Spin Structures on Affine Kac-Moody Symmetric Spaces

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Abstract. We construct certain Fréchet principal $K$-bundles over affine Kac-Moody symmetric spaces by means of the natural projection $G \to G/K$ where $G$ is a (possibly twisted) geometric affine Kac-Moody group and $K$ is a real form when $G$ is complex and a “compact” real form when $G$ is real. Then we present two methods by means of which one can find certain spin structures on affine Kac-Moody symmetric spaces of compact type. We also briefly sketch defining a Dirac-like operator on these spaces.

1. Introduction

We study symmetric spaces in the context of affine Kac-Moody geometry. Our main goal is to establish a spin geometry for such symmetric spaces. These affine Kac-Moody symmetric spaces have interesting connections with supergravity theories. It is known among M-theorists that the factor space $E_{10}/K(E_{10})$ which is the base for certain bosonic geodesic $\sigma$-models has links to bosonic dynamics in some 11-dimensional supergravity theory (see [1]). In this context, elements of the “compact” form $K(E_{10})$ of the real algebraic hyperbolic Kac-Moody group $E_{10}$ are called hidden Kac-Moody symmetries. Among these symmetries, those in $K(E_{9})$ can be understood better because of their close connection with finite dimensional compact forms of finite dimensional semisimple Lie groups. Also studying the affine factor space $E_{9}/K(E_{9})$ can provide valuable insights in understanding the geometry of the hyperbolic case, namely $E_{10}/K(E_{10})$. Also, having a spinor bundle on such spaces can help to find and understand solutions of Dirac-like wave equations arising in supergravity theories as it does in the finite dimensional cases. Constructed in [2] (see also [3, 4]), affine Kac-Moody symmetric spaces are natural infinite dimensional generalization of finite dimensional symmetric spaces. They come from a completion of algebraic affine Kac-Moody groups and enjoy a natural tame Fréchet structure (for tame Fréchet spaces see [5]). This tame Fréchet structure enables us to use functional analytical tools such as the inverse function theorem and implicit function theorem to analyze affine Kac-Moody symmetric spaces. Affine Kac-Moody symmetric spaces under study here are generally of the form $G/K$ where $G$ is a (real) complex geometric affine Kac-Moody group and $K$ a geometric (“compact” real) real form obtained from a Cartan-Chevalley involution on $G$. Here by geometric affine Kac-Moody groups we mean a certain completion of minimal algebraic affine Kac-Moody groups (for minimal algebraic Kac-Moody groups see [6]). More precisely, let $G_{R}$ be a linear semisimple Lie group with a complexification $G_{C}$. We denote the corresponding algebra to $G_{R}$ and $G_{C}$ with $g_{R}$ and $g_{C}$ respectively. In this setting, we consider $C^{*} := C\setminus \{0\}$ as the unique linear complexification of the circle $S^{1}$. Define

\[ C^{*}G_{C} := \{ f : C^{*} \to G_{C} \mid f \text{ holomorphic and } \sigma \circ f(z) = f(\omega z) \}, \]

and

\[ C^{*}G_{R} := \{ f \in C^{*}G_{C} \mid f(S^{1}) \subset G_{R} \}, \]

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where \( \sigma \) comes from a diagram automorphism of order \( n \) of \( \mathfrak{g}_C \) and \( \omega := e^{2\pi i} \). A complex geometric affine Kac-Moody group \( \hat{C}G^R_C \) is a certain \((\mathbb{C}^*)^2\)-bundle over \( \hat{C}G^*_C \) whose natural real form \( \hat{C}G^R_C \) is a certain torus fiber bundle over \( \hat{C}G^*_C \) (for details see [2]). When \( \sigma \) is the identity then \( \hat{C}G^R_C \) is referred to as untwisted and denoted by \( \hat{C}G^R_C \). Note that construction is analogous to the construction of affine Kac-Moody groups as certain two dimension extensions of polynomial loop groups. Now for a compact real form \( K_R \) of \( G_R \), we define \( \hat{C}K^*_C \) and \( \hat{C}K^R_R \) as above. It turns out these groups can be considered as natural “compact” (real) forms of \( \hat{C}G^*_C \) and \( \hat{C}G^R_C \) respectively (see [2]). By \( G \) we denote a geometric affine Kac-Moody group and by \( K \) its compact (real) form for a fixed compact (real) form \( K_R \) of \( G_R \). By the main results of [2] (see also [3, 4]) we know that all of the above groups are tame Fréchet Lie groups with a topology locally compatible with the compact-open topology. Moreover, it is also shown in [2] that the corresponding symmetric spaces, \( G/K \), are tame Fréchet manifolds. In Section 2 by means of a version of the implicit function theorem for tame Fréchet spaces and applying a generalization of the methods from [7] we prove that the natural projection

\[
\pi : G \to G/K
\]

is a locally trivial tame Fréchet principal \( K \)-bundle. In [2], the geometry considered on \( G/K \) is induced by the tame Fréchet geometry on \( G \) via \( \pi \). Now having local trivializations for the bundle \( \pi \) leads us to a better understanding of the local geometry of \( G/K \) by looking at \( G \). There has been a lot of progress in the study of loop groups and loop spaces (in the sense of [8] and [9]). By the local triviality for \( \pi \) we are now able to pull-back structures on loop spaces over our symmetric spaces which are again locally trivial. In contrast with loop spaces of compact Lie groups, there exists the concept of duality among affine Kac-Moody symmetric spaces. Moreover, geometric affine Kac-Moody groups as holomorphic completions of algebraic minimal affine Kac-Moody groups (for algebraic minimal Kac-Moody groups see [10] and [6]) are carefully chosen so that they contain important symmetries such as isometries of the algebraic Kac-Moody algebra in the algebra level (see [3] Remark 3.13)). The relation between symmetric loop spaces and our affine Kac-Moody symmetric spaces is not obvious. In Section 3 we show that for affine Kac-Moody symmetric spaces of compact type, there exists a continuous injection from our symmetric spaces to symmetric loop spaces. In this section we also provide a method by which one can produce certain loop-string structures on symmetric spaces coming from their finite dimensional part. In Section 4 we define a finite spin structure on affine Kac-Moody symmetric spaces of type \( \hat{E}_n \) (\( n \leq 8 \)) via a finite dimensional unfaithful spin representation of their “compact” forms. Again by the local triviality obtained in Section 2 one can define the Čech cohomology and benefit from its features specially its cohomological spectral sequence. Having defined Čech cohomology for these bundles, we can provide certain conditions for existence of finite spin structures on the underlying symmetric spaces. Finally in Section 5 we sketch how one can define a Dirac-like operator on affine Kac-Moody symmetric spaces for those with finite spin structure.

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### 2. Factor Spaces of Tame Fréchet-Lie Groups

We start this section by citing a famous inverse function theorem due to Nash and Moser.

**Theorem 2.1 (Nash-Moser Inverse Function Theorem).** [5] Thoerem III.1.1.1] Let \( F \) and \( G \) be two tame Fréchet spaces. Let \( \phi : U \subseteq F \to G \) be a smooth tame map where \( U \) is an open subset of \( F \). Suppose that the equation for the derivative \( D_f \phi(h) = k \) has a unique solution \( h = V_f \phi(k) \) for all \( f \in U \) and all \( k \) and the family of inverses \( V \phi : U \times G \to F \) is a smooth tame map. Then \( \phi \) is locally invertible and each local inverse \( \phi^{-1} \) is a smooth tame map.
There are many different versions of the implicit function theorem derived from the above Nash-Moser inverse function theorem (see e.g. \[5\]). Here we give a proof for a simpler version of this theorem which is more suitable for the purposes of this article.

**Corollary 2.2 (Implicit Function Theorem).** Let \( F, V \) and \( W \) be three tame Fréchet spaces. Let \( \phi : U \subseteq F \times V \to W \) be a smooth tame map where \( U \) is open. Suppose that the partial derivative \( D_y \phi(X,Y) \) has a unique smooth tame right inverse for all \((X,Y) \in U\). If \( \phi(X_0,Y_0) = 0 \) for some \((X_0,Y_0) \in U\), then there exists open sets \( U' \) and \( U'' \) in \( F \) and \( V \) respectively and a unique smooth tame map \( \psi : U' \to U'' \) such that \( \phi(X,\psi(X)) = 0 \).

**Proof.** Define
\[
\Phi : U \subseteq F \times V \to F \times W \\
(x,y) \mapsto (x,\phi(x,y)).
\]
Clearly, \( \Phi \) is a smooth tame map (see \[5\] Part II.2] and \[3\] Lemma 2.25]). Moreover, \( D\Phi \) can be represented in the following matrix form with respect to the decomposition of the domain and range of \( \Phi \):
\[
\begin{pmatrix}
\text{Id} & D_x \phi \\
0 & D_y \phi
\end{pmatrix}
\]
By hypotheses, the partial derivation \( D_y \Phi(x,y) \) has a unique tame right linear inverse, \( D_y^{-1} \Phi \), for all \((x,y) \in U\). Define \( D^{-1} \Phi : U \subseteq F \times V \to F \times W \) by the following representation:
\[
\begin{pmatrix}
\text{Id} & -D_x \phi D_y^{-1} \phi \\
0 & D_y^{-1} \phi
\end{pmatrix}
\]
which defines a unique right inverse for \( D\Phi \). Note that since \( D_y^{-1} \phi : W \to V \), the composition \( -D_x \phi D_y^{-1} \phi \) in \(2.3\) is valid. Moreover, since \( \Phi \) is a smooth tame map, the family of inverses \( D^{-1} \Phi : U \times F \times V \to F \times W \) is smooth and tame and linear with respect to its \( F \) and also \( V \) components and hence it is smooth and tame with respect to all of its components (see \[5\] Part I.3] and \[3\] Theorem 2.33]).

Now the Nash-Moser inverse function theorem, Theorem 2.1, applied to \( \Phi \) implies that \( \Phi \) is locally invertible and following similar arguments as in the proof of the classical implicit function theorem, we conclude the corollary (see for example the 2nd part of the proof of\[3\] Theorem 2.48].)

**Definition 2.3 (Slicing function).** Let \( G \) be a topological group and \( K \) be a closed subgroup. Endow \( G/K \) with the quotient topology. A **slicing function** \( \psi \) for the element \( K \) in \( G/K \) is a continuous function defined from a neighborhood \( N \) around \( K \) to \( G \) such that \( \psi(K) \) is mapped to the identity in \( G \) and \( \psi \) remains constant on every coset \( xK \in N \). In other words, \( \psi \) is a continuous cross section of the projection map \( \pi : G \to G/K \).

**Definition 2.4 (Tame Fréchet submanifold).** Let \( M \) be a tame Fréchet manifold. A closed subset \( N \) of \( M \) is called a **tame Fréchet submanifold** if for every \( x \in N \) there exists a chart \((\phi_x, U_x)\) of \( M \) containing \( x \) such that \( \phi_x : U_x \to F \times W \) for a pair of tame Fréchet spaces \( F \) and \( W \), and in addition,
\[
y \in N \cap U_x \text{ if and only if } \phi_x(y) \in 0 \times W.
\]

Similar to the method in \[7\], we prove a slicing function theorem by means of our version of the implicit function theorem in the tame Fréchet context.

**Proposition 2.5.** Let \( G \) be a tame Fréchet-Lie group. Let \( K \) be a closed tame Fréchet-Lie subgroup of \( G \) such that the quotient space \( G/K \) forms a tame Fréchet manifold with respect to the quotient topology. Then there exists a slicing for \( K \). In addition, when the projection is smooth and tame, we obtain a smooth tame slicing of \( K \).
Proof. Let \((\phi, U)\) be a submanifold chart of \(G\) for \(K\) at the identity \(e \in G\). By definition, there are two tame Fréchet spaces \(F\) and \(W\) such that
\[
\phi : U \to F \times W =: H.
\]
We can assume that \(\phi(e) = 0 \in H\) and since \(G\) is a tame Fréchet-Lie group, we can also assume that \(U \cdot U \subseteq U\). Moreover, by definition submanifold charts, \(x \in K \cap U\) if and only if \(\phi(x) \in 0 \times W\). Define \(\tilde{U} := \phi(U)\), \(X := \phi(x)\) for all \(x \in U\) and
\[
(2.4) \quad \begin{cases}
\Phi : \tilde{U} \times \tilde{U} \subseteq H \times H \to H = F \times W \\
(X, Y) \mapsto (\Phi_1(X, Y), \Phi_2(X, Y)),
\end{cases}
\]
where
\[
(2.5) \quad \begin{cases}
\Phi_1 : \tilde{U} \times \tilde{U} \to F \\
(X, Y) \mapsto \pi_F \circ \phi(x^{-1}y) =: \phi(x^{-1}y)_F,
\end{cases}
\]
and
\[
(2.6) \quad \begin{cases}
\Phi_2 : \tilde{U} \times \tilde{U} \to W \\
(X, Y) \mapsto \pi_W(Y) =: Y_W,
\end{cases}
\]
where \(\pi_F\) and \(\pi_W\) are the natural projections of \(H\) onto \(F\) and \(W\) respectively. Note that all these maps are clearly smooth and tame [5, Part II.2]).

In the above setting, the partial derivative \(D_Y \Phi(X, Y)\) has the following matrix representation:
\[
(2.7) \quad \left( \begin{array}{c}
D_Y \Phi_1^X \\
D_Y \Phi_2
\end{array} \right),
\]
where \(\Phi_1^X := \Phi(X, -)\).

First note that \(D_Y \Phi_2 : H \to W\) is just the projection onto \(W\) with a unique smooth tame right inverse \(D_i : W \to F \times W\).

Let \(DL_X : H \to F \times W\) be the derivative of the left translation in \(G\) by \(x^{-1} \in U\) which is a smooth tame linear isomorphism (Since \(G\) is a tame Fréchet-Lie group, the left and right translations are tame diffeomorphisms as well as the inverse map. We ignore the role of the inverse map in the definition of \(\Phi\) for the same reason and simplicity). We obtain the following matrix representation for \(DL_X\):
\[
(2.8) \quad \left( \begin{array}{c}
(DL_X)_F \\
(DL_X)_W
\end{array} \right).
\]

Therefore, \((DL_X)_F = D_Y \Phi_1^X\) has a unique smooth tame right inverse, namely \(D_Y^{-1} \Phi_1^X\). This enables us to define a unique smooth tame right inverse \(D_Y^{-1} \Phi(X, Y)\) for \(D_Y \Phi(X, Y)\) for all \(X, Y \in \tilde{U}\) with the following matrix representation:
\[
(2.9) \quad \left( \begin{array}{c}
D_Y^{-1} \Phi_1^X \\
D_i
\end{array} \right).
\]

By assumption, for \(X_0 := \phi(e)\) we have \(\Phi(X_0, X_0) = 0\). Now our version of the implicit function theorem, Theorem 2.2 implies that there exists a unique smooth tame map \(\Psi : \tilde{U}' \subseteq \tilde{U} \to \tilde{U}'' \subseteq \tilde{U}\) such that
\[
(2.10) \quad \Phi(X, \Psi(X)) = 0.
\]

We claim that the map
\[
(2.11) \quad \begin{cases}
\psi : \phi^{-1}(\tilde{U}') \to \phi^{-1}(\tilde{U}'') \\
\psi := \phi^{-1} \circ \Psi \circ \phi,
\end{cases}
\]
defines the desirable smooth tame slicing function for \(K\).

To see this, first note that for any \(X, Y \in \tilde{U}\) with the property \(\Phi(X, Y) = 0\) we have
- \(\phi(x^{-1}y)_F = 0 \Rightarrow x^{-1}y \in K \cap U \Rightarrow y = xk\) for some \(k \in K\);
• \( \phi(y)_G = 0 \Rightarrow y \in G \setminus K. \)

We conclude from above that by the uniqueness of \( \Psi \), \( y \) is the only element in \( xK \cap G \setminus K \). And hence, \( \psi \) remains constant along \( xK \) for every \( x \) in the domain of \( \psi \). Therefore, by reduction of the domain, \( \psi : V \to G \) can be defined where \( V := \pi(\phi^{-1}(U')) \) which remains an open subset of \( G/K \) containing \( K \) with respect to the quotient topology on \( G/K \).

**Definition 2.6** (Tame Fréchet fibre bundle). Let \( \pi : P \to M \) be a (topological) fibre bundle between tame Fréchet manifolds. We call \( P \) a tame Fréchet fibre bundle over \( M \) if \( \pi \) is a smooth tame map and the bundle satisfies the following condition:

Every \( x \in M \) contains in a chart domain \( (\phi_x, U_x) \) with values in an open subset \( U \) of a tame Fréchet space \( F \) such that there exist a tame Fréchet manifold \( N \) (the typical fibre) and a smooth tame map \( \psi : \pi^{-1}(U_x) \to U \times N \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\pi^{-1}(U_x) & \xrightarrow{\psi} & U \times N \\
\downarrow{\pi} & & \downarrow{\pi_1} \\
U_x & \xrightarrow{\phi} & U \\
\end{array}
\]

and \( \pi^{-1}(y) \cong_{\text{tame}} N \) for every \( y \in U_x \). A tame Fréchet fibre bundle is locally trivial if \( \psi \) is a tame diffeomorphism. A tame Fréchet fibre bundle is called a tame Fréchet principal \( N \)-bundle when the typical fibre is a tame Fréchet-Lie group.

**Theorem 2.7.** Let \( G \) be a tame Fréchet-Lie group. Let \( K \) be a tame Fréchet-Lie subgroup of \( G \) such that the quotient space \( G/K \) forms a tame Fréchet manifold with the quotient topology. Then the natural projection \( \pi : G \to G/K \) forms a locally trivial principal \( K \)-bundle. Moreover, when \( \pi \) is a smooth tame map, it is a locally trivial tame Fréchet principal \( K \)-bundle over \( G/K \).

**Proof.** Proposition \( \text{2.6} \) and \( \text{[4], Theorem 1} \) imply that \( \pi : G \to G/K \) is a principal \( K \)-bundle. To show that \( \pi \) is a locally trivial tame Fréchet fibre bundle by the homogeneity of \( G/K \) it suffices to show that for a chart \( (\phi, U) \) of \( K \) of \( G/K \), the tame Fréchet condition \( \text{(2.12)} \) holds for some \( \psi : \pi^{-1}(U) \to U \times K \). Under the identification \( \pi^{-1}(x) \cong_{\text{tame}} \{ x, K \} \) define

\[
\psi(x, k) = (x, s(x)^{-1} \cdot k),
\]

where \( s \) denotes the smooth tame cross section of \( K \) obtained in Proposition \( \text{2.6} \). Since by the hypothesis \( \pi \) is a smooth tame map, we conclude that \( \psi \) is a smooth tame map with a smooth tame inverse defined as follows:

\[
\psi^{-1}(x, k) = s(x) \cdot k.
\]

This completes the proof.

**Observation 2.8.** Let \( G/K \) be an affine Kac-Moody symmetric space defined in \( \text{[2]} \) (see also \( \text{[3, 4]} \)). Then \( K \) is a tame Fréchet subgroup of \( G \) by means of the explicit charts given in \( \text{[2], Chapters 3 and 4} \) (see also \( \text{[3, Section 4.2.2 and 4.5.1]} \)). Moreover, the construction of such charts also shows that \( \pi : G \to G/K \) is a smooth tame submersion (see \( \text{[5, Definition 4.4.8]} \)).

**Theorem 2.9.** Let \( G/K \) be an affine Kac-Moody symmetric space. Then

\[
\pi : G \to G/K,
\]

is a locally trivial tame Fréchet principal \( K \)-bundle. In particular, if \( G/K \) is of non-compact type, then \( G \cong G/K \times K \).

**Proof.** The theorem is a direct consequence of Proposition \( \text{2.6} \) and Observation \( \text{2.8} \). In particular, when \( G/K \) is of non-compact type by \( \text{[2, Corollary 4.4.1]} \) (see also \( \text{[3, Theorem 1.2]} \)), it is diffeomorphic to a topological vector space and hence contractible. Because \( \pi : G \to G/K \) is a locally trivial principle bundle, we conclude that \( G \cong G/K \times K \) as topological spaces.
Corollary 2.10. Let $G/K$ be an affine Kac-Moody symmetric space. Then the topological gauge group of
\begin{equation}
\pi : G \to G/K,
\end{equation}
\[\text{namely, } \text{Aut}_{G/K}(G) \text{ is isomorphic to the topological group of cross sections of the locally trivial tame Fréchet principal } K\text{-bundle}\]
\begin{equation}
\text{Ad}(\pi) : \text{Ad}(G) := G \times_{\text{Ad}} K \to G/K,
\end{equation}
where $\text{Ad}(\pi)$ is constructed by considering the conjugation action of $K$ on itself. Moreover, if $G/K$ is of non-compact type, then $\text{Aut}_{G/K}(G)$ is isomorphic to the topological mapping group $\text{Map}(G/K, K)$.

Proof. Since, by Theorem 2.9, $\pi$ is a locally trivial tame Fréchet principal $K$-bundle and $K$ is a tame Fréchet-Lie group it follows that $\text{Ad}(\pi)$ is indeed a locally trivial tame Fréchet principal $K$-bundle. Let $\Gamma(\text{Ad}(\pi))$ denote the topological group of cross sections of $\text{Ad}(\pi)$. Then
\[\text{Aut}_{G/K}(G) \cong \Gamma(\text{Ad}(\pi))\]
is an immediate consequence of [11, Remark 7.1.4 and Remark 7.1.6]. Moreover, when $G/K$ is of non-compact type, then by Theorem 2.9, $\pi$ is trivial and hence the isomorphism
\[\text{Aut}_{G/K}(G) \cong \text{Map}(G/K, K)\]
follows from [11, Proposition 7.1.7]. □

3. Pull-back Loop Structures on Affine Kac-Moody Symmetric Spaces

Let $G_R$ be a linear connected semisimple compact Lie group with its unique (up to isomorphism) complexification $G_C$. In this setting, we consider $\mathbb{C}^* := \mathbb{C}\setminus\{0\}$ as the unique linear complexification of the circle $S^1$. Define
\[\mathbb{C}^*G_C := \{ f : \mathbb{C}^* \to G_C \mid f \text{ holomorphic} \},\]
and
\[\mathbb{C}^*G_R := \{ f : \mathbb{C}^* \to G_C \mid f \text{ holomorphic and } f(S^1) \subset G_R \},\]
Since $\mathbb{C}^*$ is the complexification of $S^1$ we define
\begin{equation}
\begin{cases}
H : & \mathbb{C}^*G_R \hookrightarrow LG_R \\
& f \mapsto f|_{S^1},
\end{cases}
\end{equation}
where $LG_R$ is the free loop group of $G_R$ as an infinite dimensional, paracompact manifold modeled on $LR^\infty$ with respect to the (compact-open) $C^\infty$-topology (the topology of uniform convergence of the functions and all their partial derivatives of all orders (see [5, 12, 13])).

Lemma 3.1. $H$ as defined above is a continuous injection.

Proof. By the uniqueness theorem for holomorphic functions, any $f \in \mathbb{C}^*G_R$ is completely determined by its values on $S^1$. This implies injectivity of $H$.
To see that $H$ is continuous first note that, by [3, Theorem 4.7], the Fréchet topology on $\mathbb{C}^*G_R$ is (locally) compatible with the compact-open topology. And the compact-open topology coincides with the $C^\infty$-topology for holomorphic functions (cf. [13], see also [14]). But the topology on $LG_R$ is also the $C^\infty$-topology, hence continuity. □

Let $\mathbb{C}^*G_R/\mathbb{C}^*K_R$ be an untwisted affine Kac-Moody symmetric space of compact type defined in [2, 3]. We call the tame Fréchet manifolds $\mathbb{C}^*G_R/\mathbb{C}^*K_R$, primitive affine Kac-Moody symmetric.
spaces. We always assume that the compact form $K_R$ of $G_R$ is connected. The following commutative diagram of topological embeddings

$$
\begin{array}{ccc}
C^*G_R & \overset{H}{\longrightarrow} & LG_R \\
\downarrow \uparrow & & \downarrow \uparrow \\
C^*K_R & \overset{H}{\longrightarrow} & LK_R
\end{array}
$$

where $H$ is defined as in (3.1), yields the following continuous mapping for the corresponding quotient space:

$$
C^*G_R/C^*K_R \to LG_R/LK_R.
$$

**Lemma 3.2.** The continuous mapping (3.3) is injective.

**Proof.** Let $\gamma$ and $\delta$ be two loops in $C^*G_R$ such that $[\gamma] = [\delta]$ in $LG_R/LK_R$ via $H$ as in (3.1). Hence there exists a loop $\alpha$ in $LK_R$ such that $\gamma \cdot \alpha = \delta$ on $S^1$. Since $\gamma$ and $\delta$ are both in $C^*G_R$, $\alpha = \delta \cdot \gamma^{-1}$ has a canonical extension $\tilde{\alpha}$ to $C^*$. This implies that $[\gamma] = [\delta]$ in $C^*G_R/C^*K_R$. Hence (3.3) defines a continuous injection. \qed

**Lemma 3.3.** With the above notations, we have the following continuous injection

$$
LG_R/LK_R \hookrightarrow L(G_R/K_R),
$$

with respect to the (compact-open) $C^\infty$-topology.

**Proof.** Define

$$
\Phi : \begin{cases}
\gamma : S^1 \to G_R \\
t \mapsto \gamma(t) \\
\Phi(\gamma) : S^1 \to G_R/K_R \\
t \mapsto [\gamma(t)]
\end{cases}
$$

Since the mapping $\pi : G_R \to G_R/K_R$ is continuous, $\Phi$ is continuous. Moreover, $\Phi$ is constant on $LK_R$ as it maps any loop in $LK_R$ to the constant loop $t \mapsto K_R$ in $L(G_R/K_R)$. Therefore, we obtain the following continuous map induced by $\Phi$:

$$
\tilde{\Phi} : LG_R/LK_R \to L(G_R/K_R).
$$

Now, if $\Phi(\gamma) = \Phi(\delta)$ for a pair $\gamma, \delta \in LG_R$ then

$$
\forall t \in S^1 : [\gamma(t)] = [\delta(t)] \implies \gamma(t)K_R = \delta(t)K_R
$$

hence for all $t \in S^1$, $\delta(t)^{-1}\gamma(t) \in K_R$. Define

$$
\begin{cases}
\alpha : S^1 \to K_R \\
t \mapsto \delta(t)^{-1}\gamma(t),
\end{cases}
$$

then it follows that $[\gamma]^L = [\delta]^L$ via $\alpha$ in $LG_R/LK_R$. \qed

From the above lemma we obtain the following commutative diagram

$$
\begin{array}{ccc}
LG_R & \overset{\Phi}{\longrightarrow} & L(G_R/K_R) \\
\downarrow \Phi & & \downarrow \Phi \\
LG_R/LK_R & & 
\end{array}
$$

Now we consider the following bundle:

$$
LG_R \to LG_R/LK_R.
$$
which is an $LKR$-bundle considered as infinite dimensional manifolds with the structure groups as
infinite dimensional Lie groups in the sense of [15]. Also, by applying the loop “functor” to the natural principal $KR$-bundle
(3.8) \[ \pi : GR \to G_R/K_R. \]
we obtain the following $LKR$-bundle:
(3.9) \[ L\pi : LG_R \to L(G_R/K_R). \]
We also have the following tame Fréchet principal $C^*KR$-bundle
(3.10) \[ C^*GR \to C^*GR/C^*KR. \]
Combining these bundles with the embeddings in Lemma 3.2 and Lemma 3.3 we obtain the following
commutative diagram:
(3.11) \[ C^*KR \longrightarrow LKR \]
\[ \downarrow \]
\[ C^*GR \longrightarrow LG_R \]
\[ \downarrow \]
\[ C^*GR/C^*KR \longrightarrow LG_R/LKR \]
\[ \phi \longrightarrow \phi \]
\[ \downarrow \]
\[ \Phi \longrightarrow \Phi \]
\[ \downarrow \]
\[ L\pi \longrightarrow L(G_R/K_R) \]
Now by lifting the bundle $L\pi$ over $C^*GR/C^*KR$ via $\Phi \circ i$ in (3.11) we obtain the following principal
$LKR$-bundle:
(3.12) \[ \xi : LKR \leftrightarrow (\Phi \circ i)^*(L\pi) \to C^*GR/C^*KR. \]
Moreover, any structure on $\Pi$ and, particularly, $L\pi$ can be lifted to $\xi$.

Remark 3.4. Notice that if we replace smooth loops by continuous loops and denote the mapping space it generates by $L$, we obtain a diagram similar to (3.11) but instead, the bundle (3.7) becomes a locally trivial Banach principal $LK_R$-bundle (see [15]) as a special case of Theorem 2.7, namely
(3.13) \[ LG_R \to LG_R/LK_R. \]

Let $\mathfrak{D}$ denote a Dynkin diagram and let $G(\mathfrak{D})$ denote the split (algebraically) simply-connected
real Kac-Moody group of type $\mathfrak{D}$ (see [10]). By $K(\mathfrak{D})$ we understand the fixed point subgroup associated to the Cartan-Chevalley involution $o_\mathfrak{D}$ on $G(\mathfrak{D})$ with respect to $\mathfrak{D}$. We also call $K(\mathfrak{D})$ a maximal compact form of $G(\mathfrak{D})$. Moreover, if $\mathfrak{D}$ is an irreducible simply laced Dynkin diagram, then by [16 Theorem A], up to isomorphism, there exists a uniquely determined group $Spin(\mathfrak{D})$ with a canonical two-to-one central extension:
(3.14) \[ \mathbb{Z}_2 \to Spin(\mathfrak{D}) \to K(\mathfrak{D}). \]
For a spherical Dynkin diagram $\mathfrak{D}$, let $L^*K(\mathfrak{D})$ denote the connected component of $LK(\mathfrak{D})$. When $K(\mathfrak{D})$ is connected, it is known that the number of connected components of $LK(\mathfrak{D})$ is equal to the cardinality of the first homotopy group of $K(\mathfrak{D})$. Note that in this setting when $\mathfrak{D}$ is spherical, we always assume $K(\mathfrak{D})$ to be connected.
Before we state the next result, notice that for a spherical irreducible Dynkin diagram $\mathfrak{D}$ of rank $n \geq 3$, by the universal coefficient theorem and the Hurewicz theorem we know that
(3.15) \[ H^1(BLK(\mathfrak{D}); A) \cong \text{Hom}(\pi_1(K(\mathfrak{D})), A), \]
for any abelian group $A$.
Here we give a slightly generalized version of [8 Property 2.1].
Proposition 3.5. Let $M$ be a 1-connected orientable Riemannian manifold of finite dimension. Let $P \to M$ be a principal $K(\mathfrak{D})$-bundle where $\mathfrak{D}$ is a spherical irreducible Dynkin diagram of rank $n \geq 3$ and let $\hat{K}(\mathfrak{D})$ denote the universal covering group of $K(\mathfrak{D})$. Moreover, assume that $\pi_1(K(\mathfrak{D})) \cong \mathbb{Z}_2$. Then the following are equivalent:

1. The structure group of $LP \to LM$ is reducible to $L^0K(\mathfrak{D})$.
2. The structure group of $P \to M$ can be lifted to $\hat{K}(\mathfrak{D})$.
3. The structure group of $LP \to LM$ can be lifted to $LK(\mathfrak{D})$.

Proof. It follows from the proof of [8] Proposition 2.1] by replacing $SO(n)$ by $K(\mathfrak{D})$, replacing $A$ by $\pi_1(K(\mathfrak{D}))$ in (3.15) and using the following covering map:

$$\pi_1(K(\mathfrak{D})) \to \hat{K}(\mathfrak{D}) \to K(\mathfrak{D}).$$

Corollary 3.6. Let $\mathfrak{D}$ be a spherical irreducible Dynkin diagram of rank $n \geq 3$. Assume that $G(\mathfrak{D})$ is compact and 1-connected. Assume that $K(\mathfrak{D})$ is connected and $\pi_1(K(\mathfrak{D})) \cong \mathbb{Z}_2$. Then the primitive affine Kac-Moody symmetric space

$$C^*G(\mathfrak{D})/C^*K(\mathfrak{D}),$$

admits an $L\hat{K}(\mathfrak{D})$ structure if the structure group of the bundle

$$G(\mathfrak{D}) \to G(\mathfrak{D})/K(\mathfrak{D}),$$

can be lifted to $\hat{K}(\mathfrak{D})$, or equivalently, if the structure group of

$$LG(\mathfrak{D}) \to L(G(\mathfrak{D})/K(\mathfrak{D})),$$

can be reduced to $L^0K(\mathfrak{D})$.

Proof. Since $G(\mathfrak{D})$ is 1-connected and $K(\mathfrak{D})$ is connected, by the long exact sequence of homotopy groups induced by (3.18) we conclude that $G(\mathfrak{D})/K(\mathfrak{D})$ is 1-connected. The corollary follows from Proposition 3.5 and pulling back the $L\hat{K}(\mathfrak{D})$ structure over (3.19) on (3.17) via the commutative diagram (3.11).

Definition 3.7. For a spherical simply laced irreducible Dynkin diagram $\mathfrak{D}$, a lift of the structure group of a principal $LK(\mathfrak{D})$-bundle to $L\hat{K}(\mathfrak{D}) = L\text{Spin}(\mathfrak{D})$ (if exists) is called a loop-spin structure.

Lemma 3.8. In the above setting, if the primitive affine symmetric Kac-Moody space $C^*G(\mathfrak{D})/C^*K(\mathfrak{D})$ admits a loop-spin structure, then the affine symmetric Kac-Moody space $C^*\widehat{G}(\mathfrak{D})/C^*\widehat{K}(\mathfrak{D})$ admits a loop-spin structure as well.

Proof. Similar to the proof of [8] Theorem 4.56] adapted for affine Kac-Moody symmetric spaces of type I, $C^*G(\mathfrak{D})/C^*K(\mathfrak{D})$ can be considered as an $((\mathbb{R}^+)^2)$-bundle over $C^*G(\mathfrak{D})/C^*K(\mathfrak{D})$ (which, unlike the case in [8] Theorem 4.56], does not need to be trivial). Hence there exists a surjection

$$P: C^*\widehat{G}(\mathfrak{D})/C^*\widehat{K}(\mathfrak{D}) \to C^*G(\mathfrak{D})/C^*K(\mathfrak{D}).$$

The pull-back of the loop spin structure on $C^*G(\mathfrak{D})/C^*K(\mathfrak{D})$ on $C^*\widehat{G}(\mathfrak{D})/C^*\widehat{K}(\mathfrak{D})$ by $P$ induces a loop spin structure for $C^*\widehat{G}(\mathfrak{D})/C^*\widehat{K}(\mathfrak{D})$.

Definition 3.9. For a spherical simply laced irreducible Dynkin diagram $\mathfrak{D}$, a lift of the loop-string structure on a principal bundle to the universal extension $L\text{Spin}(\mathfrak{D}) := L\text{Spin}(\mathfrak{D}) \rtimes S^1$ is called a loop-string structure.
Proposition 3.10. In the setting of Definition 3.7, assume that $G(\mathcal{D})$ is compact and 1-connected, and $K(\mathcal{D}) \cong SO(n)$ where $n \geq 5$. Assume that the canonical principal $K(\mathcal{D})$-bundle
\[ \xi: K(\mathcal{D}) \rightarrow G(\mathcal{D}) \rightarrow G(\mathcal{D})/K(\mathcal{D}), \]
has a spin structure. If the first Pontrjagin class of $\xi$ is zero (i.e., $p_1(\xi) = 0$), then $\widehat{C^*G(\mathcal{D})}/\widehat{C^*K(\mathcal{D})}$ admits a loop-string structure.

Proof. Let
\[ \eta: \text{Spin}(n) \rightarrow Q \rightarrow G(\mathcal{D})/K(\mathcal{D}), \]
be the spin bundle associated to $\xi$. Since $p_1(\xi) = 0$ by [3] Theorem 3.1
\[ L\eta: LQ \rightarrow \text{L}(G(\mathcal{D})/K(\mathcal{D})), \]
admits a loop-string structure. Now by similar procedures as in the proof of Corollary 3.6 and Lemma 3.8 we can pull back the loop-string structure of $\text{L}(G(\mathcal{D})/K(\mathcal{D}))$ first on $\widehat{C^*G(\mathcal{D})}/\widehat{C^*K(\mathcal{D})}$ and then on $\widehat{C^*G(\mathcal{D})}/\widehat{C^*K(\mathcal{D})}$. \hfill \Box

Example 3.11 (A class of loop spin structures of type $E_9$I). For $\mathcal{D} = E_8$, we have $G(E_8)$ is compact and 1-connected, also, for $K(E_8) \cong (SU(2)\times E_7)/\mathbb{Z}_2 \cong SO(16)$. Hence by Corollary 3.6 any spin structure on $G(E_8)/K(E_8)$ can be pulled back on an affine Kac-Moody symmetric space of type $E_9$I, namely
\[ (3.21) \quad \text{LSpin}(\xi): \text{LSpin}(\mathcal{D}) \hookrightarrow \text{LS} \rightarrow \widehat{C^*G(E_8)}/\widehat{C^*K(E_8)}. \]
A string structure of the symmetric space of type (EVIII) also provides a loop string structure on the above affine Kac-Moody symmetric space. The same stands for the affine Kac-Moody symmetric space obtained from the symmetric space of type (AI) for ranks greater than 4, according to the well-known list of finite dimensional symmetric spaces.

4. Finite Spin Structures on Affine Kac-Moody Symmetric Spaces of Type $E_n$

Let $M := G/K$ denote the affine Kac-Moody symmetric space of type I constructed in [2] where $G := \hat{C}^*G_{\mathbb{R}}$ and $K := \hat{C}^*K_{\mathbb{R}}$ are as in Section 2 obtained from a Cartan-Chevalley involution. In this section, all Kac-Moody groups are of type $E_n$ for $(6 \leq n \leq 9)$ and affine symmetric spaces are untwisted and of the above form. We saw in Section 2 that
\[ (4.1) \quad \pi: G \rightarrow M, \]
is a locally trivial tame Fréchet principal $K$-bundle. In this section we present conditions under which the above bundle admits finite dimensional spin bundles.

Lemma 4.1. Let $K := \hat{C}^*K_{\mathbb{R}}$ be as above where $K_{\mathbb{R}}$ is semisimple. Then it has a finite dimensional (unfaithful) $SO(32)$-representation. This representation splits into an $SO^+(16) \times SO^-(16)$-representation.

Proof. Let $K(E_9)$ denote the compact form as in Section 3 compatible with the Cartan-Chevalley involution corresponding to $K_{\mathbb{R}}$ for the loop representation of $G(E_9)$. By [17, 18, 19, 20] $K(E_9)$ (embedded in $K(E_{10})$) admits a finite dimensional (unfaithful) representation on $SO(32)$. This representation, by the main result of [21], splits into $SO^+(16) \times SO^-(16)$. Namely
\[ (4.2) \quad \rho: K(E_9) \rightarrow SO^+(16) \times SO^-(16) \hookrightarrow SO(32). \]
Note that the ± signs are superficial. The loop representation of $K(E_9)$ is of the form $S^1 \times \text{L}_{\text{pol}}K_{\mathbb{R}} \hat{=}$ (see [22]). Also, since $K_{\mathbb{R}}$ is semisimple, $L_{\text{pol}}K_{\mathbb{R}}$ is dense in $LK_{\mathbb{R}}$ (see [15] Proposition 3.5.3]). Hence $K(E_9)$ is dense in $\text{LK}_{\mathbb{R}} \hat{=}$ $S^1 \times \text{LK}_{\mathbb{R}} \hat{=}$ (see [22] Section 2.1]). Therefore, the representation $\rho$
continuously extends to an (unfaithful) representation (denoted again by $\hat{\rho}$) of $\hat{LK}_R$ (which preserves the $SO^+(16) \times SO^-(16)$-split since $SO^\pm(16)$ are closed subgroups),

\begin{equation}
\rho : \hat{LK}_R \to SO^+(16) \times SO^-(16) \to SO(32).
\end{equation}

But by Lemma 4.2, $K$ can be continuously embedded in $\hat{LK}_R$ which results in a continuous representation (denoted again by $\hat{\rho}$)

\begin{equation}
\rho : K \to SO^+(16) \times SO^-(16) \to SO(32).
\end{equation}

The lemma follows from the fact that for $E_n$ ($n \leq 8$) we have a natural embedding of $K(E_n) \to K(E_9)$ (see e.g. [21]).

From now on, we always assume that $\hat{LK}_R$ is semisimple.

We define the associated bundle to $\pi$ with respect to $\rho$ as follows

\begin{equation}
SO(\pi) : G \times_\rho SO(32) \to M.
\end{equation}

SO($\pi$) is called the finite frame bundle on $M$ with respect to $\rho$. Similarly, we define the associated bundle to $\pi$ with respect to the $SO^+(16) \times SO^-(16)$-split of $\rho$ as follows

\begin{equation}
SO^+(\pi) : G \times_\rho SO^+(16) \times SO^-(16) \to M.
\end{equation}

SO$^+$($\pi$) is called the split finite frame bundle on $M$ with respect to $\rho$.

**Definition 4.2.** We say $M$ has a (split) **finite spin (spin$^+$)** structure if the structure group of $(SO^+(\pi))$ $SO(\pi)$ has a lift to a $(Spin(16) \times Spin(16))$ $Spin(32)$ $(Spin^+(32))$. When $M$ has a finite spin structure, define

$$Spin(\pi) := G \times_\rho Spin(32) \to M.$$ We call the following associated bundle, the **finite spinor bundle** over $M$:

\begin{equation}
S(M) := G \times_\rho S^{32} \to M,
\end{equation}

where $Clif(R^{32}) \otimes \mathbb{C} \cong \text{End}(S^{32})$. Similarly, when $M$ has a split finite spin structure, define

$$Spin^+(\pi) := G \times_\rho Spin(16) \times Spin(16) \to M.$$ We call the following associated bundle, the **split finite spinor bundle** over $M$:

\begin{equation}
S^+(M) := G \times_\rho S^{16,+} \bigoplus S^{16,-} \to M,
\end{equation}

where $Clif(R^{16}) \otimes \mathbb{C} \cong \text{End}(S^{16})$. Similarly, we define real (split) spinor bundles and denote them by $S_R(M)$ and $S_R^+(M)$. Also, we can construct spinor bundles for affine symmetric spaces with finite spin$^+$ structure in the same manner.

**Remark 4.3.** Since $\text{Ad} : Spin(32) \to SO(32)$ is a universal covering map, when $K$ is connected, having a finite spin structure is equivalent to vanishing the induced morphism

$$\rho_* : \pi_1(K) \to \pi_1(SO(32)) \cong \mathbb{Z}_2,$$

(vanishing $\rho_*$ implies vanishing the second Stiefel-Whitney class of $SO(\pi)$), and similarly, having a spin$^+$ structure is equivalent to $\pi_1(K)$ not having any 2-torsions. Note that considering the definition of $\rho$ in Lemma 4.1, to have spin (spin$^+$) structure it suffices to check the above equivalent conditions for $\hat{LK}_R$.

**Remark 4.4.** In view of Remark 4.3, it is important to find a condition under which the structure group $K$ of $\pi : G \to M$ reduces to its one-component $\bar{K}$. Note that since $K$ is a topological group, we have the following exact sequence

\begin{equation}
\bar{K} \to K \to \pi_0(K).
\end{equation}

Consequently, we obtain the following cohomological mapping

\begin{equation}
\nu_1 : H^1(M, K) \to H^1(M, \pi_0(K)).
\end{equation}
Hence, the structure group $K$ of $\pi : G \rightarrow M$ reduces to $\hat{K}$ if and only if $\nu_1(\pi) = 0$.

The next lemma might be known to experts

**Lemma 4.5.** In the above setting we have

\[(4.11) \quad \pi_0(\hat{L}K^0_R) \cong \pi_1(K^0_R) \quad \text{and} \quad \pi_1(\hat{L}K^0_R) \cong \mathbb{Z} \oplus \pi_1(\hat{L}K^0_R).\]

**Proof.** We know that $\hat{L}K^0_R \cong S^1 \times \hat{K}^0_R$ (see [22] Section 2.1 and references therein). Therefore,

\[(4.12) \quad \pi_0(\hat{L}K^0_R) = \pi_0(\hat{L}K^0_R),\]

and

\[(4.13) \quad \pi_1(\hat{L}K^0_R) = \mathbb{Z} \oplus \pi_1(\hat{L}K^0_R).\]

But by [15] Proposition 4.6.9 we have

\[(4.14) \quad \pi_0(\hat{L}K^0_R) = \pi_1(K^0_R).\]

\[\square\]

**Remark 4.6.** Similar to Remark 4.4, to reduce the structure group $\hat{L}K^0_R$ to its one-component $\hat{L}K^0_R$, we consider the following exact sequence

\[(4.15) \quad \hat{L}K^0_R \rightarrow \hat{L}K^0_R \rightarrow \pi_0(\hat{L}K^0_R).\]

By Lemma 4.5 and 4.6 yields the following exact sequence

\[(4.16) \quad \hat{L}K^0_R \rightarrow \hat{L}K^0_R \rightarrow \pi_1(K^0_R),\]

which induces the following cohomological mapping

\[(4.17) \quad \hat{\nu}_1 : H^1(M, \hat{L}K^0_R) \rightarrow H^1(M, \pi_0(\hat{L}K^0_R)).\]

Hence, the structure group $\hat{L}K^0_R$ reduces to $\hat{L}K^0_R$ if and only if $\hat{\nu}_1(\pi^H) = 0$, where

\[(4.18) \quad \pi^H : G \times_H \hat{L}K^0_R \rightarrow M,\]

is the associated (topological) bundle to $\pi$ with respect to $H$ as in [31].

**Proposition 4.7.** In the above setting, if

- $\hat{\nu}(\pi) = 0$; and,
- the first Chern class of $\hat{L}K^0_R$ corresponds to a generator; and,
- $K^0_R$ is torsion free,

then $M$ admits a finite spin$^c$ structure. In particular, if $K^0_R$ is simply connected, then $M$ admits a finite spin$^c$ structure.

**Proof.** Since $\hat{\nu}(\pi) = 0$, by Remark 4.6 it suffices to show that $\pi_1(\hat{L}K^0_R)$ is torsion free. Considering the $S^1$-fibration

\[(4.19) \quad S^1 \rightarrow \hat{L}K^0_R \rightarrow \hat{L}K^0_R,\]

we obtain the following exact sequence of homotopy groups

\[(4.20) \quad 0 \rightarrow \pi_2(\hat{L}K^0_R) \rightarrow \pi_2(\hat{L}K^0_R) \cong \pi_1(S^1) \rightarrow \pi_1(\hat{L}K^0_R) \rightarrow \pi_1(\hat{L}K^0_R) \rightarrow 0.\]

Since by our hypotheses, the first Chern class of $\hat{L}K^0_R$ corresponds to a generator, the “transgression” map $\alpha$ between $\pi_2(\hat{L}K^0_R)$ and $\pi_1(S^1)$ is surjective. Hence we deduce the following exact sequence from (4.20)

\[(4.21) \quad 0 \rightarrow \pi_1(\hat{L}K^0_R) \rightarrow \pi_1(\hat{L}K^0_R) \rightarrow 0.\]
This, in view of (4.13) in Lemma 4.5, implies that \( \pi_1(\hat{L}K_R^0) \) is torsion free if and only if \( \pi_1(LK_R^0) \) is torsion free if and only if \( \pi_1(K_R^0) \) is torsion free.

In particular, when \( K_R^0 \) is simply connected, by Lemma 4.5 we know that \( \hat{L}K_R^0 \) is connected and hence \( \hat{\nu}(\pi) = 0 \). \( \hat{L}K_R^0 \) is the universal central extension by definition (see [3, Section 4.5.1] and [15, Section 4.4]), which corresponds to a generator. In this case the “transgression” map \( \alpha \) is actually an isomorphism (see e.g., [15] and the proof of [8, Proposition 2.1]).

**Definition 4.8.** An affine symmetric space is called (split) finite spin if

• \( SO(\pi) \) has a (split) finite spin structure, and,

• there exists a covariant derivative (connection) on \( S(M) \) associated to Spin(\( \pi \)).

**Example 4.9.** Let \( M := \hat{C}^*E_6R/\hat{C}^*F_4R \) be the affine symmetric space obtained from the symmetric space of type EIV on the Cartan’s list of symmetric spaces. Since \( (F_4)_R \) is simply connected, \( M \) admits a finite spin\(^c\) structure.

5. Dirac-like Operators on Affine Kac-Moody Symmetric spaces

In the previous sections we have produced two technical methods to obtain certain spin structures on some affine Kac-Moody symmetric spaces.

First note that, in the setting of Section 3, it is straightforward to define a Dirac-like operator on a loop-string affine Kac-Moody symmetric space as presented in [9, Chapter 2].

Next, in the setting of Section 4 to define a Dirac-like operator we must produce a setup first. In what follows, we use the notations and assumptions as in Section 4. For an affine symmetric space \( M \), assume that \( K \) is the fixed point set of a Cartan-Chevalley involution \( \nu \). Define

(5.1) \[ \Psi : G \times_{Ad} g \to M, \]

via \( \pi : G \to M \) (see [4,1]) where \( G \times_{Ad} g \) is considered as the **tangent bundle** of \( G \). Define the **tangent bundle** of \( M \) to be the associated vector bundle

(5.2) \[ T(M) : G \times_{Ad} p \to M, \]

where

\[ \text{Ad} : K \to \text{Aut}(p) \]

is the restriction of the adjoint representation of \( G \) to \( K \) and its co-restriction to \( p \) where \( p \) is the \(-1\)-eigenspace of \( \nu \) (see [3, Section 5.4.3]). And finally the **vertical bundle** associated to \( \pi \) is defined as follows

(5.3) \[ V(\pi) : G \times_{Ad} \mathfrak{k} \to M. \]

By the above definitions, vector fields on \( M \) correspond to right \( K \)-equivariant maps \( X : G \to p \) satisfying \( X(gh) = \text{Ad}(h^{-1})X(g) \). We obtain the following exact sequence of vector bundles on \( M \) (see [3, Section 5.9]):

(5.4) \[ V(\pi) \to \Psi \to T(M). \]

**Definition 5.1.** Let \( M \) be an affine Kac-Moody symmetric space. A **connection** on the principal bundle \( \pi \) is a \( K \)-equivariant splitting

(5.5) \[ \nabla : T(M) \to \Psi \]

of (5.4). Such a connection defines a canonical linear projection

(5.6) \[ \omega : \Psi \to V(M) \]

which is called the **connection 1-form**. This can be seen as a \( \mathfrak{k} \)-valued 1-form on \( G \). Hence, the connection **curvature** can also be defined as follows.

(5.7) \[ \Omega := d\omega + [\omega, \omega], \]

which is a \( \mathfrak{k} \)-valued 2-form.
**Lemma 5.2.** Let $M$ be an affine Kac-Moody symmetric space. Then the principal $K$-bundle
\[ \pi : G \to M, \]
admits a connection.

**Proof.** By [3] Theorem 5.37, there exists a connection on $M$ (actually on $T(M)$ as a vector bundle) induced by the unique Levi-Civita connection on $G$. This connection on $M$ induces a $K$-equivariant splitting for \((S\mathbb{R}(M))\) by [3] Lemma 5.39). Hence the lemma.

**Proposition 5.3.** Let $M$ be an affine Kac-Moody symmetric space. If $M$ admits a finite spin structure, then $M$ is spin.

**Proof.** Since any connection on the principal $K$-bundle $\pi : G \to M$ induces a connection on $\text{Spin}(\pi)$, the proposition follows from Lemma 5.2.

By abuse of notation, let $\nabla (\nabla^R)$ denote the connection on $S(M)$ ($S\mathbb{R}(M)$) obtained in Proposition 5.3.

**Definition 5.4.** At any point $m \in M$, let $\{e_i\}$ be an orthonormal basis of $S\mathbb{R}(M)_m$. Now similar to the classical case (see [23] Section II.5) we define a **Dirac-like** operator of the space of sections of $S\mathbb{R}(M)_m$, namely $D : \Gamma(S\mathbb{R}(M)) \to \Gamma(S\mathbb{R}(M))$ as follows
\[ D_m(\sigma) := \sum_{i=1}^{2^{2d}} e_i \cdot \nabla^R_{e_i}(\sigma(m)), \]
where the dot in the formula is the Clifford multiplication.

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