Stochastic differential equations driven by fractional Brownian motion with locally Lipschitz drift and their Euler approximation

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Abstract

In this paper, we study a class of one-dimensional stochastic differential equations driven by fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. The drift term of the equation is locally Lipschitz and unbounded in the neighborhood of 0. We show the existence, uniqueness and positivity of the solutions. The estimations of moments, including the negative power moments, are given. Based on these estimations, strong convergence of the positivity preserving drift-implicit Euler-type scheme is proved, and optimal convergence rate is obtained. By using Lamperti transformation, we show that our results can be applied to interest rate models such as mean-reverting stochastic volatility model and strongly nonlinear Aït-Sahalia type model.

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1 Introduction

In this paper, we shall consider a one-dimensional stochastic differential equation (in short SDEs) driven by fractional Brownian motion:

$$dX_t = B(t, X_t)dt + \sigma dB^H_t, \quad X_0 \geq 0, \quad (1.1)$$

where $B^H_t$ is a fractional Brownian motion (fBM for short) with Hurst $H \in (1/2, 1)$ and the drift term $B(t, x)$ is only local Lipschitz in $x \in (0, \infty)$ and unbounded in the neighborhood of 0. Some general results on this type of equations has been obtained in [18] motivated by the study on Cox-Ingersoll-Ross (C-I-R for short) model in mathematical finance (see [9]) with Brownian motion replaced by fBM. Due to the
memory effects of fractional Brownian motion, it would be reasonable to replace Brownian motion by fBM if there are inert investors in this market, see for instance [23]. In fact, to handle the complexity of the market, various interest rate models have been developed besides C-I-R, see for instance [1, 7, 8]. Some of them cannot be covered by the general conditions introduced in [18]. Hence one aim of this paper is to give some general conditions to cover more interest rate models by using the Lamperti transformation, even their coefficients have super-linear growth, see e.g. Example 4.2 the Ait-Sahalia-type interest rate model for details.

On the other hand, numerical approximations of SDEs arising from finance are of great interest. For instance, strong approximation of C-I-R model based on the Euler-type method was showed in [11] and optimal convergence rate is obtained; strong convergence of Euler-Maruyama type approximations for Ait-Sahalia type model is given [24]; in [19], Euler approximations for a general mean-reverting stochastic volatility model under regime-switching is presented. There are many SDEs in mathematical finance having non-Lipschitz coefficients. For Euler scheme of SDEs without global Lipschitz coefficients, one can see [11, 24, 5, 13] and references therein. However, the numerical issues for SDEs driven by fBM have not been well studied, comparing with SDEs driven by Brownian motion. Recently, the authors in [15] obtained optimal strong convergence rate of backward Euler scheme for C-I-R model driven by fBM. For numerical scheme of fractional SDEs, one can consult [15, 16, 17, 21] and references there in. In this paper, after a general discussion on (1.1), we investigate the numerical approximation of the solution to this equation when $X_0$ is positive. Strong convergence of the numerical scheme is obtained. Based on the Lamperti transformation used as in [11, 15, 18], our results can cover interesting models in mathematical finance, such as mean-reverting stochastic volatility model (Example 4.1):

$$dY_t = (a_1 - a_2 Y_t)dt + \sigma Y_t^\gamma dB_t^H, \ Y_0 > 0,$$

where $\gamma \in [1/2, 1)$; and Ait-Sahalia type model (Example 4.2):

$$dY_t = (a_1t^{-1} - a_0 + a_1 Y_t - a_2 Y_t^{\rho}) dt + \sigma Y_t^{\rho} dB_t^H, \ Y_0 > 0,$$

where $\rho > 1$ and the stochastic integral in these two models is in the sense of pathwise Riemann-Stieltjes integral developed by Zähle in [26]. The first model was studied in [19] under regime-switching, and the convergence rate is obtained. The second model was studied in [24], where the convergence rate is not clear. Following the study in [11, 15], the positivity preserving drift-implicit Euler-type method is adopted in our paper. Here, not only the strong convergence is showed, but also the convergence rate is obtained. For concrete examples presented above, the convergence order of the mean-reverting stochastic volatility model is the Hurst parameter $H$ up to a logarithmic term, which is an extension of [11]; the convergence order of the Ait-Sahalia type model is $2H - 1 \left( \frac{1}{\rho - 1} \wedge 1 \right)$ up to a logarithmic term.

This paper is structured as follows. In Section 2, we shall recall some basic facts on fractional Brownian motion. Section 3 is devoted to general discussions on (1.1), including existence and uniqueness of solutions to the equation; (negative-power) moments and modular of continuity estimations. In Section 4, we shall present our results on the numerical approximations of (1.1) and their applications on concrete examples.
2 Preliminaries

We shall recall some basic facts about fractional Brownian motion. For more details, we refer readers to [6, 22, 25].

Let $B^H = \{B^H_t, t \in [0,T]\}$ be a fractional Brownian motion with Hurst parameter $H \in (1/2,1)$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, that is, $B^H$ is a Gaussian process which is centered with the covariance function

$$\mathbb{E}(B^H_t B^H_s) = R_H(t,s) = \frac{1}{2} (t^{2H} + s^{2H} - |t-s|^{2H}).$$

For each $t \in [0,T]$, let $\mathcal{F}_t$ be the $\sigma$-algebra generated by the random variables $\{B^H_s : s \in [0,t]\}$ and the sets of probability zero. Furthermore, one can show that $\mathbb{E}|B^H_t - B^H_s|^p = C(p)|t-s|^{pH}$ for all $p \geq 1$. As a consequence of the Kolmogorov continuity criterion, $B^H$ have $(H - \epsilon)$-order Hölder continuous paths for all $\epsilon > 0$. Indeed, the studies on the sample path property of fractional Brownian motion, see for instance [25], show that

$$|B^H_t - B^H_s| \leq A|t-s|^H \sqrt{\log (1 + (t-s)^{-1})}$$

where $A$ is a random variable depending on $H$ only and there is some $c > 0$ such that $\mathbb{E}e^{cA^2} < \infty$.

Denote by $\mathcal{E}$ the set of step functions on $[0,T]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle I_{[0,t]}, I_{[0,s]} \rangle_{\mathcal{H}} := \alpha_H \int_0^T \int_0^T 1_{[0,t]}(u) 1_{[0,s]}(v) |u-v|^{2H-2} du dv = R_H(t,s),$$

where $\alpha_H = H(2H-1)$. By the B.L.T. theorem, the mapping $I_{[0,t]} \mapsto B^H_t$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space associated with $B^H$. Denote this isometry by $\phi \mapsto B^H(\phi)$.

On the other hand, the covariance kernel $R_H(t,s)$ can be written as

$$R_H(t,s) = \int_0^{\wedge \wedge} K_H(t,r) K_H(s,r) dr,$$

where $K_H$ is a square integrable kernel given by

$$K_H(t,s) = \frac{s^{1/2-H}}{\Gamma(H-1/2)} \int_s^t r^{H-1/2}(r-s)^{H-3/2} dr 1_{[0,t]}(s)$$

in which $\Gamma(\cdot)$ is the Gamma function. Using this kernel, we could define a map from $L^2([0,T])$ to the reproducing kernel space $\mathcal{H}$ defined as follows

$$\mathcal{H} = \text{span}\{R_H(t,\cdot) : t \in [0,T]\}^{(\cdot,\cdot)}_R, \quad \langle R_H(t,\cdot), R_H(s,\cdot) \rangle_R = R_H(t,s), \quad s, t \in [0,T].$$

For any $\phi \in L^2([0,T])$, let

$$(K_H \phi)(t) = \int_0^t K_H(t,s) \phi(s) ds, \quad t \in [0,T].$$

It has been proved in [14, 10] that $K_H$ is an isomorphism from $L^2([0,T])$ to $\mathcal{H}$.
Proposition 2.1. Let $a, b \in \mathbb{R}$ with $a < b$, and let $F \in C^1(\mathbb{R})$.

(1) Suppose $f \in C^\lambda(a, b)$ and $g \in C^\mu(a, b)$, where $C^\lambda(a, b)$ and $C^\mu(a, b)$ are Hölder continuous functions with order $\lambda$ and $\mu$ respectively. If $\lambda + \mu > 1$, then the Riemann-Stieltjes integral $\int_a^b f \, dg$ exists.

(2) Suppose $f \in C^\lambda(a, b)$ such that $F' \circ f \in C^\mu(a, b)$ with $\lambda + \mu > 1$. Then

$$F(f(t)) - F(f(s)) = \int_s^t F' \circ f(r) \, dr, \quad s, t \in (a, b).$$

Finally, we shall recall a result on the relationship of stochastic integral and the Skorohod integral w.r.t. fractional Brownian motion. Let

$$|\mathcal{H}| = \left\{ \psi \in \mathcal{H} \mid \|\psi\|_{\mathcal{H}}^2 = \alpha_H \int_0^T \int_0^T |\psi(s)||\psi(t)||t - s|^{2H-2}dsdt < \infty \right\}$$
and \(|\mathcal{H}| \otimes |\mathcal{H}|\) be the set of all measurable function such that
\[
\|\psi\|^2_{|\mathcal{H}| \otimes |\mathcal{H}|} := \alpha_H^2 \int_{[0,T]^4} |\psi(u,s)||\psi(v,t)||u - v|^{2H-2}|t - s|^{2H-2}dudvdtds < \infty.
\]

For \(p > 1\), we denote by \(\mathbb{D}^{1,p}_{|\mathcal{H}|}\) all the random variable \(u\) such that \(u \in |\mathcal{H}|\) a.s., its Malliavin derivative \(Du \in |\mathcal{H}| \otimes |\mathcal{H}|\) a.s., and
\[
\mathbb{E}\|u\|^p_{|\mathcal{H}|} + \mathbb{E}\|Du\|^p_{|\mathcal{H}| \otimes |\mathcal{H}|} < \infty.
\]

Then we have the following proposition, see [22, Proposition 5.2.3] and [2].

**Proposition 2.2.** Let \(u_t\) be a stochastic process in \(\mathbb{D}^{1,2}_{|\mathcal{H}|}\) such that a.s.
\[
\int_0^T \int_0^T |D_s u_t||t - s|^{2H-2}dt ds < \infty.
\]
Then
\[
\int_0^T u_t dB^H_t = \delta(u) + \alpha_H \int_0^T \int_0^T D_s u_t|t - s|^{2H-2}dt ds.
\]

For \(p > \frac{1}{H}\),
\[
\mathbb{E} \left( \sup_{t \in [0,T]} |\delta(u1_{[0,t]})|^p \right) \leq C \left( \mathbb{E} \int_0^T |u_s|^p ds + \mathbb{E} \left( \int_0^T \int_0^T |D_r u_s|^{pH} dr ds \right)^{pH} \right).
\]

### 3 A study of SDEs driven by fractional Brownian motion

In this section, we shall consider \((1.1)\) following [18]. To get the existence and uniqueness of this equation, we introduce the following assumptions.

**(A1)** The drift term \(B : [0, \infty) \times (0, \infty) \to \mathbb{R}\) is continuous and has continuous derivative w.r.t. the second variable. There exists \(K_t \geq 0\), nondecreasing in \(t\), such that
\[
(B(t, x) - B(t, y))(x - y) \leq K_t(x - y)^2, \quad x, y \in (0, \infty), \quad t \geq 0.
\]

**(A2)** There exist \(x_1 > 0, \alpha > \frac{1}{H} - 1\) and \(h_1 \in C([0, \infty), (0, \infty))\) such that
\[
B(t, x) \geq h_1(t)x^{-\alpha}, \quad t \geq 0, \quad x \leq x_1.
\]

**(A3)** There is \(x_2 > 0\) and a nonnegative locally bonded function \(h_2\) such that
\[
B(t, x) \leq h_2(t)(x + 1), \quad x \geq x_2, \quad t \geq 0.
\]

The existence and uniqueness of solutions to \((1.1)\) follows from the existence and uniqueness of the equation below:
\[
dX_t = B(t, X_t)dt + dw_t, \quad x_0 \geq 0, \tag{3.1}
\]
where \( w \in C^\beta([0,T], \mathbb{R}) \) for all \( T > 0 \) with \( \beta \in (\frac{1}{2}, H) \) such that \( \alpha > \frac{1}{\beta} - 1 \). We say \( f \) is a \( \beta \)-Hölder continuous function on \([s,t]\) if

\[
\|f\|_{s,t,\beta} := \sup_{s \leq s' < t \leq t} \frac{|f(s') - f(t')|}{(t' - s')^{\beta}} < \infty.
\]

Sometimes, we use \( \| \cdot \|_\beta \) for simplicity’s sake. For a continuous function \( f \) on \([s,t]\), we define

\[
\|f\|_{s,t,\infty} = \sup_{s \leq r \leq t} |f_r|.
\]

Our existence and uniqueness theorem for (3.1) reads as follows.

**Theorem 3.1.** Assume that (A1)-(A3) hold.

1. For all \( X_0 > 0 \), it holds that the equation (3.1) has a unique solution \( X_t \) and

\[
X \in C^\beta([0,T], (0, \infty)), \quad T > 0.
\]

2. For \( X_0 = 0 \), if there exists \( t^0 > 0 \) such that \( B(t, \cdot) \) is non-increasing on \((0,x_1)\) for all \( 0 \leq t \leq t^0 \), then (3.1) has a unique solution \( X_t \) and \( X_t \in (0, \infty) \) for all \( t > 0 \).

**Proof.** We first prove the uniqueness. Let \( X_t^{[1]} \) and \( X_t^{[2]} \) be two solutions of equation (3.1) with the same initial values, then

\[
X_t^{[1]} - X_t^{[2]} = X_s^{[1]} - X_s^{[2]} + \int_s^t \left( B(r, X_t^{[1]}) - B(r, X_t^{[2]}) \right) dr, \quad t \geq s.
\]

Combining this with (A1), we have

\[
d \left( X_t^{[1]} - X_t^{[2]} \right)^2 = \left( B(t, X_t^{[1]}) - B(t, X_t^{[2]}) \right) \left( X_t^{[1]} - X_t^{[2]} \right) dt 
\leq K_t \left( X_t^{[1]} - X_t^{[2]} \right)^2 dt.
\]

Thus, it follows from Gronwall’s inequality that \( X_t^{[1]} - X_t^{[2]} = 0 \) for all \( t \geq 0 \).

We assume that \( X_0 > 0 \). Since \( B : [0, \infty) \times (0, \infty) \to \mathbb{R} \) is continuous and has continuous derivative w.r.t. the second variable, it is clear that (3.1) has a continuous local solution. Next, we shall prove that \( X_t \in (0, \infty) \) for all \( t > 0 \). Let

\[
\tau_0 = \inf \{ t \geq 0 \mid X_t = 0 \}, \quad \tau_n = \inf \{ t \geq 0 \mid X_t \geq n \}, \quad n \in \mathbb{N}.
\]

We shall prove \( \tau_0 = \infty \) and \( \lim_{n \to \infty} \tau_n = \infty \).

If \( \tau_0 < \infty \), then there is \( t_0 \in (0, \tau_0) \) such that \( X_t \leq x_1 \) for all \( t \in (t_0, \tau_0] \). Since \( B(t,x) > 0 \) for \( x \in (0,x_1), t \geq 0 \) and

\[
0 = X_{t_0} = X_t + \int_{t_0}^{\tau_0} B(s, X_s) ds + w_{\tau_0} - w_{t_0},
\]

it follows that

\[
X_t \leq |w_{\tau_0} - w_t| \leq \|w\|_\beta (\tau_0 - t)^\beta, \quad t \in (t_0, \tau_0).
\]
On the other hand, following from (A2), (3.2) and (3.3),
\[
\|w\|_\beta (\tau_0 - t)^\beta \geq |w_{\tau_0} - w_t| \geq \int_t^{\tau_0} B(s, X_s)ds \\
\geq \int_t^{\tau_0} h_1(s) X_s^{-\alpha} ds \geq \inf_{s \in [t_0, \tau_0]} h_1(s) \int_t^{\tau_0} X_s^{-\alpha} ds \\
\geq \frac{\inf_{s \in [t_0, \tau_0]} h_1(s)}{\|w\|_\beta} \int_t^{\tau_0} \frac{1}{(\tau_0 - s)^{\alpha\beta}} ds \\
= \frac{(\tau_0 - t)^{1-\alpha\beta} \inf_{s \in [t_0, \tau_0]} h_1(s)}{\|w\|_\beta},
\]
this, together with \(\alpha > \frac{1}{\beta} - 1\), implies that
\[
0 = \lim_{t \to \tau_0} (\tau_0 - t)^{\alpha\beta - 1} \geq \lim_{t \to \tau_0} \frac{\inf_{s \in [t_0, \tau_0]} h_1(s)}{\|w\|_\beta^{\alpha + 1}} > 0.
\]
Hence \(\tau_0 = \infty\).

If \(\tau_\infty := \lim_{n \to \infty} \tau_n < \infty\), then either there exists \(t_0\) such that \(X_{t_0} = x_2 + X_0\) and \(X_t \geq x_2 + X_0\) for all \(t \in (t_0, \tau_\infty)\), or for all \(n \in \mathbb{N}\) and \(\epsilon > 0\) there exits an interval \((t_0, t_1) \subset (\tau_\infty - \epsilon, \tau_\infty)\) such that \(X_{t_0} = x_2 + X_0\) and
\[
x_2 + X_0 \leq \inf_{t \in (t_0, t_1)} X_t \leq n \leq \sup_{t \in (t_0, t_1)} X_t.
\]
In both cases,
\[
X_t = X_{t_0} + \int_{t_0}^t B(s, X_s)ds + w_t - w_{t_0} \\
\leq x_2 + X_0 + \int_{t_0}^t h_2(s) (X_s + 1) ds + w_t - w_{t_0} \\
\leq x_2 + X_0 + \|w\|_\beta \tau_\infty + \int_0^{\tau_\infty} h_2(s) ds + \left( \sup_{s \in [0, \tau_\infty]} h_2(s) \right) \int_0^t X_s ds.
\]
where we use (A3) in the second inequality. It follows from Gronwall’s inequality that for all \(t \in (t_0, t_1)\) or \(t \in (t_0, \tau_\infty)\)
\[
X_t \leq \left( x_2 + X_0 + \|w\|_\beta \tau_\infty + \int_0^{\tau_\infty} h_2(s) ds \right) \exp \left\{ \left( t - t_0 \right) \sup_{s \in [0, \tau_\infty]} h_2(s) \right\} \\
\leq \left( x_2 + X_0 + \|w\|_\beta \tau_\infty + \int_0^{\tau_\infty} h_2(s) ds \right) \exp \left\{ \tau_\infty \sup_{s \in [0, \tau_\infty]} h_2(s) \right\}.
\]
Taking supremum of the left hand side in the above inequality: for all \(t \in (t_0, t_1)\) in the first case or for all \(t, \epsilon, n\) for the second case, the left hand side is infinite but the right hand side is a finite constant. This is a contradiction. Hence, \(\tau_\infty = \infty\).

Finally, we shall deal with the case that \(X_0 = 0\). For \(n \in \mathbb{N}\), let \(X_t^{[n]}\) be the solution of (3.1) with \(X_0 = 1/n\). For \(n, m \in \mathbb{N}\), \(n < m\), let \(\tau = \inf\{t \geq 0 \mid X_t^{[n]} = X_t^{[m]}\}\). By the uniqueness, \(X_t^{[n]} = X_t^{[m]}\) for all \(t \geq \tau\), or \(\tau = \infty\). It is clear that \(X_t^{[n]} > X_t^{[m]}\) if
Starting from any $X(3.1)$ which is positive. Thus, for all $t > 0$, the sequence $B(t, x)$ is non-increasing for $x \in (0, x_1)$, the following inequality follows from the monotone convergence theorem

$$
\lim_{n \to \infty} \int_0^{t \wedge \tau^{n_0}} B(s, X_s^{[n]}) ds = \int_0^{t \wedge \tau^{n_0}} B(s, X_s) ds.
$$

Taking into account that $X_t^{[n]}$ satisfies (3.1),

$$
X_{t \wedge \tau^{n_0}} = \int_0^{t \wedge \tau^{n_0}} B(s, X_s) ds + w_{t \wedge \tau^{n_0}} - w_0.
$$

Moreover, this inequality yields that

$$
\int_0^{t \wedge \tau^{n_0}} B(s, X_s) ds < \infty.
$$

Thus, $B(s, X_s) < \infty$ a.e. $s \in [0, t \wedge \tau^{n_0}]$. By (A2), $X_s > 0$ a.e. $s \in [0, t \wedge \tau^{n_0}]$. Starting from any $X_s > 0$ with $s \in [0, t \wedge \tau^{n_0}]$, there exists unique solution to (3.1) which is positive. Thus, $X_s > 0$ for all $s \in (0, t \wedge \tau^{n_0}]$. According to the proof above, $\{X_t\}_{t \in [0, t \wedge \tau^{n_0}]}$ can be extended to a solution for all $t > 0$ and $X_t > 0$ for all $t > 0$.

\[\Box\]

**Remark 3.1.** It is clear that $X_t$ is $\beta$-Hölder continuous on $[0, T]$ for all $T > 0$ if $X_0 > 0$. However, we should remark here that, the solution $X_t$ with $X_0 = 0$ cannot be $\beta$-Hölder continuous on interval contains $0$. Otherwise, there is $C > 0$ and $t_0 > 0$ such that $X_t \leq Ct^\beta$ for $t \in [0, t_0]$. Letting $\tau_{x_1} = \{t \geq 0 \mid X_t \geq x_1\}$, it follows from (3.1) and (A2) that

$$
X_t = X_0 + \int_0^t B(s, X_s) ds + w_t - w_0
$$

$$
\geq \int_0^t h_1(s) X_s^{-\alpha} ds - \|w\|_\beta t^\beta
$$

$$
\geq \left( \inf_{s \in [0,t]} h_1(s) \right) \int_0^t \frac{1}{C^\alpha} t^{1-\alpha} ds - \|w\|_\beta t^\beta
$$

$$
= \left( \inf_{s \in [0,t]} h_1(s) \right) \frac{t^{1-\alpha}}{C} - \|w\|_\beta t^\beta, \ t \leq \tau_{x_1}.
$$

Then, recalling that $\alpha > 1/\beta - 1$ implies $1 - \alpha \beta - \beta < 0$,

$$
C + \|w\|_\beta \geq \lim_{t \to 0^+} \frac{X_t + \|w\|_\beta t^\beta}{t^\beta} \geq \lim_{t \to 0^+} \frac{\left( \inf_{s \in [0,t]} h_1(s) \right) t^{1-\alpha-\beta}}{C} = \infty.
$$

According to this theorem, the stochastic equation (1.1) has a unique pathwise solution. Next, we shall study the Malliavin differentiability of $X_t$. 

Lemma 3.2. Assume (A1), (A2) and (A3) hold. Let $X_t$ be the solution of (1.1). Then for all $t > 0$, $X_t \in D_{[t_0, t]}^{1/2}$ with

$$D_sX_t = \sigma \exp \left\{ \int_s^t \nabla B(r, X_r) \, dr \right\} \mathbf{1}_{[0,t]}(s),$$

and the law of $X_t$ has density w.r.t. the Lebesgue measure on $\mathbb{R}$.

The proof just follows the line of [18 Theorem 3.3.], and the outline of the proof is presented here for the convenience of readers.

Proof. By (A1), we have

$$\frac{B(t, x) - B(t, y)}{x - y} \leq K_t, \quad t > 0, x \neq y,$$

which implies the derivative of $B(t, \cdot)$, denoted by $\nabla B(t, \cdot)$, is bounded from above by $K_t$. Let $\epsilon \in (0, 1)$, $h \in \mathcal{H}$ with $h_0 = 0$ and

$$X_t^\epsilon = X_0 + \int_0^t B(r, X_r)dr + \sigma B_t^H + \sigma \epsilon K_H e^{K_H^*}h(t).$$

Then

$$X_t^\epsilon - X_t = \int_0^t (B(r, X_t^\epsilon) - B(r, X_r))dr + \sigma \epsilon K_H e^{K_H^*}h(t)$$

$$= \int_0^t \nabla B \left( r, X_t^\epsilon \right) (X_t^\epsilon - X_r)dr + \sigma \epsilon K_H e^{K_H^*}h(t), \quad t > 0,$$

where $X_t^\epsilon = X_t + \xi^\epsilon_t(X_t^\epsilon - X_r)$ and $\xi^\epsilon_t \in (0, 1)$ depends on $s$ and $\epsilon$. This equality, along with (2.1) (see also [6] Lemma 2.1.9.],) implies

$$X_t^\epsilon - X_t = \sigma \epsilon \int_0^t \exp \left\{ \int_s^t \nabla B \left( r, X_t^\epsilon \right)dr \right\} (K_H e^{K_H^*}h)(ds)$$

$$= \sigma \epsilon \int_0^T K_H \left( \exp \left\{ \int_0^t \nabla B \left( r, X_t^\epsilon \right)dr \right\} \mathbf{1}_{[0,t]}(\cdot) \right) (s)K_H^*h(s)ds.$$ 

Since the continuity of $\nabla B(t, \cdot)$, (3.3) and $K_H^*h \in L^2([0,T])$, it follows from the dominated convergence theorem that the limit

$$\lim_{\epsilon \to 0^+} \frac{X_t^\epsilon - X_t}{\epsilon} = \sigma \int_0^T K_H \left( \exp \left\{ \int_0^t \nabla B \left( r, X_t \right)dr \right\} \mathbf{1}_{[0,t]}(\cdot) \right) (s)K_H^*h(s)ds$$

holds almost sure and in $L^2(\Omega)$. Consequently,

$$DX_t = \sigma \exp \left\{ \int_0^t \nabla B \left( r, X_t \right)dr \right\} \mathbf{1}_{[0,t]}(\cdot).$$

It is clear that $\|DX_t\|_{\mathcal{H}} > 0$, and $E\|DX_t\|_{\mathcal{H}}^2 < \infty$ follows from (3.3). Then the existence of density w.r.t. the Lebesgue measure follows from the classical result of Malliavin calculus, see e.g. [22 Theorem 2.1.2 or Theorem 2.1.3].
Next, we shall study the moment estimates of solutions to (1.1). To this end, we introduce the following assumption.

(A2') The condition (A2) holds. There exist $\theta > 0$ and $h_4 \in C([0, \infty), (0, \infty))$ such that
\[
B(t, x) \leq h_4(t)(1 + x + x^{-\theta}), \quad t \geq 0, x > 0.
\] (3.5) \[A2\]

It should be noted that $\theta \geq \alpha$ by (A2) and (3.5), and (A2') implies (A3). This assumption is used for positive moment estimate. To give the negative moment estimate, we introduce the following

(A3') there exists a $q > 0$ and a locally bounded nonnegative function $h_3$ such that
\[
(B(t, x))^- \leq h_3(t)(1 + x^q), \quad s \geq 0, x > 0
\] (3.6) \[A3\]

where $(B(t, x))^-$ denote the negative part of $B(t, x)$.

We first consider the negative moments for the solution to (1.1).

**Lemma 3.3.** Assume (A1)-(A3) and (A3'). Let $X_t$ be a solution to (1.1) with $X_0 > 0$.

(1) Suppose $\alpha = 1$. For $p \geq 1$ with
\[
h_1(s) \geq ((p + 1) \lor q)H s^{2H-1}e^H_0 K^*_su^p du, \quad s \in [0, T],
\] (3.7) \[inequ-h1-p-s\]

then
\[
\sup_{s \in [0, T]} \mathbb{E} X_s^{-p} < \infty.
\]

If (3.7) holds with $p$ replaced by $2(p + 2)$, then
\[
\mathbb{E} \sup_{s \in [0, T]} X_s^{-p} < \infty.
\]

(2) Suppose $\alpha > 1$. Then for all $p > 0$ and $T \geq 0$,
\[
\mathbb{E} \sup_{s \in [0, T]} X_s^{-p} < \infty.
\]

Proof. We first prove
\[
\sup_{s \in [0, T]} \mathbb{E} X_s^{-p} + \int_0^T \mathbb{E} X_s^{-p}ds < \infty,
\] (3.8) \[neg-int\]

where for $\alpha = 1$, we impose (3.7). In fact, due to the Hölder inequality, we only need to prove the claim for large $p$. Thus we assume that $p + 1 \geq q$. Since $X_t$ is $\beta$-Hölder continuous for $\beta < H$, applying Proposition 2.1, Proposition 2.2 and Lemma 3.2, we obtain that
\[
(X_t + \epsilon)^- = (X_0 + \epsilon)^- - p\int_0^t \frac{B(s, X_s)}{(\epsilon + X_s)^{p+1}}ds - \sigma p\int_0^t (\epsilon + X_s)^{-(p+1)}dB^H_s
\leq (X_0 + \epsilon)^- - p\int_0^t \frac{B(s, X_s)}{(\epsilon + X_s)^{p+1}}ds - \sigma p\int_0^t (\epsilon + X_s)^{-(p+1)}\delta B^H_s
\]
Consequently,

\[ + \sigma p(p + 1)\alpha H \int_0^t \frac{s^{2H-1}}{(\epsilon + X_s)^{p+2}} D_x X_s |s - r|^{2H-2}drds \]

\[ \leq (X_0 + \epsilon)^{-p} - \sigma \int_0^t \frac{B(s, X_s) X_s - \sigma^2(p + 1) H s^{2H-1} e^{\int_0^s K_2^+ du}}{(\epsilon + X_s)^{p+2}}ds \]

and

\[ -\frac{B(s, x)}{(\epsilon + x)^{p+2}} \leq -\frac{h_1(s)}{x^\alpha(\epsilon + x)^{p+2}} \mathbb{1}_{[x \leq \tilde{x}_1]} + \frac{h_3(s)(1 + x^q)}{(\epsilon + x)^{p+2}} \mathbb{1}_{[x \geq \tilde{x}_1]} \]

and

\[ -\frac{B(s, x) - \sigma^2(p + 1) H s^{2H-1} e^{\int_0^s K_2^+ du}}{(\epsilon + x)^{p+2}} \]

\[ \leq -\frac{h_1(s)x^{-(\alpha+1)} - \sigma^2(p + 1) H s^{2H-1} e^{\int_0^s K_2^+ du}}{(\epsilon + x)^{p+2}} \mathbb{1}_{[x \leq \tilde{x}_1]} \]

\[ + \frac{h_3(s)(1 + x^q)x + (p + 1) H s^{2H-1} e^{\int_0^s K_2^+ du}}{(\epsilon + x)^{p+2}} \mathbb{1}_{[x \geq \tilde{x}_1]} \]

\[ \leq -\frac{h_1(s)\tilde{x}_1^{-(\alpha+1)} - \sigma^2(p + 1) H s^{2H-1} e^{\int_0^s K_2^+ du}}{x^{p+2}} \mathbb{1}_{[x \leq \tilde{x}_1]} \]

\[ + (p + 1) H s^{2H-1} e^{\int_0^s K_2^+ du} \tilde{x}_1^{-(p+2)} + h_3(s) \left( \tilde{x}_1^{-(p+1)} + \tilde{x}_1^{-p+1} \right) \]

Since (3.7) and the definition of \( \tilde{x}_1 \), there exists \( C > 0 \) depending on \( \tilde{x}_1, p, q, \sigma \) such that

\[ (X_t + \epsilon)^{-p} \leq (X_0 + \epsilon)^{-p} + C \int_0^t (h_3(s) + s^{2H-1})ds - \sigma \int_0^t (\epsilon + X_s)^{-(p+1)} B_s^H. \]

Taking expectation and letting \( \epsilon \to 0 \), (3.8) is proved.

If \( \alpha > 1 \) or (3.7) holds with \( p \) replaced by \( 2(p + 2) \), then

\[ \sup_{[0,T]} \mathbb{E} X_t^{-2(p+2)} < \infty. \]

Consequently,

\[ \int_0^T \int_0^T \mathbb{E} (D_x X_t)^{-(p+2)} (D_u X_u)^{-(p+2)} |u - v|^{2H-2} |t - s|^{2H-2} du dv ds dt \]

\[ \leq \sigma^2 e^{2 \int_0^T K_2^+ du} \sup_{[0,T]} \mathbb{E} X_t^{-2(p+2)} \int_0^T \int_0^T |u - v|^{2H-2} |t - s|^{2H-2} du dv ds dt \]
Hence \( X^{-(p+1)} \in D_{[H]}^{1,2} \). By Proposition \(2.1\), Proposition \(2.2\) and Lemma \(3.2\) again, there is some \( C > 0 \) depending on \( \bar{x}_1, p, q, \sigma \) such that

\[
X_t^{-p} \leq X_0^{-p} + C \int_0^t (h_3(s) + s^{2H-1})ds - p \int_0^t X_s^{-(p+1)}dB_s^H.
\]

It follows from the maximal inequality of the Skorohod integral (see e.g. [22, Page 293] or Proposition \(2.2\)) that

\[
\left( \mathbb{E} \sup_{s \in [0,t]} \left| \int_0^t X_s^{-(p+1)}dB_s^H \right|^2 \right)^{\frac{1}{2}} \\
\leq C \left( \int_0^t \mathbb{E}X_s^{-2(p+1)}ds + \mathbb{E} \int_0^t \left( \int_0^s (p + 1) \pi X_s^{-\frac{p+2}{H}} |D_rX_s| \pi dr \right)^{2H}ds \right)^{\frac{1}{2}} \\
\leq C_{p,H}(1 + t^H)e^{\int_0^t K_d du} \left( \int_0^t \mathbb{E}X_s^{-2(p+1)}ds \right)^{\frac{1}{2}}.
\]

Then

\[
\mathbb{E} \sup_{s \in [0,T]} X_s^{-p} \leq X_0^{-p} + C \int_0^T (h_3(s) + s^{2H-1})ds \\
+ C_{p,H}(1 + T^H)e^{\int_0^T K_d du} \left( \int_0^T \mathbb{E}X_s^{-2(p+1)}ds \right)^{\frac{1}{2}},
\]

which implies the required conclusion.

\[\square\]

If \((3.5)\) holds with \( B(t, x) \) replaced by \(|B(t, x)|\), then we can obtain moment estimates of \(|X|_{0,T,\infty}\) by applying \([12\) Theorem 3.1\] to \(X_t^{1+\theta}\). However, if \((3.6)\) holds, that is we allow that \(|B(t, x)|\) has super-linear growth at infinite, then the following lemma can not be covered by \([12\). For \(g \in C([0,T], \mathbb{R}^d)\), we denote by \(M_{g,T}(-)\) the modulus of continuity of \(g\) on \([0,T]\), i.e.

\[
M_{g,T}(h) = \sup_{0 \leq s,t \leq T, |s-t| \leq h} |g_t - g_s|.
\]

\[\text{lem-psi-mom}\]

**Lemma 3.4.** Assume \((A1), (A2') and (A3')\). Let \(\{X_t\}_{t \geq 0}\) be a solution of \((1.1)\) with \(X_0 > 0\).

If \(\alpha > 1\), then for any \(p > 0\) and \(T > 0\), we have

\[
\mathbb{E}\|X\|_{0,T,\infty}^p < \infty, \tag{3.9} \text{inequ-11}
\]

and

\[
\left( \mathbb{E}M_{X,T}^p + \mathbb{E}M_{X^{-1},T}^p \right)^{\frac{1}{2}} \leq C_{p,T} \left( h + h^H \sqrt{\log(1 + 1/h)} \right). \tag{3.10} \text{inequ-22}
\]

If \(\alpha = 1\), then for \(p > 0\), there exists \(T > 0\) such that \((3.9)\) and \((3.10)\) hold.
Proof. Suppose $\alpha > 1$. We first prove that

$$\mathbb{E} \left( X_t^p + \int_0^t X_s^p ds \right) < \infty, \ t \geq 0, p > 0. \tag{3.11}$$

In fact, by chain rule, Lemma 3.2 and Proposition 2.2 for any $n > 0$

$$\frac{nX_t^p}{n + X_t^p} = \int_0^t \frac{pn^2X_s^{p-1}}{(n + X_s^p)^2} B_s d s + \int_0^t \frac{\sigma pn^2X_s^{p-1}}{(n + X_s^p)^2} dB_s^H$$

$$\leq \int_0^t \frac{pn^2h_4(s)X_s^{p-1}(1 + X_s + X_s^{-\theta})}{(n + X_s^p)^2} ds + \int_0^t \frac{\sigma pn^2X_s^{p-1}}{(n + X_s^p)^2} dB_s^H$$

$$+ \int_0^t \int_0^s \frac{\alpha \sigma pn^2X_s^{-2}(n(p - 1) + (p + 1)X_s^p)}{(n + X_s^p)^3} D_r X_s | r - s |^{2H-2} dr ds$$

$$\leq \int_0^t \left( \frac{2pnh_4(s)X_s^p}{n + X_s^p} + ph_4(s)(1 + X_s^{-\theta}) \right) ds + \int_0^t \frac{\sigma pn^2X_t^{p-1}}{(n + X_t^p)^2} dB_t^H$$

$$+ C_{H,p,K,\alpha} \int_0^t \left( h_4(s)(1 + X_s^{-\theta}) + (p - 1)^{2H-1} X_s^2 \right) ds,$$

where $C_{t,p,K,\alpha}$ is locally bounded in $t$. Then it follows from the Gronwall lemma and Lemma 3.3 that

$$\mathbb{E} \frac{nX_t^p}{n + X_t^p} \leq C_{t,p,K,\alpha} e^{\int_0^t \left( h_4(s)(1 + X_s^{-\theta}) + (p - 1)^{2H-1} X_s^2 \right) ds} < \infty,$$

which implies (3.11) by letting $n \to \infty$.

Next, we shall prove that

$$\mathbb{E} \sup_{s \in [0,t]} X_s^p < \infty, \ t \geq 0, p > 0. \tag{3.12}$$

Indeed, by chain rule, (3.11) and Lemma 3.2 we have $X_t^{p-1} \in \mathbb{D}^{1,2}_{[t]}$ and

$$X_t^p = \int_0^t X_s^{p-1} B(s, X_s) ds + \int_0^t X_s^{p-1} dB_s^H$$

$$\leq \int_0^t h_4(s)(X_s^{p-1} + X_s^p + X_s^{p-\theta-1}) ds + \int_0^t X_s^{p-1} dB_s^H$$

$$+ \sigma (p - 1) \alpha H \int_0^t \int_0^r X_s^{p-s} D_r X_s | r - s |^{2H-2} dr ds$$

$$\leq C \int_0^t h_4(s)(1 + X_s^p) ds + |\sigma| \int_0^t X_s^{p-1} dB_s^H$$

$$+ C_{H,\alpha,p} \int_0^t X_s^{p-2} | s |^{2H-1} ds. \tag{3.13}$$
The maximal inequality of Skorohod integral yields that the following inequality holds

\[
\left( \mathbb{E} \sup_{s \in [0,t]} \left| \int_0^s X_r^{p-1} \delta B^H_r \right| \right)^{\frac{1}{2}} \leq C \left( \int_0^t \mathbb{E} X_{r}^{2(p-1)} dr + \mathbb{E} \int_0^t \left( \int_0^{r} (p-1) \frac{1}{X_r^{p-2}} |D_u X_r| \frac{d| Du X_r|}{X^p} \right)^{2H} dr \right)^{\frac{1}{2}} \\
\leq C_{H,p}(1 + t^H) (\int_0^t \mathbb{E} X_{r}^{2(p-1)} dr)^{\frac{1}{2}}. 
\]

(3.14)

Combining (3.13) and (3.14) with (3.11), we get (3.12).

Next, we shall prove the estimates of modulus of continuous. By (A2') and (A3'), we have

\[
|B(s,x)| \leq (h_3 \vee h_4)(s)(1 + x^q + x^{-\theta}) \equiv \tilde{h}(s)(1 + x^q + x^{-\theta}).
\]

Then for any \(t > s \geq 0\),

\[
|X_t - X_s| \leq \int_s^t |B(r, X_r)| dr + |\sigma(B_t^H - B_s^H)| \\
\leq \int_s^t \tilde{h}(r) \left( 1 + X_r^q + X_r^{-\theta} \right) dr + |\sigma| \mathbb{M}_{B^H,T}((t - s)) \\
\leq \left( \sup_{s \leq r \leq t} \tilde{h}(r) \left( 1 + \|X\|_{q,t,\infty}^q + \|X^{-1}\|_{\theta,t,\infty}^{-\theta} \right) (t - s) \right) + |\sigma| \mathbb{M}_{B^H,T}((t - s)), 
\]

which implies for any \(p > \frac{1}{1-\beta}\),

\[
\mathbb{E} M_{X,T}(h)^p \leq C_{T,p} \left( 1 + \mathbb{E} \|X\|_{q,T,\infty}^p + \mathbb{E} \|X^{-1}\|_{\theta,T,\infty}^{-\theta} \right) h^p + C_p |\sigma|^p \mathbb{E}(\mathbb{M}_{B^H,T}(h))^p.
\]

It follows from \(\alpha > 1\), Lemma 3.3 and the modulus of continuity of \(B^H\) (see e.g. [25, Theorem 4.2] or [20, Theorem 6.3.3]) that

\[
\mathbb{E} M_{X,T}(h)^p \leq C_{p,T} \left\{ h^p + h^{pH} \left( \log \left( 1 + \frac{1}{h} \right) \right)^{\frac{pH}{2}} \right\}.
\]

By the H"{o}lder inequality and the following inequality

\[
\sup_{|t-s| \leq h, s,t \leq T} |X_t^{-1} - X_s^{-1}| \leq \sup_{0 \leq s,t \leq T} \left( \frac{1}{X_t X_s} \right) \sup_{0 \leq \xi \leq T} |X_t - X_s| \\
\leq \left( \sup_{0 \leq \xi \leq T} X_\xi^{-2} \right) M_{X,T}(h),
\]

we get the moment estimate of the modulus of continuity of \(X^{-1}\).

For \(\alpha = 1\), one can repeat the argument for \(\alpha > 1\), and takes note that negative power moments in (1) of Lemma 3.3 hold for small \(T\) depending on \(p\).
4 Numerical approximation

In this section, we shall consider the numerical approximation of the following equation
\[ dX_t = B(X_t)dt + \sigma dB_t^H, \quad X_0 > 0. \] (4.1)

The drift term \( B(\cdot) \) satisfies \((A1), (A2')\) and \((A3')\), and all these conditions are independent of time. To ensure the positivity of the numerical scheme, we shall use the backward Euler method as in [15]. Moments estimates obtained in the previous section will be used here.

In addition to \((A1), (A2')\) and \((A3')\), we shall impose the following assumptions.

\((H1)\) There is \( h_0 > 0 \) such that the following equation
\[ U(x) + c \equiv B(x)h - x + c = 0 \]
has a unique positive solution for any \( c \in \mathbb{R} \) and \( 0 < h < h_0 \).

\((H2)\) The drift term \( B \in C^2(\mathbb{R}) \), and there are nonnegative constants \( p_1, p_2 \) and \( C > 0 \) such that
\[ |\nabla B(x)| + |\nabla^2 B(x)| \leq C(1 + x^{p_1} + x^{-p_2}), \quad x > 0. \] (4.2)

Sufficient conditions to ensure \((H1)\) are that for \( 0 < h < h_0 \)
\[ \lim_{x \to 0^+} U(x) = \infty, \quad \lim_{x \to \infty} \nabla U(x) = -\infty \]
and \( \nabla U(x) < 0 \) or \( \nabla^2 U(x) \neq 0 \).

Let \( T > 0, N \in \mathbb{N} \) such that \( h := \frac{T}{N} < h_0, t_n = nh, \) and let \( \Delta B_{n+1}^H = B_{t_{n+1}}^H - B_{t_n}^H \).
Since \( X_0 > 0 \), we define
\[ X_{n+1} = X_n + B(X_{n+1})h + \sigma \Delta B_{n+1}^H, \quad n \in \mathbb{N} \cup \{0\}. \] (4.3)

Due to \((H1)\), the equation (4.3) has a unique positive solution \( X_{n+1}, n \geq 0 \). Let
\[ X_t^h = \frac{t_{n+1} - t}{t_{n+1} - t_n} X_n + \frac{t - t_n}{t_{n+1} - t_n} X_{n+1}, \quad t_n \leq t \leq t_{n+1}. \]

For a random variable \( \xi \), we denote \( \|\xi\|_p = (\mathbb{E}|\xi|^p)^{\frac{1}{p}} \). Our result on numerical approximation of (4.1) reads as follows.

**Theorem 4.1.** Assume \((A1), (A2''), (A3''), (H1)\) and \((H2)\) hold. Let \( h < h_0 \land K^{-1} \), and let \( X_{n+1} \) be defined as above.

(1) If \( \alpha > 1 \), then
\[ \mathbb{E} \sup_{0 \leq n \leq N-1} |X_{t_{n+1}} - X_{n+1}|^p \leq C_{T,X_0,\theta,H,p,B} h^{pH}, \] (4.4)

(2) If \( \alpha = 1 \), then for \( p > 0 \), there is \( T > 0 \) such that (4.4) and (4.5) hold.
Proof. We only prove the claim for $\alpha > 1$. For $\alpha = 1$, the negative power moments estimates hold for $T$ depending on the given $p > 0$ (see Lemma 3.3). Then for $T$ small enough, the arguments for $\alpha > 1$ work well in the small interval, and the claim can be obtained.

(1) We first prove (4.4). It follows from the definition of $X_{n+1}$ and the mean value theorem that that

\[ X_{tn+1} - X_{n+1} = X_{tn} - X_n + \int_{tn}^{tn+1} B(X_s)ds - B(X_{n+1})h \]

\[ = X_{tn} - X_n - \int_{tn}^{tn+1} (B(X_{n+1}) - B(X_s)) ds + (B(X_{tn+1}) - B(X_{n+1})) h \]

\[ = X_{tn} - X_n + \nabla B(X_{n+1} + \xi_{n+1}(X_{tn+1} - X_{n+1}))h(X_{tn+1} - X_{n+1}) \]

\[ - \int_{tn}^{tn+1} \left( \int_s^{tn+1} \nabla B(X_r)B(X_r)dr + \sigma \int_s^{tn+1} \nabla B(X_r)dB^H_r \right) ds, \]

where $\xi_{n+1} \in (0, 1)$. By (A1), $\nabla B(x) \leq K$ for all $x > 0$. Then, letting

\[ \Delta_{n+1} = \nabla B(X_{n+1} + \xi_{n+1}(X_{tn+1} - X_{n+1})), \]

we have

\[ 1 - \Delta_{n+1}h \geq 1 - Kh > 0 \]

holds for small $h$. On the other hand, it follows from the Fubini theorem that

\[ \int_{tn}^{tn+1} \left( \int_s^{tn+1} \nabla B(X_r)B(X_r)dr + \sigma \int_s^{tn+1} \nabla B(X_r)dB^H_r \right) ds \]

\[ = \int_{tn}^{tn+1} (r - tn) \nabla B(X_r)B(X_r)dr + \sigma \int_{tn}^{tn+1} (r - tn) \nabla B(X_r)dB^H_r. \]

Substituting this into (4.4), letting $\Upsilon_{n+1} = X_{tn+1} - X_{n+1}$ and

\[ Q_{n+1} = - \int_{tn}^{tn+1} (r - tn) \nabla B(X_r)B(X_r)dr - \sigma \int_{tn}^{tn+1} (r - tn) \nabla B(X_r)dB^H_r, \]

we get that

\[ \Upsilon_{n+1} = (1 - \Delta_{n+1}h)^{-1} \Upsilon_n + (1 - \Delta_{n+1}h)^{-1}Q_{n+1}. \]

Consequently,

\[ \Upsilon_{n+1} = \sum_{i=1}^{n+1} Q_i \prod_{k=i}^{n+1} (1 - \Delta_k h)^{-1} =: \sum_{i=1}^{n+1} Q_i \rho_i. \]

Next, we shall estimate the right hand side of the above equality. Since

\[ \prod_{k=i}^{n+1} (1 - \Delta_k h)^{-1} \leq (1 - Kh)^{-n+1} \leq e^{n \log \frac{1}{1-Kh}} \leq e^{\frac{nKh}{1-Kh}} = e^{\frac{KT}{1-Kh}}, \]

(4.8) [Inequ-de-1]
it follows from (A3') that
\[ E \sup_{1 \leq n \leq N} |Y_n|^p \leq C_{T,K}E \left( \sum_{i=1}^{N} |Q_i| \right)^p. \]

By the definition of \( Q_i \), there are two integrals to be estimated. For the ordinary integral, it follows from (A2'), (A3') and (H2) that
\[
\left\| \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} (t_{i-1} - r) \nabla B(X_r) B(X_r) dr \right\|_p \leq \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \| \nabla B(X_r) B(X_r) \|_p dr \\
\leq C_T h \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \| 1 + X_r^{-(\theta + p_1)} + X_r^{p_1 (1 + p_2)} \|_p dr \\
\leq C_{T,\theta,X_0,p_1,p_2,q,K} h. \tag{4.9} \]

For the stochastic integration, by [22, Theorem 5.2.3]
\[
\sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} |r - t_{i-1}| \nabla B(X_r) dB_r^{H_t} \\
\leq \sum_{i=1}^{N} \int_{0}^{T} \int_{0}^{r} |t - t_{i-1}| \nabla B(X_t) \mathbb{1}_{(t_{i-1}, t_i)} \delta B_r^{H_t} dr \\
\quad + \sum_{i=1}^{N} \int_{0}^{T} \int_{0}^{r} (r - t_{i-1}) \nabla^2 B(X_r) D_s X_r |s - r|^{2H - 2} \mathbb{1}_{(t_{i-1}, t_i)}(r) \mathbb{1}_{(0,t)}(s) ds dr \\
\quad = \sum_{i=1}^{N} \int_{0}^{T} \int_{0}^{r} (r - t_{i-1}) \nabla B(X_r) \mathbb{1}_{(t_{i-1}, t_i)}(r) \delta B_r dr \\
\quad + \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \int_{0}^{r} (r - t_{i-1}) |\nabla^2 B(X_r)| |D_s X_r| |s - r|^{2H - 2} ds dr \\
\quad =: I_1 + I_2. \tag{4.10} \]

For \( I_2 \), it follows from (A3') that
\[
\| I_2 \|_p \leq \left\| \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \int_{0}^{r} (r - t_{i-1}) |\nabla^2 B(X_r)| |D_s X_r| |s - r|^{2H - 2} ds dr \right\|_p \\
\leq C h \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \left\| X_r^{-p_1} + X_r^{p_2} \right\|_p \int_{0}^{s} |K du| |s - r|^{2H - 2} ds dr \\
\leq C_{T,K,H} \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \left\| X_r^{-p_1} + X_r^{p_2} \right\|_p |r|^{2H - 1} dr \\
\leq C_{T,\theta,H,X_0,K} h. \tag{4.11} \]

For \( I_1 \), it follows from Minkowski’s inequality that
\[
\| I_1 \|_p = \left\| \sum_{i=1}^{N} \int_{0}^{T} (r - t_{i-1}) \nabla B(X_r) \mathbb{1}_{(t_{i-1}, t_i)} \delta B_r^{H_t} \right\|_p.
\]
By \[22\) Proposition 1.5.8\],
\[
\mathbb{E} \left| \int_0^T (r - t_{i-1}) \nabla B(X_r) 1_{(t_{i-1}, t_i]} \delta B^H \right|^p
\]
\[
\leq C_p \left( \int_{(t_{i-1}, t_i]}^T (r - t_{i-1})(s - t_{i-1}) |\mathbb{E} \nabla B(X_r) | |\mathbb{E} \nabla B(X_s) | r - s |2H - 2| dr ds \right)^{\frac{p}{2}}
\]
\[
+ \mathbb{E} \left( \int_{t_{i-1}}^{t_i} \int_{0}^{s} (r - t_{i-1})(s - t_{i-1}) \nabla^2 B(X_r) \left| \nabla^2 B(X_s) \right| |D_u X_r| |D_v X_s| (u - v)^{2H - 2} |r - s|^{2H - 2} du dv ds dr \right)^{\frac{p}{2}}
\]
\[
\leq C_{T, p_1, p_2, K} h^p \left( \int_{(t_{i-1}, t_i]}^T |r - s|^{2H - 2} dr ds \right)^{\frac{p}{2}}
\]
\[
+ C_{T, K, H} h^p \left( \int_{(t_{i-1}, t_i]}^T \int_{0}^{s} (1 + X_r^{p_1} + X_s^{p_2}) \left( 1 + X_r^{p_1} + X_s^{p_2} \right)^{2H - 2} \right)^{\frac{p}{2}}
\]
\[
\times \left| u - v \right|^{2H - 2} |r - s|^{2H - 2} du dv ds dr \right)^{\frac{p}{2}}
\]
\[
\leq C_{T, X_0, H, K} h^{p + H} + C_{T, X_0, H, p_1, p_2, p} h^p \left( \int_{(t_{i-1}, t_i]}^T \int_{0}^{T} |u - v|^{2H - 2} |r - s|^{2H - 2} du dv ds \right)^{\frac{p}{2}}
\]
\[
\leq C_{T, X_0, H, p_1, p_2, p} h^{p + H}.
\]
Thus
\[
\| I_1 \|_p \leq C_{T, X_0, p, H, K} \sum_{i=1}^{N} h^{1 + H} \leq C_{T, X_0, p, H, K} h^H .
\]
(4.12) Inequ-11

Substituting (4.9), (4.10), (4.11) and (4.12) into (4.7), we obtain
\[
\mathbb{E} \sup_{0 \leq n \leq N - 1} |Y_{n+1}|^p \leq C_{T, X_0, q, H, p, q, K, p_1, p_2} h^{pH}.
\]

(2) For \( t \in [t_n, t_{n+1}] \),
\[
X_t - X_t^h = \left( -\frac{t - t_n}{h} X_{t_{n+1}} - \frac{t_{n+1} - t}{h} X_t + \frac{t - t_n}{h} (X_{t_{n+1}} - X_{t_{n+1}}) \right)
\]
\[
+ \frac{t_{n+1} - t}{h} (X_{t_n} - X_t)
\]
\[
\leq |X_{t_{n+1}} - X_t| + |X_{t_n} - X_t| + |Y_{n+1}| + |Y_n|
\]
\[
\leq 2 \int_{t_n}^{t_{n+1}} |B(X_r)| dr + |B^H_{t_{n+1}} - B^H_t| + |B^H_t - B^H_n|
\]
\[
+ |Y_{n+1}| + |Y_n|
\]
\[
\leq C \int_{t_n}^{t_{n+1}} \left( X_r^{q'1} + X_r^{-q} \right) dr + 2M_{H, T}(h) + |Y_{n+1}| + |Y_n|
\]
\[ \leq C \left( \lVert X \rVert_{0,T,\infty}^q h + \left( \int_0^T X_t^{-\theta} \frac{d\theta}{r} \right)^{1-H} h^H \right) \\
+ 2M_{B^H,T}(h) + 2 \sup_{1 \leq n \leq N} |Y_n| \\
= : I_1 + I_2 + I_3. \]

Since \( I_1, I_2 \) and \( I_3 \) are independent of \( n \) and \( t \), we have

\[ \mathbb{E} \sup_{t \in [0,T]} \lvert X_t - X_t^h \rvert^p \leq 3^{p-1} (\mathbb{E}I_1^p + \mathbb{E}I_2^p + \mathbb{E}I_3^p). \]

It follows from Lemma 3.3 and Lemma 3.4 that

\[ \mathbb{E}I_1^p \leq C_{K,q,T,X_0,p,\theta,H} h^{pH}. \]

The inequality (4.4) yields that

\[ \mathbb{E}I_3^p = 2^p \mathbb{E} \sup_{1 \leq n \leq N} |Y_n|^p \leq C_{K,q,T,X_0,p,\theta,H} h^{pH}. \]

The modulus of continuity of \( B^H \) (see e.g. [25, Theorem 4.2] or [20, Theorem 6.3.3]) implies that there is a constant \( C_{T,p} > 0 \) such that

\[ \mathbb{E}I_2^p = 2^p \mathbb{E}M_{B^H,T}(h) \leq C_{T,p} h^{pH} (\log(1 + 1/h))^{p/2}, \]

Therefore,

\[ \mathbb{E} \sup_{t \in [0,T]} \lvert X_t - X_t^h \rvert^p \leq C_{K,q,T,X_0,p,\theta,H} h^{pH} (\log(1 + 1/h))^{p/2}. \]

Some concrete models can be transformed to (4.1), see Example 4.1 and Example 4.2 for instance. The following corollary is crucial to getting the numerical approximation of them.

\textbf{Corollary 4.2.} Assume the hypotheses of Theorem 4.1 hold.

(1) If \( \alpha > 1 \), then for any \( l > 0 \),

\[ \left( \mathbb{E} \sup_{t \in [0,T]} \lvert X_t^l - (X_t^h)^l \rvert^p \right)^{\frac{1}{p}} \leq C h^{H(\lceil \alpha \rceil)} (\log(1 + 1/h))^{\lceil \alpha \rceil}, \quad (4.13) \]

for any \( l \in (0, \alpha] \),

\[ \left( \mathbb{E} \sup_{t \in [0,T]} \lvert X_t^{1-l} - (X_t^h)^{1-l} \rvert^p \right)^{\frac{1}{p}} \leq C_p h^{(2H-1)(\lceil \alpha \rceil)} (\log(1 + 1/h))^{\lceil \alpha \rceil}, \quad (4.14) \]

(2) If \( \alpha = 1 \), then for \( l > 0 \) and \( p > 0 \), there is \( T > 0 \) such that (4.13) holds; for \( l \in (0, 1] \) and \( p > 0 \), there is \( T > 0 \) such that (4.14) holds.
Proof. For $l \in (0, 1]$, it follows from Lemma 3.4 that
\[
\left( \mathbb{E} \sup_{t \in [0,T]} |X_t^l - (X_t^h)^l|^p \right)^{\frac{1}{p}} \leq \left( \mathbb{E} \sup_{t \in [0,T]} |X_t - X_t^h|^p \right)^{\frac{1}{p}} \leq \left( \mathbb{E} \sup_{t \in [0,T]} |X_t - X_t^h|^p \right)^{\frac{1}{p}} \leq C h^{lH} \left( \log(1 + \frac{1}{h}) \right)^{\frac{1}{2}}.
\]

For $l > 1$,
\[
\left( \mathbb{E} \sup_{t \in [0,T]} |X_t^l - (X_t^h)^l|^p \right)^{\frac{1}{p}} \leq \left( \mathbb{E} \sup_{t \in [0,T]} \left( X_t^{p(l-1)} \vee (X_t^h)^{p(l-1)} \right) |X_t - X_t^h|^p \right)^{\frac{1}{p}} \leq \left( \mathbb{E} \sup_{t \in [0,T]} \left( X_t^{2p(l-1)} \vee (X_t^h)^{2p(l-1)} \right) \right)^{\frac{1}{2}} \left( \mathbb{E} \sup_{t \in [0,T]} |X_t - X_t^h|^{2p} \right)^{\frac{1}{2p}} \leq C_{p,T} h^H \left( \log(1 + \frac{1}{h}) \right)^{\frac{1}{2}}.
\]

Hence, we have proved our first claim.

To consider the negative power approximation, we first give an estimate of $X_{n-1}$. By (4.3), there is positive constant $C$ which is independent of $n, h$ such that
\[
C (X_{n+1}^\alpha h - (X_{n+1}^q + 1)h) \leq B(X_{n+1})h \\
\leq |X_{n+1} - X_{n+1}| + |X_{n+1} - X_{n}| + |X_{n} - X_{n}| + |\sigma||B_{n+1}^H|.
\]

Then
\[
C \sup_{1 \leq n \leq N} X_{n-\alpha} \leq C \sup_{1 \leq n \leq N} (X_n^q + 1) + \frac{2}{h} \sup_{1 \leq n \leq N} |X_n - X_{n}| \\
+ \frac{1}{h} \sup_{1 \leq n \leq N} |X_{n} - X_{n}| + \frac{|\sigma|}{h} M_{B_{n+1}}(h) \\
\leq C \sup_{1 \leq n \leq N} (X_n^q + 1) + \frac{2}{h} \sup_{1 \leq n \leq N} |X_n - X_{n}| \\
+ \frac{1}{h} M_{X,T}(h) + \frac{|\sigma|}{h} M_{B_{n+1}}(h). \quad (4.15) \quad \text{add-ineq}
\]

Since (4.11), it is clear that for all $p > 0$, we have $\mathbb{E} \sup_{1 \leq n \leq N} (X_n^p) < \infty$. By Theorem 4.11
\[
\mathbb{E} \sup_{1 \leq n \leq N} |X_n - X_{n}|^p \leq h^{H_H} \left( \log(1 + \frac{1}{h}) \right)^{\frac{1}{2}}.
\]

It follows from Lemma 3.4 that
\[
\mathbb{E} M_{X,T}^p(h) \leq h^{H_H} \left( \log(1 + \frac{1}{h}) \right)^{\frac{1}{2}}.
\]
Combining these with (4.15), we get that

\[
E \sup_{1 \leq n \leq N} X_n^{-p \alpha} \leq C \left( 1 + h^{(H-1)p} \left( \log(1 + \frac{1}{h}) \right)^\frac{p}{2} \right).
\] (4.16) \[E X_n^{-p} \]

Then for \( l \in [1, \alpha] \), it follows from (4.16) that

\[
\left( E \sup_{t \in [0,T]} \left| X_t^{-l} - (X_t^h)^{-l} \right|^p \right)^\frac{1}{p} \leq \left( E \sup_{t \in [0,T]} \left| X_t^l - (X_t^h)^l \right|^p \right)^\frac{1}{p} \leq \left( E \sup_{t \in [0,T]} \left| X_t - (X_t^h) \right|^p \right)^\frac{1}{p}
\]

\[
\leq \left( E \sup_{t \in [0,T]} X_t^{-3pl} \left( E \sup_{t \in [0,T]} (X_t^h)^{-3pl} \right) \right)^\frac{1}{p} \leq C_{p,T} \left( 1 + h^{H-1} \left( \log(1 + \frac{1}{h}) \right)^\frac{1}{2} \right) h^H \left( \log(1 + \frac{1}{h}) \right)^\frac{1}{2} \leq C_{p,T} h^{2H-1} \log(1 + \frac{1}{h}).
\]

For \( l < 1 \),

\[
\left( E \sup_{t \in [0,T]} \left| X_t^{-l} - (X_t^h)^{-l} \right|^p \right)^\frac{1}{p} \leq \left( E \sup_{t \in [0,T]} \left| X_t^{-1} - (X_t^h)^{-1} \right|^p \right)^\frac{1}{p} \leq C_{p,T} h^{(2H-1)l} \left( \log(1 + \frac{1}{h}) \right)^l.
\]

Combining these two cases together, we prove our second conclusion.

\[\square\]

**Remark 4.1.** If \( \phi \) is a continuous function on \((0, \infty)\) such that

\[
|\phi(x) - \phi(y)| \leq C|x - y| \quad \text{or} \quad |\phi(x) - \phi(y)| \leq C|x^l - y^l|
\]

for \( l \) as in Corollary 4.2 and some \( C > 0 \). Then we can approximate \( \phi(X_t) \) by \( \phi(X_t^h) \).

For \( \alpha = 1 \), the convergence of the backward Euler scheme for C-I-R model driven by fractional Brownian motion has been obtained in [13]. Theorem 4.1 and Corollary 4.2 can also be applied to C-I-R model, and stronger convergence can be obtained for some \( p > 0 \) and some small \( T > 0 \) depending on \( p \). To get more specific and sharp dependencies between \( p \) and \( T \), one can follows the proof of [13, Theorem 4.1] and Theorem 4.1.

Finally, we apply our results to the two examples introduced in the introduction.

**Example 4.1.** We consider the numerical simulation of the following equation

\[
dY_t = (a_1 - a_2 Y_t)dt + \sigma Y_t^\gamma dB_t^H, X_0 > 0
\] (4.17) \[\text{equ-Y-1}\]
with $\gamma \in (\frac{1}{2}, 1)$, $a_1 > 0$, $a_2 \in \mathbb{R}$ and $\sigma \neq 0$. To study this equation, we consider

$$dX_t = (1 - \gamma)\left(a_1 X_t^{-\frac{\gamma}{1-\gamma}} - a_2 X_t\right)dt + \sigma(1 - \gamma)dB_t^H, X_0 = Y_0^{1-\gamma}.$$

Setting $B(x) = (1 - \gamma)a_1x^{-\frac{\gamma}{1-\gamma}} - a_2(1 - \gamma)x$, it is clear that (A1), (A2’) and (A3) hold with $K = -a_2$, $\theta = \alpha = -\frac{\gamma}{1-\gamma}$, and $q = 1_{\{a_2 > 0\}}$. Then this equation has a unique solution by applying Theorem 3.1. Moreover, it follows from the chain rule that $Y_t = X_t^{-\frac{\gamma}{1-\gamma}}$ and (4.17) has a uniqueness solution. It is clear that there is $h_0 > 0$ such that $1 + a_2(1 - \gamma)h > 0$ for $h \in (0, h_0)$. Then for all $c \in \mathbb{R}$, the equation

$$B(x)h - c \equiv (1 - \gamma)a_1 x^{-\frac{\gamma}{1-\gamma}}h - (1 + a_2(1 - \gamma)h)x + c = 0$$

has a unique positive solution. It follows from Corollary 4.3 that

$$\left(\mathbb{E} \sup_{t \in [0,T]} |Y_t - (X_t^h)^{-\frac{1}{1-\gamma}}|^p\right)^{\frac{1}{p}} \leq Ch^H (\log(1 + 1/h))^{\frac{1}{2}}.$$

**EXAMPLE 4.2.** In this example, we investigate the nonlinear Aït-Sahalia-type interest rate model:

$$dY_t = (a_{-1}Y_t^{-1} - a_0 + a_1Y_t - a_2Y_t^p)dt + \sigma Y_t^p dB_t^H, Y_0 > 0, \quad (4.18)$$

Aït-Sa

with $r + 1 > 2\rho$ and $r \geq 2 \wedge \rho + 1 > 2$ and $a_i > 0, i = -1, 0, 1, 2$. To study (4.18), we consider

$$dX_t = (\rho - 1)\left(a_2 X_t^{-\frac{\rho - p}{1-\rho}} - a_1 X_t + a_0 X_t^{\frac{\rho - p}{1-\rho}} - a_{-1} X_t^{-\frac{\rho}{1-\rho}}\right)dt$$

$$+ (1 - \rho)\sigma dB_t^H, X_0 = Y_0^{1-\rho}. \quad (4.19)$$

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Set

$$B(x) = (\rho - 1)\left(a_2 x^{-\frac{\rho - p}{\rho - 1}} - a_1 x + a_0 x^{\frac{\rho - p}{\rho - 1}} - a_{-1} x^{-\frac{\rho}{\rho - 1}}\right)$$

$$\equiv b_1 x^{-\frac{\rho - p}{\rho - 1}} - b_2 x + b_3 x^{\frac{\rho - p}{\rho - 1}} - b_4 x^{-\frac{\rho}{\rho - 1}}.$$

Since $\frac{\rho - p}{\rho - 1} > 1$, it is clear that (A1), (A2’) and (A3’) hold with $\theta = \alpha = \frac{\rho - p}{\rho - 1}$, $q = \frac{\rho + 1}{\rho - 1}$ and some constant $K$. Then this equation has a unique solution, and so does (4.18). Moreover $Y_t = X_t^{-\frac{\rho}{\rho - 1}}$. It is clear by $\frac{\rho - p}{\rho - 1} > 1$ and $\frac{\rho + 1}{\rho - 1} > \frac{\rho}{\rho - 1}$ that for $h > 0$

$$\lim_{x \to 0^+} (B(x)h - x) = +\infty, \quad \lim_{x \to +\infty} (B(x)h - x) = -\infty.$$

On the other hand,

$$\nabla B(x)h - 1 = -\frac{b_1(r - \rho)h}{\rho - 1} x^{-\frac{\rho + 1}{\rho - 1}} - (b_2h + 1)$$

$$+ \frac{b_3ph}{\rho - 1} x^{\frac{1}{\rho - 1}} - \frac{b_4(\rho + 1)h}{\rho - 1} x^{\frac{\rho}{\rho - 1}}.$$
Then for $0 < h < \frac{4(\rho-1)b_4(\rho+1)}{b_3 \rho^2}$, we have

$$
\frac{(b_4 \rho)^2 h^2}{(\rho - 1)^2} - 4 \left( \frac{b_1 (r - \rho) h}{\rho - 1} x_{\rho-1}^{\rho+1} + b_2 h + 1 \right) \frac{b_4 (\rho + 1) h}{\rho - 1} < 0,
$$

which implies that $\nabla B(x) h - 1 < 0$. Consequently, (H1) holds. Hence, Theorem 4.1 can be applied to (4.19).

Since $r + 1 > 2\rho$ and $r > 2 \wedge \rho + 1$, we have $\frac{1}{\rho - 1} \leq \frac{r - \rho}{\rho - 1}$. Letting $l = \frac{1}{\rho - 1}$ in Corollary 4.2, we have

$$
\left( \mathbb{E} \sup_{t \in [0, T]} |Y_t - (X_t^h)^{\frac{1}{\rho - 1}}|^p \right)^{\frac{1}{p}} \leq C_{p,T} h^{(2H-1)(\frac{1}{\rho - 1} \wedge 1)} (\log(1 + 1/h))^{\frac{1}{\rho - 1} \wedge 1},
$$

which implies that

$$
\lim_{h \to 0^+} \mathbb{E} \sup_{t \in [0, T]} |Y_t - (X_t^h)^{\frac{1}{\rho - 1}}|^p = 0.
$$

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