The resolvent algebra of non-relativistic Bose fields: sectors, morphisms, fields and their dynamics

Detlev Buchholz

Mathematisches Institut, Universität Göttingen,
Bürgerstrae 40, D-37073 Göttingen - Germany

Abstract

In a previous article it was shown that non-relativistic Hamiltonians, including largely arbitrary pair potentials, induce an automorphic action of time translations on the C*-algebra generated by the gauge invariant (particle number preserving) observables in the resolvent algebra of a non-relativistic Bose field. In this note these results are extended to the corresponding field algebra, i.e. the C*-algebra generated by the observables and a pair of particle number changing isometries. The underlying strategy of proof relies on ideas developed by Doplicher, Haag and Roberts in the sector analysis of relativistic quantum field theories. A more detailed exposition of these results is in preparation.

1 Introduction

In a recent article [2] we have established the stability of the gauge invariant (particle number preserving) subalgebra of observables of the resolvent algebra of a non-relativistic Bose field under the automorphic action of dynamics involving pair potentials. It is the aim of the present note to extend this result to a larger field algebra of operators, changing the particle number. Our approach is based on ideas of Doplicher, Haag and Roberts in a general analysis of superselection sectors in relativistic quantum field theory [3].
The resolvent algebra of non-relativistic Bose fields is faithfully represented on Fock space, where the subspaces with fixed particle number are superselection sectors for the subalgebra of observables. We will study here particle number changing isometries, which are contained in a slight extension of the resolvent algebra. Their adjoint action describes non-unital morphisms of the observable algebra. These morphisms are transportable, i.e. they are related by intertwiners (partial isometries) contained in the algebra of observables. Our main result consists of the proof that they are also transportable with regard to the action of space and time translations involving pair interactions, i.e. there exist intertwiners in the algebra of observables between the space and time translated morphisms. The stability of the corresponding field algebra, i.e. the C*-algebra generated by the isometries and the observables, under the automorphic action of these translations then follows.

2 Preliminaries

Adopting the notation and definitions in [2], we consider the family of isometries on Fock space $\mathcal{F}$, which are given by the formula

$$X_f \doteq a(f)^*(1 + N_f)^{-1/2}, \quad f \in \mathcal{D}(\mathbb{R}^s), \quad \|f\|_2 = 1,$$

where $a^*(f), a(f)$ are creation, respectively annihilation operators and $N_f = a^*(f)a(f)$. They arise from operators $a(f)^*(1 + N_f)^{-\kappa}, \kappa > 1/2$, contained in the resolvent algebra of Bose fields, and satisfy

$$X_f^*X_f = 1, \quad X_fX_f^* = E_f,$$

where $E_f$ is the projection onto the orthogonal complement of the kernel of $a(f)$ in $\mathcal{F}$. Note that this space coincides with $|f\rangle \otimes_s \mathcal{F} \subset \mathcal{F}$, where $|f\rangle \in \mathcal{F}_1$ is the single particle vector corresponding to $f \in \mathcal{D}(\mathbb{R}^s)$ and we identify $|f\rangle \otimes_s \Omega \doteq |f\rangle$. The isometries induce morphisms $\rho_f$ of the algebra $\overline{\mathcal{A}}$, given by

$$\rho_f(A) \doteq X_fAX_f^*, \quad A \in \overline{\mathcal{A}}.$$

They define representations of $\overline{\mathcal{A}}$ on the subspace $E_f\mathcal{F} \subset \mathcal{F}$. The morphisms act non-trivially only on operators which are localized in regions containing the support of $f$, i.e. they are localized in this support region.
It is not difficult to see that these morphisms define covariant representations of $\mathfrak{A}$ with regard to space and time translations. The spatial translations are determined by the momentum operator on $\mathcal{F}$,

$$P = (1/i) \int dx \ a^*(x) \partial a(x),$$

where $\partial$ denotes the gradient. The time translations are fixed by the Hamiltonians

$$H = \int dx \partial a^*(x) \partial a(x) + \int dx \int dy \ a^*(x)a^*(y) V(x - y) a(x)a(y),$$

where we restrict attention to pair potentials $V \in C_0(\mathbb{R}^s)$. As was shown in [2], the adjoint action of the unitary operators $e^{ixP}$, $x \in \mathbb{R}^s$, and $e^{itH}$, $t \in \mathbb{R}$, leaves the algebra $\mathfrak{A}$ invariant, describing spatial, respectively time translations of the observables. It is apparent that the unitary operators $X_f e^{ixP} X_f^*$ and $X_f e^{itH} X_f^*$ on $E_f \mathcal{F}$ describe these actions in the representation $\rho_f$ of $\mathfrak{A}$.

It is less trivial to show that the corresponding field algebra $\mathfrak{R}$, i.e. the $C^*$-algebra generated by $\mathfrak{A}$ and the pair $X_f, X_f^*$, is stable under the action of the spacetime translations. For the spatial translations, this is a consequence of the fact, established in [2], that the products of isometries $X_f X_g^*$ are elements of $\mathfrak{A}$ for any normalized pair $f, g \in \mathcal{D}(\mathbb{R}^s)$. Since the space of test functions $\mathcal{D}(\mathbb{R}^s)$ is stable under spatial translations $f \mapsto f_x$, it follows that

$$e^{ixP} X_f e^{-ixP} = X_{f_x} X_f = (X_{f_x} X_f^*) X_f,$$

where $X_{f_x} X_f^* \in \mathfrak{A}$, $x \in \mathbb{R}^s$. It follows that $\mathfrak{R}$ is stable under spatial translations.

We proceed in a similar manner in case of the time translations and consider the isometries

$$e^{itH} X_f e^{-itH} = (e^{itH} X_f e^{-itH} X_f^*) X_f = (e^{itH} e^{-itX_f H X_f^*}) E_f X_f.$$

Here we used again the relations $X_f^* X_f = 1$ and $X_f X_f^* = E_f$. The non-trivial step in our argument consists of the demonstration that

$$\Gamma_f(t) = e^{itH} e^{-itX_f H X_f^*} E_f \in \mathfrak{R} \quad \text{for} \quad f \in \mathcal{D}(\mathbb{R}^s), \ t \in \mathbb{R}. \quad (2.2)$$

It then follows as in case of the spatial translations that the field algebra $\mathfrak{R}$, generated by $\mathfrak{A}$ and $X_f, X_f^*$, is stable under the adjoint action of the unitary operators $e^{itH}$, $t \in \mathbb{R}$. 

3
The proof of (2.2) is given in several steps, where we make use of the particle picture by restricting the above (gauge invariant) operator \( \Gamma_t(f) \) to the subspaces \( \mathcal{F}_n \subset \mathcal{F} \), \( n \in \mathbb{N} \).

Since we will freely alternate between the field theoretic approach and the particle picture, let us recall some basic formulas. Given \( f_1, \ldots, f_n \in \mathcal{D}(\mathbb{R}^s) \) one has

\[
|f_1\rangle \otimes_s \cdots \otimes_s |f_n\rangle = (1/n!)^{1/2} a^*(f_1) \cdots a^*(f_n) \Omega \in \mathcal{F}_n.
\]

Next, let \( O_1 \) be a single particle operator on \( \mathcal{F}_1 \) with (distributional) kernel \( x, y \mapsto \langle x|O_1|y \rangle \). Its canonical lift to \( \mathcal{F}_n \), \( n \in \mathbb{N} \), obtained by forming symmetrized tensor products with the unit operator and amplifying it with the appropriate weight factor \( n \), is given by

\[
n(n-1) \left( O_1 \otimes_s 1 \otimes_s \cdots \otimes_s 1 \right) = \int dx \int dy a^*(x) \langle x|O_1|y \rangle a(y) \upharpoonright \mathcal{F}_n.
\]

The field theoretic operator on the right hand side of this equality will be called second quantization of \( O_1 \). Similarly, if \( O_2 \) is a two-particle operator acting to \( \mathcal{F}_2 \) with kernel \( x_1, x_2, y_1, y_2 \mapsto \langle x_1, x_2|O_2|y_1, y_2 \rangle \), one has, \( n \geq 2 \),

\[
n(n-2) \left( O_2 \otimes_s 1 \otimes_s \cdots \otimes_s 1 \right) = \int dx_1 \int dx_2 \int dy_1 \int dy_2 a^*(x_1) a^*(x_2) \langle x_1, x_2|O_2|y_1, y_2 \rangle a(y_1) a(y_2) \upharpoonright \mathcal{F}_n.
\]

The operator on the right hand side will be called second quantization of \( O_2 \). Recalling that the Hamiltonians of interest here have the form

\[
H = \int dx \partial a^*(x) \partial a(x) + \int dx \int dy a^*(x) a^*(y) V(x - y) a(x) a(y),
\]

the first integral is the second quantization of the single particle operator \( \mathbf{P}^2 \) and the second integral the second quantization of the two-particle operator \( V \in C_0(\mathbb{R}^s) \). Note that the kernel of proper pair potentials has the singular form

\[
x_1, x_2, y_1, y_2 \mapsto (1/2) (\delta(x_1 - y_1) \delta(x_2 - y_2) + \delta(x_1 - y_2) \delta(x_2 - y_1)) V(y_1 - y_2),
\]

which reduces the second quantization of \( V \) to a double integral. We will have occasion to discuss also less singular versions of potentials.
Given $n \in \mathbb{N}$, the restrictions of $H$ to $\mathcal{F}_n$ can be presented as

$$H_n \doteq H \upharpoonright \mathcal{F}_n = n (P^2 \otimes_s 1 \otimes_s \cdots \otimes_s 1) + n(n-1) (V \otimes_s 1 \otimes_s \cdots \otimes_s 1)$$

$$= \sum_j P_j^2 + \sum_{k \neq l} V(Q_k - Q_l), \quad j, k, l = 1, \ldots, n,$$

where the second line represents the familiar version of the operators. The first line will be useful, however, in the subsequent decompositions of these operators.

We will also make use of the second quantization $N_f$ of the one-particle operator $E_{f,1}$, the projection onto the ray of $|f\rangle$. The restrictions of this number operator to $\mathcal{F}_n$ are given by $N_{f,n} \doteq N_f \upharpoonright \mathcal{F}_n = n (E_{f,1} \otimes_s 1 \otimes_s \cdots \otimes_s 1)$. We also note that the projections $E_{f,n} \doteq E_f \upharpoonright \mathcal{F}_n$ can be expressed in terms of $E_{f,1}$ by the formula

$$E_{f,n} = 1 - (1 - E_{f,1}) \otimes_s \cdots \otimes_s (1 - E_{f,1}).$$

Hence, decomposing the tensor product into a sum of tensor products of $E_{f,1}$ and unit operators, it follows from [2, Lem. 3.3] that $E_{f,n} \in \mathcal{A} \upharpoonright \mathcal{F}_n$.

We recall that the algebra of observables $\mathcal{A}$ is isomorphic to the (bounded) inverse limit of an inverse system of approximately finite dimensional algebras, $\mathfrak{A} \doteq \{ \mathfrak{A}_n, \kappa_n \}_{n \in \mathbb{N}_0}$, satisfying the coherence condition $\kappa_n(K_n) = K_{n-1}$ for any $K = \{ K_n \}_{n \in \mathbb{N}_0} \in \mathfrak{A}$. The algebras $\mathfrak{A}_n$ are formed by sums of $n$-fold symmetric tensor products of compact operators and unit operators. The elements of the algebra $\mathcal{A}$ are all bounded operators $A$ on $\mathcal{F}$ with the defining property $K(A) \doteq \{ K_n(A) \doteq A \upharpoonright \mathcal{F}_n \}_{n \in \mathbb{N}_0} \in \mathfrak{A}$. Note that in order to show that some operator $X$ belongs to $\mathcal{A}$ one has to show that (a) $X_n \doteq X \upharpoonright \mathcal{F}_n \in \mathfrak{A}_n$, (b) $\kappa_n(X_n) = X_{n-1}$, $n \in \mathbb{N}_0$, and (c) $X$ is bounded. In view of this fact, we will deal here primarily with the inverse system $\mathfrak{A}$. 

3 Analysis

Turning to the analysis, we need to control the difference $(H - X_f H X_f^*) \upharpoonright E_f \mathcal{F}$ of the generators of the dynamics, which enters in the series expansion of the operators $\Gamma_f(t)$. Note that by restricting this operator to a subspace $\mathcal{F}_n$, one obtains $H_n$ for the first Hamiltonian, and $H_{n-1}$ for the second Hamiltonian, sandwiched between isometries. So
we must compare operators on different subspaces of $\mathcal{F}$. In our first technical lemma we relate operators, defined on $\mathcal{F}_{n-1}$, with their lifts to the space $E_f \mathcal{F}_n = \{f\} \otimes_s \mathcal{F}_{n-1} \subset \mathcal{F}_n$, $n \in \mathbb{N}$, induced by the field operators. The normalized function $f \in \mathcal{D}(\mathbb{R}^s)$ will be kept fixed throughout the subsequent discussion.

**Lemma 3.1.** Let $n \in \mathbb{N}$ and let $O_{n-1}$ be an operator whose domain $\mathcal{D}_{n-1} \subset \mathcal{F}_{n-1}$ is stable under the action of the spectral projections of $N_{f,n-1}$. Then, for any $\Phi_{n-1} \in \mathcal{D}_{n-1}$,

(i) $X_f O_{n-1} X_f^* (|f\rangle \otimes_s \Phi_{n-1}) = |f\rangle \otimes (1 + N_{f,n-1})^{-1/2} O_{n-1} (1 + N_{f,n-1})^{1/2} \Phi_{n-1}$

(ii) $|f\rangle \otimes_s O_{n-1} \Phi_{n-1} = a(f)^* O_n (1 + N_{f,n-1})^{-1} a(f) |f\rangle \otimes_s \Phi_{n-1}$.

**Proof.** (i) Pick any $\Phi_{n-1} \in \mathcal{D}_{n-1}$. Then $a(f)^* (|f\rangle \otimes_s \Phi_{n-1}) = (1 + N_{f,n-1}) \Phi_{n-1}$. It follows that $a(f) (|f\rangle \otimes_s \Phi_{n-1}) = n^{-1/2} (1 + N_{f,n-1}) \Phi_{n-1}$. Noticing that the spectral decomposition of $N_{f,n-1}$ is a finite linear combination of its spectral projections, one sees that the vector $X_f^* (|f\rangle \otimes_s \Phi_{n-1})$ is also an element of $\mathcal{D}_{n-1}$ and

$$X_f O_{n-1} X_f^* (|f\rangle \otimes_s \Phi_{n-1}) = n^{-1/2} X_f O_{n-1} (1 + N_{f,n-1})^{1/2} \Phi_{n-1}$$

$$= n^{-1/2} a^*(f) (1 + N_{f,n-1})^{-1/2} O_{n-1} (1 + N_{f,n-1})^{1/2} \Phi_{n-1}$$

$$= |f\rangle \otimes_s (1 + N_{f,n-1})^{-1/2} O_{n-1} (1 + N_{f,n-1})^{1/2} \Phi_{n-1},$$

proving the first statement.

(ii) Replacing in (i) the operator $O_{n-1}$ by $(1 + N_{f,n-1})^{1/2} O_{n-1} (1 + N_{f,n-1})^{-1/2}$ and noticing that $(1 + N_{f,n-1})^{-1} a(f) (|f\rangle \otimes_s \Phi_{n-1})$ lies in the domain of $O_{n-1}$, the second statement follows from the first one. \hfill \Box

We consider now the restrictions $H_{n-1}$ of the Hamiltonians $H$ of interest to $\mathcal{F}_{n-1}$, $n \in \mathbb{N}$. For these restrictions the spaces

$$\mathcal{D}_{n-1} = \mathcal{D}(\mathbb{R}^s) \otimes_s \cdots \otimes_s \mathcal{D}(\mathbb{R}^s) \subset \mathcal{F}_{n-1}$$

are domains of essential selfadjointness. It is also evident that these spaces are stable under the action of the spectral projections of $N_{f,n-1}$. So the first part of the preceding lemma applies to $X_f H_{n-1} X_f^* (|f\rangle \otimes \mathcal{D}_{n-1})$, which induces on $\mathcal{D}_{n-1}$ the action

$$(1 + N_{f,n-1})^{-1/2} H_{n-1} (1 + N_{f,n-1})^{1/2}.$$

We compare now the operator $H \upharpoonright \mathcal{D}_{n-1}$ with $(1 + N_f)^{-1/2} H (1 + N_f)^{1/2} \upharpoonright \mathcal{D}_{n-1}$, where the latter operator is also defined on $\mathcal{D}_{n-1}$. \hfill \Box
Lemma 3.2. Let \( n \in \mathbb{N} \). Then
\[
(H - (1 + N_f)^{-1/2}H(1 + N_f)^{1/2}) |\mathcal{D}_{n-1} = \hat{A}_{f,n-1} + \hat{B}_{f,n-1}.
\]
Here \( \hat{A}_{f,n-1} = \hat{A}_f |\mathcal{F}_{n-1} \in \mathcal{K}_{n-1} \), where \( \hat{A}_f \) is the difference between the second quantizations of one- and two-particle operators of finite rank and the corresponding transformed operators, obtained by the similarity transformation \((1 + N_f)^{-1/2} \cdot (1 + N_f)^{1/2}\).

If \( n \geq 3 \), \( \hat{B}_{f,n-1} = \hat{B}_f |\mathcal{F}_{n-1} \), where \( \hat{B}_f \) is the difference between the second quantization of a modified (localized) pair potential and its similarity transformed version. The localized pair potential \( \hat{V}_{f,2} \) is defined on \( \mathcal{F}_2 \) by
\[
\hat{V}_{f,2} = 2 (E_{f,1} \otimes s_1) V ((1 - E_{f,1}) \otimes s_1) + 2 ((1 - E_{f,1}) \otimes s_1) V (E_{f,1} \otimes s_1).
\]
The restriction of the resulting operator \( \hat{B}_f \) to \( \mathcal{F}_{n-1} \) is given by
\[
\hat{B}_{f,n-1} = \hat{V}_{f,n-1} - (1 + N_{f,n-1})^{-1/2} \hat{V}_{f,n-1} (1 + N_{f,n-1})^{1/2}
\]
where
\[
\hat{V}_{f,n-1} = (n-1)(n-2) (\hat{V}_{f,2} \otimes s_1 \otimes s_2 \cdots \otimes s_{n-1}).
\]

Remark: Since the operator \( \hat{V}_{f,2} \) is not an element of \( \mathcal{K}_2 \), it has to be treated separately. It will be crucial in the subsequent analysis that \( \hat{V}_{f,2} \) is effectively localized by the factor \((E_{f,1} \otimes s_1)\), next to \( V \).

Proof. Making use of the tensor notation, we have
\[
H_n = n (P^2 \otimes s_1 \otimes s_2 \cdots \otimes s_{n-1}) + n(n-1) (V \otimes s_1 \otimes s_2 \cdots \otimes s_{n-2}).
\]
We decompose the operator \( P^2 \), defined on \( \mathcal{F}_1 \), into
\[
P^2 = (1 - E_{f,1}) P^2 (1 - E_{f,1}) + E_{f,1} P^2 (1 - E_{f,1}) + (1 - E_{f,1}) P^2 E_{f,1} + E_{f,1} P^2 E_{f,1}.
\]
This decomposition is meaningful since \( |f\rangle \) lies in the domain of \( P^2 \). The first operator on the right hand side of this equality maps the orthogonal complement of the ray of \( |f\rangle \) into itself and the three remaining operators are of rank one. Similarly, we decompose the pair potential \( V \) on \( \mathcal{F}_2 \) into
\[
V = ((1 - E_{f,1}) \otimes s_1 (1 - E_{f,1})) V ((1 - E_{f,1}) \otimes s_1 (1 - E_{f,1})) - (E_{f,1} \otimes s E_{f,1}) V (E_{f,1} \otimes s E_{f,1})
- (E_{f,1} \otimes s E_{f,1}) V ((1 - 2E_{f,1}) \otimes s_1) - ((1 - 2E_{f,1}) \otimes s_1) V (E_{f,1} \otimes s E_{f,1})
+ 2 (E_{f,1} \otimes s_1) V ((1 - E_{f,1}) \otimes s_1) + 2 ((1 - E_{f,1}) \otimes s_1) V (E_{f,1} \otimes s_1).
\]
The first operator on the right hand side of this equality maps the orthogonal complement of $|f⟩ \otimes_s \mathcal{F}_1 \subset \mathcal{F}_2$ into itself. The second up to the fourth terms are operators of finite rank due to appearance of the factor $(E_{f,1} \otimes E_{f,1})$. The two terms in the last line form the operator $\hat{V}_{f,2}$, given in the the lemma.

Tensoring these operators with unit operators 1 and multiplying them with factors of $n$ according to their occurrence, we proceed to

$$\Delta_{n-1} = (H_{n-1} - (1 + N_{f,n-1})^{-1/2}H_{n-1} (1 + N_{f,n-1})^{1/2}).$$

Since the operators

$$(1 - E_{f,1}) P^2 (1 - E_{f,1}) \otimes_s 1 \otimes_s \cdots \otimes_s 1,$$

$$(1 - E_{f,1}) \otimes_s (1 - E_{f,1}) V (1 - E_{f,1}) \otimes_s (1 - E_{f,1}) \otimes_s 1 \otimes_s \cdots \otimes_s 1$$

commute with $N_{f,n-1}$, they do not contribute to $\Delta_{n-1}$. The remaining terms in $\Delta_{n-1}$ consist of two types. The first one is, for any $n \in \mathbb{N}$, a sum of fixed one- and two-particle operators of finite rank which are tensored with unit operators and amplified by factors of $n$. Since the operators $(1 + N_{f,n-1})^{\pm 1/2}$ appearing in the similarity transformation are elements of $\mathfrak{K}_{n-1}$, it follows that the terms $\hat{A}_{f,n-1}$ are contained in $\mathfrak{K}_{n-1}$; moreover, they are the restrictions of some global operator $\hat{A}_f$, as described in the statement. In the second type of terms contributing to $\Delta_{n-1}$ there enters the second quantization of the localized pair potential $\hat{V}_{f,2}$. The resulting operators $\hat{B}_{f,n-1}$ are the bounded restrictions of some unbounded operator $\hat{B}_f$, which describes the difference between the localized interaction operator and its similarity transformed version, $n \in \mathbb{N}$. $\square$

Next, we compare the operators $H \upharpoonright |f⟩ \otimes_s \mathcal{D}_{n-1}$ and $|f⟩ \otimes_s H \upharpoonright \mathcal{D}_{n-1}$.

**Lemma 3.3.** Let $n \in \mathbb{N}$. Then (pointwise on $\mathcal{D}_{n-1}$)

$$H \upharpoonright |f⟩ \otimes_s \mathcal{D}_{n-1} - |f⟩ \otimes_s H \upharpoonright \mathcal{D}_{n-1} = \hat{A}_{f,n} + \hat{B}_{f,n}.$$  

Here $\hat{A}_{f,n} = \hat{A}_f \upharpoonright \mathcal{F}_n \in \mathfrak{K}_n$, where $\hat{A}_f$ is the second quantization of one- and two-particle operators of finite rank, multiplied from the right by the operator $N_f^{-1} E_f$. For $n \geq 2$, $\hat{B}_{f,n} = \hat{B}_f \upharpoonright \mathcal{F}_n$, where $\hat{B}_f$ is the second quantization of the localized pair potential.
\( \hat{V}_{f,2} = V (E_{f,1} \otimes_s 1) \), multiplied from the right by \( N_f^{-1} E_f \). Its (bounded) restrictions to \( |f\rangle \otimes_s \mathcal{D}_{n-1} \) are given by

\[
\hat{B}_{f,n} = \hat{V}_{f,n} N_{f,n}^{-1} E_{f,n} \quad \text{where} \quad \hat{V}_{f,n} = n(n-1) \left( \hat{V}_{f,2} \otimes_s 1 \otimes_s \cdots \otimes_s 1 \right)_{n-2}.
\]

Proof. It suffices to establish the statement for vectors of the special form

\[
\Phi_{n-1} = |f_1\rangle \otimes_s \cdots \otimes_s |f_{n-1}\rangle,
\]

where \( f_1, \ldots, f_{n-1} \in \mathcal{D}(\mathbb{R}^s) \) are members of some orthonormal basis in \( L^2(\mathbb{R}^s) \) which includes \( f \). Making use of the fact that the Hamiltonians are symmetrized sums of one- and two-particle operators, one obtains

\[
H_n \left( |f\rangle \otimes_s |f_1\rangle \otimes_s \cdots \otimes_s |f_{n-1}\rangle \right) = |f\rangle \otimes_s H_{n-1} \left( |f_1\rangle \otimes_s \cdots \otimes_s |f_{n-1}\rangle \right)
\]

\[
= |P^2 f\rangle \otimes_s |f_1\rangle \otimes_s \cdots \otimes_s |f_{n-1}\rangle + \sum_{i=1}^{n-1} (V |f\rangle \otimes_s |f_i\rangle) \otimes_s |f_1\rangle \otimes_s \cdots \hat{\otimes}_i \cdots \otimes_s |f_{n-1}\rangle,
\]

where the symbol \( \hat{\otimes} \) indicates omission of the single particle component \( |f_i\rangle \). We must determine the operator on \( \mathcal{F}_n \) which maps the vector \( |f\rangle \otimes_s |f_1\rangle \otimes_s \cdots \otimes_s |f_{n-1}\rangle \) to the vector on the right hand side of the preceding equality. Recalling that \( f, f_1, \ldots, f_{n-1} \) are members of some orthonormal basis, we have

\[
(P^2 E_{f,1} \otimes_s 1 \otimes_s \cdots \otimes_s 1) \left( |f\rangle \otimes_s |f_1\rangle \otimes_s \cdots \otimes_s |f_{n-1}\rangle \right) = n_f / n (P^2 f) \otimes_s |f_1\rangle \otimes_s \cdots \otimes_s |f_{n-1}\rangle,
\]

where \( n_f \) is the number of factors \( |f\rangle \) appearing in the vector. This equality holds for arbitrary vectors \( \Phi_{n-1} \) if one replaces the number \( n_f \) by the operator \( N_{f,n} \). Furthermore, since the vector is an element of the space \( |f\rangle \otimes_s \mathcal{F}_{n-1} \), it does not change if one multiplies it by the projection \( E_{f,n} \). This gives, \( n \in \mathbb{N} \),

\[
|P^2 f\rangle \otimes_s |f_1\rangle \otimes_s \cdots \otimes_s |f_{n-1}\rangle
\]

\[
= n (P^2 E_{f,1} \otimes_s 1 \otimes_s \cdots \otimes_s 1) N_{n-1}^{-1} E_{f,n} \left( |f\rangle \otimes_s |f_1\rangle \otimes_s \cdots \otimes_s |f_{n-1}\rangle \right),
\]

which leads to the definiton

\[
\hat{A}_{f,n} = n (P^2 E_{f,1} \otimes_s 1 \otimes_s \cdots \otimes_s 1) N_{f,n}^{-1} E_{f,n}.
\]
Since $P^2 E_{f,1}$ has finite rank and $N_{f,n}^{-1} E_{f,n} \in \mathcal{R}_n$, we conclude that $\hat{A}_n \in \mathcal{R}_n$. Moreover, it is the restriction of an operator $\hat{A}_f$ to $\mathcal{F}_{n+1}$ which is the second quantized, localized single particle kinetic energy, multiplied by $N_f^{-1} E_f$. In a similar manner, $n \in \mathbb{N}$,

\[
\sum_{i=1}^{n-1} \left( V|f\rangle \otimes_s |f_i\rangle \right) \otimes_s |f_1\rangle \otimes_s \cdots \otimes_s |f_{n-1}\rangle
= n(n-1) \left( V(E_{f,1} \otimes_s 1) \otimes_s 1 \otimes_s \cdots \otimes_s 1 \right) N_{f,n}^{-1} E_{f,n} |f\rangle \otimes_s |f_1\rangle \otimes_s \cdots \otimes_s |f_{n-1}\rangle
\]

\[= \hat{B}_{f,n} |f\rangle \otimes_s |f_1\rangle \otimes_s \cdots \otimes_s |f_{n-1}\rangle ,\]

where the localized pair potential $\hat{V}_{f,2} \doteq V(E_{f,1} \otimes_s 1)$ on $\mathcal{F}_2$ appears in the second line. The resulting bounded pair potentials $\hat{B}_{f,n+1}$ are the restrictions of some unbounded operator $\hat{B}_{f}$ on $\mathcal{F}$, describing the second quantization of the localized interaction potential, multiplied by $N_f^{-1} E_f$. 

We have accumulated now the information needed for the description of the structure of the operator $(H - X_f H X_f^*) E_f$.

**Proposition 3.4.** Let $n \in \mathbb{N}_0$, then

\[(H - X_f H X_f^*) E_f \upharpoonright \mathcal{F}_n = (A_{f,n} + B_{f,n}),\]

where $A_{f,n} \in \mathcal{R}_n$ are the restrictions of the unbounded operator

\[A_f = \hat{A}_f + a(f)^* \hat{A}_f (1 + N_f)^{-1} a(f)\]

to $\mathcal{F}_n$. The operators $\hat{A}_f, \hat{A}_f$ were defined in Lemmas 3.2 and 3.3 respectively. In a similar manner, $B_{f,n} = B_f \upharpoonright \mathcal{F}_n$ are the bounded restrictions of the unbounded operator

\[B_f = \hat{B}_f + a(f)^* \hat{B}_f (1 + N_f)^{-1} a(f),\]

where the operators $\hat{B}_f, \hat{B}_f$ were also defined in these two lemmas.

**Proof.** Recalling that $E_f \mathcal{F}_n = |f\rangle \otimes_s \mathcal{F}_{n-1}$, one obtains for $\Phi_{n-1} \in \mathcal{D}_{n-1}$

\[(H - X_f H X_f^*) \left( |f\rangle \otimes_s \Phi_{n-1} \right) = \left( H_n (|f\rangle \otimes_s \Phi_{n-1}) - |f\rangle \otimes_s H_{n-1} \Phi_{n-1} \right) + |f\rangle \otimes_s \left( H_{n-1} - (1 + N_{f,n-1}^{-1}) H_{n-1} (1 + N_{f,n-1}^{-1})^{-1/2} \right) \Phi_n.

10
The first term on the right hand side of this equality coincides according to Lemma 3.3
with \((\hat{A}_{f,n} + \hat{B}_{f,n}) (|f\rangle \otimes_s \Phi_{n-1})\), where \(\hat{A}_{f,n} \in \mathfrak{K}_n\). In the second term we made use of the
first part of Lemma 3.1 according to which

\[ X_f H X_f^* (|f\rangle \otimes_s \Phi_{n-1}) = |f\rangle \otimes_s (1 + N_{f,n-1})^{-1/2} H_{n-1} (1 + N_{f,n-1})^{1/2} \Phi_{n-1}. \]

As has been shown in Lemma 3.2, the second term in the above equality can be presented
in the form \(|f\rangle \otimes_s (\hat{A}_{f,n-1} + \hat{B}_{f,n-1}) \Phi_{n-1}\), and Lemma 3.1(ii) implies that it coincides with
the image of \(|f\rangle \otimes \Phi_{n-1}\) under the action of \(a^*(f)(\hat{A}_{f} + \hat{B}_{f})(1 + N_f)^{-1} a(f) \upharpoonright \mathcal{F}_n\). Since
\(\hat{A}_{f,n-1}\) and \((1 + N_{f,n-1})^{-1}\) are elements of \(\mathfrak{K}_{n-1}\), the operator \(a^*(f)\hat{A}_{f}(1 + N_f)^{-1} a(f) \upharpoonright \mathcal{F}_n\)
is contained in \(\mathfrak{K}_n\). (Note that the creation and annihilation operators in this equality
can be mollified by spectral projections of the number operator \(N_f\) without affecting their
action on \(\mathcal{F}_n\), cf. also the discussion below.) Summing up the resulting contributions, the
statement follows.

We turn now to the analysis of the operator function \(t \mapsto \Gamma_f(t)\), defined above. It is
differentiable in \(t\) in the sense of sesquilinear forms between vectors in the domains of \(H\),
respectively \(X_f H X_f^*\). The derivatives are given by

\[
\frac{d}{dt} \Gamma_f(t) = ie^{itH}(H - X_f H X_f^*) e^{-itX_f H X_f^*} E_f
= ie^{itH}(H - X_f H X_f^*) E_f e^{-itX_f H X_f^*} E_f = ie^{itH}(H - X_f H X_f^*) E_f e^{-itH} \Gamma_f(t),
\]

where the second equality holds since \(X_f H X_f^*\) commutes with \(E_f\). We restrict this equality
to \(\mathcal{F}_n\). By Proposition 3.4 we have \((H - X_f H X_f^*) E_f \upharpoonright \mathcal{F}_n = A_{f,n} + B_{f,n}\), where \(A_{f,n} \in \mathcal{K}_n\)
and \(B_{f,n}\) is a bounded operator. Putting \(C_{f,n}(t) = e^{itH_n}(A_{f,n} + B_{f,n}) e^{-itH_n}\), we can solve
the above equation by the series

\[
\Gamma_{f,n}(t) \doteq \Gamma_f(t) \upharpoonright E_{f,n} \mathcal{F}_n
= \left(1 + \sum_{k=1}^{\infty} \int_0^t ds_k \int_0^{s_k} ds_{k-1} \cdots \int_0^{s_2} ds_1 C_{f,n}(s_k) \cdots C_{f,n}(s_1)\right) \upharpoonright E_{f,n} \mathcal{F}_n,
\]

where the series converges absolutely in norm since the operators \(C_{f,n}\) are bounded. Note
that the range of these operators does not lie in \(E_{f,n} \mathcal{F}_n\).

We want to show that \(\Gamma_f(t) \upharpoonright \mathcal{F}_n \in \mathfrak{K}_n, n \in \mathbb{N}_0\). As we shall see, it is sufficient to prove
that the functions \(t \mapsto \int_0^t ds C_{f,n}(s)\) have values in \(\mathfrak{K}_n\) and are norm continuous, \(t \in \mathbb{R}\). For
the summand $A_{f,n} \in \mathbb{R}_n$ of $C_{f,n}$ this property follows from the fact that the time evolution acts pointwise norm continuously on $\mathbb{R}_n$. The argument for the second summand $B_{f,n}$ is more involved since these operators are not contained in $\mathbb{R}_n$. We begin with a technical lemma about integrals of functions having values in operators, respectively linear maps.

**Lemma 3.5.** Let $\mathcal{H}_k$ be Hilbert spaces and let $\mathfrak{B}_k \subset \mathcal{B}(\mathcal{H}_k)$ be C*-algebras, $k = 1, 2$, let $s \mapsto B_1(s) \in \mathcal{B}(\mathcal{H}_1)$ be an operator function, and let $s \mapsto \lambda(s) : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ be a function with values in linear maps which are normal, i.e. continuous as maps from $\mathcal{B}(\mathcal{H}_1)$ to $\mathcal{B}(\mathcal{H}_2)$ in the strong operator (s.o.) topology, $s \in \mathbb{R}$. Moreover, let either one of the following two conditions be satisfied.

(i) $s \mapsto B_1(s)$ is continuous in the norm topology and has values in $\mathfrak{B}_1$; $s \mapsto \lambda(s)$ is, pointwise on $\mathcal{B}(\mathcal{H}_1)$, continuous in the s.o. topology of $\mathcal{B}(\mathcal{H}_2)$, bounded on compact subsets of $s \in \mathbb{R}$, and the restriction of $\int_0^t ds \lambda(s)$ to $\mathfrak{B}_1$ has values in $\mathfrak{B}_2$, $\int_0^t ds \lambda(s)(B_1(s)) \subset \mathfrak{B}_2$, $t \in \mathbb{R}$.

(ii) $s \mapsto B_1(s)$ is continuous in the s.o. topology and $\int_0^t ds B_1(s) \in \mathfrak{B}_1$, $t \in \mathbb{R}$; $s \mapsto \lambda(s)$ is norm continuous and, for any $s \in \mathbb{R}$, its restriction to $\mathfrak{B}_1$ has values in $\mathfrak{B}_2$, $\lambda(s)(\mathfrak{B}_1) \subset \mathfrak{B}_2$.

Then $s \mapsto \lambda(s)(B_1(s)) \in \mathcal{B}(\mathcal{H}_2)$ is s.o. continuous, $t \mapsto \int_0^t ds \lambda(s)(B_1(s)) \in \mathfrak{B}_2$, and the latter function is norm continuous, $t \in \mathbb{R}$. All integrals in this statement are defined in the s.o. topology. (Note that the functions in this lemma are not necessarily defined by the action of some dynamics.)

**Proof.** Assuming (i), let $s_0 \in \mathbb{R}$. Then one has on $\mathcal{H}_2$

$$
\lambda(s)(B_1(s)) - \lambda(s_0)(B_1(s_0)) = (\lambda(s) - \lambda(s_0))(B_1(s_0)) + \lambda(s)(B_1(s) - B_1(s_0)).
$$

This expression converges to 0 in the s.o. topology of $\mathcal{B}(\mathcal{H}_2)$ for $s \to s_0$. For, $\lambda(s) \to \lambda(s_0)$ pointwise on $\mathcal{B}(\mathcal{H}_1)$ in this topology, so the first term on the right hand side vanishes. Since the restriction of $\lambda(s)$ to the unit ball in $\mathcal{B}(\mathcal{H}_1)$ is bounded on compact subsets of $s \in \mathbb{R}$, the norm of the second term vanishes as well since $B_1(s) \to B_1(s_0)$ in norm. Hence $s \mapsto \lambda(s)(B(s))$ is continuous in the s.o. topology of $\mathcal{B}(\mathcal{H}_2)$. Let, without loss of generality,
where $\|\lambda\|_\infty$ denotes the supremum of the norm of $s \mapsto \lambda(s)$ on any bounded subset of $\mathbb{R}$, containing the integration interval. This bound implies that the expression on the first line tends to 0 in the limit $m \to \infty$. Since, by assumption, $\int_{(t-1)t/m}^{lt/m} ds \lambda(s)(B_1) \subseteq B_2$ and $B_2$ is a C*-algebra, it follows that $t \mapsto \int_0^t ds \lambda(s)(B(s)) \in B_2$, $t \in \mathbb{R}$. The statement about the continuity properties of this function is a consequence of the trivial estimate

$$
\| \int_0^t ds \lambda(s)(B(s)) - \int_0^{t_1} ds \lambda(s)(B(s)) \|_2 \leq \| B \|_\infty \| \lambda \|_\infty |t_2 - t_1|,
$$

where $\|B_1\|_\infty$ denotes the supremum of the norm of $s \mapsto B_1(s)$ on any bounded subset of $\mathbb{R}$, containing the integration interval.

Next, assuming that (ii) holds, let $s_0 \in \mathbb{R}$. Then one proceeds to

$$
\lambda(s)(B_1(s)) - \lambda(s_0)(B_1(s_0)) = \lambda(s_0)(B_1(s) - B_1(s_0)) + (\lambda(s) - \lambda(s_0))(B_1(s)).
$$

Since the map $\lambda(s_0)$ is normal on $\mathcal{B}(\mathcal{H}_1)$, the first term on the right hand side of this equality vanishes in the s.o. topology in the limit $s \to s_0$. The second term vanishes in this limit as well, since $\lambda(s) \to \lambda(s_0)$ in the norm topology of $\mathcal{B}(\mathcal{H}_2)$, uniformly on bounded subsets of $\mathcal{B}(\mathcal{H}_1)$. Thus $s \mapsto \lambda(s)(B_1(s))$ is continuous in the s.o. topology. As in the preceding step, we partition the integration interval, giving the estimate

$$
\| \int_0^t ds \lambda(s)(B_1(s)) - \sum_{l=1}^m \lambda(lt/m) \left( \int_{(l-1)t/m}^{lt/m} ds B_1(s) \right) \|_2
= \| \sum_{l=1}^m \int_{(l-1)t/m}^{lt/m} ds \left( (\lambda(s) - \lambda(lt/m))(B_1(s)) \right) \|_2 \leq \|B_1\|_\infty \sum_{l=1}^m \int_{(l-1)t/m}^{lt/m} ds \|\lambda(s) - \lambda(lt/m)\|_2.
$$

Because of the continuity properties of $s \mapsto \lambda(s)$, the expression on the first line tends to 0 in the limit $m \to \infty$. Since, by assumption, $\int_{(l-1)t/m}^{lt/m} ds B_1(s) \in B_1$ and $\lambda(lt/m)$ maps the C*-algebra $\mathfrak{B}_1$ into $\mathfrak{B}_2$, $1 \leq l \leq m$, one obtains again $t \mapsto \int_0^t ds \lambda(s)(B_1(s)) \in B_2$, $t \in \mathbb{R}$. The continuity of this function follows from the preceding argument. \qed
This lemma will be applied to different types of functions and has therefore been formulated in general terms. As a first application, we consider maps $\beta_{g,n}: \mathcal{B}(\mathcal{F}_{n-1}) \to \mathcal{B}(\mathcal{F}_n)$, $g \in L^2(\mathbb{R}^\ast)$, given by
\[
\beta_{g,n}(\cdot) = a^*(g) \cdot a(g) \upharpoonright \mathcal{F}_n, \quad n \in \mathbb{N}.
\]
Since $\|a(g)\|_n \leq n \|g\|_2$, hence $\|\beta_{g_1,n} - \beta_{g_2,n}\|_n \leq n^2 \|g_1 + g_2\|_2 \|g_1 - g_2\|_2$, these maps are bounded and depend norm continuously on the underlying functions $g_1, g_2 \in L^2(\mathbb{R}^\ast)$. We will make use of the fact that $\beta_{g,n}$ maps the algebra $\mathfrak{R}_{n-1} \subset \mathcal{B}(\mathcal{F}_{n-1})$ into $\mathfrak{R}_n \subset \mathcal{B}(\mathcal{F}_n)$, $n \in \mathbb{N}$. In order to see this, note that one can replace for given $n \in \mathbb{N}$ the operator $a(g) \upharpoonright \mathcal{F}_n$ by $G_n a(g) \upharpoonright \mathcal{F}_n$, where $G_n$ is the (finite) sum of the spectral projections of $N_{g,n} = \|g\|_2^2 a^*(g) a(g) \upharpoonright \mathcal{F}_n$. The operator $G_n a(g)$ is an element of the resolvent algebra $\mathfrak{R}$, and the preceding statements are also true for its adjoint $a^*(g) G_n$. Now, given any $K_{n-1} \in \mathfrak{R}_{n-1}$, there is some operator $A \in \mathfrak{A}$ such that $A \upharpoonright \mathcal{F}_{n-1} = K_{n-1}$. The gauge invariant operator $a^*(g) G_n A G_n a(g)$ is an element of $\mathfrak{A}$, and its restriction to $\mathcal{F}_n$ coincides with some operator in $\mathfrak{R}_n$, cf. [2, Lem. 3.3]. This proves that $\beta_{g,n}(\mathfrak{R}_{n-1}) \subset \mathfrak{R}_n$. It also follows from these arguments that the maps $\beta_{g,n}$ are normal.

In the subsequent corollary we deal with integrals of gauge invariant operator functions, involving the non-interacting time evolution, induced by the Hamiltonian $H_0$. We make use of the notation $s \to B^0(s) = e^{iH_0 s} B e^{-iH_0 s}$ and put $B^0_n(s) = B^0(s) \upharpoonright \mathcal{F}_n = e^{iH_0 s} B_n e^{-iH_0 s}$. Note that these functions are strong operator continuous, so their integrals are defined in this topology. In order to avoid constant repetitions of this fact, we make the following standing declaration.

**Statement:** All integrals appearing in the subsequent analysis are defined in the strong operator topology, unless otherwise stated.

**Corollary 3.6.** Let $n \in \mathbb{N}$, let $g \in L^2(\mathbb{R}^\ast)$, and let $B_{n-1} \in \mathcal{B}(\mathcal{F}_{n-1})$ be such that the function $t \mapsto \int_0^t ds B_{n-1}^0(s)$ has values in $\mathfrak{R}_{n-1}$. Then $t \mapsto \int_0^t ds \beta_{g,n}(B_{n-1}^0(s))$ is norm continuous and has values in $\mathfrak{R}_n$, $t \in \mathbb{R}$.

**Proof.** Consider the function $s \mapsto \beta_{g,n}(B_{n-1}^0(s))$. Since we are dealing with the non-interacting time evolution, we have $\beta_{g,n}(B_{n-1}^0(s)) = \beta_{g(s),n}(B_{n-1}^0(s))$, where $g(s) \in L^2(\mathbb{R}^\ast)$ denotes the time translated wave function $g$, which depends continuously on $s \in \mathbb{R}$. Thus $s \mapsto \beta_{g(s),n}$ is norm continuous, normal, and its restriction to $\mathfrak{R}_{n-1}$ has values in $\mathfrak{R}_n$, as was shown above. The function $s \mapsto B_{n-1}^0(s)$ is s.o. continuous and since by assumption $\int_0^t ds B_{n-1}^0(s) \in \mathfrak{R}_{n-1}$, $t \in \mathbb{R}$, the statement follows from Lemma 3.5(ii). \hfill \Box
In the next lemma we analyze the localized pair potentials which appear as factors in the operators $\check{B}_f$ and $\hat{B}_f$, defined in Lemmas 3.2 and 3.3.

**Lemma 3.7.** Let $\check{V}_{f,2}$ and $\hat{V}_{f,2}$ be the localized pair potentials defined in Lemmas 3.2 and 3.3, respectively. Putting $V_{f,2}$ for either one of these potentials, one has

(i) the function $t \mapsto \int_0^t ds V_{f,2}^0(s)$ on $F_2$ is norm continuous and has values in the compact operators;

(ii) for any $n \in \mathbb{N}$, $n \geq 2$, the function $t \mapsto \int_0^t ds V_{f,n}^0(s)$ on $F_n$ is norm continuous and has values in $\mathcal{K}_n$, $t \in \mathbb{R}$, where $s \mapsto V_{f,n}^0(s) = n(n-1) (V_{f,2}^0(s) \otimes_s 1 \otimes_s \cdots \otimes_s 1)$.

**Proof.** We give the proof for the potential $\hat{V}_{f,2} = V (E_{f,1} \otimes_s 1)$. Since $\check{V}_{f,2}$ also contains the localizing factor $(E_{f,1} \otimes_s 1)$, the corresponding argument is similar and therefore omitted.

(i) First, we consider potentials $V$ having compact support. Choosing some smooth characteristic function $x \mapsto \chi(x)$ which is equal to 1 for $x \in \text{supp} f \cup (\text{supp} f + \text{supp} V)$ and has compact support, we can proceed to $\check{V}_{f,2} = V_{f,\chi} (E_{f,1} \otimes_s 1)$, where the potential $x, y \mapsto V_{f,\chi}^0(x, y) = V(x - y) \chi(x)\chi(y)$ is compactly supported on the two-particle configuration space $\mathbb{R}^s \times \mathbb{R}^s$. The function $s \mapsto V_{f,\chi}^0(s)$ is continuous in the s.o. topology, $t \mapsto \int_0^t ds V_{f,\chi}^0(s)$ is norm continuous, and it has values in the compact operators on $F_2$; these facts have been established in previous work, cf. for example [1]. Furthermore, the function, having values in linear maps on $\mathcal{B}(F_2)$, given by

$$s \mapsto \lambda(s)(\cdot) \doteq (\cdot) (E_{f,1} \otimes_s 1)^0(s) = (\cdot) (E_{f,1}^0(s) \otimes_s 1),$$

is uniformly continuous (recall that $E_{f,1}$ is a one-dimensional projection), normal, and it maps compact operators on $F_2$ into compact operators. Lemma 3.5(ii) therefore implies that

$$t \mapsto \int_0^t ds \lambda(s)(V_{f,\chi}^0(s)) = \int_0^t ds (V_{f,\chi} (E_{f,1} \otimes_s 1))^0(s) = \int_0^t ds (V (E_{f,1} \otimes_s 1))^0(s)$$

is norm continuous and has values in the compact operators on $F_2$ for the restricted class of potentials. Now,

$$\|V(E_{f,1} \otimes_s 1)\|_2 \leq \sup_{x \in \text{supp} f, \ y \in \mathbb{R}^s} |V(x - y)|,$$

and this upper bound implies that the last integral in the preceding equality is norm continuous on $F_2$ with regard to $V \in C_0(\mathbb{R}^s)$. So the preceding result extends to all
potentials in \( C_0(\mathbb{R}^s) \).

(ii) By the very definition of the spaces \( K_n \), any compact operator \( C \) on \( F_2 \) gives rise to elements \( C \otimes 1 \otimes \cdots \otimes 1 \in K_n, n \in \mathbb{N} \). So the second statement follows from the preceding step. As has been mentioned, the same arguments apply to the localized pair potential \( \hat{V}_{f,2} \), completing the proof.

In the next step we show that the statement of the preceding lemma also holds for the interacting dynamics. In fact, we will prove a more general result, involving also the maps \( \beta_{g,n} \), defined above. We recall the short hand notation \( B^0(s) \doteq e^{i s H_0} B e^{-i s H_0} \) and, omitting the superscript 0, we will use an analogous notation for the interacting dynamics, \( B(s) \doteq e^{i s H} B e^{-i s H}, s \in \mathbb{R} \). We also put \( B_n(s) \doteq B(s) \upharpoonright \mathcal{F}_n = e^{i s H_n} B_n e^{-i s H_n} \).

**Lemma 3.8.** Let \( n \in \mathbb{N} \), let \( g \in \mathbb{R}^s \), and let \( B_m \in \mathcal{B}(\mathcal{F}_m) \) such that \( \int_0^t ds B^0_m(s) \in K_m \) for \( t \in \mathbb{R}, m = n, n-1 \). Then one has for the interacting dynamics

(i) \( t \mapsto \int_0^t ds B_n(s) \) is norm continuous and has values in \( K_n \).

(ii) \( t \mapsto \int_0^t ds \beta_{g,n}(B_{n-1})(s) \) is norm continuous and has values in \( K_n \).

**Proof.** Let \( \Theta_n(s) \doteq e^{i s H_0} e^{-i s H_n} = e^{i s H_0} e^{-i s H} \upharpoonright \mathcal{F}_n \) and put \( \theta_n(s) \doteq \text{Ad} \Theta_n(s), s \in \mathbb{R} \). The function \( s \mapsto \theta_n(s) \) of linear maps on \( \mathcal{B}(\mathcal{F}_n) \) is norm continuous. This is a consequence of its standard series expansion in terms of multiple integrals, cf. [2, Eq. 4.2]. It leads to the estimate, \( 0 \leq s_1 \leq s_2 \),

\[
\| (\theta_n(s_2) - \theta_n(s_1))(B_n) \|_n \leq \| B_n \|_n \sum_{k=1}^{\infty} 2^k / k! \| V_n \|_n^k \int_{s_1}^{s_2} du u^{k-1}, \quad B_n \in \mathcal{B}(\mathcal{F}_n),
\]

where \( V_n \) is the interaction operator on \( \mathcal{F}_n \). It was shown in [2, Lem. 4.3] that \( \theta_n(s) \) maps \( \mathcal{K}_n \) onto itself. (This statement was establish in that reference for a larger algebra; but making use of the fact that \( \theta_n(s) \) commutes with the permutations of particle numbers, it holds for the symmetric subalgebra \( \mathcal{K}_n \), as well.) It is also clear that the maps \( \theta_n(s) \), induced by unitary operators, are normal. Now since these maps are automorphisms, both, of \( \mathcal{B}(\mathcal{F}_n) \) and of \( \mathcal{K}_n \), all preceding statements hold also for the inverse maps, \( \theta_n^{-1}(s) \) given by the adjoint action of \( \Theta_n(s)^{-1} = e^{i s H_n} e^{-i s H_0} \), \( s \in \mathbb{R} \). Hence the maps \( s \mapsto \theta_n^{-1}(s) \) comply with all conditions given in Lemma 3.5(ii).
Turning first to statement (i), the function \( s \mapsto B^0_n(s) \) satisfies the remaining conditions in Lemma 3.5(ii) by assumption. Thus
\[
t \mapsto \int_0^t ds \, \theta_n^{-1}(s)(B^0_n(s)) = \int_0^t ds \, B_n(s)
\]
is norm continuous and has values in \( \mathfrak{R}_n \). As to statement (ii), it follows from the assumptions and Corollary 3.6 that \( s \mapsto \beta_{g,n}(B_{n-1})^0(s) \) satisfies the remaining conditions in Lemma 3.5(ii). So
\[
t \mapsto \int_0^t ds \, \theta_n^{-1}(s)((\beta_{g,n}(B_{n-1}))^0(s)) = \int_0^t ds \, (\beta_{g,n}(B_{n-1}))(s)
\]
is also norm continuous and has values in \( \mathfrak{R}_n \), completing the proof of the statement.

We apply now these results to the operator functions \( t \mapsto \int_0^t ds \, C_{f,n}(s) \) which appear in the series expansion (3.1) of \( \Gamma f(t) \upharpoonright E_{f,n} F_n \). It was shown in Proposition 3.4 that
\[
C_{f,n} = A_{f,n} + B_{f,n},
\]
where \( A_{f,n} \in \mathfrak{R}_n \). So, as a consequence of [2, Prop. 4.4] and the fact that the dynamics commutes with permutations, the function \( t \mapsto \int_0^t ds \, A_{f,n}(s) \), defined by the interacting dynamics, is norm continuous and has values in \( \mathfrak{R}_n, t \in \mathbb{R} \).

Turning to the operators \( B_{f,n} \), we have to cope with the problem that the underlying localized pair potentials are not contained in \( \mathfrak{R}_n \). According to Proposition 3.4 the operators \( B_{f,n} \) are given by
\[
B_{f,n} = \beta_{f,n}(\hat{V}_{f,n-1} - (1 + N_{f,n-1})^{-1/2} \hat{V}_{f,n-1} (1 + N_{f,n-1})^{1/2}) + \hat{V}_{f,n} N_{f,n}^{-1} E_{f,n}.
\]
where \( \hat{V}_{f,n-1} \) and \( \hat{V}_{f,n} \) were defined in Lemmas 3.2 and 3.3 respectively, and we made use of the maps \( \beta_{f,n}(\cdot) = a^*(f) \cdot a(f) \upharpoonright F_n \), introduced above.

The operator \( \hat{V}_{f,n-1} \) and its similarity transformed counterpart in the first term on the right hand side of this equality combine into a finite sum \( \sum_j K'_{n-1,j} \hat{V}_{f,n-1} K''_{n-1,j} \), where \( K'_{n-1,j}, K''_{n-1,j} \in \mathfrak{R}_{n-1} \). In order to see that the, by the interacting dynamics time translated operators integrate to elements of \( \mathfrak{R}_{n-1} \), we consider the functions with values in linear maps on \( \mathcal{B}(F_{n-1}) \), given by
\[
s \mapsto \mu_{n-1}(s)(\cdot) \doteq \sum_j K'_{n-1,j,0}(s) \cdot K''_{n-1,j,0}(s).
\]
Recalling that the action of the dynamics on \( \mathfrak{R}_{n-1} \) is pointwise norm continuous as well as the results of Lemma 3.7(ii), it is apparent that the function of maps \( s \mapsto \mu_{n-1}(s) \) and
the operator function \( s \mapsto \hat{V}_{f,n-1}^0(s) \) comply with the conditions given in Lemma 3.5(ii). Hence \( t \mapsto \int_0^t ds (\mu_{n-1}(\hat{V}_{f,n-1}))^0(s) \) is norm continuous and has values in \( \mathcal{R}_{n-1} \), where we have put \( \mu_{n-1} \equiv \mu_{n-1}(0) \). It then follows from Lemma 3.8(ii) that the function, defined by the interacting dynamics,

\[
t \mapsto \int_0^t ds (\beta_{f,n}(\mu_{n-1}(\hat{V}_{f,n-1}))) (s)
\]
is norm continuous and has values in \( \mathcal{R}_n \).

In a similar manner one deals with the second term \( \hat{V}_{f,n} N_{f,n}^{-1} E_{f,n} \) contributing to \( B_{f,n} \). The operator \( N_{f,n}^{-1} E_{f,n} \) is an element of \( \mathcal{R}_n \), on which the dynamics acts pointwise norm continuously, and the function \( t \mapsto \int_0^t ds (\hat{V}_{f,n} N_{f,n}^{-1} E_{f,n})(s) \) is norm continuous and has values in \( \mathcal{R}_n \) as a consequence of Lemmas 3.7(ii) and 3.8(i). By the preceding arguments, it follows from Lemma 3.5(ii) that also \( t \mapsto \int_0^t ds (\hat{V}_{f,n} N_{f,n}^{-1} E_{f,n})(s) \) is norm continuous and has values in \( \mathcal{R}_n \). So, to summarize, we conclude that the integral \( t \mapsto \int_0^t ds C_{f,n}(s) \), defined with regard to the interacting dynamics, is norm continuous and has values in \( \mathcal{R}_n \). This information enters in the following result concerning the operators \( \Gamma_{f,n}(t) \equiv \Gamma_f(t) \upharpoonright E_{f,n} \mathcal{F}_n \).

**Proposition 3.9.** Let \( n \in \mathbb{N}_0 \), then \( t \mapsto \Gamma_{f,n}(t) \in \mathcal{R}_n \), and this function is norm continuous, \( t \in \mathbb{R} \).

**Proof.** We make use of the series expansion (3.1). Since \( C_{f,n} \) is bounded, it follows from the argument given in Lemma 3.8 that \( t \mapsto \Gamma_{f,n}(t) \) is norm continuous, \( t \in \mathbb{R} \). For the proof that it has values in \( \mathcal{R}_n \), it suffices to show that the multiple integrals in the absolutely convergent series expansion (3.1),

\[
t \mapsto \mathcal{D}_{k,n}(t) \equiv \int_0^t ds_k \int_0^{s_k} ds_{k-1} \cdots \int_0^{s_2} ds_1 C_{f,n}(s_k) \cdots C_{f,n}(s_1), \quad k \in \mathbb{N},
\]

are norm continuous elements of \( \mathcal{K}_n \). This is accomplished by induction. For the first term, corresponding to \( k = 1 \), these properties were established in the preceding analysis. By the induction hypothesis, \( t \mapsto \mathcal{D}_{k,n}(t) \) shares these properties. For the induction step from \( k \) to \( k + 1 \), we note that \( t \mapsto \mathcal{D}_{k+1,n}(t) = \int_0^t ds C_{f,n}(s) \mathcal{D}_{k,n}(s) \). According to the induction hypothesis, \( s \mapsto \mathcal{D}_{k,n}(s) \) is norm continuous and has values in \( \mathcal{R}_n \). Moreover, the linear function (left multiplication) \( s \mapsto \lambda_n(s)(\cdot) \equiv C_{f,n}(s) \cdot \) on \( \mathcal{B}(\mathcal{F}_n) \) is normal, pointwise continuous in the s.o. topology, bounded, and \( \int_0^t ds \lambda_n(s)(\cdot) \) maps \( \mathcal{R}_n \) into itself, as was shown in the initial step, \( t \in \mathbb{R} \). Hence, according to Lemma 3.5(i), the function \( t \mapsto \mathcal{D}_{k+1,n}(t) \) has the desired properties, completing the proof. \( \square \)
Having seen that the restrictions of the operators \( \Gamma_f(t) \) to \( \mathcal{F}_n \) determine operators in \( \mathcal{K}_n, t \in \mathbb{R} \), we must show now that these operators form coherent sequences. There the inverse maps \( \kappa_n : \mathcal{K}_n \to \mathcal{K}_{n-1} \) enter, \( n \in \mathbb{N}_0 \). We recall some important properties of these maps, established in [2]. Given any \((C \otimes_s 1 \otimes_s \cdots \otimes_s 1) \in \mathfrak{K}_n\), where \( C \) is a compact operator on \( \mathcal{F}_m \), one has

\[
\kappa_n(C \otimes_s 1 \otimes_s \cdots \otimes_s 1) = (n-m)/n \ (C \otimes_s 1 \otimes_s \cdots \otimes_s 1), \quad 0 \leq m \leq n.
\]

The maps \( \kappa_n \) are *-homomorphisms, mapping \( \mathfrak{K}_n \) onto \( \mathfrak{K}_{n-1} \). In particular, they are norm continuous, \( \|\kappa_n(K_n)\|_{n-1} \leq \|K_n\|_n \). \( K_n \in \mathfrak{K}_n \). A sequence \( \{K_n \in \mathfrak{K}_n\}_{n \in \mathbb{N}_0} \) is said to be coherent if \( \kappa_n(K_n) = K_{n-1}, \ n \in \mathbb{N}_0 \). Such coherent sequences are the elements of the (bounded) inverse limit \( \mathfrak{K} \) of the inverse system \( \{\mathfrak{K}_n, \kappa_n\}_{n \in \mathbb{N}_0} \).

In order to establish the desired result, we make use again of the series expansion (3.2). The essential step in our argument consists of proving the relation

\[
\kappa_n\left( \int_0^t ds C_n(s)D_n(s) \right) = \int_0^t ds C_{n-1}(s)\kappa_n(D_n(s))
\]

for any norm continuous function \( s \mapsto D_n(s) \) with values in \( \mathfrak{K}_n, \ n \in \mathbb{N}_0 \). Since the functions \( s \mapsto C_n(s) \) are not contained in that algebra, this requires some work. We begin with the following simple result.

**Lemma 3.10.** Let \( m = 1, 2 \) and let \( O \) be the second quantization of an \( m \)-particle operator such that \( O \upharpoonright \mathcal{F}_m \) is compact. Then \( O_n \upharpoonright \mathcal{F}_n \in \mathfrak{K}_n \) and \( \kappa_n(O_n) = O_{n-1}, \ n \in \mathbb{N}_0 \).

**Proof.** The second quantizations of one- and two-particle operators and their restrictions to \( \mathcal{F}_n \) were discussed in the beginning of this note. So let \( O \) be the second quantization of a compact one particle operator. Then \( O_n = n(O_1 \otimes_s 1 \otimes_s \cdots \otimes_s 1) \in \mathfrak{K}_n \) and

\[
\kappa_n(O_n) = ((n-1)/n) \ n(O_1 \otimes_s 1 \otimes_s \cdots \otimes_s 1) = O_{n-1} \in \mathfrak{K}_{n-1}.
\]

Similarly, if \( O \) is the second quantization of a compact two-particle operator, one obtains

\[
O_n = n(n-1)(O_2 \otimes_s 1 \otimes_s \cdots \otimes_s 1) \in \mathfrak{K}_n \quad \text{and}
\]

\[
\kappa_n(O_n) = ((n-2)/n) \ n(n-1)(O_2 \otimes_s 1 \otimes_s \cdots \otimes_s 1) = O_{n-1} \in \mathfrak{K}_{n-1},
\]

completing the proof. \( \square \)
The non-interacting time evolution does not mix tensor factors and hence maps two-particle operators into themselves. Adopting as before the notation \( V_{f,2} \) for either one of the localized pair potentials \( \tilde{V}_{f,2} \) and \( \hat{V}_{f,2} \), it follows from Lemma 3.11 and the preceding lemma that the second quantizations of the compact operators \( \int_0^t ds V_{f,2}^0(s) \) satisfy

\[
\kappa_n \left( \int_0^t ds V_{f,n}^0(s) \right) = \int_0^t ds V_{f,n-1}^0(s), \quad n \in \mathbb{N}.
\]

We need, however, stronger results for integrals involving the interacting dynamics, where the localized potentials are sandwiched between operators in \( \mathfrak{X}_n \) and acted upon by the maps \( \beta_{g,n} \), defined above. The relation between the maps \( \beta_{f,n} \) and the maps \( \kappa_n \) is established in the subsequent lemma. There we rely on results in [2, Lem. 3.4], which were established by making use of the quasilocal structure of the algebra \( \mathfrak{A} \).

**Lemma 3.11.** Let \( n \in \mathbb{N} \) and let \( g \in L^2(\mathbb{R}^+) \). Then \( \kappa_n \circ \beta_{g,n} = \beta_{g,n-1} \circ \kappa_{n-1} \) on \( \mathfrak{X}_n \).

**Proof.** Given \( K_{n-1} \in \mathfrak{X}_{n-1} \), let \( A \in \mathfrak{A} \) such that \( A \upharpoonright F_{n-1} = K_{n-1} \), cf. [2, Lem. 3.3]. As has been shown in [2, Lem. 3.4], one has \( A \upharpoonright F_m = K_m = \mathfrak{X}_m \) and \( \kappa_m(K_m) = K_{m-1}, \ m \in \mathbb{N}_0 \).

Now, as was explained subsequent to Lemma 3.5, \( \beta_{g,n}(K_{n-1}) = a^*(g) G_n AG_n a(g) \upharpoonright F_n \), where \( a^*(g) G_n AG_n a(g) \in \mathfrak{A} \). Hence, by another application of [2, Lem. 3.4], the operators \( L_m = \beta_{g,m}(K_{m-1}) = a^*(g) G_n AG_n a(g) \upharpoonright F_m \) also satisfy \( \kappa_m(L_m) = L_{m-1}, 0 \leq m \leq n \).

Thus

\[
\kappa_n(\beta_{g,n}(K_{n-1})) = \kappa_n(L_n) = L_{n-1} = \beta_{g,n-1}(K_{n-2}) = \beta_{g,n-1}(\kappa_{n-1}(K_{n-1})), \quad K_{n-1} \in \mathfrak{X}_{n-1},
\]

as claimed.

We can determine now the action of \( \kappa_n \) on integrals involving the localized pair potentials and acted upon by the interacting dynamics.

**Lemma 3.12.** Let \( V_{f,2} \) be either one of the localized pair potentials \( \tilde{V}_{f,2} \) and \( \hat{V}_{f,2} \), defined in Lemmas 3.2 and 3.3, respectively, let \( n \in \mathbb{N} \), and let \( K_n', K_n'' \in \mathfrak{X}_n \). Then

\[
\kappa_n \left( \int_0^t ds \left( K_n' V_{f,n} K_n'' \right)(s) \right) = \int_0^t ds \left( \kappa_n(K_n') V_{f,n-1} \kappa_n(K_n'') \right)(s).
\]

Moreover, if \( K_{n-1}', K_{n-1}'' \in \mathfrak{X}_{n-1} \), one has

\[
\kappa_n \left( \int_0^t ds \beta_{f,n}(K_{n-1}' V_{f,n-1} K_{n-1}'') \right) = \int_0^t ds \beta_{f,n-1}(\kappa_{n-1}(K_{n-1}' V_{f,n-2} \kappa_{n-1}(K_{n-1}'')) \right)(s).
\]
Proof. The argument is identical for the potentials \( \hat{V}_{f,2} \) and \( \hat{V}_{f,2} \), so we do not need to distinguish between them. In a first step we establish the two statements for the non-interacting time evolution. Turning to the first statement, we approximate as in preceding arguments the first integral \( \int_0^t ds \left( (K'_{n} V_{f,n} K''_{n})^0(s) \right) \in \mathfrak{R}_n \) by the, in the limit of large \( m \), norm convergent sum

\[
\sum_{l=1}^{m} K'_{n}^0(lt/m) \int_{(l-1)t/m}^{lt/m} ds \ V_{f,n}^0(s) \ K''_{n}^0(lt/m). 
\]

Applying to this sum the (norm continuous) homomorphisms \( \kappa_n \), we obtain

\[
\sum_{l=1}^{m} \kappa_n((K'_{n})^0(lt/m) \int_{(l-1)t/m}^{lt/m} ds \ V_{f,n-1}^0(s) \ \kappa_n((K''_{n})^0(lt/m)),
\]

where we used the relation \( \kappa_n \circ \alpha_n(0)(s) = \alpha_n(0) \circ \kappa_n, s \in \mathbb{R} \); it follows from the fact that the non-interacting dynamics does not mix tensor factors. Going back to the limit of large \( m \), the sum converges in norm to the second integral \( \int_0^t ds \left( \kappa_n((K'_{n}) V_{f,n-1} \ \kappa_n((K''_{n}))^0(s) \right) \in \mathfrak{R}_{n-1} \), proving the first relation in the absence of interaction.

Turning to the second relation, we proceed as in Corollary 3.6 and put

\[
(\beta_{f,n}(K'_{n-1} V_{f,n-1} K''_{n-1})^0(s) = \beta_{f(s),n}((K'_{n-1} V_{f,n-1} K''_{n-1})^0(s)).
\]

The function \( s \mapsto \beta_{f(s),n} \) acts norm continuously on \( \mathcal{B}(\mathcal{F}_{n-1}) \), is normal, and it maps \( \mathfrak{R}_{n-1} \) into \( \mathfrak{R}_n \); the function \( s \mapsto (K'_{n-1} V_{f,n-1} K''_{n-1})^0(s) \) is s.o. continuous and its integral has values in \( \mathfrak{R}_{n-1} \). Hence, according to Lemma 3.5(ii), we can approximate the first integral in the second relation of the statement by the for large \( m \) norm convergent sum

\[
\sum_{l=1}^{m} \beta_{f(lt/m),n} \left( \int_{(l-1)t/m}^{lt/m} ds \ (K'_{n-1} V_{f,n-1} K''_{n-1})^0(s) \right).
\]

Applying to this relation the homomorphism \( \kappa_n \), we obtain according to Lemma 3.11 and the preceding step

\[
\sum_{l=1}^{m} \beta_{f(lt/m),n-1} \left( \int_{(l-1)t/m}^{lt/m} ds \ (\kappa_{n-1}(K'_{n-1}) V_{f,n-2} \kappa_{n-1}(K''_{n-1}))^0(s) \right).
\]

Proceeding again to the limit of large \( m \), this gives the second integral in the second relation of the statement, thereby completing its proof in the absence of interaction.
In order to extend these results to the interacting dynamics, we make use of the maps \( \theta_n(s) \), introduced in the proof of Lemma 3.8. We recall that they were defined by the adjoint action of \( e^{isH_n}e^{-isH_n} \), \( s \in \mathbb{R} \). Since \( s \mapsto \theta_n(s) \) and its inverse are norm continuous and map \( \mathfrak{R}_n \) onto \( \mathfrak{R}_n \), we can apply Lemma 3.5(ii) and approximate the integral

\[
\int_0^t ds \left( K'_n V_{f,n} K''_n \right)(s) = \int_0^t ds \theta^{-1}_n(s) \left( (K'_n V_{f,n} K''_n)^0(s) \right) \in \mathfrak{R}_n
\]

by the norm convergent sum

\[
\sum_{l=1}^m \theta^{-1}_n(lt/m) \left( \int_{(l-1)t/m}^{lt/m} ds \left( K'_n V_{f,n} K''_n \right)^0(s) \right).
\]

Applying to these sums the map \( \kappa_n \) and making use of the results in the preceding step as well as the relation \( \kappa_n \circ \theta_n(s) = \theta_{n-1}(s) \circ \kappa_n \), established in \([2\text{, Lem. 4.5}]\), we obtain

\[
\sum_{l=1}^m \theta^{-1}_{n-1}(lt/m) \left( \int_{(l-1)t/m}^{lt/m} ds \left( \kappa_n(K'_n) V_{f,n-1} \kappa_n(K''_n) \right)^0(s) \right).
\]

This expression converges in the limit of large \( m \) in norm to the integral

\[
\int_0^t ds \theta^{-1}_{n-1}(s) \left( (\kappa_n(K'_n) V_{f,n-1} \kappa_n(K''_n))^0(s) \right) = \int_0^t ds \left( \kappa_n(K'_n) V_{f,n-1} \kappa_n(K''_n) \right)(s) \in \mathfrak{R}_{n-1}.
\]

establishing the first relation in the presence of interaction. The argument for the second relation is identical, completing the proof. \( \square \)

These results put us into the position to determine the action of the homomorphisms \( \kappa_n \) on the operators \( \Gamma_{f,n}(t) \). Here we rely again on the expansion (3.1). Let us recall the information which we have about the operators \( C_n \), entering into this expansion. According to Proposition 3.4 and its preceding Lemmas 3.2 and 3.3 they have the structure, \( n \in \mathbb{N} \),

\[
C_{f,n} = (O_{1,2} n + \hat{V}_{f,n}) N^{-1}_{f,n} E_{f,n} + \beta_{f,n} \left( (O_{1,2} n-1 + \hat{V}_{f,n-1})(1 + N_{f,n-1})^{-1} \right) - \beta_{f,n} \left( (1 + N_{f,n-1})^{-1/2}(O_{1,2} n + \hat{V}_{f,n-1})(1 + N_{f,n-1})^{-1/2} \right).
\]

(3.3)

Here the symbols \( O_{1,2} n-1 \), \( O_{1,2} n \) denote the restrictions to \( \mathcal{F}_n \), respectively \( \mathcal{F}_{n-1} \), of the second quantizations of compact one- and two-particle operators, defined in the abovementioned lemmas. The operators \( \hat{V}_{f,n-1} \) and \( \hat{V}_{f,n} \) are the restrictions of the second quantizations of localized pair potentials \( \hat{V}_{f,2} \) and \( \hat{V}_{f,2} \), which were also specified in these lemmas.
Lemma 3.13. For \( n \in \mathbb{N} \), let \( C_n \) be the operators given in (3.3), and let \( s \mapsto D_n(s) \in \mathfrak{A}_n \) be norm continuous. Then
\[
\kappa_n \left( \int_0^t ds \, C_n(s) \, D_n(s) \right) = \int_0^t ds \, C_{n-1}(s) \, \kappa_n(D_n(s)) , \quad t \in \mathbb{R} ,
\]
where the integrals have values in \( \mathfrak{A}_n \), respectively \( \mathfrak{A}_{n-1} \).

Proof. We begin by proving the statement for the constant function \( s \mapsto D_n(s) = 1 \mapsto \mathcal{F}_n \) and consider first the contributions to \( s \mapsto C_n(s) \) containing the operators \( O_{1,2,m} \) as a factor, \( m = n, n-1 \). These contributions depend norm continuously on \( s \in \mathbb{R} \) since they are elements of \( \mathfrak{A}_m \) and are sandwiched between operators from these spaces. We also recall that \( \beta_{f,n} \) maps \( \mathfrak{A}_{n-1} \) into \( \mathfrak{A}_n \). Hence one can interchange in these terms the action of \( \kappa_n \) with the integration.

The action of \( \kappa_m \) on the operators \( O_{1,2,m} \) is given by \( \kappa_m(O_{1,2,m}) = O_{1,2,m-1} \), cf. Lemma 3.10. Furthermore, recalling that \( N_{f,m} \) is the restriction of the second quantization of the one-particle operator \( E_{f,1} \) to \( \mathcal{F}_m \) and that \( \kappa_m \) are homomorphisms, one obtains for any continuous function \( h \) the relation \( \kappa_m(h(N_{f,m})) = h(N_{f,m-1}) \), \( m = n, n-1 \). Finally, representing the projections \( E_{f,m} \in \mathfrak{A}_m \) in standard form, one gets
\[
E_{f,m} = \sum_{l=1}^m (-1)^{l+1} \left( \begin{array}{c} m \\ l \end{array} \right) E_{f,1} \otimes_s \cdots \otimes_s E_{f,1} \otimes_{m-l} 1 .
\]
Applying to them the map \( \kappa_m \), gives
\[
\kappa_m(E_{f,m}) = \sum_{l=1}^m (-1)^{l+1} \left( \begin{array}{c} m-1 \\ l-1 \end{array} \right) \left( \begin{array}{c} m-1 \\ l-1 \end{array} \right) E_{f,1} \otimes_{m-l-1} 1 = E_{f,m-1} .
\]
Making use of these relations and Lemma 3.11 it follows that the statement of the lemma holds for all contributions to \( C_{f,n} \), containing the operators \( O_{1/2,m} \), \( m = n, n-1 \).

For the contributions containing the localized potentials \( \tilde{V}_{f,n-1}, \tilde{V}_{f,n} \), one must integrate the corresponding operators first, since otherwise the action of \( \kappa_n \) is not defined. For the integrated operators, the statement follows directly from the results established in Lemma 3.12 and the relations obtained in the preceding step. This completes the proof of the statement for the constant function \( s \mapsto D_n(s) \).

Let us turn now to the statement for arbitrary norm continuous functions \( s \mapsto D_n(s) \) with values in \( \mathfrak{A}_n \). There we proceed as in the proof of Proposition 3.9 and consider again
the linear function (left multiplication) \( s \mapsto \lambda_n(s)(\cdot) = C_{f,n}(s) \cdot \) on \( B(F_n) \). It is normal, pointwise continuous in the s.o. topology, bounded, and \( \int_0^t ds \lambda_n(s)(\cdot) \) maps \( \mathfrak{A}_n \) into itself, \( t \in \mathbb{R} \). Hence, according to Lemma 3.5(i), we can approximate the integral on the left hand side of the stated equality by the norm convergent sum

\[
\sum_{l=1}^{m} \int_{(l-1)t/m}^{lt/m} ds C_n(s) D_n(lt/m) .
\]

Applying the homomorphism \( \kappa_n \), we obtain

\[
\sum_{l=1}^{m} \int_{(l-1)t/m}^{lt/m} ds C_{n-1}(s) \kappa_n(D_n(lt/m)) ,
\]

where we made use of the result obtained in the preceding step. Proceeding in this expression to the limit of large \( m \), we arrive at the integral on the right hand side of the stated equality, completing the proof of the lemma.

The preceding lemma is a key ingredient in the proof of the following proposition, which is the main result of this note.

**Theorem 3.14.** Let \( \Gamma_f(t) \) be the operators defined in equation (2.2). Their restrictions \( \Gamma_{f,n}(t) = \Gamma_f(t) \mid F_n \) are elements of \( \mathfrak{A}_n \) which satisfy \( \kappa_n(\Gamma_{f,n}(t)) = \Gamma_{f,n-1}(t) \), and they are uniformly bounded, \( n \in \mathbb{N}_0 \). Thus \( \Gamma_f(t) \in \mathfrak{A}_n \), \( t \in \mathbb{R} \).

**Proof.** It is apparent that the operators \( \Gamma_{f,n}(t) \) are uniformly bounded in \( n \). The statement that \( \Gamma_{f,n}(t) \in \mathfrak{A}_n \) was established in Proposition 3.9, \( n \in \mathbb{N}_0 \). So it remains to verify the coherence condition. There we make use again of the expansion (3.1). We need to show that the multiple integrals \( t \mapsto D_{k,n}(t) \) involving the operators \( C_n \), cf. equation (3.2), are mapped by \( \kappa_n \) into the corresponding integrals with the operators \( C_{n-1} \). The statement then follows from the norm convergence of the series. For the proof we make use of the inductive argument given in the proof of Proposition 3.9. We have shown in the preceding lemma that

\[
\kappa_n(D_{1,n}(t)) = \kappa_n\left( \int_0^t ds C_n(s) \right) = \int_0^t ds C_{n-1}(s) = D_{1,n-1}(t) , \quad n \in \mathbb{N}_0 .
\]

Assuming that the analogous relation holds for the \( k \)-fold integrals, involving \( C_n \), we represent the \((k + 1)\)-fold integral in the form \( t \mapsto D_{k+1,n}(t) = \int_0^t ds C_n(s) D_{k,n}(s) \), where
$s \mapsto D_{k,n}(s) \in \mathcal{K}_n$ is norm continuous. Thus it follows from Lemma 3.13 that

\[
\kappa_n(D_{k+1,n}(t)) = \kappa_n\left( \int_0^t ds \ C_n(s) \ D_{k,n}(s) \right) = \int_0^t ds \ C_{n-1}(s) \kappa_n(D_{k,n}(s)) = \int_0^t ds \ C_{n-1}(s) \ D_{k,n-1}(s),
\]

where in the last equality we made use of the induction hypothesis. This establishes the coherence condition and thereby completes the proof. \ \Box

It follows from these results that the field algebra $\overline{\mathbb{K}}$, i.e. the $C^*$-algebra generated by $\mathcal{R}$ and any given pair of isometries $X_f, X_f^*$ for some normalized $f \in \mathcal{D}(\mathbb{R}^s)$, is stable under the adjoint action of the unitary operators $e^{itH}, t \in \mathbb{R}$, for all Hamiltonians $H$ with pair potentials $V \in C_0(\mathbb{R}^s)$. In analogy to previous results for the observables, one can also establish continuity properties of the corresponding action on $\overline{\mathbb{R}}$ with regard to a locally convex topology induced by a countable family of seminorms.

References

[1] Detlev Buchholz, Hendrik Grundling: Quantum systems and resolvent algebras, Lect. Notes Phys. **899** 33-45; e-print arXiv:1306.0860, 2013

[2] Detlev Buchholz: The resolvent algebra of non-relativistic Bose fields: observables, dynamics and states, Commun. Math. Phys. https://doi.org/10.1007/s00220-018-3144-6; e-print arXiv:1709.08107

[3] Sergio Doplicher, Rudolf Haag, John E. Roberts: Local observables and particle statistics I, Commun. Math. Phys. **23** 1971, 199230; Local observables and particle statistics II, Commun. Math. Phys. **35** 1974, 49-85

25