamma^{(2)} \), we obtain the additional inequalities
\[ v_3 > 0, \quad \text{(B.16)} \]
\[ -4 \frac{r_2 - 2}{r_2 + 2} v_1v_3 > v_1^2 + v_2^2 + v_3^2 - 2(v_1v_2 + v_2v_3 + v_1v_3), \quad \text{(B.17)} \]
\[ -4 \frac{r_1 - 2}{r_1 + 2} v_2v_3 > v_1^2 + v_2^2 + v_3^2 - 2(v_1v_2 + v_2v_3 + v_1v_3). \quad \text{(B.18)} \]
If Condition \( (C) \) is satisfied, then there exists \( v_1, v_2, v_3 \) satisfying \( \text{(B.14)} \)--\( \text{(B.18)} \). Let us define for \( i = 1, 2, 3, \quad \nu_i = \frac{r_i - 2}{r_i + 2} \in (0, 4). \) If we assume that moreover, \( \gamma_3 v_1v_2 = \gamma_1 v_2v_3 = \gamma_2 v_1v_3, \) that is \( \frac{\nu_1}{v_3} = \frac{\nu_2}{v_1} = \frac{v_2}{v_3}, \) then we have that
\[ 0 > \gamma_1^2 + \gamma_2^2 + \gamma_3^2 - 2(\gamma_1\gamma_2 + \gamma_2\gamma_3 + \gamma_1\gamma_3) + \gamma_1\gamma_2\gamma_3. \quad \text{(B.19)} \]
In the case \( \frac{\nu_3 - 2}{r_3 + 2} \geq \frac{\nu_2 - 2}{r_2 + 2} + \frac{\nu_1 - 2}{r_1 + 2}, \) by Lemma B.9, Inequality \( \text{(B.10)} \) holds. We show in Lemma B.11 that under the assumption \( \frac{\nu_3 - 2}{r_3 + 2} \leq \frac{\nu_2 - 2}{r_2 + 2} + \frac{\nu_1 - 2}{r_1 + 2}, \) Condition \( (C) \) implies Inequality \( \text{(B.19)} \). To conclude the proof of \( (iii) \Rightarrow (i) \) and therefore the proof of Proposition B.7, we show in Lemma B.10 that \( \text{(B.19)} \) implies \( \text{(B.10)} \).

**Lemma B.10.** *Let us assume that \( 2 < r_1 \leq r_2 \leq r_3, \) then Inequality \( \text{(B.19)} \) implies Inequality \( \text{(B.10)} \).*

**Proof.** We see the term on the r.h.s. of Inequality \( \text{(B.19)} \) as a second degree polynomial in the variable \( \gamma_3 \), which has two distinct roots \( z_- < z_+ \). Indeed, \( \gamma_1, \gamma_2 \in (0, 4) \), and the discriminant of the
polynomial is \( \gamma_1 \gamma_2 (4 - \gamma_1) (4 - \gamma_2) > 0 \). As \( \gamma_3 = 4^{r_3 - 2} \), Inequality (B.19) is equivalent to \( r_3 \in \left( \frac{8 + 2z_-}{4 - z_-}, \frac{8 + 2z_+}{4 - z_+} \right) \), where \( z_{\pm} = 16 \sqrt{\frac{\sqrt{r_1 - 2} \pm \sqrt{r_2 - 2}}{(r_1 + 2)(r_2 + 2)}} \), and

\[
\frac{8 + 2z_{\pm}}{4 - z_{\pm}} = 16 \left( \frac{\sqrt{r_1 - 2} \pm \sqrt{r_2 - 2}}{(r_1 - 2) (r_2 - 2) \mp 4} \right)^2 + 2.
\]

By Lemma B.9, we conclude that Inequality (B.19) implies Inequality (B.10).

**Lemma B.11.** Let us assume that \( 2 < r_1 \leq r_2 \leq r_3 \), and that \( \gamma_3 < \gamma_1 + \gamma_2 \). If Condition (C) is satisfied then Inequality (B.19) holds.

**Proof.** Let us define the function \( f : (\mathbb{R}_+^*)^3 \to \mathbb{R} \) by \( f(a_1, a_2, a_3) = 2(a_1 + a_2 + a_3) - \frac{a_1 a_2}{a_3} - \frac{a_2 a_3}{a_1} - \frac{a_1 a_3}{a_2} \). Reformulating Inequalities (B.15)–(B.18) with the change of variables \( a_i = \frac{1}{\zeta_i}, i = 1, 2, 3 \), we obtain that

\[
f(a_1, a_2, a_3) > \max \{ \gamma_1 a_1, \gamma_2 a_2, \gamma_3 a_3 \}.
\]

Under the assumption that \( 2 < r_1 \leq r_2 \leq r_3 \), we have that \( 0 < \gamma_1 \leq \gamma_2 \leq \gamma_3 \), and let us remark that

\[
\max \{ \gamma_1 a_1, \gamma_2 a_2, \gamma_3 a_3 \} \geq \max \{ \gamma_3 a_{(1)}, \gamma_2 a_{(2)}, \gamma_1 a_{(3)} \},
\]

where \( (a_{(1)}, a_{(2)}, a_{(3)}) \) is the nondecreasing reordering of \( (a_1, a_2, a_3) \). Let \( \mathcal{R} \) be the set of elements \( (a_1, a_2, a_3) \in (\mathbb{R}_+^*)^3 \) such that \( a_1 \leq a_2 \leq a_3 \) and

\[
f(a_1, a_2, a_3) > \max \{ \gamma_1 a_1, \gamma_2 a_2, \gamma_3 a_1 \}.
\]

If Condition (C) is satisfied then \( \mathcal{R} \) is not empty. Let us remark first that as both sides of (B.20) are homogeneous of order 1, \( \mathcal{R} \) is stable by scaling: if \( (a_1, a_2, a_3) \in \mathcal{R} \) then for \( \zeta > 0 \), \( (\zeta a_1, \zeta a_2, \zeta a_3) \in \mathcal{R} \). Moreover, if we assume that there exists \( (a_1, a_2, a_3) \in \mathcal{R} \) such that \( \gamma_1 a_3 = \gamma_2 a_2 = \gamma_3 a_1 =: \Delta \), then we can check that Inequality (B.19) is satisfied. Indeed, we have that \( a_3 = \frac{\Delta}{\gamma_1}, a_2 = \frac{\Delta}{\gamma_2}, a_3 = \frac{\Delta}{\gamma_1} \), and

\[
f \left( \frac{\Delta}{\gamma_1}, \frac{\Delta}{\gamma_2}, \frac{\Delta}{\gamma_3} \right) > \Delta.
\]

Multiplying both sides of (B.21) by \( \frac{\frac{\Delta}{\gamma_1} \frac{\Delta}{\gamma_2} \frac{\Delta}{\gamma_3}}{\Delta} \), we obtain (B.19).

Let \( (q_1, q_2, q_3) \in \mathcal{R} \). To prove that there exists \( (u_1, u_2, u_3) \in \mathcal{R} \), such that \( \gamma_1 u_3 = \gamma_2 u_2 = \gamma_3 u_1 \), and conclude with the previous argument, we construct a path included into \( \mathcal{R} \) that goes from \( (q_1, q_2, q_3) \) to \( (u_1, u_2, u_3) \). We now distinguish the three cases.

Case 1: \( \gamma_3 q_1 \leq \max \{ \gamma_2 q_2, \gamma_1 q_3 \} = \gamma_1 q_3 \). Let us remark that if \( a_1 \leq a_2 \leq a_3 \), the partial derivative \( \partial_{a_1} f \) satisfies

\[
\partial_{a_1} f(a_1, a_2, a_3) = \frac{a_2 a_3}{a_1^2} - \left( \sqrt{\frac{a_2}{a_3}} - \sqrt{\frac{a_3}{a_2}} \right)^2 \geq \frac{a_3}{a_2} \left( \left( \frac{a_2}{a_1} \right)^2 - 1 \right) \geq 0.
\]
For $x \in [q_1, \min(q_2, \frac{\gamma_1}{\gamma_3} q_3)]$, $f(x, q_2, q_3) \geq f(q_1, q_2, q_3)$. If $\min(q_2, \frac{\gamma_1}{\gamma_3} q_3) = \frac{\gamma_1}{\gamma_3} q_3$, then for $\tilde{q}_1 = \frac{\gamma_1}{\gamma_3} q_3$, we have that

$$f(\tilde{q}_1, q_2, q_3) \geq f(q_1, q_2, q_3) > \gamma_1 q_1 = \gamma_3 \tilde{q}_1 \geq \gamma_2 q_2. \tag{B.23}$$

We scale (B.23) and define $\zeta = \frac{\gamma_1}{q_1} = \frac{\gamma_3}{q_3} > 0$ so that $\zeta \tilde{q}_1 = \gamma_1$ and $\zeta q_3 = \gamma_3$. We have that

$$f(\zeta \tilde{q}_1, \zeta q_2, \zeta q_3) \geq f(\zeta \tilde{q}_1, \zeta q_2, \zeta q_3) > \gamma_3 \gamma_1 \zeta^{-1} \geq \gamma_2 \zeta q_2. \tag{B.24}$$

We now increase $q_2$ in (B.24). For $a_1, a_2, a_3 > 0$, such that $a_1 \leq a_2 \leq a_3$ and $a_2 \leq \frac{a_3 a_1}{a_3 - a_1}$ we have that

$$\partial_{a_2} f(a_1, a_2, a_3) = a_1 a_3 \left( \frac{1}{a_2^2} - \left( \frac{1}{a_1} - \frac{1}{a_3} \right)^2 \right) \geq 0.$$ 

By hypothesis,

$$q_2 \leq \frac{\gamma_1}{\gamma_3} \frac{q_3}{\zeta - \gamma_1} = \frac{\gamma_3 q_1}{q_3 - \tilde{q}_1},$$

so for $z \in [q_2, \frac{\gamma_1}{\gamma_3} q_3]$, $f(\zeta \tilde{q}_1, \zeta z, \zeta q_3) \geq f(\zeta \tilde{q}_1, \zeta q_2, \zeta q_3)$ and in particular for $\tilde{q}_2 = \frac{\gamma_1}{\gamma_3} q_3$, we have that

$$f(\zeta \tilde{q}_1, \zeta \tilde{q}_2, \zeta q_3) > \gamma_1 \zeta q_3 = \gamma_3 \zeta \tilde{q}_1 = \gamma_2 \zeta \tilde{q}_2,$$

so we obtain (B.19). If $\min(q_2, \frac{\gamma_1}{\gamma_3} q_3) = q_2$, as the function $x \rightarrow f(x, q_2, q_3)$ is nondecreasing for $x \in [q_1, q_2]$, we have that

$$f(q_2, q_2, q_3) \geq f(q_1, q_2, q_3) > \gamma_1 q_3 \geq \gamma_3 q_2 \geq \gamma_2 q_2.$$

As the function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $g(a_2, a_3) = f(a_2, a_2, a_3)$ satisfies $\partial_{a_2} g(a_2, a_3) = 4 - 2 \frac{a_2}{a_3} \geq 0$ if $a_2 \leq a_3$, we have that for $y \in [q_2, \frac{\gamma_1}{\gamma_3} q_3]$, $f(y, y, q_3) \geq f(q_1, q_2, q_3)$. In particular for $\tilde{q}_1 = \tilde{q}_2 = \frac{\gamma_1 q_1}{\gamma_3}$, we have

$$f(\tilde{q}_1, \tilde{q}_2, q_3) \geq f(q_1, q_2, q_3) > \gamma_1 q_3 \geq \gamma_3 \tilde{q}_1 \geq \gamma_2 \tilde{q}_2. \tag{B.25}$$

and (B.25) is treated in the same way as (B.23).

Case 2: $\gamma_3 q_1 \leq \max(\gamma_2 q_2, \gamma_1 q_3) = \gamma_2 q_2$. Using (B.22), as $\partial_{a_1} f(x, q_2, q_3) \geq 0$ for $x \in [q_1, q_2]$, we have that for $\tilde{q}_1 = \frac{\gamma_1}{\gamma_3} q_3 \in [q_1, q_2]$,

$$f(\tilde{q}_1, q_2, q_3) \geq f(q_1, q_2, q_3) > \gamma_2 q_2 = \gamma_3 \tilde{q}_1 \geq \gamma_1 q_3.$$

For $\theta = \frac{\gamma_1}{\gamma_2} = \frac{\gamma_3}{\tilde{q}_1}$, we have that

$$f(\theta \tilde{q}_1, \theta q_2, \theta q_3) > \gamma_3 \gamma_2 \geq \gamma_1 \theta q_3.$$

As $\gamma_3 - \gamma_2 < \gamma_1$, we have that

$$q_3 \leq \frac{\gamma_2 \gamma_3}{\theta \gamma_1} < \frac{1}{\theta} \frac{\gamma_2 \gamma_3}{\gamma_3 - \gamma_2} = \frac{\tilde{q}_1 q_2}{q_2 - \tilde{q}_1}.$$
Moreover, for $0 < a_1 \leq a_2 \leq a_3$ such that $a_3 \leq \frac{a_2 a_1}{a_2 - a_1}$,
\[
\partial_{a_3} f(a_1, a_2, a_3) = a_1 a_2 \left( \frac{1}{a_3^2} - \left( \frac{1}{a_1} - \frac{1}{a_2} \right)^2 \right) \geq 0,
\] (B.26)
so for $\hat{q}_3 = \frac{\gamma_2 \gamma_3}{\gamma_1}$, we obtain
\[
f(\theta \tilde{q}_1, \theta q_2, \theta \tilde{q}_3) \geq f(\theta \tilde{q}_1, \theta q_2, \gamma q_3) > \gamma_2 \theta q_2 = \gamma_3 \theta \tilde{q}_1 = \gamma_1 \theta \tilde{q}_3,
\]
and we deduce (B.19).

Case 3: $\gamma_3 q_1 > \max(\gamma_2 q_2, \gamma_1 q_3)$. We show that there exists $\tilde{q}_3 \geq q_3$ such that
\[
f(q_1, q_2, \tilde{q}_3) \geq f(q_1, q_2, q_3) > \gamma_3 q_1 = \gamma_1 \tilde{q}_3 \geq \gamma_2 q_2.
\] (B.27)
Let $\eta = \frac{\gamma_1}{q_1}$ so that $\eta q_1 = \gamma_1$. We have that
\[
f(\eta q_1, \eta q_2, \gamma q_3) \geq \gamma_3 \gamma_1 > \gamma_1 \eta q_3 = \gamma_1 \gamma q_2.
\]
As $q_2 < \frac{\gamma_1}{\eta q_2}$ and $\gamma_3 - \gamma_2 < \gamma_1$, we have that $\frac{\gamma_1}{\gamma_3 - \gamma_2} > 1$ and
\[
q_3 \leq \frac{\gamma_3}{\eta} < \frac{1}{\eta} \frac{\gamma_1 \gamma_3}{\gamma_3 - \gamma_2} = \frac{q_1 q_2}{\eta q_2 - q_2} < \frac{q_1 q_2}{q_2 - q_1}.
\]
Using (B.26), we deduce that for $y \in [q_3, \frac{\gamma_1}{\eta}]$, the function $y \rightarrow f(\eta q_1, \eta q_2, \eta y)$ is non decreasing and for $\tilde{q}_3 = \frac{\gamma_3}{\eta}$, we obtain
\[
f(\eta q_1, \eta q_2, \eta \tilde{q}_3) \geq f(\eta q_1, \eta q_2, \eta q_3) > \gamma_3 \eta q_1 = \gamma_1 \eta \tilde{q}_3 \geq \gamma_2 \eta q_2,
\]
which is equivalent to (B.27) and we conclude in the same manner as for (B.23) in Case 1. □

**APPENDIX C: PROOFS OF SECTION 3.1**

**C.1 Proof of Lemma 3.1**

*Proof.* Let $v \in \mathcal{P}(\mathbb{R})$ and let $u$ be a solution to $LV(v)$. As $\bar{\sigma}_{\text{Dup}} \in L^\infty([0, T], W^{1,\infty}(\mathbb{R}))$ and $u \in L^2_{\text{loc}}((0, T]; H^1(\mathbb{R}))$, we have that $dt$-a.e. on $(0, T]$, $(\bar{\sigma}^2_{\text{Dup}} u)(t, \cdot) \in H^1(\mathbb{R})$ and $\bar{\sigma}^2_{\text{Dup}}(t, \cdot) \partial_x u(t, \cdot) = \partial_x (\bar{\sigma}^2_{\text{Dup}} u)(t, \cdot) - (u \partial_x \bar{\sigma}^2_{\text{Dup}})(t, \cdot)$ in the sense of distributions on $\mathbb{R}$. Then for any function $\phi$ defined for $(t, x) \in [0, T] \times \mathbb{R}$ by $\phi(t, x) := g_1(t) g_2(x)$, with $g_1 \in C^\infty_c((0, T])$ and $g_2 \in C^\infty_c(\mathbb{R})$, the Borel measure $d m := udxdt$ satisfies the equality,
\[
\int_{(0, T) \times \mathbb{R}} \left[ \partial_t \phi + \left( r - \frac{1}{2} \bar{\sigma}^2_{\text{Dup}} \right) \partial_x \phi + \frac{1}{2} \bar{\sigma}^2_{\text{Dup}} \partial^2_{xx} \phi \right] d m = 0.
\]
By density of the space spanned by the functions of type $g_1(t) g_2(x)$ in $C_c^\infty((0, T) \times \mathbb{R})$ for the norm $\phi \in C_c^\infty((0, T) \times \mathbb{R}) \rightarrow ||\phi||_{\infty} + ||\partial_x \phi||_{\infty} + ||\partial^2_{xx} \phi||_{\infty} + ||\partial_t \phi||_{\infty}$, the previous equality is also satisfied for any function $\phi \in C_c^\infty((0, T) \times \mathbb{R})$. The variational formulation of the PDE (8) as defined in Bogachev, Röckner, and Shaposhnikov (2009, Equality 1.5) is then satisfied.
As \( h_1(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \) and it is easy to check that \( h_1 \in H^1(\mathbb{R}) \). As \((u(t), h_1)_{t \to 0} \int h_1 d\nu > 0\) and as \( u \) is nonnegative, we obtain that for any \( \tau \in (0, T) \), \( \int_0^\tau (u(t), h_1)_{t \to 0} d\tau > 0 \), so \( \text{ess sup} u(t, x) > 0 \).

In addition, with Assumption \((B)\), the functions \( \frac{1}{2} \sigma^2_{D_u} \), \( \partial_x \left( \frac{1}{2} \sigma^2_{D_u} \right) \) and \( (r - \frac{1}{2} \sigma^2_{D_u} - \partial_x \left( \frac{1}{2} \sigma^2_{D_u} \right)) \) are uniformly bounded, and \( \sigma^2 \leq \sigma^2_{D_u} \) a.e. on \([0, T] \times \mathbb{R}\), so by Bogachev et al. (2009, Corollary 3.1) we obtain that \( u \) is continuous and positive on \((0, T) \times \mathbb{R}\).

\[
C.2 \quad \text{Proof of Proposition 3.2}
\]

\textbf{Proof.} The main ingredient to obtain uniqueness to \( LV(\nu) \) is Figalli (2008, Proposition 4.2). In the proof of Theorem 2.4, we obtained existence of a solution to \( LV(\nu) \). Moreover, if \( u \) is a solution to \( LV(\nu) \), then with the same arguments as in the proof of Lemma 3.1, we show that the Borel measure \( u dt \) solves PDE (8) with initial condition \( \nu \) in the sense of distributions, which means that for any \( \phi \in C^\infty_c(\mathbb{R}) \), the equality

\[
\frac{d}{dt} \int \phi(x) u(t, x) dx = \int \left[ \left( r - \frac{1}{2} \sigma^2_{D_u}(t, x) \right) \partial_x \phi(x) + \frac{1}{2} \sigma^2_{D_u}(t, x) \partial^2_x \phi(x) \right] u(t, x) dx,
\]

holds in the sense of distributions on \((0, T)\), and \( u \) converges to \( \nu \) in duality with \( C^\infty_c(\mathbb{R}) \) as \( t \to 0 \).

Under Assumptions \((B)\) and \((H)\), by Figalli (2008, Proposition 4.2), the measure \( u dt \) is the unique solution to PDE (8) with initial condition \( \nu \) in the sense of distributions and therefore \( u \) is the unique solution to \( LV(\nu) \).

It is then sufficient to exhibit a solution \( u dt \) to PDE (8) in the distributional sense with initial condition \( \nu \) and such that \( \int u^2(t, x) dx \leq \frac{C}{\sqrt{t}} \) for a.e. \( t \in (0, T) \), where \( C > 0 \) is a constant that does not depend on \( \nu \). Under Assumptions \((B)\) and \((H)\), the martingale problem associated to SDE (4) is well posed by Stroock and Varadhan (1979, Theorems 6.1.7 and 7.2.1), and as mentioned in Menozzi & Lemaire (2010, Paragraph 4.1), there exists two constants \( c := c(\sigma, \chi) \) and \( C := C(T, \sigma, H_0, \chi) \) such that for \( y \in \mathbb{R} \), the solution \( (X^y_t)_{t \geq 0} \) to SDE (4) with initial distribution \( \delta_y \) has the density \( p^y(t, x) \) that satisfies \( \forall (t, x) \in (0, T] \times \mathbb{R} \), \( p^y(t, x) \leq C u_c(t, x, y) \), with \( u_c(t, x, y) := \sqrt{\frac{c}{2\pi t}} \exp(-e^{-(x-y)^2/2t}) \) for \( t \in (0, T] \), \( x, y \in \mathbb{R} \), and the function \( y \to p^y(t, x) dx \) is measurable. For \( \phi \in C^\infty_c(\mathbb{R}) \), using Itô’s lemma on the process \( t \to \phi(X^y_t) \) and taking the expectation, we have that in the sense of distributions,

\[
\frac{d}{dt} \int \phi(x) p^y(t, x) dx = \int \left[ \left( r - \frac{1}{2} \sigma^2_{D_u}(t, x) \right) \phi'(x) + \frac{1}{2} \sigma^2_{D_u}(t, x) \phi''(x) \right] p^y(t, x) dx,
\]

and \( \lim_{t \to 0} \int \phi(x) p^y(t, x) dx = \phi(y) \). We set \( u := \int p^y \nu(dy) \) and we check that \( u dt \) solves PDE (8) with initial condition \( \nu \) in the distributional sense. Indeed, for \( \psi \in C^\infty_c((0, T), \mathbb{R}) \), we check, using Fubini’s theorem, that

\[
- \int_0^T \psi'(t) \int \phi(x) u(t, x) dx dt = \int_0^T \psi(t) \int \left[ \left( r - \frac{1}{2} \sigma^2_{D_u}(t, x) \right) \phi'(x) + \frac{1}{2} \sigma^2_{D_u}(t, x) \phi''(x) \right] u(t, x) dx dt,
\]
and using Lebesgue’s theorem, that \( \lim_{t \to 0} \int_{\mathbb{R}^2} \phi(x)p^\nu(t, x)dx \nu(dy) = \int_{\mathbb{R}} \left( \lim_{t \to 0^+} \int_{\mathbb{R}} \phi(x)p^\nu(t, x)dx \right) \nu(dy) \) and we conclude that \( u \) is the unique solution to \( LV(v) \) and that moreover \( u \) coincides with the time marginals of the solution to SDE (4) with initial distribution \( \nu \). Finally we define \( \zeta := \frac{C^2 \sqrt{\epsilon}}{2 \sqrt{\pi}} \) and for a.e. \( t \in (0, T] \), by Jensen’s inequality, \( \int_{\mathbb{R}} u^2(t, x)dx \leq \int_{\mathbb{R}^2} (p^\nu(t, x))^2 \nu(dy)dx \leq \frac{C^2}{2 \pi t} \int_{\mathbb{R}} \exp(-c^2 \frac{t}{x})dx \leq \frac{c}{\sqrt{t}}. \)

\[ \square \]

Appendix D: Proof of Theorem 4.2

Let us define \( S := \mathbb{R} \times \mathcal{Y} \), \( \bar{b} := \max_{i \in \{1, \ldots, d\}} ||b||_{\infty}, \bar{a} := \max_{i \in \{1, \ldots, d\}} ||a||_{\infty}, \) and \( \bar{q} := \max_{i, j \in \{1, \ldots, d\}} ||q_{ij}||_{\infty}. \) For \( (x, y) \in S \) we denote by \( \delta_{x,y} \) the Dirac distribution on \( \{(x, y)\} \). Lemma D.1 is a consequence of the constant expectation of a martingale combined with Fubini’s theorem.

Lemma D.1. Let \( \{\nu_{x,y}\}_{(x,y) \in S} \) be a measurable family of probability measures on \( E \) such that for \( \mu_0 := \nu \), \( \nu \) is a martingale solution to SDE (36) with initial distribution \( \delta_{x,y} \). Define the measure \( \mu_i(t, \cdot) \) for \( 1 \leq i \leq d \) and \( t \in (0, T) \) by

\[
\int_{\mathbb{R}} \psi(x)\mu_i(t, dx) := \int_{E \times S} \psi(X_t)1_{\{Y_{i-1} = \bar{i}\}}d\nu_{x,y}(X, Y)\mu_0(dx, dy)
\]

for any function \( \psi \in C^2_b(\mathbb{R}) \). Then \( \mu_i(t, \cdot, \cdot, \cdot, \cdot)_t \) is a solution to the PDS (33) and (34).

We now prove Theorem 4.2.

Step 1: We first establish the result for a regularized version of PDS (33) and (34). Let \( \rho_X \) and \( \rho_T \) be convolution kernels defined by \( \rho_X(x) = \rho_T(x) = C_0 e^{-\sqrt{\epsilon} x^2} \), for \( x \in \mathbb{R} \), and where \( C_0 = (\int_{\mathbb{R}} e^{-\sqrt{\epsilon} x^2}dx)^{-1} \). For \( \epsilon > 0 \) and \( (x, t) \in \mathbb{R}^2 \), we define the functions \( \rho_X^\epsilon(x) := \frac{1}{\epsilon} \rho_X(\frac{x}{\epsilon}) \), \( \rho_T^\epsilon := \frac{1}{\epsilon} \rho_T(\frac{t}{\epsilon}) \) and \( \rho^\epsilon(t, x) := \rho_T^\epsilon(t)\rho_X^\epsilon(x) \). For \( 1 \leq i \leq d \) and \( t \in [0, T] \), we extend the definition of \( \mu_i(t, \cdot) \) to \( \mathbb{R}^2 \) by setting \( \mu_i(t, \cdot) = \mu_0(\cdot, \{i\}) \) and \( \mu_i(T, \cdot) = \mu_i(T, \cdot) \). As a consequence, given \( \psi \in C^2_b(\mathbb{R}) \) and \( 1 \leq i \leq d \), the extended function \( \tau \to \int_{\mathbb{R}} \psi(x)\mu_i(t, dx) \) is now continuous and bounded on \( \mathbb{R} \). For \( 1 \leq i \leq d \), we also extend the definitions of the functions \( (a_i)_{t \in [0, T]} \) on \( \mathbb{R}^2 \) by setting \( a_i(t, \cdot) = 0 \) if \( t \not\in [0, T] \). In the same way, we extend the definitions \( \{b_i\}_{t \in [0, T]} \) and \( \{q_{ij}\}_{t \in [0, T]} \). Then the family \( (\mu_i(t, \cdot), \cdot, \cdot, \cdot, \cdot, \cdot, \cdot)_{t \in [0, T]} \) satisfies Equality (35) in the distributional sense on \( \mathbb{R} \). We define \( a_i^\epsilon = \frac{(a_i)_{t \in [0, T]}}{\mu_i} \), \( b_i^\epsilon = \frac{(b_i)_{t \in [0, T]}}{\mu_i} \), \( q_{ij}^\epsilon = \frac{(q_{ij})_{t \in [0, T]}}{\mu_i} \) for \( 1 \leq i, j \leq d \). The family \( (\mu_i^\epsilon(t, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot)_{t \in [0, T]} \) is a smooth solution of the following PDS, denoted by \( (PDS)^\epsilon \), where for \( 1 \leq i \leq d \),

\[
\frac{\partial}{\partial t} \mu_i^\epsilon + \partial_x (b_i^\epsilon \mu_i^\epsilon) - \frac{1}{2} \partial_{xx} (a_i^\epsilon \mu_i^\epsilon) - \sum_{j=1}^{d} q_{ij}^\epsilon \mu_i^\epsilon = 0
\]

\[
\mu_i^\epsilon(0) = \mu_i \ast \rho^\epsilon(0, \cdot).
\]

Without loss of generality, we suppose that for \( 1 \leq i \leq d \), \( \mu_i^\epsilon \) has a positive density on \( \mathbb{R}^2 \). If not, then \( \mu_i(t, \mathbb{R}) \) is equal to zero for all \( t \in (0, T) \) and it is sufficient in that case to consider PDS (33) and (34) without the state \( i \). Under this assumption, the functions \( a_i^\epsilon, b_i^\epsilon, q_{ij} \) for \( 1 \leq i, j \leq d \) are well defined and \( ||a_i^\epsilon||_{\infty} \leq ||a_i||_{\infty}, ||b_i^\epsilon||_{\infty} \leq ||b_i||_{\infty}, ||q_{ij}^\epsilon||_{\infty} \leq ||q_{ij}||_{\infty} \). It is easy to check that for \( x \in \mathbb{R} \), and \( k \geq 1 \), there exists constants \( C_{X,k} \) s.t. \( |\partial_{x_k}^\nu(x)| \leq C_{X,k} |\rho_X(x)| \), so \( a_i^\epsilon, b_i^\epsilon, q_{ij} \).
are continuous and bounded on $\mathbb{R}^2$, as well as their spatial derivatives. Let us denote by $(SDE)_c$, the SDE $dX^c_t = b^c_{Y^c}(t, X^c_t)dt + \sqrt{a^c_{Y^c}(t, X^c_t)}dW_t$, where $Y^c_t$ is a stochastic process with values in $\mathcal{Y}$, and that satisfies $\mathbb{P}(Y^c_{t+dt} = j|(X^c_s, Y^c_s)_{0\leq s \leq t}) = q^c_{Y^c,j}(t, X^c_t)dt + O((dt)^2)$, for $j \neq Y^c_t$. As the functions $(a^c_{Y^c})_{1\leq i \leq d}, (b^c_{Y^c})_{1\leq i \leq d}, (q^c_{Y^c,j})_{1\leq i,j \leq d}$ are continuous, Lipschitz and bounded, by Menaldi (2014, Theorem 5.3) and the Kunita–Watanabe theorem, for any $(x, y) \in S$ there exists a unique martingale solution $\nu_{X,Y}$ to $(SDE)_c$ with initial distribution $\delta_{(x,y)}$ and by Menaldi (2014, Proposition 5.52), the function $(x, y) \mapsto \nu^c_{X,Y}$ is measurable. We define $\nu^c := \sum_{i=1}^d \int_{\mathbb{R}} \nu_{X,i}(\mu_i * \rho^c)(0, x)dx$. For $t \in (0, T]$, $1 \leq i \leq d$, we define the measure $\mu^c_i(t, \cdot)$ by

$$\forall \psi \in C^2_b(\mathbb{R}), \quad \int_{\mathbb{R}} \psi(x)\mu^c_i(t, dx) = \int_E \psi(X_i)1_{\{|Y_{t,i}|=1\}} d\nu^c(X,Y).$$

By Lemma D.1, $(\mu^c_i)_{1\leq i \leq d}$ solves $(PDS)_c$ with initial condition $(\mu_i * \rho^c(0, \cdot))_{1\leq i \leq d}$. As $(PDS)_c$ has a unique solution by Proposition D.2, we obtain that for $t \in (0, T]$, $1 \leq i \leq d$, $\mu^c_i(t, \cdot) = \mu^c_i(t, \cdot)$,

$$\forall \psi \in C^2_b(\mathbb{R}), \quad \int_{\mathbb{R}} \psi(x)\mu^c_i(t, dx) = \int_E \psi(X_i)1_{\{|Y_{t,i}|=1\}} d\nu^c(X,Y). \quad (D.2)$$

Step 2: Let $(\epsilon_n)_{n \geq 0}$ be a positive sequence decreasing to 0. We check that the family of measures $(\nu^c_n)_{n \geq 0}$ has a converging subsequence. For $n \geq 0$, as we might not have $\sum_{i=1}^d \int_{\mathbb{R}} \mu_i * \rho^c(0, x)dx = 1$, we define $\nu_n := \sum_{i=1}^d \int_{\mathbb{R}} \mu_i * \rho^c(0, x)dx$, so that $\nu_n$ is a probability measure. Moreover, we check that

$$\int_{\mathbb{R}} \phi(x)\nu^c_n(t, dx) \to_{n \to \infty} \int_{\mathbb{R}} \phi(x)\mu_i(t, dx), \quad (D.3)$$

for $t \in \mathbb{R}$, $1 \leq i \leq d$ and $\phi \in C^2_b(\mathbb{R})$. The convergence (D.3) can be justified using the representation $\int_{\mathbb{R}} \phi(x)\mu^c_n(t, dx) = \int_{\mathbb{R}^2} \rho^c_X(t-s)\phi(s, \mu_i(s, dx)ds$. Indeed, as $\partial_s \phi$ is bounded, we have that $\phi * \rho^c_X$ converges uniformly to $\phi$ as $\epsilon_n \to 0$ and the convolution in time does not carry difficulty as the convolution parameter goes to 0 because the function $t \to \int \phi(x)\mu_i(t, dx)$ is continuous and bounded, as $\phi$ and $s \to \mu(s, \mathbb{R})$ are bounded functions. With (D.3), we have that $\sum_{i=1}^d \int_{\mathbb{R}} \mu_i * \rho^c(0, x)dx \to 1$, so it is sufficient to check that the family $(\nu^c_n)_{n \geq 0}$ has a converging subsequence, as any limit point of $(\nu^c_n)_{n \geq 0}$ is also a limit point of $(\nu^c_n)_{n \geq 0}$. To do so, the tightness of the family $(\nu^c_n)_{n \geq 0}$ can be obtained using Aldous’ criterion and classical arguments taking advantage of the boundedness of the coefficients $a_i, b_i, q_{ij}$ for $1 \leq i, j \leq d$. We then conclude by Prohorov’s theorem, as $E$ is Polish. For notational simplicity, we assume in what follows that $(\nu^c_n)_{n \geq 0}$ and $(\nu^c_n)_{n \geq 0}$ converge to a limit measure $\nu$ as $n \to \infty$. By Ethier and Kurtz (1986, Lemma 7.7 and Theorem 7.8), $D(\nu) := \{t \in [0, T], \nu((X_t, Y_t) = (X_t, Y_t)) = 1\}$ has a complement in $[0, T]$ which is at most countable. Moreover, for $t \in D(\nu)$, $1 \leq i \leq d$, and $\phi \in C^\infty_c(\mathbb{R})$,

$$\int_E \phi(X_t)1_{\{|Y_{t,i}|=1\}} d\nu^c(X,Y) \to_{n \to \infty} \int_E \phi(X_t)1_{\{|Y_{t,i}|=1\}} d\nu(X,Y).$$

With (D.2) and (D.3) we deduce the following equality, for $t \in D(\nu)$:

$$\int_{\mathbb{R}} \phi(x)\mu_i(t, dx) = \int_E \phi(X_t)1_{\{|Y_{t,i}|=1\}} d\nu(X,Y). \quad (D.4)$$

As the function $t \to \int_E \phi(X_t)1_{\{|Y_{t,i}|=1\}} d\nu(X,Y)$ is right-continuous and the function $t \to \int_{\mathbb{R}} \phi(x)\mu_i(t, dx)$ is continuous, Equality (D.4) holds for $t \in (0, T]$. 


Step 3: we check that \( \nu \) is a martingale solution to SDE (36) with initial distribution \( \mu_0 \). For \( t \in [0, T] \), \( 1 \leq i \leq d \) and \( x \in \mathbb{R} \), we define \( L_t \phi(x, i) := b_i(t, x) \partial_x \phi(x, i) + \frac{1}{2} a_i(t, x) \partial^2_{xx} \phi(x, i) + \sum_{j=1}^d q_{ij}(t, x) \phi(x, j) \). For \( 1 \leq i, j \leq d \), let \( \bar{b}_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \), \( \bar{a}_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \), \( \bar{q}_{ij} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be bounded and continuous functions. Let \( (\bar{b}^n_i, \bar{a}^n_i, \bar{q}^n_{ij}) \) be built from \( (\bar{b}_i, \bar{a}_i, \bar{q}_{ij}) \) analogously to the construction of \( (b^n_i, a^n_i, q^n_{ij}) \) from \( (b_i, a_i, q_{ij}) \). For \( t \in [0, T] \), let us introduce the operators \( \bar{L}^n_t \), \( \bar{L}_t^\epsilon \), and \( \bar{L}^\epsilon_n \).

Their action on a test function \( \phi \in C^\infty_c(\mathbb{R}) \) is the same as \( L_t \) but replacing \( (b_i, a_i, q_{ij}) \) by, respectively, \( (\bar{b}^n_i, \bar{a}^n_i, \bar{q}^n_{ij}) \) for \( i, j \leq d \) and \( (\bar{b}^\epsilon_i, \bar{a}^\epsilon_i, \bar{q}^\epsilon_{ij}) \) for \( i, j \leq d \). Let \( s \in [0, T] \), \( m \in \mathbb{N} \), \( 0 \leq s_1 \leq \cdots \leq s_m \leq s \) and let \( \psi_1, \ldots, \psi_m \) be bounded and continuous functions on \( S \), with \( \| \psi_i \|_{\infty} \leq 1 \) for \( 1 \leq i \leq m \). As for \( n \geq 0 \), \( \tilde{\nu}^n \) is a martingale solution to \( (SDE)^n \), and \( \nu^n = \sum_{i=1}^d \int_{\mathbb{R}} \mu_i * \rho^n(0, x) \mathbb{d}x \tilde{\nu}^n \), we have that for \( t \geq s \),

\[
\int_E \left[ \phi(X_t, Y_t) - \phi(X_s, Y_s) - \int_s^t \bar{L}_u \phi(X_u, Y_u) \mathbb{d}u \right] \prod_{i=1}^m \psi_i(X_{s_i}, Y_{s_i}) \mathbb{d}\nu^n(X, Y) \tag{D.5}
\]

Our goal now is to let \( n \to \infty \) in the Inequality (D.5), and obtain that for any \( s_1, \ldots, s_m, s, t \in [0, T] \), satisfying \( 0 \leq s_1 \leq \cdots \leq s_m \leq s \leq t \),

\[
\int_E \left[ \phi(X_t, Y_t) - \phi(X_s, Y_s) - \int_s^t \bar{L}_u \phi(X_u, Y_u) \mathbb{d}u \right] \prod_{i=1}^m \psi_i(X_{s_i}, Y_{s_i}) \mathbb{d}\nu(X, Y) \tag{D.6}
\]

\[
\leq \sum_{i=1}^d \int_s^t \int_{\mathbb{R}} \left( \frac{1}{2} |a_i(u, x) - \bar{a}_i(u, x)| |\partial^2_{xx} \phi(x, i)| + |b_i(u, x) - \bar{b}_i(u, x)| |\partial_x \phi(x, i)| \right) \mu_i(u, d\mathbb{x}) \mathbb{d}u \tag{D.7}
\]

\[
+ \sum_{i,j=1}^d \int_s^t \int_{\mathbb{R}} |q_{ij}(u, x) - \bar{q}_{ij}(u, x)| |\partial_x \phi(x, j)| \mu_i(u, d\mathbb{x}) \mathbb{d}u \tag{D.8}
\]

Here we explain how to conclude the proof of Theorem 4.2, if we suppose that (D.6)–(D.8) hold. Let us remark that the term \( 1 \int_{E} \int_s^t (\bar{L}_u - L_u) \phi(X_u, Y_u) \mathbb{d}u \prod_{i=1}^m \psi_i(X_{s_i}, Y_{s_i}) \mathbb{d}\nu(X, Y) \) is also dominated by the sum of the terms in the lines (D.7) and (D.8) By Cannarsa and D’Aprile (2006, Theorem 3.45), for \( 1 \leq i \leq d \), we can choose sequences of continuous functions \( (\tilde{a}^k_i)_{k \geq 0}, (\tilde{b}^k_i)_{k \geq 0}, \) and \( (\tilde{q}^k_{ij})_{k \geq 0} \) for \( 1 \leq j \leq d \) converging, respectively, to \( a_i, b_i, q_{ij} \) in \( L^1([0, T] \times \mathbb{R}, \mu_i(t, \cdot) dt) \). Then we obtain that

\[
\int_E \left[ \phi(X_t, Y_t) - \phi(X_s, Y_s) - \int_s^t L_u \phi(X_u, Y_u) \mathbb{d}u \right] \prod_{i=1}^m \psi_i(X_{s_i}, Y_{s_i}) \mathbb{d}\nu(X, Y) = 0.
\]

This is enough to conclude that \( \nu \) is a martingale solution to SDE (36) with initial condition \( \mu_0 \) and finish the proof of Theorem 4.2. We now show how to obtain (D.6)–(D.8). As the function

\[(s_1, \ldots, s_m, s, t) \to \int_E \left[ \phi(X_t, Y_t) - \phi(X_s, Y_s) - \int_s^t \bar{L}_u \phi(X_u, Y_u) \mathbb{d}u \right] \prod_{i=1}^m \psi_i(X_{s_i}, Y_{s_i}) \mathbb{d}\nu(X, Y)
\]

is right-continuous and as the terms in (D.7) and (D.8) are continuous in the variables \((s_1, \ldots, s_m, s, t)\), it is sufficient to show that the inequality in (D.6)–(D.8) holds if we moreover assume that \( s_1, \ldots, s_m, s, t \in D(\nu) \). Let us show that for \( s_1, \ldots, s_m, s, t \in D(\nu) \),
\[
\int_E \left[ \phi(X_t, Y_t) - \phi(X_s, Y_s) - \int_s^t \tilde{L}_u^\epsilon \phi(X_u, Y_u) du \right] \prod_{i=1}^m \psi_i \left( X_{s_i}, Y_{s_i} \right) d\nu^\epsilon_n(X, Y) \\
\rightarrow_{n \to \infty} \int_E \left[ \phi(X_t, Y_t) - \phi(X_s, Y_s) - \int_s^t \tilde{L}_u \phi(X_u, Y_u) du \right] \prod_{i=1}^m \psi_i \left( X_{s_i}, Y_{s_i} \right) d\nu(X, Y). \quad (D.9)
\]

By Ethier and Kurtz (1986, Lemma 7.7 and Theorem 7.8), we have convergence of the finite-dimensional distributions of \((\widetilde{\nu}_n)_{n \geq 0}\) corresponding to times \(s_1, \ldots, s_m, s, t\). Recalling that \(\nu^\epsilon_n = (\sum_{i=1}^d \int_\mathbb{R} \mu_i * \rho^\epsilon_n(0, x) dx)_n\) with \(\sum_{i=1}^d \int_\mathbb{R} \mu_i * \rho^\epsilon_n(0, x) dx \to \infty\), the following convergence holds:

\[
\int_E \left[ \phi(X_t, Y_t) - \phi(X_s, Y_s) \right] \prod_{i=1}^m \psi_i \left( X_{s_i}, Y_{s_i} \right) d\nu^\epsilon_n(X, Y) \\
\rightarrow_{n \to \infty} \int_E \left[ \phi(X_t, Y_t) - \phi(X_s, Y_s) \right] \prod_{i=1}^m \psi_i \left( X_{s_i}, Y_{s_i} \right) d\nu(X, Y). \quad (D.10)
\]

Using the decomposition

\[
\int_s^t \left[ \int_E \tilde{L}_u^\epsilon \phi(X_u, Y_u) \prod_{i=1}^m \psi_i \left( X_{s_i}, Y_{s_i} \right) d\nu^\epsilon_n(X, Y) - \int_E \tilde{L}_u \phi(X_u, Y_u) \prod_{i=1}^m \psi_i \left( X_{s_i}, Y_{s_i} \right) d\nu(X, Y) \right] du \\
\leq \int_s^t \left[ \int_E \tilde{L}_u \phi(X_u, Y_u) \prod_{i=1}^m \psi_i \left( X_{s_i}, Y_{s_i} \right) d\nu^\epsilon_n(X, Y) - \int_E \tilde{L}_u \phi(X_u, Y_u) \prod_{i=1}^m \psi_i \left( X_{s_i}, Y_{s_i} \right) d\nu(X, Y) \right] du \\
+ \int_s^t \int_E |\tilde{L}_u^\epsilon \phi(X_u, Y_u) - \tilde{L}_u \phi(X_u, Y_u)| \prod_{i=1}^m |\psi_i \left( X_{s_i}, Y_{s_i} \right)| d\nu^\epsilon_n(X, Y) du,
\]

we have that the term on line (D.11) converges to zero as \(n \to \infty\), by the same argument used to obtain (D.10) and Lebesgue’s theorem. For the term (D.12), as \(||\psi_i||_\infty \leq 1\) for \(1 \leq i \leq m\), we have that

\[
\int_s^t \int_E |\tilde{L}_u^\epsilon \phi(X_u, Y_u) - \tilde{L}_u \phi(X_u, Y_u)| \prod_{i=1}^m |\psi_i \left( X_{s_i}, Y_{s_i} \right)| d\nu^\epsilon_n(X, Y) du \\
\leq \sum_{i=1}^d \int_s^t \int_\mathbb{R} |\tilde{L}_u^\epsilon \phi(x, i) - \tilde{L}_u \phi(x, i)| \mu_i^\epsilon(u, x) d\nu^\epsilon_n du.
\]

As the terms \(\tilde{a}_i, \tilde{b}_i, \tilde{q}_{ij}\) for \(1 \leq i, j \leq d\) are continuous and bounded, to obtain convergence of the r.h.s. to 0 as \(\epsilon_n \to 0\), it is sufficient to show that for \(1 \leq i \leq d\), a continuous and bounded function \(z_i : \mathbb{R}^2 \to \mathbb{R}\) and \(z_i^\epsilon : \frac{(z_i \mu_i^\epsilon \rho^\epsilon_n)}{\mu_i^\epsilon}\) for \(\epsilon > 0\), the following convergence holds: \(I_{z_i^\epsilon} := \int_s^t \int_\mathbb{R} |z_i^\epsilon(u, x) - z_i(u, x)| \mu_i^\epsilon(u, x) d\nu^\epsilon_n du \to 0\). The previous term is bounded by

\[
I_{z_i^\epsilon} \leq \int_s^t \int_\mathbb{R} \rho^\epsilon_T(u-v) \left( \left( \int_\mathbb{R} \rho_X^\epsilon(x-y) |z_i(v, y) - z_i(u, x)| dx \right) \mu_i(v, dy) \right) dvdu. \quad (D.13)
\]
Let us note that by the weak continuity and boundedness of the family of measures \((\mu_i(t, dx))_{t \in [0,T]}\),

\[
\lim_{M \to \infty} \sup_{t \in [0,T]} \mu_i(t, \mathbb{R}\setminus(-M, M)) = \lim_{M \to \infty} \sup_{t \in [0,T]} \mu_i(t, \mathbb{R}\setminus(-M, M)) = 0.
\]

It is then sufficient to reason over a large enough compact set of \(\mathbb{R}^2\) and take advantage of the uniform continuity and the boundedness of the function \(z_i\). Moreover, as the mass carried by the measure outside that compact can be set as small as possible, this ensures the convergence of the r.h.s. of (D.13) to 0 as \(\epsilon \to 0\).

We now upper bound the term in the r.h.s. of Inequality (D.5) by (D.7) and (D.8). Let us define the function \(g : \mathbb{R} \to \mathbb{R}\) by \(g(v) = \int_{\mathbb{R}} |a_i(v, x) - \bar{a}_i(v, x)| |\partial^2_x \phi(x, i)| \mu_i(v, dx)\) and for \(\epsilon > 0\), the function \(g^\epsilon : \mathbb{R} \to \mathbb{R}\) by \(g^\epsilon(v) = \int_{\mathbb{R}} |a_i(v, x) - \bar{a}_i(v, x)| (\rho^e_X | |\partial^2_x \phi(x, i)|) \mu_i(v, dx)\). It is sufficient to check that

\[
\int_s^t \left( \int_{\mathbb{R}} \rho^e_T(u - v) g^\epsilon(v)dv \right) du \to_\epsilon \int_s^t g(u)du,
\]

and the convergence holds, as \(\rho^e_X | |\partial^2_x \phi| \to |\partial^2_x \phi|\) uniformly and thus the family \((g^\epsilon)_{\epsilon > 0}\) converges to \(g\) uniformly. A similar argument applies by replacing \(a_i\) by \(b_i\) or \(q_{ij}\) in the definitions of \(g\) and \(g^\epsilon\), so that (D.6)–(D.8) hold, and this concludes the proof.

**Proposition D.2.** For \(\epsilon > 0\), \((PDS)_{\epsilon}\) has a unique solution in the sense of Definition 4.1.

**Proof.** For \(0 < s \leq t \leq T\), let us define \((X^\epsilon_t(s, x, y), Y^\epsilon_t(s, x, y))\) the value at \(t\) of the solution to \((SDE)_{\epsilon}\) starting from \((x, y) \in S\) at \(s\). For \(\phi : S \to \mathbb{R}\) such that for \(i \in \{1, \ldots, d\}\), \(\phi(\cdot, i) \in C^\infty(\mathbb{R})\), we define \(T^\epsilon_t\phi(s, x, y) = \mathbb{E}[\phi(X^\epsilon_t(s, x, y), Y^\epsilon_t(s, x, y))]\), and for \(f : [0, t] \times S \to \mathbb{R}\), such that for \(1 \leq i \leq d\), \(f(\cdot, i) \in C^{1,2}_b([0, t] \times \mathbb{R})\), we define \(L^\epsilon f(s, x, i) := b^\epsilon_i(s, x) \partial_x f(s, x, i) + \frac{1}{2} a^2_i(s, x) \partial^2_{xx} f(s, x, i) + \sum_{j=1}^d q^\epsilon_{ij}(s, x) f(s, x, j)\). By Yin and Zhu (2010, Theorem 5.2), for \(1 \leq i \leq d\), the function \((s, x) \to (T^\epsilon_t\phi)(s, x, i)\) belongs to \(C^{1,2}_b([0, t] \times \mathbb{R})\), and the function \((s, x, i) \to T^\epsilon_t\phi(s, x, i)\) satisfies the backward Kolmogorov equation \(\partial_s (T^\epsilon_t\phi) + L^\epsilon (T^\epsilon_t\phi) = 0\). It is sufficient to prove that for \((\mu^\epsilon_1, \ldots, \mu^\epsilon_d)\) a solution to \((PDS)_{\epsilon}\),

\[
\frac{d}{ds} \sum_{i=1}^d \int_{\mathbb{R}} (T^\epsilon_t\phi)(s, x, i) \mu^\epsilon_i(s, dx) = \sum_{i=1}^d \int_{\mathbb{R}} \left[ \partial_s (T^\epsilon_t\phi)(s, x, i) + L^\epsilon (T^\epsilon_t\phi)(s, x, i) \right] \mu^\epsilon_i(s, dx) = 0
\]

(D.14)

to obtain uniqueness. Indeed, taking \(\mu^\epsilon\) and \(\tilde{\mu}^\epsilon\) two solutions to \((PDS)_{\epsilon}\) satisfying \(\sup_{t \in (0,T]} \mu^\epsilon(t, \mathbb{R}) + \tilde{\mu}^\epsilon(t, \mathbb{R})) \leq B\) for some \(B > 0\) and using the decomposition

\[
\left| \sum_{i=1}^d \int_{\mathbb{R}} (T^\epsilon_t\phi)(s, x, i)(\mu_i - \mu^\epsilon_i)(s, dx) \right| 
\leq \left| \sum_{i=1}^d \int_{\mathbb{R}} ((T^\epsilon_t\phi)(s, x, i) - (T^\epsilon_t\phi)(0, x, i))(\mu_i - \mu^\epsilon_i)(s, dx) \right|
\]
\[ + \sum_{i=1}^{d} \int_{\mathbb{R}} (T^c_t \phi)(0, x, i)(\mu_t - \mu^c_t)(s, dx) \]
\[ \leq Bs \sum_{i=1}^{d} ||\partial_s T^c_t \phi(\cdot, \cdot, i)||_{\infty} + \left| \sum_{i=1}^{d} \int_{\mathbb{R}} (T^c_t \phi)(0, x, i)(\mu_t - \mu^c_t)(s, dx) \right|, \]
we obtain that the function \( s \rightarrow \sum_{i=1}^{d} \int (T^c_t \phi)(s, x, i)(\mu_t - \tilde{\mu})_t(s, dx) \) has the limit 0 when \( s \to 0 \), as \( \mu^c_t \) and \( \tilde{\mu}^c_t \) have the same initial condition and is constant on \((0, t)\) if (D.14) holds. Finally, letting \( s \to t \), we have that \( \sum_{i=1}^{d} \int \phi(t, x, i)(\mu_t - \tilde{\mu}^c_t)(t, dx) = 0 \) for \( t \in [0, T] \), hence the uniqueness.

Finally, to prove the Equality (D.14), we use a classical density argument. For \( \psi, \phi \) two real-valued functions defined on \( S \) such that for \( i \in \{1, \ldots, d\} \), \( \phi(\cdot, i) \in C^\infty_c((0, t)) \) and \( \psi(\cdot, i) \in C^\infty_c(\mathbb{R}) \), by Definition 4.1, we have in the sense of distributions that
\[
\frac{d}{ds} \sum_{i=1}^{d} \int \phi(s, i) \psi(x, i) \mu_t^c(s, dx) = \sum_{i=1}^{d} \int \left[ \partial_s \phi(s, i) \psi(x, i) + L^c(\phi \psi)(s, x, i) \right] \mu_t^c(s, dx).
\] (D.15)

As \( \lim_{M \to \infty} \sum_{i=1}^{d} \int_{\mathbb{R} \setminus |M, M|} \mu_t^c(s, dx) < \infty \), we reason on a compact subset \( K \subset \mathbb{R} \) large enough and we take advantage of the density of functions \( (s, x) \in (0, t) \times \mathbb{R} \to \bar{\phi}(s) \bar{\psi}(x) \), where \( \bar{\phi} \in C^\infty_c((0, t)) \) and \( \bar{\psi} \in C^\infty_c(\mathbb{R}) \) in \( C^\infty_c((0, t) \times \mathbb{R}) \), to obtain the Equality (D.14) and conclude the proof.

\begin{lemma}
Let \( c > 0 \) and let \((X^c_t, Y^c_t)\) be the solution to \((SDE)_c\). Then \( \mathbb{E}[\sup_{t \in [0, T]} |X^c_t - X^c_0|] \leq T \tilde{b} + 2\sqrt{T} \tilde{a}^{1/2} \).
\end{lemma}

\begin{proof}
For \( t \in [0, T] \), we have the inequality
\[
|X^c_t - X^c_0| \leq \int_0^t |b^c_{Y^c_s}(X^c_s)| ds + \sup_{u \leq t} \int_0^u \sqrt{v^c_{Y^c_s}(X^c_s) dW^s} \leq T \tilde{b} + \sup_{u \leq T} \int_0^u \sqrt{a^c_{Y^c_s}(X^c_s) dW^s}.
\]
Moreover, using Doob’s inequality, \( \mathbb{E}[\sup_{u \leq T} \int_0^u \sqrt{a^c_{Y^c_s}(X^c_s) dW^s}] \leq 2\sqrt{T} \tilde{a}^{1/2} \). We conclude the proof by taking the supremum on \( t \in [0, T] \) and the expectation.
\end{proof}