Embedding Operators in Sobolev-Lions Spaces and Applications

Department of Mechanical Engineering, Okan University, Akfirat, Tuzla 34959, Istanbul, Turkey,

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Abstract

The embedding theorems in Sobolev-Lions type anisotropic weighted spaces $W^{l,p}_{\gamma}(\Omega; E_0, E)$ are studied, here $E_0$ and $E$ are two Banach spaces. The most regular interpolation spaces $(E_0, E)_{p,\theta,\alpha}$ between $E_0$ and $E$ are found such that the mixed differential operators $\bar{D}^\alpha$ are bounded from $W^{l,p}_{\gamma}(\Omega; E_0, E)$ to $L^{p,\gamma}(\Omega; (E_0, E)_{p,\theta,\alpha})$, where $L^{p,\gamma}$ denotes weighted abstract Lebesgue space with mixed norm and

$$\alpha = (\alpha_1, \alpha_2, ..., \alpha_n), \quad l = (l_1, l_2, ..., l_n), \quad \theta_\alpha = \sum_{k=1}^{n} \frac{\alpha_k}{l_k}.$$ 

By applying this result separability properties of degenerate anisotropic differential operator equations, well-posedeness and Strichartz type estimates for solution of corresponding parabolic problem are established.

Key Words: Sobolev spaces, Embedding operators, vector-valued spaces, Differential operator equations, Interpolation of Banach spaces

1. Introduction

Embedding of function spaces were studied in a series of books and papers (see, for example [2, 4, 14, 15]). The embedding properties of abstract function spaces have been considered e.g. in [1, 6, 12, 16-21]. Lions-Peetre [12] showed that if $u \in L^2(0, T; H_0)$ and $u^{(m)} \in L^2(0, T; H)$, then $u^{(i)} \in L^2(0, T; [H, H_0]_{\theta, i})$, $i = 1, 2, ... m - 1$, where $H_0$, $H$ are Hilbert spaces, $H_0$ is continuously and densely embedded into $H$, and $[H_0, H]_\theta$ is an interpolation space between $H_0$, $H$ for $0 \leq \theta \leq 1$. The similar questions for $H$-valued Sobolev spaces $W^l_2(\Omega; H_0, H)$ studied in [24], where $\Omega \subset R^n$. Then, the boundedness of differential operator $u \rightarrow D^\alpha u$ from $W^l_p(\Omega; H_0, H)$ to $L^p (\Omega; (H_0, H)_{p,|\alpha|l})$ were considered in [18, 19]. This question is generalized for corresponding weighted spaces in [13]. Later,
such type embedding results in \(E\)-valued function spaces \(W_{\gamma}^l (\Omega; E_0, E)\) and its weighted versions studied in [18 – 21]. In this paper, we prove the continuity and compactness of embedding operators in weighted anisotropic function spaces \(W_{\gamma}^l (\Omega; E_0, E)\) for mixed \(p\), which will be defined in bellow.

Here \(l = (l_1, l_2, \ldots, l_n)\), \(l_k\) are positive integers and \(\gamma (x)\) is a positive measurable function on \(\Omega \subset \mathbb{R}^n\). Let \(\alpha_1, \alpha_2, \ldots, \alpha_n\) be nonnegative integers, \(p = (p_1, p_2, \ldots, p_n)\) and \(q = (q_1, q_2, \ldots, q_n)\) for \(1 \leq p_k \leq q_k < \infty\),

\[
\kappa = \sum_{k=1}^n \frac{\alpha_k + \frac{1}{p_k} - \frac{1}{q_k}}{l_k}, \quad D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n} = \frac{\partial^\alpha}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.
\]

Let \(A\) be a positive operator in \(E\), then there are fractional powers of the operator \(A\) (see [22], §1.15.1) and for each powers \(A^\theta\) of \(A\) let \(E (A^\theta)\) denote the domain \(D (A^\theta)\) of \(A^\theta\) with graphical norm. Under certain assumptions to be stated later, we prove that differential operators \(u \to D^\alpha u\) are bounded from \(W_{\gamma}^l (\Omega; E(A), E)\) to \(L_{q, \gamma} (\Omega; (E(A), E)_{\kappa + \mu, \sigma})\), i.e embedding

\[
D^\alpha W_{\gamma}^l (\Omega; E(A), E) \subset L_{q, \gamma} (\Omega; (E(A), E)_{\kappa + \mu, \sigma})
\]

is continuous. More precisely, we prove the following uniform sharp estimate

\[
\|D^\alpha u\|_{L_{q, \gamma} \Omega; E(A), E)_{\kappa + \mu, \sigma}} \leq C_{\mu} (h^\mu \|u\|_{W_{\gamma}^l (\Omega; E(A), E)} + h^{-\mu} \|u\|_{L_{q, \gamma} \Omega; E)}
\]

for \(u \in W_{\gamma}^l (\Omega; E(A), E)\), \(0 \leq \mu < 1 - \kappa\) and \(h > 0\), where \(\sigma = \max \left\{ p_k \right\}\).

The constant \(C_{\mu}\) is independent of \(u\) and \(h\). Further, we prove compactness of embedding operator. These kind of embedding theorems occur in the investigation of boundary value problems for anisotropic elliptic differential-operator equations

\[
\sum_{k=1}^n (-1)^{l_k} t_k \frac{\partial^{2l_k} u}{\partial x_k^{2l_k}} + (A + \lambda) u(x) + \sum_{|\alpha| < \kappa} \prod_{k=1}^n t_k^{\alpha_k} A_\alpha (x) D^\alpha u(x) = f(x),
\]

where \(A, A_\alpha (x)\) are linear operators in a Banach space \(E\), \(t_k\) are positive parameters and \(\lambda\) is a complex number. By using the above embedding results and operator valued multiplier theorems we obtain that problem (1.1) is uniform separable in \(L_{p, \gamma} (\mathbb{R}^n; E)\), i.e. for \(f \in L_{p, \gamma} (\mathbb{R}^n; E)\) problem (1.1) has a unique solution \(u \in W_{\gamma}^l (\mathbb{R}^n; (E(A), E))\) and the following uniform coercive estimate holds

\[
\sum_{k=1}^n t_k \left\| \frac{\partial^{2l_k} u}{\partial x_k^{2l_k}} \right\|_{L_{p, \gamma} (\mathbb{R}^n; E)} + \|Au\|_{L_{p, \gamma} (\mathbb{R}^n; E)} \leq C \|f\|_{L_{p, \gamma} (\mathbb{R}^n; E)},
\]

where the constant \(C\) depend only on \(p\) and \(l\).
Moreover, we get the following uniform sharp resolvent estimate

\[
\sum_{k=1}^{n} \sum_{i=0}^{2l_k} \frac{t_k}{|\lambda|^{1-l_k}} \left\| \frac{\partial^i}{\partial x_k^i} (O_t + \lambda)^{-1} \right\| + \left\| A (O_t + \lambda)^{-1} \right\| \leq C,
\]

where \(O_t\) is the operator generated by problem (1.1).

For \(l_1 = l_2 = \ldots = l_n = m\) we get the elliptic differential-operator equation

\[
\sum_{k=1}^{n} (-1)^{m_k} \frac{\partial^{2m_k} u}{\partial x_k^{2m_k}} + Au(x) + \sum_{|\alpha| < 2m} \prod_{k=1}^{n} t_k^{\alpha_k} A_{\alpha}(x) D_{\alpha} u(x) = f(x).
\]

Then, by using regularity properties of (1.1) the well-posedeness and uniform Strichartz type estimates are established for the solution of abstract parabolic problem

\[
\frac{\partial u}{\partial t} + \sum_{k=1}^{n} (-1)^{l_k} \varepsilon_k \frac{\partial^{2l_k} u}{\partial x_k^{2l_k}} + Au = f(t, x), \quad \text{for } u(0, x) = 0, \ x \in \mathbb{R}^n, \ t \in (0, T),
\]

where \(A\) is a linear operator in a Banach space \(E\) and \(\varepsilon_k\) are small positive parameters.

In this direction we should mention e.g. the works presented in [18 – 21], [22], [26].

Modern analysis methods, particularly abstract harmonic analysis, the operator theory, interpolation of Banach spaces, theory of semigroups and perturbation theory of linear operators are the main tools implemented to carry out the analysis.

2. Notations and definitions

Let \(\mathbb{R}, \mathbb{C}\) be the sets of real and complex numbers, respectively. Let \(E_1\) and \(E_2\) be two Banach spaces and \(L(E_1, E_2)\) denotes the spaces of bounded linear operators from \(E_1\) to \(E_2\). For \(E_1 = E_2 = E\) we denote \(L(E, E)\) by \(L(E)\). We will sometimes write \(A + \xi\) instead of \(A + \xi I\) for a scalar \(\xi\) and \((A + \xi I)^{-1}\) denotes the resolvent of operator \(A\), where \(I\) is the identity operator in \(E\).

Let

\[
S_\varphi = \{ \xi \in \mathbb{C}, |\arg \xi| \leq \varphi \} \cup \{0\}, \ 0 < \varphi \leq \pi.
\]

**Definition 1.** A linear operator \(A\) is said to be \(\varphi\)-positive in a Banach space \(E\), if \(D(A)\) is dense on \(E\) and

\[
\left\| (A + \xi I)^{-1} \right\|_{L(E)} \leq M (1 + |\xi|)^{-1}
\]

with \(\xi \in S_\varphi\), where \(M\) is a positive constant.

**Definition 2.** For \(-\infty < \theta < \infty\) let

\[
E^{\theta}(A^\theta) = \left\{ u \in D(A^\theta) : \left\| u \right\|_{E(A^\theta)} = \left\| A^\theta u \right\|_E + \left\| u \right\|_E < \infty \right\}.
\]
Let $\gamma = \gamma (x)$ be a positive measurable function on $\Omega \subset \mathbb{R}^n$.

**Definition 3.** $L_{p,\gamma}^p (\Omega; E)$ denotes the space of strongly measurable $E$-valued functions such that are defined on $\Omega$ with the norm

$$\|u\|_{L_{p,\gamma}^p (\Omega; E)} = \left( \int_{\Omega} \|u\|_E^p \gamma (x) \, dx \right)^{1/p} < \infty, \quad 1 \leq p < \infty.$$  

For $\gamma (x) \equiv 1$ we denote $L_{p,\gamma}^p (\Omega; E)$ by $L_p^p (\Omega; E)$. For $p = (p_1, p_2, ..., p_n)$, $1 \leq p_k < \infty$ we denote by $L_{p,\gamma}^p (\mathbb{R}^n; E)$ the space of all $E$-valued strongly measurable on $\mathbb{R}^n$ functions with mixed norm

$$\|f\|_{L_{p,\gamma}^p (\mathbb{R}^n; E)} = \left( \left( \prod_{k=1}^n \left( \int_{\mathbb{R}^n} \|f\|_E^p \gamma (x) \, dx \right)^{\frac{p_k}{p_k - 1}} \right)^{\frac{1}{p_n}} \right)^{\frac{1}{p_n}} < \infty.$$  

For $\gamma (x) \equiv 1$ we denote $L_{p,\gamma}^p (\mathbb{R}^n; E)$ by $L_p^p (\mathbb{R}^n; E)$.

The weight $\gamma (x)$ satisfies an $A_p$ condition; i.e., $\gamma (x) \in A_p$, $p \in (1, \infty)$ if there is a positive constant $C$ such that

$$\left( \frac{1}{|Q|} \int_Q \gamma (x) \, dx \right) \left( \frac{1}{|Q|} \int_Q \gamma^{p - 1} (x) \, dx \right)^{p - 1} \leq C$$  

for all compacts $Q \subset \mathbb{R}^n$.

**Remark 2.1.** The result [23] implies that the space $l_p$, $p \in (1, \infty)$ satisfies the multiplier condition with respect to $p$ and the weight functions

$$\gamma = |x|^\alpha, \quad -1 < \alpha < p - 1, \quad \gamma = \prod_{k=1}^N \left( 1 + \sum_{j=1}^n |x_j|^{\alpha_{jk}} \right)^{\beta_k},$$

$$\alpha_{jk} \geq 0, \quad N \in \mathbb{N}, \quad \beta_k \in \mathbb{R},$$

Suppose that $S = S (\mathbb{R}^n)$ is Schwartz space of test functions and $S' (E) = S (\mathbb{R}^n; E)$ is the space of linear continued mapping from $S$ into $E$ and is called $E$- valued Schwartz distributions. For $\varphi \in S$ the Fourier transform $\hat{\varphi}$ and inverse Fourier transform $\check{\varphi}$ are defined by the relations

$$\hat{\varphi} (\xi) = (F \varphi) (\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \varphi (x) e^{-i\xi x} \, dx,$$

$$\check{\varphi} (x) = (F^{-1} \varphi) (x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \varphi (\xi) e^{i\xi x} \, d\xi,$$
where
\[ \xi = (\xi_1, \xi_2, \ldots, \xi_n), \quad x = (x_1, x_2, \ldots, x_n), \quad \xi x = \xi_1 x_1 + \xi_2 x_2 + \ldots + \xi_n x_n. \]

The Fourier transformation and the inverse Fourier transformation of \( E \)-valued generalized functions \( f \in S'(\mathbb{R}^n; E) \) are defined by the relations.

\[ \langle f^\wedge, \varphi \rangle = \langle f, \hat{\varphi} \rangle \quad \text{and} \quad \langle \hat{f}, \varphi \rangle = \langle f, \hat{\varphi} \rangle, \]

where \( \langle f, \varphi \rangle \) means the value of generalized function \( f \in S'(\mathbb{R}^n; E) \) on the \( \varphi \in S(\mathbb{R}^n) \).

**Definition 4.** Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \), where \( \alpha_i \) are positive integers. The \( E \)-values generalized functions \( D^\alpha f \) is called the generalized derivatives in the sense of Schwartz distributions of the generalized function \( f \in S'(\mathbb{R}^n; E) \) if the relation

\[ \langle D^\alpha f, \varphi \rangle = (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle \]

holds for all \( \varphi \in S \).

It is known for all \( \varphi \in S \) the relations

\[ F(D^\alpha_x \varphi) = (i\xi_1)^{\alpha_1} \cdots (i\xi_n)^{\alpha_n} \varphi, \quad D^\alpha_\xi [F(\varphi)] = F[(-ix_1)^{\alpha_1} \cdots (-ix_n)^{\alpha_n} \varphi] \quad (2.1) \]

holds.

Let \( \eta \) be a infinitely differentiable function with polynomial structure and \( f \in S'(\mathbb{R}^n; E) \). Then \( \eta f \in S'(\mathbb{R}^n, E) \) is a generalized function defined by the relation

\[ \langle \eta f, \varphi \rangle = \langle f, \eta f \rangle \quad \forall \varphi \in S(\mathbb{R}^n). \]

By using Definition 4 and relations (2.1) we get

\[ F(D^\alpha_x f) = (i\xi_1)^{\alpha_1} \cdots (i\xi_n)^{\alpha_n} f, \quad D^\alpha_\xi [F(f)] = F[(-ix_1)^{\alpha_1} \cdots (-ix_n)^{\alpha_n} f] \quad (2.2) \]

for \( f \in S'(\mathbb{R}^n; E) \).

The Banach space \( E \) is called an UMD-space if the Hilbert operator

\[ (Hf)(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} \, dy \]

is bounded in \( L^p(\mathbb{R}, E) \) for \( p \in (1, \infty) \) (see. e.g. [2]). UMD spaces include e.g. \( L^p, l_p \) spaces and Lorentz spaces \( L_{pq} \) for \( p, q \in (1, \infty) \).

\( C^m(\Omega; E) \) will denote the spaces of \( E \)-valued bounded uniformly strongly \( m \) times continuously differentiable functions on \( \Omega \). Assume \( \gamma \) is such that \( S(\mathbb{R}^n; E_1) \) is dense in \( L_{p,\gamma}(\mathbb{R}^n; E_1) \).
investigated e.g. in [9-11].
e.g. in [5, 6].

\[ \mathcal{U} : L_{p,q}(R^n;E_2) \rightarrow L_{q,q}(R^n;E_2). \]

We denote the set of all multipliers from \( L_{p,q}(R^n;E_1) \) to \( L_{q,q}(R^n;E_2) \) by \( M_{p,q}^{q,q}(E_1,E_2) \). For \( E_1 = E_2 \) it denotes by \( M_{p,q}^{q,q}(E) \).

Let \( \Phi_h = \{ \Psi_h \in M_{p,q}^{q,q}(E_1,E_2), h \in Q \} \) denote a collection of multipliers depending on the parameter \( h \).

We say that \( W_h \) is a uniform collection of multipliers if there exists a positive constant \( M \) independent of \( h \in Q \) such that
\[
\| F^{-1} \Psi_h F u \|_{L_{q,q}(R^n;E_2)} \leq M \| u \|_{L_{p,q}(R^n;E_1)}
\]
for all \( u \in S(R^n;E_1) \) and \( h \in Q \).

Note that, Fourier multiplier theorems in complex valued weighted \( L_p \) spaces investigated e.g. in [9-11]. In Banach space-valued classes this question studied e.g. in [5, 6].

Let
\[
U_n = \beta = (\beta_1, \beta_2, ..., \beta_n), \beta_i \in \{0,1\}, \sigma_k = \left( \frac{1}{p_k} - \frac{1}{q_k} \right),
\]
\[
V_n = \{ \xi = (\xi_1, \xi_2, ..., \xi_n) \in R^n, \xi_i \neq 0, i = 1, 2, ..., n \}.
\]

**Definition 5.** A function \( \Psi \in C^{(m)}(R^n;L(E_1;E_2)) \) is called a multiplier from \( L_{p,q}(R^n;E_1) \) to \( L_{q,q}(R^n;E_2) \) if the map \( u \rightarrow \Lambda u = F^{-1} \Psi(\xi) F u, u \in S(R^n;E_1) \) is well defined and extends to a bounded linear operator
\[
\Lambda : L_{p,q}(R^n;E_1) \rightarrow L_{q,q}(R^n;E_2).
\]

We denote the set of all multipliers from \( L_{p,q}(R^n;E_1) \) to \( L_{q,q}(R^n;E_2) \) by \( M_{p,q}^{q,q}(E_1,E_2) \). For \( E_1 = E_2 \) it denotes by \( M_{p,q}^{q,q}(E) \).

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We say that \( W_h \) is a uniform collection of multipliers if there exists a positive constant \( M \) independent of \( h \in Q \) such that
\[
\| F^{-1} \Psi_h F u \|_{L_{q,q}(R^n;E_2)} \leq M \| u \|_{L_{p,q}(R^n;E_1)}
\]
for all \( u \in S(R^n;E_1) \) and \( h \in Q \).

Note that, Fourier multiplier theorems in complex valued weighted \( L_p \) spaces investigated e.g. in [9-11]. In Banach space-valued classes this question studied e.g. in [5, 6].

Let
\[
U_n = \beta = (\beta_1, \beta_2, ..., \beta_n), \beta_i \in \{0,1\}, \sigma_k = \left( \frac{1}{p_k} - \frac{1}{q_k} \right),
\]
\[
V_n = \{ \xi = (\xi_1, \xi_2, ..., \xi_n) \in R^n, \xi_i \neq 0, i = 1, 2, ..., n \}.
\]

**Definition 6.** The Banach space \( E \) satisfies the multiplier condition with respect to \( p \) and \( q \) (or with respect to \( p \) in the case of \( p = q \)) and with respect to weighted function \( \gamma \) if for all \( \Psi \in C^{(n)}(R^n;L(E)) \) with \( \beta \in U_n, \xi \in V_n \) the inequality
\[
\| D^\beta \Psi(\xi) \|_{L(E)} \leq C \prod_{k=1}^{n} |\xi_k|^{-\sigma_k} (\beta_k + \sigma_k)^{\gamma_k}
\]
implies \( \Psi \in M_{p,q}^{q,q}(E) \).

Note that, if \( E_1 \) and \( E_2 \) are UMD spaces, \( \gamma(x) \equiv 1 \) and \( p_1 = p_2 = ... = p_n = p \), then by virtue of operator valued multiplier theorems (see e.g [7], [24]) we obtain that \( \Psi \) is a Fourier multiplier in \( L_p(R^n;E) \). It is well known (see [12]) that any Hilbert space satisfies the multiplier condition for \( \gamma(x) \equiv 1 \) with respect to any \( p \) and \( q \) with \( 1 < p \leq q < \infty \). However, there are Banach spaces which are not Hilbert spaces but satisfy the multiplier condition, for example UMD spaces, \( \xi \)–convex Banach lattice spaces (see [7], [24], [5], [6]).

Assume \( \gamma_k \) are positive measurable functions on \( \mathbb{R} \) and
\[
D_k^{[i]} = \left( \gamma_k(x_k) \frac{\partial}{\partial x_k} \right)^i.
\]

**Definition 7.** Consider the following spaces:
\[
W_{p,q}^l(\Omega;E_0,E) = \left\{ u \in L_{p,q}(\Omega;E_0), D_k^{[i]} u \in L_{p,q}(\Omega;E), k = 1, 2, ..., n \right\},
\]
First of all, in a similar way as in [21, Lemma 3.1] we have \( W \) parametrized norms in \( C \) positive numbers \( E \) nonnegative and \( l \). Then for any \( h > t \) \( \Psi \) Lemma A 1. Assume \( A \) is a \( \varphi \)– positive linear operator on a Banach space \( E \). Then for any \( h > 0 \) and \( 0 < \mu \leq 1 - \kappa \) the operator-function

\[
\Psi_t (\xi) = \Psi_{t,h,\mu} (\xi) = T (t) |\xi|^{\alpha + \sigma} A^{-1 - \kappa - \mu} h^{-\mu} \left[ A + \sum_{k=1}^{n} t_k |\xi_k|^{l_k} + h^{-1} \right]^{-1}
\]

For \( \gamma (x) \equiv 1 \) we will denote \( W^{[i]}_{p,\gamma} (\Omega, E_0, E) \) and \( W^{[i]}_{p,\gamma} (\Omega, E_0, E) \) by \( W^{[i]}_p (\Omega, E_0, E) \).

Let \( t_k \) be positive parameters and \( t = (t_1, t_2, ..., t_n) \). We define the following parametrized norms in \( W^{[i]}_{p,\gamma} (\Omega; E_0, E) \) and \( W^{[i]}_{p,\gamma} (\Omega; E_1, E) \) such that

\[
\|u\|_{W^{[i]}_{p,\gamma} (\Omega; E_0, E)} = \|u\|_{L_{p,\gamma} (\Omega; E_0)} + \sum_{k=1}^{n} \|D_{k} D_{k}^{[i]} u\|_{L_{p,\gamma} (\Omega; E)} < \infty,
\]

\[
\|u\|_{W^{[i]}_{p,\gamma} (\Omega; E_0, E)} = \|u\|_{L_{p,\gamma} (\Omega; E_0)} + \sum_{k=1}^{n} \|D_{k} D_{k}^{[i]} u\|_{L_{p,\gamma} (\Omega; E)} < \infty.
\]

For two elements \( u, v \in E \) the expression \( \|u\| \sim \|v\| \) means that there exist positive numbers \( C_1 \) and \( C_2 \) such that

\[
C_1 \|u\| \leq \|v\| \leq C_2 \|u\|.
\]

3. Embedding theorems

In this section, we prove that the generalized derivative operator \( D^\alpha \) generates a continuous embedding in Sobolev spaces of vector-functions. Let \( \alpha_k \) be nonnegative and \( l_k \) positive integers and

\[
\alpha = (\alpha_1, \alpha_2, ..., \alpha_n), \quad l = (l_1, l_2, ..., l_n), \quad \sigma = (\sigma_1, \sigma_2, ..., \sigma_n),
\]

\[
t = (t_1, t_2, ..., t_n), \quad t_k > 0, \quad \sigma_k > 0,
\]

\[
1 < p_k \leq q_k < \infty, \quad p = (p_1, p_2, ..., p_n), \quad q = (q_1, q_2, ..., q_n),
\]

\[
\kappa = \left( \left( \alpha + \frac{1}{p} - \frac{1}{q} \right) : l \right) = \sum_{k=1}^{n} \frac{\alpha_k + \sigma_k}{l_k},
\]

\[
\xi = (\xi_1, \xi_2, ..., \xi_n) \in R^n, \quad |\xi|^{\alpha + \sigma} = \prod_{k=1}^{n} |\xi_k|^{|\alpha_k + \sigma_k|}, \quad T (t) = \prod_{k=1}^{n} t_k^{|\alpha_k + \sigma_k|}.
\]

First of all, in a similar way as in [21, Lemma 3.1] we have

Lemma A 1. Assume \( A \) is a \( \varphi \)– positive linear operator on a Banach space \( E \). Then for any \( h > 0 \) and \( 0 < \mu \leq 1 - \kappa \) the operator-function
is bounded in $E$ uniformly with respect to $\xi \in \mathbb{R}^n$, $h > 0$ and $t$, i.e. there exists a constant $C_\mu$ such that

$$
\|\Psi_{t,h,\mu}(\xi)\|_{L(E)} \leq C_\mu
$$

(3.1)

for all $\xi \in \mathbb{R}^n$ and $h > 0$.

In a similar way as in [19, Theorem 2] we obtain the following

**Theorem A.** Assume $E_0$, $E$ are two Banach spaces and the embedding $E_0 \subset E$ is compact. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and $\gamma \in A_{p_k}$ for $p_k \in (1, \infty)$. Then the embedding

$$
W_{p,\gamma}^l(\Omega; E_0, E) \subset L_{p,\gamma}(\Omega; E)
$$

is compact.

Let

$$
X = L_{p,\gamma}(\mathbb{R}^n; E), \quad Y = W_{p,\gamma}^l(\mathbb{R}^n; E(A), E).
$$

One of main result of this section is the following:

**Theorem 3.1.** Assume $E$ is a Banach space satisfying the multiplier condition with respect to $p$, $q$ and weighted function $\gamma$. Suppose $A$ is a $\varphi$-positive operator in $E$. Then for $0 \leq \mu \leq 1 - \kappa$, $1 < p_k \leq q_k < \infty$ or $0 < \mu < 1 - \kappa$ for $1 \leq p_k \leq q_k \leq \infty$ the embedding

$$
D^\alpha \Psi_{t,\gamma,\mu}^l(\mathbb{R}^n; E(A), E) \subset L_{q,\gamma}(\mathbb{R}^n; E(A^{1-\kappa-\mu}))
$$

is a continuous and there exists a constant $C_\mu > 0$ depending only on $\mu$ such that

$$
T(t) \|D^\alpha u\|_{L_{q,\gamma}(\mathbb{R}^n; E(A^{1-\kappa-\mu}))} \leq C_\mu \left[ h^\mu \|u\|_{W_{p,\gamma}^l(\mathbb{R}^n; E(A), E)} + h^{-(1-\mu)} \|u\|_{X} \right]
$$

(3.2)

for $u \in Y$ and $h > 0$.

**Proof.** We have

$$
\|D^\alpha u\|_{L_{q,\gamma}(\mathbb{R}^n; E(A^{1-\kappa-\mu}))} < \infty
$$

(3.3)

$$
\left( \left( \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \|D^\alpha u\|_{E(A^{1-\kappa-\mu})} \gamma(x) \, dx_1 \right)^{\frac{q_1}{2}} \, dx_2 \right)^{\frac{q_2}{2}} \, dx_3 \right)^{\frac{q_3}{2}} \, dx_4 \right)^{\frac{q_4}{q}}
$$

$$
\left( \left( \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \|A^{1-\kappa-\mu} D^\alpha u\|_{E} \gamma(x) \, dx_1 \right)^{\frac{q_1}{2}} \, dx_2 \right)^{\frac{q_2}{2}} \, dx_3 \right)^{\frac{q_3}{2}} \, dx_4 \right)^{\frac{q_4}{q}}
$$

for all $u$ such that

$$
\|D^\alpha u\|_{L_{q,\gamma}(\mathbb{R}^n; E(A^{1-\kappa-\mu}))} < \infty.
$$
On the other hand, it is clear to see that

$$A^{1-\alpha-\mu} D^\alpha u = F^{-1} F A^{1-\alpha-\mu} D^\alpha u = F^{-1} A^{1-\alpha-\mu} F D^\alpha u =$$

$$(3.4)$$

$$F^{-1} A^{1-\alpha-\mu} (i \xi)^\alpha F u = F^{-1} (i \xi)^\alpha A^{1-\alpha-\mu} F u.$$ 

Hence, denoting $Fu$ by $\hat{u}$, we get from relations (3.3) and (3.4)

$$\|D^\alpha u\|_{L^{q,\gamma}(R^n; E(A^{1-\alpha-\mu}))} \preceq \|F^{-1} (i \xi)^\alpha A^{1-\alpha-\mu} \hat{u}\|_{L^{q,\gamma}(R^n; E)}.$$ 

Similarly, in view of Definition 7 for $u \in Y$ we have

$$\|u\|_{W^{1,q,\gamma}(R^n; E(A), E)} = \|u\|_{L^{q,\gamma}(R^n; E(A))} + \sum_{k=1}^n \|t_k D^k \hat{u}\|_X =$$

$$\|F^{-1} \hat{u}\|_X + \sum_{k=1}^n \|t_k F^{-1} [(i \xi_k)^l \hat{u}]\|_X \preceq \|F^{-1} A \hat{u}\|_X + \sum_{k=1}^n \|t_k F^{-1} [(i \xi_k)^l \hat{u}]\|_X.$$ 

Therefore, for proving the inequality (3.2) it suffices to show

$$T(t) \|F^{-1} (i \xi)^\alpha A^{1-\alpha-\mu} \hat{u}\|_{L^{q,\gamma}(R^n; E)} \leq$$

$$C_\mu (h^n \|F^{-1} A \hat{u}\|_X + \sum_{k=1}^n \|t_k F^{-1} [(i \xi_k)^l \hat{u}]\|_X + h^{-(1-\mu)} \|F^{-1} \hat{u}\|_X). \quad (3.5)$$

Therefore, the inequality (3.2) will follow if we prove the following estimate

$$T(t) \|F^{-1} [\xi^{\alpha+\sigma} A^{1-\alpha-\mu} \hat{u}]\|_{L^{q,\gamma}(R^n; E)} \leq C_\mu \|F^{-1} G(\xi) \hat{u}\|_X.$$ \quad (3.6)

for $u \in Y$, where

$$G(\xi) = h^n \left[ A + \sum_{k=1}^n t_k |\xi_k|^l + h^{-(1-\mu)} \right].$$

Due to positivity of $A$, the operator function $G(\xi)$ has a bounded inverse in $E$ for all $\xi \in R^n$. So, we can set

$$T(t) F^{-1} \xi^{\alpha+\sigma} A^{1-\alpha-\mu} \hat{u} =$$

$$T(t) F^{-1} \xi^{\alpha+\sigma} A^{1-\alpha-\mu} G^{-1}(\xi) \left[ h^n \left( A + \sum_{k=1}^n t_k |\xi_k|^l \right) + h^{-(1-\mu)} \right] \hat{u}. \quad (3.7)$$

By Definition 6 it is clear to see that the inequality (3.6) will follow immediately from (3.7) if we can prove that the operator-function

$$\Psi_{t, h, \mu} = \xi^{\alpha+\sigma} A^{1-\alpha-\mu} \left[ h^n (A + \sum_{k=1}^n t_k |\xi_k|^l) + h^{-(1-\mu)} \right]^{-1}$$
is a multiplier in $M^{0,\gamma}_{p} (E)$ uniformly with respect to $h$ and $t$. So, it suffices to show that for all $\beta \in U_n$ and $\xi \in V_n$ there exists a constant $C_\mu > 0$ such that the following uniform estimate holds

$$
\| D^\beta \Psi_{t,h,\mu} (\xi) \|_{L(E)} \leq C_\mu \prod_{k=1}^{n} |\xi_k|^{-(\beta_k + \sigma_k)}.
$$

(3.8)

To see this, by applying the Lemma A_1 for all $\xi \in \mathbb{R}^n$ we get a constant $C_\mu > 0$ depending only on $\mu$ such that

$$
|\Psi_{t,h,\mu} (\xi) |_{L(E)} \leq C_\mu \prod_{k=1}^{n} |\xi_k|^{-\sigma_k}.
$$

(3.9)

This shows that the inequality (3.8) is satisfies for $\beta = (0,...,0)$. Now, we next consider (3.8) for $\beta = (\beta_1,...,\beta_n)$ where $\beta_k = 1$ and $\beta = 0$ for $j \neq k$. Then, by using the positivity properties of $A$ we obtain

$$
\left| \frac{\partial}{\partial \xi_k} \Psi_t (\xi) \right| \leq \prod_{k=1}^{n} \left| t_k \right|^{\alpha_k} \left| \alpha_k \right| \left| \xi_1^{\alpha_1} \cdots \xi_{k-1}^{\alpha_{k-1}} \xi_k^{\alpha_k-1} \cdots \xi_n^{\alpha_n} \right|
$$

$$
\left| A^{1-\alpha-\mu} \left( h^\mu \left( A + \sum_{k=1}^{n} t_k |\xi_k|^{l_k} \right) + h^{-(1-\mu)} \right) \right|^{-1} + \left| A^{1-\alpha-\mu} \left( h^\mu \left( A + \sum_{k=1}^{n} t_k |\xi_k|^{l_k} \right) + h^{-(1-\mu)} \right) \right|^{-2} \leq C_\mu |\xi_k|^{-1} \sum_{j=1}^{n} |\xi_j|^{-\sigma_j} , \ k = 1,2...n.
$$

Repeating the above process, we obtain that for all $\beta \in U_n$, $\xi \in V_n$ there exists a constant $C_\mu > 0$ depending only $\mu$ such that

$$
\| D^\beta \Psi_t (\xi) \|_{L(E)} \leq C_\mu |\xi|^{-(\beta + \sigma)}.
$$

Therefore, the operator-function $\Psi_{t,h,\mu} (\xi)$ is a uniform multiplier with respect to $h$ and $t$, i.e.,

$$
\Psi_{t,h,\mu} \in M^{0,\gamma}_{p} (E) , \ \text{for} \ t_k, h, \mu \in \mathbb{R}_+.
$$

This completes the proof of the Theorem 3.1.

It is possible to state Theorem 3.1 in a more general setting. For this aim, we use the concept of extension operator.

**Condition 3.1.** Let $A$ be positive operator in Banach spaces $E$ satisfying multiplier condition with respect to $p$ and weighted function $\gamma$. Assume a region $\Omega \subset \mathbb{R}^n$ such that there exists bounded linear extension operator $B$ from $W^{l_i}_{p,\gamma} (\Omega, E(A), E)$ to $W^{l_i}_{p,\gamma} (R^n, E(A), E)$ for $1 \leq p_k \leq \infty$. 

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Remark 3.1. If $\Omega \subset R^n$ is a region satisfying the strong 1-horn condition (see [3], p.117), $E = C$, $A = I$ and $\gamma(x) \equiv 1$ then for $1 < p < \infty$ there exists a bounded linear extension operator from $W^1_p(\Omega; R)$ to $W^1_p(R^n; R)$.

Theorem 3.2. Assume conditions of Theorem 3.1 and Condition 3.1 are hold. Then for $0 \leq \mu \leq 1 - \kappa$ the embedding

$$D^\alpha W^{l}_{p,\gamma}(\Omega; E(A), E) \subset L^{q,\gamma}_{\alpha}(\Omega; E(A^{1-\kappa-\mu}))$$

is continuous and there exists a constant $C_\mu$ depending only on $\mu$ such that

$$T(t) \|D^\alpha u\|_{L^{q,\gamma}_{\alpha}(\Omega; E(A^{1-\kappa-\mu}))} \leq$$

$$C_\mu \left[ h^\mu \|u\|_{W^{l}_{p,\gamma}(\Omega; E(A), E)} + h^{-(1-\mu)} \|u\|_{L^{q,\gamma}_{\alpha}(\Omega; E)} \right]$$

for $u \in W^{l}_{p,\gamma}(\Omega; E(A), E)$ and $h > 0$.

Proof. It is suffices to prove the estimate (3.10). Let $B$ be a bounded linear extension operator from $W^{l}_{p,\gamma}(\Omega; E(A), E)$ to $W^{l}_{p,\gamma}(R^n; E(A), E)$, and let $B_\Omega$ be the restriction operator from $R^n$ to $\Omega$. Then for any $u \in W^{l}_{p,\gamma}(\Omega; E(A), E)$ we have

$$T(t) \|D^\alpha u\|_{L^{q,\gamma}_{\alpha}(\Omega; E(A^{1-\kappa-\mu}))} = T(t) \|D^\alpha B_\Omega B u\|_{L^{q,\gamma}_{\alpha}(\Omega; E(A^{1-\kappa-\mu}))} \leq$$

$$CT(t) \|D^\alpha B u\|_{L^{q,\gamma}_{\alpha}(R^n; E(A^{1-\kappa-\mu}))} \leq$$

$$C_\mu \left[ h^\mu \|B u\|_{W^{l}_{p,\gamma}(R^n; E(A), E)} + h^{-(1-\mu)} \|B u\|_{L^{q,\gamma}_{\alpha}(R^n; E)} \right] \leq$$

$$C_\mu \left[ h^\mu \|u\|_{W^{l}_{p,\gamma}(\Omega; E(A), E)} + h^{-(1-\mu)} \|u\|_{L^{q,\gamma}_{\alpha}(\Omega; E)} \right].$$

Result 3.1. Assume the conditions of Theorem 3.2 are satisfied. Then for $u \in W^{l}_{p,\gamma}(\Omega; E(A), E)$ we have the following multiplicative estimate

$$\|D^\alpha u\|_{L^{q,\gamma}_{\alpha}(\Omega; E(A^{1-\kappa-\mu}))} \leq C \|u\|_{W^{l}_{p,\gamma}(\Omega; E(A), E)} \|u\|_{L^{q,\gamma}_{\alpha}(\Omega; E)}.$$  (3.11)

Indeed, setting

$$h = \|u\|_{L^{q,\gamma}_{\alpha}(\Omega; E)} \|u\|_{W^{l}_{p,\gamma}(\Omega; E(A), E)}^{-1}$$

in (3.10) we obtain (3.11).

Theorem 3.3. Assume that the conditions of Theorem 3.2 are satisfied. Suppose that $\Omega$ is a bounded domain in $R^n$ and $A^{-1}$ is a compact operator in $E$. Let $\gamma \in A_{p_k}$ for $p_k \in (1, \infty)$. Then for $0 < \mu \leq 1 - \kappa$ the embedding

$$D^\alpha W^{l}_{p,\gamma}(\Omega; E(A), E) \subset L^{q,\gamma}_{\alpha}(\Omega; E(A^{1-\kappa-\mu}))$$

is compact.

Proof. By virtue of Theorem A we get that the embedding

$$W^{l}_{p,\gamma}(\Omega; E(A), E) \subset L^{q}_{\alpha}(\Omega; E)$$
is compact. Then by (3.11) we obtain the assertion of Theorem 3.3.

Let
\[ \sigma = \max_k p_k. \]

**Theorem 3.4.** Suppose conditions of Theorem 3.1 are hold. Then for \( 0 < \mu < 1 - \kappa \) the embedding
\[ D^\alpha W^l_{p,q} (R^n; E (A) , E) \subset L_{q,\gamma} \left( R^n; (E (A) , E)_{\kappa+\mu,\sigma} \right) \]
is continuous and there exists a constant \( C_\mu \) depending only on \( \mu \) such that
\[ \| D^\alpha u \|_{L_{q,\gamma} (R^n; (E (A) , E)_{\kappa+\mu,\sigma})} \leq \| u \|_{L_{p,\gamma} (R^n; E)} + h^{- (1 - \mu)} \| u \|_{L_{p,\gamma} (R^n; E)} \]
for \( u \in Y \) and \( 0 < h \leq h_0 < \infty \).

**Proof.** It is sufficient to prove the estimate (3.12) for \( u \in Y \). By definition of interpolation spaces \( (E (A) , E)_{\kappa+\mu,\sigma} \) (see [22, §1.14.5]) the estimate (3.12) is equivalent to the inequality
\[ \left\| F^{-1} y^{1 - \kappa - \mu - \frac{1}{r}} \left[ A^{\kappa+\mu} (A + y)^{-1} \right] \xi^\alpha \hat{u} \right\|_{L_{p,\gamma} (R^n; E)} \]
\[ \leq C_\mu \left\| F^{-1} \left[ h^\mu \left( A + \sum_{k=1}^n A + \sum_{k=1}^n |\xi_k|^l + h^{- (1 - \mu)} \right) \hat{u} \right] \right\|_{L_{p,\gamma} (R^n; E)} \]  
(3.13)

By multiplier properties, the inequality (3.13) will follow immediately if we will prove that the operator-function
\[ \Psi = (i\xi)^\alpha y^{1 - \kappa - \mu - \frac{1}{r}} A^{\kappa+\mu} (A + y)^{-1} \left[ h^\mu \left( A + \sum_{k=1}^n |\xi_k|^l \right) + h^{- (1 - \mu)} \right]^{-1} \]
is a multiplier from \( L_{p,\gamma} (R^n; E) \) to \( L_{p,\gamma} (R^n; L_\sigma (R^n; E)) \). This fact is proved by the same manner as Theorem 3.1. Therefore, we get the estimate (3.12).

In a similar way, as the Theorem 3.2 we obtain

**Theorem 3.5.** Suppose conditions of Theorem 3.2 are hold. Then for \( 0 < \mu < 1 - \kappa \) the embedding
\[ D^\alpha W^l_{p,q} (\Omega; E (A) , E) \subset L_{q,\gamma} \left( \Omega; (E (A) , E)_{\kappa+\mu,\sigma} \right) \]
is continuous and there exists a constant \( C_\mu \) depending only on \( \mu \) such that
\[ \| D^\alpha u \|_{L_{q,\gamma} (\Omega; (E (A) , E)_{\kappa+\mu,\sigma})} \leq C_\mu \left[ h^\mu \| u \|_{W^l_{p,q} (\Omega; (E (A) , E))} + h^{- (1 - \mu)} \| u \|_{L_{p,\gamma} (\Omega; E)} \right] \]
(3.14)
Indeed setting

\[ \|\mathcal{D}^a u\|_{L_{q,\gamma}(\Omega; (E(A), E))} \leq C_{\mu} \|u\|_{W_{p,\gamma}^{1,\mu}(\Omega; (E(A), E))} \|u\|_{L_{p,\gamma}(\Omega; E)} \]  

(3.15)

Indeed setting \(\|u\|_{L_{p,\gamma}(\Omega; E)} \cdot \|u\|_{W_{p,\gamma}^{1,\mu}(\Omega; (E(A), E))}\) in (3.14) we obtain (3.15).

From the estimate (3.15) and Theorem A, in a similar way as Theorem 3.3 we obtain

**Theorem 3.6.** Assume that the conditions of Theorem 3.2 are satisfied. Suppose \(\Omega\) is a bounded domain in \(\mathbb{R}^n\) and \(A^{-1}\) is a compact operator in \(E\). Let \(\gamma \in A_{p_k}\) for \(p_k \in (1, \infty)\). Then for \(0 < \mu \leq 1 - \sigma\) the embedding

\[ \mathcal{D}^a W_{p,\gamma}^{l} (\Omega; (E(A), E)) \subset L_{q,\gamma} (\Omega; (E(A), E)) \]

is compact.

From Theorem 3.2 we obtain

**Result 3.2.** Assume the conditions of Theorem 3.2 are satisfied for \(l_1 = l_2 = \ldots = l_n = m\). Then for \(0 \leq \mu \leq 1 - \sigma\) the embedding

\[ \mathcal{D}^a W_{p,\gamma}^{m} (\Omega; (E(A), E)) \subset L_{q,\gamma} (\Omega; (E(A^{1-\sigma-\mu}), E)) \]

is continuous and there exists a constant \(C_{\mu}\) depending only on \(\mu\) such that

\[ T(t) \|\mathcal{D}^a u\|_{L_{q,\gamma}(\Omega; (E(A^{1-\sigma-\mu})))} \leq C_{\mu} \left[ h^\mu \|u\|_{W_{p,\gamma}^{m}(\Omega; (E(A), E))} + h^{-(1-\mu)} \|u\|_{L_{p,\gamma}(\Omega; E)} \right] \]

for \(u \in W_{p,\gamma}^{m}(\Omega; (E(A), E))\) and \(h > 0\), where

\[ \sigma = \frac{1}{m} \left( |\alpha| + \sum_{k=1}^{n} \sigma_k \right). \]

**Result 3.3.** If \(E = H\), where \(H\) is a Hilbert space and \(p_k = q_k = 2\), \(\Omega = (0, T)\), \(l_1 = l_2 = \ldots = l_n = m\), \(A = A^\infty \geq cI\), \(\gamma(x) \equiv 1\) then we obtain the well known Lions-Peetre [11] result. Moreover, the result of Lions-Peetre are improving even in the one dimensional case for the non selfadjoint positive operators \(A\).

From Theorems 3.2, 3.3 we obtain

**Result 3.4.** Suppose the conditions of Theorem 3.2 are satisfied for \(\gamma(x) \equiv 1\). Then for \(0 \leq \mu \leq 1 - \sigma\) the embedding

\[ \mathcal{D}^a W_{p}^{l} (\Omega; (E(A), E)) \subset L_{q} (\Omega; (E(A^{1-\sigma-\mu}))) \]

is a continuous and there exists a constant \(C_{\mu} > 0\), depending only on \(\mu\) such that

\[ T(t) \|\mathcal{D}^a u\|_{L_{q}(\Omega; (E(A^{1-\sigma-\mu})))} \leq C_{\mu} \left[ h^\mu \|u\|_{W_{p}^{l}(\Omega; (E(A), E))} + h^{-(1-\mu)} \|u\|_{L_{p}(\Omega; E)} \right] \]
for $u \in W^l_p (\Omega; E (A), E)$ and $h > 0$.

Moreover, if $\Omega$ is a bounded domain in $\mathbb{R}^n$ and $A^{-1}$ is a compact operator in $E$, then for $0 < \mu \leq 1 - \varkappa$ the embedding

$$D^\alpha W^l_p (\Omega; E (A), E) \subset L^q (\Omega; E (A^{1-\varkappa-\mu}))$$

is compact.

If $E = \mathbb{C}$, $A = I$, $\gamma (x) \equiv 1$ we get the embedding $D^\alpha W^l_p (\Omega) \subset L^q (\Omega)$ proved in [3] for numerical Sobolev spaces $W^l_p (\Omega)$.

4. Application

Let $s > 0$. Consider the following sequence space (see e.g. [22, § 1.18])

$$l^s_q = \{ u = \{ u_i \}, \ i = 1, 2, ..., \infty, \ u_i \in \mathbb{C} \}$$

with the norm

$$\| u \|_{l^s_q} = \left( \sum_{i=1}^{\infty} 2^{i\nu s} | u_i |^p \right)^{\frac{1}{q}} < \infty, \ \nu \in (1, \infty).$$

Note that, $l^0_{\nu} = l_{\nu}$. Let $A$ be infinite matrix defined in $l_{\nu}$ such that $D (A) = l^s_{\nu}$, $A = [ \delta_{ij} 2^{s i}]$, where $\delta_{ij} = 0$, when $i \neq j$, $\delta_{ij} = 1$, when $i = j = 1, 2, ..., \infty$.

It is clear to see that, the operator $A$ is positive in $l_{\nu}$. Then from Theorem 3.2 and Theorem 3.3 we obtain the following results

**Result 4.1.** Suppose the conditions of Theorem 3.2 are satisfied for $E = \mathbb{C}$. Then for $0 \leq \mu \leq 1 - \varkappa$, $1 < p_k \leq q_k < \infty$ or $0 < \mu < 1 - \varkappa$ for $1 \leq p_k \leq q_k \leq \infty$ the embedding

$$D^\alpha W^l_{p,\gamma} (\Omega, l^s_q, l_{\nu}) \subset L^q_{p,\gamma} (\Omega, l^s_q, (1-\varkappa-\mu)_{\nu})$$

is a continuous and there exists a constant $C_{\mu} > 0$, depending only on $\mu$ such that

$$T (t) \| D^\alpha u \|_{L^q_{p,\gamma} (\Omega; l^s_q, (1-\varkappa-\mu)_{\nu})} \leq C_{\mu} \left[ \Phi^{\mu} (\| u \|_{W^l_{p,\gamma} (\Omega; l^s_q, l_{\nu})}) + h^{-1-\mu} \| u \|_{L^q_{p,\gamma} (\Omega; l_{\nu})} \right]$$

for $u \in W^l_{p,\gamma} (\Omega, l^s_q, l_{\nu})$ and $h > 0$.

**Result 4.2.** Suppose the conditions of Theorem 3.3 are hold for $E = \mathbb{C}$. Then for $0 < \mu \leq 1 - \varkappa$, $1 < p_k \leq q_k < \infty$ or $0 < \mu < 1 - \varkappa$ for $1 \leq p_k \leq q_k \leq \infty$ the embedding

$$D^\alpha W^l_{p,\gamma} (\Omega, l^s_q, l_{\nu}) \subset L^q_{p,\gamma} (\Omega, l^s_q, (1-\varkappa-\mu)_{\nu})$$

is compact.
**Result 4.3.** For $0 \leq \mu \leq 1 - \kappa$, $1 < p_k \leq q_k < \infty$ or $0 < \mu < 1 - \kappa$ for $1 \leq p_k \leq q_k \leq \infty$ the embedding

$$D^\alpha W^l_p (\Omega, l^s_t, l^v_\nu) \subset L_q \left( \Omega, l^{(1-\kappa-\mu)}_t, l^v_\nu \right)$$

is a continuous and there exists a constant $C_\mu > 0$, depending only on $\mu$ such that

$$T(t) \| D^\alpha u \|_{L_q(\Omega; l^{(1-\kappa-\mu)}_t, l^v_\nu)} \leq C_\mu \left[ h^\mu \| u \|_{W^l_p(\Omega; l^s_t, l^v_\nu)} + h^{-(1-\mu)} \| u \|_{L_q(\Omega; l^v_\nu)} \right]$$

for $u \in W^l_p (\Omega, l^s_t, l^v_\nu)$ and $h > 0$.

**Result 4.4.** For $0 < \mu \leq 1 - \kappa$, $1 < p_k \leq q_k < \infty$ or $0 < \mu < 1 - \kappa$ for $1 \leq p_k \leq q_k \leq \infty$ the embedding

$$D^\alpha W^l_p (\Omega, l^s_t, l^v_\nu) \subset L_q \left( \Omega, l^{(1-\kappa-\mu)}_t, l^v_\nu \right)$$

is compact.

Note that, these results haven’t been obtained with classical method until now.

5. Separable degenerate abstract differential operators

Let us consider the problem

$$\sum_{k=1}^{n} (-1)^k t_k D^{[k]}_k u (x) + (A + \lambda) u (x) + \sum_{\alpha; 2|\alpha| < 1} \prod_{k=1}^{n} t_k^{\alpha_k} A_\alpha (x) D^{[\alpha]} u (x) = f (x), \quad x \in \mathbb{R}^n,$$

considered in $L_p (\mathbb{R}^n; E)$, where $A, A_\alpha$ are linear operators in a Banach space $E$, $t_k$ are positive and $\lambda$ is a complex parameter.

Let

$$X = L_p, \tilde{\gamma} (\mathbb{R}^n; E), \quad Y = W_p^{2l}, \tilde{\gamma} (\mathbb{R}^n; E (A), E).$$

**Remark 5.1.** Under the substitution

$$\tau_k = \int_0^{x_k} \gamma^{-1} (y) dy$$

the spaces $L_p (\mathbb{R}^n; E)$, $W^{2l}_p, \tilde{\gamma} (\mathbb{R}^n; E (A), E)$ are mapped isomorphically onto the weighted spaces $X$ and $Y$, where

$$\tilde{\gamma} = \tilde{\gamma} (\tau) = \prod_{k=1}^{n} \gamma_k (x_k (\tau_k)), \quad \tau = (\tau_1, \tau_2, ..., \tau_n).$$
Moreover, under this transformation the problem (5.1) is mapped to the following undegenerate problem

\[ \sum_{k=1}^{n} (-1)^{l_k} t_k D_k^{2l_k} \tilde{u}(\tau) + (A + \lambda) \tilde{u}(\tau) + \sum_{|\alpha|: 2|\alpha| < 1} \prod_{k=1}^{n} \left( \frac{\partial}{\partial x_k} \right)^{\alpha_k} \tilde{A}_\alpha(x) D^\alpha \tilde{u}(\tau) = \tilde{f}(\tau), \ \tau \in \mathbb{R}^n, \]

considered in the weighted space \( L_p, \tilde{\gamma}(\mathbb{R}^n; \mathcal{E}) \), where

\( \tilde{u}(\tau) = u(x_1(\tau_1), x_2(\tau_2), ..., x_n(\tau_n)), \ \tilde{f}(\tau) = f(x_1(\tau_1), x_2(\tau_2), ..., x_n(\tau_n)) \).

By redenoting \( u = \tilde{u}(\tau) \) and \( f = \tilde{f}(\tau) \) we get

\[ \sum_{k=1}^{n} (-1)^{l_k} t_k D_k^{2l_k} u(\tau) + (A + \lambda) u(\tau) + \sum_{|\alpha|: 2|\alpha| < 1} \prod_{k=1}^{n} \left( \frac{\partial}{\partial x_k} \right)^{\alpha_k} A_\alpha(x) D^\alpha u(\tau) = f(\tau), \ \tau \in \mathbb{R}^n. \] (5.3)

Consider first of all, the principal part of (5.3), i.e. consider the problem

\[ \sum_{k=1}^{n} (-1)^{l_k} t_k D_k^{2l_k} u(\tau) + (A + \lambda) u(\tau) = f(\tau), \ \tau \in \mathbb{R}^n. \] (5.4)

**Theorem 5.1.** Assume the following conditions are satisfied:

1. \( t_k > 0, \ p = (p_1, p_2, ..., p_n), \ 1 < p_k < \infty, \ k = 1, 2, ..., n; \)
2. \( E \) is Banach space satisfying multiplier condition with respect to \( p \) and weighted function \( \tilde{\gamma}; \)
3. \( A \) is a \( \varphi \)-positive operator in Banach space \( E \) for \( 0 \leq \varphi < \pi \).

Then for \( f \in X \) and \( \lambda \in S(\varphi) \) problem (5.4) has a unique solution \( u(x) \) that belongs to \( Y \) and the uniform coercive estimate holds

\[ \sum_{k=1}^{n} \sum_{i=0}^{2l_k} t_k^{\frac{\alpha_k}{p_k}} |\lambda|^{1 - \frac{\alpha_k}{p_k}} \left\| D_k^{[i]} u \right\|_X + \|Au\|_X \leq C \|f\|_X. \] (5.5)

**Proof.** By applying Fourier transform to the equation (5.4) we obtain

\[ \sum_{k=1}^{n} t_k \xi_k^{2l_k} \hat{u}(\xi) + (A + \lambda) \hat{u}(\xi) = \hat{f}(\xi). \] (5.6)
It is clear that \( \sum_{k=1}^{n} t_k \xi_k^{2l_k} \geq 0 \) for all \( \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \). Therefore, we get that \( \lambda + \sum_{k=1}^{n} t_k \xi_k^{2l_k} \in S(\varphi) \) for all \( \xi \in \mathbb{R}^n \). Since \( A \) is \( \varphi \)-positive, we deduce that the operator function
\[
\Psi(\xi) = A + \lambda + \sum_{k=1}^{n} t_k \xi_k^{2l_k}
\]
has a bounded inverse \( \Psi^{-1}(\xi) \) in \( E \), for all \( \xi \in \mathbb{R}^n \). Hence from (5.6) we obtain that the solution of (5.4) can be represented in the form
\[
u(x) = F^{-1}\left(\Psi^{-1}(\xi) \hat{f}(\xi)\right).
\]
Moreover, we have
\[
\|D^i_k u\|_X = \left\|F^{-1}(i\xi_k)^i \hat{u}\right\|_X = \left\|F^{-1}\xi_k^i \Psi^{-1}(\xi) \hat{f}\right\|_X
\]
and
\[
\|Au\|_X = \left\|F^{-1}A\hat{u}\right\|_X = \left\|F^{-1}A\Psi^{-1}(\xi) \hat{f}\right\|_X.
\]
By virtue of (5.7) and (5.8) for proving (5.5) it is suffices to show the following estimate
\[
\sum_{k=1}^{n} \sum_{i=0}^{2l_k} t_k^{2i} |\lambda|^{1-\frac{1}{2l_k}} \left\|(i\xi_k)^i \hat{u}\right\|_X + \|Au\|_X \leq C \|f\|_X
\]
for all \( u \in Y \). For this aim, it sufficient to show that the operator functions
\[
\varphi_{\lambda,t}(\xi) = \Psi^{-1}(\xi), \quad \varphi_{ki}(\xi) = \sum_{i=0}^{2l_k} t_k^{2i} |\lambda|^{1-\frac{1}{2l_k}} \xi_k^i \Psi^{-1}(\xi)
\]
are multipliers in \( X \) uniformly with respect to \( t_k \) and \( \lambda \). Firstly, show that \( \varphi_{\lambda}(\xi) = \Psi^{-1}(\xi) \) is a multiplier in \( X \) uniformly in \( \lambda \) and \( t_k \). Indeed, for all \( \xi \in \mathbb{R}^n \) and \( \lambda \in S(\varphi) \) we get
\[
\|\varphi_{\lambda}(\xi)\|_{L(E)} \leq M \left(1 + \left|\sum_{k=1}^{n} t_k \xi_k^{2l_k}\right|\right)^{-1} \leq M_0.
\]
It is clear that
\[
\frac{\partial}{\partial \xi_k} \varphi_{\lambda,t}(\xi) = 2l_k t_k \left[A + \lambda + \sum_{k=1}^{n} t_k \xi_k^{2l_k}\right]^{-2} \xi_k^{2l_k-1}.
\]
Hence,
\[
\left\|\xi_k \frac{\partial}{\partial \xi_k} \varphi_{\lambda,t}\right\|_{L(E)} \leq 2l_k t_k \xi_k^{2l_k} \left[A + \lambda + \sum_{k=1}^{n} t_k \xi_k^{2l_k}\right]^{-2} \leq
\]
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\[ 2l_k t_k \xi_k^{2l_k} \left( 1 + \left| \lambda + \sum_{k=1}^{n} t_k \xi_k^{2l_k} \right| \right)^{-2} \leq M. \quad (5.10) \]

Using the estimate (5.10) we show the following uniform estimate

\[ |\xi_1|^{\beta_1} \cdots |\xi_n|^{\beta_n} \left\| D_{\xi}^\beta \varphi_{\lambda,t} (\xi) \right\|_{L(E)} \leq C \quad (5.11) \]

for \( \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{U}_n \) and \( \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{V}_n \). In similar way, we prove that

\[ |\xi_1|^{\beta_1} \cdots |\xi_n|^{\beta_n} \left\| D_{\xi}^\beta \varphi_{ki} (\xi) \right\|_{L(E)} \leq C. \quad (5.12) \]

Since Banach space \( E \) satisfies multiplier condition with respect to \( p \) and \( \gamma \), in view of estimates (5.11) and (5.12) we obtain that the operator-functions \( \varphi_{\lambda,t} \), \( \varphi_{ki} \) are multipliers in \( X \). So, we obtain the estimate (5.9) which in turn gives the estimate (5.5). That is we obtain the assertion.

Consider the operator \( \tilde{O}_0 \) in \( X \) generated by the problem (5.4) that is

\[ D \left( \tilde{O}_0 \right) = Y \quad \text{and} \quad \tilde{O}_0 u = \sum_{k=1}^{n} (-1)^k t_k D_{k}^{2l_k} u + Au. \]

From Theorem 5.1 we obtain

**Result 5.1.** Assume conditions of Theorem 5.1 are satisfied. Then the operator \( \tilde{O}_0 \) is positive in \( X \).

**Theorem 5.2.** Suppose the conditions of Theorem 5.1 are satisfied and \( A_\alpha (x) A^{-1-|\alpha|2(-\mu)} \in L_\infty (\mathbb{R}^n, L(E)) \) for \( 0 < \mu < 1 - |\alpha|2| \). Then for all \( f \in X \) and \( \lambda \in S (\varphi) \) problem (5.3) has a unique solution \( u (x) \in Y \) and the uniform coercive estimate holds

\[ \sum_{k=1}^{n} \sum_{i=0}^{2l_k} t_k^{2l_k} |\alpha|^{1-\frac{\kappa}{\lambda}} \left\| D_{k}^i u \right\|_X + \|Au\|_X \leq C \left\| f \right\|_X. \quad (5.13) \]

**Proof.** Consider the problem (5.3). We denote by \( \tilde{O}_1 \) the operator in \( L_{p,q} (\mathbb{R}^n; E) \) generated by problem (5.3). Namely

\[ D \left( \tilde{O}_1 \right) = Y, \quad \tilde{O}_1 u = \tilde{O}_0 u + \tilde{O}_1 u, \quad (5.15) \]

where

\[ \tilde{O}_1 u = \sum_{|\alpha|2| < 1} \prod_{k=1}^{n} t_k^{\frac{\alpha_k}{\lambda}} A_\alpha D^\alpha u. \]

The estimate (5.5) implies that the operator \( \tilde{O}_0 + \lambda \) has a bounded inverse from \( X \) into \( Y \). By Theorem 3.1 for all \( u \in Y \) we get

\[ \left\| \tilde{O}_1 u \right\|_X \leq \sum_{|\alpha|2| < 1} \prod_{k=1}^{n} t_k^{\frac{\alpha_k}{\lambda}} \left\| A_\alpha (x) D^\alpha u \right\|_X \leq \sum_{|\alpha|2| < 1} \prod_{k=1}^{n} t_k^{\frac{\alpha_k}{\lambda}} \left\| A^{1-|\alpha|2(-\mu)} D^\alpha u \right\|_X \leq \]

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C \left[ h^\mu \left( \sum_{k=1}^n t_k \left\| D_k^{[2]} u \right\|_X + \| Au \|_X \right) + h^{-(1-\mu)} \| u \|_X \right] . \quad (5.16)

Then from (5.16) for \( u \in Y \) we obtain

\[ \left\| \hat{O}_1 u \right\|_X \leq C \left[ h^\mu \left( \left( \hat{O}_0 + \lambda \right) u \right) \right]_X + h^{-(1-\mu)} \| u \|_X . \quad (5.17) \]

It is clear that

\[ \| u \|_X = \frac{1}{\lambda} \left\| \left( \hat{O}_0 + \lambda \right) u - \hat{O}_0 u \right\|_X , \ u \in Y. \]

By Definition 1 we get

\[ \| u \|_X \leq \frac{1}{|\lambda|} \left( \left( \hat{O}_0 + \lambda \right) u \right) + \| \hat{O}_0 u \|_X \leq \frac{1}{|\lambda|} \left( \left( \hat{O}_0 + \lambda \right) u \right) + \]

\[ + \frac{1}{|\lambda|} \left[ \sum_{k=1}^n t_k \left\| D_k^{[2]} u \right\|_X + \| Au \|_X \right] . \quad (5.18) \]

From Theorem 5.1 and estimates (5.16) – (5.18) for \( u \in Y \) we obtain

\[ \left\| \hat{O}_1 u \right\|_X \leq C h^\mu \left( \left( \hat{O}_0 + \lambda \right) u \right) \|_X + C_1 |\lambda|^{-1} h^{-(1-\mu)} \left( \left( \hat{O}_0 + \lambda \right) u \right) \|_X . \quad (5.19) \]

Then choosing \( h \) and \( \lambda \) such that \( C h^\mu < 1, \ C_1 |\lambda|^{-1} h^{-(1-\mu)} < 1 \) from (5.19) we obtain that

\[ \left\| \hat{O}_1 \left( \hat{O}_0 + \lambda \right)^{-1} \right\|_{L(X)} < 1. \quad (5.20) \]

Using the relation (5.15), Theorem 5.1, (5.20) and perturbation theory of linear operators (see for instance [8]) we obtain that the operator \( \hat{O} + \lambda \) is invertible from \( X \) into \( Y \). It is implies that for all \( f \in X \) problem (5.3) have a unique solution \( u \in Y \) and the estimate (5.13) holds.

Let \( O_t \) denotes the operator in \( L_p (R^n, E) \) generated by problem (5.1), i.e.

\[ D(O_t) = W_{p; \gamma}^{[2]} (R^n, E (A), E) , \]

\[ O_t u = \sum_{k=1}^n (-1)^{|\alpha|} t_k D_k^{[2]} u + A u + \sum_{|\alpha|:|2|<1} \prod_{k=1}^n \frac{\alpha_k}{2} A_{\alpha} D_{[\alpha]} u. \]

From Theorem 5.1 and Remark 5.1 we obtain the following

**Result 5.2.** Assume conditions of Theorem 5.2 are satisfied. Then for all \( f \in L_p (R^n; E) \) and \( \lambda \in S(\varphi) \) the equation (5.1) has a unique solution \( u(x) \) that belongs to \( W_{p; \gamma}^{[2]} (R^n, E (A), E) \). Moreover, the uniform coercive estimate holds

\[ \sum_{k=1}^n \sum_{i=0}^{2k} t_k \left| \lambda \right|^{1-\frac{i}{n}} \left\| D_k^{[i]} u \right\|_{L_p (R^n; E)} + \| Au \|_{L_p (R^n; E)} \leq C \| f \|_{L_p (R^n; E)} . \quad (5.21) \]
Result 5.3. Assume the conditions of Theorem 5.2 are satisfied. Then the resolvent of operator $O_t$ satisfies the following sharp uniform coercive estimate

$$
\sum_{k=1}^{n} \sum_{i=0}^{2l_k} \frac{\pi^k}{\pi^k} \left| \lambda \right|^{1 - \frac{i}{2l_k}} \left\| D_k^{[i]} \left( O_t + \lambda \right)^{-1} \right\|_{B_p} + \left\| A \left( O_t + \lambda \right)^{-1} \right\|_{B_p} \leq C,
$$

where

$$B_p = L \left( L_p \left( \mathbb{R}^n ; E \right) \right).$$

From the Result 5.3 and theory of semigroup (see e.g. [22, §1.14.5]) we obtain Result 5.4. Assume conditions of Theorem 5.2 are satisfied for $\phi \in \left( \frac{\pi}{2}, \pi \right)$. Then the operator $O_t$ is a generator of analytic semigroup in $X$.

Remark 5.2. There are a lot of positive operators in the different concrete Banach spaces. Therefore, putting the concrete Banach spaces instead of $E$ and concrete positive differential, pseudodifferential operators, or finite, infinite matrices instead of $A$ in (1.1), by virtue of Theorem 5.2 we obtain the separability properties of different class of degenerate partial differential equations or system of equations.

6. Abstract Cauchy problem for anisotropic parabolic equation with parameters

Consider now, the Cauchy problem (1.2). In this section, we obtain the existence and uniqueness of the maximal regular solution of problem (1.2) in mixed $L_{p,\tilde{\gamma}} \left( \mathbb{R}^n ; E \right)$ norms.

Let $O_{\varepsilon}$ denote differential operator generated by problem (1.1) for $t_k = \varepsilon_k$, $A_{\alpha}(x) = 0$ and $\lambda = 0$, where $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$. Let

$$X = L_{p,\tilde{\gamma}} \left( \mathbb{R}^n ; E \right).$$

Theorem 6.1. Assume $E$ is Banach space satisfying multiplier condition with respect to $p = (p_1, p_2, ..., p_n)$ and weighted function $\tilde{\gamma}$. Suppose $A$ is a $\varphi$-positive operator in Banach space $E$ for $0 \leq \varphi < \pi$. Then the operator $O_{\varepsilon}$ is uniformly $R$-positive in $X$.

Proof. The Result 5.2 implies that the operator $O_{\varepsilon}$ is uniformly positive in $X$. We have to prove the $R$-boundedness of the set

$$
\sigma_t (\xi, \lambda) = \left\{ \lambda (O_{\varepsilon} + \lambda)^{-1} : \lambda \in S_{\varphi} \right\}.
$$

From the Theorem 5.1 we have

$$\lambda (O_{\varepsilon} + \lambda)^{-1} f = F^{-1} \Phi_{\varepsilon} (\xi, \lambda) \hat{f},$$

for $f \in X$, where

$$\Phi_{\varepsilon} (\xi, \lambda) = \lambda (A + L_{\varepsilon} (\xi) + \lambda)^{-1}, \quad L_{\varepsilon} (\xi) = \sum_{k=1}^{n} \varepsilon_k \xi_k^{2l_k}. \quad \tilde{\gamma}$$
By definition of $R$-boundedness, it is sufficient to show that the operator function $\lambda \left[ A (\xi) + L_\varepsilon (\xi) + \lambda \right]^{-1}$ (depended on variable $\lambda$ and parameters $\xi, \varepsilon$) is uniformly bounded multiplier in $X$. In a similar manner as in Theorem 5.1 one can easily show that $\Phi_t (\xi, \lambda)$ is uniformly bounded multiplier in $X$. Then, by definition of $R$-boundedness we have

$$
\int_0^1 \left\| \sum_{j=1}^m r_j (y) \lambda_j (O_\varepsilon + \lambda_j)^{-1} f_j \right\|_X dy = \int_0^1 \left\| \sum_{j=1}^m r_j (y) F^{-1} \Phi_\varepsilon (\xi, \lambda_j) \hat{f}_j \right\|_X dy = 
$$

for all $\xi_1, \xi_2, ..., \xi_m \in R^n$, $\lambda_1, \lambda_2, ..., \lambda_m \in S_\sigma$, $f_1, f_2, ..., f_m \in X$, $m \in N$, where $\{r_j\}$ is a sequence of independent symmetric $\{-1, 1\}$-valued random variables on $[0, 1]$. Hence, the set $\sigma_\varepsilon (\xi, \lambda)$ is uniformly $R$-bounded.

Now we are ready to state the main result of this section. Let $G = (0, T) \times R^n$ and $\tilde{p} = (p, p_0)$. Let

$$
L_{p, \gamma} (G; E) = L_{p_0} (0, T; X).
$$

**Theorem 6.2.** Assume the conditions of Theorem 6.1 are satisfied for $\varphi \in (\frac{\pi}{2}, \pi)$. Then for $f \in L_{p, \gamma} (G; E)$ problem (1.2) has a unique solution $u \in W^{1,2l}_{p, \gamma} (G; E \langle A \rangle, E)$ and the following uniform coercive estimate holds

$$
\left\| \frac{\partial u}{\partial t} \right\|_{L_{p, \gamma} (G; E)} + \sum_{k=1}^n \varepsilon_k \left\| D_{k, u}^2 u \right\|_{L_{p, \gamma} (G; E)} + \| Au \|_{L_{p, \gamma} (G; E)} \leq C \| f \|_{L_{p, \gamma} (G; E)}.
$$

**Proof.** The problem (1.2) can be expressed as the following Cauchy problem

$$
\frac{du}{dt} + O_\varepsilon u (t) = f (t), \ u (0) = 0. \quad (6.1)
$$

Theorem 6.1 implies that the operator $O_\varepsilon$ is $R$-positive and by Result 5.4 it is a generator of an analytic semigroup in $X$. Then by virtue of [24, Theorem 4.2] we obtain that for $f \in L_{p_0} (0, T; X)$ problem (6.1) has a unique solution $u \in W^{1,0}_{p_0} (0, T; D (O_\varepsilon), X)$ and the following uniform estimate holds

$$
\left\| \frac{du}{dt} \right\|_{L_{p_0} (0, T; X)} + \| O_\varepsilon u \|_{L_{p_0} (0, T; X)} \leq C \| f \|_{L_{p_0} (0, T; X)}. \quad (6.2)
$$

Since $L_{p_0} (0, T; X) = L_{p_\gamma} (G; E)$, by Theorem 5.1 we have

$$
\| O_\varepsilon u \|_{L_{p_0} (0, T; X)} = \| O_\varepsilon u \|_{L_{\tilde{p}} (G; E)}.
$$

This relation and the estimate (6.2) implies the assertion.
Consider now, the Cauchy problem for degenerate parabolic equation
\[ \frac{\partial u}{\partial t} + \sum_{k=1}^{n} (-1)^{k} \varepsilon_{k} D^{[2k]} u + A u = f(t,x), \]  
(6.3)

\[ u(0,x) = 0, \quad x \in \mathbb{R}^{n}, \quad t \in (0, T), \]

where \( A \) is a linear operator in a Banach space \( E \) and \( \varepsilon_{k} \) are small positive parameters.

From Theorem 6.2, and Remark 5.1 we obtain

**Result 6.1.** Assume conditions of Theorem 6.1 are satisfied for \( \varepsilon \in (\frac{\pi}{2}, \pi) \). Then for \( f \in L_{\beta}(G; E) \) problem (6.3) has a unique solution \( u \in W_{\beta}^{1,2}\) \( (G; E(A), E) \) and the following uniform coercive estimate holds

\[ \left\| \frac{\partial u}{\partial t} \right\|_{L_{\beta}(G; E)} + \sum_{k=1}^{n} \left\| D^{[2k]} u \right\|_{L_{\beta}(G; E)} + \left\| A u \right\|_{L_{\beta}(G; E)} \leq C \left\| f \right\|_{L_{\beta}(G; E)}. \]

7. System of parabolic equation of infinite order with small parameters

Consider the infinity systems of Cauchy problem for the degenerate anisotropic parabolic equation
\[ \frac{\partial u_{m}}{\partial t} + \sum_{k=1}^{n} (-1)^{k} \varepsilon_{k} D^{[2k]} u_{m} + \sum_{j=1}^{\infty} d_{j}(x) u_{j} = f_{m}(t,x), \]  
(7.1)

\[ u_{m}(0,x) = 0, \quad x \in \mathbb{R}^{n}, \quad t \in (0, T), \quad m = 1, 2, ..., \infty, \]  
(7.2)

where \( d_{j} \) are complex valued functions, \( \varepsilon_{k} \) are small positive parameters

\[ B = \{d_{m}\}, \quad d_{m} > 0, \quad u = \{u_{m}\}, \quad Bu = \{d_{m}u_{m}\}, \quad m = 1, 2, ..., \infty, \]

\[ l_{q}(B) = \left\{ u \in l_{q}, \left\| u \right\|_{l_{q}(D)} = \left\| Bu \right\|_{l_{q}} = \left( \sum_{m=1}^{\infty} |d_{m}u_{m}|^{q} \right)^{\frac{1}{q}} < \infty \right\}. \]

In this section we show the following result:

**Theorem 7.1.** For \( f(t,x) = \{f_{m}(t,x)\}^{\infty}_{1} \in L_{\beta}(G;l_{q}) \) problem (7.1) – (7.2) has a unique solution \( u \in W_{\beta}^{1,2} (G, l_{q}(D), l_{q}) \) and the following coercive uniform estimate holds

\[ \left\| \frac{\partial u}{\partial t} \right\|_{L_{\beta}(G;l_{q})} + \sum_{k=1}^{n} \left\| D^{[2k]} u \right\|_{L_{\beta}(G;l_{q})} + \left\| A u \right\|_{L_{\beta}(G;l_{q})} \leq C \left\| f \right\|_{L_{\beta}(G;l_{q})}. \]  
(7.3)

**Proof.** Assume \( E = l_{q} \) and \( A \) is such that

\[ A = [d_{m}\delta_{mj}], \quad m, j = 1, 2, ..., \infty. \]
It is clear that the operator $A$ is $R$-positive in $l_q$. Then, from Result 6.1 we obtain the assertion. Now, consider the following Cauchy problem

$$\frac{\partial u_m}{\partial t} + \sum_{k=1}^{n} (-1)^k \epsilon_k \frac{\partial^{2l_k} u_m}{\partial x_k^{2l_k}} + \sum_{j=1}^{\infty} d_j (x) u_j = f_m (t, x), \quad (7.4)$$

$$u_m (0, x) = 0, \quad x \in R^n, \quad m = 1, 2, ..., \infty. \quad (7.5)$$

From Theorem 7.1 and Remark 5.1 we obtain

**Result 7.1.** For $f(t, x) = \{ f_m (t, x) \}_{1}^{\infty} \in L_p (G; l_q)$ problem (7.4) - (7.5) has a unique solution $u \in W^{1, 2l} (G, l_q (D), l_q)$ and the following coercive uniform estimate holds

$$\left\| \frac{\partial u}{\partial t} \right\|_{L_{p, \gamma} (G; l_q)} + \sum_{k=1}^{n} \left\| \frac{\partial^{2l_k} u}{\partial x_k^{2l_k}} \right\|_{L_{p, \gamma} (G; l_q)} + \| Au \|_{L_{p, \gamma} (G; l_q)} \leq C \| f \|_{L_{p, \gamma} (G; l_q)}.$$

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