Strongly Goldie Dimension

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Abstract. Let $R$ be an associative ring with identity. A unital right $R$-module $M$ is called strongly finite dimensional if $\operatorname{Sup}\{\operatorname{G.dim}(M/N) \mid N \leq M\} < +\infty$. Properties of strongly finite dimensional modules are explored. It is also proved that: (1) If $R$ is left $F$-injective and strongly right finite dimensional, then $R$ is left finite dimensional. (2) If $R$ is right $F$-injective, then $R$ is right finite dimensional if and only if $R$ is semilocal. Thus the Faith-Menal conjecture is true if $R$ is strongly right finite dimensional. Some known results are obtained as corollaries.

1. Introduction

Throughout this paper rings are associative with identity and all modules are unital right $R$-modules. For a subset $X$ of a ring $R$, the left annihilator of $X$ in $R$ is $I(X) = \{r \in R : rx = 0 \text{ for all } x \in X\}$. For any $a \in R$, we write $I(a)$ for $I(\{a\})$. Right annihilators are defined analogously. $N \leq_{\text{ess}} M$ means $N$ is an essential submodule of $M$. Let $\mathcal{A}$ be a family of modules, we use $|\mathcal{A}|$ to denote the cardinality of $\mathcal{A}$.

An $R$-module $M$ has Goldie dimension $n$ (written $\operatorname{G.dim} M = n$) if there is an essential submodule $V \leq_{\text{ess}} M$ that is a direct sum of $n$ uniform submodules. If, on the other hand, no such integer $n$ exists, we write $\operatorname{G.dim} M = +\infty$. We call an $R$-module $M$ is finite dimensional if $\operatorname{G.dim} M < +\infty$. A ring $R$ is called right finite dimensional if it is finite dimensional as a right $R$-module. Left dimensional rings can be defined similarly. In this article, strongly Goldie dimension is introduced. An $R$-module $M$ is with strongly Goldie dimension $n$ (written $\operatorname{SG.dim} M = n$) if $\operatorname{Sup}\{\operatorname{G.dim}(M/N) \mid N \leq M\} = n$. Otherwise, $\operatorname{SG.dim} M = +\infty$. $M$ is called strongly finite dimensional if $\operatorname{SG.dim}$
Properties of strongly finite dimensional modules are explored. As applications, we show that the Faith-Menal conjecture is true if $R_R$ is strongly finite dimensional. The Faith-Menal conjecture was raised by Faith and Menal in [3]. It says that every strongly right Johns ring is QF. Recall that a ring $R$ is called QF if it is one-sided noetherian and one-sided self-injective. A ring $R$ is right Johns if $R$ is right noetherian and every right ideal is an annihilator. Right Johns rings were characterized by Johns in [4], but he used a false result of Kurshan [5, Theorem 3.3] to show that right Johns rings are right artinian. In [2], Faith and Menal gave a counter example to show that right Johns rings need not be right artinian. Later (see [3]) they defined strongly right Johns ring (the matrix ring $M_{n \times n}(R)$ is right Johns for all $n \geq 1$) and characterized such rings as right noetherian and left $FP$-injective rings. But they didn’t know whether a strongly right Johns ring is QF.

2. Characterizations of strongly finite dimensional modules

Definition 2.1. An $R$-module $M$ is with strongly Goldie dimension $n$ (written $\text{SG.dim } M = n$) if $\text{Sup}\{\text{G.dim}(M/N) \mid N \leq M\} = n$. Otherwise, $\text{SG.dim } M = +\infty$. $M$ is called strongly finite dimensional if $\text{SG.dim } M < +\infty$. A ring $R$ is called strongly right finite dimensional if it is strongly finite dimensional as a right $R$-module. Strongly left finite dimensional rings are defined similarly.

Example 2.2. If a ring $R$ is strongly right finite dimensional, then it is right finite dimensional. But the converse is not true, even if $R$ is a commutative noetherian ring.

Proof. For example, let $R = \mathbb{Z}$, then $R$ is a commutative noetherian ring. But it is not strongly finite dimensional. For every ideal of $\mathbb{Z}$ is of the form $n\mathbb{Z}$ where $n = p_1^{m_1} \cdots p_k^{m_k}$ is a product of powers of prime numbers. Write $\mathbb{Z}_l = \mathbb{Z}/l\mathbb{Z}, l$ is a positive integer. Then it is well known that $\mathbb{Z}_n \simeq \mathbb{Z}_{p_1^{m_1}} \times \cdots \times \mathbb{Z}_{p_k^{m_k}}$, which implies that $\text{G.dim } \mathbb{Z}_n \geq k$. Since $n$ is arbitrary, $\text{SG.dim } \mathbb{Z} = +\infty$. □
Proposition 2.3. If $N$ is a quotient module of $M$, then $SG.\dim N \leq SG.\dim M$.

Proof. By definition. \qed

Lemma 2.4. (See [6, Theorem 6.37]) Let $0 \to A \to B \to C \to 0$ be an exact sequence of modules. Then $G.\dim B \leq G.\dim A + G.\dim C$.

Proposition 2.5. Let $0 \to A \to B \to C \to 0$ be an exact sequence of modules. Then $SG.\dim B \leq SG.\dim A + SG.\dim C$.

Proof. For any $K \leq B$, by the above lemma, $G.\dim \frac{B}{K} \leq G.\dim \frac{A+K}{K} + G.\dim \frac{B}{A+K} \leq SG.\dim A + SG.\dim \frac{B}{A} = SG.\dim A + SG.\dim C$. Since $K$ is arbitrary, we have $SG.\dim B \leq SG.\dim A + SG.\dim C$. \qed

Corollary 2.6. $SG.\dim (A+B) \leq SG.\dim A + SG.\dim B$

Proof. Since $0 \to A \to A+B \to B/A \cap B \to 0$ is an exact sequence, by the above proposition, $SG.\dim (A+B) \leq SG.\dim A + SG.\dim \frac{B}{A \cap B} \leq SG.\dim A + SG.\dim B$. \qed

Corollary 2.7. $SG.\dim (M_1 \oplus M_2 \oplus \cdots \oplus M_n) = SG.\dim M_1 + SG.\dim M_2 + \cdots + SG.\dim M_n$.

Proof. We only prove for $n = 2$, the others are similarly. For any quotient module $K_i$ of $M_i$, $i=1,2$. It is clear that $K_1 \oplus K_2$ is a quotient module of $M_1 \oplus M_2$. Then by definition, $SG.\dim M_1 + SG.\dim M_2 \leq SG.\dim (M_1 \oplus M_2)$. On the other side, $SG.\dim M_1 + SG.\dim M_2 \geq SG.\dim (M_1 \oplus M_2)$ by the above corollary. \qed

A nonzero module is called uniform if every nonzero submodule is an essential submodule. A module is called uniserial if its submodules are linearly ordered by inclusion. It is obvious that every uniserial module is uniform. But the converse is not true. For example, consider $\mathbb{Z}$ as a $\mathbb{Z}$-module, then $\mathbb{Z}$ is uniform but not uniserial. A ring $R$ is called right serial if $R_R$ is a direct sum of uniserial modules.
Corollary 2.8. If $R$ is right serial, then $R$ is strongly right finite dimensional and $\text{SG.dim } R_R = \text{G.dim } R_R$.

Remark 2.9. Since there are right artinian rings which are not right serial, strongly finite dimensional rings may not be right serial.

It is obvious that $\text{SG.dim } M \geq 1$ for every nonzero module $M$.

If $\text{SG.dim } M = 1$, we have

Theorem 2.10. The following are equivalent for an $R$-module $M$:

1. $\text{SG.dim } M = 1$.
2. $M$ is uniserial.
3. Every nonzero quotient module of $M$ is uniform.

Proof. It is obvious that every quotient module of a uniserial module is also uniserial. So (2) $\Rightarrow$ (3) $\Rightarrow$ (1). Now we assume (1), if $M$ is not uniserial, then there exists two different submodules $A$ and $B$ of $M$, neither $A \subset B$ nor $B \subset A$. Then it is clear that $\frac{A}{A \cap B}$ and $\frac{B}{A \cap B}$ are nonzero submodules of $\frac{M}{A \cap B}$. Hence there is an inclusion map: $\frac{A}{A \cap B} \oplus \frac{B}{A \cap B} \hookrightarrow \frac{M}{A \cap B}$, which implies $\text{SG.dim } M \geq 2$, a contradiction. \hfill $\square$

Example 2.11. A strongly finite dimensional module $M$ may not be noetherian, even if $\text{SG.dim } M = 1$.

Proof. For example, let $p$ be a positive prime number. Then $M = \{a/p^n \in \mathbb{Q} \mid a \in \mathbb{Z} \text{ and } n \in \mathbb{N}\}$ is an additive subgroup of $\mathbb{Q}$ with subgroup $\mathbb{Z}$. Denote the factor group $M/\mathbb{Z}$ by $\mathbb{Z}_{p^\infty}$. It is clear that every subgroup of $\mathbb{Z}_{p^\infty}$ is cyclic and spanned by $1/p^n$ for some $n$, which implies that $\mathbb{Z}_{p^\infty}$ is a uniserial but not noetherian $\mathbb{Z}$-module. \hfill $\square$

Example 2.12. A strongly finite dimensional module $M$ may not be artinian, even if $\text{SG.dim } M = 1$. For example (see [3, Example 2.12]), let $\mathbb{Z}[i]$ be the ring of Gaussian’s integers, $V = \mathbb{Z}[i]_{(2-i)}$ be its localization at the prime ideal generated by $2 - i$, $\sigma$ be the complex conjugation and $R$ be the skew power
Theorem 2.13. The following are equivalent for an $R$-module $M$:

1. $\text{SG.dim} M = n$.
2. $\text{Sup \{\lvert A \rvert \mid A \text{ is a coindependent family of } N, \text{ where } N \text{ is any submodule of a quotient module of } M\} = n}$

Proof. (1)$\implies$ (2). Assume there exists a nonzero submodule $K$ of a quotient module of $M$ and a coindexdependent family $\{K_i, 1 \leq i \leq m\}$ of $K$ such that $m \geq n+1$. Then we have a canonical inclusion $f : \bigoplus_{i=1}^{m} K_i \hookrightarrow K_1 \oplus \cdots \oplus K_m$ with $f(k + \bigcap_{i=1}^{m} K_i) = (k+K_1, \cdots, k+K_m), \forall k \in K$. Now we set $N_i = \bigcap_{j \neq i} K_j, 1 \leq i \leq m$. Then by the definition of coindexdependent family, $K_i + N_i = K, 1 \leq i \leq m$. Thus for each $i$, $f(N_i)$ is not zero and $f(N_i) = (0, \cdots, x+K_i, \cdots, 0)$, $x \in N_i$. Hence $f(N_1) + \cdots + f(N_m)$ is a direct sum. Since $f$ is monic, $\text{G.dim} \left( \bigoplus_{i=1}^{m} K_i \right) \geq m$, which shows that $\text{G.dim} \left( \bigoplus_{i=1}^{m} K_i \right) \geq m > n$, a contradiction.

(2)$\implies$ (1). If $\text{SG.dim} M > n$, then there exists a nonzero quotient module $N$ of $M$ with a family $\{K_i, 1 \leq i \leq m > n\}$ of independent submodules of $N$. Now take $N_i = \sum_{k \neq i} K_i, 1 \leq i \leq m$. It is clear that $\{N_i, 1 \leq i \leq m\}$ is a coindexdependent family of $\sum K_i$, which is a contradiction. So $\text{SG.dim} M \leq n$. By hypothesis, there exists a nonzero module $A$, which is a submodule of a quotient module $B$ of $M$. And $A$ has a coindependent family $\{A_i, 1 \leq i \leq n\}$ of proper submodules of $B$. Then by the construction in the proof of (1)$\implies$ (2), $\text{G.dim} \left( \bigoplus_{i=1}^{m} A_i \right) \geq n$. So $\text{SG.dim} M \geq \text{G.dim} \left( \bigoplus_{i=1}^{m} A_i \right) \geq \text{G.dim} \left( \bigoplus_{i=1}^{m} A_i \right) \geq n$. Hence $\text{SG.dim} M = n$. □
A module $M$ is said to be of finite length (see [1]) if there exists a finite chain of submodules of $M$: $M = M_0 > M_1 > \cdots > M_n = 0$ where $M_{i-1}/M_i$ is simple $(i = 1, 2, \ldots, n)$. Write $c(M) = n$ for the composition length of $M$.

Lemma 2.14. [The Schreier Refinement Theorem]
If $M$ is a module of finite length and if $M = N_0 > N_1 > \cdots > N_p = 0$ is a chain of submodules of $M$, then there is a composition series for $M$ whose terms include $N_0, N_1, \ldots, N_p = 0$.

Lemma 2.15. [Corollary 6.7 (2)] If $M$ is a module of finite length, then $G.dim \, M \leq c(M)$. $G.dim \, M = c(M) < +\infty$ if and only if $M$ is semisimple.

Proposition 2.16. If $M$ is a module of finite length, then $SG.dim \, M \leq c(M)$. $SG.dim \, M = c(M) < +\infty$ if and only if $M$ is semisimple.

Proof. For any quotient module $N$ of $M$, $c(N) \leq c(M)$ by Lemma 2.14. Thus by Lemma 2.15, $SG.dim \, M \leq c(M)$. If $M$ is a semisimple module of finite length, then it is easy to get that $SG.dim \, M = c(M) < +\infty$. On the converse, if $SG.dim \, M = c(M) < +\infty$, then there exists a quotient module $N_0$ of $M$ such that $G.dim \, N_0 = SG.dim \, M = c(M)$. But by Lemma 2.14, $c(N_0) < c(M)$ if $N_0$ is not isomorphic to $M$. Thus $N_0$ must be isomorphic to $M$ and $G.dim \, N_0 = c(N_0)$. Hence $M \simeq N_0$ is semisimple by Lemma 2.15.

A ring $R$ is called right semiartinian if every right $R$-module has an essential right socle.

Theorem 2.17. A ring $R$ is right artinian if and only if $R$ is right semiartinian and strongly right finite dimensional.

Proof. If $R$ is right artinian, then it is right semiartinian. Since any right artinian ring is right noetherian, $R_R$ is of finite length by [1, Proposition 11.1]. Thus by the above proposition, $R$ is strongly finite dimensional. On the other hand, if $R$ is right semiartinian and strongly right finite dimensional, then every cyclic right $R$-module is finitely cogenerated. So $R$ is right artinian by Vámos Lemma (see [8, Lemma 1.52]).
Remark 2.18. From Example 2.12, we see that right semiartinian condition is necessary in the above theorem.

Example 2.19. There exists strongly right finite dimensional rings which are not strongly left finite dimensional. Since every one sided artinian ring is left and right semiartinian, by Theorem 2.15, we only need to find right artinian rings which are not left artinian.

3. Applications

Recall that a ring $R$ is called right $F$-injective if, for every $R$-homomorphism from a finitely generated right ideal to $R_R$ can be extended from $R_R$ to $R_R$. $R$ is called right $FP$-injective if, for any free right $R$-module $F$ and any finitely generated $R$-submodule $N$ of $F$, every $R$-homomorphism $f : N \to R$ can be extended to an $R$-homomorphism $g : F \to R$. It is obvious that right $FP$-injective rings are right $F$-injective. But it is still unknown whether a right $F$-injective ring is right $FP$-injective. Left $F$-injective and left $FP$-injective rings can be defined similarly.

Lemma 3.1. (see [8, Lemma 1.37]) A ring $R$ is left $F$-injective if and only if it satisfies the following conditions:

1. $r(T \cap T') = r(T) + r(T')$ for all finitely generated left ideals $T$ and $T'$ of $R$.
2. $rl(a) = aR$ for all $a \in R$.

Theorem 3.2. If $R$ is left $F$-injective and strongly right finite dimensional, then $R$ is left finite dimensional.

Proof. If $R$ is not left finite dimensional, then there exist $a_i \in R, i = 1, 2, 3, \ldots$, such that $\sum_{i=1}^{\infty} Ra_i$ is a direct sum. Since $R$ is left $F$-injective, by the above lemma, for any integers $m$ and any finite subset $I \subset \mathbb{N} \setminus m$, $R = r(0) = r(Ra_m \cap \sum_{i \in I} Ra_i) = r(a_m) + r(\sum_{i \in I} Ra_i) = r(a_m) + \cap_{i \in I} r(a_i)$. It is also clear $r(i) \neq r(j), i \neq j$. Thus $\{r(a_i), i = 1, 2, 3, \ldots\}$ is a coindependent family of $R_R$. So by Theorem 2.13, $\text{SG.dim}(R_R) = +\infty$, a contradiction. \(\Box\)
Theorem 3.3. The Faith-Menal conjecture is true if $R$ is strongly right finite dimensional.

Proof. Since $R$ is strongly right Johns, $R$ is left $FP$-injective. So $R$ is left finite dimensional by the above theorem. Thus $R$ is $QF$ by [3, Corollary 1.3]. □

Theorem 3.4. If $R$ is right $F$-injective, then $R$ is semilocal if and only if $R$ is right finite dimensional.

Proof. If $R$ is semilocal, then by [7, Theorem 1.3, Corollary 3.2], $R R$ does not contain an infinite coindependent family of submodules. Then from the proof in Theorem 3.2, we have that $R$ is right dimensional. On the converse, since $R$ is right $F$-injective, $R$ is right $C2$ (every right ideal is essential in a direct summand of $R R$) by [8, Proposition 5.10]. Thus $R$ is semilocal by [8, Corollary C.3]. □

Corollary 3.5. If $R$ is right $FP$-injective or right self-injective, then $R$ is semilocal if and only if $R$ is right finite dimensional.

Corollary 3.6. (see [8, Theorem 5.56]) The following are equivalent for a ring $R$:

1. $R$ is semilocal, right $FP$-injective, and right Kasch.
2. $R$ is right finite dimensional, right $FP$-injective, and right Kasch.

A ring $R$ is called right $PF$ if $R$ is a semilocal and right self-injective ring with an essential right socle.

Corollary 3.7. $R$ is right $PF$ if and only if $R$ is a right finite dimensional and right self-injective ring with an essential right socle.

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