On Interval-Valued Intuitionistic Fuzzy $B$-Subalgebras of $B$-Algebra

Narciso C. Gonzaga Jr$^*$ and Jocelyn P. Vilela

Department of Mathematics and Statistics, College of Science and Mathematics, Mindanao State University - Iligan Institute of Technology, Andres Bonifacio Ave., Tibanga, Iligan City - 9200, Philippines; narzie.math0725@gmail.com

Abstract

In this paper, we introduce and investigate the notion of interval-valued intuitionistic fuzzy $B$-subalgebra of a $B$-algebra. We further investigate some of the properties of interval-valued intuitionistic fuzzy $B$-subalgebra under a $B$-homomorphism.

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1. Introduction

The theory of fuzzy sets was first proposed by$^{14}$. His seminal work became a useful tool for describing the behavior of systems that are too complex or ambiguous to give precise mathematical analysis by classical tools and methods. Since then, fuzzy sets has become a fertile area of research in engineering, physics, medical and social sciences, statistics, etc.$^{12}$.

After the introduction of fuzzy sets, there have been a number of extensions of this fundamental concept. Among them are the Intuitionistic Fuzzy Sets (IFSs) which were introduced by Atanassov$^{1,2}$, and Interval-Valued Fuzzy sets (IVFSs) which were introduced by$^{15}$. It should be noted that, due to$^3$, IFSs and IVFSs are equipollent generalizations of fuzzy sets. Moreover, they introduced the concept of Interval-Valued Intuitionistic Fuzzy Sets (IVIFs) as a generalization of both IFSs and IVFSs.

In$^8$ introduced two classes of algebras of logic, namely, $BCI$- and $BCK$-algebras. It is known that the class of $BCI$-algebra is a proper subclass of the class of $BCK$-algebra. Since the introduction of $BCI/BCK$-algebras, many researches were conducted to give light to their generalizations. In$^5$ introduced the notion of $BCH$-algebras and showed that the class of $BCI$-algebra is a proper subclass of the class of $BCH$-algebra. In$^7$ introduced the concept of $BH$-algebra as a generalization of $BCI/BCK/BCH$-algebras. Meanwhile, In$^9$ introduced and investigated a class of algebra – $B$-algebra – which is related to some classes of algebras such as $BCI/BCK/BCH$-algebra, and which seems to have rather profound properties without otherwise being complicated.

Recently, there are numerous researches that present the applications of fuzzy sets to abstract algebra. In particular, the applications of fuzzy sets to $B$-algebra are being studied. In$^8$ introduced the concept of fuzzy $B$-algebra and some related properties were investigated. In$^10$ meanwhile applied the concept of fuzzy sets to ideals and closed ideals of $B$-algebra. On the other hand, the applications of the extensions of fuzzy sets are also being considered. In$^{11}$ introduced the notion of Interval-Valued (IV) $B$-subalgebra of a $B$-algebra. Also, In$^{10}$ investigated the concept of Intuitionistic Fuzzy (IF) $B$-algebra and some of its properties are explored.

In this paper, the concept of Interval-Valued Intuitionistic Fuzzy (IVIF) $B$-subalgebra of a $B$-algebra is introduced. In Section 2, we present the preliminaries that are essential to this paper. In Section 3, we define the IVIF $B$-subalgebra of a $B$-algebra and discuss some of its properties. Furthermore in Section 4, some properties of IVIF $B$-subalgebra under a $B$-homomorphism are investigated. Finally in Section 5, a conclusion of the proposed work is given.

*Author for correspondence
2. Preliminaries

In this section, we recall some definitions and results that will be used in this paper.

**Definition 2.1** In a B-algebra is an ordered triple \((X, *, 0)\) where X is a nonempty set with a constant "0" and a binary operation "*" satisfying the following axioms: for all \(x, y, z \in X\).
1. \(x * x = 0\).
2. \(x * 0 = x\).
3. \((x * y) * z = x * (z * (0 * y))\).

**Definition 2.2** In a nonempty subset \(Y\) of a B-algebra \(X\) is called a B-subalgebra of \(X\) if \(x * y \in Y\) for all \(x, y \in Y\).

**Lemma 2.3** In \(\mathbb{R}\) let \(X\) be a B-algebra. Then \(x * y = x * (0 * (0 * y))\) for all \(x, y \in X\).

**Definition 2.4** In \(\mathbb{R}\) let \(X\) and \(Y\) be B-algebras and \(f : X \to Y\) a mapping. We say that \(f\) is a B-homomorphism if \(f(x * y) = f(x) * f(y)\) for all \(x, y \in X\).

Observe that if \(f\) is a B-homomorphism, then \(f(0)_X = 0_Y\), where \(0_X\) and \(0_Y\) are the constant element of \(X\) and \(Y\), respectively.

**Definition 2.5** In \(\mathbb{R}\) let \(X\) be a nonempty set. A fuzzy set \(A\) on \(X\) is characterized by a membership function \(\mu_A : X \to [0, 1]\). Alternatively, a fuzzy set is an object \(A = \{(x, \mu_A(x)) : x \in X\}\) (1)

where \(0 \leq \mu_A(x) \leq 1\) denote the membership value of \(x \in X\) in \(A\).

**Definition 2.6** In \(\mathbb{R}\) let \(A\) be a nonempty set. An Intuitionistic Fuzzy Set (IFS) \(A\) on \(X\) is a fuzzy set equipped with a non-membership function \(\gamma_A : X \to [0, 1]\), that is, it is an object \(A = (\mu_A, \gamma_A) = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}\) (2)

such that \(0 \leq \mu_A(x) + \gamma_A(x) \leq 1\) for all \(x \in X\).

**Definition 2.7** In \(\mathbb{R}\) an IFS \(A = (\mu_A, \gamma_A)\) over a B-algebra \(X\) is an Intuitionistic Fuzzy (IF) B-algebra on \(X\) if for all \(x, y \in X\),
1. \(\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}\),
2. \(\gamma_A(x * y) \leq \max\{\gamma_A(x), \gamma_A(y)\}\).

**Definition 2.8** In \(\mathbb{R}\) by an interval number \(D\) on \([0, 1]\), we mean a closed subinterval \([a, a']\) of \([0, 1]\). We denote the collection of all interval numbers on \([0, 1]\) as \(D[0, 1]\).

**Definition 2.9** In \(\mathbb{R}\) let \(D_1 = [a^-_1, a^+_1]\) and \(D_2 = [a^-_2, a^+_2]\) be interval numbers.
1. We say that \(D_1 \leq D_2\) if and only if \(a^-_1 \leq a^-_2\) and \(a^+_1 \leq a^+_2\) (the same result holds if we interchange \(\leq\) by \(\geq\) or \(\sim\)).
2. \(\min\{D_1, D_2\} = \left[\min\{a^-_1, a^-_2\}, \min\{a^+_1, a^+_2\}\right]\).
3. \(\max\{D_1, D_2\} = \left[\max\{a^-_1, a^-_2\}, \max\{a^+_1, a^+_2\}\right]\).

Let \(\{D_i = [a^-_i, a^+_i] : i \in I\} \subseteq D[0, 1]\). Then
4. \(\text{rsup}_{i \in I} D_i = \left[\sup_{i \in I} (a^-_i), \sup_{i \in I} (a^+_i)\right]\).
5. \(\text{rinf}_{i \in I} D_i = \left[\inf_{i \in I} (a^-_i), \inf_{i \in I} (a^+_i)\right]\).

**Definition 2.10** In \(\mathbb{R}\) let \(X\) be a nonempty set. An Interval-Valued Intuitionistic Fuzzy set (IVIFS) \(A\) on \(X\) is characterized by its interval-valued degree of membership for each element of \(X\), that is, it is an object of the form \(A = \{(x, [\mu^-_A(x), \mu^+_A(x)]) : x \in X\}\) (3)

where \(\mu^-_A\) and \(\mu^+_A\) are membership functions for \(X\) with \(\mu^-_A \leq \mu^+_A\).

For each \(x \in X\), let \(\widetilde{\mu}_A(x) = [\mu^-_A(x), \mu^+_A(x)]\). Clearly, \(\widetilde{\mu}_A(x) \in D[0, 1]\) whenever \(\mu^+_A(x) \leq \mu^-_A(x)\). Also, it is clear that if \(\mu^+_A(x) = k = \mu^-_A(x)\), then \(\widetilde{\mu}_A(x) = [k, k] \in D[0, 1]\). Thus, it follows that \(\widetilde{\mu}_A(x) \in D[0, 1]\) for all \(x \in X\). Hence, the IVIFS \(A\) can be written as \(A = \{(x, \widetilde{\mu}_A(x)) : x \in X\}\) (4)

where \(\widetilde{\mu}_A : X \to [0, 1]\) denotes the interval-valued membership function for \(X\) in \(A\).

**Definition 2.11** In \(\mathbb{R}\) let \(X\) be a nonempty set. An Interval-Valued Intuitionistic Fuzzy set (IVIFS) \(A\) on \(X\) is an object \(A = \{(x, [\mu^-_A(x), \mu^+_A(x)], [\gamma^-_A(x), \gamma^+_A(x)]) : x \in X\}\) (5)

such that \(\mu^-_A, \mu^+_A\) and \(\gamma^-_A, \gamma^+_A\) are membership and
non-membership functions for $X$ where $\mu_i^x \leq \mu_i^x$ and $\gamma_i^x \leq \gamma_i^x$ with $0 \leq \mu_i^x(x) + \gamma_i^x(x) \leq 1$ for all $x \in X$.

Similar to the case of Definition 2.10, if $\mu_i(x) = [\mu_i^x(x), \mu_i^x(x)]$ and $\gamma_i(x) = [\gamma_i^x(x), \gamma_i^x(x)]$ for each $x \in X$, then $\mu_i(x), \gamma_i(x) \in D[0,1]$ for all $x \in X$. Therefore we may write the IVIFS $A$ as

$$A = (\overline{\mu_i}, \overline{\gamma_i}) = \{ (x, \mu_i(x), \gamma_i(x)) : x \in X \}$$

where $\overline{\mu_i} : X \rightarrow D[0,1]$ and $\overline{\gamma_i} : X \rightarrow D[0,1]$ are the interval-valued membership and non-membership functions for $X$ in $A$, respectively.

**Definition 2.12** In let $A_i = (\overline{\mu_i}, \overline{\gamma_i})$ and $A_2 = (\overline{\mu_2}, \overline{\gamma_2})$ be IVIFSs on a nonempty set $X$. The intersection of $A_1$ and $A_2$ is the IVIFS $A_1 \cap A_2 = (\overline{\mu_{1 \cap 2}}, \overline{\gamma_{1 \cap 2}})$ where for all $x \in X$,

$$\overline{\mu_{1 \cap 2}}(x) = \min \{ \mu_1^x(x), \mu_2^x(x) \},$$

and

$$\overline{\gamma_{1 \cap 2}}(x) = \max \{ \gamma_1^x(x), \gamma_2^x(x) \}.$$  

**Definition 2.13** In the necessity and possibility operators, denoted respectively by “□” and “◊”, are the IVIFS $\square A = (\overline{\mu_i}, \overline{\gamma_i})$ and $\Diamond A = (\overline{\gamma_i}, \overline{\gamma_i})$, respectively with

$$\mu_i^x = [1 - \mu_i^x(x), 1 - \mu_i^x(x)],$$

and

$$\gamma_i^x = [1 - \gamma_i^x(x), 1 - \gamma_i^x(x)]$$

for all $x \in X$.

**Definition 2.14** In let $f : X \rightarrow Y$ be a mapping and $B = (\overline{\mu_b}, \overline{\gamma_b})$ an IVIFS on $A$. The preimage of $B$ under $f$ is defined as

$$f^{-1}(B) = \{ (x, f^{-1}(\overline{\mu_b})(x), f^{-1}(\overline{\gamma_b})(x)) : x \in X \}.$$

**Definition 2.15** In let $f : X \rightarrow Y$ be a mapping and $A = (\overline{\mu_i}, \overline{\gamma_i})$ an IVIFS on $X$. The image of $A$ under $f$ is defined as

$$f(A) = \{ (y, f_{\sup}(\overline{\mu_i})(y), f_{\inf}(\overline{\gamma_i})(y)) : y \in Y \}.$$
\[ \mu_A(1*3) = \overline{\mu_A}(1) \geq [0.3, 0.9] \\
> [0.1, 0.6] \\
= \min(0.1, 0.1), \min(0.6, 0.6) \\
= \min(\mu_A(1), \gamma_A(3)) \]

and
\[ \gamma_A(1*3) = \overline{\gamma_A}(1) = [0.07, 0.085] \\
< [0.08, 0.2] \\
= \max(0.08, 0.08), \max(0.2, 0.2) \\
= \max(\gamma_A(1), \gamma_A(3)). \]

Furthermore, it follows that for all \( x, y \in X \), \( \mu_A(x*y) \geq \min(\mu_A(x), \gamma_A(y)) \) and \( \gamma_A(x*y) \leq \max(\gamma_A(x), \gamma_A(y)) \)

Hence, \( A \) is an IVIF \( B \)-subalgebra of \( X \).

**Lemma 3.3** Let \( A = (\overline{\mu_A}, \overline{\gamma_A}) \) be an IVIF \( B \)-subalgebra of \( X \). Then for all \( x, y \in X \), \( \overline{\mu_A}(0) \geq \mu_A(x) \) and \( \overline{\gamma_A}(0) \leq \gamma_A(x) \).

Proof. Let \( x \in X \). Then we have \( x*y = 0 \). Thus,
\[ \overline{\mu_A}(0) = \overline{\mu_A}(x*y) \geq \min(\mu_A(x), \gamma_A(x)) = \mu_A(x) \]
and
\[ \overline{\gamma_A}(0) = \overline{\gamma_A}(x*y) \leq \max(\gamma_A(x), \gamma_A(x)) = \gamma_A(x). \]

This completes the proof.

For any \( x, y \in X \), let \( \prod x*y \) denotes the expression \( x*(...*(x*y)) \where \( x \) occurred \( n \) times.

**Proposition 3.4** Let \( A = (\overline{\mu_A}, \overline{\gamma_A}) \) be an IVIF \( B \)-subalgebra of \( X \). Then for all \( x \in X \),
1. \( \overline{\mu_A}(\prod x*y) \geq \mu_A(x) \) and \( \overline{\gamma_A}(\prod x*y) \leq \gamma_A(x) \)
if \( n \) is odd,
2. \( \overline{\mu_A}(\prod x*y) = \mu_A(x) \) and \( \overline{\gamma_A}(\prod x*y) = \gamma_A(x) \)
if \( n \) is even.

Proof. Let \( x \in X \) and assume that \( n \) is odd. Then \( n = 2k - 1 \) for some positive integer \( k \). We proceed with induction on \( k \). If \( k = 1 \), then the inequalities hold by Lemma 3.3. So suppose
\[ \overline{\mu_A}(\prod x*y) \geq \mu_A(x) \] and \( \overline{\gamma_A}(\prod x*y) \leq \gamma_A(x) \)
for a positive integer \( k \). Then by (1) and (2) of Definition 2.1,
\[ \overline{\mu_A}(\prod_{k=1}^{2k-1} x*y) = \overline{\mu_A}(\prod_{k=1}^{2k-1} x*y) \]
\[ = \overline{\mu_A}(\prod_{k=1}^{2k-1} x*y) \]
\[ = \overline{\mu_A}(\prod_{k=1}^{2k-1} x*y) \]
\[ \geq \mu_A(x) \]
and
\[ \overline{\gamma_A}(\prod_{k=1}^{2k-1} x*y) = \overline{\gamma_A}(\prod_{k=1}^{2k-1} x*y) \]
\[ = \overline{\gamma_A}(\prod_{k=1}^{2k-1} x*y) \]
\[ \leq \gamma_A(x). \]

Hence, the inequalities hold for \( k+1 \). We can prove (2) similarly.

**Lemma 3.5** Let \( A = (\overline{\mu_A}, \overline{\gamma_A}) \) be an IVIF \( B \)-subalgebra of \( X \). Then \( \overline{\mu_A}(0) = [1,1] \) and \( \overline{\gamma_A}(0) = [0,0] \) if and only if there exists a sequence \( \langle x_n \rangle \) in \( X \) such that
\[ \lim_{n \to \infty} \mu_A(x_n) = [1,1] \] and \( \lim_{n \to \infty} \gamma_A(x_n) = [0,0] \).

Proof. We will first prove the sufficiency. By Lemma 3.3, we have \( \mu_A(0) \geq \mu_A(x_n) \) and \( \gamma_A(0) \leq \gamma_A(x_n) \) for all \( n \in \mathbb{N} \). This means that \( \mu_A(0) \geq \mu_A(x_n) \) and \( \gamma_A(0) \leq \gamma_A(x_n) \). Thus, we have
\[ [1,1] \geq \mu_A(0) \geq \mu_A(x_n) = [1,1] \] and \( [0,0] \leq \gamma_A(0) \leq \gamma_A(x_n) = [0,0] \).

Hence, \( \mu_A(0) = [1,1] \) and \( \gamma_A(0) = [0,0] \).

Conversely, suppose \( \mu_A(0) = [1,1] \) and \( \gamma_A(0) = [0,0] \). Consider the sequence \( \langle x_n \rangle = \langle 0,0,...,0,... \rangle \)
Then we get two sequences of interval numbers:
\[ \langle \mu_A(x) \rangle = \langle [1,1],[1,1],... \rangle \] and \( \langle \gamma_A(x) \rangle = \langle [0,0],[0,0],... \rangle \).

So, we have
\[ \lim_{n \to \infty} \mu_A(x_n) = \lim_{n \to \infty} [1,1] = [1,1] \] and \( \lim_{n \to \infty} \gamma_A(x_n) = \lim_{n \to \infty} [0,0] = [0,0] \)
and so the proof is complete.
Theorem 3.6 An IVIFS $A = (\mu_A, \nu_A)$ on $X$ is an IVIF $B$-subalgebra of $X$ if and only if for any $x, y \in X$,

1. $\mu_A(0 \ast x) \leq \mu_A(x)$ and $\nu_A(0 \ast y) \leq \nu_A(y)$,
2. $\mu_A(x \ast (0 \ast y)) \geq \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x \ast (0 \ast y)) \leq \max\{\nu_A(x), \nu_A(y)\}$.

Proof. Suppose $A$ is an IVIF $B$-subalgebra of $X$. Then we have

$$\mu_A(x \ast (0 \ast y)) \geq \min\{\mu_A(x), \mu_A(y)\}$$
and
$$\nu_A(x \ast (0 \ast y)) \leq \max\{\nu_A(x), \nu_A(y)\}.$$  

Also, it follows that

$$\mu_A(x \ast (0 \ast y)) \geq \min\{\mu_A(x), \mu_A(y)\} \geq \min\{\mu_A(0), \mu_A(0)\}$$
and
$$\nu_A(x \ast (0 \ast y)) \leq \max\{\nu_A(x), \nu_A(y)\} \leq \max\{\nu_A(0), \nu_A(0)\}.$$  

This completes the proof of the necessity.

Conversely, assume that (1) and (2) holds. Note that by Lemma 2.3, $x \ast y = x \ast (0 \ast (0 \ast y))$ for all $x, y \in X$. Thus,

$$\mu_A(x \ast y) = \mu_A(x \ast (0 \ast (0 \ast y)))$$
and
$$\nu_A(x \ast y) = \nu_A(x \ast (0 \ast (0 \ast y))).$$

Hence, $A$ is an IVIF $B$-subalgebra and the proof is complete.

Theorem 3.7 An IVIFS $A = (\mu_A, \nu_A)$ on $X$ is an IVIF $B$-subalgebra of $X$ if and only if $A' = (\mu_A', \nu_A')$ and $A'' = (\mu_A'', \nu_A'')$ are IF $B$-algebras on $X$.

Proof. Suppose $A'$ and $A''$ are IF $B$-algebras on $X$.

By Definition 2.7, we have

$$\mu_A'(x \ast y) \geq \min\{\mu_A(x), \mu_A(y)\} \geq \min\{\mu_A'(x), \mu_A'(y)\}$$
and
$$\nu_A'(x \ast y) \leq \max\{\nu_A(x), \nu_A(y)\} \leq \max\{\nu_A'(x), \nu_A'(y)\}.$$  

for all $x, y \in X$.

Thus,

$$\mu_A(x \ast y) = [\mu_A(x \ast y), \mu_A(x \ast y)]$$
and
$$\nu_A(x \ast y) = [\nu_A(x \ast y), \nu_A(x \ast y)]$$
are IF $B$-algebras on $X$.

Hence, $A$ is an IVIF $B$-subalgebra and the proof is complete.
by Definition 2.7, \( A^+ = (\mu_1^+, \gamma_1^+) \) and \( A^- = (\mu_1^-, \gamma_1^-) \) are IF-B-algebras on X.

**Theorem 3.8** Let \( A_1 = (\mu_{\gamma_1}, \gamma_{\gamma_1}) \) and \( A_2 = (\mu_{\gamma_2}, \gamma_{\gamma_2}) \) be IVIF B-subalgebras of X. Then \( A_1 \cap A_2 \) is an IVIF B-subalgebra of X.

**Proof.** Let \( x, y \in X \). Then by Theorem 3.7 and Definitions 2.7 and 2.12, we have

\[
\overline{\mu}_{A_1 \cap A_2}(x * y) = \min\{\mu_{A_1}(x * y), \mu_{A_2}(x * y)\},
\]

\[
\overline{\gamma}_{A_1 \cap A_2}(x * y) = \max\{\gamma_{A_1}(x * y), \gamma_{A_2}(x * y)\},
\]

and

\[
\overline{\mu}_{A_1 \cap A_2}(x * y) \leq \min\{\mu_{A_1}(x * y), \mu_{A_2}(x * y)\}.
\]

Therefore, \( A_1 \cap A_2 \) is an IVIF B-subalgebra of X.

**Corollary 3.9** Let \( \{A_i : i \in I\} \) be a family of IVIF B-subalgebras of X. Then \( \bigcap_{i \in I} A_i \) is an IVIF B-subalgebra of X.

**Proposition 3.10** Let \( A = (\overline{\mu}_A, \overline{\gamma}_A) \) be an IVIF B-subalgebra of X. Then \( \square_A \) and \( \Diamond_A \) are IVIF B-subalgebras of X.

**Proof.** It suffices to show that \( \overline{\gamma}_A(x * y) \geq r \min\{\overline{\gamma}_A(x), \overline{\gamma}_A(y)\} \) and

\[
\overline{\mu}_A(x * y) \leq r \max\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}
\]

for all \( x, y \in X \). Now by Theorem 3.7,

\[
\overline{\gamma}_A(x * y) = [1 - \gamma_1(x * y), 1 - \gamma_2(x * y)]
\]

\[
\overline{\mu}_A(x * y) = [\mu_1(x * y) + \mu_2(x * y), 1 - \mu_2(x * y)]
\]

and

\[
\overline{\mu}_A(x * y) = [1 - \mu(x * y), 1 - \mu(x * y)]
\]

\[
\overline{\gamma}_A(x * y) = [\gamma(x * y), \gamma_2(x * y)]
\]

Therefore, \( \square_A \) and \( \Diamond_A \) are IVIF B-subalgebras of X.

**Lemma 3.11** Let \( A = (\overline{\mu}_A, \overline{\gamma}_A) \) be an IVIF B-subalgebra of X. Then \( I_{\overline{\mu}_A} = \{x \in X : \overline{\mu}_A(x) = \overline{\mu}_A(0)\} \) and \( I_{\overline{\gamma}_A} = \{x \in X : \overline{\gamma}_A(x) = \overline{\gamma}_A(0)\} \) are B-subalgebras of X.

**Proof.** Clearly, \( 0 \in I_{\overline{\mu}_A} \) and \( 0 \in I_{\overline{\gamma}_A} \); hence, \( I_{\overline{\mu}_A} \neq \emptyset \) and \( I_{\overline{\gamma}_A} \neq \emptyset \). Let \( x, y \in I_{\overline{\mu}_A} \). Then \( \mu_A(x) = \mu_A(0) \) and \( \mu_A(x * y) = \mu_A(0) \). Thus, it follows that \( \overline{\mu}_A(x * y) \leq \max\{\overline{\mu}_A(x), \overline{\mu}_A(y)\} \) and \( \overline{\gamma}_A(x * y) = \overline{\gamma}_A(0) \) and \( \mu_A(x * y) = \mu_A(0) \).

**Theorem 3.12** Let \( Y \) be a nonempty subset of X. Suppose \( A = (\overline{\mu}_A, \overline{\gamma}_A) \) is an IVIFS on X defined by

\[
\overline{\mu}_A(x) = \begin{cases} [u^1_i, u^2_i] & \text{if } x \in Y, \\ [v^1_i, v^2_i] & \text{otherwise} \end{cases}
\]

and

\[
\overline{\gamma}_A(x) = \begin{cases} [u^1_i, u^2_i] & \text{if } x \in Y, \\ [v^1_i, v^2_i] & \text{otherwise} \end{cases}
\]

where \( [u^1_i, u^2_i], [v^1_i, v^2_i] \in D[0,1] \) for \( i = 1, 2 \) such that \( u^1_i, u^2_i \geq v^1_i, v^2_i \) and \( u^1_i + u^2_i \leq 1 \) and \( v^1_i + v^2_i \leq 1 \). Then A is an IVIF B-subalgebra of X and only if Y is a B-subalgebra of X. In particular, \( I_{\overline{\mu}_A} = Y = I_{\overline{\gamma}_A} \).

**Proof.** Let \( y_1, y_2 \in Y \) and assume that A is an IVIF B-subalgebra of X. Then \( \mu_A(y_1) = [u^1_i, u^2_i] = \mu_A(y_2) \), implying that

\[
[u^1_i, u^2_i] \geq \mu_A(y_1 * y_2), \quad \overline{\mu}_A(y_1 * y_2) = \overline{\mu}_A(y_1) \Rightarrow \overline{\mu}_A(y_1 * y_2) = [u^1_i, u^2_i].
\]

This means that \( \overline{\mu}_A(y_1 * y_2) = [u^1_i, u^2_i] \), that is, \( y_1 * y_2 \in Y \). Hence, Y is a B-subalgebra of X.
Conversely, assume that $Y$ is a $B$-subalgebra of $X$. Let $x, y \in X$. If $x, y \in Y$, then $x \ast y \in Y$ and so we have

$$\overline{\mu}_A(x \ast y) = [u^1, u^2] = \{u^1, u^2\}$$

and

$$\overline{\gamma}_A(x \ast y) = \{u^1, u^2\} = \{u^1, u^2\} = \max\{\overline{\gamma}_A(x), \overline{\gamma}_A(y)\}.$$ 

If $x \not\in Y$ or $y \not\in Y$, then either $x \ast y \in Y$ or $x \ast y \in Y$ (the latter is possible if $x = y$). Either of these, it follows that

$$\overline{\mu}_A(x \ast y) \geq [v^1, v^2] = \min\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}$$

and

$$\overline{\gamma}_A(x \ast y) \leq [v^1, v^2] = \max\{\overline{\gamma}_A(x), \overline{\gamma}_A(y)\}.$$ 

Hence in either of the cases, $A$ is an IVIF $B$-subalgebra of $X$.

For the second statement, observe that $0 \in Y$ since $Y$ is an $X$-subalgebra of $X$. Therefore,

$$I_{\overline{\mu}_A} = \{x \in X : \overline{\mu}_A(x) = \overline{\mu}_A(0)\} = \{x \in X : \overline{\mu}_A(x) = [u^1, u^2]\} = Y$$

and

$$I_{\overline{\gamma}_A} = \{x \in X : \overline{\gamma}_A(x) = \overline{\gamma}_A(0)\} = \{x \in X : \overline{\gamma}_A(x) = [u^1, u^2]\} = Y.$$ 

**Definition 3.13** Let $A = (\overline{\mu}_A, \overline{\gamma}_A)$ be an IVIFS on $X$ and $D_1, D_2 \in D[0, 1]$. The subsets $\overline{\mu}_A; D_1 = \{x \in X : \overline{\mu}_A(x) \geq D_1\}$ and $\overline{\gamma}_A; D_2 = \{x \in X : \overline{\gamma}_A(x) \leq D_2\}$ of $X$ are called the **upper $D_1$- and lower $D_2$-level sets** for $A$, respectively.

**Theorem 3.14** An IVIFS $A = (\overline{\mu}_A, \overline{\gamma}_A)$ on $X$ is an IVIF $B$-subalgebra of $X$ if and only if $U(\overline{\mu}_A ; D_1)$ and $L(\overline{\gamma}_A ; D_2)$ are either empty or $B$-subalgebras of $X$ for all $D_1, D_2 \in D[0, 1]$.

**Proof.** Suppose $A$ is an IVIF $B$-subalgebra of $X$. Let $D_1, D_2 \in D[0, 1]$ and assume that $U(\overline{\mu}_A ; D_1) \neq \emptyset$ and $L(\overline{\gamma}_A ; D_2) \neq \emptyset$. Let $x, y \in U(\overline{\mu}_A ; D_1)$ and $\overline{\mu}_A(x) \geq D_1$ and $\overline{\mu}_A(y) \geq D_1$, implying that

$$\overline{\mu}_A(x \ast y) \geq \min\{\overline{\mu}_A(x), \overline{\mu}_A(y)\} \geq \min\{D_1, D_1\} = D_1.$$ 

Thus, $x \ast y \in U(\overline{\mu}_A ; D_1)$ and so $U(\overline{\mu}_A ; D_1)$ is a $B$-subalgebra of $X$. Further, if $x, y \in L(\overline{\gamma}_A ; D_2)$, then $\overline{\gamma}_A(x) \leq D_2$ and $\overline{\gamma}_A(y) \leq D_2$. So, we have

$$\overline{\gamma}_A(x \ast y) \leq \max\{\overline{\gamma}_A(x), \overline{\gamma}_A(y)\} \leq \max\{D_2, D_2\}.$$ 

Thus, $x \ast y \in L(\overline{\gamma}_A ; D_2)$. Hence, $L(\overline{\gamma}_A ; D_2)$ is a $B$-subalgebra of $X$.

Conversely, let $x, y \in X$ and set $D = \min\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}$. Then $\overline{\mu}_A(x) \geq D$ and $\overline{\mu}_A(y) \geq D$. Thus, we have $x, y \in U(\overline{\mu}_A ; D)$. If $\overline{\mu}_A(x) \geq D$ and so $x \neq \emptyset$. So by hypothesis, $x \ast y \in U(\overline{\mu}_A ; D)$.

Hence,

$$\overline{\mu}_A(x \ast y) \geq D = \min\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}.$$ 

Furthermore, set $D' = \max\{\overline{\gamma}_A(x), \overline{\gamma}_A(y)\}$. Then $\overline{\gamma}_A(x) \leq D'$ and $\overline{\gamma}_A(y) \leq D'$. This implies that $x, y \in L(\overline{\gamma}_A ; D')$ and so $L(\overline{\gamma}_A ; D') \neq \emptyset$. Again by hypothesis, $x \ast y \in L(\overline{\gamma}_A ; D')$. Therefore,

$$\overline{\gamma}_A(x \ast y) \leq D' = \max\{\overline{\gamma}_A(x), \overline{\gamma}_A(y)\}.$$ 

Consequently, $A$ is an IVIF $B$-subalgebra of $X$.

Let $Y$ be a $B$-subalgebra of $X$ and $A = (\overline{\mu}_A, \overline{\gamma}_A)$ an IVIFS on $X$ defined as follows: for all $x \in X$,

$$\overline{\mu}_A(x) = \begin{cases} [r^-, r^+] & \text{if } x \in Y, \\ [0, 0] & \text{otherwise} \end{cases}$$

and

$$\overline{\gamma}_A(x) = \begin{cases} [s^-, s^+] & \text{if } x \in Y, \\ [1, 1] & \text{otherwise}. \end{cases}$$

with $[r^-, r^+], [s^-, s^+] \in D[0, 1]$ such that $r^+ + s^+ \leq 1$. By Theorem 3.12, $A$ is an IVIF $B$-subalgebra of $X$. Note that $U(\overline{\mu}_A ; [r^-, r^]) \in Y \in L(\overline{\gamma}_A ; [s^-, s^])$. Hence, we have the following result:

**Corollary 3.15** Any $B$-subalgebra of $X$ can be realized as both upper and lower level set for some IVIF $B$-subalgebra of $X$. 

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4. B-Homomorphism and IVIF B-Subalgebra

In this section, we investigate some results on IVIF B-subalgebra of a B-algebra under a B-homomorphism. Observe that if \( f : X \rightarrow Y \) is a B-homomorphism and \( A = (\mu_A, \gamma_A) \) is an IVIF B-subalgebra of \( X \) (resp. \( Y \)), then we have \( f^{-1}(\mu_A)(0) = \mu_A(0) \) and \( f^{-1}(\gamma_A)(0) = \gamma_A(0) \) (resp. \( f_{\sup}(\mu_A)(0) = \mu_A(0) \) and \( f_{\inf}(\gamma_A)(0) = \gamma_A(0) \).

**Theorem 4.1** Let \( X \) and \( Y \) be B-algebras and \( f : X \rightarrow Y \) a B-homomorphism. If \( A = (\mu_A, \gamma_A) \) is an IVIF B-subalgebra of \( Y \), then \( f^{-1}(A) \) is an IVIF B-subalgebra of \( X \). If \( f \) is surjective, then the converse holds.

**Proof.** Assume that \( A \) is an IVIF B-subalgebra of \( Y \). Let \( x_1, x_2 \in X \). Then we have

\[
\begin{align*}
f^{-1}(\mu_A)(x_1 \ast x_2) &= \mu_A(f(x_1) \ast f(x_2)) \\
&= \min\{\mu_A(f(x_1)), \mu_A(f(x_2))\} \\
&= \min\{f^{-1}(\mu_A)(x_1), f^{-1}(\mu_A)(x_2)\}
\end{align*}
\]

and

\[
\begin{align*}
f^{-1}(\gamma_A)(x_1 \ast x_2) &= \gamma_A(f(x_1) \ast f(x_2)) \\
&= \gamma_A(f(x_1)) \ast \gamma_A(f(x_2)) \\
&\leq \max\{\gamma_A(f(x_1)), \gamma_A(f(x_2))\} \\
&= \max\{f^{-1}(\gamma_A)(x_1), f^{-1}(\gamma_A)(x_2)\}
\end{align*}
\]

Hence, \( f^{-1}(A) \) is an IVIF B-subalgebra of \( X \).

Conversely, suppose \( f^{-1}(A) \) is an IVIF B-subalgebra of \( X \). Let \( y_1, y_2 \in Y \). Then \( y_1 = f(x_1) \) and \( y_2 = f(x_2) \) for some \( x_1, x_2 \in X \). Thus,

\[
\begin{align*}
\mu_A(y_1 \ast y_2) &= \mu_A(f(x_1) \ast f(x_2)) \\
&= \mu_A(f(x_1) \ast x_2) \\
&= \mu_A(f(x_1)) \ast \mu_A(f(x_2)) \\
&\geq \min\{f^{-1}(\mu_A)(x_1), f^{-1}(\mu_A)(x_2)\} \\
&= \min\{\mu_A(y_1), \mu_A(y_2)\}
\end{align*}
\]

and

\[
\begin{align*}
\gamma_A(y_1 \ast y_2) &= \gamma_A(f(x_1) \ast f(x_2)) \\
&= \gamma_A(f(x_1)) \ast \gamma_A(f(x_2)) \\
&\leq \max\{\gamma_A(f(x_1)), \gamma_A(f(x_2))\} \\
&= \max\{\gamma_A(y_1), \gamma_A(y_2)\}
\end{align*}
\]

Therefore, \( A \) is an IVIF B-subalgebra of \( Y \).

**Definition 4.2** An IVIFS \( A = (\mu_A, \gamma_A) \) on \( X \) is said to have the rsup - rinfin property if for any \( \emptyset \neq T \subseteq X \), there exists \( x_0 \in T \) such that

\[
\mu_A(x_0) = \sup_{x \in T} \mu_A(x) \quad \text{and} \quad \gamma_A(x_0) = \inf_{x \in T} \gamma_A(x).
\]

**Theorem 4.3** Let \( X \) and \( Y \) be B-algebras and \( f : X \rightarrow Y \) a B-homomorphism. If \( A = (\mu_A, \gamma_A) \) is an IVIF B-subalgebra of \( X \) having the rsup - rinfin property, then \( f(A) \) is an IVIF B-subalgebra of \( Y \).

**Proof.** Suppose \( A \) is an IVIF B-subalgebra of \( X \) having the rsup - rinfin property. Let \( y_1, y_2 \in Y \). If \( f^{-1}(y_1) = \emptyset \) or \( f^{-1}(y_2) = \emptyset \), then either \( f_{\sup}(\mu_A)(y_1) = [0, 0] \) or \( f_{\inf}(\mu_A)(y_1) = [1, 1] \), and either \( f_{\sup}(\gamma_A)(y_1) = [0, 0] \) or \( f_{\inf}(\gamma_A)(y_1) = [1, 1] \). Either of these, we have

\[
\begin{align*}
f_{\sup}(\mu_A)(y_1 \ast y_2) &\geq [0, 0] = \min\{f_{\sup}(\mu_A)(y_1), f_{\sup}(\mu_A)(y_2)\} \\
&\leq \max\{f_{\inf}(\mu_A)(y_1), f_{\sup}(\mu_A)(y_2)\}
\end{align*}
\]

and

\[
\begin{align*}
f_{\inf}(\gamma_A)(y_1 \ast y_2) &\leq [1, 1] = \max\{f_{\inf}(\gamma_A)(y_1), f_{\inf}(\gamma_A)(y_2)\}
\end{align*}
\]

So suppose \( f^{-1}(y_1) \neq \emptyset \) and \( f^{-1}(y_2) \neq \emptyset \). Consider

\[
U = \{x_1 \ast x_2 : x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2)\}.
\]

Then \( U \neq \emptyset \). We claim that \( U \subseteq f^{-1}(y_1 \ast y_2) \). Let \( u \in U \). Then \( u = x_1 \ast x_2 \) for some \( x_1 \in f^{-1}(y_1) \) and \( x_2 \in f^{-1}(y_2) \). Now,

\[
f(u) = f(x_1 \ast x_2) = f(x_1) \ast f(x_2) = y_1 \ast y_2
\]

and so \( u \in f^{-1}(y_1 \ast y_2) \). This proves the claim. The claim implies that \( f^{-1}(y_1 \ast y_2) \neq \emptyset \). Let \( x_0 \in f^{-1}(y_1) \) and \( x_0' \in f^{-1}(y_2) \) such that
\[ \mu_A(x_0) = \sup_{x \in f^{-1}(y_1)} \mu_A(x), \quad \gamma_A(x_0) = \inf_{x \in f^{-1}(y_1)} \gamma_A(x), \]

and

\[ \mu_A(x_0') = \sup_{x \in f^{-1}(y_2)} \mu_A(x), \quad \gamma_A(x_0') = \inf_{x \in f^{-1}(y_2)} \gamma_A(x). \]

Then \( x_0 \ast x_0' \in f^{-1}(y_1 \ast y_2) \). Thus,

\[ f_{\text{sup}} \left( \mu_A \right)(y_1 \ast y_2) = \sup_{x \in f^{-1}(y_1) \ast f^{-1}(y_2)} \mu_A(x) \]

\[ \geq \mu_A(x_0 \ast x_0') \]

\[ \geq \min \{ \mu_A(x_0), \mu_A(x_0') \} \]

\[ = \min \left\{ \sup_{x \in f^{-1}(y_1)} \mu_A(x), \sup_{x \in f^{-1}(y_2)} \mu_A(x) \right\} \]

\[ = \min \left\{ \sup_{x \in f^{-1}(y_1)} \mu_A(x), \sup_{x \in f^{-1}(y_2)} \mu_A(x) \right\} \]

and

\[ f_{\text{inf}} \left( \gamma_A \right)(y_1 \ast y_2) = \inf_{x \in f^{-1}(y_1) \ast f^{-1}(y_2)} \gamma_A(x) \]

\[ \leq \gamma_A(x_0 \ast x_0') \]

\[ = \min \{ \gamma_A(x_0), \gamma_A(x_0') \} \]

\[ = \max \left\{ \inf_{x \in f^{-1}(y_1)} \gamma_A(x), \inf_{x \in f^{-1}(y_2)} \gamma_A(x) \right\} \]

\[ = \max \left\{ \inf_{x \in f^{-1}(y_1)} \gamma_A(x), \inf_{x \in f^{-1}(y_2)} \gamma_A(x) \right\} \]

Hence, \( f(A) \) is an IVIF B-subalgebra of \( Y \).

**Corollary 4.4** Let \( X \) and \( Y \) be \( B \)-algebras where \( X \) is finite and \( f : X \rightarrow Y \) a \( B \)-homomorphism. If \( A = (\mu_A, \gamma_A) \) is an IVIF \( B \)-subalgebra of \( X \), then \( f(A) \) is an IVIF \( B \)-subalgebra of \( Y \).

The following characterization shows that the rsup - rinf property of an IVIFS \( A = (\mu_A, \gamma_A) \) on \( X \) can be inherited upon performing the operations “\( \Box \)” and “\( \Diamond \)” on \( A \).

**Lemma 4.5** Let \( A = (\mu_A, \gamma_A) \) be an IVIFS on \( X \) having the rsup - rinf property. Then \( \Box A \) and \( \Diamond A \) are IVIFSs on \( X \) having the rsup - rinf property.

**Proof.** Let \( T \) be a nonempty subset of \( X \) and \( x_0 \in T \) be such that

\[ \mu_A(x_0) = \sup_{x \in T} \mu_A(x) \quad \text{and} \quad \gamma_A(x_0) = \inf_{x \in T} \gamma_A(x). \]

Then

\[ \mu_A(x_0) = [1 - \mu_A(x_0), 1 - \mu_A(x_0)] \]

\[ = \inf_{x \in T} \{ 1 - \mu_A(x), 1 - \mu_A(x) \} \]

\[ = \inf_{x \in T} \{ 1 - \mu_A(x), 1 - \mu_A(x) \} \]

and

\[ \gamma_A(x_0) = [1 - \gamma_A(x_0), 1 - \gamma_A(x_0)] \]

\[ = \inf_{x \in T} \{ 1 - \gamma_A(x), 1 - \gamma_A(x) \} \]

\[ = \inf_{x \in T} \{ 1 - \gamma_A(x), 1 - \gamma_A(x) \} \]

\[ = \inf_{x \in T} \{ 1 - \gamma_A(x), 1 - \gamma_A(x) \} \]

Therefore, \( \Box A \) and \( \Diamond A \) are IVIFSs on \( X \) having the rsup - rinf property.

The next result is immediate to Proposition 3.10 and Lemma 4.6.

**Corollary 4.7** If \( f : X \rightarrow Y \) is a \( B \)-homomorphism and \( A = (\mu_A, \gamma_A) \) is an IVIF \( B \)-subalgebra of \( X \) having the rsup - rinf property, then \( \Box A \) and \( \Diamond A \) are IVIF \( B \)-subalgebras of \( Y \).

### 5. Conclusion

Zadeh introduced the (interval-valued) fuzzy sets as an aid in describing complex and ambiguous systems that classical mathematics cannot directly be analysed and interpreted. Since then, various applications of fuzzy sets in different fields (e.g., physics, engineering, medical science) were studied. Atanassov meanwhile introduced the notion of Intuitionistic Fuzzy Sets (IFSs) and, together with Gargov, developed the Interval-Valued Intuitionistic Fuzzy Sets (IVIFSs); both of them are generalizations of Zadeh’s fuzzy sets. Like fuzzy sets, they are being vigorously...
studied for their applications to the real world. In this paper, we applied the concepts of IVIFSs to $B$-algebras. We have introduced the notion of IVIF $B$-subalgebras of a $B$-algebra and studied some of its properties. A natural extension of this work is connected with the study of IVIF ideals and closed ideals of $B$-algebras.

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7. References

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