TOPOLOGICAL LOWER BOUNDS ON THE SIZES OF SIMPLICIAL COMPLEXES AND SIMPLICIAL SETS

SERGEY AVVAKUMOV ♠ AND ROMAN KARASEV ♣

Abstract. We prove that if an $n$-dimensional space $X$ satisfies certain topological conditions then any triangulation of $X$ as well as any its representation as a simplicial set with contractible faces has at least $2^n$ faces of dimension $n$.

One example of such $X$ is the $n$-dimensional torus $(S^1)^n$.

1. Introduction

We establish a lower bound on the number of top-dimensional simplices in a simplicial complex or a simplicial set homeomorphic (or homotopically equivalent) to the given topological space, provided that the topology of the space is rich in a certain sense.

Theorem 1.1. Let $G$ be a finite group and let $V$ be an $n$-dimensional real $G$-representation. Let $X$ be an $n$-dimensional simplicial set with a simplicial action of $G$. Assume that

1. the action is such that for every face $\sigma$ of $X$ and every $g \neq e \in G$ the intersection $\sigma \cap g\sigma$ is empty;
2. the image of every $G$-equivariant map $X \to V$ contains the origin in $V$ (the Borsuk–Ulam type property).

Then $X$ has at least $2^n$ $G$-orbits of faces of dimension $n$.

Here is an example of use of Theorem 1.1:

Corollary 1.2. Let $X$ be a simplicial set such that all its closed faces are contractible. Let $X$ have cohomology classes $\xi_1, \ldots, \xi_n \in H^1(X; \mathbb{F}_2)$ with non-zero product. Then $X$ has at least $2^n$ faces of dimension $n$.

Example 1.3. In particular, any simplicial set with all faces contractible (this is a non-trivial requirement for a simplicial set) and homeomorphic (or homotopically equivalent) to the $n$-torus $(S^1)^n$ or to the $n$-dimensional real projective space $\mathbb{R}P^n$ has at least $2^n$ faces of dimension $n$.

For simplicial sets representing $\mathbb{R}P^n$ this bound is sharp. By identifying the opposite points of the boundary of the $(n+1)$-dimensional crosspolytope we get a simplicial set homeomorphic to $\mathbb{R}P^n$ with exactly $2^n$ faces of dimension $n$, all whose faces are embedded simplices and so contractible.

Theorem 1.1 can be seen as a generalization of the result of Bárany and Lovász [3], who proved that any centrally symmetric triangulation of the $n$-sphere has at least $2^{n+1}$ faces of dimension $n$.

One powerful tool for face enumeration in manifolds (and simplicial complexes) is the manifold $g$-conjecture which was recently proved by Adiprasito [1], see also http://www.math.huji.ac.il/~adiprasito/bpa.pdf for the most general statement of the manifold lower bound theorem, and see also [7] and [2] for another proof. Using the conjecture one can, under some assumptions, bound from below the

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number of faces in a triangulation of a manifold in terms of its Betti numbers, see [4, Theorem 4.4]. From personal communication with Karim Adiprasito we learned that the assumption of the coefficients field being infinite in [4, Theorem 4.4] is most probably not required. A finite field can be extended to an infinite one and Betti numbers are invariant under field extension.

Compared to [4, Theorem 4.4], our Theorem 1.1 can be applied to simplicial sets as well as simplicial complexes; it also has a rather short, elementary, and self-contained proof. On the other hand, [4, Theorem 4.4] sometimes gives stronger (though still only exponential) bounds, for instance for the $n$-torus, and involves Betti numbers which might be easier to deal with than checking the topological condition of Theorem 1.1.

Another notable approach to general lower bounds on the size of a triangulation is to exploit the natural similarity between the number of top-dimensional simplices and the continuous volume. The latter can be estimated in terms of the systole using the classical Gromov’s systolic inequality. For an example of this approach see [6]. Unfortunately, systolic inequalities produce good bounds only when the systole is long enough. In our discrete setting the systole corresponds to the edge-length of the shortest non-contractible path in the given space and without additional assumptions can be as short as 3 in simplicial complexes or 2 in simplicial sets of the kind we study, which is not enough for a non-trivial bound.

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2. Proofs

Proof of Theorem 1.1. Assuming that the number of $G$-orbits of $n$-faces of $X$ is strictly less than $2^n$, we are going to construct a $G$-equivariant map $X \to V$ that misses the origin contradicting Assumption 2 of the theorem.

The proof basically follows the classical argument (which we learned from Arseniy Akopyan) showing that a probability that a random $n$-simplex contains the origin is $2^{-n}$, assuming that its vertices are mutually independent and the distribution of any single vertex is absolutely continuous and centrally symmetric.

Let $v_1,\ldots,v_N$ be the representatives of all the $G$-orbits of vertices of $X$. To construct a $G$-equivariant map $F: X \to V$ it is sufficient to choose the points $F(v_k) \in V$, extend this $F$ to other vertices of $X$ equivariantly (note that by Assumption 1 of the theorem the action of $G$ on vertices is free), and then extend $F$ to each face of $X$ linearly.

For any vector of signs $e \in \{-1,1\}^N$, we consider the modification of $F$, that is defined in the similar way, but starting with

$$F_e(v_k) = e_k F(v_k).$$

Note that by equivariance $F_e(gv_k) = ge_k F(v_k) = e_k g F(v_k) = e_k F(gv_k)$.

Let us require that all such $F_e$ are generic, that is the $F_e$ images of any $n$ vertices of $X$ from different $G$-orbits are linearly independent and the images of any $n+1$ vertices of $X$ from different $G$-orbits are affinely independent. The set of the initial maps $F$ that produce a non-generic $F_e$ for some sign vector $e$ is indeed a proper algebraic subset of the set of all possible maps $F$, hence we may restrict our consideration to families of generic $F_e$.

Let $\sigma$ be an $n$-face of $X$. Consider the case when it is degenerate in the sense that some of its vertices coincide in $X$ (this may happen with a simplicial set). Let $w_0,\ldots,w_k$ be its distinct vertices. By Assumption 1 they belong to different $G$-orbits and therefore by the genericity their $F_e$-images are linearly independent. Since $k < n$ in the presence of coincidences, $F_e(\sigma)$ does not touch the origin.
Now consider the case when $\sigma$ has $n+1$ distinct vertices in $X$, all belonging to different $G$-orbits by Assumption 1. Let $w_0, \ldots, w_n$ be the $F$-images of the vertices of $\sigma$, for a generic $F$ they are affinely independent. Write $0 \in V$ as the unique (up to a multiplication by a non-zero number) linear combination

$$0 = a_0w_0 + \cdots + a_nw_n.$$  

For a generic $F$, we also have that $a_i \neq 0$ for all $i$. Evidently, $0 \in F(\sigma)$ if and only if all the coefficients $a_i$ are of the same sign.

Choose $e \in \{-1,1\}^N$ uniformly at random. Considering the $F_e$-images $e_{k(0)}w_0, \ldots, e_{k(n)}w_n$ of the vertices of $\sigma$ (here $k(i)$ is the number of the $G$-orbit of the $i$th vertex of $\sigma$), the linear combination above becomes

$$0 = a_0e_{k(0)}(e_{k(0)}w_0) + \cdots + a_ne_{k(n)}(e_{k(n)}w_n),$$

that is, its coefficients change to $e_{k(0)}a_0, \ldots, e_{k(n)}a_n$. Because all the vertices of $\sigma$ belong to distinct $G$-orbits, the indices $k(0), \ldots, k(n)$ are distinct and the corresponding signs $e_{k(i)}$ are all independent.

Tracing the signs of the coefficients $e_{k(0)}a_0, \ldots, e_{k(n)}a_n$, we conclude that precisely $2^{-n}$ fraction of the simplices $F_e(\sigma)$ have the coefficients in the linear combination (2.1) of the same sign, meaning that $F_e(\sigma)$ covers the origin with probability $2^{-n}$.

Choose the representatives $\sigma_1, \ldots, \sigma_m$ of $G$-orbits of $n$-faces of $X$. If $m < 2^n$ then with positive probability none of $F_e(\sigma_i)$ contain the origin. From $G$-equivariance of $F_e$ there is no image of an $n$-face of $X$ containing the origin, that is $F_e(X) \neq 0$ at all, since the dimension of $X$ is $n$. This contradiction shows that $m \geq 2^n$.

\textbf{Proof of Corollary 1.2.} In this proof we use some topology and the reader is referred to the textbook [5] that contains the necessary basics.

Every $\xi_i$ corresponds to a classifying map $f_i : X \rightarrow K(\mathbb{Z}_2, 1) = \mathbb{RP}^\infty$ and the pullback of the double covering $S^\infty \rightarrow \mathbb{RP}^\infty$ is a double covering $\tilde{X}_i \rightarrow X$. The fiber-wise product of these double coverings is a covering $\tilde{X} \rightarrow X$ corresponding to a map $\pi_1(X) \rightarrow \mathbb{Z}_2$. Put $G = \mathbb{Z}_2$ and let $g_1, \ldots, g_n$ be its generators so that $g_i$ corresponds to the $i$th factor in the Cartesian product.

Note that a class $\xi_i$ is the pullback of the generator $\eta \in H^1(\mathbb{RP}^\infty; F_2)$ under $f_i$. Consider $V = \mathbb{R}^n$ as a representation of $G = \mathbb{Z}_2^n$ so that $g_i$ flips the sign of the $i$th coordinate of $\mathbb{R}^n$ and preserves the other coordinates.

Now we want to apply Theorem 1.1 to $\tilde{X}$ and the action of $G$ on it. Assumption 1 of Theorem 1.1 is satisfied since for every face $\sigma$ of $X$ (from its contractibility) its preimage in $\tilde{X}$ is covered by a disjoint set of $G$-shifted copies of $\sigma$. Moreover, so we establish that such copies of all faces of $X$ in $\tilde{X}$ constitute a representation of $\tilde{X}$ as a simplicial set.

As required by Theorem 1.1, let us drop all faces of $X$ (and respectively $\tilde{X}$) of dimension higher that $n$, that is, pass to the $n$-skeleton. This does not influence the cohomology product inequality $\xi_1 \cdots \xi_n \neq 0$ and keeps Assumption 1 of Theorem 1.1 valid. It remains to show that there cannot be a $G$-invariant map $F : \tilde{X} \rightarrow V$ for the $n$-dimensional $\tilde{X}$, thus checking Assumption 2 of Theorem 1.1.

Put $\tilde{Z}_i = \{x \in \tilde{X} \mid F_i(x) = 0\}$ for $i = 1, \ldots, n$, where $F_i$ are coordinates of $F$. Any set $\tilde{Z}_i$ is $G$-invariant and projects to $Z_i \subseteq X$, let $U_i = X \setminus Z_i$. Note that the inequalities $F_i(x) > 0$ and $F_i(x) < 0$ split $\tilde{X} \setminus \tilde{Z}_i$ in two parts, interchanged by the involution $g_i \in G$. Projecting this splitting to the two-sheet covering $\tilde{X}_i \rightarrow X$, we see that this covering $\tilde{X}_i \rightarrow X$ trivializes over $U_i$. This means that the composition of the inclusion $U_i \rightarrow X$ and $f_i : X \rightarrow \mathbb{RP}^\infty$ is null-homotopic, which implies $\xi_i|_{U_i} = (f_i|_{U_i})^*\eta = 0$. 
If \( U_1 \cup \cdots \cup U_n = X \) then one would have \( \xi_1 \cdots \xi_n = 0 \) over \( X \) by the standard property of the cohomology product. Since this is not the case, we have \( Z_1 \cap \cdots \cap Z_n \neq \emptyset \). Because the sets \( \tilde{Z}_i \) are \( G \)-equivariant and so the lifting of each \( Z_i \) in \( \tilde{X} \) is precisely \( \tilde{Z}_i \), we get that \( \tilde{Z}_1 \cap \cdots \cap \tilde{Z}_n \neq \emptyset \), and so \( F^{-1}(0) \neq \emptyset \).

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Sergey Avvakumov, School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel

Email address: savvakumov@gmail.com

Roman Karasev, Institute for Information Transmission Problems RAS, Bolshoy Karetny per. 19, Moscow, Russia 127994 and Moscow Institute of Physics and Technology, Institutskiy per. 9, Dolgoprudny, Russia 141700

Email address: roman.karasev@mail.ru

URL: [http://www.rkarasev.ru/en/](http://www.rkarasev.ru/en/)