No Broadcasting of Quantum Correlation

Sourav Chatterjee  
Center for Computational Natural Sciences and Bioinformatics,  
International Institute of Information Technology-Hyderabad, Gachibowli, Hyderabad-500032, India.

Sk Sazim  
Institute of Physics, Sainik School Post, Bhubaneswar-751005, Odisha, India

Indranil Chakrabarty  
Center for Security, Theory and Algorithmic Research,  
International Institute of Information Technology-Hyderabad, Gachibowli, Hyderabad, India.

In this work, we extensively study the problem of broadcasting of quantum correlations. This includes broadcasting of quantum entanglement as well as correlations that go beyond the notion of entanglement. In order to have a holistic view of broadcasting, we investigate the problem by starting with most general representation of two qubit mixed states in terms of the Bloch vectors. As a cloning transformation we have used universal symmetric Buzek-Hillery (B-H) cloning transformation both locally (with optimal cloner) and non locally. More specifically, we obtain a set of ranges in terms of Bloch vectors for which broadcasting of entanglement will be possible. To set examples in support of our result, we also calculate the broadcasting range for special states like werner-like state and bell diagonal state. In addition to the idea of broadcasting of entanglement for general two qubit mixed states, we explore the broadcasting of quantum correlations that go beyond entanglement with the help of local and nonlocal cloners. Remarkably, we see that it is impossible to broadcast such correlations by using local quantum copying machines. Taking two different types Buzek-Hillery quantum cloners (state dependent and state independent) we analytically prove the impossibility of broadcasting and present our result in the form of several theorems. This result brings out a fundamental difference between the correlation defined from the perspective of entanglement and the correlation measure which claims to go beyond entanglement (here we use geometric discord as a measure of such correlations).

1. INTRODUCTION

The impossibility to clone quantum states is regarded as one of the most fundamental restriction that nature provides us [1]. In other words the “No cloning theorem” states that there does not exist any quantum mechanical process, which can take two distinct non-orthogonal quantum states (|ψ⟩, |ϕ⟩) into states |ψ⟩ ⊗ |ψ⟩, |ϕ⟩ ⊗ |ϕ⟩ respectively. Even though we cannot copy an unknown quantum state perfectly but quantum mechanics never rules out the possibility of cloning it approximately [1][9]. It also doesn’t prohibit probabilistic cloning as one can always clone an arbitrary quantum state perfectly with some non-zero probability of success [9][10].

In the year 1996, Buzek et al. went beyond the idea of perfect cloning and introduced the concept of approximate cloning with certain fidelity of achievement. Not only that, they also created the state independent quantum copying machine by keeping the fidelity of cloning machine independent of the input state parameters. This machine is popularly known as universal quantum cloning machine (UQCM) [2]. Later this machine was shown to be optimal [3][11]. Other than this state independent quantum cloning machine (QCM), there are also a set of state dependent QCMs (i.e., for which the quality of copies depend on the input state) [3][9]. Various probabilistic quantum cloning machines were also proposed [3][10].

Quantum entanglement [12] which lies at the heart of quantum information theory is the significant reason behind the better achievement of fidelity of QCMs [13]. Not only that, it also plays a significant role in computational and communicational processes like quantum key generation [14][15], secret sharing [16], teleportation [17], superdense coding [18], entanglement swapping [19][20], remote entanglement distribution [21] and many more tasks [22]. The more “pure” is entanglement, more “valuable” is the given two-particle state atleast in the context of quantum information processing tasks. Therefore, to extract pure quantum entanglement from a partially entangled state, researchers had done a lot of work in the past years on purification procedures [23]. In other words, it is possible to compress locally an amount of quantum information. Now at this point a question arises: whether the opposite is true or not i.e. can quantum correlations be “decompressed”? This question was tackled by several researchers [24][26] using the concept of “Broadcasting of quantum inseparability”. Broadcasting is nothing but a local copying of non-local quantum correlations [24]. That is the entanglement originally shared by a single pair is transferred into two less entangled pairs using only local operations as well as nonlocal operations.

In general, the term broadcasting can be used in different contexts. In classical theory one can always broadcast the information, however in quantum theory...
not all states are eligible for broadcasting. In this context, Barnum et al. were the first to show that non-commuting mixed states do not meet the criteria of broadcasting [27]. Quite recently, many authors showed by using sophisticated methods that correlations in a single bipartite state can be locally broadcast if and only if the states are classical (i.e. having classical correlation) [28][31]. In the previous cases, we generally talked about broadcasting of a general quantum state. But when we refer broadcasting of an entangled state, we generally talk about creating more pairs of less entangled state from a given entangled state. This is done by applying local cloning operation on each qubit of the given entangled state, or sometimes by applying global cloning operations on the entangled state itself [1][24][26]. Buzek et al. showed that the decompression of initial quantum entanglement is possible, i.e. from a pair of entangled particles, two less entangled pairs can be obtained by local operation [24]. Further, Bandyopadhyay et al. [26] studied the broadcasting of entanglement and showed that only those universal quantum cloners whose fidelity is greater than \(1/2(1+\sqrt{1})\) are suitable because only then the non-local output states becomes inseparable for some values of the input parameter \(\alpha\). They proved that an entanglement is optimally broadcast only when optimal quantum cloners are used for local copying and also showed that broadcasting of entanglement into more than two entangled pairs is not possible using only local operations. Ghiu investigated the broadcasting of entanglement by using local 1 \(\rightarrow\) 2 optimal universal asymmetric Pauli machines and showed that the inseparability is optimally broadcast when symmetric cloners are applied [32]. In other works, authors investigated the problem of secretly broadcasting of three-qubit entangled state between two distant partners with universal quantum cloning machine and then the result is generalized to generate secret entanglement among three parties [33]. Various other works on broadcasting of entanglement depending on the types of QCMs were also done in the later period [34][35].

In this work, we mainly investigate the problem of broadcasting of quantum correlation. Traditionally, by quantum correlation we refer to entanglement. First part of our study is about broadcasting of quantum entanglement for a general two qubit mixed states. For the first time in the existing research on broadcasting, we provide the broadcasting range for general two qubit state in terms of Bloch vectors. For this we apply the B-H cloning machine, both locally and non-locally. We separately provide broadcasting ranges for werner-like and bell-diagonal as examples. In the second part of our work, while exploring the possibility of broadcasting of quantum correlation that go beyond entanglement (geometric discord), remarkably we find it is impossible to broadcast such correlation with the help of local and nonlocal cloners. We analytically prove this by taking different cloners and present the same in form of different theorems. This is indeed one such result which highlights how fundamentally two approaches to quantify quantum correlations are different.

2. Broadcasting of Quantum Entanglement

In this section, we consider broadcasting of quantum entanglement (inseparability) with help of both local and nonlocal cloning operations. Let us begin with a situation where we have two distant parties A and B and they share a two qubit mixed state \(\rho_{12}\) which can be canonically expressed as [11]:

\[
\rho_{12} = \frac{1}{4} \left[ I_{2 \times 2} \otimes I_{2 \times 2} + \sum_{i=1}^{3} (x_i \sigma_i \otimes I) + \sum_{i=1}^{3} (y_i I \otimes \sigma_i) + \sum_{i,j=1}^{3} (t_{ij} \sigma_i \otimes \sigma_j) \right] = \{ \vec{X}, \vec{Y}, T_{3 \times 3} \}, (1)
\]

where \(\vec{X} = \{ x_1, x_2, x_3 \} \) and \(\vec{Y} = \{ y_1, y_2, y_3 \} \) are Bloch vectors with \(0 \leq \| \vec{X} \| \leq 1\) and \(0 \leq \| \vec{Y} \| \leq 1\). Here, \(t_{ij}\)’s \((i, j = 1, 2, 3)\) are elements of the correlation matrix \(T = [t_{ij}]_{3 \times 3}\), \(\sigma_i = (\sigma_1, \sigma_2, \sigma_3)\) are the Pauli matrices and \(I\) is the identity matrix.

Our basic objective is to broadcast the amount of entanglement present in the given input pair to many pairs. For that we start with a two qubit state \(\rho_{12}\) and then apply cloning operations to produce a composite system \(\hat{\rho}_{1234}\). The broadcasting of quantum entanglement will be possible if we are able to produce more entangled pairs from it. In other words, if local outputs states \(\hat{\rho}_{13}\) and \(\hat{\rho}_{24}\) are separable and nonlocal output states \(\hat{\rho}_{14}\) and \(\hat{\rho}_{23}\) are inseparable, then we will conclude that we are able to create more entangled pairs \(\hat{\rho}_{14}, \hat{\rho}_{23}\) from the initial pair \(\rho_{12}\).

In order to test the separability as well as inseparability for the output states, we generally use Peres-Horodecki criteria. This is a necessary and sufficient condition for detection of entanglement for bipartite systems with dimension \(2 \otimes 2\) and \(2 \otimes 3\).

Peres-Horodecki Theorem [29]: The necessary and sufficient condition for the state \(\rho (2 \otimes 2 \text{ or } 2 \otimes 3)\) to be inseparable is that at least one of the eigenvalues of the partially transposed operator \(\rho^T\) is negative. We can always represent the density operator \(\rho\) in form a density matrix \(\rho_{m,n}^{T}\) and its partially transposed density operator as \(\rho_{m,n}^{T} = \rho_{m,n}^{T}\). Here, \(\rho_{m,n}\) can be mathematically expressed as,

\[
\rho_{m,n} = \langle e_m | (f_{|\mu|}) | e_n \rangle | f_{\nu} \rangle,
\]

where \(|e_m\rangle (|\nu\rangle\rangle\) denotes the orthonormal basis in the Hilbert space of the first (second) subsystem of \(2 \otimes 2\) or \(2 \otimes 3\) dimension of the composite system. This can be
equivalently expressed by the condition that at least one of the two determinants

\[
W_3 = \begin{vmatrix}
\rho_{00,00} & \rho_{01,00} & \rho_{00,10} \\
\rho_{00,01} & \rho_{01,01} & \rho_{00,11} \\
\rho_{10,00} & \rho_{11,00} & \rho_{10,10}
\end{vmatrix}
\]

and

\[
W_4 = \begin{vmatrix}
\rho_{00,00} & \rho_{01,00} & \rho_{00,10} & \rho_{01,10} \\
\rho_{00,01} & \rho_{01,01} & \rho_{00,11} & \rho_{01,11} \\
\rho_{10,00} & \rho_{11,00} & \rho_{10,10} & \rho_{11,10} \\
\rho_{10,01} & \rho_{11,01} & \rho_{10,11} & \rho_{11,11}
\end{vmatrix}
\]

is negative; with

\[
W_2 = \begin{vmatrix}
\rho_{00,00} & \rho_{01,00} \\
\rho_{00,01} & \rho_{01,01}
\end{vmatrix}
\]

being simultaneously non-negative.

### 2.1. Broadcasting of entanglement via local cloning

In this subsection, we deal with the problem of broadcasting of quantum entanglement by using local cloning transformation.

Here, once again we start with a two qubit state \(\rho_{12}\) (given in Eq. \(1\)) shared between two parties \(A\) and \(B\). The first qubit ‘1’ belongs to \(A\) and the second qubit ‘2’ belongs to \(B\). Each of them now individually apply a local copying operation \([4, 24]\) on their own qubit to produce the state \(\tilde{\rho}_{12}\). In general the B-H state independent optimal cloning transformation for a given input state ‘a’ and blank state ‘b’, is given by,

\[
U_{bhsi}^1 |0\rangle_a |\Sigma\rangle_b |Q\rangle_x \rightarrow \sqrt{\frac{2}{3}} |00\rangle_{ab} |0\rangle_x + \sqrt{\frac{1}{3}} |+\rangle_{ab} |1\rangle_x
\]

\[
U_{bhsi}^1 |1\rangle_a |\Sigma\rangle_b |Q\rangle_x \rightarrow \sqrt{\frac{2}{3}} |11\rangle_{ab} |1\rangle_x + \sqrt{\frac{1}{3}} |+\rangle_{ab} |0\rangle_x
\]

where \(|+\rangle_{ab} = \frac{1}{\sqrt{2}} (|0\rangle_{ab} + |1\rangle_{ab})\) and \(x\) denotes the machine state. In our case, the input states \(a, b\) are the qubits ‘1’ and ‘2’.

**Definition 2.1** \([24, 25]\): An entangled state \(\rho_{12}\) is said to be broadcast after the application of local cloning operation \(U_1 \otimes U_2\), each of the type given by Eq. \(5\), on the qubits 1 and 2 respectively, if for some values of the input state parameters,

- the non-local output states between \(A\) and \(B\)

\[
\tilde{\rho}_{14} = Tr_{23} [U_1 \otimes U_2 (\rho_{12})]
\]

\[
\tilde{\rho}_{23} = Tr_{14} [U_1 \otimes U_2 (\rho_{12})]
\]

are inseparable, \(6\)

- the local output states of \(A\) and \(B\)

\[
\tilde{\rho}_{13} = Tr_{24} [U_1 \otimes U_2 (\rho_{12})]
\]

\[
\tilde{\rho}_{24} = Tr_{13} [U_1 \otimes U_2 (\rho_{12})]
\]

are separable. \(7\)

It is quite obvious that the state \(\rho_{12}\) is a general mixed state and is not going to be entangled for all values of the input state parameters. However, the broadcasting of entanglement is going to be relevant only when \(\rho_{12}\) is inseparable. The range of input state parameters for which broadcasting will be possible is always going to be a subclass of the range of the input state parameters for which \(\rho_{12}\) is entangled.

After we obtain the composite system \(\tilde{\rho}_{1234}\), we trace out the qubits 2, 4 and 1, 3 to obtain the local output states \(\tilde{\rho}_{13}\) on \(A\)’s side and \(\tilde{\rho}_{24}\) on \(B\)’s side respectively. They are given in canonical representation by,

\[
\tilde{\rho}_{13} = \left\{ \frac{2}{3} \hat{X}, \frac{2}{3} \hat{X}, \frac{1}{3} I_{3 \times 3} \right\},
\]

\[
\tilde{\rho}_{24} = \left\{ \frac{2}{3} \hat{Y}, \frac{2}{3} \hat{Y}, \frac{1}{3} I_{3 \times 3} \right\}.
\]

In the above equation, \(\hat{X} = \{ x_1, x_2, x_3 \}\) and \(\hat{Y} = \{ y_1, y_2, y_3 \}\) are the Bloch vectors and the correlation matrix \(T = \frac{1}{4} I\) , where \(I\) is the \(3 \times 3\) identity matrix.

Next, we apply Peres-Horodecki criterion to investigate whether these local output states on either side of these two parties are separable or not. After evaluating determinants \(W_2, W_3\) and \(W_4\) (as given in Eq. \(1\) and Eq. \(3\)) we obtain a range involving input state parameters within which the local outputs, \(\tilde{\rho}_{13}\) and \(\tilde{\rho}_{24}\), are
separable. The range obtained for each of these states are given by,
\[
\left(0 \leq \|x\| \leq \frac{3}{4} \text{ and } \|x\| \leq 1 + x_3^2 + x_3 \right. \\
\left. \text{and } \|x\| - x_3^2 > 2 + 2x_3\right)
\]
as well as
\[
\left(0 \leq \|y\| \leq \frac{3}{4} \text{ and } \|y\| \leq 1 + y_3^2 + y_3 \right. \\
\left. \text{and } \|y\| - y_3^2 > 2 + 2y_3\right)
\]
respectively.

Similarly, after tracing out the local output states from the composite system, we have the nonlocal output states \(\tilde{\rho}_{14}\) and \(\tilde{\rho}_{23}\) as
\[
\tilde{\rho}_{14} = \tilde{\rho}_{23} = \left\{ \frac{2}{3} X, \frac{2}{3} Y, \frac{4}{9} T \right\},
\]
where \(X = \{x_1, x_2, x_3\}, Y = \{y_1, y_2, y_3\}\) and \(T = [t_{ij}]\) with \(i, j = \{1, 2, 3\}\) is the 3 \(\times\) 3 correlation matrix.

Similarly, over here also we take the help of Peres-Horodecki criterion to find out the condition under which the nonlocal output states will be inseparable. This condition for inseparability of the states \(\tilde{\rho}_{14}\) and \(\tilde{\rho}_{23}\) involving input state parameters is given by Eq. (51) in Appendix-1.

Now combining each of these two ranges determining the separability of the local states given by Eq. (9) and inseparability of the nonlocal states given by Eq. (51), we obtain the range for broadcasting of entanglement as,
\[
\left(\text{with } W_4 < 0 \text{ or } W_4^I < 0 \text{ and } W_4^I \geq 0\right) \quad \text{and}
\left(0 \leq \|x\| \leq \frac{3}{4} \text{ and } \|x\| \leq 1 + x_3^2 + x_3 \right. \\
\left. \text{and } \|x\| - x_3^2 > 2 + 2x_3\right)
\]
as well as
\[
\left(0 \leq \|y\| \leq \frac{3}{4} \text{ and } \|y\| \leq 1 + y_3^2 + y_3 \right. \\
\left. \text{and } \|y\| - y_3^2 > 2 + 2y_3\right).
\]
Here, the expressions of \(W_4^I, W_4^I\) and \(W_4^I\) are given by Eq. (60).

Next, we give examples of two qubit mixed states in form of werner-like states \([38, 39]\) and bell-diagonal states \([19, 40]\) and separately state their broadcasting ranges.

**Example 2.1: Werner-like States**

First of all, we consider the example of werner-like states. These states can more formally be expressed as,
\[
\rho_{12}^w = \{X, X, T\},
\]
where \(X = \{0, 0, p(\alpha^2 - \beta^2)\}\) are Bloch vectors and the elements \((t_{ij})\) of the correlation matrix \(T = [t_{ij}]_{3 \times 3}\) are \([2p\alpha\beta, 0, 0; 0, -2p\alpha\beta, 0; 0, 0, p]\).

In another way, the werner-like states can also be represented as,
\[
\rho_{12}^w = \left(\frac{1 - p}{4}\right) I \otimes I + p |\varphi\rangle \langle \varphi|,
\]
where \(|\varphi\rangle = \alpha |00\rangle + \beta |11\rangle\). Here, \(p\) denotes the classical randomness \((0 \leq p \leq 1)\). In the above equation, \(\alpha\) and \(\beta\) are the probability amplitudes, satisfying the condition \(\alpha^2 + \beta^2 = 1\). The local output states obtained after applying cloning operation on both the qubits 1 and 2 are given by,
\[
\rho_{13} = \rho_{24} = \left\{X, X, \frac{1}{3} I_{3 \times 3}\right\},
\]
where \(X = \{0, 0, \frac{2p(\alpha^2 - \beta^2)}{3}\}\) are Bloch vectors and the correlation matrix \(T = \frac{1}{3} I_{3 \times 3}\).

From Peres-Horodecki theorem, if follows that by using Eq. (3), the local output states will be separable if either of the following two conditions are satisfied,
\[
\begin{align*}
\bullet & \quad 0 \leq p \leq \frac{\sqrt{3}}{2} \quad \text{and} \quad 0 \leq \alpha^2 \leq 1, \\
\bullet & \quad \frac{\sqrt{3}}{2} < p \leq 1 \quad \text{and} \quad \frac{2p - \sqrt{3}}{4p} \leq \alpha^2 \leq \frac{\sqrt{3} + 2p}{4p}.
\end{align*}
\]

Similarly, we have the nonlocal output states as,
\[
\tilde{\rho}_{14} = \tilde{\rho}_{23} = \{X, X, Z_{1 \times 3} I_{3 \times 3}\}
\]
where \(X = \{0, 0, \frac{2p(\alpha^2 - \beta^2)}{3}\}\) are Bloch vectors and the correlation matrix \(T = Z_{1 \times 3} I_{3 \times 3}\) with \(Z = \frac{8p\alpha\beta}{9}; -\frac{8p\alpha\beta}{9}; \frac{4p}{9}\).

Using Peres-Horodecki theorem, the inseparability range of these nonlocal output states turn out to be,
\[
\frac{3}{4} < p \leq 1 \quad \text{and} \quad \frac{8 - L}{16} < \alpha^2 < \frac{8 + L}{16},
\]
where \(L = \sqrt{48 - \frac{8L}{p} + \frac{16}{p}}\). On merging this inseparable zone along with the separable zone given by Eq. (15), we discover that the broadcasting range is exactly same as the inseparability range given by Eq. (15). In FIG. 2, we depict this broadcastable zone (given by Eq. (17)) among the allowed region of input state parameters given by Eq. (13).

Next we provide two different tables for detailed analysis of the above broadcasting range. In TABLE I we give the broadcasting range of the werner-like states in terms \(p\) for the different values of the input state parameter \(\alpha^2\).
Note 2: Similarly we note that for \( \alpha = \beta = \frac{1}{\sqrt{3}} \) (i.e. when \( |\varphi\rangle_{12} \) is maximally entangled) Eq. (13) reduces to the Werner state \([38]\), for which the range for broadcasting of entanglement becomes,

\[
\frac{3}{4} < p \leq 1.
\]

**Example 2.2: Bell-diagonal States**

Next, we consider the broadcasting of the bell-diagonal states via local copying.

Our input bell-diagonal state to the local cloner can be more formally expressed as,

\[
\rho_{12}^b = \left\{ \overrightarrow{X}, \overrightarrow{X}, Z_{1x3}.I_{3x3} \right\}
\]

where \( \overrightarrow{X} = \{0, 0, 0\} \) are Bloch vectors and the correlation matrix \( T = Z_{1x3}.I_{3x3} \)

The above input bell-diagonal state can be rewritten as \([19, 40]\),

\[
\rho_{12}^b = \frac{1}{4} \left( I \otimes I + \sum_{i=1}^{3} (c_i \sigma_i \otimes \sigma_i) \right)
\]

where \(-1 \leq c_i \leq 1 \) and \( \sigma_i \)’s are the Pauli operators with \( i = \{1, 2, 3\} \). Here the four Bell states \( |\gamma_{mn}\rangle = (|0, n\rangle + (-1)^m |1, 1 \oplus n\rangle) / \sqrt{2} \) represents the eigenstates of \( \rho_{12}^b \) with eigenvalues,

\[
\lambda_{mn} = \frac{1}{4} \left[ 1 + (-1)^m c_1 - (-1)^{(m+n)} c_2 + (-1)^n c_3 \right].
\]

Also, for \( \rho_{12}^b \) to be a valid density operator, its eigenvalues have to be positive, i.e. \( \lambda_{mn} \geq 0 \).

Once again by applying local cloning and tracing out the qubits we get the local output states as:

\[
\tilde{\rho}_{13} = \tilde{\rho}_{24} = \left\{ \overrightarrow{X}, \overrightarrow{X}, \frac{1}{3} I_{3x3} \right\}
\]

where \( \overrightarrow{X} = \{0, 0, 0\} \) are Bloch vectors and the correlation matrix \( T = \frac{1}{3} I_{3x3} \).

It turns out that for these local output states both \( W_3 \) as well as \( W_4 \) given by Eq. (3) are non-negative and independent of the input state parameters \( (c_i \)’s). Hence, \( \tilde{\rho}_{13} \) and \( \tilde{\rho}_{24} \) always remain separable.

On the other hand, the nonlocal outputs are given by,

\[
\tilde{\rho}_{14} = \tilde{\rho}_{23} = \left\{ \overrightarrow{X}, \overrightarrow{X}, \frac{4}{9} Z_{1x3}.I_{3x3} \right\}
\]

where \( \overrightarrow{X} = \{0, 0, 0\} \) are Bloch vectors and the correlation matrix \( T = \frac{4}{9} Z_{1x3}.I_{3x3} \) with \( Z = |c_1; c_2; c_3\).
The inseparability range for these nonlocal output states of the input bell-diagonal state $\rho_{12}$ in terms of $c_i$'s, is given by
\[
\left(-1 \leq c_1 < -\frac{1}{4} \text{ and } (c_1 + c_2 + c_3 < -\frac{9}{4} \text{ or } c_1 - c_3 + \frac{9}{4} < c_2 \leq 1) \right) \text{ or } \left(\frac{1}{4} < c_1 \leq 1 \text{ and } (\frac{9}{4} - c_1 + \frac{9}{4} < c_2) \right). 
\]
along with the condition that $\lambda_{mn} \geq 0$. It is evident that the broadcasting range of the bell-diagonal state is same as the inseparability range in Eq. (25) since the local output states in this case are always separable.

In FIG. 3, we depict the above broadcastable zone (given by Eq. (25)) within the permissible region of the input state parameters, specified by the 3-tuple $(c_1, c_2, c_3)$ from Eq. (24). Now for $-1 \leq c_i \leq 1$, the condition that $\rho_{12}$ is necessarily a positive operator, i.e. $\lambda_{mn} \geq 0$, results in giving a tetrahedral geometrical representation of bell-diagonal states $\mathcal{T}$ whose four vertices are the four Bell states or the eigenstates $|\gamma_{mn}\rangle$. The separable part within the geometry of bell-diagonal states $\mathcal{T}$ comes out to be an octahedron $\mathcal{O}$ which is specified by the relation $|c_1| + |c_2| + |c_3| \leq 1$ or $\lambda_{mn} \leq \frac{1}{2}$. Within the tetrahedron $\mathcal{T}$, the four entangled (inseparable) zones lie outside the octahedron $\mathcal{O}$, one from each vertex of $\mathcal{T}$ with the value of $\lambda_{mn}$ being greatest at the vertex points for each of them [40]. Interestingly, we discover that the broadcastable zone procured by using the above broadcasting condition in Eq. (25) turns out to be cones $\mathcal{C}$'s, fitting as small caps on these entangled zones of the tetrahedron $\mathcal{T}$. It is also consistent with the fact that the maximally entangled states $|\gamma_{mn}\rangle$ lie at the vertices of $\mathcal{T}$, so the broadcastable regions start from that and vanish on the way towards the separable part $\mathcal{O}$. This is because the amount of entanglement keeps decreasing in the same direction. In other words, the states beyond the conic regions ($\mathcal{C}$'s) lack the amount of initial entanglement required to be able to broadcast the same by local cloning operations.

It is interesting to observe that if $c_1 = -1$ then $c_j = c_k$ and if $c_i = 1$ then $c_j = -c_k$ where for each case $-1 \leq c_j (c_k) < -\frac{9}{8}$ or $\frac{1}{2} < c_j (c_k) \leq 1$ with $i \neq j \neq k$ and $i, j, k = \{1, 2, 3\}$. This happens due to the symmetry of the bell-diagonal states and that of the conic broadcasting zones as depicted in FIG. 3. For the same reason, we also find that the four $\mathcal{C}$'s or the conic zones grow symmetrically and uniformly from $c_i$'s = $-1$ (1) and cease to exist for any value equal or beyond $-\frac{5}{8}$ ($\frac{5}{8}$). Hence in the TABLE III, we give the broadcasting range of bell-diagonal states $\rho_{12}$ for different values of the first two input state parameters $c_1$, $c_2$ and variable over the third $c_3$, between the valid zone from $-1$ to $-5/8$ or $5/8$ to 1. In this table, we restrict our results only to the negative range of inputs for $c_1$ and $c_2$ as the result of the broadcasting range in terms of $c_3$ remains unchanged when corresponding positive values of $c_1$ and $c_2$ are substituted in Eq. (25).

| $c_1$ | $c_2$ | Broadcasting Range |
|---|---|---|
| $-1$ | $-1$ | $-1 \leq c_3 \leq -\frac{5}{8}$ |
| $-1$ | $-\frac{5}{8}$ | $-1 \leq c_3 < -\frac{5}{8}$ |
| $-\frac{5}{8}$ | $-\frac{5}{8}$ | $-1 \leq c_3 < -\frac{5}{8}$ |
| $-\frac{5}{8}$ | $1$ | $-1 \leq c_3 < -\frac{5}{8}$ |

TABLE III: Broadcasting ranges obtained with local cloners for different valid values of input state parameters $c_1$ and $c_2$.

2.2. Broadcasting of entanglement via nonlocal cloning

In this subsection, we reconsider the problem of broadcasting of quantum entanglement but this time by using nonlocal cloning transformation. This situation is analogous to the previous case where we have used local cloning operations. Here, the basic idea is that the entire state $\rho_{12}$ (given in Eq. (1)) is with one person and he wishes to have more than one copy of the state. In that process, he applies a global unitary operation $U_{12}$ to
produce $\tilde{\rho}_{1234}$. The cloning transformation, is given by,

$$U_{nl}^{cl} |\Psi_i\rangle_{12} |Z\rangle_{34} |X\rangle_{xy} \rightarrow c |\Psi_i\rangle_{12} |\Psi_i\rangle_{34} |X_i\rangle_{xy}$$

$$+d \sum_{j \neq i} (|\Psi_i\rangle_{12} |\Psi_j\rangle_{34} + |\Psi_j\rangle_{12} |\Psi_i\rangle_{34}) |X_j\rangle_{xy}; \quad (26)$$

where $i,j \in \{1, \ldots, M\}$. Here $M = 4$ and correspondingly the basis vectors are $|\Psi_1\rangle = |00\rangle$, $|\Psi_2\rangle = |01\rangle$, $|\Psi_3\rangle = |10\rangle$ and $|\Psi_4\rangle = |11\rangle$. The subscripts 1 and 2 denote the two qubit state of the system to be copied; 3 and 4 denote the blank states; $x$ and $y$ denote the machine states respectively. The constants $c$ and $d$ are real and in this case, take the values $\sqrt{2} / 3$ and $\frac{1}{10}$ respectively.

**STEP 1:**

![Diagram of cloning process](image)

**STEP 2**:

![Diagram of cloning process](image)

**FIG. 4:** The figure shows the broadcasting of the state $\rho_{12}$ into $\tilde{\rho}_{12}$ and $\tilde{\rho}_{34}$ through application of a nonlocal (global) cloning unitary $U_{12}$.

**Definition 2.2 [24, 25]:** An entangled state $\rho_{12}$ is said to be broadcast after the application of nonlocal cloning operation $U_{12}$ (given by Eq. (26)) together on the qubits 1 and 2, if for some values of the input state parameters,

- the desired output states $\tilde{\rho}_{12} = Tr_{34} [U_{12} (\rho_{12})]$, $\tilde{\rho}_{34} = Tr_{12} [U_{12} (\rho_{12})]$ are inseparable,

- and the remaining output states $\tilde{\rho}_{13} = Tr_{24} [U_{12} (\rho_{12})]$, $\tilde{\rho}_{24} = Tr_{13} [U_{12} (\rho_{12})]$ are separable.

We could have chosen either the diagonal pairs ($\tilde{\rho}_{14}$ & $\tilde{\rho}_{23}$) instead of choosing the pairs: $\tilde{\rho}_{12}$ & $\tilde{\rho}_{34}$ as our desired pairs in the above definition. However, we refrain ourselves from choosing the pairs $\tilde{\rho}_{13}$ & $\tilde{\rho}_{24}$ as the desired pairs.

Once we have the composite system $\tilde{\rho}_{1234}$, we trace out the qubits 3 and 4 to obtain the output state $\tilde{\rho}_{12}$ or the qubits 1 and 2 to obtain $\tilde{\rho}_{34}$. These two density operators are identical and they can be represented as,

$$\tilde{\rho}_{12} = \tilde{\rho}_{34} = \left\{ \frac{3}{5} \mathbf{X}, \frac{3}{5} \mathbf{Y}, \frac{3}{5} \mathbf{T} \right\} \quad (27)$$

where $\mathbf{X} = \{x_1, x_2, x_3\}$, $\mathbf{Y} = \{y_1, y_2, y_3\}$ and $\mathbf{T} = [T_{ij}]$ with $i, j \in \{1, 2, 3\}$, is the $3 \times 3$ correlation matrix.

We apply the Peres-Horodecki criteria to find out the condition under which the above output states will be inseparable. The condition for inseparability of the states $\tilde{\rho}_{12}$ and $\tilde{\rho}_{34}$ involving input state parameters is given by Eq. (53) in Appendix-2.

Next, proceeding in similar manner, we obtain the remaining states $\tilde{\rho}_{13}$ and $\tilde{\rho}_{24}$ by tracing out the qubits 2, 4 and 1, 3 from $\tilde{\rho}_{1234}$ respectively. The output states $\tilde{\rho}_{13}$ and $\tilde{\rho}_{24}$ are given by,

$$\tilde{\rho}_{13} = \left\{ \frac{2}{3} \mathbf{X}, \frac{2}{3} \mathbf{Y}, \frac{1}{3} \mathbf{I}_{3 \times 3} \right\},$$

$$\tilde{\rho}_{24} = \left\{ \frac{2}{3} \mathbf{X}, \frac{2}{3} \mathbf{Y}, \frac{1}{3} \mathbf{I}_{3 \times 3} \right\}. \quad (28)$$

In the above equation, $\mathbf{X} = \{x_1, x_2, x_3\}$ and $\mathbf{Y} = \{y_1, y_2, y_3\}$ are the Bloch vectors and the correlation matrix $\mathbf{T} = \frac{1}{3} \mathbf{I}$, where $\mathbf{I}$ is the $3 \times 3$ identity matrix.

Similarly, here also we apply the Peres-Horodecki criterion to see whether these output states are separable or not. After evaluating determinants $W_2$, $W_3$ and $W_4$ (as given in Eq. (1) and Eq. (3)) we obtain a range involving input state parameters for which the output states, $\tilde{\rho}_{13}$ and $\tilde{\rho}_{24}$, are separable. This range is given by,

$$0 \leq ||x|| \leq \frac{8}{9} \quad \text{and} \quad ||x|| - x_3^2 \leq \frac{4}{3} (1 + x_3)$$

as well as

$$0 \leq ||y|| \leq \frac{8}{9} \quad \text{and} \quad ||y|| - y_3^2 \leq \frac{4}{3} (1 + y_3) \quad (29)$$

respectively.

Now, clubbing each of these two ranges determining the separability given by Eq. (29) and inseparability given by Eq. (53) of the desired states, we obtain the range for broadcasting of entanglement as,

$$\left[ W_3^{nl} < 0 \quad \text{or} \quad W_4^{nl} < 0 \quad \text{and} \quad W_2^{nl} > 0 \right]$$

$$\left[ 0 \leq ||x|| \leq \frac{8}{9} \quad \text{and} \quad ||x|| - x_3^2 \leq \frac{4}{3} (1 + x_3) \right]$$

as well as

$$0 \leq ||y|| \leq \frac{8}{9} \quad \text{and} \quad ||y|| - y_3^2 \leq \frac{4}{3} (1 + y_3) \quad (30)$$
Here, the expressions of $W_{2}^{nl}$, $W_{3}^{nl}$ and $W_{4}^{nl}$ are given by Eq. (32).

Next, in order to exemplify we look into the broadcasting range of two different types of input states: (a) werner-like states [38, 39] and (b) bell-diagonal states [19, 40].

**Example 3.1: Werner-Like State**

Quite similar to the previous section, here also we consider a bigger class of mixed entangled state, the werner-like states given earlier by Eq. (12).

After cloning, the desired output states in this case are given by,

$$\hat{\rho}_{12} = \hat{\rho}_{34} = \left\{ \vec{X}, \vec{X}, Z_{1\times3}, I_{3\times3} \right\}$$  \hspace{1cm} (31)

where $\vec{X} = \{0, 0, \frac{7}{5}p (\alpha^2 - \beta^2)\}$ are Bloch vectors and the correlation matrix $T = Z_{1\times3}, I_{3\times3}$ with $Z = \left[ \begin{array}{ccc} 6p\alpha\beta & -6p\alpha\beta & 6p \end{array} \right]$. The inseparability range of the desired output states is given by,

$$\frac{5}{9} < p \leq 1 \quad \text{and} \quad 1 - H < \alpha^2 < \frac{1}{2} + H,$$  \hspace{1cm} (32)

where $H = \sqrt{\frac{27p^2 + 30p - 25}{144p}}$.

The remaining output states are given by,

$$\hat{\rho}_{13} = \hat{\rho}_{24} = \left\{ \vec{X}, \vec{X}, \frac{1}{3} I_{3\times3} \right\}$$  \hspace{1cm} (33)

where $\vec{X} = \{0, 0, \frac{3}{5}p (\alpha^2 - \beta^2)\}$ are Bloch vectors and the correlation matrix $T = \frac{1}{3} I_{3\times3}$.

These output states will be separable if either of the following two conditions is satisfied,

- $0 \leq p \leq \sqrt{\frac{8}{9(1-2\alpha^2)^2}}$ and $0 \leq \alpha^2 \leq \frac{3-2\sqrt{2}}{6}$ or $\frac{3+2\sqrt{2}}{6} < \alpha^2 \leq 1$,

- $0 \leq p \leq 1$ and $\frac{3-2\sqrt{2}}{6} < \alpha^2 \leq \frac{3+2\sqrt{2}}{6}$.

$$\text{where} \quad \alpha^2 \leq 1$$ \hspace{1cm} (34)

After merging the separability and inseparability conditions given by Eq. (31) and Eq. (32) respectively, the broadcasting range of the werner-like state turns out to be same as the inseparability range and is thus given by Eq. (32).

In FIG. 5, we show this broadcastable zone, given by Eq. (32), among the prescribed region of input state parameters given by Eq. (13).

| Input State Parameter $\alpha^2$ | Broadcasting Range |
|---------------------------------|---------------------|
| 0.2                             | 0.64 < $p$ < 1      |
| 0.4                             | 0.56 < $p$ < 1      |
| 0.5                             | 0.55 < $p$ < 1      |
| 0.6                             | 0.56 < $p$ < 1      |
| 0.8                             | 0.64 < $p$ < 1      |

**TABLE IV:** The broadcasting ranges obtained with a nonlocal cloner for different values of the input state parameter ($\alpha^2$).

In TABLE IV we give the broadcasting range in terms of the classical mixing parameter $p$ for given values of input state parameter $\alpha^2$.

| Input State Parameter $p$ | Broadcasting Range |
|---------------------------|---------------------|
| 0.56                      | 0.42 < $\alpha^2$ < 0.58 |
| 0.65                      | 0.19 < $\alpha^2$ < 0.81 |
| 0.75                      | 0.10 < $\alpha^2$ < 0.90 |
| 0.85                      | 0.06 < $\alpha^2$ < 0.94 |
| 0.95                      | 0.04 < $\alpha^2$ < 0.96 |
| 1                         | 0.03 < $\alpha^2$ < 0.97 |

**TABLE V:** The broadcasting ranges obtained with a nonlocal cloner for different values of the classical mixing parameter ($p$).
Note 3: We note that for $p = 1$ case Eq. (13) reduces to
a non-maximally entangled state, for which the range for
broadcasting of entanglement comes out to be [4][26]
\[
\frac{1}{6} \left( 3 - 2\sqrt{2} \right) < \alpha^2 < \frac{1}{6} \left( 3 + 2\sqrt{2} \right).
\] (35)

Note 4: Again for $\alpha = \beta = \frac{1}{\sqrt{2}}$ (i.e. when $|\psi\rangle_{12}$ is max-
immally entangled) Eq. (13) reduces to the Werner state
[38], for which the range for broadcasting of entanglement
becomes,
\[
\frac{5}{9} < p \leq 1.
\] (36)

Example 3.2: Bell-diagonal states

Lastly, we once again consider the bell-diagonal states
(given earlier by Eq. (20)) as our initial entangled state.

Once the nonlocal cloner is applied to it we have the
desired output states as,
\[
\tilde{\rho}_{12} = \tilde{\rho}_{34} = \left\{ \vec{X}, \vec{X}, \frac{3}{5}Z_{1x3}, I_{3x3} \right\}
\] (37)

where $\vec{X} = \{0, 0, 0\}$ are Bloch vectors and the corre-
lation matrix $T = \frac{1}{3}Z_{1x3}, I_{3x3}$ with $Z = [c_1; c_2; c_3]$.

The inseparability range of the desired output states is
given by,
\[
(6c_1 - 3H + 5)(3H - 6c_2 - 5)(3H - 6c_2 - 5)(3H + 5) < 0 \
\] (38)

where $H = Tr(T)$ where $T = [\lambda_{ij}]_{3x3} = 
[c_1, 0, 0; 0, c_2, 0; 0, 0, c_3]$ along with the condition
that $\lambda_{mn} \geq 0$ from the positivity of input density
operator $\rho_{12}$.

The remaining output states are given by,
\[
\tilde{\rho}_{13} = \tilde{\rho}_{24} = \left\{ \vec{X}, \vec{X}, \frac{1}{5}I_{3x3} \right\}
\] (39)

where $\vec{X} = \{0, 0, 0\}$ are Bloch vectors and the corre-
lation matrix $T = \frac{1}{5}I_{3x3}$.

These output states are independent of the input state
parameter ($c_i$’s) and will be always separable since for
them the $W_3$ and $W_4$ from Eq. (3) comes out to be a
positive number. Hence, the broadcasting range of the
bell-diagonal state is same as the inseparability range as
given in Eq. (38).

Quite analogous to our geometric analysis in local copy-
ing case of the broadcasting region of bell-diagonal state,
in FIG. 6 we depict the above broadcastable zone (given
by Eq. (38)) among the allowed region of the input state
parameters, specified by the 3-tuple ($c_1$, $c_2$, $c_3$) from
Eq. (21). As earlier, we will have the same tetrahedral
representation of bell-diagonal states $\mathcal{T}$ whose four ver-
tices host the four Bell states or the eigenstates $|\gamma_{mn}\rangle$.

The separable part within $\mathcal{T}$ again comes out to be the
same octahedron $\mathcal{O}$, which here also is specified by the
relation $|c_1| + |c_2| + |c_3| \leq 1$. Similar to the previous
case, here within the tetrahedron $\mathcal{T}$, the four entangled
(inseparable) zones are present outside the octahedron
$\mathcal{O}$, extending from each vertex of $\mathcal{T}$ with the value of
$\lambda_{mn}$ being greatest at the vertex points for each of them
[40]. The broadcastable zone procured by using the above
broadcasting condition in Eq. (38) like in previous case
turns out to be cones $\mathcal{C}$’s, fitting as small caps on these
inseparable zones of $\mathcal{T}$. In this occasion, the height of
the cones $\mathcal{C}$’s, contrary to $\mathcal{O}$’s in FIG. 3 are much more
giving the impression that the broadcastable zone extends
much wider when a nonlocal copying machine is used
to copy entanglement for the same input state. Hence,
we once again infer that when the maximally entangled
states $|\gamma_{mn}\rangle$ lie at the vertices of $\mathcal{T}$, the broadcastable
regions start from those and vanish on the way towards
the separable part $\mathcal{O}$. The reason being same that the
amount of entanglement keeps withering in the same di-
rection. In other words, the states beyond the conic re-
gions ($\mathcal{C}$’s) donot posses the amount of initial entangle-
ment needed to be able to broadcast the same by nonlocal
cloning operations.

Similarly as in the case with local cloners, here also if
$c_i = -1$ then $c_j = c_k$ and if $c_i = 1$ then $c_j = -c_k$ while
for each case $-1 \leq c_j (c_k) < -\frac{1}{3}$ or $\frac{2}{3} < c_j (c_k) \leq 1
with i \neq j \neq k$ and $i, j, k = \{1, 2, 3\}$. This happens
due to the symmetry of the bell-diagonal states and that
of the conic broadcasting zones as depicted in FIG. 6.
For the same reason, we also find that the four $\mathcal{O}$’s or the
conic zones grow symmetrically and uniformly from
$c_i$’s $= -1$ (1) and ceases to exist for any value equal
or beyond $-\frac{1}{3}$ (\frac{2}{3}). Hence in TABLE VI we give the
broadcasting range of bell-diagonal states $\rho_{12}$ for different
values of the first two input state parameters $c_1$, $c_2$ and
variable over the third $c_3$, between the valid zone from
$-1$ to $-\frac{1}{3}$ or $\frac{2}{3}$ to 1. In this table, we restrict our results
only to the negative range of inputs for $c_1$ and $c_2$ as the
result of the broadcasting range in terms of $c_3$ remains
unchanged when corresponding positive values of $c_1$ and
$c_2$ are substituted in Eq. (25).

| $c_1$ | $c_2$ | Broadcasting Range |
|------|------|---------------------|
| $\frac{5}{9}$ | $\frac{4}{9}$ | $-1 \leq c_3 \leq -\frac{1}{3}$ |
| $\frac{2}{9}$ | $\frac{7}{9}$ | $-1 \leq c_3 \leq -\frac{1}{3}$ |
| $\frac{7}{9}$ | $\frac{2}{9}$ | $-1 \leq c_3 \leq -\frac{1}{3}$ |
| $\frac{1}{3}$ | $\frac{2}{3}$ | $-1 \leq c_3 \leq -\frac{1}{3}$ |
| $\frac{2}{3}$ | $\frac{1}{3}$ | $-1 \leq c_3 \leq -\frac{1}{3}$ |

TABLE VI: Broadcasting ranges obtained with nonlocal cloners for
different valid values of input state parameters $c_1$ and $c_2$. 
of which a (brown) cone ρ using nonlocal cloning operations within the geometry of bell-diagonal local cloning [26].

It is not surprising that nonlocal cloning will produce a subsystem separately gets entangled with a cloning machine, whereas in local cloning each individual bipartite system as a whole gets entangled with a single implement gives us a much wider broadcasting range for entanglement. So indeed the entanglement of the nonlocal outcome, cloning operations are applied on the individual subsystems only. The entanglement of the nonlocal output comes as a by-product of it. As a result of that the height broadcastable conic region has increased considerably compared to that obtained in FIG. 3 with local cloners.

Interestingly, here we find for the above two cases that the use of a nonlocal cloner despite being difficult to implement gives us a much wider broadcasting range for entanglement. Though local operations (if not unitary) leads to inevitable loss of entanglement however there are no such restrictions for nonlocal operations. In one word, for the nonlocal case, the entanglement of the system is much more copied. In local cloning of entanglement, cloning operations are applied on the individual subsystems only. The entanglement of the nonlocal output comes as a by-product of it. As a result of that the bipartite system as a whole gets entangled with a single cloning machine, whereas in local cloning each individual subsystem separately gets entangled with a cloning machine. A larger amount of entanglement transfer to the machine takes place in the local cloning case. So indeed it is not surprising that nonlocal cloning will produce a wider range for broadcasting of entanglement than the local cloning [26].

3. BROADCASTING OF QUANTUM CORRELATION BEYOND ENTANGLEMENT

In this section, we consider broadcasting of quantum correlations which go beyond the notion of entanglement. More specifically, we analyse the possibility of creating more number of correlated quantum states from an initial quantum state using cloning operations. In the first subsection, we discuss about quantum correlation beyond entanglement. In the second subsection, we briefly describe the cloning machines which we are going to use for the purpose of broadcasting. Finally, in the third subsection, we show that it is impossible to broadcast quantum states using these cloning machines in the form of different theorems.

3.1. Quantum correlation beyond entanglement

Though quantum correlation is synonymous to entanglement for pure two qubit quantum states, however precise nature of the quantum correlations is not well understood for two-qubit mixed states and multipartite states [38, 41]. It has been suggested that quantum correlations go beyond the simple idea of entanglement [32]. The basic idea of quantum discord and other measures is to quantify all types of quantum correlations including entanglement [43–45]. Physically, quantum discord captures the amount of mutual information in multipartite systems which are locally inaccessible [46]. There is another approach to quantify quantum correlations. This is done by distance based measures. Distance-based discord is defined as the minimal distance between a quantum state and all other states with zero discord [47–49]. It is similar to the geometric measure of quantum entanglement [40]. As a result, this kind of measure is also called the geometric measure of quantum discord (or simply geometric discord). Here we use this measure of discord to quantify the amount of quantum correlation beyond entanglement present in between a pair of qubits.

Geometric Discord [49]: Geometric discord or square norm-based discord [47–48] of any general two qubit state ρ (of the form given by Eq. (1)) is defined as,

\[ D_G(\rho) = \min_{\chi} \| \rho - \chi \|^2, \]  

where the minimum is over all possible classical states χ which is of the form \( p_1 |\psi_1\rangle \langle \psi_1| \otimes \rho_1 + p_2 |\psi_2\rangle \langle \psi_2| \otimes \rho_2 \) with probabilities \( p_1 + p_2 = 1 \). Here, \( |\psi_1\rangle \) and \( |\psi_2\rangle \) are two orthonormal basis of subsystems A. The states \( \rho_1 \) and \( \rho_2 \) are two density matrices of subsystem B. In the above equation, \( \| \rho - \chi \|^2 = Tr(\rho - \chi)^2 \) is referred to as the square norm of the Hilbert-Schmidt space. Particularly, for an arbitrary two-qubit system (given by Eq. (1)), an analytical expression of geometric discord has been obtained [48]. The geometric discord of such a two-qubit state ρ is

\[ D_G(\rho) = \frac{1}{4}(\| \vec{x} \|^2 - \| T \|^2 - \lambda_{\max}), \]  

where \( \lambda_{\max} \) is the maximal eigenvalues of matrix \( \Omega = \vec{x} \vec{x}^T - TT^T \). Here the superscript t stands for transpose of a vector or matrix, \( \vec{x} \) is a column vector with norm
\[ \| \vec{x} \| = x_1^2 + x_2^2 + x_3^2. \] The symbol \( T = t_{ij} = Tr(\sigma_i \otimes \sigma_j) \) is a 3 \times 3 matrix, where \( \sigma_{i(j)} \) represent Pauli matrices.

### 3.2. Quantum cloning machines beyond No-cloning theorem

No-cloning theorem forbids the cloning of arbitrary quantum states. However, it doesn’t rule out the possibility of cloning approximately with certain fidelity. More precisely, quantum cloning is a completely positive (CP) trace preserving map between two quantum systems, supported by an ancilla [3, 9].

In this subsection, we briefly describe those cloning machines which we are going to use later for showing the impossibility of broadcasting of a correlation.

#### 3.2.1. Buzek-Hillery (B-H) state dependent cloning machines

The B-H state-dependent cloner was developed from B-H quantum cloning transformation by relaxing the universality condition namely \( \frac{D_{ab}}{\alpha \beta} = 0 \) where \( D_{ab} = Tr \left[ (\rho_{ab}^{(out)} - \rho_{a}^{(in)} \otimes \rho_{b}^{(in)})^2 \right] \) and \( \alpha, \beta \) represents the input state parameter with \( \alpha^2 + \beta^2 = 1 \) [25]. The output states of the cloner is \( \rho_{ab}^{(out)} \) when the input states of the cloner are \( \rho_{a}^{(in)} \) and \( \rho_{b}^{(in)} \) in modes “a” and “b” respectively [23]. In general, the B-H quantum cloning transformation \( U_{bhsd} \) is given by

\[
\begin{align*}
U_{bhsd} |0\rangle |\gamma\rangle |X\rangle &\rightarrow |0\rangle |0\rangle |X_0\rangle + (|0\rangle |1\rangle + |1\rangle |0\rangle) |Y_0\rangle \quad (42) \\
U_{bhsd} |1\rangle |\gamma\rangle |X\rangle &\rightarrow |1\rangle |1\rangle |X_1\rangle + (|0\rangle |1\rangle + |1\rangle |0\rangle) |Y_1\rangle \quad (43)
\end{align*}
\]

where \( |\gamma\rangle \) and \( |X_i\rangle \) \( (i = 0, 1) \) are the blank and machines states respectively. The unitarity constraints give rise to the following conditions on the output machine states,

\[
\begin{align*}
\langle X_i | X_i \rangle + 2 \langle Y_i | Y_i \rangle &= 1, \quad i = 0, 1 \\
\langle Y_0 | Y_1 \rangle &= \langle Y_1 | Y_0 \rangle = 0.
\end{align*}
\]

Also here we assume that

\[
\begin{align*}
\langle X_0 | Y_0 \rangle &= \langle X_1 | Y_1 \rangle = \langle X_1 | Y_0 \rangle = 0.
\end{align*}
\]

The distortion \( D_{ab} \) which describes the distance between the input and output state is given by [25],

\[
D_{ab} = Tr \left[ (\rho_{ab}^{(out)} - \rho_{a}^{(in)} \otimes \rho_{b}^{(in)})^2 \right], \quad (47)
\]

where, \( \rho_{ab}^{(out)} \) describes the output state of the cloner when the input states of the cloner are \( \rho_{a}^{(id)} \) and \( \rho_{b}^{(id)} \) in modes “a” and “b” respectively. Thus, we can make the cloner input state dependent by ensuring that the cloning transformations in Eq. (43) is input state dependent with \( \frac{D_{ab}}{\alpha \beta} \neq 0 \).

#### 3.2.2. Buzek-Hillery (B-H) state independent cloning machines

B-H state independent cloning machines \( U_{bhsi} \) is a \( M \)-dimensional quantum copying machine acting on a state \( |\Psi_i\rangle_{a_0} \) \( (i = 1, ..., M) \). This state is to copied on a blank state \( |0\rangle_{a_1} \). The copier is initially prepared in state \( |X_i\rangle_x \) which subsequently get transformed into another set of vectors \( |X_i\rangle_x \) as a result of application of these cloners. The transformation scheme \( U_{bhsi} \), in this case, is given by [3].

\[
\begin{align*}
U_{bhsi} |\Psi_i\rangle_{a_0} |0\rangle_{a_1} |X_i\rangle_x &\rightarrow c |\Psi_i\rangle_{a_0} |\Psi_i\rangle_{a_1} |X_i\rangle_x \\
&+ d \sum_{j \neq i} (|\Psi_i\rangle_{a_0} |\Psi_j\rangle_{a_1} + |\Psi_j\rangle_{a_0} |\Psi_i\rangle_{a_1}) |X_j\rangle_x, \quad (48)
\end{align*}
\]

where \( i, j = \{1, ..., M\} \) and the coefficients \( c \) and \( d \) are real. From the unitarity and normalization condition we have,

\[
c^2 = \frac{2}{M + 1}; \quad d^2 = \frac{1}{2(M + 1)}.
\]

In particular, we consider \( M = 2^m \) where \( m \) is the number of qubits in a given quantum register. With the help of the scaling property it is ensured that the quality of the cloning doesn’t depend on the particular state which is going to be copied [3].

The cloning transformation given in Eq. (48) reduces to the one used for local copying, given by Eq. (5), for \( M = 2 \) case and to the other one used for nonlocal copying, given by Eq. (26), for \( M = 4 \) case. From Eq. (49) it can be easily observed that the corresponding values of coefficients \( c \) and \( d \) are \( \sqrt{\frac{1}{2}} \) and \( \sqrt{\frac{1}{2}} \) respectively, for the \( M = 2 \) case and are \( \sqrt{\frac{2}{3}} \) and \( \sqrt{\frac{2}{3}} \) respectively, for \( M = 4 \) case.

### 3.3. Broadcasting of correlations beyond entanglement via local and nonlocal cloning operations

In this subsection, we investigate the problem of broadcasting of quantum correlations by using state independent and state dependent local as well as nonlocal B-H cloning machines. Here we present several theorems in support of our claim that quantum correlation in general (at least in the context of geometric discord) cannot be broadcast by using the above mentioned cloning machines. We further conjecture that given a correlation measure \( Q \) and a cloning operation \( U_C \), it is impossible to broadcast total quantum correlation that goes beyond entanglement by these cloning operations. In this work, we explain what we exactly mean by broadcasting of quantum correlation and how it is different from broadcasting of entanglement. We have already stated that for entanglement to be broadcast, the local output
states at the end of the cloning operation have to be separable while the nonlocal ones have to be inseparable. Similarly, if we want to broadcast quantum correlation, the local output states have to be uncorrelated and while the nonlocal ones have to quantum mechanically correlated.

**Definition 3.3.1:** An correlated state $\rho_{12}$ is said to be broadcast after the application of local cloning operation ($U_1 \otimes U_2$ on the qubits ‘1’ and ‘2’ respectively) or nonlocal cloning operation ($U_{12}$ on the two qubit state), if for some values of the input state parameters,

- the non-local output states between A and B,
  
  for the local cloning case:
  
  - $\hat{\rho}_{14} = \text{Tr}_{23} [U_1 \otimes U_2 (\rho_{12})]$,  
  - $\hat{\rho}_{24} = \text{Tr}_{14} [U_1 \otimes U_2 (\rho_{12})]$  
  have a non-vanishing geometric discord, i.e. $D_G(\hat{\rho}_{14}) > 0$ ($D_G(\hat{\rho}_{24}) > 0$) from Eq. (11); whereas

- for the nonlocal cloning case:
  
  - $\hat{\rho}_{12} = \text{Tr}_{34} [U_{12} (\rho_{12})]$,  
  - $\hat{\rho}_{34} = \text{Tr}_{12} [U_{12} (\rho_{12})]$  
  have a non-vanishing geometric discord, i.e. $D_G(\hat{\rho}_{12}) > 0$ ($D_G(\hat{\rho}_{34}) > 0$); and

- the local output states of A and B,

  $\hat{\rho}_{13} = \text{Tr}_{24} [U_1 \otimes U_2 (\rho_{12})]$,  
  $\hat{\rho}_{24} = \text{Tr}_{13} [U_1 \otimes U_2 (\rho_{12})]$  
  have a vanishing geometric discord, i.e. $D_G(\hat{\rho}_{13}) = 0$ ($D_G(\hat{\rho}_{24}) = 0$).

Here, we intend to analyze whether by any such copy operations, we can broadcast correlation which lie beyond entanglement starting with a most general two qubit state as the input to the cloner. More specifically, for any permitted range of input state parameter, whether the output local two qubits states have zero discord when the discord between the nonlocal ones are non-vanishing. For this purpose, we consider two specific types of QCMs, illustrated in the above subsection, namely: (a) B-H state independent optimal cloner and (b) B-H state dependent cloner.

### 3.3.1. Broadcasting of correlations using Buzek-Hillery (B-H) local cloners

Here we use B-H state independent and state dependent cloning operation locally (given by Eq. (5) and Eq. (43)) and we find that it is impossible to broadcast quantum correlations by such methods.

**Lemma 3.1** The correlations beyond entanglement within any state of the form $\rho_{12}$ (given by Eq. (1)) cannot be broadcast into two lesser correlated states $\rho_{14}$ and $\rho_{23}$ (given by Eq. (6)) if B-H state independent optimal cloning transformation ($U_{bhsi}^l$) is used to clone the input qubits locally.

**Proof:** When B-H state independent cloning transformation $U_{bhsi}^l$ (given by Eq. (5)) is applied to locally clone the qubits ‘1 $\rightarrow$ 3’ and ‘2 $\rightarrow$ 4’ of the input state $\rho_{12}$ (given in Eq. (1)), we have the local states $\hat{\rho}_{13}$, $\hat{\rho}_{24}$ (Eq. (7)) and the nonlocal outputs $\hat{\rho}_{14}$, $\hat{\rho}_{23}$ (Eq. (6)). The exact form of local output states are given by Eq. (8) and that of the nonlocal output states by Eq. (10).

Now the geometric discord $D_G$, calculated using Eq. (11), of the local output states:

(a) $\hat{\rho}_{13}$ on A’s side turn out to be $D_G(\hat{\rho}_{13}) = \frac{1}{5} (1 + 2||x||)$ and

(b) $\hat{\rho}_{24}$ on B’s side turn out to be $D_G(\hat{\rho}_{24}) = \frac{1}{5} (1 + 2||y||)$, 

which never vanishes for all, $0 \leq ||x|| \leq 1$ and $0 \leq ||y|| \leq 1$, with $||x|| = x_1^2 + x_2^2 + x_3^2$ and $||y|| = y_1^2 + y_2^2 + y_3^2$.

However, as given by the definition of broadcasting of correlations, it is necessary that $D_G(\hat{\rho}_{13}) = 0$ ($D_G(\hat{\rho}_{24}) = 0$) for some allowed range of input state parameters. Thus, it is proved that we cannot broadcast correlations beyond entanglement by using state independent optimal B-H local copying machine as the local cloner.

**Lemma 3.2** It is impossible to broadcast correlations beyond entanglement within any state of the form $\rho_{12}$ (given by Eq. (1)) into two lesser correlated states $\rho_{14}$ and $\rho_{23}$ (given by Eq. (6)) if B-H state dependent cloning transformation ($U_{bhsd}^l$) is used to clone the input qubits locally.

**Proof:** If the B-H state dependent cloning transformation $U_{bhsd}^l$ (given by Eq. (43)) is applied to locally clone the qubits ‘1 $\rightarrow$ 3’ and ‘2 $\rightarrow$ 4’ of an input most general mixed quantum state $\rho_{12}$ (given in Eq. (1)), then we have the local output states as,

(a) $\hat{\rho}_{13} = \{(1 - 2\lambda)^2X, (1 - 2\lambda)^2Z, (1 - 2\lambda)^2Z_1x, I_3x, 3\}$; where $Z = [2\lambda; 2\lambda; 1 - 4\lambda]$ with $X = [x_1, x_2, x_3]$ and

(b) $\hat{\rho}_{24} = \{(1 - 2\lambda)^2Y, (1 - 2\lambda)^2Y, (1 - 2\lambda)^2Z_1x, I_3x, 3\}$; where $Z = [2\lambda; 2\lambda; 1 - 4\lambda]$ with $Y = [y_1, y_2, y_3]$ and the nonlocal output states as,

$\hat{\rho}_{14} = \hat{\rho}_{23} = \{(1 - 2\lambda)^2X, (1 - 2\lambda)^2Y, (1 - 2\lambda)^2T_3x\};$ where $T = [t_{ij}]$ with $i, j = \{1, 2, 3\}$, $X = \{x_1, x_2, \frac{1}{2}(1 - 2\lambda)x_3\}$ and $Y = \{y_1, y_2, \frac{1}{2}(1 - 2\lambda)y_3\}$.  

Here $I$ represent the identity matrix and $T$ denotes the correlation matrix.
The geometric discord $D_G$, of the local output states are given by:

\[
\begin{align*}
(a) & \quad \hat{\rho}_{13} \text{ on } A's \text{ side turn out to be } D_G(\hat{\rho}_{13}) = \\
&\quad \frac{1}{2} \left( 1 + (1 - 2\lambda)^2 \|x\|- 8\lambda + 20\lambda^2 \right) \quad \text{and} \\
(b) & \quad \hat{\rho}_{24} \text{ on } B's \text{ side turn out to be } D_G(\hat{\rho}_{24}) = \\
&\quad \frac{1}{2} \left( 1 + (1 - 2\lambda)^2 \|y\|- 8\lambda + 20\lambda^2 \right),
\end{align*}
\]

which always remains non-vanishing for $0 \leq \lambda \leq \frac{1}{2}$.

\[
\begin{align*}
\text{Theorem 3.3} & \quad \text{Given a two qubit general mixed state } \rho_{12} \text{ (of the form in Eq. (1)) and B-H state independent local cloning transformations (} U_{bhsi}^{nl} \text{), it is impossible to broadcast the correlation within the state } \rho_{12} \text{ into two quantum correlated states:} \\
&\quad \text{(a) } \hat{\rho}_{13} = Tr_{13}[U_1 \otimes U_2 (\rho_{12})] \quad \text{and} \\
&\quad \text{(b) } \hat{\rho}_{23} = Tr_{12}[U_1 \otimes U_2 (\rho_{12})],
\end{align*}
\]

\[
\text{(given by Eq. (6)) by local application of the cloners on each of these qubits.}
\]

**Proof:** Therefore, by combining Lemma 3.1 and Lemma 3.2 we can conclude that it is indeed impossible to broadcast correlations beyond entanglement by applying B-H cloning transformations of state independent or state dependent type locally on a mixed state of the form of $\rho_{12}$ (given by Eq. (1)), into two lesser correlated states of type $\hat{\rho}_{13}$ and $\hat{\rho}_{23}$ (given by Eq. (6)). This proves Theorem 3.3.

3.3.2. Broadcasting of correlations using Buzek-Hillery (B-H) nonlocal cloners

In this approach, we use B-H state independent nonlocal cloning operation (given by Eq. (26)) and we find that, here too it is impossible to broadcast quantum correlations by such approaches.

\[
\begin{align*}
\text{Theorem 3.4} & \quad \text{Given a two qubit general mixed state } \rho_{12} \text{ (of the form in Eq. (1)) and B-H state independent nonlocal cloning transformations (} U_{bhsi}^{nl} \text{), it is impossible to broadcast the correlation within the state } \rho_{12} \text{ into two quantum correlated states:} \\
&\quad \text{(a) } \hat{\rho}_{12} = Tr_{34}[U_{12} (\rho_{12})], \\
&\quad \text{(b) } \hat{\rho}_{34} = Tr_{12}[U_{12} (\rho_{12})],
\end{align*}
\]

\[
\text{(given by Eq. (27)) by nonlocal application of the cloner on the input two qubit state.}
\]

**Proof:** When B-H state independent nonlocal cloning transformation $U_{bhsi}^{nl}$ (given by Eq. (26)) is applied to clone the qubits ‘12 → 34’ of an input most general mixed quantum state $\rho_{12}$ (given in Eq. (1)), then we have the local output states as $\hat{\rho}_{13}$, $\hat{\rho}_{24}$ (Eq. (28)) and the nonlocal output states as $\hat{\rho}_{12}$, $\hat{\rho}_{34}$ (Eq. (26)). The exact form of local output states are given by Eq. (28) and that of the nonlocal output states are given by Eq. (27).

Now the geometric discord $D_G$ for these output states are:

\[
\begin{align*}
&\quad (a) \quad \hat{\rho}_{13} \text{ on } A's \text{ side turn out to be } D_G(\hat{\rho}_{13}) = \\
&\quad \frac{1}{50} (2 + 9 \|x\|), \\
&\quad (b) \quad \hat{\rho}_{24} \text{ on } B's \text{ side turn out to be } D_G(\hat{\rho}_{24}) = \\
&\quad \frac{1}{50} (2 + 9 \|y\|),
\end{align*}
\]

which never vanishes since $0 \leq \|x\| \leq 1$ and $0 \leq \|y\| \leq 1$; with $\|x\| = x_1^2 + x_2^2 + x_3^2$ and $\|y\| = y_1^2 + y_2^2 + y_3^2$.

For broadcasting of correlations, as per our definition, it is necessary that $D_G(\hat{\rho}_{13}) = 0 \quad (D_G(\hat{\rho}_{24}) = 0)$ for some allowed range of input state parameters. So, it is shown that we cannot broadcast correlations beyond entanglement by using state independent B-H nonlocal copying machine $U_{bhsi}^{nl}$ as the cloner.

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[3] D. Bruß, P.D. DiVincenzo, A. Ekert, C. A. Fuchs, C. Macchiavello and J. A. Smolin, Phys. Rev. A 57, 2368 (1998).
Appendix-1: Inseparability range of nonlocal outputs obtained using local cloners

In this part, we evaluate the determinants $W_2$, $W_3$ and $W_4$ (as given in Eq. (4) and Eq. (3)) of the Peres-Horodecki criterion for the states $\rho_{14}$ and $\rho_{23}$ given by Eq. (10), and denote them as $W_2^l$, $W_3^l$ and $W_4^l$ respectively. The mathematical expressions of these determinants are given as follow,

$$W_2^l = -\frac{1}{6^4} \left[ 4 \sum_{i=1}^{2} (t_{3i} + 3y_i)^2 - Q (Q - 12x_3 - 18) \right],$$

$$W_3^l = 3^6 + 2^6 \left[ 2 \left( \delta + \sum_{i,j=1}^{t_{i,j}t_{3i}t_{3j}} + t_{33} \left( \sum_{i=1}^{3} t_{ij}^2 + \sum_{i,j=1}^{2} - \sum_{i,j=1}^{2} t_{ij}^2 + 3 \left( \sum_{i,j=1}^{2} (t_{ij} - t_{jj})(x_{i}t_{3i} + y_{i}t_{3i}) \right) \right) \right] + \frac{3}{2} \left( \sum_{i,j=1}^{3} (x_{i} + y_{3}) \right),$$

$$W_4^l = \frac{1}{6^8} \left[ 3^8 + 2^6 \left\{ (||x|| - ||y||)^2 - \left( \sum_{i,j=1}^{t_{ij}^2} - 4 \left( \sum_{i,j=1}^{t_{ij}^2} y_{ij} \right) + \frac{9}{4} (||x|| - ||y||) \right) \right\} + \frac{2}{9} \left( \sum_{i,j=1}^{t_{ij}^2} t_{ij}^2 + \sum_{i=1}^{t_{2i}^2t_{3i}} \right) \right] + \sum_{i,j=1}^{3} \left( (1 - \delta)_{i,j}^2p_{j}^2\right) t_{ij}^2 + 4 \left( \delta - \sum_{i,j=1}^{3} (x_{i}x_{j}^2 - \sum_{i,j=1}^{3} t_{ij}t_{jp} + y_{ip}t_{j}^2p_{j} + \sum_{i,j=1}^{3} x_{i}y_{j} \right),$$

$$\sum_{i,j=1}^{3} \left( (1 - \delta)_{i,j}^2p_{j}^2\right) t_{ij}^2 + 4 \left( \delta - \sum_{i,j=1}^{3} (x_{i}x_{j}^2 - \sum_{i,j=1}^{3} t_{ij}t_{jp} + y_{ip}t_{j}^2p_{j} + \sum_{i,j=1}^{3} x_{i}y_{j} \right),$$

$$W_3^l < 0 \text{ or } W_4^l < 0 \text{ and } W_2^l \geq 0.$$}

Appendix-2: Inseparability range of desired outputs obtained using nonlocal cloners

Here, we again evaluate the determinants $W_2$, $W_3$ and $W_4$ (as given in Eq. (4) and Eq. (3)) of the Peres-Horodecki criterion for the states $\rho_{12}$ and $\rho_{23}$ given by Eq. (27), and denote them as $W_2^{nl}$, $W_3^{nl}$ and $W_4^{nl}$ respectively. The mathematical expressions of these determinants turn out to be the following,

$$W_2^{nl} = \frac{1}{20^2} \left[ 25 - 9 \left( \sum_{i=1}^{3} t_{3i}^2 - x_{3}^2 \right) + 30x_{3} - 9 \sum_{i=1}^{3} y_{i} (2t_{3i} + y_{i}) \right],$$

where $Q = 4t_{33} + 6x_3 + 6y_3 + 9$ and $\delta = t_{33} (t_{12}t_{23} - t_{13}t_{22}) + t_{32} (t_{33} - t_{11}t_{23} + t_{33} (t_{11}t_{23} - t_{12}t_{22})$. The symbols $||x|| = \sum_{i=1}^{3} t_{3i}^2$ or $||y|| = \sum_{i=1}^{3} t_{3i}^2$ represents the norms of the respective vectors $X$ and $Y$ given in Eq. (10). These nonlocal outputs $\rho_{14}$ and $\rho_{23}$ will be inseparable when,

$$W_2^{nl} < 0 \text{ or } W_4^{nl} < 0 \text{ and } W_2^{nl} \geq 0.$$
\[ W_{3n} = \frac{9}{20^3} \left[ \left\{ (z - 10 - 3y_3) \left( \sum_{i=1}^{3} (t_{3i} + y_i)^2 - \left( \frac{5 + 3x_3}{3} \right)^2 \right) \right\} - \left\{ (5 - 3t_{33}) \left( \sum_{i=1}^{3} t_{3i}^2 + \sum_{i=1}^{3} x_i^2 \right) + (10 - 6t_{33}) \right\} \right] \\
\sum_{i=1}^{2} t_{3i} x_i + 3x_3 \sum_{i=1}^{2} (t_{3i} + x_i)^2 + z \sum_{i,j=1;i \neq j}^{2} t_{ij}^2 - 6 \sum_{i,j=1;i \neq j}^{2} (t_{3i} + x_i) + (t_{3j} + y_j) \sum_{i,j=1;i \neq j}^{2} t_{ij} + 2t_{12}t_{23} z + \sum_{i=1}^{2} t_{ii} (-1)^{i+1} \left( \sum_{j=1}^{2} (t_{j3} + x_j) (t_{j3} + y_j) (-1)^i \right) + 3 \left( \sum_{i=1}^{3} \sum_{i \neq 3} \sum_{i \neq j} t_{i1}^2 (-1)^{i+j} + \sum_{i=1}^{3} t_{2i}^2 (-1)^{i+j} \right) - \sum_{i=1}^{3} x_i^2 \\
-2 \sum_{i=1}^{3} t_{3i} x_i + 2 (t_{12}t_{23} - t_{11}t_{22}) x_3 \right] \]

\[ W_{4n} = \frac{1}{20^4} \left[ -25 \left\{ \left( \sum_{i,j=1;i \neq j}^{3} t_{ij}^2 + \sum_{i=1}^{3} x_i^2 + x_1^2 \right) \right\} + 9 \left\{ 9t_{11}^4 + 2t_{11}^2 (-1)^{(7n)+1} - \sum_{i=1}^{3} \sum_{i,j=1;i \neq j}^{3} t_{ij}^2 \left( (-1)^{(7n)+1} + \sum_{i=1}^{3} y_i^2 (-1)^{(7n)+1} \right) \right\} + 9t_{12}^4 \left( \sum_{i=1}^{3} t_{1i}^2 (i-1) - 2 \left( \sum_{i=1}^{2} t_{2i}^2 (-1)^{i+1} \right) + \sum_{i=1}^{3} \sum_{i,j=1;i \neq j}^{3} t_{ij}^2 (-1)^{(7n)+1} - \sum_{i=1}^{3} y_i^2 (-1)^{(7n)+1} \right) \right\} + \sum_{i=1}^{3} t_{3i} y_i \right] \\
+ \sum_{i,j=2;i \neq j}^{3} t_{ij} x_j y_i - \sum_{i,j=2;i \neq j}^{3} t_{ij} x_j y_i - 2 \sum_{i=2}^{3} \sum_{i,j=1;i \neq j}^{3} t_{ij} - y_1 y_i \right) \right] \right\} + t_{12} \left\{ \sum_{i,j=1;i \neq j}^{3} t_{ij} \right\} \right] \right\} \right\} + 9 \left\{ 9 \left( t_{13} + \sum_{i=1}^{3} t_{3i}^2 \right) + \sum_{i=1}^{3} t_{3i} + \sum_{i=1}^{3} x_i^4 + \sum_{i=1}^{3} y_i^4 \right\} + 18 \left( \sum_{i=1}^{3} (x_1^2 - x_3^2) + \sum_{i=1}^{3} t_3 x_i^2 + \sum_{i=1}^{3} t_3 x_i^2 (-1)^{i+1} - \sum_{i=1}^{3} x_i^2 \right) y_2 + \sum_{i=1}^{3} t_{3i} \sum_{i=1}^{3} t_{3i}^2 + \sum_{i=1}^{3} t_{2i}^2 t_{3i}^2 + x_2^2 (x_2^2 + x_3^2) + y_1^2 y_2^2 + \sum_{i=2}^{3} x_i^2 + \sum_{i=1}^{3} x_i^2 + \sum_{i=1}^{3} y_i^2 \right\} + 120 (t_{22} x_2 y_2 \left\{ \sum_{i=1}^{3} t_{3i} y_i \right\} + 18 \left\{ \left( \sum_{i=1}^{3} t_{2i}^2 + t_{2i}^2 + t_{2i}^2 \right) (||x|| - 2x_2) + \sum_{i=1}^{3} t_{3i}^2 (-1)^{i+1} + \sum_{i=1}^{3} t_{3i}^2 (-1)^{i+1} + \sum_{i=1}^{3} t_{3i}^2 (-1)^{i+1} \right\} + 25 t_3 \left( t_{33} (x_2 x_3 + t_{23} x_2) + x_1 \right) + t_{32} \left( t_{31} y_1 y_2 + t_{23} x_2 + t_{23} x_1 y_1 \right) + 9 y_3 \sum_{i=1}^{3} t_{3i}^2 (-1)^{i} + \sum_{i=1}^{3} t_{3i} - \sum_{i=1}^{3} y_i^2 - ||x|| + ||y|| \right\} + 24 \left( \sum_{i=1}^{3} t_{3i} x_i \right) - 36 \left( \sum_{i=2}^{3} t_{3i} y_i - t_{22} (t_{31} x_1 - t_{23} y_2) \right) + 9 y_3 \sum_{i=1}^{3} t_{3i}^2 (-1)^{i} - \sum_{i=1}^{3} t_{3i}^2 (-1)^{i+1} + \sum_{i=1}^{3} t_{3i}^2 (-1)^{i+1} + 3x_1 \sum_{i=2}^{3} t_{3i} x_i + 3y_1 \sum_{i,j=2;i \neq j}^{3} t_{ij} x_j (-1)^{i} + 3y_2 \sum_{i,j=2;i \neq j}^{3} t_{ij} x_j (-1)^{i} + 5x_1 y_3 \right\} + 25 \left( 3 \sum_{i=1}^{3} t_{2i}^2 (-1)^{4i} + \sum_{i=1}^{3} t_{3i}^2 (-1)^{4i} + \sum_{i=1}^{3} x_i^2 (-1)^{7i+1} \right) - \sum_{i=1}^{3} y_i (-1)^{4i} \right\} \right] \]
where \( z = 5 + 3t_{33} + 3x_3 \). The symbols \( \|x\| = x_1^2 + x_2^2 + x_3^2 \) or \( \|y\| = y_1^2 + y_2^2 + y_3^2 \) represents the norms of the respective vectors \( \vec{X} \) and \( \vec{Y} \) given in Eq. (27) and \( \% \) denotes the modulo operator. These desired output states \( \tilde{\rho}_{12} \) and \( \tilde{\rho}_{34} \) will be inseparable when,

\[
W_{nl}^3 < 0 \text{ or } W_{nl}^4 < 0 \text{ and } W_{nl}^2 \geq 0.
\]

(53)