A comparison of fitting criteria for circle arc measurement applications

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Abstract: Measuring circular shape is a main task of dimensional metrology, characterized by circle diameter and its roundness, both for full circles and circle arcs. Point coordinates allows measuring both arcs and full circle, by fitting to substitution geometry, in many cases by least-squares criteria and fitting. Nevertheless, circle shape can be also characterized by the minimum zone, minimum circumscribed and maximum inscribed circles. This research presents a systematic experimental analysis of results of normal distributed points around the substitution circle through simulation, for different circle arc angles to the full circle and for the four mentioned fitting criteria. The results show the influence of arc angle in the variability of the results across criteria and the different behaviour depending on the arc amplitude. The results confirm the good stability and behaviour of least squares and minimum zone criteria, while warns the use of minimum circumscribed and maximum inscribed circles over half circumference. Experimental regression facilitates estimation of the minimum zone criteria from the least squares fitting that are independently verified with literature datasets with good results.

Keywords: Least squares circle, Minimum zone circle, Minimum circumscribed circle, Maximum inscribed circle.

1. Introduction

Circle fitting from point coordinates is a fundamental geometry feature in dimensional metrology. Circular shapes in mechanical engineering are relevant for performance in many systems, so it requires measuring circle diameter and/or circular shape deviation from circularity or roundness. Four criteria of roundness are standardized: least-squares circle (LS), minimum zone circle (MZ), and the derived minimum circumscribed circle (MC) or maximum inscribed circle (MI).

LS is an integral method used in calibrations, providing robust results with minimum uncertainty from the sample on the average diameter. Nevertheless, minimum zone criteria (MZC and the derived MCC and MIC) are recommended by standards [1,2] for roundness measurement in order to assess compliance of functional tolerances. The minimum zone criteria are based on the $L^\infty$ or Chebyshev metric, instead of the $L^2$ of the least squares, but the algorithms are more complex that the least-squares fitting, while the fitting is more sensible to outliers than the least-squares fitting [3].

A great advantage of least-squares fitting is that minimizes the mean square error of the deviations around the average circle. It is the maximum likelihood circle in statistical terms, so it minimizes the estimated uncertainty of the substitution circle parameters obtained from the dataset of sample points.
The least-squares criterion minimizes the sum residuals in the estimation of the substitution geometry, so an estimation of the spread from the ideal circle shape is obtained directly from the fitting. A first approximation of the goodness of the fitting is also the residuals or the root mean square error (RMSE) of the fitting that is minimum by the algorithm objective, so both the geometry substitution and the goodness of the fitting are obtained naturally by facing the problem of least squares. Both, the diameter and roundness are commonly determined from least-squares criteria, even when it is assumed that the roundness got from the least squares fitting is not minimal. The dimensional features of diameter or roundness should be offered through its best value and the estimation of uncertainty. The GUM (guide to the expression of uncertainty) from ISO standards establishes uncertainty estimation in the spread of the measurement realization around the reference value (unknown, but conventional agreed as true value), so the least square fitting results in the minimum spread around the circle of substitution, so providing the better estimation of uncertainty contribution of the sampling dataset. The uncertainty from sampling is obtained from the fitting itself and through the law of uncertainty propagation [GUM] can be easily aggregated along the rest of relevant uncertainty factors (instrument calibration or verification of CMM or environmental effects like temperature or humidity). Even though, the least-squares circle has an intrinsic trend to result in bigger fitted circles that the original circle diameter from which the sample is taken [4]. The essential bias at the leading order of the fitting circle is null for the centre but is about \( \frac{s^2}{2R} \) for the radius \( R \), so giving slightly bigger fitted circles than the true circle.

In spite of the numerous advantages pointed out, the minimum zone tolerance is preferred by ISO or ANSI standards as the criteria for roundness measurement. It provides the minimum total deviation of the measured points from the circle of substitution geometry. Unfortunately, the minimum zone criteria do not deal directly with the minimization of the residuals. In consequence, the uncertainty will be higher than that obtained from the least-square fitting. Uncertainty estimation using results spread around the substitution geometry will give higher uncertainty estimation that least squares, so for a good estimation, the uncertainty should be estimated in a second step after the minimum zone circle has been fitted. The uncertainty of point coordinates propagation in the fitting model can be estimated through the Monte Carlo Method. The minimum circumscribed circle and the maximum inscribed circle are also derived from the Chebyshev metric, but only one bound. They can provide the substitution geometry in parts for mechanical fits. When the part is a hub, the MC provides the minimum hole diameter in the fitting. In a similar way, MI applied to holes provide the maximum hub diameter that can be fitted on it. The difficulties of the Chebyshev algorithms and the extra effort for proper uncertainty estimation make the minimum zone criteria little popular in industrial measuring, in spite of its recommendation to provide better roundness estimation. In addition, the fact that the calibration processes are performed under the least squares criteria, limits minimum zone criteria to roundness, so the traceable diameter in precise roundness standards, like glass hemispheres, can be found typically in calibrations under least-squares criteria.

The comparison of fitting criteria has been previously faced, but scarcely. A main difficulty of criteria comparison is the different nature of the problem formulation, combined with the random effects that makes difficult getting conclusions. In fact, the circle fitted under any of the four criteria is the same as a measurand, but the use of different fitting criteria can yield quite different quantitative results, so a main concern should be initially to select the proper fitting for the fitting purpose. Even though, metrological practice establishes the prevalence of least squares fitting with good proved algorithms that can get the benefit of its robustness in the presence of noise, in particular for industrial or field measurements with limited metrological infrastructures. Small deviation from ideal circle geometry can be due to random noise uncertainty and not to real form deviation. Conversely, significant deviations can be beyond noise and represent real shape deviation from a circle, so minimum zone could represent better the tolerance of form. Research on quantitative expectations in the different results obtained by the used of each criteria is a worthy objective of practical significance. Among the work of direct focus on the difference of the arc circle criteria, it can be mentioned early works [5] or the attempts to quantify by inequality the fitting relationships obtained by [6]. Some other authors put the focused just on the quantitative difference results [7]. Circle arc measuring is relevant in field and research applications in
manufacturing. In particular, in machining, those for measuring for the nose tool radius that influences tool wear and surface roughness of machined surfaces and the tool radius compensation in CNC programming or tool life [8].

This work researches and discusses the comparison of those criteria and looks for some quantitative experimental relationships of potential use for practical purposes. Based on former studies on circle characterization and algorithms [9,10], but adapting and extending the full circle algorithms to the fitting of circle arcs. In Section 2, the methodology, algorithms and range of experimental simulation are established. Next, Section 3 includes the simulation results and findings. Finally, Section 4 recapitulate main findings and prospect for further research.

2. Methodology
The experimental setup includes least-squares method that minimizes the squares of the radial difference of \( n \) sample points with the geometry of substitution (1). The well-known algorithm Levenberg-Marquardt [11,12] is implemented in Matlab for direct use with point coordinates.

\[
\min \left\{ \sum_{i=1}^{n} \left[ R_i - R_i^{'} \right]^2 \right\} ; \quad R_i = \sqrt{(x_i - a)^2 + (y_i - b)^2} ; \quad i = 1...n \text{ points; solution } \{ R_i, \text{ circle center } (a,b) \}
\]  \hspace{1cm} (1)

The minimum zone algorithm [9,10] was originally developed for full circles, but now adapted in computational details for circle arcs. The minimum zone circle problems are determined by four critical points. The formulation in rectangular coordinates follows for the minimum zone circle (2), minimum circumscribed (3) and maximum inscribed circle (4).

\[
\min \left\{ \max_{i=1}^{n} \left| R_i - R_{mc} \right| \right\} ; \quad R_i = \sqrt{(x_i - a)^2 + (y_i - b)^2} ; \quad i = 1...n \text{ point; solution } \{ R_{mc}, \text{ circle center } (a,b) \}
\]  \hspace{1cm} (2)

\[
\min \left\{ \frac{R_{mc}}{R_{mi}} \right\} \geq R_i ; \quad R_i = \sqrt{(x_i - a)^2 + (y_i - b)^2} ; \quad i = 1...n \text{ points; solution } \{ R_{mi}, \text{ circle center } (a,b) \}
\]  \hspace{1cm} (3)

\[
\max \left\{ \frac{R_{mi}}{R_{mc}} \right\} \leq R_i ; \quad R_i = \sqrt{(x_i - a)^2 + (y_i - b)^2} ; \quad i = 1...n \text{ points; solution } \{ R_{mi}, \text{ circle center } (a,b) \}
\]  \hspace{1cm} (4)

In order to simulate a range of cases representative it is considered a circle of radius 1. When considering the deviation of diameter or overall circular shape from a perfect circle, the least squares algorithm can initially have considered the reference, because when deviation is due to pure random effects, the maximum likelihood circle is the least-squares one. The experimental setup for simulation includes different levels of standard deviation to radius ration \( s/R \) in the distribution of points around the true circle of radius 1. The base point in the circumference are evenly distributed every \( 15^\circ \), so up to 24 partial central angle arcs are considered for each circumference. The simulation deviated points are generated by adding to the nominal coordinate position \( (x,y) \) the independent shot to a Gaussian distribution \( N(0,s) \) for each coordinate for each setup trial.

The range of interest of the standard deviation is sought with physical meaning with respect to mechanical tolerances. The standard ranges of ISO tolerance classes [13] is a reference point to establish significant order of magnitude of the part size radius \( R \) and spread in the interval of tolerance by setting the standard deviation \( s \) in the simulation. The relevance of circularity measurement has evolved from more theoretical approaches [14,15] to the newer more focused on its practical assessment [16,17], in line with our work. The ratio \( s/R \) will be representative of the real measuring processes. Considering the ISO ranges of quality from \( IT 5 \) to \( IT 11 \) and size in the intervals from 6-10 mm to 400-500 mm the order of magnitude of \( s \) from \( 10^{-5} \) to \( 10^{-3} \) mm covers an interesting range of variability. Note that the generation of coordinate point from \( N(0,s) \) in both \( x \) and \( y \) generates deviations that can be at maximum 3s with a coverage factor at 95% confidence of 99%. In the simulation, five levels of \( s/R \) are used: \( 3.10^{-5}, 1.10^{-4}, 3.10^{-4}, 1.10^{-3} \) and \( 5.10^{-3} \).

Each simulation of the circle arc contains up to 24 base point position from small arc to the full circle, and 1,000 set of randomly generated coordinates point are solved for each base point position.
3. Experimental simulation results and discussion

3.1. Circle radius measurement

The fitting results of up to 24,000 circle points are enclosed at each graph in figure 1, so every point in the plot is the average result of 1,000 points, randomly generated by $N(0,s)$. The baseline circle or true circle is $R=1$. The radius of the LS, MZ, MC and MI criteria are represented. Even when 5 different levels of $s/R$ are used, from $3 \times 10^{-5}$ to $5 \times 10^{-3}$, only 3 levels are represented for paper space limitations.

![Figure 1. Circle arc radius for different fitting criteria (LS, MZ, MC, MI vs. spread variability s/R).](image)

The results show how in small arcs under approximately 1 rad (about 57°) there is a great dispersion of results and the radius obtained from the different algorithms for the radius is quite different. It must be remarked the spread for small arcs, where the defining spread of points can determine very different arc curvature in the fitting depending on the criteria. In overall terms, both LS and MZ give consistently the expected baseline true radius $R=1$ with small differences that will be quantified in this later. Meanwhile, the results of the mean radius for MC and MI are quite similar to those of LS and MZ up to 180° (half circle) approximately. Once the curvature is enforced to fit over a half circle the mean radius separates significantly.

![Table 1. Full circle radius for different fitting criteria. Average from 1000 points sample.](image)

| s/R  | 3.E-05 | 1.E-04 | 3.E-04 | 1.E-03 | 5.E-03 |
|------|--------|--------|--------|--------|--------|
| R_ls | 1.00000011 | 1.00000007 | 1.00000006 | 1.00000115 | 1.00002506 |
| R_mz | 1.00000032 | 1.00000027 | 1.00000018 | 1.00000896 | 1.00002586 |
| R_mc | 1.00007075 | 1.00023582 | 1.00070716 | 1.00235987 | 1.01179034 |
| R_mi | 0.99992982 | 0.99976503 | 0.99929267 | 0.99766791 | 0.99119472 |

In table 1, the anticipated effect of a slight bias of $R_{ls}$ towards big circles is appreciated. While the relative size of the minimum zone criteria is $R_{mc} > R_{mz} > R_{mi}$, the relative size of the expected (1,000 samples mean) circle with respect to the least-squares criteria for the full circle is observed from the experimental results that becomes consistently for each level of $s/R$, $R_{mz} > R_{ls}$. Note that the distribution in the simulation is Gaussian, but in field measurement MZ is very sensitive to the presence of outliers so this relationship cannot be ensured in general (see for instance the results of table 2).

In addition to the average results, the variability of the 1,000 samples is characterized from the root mean square error (RMSE). By algorithm nature, the minimum RMSE is got through the least-squares fitting, but it is remarkable that even when the minimum zone fitting yield a much higher RMSE (in a
trade-off with its lower tolerance zone), the experimental results are almost insensitive to the baseline standard deviation \( s \), to across the 3 orders of magnitude from \( 3 \times 10^{-5} \) to \( 5 \times 10^{-3} \). Note that from partial arc up to full circle, the curve of RMSE are practically identical, showing a maximum of variability around \( \frac{3}{4} \) of the full circle, figure 2.

**Figure 2.** Root mean square error across arc angle for R\_mz.

With respect to the MC and MI criteria, the results show high variability (RMSE) in the range from 90 to 180° (see figure 3), and RMSE grows over the range of \( s/R \). RMSE is in every point of the plot the mean value over 1,000 samples obtained from the RMSE of each fit. In spite the close average agreement in that range of all fitting criteria (see figure 1), high values of RMSE can make little practical the reproducibility of MC and MI criteria based on a limited number of samples.

**Figure 3.** Root mean square error (RMSE) across arc angle for R\_mc and R\_mi.

3.2. *Roundness measurement by tolerance zone*

Establishing an estimative quantitative relationship between least squares and minimum zone algorithm can be worthy to understand the expected difference between criteria. In figure 4 the variation of the tolerance zone for LS and MZ criteria are represented across arc angle and \( s/R \). The tolerance zone grows as the arc circle angle grows. Very small angles under 15° for low \( s/R \) and up to 60° for high \( s/R \), should be avoided for consistent results, so the curvature could be mismatched, as previously remarked in figure 1. The fact that the tolerance zone grows with arc angle can recommend measuring arc circle for functional purposes only when the circular surface of the arc is functional, so extending measurement
will only increase the value of tolerance zone result. The difference between both criteria is consistent over both ranges, suggesting that the evaluation of one of them could allow estimating the other.

3.3. The minimum zone fitting estimated from the least squares fitting

The LS criteria is generally adopted for several reasons previously discussed. Nevertheless, for applications where roundness determination is critical, MZ criteria should be selected. When functionality is that of a hub or a hole, circle arc might be better theoretically assessed by MC and MI respectively. Although, it has been shown that both MC and MI are very variable in some ranges, so big samples could be necessary to determine consistently the radius under those criteria. Searching the experimental relationship between LS and MZ can reduce the risk of diameter and roundness evaluation based only just one criteria.

An experimental multivariate regression analysis has been conducted over the fitting results of LS and MZ, for different arc angles and across the whole range of s/R, from $3 \times 10^{-5}$ to $5 \times 10^{-3}$. In addition to the regression coefficient $r^2$, it has been analyses the studentized residuals. Regression has been accomplished from 120 values, when each one represent the mean of 1,000 shots on the $N(0,s)$, as pointed before. The goodness of the regression is very good with coefficient of regression over 99.99%. The p-test for the regression parameters is under 0.05 showing significance of the model estimators and the studentized residuals are between ±2, with very few outliers. In addition, the interval at 95% confidence limit of the regression estimator in the model is included in brackets. The resulting regression equations (5-8) allow the estimation of the minimum zone radius from the least squares fit in the range of the experimental data. In the expressions the arc angle is noted A [°].

$$R_{_mz} = 0.999889 \cdot R_{_ls}; \text{ at 95% conf.}[0.999737,1.00004]; \quad r^2 = 99.9999\%$$  \hspace{1cm} (5)  

$$R_{_mi} = 0.997727 \cdot R_{_ls}; \text{ at 95% conf.}[0.996469,0.998985]; \quad r^2 = 99.9953\%$$  \hspace{1cm} (6)  

$$R_{_mc} = 1.00255 \cdot R_{_ls}; \text{ at 95% conf.}[1.00117,1.00393]; \quad r^2 = 99.9944\%$$  \hspace{1cm} (7)  

$$TZ_{_mz} = 0.90609 \cdot TZ_{_ls} \left(1 + 0.011118 \cdot A^{0.2}\right) \text{ at 95% conf.}[0.902947,0.909239]$$  \hspace{1cm} and \hspace{1cm} [0.00904894,0.0112049];  

$$r^2 = 99.9997\%; \quad r^2_{adjusted} = 99.9997\%.$$  

(8)

It must be noted that the functional relationship under regression is intrinsically heteroscedastic, since the variability RMSE grows with the s/R of the set of simulation, so the variance of fitting results naturally grows as s/R increases. Under heteroscedasticity the regression could be weighted by the reverse of the variance for normalization purposes, with the possibility of doing hypothesis contrast with
confidence limits based on the regression. Since that purpose is beyond the scope of this work, ordinary multivariate regression is maintained, being aware that the confidence limits for high values of $s/R$ can be surpassed, but the average values of the prediction model are not affected by heterodasticity.

From the quantitative regression results, the regression study across circle arcs (more the 200,000 simulation shots) shows a difference of no more that 10% between the tolerance zone from the LS and the minimum zone obtained by the MZ criteria. In addition, the MZ, MC and MI circles are bounded with respect the LS circle and the relative size differs less than 1% from LS circle.

3.4. Trial tests
In order to assess the behaviour of the regression models developed for the LS and MZ criteria, first trials of circle fitting are selected from literature sample point dataset circles, Table 2. The precise fitting circles are calculated for each criteria and the estimated fitting in the table are calculated from (5-8). The error (%) of the estimation is also included. It can be appreciated a general low error from the estimation. The higher deviation is observed in TZ_mz, under 4%. Noteworthy, the resulting distribution of radius under least-squares criteria has been tested for normality through the Chi-test with acceptance of the null hypothesis. The estimator (5-8) in the range of $s/R$, from $3.10^{-5}$ to $5.10^{-3}$ can allow an affordable estimation of minimum zone fitting from the common least squares fitting.

### Table 2. LS and MZ fitting and regression estimations from selected datasets.

| Dataset - no. Points | $s/R$ | $R_{ls}$ | RMSE_{ls} | TZ_{ls} | $R_{mz}$ | TZ_{mz} | $R_{mc}$ | $R_{mi}$ |
|----------------------|------|--------|---------|------|--------|-------|--------|--------|
| Jywe and Liu [14] - 39 | fitting | 1.000511 | 0.002573 | 0.009195 | 1.000225 | 0.008537 | 1.004422 | 0.996030 |
| 5.E-05 | estimated fitting | 1.000400 | 0.008624 | 0.998237 | 0.998237 |
| Sui and Zhang [15] - 20 | fitting | 50.000110 | 0.000003 | 0.003021 | 50.001573 | 0.002739 | 50.001314 | 49.998921 |
| 8.E-4 | estimated fitting | 49.994560 | 0.002833 | 49.886460 | 49.886460 |
| Jiang et al. [16] - 40 | fitting | 7.500249 | 0.131151 | 0.025814 | 7.498217 | 0.025118 | 7.510425 | 7.488079 |
| 1.E-03 | estimated fitting | 7.499416 | 0.024212 | 7.483201 | 7.483201 |
|       | error (%) | -0.02 | 3.45 | -0.23 | -0.22 |

4. Conclusions
An experimental simulation research on the LS, MZ, MC and MI fitting circles has been developed with Gaussian distribution around the reference circle. The results show very variable fittings radius under the different criteria for small arcs (under 15° in the low range of $s/R$ and under 60° for high $s/R$, in the range of study $s/R$ from $3.10^{-5}$ to $5.10^{-3}$). In addition, high sensitivity in the results of MC and MI over half circle are observed, while the results up to 180° is quite similar, but with high RMSE, so making MI and MC little robust for low sample size. The results obtained from LS and MZ maintained a consistent difference all across the different arc angles. This suggested attempting the multivariate regression models that has been developed in order to support minimum zone fitting estimation from least squares estimation in the range of $s/R$ application of Gaussian spread around the true circle. Regression estimations show a maximum improvement about 10% in the reduction of the tolerance zone, when applying the minimum zone criteria instead the least-squares one. This statistical estimation could fail for outliers in the measuring process, but it establishes a reference estimation to assess the convenience of using MZ instead LS depending on the application.

The experimentation shows insensitivity of the RMSE of $R_{mz}$ for different levels of $s/R$ that would require further research to understand its origin. In addition, a future study of the relationship between LS, MZ, MC and MI for other distributions instead the Gaussian can yield practical insights about the connection among the different circle fitting criteria for researchers and practitioners.

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