Generalized Minimum Distance Estimators in Linear Regression with Dependent Errors

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Abstract

This paper discusses minimum distance estimation method in the linear regression model with dependent errors which are strongly mixing. The regression parameters are estimated through the minimum distance estimation method, and asymptotic distributional properties of the estimators are discussed. A simulation study compares the performance of the minimum distance estimator with other well celebrated estimator. This simulation study shows the superiority of the minimum distance estimator over another estimator. 

KoulMde (R package) which was used for the simulation study is available online. See section 4 for the detail.

Keywords: Dependent errors; Linear regression; Minimum distance estimation; Strongly mixing

1 Introduction

Consider the linear regression model

\( y_i = x_i' \beta + \varepsilon_i, \)  

(1.1)

where \( E\varepsilon_i \equiv 0, \ x_i = (1, x_{i2}, \ldots, x_{ip})' \in \mathbb{R}^p \) with \( x_{ij}, j = 2, \ldots, p, i = 1, \ldots, n \) being non random design variables, and where \( \beta = (\beta_1, \ldots, \beta_p)' \in \mathbb{R}^p \) is the parameter vector of interest. The methodology where the estimators are obtained by minimizing some dispersions or pseudo distances between the data and the underlying model is referred to as the minimum distance (m.d.) estimation method. In this paper we estimate regression parameter vector \( \beta \) by the m.d. estimation method when the collection of \( \varepsilon_i \) in the model (1.1) is a dependent process.

Let \( T_1, \ldots, T_n \) be independent identically distributed (i.i.d.) random variables (r.v.’s) with distribution function (d.f.) \( G_\vartheta \) where \( \vartheta \) is unknown. The classical m.d. estimator of \( \vartheta \) is obtained by minimizing following Cramèr-von Mises (CvM) type \( L_2 \)-distance

\( \int \{ G_n(y) - G_\vartheta(y) \}^2 dH(y) \)  

(1.2)
where $G_n$ is an empirical d.f. of $T_i$’s and $H$ is a integrating measure. There are multiple reasons as to why CvM type distance is preferred, including the asymptotic normality of the corresponding m.d. estimator; see, e.g., Parr and Schucany (1980), Parr (1981) and Millar (1981). Many researchers have tried various $H$’s to obtain the m.d. estimators. Anderson and Darling (1952) proposed Anderson-Darling estimator obtained by using $dG_\varphi / \{G_\varphi (1 - G_\varphi )\}$ for $dH$. Another important example includes $H(y) \equiv y$, giving a rise to Hodges - Lehmann type estimators. If $G_n$ and $G_\varphi$ of the integrand are replaced with kernel density estimator and assumed density function of $T_i$, the Hellinger distance estimators will be obtained; see Beran (1977).

Departing from one sample setup, Koul and DeWet (1983) extended the domain of the application of the m.d. estimation to the regression setup. On the assumption that $\varepsilon_i$’s are i.i.d. r.v.’s with a known d.f. $F$, they proposed a class of the m.d. estimators by minimizing $L_2$-distances between a weighted empirical d.f. and the error d.f. $F$. Koul (2002) extended this methodology to the case where error distribution is unknown but symmetric around zero. Furthermore, it was shown therein that when the regression model has independent non-Gaussian errors the m.d. estimators of the regression parameters — obtained by minimizing $L_2$-distance with various integrating measures — have the least asymptotic variance among other estimators including Wilcoxon rank, the least absolute deviation (LAD), the ordinary least squares (OLS) and normal scores estimators of $\beta$: e.g. the m.d. estimators obtained with a degenerate integrating measure display the least asymptotic variance when errors are independent Laplace r.v.’s.

However, the efficiency of the m.d. estimators depends on the assumption that errors are independent; with the errors being dependent, the m.d. estimation method will be less efficient than other estimators. Examples of the more efficient methods include the generalized least squares (GLS); GLS is nothing but regression of transformed $y_i$ on transformed $x_i'$. The most prominent advantage of using the GLS method is “decorrelation” of errors as a result of the transformation. Motivated by efficiency of m.d. estimators — which was demonstrated in the case of independent non-Gaussian errors — and the desirable property of the GLS (decorrelation of the dependent errors), the author proposes generalized m.d. estimation method which is a mixture of the m.d. and the GLS methods: the m.d. estimation will be applied to the transformed variables. “Generalized” means the domain of the application of the m.d. method covers the case of dependent errors; to some extent, the main result of this paper generalizes the work of Koul (2002). As the efficiency of the m.d. method is demonstrated in the case of independent errors, the main goal of this paper is to show that the generalized m.d. estimation method is still competitive when the linear regression model has dependent errors; indeed, the simulation study empirically shows that the main goal is achieved.
The rest of this article is organized as follows. In the next section, characteristics of dependent errors used through this paper is studied. Also, the CvM type distance and various processes — which we need in order to obtain the estimators of $\beta$ — will be introduced. Section 3 describes the asymptotic distributions and some optimal properties of the estimators. Findings of a finite sample simulations are described in Section 4. All the proofs are deferred until Appendix. In the remainder of the paper, an Italic and boldfaced variable denotes a vector while a non-Italic and boldfaced variable denotes a matrix. An identity matrix will carry a suffix showing its dimension: e.g. $I_{n \times n}$ denotes a $n \times n$ identity matrix.

For a function $f : \mathbb{R} \to \mathbb{R}$, let $|f|_{H}^{2}$ denote $\int f^{2}(y)\,dH(y)$. For a real vector $\mathbf{u} \in \mathbb{R}^{p}$, $\|\mathbf{u}\|$ denotes Euclidean norm. For any r.v. $Y$, $\|Y\|_{p}$ denotes $(E|Y|^{p})^{1/p}$. For a real matrix $\mathbf{W}$ and $y \in \mathbb{R}$, $\mathbf{W}(y)$ means that its entries are functions of $y$.

2 Strongly mixing process & CvM type distance

Let $\mathcal{F}_{m}$ be the $\sigma$-field generated by $\varepsilon_{m}, \varepsilon_{m+1}, \ldots, \varepsilon_{l}$, $m \leq l$. The sequence $\{\varepsilon_{j}, j \in \mathbb{Z}\}$ is said to satisfy the strongly mixing condition if

$$
\alpha(k) := \sup \{|P(A \cap B) - P(A)P(B)| : A \in \mathcal{F}_{-\infty}^{0}, B \in \mathcal{F}_{k}^{\infty}\} \to 0,
$$

as $k \to \infty$. $\alpha$ is referred to as mixing number. Chanda (1974), Gorodetskii (1977), Koul (1977), and Withers (1979) investigated the decay rate of the mixing number. Having roots in their works, Section 3 defines the decay rate assumed in this paper; see, e.g., the assumption (a.8). Hereinafter the errors $\varepsilon_{i}$’s are assumed to be strongly mixing with mixing number $\alpha$. In addition, $\varepsilon_{i}$ is assumed to be stationary and symmetric around zero.

Next, we introduce the basic processes and the distance which are required to obtain desired result. Recall the model (1.1). Let $\mathbf{X}$ denote the $n \times p$ design matrix whose $i$th row vector is $\mathbf{x}_{i}$. Then the model (1.1) can be expressed as

$$
\mathbf{y} = \mathbf{X}\beta + \varepsilon,
$$

where $\mathbf{y} = (y_{1}, y_{2}, \ldots, y_{n})' \in \mathbb{R}^{n}$ and $\varepsilon = (\varepsilon_{1}, \varepsilon_{2}, \ldots, \varepsilon_{n})' \in \mathbb{R}^{n}$ Let $\mathbf{Q}$ be any $n \times n$ real matrix so that the inverse of $\mathbf{Q}^{2}$ is a positive definite symmetric matrix. Note that the diagonalization of positive definite symmetric matrix guarantees the existence of $\mathbf{Q}$ which is also a symmetric matrix. Let $\mathbf{q}_{i}' = (q_{i1}, \ldots, q_{in})$ for $1 \leq i \leq n$ denote the $i$th row vector of $\mathbf{Q}$. Define transformed variables

$$
\tilde{y}_{i} = \mathbf{q}_{i}'\mathbf{y}, \quad \tilde{x}_{i}' = \mathbf{q}_{i}'\mathbf{X}, \quad \tilde{\varepsilon}_{i} = \mathbf{q}_{i}'\varepsilon, \quad 1 \leq i \leq n.
$$
As in the GLS method, $Q$ obtained from covariance matrix of $\varepsilon$ transforms dependent errors into uncorrelated ones, i.e., “decorrelates” the errors. However, the GLS obtains $Q$ in a slightly different manner. Instead of using $Q^2$, the GLS equates $Q'Q$ to the inverse of the covariance matrix, i.e., the GLS uses Cholesky decomposition. The empirical result in Section 4 describes that $Q$ from the diagonalization yields better estimators. Here we propose the class of the generalized m.d. estimators of the regression parameter upon varying $Q$. We impose Noether (1949) condition on $QX$. Now let $A = (X'Q^2X)^{-1/2}$ and $a_j$ denote $j$th column of $A$. Let $D = ((d_{ik}))$, $1 \leq i \leq n$, $1 \leq k \leq p$, be an $n \times p$ matrix of real numbers and $d_j$ denote $j$th column of $D$. As stated in Koul (2002, p.60), if $D = QXA$ (i.e., $d_{ik} = q'_iXa_k$), then under Noether condition,

\begin{equation}
(2.1) \quad \sum_{i}^{n} d_{ik}^2 = 1, \quad \max_{1\leq i\leq n} d_{ik}^2 = o(1) \quad \text{for all } 1 \leq k \leq p.
\end{equation}

Next, define CvM type distance from which the generalized m.d. estimator are obtained. Let $f_i$ and $F_i$ denote the density function and the d.f. of $\tilde{\varepsilon}_i$, respectively. Analogue of (1.2) — with $G_n$ and $G_\theta$ being replaced by empirical d.f. of $\tilde{\varepsilon}_i$ and $F_i$ — will be a reasonable candidate. However, the d.f. $F_i$ is rarely known. Since the original regression error $\varepsilon_i$’s are assumed to be symmetric, the transformed error $\tilde{\varepsilon}_i$’s are also symmetric; therefore we introduce, as in Koul (2002; Definition 5.3.1),

\begin{align*}
U_k(y, b; Q) &:= \sum_{i=1}^{n} d_{ik} \{ I(q'_iy - q'_iXb \leq y) - I(-q'_iy + q'_iXb < y) \}, \\
U(y, b; Q) &:= (U_1(y, b; Q), ..., U_p(y, b; Q))', \quad y \in \mathbb{R}, \\
L(b; Q) &:= \int \|U(y, b; Q)\|^2 dH(y), \quad b \in \mathbb{R}^p,
\end{align*}

$$
= \sum_{k=1}^{p} \int \left[ \sum_{i=1}^{n} d_{ik} \{ I(q'_iy - q'_iXb \leq y) - I(-q'_iy + q'_iXb < y) \} \right]^2,
$$

where $I(\cdot)$ is an indicator function, and $H$ is a $\sigma$–finite measure on $\mathbb{R}$ and symmetric around 0, i.e., $dH(-x) = -dH(x)$, $x \in \mathbb{R}$. Subsequently, define $\hat{\beta}$ as

$$
L(\hat{\beta}; Q) = \inf_{b \in \mathbb{R}^p} L(b; Q).
$$

Next, define

$$
\tilde{Q}_i := \begin{bmatrix} q'_i \\ 0' \\ \vdots \\ 0' \end{bmatrix}, \quad \tilde{Q} := \begin{bmatrix} \tilde{Q}_1 \\ \tilde{Q}_2 \\ \vdots \\ \tilde{Q}_n \end{bmatrix},
$$
where $q'_i$ is the $i$th row vector of $Q$ and $0 = (0, ..., 0)' \in \mathbb{R}^n$; observe that $\tilde{Q}_i$ and $\tilde{Q}$ are $n \times n$ and $n^2 \times n$ matrices, respectively. Define a $n \times n^2$ matrix $I_f(y)$ so that its $(i,j)$th entry is $f_i(y)I(j = n(i - 1) + 1)$: e.g., $(k,k(k - 1) + 1)$th entry is $f_k(y)$ for all $1 \leq k \leq n$ and all other entries are zeros. Finally, define following matrices:

\begin{equation}
\Sigma_D := \int I_f(y)DD'I_f(y) \, dH(y), \quad \Sigma := AX'\tilde{Q}'\Sigma_D\tilde{Q}XA,
\end{equation}

which are needed for the asymptotic properties of $\hat{\beta}$. Let $f_{ij}^H := \int f_i f_j \, dH$ and $d_{ij}^* := \sum_{k=1}^p d_{ik}d_{jk}$. Note that

$$\Sigma = AX'\left[ \sum_{i=1}^n \sum_{j=1}^n d_{ij}^* f_{ij}^H q_i q'_j \right]XA.$$

3 Asymptotic distribution of $\hat{\beta}$

In this section we investigate the asymptotic distribution of $\hat{\beta}$ under the current setup. Note that minimizing $L(\cdot; Q)$ does not have the closed form solutions; only numerical solutions can be tried, and hence it would be impracticable to derive asymptotic distribution of $\hat{\beta}$. To redress this issue, define for $b \in \mathbb{R}^p$

$$L^*(b; Q) = \int \|U(y, \beta; Q) + 2\Sigma_{DA}(y)A^{-1}(b - \beta)\|^2 \, dH(y),$$

where $\Sigma_{DA}(y) := D'I_f(y)\tilde{Q}XA$ is a $p \times p$ matrix. Next, define

$$L^*(\tilde{\beta}; Q) = \inf_{b \in \mathbb{R}^p} L^*(b; Q).$$

Unlike $L(\cdot; Q)$, minimizing $L^*(\cdot; Q)$ has the closed form solution. Therefore, it is not unreasonable to approximate the asymptotic distribution of $\tilde{\beta}$ by one of $\tilde{\beta}$ if $L(\cdot; Q)$ can be approximated by $L^*(\cdot; Q)$. This idea is plausible under certain conditions which are called uniformly locally asymptotically quadratic; see Koul (2002, p.159) for the detail. Under these conditions, it was shown that difference between $\hat{\beta}$ and $\tilde{\beta}$ converges to zero in probability; see theorem 5.4.1. The basic method of deriving the asymptotic properties of $\tilde{\beta}$ is similar to that of sections 5.4, 5.5 of Koul (2002). This method amounts to showing that $L(\beta + Au; Q)$ is uniformly locally asymptotically quadratic in $u$ belonging to a bounded set and $\|A^{-1}(\tilde{\beta} - \beta)\| = O_p(1)$. To achieve these goals we need the following assumptions which in turn have roots in section 5.5 of Koul (2002).
The matrix $X'Q^2X$ is nonsingular and, with $A = (X'Q^2X)^{-1/2}$, satisfies
\[
\limsup_{n \to \infty} n \max_{1 \leq j \leq p} \|d_j\|^2 < \infty.
\]

The integrating measure $H$ is $\sigma-$finite and symmetric around 0, and
\[
\int_0^\infty (1 - F_i)^{1/2} dH < \infty, \quad 1 \leq i \leq n.
\]

For any real sequences $\{a_n\}, \{b_n\}$, $b_n - a_n \to 0$,
\[
\limsup_{n \to \infty} \int_{b_n}^{a_n} \int f_i(y + x) dH(y) dx = 0, \quad 1 \leq i \leq n.
\]

For $a \in \mathbb{R}$, define $a^+ := \max(a, 0)$, $a^- := a^+ - a$. Let $\theta_i := \|q_i'XA\|$. For all $u \in \mathbb{R}^p$, $\|u\| \leq b$, for all $\delta > 0$, and for all $1 \leq k \leq p$,
\[
\limsup_{n \to \infty} \int \left[ \sum_{i=1}^n d_{ik}^+ \{ F_i(y + q_i'XAu + \delta \theta_i) - F_i(y + q_i'XAu - \delta \theta_i) \} \right]^2 dH(y) \leq c\delta^2,
\]
where $c$ does not depend on $u$ and $\delta$.

For each $u \in \mathbb{R}^p$ and all $1 \leq k \leq p$,
\[
\int \left[ \sum_{i=1}^n d_{ik} \{ F_i(y + q_i'XAu) - F_i(y) - q_i'XAu f_i(y) \} \right]^2 dH(y) = o(1).
\]

$F_i$ has a continuous density $f_i$ with respect to the Lebesgue measure on $(\mathbb{R}, \mathcal{B})$ for $i = 1, 2, ..., n$.

$0 < \int_0^\infty f_i^r dH < \infty$, for $r = 1/2, 1, 2$ and $i = 1, 2, ..., n$.

The $\{\varepsilon_i\}$ in the model (1.1) is strongly mixing with mixing number $\alpha(\cdot)$ satisfying
\[
\limsup_{n \to \infty} \sum_{k=1}^{n-1} k^2 \alpha(k) < \infty.
\]

Remark 3.1. Note that (a.1) implies Noether condition and (a.2) implies $\int_0^\infty (1 - F_i) dH < \infty$. From Corollary 5.6.3 of Koul (2002), we note that in the case of i.i.d. errors, the asymptotic normality of $\hat{\beta}$ was established under the weaker conditions: Noether condition and $\int_0^\infty (1 -
\( F_i dH < \infty \). The dependence of the errors now forces us to assume two stronger conditions (a.1) and (a.2).

**Remark 3.2.** Here we discuss examples of \( H \) and \( F \) that satisfy (A.2). Clearly it is satisfied by any finite measure \( H \). Next consider the \( \sigma \)-finite measure \( H \) given by \( dH \equiv \{ F_i(1-F_i) \}^{-1} dF_i \), \( F \) a continuous d.f. symmetric around zero. Then \( F_i(0) = 1/2 \) and

\[
\begin{align*}
\int_0^\infty (1-F_i)^{1/2} dH &= \int_0^\infty \frac{(1-F_i)^{1/2}}{F_i(1-F_i)} dF_i \\
&\leq 2 \int_{1/2}^1 (1-u)^{-1/2} du < \infty.
\end{align*}
\]

Another useful example of a \( \sigma \)-finite measure \( H \) is given by \( H(y) \equiv y \). For this measure, (a.2) is satisfied by many symmetric error d.f.s including normal, logistic, and Laplace. For example, for normal d.f., we do not have a closed form of the integral, but by using the well celebrated tail bound for normal distribution — see e.g., Theorem 1.4 of Durrett (2005) — we obtain

\[
\int_0^\infty \{1 - F_i(y)\}^{1/2} dy \leq (2\pi)^{-1/2} \int_0^\infty y^{-1/2} \exp(-y^2/4) dy = (2/\pi)^{1/2} \Gamma(1/4).
\]

Recall from Koul (2002) that the \( \hat{\beta} \) corresponding to \( H(y) \equiv y \) is the extensions of the one sample Hodges-Lehmann estimator of the location parameter to the above regression model.

**Remark 3.3.** Consider condition (a.7). If \( f_i \)'s are bounded then \( \int f_i^{1/2} dH < \infty \) implies the other two conditions in (a.7) for any \( \sigma \)-finite measure \( H \). For \( H(y) \equiv y \), \( \int f_i^{1/2}(y) dy < \infty \) when \( f_i \)'s are normal, logistic or Laplace densities. In particular, when \( dH = \{ F_i(1-F_i) \}^{-1} dF_i \) and \( F_i \)'s are logistic d.f.'s, so that \( dH(y) \equiv dy \), this condition is also satisfied.

We are ready to state the needed results. The first theorem establishes the needed uniformly locally asymptotically quadraticity while the corollary shows the boundedness of a suitably standardized \( \hat{\beta} \). Theorem 3.1 and Corollary 3.1 are counterparts of conditions (A\( \tilde{\text{A}} \)) and (A5) in theorem 5.4.1 of Koul (2002), respectively. Note that condition (A4) in theorem 5.4.1 is met by (A.7) in the Appendix; condition (A6) in theorem 5.4.1 is trivial.

**Theorem 3.1.** Let \( \{y_i, 1 \leq i \leq n\} \) be in the model (1.1). Assume that (a.1)-(a.8) hold. Then, for any \( 0 < c < \infty \),

\[
E \sup_{\|A^{-1}(b-\beta)\| \leq \epsilon} \|\mathcal{L}(b; Q) - \mathcal{L}^*(b; Q)\| = o(1).
\]

**Proof.** See Appendix. \( \Box \)

**Corollary 3.1.** Suppose that the assumptions of Theorem 3.1 hold. Then for any \( \epsilon > 0 \),
\[ 0 < M < \infty \text{ there exists an } N, \text{ and } 0 < c < \infty \text{ such that} \]

\[ P\left( \inf_{\|A^{-1}(b-\beta)\| \geq c} L(b; Q) \geq M \right) \geq 1 - \epsilon, \quad \forall n \geq N. \]  

**Proof.** See Appendix. \[\square\]

**Theorem 3.2.** Under the assumptions of Theorem 3.1,

\[ A^{-1}(\tilde{\beta} - \beta) = -\frac{1}{2} \Sigma^{-1} AX' \tilde{Q}' \int I'_f(y)DU(y, \beta) dH(y) + o_p(1), \]

where \( \Sigma \) is as in (2.2).

**Proof.** Note that the first term in the right-hand side is nothing but \( A^{-1}(\tilde{\beta} - \beta) \). Therefore, the proof follows from Theorem 3.1 and Corollary 3.1, as in i.i.d. case illustrated in the theorem 5.4.1 of Koul (2002).

Next, define

\[ \psi_i(x) := \int_{-\infty}^{-x} f_i(y) dH(y) - \int_{-\infty}^{x} f_i(y) dH(y), \]

\[ Z_n := \int I'_f(y)DU(y, \beta) dH(y). \]

Symmetry of the \( F_i \) around 0 yields \( E\psi_i(\tilde{\epsilon}_j) = 0 \) for \( 1 \leq i, j \leq n \). Let \( \Sigma_{ZZ} \) denote covariance matrix of \( AX'\tilde{Q}'Z_n \). Define a \( n \times n \) matrix \( \Sigma_{\psi} \) and write \( \Sigma_{\psi} = ((\gamma_{ij})) \), \( 1 \leq i \leq p, 1 \leq j \leq p \) where

\[ \gamma_{ij} = \sum_{l=1}^{n} \sum_{h=1}^{n} d^*_l d^*_h E [\psi_i(q'_l \tilde{\epsilon}) \psi_j(q'_h \tilde{\epsilon})]. \]

Observe that

\[ \Sigma_{ZZ} = E(AX'\tilde{Q}'Z_n Z'_n \tilde{Q}XA) = AX'Q'\Sigma_{\psi}QXA. \]

Now, we are ready to state the asymptotic distribution of \( \hat{\beta} \).

**Lemma 3.1.** Assume \( \Sigma_{ZZ} \) is positive definite for all \( n \geq p \). In addition, assume that

\[ \sup_{u \in \mathbb{R}^p, \|u\| = 1} u' \Sigma_{ZZ}^{-1} u = O(1). \]

Then

\[ \Sigma_{ZZ}^{-1/2} AX'\tilde{Q}'Z_n \rightarrow_d N(0, I_{p \times p}), \]

where \( 0 = (0, ..., 0)' \in \mathbb{R}^p \) and \( I_{p \times p} \) is the \( p \times p \) identity matrix.
**Proof.** To prove the claim, it suffices to show that for any \( \lambda \in \mathbb{R}^p \), \( \lambda' \Sigma_{ZZ}^{-1/2}AX'\tilde{Q}'Z_n \) is asymptotically normally distributed. Note that

\[
\lambda' \Sigma_{ZZ}^{-1/2}AX'\tilde{Q}'Z_n = \sum_{i=1}^n \left[ \left( \lambda' \Sigma_{ZZ}^{-1/2}AX'q_i \right) \sum_{j=1}^n d_{ij}^* \psi_i(q_j \varepsilon) \right],
\]

which is the sum as in the theorem 3.1 from Mehra and Rao (1975) with \( c_{ni} = \lambda' \Sigma_{ZZ}^{-1/2}AX'q_i \) and \( \xi_{ni} = \sum_{j=1}^n d_{ij}^* \psi_i(q_j \varepsilon) \). Note that

\[
\tau_c^2 := \sum_{i=1}^n c_{ni}^2 = \lambda' \Sigma_{ZZ}^{-1} \lambda, \quad \sigma_n^2 := E \left\{ \sum_{i=1}^n \left[ \left( \lambda' \Sigma_{ZZ}^{-1/2}AX'q_i \right) \sum_{j=1}^n d_{ij}^* \psi_i(q_j \varepsilon) \right] \right\}^2 = \| \lambda \|^2.
\]

Also, observe that

\[
\max_{1 \leq i \leq n} c_{ni}^2/\tau_c^2 \leq \max_{1 \leq i \leq n} \frac{\| \lambda' \Sigma_{ZZ}^{-1/2} \| \| AX'q_i \| ^2}{\lambda' \Sigma_{ZZ}^{-1} \lambda} = \max_{1 \leq i \leq n} \| AX'q_i \| ^2 \rightarrow 0,
\]

by assumption (A.1). Finally, we obtain

\[
\lim \inf_{n \rightarrow \infty} \frac{\sigma_n^2}{\tau_c^2} \geq \| \lambda \|^2/(\lim \sup \lambda' \Sigma_{ZZ}^{-1} \lambda) > 0,
\]

by the assumption that the terms in the denominator is \( O(1) \). Hence, the desired result follows from the theorem 3.1 of Mehra and Rao (1975). \( \square \)

**Corollary 3.2.** In addition to the assumptions of Theorem 3.1, let the assumption of Lemma 3.1 hold. Then

\[
\Sigma_{ZZ}^{-1/2} \Sigma A^{-1}(\hat{\beta} - \beta) \rightarrow_d 2^{-1} N(0, I_{p \times p}).
\]

**Proof.** Claim follows from Lemma 3.1 upon noting that

\[
Z_n = \int Y_f(y)DU(y, \beta) dH(y).
\]

\( \square \)

**Remark 3.4.** Let \( \text{Asym}(\hat{\beta}) \) denote the asymptotic variance of \( \hat{\beta} \). Then we have

\[
\text{Asym}(\hat{\beta}) = 4^{-1} A \Sigma^{-1} \Sigma_{ZZ} \Sigma^{-1} A = 4^{-1} A (AX'\tilde{Q}'\Sigma_{D} \tilde{Q}XA)^{-1}(AX'Q'\Sigma_{\psi} QXA)(AX'\tilde{Q}' \Sigma_{D} \tilde{Q}XA)^{-1} A
\]

Observe that if all the transformed errors have the same distribution, i.e., \( f_1 = f_2 = \cdots = f_n \),
we have
\[ AX'\tilde{Q}'\Sigma D\tilde{Q}XA = (|f_1|^2_H)^{-1}I_{p\times p}. \]
Therefore, \( Asym(\tilde{\beta}) \) will be simplified as
\[ (2|f_1|^2_H)^{-2}A(AX'Q'\Sigma QXA)A. \]
Moreover, if all the transformed errors are uncorrelated as a result of the transformation, \( Asym(\tilde{\beta}) \) can be simplified further as
\[ \tau(2|f_1|^2_H)^{-2}(X'Q^2X)^{-1}, \]
where \( \tau = Var(\psi_1(q'_1\varepsilon)) \).

4 Simulation studies

In this section the performance of the generalized m.d. estimator is compared with one of the GLS estimators. Let \( \Omega := E(\varepsilon\varepsilon') \) and \( \hat{\Omega} \) denote covariance matrix of the errors and its estimate, respectively. Consequently we obtain the GLS estimator of \( \beta \)
\[ \hat{\beta}_{GLS} = (X'\hat{\Omega}^{-1}X)(X'\hat{\Omega}^{-1}y). \]

In order to obtain the generalized m.d. estimator, we try two different \( Q \)'s: \( Q_s \) and \( Q_c \) where
\[ Q_s^2 = \hat{\Omega}^{-1}, \quad Q_c'Q_c = \hat{\Omega}^{-1}. \]

We refer to the generalized m.d. estimators corresponding to \( Q_s \) and \( Q_c \) as GMD1 and GMD2 estimators, respectively.

In order to generate strongly mixing process for the dependent errors, the several restrictive conditions are required so that the mixing number \( \alpha \) decays fast enough — i.e., the assumption (a.8) is met. Withers (1981) proposed the upperbound and the decay rate of the mixing number \( \alpha \). For the sake of completeness, we reproduce Theorem and Corollary 1 here.

Lemma 4.1. Let \( \{\xi_i\} \) be independent r.v.s on \( R \) with characteristic functions \( \{\phi_i\} \) such that
\[ (2\pi)^{-1}\max_i \int |\phi_i(t)|dt < \infty \]
and
\[ \max_i E|\xi_i|^{\delta} < \infty \quad \text{for some } \delta > 0. \]
Let \( \{g_v : v = 0, 1, 2, \ldots\} \) be a sequence of complex numbers such that
\[
\left\{ S_t(\min(1, \delta)) \right\}^{\max(1, \delta)} \to 0 \quad \text{as } t \to \infty
\]
where
\[
S_t(\lambda) = \sum_{v=t}^{\infty} |g_v|^\lambda.
\]
Assume that
\[
g_v = O(v^{-\kappa}) \quad \text{where } \kappa > 1 + \delta^{-1} + \max(1, \delta^{-1}).
\]
Then the sequence \( \{\varepsilon_n : \varepsilon_n = \sum_{v=0}^{\infty} g_v \xi_{n-v} \} \) is strongly mixing with mixing number \( \alpha_t(k) = O(k^{-\eta}) \) where
\[
\eta = (\kappa \delta - \max(\delta, 1))(1 + \delta)^{-1} - 1 > 0.
\]
To generate strongly mixing process by Lemma 4.1, we consider four independent \( \xi_i \)'s: normal, Laplace, logistic, and mixture of the two normals (MTN). Note that all the \( \xi_i \)'s have the finite second moments, and hence, we set \( \delta \) at 2. It can be easily seen that for any \( \kappa > 7 \) we have \( \eta > 3 \), and hence the assumption (a.8) is satisfied. Then for \( \epsilon > 0 \)
\[
(4.1) \quad \varepsilon_n = \sum_{v=0}^{\infty} v^{-(7+\epsilon)} \xi_{n-v}
\]
satisfies the strongly mixing condition with \( \alpha(k) = O(k^{-(3+\epsilon)}) \). We let \( \epsilon = 0.5 \), or equivalently, \( \kappa = 7.5 \).

The \( \xi \) has a Laplace distribution if its density function is
\[
f_{Lo}(x) := (2s_1)^{-1} \exp(-|x - \mu_1|/s_1)
\]
while the density function of Logistic innovation is given by
\[
f_{Lo}(x) := s_2^{-1} \exp(-|x - \mu_2|/s_2)/(1 + \exp(-|x - \mu_2|/s_2))^2.
\]
When we generate \( \{\xi_i\}_{i=1}^{n} \), we set mean of normal, Laplace, and logistic innovations at 0 (i.e., \( \mu_1 = \mu_2 = 0 \)) since we assumed the \( \varepsilon \), the sum of \( \xi_i \)'s, is symmetric. We set the standard deviation of normal \( \xi \) at 2 while both \( s_1 \) and \( s_2 \) are set at 5 for Laplace and logistic, respectively. For MTN, we consider \( (1 - \epsilon)N(0, 2^2) + \epsilon N(0, 10^2) \) where \( \epsilon = 0.1 \). In each \( \xi \), we subsequently generate \( \{\varepsilon_i\}_{i=1}^{n} \) using (4.1).

Next, we set the true \( \beta = (-2, 3, 1.5, -4.3)' \), i.e., \( p = 4 \). For each \( k = 2, 3, 4 \), we obtain \( \{x_{ik}\}_{i=1}^{n} \) in (1.1) as a random sample from the uniform distribution on [0, 50]; \( \{y_i\}_{i=1}^{n} \) is subsequently generated using models (1.1). We estimate \( \beta \) by the generalized m.d. and
the GLS methods. We report empirical bias, standard error (SE), and mean squared error (MSE) of these estimators. We use the Lebesgue integrating measure, i.e., $H(y) \equiv y$. To obtain the generalized m.d. estimators, the author used R package KoulMde. The package is available from Comprehensive R Archive Network (CRAN) at https://cran.r-project.org/web/packages/KoulMde/index.html. Table 1 and 2 report biases, SE's and MSE's of estimators for the sample sizes 50 and 100, each repeated 1,000 times. The author used High Performance Computing Center (HPCC) to accelerate the simulations. All of the simulations were done in the R-3.2.2.

|      | GLS | GMD1 | GMD2 |
|------|-----|------|------|
|      | bias| SE   | MSE  | bias| SE | MSE  | bias| SE | MSE  |
| $\beta_1$ | 0.0011 | 0.0798 | 0.0064 | 0.0014 | 0.0797 | 0.0064 | 8e-04 | 0.0805 | 0.0065 |
| N    | 0.0036 | 0.0753 | 0.0057 | 0.0032 | 0.0756 | 0.0057 | 0.0036 | 0.0761 | 0.0058 |
| $\beta_3$ | -0.0023 | 0.0784 | 0.0062 | -0.0024 | 0.0784 | 0.0062 | -0.0023 | 0.0787 | 0.0062 |
| $\beta_4$ | 0.0014 | 0.0772 | 0.0060 | 0.0016 | 0.0773 | 0.0060 | 0.0013 | 0.0775 | 0.0062 |
| $\beta_1$ | -3e-04 | 0.1085 | 0.0118 | -3e-04 | 0.107 | 0.0115 | -6e-04 | 0.1073 | 0.0115 |
| La   | -0.0011 | 0.1145 | 0.0131 | -0.0011 | 0.1137 | 0.0129 | -0.001 | 0.1147 | 0.0132 |
| $\beta_3$ | -0.0011 | 0.1129 | 0.0127 | -0.001 | 0.1121 | 0.0126 | -0.0011 | 0.1127 | 0.0127 |
| $\beta_4$ | 7e-04 | 0.1193 | 0.0142 | 7e-04 | 0.119 | 0.0142 | 4e-04 | 0.1194 | 0.0143 |
| $\beta_1$ | -0.0111 | 0.1438 | 0.0208 | -0.0113 | 0.1429 | 0.0205 | -0.0108 | 0.144 | 0.0209 |
| Lo   | -0.0034 | 0.1516 | 0.023 | -0.0033 | 0.1513 | 0.0229 | -0.0033 | 0.1515 | 0.023 |
| $\beta_3$ | -0.0027 | 0.1465 | 0.0215 | -0.002 | 0.1461 | 0.0213 | -0.0024 | 0.1465 | 0.0215 |
| $\beta_4$ | 0.003 | 0.1485 | 0.0221 | 0.0027 | 0.1481 | 0.0219 | 0.0029 | 0.1478 | 0.0218 |

Table 1: Bias, SE, and MSE of estimators with $n = 50$.

As we expected, both biases and SE's of all estimators decrease as $n$ increases. First, we consider the normal $\xi$'s. When $\xi$'s are normal, the GLS and GMD1 estimators display the best performance; GLS and GMD1 show similar biases, SE's, and hence MSE's. GMD2 estimators show slightly worse performance than aforementioned ones; they display similar or smaller bias — e.g. estimators corresponding to $n = 50$ and $\beta_1, \beta_3, \beta_4$ — while they always have larger SE's which in turn cause larger MSE's. Therefore, we conclude that GLS and GMD1 show similar performance to each other but better one than GMD2 when $\xi$'s are normal.

For non-Gaussian $\xi$'s, we come up with a different conclusion: the GMD1 estimators outperform all other estimators while The GLS and GMD2 estimators display the similar
Table 2: Bias, SE, and MSE of estimators with \( n = 100. \)

|     | GLS | GMD1 | GMD2 |
|-----|-----|------|------|
| \( \beta_1 \) | -3e-04 | 0.0518 | 0.0027 |
| \( \beta_2 \) | 0.0014 | 0.049 | 0.0024 |
| \( \beta_3 \) | -0.0032 | 0.0498 | 0.0025 |
| \( \beta_4 \) | 2e-04 | 0.0495 | 0.0024 |
| \( \beta_1 \) | 0.0013 | 0.0731 | 0.0053 |
| \( \beta_2 \) | -0.0027 | 0.0703 | 0.0049 |
| \( \beta_3 \) | -0.0039 | 0.0715 | 0.0051 |
| \( \beta_4 \) | -0.0015 | 0.0672 | 0.0045 |
| \( \beta_1 \) | -5e-04 | 0.0913 | 0.0083 |
| \( \beta_2 \) | -8e-04 | 0.0925 | 0.0086 |
| \( \beta_3 \) | 5e-04 | 0.0707 | 0.005 |
| \( \beta_4 \) | 0.001 | 0.0676 | 0.0046 |

Note that weighing the merits of the GLS, the GMD1, and the GMD2 estimators in terms of bias is hard. For example, for the Laplace \( \xi \) when \( n = 50 \), the GLS and GMD1 estimators of all \( \beta_i \)'s show the almost same biases; the GMD2 estimator of \( \beta_1 \) (\( \beta_4 \)) show smaller (larger) bias than the GLS and the GMD1 estimators. When we consider the SE, the GMD1 estimators display the least SE's regardless of \( n \)'s and \( \xi \)'s. The GLS and the GMD2 estimators show somewhat similar SE's when \( \xi \) is Laplace or logistic; however, the GMD2 estimators have smaller SE's than the GLS ones when \( \xi \) is MTN. As a result, the GMD1 estimators display the least MSE for all non-Gaussian \( \xi \)'s and \( n \)'s; the GMD2 and the GLS — corresponding to Laplace or logistic \( \xi \)'s — show similar MSE's while the GMD2 estimators show smaller MSE than the GLS ones when \( \xi \) is MTN.

Appendix

Proof of Theorem 3.1. Section 5.5 of Koul (2002) illustrates (3.1) holds for independent errors. Proof of the theorem, therefore, will be similar to the one of Theorem 5.5.1 in that
section. Define for \( k = 1, 2, ..., p, u \in \mathbb{R}^p, y \in \mathbb{R}, \)

\[
J_k(y, u) := \sum_{i=1}^{n} d_{ik} F_i(y + q_i' X A u), \quad Y_k(y, u) := \sum_{i=1}^{n} d_{ik} I(q_i' \varepsilon \leq y + q_i' X A u),
\]

\[
W_k(y, u) := Y_k(y, u) - J_k(y, u).
\]

Rewrite

\[
L(\beta + A u; Q) = \sum_{k=1}^{p} \int \left[ \{W_k(y, u) - W_k(y, 0)\} + \{W_k(-y, u) - W_k(-y, 0)\} 
\right.
\]

\[
+ \{(J_k(y, u) - J_k(y, 0)) - \sum_{i=1}^{n} d_{ik} q_i' X A u f_i(y)\}
\]

\[
+ \{(J_k(-y, u) - J_k(-y, 0)) - \sum_{i=1}^{n} d_{ik} q_i' X A u f_i(y)\}
\]

\[
+ \{U_k(y, \beta) + 2 \sum_{i=1}^{n} d_{ik} q_i' X A u f_i(y)\}^2 dH(y).
\]

where \( 0 = (0, 0, ..., 0)' \in \mathbb{R}^p. \) Note that the last term of the integrand is the \( k \)th coordinate of \( U(y, \beta; Q) + 2 \sum_{i=1}^{n} d_{ik} q_i' X A u f_i(y) \) vector in \( L^*(b; Q). \) If we can show that suprema of \( L_H^2 \) norms of the first four terms of the integrand are \( o_p(1), \) then applying Cauchy-Schwarz (C-S) inequality on the cross product terms in (A.3) will complete the proof. Therefore to prove theorem it suffices to show that for all \( k = 1, 2, ..., p \)

\[
E \sup \int |W_k(\pm y, u) - W_k(\pm y, 0)|^2 dH(y) = o(1),
\]

\[
\sup \int |(J_k(\pm y, u) - J_k(\pm y, 0)) - \sum_{i=1}^{n} d_{ik} q_i' X A u f_i(y)|^2 dH(y) = o(1),
\]

\[
E \sup \int |U_k(y, \beta) + 2 \sum_{i=1}^{n} d_{ik} q_i' X A u f_i(y)|^2 dH(y) = O(1).
\]

where sup is taken over \( \|u\| \leq b. \) Here we consider the proof of the case \(+y\) only. The similar facts will hold for the case \(-y\).

Observe that (A.2) implies

\[
E \int U_k(y, \beta)^2 dH(y) \leq 2n \max_{1 \leq i \leq n} \|d_i\|^2 \max_{1 \leq i \leq n} \int (1 - F_i) dH < \infty.
\]
Therefore, (A.6) immediately follows from (A.2) and (A.7). The proof of (A.5) does not involve the dependence of errors, and hence, it is the same as the proof of (5.5.11) of Koul (2002). Thus, we shall prove (A.4), thereby completing the proof of theorem.

To begin with let $J_{\pm}^k(\cdot, \cdot), Y_{\pm}^k(\cdot, \cdot)$, and $W_{\pm}^k(\cdot, \cdot)$ denote $J_k(\cdot, \cdot, u)$, $Y_k(\cdot, \cdot, u)$ and $W_k(\cdot, \cdot, u)$ in (A.2) when $d_{ik}$ is replaced with $d_{\pm}^k$ so that $J_k = J^+_k - J^-_k$, $Y_k = Y^+_k - Y^-_k$, and $W_k = W^+_k - W^-_k$. Define for $x \in \mathbb{R}^p$, $u \in \mathbb{R}^p$, $y \in \mathbb{R}$,

$$p_i(y, u; X) := F_i(y + q'_iXu) - F(y),$$

$$B_{ni} := I(q'_i\varepsilon \leq y + q'_iXu) - I(q'_i\varepsilon \leq y) - p_i(y, u; X).$$

Rewrite

$$W_{k0}^+ - W_{k0}^- = \sum_{i=1}^n d_{\pm}^k \{I(q'_i\varepsilon \leq y + q'_iXu) - I(q'_i\varepsilon \leq y) - p_i(y, u; X)\}.$$ 

Note that

(A.8) \[ E B_{ni}^2 \leq F_i(y + \theta_i\|u\|) - F_i(y) \]

Recall a lemma from Deo(1973).

**Lemma A.2.** Suppose for each $n \geq 1$, \{\xi_{nj}, 1 \leq j \leq n\} are strongly mixing random variables with mixing number $\alpha_n$. Suppose $X$ and $Y$ are two random variables respectively measurable with respect to $\sigma\{\xi_{n1}, \ldots, \xi_{nk}\}$ and $\sigma\{\xi_{nk+m}, \ldots, \xi_{nm}\}$, $1 \leq m$, $m+k \leq n$. Assume $p, q$ and $r$ are such that $p^{-1} + q^{-1} + r^{-1}$, and $\|X\|_p \leq \infty$ and $\|Y\|_q \leq \infty$. Then for each $1 \leq m$, $k + m \leq n$

(A.9) \[ |E(XY) - E(X)E(Y)| \leq 10 \cdot \alpha_n^{1/r} \|X\|_p \|Y\|_q. \]

Consequently if $\|X\|_\infty = B < \infty$ then for $q > 1$ and each $1 \leq m$, $k + m \leq n$

(A.10) \[ |E(XY) - E(X)E(Y)| \leq 10 \cdot \alpha_n^{1-1/q} \|Y\|_q. \]

In addition, consider following lemma.

**Lemma A.3.** For $1 < r < 3,$

(A.11) \[ n^{-1} \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \alpha_1^{1/r} = O(1). \]
Proof. For given $r$, let $p$ such that $1/r + 1/p = 1$. Note that

$$\frac{p}{r} = \frac{1}{r-1} > \frac{1}{2}.$$  

Therefore, by Hölder’s inequality with $p$ and $r$, we have

$$n^{-1} \sum_{i=1}^{n} \sum_{k=1}^{n-i} \alpha^{1/r}(k) \leq \left( \sum_{k=1}^{n-1} \frac{(n-k)^p}{n^p} \cdot \frac{1}{k^{2p/r}} \right)^{1/p} \left( \sum_{k=1}^{n-1} k^2 \alpha(k) \right)^{1/r} < \infty. \quad (A.12)$$

The last inequality follows from the assumption (A.8.6), thereby completing the proof of lemma.

Now, we consider the cross product terms of $E|W_{ku} - W_{k0}|^2_H$.

$$\left| E \int \sum_{i=1}^{n} \sum_{j=i+1}^{n} d_{ik} d_{jk} \{ I(q_i' \varepsilon \leq y + q_i' X Au) - I(q_i' \varepsilon \leq y) - p_i(y, u; X) \} \right|$$

$$\times \left\{ I(q_j' \varepsilon \leq y + q_j' X Au) - I(q_j' \varepsilon \leq y) - p_j(y, u; X) \right\} \ dH(y)$$

$$\leq \sum_{i=1}^{n} \sum_{j=i+1}^{n} d_{ik} d_{jk} \int |E_{ni} B_{nj}| \ dH$$

$$\leq 10 \sum_{i=1}^{n} \sum_{j=i+1}^{n} d_{ik} d_{jk} \cdot \alpha^{1/2}(j-i) \cdot \int \|B_{nj}\|_2 \ dH$$

$$\leq 10 b^{1/2} \{ n \max_i d_{ik}^2 \} \cdot \{ \max_i \theta_i \}^{1/2} \cdot n^{-1} \sum_{i=1}^{n} \sum_{m=1}^{n-i} \alpha^{1/2}(m) \int f_i^{1/2} \ dH \rightarrow 0.$$  

The second inequality follows from Lemma A.2, and the convergence to zero follows from the Lemma A.3 with $r = 2$, (A.1), and (A.7). Consequently, by Fubini’s Theorem together with (A.3), we obtain, for every fixed $\|u\| \leq b$,

$$\limsup_{n \to \infty} E|W_{ku} - W_{k0}|^2_H \leq \limsup_{n \to \infty} \int \sum_{i=1}^{n} d_{ik}^2 |F_i(y + q_i' X Au) - F(y)| \ dH(y)$$

$$\leq \limsup_{n \to \infty} \{ n \max_i d_{ik}^2 \} \int_{-a_n}^{a_n} \int f_i(y + s) dH(y) ds$$

$$= 0,$$

where $a_n = b \max_i \theta_i \rightarrow 0$.

To complete the proof of (A.4), it suffices to show that for all $\epsilon > 0$, there exists a $\delta > 0$
such that for all \( v \in \mathbb{R}^p, \| u - v \| \leq \delta \),

\[(A.15) \quad \limsup_{n \to \infty} E \sup_{\| u - v \| \leq \delta} |\mathcal{K}_{ku} - \mathcal{K}_{kv}| \leq \epsilon,
\]

where

\[\mathcal{K}_{ku} := |W_{ku} - W_{k0}|^2_H, \quad u \in \mathbb{R}^p, \quad 1 \leq k \leq p.\]

(A.15) follows from (5.5.5) of Koul (2002), thereby completing the proof of theorem.

\[\square\]

**Proof of Corollary 3.1.** The proof of (3.2) for independent errors can again be found in the section 5.5 of Koul (2002). The difference between the proof in the section 5.5 and one here arises only in the part which involves the dependence of the error. Thus, we present only the proof of an analogue of (5.5.27) in Koul (2002). Let

\[L_k := \int \left[ W_k(y, 0) + W_k(-y, 0) + \left\{ J_k(y, 0) + J_k(-y, 0) - \sum_{i=1}^n d_{ik} \right\} \right] f_1(y) \, dH(y)\]

\[= \int \left[ \sum_{i=1}^n d_{ik} \{ I(q_i \epsilon \leq y) - I(-q_i \epsilon < y) \} \right] f_1(y) \, dH(y),\]

\[L := (L_1, ..., L_p).\]

Note that \( L_k = \int U_k(y, \beta; Q) f_1(y) \, dH(y) \). By the symmetry of \( H \) and Fubini’s theorem, we obtain

\[E \int \{ I(q_i \epsilon \leq y) - I(-q_i \epsilon < y) \}^2 \, dH(y) = 4 \int_0^\infty (1 - F_i) \, dH, \quad 1 \leq i \leq n.\]

In addition, Lemma A.2 yields, for \( j > i \),

\[E \int \{ I(q_i \epsilon \leq y) - I(-q_i \epsilon < y) \} \{ I(q_j \epsilon \leq y) - I(-q_j \epsilon < y) \} \, dH(y) \leq 20 \sqrt{2} \alpha^{1/2} (j - i) \max_{1 \leq i \leq n} \int_0^\infty (1 - F_i)^{1/2} \, dH.\]

Together with the fact that \( (1 - F_i) \leq (1 - F_i)^{1/2} \), by (A.1), (A.2), (A.7), and Lemma A.3,
we obtain, for some $0 < C < \infty$,
\[
E \|L\|^2 \leq \sum_{k=1}^{p} |f_{1H}|^2 \left\{ 4 \sum_{i=1}^{n} d_{ik}^2 \int_{0}^{\infty} (1 - F_i) \, dH + 40 \sqrt{2} \, n \max_{1 \leq i \leq n} d_{ik}^2 \right\}
\times n^{-1} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \alpha^{1/2} (j - i) \max_{1 \leq i \leq n} \int_{0}^{\infty} (1 - F_i)^{1/2} \, dH \right\}
< C_p |f_{1H}|^2 \max_{1 \leq i \leq n} \int_{0}^{\infty} (1 - F_i)^{1/2} \, dH.
\]

Using $E L_k = 0$ for $k = 1, \ldots, p + 1$ and Chebyshev inequality, for all $\epsilon > 0$ there exists $N_1$ and $c_\epsilon$ such that
\[
(A.16) \quad P(\|L\| \leq c_\epsilon) \geq 1 - \frac{C_p |f_{1H}|^2 \max_{1 \leq i \leq n} \int_{0}^{\infty} (1 - F_i)^{1/2} \, dH}{c_\epsilon} \geq 1 - \epsilon/2, \quad n \geq N_1.
\]

The rest of the proof will be the same as the proof of Lemma 5.5.4 of Kou\l{} (2002).

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