Strings with Discrete Target Space

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We investigate the field theory of strings having as a target space an arbitrary discrete one-dimensional manifold. The existence of the continuum limit is guaranteed if the target space is a Dynkin diagram of a simply laced Lie algebra or its affine extension. In this case the theory can be mapped onto the theory of strings embedded in the infinite discrete line \( \mathbb{Z} \) which is the target space of the SOS model. On the regular lattice this mapping is known as Coulomb gas picture. Introducing a quantum string field \( \Psi_x(\ell) \) depending on the position \( x \) and the length \( \ell \) of the closed string, we give a formal definition of the string field theory in terms of a functional integral. The classical string background is found as a solution of the saddle-point equation which is equivalent to the loop equation we have previously considered [1]. The continuum limit exists in the vicinity of the singular points of this equation. We show that for given target space there are many ways to achieve the continuum limit; they are related to the multicritical points of the ensemble of surfaces without embedding. Once the classical background is known, the amplitudes involving propagation of strings can be evaluated by perturbative expansion around the saddle point of the functional integral. For example, the partition function of the noninteracting closed string (toroidal world sheet) is the contribution of the gaussian fluctuations of the string field. The vertices in the corresponding Feynman diagram technique are constructed as the loop amplitudes in a random matrix model with suitably chosen potential.

Nuclear Physics B 376 (1992) 539-598

SPhT/91-142

9/91

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1. Introduction

The recent progress in the theory of non-critical strings has been prepared by the conjecture that the functional integral of 2d gravity can be discretized as a sum over planar graphs [2]. Such a discretization allowed to apply powerful methods of calculation borrowed from the theory of random matrices [3]. Some of the corresponding matrix models can be solved exactly after being reduced to a problem of non-interacting fermions in a common potential. This fermionic representation proved to be very helpful in practical calculations but its connection with the original model seems quite formal. It is not clear, for example, whether the nonperturbative phenomena in the fermion problem can be given an interpretation in terms of strings.

It is therefore desirable to have a formalism in which the string excitations appear explicitly. An example of such a formalism applied to the string propagating in the one-dimensional space $\mathbb{R}$ is the collective field approach of Das and Jevicki [4].

In this paper we develop a similar formalism for string theories with discrete target spaces. Bosonic string with discrete target spaces are interesting mainly because of their interpretation as theories of two-dimensional gravity coupled to matter. The (discrete) degrees of freedom of the matter field are the points of the target space. Because of the enormous symmetry of the problem, it is much easier to investigate the critical behaviour of statistical systems on a fluctuating surface than on the plane. The exact results obtained for a fluctuating surface can be translated according to [5], [6] to the case of a frozen geometry of the world sheet.

The string models we are investigating are a generalization of the $ADE$ strings constructed in ref. [1]. They are closely related to the SOS and RSOS solvable statistical models on a regular planar lattice [7]. The target spaces of these models are one-dimensional discrete manifolds representing Dynkin diagrams of simply laced Lie algebras or their affine extensions. All such target spaces can be mapped onto a master target space which is the infinite discretized line $\mathbb{Z}$.

Since in a one-dimensional embedding space there is no room for transverse excitations, the states of the closed string is completely determined by the two global modes - the center of masses $x$ and its length $L$. The string field $\Psi(L)$ is an operator creating on the world sheet a boundary of length $L$ and position $x$ in the target space. The path integral for the correlation functions of the string field involve embeddings of the world sheet such that each connected component of its boundary is mapped into a single point of the target space.

The string models related to the SOS model are remarkable with the possibility to reduce the string path integral to a simpler one involving only the global modes $x$ and $L$, even before going to the continuum limit. As a result one obtains an effective two-dimensional QFT for the field $\Psi_x(L)$.

The reduction of the string path integral stems from the loop, or polygon representation of the SOS-type statistical systems [7], which has been generalized to the case of irregular lattices in [1]. A map of the world sheet of the string in the target space can be represented by a collection of nonintersecting loops (domain walls) separating the domains of constant coordinate $x$. We assume that the discontinuity across a domain wall is ±1. Each loop configuration is characterized by its topology, the lengths of the loops, and the
coordinates of the connected domains. It can be imagined as a “stroboscopic picture” of a
the evolution of one or several closed strings, the time direction on the world sheet being
at all points orthogonal to the domain walls. The evolution of the string is decomposed
into elementary processes represented by the domain walls and connected domains. The
domain walls describe elementary steps in the $X$-space without changing the world-sheet
geometry. On the other hand, the connected domains describe topology-changing processes
without propagation in the embedding space. This nice factorization renders the model
exactly solvable.

A loop configuration is represented by a graph with lines corresponding to domain
walls and vertices corresponding to the connected domains on the world sheet. There
are vertices of all orders including tadpoles describing processes of creation or decay of a
closed-string state. Each vertex is characterized also by the number $H$ of handles of the
connected domain on the world sheet. Vertices with $H > 0$ describe processes involving
creation and annihilation of $H$ virtual closed strings at the same point $x$ of the embedding
space.

The string path integral can be written as a sum over all loop configurations, and
the entropy of a loop configuration factorizes to a product of weights associated with
the domains. The weight of each domain is proportional to the entropy of a nonembedded
random surface with the corresponding topology. The latter can be calculated, for example,
by using the matrix model formalism. In this way the the graphs representing the loop
configurations can be interpreted as Feynman diagrams for an ordinary field theory in the
space of global modes $(x, L)$ and the string path integral is reduced to a functional integral
for the field $\Psi_x(L)$.

The first (and in a sense the most difficult) problem to solve is to find the classical
string background $\Psi^{cl}_x(L)$ which is the solution of the classical equation of motion in
the effective field theory. Then the propagation of the noninteracting closed string is
determined by the gaussian fluctuations of the string field around the classical solution
and the vertices are given by the higher order terms. The accomplishment of this program
will be the subject of this paper.

We are going to present in details the results announced in three short publications
[8], [9], and [10], as well as a few new results as the microscopic construction of the dilute
$ADE$ models and a description of their multicritical points [11].

The paper is organized as follows.

In section 2 we define the path integral for a string theory with discrete target space
and establish its equivalence with an ordinary field theory for the string field $\Psi_x(L)$. The
measure in the space of world sheet geometries will be discretized by planar graphs. Therefore we have first to define the statistical systems related to the SOS model (known also as interaction-round-a-face, or IRF models [7]) on an arbitrary irregular lattice $S$. In sect. 2.1 we present a generalized version of the construction of these models worked out in
[1]. The present version allows two nearest neighbour sites $s$ and $s'$ of $S$ to have coordinates
(heights) $x$ and $x'$ which either or coincide or are nearest neighbours in the $X$ space. In
this way we are able to achieve both the dense and dilute phases of the IRF models. The
important special case $X = \mathbb{Z}$ is discussed in sect. 2.2, where we consider a nonunitary
version of the SOS model with complex Boltzmann weights. This model renormalizes at
large distances onto a gaussian field with a linear term coupled to the local curvature of the
discretized world sheet. The mapping of the IRF models onto this SOS model known as
Coulomb gas picture, can be constructed in the same way as in the case of a lattice without
curvature [7], [12], [13]. The Coulomb gas picture is based on the loop expansion which is
explained in sect. 2.3. The partition function of the statistical system with target space
X equals the sum over the embeddings of a world sheet with fixed geometry. In section
2.4 we define the string path integral as a sum over the world sheet geometries and review
its description by means of a two-component gaussian field [14] [5] [6]. The normalization of
the electric and magnetic charges is fixed by the correspondence with the SOS model. In
section 2.5 we establish the equivalence between the string path integral and the Feynman
diagram series for the field $\Psi_x(L)$ describing the dynamics of the global modes of the
string. By means of a shift $\Psi_x(L) \rightarrow \Psi_x(L) + \Psi_x^{cl}(L)$ we define an improved diagram
technique containing no tadpoles. The classical string background $\Psi_x^{cl}(L)$ is determined
by the saddle-point equation. In sect. 2.6 we show that this equation is equivalent to the
loop equation considered in [1].

In section 3 we study the continuum limit of the string field theory. For this purpose
we introduce a cutoff $a$ with dimension of length and replace the dimensionless length
$L$ with a renormalized length $\ell = aL$. The continuum limit can be achieved if the bare
parameters of the string are tuned to a critical point. The critical points can be found
as the singularities of the loop equation for the classical string background. The latter
allows an interpretation as the partition function of a gas of nonintersecting loops on a
fluctuating surface with the topology of a disc. In the vicinity of a critical point the volume
of the world sheet always diverges and the different critical regimes are distinguished by
the behaviour of the loops. The latter is governed by a parameter (energy) coupled to
the total length of the loops. If the energy exceeds the entropy, then the loops remain
small, and the possible critical regimes are the $m$-critical points of nonembedded surfaces
(sect. 3.1). This is the phase of noncritical loops. In the opposite case, considered in sect.
3.2, the energy of the loops is not sufficient to compensate their entropy, and the loops
form a dense critical phase filling the world sheet (sect. 3.2). Finally, the dilute phase of
the loop gas (sect. 3.3) is achieved when the energy of the loops is tuned to its critical
value. In the dilute phase the loops are still critical but they cover only a small fraction
of the points of the lattice. In this phase the critical behaviour is sensitive to the choice of
the integration measure in the space of world sheet geometries. By tuning the Boltzmann
weights of the planar graphs we can achieve an infinite sequence of multicritical points of
the dilute phase. The continuum limit of the loop equation (sect. 3.4) is universal; the
different critical regimes are classified by the possible asymptotics of the solution at small
lengths.

Once the string background is known, it is not difficult to find explicit expressions for
the Feynman rules in the continuum limit. The vertices in the improved diagram technique
(i.e., dressed by tadpoles) can be calculated as the loop correlators in a special matrix
model constructed in sect. 4.1. The physical meaning of the coupling constant of this
matrix model is discussed in sect. 4.2. It is coupled to a local operator which has negative
dimension if the theory is not unitary. Its dimension is related to the fractal dimension
of the connected domains on the world sheet. In sect. 4.3 we give explicit expressions for
the planar \((H = 0)\) vertices in the continuum limit. Since the nonplanar loop amplitudes in the one-matrix model are not yet known, we have only fixed the general structure of the nonplanar vertices. The expression for a vertex with \(H\) handles and \(n\) legs contains \(3H + n - 3\) unknown coefficients. The Feynman rules simplify in the momentum space \((p, E)\) where the propagator of the string field diagonalizes. The spectrum of on-shells states discussed in sect. 4.4 is given by the light cone in the \((p, E)\) space. The latter has the geometry if a half-infinite cylinder since the discreteness of the \(X\) space leads to periodicity in the \(p\)-direction. Finally, in sect. 4.5 we check that the partition function of the noninteracting string (closed surfaces with the topology of a torus) is obtained by integrating over the gaussian fluctuations around the saddle point.

2. Statistical systems on planar graphs and kinematics of strings with discrete target space

2.1. The IRF height models on an arbitrary irregular lattice

It is well known \([15]\) that the two-dimensional rational conformal-invariant QFT with central charge \(C < 1\) are classified by the simply laced Lie algebras (i.e., these of the classical series \(A_n, D_n, E_6, E_7, E_8\)). Each of these theories can be constructed microscopically as a lattice statistical model whose local degrees of freedom are labeled by the points of the Dynkin diagram of the corresponding Lie algebra \([10]\). The statistical models associated with dynkin diagrams of \(A, D, E\) type, or shortly \(ADE\) models represent a natural generalization the RSOS face models considered by Andrews, Baxter and Forrester \([17]\). The local degrees of freedom are attached to the sites of the lattice and interact through “interactions-round-a-face” (IRF) around each plaquette.

Similarly, the extended \(\hat{A} \hat{D} \hat{E}\) Dynkin diagrams describe conformal invariant QFT with \(C = 1\) and discrete spectrum of conformal dimensions. In fact, Pasquier’s construction \([16]\) can be applied to any one-dimensional discrete manifold \(X\), i.e., a set of points \(x\) and links \(\langle xx'\rangle\) with the structure of a one-dimensional simplicial complex. The requirement that \(X\) is an (extended) Dynkin diagram of \(ADE\) type guarantees the existence of a scaling limit. Another important target space is the infinite discretized line \(\mathbb{Z}\). This is the target space of the model known as SOS (solid-on-solid) model.

Before presenting the definition of the IRF model with target space \(X\) we are going to describe the space of excitations of this model.

The one-dimensional discrete manifold \(X\) is defined by the set of its points \(x\) and the adjacency matrix \(C_{xx'}\), \(x, x' \in X\)

\[
C_{xx'} = \text{[the number of links connecting } x \text{ and } x']
\]  

(2.1)

We assume that the target space \(X\) is represented by a nonoriented graph which implies symmetric adjacency matrix.

The Hilbert space of states of the \(X\)-field in a fixed “time slice” consists of all closed paths in \(X\) with given length. Therefore in order to identify the ground state and the excited states we have to solve the problem of random motion in \(X\). The propagation
kernel for a random walk on \( X \) consisting of \( n \) steps is just the \( n \)-th power of the adjacency matrix \( C \). Introducing the eigenvectors \( V^x_{(p)} \)

\[
\sum_{x'} C_{xx'} V^x_{(p)} = \beta_p V^x_{(p)}
\]

we can write the kernel \( K_{xx'} = (C^n)_{xx'} \) as a sum of projectors on the eigenstates

\[
(K^n)_{xx'} = \sum_p (\beta_p)^n V^x_{(p)} V^{x'}_{(p)}
\]

If \( X \) is an ADE Dynkin diagram (Fig.1), then the eigenvalues of \( C \) have the form

\[
\beta_p = 2 \cos(\pi p), \quad p = m/h
\]

where \( h \) is the Coxeter number and the integer \( m \) is one of the Coxeter exponents of the Dynkin diagram \( X \). The eigenvectors \( V^x_{(p)} \) define the Fourier transform from coordinates \( x \) to the discrete momenta \( p = m/h \). The ground state corresponds to the maximal eigenvalue of the adjacency matrix \( C \)

\[
\beta \equiv \beta_{p_0} = 2 \cos(\pi p_0), \quad p_0 = 1/h \quad (A, D, E)
\]

If \( X \) is an extended \( \hat{A}\hat{D}\hat{E} \) Dynkin diagram, then the spectrum of Coxeter exponents includes \( m = 0 \) and the maximal eigenvalue of \( C \) is

\[
\beta = 2, \quad p_0 = 0 \quad (\hat{A}, \hat{D}, \hat{E})
\]

The inverse statement reads: if all the eigenvalues of the adjacency matrix are less than 2, then the graph \( X \) is an either an ADE dynkin diagram or a quotient \( A_{2n}/Z_2 \) \[13\]
In order to simplify the notations we shall always denote the ground state wavefunction by
\[ S_x = V_{(p_0)}^x \]  \hfill (2.7)

Since the entities of the adjacency matrix are nonnegative, by the Peron-Frobenius theorem \( S_x \geq 0 \) for all \( x \in X \).

The propagation kernel \((2.3)\) becomes in the limit of large proper time \( n \) a single projection operator to the ground state \( S_x \)
\[(K^n)_{xx'} \sim S_x \quad (\beta^n) \quad S_{x'} \]  \hfill (2.8)

It is convenient to define the wavefunctions of the excited states as
\[ \chi^x_{(p)} = V^x_{(p)} / S_x \]  \hfill (2.9)

They satisfy a closed algebra
\[ \chi^x_{(p)} \chi^x_{(p')} = \sum_{p''} C_{pp'p''} \chi^x_{p''} \]  \hfill (2.10)

and orthogonality conditions of the form
\[ \sum_x S_x^2 \chi^x_{(p)} \chi^x_{(p')} = \delta_{pp'}, \quad \sum_p \chi^x_{(p)} \chi^x_{(p')} = \delta_{x,x'} (S_x S_{x'})^{-1} \]  \hfill (2.11)

Let us give two simplest examples.

i) The Dynkin diagram of the algebra \( A_{h=1} \) is the chain of \( h-1 \) points \( x = 1, 2, ..., h-1 \). This is the target space of the RSOS models [17]. The eigenvectors of \( C \) are
\[ V^x_{(p)} = \sqrt{\frac{2}{h}} \sin(\pi px), \quad p = 1/h, 2/h, ..., (h-1)/h; \quad S_x = \sqrt{\frac{2}{h}} \sin(\pi x/h) \]  \hfill (2.12)

In this case the ground state corresponds to \( p_0 = 1/h \).

ii) The extended Dynkin diagram corresponding to the affine Kac-Moody algebra \( \hat{A}_{2h-1} \) is the ring \( \mathbb{Z}_{2h} \) made of \( 2h \) points. It can be represented by the set of integers modulo \( 2h \). Then the connectivity matrix \( \hat{C}_{xx'} \) reads
\[ \hat{C}_{xx'} = \delta^{(2h)}_{x,x'+1} + \delta_{x,x'-1}^{(2h)} \]  \hfill (2.13)

where \( \delta^{(2h)} \) is the Kronecker symbol modulo \( 2h \). The eigenvectors of the matrix \( \hat{C} \) are
\[ V^x_{(p)} = (2h)^{-1/2} \exp(i\pi px), \quad p = 0, \pm 1/h, ..., \pm (h-1)/h, 1. \]  \hfill (2.14)

The target space is translational invariant and the spectrum of momenta contains the point \( p = 0 \). The ground state is \( S^x = V^x_{(0)} = 1/\sqrt{2h} \). The vertex operators (order parameters)
\[ \chi^x_{(p)} = e^{i\pi px} \]  \hfill (2.15)
satisfy a closed algebra with fusion rules representing the momentum conservation modulo the period 2 of the momentum space

\[ C_{pp'p''} = \delta^{(2)}(p + p' + p'') \] (2.16)

The microscopic definition of the IRF models on a general irregular lattice was presented in [1] as a direct generalization of Pasquier’s construction [16], the only new point being the explicit dependence of the Boltzmann weights on the scalar curvature. The effects of the curvature have been also studied in [13]. Below we present a more general construction including both the dense and the dilute versions of the IRF models.

Let \( S \) be a two-dimensional discrete manifold such that all its faces are squares (we shall call them also plaquettes). It can be constructed from an arbitrary planar graph and its dual one by taking the points of both graphs and adding bonds connecting all pairs of points which serve as extremities of mutually dual lines. A small section of such lattice is shown in Fig. 2

Each configuration of the \( x \)-field on \( S \) defines a map \( S \to X \). Sometimes the local values of the \( x \)-field are called heights. The map is continuous in the sense that nearest neighbours in \( X \) are images of nearest neighbours in \( S \). In the original papers a stronger condition has been imposed, namely, that the links \( \langle ss' \rangle \) of \( S \) are mapped into links \( \langle xx' \rangle \) of \( X \). Here we relax this condition by allowing two nearest neighbours in \( S \) to be mapped into the same point \( x \in X \).

![Fig. 2: A section of the discretized world sheet](image)

The statistical weight of each field configuration is a product of factors associated with the plaquettes and sites of the lattice \( S \). We assume that the weights of the plaquettes are symmetric under cyclic permutations. This property will be of crucial importance for the generalization of the models to the case of an irregular lattices. The weight of the plaquette \( (x_2 x_4)_{(1 s_3)} \) is defined by

\[
\Omega(x_1, x_2, x_3, x_4) = x_1 x_2 x_3 x_4 + x_1 x_2 x_3 x_4 + x_1 x_2 x_3 x_4 + x_1 x_2 x_3 x_4
\]

\[ + x_1 x_2 x_3 x_4 + x_1 x_2 x_3 x_4 + x_1 x_2 x_3 x_4 \] (2.17)
with
\[
\begin{align*}
(x_1, x_2) & = \delta_{x_1 x_2} \delta_{x_2 x_3} \delta_{x_3 x_4} \delta_{x_4 x_1} \\
(x_3, x_4) & = \frac{1}{T} C_{x_1 x_3} C_{x_3 x_4} \delta_{x_4 x_2} \delta_{x_2 x_1} \sqrt{\frac{S_{x_1}}{S_{x_3}}} \\
(x_1, x_2) & = \frac{1}{T^2} C_{x_1 x_3} C_{x_3 x_4} C_{x_4 x_1} C_{x_1 x_2} \delta_{x_2 x_4} \sqrt{\frac{S_{x_1} S_{x_3}}{S_{x_2} S_{x_4}}}
\end{align*}
\]

where \( T \) is the temperature of the statistical system.

The factors associated with the sites \( s \in S \) reflect the response of the \( x \)-field to the local curvature \( \hat{R}_s \) which is concentrated at the sites. On an irregular lattice the local Gaussian curvature at the point \( s \) is defined as twice the deficit angle
\[
\hat{R}_s = \pi (4 - q_s)
\]

where \( q_s \) is the coordination number at the site \( s \) (= the number of plaquettes sharing this site). We use the standard normalization such that
\[
\sum_{s \in S} \hat{R}_s = 4\pi \chi
\]

where \( \chi = 2 - 2H \) is the Euler characteristic of the surface \( S \). The weight associated with a site \( s \) is
\[
\Omega(x) = (S_x)^{\hat{R}_s/4\pi}
\]

The factors (2.21) are trivial on a regular square lattice where all coordination numbers are equal to 4.

Collecting all factors, we define the partition function of the model with target space \( X \) on the lattice \( S \) as the sum over all maps \( S \rightarrow X \)
\[
F[S] = \sum_{S \rightarrow X} \prod_s \Omega(x(s)) \prod_{(s_1, s_2, s_3, s_4)} \Omega(x(s_1), x(s_2), x(s_3), x(s_4))
\]

In the limit \( T \rightarrow 0 \) the r.h.s. of (2.22) is the partition function of the critical ADE models considered in [1]. In the high temperature limit \( T \rightarrow \infty \) the \( x \)-field freezes to a constant and there are no long range correlations. At some critical temperature \( T_c \) there should be a phase transition between the low-temperature (critical) and the high-temperature (noncritical) phase. At the point \( T_c \) a new critical regime can be achieved. We call this phase, by reasons which will become clear later, dilute phase. The critical low-temperature phase will be referred also as the dense phase of the IRF models.
2.2. The nonrestricted SOS model with a false vacuum

It is known [19], [20], that the dynamics of the two-dimensional conformal theories can be described in terms of a Gaussian massless field with a linear term coupled to the local curvature. Let $\xi_a, \ a = 1, 2,$ be the local coordinates on a surface $S$ with metric $\hat{g}_{ab}(\xi); a = 1, 2$ and local curvature $\hat{R}(\xi)$. Then the gaussian field $x(\xi)$ is defined by the action

$$A[x; \hat{g}] = \frac{1}{4} \int d^2 \xi \sqrt{\det \hat{g}} \left[ \pi g \hat{g}^{ab} \partial_a x(\xi) \partial_b x(\xi) + i \alpha_0 x(\xi) \hat{R}(\xi) \right]$$

(2.23)

where $g$ is the coupling constant and $\alpha_0$ is the background electric charge.

One can map all conformal field theories with conformal anomaly $C \leq 1$ onto this gaussian field in the sense that the partition and correlation functions can be interpreted in terms of distributions of electric and magnetic charges in the gaussian theory. This mapping is known as Coulomb gas picture [21], [22], [23], [24]. The electric charge $e$ is carried by the vertex operator $V_\alpha(\xi) = \exp(i \pi \alpha x(\xi))$ and the charge $\mu$ at the point $\xi$ creates a discontinuity $2\mu$ along a line starting at the point $\xi$ (in fact, the discontinuity can be distributed among several lines forming a star with center at $\xi$).

The $(++)$-component of the energy-momentum tensor is

$$T(\xi) = -g \pi^2 \partial_\xi x \partial_\xi x + i \pi \alpha_0 \partial_\xi^2 x(\xi)$$

(2.24)

where we have introduced the complex variable $\xi_\pm = \xi_1 \pm i \xi_2$. From the o.p.e. [19], [23]

$$T(\xi)T(\xi') = \frac{1}{2} \frac{1}{(\xi - \xi')^4} \left[ 1 - 6 \frac{\alpha_0^2}{g} \right] + \frac{2}{(\xi - \xi')^2} T(\xi_2) + ...$$

(2.25)

$$T(\xi)V_\alpha(\xi') = \frac{\alpha(2\alpha_0 - \alpha)}{4g} \frac{1}{(\xi - \xi')^2} V_\alpha(\xi') + ...$$

(2.26)

one reads that the conformal anomaly $C$ of the field (2.23) and the conformal dimensions of the vertex operators $V_\alpha$ are

$$c = 1 - 6 \frac{\alpha_0^2}{g}; \quad \Delta_\alpha = \frac{\alpha(\alpha - 2\alpha_0)}{4g}$$

(2.27)

The operators $V_\alpha$ and $V_{2\alpha_0 - \alpha}$ have the same conformal dimension and therefore can be considered as related by charge conjugation. Their two-point function is

$$\langle V_\alpha(\xi)V_{2\alpha_0 - \alpha}(\xi') \rangle = |\xi - \xi'|^{\alpha(2\alpha_0 - \alpha)/g}$$

(2.28)

where $g$ is the coupling constant in (2.23). The charge neutrality is restored by the presence of the background charge $-2\alpha_0$. The construction of higher correlation functions involves the so called screening operators that carry nonzero electric charge and have conformal dimension 1. By eq. (2.27) there are two such operators which are charge-conjugated to each other

$$\alpha_\pm(\alpha_\pm - 2\alpha_0) = 4g; \quad \alpha_+ + \alpha_- = 2\alpha_0$$

(2.29)
The allowed charges are labeled according to the number of screening charges needed to neutralize the 4-point function \[23\]
\[\alpha_{rs} = \frac{1 - r}{2} \alpha_+ + \frac{1 - s}{2} \alpha_- \]  
(2.30)

The conformal dimensions of the corresponding vertex operators form the Kac spectrum
\[\Delta_{rs} = \frac{(r \alpha_+/2 + s \alpha_-/2)^2 - \alpha_0^2}{4g} \]  
(2.31)

Finally, the vortex operator with magnetic charge (discontinuity) \(\mu\) has conformal dimension
\[\tilde{\Delta}_\mu = \frac{g^2 \mu^2 / 4 - \alpha_0^2}{4g} \]  
(2.32)

The gaussian field dominance in the two-dimensional critical phenomena has its microscopic equivalent. It happens that all target spaces with dimension not greater than one can be mapped onto the target space of the nonrestricted SOS model which is the discretized real line \(Z\). The corresponding connectivity matrix
\[C_{xx'} = \delta_{x,x'+1} + \delta_{x,x'-1} \]  
(2.33)

has a continuous spectrum of excitations
\[\beta_p = 2 \cos(\pi p), \quad -1 < p \leq 1 \]  
(2.34)

and its eigenvectors are plane waves
\[V^x_{(p)} = e^{i\pi px} \]  
(2.35)

The momentum space (the dual of the SOS target space) is therefore a circle with perimeter equal to two.

The large distance behaviour of this SOS model is argued \[23\] \[24\] to be described by the action (2.23) with an appropriate choice of the background charge \(\alpha_0\) and the coupling constant \(g\). The background charge in the SOS model is introduced by taking an excited state with momentum \(p_0\) as a vacuum state
\[S_x = V^x_{(p_0)} = e^{i\pi p_0 x} \]  
(2.36)

Then the curvature dependent-factor in the definition of the partition function can be written as an exponent
\[\prod_s (S_x)^{\hat{R}(s)/4\pi} = \exp \left(i \frac{p_0}{4} \sum_s \hat{R}(s) x(s) \right) \]  
(2.37)
which is the microscopic realization of the curvature-dependent term in (2.23). Knowing that the global curvature does not renormalize, we conclude that

$$\alpha_0 = p_0 + \text{even integer} \quad (2.38)$$

Since the charge in the SOS model is determined up to an even integer, we can assume that $\alpha_0 = p_0$.

Now it remains to fix the coupling constant $g$ in the gaussian theory.

First we observe that the spectrum of allowed momenta in the target space of the SOS model is $p = mp_0$, $m = \text{integer}$. The operators $\text{(2.35)}$ corresponding to momenta outside this spectrum have vanishing correlators because the electric neutrality is not fulfilled. On the other hand, due to the discreteness of the target space of the SOS model all charges of the form $\alpha \pm 2n$ in the gaussian model are indistinguishable in the SOS model. Therefore, the spectrum of allowed electric charges in the gaussian model is

$$\alpha = rp_0 + 2n; \quad r, n \in \mathbb{Z} \quad (2.39)$$

The way the coupling constant $g$ depends on the background charge $\alpha_0$ is fixed by the compatibility of $\text{(2.29)}$ and $\text{(2.39)}$

$$\alpha_0 = g - 1; \quad \alpha_+ = 2g, \quad \alpha_- = -2 \quad (2.40)$$

Then the spectrum of allowed charges $\text{(2.39)}$ coincides with $\text{(2.30)}$ and the dimensions of the Kac spectrum $\text{(2.31)}$ read

$$\Delta_{rs} = \frac{(rg - s)^2 - (g - 1)^2}{4g} \quad (2.41)$$

The corresponding central charge $\text{(2.27)}$

$$C = 1 - 6(g - 1)^2/g \quad (2.42)$$

is symmetric under $g \to 1/g$.

By $\text{(2.40)}$ the coupling constant $g$ of the gaussian field is related to the “vacuum”eigenvalue $\beta$ of the connectivity matrix

$$\beta = 2 \cos(\pi p_0) = -2 \cos(\pi g) \quad (2.43)$$

The branch of the multivalued function $g = \frac{1}{\pi} \arccos(-\beta)$ is determined by the dynamics, that is, by the choice of the critical regime for the SOS model. The analysis of the $O(n)$ model $\text{(2.24)}$ which is in a sense dual to the SOS model with $\beta = n$, suggests that the interval $0 < g < 1$ describes the dense phase and the interval $1 < g < 2$ describes the dilute phase of the SOS model. Sasha Zamolodchikov conjectured $\text{(2.25)}$ that the other branches of $\text{(2.43)}$ ($2 < g < \infty$) correspond to multicritical regimes of the SOS model. This is shown to be the case for the SOS model on a fluctuating lattice $\text{(11)}$. We will give a sketch of the
analysis of ref. [11] in one of the next sections. The $m$-critical point corresponds to $g$ in the interval $(m-1, m)$.

The order parameters of the SOS model are represented by vertex operators in the gaussian theory

$$
\chi^{(s)}_p = e^{i\pi(p-p_0)x^{(s)}} \leftrightarrow V_\alpha(\xi) = e^{i\pi\alpha x(\xi)}, \alpha = p - p_0
$$

The operators carrying charges $p = rp_0, r = 0, \pm 1, \ldots$ have conformal dimensions in the Kac spectrum

$$
\Delta_{\text{electric}}^{(p)} = \frac{p^2 - p_0^2}{4g} = \frac{(g-1)^2(r^2-1)}{4g} = \Delta_{rr}
$$

The vortex operators in the SOS model can have only integer magnetic charge. The conformal dimension of a magnetic operator with charge $\mu = m, m = 0, \pm 1, \ldots$ is

$$
\Delta_{\text{magnetic}}^m = \frac{(mg/2)^2 - (g-1)^2}{4g} = \Delta_{m0}
$$

The presence of negative dimensions means that the theory is not unitary. There are two charge conjugated identity operators ($p = \pm p_0$).

2.3. The loop expansion

The partition functions of the IRF models on a surface with the topology of a sphere is equal to the partition function of a gas of nonintersecting loops on the surface [22], [16]. The loops have the meaning of domain walls separating domains of constant $x$.

The microscopic construction of the loop expansion goes as follows. Write the partition function as a sum of monomials, choosing for each plaquette only one of the seven terms on the r.h.s. of (2.17). The ensemble of field configurations $S \rightarrow X$ is divided in this way into subsets, each characterized by a given decoration of the plaquettes of $S$. The decorations of all plaquettes form a set of nonintersecting polygons on the lattice $S^*$ dual to $S$ (Fig. 3). The polygons play the role of boundaries of the domains of constant $x$. 

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The partition function is equal to the sum over all configurations of polygons on the world sheet and all allowed values of $x$ in the connected domains bounded by polygons. The Boltzmann weight of each configuration is equal to $T^{-L}$ where $L$ is the total length of all polygons, times a product of factors associated with the domains bounded by polygons.

Let us consider for simplicity a lattice with the topology of a disc with and impose a Dirichlet boundary condition $x = constant$. The weight of a domain of constant height $x$ with $n$ boundaries is $(S_x)^{2-n}$. To prove this we can use the freedom to redistribute the Boltzmann weights between plaquettes and vertices. The new plaquette weights are given by (2.18) with

$$\delta_{x_1x_3}\delta_{x_3x_4}\delta_{x_4x_2}\delta_{x_2x_1} \frac{1}{S_{x_4}}$$

$$= \frac{1}{T} C_{x_1x_3} C_{x_3x_4} \delta_{x_4x_2} \delta_{x_2x_1} \frac{1}{S_{x_4}}$$

$$= C_{x_1x_3} C_{x_3x_4} C_{x_4x_2} C_{x_2x_1} \frac{1}{S_{x_4}}$$

and the new vertex weights are

$$\Omega(x) = S_x$$

Using the fact that the number of bonds is twice the number of faces, one can easily show that the total power of $S_x$ is equal to the Euler characteristic $\chi = 2 - n$ of the domain. It is convenient to distribute the weight $S_x^{2-n}$ as follows: a factor $S_x$ for the outer boundary and a factor $S_x^{-1}$ for each of the inner boundaries.

Next, for a given configuration of polygons we wish to sum over all possible heights of the domains bounded by these polygons. Let us start with the domains with the topology of a disc, i.e., having no inner boundaries. The sum over the allowed heights of such domain is performed using the definition (2.2). The result is $\beta S_x$ where $x$ is the height of the surrounding domain. The factor $S_x$ cancels with the factor $S_x^{-1}$ in the Boltzmann weight of the surrounding domain, associated with this boundary. Proceeding this way, from the inside out, we find that the Boltzmann weight of a polygon configuration is
obtained by assigning to each polygon a factor $\beta = 2 \cos(\pi p_0)$. There will be also an overall factor $S_x$ where $x$ is the height of the most outer domain. Note that the polygons have no orientation. One can introduce an orientation and split the weight of a polygon into two phase factors $\exp(\pm i\pi p_0)$ for the two possible orientations. Note that the Boltzmann weights of the loops do not depend on the local curvature.

The correlation function of two order parameters, or vertex operators (2.9) is equal, up to a normalization, by the partition function of the loop gas where the loops enclosing only one of the two points have different fugacity $\beta_p$. The correlation function of two disorder parameters, or vortex operators with magnetic charge $m$ is given by the partition function of the loop gas in presence of $m$ nonintersecting lines having as extremities the two points. We have seen that the partition function on a surface with the topology of a disc or sphere depends on the model only through the momentum $p_0$ of the ground state in the target space. The mapping onto the SOS model is trivial in this case. For higher genus surfaces the partition function will depend on the spectrum of allowed momenta (torus), fusion rules (double torus), etc. The mapping onto the SOS model then becomes more and more involved and requires the introduction of a system of distributed electric and magnetic charges [26] [13]. For more complicated geometries it is convenient to formulate the loop expansion using a special diagram technique which will be considered later in this section.

2.4. Summing over the world-sheet geometries

Now we come to the problem of the evaluation of the string path integral. The partition function of the string is defined formally as an integral in the space of all embedded surfaces. In our case the integration with respect to the intrinsic geometry is equivalent to the average of the partition function in the ensemble of all possible lattices $\mathcal{S}$. The dual graphs $\mathcal{S}^*$ are generated by the perturbative expansion of a $\phi^4$ matrix field theory. In this approach the length of a loop on the graph is an integer (all links of the graph are supposed to be of unit length). In what follows we prefer to consider the length $L$ as a continuous parameter, in order to simplify the notations. However, the equivalence with the discrete formulation is complete.

For the moment both the length $L$ and the area $A$ are dimensionless quantities. Since we are interested in the scaling limit $A \to \infty, L \to \infty$, later we are going to introduce the renormalized quantities

$$\ell = aL, \quad A = a^{2\nu}A$$

(2.49)

where $a$ will be a cut-off parameter (elementary length). It is natural to assume that the dimension of the world sheet is 2; then $1/\nu$ gives the fractal dimension of the loops which is determined by the dynamics. This is also the fractal dimension $D_B$ of the boundary of the world sheet with Dirichlet boundary condition on $x$. It can be any number between 1 and infinity.

Let $\mathcal{S}$ be any closed connected discrete manifold whose cells are squares. Introducing the bare “cosmological constant” $K_0$ coupled to the area ($= \text{number of plaquettes}$) $A(\mathcal{S})$ and the bare “string interaction constant” $\kappa_0$ coupled to the genus ($= \text{number of handles}$)
$H(S)$, we define the canonical partition function as a sum over all such $S$

$$F(K_0, \kappa_0) = \sum_S \kappa_0^{2-2H(S)} e^{-K_0 A(S)} F(S)$$  \hspace{1cm} (2.50)$$

where $F(S)$ is the partition function for frozen geometry defined by (2.22). This partition function is well defined for $K_0$ larger than some critical $K_*$ where the continuum limit may exist. In what follows the cosmological constant will be introduced implicitly through the measure in the space of empty surfaces.

The loop expansion acquires a new significance in the case of fluctuating geometry. It helps to reduce the problem to the problem of surfaces without embedding. The condition for this is the factorization of the Boltzmann weight of a polygon configuration to a product of weights associated with the connected domains. This property is very restrictive. It excludes, for example, terms in the action quadratic in the scalar curvature.

The integration measure over surfaces is controlled by two parameters: the cosmological constant $K_0$ coupled to the area of the world sheet and the string interaction constant $\kappa_0$ coupled to its topology. If $F^{(H)}(A)$ is the partition function of surfaces with fixed area $A$ and topology of a sphere with $H$ handles, then the canonical partition function reads

$$F(K_0, \kappa_0) = \sum_{H=0}^{\infty} \kappa_0^H \int_0^{\infty} dA \ e^{-K_0 A} F^{[H]}(A)$$  \hspace{1cm} (2.51)$$

Note that $K_0$ and $\kappa_0$ are dimensionless constants which will be renormalized in the continuum limit $A \to \infty$. There are different ways to achieve the continuum limit depending on the choice of the measure in the space of surfaces.

The mapping of the SOS model onto a gaussian field can be constructed also in the case of fluctuating geometry. The sum over the geometries means functional integration w.r. to the metric $\hat{g}_{ab}$ in (2.23). In conformal gauge

$$\hat{g}^{ab}(\xi) = \hat{g}_0^{ab}(\xi)e^{2\pi\nu\phi(\xi)}$$  \hspace{1cm} (2.52)$$

where $\hat{g}_0^{ab}(\xi)$ is some fiducial metric, the functional integral leads to a theory of Liouville gravity coupled to the matter fields [27]. As has been demonstrated by David, Distler and Kawai [3], it is consistent to treat the Liouville field $\phi$ as a gaussian field (with renormalized parameters) and the Liouville interaction as a perturbation. The fields $x(\xi)$ and $\phi(\xi)$ combine into a two-component gaussian field with conformal anomaly 26 defined by the action

$$\mathcal{A}[x, \phi] = \frac{1}{4} \int d^2 \xi \sqrt{\det \hat{g}_0} \left[ \pi g \ \hat{g}_0^{ab}(\partial_a x \partial_b x + \partial_a \phi \partial_b \phi) + \hat{R}(\xi)(ip_0 x(\xi) - \varepsilon_0 \phi(\xi)) \right]$$

$$+ \Lambda \int d^2 \xi \sqrt{\det \hat{g}_0} e^{2\pi\nu\phi(\xi)}$$  \hspace{1cm} (2.53)$$

where $\Lambda \sim K_0 - K_0^*$ is the renormalized cosmological constant.
The vertex operators dressed by the fluctuations of the metric are

\[ V_{(p,\varepsilon)}(\xi) = e^{i\pi (p-p_0) x(\xi) - \pi (\varepsilon(p) - \varepsilon_0) \phi(\xi)} \]  

(2.54)

In particular, the puncture operator ( = identity operator + gravitational dressing) is represented by

\[ \mathcal{P}(\xi) = e^{-\pi (\varepsilon(p_0) - \varepsilon_0) \phi(\xi)} = e^{2\pi \nu \phi(\xi)} \]  

(2.55)

The conformal anomalies of the two components of the gaussian field are \( c_x = 1 - 6\alpha_0^2/g, \) \( c_\phi = 1 + 6\varepsilon_0^2/g \) and the condition \( c_x + c_\phi = 26 \) implies

\[ \varepsilon_0^2 - p_0^2 = 4g \]  

(2.56)

We choose the positive solution

\[ \varepsilon_0 = g + 1, \ p_0 = g - 1 \]  

(2.57)

so that the two screening charges can be represented as

\[ \alpha_\pm = p_0 \pm \varepsilon_0 = g - 1 \pm (g + 1) \Rightarrow \alpha_+ = 2g, \alpha_- = -2 \]  

(2.58)

The condition that the conformal dimension of the operator (2.54) is one

\[ \Delta_x + \Delta_\phi = \frac{p^2 - p_0^2}{4g} - \frac{\varepsilon(p)^2 - \varepsilon_0^2}{4g} = 1 \]  

(2.59)

combined with (2.56) leads to the relation

\[ \varepsilon(p)^2 - p^2 = 0 \]  

(2.60)

which can be interpreted as a mass-shell condition of the same form as the mass-shell condition for a light particle in a two-dimensional space-time.

In principle, the vertex operator (2.44) dressed by the fluctuations of the metric is a linear combination of two operators (2.54) corresponding to the two solutions of (2.60)

\[ V_{(p)}(\xi) \to A_+ V_{p,\varepsilon}(\xi) + A_- V_{p,-\varepsilon}(\xi) \]  

(2.61)

The two terms will have the same dimension if the coefficient \( A_- \) contains a positive power \( a^{2|p|} \) of the cutoff \( a \). Therefore the second term is irrelevant when \( p \neq 0 \). Since the minimal momentum in the theories with \( C < 1 \) is positive (\( p_0 = 1/h \)), all physical operators correspond to the positive branch of (2.60)

\[ \varepsilon(p) = |p| \]  

(2.62)

Another motivation for the choice (2.62) is based on the quasiclassical treatment of the Liouville theory [28].
In a theory with $C = 1$ ($g = 1$, $p_0 = 0$) both terms become essential in the limit $p \to 0$ and the Liouville interaction in (2.53) which comes from the gravitational dressing of the identity operator can be not simply an exponential [25]. By (2.57) and (2.62) the Liouville charge of the identity operator (2.55) in the area term of (2.53) equals $\varepsilon_0 - \varepsilon(p_0) = 2\nu$ where

$$\nu = \frac{1}{2} (g + 1 - |g - 1|) = \begin{cases} g, & \text{if } g < 1 ; \\ 1, & \text{if } g > 1 \end{cases}$$

(2.63)

The gravitational dimension of the vertex operator with electric charge $\alpha_{rs} = p_{rs} + p_0 = r - sg + (g - 1)$ is [3], [4],

$$\delta_{rs} = 1 - \frac{\varepsilon_0 - \varepsilon(p_{rs})}{\varepsilon_0 - \varepsilon(p_0)} = \frac{|r - gs| - |g - 1|}{|g + 1| - |g - 1|}$$

(2.64)

In particular,

$$\delta^{\text{electric}} = \frac{|p| - |p_0|}{2\nu}, \quad \delta^{\text{magnetic}} = \frac{mg/2 - |p_0|}{2\nu}$$

(2.65)

Finally, the so-called string susceptibility exponent $\gamma_{\text{str}}$ is related to the conformal anomaly of the matter field by

$$C = 1 - 6 \frac{\gamma_{\text{str}}^2}{1 - \gamma_{\text{str}}}$$

(2.66)

It can be determined as twice the dimension $\delta_{00}$ in the Kac spectrum of gravitational dimensions (2.64)

$$\gamma_{\text{str}} = -2 \frac{|g - 1|}{g + 1 - |g - 1|} = -\frac{\varepsilon(p_0)}{\nu}; \quad \nu(2 - \gamma_{\text{str}}) = \varepsilon_0$$

(2.67)

2.5. Diagram technique

We have seen that a map $S \to X$ can be described by a collection of self-avoiding nonintersecting loops dividing the world sheet into domains of constant $x$. The energy of each such loop (domain wall) is proportional to its length $L$. In order to simplify the notations, from now we shall consider the length as a continuous quantity. The Boltzmann weight of a domain wall is

$$T^{-L} = \exp(-2P_0L)$$

(2.68)

Further, the weight of each domain of constant height $x$ depends on the geometry of the domain through its Euler characteristic. Indeed, the product of the Boltzmann weights (2.47) associated with the sites and squares of the domain is

$$(S_x)^{\text{sites-squares}} = (S_x)^{2-2H-n}$$

(2.69)

where $H$ is the number of handles and $n$ is the number of boundaries of the domain.

In this way the partition function of an $ADE$ model on a surface $S$ with arbitrary geometry has been reformulated as the partition function of a gas of nonintersecting loops
on this surface. The Boltzmann weights of the loops depend on the geometry of the surface only through homotopic invariants. In particular, they do not feel the local curvature. The loops sensible to the topology of the surface are the noncontractible ones that wrap around the handles of the surface $S$. Proceeding as in subsec. 2.3 one can check that all contractible loops have fugacity $\beta = 2 \cos(\pi p_0)$ while the fugacity of the loops wrapping around a handle is $\beta_p = 2 \cos(\pi p)$ where $p$ is the momentum of the matter-field excitation propagating along this handle. Thus the mapping of the $ADE$ models onto the loop gas depends, in the case of a surface with arbitrary topology, both on the spectrum of the order parameters and the way they interact. This mapping can be systematically formulated using a special diagram technique which has been first considered in [16] and developed further in [1]. In the case of fluctuating world-sheet geometry this diagram technique gives rise to the Feynman rules for the corresponding string field theory, to be discussed below.

The sum over the embeddings of a world sheet with fixed geometry is a formidable problem. It can be avoided by reorganizing the measure over embedded surfaces. First we take the sum over the world-sheet geometries with given configuration of domain walls. Each domain-wall configuration is determined by its topology and the lengths of the loops. It can be described by a Feynman diagram with vertices corresponding to the domains and lines corresponding to the domain walls (Fig. 4).

![Fig. 4: A loop configuration on a toroidal surface and the corresponding Feynman diagram](image)

The heights $x$ and the lengths $L$ are associated with the vertices and lines, correspondingly. The partition function (2.51) of the string is equal to the sum of all connected diagrams without external lines.

The weight of a diagram is a product of the weights associated with its vertices and propagators. A vertex is defined by the coordinate $x$, the number of handles $H$, and the lengths $L_1, \ldots, L_n$ of the boundaries of the corresponding domain. Its weight factorizes to a coordinate- and length-dependent components

$$V_o^{(H)}(x|L_1, \ldots, L_n) = (\kappa_0/S_x)^n - 2 + 2^H e^{-p_0(L_1 + \ldots + L_n)} W_o^{(H)}(L_1, \ldots, L_n)$$

(2.70)

where $W_o^{(H)}(L_1, \ldots, L_n)$ is the partition function of nonembedded (empty of loops) surfaces with $H$ handles and $n$ boundaries with lengths $L_1, \ldots, L_n$.

The “propagator” associated with a line $L$ is

$$G_0(x, L|x', L') = C_{xx'} \delta(L - L')$$

(2.71)
The integration over the lengths is performed with the measure $dL/L$. This measure is consistent with the convention that each boundary has a marked point on it.

It is convenient to introduce a quantum field theory in which all these graphs appear as Feynman diagrams. The latter can be formulated as a functional integral over the configurations of a loop field $\Psi_x(L)$ defined in the direct product of the target space $X$ and the positive real axis. The interactions of the loop field are given by the vertices (2.70).

It is not difficult to see that the grand canonical partition function $Z[J] = \exp(F[J])$ (i.e., the one where also disconnected surfaces are allowed) can be written as the following functional integral

$$Z[J] = \int D\Psi e^{S[\Psi]}$$

$$S(\Psi) = -\frac{1}{2} \sum_{x,x'} \int_0^\infty \frac{dL}{L} \Psi_x(L) \left(C^{-1}\right)_{xx'} \Psi_{x'}(L)$$

$$+ \sum_x \sum_{n=1}^\infty \sum_{H=0}^\infty \frac{1}{n!} \left[ \int_0^\infty dL J_x(L) \Psi_x(L) \right]$$

$$+ \sum_x \int_0^\infty \frac{dL_1}{L_1} ... \frac{dL_n}{L_n} V_o(H)(x|L_1, ..., L_n) \prod_{k=1}^n \Psi_x(L_k)$$

(2.72)

The multiloop correlation functions are formally obtained as derivatives with respect to the source $J(L)$

$$\langle \Psi_{x_1}(L_1) ... \Psi_{x_n}(L_n) \rangle = \left( \prod_{k=1}^n \frac{\delta}{\delta J_{x_k}(L_k)} F[J] \right)_{J=0}$$

(2.73)

The functional integration measure corresponds to the norm

$$\|\Psi\|^2 = \sum_x \int_0^\infty \frac{dL}{L} (\Psi_x(L))^2$$

(2.74)

Formally, eq. (2.72) defines the string field theory corresponding to the target space $X$. However, in order to make this definition more explicit, a lot of work has to be done. First, the diagram technique following from the domain wall representation contains tadpoles. This means that the string field has a nontrivial background which can be found as the solution of the saddle-point equation

$$\Psi_x^{cl}(L) = \left( \frac{\delta F[J]}{\delta J_x(L)} \right)_{\kappa_0=0}$$

(2.75)

This is the partition function of surfaces with the topology of the disc and bounded by a contour of length $\ell$ situated at the point $x$. 

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It is easier to solve the classical equation of motion (2.75) after going to the momentum space. For this we expand the string field as a linear combination of eigenstates (2.2)

\[ \Psi_x(L) = \sum_p V^x(p) \Psi(p)(L) \]  

Let us denote by \( \mathcal{V}^H_o(p_1, \ldots, p_n|L_1, \ldots, L_n) \) the vertices (2.70) in the momentum space. They again have a factorized form

\[ \mathcal{V}^H_o(p_1, \ldots, p_n|L_1, \ldots, L_n) = \kappa_0^{n-2+2H} N^{(H)}_{p_1 \ldots p_n} e^{-P_0(L_1+\ldots+L_n)} W^H_o(L_1, \ldots, L_n) \]  

where

\[ N^{(H)}_{p_1 \ldots p_n} = \sum_x \prod_{k=1}^n V^x(p_k)(S_x)^{2-n-2H} = \sum_x (S_x)^2 \mathcal{X}_{(p_1)} \ldots \mathcal{X}_{(p_n)} \left( \sum_p \mathcal{X}_{(p)} \mathcal{X}_{(p)} \right)^H \]  

The \( p \)-dependent part of the vertices (2.78) has a natural interpretation in terms of string states. The factor \( (S_x)^2 \) is related to the measure in the \( X \) space, each leg is multiplied by an order parameter \( \mathcal{X}_{(p)} \) and each handle can be considered as the result of contracting two legs \(^2\) (in the last line of (2.78) we have used the orthogonality relations (2.11)). For all target spaces

\[ N^{(0)}_{p_1 \ldots p_n} = \delta(p, p'), N^{(0)}_{p p' p''} = C_{p p' p''} \]  

where \( C_{p p' p''} \) are the fusion rules (2.10). By applying several times the fusion (2.10) the coefficient (2.78) can be represented as a sum of products of fusion coefficients. For example,

\[ N^{(1)}_{p} = \sum_{p'} C_{p p'}, \quad N^{(1)}_{p_1 p_2} = \sum_{p, p'} C_{p_1 p_2 p} C_{p p' p'}, \quad N^{(0)}_{p_1 p_2 p_3 p_4} = \sum_{p} C_{p_1 p_2 p} C_{p p_3 p_4} \]  

In general, the expression for the coefficient \( N^{(H)}_{p_1 \ldots p_n} \) can be represented by a “Feynman diagram” with \( n-2+2H \) vertices (to each vertex is assigned a fusion coefficient) and \( H \) loops.

\(^2\) Each handle therefore contributes a factor proportional to the volume of the target space. In order to define the topological expansion for the string embedded in \( \mathbb{Z} \), we have first to introduce a cutoff in the dual momentum space. The simplest way to do this is to replace the continuum spectrum of momenta by a discrete one with a small spacing \( \delta p \). Such a discretization appears automatically in the matrix-model realization of the string embedded in \( \mathbb{R} \) [30]. The fact that the vertices depend explicitly on the cutoff does not contradict the general covariance. Indeed, the renormalized string interaction constant \( \kappa \) depends explicitly on the cutoff \( a \) in the space of lengths \( \ell \): it vanishes as \( 1/\log a \). If we choose \( \delta p \sim 1/\log a \), then the partition function will be cutoff-independent.
Thus the $p$-dependent component of a multiple vertex can be decomposed into elementary interactions involving only three strings. Note that all “Feynman diagrams” representing the possible decompositions of a multiple vertex have the same contribution (duality property).

Now we can write the functional integral (2.72) in the form

$$e^{F[J]} = \int d\Psi e^{\mathcal{A}[\Psi]}$$

$$\mathcal{A}[\Psi] = -\frac{1}{2} \sum_p \int_0^\infty \Psi(p)(L)(2\cos(\pi p))^{-1}\Psi(p)(L) \frac{dL}{L} + \sum_{n=1}^\infty \sum_{H=0}^\infty \sum_{p_1,...,p_n} \frac{1}{n!} \int_0^\infty V^{(H)}_0(p_1,...,p_n|L_1,...,L_n) \prod_{k=1}^n \Psi(p_k)(L_k) \frac{dL_k}{L_k}$$

and the classical equations of motion (2.75) are equivalent to

$$\Psi^{cl}_{(p)}(L) = \delta_{p,p_0} W^{(0)}(L) \quad (2.82)$$

$$W(L) = \sum_{n=0}^\infty \frac{\beta^n}{n!} \int_0^\infty \prod_{k=1}^n dL_k \frac{L_k}{L_k} e^{-P_0 L_k} W(L_k) W^{(H=0)}(L,L_1,...,L_n) \quad (2.83)$$

By the loop expansion of subsec. 2.3 the quantity $W^{(H)}(L)$ is the partition function of the gas of nonintersecting loops on a disc with boundary of length $L$ and a fluctuating metric. The index $(0)$ means planar topology of the world sheet ($H = 0$). Eq. (2.83) is graphically represented in Fig. 5.

Fig. 5: A diagrammatic representation of the saddle-point equation.

Suppose that we have solved the equation for the string background (2.82). Shifting the field in the functional integral (2.81) by its classical value

$$\Psi_{(p)}(L) \rightarrow \Psi^{cl}_{(p)}(L) + \Psi_{(p)}(L) \quad (2.84)$$

we arrive at a new diagram technique without tadpoles. The new vertices are obtained from the original ones by dressing with tadpoles (Fig. 6).

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\[3\] We believe that the $\ell$-dependent component of a multiple vertex can be decomposed as well into elementary interactions by means of Witten’s factorization arguments [31]
\[ V^{(H)}(p_1, \ldots, p_n | L_1, \ldots, L_n) = n_0^{-2+2H} N^{(H)}_{p_1 \ldots p_n} W^{(H)}(L_1, \ldots, L_n) \] (2.85)

\[ W^{(H)}(L_1, \ldots, L_n) = \sum_{k=0}^{n} \frac{1}{k!} \int_0^\infty \prod_{r=n+1}^{n+k} W^{(0)}(L_r) e^{-P_0 L_r} \frac{dL_r}{L_r} \]

\[ e^{-P_0(L_1+\ldots+L_n)} W^{(H)}_0(L_1, \ldots, L_n, L_{n+1}, \ldots, L_{n+k}) \] (2.86)

Note that for \( n = 1 \) (2.86) coincides with the saddle-point equation (2.83). The quantity \( W^{(H)}(L_1, \ldots, L_n) \) is the partition function of the loop gas on a fluctuating surface with \( H \) handles and \( n \) boundaries with the condition that only contractible loops are allowed.

The gaussian part of the new action (the inverse propagator) is equal to the sum of the inverse original propagator (2.71) and the two-point vertex \( V^{(0)}(p, p'|L, L') = \delta_{pp'} W^{(0)}(L, L') \). (It is convenient to treat the vertices \( V^{(H)}(p_1, p_2 | L_1, L_2) \) with \( H > 0 \) as part of the interaction.) The propagator \( G_p(L, L') \) is therefore determined by the equation

\[ G_p(L', L'') = 2 \cos(\pi p) \left[ \delta(l', l'') + 2 \cos(\pi p) \int \frac{dL}{L} W^{(0)}(L', L) G_p(L, L'') \right] \] (2.87)

whose solution is graphically represented in Fig. 7.

The symmetric function \( W^{(0)}(L, L') \) can be considered as a kernel of a symmetric operator in the space of the real square-integrable functions defined on the positive real line. This operator describes the evolution of the string in the space of the world-sheet geometries. Let us introduce its eigenstates which we assume orthonormalized

\[ \int_0^\infty \frac{dL}{L} W^{(0)}(L, L') \langle L | E \rangle = \Omega(E) \langle L' | E \rangle \] (2.88)
Then the propagator (2.87) is diagonalized in the same basis and its eigenstates are

\[ \tilde{G}(E, p) = \frac{2 \cos(\pi p)}{1 - 2 \cos(\pi p) \Omega(E)} \]  

(2.89)

The eigenstates \( \langle L | E \rangle \) describe the excitations in the \( L \)-space in the same way as the eigenstates of the connectivity matrix describe the excitations in the \( x \)-space. The wave functions of the closed string states are labeled by the “energy-momentum” \((E, p)\) and have a factorized form

\[ \langle E, p | L, x \rangle = V_{(p)}^{x} \langle E | L \rangle \]  

(2.90)

We have seen that the string background depends on the structure of the \( X \)-space only through the momentum \( p_{0} \) of the ground state \((p_{0} = 1/h \) for the Dynkin diagrams and \( p_{0} = 0 \) for the extended Dynkin diagrams\) and therefore the same factorization takes place for the vertices (2.85) describing interactions due to joining and splitting of closed strings. This is the miracle which makes possible to solve exactly the string theory in our formalism. Note that in our case the separation of the two global modes occurs before taking the continuum limit, which is not the case for the standard discretization of the \( D = 1 \) string path integral [32]. It is worth to try to understand better the origin of this symmetry.

Let us mention that there exists a simpler model such that the only allowed momentum is \( p_{0} \). This means that the vertices have trivial coordinate part. A microscopic definition of this model is given by the \( O(n) \) model on a fluctuating lattice with \( n = -2 \cos(\pi g) \) [33].

2.6. Loop equation

In order to find the explicit expressions for the dressed vertices (2.86) we have first to solve the saddle-point equation (2.82). Imagine that the \( n \)-loop amplitudes for the empty random surface are constructed as the connected correlation functions of an \( N \times N \) Hermitean random matrix \( \Phi \) with potential \( U_{o}(\Phi) \)

\[ \sum_{H=0}^{\infty} N^{2-2H} W_{o}^{(H)}(L_{1}, ..., L_{n}) = \prod_{k=1}^{n} \frac{\delta}{\delta L_{k}} F_{o}[J] \]  

(2.91)

\[ e^{F_{o}[J]} = \int d\Phi e^{NTr U_{o}(\Phi)} + \int_{0}^{\infty} dL Tr \exp(L\Phi) \]  

(2.92)

Then the loop amplitudes \( W_{o}^{(H)}(L_{1}, ..., L_{n}) \) satisfy the following system of integral equations [34], [35]

\[ U_{o}^{H}(\partial/\partial L_{i}) W_{o}^{(H)}(L, L_{1}, ..., L_{n}) \]

\[ + \sum_{H'=0}^{H} \sum_{I+J=\{1, ..., n\}} \int_{0}^{L} dL' W_{o}^{(H')} (L_{i} | i \in I) W_{o}^{(H-H')} (L-L'; L_{j} | j \in J) \]

\[ W^{(H-1)}(L-L', L_{1}, ..., L_{n}) + \sum_{k=1}^{n} W_{o}^{(H)} (L+L_{k}; L_{s} | s \neq k) = 0 \]  

(2.93)
Inserting (2.92) into (2.83) we arrive at the following closed equation for the classical string field \[8\], \[9\]

\[
U'_\circ(\partial/\partial L)W^{(0)}(L) = \int_0^L dL'W^{(0)}(L')W^{(0)}(L - L') + \beta \int_0^\infty dL'W^{(0)}(L')W^{(0)}(L + L')e^{-2P_0L'}
\] (2.94)

which should be completed with the condition that \(W^{(0)}(L)\) is analytic at the point \(L = 0\)

\[
W^{(0)}(L) = \sum_{k=0}^\infty \frac{L^k}{k!}W_k \] (2.95)

In the limit \(L \to \infty\) the function \(W(L)\) is expected to behave as

\[
W^{(0)}(L) \sim L^{-b}e^{P_R L} \] (2.96)

where the entropy per unit length \(P_R\) depends on the parameters of the theory.

Eq. (2.94) has a transparent geometrical meaning \[36\]. A small deformation of the world sheet at the marked point is equivalent to a local deformation of the boundary. Its exact form depends on the measure in the space of the world-sheet geometries. In general, a local deformation of the boundary is equivalent to a differential operator

\[
U'_\circ(\partial/\partial L) = g_2\partial/\partial L + g_3\partial^2/\partial L^2 + ... \] (2.97)

On the other hand, the deformation of the world sheet cancels part of the integration measure in the space of surfaces and produces boundary (or contact) terms. The standard contact term, which also appears in pure gravity, is due to the surfaces with degenerated world sheet such that another point of the boundary approaches the marked point. Here we encounter a new contact term due to one of the loops approaching the marked point at the boundary.

The loop equation (2.94) has been derived originally \[1\] in terms of the coefficients \(W_k\)

\[
\sum_{k \geq 2} g_k W_{n+k-2} = \sum_{k=0}^{n-2} W_k W_{n-2-k} + \beta \sum_{p,q=0}^\infty \frac{(p+q)!}{p!q!} e^{-2P_0(p+q+1)} W_p W_{n+q-1} \] (2.98)

We arrive at this form of the loop equation if the gas of loops is considered on a random graph, the length of the loops being the number of bonds they occupy. The formulation in terms of continuous lengths has been proposed by Kazakov \[36\].

It is convenient to introduce the Laplace image

\[
\hat{W}(P) = \int_0^\infty dL W^{(0)}(L)e^{-PL} \] (2.99)
which can be interpreted as the partition function of surfaces with the topology of a disc and “boundary cosmological constant” \( P \). In order to simplify the notations we do not assign an index (0) to \( \hat{W}(P) \). The condition of analyticity (2.95) implies

\[
\hat{W}(P) = \frac{W_0}{P} + \frac{W_1}{P^2} + \ldots, \quad P \to \infty
\]  

(2.100)

(In what follows we assume the standard normalization \( W_0 = 1 \).) The asymptotics (2.90) means that the series (2.100) converges for \( |P| > P_R \); in the vicinity of the point \( P_R \) it behaves as \((P - P_R)^{b-1}\).

We are interested in the solutions of the loop equation having a single cut \([P_L, P_R] \) along the real \( P \)-axis. The loop equation (2.94) in terms of \( P \)-variables reads

\[
\hat{W}^2(P) = \oint_{\mathcal{C}} \frac{dP'\hat{W}(P')}{2\pi i(P - P')} [\beta \hat{W}(2P_0 - P') - U'_\circ(P')] \]  

(2.101)

where the contour \( \mathcal{C} \) encloses the cut \([P_L, P_R] \) of the Riemann surface of the function \( \hat{W}(P) \). The contour integral makes sense only if the two cuts \([P_L, P_R] \) and \([2P_0 - P_R, 2P_0 - P_L] \) of the Riemann surface of the integrand function do not overlap, that is, for \( P_0 > P_R \). The positions \( P_L \) and \( P_R \) of the branchpoints are functions of the potential \( U_\circ(P) \) and the temperature \( P_0 \) of the loop gas. Eqn. (2.101) implies the following conditions on the real and imaginary part of \( \hat{W}(P) \) [1]

\[
2\text{Re}\hat{W}(P) + \beta \hat{W}(2P_0 - P) + U'_\circ(P) = 0, P \in [P_L, P_R]
\]

\[
\text{Im}\hat{W}(P) = 0, P \notin [P_L, P_R]
\]  

(2.102)

3. Critical behaviour

3.1. Critical surfaces, noncritical loops

The partition function (2.50) is defined in certain domain \( \mathcal{D} \) in the space of the parameters \( P_0, K_0, \ldots \). The critical behaviour is achieved at the boundary \( \partial\mathcal{D} \) of this physical domain, where the partition function and the observables develop singularities. There are two types of singularities in the space of parameters. The first is related to the diverging area of the fluctuating surface. It appears when the coupling constants \( g_1, g_2, \ldots \) in the potential \( U_\circ(P) \) which defines the integration measure in the space of nonembedded surfaces, are tuned in a special way. The loops on the world sheet are small and do not affect the scaling properties. The critical behaviour of the model is the (multicritical) behaviour of pure gravity. The \( m \)-critical points of the nonembedded random surface [33] have been interpreted [37], [38], [39], [40] as pure gravity coupled to a conformal field theory \((2, 2m - 1)\). It can be mapped onto the Coulomb gas (2.53) with \( g = m - 1/2 \). The \( m \)-critical potential has been chosen in [36] as an even polynomial of degree \( 2m \)

\[
U_\circ(P) = \sum (-)^{k-1} \frac{(k - 1)!m!}{(2k)!(m - k)!} P^{2k}
\]  

(3.1)
However, if the symmetry $P \leftrightarrow -P$ is abandoned, it suffices to take a polynomial of degree $m + 1$. The scaling limit is achieved by replacing $L, P$ by the renormalized quantities $\ell, z$

$$L = \ell/a, \quad P = P_* + az$$  \hspace{1cm} (3.2)

At the $m$-critical point the scaling part $\hat{w}(z)$ of the planar loop amplitude is defined by

$$\hat{W}(P) = \hat{W}(P_*) + a^{m-1/2} \hat{w}(z)$$  \hspace{1cm} (3.3)

In the matrix-model realization of the multicritical points the renormalised cosmological constant $\lambda$ scales as $a^{-m}$ which could mean that the fractal dimension of the boundary of the world sheet is $2/m$. For $m > 2$ this is less than the classical dimension 1. The resolution of this paradox comes from the observation [37] [38] that for $m > 2$ $\lambda$ is coupled to the operator with minimal (negative) gravitational dimension $\delta_{m1} = 1 - m/2$ and therefore does not measure the area of the world sheet. The true cosmological constant $\Lambda$ has dimension $a^{-2}$ for $m \geq 2$ and $a^{-1}$ for $m = 1$. Finally, the string interaction constant scales as

$$\kappa_0 = \kappa a^{1-\gamma_{\text{str}}/2}; \quad \gamma_{\text{str}} = 3/2 - m$$  \hspace{1cm} (3.4)

3.2. Dense phase

When $P_0$ approaches its critical value $P_*$, long-range effects due to the diverging size of the loops change the critical behaviour of the model. For generic potential $U_o$ the random surface grows only due to the critical loops which are densely packed on the world sheet. Its area is essentially equal to the total length of the loops. This is the dense phase of the loop gas. If both the potential $U_o$ and the temperature $P_0$ are tuned, a more complicated picture arises.

The regime of critical loops coupled to noncritical surfaces describes the dense phase of the loop gas model. In this regime the loops are densely packed and fill almost all surface of the world sheet. The nonrenormalized area of the world sheet is therefore equal to the total (nonrenormalized) length of the loops: $A = L_{\text{tot}}$ and $P_o$ plays the role of a cosmological constant.

It is easy to see that the condition for complete compensation of the entropy and the energy of the loops is $P_R = P_0$. First let us remind that $W(L)$ behaves when $L$ is large according to (2.96). Let us choose one of the random loops on a surface with the topology of a sphere. It splits the sphere into two discs, each contributing a factor $e^{P_R L}$ when $L$ goes to infinity. Thus the entropy $2P_R L$ of the loop totally compensate its energy $2P_0 L$ when $P_R = P_0$.

The vicinity of the critical point is defined by introducing a cutoff parameter $a$ with dimension of length. The variable $z$ dual to the renormalized length $\ell$ (the renormalized boundary cosmological constant) is introduced by

$$P = P_* + az, \quad P_R = P_* - aM$$  \hspace{1cm} (3.5)

The parameter $M$ is the contribution of the fluctuations of the surface to the renormalized boundary cosmological constant. It defines the position of the cut in the $z$-plane.
The singular part of the loop amplitude $W(P)$ behaves at the critical point $P_R = P_0$ ($M = 0$) as $z^g$ where $g$ is the solution of $2\cos(\pi g) + \beta = 0$, $0 < g < 1$ \cite{11,1}. Therefore we define the scaling part $\hat{w}(z)$ of the loop amplitude as

$$\hat{W}(P) = \hat{W}_* + (\text{constant})a^g\hat{w}(z)$$

(3.6)

where $\hat{W}_*$ is the value of the bare loop amplitude for $P = P_*, P_0 = P_*$. 

Now we are going to find the scaling law for the cosmological constant $\Lambda$ which is defined as

$$P_0 = P_* - (\text{constant})a^{2\nu}\Lambda$$

(3.7)

If we assume that the dimension of the world sheet is 2, then the exponent $\nu$ has the meaning of inverse fractal dimension of the boundary of the world sheet. The induced boundary cosmological constant $M$ is a function of $P_0$ which vanishes at $P_0 = P_*$. Therefore the renormalized bulk and boundary cosmological constants are related by

$$\Lambda = (\text{constant})M^{2\nu}$$

(3.8)

The scaling of $\Lambda$ can be extracted from a nonlinear algebraic equation which follows from the loop equation (2.101). In order to get rid of the contour integration we use the following trick. First write again eq. (2.101) with the variables $P, P'$ replaced by their images w.r.t. the reflection $P \rightarrow 2P_0 - P$. The integrand remains the same but the contour $C$ is replaced by a contour $\bar{C}$ enclosing the cut $[2P_0 - P_R, 2P_0 - P_L]$ of $\hat{W}(2P_0 - P)$. Then add the two equations and apply the Cauchy theorem to the integral along the contour $C + \bar{C}$. The result is the following functional equation for $\hat{W}(P)$

$$[-U'_o(P)\hat{W}(P) - \sum_{k \geq 0} P^k (g_{k+2}W_0 + g_{k+3}W_1 + ...) - \hat{W}^2(P)]$$

$$+ [P \rightarrow 2P_0 - P] = \beta\hat{W}(P)\hat{W}(2P_0 - P)$$

(3.9)

At the point $P = P_0$ this functional equation becomes algebraic

$$(2 + \beta)\hat{W}^2(P_0) + 2U'_o(P_0)\hat{W}(P_0) - 2 \sum_{k \geq 0} P^k_0 (g_{k+2} + g_{k+3}W_1 + ...) = 0$$

(3.10)

Eq. (3.10) becomes very simple in the limit $P_0 \rightarrow -\infty$ where the only allowed loop configurations are those with no space between loops. This extremely dense limit can be achieved by choosing gaussian potential ($g_2 = -1, g_3 = \ldots = 0$). Then eq. (3.10) involves only the coefficient $W_0 = 1$ of the expansion (2.100) and can be solved instantly

$$\hat{W}(P_0) = 2/(P_0 + \sqrt{P_0^2 - 2(2 + \beta)})$$

(3.11)

This quantity has a singularity at $P_0 = \sqrt{2(2 + \beta)} = P_*$ in accord with the exact result obtained with more elaborated technique \cite{11}. 27
By the definitions (3.6) and (3.7) we find
\[ W(P_0) = W_\ast + (\text{const.})a^g \hat{w}(0) = W_\ast - (\text{const.})\sqrt{\Lambda} \]  
(3.12)
By its dimension \( \hat{w}(0) \sim M^g \) and eq. (3.12) implies \( \nu = g \). From now we will fix the normalizations of \( \Lambda \) and \( M \) so that
\[ \hat{w}(0) = -\cos(\pi g/2)M^g, \quad \Lambda = M^{2g} \]  
(3.13)
(Note that the sign of \( \hat{w}(M) \) should be negative.) This scaling law persists for generic potential \( U_\circ(P) \), i.e., in a domain \( \mathcal{D}_1 \) of codimension zero in the space of coupling constants \( g_2, g_3, \ldots \). Indeed, let us repeat the above argument for a generic potential \( U_\circ(P) = g_2P^2 + g_3P^3 + \ldots \). Expanding eq. (3.10) around the point \( P_0 = P_\ast \) we find a relation between \( \hat{w}(0) = -\cos(\pi g/2)M^g \) and \( \Lambda \). In the limit \( a \to 0 \) the most singular part of the coefficients \( W_k \) is the v.e.v. of the puncture operator \( \mathcal{P} = -\partial/\partial \Lambda \)
\[ W_k \sim a^{2\nu(1-\gamma_{\text{str}})}\langle \mathcal{P} \rangle \sim (a^{2\nu}\Lambda)^{1-\gamma_{\text{str}}} \]  
(3.14)
Inserting this in (3.10) we find a relation of the type
\[ (aM)^{2g} = \sum_{k \geq 1} A_k (a^{2\nu}\Lambda)^k + B (a^{2\nu}\Lambda)^{1-\gamma_{\text{str}}} \]  
(3.15)
where \( A_1, A_2, \ldots \) and \( B \) are numerical coefficients.

In the limit \( a \to 0 \) only the smallest power in \( a \) will survive. For a generic potential all the coefficients \( A_k \) are nonzero. The strongest singularity is that of the \( A_1 \)-term and the matching of powers gives
\[ 2g = \min[2\nu, 2\nu(1-\gamma_{\text{str}})] \]  
(3.16)
Since the string susceptibility exponent is always non-positive, this implies \( \nu = g \). In order to fix the string susceptibility exponent, we use the fact that it is related to the dimension of the loop amplitude
\[ \nu(2 - \gamma_{\text{str}}) - 1 = g \]  
(3.17)
Thus in the dense phase
\[ \nu = g, \quad \gamma_{\text{str}} = 1 - 1/g; \quad 0 < g < 1 \quad (\text{dense phase}) \]  
(3.18)
In the interval \( 1/2 < g < 1 \) the difference \( P_\ast - P_0 \) vanishes faster than \( a \) and the point \( P_0 \) in the definition of the scaling variables can be replaced with \( P_\ast \). This is not the case, however, if \( g < 1/2 \). In this case the cut of the loop amplitude appears at \( z = -M - a^{2g-1}\Lambda \) and in the limit \( a \to 0 \) the singularity of the loop amplitude is gone to \( -\infty \). We have a very poor understanding of this case at the moment. Let us only mention that the fugacity of the loops is negative and the fractal dimension of the boundary is larger than two. The points \( g = 1/m, \quad m = 1, 2, 3, \ldots \) have integer \( \gamma_{\text{str}} \) and should describe topological theories. A very interesting discussion on the physics at these points has been given by H. Saleur [12] (see also [11]).

The limiting case \( g = 1/2 \) corresponds to the gaussian model. In this case there is no singularity at all and the choice of the distance \( M \) is arbitrary.
3.3. Dilute phase

Now let us consider the case when both kinds of singularities are present. If the potential $U_\circ(P)$ is critical ($m = 2$), then at the point $M = M_*$ both the length of the loops and the area of the space between them become infinite. This phase is known as the dilute phase of the loop gas. The multicritical potentials will lead, further, to new types of critical behaviour of the gas of loops [11] which we are going to discuss briefly.

As we have seen above, the dense phase exists in a domain $D_1$ of codimension zero in the space of coupling constants $g_1, g_2, ...$. The generic point on the boundary $D_2 = \partial D_1$ of this domain corresponds to the next critical regime. The simplest way to achieve it is to adjust the constant $g_3$ in a cubic potential so that the coefficient $A_1$ in (3.15) vanishes. Then the condition of matching of powers reads

$$2g = \min[4\nu, 2\nu(1 - \gamma_{str})]$$

If we are in the interval $1 < 1 - \gamma_{str} < 2$, this condition means $2g = 2\nu(1 - \gamma_{str})$. Then from (3.17) we find

$$\nu = 1, \quad \gamma_{str} = 1 - g; \quad 1 < g < 2 \text{ (dilute phase)}$$

Proceeding in the same way, we can achieve the $m$-critical point by adjusting the coupling constants $g_3, ..., g_{m+1}$ so that the coefficients $A_1, ..., A_{m-1}$ vanish. The condition of matching of powers then gives [11]

$$2g = \min[2m\nu, 2\nu(1 - \gamma_{str})]$$

Combining (3.21) with (3.17) we find

$$\nu = 1, \quad \gamma_{str} = 1 - g; \quad m - 1 < g < m \text{ (m-critical dilute phase)}$$

Let us assume that the loop gas with given $g$ describes at the critical point a conformal field theory coupled to gravity. Then, by the KPZ-DDK argument [5], [6] its conformal anomaly is given by

$$C = 1 - 6\frac{\gamma_{str}^2}{1 - \gamma_{str}} = 1 - 6\frac{(g - 1)^2}{g}$$

Thus the interval $0 < g < \infty$ covers twice the spectrum $-\infty < C < 1$ of the central charge. Note that even if the central charge is symmetric w.r.t. $g \to 1/g$, the fractal dimension of the boundary is not. In the dilute phase it takes the “classical” value 1 and in the dense phase it is greater than one.

The $m$-critical quantum gravity coupled to a gas of critical loops exhibits a specific critical behaviour characterized by the fugacity $\beta$ of the loops. One can say that the $(2m+1, 2)$-conformal theory of matter coupled to the gas of loops with fugacity $\beta$ produce an effective matter field with central charge $C = 1 - 6(g - 1)^2/g$ where $g$ is the branch of the function $g = -(1/\pi) \arccos(\beta/2)$ determined by $m - 1 < g < m$. 

29
The change of the critical regime occurs at the integer points \( g = m, \ m = 1, 2, 3, \ldots \). Let us consider the vicinity of the point \( g = m \) dividing the \( m \)-critical and \( m + 1 \)-critical dilute phases of the loop gas. In this case eq. (3.17) reads

\[
M^2 = \Lambda \left[ A_m (a^2 \Lambda)^{m-g} + B \right]^{1/g}
\]

(3.24)

The first term on the r.h.s. vanishes when \( g < m \) because the power of \( a \) is positive as well as when \( g < m \) because then \( A_m = 0 \). However, when \( g = m \) both terms survive and our previous arguments may not be applicable. The limits \( g \to m \) and \( a \to 0 \) do not commute and one has to solve the loop equation for \( g = m \) and then take the continuum limit \( a \to 0 \).

Exact solutions are known for the cases \( g = 1 \) and \( g = 2 \), to be discussed below. At \( g = 2 \) the scaling is the same as in the whole dilute phase, \( \Lambda = M^2 \). At the point \( g = 1 \) which is the endpoint of the dilute phase, the scaling law receives logarithmic corrections:

\[
\Lambda = (M \log M)^2.
\]

3.4. The loop equation and the string background in the continuum limit

We are going to solve the loop equation (2.101) in the continuum limit. Let us define the vicinity of the critical point as follows

\[
P = P_0 + az
\]

\[
P_R = P_0 - aM
\]

\[
\tilde{W}(P) = \frac{(2U'_0(P_0 + az) - \beta U'(P_0 - az))}{(4 - \beta^2)} + (\text{const.}) a^g \hat{w}(z)
\]

Here we used as a reference point \( P_0 \) instead of \( P_* \); we shall see that if \( 1/2 < g < 1 \) this makes no difference in the limit \( a \to 0 \).

After the vicinity of the critical point is blown up by the change of variables (3.25), the cut \( [P_L, P_R] \) of the Riemann surface of \( W(P) \) is replaced by the cut \( [-\infty, -M] \) along the negative \( z \)-axis. Eq. (2.102) implies the following conditions on the real and imaginary parts of \( \hat{w}(z) \)

\[
2 \text{Re} \hat{w}(z) + \beta \hat{w}(-z) = 0, \quad z \leq -M
\]

\[
\text{Im} \hat{w}(z) = 0, \quad z \geq -M
\]

(3.26)

Eq (3.26) becomes more transparent when written in terms of the variable \( \tau \) which we have introduced in [8], [9]

\[
z = M \cosh(\tau)
\]

(3.27)

Then the \( z \)-plane cut along the interval \(-\infty < z < -M\) is mapped into the half-strip

\[
\text{Re } \tau \geq 0, \quad -\pi \leq \text{Im } \tau \leq \pi
\]

(3.28)

The two sides of the cut are mapped to the horizontal boundaries \([\tau = t \pm i\pi, t > 0] \) of the \( \tau \)-strip. Therefore, if \( z \) is real and positive, then \(-z \pm i0 = M \cosh(\tau \pm i\pi)\). Since \( \hat{w}(z) \) is a real function \((\hat{w}(z)^* = \hat{w}(z^*))\), the first of the eqs. (3.26) can be written as

\[
[\cos(\pi \partial/\partial \tau) - \cos(\pi g)] \hat{w}(z(\tau)) = 0
\]

(3.29)
Eq. (3.29) has a single solution (up to a constant factor) which is real along the vertical boundary of the \( \tau \)-strip and behaves at infinity as \( z^g \)

\[ \hat{w}(z) = -M^g \cosh(g\tau) \] (3.30)

If we return to the original variable \( z \), the solution reads

\[ \hat{w}(z) = -\frac{1}{2}[(z + \sqrt{z^2 - M^2})^g + (z - \sqrt{z^2 - M^2})^g] \] (3.31)

It can be expanded as an infinite series in fractional (in general) negative powers of \( z \) with radius of convergence \( 1/z = 1/M \)

\[ \hat{w}(z) = \sum_{n \geq 0} w_n^\pm M^g(M/z)^{2n\pm g} \] (3.32)

and dimensionless coefficients

\[
\begin{align*}
    w_n^\pm &= \pm \frac{\Gamma(2n \pm g)}{\Gamma(n + 1 \pm g)n!} \\
    &= \pm \frac{(2n \pm g)(2n \pm g - 1)\ldots(n + 2 \pm g)}{n!}
\end{align*}
\] (3.33)

Even if each of the terms in (3.32) has a cut \(-\infty < z < 0\), the whole series defines a function which is analytic for \( |z| < M \).

The v.e.v. of the loop operator is determined by the same equation (3.26) both in the dense and dilute phases. It is given by the same analytic function (3.31) with \( g \) ranging from 0 to \( \infty \). The \( m \)-critical behaviour of the loop amplitude is described by the branch \( m - 1 < g < m \) of the parametrization (2.43). The function \( \hat{w}(z) \) has a square-root singularity at \( z = -M \). It is meromorphic in the \( z \)-plane with a cut from \( z = -M \) to \( z = -\infty \) where it behaves as \( z^g \). For nonrational values of \( g \) the Riemann surface of \( \hat{w}(z) \) is infinitely foliated and has two cuts \(-\infty < p < -M \) and \( M < p < \infty \). All sheets except the first one have two cuts. We will always consider the function \( \hat{w}(z) \) on the first sheet.

For \( g = p/q \) the Riemann surface of \( w(z) \) has a branch point of order \( q \) at infinity. Then the first and the last sheet have only one cut.

If we introduce a renormalized length \( \ell = aL \), then the inverse Laplace image of \( \hat{w}(z) \)

\[
    w(\ell) = \int_{-i\infty}^{i\infty} dz e^{\ell z} \hat{w}(z); \quad \hat{w}(z) = \int_0^{\infty} d\ell \ e^{-z\ell} w(\ell)
\] (3.34)

is the Bessel function \[15\]

\[ w(\ell) = \frac{M^g}{\ell} K_g(M\ell). \] (3.35)

It satisfies the following loop equation

\[
\int_0^{\infty} d\ell' w(\ell') w(\ell - \ell') + \beta \int_0^{\infty} d\ell' w(\ell') w(\ell + \ell') = 0
\] (3.36)
which is the renormalized version of (2.94). The renormalized string background (3.34) behaves differently at large and small lengths

\[ w(\ell) = \ell^{-1} M^g K_g(M\ell) \sim \begin{cases} M^{g-1/2} \ell^{-3/2} e^{-M\ell}, & \text{if } \ell \gg M^{-1}; \\ \ell^{-g-1}, & \text{if } \ell \ll M^{-1} \end{cases} \quad (3.37) \]

The small-$\ell$ asymptotics (3.37) can be taken as a subsidiary condition to eq. (3.36) in order to have a unique solution. The large-$\ell$ asymptotics describes a loop of length much larger than the characteristic length $\ell = 1/M$. The area of the world sheet is zero in this limit and the power of $\ell$ is the same as in the topological gravity ($g = 1/2$). The small-$\ell$ asymptotics describes the critical point where the area of the world sheet and the characteristic length of the loops are infinite. We see that the square-root singularity changes generically to a nonrational one in the limit $M \to 0$.

In order to reproduce the regime of critical surfaces, noncritical loops we have to take the limit $\Lambda \to 0$, keeping $M$ finite, i.e., to leave the trajectory $\Lambda = M^{2\nu}$. However, we can achieve the critical points of nonembedded surfaces by tending the fugacity $\beta$ of the loops to zero. This limit is achieved at the half-integer values of $g$. By the scaling of the loop amplitude $\hat{w}(z)$ these points can be identified with the multicritical points of pure gravity [36]. The explicit expression for the loop average $\hat{w}(z)$ in the $m$-critical point is

\[ \hat{w}(z) = - \sum_{k=0}^{m} (1 + (-1)^k) 2^{m-1/2} \frac{(2m-1)!}{k!(2m-1-k)!} \left( z + M \right)^{m-1/2-k/2} \left( z - M \right)^{k/2} \quad (3.38) \]

Eq. (3.38) reproduces the known results (see, for example, [34]) for $g = 3/2$ (pure gravity) and $g = 1/2$ (gaussian model)

\[ \hat{w}(z) = \begin{cases} -2(2z-M) \sqrt{(z+M)/2}, & g = 3/2 \\ -2\sqrt{(z+M)/2}, & g = 1/2 \end{cases} \quad (3.39) \]

At the integer points $g = m$, $m = 1, 2, 3, 4, \ldots$ the planar loop amplitude (3.31) is a polynomial and has no cut at all. However, this is not always the physical solution because the limit $a \to 0$ is taken prior to the limit $g \to m$. The exact solution for the case $g = 2$ [44] lead to the following expression in the continuum limit 4

\[ \hat{w}(z)_{g=2} = \left[ \frac{d}{dg} \hat{w}(z) \right]_{g=2}, \quad \Lambda = M^2 \quad (3.40) \]

---

4 The equivalence between the gas of dilute loops with fugacity $\beta = -2$ ($g = 2$) and the bosonic string embedded in -2 dimensions can be established as follows. If the loops are considered as oriented, then their fugacity is -1. Further, it is easy to see that the condition of nonintersection can be abandoned, since the total contribution of the configurations containing intersecting loops vanishes because of cancellations. Therefore the partition function can be written as the exponential of minus the entropy of a single oriented loop, with no restriction to its configurations. This is exactly the determinant of the Laplace operator in the lattice $S$, which can be written as a Gaussian integral with respect to a couple of grassmanian fields $x, \bar{x}$ defines on $S$. Taking the derivative w.r. to $g$ we subtract the zero mode of the Laplacian. By the Kirchoff theorem,
The case $g = 1$ has been solved by M. Gaudin [43] for the gaussian potential $U_0 = -1/2\Phi^2$. The imaginary part of $\hat{w}(z)$ along the cut $-\infty < z < -M$ has been found as the solution of an integral equation with Cauchy kernel. From this it is not difficult to reconstruct the meromorphic function $\hat{w}(z)$

$$\hat{w}(z)_{g=1} = -M[\tau^2 \cosh \tau + 2\tau \log(aM) \sinh \tau]$$

$$= -2\sqrt{z^2 - M^2} \log(aM) \log \left( \frac{z + \sqrt{z^2 - M^2}}{M} \right)$$

$$- z \left[ \log \left( \frac{z + \sqrt{z^2 - M^2}}{M} \right) \right]^2$$

$$= \left[ \frac{d^2}{dg^2} \hat{w}(z) \right]_{g=1} + (\text{const}) \ z, \quad \Lambda = [M \log(aM)]^2$$

The origin of the logarithmic corrections in these two cases is quite different. In the case $g = 1$ this is the appearance of a massless excitation (the “tachyon”). The divergences due to this zero mode produce the logarithmic factor in the relation between $\Lambda$ and $M$. The logarithmic scaling violation is a well known phenomenon in the string theory with $X = \mathbb{R} [36], [47]$. In the case $g = -2$ there are no divergences which can produce logarithmic corrections in the scaling of $M$. The logarithmic factors in the loop amplitude are related to the fact that the partition function vanishes and therefore the loop amplitude is actually a derivative [42]. Within the matrix-model representation, the vanishing if the partition function can be explained with the occurrence of a supersymmetry at the point $g = -2$ [48], [44]. This has nothing to do with the sum over geometries. The partition function for fixed world-sheet geometry is equal to the determinant of the Laplace operator on the corresponding lattice and vanishes due to its zero mode.

As for the higher integer points, $g = 3, 4, \ldots$, it is possible that there are no logarithmic singularities, these points will be discussed in [11].

We would like to understand why the loop amplitude of the $g = 1$ string is equal to the second derivative in $g$ of the generic solution. For the moment this is just an experimental fact.

4. The multi-loop amplitudes in the scaling limit

4.1. An effective matrix model for the dressed vertices

In order to be able to exploit the functional integral formulation (2.72) of the ADE

the finite part of the determinant is equal to the sum of connected spanning trees on the world sheet. Since each spanning tree defines a dense loop on the world sheet [16], the derivative of the partition function can be also expressed in terms of the gas of dense loops with fugacity $\beta \rightarrow 0$. We see that the derivatives of the models $g = 1/2$ and $g = 2$ are identical and describe the string in -2 dimensions.

Saleur [42] considered the topological points $g = 1/2, 1/3, \ldots$ which are in a dual to the points $g = 2, 3, \ldots$. He argued that a self-consistent definition of the theory as a derivative in $g$ exists only for $g = 1/2$. 

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and SOS strings we need explicit expressions for the vertices (2.83). Using the random-matrix representation (2.91), (2.92) for the bare vertices, the $L$-dependent part (2.86) of the dressed vertices can be calculated by means of a random matrix model with a special potential. Namely, the generating function of the dressed vertices

$$U[J] = \sum_{H=0}^{\infty} \sum_{n=0}^{\infty} \frac{N^{2-2H-n}}{n!} \int W(H)(L_1, ..., L_n) \prod_{k=1}^{n} J(L_k) \frac{dL_k}{L_k}$$

(4.1)

is equal to the free energy of the effective matrix model

$$e^{U[J]} = \int d\Phi e^{(N \text{tr} U(\Phi)) + \int_0^\infty \text{tr} \Phi(L) J(L) dL/L}$$

(4.2)

with non-polynomial potential

$$U(\Phi) = U_0(\Phi) + \beta \int_0^\infty \frac{dL}{L} e^{-2P_0L} W(0)(L) \text{tr} e^{L\Phi}$$

(4.3)

The potential $U(P)$ is analytic in the complex plane cut along the line $2P_0 - P_R < P < 2P_0 - P_L$; it can be represented as an infinite series in $\Phi$ with radius of convergence $2P_0 - P_R$. Its derivative is related to the planar loop amplitude by

$$U'(\Phi) = \beta \hat{W}(0)(2P_0 - \Phi) + U'_0(\Phi)$$

(4.4)

Geometrically, the free energy $U[J]$ of the effective matrix model is the partition function of the gas of loops on a surface with fluctuating geometry with the condition that all noncontractible loops are suppressed. For example, loops going around a handle or a boundary are forbidden.

The one-loop amplitude in the matrix model (4.2) coincides in the planar limit $N \to \infty$ with the string background (2.82) (the partition function of the loop gas on a disc)

$$\langle N^{-1} \text{tr} e^{L\Phi} \rangle = W(0)(L), \quad N \to \infty$$

(4.5)

In order to prove this, it will be sufficient to show that (4.3) satisfies the loop equation (2.94). Indeed, the planar loop amplitude $\hat{W}(P)$ in any matrix model is related to the potential $U(P)$ by

$$\text{Re} W(P) = -\frac{1}{2} U'(P), \quad P_L < P < P_R$$

$$\text{Im} W(P) = 0, \quad P < P_L \text{ or } P > P_R$$

(4.6)

With the choice (4.4) of the potential, (4.6) is equivalent to (2.102) and therefore to (2.94).

We are interested in the continuum limit of the dressed vertices

$$w^{(H)}(\ell_1, ... \ell_n) = a^{-(g+1)(n-2+2H)} W^{(H)}(\ell_1/a, ..., \ell_n/a), \quad a \to 0$$

(4.7)

Below we give a method for their evaluation using the random matrix representation (4.2). The idea is to replace the potential (4.3) with a simpler one such that the corresponding matrix model will generate directly the scaling limit (4.7) of the dressed vertices.
Such a potential will represent a function \( u(z, M) \) with the following scaling property

\[
  u(z, M) = \rho^\alpha u(\rho z, \rho M) \tag{4.8}
\]

Comparing (4.6) and the functional equations (3.26) for the renormalized classical loop field \( \hat{w}(z) \), we see that a possible choice of the scaling potential (4.8) is

\[
  u(z) = \beta \int_0^\infty \frac{d\ell}{\ell} w(\ell) e^{z\ell}; \quad \frac{d}{dz} u(z) = \beta \hat{w}(-z) \tag{4.9}
\]

Let us mention that the inverse problem of finding the potential given the loop amplitude has many solutions. For example, the multicritical points of pure gravity can be obtained from the polynomial potentials of Kazakov [36] as well as from the non-polynomial singular potentials of Gross and Migdal [49].

We have shown that the dressed vertices in the scaling limit (4.7) can be calculated as the multi-loop correlators in the effective matrix model with potential (4.9)

\[
  \sum_{H=0}^{\infty} w^{(H)}(\ell_1, \ldots, \ell_n) N^{2-n-2H} = \int d\Phi e^{\beta N \int_0^{\infty} (d\ell/\ell) w(\ell) \text{tr} \exp(\ell \Phi)} \prod_{k=1}^n \text{tr} e^{\ell_k \Phi} \tag{4.10}
\]

The one- and two-loop correlators in the planar limit are

\[
  w^{(0)}(\ell) = \frac{1}{\ell} M^g K_g(M \ell) \tag{4.11}
\]

\[
  w^{(0)}(\ell, \ell') = \frac{\sqrt{\ell \ell'}}{\ell + \ell'} e^{-M(\ell + \ell')} \tag{4.12}
\]

The general formula for the planar multi-loop correlators in the matrix model [50] [45] reads

\[
  w^{(0)}(\ell_1, \ldots, \ell_n) = \sqrt{\ell_1 \cdots \ell_n} \left( -\frac{\partial}{\partial y} \right)^{n-2} e^{-f(y; M)(\ell_1 + \cdots + \ell_n)}
  = \sqrt{\ell_1 \cdots \ell_n} \frac{\partial^{n-2}}{\partial y^{n-2}} \int_{f(y; M)}^{\infty} dz e^{-z(\ell_1 + \cdots + \ell_n)} \tag{4.13}
\]

where the derivatives are taken at the point \( y = 0 \) and the function \( f(y; M) \) is determined by the diffusion equation (Appendix A) corresponding to the potential (4.9)

\[
  y = \lim_{N \to \infty} \beta \int_0^\infty d\ell \ w(\ell) \langle y | e^{z(-f(t) + (d/dt)^2)} | y \rangle
  = \int_0^\infty \frac{d\ell}{\ell} \left[ \beta M^g K_g(M \ell) \right] \left[ \frac{e^{-f(y; M) \ell}}{\sqrt{\ell}} \right] \tag{4.14}
\]

We assume that the infinite constant produced by the singularity at small distances (\( \ell \sim a \)) is subtracted so that

\[
  f(0; M) = M \tag{4.15}
\]

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The finite part of the integral can be extracted by means of analytic continuation w.r. to a regulating parameter \( \alpha \). First we multiply the integrand by \( \ell^{\alpha+1/2} \), then calculate the integral (4.14) in the domain \( \text{Re} \alpha > g \) where it converges, using the formula

\[
\int_0^\infty d\ell \ell^{\alpha-1}e^{-z\ell}K_g(M\ell) = \sqrt{\frac{\pi}{2M}}(M+z)^{1/2-\alpha}\frac{\Gamma(\alpha+g)\Gamma(\alpha-g)}{\Gamma(\alpha+1/2)}
\]

\[
2F_1(1/2+g,1/2-g;1/2+\alpha;\frac{M-z}{2M})
\]

and finally perform analytic continuation to \( \alpha = 1/2 \).

The result is

\[
y = \beta M^g \sqrt{\frac{\pi}{2M}}(M+f)\sum_{k=0}^\infty \frac{\Gamma(1/2+g+k)\Gamma(1/2-g+k)}{\Gamma(1/2+\alpha+k)k!}(\frac{M-f}{2M})^k
\]

(4.16)

(4.17)

\[
\sum_{k=1}^\infty \frac{1}{k!(k-1)!} \frac{\Gamma(k-1/2+g)\Gamma(k-1/2-g)}{\Gamma(1/2+g)\Gamma(1/2-g)}(\frac{M-f}{2M})^k
\]

\[
= 2(2\pi)^{3/2}(g^2 - \frac{1}{4})M^{g+1/2}
\]

\[
= 2(2\pi)^{3/2}(g^2 - \frac{1}{4})M^{g+1/2}
\]

The parameter \( y \) is the coupling constant of the matrix model; in terms of the loop gas it is the coupling of a special local operator, to be discussed below.

4.2. Some critical exponents

Until this moment we have considered two of the critical exponents of the loop gas model - the string susceptibility \( \gamma_{s \text{tr}} \) and the fractal dimension of the loops \( D_B = 1/\nu \) which is the same as the fractal dimension of the boundary of the world sheet. Now we are going to introduce a third exponent \( D_C \) which has the meaning of fractal dimension of the connected domains bounded by loops.

By construction, the vacuum energy of the matrix model in the planar limit is the partition function of the gas of nonintersecting loops on the sphere, with one labeled domain. One can imagine it as the partition function of the gas of small spots (baby universes) floating in the labeled domain. A spot of perimeter \( \ell \) represents the partition function of the loop gas on a disc and have a Boltzmann weight \( u(\ell) = \beta w(\ell) \). The size of these baby universes is of the order of the cut-off \( a \) because of their entropy being divergent at \( \ell \to 0 \). The measure in the space of world-sheet metrics is tuned to the \( m \)-critical phase of nonembedded random surfaces. The parameter \( y \) makes sense of “cosmological constant” for the labeled domain.
Let $A_C$ and $A$ be the area of the labeled connected domain and the total area of the world sheet, correspondingly. (Of course, the labeled domain is equivalent to any other connected domain on the world sheet.) Then the fractal dimension of a single connected domain is determined by the way these two areas grow near the critical point $M = 0$

$$A_C \sim A^{D_C/2} \quad (4.18)$$

The area of a single connected domain is measured by the matrix-model puncture operator

$$\mathcal{P}_C = -\frac{\partial}{\partial y} \quad (4.19)$$

The coupling constant $y$ of the matrix model scales as $M^{g+1/2}$; this is sufficient to determine the fractal dimension $D_C$. It is convenient to consider the dense and the dilute phases separately.

(i) Dense phase

In this phase

$$\mathcal{P}_C = -\frac{\partial}{\partial y} \Rightarrow A_C, \quad \mathcal{P} = \frac{\partial}{\partial \Lambda} \Rightarrow A \quad (4.20)$$

The cosmological constant $\Lambda$ is related to $y$ by $y = M^{g+1/2} = \Lambda^{1/2+1/(4g)}$ and the fractal dimension of the connected area is

$$D_C = 1 + \frac{1}{2g} \quad (4.21)$$

For positive $\beta \ (g > 1/2)$ all Boltzmann weights are also positive and the gas of loops allows a statistical interpretation. Then the area of a connected domain diverges more slowly than the total area

$$A_{\text{conn}} \sim A^{1/(2+1/4g)} \quad (4.22)$$

In the dense phase $A$ coincides with the total length of the loops on the world sheet, and $A_{\text{conn}}$ - with the total length of the loops forming the boundary (hull) of a connected domain.

The scaling (4.22) is confirmed by the following argument based on the Coulomb gas picture.

The susceptibility $\frac{\partial^2 U}{\partial y^2}$ of the matrix model can be interpreted as the two-point function of a special local operator defined on the whole world sheet and not only on the labeled connected domain. The two-point function of this operator is zero when the two points are in different domains and one if they are in the same domain. This operator can be constructed as a vertex operator with electric charge $e = g - 1 \pm 1/2$ [52]. The corresponding gravitational scaling dimension

$$\delta_{0,1/2} = \frac{1/2 - |g - 1|}{2g} = \frac{1/2 - 1/(4g)} {4.23}$$

is positive when $1/2 < g < 1$. The dimension $1 - \delta_{0,1/2} = D_C/2$ of the coupling constant of this operator indeed coincides with the dimension of $y$. 

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(ii) Dilute phase

In this phase the puncture operator $P_C$ of the matrix model scales as $\Lambda^{-(g+1/2)/2}$. In the $m$-critical phase $P_C = A_C^{m/2}$ and the two areas are related by

$$A_C \sim A^{D_C/2}; \quad D_C = (2g+1)/m$$

In the Coulomb gas picture, the constant $y$ is coupled to the same local operator with gravitational scaling dimension

$$\delta_{0,1/2} = \frac{1/2 - |g-1|}{2} = \frac{3}{4} - g/2$$

This time $1 - \delta_{0,1/2} = (m/4)D_C$.

Knowing the area of a connected domain, we can evaluate the characteristic number of connected domains on the world sheet. By the Euler theorem, this is also the number of loops $N$ (including the most external loop representing the boundary). Therefore

$$N = A/A_C \sim A^{1-D_C}$$

$$= \begin{cases} A^{1/2-1/(4g)}, & \text{dense phase} \\ A^{(m-g-1/2)/m}, & \text{$m$-critical dilute phase} \end{cases}$$

In the interval $1/2 < g < 3/2$ where the loop gas has statistical interpretation (all Boltzmann weights are positive, including these related to the measure over random surfaces) the total number of loops tends to infinity when the critical point $M = 0$ is approached. The density of loops is proportional to $1/A_C$ and goes down to zero at the critical point.

The number of loops becomes one at the half-integer points $g = m - 1/2, m = 1, 2, 3, \ldots$ which are the solutions of the equation $\beta = 0$. These are the multicritical points of pure gravity where the only loop is the boundary of the world sheet. At $m = 1$ and $m = 2$ the dimension $\delta_{0,1/2}$ of the local operator coupled to $y$ is zero and therefore $P_C = P$. In the higher multicritical points with $\beta = 0$ the dimension of the operator $P_C$ fits the dimension $\delta_{m-1,1}$ of the Kac table of the model $(2,2m-1)$

$$\delta_{0,1/2} = \frac{1/2 - |m-3/2|}{2} = \frac{(m-1) - (m-1/2) - |m-3/2|}{2} = \delta_{m-1,1}$$

Therefore $P_C$ can be identified with the operator with the most negative dimension in the $m$-critical regime of pure gravity.

4.3. Feynman rules

Now we are able to formulate the Feynman diagram technique for the $ADE$ strings in a more explicit way.

The vertices (2.77) and (2.85) read, in the continuum limit

$$v^{(H)}(x|\ell_1, \ldots, \ell_n) = (\kappa S_x)^{2H-2+n} w^{(H)}(\ell_1, \ldots, \ell_n)$$
\[ v(H)(p_1, \ldots, p_n|\ell_1, \ldots, \ell_n) = \kappa^{n-2+2H} N_{p_1 \ldots p_n}^{(H)}(\ell_1, \ldots, \ell_n) \]  
(4.29)

where \( w(H) \) are the loop amplitudes in the one-matrix model with potential \( u(z) = \beta \hat{w}(-z) \), \( \hat{w}(z) \) given by (3.31), and the renormalized string interaction constant \( \kappa \) is defined by

\[ \kappa = a^{-\nu(2-\gamma_{str})} \kappa_0 = a^{-g-1} \kappa_0 \]  
(4.30)

The explicit form of the vertices without handles \((H = 0)\) is given by (1.17) and (1.13). To our knowledge, the explicit formula for the general macroscopic loop amplitudes in the one-matrix model is not yet found. Below we give an argument allowing to imagine the general form of the matrix-model loop amplitudes with \( H > 0 \).

In the general case the loop amplitudes in the effective model can be considered as loop amplitudes for the gaussian model dressed by tadpoles representing the nongaussian part of the potential (see eq. (2.86)). From the explicit form of the gaussian loop amplitudes (Appendix B) we conclude that

\[ w(H)(\ell_1, \ldots, \ell_n) = \sqrt{\ell_1 \ldots \ell_n} P_n^{(H)}(\ell_1 + \ldots + \ell_n) e^{-M(\ell_1 + \ldots + \ell_n)} \]  
(4.31)

where \( P_n^{(H)}(\ell) \) is a polynomial of degree \( n - 3 + 3H \) with coefficients depending on \( g \) and \( M \)

\[ P_n^{(H)}(\ell) = \sum_{k=0}^{n-3+3H} A_{n,k}^{(H)}(g, M) \ell^k \]  
(4.32)

(Note that \( n - 3 + 3H \) is the complex dimension of the moduli space for surfaces with \( H \) handles and \( n \) punctures).

Let us denote the \( n \)-loop amplitude in the continuum limit by

\[ \langle \prod_{k=1}^{n} \Psi_{x_k}(\ell_k) \rangle \]  
(4.33)

where \( \Psi_x(\ell) \) is the loop operator creating a boundary of length \( \ell \) at the point \( x \). For the tree approximation \((H = 0)\) we will use the symbol \( \langle \ldots \rangle_0 \). The v.e.v. of the loop operator is, in the tree approximation,

\[ \langle \Psi_x(\ell) \rangle = \frac{1}{\ell} M^g K_{g}(M\ell) S_x \]  
(4.34)

The two-loop correlator satisfies the equation

\[ \langle \Psi_{x_1}(\ell_1) \Psi_{x_2}(\ell_2) \rangle = \sum_x C_{x_1 x} \int_0^\infty d\ell \ w^{(0)}(\ell_1, \ell) \langle \Psi_x(\ell) \Psi_{x_2}(\ell_2) \rangle \]  
(4.35)

where \( w^{(0)}(\ell, \ell') \) is the two-loop amplitude in the effective matrix model. The latter is diagonalized by Bessel functions [51]

\[ w(\ell, \ell') = \frac{\sqrt{\ell \ell'}}{\ell + \ell'} e^{-M(\ell + \ell')} = \int_0^\infty dE \langle \ell|E \rangle \frac{1}{2 \cosh \pi E} \langle E|\ell' \rangle \]  
(4.36)
\[ \langle \ell | E \rangle = \frac{2}{\pi} \sqrt{\pi E} \sinh(\pi E) K_{iE}(M\ell) ; \quad K_{iE}(M\ell) = \int_0^\infty d\tau e^{-M\ell \cosh \tau \cos(E\tau)} \] (4.37)

where the states \( \langle \ell | E \rangle \) form a complete orthonormal system [45]. On the other hand, the connectivity matrix is diagonalized by

\[ C_{xx'} = \sum_p V^x_p 2 \cos(\pi p) V^{x'}_{(p)} \] (4.38)

where the sum includes the allowed momenta \( p = m/h \). Thus the wave functions of the closed string states diagonalizing the two-loop correlator are labeled by the “energy-momentum” \( (E, p) \) and have a factorized form

\[ \langle E, p | \ell, x \rangle = V^x_{(p)} \sqrt{E \sinh(\pi E)} K_{iE}(M\ell) \] (4.39)

The equation for the two-loop correlator becomes algebraic after the diagonalization and its solution is [10]

\[ \langle \Psi_x(\ell) | \Psi_{x'}(\ell') \rangle_0 = \int_0^\infty dE \sum_p \langle \ell, x | E, p \rangle \frac{1}{2 \cosh(\pi E) - 2 \cos(\pi p)} \langle E, p | \ell', x' \rangle \] (4.40)

For \( p = 1/2 \) (pure gravity) and \( p = 1/3 \) (Ising model) the two-loop correlator has been calculated from matrix models [45], [53]. One can check that (4.40) reproduces the expressions found for these particular cases.

Remarkably, the two-loop correlator is universal; it depends on the model only through its spectrum of allowed momenta. This makes possible to map the Hilbert spaces of the \( ADE \) strings onto the Hilbert space of the SOS string whose spectrum of excitations covers the whole interval \(-1 < p \leq 1\). The two-loop amplitude is essentially the propagator in the string diagram technique. The latter is given by the continuum limit of (2.89)

\[ G_{(p)}(\ell, \ell') = \int_0^\infty dE \langle \ell | E \rangle \tilde{G}(E, p) \langle E | \ell' \rangle \] (4.41)

\[ \tilde{G}(E, p) = \frac{2 \cosh(\pi E) \cos(\pi p)}{\cosh(\pi E) - \cos(\pi p)} \] (4.42)

The \((E, p)\) space has the geometry of a semi-infinite cylinder. It is represented by the infinite \((E, p)\) plane factored by the relation of equivalence

\[ (E, p) \equiv (E, p + 2) \] (4.43)

The periodicity of the propagator (4.39) is a direct consequence of the discreteness of the coordinate space \( X \).

It follows from the explicit form of the eigenstates (4.37) that the \( \tau \)-parametrization of the boundary cosmological constant \( z \) (eq. (3.27)) provides a local coordinate dual to the energy \( E \).
The configuration space of the string field theory is a direct product of the $X$ and $\tau$ spaces. The relation between the $\ell$ and $\tau$ representations has been discussed in ref. [53]. It has been argued that the $\tau$ variable makes sense of the time variable in the Das-Jevicki approach, and that the transformation from $\ell$ to $\tau$ is analogous to the Bäcklund transformation in the Liouville theory. This transformation reads explicitly [53]

\[
K_{iE}(M\ell) = \int_0^\infty d\tau e^{-M\ell \cosh \tau} \cos(E\tau) \\
\frac{\cos E\tau}{E \sinh \pi E} = \int_0^\infty \frac{d\ell}{\ell} e^{-M\ell \cosh \tau} K_{iE}(M\ell)
\] (4.44)

and does not relate delta-function normalized bases of wavefunctions.

Let us consider the case when the target space is the discretized real line $\mathbb{Z} \subset \mathbb{R}$. The inverse propagator (4.42) acts in the $(x, \tau)$ space as a differential operator of infinite order

\[
\hat{G}^{-1} = [\cosh(\pi \partial / \partial x)]^{-1} - [\cos(\partial / \partial \tau)]^{-1}
\] (4.45)

This operator can be also interpreted as a finite-difference operator in the $x$ and $i\tau$ directions. Thus Minkowski rotation $\tau \to t = i\tau$ of the $\mathbb{Z}$ string is described by a lattice Hamiltonian. The saddle-point equation (3.29) means that the classical string background is annihilated by the Hamiltonian (4.45).

It will be convenient to apply the Feynman rules directly in the $(E, p)$ space. The Fourier image of the $\ell$-dependent part of the vertices (4.31) can be found using the integration formula

\[
\int_0^\infty \frac{d\ell}{\ell} \sqrt{\ell e^{-M\ell \cosh \tau}} K_{iE}(M\ell) = \frac{\sqrt{\pi}}{(2M)^{k+1/2}} \frac{\Gamma(k + 1/2 + iE)\Gamma(k + 1/2 - iE)}{\Gamma(k + 1)}
\] (4.46)

which follows from (4.16). The result is

\[
\tilde{v}^{(H)}(p_1, \ldots, p_n | E_1, \ldots, E_n) = \kappa^{n-2+2H} N_{p_1 \ldots p_n} \tilde{w}^{(H)}(E_1, \ldots, E_n) \] (4.47)

\[
\tilde{w}^{(H)}(E_1, \ldots, E_n) = \int_0^\infty \prod_{k=1}^n \frac{d\ell_k}{\ell_k} \langle E_k | \ell_k \rangle w^{(H)}(\ell_1, \ldots, \ell_n)
\]

\[
= \sum_{m=0}^{n-3+3H} \sum_{m_1 + \ldots + m_n = m; m_k \geq 0} m! \ A_{n,m}^{(H)}(g, M)
\] (4.48)

\[
\times \prod_{k=1}^n \sqrt{E_k \sinh(\pi E_k)} \frac{\Gamma(1/2 + m_k + iE)\Gamma(1/2 + m_k - iE)}{m_k! m_k!}
\]

The vertices in the $(E, p)$ space are given by the product of the r.h.s. of (4.48) and (2.78). Each vertex represents a symmetric polynomial in $E_1, \ldots, E_n$ times a factor $\sqrt{\pi E_k \sinh(\pi E_k) / \cosh(\pi E_k)}$ for each line.
It is convenient to absorb the non-polynomial factors in the propagators; then the new propagator and vertices have the following form

\[
\tilde{G}_{\text{new}}(E,p) = \frac{\sqrt{\sinh(\pi E)}}{\sqrt{\pi E \cosh(\pi E) - \cos(\pi p)}} \frac{2 \cos(\pi p) \cosh(\pi E)}{\cosh(\pi E)} \frac{\sinh(\pi E)}{E} = G_{\varphi}(E,p) - G_{\varphi}(E,1/2) \tag{4.49}
\]

\[
G_{\varphi}(E,p) = \frac{\sinh(\pi E)}{\pi E} \frac{2}{\cosh(\pi E) - \cos(\pi p)} = \sum_{n=-\infty}^{\infty} \frac{1}{E^2 + (p + 2n)^2} \tag{4.50}
\]

\[
\tilde{V}^{(H)}_{\text{new}}(p_1, ..., p_n|E_1, ..., E_n) = N^{(H)}_{p_1 ... p_n} \sum_{m=0}^{n-3+3H} \sum_{m_1 + ... + m_n = m; m \geq 0} m! A^{(H)}_{n,m} \prod_{k=1}^{n} \frac{E_k}{(m_k!)^2} \prod_{s=0}^{m_k-1} \left[ E_k^2 + (s + 1/2)^2 \right] \tag{4.51}
\]

With the Feynman rules (4.49), (4.51), the evaluation of the loop amplitudes can be done as in an ordinary quantum field theory. The Greens functions of this effective field theory are the correlation functions of the Fourier-transformed loop field

\[
\tilde{\Psi}(E,p) = \sum_x (S_x)^2 \int_0^\infty d\ell \frac{\ell}{\ell} \chi_x^{(p)}(E) \Psi_x(\ell) \tag{4.52}
\]

\[
\tilde{\Psi}(E,p)_{\text{new}} = \frac{\sqrt{E \cosh(\pi E)}}{\sqrt{\sinh(\pi E)}} \tilde{\Psi}(E,p) \tag{4.53}
\]

Let us calculate, for example, the three-loop correlator of the loop field \(\Psi^{(p)}(\ell) = \sum_x \chi_x^{(p)}(E) \Psi_x(\ell)\). We start with the corresponding Greens function in the \((E,p)\) space which is given by a single Feynman diagram

\[
\langle \prod_{k=1}^{3} \Psi(E_k, p_k) \rangle_0 = C_{p_1p_2p_3} M^{-(1+g)} \prod_{k=1}^{3} \tilde{G}(E_k, p_k) \tag{4.54}
\]

Then we integrate w.r. to the \(E\) variables associated with the three legs according to the formula

\[
\int_0^\infty dE \frac{E \sinh(\pi E)}{\cosh(\pi E) - \cos(\pi p)} K_{iE}(M\ell) = M\ell K_{1-|p|}(M\ell) \tag{4.55}
\]

to find

\[
\langle \prod_{k=1}^{3} \Psi_{(p_k)}(\ell_k) \rangle = \Lambda^{-(1-\gamma_{str}/2)} C_{p_1p_2p_3} \prod_{k=1}^{3} \ell_k K_{1-|p_k|}(M\ell_k) \tag{4.56}
\]

In the particular case \(p_1 = p_2 = p_3 = p_0\) corresponding to three loops associated with the identity operator, eq. (4.56) is in accordance with the the formula conjectured in [1].

---

6 The form of the propagator (4.49) suggests an interpretation of the string field as a collection of two modes: a bosonic relativistic particle with compactified momentum space in the space-like direction and a ghost excitation (twist) with spectrum of momenta \(p = \pm 1/2\)
4.4. Spectroscopy

The spectrum of on-shell states is given by the poles of $\tilde{G}(E,p)$. They form the light cone in the $(E,p)$ cylinder (Fig. 8)

$$E = \pm i\varepsilon_n(p), \quad \varepsilon_n(p) = |p + 2n|; \quad n = 0, \pm 1, \pm 2, \ldots \quad (4.57)$$

The periodicity in the momentum space is a consequence of the discreteness of the coordinate $X$ space. If the $X$-space is compact, then the momentum space is discrete as well and the on-shell states form an infinite two-dimensional lattice on the space-time cylinder.

Fig. 8: The light cone for a compact momentum space.

The edges of the strip correspond to identical momenta $p = \pm 1$.

Each on-shell state creates a “microscopic loop” on the world sheet and corresponds to a local scaling operator. The sequence of poles corresponding to a given momentum $p$ defines the set of local operators $O_{(p);n}; n = 0, \pm 1, \pm 2, \ldots$ having nonvanishing correlations with the vertex operator

$$V(p) \equiv O_{(p);0} = \sum_x \tilde{\chi}_{(p)}^x P_x \quad (4.58)$$

where $P_x$ is the operator creating a puncture in the world sheet with coordinate $x$. These operators are the coefficients in the expansion of the off-shell loop field $\Psi_{(p)}(\ell)$ as a series of on-shell wave functions $\Psi_{(p)}(\ell)$

$$\Psi_{(p)}(\ell) = \sum_{n=-\infty}^{\infty} (|p| + 2n)M{||p|+2n|}I_{|p|+2n}(M\ell)O_{(p);n} \quad (4.59)$$

The expansion of the two-loop correlator as a sum of products of on-shell wave functions reads

$$\langle \Psi_{(p)}(\ell) \Psi_{(p)}(\ell') \rangle = \sum_{n=-\infty}^{\infty} \frac{|p| + 2n}{\sin(\pi p)} K_{|p|+2n}(M\ell) I_{|p|+2n}(M\ell'), \quad \ell' < \ell \quad (4.60)$$
Eqs. (4.59) and (4.60) imply
\[
\langle \mathcal{O}_p; n \Psi(p') (\ell) \rangle = M|p| + 2n |K|p| + 2n (M\ell) \tag{4.61}
\]
A direct derivation of (Laplace transform of ) (4.61) using the loop equations can be found in [1] and [9].

The gravitational dimension \( \delta_n(p) \) of the local operator \( \mathcal{V}_p; n \) can be found from (4.61)
\[
M|p| + 2n |K|p| = \Lambda \delta_n(p) - \gamma \text{str} / 2 \Rightarrow \delta_n(p) = |p| + 2n | - |p_0| \tag{4.62}
\]
where \( p_0 = g - 1 \) is the background momentum (2.57) (see (3.18), (3.22)).

The dimensions of the operators \( \mathcal{O}_p; n \) with \( p = kp_0 \), \( k \) - integer, are contained in the KPZ-DDK spectrum (2.64)
\[
\delta(\mathcal{O}_{r_0; n}) = \delta_{k+2n,k} \tag{4.63}
\]

4.5. Partition function of the noninteracting string

The Feynman rules in the continuum limit are generated by the functional integral
\[
e^{F[J]} = \int d\Psi e^{\mathcal{A}[\Psi]}
\]
\[
\mathcal{A}[\Psi] = -\frac{1}{2} \sum_p \int_0^\infty d\ell \Psi_p(\ell) \int_0^\infty dE \langle \ell | E \rangle \left( \frac{1}{\cos \frac{\pi p}{2p}} - \frac{1}{\cosh \frac{\pi E}{2p}} \right) \langle E | \ell' \rangle \Psi_{p'}(\ell') + \sum_n \sum_p \sum_{\ell_1, \ldots, \ell_n} \frac{1}{n!} \int_0^\infty d\ell_{\ell} \Psi_{(p_1 \ldots p_n)}(\ell) \prod_{k=1}^n \Psi_{(p_k)}(\ell_k) \frac{d\ell_k}{\sqrt{\ell_k}} \tag{4.64}
\]
which formally defines the string field theory with target space \( X \). The partition function \( F^{(1)} \) of the free string (topology of a torus) is closely related to the vacuum energy of this field theory.

The loop expansion for the embedded surfaces with the topology of a torus involves loop configurations with contractible and noncontractible loops. It is easy to see that the logarithm of the determinant of the kernel \( G_{xx'}(\ell, \ell') \) is equal to twice the contribution of the surfaces in the loop expansion containing at least one noncontractible loop. Indeed, if we return to the original diagram technique (subsec. 2.5) and retain only the gaussian part of the action, the contribution of the gaussian fluctuations of the \( \Psi \)-field to the vacuum energy will be given by the sum of all Feynman diagrams with the topology of a tree with one cycle. Each cycle corresponds to a closed path in \( X \) and consists of even number of points (bare vertices). A cycle with \( 2k \) points should be taken with a symmetry factor \( 2k \).
Let us denote by $F_{[k;p]}$ the contribution of the toroidal surfaces ($H = 1$) containing exactly $k$ noncontractible loops with momentum $p$. Then the partition function for fixed momentum $p$ is equal to

$$F(p) = \sum_{k=0}^{\infty} F_{[k;p]}$$

$$= F_0 + \frac{1}{2} \sum_{k=1}^{\infty} \sum_{p} \int_{a}^{\infty} \frac{d\ell}{\ell} \int_{0}^{\infty} \frac{dE}{\pi} \langle \ell|E \rangle \left( \frac{\cos(\pi p)}{\cosh(\pi E)} \right)^{2k} \langle E|\ell \rangle$$

$$= F_0 - \int_{a}^{\infty} \frac{d\ell}{\ell} |\langle \ell|E \rangle|^2 \log \left( 1 - \frac{\cos^2(\pi p)}{\cosh^2(\pi E)} \right)$$

(4.65)

The contribution of the surfaces without noncontractible loops is equal to the trace of the identity operator in the $X$-space (= the number of its points) times the $1/N^2$-correction to the free energy of the effective matrix model considered in the beginning of this section. A simple calculation (see, for example Appendix C of [49]) leads to

$$F_0 = \frac{g - 1/2}{24} \log(aM)$$

(4.66)

The integral over the length $\ell$ in (4.65) yields a factor $\log[1/(aM)]$ (this is the diagonal value of the kernel of the regularized identity operator in the $E$-space) and the $E$-integration gives

$$\int_{0}^{\infty} dE \log \left( 1 - \frac{\cos^2(\pi p)}{\cosh^2(\pi E)} \right) = \frac{\pi}{2} (1/2 - |p|)^2$$

(4.67)

Inserting this in (4.65) we find for contribution to the partition function of an excitation with momentum $p$

$$F(p) = \log(aM) \frac{g - 1/2 + 6(1/2 - |p|)^2}{24}$$

(4.68)

Let us give some examples.

(i) The SOS string compactified on $\mathbb{Z}_{2h}$

Summing over the allowed momenta $p = m/h; \ m = 0, \pm 1, \ldots, \pm(h - 1), h$ in the embedding space $X = \mathbb{Z}_{2h}$ we find

$$F_{2h}(g) = \log(aM) \left( \frac{h^2}{24h} + 2h \frac{g - 1/2}{24} \right) = \log(aM) \left( \frac{gh + 1/h}{12} \right)$$

(4.69)

The $\mathbb{Z}_{2h}$ string is well defined if the background momentum $p_0$ belongs to the spectrum of allowed momenta. Therefore the possible values of $g$ are

$$g = m \pm \frac{k}{h}; \ m = \text{integer}, 0 \leq k < h$$

(4.70)
The partition function of the string embedded in $\mathbb{Z}_{2h}$ is symmetric under the duality transformation
\[ g \to 1/g, \ h \to 1/h, \ M \to M^9 \] (4.71)
relating the dense and dilute phases. Due to the different scaling in these two phases, the cosmological constant is invariant under the duality transformation. The self-dual point is $g = 1, h = 1$. It corresponds to the level-one representation of an $SU_2$ current algebra. The Kosterlitz-Thouless transition may occur at $g = 1, h = 2$ and its dual point $g = 1, h = 1/2$. The case $h = 2$ describes the $\mathbb{Z}_4$ model which is a particular case of the Ashkin-Teller model. At this point the magnetic (vortex) operator with discontinuity $m = 4$ (= the period in the $X$ space) becomes marginal: its conformal dimension (2.46) is 1. The dual model is formally the $\mathbb{Z}_1$ model having as a target space a graph with one point and one loop. The space of paths in this space is not trivial because each step is made by traversing the loop in one of the two possible directions. This $\mathbb{Z}_1$ string is equivalent to the limit of the $O(n)$ string introduced in [33]. The vortex operators in the $O(n)$ string are the electric operators in the $\mathbb{Z}_1$ string. The spectrum of momenta consists of all integers and the marginal operator is the vertex operator with $p = 2$ (= the period in the momentum space).

(ii) The RSOS strings

The embedding space $X = A_{h-1}$ has spectrum of momenta $p = m/h; \ m = 1, 2, ..., h-1$ and the partition function reads
\[
F_{A_{h-1}} = \frac{(h-1)(h-2)}{12h} \log(aM) \\
= \frac{h-2}{24} \log(a^2(1-1/h)\Lambda), \ \text{dense phase (}g = 1 - 1/h) \\
F_{A_{h-1}} = \frac{h(h-1)}{12h} \log(aM) \\
= \frac{h-1}{24} \log(a^2\Lambda), \ \text{dilute phase (}g = 1 + 1/h) \tag{4.73}
\]

In both critical regimes the partition function of the $A_{h-1}$ string is equal to the difference between the partition functions of the $\mathbb{Z}_{2h}$ and $\mathbb{Z}_2$ strings
\[
F_{A_{h-1}} = \frac{1}{2} \left( F_{2h}(g = 1 \pm 1/h) - F_2(g = 1 \pm 1/h) \right) \tag{4.74}
\]
The relation (4.74) takes place even before performing the sum over the world sheet geometries [1].

In a similar way one can calculate the partition functions of the $D$ and $E$ strings. In the $A$ and $D$ cases the expressions will coincide with these obtained from the matrix models in the formulation of M. Douglas [44] by Di Francesco and Kutasov [55]. All these partition functions can be expressed as linear combinations of partition functions of $\mathbb{Z}_n$ strings [1]. The formulas are the same as these for the regular lattice [12]. The Coulomb gas calculation of the genus-one partition function of the $ADE$ strings was presented in [56].

\[7\] Here we restrict ourselves to the unitary theories with $0 < g < 2$. 

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5. Concluding remarks

We have shown that the propagation and interactions of strings embedded in a discrete one-dimensional target space are described by string field theory with propagator (4.42) and vertices (4.48). We were not able to calculate explicitly the vertices with higher topology but they are defined unambiguously by the matrix integral (4.2).

Our diagram technique based on the loop-gas representation of the IRF models, provides a natural decomposition of the moduli space into elementary cells. The vertices represent the topology-changing amplitudes involving no propagation in the target space. The fact that the vertices are themselves loop amplitudes guarantees the “stringy” large order behaviour [57]. The vertices included in a Feynman diagram cut a number of holes in the moduli space such that the rest of it is a direct product of one-dimensional spaces (one for each propagator). The degenerated surfaces appear when the proper time for some of the propagators becomes infinite (the proper time is measured by the number of the noncontractible loops along the corresponding cylindric surfaces). This decomposition of the moduli space is well defined only if all proper times are nonzero (i.e., if there is at least one noncontractible loop in each of the channels. The situation when one of the proper times vanish is taken into account by replacing the two involved vertices with a new vertex of higher topology. Thus the interaction of the closed string is described by a collection infinitely many vertices, one for each topology. This is possibly a general feature of the perturbative expansion for any string theory [58].

It would be interesting to compare our diagram technique for the loop amplitudes to the one for the string embedded in IR. The two-loop correlator for the string embedded in IR [45][59], [53],

$$\tilde{G}(E, p) = \frac{E}{\sinh(\pi E)} \frac{1}{E^2 + p^2}$$

is compatible with (4.40) only in the limit of small $p$ and $E$. The propagator of the IR-string contains an infinite sequence of poles at integer imaginary values of $E$ which have been interpreted as the special stationary states [60] discovered by A. Polyakov [60]. These states are created by infinitesimal motion in the $X$ space; in the Coulomb gas picture they are constructed by taking the derivatives of the $x$-field. Therefore, such states should not exist in a string theory with discrete target space. The special states in the IR-string are related to the closed classical trajectories in the imaginary time direction.

The infinite spectrum of states in our case comes from the fact that the discreteness of the $X$-space leads to periodicity in the momentum space. As a consequence, the light cone maps an infinite set of energies to the same momentum. These energies make the tower of gravitational descendents of the vertex operator with given momentum. The classical motion in the imaginary time direction is governed by a finite-difference Hamiltonian.

The fact that the difference between the embedding spaces $\mathbb{Z}$ and $\mathbb{R}$ survives in the continuum limit can be explained. If we construct the D=1 string as an infinite chain of coupled matrices [61] [62], then the IR-string will describe the low temperature phase

8 Since these “states” can be eliminated by redefinition of the vertices, their physical meaning is not very clear to us.
of the matrix chain and the \( \mathbb{Z} \)-string will describe the Kosterlitz-Thouless point where the distance between two nearest neighbours on the chain remains finite in the continuum limit. The change in the critical behaviour of the matrix chain near the Kosterlitz-Thouless point is due to the liberation of the angular excitations \([63][64]\).

**Acknowledgements**

I thank M. Douglas, P. Di Francesco, B. Duplantier, G. Moore, A. Polyakov, S. Shenker, M. Staudacher and A. Zamolodchikov for many stimulating discussions. I am grateful to J.-B. Zuber for a critical reading of the manuscript.

**Appendix A. The method of orthogonal polynomials**

The integral over Hermitean \( N \times N \) matrices

\[
e^{N^2 F} = \int d\Phi e^{NtrU(\Phi)}
\]

(A.1)
can be written as an integral w.r.t. the eigenvalues \( \phi_n, n = 1, ..., N \), of the matrix \( \Phi \)

\[
e^{N^2 F} = \int \prod_{k=1}^{N} d\phi_k e^{U(\phi_k)} \prod_{i<k}(\phi_i - \phi_k)^2
\]

(A.2)

This integral can be interpreted as the scalar product of the wave function of \( N \) fermions with itself. The one-fermion wave functions are of the form

\[
\langle \phi | n \rangle = P_n(\phi)e^{U(\phi)/2}
\]

(A.3)

where \( P_n(\phi) \) is a polynomial of degree \( n \). The polynomials are fixed by the condition of orthonormality

\[
\langle m | n \rangle = \int d\phi P_m(\phi)P_n(\phi)e^{U(\phi)} = \delta_{mn}
\]

(A.4)

The ground state of the system of fermions is equal to the antisymmetrized product of the states \( |n\rangle, n = 1, ..., N \)

\[
\Xi(\phi_1, ..., \phi_N) = \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^{[\sigma]} \langle 1 | \phi_{\sigma_1} | ... | N | \phi_{\sigma_N} \rangle
\]

(A.5)

The operator \( \Phi \) is represented in the space of one-fermion states is represented by the tridiagonal Jacobi matrix which we denote by the same letter

\[
\Phi | n \rangle = R_{n+1} | n + 1 \rangle + R_n | n - 1 \rangle + S_n | n \rangle
\]

(A.6)
The nonzero matrix elements \( S_n \) and \( R_n = A_{n-1}/A_n \) of this matrix are fixed by the recurrence relations

\[
\langle n | \Phi U'(\Phi) | n \rangle = (2n + 1)/N
\]

(A.7)
which can be interpreted in terms of diffusion of a particle with \( n \)-dependent hopping parameter. In the limit \( N \to \infty \) we can replace \( R_n, S_n \) by

\[
R = \lim_{n \to \infty} R_n, \quad S = \lim_{n \to \infty} S_n
\]

and the resolvent of the Jacobi matrix is given by the propagator of a one-dimensional relativistic particle

\[
\mathcal{R}_{nn'}(P) = \langle n| \frac{1}{P - \Phi} |n' \rangle = \left( \frac{P - S - \sqrt{(P - S)^2 - 4R^2}}{2R} \right)^{|n-n'|} \frac{1}{\sqrt{(P - S)^2 - 4R^2}} \quad (A.10)
\]

Another useful quantity is the inverse Laplace transform of (A.10)

\[
\langle n| e^{L\Phi} |n' \rangle = e^{SL} I_{|n-n'|}(2RL) \quad (A.11)
\]

where \( I_n(z) \) is a modified Bessel function. All quantities characterizing the radial part of the random matrix can be expressed in terms of (A.10) or (A.11). For example, the one- and two-loop correlators are given by

\[
W(L) = \frac{1}{N} \sum_{n=1}^{N} \langle n| e^{L\Phi} |n \rangle = e^{SL} \frac{1}{N} \sum_{n=1}^{N} I_0(2R_nL) \quad (A.12)
\]

\[
W(L, L') = \sum_{n>N, n'<N} \langle n| e^{L\Phi} |n' \rangle \langle n'| e^{L'\Phi} |n \rangle \quad (A.13)
\]

In the limit \( N \to \infty \) we can introduce the continuous variable

\[
y = 1 - \frac{n}{N} \quad (A.14)
\]

and consider \( R_n \) and \( S_n \) as functions of \( y \):

\[
R_n = R(y), \quad S_n = S(y); \quad R(0) = R, \quad S(0) = S \quad (A.15)
\]

These functions are determined by (A.7), (A.8). Using the explicit expression (A.10) for the resolvent we find

\[
\int_{P_L}^{P_R} dP \frac{PU'(P)}{\sqrt{(P - S)^2 - 4R^2}} = \pi(1 - y) \quad (A.16)
\]

\[
\int_{P_L}^{P_R} dP \frac{U'(P)}{\sqrt{(P - S)^2 - 4R^2}} = 0 \quad (A.17)
\]
The positions of the branchpoints of $W$ are therefore given by

$$P_L = S - 2R, \quad P_R = S + 2R \quad (A.18)$$

It follows by (A.12) that

$$W(L) = -\int_0^1 dy \ e^{S(y)L} I_0(2R(y)L) \quad (A.19)$$

$$W(P) = -\int_0^1 dy \ \frac{1}{\sqrt{(P - P_L(y))(P - P_R(y))}} \quad (A.20)$$

The two-loop correlator is obtained immediately from (A.13)

$$W(L, L') = e^{S(L + L')} \sum_{n=1}^{\infty} n I_n(2RL) I_n(2RL') \quad (A.21)$$

Its Laplace image depends only on the positions of the endpoints of the cut and is given by

$$W(P, P') = \sqrt{\frac{1}{(P - P_R)(P' - P_L)}} + \sqrt{\frac{1}{(P' - P_R)(P - P_L)}} - 2 \quad (A.22)$$

In the continuum limit

$$L = \ell/a, \quad P = P_R^* + az, \quad P_R(y) = P_R^* - af(y), \quad f(0) = M \quad (A.23)$$

eqs. (A.16) and (A.17) imply

$$\int_f^\infty dz \frac{u'(-z)}{\sqrt{a(z + M)}} = y \quad (A.24)$$

where $u(z) = U(P_R^* + az)$. If the potential $u(z)$ scales as $a^\alpha$, the dependence on the cutoff $a$ can be eliminated by rescaling $u(z) \to a^{-\alpha}u(z), y \to a^{\alpha-1/2}y$. Eq. (A.24) is then the Laplace image of (4.14) for the potential (4.9).

Finally, the two-loop correlator in the scaling limit reads

$$w(\ell, \ell') = \sqrt{\ell} \ \frac{e^{-M(\ell + \ell')}}{\ell + \ell'} \quad (A.25)$$

$$w(z, z') = \frac{\sqrt{z + M} + \sqrt{z' + M}}{4 (z - z')^2} - 2 = -\frac{\partial}{\partial z} \frac{\partial}{\partial z'} \log(\sqrt{z + M} + \sqrt{z' + M}) \quad (A.26)$$

In the case of the gaussian potential $U(\Phi) = -\Phi^2/2$ the one-loop amplitude can be calculated explicitly in all orders in $\kappa$ in the continuum limit (65) (66)

$$w(\ell) = \int_{-\infty}^{-M} dy \langle y | e^{y-k^2(\partial/\partial y)^2} | y \rangle = \ell^{-3/2} e^{-M\ell + \kappa \ell^3} \quad (A.27)$$
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