Degenerations of abelian surfaces and Hodge structures

K. Hulek and J. Spandaw

1 Introduction

The starting point of this note is twofold: In [10] the first author together with Kahn and Weintraub constructed and described a toroidal compactification of the moduli space of $(1,p)$-polarized abelian surfaces with a (canonical) level structure. Moreover, Mumford’s construction was used to associate to each boundary point a degenerate abelian surface. On the other hand Carlson, Cattani and Kaplan gave in [3] an interpretation of toroidal compactifications of moduli spaces of abelian varieties in terms of mixed Hodge structures. Here we want to discuss a connection between [10] and [3]. More precisely, we restrict ourselves to corank 1 degenerations in the sense of [10], or equivalently to type II degenerations, i.e. to cycles of elliptic ruled surfaces. Our main result says that a degenerate abelian surface associated to a boundary point is (almost) completely determined by the boundary point (for a precise formulation see theorem 26). The crucial ingredient in the proof is the Local Invariant Cycle theorem which relates the variation of Hodge structure (VHS) to the mixed Hodge structure (MHS) on the singular surface.

In section 2 we collect some basic facts about semi-stable degenerations of abelian surfaces, some of which are well known. The MHS of a cycle of elliptic ruled surfaces is computed in section 3. Section 4 starts with a review of corank 1 boundary points in moduli spaces of $(1,p)$-polarized abelian surfaces with a level structure. Next the VHS of a specific family associated to such a boundary point is described. Using the Local Invariant Cycle theorem, this and the result from section 3 give theorem 26.

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2 Preliminaries on degenerations

In this section we want to summarize basic facts on degenerations of abelian surfaces some of which are well known ([5, 13]).

2.1 Cycles of elliptic surfaces

Let \( \Delta = \{ z \in \mathbb{C} : |z| < 1 \} \) be the unit disk. We consider proper, flat families

\[
\begin{array}{ccc}
X & \xrightarrow{p} & \Delta \\
\downarrow & & \\
\Delta & & 
\end{array}
\]

with \( X_t \) smooth abelian for \( t \neq 0 \). We shall always assume the total space \( X \) to be smooth and Kähler, and the components of the singular fibre \( X_0 \) to be algebraic. We also assume \( X \) to be relatively minimal.

If \( X_0 \) has global normal crossings and no triple points then Persson [13, proposition 3.3.1] or [5, p. 11 and 17] has shown, that \( X_0 \) is smooth abelian or a cycle of elliptic ruled surfaces, i.e.

\[
X_0 = Y_1 \cup \ldots \cup Y_N.
\]

The \( Y_i \) are smooth ruled elliptic surfaces. \( Y_i \) and \( Y_{i+1} \) intersect transversally along a smooth curve which is a section of both \( Y_i \) and \( Y_{i+1} \). In particular, the \( Y_i \) are all ruled surfaces over the same base curve \( C \) with two disjoint sections. The situation can be envisaged as in figure 1. We take \( Y_1 \) to be the bottom ruled surface and \( Y_n \) the top ruled surface, and call \( s \) the gluing parameter or shift of \( X_0 \). A careful analysis of Persson’s proof shows that, if one replaces the hypothesis of global normal crossings by local normal crossings, than the only additional case is the following: \( X_0 \) is irreducible, its normalization is an elliptic ruled surface with two disjoint sections and \( X_0 \) arises from this by gluing these two sections with a shift \( s \). We shall always refer to this situation (including the case \( N = 1 \)) as a cycle of elliptic ruled surfaces.
Figure 1: A cycle of ruled surfaces over $C$ glued with a shift $s$
The central fibre of such a degeneration is determined by the following data:

1. the number $N$ of components
2. the base curve $C$
3. line bundles $L_i$ with $\mathbb{P}(\mathcal{O} \oplus L_i) \cong Y_i$\[\text{1}\]
4. the gluing parameter $s$.

We first want to show that these data are not independent of each other. Before we can do this, we fix some more notation. First assume that $N \geq 2$. We denote the intersection of $Y_i$ and $Y_{i+1}$ by $C_i$. Furthermore we normalize the line bundle $L_i$ in such a way that $\mathcal{O}_{Y_i}(C_i-1) \mid_{C_i-1} = L_i$ and hence $\mathcal{O}_{Y_i}(C_i) \mid_{C_i} = L_i^{-1}$. Under this assumption the line bundle $L_i$ is uniquely determined. All indices have to be read modulo $N$. The above discussion also makes sense in case $N = 1$, if we replace $Y_1$ by its normalization. The next two propositions are special cases of results proved by Persson \[\text{13}\].

**Proposition 1** $K_X = \mathcal{O}_X$.

**Proof.** Since $K_X \mid_{X_t} = \mathcal{O}_{X_t}$ for $t \neq 0$ it follows that

$$K_X = \sum_{i=1}^{N} m_i Y_i$$

where the $m_i$ are uniquely defined up to a common summand. After possibly relabelling the components of $X_0$ we can assume $m_1$ to be maximal. Moreover by adding multiples of a fibre of $p$ we may assume that $m_1 = 0$. This already gives the result for $N = 1$. Now assume $N \geq 2$.

Adjunction gives

$$(K_X + Y_i) \mid_{Y_i} = K_{Y_i}.$$\[\text{2}\]

Since

$$K_{Y_i} = -C_{i-1} - C_i + a_i f_{P_i}$$

for a suitable ruling $f_{P_i}$ over a point $P_i \in C$ and

$$Y_i \mid_{Y_i} = -C_{i-1} - C_i$$

\[1\text{Strictly speaking for } N = 1 \text{ we have to take the normalization.}\]
this implies
\[(m_{i-1} - m_i)C_{i-1} + (m_{i+1} - m_i)C_i - a_if_{P_i} = 0.\]

Since \(m_1 = 0\) we find
\[m_0C_0 + m_2C_1 = a_1f_{P_1}.\]

Since, moreover, all \(m_i \leq 0\) this shows that \(m_0 = m_2 = a_1 = 0\). Continuing in this way we find \(m_i = a_i = 0\) for all \(i\).

\[\square\]

**Proposition 2**

(i) All line bundles \(L_i\) are isomorphic.

(ii) \(\text{deg} L_i = 0\) for all \(i\).

**Proof.** (i) For the normal bundle of \(C_i\) in \(X\) we have

\[\mathcal{N}_{C_i/X} = \mathcal{N}_{C_i/Y_i} \oplus \mathcal{N}_{C_i/Y_{i+1}}\]
\[= L_i^{-1} \oplus L_{i+1}.\]

On the other hand, since \(K_X = \mathcal{O}_X\) and since \(C_i\) is an elliptic curve, we find again by adjunction that \(\det \mathcal{N}_{C_i/X} = \mathcal{O}_{C_i}\). This shows (i).

(ii) Assume that \(\text{deg} L_i = m \neq 0\). Then \(|(C_i)_{Y_i}^2 = m\) and \((C_i)_{Y_{i+1}}^2 = -(C_i)_{Y_i}^2\) for all \(i\). But then a topological argument (see [13, p. 94]) shows that
\[H_1(X_t, \mathbb{Z}) = \mathbb{Z}^3 \oplus \mathbb{Z}_m\]
for general \(t\). This contradicts the fact that \(X_t\) is abelian.

\[\square\]

### 2.2 Additional structures

We shall now consider further structures on the family \(p : X \to \Delta\). First of all we consider degenerations of \emph{polarized} abelian surfaces, i.e. we assume that a line bundle \(\mathcal{O}_X(1)\) exists on \(X\) such that \(\mathcal{O}_{X_t}(1) = \mathcal{O}(1)|_{X_t}\) for \(t \neq 0\) is a polarization on the smooth abelian surface \(X_t\), and that \(\mathcal{O}_{X_0}(1)\) is ample. We are particularly interested in the case where \(\mathcal{O}_{X_t}(1)\) represents a polarization of type \((1,p)\) where \(p \geq 3\) is a prime number.

We also want to assume that \(p : X \to \Delta\) is a degeneration of polarized abelian surfaces with a (canonical) level structure. For the concept of (canonical) level structure on \((1,p)\)-polarized abelian surfaces see [10, I.1].
Before we give a formal definition recall the following: Let $Y = \mathbb{P}(\mathcal{O} \oplus \mathcal{L})$ be a $\mathbb{P}^1$-bundle over an elliptic curve $C$ with $\deg \mathcal{L} = t$. Let $Y^0$ be the open part of $Y$ which is given by removing the two sections defined by line bundles $\mathcal{O}$, resp. $\mathcal{L}$. Then $Y^0$ is a $\mathbb{C}^*$-bundle over the base curve $C$. More precisely $Y^0$ carries the structure of a commutative complex Lie group, and as such it is an extension of the form

$$1 \longrightarrow \mathbb{C}^* \longrightarrow Y^0 \longrightarrow C \longrightarrow 0.$$ 

In fact $Y^0$ is a semi-abelian surface of rank 1.

**Definition 3** Let $p : X \to \Delta$ be a degeneration of $(1,p)$-polarized abelian surfaces. We say that this is a degeneration of $(1,p)$-polarized abelian surfaces with a (canonical) level structure if the following holds:

(i) There exists an open subset $X^0 \subset X$ such that $X^0 \to \Delta$ is a family of abelian Lie groups with the following property: $X^0_t = X_t$ for $t \neq 0$ and $X^0_0$ is the smooth part of a component of $X_0$ and as such carries the structure of a semi-abelian surface.

(ii) There exists an action of $\mathbb{Z}_p \times \mathbb{Z}_p$ on $X$ over $\Delta$ with the following properties: It leaves $\mathcal{O}_X(1)$ invariant and defines a (canonical) level structure on $X_t$ for $t \neq 0$ (in particular it operates on $X_t$ by translation by elements of order $p$). Moreover, the subgroup of $\mathbb{Z}_p \times \mathbb{Z}_p$ which stabilizes $X^0_0$ acts on $X^0_0$ as a subgroup.

The two next results show that the presence of a polarization and a level structure imposes strong conditions on the singular fibre $X_0$.

**Lemma 4** Let $p : X \to \Delta$ be a degeneration of abelian surfaces with $(1,p)$-polarization and a (canonical) level-structure. Then there are only two possibilities:

(i) The central fibre consists of one component $X_0$. If $\tilde{X}_0$ is its normalization, then $\mathcal{O}_{\tilde{X}_0}(1) = \mathcal{O}_{\tilde{X}_0}(C_0 + pf_P)$ for a suitable point $P$ in the base curve $C$.

(ii) The central fibre consists of $p$ components $Y_i$, $i = 1, \ldots, p$. In this case $\mathcal{O}_{Y_i}(1) = \mathcal{O}_{Y_i}(C_{i-1} + f_{P_i})$ for suitable points $P_i$.

**Proof.** We have

$$\mathcal{O}_{Y_i}(1) = \mathcal{O}_{Y_i}(a_iC_{i-1} + b_i f_{P_i})$$
for suitable integers $a_i$ and $b_i$. Since these line bundles glue together to give a line bundle on $X_0$ it follows immediately that all $b_i$ are equal. We denote this number by $b$. Since $\mathcal{O}_{X_0}(1)$ is ample, and since we are dealing with a degeneration of abelian surfaces with a $(1, p)$-polarization it follows that

$$2b \sum_{i=1}^{N} a_i = 2p$$

(1)

with all $a_i > 0$.

We now consider the action of $\mathbb{Z}_p \times \mathbb{Z}_p$ on the set $\{Y_1, \ldots, Y_N\}$. Let $G$ be the stabilizer of $Y_1$. Since $p$ is a prime number, there are three possibilities:

(i) $G = \{1\}$. Then $N \geq p^2$ and this contradicts formula (1).

(ii) $G = \mathbb{Z}_p$. In this case $N \geq p$. It follows from formula (1) that $N = p$ and $a_i = b = 1$.

(iii) $G = \mathbb{Z}_p \times \mathbb{Z}_p$. The group $\mathbb{Z}_p \times \mathbb{Z}_p$ acts on $X_0$ as a group of automorphisms, hence it must leave its singular locus invariant. Since $p$ is an odd prime number the group $G$ must leave the curve $C_0$ invariant.

As a subgroup of $X_0^0$ the group $G$ acts by translation on $C_0$. The multiplicative group $\mathbb{C}^*$ contains no subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$, hence the group $G$ must contain at least a subgroup $\mathbb{Z}_p$ which acts non-trivially on $C_0$. Since $\mathcal{O}_X(1)$ is invariant under $G$ it follows that the degree of $\mathcal{O}_X(1)$ restricted to $C_0$ must be divisible by $p$. I.e. $b$ must be divisible by $p$. By formula (1) it follows that $b = p$, $N = 1$ and $a_1 = 1$. \qed

**Remark 5** It is easy to construct degenerations $p : X \to \Delta$ of $(1, p)$-polarized abelian surfaces without a level structure, such that the polarization on different components $Y_i$ is numerically different.

In lemma 4 we have seen that the central fibre of a degeneration $p : X \to \Delta$ of $(1, p)$-polarized abelian surfaces with a level-structure has either 1 or $p$ components. Recall that $Y_i = \mathbb{P}(\mathcal{O} \oplus L_i)$ and that all $L_i$ are isomorphic and of degree 0 (proposition 2). We shall denote this line bundle by $L$. Moreover, assume that we have chosen an origin $O$ on the base curve $C$. Then we can consider the shift $s$ as a point on $C$.

**Proposition 6** Let $p : X \to \Delta$ be a degeneration of $(1, p)$-polarized abelian surfaces with a (canonical) level structure, and denote the shift of $X_0$ by $s$. Then there are two possibilities:
(i) $N = 1$ and $\mathcal{L} = \mathcal{O}_C(\sqrt{f - \mathcal{O}})$

(ii) $N = p$ and there exist a point $s' \in C$ with $s = ps'$ such that $\mathcal{L} = \mathcal{O}(f' - \mathcal{O})$.

Proof. (i) Assume $N = 1$. Then $\mathcal{O}_{\tilde{X}_0}(1) = \mathcal{O}_{\tilde{X}_0}(C_0 + pf_P)$ and hence

$$\mathcal{O}_{\tilde{X}_0}(1)|_{C_0} = \mathcal{L} \otimes \mathcal{O}_C(pP), \quad \mathcal{O}_{\tilde{X}_0}(1)|_{C_1} = \mathcal{O}_C(pP).$$

Since $C_1$ and $C_0$ are glued with the shift $s$ a necessary and sufficient condition for $\mathcal{O}_{\tilde{X}_0}(1)$ to descend to a line bundle on $X_0$ is

$$\mathcal{L} \otimes \mathcal{O}_C(pP) = \mathcal{O}_C(pP) \otimes \mathcal{O}_C(s - O)^\otimes p$$

which gives the claim.

(ii) Assume $N = p$. Then $\mathcal{O}_{Y_1}(1) = \mathcal{O}_{Y_1}(C_{i-1} + f_{P_i})$ and hence

$$\mathcal{O}_{Y_1}(1)|_{C_1} = \mathcal{O}_C(P_1), \quad \mathcal{O}_{Y_2}(1)|_{C_1} = \mathcal{L} \otimes \mathcal{O}_C(P_2).$$

From gluing $Y_1$ and $Y_2$ we obtain

$$\mathcal{O}_C(P_2) = \mathcal{L}^{-1} \otimes \mathcal{O}_C(P_1).$$

Continuing in this way, we get

$$\mathcal{O}_C(P_p) = \mathcal{L}^{-(p-1)} \otimes \mathcal{O}_C(P_1).$$

Finally gluing $C^p$ and $C_0$ with a shift $s$ gives the condition

$$\mathcal{L} \otimes \mathcal{O}_C(P_1) = \mathcal{L}^{-(p-1)} \otimes \mathcal{O}_C(P_1) \otimes \mathcal{O}_C(s - O)$$

i.e.

$$\mathcal{L}^p = \mathcal{O}_C(s - O)$$

as claimed.

Remark 7 In [10, part II] a number of explicit examples of degenerations of $(1, p)$-polarized abelian surfaces with a level structure were constructed. It was shown that both types of degenerations which were discussed above actually occur. Moreover, all elliptic curves $C$ and all shifts $s$ can be realised.
3 The mixed Hodge structure on $H^1(X_0)$

Let $X_0 = \bigcup_{i \in \mathbb{Z}_N} Y_i$ be a cycle of elliptic ruled surfaces as in the previous section. In this section we calculate the MHS on $H^1(X_0)$ and show that the base curve $C$ and the shift $s$ can be recovered from it.

3.1 The spectral sequence associated to $X_0$

For technical reasons we want that $X_0$ has global normal crossings. Therefore, we shall first assume $N \geq 2$. However, this is not an essential hypothesis (see remark (14)).

The MHS on $H^q(X_0)$ can be computed as follows (see [7, p.103]). Let $Y^0 = \bigsqcup_{i \in \mathbb{Z}_N} Y_i$ and $Y^1 = \bigsqcup_{i \in \mathbb{Z}_N} C_i$, where $C_i = Y_i \cap Y_{i+1}$. The maps $\alpha_i$ and $\beta_i$ are defined as the inclusions

$$\alpha_i : C_i \hookrightarrow Y_i \text{ and } \beta_i : C_i \hookrightarrow Y_{i+1}.$$ 

Consider the double complex

$$A^{pq} = A^q(Y^p)$$

of global $C^\infty$ differential forms, where $d : A^{pq} \to A^{p,q+1}$ is the exterior derivative and $\delta : A^{pq} = \bigoplus_{i \in \mathbb{Z}_N} A^q(Y_i) \to A^{q+1} = \bigoplus_{i \in \mathbb{Z}_N} A^q(C_i)$ on $A^q(Y_i)$ is given by $(-\beta_i^*, \alpha_i^*): A^q(Y_i) \to A^q(C_{i-1}) \oplus A^q(C_i)$. These coboundary maps satisfy the relations $d^2 = 0$, $d\delta = \delta d$ and $\delta^2 = 0$. There is a single complex $(A^\bullet, D)$ associated to $(A^{\bullet \bullet}, d, \delta)$:

$$A^k = \bigoplus_{p+q=k} A^{pq} \quad D = (-1)^p d + \delta \text{ on } A^{pq}.$$ 

We call $\mathbb{H}^k := H^k_D(A^\bullet)$ the hypercohomology of the double complex.

Lemma 8 The hypercohomology $\mathbb{H}^k$ is canonically isomorphic to $H^k(X_0)$.

Proof. Let $i_p : Y^p \to X_0$ be the natural map. Let $A^{pq}$ be the sheaf on $X_0$ defined by

$$A^{pq}(U) = A^q(i_p^{-1}U).$$

Set $A^k := \bigoplus_{p+q=k} A^{pq}$ and let $D$ be the sheafified version of the coboundary operator $D$ above. It is shown in [8, lemma 4.6] that the complex $(A^\bullet, D)$ is an acyclic resolution of the constant sheaf $\mathbb{C}_{X_0}$, hence $H^k(X_0, \mathbb{C}) = H^k(\Gamma(X_0, A^\bullet)) = H^k_D(A^\bullet) = \mathbb{H}^k$. □
There exists a spectral sequence $E_{pq}^r$ with $E_{0q}^{pq} = A^{pq}$ and $d_0 = d$. The map $d_1$ is induced by $\delta$ and, more generally, $d_r : E_{r+q-r+1}^{pq} \to E_{pq}^{pq}$ is induced by $D$. Notice that $E_{1q}^{pq} = H^q(Y^p)$. Since $A^{pq} = 0$ for $p < 0$ and $p > 1$, the spectral sequence degenerates at $E_2$, i.e. $E_2^{pq} = E_\infty^{pq}$.

The weight filtration $W_\bullet$ on $H^k$ is defined to be the filtration induced by $W_k(A^\bullet) = \oplus_{q \leq k} A^{a,q}$. By the theory of spectral sequences we have $E_\infty^{pq} = \text{Gr}^W_{E_1^{pq}} H^p + q$. Since $A^{pq} = 0$ for $p < 0$ and $p > 1$, this boils down to a commutative diagram with exact rows

$$
\begin{array}{cccccc}
0 & \longrightarrow & W_0 & \longrightarrow & H^1(X_0) & \longrightarrow & \text{Gr}_1 W \longrightarrow & 0 \\
& & \| & & \| & & \| & & \\
0 & \longrightarrow & E_\infty^{10} & \overset{i}{\longrightarrow} & \mathbb{H}^1 & \overset{\pi}{\longrightarrow} & E_\infty^{01} & \longrightarrow & 0.
\end{array}
$$

The map $i$ is induced by the inclusion $A^{10} \to A^{10} \oplus A^{01}$; the map $\pi$ is induced by the projection $A^{10} \oplus A^{01} \to A^{01}$.

The Hodge filtration on $\mathbb{H}^k$ is induced by $F^p(A^k) = \oplus_{a+b=k} F^p(A^{a,b})$. In our case we have

$$F_1^{1 \mathbb{H}^1} = \text{Ker}(D|_{F^1(Y^0)}) = \text{Ker}(\delta : F^1(Y^0) \to F^1(Y^1)).$$

**Lemma 9** Let $X_0$ be a cycle of ruled surfaces over a smooth curve $C$. The weight filtration on $H^1(X_0)$ takes the form

$$0 \subset W_0 \subset W_1 = H^1(X_0).$$

Furthermore, $W_0$ is the unique 1-dimensional Hodge structure $T(0)$ of type $(0,0)$ and $\text{Gr}_1^W$ is canonically isomorphic to $H^1(C)$ (as a Hodge structure).

**Proof.** Since $E_{1q}^{pq} = H^q(Y^p)$ we have $W_0 = E_{0q}^{10} = E_2^{10} = \text{Coker}(\delta : H^0(Y^0) \to H^0(Y^1))$. Now it is easy to see that

$$
\begin{array}{cccccc}
H^0(Y^0) & \overset{\delta}{\longrightarrow} & H^0(Y^1) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
\| & & \| & & \| & & \\
\oplus H^0(X_i) & \overset{\oplus}{\longrightarrow} & \oplus \mathbb{Z},
\end{array}
$$

where $S$ is the summation map, is commutative and has an exact top row, hence $W_0$ is indeed the trivial Hodge structure $T(0)$.
To show that \( \text{Gr}_1^W = E_\infty = E_2^0 = \text{Ker}(\delta : H^1(Y^0) \to H^1(Y^1)) \) is isomorphic to \( H^1(C) \), we introduce the following maps: Let
\[
\rho_i : Y_i \to C_i \quad \text{and} \quad \sigma_i : Y_{i+1} \to C_i
\]
be the projections and let \( \tau_i = \sigma_0 \circ (\alpha_1 \circ \sigma_1) \circ (\alpha_2 \circ \sigma_2) \circ \cdots \circ (\alpha_{i-1} \circ \sigma_{i-1}) : Y_i \to C_0 = C \). Then one easily shows that the following sequence is exact

\[
0 \longrightarrow H^1(C) \xrightarrow{\tau^*} H^1(Y^0) \xrightarrow{\delta} H^1(Y^1),
\]
where \( \tau^* := (\tau_1^*, \ldots, \tau_N^*) : H^1(C) \to H^1(Y^0) = H^1(Y_1) \oplus \cdots \oplus H^1(Y_N) \).

### 3.2 The base curve and the extension class

The mixed Hodge structure \( H^1(X_0) \) can be regarded as an extension of the pure Hodge structures \( W_0 \) and \( \text{Gr}_1^W \). We want to show how one can recover the base curve \( C \) and the shift \( s \) from this extension. In [2] an object \( \text{Ext}(\text{Gr}_1^W, W_0) \) was introduced which parametrizes all such extensions. In our case it is an abelian variety, namely the base curve \( C \).

**Lemma 10** Let \( W_\bullet \) be as above. Then there exists a canonical isomorphism between \( \text{Ext}(\text{Gr}_1^W, W_0) \) and \( \text{Alb}(C) \).

**Proof.** For any Hodge structure \( H \) we define \( J^0 H = H_C/H_Z + F^0 H \). If \( H_1 \) and \( H_2 \) are two Hodge structures, then we can apply this to \( H = \text{Hom}(H_1, H_2) \) and we get \( \text{Ext}(H_1, H_2) = J^0 \text{Hom}(H_1, H_2) \) (see [2, prop. 2]). Now if \( H_1 = H^1(X) \), where \( X \) is a smooth projective variety and \( H_2 = T(0) \), then \( \text{Hom}(H_1, H_2) \) is the dual Hodge structure of \( H^1(X) \) and has weight \(-1\). Its Hodge filtration has the form
\[
0 = F^1 \subset F^0 \subset F^{-1} = H^1(X)^*,
\]
where \( F^0(H^1(X)^*) = (H^{0,1}(X))^* \). Hence \( H^1(X)^*/F^0(H^1(X)^*) = H^{1,0}(X)^* \) and \( J^0(H^1(X)^*) = H^{1,0}(X)^*/H_1(X, \mathbb{Z}) = \text{Alb}(X) \).

In [2] we find the following algorithm to calculate the extension class \( e \in J^0 \text{Hom}(B, A) \) of an extension
\[
0 \longrightarrow A \xrightarrow{i} H \xrightarrow{\pi} B \longrightarrow 0.
\]
1. Choose an integral retraction $r : H \to A$, i.e. a map defined over $\mathbb{Z}$ satisfying $r \circ i = 1_A$.

2. Define two Hodge filtrations on $A \oplus B$:
   $$\tilde{F}_\bullet := (r, \pi)(F^\bullet H)$$
   and
   $$\tilde{F}_0^\bullet := F^\bullet A \oplus F^\bullet B.$$

3. Find a $\psi \in \text{Hom}(B, A)_C$ such that
   $$\left( \begin{array}{cc} 1_A & \psi \\ 0 & 1_B \end{array} \right) \tilde{F}_0^\bullet = \tilde{F}_\infty^\bullet.$$  

Then $e = [\psi] \in J^0 \text{Hom}(B, A)$.

Let $X_0$ be a cycle of ruled surfaces over the elliptic curve $C = V/\Lambda$ with shift $s \in \text{Alb}(C)$. We now describe the $\mathbb{Z}$-splitting of the exact sequence
$$0 \to W_0 \to \mathbb{H}^1 \to \text{Gr}^W_1 \to 0,$$ as required in the first step of this algorithm, explicitly in terms of $C$ and $s$. Let $\{\lambda_1, \lambda_2\}$ be a $\mathbb{Z}$-basis for $\Lambda$ and let $x_i : V \to \mathbb{R}$ be the corresponding coordinate functions ($i = 1, 2$). Let $s_i = x_i(s)$, i.e.
$$s = s_1 \lambda_1 + s_2 \lambda_2.$$ (The numbers $s_i$ are defined only up to integers.) Let $\phi \in H^1(C)$. Since $\{dx_1, dx_2\}$ is a basis for $H^1(C)$, we may represent $\phi$ by $\xi_1 dx_1 + \xi_2 dx_2 \in A^1(C)$, which we will also denote by $\phi$. Consider
$$\tau^* \phi := (\tau_1^*, \ldots, \tau_N^*) \phi \in E^{01}_0.$$ Recall that $\tau^* \phi$ represents an element in $E^{01}_\infty$. Set
$$f = (f_1, \ldots, f_N) \in E^{10}_0,$$
where
$$f_i = \begin{cases} 0 & 1 \leq i < N \\ -\xi_1 s_1 - \xi_2 s_2 & i = N \end{cases}$$
is a constant function on $C_i$.

**Lemma 11** The pair $(\tau^* \phi, f) \in A^{01} \oplus A^{10}$ represents a class in $\mathbb{H}^1$.

**Proof.** First of all, $d(\tau^* \phi) = 0$ since $d \phi = 0$. Furthermore, $\delta(\tau^* \phi) = (0, \ldots, 0, (t_{-s} - 1_C)^* \phi)$, where $t_{-s} : x \mapsto x - s : C \to C$. Let $g = \xi_1 x_1 + \xi_2 x_2 \in \text{Hom}_\mathbb{R}(V, \mathbb{R})$. Then $\phi = dg$ and $f_N = (t_{-s} - 1_C)^* g$. \qed
We write \((\tau^*\phi, f) = \sigma(\phi)\). By the lemma above, the map \(\sigma : H^1(C) \to E^{01}_0 \oplus E^{10}_0\) induces a map

\[\bar{\sigma} : \text{Gr}^{W}_1 = H^1(C) \to \mathbb{H}^1.\]

Since \(\pi : \mathbb{H}^1 \to \text{Gr}^{W}_1\) is induced by the projection \(A^{01} \oplus A^{10} \to A^{01}\), \(\pi \circ \bar{\sigma} = 1_{\text{Gr}^{W}_1}\), i.e. \(\bar{\sigma}\) is a splitting over \(\mathbb{R}\). In fact, \(\bar{\sigma}\) is defined over \(\mathbb{Z}\):

**Lemma 12** The map \(\bar{\sigma}\) splits the sequence

\[0 \longrightarrow W_0 \xrightarrow{i} \mathbb{H}^1 \xrightarrow{\pi} \text{Gr}^{W}_1 \longrightarrow 0\]

over \(\mathbb{Z}\).

Assuming this for the moment, we show that the extension class is identified with the shift.

**Corollary 13** The natural isomorphism \(\text{Ext}(\text{Gr}^{W}_1, W_0) \cong \text{Alb}(C)\) identifies the extension class \(e = [\psi]\) with the shift \(s\).

**Proof.** Let \(r : \mathbb{H}^1 \to W_0\) be the map such that

\[(r, \pi) = (i, \bar{\sigma})^{-1} : \mathbb{H}^1 \to W_0 \oplus \text{Gr}^{W}_1.\]

As before we set

\[\bar{F}^{1}_{\infty} = (r, \pi)F^{1}\mathbb{H}^1\]

and \(\tilde{F}^{1}_{0} = F^{1}(W_0) \oplus F^{1}(\text{Gr}^{W}_1)\) in \(W_0 \oplus \text{Gr}^{W}_1\). The extension class is represented by a map \(\psi : \text{Gr}^{W}_1 \to W_0\) such that

\[
\begin{pmatrix}
1_{W_0} & -\psi \\
0 & 1_{\text{Gr}^{W}_1}
\end{pmatrix}
\tilde{F}^{1}_{\infty} = \tilde{F}^{1}_{0}.
\]

Since \(\tilde{F}^{1}_{0} \cap W_0 = F^{1}(W_0) = 0\) we get

\[r(\omega) - \psi(\pi(\omega)) = 0\]

for all \(\omega \in F^{1}\mathbb{H}^1\).

Now assume that \(C = \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})\) and let \(\phi = \tau dx_1 + dx_2\), where \(x_1, x_2\) are the coordinates dual to the basis \(\lambda_1 = \tau, \lambda_2 = 1\). Then \(F^{1}H^{1}(C) = \mathbb{C}\phi\) (see §4.2). Recall that \((\tau^*\phi, 0) \in A^{01} \oplus A^{10}\) represents an element \(\omega \in F^{1}\mathbb{H}^1\)
such that \( \pi(\omega) = \phi \). Consider \( \sigma(\phi) = (\tau^* \phi, f) \). Since \( i: W_0 \to \mathbb{H}^1 \) is induced by the inclusion \( A^{10} \to A^{01} \oplus A^{10} \),

\[
\omega = i(-f) + \sigma(\phi),
\]

hence

\[
\psi(\tau dx_1 + dx_2) = \psi(\pi(\omega)) = r(\omega) = -f
\]

in \( W_0 \). Identifying \( W_0 \) with \( \mathbb{Z} \) as in the proof of lemma \( 3 \) we find

\[
\psi(\tau dx_1 + dx_2) = -\sum f_i = -f_N = \tau s_1 + s_2 = \int_0^s \tau dx_1 + dx_2.
\]

Hence \( \psi = \int_0^s \) on \( F^1H^1(C) \), i.e. \( \text{Ext}(\text{Gr}_W, W_0) \cong \text{Alb}(C) \) identifies the extension class \( e = [\psi] \) with the shift \( s \).

\begin{remark}
The above result is also true for \( N = 1 \). In this case let \( Y_0 \) be the normalization of \( X_0 \). Let \( X_0' \) be the cycle of elliptic ruled surfaces consisting of two copies of \( Y_0 \) and glued with the same shift \( s \) as \( X_0 \). There exists a projection map \( X_0' \to X_0 \) given by contracting one of the components of \( X_0' \). By functoriality, the isomorphism between the vector spaces \( H^1(X_0) \) and \( H^1(X_0') \) induced by this map is in fact an isomorphism of mixed Hodge structures. Hence we can work with \( X_0' \) instead of \( X_0 \).
\end{remark}

\begin{remark}
Instead of cycles of elliptic ruled surfaces we can also consider cycles of \( \mathbb{P}^1 \)-bundles over an abelian variety \( Z \). Again the Hodge filtration takes on the form \( 0 = W_{-1} \subset W_0 \subset W_1 = H^1(X_0) \) where \( W_0 \) is the 1-dimensional Hodge structure of type \((0, 0)\). As before \( \text{Gr}^W_1 = H^1(Z) \) and hence \( \text{Ext}(\text{Gr}^W_1, W_0) = \text{Alb}(Z) \). The same proof as before also shows that the extension class in \( \text{Ext}(\text{Gr}^W_1, W_0) \) again coincides with the shift \( s \).
\end{remark}
3.3 Proof of lemma

Let $c : [0, 1] \to X_0$ be the loop described in figure 2 [3, 23.8]. Clearly, $H_1(X_0, \mathbb{Z}) = H_1(C, \mathbb{Z}) \oplus \mathbb{Z}c$. Using the Kronecker product (see [3]) the loop $c$ determines a map

$$c : H^1(X_0, \mathbb{Z}) \to \mathbb{Z}.$$ 

Since $\bar{\sigma}$ splits the sequence over $\mathbb{R}$, all we have to do is show that $c(\bar{\sigma}(dx_i)) \in \mathbb{Z}$ for $i = 1, 2$. In fact, we will prove that $c \circ \bar{\sigma} = 0$.

In order to understand how the loop $c$ operates on $\mathbb{H}^1$, we want to identify $\mathbb{H}^1 = H^1(\Gamma(X_0, \mathcal{A}^\bullet))$ with the simplicial cohomology $H^1_\nabla(X_0)$. In [3, p.95] we find a canonical isomorphism

$$H^k(\Gamma(X_0, \mathcal{B}^\bullet)) \to H^k_\nabla(X_0),$$

induced by integration, where

$$\mathcal{B}^k := \text{Ker}(\delta : \mathcal{A}^{0k} \to \mathcal{A}^{1k}).$$

One easily checks that the complex $(\mathcal{B}^\bullet, d)$ is also an acyclic resolution of the constant sheaf $\mathbb{C}_{X_0}$. The inclusions $j_k : \mathcal{B}^k \hookrightarrow \mathcal{A}^{0k} \hookrightarrow \mathcal{A}^k$ commute with the coboundary operators, hence the canonical isomorphism between $H^k(\Gamma(X_0, \mathcal{B}^\bullet))$ and $H^k(\Gamma(X_0, \mathcal{A}^\bullet))$ is induced by $j_k$ (see [15, 5.24]).
Let \( \phi = \xi_1 dx_1 + \xi_2 dx_2 \) and \( (\omega, f) = \bar{s}(\phi) \), i.e.
\[
\omega = (\omega_1, \ldots, \omega_N) \quad \omega_i = \tau_i^* \phi \\
f = (0, \ldots, 0, f_N) \quad f_N = -\xi_1 s_1 - \xi_2 s_2
\]
Let \( g = (g_1, \ldots, g_N) \in \Gamma(X_0, A^0) \) be such that
\[
g_i|_{C_j} = \begin{cases} 
-f_N & (i, j) = (N, 0) \\
0 & \text{otherwise}
\end{cases}
\]
Since \( \delta g = -f \), we have that \( \omega' = \omega + dg \in \Gamma(X_0, B^1) \). Since \( (\omega', 0) = (\omega, f) + (df, \delta f) \) we have \( j_1[\omega'] = \bar{s}(\phi) \) in \( \mathbb{H}^1 \).

Now we view the loop \( c \) as an element of \( \text{Hom}_{\mathbb{Z}}(H^1_{\bar{\mathbb{C}}}(X_0, \mathbb{Z}), \mathbb{Z}) \). Using the defining properties of \( g_i \) and the fact that integrals of \( \omega_i \) along the rulings vanish, we get
\[
c(\bar{s}(\phi)) = \sum_{i=1}^{N} \int_{p_{i-1}}^{p_i} \omega_i' = \int_{p_{N-1}}^{p_N} \omega_N' = g_N(p_N) - g_N(p_{N-1}) + \int_{p_N + s}^{p_N} \omega_N = (s_1 \xi_1 + s_2 \xi_2) - (s_1 \xi_1 + s_2 \xi_2) = 0.
\]

4 Variation of Hodge structure

In this section we want to compute the VHS associated to boundary points of the moduli space of \((1, p)\)-polarized abelian surfaces with a (canonical) level structure (cf. definition 23 and theorem 26).

4.1 \( D \)-polarized abelian varieties

Our main reference is [12]. Fix a type \( D \), i.e. an ordered sequence \((d_1, \ldots, d_g)\) of positive integers satisfying \( d_i | d_{i+1} \) \( (i = 1, \ldots, g - 1) \). We will often write
\[ D = \text{diag}(d_1, \ldots , d_g) \]. Let \( \mathbb{H}_g \) be the Siegel space of degree \( g \). To \( \tau \in \mathbb{H}_g \) we associate an abelian variety \( X_{\tau,D} \) of dimension \( g \) and a polarization \( E_{\tau,D} \) of type \( D \) as follows. First,

\[ X_{\tau,D} = \mathbb{C}^g / \Lambda_{\tau,D}, \]

where

\[ \Lambda_{\tau,D} = (\tau, D)\mathbb{Z}^{2g}. \]

Let \( \lambda_i \ (i = 1, \ldots , 2g) \) be the \( i \)-th column of \( (\tau, D) \). Then we define \( E_{\tau,D} \) to be the map \( \Lambda \times \Lambda \rightarrow \mathbb{Z} \) given by the matrix

\[
E_{\tau,D} = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}
\]

with respect to the basis \( \{ \lambda_1, \ldots , \lambda_{2g} \} \) of \( \Lambda_{\tau,D} \). For this reason we will refer to this basis as the symplectic basis. As an alternating map \( E \) can be regarded as an element in \( H^2(X_{\tau,D}, \mathbb{Z}) \). It is standard knowledge that (up to sign) it is in fact the first Chern class of an ample line bundle on \( X_{\tau,D} \), i.e. a polarization.

Every \( D \)-polarized abelian variety is isomorphic to \( (X_{\tau,D}, E_{\tau,D}) \) for some \( \tau \in \mathbb{H}_g \). Furthermore, we can construct a “universal \( D \)-polarized abelian variety” over \( \mathbb{H}_g \) as follows:

\[ X_D = \mathbb{Z}^{2g} \backslash (\mathbb{C}^g \times \mathbb{H}_g), \]

where \( \mathbb{Z}^{2g} \) operates on \( \mathbb{C}^g \times \mathbb{H}_g \) by

\[ l(v, \tau) = (v + (\tau, D)l, \tau) \]

for \( l \in \mathbb{Z}^{2g} \) and \( (v, \tau) \in \mathbb{C}^g \times \mathbb{H}_g \). The projection \( \mathbb{C}^g \times \mathbb{H}_g \rightarrow \mathbb{H}_g \) induces a projection

\[ \pi : X_D \rightarrow \mathbb{H}_g, \]

such that \( \pi^{-1}(\tau) = X_{\tau,D} \). \( X_D \) is a complex manifold and carries a universal polarization \( L \) (see [12, lemma 8.7.1]).

We are only interested in \( D = (1,p) \), \( D = (1) \) and \( D = (p) \) and will often omit \( D \) if confusion seems unlikely.
4.2 The polarized Hodge structure of \((X_{\tau,D}, E_{\tau,D})\)

We can describe the Hodge structure of the abelian variety \(X_{\tau,D}\) very explicitly. For any abelian variety \(X = V/\Lambda\) we have

\[
F^1(H^1(X)) = H^{1,0}(X) \subset H^1(X, \mathbb{C}) \quad \| \quad V^* = \text{Hom}_\mathbb{C}(V, \mathbb{C}) \subset \text{Hom}_\mathbb{R}(V, \mathbb{C}).
\]

Now consider \(X = X_{\tau,D}\), let \(\{e_1, \ldots, e_g\}\) be the standard basis of \(V = \mathbb{C}^g\) and let \(\{\lambda_1, \ldots, \lambda_{2g}\}\) be the symplectic basis of \(\Lambda = \Lambda_{\tau,D}\). Then \(\{\lambda_1^*, \ldots, \lambda_{2g}^*\}\) is a \(\mathbb{Z}\)-basis of \(H^1(X, \mathbb{Z})\) and \(\{e_1^*, \ldots, e_g^*\}\) is a \(\mathbb{C}\)-basis of \(F^1(H^1(X))\). Using the coordinates of \(H^1(X, \mathbb{C}) = H^1(X, \mathbb{Z}) \otimes \mathbb{C}\) determined by \(\{\lambda_1^*, \ldots, \lambda_{2g}^*\}\), \(e_i^*\) is given by the \(i\)-th row of the period matrix \((\tau, D)\).

A polarization \(\omega\) on a compact complex manifold \(X\) induces a bilinear form \(Q\) on \(H^n(X)\):

\[
Q(\phi, \psi) = (-1)^{n(n-1)/2} \int_X \phi \wedge \psi \wedge \omega^{d-n},
\]

where \(d = \text{dim} X\). The pair \((H^n(X), Q)\) is a polarized Hodge structure of weight \(n\) (see [7, p.7]).

We now calculate the polarization on \(H^1(X_{\tau,D})\) induced by \(E_{\tau,D}\). Let \(x_i : V \to \mathbb{R} (i = 1, \ldots, 2g)\) be the coordinates with respect to the real basis \(\{\lambda_1, \ldots, \lambda_{2g}\}\) of \(V\). Then \(dx_i\) corresponds to \(\lambda_i^*\) under the natural isomorphism \(H^1_{\text{de Rham}}(X) \cong \text{Hom}(\Lambda, \mathbb{Z})\). Furthermore \(E_{\tau,D} \in H^2(X, \mathbb{Z})\) is represented by the 2-form

\[
\omega = - \sum_{i=1}^g d_i dx_i \wedge dx_{i+g}
\]

and

\[
\int_X dx_1 \wedge dx_{g+1} \wedge \cdots \wedge dx_g \wedge dx_{2g} = 1
\]

(see [12 lemmas 3.6.4 and 3.6.5]). It follows that \(Q\) is given by the matrix

\[
(g-1)! \begin{pmatrix}
0 & -\hat{D} \\
\hat{D} & 0
\end{pmatrix}
\]

\[
\hat{D} = \left( \prod_{i=1}^{2g} d_i \right) D^{-1}
\]

with respect to the basis \(\{\lambda_1^*, \ldots, \lambda_{2g}^*\}\) of \(H^1(X, \mathbb{Z}) = \text{Hom}(\Lambda, \mathbb{Z})\).
4.3 **Boundary points of** \( \mathcal{A}^*(1, p) \)**

We have to recall briefly some facts about compactifications of moduli spaces of abelian surfaces. All relevant details can be found in [10]. By \( \mathcal{A}(\infty, \sqrt{\_}) \) we denote the moduli space of \((1, p^-)\)-polarized abelian surfaces with a (canonical) level structure \((p \geq 3, \text{prime})\). Recall that

\[
\mathcal{A}(\infty, \sqrt{\_}) = \mathbb{H} / \Gamma_{\infty, \sqrt{\_}}
\]

where

\[
\Gamma_{1, p} = \left\{ g \in \text{Sp}(4, \mathbb{Z}) ; g - 1 \in \begin{pmatrix}
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\
p\mathbb{Z} & p\mathbb{Z} & p\mathbb{Z} & p^2\mathbb{Z} \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z}
\end{pmatrix} \right\}
\]

acts on Siegel space \( \mathbb{H}_2 \) by

\[
g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \tau \mapsto (A\tau + B)(C\tau + D)^{-1}.
\]

In [10] a torodial compactification \( \mathcal{A}^*(1, p) \) of \( \mathcal{A}(\infty, \sqrt{\_}) \) was constructed. To compactify \( \mathcal{A}(\infty, \sqrt{\_}) \) one has to add (non-compact) boundary surfaces (corank 1 boundary points) and boundary curves (corank 2 boundary points). Here we shall restrict ourselves exclusively to the boundary surfaces. These surfaces are indexed by the vertices of the Tits building of \( \Gamma_{1, p} \) which correspond to lines \( l \subset \mathbb{Q}^4 \). According to [10] there are two types of boundary surfaces, namely one central boundary surface \( D(l_0) \) and \( p(p - 1)/2 \) peripheral boundary surfaces \( D(l_{(a,b)}) \) where \((a, b) \in (\mathbb{Z}_p \times \mathbb{Z}_p \setminus \{0\})/(\pm 1)\). The group \( \text{SL}(2, \mathbb{Z}_p) \) acts on \( \mathcal{A}^*(1, p) \) and permutes the peripheral boundary surfaces of \( \mathcal{A}^*(1, p) \) transitively. Therefore it is enough to consider one of them, namely \( D(l_{(0,1)}) \).

From now on let \( l = l_0 \) or \( l_{(0,1)} \). The stabilizer \( P(l) \) of \( l \) in \( \Gamma_{1, p} \) is an extension of the form

\[
1 \longrightarrow P'(l) \longrightarrow P(l) \longrightarrow P''(l) \longrightarrow 1
\]
where $P'(l)$ is a rank 1 lattice. The compactification procedure requires that one first takes the partial quotient of $\mathbb{H}_2$ with respect to $P'(l)$. For $l = l_0$

$$P'(l) = \left\{ \begin{pmatrix} 1 & n & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} ; n \in \mathbb{Z} \right\}$$

and for $l = l_{(0,1)}$:

$$P'(l) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & np^2 & 0 \\ 0 & 1 & 1 \end{pmatrix} ; n \in \mathbb{Z} \right\}$$

The partial quotient map $e(l) : \mathbb{H}_2 \to \mathbb{H}_2/P'(l)$ is then given by

$$e(l_0) : \mathbb{H}_2 \to \mathbb{C}^* \times \mathbb{C} \times \mathbb{H}_1, \quad \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} \mapsto (e^{2\pi i \tau_1}, \tau_2, \tau_3)$$

resp.

$$e(l_{(0,1)}) : \mathbb{H}_2 \to \mathbb{H}_1 \times \mathbb{C} \times \mathbb{C}^*, \quad \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} \mapsto (\tau_1, \tau_2, e^{2\pi i \tau_3/p^2}).$$

Here $\mathbb{H}_1$ denotes the usual upper half plane. This maps $\mathbb{H}_2$ to an interior neighbourhood of $\{0\} \times \mathbb{C} \times \mathbb{H}_1$, resp. $\mathbb{H}_1 \times \mathbb{C} \times \{0\}$, i.e. the interior of the closure of the image, $X(l) = (e(\mathbb{H}_2))^o$ is an open neighbourhood of $\{0\} \times \mathbb{C} \times \mathbb{H}_1$, resp. $\mathbb{H}_1 \times \mathbb{C} \times \{0\}$. The partial compactification in the direction of $l$ then consists of adding the set $\{0\} \times \mathbb{C} \times \mathbb{H}_1$, resp. $\mathbb{H}_1 \times \mathbb{C} \times \{0\}$. There is a natural map $X(l) \to \mathcal{A}^*(1, p)$ given by dividing out the extended action of $P''(l_0)$ on $X(l)$, which maps $\{0\} \times \mathbb{C} \times \mathbb{H}_1$, resp. $\mathbb{H}_1 \times \mathbb{C} \times \{0\}$ to the boundary surface $D(l_0)$, resp. $D(l_{(0,1)})$. 

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4.4 Two 1-parameter families

We consider two 1-parameter families of $\mathbb{1}(1,p)$-polarized abelian surfaces which are closely related to the central, resp. peripheral boundary components (see proof of theorem [24]). For $M$ a positive integer we set

$$\mathbb{H}_1(M) := \{ \tau \in \mathbb{H}_1 \ ; \ \text{Im} \tau > M \}$$

and

$$\Delta^*(M) := \{ t \in \mathbb{C} \ ; \ 0 < |t| < e^{-2\pi i M} \}.$$

First we fix a pair $(\tau_2, \tau_3) \in \mathbb{C} \times \mathbb{H}_1$. For sufficiently large $M$ we have a map

$$\mathbb{H}_1(M) \to \mathbb{H}_2, \ \tau \mapsto \begin{pmatrix} \tau & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix}.$$

The first 1-parameter family which we want to consider is the pull back of the universal family $\pi : X_D \to \mathbb{H}_2$ to a family $\pi_1 : X_D \to \mathbb{H}_1(M)$ via this map. For the second family we fix a pair $(\tau_1, \tau_2) \in \mathbb{H}_1 \times \mathbb{C}$ and pull back the universal family $\pi : X_D \to \mathbb{H}_2$ to a family $\pi_2 : X_D \to \mathbb{H}_1(M)$ via the map

$$\mathbb{H}_1(M) \to \mathbb{H}_2, \ \tau \mapsto \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

where $M$ is again chosen sufficiently large.

We denote the polarized abelian surface associated to $\tau \in \mathbb{H}_1(M)$ by $(X_\tau, E_\tau)$. Since in both cases $(X_\tau, E_\tau)$ depends only on $\tau \mod \mathbb{Z}$, the family $\pi_i$ is the pull back of a family $p_i : X_D \to \Delta^*(M)$ via the map

$$\mathbb{H}_1(M) \to \Delta^*(M), \ \tau \mapsto t = e^{2\pi i \tau}.$$

Consider the maps

$$\tilde{\phi}_1 : \mathbb{H}_1(M) \to \text{Grass}(2,4), \ \tau \mapsto \begin{pmatrix} \tau & \tau_2 & 1 & 0 \\ \tau_2 & \tau_3 & 0 & p \end{pmatrix},$$

and

$$\tilde{\phi}_2 : \mathbb{H}_1(M) \to \text{Grass}(2,4), \ \tau \mapsto \begin{pmatrix} \tau_1 & \tau_2 & 1 & 0 \\ \tau_2 & p^2 \tau & 0 & p \end{pmatrix}.$$
\[ T_i = \left( \begin{array}{cc} 1 & n_i \\ 0 & 1 \end{array} \right) \text{, where } n_1 = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \text{ and } n_2 = \left( \begin{array}{cc} 0 & 0 \\ 0 & p \end{array} \right) \] By construction \( \tilde{\phi}_i \) descends to a map \( \bar{\phi}_i : \Delta^\ast(M) \to \langle T_i \rangle / D_2 : \\
\begin{array}{cc}
\mathbb{H}_1 & \xrightarrow{t} \\
\downarrow & \\
\Delta^\ast(M) & \xrightarrow{\phi_i} \langle T_i \rangle / D_2 
\end{array} \\
If we identify \( H^1(X, \mathbb{C}) \) with \( \mathbb{C}^4 \) using the coordinates induced by the basis \( \{ \lambda_1^\ast, \ldots, \lambda_4^\ast \} \) of \( H^1(X, \mathbb{Z}) \), then by the description of the Hodge structure on the first cohomology of an abelian variety given in the previous section, \( \phi_i \) is the VHS associated to the family \( p_i : X_D \to \Delta^\ast(M) \).

### 4.5 The limit mixed Hodge structure

The family \( p_i : X_D \to \Delta^\ast(M) \) \( (i = 1, 2) \) induces a MHS on \( H^1(X_i) \) \( (t \in \Delta^\ast(M) \) arbitrary but fixed; see [7, Chapter IV]). Its weight filtration is of the form \( 0 = W_{-1} \subset W_0 \subset W_1 \subset W_2 = H^1(X_i) \). It is is given by 
\[ W_0 = \text{Im} N \text{ and } W_1 = \text{Ker} N, \]
where \( N = \log T = T - 1 = \left( \begin{array}{cc} 0 & n_i \\ \mathbf{0} & 0 \end{array} \right) \). (We have written \( T \) instead of \( T_i \) etc.)

The limit Hodge filtration \( F^\ast_{\infty} \subset H^1(X_t) \) is calculated as follows: first define \( \bar{\psi} : \mathbb{H}_1(M) \to \mathcal{D}_2 \subset \text{Grass}(2, 4) \) (where \( \mathcal{D}_2 \) is the compact dual of \( D_2 \)) by
\[ t^\ast \bar{\psi} (\tau) = T^{-\tau t} \bar{\phi}(\tau) = (1 - \tau N)^{t \bar{\phi}(\tau)}. \]

By construction \( \bar{\psi}(\tau + 1) = \bar{\psi}(\tau) \), hence \( \bar{\psi} \) descends to a map \( \psi : \Delta^\ast(M) \to \mathcal{D}_2 \). By the work of Griffiths and Schmid \( \psi \) extends to \( \Delta \) and \( F^\ast_{\infty} := \psi(0) \) together with the monodromy weight filtration defines a MHS on \( H^1(X_t) \).

From now on we will write \( e_i \) instead of \( \lambda_i^\ast \) for the basis vectors of \( H^1(X, \mathbb{Z}) \). In our case, an immediate calculation shows that \( \bar{\psi} \) is constant and 
\[ W_0 = [e_1], \quad W_1 = [e_1, e_2, e_4], \quad F^1_{\infty} = \left[ \begin{array}{ccc} 0 & \tau_2 & 1 \\ \tau_2 & \tau_3 & 0 \\ \tau_2 & 0 & p \end{array} \right]. \]

for the family \( p_1 : X_D \to \mathbb{H}_1(M) \) and 
\[ W_0 = [e_2], \quad W_1 = [e_1, e_2, e_3], \quad F^1_{\infty} = \left[ \begin{array}{ccc} \tau_1 & \tau_2 & 1 \\ \tau_2 & 0 & 0 \\ \tau_2 & 0 & p \end{array} \right]. \]
for the family $p_2 : X_\mathcal{P} \to \mathcal{H}_1(M)$.

We shall now restrict our attention to $W_1$. Here the Hodge filtration is given by

$$F^1_{\infty} \cap W_1 = [\tau_2 e_1 + \tau_3 e_2 + pe_4]$$

resp.

$$F^1_{\infty} \cap W_1 = [\tau_1 e_1 + \tau_2 e_2 + e_3].$$

By abuse of notation we will denote this by $F^1_{\infty}$, too. Finally, recall that all elements of the nilpotent orbit $\{T^* F^1_{\infty}\}$ of Hodge filtrations on $W_2 = H^1(X_\mathcal{P})$ induce the same MHS on $W_1$ (cf. [7, p.84]).

4.6 Calculation of $\text{Gr}_W^1$ and the extension class

In the previous section we computed the limit MHS on $H^1(X_\mathcal{P})$ determined by $p_i : X \to \Delta^*(M)$ ($i = 1, 2$). We now want to compute $\text{Gr}_W^1$ and the extension class.

For any integer $n$ and $\tau \in \mathcal{H}_1$ we define

$$X_{\tau,n} = V/\Lambda = \mathbb{C}f/(\mathbb{Z}\mu_1 + \mathbb{Z}\mu_2)$$

where $\mu_1 = \tau \bar{f}$ and $\mu_2 = nf$.

**Proposition 16** (i) In the central case ($i = 1$) $\text{Gr}_W^1 = H^1(C)$ where $C = X_{\tau_3,p}$.  

(ii) In the peripheral case ($i = 2$) $\text{Gr}_W^1 = H^1(C)$ where $C = X_{\tau_1,1}$.

**Proof.** (i) Setting $\mu_1^* = e_2$ and $\mu_2^* = e_4$ we have

$$F^1 = [f^*] = [\tau_3 \mu_1^* + p \mu_2^*] = [\tau_3 e_2 + pe_4]$$

and the claim follows immediately from the calculations of 4.5.

(ii) Setting $\mu_1^* = e_1$ and $\mu_2^* = e_3$ we have

$$F^1 = [f^*] = [\tau_1 \mu_1^* + \mu_2^*] = [\tau_1 e_1 + e_3]$$

and the result follows as before. \qed
Next we want to calculate the class of the extensions

\[ 0 \rightarrow W_0 \xrightarrow{i} W_1 \xrightarrow{\pi} \text{Gr}_1^W \rightarrow 0. \] (2)

The recipe for finding the extension class of (2) demands that we choose an integral retraction \( r : W_1 \rightarrow W_0 \) and then transfer \( F_\infty^1 \) to \( W_0 \oplus \text{Gr}_1^W \) via \((r, \pi)\). In the central case we take \( r(e_1) = e_1 \) and \( r(e_2) = r(e_4) = 0 \) and in the peripheral case \( r(e_2) = e_2 \) and \( r(e_1) = r(e_3) = 0 \). Then, in both cases, \((r, \pi) = 1_{\text{Gr}_1^W}\), hence \( \tilde{F}_\infty^1 = F_\infty^1 \). Next we define the trivial extension

\[ \tilde{F}_0^1 = F^1(W_0) \oplus F^1(\text{Gr}_1^W) = F^1(\text{Gr}_1^W) = [\tau_3 e_2 + pe_4] \]

for the first family and similarly

\[ \tilde{F}_0^1 = [\tau_1 e_1 + e_3] \]

for the second family. The extension class \( e \in J^0 \text{Hom}(\text{Gr}_1^W, W_0) \) is represented by any map \( \psi : \text{Gr}_1^W \rightarrow W_0 \) such that

\[
\begin{pmatrix}
1_{W_0} & \psi \\
0 & 1_{\text{Gr}_1^W}
\end{pmatrix}
\tilde{F}_0^1 = \tilde{F}_\infty^1.
\]

For the first family we can take \( \psi = (\tau_2/p)e_4^* \otimes e_1 \) and for the second family \( \psi = \tau_2 e_3^* \otimes e_2 \).

After identifying \( W_0 = \mathbb{Z} \) and \( \text{Gr}_1^W = H^1(C) \), where \( C \) is as in proposition 16, the extension class \( e = [\psi] \) is well defined in

\[
\text{Ext}(\text{Gr}_1^W, W_0) = J^0 \text{Hom}(\text{Gr}_1^W, W_0)
= H^1(C)^*/(H^0,1(C))^* + H^1(C, \mathbb{Z})^*
= H^{1,0}(C)^*/H_1(C, \mathbb{Z})
= \text{Alb}(C)
\]

and is represented by \( \psi|_{H^{1,0}} \).

**Proposition 17** Under the identification \( \text{Ext}(\text{Gr}_1^W, W_0) = \text{Alb}(C) \) the extension class of (3) corresponds to \([\tau_2 f] \in \text{Alb}(C)\).
Proof. Notice that if \( C = V/\Lambda \), then \( H^1(C, \mathcal{C}) = \text{Hom}_\mathbb{R}(V, \mathcal{C}) \) and

\[
\begin{array}{ccc}
H^1(C, \mathcal{C}) & \xrightarrow{f_0^{[\varphi]}} & \mathbb{C}/\text{periods} \\
\| & & \uparrow \\
\text{Hom}_\mathbb{R}(V, \mathcal{C}) & \xrightarrow{\text{ev}(\nu)} & \mathbb{C}
\end{array}
\]

commutes.

The extension class \( e \in \text{Ext}(\text{Gr}^W_1, W_0) \) corresponds to \([\tau_2 f] \in \text{Alb}(C)\) since

\[
\int_0^{[\tau_2 f]} (\tau_3 e_2 + pe_4) = \int_0^{[\tau_2 f]} (f^*) = \tau_2 = \psi(\tau_3 e_2 + pe_4)
\]

in the central case and

\[
\int_0^{[\tau_2 f]} (\tau_1 e_1 + e_3) = \int_0^{[\tau_2 f]} (f^*) = \tau_2 = \psi(\tau_1 e_2 + e_3)
\]

in the peripheral case. \( \square \)

4.7 The number of components

In this section we will proof the following proposition.

**Proposition 18** Let \( \pi: X \to \Delta \) be a semi-stable degeneration of \((1, p)\)-polarized abelian surfaces with (canonical) level structure. Assume that the central fibre \( X_0 \) is a cycle of \( N \) ruled surfaces. Then the number \( N \) is determined by the polarized VHS (PVHS) on \( \Delta^* \).

The polarization on the VHS is induced by the polarization \( \mathcal{L} \) in the following way: For \( Y \subset X \) define

\[
c_Y = c_1(\mathcal{L}|_Y) \in H^2(Y)
\]

25
and

\[ Q_Y: (a, b) \mapsto a \cup b \cup c_Y: H^1(Y) \times H^1(Y) \to H^4(Y). \]

We are mostly interested in the case where \( Y \) is \( X, X_0 \) or \( X_t \). We write \( c_0 \) for \( c_{X_0} \) and \( Q_0 \) for \( Q_{X_0} \). Similarly, we write \( c_t \) for \( c_{X_t} \) and \( Q_t \) for \( Q_{X_t} \).

The existence of a global level structure is needed to insure that \( \deg(\mathcal{L}|_{F_k}) \), where \( F_k \) is the fibre of \( Y_k \), is independent of \( k \). Indeed, by lemma 4 of §2.2 it is always 1.

For the rest of the section \( \pi: X \to \Delta \) is as in proposition \( \text{[18]} \). As in §3.1 we identify \( C \) with \( C_0 = Y_0 \cap Y_1 \). Let \( Y_k \) and \( \tau_k: Y_k \to C \) be as in §3.1 and let \( i: C \hookrightarrow X_0 \) and \( j_k: Y_k \hookrightarrow X_0 \) be the inclusion maps.

**Lemma 19** Under the isomorphism \( (\tau_k^*): H^1(C) \to \mathrm{Gr}^W_1 \), the induced map \( (j_k^*): W_1 \to \mathrm{Gr}^W_1 \) corresponds to the map \( i^* : H^1(X_0) \to H^1(C) \) induced by the inclusion \( i: C \hookrightarrow X_0 \).

**Proof.** This follows immediately from \( j_k^* = \tau_k^* i^*: H^1(X_0) \to H^1(Y_k) \), which in turn follows from the following facts: \( \rho_k^* \) is the inverse of \( \alpha_k^* \), \( \sigma_k^* \) is the inverse of \( \beta_k^* \), \( i = i_0 = j_1 \beta_0 \), and \( j_{k+1} \beta_k = j_k \alpha_k \).

**Proof of proposition \( \text{[18]} \).** The PVHS on \( \Delta^* \) determines the bilinear form \( Q_t \) on \( W_2 = H^1(X_t) \). Since the maps on cohomology induced by an inclusion \( Y' \subset Y \) are compatible with \( Q_Y \) and \( Q_{Y'} \), \( Q_t \) induces \( Q_0 \) on \( H^1(X_0) = W_1 \subset W_2 \), i.e., we have a commutative diagram

\[
\begin{array}{ccc}
H^1(X_t) \otimes H^1(X_t) & \xrightarrow{Q_t} & H^4(X_t) \\
\uparrow k_t^* \circ k_t^* & & \uparrow k_t^* \\
H^1(X) \otimes H^1(X) & \xrightarrow{Q} & H^4(X) \\
\uparrow k_0^* \circ k_0^* & & \uparrow k_0^* \\
H^1(X_0) \otimes H^1(X_0) & \xrightarrow{Q_0} & H^4(X_0).
\end{array}
\]

For \( t \in \Delta \) let \( k_t: X_t \hookrightarrow X \) be the inclusion map and let

\[ \delta: H^4(X_0) \to H^4(X_t) \]

be \( k_t^*(k_0^*)^{-1} \) (recall that \( k_0^* \) is an isomorphism). Since the monodromy is compatible with \( Q_t \), i.e., \( Q_t(Ta, Tb) = Q_t(a, b) \), so is \( N = T - 1 \), i.e.
$Q_t(Na, b) = Q_t(a, Nb)$. Since $W_1 = \text{Ker } N$ and $W_0 = \text{Im } N \ Q_t(W_1, W_0) = 0$. In other words, there exists a bilinear form $q$ on $H^1(C)$ making the diagram

$$H^1(X_0) \times H^1(X_0) \xrightarrow{\delta Q_0} H^4(X_t) = \mathbb{Z}
$$

$$i^* \times i^*
\downarrow q
\downarrow
H^1(C) \times H^1(C)
$$

commute. Furthermore, since $i^* : H^1(X_0) \to H^1(C)$ is surjective, $q$ is determined by $\epsilon Q_0$ where

$$\epsilon : H^2(C) \to H^4(X_t)
$$

is the isomorphism which sends the generator of $H^2(C)$ corresponding to the canonical orientation of $C$ to the same thing on $X_t$. Let

$$\gamma : H^1(X) \times H^1(C) \to H^2(C)
$$

be the intersection form. We will show in corollary 22 that $q = N \epsilon \gamma$. This means that the data over $\Delta^*$ determine $N$. \qed

For convenience, we shall first assume that $N \geq 2$. The case $N = 1$ requires minor modifications and is dealt with later.

Since the line bundle determining $Y_k$ has degree 0, there exists a continuous map $q_k : Y_k \to \mathbb{P}^1$ such that

$$(\tau_k, q_k) : Y_k \to C \times \mathbb{P}^1
$$

is a homeomorphism. Let $f \in H^2(\mathbb{P}^1)$ be the canonical generator, determined by the complex structure. We define $\beta$ by demanding that the diagram

$$H^2(C) \xrightarrow{\beta} H^4(X_0)
$$

$$(1, \ldots, 1) \otimes f
\downarrow
\oplus H^2(C) \otimes H^2(\mathbb{P}^1) \xrightarrow{\beta \otimes q_k} \oplus H^4(Y_k)
$$

be commutative. (Notice that $(j_k^*)$ is an isomorphism.)

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Lemma 20  The diagram

\[
\begin{array}{ccc}
H^1(X_0) \times H^1(X_0) & \xrightarrow{Q_0} & H^4(X_0) \\
\downarrow{i^* \times i^*} & & \uparrow{\beta} \\
H^1(C) \times H^1(C) & \xrightarrow{\gamma} & H^2(C)
\end{array}
\]

commutes.

Proof. By definition of \(\beta\) we have to show that

\[
\begin{array}{ccc}
H^1(X_0) \times H^1(X_0) & \xrightarrow{Q_0} & H^4(X_0) \\
\downarrow{i^* \times i^*} & & \uparrow{\beta} \\
H^1(C) \times H^1(C) & \xrightarrow{(j_k^*, \ldots, j_k^*)} & \oplus H^4(Y_k) \\
\downarrow{(\gamma, \ldots, \gamma)} & & \uparrow{(\tau_k^* \cup q_k^*)} \\
\oplus H^2(C) & \otimes f & \oplus H^2(C) \otimes H^2(\mathbb{P}^1)
\end{array}
\]

commutes. In lemma 4 of \(\S 2\) we showed that \(\deg(C|_{F_k}) = 1\) for all \(k\), where \(F_k\) is the ruling of \(Y_k\). Hence we can write \(j_k^*c_0) = \tau_k^*(d_k) + q_k^*(f)\) for some \(d_k \in H^2(C)\). Since \(j_k^* = \tau_k^* i^*\) on \(H^1(X_0)\) (see proof of lemma 19),

\[
j_k^*Q_0(a, b) = \tau_k^* i^*(a \cup b) \cup (\tau_k^*(d_k) + q_k^*(f)) = \tau_k^* i^*(a \cup b) \cup q_k^*(f),
\]

because \(i^*(a \cup b) \cup d_k \in H^4(C) = 0\).

The orientations of the components of the singular fibre are compatible with the orientation of the general fibre in the following sense:

Lemma 21  If \(N\) is the number of components of \(X_0\), then \(\delta \beta = N\epsilon\).

Proof. By definition of \(\beta\), it suffices to show that the diagram

\[
\begin{array}{ccc}
H^4(X_0) & \xrightarrow{k_0^*} & H^4(X) \\
(j_k^*) & \sim & \uparrow{k_i^*} \\
\oplus H^4(Y_k) & \xrightarrow{(1, \ldots, 1)} & H^4(X_i)
\end{array}
\]
commutes. The inverse of \( k_0^* \) is induced by a retraction \( r: X \to X_0 \), which exhibits \( X_0 \) as a strong deformation retract of \( X \). This retraction restricts to \( r_t: X_t \to X_0 \). Let \( Z_k \subset X_t \) be the inverse image of \( Y_k \). Then \( S_k := Z_k \cap Z_{k+1} \) is the inverse image of \( T_k := Y_k \cap Y_{k+1} \). Pick any \( \kappa \in \{1, \ldots, N\} \). Let \( Y' = Y_\kappa, Y'' = \cup_{k \neq \kappa} Y_k, Z' = Z_\kappa \) and \( Z'' = \cup_{k \neq \kappa} Z_k \). Let \( S = Z' \cap Z'' = S_{\kappa-1} \cup S_{\kappa} \) and \( T = Y' \cap Y'' = T_{\kappa-1} \cup T_{\kappa} \).

We will need below that \( H^4(Z') = H^4(Z'') = 0 \). To see this, we use that the retraction \( r_t: Z_k \to Y_\kappa \) is the real oriented blow up along \( T_{\kappa-1} \cup T_{\kappa} \). The “exceptional divisors” \( S_k = r_t^{-1}(T_k) \) are \( S^1 \)-bundles over \( T_k \). They are trivial, because the triple \((Y_k, T_{k-1}, T_k)\) is homeomorphic to \((B \times S^2, B \times \{0\}, B \times \{\infty\})\), where \( B = S^1 \times S^1 \). It follows that the triple \((Z_k, S_{k-1}, S_k)\) is homeomorphic to \((B \times S^1 \times [0,1], B \times S^1 \times \{0\}, B \times S^1 \times \{1\})\). In particular, \( H^4(Z_k) = 0 \) and \( H^3(Z_k) \to H^3(S_k) \) and \( H^3(Z_k) \to H^3(S_{k-1}) \) are surjective for all \( k \). Mayer-Vietoris now implies that \( H^4(Z'') = \oplus_{k \neq \kappa} H^4(Z_k) \) and this proves the claim.

Consider the commutative diagram

\[
\begin{array}{ccc}
H^4(Y') & \leftarrow & H^4(X_0) \xrightarrow{r_t^*} H^4(X_t) \\
\sim & & \sim \\
H^4(Y', T) & \leftarrow & H^4(X_0, Y'') \xrightarrow{r_t^*} H^4(X_t, Z'')
\end{array}
\]

The map \( H^4(X_0, Y'') \to H^4(X_t, Z'') \) is an isomorphism by Alexander duality since \( r_t \) is an homeomorphism \( X_0 \setminus Y'' \to X_t \setminus Z'' \) (cf. [11, p.23]). Similarly, \( H^4(X_0, Y'') \to H^4(Y', T) \) is an isomorphism. The left vertical map is an isomorphism because \( T \) has topological dimension 2. The right vertical map is surjective because \( H^4(Z'') = 0 \), as we have seen above. But a surjective map form \( Z \) to \( Z \) is automatically injective.

It follows from [3, p.99] that the map \( H^4(Y'') \to H^4(X_0) \) via \( H^4(Y', T) \) and \( H^4(X_0, Y'') \) composed with the Mayer-Vietoris isomorphism \( H^4(Y') \oplus H^4(Y'') \cong H^4(X_0) \) is just the inclusion of \( H^4(Y'') \) into \( H^4(Y') \oplus H^4(Y'') \). We have to show that the composition of this map with \( r_t^*: H^4(X_0) \to H^4(X_t) \) maps the orientation class to the orientation class. As before, we may identify the relative cohomology groups with the cohomology with compact supports of the respective complements. But \( r_t: X_t \setminus Z'' \to Y' \setminus T \) is a complex isomorphism. In particular, it respects the orientation.
Corollary 22 \( q = N\epsilon\gamma \).

Proof. By lemma 20 \( q = \delta\beta\gamma \). Now the result follows immediately from the previous lemma 21.

Finally, we consider the case \( N = 1 \). We proceed as in the case \( N \geq 2 \). First we have to define \( \beta : H^2(C) \to H^4(X_0) \). To do this, notice that we have an exact Mayer-Vietoris sequence

\[ H^{q-1}(\tilde{C}) \to H^q(X_0) \to H^q(\tilde{X}_0) \oplus H^q(C) \to H^q(\tilde{C}), \]

where \( \tilde{X}_0 \) is the normalization of \( X_0 \) and \( \tilde{C} = \pi^{-1}(C) \) (see [2, p.120]). Hence the normalization map \( \pi : \tilde{X}_0 \to X_0 \) induces an isomorphism \( H^4(X_0) \to H^4(\tilde{X}_0) \). We identify \( H^4(X_0) \) with \( H^4(\tilde{X}_0) \) via \( \pi^* \) and define \( \beta \) by demanding that it maps the orientation class of \( C \) to the orientation class of \( \tilde{X}_0 \). With this definition of \( \beta \) and using that the degree \( \deg(\pi^*(L)|_F) \) of \( L \) along the fibre \( F \) in the normalization is 1 again by lemma 4 of §2, the diagram in lemma 20 again commutes. The proof is an easy adaption of the proof given for \( N \geq 2 \) and is left to the reader.

To prove lemma 21 for \( N = 1 \), one argues exactly as in the case \( N \geq 2 \) using the commutative diagram

\[
\begin{array}{ccc}
H^4(\tilde{X}_0) & \longleftarrow & H^4(X_0) \\
\uparrow & & \uparrow \\
H^4(\tilde{X}_0, \tilde{C}) & \longleftarrow & H^4(X_0, C) \\
\end{array}
\quad
\begin{array}{ccc}
H^4(X_t) & \longrightarrow & H^4(X_t) \\
\uparrow & & \uparrow \\
H^4(X_t, S) & \longrightarrow & H^4(X_t, S), \\
\end{array}
\]

where \( S = r_t^{-1}(C) \). (This time, all maps are isomorphisms.)

5 A uniqueness result

Here we combine the results of sections 3 and 4 to prove a uniqueness result for degenerate abelian surfaces.

In [1,4] we discussed central and peripheral corank 1 boundary points of \( \mathcal{A}^*(1,p) \). Note that the universal family \( \pi : X_D \to \mathbb{H}_2 \) descends to a family \( \pi' : X_D \to e(\mathbb{H}_2) \). By [10, part II.4] this family can be extended to a family \( \bar{\pi} : \bar{X}_D \to X(l) \). The fibres of \( \bar{X}_D \) over the ”boundary” \( \{0\} \times \mathbb{C} \times \mathbb{H}_1 \), resp. \( \mathbb{H}_1 \times \mathbb{C} \times \{0\} \) are cycles of elliptic ruled surfaces with \( N = 1 \), resp. \( N = p \).
Definition 23 Let \( [q] \in \mathcal{A}^*(1, p) \) be a corank 1 boundary point. We say that a surface \( X_0 \) is a *degenerate abelian surface associated to \([q]\)* if the following holds: There exists a 1-parameter degeneration of polarized abelian surfaces with (canonical) level structure \( p : X \to \Delta \) as in section 2.2 with central fibre \( p^{-1}(0) = X_0 \) and an embedding \( f : \Delta \to X(l) \) such that:

(i) \( f(\Delta) \) meets \( \{0\} \times \mathbb{C} \times \mathbb{H}_1 \) resp. \( \mathbb{H}_1 \times \mathbb{C} \times \{0\} \) transversely in the point \( q = f(0) \) which is mapped to \([q]\).

(ii) The restriction of \( p : X \to \Delta \) to \( \Delta^* \) is the pull back of the family \( \pi' : X_D \to \mathcal{E}(\mathbb{H}_2) \) via \( f \).

Remark 24 The reason why we work with \( X(l) \) rather than with the compactification \( \mathcal{A}^*(1, p) \) itself is that the map \( \mathbb{H}_2 \to \mathcal{A}(\infty, \sqrt{\cdot}) \) is branched over two Humbert surfaces \( H_1 \) and \( H_2 \). Near these surfaces, resp. their closure in \( \mathcal{A}^*(1, p) \) one has no universal family.

Lemma 25 If \( X_0 \) is a degeneration with only local normal crossing singularities associated to a corank 1 boundary point \([q]\), then the MHS on \( H^1(X_0) \) is determined by this point.

Proof. This follows from the Local Invariant Cycle theorem [7, Ch. VI] (which also holds when we have local normal crossing rather than global normal crossing) together with the existence of the extended family \( \bar{\pi} : \bar{X}_D \to X(l) \): Indeed, compare the family \( p : X \to \Delta \) as in definition (23) with the family \( f^*X_D \to \Delta \). On \( \Delta^* \) these two families agree (they are both the pull back of the universal family), hence they determine the same VHS and thus the same limit MHS on \( H^1(X_t) \) \((t \in \Delta^* \) arbitrary but fixed). But by the Local Invariant Cycle theorem the MHS on the central fibre is determined by this limit MHS at least over \( \mathbb{Q} \): more precisely, \( H^1(X_0) = W_1(H^1(X_t)) = H^1(\bar{\pi}^{-1}(q)) \) as \( \mathbb{Q} \)-MHS. It remains to show that

\[
0 \to H^1(X) \to H^1(X_t) \xrightarrow{N} H^1(X_t)
\]

is exact over \( \mathbb{Z} \). For \( n \gg 0 \) the zero locus \( Y \subset X \) of a general element of \( \Gamma(X, \mathcal{L}^n) \) is a semi-stable degeneration of the smooth curve \( Y_t = Y \cap X_t \). By the Picard-Lefschetz theorem [4, theorem III. 14.1]

\[
0 \to H^1(Y) \to H^1(Y_t) \xrightarrow{N} H^1(Y_t)
\]
is exact over $\mathbb{Z}$. By the Lefschetz hyperplane theorem $H^1(X_t) \rightarrow H^1(Y_t)$ is injective, hence so is $H^1(X) \rightarrow H^1(Y)$. We have now reduced our problem to showing that $H^1(Y)/H^1(X)$ is torsion free. Since $H^1(Y)/H^1(X) \hookrightarrow H^2(X,Y)$ it is sufficient to show that $H^2(X,Y)$ or, equivalently $H_1(X,Y) = \text{Coker} (H_1(Y) \rightarrow H_1(X))$ is torsion free. Since $\dim \ker N = 3$ the central fibre $X_0$ is not smooth abelian. Hence by Persson’s theorem $X_0$ is a cycle of elliptic ruled surfaces. Assume that $X_0$ has at least two components. (The remaining case is analogous.) Let $C$ be one of the double curves and $X_i$ one of the components of $X_0$. Then

$$H_1(X) = H_1(X_0) \cong H_1(C) \oplus \mathbb{Z}c$$

where $c \in H_1(X_0)$ is as in 3.3. Let $Y_i = X_i \cap Y$. Then $X_i$ and $Y_i$ are smooth and irreducible. We have a commutative diagram

$$\begin{array}{ccc}
H_1(C) & = & H_1(X_i) \\
\uparrow & & \uparrow \\
H_1(Y_i) & \rightarrow & H_1(Y).
\end{array}$$

Since $c$ is clearly induced from $Y$ up to $H_1(C)$, it is sufficient to prove that

$$H_1(Y_i) \rightarrow H_1(X_i)$$

is surjective. But this follows from the Lefschetz hyperplane theorem since $Y_i$ is ample on $X_i$. This shows that $H^1(X_0) = W_1(H^1(X_t))$ depends only on the point $[q]$.

**Theorem 26** Let $X_0$ be a degeneration associated to a corank 1 boundary point $[q]$ of $\mathcal{A}^*(1,p)$ with only local normal crossing singularities and no triple points. Then $X_0$ is a cycle of ruled surfaces (possibly with only one component) over an elliptic curve $C$. There exists a $\mathcal{L} \in \text{Pic}^0(C)$ such that all components are isomorphic to $\mathbb{P}^1(\mathcal{O} \oplus \mathcal{L})$. If $q$ is a central boundary point then $X_0$ is completely determined by $q$. If $q$ is peripheral, then the number $N$ of components, the base curve $C$ and the shift $s$ are uniquely determined by the point $[q]$. The line bundle $\mathcal{L} \in \text{Pic}^0(C)$ determining the elliptic ruled surfaces of the cycle is determined (at least) up to a $p$-torsion point of $\text{Pic}^0(C)$. More precisely:
1. If \([q] = [\tau_2, \tau_3]\) is a central boundary point, then \(N = 1\), \(C = X_{\tau_3, p}\), \(s = [\tau_2]\) and \(L = \mathcal{O}(s - O)\).

2. If \([q] = [\tau_1, \tau_2]\) is a peripheral boundary point, then \(N = p\), \(C = X_{\tau_1, 1}\), \(s = [\tau_2]\) and there exists a point \(s' \in C\) such that \(L = \mathcal{O}(s' - O)\).

Proof. We first note that the pair \((X_{\tau_3, p}, [\tau_2])\), resp. \((X_{\tau_1, 1}, [\tau_2])\) only depends on the point \([q]\) but not on its representative \([10, \text{part I.3}]\).

If \(X_0\) has only global normal crossings then Persson’s result \([13, \text{proposition 3.3.1}]\) says that \(X_0\) is a cycle of elliptic ruled surfaces.

The previous lemma asserts that the MHS on \(H^1(X_0)\) is determined by \(q\). Furthermore, this MHS is the weight 1 part of the limit MHS on \(H^1(X_t)\) of the family \(p_1\) (resp. \(p_2\)) defined in \(\S 4.4\) in the central (resp. peripheral) case. Indeed, \(p_1\) (resp. \(p_2\)) is the restriction of the universal family to a small disk \(\Delta \times (\tau_2, \tau_3)\) (resp. \((\tau_1, \tau_2) \times \Delta\)). By lemma 8 the base curve \(C\) is determined by the equation \(H^1(C) = \text{Gr}^W_1(H^1(X_0)) = \text{Gr}^W_1(H^1(X_t))\) and this equation was solved in proposition 16. Corollary 13 shows how to recover the shift \(s\) from \(H^1(X_0) = W_1(H^1(X_t))\) and the result for the families \(p_1\) and \(p_2\) is given in proposition 17.

Applying proposition 18 of \(\S 4.7\) to the family \(X\) defining \(X_0\) and to the family constructed in \([10]\), one sees that the number of components of \(X_0\) is independent of the choice of the disk \(\Delta \hookrightarrow X(l)\) (cf. proof of lemma 25). But it follows from our calculations in \(\S 4\) that the polarization induces \(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\) (with respect to the basis \(\{e_2, e_4\} \mod e_1\) of \(\S 4.3\)) in the central case and \(\begin{pmatrix} 0 & -p \\ p & 0 \end{pmatrix}\) (with respect to the basis \(\{e_1, e_3\} \mod e_2\)) in the peripheral case. It follows from proposition 18 of \(\S 4.7\) that the number of components is 1 and \(p\) respectively.

The statement about the form of \(L \in \text{Pic}^0(C)\) now follows directly from proposition 6 of \(\S 2.2\). \(\Box\)

Remark 27 Note that in the proof of theorem 26 the level structure is not needed to determine \(C\) and \(s\).

Remark 28 It seems reasonable to expect that also in the peripheral case \(L\) is completely determined by the boundary point \(q\). To show this in the
spirit of this paper, one probably has to translate the notion of (canonical) level structure to Hodge structures.

**Remark 29** This result is compatible with [10]. Notice that we only used the existence of the degeneration $X_D$ as constructed there in order to prove our uniqueness result, but not the precise description of its singular fibres.

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