SURFACES OF PRESCRIBED MEAN CURVATURE IN QUASI-FUCHSIAN MANIFOLDS

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Abstract. Let $M$ be a quasi-Fuchsian three-manifold that contains a closed incompressible surface with principal curvatures within the range of the unit interval, for a prescribed function $H$ (with mild conditions) on $M$, we construct a closed incompressible surface with mean curvature $H$. A direct application is the existence of embedded surfaces of prescribed constant mean curvatures with constants in $(-2, 2)$.

1. Introduction

1.1. Main results. By hyperbolic space, we mean a Riemannian manifold of constant sectional curvature $-1$. We are particularly interested in the dimension three. The quasi-Fuchsian manifold is an important class of hyperbolic three-manifolds, and the space of these manifolds is the quasi-Fuchsian space, a complex manifold of (complex) dimension $6g - 6$, where we always assume any incompressible surface in a quasi-Fuchsian manifold has genus $g \geq 2$. Graphically, the quasi-Fuchsian space can be viewed naturally as a “higher” Teichmüller space, a square with the Fuchsian locus sitting inside as a diagonal.

Definition 1.1. A closed surface $S$ is called to have small curvatures if its principal curvatures $\{\lambda_j\}$ satisfy

\[ |\lambda_j(x)| < 1, \forall x \in S, j = 1, 2. \]

It is of great interest to understand the structures, such as Riemannian, topological, and complex structures, of the quasi-Fuchsian space. A special subspace, the almost Fuchsian space, consists of what we call almost Fuchsian manifolds:

Definition 1.2. A quasi-Fuchsian manifold is called almost Fuchsian if it contains an incompressible minimal surface of small curvatures in the sense of (1.1). Here an incompressible surface is a smooth closed surface in a three-manifold which induces an injection between their fundamental groups.

It is evident that incompressible surfaces of small curvatures play important role in three dimensional geometry and topology, see for example [Thu82], [Eps84], [Rub05] and [GHW09]. The space of almost Fuchsian manifolds is an open subspace of the same dimension of the quasi-Fuchsian space, and it contains the space of Fuchsian manifolds which is diffeomorphic to Teichmüller space ([Uhl83]). Recent progress in GHMC (globally

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hyperbolic maximally compact) $AdS_3$ space ([Mes07, KS07]) resembles strongly to the geometry of quasi-Fuchsian three-manifolds.

A fundamental fact about a quasi-Fuchsian three-manifold $M$ is that it admits an immersed incompressible minimal surface ([SY79], [SU82]). When $M$ is further almost Fuchsian, Uhlenbeck showed that $M$ only admits one incompressible minimal surface, and it is necessarily embedded ([Uhl83]). Furthermore, we are able to use the mean curvature flow to deform a somewhat arbitrary graphical surface of a surface of small curvatures to this minimal surface with exponential convergence ([HW09b]).

We consider a larger class of quasi-Fuchsian three-manifolds and obtain results of existence of embedded incompressible surfaces of special geometry:

**Definition 1.3.** A quasi-Fuchsian manifold $M$ is called nearly almost Fuchsian if it contains an incompressible surface of small curvatures in the sense of (1.1).

Obviously this class contains all almost Fuchsian manifolds, though the number of incompressible minimal surfaces might be larger than one.

**Basic assumptions:** Throughout this paper, we denote $M$ a nearly almost Fuchsian manifold, and $S$ a closed incompressible surface of $M$ which has small curvatures in the sense of (1.1). All surfaces we encounter are assumed to be closed, incompressible, of genus at least two.

We are greatly inspired by a beautiful paper of Ecker-Huisken ([EH91]), where they studied the problem of prescribed mean curvature in cosmological space-times. They settled the problem with assumptions of the existence of the barrier surfaces and certain monotonicity conditions on the prescribed function, as well as a time-like convergence condition on the space-times. As they pointed out, the mean curvature flow in a Lorentzian manifold behaves more regularly than in a Riemannian manifold.

In the present work, we construct an incompressible surface in $M$ with prescribed mean curvature function $\mathcal{H} : M \to \mathbb{R}$ which satisfies some mild conditions. To make it more precise, let $S$ be any incompressible surface in $M$, with $|\lambda_j(S)| < 1$, where $\{\lambda_j(S)\}_{j=1,2}$ are principal curvatures of $S$. The normal flow from $S$ forms a foliation of parallel surfaces (or the equidistant foliation) $\{S(r)\}_{r \in \mathbb{R}}$, see ([Uhl83] or [HW09a]), for the three-manifold $M$. Therefore any point $P \in M$ can be uniquely represented as $P(x,r)$, where $x$ is the conformal coordinate on $S$.

**Theorem 1.4.** Let $M$ be a nearly almost Fuchsian manifold and it contains a closed surface $S$ with $|\lambda_j(S)| < 1$ for $j = 1, 2$. Let $\mathcal{H} : M \to \mathbb{R}, (x,r) \mapsto \mathcal{H}(x,r) \subset [a,b] \subset (-2,2)$ be a smooth function with bounded gradients $|\nabla \mathcal{H}|$ and $|\nabla^2 \mathcal{H}|$ in $M$, then there exists an embedded incompressible surface $S(\infty)$ on $M$ such that $H(S(\infty)) = \mathcal{H}|_{S(\infty)}$.

An immediate application is the following corollary:

**Corollary 1.5.** For any constant $c \in (-2,2)$, there exists an embedded incompressible surface $S(\infty)$ on $M$ with constant mean curvature $c$. In particular, $M$ admits an embedded incompressible minimal surface.
By using the usual mean curvature flow, we \([\text{HW09b}]\) have shown that if \(M\) is nearly almost Fuchsian, then \(M\) admits (at least) one embedded minimal surface.

This paper is a continuation of our study of geometric evolution equations in quasi-Fuchsian three-manifolds, begun in \([\text{Wan08}]\) and \([\text{HW09a}]\). The geometric evolution equations of the evolution of hypersurfaces by their mean curvature have been studied extensively in various ambient Riemannian manifolds, (see for instance \([\text{Bra78}]\), \([\text{Hui84}]\), \([\text{Hui86}]\), \([\text{And02}]\), and many others) as well as in Lorentzian spaces (see \([\text{Ger83}]\), \([\text{Bar84}]\), \([\text{EH91}]\), \([\text{Eck03}]\), etc.).

The mean curvature flow equation of a forcing term \(f\) has the following form:

\[
\frac{\partial}{\partial t} F(x, t) = (f(x, t) - H(x, t))\nu(x, t)
\]

\(\quad F(\cdot, 0) = F_0\),

where all terms will be made transparent in section \(\S 2.3\). Obviously, this formulation is a generalization of the mean curvature flow (when the forcing term \(f \equiv 0\)), and the volume preserving mean curvature flow (when \(f = \int_{S(t)} h du |S(t)|\), the average mean curvature of the evolving surface \(S(t)\)). The proof of the Theorem 1.4 is based on solving the initial value problem \((1.2)\) with the prescribed function \(\mathcal{H}(F(\cdot, t))\) as the forcing term. It should be noted that the presence of a global forcing term makes quantities of the evolution equations involved in the calculations more delicate.

Our proof of the main Theorem 1.4 roughly goes as follows: we start with a fiber \(S(r)\) of the equidistant foliation \(\{S(r)\}_{r \in \mathbb{R}}\) of \(M\), which is a graph over a fixed surface \(S\) of small curvatures, then use it as an initial immersion of the mean curvature flow \((1.2)\) with \(\mathcal{H}(F(\cdot, t))\) as the forcing term, and we show the long time existence of the solution for the equation. The limiting surface \(S(\infty)\) is embedded since it is also a graph over \(S\). Our estimates rely on the basic fact that the graph functions behave quite regularly in hyperbolic spaces under evolution equations (see for example \([\text{EH89}, \text{Unt03}, \text{CM04}]\)).

1.2. Notation. This subsection recalls notation that will be employed in our paper. The notation is quite similar to that introduced in \([\text{HW09b}]\), with a few additions:

- \(\mathbb{H}^3\): hyperbolic 3-space, with isometry group \(\text{PSL}(2, \mathbb{C})\);
- \(M\): a nearly almost Fuchsian three-manifold (Definition 1.3);
- \(S\): a surface of \(M\) of small curvatures in the sense of \((1.1)\);
- \(\lambda_j(S)\): the principal curvatures of \(S\), and \(|\lambda_j(S)| < 1, j = 1, 2\);
- \(\mathbf{n}\): the unit normal vector field to \(S\);
- \(S(r)\): the parallel surface to \(S\) of hyperbolic distance \(r\); these surfaces form the equidistant foliation of \(M\);
- \(\mu_j(x, r)\): the principal curvatures of \(S(r)\) at the point \((x, r), j = 1, 2\);
- \(S(t)\): the evolving surface under the mean curvature flow \((1.2)\);
- \(g(\cdot, t) = \{g_{ij}\}\): the induced metric of \(S(t)\);
- \(A(\cdot, t) = \{a_{ij}\}\): the second fundamental form of \(S(t)\);
- \(H(\cdot, t) = g^{ij}h_{ij}\): the mean curvature of \(S(t)\) with respect to the normal pointing to \(S\);
\[ |A|^2 = g^{ij}g^{kl}h_{ik}h_{jl}: \] the square norm of \( A(t) \) for \( S(t) \);

\[ \Delta = g^{ij}\nabla_i \nabla_j: \] the Laplacian on \( S(t) \);

\[ \nabla: \] the covariant derivative of \( S(t) \);

\[ \nu(t): \] the unit normal vector field to \( S(t) \);

\[ d\mu: \] the area element for \( S(t) \);

\[ |S(t)|: \] the surface area of \( S(t) \);

\[ u(\cdot, t): \] the height function measuring the distance to the reference surface;

\[ \Theta(\cdot, t) = \langle \nu(\cdot, t), n \rangle: \] the gradient function.

We add a bar on top for each quantity or operator with respect to \((M, \bar{g}_{\alpha\beta})\).

**Plan of the paper.** We provide necessary background material in §2, especially the almost Fuchsian manifolds, the equidistant foliation, and the mean curvature flow. We prove the Theorem 1.4 (prescribing mean curvature function) in §3 (long-time existence) and §4 (convergence), by the way of proving Theorem 3.1.

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## 2. Preliminaries

In this section, we briefly introduce some preliminary facts that will be used in this paper. We start with the geometry of quasi-Fuchsian three-manifold in §2.1, and define the equidistant foliations for \( M \) in §2.2, which is our point of departure for our analysis; introduce the general mean curvature flow equations and their corresponding evolution equations for the metrics and the second fundamental forms in §2.3.

### 2.1. Quasi-Fuchsian three-manifolds.

For any unexplained terminology and more detailed references on Kleinian groups and low dimensional topology, we refer to [Mar74, Thu82].

The universal cover of a hyperbolic three-manifold is \( \mathbb{H}^3 \), and the deck transformations induce a representation of the fundamental group of the manifold in \( Isom(\mathbb{H}^3) = PSL(2, \mathbb{C}) \), the (orientation preserving) isometry group of \( \mathbb{H}^3 \). A subgroup \( \Gamma \subset PSL(2, \mathbb{C}) \) is called a **Kleinian group** if \( \Gamma \) acts on \( \mathbb{H}^3 \) properly discontinuously.

For any Kleinian group \( \Gamma, \forall p \in \mathbb{H}^3 \), the orbit set \( \Gamma(p) = \{ \gamma(p) \mid \gamma \in \Gamma \} \) has accumulation points on the boundary \( S^2_\infty = \partial \mathbb{H}^3 \), and these points are the **limit points** of \( \Gamma \), and the closed set of all these points is called the **limit set** of \( \Gamma \), denoted by \( \Lambda_\Gamma \). In the case when \( \Lambda_\Gamma \) is contained in a circle \( S^1 \subset S^2 \), the quotient \( \mathbb{H}^3/\Gamma \) is called **Fuchsian**, and it is isometric to a product space of a totally geodesic hyperbolic surface and the real line. Clearly, the space of Fuchsian manifolds is isometric to the space of hyperbolic metrics on a closed surface, or Teichmüller space.

If the limit set \( \Lambda_\Gamma \) lies in a Jordan curve, the quotient \( \mathbb{H}^3/\Gamma \) is called **quasi-Fuchsian**, and it is topologically a product space \( S \times \mathbb{R} \), where \( S \) is a closed surface. A quasi-Fuchsian is a complete hyperbolic three-manifold which is quasi-isometric to a Fuchsian manifold.
An extraordinary fact about quasi-Fuchsian manifolds is the Bers’ simultaneous uniformization theorem ([Ber72]), which states that, for each pair of points in Teichmüller space, there is one quasi-Fuchsian manifold with these two conformal structures as conformal boundaries. Therefore, the deformation theory of Kleinian groups is deeply related to the deformation theory of Riemann surfaces.

We are particularly interested in a class of quasi-Fuchsian manifolds: nearly almost Fuchsian manifolds which contain a closed surface of small curvatures. By the work of [SY79] and [SU82], such a nearly almost Fuchsian manifold $M$ admits at least one immersed minimal surface. The importance of this particular class is two-fold: constant curvature of the space simplifies the evolution equations involved in the calculation, and admission of the equidistant foliation provides geometric bounds for the evolving surfaces in the mean curvature flow.

2.2. Equidistant foliation for $M$. We now follow the notation introduced in §1.2. The induced metric on the reference surface $S$ is given by $g_{ij}(x) = e^{2v(x)}\delta_{ij}$, where $v(x)$ is a smooth function on $S$, and the second fundamental form of $S$ is $A(x) = [h_{ij}]_{2 \times 2}$, here $h_{ij}$ is given by, for $1 \leq i, j \leq 2$,

$$h_{ij} = \langle \nabla e_i, \nu', e_j \rangle = -\langle \nabla e_i, e_j \rangle,$$

where $\{e_1, e_2\}$ is a basis on $S$, $\nu'$ is the unit normal vector to $S$, and $\nabla$ is the Levi-Civita connection on $(M, \bar{g}_{\alpha\beta})$.

Let $\lambda_1(x)$ and $\lambda_2(x)$ be the eigenvalues of $A(x)$ and $|\lambda_j(x)| < 1$ for $j = 1, 2$. They are the principal curvatures of $S$, and we denote $H(x) = \lambda_1(x) + \lambda_2(x)$ as the mean curvature function of $S$, and $|H(S)| < 2$.

Let $S(r)$ be the family of equidistant surfaces with respect to $S$, i.e., $S(r)$ is a parallel surface to $S$ by moving hyperbolic distance $r$ away from $S$ in the normal direction:

$$S(r) = \{\exp_x(r\nu) \mid x \in S\}, \quad r \in (-\varepsilon, \varepsilon).$$

The induced metric on $S(r)$ is denoted by $g(x, r) = g_{ij}(x, r)$, and the second fundamental form is denoted by $A(x, r) = [h_{ij}(x, r)]_{1 \leq i, j \leq 2}$. The mean curvature on $S(r)$ is thus given by $H(x, r) = g^{ij}(x, r)h_{ij}(x, r)$. We collect the following lemma:

**Lemma 2.1** ([Uhl83], [HW09a]). The induced metric $g(x, r)$ on $S(r)$ has the form

$$g(x, r) = e^{2v(x)}[\cosh r \| + \sinh r e^{-2v(x)}A(x)]^2,$$

The metric is of non-singular for all $r \in \mathbb{R}$. The principal curvatures of the surface $S(r)$ are given by

$$\mu_j(x, r) = \frac{\tanh r + \lambda_j(x)}{1 + \lambda_j(x) \tanh r}, \quad j = 1, 2,$$

and mean curvature is

$$H(x, r) = \frac{2(1 + \lambda_1 \lambda_2) \tanh r + (\lambda_1 + \lambda_2)(1 + \tanh^2 r)}{1 + (\lambda_1 + \lambda_2) \tanh r + \lambda_1 \lambda_2 \tanh^2 r}.$$

Therefore $\{S(r)\}_{r \in \mathbb{R}}$ forms a foliation of surfaces parallel to $S$, called the equidistant foliation or the normal flow. It is easy to verify.
Lemma 2.2. For the equidistant foliation \( \{ S(r) \}_{r \in \mathbb{R}} \), we have the following:

(i) Each \( S(r) \) has small curvatures: \(|\mu_j(x,r)| < 1\) for all \( x \in S, r \in \mathbb{R} \);

(ii) For fixed \( x \in S \), \( \mu_j(x,r) \) is an increasing function of \( r \). Moreover, \( \mu_j(x,r) \to \pm 1 \) as \( r \to \pm \infty \);

(iii) For fixed \( x \in S \), \( H(x,r) \) is an increasing function of \( r \). Moreover, \( H(x,r) \to \pm 2 \) as \( r \to \pm \infty \).

We note that, if \( M \) is not nearly almost Fuchsian, then the metric \( g(x,r) \) on \( S(r) \) will develop singularity quickly.

The following lemma is the well-known Hopf’s maximum principle for tangential hypersurfaces in Riemannian geometry:

Lemma 2.3 (\cite{Hop89}). Let \( \Sigma_1 \) and \( \Sigma_2 \) be two hypersurfaces in a Riemannian manifold which intersect at a common point \( p \) tangentially. If \( \Sigma_2 \) lies in positive side of \( \Sigma_1 \) around \( p \), then \( H_1 \leq H_2 \), where \( H_i \) is the mean curvature of \( \Sigma_i \) at \( p \) for \( i = 1, 2 \).

2.3. Mean curvature flow with a forcing term. Let \( F_0 : S \to M \) be the immersion of \( S \) in \( M \) such that \( S_0 = F_0(S) \) is contained in the positive side of \( S \), and is a graph over \( S \) with respect to \( n \), i.e., \( \langle n, \nu \rangle \geq c > 0 \), here \( n \) is the unit normal vector on \( S \) and \( \nu \) is the unit normal vector on \( S_0 \) and \( c \) is a constant depending only on \( S_0 \).

We consider a family of immersions of surfaces in \( M \),

\[
F : S \times [0, T) \to M, \quad 0 \leq T \leq \infty
\]

with \( F(\cdot, 0) = F_0 \). For each \( t \in [0, T) \), \( S(t) = \{ F(x,t) \in M \mid x \in S \} \) is the evolving surface at time \( t \), and \( H(x,t) \) its mean curvature.

The mean curvature flow equation (\cite{EH91}) with a forcing term \( f \) is given by, as in (1.2):

\[
\begin{align*}
\frac{\partial}{\partial t} F(x,t) &= (f(x,t) - H(x,t))\nu(x,t), \\
F(\cdot, 0) &= F_0,
\end{align*}
\]

Here \( -\nu \) points to the surface \( S \).

The equation (1.2) is parabolic, and Huisken proved the short-time existence of the solutions and found initial compact surface quickly develops singularities along the flow, moreover, he showed the blow-up of the norm of the second fundamental forms if the singularity occurs in finite time.

Theorem 2.4 (\cite{Hui84}, \cite{Hui86}). If the initial surface \( S_0 \) is smooth, then the equation (1.2) has a smooth solution on some maximal open time interval \( 0 \leq t < T \), where \( 0 < T \leq \infty \). If \( T < \infty \), then \( |A|_{\text{max}}(t) \equiv \max_{x \in S} |A|(x,t) \to \infty \) as \( t \to T \).

Therefore, the key to the existence of long time solution is to find uniform bounds for the square norm of the second fundamental forms on the evolving surfaces along the flow.

3. Prescribing mean curvature

This section is devoted to proving the existence of long-time solution part of the Theorem 1.1, by analyzing the mean curvature flow equation with the prescribed mean curvature
function as the forcing term:

\[
\begin{cases}
\frac{\partial}{\partial t} F(x,t) = (\mathcal{H} - H(x,t))\nu(x,t), \\
F(\cdot,0) = F_0,
\end{cases}
\]

where \(\mathcal{H}\) is the prescribed mean curvature function on \(M = \{S(r)\}_{r \in \mathbb{R}}\). Writing any point \(P \in M\) as a pair \((x,r)\), where \(x \in S\), the function \(\mathcal{H}\) is smooth, and with bounded gradient with respect to \(\nabla\). Moreover, we require \(-2 < a \leq \mathcal{H} \leq b < 2\) for some constants \(a\) and \(b\). Our strategy is to show the long-time existence of the solution to (3.1), as well as the convergence and the uniqueness of the limiting surface.

Our major goal is to establish the following theorem, which will imply our main theorem 1.4:

**Theorem 3.1.** Let \(M\) and \(S\) be as in Theorem 1.4. Suppose a smooth closed surface \(S_0 = S(r)\) for some \(r\), where \(\{S(r)\}_{r \in \mathbb{R}}\) forms the equidistant foliation for \(M\) as in §2.2. Then

(i) the mean curvature flow equation (3.1) with initial surface \(S(0) = S_0\) has a long time solution;

(ii) the evolving surfaces \(\{S(t)\}_{t \in \mathbb{R}}\) stay smooth and they remain as graphs over \(S\) for all time;

(iii) For every sequence \(\{t_i\} \rightarrow \infty\), there is a subsequence \(\{t'_i\} \rightarrow \infty\) such that the surfaces \(\{S(t_i)\}\) converges uniformly in \(C^\infty\) to a smooth embedded surface \(S(\infty)\) satisfying \(H(S(\infty)) = \mathcal{H}|_{S(\infty)}\).

### 3.1. Some evolution equations.

In this subsection, we collect and derive a number of evolution equations of some quantities and operators on \(S(t), t \in [0,T]\) which are involved in our calculations.

Proceeding as in [Hui84, Lemmas 3.2, 3.3, and Theorem 3.4], and keeping track of terms involving \(\mathcal{H}\), we find:

**Proposition 3.2.** The evolution equations of the induced metric \(g_{ij}\), the normal vector field \(\nu\), and the area element \(d\mu\) are given by

\[
\begin{align*}
\frac{\partial}{\partial t} g_{ij} &= 2(\mathcal{H} - H)h_{ij}, \\
\frac{\partial}{\partial t} \nu &= \nabla(H - \mathcal{H}), \\
\frac{\partial}{\partial t} d\mu &= H(\mathcal{H} - H)d\mu.
\end{align*}
\]

We will need the evolution equation for the mean curvature \(H(\cdot,t)\):

**Lemma 3.3.**

\[
\frac{\partial}{\partial t} H = \Delta(H - \mathcal{H}) + (H - \mathcal{H})(|A|^2 - 2).
\]

**Proof.** Consider the well-known Simons’ identity (see [Sim68, SSY75]), satisfied by the second fundamental form \(h_{ij}\):

\[
\Delta h_{ij} = \nabla_i \nabla_j H - (|A|^2 - 2)h_{ij} + H(h_{ii}h_{ij} + g_{ij}).
\]
Here we used our special situation: $M$ has constant sectional curvature $-1$, and $M$ has dimension three, hence Ricci curvature $Ric(\nu, \nu) = -2$, and $\bar{R}_{ijk} = -g_{ij}$ for $1 \leq i, j \leq 2$.

Also we have

$$\frac{\partial}{\partial t} h_{ij} = \nabla_i \nabla_j (H - \mathcal{K}) + (\mathcal{K} - H)(h_{ij}h_{ij} + g_{ij}).$$

Now we proceed the calculation using $H = g^{ij}h_{ij}$ and (3.2) to obtain (3.3).

We are now arriving at the key evolution equation:

**Lemma 3.4.** We have the following equation for the square norm of the second fundamental form:

$$\frac{\partial}{\partial t} |A|^2 = \Delta |A|^2 - 2|A|^2 - 2h_{ij}\nabla_i \nabla_j \mathcal{K}$$

$$+ 2|A|^2(|A|^2 + 2) - 2\mathcal{K}Tr A^3 + 2H(\mathcal{K} - 2H),$$

where $Tr(A^3) = h_{ij}h_{ij}h_{ii}$.

**Proof.** Using again the fact that $M$ is a hyperbolic three-manifold, the equation (3.6) is obtained by proceeding the calculations in [Hui86] and [HY96].

It will be very important for our a priori estimates for $|A|^2$ to recall the height function $u(\cdot, t)$ and the gradient function $\Theta(\cdot, t)$ on $S(t)$ from the introduction:

$$u(x, t) = \ell(F(x, t))$$

(3.7)

$$\Theta(\cdot, t) = \langle \nu(\cdot, t), n \rangle,$$

(3.8)

for all $(x, t) \in S \times [0, T_{\text{max}})$. Here $T_{\text{max}}$ is the right endpoint of the maximal time interval on which the solution to (3.1) exists, and $\ell(p) = \pm \text{dist}(p, S)$ for all $p \in M$, the distance to the reference surface $S$. It is clear that the surface $S(t)$ becomes a graph over $S$ if $\Theta(\cdot, t) > 0$ on $S(t)$.

The evolution equations of $u(\cdot, t)$ and $\Theta(\cdot, t)$ have the following forms:

**Proposition 3.5 ([Bar84, EH91]).**

$$\frac{\partial}{\partial t} u = (\mathcal{K} - H)\Theta$$

(3.9)

$$\frac{\partial}{\partial t} \Theta = \Delta \Theta + (|A|^2 - 2)\Theta + n(H_n) - \langle n, \nabla \mathcal{K} \rangle$$

$$+ (\mathcal{K} - H)\langle \nabla_\nu n, \nu \rangle,$$

(3.11)

where $\text{div}$ is the divergence on $S(t)$, and $n(H_n)$ is the variation of mean curvature function of $S(t)$ under the deformation vector field $n$.

**Proof.** We only show (3.11). By the definition of the gradient function, we have

$$\frac{\partial \Theta}{\partial t} = \langle n, \nabla(H - \mathcal{K}) \rangle + (\mathcal{K} - H)\langle \nabla_\nu n, \nu \rangle.$$

On the other hand, we have

$$\Delta \Theta = -(|A|^2 - 2)\Theta + \langle n, \nabla H \rangle - n(H_n),$$
ompleting the proof.

3.2. Estimates for $\Theta(\cdot, t)$. It appears to be very difficult to bound $|A|^2$ simply from the equation (3.6). In what follows, we show the positivity of the gradient function $\Theta(\cdot, t)$ and use the evolution equation for $\Theta^{-\delta}(\cdot, t)$, for some $\delta > 2$ to add enough negative terms to (3.6). This subsection is to serve this purpose.

Our first technical lemma is the following: given positive lower bound on the gradient function on the initial surface, the function stays positive along the mean curvature flow, i.e.,

**Proposition 3.6.** If $\Theta(\cdot, 0) \geq C > 0$, where $C$ is any positive constant, then $\Theta(\cdot, t) \geq \Theta_0 > 0$ for some constant $\Theta_0$, depending only on $S_0$ and $T$, for $t \in [0, T)$.

The statement of this lemma is slightly stronger than what we need: our choice of the initial surface $S(0)$ is a parallel surface of the reference surface $S$, hence $\Theta(\cdot, 0) = 1$.

**Proof.** We proceed as in [HW09b, Lemma 3.4], with special cares on terms involving $H$. Let $\Theta_{\min}(t) = \min_{x \in S} \Theta(x, t)$.

We estimate the terms in the equation (3.11), starting with the expression $n(H_n)$, from (Bar84, Eq. (2.10)):

$$|n(H_n)| \leq C_1 (\Theta^3 + |A|),$$

for some $C_1 > 0$.

We also have the following estimate from (Eck03, Page 187):

$$|\langle \nabla_n n, \nu \rangle| \leq C_2 \Theta^2,$$

where $C_2 = \|\nabla n\| > 0$.

Thirdly, we have the following estimate from (EH91, Page 602):

$$|\langle n, \nabla H \rangle| \leq \Theta^2 \|\nabla H\| \leq \|\nabla H\|,$$

which is bounded since $H$ has bounded gradient in $M$.

Collecting these estimates, and we obtain from the equation (3.11):

$$\frac{d}{dt} \Theta_{\min} \geq (|A|^2 - 2) \Theta_{\min} - C_1 (\Theta_{\min}^3 + |A| \Theta_{\min}^2)$$

$$- \|\nabla H\| \Theta_{\min}^2 - C_2 (|H| + |H|) \Theta_{\min}^2$$

$$\geq (|A|^2 - 2) - C_1 (1 + |A|) - \|\nabla H\| - C_2 (|H| + \sqrt{2} |A|) \Theta_{\min}$$

$$= (|A|^2 - (C_1 + C_2 \sqrt{2}) |A| - (2 + C_1 + \|\nabla H\| + |H| C_2)) \Theta_{\min}$$

$$\geq - \left( \frac{(C_1 + C_2 \sqrt{2})^2}{4} + 2 + C_1 + \|\nabla H\| + |H| C_2 \right) \Theta_{\min}$$

Since $\Theta_{\min}(0) \geq C > 0$, then

$$\Theta_{\min}(t) \geq C \exp(-C_3 t) \quad \text{on } [0, T),$$

where

$$C_3 = \frac{(C_1 + C_2 \sqrt{2})^2}{4} + 2 + C_1 + \|\nabla H\| + |H| C_2.$$
We can choose \( \Theta_0 = C \exp(-C_3 T) \) to complete the proof.

We also record the following proposition on the derivatives of \( \Theta(\cdot, t) \), postponing its proof in the next subsection as we need extra bounds on \( u(\cdot, t) \) and its derivatives.

**Proposition 3.7.** Suppose the mean curvature flow equation (3.1) has a solution on \([0, T)\), \( 0 < T \leq \infty \), then there exists a constant \( 0 < C_4 < \infty \) depending only on \( S_0 \) such that
\[
|\nabla \Theta|^2 \leq C_4
\]
on \( S(t) \), for \( 0 \leq t < T \).

### 3.3. Estimates on \( u(\cdot, t) \)

We now turn our attention to the following estimates on the height function, where we rely on, and make use of the properties of the mean curvatures along the equidistant foliation on \( M \), as well as the bounds \([a, b] \subset (-2, 2)\) on the prescribed mean curvature \( \mathcal{H} \).

**Theorem 3.8.** Suppose the mean curvature flow (3.1) has a solution on \([0, T)\), \( 0 < T \leq \infty \), then \( u(\cdot, t) \) is uniformly bounded on \( S \times [0, T) \), i.e.,
\[
0 < C_5 \leq u(x, t) \leq C_6 < \infty , \quad \forall (x, t) \in S \times [0, T) ,
\]
where \( C_5 \) and \( C_6 \) are constants depending only on the surface \( S_0 \).

**Proof.** At each time \( t \in [0, T) \), let \( x(t) \in S \) be the point such that
\[
u_{\text{max}}(t) \equiv \max_{x \in S} u(x, t) = u(x(t), t) ,
\]
and let \( y(t) \in S \) be the point such that
\[
u_{\text{min}}(t) \equiv \min_{y \in S} u(y, t) = u(y(t), t) .
\]

By the equation (3.9) and the positivity of \( \Theta \) along the flow (Lemma 3.6), the part of \( S(t) \) with \( H < \mathcal{H}|_{S(t)} \) will move along the positive direction of \( n \) while the part of \( S(t) \) with \( H > \mathcal{H}|_{S(t)} \) will move along the negative direction of \( n \), therefore we can assume that \( \nu_{\text{max}}(t) \) is increasing and \( \nu_{\text{min}}(t) \) is decreasing, after some \( t_0 \in (0, T) \).

In order to show that \( u \) is uniformly bounded along the flow for \( t \in [t_0, T) \), we must exclude the following cases as \( t \to T \):

- (i) \( \nu_{\text{min}}(t) \to -\infty \) and \( \nu_{\text{max}}(t) \to \infty \);
- (ii) \( \nu_{\text{min}}(t) \to \infty \) and \( \nu_{\text{max}}(t) \to \infty \);
- (iii) \( \nu_{\text{min}}(t) \to -\infty \) and \( \nu_{\text{max}}(t) \to -\infty \);
- (iv) \( \nu_{\text{min}}(t) \) is uniformly bounded, while \( \nu_{\text{max}}(t) \to \infty \);
- (v) \( \nu_{\text{min}}(t) \to -\infty \), while \( \nu_{\text{max}}(t) \) is uniformly bounded.

We consider, at \( F(x(t), t) \), \( \Theta = \langle n, \nu \rangle = 1 \), then
\[
0 \leq \frac{\partial u}{\partial t} = \mathcal{H} - H .
\]

By the maximum principle, we have
\[
b \geq \mathcal{H}(x(t), t) \geq H(x(t), t) .
\]
Now we recall from Lemma 2.2, the principal curvatures $\mu_j(p, r)$ of the fiber surfaces for the equidistant foliation are increasing functions of $r$ with limits $\pm 1$ as $r \to \pm \infty$ and the mean curvatures approach $\pm 2$ as $r \to \pm \infty$. Now if $u_{\max}(t) \to \infty$ as $t \to T$, then we will have $b \geq 2$, which contradicts our assumption that $b \in (-2, 2)$.

Similarly, at the point $F(y(t), t)$, we have

\[
a \leq \mathcal{H}(y(t), t) \leq H(y(t), t),
\]

and if $u_{\min}(t) \to -\infty$ as $t \to T$, then we will have $a \leq -2$. This is also impossible since $a, b \in (-2, 2)$.

So the mean curvature flow is uniformly bounded by two parallel surfaces $S(r_1)$ and $S(r_2)$ with $0 < r_1 < r_2 < +\infty$ on $[0, T)$.

This theorem guarantees the evolving surfaces stay in compact region in $M$ along the entirety of the flow.

**Proposition 3.9.** If the mean curvature flow equation (3.1) has a solution on $[0, T)$, $0 < T \leq \infty$, then

\[
|\nabla^\ell u| \leq K_\ell < \infty,
\]

for all $\ell = 1, 2, \ldots$, where $\{K_\ell\}_{\ell=1}^\infty$ is the collection of constants depending only on $\ell$, the initial data and the maximal time $T$.

**Proof.** From Lemma 3.6, evolving surfaces are graphs of the height function $u(\cdot, t)$ to the reference surface $S$. Therefore $\Theta(\cdot, t) = 1/\sqrt{1 + |\nabla u|^2}$ ([Hui86]), and then $|\nabla u|$ is uniformly bounded from above by a constant depending only on the initial data and $T$.

We observe that equation (3.10) is a single quasilinear parabolic equation for the height function $u(\cdot, t)$, and $u$ is uniformly bounded by the Theorem 3.8. This enables us to apply the standard regularity results in quasilinear second order parabolic equations ([Fri64, Lie96]) to find

\[
|\nabla^\ell u| \leq K_\ell < \infty,
\]

for all $\ell = 1, 2, \ldots$, where $\{K_\ell\}_{\ell=1}^\infty$ is the collection of constants depending only on $\ell$, the initial data and $T$. \qed

**Proof of Proposition 3.7.** The conclusion is immediate by the relation $\Theta(\cdot, t) = 1/\sqrt{1 + |\nabla u|^2}$, and upper bounds on the derivatives of $u(\cdot, t)$ (Proposition 3.9). \qed

### 3.4. Estimates on $|A|^2$ and $H^2$

In this subsection we establish uniform bounds for $|A|^2$, which is crucial in showing the long-time existence of the solution to the mean curvature flow (3.1).

**Theorem 3.10.** If the mean curvature flow (3.1) has a solution on $[0, T)$, $0 < T \leq \infty$, then there exists a constant $C_7 < \infty$ depending only on $S_0$ such that

\[
|A(\cdot, t)|^2 \leq C_7 < \infty
\]

on $S(t)$, for $0 \leq t < T$. Moreover,

\[
\sup_{S_0 \times [0, T)} |\nabla^m A(\cdot, t)|^2 \leq C(m) < \infty
\]
for all $m \geq 1$.

**Proof.** In the first part, we follow closely to the proof of [HW09b, Theorem 4.1], with special cares to terms involving $\mathcal{H}$.

We introduce the function $\eta(\cdot, t) = \frac{1}{\Theta(\cdot, t)}$, and we want to use the evolution equation for some power of $\eta(\cdot, t)$ to add enough negative terms to the right-hand side of (3.16). We have

$$
\frac{\partial}{\partial t} \eta = -\eta^2 \Theta_t
$$

$$
= \Delta \eta - 2\Theta |\nabla \eta|^2 - (|A|^2 - 2)\eta - \eta^2 J',
$$

where we denote $J' = n(H_n) + (\mathcal{H} - H)\langle \nabla_\nu n, \nu \rangle - \langle \nabla \mathcal{H}, n \rangle$.

We also have

$$
L(\eta^2) = -6|\nabla \eta|^2 - 2\eta^2(|A|^2 - 2) - 2\eta^3 J',
$$

where we introduce the operator $L = \frac{\partial}{\partial t} - \Delta$ to simplify our notation.

We also have

$$
L(\eta^4) = -20\eta^2|\nabla \eta|^2 - 4\eta^4(|A|^2 - 2) - 4\eta^5 J'.
$$

We consider the function $f(\cdot, t) = |A|^2\eta^4$, and compute

$$
\frac{\partial}{\partial t} f = \eta^4 \frac{\partial}{\partial t} (|A|^2) + |A|^2 \frac{\partial}{\partial t} (\eta^4),
$$

We apply (3.6) and (3.17) to above and find

$$
L(f) = -2\eta^4|A|^4 - 2\eta^4|\nabla A|^2 + H\mathcal{H}(3|A|^2 - H^2 - 2)\eta^4 - 2h_{ij}\nabla_i \nabla_j \mathcal{H}
$$

$$
-2\nabla|A|^2 \cdot \nabla(\eta^4) + 4(3|A|^2 - H^2)\eta^4 - 20|A|^2\eta^2|\nabla \eta|^2 - 4|A|^2\eta^5 J',
$$

where the dominant term on the right is $-2\eta^4|A|^4 = -2\Theta^4 f^2$, for large $|A|^2$.

Note that we applied the assumption that $\mathcal{H}$ have bounded gradients $|\nabla \mathcal{H}|$ and $|\nabla^2 \mathcal{H}|$ in $M$, and the following bound [EH91, p. 604]:

$$
|\nabla^2 \mathcal{H}| \leq \Theta^2 |\nabla^2 \mathcal{H}| + \Theta |A||\nabla \mathcal{H}| \leq |\nabla^2 \mathcal{H}| + |A||\nabla \mathcal{H}|.
$$

Now suppose $|A|^2$ is not uniformly bounded, then

$$
|A|^2_{\text{max}}(\cdot, t) \to \infty \text{ as } t \to T.
$$

Since $f(\cdot, t) = |A|^2\eta^4 \geq |A|^2$, we have:

$$
f_{\text{max}}(\cdot, t) \to \infty \text{ as } t \to T.
$$

Therefore there exists a $t_1 \in (0, T)$ such that when $t > t_1$, we have

$$
\frac{d}{dt} f_{\text{max}} \leq -2\eta^4|A|^4 - 2\eta^4|\nabla A|^2 + 4(3|A|^2 - H^2)\eta^4
$$

$$
-2\nabla|A|^2 \cdot \nabla(\eta^4) - 12|A|^2\eta^2|\nabla \eta|^2 - 4|A|^2\eta^5 J'.
$$

From Proposition 3.6, at the point where $f_{\text{max}}$ occurs, we have

$$
-2\eta^4|A|^4 = -2\Theta^4 f^2_{\text{max}} \leq -2\Theta^4 f^2_{\text{max}}.
$$
From the proof of Proposition 3.6, we estimate the term $J'$:

$$\left| J' \right| = \left| n(H_n) + (\mathcal{H} - H)(\nabla_{\nu} n, \nu) - (\nabla_{\nu} \mathcal{H}, n) \right| \leq C_1 \Theta^3 + \left( C_1 |A| + C_2 (2 + |H|) + \|\nabla \mathcal{H}\| \right) \Theta^2.$$ 

Since $\Theta_0 \leq \Theta < 1$, for $t > t_1$, now we have

$$\frac{d}{dt} f_{\text{max}} \leq -2 \Theta_0^4 f_{\text{max}}^2 + \text{lower order terms}.$$ 

This is a contradiction since $\frac{d}{dt} f_{\text{max}} > 0$. Therefore $f(\cdot, t) = |A|^2 \eta^4$ is uniformly bounded, which also bounds $|A|^2$ from above.

The second part of the theorem is a standard induction argument, similar to the proof of [HW09b, Proposition 4.6], where we applied the argument in [Ham82] and [Hui84].

4. Prescribing mean curvature: conclusion

With all the pieces in place, we, in this section, prove Theorem 3.1: we establish long-time existence, examine convergence, and investigate the limiting surfaces.

**Proof.** of Theorem 3.1 (i) and (ii) (long-time existence): Suppose the maximal time $T < \infty$, and denote

$$(4.1) \quad S(T) = \lim_{t \to T} S(t) = \{ \lim_{t \to T} F(\cdot, t) \}.$$ 

By the Theorem 3.6, the height function $u(\cdot, t)$ is uniformly bounded, therefore, the surfaces $\{S(t)\}_{t \in [0, T]}$ stay in a compact smooth region in $M$, hence (4.1) is well-defined.

Applying the uniform bound on $|A(\cdot, t)|^2$ (Theorem 3.10), and the elementary inequality $|A|^2 \geq \frac{1}{2} H^2$, we find that $|H(\cdot, t)|^2$ is also uniformly bounded. The bounds on $|A(\cdot, t)|^2$, $|H(\cdot, t)|^2$ and $\mathcal{H}$, together with equation (3.2), imply the following:

$$\int_0^T \max_S \left| \frac{\partial}{\partial t} g_{ij}(\cdot, t) \right| dt \leq C_8 < \infty,$$

for some positive constant $C_8$. This enables us to apply [Ham82, Lemma 14.2], and find that $S(T)$ is a surface. It is also smooth since all derivatives of $A(\cdot, t)$ are bounded by the second part of Theorem 3.10.

The evolving surfaces $\{S(t)\}$ stay as graphs of the reference surface $S$ by Propositions 3.6 and 3.7. Hence the limiting smooth surface $S(T)$, is again, a graph over $S$. Now $S(T)$ satisfies the initial conditions for the mean curvature flow equation: smooth, closed, incompressible, and graph over $S$, therefore we use it as our initial surface in the equation (3.1) to extend the flow beyond the maximal time $T$, by the existence of short-time solutions for the parabolic equation (Theorem 2.4).

Therefore, the solution to (3.1) exists for all time, i.e., $T = \infty$, and each evolving surface $S(t)$ stays as a graph over $S$.

We are interested in the limiting behavior of the mean curvature $H(\cdot, t)$:

**Theorem 4.1.** The following holds:

$$\sup_{t \to \infty} |H(\cdot, t) - \mathcal{H}| = 0.$$
Proof. Let $M_t \subset M$ be the region bounded by the reference surface $S$ and $S(t)$. By Proposition 3.6, surfaces $\{S(t)\}$ are bounded in a compact region, hence the area $|S(t)|$ and the volume $|M_t|$ are uniformly bounded in $t$.

Applying the divergence theorem, we compute:

\[
\frac{d}{dt} \int_{M_t} \mathcal{H} dV = \int_{M_t} \text{div} \left( \mathcal{H} \frac{\partial}{\partial t} F \right) dV = \int_{S(t)} \left\langle \frac{\partial}{\partial t} F, \nu \right\rangle \mathcal{H} d\mu,
\]

here $\text{div}$ is the divergence on $M$. Denote the function

\[
\alpha(t) = |S(t)| - \int_{M_t} \mathcal{H} dV.
\]

Then from (3.4) and (4.3), we have

\[
\frac{d}{dt} \alpha(t) = \int_{S(t)} H(\mathcal{H} - H) d\mu - \int_{S(t)} \mathcal{H}(\mathcal{H} - H) d\mu
\]

\[
= - \int_{S(t)} (H - \mathcal{H})^2 d\mu.
\]

Therefore the function $\alpha(t)$ is non-increasing along the flow. The integral

\[
\int_{0}^{\infty} \int_{S(t)} (H - \mathcal{H})^2 d\mu dt = \alpha(0) - \alpha(\infty) < \infty,
\]

and we find that the integral $\int_{S(t)} (H - \mathcal{H})^2 d\mu$ is uniformly bounded.

We compute the $t$–derivative of this integral:

\[
\frac{d}{dt} \int_{S(t)} (H - \mathcal{H})^2 d\mu = \int_{S(t)} 2(H - \mathcal{H})(H_t - \mathcal{H}_t) - H(H - \mathcal{H})^3 d\mu
\]

\[
= \int_{S(t)} -2|\nabla(H - \mathcal{H})|^2 + 2(H - \mathcal{H})^2 (|A|^2 - 2) d\mu
\]

\[
+ \int_{S(t)} -H(H - \mathcal{H})^3 - 2\mathcal{H}(H - \mathcal{H}) d\mu.
\]

Here we applied (3.5).

The absolute value can be bounded by:

\[
\left| \frac{d}{dt} \int_{S(t)} (H - \mathcal{H})^2 d\mu \right| \leq \sup_{S(t)} (|H(H - \mathcal{H})| + 2|A|^2) \int_{S(t)} (H - \mathcal{H})^2 d\mu
\]

\[
+ 2 \int_{S(t)} (|\nabla(H - \mathcal{H})|^2 + |\langle \nabla \mathcal{H}, \nu \rangle||H - \mathcal{H}|^2) d\mu.
\]

where we used

\[
\int_{S(t)} \mathcal{H}_t(H - \mathcal{H}) d\mu \leq \int_{S(t)} |\langle \nabla \mathcal{H}, \nu \rangle||H - \mathcal{H}|^2 d\mu.
\]

Recall that the uniform bounds for $H(\cdot, t)$ follows from Theorem 3.10 and the inequality $2|A|^2 \geq H^2$. Its gradient bound is standard from the uniform bounds of $|A|^2$ and $|\nabla A|^2$. Now using the uniform bounds on $H$, $\mathcal{H}$, $|A|^2$, and their gradients, as well as Theorem 3.8,
we find that the term $\left| \frac{d}{dt} \int_{S(t)} (H - \mathcal{H})^2 d\mu \right|$ is also uniformly bounded in $t$. Therefore it must tend to zero as $t \to \infty$. This together with (4.5) implies the $L^2$ bound:

$$\sup_{t \to \infty} \| H(\cdot, t) - \mathcal{H} \|_{L^2} = 0.$$  

This $L^2$-estimate in conjunction with uniform bound on $|\nabla (H - \mathcal{H})|$ allow us to apply the standard interpolation argument (see for example, [Aub98], pp. 88–95) to show the $L^\infty$ bound, i.e., (4.2). 

**Proof of Theorem 3.1 (iii).** Since all evolving surfaces stay in a compact region (Theorem 3.8), and we have obtained uniform bounds for $|A|^2$ and its derivatives (Theorem 3.10), we can employ the theorem of Arzela-Ascoli to extract a subsequence of $S(t_i)$ converging to a limiting surface, of mean curvature function $\mathcal{H}$ (Theorem 4.1).

The evolving surfaces $\{S(t)\}_{t \geq 0}$ remain as graphs over the reference surface, therefore, by the estimates on the height function $u(\cdot, t)$ and its derivatives (Theorem 3.8, Proposition 3.9), the limiting surface is also a graph over the reference surface, hence embedded.  

The Corollary 1.5 is a direct application of the main theorem, by taking constant functions as the prescribed mean curvatures. One shall note that Theorem 1.4 does not directly apply to the case of volume preserving mean curvature flow, the situation we treated in [HW09a].

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