QUALITATIVE UNCERTAINTY PRINCIPLES FOR THE WINDOWED OPDAM–CHEREDNIK TRANSFORM ON WEIGHTED MODULATION SPACES

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Abstract. The aim of this paper is to establish a few qualitative uncertainty principles for the windowed Opdam–Cherednik transform on weighted modulation spaces associated with this transform. In particular, we obtain the Cowling–Price’s, Hardy’s and Morgan’s uncertainty principles for this transform on weighted modulation spaces. The proofs of the results are based on versions of the Phragmén–Lindelöf type result for several complex variables on weighted modulation spaces and the properties of the Gaussian kernel associated with the Jacobi–Cherednik operator.

1. Introduction

The classical uncertainty principle states that a non-zero function and its Fourier transform cannot both be sharply localized. Several forms of the uncertainty principle can be formulated depending on various ways of measuring the localization of the function. Mainly, there are two types of uncertainty principle: qualitative and quantitative uncertainty principles. Qualitative uncertainty principles imply the vanishing of a function under some strong conditions on the function. In particular, Cowling and Price [1], Morgan [2], Hardy [3], and Beurling [4] theorems are examples of qualitative uncertainty principles. On the other side, quantitative uncertainty principles tell us information about how a function and its Fourier transform are related to each other. For example, Donoho and Stark [5], Slepian and Pollak [6], and Benedicks [7] theorems are quantitative uncertainty principles.

One of the celebrated uncertainty principles in harmonic analysis is Hardy’s theorem [3]. This theorem is about the decay of a measurable function and its Fourier transform at infinity. More precisely, let a and b be two positive constants and suppose that \( f \) is a measurable function on \( \mathbb{R} \) such that

\[
|f(x)| \leq Ce^{-ax^2} \quad \text{and} \quad |\hat{f}(\xi)| \leq Ce^{-b\xi^2},
\]

for some constants \( C > 0 \) and where \( \hat{f} \) is the Fourier transform of \( f \) formally defined by

\[
\hat{f}(\xi) = \int_{-\infty}^{\infty} f(t)e^{-2\pi i \xi t} \, dt.
\]

Then \( f = 0 \) almost everywhere if \( ab > \frac{1}{4} \), \( f(x) = Ce^{-ax^2} \) for some constant \( C \) if \( ab = \frac{1}{4} \), and there are infinitely many non-zero functions satisfying the assumptions if \( ab < \frac{1}{4} \). Later, an \( L^p \)-version of this theorem was proved by Cowling and Price in [1]. It states that: let \( 1 \leq p, q \leq \infty \) with \( \min(p, q) \) is finite, and \( f \) be a measurable function on \( \mathbb{R} \) such that

\[
\|e^{ax^2}f\|_p < \infty \quad \text{and} \quad \|e^{b\xi^2}\hat{f}\|_q < \infty.
\]

Then \( f = 0 \) almost everywhere if \( ab \geq \frac{1}{4} \), and there are infinitely many non-zero functions satisfying the assumptions if \( ab < \frac{1}{4} \).

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The Hardy’s and Cowling–Price’s uncertainty principles were extended to different settings by many authors (see [8]). Particularly, Morgan in [2] obtained the following uncertainty principle by replacing the function $e^{a x^2}$ by $e^{a |x|^2}$, where $a > 2$ in Hardy’s theorem. It states that for two positive real numbers $\alpha, \beta$ such that $\alpha > 2$ and $1/\alpha + 1/\beta = 1$, if

$$e^{a |x|^2} f \in L^\infty(\mathbb{R}) \quad \text{and} \quad e^{b \lambda^\beta} \hat{f} \in L^\infty(\mathbb{R}),$$

then $f = 0$ almost everywhere for $(\alpha a)^{1/\alpha} (b \hat{\lambda})^{1/\beta} > (\sin (\frac{\pi}{2} (\beta - 1)))^{1/\beta}$. Further, $L^p$–$L^q$-version of Morgan’s theorem was proved by Ben Farah and Mokni in [9]. For a more detailed study on the history of the uncertainty principle, and many other generalizations and variations of the uncertainty principle, we refer to the book of Havin and Jöricke [10], and the excellent survey of Folland and Sitaram [11].

Considerable attention has been devoted to finding generalizations to new contexts for the Hardy’s, Cowling–Price’s, and Morgan’s uncertainty principles. For example, these theorems were investigated in [12] for the generalized Fourier transform, and in [13] for the Heisenberg group. Further, an $L^p$ version of Hardy’s theorem was proved for the Dunkl transform in [14]. In [15], Daher et al. have obtained some uncertainty principles for the Cherednik transform as a generalization of Euclidean uncertainty principles for the Fourier transform. These results are further extended to the Opdam–Cherednik transform in [16] using composition properties of the Opdam–Cherednik transform and classical uncertainty principles for the Fourier transform. Moreover, these types of uncertainty principles for the Opdam–Cherednik transform on modulation spaces were studied by the second author in [17]. Recently, the second author introduced the windowed Opdam–Cherednik transform and discussed the time-frequency analysis of localization operators associated with this transform on modulation spaces in [18]. Further, we have investigated some quantitative uncertainty principles for the windowed Opdam–Cherednik transform in [19]. However, up to our knowledge, qualitative uncertainty principles for this transform have not been studied in weighted modulation spaces. In this paper, we extend the Cowling–Price’s, Hardy’s, and Morgan’s uncertainty principles for the windowed Opdam–Cherednik transform on weighted modulation spaces associated with this transform.

A common key to obtain uncertainty principles for the Opdam–Cherednik transform or some other generalized transform is to use the Hölder inequality and show that this transform is an entire function on $\mathbb{C}$ (see [17]). In the case of the windowed Opdam–Cherednik transform, the main difficulty is that, the time-frequency shift of the window function in the integral representation of this transform does not satisfy the exponential decay condition, whereas for the Opdam–Cherednik transform the eigenfunction in the integral representation satisfies the decay condition. To overcome this difficulty, we consider the non-zero window function $g$ from a suitable modulation space and apply Hölder’s inequality to show that this transform is an entire function on $\mathbb{C}^2$.

An important motivation to prove these types of qualitative uncertainty principles for the windowed Opdam–Cherednik transform on weighted modulation spaces arises from the classical uncertainty principles for the Fourier transform on the Lebesgue spaces. Over the years, modulation spaces have become one of the most active branches of research in modern contemporary mathematics due to their appearances in current topics such as pseudo-differential operators, partial differential equations, etc., and used extensively in several areas of analysis, engineering, and physics. Uncertainty principles have implications in two main areas: quantum mechanics and signal analysis, and weighted modulation spaces are broadly used in these areas. We hope that the study of uncertainty principles for the weighted modulation spaces makes a significant impact in these areas. Another important motivation to study the Jacobi–Cherednik operators arises from their relevance in the algebraic description of exactly solvable quantum many-body systems of Calogero–Moser–Sutherland type (see [20, 21]) and they provide a useful tool in the study of special functions with root systems (see [22, 23]). These describe algebraically integrable systems in one dimension and have gained significant interest in mathematical physics. Other motivation for the investigation of uncertainty principles for the windowed Opdam–Cherednik
Cherednik transform is defined as follows. Where

\[ \text{Opdam–Cherednik transform of } f \]

is a slice formula, that is, this transform is decomposed as a composition of the classical Fourier transform and the Jacobi–Cherednik intertwining operator (see [16]). However, without using a slice formula, we obtain uncertainty principles for the windowed Opdam–Cherednik transform by using an estimate for the Gaussian kernel [24].

Since weighted modulation spaces are much larger spaces than the weighted Lebesgue spaces, a natural question to ask is: can we determine the functions \( f \) such that \( f \) and the windowed Opdam–Cherednik transform of \( f \) satisfying the conditions of Hardy’s, Cowling–Price’s, and Morgan’s theorems for the weighted modulation spaces? In this paper, we give affirmative answers to all of these questions. The natural key to obtaining extensions of uncertainty principles for the Opdam–Cherednik transform is a slice formula, that is, this transform is decomposed as a composition of the classical Fourier transform and the Jacobi–Cherednik intertwining operator (see [16]). However, without using a slice formula, we obtain uncertainty principles for the windowed Opdam–Cherednik transform by using an estimate for the Gaussian kernel [24].

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Apart from introduction, the paper is organized as follows. In Section 2, we recall some basic facts about the Jacobi–Cherednik operator and give the main results for the Opdam–Cherednik transform. Then, we discuss the results related to the windowed Opdam–Cherednik transform and give some properties of the Gaussian kernel associated with the Jacobi–Cherednik operator. In Section 3, we study the weighted modulation spaces associated with the windowed Opdam–Cherednik transform. In this section, we collect the necessary definitions and results from the harmonic analysis related to the windowed Opdam–Cherednik transform. For a detailed discussion on this transform, we refer to [18].

Let \( T_{\alpha,\beta} \) denote the Jacobi–Cherednik differential–difference operator (also called the Dunkl–Cherednik operator)

\[
T_{\alpha,\beta} f(x) = \frac{d}{dx} f(x) + \left[ (2\alpha + 1) \coth x + (2\beta + 1) \tanh x \right] \frac{f(x) - f(-x)}{2} - \rho f(-x),
\]

where \( \alpha, \beta \) are two parameters satisfying \( \alpha \geq \beta \geq -\frac{1}{2}, \alpha > -\frac{1}{2}, \) and \( \rho = \alpha + \beta + 1 \). Let \( \lambda \in \mathbb{C} \). The Opdam hypergeometric functions \( G_{\lambda}^{\alpha,\beta} \) on \( \mathbb{R} \) are eigenfunctions \( T_{\alpha,\beta} G_{\lambda}^{\alpha,\beta}(x) = i\lambda G_{\lambda}^{\alpha,\beta}(x) \) of \( T_{\alpha,\beta} \) that are normalized such that \( G_{\lambda}^{\alpha,\beta}(0) = 1 \). The eigenfunction \( G_{\lambda}^{\alpha,\beta} \) is given by

\[
G_{\lambda}^{\alpha,\beta}(x) = \varphi_{\lambda}^{\alpha,\beta}(x) - \frac{1}{\rho - i\lambda} \frac{d}{dx} \varphi_{\lambda}^{\alpha,\beta}(x) = \varphi_{\lambda}^{\alpha,\beta}(x) + \frac{\rho + i\lambda}{4(\alpha + 1)} \sinh 2x \varphi_{\lambda}^{2\alpha+1,\beta+1}(x),
\]

where \( \varphi_{\lambda}^{\alpha,\beta}(x) = 2F_1 \left( \frac{\alpha + \beta + 1}{2}, \frac{\alpha + \beta}{2}; \alpha + 1; -\sinh^2 x \right) \) is the classical Jacobi function. For any \( \lambda \in \mathbb{C} \) and \( x \in \mathbb{R} \), the eigenfunction \( G_{\lambda}^{\alpha,\beta} \) satisfy \( |G_{\lambda}^{\alpha,\beta}(x)| \leq C e^{-\rho|x|} e^{\operatorname{Im}(\lambda)||x||} \), where \( C \) is a positive constant. Since \( \rho > 0 \), we have \( |G_{\lambda}^{\alpha,\beta}(x)| \leq C e^{\operatorname{Im}(\lambda)||x||} \).

Let \( C_{c}(\mathbb{R}) \) denotes the space of continuous functions on \( \mathbb{R} \) with compact support. The Opdam–Cherednik transform is defined as follows.
Definition 2.1. Let $\alpha \geq \beta \geq -\frac{1}{2}$ with $\alpha > -\frac{1}{2}$. The Opdam–Cherednik transform $\mathcal{H}_{\alpha,\beta} f$ of a function $f \in C_c(\mathbb{R})$ is defined by

$$
\mathcal{H}_{\alpha,\beta} f(\lambda) = \int_{\mathbb{R}} f(x) G^{\alpha,\beta}_{\lambda} (-x) A_{\alpha,\beta}(x) dx \quad \text{for all } \lambda \in \mathbb{C},
$$

where $A_{\alpha,\beta}(x) = (\sinh |x|)^{2\alpha+1}(\cosh |x|)^{2\beta+1}$. The inverse Opdam–Cherednik transform for a suitable function $g$ on $\mathbb{R}$ is given by

$$
\mathcal{H}^{-1}_{\alpha,\beta} g(x) = \int_{\mathbb{R}} g(\lambda) G^{\alpha,\beta}_{\lambda}(x) d\sigma_{\alpha,\beta}(\lambda) \quad \text{for all } x \in \mathbb{R},
$$

where

$$
d\sigma_{\alpha,\beta}(\lambda) = \left(1 - \frac{\rho}{i\lambda}\right) \frac{d\lambda}{8\pi |C_{\alpha,\beta}(\lambda)|^2}
$$

and

$$
C_{\alpha,\beta}(\lambda) = \frac{2^{\rho-i\lambda}(\alpha + 1)\Gamma(i\lambda)}{\Gamma\left(\frac{2\alpha+1+i\lambda}{2}\right) \Gamma\left(\frac{2\beta+1+i\lambda}{2}\right)}, \quad \lambda \in \mathbb{C} \setminus i\mathbb{N}.
$$

The Plancherel formula is given by

$$
\int_{\mathbb{R}} |f(x)|^2 A_{\alpha,\beta}(x) dx = \int_{\mathbb{R}} \mathcal{H}_{\alpha,\beta} f(\lambda) \overline{\mathcal{H}_{\alpha,\beta} f(-\lambda)} d\sigma_{\alpha,\beta}(\lambda), \quad \text{(1)}
$$

where $\hat{f}(x) := f(-x)$.

Let $L^p(\mathbb{R}, A_{\alpha,\beta})$ (resp. $L^p(\mathbb{R}, \sigma_{\alpha,\beta})$), $p \in [1, \infty]$, denote the $L^p$-spaces corresponding to the measure $A_{\alpha,\beta}(x) dx$ (resp. $d|\sigma_{\alpha,\beta}|(x)$). The Schwartz space $\mathcal{S}_{\alpha,\beta}(\mathbb{R}) = (\cosh x)^{-\rho} \mathcal{S}(\mathbb{R})$ is defined as the space of all differentiable functions $f$ such that

$$
\sup_{x \in \mathbb{R}} (1 + |x|)^m e^{\rho|x|} \left| \frac{d^m}{dx^n} f(x) \right| < \infty,
$$

for all $m, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, equipped with the obvious seminorms. The Opdam–Cherednik transform $\mathcal{H}_{\alpha,\beta}$ and its inverse $\mathcal{H}^{-1}_{\alpha,\beta}$ are topological isomorphisms between the space $\mathcal{S}_{\alpha,\beta}(\mathbb{R})$ and the space $\mathcal{S}(\mathbb{R})$ (see [25], Theorem 4.1).

The generalized translation operator associated with the Opdam–Cherednik transform is defined by [26]

$$
\tau_{x}^{(\alpha,\beta)} f(y) = \int_{\mathbb{R}} f(z) d\mu_{x,y}^{(\alpha,\beta)}(z), \quad \text{(2)}
$$

where $d\mu_{x,y}^{(\alpha,\beta)}$ is given by

$$
d\mu_{x,y}^{(\alpha,\beta)}(z) = \begin{cases} 
K_{\alpha,\beta}(x, y, z) A_{\alpha,\beta}(z) dz & \text{if } xy \neq 0 \\
\delta_x(z) & \text{if } y = 0 \\
\delta_y(z) & \text{if } x = 0 
\end{cases} \quad \text{(3)}
$$

and

$$
K_{\alpha,\beta}(x, y, z) = M_{\alpha,\beta} |\sinh x \cdot \sinh y \cdot \sinh z|^{-2\alpha} \int_{0}^{\pi} g(x, y, z, \chi)_{+}^{\alpha-\beta-1} 
\times \left[ 1 - \sigma_{x,y,z} + \sigma_{x,z,y} + \sigma_{z,y,x} + \frac{\rho}{\beta + \frac{1}{2}} \coth x \cdot \coth y \cdot \coth z (\sin \chi)^2 \right] \times (\sin \chi)^{2\beta} d\chi,
$$

where

$$
M_{\alpha,\beta} = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha - \beta) \Gamma(\beta + \frac{1}{2})}.
$$
The windowed Opdam–Cherednik transform satisfies the following properties (see \[25,27\]). Let for every \(x, y \in \mathbb{R}\), \(\xi \in \mathbb{R}\), using the Plancherel formula (1) and the translation invariance of the Plancherel measure \(d\alpha,\beta\), we get \(\|\mathcal{M}_\xi^{(\alpha,\beta)} g\|_{L^2(\mathbb{R}, A_{\alpha,\beta})} = \|g\|_{L^2(\mathbb{R}, A_{\alpha,\beta})}\). Now, for a non-zero window function \(g \in L^2(\mathbb{R}, A_{\alpha,\beta})\) and \((x, \xi) \in \mathbb{R}^2\), we define the function \(g_{x,\xi}^{(\alpha,\beta)}\) by

\[
g_{x,\xi}^{(\alpha,\beta)} = \mathcal{M}_\xi^{(\alpha,\beta)} g.
\]

For any \(f \in L^2(\mathbb{R}, A_{\alpha,\beta})\), the windowed Opdam–Cherednik transform is defined by

\[
\mathcal{W}_g^{(\alpha,\beta)}(f)(x, \xi) = \int_{\mathbb{R}} f(s) \bar{g}_{x,\xi}^{(\alpha,\beta)}(-s) A_{\alpha,\beta}(s) \, ds, \quad (x, \xi) \in \mathbb{R}^2.
\]

We define the measure \(A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta}\) on \(\mathbb{R}^2\) by

\[
d(A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})(x, \xi) = A_{\alpha,\beta}(x) dx \, d|\sigma_{\alpha,\beta}|(\xi).
\]

The windowed Opdam–Cherednik transform satisfies the following properties (see \[18\]).

**Proposition 2.2.**

1. (Plancherel’s formula) Let \(g \in L^2(\mathbb{R}, A_{\alpha,\beta})\) be a non-zero window function. Then for every \(f \in L^2(\mathbb{R}, A_{\alpha,\beta})\), we have

\[
\|\mathcal{W}_g^{(\alpha,\beta)}(f)\|_{L^2(\mathbb{R}^2, A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})} = \|f\|_{L^2(\mathbb{R}, A_{\alpha,\beta})} \|g\|_{L^2(\mathbb{R}, A_{\alpha,\beta})}.
\]

2. (Reconstruction formula) Let \(g \in L^2(\mathbb{R}, A_{\alpha,\beta})\) be a non-zero positive window function. Then for every \(F \in L^2(\mathbb{R}^2, A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})\), we have

\[
\mathcal{W}_g^{(\alpha,\beta)}(F) = \frac{1}{\|g\|_{L^2(\mathbb{R}, A_{\alpha,\beta})}^2} \int_{\mathbb{R}^2} F(x, \xi) g_{x,\xi}^{(\alpha,\beta)}(-\cdot) d(A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})(x, \xi),
\]

weakly in \(L^2(\mathbb{R}, A_{\alpha,\beta})\).

Let \(t > 0\). The Gaussian kernel \(E_t^{\alpha,\beta}\) associated with the Jacobi–Cherednik operator is defined by

\[
E_t^{\alpha,\beta}(s) = \mathcal{W}_g^{(\alpha,\beta)}(e^{-t(\lambda^2+\mu^2)})(s), \quad \text{for all } s \in \mathbb{R}.
\]

For all \(t > 0\), \(E_t^{\alpha,\beta}\) is a \(C^\infty\)-function on \(\mathbb{R}\). Moreover, for all \(t > 0\) and all \(\lambda, \mu \in \mathbb{R}\), we have

\[
\mathcal{W}_g^{(\alpha,\beta)}(E_t^{\alpha,\beta})(\lambda, \mu) = e^{-t(\lambda^2+\mu^2)}.
\]

The kernel \(h_{x,y,z}^{K_{\alpha,\beta}}(x,y,z)\) satisfies the following symmetry properties:

\[
h_{x,y,z}^{K_{\alpha,\beta}}(x,y,z) = h_{x,y,z}^{K_{\alpha,\beta}}(y,x,z), h_{x,y,z}^{K_{\alpha,\beta}}(x,y,z) = h_{x,y,z}^{K_{\alpha,\beta}}(-z,y,x), h_{x,y,z}^{K_{\alpha,\beta}}(x,y,z) = h_{x,y,z}^{K_{\alpha,\beta}}(-x,z,y).
\]

For every \(x, y \in \mathbb{R}\), we have \(\tau_x^{(\alpha,\beta)} f(y) = \tau_y^{(\alpha,\beta)} f(x)\), and \(H_{\alpha,\beta}(\tau_x^{(\alpha,\beta)} f)(\lambda) = C_{\alpha,\beta}^{(\alpha,\beta)}(x) H_{\alpha,\beta}(f)(\lambda)\), with \(f \in C_c(\mathbb{R})\). For a more detailed study on the Opdam–Cherednik transform, we refer to \[25,27\].

Let \(g \in L^2(\mathbb{R}, A_{\alpha,\beta})\) and \(\xi \in \mathbb{R}\), the modulation operator of \(g\) associated with the Opdam–Cherednik transform is defined by

\[
\mathcal{M}_{\xi}^{(\alpha,\beta)} g = H_{\alpha,\beta}^{-1} \left( \frac{\sqrt{\sigma^{(\alpha,\beta)}(\mathcal{H}_{\alpha,\beta}(g))}}{\mathcal{H}_{\alpha,\beta}(g)} \right).
\]
We refer to [28] for further properties of the Gaussian kernel $E_t^{\alpha,\beta}$. From (24), Theorem 3.1, there exist two real numbers $\mu_1$ and $\mu_2$, such that

$$
\frac{e^{\mu_1 t}}{2^{\alpha+1} \Gamma(\alpha+1)t^{\alpha+1}\sqrt{B_{\alpha,\beta}(x)}} \leq \frac{e^{\mu_2 t}}{2^{\alpha+1} \Gamma(\alpha+1)t^{\alpha+1}\sqrt{B_{\alpha,\beta}(x)}}, \quad \forall x \in \mathbb{R},
$$

(8)

where $B_{\alpha,\beta}(x) = (\sinh |x|/|x|)^{2\alpha+1}(\cosh |x|)^{2\beta+1}$ for all $x \in \mathbb{R} \setminus \{0\}$ and $B_{\alpha,\beta}(0) = 1$. Further, we have $A_{\alpha,\beta}(x) = |x|^{2\alpha+1}B_{\alpha,\beta}(x)$ and for all $x \in \mathbb{R}$, $B_{\alpha,\beta}(x) \geq 1$.

3. Weighted modulation spaces associated with the windowed Opdam–Cherednik transform

For $x, \xi \in \mathbb{R}$, let $M_\xi$ and $T_x$ denote the operators of modulation and translation. Then, the short-time Fourier transform (STFT) of a function $f$ with respect to a window function $g \in \mathcal{S}(\mathbb{R})$ is defined by

$$V_g f(x, \xi) = \langle f, M_\xi T_x g \rangle = \int_\mathbb{R} f(t) g(t-x) e^{-2\pi i t \xi} dt, \quad (x, \xi) \in \mathbb{R}^2.$$ 

The modulation spaces were introduced by Feichtinger [29,30], by imposing integrability conditions on the STFT of tempered distributions. Here, we are interested in weighted modulation spaces with respect to the measure $A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta}$. We define the measure $A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta}$ on $\mathbb{R}^2$ by $d(A_{\alpha,\beta} \otimes A_{\alpha,\beta})(x, \xi) = A_{\alpha,\beta}(x) d\xi$.

**Definition 3.1.** Let $m$ be a non-negative function on $\mathbb{R}^2$, $g \in \mathcal{S}(\mathbb{R})$ be a fixed non-zero window function, and $1 \leq p, q \leq \infty$. Then the weighted modulation space $M_{p,q}^m(\mathbb{R}, A_{\alpha,\beta})$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R})$ such that $V_g f \in L_{p,q}^m(\mathbb{R}^2, A_{\alpha,\beta} \otimes A_{\alpha,\beta})$. The norm on $M_{p,q}^m(\mathbb{R}, A_{\alpha,\beta})$ is

$$
\|f\|_{M_{p,q}^m(\mathbb{R}, A_{\alpha,\beta})} = \|V_g f\|_{L_{p,q}^m(\mathbb{R}^2, A_{\alpha,\beta} \otimes A_{\alpha,\beta})} = \left( \int_\mathbb{R} \left( \int_\mathbb{R} |V_g f(x, \xi)|^p |m(x, \xi)|^p A_{\alpha,\beta}(x) dx \right)^{\frac{q}{p}} A_{\alpha,\beta}(\xi) d\xi \right)^{\frac{1}{q}} < \infty,
$$

with the usual adjustments if $p$ or $q$ is infinite.

If $p = q$, then we write $M_p^m(\mathbb{R}, A_{\alpha,\beta})$ instead of $M_{p,p}^m(\mathbb{R}, A_{\alpha,\beta})$. When $m = 1$ on $\mathbb{R}^2$, then we write $M_{p,q}^1(\mathbb{R}, A_{\alpha,\beta})$ and $M_{p,p}^1(\mathbb{R}, A_{\alpha,\beta})$ for $M_{p,q}^m(\mathbb{R}, A_{\alpha,\beta})$ and $M_{p,p}^m(\mathbb{R}, A_{\alpha,\beta})$ respectively. Also, we denote by $M_0^m(\mathbb{R}, \sigma_{\alpha,\beta})$ the weighted modulation space corresponding to the measure $d(\sigma_{\alpha,\beta})(x)$ and $M_0^m(\mathbb{R})$ the weighted modulation space corresponding to the Lebesgue measure $dx$.

We define the measure $(A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta}) \ast (A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})$ on $\mathbb{R}^2 \times \mathbb{R}^2$ by

$$d((A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta}) \ast (A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta}))(x_1, \xi_1, x_2, \xi_2) = d(A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})(x_1, \xi_1) d(A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})(x_2, \xi_2).$$

Then $M_{p,q}^m(\mathbb{R}^2, A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})$ denotes the weighted modulation space on $\mathbb{R}^2$ with respect to the measure $A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta}$. The norm on $M_{p,q}^m(\mathbb{R}^2, A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})$ is given by

$$
\|f\|_{M_{p,q}^m(\mathbb{R}^2, A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})} = \|V_g f\|_{L_{p,q}^m(\mathbb{R}^4, (A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta}) \ast (A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})))} = \left( \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |V_g f(x_1, \xi_1, x_2, \xi_2)|^p |m((x_1, \xi_1), (x_2, \xi_2))|^p d(A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})(x_1, \xi_1) \right)^{\frac{q}{p}} \times d(A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})(x_2, \xi_2) \right)^{\frac{1}{q}} < \infty,
$$

with the usual modification when $p = \infty$ or $q = \infty$.

The definitions of $M_{p,q}^m(\mathbb{R}, A_{\alpha,\beta})$ and $M_{p,p}^m(\mathbb{R}^2, A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})$ are independent of the choice of $g$ in the sense that each different choice of $g$ defines equivalent norms on $M_{p,q}^m(\mathbb{R}, A_{\alpha,\beta})$ and $M_{p,p}^m(\mathbb{R}^2, A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})$ respectively. Each weighted modulation space is a Banach space. For $p = q = 2$, we have $M_2^2(\mathbb{R}, A_{\alpha,\beta}) = L_2^2(\mathbb{R}, A_{\alpha,\beta})$. For other $p = q$, the space $M_p^p(\mathbb{R}, A_{\alpha,\beta})$ is not
L^p_m(\mathbb{R}, A_{\alpha, \beta}).  In fact for p = q > 2, the space M^p_m(\mathbb{R}, A_{\alpha, \beta}) is a superset of L^q_m(\mathbb{R}, A_{\alpha, \beta}).  We have the following inclusion
\[ S(\mathbb{R}) \subset M^1_m(\mathbb{R}, A_{\alpha, \beta}) \subset M^2_m(\mathbb{R}, A_{\alpha, \beta}) = L^2_m(\mathbb{R}, A_{\alpha, \beta}) \subset M^\infty(\mathbb{R}, A_{\alpha, \beta}) \subset S'(\mathbb{R}). \]

In particular, we have \( M^p_m(\mathbb{R}, A_{\alpha, \beta}) \hookrightarrow L^p_m(\mathbb{R}, A_{\alpha, \beta}) \) for 1 \leq p \leq 2, and \( L^p_m(\mathbb{R}, A_{\alpha, \beta}) \hookrightarrow M^p_m(\mathbb{R}, A_{\alpha, \beta}) \) for 2 \leq p \leq \infty. Moreover, the dual of a weighted modulation space is also a weighted modulation space, if \( p < \infty, q < \infty, (M^p_m(\mathbb{R}, A_{\alpha, \beta}))' = M^{p'}(\mathbb{R}, A_{\alpha, \beta}), \) where \( p', q' \) denote the dual exponents of \( p \) and \( q \), respectively. For further properties and uses of weighted modulation spaces, we refer to [31].

4. Qualitative uncertainty principles for the windowed Opdam–Cherednik transform

In this section, we obtain the Cowling–Price’s, Hardy’s and Morgan’s uncertainty principles for the windowed Opdam–Cherednik transform on weighted modulation spaces associated with this transform. From onwards, we consider the weight function \( m \) such that \( m(x, \xi) \geq 1 \) on \( \mathbb{R}^2 \) (resp. \( \mathbb{R}^4 \)). We begin with the following lemma.

Lemma 4.1. Let \( f(t_1, t_2) = 1 \) and \( g(t_1, t_2) = e^{-\pi(t_1^2 + t_2^2)}. \) Then
\[ V_g f((x_1, \xi_1), (x_2, \xi_2)) = e^{-2\pi i(x_1 x_2 + \xi_1 \xi_2)} e^{-\pi(x_2^2 + \xi_2^2)}. \]

Also, for \( p \in [1, \infty) \) and \( \rho_1, \rho_2, \sigma > 0, \) we have
\[ \|f\|_{M^p_m((|\rho_1 \rho_1(\sigma + 1)| \times |\rho_2 \rho_2(\sigma + 1)|)} \leq \rho_1^\frac{2}{p} \rho_2^\frac{2}{p}. \]

Proof. Using the definition of the STFT, we get
\[
V_g f((x_1, \xi_1), (x_2, \xi_2)) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\pi[(t_1-x_1)^2+(t_2-x_2)^2]} e^{-2\pi i(x_1 t_1 + \xi_1 t_2)} dt_1 dt_2
= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\pi(s_1^2 + s_2^2)} e^{-2\pi i(x_1 s_1 + \xi_1 s_2 + x_2 s_1 + \xi_2 s_2)} ds_1 ds_2
= e^{-2\pi i(x_1 x_2 + \xi_1 \xi_2)} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\pi(s_1^2 + s_2^2)} e^{-2\pi i(x_2 s_1 + \xi_2 s_2)} ds_1 ds_2
= e^{-2\pi i(x_1 x_2 + \xi_1 \xi_2)} e^{-\pi(x_2^2 + \xi_2^2)}.
\]

Further, we have
\[
\|f\|_{M^p_m((|\rho_1 \rho_1(\sigma + 1)| \times |\rho_2 \rho_2(\sigma + 1)|)} = \|V_g f\|_{L^p_m((|\rho_1 \rho_1(\sigma + 1)| \times |\rho_2 \rho_2(\sigma + 1)|) \times |\rho_1 \rho_1(\sigma + 1)| \times |\rho_2 \rho_2(\sigma + 1)|)}
\leq \left( \rho_1^{\rho_1(\sigma + 1)} \rho_2^{\rho_2(\sigma + 1)} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\pi p(x_2^2 + \xi_2^2)} dx_1 d\xi_1 dx_2 d\xi_2 \right)^\frac{1}{p}
\leq \left( \rho_1^{\rho_1(\sigma + 1)} \rho_2^{\rho_2(\sigma + 1)} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\pi p(x_2^2 + \xi_2^2)} dx_1 d\xi_1 dx_2 d\xi_2 \right)^\frac{1}{p}
= \rho_1^\frac{2}{p} \rho_2^\frac{2}{p}.
\]
4.1. Cowling–Price’s theorem for the windowed Opdam–Cherednik transform. In this subsection, we obtain an $M_m^p$–$M_m^q$-version of Cowling–Price’s theorem for the windowed Opdam–Cherednik transform. First, we establish the following lemma of Phragmén–Lindelöf type using a similar technique as in [1]. This lemma plays a crucial role in the proof of Cowling–Price’s theorem. An $L^p$-version of the following lemma proved in [1], however here we prove the lemma for the weighted modulation space $M_m^p(\mathbb{R}^2, A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})$.

**Lemma 4.2.** Let $\Phi$ be analytic in the region $Q = \{(r_1e^{i\theta_1}, r_2e^{i\theta_2}) : r_1, r_2 > 0, 0 < \theta_1, \theta_2 < \frac{\pi}{2}\}$ and continuous on the closure $\bar{Q}$ of $Q$. Assume that for any $p \in [1, \infty]$ and constants $A, a > 0$, we have

$$|\Phi(x + iy, u + iv)| \leq A \, e^{a(x^2 + u^2)} \quad \text{for } x + iy, u + iv \in \bar{Q},$$

and

$$\|\Phi|_{\mathbb{R}^2}\|_{M_m^p(\mathbb{R}^2, A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})} \leq A.$$

Then

$$\int_{r_1}^{r_2} \int_{\sigma}^{\sigma + 1} |\Phi(\rho_1 e^{i\psi_1}, \rho_2 e^{i\psi_2})| \, d\rho_1 d\rho_2 \leq A \max \left\{ e^{2a}, (\sigma + 1)^{q/2} \right\}$$

for $\psi_1, \psi_2 \in [0, \frac{\pi}{2}]$ and $\sigma \in \mathbb{R}^+$.\[\]

**Proof.** Using the definition of $M_m^p(\mathbb{R}^2, A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})$ and the fact that there is a constant $k_1 > 0$ such that $|C_{\alpha,\beta}(\lambda)|^{-2} \geq k_1 |\lambda|^{2\sigma+1}$ for all $\lambda \in \mathbb{R}$ with $|\lambda| \geq 1$ (see [32], page 157), and $A_{\alpha,\beta}(x) \geq 1$ for any $x \in \mathbb{R}$, we get

$$\|\Phi|_{\mathbb{R}^2}\|_{M_m^p(\mathbb{R}^2, A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})} = \|V_h \Phi\|^p_{L_m^p(\mathbb{R}^d, (A_{\alpha,\beta} \otimes \sigma_{\alpha,\beta})^*)}$$

\begin{align*}
\geq & \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{|\xi_1| \geq 1} \int_{|\xi_2| \geq 1} |V_h \Phi((x_1, \xi_1), (x_2, \xi_2))|^p |m((x_1, \xi_1), (x_2, \xi_2))|^p \\
& \times A_{\alpha,\beta}(x_1) dx_1 A_{\alpha,\beta}(x_2) dx_2 d|\sigma_{\alpha,\beta}|(\xi_1) d|\sigma_{\alpha,\beta}|(\xi_2) \\
\geq & \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{|\xi_1| \geq 1} \int_{|\xi_2| \geq 1} |V_h \Phi((x_1, \xi_1), (x_2, \xi_2))|^p |m((x_1, \xi_1), (x_2, \xi_2))|^p dx_1 dx_2 \\
& \times \left| 1 - \frac{\rho}{i\xi_1} \frac{d\xi_1}{8\pi |C_{\alpha,\beta}(\xi_1)|^2} \right| \left| 1 - \frac{\rho}{i\xi_2} \frac{d\xi_2}{8\pi |C_{\alpha,\beta}(\xi_2)|^2} \right| \\
\geq & \frac{k_1^2 k_2^2}{64\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{|\xi_1| \geq 1} \int_{|\xi_2| \geq 1} |V_h \Phi((x_1, \xi_1), (x_2, \xi_2))|^p |m((x_1, \xi_1), (x_2, \xi_2))|^p \\
& \times dx_1 dx_2 |\xi_1|^{2\sigma+1} |\xi_2|^{2\sigma+1} d\xi_1 d\xi_2 \\
\geq & \frac{k_1^2 k_2^2}{64\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{|\xi_1| \geq 1} \int_{|\xi_2| \geq 1} |V_h \Phi((x_1, \xi_1), (x_2, \xi_2))|^p |m((x_1, \xi_1), (x_2, \xi_2))|^p \\
& \times dx_1 dx_2 d\xi_1 d\xi_2,
\end{align*}

where $h \in \mathcal{S}(\mathbb{R}^2)$. This shows that $\|\Phi|_{\mathbb{R}^2}\|_{M_m^p(\mathbb{R}^2)} \leq A$. Next, we define a function $f$ on $\bar{Q}$ by

$$f(z_1, z_2) = \Phi(z_1, z_2) \exp \left( i\varepsilon e^{i\theta} (z_1^{(\pi-2\varepsilon)/\theta} + z_2^{(\pi-2\varepsilon)/\theta}) + ia \cot(\theta) (z_1^2 + z_2^2) / 2 \right),$$

for $\theta \in (0, \pi/2)$ and $\varepsilon \in (0, \pi/2 - \theta)$. Then for $\psi_1, \psi_2 \in [0, \theta]$, we get

$$|f(\rho_1 e^{i\psi_1}, \rho_2 e^{i\psi_2})| \leq A \exp \left\{ a \cos^2(\psi_1) \rho_1^2 - \varepsilon \sin(\varepsilon + (\pi - 2\varepsilon)\psi_1/\theta) \rho_1^{(\pi - 2\varepsilon)/\theta} - a \cot(\theta) \sin(2\psi_1) \rho_1^2/2 \right\}$$

\begin{align*}
& \times \exp \left\{ a \cos^2(\psi_2) \rho_2^2 - \varepsilon \sin(\varepsilon + (\pi - 2\varepsilon)\psi_2/\theta) \rho_2^{(\pi - 2\varepsilon)/\theta} - a \cot(\theta) \sin(2\psi_2) \rho_2^2/2 \right\} \\
& \leq A \exp \left\{ a(\rho_1^2 + \rho_2^2) - \varepsilon \sin(\varepsilon + (\pi - 2\varepsilon)\psi_1/\theta) \rho_1^{(\pi - 2\varepsilon)/\theta} - \varepsilon \sin(\varepsilon + (\pi - 2\varepsilon)\psi_2/\theta) \rho_2^{(\pi - 2\varepsilon)/\theta} \right\}.
\end{align*}

Applying a similar approach as in [1] to the function $f$, we get the subsequent estimates.
For $\rho_1, \rho_2 > 0$, $|f(\rho_1 e^{i\psi_1}, \rho_2 e^{i\psi_2})| \leq A$ and thus we obtain
\[
\int_{\sigma}^{\sigma+1} \int_{\sigma}^{\sigma+1} |f(\rho_1 \tau_1, \rho_2 \tau_2)| \, d\tau_1 d\tau_2 \leq \int_{\sigma}^{\sigma+1} \int_{\sigma}^{\sigma+1} A \, d\tau_1 d\tau_2 = A.
\]
Similarly, if $\rho_1, \rho_2 \in [0, (\sigma + 1)^{-1}]$, then
\[
\int_{\sigma}^{\sigma+1} \int_{\sigma}^{\sigma+1} |f(\rho_1 \tau_1, \rho_2 \tau_2)| \, d\tau_1 d\tau_2 \leq \sup\{|f(\rho_1, \rho_2)| : \rho_1, \rho_2 \leq 1\} \leq A e^{2a}.
\]
Finally, for $\rho_1, \rho_2 > (\sigma + 1)^{-1}$, using Hölder’s inequality for $M^p_m$ and Lemma 4.1, we get
\[
\begin{align*}
\int_{\sigma}^{\sigma+1} \int_{\sigma}^{\sigma+1} |f(\rho_1 \tau_1, \rho_2 \tau_2)| \, d\tau_1 d\tau_2 &= \frac{1}{\rho_1 \rho_2} \int_{\rho_1 \rho_2}^{\rho_1 (\sigma+1)} \int_{\rho_2 (\sigma+1)} f(\tau_1, \tau_2) \, d\tau_1 d\tau_2 \\
&\leq \frac{1}{\rho_1 \rho_2} \|f\|_{M^p_m([\rho_1 \rho_1 (\sigma+1)|\times[\rho_2 \rho_2 (\sigma+1)]}) \|1\|_{M^q_m([\rho_1 \rho_1 (\sigma+1)|\times[\rho_2 \rho_2 (\sigma+1))]} \\
&\leq (\rho_1 \rho_2)^{2-q} \|\Phi\|_{M^p_m([\rho_1 \rho_1 (\sigma+1)|\times[\rho_2 \rho_2 (\sigma+1)])} \\
&\leq A (\sigma + 1)^{\frac{2}{\sigma} - 2}.
\end{align*}
\]

Now the remaining part of the proof follows similarly as in [1].

\textbf{Theorem 4.3.} Let $g \in M^1_m(\mathbb{R}, A_{\alpha, \beta})$ be a non-zero window function and $1 \leq p, q \leq \infty$ with at least one of them finite. Suppose that $f$ is a measurable function on $\mathbb{R}$ such that
\[
e^{ax^2} f \in M^p_m(\mathbb{R}, A_{\alpha, \beta}) \quad \text{and} \quad e^{b(\lambda^2 + \mu^2)} W^{(\alpha, \beta)}_g(f) \in M^q_m(\mathbb{R}^2, A_{\alpha, \beta} \otimes \sigma_{\alpha, \beta}),
\]
for some constants $a, b > 0$. Then the following results hold:
\begin{enumerate}[(i)]
\item If $ab \geq \frac{1}{4}$, then $f = 0$ almost everywhere.
\item If $ab < \frac{1}{4}$, then for all $t \in (b, \frac{1}{4a})$, the functions $f = E_t^{\alpha, \beta}$ satisfy the relations (10).
\end{enumerate}

\textbf{Proof.} We divide the proof into three steps.

Step 1: Assume that $ab > \frac{1}{4}$. The function
\[
W^{(\alpha, \beta)}_g(f)(\lambda, \mu) = \int_{\mathbb{R}} f(s) e^{\lambda s} (-s) A_{\alpha, \beta}(s) \, ds,
\]
is well defined, entire on $\mathbb{C}^2$, and satisfies the condition
\[
|W^{(\alpha, \beta)}_g(f)(\lambda, \mu)| \\
\leq \int_{\mathbb{R}} |f(s)| e^{\lambda s} (-s) A_{\alpha, \beta}(s) ds \\
e^{\frac{|\mathrm{Im}(\lambda)|^2 + |\mathrm{Im}(\mu)|^2}{4a^2}} \int_{\mathbb{R}} e^{ax^2} f(s) e^{-a\left(\frac{|\mathrm{Im}(\lambda)|^2 + |\mathrm{Im}(\mu)|^2}{4a^2}\right)} |g^{(\alpha, \beta)}_{\lambda, \mu}(-s)| \, A_{\alpha, \beta}(s) ds,
\]
so by Hölder’s inequality,
\[
\leq e^{\frac{|\mathrm{Im}(\lambda)|^2 + |\mathrm{Im}(\mu)|^2}{4a^2}} \left\|e^{ax^2} f\right\|_{M^p_m(\mathbb{R}, A_{\alpha, \beta})} \left\|e^{-a\left(\frac{|\mathrm{Im}(\lambda)|^2 + |\mathrm{Im}(\mu)|^2}{4a^2}\right)} |g^{(\alpha, \beta)}_{\lambda, \mu}|\right\|_{M^q_m(\mathbb{R}, A_{\alpha, \beta})},
\]
where \( p' \) is the conjugate exponent of \( p \). Since \( M^{p'}_m(\mathbb{R}, A_{\alpha, \beta}) \) is invariant under translations and modulations, we get
\[
\left\| e^{-a \left( \frac{(x^2 + \text{Im}(x))^{2} + \text{Im}(x)^2}{4a^2} \right)} |g_{\lambda, \mu}^{(\alpha, \beta)}| \right\|_{M^{p'}_m(\mathbb{R}, A_{\alpha, \beta})} \leq \left\| e^{b(\lambda^2 + \mu^2)} \right\|_{M^{p'}_m(\mathbb{R}, A_{\alpha, \beta})} < \infty.
\]

We consider the function \( \Phi \) defined on \( \mathbb{C}^2 \) by
\[
\Phi(\lambda, \mu) = e^{-\frac{a^2}{4n}} W_g^{(\alpha, \beta)}(f)(\lambda, \mu).
\]
Then \( \Phi \) is an entire function on \( \mathbb{C}^2 \) and using the relation (11), we find that there exists a constant \( A \) for which
\[
|\Phi(\lambda, \mu)| \leq A e^{-\frac{(\text{Re}(\lambda))^2 + (\text{Re}(\mu))^2}{4n}}, \quad \text{for all } \lambda, \mu \in \mathbb{C}.
\]

In the following, we consider two cases.

(i) Let \( q < \infty \). Using \( ab > \frac{4}{7} \) and the hypothesis (10), we get
\[
\left\| \Phi \right\|_{\mathbb{R}^2} \leq \left\| e^{b(\lambda^2 + \mu^2)} W_g^{(\alpha, \beta)}(f) \right\|_{M^{\infty}_m(\mathbb{R}^2, A_{\alpha, \beta} \otimes \sigma_{\alpha, \beta})} \leq \left\| e^{b(\lambda^2 + \mu^2)} W_g^{(\alpha, \beta)}(f) \right\|_{M^{\infty}_m(\mathbb{R}^2, A_{\alpha, \beta} \otimes \sigma_{\alpha, \beta})} < A.
\]

By applying Lemma 4.2 to the functions \( \Phi(\lambda, \mu), \Phi(-\lambda, -\mu), \overline{\Phi(\lambda, \mu)} \) and \( \overline{\Phi(-\lambda, -\mu)} \), we obtain that for all \( \psi_1, \psi_2 \in [0, 2\pi] \) and large \( \sigma \)
\[
\int_{\sigma}^{\sigma+1} \int_{\sigma}^{\sigma+1} |\Phi(\rho_1 e^{i\psi_1}, \rho_2 e^{i\psi_2})| \, d\rho_1 d\rho_2 \leq B(\sigma + 1)^{-\frac{4}{7} - 2},
\]
for some constant \( B \). Now, applying Cauchy’s integral formula for several complex variables (see [33], Theorem 1.3.3), we get
\[
|D^{(n)} \Phi(0)| \leq n!(2\pi)^{-2} \int_0^{2\pi} \int_0^{2\pi} \left| \Phi(\rho_1 e^{i\psi_1}, \rho_2 e^{i\psi_2}) \right| \rho_1^{-n} \rho_2^{-n} d\psi_1 d\psi_2.
\]
Hence, for large \( \sigma \),
\[
|D^{(n)} \Phi(0)| \leq n!(2\pi)^{-2} \int_0^{2\pi} \int_0^{2\pi} \left| \Phi(\rho_1 e^{i\psi_1}, \rho_2 e^{i\psi_2}) \right| \rho_1^{-n} \rho_2^{-n} d\psi_1 d\psi_2 \\
\leq Bn! \sigma^{-2n}(\sigma + 1)^{-\frac{4}{7} - 2}.
\]

Let \( \sigma \to \infty \). If \( q \geq 2 \), then there exists a constant \( B_1 \) such that \( (\sigma + 1)^{-\frac{4}{7} - 1} \leq B_1 \), and consequently \( D^{(n)} \Phi(0) = 0 \) for \( n \geq 1 \). Thus \( \Phi(\lambda, \mu) = D \), for some constant \( D \). From (14), \( \Phi(\lambda, \mu) = 0 \) for all \( \lambda, \mu \in \mathbb{C} \). Further, if \( q < 2 \), then \( D^{(n)} \Phi(0) = 0 \) for \( n \geq 2 \). Hence \( \Phi(\lambda, \mu) = C_1 \lambda + C_2 \mu + D \), for some constants \( C_1, C_2 \), and \( D \). From (13) and (14), \( \Phi(\lambda, \mu) = 0 \) for all \( \lambda, \mu \in \mathbb{C} \). Therefore \( W_g^{(\alpha, \beta)}(f)(\lambda, \mu) = 0 \) for all \( \lambda, \mu \in \mathbb{R} \). Thus \( f = 0 \) almost everywhere on \( \mathbb{R} \) by (5).

(ii) Let \( q = \infty \). As \( ab > \frac{4}{7} \), then from (10), we have
\[
\left\| \Phi \right\|_{\mathbb{R}^2} \leq \left\| e^{b(\lambda^2 + \mu^2)} W_g^{(\alpha, \beta)}(f) \right\|_{M^{\infty}_m(\mathbb{R}^2, A_{\alpha, \beta} \otimes \sigma_{\alpha, \beta})} < \infty.
\]
If \( q = \infty \), then we can refined the estimate obtained in Lemma 4.2 such that \( \max\{e^{2n}, (\sigma + 1)^{-\frac{4}{7} - 2}\} \) is replaced by 1. From (15), we get
\[
|D^{(n)} \Phi(0)| \leq A n! \sigma^{-2n}.
\]
Then $D^{(n)} \Phi(0) = 0$ for $n \geq 1$, and this implies that $\Phi(\lambda, \mu) = C$ for all $\lambda, \mu \in \mathbb{C}$ and for some constant $C$. Therefore
\[
e^{b(\lambda^2 + \mu^2)} W_g^{(\alpha, \beta)}(f)(\lambda, \mu) = C \ e^{b - \frac{1}{4m}(\lambda^2 + \mu^2)}
\]
for all $\lambda, \mu \in \mathbb{R}$. Since $ab > \frac{1}{4}$, this function satisfies the relation (16) implies that $C = 0$. Thus from (5), we get $f = 0$ almost everywhere on $\mathbb{R}$.

Step 2: Assume that $ab = \frac{1}{4}$.

(a) If $q < \infty$, by the same proof as for the point (i) of the first step, we get $f = 0$ almost everywhere on $\mathbb{R}$.

(b) Let $q = \infty$ and $1 \leq p < \infty$. We have $\|\Phi\|_{L^p(\mathbb{R})} \leq \infty$. Then by the point (ii) of the first step, the relation (12), and the property (7) of the Gaussian kernel $\mathcal{F}_m^{\alpha, \beta}$, we obtain
\[
W_g^{(\alpha, \beta)}(f)(\lambda, \mu) = C \ e^{-\frac{(\lambda^2 + \mu^2)}{4m}} = C \ W_g^{(\alpha, \beta)}(\mathcal{F}_m^{\alpha, \beta})(\lambda, \mu), \text{ for all } \lambda, \mu \in \mathbb{R},
\]
for some constant $C$. Thus, using the injectivity of $W_g^{(\alpha, \beta)}$, we get
\[
f(x) = C \ E_\lambda^{\alpha, \beta}(x), \quad \text{a.e. } x \in \mathbb{R}.
\]

By using the relations (8) and (18), we obtain
\[
\frac{2C e^{\frac{\alpha}{4m} a^{\alpha+1}}}{\Gamma(\alpha + 1) B_{\alpha, \beta}(x)} \leq e^{ax^2} f(x), \quad \text{for all } x \in \mathbb{R}.
\]

For finite $p$, using the properties of the functions $A_{\alpha, \beta}$ and $B_{\alpha, \beta}$, we get
\[
\left\| \frac{1}{B_{\alpha, \beta}(x)} \right\|_{L^p(\mathbb{R}, A_{\alpha, \beta})} = \infty.
\]

Moreover, from (10) we have $\|e^{ax^2} f\|_{M^p(\mathbb{R}, A_{\alpha, \beta})} < \infty$, this is impossible unless $C = 0$. Then we obtain from (18) that $f = 0$ almost everywhere on $\mathbb{R}$.

Step 3: Assume that $ab < \frac{1}{4}$. Let $t \in (b, \frac{1}{4m})$ and $f = E_t^{\alpha, \beta}$. From the relation (8), we get
\[
K_1 e^{-\frac{4m}{4m-a} t x^2} \leq e^{ax^2} f(x) \leq K_2 e^{-\frac{4m}{4m-a} t x^2}, \quad \text{for all } x \in \mathbb{R},
\]
for some constants $K_1, K_2 > 0$. As $t < \frac{1}{4m}$, we deduce that $e^{ax^2} f \in M^p(\mathbb{R}, A_{\alpha, \beta})$. Using the relation (6), we get
\[
e^{b(\lambda^2 + \mu^2)} W_g^{(\alpha, \beta)}(f)(\lambda, \mu) = e^{-(t-b)(\lambda^2 + \mu^2)}, \quad \text{for all } \lambda, \mu \in \mathbb{R}.
\]
The condition $t > b$ and the inequality $|C_{\lambda, \beta}(\lambda)|^{-2} \leq k_2 |\lambda|^{2\alpha+1}$ at infinity (see [32], page 157) imply that $e^{b(\lambda^2 + \mu^2)} W_g^{(\alpha, \beta)}(f) \in M^p(\mathbb{R}^2, A_{\alpha, \beta} \otimes \sigma_{\alpha, \beta})$. This completes the proof.

4.2. Hardy’s theorem for the windowed Opdam–Cherednik transform. Here, we obtain an analogue of the classical Hardy’s theorem for the windowed Opdam–Cherednik transform on weighted modulation spaces associated with this transform. In particular, we determine the functions $f$ satisfying the relations (10) in the special case $p = q = \infty$.

**Theorem 4.4.** Let $g \in M^1(\mathbb{R}, A_{\alpha, \beta})$ be a non-zero window function and $f$ be a measurable function on $\mathbb{R}$ such that
\[
e^{ax^2} f \in M_n(\mathbb{R}, A_{\alpha, \beta}) \quad \text{and} \quad e^{b(\lambda^2 + \mu^2)} W_g^{(\alpha, \beta)}(f) \in M_n(\mathbb{R}^2, A_{\alpha, \beta} \otimes \sigma_{\alpha, \beta}),
\]
for some constants $a, b > 0$. Then
(i) If $ab > \frac{1}{4}$, we have $f = 0$ almost everywhere.
(ii) If $ab = \frac{1}{4}$, the function $f$ is of the form $f = CE^{\alpha, \beta}$, for some real constant $C$.
(iii) If $ab < \frac{1}{4}$, there are infinitely many nonzero functions $f$ satisfying the conditions (19).
Proof. (i) If $ab > \frac{1}{4}$, the result follows from point (ii) of the first step of the proof of Theorem 4.3.

(ii) If $ab = \frac{1}{4}$ and $\|e^{b(2u^2 + \rho^2)}W_g^{(\alpha, \beta)}(f)\|_{L_m^\infty(\mathbb{R}^2, A_{\alpha, \beta} \otimes \sigma_{\alpha, \beta})} < \infty$, then from (18) and Step 2(b) of the proof of Theorem 4.3, we have $f = CE_{\alpha, \beta}^\frac{a}{4}$, for some real constant $C$. Using the property $B_{\alpha, \beta}(x) \geq 1$, from relations (8) and (18), we get

$$e^{ax^2}f(x) \leq \frac{2Ce^{\frac{a}{4}x^2}}{\Gamma(\alpha + 1)} \sqrt{B_{\alpha, \beta}(x)}, \text{ for all } x \in \mathbb{R}.$$ 

Moreover, from (19) we have $\|e^{ax^2}f\|_{L_m^\infty(\mathbb{R}^2, A_{\alpha, \beta})} < \infty$, this is impossible unless $f = CE_{\alpha, \beta}^\frac{a}{4}$. This completes the result of point (ii).

(iii) If $ab < \frac{1}{4}$, the functions $f = E_{\alpha, \beta}^\frac{a}{4}, t \in (b, \frac{1}{4a})$, satisfy the conditions (19). This completes the proof of the theorem. \qed

4.3. Morgan’s theorem for the windowed Opdam–Cherednik transform. The aim of this subsection is to prove an $M_m^n - M_m^q$-version of Morgan’s theorem for the windowed Opdam–Cherednik transform. Before we prove the main result of this subsection, we first need the following lemma.

**Lemma 4.5** ([9], Lemma 2.3). Suppose that $\rho \in (1, 2)$, $q \in [1, \infty)$, $\eta > 0$, $M > 0$ and $B > \eta \sin \frac{\pi}{2}(\rho - 1)$. If $\Phi$ is an entire function on $\mathbb{C}^2$ satisfying the conditions

(i) $|\Phi(x + iy, u + iv)| \leq Me^{\eta |y|^{\rho} + |v|^{\rho}}$, for any $x, y, u, v \in \mathbb{R}$,

(ii) $e^{B(|x|^{\rho} + |u|^{\rho})}\Phi|_{\mathbb{R}^2} \in L^q(\mathbb{R}^2),$

then $\Phi = 0$.

As an application of the above lemma, in the following, we obtain a version of the Phragmén–Lindlöf type result for the weighted modulation spaces on $\mathbb{R}^2$.

**Lemma 4.6.** Suppose that $\rho \in (1, 2)$, $q \in [1, \infty)$, $\eta > 0$, $M > 0$ and $B > \eta \sin \frac{\pi}{2}(\rho - 1)$. If $\Phi$ is an entire function on $\mathbb{C}^2$ satisfying the conditions

(i) $|\Phi(x + iy, u + iv)| \leq Me^{\eta |y|^{\rho} + |v|^{\rho}}$, for any $x, y, u, v \in \mathbb{R}$,

(ii) $e^{B(|x|^{\rho} + |u|^{\rho})}\Phi|_{\mathbb{R}^2} \in M_m^q(\mathbb{R}^2, A_{\alpha, \beta} \otimes \sigma_{\alpha, \beta})$, 

then $\Phi = 0$.

**Proof.** Let $R > 0$ be such that 

$$B > \eta((R + 1)/R)^\rho \sin \frac{\pi}{2}(\rho - 1).$$

Let us consider the entire function $F$ on $\mathbb{C}^2$ by

$$F(z_1, z_2) = \int_R^{R + 1} \int_R^{R + 1} \Phi(t_1 z_1, t_2 z_2) \, dt_1 dt_2.$$

Then using Cauchy’s integral formula for several complex variables (see [33], Theorem 1.3.3), we obtain that the derivatives of $F$ satisfy the condition

$$D^{(n)}F(0) = \left[\left((R + 1)^{n+1} - R^{n+1}\right)/(n + 1)\right]^2 D^{(n)}\Phi(0), \text{ for any } n \in \mathbb{N}.$$

Therefore, $\Phi = 0$ if and only if $F = 0$. By assumption (i), we get

$$|F(x + iy, u + iv)| \leq Me^{(R + 1)^\rho \eta |y|^{\rho} + |v|^{\rho}}$$

for any $x, y, u, v \in \mathbb{R}$. \hfill (20)

Let $x, u \in \mathbb{R} \setminus \{0\}$. Then using the change of variables $x_1 = xt_1$ and $u_1 = ut_2$, we obtain

$$F(x, u) = \frac{1}{xu} \int_{Rx}^{(R + 1)x} \int_{Ru}^{(R + 1)u} \Phi(x_1, u_1) \, dx_1 du_1.$$
Hence,

$$|F(x,u)| \leq \frac{1}{|x|\|u|} \int_{|x|<1} \int_{|u|<1} |\Phi(x_1,u_1)| \ dx_1 \ du_1$$

$$= \frac{1}{|x|\|u|} \int_{|x|<1} \int_{|u|<1} e^{B(|x|^p+|u|^p)} e^{-B(|x|^p+|u|^p)} |\Phi(x_1,u_1)| \ dx_1 \ du_1$$

$$\leq \frac{1}{|x|\|u|} e^{-BR^p(|x|^p+|u|^p)} \int_{|x|<1} \int_{|u|<1} e^{B(|x|^p+|u|^p)} |\Phi(x_1,u_1)| \ dx_1 \ du_1.$$ 

Using Hölder’s inequality and the relation (9), we get

$$|F(x,u)| \leq \frac{1}{|x||u|} e^{-BR^p(|x|^p+|u|^p)} \|e_B \Phi\|_{M^p_m(\mathbb{R}^2)} \|1\|_{M_q^p((|x|, (R+1)x) \times (|u|, (R+1)u))}$$

where $e_B(x,u) = e^{B(|x|^p+|u|^p)}$ and $q'$ is the conjugate exponent of $q$. Since

$$\|1\|_{M_q^p((|x|, (R+1)x) \times (|u|, (R+1)u))} \leq C|x|^{\frac{2}{q'}} |u|^{\frac{2}{q'}}$$

for some constant $C > 0$, we have

$$|F(x,u)| \leq \frac{C}{|x|^{1-2/q'} |u|^{1-2/q'}} e^{-BR^p(|x|^p+|u|^p)} \|e_B \Phi\|_{M^p_m(\mathbb{R}^2, A_{\alpha}, \sigma_{\alpha}, \beta)}.$$ 

Since $F$ is continuous on $\mathbb{R}^2$, using assumption (ii), we obtain

$$e^{BR^p(|x|^p+|u|^p)} F(x,u) \in L^\infty(\mathbb{R}^2). \quad (21)$$

Using the inequalities (20) and (21), and applying Lemma 4.5 for $q = \infty$ to $F$, we get $F = 0$, thus $\Phi = 0$. This completes the proof of the lemma. \hfill \Box

**Theorem 4.7.** Let $g \in M^1_m(\mathbb{R}, A_{\alpha}, \beta)$ be a non-zero window function, $p \in [1, \infty]$, $q \in [1, \infty)$, $a > 0$, $b > 0$, and let $\alpha, \beta$ be positive real numbers satisfying $\alpha > 2$ and $1/\alpha + 1/\beta = 1$. Suppose that $f$ is a measurable function on $\mathbb{R}$ such that

$$e^{a|x|\alpha} f \in M^p_m(\mathbb{R}, A_{\alpha}, \beta) \quad \text{and} \quad e^{b(|\lambda|^\beta + |\mu|^\beta)} W_g^{(\alpha, \beta)}(f) \in M^q_m(\mathbb{R}^2, A_{\alpha}, \sigma_{\alpha}, \beta).$$

If

$$(a\alpha)^{1/\alpha} (b\beta)^{1/\beta} > \left( \sin \left( \frac{\pi}{2} (\beta - 1) \right) \right)^{1/\beta},$$

then $f = 0$.

**Proof.** Assume that $f$ is a measurable function on $\mathbb{R}$ such that

$$e^{a|x|\alpha} f \in M^p_m(\mathbb{R}, A_{\alpha}, \beta) \quad (22)$$

and

$$e^{b(|\lambda|^\beta + |\mu|^\beta)} W_g^{(\alpha, \beta)}(f) \in M^q_m(\mathbb{R}^2, A_{\alpha}, \sigma_{\alpha}, \beta). \quad (23)$$

To prove that the windowed Opdam–Cherednik transform of $f$ satisfies the conditions (i) and (ii) of Lemma 4.6, we use conditions (22) and (23), and we deduce that $f = 0$ almost everywhere. The function

$$W_g^{(\alpha, \beta)}(f)(\lambda, \mu) = \int_{\mathbb{R}} f(s) \overline{g_{\lambda, \mu}^{(\alpha, \beta)}(s)} A_{\alpha, \beta}(s) \ ds$$
is well defined, entire on $\mathbb{C}^2$, and satisfies the condition
\[
|\mathcal{W}_g^{(\alpha,\beta)}(f)(\lambda,\mu)| \leq \int_{\mathbb{R}} |f(s)| |g^{(\alpha,\beta)}(\lambda,\mu)(-s)| A_{\alpha,\beta}(s) ds
\]
\[
\leq \left\| e^{-a|s|^\alpha} \right\|_{M^p(|\mathbb{R}|,A_{\alpha,\beta})} \left\| e^{-a|s|^\alpha} \right\|_{M^p(|\mathbb{R}|,A_{\alpha,\beta})} , \text{by Hölder’s inequality,}
\]
\[
\leq C \left\| e^{-a|s|^\alpha} \right\|_{M^p(|\mathbb{R}|,A_{\alpha,\beta})}, \text{ by (22)},
\]
where $C$ is a constant and $p'$ is the conjugate exponent of $p$.

Let
\[
C \in I = \left( \left( b\beta \right)^{-1/\beta} \left( \sin \left( \frac{\pi}{2} (\beta - 1) \right) \right)^{1/\beta}, (aa)^{1/\alpha} \right).
\]

Now
\[
\left\| e^{-a|s|^\alpha} g^{(\alpha,\beta)}(\lambda,\mu) \right\|_{M^p(|\mathbb{R}|,A_{\alpha,\beta})}
\]
\[
= \left\| e^{-a|s|^\alpha} e^{-s(|\lambda|^{\beta}+|\mu|^{\beta})^{\frac{1}{\beta}}} e^{-s(|\lambda|^{\beta}+|\mu|^{\beta})^{\frac{1}{\beta}}} g^{(\alpha,\beta)}(\lambda,\mu) \right\|_{M^p(|\mathbb{R}|,A_{\alpha,\beta})}
\]
\[
\leq \left\| e^{-a|s|^\alpha} e^{-s(|\lambda|^{\beta}+|\mu|^{\beta})^{\frac{1}{\beta}}} g^{(\alpha,\beta)}(\lambda,\mu) \right\|_{M^p(|\mathbb{R}|,A_{\alpha,\beta})}.
\]

Applying the convex inequality
\[
|ty| \leq \left( \frac{1}{\alpha} \right) |t|^\alpha + \left( \frac{1}{\beta} \right) |y|^\beta
\]

to the positive numbers $C|t|$ and $|y|/C$, we get
\[
|ty| \leq \frac{C}{\alpha} |t|^\alpha + \frac{1}{\beta C^\beta} |y|^\beta,
\]

and thus
\[
\left\| e^{-a|s|^\alpha} e^{-s(|\lambda|^{\beta}+|\mu|^{\beta})^{\frac{1}{\beta}}} g^{(\alpha,\beta)}(\lambda,\mu) \right\|_{M^p(|\mathbb{R}|,A_{\alpha,\beta})}
\]
\[
\leq e^{-\frac{|\lambda|^{\beta}+|\mu|^{\beta}}{\beta C^\beta}} \left\| e^{-(a-C^{\alpha}/\alpha)|s|^\alpha} g^{(\alpha,\beta)}(\lambda,\mu) \right\|_{M^p(|\mathbb{R}|,A_{\alpha,\beta})};
\]

Since $C \in I$, it follows that $a > C^{\alpha}/\alpha$, and thus
\[
\left\| e^{-(a-C^{\alpha}/\alpha)|s|^\alpha} g^{(\alpha,\beta)}(\lambda,\mu) \right\|_{M^p(|\mathbb{R}|,A_{\alpha,\beta})} \leq \left\| g^{(\alpha,\beta)}(\lambda,\mu) \right\|_{M^p(|\mathbb{R}|,A_{\alpha,\beta})}
\]
\[
\leq \| g \|_{M^p(|\mathbb{R}|,A_{\alpha,\beta})} < \infty.
\]

From (25), (26) and (27), we get
\[
\left\| e^{-a|s|^\alpha} g^{(\alpha,\beta)}(\lambda,\mu) \right\|_{M^p(|\mathbb{R}|,A_{\alpha,\beta})} < \infty.
\]

Moreover,
\[
|\mathcal{W}_g^{(\alpha,\beta)}(f)(\lambda,\mu)| \leq \text{Const. } e^{\frac{|\lambda|^{\beta}+|\mu|^{\beta}}{\beta C^\beta}} \text{ for any } \lambda,\mu \in \mathbb{C}.
\]

(28)
Using the condition (23) and inequality (28), we obtain that the function \( \Phi(\lambda, \mu) = W_{q}^{(a, \beta)}(f)(\lambda, \mu) \) satisfies the assumptions (i) and (ii) of Lemma 4.6 with \( \rho = \beta, \eta = 1/(\beta C^3) \), and \( B = b \). The condition \( C \in I \) implies the inequality
\[
b > \frac{1}{\beta C^3} \sin \left( \frac{\pi}{2} (\beta - 1) \right),
\]
which gives \( W_{q}^{(a, \beta)}(f) = 0 \) by Lemma 4.6, then \( f = 0 \) by (5). This completes the proof of the theorem. \( \square \)

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**Declarations**

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**References**

[1] Cowling MG, Price JF. Generalizations of Heisenberg’s inequality, In: Harmonic Analysis (Mauceri G, Ricci F, Weiss G. (eds.)), Vol. 992, Lecture Notes in Mathematics. Berlin: Springer; 1983. p. 443–449. [1, 8, 9]

[2] Morgan GW. A note on Fourier transforms. J Lond Math Soc. 1934;9:188–192. [1, 2]

[3] Hardy GH. A theorem concerning Fourier transforms. J London Math Soc. 1933;8:227–231. [1]

[4] Hörmander L. A uniqueness theorem of Beurling for Fourier transform pairs. Ark Mat. 1991;29:237–240. [1]

[5] Donoho DL, Stark PB. Uncertainty principles and signal recovery. SIAM J Appl Math. 1989;49(3):906–931. [1]

[6] Slepičan D, Pollak HO. Prolate spheroidal wave functions, Fourier analysis and uncertainty I. Bell Syst Tech J. 1961;40:43–63. [1]

[7] Benedicks M. On Fourier transforms of functions supported on sets of finite Lebesgue measure. J Math Anal Appl. 1985;106:180–183. [1]

[8] Thangavelu S. An introduction to the Uncertainty Principle. Progr Math 217. Basel: Birkhäuser; 2004. [2]

[9] Farah SB, Mokni K. Uncertainty principle and the \( L^p–L^q \)-version of Morgan’s theorem on some groups. Russian J Math Phys. 2003;10(3):245–260. [2, 12]

[10] Havin V, Jöricke B. The uncertainty principle in harmonic analysis, In: A Series of Modern Surveys in Mathematics, Vol. 28. Berlin: Springer–Verlag; 1994. [2]

[11] Folland GB, Sitaram A. The uncertainty principle: A mathematical survey. J. Fourier Anal. Appl. 1997;3:207–238. [2]

[12] Mejjaoli H, Trimèche K. Qualitative uncertainty principles for the generalized Fourier transform associated to a Dunkl type operator on the real line. Anal Math Phys. 2016;6:141–162. [2, 3]

[13] Thangavelu S. An analogue of Hardy’s theorem for the Heisenberg group. Colloq. Math. 2001;87:137–145. [2]

[14] Gallardo L, Trimèche K. An \( L^p \) version of Hardy’s theorem for the Dunkl transform. J Aust Math Soc. 2004;77(3):371–385. [2]

[15] Daher R, Hamad SL, Kawazoe T, Shimeno N. Uncertainty principles for the Cherednik transform. Proc Indian Acad Sci Math Sci. 2012;122(3):429–436. [2]

[16] Mejjaoli H. Qualitative uncertainty principles for the Opdam–Cherednik transform. Integral Transforms Spec Funct. 2014;25(7):528–546. [2, 3]

[17] Poria A. Uncertainty principles for the Opdam–Cherednik transform on modulation spaces. Integral Trans Spec Funct. 2021;32(3):191–206. [2]

[18] Poria A. Localization operators associated with the windowed Opdam–Cherednik transform on modulation spaces. 2021. arXiv:2104.15112. [2, 3, 5]

[19] Mondal SS, Poria A. Uncertainty principles for the windowed Opdam–Cherednik transform. 2021. preprint. [2]

[20] Van Diejen JF, Vinet L. Calogero–Moser–Sutherland Models, CRM Series in Mathematical Physics. New York: Springer; 2000. [2]

[21] Hikami K. Dunkl operators formalism for quantum many-body problems associated with classical root systems. Phys Soc Japan. 1996;65:394–401. [2]

[22] Dunkl CF. Hankel transforms associated to finite reflection groups. Contemp Math. 1992;138:123–138. [2]
[23] Heckman GJ. An elementary approach to the hypergeometric shift operators of Opdam. Invent Math. 1991;103:341–350. [2]

[24] Fitouhi A. Heat polynomials for a singular differential operator on \((0, \infty)\). J Constr Approx. 1989;5(2):241–270. [3, 6]

[25] Schapira B. Contributions to the hypergeometric function theory of Heckman and Opdam: sharp estimates, Schwartz space, heat kernel. Geom Funct Anal. 2008;18:222–250. [4, 5]

[26] Anker JP, Ayadi F, Sifi M. Opdam’s hypergeometric functions: product formula and convolution structure in dimension 1. Adv. Pure Appl. Math. 2012;3(1):11–44. [4]

[27] Opdam EM. Lecture notes on Dunkl operators for real and complex reflection groups, In: Mem Math Soc Japan, Vol. 8. Tokyo: 2000. [5]

[28] Chouchane F, Mili M, Trimeche K. Positivity of the intertwining operator and harmonic analysis associated with the Jacobi–Dunkl operator on \(\mathbb{R}\). Anal Appl. 2003;1(4):387–412. [6]

[29] Feichtinger HG. Modulation spaces on locally compact abelian groups, In: Wavelets and their Applications (Krishna M, Radha R, Thangavelu S. (eds.)). New Delhi: Allied Publishers; 2003. p. 1–56. [6]

[30] Feichtinger HG, Gröchenig K. Gabor frames and time–frequency analysis of distributions. J Funct Anal. 1997;146(2):464–495. [6]

[31] Gröchenig K. Foundations of Time–Frequency Analysis. Boston: Birkhäuser; 2001. [7]

[32] Trimeche K. Generalized Wavelets and Hypergroups. Amsterdam (The Netherlands): Gordon and Breach Science Publishers; 1997. [8, 11]

[33] Scheidemann V. Introduction to Complex Analysis in Several Variables. Berlin: Birkhäuser; 2005. [10, 12]

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