Extended black hole solutions in self-interacting Brans-Dicke theory

M Sharif* and Amal Majid

Department of Mathematics, University of the Punjab, Quaid-e-Azam Campus, Lahore-54590, Pakistan

* Author to whom any correspondence should be addressed.

E-mail: msharif.math@pu.edu.pk and amalmajid89@gmail.com

Keywords: Brans-Dicke theory, black holes, gravitational decoupling

Abstract

In this paper, we formulate black hole solutions through extended gravitational decoupling scheme in the framework of self-interacting Brans-Dicke theory. The addition of a new source in the matter distribution increases the degrees of freedom in the system of field equations. Transformations in radial as well as temporal metric functions split the system into two arrays. Each array includes the effects of only one source (either seed or additional). The seed source is assumed to be a vacuum and the corresponding system is specified through the Schwarzschild metric. In order to construct a suitable solution of the second system, constraints are applied on the metric potentials and energy-momentum tensor of the additional source. We obtain three solutions corresponding to different values of the decoupling parameter in the presence of a massive scalar field. The extra source is classified as normal or exotic through energy conditions. It is found that two solutions agree with the energy bounds and thus have normal matter as their source.

1. Introduction

The universe is a complex structure consisting of mysterious forces and unknown matter components. In order to gain insight into the evolution and mechanism of the cosmos, it is essential to study the arrangement and physical features of astrophysical objects. Black hole (BH) is one of the self-gravitating systems with a singularity hidden behind the event horizon. These cosmic objects have a strong gravitational field and serve as excellent laboratories to test relativistic theories in the strong-field regime. The existence of BHs has been strengthened due to the recent detection of gravitational waves [1]. Moreover, the study of BH shadow [2] has contributed to the significance of BH solutions with well-behaved physical characteristics.

Schwarzschild obtained the first BH solution for a vacuum spacetime [3]. General relativity (GR) formulates surprisingly simple solutions for BHs in accordance with the no-hair conjecture (BH solutions cannot carry additional charges [4]) with three prominent features: mass, charge and angular momentum [5]. The validity of no-hair theorem is now being tested with improved studies and observations of BH systems. In fact, different setups have been constructed to evade the no-hair theorem [6]. Recent studies suggest that BHs, as sources of extreme gravity, can possess soft quantum hair [7]. However, the derivation of new solutions representing BHs is hindered by the non-linearity as well as high degree of freedom in the field equations.

Ovalle [8] devised the method of gravitational decoupling to obtain new models representing relativistic objects with well-determined characteristics. In this scheme, known isotropic solutions (or seed solutions) are extended to their anisotropic versions by involving an additional source in the seed solution. The array of field equations is divided into two systems through a minimal geometric deformation (MGD), i.e., a transformation in the radial metric component. A viable solution represents the set corresponding to the original source whereas physically suitable constraints are applied to formulate a solution associated with the additional source. Combining both solutions yields the required anisotropic extension of the known solution. The key feature of this technique is that the seed and extra sources must be conserved individually as they interact gravitationally only. Ovalle first used the MGD approach to construct spherically symmetric solutions in the framework of Randall-Sundrum braneworld [8].
The procedure of MGD was adopted to analyze the compactness of self-gravitating objects by computing the analogue of Tolman IV in braneworld [9]. Ovalle et al [10] followed this method to evaluate three anisotropic extensions of Tolman IV in GR. Different anisotropic BH solutions were obtained by applying this approach to a vacuum Schwarzschild solution [11]. Gabbanelli et al [12] evaluated anisotropic extension of the Durgapal-Fuloria solution and discussed its viability. Estrada and Tello-Ortiz [13] formulated two anisotropic solutions by extending Heintzmann solution. Sharif and Sadiq [14] investigated the effects of electromagnetic field on extended Krori-Barua solution. Pérez Graterol [15] formulated new well-behaved anisotropic solutions by introducing deformations in the radial metric component of Buchdahl spacetime. Sharif and Ama-Tul-Mughani [16] obtained anisotropic extensions of string cloud. In a recent paper [17], Tello-Ortiz et al built anisotropic compact models by combining the techniques of embedding class-one and MGD.

The approach of gravitational decoupling through MGD has proved highly beneficial in obtaining physically relevant anisotropic solutions. However, this method applies to the scenario in which the considered sources do not exchange energy. Casadio et al [18] overcame this drawback by deforming temporal as well as radial components of the spacetime. However, this modification of MGD approach gives valid results in the case of vacuum only and fails to satisfy the conservation law related to astrophysical objects filled with fluid. Recently, Ovalle [19] introduced decoupling through deforming both (radial/temporal) metric potentials. Decoupling through extended geometric deformation (EGD) disintegrates the system without restricting the type of matter distribution. Contreras and Bargueño [20] applied this method to vacuum BTZ solution in $2+1$-dimensions. The EGD scheme has also been applied to evaluate anisotropic versions of Tolman IV [21] and Krori-Barua [22] solutions. Recently, Ovalle et al [23] formulated hairy BHs by extending Schwarzschild spacetime through the EGD technique. There are also some attempts [24] to obtain anisotropic solutions in modified theories through MGD as well as EGD schemes.

Brans and Dicke in 1961 [25] proposed a theory based on Dirac hypothesis (gravitational constant ($G$) is not a constant) as well as Mach principle (inertia arises from the acceleration of matter distribution) and introduced a long-range massless scalar field ($\phi$) that acts as the inverse of dynamical gravitational constant. A tunable parameter $\omega_{BD}$ minimally couples the scalar field to fluid distribution and is termed as the coupling constant. The role of this dimensionless parameter is reduced during the inflationary era of the Universe leading to smaller values of $\omega_{BD}$ during this phase [26]. Moreover, the weak-field tests are satisfied for $\omega_{BD} > 40,000$ [27] which results in a conflict to determine the correct range of the coupling parameter. For this purpose, the Brans-Dicke (BD) theory is remodeled to more sophisticated versions of itself such as self-interacting BD (SBD) theory. This theory involves a massive scalar field ($\chi$) and a potential function $V(\chi)$ that re-adjusts the values of $\omega_{BD}$ [28]. The coupling parameter can assume values greater than $-\frac{1}{2}$ when the scalar field is more massive than $2 \times 10^{-25}$ GeV [29].

Researchers have derived different solutions in BD theory as well as its modifications to discuss different astrophysical phenomena. Thorne and Dykla [30] studied BHs in three dimensions and concluded that 3-dimensional BD BHs are identical to their counterparts in GR. Johnson [31] investigated the conditions under which static BHs in BD gravity reduce to Schwarzschild solution. Hawking [32] showed that a stationary BH metric satisfies the BD field equations if and only if it is also a solution of GR. Geroch method [33] was employed by Sneddon and McIntosh [34] to discuss vacuum models. Bruckman and Kazes [35] considered a perfect matter source with a linear equation of state (EoS) to formulate solutions for a spherically symmetric spacetime. The BD field equations were converted to Einstein field equations by Goswami [36] to evaluate vacuum solutions. Riazi and Askari [37] used numerical techniques to approximate solution for an empty sphere and studied the trend of rotation curves. Kim [38] studied thermodynamics of BHs in BD gravity to discriminate between trivial and non-trivial BHs. Campanelli and Lousto [39] studied a family of BD solutions and determined the range of parameters yielding BH solutions different from GR. Recently, we have used MGD and EGD techniques to generate astrophysical and cosmological solutions, respectively in the context of SBD theory [40].

In this paper, we derive BH solutions in SBD theory by implementing EGD method on Schwarzschild spacetime. The additional gravitational source is decoupled from the original source (vacuum) by means of deformations in the metric components. The new black hole solution is derived for three cases of linear EoS by considering the extra source as a tensor-vacuum. Out of these solutions, two obey the energy conditions and are viable. We have also checked the asymptotic behavior of the extended solutions. The paper is organized as follows. The field equations incorporating an additional source in matter distribution are formulated in section 2. In section 3, deformations on the metric components are applied to decouple the field equations. The physical characteristics of BH solutions are studied in section 4. In the last section, a summary of the main results is presented.
2. Self-interacting Brans-Dicke theory

In this section, we formulate the SBD field equations by adding an extra source \((\Theta_{\gamma\delta})\) in the action (in relativistic units) as

\[
S = \int \sqrt{-g} (\mathcal{R} Y - \frac{\omega_{BD}}{Y} \nabla^\gamma \nabla_\gamma Y - V(Y) + L_m + \alpha L_{\Theta}) d^4x,
\]

(1)

where the Ricci scalar is denoted by \(\mathcal{R}\) whereas the Lagrangian densities of matter and new source are represented by \(L_m\) and \(L_{\Theta}\), respectively. The extra source may include scalar, vector or tensor fields which are generally responsible for inducing anisotropy in the fluid distribution. The dimensionless parameter \(\alpha\) tracks the strength of the coupling between the two matter sources \(T^m_{\gamma\delta}\) and \(\Theta_{\gamma\delta}\). The SBD action provides the following field and wave equations

\[
G_{\gamma\delta} = T^{(\Theta)}_{\gamma\delta} + \frac{\alpha}{Y} \Theta_{\gamma\delta} = \frac{1}{Y} (T^m_{\gamma\delta} + T^\Theta_{\gamma\delta} + \alpha \Theta_{\gamma\delta}),
\]

(2)

\[
\Box Y = \frac{g^{\gamma\delta} (\alpha \Theta_{\gamma\delta} + T^{(m)}_{\gamma\delta})}{3 + 2\omega_{BD}} + \frac{1}{3 + 2\omega_{BD}} \left( \frac{dV(Y)}{dY} - 2V(Y) \right),
\]

(3)

where \(\Box\) denotes the d'Alembertian operator. We consider perfect fluid as the seed source described by the following energy-momentum tensor

\[
T^m_{\gamma\delta} = (p + \rho) u_\gamma u_\delta - pg_{\gamma\delta},
\]

(4)

where \(p\), \(\rho\) and \(u_\gamma\) indicate isotropic pressure, energy density and four velocity, respectively. The energy-momentum tensor associated with the massive scalar field is expressed as

\[
T^{\Theta}_{\gamma\delta} = \Theta_{\gamma\delta} \Box Y + \frac{\omega_{BD}}{Y} \left( \frac{Y}{2} \frac{dV(Y)}{dY} - \frac{V(Y)}{2} g_{\gamma\delta} \right).
\]

(5)

The metric for a static spherically symmetric configuration is written as

\[
ds^2 = e^{2r} dt^2 - e^{2\theta} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2).
\]

(6)

The field equations linking geometry of the system to matter distribution are formulated via equations (2)–(6) as

\[
\frac{1}{r^2} - e^{-\xi} \left( \frac{1}{r^2} - \frac{\mu^l}{r} \right) = \frac{1}{Y} (\rho + \alpha \Theta^0_0 + T^{00}_0),
\]

(7)

\[
\frac{1}{r^2} + e^{-\xi} \left( \frac{1}{r^2} + \frac{\mu^r}{r} \right) = \frac{1}{Y} (p - \alpha \Theta^1_1 - T^{11}_1),
\]

(8)

\[
\frac{e^{-\xi}}{4} \left( 2 \mu^r - \varepsilon' \mu^l + 2 \frac{\mu^l - \varepsilon'}{r} \right) = \frac{1}{Y} (p - \alpha \Theta^2_2 - T^{22}_2),
\]

(9)

where

\[
T^{00}_0 = e^{-\xi} \left[ Y'' + \left( \frac{2}{r} - \frac{\varepsilon'}{2} \right) Y' + \frac{\omega_{BD}}{2Y} Y'^2 - \frac{V(Y)}{2} \right],
\]

\[
T^{11}_1 = e^{-\xi} \left[ Y'' + \left( \frac{2}{r} + \frac{\mu^l}{2} \right) Y' - \frac{\omega_{BD}}{2Y} Y'^2 - \frac{V(Y)}{2} \right],
\]

\[
T^{22}_2 = e^{-\xi} \left[ Y'' + \left( \frac{1}{r} - \frac{\varepsilon'}{2} + \frac{\mu^l}{2} \right) Y' + \frac{\omega_{BD}}{2Y} Y'^2 - \frac{V(Y)}{2} \right].
\]

Here prime represents derivative with respect to the radial coordinate. The evolution of the scalar field in equation (3) for a spherical system is evaluated as

\[
\Box Y = -e^{-\xi} \left[ \left( \frac{2}{r} - \frac{\varepsilon'}{2} + \frac{\mu^l}{2} \right) Y' + Y'' \right].
\]

(10)

An inspection of the system (7)–(9) reveals that anisotropy of the static source does not vanish unless \(\Theta^1_1 = \Theta^2_2\). Thus, the additional source plays a crucial role in extending isotropic solutions to the domain of anisotropic solutions.
3. Gravitational decoupling

The four differential equations (7)–(10) give rise to a system containing nine unknowns: five matter variables ($p, \mu, \Theta_0^2, \Theta_1^2, \Theta_2^2$); two metric potentials ($\mu, e$); massive scalar field and a potential function. The novel approach of EGD [19] reduces the degrees of freedom in the underdetermined system by transforming the temporal as well as radial metric potentials. These deformations are controlled by the parameter $\alpha$ as

$$\mu(r) \mapsto a(r) + \alpha h_1(r),$$

$$e^{-\omega(r)} \mapsto e^{-\omega(r)} + \alpha h_2(r),$$

where $h_1(r)$ and $h_2(r)$ determine the variations in temporal and radial metric functions, respectively. It is noteworthy that the above transformations do not alter the spherical symmetry of the static configuration. The system of field equations (7)–(9) disintegrates into two independent sets under the influence of transformed metric functions. The first set corresponds to the original SBD system with $\alpha = 0$ and encodes the effects of the isotropic source only as

$$\rho = \frac{1}{2r^2T(r)} \{e^{-\omega(r)}(r^2e^{\omega(r)}V(T)\dot{Y}(r) + r^2(-\omega_{BD})\dot{T}^2(r) + (rb'(r) - 4)\dot{Y}(r) - 2r\dot{T}''(r))rT(r) + 2T^2(r)(rb'(r) + e^{\omega(r)} - 1))\},$$

$$p = \frac{1}{2}\left\{\frac{1}{r^2T(r)}(e^{-\omega(r)}(-r^2\omega_{BD}\dot{T}^2(r) + T^2(r)(2\alpha'(r) - 2e^{\omega(r)} + 2) + r\dot{T}(r)(\alpha'(r) + 4)\dot{T}'(r)) - V(T)\right\},$$

$$p = \frac{1}{4rT(r)}(e^{-\omega(r)}(2\dot{T}(r)(\alpha'(r) - \beta'(r) + 2) + 2r\dot{T}''(r)) + \dot{T}^2(r))$$

$$\times(2\alpha'(r) + \alpha''(r) - \beta''(r) + \alpha\beta''(r) + 2\beta'(r) - 2r(e^{\omega(r)}\dot{T}(r)V(T) + 2r\omega_{BD}\dot{T}^2(r))\}.$$

The conservation equation of the new source $Q_d$ is given as

$$T_1^{\text{(eff)}} - \frac{d'(r)}{2}(T_0^{\text{(eff)}} - T_0^{\text{(eff)}}) = 0.$$
where

\[
\Theta_0^{(\text{eff})} = \frac{1}{4} \left( \Theta_0^0 + \frac{1}{2} h_2'(r) \nabla Y' r + h_2(r) Y'' + \frac{\omega_{\text{BD}} h_2(r) \nabla Y_2}{2 Y} + \frac{2 h_2(r) \nabla Y(r)}{r} \right),
\]

\[
\Theta_1^{(\text{eff})} = \frac{1}{4} \left( \Theta_1^0 + \frac{1}{2} e^{-b(r)} \nabla Y'(r) (h_2(r) e^{b(r)} Y(r) Y'(r) + \nabla Y(r)(\mu'(r) + 4) - r \omega_{\text{BD}} \right) \times \nabla Y(r) + r \nabla (r h_1'(r))),
\]

\[
\Theta_2^{(\text{eff})} = \frac{1}{4} \left( \Theta_2^0 + \frac{1}{2} e^{-b(r)} (r \nabla Y'(r) (e^{b(r)} h_2'(r) + h_1'(r)) + h_2(r) \right) \times e^{b(r)} (r \nabla Y(r)((\mu'(r) + 2) \nabla Y(r) + 2 r \nabla Y''(r)) + r \omega_{\text{BD}} \nabla T_2^2(r))).
\]

The effective energy-momentum tensor $T_3^{(\text{eff})}$ is conserved in $(\mu, \varepsilon)$-coordinate system as

\[
\nabla_{\gamma} T_{\beta}^{(\text{eff})} = \nabla_{(a,b)} T_{(a,b)}^{(\text{eff})} - \frac{h_1'(r)}{2} (T_0^{(\text{eff})} - T_1^{(\text{eff})}) \delta_{\beta}^a,
\]

(21)

where $\nabla_{(a,b)}$ denotes the divergence in $(a,b)$-frame of reference. Moreover, equations (16) and (20) yield

\[
\nabla_{(a,b)} T_{(a,b)}^{(\text{eff})} = 0, \quad \nabla_{(a,b)} \Theta_{(a,b)}^{(\text{eff})} = \frac{h_1'(r)}{2} (T_0^{(\text{eff})} - T_1^{(\text{eff})}) \delta_{\beta}^a.
\]

(22)

It can be directly deduced from equations (21) and (22) that the sources $T_{(\mu,\varepsilon)}$ and $\Theta_{(\mu,\varepsilon)}$ are not conserved allowing the exchange of energy between them. However, the total energy and momentum of the setup remain unaltered. Thus, the EGD technique can be used to decouple these sources as long as energy transfer from one setup to the other is possible. This feature differentiates the EGD approach from MGD technique where a radial transformation decouples the two individually conserved sources. However, in the case of vacuum or barotropic fluid, EGD can successfully decouple the matter sources interacting only gravitationally. Combining the two systems (13)–(15) and (17)–(19) yields the state determinants of the extended solution as $\rho + \alpha \Theta_0^0$, $P_\parallel = p - \alpha \Theta_1^0$, $P_\perp$. Shifting the components of $\Theta_1^0$ to the other side, the density and pressure of the extended solution are completely determined as

\[
\rho = \frac{e^{-b(r)}}{2 r^2 Y} \left( -r \nabla Y'(r)(\alpha h_2'(r) + 4 \alpha h_2(r) e^{b(r)} - r b'(r) + 4) \
+ 2 r Y''(r)(\alpha h_2(r) e^{b(r)} + 1) - 2 \nabla Y Y'(r)((\alpha e^{b(r)} h_2'(r) + \alpha h_2(r) e^{b(r)} - r b'(r) - e^{b(r)} + 1) + r^2 (-\omega_{\text{BD}} \nabla T_2^2(r)(\alpha h_2(r) e^{b(r)} + 1) + r^2 e^{b(r)} Y(r) V Y(r)),
\]

(23)

\[
P_\parallel = \frac{Y'(r)}{r^2} \left( (\alpha h_2'(r) + e^{-b(r)})(\alpha h_2(r) + a'(r) + 1) - 1 - \frac{1}{2 r Y(r)} \right) \times (r \nabla Y Y'(r)(\alpha h_2(r) + e^{-b(r)}(r \omega_{\text{BD}} \nabla Y(r) - Y(r)(\alpha h_2'(r) + ra'(r) + 4)))
- V Y(r),
\]

(24)

\[
P_\perp = (\alpha h_2(r) + e^{-b(r)} \left( \frac{1}{2} r \nabla Y(r)(\alpha e^{b(r)} h_2'(r) - b'(r)) \alpha h_2(r) e^{b(r)} + a'(r) + a'(r) \
+ \frac{2}{r} \right) + \frac{\omega_{\text{BD}} \nabla T_2^2(r)}{2 Y(r)} + \frac{1}{2} Y(r)(\alpha h_2(r) + e^{-b(r)}(1 + (e^{b(r)} h_2'(r) - b'(r)) (2 \alpha h_2(r) e^{b(r)} + 2) - \frac{1}{r} (\alpha e^{b(r)} h_2'(r) - b'(r)) \alpha h_2(r) e^{b(r)} + 1 \
+ a'(r) + a'(r) + \alpha h_2'(r) + a'(r) + 2) + a'(r) - V Y(r) \right).}
\]

(25)

4. Extended schwarzschild solutions

The process of extracting solutions of the field equations is simplified by splitting the original system into two sets. The first set governed by the isotropic source is completely specified by assuming a suitable metric leaving fewer unknowns in the second set. For this purpose, we consider the Schwarzschild metric expressed as
In the subsequent subsections, a decoupling method is used to obtain the extended solution. In order to have a well-de

determined massive scalar field mass. The restriction on the values of the coupling parameter due to weak-field tests are waived off for $m_{\text{F}} > 10^{-4}$ (in dimensionless units). Thus, we determine the massive scalar field by solving the wave equation numerically for $m_{\text{F}} = 0.1$ and $\omega_{\text{BD}} = 60$. The behavior of the obtained BH solutions is checked for $\alpha = -0.4$, $-0.5$, $-0.7$. It is noteworthy to mention here that under the applied constraints, the positive behavior of density is achieved for negative values of the decoupling parameter only.

\[ ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \frac{1}{\left(1 - \frac{2M}{r}\right)}dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2), \]  

with a singularity at $r = 0$ hidden behind an event horizon at $r = 2M$. Consequently, $e^\varphi = e^{-\varphi} = 1 - \frac{2M}{r}$. According to the no-hair theorem, information about the BH is lost behind the event horizon as physical state of matter is unknown beyond this boundary. However, efforts have been made to study BHs in different perspectives in order to avoid this theorem [6]. The addition of scalar field [41] or another generic source of matter [23] in the vacuum leads to different BH solutions known as hairy BHs (with mass $M$ and a discrete set of charges as primary hair). The major benefit of the EGD scheme is the transformation in temporal as well as radial metric components which increases the probability of hairy BH solutions with different horizons.

As there are five unknowns ($h_1(r), h_2(r), g_0, \Theta_1, \Theta_2$) in the system (17)-(19), we apply two constraints to obtain the extended solution. In order to have a well-defined causal structure of the resultant spacetime, it is necessary that the causal horizon ($e^{-\varphi} = 0$) either covers the Killing horizon ($e^\varphi = 0$) or coincides with it. Therefore, the first constraint is applied to the metric potentials as

\[ \mu = -\varepsilon, \]  

which leads to coinciding Killing and causal horizons. Setting $e^{-\varepsilon} = 0$ implies that both Killing and causal horizons occur at $r = 2M$. The second constraint is applied to $\Theta$-components through a linear EoS expressed as

\[ g_0 = a_1 \Theta_1^a + a_2 \Theta_2^a, \]  

where $a_1$ and $a_2$ are constants. This EoS has been considered previously to evaluate extended solutions through the decoupling method [11]. Applying the condition (27) to equations (11) and (12) yields the following relation between the deformation functions

\[ h_2(r) = \frac{(2M - r)(e^{\omega h(r)} - 1)}{\alpha r}. \]  

The presence of an essential singularity is confirmed through Kretschmann scalar which is evaluated as

\[ K = \frac{1}{r^6} \left[ \alpha^2 r^4 (r - 2M)^2 e^{2\omega h(r)} h_1^{m2} + 8\alpha M r^2 (2M - r) e^{2\omega h(r)} h_1^{02} + \alpha^2 r^4 (r - 2M)^2 e^{2\omega h(r)} h_1^{14} - 8\alpha^3 M r^3 (2M - r) e^{2\omega h(r)} h_1^{13} + 2\alpha^2 r^2 e^{2\omega h(r)} h_1^{12} (r^2 - 2M)^2 h_1^{m0} (r + 2r) + (2M - r) h_1^{m0} (r + 2M - 2r) + 4(12M^2 - 4Mr + r^2) e^{2\omega h(r)} + 2r (2M - r) e^{\omega h(r)} + r^2 \right]. \]  

In the subsequent subsections, $K$ is plotted to indicate the presence of a singularity at $r = 0$. We compute decoupled solutions by choosing different values of the constants $a_1$ and $a_2$ corresponding to different scenarios. Moreover, we set $V(T) = \frac{1}{2} m_T^2 T^2$, where $m_T$ is the scalar field mass. The restriction on the values of the coupling parameter due to weak-field tests are waived off for $m_T > 10^{-4}$ (in dimensionless units). Thus, we determine the massive scalar field by solving the wave equation numerically for $m_T = 0.1$ and $\omega_{\text{BD}} = 60$. The behavior of the obtained BH solutions is checked for $\alpha = -0.4$, $-0.5$, $-0.7$. It is noteworthy to mention here that under the applied constraints, the positive behavior of density is achieved for negative values of the decoupling parameter only.
4.1. Case I: Traceless $\Theta_0^2$

The additional source has a traceless energy-momentum tensor when

$$\Theta_0^0 + \Theta_1^1 = -2\Theta_2^2,$$

(since $\Theta_2^2 = \Theta_1^1$), i.e., $\Theta_0^2$ is traceless when $a_1 = -1$ and $a_2 = -2$ in equation (28) which yields

$$\frac{1}{r^3} (e^{-b(r)}(\alpha T(r)) r (3T'(r) r (e^{b(r)} h_2'(r) + h_1'(r)) + T(r) (e^{b(r)} h_2'(r))
\times (\alpha h_1'(r) + h_2'(r) + 4a h_2'(r) + h_1'(r) \alpha r h_1(r) + 2a r h_1'(r) - r
\times b'(r) + 4)) + \alpha h_2(r) e^{b(r)} (3r T(r) (\alpha h_1'(r) + h_2'(r) + 4a h_2'(r) + h_1'(r) \alpha r h_1(r) + 2a h_1'(r) (r a'(r) + 2) + 4 a'(r) + 4) + 2 r^2 \omega_{BD} T'(r)))) = 0.$$  

(30)

The deformation function $h_1(r)$ is evaluated by solving the above equation numerically along with the wave equation subject to the initial conditions $\dot{Y}(2M) = 0.8, \ T(2M) = 0.1, \ h_1(2M) = 1$ and $h_1'(2M) = 0.1$. The Kretschmann scalar shown in figure 1 approaches to infinity when $r \to 0$. Thus, a singularity exists at $r = 0$. The graphical analysis of state variables is done in the region accessible to an outer-observer with $M = 1$ (figure 2).

The energy density and tangential pressure increase as the decoupling parameter decreases. However, radial pressure is directly proportional to $\alpha$. It is noted that positive energy density is obtained when radial pressure is negative. The matter source is normal if the state determinants adhere to four energy bounds. The energy conditions (null, weak, strong and dominant) in the framework of SBD theory are, respectively, expressed as $[42]$

- NEC: $\rho + p_r \geq 0, \ \rho + p_\perp \geq 0$,
- WEC: $\rho \geq 0, \ \rho + p_r \geq 0, \ \rho + p_\perp \geq 0$,
- SEC: $\rho + p_r + 2p_\perp \geq 0$,
- DEC: $\rho - p_r \geq 0, \ \rho - p_\perp \geq 0$.

When the above conditions are violated the matter is termed as exotic. Figure 3 shows that all bounds on energy density and pressure components are satisfied in the considered setup. The plot of metric potentials in figure 4 indicates that the new solution preserves asymptotic flatness for large values of the radial coordinate.
4.2. Case II: barotropic equation of state

The new source is polytropic if it satisfies the EoS
\[ Q = Q_G K^{1/n} \]

where \( n = \frac{1}{\gamma} \) is the polytropic index and \( K \) is a positive parameter containing information about temperature. Different values of the polytropic index correspond to different types of fluids. We proceed by...
taking the simplest case of barotropic fluid (isothermal sphere of gas) associated with $\Gamma = 1$. The resulting EoS is equivalent to equation (28) for $a_1 = -\frac{1}{K}$ and $a_2 = 0$ which is expressed as

$$
\frac{1}{Kr\bar{T}(r)}(\alpha e^{-h(r)}(r\bar{T}'(r)(r\bar{T}'(r) + 2\bar{T}(r))(Ke^{h(r)}h_2'(r) + h_1'(r))
+ h_2(r)e^{h(r)}(r\bar{T}'(r)(\alpha rh_1'(r) + 4K + ra'(r) + 4) + 2Kr\bar{T}'(r)))
+ 2T^2(r)(\alpha rh_1'(r) + K + ra'(r) + 1) + (K - 1)r^2\omega_{BD} T^2(r))) = 0.
$$

Employing equation (29) and the initial conditions used in case I, equations (3) and (31) are solved simultaneously for $\bar{T}(r)$ and $h_1(r)$, respectively with $M = 1$.

Figure 5 demonstrates the presence of a singularity at $r = 0$ in the current setup. The plots of energy density and pressure components are displayed in figure 6 for $K = 0.01$. The celestial object becomes less dense for higher values of $\alpha$. The density increases to a maximum and then decreases monotonically for $r > 2M$. Moreover, the radial pressure decreases while tangential pressure increases as the decoupling parameter takes on...
higher values. The plots in figure 7 indicate that the extended solution fails to satisfy the energy conditions as 
\( \rho + p_r < 0 \) and \( \rho + p_r + 2p_\perp < 0 \) for the chosen values of \( \alpha \). Thus, the unknown source \( \Theta^j_i \) can be treated as exotic matter in this case. Finally, the metric potentials representing the spacetime of this setup are shown in figure 8. It can be clearly observed that trends of \( e^{\mu} \) and \( e^{\nu} \) do not approach 1 and thus, disobey the criterion of asymptotic flatness.

Figure 7. Energy conditions for case II.

Figure 8. Metric potentials for case II.
4.3. Case III: a particular solution

Here, we evaluate a specific solution by inserting \( a_1 = 1.4 \) and \( a_2 = 3 \) in equation \((28)\) leading to

\[
\frac{1}{r} \left( \alpha e^{-b(r)} (r \overline{T}(r)(r \overline{T}'(r))(e^{\overline{b}(r)}h_1'(r) + 2.2h_1''(r)) + \overline{T}(r)(e^{\overline{b}(r)}h_2'(r))
\right)
\times (0.75\alpha rh_1'(r) + 0.75r\alpha'(r) + 0.5) + 1.5rh_1''(r) + h_1'(r)(0.75\alpha rh_1'(r) + 1.5r\alpha'(r) - 0.75r\alpha'(r) + 2.9)) + h_2(r)e^{\overline{b}(r)}(r \overline{T}(r)(r \overline{T}'(r))(2.2\alpha rh_1''(r) + 2.2\alpha'(r) + 3.8) + 2r\overline{T}'(r)) + \overline{T}(r)(r (1.5\alpha rh_1''(r) + 0.75\alpha^2 rh_1''(r) + \alpha h_1'(r)(1.5r\alpha'(r) + 2.9) + 1.5r\alpha'(r) + 0.75r\alpha'(r) + 2.9\alpha'(r)) + 0.4) + 0.3r^2 \omega_{h_1} \overline{T}(r))) = 0.
\]

The extended solution is formulated by plugging the values of \( b(r), \alpha(r) \) and \( h_2(r) \) in equations \((3)\) and \((32)\) and solving them for initial conditions of case I with \( M = 1 \). A singularity exists at \( r = 0 \) since the plot of \( K \) in figure 9 tends to infinity for \( r \to 0 \). The state variables of the solution are plotted in figure 10 for chosen values of the
decoupling parameter. The energy density and tangential pressure are maximum at the horizon and decrease monotonically as \( r \) increases. The decrease in the values of decoupling parameter causes an increase in density and transverse pressure whereas the radial pressure increases with increase in \( \alpha \). The energy constraints are satisfied by \( \Theta_i^j \) (figure 11) ensuring the presence of normal matter. The metric functions represent a spacetime that is asymptotically flat as shown in figure 12.

Figure 11. Energy conditions for case III.

Figure 12. Metric potentials for case III.
5. Conclusions

In this paper, we have focused on developing new solutions through the EGD technique in the background of SBD theory. An additional source is incorporated in the simple seed source such as vacuum or isotropic fluid. The EGD approach reduces the degrees of freedom in the system of field equations by introducing geometric deformations in the metric functions related to the seed source. With the help of these transformations, the original system is disintegrated into two sets which exclusively correspond to the seed and new sources. We have considered the seed source as a vacuum and employed the metric potentials of Schwarzschild spacetime to specify the associated system. The extensions of this solution in the presence of \( Q \) are evaluated by imposing constraints on the metric functions and the \( \Theta \) sector. The condition \( \mu = - \varepsilon \) is implemented to ensure that Killing and causal horizons coincide at \( r = 2M \). The number of unknown constants has further been reduced by imposing the EoS \( \Theta_0 = a_1 \Theta_1 + a_2 \Theta_2 \). Three solutions have been generated corresponding to

- \( a_1 = -1 \) and \( a_2 = -2 \) (represents a traceless additional source).
- \( a_1 = -\frac{1}{2} \) and \( a_2 = 0 \) (provides a barotropic fluid distribution).
- \( a_1 = 1.4 \) and \( a_2 = 3 \) (yields a particular solution).

It is worth mentioning here that the linear EoS provides the Schwarzschild or Kiselev BH when EGD approach is applied to Schwarzschild metric in GR [23]. The massive scalar field in SBD gravity naturally induces anisotropy in the solution. Therefore, metrics different than Schwarzschild are obtained in this theory. The massive scalar field has been obtained by solving the wave equation \( V(\Gamma) = \frac{1}{2} m_\Gamma^2 \Gamma^2 \) with \( m_\Gamma = 0.1 \). The behavior of physical parameters of the extended solutions has been investigated for \( \alpha = -0.4, -0.5, -0.7 \) and \( \omega_{BD} = 60 \).

The energy density in all cases is positive for negative values of the decoupling parameter only which results in negative radial pressure. This behavior of matter variables is consistent with the work in [11]. Moreover, higher energy density is observed for lower values of \( \alpha \). The nature of the extra source has also been checked through energy conditions. The analysis of state parameters has revealed that the extended models corresponding to cases I and III are consistent with energy conditions. This implies that \( \Theta_0 \) is sourced by normal matter. However, the extended model obtained for a barotropic EoS violates the energy conditions indicating the presence of exotic matter. Moreover, the asymptotic behavior of the extended solutions has been checked through the behavior of \( e^\alpha \) and \( e^\varepsilon \). The metrics formulated for I and III are asymptotically flat whereas the metric obtained for II does not approach to a flat spacetime when \( r \to \infty \). All the BH solutions mentioned above have a singularity covered by the horizon at \( r = 2M \). Moreover, the extended Schwarzschild solutions obtained through the technique of MGD in GR violate the dominant energy condition. However, in the context of SBD gravity two solutions have been generated that adhere to all the energy constraints. It is interesting to mention here that all the results of GR can be retrieved for \( T = \text{constant} \) and \( \omega_{BD} \to \infty \).

ORCID iDs

M Sharif @ https://orcid.org/0000-0001-6845-3506

References

[1] Abott B P et al 2016 Phys. Rev. Lett. 116 061102
[2] Akiyama K et al 2019 Astrophys. J. 875 11
[3] Schwarzschild K 1916 Math. Phys. 1916 189–96
[4] Ruffini R and Wheeler J 1972 Phys. Today 24 50
[5] Hawking S W 1972 Commun. Math. Phys. 25 152
[6] Hawking S W, Perry M J and Strominger A 2016 Phys. Rev. Lett. 116 231301
[7] Antoniou G, Bakopoulos A and Kanti P 2018 Phys. Rev. Lett. 120 131102
[8] Grumiller D et al 2020 Phys. Rev. Lett. 124 041601
[9] Ovalle J 2008 Mod. Phys. Lett. A 23 1247
[10] Ovalle J and Linares F 2013 Phys. Rev. D 88 104026
[11] Ovalle J et al 2018 Eur. Phys. J. C 78 122
[12] Gabbanelli L, Rincón A and Rábido C 2018 Eur. Phys. J. C 78 370
[13] Estrada M and Tello-Ortiz F 2018 Eur. Phys. J. Plus 133 453
[14] Sharif M and Sadiq S 2018 Eur. Phys. J. C 78 410
[15] Pérez Gratero R 2018 Eur. Phys. J. Plus 133 244
[16] Sharif M and Ama-Tul-Mughani Q 2019 Int. J. Geom. Methods Mod. Phys. 16 1950187
Sharif M and Ama-Tul-Mughani Q 2020 Mod. Phys. Lett. A 35 2050091
[17] Tello-Ortiz F, Maurya S K and Gomez-Leyton Y 2020 Eur. Phys. J. C 80 324
[18] Casadio R, Ovalle J and da Rocha R 2015 Class. Quantum Grav. 32 215020
[19] Ovalle J 2019 Phys. Lett. B 788 213
[20] Contreras E and Bargueño P 2019 Class. Quantum Grav. 36 215009
[21] Sharif M and Ama-Tul-Mughani Q 2020 Ann. Phys. 415 168122
[22] Sharif M and Ama-Tul-Mughani Q 2020 Chin. J. Phys. 65 207
[23] Ovalle J et al 2021 Phys. Dark Universe 31 100744
[24] Sharif M and Saba S 2018 Eur. Phys. J. C 78 921
Sharif M and Waseem A 2019 Ann. Phys. 405 14
Sharif M and Waseem A 2019 Chin. J. Phys. 60 426
Sharif M and Saba S 2020 Int. J. Mod. Phys. D 29 1050014
[25] Brans C and Dicke R H 1961 Phys. Rev. 124 298
[26] Weinberg E J 1989 Phys. Rev. D 40 3950
[27] Will C M 2001 Living Rev. Rel. 4 4
[28] Khoury J and Weltman A 2004 Phys. Rev. D 69 044026
[29] Perivolaropoulos I, 2010 Phys. Rev. D 81 047501
[30] Thorne K S and Dykla J 1971 Astrophys. J. 166 335
[31] Johnson M 1972 Lett. Nuovo Cimento 4 327
[32] Hawking S W 1972 Commun. Math. Phys. 25 167
[33] Geroch R 1971 J. Math. Phys. 12 918
[34] Sneddon G E and McIntosh C B G 1974 Aust. J. Phys. 27 411
[35] Bruckman W F and Kazes E 1977 Phys. Rev. D 16 2
[36] Goswami G K 1978 J. Math. Phys. 19 442
[37] Rizzi N and Askari H R 1993 Mon. Not. R. Astron. Soc. 261 229
[38] Kim H 1997 Nuovo Cimento B 112 329
[39] Campanelli M and Lousto C O 1993 Int. J. Mod. Phys. 2 451
[40] Sharif M and Majid A 2020 Astrophys. Space Sci. 365 42
Sharif M and Majid A 2020 Phys. Dark Universe 30 100610
[41] Martinez C, Troncoso R and Zanelli J 2004 Phys. Rev. D 70 084035
Herdeiro C A and Radu E 2015 Int. J. Mod. Phys. D 24 1542014
Sotiriou T P 2015 Class. Quantum Grav. 32 214002
[42] Fujii Y and Maeda K 2003 The Scalar-Tensor Theory of Gravitation (Cambridge: Cambridge University Press)