DIRICHLET SERIES CONSTRUCTED FROM PERIODS OF AUTOMORPHIC FORMS

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Abstract. We consider certain Dirichlet series of Selberg type, constructed from periods of automorphic forms. We study analytic properties of these Dirichlet series and show that they have analytic continuation to the whole complex plane.

1. Introduction

1.1. Introduction. Let $k$ be a fixed natural number, $\Gamma$ be a co-finite torsion-free discrete subgroup of $SL(2, \mathbb{R})$. In this article, we consider certain Dirichlet series constructed from periods of automorphic forms for $\Gamma$ of weight $4k$.

Let us recall the definition of periods of automorphic forms. For a hyperbolic element $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{R})$, put $Q_\gamma(z) = cz^2 + (d - a)z - b$.

Definition 1.1 (Periods of automorphic forms). Let $g$ be a weight $4k$ holomorphic cusp form for $\Gamma$ and $\gamma$ be a hyperbolic element in $\Gamma$. The period integral of $g$ over the closed geodesic associated to $\gamma$ is defined by

$$\alpha_{2k}(\gamma, g) = \int_{z_0}^{\gamma z_0} Q_\gamma(z)^{2k-1} g(z) \, dz.$$  

This integral does not depend on the choice on the point $z_0 \in \mathbb{H}$ and the path from $z_0$ and $\gamma z_0$.

It is known that these periods $\{\alpha_{2k}(\gamma, g) \mid \gamma \in \Gamma: \text{hyperbolic}\}$ determine automorphic form $g$ uniquely, and can be expressed by the Petersson scalar product with relative Poincaré series associated to hyperbolic elements. (Cf. Katok [6]) The relative Poincaré series have been studied by several authors in connection with the problem of construction of cusp forms and choosing spanning sets for the space of cusp forms $S_{4k}(\Gamma)$.

Let $\text{Prim}(\Gamma)$ be the set of primitive hyperbolic conjugacy classes of $\Gamma$. For a hyperbolic element $\gamma \in \Gamma$, put $\ell(\gamma)$ be the length of the closed geodesic associated to $\gamma$ and $N(\gamma) = \exp(\ell(\gamma))$. 

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Definition 1.2 (Dirichlet series $\Xi_\Gamma(s; g)$). For $g \in S_{4k}(\Gamma)$ and $s \in \mathbb{C}$ with $\Re s > 1$, define

\begin{align}
\Xi_\Gamma(s; g) := \sum_{\gamma \in \text{Prim}(\Gamma)} \sum_{m=1}^{\infty} \beta_{2k}(\gamma, g) N(\gamma)^{-ms} \\
= \sum_{\gamma \in \text{Prim}(\Gamma)} \beta_{2k}(\gamma, g) \frac{N(\gamma)^{-s}}{1 - N(\gamma)^{-s}}
\end{align}

with

\begin{equation}
\beta_{2k}(\gamma, g) = \frac{\alpha_{2k}(\gamma, g)}{2^{6k-3} \sinh^{2k-1}(\ell(\gamma)/2)}. \tag{1.3}
\end{equation}

This series is absolutely convergent for $\Re s > 1$.

In this article, we investigate analytic properties of $\Xi_\Gamma(s; g)$. Our main result is the following theorem. (Theorem 5.6)

Theorem 1.3. Let $\Gamma$ be a co-compact torsion-free discrete subgroup of $SL(2, \mathbb{R})$ and $g \in S_{4k}(\Gamma)$. The function $\Xi_\Gamma(s; g)$, defined for $\Re s > 1$, has the analytic continuation as a meromorphic function on the whole complex plane. $\Xi_\Gamma(s; g)$ has at most simple poles located at:

(i) $s = \frac{1}{2} - j + ir_n$ ($j \in \{0, 1\}$, $n \geq 1$) when $k = 1$, with the residue

$$-4(-1)^j \frac{1}{(\pm 2ir_n - j)(\pm 2ir_n - j + 1)} \langle \varphi_n^{(1)}(1), g \rangle,$$

(ii) $s = \frac{1}{2} - j + ir_n$ ($j \geq 0$, $n \geq 1$) when $k \geq 2$, with the residue

$$4(-1)^{k+j} \langle \varphi_n^{(k)}(1), g \rangle \sum_{h=\max(0, j-2k+1)}^{j} \frac{(-1)^h}{h} \frac{(2k-3-h)}{h} \frac{(2k-1)}{j-h} \frac{1}{(\pm 2ir_n - j + h)} \prod_{m=0}^{2k-1} \frac{1}{(\pm 2ir_n - j + m)}.$$

There are no poles other than described as above. Here, $\{1/4 + r_n^2\}_{n=0}^{\infty}$ are eigenvalues of the Laplacian $-y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$ acting on $L^2(\Gamma \backslash \mathbb{H})$, and $\{\varphi_n\}_{n=0}^{\infty}$ is the orthonormal basis of $L^2(\Gamma \backslash \mathbb{H})$ such that $-y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \varphi_n = (1/4 + r_n^2) \varphi_n$. Besides, we put $\partial_2 j = y^{-2j} \frac{\partial}{\partial y} y^{2j}$ and define

$$\varphi_n^{(k)} = \left[ \partial_{2k-2} \cdots \partial_2 \partial_0 \right] \varphi_n, \quad \varphi_n^{(k)} = \left[ \partial_{2k-2} \cdots \partial_2 \partial_0 \right] \varphi_n.$$

To study the Dirichlet series $\Xi_\Gamma(s; g)$, we define certain functions $\Psi_\Gamma(s; g)$ also by using the periods of automorphic forms. We remark that if $k = 1$, $\Psi_\Gamma(s; g)$ is identified with the first variation of the Selberg zeta function for $\Gamma$ in the Teichmüller space of the Riemann surface $\Gamma \backslash \mathbb{H}$. (See [3]) Here, $\mathbb{H}$ is the upper half plane.

Definition 1.4. For $g \in S_{4k}(\Gamma)$ and a fixed point $s \in \mathbb{C}$ with $\Re s > 1$, Put

\begin{equation}
\Psi_\Gamma(s; g) := \sum_{\gamma \in \text{Prim}(\Gamma)} \frac{\beta_{2k}(\gamma, g)}{\ell(\gamma)} \left\{ \sum_{j=1}^{2k} p_j(s) \frac{d}{ds} \log Z^{(j)}(s) \right\}, \tag{1.4}
\end{equation}
and the sum is absolutely convergent. Here, \( Z_{\gamma}^{(j)}(s) \): the local higher Selberg zeta function of rank \( j \) associated to \( \gamma \) and the polynomial \( p_j(s) \in \mathbb{Z}[2s] \) are given by

\[
Z_{\gamma}^{(j)}(s) := \prod_{m=0}^{\infty} \left( 1 - N(\gamma)^{-s+m} \right)^{(j+m-1)} \quad (j \in \mathbb{Z})
\]

\[
p_j(s) := (j-1)! \left( \frac{2k-1}{j-1} \right) \left( \frac{2k+j-2}{j-1} \right) \frac{2k}{\prod_{i=j+1}^{\infty} (2s-i)}.
\]

The higher Selberg zeta function of rank \( j \) \((j \in \mathbb{Z})\), defined by the following absolutely convergent Euler product (for \( \text{Re} s > 1 \))

\[
Z^{(j)}_\Gamma(s) := \prod_{\gamma \in \text{Prim}(\Gamma)} Z_{\gamma}^{(j)}(s) = \prod_{\gamma \in \text{Prim}(\Gamma)} \prod_{m=0}^{\infty} \left( 1 - N(\gamma)^{-s+m} \right)^{(j+m-1)},
\]

is introduced and studied by Kurokawa, Wakayama and Hashimoto \([4, 5]\). For \( j \in \mathbb{Z} \), this zeta function \( Z^{(j)}_\Gamma(s) \) also has a meromorphic continuation to the whole complex plane.

We show that analytic properties of \( \Xi_\Gamma(s; g) \) is reduced to that of \( \Psi_\Gamma(s; g) \). (Propositions \( \text{Propositions 5.1 and 5.4} \).) Thus we investigate analytic properties of \( \Psi_\Gamma(s; g) \). We state the result on analytic properties of \( \Psi_\Gamma(s; g) \).

**Theorem 1.5.** Let \( \Gamma \) be a co-compact torsion-free discrete subgroup of \( SL(2, \mathbb{R}) \) and \( g \in S_{4k}(\Gamma) \). The function \( \Psi_\Gamma(s; g) \), defined for \( \text{Re} s > 1 \), has the analytic continuation as a meromorphic function on the whole complex plane. \( \Psi_\Gamma(s; g) \) has at most simple poles located at:

\[
s = \frac{1}{2} \pm in \quad (n \geq 1).
\]

There are no poles other than described as above. \( \Psi_\Gamma(s; g) \) satisfies the functional equation

\[
\Psi_\Gamma(1-s; g) = \Psi_\Gamma(s; g).
\]

This theorem is proved by the resolvent type trace formulas. (Theorem 4.5)

2. Preliminaries

In this section we introduce basic objects and fix notations.

2.1. The resolvent of the Laplacian. For an element \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}) \) and a point \( z \in \mathbb{H} \), put

\[
\gamma z = \frac{az+b}{cz+d}, \quad j(\gamma, z) = cz + d.
\]

Let \( X \) be a Riemann surface of type \((g, n)\) with \( 2g + n > 2 \) and \( \Gamma \) be a co-finite torsion-free discrete subgroup of \( SL(2, \mathbb{R}) \) such that \( X \cong \Gamma \backslash \mathbb{H} \). Here, \( \mathbb{H} = \{ z \in \mathbb{C} \mid \text{Im} z > 0 \} \) is the upper half plane with the Poincaré metric \( y^{-2}(dx^2 + dy^2) \). The group \( \Gamma \) is generated by \( 2g \) hyperbolic elements \( A_1, B_1, \ldots, A_g, B_g \) and \( n \) parabolic elements \( S_1, \ldots, S_n \) satisfying the single relation

\[
A_1 B_1 A_1^{-1} B_1^{-1} \cdots A_g B_g A_g^{-1} B_g^{-1} S_1 \cdots S_n = e.
\]

Let \( k, \ell \) be two integers. A smooth complex valued function \( f \) on \( \mathbb{H} \) is called an automorphic form of weight \((2k, 2\ell)\) with respect to the group \( \Gamma \) if for any \( z \in \mathbb{H} \) and \( \gamma \in \Gamma \),

\[
f(\gamma z) = j(\gamma, z)^{2k} j(\gamma, z)^{2\ell} f(z).
\]
An automorphic form of weight $2k$ is meant for that of weight $(2k, 0)$.

We remark that automorphic forms of weight $(2k, 2\ell)$ correspond to tensors of type $(k, \ell)$ on the Riemann surface $X \cong \Gamma \backslash \mathbb{H}$. We denote by $H^{k,\ell}$ the Hilbert space of automorphic forms of weight $(2k; 2\ell)$ with the scalar product
\begin{equation}
(f, g) = \int_{\Gamma \backslash \mathbb{H}} f(z) \overline{g(z)} y^{2k+2\ell} \frac{dx dy}{y^2}.
\end{equation}
For each integer $k$ we consider the Laplacian
\begin{equation}
\triangle_k = \overline{\partial_k} \partial_k = -y^{2-2k} \frac{\partial}{\partial z} y^{2k} \frac{\partial}{\partial \bar{z}} = -\frac{1}{4} \left[ y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - 2\sqrt{-1}ky \frac{\partial}{\partial x} \right]
\end{equation}
in the Hilbert space $H^k = H^{k,0}$. Here, $\overline{\partial_k} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} - \sqrt{-1} \frac{\partial}{\partial y} \right)$ is consider as an operator from $H^k$ to $H^{k,1}$, and $\partial_k^* = -y^{2-2k} \frac{\partial}{\partial z} y^{2k}$ is the adjoint operator to $\overline{\partial_k}$ in the scalar product (2.1), acting from $H^{k,1}$ to $H^k$, where $\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - \sqrt{-1} \frac{\partial}{\partial y} \right)$. The operator $\triangle_k$ is self-adjoint and non-negative in $H^k$. We denote by $\Omega^k(X) = S_{2k}(\Gamma)$ the subspace $\ker \triangle_k = \ker \overline{\partial_k}$ in $H^k$, consisting of holomorphic cusp forms of weight $2k$.

Let us denote by $Q^{(k)}_s(z, z')$ the resolvent kernel of the Laplacian $\triangle_k$ on the upper half-plane $\mathbb{H}$, i.e. $Q^{(k)}_s(z, z')$ is the kernel of the operator $(\triangle_k + \frac{1}{4}(s-2k)(s-1))^{-1}$ for $\Re s \geq 1$. The kernel $Q^{(k)}_s(z, z')$ is smooth for $z \neq z'$ and is holomorphic in $s$ on the whole complex plane. The kernel $Q^{(k)}_s$ has an important property that $Q^{(k)}_s(\sigma z, \sigma z') = Q^{(k)}_s(z, z')$ for any $\sigma \in SL(2, \mathbb{R})$ and $z, z' \in \mathbb{H}$. For $k = 0$ the kernel $Q^{(0)}_s$ is given by the explicit formula
\begin{equation}
Q^{(0)}_s(z, z') = \frac{\Gamma(s)^2}{\pi \Gamma(2s)} \left( 1 - \left| \frac{z-z'}{z-z'} \right|^2 \right)^s \mathrm{HypF} \left( s, s; 2s; 1 - \left| \frac{z-z'}{z-z'} \right|^2 \right)
\end{equation}
where $\mathrm{HypF}(a, b; c; z)$ is the hypergeometric function.

We denote by $G^{(k)}_s(z, z')$ the resolvent kernel of the Laplacian $\triangle_k$ on the Riemann surface $X$, i.e. $G^{(k)}_s(z, z')$ is the kernel of the operator $(\triangle_k + \frac{1}{4}(s-2k)(s-1))^{-1}$ on the Riemann surface $X = \Gamma \backslash \mathbb{H}$. For $\Re s > 1$ and $z \neq \gamma z', \gamma \in \Gamma$, the kernel $G^{(0)}_s$ is given by the absolute convergent series
\begin{equation}
G^{(0)}_s(z, z') = \sum_{\gamma \in \Gamma} Q^{(0)}_s(z, \gamma z'),
\end{equation}
which admits term-by-term differentiation with respect to the variables $z$ and $z'$. The kernel $G^{(0)}_s(z, z')$ is smooth for $z \neq \gamma z', \gamma \in \Gamma$, admits a meromorphic continuation in $s$ on the whole complex plane.

3. The function $\Psi_\Gamma(s; g)$

3.1. Poincaré series $F^{(k)}_s(z)$. Let $\Gamma$ be as in the previous section. We introduce a certain Poincaré series $F^{(k)}_s(z)$ constructed from a $2k$-th derivative of the resolvent kernel $Q^{(0)}_s(z, z')$.

**Definition 3.1.** Put $L_{2j} = (yy')^{-2j} \frac{\partial}{\partial z} \frac{\partial}{\partial z'} (yy')^{2j}$. The function $F^{(k)}_s$ on $\mathbb{H}$ is defined for $\Re s > 1$ by
\begin{equation}
F^{(k)}_s(z) := L_{2k-2} L_{2k-4} \cdots L_2 L_0 \left( G^{(0)}_s(z, z') - Q^{(0)}_s(z, z') \right) \Big|_{z' = z},
\end{equation}
where $G_s^{(0)}$ and $Q_s^{(0)}$ are the resolvent kernels \([2,3]\), \([2,4]\) of the Laplacian on the Riemann surface $X$ and on the upper half plane $\mathbb{H}$ respectively.

Since $G_s(z, z')$ admits term-by-term differentiation with respect to the variables $z$ and $z'$, we have

\[
F_s^{(k)}(z) = \sum_{\gamma \in \Gamma \setminus \{e\}} L_{2k-2} L_{2k-4} \cdots L_2 L_0 \left( Q_s^{(0)}(z, \gamma z') \right) \bigg|_{z' = z}.
\]

From the assumption on $\Gamma$, we have $\Gamma \setminus \{e\} = \Gamma_{\text{hyp}} \cup \Gamma_{\text{par}}$. Here, $\Gamma_{\text{hyp}}$ and $\Gamma_{\text{par}}$ are the set of hyperbolic elements of $\Gamma$ and the set of parabolic elements of $\Gamma$ respectively. We also define two functions $H_s^{(k)}$ and $P_s^{(k)}$ on $H$ for Re $s > 1$ by

\[
H_s^{(k)}(z) := \sum_{\gamma \in \Gamma_{\text{hyp}}} L_{2k-2} L_{2k-4} \cdots L_2 L_0 \left( Q_s^{(0)}(z, \gamma z') \right) \bigg|_{z' = z},
\]

\[
P_s^{(k)}(z) := \sum_{\gamma \in \Gamma_{\text{par}}} L_{2k-2} L_{2k-4} \cdots L_2 L_0 \left( Q_s^{(0)}(z, \gamma z') \right) \bigg|_{z' = z}.
\]

By definition, we have $F_s^{(k)} = H_s^{(k)} + P_s^{(k)}$.

We collect fundamental properties of $F_s^{(k)}$, $H_s^{(k)}$ and $P_s^{(k)}$ by using the explicit formula \([2,3]\) for $Q_s^{(0)}$.

**Proposition 3.2.**

(i) The function $F_s^{(k)}(z)$ can be written as

\[
F_s^{(k)}(z) = (-1)^k \frac{1}{\pi} \sum_{\gamma \in \Gamma \setminus \{e\}} \frac{1}{j(\gamma, z)^{2k}} \frac{1}{(z - \gamma z)^{2k}} \frac{\Gamma(s + k)^2}{\Gamma(2s)} x(r - 1)^{2k} r^{s-k} \gamma_2 F_1(s + k, s + k; 2s; r)
\]

with $r = r(z, \gamma z) = 1 - \frac{|z - \gamma z|^2}{|z - \gamma|^2}$. Two functions $H_s^{(k)}(z)$ and $P_s^{(k)}(z)$ have the same expression by replacing $\Gamma \setminus \{e\}$ with $\Gamma_{\text{hyp}}$ or $\Gamma_{\text{par}}$.

(ii) The Poincaré series $F_s^{(k)}(z)$, $H_s^{(k)}(z)$ and $P_s^{(k)}(z)$ are smooth automorphic forms of weight $4k$ for the Fuchsian group $\Gamma$, i.e. $F_s^{(k)}, H_s^{(k)}, P_s^{(k)} \in \mathcal{H}^{2k}$.

**Proof.** We show that for $F_s^{(k)}$. The other two case are quite similar.

(i) It is sufficient to show that

\[
L_{2k-2} \cdots L_2 L_0 Q_s^{(0)}(z, z') = \frac{(-1)^k \pi^{-1} \Gamma(s + k)^2}{(z - z')^{2k}} \frac{1}{\Gamma(2s)} (r - 1)^{2k} r^{s-k} \gamma_2 F_1(s + k, s + k; 2s; r)
\]
with \( r = r(z, z') = 1 - \frac{|z - z'|^2}{|\bar{z} - \bar{z}'|^2} \). We prove the formula (3.6) by induction on \( k \). For a smooth function \( f(r) = f(r(z, z')) \) with \( r(z, z') = 1 - \frac{|z - z'|^2}{|\bar{z} - \bar{z}'|^2} \), we can easily check that

\[
\frac{\partial^2}{\partial z \partial \bar{z}} \left( \frac{(yy')^{2k}}{(z - z')^{2k+2}} f(r) \right) = -2k \frac{(yy')^{2k}}{(z - z')^{2k+2}} \frac{r + 2k}{r} - 4k \frac{(yy')^{2k}}{(z - z')^{2k+2}} (1 - r) f'(r) - \frac{(yy')^{2k}}{(z - z')^{2k+2}} (1 - r)^2 \{ r f''(r) + f'(r) \}.
\]

(3.7)

Therefore, (2.3): the explicit formula of \( Q_s^{(0)}(r(z, z')) \) gives

\[
L_0 Q_s^{(0)}(r) = \frac{\partial^2}{\partial z \partial \bar{z}} Q_s^{(0)}(r(z, z')) = -\frac{\pi^{-1}}{(z - z')^2} (r - 1)^2 \left\{ r \frac{d^2}{dr^2} + \frac{d}{dr} \right\} \sum_{n=0}^{\infty} \frac{\Gamma(s + n)^2 r^{s+n}}{\Gamma(2s + n) n!}.
\]

(3.8)

So (3.6) of the case \( k = 1 \) is proved. Put

\[
f_k(r) := (-1)^k \pi^{-1} (1 - r)^{2k} r^{s-k} \frac{\Gamma(s+k)^2}{\Gamma(2s)} \, _2F_1(s+k, s+k; 2s; r),
\]

\[
D_k := -2k \frac{r + 2k}{r} - 4k(1 - r) \frac{d}{dr} - (1 - r)^2 \left\{ r \frac{d^2}{dr^2} + \frac{d}{dr} \right\}
\]

(3.10)

Then we have only to show that

\[
D_k f_k(r) = f_{k+1}(r).
\]

(3.11)

We observe that

\[
D_{k,1} f_k(r) = 2k(-1)^{k+1} \pi^{-1} (1 - r)^{2k} (r + 2k) r^{s-k-1} \frac{\Gamma(s+k)^2}{\Gamma(2s)} \, _2F_1(s+k, s+k; 2s; r).
\]

(3.12)

\[
D_{k,2} f_k(r) = 4k(-1)^{k+1} \pi^{-1} \frac{\Gamma(s+k)^2}{\Gamma(2s)} (1 - r) \frac{d}{dr} \left[ r^{s-k-1} \, _2F_1(s-k, s-k; 2s; r) \right]
\]

\[
= 4k(-1)^{k+1} \pi^{-1} \frac{\Gamma(s+k)^2}{\Gamma(2s)} (1 - r) \frac{\Gamma(2s)}{\Gamma(s-k)^2} \frac{\Gamma(s-k) \Gamma(s-k+1)}{\Gamma(2s)} r^{s-k-1} \, _2F_1(s-k, s-k+1; 2s; r)
\]

(3.13)

\[
= 4k(-1)^{k+1} \pi^{-1} (1 - r)^{2k} r^{s-k-1} (s-k) \frac{\Gamma(s+k)^2}{\Gamma(2s)} \, _2F_1(s+k, s+k-1; 2s; r).
\]
\[
D_{k,3} f_k(r) \\
= (-1)^{k+1} \pi^{-1} \frac{\Gamma(s + k)^2}{\Gamma(2s)} (1 - r)^2 r^{-1} \left( r \frac{d}{dr} \right)^2 [r^{s-k} \, _2F_1(s - k, s - k; 2s; r)] \\
= (-1)^{k+1} \pi^{-1} \frac{\Gamma(s + k)^2}{\Gamma(2s)} (1 - r)^2 \\
\times \frac{\Gamma(2s)}{\Gamma(s - k)^2} \frac{\Gamma(s - k + 1)^2}{\Gamma(2s)} r^{s-k-1} \, _2F_1(s - k + 1, s - k + 1; 2s; r) \\
(3.14) = (-1)^{k+1} \pi^{-1} (1 - r)^{2k} r^{s-k-1} (s - k)^2 \frac{\Gamma(s + k)^2}{\Gamma(2s)} \, _2F_1(s + k - 1, s + k - 1; 2s; r).
\]

Therefore, we have

\[
\left\{ (-1)^{k+1} \pi^{-1} (1 - r)^{2k} r^{s-k-1} \frac{\Gamma(s + k)^2}{\Gamma(2s)} \right\}^{-1} D_k f_k(r) \\
= 2k(r + 2k) \, _2F_1(s + k, s + k; 2s; r) + 4k(s - k) \, _2F_1(s + k, s + k - 1; 2s; r) \\
+ (s - k)^2 \, _2F_1(s + k - 1, s + k - 1; 2s; r) \\
(3.15) = (s + k)^2 (1 - r)^2 \, _2F_1(s + k + 1, s + k + 1; 2s; r).
\]

We require a lemma for further calculation.

**Lemma 3.3.**

\[
2k(r + 2k) \, _2F_1(s + k, s + k; 2s; r) + 4k(s - k) \, _2F_1(s + k, s + k - 1; 2s; r) \\
+ (s - k)^2 \, _2F_1(s + k - 1, s + k - 1; 2s; r) \\
(3.16) = (s + k)^2 (1 - r)^2 \, _2F_1(s + k + 1, s + k + 1; 2s; r).
\]

**Proof.** To simplify the notation, we write

\[
_2F_1(a, b; c; z) \equiv F, \quad _2F_1(a \pm 1, b; c; z) \equiv F(a \pm 1), \\
_2F_1(a, b \pm 1; c; z) \equiv F(b \pm 1), \quad _2F_1(a, b; c \pm 1; z) \equiv F(c \pm 1).
\]

We use the following contiguous relation (Cf. (9.2.4) in \[\text{[3]}\] p.242): \[\text{(3.17)}\]

\[
(c - a - b) F + a(1 - z) F(a + 1) - (c - b) F(b - 1) = 0.
\]

Let \(G(r)\) be the left hand side of this lemma to prove.

By using the formula \[\text{(3.17)}\] for \(a = s + k - 1, b = s + k, c = 2s,\) we have

\[
G(r) = 2k(r + 2k) F(s + k, s + k; 2s; r) + 4k(s - k) F(s + k, s + k - 1; 2s; r) \\
+ (s - k)^2 F(s + k - 1, s + k - 1; 2s; r) \\
= 2k(r + 2k) F(s + k, s + k; 2s; r) + 4k(s - k) F(s + k, s + k - 1; 2s; r) \\
+ (s - k) \left\{ (1 - 2k) F(s + k - 1, s + k; 2s; r) \\
+ (s + k - 1)(1 - r) F(s + k, s + k; 2s; r) \right\} \\
= \{2k(r + 2k) + (s + k - 1)(s - k)(1 - r)\} F(s + k, s + k; 2s; r) \\
+ (1 + 2k)(s - k) F(s + k, s + k - 1; 2s; r). \\
(3.18)
\]
By using the formula (3.17) for $a = s + k$, $b = s + k$, $c = 2s$, we have
\[
G(r) = \left\{ 2k(r + 2k) + (s + k - 1)(s - k)(1 - r) \right\} F(s + k, s + k; 2s; r)
\]
\[+ (1 + 2k) \left\{ -2k F(s + k, s + k; 2s; r) + (s + k)(1 - r) F(s + k + 1, s + k; 2s; r) \right\}
\]
\[= (s + k)(1 - r) \left\{ (1 + 2k) F(s + k + 1, s + k; 2s) \right\}
\]
\[+ (s + k - 1) F(s + k, s + k; 2s; r) \right\}.
\]
(3.19)

By using the formula (3.17) for $a = s + k$, $b = s + k + 1$, $c = 2s$, we have
\[
G(r) = (s + k)(1 - r) \left\{ (1 + 2k) F(s + k + 1, s + k; 2s) \right\}
\]
\[+ (s + k - 1) F(s + k, s + k; 2s; r) \right\}
\]
\[= (s + k)(1 - r) \left\{ (1 + 2k) F(s + k + 1, s + k; 2s) - (2k + 1) F(s + k, s + k + 1; 2s; r) \right\}
\]
\[+ (s + k)(1 - r) F(s + k + 1, s + k + 1; 2s; r) \right\}
\]
(3.20) $= (s + k)^2 (1 - r)^2 F_1(s + k + 1, s + k + 1; 2s; r)$.

It completes the proof of the lemma.

Let us complete the proof of Proposition 3.2 (i). By the above lemma and (3.15), we have
\[
\left\{ (-1)^{k+1} \pi^{-1} (1 - r)^{2k} (s - k - 1) \Gamma(s + k)^2 \right\}^{-1} D_k f_k(r)
\]
\[= (s + k)^2 (1 - r)^2 2 F_1(s + k + 1, s + k + 1; 2s; r).
\]
(3.21)

At last we have
\[
D_k f_k(r)
\]
\[= (-1)^{k+1} \pi^{-1} (1 - r)^{2(k+1)} (s - k - 1) \Gamma(s + k + 1)^2 \Gamma(2s) 2 F_1(s + k + 1, s + k + 1; 2s; r)
\]
(3.22) $= f_{k+1}(r)$.

By the assumption of the induction, (3.6) is proved. Finally, since $\frac{\partial}{\partial z}(\gamma z) = j(\gamma, z)^{-2}$, we have
\[
F^{(k)}_s(z) = \sum_{\gamma \in \Gamma \setminus \{e\}} L_{2k-2} L_{2k-4} \cdots L_2 L_0 \left( Q_s^{(0)}(z, \gamma z') \right) \bigg|_{z' = z}
\]
\[= (-1)^k \pi \sum_{\gamma \in \Gamma \setminus \{e\}} \frac{1}{j(\gamma, z)^{2k}} \frac{1}{(z - \gamma z)^{2k}} \frac{\Gamma(s + k)^2}{\Gamma(2s)}
\]
\[\times (r - 1)^{2k} r^{s - k} 2 F_1(s + k, s + k; 2s; r)
\]
(3.23)

with $r = r(z, \gamma z) = 1 - \left| \frac{z - \gamma z}{z - \gamma z} \right|^2$ and $L_{2j} = (yy')^{-2j} \frac{\partial^2}{\partial z \partial z'} (yy')^{2j}$.

(ii) It is clear from the expression (3.5) for $F^{(k)}_s$ by using the following lemma. □
Lemma 3.4. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, put $Q_\gamma(z) = j(\gamma, z) (z - \gamma z)$, i.e.

$$Q_\gamma(z) = cz^2 + (d - a)z - b.$$ 

Then the polynomial $Q_\gamma(z)$ satisfies the following formula for $\sigma \in \Gamma$.

$$Q_\gamma(\sigma z) = j(\gamma, \sigma z)(\sigma z - \gamma z)$$

$$= j(\gamma, \sigma z)j(\sigma, z)^{-1} \cdot j(\sigma, z)^{-1} j(\sigma, \sigma^{-1}\gamma \sigma z)^{-1}(z - \sigma^{-1}\gamma \sigma z)$$

$$= j(\sigma, z)^{-2} j(\gamma, z) \left\{ j(\gamma, z) j(\sigma^{-1}\gamma \sigma z) \right\}^{-1}(z - \sigma^{-1}\gamma \sigma z)$$

(3.24)

$$= j(\sigma, z)^{-2} Q_{\sigma^{-1}\gamma \sigma}(z).$$

So we have the desired formula.

\[ \square \]

3.2. Inner product formula. We study the scalar product $\langle F_s^{(k)}, g \rangle$ in detail. We firstly write down this scalar product as the sum over the hyperbolic conjugacy classes of $\Gamma$. Next we investigate the local term appearing in the sum, more concretely.

Using an explicit formula of Poincaré series $F_s^{(k)}$ in Proposition 3.2, we have the following formula for $\langle F_s^{(k)}, g \rangle$.

Lemma 3.5. Let $\mathcal{H}(\Gamma)$ be the set of the $\Gamma$-conjugacy classes in $\Gamma_{hyp}$, the hyperbolic elements of $\Gamma$. We denote $P_0$ be the primitive hyperbolic element for a given hyperbolic element $P$. Put $\mathcal{F}_0 = \{ z \in \mathbb{H} | 1 \leq |z| < N(P_0) \}$. Here $N(P_0)$ is the norm of $P_0$. Let $\tau \in SL(2, \mathbb{R})$ such that $\tau P_0 \tau^{-1} = \text{diag}(N(P_0)^{1/2}, N(P_0)^{-1/2})$. Then we have the following formula for $\langle F_s^{(k)}, g \rangle$ with $g \in S_{4k}(\Gamma)$,

$$(-1)^k \int_X F_s^{(k)}(z) \overline{g(z)} y^{4k} \frac{dx dy}{y^2}$$

$$= \sum_{P \in \mathcal{H}(\Gamma)} \int_{\mathcal{F}_0} \frac{N(P)^k}{(N(P) - 1)^{2k} z^{2k}} f_s^{(k)}(r(z, N(P)z)) \overline{j(\tau^{-1}, z)^{4k} \gamma(\tau^{-1}z) y^{4k} \frac{dx dy}{y^2}}.$$

Here, we write $F_s^{(k)}(z) = \sum_{\gamma \in \Gamma_{hyp}} \frac{1}{j(\gamma, z)^{2k}} \frac{1}{(z - \gamma z)^{2k}} f_s^{(k)}(r(z, \gamma z))$ in Proposition 3.2, i.e.

$$f_s^{(k)}(r(z, \gamma z)) = (-1)^k \pi^{-1} \Gamma(s + k)^2 \Gamma(2s)^{-1} (r - 1)^{2k} r^{-2k} F_1(s + k, s + k; 2s; r)$$

with $r = r(z, \gamma z) = 1 - \left| \frac{z \gamma z}{z - \gamma z} \right|^2$. 


Proof. Set $\mathcal{F} = \Gamma \setminus \mathbb{H}$ the fundamental domain for $\Gamma$ and $Z(P)$ be the centralizer of $P$ in $\Gamma$.

\[
\int_{\mathcal{F}} F_s^{(k)}(z) g(z) y^{4k} \frac{dx dy}{y^2} = \sum_{P \in \mathcal{H}(\Gamma)} \sum_{\sigma \in Z(P) \setminus \Gamma} \int_{\mathcal{F}} \frac{1}{\sigma(P, z)} f_s^{(k)}(r(z, \sigma^{-1} P \sigma z)) \overline{g(z)} y^{4k} \frac{dx dy}{y^2} \\
= \sum_{P \in \mathcal{H}(\Gamma)} \sum_{\sigma \in Z(P) \setminus \Gamma} \int_{\mathcal{F}} \frac{1}{\sigma(z)} f_s^{(k)}(r(z, P z)) \overline{g(z)} y^{4k} \frac{dx dy}{y^2}.
\]

(3.26)

Since $j(\sigma, z)^{-1} j(\sigma, z)^{-1} = \text{Im}(\sigma z)$, the above expression equals

\[
\sum_{P \in \mathcal{H}(\Gamma)} \sum_{\sigma \in Z(P) \setminus \Gamma} \int_{\mathcal{F}} \frac{1}{\sigma(z)} f_s^{(k)}(r(z, P z)) \overline{g(z)} y^{4k} \frac{dx dy}{y^2}.
\]

(3.27)

Here, we set $\mathcal{F}_P = \bigcup_{\sigma \in Z(P) \setminus \Gamma} \sigma(\mathcal{F})$. For a hyperbolic element $P$, there is a $\tau \in SL(2, \mathbb{R})$ such that $\tau P \tau^{-1} = D_P := \text{diag}(N(P)^{1/2}, N(P)^{-1/2})$ with $N(P) > 1$; the norm of $P$. Then $P z = \tau^{-1} D_P \tau z$ and $D_P \tau z = N(P)^{1/2} \tau z$. $\tau(\mathcal{F}_P)$ is a fundamental domain of the centralizer $Z(D_P) = \tau Z(P) \tau^{-1}$. There is a primitive element $P_0$ such that $P = P_0^n$ with $n \in \mathbb{N}$ for $P$. Thus we can replace $\tau(\mathcal{F}_P)$ by $\mathcal{F}_0 = \{ z \in \mathbb{H} \mid 1 \leq |z| < N(P_0) \}$ the fundamental domain for the group generated by $\tau P_0 \tau^{-1} = D_{P_0}$. Note that $j(\tau, z) = j(\tau^{-1}, \tau z)^{-1}$, then the proof is finished.

Let us investigate the local integral corresponding to a hyperbolic conjugacy class appearing in the sum (3.25) in Lemma 3.5.

**Definition 3.6.** For $P \in \mathcal{H}(\Gamma)$ and $g \in S_{4k}(\Gamma)$, we define

\[
I_s^{(k)}(P; g) = \frac{N(P)^k}{(N(P) - 1)^{2k}} \int_{\mathcal{F}_P} \frac{1}{z^{2k}} f_s^{(k)}(r(z, N(P) z)) \overline{g(z)} y^{4k} \frac{dx dy}{y^2}.
\]

(3.28)

Here, $f_s^{(k)}$ is defined in Lemma 3.5 and $g_r(z) = j(\tau^{-1}, z)^{-4k} g(\tau^{-1} z)$.

So we have a paraphrase of Lemma 3.5, i.e.

\[
\langle F_s^{(k)}, g \rangle = \int_{\Gamma \setminus \mathcal{H}} F_s(z) \overline{g(z)} y^{4k} \frac{dx dy}{y^2} = \sum_{P \in \mathcal{H}(\Gamma)} I_s^{(k)}(P; g).
\]

(3.29)

We claim that the local term $I_s^{(k)}(P; g)$ is the multiple of the periods of automorphic forms over the simple closed geodesic associated to $P_0$. 

Proposition 3.7. For $P \in \mathcal{H}(\Gamma)$, let $P_0$ be the primitive element such that $P_0^n = P$ with $n \in \mathbb{N}$. Then we have

$$I_s(k; g) = \int_{\mathcal{H}} P^k(z) \left( \frac{g(z)}{Q(z)} \right) d\mu(z)$$

where $\mu$ is the Haar measure on $\mathcal{H}$.

Proof. By using the polar coordinate for $z = Re^{i\theta} \in \mathcal{F}_0 = \{z \in \mathbb{H} | 1 \leq |z| < N(P_0)\}$,

$$I_s(k; g) = \frac{N(P)^k}{(N(P) - 1)^{2k}} \int_0^\pi \frac{f_s(z)}{P^k} R^{-2k} e^{-2ki\theta} f_s^{(k)} \left( \frac{4N(P) \sin^2 \theta}{(N(P) - 1)^2 \cos^2 \theta + (N(P) + 1)^2 \sin^2 \theta} \right) \sin^{4k-2} \theta d\theta.$$  \hspace{1cm} (3.31)

Let us consider the complex conjugate of the integral in the square bracket of the last formula. Put $z_0 = e^{i\theta}$ and use the fact that $(z - N(P_0)z) j(D_{P_0}, z) = (1 - N(P_0)) N(P_0)^{-1/2} z$, then it equals

$$\int_{z_0}^{D_{P_0}} z^{2k-1} g\tau(z) d\tau = \frac{N(P_0)^{k-\frac{1}{2}}}{(1 - N(P_0))^{2k-1}} \int_{z_0}^{D_{P_0}} Q_{D_{P_0}}(z)^{2k-1} g\tau(z) d\tau.$$  \hspace{1cm} (3.32)

Recall that $D_{P_0} = \text{diag}(N(P_0)^{1/2}, N(P_0)^{-1/2}) = \tau P_0 \tau^{-1}$. By using Lemma 3.3 on $Q_{P_0}$ and fact that $g \in S_{4k}(\Gamma)$, the differential form satisfies that

$$Q_{\tau P_0 \tau^{-1}}(z)^{2k-1} g\tau(z) d\tau = Q_{P_0}(\tau^{-1} z)^{2k-1} j(\tau^{-1} z)^{4k-2} j(\tau^{-1} z)^{-4k} g(\tau^{-1} z) j(\tau^{-1} z) d(\tau^{-1} z).$$

So we have

$$\int_{z_0}^{D_{P_0}} Q_{\tau P_0 \tau^{-1}}(z)^{2k-1} g\tau(z) d\tau = \int_{\tau^{-1} z_0}^{\tau^{-1}} Q_{P_0}(z)^{2k-1} g(z) d\tau.$$  \hspace{1cm} (3.33)

with $z_1 = \tau^{-1} z_0$. The rest of the proof follows from the fact that the differential form $Q_{P_0}(z)^{2k-1} g(z) d\tau$ is holomorphic on $H$ and $\langle P_0 \rangle$-invariant. Thus the period $\alpha_{2k}(P_0, g)$ is well-defined and the proof is finished. \hfill $\square$

Let us calculate the definite integral on $\theta$ in (3.31) in Proposition 3.7. Then we have an explicit formula for $I_s^{(k)}(P; g)$ by the following proposition.
Proposition 3.8. Put
\begin{equation}
J_s^{(k)}(P) = \int_0^\pi f_s^{(k)} \left( \frac{4N(P) \sin^2 \theta}{(N(P) - 1)^2 \cos^2 \theta + (N(P) + 1)^2 \sin^2 \theta} \right) \sin^{4k-2} \theta d\theta.
\end{equation}

Then we have
\begin{equation}
J_s^{(k)}(P) = \frac{(-1)^{k-1} \Gamma(2s - 1)}{2^{4k-2} \Gamma(2s - 2k)} (N(P) - 1)^{2k-1} N(P)^{-s-k+1} \times_2 F_1 \left( -2k, 2k; 2 - 2s; \frac{1}{1 - N(P)^{-1}} \right).
\end{equation}

Proof. By definition,
\begin{equation}
J_s^{(k)}(P) = (-1)^k \frac{1}{\pi} \frac{\Gamma(s + k)^2}{\Gamma(2s)} \int_0^\pi \frac{4N(P) \sin^2 \theta}{(r - 1)^{2k}\sin^2 \theta} \mathrm{d}\theta.
\end{equation}

\begin{equation}
r = r(\theta) = \frac{4N(P) \sin^2 \theta}{(N(P) - 1)^2 \cos^2 \theta + (N(P) + 1)^2 \sin^2 \theta} = \frac{4N(P)}{(N(P) - 1)^2 \cot^2 \theta + (N(P) + 1)^2}
\end{equation}

and
\begin{equation}
r - 1 = -\frac{(N(P) - 1)^2 (\sin \theta)^{-2}}{(N(P) - 1)^2 \cot^2 \theta + (N(P) + 1)^2}.
\end{equation}

Substituting \(\cot \theta = 1 + t\), then we have
\begin{equation}
r = r(t) = \frac{4N(P)}{(N(P) + 1)^2} \frac{1}{1 + t^2},
\end{equation}

\begin{equation}
-\frac{d\theta}{\sin^2 \theta} = \frac{N(P) + 1}{N(P) - 1} \mathrm{d}t \quad \text{and} \quad (r - 1)^{2k} \sin^{4k-2} \theta \mathrm{d}\theta = \frac{(N(P) - 1)^{4k-1}}{(N(P) + 1)^{4k-1}} \frac{1}{(1 + t^2)^{2k}} \mathrm{d}t.
\end{equation}

Thus we have
\begin{equation}
J_s^{(k)}(P) = (-1)^k \frac{1}{\pi} \frac{\Gamma(s + k)^2}{\Gamma(2s)} \Gamma(2s) \left( \frac{4N(P)}{(N(P) + 1)^2} \right)^{s-k} \times_2 F_1 \left( s + k, s + k; 2s; \frac{1}{1 + t^2} \right) dt.
\end{equation}

Using the power series expansion of the hypergeometric function, we carry out the integral term by term: this is permissible by dominated convergence theorem.
Lemma 3.9.

We require a lemma on the hypergeometric series:

\[
\begin{align*}
2F1\left(s + k - \frac{1}{2}; s + k; 2s; \frac{4N(P)}{(N(P) + 1)^2}\right) \\
= \left(\frac{N(P) + 1}{N(P)}\right)^{2s + 2k - 1} 2F1(2s + 2k - 1, 2k; 2s; N(P)^{-1}).
\end{align*}
\]

Proof. We use the following formula on quadratic transformations of the hypergeometric function (Cf. [8] (9.6.5), p.251):

\[
\begin{align*}
2F1\left(a, a + \frac{1}{2}; c; z\right) = \left(1 + \sqrt{1 - z}\right)^{-2a} 2F1\left(2a, 2a - c + 1; c; \frac{1 - \sqrt{1 - z}}{1 + \sqrt{1 - z}}\right)
\end{align*}
\]
with $|\arg(1-z)| < \pi$. Here, $\sqrt{1-z}$ is meant the branch which is positive for real $z$ in the interval $(0,1)$. Put $a = s + k - \frac{1}{2}, \ c = 2s$ and $z = \frac{4N(P)}{(N(P)+1)^2}$ in (3.41), then we have

$$
2F_1\left(s + k - \frac{1}{2}, s + k; 2s; \frac{4N(P)}{(N(P)+1)^2}\right)
= \left(\frac{1}{2} + \frac{1}{2} \frac{N(P) - 1}{N(P) + 1}\right)^{-2s+k-1} 2F_1\left(2s + 2k - 1, 2k; 2s; \frac{1 - (N(P) - 1)(N(P) + 1)^{-1}}{1 + (N(P) - 1)(N(P) + 1)^{-1}}\right)
= \left(\frac{N(P) + 1}{N(P)}\right)^{2s+2k-1} 2F_1\left(2s + 2k - 1, 2k; 2s; N(P)^{-1}\right).
$$

This completes the proof.

We evaluate the hypergeometric series in the above formula.

**Lemma 3.10.**

$$
2F_1\left(2s + 2k - 1, 2k; 2s; N(P)^{-1}\right)
= (1 - N(P)^{-1})^{-2k} \frac{\Gamma(2s)}{\Gamma(2s + 2k - 1)} \frac{\Gamma(2s - 1)}{\Gamma(2s - 2k)} 2F_1\left(-2k - 1, 2k; 2 - 2s; \frac{1}{1 - N(P)^{-1}}\right).
$$

**Proof.** Set $x = N(P)^{-1}$. We have

$$(3.42) \quad 2F_1(2s + 2k - 1, 2k; 2s; x) = (1 - x)^{-(4k-1)} 2F_1(-(2k - 1), 2s - 2k; 2s; x),$$

by the formula (Cf. [S] (9.5.3), p.248]):

$$
2F_1(a, b; c; z) = (1 - z)^{c-a-b} 2F_1(c - a, c - b; c; z).
$$

Next we use the following formula on linear transformations of the hypergeometric function (Cf. [S] (9.5.8), p.249)):

$$
2F_1(a, b; c; z) = (1 - z)^{-a} \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)} 2F_1\left(a, c-b; 1 + a - b; \frac{1}{1 - z}\right)
+ (1 - z)^{-b} \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)} 2F_1\left(c-a, b; 1 - a + b; \frac{1}{1 - z}\right).
$$

(3.43)

Put $a = -(2k - 1), \ b = 2s - 2k$ and $c = 2s$ in (3.43), then we have

$$
2F_1(-(2k - 1), 2s - 2k; 2s; x)
= (1 - x)^{2k-1} \frac{\Gamma(2s)\Gamma(2s - 1)}{\Gamma(2s + 2k - 1)\Gamma(2s - 2k)} 2F_1\left(-(2k - 1), 2k; 2 - 2s; \frac{1}{1 - x}\right),
$$

(3.44)

By (3.42) and (3.44), we have the desired formula.

Let us complete the proof of Proposition 3.8. \qed
Note that $\Gamma(2s + 2k - 1) = \pi^{-1/2} 2^{2s + 2k - 2} \Gamma(s + k - 1/2) \Gamma(s + k)$ by the duplication formula of the Gamma function. By the above two lemmas, we have

\[
J_s^{(k)}(P) = (-1)^{k} \pi^{-\frac{1}{2}} \frac{(N(P) - 1)^{4k - 1}}{(N(P) + 1)^{4k - 1}} \frac{4N(P)}{(N(P) + 1)^2} \left( \frac{2s + 2k - 1}{N(P) - 1} \right)^{2k - 2} \times \frac{\Gamma(s + k - \frac{1}{2}) \Gamma(s + k) \left( \frac{N(P) + 1}{N(P)} \right)^{2s + 2k - 1} \left( \frac{N(P)}{N(P) - 1} \right)^{2k}}{\Gamma(2s) \Gamma(2s - 1) \Gamma(2s - 2k) 2F_1 \left( -\frac{2k}{2s - 1}, \frac{2k}{2s - 1}; -\frac{1}{N(P) - 1} \right)}
\]

\[
\times 2F_1 \left( -\frac{2k}{2s - 1}, \frac{2k}{2s - 1}; \frac{N(P)}{N(P) - 1} \right).
\]

(3.45)

So we get the desired formula. □

By Proposition 3.7 and Proposition 3.8, we obtain an explicit formula for $I_s^{(k)}(P; g)$, i.e.

\[
I_s^{(k)}(P; g) = -\frac{N(P)^k}{(N(P) - 1)^{2k}} \frac{N(P)}{N(P) - 1}^{2k - 1} \alpha_{2k}(P_0, g) J_s^{(k)}(P)
\]

\[
= \frac{(-1)^k}{2^{k-2}} \frac{N(P)^k}{(N(P) - 1)^{2k - 1} \alpha_{2k}(P_0, g)} \frac{\Gamma(2s - 1)}{\Gamma(2s - 2k)} \frac{N(P)}{N(P) - 1} \times 2F_1 \left( -\frac{2k}{2s - 1}, \frac{2k}{2s - 1}; -\frac{1}{N(P) - 1} \right)
\]

(3.46)

Theorem 3.11. For $g \in S_{4k}(\Gamma)$ and a fixed point $s \in \mathbb{C}$ with Re $s > 1$, we have

(3.47)

\[
\langle I_s^{(k)}(P; g) \rangle = \int_{\Gamma \backslash \mathcal{H}} \mathcal{F}_s^{(k)}(z)g(z) y^{4k} \frac{dx dy}{y^2} = \sum_{P \in \mathcal{H}(\Gamma)} I_s^{(k)}(P; g)
\]

and $I_s^{(k)}(P; g)$ is given by

\[
I_s^{(k)}(P; g) = (-1)^k \frac{\alpha_{2k}(P_0, g)}{2^{k-3} \sinh(2k - 1)(2 - 1 \log N(P_0))} \frac{\Gamma(2s - 1)}{\Gamma(2s - 2k)} \times 2F_1 \left( -\frac{2k}{2s - 1}, \frac{2k}{2s - 1}; -\frac{1}{N(P) - 1} \right) \frac{N(P)}{N(P) - 1} \times 2F_1 \left( -\frac{2k}{2s - 1}, \frac{2k}{2s - 1}; \frac{N(P)}{N(P) - 1} \right)
\]

(3.48)

The series (3.46) is absolutely convergent.

We can rewrite the above theorem by using local higher Selberg zeta functions of rank $j$ ($1 \leq j \leq 2k$). Then, we find that the function $\Psi_{\Gamma}(s; g)$ in Definition 3.4 equals that the inner product in the above theorem, i.e.

(3.49) $\Psi_{\Gamma}(s; g) = (-1)^k \langle I_s^{(k)}(P; g) \rangle$. 

Theorem 3.12. For \( g \in S_{4k}(\Gamma) \) and a fixed point \( s \in \mathbb{C} \) with \( \text{Re} s > 1 \), we have
\[
\Psi_\Gamma(s; g) = (-1)^k (F_{s}^{(k)}, g) = \sum_{P \in \mathcal{H}_s(\Gamma)} (-1)^k I_{s}^{(k)}(P; g)
\]
(3.50)
\[
= \sum_{\gamma \in \text{Prim}(\Gamma)} \frac{\alpha_{2k}(\gamma, g)}{2^{6k-3} \ell(\gamma) \sinh^{2k-1}(\ell(\gamma)/2)} \left\{ \sum_{j=1}^{2k} p_j(s) \frac{d}{ds} \log Z_{\gamma}^{(j)}(s) \right\}.
\]
Here, \( \ell(\gamma) = \log N(\gamma) \), \( Z_{\gamma}^{(j)}(s) \), the local higher Selberg zeta function of rank \( j \), and the polynomial \( p_j(s) \in \mathbb{Z}[2s] \) are given by
\[
Z_{\gamma}^{(j)}(s) = \prod_{m=0}^{\infty} \left( 1 - N(\gamma)^{-s+m} \right)^{(j+m-1)/m},
\]
(3.51)
\[
p_j(s) = (j-1)! \binom{2k-1}{j-1} \binom{2k+j-2}{j-1} 2^k \prod_{i=j+1} \left( 2s - i \right).
\]
(3.52)
The series (3.50) is absolutely convergent.

Proof. By (3.46) and (3.48) in Theorem 3.11 we have
\[
(-1)^k (F_{s}^{(k)}, g)
\]
(3.53)
\[
= \sum_{P \in \mathcal{H}_s(\Gamma)} \frac{\alpha_{2k}(P, g)}{2^{6k-3} \sinh^{2k-1}(\ell(P)/2)} \left\{ \sum_{j=1}^{2k} p_j(s) \left( \frac{N(P)}{N(P) - 1} \right)^j \right\} N(P)^{-s}.
\]
Here, we put \( p_j(s) = \frac{\Gamma(2s-1) (1-2k)_{j-1} (2k)_{j-1}}{\Gamma(2s-2k) (j-1)! (2-2s)_{j-1}} \). The relation \( \log(1 - x) = - \sum_{k=1}^{\infty} \frac{x^k}{k} \) for \( |x| < 1 \) and the definition of \( Z_{P_0}(s) \) imply
\[
\log Z_{P_0}^{(j)}(s) = - \left( j + m - 1 \right) \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \frac{1}{k} N(P_0)^{-k(m+s)}.
\]
(3.54)
Next, differentiating (3.54) with respect to \( s \), we find that
\[
\frac{1}{\log N(P_0)} \frac{d}{ds} \log Z_{P_0}^{(j)}(s) = \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \left( j + m - 1 \right) N(P_0)^{-km-ks}
\]
(3.55)
\[
= \sum_{k=1}^{\infty} \frac{1}{(1 - N(P_0)^{-k})^j} N(P_0)^{-ks} = \sum_{k=1}^{\infty} \left( \frac{N(P_0)^{k}}{N(P_0)^{k-1}} \right)^j N(P_0)^{-ks}.
\]
We can easily check that
\[
p_j(s) = \frac{\Gamma(2s-1) (1-2k)_{j-1} (2k)_{j-1}}{\Gamma(2s-2k) (j-1)! (2-2s)_{j-1}}
\]
(3.56)
\[
= (j-1)! \binom{2k-1}{j-1} \binom{2k+j-2}{j-1} 2^k \prod_{i=j+1} \left( 2s - i \right).
\]
Substituting (3.55) and (3.56) into (3.53), we have the desired formula. The proof of convergence is assured by the following corollary. \( \square \)
Corollary 3.13. For \( g \in S_{4k}(\Gamma) \) and \( s \in \mathbb{C} \) with \( \text{Re} \, s > 1 \), we have the following estimate.

\[
\left| \Psi_\Gamma(s; g) \right| \leq \frac{1}{2k - 2} \| y^{2k} g \|_\infty \left\{ \sum_{j=1}^{2k} \left| p_j(s) \frac{d}{ds} \log Z_\Gamma^{(j)}(s) \right| \right\}_{s = \text{Re} \, s}.
\]

Here, \( Z_\Gamma^{(j)}(s) \) \( (j = 1, 2, \ldots, 2k) \) are the higher Selberg zeta function of rank \( j \) for \( \Gamma \), defined by the following absolutely convergent Euler products for \( \text{Re} \, s > 1 \),

\[
Z_\Gamma^{(j)}(s) = \prod_{\gamma \in \text{Prim}(\Gamma)} \prod_{m=0}^{\infty} \left( 1 - N(\gamma)^{- (m + s)} \right)^{(j + m - 1)}.
\]

Proof. Firstly we estimate \( \alpha_{2k}(\gamma, g) \) for \( \gamma \in \text{Prim}(\Gamma) \) and \( g \in S_{4k}(\Gamma) \). Let \( \tau \in SL(2, \mathbb{R}) \) such that \( \tau \gamma \tau^{-1} = \text{diag}(N(\gamma)^{1/2}, N(\gamma)^{-1/2}) \). Take the point \( z_0 \in \mathbb{H} \) such that \( \tau^{-1}(z_0) = i \). Since the geodesic connecting \( i \) and \( i \sqrt{N(\gamma)} \) is the imaginary axis, we have

\[
\alpha_{2k}(\gamma, g) = \int_{\gamma z_0}^{\gamma z_1} Q_\gamma(z) z^{2k-1} g(z) \, dz = \int_{1}^{\sqrt{N(\gamma)}} \left\{ N(\gamma)^{- \frac{1}{2}} - N(\gamma)^{\frac{1}{2}} \right\} z^{2k-1} g_\tau(z) \, dz
\]

\[
= -2^{2k-1} \sinh^{2k-1}(\ell(\gamma)/2) \int_{1}^{\sqrt{N(\gamma)}} (iy)^{2k-1} g_\tau(iy) \, dy
\]

\[
= 2^{k-1} \sinh^{2k-1}(\ell(\gamma)/2) \int_{1}^{\sqrt{N(\gamma)}} y^{2k} g_\tau(iy) \, dy
\]

Therefore, we have

\[
|\alpha_{2k}(\gamma, g)| \leq 2^{2k-1} \sinh^{2k-1}(\ell(\gamma)/2) \left( \sup_{y > 0} |\text{Im} \, (\tau^{-1}(iy))^{2k} g(\tau^{-1}(iy))| \right) \int_{1}^{\sqrt{N(\gamma)}} \frac{dy}{y}
\]

\[
\leq 2^{2k-1} \ell(\gamma) \sinh^{2k-1}(\ell(\gamma)/2) \| y^{2k} g \|_\infty.
\]

Secondly in (3.58), for \( s = \sigma + it \in \mathbb{C} \) with \( \sigma > 1 \) and a hyperbolic element \( P \),

\[
|p_j(s) \left( \frac{N(P)}{N(P) - 1} \right)^j N(P)^{-s}| = |p_j(s)| \left( \frac{N(P)}{N(P) - 1} \right)^j N(P)^{-\sigma}.
\]

From (3.60) and (3.61), we complete the proof. \( \square \)

4. Analytic continuation of \( \Psi_\Gamma(s; g) \)

Hereafter, we assume that \( \Gamma \) is co-compact torsion-free, i.e. \( X \) is a compact Riemann surface of genus \( g \geq 2 \).

4.1. A variant of the resolvent trace formula. Since we assume that \( \Gamma \) is co-compact, the Laplacian \( \Delta_0 \) has no continuous spectrum on \( L^2(\Gamma \backslash \mathbb{H}) \). The eigenvalues of

\[
4 \Delta_0 = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)
\]

forms a countable subset of non-negative real numbers enumerated as

\[
0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots
\]

so that each eigenvalue occurs in this sequences with its multiplicity.
Let \( \{ \varphi_n \}_{n \geq 0} \) be the orthonormal basis of \( L^2(\Gamma \setminus \mathbb{H}) \) such that \( \varphi_n \in C^\infty(\Gamma \setminus \mathbb{H}) \) and \( 4 \Delta_0 \varphi_n = \lambda_n \varphi_n \). Put \( \lambda_n = 1/4 + r_n^2 \) for each \( n \). Recall that \( G_s^{(0)} \) is the kernel function of the operator \( (\Delta_0 - 1/4 s(1 - s))^{-1} \) on the Riemann surface \( X = \Gamma \setminus \mathbb{H} \).

**Proposition 4.1.** Let \( m \in \mathbb{N} \) and \( s \in \mathbb{C} \) be such that \( m \geq 1 \) and \( \text{Re } s > 1 \), then the function

\[
\left( -\frac{1}{2s - 1} \right)^m G_s^{(0)}(z, z') = \sum_{n=0}^{\infty} \left( \frac{4}{(s - \frac{1}{2})^2 + r_n^2} \right)^m \frac{\varphi_n(z)\varphi_n(z')}{m!}.
\]

Here the right hand side of this identity converges uniformly in \( (z, z') \in (\Gamma \setminus \mathbb{H})^2 \).

**Proof.** Let \( s, a \in \mathbb{C} \), \( \text{Re } s > 1 \), \( \text{Re } a > 1 \). We have (Cf. [1, Theorem 2.1.2, p.46])

\[
G_s^{(0)}(z, z') - G_a^{(0)}(z, z') = \sum_{n=0}^{\infty} \left( \frac{4}{(s - \frac{1}{2})^2 + r_n^2} - \frac{4}{(a - \frac{1}{2})^2 + r_n^2} \right) \varphi_n(z)\varphi_n(z').
\]

By differentiating (4.2) with respect to \( s \), we have formally

\[
\frac{1}{m!} \left( \frac{4}{(s - \frac{1}{2})^2 + r_n^2} \right)^m G_s^{(0)}(z, z') = \sum_{n=0}^{\infty} \left( \frac{4}{(s - \frac{1}{2})^2 + r_n^2} \right)^m \varphi_n(z)\varphi_n(z').
\]

The series

\[
\sum_{n=0}^{\infty} \left( \frac{4}{(s - \frac{1}{2})^2 + r_n^2} \right)^m \left( \int_{\Gamma \setminus \mathbb{H}} \frac{|\varphi_n(z)|^2}{y^2} \frac{dxdy}{y^2} \right)^{\frac{1}{2} \times 2}
\]

converges absolutely since \( \sum_{n \geq 1} \lambda_n^{-2} \) converges. The proof is finished. See also Proposition 4.2.1 in [2].

**Proposition 4.2.** Let \( m \in \mathbb{N} \) and \( s \in \mathbb{C} \) be such that \( m \geq 2k + 1 \) and \( \text{Re } s > 1 \). Put \( \partial_{2j} = y^{-2j} \frac{\partial}{\partial z} y^{2j} \) and \( \partial_{2j}^2 = y^{-2j} \frac{\partial^2}{\partial z^2} y^{2j} \). Then

\[
\frac{1}{m!} \left[ \partial_{2k-2} \cdots \partial_{2} \partial_0 \right] \left[ \partial_{2k-2} \cdots \partial_{2} \partial_0 \right] \left( -\frac{1}{2s - 1} \right)^m G_s^{(0)}(z, z')
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{4}{(s - \frac{1}{2})^2 + r_n^2} \right)^m \left[ \partial_{2k-2} \cdots \partial_{2} \partial_0 \right] \varphi_n(z) \left[ \partial_{2k-2} \cdots \partial_{2} \partial_0 \right] \varphi_n(z').
\]

Here the right hand side of this identity converges uniformly in \( (z, z') \in (\Gamma \setminus \mathbb{H})^2 \).

**Proof.** Operate the differential operators on the both sides of (4.1), we have the desired formula (4.5) formally. The series

\[
\sum_{n=0}^{\infty} \frac{4}{(s - \frac{1}{2})^2 + r_n^2} \left( \int_{\Gamma \setminus \mathbb{H}} \left| \partial_{2k-2} \cdots \partial_{2} \partial_0 \varphi_n(z) \right|^2 y^{2k} \frac{dxdy}{y^2} \right)^{\frac{1}{2} \times 2}
\]

\[
= \sum_{n=0}^{\infty} \frac{4\lambda_n^{2k}}{(s - \frac{1}{2})^2 + r_n^2} \left( \int_{\Gamma \setminus \mathbb{H}} \left| \partial_{2k-2} \cdots \partial_{2} \partial_0 \varphi_n(z) \right|^2 y^{2k} \frac{dxdy}{y^2} \right)^{\frac{1}{2} \times 2}
\]

converges absolutely. The proof is finished.
Proposition 4.3. Let \( m \in \mathbb{N} \) and \( s \in \mathbb{C} \) be such that \( m \geq 2k + 1 \) and \( \text{Re} \, s > 1 \). Then

\[
\lim_{z' \to z} \left( -\frac{1}{2s - 1} d \right)^m L_{2k-2} L_{2k-4} \cdots L_2 L_0 Q_s^{(0)}(z, z') = 0, \quad z \in \mathbb{H}.
\]

Proof. Put \( q_s^{(k)}(z, z') = L_{2k-2} L_{2k-4} \cdots L_2 L_0 Q_s^{(0)}(z, z') \). By using (3.6) in the proof of Proposition 3.2,

\[
q_s^{(k)}(z, z') = (-1)^k \frac{\pi^{-1}}{(z - z')^{2k}} (r - 1)^{2k} \frac{\Gamma(s + k)^2}{\Gamma(2s)} r^{s-k} {}_2F_1(s + k, s + k; 2s; r)
\]

with \( r(z, z') = 1 - \frac{|z - z'|^2}{|z - z'|^2} \) and the formula (it can be derived from (9.7.5) and (9.7.6) in [8] p.257):

\[
{}_2F_1(s + k, s + k; 2s; r) = \frac{\Gamma(2s)}{\Gamma(s + k)^2} (1 - r)^{-2k} \sum_{n=0}^{2k-1} \frac{(-1)^n (2k - 1 - n)! (s - k)_n (s - k)_n}{n!} (1 - r)^n \\
- \frac{\Gamma(2s)}{\Gamma(s - k)^2} \sum_{n=0}^{\infty} \frac{(s + k)_n (s + k)_n}{n!(2k + n)!} \left[ \log(1 - r) + 2\psi(s + k + n) - \psi(n + 1) \right] (1 - r)^n
\]

for \( |r - 1| < 1 \), \( |\arg(1 - r)| < \pi \).

Here, \( (\alpha)_n = \Gamma(\alpha + n)/\Gamma(\alpha) \) and \( \psi(z) = \Gamma'(z)/\Gamma(z) \). Therefore, we have

\[
\left\{ (-1)^k \frac{\pi^{-1}}{(z - z')^{2k}} r^{s-k} \right\}^{-1} q_s^{(k)}(z, z') = \sum_{n=0}^{2k-1} \frac{(-1)^n (2k - 1 - n)! (s - k)_n (s - k)_n}{n!} (1 - r)^n \\
- \frac{\Gamma(s + k)^2}{\Gamma(s - k)^2} \sum_{n=0}^{\infty} \frac{(s + k)_n (s + k)_n}{n!(2k + n)!} \left[ \log(1 - r) \right.
\]

\[
\left. + 2\psi(s + k + n) - \psi(n + 1) - \psi(2k + n + 1) \right] (1 - r)^{n+2k}.
\]

Note that

\[
r^{s-k} = \left\{ 1 + (r - 1) \right\}^{s-k}
\]

\[
= \sum_{j=0}^{2k} \binom{s - k}{j} (-1)^j (1 - r)^j + O((1 - r)^{2k}) \quad (r \to 1 - 0),
\]

then we have (after some calculation, see also Proposition 3.1.3 in [2])

\[
q_s^{(k)}(z, z') = (-1)^k \frac{\pi^{-1}}{(z - z')^{2k}} \left\{ \sum_{j=0}^{2k-1} a_j (s - 1 - r)^j + b(s) (1 - r)^{2k} \log(1 - r) \right\}
\]

\[
+ O((1 - r)^{2k}) \quad (r \to 1 - 0)
\]
with
\begin{equation}
 a_j(s) = (-1)^j \frac{(2k - 1 - j)!}{j!} \prod_{i=0}^{j-1} \{s^2 - s - k^2 - (2i - 1)k + i(i + 1)\},
\end{equation}
\begin{equation}
 b(s) = -\frac{1}{(2k)!} \prod_{i=0}^{k-1} \{s^2 - s - i(i + 1)\}^2.
\end{equation}
We find that
\begin{align*}
 a_j(1 - s) &= a_j(s), \quad \deg a_j(s) = 2j \quad (0 \leq j \leq 2k - 1),
 b(1 - s) &= b(s), \quad \deg b(s) = 4k.
\end{align*}
So we have
\begin{equation}
\left( -\frac{1}{2s - 1} \frac{d}{ds} \right) a_j(s) = \left( -\frac{1}{2s - 1} \frac{d}{ds} \right)^{2k+1} b(s) = 0.
\end{equation}
Therefore, if \( m \geq 2k + 1 \) then we have
\begin{equation}
\lim_{z' \to z} \left( -\frac{1}{2s - 1} \frac{d}{ds} \right)^m q_s^{(k)}(z, z') = 0.
\end{equation}
Here we used the fact
\begin{equation}
\lim_{z' \to z} \frac{1}{(z - z')^2(1 - r)^2} = \lim_{z' \to z} \frac{(\bar{z} - z')^2}{(\bar{z} - z')^2(z - \bar{z})^2} = 0.
\end{equation}
This completes the proof. \( \square \)

**Proposition 4.4.** Let \( m \in \mathbb{N} \) and \( s \in \mathbb{C} \) be such that \( m \geq 2k + 1 \) and \( \Re s > 1 \). Then
\begin{equation}
\lim_{z' \to z} L_{2k-2}L_{2k-4} \cdots L_2L_0 \left( -\frac{1}{2s - 1} \frac{d}{ds} \right)^m G_s^{(0)}(z, z')
\end{equation}
\begin{equation}
= \left( -\frac{1}{2s - 1} \frac{d}{ds} \right)^m \sum_{\gamma \in \Gamma \setminus \{e\}} L_{2k-2}L_{2k-4} \cdots L_2L_0 \left( Q_s^{(0)}(z, \gamma z') \right) \bigg|_{z' = z}
\end{equation}
\begin{equation}
= \left( -\frac{1}{2s - 1} \frac{d}{ds} \right)^m F_s^{(k)}(z).
\end{equation}

**Proof.** By using Proposition 4.3 and interchanging the order of differentiation. By Theorem 3.11 and Propositions 4.2 and 4.4, we have the following formula. \( \square \)

**Theorem 4.5.** Let \( g \in S_{4k}(\Gamma) \). Define \( \partial_2 \partial_0 = y^{-2j} \frac{\partial}{\partial y} y^{2j} \) and put
\begin{equation}
\varphi_n^{(k)} := \left[ \partial_{2k-2} \cdots \partial_2 \partial_0 \right] \varphi_n, \quad \overline{\varphi_n^{(k)}} := \left[ \partial_{2k-2} \cdots \partial_2 \partial_0 \right] \overline{\varphi_n}.
\end{equation}
If \( m \geq 2k + 1 \) and \( \Re s > 1 \), then we have
\begin{equation}
(-1)^k \sum_{n=1}^{\infty} \frac{4}{((s - 1/2)^2 + n^2)^{m+1}} \langle \varphi_n^{(k)}, \overline{\varphi_n^{(k)}}, g \rangle = \frac{1}{m!} \left( -\frac{1}{2s - 1} \frac{d}{ds} \right)^m \Psi_\Gamma(s; g)
\end{equation}
with
\begin{equation}
\Psi_\Gamma(s; g) = \sum_{\gamma \in \text{Prim}(\Gamma)} \frac{\alpha_{2k}(\gamma, g)}{2^{6k-3} \ell(\gamma) \sinh^{2k-1}(\ell(\gamma)/2) \left\{ \sum_{j=1}^{2k} p_j(s) \frac{d}{ds} \log Z_\gamma^{(j)}(s) \right\}}.
\end{equation}
4.2. **Analytic continuation of** \( \Psi_\Gamma(s;g) \). We study analytic properties of \( \Psi_\Gamma(s;g) \). By using Theorem 4.5, we have the following theorem.

**Theorem 4.6.** The function \( \Psi_\Gamma(s;g) \), defined for \( \text{Re } s > 1 \), has the analytic continuation as a meromorphic function on the whole complex plane. \( \Psi_\Gamma(s;g) \) has at most simple poles located at:

\[
s = \frac{1}{2} \pm i r_n \quad (n \geq 1).
\]

There are no poles other than described as above. \( \Psi_\Gamma(s;g) \) satisfy the functional equation

\[
(4.21) \quad \Psi_\Gamma(1 - s;g) = \Psi_\Gamma(s;g).
\]

**Proof.** By using Theorem 4.5, the left hand side of (4.19) is a meromorphic function of \( s \in \mathbb{C} \) and its poles are located at the points \( s = \frac{1}{2} \pm r_n \) with order \( m + 1 \). Hence, \( \Psi_\Gamma(s;g) \) is a meromorphic function with at simple poles only at \( s = \frac{1}{2} \pm r_n \). This completes the proof. \( \square \)

5. **Dirichlet series** \( \Xi_\Gamma(s;g) \)

We study analytic properties of the Dirichlet series \( \Xi_\Gamma(s;g) \). We show that \( \Xi_\Gamma(s;g) \) are related to \( \Psi_\Gamma(s;g) \) and find the relations between them. As a result, analytic properties of \( \Xi_\Gamma(s;g) \) are derived from that of \( \Psi_\Gamma(s;g) \).

5.1. **The difference of** \( \Psi_\Gamma(s;g) \). We consider the difference of \( \Psi_\Gamma(s;g) \).

**Proposition 5.1.** For \( 0 \leq l \leq 2k - 1 \), \( g \in S_{4k}(\Gamma) \) and a fixed point \( s \in \mathbb{C} \) with \( \text{Re } s > 1 \), Put

\[
\begin{align*}
\Psi_\Gamma^{[0]}(s;g) &:= \Psi_\Gamma(s;g), \\
\Psi_\Gamma^{[l+1]}(s;g) &:= \frac{1}{2s + l} \left\{ \Psi_\Gamma^{[l]}(s;g) - \Psi_\Gamma^{[l]}(s + 1;g) \right\} \quad (0 \leq l \leq 2k - 2).
\end{align*}
\]

Then, we have

\[
\Psi_\Gamma^{[l]}(s;g) = \sum_{\gamma \in \text{Prim}(\Gamma)} \frac{\alpha_{2k}(\gamma, g)}{2^{6k-3} \ell(\gamma) \sinh^{2k-1}(\ell(\gamma)/2)} \left\{ \sum_{j=1}^{2k-l} p_j^{[l]}(s) \frac{d}{ds} \log Z_\gamma^{(j-l)}(s) \right\}
\]

with

\[
p_j^{[l]}(s) = (j - 1)! \binom{2k - 1 - l}{j - 1} \binom{2k + j - 2 - l}{j - 1} \prod_{i=j+1}^{2k-l} (2s + l - i).
\]

**Proof.** We prove by induction on \( l \). It is clear for \( l = 0 \). Let

\[
F_\gamma^{[l]}(s) = \sum_{j=1}^{2k-l} p_j^{[l]}(s) f_{j-l}(s)
\]
with \( f_j(s) = \frac{d}{ds} \log Z^{(j)}(s) \). Firstly we note that

\[
p_j^{[l]}(s) - p_j^{[l]}(s + 1)
= (j - 1)! \binom{2k - 1 - l}{j - 1} \binom{2k + j - 2 - l}{j - 1}
\prod_{i=j+3}^{2k-l-2} (2s + l - i)
\times \left\{ (2s - 2k + 2l + 1)(2s - 2k + 2l) - (2s + l - j + 1)(2s + l - j) \right\}
= (j - 1)! \binom{2k - 1 - l}{j - 1} \binom{2k + j - 2 - l}{j - 1}
\prod_{i=j+3}^{2k-l-2} (2s + l - i)
\times \left\{ -2(2k - l - j)(2s) + (2k - l - j)(2k - 1 + j - 3l) \right\}.
\]

Secondly by using the fact:

\[
f_j(s) - f_j(s + 1) = \frac{d}{ds} \log Z^{(j)}(s) - \frac{d}{ds} \log Z^{(j-1)}(s)
= \ell(\gamma) \sum_{k=1}^{\infty} \left( \frac{N(\gamma)^k}{N(\gamma)^k - 1} \right)^j N(\gamma)^{-ks} - \ell(\gamma) \sum_{k=1}^{\infty} \left( \frac{N(\gamma)^k}{N(\gamma)^k - 1} \right)^j N(\gamma)^{-k(s+1)}
= \ell(\gamma) \sum_{k=1}^{\infty} \left( \frac{N(\gamma)^k}{N(\gamma)^k - 1} \right)^{j-1} N(\gamma)^{-ks} = f_{j-1}(s).
\]

Thus we have

\[
F^{(l)}(s) - F^{(l)}(s + 1)
= \sum_{j=1}^{2k-l} \left[ p_j^{[l]}(s + 1) \left\{ f_{j-l}(s) - f_{j-l}(s + 1) \right\} \right]
\times \left\{ -2(2s) + (2k - 1 + j - 3l) \right\} f_{j-l}(s)
= \sum_{j=1}^{2k-l} \left[ p_j^{[l]}(s + 1) f_{j-l-1}(s) \right]
\times \sum_{j=1}^{2k-l-1} (j - 1)! \binom{2k - 1 - l}{j - 1} \binom{2k + j - 2 - l}{j - 1}
\prod_{i=j+3}^{2k-l-2} (2s + l - i)
\times \left\{ -2(2s) + (2k - 1 + j - 3l) \right\} f_{j-l}(s).
\]
Let \( a_j(s) \) be the coefficient function of \( f_{j-l-1}(s) \) in the last formula. Then we have

\[
a_j(s) = (j - 1)! \left( \frac{2k - 1 - l}{j - 1} \right) \left( \frac{2k + j - 2 - l}{j - 1} \right) \prod_{i=j+1}^{2k-l} (2s + 2l + i - 1)
\]

\[
+ (j - 2)! (2k - l - j + 1) \left( \frac{2k - 1 - l}{j - 2} \right) \left( \frac{2k + j - 3 - l}{j - 2} \right)
\]

\[
\times \prod_{i=j+2}^{2k-l-2} (2s + 2l - i) \left\{ -2(2s) + (2k - 2 + j - 3l) \right\}
\]

\[
= \prod_{i=j+2}^{2k-l-2} (2s + 2l - i) \left[ (j - 1)! \left( \frac{2k - 1 - l}{j - 1} \right) \left( \frac{2k + j - 2 - l}{j - 1} \right) (2s + 2l + j + 1)
\]

\[
+ (j - 2)! (2k - l - j + 1) \left( \frac{2k - 1 - l}{j - 2} \right) \left( \frac{2k + j - 3 - l}{j - 2} \right)
\]

\[
\times \left\{ -2(2s + l) + (2k - 2 + j - l) \right\}
\]

\[(5.8) \]

We show that \( a_j(s) \) is divisible by \( 2s + l \).

\[
a_j(s) = \prod_{i=j+2}^{2k-l-2} (2s + 2l - i) \left( \frac{2k + j - 3 - l}{j - 1} \right) (2s + 2l - i + 1)
\]

\[
-2(j - 2)! (2k - l - j + 1) \left( \frac{2k - 1 - l}{j - 2} \right) \left( \frac{2k + j - 3 - l}{j - 2} \right)
\]

\[
\times \left\{ -(j - 1)(j - 1)! \left( \frac{2k - 1 - l}{j - 1} \right) \left( \frac{2k + j - 3 - l}{j - 1} \right)
\]

\[
+ (j - 2)! (2k - l - j + 1) \left( \frac{2k - 1 - l}{j - 2} \right) \left( \frac{2k + j - 3 - l}{j - 2} \right) (2s + 2l + j - l)
\]

\[
= \prod_{i=j+2}^{2k-l-2} (2s + 2l - i) \left( \frac{2k + j - 3 - l}{j - 1} \right) \left( \frac{2k + j - 3 - l}{j - 1} \right)
\]

\[
= (2s + l) (j - 1)! \left( \frac{2k - l - 2}{j - 1} \right) \left( \frac{2k + j - l - 3}{j - 1} \right) \prod_{i=j+1}^{2k-l-1} (2s + l + i - 1)
\]

\[(5.9) \]

At last we have

\[
F_\gamma(s) - F_\gamma(s + 1) = (2s + l) \sum_{j=1}^{2k-l-1} P_j^{[l+1]} f_{j-l-1}(s).
\]

(5.10)

This completes the proof.

**Lemma 5.2.** For \( 0 \leq l \leq 2k - 1 \), we have

\[
\Psi^{[l]}_\Gamma(s; g) = \sum_{j=0}^{l} c_j^{[l]}(s) \Psi_\Gamma(s + j; g)
\]

(5.11)
with

\[
(5.12) \quad c_j^l(s) = \frac{(-1)^j \binom{l}{j}}{\prod_{i=0}^{l} (2s+j-i)}.
\]

Proof. By the assumption of the induction on \(l\),

\[
(2s + l) \Psi_G^{l+1}(s; g) = \Psi_G^l(s; g) - \Psi_G^l(s+1; g)
= \sum_{j=0}^{l+1} (-1)^j \binom{2s+j+l}{j} \prod_{i=0, i \neq j}^{l+1} (2s+j-i) \Psi_G(s+j; g)
\]

\[
= \sum_{j=0}^{l+1} (-1)^j (2s + l) \binom{l+1}{j} \prod_{i=0, i \neq j}^{l+1} (2s+j-i) \Psi_G(s+j; g)
\]

\[
(5.13)
\]

This completes the proof. \(\Box\)

By using Theorem 4.6, Proposition 5.1 and Lemma 5.2, we have the following theorem.

**Theorem 5.3.** For \(0 \leq l \leq 2k - 1\), the function

\[
(5.14) \quad \Psi_G^l(s; g) = \sum_{\gamma \in \text{Prim}(\Gamma)} \frac{\alpha_{2k} \gamma, \gamma \gamma}{2^{2k-3} \ell(\gamma) \sinh^{2k-1}(\ell(\gamma)/2)} \left\{ \sum_{j=1}^{2k-l} p_j^l(s) \frac{d}{ds} \log Z_G^{(j-l)}(s) \right\}
\]

with

\[
p_j^l(s) = (j-1)! \binom{2k-1-l}{j-1} \binom{2k-j-2-l}{j-1} \prod_{i=j+1}^{2k-l} (2s+l-i),
\]

defined for \(\text{Re } s > 1\), has the analytic continuation as a meromorphic function on the whole complex plane. \(\Psi_G^l(s; g)\) has at most simple poles located at:

\[
s = \frac{1}{2} - j \pm ir_n \quad (0 \leq j \leq l, \ n \geq 1),
\]

and its residue at \(s = 1/2 - j \pm ir_n\) is given by

\[
(5.15) \quad \frac{(-1)^j \binom{l}{j}}{\prod_{m=0, m \neq j}^{l} (\pm 2ir_n - j + m)} \text{Res}_{s=1/2 \pm ir_n} \Psi_G(s; g)
= 4(-1)^k \frac{(-1)^j \binom{l}{j}}{\prod_{m=0}^{l} (\pm 2ir_n - j + m)} \langle \varphi_{n}^{(k)}, \varphi_{n}^{(k)} \rangle_{g}
\]

There are no poles other than described as above. \(\Psi_G^l(s; g)\) satisfy the functional equation

\[
(5.16) \quad \Psi_G^l(1-l-s; g) = \Psi_G^l(s; g).
\]
Next we introduce certain functions $\Psi_{\Gamma}^{[2k-1,p]}(s; g)$ for $1 \leq p \leq 2k - 1$.

**Proposition 5.4.** For $g \in S_{4k}(\Gamma)$ and a fixed point $s \in \mathbb{C}$ with $\text{Re} s > 1$, put

$$
\Psi_{\Gamma}^{[2k-1,0]}(s; g) := \Psi_{\Gamma}^{[2k-1]}(s; g),
\Psi_{\Gamma}^{[2k-1,p]}(s; g) := \sum_{j=0}^{\infty} \Psi_{\Gamma}^{[2k-1,p-1]}(s + j; g)
$$

(5.17)

Then, we have

$$
\Psi_{\Gamma}^{[2k-1,p]}(s; g) = \sum_{\gamma \in \text{Prim}(\Gamma)} \frac{\alpha_{2k}(\gamma, g)}{2^{2k-3} \ell(\gamma) \sin^{2k-1}(\ell(\gamma)/2)} \frac{d}{ds} \log Z_{\gamma}^{(2k-p)}(s).
$$

(5.18)

Besides, $\Psi_{\Gamma}^{[2k-1,p]}(s; g)$ $(1 \leq p \leq 2k - 1)$ has the analytic continuation as a meromorphic function on the whole complex plane and has at most simple poles located at:

$$
s = \frac{1}{2} - j \pm ir_n \quad (j \in \{0\} \cup \mathbb{N}, \, n \in \mathbb{N}),
$$

and its residue at $s = 1/2 - j \pm ir_n$ is given by

$$
4(-1)^{k+j} \langle \varphi_n^{(k)}, \varphi_n^{(k)} \rangle \sum_{h=\max(0,j-2k+1)}^{j} \frac{(-1)^h (p+h-1)(2k-1)}{2k-1} \prod_{m=0}^{\infty} (\pm 2ir_n - j + m).
$$

(5.19)

There are no poles other than described as above.

**Proof.** For a primitive hyperbolic $\gamma \in \Gamma$, we have

$$
\frac{1}{\ell(\gamma)} \sum_{j=0}^{\infty} \frac{d}{ds} \log Z_{\gamma}^{(m)}(s + j) = \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \left( \frac{N(\gamma)^k}{N(\gamma)^k - 1} \right)^m N(\gamma)^{-k(s+j)}
$$

$$
= \sum_{k=1}^{\infty} \left( \frac{N(\gamma)^k}{N(\gamma)^k - 1} \right)^{m+1} N(\gamma)^{-k} = \frac{1}{\ell(\gamma)} \frac{d}{ds} \log Z_{\gamma}^{(m+1)}(s).
$$

(5.18)

Therefore, we have (5.18). We can check that $\Psi_{\Gamma}^{[2k-1,p]}(s; g)$ is absolutely convergent for $\text{Re} s > 1$ by the expression (5.18). By Theorem 5.3, $\Psi_{\Gamma}^{[2k-1,p]}(s; g)$ has the analytic continuation as a meromorphic function on the whole complex plane. $\Psi_{\Gamma}^{[2k-1,p]}(s; g)$ has at most simple poles at $s = \frac{1}{2} - j \pm ir_n$ $(j \geq 0, \, n \geq 1)$ by (5.17). We completes the proof.

**5.2.** Dirichlet series $\Xi_{\Gamma}(s; g)$. Finally we have a paraphrase of analytic properties of $\Psi_{\Gamma}(s; g)$ in terms of the Dirichlet series $\Xi_{\Gamma}(s; g)$.
**Definition 5.5.** For $g \in S_{4k}(\Gamma)$ and $s \in \mathbb{C}$ with $\Re{s} > 1$, define

$$
\Xi_{\Gamma}(s; g) = \sum_{\gamma \in \text{Prim}(\Gamma)} \sum_{m=1}^{\infty} \beta_{2k}(\gamma, g) N(\gamma)^{-ms}
$$

with

$$
\beta_{2k}(\gamma, g) = \frac{\alpha_{2k}(\gamma, g)}{2^{6k-3} \sinh^{2k-1}(\ell(\gamma)/2)}.
$$

**Theorem 5.6.** The function $\Xi_{\Gamma}(s; g)$, defined for $\Re{s} > 1$, has the analytic continuation as a meromorphic function on the whole complex plane. $\Xi_{\Gamma}(s; g)$ has at most simple poles located at:

1. $s = \frac{1}{2} - j \pm ir_n$ ($j \in \{0, 1\}$, $n \geq 1$) when $k = 1$, with the residue

$$
-4(-1)^j \frac{(\pm 2ir_n - j)(\pm 2ir_n - j + 1)}{(\pm 2ir_n - j)(\pm 2ir_n - j + 1)} \langle \varphi_n^{(1)} \varphi_n^{(1)}, g \rangle,
$$

2. $s = \frac{1}{2} - j \pm ir_n$ ($j \geq 0$, $n \geq 1$) when $k \geq 2$, with the residue

$$
4(-1)^{k+j} \langle \varphi_n^{(k)} \varphi_n^{(k)}, g \rangle \sum_{h=\max(0,j-2k+1)}^{j} \frac{(-1)^h (2k+h-3)(2k-1)(j-h)}{2k-1} \prod_{m=0}^{2k-1} (\pm 2ir_n - j + m).
$$

There are no poles other than described as above.

**Proof.** Put $p = 2k - 2$ in Proposition 5.4 and note that

$$
\frac{1}{\ell(\gamma)} \frac{d}{ds} \log Z_{\gamma}^{(0)}(s) = \sum_{k=1}^{\infty} N(\gamma)^{-ks} = \frac{N(\gamma)^{-s}}{1 - N(\gamma)^{-s}}.
$$

Then we have

$$
\Xi_{\Gamma}(s; g) = \Psi_{\Gamma}^{[2k-1, 2k-2]}(s; g).
$$

The proof is finished. \(\square\)

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