A QUADRATIC LOWER Bound For THE ConVERGENCE RATE IN THE ONE-DIMENSIONAL Hegselmann-Krause BOUNDED CONFIDENCE DYNAMICS

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Abstract. Let $f_k(n)$ be the maximum number of time steps taken to reach equilibrium by a system of $n$ agents obeying the $k$-dimensional Hegselmann-Krause bounded confidence dynamics. Previously, it was known that $\Omega(n) = f_1(n) = O(n^3)$. Here we show that $f_1(n) = \Omega(n^2)$, which matches the best-known lower bound in all dimensions $k \geq 2$.

1. Introduction

The Hegselmann-Krause bounded confidence model of opinion dynamics (HK-model for brevity), introduced in \cite{HK}, works as follows. We have a finite number $n$ of agents, indexed by the integers $1, 2, \ldots, n$. Time is measured discretely and the opinion of agent $i$ at time $t \in \mathbb{N} \cup \{0\}$ is represented by a vector $x_t(i) \in \mathbb{R}^k$, where $k$ is fixed. There is a fixed parameter $r > 0$ such that the dynamics are given by

\begin{equation}
    x_{t+1}(i) = \frac{1}{|\mathcal{N}_t(i)|} \sum_{j \in \mathcal{N}_t(i)} x_t(j),
\end{equation}

where $\mathcal{N}_t(i) = \{j : ||x_t(j) - x_t(i)|| \leq r\}$ and $|| \cdot ||$ is the Euclidean norm on $\mathbb{R}^k$. The sociological interpretation of (1.1) is then that there are $k$ “issues” on which agents have opinions, and each agent is only willing to compromise at any time with those whose opinions on every issue lie close to their own. Since the dynamics are obviously unaffected by rescaling all opinions and the confidence bound $r$ by a common factor, we can assume without loss of generality that $r = 1$.

Let $(x(1), \ldots, x(n))$ be a configuration of opinion vectors. We say that agents $i$ and $j$ agree if $x(i) = x(j)$. A maximal set of agents that agree is called a cluster, and the number of agents in a cluster is called its size. The configuration is said to be frozen if $||x(i) - x(j)|| > 1$ whenever $x(i) \neq x(j)$. Clearly, if the configuration is frozen then $x_{t+1}(i) = x_t(i)$ for all $i$, and it is easy to see that the converse also holds.

Perhaps the most fundamental result about the HK-dynamics is that any configuration of opinions will freeze in finite time. Indeed, the same is true of a wide class of models including HK as a simple prototype, see \cite{C}. Let $f_k(n)$ denote the maximum number of time steps taken to freeze by a configuration of $n$ agents with opinions in $\mathbb{R}^k$ obeying (1.1). It turns out that $f_k(n)$ is bounded by a universal polynomial function of $n$ and $k$. This was established in \cite{BBCN}, who gave the bound $f_k(n) = O(n^{10}k^2)$. Much better bounds are known in one dimension, however, which was in fact the only case originally considered in \cite{HK}. The first polynomial bound $f_1(n) = O(n^5)$ was established in \cite{MBCF} and the current record is $f_1(n) = O(n^3)$, also due to \cite{BBCN}. An important fact which makes the one-dimensional model much simpler to analyse is that, as soon as two agents become separated by a distance greater than one, they will never again interact. This is not always the case in higher dimensions.\textsuperscript{2}

\textsuperscript{1}Other terms used in the literature are “in equilibrium” or “has converged”. We think our term captures the point with the least possible room for misinterpretation, however.

\textsuperscript{2}As an example in $\mathbb{R}^2$, consider three agents $a, b, c$ initially placed at $(0, -0.5), (0, 0.5)$ and $(1, 0)$ respectively, so that $c$ has no agents within distance 1. At $t = 0$ only $a$ and $b$ will interact, but this first interaction will bring them both to $(0, 0)$ where they are close enough to $c$ to interact at $t = 1.$

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Regarding lower bounds for $f_k(n)$, the evidence so far has also suggested that dimension one might be special. It was first noted in [MBCF] that $f_1(n) = \Omega(n)$. For suppose we start from the configuration $E_n = (1, 2, \ldots, n)$, so opinions are equally spaced with gaps equal to the confidence bound. Then it is not hard to see that, as the configuration updates, if $i < n/2$ then the opinions of agents $i$ and $(n + 1) - i$ will remain constant as long as $t < i$, while both will change at $t = i$. Hence, the time taken for the configuration $E_n$ to freeze is at least $n/2$. In fact, this configuration freezes in time $5n/6 + O(1)$, see [HW].

Already in two dimensions, however, a quadratic lower bound is known. This was also proven in [BBCN]. Their example, which we denote $F_n$, places the $n$ agents at the vertices of a regular $n$-gon of side-length one, and they show that the system requires at least $n^2/28$ steps to freeze. The configuration $F_n$ seems, at least in hindsight, like a natural “two-dimensional version” of $E_n$. It is not really clear how far one can push this idea, however, as the upper bound of $O(n^{10}k^2)$ for all dimensions makes immediately clear. Indeed, there is no example known in dimensions $k \geq 3$ which takes longer to freeze than $F_n$, now considered as a configuration on a plane in $\mathbb{R}^k$.

In this paper, we will prove that $f_1(n) = \Omega(n^2)$ by exhibiting an explicit sequence $D_n$ of configurations which take this long to freeze. In fact, we shall abuse notation slightly. Though we could define a suitable configuration for any number $n$ of agents, in order to simplify the appearance of certain formulas we will assume that $n$ is even and let $D_n$ denote a certain configuration on $3n + 1$ agents. Our construction basically combines the chain $E_n$ with an example of Kurz [K], and is defined as follows:

**Definition 1.1.** Let $n$ be a positive, even integer. The configuration $D_n$ consists of $3n + 1$ agents whose opinions are given by

$$
(1.2) \quad x(i) = \begin{cases} 
-\frac{1}{n}, & \text{if } 1 \leq i \leq n, \\
 i - (n + 1), & \text{if } n + 1 \leq i \leq 2n + 1, \\
n + \frac{1}{n}, & \text{if } 2n + 2 \leq i \leq 3n + 1.
\end{cases}
$$

The configuration is represented pictorially in Figure 1. It has the shape of a dumbbell.

![Figure 1. Schematic representation of the configuration $D_n$. Each dumbbell has weight $n$.](image1)

Indeed, someone familiar with the theory of Markov chains might consider this a natural candidate for maximising the freezing time. There is a subtlety, however. Along the “bar” of the dumbbell, opinions are equally spaced at distance one, whereas the two dumbbell clusters themselves are positioned much closer, at distance $1/n$, to the ends of the bar. The latter is what raises the freezing time from $\Theta(n)$ to $\Theta(n^2)$, as will become evident from the proof below. In fact, this is just one of at least three ways of considering our construction as a modification of others previously known which all freeze in linear time. A second way would be to think of it

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3In the general theory of irreducible Markov chains on graphs, dumbbell-like graphs are known to have the longest mixing times. See, for example, [LPW].
as starting from $E_n$, which freezes in time $O(n)$, and then adding the dumbbells. A third would be to start from the configuration in [K], which consists of the two dumbbells placed at distance $1/n$ from their respective solitary agents, but then without the long intermediate chain. Kurz showed that his configuration took time $\Omega(n)$ to freeze and as a by-product of our method, it can be easily shown to freeze in time $O(n)$.

Let us now formally state our result.

**Theorem 1.2.** The configuration $\mathcal{D}_n$ freezes after time $\Theta(n^2)$.

The proof will be given in the next section, but let’s give some intuition here. At $t = 0$, the leftmost solitary agent $n + 1$ will do nothing, as the influence of the cluster of size $n$ at distance $1/n$ to his left is exactly balanced by that of agent $n + 2$ at distance one to his right. The cluster, on the other hand, will start to move toward agent $n + 1$ at speed $\approx 1/n^2$. As the cluster has almost complete influence over agent $n + 1$, he will respond by also moving to the right at speed $\approx 1/n^2$. The cluster and the solitary agent influence each other in such a way as to essentially maintain a constant speed to the right. This induces a wave of motion to the right for all the agents in the left half of the chain and, by symmetry, a similar wave of motion to the left in the right half of the configuration. Agent $\frac{3n}{2} + 1$, who is right in the middle, stays perfectly still. Now, if communication along the chain were perfect, the speed of each agent would be the average of those of its two neighbours. Since the outermost agents move with speed $\approx 1/n^2$ and the middle agent is stationary, this would imply that the relative speed between agents $n + 1$ and $n + 2$ would be $\approx 2/n^3$. The fact that agents are uniformly spaced along the chain initially suggests that communication should be “good enough” for this approximation to be viable. Hence, to summarise the intuition: The cluster on the left and agent $n + 1$ both move inwards at speed $\approx 1/n^2$ and catch up with agent $n + 2$ at a rate of $\approx 2/n^3$. Since the initial distance between the cluster and agent $n + 2$ is $1 + 1/n$, this means that agent $n + 2$ will be visible to the cluster at time $\approx n^2/2$. But when that happens, agent $n + 2$ will be yanked to the left, causing it to lose sight of $n + 3$. Agents $n + 1$ and $n + 2$ will then immediately merge with the cluster of size $n$ to form a cluster of size $n + 2$, which is then frozen. The same thing happens on the right. In the remaining chain of $n - 4$ agents, the gaps will all still be close to one, so this chain will freeze in time $O(n)$ - we will use the results of [HW] to prove this last step. Hence, the entire configuration will freeze after time $\Theta(n^2)$.

In fact this heuristic is a little off. It suggests that the distance between agents $n + 1$ and $n + 2$ at time $t$ will be $\approx 1 - 2t/n^3$ and that they will come into contact at time $\approx n^2/2$. In fact, the distance seems to decrease as $\approx 1 - 2\sqrt{t}/n^3$ so that contact is established at time $\approx n^2/4$. See Remark 2.3 below.

We finish this section by giving some more fairly standard terminology to be used below. Let $(x(1), \ldots, x(n))$ be a configuration of one-dimensional opinions, obeying the convention that $x(i) \leq x(j)$ whenever $i \leq j$. We can define a receptivity graph $G$, whose nodes are the $n$ agents and where an edge is placed between agents $i$ and $j$ whenever $|x(i) - x(j)| \leq 1$. We say that agents $i$ and $j$ are connected if they are in the same connected component of the receptivity graph. Observe that every connected component of $G$ is an interval of agents and that $i$ is disconnected from $i + 1$ if and only if $x(i + 1) > x(i) + 1$.

2. Proof of Theorem 1.2

**Lemma 2.1.** Let $n \geq 2$ and let $\mathcal{P}_n$ denote the path on $n$ vertices, indexed from left-to-right by the integers $1, \ldots, n$. Let $X_0, X_1, \ldots$ be a random walk on $\mathcal{P}_n$ with transition probabilities $p_{ij}$

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In fact, in the Markov chain literature, this configuration is commonly termed a dumbbell, whereas ours would be referred to as a “dumbbell with a chain in between”. We hope the reader is not confused!
Let us consider instead a cycle $C_t$ node 1 hits 1 itself up to time $h$ that in turn which binomial coefficients at level $n$ than $2^{h}j$ $t$ $\leq$ or just do it “by hand”). Unwinding our argument, what we have shown is that, provided $\kappa < 1$ is an absolute constant (one can use a Chernoff-type inequality to prove this, or just do it “by hand”). Unwinding our argument, what we have shown is that, provided $1 \leq t \leq n^2$ and standing still. The walk will be back at node 1 at time $t$.

For any $i$, $j$ and $t \geq 0$, let $h_{i,j}(t)$ denote the expected number of times a walk started at $i$ will hit $j$ up to and including time $t$, i.e.:

\[
(2.2) \quad h_{i,j}(t) = \mathbb{E}[\# s : X_s = j, 0 \leq s \leq t \mid X_0 = i].
\]

Then $h_{i,j}(t) \leq c_1 \cdot \sqrt{t}$ for all $i$, $j$ and all $1 \leq t \leq n^2$, where $c_1 > 0$ is an absolute constant.

Proof. This result surely follows from standard textbook facts about random walks on graphs, but since we cannot point to a reference for the precise result, we shall outline a proof in any case.

We shall obtain an appropriate bound when $(i, j) = (1, 1)$. It will be clear that a similar argument can be applied for any pair $(i, j)$ and yield a constant $c_1$ independent of $i$ and $j$. Let us consider instead a cycle $C_{n+1}$ of length $n + 1$, with vertices indexed by $1, 2, \ldots, n + 1$, and a random walk on the cycle for which the transition probabilities are $p'_{i,j} = 1/3$ if $|i - j| (\text{mod } n + 1) \leq 1$ and $p'_{i,j} = 0$ otherwise. Let $h'(t)$ denote the expected number of times a walk started at node 1 hits either 1 or 2 up to and including time $t$. It is quite clear that $h_{1,1}(t) \leq h'(t)$. Furthermore, it is easy to see (indeed, the argument to follow will explain why) that in turn $h'(t) \leq 2 \cdot h''(t)$, where the latter is the expected number times the walk started at node 1 hits 1 itself up to time $t$ (including $t = 0$). It thus suffices to prove that $h''(t) = O(\sqrt{t})$ for all $1 \leq t \leq n^2$. We go one step further. Let $q(t)$ denote the probability that the walk on $C_{n+1}$, started at node 1, is also at node 1 at time $t$. By linearity of expectation, it suffices to prove that $q(t) = O(1/\sqrt{t})$ for all $1 \leq t \leq n^2$.

So fix a time $t \geq 1$. Any walk consists of steps of three types: clockwise, anticlockwise and standing still. The walk will be back at node 1 at time $t$ if and only if the numbers of clockwise and anticlockwise steps among the first $t$ steps are congruent modulo $n + 1$. The expected number of standing still steps is $t/3$ and, up to an error of order $e^{-\alpha t}$, where $\alpha > 0$ is an absolute constant, we can ignore all walks where the number of standing still steps is greater than $t/2$ say. Conditioned on the number $l$ of such steps and their timings, there are $2^{l-1}$ possible walks. The number of these which have $c$ clockwise steps is $\binom{t-l}{c}$, which is less than $\frac{2^{l-1}}{\sqrt{l}}$ for any $c$ and maximised at $c = \lfloor \frac{t}{2} \rfloor$. Since we’re assuming $l \leq t/2$, it follows that every binomial coefficient is less than $2^{t-l} \sqrt{l}$. The ones that contribute to $q(t)$ are those such that $2c \equiv t - l \pmod{n+1}$. The gap between any two such values of $c$ is at least $(n+1)/2$ which, since $t \leq n^2$, is $\Omega(\sqrt{t})$. Now it is easy to see that, for any positive integer $n$, the rate at which binomial coefficients at level $n$ decrease as one moves away from $\lfloor n/2 \rfloor$ satisfy

\[
(2.3) \quad \binom{n}{\lfloor n/2 \rfloor + \xi \sqrt{n}} \leq \kappa^\xi \binom{n}{\lfloor n/2 \rfloor},
\]

where $\kappa < 1$ is an absolute constant (one can use a Chernoff-type inequality to prove this, or just do it “by hand”). Unwinding our argument, what we have shown is that, provided $1 \leq t \leq n^2$ and conditioning on the number and timing of all standing still steps up to time $t$, the probability of the walk being back at node 1 is $O(1/\sqrt{t}) + O(e^{-\alpha t}) = O(1/\sqrt{t})$. Hence, $q(t) = O(1/\sqrt{t})$, as desired. 

\[
\]
Lemma 2.2. Let \( n \in \mathbb{N}, \kappa \in \mathbb{R}_{>0} \) and, for \( t \geq 0 \), let \( \delta_t = (\delta_{1,t}, \ldots, \delta_{n,t}) \) be a sequence of vectors in \( \mathbb{Z}^n_{\geq 0} \) defined recursively as follows:

\[
\begin{align*}
\delta_0 &= (0, \ldots, 0), \\
\delta_{1,t+1} &= \kappa + \frac{2}{3} \delta_{1,t} + \frac{1}{3} \delta_{2,t}, \\
\delta_{n,t+1} &= \kappa + \frac{2}{3} \delta_{n,t} + \frac{1}{3} \delta_{n-1,t}, \\
\delta_{i,t} &= \frac{1}{3} (\delta_{i-1,t} + \delta_{i,t} + \delta_{i+1,t}), \quad \forall 2 \leq i \leq n-1.
\end{align*}
\]

Then there is an absolute constant \( c_2 > 0 \) such that \( \delta_{i,t} \leq c_2 \cdot \kappa \cdot \sqrt{t} \) for all \( i \) and all \( t \leq n^2 \).

Proof. For any \( t \), it is clear that \( \delta_{i,t} = \delta_{(n+1)-i,t} \) and that \( \delta_{i,t} \geq \delta_{i+1,t} \) for all \( i < n/2 \). It thus suffices to prove that \( \delta_{1,t} = O(\kappa \sqrt{t}) \) for all \( t \leq n^2 \).

The recursion can be written in matrix form as

\[
\begin{align*}
\delta_0 &= 0, \\
\delta_{t+1} &= v + P \cdot \delta_t,
\end{align*}
\]

where \( v = (\kappa, 0, 0, \ldots, 0, \kappa)^T \) and \( P = (p_{i,j}) \) is the transition matrix of (2.1). It follows easily from (2.8) and (2.9) that, for any \( t > 0 \),

\[
\delta_t = (I + P + \cdots + P^{t-1}) v.
\]

Hence,

\[
\delta_{1,t} = \kappa \cdot (h_{1,1}(t) + h_{1,n}(t))
\]

and so the result follows from Lemma 2.1.

Proof of Theorem 1.2. Let \( x_0 = D_n \in \mathbb{R}^{3n+1} \) and for all \( t > 0 \) let the updates \( x_t = (x_t(1), \ldots, x_t(3n+1)) \) be generated according to (1.1). So \( x_t \) represents the positions of the agents at time \( t \). We will find it more convenient to work instead with the vectors of gaps \( y_t = (y_{0,t}, \ldots, y_{n+1,t}) \in \mathbb{R}^{n+2} \) given by

\[
y_{i,t} = x_t(n+1+i) - x_t(n+i), \quad 0 \leq i \leq n+1.
\]

Observe that \( y_0 = (\frac{1}{n}, 1, \ldots, 1, \frac{1}{n}) \). Let \( G_t \) denote the receptivity graph at time \( t \). For as long as \( G_t = G_0 \), it is easily checked that \( y_{i+1,t} = M \cdot y_t \) where \( M = (m_{i,j}) \) is an \( (n+2) \times (n+2) \) matrix whose upper left 2 \( \times \) 3 block is

\[
\begin{pmatrix}
\frac{n}{(n+1)(n+2)} & \frac{n+2}{2n+1} & 0 \\
\frac{n}{n+2} & \frac{1}{3(n+2)} & 0 \\
\end{pmatrix},
\]

which is symmetric about its midpoint, i.e.:

\[
m_{i,j} = m_{(n+3)-i,(n+3)-j}
\]

and which, for \( 3 \leq i \leq n \), satisfies

\[
m_{i,j} = \begin{cases} 1/3, & \text{if } |i-j| \leq 1, \\ 0, & \text{otherwise.} \end{cases}
\]

We define auxiliary vectors \( \delta_t = (\delta_{0,t}, \ldots, \delta_{n+1,t}) \) as follows:

\[
y_{i,t} = \frac{1}{n} - \frac{\delta_{i,t}}{n^2}, \quad \text{if } i = 0 \text{ or } i = n+1,
\]

\[
y_{i,t} = 1 - \frac{\delta_{i,t}}{n^2}, \quad \text{for } 1 \leq i \leq n.
\]
Observe that \( \delta_0 = 0 \) and \( \delta_{i,t} = \delta_{(n+1)-i,t} \) for all \( i \) and \( t \). As long as \( G_t = G_0 \) one checks that the following recursion is satisfied:

\[
\delta_{0,t+1} \leq 1 + \frac{1}{n} (\delta_{0,t} + \delta_{1,t} - 1),
\]

\[
\delta_{1,t+1} \leq \delta_{0,t} + \frac{2}{3} \delta_{1,t} + \frac{1}{3} \delta_{2,t},
\]

\[
\delta_{i,t+1} = \frac{1}{3} (\delta_{i-1,t} + \delta_{i,t} + \delta_{i+1,t}) \quad \text{for} \quad 2 \leq i \leq n-1.
\]

From Lemma 2.2 it is easy to deduce that, for some absolute constant \( c_3 > 0 \) and all \( t \leq c_3 \cdot n^2 \), the solution to \( (2.18)-(2.20) \) with initial condition \( \delta_0 = 0 \) will satisfy

\[
\delta_{0,t} \leq 2, \quad \delta_{n+1,t} \leq 2, \quad \delta_{i,t} < n-2 \quad \text{for} \quad 1 \leq i \leq n.
\]

But this in turn implies, from \( (2.16) \) and \( (2.17) \), that indeed it is true that \( G_t = G_0 \) for all \( t \leq c_3 \cdot n^2 \) and \( y_{0,t} + y_{1,t} > 1 \) for all such \( t \). In other words, agent \( n+2 \) will not be visible to the cluster on the left before time \( c_3 \cdot n^2 \), which proves that the configuration will take at least this long to freeze.

To prove that the configuration does indeed freeze in time \( O(n^2) \), we just need to turn the whole argument around somewhat. It is easy to deduce instead from the above relations that

\[
\delta_{0,t} \geq 1/2 \quad \text{for all} \quad t > 0 \quad \text{and hence, instead of} \quad (2.19), \quad \text{that}
\]

\[
\delta_{1,t+1} \geq \frac{1}{4} + \frac{2}{3} \delta_{1,t} + \frac{1}{3} \delta_{2,t}.
\]

The argument in Lemma 2.2 can then be turned on its head to deduce that \( \delta_{1,t} = \Omega(h_{1,1}(t)) \), while it is almost trivial that \( h_{1,1}(t) = \Omega(\frac{t}{n}) \). What all of this implies is that agent \( n+2 \) will indeed become visible to the cluster on the left at time \( t^* = \Theta(n^2) \), and it will then immediately disconnect from agent \( n+3 \). We then just need to consider the subsequent evolution of the chain \( \mathcal{C} \) of agents \( n+3, \ldots, 2n-2 \). Since \( \delta_{i,t} = O(n) \) for every \( i \), it follows from \( (2.16) \) and \( (2.17) \) that the gaps between consecutive agents in \( \mathcal{C} \) are all greater than \( 1 - O(1/n) \). Hence the chain will freeze in time \( 5n/6 + O(1) \). This last deduction follows from the results in [HW], more precisely from Theorem 1.1 and remarks at the outset of Section 3 in that paper.

This completes the proof of Theorem 1.2.

**Remark 2.3.** We have not tried to compute accurate constant factors in the proof of Theorem 1.2. However, a combination of simulations and the Ockham’s razor principle lead us to believe that the freezing time for \( \mathcal{D}_n \) is \( (1 + o(1)) \frac{n^2}{4} \). The factor of \( 4 = 2^2 \) comes from the fact that the numbers \( \delta_{1,t} \) in \( (2.17) \) seem to grow like \( 2 \sqrt{t} \).

Note that, if our hypothesis is correct, then the configuration \( \mathcal{D}_n \) still grows more slowly, at least for \( n \gg 0 \), than the two-dimensional configuration \( \mathcal{F}_{3n+1} \). These are also two quite different types of configurations. Hence, our work seems to make it even less clear what the right estimate for the function \( f_k(n) \) might be in higher dimensions.

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