POSITIVITY OF $|p|^a|q|^b + |q|^b|p|^a$

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Abstract. We show that
\[ J_{a,b,n} := \frac{1}{2}(|p|^a|q|^b + |q|^b|p|^a) \]
is positive under suitable conditions on the exponents $a$ and $b$ and
the underlying dimension $n$. (Here $q$ is the multiplication by $x$ and $p := i^{-1}\nabla$.)
Furthermore we show a generalization of the generalized Hardy inequalities
for the fractional Laplacians.

1. Introduction

The classical Hardy inequality ([3, Formula (4)])
\[ \int_a^\infty \left( \frac{F}{x} \right)^\kappa dx \leq \left( \frac{\kappa}{\kappa - 1} \right)^\kappa \int_a^\infty f^\kappa dx \]
with $F(x) = \int_a^x f(t)dt$ and $\kappa > 1$ is one of the longest known inequalities allowing to
bound the weighted $L^\kappa$-norm of a decaying function by the $L^\kappa$-norm of its gradient
(Hardy [3]). In modern textbooks, see, e.g., Reed and Simon [7, p. 169], this occurs
($\kappa = 2$) as the quantum mechanical uncertainty principle lemma and is written in
three dimensions as
\[ \int_{\mathbb{R}^3} |\nabla \psi|^2 \geq \frac{1}{4} \int_{\mathbb{R}^3} |\psi(x)|^2 dx. \]
This, in turn was generalized by Herbst [4] (see also Yafaev [8] and Frank et al [1])
to fractional Laplacians (see [19]).

In a seemingly different context, the excess charge problem of atoms, Lieb [5]
needed
\[ |q||p|^2 + |p|^2|q| > 0 \]
which, however, turned out to be equivalent to the quantum mechanical uncertainty
principle. Here $p = -i\nabla$ is the momentum operator and $q$ (multiplication by $x$) is
the position operator. Lieb [5] showed in fact, that also
\[ |q||p| + |p||q| > 0 \]
in three dimensions by reducing it to (2).

With the advent of graphene physics, two-dimensional versions of Lieb’s inequality
became of physical interest which, however, could not simply be reduced to (2).
Instead, [8] was directly proven [2].

The purpose of this article is to show, that the positivity of the Jordan product
$J_{a,b,n} := \frac{1}{4}(|p|^a|q|^b + |q|^b|p|^a)$ is in fact a generalization which reduces, for $b = n - a$
to Hardy inequalities for fractional Laplacians. Here $a$ and $b$ are positive constants
and $n$ is the underlying dimension of the appropriate function space.

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2. Positivity and Relation to Generalized Hardy Inequalities

Our basic result is the following operator inequality on $L^2(\mathbb{R}^n)$ for the momentum operator $p = -i\nabla$ and the position operator $q$ (multiplication by $x$).

**Theorem 1.** Assume $n \geq a + b$ and $\min\{a, b\} \in [0, 2]$. Then on $C_0^\infty(\mathbb{R}^n)$

\[ 0 < J_{a,b,n} := \frac{1}{2}|p|^a q^b + |q|^b p^a. \]

In fact, our proof shows more, namely

\[ |q|^{b/2} H_{a,n} q^{b/2} \leq J_{a,b,n} \]

where $H_{a,n}$ is the Hardy operator of $[19]$.

As indicated in the introduction, the case $n = 3$, $a = 2$, and $b = 1$ has an important consequence in atomic physics: it is an essential ingredient in bounding the total number of electrons that atoms can bind: the number of electrons that an atom can bind can never exceed twice its nuclear charge. This special case was proven and applied in this context by Lieb [5]. The case $n = 2$ and $a = b = 1$ plays a similar role in investigating how many electrons a magnetic quantum dot in a graphene layer can bound and was proved and applied in that context (Handerek and Siedentop [2]).

**Proof.** For the proof we can assume that $a \leq b$, since, if not, we use the Fourier transform to exchange the role of $p$ and $q$.

We first treat the case, that $a < 2$. In this case we follow the strategy of [2] and use the identity (20). Thus, by polarization

\[ t := \frac{1}{2}(\psi, (|p|^a q^b + |q|^b p^a)\psi) \]

\[ = \alpha_{a,n} R \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy \left( \frac{\psi(x) - \psi(y)}{|x - y|^{n+a}} \right)^2 \]

\[ = \alpha_{a,n} R \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy \left( \frac{2|x|^b |\psi(x)|^2 - (|x|^b + |y|^b) \psi(x) \psi(y)}{|x - y|^{n+a}} \right). \]

Now, setting $\psi = g/|.|^{(n+b-a)/2}$ we get

\[ \frac{t}{\alpha_{a,n}} = \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy \left( \frac{|x|^b |g(x)|^2}{|x|^{(n+b-a)/2}} + \frac{|y|^b |g(y)|^2}{|y|^{(n+b-a)/2}} - \frac{2b|g(x)g(y)| |y|^b}{|x - y|^{n+a}} \right) \]

\[ = \int_{\mathbb{R}^n} \frac{dx}{|x|^n} [g(x)]^2 \int_{\mathbb{R}^n} dy \frac{2|x|^a - |x|^{a+b} |y|^{n+a} + |x|^{a+b} |y|^{-n-a}}{|x - y|^{n+a}} \frac{2|x|^a - |x|^{a+b} |y|^{n+a} + |x|^{a+b} |y|^{-n-a}}{|x - y|^{n+a}} \]

\[ + \int_{\mathbb{R}^n} \frac{dx}{|x|^{(n+b-a)/2}} \frac{(|x|^b + |y|^b) |g(x) - g(y)|^2}{|x - y|^{(n+b-a)/2}}. \]

At this point we could simply drop the last term, since it is positive. However, with minimal extra effort we estimate the last term using $|x|^b + |y|^b \geq 2|x|^{b/2} |y|^{b/2}$ and
obtain using (21)

\begin{align*}
(8) \quad t & \geq \alpha_{a,n} \int_{\mathbb{R}^n} dx \frac{|g(x)|^2}{|x|^n} \int_{\mathbb{R}^n} dy \frac{2 - |y|^{\frac{n+a-b}{2}} - |y|^{\frac{n-a-b}{2}}}{(2|y|)^{\frac{n+a-b}{2}} \left( \begin{array}{c} \frac{b}{2} + 1 \\ \frac{n-a-b}{2} \end{array} \right)} + (\psi, |q|^{b/2} H_{a,n} |q|^{b/2} \psi) \\
& = \frac{\alpha_{a,n}}{2} \int_{\mathbb{R}^n} dx \frac{|g(x)|^2}{|x|^n} \int_{\mathbb{R}^n} dy \frac{2|y|^{\frac{n-a}{2}} - |y|^{b/2} - |y|^{b/2}}{(2|y|)^{\frac{n-a}{2}} \- \omega \cdot \epsilon} + (\psi, |q|^{b/2} H_{a,n} |q|^{b/2} \psi) \\
& \geq 0
\end{align*}

assuming – in the last line – that \( \psi \) is not identical zero. The positivity, i.e., the last inequality, follows from the positivity of the numerator of the last integral which is a consequence of the fact that the function \( f(\alpha) := r^\alpha + r^{-\alpha} \) is strictly monotone increasing for positive \( r \) and \( n - a \geq b \).

We now supply the missing case that \( \min\{a, b\} = 2 \). Again we may assume that \( a \leq b \) without loss of generality. An easy calculation shows

\begin{align*}
(9) \quad & \frac{1}{2} (|p|^2 |q|^b + |q|^b |p|^2) \\
& = |q|^2 \left( (|p|^2 + \frac{1}{2} |q|^{-\frac{1}{2}} (|p|^2, |q|^{\frac{1}{2}}) |q|^{-\frac{1}{2}}) |q|^\frac{1}{2} \right) \\
& = |q|^2 \left( |p|^2 - \frac{b^2}{4} |q|^{-2} \right) |q|^\frac{b}{2} \\
& \geq |q|^2 H_{2,n} |q|^{\frac{b}{2}} > 0,
\end{align*}

because \( b \leq n - 2 \). Since the first inequality is actually an equality in the case \( b = n - 2 \), it shows that our assumption \( a + b \leq n \) is critical, since Herbst’s inequalities are sharp.

\[ \square \]

3. **Ground State Representation**

The result of the previous section can be viewed as a warmup for the following result.

**Theorem 2.** Assume \( a, b \in (0, \infty) \) with \( a + b \leq n \), \( \min\{a, b\} \in (0, 2) \), and \( \psi \in C_0^\infty (\mathbb{R}^n) \). Then

\begin{align*}
(10) \quad (\psi, (J_{a,b,n} - L_{a,b,n} |q|^{b-a}) \psi) &= (\psi, |q|^{\frac{b}{2}} H_{a,n} |q|^{\frac{b}{2}} \psi) \\
& + \alpha_{a,n} \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy \frac{(|x|^{\frac{b}{2}} - |y|^{\frac{b}{2}})^2 |\psi(x)| |x|^{\frac{b}{2}} - |\psi(y)| |y|^{\frac{b}{2}})^2}{2|x|^\gamma |x - y|^{n+a} |y|^\gamma}
\end{align*}

where \( \gamma = (n + b - a)/2 \) and

\begin{align*}
(11) \quad L_{a,b,n} &= 2^n \frac{\Gamma \left( \frac{n-b+a}{4} \right) \Gamma \left( \frac{n+b+a}{4} \right)}{\Gamma \left( \frac{a+b-a}{4} \right) \Gamma \left( \frac{a-b-a}{4} \right)}.
\end{align*}

**Monotony of** \( L_{a,b,n} \): Note that \( L_{a,b,n} \) is a strictly monotone decreasing function in \( b \) on the interval \( [0, n - a] \) and vanishes at \( n - a \). The second claim is obvious, since \( \lim_{x \to 0} \Gamma(x) = 0 \). For the first claim we use the log convexity of the Gamma function (Bohr and Mollerup).

**Sharpness:** Formula (10) implies the inequality

\begin{align*}
(12) \quad J_{a,b,n} & > L_{a,b,n} |q|^{b-a} + |q|^{b/2} H_{a,n} |q|^{b/2}
\end{align*}
is strict under the assumptions of the theorem, since the remainder term in (10) vanishes, if and only if $\psi(x) = c|x|^{-\gamma}$ which is only in $L^2$ when $c = 0$. However, the remainder can be made arbitrarily small by a smooth cut-off tending to infinity.

If $a = 2$, equality holds in (12) because of the calculation (9).

**Proof.** Pick $\gamma := \frac{n+b-a}{2}$. By Fourier transform of $|\cdot|^{-\alpha}$ (see (13)), we know that

\begin{equation}
\begin{aligned}
(|\psi|^2|x|^\gamma, J_{a,b,n}|x|^{-\gamma}) &= \frac{1}{2} \int_{\mathbb{R}^n} d\xi |\xi|^a \left( (|\psi(x)|^2 |x|^{\gamma})^{\wedge} (|x|^{b-\gamma})^{\wedge} (\xi) + (|\psi(x)|^2 |x|^{\gamma})^{\wedge} (\xi) \right) \\
&= \frac{1}{2} \left( \frac{B_{n-(\gamma-b)}}{B_{\gamma-b}} \int_{\mathbb{R}^n} d\xi |\xi|^{a-n+\gamma-b} (|\psi(x)|^2 |x|^{\gamma})^{\wedge} (\xi) + \frac{B_{n-\gamma}}{B_{\gamma}} \int_{\mathbb{R}^n} d\xi |\xi|^{a-n+\gamma} (|\psi(x)|^2 |x|^{\gamma})^{\wedge} (\xi) \right) \\
&= \frac{1}{2} \left( \frac{B_{n+a+b-\gamma} B_{n-(\gamma-b)} B_{\gamma-b}}{B_{n-a-\gamma} B_{\gamma-b}} \int_{\mathbb{R}^n} dx |\psi(x)|^2 |x|^{b-a} \right) \\
&= L_{a,b,n} \int_{\mathbb{R}^n} dx |\psi(x)|^2 |x|^{b-a}.
\end{aligned}
\end{equation}

(Note that we refrained from doing obvious mollifications.) We have a similar computation for the operator $|q|^\frac{2}{2} |p|^\alpha |q|^\frac{2}{2} |x|^{-\gamma}$,

\begin{equation}
\begin{aligned}
(|\psi|^2|x|^\gamma, |q|^\frac{2}{2} |p|^\alpha |q|^\frac{2}{2} |x|^{-\gamma}) &= \int_{\mathbb{R}^n} d\xi |\xi|^a \left( (|\psi(x)|^2 |x|^{\gamma})^{\frac{2}{2}} (|x|^{|\gamma|})^{\wedge} (\xi) \right) \\
&= \frac{B_{n-(\gamma-b)}}{B_{\gamma-b}} \int_{\mathbb{R}^n} d\xi |\xi|^{a-n+\gamma-b} (|\psi(x)|^2 |x|^{\gamma})^{\frac{2}{2}} (\xi) \\
&= \frac{B_{n+a+b-\gamma} B_{n-(\gamma-b)} B_{\gamma-b}}{B_{n-a-\gamma} B_{\gamma-b}} \int_{\mathbb{R}^n} dx |\psi(x)|^2 |x|^{b-a} \\
&= 2^a \frac{\Gamma\left(\frac{n+a}{2}\right) \Gamma\left(\frac{n+b}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \int_{\mathbb{R}^n} dx |\psi(x)|^2 |x|^{b-a}.
\end{aligned}
\end{equation}

On the other hand, by using (20) and polarization we can compute the above quantities again and obtain

\begin{equation}
\begin{aligned}
(|\psi|^2|x|^\gamma, J_{a,b,n}|x|^{-\gamma}) &= \frac{1}{2} \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{dx dy}{|x-y|^{n+a}} \left( (|\psi(x)|^2 |x|^{\gamma} - |\psi(y)|^2 |y|^{\gamma})(|x|^{b-\gamma} - |y|^{b-\gamma}) + (|\psi(x)|^2 |x|^{b-\gamma} + |\psi(y)|^2 |y|^{b+\gamma})(|x|^{-\gamma} - |y|^{-\gamma}) \right) \right) \\
&= \frac{1}{2} \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{dx dy}{|x-y|^{n+a}} \left( 2|\psi(x)|^2 |x|^{b} + 2|\psi(y)|^2 |y|^{b} \right. \\
&\quad \left. - |\psi(x)|^2 |x|^{b-\gamma} |y|^{-\gamma} - |\psi(y)|^2 |x|^{b+\gamma} |y|^{\gamma} - |\psi(x)|^2 |x|^{b-\gamma} |y|^{-\gamma} - |\psi(y)|^2 |x|^{b+\gamma} |y|^{\gamma} \right) \right).
\end{aligned}
\end{equation}
By (6) and subtraction and addition of $2\Re \psi(x)\psi(y)|y|^b + 2\Re \psi(x)\overline{\psi(y)}|x|^b$ in the above braces we get
\begin{equation}
(16) \quad \left( |\psi|^2|x|^\gamma, J_{a,b,n}|x|^{-\gamma} \right) \\
= \alpha_{a,n} \Re \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{dxdy}{|x-y|^{n+a}} \left( \psi(x) - \psi(y) \right) \left( |x|^b \psi(x) - |y|^b \psi(y) \right) \\
+ \frac{1}{2} \alpha_{a,n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{dxdy}{|x-y|^{n+a}} \left( 2\Re \psi(x)\psi(y)|y|^b + 2\Re \psi(x)\overline{\psi(y)}|x|^b \\
- |\psi(x)|^2 |x|^\gamma |y|^{b-\gamma} - |\psi(y)|^2 |x|^\gamma |y|^b \right) \\
- |\psi(x)|^2 |x|^b |y|^{-\gamma} - |\psi(y)|^2 |x|^b |y|^b \right)
= (\psi, J_{a,b,n} \psi) - \alpha_{a,n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{2} \frac{(|x|^b + |y|^b) dxdy}{|x-y|^{n+a}} |\psi(x)| |x|^\gamma - |\psi(y)||y|^\gamma^2.
\end{equation}

The sesquilinear form of $|q|^{\frac{a}{2}} |p|^{\alpha} |q|^{\frac{b}{2}}$ is
\begin{equation}
(17) \quad (|\psi|^2|x|^\gamma, |q|^{\frac{a}{2}} |p|^{\alpha} |q|^{\frac{b}{2}}|x|^{-\gamma}) \\
= \alpha_{a,n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{dxdy}{|x-y|^{n+a}} \left( |\psi(x)|^2 |x|^\frac{\gamma}{2} + |\psi(y)|^2 |y|^\frac{\gamma}{2} \right)(|x|^\frac{\gamma}{2} - |y|^\frac{\gamma}{2}) \\
= \alpha_{a,n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{dxdy}{|x-y|^{n+a}} \left( |\psi(x)|^2 |x|^b + |\psi(y)|^2 |y|^b \\
- 2\Re \psi(x)\psi(y)|x|^\frac{b}{2} |y|^\frac{b}{2} + 2\Re \psi(x)\overline{\psi(y)}|x|^\frac{b}{2} |y|^\frac{b}{2} \\
- |\psi(x)|^2 |x|^\frac{b}{2} |y^\gamma - |\psi(y)|^2 |x|^\frac{b}{2} |y|^b \right)
= (\psi, |q|^{\frac{a}{2}} |p|^{\alpha} |q|^{\frac{b}{2}} \psi) - \alpha_{a,n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{dxdy}{|x-y|^{n+a}} \frac{|\psi(x)||x|^\gamma - |\psi(y)||y|^\gamma^2}{|x-y|^{\frac{n+a}{2}}}. \tag{17}
\end{equation}

A combination of the computations (13) to (17) and the ground state representation (21) gives us the desired result. \hfill \Box

\section*{Appendix A. Auxiliary Facts}

For the reader’s convenience we collect some helpful known facts:

\begin{itemize}
  \item \textbf{Fourier transforms of powers:} For $\alpha \in (0, n)$
  \begin{equation}
  B_{\alpha} \mathcal{F}(| \cdot |^{-\alpha}) = B_{n-\alpha} | \cdot |^{-n + \alpha}
  \end{equation}
  with $B_{\alpha} := 2^{\frac{\alpha}{2}} \Gamma(\alpha/2)$ (see, e.g., Lieb and Loss \cite[Theorem 5.9]{LiebLoss2001}).
  \item \textbf{Generalized Hardy Inequalities (Herbst \cite{Herbst1985}):} Assume $\alpha \in (0, n)$. Then, on $H^{\alpha/2}(\mathbb{R}^n)$
  \begin{equation}
  \mathcal{H}_{\alpha,n} := |p|^{\alpha} - 2^{\alpha} \left[ \frac{\Gamma\left(\frac{n+\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)} \right] |q|^{-\alpha} > 0.
  \end{equation}
  The inequality is sharp in the sense that there is no smaller constant in front of $|q|^{-\alpha}$ which allows this inequality on $C_0^\infty(\mathbb{R}^n)$.
  Hardy’s classical inequality is obtained for $\alpha = 2$, Kato’s inequality is the case $\alpha = 1$.
  \item \textbf{Fractional Laplacian:} For $\psi \in H^{\alpha/2}(\mathbb{R}^n)$
  \begin{equation}
  (\psi, |p|^{\alpha} \psi) = \alpha_{a,n} \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy \frac{|\psi(x) - \psi(y)|^2}{|x-y|^{n+a}}
  \end{equation}
  with
  \begin{equation}
  \alpha_{a,n} = \frac{2^{a-1} \Gamma\left(\frac{n+\alpha}{2}\right)}{\pi^{n/2} |\Gamma(-\frac{\alpha}{2})|}.
  \end{equation}
\end{itemize}
Ground State transformed Hardy Operator (Frank et al [1]): For all $\psi \in C^0_0(\mathbb{R}^n \setminus \{0\})$

\begin{equation}
(\psi, H_{a,n}\psi) = \alpha_{a,n} \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy \frac{\left| \psi(x)|x|^{-a} - \psi(y)|y|^{-a} \right|^2}{|x|^{-2a} |x-y|^{n+a} |y|^{-2a}}.
\end{equation}

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