Corrigendum for “Almost vanishing polynomials and an application to the Hough transform” *

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In this note we correct a technical error occurred in [1]. This affects the bounds given in that paper, even though the structure and the logic of all proofs remain fully unchanged. The error is due to a repeated wrong use of Hölder’s inequality (a transpose of a matrix was missed). The first time it occurs is in the first inequality of formula (8) of [1]. Indeed, the correct version of that formula is:

\[ |\text{Jac}_f(p^*(p) - p^*)^t| = |\text{Jac}_f(p)^t\mathcal{E}^{-1}\mathcal{E}(p^*(p) - p)^t| \]
\[ \leq \|\text{Jac}_f(p)^t\mathcal{E}^{-1}\mathcal{E}(p^*)^t\|_1\|\mathcal{E}(p^*(p) - p)^t\|_{\infty} \]
\[ \leq \|\mathcal{E}^{-1}\text{Jac}_f(p)^t\|_1 \leq \|\text{Jac}_f(p)^t\|_1\|\mathcal{E}^{-1}\|_1. \]

We refer to our paper [1], using the same notation. Here, we confine ourselves to state the correct versions of Proposition 2.1, Proposition 2.5, Proposition 3.2, and Proposition 4.3 of [1], respectively. Up to the error pointed out and corrected as above, the proofs go parallel to those in [1]. In the following, \( P \) denotes the multivariate polynomial ring \( \mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n] \).

**Proposition 2.1** Let \( f = f(x) \) be a non-constant polynomial of \( P \), let \( p \) be a point of \( \mathbb{R}^n \), and let \( C(p) \subseteq B(p) \) be an \((\infty, \varepsilon)\)-unit cell centered at \( p \). If

\[ |f(p)| > \|\text{Jac}_f(p)^t\|_1\varepsilon_{\text{max}} + \frac{n}{2}\varepsilon_{\text{max}}^2 := B_1, \]

then the hypersurface of equation \( f = 0 \) does not cross \( C(p) \).

**Proposition 2.5** Let \( f(x) \) be a degree \( \geq 2 \) polynomial of \( P \). Let \( p \) be a point of \( \mathbb{R}^n \) and let \( C(p) \subseteq B(p) \) be an \((\infty, \varepsilon)\)-unit cell centered at \( p \). If

\[ |f(p)| > \|\text{Jac}_f(p)^t\|_1\varepsilon_{\text{max}} + \frac{n}{2}\|H_f(p)\|_{\infty}\varepsilon_{\text{max}}^2 := B_1', \]

then the hypersurface of equation \( f = 0 \) does not cross \( C(p) \) neglecting contributions of order \( O(\varepsilon_{\text{max}}^3) \).

**Proposition 3.2** Let \( f = f(x) \) be a degree \( \geq 2 \) polynomial of \( P \), let \( p \) be a point of \( \mathbb{R}^n \) such that \( \text{Jac}_f(p) \) is not the zero vector, and let \( C(p) \subseteq B(p) \) be an \((\infty, \varepsilon)\)-unit cell centered at \( p \). Let \( R \) be a positive real number such that \( R < \min\{\varepsilon_{\text{min}}, \frac{\|\text{Jac}_f(p)^t\|_1}{\|H_f(p)^t\|}\} \). Set \( c := \max\{2, \sqrt{n}\} \). If

\[ |f(p)| < \frac{2R}{J(c + \n^5/2HJR)} := B_2, \]

then the hypersurface of equation \( f = 0 \) crosses \( C(p) \).

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Proposition 4.3 Let \( f = f(x) \) be a degree \( \geq 2 \) polynomial of \( P \), let \( p \) be a point of \( \mathbb{R}^n \) such that Jacobian \( \text{Jac}_f(p) \) and the Hessian matrix \( H_f(p) \) are nontrivial, and let \( C(p) \subseteq B(p) \) be an \((\infty, \varepsilon)\)-unit cell centered at \( p \). Let \( R \) be a positive real number such that \( R < \min \left\{ \varepsilon \min, \frac{\| \text{Jac}_f(p) \|_1}{\| H_f(p) \|_{\infty}} \right\} \), let \( c := \max \{ 2, \sqrt{n} \} \) and set

\[
\Theta := \| \text{Jac}_f(p) \|_{\infty} + n^2 (1 + 2\sqrt{n}) \frac{\| H_f(p) \|_{\infty} R}{\| \text{Jac}_f(p) \|_1}.
\]

If

\[
|f(p)| < \frac{2R}{\Theta(c + n^{9/2} \| H_f(p) \|_{\infty} \Theta R)} =: B'_2,
\]

then the hypersurface of equation \( f = 0 \) crosses \( C(p) \) neglecting order \( O(R^2) \) contributions.

Remark. More accurate and general estimates [2], when specialized to the case of hypersurfaces and \( \infty \)-norm, allow us to improve the bounds above, this also assuring that the applications discussed in [1, Section 6] still remain meaningful. Precisely, in [2, Theorem 3.2], we can in fact show that the bound \( B_2 \) goes as \( \frac{1}{n} \) instead of \( \frac{1}{n^{9/2}} \), weakening the assumption. Similarly, one can shows that the bound \( B'_2 \) in [2, Theorem 4.6] goes as \( \frac{1}{n^{9/2}} \) instead of \( \frac{1}{n^{7/2}} \).

References

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Almost vanishing polynomials and an application to the Hough transform

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Abstract

We consider the problem of deciding whether or not an affine hypersurface of equation $f = 0$, where $f = f(x_1, \ldots, x_n)$ is a polynomial in $\mathbb{R}[x_1, \ldots, x_n]$, crosses a bounded region $T$ of the real affine space $\mathbb{A}^n$. We perform a local study of the problem, and provide both necessary and sufficient numerical conditions to answer the question. Our conditions are based on the evaluation of $f$ at a point $p \in T$, and derive from the analysis of the differential geometric properties of the hypersurface $z = f(x_1, \ldots, x_n)$ at $p$. We discuss an application of our results in the context of the Hough transform, a pattern recognition technique for the automated recognition of curves in images.

Introduction

We study conditions in order to decide whether or not a given affine hypersurface intersects a bounded region. More precisely, let $f = f(x)$ be a polynomial in $\mathbb{R}[x]$, where $x = (x_1, \ldots, x_n)$ denotes the variables. Intuitively, one may think that the evaluation of $f$ at a point $p$ of the real affine space $\mathbb{A}^n$ be sufficient to determine if either or not the hypersurface of equation $f = 0$ passes through a given region containing the point $p$. That is, if $f(p)$ is sufficiently “small” then the hypersurfaces crosses the region, while, if $f(p)$ is “large”, the hypersurface doesn’t cross the region. Indeed, this is not the case, as the following examples show.

First, look at the polynomial $f(x,y) = x^2 + \frac{1}{100} y^2 - \frac{1}{100} \in \mathbb{R}[x,y]$ and the point $p = (0,2)$. We have $|f(p)| = 0.03$ which we may consider a small evaluation. Nevertheless the point $p$ lies far from the curve $f = 0$ (the minimal Euclidean distance of $p$ from points of the curve $f = 0$ is 1, as Figure 1 shows). The reason has to be found in the local, differential geometric properties of the surface $z = f(x,y)$, as it is shown in Figure 2.

Next, we consider the polynomial $f(x,y) = y - 10x^2 \in \mathbb{R}[x,y]$ and the point $p = (1.1,10)$. We have $|f(p)| = 2.1$ which we may think as a large evaluation. Nevertheless the point $p$ lies close to the curve $f = 0$ (the minimal Euclidean distance of $p$ from points of the curve $f = 0$ is about 0.1, as Figure 3 shows). Again, the reason has to be found in the local, differential geometric properties of the surface $z = f(x,y)$, as it is shown in Figure 4.

Such examples make clear a sort of ambiguity of the expression “almost vanishing polynomials”, motivating our interest to treat the matter. The paper is organized a follows. In Section 1 we recall the definitions of different norms and their basic properties, and a few facts about the analytic nature of polynomials. In Section 2 we provide necessary numerical conditions for an affine hypersurface $f = 0$ to cross a bounded region containing a given point $p$ of $\mathbb{R}^n$. While in Section 3 we provide sufficient conditions. Both the conditions (see propositions 2.1 and 3.2) are expressed in terms of the Jacobian and the Hessian matrices, and depend on the quantities $H$ and $J$ (defined in (5) and (10), respectively) which express exact bounds, but not easy to be computed.

The question to avoid the computations of $H$ and $J$, by confining ourself to find numerical conditions valid up to a higher-order analysis, that is, up to small values of a given tolerance and...
disregarding higher-order contributions, becomes then natural. In the case of necessary conditions we settle that question, up to a second-order analysis, that is, disregarding third-order contributions, in Proposition 2.5. The analogous result for sufficient conditions looks much harder, and it works up to a first-order analysis. We devote the entire Section 4 to discuss and fix this case (see Proposition 4.3).

Unfortunately, the above conditions do not fit together to give “if and only if” statements. However, because of the local nature of the results, a more accurate analysis, performed by iteratively considering smaller subregions of the given region may overcome that problem (see Remark 4.6).

In Section 5 we summarize the results discussed above in an explanatory algorithmic way. An implementation of the algorithms (available at the web page http://www.dima.unige.it/~torrente/recognitionAlgorithm.cocoa5) has been done using CoCoA5 (see [3]).

We observe that all the above mentioned conditions use the $\| \cdot \|_1$ and $\| \cdot \|_\infty$ norms, instead of the perhaps more popular $\| \cdot \|_2$ norm. The reason for that consists in the application we have in mind, a true guideline of our work, we deal with in Section 6. We discuss there an explicit way of how using our results in a case of special interest, that is, the Hough transform technique. The Hough transform is a pattern recognition technique for the automated recognition of curves in images (see [4], [1] and [2]). In this context, given a suitable family $\mathcal{F}$ of curves in the image plane $\mathbb{A}_x^2$, the Hough transform $\Gamma_p(\mathcal{F})$ of a point $p$ in $\mathbb{A}_x^2$ with respect to $\mathcal{F}$ is a hypersurface in an
affine parameter space $\mathbb{R}^t$ (see Definition 6.1), and $\Gamma_p(F)$ plays the role of the affine hypersurface of equation $f = 0$ we started from. The core of the recognition algorithm based on the Hough transforms is to count how many hypersurfaces $\Gamma_p(F)$ cross a given cell of a suitable discretization of the parameter space $\mathbb{R}^t$, and such a cell is usually defined in $\|\cdot\|_\infty$ norm. The center of the cell which counts the maximum number of crossings is then used to detect the recognized curve. A relevant issue is to validate the outputs of our algorithm by comparing with pattern recognition techniques exploited, e.g., in [9]; to this aim we provide explicit examples which give some hopeful evidence. A novelty here is that our approach works for any number $t$ of parameters, while, as far as we know, the Hough transform recognition technique typically requires three parameters at most.

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1 Background material

In this section we recall basic definitions and concepts from numerical algebra and some analytic properties of polynomials systematically used throughout the paper. We start recalling the definition of different norms on the space of matrices and their basic properties (for proofs and more details we refer to [6] and [14]).

For $m$, $n$ positive integers, we let $\text{Mat}_{m \times n}(\mathbb{R})$ be the set of $m \times n$ matrices with entries in $\mathbb{R}$; if $m = n$ we simply write $\text{Mat}_n(\mathbb{R})$. For any $M \in \text{Mat}_{m \times n}(\mathbb{R})$, we will denote by $M^t$ its transpose.

**Definition 1.1** Let $v$ be an element of $\text{Mat}_{n \times 1}(\mathbb{R})$ and let $r \geq 1$ be a real number. Set $v^t := (v_1, \ldots, v_n)$. The $r$-norm\(^3\) of $v$ is defined by

$$
\|v\|_r := \left(\sum_{i=1}^n |v_i|^r\right)^{\frac{1}{r}}.
$$

In particular, if $r = 1$, we get the expression $\|v\|_1 = \sum_{i=1}^n |v_i|$. If $r = 2$ we get the well-known Euclidean norm $\|v\|_2 = (\sum_{i=1}^n |v_i|^2)^{1/2}$. While, if $r \to \infty$, the $r$-norm approaches the $\infty$-norm defined by $\|v\|_\infty := \max_{i=1,\ldots,n} \{ |v_i| \}$.

**Proposition 1.2** (Holder’s inequality) Let either $q > r \geq 1$ be real numbers such that $\frac{1}{q} + \frac{1}{r} = 1$ or $q = \infty$ and $r = 1$. Then for each $v, w \in \text{Mat}_{n \times 1}(\mathbb{R})$ we have the inequality

$$
|v^t w| \leq \|v\|_q \|w\|_r.\]

Apart from $r$-norms, other very useful norms on $\text{Mat}_{n \times 1}(\mathbb{R})$ are the weighted $r$-norms, in which case each component of any column vector $v \in \text{Mat}_{n \times 1}(\mathbb{R})$ is rescaled according to a given weight (see [14]).

**Definition 1.3** Let $W$ be a positive diagonal matrix in $\text{Mat}_n(\mathbb{R})$ and let $\|\cdot\|$ be any norm on $\text{Mat}_{n \times 1}(\mathbb{R})$. The $W$-weighted norm on $\text{Mat}_{n \times 1}(\mathbb{R})$ is then defined by the formula

$$
\|v\|_W := \|Wv\|.
$$

where $v \in \text{Mat}_{n \times 1}(\mathbb{R})$ and $Wv$ denotes the usual product of matrices. If either $\|\cdot\| = \|\cdot\|_r$ or $\|\cdot\| = \|\cdot\|_\infty$, we usually write

$$
\|\cdot\|_{r,W} =: \|\cdot\|_{r,W} \quad \text{and} \quad \|\cdot\|_{\infty,W} =: \|\cdot\|_{\infty,W}.
$$

\(^3\)We will only use matrix norms; however, let us mention that in the literature one also refer to this norm as the “r-norm of the vector $(v_1, \ldots, v_n)$ in $\mathbb{R}^n$.”
The geometry of the closed unit ball (the set of all the vectors of norm less or equal to 1) clearly depends on the norm we use. Here we consider the case of weighted $r$-norms.

**Definition 1.4** Let $W$ be a positive diagonal matrix in $\text{Mat}_n(\mathbb{R})$, let either $r \geq 1$ be a positive real number or $r = \infty$, and let $p$ be a point of $\mathbb{R}^n$. The $(r,W)$-unit ball centered at $p$, denoted by $B_{r,W}(p)$, is the closed convex set defined as

$$B_{r,W}(p) = \{ x \in \mathbb{R}^n \text{ such that } \| (x - p)^t \|_{r,W} \leq 1 \}. $$

For simplicity of notation, and when no confusion will arise, the $(r,W)$-unit ball centered at $p$ will be simply called unit ball centered at $p$ and denoted by $B(p)$.

Now, we recall the definition of matrix norm on $\text{Mat}_{m \times n}(\mathbb{R})$ induced by a given $r$-norm on $\text{Mat}_{n \times 1}(\mathbb{R})$ and some basic facts about them.

**Definition 1.5** Let $M = (m_{ij})$ be a matrix in $\text{Mat}_{m \times n}(\mathbb{R})$. The $r$-matrix norm is the norm on $\text{Mat}_{m \times n}(\mathbb{R})$ induced by the $r$-norm on $\text{Mat}_{n \times 1}(\mathbb{R})$, and defined by the formula

$$\| M \|_r := \max_{\| v \|_r = 1} \| Mv \|_r,$$

where $v \in \text{Mat}_{n \times 1}(\mathbb{R})$. In particular, one has $\| M \|_1 = \max_{i=1,...,n} \left\{ \sum_{j=1,...,m} |m_{ij}| \right\}$ for $r = 1$.

If $r = 2$, denoting by $\lambda_i(\cdot)$ the $i$-th eigenvalue, we have $\| M \|_2 = (\max_{i=1,...,n} \lambda_i(M^tM))^{1/2}$. While, if $r \to \infty$, the matrix $r$-norm approaches the $\infty$-matrix norm defined by $\| M \|_\infty := \max_{i=1,...,m} \left\{ \sum_{j=1,...,n} |m_{ij}| \right\}$.

**Remark 1.6** Let us point out the following consequence of Definition 1.5, every so often used throughout the paper. For any vector $w \in \text{Mat}_{1 \times n}(\mathbb{R})$, one has

$$\| w \|_1 = \| w^t \|_\infty.$$

Next, let us introduce one more matrix norm we need.

**Definition 1.7** Let $M = (m_{ij})$ be a matrix in $\text{Mat}_{m \times n}(\mathbb{R})$. The max-norm is the norm on $\text{Mat}_{m \times n}(\mathbb{R})$ defined by

$$\| M \|_{\text{max}} := \max_{i,j} |m_{ij}|.$$

Note that, via the natural identification of $\text{Mat}_n(\mathbb{R})$ with $\text{Mat}_{n^2 \times 1}(\mathbb{R})$, a matrix $M \in \text{Mat}_n(\mathbb{R})$ can be viewed as an element, that we will denoted by $M^{(v)}$ to avoid confusion, of $\text{Mat}_{n^2 \times 1}(\mathbb{R})$. Moreover,

$$\| M \|_{\text{max}} = \| M^{(v)} \|_\infty. \hspace{1cm} (1)$$

We finally recall some useful relations between the norms introduced above, and the sub-multiplicative property (see [6, §2.3.1]).

**Proposition 1.8** For each $M \in \text{Mat}_{m \times n}(\mathbb{R})$ the following inequalities hold true:

1. $\frac{1}{\sqrt{n}} \| M \|_1 \leq \| M \|_2 \leq \sqrt{m} \| M \|_\infty.$
2. $\frac{1}{\sqrt{m}} \| M \|_1 \leq \| M \|_2 \leq \sqrt{n} \| M \|_1.$
3. $\| M \|_{\text{max}} \leq \| M \|_2 \leq \sqrt{mn} \| M \|_{\text{max}}.$
Proposition 1.9 Let either \( r \geq 1 \) be a positive real number or \( r = \infty \). The \( r \)-matrix norm induced by the \( r \)-norm is a sub-multiplicative norm, that is, for each \( A \in \text{Mat}_{m \times n}(\mathbb{R}), B \in \text{Mat}_{n \times t}(\mathbb{R}) \), one has
\[
\|AB\|_r \leq \|A\|_r \|B\|_r.
\]
In particular, for each \( A \in \text{Mat}_{m \times n}(\mathbb{R}) \) and \( v \in \text{Mat}_{n \times 1}(\mathbb{R}) \), one has \( \|Av\|_r \leq \|A\|_r \|v\|_r \).

Next, we collect some definitions and basic facts of analytic nature about polynomials. The notation is borrowed from [12]. In particular, we let \( x_1, \ldots, x_n \) be indeterminates and most of the times we use for simplicity the notation \( x = (x_1, \ldots, x_n) \). The multivariate polynomial ring \( \mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n] \) is denoted by \( P \). Given \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \), we denote by \( |\alpha| \) the number \( \alpha_1 + \cdots + \alpha_n \), by \( \alpha! \) the number \( \alpha_1! \cdots \alpha_n! \), by \( x^{\alpha} \) the power product \( x_1^{\alpha_1} \cdots x_n^{\alpha_n} \), and by
\[
\frac{\partial^{\alpha} f}{\partial x_1 \cdots \partial x_n} := \frac{\partial^{\alpha} f}{\partial x_1 \cdots \partial x_n} \text{ the } \alpha\text{-partial derivative of a polynomial } f = f(x) \in P.
\]

Moreover, following the standard notation, we denote by \( \text{Jac}_f(x) := \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right) \) the Jacobian (or gradient) of \( f \), and by \( H_f(x) := \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j=1,\ldots,n} \) the \( n \times n \) symmetric Hessian matrix of \( f \).

Definition 1.10 Let \( p \) be a point of \( \mathbb{R}^n \) and let \( f = f(x) \) be a polynomial in \( P \). Let \( k \) be a non-negative integer. The \( k \)-th Taylor polynomial \( p_k(x) \) and the \( k \)-th remainder term \( R_k(x) \) of \( f(x) \) at \( p \) are defined respectively as
\[
p_k(x) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} \frac{\partial^{\alpha} f}{\partial x^{\alpha}} (p)(x-p)^{\alpha} \quad \text{and} \quad R_k(x) = \sum_{|\alpha| > k} \frac{1}{\alpha!} \frac{\partial^{\alpha} f}{\partial x^{\alpha}} (p)(x-p)^{\alpha},
\]
so that the polynomial \( f(x) \) can always be expressed as \( f(x) = p_k(x) + R_k(x) \).

We use the following formulation of Taylor’s theorem. We recall that, given a real value \( \eta \ll 1 \) and a real function \( \omega : \mathbb{R}^n \to \mathbb{R} \), we write \( \omega(x) = \Theta(\eta^m) \), \( m \in \mathbb{N} \), to mean that \( \omega(x)/\eta^m \) is bounded near the origin.

Proposition 1.11 (Taylor’s theorem) Let \( k \) be a non-negative integer, let \( p \) be a point of \( \mathbb{R}^n \) and let \( f(x) \) be a polynomial in \( P \). Then:

1. For each \( \alpha \in \mathbb{N}^n \) such that \( |\alpha| = k \) there exists a polynomial \( h_\alpha(x) \) in \( P \) such that the \( k \)-th remainder term \( R_k(x) \) of \( f(x) \) at \( p \) can be expressed as
\[
R_k(x) = \sum_{|\alpha| = k} h_\alpha(x)(x-p)^{\alpha} \quad \text{and} \quad \lim_{x \to p} h_\alpha(x) = 0.
\]

In particular, \( R_k(x) = O(\|x-p\|^{k+1}) \) for any norm \( \| \cdot \| \).

2. For every point \( q \in \mathbb{R}^n \) there exists a point \( \xi \in \mathbb{R}^n \) of the line segment from \( p \) to \( q \) such that the evaluation of the \( k \)-th remainder term \( R_k(x) \) at \( q \) is
\[
R_k(q) = \sum_{|\alpha| = k+1} \frac{1}{\alpha!} \frac{\partial^{\alpha} f}{\partial x^{\alpha}} (\xi)(q-p)^{\alpha}.
\]

The expression in Proposition 1.11(2) is known as the Lagrange form of the remainder.

Finally, we discuss a generalization of the Mean Value Theorem for the case we need of \( \infty \)-matrix norm. The content of this section may be known to experts and it is based on the following version of the Mean Value Theorem for vector valued real functions (see the proof presented in [7]). We nevertheless include details for lack of reference.
Proposition 1.12 Let $U \subseteq \mathbb{R}^n$ be a convex open set and let $p \in U$. Let $\phi: U \rightarrow \mathbb{R}^m$ be a differentiable vector valued function on $U$ and denote by $D\phi(x)$ the $m \times n$ matrix of first order derivatives of each component of $\phi$, that is,

$$D\phi(x) = \begin{pmatrix} \frac{\partial \phi_1}{\partial x_1} & \cdots & \frac{\partial \phi_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_m}{\partial x_1} & \cdots & \frac{\partial \phi_m}{\partial x_n} \end{pmatrix}$$

Let either $r \geq 1$ be a real number or $r = \infty$. Then, for each $x \in U$, one has

$$\| (\phi(x) - \phi(p))^t \|_r < \sup_{0 < \nu < 1} \| D\phi(p + \nu(x - p)) \|_r \| (x - p)^t \|_r.$$ 

Let $U \subseteq \mathbb{R}^n$ be a convex open set, and let $M(x) = (m_{ij}(x))$ be a matrix whose entries are the evaluations at $x \in U$ of differentiable vector valued functions $m_{ij}: U \rightarrow \mathbb{R}^n$. Hence, in particular, $M(p) \in \text{Mat}_n(\mathbb{R})$ for each given point $p \in U$. We will use the following special case of Proposition 1.12.

Lemma 1.13 Let $U \subseteq \mathbb{R}^n$ be a convex open set. Fix a point $p$ of $U$ and let $M(x) = (m_{ij}(x))$ be a matrix as above. For each $x \in U$, we have

$$\| M(x) \|_\infty < n^2 \| M(p) \|_\infty + O(\| (x - p)^t \|_\infty).$$

Proof. By combining statements (1) and (3) of Proposition 1.8 it follows that

$$\| M(x) \|_\infty \leq n^{3/2} \| M(\nu)(x) \|_\infty. \quad (2)$$

Consider the vector valued function $\phi = (M(\nu))^t: U \rightarrow \mathbb{R}^{n^2}$ defined by $\phi(x) := (M(\nu))^t(\nu)$. Clearly, $\phi$ is differentiable on $U$, so we can apply Proposition 1.12 (with $r = \infty$) to get

$$\| M(x)^{\nu} - M(p)^{\nu} \|_\infty < \sup_{0 < \nu < 1} \| D(M(p + \nu(x - p))(\nu))^{\nu} \|_\infty \| (x - p)^t \|_\infty = O(\| (x - p)^t \|_\infty).$$

Combining the previous inequality with

$$\| M(x)^{\nu} \|_\infty - \| M(p)^{\nu} \|_\infty \leq \| M(x)^{\nu} - M(p)^{\nu} \|_\infty$$

(a consequence of the usual triangular inequality), we obtain

$$\| M(x)^{\nu} \|_\infty < \| M(p)^{\nu} \|_\infty + O(\| (x - p)^t \|_\infty) = \| M(p) \|_\max + O(\| (x - p)^t \|_\infty).$$

From statements (1) and (3) of Proposition 1.8 we then find

$$\| M(x)^{\nu} \|_\infty < \| M(p) \|_\max + O(\| (x - p)^t \|_\infty)$$

$$\leq \| M(p) \|_2 + O(\| (x - p)^t \|_\infty)$$

$$\leq \sqrt{n} \| M(p) \|_\infty + O(\| (x - p)^t \|_\infty). \quad (3)$$

Combining (3) with (2) we are done. Q.E.D.

2 Necessary crossing cell conditions

In this section we provide necessary numerical conditions for an affine hypersurface to cross a bounded region containing a given point of $\mathbb{R}^n$.

We need to fix some notation. Let $x = (x_1, \ldots, x_n)$ be indeterminates and denote by $P$ the multivariate polynomial ring $\mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n]$. Let $f = f(x)$ be a polynomial of $P$ and let $p = (p_1, \ldots, p_n)$ be a point of $\mathbb{R}^n$. From now on through the paper, we make the blanket
assumption that the zero locus $f = 0$ is of dimension $n - 1$, that is, it is a (not necessarily irreducible) hypersurface in $\mathbb{A}_d^n$, even if most of the results extends to case (not of interest for our purposes) when the locus $f = 0$ has real components of lower dimension.

Let $\varepsilon_1, \ldots, \varepsilon_n$ be positive real numbers. Set

$$\varepsilon := (\varepsilon_1, \ldots, \varepsilon_n), \quad \varepsilon_{\min} := \min\{\varepsilon_1, \ldots, \varepsilon_n\}, \quad \varepsilon_{\max} := \max\{\varepsilon_1, \ldots, \varepsilon_n\},$$

and let $E \in \text{Mat}_n(\mathbb{R})$ be the positive diagonal matrix with entries $1/\varepsilon_1, \ldots, 1/\varepsilon_n$. We also say that $\varepsilon$ is the tolerance vector. Throughout this section we shall use the $E$-weighted $\infty$-norm on $\mathbb{R}^n$ (see Definition 1.3) and we consider the corresponding closed $(\infty, \varepsilon)$-unit ball $B(p)$ centered at $p$ (see Definition 1.4). The results of this section apply to any set obtained from $B(p)$ by removing parts of its boundary as follows. Consider the hyperplanes

$$L^\pm_k : x_k = p_k \pm \varepsilon_k, \quad k = 1, \ldots, n,$$

and let $I^+, I^-$ be (possibly and not necessarily distinct) subsets of $\{1, \ldots, n\}$. Then define

$$C(I^+, I^-)(p) := B(p) \setminus \bigcup_{k \in I^+, k \in I^-} (L^+_k \cup L^-_k).$$

We simply write $C(p) := C(I^+, I^-)(p)$ whenever there is no matter what the indexes $k \in I^+, k \in I^-$ are considered. We refer to $C(p)$ as an $(\infty, \varepsilon)$-unit cell centered at $p$.

In the following proposition we provide a necessary condition on $|f(p)|$ for an affine hypersurface of equation $f = 0$ to cross an $(\infty, \varepsilon)$-unit cell $C(p)$ centered at $p$. Such a condition is expressed in terms of the quantity (depending on the unit ball $B(p)$)

$$H := \max_{x \in B(p)} \|H_f(x)\|_{\infty}.$$  \hfill (5)

**Proposition 2.1** Let $f = f(x)$ be a non-constant polynomial of $P$, let $p$ be a point of $\mathbb{R}^n$, and let $C(p) \subseteq B(p)$ be an $(\infty, \varepsilon)$-unit cell centered at $p$. If

$$|f(p)| > \|\text{Jac}_f(p)\|_1 \varepsilon_{\max} + \frac{H}{2} \varepsilon_{\max}^2 =: B_1,$$

then the hypersurface of equation $f = 0$ does not cross $C(p)$.

**Proof.** Suppose by contradiction that the hypersurface $f = 0$ crosses the region $C(p)$, that is, suppose there exists a point $p^* \in C(p)$ such that $f(p^*) = 0$. From the formal Taylor expansion of $f(x)$ at $p$ (see Definition 1.10) it follows that

$$f(x) = p_1(x) + R_1(x) = f(p) + \text{Jac}_f(p)(x - p)^t + R_1(x),$$

where $R_1(x)$ is the 1-st remainder term of $f(x)$ at $p$. If we evaluate the former expression at $p^*$ and apply Taylor’s theorem (see Proposition 1.11(2)) we get

$$0 = f(p^*) = f(p) + \text{Jac}_f(p)(p^* - p)^t + \frac{1}{2}(p^* - p)H_f(\xi)(p^* - p)^t,$$

where $\xi$ is a point of the line that connects $p$ to $p^*$. Hence,

$$|f(p)| \leq |\text{Jac}_f(p)(p^* - p)^t| + \frac{1}{2}|(p^* - p)H_f(\xi)(p^* - p)^t|.$$  \hfill (7)

In the following computations we systematically use Hölder’s inequality and the sub-multiplicative property of the matrix norms (see propositions 1.2 and 1.9). Recalling that $E = \text{diag}(1/\varepsilon_1, \ldots, 1/\varepsilon_n)$, from the present assumption $p^* \in C(p)$ we obtain

$$|\text{Jac}_f(p)(p^* - p)^t| = \|\text{Jac}_f(p)E^{-1}E(p^* - p)^t\|_1 \leq \|\text{Jac}_f(p)E^{-1}\|_1 \|E(p^* - p)^t\|_\infty \leq \|\text{Jac}_f(p)E^{-1}\|_1 \leq \|\text{Jac}_f(p)\|_1 \|E^{-1}\|_1.$$  \hfill (8)
Analogously, we find
\[
| (p^* - p) H_f(\xi) (p^* - p)^t | = | (p^* - p) H_f(\xi) E^{-1} E (p^* - p)^t |
\leq \| (p^* - p) H_f(\xi) E^{-1} \|_1 \| E (p^* - p)^t \|_\infty \\
\leq \| (p^* - p) E E^{-1} H_f(\xi) E^{-1} \|_1 \\
\leq \| (p^* - p) E \|_1 \| E^{-1} H_f(\xi) E^{-1} \|_1 \\
= \| E (p^* - p)^t \|_\infty \| E^{-1} H_f(\xi) E^{-1} \|_1 \leq \| E^{-1} \|_f \| H_f(\xi) \|_1.
\]
(9)

Noting that \( \| E^{-1} \|_1 = \varepsilon_{\text{max}} \) and combining with inequalities (7), (8), (9), we get
\[
| f(p) | \leq \| \text{Jac}_f(p) \|_1 \varepsilon_{\text{max}} + \frac{1}{2} \| H_f(\xi) \|_1 \varepsilon_{\text{max}}^2.
\]

Since \( H_f(\xi) \) is a symmetric matrix, one has \( \| H_f(\xi) \|_1 = \| H_f(\xi) \|_\infty \). Furthermore, by definition, \( \| H_f(\xi) \|_\infty \leq H \). Thus the assertion follows. Q.E.D.

**Example 2.2** In \( P = \mathbb{R}[x,y] \) we consider the polynomial \( f(x,y) = x^2 + \frac{1}{100} y^2 - \frac{1}{100} \) and the point \( p = (0,2) \), the first example in the Introduction. We let \( \varepsilon = (0.05,0.1) \), so \( \varepsilon_{\text{max}} = 0.1 \). Let \( B(p) \) be the \((\infty, \varepsilon)\)-unit cell centered at \( p \). We have
\[
\text{Jac}_f(x,y) = \left( 2x, \frac{1}{50} y \right) \quad \text{and} \quad H_f(x,y) = \left( \begin{array}{cc} 2 & 0 \\ 0 & \frac{1}{50} \end{array} \right),
\]
so that \( \| \text{Jac}_f(p) \|_1 = \frac{1}{25} \) and \( \| H_f(x,y) \|_\infty = 2 \) for each \((x,y)\). Therefore \( H = 2 \). The magnitude \( |f(p)| \) of \( f \) at \( p \) is \( |f(p)| = 0.03 \), which is strictly greater than the bound \( B_1 = 0.014 \) provided in Proposition 2.1. We then conclude that the curve \( f = 0 \) does not cross \( B(p) \).

**Example 2.3** In \( P = \mathbb{R}[x,y] \) we consider the polynomial \( f(x,y) = 4x^2 + y^2 - 4x \). Then
\[
\text{Jac}_f(x,y) = (8x - 4, 2y) \quad \text{and} \quad H_f(x,y) = \left( \begin{array}{cc} 8 & 0 \\ 0 & 2 \end{array} \right).
\]
In particular \( \| H_f(x,y) \|_\infty = 8 \) for each \((x,y)\), so that \( H = 8 \). We let \( \varepsilon = (0.1,0.1) \), whence \( \varepsilon_{\text{max}} = 0.1 \). First, consider the point \( p_1 = (\frac{1}{4}, \frac{1}{2}) \) and the \((\infty, \varepsilon)\)-unit cell \( B(p_1) \). We have \( |f(p_1)| = \frac{1}{2} \), which is strictly greater than the bound \( B_1 = 0.24 \) provided in Proposition 2.1. We conclude that the curve \( f = 0 \) does not cross \( B(p_1) \). Next, consider the point \( p_2 = (\frac{1}{3}, \frac{1}{2}) \) and the unit cell \( B(p_2) \). We have \( |f(p_2)| = 0.0775 \), which is strictly less than \( B_1 = 0.28 \). Therefore the assumptions of Proposition 2.1 are not satisfied.

**Example 2.4** In the particular case of a degree 2 polynomial \( f(x) \) of \( P \) the quantity \( H = \max_{x \in B(p)} \| H_f(x) \|_\infty \) can be easily computed.

Let \( f_2(x) = \sum_{i \leq j} c_{ij} x_i x_j \) be the homogeneous component of \( f(x) \) of degree 2. Let \( p \) be a point of \( \mathbb{R}^n \) and let \( B(p) \) be the \((\infty, \varepsilon)\)-unit ball centered at \( p \). Then
\[
H = \max_{i=1,...,n} \left( 2|c_{ii}| + \sum_{j \neq i} |c_{ij}| \right).
\]
To see this, recall Definition 1.5 and note that \( H_f(x) = (h_{ij}) \) is the symmetric matrix with entries \( h_{ij} = c_{ij} \) if \( i \neq j \) and \( h_{ij} = 2c_{ii} \) for \( i = j \).

It is natural to ask for a statement analogous to Proposition 2.1 which avoids the computation of the quantity \( H \), and provides a non-crossing cell condition simply evaluating the Hessian matrix at a given point. The following statement settles the question for small values of \( \varepsilon \) of the (components of) the tolerance vector \( \varepsilon \) and disregarding third-order contributions. To this purpose, for the rest of this section, we assume \( \varepsilon_{\text{max}} \ll 1 \).
Proposition 2.5 Let $f(x)$ be a degree $\geq 2$ polynomial of $P$. Let $p$ be a point of $\mathbb{R}^n$ and let $C(p) \subseteq B(p)$ be an $(\infty, \varepsilon)$-unit cell centered at $p$. If

$$|f(p)| > \|\text{Jac}_f(p)\|_1 \varepsilon_{\text{max}} + \frac{1}{2} \|H_f(p)\|_{\infty} \varepsilon_{\text{max}}^2 := B'_1,$$

then the hypersurface of equation $f = 0$ does not cross $C(p)$ neglecting contributions of order $O(\varepsilon_{\text{max}}^3)$.

**Proof.** Suppose by contradiction that the hypersurface $f = 0$ crosses the cell $C(p)$, that is, suppose there exists a point $p^* \in C(p)$ such that $f(p^*) = 0$. From the Taylor expansion of $f(x)$ at $p$ (see Definition 1.10) it follows that

$$f(x) = p_2(x) + R_2(x) = f(p) + \text{Jac}_f(p)(x - p)^t + \frac{1}{2}(x - p)H_f(p)(x - p)^t + R_2(x),$$

where $R_2(x)$ is the 2-nd remainder term of $f(x)$ at $p$. Since $p^* \in C(p)$, evaluating the former expression at $p^*$ and applying Taylor’s theorem (see Proposition 1.11(1)), we obtain $R_2(p^*) = O(\|p^* - p\|^3 \varepsilon_{\text{max}})$. Thus we find

$$0 = f(p^*) = f(p) + \text{Jac}_f(p)(p^* - p)^t + \frac{1}{2}(p^* - p)H_f(p)(p^* - p)^t + O(\|p^* - p\|^3 \varepsilon_{\text{max}}).$$

Up to the term $O(\|p^* - p\|^3 \varepsilon_{\text{max}})$, the previous expression differs from (6) only because $H_f(\xi)$ has been here replaced by $H_f(p)$. For this reason, and recalling that

$$\|p^* - p\|^t \leq \|\xi - p\|^t \|\xi - p\|_{\infty} \leq \|\xi - p\|_{\infty} = \varepsilon_{\text{max}},$$

the same argument as in the proof of Proposition 2.1 applies to give

$$|f(p)| \leq \|\text{Jac}_f(p)\|_1 \varepsilon_{\text{max}} + \frac{1}{2} \|H_f(p)\|_{\infty} \varepsilon_{\text{max}}^2 + O(\varepsilon_{\text{max}}^3).$$

Thus, up to a tolerance of $O(\varepsilon_{\text{max}}^3)$, we conclude that $|f(p)| \leq B'_1$. This completes the proof.

Q.E.D.

In the following example we compare the conditions provided in proposition 2.1 and 2.5. The bound $B_1$ is precise but harder than $B'_1$ to be computed. On the other hand, the bound $B'_1$, since it is effected by a second-order error analysis, it is only reliable for small values of the tolerance vector $\varepsilon$.

**Example 2.6** We consider the polynomial $f = y - 10x^4 \in \mathbb{R}[x, y]$ and the point $p = (0, 1)$. We let $\varepsilon = (0.75, 0.75)$ and let $B(p)$ be the $(\infty, \varepsilon)$-unit ball centered at $p$. We observe that the plane curve of equation $f = 0$ crosses the neighborhood $B(p)$. In order to verify the conditions of propositions 2.1 and 2.5, evaluating $f$ at $p$ gives $|f(p)| = 1$. Moreover,

$$\text{Jac}_f(x, y) = (-40x^3, 1) \quad \text{and} \quad H_f(x, y) = \begin{pmatrix} -120x^2 & 0 \\ 0 & 0 \end{pmatrix}.$$  

We have $\|\text{Jac}_f(x, y)\|_1 = 1$, $\|H_f(p)\|_{\infty} = 0$, $H = \max_{(x,y) \in B(p)} \|H_f(x, y)\|_{\infty} = \frac{125}{2}$. Consequently, the bounds $B_1$ and $B'_1$ are

$$B_1 = \frac{1263}{64} \approx 19.7 \quad \text{and} \quad B'_1 = 0.75.$$  

The condition of Proposition 2.1 is not satisfied, so no conclusion can be made on the intersection of the curve of equation $f = 0$ with $B(p)$. On the other hand, the condition of Proposition 2.5 is verified, so, up to a second-order error analysis, we would wrongly conclude that the curve $f = 0$ does not cross $B(p)$ (in this case, the value 0.75 of the components of $\varepsilon$ is not small enough).

We end this section by comparing the bounds provided by propositions 2.1 and 2.5.

**Lemma 2.7** Let $B_1, B'_1$ be the bounds as above. Thus $B_1 \geq B'_1$.

**Proof.** It follows from the definition of $B_1$ and $B'_1$, and from the obvious inequality $H = \max_{x \in B(p)} \|H_f(x)\|_{\infty} \geq \|H_f(p)\|_{\infty}$. Q.E.D.
3 Sufficient crossing cell conditions I

In this section we provide sufficient numerical conditions for an affine hypersurface to cross a bounded region containing a given point of \( \mathbb{R}^n \).

For each \( x = (x_1, \ldots, x_n) \) such that \( \text{Jac}_f(x) \) is not zero, we consider the pseudo-inverse matrix of \( \text{Jac}_f(x) \), defined by

\[
\text{Jac}_f^t(x) := \text{Jac}_f(x)^t \left( \text{Jac}_f(x) \text{Jac}_f(x)^t \right)^{-1} = \frac{\text{Jac}_f(x)^t}{\| \text{Jac}_f(x) \|_2^2},
\]

Note that \( \text{Jac}_f^t(x) \) is the right inverse of \( \text{Jac}_f(x) \), that is, \( \text{Jac}_f(x) \text{Jac}_f^t(x) = 1 \).

For any positive real number \( R \), set

\[
\mathcal{D}(p, R) := \{ x \in \mathbb{R}^n \text{ such that } \| (x - p)^t \|_\infty < R \}.
\]

Note that \( \mathcal{D}(p, R) \subseteq C(p) \) as soon as \( R < \varepsilon_{\min} \). Then set

\[
J := \sup_{x \in \mathcal{D}(p, R)} \| \text{Jac}_f^t(x) \|_\infty. \tag{10}
\]

We need the following technical lemma.

**Lemma 3.1** Let \( f = f(x) \) be a degree \( \geq 2 \) polynomial of \( P \) and let \( p \) be a point of \( \mathbb{R}^n \) such that the Jacobian \( \text{Jac}_f(p) \) is nontrivial. Let \( R \) be a positive real number such that \( R < \varepsilon_{\min} \). If \( R < \frac{1}{\| \text{Jac}_f(p) \|_1} \), then \( \text{Jac}_f(x) \) is nontrivial for \( x \in \mathcal{D}(p, R) \).

**Proof.** First we note that since \( R < \varepsilon_{\min} \), we have \( \mathcal{D} := \mathcal{D}(p, R) \subseteq B(p) \). Moreover, for each point \( x \in \mathcal{D} \), one has the triangular inequality relation

\[
\| \| \text{Jac}_f(x) \|_1 - \| \text{Jac}_f(p) \|_1 \| \leq \| \text{Jac}_f(x) - \text{Jac}_f(p) \|_1. \tag{11}
\]

We consider the vector valued function \( \phi = \text{Jac}_f : \mathcal{D} \to \mathbb{R}^n \); we observe that \( \phi \) is differentiable on the open convex set \( \mathcal{D} \) and that \( D\phi(x) = H_f(x) \). Applying Proposition 1.12 to \( \phi \) with \( r = \infty \) and using the fact that \( x \in B(p) \), we then have

\[
\| \text{Jac}_f(x)^t - \text{Jac}_f(p)^t \|_\infty < \sup_{0 < \nu < 1} \| H_f(p + \nu(x - p)) \|_\infty \| (x - p)^t \|_\infty < \sup_{0 < \nu < 1} \| H_f(p + \nu(x - p)) \|_\infty R. \tag{12}
\]

Recalling definition (5) of the quantity \( H \), we get \( \| \text{Jac}_f(x)^t - \text{Jac}_f(p)^t \|_\infty < HR \). Since \( \| \text{Jac}_f(x)^t - \text{Jac}_f(p)^t \|_\infty = \| \text{Jac}_f(x) - \text{Jac}_f(p) \|_1 \) (see Remark 1.6), from inequality (11) we then find \( \| \text{Jac}_f(x) \|_1 > \| \text{Jac}_f(p) \|_1 - HR > 0 \), so that the Jacobian \( \text{Jac}_f(x) \) is nontrivial for \( x \in \mathcal{D} \).

Q.E.D.

We have the following sufficient condition to conclude that the hypersurface of equation \( f = 0 \) crosses a given unit cell \( C(p) \).

**Proposition 3.2** Let \( f = f(x) \) be a degree \( \geq 2 \) polynomial of \( P \), let \( p \) be a point of \( \mathbb{R}^n \) such that \( \text{Jac}_f(p) \) is not the zero vector, and let \( C(p) \subseteq B(p) \) be an \( (\infty, \varepsilon) \)-unit cell centered at \( p \). Let \( R \) be a positive real number such that \( R < \min \{ \varepsilon_{\min}, \frac{\| \text{Jac}_f(p) \|_1}{R} \} \). Set \( c := \max\{2, \sqrt{n}\} \). If

\[
|f(p)| < \frac{2R}{J(c + \sqrt{n}HR)} =: B_2,
\]

then the hypersurface of equation \( f = 0 \) crosses \( C(p) \).
Proof. If \( f(p) = 0 \) there is nothing to prove. From Lemma 3.1 we know that the Jacobian \( \text{Jac}_f(x) \) is nonzero for \( x \in D := D(p, R) \). Moreover, since \( R < \varepsilon_{\min} \), one has \( D \subseteq C(p) \).

We now construct a sequence of points \( \{p_k\}_{k \in \mathbb{N}} \) as follows. We let \( p_0 = p \) and, for each \( k \geq 0 \), we define

\[
s_k := -\text{Jac}_{f}^{i}(p_k)f(p_k) = -\frac{f(p_k)}{\|\text{Jac}_{f}(p_k)\|_2}\text{Jac}_{f}(p_k)^i \quad \text{and} \quad p_{k+1} := p_k + s_k. \tag{13}
\]

The construction of the points \( p_k \) draws back to the Normal Flow algorithm (see [15]), an iterative method mainly used in homotopy and continuation problems. Obviously, \( p = p_0 \in D \). We prove by induction that the points \( p_k \)'s all lie in \( D \) and satisfy the inequality

\[
|f(p_k)| < |f(p_{k-1})| \quad \text{for each} \quad k \geq 1. \tag{14}
\]

**Step I** (The \( k = 1 \) case). From the definitions of \( s_0 \) and \( J \) (see (10)) we have

\[
\|s_0\|_\infty = \|\text{Jac}_{f}^1(p)\|_\infty |f(p)| \leq J|f(p)|.
\]

Moreover, by assumption, it follows that \( |f(p)| < B_2 < \frac{2H}{J} \leq \frac{R}{J} \). Thus \( \|s_0\|_\infty < R \) (since \( D \) is defined in \( \|\cdot\|_\infty \) norm), showing that \( p_1 \in D \).

From the formal Taylor expansion of \( f(x) \) at \( p \) (see Definition 1.10) it follows that

\[
f(x) = f(p) + \text{Jac}_{f}(p)(x - p)^t + R_1(x),
\]

where \( R_1(x) \) is the 1-st remainder term of \( f(x) \) at \( p \). If we evaluate the former expression at \( p_1 \) and apply Taylor’s theorem (see Proposition 1.11(2)) we get

\[
f(p_1) = f(p) + \text{Jac}_{f}(p)(p_1 - p)^t + 1\frac{(p_1 - p)H_f(\xi)(p_1 - p)^t}{2},
\]

where \( \xi \) is a point of the line that connects \( p \) to \( p_1 \). Therefore, by definitions (13), we get

\[
f(p_1) = f(p) + \text{Jac}_{f}(p)s_0 + 1\frac{s_0^tH_f(\xi)s_0}{2}.
\]

Let us upper bound the absolute value of the quantity

\[
Q := 1\frac{|f(p)|}{2}\frac{\|\text{Jac}_{f}(p)\|_2}{\|	ext{Jac}_{f}(p)\|_2}H_f(\xi)\text{Jac}_{f}(p)^t.
\]

To this end, use Hölder’s inequality (see Proposition 1.2), Proposition 1.8(1) and Remark 1.6, and recall the definitions of \( H, J \) (see (5), (10) respectively) to get:

\[
|Q| \leq 1\frac{|f(p)|}{2}\frac{\|\text{Jac}_{f}(p)\|_2^2}{\|	ext{Jac}_{f}(p)\|_2^2}||H_f(\xi)||_2 \leq 1\frac{|f(p)|}{2}\frac{|f(p)|}{\|	ext{Jac}_{f}(p)\|_2^2}\sqrt{n}||H_f(\xi)||_\infty
\]

\[
\leq 1\frac{|f(p)|}{2}\frac{\sqrt{n}|f(p)|}{\|	ext{Jac}_{f}(p)\|_2^2}\frac{H}{\|	ext{Jac}_{f}(p)\|_2^2}||\text{Jac}_{f}(p)^t||_\infty
\]

\[
= 1\frac{|f(p)|}{2}\frac{\sqrt{n}|f(p)||\text{Jac}_{f}(p)||_\infty}{\|	ext{Jac}_{f}(p)\|_1}\frac{H}{\|	ext{Jac}_{f}(p)\|_1} \leq 1\frac{|f(p)|}{2}\frac{\sqrt{n}|f(p)|}{\|	ext{Jac}_{f}(p)\|_1}. \tag{15}
\]
By the assumption on \( R \) we thus obtain \(|Q| < \frac{1}{2}\sqrt{\pi}|f(p)|\frac{1}{R}\). On the other hand, \(|f(p)| < B_2 < \frac{2R}{\sqrt{\pi}}\). Therefore \(|Q| < 1\), so that equality (15) reads \(|f(p_1)| < |f(p)|\), showing condition (14) for \( k = 1 \).

**Step II** (The inductive step). Suppose that the points \( p, p_1, \ldots, p_k \) of the sequence lie in \( D \) and that \( 0 < |f(p_k)| < \cdots < |f(p)| \). Hence, in particular, the points \( p, p_1, \ldots, p_k \) are all distinct, so that, by definition, \( \|s_{i-1}\|_\infty \neq 0 \) for \( i = 1, \ldots, k \).

First we show that \( p_{k+1} \in D \). For \( i = 1, \ldots, k \), the formal Taylor expansion of \( f(x) \) at \( p_{i-1} \) (see Definition 1.10) yields

\[
f(x) = f(p_{i-1}) + \text{Jac}_f(p_{i-1})(x - p_{i-1}) + R_1(x),
\]

where \( R_1(x) \) is the 1-st remainder term of \( f(x) \) at \( p_{i-1} \). If we evaluate the former expression at \( p_i \) and apply Taylor’s theorem (see Proposition 1.11(2)) we get

\[
f(p_i) = f(p_{i-1}) + \text{Jac}_f(p_{i-1})(p_i - p_{i-1}) + \frac{1}{2}(p_i - p_{i-1})H_f(\xi_i)(p_i - p_{i-1})^t,
\]

where \( \xi_i \) is a point of the line that connects \( p_{i-1} \) to \( p_i \). On the other hand, by definition of \( s_{i-1} \) and recalling that \( \text{Jac}_f(x)\text{Jac}_f^t(x) = 1 \), we have \( \text{Jac}_f(p_{i-1})s_{i-1} = -f(p_{i-1}) \), whence

\[
f(p_{i-1}) = -\text{Jac}_f(p_{i-1})s_{i-1} = -\text{Jac}_f(p_{i-1})(p_i - p_{i-1})^t.
\]

By combining (16) and (17) with Hölder’s inequality, we get

\[
|f(p_i)| = \frac{1}{2}|(p_i - p_{i-1})H_f(\xi_i)(p_i - p_{i-1})^t| \\
\leq \frac{1}{2}||p_i - p_{i-1}||H_f(\xi_i)||1||H_f(\xi_i)||1||p_i - p_{i-1}||^t||\infty \\
\leq \frac{1}{2}||p_i - p_{i-1}||1||H_f(\xi_i)||1||p_i - p_{i-1}||^t||\infty \\
= \frac{1}{2}||H_f(\xi_i)||1||p_i - p_{i-1}||^t||2||\infty.
\]

Since the Hessian matrix is symmetric we have \( ||H_f(\xi_i)||1 = ||H_f(\xi_i)||\infty \). Thus, by definition of \( H \), the previous relation yields

\[
|f(p_i)| \leq \frac{1}{2}||H_f(\xi_i)||\infty ||p_i - p_{i-1}||^t||^2||\infty \leq \frac{1}{2}||H||s_{i-1}||^2||\infty.
\]

Now, define \( \tau_i := \frac{||s_{i-1}||\infty}{||s_{i-1}||\infty} \). Therefore inequality (19) gives

\[
\|s_i\|\infty = \|\text{Jac}_f(p_i)\|\infty \|f(p_i)\| \leq \|J||f(p_i)\| \leq \frac{1}{2}||JH||s_{i-1}||^2||\infty.
\]

Thus

\[
\tau_i = \frac{\|s_i\|\infty}{\|s_{i-1}\|\infty} \leq \frac{1}{2}||JH||s_{i-1}||\infty \leq \frac{1}{2}||J^2H||f(p_{i-1})|| \leq \frac{1}{2}||J^2H||f(p)||.
\]

Since \( |f(p)| < B_2 < \frac{2R}{\sqrt{\pi}} \) \( \leq \frac{2R}{\sqrt{\pi}} \), it must be \( \tau_i < 1 \) by the above inequality. Let \( \tau := \max_{i=1, \ldots, k} \{\tau_i\} \). We bound \( \|f(p_{k+1} - p)\|\infty \) as follows:

\[
\|f(p_{k+1} - p)^t\|\infty \leq \|s_0\|\infty + \|s_1\|\infty + \cdots + \|s_k\|\infty \\
= \|s_0\|\infty + \tau_1\|s_0\|\infty + \tau_1\tau_2\|s_0\|\infty + \cdots + \tau_1\tau_2\cdots\tau_k\|s_0\|\infty \\
= \|s_0\|\infty (1 + \tau_1 + \tau_1\tau_2 + \cdots + \tau_1\tau_2\cdots\tau_k) \\
\leq \|s_0\|\infty \sum_{i=0}^k \tau^i < \|s_0\|\infty \sum_{i=0}^\infty \tau^i = \|s_0\|\infty \frac{1}{1 - \tau} \leq \|J||f(p)|| \frac{1}{1 - \tau}.
\]
Then, by inequality (20) and the assumption $|f(p)| < B_2$, we find
\[
\| (p_{k+1} - p)_t \|_\infty < J |f(p)| \frac{1}{1 - \frac{1}{2} ||J^t H||} = \frac{2||f(p)||}{2 - \frac{1}{2} ||J^t H||} < R,
\]
therefore $p_{k+1} \in D$.

Now, let us prove that $|f(p_{k+1})| < |f(p_k)|$. To this purpose we observe that relation (15) can be easily adapted to the pair of points $p_k, p_{k+1}$ in the form
\[
f(p_{k+1}) = |f(p_k)| \left( \frac{1}{2} \frac{|f(p_k)|}{\|J^t Jac_f(p_k)\|} J^t Jac_f(p_k) H(f(\xi_k) Jac_f(p_k) t),
\]
where $\xi_k$ is a point of the line connecting $p_k$ to $p_{k+1}$. Let us upper bound the absolute value of the quantity
\[
Q_k := \frac{1}{2} \frac{|f(p_k)|}{\|J^t Jac_f(p_k)\|} J^t Jac_f(p_k) H(f(\xi_k) Jac_f(p_k) t).
\]
As previously done to upper bound the quantity $|Q|$, by using now Hölder’s inequality, the first two statements of Proposition 1.8, and the definition of $H$, we get
\[
|Q_k| \leq \frac{1}{2} \sqrt{n} \frac{|f(p_k)|}{\|J^t Jac_f(p_k)\|^2} \|H_J(\xi_k)\| \leq \frac{1}{2} \sqrt{n} \frac{|f(p_k)|}{\|J^t Jac_f(p_k)\|^2} H
\]
\[
\leq \frac{1}{2} \sqrt{n} \frac{|f(p_k)|}{\|J^t Jac_f(p_k)\|^2} H < \frac{1}{2} \sqrt{n} \frac{|f(p_k)|}{\|J^t Jac_f(p_k)\|^2} H,
\]
where the last inequality comes from the inductive hypothesis $|f(p_k)| < |f(p)|$. Since $Jac_f(p_k) Jac_f(p_k)^t = 1$, Hölder’s inequality and the definition of $J$ give
\[
1 = |Jac_f(p_k) Jac_f(p_k)^t| \leq \|Jac_f(p_k)\| \|Jac_f(p_k)^t\| \leq \|Jac_f(p_k)\| H.
\]

Therefore inequality (22) becomes $|Q_k| < \frac{1}{2} \sqrt{n} |f(p)| H / 2$. On the other hand, $|f(p)| < B_2 < \frac{2}{\sqrt{n} H J^t}$. Thus we find $|Q_k| < 1$, so that equality (21) yields $|f(p_{k+1})| < |f(p_k)|$, as we want.

**Step III** (Conclusion). If there exists $k \in \mathbb{N}$ such that $f(p_k) = 0$ we are done. Otherwise, we know from Step II that $\tau_k := \frac{\|s_k\|_\infty}{\|s_{k-1}\|_\infty} < 1$ for $k \in \mathbb{N}$. Then, by D’Alembert criterion, the series $\sum_{i=1}^\infty \|s_i\|_\infty$ converges, so that $\lim_{k \to \infty} \left( \sum_{i=k+1}^\infty \|s_i\|_\infty \right) = 0$. Define $p^*: = p + \sum_{i=1}^\infty s_i$. Then, since $p_k^* = p^* + \sum_{i=1}^k s_i$, one has
\[
\lim_{k \to \infty} \|p_k - p^*\|_\infty = \lim_{k \to \infty} \left( \sum_{i=k+1}^\infty s_i \right) = \lim_{k \to \infty} \left( \sum_{i=k+1}^\infty s_i \right) = 0.
\]
Thus the sequence of points $\{p_k\} \in \mathbb{N}$ converges to the point $p^*$. Since the $p_k$’s belong to $D$, the point $p^*$ belongs to the closure $\overline{D} \subseteq C(p)$. We also know that $\|s_k\|_\infty = \tau_1 \tau_2 \ldots \tau_k \|s_0\|_\infty < \tau^k \|s_0\|_\infty$, where $\tau = \sup_{k \in \mathbb{N}} \{\tau_k\}$. Therefore $\lim_{k \to \infty} \|s_k\|_\infty < \lim_{k \to \infty} \tau^k \|s_0\|_\infty = 0$. From inequality (19), we then conclude that $|f(p)| = \lim_{i \to \infty} |f(p_i)| \leq \frac{1}{2} H \lim_{i \to \infty} \|s_{i-1}\|_\infty^2 = 0$. This completes the proof.

Q.E.D.

**Remark 3.3** (The linear case) Notation as above. It is just the case noting that, if the polynomial $f = f(x)$ is linear, the bound $B_1$ of Proposition 2.1 simply becomes $B_1 = \|Jac_f(p)\| \|e_{\max}\|$. Concerning the results of the present section, in the linear case, Lemma 3.1 holds true under the only assumption that $R < e_{\min}$. Similarly, Proposition 3.2 holds true under the only assumption that $R < e_{\min}$ as well. Moreover, the bound $B_2$ simply becomes $B_2 = \frac{3}{2}$. Indeed, for a linear polynomial $f$, the same argument as in Step I of the proposition shows that the hyperplane $f = 0$ crosses the cell $C(p)$ as soon as $|f(p)| < \frac{R}{2}$ (since equation (15) yields $f(p_1) = 0$ in that case).
Example 3.4 We consider the polynomial $f = y - 10x^2$ and the point $p = (1.1,10)$, the second example in the Introduction. We let $\varepsilon = (0.13,0.13)$ and consider the $(\infty,\varepsilon)$-unit ball $B(p)$ centered at $p$. We have
\[ \text{Jac}_f(x) = (-20x, 1) \quad \text{and} \quad \text{Jac}_f^1(x) = \frac{1}{1 + 400x^2} \begin{pmatrix} -20x \\ 1 \end{pmatrix}, \]
whence $\|\text{Jac}_f(p)\|_1 = 22 > 0$. Furthermore $\|H_f(x)\|_\infty = 20$, for each $x$, so that $H = 20$. We let $R = 0.12$ and compute $J \approx 0.05$. We have that $\|f(p)\| = 2.1$ is strictly smaller than $\frac{2}{\sqrt{\sup_{|x| \leq \sqrt{R^2}} H_f(x)}} R \approx 2.21$, which is the bound provided in Proposition 2.1. We conclude that the curve of equation $f = 0$ goes through the ball $B(p)$.

Example 3.5 We consider the polynomial $f = 4x^2 + y^2 - 4x$ as in Example 2.3 and the point $p_2 = (\frac{1}{2}, \frac{3}{4})$. We let $\varepsilon = (0.1,0.1)$ and consider the $(\infty,\varepsilon)$-unit ball $B(p)$ centered at $p$. We have $H = 8$ and $\|\text{Jac}_f(p_2)\|_1 = \|(-2.4,1.5)\|_1 = 2.4 > 0$. We choose $R = 0.075 < \min\{0.1,0.3\}$. Direct computations show that $J = \sup_{(x,y) \in D} \|\text{Jac}_f^1(x,y)\|_1 = \frac{18}{16}$. Therefore for the bound $B_2$ of Proposition 3.2 we find $B_2 = \frac{3^2}{2^2 \sqrt{\sup_{|x| \leq \sqrt{R^2}} H_f(x)}} \approx 0.18$. Since $|f(p)| = 0.0775$ is strictly smaller than $B_2$, by using Proposition 2.1 we conclude that the curve of equation $f = 0$ crosses the ball $B(p)$.

4 Sufficient crossing cell conditions II

In this section we provide sufficient numerical conditions, working up to a first-order error analysis, for an affine hypersurface to cross a bounded region containing a given point of $\mathbb{R}^n$.

To begin with, let us upper bound the quantity $J$ defined in (10).

Proposition 4.1 Let $f(x)$ be a degree $\geq 2$ polynomial of $P$ and let $p$ be a point of $\mathbb{R}^n$. Let $R$ be a positive real number and suppose that the Jacobian $\text{Jac}_f(x)$ is nonzero for each $x \in D(p,R)$. Then
\[ J < \|\text{Jac}_f^1(p)\|_\infty + n^2(1 + \sqrt{n}) \|H_f(p)\|_\infty \|\text{Jac}_f(p)\|_1^{-1} R + O(R^2). \]

Proof. Let $x \in D := D(p,R)$. Consider the vector valued function $(\text{Jac}_f^1)^t : D \to \mathbb{R}^n$. Since by hypothesis $\text{Jac}_f(x)$ has full row rank in $D$, it follows that $(\text{Jac}_f^1)^t$ is differentiable on the open convex set $D$. We apply Proposition 1.12 with $r = \infty$ to get
\[ \|\text{Jac}_f^1(x) - \text{Jac}_f^1(p)\|_\infty < \sup_{0 < \nu < 1} \|D\text{Jac}_f^1(p + \nu(x - p))\|_\infty \|\nu\|_\infty \|x - p\|_\infty. \tag{23} \]

Combining (23) with $\|\|\text{Jac}_f^1(x)\|_\infty - \|\text{Jac}_f^1(p)\|_\infty \| \leq \|\text{Jac}_f^1(x) - \text{Jac}_f^1(p)\|_\infty$ (the usual consequence of the triangular inequality), we have
\[ \|\text{Jac}_f^1(x)\|_\infty < \|\text{Jac}_f^1(p)\|_\infty + \sup_{0 < \nu < 1} \|D\text{Jac}_f^1(p + \nu(x - p))\|_\infty \|\nu\|_\infty \|x - p\|_\infty. \tag{24} \]

Applying Lemma 1.13 to the matrix $M(x) = D\text{Jac}_f^1(x)$, one has
\[ \sup_{0 < \nu < 1} \|D\text{Jac}_f^1(p + \nu(x - p))\|_\infty < n^2 \|D\text{Jac}_f^1(p)\|_\infty + O(R). \tag{25} \]

We explicitly express $D\text{Jac}_f^1(x)$ by computing the partial derivatives of each component of $\text{Jac}_f^1(x)$. That is,
\[ D\text{Jac}_f^1(x) = \frac{1}{\|\text{Jac}_f(x)\|_2^2} \left( \|\text{Jac}_f(x)\|_2^2 H_f(x) - 2\text{Jac}_f(x)^t\text{Jac}_f(x)H_f(x) \right). \]
We now upper bound $\|DJac^t_f(p)\|_\infty$ by

$$
\|DJac^t_f(p)\|_\infty \leq \frac{1}{\|Jac_f(p)\|_2} (\|Jac_f(p)\|_2^2 + 2\|Jac_f(p)\|_\infty \|H_f(p)\|_\infty)
$$

where the last equality follows from Remark 1.6. Proposition 1.8 gives $\|Jac_f(p)\|_2 \leq \|Jac_f(p)\|_2^2$, as well as $\|Jac_f(p)\|_1 \leq \sqrt{n} \|Jac_f(p)\|_2$. Thus, noting that $\|Jac_f(p)\|_2 = \|Jac_f(p)\|_2^2$, and using again Proposition 1.8(2), one has

$$
\|DJac^t_f(p)\|_\infty \leq \frac{1}{\|Jac_f(p)\|_2} (\|Jac_f(p)\|_2^2 + 2\|Jac_f(p)\|_\infty \|H_f(p)\|_\infty)
$$

By combining (25) and (26) and recalling that $x \in D$ implies $\|(x - p)^t\|_\infty < R$, inequality (24) yields

$$
\|Jac^t_f(x)\|_\infty < \|Jac^t_f(p)\|_\infty + n^2(1 + 2\sqrt{n}) \frac{\|H_f(p)\|_\infty}{\|Jac_f(p)\|_1^2} R + O(R^2).
$$

By definition of $J$ we are done. Q.E.D.

Next, we need a technical result (valid up to a first-order error analysis).

**Lemma 4.2** Let $f = f(x)$ be a degree $\geq 2$ polynomial of $P$ and let $p$ be a point of $\mathbb{R}^n$ such that both the Jacobian $Jac_f(p)$ and the Hessian matrix $H_f(p)$ are nontrivial. Let $R$ be a positive real number such that $R < \epsilon_{\min}$. If $R < \frac{\|Jac_f(p)\|_\infty}{n^2\|H_f(p)\|_\infty}$, then, neglecting contributions of order $O(R^2)$, the Jacobian $Jac_f(x)$ is nonzero for each $x \in D(p, R)$.

**Proof.** Since $R < \epsilon_{\min}$, we have $D := D(p, R) \subseteq B(p)$. Lemma 1.13 applied to the Hessian matrix $M(x) = H_f(x)$ gives, for each $x \in D$,

$$
\|H_f(p + \nu(x - p))\|_\infty < \frac{n^2\|H_f(p)\|_\infty}{\|Jac_f(p)\|_1^2} + O(R).
$$

On the other hand, inequality (12) still holds true. Thus we find $\|Jac_f(x)^t - Jac_f(p)^t\|_\infty < \frac{n^2\|H_f(p)\|_\infty}{\|Jac_f(p)\|_1^2} R + O(R^2)$, so that the same conclusion as in the proof of Lemma 3.1 gives now

$$
\|Jac_f(x)^t\|_1 > \|Jac_f(p)^t\|_1 - \frac{n^2\|H_f(p)\|_\infty}{\|Jac_f(p)\|_1^2} R - O(R^2).
$$

Therefore, up to a first-order error analysis, $Jac_f(x)$ is nonzero for each $x \in D$, as we want. Q.E.D.

As in the case of necessary conditions, it is natural to ask for a statement analogous to Proposition 3.2 which avoids the computation of the quantities $J$ and $H$, providing a crossing cell condition simply evaluating the Jacobian and the Hessian matrix at a given point. The following statement settles the question, for small values of the tolerance vector $\epsilon$, up to a first-order error analysis, that is, disregarding second-order contributions. To this purpose, for the rest of this section, we assume $\epsilon_{\max} \ll 1$.

**Proposition 4.3** Let $f = f(x)$ be a degree $\geq 2$ polynomial of $P$, let $p$ be a point of $\mathbb{R}^n$ such that Jacobian $Jac_f(p)$ and the Hessian matrix $H_f(p)$ are nontrivial, and let $C(p) \subseteq B(p)$ be an $(\infty, \epsilon)$-unit cell centered at $p$. Let $R$ be a positive real number such that $R < \min \{\epsilon_{\min}, \frac{\|Jac_f(p)\|_1}{n^2\|H_f(p)\|_\infty}\}$, let $c := \max\{2, \sqrt{n}\}$ and set

$$
\Theta := \|Jac^t_f(p)\|_\infty + n^2(1 + 2\sqrt{n}) \frac{\|H_f(p)\|_\infty}{\|Jac_f(p)\|_1^2} R.
$$
If
\[
|f(p)| < \frac{2R}{\Theta(c + n^{5/2}\|H_f(p)\|_{\infty}\Theta R)} =: B'_2,
\]
then the hypersurface of equation \( f = 0 \) crosses \( C(p) \) neglecting order \( O(R^2) \) contributions.

**Proof.** The proof runs parallel to that of Proposition 3.2. First, note that if \( |f(p)| = O(R^2) \) there is nothing to prove, so we can assume \( |f(p)| > O(R^2) \). From Lemma 4.2 we know that the Jacobian \( \text{Jac}_f(x) \) is nonzero, for \( x \in \mathcal{D} := \mathcal{D}(p, R) \), up to a first-order analysis. Moreover, since \( R < \varepsilon_{\min} \), one has \( \mathcal{D} \subseteq C(p) \).

We now consider the sequence of points \( \{p_k\} \) and the column vectors \( s_k, k \in \mathbb{N} \), defined as in (13). Obviously, \( p := p_0 \in \mathcal{D} \). We prove by induction that, for each \( k \geq 1 \), the points \( p_k \)'s all lie in \( \mathcal{D} \) up to \( O(R^2) \), that is,
\[
\|(p_k - p)^t\|_{\infty} < R + O(R^2),
\]
and satisfy the inequality
\[
|f(p_k)| < |f(p_{k-1})| + O(R^2).
\]
**Step I** (The \( k = 1 \) case). From the definitions of \( s_0 \) and \( \Theta \) we have
\[
\|s_0\|_{\infty} = \|\text{Jac}_f(p)\|_{\infty}|f(p)| < \Theta|f(p)|.
\]
Furthermore, since \( c \geq 2 \), one has by assumption \( |f(p)| < B'_2 < \frac{2R}{\Theta(c + R)} \leq \frac{R}{\Theta} \). Thus \( \|s_0\|_{\infty} < R \), that is \( p_1 \in \mathcal{D} \), showing condition (28) for \( k = 1 \).

From the formal Taylor expansion of \( f(x) \) at \( p \) (see Definition 1.10) it follows that
\[
f(x) = f(p) + \text{Jac}_f(p)(x - p)^t + \frac{1}{2}(x - p)H_f(p)(x - p)^t + R_2(x),
\]
where \( R_2(x) \) is the 2-nd remainder term of \( f(x) \) at \( p \). If we evaluate the former expression at \( p_1 \) and apply Taylor’s theorem (see Proposition 1.11(1)) we get
\[
f(p_1) = f(p) + \text{Jac}_f(p)(p_1 - p)^t + \frac{1}{2}(p_1 - p)H_f(p)(p_1 - p)^t + O(R^3).
\]
Therefore, by definitions (13) with \( k = 0 \), we get
\[
f(p_1) = f(p) + \text{Jac}_f(p)s_0 + \frac{1}{2}s_0^tH_f(p)s_0 + O(R^3)
\]
\[
= f(p) - f(p) - f(p)\frac{\text{Jac}_f(p)^t\text{Jac}_f(p)}{\|\text{Jac}_f(p)\|_2^2} + \frac{1}{2}\frac{|f(p)|^2}{\|\text{Jac}_f(p)\|_2^2} \text{Jac}_f(p)^tH_f(p)\text{Jac}_f(p)^t + O(R^3)
\]
\[
= f(p) - f(p) + \frac{1}{2}\frac{|f(p)|^2}{\|\text{Jac}_f(p)\|_2^2} \text{Jac}_f(p)^tH_f(p)\text{Jac}_f(p)^t + O(R^3)
\]
\[
= |f(p)|\left(\frac{1}{2}\frac{|f(p)|^2}{\|\text{Jac}_f(p)\|_2^2} \text{Jac}_f(p)^tH_f(p)\text{Jac}_f(p)^t\right) + O(R^3).
\]
Let us upper bound the absolute value of the quantity
\[
Q := \frac{1}{2}\frac{|f(p)|^2}{\|\text{Jac}_f(p)\|_2^2} \text{Jac}_f(p)^tH_f(p)\text{Jac}_f(p)^t.
\]
To this end, use Hölder’s inequality, Proposition 1.8(1) and Remark 1.6, to get:
\[
|Q| \leq \frac{1}{2}\frac{|f(p)|^2}{\|\text{Jac}_f(p)\|_2^2} \|\text{Jac}_f(p)^t\|_{\infty} \|H_f(p)\|_2^2
\]
\[
\leq \frac{1}{2}\frac{|f(p)|^2}{\|\text{Jac}_f(p)\|_2^2} \sqrt{n}\|H_f(p)\|_{\infty}
\]
\[
= \frac{1}{2}\sqrt{n}|f(p)| \|\text{Jac}_f(p)^t\|_{\infty} \|H_f(p)\|_{\infty} \|\text{Jac}_f(p)\|_2 \|\text{Jac}_f(p)^t\|_{\infty}
\]
\[
= \frac{1}{2}\sqrt{n}|f(p)| \|\text{Jac}_f(p)^t\|_{\infty} \|H_f(p)\|_{\infty} \|\text{Jac}_f(p)\|_1.
\]
Recalling the definition of $\Theta$ and the assumption on $R$, we find

$$|Q| < \frac{1}{2} \sqrt{n} |f(p)|_\Theta \frac{\|H_f(p)\|_\infty}{\|\text{Jac}_f(p)\|_1} < \frac{1}{2} \sqrt{n} |f(p)|_R.$$  

On the other hand, since $c \geq \sqrt{n}$, one has by assumption that $|f(p)| < B_2' < \frac{2n}{c^2} \leq \frac{2n}{\sqrt{\Theta^2}}$. Therefore $|Q| < 1$, so that equality (30) proves condition (29) for $k = 1$.

**Step II** (Inductive step). Suppose that the points $p, p_1, \ldots, p_k$ of the sequence meet conditions (28) and (29). Further assume $|f(p_i)| > O(R^2)$, whence, in particular, $\|s_i\|_\infty = \|\text{Jac}_f(p_i)\|_\infty |f(p_i)| > O(R^2)$, $i = 1, \ldots, k$. We want to show that the point $p_{k+1}$ satisfies such conditions as well. From the formal Taylor expansion of $f(x)$ at $p_{k-1}$ (see Definition 1.10) it follows that

$$f(x) = f(p_{k-1}) + \text{Jac}_f(p_{k-1})(x - p_{k-1})^t + \frac{1}{2} (x - p_{k-1}) H_f(p_{k-1})(x - p_{k-1})^t + R_2(x),$$

where $R_2(x)$ is the 2-nd remainder term of $f(x)$ at $p_{k-1}$, $i = 1, \ldots, k$. If we evaluate the former expression at $p_i$ and apply Taylor’s theorem (see Proposition 1.11(1) we get

$$f(p_i) = f(p_{k-1}) + \text{Jac}_f(p_{k-1})(p_i - p_{k-1})^t + \frac{1}{2} (p_i - p_{k-1}) H_f(p_{k-1})(p_i - p_{k-1})^t + O(R^3).$$

(31)

Then the same argument giving inequality (18) in the proof of Proposition 3.2 yields now

$$|f(p_i)| \leq \frac{1}{2} \|H_f(p_{k-1})\|_1 \|(p_i - p_{k-1})^t\|_\infty^2 + O(R^3).$$

Since the Hessian matrix is symmetric one has $\|H_f(p_{k-1})\|_1 = \|H_f(p_{k-1})\|_\infty$. Moreover, from Lemma 1.13 applied to $M(x) = H_f(x)$, we have $\|H_f(p_{k-1})\|_\infty < n^2\|H_f(p)\|_\infty + O(R)$. Thus the previous relation yields

$$|f(p_i)| \leq \frac{1}{2} n^2 \|H_f(p)\|_\infty \|s_{i-1}\|_\infty^2 + \frac{1}{2} \|s_{i-1}\|_\infty \|f(p_i)\|_\infty + O(R^3)$$

$$\leq \frac{1}{2} n^2 \|H_f(p)\|_\infty \|s_{i-1}\|_\infty^2 + O(R^3),$$

(32)

where the last inequality is a consequence of the inductive assumption

$$\|(p_i - p_{i-1})^t\|_\infty \leq \|(p_i - p)^t\|_\infty + \|(p_{i-1} - p)^t\|_\infty < 2R + O(R^2), \ i = 1, \ldots, k.$$

Proposition 4.1 reads $J < \Theta + O(R^2)$. Then inequality (32) gives

$$\|s_i\|_\infty = \|\text{Jac}_f(p_i)\|_\infty |f(p_i)| < (\Theta + O(R^2)) \left( \frac{1}{2} n^2 \|H_f(p)\|_\infty \|s_{i-1}\|_\infty^2 + O(R^3) \right)$$

$$\leq \frac{1}{2} n^2 \Theta \|H_f(p)\|_\infty \|s_{i-1}\|_\infty^2 + O(R^3).$$

Now, define $\tau_i := \frac{\|s_i\|_\infty}{\|s_{i-1}\|_\infty}$. Recalling the inductive assumption $|f(p_i)| < |f(p_{i-1})| + O(R^2)$, $i = 1, \ldots, k$, it thus follows that

$$\tau_i = \frac{\|s_i\|_\infty}{\|s_{i-1}\|_\infty} \leq \frac{1}{2} n^2 \Theta \|H_f(p)\|_\infty \|s_{i-1}\|_\infty + \frac{O(R^3)}{\|s_{i-1}\|_\infty}$$

$$< \frac{1}{2} n^2 \Theta \|H_f(p)\|_\infty \|s_{i-1}\|_\infty + O(R)$$

$$< \frac{1}{2} n^2 \Theta^2 \|H_f(p)\|_\infty |f(p_{i-1})| + O(R) < \frac{1}{2} n^2 \Theta^2 \|H_f(p)\|_\infty |f(p)| + O(R).$$

(33)

Setting $T := \frac{1}{2} n^2 \Theta^2 \|H_f(p)\|_\infty |f(p)|$, and recalling that

$$|f(p)| < B_2' < \frac{2n}{\sqrt{\Theta^2}}.$$

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we then have \( T < \frac{1}{\sqrt{n}} \leq 1 \), so that \( \tau_i < T + O(R) \) by inequality (33). Thus, as in the corresponding step in the proof of Proposition 3.2, we get the bound:

\[
\| (p_{k+1} - p)^f \|_\infty \leq \| s_0 \|_\infty (1 + \tau_1 + \tau_1 \tau_2 + \ldots + \tau_1 \tau_2 \ldots \tau_k) < \| s_0 \|_\infty \sum_{i=0}^{k} (T + O(R))^i = \| s_0 \|_\infty \left( \sum_{i=0}^{k} T^i + O(R) \right)
\]

\[
\leq \| s_0 \|_\infty \sum_{i=0}^{\infty} T^i + O(R^2) = \| s_0 \|_\infty \frac{1}{1-T} + O(R^2)
\]

\[
= \| \text{Jac}_f(p) \|_\infty |f(p)| + O(R^2) < \frac{\Theta |f(p)|}{1-T} + O(R^2).
\]

By definition of \( T \) we then find

\[
\| (p_{k+1} - p)^f \|_\infty < \frac{\Theta |f(p)|}{1 - \frac{1}{2} n^2 \Theta^2 \| H_f(p) \|_\infty |f(p)|} + O(R^2)
\]

\[
= \frac{2\Theta |f(p)|}{2 - n^2 \Theta^2 \| H_f(p) \|_\infty |f(p)|} + O(R^2) < R + O(R^2),
\]

where the last inequality rewrites as \( |f(p)| < \frac{2n^2 \Theta^2 \| H_f(p) \|_\infty |f(p)|}{2 - n^2 \Theta^2 \| H_f(p) \|_\infty |f(p)|} \), which follows from the assumption \( |f(p)| < B_2^* \). We then conclude that \( p_{k+1} \in \mathcal{D} \) (up to \( O(R^2) \)).

Moreover, we observe that relation (30) can be easily adapted to the pair of points \( p_k, p_{k+1} \) in the form

\[
f(p_{k+1}) = |f(p_k)| \left( \frac{1}{2} \frac{|f(p_k)|}{\| \text{Jac}_f(p_k) \|_2^2} \text{Jac}_f(p_k) H_f(p_k) \text{Jac}_f(p_k)^t \right) + O(R^3).
\]

Let us upper bound the absolute value of the quantity

\[
Q_k := \frac{1}{2} \frac{|f(p_k)|}{\| \text{Jac}_f(p_k) \|_2^4} \| H_f(p_k) \|_\infty \text{Jac}_f(p_k) H_f(p_k) \text{Jac}_f(p_k)^t.
\]

As in Step I, by using Hölder’s inequality and Proposition 1.8(1), we obtain

\[
|Q_k| \leq \frac{1}{2} \sqrt{n} \frac{|f(p_k)|}{\| \text{Jac}_f(p_k) \|_2^2} \| H_f(p_k) \|_\infty \leq \frac{1}{2} \sqrt{n} \frac{|f(p_k)|}{\| \text{Jac}_f(p_k) \|_2^4} \| H_f(p_k) \|_\infty.
\]

By Lemma 1.13 applied to the matrix \( M(x) = H_f(x) \), we then have

\[
|Q_k| < \frac{1}{2} \sqrt{n} \frac{|f(p_k)|}{\| \text{Jac}_f(p_k) \|_2^4} (n^2 \| H_f(p) \|_\infty + O(R))
\]

\[
\leq \frac{1}{2} n^{5/2} \frac{|f(p_k)|}{\| \text{Jac}_f(p_k) \|_1^2} \| H_f(p) \|_\infty + O(R^2),
\]

(35)

where the last inequality comes from \( |f(p_k)| < |f(p)| + O(R^2) < \frac{2n}{\Theta} + O(R^2) \), a consequence of the inductive hypothesis and the assumption \( |f(p)| < B_2^* \). By Hölder’s inequality and the definition of \( \Theta \) we find

\[
1 = | \text{Jac}_f(p_k) \text{Jac}_f(p_k)^t | \leq \| \text{Jac}_f(p_k) \|_1 \| \text{Jac}_f(p_k)^t \|_\infty \leq \| \text{Jac}_f(p_k) \|_1 (\Theta + O(R^2)).
\]

Therefore inequality (35) becomes:

\[
|Q_k| < \frac{1}{2} n^{5/2} |f(p_k)| \| H_f(p) \|_\infty (\Theta + O(R^2))^2 + O(R^2)
\]

\[
< \frac{1}{2} n^{5/2} (|f(p)| + O(R^2)) \| H_f(p) \|_\infty \Theta^2 + O(R^2)
\]

\[
= \frac{1}{2} n^{5/2} \| H_f(p) \|_\infty \Theta^2 |f(p)| + O(R^2).
\]
On the other hand, $|f(p)| < B_2' < \frac{2}{n_5/2\|Hf(p)\|_\infty} \Theta R$. Thus we find $|Q_k| < 1 + O(R^2)$, so that equality (34) yields the desired condition $|f(p_{k+1})| < |f(p_k)| + O(R^2)$.

**Step III (Conclusion).** If there exists $k \in \mathbb{N}$ such that $|f(p_k)| = O(R^2)$ we are done. Otherwise, we know from Step II that $\tau_k := \frac{\|s_k\|_\infty}{\|s_{k-1}\|_\infty} < T + O(R) < 1 + O(R)$ for $k \in \mathbb{N}$, whence $\tau_k < 1$ for $R \ll 1$. Then, the same argument as in Step III of the proof of Proposition 3.2 applies to say that the sequence of points $\{p_k\}_{k \in \mathbb{N}}$ converges to a point $p^*$. Since the $p_k$’s belong to $\mathcal{D}$ up to $O(R^2)$, the point $p^*$ belongs to the closure $\mathcal{D} \subseteq \mathcal{C}(p)$ up to $O(R^2)$. We also know that $\|s_k\|_\infty \approx \tau_1 \tau_2 \ldots \tau_k \|s_0\|_\infty < \tau^k \|s_0\|_\infty$, where $\tau = \sup_{p_k \in \mathbb{N}}(\tau_k \cdots)$. Therefore $\lim_{k \to \infty} \|s_k\|_\infty < \lim_{k \to \infty} \tau^k \|s_0\|_\infty = 0$. From inequality (32), we then conclude that

$$|f(p^*)| = \lim_{i \to \infty} |f(p_i)| \leq \frac{1}{2} n_2^2 \|Hf(p)\|_\infty \lim_{i \to \infty} \|s_{i-1}\|_2^2 + O(R^3) = O(R^3),$$

so that the hypersurface $f = 0$ crosses the cell $\mathcal{C}(p)$ neglecting order $O(R^3)$ (hence order $O(R^2)$) contributions.

Q.E.D.

**Example 4.4** We consider the polynomial $f(x, y) = y^2 + x^3 - x - 3$ and the point $p = (0, 1.7)$. We let $\varepsilon = (0.06, 0.06)$ and consider the $(\infty, \varepsilon)$-unit ball $B(p)$ centered at $p$. Direct computations show that $\|\text{Jac}_f(p)\|_1 = 3.4 \neq 0$ and $\|Hf(p)\|_\infty = 2$, whence $\frac{\|\text{Jac}_f(p)\|_1}{n_5^2 \|Hf(p)\|_\infty} = 0.425$. We choose $R = 0.05 < \min\{0.06, 0.425\}$. Further, we compute $\Theta \approx 0.403$, so for the bound $B_2'$ of Proposition 4.3 we find

$$B_2' = \frac{2R}{\Theta(2 + n_5^2 \|Hf(p)\|_\infty)} \approx 0.1113.$$

Since $|f(p)| = \frac{11}{10} = 0.11$ is strictly smaller than $B_2'$, by using Proposition 4.3 we conclude that the elliptic curve of equation $f = 0$ crosses the ball $B(p)$.

Keeping the assumptions and notation as in propositions 2.5, 3.2 and 4.3 we conclude this section comparing the bounds $B_1'$, $B_2$ and $B_2'$ provided by such propositions.

**Proposition 4.5** Notation and assumptions as above. Further assume $\varepsilon_{\max} \ll 1$ and let $R$ be a positive real number such that

$$R < \min \left\{ \varepsilon_{\min}, \frac{\|\text{Jac}_f(p)\|_1}{H}, \frac{\|\text{Jac}_f(p)\|_1}{n_2^2 \|Hf(p)\|_\infty} \right\}.$$

Then $B_1' + O(R^3) < B_2' < B_2$.

**Proof.** Recalling the definitions of the quantities $B_2$, $J$, $H$, $\Theta$ we get

$$B_2 = \frac{2R}{J(c + \sqrt{n}HR)} > \frac{2R}{(\Theta + O(R^2))(c + \sqrt{n}H(\Theta + O(R^2))R)} = \frac{2R}{\Theta(c + \sqrt{n}H\Theta R) + O(R^2)}.$$

Lemma 1.13, applied to the Hessian matrix in the open convex set $\mathcal{D} = \mathcal{D}(p, R)$, yields $H < n^2 \|Hf(p)\|_\infty + O(R)$, so that

$$B_2 > \frac{2R}{\Theta(c + \sqrt{n}(n^2 \|Hf(p)\|_\infty + O(R))R)} + O(R^2) = \frac{2R}{\Theta(c + n_5^2 \|Hf(p)\|_\infty)R) + O(R^2)}.$$

Write the right-hand side term of (36) as

$$\frac{2R}{\Theta(c + n_5^2 \|Hf(p)\|_\infty)R) + O(R^2)} \frac{1}{1 + \frac{O(R^2)}{c + n_5^2 \|Hf(p)\|_\infty)R)} = \frac{1}{1 + \frac{O(R^2)}{c + n_5^2 \|Hf(p)\|_\infty)R)}.$$
Noting that \( \left| \frac{O(R^2)}{c + n^{5/2} \| H_f(p) \|_\infty \Theta R} \right| < 1 \) for \( R \ll 1 \), the quantity in (37) rewrites as
\[
\frac{2R}{\Theta(c + n^{5/2} \| H_f(p) \|_\infty \Theta R)} \left( 1 - \frac{O(R^2)}{c + n^{5/2} \| H_f(p) \|_\infty \Theta R} + \cdots \right) = \frac{2R}{\Theta(c + n^{5/2} \| H_f(p) \|_\infty \Theta R)} + O(R^2),
\]
By definition of \( B'_2 \), the lower bound of \( B_2 \) then easily follows.

From Lemma 4.2 we know that the Jacobian \( \text{Jac}_f(x) \) is nonzero in \( D \), so that \( \text{Jac}_f(p) \text{Jac}_f^\dagger(p) = 1 \). Thus Hölder’s inequality yields
\[
1 \leq \| \text{Jac}_f(p) \|_1 \| \text{Jac}_f^\dagger(p) \|_\infty \leq \| \text{Jac}_f(p) \|_1 J. \]
Therefore, recalling the assumption on \( R \), we upper bound \( B_2 \) by
\[
B_2 = \frac{2R}{\sqrt[2]{\| H_f(p) \|_\infty}} < \frac{2R}{cJ} \leq \frac{R}{J} \leq \| \text{Jac}_f(p) \|_1 R < \| \text{Jac}_f(p) \|_1 \varepsilon_{\max}. \tag{38}
\]
Recalling the definition of \( B'_1 \) we get the desired upper bound. Q.E.D.

**Remark 4.6** The inequality \( B_1 > B_2 \) holds true. For degree \( \geq 2 \) polynomials, it is a consequence of Lemma 2.7 and Proposition 4.5; in the linear case it follows from Remark 3.3 and equality (38). It then may happen that \( |f(p)| \) belongs to the interval \( (B_2, B_1) \). In this case, with the only use of propositions 2.1 and 3.2, we cannot conclude whether or not the hypersurface \( f = 0 \) crosses a given unit cell \( C(p) \). Because of the local nature of the previous results, a more accurate analysis, performed by iteratively considering smaller cells, may overcome that problem. Correspondingly, up to a second-order analysis, Proposition 4.5 gives \( B'_1 > B'_2 \).

## 5 The crossing area algorithm

We keep the notation of the previous sections. In particular, letting \( x = (x_1, \ldots, x_n) \) be indeterminates, we recall that \( f = f(x) \) is a non-costant polynomial of \( P \), \( p \) is a point of \( \mathbb{R}^n \), \( \varepsilon_1, \ldots, \varepsilon_n \) are positive real numbers, \( \varepsilon \) denotes the tolerance vector \( (\varepsilon_1, \ldots, \varepsilon_n) \), and \( C(p) \subseteq B(p) \) is an \( (\infty, \varepsilon) \)-unit cell centered at \( p \) (see Section 2). In this section, we describe an algorithm to decide whether or not a hypersurface of equation \( f = 0 \) intersect a given bounded region in the affine space \( A^2_n(\mathbb{R}) \).

Let’s start with a local version of such a crossing problem. That is, we introduce an algorithm that, given a non-costant polynomial \( f \) and a unit cell \( C(p) \), returns a value which describes the intersection of the hypersurface of equation \( f = 0 \) with \( C(p) \). Namely,

- 0 if the hypersurface \( f = 0 \) does not cross \( C(p) \);
- 1 if the hypersurface \( f = 0 \) crosses \( C(p) \);
- \( \zeta \) (unknown) if neither Proposition 2.1 nor Proposition 3.2 applies.

Summarizing, we have:

**The CROSSING CELL algorithm**

Given a non-costant polynomial \( f = f(x) \in P \), a point \( p \in \mathbb{R}^n \) such that both the Jacobian \( \text{Jac}_f(p) \) and the Hessian matrix \( H_f(p) \) are nontrivial at \( p \), and a tolerance vector \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \), the algorithm returns an element of \( \{0, 1, \zeta\} \).

1. Compute \( |f(p)| \), and the bounds \( B_1 \) and \( B_2 \) from propositions 2.1 and 3.2 (see also Remark 3.3).
2. If \( |f(p)| > B_1 \) return 0; if \( |f(p)| < B_2 \) return 1; else return \( \zeta \).

**Remark 5.1** We observe that there may be variants of the previous algorithm. For instance:
1. The crossing cell algorithm could be performed up to a first-order error analysis simply replacing the bounds $B_1$ and $B_2$ by $B'_1$ and $B'_2$ (defined in propositions 2.5 and 4.3).

2. In the case $B_2 < f(p) < B_1$, in order to limit the problem of indeterminacy (already pointed out in Remark 4.6), the crossing cell algorithm could be performed by iteratively considering smaller unit cells $C(p)$, simply obtained by a subdivision procedure ending as soon as such cells are sufficiently small to solve the problem in the given context.

Now, we consider a more general version of the crossing problem. Let $f = f(x) \in P$ as above, and let $T$ be a nontrivial bounded region of $\mathbb{A}^n(\mathbb{R})$ of type $\mathcal{T} := [a_1, b_1] \times \cdots \times [a_n, b_n]$, with $a_k, b_k \in \mathbb{R}$ and $a_k < b_k$, for each $k = 1, \ldots, n$. In order to reduce the problem of studying the intersection of the hypersurface $f = 0$ with the region $T$ to a local crossing cell problem, we perform a discretization of the region $T$ as follows. Let $d_k$ be the sampling distance with respect to the component $x_k$, $k = 1, \ldots, n$. For each $k = 1, \ldots, n$, we define

$$J_k := \left[ \frac{b_k - a_k - \frac{d_k}{2}}{d_k} \right] + 1 \quad \text{and} \quad x_{k,j} := a_k + j_k d_k,$$

(39)

where $[x] = \min\{z \in \mathbb{N} | z \geq x\}$ and $j_k = 0, \ldots, J_k - 1$. Here $J_k$ denotes the number of considered samples for each component, and $j_k$ the index of the sample. We denote by $j$ the multi-index $(j_1, \ldots, j_n)$, by $x_j := (x_{1,j}, \ldots, x_{n,j})$ the j-th sampling point, and by

$$C(j) := \left\{ x = (x_1, \ldots, x_n) \in \mathbb{A}^n(\mathbb{R}) \mid x_k \in \left[ x_{k,j} - \frac{d_k}{2}, x_{k,j} + \frac{d_k}{2} \right], \ k = 1, \ldots, n \right\}$$

the cell centered at (and represented by) the point $x_j$. The discretization of $T$ is given by the $J_1 \times \cdots \times J_n$ cells of type $C(j)$ which are a covering of the region $T$. Let us stress the fact that the discretization is defined by relations (39), that is, by the initializing point $a = (a_1, \ldots, a_n) \in T$ and the discretization step $d = (d_1, \ldots, d_n)$.

We define a multi-matrix $A = (a_{j_1,j_2,\ldots,j_n})$ of type $J_1 \times \cdots \times J_n$, called the crossing area matrix (with respect to the given discretization $\{a, d\}$ defined by $a$ and $d$). Each entry $A(j)$ of $A$ contains informations about whether or not the hypersurface of equation $f = 0$ crosses the cell $C(j)$. Namely,

- $A(j) = 0$ if the hypersurface $f = 0$ does not cross $C(j)$;
- $A(j) = 1$ if the hypersurface $f = 0$ crosses $C(j)$;
- $A(j) = \zeta$ (unknown) if neither Proposition 2.1 nor Proposition 3.2 applies.

In order to use propositions 2.1 and 3.2, we interpret each cell $C(j)$ in terms of the unit ball of an appropriate normed space $\mathbb{R}^n$, where the appropriate norm needs to be defined. To this end, let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) := \left( \frac{d_1}{2}, \ldots, \frac{d_n}{2} \right) =: \frac{d}{2}$, let $E \in \text{Mat}_n(\mathbb{R})$ be the positive diagonal matrix with entries $1/\varepsilon_1, \ldots, 1/\varepsilon_n$, and consider the $E$-weighted $\infty$-norm on $\mathbb{R}^n$ (see Definition 1.3). Since 

$$\|E(x - x_j)^t\|_{\infty, \varepsilon} = \|E(x - x_j)^t\|_{\infty} = \max_{k=1,\ldots,n} \left\{ \frac{2(x_k - x_{k,j})}{d_k} \right\},$$

we can express the $(\infty, \varepsilon)$-unit ball centered at $x_j$ as

$$B(j) = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{A}^n(\mathbb{R}) \mid \max_{k=1,\ldots,n} \left\{ \left| x_k - x_{k,j} \right| \right\} \leq \frac{d_k}{2} \right\}.$$

Therefore

$$C(j) = B(j) \setminus \bigcup_{k=1,\ldots,n} L_k^+,$$

where $L_k^+$ is the hyperplane of equation $L_k^+ : x_k = x_{k,n_k} + \varepsilon_k, \ k = 1, \ldots, n$. According to the notation settled at the beginning of Section 2, it follows that $C(j)$ is an $(\infty, \varepsilon)$-unit cell centered at $x_j$. The whole procedure is gathered in the following algorithm.
The CROSSING AREA algorithm

Given a non-costant polynomial \( f = f(x) \in \mathbb{P} \), a region \( T := [a_1, b_1] \times \cdots \times [a_n, b_n] \) of \( \mathbb{R}^n \), and a discretization step \( d = (d_1, \ldots, d_n) \), the algorithm returns a multi-matrix \( \mathcal{A} \) with values in \( \{0, 1, \zeta\} \).

1. Let \( a = (a_1, \ldots, a_n) \in T \) be the initializing point. Using relations (39) construct the discretization \( \{a, d\} \) of the region \( T \) and the multi-matrix \( \mathcal{A} \) of size \( J_1 \times \cdots \times J_n \).

2. For each multi-index \( j = (j_1, \ldots, j_n) \) assign to \( \mathcal{A}(j) \) the output of the CROSSING CELL algorithm applied to the polynomial \( f \), the point \( p = x_j \) and the tolerance vector \( \varepsilon = \frac{d}{2} \).

3. Return \( \mathcal{A} \).

An implementation of the CROSSING CELL and the CROSSING AREA algorithms has been done using CoCoA5 (see [3]) and is available at http://www.dima.unige.it/~torrente/recognitionAlgorithm.cocoa5.

6 An application to the Hough transform

In this section we discuss an explicit manner of how using the CROSSING AREA algorithm in a case of special interest, that is, the Hough transform technique. The Hough transform is a pattern recognition technique, based on algebraic geometry arguments, for the automated recognition of curves in images. We refer to [4], [1] and [2] for background material and complete details, and to [9], [11] for applications and further developments. Here, we restrict ourselves to just recall few basic definitions and properties.

Most of the results in this section hold over an infinite integral ring \( K \). However, let us restrict to the case of interest in the applications, assuming \( K = \mathbb{R} \).

For every \( t \)-tuples of independent parameters \( \lambda := (\lambda_1, \ldots, \lambda_t) \in \mathbb{R}^t \), let

\[
 f_\lambda(x) = \sum_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n} g_{i_1, \ldots, i_n}(\lambda), \quad i_1 + \cdots + i_n \leq d, \quad (40)
\]

be a family \( \mathcal{P} \) of irreducible polynomials in the indeterminates \( x := (x_1, \ldots, x_n) \), of a given degree \( d \) (not depending on \( \lambda \)), whose coefficients \( g_{i_1, \ldots, i_n}(\lambda) \) are expressed polynomially in \( \lambda \). Let \( \mathcal{F} \) be the corresponding family of the zero loci \( \mathcal{H}_\lambda = \{x \in \mathbb{A}_n^t(\mathbb{R}) \mid f_\lambda(x) = 0\} \), and let assume that \( \mathcal{H}_\lambda \) is a hypersurface for each parameter \( \lambda \) belonging to an euclidean open subset \( \mathcal{U} \subseteq \mathbb{R}^t \) (of course, this is always the case if the base field \( K \) is algebraically closed). Clearly, such hypersurfaces are irreducible, since the polynomials of the family \( \mathcal{P} \) are assumed to be irreducible in \( \mathbb{R}[x] \). So, we want \( \mathcal{F} \) to be a family of irreducible hypersurfaces which share the degree.

**Definition 6.1** Let \( \mathcal{F} \) be a family of hypersurfaces \( \mathcal{H}_\lambda \) as above, and let \( p = (x_1(p), \ldots, x_n(p)) \) be a point in the image space \( \mathbb{A}_n^{t}(\mathbb{R}) \). Let \( \Lambda := (\Lambda_1, \ldots, \Lambda_t) \) be indeterminates, and let \( \Gamma_p(\mathcal{F}) \) be the hypersurface defined in the affine \( t \)-dimensional parameter space \( \mathbb{A}_n^{t}(\mathbb{R}) \) by the polynomial equation

\[
 f_p(\Lambda) = \sum_{i_1, \ldots, i_n} x_1(p)^{i_1} \cdots x_n(p)^{i_n} g_{i_1, \ldots, i_n}(\Lambda) = 0, \quad i_1 + \cdots + i_n \leq d.
\]

We say that \( \Gamma_p(\mathcal{F}) \) is the Hough transform of the point \( p \) with respect to the family \( \mathcal{F} \). If no confusion will arise, we simply say that \( \Gamma_p(\mathcal{F}) \) is the Hough transform of \( p \).

Summarizing, the polynomials family defined by (40) gives rise to a polynomial \( F(x; \Lambda) \in \mathbb{R}[x; \Lambda] \) giving, for each \( \lambda \in \mathcal{U} \) and for each point \( p \in \mathbb{A}_n^{t}(\mathbb{R}) \),

\[
 H_\lambda : F(x; \lambda) = f_\lambda(x) = 0 \quad \text{and} \quad \Gamma_p(\mathcal{F}) : F(p; \Lambda) = f_p(\Lambda) = 0.
\]

The following general facts hold true (see [1, Theorem 2.2, Lemma 2.3], [2, Section 3]).
1. The Hough transforms $\Gamma_p(F)$ of the pairs $(H_\lambda, p)$, when $p$ varies on $H_\lambda$, all pass through the point $\lambda$.

2. Assume that the Hough transforms $\Gamma_p(F)$, when $p$ varies on $H_\lambda$, have a point in common other than $\lambda$, say $\lambda'$. Thus the two hypersurfaces $H_\lambda, H_{\lambda'}$ coincide.

3. (Regularity property) The following conditions are equivalent:

   (a) For any hypersurfaces $H_\lambda$, $H_{\lambda'}$ in $F$, the equality $H_\lambda = H_{\lambda'}$ implies $\lambda = \lambda'$.

   (b) For each hypersurface $H_\lambda$ in $F$, one has $\bigcap_{p \in H_\lambda} \Gamma_p(F) = \lambda$.

A family $F$ which meets one of the above equivalent conditions is said to be Hough regular.

Let us consider the case $n = 2$ we are interested in. Given a profile of interest in the image plane $\Lambda^2_\Xi(\mathbb{R})$, the Hough approach detects a curve of the family $F = \{H_\lambda\}$ best approximating it. Under the assumption of Hough regularity on $F$, the detection procedure can be highlighted as follows.

I. Choose a set $\Xi = \{p_1, \ldots, p_\nu\}$ of points of interest in the image plane $\Lambda^2_\Xi(\mathbb{R})$.

II. In the parameter space $\Lambda^t_\Lambda(\mathbb{R})$ find the (unique) intersection of the Hough transforms corresponding to the points $p_\nu$, that is, compute $\lambda = \bigcap_{\nu=1}^\nu \Gamma_{p_\nu}(F)$.

III. Return the curve $H_\lambda$ uniquely determined by the parameter $\lambda$.

From a practical point of view, Step II is usually performed using the so called “voting procedure”, a discretization approach for the (not easy) problem of computation of the intersection point $\lambda$. Its core consists of the following three steps. Find a proper discretization of a suitable bounded region $T$ contained in the open set $U \subset \mathbb{R}^t$ of the parameter space. Construct on it an accumulator function, that is, a function that, for each Hough transform $\Gamma_{p_\nu}(F)$ and for each cell of the discretized region, records and sums the “vote” 1, if $\Gamma_{p_\nu}(F)$ crosses the cell, and the “vote” 0 otherwise. Optimize the accumulator function by computing the cell corresponding to the (local) maximum; as suggested by the general results recalled above, the center of that cell is an approximation of the coordinates of the intersection point $\lambda$ (see [1, Section 6] and [9, Section 4]). Of course, such an approximation is determined up to the chosen discretization of $T$.

Let us stress the fact that the results of previous sections 2, 3, and 4 can be used for the construction of the accumulator function, as shown in the following algorithm.

---

### The RECOGNITION algorithm

Given, in the image space $\Lambda^2_\Xi(\mathbb{R})$, a Hough regular family $F$ of irreducible curves of the same degree and a set $\Xi = \{p_1, \ldots, p_\nu\}$ of points of interest; given, in the parameter space $\Lambda^t_\Lambda(\mathbb{R})$, a region $T := [a_1, b_1] \times \cdots \times [a_t, b_t]$ and a discretization, defined by the initializing point $a = (a_1, \ldots, a_t) \in T$ and a discretization step $d = (d_1, \ldots, d_t)$, the algorithm returns a point $\lambda \in \Lambda^t_\Lambda(\mathbb{R})$.

1. For each $p_\nu \in \Xi$, let $A_\nu$ be the output of the CROSSING AREA algorithm applied to the Hough transform $\Gamma_{p_\nu}(F)$ w.r.t. to the region $T$ and the discretization $\{a, d\}$.

2. Compute the multi-matrix $A = \sum_{\nu=1}^\nu A_\nu$.

3. In the fixed discretization of $T$ find the cell corresponding to the unique local maximum of $A$; call its center $\lambda$ and return it.

---

We observe that this approach works for any number $t$ of parameters, while, as far as we know, the Hough transform recognition technique for detection of curves hardly handles more than three parameters.

In the rest of the section we discuss some illustrative examples, in which our approach is effectively used to compute the accumulator function, which is the core of the recognition algorithm.
based on the Hough transform. In examples 6.3 and 6.4 below, our outputs are compared with the results obtained by using well-established pattern recognition techniques for the detection of curves in images (see [1, Sections 6, 7] and also [9, Sections 4, 5]). Our aim is simply to show that our approach may be successfully used, in a complementary way, in this context too. Note that all the computations have been performed on an Intel Core 2 Duo processor (at 1.86 GHz), and using the CoCo A5 implementation of the CROSSING AREA algorithm. We follow the approach suggested in Remark 5.1: in particular, we exploit the bounds $B_1$ and $B_2$ from propositions 2.5 and 4.3 for degree $\geq 2$, and the bounds $B_1$, $B_2$ from propositions 2.1, 3.2 and Remark 3.3 in the linear case.

All the families $\mathcal{F} = \{C_\lambda\}$ of curves in the examples below meet the regularity property mentioned above (see [9] for details).

**Example 6.2 (Conchoid of Sl"use)** In the affine plane $\mathbb{R}^2$ consider the family $\mathcal{F} = \{C_{a,b}\}$ of rational cubic curves defined by the equation

$$C_{a,b} : a(x - a)(x^2 + y^2) = b^2x^2,$$

(41)

for some positive real numbers $a$, $b$. Such a cubic is classically known as *conchoid of Sl"use* of parameters $a$, $b$ (see [13, p. 130]). The conchoid of Sl"use is an unbounded rational curve with a single singular point (the origin $O = (0, 0)$) and a single vertical asymptote (the line $x = a$), and a single axis of symmetry (the line $y = 0$). Up to the isolated point $O$, the curve $C_{a,b}$ lies in the region of the plane $\mathbb{R}^2$ defined by $a < x \leq \frac{a^2 + b^2}{a}$.

For any point $p = (x(p), y(p))$ of the image plane the Hough transform is a conic (an ellipse centered at $\left(\frac{x(p)}{2}, 0\right)$) in the parameter plane $\mathbb{R}^2$ of equation

$$\Gamma_p(\mathcal{F}) : (x(p)^2 + y(p)^2)A^2 + x(p)^2 B^2 - x(p)(x(p)^2 + y(p)^2)A = 0.$$

In this example we aim to recognize the conchoid of Sl"use of parameters $(a, b) = (\frac{1}{4}, 1)$. To this end, we choose a set $\mathcal{X}$ of 20 points which lie close to such a curve in the following way. We divide the interval $\left(\frac{1}{2}, \frac{17}{4}\right]$ in 20 identical parts: the value of each node represents the $x$-coordinate of each point of $\mathcal{X}$. We then obtain the value of the $y$-coordinates by simply solving equation (41) in $y$ (with $a = \frac{1}{4}$, $b = 1$), and picking the rational approximation (with an error of $10^{-1}$) of one of its two (symmetric) solutions (this is computed with CoCo A5, using the function \texttt{RealRootsApprox}). The points of $\mathcal{X}$ are represented in Figure 5.

![Figure 5: The set $\mathcal{X}$ of selected points.](image)

In the parameter plane, we choose the region $\mathcal{T} = [0.1, 0.5] \times [0.1, 1.1]$ and the discretization step $d = (0.025, 0.025)$. We apply the RECOGNITION algorithm to $\mathcal{F}$, $\mathcal{X}$, $\mathcal{T}$, $d$ and we find the
The maximum entry of $A$ is 20 (which is exactly the cardinality of $X$), and it corresponds to the cell centered at $(A, B) = (\frac{1}{4}, 1)$, that is, exactly to the point of the parameter plane which identifies the curve $C_{4,1}$ we started from.

The examples below consist of an application of the recognition algorithm to detect profiles of interest in medical imaging. They comes from X-ray Computerized Tomography images studied in [9]: see in particular Figure 3 (detection of a lumbar vertebra profile) and Figure 4 (detection of the canal spynal) of that paper.

**Example 6.3 (Curve with 3 convexities)** In the affine plane $A^2(x,y)(\mathbb{R})$ consider the family $\mathcal{F} = \{C_{a,b}\}$ of sextic curves

$$C_{a,b} : (x^2 + y^2)^3 = (a(x^2 + y^2) - b(x^3 - 3xy^2))^2,$$

for positive real numbers $a, b$, with $b < 1$. Such curves, classically known as *curve with 3 convexities* (see [13, p. 183]), are bounded and contained in the circular crown of radii $\frac{a}{1+b}$ and $\frac{b}{1+a}$. The origin $O = (0,0)$ is an isolated point with multiplicity 4, with complex conjugates tangent lines of equation $(x^2 + y^2)^2 = 0$. Further, $O$ has some special complexity: a more detailed local study of the curve at $O$ shows that in fact $C_{a,b}$ is rational.

For any point $p = (x(p), y(p))$ of the image plane the Hough transform is the degenerate conic (i.e., the union of two parallel lines) in the parameter plane $A^2(A,B)(\mathbb{R})$ of equation

$$\Gamma_p(\mathcal{F}) : (x(p)^2 + y(p)^2)^3 = (A(x(p)^2 + y(p)^2) - B(x(p)^3 - 3xy(p)^2))^2.$$  

In the image plane $A^2(x,y)(\mathbb{R})$ we consider 1170 points $Y$ represented in Figure 7 (the data set $Y$ has been taken from [9], see Section 5 and Figure 4, after applying a standard edge detection algorithm). We zoom in the image to consider the portion of interest, and extract from $Y$ the set of points lying in the box $[-1.5,1.5] \times [-1.5,1.5]$. Such a set is denoted by $X$ and consists of 320 points, represented in Figure 8. Taking into account the variance of the curve, we choose in the parameter plane the region $T = [0.35,0.9] \times [0.175,0.5]$ and the discretization step $d = (0.015,0.015)$. Applying the RECOGNITION algorithm to $\mathcal{F}$, $X$, $T$, $d$ we get a two-dimensional matrix $A$ of size 19 $\times$ 12. The (unique) maximum entry of $A$ corresponds to the cell centered at $(A, B) = (0.53,0.445)$, which yields the red curve represented in Figure 8.

Consider now the family $\mathcal{F} = \{C_{a,b,m}\}$ of sextic curves

$$C_{a,b,m} : (mx^2 + y^2)^3 = (a(mx^2 + y^2) - b(x^3 - 3xy)^2))^2,$$

for real positive parameters $a, b, m$, with $b < 1$. This is a slight variant of the curve with 3 convexities which corresponds to the case $m = 1$.

For any point $p = (x(p), y(p))$ of the image plane the Hough transform is the quartic surface in the parameter space $A^3(A,B,M)(\mathbb{R})$ of equation

$$\Gamma_p(\mathcal{F}) : (Mx(p)^2 + y(p)^2)^3 = (A(Mx(p)^2 + y(p)^2) - B(x(p)^3 - 3xy(p)^2))^2.$$
In the image plane $\mathbb{A}_2^2(x,y)(\mathbb{R})$ we consider 2433 points $\mathbb{Y}$ represented in Figure 9 (the data set $\mathbb{Y}$ has been taken from [9, Figure 4, Top case], after an edge detection processing). We zoom in the image to consider the portion of interest, and extract from $\mathbb{Y}$ the set of points lying in the box $[-1.5, 1.5] \times [-1.5, 1.5]$. Such a set is denoted by $\mathbb{X}$ and consists of 132 points, represented in Figure 10. Taking into account the variance of the curve, we choose in the parameter plane the region $T = [0.7, 1] \times [0.18, 0.9] \times [1.1]$ and the discretization step $d = (0.02, 0.02, 0.02)$. Applying the RECOGNITION algorithm to $\mathcal{F}$, $\mathbb{X}$, $T$, $d$, we get a three-dimensional matrix $\mathcal{A}$ of size $9 \times 6 \times 6$. The (unique) maximum entry of $\mathcal{A}$ corresponds to the cell centered at $(A, B, M) = (0.82, 0.04, 1.1)$ which yields the red curve represented in Figure 10.

Example 6.4 (Elliptic curves) In the affine plane $\mathbb{A}_2^2(x,y)(\mathbb{R})$ consider the family $\mathcal{F} = \{C_{a,b,m,n}\}$ of cubic curves of equation

$$C_{a,b,m,n} : x^2 = -my^3 + ny^2 - ay + b,$$

for real parameters $a$, $b$, $m$, $n$ with $m$ positive. The general curve of the family is non-singular, so that it is an elliptic curve. E.g., for $m = 1$ and $n = 0$, we get the so called Weierstrass equation. For special values of the parameters $a$, $b$, $m$, $n$, the cubic has either a nodal or a cuspidal double point (as clearly it happens e.g. for $a = b = n = 0$).
For any point \( p = (x(p), y(p)) \) of the image plane the Hough transform is the hyperplane in the parameter space \( \mathcal{A}^4_{(A,B,M,N)}(\mathbb{R}) \) of equation

\[
\Gamma_p(\mathcal{F}) : My(p)^3 - Ny(p)^2 + Ay(p) - B + x(p)^2 = 0.
\]

In the image plane \( \mathcal{A}^2_{(x,y)}(\mathbb{R}) \) we consider 1084 points \( \mathcal{Y} \) represented in Figure 11 (the data set \( \mathcal{Y} \) has been taken from [9], see Figure 3 (upper panels), after an edge detection processing). We zoom in the image to consider the portion of interest, and extract from \( \mathcal{Y} \) the set of points lying in the box \([-3, 3] \times [-3, 3]\). Such a set is denoted by \( \mathcal{X} \) and consists of 636 points, represented in Figure 12. After looking at the variance of the curve, we choose the region in the parameter space \( \mathcal{T} = [-1.02, 0.206] \times [1.96, 2.89] \times [0.8, 1.2] \times [-0.2, 0.2] \) and the discretization step \( d = (0.1, 0.1, 0.1, 0.1) \). We apply the RECOGNITION algorithm to \( \mathcal{F}, \mathcal{X}, \mathcal{T}, d \) and find a four-dimensional matrix \( A \) of size \( 7 \times 6 \times 3 \times 3 \) which exhibits a unique maximum. Its value corresponds to the cell centered at \( (A, B, M, N) = (-0.42, 2.76, 0.8, 0) \) which yields the red curve represented in Figure 12.

In closing, we propose a comparison attempt of the outputs of our algorithm and the pattern recognition techniques used in [9], where curves from the families \( \mathcal{F} \) studied in examples 6.3 and 6.4 have been used to detect vertebrae profiles. The black curves in figures 13 and 14 below are those detected in Figures 4 (Bottom case) and Figure 3 (panel (d)) of [9], respectively.
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