Quantum phases in entropic dynamics*

Nicholas Carrara and Ariel Caticha
Department of Physics, University at Albany–SUNY, Albany, NY 12222, USA

Abstract

In the Entropic Dynamics framework the dynamics is driven by maximizing entropy subject to appropriate constraints. In this work we bring Entropic Dynamics one step closer to full equivalence with quantum theory by identifying constraints that lead to wave functions that remain single-valued even for multi-valued phases by recognizing the intimate relation between quantum phases, gauge symmetry, and charge quantization.

1 Introduction

In the Entropic Dynamics (ED) framework the Schrödinger equation is derived as an application of entropic methods of inference and, as always with inference, the first and most crucial step is to be clear about what we want to infer. What microstates are we talking about? This defines the ontology of the model. Once that choice is made the dynamics is driven by entropy subject to information expressed by constraints.

ED takes the epistemic view of the wave function $\Psi$ to its logical conclusion. Within an inferential framework it is not sufficient to just state that the probability $|\Psi|^2$ reflects a state of knowledge; it is also necessary to demand that the phase receive an epistemic interpretation, and that all changes in $\Psi$ be dictated by the maximum entropy and Bayesian updating rules. Thus, the ED framework is very restrictive: it must account for both the unitary time evolution described by the Schrödinger equation and the collapse of the wave function during measurement.

But even after ED succeeds in accomplishing these tasks a challenge still remains: is ED fully equivalent to quantum mechanics (QM) or does it merely reproduce a subset of its solutions? Problems of this kind were pointed out long ago by Takabayasi in the context of the hydrodynamical interpretation of QM, and later revived by Wallstrom in the context of Nelson’s stochastic

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1 The principle of maximum entropy as a method for inference can be traced to the pioneering work of E. T. Jaynes. For a pedagogical overview of Bayesian and entropic inference and further references see [1].
mechanics. Wallstrom’s objection is that stochastic mechanics leads to phases and wave functions that are either both multi-valued or both single-valued. Both alternatives are unsatisfactory because on one hand QM requires single-valued wave functions, while on the other hand single-valued phases exclude states that are physically relevant (e.g., states with non-zero angular momentum).

In previous work the constraints that drive the dynamics where introduced in two different ways, either by postulating some extra variables \[2\]–[12], or by the explicit introduction of a “drift” potential \[3\]–[6]. One of the goals of this paper is to show that these two types of constraint can be imposed simultaneously which lends the theory greater flexibility and expands the range of future potential applications. We identify constraints that lead to single-valued wave functions, but nevertheless allow for multi-valued phases\(^2\) and naturally lead to the local gauge symmetry required for electromagnetic interactions. Our argument involves two ingredients. The first is the recognition that a deeper understanding of the phase of the wave function must consider the intimate relation between quantum phases and gauge symmetry. The second ingredient is the recognition that in order for ED to agree with experiment it is necessary that the dynamics be linear. ED differs from standard QM in many crucial ways but its demand for linearity is not one of them. The demand that the linear and the probabilistic structures be compatible with each other implies that ED constraints must lead to single-valued wave functions \[11\].

Here we will focus on the derivation of the Schrödinger equation but the ED approach has been applied to a variety of other topics including the quantum measurement problem \[13]\[14\]; momentum and uncertainty relations \[15\]; the Bohmian limit \[6]\[16\] and the classical limit \[17\]; the extensions to curved spaces \[18\] and to relativistic fields \[19]\[20\].

2 Entropic Dynamics — a brief overview

The statistical model— We consider \(N\) particles living in a flat Euclidean space \(X\) with metric \(\delta_{ab}\). The first important assumption is that position plays a distinguished role: it defines the ontic state of the system. The fact that at all times particles have definite positions deviates from the standard Copenhagen interpretation according to which definite values are created by measurement\(^3\).

In ED positions are in general unknown; they are the quantities to be inferred. The position of each particle will be denoted by \(x^a_n\) where the index \(n = 1 \ldots N\) labels the particle and \(a = 1, 2, 3\) its spatial coordinates. The position of the system in configuration space \(X_N = X \times \ldots \times X\) will be denoted either

\(^2\)A hint towards a satisfactory resolution of Wallstrom’s objection is found in Takabayasi’s later work which incorporates spin into his hydrodynamical approach \[10\]. Although here we focus on non-spinning particles our choice of constraints can be generalized to particles with spin \(1/2\) — a project to be addressed in a future publication.

\(^3\)On the other hand, in ED — just as in the Copenhagen interpretation — other observables such as energy or momentum do not in general have definite values; their values are created by the act of measurement. These other quantities are epistemic in that they do not reflect properties of the particles but of the wave function.
by $x$ or by the components $x^A$ where $A = (n,a)$, and the corresponding volume element is $d^{3N}x = dx$.

The second assumption is that in addition to the particles there also exist some other variables denoted $y$ (2) [12]. This assumption is not unreasonable: the world does contain stuff beyond the $N$ particles of interest. It is also most fortunate that we need not be too specific about these $y$ variables. It turns out that the relevant information is conveyed by their entropy,

$$S(x) = - \int dy \, p(y|x) \log \frac{p(y|x)}{q(y)},$$

where we assume that the probability distribution $p(y|x)$ depends on the location $x$ of the particles and $q(y)$ is some unspecified underlying measure.

Having identified the microstates $(x,y) \in \mathbf{X}_N \times \mathbf{Y}$ we tackle the dynamics. The goal is to find the probability density $P(x'|x)$ for the transition from an initial $x$ to a new $x'$. Since both $x'$ and the corresponding $y'$ are unknown the relevant space is not just $\mathbf{X}_N$ but $\mathbf{X}_N \times \mathbf{Y}$. The distribution we seek is the joint distribution $P(x', y'|x, y)$. It is found by maximizing the appropriate entropy,

$$S[P,Q] = - \int dx' dy' \, P(x', y'|x, y) \log \frac{P(x', y'|x, y)}{Q(x', y'|x, y)},$$

relative to a joint prior $Q(x', y'|x, y)$ and subject to the appropriate constraints.

**The prior**— We adopt a prior $Q(x', y'|x, y)$ that represents a state of extreme ignorance: knowledge of $x'$ tells us nothing about $y'$ and vice versa. This is a product, $Q(x', y'|x, y) = Q(x'|x, y)Q(y'|x, y)$, in which $Q(x'|x, y)dx'$ and $Q(y'|x, y)dy'$ are uniform\(^4\) that is, proportional to the respective volume elements, $d^{3N}x = dx$ and $q(y)dy$. Since proportionality constants have no effect on the entropy maximization, the joint prior is

$$Q(x', y'|x, y) = q(y').$$

**The constraints**— We first write the posterior as a product,

$$P(x', y'|x, y) = P(x'|x, y)P(y'|x', x, y).$$

We require that the new $x'$ depends only on $x$ so we set $P(x'|x, y) = P(x'|x)$. We also require that the uncertainty in $y'$ depends only on $x'$, $P(y'|x', x, y) = p(y'|x')$. Therefore, the first constraint is

$$P(x', y'|x, y) = P(x'|x)p(y'|x').$$

To implement it substitute (3) and (4) into (2),

$$S[P,Q] = - \int dx' \, P(x'|x) \log P(x'|x) + \int dx' \, P(x'|x)S(x').$$

\(^4\)Strictly uniform non-normalizable priors can be mathematically problematic but here no such difficulties arise. By “uniform” we actually mean any distribution that is essentially flat over the support of the posterior which in our case will be infinitesimally narrow.

where $S(x)$ is given in eq.(11). Next, the continuity of the motion is enforced by requiring that the steps $\Delta x^n_a$ from $x^n_a$ to $x^n_a + \Delta x^n_a$ taken by each individual particle be infinitesimally short. This is implemented by imposing $N$ independent constraints,

$$
\int dx' P(x'|x) \Delta x^n_a \Delta x^n_b \delta_{ab} = \langle \Delta x^n_a \Delta x^n_b \rangle \delta_{ab} = \kappa_n, \quad (n = 1 \ldots N).
$$

where repeated indices are summed over and we eventually take the limit $\kappa_n \to 0$. The $\kappa_n$'s are chosen to be constant to reflect the translational symmetry of the space $X$ and they are $n$-dependent in order to accommodate non-identical particles.

The transition probability—Varying $P(x'|x)$ to maximize (6) subject to (7) and normalization gives

$$
P(x'|x) = \frac{1}{\zeta} \exp \left[ S(x') - \frac{1}{2} \sum_n \alpha_n \delta_{ab} \Delta x^n_a \Delta x^n_b \right],
$$

where $\zeta$ is a normalization constant and the Lagrange multipliers $\alpha_n$ are chosen to implement the constraints eq.(7). In eq.(8) it is clear that the infinitesimally short steps are obtained in the limit of large $\alpha_n$. It is therefore useful to Taylor expand,

$$
S(x') = S(x) + \sum_n \Delta x^n_a \partial S/\partial x^n_a + \ldots
$$

and rewrite $P(x'|x)$ as

$$
P(x'|x) = \frac{1}{Z} \exp \left[ -\frac{1}{2} \sum_n \alpha_n \delta_{ab} (\Delta x^n_a - \langle \Delta x^n_a \rangle) (\Delta x^n_b - \langle \Delta x^n_b \rangle) \right],
$$

where $Z$ is a new normalization constant and $\Delta x^n_a$ is given by eq.(12) below.

To find how these short steps accumulate we introduce time as a bookkeeping device. As discussed in [2]-[5] entropic time is measured by the fluctuations themselves (see eq.(13) below) which leads to the choice

$$
\alpha_n = \frac{m_n}{\eta \Delta t},
$$

where $\Delta t$ is the time taken by the short step, the $m_n$ are particle-specific constants that will be called “masses”, and $\eta$ is a constant that fixes the units of time relative to those of length and mass. A generic displacement is then expressed as an expected drift plus a fluctuation,

$$
\Delta x^n_a = \Delta x^A = b^A \Delta t + \Delta w^A,
$$

where $b^A(x)$ is the drift velocity,

$$
\langle \Delta x^A \rangle = b^A \Delta t \quad \text{with} \quad b^A = \frac{\eta}{m_n} \delta^{AB} \partial_B S = \eta m^A B \partial_B S,
$$

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and \( \partial_A = \partial/\partial x^a_n; m_{AB} = m_n \delta_{AB} \) is the “mass” tensor and \( m^{AB} = \delta^{AB}/m_n \) is its inverse. The fluctuations \( \Delta w^A \) satisfy,

\[
\langle \Delta w^A \rangle = 0 \quad \text{and} \quad \langle \Delta w^A \Delta w^B \rangle = \eta \frac{m^{AB}}{m_n} \delta_{AB} \Delta t = \eta m^{AB} \Delta t .
\]

Thus ED leads to the non-differentiable trajectories that are characteristic of a Brownian motion.

The Fokker-Planck equation— Once the probability for a single short step is found, eq. (10), the accumulation of many short steps leads to a probability distribution \( \rho(x,t) \) in configuration space that obeys a Fokker-Planck equation (FP),

\[
\frac{\partial \rho}{\partial t} = -\sum_a \partial_a (\rho v^a) = -\partial_A (\rho v^A),
\]

where \( v^A \) is the velocity of the probability flow in configuration space or current velocity. It is given by

\[
v^A = m^{AB} \partial_B \Phi_0 \quad \text{and} \quad \Phi_0 = \eta S - \eta \log \rho^{1/2}
\]

where \( \Phi_0 \) will be called the phase.

Hamiltonian entropic dynamics— The FP eq. (15) describes a standard diffusion of a single dynamical field, \( \rho(x) \), that evolves in response to a non-dynamical field given by the entropy \( S(x) \). In contrast, a quantum dynamics includes a second dynamical field, the phase of the wave function. In ED this evolving phase is introduced by continuously updating the constraint (5) which allows the entropy \( S(x) \), or equivalently the phase \( \Phi_0(x) \), to become dynamical.

First we note that without loss of generality we can always find a functional \( \tilde{H}[\rho, \Phi_0] \) so that \( \partial_t \rho = \delta \tilde{H}/\delta \Phi_0 \) reproduces the FP equation (15). The specific updating rule for \( S \) or \( \Phi_0 \) is inspired by an idea of Nelson’s [21]: requiring that \( \Phi_0 \) be updated in such a way that the functional \( \tilde{H}[\rho, \Phi_0] \) be conserved leads to Hamilton’s equations

\[
\partial_t \rho = \frac{\delta \tilde{H}}{\delta \Phi_0} \quad \text{and} \quad \partial_t \Phi_0 = \frac{\delta \tilde{H}}{\delta \rho} .
\]

\( \tilde{H}[\rho, \Phi_0] \) is the “ensemble” Hamiltonian. The second equation in (17) is a Hamilton-Jacobi equation (HJ). Additional arguments from information geometry [4] can then be invoked to suggest that the natural choice of \( \tilde{H} \) is

\[
\tilde{H}[\rho, \Phi_0] = \int dx \rho \left[ \frac{1}{2} m^{AB} \partial_A \Phi_0 \partial_B \Phi_0 + V + \xi m^{AB} \frac{1}{\rho^2} \partial_A \rho \partial_B \rho \right] .
\]

The first term in the integrand is the “kinetic” term that reproduces the FP equation (15). The second term represents the simplest non-trivial interaction and introduces the standard potential \( V(x) \). The third term, motivated by information geometry, is the trace of the Fisher information and is called the “quantum” potential. The parameter \( \xi \) controls the relative contributions of the two potentials: \( \xi = 0 \) leads to a stochastic classical mechanics; \( \xi > 0 \) leads to quantum theory — in fact, \( \xi \) defines Planck’s constant as \( \hbar = (8 \xi)^{1/2} \).
The Schrödinger equation— To conclude this brief review of ED we note that at this point the dynamics is fully specified by equations (17) and (18). We can combine $\rho$ and $\Phi_0$ into a single complex function, $\Psi_0 = \rho^{1/2} \exp(i\Phi_0/h)$. Then the pair of Hamilton’s equations (17) can be rewritten as a single complex Schrödinger equation that is explicitly linear,

$$i\hbar \partial_t \Psi_0 = -\frac{\hbar^2}{2m} \partial_A \partial_B \Psi_0 + V \Psi_0.$$ (19)

However, even though eqs.(17) can be written in the form (19), this does not mean that they are equivalent to the full quantum theory. The problem is that eqs.(17) only reproduce a subset of all the wave functions required by quantum mechanics. More specifically since both $S(x)$ and $\rho(x)$ are single-valued — the total change as one moves in a closed path vanishes,

$$\Delta S = \oint_{\Gamma} d\ell \partial_A S = 0 \quad \text{and} \quad \Delta \rho = \oint_{\Gamma} d\ell \partial_A \rho = 0,$$ (20)

so that both $\Phi_0$ and $\Psi_0$ are single-valued too. The single-valuedness of $\Psi_0$ is precisely what we want, but the single-valuedness of $\Phi_0$ is too restrictive. It excludes, for example, eigenstates of angular momentum that have manifestly multi-valued phases ($\Psi \propto e^{im\phi}$ where $\phi$ is the azimuthal angle and $m$ is an integer).

### 3 Gauge symmetry and multi-valued phases

A minimal ED was derived in the previous section. A richer dynamics that allows additional interactions can be achieved by imposing additional constraints.

**Additional constraints**— We assume that the motion of each particle is affected by an additional potential field $\varphi(x)$ where $x \in \mathbf{X}$ is a point in 3D space with the topological properties of an angle ($\varphi(x)$ and $\varphi(x) + 2\pi$ describe the same angle). We further assume that these angles can be redefined by different amounts $\chi(x)$ at different places, that is, the origin from which these angles are measured can be set independently at each $x$. This is a local gauge symmetry and it immediately raises the question of how one compare angles at different locations in order to define derivatives. The answer is well known: introduce a connection field, a vector potential $A_a(x)$ that defines which angle at $x + \Delta x$ is the “same” as the angle at $x$. This is implemented by imposing that as $\varphi \to \varphi + \chi$ then the connection transforms as $A_a \to A_a + \partial_a \chi$ so that the corrected derivative $\partial_a \varphi - A_a$ remains invariant.

To derive an ED that incorporates interactions gauge invariant interactions with these potentials, in addition to (7) and normalization, for each particle we impose the constraint

$$\langle \Delta x^a \rangle [\partial_a \varphi(x_n) - A_a(x_n)] = \kappa_n(x_n), \quad (n = 1 \ldots N)$$ (21)

Note that since $\varphi$ is dimensionless the vector potential $A_a$ has units of inverse length and this implicitly defines the units of electric charge. These are not the units conventionally adopted in electromagnetism.
where \( \kappa_n'(x_n) \) are functions to be specified below.

The transition probability \( P(x'|x) \) that maximizes the entropy \( S[P, Q] \) in (6) is

\[
P(x'|x) = \frac{1}{\zeta} \exp \left[ S(x') - \sum_n \left( \frac{\alpha_n}{2} \delta_{ab} \Delta x_n^a \Delta x_n^b - \beta_n \left( \partial_{na} \varphi(x_n) - A_n(x_n) \right) \Delta x_n^a \right) \right]
\]

(22)

where \( \partial_{na} = \partial / \partial x_n^a \), \( \alpha_n \) and \( \beta_n \) are Lagrange multipliers, and \( \zeta \) is a normalization constant. For large \( \alpha_n \) Taylor expand \( S(x') \) about \( x \), and use eq.(11), then, as in eq.(12) a generic displacement \( \Delta x_A \) can be expressed in terms of a expected drift plus a fluctuation, \( \Delta x_A = b_A \Delta t + \Delta w_A \), but the drift velocity (13) now includes a new term,

\[
b_n^a = \frac{m}{m_n} \delta^{ab} \left[ \partial_{nb} \{ S(x) + \beta_n \varphi(x_n) \} - \beta_n A_b(x_n) \right],
\]

(23)

while the fluctuations \( \Delta w_A \), eq.(14), remain unchanged.

Hamilton’s equations – As before, the accumulation of many short steps leads to the FP equation (15), but now the current velocity \( v^A = v_n^a \) must be suitably modified,

\[
v^A = m^{AB} \left( \partial_B \Phi - \bar{A}_B \right) \quad \text{with} \quad \Phi = \eta(S + \bar{\varphi} - \log \rho^{1/2}),
\]

(24)

where we introduced the configuration space quantities,

\[
\bar{A}_A(x) = \eta \beta_n A_a(x_n) \quad \text{and} \quad \bar{\varphi}(x) = \sum_n \beta_n \varphi(x_n).
\]

(25)

where \( A = (n, a) \). Note that \( v^A \) is gauge invariant. The new ensemble Hamiltonian \( \hat{H} \), eq.(18), is

\[
\hat{H}[\rho, \Phi] = \int dx \left[ \frac{\rho^{AB}}{2} \left( \partial_A \Phi - \bar{A}_A \right) \left( \partial_B \Phi - \bar{A}_B \right) + \rho V + \frac{\hbar^2}{2m^{AB}} \partial_A \rho \partial_B \rho \right],
\]

and the new FP equation now reads,

\[
\partial_t \rho = -\partial_A \left[ \rho m^{AB} \left( \partial_B \Phi - \bar{A}_B \right) \right] = \frac{\delta \hat{H}}{\delta \Phi}.
\]

(27)

As before, the requirement that \( \hat{H} \) be conserved for arbitrary initial conditions amounts to imposing the conjugate Hamilton equation, eq.(17), which leads to the Hamilton-Jacobi equation,

\[
\partial_t \Phi = -\frac{\delta \hat{H}}{\delta \rho} = - \frac{1}{2} m^{AB} \left( \partial_A \Phi - \bar{A}_A \right) \left( \partial_B \Phi - \bar{A}_B \right) - V + \frac{\hbar^2}{2m^{AB}} \partial_A \partial_B \rho^{1/2} \frac{1}{\rho^{1/2}}.
\]

(28)

Finally, we combine \( \rho \) and \( \Phi \) into a single wave function, \( \Psi = \rho^{1/2} \exp(i\Phi/\hbar) \), to obtain the linear Schrödinger equation,

\[
\hbar \partial_t \Psi = -\sum_n \frac{\hbar^2}{2m_n} \delta^{ab} \left( \frac{\partial}{\partial x_n^a} - \frac{i}{\hbar} \eta \beta_n A_a(x_n) \right) \left( \frac{\partial}{\partial x_n^b} - \frac{i}{\hbar} \eta \beta_n A_b(x_n) \right) \Psi + V \Psi.
\]

(29)
4 Discussion

Electric charges are Lagrange multipliers—Recalling the standard expression for covariant derivatives,

\[
\frac{\partial}{\partial x_n^a} - \frac{iq_n}{\hbar c} A_a(x_n),
\]

(30)

\((q_n\) is the electric charge of particle \(n\) and \(c\) is the speed of light\) shows that \((29)\) is indeed the Schrödinger equation provided the multipliers \(\beta_n\) are chosen to be particle-dependent constants that are related to electric charges by

\[
\beta_n = \frac{q_n}{\eta c} \quad \text{or} \quad q_n = c\eta \beta_n.
\]

(31)

Thus, in ED electric charges are Lagrange multipliers that measure the strength of the particles’ coupling to the \(\varphi\) and \(A_a\) potentials.

Single-valued wave functions, quantized circulation, and quantized charges—The success of any framework for inference such as ED depends on identifying the correct constraints. The choice of constraints in section 2 succeeds in reproducing many of the features of quantum theory including a linear Schrödinger equation but is ultimately unsatisfactory because it leads to single-valued wave functions with single-valued phases that fail to include all quantum states.

The choice of constraints adopted in section 3 represent an improvement because they take into account the relation between quantum phases and gauge symmetry. However, the wave functions \(\Psi\) obtained for generic choices of the multipliers \(\beta_n\) are also problematic in that they give multi-valued phases \(\Phi\), eq.(24), that lead to multi-valued wave functions. Indeed, since \(\varphi\) is an angle the integral over a closed loop \(\Gamma_n\) in which all particles except \(n\) are kept fixed gives

\[
\Delta \varphi = \oint_{\Gamma_n} d\ell_n^a \partial_{na} \varphi = 2\pi \nu(\Gamma_n),
\]

(32)

where \(\nu(\Gamma_n)\) is an integer that depends on the loop \(\Gamma_n\). Since \(S\) and \(\log \rho\) are single-valued, from \((24)\), we have

\[
\Delta \Phi = \oint_{\Gamma_n} d\ell_n^a \partial_{na} \Phi = \eta \beta_n \oint_{\Gamma_n} d\ell_n^a \partial_{na} \varphi = \frac{\eta \beta_n}{\hbar} 2\pi \nu(\Gamma_n),
\]

(33)

so that \(\Psi\) is not single-valued.

Unfortunately, this means that even though \((29)\) is linear, its linearity is in conflict with the underlying probabilistic structure. To see the problem consider two multivalued ED solutions of \((29)\), \(\Psi_1\) and \(\Psi_2\). Their magnitudes \(|\Psi_1|^2 = \rho_1\) and \(|\Psi_2|^2 = \rho_2\) are single-valued because they are probability densities. However, even though \(\alpha_1 \Psi_1 + \alpha_2 \Psi_2 = \Psi_3\) is also a solution, it turns out that its magnitude \(|\Psi_3|^2\) will in general turn out to be multivalued which precludes a probabilistic interpretation \([11]\). Mere linearity is not enough. The condition
for the linear and probabilistic structures to be compatible with each other is that wave functions be single-valued.

Inspection of Eq. (33) for arbitrary loops shows that the choice of constraint — that is, the choice of $\beta_n$ — that leads to single-valued wave functions is

$$\frac{\eta \beta_n}{\hbar} = \mu$$

(34)

where $\mu$ is an integer.

Equation (31) then shows that electric charges must be quantized in units of a basic charge $q = \hbar c$,

$$\frac{\eta \beta_n}{\hbar} = q_n \frac{\hbar}{\hbar c} = \mu \quad \text{or} \quad q_n = \mu q .$$

(35)

Changing to conventional units for charges and potentials is straightforward; just rescale $\lambda q_n = q'_n$ and $A_a/\lambda = A'_a$ so that $q_n A_a = q'_n A'_a$.

**Conclusion**— The equivalence of ED and quantum mechanics with wave functions that remain single-valued even for multi-valued phases is achieved by imposing constraints that recognize the intimate relation between quantum phases and gauge symmetry. The condition for compatibility between the probabilistic and linear structures is that charges be quantized.

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