Research Article

Semiclassical Solutions for a Kind of Coupled Schrödinger Equations

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In this paper, we are concerned with the following coupled Schrödinger equations

\[
\begin{aligned}
-\lambda^2 u_t + a_1(x)u &= c(x)v + a_2(x)|u|^{p-2}u + a_3(x)|u|^{2^*-2}u, \quad x \in \mathbb{R}^N, \\
-\lambda^2 v_t + b_1(x)v &= c(x)u + b_2(x)|v|^{p-2}v + b_3(x)|v|^{2^*-2}v, \quad x \in \mathbb{R}^N,
\end{aligned}
\]

where \(2 < p < 2^*, 2 < q < 2^*, 2^* = 2N/(N-2)\), and \(N \geq 3\). In this paper, we are concerned with the following coupled Schrödinger equations

\[
\begin{aligned}
-\lambda^2 u_t + a_1(x)u &= c(x)v + a_2(x)|u|^{p-2}u + a_3(x)|u|^{2^*-2}u, \quad x \in \mathbb{R}^N, \\
-\lambda^2 v_t + b_1(x)v &= c(x)u + b_2(x)|v|^{p-2}v + b_3(x)|v|^{2^*-2}v, \quad x \in \mathbb{R}^N,
\end{aligned}
\]

where \(2 < p < 2^*, 2 < q < 2^*, 2^* = 2N/(N-2)\) are the Sobolev critical exponent; \(\lambda > 0\) is a parameter; and \(a_1, a_2, a_3, b_1, b_2, b_3, c \in C(\mathbb{R}^N, \mathbb{R})\) and \(u, v \in H^1(\mathbb{R}^N)\).

As it is known in [1], this type of systems arises in nonlinear optics. In the past years, under different kinds of assumptions on the potential \(V\) and the nonlinearity \(f\), many authors [2–8] focus on the following kind of Schrödinger equation:

\[
-\lambda^2 u + V(x)u = f(x, u), \quad x \in \mathbb{R}^N.
\]

As one knows, single-mode optical fibers are not really “single mode” but actually bimodal because of the presence of birefringence. So recently, the coupled Schrödinger systems are investigated by the authors [9–12]. For more related results and physical background on Schrödinger systems, please see [13–23] and references therein.

In [11], the authors investigated standing waves for the following kind of coupled Schrödinger equations:

\[
\begin{aligned}
-\lambda^2 u_t + a_1(x)u &= cv + |u|^{p-2}u, \quad x \in \mathbb{R}^N, \\
-\lambda^2 v_t + b_1(x)v &= cu + |v|^{2^*-2}v, \quad x \in \mathbb{R}^N,
\end{aligned}
\]

where \(a_1, b_1 \in C(\mathbb{R}^N, \mathbb{R}), N \geq 3, u, v > 0, u, v \in H^1(\mathbb{R}^N), u(x), v(x) \to 0\) as \(|x| \to \infty\). Under the following conditions:

\(\text{A0}\) there exist positive constants \(a_1^0 > 0\) and \(b_1^0 > 0\) such that \(a_1(x) \geq a_1^0, b_1(x) \geq b_1^0, 0 < c \leq \sqrt{a_1^0 b_1^0};\) they obtained the existence of a positive solution for (3) if \(\lambda\) is sufficiently small. But, if \(a_1^0 = \inf a_1 = 0\) or \(b_1^0 = \inf b_1 = 0\), then \(0 < c \leq \sqrt{a_1^0 b_1^0}\) cannot hold. So in the very recent paper [12], Peng et al. investigated the following coupled Schrödinger equations and generalize the result in [11]:
\[
\begin{align*}
-\Delta u + a_1(x)u &= c(x)v + |u|^{p-2}u, \quad x \in \mathbb{R}^N, \\
-\Delta v + b_1(x)v &= c(x)u + |v|^{q-2}v, \quad x \in \mathbb{R}^N,
\end{align*}
\]  
(4)

where \(a_1, b_1\) are the same as in (3), \(N \geq 3\). Under the following conditions,

(A1) \(a_1(x) \geq a_1(0) = 0\) and \(b_1(x) \geq 0\), and there exist constants \(a_1^0 > 0\) and \(b_1^0 > 0\) such that the measure of the sets \(A_1^0 := \{x : a_1(x) < a_1^0\}\) and \(B_1^0 := \{x : b_1(x) < b_1^0\}\) are finite.

(A2) there exists a constant \(\theta \in (0, 1)\) such that \(|c(x)|^2 \leq \theta a_1(x)b_1(x)\) for all \(x \in \mathbb{R}^N\); Peng et al. proved that system (4) has at least one nontrivial solution. An interesting question is what will happen if the nonlinearity is also critical growth in system (4)? Motivated mainly by the abovementioned results, we will answer this question and prove that system (1), under conditions (A1) and (A2), and

(A3) there exist constants \(a_2^0, a_3^0, a_4^0, a_5^0, a_6^0, b_1^0, b_2^0, b_3^0, b_4^0 > 0\) such that

\[
\begin{align*}
\frac{a_2^0}{a_2(x)} &\leq a_2^0, \quad \frac{a_3^0}{a_3(x)} \leq a_3^0, \quad \frac{a_4^0}{a_4(x)} \leq b_1(x) \leq b_2^0, \\
\frac{b_3^0}{b_3(x)} &\leq b_3^0, \quad \forall x \in \mathbb{R}^N,
\end{align*}
\]  
(5)

possesses nontrivial solutions if \(\lambda \in (0, \lambda_0)\), where \(\lambda_0\) is related to \(a_1, a_2, \ldots, a_6, b_1, b_2, \ldots,\) and \(N\). As far as we know, similar results for system (1) with a critical exponent have not been investigated by variational methods in the literature. The following condition is similar to condition (A1):

(A1') \(b_1(x) \geq b_1(0) = 0\) and \(a_1(x) \geq 0\), and there exist constants \(a_1^0 > 0\) and \(b_1^0 > 0\) such that the measure of the sets \(A_1^0 := \{x \in \mathbb{R}^N : a_1(x) < a_1^0\}\) and \(B_1^0 := \{x \in \mathbb{R}^N : b_1(x) < b_1^0\}\) are finite.

Since \((q - 2)N - 2q < 0\) and \((p - 2)N - 2p < 0\), one can choose \(d_0 \geq 1\) such that

\[
C_1a_1^{1/2} + C_2b_1^{1/2} + C_3a_2^{1/2} + C_4b_2^{1/2} + C_5b_3^{1/2} + C_6b_4^{1/2} \leq \frac{1}{2}(1 - \theta),
\]  
(6)

where

\[
\begin{align*}
\alpha &= \frac{\omega_Na_1^2(p-2)}{2Np}, \\
\beta &= \frac{\omega_Nb_1^2(q-2)}{2Nq}, \\
\gamma &= \frac{\omega_Nb_2^2(q-2)}{2Nq}, \\
\delta &= \frac{\omega_Nb_3^{2-N/2}b_4^{2-N/2}}{(2-N/2)^{2-N/2}(p-2)}, \\
\zeta &= \frac{\omega_Nb_2^{2-N/2}b_4^{2-N/2}}{(2-N/2)^{2-N/2}(q-2)}, \\
\eta_0 \text{ and } \eta_1 \text{ are embedding constants and } \omega_N \text{ is the volume of the unit ball in } \mathbb{R}^N. \text{ From (A1') and (A1), using } b_1(0) = 0 \text{ and } a_1(0) = 0, \text{ one can let } \mu_0 > 1 \text{ such that}
\]

\[
\sup_{\mu^2 \geq 1} |b_1(x)| \leq d_0^{-2}, \quad \sup_{\mu^2 \geq 1} |a_1(x)| \leq d_0^{-2}, \quad \forall \mu \geq \mu_0.
\]  
(8)

Let \(w = (u, v)\) and \(\lambda^2 = \mu\), then system (1) can be rewritten as

\[
\begin{align*}
-\Delta u + \mu a_1(x)u &= \mu c(x)v + \mu a_2(x)|u|^{p-2}u + \mu a_3(x)|u|^{q-2}u, \quad x \in \mathbb{R}^N, \\
-\Delta v + \mu b_1(x)v &= \mu c(x)u + \mu b_2(x)|v|^{p-2}v + \mu b_3(x)|v|^{q-2}v, \quad x \in \mathbb{R}^N,
\end{align*}
\]  
(9)

and the functional of (9) is given by

\[
S_\mu(w) = \frac{1}{2} \int_{\mathbb{R}^N} \left[ |\Delta u|^2 + |\Delta v|^2 + \mu a_1(x)|u|^2 + \mu b_1(x)|v|^2 \right] dx
\]

\[
- \frac{\mu}{p} \int_{\mathbb{R}^N} a_2(x)|u|^p dx - \frac{\mu}{q} \int_{\mathbb{R}^N} b_2(x)|v|^q dx
\]

\[
- \frac{\mu}{2} \int_{\mathbb{R}^N} a_3(x)|u|^r dx - \frac{\mu}{2} \int_{\mathbb{R}^N} b_3(x)|v|^r dx
\]

\[
- \mu \int_{\mathbb{R}^N} c(x)uv dx.
\]  
(10)

As is known, the solutions of (1) are the critical points of \(S_\mu\). The main results are the following.

**Theorem 1.** Suppose that (A1)–(A3) or (A1')–(A3) hold. Then, (9) possesses at least one nontrivial solution \(w_\mu = (u_\mu, v_\mu)\) such that \(0 < S_\mu(w_\mu) \leq \beta \mu^{1-N/2} \) for \(\mu \geq \mu_0^0\).

**Theorem 2.** Suppose that (A1)–(A3) or (A1')–(A3) hold. Then, (1) possesses at least one nontrivial solution \(w_\mu = (u_\mu, v_\mu)\) such that \(0 < S_\mu(w_\mu) \leq \beta \lambda^{N/2} \) for \(0 < \lambda \leq \lambda_0\).

**Remark 3.** Since the presence of the terms \(a_3(x)|u|^{2-2}u, b_3(x)|v|^{2-2}v, b_3(x)|v|^{p-2}v,\) and \(b_3(x)|v|^{q-2}v\), system (1) is more general than (4), and it is more difficult to deal with the nontrivial solutions. In order to prove that system (1) has nontrivial solutions, we need to find some conditions to restrict \(a_3(x), a_4(x), b_3(x),\) and \(b_4(x)\). It seems that there is no literature considering system (1).

**2. Preliminaries**

Let

\[
E = \left\{ (u, v) : \int_{\mathbb{R}^N} \left[ a_1(x)|u|^2 + b_1(x)|v|^2 \right] dx < \infty, u, v \in H^1(\mathbb{R}^N) \right\},
\]  
(11)
\[ \|w\|_{H^s}^2 := \left\{ \int_{\mathbb{R}^N} \left( |\Delta u|^2 + |\Delta v|^2 + |u|^2 + |v|^2 \right) dx \right\}^{1/2}, \quad \forall w = (u, v) \in E. \]  

(12)

From Lemma 1 of [17], by (A1) or (A1') and the Sobolev inequality, there exists a positive constant \( \eta_0 > 0 \) independent of \( \mu \) such that

\[ \|w\|_{H^1} = \left\{ \int_{\mathbb{R}^N} \left[ |\Delta u|^2 + |\Delta v|^2 + |u|^2 + |v|^2 \right] dx \right\}^{1/2} \leq \eta_0 \|w\|_{\mu^*}, \quad \forall w \in E, \mu \geq 1, \]  

(13)

where \( H^1 = H^1(\mathbb{R}^N) \). Then, \( (E, \|\cdot\|_{\mu^*}) \) is a Banach space for \( \mu \geq 1 \) equipped with the norm given by (12). Moreover, for \( s \in [2, 2^*] \), one has

\[ \|w\|_s \leq \eta_s \|w\|_{H^1} \leq \eta_s \eta_0 \|w\|_{\mu^*}, \quad \forall w \in E, \mu \geq 1, \]  

(14)

where \( \|w\|_s \) is the usual norm in space \( L^s(\mathbb{R}^N) \). From (12), we rewrite \( S_\mu \) as

\[ S_\mu(w) = \frac{1}{2} \|w\|_{\mu^*}^2 - \mu \int_{\mathbb{R}^N} a(x)|u|^2 dx - \mu \int_{\mathbb{R}^N} b_2(x)|v|^2 dx - \mu \int_{\mathbb{R}^N} \frac{c(x)}{2} |u|^2 \, dx - \mu \int_{\mathbb{R}^N} b_2(x)|v|^2 \, dx - \mu \int_{\mathbb{R}^N} c(x)uv \, dx, \quad \forall w \in E. \]  

(15)

It is not difficult to see that \( S_\mu \in C^1(E, \mathbb{R}) \) and

\[ \left\langle S'_\mu(w), \bar{w} \right\rangle = \int_{\mathbb{R}^N} (\Delta u \bar{u} + \Delta v \bar{v} + \mu a_1(x) |u|^2 \bar{u} + \mu b_1(x) v \bar{v}) \, dx - \mu \int_{\mathbb{R}^N} b_2(x)|v|^2 \, dx - \mu \int_{\mathbb{R}^N} \frac{c(x)}{2} |u|^2 \, dx - \mu \int_{\mathbb{R}^N} b_2(x)|v|^2 \, dx - \mu \int_{\mathbb{R}^N} c(x)uv \, dx, \quad \forall w = (u, v), \bar{w} = (\bar{u}, \bar{v}) \in E. \]  

(16)

As in [12, 22], let

\[ \theta(x) = \begin{cases} \frac{1}{d_0}, & |x| \leq d_0, \\ d_0^{N-1} \left[ |x|^{-N} - (2d_0)^{-N} \right], & d_0 < |x| \leq 2d_0, \\ 0, & |x| > 2d_0. \end{cases} \]  

(17)

Then, \( \theta \in H^1(\mathbb{R}^N) \); moreover,

\[ \|\nabla \theta\|_{2}^2 = \int_{\mathbb{R}^N} |\nabla \theta(x)|^2 \, dx \leq \frac{N\omega_N d_0^{N-4}}{(N+2)(1-2^{-N})}, \]  

(18)

\[ \|\theta\|_{2}^2 = \int_{\mathbb{R}^N} |\theta(x)|^2 \, dx \leq \frac{2\omega_N d_0^{N-2}}{N(1-2^{-N})}. \]  

(19)

In the next section, we will prove the main results.

### 3. Proof of the Main Results

**Proof of Theorem 1.** The proof of Theorem 1 is divided into four steps.

**Step 1.** We first prove that for any \( \mu \geq \mu_0 > 1 \), one has

\[ \sup \left\{ S_\mu(0, t_\mu) : t \geq 0 \right\} \leq \beta \mu^{-N/2}, \quad \sup \left\{ S_\mu(t_\mu, 0) : t \geq 0 \right\} \leq \alpha \mu^{-N/2}, \]  

(20)

where \( \epsilon_\mu(x) = \theta(\mu^{1/2}x) \). From (8), (9), (17), (18), (19), and (A3), we have

\[ S_\mu(0, t_\mu) = \frac{1}{2} \int_{\mathbb{R}^N} \left[ \|
abla \epsilon_\mu\|_{\mu^*}^2 + \epsilon_\mu(0) \|\tilde{\epsilon}_\mu\|_{\mu^*}^2 \right] \, dx - \frac{1}{2} \int_{\mathbb{R}^N} b_2(x)\|\tilde{\epsilon}_\mu\|_{\mu^*}^2 \, dx - \mu \int_{\mathbb{R}^N} b_2(x)|\tilde{\epsilon}_\mu|^2 \, dx - \mu \int_{\mathbb{R}^N} \frac{c(\mu^{1/2}x)}{2} |\tilde{\epsilon}_\mu|^2 \, dx \]  

\[ \leq \frac{1}{2} \left( \|
abla \tilde{\epsilon}_\mu\|_{\mu^*}^2 + \|\tilde{\epsilon}_\mu\|_{\mu^*}^2 \right) \leq b_1, \]  

(21)

Similarly, from (8), (9), (17), (18), (19), and (A3), we have

\[ S_\mu(t_\mu, 0) \leq \mu^{1-N/2} \omega_0 d_0^{N(d-2)/(d-2)} \left( \frac{N^2 + 2(N+2)}{(N+2)(1-2^{-N})} \right) \left( \frac{d_0^{(d-2)}(2d_0^{-2})}{2d_0^{(d-2)}} \right) \]  

(22)

which together with (21) implies that (20) holds.
Step 2. Let $c^*_\mu = \min \{S_\mu(\nu_s, 0), S_\mu(0, \nu_s)\}$, we should prove that there exists a constant $c^*_\mu \in (0, c^*_\mu]$ and a sequence $\{w_n\} \subset E$ satisfying

$$S_\mu(w_n) \to c^*_\mu, \quad \|S'_\mu(w_n)\|_{E'} \left(1 + \|w_n\|_{E'}\right), \quad \text{as } n \to \infty.$$ (23)

By a standard argument, one can obtain (23) by employing the mountain-pass lemma without the (PS) condition, so we omit the details here.

Step 3. We prove that any sequence $\{w_n\} \subset E$ satisfying (23) is bounded in $E$. From (A2) and Young’s inequality, we have

$$\mu \int_{\mathbb{R}^N} |c(x) u_n v_n| dx \leq \mu \theta \int_{\mathbb{R}^N} \left( \sqrt{a_1(x)} b_1(x) |u_n v_n| \right) dx,$$

$$\leq \frac{9}{2} \int_{\mathbb{R}^N} \left[ \mu a_1(x) u_n^2 + \mu b_1(x) v_n^2 \right] dx \quad (24)$$

For $2 < p \leq q < 2^*$, from (15), (16), (23), and (24), we have

$$c^*_\mu + o(1) = S_\mu(w_n) - \frac{1}{p} \left( \frac{1}{p} - \frac{1}{q} \right) \mu \int_{\mathbb{R}^N} b_2(x) |v_n|^{2q} dx + \left( \frac{1}{p} - \frac{1}{2} \right) \mu \int_{\mathbb{R}^N} a_3(x) |v_n|^{2p} dx$$

$$+ \left( \frac{1}{p} - \frac{1}{2} \right) \mu \int_{\mathbb{R}^N} b_3(x) |v_n|^{2p} dx - \left( \frac{1}{p} - \frac{1}{2} \right) \mu \int_{\mathbb{R}^N} c(x) u_n v_n dx$$

$$\geq \left( \frac{1}{2} - \frac{1}{p} \right) (1 - \theta) \|w_n\|_{E'}^2.$$ (25)

For $2 < q \leq p < 2^*$, from (15), (16), (23), and (24), we obtain

$$c^*_\mu + o(1) = S_\mu(w_n) - \frac{1}{q} \left( \frac{1}{p} - \frac{1}{q} \right) \mu \int_{\mathbb{R}^N} a_2(x) |v_n|^{2q} dx$$

$$+ \left( \frac{1}{p} - \frac{1}{q} \right) \mu \int_{\mathbb{R}^N} a_3(x) |v_n|^{2p} dx + \left( \frac{1}{q} - \frac{1}{2} \right) \mu \int_{\mathbb{R}^N} c(x) u_n v_n dx$$

$$+ \left( \frac{1}{q} - \frac{1}{2} \right) \mu \int_{\mathbb{R}^N} b_3(x) |v_n|^{2p} dx$$

$$\geq \left( \frac{1}{2} - \frac{1}{q} \right) (1 - \theta) \|w_n\|_{E'}^2.$$ (26)

It follows from (25) and (26) that $\{w_n\}$ is bounded in $E$. Step 4. We show that there exists a nontrivial solution. By Steps 1–3, we know that there exists a bounded sequence $\{w_n\} \subset E$ satisfying (23) with

$$c^*_\mu \leq c^*_{\mu}, \quad \forall \mu \geq \mu_U.$$ (27)

Passing to a subsequence, one can suppose that $w_n = (u_n, v_n) \to (u_\mu, v_\mu)$ in $(E, || \cdot ||_E)$ and $S'_\mu(w_n) \to 0$, as $n \to \infty$. Now, we verify that $w_\mu \neq (0, 0)$. Arguing by contradiction, assume that $w_\mu = (0, 0)$, that is, $w_n \to (0, 0)$ in $E$, so by [24], we have $w_n \to (0, 0)$ in $L^1_{\text{loc}}(\mathbb{R}^N), s \in [2^*, 2]$, and $w_n \to (0, 0)$ a.e. on $\mathbb{R}^N$. Since $A_{\alpha i}$ and $B_{\beta j}$ are sets with finite measure, we have

$$\|u_n\|_{E'}^2 = \int_{\mathbb{R}^N \setminus A_{\alpha i}} |u_n|^2 dx + \int_{A_{\alpha i}} |u_n|^2 dx = \int_{\mathbb{R}^N \setminus A_{\alpha i}} |u_n|^2 dx$$

$$+ \int_{A_{\alpha i}} |u_n|^2 dx \leq \int_{\mathbb{R}^N \setminus A_{\alpha i}} \frac{1}{\mu a_{\alpha i}} \|u_\mu\|_{E'}^2 dx + \int_{A_{\alpha i}} |u_n|^2 dx \leq \left( \frac{1}{\mu a_{\alpha i}} \right) \|u_\mu\|_{E'}^2 + o(1),$$ (28)

$$\|v_n\|_{E'}^2 = \int_{\mathbb{R}^N \setminus B_{\beta i}} |v_n|^2 dx + \int_{B_{\beta i}} |v_n|^2 dx = \int_{\mathbb{R}^N \setminus B_{\beta i}} |v_n|^2 dx$$

$$+ \int_{B_{\beta i}} |v_n|^2 dx \leq \int_{\mathbb{R}^N \setminus B_{\beta i}} \frac{1}{\mu b_{\beta i}} \|v_n\|_{E'}^2 dx + \int_{B_{\beta i}} |v_n|^2 dx \leq \left( \frac{1}{\mu b_{\beta i}} \right) \|w_n\|_{E'}^2 + o(1).$$ (29)

Similar to [12], from (14), (28), (29), and the Hölder inequality, we obtain

$$\|u_n\|_s^2 = \int_{\mathbb{R}^N} |u_n|^s dx \geq \left( \int_{\mathbb{R}^N} |u_n|^{2(2-s)/(2-2)} dx \right)$$

$$\cdot \left( \int_{\mathbb{R}^N} |u_n|^{2(2-s)/(2-2)} dx \right)^{2-s} \leq \left( \eta_\alpha \eta_{2^*} \right)^{2-s} \|w_n\|_{E'}^2 + o(1), \quad s \in (2, 2^*).$$ (30)

$$\|v_n\|_s^2 = \int_{\mathbb{R}^N} |v_n|^s dx \geq \left( \int_{\mathbb{R}^N} |v_n|^{2(2-s)/(2-2)} dx \right)$$

$$\cdot \left( \int_{\mathbb{R}^N} |v_n|^{2(2-s)/(2-2)} dx \right)^{2-s} \leq \left( \eta_\beta \eta_{2^*} \right)^{2-s} \|w_n\|_{E'}^2 + o(1), \quad s \in (2, 2^*).$$ (31)
It follows from (15), (16), (23), and (A3) that
\[ c_\mu + o(1) = S_\mu(w_n) - \frac{1}{2}\left(S_{\mu}^*(w_n), w_n\right) = (1 - \frac{1}{2p})\mu \int_{\mathbb{R}^N} a_\mu(x) |v_n|^p dx + \left(\frac{1}{2} - \frac{1}{q}\right) \mu \int_{\mathbb{R}^N} b_{\mu}(x)|v_n|^q dx + \left(\frac{1}{2} - \frac{1}{2q}\right) \mu \int_{\mathbb{R}^N} b_{\mu}(x)|v_n|^{2q} dx \]
\[ + \mu \int_{\mathbb{R}^N} b_{\mu}(x) |v_n|^2 dx \geq \frac{(p-2)\mu \theta_3^2}{2p} ||u_n||_p^p + \frac{(q-2)\mu \theta_3^2}{2q} ||v_n||_q^q \]
\[ + \frac{(2\mu - 2)\mu \theta_3^2}{22^2} ||u_n||_p^p + \frac{(2\mu - 2)\mu \theta_3^2}{22^2} ||v_n||_q^q. \]

(32)

From (14), (30), (31), and (32), we have
\[ \mu ||u_n||_p^p = \mu ||u_n||_p^p - \frac{1}{2} \mu \left( \mu d_{\mu}^{(p-2)} \right) \left( \mu d_{\mu}^{(q-2)} \right) \left( (p-2)(q-2) \right) \]
\[ + o(1) = d_{\mu}^{(p-2)} \left( \mu d_{\mu}^{(q-2)} \right) \left( (p-2)(q-2) \right) \]

(33)

From (14) and (32), we have
\[ \mu ||v_n||_q^q = \mu ||v_n||_q^q - \frac{1}{2} \mu \left( \mu d_{\mu}^{(p-2)} \right) \left( \mu d_{\mu}^{(q-2)} \right) \left( (p-2)(q-2) \right) \]
\[ + o(1) = d_{\mu}^{(p-2)} \left( \mu d_{\mu}^{(q-2)} \right) \left( (p-2)(q-2) \right) \]

(34)

From (15), (23), and (38), we have
\[ 0 < \mu = \lim_{n \to \infty} S_\mu(w_n) \leq \frac{1}{2} ||w_n||_p^p = 0, \]

(39)

It is easy to see that Theorem 2 is a direct consequence of Theorem 1.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

No potential conflict of interest was reported by the authors.

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