\textbf{\textit{\(\rho\)}-White noise solution to 2D stochastic Euler equations}

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\textbf{Abstract}
A stochastic version of 2D Euler equations with transport type noise in the vorticity is considered, in the framework of Albeverio–Cruzeiro theory (Commun Math Phys 129:431–444, 1990) where the equation is considered with random initial conditions related to the so called enstrophy measure. The equation is studied by an approximation scheme based on random point vortices. Stochastic processes solving the Euler equations are constructed and their density with respect to the enstrophy measure is proved to satisfy a Fokker–Planck equation in weak form. Relevant in comparison with the case without noise is the fact that here we prove a gradient type estimate for the density. Although we cannot prove uniqueness for the Fokker–Planck equation, we discuss how the gradient type estimate may be related to this open problem.

\textbf{Keywords} White noise · 2D Euler equations · Multiplicative noise · Fokker–Planck equation · Gradient estimates

\textbf{Mathematics Subject Classification} 60H15 · 60H40 · 35Q31 · 35Q84 · 76D05

\section{1 Introduction}
This work is devoted to the investigation of 2D Euler equations with a Gaussian distributed initial condition and perturbed by multiplicative noise in transport form. Besides its intrinsic interest as a model of stochastic fluid mechanics, this topic lies

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at the intersection of several research lines of recent interest, a fact that was our main motivation. On one side, relevant classes of PDEs, of dispersive type, have been solved recently in spaces of low regularity, replacing arbitrary initial conditions by almost every initial condition with respect to a suitable measure, see [30] for a review. Solvability of deterministic equations in infinite dimensional spaces in a probabilistic sense with respect to initial conditions has also been approached by means of the associated infinite dimensional continuity equation, see for instance [4,5,9,11,12,15,20]. On the other side, multiplicative transport noise has been proven to regularize certain singular PDEs, see the review [16]; in particular, related to the present work, it regularizes the dynamics of Euler point vortices, which is well posed in the deterministic case only for almost every initial configuration with respect to Lebesgue measure, while it is for all initial conditions when a suitable noise is added to Euler equations, see [13,19] for a related result on Vlasov–Poisson equations. That a suitable transport noise regularizes 2D Euler equations is an open problem, see [17].

The approach presented here does not solve this question yet but poses the basis for further investigations on this regularization by noise question, due to the gradient type estimates on the density. Essential for this purpose is to go beyond pure white noise solutions and consider what we call \( \rho \)-white noise solutions, since it is precisely in the investigation of uniqueness for the Fokker–Planck equation satisfied by \( \rho_t \) (the density of the law of the process with respect to the underlying Gaussian measure) that lies our hope to attack the uniqueness problem. In the Appendix we present an informal discussion about the fundamental role of a gradient estimate in the attempt to prove uniqueness of \( \rho_t \). It happens moreover that the Kolmogorov and the Fokker–Planck equations have the same structure, up to a change of sign in the drift term, due to the \( L^2 \) duality with respect to the underlying Gaussian measure. The results, in particular the gradient estimate, proved in the paper for \( \rho_t \) then hold true also for the solution of Kolmogorov equation and may provide other tools to approach uniqueness of \( \rho_t \), as explained in the Appendix. As a final comment on this topic, a uniqueness result for this stochastic 2D Euler equations would be interpreted as a regularization by noise result, because at this poor level of regularity of solutions no uniqueness result is known (the uniqueness theory for 2D Euler equations stops at the level of \( L^\infty \) vorticity fields).

Let us now describe in detail the contribution of the present paper to the previous range of topics. Let \( \mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2 \) be the two dimensional torus. The two dimensional Euler equations in vorticity form reads as

\[
\partial_t \omega_t + u_t \cdot \nabla \omega_t = 0, \quad \omega|_{t=0} = \omega_0, \quad (1.1)
\]

where \( u_t = (u^1_t, u^2_t) \) is the divergence free velocity field and \( \omega_t = \partial_2 u^1_t - \partial_1 u^2_t \) is the vorticity. We refer the reader to the introduction of [18] for a list of well posedness results on (1.1) under different regularity assumptions on \( \omega_0 \).

We consider the Eq. (1.1) perturbed by random noises:

\[
\mathrm{d}\omega_t + u_t \cdot \nabla \omega_t \mathrm{d}t + \sum_{j=1}^\infty \sigma_j \cdot \nabla \omega_t \circ \mathrm{d}W^j_t = 0, \quad (1.2)
\]
where \( \{ \sigma_j : j \in \mathbb{N} \} \) and \( \{ (W_t^j)_{t \geq 0} : j \in \mathbb{N} \} \) are, respectively, a family of smooth divergence free vector fields on \( \mathbb{T}^2 \) and a family of independent real Brownian motions defined on a filtered probability space \( (\Theta, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}) \). Equations of the form (1.2) have been studied in [10], where it was proved that, under suitable conditions on the vector fields \( \{ \sigma_j : j \in \mathbb{N} \} \), if the initial condition \( \omega_0 \in L^\infty(\mathbb{T}^2) \), then (1.2) admits a pathwise unique \( L^\infty \)-solution. The weak formulation of (1.2) is

\[
\langle \omega_t, \phi \rangle = \langle \omega_t, u_t \cdot \nabla \phi \rangle \, dt + \sum_{j=1}^{\infty} \langle \omega_t, \sigma_j \cdot \nabla \phi \rangle \circ dW_t^j, \quad (1.3)
\]

where \( \phi \in C^\infty(\mathbb{T}^2) \) and \( \langle , \rangle \) is the duality between the space \( C^\infty(\mathbb{T}^2)' \) of distributions and \( C^\infty(\mathbb{T}^2) \). The Itô form of the above equation is given by

\[
\langle \omega_t, \phi \rangle = \langle \omega_t, u_t \cdot \nabla \phi \rangle \, dt + \sum_{j=1}^{\infty} \langle \omega_t, \sigma_j \cdot \nabla \phi \rangle \, dW_t^j + \frac{1}{2} \sum_{j=1}^{\infty} \langle \omega_t, \sigma_j \cdot \nabla (\sigma_j \cdot \nabla \phi) \rangle \, dt.
\]

This equation can be rewritten in the weak vorticity formulation by using the Biot–Savart kernel \( K(x - y) \) on the torus. It is known that (see [27]) \( K \) is smooth for \( x \neq y \), \( K(y - x) = -K(x - y) \) and \( |K(x - y)| \leq C/|x - y| \) for \( |x - y| \) small enough. By the Biot–Savart law,

\[
u_t(x) = \int_{\mathbb{T}^2} K(x - y) \, \omega_t(dy).
\]

Therefore,

\[
\langle \omega_t, \nabla \phi \rangle = \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} K(x - y) \cdot \nabla \phi(x) \, \omega_t(dx) \, \omega_t(dy).
\]

Since \( K(y - x) = -K(x - y) \), we can rewrite the above quantity in the symmetric form:

\[
\langle \omega_t, \nabla \phi \rangle = \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} H_\phi(x, y) \, \omega_t(dy) \, \omega_t(dx) = \langle \omega_t \otimes \omega_t, H_\phi \rangle,
\]

where

\[
H_\phi(x, y) = \frac{1}{2} K(x - y) \cdot (\nabla \phi(x) - \nabla \phi(y)). \tag{1.5}
\]

It follows from the properties of the Biot–Savart kernel that \( H_\phi \) is smooth outside the diagonal, symmetric and

\[
|H_\phi(x, y)| \leq C \| \nabla^2 \phi \|_\infty \quad \text{for all } x, y \in \mathbb{T}^2, x \neq y. \tag{1.6}
\]

We make the convention that \( H_\phi(x, x) = 0 \) for all \( x \in \mathbb{T}^2 \). Now we obtain the weak vorticity formulation of the 2D stochastic Euler equation:
\[ d\langle \omega_t, \phi \rangle = \langle \omega_t \otimes \omega_t, H_\phi \rangle \, dt + \sum_{j=1}^{\infty} \langle \omega_t, \sigma_j \cdot \nabla \phi \rangle \, dW_j^t + \frac{1}{2} \sum_{j=1}^{\infty} \langle \omega_t, \sigma_j \cdot \nabla (\sigma_j \cdot \nabla \phi) \rangle \, dt. \] (1.7)

We need some notations in order to introduce the notion of solution to (1.7). For any \( s \in \mathbb{R} \), we write \( H^s(\mathbb{T}^2) \) for the usual Sobolev space on \( \mathbb{T}^2 \), and \( H^{-1-\delta}(\mathbb{T}^2) = \cap_{\delta>0} H^{-1-\delta}(\mathbb{T}^2) \). Let \( \omega_{WN} : \Theta \to C^\infty(\mathbb{T}^2)' \) be the white noise on \( \mathbb{T}^2 \), which is by definition a Gaussian random distribution such that

\[ \mathbb{E}[\langle \omega_{WN}, \phi \rangle \langle \omega_{WN}, \psi \rangle] = \langle \phi, \psi \rangle, \quad \text{for all } \phi, \psi \in C^\infty(\mathbb{T}^2), \]

where \( \langle \cdot, \cdot \rangle \) on the r.h.s. is the inner product in \( L^2(\mathbb{T}^2, dx) \). The law of the white noise \( \omega_{WN} \), called the enstrophy measure and denoted by \( \mu \), is supported by \( H^{-1-\delta}(\mathbb{T}^2) \). It is proven in [18, Theorem 8] that \( \langle \omega_{WN} \otimes \omega_{WN}, H_\phi \rangle \) can be defined as the limit in \( L^2(\Theta, \mathbb{P}) \) of certain approximating sequences.

**Definition 1.1 (\( \rho \)-white noise solution)** Let \( \rho : [0, T] \times H^{-1-\delta}(\mathbb{T}^2) \rightarrow [0, \infty) \) satisfy \( \int \rho_t^q \, d\mu \leq C \) for some constants \( C > 0, q > 1 \), and \( \int \rho_t \, d\mu = 1 \) for every \( t \in [0, T] \).

We say that a measurable map \( \omega : \Theta \times [0, T] \rightarrow C^\infty(\mathbb{T}^2)' \), which has trajectories of class \( C([0, T], \mathbb{H}^{-1-\delta}(\mathbb{T}^2)) \) and is adapted to \( (\mathcal{F}_t)_{t \geq 0} \), is a \( \rho \)-white noise solution of the stochastic Euler Eq. (1.2) if \( \omega_t \) has law \( \rho_t \mu \) at every time \( t \in [0, T] \), and for every \( \phi \in C^\infty(\mathbb{T}^2) \), the following identity holds \( \mathbb{P} \)-a.s., uniformly in time,

\[ \langle \omega_t, \phi \rangle = \langle \omega_0, \phi \rangle + \int_0^t \langle \omega_s \otimes \omega_s, H_\phi \rangle \, ds + \sum_{j=1}^{\infty} \int_0^t \langle \omega_s, \sigma_j \cdot \nabla \phi \rangle \, dW_j^s \]

\[ + \frac{1}{2} \sum_{j=1}^{\infty} \int_0^t \langle \omega_s, \sigma_j \cdot \nabla (\sigma_j \cdot \nabla \phi) \rangle \, ds. \] (1.8)

Before presenting the main results of this paper, we introduce our assumptions on the vector fields \( \{\sigma_j : j \in \mathbb{N}\} \):

**(H1)** For all \( j \in \mathbb{N} \), the vector fields \( \sigma_j \) are periodic, smooth and \( \text{div}(\sigma_j) = 0 \).

**(H2)** The series \( \sum_{j=1}^{\infty} ||\sigma_j||^2_2 \) and \( \sum_{j=1}^{\infty} ||\sigma_j \cdot \nabla \sigma_j||_\infty \) are convergent.

According to the following remarks, all the terms on the r.h.s. of (1.8) make sense under the conditions (H1) and (H2).

**Remark 1.2** Under the conditions of Definition 1.1, we have \( \langle \omega_s \otimes \omega_s, H_\phi \rangle \in L^1(\Theta, L^1([0, T])) \) by [18, Theorem 15]. Moreover, we deduce from (H2) that the martingale part in (1.8) is a square integrable martingale. Indeed, since \( \omega_s \) is distributed as \( \rho_s \mu \), by Hölder’s inequality,

\[ \mathbb{E}(\langle \omega_s, \sigma_j \cdot \nabla \phi \rangle^2) = \mathbb{E}_\mu(\rho_s \langle \omega, \sigma_j \cdot \nabla \phi \rangle^2) \leq (\mathbb{E}_\mu \rho_s^q)^{1/q} (\mathbb{E}_\mu \langle \omega, \sigma_j \cdot \nabla \phi \rangle^{2q'})^{1/q'}, \]

where \( \mathbb{E}_\mu \) denotes the expectation on \( H^{-1-\delta}(\mathbb{T}^2) \) w.r.t. the enstrophy measure \( \mu \). Recall that if \( \xi \sim N(0, \sigma^2) \), then for any \( p > 1 \), one has \( \mathbb{E}(|\xi|^p) \leq C_p \sigma^p \) for some
constant $C_p > 0$. Under $\mu$, $\langle \omega, \sigma_j \cdot \nabla \phi \rangle$ is a centered Gaussian r.v. with variance $\int_{\mathbb{T}^2} |\sigma_j \cdot \nabla \phi|^2 \, dx \leq \|\sigma_j\|_{\infty}^2 \|\nabla \phi\|_{\infty}^2$. Combining these facts with the property of $\rho_s$ yields
\[
\mathbb{E}(\langle \omega_s, \sigma_j \cdot \nabla \phi \rangle^2) \leq C^{1/q} C_q \|\sigma_j\|_{\infty}^2 \|\nabla \phi\|_{\infty}^2.
\]
This together with (H2) gives us
\[
\sum_{j=1}^{\infty} \int_0^t \mathbb{E}(\langle \omega_s, \sigma_j \cdot \nabla \phi \rangle^2) \, ds \leq C_q t \|\nabla \phi\|_{\infty}^2 \sum_{j=1}^{\infty} \|\sigma_j\|_{\infty}^2 < \infty,
\]
which implies the claim. In the same way, one can show that
\[
\sum_{j=1}^{\infty} \int_0^t \mathbb{E}|\langle \omega_s, \sigma_j \cdot \nabla (\sigma_j \cdot \nabla \phi) \rangle| \, ds \leq C_q t \sum_{j=1}^{\infty} (\|\nabla^2 \phi\|_{\infty} \|\sigma_j\|_{\infty}^2 + \|\nabla \phi\|_{\infty} \|\sigma_j \cdot \nabla \sigma_j\|_{\infty})
\]
\[
< \infty.
\]

Now we can present the first main result.

**Theorem 1.3** (Existence) Let $\rho_0 \in C_p\left(H^{-1-}(\mathbb{T}^2)\right)$ such that $\rho_0 \geq 0$ and $\int \rho_0 \, d\mu = 1$. Under the assumptions (H1) and (H2), there exist a bounded measurable function $\rho : [0, T] \times H^{-1-}(\mathbb{T}^2) \to [0, \|\rho_0\|_{\infty}]$, and a filtered probability space $(\Theta, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ on which there are defined an $(\mathcal{F}_t)$-adapted process $\omega : \Theta \times [0, T] \to C^\infty(\mathbb{T}^2)'$ and a sequence of $(\mathcal{F}_t)$-Brownian motions $\{(W_t^j)_{t \geq 0} : j \in \mathbb{N}\}$, such that $\omega$ is a $\rho$-white noise solution of the stochastic Euler Eq. (1.2).

Our next result is concerned with the regularity properties of the density $\rho_t$, for which we need some more notations. Given two elements $\omega, \eta \in C^\infty(\mathbb{T}^2)'$ and a function $G : C^\infty(\mathbb{T}^2)' \to \mathbb{R}$, we write $\langle \eta, D_\omega G(\omega) \rangle$ for the limit
\[
\langle \eta, D_\omega G(\omega) \rangle = \lim_{\varepsilon \to 0} \frac{G(\omega + \varepsilon \eta) - G(\omega)}{\varepsilon}
\]
when it exists. For instance, if $G$ is taken from
\[
\mathcal{F}C_p = \{ G : C^\infty(\mathbb{T}^2)' \to \mathbb{R} \mid G(\omega) = g(\langle \omega, \phi_1 \rangle, \ldots, \langle \omega, \phi_n \rangle) \text{ for some } n \in \mathbb{N} \text{ and } g \in C^\infty_p(\mathbb{R}^n), \phi_1, \ldots, \phi_n \in C^\infty(\mathbb{T}^2) \},
\]
where $C^\infty_p(\mathbb{R}^n)$ is the space of smooth functions on $\mathbb{R}^n$ having polynomial growth together with all the derivatives, then
\[
\langle \eta, D_\omega G(\omega) \rangle = \sum_{j=1}^{n} \partial_j g(\langle \omega, \phi_1 \rangle, \ldots, \langle \omega, \phi_n \rangle) \langle \eta, \phi_j \rangle.
\]
We will also write $D_\omega G(\omega) = \sum_{j=1}^{n} \partial_j g(\langle \omega, \phi_1 \rangle, \ldots, \langle \omega, \phi_n \rangle) \phi_j$. 
For our purpose, we shall need test functions which depend on time. Hence we denote by

\[ \mathcal{FC}_{P,T} = \left\{ F : [0, T] \times C^\infty(T^2) \rightarrow \mathbb{R} \mid F(t, \omega) = \sum_{i=1}^{m} g_i(t) f_i(\omega) \text{ for some } m \in \mathbb{N} \right\} \]

and \( g_i \in C^1([0, T]), f_i \in \mathcal{FC}_P, 1 \leq i \leq m \).

For \( F \in \mathcal{FC}_{P,T} \) given by \( F(t, \omega) = \sum_{i=1}^{m} g_i(t) f_i(\langle \omega, \phi_1 \rangle, \ldots, \langle \omega, \phi_n \rangle) \), we have

\[ D_\omega F(t, \omega) = \sum_{i=1}^{m} g_i(t) \sum_{j=1}^{n} \partial_j f_i(\langle \omega, \phi_1 \rangle, \ldots, \langle \omega, \phi_n \rangle) \phi_j. \]

Set

\[ \langle b(\omega), D_\omega F(t, \omega) \rangle := \sum_{i=1}^{m} g_i(t) \sum_{j=1}^{n} \partial_j f_i(\langle \omega, \phi_1 \rangle, \ldots, \langle \omega, \phi_n \rangle) \langle \omega \otimes \omega, H_{\phi_j} \rangle, \]

where \( \langle \omega \otimes \omega, H_{\phi_j} \rangle, j = 1, \ldots, n, \) are limits of Cauchy sequences in \( L^2(H^{-1-}(T^2), \mu) \) (see [18, Theorem 8]). Hence \( \langle b(\omega), D_\omega F(t, \omega) \rangle \) belongs to \( C([0, T], L^r(H^{-1-}(T^2), \mu)) \) for all \( r \in [1, 2) \).

**Theorem 1.4** (Regularity) Let \( \rho : [0, T] \times H^{-1-}(T^2) \rightarrow \mathbb{R}, \| \rho_0 \|_\infty \) be the density function given in Theorem 1.3.

(i) For any \( F \in \mathcal{FC}_{P,T} \) with \( F(T, \cdot) = 0 \), the function \( \rho \) satisfies

\[ 0 = \int F(0, \omega) \rho_0(\omega) \mu(d\omega) \\
+ \int_0^T \int \left[ (\partial_t F)(t, \omega) + \langle b(\omega), D_\omega F(t, \omega) \rangle \right] \rho_t(\omega) \mu(d\omega) dt \\
+ \frac{1}{2} \sum_{k=1}^{\infty} \int_0^T \int \{ \sigma_k \cdot \nabla \omega, D_\omega \langle \sigma_k \cdot \nabla \omega, D_\omega F(t, \omega) \rangle \} \rho_t(\omega) \mu(d\omega) dt. \quad (1.9) \]

(ii) For every \( k \in \mathbb{N} \), \( \langle \sigma_k \cdot \nabla \omega, D_\omega \rho_t(\omega) \rangle \) exists in the distributional sense and the gradient estimate holds:

\[ \sum_{k=1}^{\infty} \int_0^T \left\{ \sigma_k \cdot \nabla \omega, D_\omega \rho_t(\omega) \right\}^2 \mu(d\omega) dt \leq \| \rho_0 \|_\infty^2. \quad (1.10) \]
Remark 1.5 (1) We briefly explain the meaning of the second order term in \((1.9)\). The distribution \(\sigma_k \cdot \nabla \omega\) is understood as follows:

\[
\langle \sigma_k \cdot \nabla \omega, \phi \rangle := - \langle \omega, \sigma_k \cdot \nabla \phi \rangle, \quad \phi \in C^\infty(\mathbb{T}^2),
\]

since we assume \(\sigma_k\) is smooth and divergence free. Given \(G \in \mathcal{FC}_p\) of the form \(G(\omega) = g(\langle \omega, \phi_1 \rangle, \ldots, \langle \omega, \phi_n \rangle)\), we consider the new functional \(H : C^\infty(\mathbb{T}^2)' \to \mathbb{R}\) defined by

\[
H(\omega) = \langle \sigma_k \cdot \nabla \omega, D_\omega G(\omega) \rangle = - \sum_{j=1}^n \partial_j g(\langle \omega, \phi_1 \rangle, \ldots, \langle \omega, \phi_n \rangle) \langle \omega, \sigma_k \cdot \nabla \phi_j \rangle.
\]

Then \(H \in \mathcal{FC}_p\). In Lemma 4.1 below we compute explicitly the term \(\langle \sigma_k \cdot \nabla \omega, D_\omega H(\omega) \rangle\) (see also Remark 4.2).

(2) We explain here what we mean by \(\langle \sigma_k \cdot \nabla \omega, D_\omega \rho_t(\omega) \rangle\) exists in the distributional sense for all \(k \in \mathbb{N}\). It comes from the equality \((4.38)\) which looks like an integration by parts formula. Thanks to \((4.38)\) and the fact that \(\text{div}_\mu(\sigma_k \cdot \nabla \omega) = 0\) (see Lemma 4.5), it is natural to define \(\langle \sigma_k \cdot \nabla \omega, D_\omega \rho_t(\omega) \rangle = G_k(t, \omega)\) with some \(G \in L^2(\mathbb{N} \times [0, T] \times H^{-1-}, \# \otimes dt \otimes \mu)\), where \# is the counting measure on the set \(\mathbb{N}\) of natural numbers.

At the heuristic level, the gradient estimate \((1.10)\) can be guessed by an energy-type computation on \(\rho_t\), using skew-symmetry with respect to \(\mu\) of certain differential operators. However, energy-type computations cannot be performed rigorously on weak solutions satisfying \((1.9)\). Our strategy will be to prove a gradient estimate for the density associated to the point vortex approximation and then pass to the limit.

The paper is organized as follows. In Sect. 2, we recall some facts about the stochastic dynamics of \(N\)-point vortices. More precisely, Sect. 2.1 is concerned with stochastic point vortices with an initial distribution which converges weakly to the white noise measure \(\mu\), and Sect. 2.2 studies the case of general initial distributions, which is the basis for the approximation argument in later parts of the paper. We provide the proof of Theorem 1.3 in Sect. 3 which mainly follows the arguments in \([18, \text{Section 4.2}]\). The two assertions of Theorem 1.4 will be proved in Sects. 4.1 and 4.2 respectively. In particular, the proof of assertion (ii) constitutes the main technical part of the current work, and it is done by first approximating the singular Biot–Savart kernel \(K\) with smooth ones, and then letting the number \(N\) of point vortices tend to infinity. In “Appendix”, we discuss some possible approaches towards the uniqueness of the Eq. \((1.9)\), with an emphasis on the role of gradient type estimate \((1.10)\).

2 Stochastic point vortex dynamics

According to \([25, \text{Section 4.4}]\), in the singular case that the vorticity \(\omega_0\) is given by \(N \geq 2\) point vortices, the Euler Eq. \((1.1)\) can be interpreted as the finite dimensional dynamics in \((\mathbb{T}^2)^N\):
\[
\frac{dX_{i,N}^t}{dt} = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \xi_j K(X_{i,N}^t - X_{j,N}^t), \quad i = 1, \ldots, N,
\]

with initial condition \((X_{1,N}^0, \ldots, X_{N,N}^0) \in (\mathbb{T}^2)^N \setminus \Delta_N\), where \(\xi = (\xi_1, \ldots, \xi_N) \in (\mathbb{R} \setminus \{0\})^N\) and

\[
\Delta_N = \{ (x_1, \ldots, x_N) \in (\mathbb{T}^2)^N : \text{there are } i \neq j \text{ such that } x_i = x_j \}
\]
is the generalized diagonal. One can find an example in [25, Section 4.2] which shows that three different vortex points starting from certain positions collapse to one point in finite time. Nevertheless, the above system of equations admits a unique solution for \((\text{Leb} \otimes \lambda_{N,T^2})\)-a.e. starting point in \((\mathbb{T}^2)^N \setminus \Delta_N\).

For the stochastic Euler Eq. (1.2), the random version of the point vortex dynamics is given by

\[
\begin{align*}
\frac{dX_{i,N}^t}{dt} &= \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \xi_j K(X_{i,N}^t - X_{j,N}^t) \, dt + \sum_{j=1}^{N} \sigma_j(X_{i,N}^t) \circ dW_j^t, \quad i = 1, \ldots, N. \\
\end{align*}
\]

(2.1)

Here, we use only a finite number of noises, because the stochastic equations with infinitely many noises may not admit a solution under the assumptions (H1) and (H2). One can of course use a different number of noises, but the intuition is that this number should tend to \(\infty\) together with \(N\). A heuristic discussion of the relationship between (2.1) and

\[
\begin{align*}
d\omega_t + u_t \cdot \nabla \omega_t \, dt + \sum_{j=1}^{N} \sigma_j \cdot \nabla \omega_t \circ dW_t^j &= 0 \\
\end{align*}
\]

(2.2)
can be found in [19, Section 2.3]. Roughly speaking, let \((X_{1,N}^t, \ldots, X_{N,N}^t)\) be the solution of (2.1) and set

\[
\omega_t^N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \xi_i \delta_{X_{i,N}^t},
\]

then for any \(\phi \in C^\infty(\mathbb{T}^2)\), by applying the Itô formula, \(\omega_t^N\) satisfies

\[
\begin{align*}
d(\omega_t^N, \phi) &= \langle \omega_t^N, u_t^N \cdot \nabla \phi \rangle \, dt + \sum_{j=1}^{N} \langle \omega_t^N, \sigma_j \cdot \nabla \phi \rangle \circ dW_t^j, \\
\end{align*}
\]

(2.3)

where \(u_t^N\) and \(\omega_t^N\) are related by (1.4).

Fix any \(N \in \mathbb{N}\) and denote by \(\lambda_N = \text{Leb}_{\mathbb{T}^2} \otimes \lambda_{N,T^2}\) which is a probability measure on \((\mathbb{T}^2)^N\).
For every $(\xi_1, \ldots, \xi_N) \in (\mathbb{R}\setminus\{0\})^N$ and for $\lambda_N$-a.e. $(X_{0,0}^{1,N}, \ldots, X_{0,0}^{N,N}) \in (\mathbb{T}^2)^N \setminus \Delta_N$, almost surely, the system (2.1) has a unique strong solution $(X_{t,0}^{1,N}, \ldots, X_{t,0}^{N,N})$ for all $t \geq 0$. Moreover, if the initial data $(X_{0,0}^{1,N}, \ldots, X_{0,0}^{N,N})$ is a random variable distributed as $\lambda_N$, but is independent of the Brownian motions $\{W_t^j\}_{t \geq 0} : 1 \leq j \leq N$, then $(X_{t,0}^{1,N}, \ldots, X_{t,0}^{N,N})$ is a stationary process with invariant marginal law $\lambda_N$.

**Proof** Note that our hypothesis (H1) is the same as the first one of [19, Hypothesis 1], hence the first result follows from [19, Theorem 8]. We denote by $\varphi_t(X_0)$ the strong solution to (2.1) with initial condition $X_0 \in (\mathbb{T}^2)^N \setminus \Delta_N$ when the solution exists. We remark that we do not need the ellipticity assumption in [19, Hypothesis 1], since the existence of solution to (2.1) for a.e. starting point is enough for our purpose.

For proving the second assertion, let $K^\delta$ be the approximation of $K$ given in [19, Section 3.2] and $\varphi_t^\delta$ the flow of diffeomorphisms generated by (2.1) with $K$ replaced by $K^\delta$, see [22, Section 4.7]. Since the Jacobian associated with the flow diffeomorphism $\varphi_t^\delta$ is 1 almost surely, for any smooth function $h$ on $(\mathbb{T}^2)^N$, we have (cf. [19, Lemma 3])

$$\int_{(\mathbb{T}^2)^N} h(\varphi_t^\delta(X_0)) \, dX_0 = \int_{(\mathbb{T}^2)^N} h(Y) \, dY, \quad t \geq 0.$$ 

Therefore,

$$\mathbb{E} \int_{(\mathbb{T}^2)^N} h(\varphi_t^\delta(X_0)) \, dX_0 = \int_{(\mathbb{T}^2)^N} h(Y) \, dY, \quad t \geq 0.$$ 

For $(\lambda_N \otimes \mathbb{P})$-a.s. $(X_0, \theta) \in (\mathbb{T}^2)^N \times \Theta$, we have $\varphi_t^\delta(X_0, \theta) \to \varphi_t(X_0, \theta)$ as $\delta \to 0$, see the proof of [19, Theorem 8]. Letting $\delta \to 0$ in the above equality leads to

$$\int_{(\mathbb{T}^2)^N} h(Y) \, dY = \mathbb{E} \int_{(\mathbb{T}^2)^N} h(\varphi_t(X_0)) \, dX_0 = \int_{(\mathbb{T}^2)^N} P_t^N h(X_0) \, dX_0,$$

where $P_t^N$ is the semigroup associated to the system (2.1). This implies that $\lambda_N$ is the invariant measure of $P_t^N$. The stationarity follows from the fact that the Eq. (2.1) are of time-homogeneous Markovian type.

**2.1 Stochastic point vortices with initial distribution converging to white noise**

On the probability space $(\Theta, \mathcal{F}, \mathbb{P})$, let $\{\xi_n\}$ be an i.i.d. sequence of $N(0, 1)$ r.v.’s and $\{X_0^n\}$ be an i.i.d. sequence of $\mathbb{T}^2$-valued r.v.’s, independent of $\{\xi_n\}$ and uniformly distributed. Both families are independent of the Brownian motions $\{W_t^j\}_{t \geq 0} : j \in \mathbb{N}$. For every $N \in \mathbb{N}$, denote by

$$\lambda_N^0 = (N(0, 1) \otimes \text{Leb}_{\mathbb{T}^2})^\otimes N$$
the law of the random vector 
\[(\xi_1, X_0^1), \ldots, (\xi_N, X_0^N)\].

Let us consider the measure-valued vorticity field
\[\omega_0^N = \frac{1}{\sqrt{N}} \sum_{n=1}^N \xi_n \delta_{X_0^n}.\]

As mentioned in [18, Remark 20], \(\omega_0^N\) can be regarded as a r.v. taking values in the space \(H^{-1}(\mathbb{T}^2)\) whose law is denoted by \(\mu_0^N\). Denote by \(\mathcal{M}(\mathbb{T}^2)\) the space of signed measures on \(\mathbb{T}^2\) with finite variation, and
\[\mathcal{M}_N(\mathbb{T}^2) = \{\mu \in \mathcal{M}(\mathbb{T}^2) | \exists X \subset \mathbb{T}^2 \text{ such that } \#(X) = N \text{ and } \text{supp}(\mu) = X\}.\]

We can define the map \(T_N : (\mathbb{R} \times \mathbb{T}^2)^N \to \mathcal{M}_N(\mathbb{T}^2) \subset H^{-1}(\mathbb{T}^2)\) as
\[\left((\xi_1, X_0^1), \ldots, (\xi_N, X_0^N)\right) \mapsto \omega_0^N = \frac{1}{\sqrt{N}} \sum_{n=1}^N \xi_n \delta_{X_0^n},\] (2.4)
then it holds that
\[\mu_0^N = (T_N)_\# \lambda_N^0 = \lambda_N^0 \circ T_N^{-1}.\]

It is proved in [18, Proposition 21] that, for any \(\delta > 0\), as \(N \to \infty\), \(\omega_0^N\) converges in law on \(H^{-1-\delta}(\mathbb{T}^2)\) to the white noise \(\omega_{WN}\).

**Proposition 2.2** As \(N \to \infty\), the probability measures \(\mu_0^N\) converge weakly to \(\mu\) on \(H^{-1}(\mathbb{T}^2)\).

**Proof** Step 1 We first show that \(\{\mu_0^N : N \in \mathbb{N}\}\) is tight on \(H^{-1}(\mathbb{T}^2)\). Fix an arbitrary \(\varepsilon > 0\). For every \(n \in \mathbb{N}\), since \(\mu_0^N\) converges weakly to \(\mu\) on \(H^{-1-1/n}(\mathbb{T}^2)\), it follows from [6, p. 60, Theorem 5.2] that the family \(\{\mu_0^N : N \in \mathbb{N}\}\) is tight on \(H^{-1-1/n}(\mathbb{T}^2)\). Therefore, there exists a compact set \(K_{\varepsilon,n} \subset H^{-1-1/n}(\mathbb{T}^2)\) such that
\[\sup_{N \in \mathbb{N}} \mu_0^N(\{H^{-1-1/n}(\mathbb{T}^2) \setminus K_{\varepsilon,n}\}) < \frac{\varepsilon}{2^n}.\]

Let \(K_\varepsilon = \bigcap_{n \in \mathbb{N}} K_{\varepsilon,n}\); then \(K_\varepsilon \subset \bigcap_{n \in \mathbb{N}} H^{-1-1/n}(\mathbb{T}^2) = H^{-1}(\mathbb{T}^2)\). By the above inequality, we have for all \(N \in \mathbb{N}\) that
\[\mu_0^N(\{H^{-1}(\mathbb{T}^2) \setminus K_\varepsilon\}) \leq \mu_0^N\left(\bigcup_{n=1}^\infty \{H^{-1-1/n}(\mathbb{T}^2) \setminus K_{\varepsilon,n}\}\right) \leq \sum_{n=1}^\infty \frac{\varepsilon}{2^n} = \varepsilon.\]
Then the tightness of \( \{ \mu^0_N : N \in \mathbb{N} \} \) on \( H^{-1}_- (\mathbb{T}^2) \) will follow if we can show that \( K_\varepsilon \) is compact in \( H^{-1}_- (\mathbb{T}^2) \). It is equivalent to show that \( K_\varepsilon \) is sequentially compact in itself. Let \( \{ \omega_n : n \in \mathbb{N} \} \subset K_\varepsilon \) be an arbitrary sequence which will also be denoted by \( \{ \omega_{0,n} : n \in \mathbb{N} \} \).

Since \( K_{\varepsilon,1} \) is compact in \( H^{-2} (\mathbb{T}^2) \) and \( \{ \omega_{0,n} : n \in \mathbb{N} \} \subset K_{\varepsilon,1} \), we can find a subsequence \( \{ \omega_{1,n} : n \in \mathbb{N} \} \) of \( \{ \omega_{0,n} : n \in \mathbb{N} \} \), such that \( \omega_{1,n} \) converges with respect to the norm \( \| \cdot \|_{H^{-2}} \) to some \( \omega_{1,0} \in K_{\varepsilon,1} \).

Repeating this procedure inductively, for every \( m \in \mathbb{N} \), we can find a subsequence \( \{ \omega_{m,n} : n \in \mathbb{N} \} \) of \( \{ \omega_{m-1,n} : n \in \mathbb{N} \} \) such that \( \omega_{m,n} \) converges with respect to the norm \( \| \cdot \|_{H^{-1-1/m}} \) to some \( \omega_{m,0} \in K_{\varepsilon,m} \).

We claim that \( \omega_{m,0} = \omega_{m+1,0} \) for all \( m \in \mathbb{N} \). Indeed, on the one hand, since \( \omega_{m+1,n} \) converges to \( \omega_{m+1,0} \) with respect to the norm \( \| \cdot \|_{H^{-1-1/(m+1)}} \), it also converges to \( \omega_{m+1,0} \) with respect to the weaker norm \( \| \cdot \|_{H^{-1-1/m}} \). On the other hand, as a subsequence of \( \{ \omega_{m,n} : n \in \mathbb{N} \} \), \( \{ \omega_{m+1,n} : n \in \mathbb{N} \} \) also converge in \( H^{-1-1/m} (\mathbb{T}^2) \) to \( \omega_{m,0} \). By the uniqueness of limit, we obtain \( \omega_{m+1,0} = \omega_{m,0} \).

Therefore we can denote by \( \omega_0 \) the common limit of all the subsequences, which belongs to all \( K_{\varepsilon,m} \), and hence is in \( K_\varepsilon \). Now taking the diagonal subsequence \( \{ \omega_{n,n} : n \in \mathbb{N} \} \), we see that \( \omega_{n,n} \) tends to \( \omega_0 \) with respect to all the norms \( \| \cdot \|_{H^{-1-1/m}} \), \( m \geq 1 \), hence the convergence holds in \( H^{-1}_- (\mathbb{T}^2) \) too. This shows that \( K_\varepsilon \) is sequentially compact in itself.

Step 2 Let \( \{ \mu^0_{N_k} : k \in \mathbb{N} \} \) be a subsequence converging weakly to some \( \nu \) on \( H^{-1}_- (\mathbb{T}^2) \). Then we have \( \nu = \mu \). Indeed, for any bounded continuous function \( F \) on \( H^{-1-\delta} (\mathbb{T}^2) \), it is also continuous on \( H^{-1}_- (\mathbb{T}^2) \), hence \( \lim_{k \to \infty} \int_{H^{-1-\delta}} F \, d\mu^0_{N_k} = \int_{H^{-1-\delta}} F \, d\nu \). We conclude that \( \mu^0_{N_k} \) converges weakly to \( \nu \) on \( H^{-1-\delta} (\mathbb{T}^2) \) for any \( \delta > 0 \). This implies \( \nu = \mu \). By the corollary of [6, Theorem 5.1], the whole sequence \( \{ \mu^0_N : N \in \mathbb{N} \} \) converge weakly to \( \mu \) on \( H^{-1}_- (\mathbb{T}^2) \).

As a consequence of Theorem 2.1, we can prove (cf. [18, Proposition 22] for the proof)

**Proposition 2.3** Consider the stochastic point vortex dynamics (2.1) with random intensities \( (\xi_1, \ldots, \xi_N) \) and random initial positions \( (X^1_0, \ldots, X^N_0) \) distributed as \( \lambda^0_0 \).

For a.s. value of \( (\xi_1, X^1_0), \ldots, (\xi_N, X^N_0) \), the stochastic dynamics \( (X^1_t, \ldots, X^N_t, \tilde{N}) \) is well defined in \( \Delta^c_N \) for all \( t \geq 0 \), and the associated measure-valued vorticity \( \omega^N_t \) satisfies the stochastic weak vorticity formulation of (2.3): for all \( \phi \in C^\infty (\mathbb{T}^2) \),

\[
\langle \omega^N_t, \phi \rangle = \langle \omega^0_N, \phi \rangle + \int_0^t \langle \omega^N_s \otimes \omega^N_s, H_\phi \rangle \, ds + \sum_{j=1}^N \int_0^t \langle \omega^N_s, \sigma_j \cdot \nabla \phi \rangle \, dW^j_s + \frac{1}{2} \sum_{j=1}^N \int_0^t \langle \omega^N_s, \sigma_j \cdot \nabla (\sigma_j \cdot \nabla \phi) \rangle \, ds.
\]

(2.5)

The stochastic process \( \omega^N_t \) is stationary in time, with the law \( \mu^0_N \) at any time \( t \geq 0 \).
The following integrability properties of $\omega_t^N$ are proved in [18, Lemma 23] (except the second estimate, whose proof is similar to that of the first one).

**Lemma 2.4** Assume $f : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{R}$ and $g : \mathbb{T}^2 \to \mathbb{R}$ are bounded and measurable, and $f$ is symmetric. Then, for every $p \geq 1$ and $\delta > 0$, there are constants $C_p, C_{p, \delta} > 0$ such that for all $t \in [0, T]$,

$$
\mathbb{E}[\|\omega_t^N \otimes \omega_t^N, f\|_p^p] \leq C_p \|f\|_\infty^p, \quad \mathbb{E}[\|\omega_t^N, g\|_p^p] \leq C_p \|g\|_\infty^p, \quad \mathbb{E}[\|\omega_t^N\|_{H^{-1, \delta}}^p] \leq C_{p, \delta}.
$$

Moreover,

$$
\mathbb{E}[\|\omega_t^N \otimes \omega_t^N, f\|^2] = \frac{3}{N} \int f^2(x, x) \, dx + \frac{N - 1}{N} \left[ \int f(x, x) \, dx \right]^2 + \frac{2(N - 1)}{N} \int \int f^2(x, y) \, dx \, dy.
$$

### 2.2 Stochastic point vortices with general initial distribution

In this part, we shall consider stochastic point vortex dynamics (2.1) with more general initial distribution. Recall the definitions of $\lambda_N^0, \mu_N^0$ and $T_N : (\mathbb{R} \times \mathbb{T}^2)^N \to \mathcal{M}(\mathbb{T}^2)$ in Sect. 2.1. The next lemma is taken from [18, Lemma 29].

**Lemma 2.5** Let $\rho : H^{-1, -1}(\mathbb{T}^2) \to [0, \infty)$ be a measurable function with $\int_{H^{-1, -1}} \rho(\omega) \mu_N^0(\omega) \, d\omega < \infty$. Under the mapping $T_N$, the measure $\lambda_N^0 = (\rho \circ T_N) \mu_N^0$ has the image measure $\mu_N^\rho = \rho \mu_N^0$.

**Proof** We denote by $(a, x)$ a typical element in $(\mathbb{R} \times \mathbb{T}^2)^N = \mathbb{R}^N \times (\mathbb{T}^2)^N$, where $a = (a_1, \ldots, a_N) \in \mathbb{R}^N$, $x = (x_1, \ldots, x_N) \in (\mathbb{T}^2)^N$. For every non-negative measurable function $F$, the change-of-variable formula yields

$$
\int_{H^{-1, -1}(\mathbb{T}^2)} F(\omega) \mu_N^\rho \, d\omega = \int_{(\mathbb{R} \times \mathbb{T}^2)^N} F(T_N(a, x)) \lambda_N^0(\omega) \, d\omega
$$

$$
= \int_{(\mathbb{R} \times \mathbb{T}^2)^N} F(T_N(a, x)) \rho(T_N(a, x)) \lambda_N^0(\omega) \, d\omega
$$

$$
= \int_{H^{-1, -1}(\mathbb{T}^2)} F(\omega) \rho(\omega) \mu_N^0 \, d\omega.
$$

\qed
on Borel sets of \((\mathbb{R} \times \mathbb{T}^2)^N\). By Lemma 2.5 its image measure on \(H^{-1}(-\mathbb{T}^2)\) under the map \(T_N\) is \(C_N \rho_0 \mu_0^N\). Recall that the stochastic point vortex dynamics (2.1) is well defined for \(\lambda^*_N\)-a.e. \(((\xi_1, X_0^1), \ldots, (\xi_N, X_0^N)) \in (\mathbb{R} \times \mathbb{T}^2)^N\), see Proposition 2.3. Hence it is well defined for a.e. \(((\xi_1, X_0^1), \ldots, (\xi_N, X_0^N)) \in (\mathbb{R} \times \mathbb{T}^2)^N\) with respect to \(C_N (\rho_0 \circ T_N) \lambda^*_N\). Denote by \(\omega_{\rho_0,t}^N\) the vorticity of this point vortex dynamics; the law of \(\omega_{\rho_0,0}^N\) on \(\mathcal{M}_N(\mathbb{T}^2) \subset H^{-1}(\mathbb{T}^2)\) is \(C_N \rho_0 \mu_0^N\).

**Lemma 2.6** For any non-negative measurable function \(F\) on \(H^{-1}(-\mathbb{T}^2)\), one has

\[
\mathbb{E}\left[F(\omega_{\rho_0,t}^N)\right] \leq C_N \|\rho_0\|_\infty \int_{\mathcal{M}_N(\mathbb{T}^2)} F(\omega) \mu_0^N(d\omega).
\]

In particular, the law of \(\omega_{\rho_0,t}^N\) on \(\mathcal{M}_N(\mathbb{T}^2)\) has a density \(\rho_t^N\) w.r.t. \(\mu_0^N\).

**Proof** A given \(\omega \in \mathcal{M}_N(\mathbb{T}^2)\) corresponds to \(N!\) different elements \((a, x) \in (\mathbb{R} \times \mathbb{T}^2)^N\). These elements differ from each other by a permutation. However, by changing accordingly the order of the equations in the system (2.1), the solutions give rise to the same random measure-valued vorticity field at any time \(t > 0\). Thus, there exists a unique stochastic process \(\Phi_t^N(\omega)\) associated to the system (2.1), which is well defined for \(\mu_0^N\)-a.e. \(\omega \in \mathcal{M}_N(\mathbb{T}^2)\). For any nonnegative measurable function \(F : \mathcal{M}_N(\mathbb{T}^2) \to \mathbb{R}_+\), by the last assertion of Proposition 2.3,

\[
\mathbb{E}\left[\int_{\mathcal{M}_N(\mathbb{T}^2)} F(\Phi_t^N(\omega)) \mu_0^N(d\omega) = \int_{\mathcal{M}_N(\mathbb{T}^2)} F(\omega) \mu_0^N(d\omega)\right].
\]

Now, note that \(\omega_{\rho_0,t}^N = \Phi_t^N(\omega_{\rho_0,0}^N)\) where \(\omega_{\rho_0,0}^N\) is distributed as \(C_N \rho_0 \mu_0^N\). Therefore,

\[
\mathbb{E}\left[F(\omega_{\rho_0,t}^N)\right] = \mathbb{E}\left[F(\Phi_t^N(\omega_{\rho_0,0}^N))\right] = \int_{\mathcal{M}_N(\mathbb{T}^2)} \mathbb{E}\left[F(\Phi_t^N(\omega))\right] C_N \rho_0(\omega) \mu_0^N(d\omega)
\leq C_N \|\rho_0\|_\infty \int_{\mathcal{M}_N(\mathbb{T}^2)} \mathbb{E}[F(\Phi_t^N(\omega))] \mu_0^N(d\omega)
= C_N \|\rho_0\|_\infty \int_{\mathcal{M}_N(\mathbb{T}^2)} F(\omega) \mu_0^N(d\omega),
\]

where the last equality follows from (2.6).

We have the following useful estimates.

**Corollary 2.7** Assume \(f : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{R}\) and \(g : \mathbb{T}^2 \to \mathbb{R}\) are bounded and measurable, and \(f\) is symmetric. Then for any \(p \geq 1\) and \(\delta \in (0, 1)\), there exist \(C_{\rho_0,p}, C_{\rho_0,p,\delta} > 0\) such that for all \(t \in [0, T]\),

\[
\mathbb{E}\left[\|\omega_{\rho_0,t}^N \otimes \omega_{\rho_0,t}^N, f\|_p^p\right] \leq C_{\rho_0,p} \|f\|_p^p,
\]
\[
\mathbb{E}\left[\|\omega_{\rho_0,t}^N, g\|_p^p\right] \leq C_{\rho_0,p} \|g\|_p^p,
\]
\[
\mathbb{E}\left[\|\omega_{\rho_0,t}^N\|_{H^{-1-s}}^p\right] \leq C_{\rho_0,p,\delta}.
\]
Proof Since \( \lim_{N \to \infty} C_N = 1 \), we have \( C_0 = \sup_{N \geq 1} C_N < \infty \). Applying Lemma 2.6 with \( F(\omega) = |\langle \omega \otimes \omega, f \rangle|^p \), then we deduce the first result from the estimate in Lemma 2.4 with \( C_{p_0, p} = C_0\|\rho_0\|_\infty C_p \). The last two estimates follow in the same way.

3 Proof of Theorem 1.3

For simplification of notations, we shall write in the sequel \( \omega_t^N \) instead of \( \omega_{\rho_0, t}^N \) given in Sect. 2.2, since \( \rho_0 \) is fixed.

The difference of the proof, compared to that of [18, Theorem 24], is that the process \( \langle \omega_t^N, \phi \rangle \) does not have differentiable trajectories, hence we shall use fractional Sobolev spaces and apply another compactness criterion proved in [28, p. 90, Corollary 9]. We state it here in our context.

Take \( \delta \in (0, 1) \) and \( \kappa > 5 \) (this choice is due to estimates below) and consider the spaces

\[
X = H^{-1-\delta/2}(\mathbb{T}^2), \quad B = H^{-1-\delta}(\mathbb{T}^2), \quad Y = H^{-\kappa}(\mathbb{T}^2).
\]

Then \( X \subset B \subset Y \) with compact embeddings and we also have, for a suitable constant \( C > 0 \) and for

\[
\theta = \frac{\delta/2}{\kappa - 1 - \delta/2},
\]

the interpolation inequality

\[
\|\omega\|_B \leq C\|\omega\|_X^{1-\theta}\|\omega\|_Y^\theta, \quad \omega \in X.
\]

These are the preliminary assumptions of [28, p. 90, Corollary 9]. We consider here a particular case:

\[
S = L^{p_0}(0, T; X) \cap W^{1/3, 4}(0, T; Y),
\]

where for \( 0 < \alpha < 1 \) and \( p \geq 1 \),

\[
W^{\alpha, p}(0, T; Y) = \left\{ f : f \in L^p(0, T; Y) \text{ and } \int_0^T \int_0^T \frac{\|f(t) - f(s)\|_Y^p}{|t-s|^\alpha p + 1} \, dt \, ds < \infty \right\}.
\]

Lemma 3.1 Let \( \delta \in (0, 1) \) and \( \kappa > 5 \) be given. If

\[
p_0 > \frac{12(\kappa - 1 - 3\delta/2)}{\delta},
\]

then \( S \) is compactly embedded into \( C([0, T], H^{-1-\delta}(\mathbb{T}^2)) \).

Proof Recall that \( \theta \) is defined in (3.1). In our case, we have \( s_0 = 0, r_0 = p_0 \) and \( s_1 = 1/3, r_1 = 4 \). Hence \( s_\theta = (1-\theta)s_0 + \theta s_1 = \theta/3 \) and

\[\square\]
Lemma 3.2 \(\frac{1}{r_0} = \frac{1 - \theta}{r_0} + \frac{\theta}{r_1} = \frac{1 - \theta}{p_0} + \frac{\theta}{4}\).

It is clear that for \(p_0\) given above, it holds \(s_\theta > 1/r_\theta\), thus the desired result follows from the second assertion of [28, Corollary 9].

For \(N \geq 1\), let \(Q^N\) be the law of \(\omega^N\) on \(\mathcal{X} := C([0, T], H^{-1-\delta}(\mathbb{T}^2))\). We want to prove that the family \(\{Q^N\}_{N \geq 1}\) is tight in \(\mathcal{X}\).

**Lemma 3.2** The family \(\{Q^N\}_{N \geq 1}\) is tight in \(\mathcal{X}\) if and only if it is tight in \(C([0, T], H^{-1-\delta}(\mathbb{T}^2))\) for any \(\delta > 0\).

The proof is similar to Step 1 of the proof of Proposition 2.2. In view of the above two lemmas, it is sufficient to prove that \(\{Q^N\}_{N \geq 1}\) is bounded in probability in \(W^{1/3,4}(0, T; H^{-\kappa}(\mathbb{T}^2))\) and in each \(L^{p_0}(0, T; H^{-1-\delta}(\mathbb{T}^2))\) for any \(p_0 > 0\) and \(\delta > 0\).

First we show that the family \(\{Q^N\}_{N \geq 1}\) is bounded in probability in \(L^{p_0}(0, T; H^{-1-\delta}(\mathbb{T}^2))\). We have by Corollary 2.7 that

\[
\mathbb{E}\left[ \int_0^T \|\omega_t^N\|_{H^{-1-\delta}}^{p_0} \, dt \right] = \int_0^T \mathbb{E}[\|\omega_t^N\|_{H^{-1-\delta}}^{p_0}] \, dt \leq C_{p_0, p_0, \delta} T, \quad \text{for all } N \geq 1.
\]

By Chebyshev’s inequality, we obtain the boundedness in probability of the family \(\{Q^N\}_{N \geq 1}\) in \(L^{p_0}(0, T; H^{-1-\delta}(\mathbb{T}^2))\).

Next, we prove the boundedness in probability in \(W^{1/3,4}(0, T; H^{-\kappa}(\mathbb{T}^2))\). Again by the Chebyshev inequality, it suffices to show that

\[
\sup_{N \geq 1} \mathbb{E}\left[ \int_0^T \|\omega_t^N\|_{H^{-\kappa}}^4 \, dt + \int_0^T \int_0^T \frac{\|\omega_t^N - \omega_s^N\|_{H^{-\kappa}}^4}{|t-s|^{7/3}} \, dt \, ds \right] < \infty.
\]

In view of (3.2), we see that it is sufficient to establish a uniform estimate on the expectation \(\mathbb{E}\|\omega_t^N - \omega_s^N\|_{H^{-\kappa}}^4\).

**Lemma 3.3** Under the assumption (H2), for any \(\phi \in C^{\infty}(\mathbb{T}^2)\), we have

\[
\mathbb{E}\left[ (\omega_t^N - \omega_s^N, \phi)^4 \right] \leq C (t - s)^2 (\|\nabla \phi\|_\infty^4 + \|\nabla^2 \phi\|_\infty^4).
\]

**Proof** The Eq. (2.5) holds for \((\omega_t^N)_{0 \leq t \leq T}\), thus

\[
\langle \omega_t^N - \omega_s^N, \phi \rangle = \int_s^t \langle \omega_r^N \otimes \omega_r^N, H_\phi \rangle \, dr + \sum_{j=1}^N \int_s^t \langle \omega_r^N, \sigma_j \cdot \nabla \phi \rangle \, dW_t^j + \frac{1}{2} \sum_{j=1}^N \int_s^t \langle \omega_r^N, \sigma_j \cdot \nabla (\sigma_j \cdot \nabla \phi) \rangle \, dr.
\]
First, Hölder’s inequality and Corollary 2.7 lead to

\[
\mathbb{E} \left[ \left( \int_s^t \langle \omega_r^N \otimes \omega_r^N, H_\phi \rangle \, dr \right)^4 \right] \leq (t-s)^3 \mathbb{E} \left[ \left( \int_s^t \langle \omega_r^N \otimes \omega_r^N, H_\phi \rangle^4 \, dr \right) \right]
\]
\[
\leq (t-s)^3 \int_s^t C \|H_\phi\|_\infty^4 \, dr \leq C(t-s)^4 \|\nabla^2 \phi\|_\infty^4.
\]

(3.4)

where the last inequality follows from (1.6). Next, by Burkholder’s inequality,

\[
\mathbb{E} \left[ \left( \sum_{j=1}^N \int_s^t \langle \omega_r^N, \sigma_j \cdot \nabla \phi \rangle \, dW_r^j \right)^4 \right] \leq C \mathbb{E} \left[ \left( \int_s^t \sum_{j=1}^N \langle \omega_r^N, \sigma_j \cdot \nabla \phi \rangle^2 \, dr \right)^2 \right]
\]
\[
\leq C(t-s) \int_s^t \mathbb{E} \left[ \left( \sum_{j=1}^N \langle \omega_r^N, \sigma_j \cdot \nabla \phi \rangle^2 \right)^2 \right] \, dr.
\]

We have by Cauchy’s inequality and Corollary 2.7 that

\[
\mathbb{E} \left[ \left( \sum_{j=1}^N \langle \omega_r^N, \sigma_j \cdot \nabla \phi \rangle^2 \right)^2 \right] = \sum_{j,k=1}^N \mathbb{E} \left[ \langle \omega_r^N, \sigma_j \cdot \nabla \phi \rangle^2 \langle \omega_r^N, \sigma_k \cdot \nabla \phi \rangle^2 \right]
\]
\[
\leq \sum_{j,k=1}^N \left[ \mathbb{E} \langle \omega_r^N, \sigma_j \cdot \nabla \phi \rangle^4 \right]^{1/2} \left[ \mathbb{E} \langle \omega_r^N, \sigma_k \cdot \nabla \phi \rangle^4 \right]^{1/2}
\]
\[
\leq C \left( \sum_{j=1}^N \| \sigma_j \cdot \nabla \phi \|_\infty^2 \right)^2 \leq \tilde{C} \| \nabla \phi \|_\infty^4,
\]

where the last inequality follows from (H2). Substituting this estimate into the above inequality yields

\[
\mathbb{E} \left[ \left( \sum_{j=1}^N \int_s^t \langle \omega_r^N, \sigma_j \cdot \nabla \phi \rangle \, dW_r^j \right)^4 \right] \leq C(t-s)^2 \| \nabla \phi \|_\infty^4.
\]

(3.5)

Finally, by Hölder’s inequality,

\[
\mathbb{E} \left[ \left( \sum_{j=1}^N \int_s^t \langle \omega_r^N, \sigma_j \cdot \nabla (\sigma_j \cdot \nabla \phi) \rangle \, dr \right)^4 \right] \leq (t-s)^3 \int_s^t \mathbb{E} \left[ \left( \sum_{j=1}^N \langle \omega_r^N, \sigma_j \cdot \nabla (\sigma_j \cdot \nabla \phi) \rangle \right)^4 \right] \, dr
\]
\[
\leq (t-s)^3 \int_s^t \left[ \sum_{j=1}^N \left( \mathbb{E} \langle \omega_r^N, \sigma_j \cdot \nabla (\sigma_j \cdot \nabla \phi) \rangle^4 \right)^{1/2} \right]^4 \, dr.
\]
Since 

\[
\left( \mathbb{E}\left| \omega^{N}_{t} \cdot \sigma_{j} \cdot \nabla (\sigma_{j} \cdot \nabla \phi) \right|^{4} \right)^{\frac{1}{4}} \leq C \left\| \sigma_{j} \cdot \nabla (\sigma_{j} \cdot \nabla \phi) \right\|_{\infty}
\leq C \left( \left\| \sigma_{j} \right\|_{\infty}^{2} \left\| \nabla^{2} \phi \right\|_{\infty} + \left\| \sigma_{j} \cdot \nabla \sigma_{j} \right\|_{\infty} \left\| \nabla \phi \right\|_{\infty} \right),
\]

we have by (H2) that

\[
\mathbb{E} \left[ \left( \sum_{j=1}^{N} \int_{s}^{t} \langle \omega^{N}_{r} , \sigma_{j} \cdot \nabla (\sigma_{j} \cdot \nabla \phi) \rangle \, dr \right)^{4} \right] \leq C(t-s)^{2} \sum_{k} (1 + |k|^{2})^{-\kappa} \mathbb{E} \left[ \left| \omega^{N}_{t} - \omega^{N}_{s} , e_{k} \right|^{4} \right] \leq \tilde{C} (t-s)^{4} \sum_{k} (1 + |k|^{2})^{-\kappa} |k|^{8} \leq \tilde{C} (t-s)^{2},
\]

since $2\kappa - 8 > 2$ due to the choice of $\kappa$. Consequently,

\[
\mathbb{E} \left[ \left\| \omega^{N}_{t} - \omega^{N}_{s} \right\|_{H^{-\kappa}}^{4} \right] = \mathbb{E} \left[ \left( \sum_{k} (1 + |k|^{2})^{-\kappa} \left| \omega^{N}_{t} - \omega^{N}_{s} , e_{k} \right|^{2} \right)^{2} \right] \leq \left( \sum_{k} (1 + |k|^{2})^{-\kappa} \right) \sum_{k} (1 + |k|^{2})^{-\kappa} \mathbb{E} \left[ \left| \omega^{N}_{t} - \omega^{N}_{s} , e_{k} \right|^{4} \right] \leq \tilde{C} (t-s)^{2} \sum_{k} (1 + |k|^{2})^{-\kappa} |k|^{8} \leq \tilde{C} (t-s)^{2},
\]

Applying Lemma 3.3 with $\phi(x) = e^{2\pi i k \cdot x}$ leads to

\[
\mathbb{E} \left[ \left| \omega^{N}_{t} - \omega^{N}_{s} , e_{k} \right|^{4} \right] \leq C (t-s)^{2} |k|^{8}, \quad k \in \mathbb{Z}_{0}^{2} = \mathbb{Z}^{2} \setminus \{0\}.
\]

As a result, by Cauchy’s inequality,

\[
\mathbb{E} \left[ \left( \sum_{j=1}^{N} \int_{s}^{t} \langle \omega^{N}_{r} , \sigma_{j} \cdot \nabla (\sigma_{j} \cdot \nabla \phi) \rangle \, dr \right)^{4} \right] \leq C(t-s)^{3} \int_{s}^{t} \left[ \sum_{j=1}^{N} \left( \left\| \sigma_{j} \right\|_{\infty}^{2} \left\| \nabla^{2} \phi \right\|_{\infty} + \left\| \sigma_{j} \cdot \nabla \sigma_{j} \right\|_{\infty} \left\| \nabla \phi \right\|_{\infty} \right) \right]^{4} \, dr \leq C(t-s)^{3} \left( \left\| \nabla^{2} \phi \right\|_{\infty} + \left\| \nabla \phi \right\|_{\infty} \right)^{4}.
\]

Combining this estimate together with (3.3)–(3.5), we obtain the desired estimate.

The proof of the boundedness in probability of \( \{ Q^{N} \}_{N \geq 1} \) in \( W^{1,4}(0, T; H^{-\kappa}(\mathbb{T}^{2})) \) is complete.

We have shown that the family \( \{ Q^{N} \}_{N \in \mathbb{N}} \) is bounded in probability in \( L^{p_{0}}(0, T; H^{-1-\delta/2}) \cap W^{1,4}(0, T; H^{-\kappa}) \) for any \( p_{0} > 0 \) and \( \delta > 0 \), hence it is tight in \( C([0, T], H^{-1-\delta}) \) for any \( \delta > 0 \). Lemma 3.2 implies that \( \{ Q^{N} \}_{N \in \mathbb{N}} \) is tight in \( \mathcal{X} = C([0, T], H^{-1-\delta}) \).
Since we are dealing with the SDEs (2.1), we need to consider $Q^N$ together with the distribution of Brownian motions. Although we use only finitely many Brownian motions in (2.1), here we consider for simplicity the whole family $\{(W_t^j)_{0 \leq t \leq T} : j \in \mathbb{N}\}$. To this end, we assume $\mathbb{R}^\infty$ is endowed with the metric

$$d_\infty(a, b) = \sum_{n=1}^{\infty} \frac{|a_n - b_n| \wedge 1}{2^n}, \quad a, b \in \mathbb{R}^\infty.$$ 

Then $(\mathbb{R}^\infty, d_\infty(\cdot, \cdot))$ is separable and complete (see [6, p. 9, Example 1.2]). The distance in $\mathcal{Y} := C([0, T], \mathbb{R}^\infty)$ is given by

$$d_\mathcal{Y}(w, \hat{w}) = \sup_{t \in [0, T]} d_\infty(w(t), \hat{w}(t)), \quad w, \hat{w} \in \mathcal{Y},$$

which makes $\mathcal{Y}$ a Polish space. Denote by $\mathcal{W}$ the law on $\mathcal{Y}$ of the sequence of independent Brownian motions $\{(W_t^j)_{0 \leq t \leq T} : j \in \mathbb{N}\}$.

To simplify the notations, we write $W_t = (W_t^j)_{0 \leq t \leq T}$ for the whole sequence of processes $\{(W_t^j)_{0 \leq t \leq T} : j \in \mathbb{N}\}$ in $\mathcal{Y}$. Denote by $P^N$ the joint law of $\{(\omega^N, W_t) : t \in \mathcal{X} \times \mathcal{Y}, N \geq 1\}$. Since the marginal laws $\{Q^N\}_{N \in \mathbb{N}}$ and $\{\mathcal{W}\}$ are respectively tight on $\mathcal{X}$ and $\mathcal{Y}$, we conclude that $\{P^N\}_{N \in \mathbb{N}}$ is tight on $\mathcal{X} \times \mathcal{Y}$. By Skorokhod’s representation theorem, there exist a subsequence $\{N_k\}_{k \in \mathbb{N}}$ of integers, a probability space $(\hat{\Theta}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ and stochastic processes $(\hat{\omega}^N, \hat{W}^N)$ on this space with the corresponding laws $P^{N_k}$, and converging $\hat{\mathbb{P}}$-a.s. in $\mathcal{X} \times \mathcal{Y}$ to a limit $(\hat{\omega}, \hat{W})$. We are going to prove that $(\hat{\omega}, \hat{W})$, or more precisely another closely defined process, is the solution claimed by Theorem 1.3.

Similar to the discussions above [18, Lemma 28], we need to enlarge the probability space $(\hat{\Theta}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ so that it contains certain independent r.v.’s we need. The rough idea is to apply a random permutation to an $(\mathbb{R} \times \mathbb{T}^2)^N$-valued r.v. which corresponds, via the mapping (2.4), to a r.v. with values in $\mathcal{M}_N(\mathbb{T}^2)$, see the end of Step 1 in the proof of [18, Lemma 28] for more details. Denote by $(\tilde{\Theta}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ a probability space on which, for every $N \geq 1$, we define a uniformly distributed random permutation $\tilde{s}_N : \tilde{\Theta} \to \Sigma_N$, where $\Sigma_N$ is the permutation group of order $N$. Define the product probability space

$$(\tilde{\Theta}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) = (\tilde{\Theta} \times \tilde{\Theta}, \tilde{\mathcal{F}} \otimes \tilde{\mathcal{F}}, \tilde{\mathbb{P}} \otimes \tilde{\mathbb{P}})$$

and the new processes

$$(\omega^{N_k}, W^{N_k}) = (\hat{\omega}^{N_k}, \hat{W}^{N_k}) \circ \pi_1, \quad (\omega, W) = (\hat{\omega}, \hat{W}) \circ \pi_1, \quad s_N = \tilde{s}_N \circ \pi_2,$$

where $\pi_1$ and $\pi_2$ are the projections on $\hat{\Theta} \times \tilde{\Theta}$. Here, we slightly abuse the notations by denoting the final probability spaces and processes like the original ones. We shall clarify in the sequel which ones we are investigating.

First, we have the following simple result.
Lemma 3.4 For every \( t \in [0, T] \), the law \( \mu_t \) of \( \omega_t \) on \( H^{-1-}(\mathbb{T}^2) \) is absolutely continuous with respect to the law \( \mu \) of white noise, with a bounded density denoted by \( \rho_t \).

**Proof** Note that \( \omega_t \) is defined on the product probability space (3.6) but it has the same law with \( \hat{\omega} \). Hence it suffices to prove the assertion for \( \hat{\omega}_t, \ t \in [0, T] \).

For every non-negative \( F \in C_b(H^{-1-}(\mathbb{T}^2)) \), since \( \hat{\omega}_t^{N_k} \) converges to \( \hat{\omega}_t \) a.s., one has

\[
\int F(\omega) \, d\mu_t(\omega) = \hat{\mathbb{E}} \left[ F(\hat{\omega}_t) \right] = \lim_{k \to \infty} \hat{\mathbb{E}} \left[ F(\hat{\omega}_t^{N_k}) \right] = \lim_{k \to \infty} \mathbb{E} \left[ F(\omega_t^{N_k}) \right]
\]

(3.7)

where \( \hat{\mathbb{E}} \) is the expectation on \( (\hat{\Theta}, \hat{\mathcal{F}}, \hat{\mathbb{P}}) \) and \( \mathbb{E} \) the one on the original probability space. By Lemma 2.6 and Proposition 2.2,

\[
\int F(\omega) \, d\mu_t(d\omega) \leq \lim_{k \to \infty} C_{N_k} \| \rho_0 \|_\infty \int F(\omega) \, d\mu_0^{N_k}(d\omega) = \| \rho_0 \|_\infty \int F(\omega) \, d\mu(\omega).
\]

This implies that \( \mu_t \ll \mu \) with a density bounded by \( \| \rho_0 \|_\infty \). \( \square \)

The following result identifies the structure of \( \omega_t^{N_k} \) as a sum of Dirac masses.

Lemma 3.5 The process \( \omega_t^{N_k} \) on the new probability space can be represented in the form

\[
\frac{1}{\sqrt{N_k}} \sum_{i=1}^{N_k} \xi_i \delta_{X_t^{i,N_k}},
\]

where

\[
\left( \left( \xi_1, X_t^{1,N_k} \right), \ldots, \left( \xi_{N_k}, X_t^{N_k,N_k} \right) \right)
\]

is a random vector with law \( C_{N_k}(\rho_0 \circ T_{N_k}) \lambda_{N_k}^0 \) and \( (X_t^{1,N_k}, \ldots, X_t^{N_k,N_k}) \) solves the stochastic system (2.1) with the initial condition \( (X_0^{1,N_k}, \ldots, X_0^{N_k,N_k}) \) and new Brownian motions \( \left\{ (W_t^{N_k,j}) : 1 \leq j \leq N_k \right\} \) defined above.

**Proof** Repeating Step 1 of the proof of [18, Lemma 28], we can find a family of random elements \( (\hat{\xi}_1, \hat{X}_t^{1,N_k}), \ldots, (\hat{\xi}_{N_k}, \hat{X}_t^{N_k,N_k}) \) in \( \mathbb{R} \times C([0, T], \mathbb{T}^2) \), such that

\[
\hat{\omega}_t^{N_k} = \frac{1}{\sqrt{N_k}} \sum_{i=1}^{N_k} \hat{\xi}_i \delta_{\hat{X}_t^{i,N_k}}.
\]

The first claim will be proved after a redefinition of the random elements.

Next we follow the arguments of Krylov [21, Section 2.6, p. 89]. Consider the filtration defined on the original probability space \( (\Theta, \mathcal{F}, \mathbb{P}) \):

\[
\mathcal{F}_t = \sigma \left( (\xi_n, X_0^n) : n \in \mathbb{N} \right) \vee \sigma \left( W_s : s \leq t \right), \quad t \in [0, T],
\]

where \( (\xi_n, X_0^n), \ n \in \mathbb{N} \) are given at the beginning of Sect. 2.1. Recall that we denote by \( W_t \) the sequence of Brownian motions \( \{ W_t^j : j \in \mathbb{N} \} \). The processes \( \left( \omega_t^N, W_t \right) \) are adapted to the filtration \( (\mathcal{F}_t)_{0 \leq t \leq T} \). Fix any \( t_0 \in [0, T) \). The increments of \( W_s \) after the time \( t_0 \) is independent on \( \mathcal{F}_{t_0} \). Therefore, the processes \( \left( \omega_t^N, W_t \right) (t \leq t_0) \) do not depend on the increments of \( W_s \) after the time \( t_0 \). Due to the coincidence of
finite dimensional distributions, the processes \((\hat{\omega}^{N_k}_t, \hat{W}^{N_k}_t)\) \((t \leq t_0)\) do not depend on
the increments of \(\hat{W}^{N_k}_s\) after the time \(t_0\). This property passes in the limit procedure
to the process \((\hat{\omega}, \hat{W})\). For the sake of convenience, we also denote \((\hat{\omega}, \hat{W})\) by
\((\hat{\omega}^{N_0}_t, \hat{W}^{N_0}_t)\). The above arguments imply that, for all \(k \geq 0\) and any \(j \in \mathbb{N}\), \(\hat{W}^{N_k,j}_t\) is a Brownian motion with respect to the filtration \(\hat{\mathcal{F}}^{N_k}_t\), which is the completion of
\(\sigma(\hat{\omega}^{N_k}_s, \hat{W}^{N_k}_s : s \leq t), t \in [0, T]\). Moreover, for all \(k \geq 0\) and \(s \leq t\), \(\hat{\omega}^{N_k}_s\) is \(\hat{\mathcal{F}}^{N_k}_t\)
measurable. Since \(\hat{\omega}^{N_k}_s\) is continuous with respect to \(s\), it is a progressively measurable
process with respect to \(\hat{\mathcal{F}}^{N_k}_t\). Therefore, the stochastic integrals involved below make
sense.

For any \(k \geq 1\), the original process \(\omega^{N_k}_t\) satisfies (2.5), which implies

\[
\mathbb{E}\left[\sup_{t \in [0,T]} \left| \langle \omega^{N_k}_t, \phi \rangle - \langle \omega^{N_k}_0, \phi \rangle - \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \nabla \phi(x) \cdot K(x - y) \omega^{N_k}_s(dx) \omega^{N_k}_s(dy) ds \right| \right] = 0
\]

for all \(\phi \in C^\infty(\mathbb{T}^2)\). The same property holds for the new processes \((\hat{\omega}^{N_k}_t, \hat{W}^{N_k}_t)\), because they have the same finite dimensional distributions with \((\omega^{N_k}_t, W_t)\). Hence, \(\hat{\mathbb{P}}\)-a.s., it holds

\[
\sup_{t \in [0,T]} \left| \langle \hat{\omega}^{N_k}_t, \phi \rangle - \langle \hat{\omega}^{N_k}_0, \phi \rangle - \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \nabla \phi(x) \cdot K(x - y) \hat{\omega}^{N_k}_s(dx) \hat{\omega}^{N_k}_s(dy) ds \right| = 0
\]

on a dense countable set of \(\phi \in C^\infty(\mathbb{T}^2)\). Using the structure
\(\hat{\omega}^{N_k}_t = \frac{1}{\sqrt{N_k}} \sum_{i=1}^{N_k} \hat{\epsilon}_i \delta_{\hat{X}_i^{N_k}, t}\), we conclude that \((\hat{X}^{1,N_k}_t, \ldots, \hat{X}^{N_k,N_k}_t)\) solves the stochastic system (2.1) with the Brownian motions \(\hat{W}^{N_k,j}_t\) for all \(1 \leq j \leq N_k\).

At this stage, we can get the final assertion by applying the so-called shuffling procedure, which amounts to redefining the r.v.’s and processes on the product probability space (3.6) by composition with random permutations given before Lemma 3.4. The remaining part of the proof is similar to the end of Step 2 of the proof of [18, Lemma 28], with the exception that the random vector (3.8) has law \(C_{N_k}(\rho_0 \circ T_{N_k}) \lambda^0_{N_k}\), thus we omit it here.

Finally, we show that the processes \((\omega, W)\) defined on the new probability space (3.6) is a \(\rho\)-white noise solution to the stochastic Euler equation.

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Proposition 3.6 For any $\phi \in C^\infty(\mathbb{T}^2)$ and $t \in [0, T]$,

$$
\mathbb{E}\left[\langle \omega_t, \phi \rangle - \langle \omega_0, \phi \rangle - \int_0^t \langle \omega_s \otimes \omega_s, H_\phi \rangle \, ds - \sum_{j=1}^{\infty} \int_0^t \langle \omega_s, \sigma_j \cdot \nabla \phi \rangle \, dW_s^j \right]
- \frac{1}{2} \sum_{j=1}^{\infty} \int_0^t \langle \omega_s, \sigma_j \cdot \nabla (\sigma_j \cdot \nabla \phi) \rangle \, ds \right\rceil_1 = 0.
$$

This implies that (1.8) holds a.s. at time $t$. Since the processes are continuous, we see that the identity holds uniformly in time, with probability one on the product space (3.6). This will prove the assertion of Theorem 1.3.

Proof of Proposition 3.6 We denote by $I$ the expectation on the left hand side of the identity. Recall the definition of $(\omega^N_k, W^N_k)$ before Lemma 3.4. This process has the same distribution as that of $(\bar{\omega}^N_k, \bar{W}^N_k)$. Thus it follows from (3.9) that for every $k \in \mathbb{N}$, it holds $\mathbb{P}$-a.s.,

$$
\langle \omega^N_k, \phi \rangle - \langle \omega^N_0, \phi \rangle - \int_0^t \langle \omega^N_k \otimes \omega^N_k, H_\phi \rangle \, ds - \sum_{j=1}^{N_k} \int_0^t \langle \omega^N_k, \sigma_j \cdot \nabla \phi \rangle \, dW^N_{k,j} \rangle \right\rceil_1 = 0.
$$

Consequently, using the simple inequality $|a + b| \wedge 1 \leq |a| \wedge 1 + |b| \wedge 1$ leads to

$$
I \leq \mathbb{E}\left[|\langle \omega_t, \phi \rangle - \langle \omega^N_t, \phi \rangle| \wedge 1\right] + \mathbb{E}\left[|\langle \omega_0, \phi \rangle - \langle \omega^N_0, \phi \rangle| \wedge 1 \right]
+ \mathbb{E}\left[\left|\int_0^t \langle \omega_s \otimes \omega_s, H_\phi \rangle \, ds - \int_0^t \langle \omega^N_s \otimes \omega^N_s, H_\phi \rangle \, ds \right| \wedge 1 \right]
+ \mathbb{E}\left[\left|\sum_{j=1}^{\infty} \int_0^t \langle \omega_s, \sigma_j \cdot \nabla \phi \rangle \, dW_s^j - \sum_{j=1}^{N_k} \int_0^t \langle \omega^N_s, \sigma_j \cdot \nabla \phi \rangle \, dW^N_{s,j} \right| \wedge 1 \right]
+ \mathbb{E}\left[\left|\frac{1}{2} \sum_{j=1}^{\infty} \int_0^t \langle \omega_s, \sigma_j \cdot \nabla (\sigma_j \cdot \nabla \phi) \rangle \, ds - \frac{1}{2} \sum_{j=1}^{N_k} \int_0^t \langle \omega^N_s, \sigma_j \cdot \nabla (\sigma_j \cdot \nabla \phi) \rangle \, ds \right| \wedge 1 \right].
$$

We denote by $I^N_k, i = 1, \ldots, 5$ the terms on the right hand side of the above inequality.

First, by the a.s. convergence of $\omega^N_k$ to $\omega$ in $C([0, T], H^{-1}(\mathbb{T}^2))$ we immediately get

$$
\lim_{k \to \infty} I^N_1 = \lim_{k \to \infty} I^N_2 = 0.
$$

Next, to show that $I^N_3$ tends to $0$, we consider a smooth approximation $H^\delta_\phi$ of $H_\phi$ (see [18, Remark 9]), with $H^\delta_\phi(x, x) = 0$ for all $x \in \mathbb{T}^2$ and $\delta > 0$. The a.s.
convergence of $\omega_{N_k}$ to $\omega$ in $C([0, T], H^{-1-}(\mathbb{T}^2))$ implies that of $\omega_{N_k} \otimes \omega_{N_k}$ to $\omega \otimes \omega$ in $C([0, T], H^{-2-}((\mathbb{T}^2 \times \mathbb{T}^2))$. Hence, for all $\delta > 0$,

$$\lim_{k \to \infty} \mathbb{E} \left[ \left| \int_0^t \langle \omega_s \otimes \omega_s, H^\delta \phi \rangle \, ds - \int_0^t \langle \omega_{N_k}^s \otimes \omega_{N_k}^s, H^\delta \phi \rangle \, ds \right| \wedge 1 \right] = 0.$$

As a result,

$$\lim_{k \to \infty} I_{3k} \leq \mathbb{E} \left[ \left| \int_0^t \langle \omega_s \otimes \omega_s, H^\delta \phi - H \phi \rangle \, ds \right| \wedge 1 \right] + \limsup_{k \to \infty} \mathbb{E} \left[ \left| \int_0^t \langle \omega_{N_k}^s \otimes \omega_{N_k}^s, H^\delta \phi - H \phi \rangle \, ds \right| \wedge 1 \right] =: I_{3,1} + I_{3,2}. \quad (3.10)$$

We have

$$I_{3,1} \leq \int_0^t \mathbb{E} |\langle \omega_s \otimes \omega_s, H^\delta \phi - H \phi \rangle| \, ds \leq \int_0^t \left[ \mathbb{E} \langle \omega_s \otimes \omega_s, H^\delta \phi - H \phi \rangle^2 \right]^{1/2} \, ds.$$

Thus by Lemma 3.4 and [18, Theorem 8],

$$I_{3,1} \leq \int_0^t \left[ \mathbb{E} \mu(\rho_s(\omega), H^\delta \phi - H \phi)^2 \right]^{1/2} \, ds \leq t \| \rho_0 \|_2 \left[ \mathbb{E} \mu(\omega \otimes \omega, H^\delta \phi - H \phi)^2 \right]^{1/2} \to 0 \quad \text{as } \delta \to 0. \quad (3.11)$$

Here $\mathbb{E}_\mu$ is the expectation on $H^{-1-}$ w.r.t. the white noise measure $\mu$. Similarly, using Lemma 2.6 we can show that

$$I_{3,2} \leq \limsup_{k \to \infty} \int_0^t \left[ \mathbb{E} \langle \omega_{N_k}^s \otimes \omega_{N_k}^s, H^\delta \phi - H \phi \rangle^2 \right]^{1/2} \, ds \leq \limsup_{k \to \infty} \int_0^t C_{N_k} \| \rho_0 \|_2 \int_{H^{-1-}(\mathbb{T}^2)} |\omega \otimes \omega, H^\delta \phi - H \phi|^2 \mu_{N_k}^0 (d\omega) \right]^{1/2} \, ds.$$

Now by the last assertion of Lemma 2.4 and the fact that $H_\phi(x, x) = H^\delta_\phi(x, x) = 0$,

$$I_{3,2} \leq t \sqrt{2} \| \rho_0 \|_2 \left[ \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} (H^\delta \phi - H \phi)^2 (x, y) \, dx \, dy \right]^{1/2} \to 0 \quad \text{as } \delta \to 0.$$

Combining this result with (3.10) and (3.11), we obtain

$$\lim_{k \to \infty} I_{3k} = 0.$$

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We turn to the simpler term $I_{5}^{N_k}$. Fix some big integer $J$. For $N_k > J$, we have

$$2I_{5}^{N_k} \leq \mathbb{E} \left[ \left| \sum_{j=1}^{J} \int_{0}^{t} \left( \omega_s, \sigma_j \cdot \nabla (\sigma_j \cdot \nabla \phi) \right) \right| \right] + \mathbb{E} \left[ \left| \sum_{j=J+1}^{\infty} \int_{0}^{t} \left( \omega_s, \sigma_j \cdot \nabla (\sigma_j \cdot \nabla \phi) \right) \right| \right]$$

$$=: I_{5.1}^{N_k} + I_{5.2}^{N_k} + I_{5.3}^{N_k}.$$  \hfill (3.12)

Analogous to $I_{1}^{N_k}$ and $I_{2}^{N_k}$, since $\sum_{j=1}^{J} \sigma_j \cdot \nabla (\sigma_j \cdot \nabla \phi)$ is smooth on $\mathbb{T}^2$, we have

$$\lim_{k \to \infty} I_{5.1}^{N_k} = 0. \quad \hfill (3.13)$$

Next, by Lemma 3.4,

$$I_{5.2}^{N_k} \leq \sum_{j=J+1}^{\infty} \int_{0}^{t} \mathbb{E} \left| \left( \omega_s, \sigma_j \cdot \nabla (\sigma_j \cdot \nabla \phi) \right) \right| ds \leq \|\rho_0\|_{\infty} \sum_{j=J+1}^{\infty} \int_{0}^{t} \mathbb{E}_{\mu} \left| \left( \omega_s, \sigma_j \cdot \nabla (\sigma_j \cdot \nabla \phi) \right) \right| ds.$$  

By Cauchy’s inequality and the definition of the white noise measure $\mu$, we have

$$\mathbb{E}_{\mu} \left| \left( \omega_s, \sigma_j \cdot \nabla (\sigma_j \cdot \nabla \phi) \right) \right| \leq \left( \mathbb{E}_{\mu} \left| \left( \omega_s, \sigma_j \cdot \nabla (\sigma_j \cdot \nabla \phi) \right) \right|^2 \right)^{1/2}$$

$$= \left( \int_{\mathbb{T}^2} \left| \sigma_j \cdot \nabla (\sigma_j \cdot \nabla \phi) \right|^2 dx \right)^{1/2}$$

$$\leq \|\sigma_j\|_{\infty}^2 \|\nabla^2 \phi\|_{L^2(\mathbb{T}^2)} + \|\sigma_j \cdot \nabla \sigma_j\|_{\infty} \|\nabla \phi\|_{L^2(\mathbb{T}^2)}.$$  

Therefore, for any $k$,

$$I_{5.2}^{N_k} \leq C_{\phi} T \|\rho_0\|_{\infty} \sum_{j=J+1}^{\infty} \left( \|\sigma_j\|_{\infty}^2 + \|\sigma_j \cdot \nabla \sigma_j\|_{\infty} \right). \quad \hfill (3.14)$$

In the same way, using Lemma 2.6, we can prove that, for all $N_k > J$,

$$I_{5.2}^{N_k} \leq C'_{\phi} T \|\rho_0\|_{\infty} \sum_{j=J+1}^{\infty} \left( \|\sigma_j\|_{\infty}^2 + \|\sigma_j \cdot \nabla \sigma_j\|_{\infty} \right).$$
Combining this estimate with (H2) and (3.12)–(3.14), first letting $k \to \infty$ in (3.12), and then $J \to \infty$, we obtain
\[
\lim_{k \to \infty} I_{5j}^{N_k} = 0.
\]
It remains to deal with the more difficult term $I_{4j}^{N_k}$. Fix again $J \in \mathbb{N}$. We have, for all $N_k > J$,
\[
I_{4j}^{N_k} \leq \mathbb{E} \left[ \left| \sum_{j=J+1}^{N_k} \int_0^t \langle \omega_s, \sigma_j \cdot \nabla \phi \rangle \, dW_j^s \right|^2 \right] + \mathbb{E} \left[ \left| \sum_{j=J+1}^{N_k} \int_0^t \langle \omega_{s}^{N_k}, \sigma_j \cdot \nabla \phi \rangle \, dW_{N_k}^j \right|^2 \right]
\]
\[+ \mathbb{E} \left[ \left| \sum_{j=1}^{J} \int_0^t \langle \omega_s, \sigma_j \cdot \nabla \phi \rangle \, dW_j^s - \sum_{j=1}^{J} \int_0^t \langle \omega_{s}^{N_k}, \sigma_j \cdot \nabla \phi \rangle \, dW_{N_k}^j \right|^2 \right]
\]
\[=: I_{4j,1}^{N_k} + I_{4j,2}^{N_k} + I_{4j,3}^{N_k}. \tag{3.15}\]
By the Cauchy inequality and Itô isometry,
\[
I_{4j,1}^{N_k} \leq \left\{ \mathbb{E} \left[ \left| \sum_{j=J+1}^{N_k} \int_0^t \langle \omega_s, \sigma_j \cdot \nabla \phi \rangle \, dW_j^s \right|^2 \right] \right\}^{1/2}
\]
\[= \left\{ \int_0^t \sum_{j=J+1}^{\infty} \mathbb{E} \langle \omega_s, \sigma_j \cdot \nabla \phi \rangle^2 \, ds \right\}^{1/2}. \tag{3.16}\]
Lemma 3.4 implies that
\[
I_{4j,1}^{N_k} \leq \left\{ \int_0^t \sum_{j=J+1}^{\infty} \|\rho_0\|_{\infty} \mathbb{E}_{\mu} \langle \omega, \sigma_j \cdot \nabla \phi \rangle^2 \, ds \right\}^{1/2}
\]
\[= \sqrt{t}\|\rho_0\|_{\infty} \left( \sum_{j=J+1}^{\infty} \int_{T^2} |\sigma_j \cdot \nabla \phi|^2 \, dx \right)^{1/2}
\]
\[\leq \sqrt{t}\|\rho_0\|_{\infty} \|\nabla \phi\|_{L^2(T^2)} \left( \sum_{j=J+1}^{\infty} \|\sigma_j \|_{\infty}^2 \right)^{1/2}. \tag{3.17}\]
Similarly, by Corollary 2.7,
\[
I_{4j,2}^{N_k} \leq \left\{ \int_0^t \sum_{j=J+1}^{N_k} \mathbb{E} \langle \omega_{s}^{N_k}, \sigma_j \cdot \nabla \phi \rangle^2 \, ds \right\}^{1/2}
\]
\[\leq \left\{ \int_0^t \sum_{j=J+1}^{\infty} C\|\rho_0\|_{\infty} \|\sigma_j \cdot \nabla \phi\|_{\infty} \|\sigma_j \|_{\infty}^2 \, ds \right\}^{1/2}
\]
\[\leq \sqrt{C}t\|\rho_0\|_{\infty} \|\nabla \phi\|_{\infty} \left( \sum_{j=J+1}^{\infty} \|\sigma_j \|_{\infty}^2 \right)^{1/2}. \tag{3.17}\]
Finally, we consider the quantity $I_{4,3}^{N_k}$. Denote by $\eta_s = (\langle \omega_s, \sigma_1 \cdot \nabla \phi \rangle, \ldots, \langle \omega_s, \sigma_J \cdot \nabla \phi \rangle)$; then

$$\mathbb{E}(|\eta_s|^4) = \mathbb{E}\left( \left( \sum_{j=1}^J (\omega_s, \sigma_j \cdot \nabla \phi)^2 \right)^2 \right) = \sum_{j,l=1}^J \mathbb{E}(\langle \omega_s, \sigma_j \cdot \nabla \phi \rangle^2 \langle \omega_s, \sigma_l \cdot \nabla \phi \rangle^2) \leq \sum_{j,l=1}^J (\mathbb{E}(\langle \omega_s, \sigma_j \cdot \nabla \phi \rangle^4)^{1/2} (\mathbb{E}(\langle \omega_s, \sigma_l \cdot \nabla \phi \rangle^4)^{1/2} = \left\{ \sum_{j=1}^J (\mathbb{E}(\langle \omega_s, \sigma_j \cdot \nabla \phi \rangle^4)^{1/2} \right)^2.$$

Again by Lemma 3.4,

$$\mathbb{E}(|\eta_s|^4) \leq \left\{ \sum_{j=1}^J (\|\rho_0\|_{\infty} C \|\sigma_j \cdot \nabla \phi\|_{\infty}^4)^{1/2} \right\} \leq C \|\rho_0\|_{\infty} \|\nabla \phi\|_{\infty}^4 \left\{ \sum_{j=1}^J \|\sigma_j\|_{\infty}^2 \right\}^2.$$

As a result, $\int_0^T \mathbb{E}(|\eta_s|^4) \, ds < \infty$. Similarly, setting $\eta_s^k = (\langle \omega_s^{N_k}, \sigma_1 \cdot \nabla \phi \rangle, \ldots, \langle \omega_s^{N_k}, \sigma_J \cdot \nabla \phi \rangle)$ and using Corollary 2.7, we can show that

$$\sup_{k \in \mathbb{N}} \int_0^T \mathbb{E}(|\eta_s^k|^4) \, ds < \infty.$$

Since $(\omega^{N_k}, W^{N_k})$ converge to $(\omega, W)$ a.s., we can apply [24, Lemma 3.2] to get

$$\lim_{k \to \infty} \mathbb{E}\left[ \left| \sum_{j=1}^J \int_0^T \langle \omega_s, \sigma_j \cdot \nabla \phi \rangle \, dW^j_s - \sum_{j=1}^J \int_0^T \langle \omega_s^{N_k}, \sigma_j \cdot \nabla \phi \rangle \, dW^{N_k,j}_s \right|^2 \right] = 0.$$

Therefore, first letting $k \to \infty$ and then $J \to \infty$ in (3.15), we deduce from the above limit and (3.16), (3.17) that

$$\lim_{k \to \infty} I_{4,3}^{N_k} = 0.$$

We have shown that all the terms $I_{i}^{N_k}, i = 1, \ldots, 5$ tend to 0 as $k \to \infty$. The proof is complete.

\section*{4 Proof of Theorem 1.4}

We prove the two assertions of Theorem 1.4 in the following two subsections respectively.
4.1 Proof of assertion (i)

Let $\omega$ be a solution of the stochastic Euler Eq. (1.2) given by Theorem 1.3, with the associated density $\rho$. Let $F \in \mathcal{F}_{\rho,T}$ be of the form $F(t, \omega) = \sum_{i=1}^{m} g_i(t) f_i(\langle \omega, \phi_1 \rangle, \ldots, \langle \omega, \phi_n \rangle)$. For every $j = 1, \ldots, n$, we have

\[
d\langle \omega_t, \phi_j \rangle = \langle \omega_t \otimes \omega_t, H_{\phi_j} \rangle \, dt + \sum_{k=1}^{\infty} \langle \omega_t, \sigma_k \cdot \nabla \phi_j \rangle \, dW_t^k + \frac{1}{2} \sum_{k=1}^{\infty} \langle \omega_t, \sigma_k \cdot \nabla (\sigma_k \cdot \nabla \phi_j) \rangle \, dt.
\]

To simplify the notations, we denote by $\Phi = (\phi_1, \ldots, \phi_n)$ and $\langle \omega, \Phi \rangle = (\langle \omega, \phi_1 \rangle, \ldots, \langle \omega, \phi_n \rangle)$. Then, the Itô formula leads to

\[
d f_i(\langle \omega_t, \Phi \rangle) = \left[ \sum_{j=1}^{n} \partial_j f_i(\langle \omega_t, \Phi \rangle) \right] \langle \omega_t \otimes \omega_t, H_{\phi_j} \rangle \, dt + \sum_{k=1}^{\infty} \langle \omega_t, \sigma_k \cdot \nabla \phi_j \rangle \, dW_t^k + \frac{1}{2} \left[ \sum_{k=1}^{\infty} \langle \omega_t, \sigma_k \cdot \nabla (\sigma_k \cdot \nabla \phi_j) \rangle \right] \, dt.
\]

By the definition of $\langle D_\omega F(t, \omega), b(\omega) \rangle$,

\[
d F(t, \omega_t) = \sum_{i=1}^{m} g_i'(t) f_i(\langle \omega_t, \Phi \rangle) \, dt + \sum_{i=1}^{m} g_i(t) \, d f_i(\langle \omega_t, \Phi \rangle)
\]

\[
= \left[ (\partial_t F)(t, \omega_t) + \langle b(\omega_t), D_\omega F(t, \omega_t) \rangle \right] \, dt + dM(t)
\]

\[
+ \frac{1}{2} \sum_{k=1}^{\infty} \sum_{l=1}^{m} g_l(t) \left[ \sum_{j=1}^{n} \partial_j f_i(\langle \omega_t, \Phi \rangle) \langle \omega_t, \sigma_k \cdot \nabla (\sigma_k \cdot \nabla \phi_j) \rangle \right] \, dt,
\]

where the martingale part

\[
d M(t) = \sum_{i=1}^{m} g_i(t) \sum_{j=1}^{n} \partial_j f_i(\langle \omega_t, \Phi \rangle) \sum_{k=1}^{\infty} \langle \omega_t, \sigma_k \cdot \nabla \phi_j \rangle \, dW_t^k.
\]
Lemma 4.1 Assume that $G \in \mathcal{FC}_p$ has the form $G(\omega) = g(\langle \omega, \Phi \rangle) = g(\langle \omega, \phi_1 \rangle, \ldots, \langle \omega, \phi_n \rangle)$. Then

$$\{ \sigma_k \cdot \nabla \omega, D_\omega \{ \sigma_k \cdot \nabla \omega, D_\omega G \} \} = \sum_{j=1}^n \partial_j g(\langle \omega, \Phi \rangle) \{ \omega, \sigma_k \cdot \nabla (\sigma_k \cdot \nabla \phi_j) \}$$

$$+ \sum_{j,l=1}^n \partial_{j,l} g(\langle \omega, \Phi \rangle) \{ \omega, \sigma_k \cdot \nabla \phi_j \} \{ \omega, \sigma_k \cdot \nabla \phi_l \}.$$

**Proof** Since $D_\omega G = \sum_{j=1}^n \partial_j g(\langle \omega, \Phi \rangle) \phi_j$, we have

$$\{ \sigma_k \cdot \nabla \omega, D_\omega G \} = \sum_{j=1}^n \partial_j g(\langle \omega, \Phi \rangle) \{ \omega, \sigma_k \cdot \nabla \phi_j \} = - \sum_{j=1}^n \partial_j g(\langle \omega, \Phi \rangle) \{ \omega, \sigma_k \cdot \nabla \phi_j \},$$

where the last equality is due to $\text{div}(\sigma_k) = 0$. Therefore,

$$D_\omega \{ \sigma_k \cdot \nabla \omega, D_\omega G \} = - \sum_{j=1}^n \{ \omega, \sigma_k \cdot \nabla \phi_j \} \sum_{l=1}^n \partial_{l,j} g(\langle \omega, \Phi \rangle) \phi_l$$

$$- \sum_{j=1}^n \partial_j g(\langle \omega, \Phi \rangle) \{ \sigma_k \cdot \nabla \phi_j \}.$$

As a result,

$$\{ \sigma_k \cdot \nabla \omega, D_\omega \{ \sigma_k \cdot \nabla \omega, D_\omega G \} \} = - \sum_{j,l=1}^n \partial_{l,j} g(\langle \omega, \Phi \rangle) \{ \omega, \sigma_k \cdot \nabla \phi_j \} \{ \sigma_k \cdot \nabla \omega, \sigma_k \cdot \nabla \phi_l \}$$

$$- \sum_{j=1}^n \partial_j g(\langle \omega, \Phi \rangle) \{ \sigma_k \cdot \nabla \omega, \sigma_k \cdot \nabla \phi_j \}.$$

This immediately leads to the desired result by integration by parts. □

Using the above lemma, we obtain

$$dF(t, \omega_t) = \left[ (\partial_t F)(t, \omega_t) + \langle b(\omega_t), D_\omega F(t, \omega_t) \rangle \right] dt + dM(t)$$

$$+ \frac{1}{2} \sum_{k=1}^\infty \{ \sigma_k \cdot \nabla \omega_t, D_\omega \{ \sigma_k \cdot \nabla \omega_t, D_\omega F(t, \omega_t) \} \} dt. \quad (4.3)$$

Following the arguments in Remark 1.2 we can show that $M(t)$ is a square integrable martingale. Indeed, by the expression (4.2) of $M(t)$, it is sufficient to show that for each $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, one has
\[ I := \sum_{k=1}^{\infty} \mathbb{E} \int_{0}^{T} \left| g_i(t) \partial_j f_i(\langle \omega_t, \Phi \rangle) \langle \omega_t, \sigma_k \cdot \nabla \phi_j \rangle \right|^2 dt < \infty. \]

Since the law of \( \omega_t \) is \( \rho_t \mu \) and \( \| \rho_t \|_\infty \leq \| \rho_0 \|_\infty \) for all \( t \in [0, T] \), we have

\[ I \leq \| g_i \|_\infty^2 \| \rho_0 \|_\infty T \sum_{k=1}^{\infty} \mathbb{E}_\mu \left( \left| \partial_j f_i(\langle \omega, \Phi \rangle) \langle \omega, \sigma_k \cdot \nabla \phi_j \rangle \right|^2 \right), \]

where \( \mathbb{E}_\mu \) is the expectation on \( H^{-1/2} \) w.r.t. \( \mu \). By Cauchy’s inequality,

\[ I \leq C \sum_{k=1}^{\infty} \left( \mathbb{E}_\mu \left| \partial_j f_i(\langle \omega, \Phi \rangle) \right|^4 \right)^{1/2} \left( \mathbb{E}_\mu \left| \langle \omega, \sigma_k \cdot \nabla \phi_j \rangle \right|^4 \right)^{1/2} \]

\[ \leq CC_1 \sum_{k=1}^{\infty} (C_2 \| \sigma_k \cdot \nabla \phi_j \|_\infty^4)^{1/2} \leq C' \| \nabla \phi_j \|_\infty^2 \sum_{k=1}^{\infty} \| \sigma_k \|_\infty^2 < \infty, \]

where the second inequality is due to the facts that the function \( \partial_j f_i \) has polynomial growth and \( \langle \omega, \Phi \rangle \) is a centered Gaussian random vector under \( \mu \).

Integrating (4.3) from 0 to \( T \) and taking expectation, we deduce from \( F(T, \cdot) = 0 \) that

\[ 0 = \mathbb{E} F(0, \omega_0) + \int_{0}^{T} \mathbb{E} \left[ (\partial_t F)(t, \omega_t) + \langle b(\omega_t), D_\omega F(t, \omega_t) \rangle \right] dt \]

\[ + \frac{1}{2} \sum_{k=1}^{\infty} \int_{0}^{T} \mathbb{E} \left[ \sigma_k \cdot \nabla \omega_t, D_\omega \langle \sigma_k \cdot \nabla \omega_t, D_\omega F(t, \omega_t) \rangle \right] dt \]

\[ = \int F(0, \omega) \rho_0(\omega) \mu(d\omega) + \int_{0}^{T} \int \left[ (\partial_t F)(t, \omega) + \langle b(\omega), D_\omega F(t, \omega) \rangle \right] \rho_t(\omega) \mu(d\omega) dt \]

\[ + \frac{1}{2} \sum_{k=1}^{\infty} \int_{0}^{T} \left| \sigma_k \cdot \nabla \omega, D_\omega \langle \sigma_k \cdot \nabla \omega, D_\omega F(t, \omega) \rangle \right| \rho_t(\omega) \mu(d\omega) dt. \]

The proof of assertion (i) is complete.

**Remark 4.2** We remark that each integral on the r.h.s. of the above equation is finite. For instance, since \( \rho_t \) is bounded on \( H^{-1/2} \), uniformly in \( t \in [0, T] \), by the assertion above Theorem 1.4, we see that

\[ \int_{0}^{T} \int \left| \langle b(\omega), D_\omega F(t, \omega) \rangle \rho_t(\omega) \right| \mu(d\omega) dt < \infty. \]

Next, to prove the finiteness of the last integral, by (4.1) and (4.3), it is enough to show that for any fixed \( i \in \{1, \ldots, m\} \) and \( j, l \in \{1, \ldots, n\} \),

\[ J_1 := \sum_{k=1}^{\infty} \int_{0}^{T} \left| \partial_j f_i(\langle \omega, \Phi \rangle) \langle \omega, \sigma_k \cdot \nabla (\sigma_k \cdot \nabla \phi_j) \rangle \right| \rho_t(\omega) \mu(d\omega) dt < \infty \]

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and

\[ J_2 := \sum_{k=1}^{\infty} \int_0^T \int |\partial_{j,l} f_i (\langle \omega, \Phi \rangle, \omega, \sigma_k \cdot \nabla \phi_j) \langle \omega, \sigma_k \cdot \nabla \phi_l \rangle \rho_t (\omega) | \mu (d\omega) dt < \infty. \]

Here we only prove the second estimate. We have

\[ J_2 \leq \| \rho_0 \|_\infty T \sum_{k=1}^{\infty} \mathbb{E}_\mu |\partial_{j,l} f_i (\langle \omega, \Phi \rangle, \omega, \sigma_k \cdot \nabla \phi_j) \langle \omega, \sigma_k \cdot \nabla \phi_l \rangle| \]

\[ \leq C \sum_{k=1}^{\infty} \left( \mathbb{E}_\mu |\partial_{j,l} f_i (\langle \omega, \Phi \rangle)|^2 \right)^{1/2} \left( \mathbb{E}_\mu |\omega, \sigma_k \cdot \nabla \phi_j|^4 \right)^{1/4} \left( \mathbb{E}_\mu |\omega, \sigma_k \cdot \nabla \phi_l|^4 \right)^{1/4} \]

\[ \leq C C_1 \sum_{k=1}^{\infty} C_2 \| \sigma_k \cdot \nabla \phi_j \|_\infty \| \sigma_k \cdot \nabla \phi_l \|_\infty < \infty, \]

where the third inequality follows from the fact that \( \partial_{j,l} f_i \) has polynomial growth, and the last one is due to (H2).

### 4.2 Proof of assertion (ii)

Our strategy is to prove the assertion in three steps:

1. Fix \( N \in \mathbb{N} \). Prove the gradient estimate of the form (4.6) and (4.12) on \((\mathbb{T}^2)^N\) in the smooth case, i.e. the kernel \( K \) in (2.1) is replaced by some smooth \( K^\delta \).
2. Let \( \delta \to 0 \) to get the gradient estimate in the case of the singular Biot–Savart kernel \( K \), and rewrite it in terms of the density \( \rho_t^N \) of point vortices.
3. Let \( N \to \infty \) to obtain the desired result.

**Step 1: Smooth kernel \( K^\delta \)** We fix \( N \geq 1 \) and let \( K^\delta \) be the smooth kernel given in [19, Section 3.2]. For the moment we fix a family of vortex intensities \( \xi = (\xi_1, \ldots, \xi_N) \). Consider (2.1) with \( K \) replaced by \( K^\delta \) and denote the solution flow by \( X^\delta_t = (X^\delta_t, 1, \ldots, X^\delta_t, N) \). It is well known that \( X^\delta_t \) is a stochastic flow of diffeomorphisms on \((\mathbb{T}^2)^N\) (cf. [22, Section 4.7]).

Define the vector fields \( A_k^{(N)} \) on \((\mathbb{T}^2)^N\) as follows: for \( x = (x_1, \ldots, x_N) \in (\mathbb{T}^2)^N \),

\[ A_k^{(N)} (x) = A_{\sigma_k}^{(N)} (x) = (\sigma_k (x_1), \ldots, \sigma_k (x_N)). \]

For simplicity we shall write \( A_k \), \( k \in \mathbb{N} \). We also define the drift vector field \( A_0^\delta : (\mathbb{T}^2)^N \to (\mathbb{R}^2)^N \) by

\[ (A_0^\delta)_i (x) = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \xi_j K^\delta (x_i - x_j), \quad x \in (\mathbb{T}^2)^N, \quad 1 \leq i \leq N. \]
Then the Eq. (2.1) can be simply written as

$$
\begin{align*}
  dX_\delta t &= A_0^\delta (X_\delta t) dt + \sum_{k=1}^{N} A_k(X_\delta t) \circ dW^k_t, \quad X_0^\delta = x \in (\mathbb{T}^2)^N.
\end{align*}
$$

**Smooth initial condition** Let $v_0 : (\mathbb{T}^2)^N \to \mathbb{R}$ be a smooth function. For $x \in (\mathbb{T}^2)^N$, define $v_\delta^t(x) = v_0(X_\delta^t, -1)(x)$, where $X_\delta^t, -1$ is the inverse flow. We have (see [7, pp. 103–106])

$$
\begin{align*}
  dv_\delta t &= -(A_0^\delta \cdot \nabla v_\delta^t) dt - \sum_{k=1}^{N} (A_k \cdot \nabla v_\delta^t) dW^k_t + \frac{1}{2} \sum_{k=1}^{N} [A_k \cdot \nabla (A_k \cdot \nabla v_\delta^t)] dt.
\end{align*}
$$

Here $\nabla = \nabla^{2N}$ is the gradient on $(\mathbb{T}^2)^N$. For $\phi \in C^{1,2}([0, T] \times (\mathbb{T}^2)^N)$, Itô’s formula leads to

$$
\begin{align*}
  d(\phi, v_\delta^t) &= \phi_t v_\delta^t dt - \phi_t (A_0^\delta \cdot \nabla v_\delta^t) dt - \phi_t \sum_{k=1}^{N} (A_k \cdot \nabla v_\delta^t) dW^k_t \\
  &\quad + \frac{1}{2} \phi_t \sum_{k=1}^{N} [A_k \cdot \nabla (A_k \cdot \nabla v_\delta^t)] dt.
\end{align*}
$$

Integrating from 0 to $T$ yields

$$
\begin{align*}
  \phi_T v_\delta^T &= \phi_0 v_0^\delta + \int_{0}^{T} \phi_t v_\delta^t dt - \int_{0}^{T} \phi_t (A_0^\delta \cdot \nabla v_\delta^t) dt - \sum_{k=1}^{N} \int_{0}^{T} \phi_t (A_k \cdot \nabla v_\delta^t) dW^k_t \\
  &\quad + \frac{1}{2} \sum_{k=1}^{N} \int_{0}^{T} \phi_t [A_k \cdot \nabla (A_k \cdot \nabla v_\delta^t)] dt.
\end{align*}
$$

Define

$$
\begin{align*}
  u_\delta^t(x) &= \mathbb{E} v_\delta^t(x) = \mathbb{E} [v_0(X_\delta^t, -1)(x)], \quad (t, x) \in [0, T] \times (\mathbb{T}^2)^N.
\end{align*}
$$

Denote by $\langle \cdot, \cdot \rangle$ the inner product in $L^2((\mathbb{T}^2)^N, \lambda_N)$, where $\lambda_N = \text{Leb}_{\mathbb{T}^2}^\otimes N$. By integrating (4.4) on $(\mathbb{T}^2)^N$ and taking expectation we obtain
Note that \( u_\delta \in C^1([0, T], C^\infty((\mathbb{T}^2)^N)) \). Choosing \( \phi = u_\delta \) leads to

\[
\| u_\delta T \|_{L^2_x}^2 = \| u_0 \|_{L^2_x}^2 + \frac{1}{2} \int_0^T \frac{d}{dt} \| u_\delta t \|_{L^2_x}^2 \, dt - \int_0^T \{ u_\delta t, A_0^\delta \cdot \nabla u_\delta t \} \, dt \\
+ \frac{1}{2} \sum_{k=1}^N \int_0^T \{ u_\delta t, A_k \cdot \nabla (A_k \cdot \nabla u_\delta t) \} \, dt.
\]

Since \( \text{div}_{2N} (A_0^\delta) = 0 \), the third term on the r.h.s. vanishes. Applying the integration by parts in the last term yields

\[
\sum_{k=1}^N \int_0^T \| A_k \cdot \nabla u_\delta t \|_{L^2_x}^2 \, dt = \| u_\delta 0 \|_{L^2_x}^2 - \| u_\delta T \|_{L^2_x}^2 \leq \| v_0 \|_{\infty}^2.
\]

(4.6)

Therefore, we obtain the gradient estimate in the case that \( v_0 \in C^\infty((\mathbb{T}^2)^N) \).

**Continuous initial condition** Assume now \( v_0 \in C((\mathbb{T}^2)^N) \). We take a sequence of smooth functions \( v_n \) which converge uniformly to \( v_0 \), such that \( \| v_n \|_{\infty} \leq \| v_0 \|_{\infty} \). Then \( u_\delta t^{n,n}(x) = \mathbb{E}[v_n(X_t^{\delta,n}(x))] \) satisfies (4.6), i.e.

\[
\sum_{k=1}^N \int_0^T \| A_k \cdot \nabla u_\delta t^{n,n} \|_{L^2_x}^2 \, dt \leq \| v_n \|_{\infty}^2 \leq \| v_0 \|_{\infty}^2.
\]

(4.7)

Define the finite set of integers \( S_N = \{ 1, \ldots, N \} \). The above inequality shows that the sequence \( \{ (A_k \cdot \nabla u_\delta t^{n,n})(x) \mid (k, t, x) \in S_N \times [0, T] \times (\mathbb{T}^2)^N \}_n \) is bounded in \( L^2(S_N \times [0, T] \times (\mathbb{T}^2)^N, \# \otimes dt \otimes \lambda_N) \), where \( \# \) is the counting measure on \( S_N \) and \( \lambda_N = \text{Leb}_{\mathbb{T}^2}^{\otimes N} \). We denote this Hilbert space by \( L^2(S_N \times [0, T] \times (\mathbb{T}^2)^N) \) for simplicity. Then, there exists a subsequence \( \{ (A_k \cdot \nabla u_\delta t^{i,n}) \}_i \) which converges weakly to some \( \alpha^\delta \in L^2(S_N \times [0, T] \times (\mathbb{T}^2)^N) \), satisfying

\[
\sum_{k=1}^N \int_0^T \| \alpha_k^\delta (t) \|_{L^2_x}^2 \, dt \leq \| v_0 \|_{\infty}^2.
\]

(4.8)

Define the space of test functions by

\[
C_T(N) = \{ \beta = (\beta_1, \ldots, \beta_N) \mid \beta_k \in C^{0,1}([0, T] \times (\mathbb{T}^2)^N) \text{ for all } 1 \leq k \leq N \}.
\]

(4.9)
Then for any $\beta \in C_T(N)$,

$$\lim_{i \to \infty} \sum_{k=1}^{N} \int_{0}^{T} \int_{(T^2)^N} (A_k \cdot \nabla u_i^{\delta,n_i})(x) \beta_k(t,x) \, dx \, dt = \sum_{k=1}^{N} \int_{0}^{T} \int_{(T^2)^N} \alpha_k^{\delta}(t,x) \beta_k(t,x) \, dx \, dt.$$

Using the fact that $\text{div}_2 N(A_k) \equiv 0$ and integrating by parts give us

$$\sum_{k=1}^{N} \int_{0}^{T} \int_{(T^2)^N} (A_k \cdot \nabla u_i^{\delta,n_i})(x) \beta_k(t,x) \, dx \, dt = -\sum_{k=1}^{N} \int_{0}^{T} \int_{(T^2)^N} u_i^{\delta,n_i}(x)(A_k \cdot \nabla \beta_k(t))(x) \, dx \, dt$$

as $i \to \infty$, since $v_{n_i}$ converges uniformly to $v_0$. Here, $u_i^{\delta}(x) = E[v_0(X_t^{\delta,-1}(x))]$. Combining the two limits above yields

$$\sum_{k=1}^{N} \int_{0}^{T} \int_{(T^2)^N} \alpha_k^{\delta}(t,x) \beta_k(t,x) \, dx \, dt = -\sum_{k=1}^{N} \int_{0}^{T} \int_{(T^2)^N} u_i^{\delta}(x)(A_k \cdot \nabla \beta_k(t))(x) \, dx \, dt,$$

which holds for any $\beta \in C_T(N)$. This equality implies the weak limit $\alpha^{\delta}$ is independent of the choices of the sequence $\{v_n\}_{n \in \mathbb{N}}$ of smooth approximating functions and the subsequence $\{n_i\}_{i \in \mathbb{N}}$. Moreover, for any fixed $k \in S_N$, taking $\beta \in C_T(N)$ such that $\beta_j \equiv 0$ for all $j \neq k$, we obtain

$$\int_{0}^{T} \int_{(T^2)^N} \alpha_k^{\delta}(t,x) \beta_k(t,x) \, dx \, dt = -\int_{0}^{T} \int_{(T^2)^N} u_i^{\delta}(x)(A_k \cdot \nabla \beta_k(t))(x) \, dx \, dt.$$

Since the vector fields $\{A_k\}_{k \in S_N}$ are divergence free, we see that the following equalities

$$A_k \cdot \nabla u_i^{\delta} = \alpha_k^{\delta}, \quad k \in S_N$$

hold in the distributional sense. Combining this fact with (4.8) yields the gradient estimate

$$\sum_{k=1}^{N} \int_{0}^{T} \|A_k \cdot \nabla u_i^{\delta}\|_{L^2}^2 \, dt \leq \|v_0\|_{\infty}^2.$$

(4.12)
Step 2: Non-smooth kernel $K$ In this step we aim to extend the above gradient estimate to the case where $K$ is the singular Biot–Savart kernel. The proof is similar to the passage to the limit from smooth initial conditions to continuous ones.

Let $v_0 \in C((\mathbb{T}^2)^N, \mathbb{R}_+)$ be the initial probability density function of $X_0^\delta$. For any nonnegative continuous function $F$ on $(\mathbb{T}^2)^N$, we have

$$
\mathbb{E}[F(X_t^\delta)] = \int_{(\mathbb{T}^2)^N} \mathbb{E}[F(X_t^\delta(x))] v_0(x) \, dx
$$

$$
= \mathbb{E} \int_{(\mathbb{T}^2)^N} F(y) v_0(X_{t,}^\delta)^{-1}(y) \, dy \leq \|v_0\|_{\infty} \int_{(\mathbb{T}^2)^N} F(y) \, dy,
$$

(4.13)

where the second equality is due to the fact that $X_t^\delta$ preserves the volume measure of $(\mathbb{T}^2)^N$. By the proof of [19, Theorem 8], for $\lambda_N$-a.e. $x \in (\mathbb{T}^2)^N$, we have a.s. $X_t^\delta(x) \to X_t(x)$ as $\delta \to 0$ for all $t \in [0, T]$. The dominated convergence theorem yields

$$
\lim_{\delta \to 0} \mathbb{E}[F(X_t^\delta)] = \mathbb{E} \int_{(\mathbb{T}^2)^N} F(X_t(x)) v_0(x) \, dx
$$

$$
= \int_{(\mathbb{T}^2)^N} \mathbb{E}[F(X_t(x))] v_0(x) \, dx = \mathbb{E} F(X_t).
$$

(4.14)

Combining (4.13) and (4.14) we conclude that the law $\mu_t$ of $X_t$ is absolutely continuous w.r.t. $\lambda_N = \text{Leb}_{\mathbb{T}^2}$, with a density function $u_t$ bounded by $\|v_0\|_{\infty}$. Moreover, the second equality in (4.13) shows that $u_t^\delta(x) = \mathbb{E}[v_0(X_t^\delta)^{-1}(x))]$ is the density function of $X_t^\delta$. For general bounded continuous function $F$ on $(\mathbb{T}^2)^N$, analogous to (4.14), we have

$$
\lim_{\delta \to 0} \int_{(\mathbb{T}^2)^N} F(x) u_t^\delta(x) \, dx = \int_{(\mathbb{T}^2)^N} F(x) u_t(x) \, dx
$$

(4.15)

which means that $u_t^\delta$ converges weakly to $u_t$ as $\delta \to 0$.

Recall that $S_N = \{1, \ldots, N\}$. By (4.12), the family $\{(A_k \cdot \nabla u_t^\delta)(x) \mid k, t, x \in S_N \times [0, T] \times (\mathbb{T}^2)^N\}_{\delta \geq 0}$ is bounded in the Hilbert space $L^2(S_N \times [0, T] \times (\mathbb{T}^2)^N)$. Thus, there exists a subsequence $\{(A_k \cdot \nabla u_t^\delta)(x)\}_{i \in \mathbb{N}}$ which converges weakly to some function $\alpha \in L^2(S_N \times [0, T] \times (\mathbb{T}^2)^N)$, satisfying

$$
\sum_{k=1}^N \int_0^T \|\alpha_k(t)\|^2_{L^2} \, dt \leq \|v_0\|^2_{\infty}.
$$

(4.16)

Consequently, for any $\beta \in C_T(N)$ [see (4.9)],

$$
\lim_{i \to \infty} \sum_{k=1}^N \int_0^T \int_{(\mathbb{T}^2)^N} (A_k \cdot \nabla u_t^\delta)(x) \beta_k(t, x) \, dx \, dr = \sum_{k=1}^N \int_0^T \int_{(\mathbb{T}^2)^N} \alpha_k(t, x) \beta_k(t, x) \, dx \, dr.
$$
By (4.10) and (4.11),

\[
\sum_{k=1}^{N} \int_{0}^{T} \int_{(T^2)^N} \left( A_k \cdot \nabla u^i_t \right)(x) \beta_k(t, x) \, dx \, dt
\]

\[
= - \sum_{k=1}^{N} \int_{0}^{T} \int_{(T^2)^N} u^i_t(x)(A_k \cdot \nabla \beta_k(t))(x) \, dx \, dt
\]

\[
\rightarrow - \sum_{k=1}^{N} \int_{0}^{T} \int_{(T^2)^N} u_t(x)(A_k \cdot \nabla \beta_k(t))(x) \, dx \, dt
\]

as \( i \to \infty \), where the last step follows from (4.15). Combining the two limits above yields that, for any \( \beta \in C_T(N) \),

\[
\sum_{k=1}^{N} \int_{0}^{T} \int_{(T^2)^N} \alpha_k(t, x) \beta_k(t, x) \, dx \, dt = - \sum_{k=1}^{N} \int_{0}^{T} \int_{(T^2)^N} u_t(x)(A_k \cdot \nabla \beta_k(t))(x) \, dx \, dt.
\]

As above, we deduce from this equality that \( \alpha \) does not depend on the choice of the subsequence \( \{(A_k \cdot \nabla u^i_t)\}_{i \in \mathbb{N}} \), and for all \( k \in S_N \),

\[
A_k \cdot \nabla u = \alpha_k
\]

holds in the distribution sense. Moreover, we deduce from (4.16) the gradient estimate

\[
\sum_{k=1}^{N} \int_{0}^{T} \left\| A_k \cdot \nabla u_t \right\|_{L^2}^2 \, dt \leq \| v_0 \|_{\infty}^2.
\]

Random intensity vector \( \xi = (\xi_1, \ldots, \xi_N) \) In the above discussions we assumed the intensity vector \( \xi = (\xi_1, \ldots, \xi_N) \) is fixed. To be more precise, we shall write in the sequel \( X^\xi_t(x) = (X^\xi_{1,1}(x), \ldots, X^\xi_{N,N}(x)) \) for the strong solution of (2.1) which is well defined for a.e. \( x \in \Delta_N \), and \( u^\xi_t \) the density of \( X^\xi_t \) starting from the initial density \( v_0 \in C((T^2)^N) \). Recall that \( \Delta_N \) is the generalized diagonal defined at the beginning of Sect. 2.

Now we suppose \( \xi \) is a random vector and the joint law of \( (\xi, X_0) \) is

\[
\rho(a, x) \lambda_N^0(da, dx) = \rho(a, x) p_N(a) \, da \, dx,
\]

where \( \rho : (\mathbb{R} \times T^2)^N \to \mathbb{R}_+ \) is a bounded continuous probability density function w.r.t. \( \lambda_N^0 \), and \( p_N(a) = (2\pi)^{-N/2} e^{-|a|^2/2} \). Then the marginal distribution of \( \xi \) is

\[
\tilde{\rho}(a) = p_N(a) \int_{(T^2)^N} \rho(a, x) \, dx,
\]
and the conditional distribution of $X_0$ given $\xi = a$ is

$$v_a(x) = \frac{\rho(a, x) \rho_N(a)}{\tilde{\rho}(a)} = \frac{\rho(a, x)}{\int_{\mathbb{T}^2} \rho(a, x) \, dx}.$$  \hspace{1cm} (4.20)

Therefore, under the probability measure $\rho(a, x)\lambda_N^0(\,da, \,dx)$ and given $\xi = a$, $v_a$ is the initial density of $X_0^a$. Let $u_t^a$ be the density of $X_t^a$ when the initial density of $X_0^a$ is given by $v_a(x)$ in (4.20). According to (4.17) and (4.18), for all $k \in \{1, \ldots, N\}$ and $\beta \in C^{0,1}([0, T] \times \mathbb{T}^2)^N$,

$$\int_0^T \int_{\mathbb{T}^2} (A_k \cdot \nabla u_t^a)(x) \beta(t, x) \, dx \, dt = - \int_0^T \int_{\mathbb{T}^2} u_t^a(x)(A_k \cdot \nabla \beta(x))(x) \, dx \, dt.$$ \hspace{1cm} (4.21)

Applying (4.19) leads to

$$\sum_{k=1}^N \int_0^T \|A_k \cdot \nabla u_t^a\|^2_{L^2} \, dt \leq \|v_a\|^2_{\infty}. \hspace{1cm} (4.22)$$

We remark that if $\tilde{\rho}(a) = 0$ for some $a \in \mathbb{R}^N$, then $\rho(a, x) = 0$ for all $x \in (\mathbb{T}^2)^N$ since $\rho$ is continuous. In this case it is natural to set $v_a(x) = u_t^a(x) = 0$, and the properties (4.21) and (4.22) hold as well.

Now we compute the joint law of $(\xi, X_t)$ when the initial variables $(\xi, X_0^0)$ are distributed as $\rho(a, x)\lambda_N^0(\,da, \,dx)$. For any bounded measurable function $F$ on $(\mathbb{R} \times \mathbb{T}^2)^N$,

$$\mathbb{E}[F(\xi, X_t^\xi)] = \int_{\mathbb{R}^N} \mathbb{E}[F(\xi, X_t^\xi) | \xi = a] \tilde{\rho}(a) \, da = \int_{\mathbb{R}^N} \mathbb{E}[F(a, X_t^a)] \tilde{\rho}(a) \, da$$

$$= \int_{\mathbb{R}^N} \int_{(\mathbb{T}^2)^N} F(a, x) u_t^a(x) \tilde{\rho}(a) \, dx \, da.$$

Thus, the joint distribution of $(\xi, X_t^\xi)$ is $u_t^a(x) \tilde{\rho}(a) \, dx \, da$, and its density w.r.t. $\lambda_N^0$ is

$$\tilde{u}_t(a, x) = u_t^a(x) \int_{(\mathbb{T}^2)^N} \rho(a, x) \, dx.$$  

Now we can transfer the property (4.21) and the gradient estimate (4.22) to the density $\tilde{u}_t(a, x)$. First, for any $\beta \in C^{0,1}([0, T] \times (\mathbb{T}^2)^N)$, multiplying both sides of (4.21) by $\int_{(\mathbb{T}^2)^N} \rho(a, x) \, dx$ and integrating on $\mathbb{R}^N$ w.r.t. $\rho_N(\,da)$ lead to

$$\int_0^T \int_{(\mathbb{R} \times \mathbb{T}^2)^N} (A_k(x) \cdot \nabla_{2N} \tilde{u}_t(a, x)) \beta(t, x) \lambda_N^0(\,da, \,dx) \, dt$$

$$= - \int_0^T \int_{(\mathbb{R} \times \mathbb{T}^2)^N} \tilde{u}_t(a, x) (A_k(x) \cdot \nabla_{2N} \beta(t, x)) \lambda_N^0(\,da, \,dx) \, dt,$$ \hspace{1cm} (4.23)
where $\nabla_{2N}$ is the gradient w.r.t. the $x$ variable. Next, multiplying both sides of (4.22) by $(\int_{(T^2)^N} \rho(a, x) \, dx)^2$, we obtain

$$
\sum_{k=1}^{N} \int_{0}^{T} \int_{(T^2)^N} (A_k \cdot \nabla_{2N} \tilde{u}_t(a, \cdot))^2 \, dx \, dt \leq \left( \int_{(T^2)^N} \rho(a, x) \, dx \right)^2 \frac{\|\rho(a, \cdot)\|_\infty^2}{\left( \int_{(T^2)^N} \rho(a, x) \, dx \right)^2} \|\rho(a, \cdot)\|_\infty^2 \leq \|\rho\|_\infty^2.
$$

Integrating w.r.t. $\rho_N(a) \, da$ on $\mathbb{R}^N$ yields

$$
\sum_{k=1}^{N} \int_{0}^{T} \|A_k \cdot \nabla_{2N} \tilde{u}_t\|_{L^2(\rho_0^0)}^2 \, dt \leq \|\rho_0\|_\infty^2.
$$

(4.24)

**Transfer to gradient estimate on the density of vorticity**

Let $\rho_0 : \mathcal{M}_N(T^2) \to \mathbb{R}^+_+$ be a bounded continuous function such that $\int_{\mathcal{M}_N(T^2)} \rho_0 \, d\mu_0^0 = 1$. We consider the stochastic point vortex dynamics (2.1) with $(\xi, X_0)$ distributed as $(\rho_0 \circ T_N)(a, x) \lambda_0^0(da, dx)$, where $T_N$ is defined in (2.4). Denoting again by $\tilde{u}_t(a, x)$ the joint density of $(\xi, X_\xi, t)$ w.r.t. $\lambda_0^N$, the gradient estimate (4.24) becomes

$$
\sum_{k=1}^{N} \int_{0}^{T} \|A_k \cdot \nabla_{2N} \tilde{u}_t\|_{L^2(\rho_0^0)}^2 \, dt \leq \|\rho_0\|_\infty^2.
$$

(4.25)

We intend to transform the above gradient estimate to the density function $\rho_t^N$ of

$$
\omega_t^N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \xi_i \delta_{X_t^{\xi,i}}.
$$

The existence of $\rho_t^N$ is due to Lemma 2.6. We shall show that, for every $k \in S_N = \{1, \ldots, N\}$, $\{\sigma_k \cdot \nabla \omega, D_\omega \rho_t^N\}$ exists in the distributional sense, that is, there exists some $g_k \in L^2([0, T] \times \mathcal{M}_N, dt \otimes \mu_0^0)$ such that for all $f \in \mathcal{F}\mathcal{C}_{P,T}$,

$$
\int_{0}^{T} \int_{\mathcal{M}_N} g_k(t, \omega) f(t, \omega) \mu_0^0(d\omega) dt = -\int_{0}^{T} \int_{\mathcal{M}_N} \rho_t^N(\omega) (\sigma_k \cdot \nabla \omega, D_\omega f(t, \omega)) \mu_0^0(d\omega) dt,
$$

(4.26)

where $\mathcal{M}_N = \mathcal{M}_N(\mathbb{T}^2)$ and $D_\omega f(t, \omega)$ is defined before Theorem 1.4. $\mathcal{F}\mathcal{C}_{P,T}$ is the family of test functionals defined in the introduction, which can also be regarded as smooth functionals on $\mathcal{M}_N$. To this end, we need the following simple facts.
Lemma 4.3 For any \( G \in \mathcal{FC}_P \), under the map \((\mathbb{R} \times \mathbb{T}^2)^N \ni (a, x) \rightarrow \omega = T_N(a, x) \in \mathcal{M}_N\),

\[
A_k(x) \cdot \nabla_{2N}(G \circ T_N)(a, x) = -\langle \sigma_k \cdot \nabla \omega, D_\omega G \rangle.
\]

Moreover, \( \text{div}_N^0 (\sigma_k \cdot \nabla \omega) = 0 \) in the sense of distribution; that is, for all \( G \in \mathcal{FC}_P \),

\[
\int_{\mathcal{M}_N} \langle \sigma_k \cdot \nabla \omega, D_\omega G \rangle d\mu_0^0 = 0. \tag{4.27}
\]

**Proof** Assume that \( G \in \mathcal{FC}_P \) has the form \( G(\omega) = g(\langle \omega, \phi_1 \rangle, \ldots, \langle \omega, \phi_n \rangle) \); then

\[
(G \circ T_N)(a, x) = g \left( \frac{1}{\sqrt{N}} \sum_{i=1}^N a_i \phi_1(x_i), \ldots, \frac{1}{\sqrt{N}} \sum_{i=1}^N a_i \phi_n(x_i) \right).
\]

Recall the notation \( \langle \omega, \Phi \rangle = (\langle \omega, \phi_1 \rangle, \ldots, \langle \omega, \phi_n \rangle) \) used in Sect. 4.1. By direct computation,

\[
A_k(x) \cdot \nabla_{2N}(G \circ T_N)(a, x) = \sum_{i=1}^N \sigma_k(x_i) \cdot \partial_{x_i} (G \circ T_N)(a, x)
\]

\[
= \sum_{i=1}^N \sigma_k(x_i) \cdot \left( \sum_{j=1}^n \partial_j g(\langle \omega, \Phi \rangle) \frac{a_j}{\sqrt{N}} \nabla \phi_j(x_i) \right)
\]

\[
= \sum_{j=1}^n \partial_j g(\langle \omega, \Phi \rangle) \langle \omega, \sigma_k \cdot \nabla \phi_j \rangle = -\langle \sigma_k \cdot \nabla \omega, D_\omega G \rangle.
\]

Now we can prove the second assertion. Since \( \mu_0^0 = (T_N)_# \lambda_N^0 \), we have

\[
\int_{\mathcal{M}_N} \langle \sigma_k \cdot \nabla \omega, D_\omega G \rangle d\mu_0^0(d\omega) = -\int_{(\mathbb{R} \times \mathbb{T}^2)^N} A_k(x) \cdot \nabla_{2N}(G \circ T_N)(a, x) \lambda_N^0(da, dx)
\]

\[
= -\int_{\mathbb{R}^N} p_N(a) da \int_{(\mathbb{T}^2)^N} A_k(x) \cdot \nabla_{2N}(G \circ T_N)(a, x) dx
\]

\[
= 0,
\]

due to \( \text{div}_{2N}(A_k)(x) = 0 \). Here \( p_N(a) \) is the density function of the standard Gaussian distribution on \( \mathbb{R}^N \). \( \square \)
Note that the law $\mu_t^N(d\omega) = \rho_t^N(\omega)\mu_0^0(d\omega)$ of $\omega_t^N$ is the image of that of $(\xi_t, X_t^\xi)$ under the map $T_N : (\mathbb{R} \times T^2)^N \rightarrow \mathcal{M}_N$. Therefore,

$$
\int_0^T \int_{\mathcal{M}_N} \rho_t^N(\omega)\langle \sigma_k \cdot \nabla \omega, D_\omega f(t) \rangle \mu_0^0(d\omega) dt
= \int_0^T \int_{(\mathbb{R} \times T^2)^N} \langle \sigma_k \cdot \nabla \omega, D_\omega f(t) \rangle|_{\omega = T_N(a, x)} \tilde{u}_t(a, x) \lambda_0^0(da, dx) dt
= - \int_0^T \int_{(\mathbb{R} \times T^2)^N} [A_k(x) \cdot \nabla_{2N} (f(t) \circ T_N)(a, x)] \tilde{u}_t(a, x) \lambda_0^0(da, dx) dt,
$$

where in the last step we applied Lemma 4.3. The integration by parts formula (4.23) yields

$$
\int_0^T \int_{\mathcal{M}_N} \rho_t^N(\omega)\langle \sigma_k \cdot \nabla \omega, D_\omega f(t) \rangle \mu_0^0(d\omega) dt
= \int_0^T \int_{(\mathbb{R} \times T^2)^N} (f(t) \circ T_N)(a, x) [A_k(x) \cdot \nabla_{2N} \tilde{u}_t(a, x)] \lambda_0^0(da, dx) dt
= \int_0^T \int_{(\mathbb{R} \times T^2)^N} (f(t) \circ T_N)(a, x) \mathbb{E}\left( [A_k(x) \cdot \nabla_{2N} \tilde{u}_t(a, x)] | \mathcal{G} \right) \lambda_0^0(da, dx) dt,
$$

where the conditional expectation is taken w.r.t. the probability measure $\lambda_0^0$, and $\mathcal{G}$ is the sub-\(\sigma\)-field of the Borel field $\mathcal{B}((\mathbb{R} \times T^2)^N)$ defined as

$$
\mathcal{G} = \sigma\left( \{ F \circ T_N \mid F : \mathcal{M}_N \rightarrow \mathbb{R} \text{ is measurable} \} \right).
$$

There exists some $g_k(t) : \mathcal{M}_N \rightarrow \mathbb{R}$ such that

$$
\mathbb{E}\left( [A_k(x) \cdot \nabla_{2N} \tilde{u}_t(a, x)] | \mathcal{G} \right) = - (g_k(t) \circ T_N)(a, x) \quad \text{for all } k \in \{1, \ldots, N\}. \quad (4.29)
$$

Since the conditional expectation is an orthogonal projection from $L^2((\mathbb{R} \times T^2)^N, \mathcal{B}((\mathbb{R} \times T^2)^N), \lambda_0^0)$ to $L^2((\mathbb{R} \times T^2)^N, \mathcal{G}, \lambda_0^0)$, we have

$$
\|g_k(t)\|_{L^2(\mathcal{M}_N)} = \|g_k(t) \circ T_N\|_{L^2(\lambda_0^0)} \leq \|A_k \cdot \nabla_{2N} \tilde{u}_t\|_{L^2(\lambda_0^0)}.
$$

Combining this with the gradient estimate (4.25), we have

$$
\sum_{k=1}^N \int_0^T \|g_k(t)\|_{L^2(\mathcal{M}_N)}^2 dt \leq \|\rho_0\|_{L^\infty}^2. \quad (4.30)
$$
Substituting (4.29) into (4.28), we obtain

\[
\int_0^T \int_{\mathcal{M}_N} \rho_i^N(\omega)[\sigma_k \cdot \nabla \omega, D_\omega f(t)] \mu_N^0(d\omega)dt = - \int_0^T \int_{(\mathbb{R} \times \mathbb{T}^2)^N} (f(t) \circ T_N)(a, x)(g_k(t) \circ T_N)(a, x) \lambda_N^0(da, dx)dt = - \int_0^T \int_{\mathcal{M}_N} g_k(t, \omega) f(t, \omega) \mu_N^0(d\omega)dt.
\]

This is the desired equality (4.26). Moreover, thanks to the fact \(\text{div} \mu_N^0(\sigma_k \cdot \nabla \omega) = 0\) proved in Lemma 4.3, we conclude the existence of \(\{\sigma_k \cdot \nabla \omega, D_\omega \rho_i^N\}\) in the distributional sense, and

\[
\{\sigma_k \cdot \nabla \omega, D_\omega \rho_i^N\} = g_k(t, \omega), \quad k \in \{1, \ldots, N\}. \tag{4.31}
\]

Combining this equality with (4.30) yields the gradient estimate below:

\[
\sum_{k=1}^N \int_0^T \int_{\mathcal{M}_N} \{\sigma_k \cdot \nabla \omega, D_\omega \rho_i^N\}^2 \mu_N^0(d\omega)dt \leq \|\rho_0\|_\infty^2. \tag{4.32}
\]

\textbf{Step 3: Letting }\(N \to \infty\)\textbf{.} Now suppose that we are given }\(\rho_0 \in C_b(\mathbb{H}^{-1}(\mathbb{T}^2), \mathbb{R}_+)\) such that }\(\int \rho_0 \, d\mu = 1\), where }\(\mu\)\textbf{ is the law of the white noise}. Let }\(\rho_i\)\textbf{ be given in Theorem 1.3. Our purpose in this step is to prove the gradient estimate (1.10) on }\(\rho_i\).

For any }\(N \in \mathbb{N}\), consider the restriction }\(\rho_0^N\)\textbf{ of }\(\rho_0\)\textbf{ to }\(\mathcal{M}_N\) \((\subset \mathbb{H}^{-1})\)\textbf{ and the stochastic point vortex dynamics starting from }\(C_N \rho_0^N(\omega)\mu_N^0(d\omega)\), where }\(C_N\)\textbf{ is the normalizing constant: }\(C_N = (\int \rho_0^N \, d\mu_N^0)^{-1}\). Since }\(\mu_N^0\)\textbf{ converges weakly to }\(\mu\), we have }\(\lim_{N \to \infty} C_N = 1\). By (4.32), we know that the density }\(\rho_i^N\)\textbf{ of the stochastic point vortices }\(\omega_i^N\)\textbf{ satisfies the gradient estimate}

\[
\sum_{k=1}^N \int_0^T \int_{\mathcal{M}_N} \{\sigma_k \cdot \nabla \omega, D_\omega \rho_i^N\}^2 \mu_N^0(d\omega)dt \leq C_N^2 \|\rho_0\|_\infty^2 \leq C_N^2 \|\rho_0\|_\infty^2, \quad N \in \mathbb{N}. \tag{4.33}
\]

For every }\(k > N\), we define }\(\{\sigma_k \cdot \nabla \omega, D_\omega \rho_i^N\} = 0\text{ for all } (t, \omega) \in [0, T] \times \mathcal{M}_N\). We want to show that the family }\(\{\{\sigma_k \cdot \nabla \omega, D_\omega \rho_i^N\}| (k, t, \omega) \in \mathbb{N} \times [0, T] \times \mathcal{M}_N\}\_{N \in \mathbb{N}}\text{ has a subsequence which converges in a certain sense to some } G \in L^2(\mathbb{N} \times [0, T] \times H^{-1}, \# \otimes dt \otimes \mu), \text{ where } \# \text{ is the counting measure on } \mathbb{N}\text{. To this end, we denote by}

\[
\mathcal{F}C_{P,T}(\mathbb{N}) = \left\{ f : \mathbb{N} \times [0, T] \times H^{-1} \to \mathbb{R} \mid \exists n_f \in \mathbb{N} \text{ s.t. } f_k \in \mathcal{F}C_{P,T} \text{ for } k \leq n_f, \text{ and } f_k \equiv 0 \text{ for } k > n_f \right\}. \tag{4.34}
\]
It is a dense linear subspace of \( L^2(\mathbb{N} \times [0,T] \times H^{-1-}, \# \otimes dt \otimes \mu) \). Fix an \( f \in \mathcal{F}C_{p,T}(\mathbb{N}) \), by the definition of \( \{\sigma_k \cdot \nabla \omega, D\omega \rho^N_t\} \) [cf. (4.26) and (4.31)], we have, for all \( N > n_f \),

\[
\sum_{k=1}^{\infty} \int_0^T \int_{\mathcal{M}_N} \langle \sigma_k \cdot \nabla \omega, D\omega \rho^N_t \rangle f_k(t,\omega) \, \mu^0_N(d\omega) \, dt
\]

\[
= - \sum_{k=1}^{\infty} \int_0^T \int_{\mathcal{M}_N} \rho^N_t(\omega) \langle \sigma_k \cdot \nabla \omega, D\omega f_k(t,\omega) \rangle \, \mu^0_N(d\omega) \, dt.
\]

(4.34)

Note that the sums over \( k \in \mathbb{N} \) on both sides are indeed finite sums. To proceed further, we need some preparations.

**Lemma 4.4**

(1) Let \( \{N_i\}_{i \in \mathbb{N}} \) be the subsequence obtained before Lemma 3.4; then for any \( F \in \mathcal{F}C_{p,T} \),

\[
\lim_{i \to \infty} \int_0^T \int_{\mathcal{M}_{N_i}} \rho^N_{N_i}(\omega) F(t,\omega) \, \mu^0_{N_i}(d\omega) \, dt = \int_0^T \int_{H^{-1-}} \rho_t(\omega) F(t,\omega) \, \mu(\omega) \, d\omega.
\]

(2) For any \( G \in \mathcal{F}C_p \),

\[
\lim_{N \to \infty} \int_{\mathcal{M}_N} G(\omega) \, \mu^0_N(d\omega) = \int_{H^{-1-}} G(\omega) \, \mu(d\omega).
\]

**Proof**

(1) It suffices to prove the limit for \( F(t,\omega) = f(\tau) G(\omega) \), where \( f \in C([0,T]) \) and \( G(\omega) = g(\langle \omega, \phi_1 \rangle, \ldots, \langle \omega, \phi_n \rangle) \in \mathcal{F}C_p \). We have

\[
\int_0^T \int_{\mathcal{M}_{N_i}} \rho^N_{N_i}(\omega) F(t,\omega) \, \mu^0_{N_i}(d\omega) \, dt = \int_0^T f(\tau) \hat{E}[G(\hat{\omega}^N_{N_i})] \, d\tau,
\]

where \( \hat{\mathbb{E}} \) is the expectation on the probability space \((\hat{\Theta}, \hat{\mathcal{F}}, \hat{\mu})\), which comes from the Skorokhod’s representation theorem in Sect. 3. If \( G \) is bounded, by (3.7) and the dominated convergence theorem, we see that the limit holds true. Using the method of truncation, it is sufficient to show that \( \{G(\hat{\omega}^N_{N_i})\}_{i \in \mathbb{N}} \) is bounded in \( L^2([0,T] \times \hat{\Theta}) \).

By Lemma 2.6,

\[
\hat{\mathbb{E}}[G^2(\hat{\omega}^N_{N_i})] = \int_{\mathcal{M}_{N_i}} G^2(\omega) \rho^N_{N_i}(\omega) \, \mu^0_{N_i}(d\omega) \leq C_{N_i} \|\rho_0\|_{\infty} \int_{\mathcal{M}_{N_i}} G^2(\omega) \, \mu^0_{N_i}(d\omega).
\]

Note that \( G(\omega) = g(\langle \omega, \phi_1 \rangle, \ldots, \langle \omega, \phi_n \rangle) \) and \( g \) has polynomial growth. Combining this fact with the definition of \( \mu^0_N \) in Sect. 2.1, some simple calculations lead to the desired result.

(2) The proof follows as above; the only difference is that we replace the limit (3.7) by the weak convergence of \( \mu^0_N \) to \( \mu \) proved in Proposition 2.2. \( \square \)
By Remark 1.5(1), we have \( \sum_{k=1}^{\infty} \langle \sigma_k \cdot \nabla \omega, D_\omega f_k(t, \omega) \rangle = \sum_{k=1}^{n_f} \langle \sigma_k \cdot \nabla \omega, D_\omega f_k(t, \omega) \rangle \in \mathcal{FC}_{P, T} \). The first assertion of Lemma 4.4 leads to

\[
\lim_{i \to \infty} \sum_{k=1}^{\infty} \int_{0}^{T} \int_{M_{N_i}} \rho^N_i(\omega) \langle \sigma_k \cdot \nabla \omega, D_\omega f_k(t, \omega) \rangle \mu^0_{N_i}(d\omega) dt = \sum_{k=1}^{\infty} \int_{0}^{T} \int_{H^{-1}} \rho_i(\omega) \langle \sigma_k \cdot \nabla \omega, D_\omega f_k(t, \omega) \rangle \mu(d\omega) dt.
\]

(4.35)

From (4.34) and (4.35) we see that, for all \( f \in \mathcal{FC}_{P, T}(\mathbb{N}) \), the limit

\[
\lim_{i \to \infty} \sum_{k=1}^{\infty} \int_{0}^{T} \int_{M_{N_i}} \langle \sigma_k \cdot \nabla \omega, D_\omega f_k(t, \omega) \rangle f_k(t, \omega) \mu^0_{N_i}(d\omega) dt
\]

exists, and denoting it by \( L(f) \), we have

\[
L(f) = - \sum_{k=1}^{\infty} \int_{0}^{T} \int_{H^{-1}} \rho_i(\omega) \langle \sigma_k \cdot \nabla \omega, D_\omega f_k(t, \omega) \rangle \mu(d\omega) dt.
\]

(4.36)

The above equality clearly implies that \( L \) is an linear functional on \( \mathcal{FC}_{P, T}(\mathbb{N}) \), which is a dense subspace of \( L^2(\mathbb{N} \times [0, T] \times H^{-1}, \# \otimes dt \otimes \mu) \). Moreover, by Cauchy’s inequality,

\[
|L(f)| = \lim_{i \to \infty} \left| \sum_{k=1}^{\infty} \int_{0}^{T} \int_{M_{N_i}} \langle \sigma_k \cdot \nabla \omega, D_\omega f_k(t, \omega) \rangle \mu^0_{N_i}(d\omega) dt \right|
\]

\[
\leq \liminf_{i \to \infty} \left\| \langle \sigma_k \cdot \nabla \omega, D_\omega f_k(t, \omega) \rangle \right\|_{L^2(\mathbb{N} \times [0, T] \times M_{N_i})} \|f\|_{L^2(\mathbb{N} \times [0, T] \times M_{N_i})}
\]

\[
\leq \|\rho_0\|_\infty \|f\|_{L^2(\mathbb{N} \times [0, T] \times H^{-1})},
\]

where in the last step we have used (4.33) and the second assertion of Lemma 4.4, by regarding \( \sum_{k=1}^{\infty} f_k^2(t, \omega) = \sum_{k=1}^{n_f} f_k^2(t, \omega) \in \mathcal{FC}_{P, T} \) as a test function.

Summarizing the above arguments, we see that \( L : \mathcal{FC}_{P, T}(\mathbb{N}) \to \mathbb{R} \) is a bounded linear functional, and thus it can be extended to the whole space \( L^2(\mathbb{N} \times [0, T] \times H^{-1}, \# \otimes dt \otimes \mu) \) as a bounded linear functional with the norm \( \|L\| \leq \|\rho_0\|_\infty \). By Riesz’s representation theorem, there exists \( G \in L^2(\mathbb{N} \times [0, T] \times H^{-1}, \# \otimes dt \otimes \mu) \) such that

\[
\|G\|_{L^2(\mathbb{N} \times [0, T] \times H^{-1})} = \|L\| \leq \|\rho_0\|_\infty,
\]

(4.37)

and for all \( f \in L^2(\mathbb{N} \times [0, T] \times H^{-1}, \# \otimes dt \otimes \mu) \),

\[
L(f) = \sum_{k=1}^{\infty} \int_{0}^{T} \int_{H^{-1}} G_k(t, \omega) f_k(t, \omega) \mu(d\omega) dt.
\]
In particular, for every fixed \( k \in \mathbb{N} \), taking \( f \in \mathcal{F}\mathcal{C}_P, T(\mathbb{N}) \) such that \( f_j \equiv 0 \) for all \( j \neq k \), we have by (4.36) that

\[
\int_0^T \int_{H^{-1}} G_k(t, \omega) f_k(t, \omega) \mu(d\omega) dt = - \int_0^T \int_{H^{-1}} \rho_t(\omega) \langle \sigma_k \cdot \nabla \omega, D_\omega f_k(t, \omega) \rangle \mu(d\omega) dt.
\]

(4.38)

We need the final preparation.

**Lemma 4.5** For all \( k \in \mathbb{N} \), \( \text{div}_\mu(\sigma_k \cdot \nabla \omega) = 0 \) in the distributional sense, i.e. for all \( G \in \mathcal{F}\mathcal{C}_P \),

\[
\int_{H^{-1}} \langle \sigma_k \cdot \nabla \omega, D_\omega G \rangle \mu(d\omega) = 0.
\]

**Proof** By (1) of Remark 1.5, we have \( \langle \sigma_k \cdot \nabla \omega, D_\omega G \rangle \in \mathcal{F}\mathcal{C}_P \). The desired result follows from (4.27) and the second assertion of Lemma 4.4. \( \Box \)

Combining Lemma 4.5 and (4.38), we see that \( \langle \sigma_k \cdot \nabla \omega, D_\omega \rho_t(\omega) \rangle \) exists in the distributional sense and

\[
\langle \sigma_k \cdot \nabla \omega, D_\omega \rho_t(\omega) \rangle = G_k(t, \omega), \quad k \in \mathbb{N}.
\]

Substituting this equality into (4.37) eventually leads to the gradient estimate

\[
\sum_{k=1}^\infty \int_0^T \int_{H^{-1}} \langle \sigma_k \cdot \nabla \omega, D_\omega \rho_t(\omega) \rangle^2 \mu(d\omega) dt \leq \| \rho_0 \|_\infty^2.
\]

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### 5 Appendix

The notion of uniqueness for a stochastic equation has several faces: beyond the most classical ones, namely pathwise uniqueness and uniqueness in law, there are others, like uniqueness of the Lagrangian flow and uniqueness of the solution of the Fokker–Planck equation. Due to the outstanding difficulty of the particular case of stochastic Euler equations, in the following discussion let us concentrate on the weakest one of the previous concepts, namely uniqueness of the solution of the Fokker–Planck equation; when it is true, in some cases one can prove more (see for instance [3] for Lagrangian flows and [29] for uniqueness in law).

We outline below a few approaches to uniqueness of a Fokker–Planck equation of the form

\[
\partial_t v_t = \mathcal{L}_x^* v_t, \quad v|_{t=0} = v_0,
\]

(5.1)
where \( \mathcal{L}^* \) is an operator acting on measures, which is the formal dual of a certain linear operator \( \mathcal{L} \) acting on functions \( f \), namely, \( \langle \mathcal{L}^* \nu, f \rangle = \langle \nu, \mathcal{L} f \rangle \) (we write \( \langle \nu, g \rangle \) for \( \int g(x) \nu(dx) \)). The particular case considered in this paper has the generator defined as

\[
(\mathcal{L} f)(\omega) = \langle b(\omega), D_\omega f(\omega) \rangle + \frac{1}{2} \sum_{k=1}^{\infty} \langle \sigma_k \cdot \nabla \omega, D_\omega \langle \sigma_k \cdot \nabla \omega, D_\omega f(\omega) \rangle \rangle
\]

with the notations \( b(\omega), D_\omega f(\omega) \) etc. introduced in the previous sections; the function \( f \) is in \( FC_P \).

As we have already remarked in the introduction, at present we are unable to prove uniqueness for Eq. (5.1). The aim of the following sections is only to show that gradient estimates are at the core of this question.

Before we continue, let us notice that, in our case, if we denote by \( \mu \) the enstrophy measure and by \( \mathcal{L}^*_\mu \) the formal dual of \( \mathcal{L} \) in \( L^2(H^{-1}-, \mu) \), then

\[
(\mathcal{L}^*_\mu f)(\omega) = - \langle b(\omega), D_\omega f(\omega) \rangle + \frac{1}{2} \sum_{k=1}^{\infty} \langle \sigma_k \cdot \nabla \omega, D_\omega \langle \sigma_k \cdot \nabla \omega, D_\omega f(\omega) \rangle \rangle,
\]

hence the structure of \( \mathcal{L} \) and \( \mathcal{L}^*_\mu \) is similar, up to a sign in the drift term. If \( \nu_t \) has a density \( \rho_t \) w.r.t. \( \mu \), we write the Eq. (5.1) as

\[
\partial_t \rho_t = \mathcal{L}^*_\mu \rho_t, \quad \rho|_{t=0} = \rho_0.
\]

5.1 Lions theorem

We first describe the difficulties in the attempt to apply a classical variational theorem of J. L. Lions [23, Chap. 4]. Probably this is one of the most restrictive approaches, compared for instance to duality or semigroup theory. However, it clarifies in the easiest possible way some of the difficulties.

We need a Gelfand triple \( V \subset H \subset V' \) and a bilinear map \( a : V \times V \to \mathbb{R} \), such that Eq. (5.2) is interpreted in the form

\[
\langle \rho_t, f \rangle + \int_0^t a(\rho_s, f) \, ds = \langle \rho_0, f \rangle,
\]

where \( \langle \,, \rangle \) is the inner product in \( L^2(H^{-1}-, \mu) \). When \( a \) is continuous and coercive on \( V \), existence and uniqueness is true. In our case the natural choice for \( H \) is \( L^2(H^{-1}-, \mu) \), with \( \| f \|^2_H = \langle f, f \rangle \); for \( V \), it is the space of all \( f \in H \) such that the derivatives \( \langle \sigma_k \cdot \nabla \omega, D_\omega f(\omega) \rangle \) exist in \( H \) for all \( k \in \mathbb{N} \) in the distributional sense described in the paper and

\[
\| f \|^2_V := \| f \|^2_H + \sum_{k=1}^{\infty} \int_{H^{-1}-} \langle \sigma_k \cdot \nabla \omega, D_\omega f(\omega) \rangle^2 \mu(d\omega) < \infty;
\]
and the bilinear form is given by

\[ a(f, g) = a_0(f, g) + a_1(f, g), \]

\[ a_0(f, g) = \frac{1}{2} \sum_{k=1}^{\infty} \int_{H^{-1}} \langle \sigma_k \cdot \nabla \omega, D_{\omega} f(\omega) \rangle \langle \sigma_k \cdot \nabla \omega, D_{\omega} g(\omega) \rangle \mu(d\omega), \]

\[ a_1(f, g) = -\int_{H^{-1}} f(\omega) \langle b(\omega), D_{\omega} g(\omega) \rangle \mu(d\omega). \]

Coercivity is true, since

\[ a_0(f, f) = \frac{1}{2}(\| f \|_V^2 - \| f \|_H^2), \]

\[ a_1(f, f) = 0. \]

What is not clear is continuity (and even the fact that the term \( a_1(f, g) \) is well defined on \( V \times V \)), which requires

\[ |a_1(f, g)| \leq C \| f \|_V \| g \|_V \tag{5.3} \]

for all \( f, g \in V \).

We do not claim that property (5.3) is not true; simply that our understanding is still too poor. At present, we do not know how to use efficiently the fact that \( f \) can be estimated in the \( V \) topology, when dealing with property (5.3). Using the non-optimal inequality \( \int_{H^{-1}} f^2(\omega) \mu(d\omega) \leq \| f \|_V^2 \) we have

\[ |a_1(f, g)| \leq \| f \|_V \left( \int_{H^{-1}} \langle b(\omega), D_{\omega} g(\omega) \rangle^2 \mu(d\omega) \right)^{1/2}, \]

and thus we are faced to prove

\[ \int_{H^{-1}} \langle b(\omega), D_{\omega} g(\omega) \rangle^2 \mu(d\omega) \leq C \| g \|_V^2. \]

If \( \sigma_k(x) = e^{2\pi i k \cdot x} \frac{k}{|k|^\gamma} \) for \( k \in \mathbb{Z}_0 := \mathbb{Z}^2 \setminus \{0\} \) (for \( \gamma > 2 \), since \( \sigma_k \cdot \nabla \sigma_k = 0 \), these vector fields satisfy our assumptions) we may write

\[ \langle b(\omega), D_{\omega} g(\omega) \rangle = 2\pi i \sum_{k \in \mathbb{Z}_0} |k|^{\gamma - 2} \langle \omega, e^{2\pi i k \cdot x} \rangle \langle \sigma_k \cdot \nabla \omega, D_{\omega} g(\omega) \rangle. \]
Even the trivial $L^1(\mu)$-estimate

$$\int_{H^{-1}} |\langle b(\omega), D_\omega g(\omega) \rangle| \mu(d\omega) \leq 2\pi \left( \sum_{k \in \mathbb{Z}^2_0} |k|^{2\gamma - 4} \int_{H^{-1}} |\langle \omega, e^{2\pi ik \cdot x} \rangle|^2 \mu(d\omega) \right)^{1/2} \|g\|_V$$

$$= 2\pi \left( \sum_{k \in \mathbb{Z}^2_0} |k|^{2\gamma - 4} \right)^{1/2} \|g\|_V$$

requires too much on $\gamma$, namely $\gamma < 1$. The previous elementary computation fails to prove (5.3).

### 5.2 Duality

A widely used general idea to prove uniqueness of a linear equation is to prove existence for a suitable dual equation; a reference in Probability theory is [29]. With our notations, an example of dual equation is

$$\partial_t u + Lu = 0 \text{ on } [0, t_0], \quad u|_{t=t_0} = \phi$$

(5.4)

where $t_0$ is arbitrary in $[0, T]$. The heuristic argument is that, by ordinary calculus and the duality between $L$ and $L^*$,

$$\langle v_t, \phi \rangle - \langle v_0, u_0 \rangle = \int_0^{t_0} \left( \langle L^* v_s, u_s \rangle - \langle v_s, Lu_s \rangle \right) ds = 0.$$

Assume that $\phi$ may vary in a determining class (a class with the property that two measures coinciding over it are the same measure). Then the action of $v_{t_0}$ on test functions $\phi$ is identified by the initial condition $v_0$ and by the solution $u$:

$$\langle v_{t_0}, \phi \rangle = \langle v_0, u_0 \rangle.$$  

(5.5)

The existence of $u$ identifies the measure $v_{t_0}$, providing uniqueness for (5.1).

Having described the general idea, let us see now the technical problems arising in its implementation in our particular case. Let us recall we have proved the existence of a solution of (5.1) in the sense of the weak identity ($v_t = \rho_t \mu$ and $\langle ., . \rangle$ now denotes the inner product in $L^2(H^{-1}, \mu)$)

$$\langle \rho_t, F_t \rangle - \langle \rho_0, F_0 \rangle = \int_0^t \left( \int_{H^{-1}} \rho_s (L F_s + \partial_s F_s) d\mu \right) ds$$

(5.6)

for every function $F$ of class $\mathcal{F}C_{p,T}$; solution satisfying the gradient estimate. Exactly with the same proof (due to the analogy between $L$ and $L^* = L^*_\mu$) one can prove existence of a solution of (5.4) in the sense of the weak identity.
\[ \langle u_{t_0}, F_{t_0} \rangle - \langle u_t, F_t \rangle + \int_{t}^{t_0} \left( \int_{H^{-1-}} u_s \left( \mathcal{L}^* F_s - \partial_s F_s \right) d\mu \right) ds = 0 \]

for every \( F \in \mathcal{FC}_{P, t_0} \). Now the question is how to prove rigorously (5.5) starting from the previous two identities. No one of the functions \( \rho \) or \( u \) has sufficient regularity to be used as a test function in the weak formulation of the other function. Using the gradient estimate known for \( \rho \) we may rewrite the weak formulation (5.6) as

\[ \langle \rho_t, F_t \rangle + \int_0^t a_0(\rho_s, F_s) \, ds = \langle \rho_0, F_0 \rangle + \int_0^t \left( \int_{H^{-1-}} \rho_s \left( \partial_s F_s + \langle b, D F_s \rangle \right) d\mu \right) ds. \]

This formulation is less demanding in terms of regularity of \( u \) and \( F \) since only first order differential operators are applied to them. But, in order to take \( F = u \), even taking advantage of the additional property that \( \rho \) is bounded when \( \rho_0 \in L^\infty(H^{-1-}, \mu) \), we need

\[ \langle b, Du \rangle \in L^1(0, T; L^1(H^{-1-}, \mu)). \]  

This requirement is weaker than that in the Lions approach but, as described in the previous subsection, we do not know if it holds true under our assumptions.

Thus a direct substitution of \( u \) as a test function in the equation of \( \rho \) meets problems similar to (although weaker than) those of the Lions approach. This is not the end of the story: a general idea, with plenty of possible implementations, consists in introducing a sequence \( (u^N_t) \) with the following properties:

- \( u^N_t \) is an admissible test function for the \( \rho_t \)-equation (5.6),
- \( u^N_t \) converges to \( u_t \) and
- \( \partial_t u^N_t + L u^N_t \) converges to zero

in suitable topologies. At present we have not found a solution to this question. It is also strongly related to the next two strategies: for instance, one may construct \( u^N_t \) by space-mollifiers which lead to the fact that \( \partial_t u^N_t + L u^N_t \) is equal to a commutator, as in Sect. 5.4 below.

**Remark 5.1** A well known method to prove uniqueness in law for a stochastic differential equation based on the existence of a solution to Kolmogorov equation is closely related to the topic of this section. In our case we should take a solution \( \omega_t \) of the stochastic Euler equation and a solution \( u_t \) of the backward Kolmogorov equation (5.4) and compute

\[ du(t, \omega_t), \]

realizing that it is given by the local martingale (in differential notation)

\[ \sum_{k=1}^{\infty} \langle \sigma_k \cdot \nabla \omega_t, D_{\omega_t} u(t, \omega_t) \rangle \, dW^k_t. \]
One can easily recognize the structure of the duality argument and the role of a gradient type estimate. The difficulty to make this argument rigorous is similar to what described above: the regularity of $u$ is not sufficient to apply the classical Itô formula, hence a regularization is needed, leading to the same difficulties mentioned above, and below in Sect. 5.4.

5.3 Dense range conditions

In semigroup theory a powerful theorem providing existence and uniqueness of solutions to a linear differential equation is Lumer–Phillips theorem. It is based on two assumptions: dissipativity, which is natural for Fokker–Planck type equations; plus a dense range condition (see [26] for several versions and details). A version of this argument for Fokker–Planck equations in infinite dimensions, interpretable also as a result of duality, is given by [8].

The version of this strategy we invoke here is simply based on identity (5.6), hence it is strongly related to the previous section. But the question here is: define a domain $D$ for the operator

$$(\partial_t + \mathcal{L}) : D \rightarrow L^1 \left( 0, T ; L^1 \left( H^{-1}, \mu \right) \right)$$

which includes the final time condition $u|_{t=T} = 0$ and prove that

$$(\partial_t + \mathcal{L}) (D) \text{ is dense in } L^1 \left( 0, T ; L^1 \left( H^{-1}, \mu \right) \right).$$

(5.8)

If we succeed, then we use the additional property that $\rho$ is bounded when $\rho_0 \in L^\infty \left( H^{-1}, \mu \right)$, from identity (5.6) we deduce that $\rho$ is identified.

One way of proving the dense range condition is by replacing the operator $\mathcal{L}$ with a simpler one $\mathcal{L}^N$ such that

$$(\partial_t + \mathcal{L}^N) (D) \text{ is dense in } L^1 \left( 0, T ; L^1 \left( H^{-1}, \mu \right) \right).$$

Assume this happens (in itself this can be done in various ways). Take $G \in L^1 \left( 0, T ; L^1 \left( H^{-1}, \mu \right) \right)$, $\epsilon > 0$ and $G_N \in L^1 \left( 0, T ; L^1 \left( H^{-1}, \mu \right) \right)$ with distance less than $\epsilon$ from $G$ and equal to $(\partial_t + \mathcal{L}^N) F_N$ for a certain $F_N \in D$. Then

$$(\partial_t + \mathcal{L}) F_N = G_N + \left( \mathcal{L} F_N - \mathcal{L}^N F_N \right).$$

If we prove

$$\liminf_{N \to \infty} \int_0^T \int_{H^{-1}} \left| \mathcal{L} F_N - \mathcal{L}^N F_N \right| d\mu ds = 0$$
then we have proved the dense range condition (5.8) and thus uniqueness for Eq. (5.1).

A strategy is to take \( L^N \) of the form

\[
\left( L^N f \right)(\omega) = \left\{ b^N(\omega) , D_\omega f(\omega) \right\} + \frac{1}{2} \sum_{k=1}^{\infty} \left\{ \sigma_k \cdot \nabla \omega , D_\omega (\sigma_k \cdot \nabla \omega , D_\omega f(\omega)) \right\},
\]

so that we have only to prove

\[
\liminf_{N \to \infty} \int_{t_0}^{t} \int_{H^{-1}} \left| \left\{ b(\omega) - b^N(\omega) , D_\omega F_N(\omega) \right\} \right| d\mu ds = 0.
\]

We see again that the key property is a gradient type condition, similar to (5.7).

**5.4 Renormalized solutions**

Instead of using duality, heuristically one could use the following energy type argument. Let \( \rho^{(i)}_t, i = 1, 2 \), be two solutions of Eq. (5.6). Then \( \rho_t = \rho_t^{(1)} - \rho_t^{(2)} \) satisfies identity (5.6) with \( \rho_0 = 0 \). If we may choose \( F = \rho \), we get (using also \( \int_{0}^{t} \rho_s \partial_s \rho_s ds = \frac{1}{2} \int_{0}^{t} \rho_s^2 ds = \frac{1}{2} \rho_t^2 \))

\[
\frac{1}{2} \int_{H^{-1}} \rho_t^2 d\mu + \int_{0}^{t} a_0(\rho_s, \rho_s) ds = 0,
\]

hence

\[
\int_{H^{-1}} \rho_t^2 d\mu \leq 0 \quad (5.9)
\]

which implies \( \rho_t = 0 \).

Rigorously speaking, the problem is to pass from (5.6) to (5.9). Notice that an analog of (5.9) is known for each \( \rho^{(i)}_t, i = 1, 2 \), but not for the difference.

A classical way (see [2,14]) to go from (5.6) to (5.9) is to mollify \( \rho_t \) so that computations can be performed rigorously. Call generically

\[
\rho^\epsilon_t := P_\epsilon \rho_t
\]

a smoothed version of \( \rho_t \) obtained by the application of a smoothing linear operator (see examples in [4,12,15]). We have (taking \( P_\epsilon^* \rho^\epsilon_t \) as test function)

\[
\langle \rho^\epsilon_t, \rho^\epsilon_t \rangle = \int_{0}^{t} \left( \int_{H^{-1}} \rho^\epsilon_s \left( \mathcal{L} \rho^\epsilon_s + \partial_s \rho^\epsilon_s \right) d\mu \right) ds
\]

\[
+ \int_{0}^{t} \left( \int_{H^{-1}} \rho_s \left( \mathcal{L} P_\epsilon^* - P_\epsilon^* \mathcal{L} \right) \rho^\epsilon_s d\mu \right) ds.
\]
The commutator $\left[ L, P^* \right] := L P^* - P^* L$ arises. We deduce

$$\frac{1}{2} \int_{H^{-1}} (\rho_t^*)^2 \, d\mu + \int_0^t \mathbb{V} \left( \int_{H^{-1}} \rho_s \left( L P^* - P^* L \right) \rho_s^* \, d\mu \right) \, ds.$$ 

If we can prove that the right-hand side converges to zero, then we deduce $\rho_t = 0$. Convergence to zero of commutators is a very technical subject, well understood in finite dimensions ([2,14] and several subsequent references) with a number of results in the infinite dimensional case ([4,5,12,15,20]). The present known conditions in infinite dimensions cannot be applied to our case; however, some degree of differentiability of either the solution or the drift is needed in the estimates and here, in our case, we have a mild form of differentiability provided by the gradient estimates for $\rho^{(i)}_t$, $i = 1, 2$. This direction, as the previous ones, may deserve further study.

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