ON SOME MEAN VALUE RESULTS FOR THE ZETA-FUNCTION IN SHORT INTERVALS

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ABSTRACT. Let $\Delta(x)$ denote the error term in the Dirichlet divisor problem, and let $E(T)$ denote the error term in the asymptotic formula for the mean square of $|\zeta(1/2 + it)|$. If $E^*(t) := E(t) - 2\pi \Delta^*(t/(2\pi))$ with $\Delta^*(x) = -\Delta(x) + 2\Delta(2x) - 1/2 \Delta(4x)$ and $\int_0^T E^*(t) \, dt = \frac{3}{4} \pi T + R(T)$, then we obtain a number of results involving the moments of $|\zeta(1/2 + it)|$ in short intervals, by connecting them to the moments of $E^*(T)$ and $R(T)$ in short intervals. Upper bounds and asymptotic formulas for integrals of the form

$$\int_T^{2T} \left( \int_{t-H}^{t+H} |\zeta(1/2 + iu)|^2 \, du \right)^k \, dt \quad (k \in \mathbb{N}, 1 \ll H \leqslant T)$$

are also treated.

1. INTRODUCTION

As usual, let

$$\Delta(x) := \sum_{n \leqslant x} d(n) - x(\log x + 2\gamma - 1)$$

(1.1)

denote the error term in the classical Dirichlet divisor problem. Also let

$$E(T) := \int_0^T |\zeta(1/2 + it)|^2 \, dt - T \left( \log \left( \frac{T}{2\pi} \right) + 2\gamma - 1 \right)$$

(1.2)

denote the error term in the mean square formula for $|\zeta(1/2 + it)|$. Here $d(n)$ is the number of divisors of $n$, $\zeta(s)$ is the Riemann zeta-function, and $\gamma = -\Gamma'(1) = 0.085$. 

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0.577215... is Euler’s constant. In view of F.V. Atkinson’s classical explicit formula for $E(T)$ (see [1], [4, Chapter 15] and [5, Chapter 2]) it was known long ago that there are analogies between $\Delta(x)$ and $E(T)$. However, in this context it seems that instead of the error-term function $\Delta(x)$ it is more exact to work with the modified function $\Delta^*(x)$ (see M. Jutila [14], [15] and T. Meurman [17]), where

$$\Delta^*(x) := -\Delta(x) + 2\Delta(2x) - \frac{1}{2} \Delta(4x)$$

(1.3)

$$= \frac{1}{2} \sum_{n \leq 4x} (-1)^n d(n) - x(\log x + 2\gamma) - 1,$$

since it turns out that $\Delta^*(x)$ is a better analogue of $E(T)$ than $\Delta(x)$. Namely, M. Jutila (op. cit.) investigated both the local and global behaviour of the difference

$$E^*(t) := E(t) - 2\pi \Delta^*(\frac{t}{2\pi}),$$

and in particular in [14] he proved that

$$\int_T^{T+H} (E^*(t))^2 \, dt \ll_{\varepsilon} HT^{1/3} \log^3 T + T^{1+\varepsilon} \quad (1 \leq H \leq T).$$

(1.4)

Here and later $\varepsilon$ denotes positive constants which are arbitrarily small, but are not necessarily the same ones at each occurrence, while $a \ll_{\varepsilon} b$ (same as $a = O_{\varepsilon}(b)$) means that the $\ll$-constant depends on $\varepsilon$. The significance of (1.4) is that, in view of (see e.g., [4])

$$\int_0^T (\Delta^*(t))^2 \, dt \sim AT^{3/2},$$

$$\int_0^T E^2(t) \, dt \sim BT^{3/2} \quad (A, B > 0, T \to \infty),$$

it transpires that $E^*(t)$ is in the mean square sense of a lower order of magnitude than either $\Delta^*(t)$ or $E(t)$.

Later works provided more results on the mean values of $E^*(T)$. Thus in [9] the author sharpened (1.4) (in the case when $H = T$) to the asymptotic formula

$$\int_0^T (E^*(t))^2 \, dt = T^{4/3} P_3(\log T) + O_{\varepsilon}(T^{7/6+\varepsilon}),$$

(1.5)

where $P_3(y)$ is a polynomial of degree three in $y$ with positive leading coefficient, and all the coefficients may be evaluated explicitly. This, in particular, shows that (1.4) may be complemented with the lower bound

$$\int_T^{T+H} (E^*(t))^2 \, dt \gg HT^{1/3} \log^3 T \quad (T^{5/6+\varepsilon} \leq H \leq T),$$

(1.6)
which is implied by (1.5). It seems likely that the error term in (1.5) is $O_\varepsilon T^{1+\varepsilon}$, but this seems difficult to prove. In [12] the author showed that (1.6) remains true for $T^{2/3+\varepsilon} \leq H \leq T$.

In what concerns higher moments of $E^*(T)$ the author proved ([6, Part 4])

$$
(1.7) \quad \int_0^T |E^*(t)|^3 \, dt \ll_\varepsilon T^{3/2+\varepsilon},
$$
in [6, Part 2] that

$$
(1.8) \quad \int_0^T |E^*(t)|^5 \, dt \ll_\varepsilon T^{2+\varepsilon},
$$
so that by the Cauchy-Schwarz for integrals (1.7) and (1.8) yield

$$
(1.9) \quad \int_0^T (E^*(t))^4 \, dt \ll_\varepsilon T^{7/4+\varepsilon}.
$$

In part [6, Part 3] the error-term function $R(T)$ was introduced by the relation

$$
(1.10) \quad \int_0^T E^*(t) \, dt = \frac{3\pi}{4} T + R(T).
$$

It was shown, by using an estimate for two-dimensional exponential sums, that

$$
(1.11) \quad R(T) = O_\varepsilon (T^{593/912+\varepsilon}), \quad \frac{593}{912} = 0, 6502129 \ldots .
$$

In the same paper it was also proved that

$$
(1.12) \quad \int_0^T R^2(t) \, dt = T^2 p_3(\log T) + O_\varepsilon (T^{11/6+\varepsilon}),
$$
where $p_3(y)$ is a cubic polynomial in $y$ with positive leading coefficient, whose all coefficients may be explicitly evaluated, and

$$
(1.13) \quad \int_0^T R^4(t) \, dt \ll_\varepsilon T^{3+\varepsilon}.
$$

The asymptotic formula (1.12) bears resemblance to (1.5), and it is proved by a similar technique. The exponents in the error terms are, in both cases,
less than the exponent of $T$ in the main term by $1/6$. From (1.5) one obtains that $E^*(T) = \Omega(T^{1/6}(\log T)^{3/2})$, which shows that $E^*(T)$ cannot be too small ($f(x) = \Omega(g(x))$ means that $f(x) = o(g(x))$ does not hold as $x \to \infty$). Likewise, (1.12) yields

$$R(T) = \Omega\left(T^{1/2}(\log T)^{3/2}\right).$$

It seems plausible that the error term in (1.12) should be $O_\varepsilon(T^{5/3+\varepsilon})$, and one may conjecture that

$$R(T) = O_\varepsilon(T^{1/2+\varepsilon})$$

holds, which is supported by (1.12). In [12] it was proved that, in the range $T^{2/3+\varepsilon} \leq H \leq T$, we have

$$\int_T^{T+H} R^2(t) \, dt \gg HT \log^3 T,$$

and, for $T^{\varepsilon} \leq H \leq T$,

$$\int_T^{T+H} R^2(t) \, dt \ll_{\varepsilon} HT \log^3 T + T^{5/3+\varepsilon}.$$ 

2. Statement of results

Mean values (or moments) of $|\zeta(\frac{1}{2} + it)|$ represent one of the central themes in the theory of $\zeta(s)$. There are two monographs dedicated solely to it: the author’s [5], and that of K. Ramachandra [18]. Our results connect bounds for the moments of $|\zeta(\frac{1}{2} + it)|$, $E^*(t)$ and $R(t)$ in short intervals. The meaning of “short interval” is that $[T, T + H]$ is such an interval where one can have $H$ much smaller than $T$, namely $H = o(T)$ as $T \to \infty$. The results are contained in

THEOREM 1. For $k \in \mathbb{N}$ fixed, $T^{1/3} \leq H = H(T) \leq T$, we have

$$\int_T^{T+H} |\zeta(\frac{1}{2} + it)|^{2k+2} \, dt \ll_k (\log T)^{k+2} \int_{T-H}^{T+2H} |E^*(t)|^k \, dt.$$ 

and

$$\int_T^{T+H} |E^*(t)|^{2k} \, dt \ll_k (\log T)^{k+2} \int_{T-H}^{T+2H} |R(t)|^k \, dt.$$
THEOREM 2. Let $k \in \mathbb{N}$ be fixed and $T^\varepsilon \leq H = H(T) \leq T$. If
\begin{equation}
\int_0^T |E^*(t)|^k \, dt \ll_{\varepsilon, k} T^{A(k)+\varepsilon}
\end{equation}
for some constant $A(k)$, then we must have $A(k) \geq 1 + k/6$, and
\begin{equation}
\int_0^{2T} \left( \int_{t-H}^{t+H} |\zeta(\frac{1}{2} + iu)|^2 \, du \right)^k \, dt \ll_{\varepsilon, k} T^{A(k)+\varepsilon} + TH^k (\log T)^k.
\end{equation}

When $k = 1$ or $k = 2$ a much more precise result can be obtained for the integral in (2.4). This is contained in

THEOREM 3. For $T^\varepsilon \leq H = H(T) \leq T$ we have
\begin{equation}
\int_0^{2T} \left( \int_{t-H}^{t+H} |\zeta(\frac{1}{2} + iu)|^2 \, du \right) \, dt = 2H \left( T \log \left( \frac{2T}{e\pi} \right) \right) + O(H^2) + O(T^{3/4}).
\end{equation}

and
\begin{equation}
\int_0^{2T} \left( \int_{t-H}^{t+H} |\zeta(\frac{1}{2} + iu)|^2 \, du \right)^2 \, dt \ll H^2 T (\log T)^2.
\end{equation}

For $T^\varepsilon \leq H = H(T) \leq T^{1/2-\varepsilon}$ we have the asymptotic formula
\begin{equation}
\int_0^{2T} \left( \int_{t-H}^{t+H} |\zeta(\frac{1}{2} + iu)|^2 \, du \right)^2 \, dt = H^2 T (4 \log^2 T + e_1 \log T + e_0)
\end{equation}

\begin{equation}
+ HT \sum_{j=0}^3 d_j \log^j \left( \frac{\sqrt{T}}{2H} \right) + O_\varepsilon (T^{1/2+\varepsilon} H^2) + O_\varepsilon (T^{1+\varepsilon} H^{1/2}).
\end{equation}

where the $d_j$'s and $e_1, e_0$ are suitable constants ($d_3 > 0$).

The proofs of Theorem 1, Theorem 2 and Theorem 3 will be given in Section 3. In Section 4 we shall provide some corollaries and remarks to these theorems.
3. Proofs of the Theorems

In (2.1) of Theorem 1 we have an estimate for the moments of $|\zeta(\frac{1}{2} + it)|$. In order to deal with these moments we shall use the standard large values technique (see e.g., [4, Chapter 8]). To transform discrete sums into sums of integrals one uses the bound

\begin{equation}
|\zeta(\frac{1}{2} + it)|^k \ll \log t \int_{t-1}^{t+1} |\zeta(\frac{1}{2} + ix)|^k \, dx + 1, \quad (k \in \mathbb{N} \text{ fixed})
\end{equation}

which is Theorem 1.2 of [5] (see also Lemma 7.1 of [4]).

We begin (henceforth let $L = \log T$ for brevity) by noting that, for $T^\varepsilon \ll G \leq T$,

\begin{align*}
\int_{T-G}^{T+G} |\zeta(\frac{1}{2} + it)|^2 \, dt &= \int_{-G}^{G} |\zeta(\frac{1}{2} + iT + iu)|^2 \, du \\
&\leq e \int_{-\infty}^{\infty} |\zeta(\frac{1}{2} + iT + iu)|^2 e^{-\left(u/G\right)^2} \, du \\
&= e \int_{-GL}^{GL} |\zeta(\frac{1}{2} + iT + iu)|^2 e^{-\left(u/G\right)^2} \, du + O(e^{-\frac{1}{4}L^2}).
\end{align*}

In view of (1.2) we further have, on integrating by parts,

\begin{align*}
\int_{-GL}^{GL} |\zeta(\frac{1}{2} + iT + iu)|^2 e^{-\left(u/G\right)^2} \, du &= \int_{-GL}^{GL} e^{-\left(u/G\right)^2} \, dE(T+u) + O(GL) \\
&= 2 \int_{-GL}^{GL} uG^{-2} e^{-\left(u/G\right)^2} E(T+u) \, du + O(GL).
\end{align*}

By the definition of $E^*(T)$ the last integral becomes

\begin{equation}
\frac{1}{G^2} \int_{-GL}^{GL} xE^*(T+x) e^{-\left(x/G\right)^2} \, dx + \frac{2\pi}{G^2} \int_{-GL}^{GL} x\Delta^* \left(\frac{T+x}{2\pi}\right) e^{-\left(x/G\right)^2} \, dx.
\end{equation}

To bound the integral containing the $\Delta^*$ function, we shall use the estimate

\begin{equation}
\sum_{x^\varepsilon n \leq x+h} d(n) \ll h \log x \quad (x^\varepsilon \leq h \leq x),
\end{equation}

which follows from a general result of P. Shiu [19] on multiplicative functions. Write

\begin{equation}
\int_{-GL}^{GL} x\Delta^* \left(\frac{T+x}{2\pi}\right) e^{-\left(x/G\right)^2} \, dx = \int_{-GL}^{0} \cdots \, dx + \int_{0}^{GL} \cdots \, dx,
\end{equation}
and make the change of variable $y = -x$ in the first integral on the right-hand side. Then (3.3) becomes

$$
\int_0^{GL} y \Delta^* \left( \frac{T - y}{2\pi} \right) e^{-\left(\frac{y}{G}\right)^2} \, dy + \int_0^{GL} x \Delta^* \left( \frac{T + x}{2\pi} \right) e^{-\left(\frac{x}{G}\right)^2} \, dx
$$

$$
= \int_0^{GL} x \left\{ \Delta^* \left( \frac{T + x}{2\pi} \right) - \Delta^* \left( \frac{T - x}{2\pi} \right) \right\} e^{-\left(\frac{x}{G}\right)^2} \, dx.
$$

For $|x| \leq T^{\varepsilon/3}$ we use the trivial bound (coming from $d(n) \ll \varepsilon n^{\varepsilon/3}$)

$$
\Delta^* \left( \frac{T + x}{2\pi} \right) - \Delta^* \left( \frac{T - x}{2\pi} \right) \ll \varepsilon T^{2\varepsilon/3},
$$

while for $T^{\varepsilon/3} < |x| \leq GL$ we use (3.2). This yields

$$
\int_0^{GL} x \left\{ \Delta^* \left( \frac{T + x}{2\pi} \right) - \Delta^* \left( \frac{T - x}{2\pi} \right) \right\} e^{-\left(\frac{x}{G}\right)^2} \, dx
\ll \varepsilon T^{2\varepsilon/3} G^2 + G^3 L \ll G^3 L,
$$

since $G \gg T^{\varepsilon}$. Therefore (3.4) furnishes the bound

$$
\frac{2\pi}{G^2} \int_{-GL}^{GL} x \Delta^* \left( \frac{T + x}{2\pi} \right) e^{-\left(\frac{x}{G}\right)^2} \, dx \ll GL,
$$

and we obtain the starting point for the proof of (2.1), which we formulate as

**Lemma 1.** For $T^{\varepsilon} \leq G = G(T) \leq T, L = \log T$ we have

$$
\int_{T-G}^{T+G} |\zeta(\frac{1}{2} + it)|^2 \, dt \leq \frac{2e}{G^2} \int_{-GL}^{GL} x E^*(T + x) e^{-\left(\frac{x}{G}\right)^2} \, dx + O(GL).
$$

We return to the proof of (2.1) and suppose now that $\{t_r\}_{r=1}^R$ is a set of points satisfying

$$
T < t_1 < \ldots < t_R \leq T + H, |\zeta(\frac{1}{2} + it_r)| \geq V, |t_r - t_s| \geq 1 \ (r = 1, \ldots, R).
$$

We use (3.1) and group the intervals $[t_r - 1, t_r + 1]$ into disjoint subintervals of the form

$$
[t_s - G, t_s + G] \ (s = 1,2,\ldots, S \leq R, G \ll H).
$$
Then by Lemma 1 we obtain (we may suppose that the sum over $s$ below is the largest of several sums of the same type)

$$RV^2 \ll L \sum_{s=1}^{S} \int_{T_s-G}^{T_s+G} |\zeta(\frac{1}{2} + it)|^2 \, dt$$

(3.7)

$$\leq 2eL \sum_{s=1}^{S} \frac{1}{G^2} \int_{-GL}^{GL} xe^*(\tau_s + x)e^{-(x/G)^2} \, dx,$$

provided that, for some sufficiently small $c > 0$, we choose

$$G = cV^2/L.$$

By bounds for $|\zeta(\frac{1}{2} + it)|$ we obtain $G \ll T^{1/3} \ll H$, and we choose a representative set of points $\tau_\ell, \ell = 1, \ldots, S' (\leq S)$ from the set $\{\tau_s\}_{s=1}^{S}$ such that the intervals $(\tau_\ell - GL, \tau_\ell + GL)$ are disjoint for $\ell = 1, \ldots, S'$. Therefore it follows by Hölder’s inequality for integrals that

$$RV^2 \ll L \sum_{\ell=1}^{S'} G^{-2} \int_{-GL}^{GL} |xe^*(\tau_\ell + x)|e^{-(x/G)^2} \, dx$$

$$\ll LG^{-2} \sum_{\ell=1}^{S'} \left( \int_{-GL}^{GL} |e^*(\tau_\ell + x)|^k e^{-(x/G)^2} \, dx \right)^{\frac{1}{k}} \left( \int_{-GL}^{GL} |x|^k e^{-(x/G)^2} \, dx \right)^{1 - \frac{1}{k}}$$

$$\ll LG^{-2} \left( \sum_{\ell=1}^{S'} \int_{-GL}^{GL} |e^*(\tau_\ell + x)|^k e^{-(x/G)^2} \, dx \right)^{\frac{1}{k}} G^{2 - \frac{1}{k}} (S')^{1 - \frac{1}{k}}$$

$$\ll G^{-\frac{1}{k}} LS^{1 - \frac{1}{k}} \left( \int_{T-H}^{T+2H} |E^*(x)|^k \, dx \right)^{\frac{1}{k}}.$$

Since $S \leq R$, in view of (3.8) this gives

$$R \ll V^{-2k}G^{-1}L^k \int_{T-H}^{T+2H} |E^*(x)|^k \, dx$$

(3.9)

$$\ll V^{-2k-2}L^{k+1} \int_{T-H}^{T+2H} |E^*(x)|^k \, dx.$$

This is somewhat sharper than the bound proved by the author in [6, Part II], which contained $T^c$ instead of a log-power, and the result was stated for the “long”
interval $[T, 2T]$. The bound in (2.1) follows if the integral on the left-hand side is split into $O(\log T)$ subintegrals where $V \leq |\zeta(1/2 + it)| \leq 2V$. Denoting each such integral as $I_V$, we estimate it as

$$I_V \ll \sum_{r=1}^{RV} |\zeta(\frac{1}{2} + it_r)|^{2k+2} \ll RV^{2k+2} \ll L^{k+1} \int_{T-H}^{T+2H} |E^*(x)|^k \, dx,$$

where the points $t_r$ are chosen in such a way that $|t_r - t_s| \geq 1$ for $r \neq s$. Then (2.1) follows at once.

To prove (2.2) we need ($C$ denotes generic positive constants)

**Lemma 2.** For $T^\varepsilon \leq G = G(T) \leq T$, $t \asymp T$, $L = \log T$ we have

(3.10) $$E^*(t) \leq \frac{C}{G} \int_t^{t+G} \varphi_+(u)E^*(u) \, du + CGL,$$

and

(3.11) $$E^*(t) \geq \frac{C}{G} \int_{t-G}^t \varphi_-(u)E^*(u) \, du - CGL.$$

Here $\varphi_+$ is a non-negative, smooth function supported in $[t, t+G]$ such that $\varphi_+(u) = 1$ for $t + G/4 \leq u \leq t + 3G/4$. Similarly, in (3.11) $\varphi_-$ is a non-negative, smooth function supported in $[t - G, t+G]$ such that $\varphi_-(u) = 1$ for $t - 3G/4 \leq u \leq t - G/4$.

The proof of these inequalities is similar, so it suffices only to prove (3.10). From (1.2) we have, for $0 \leq u \ll T$,

$$0 \leq \int_T^{T+u} |\zeta(\frac{1}{2} + it)|^2 \, dt = (T + u) \left( \log \left( \frac{T + u}{2\pi} \right) + 2\gamma - 1 \right) - T \left( \log \left( \frac{T}{2\pi} \right) + 2\gamma - 1 \right) + E(T + u) - E(T).$$

By the mean-value theorem this implies

$$E(T) \leq E(T + u) + O(u \log T),$$

giving by integration and change of notation

(3.12) $$E(t) \leq \frac{C}{G} \int_t^{t+G} \varphi_+(u)E(u) \, du + CG \log T \quad (1 \ll G \ll T, \ C > 0, \ t \asymp T).$$
By using (3.2) again it is established that, for $T^\varepsilon \leq G \leq T$, $t \asymp T$,

$$
2\pi \Delta^* \left( \frac{t}{2\pi} \right) = \frac{C}{G} \int_t^{t+G} \varphi_+(u) \Delta^* \left( \frac{u}{2\pi} \right) \, du + O(G \log T).
$$

(3.13)

Therefore by combining (3.12) and (3.13) one obtains (3.10), since

$$
E^*(t) = E(t) - 2\pi \Delta^* \left( \frac{t}{2\pi} \right).
$$

In proving (2.2) we use (3.10) if $E^*(t) > 0$, and (3.11) otherwise. Suppose $E^*(t) > 0$. Then by integrations by parts we obtain from (3.10)

$$
E^*(t) \leq \frac{C}{G} \int_t^{t+G} \varphi_+(u) E^*(u) \, du + CGL
$$

(3.14)

$$
= - \frac{C}{G} \int_t^{t+G} \left( \frac{3\pi}{4} u + R(u) \right) \varphi'_+(u) \, du + CGL
$$

$$
= - \frac{C}{G} \int_t^{t+G} \left( u \varphi_+(u) \bigg|_{u=t}^{t+H} - \int_t^{t+G} \varphi_+(u) \, du \right) - \frac{C}{G} \int_t^{t+G} R(u) \varphi'_+(u) \, du + CGL
$$

$$
= O(GL) - \frac{C}{G} \int_t^{t+G} R(u) \varphi'_+(u) \, du.
$$

Combining (3.14) with the corresponding lower bound and using the fact that

$$
\varphi'_+(u) \ll \frac{1}{G},
$$

it follows that we have proved

LEMMA 3. For $T^\varepsilon \leq G = G(T) \leq T$, $t \asymp T$, we have

$$
|E^*(t)| \ll \frac{1}{G^2} \int_{t-G}^{t+G} |R(u)| \, du + CGL.
$$

(3.15)

If we suppose that $R(T) \ll_t T^{\alpha+\varepsilon}$ then from (3.15), (3.5) of Lemma 1 and (3.1) we obtain

$$
\zeta \left( \frac{1}{2} + it \right) \ll_t t^{\alpha/4+\varepsilon}, \quad E^*(T) \ll_t T^{\alpha/2+\varepsilon},
$$

(3.16)
so that with the value \( \alpha = \frac{593}{912} = 0.6502129 \ldots \) (see (1.11)) we have the bounds

\[
(3.17) \quad \zeta\left(\frac{1}{2} + it\right) \ll \varepsilon |t|^\frac{593}{3648}, \quad 593/3648 = 0.16255 \ldots,
\]
\[
E^*(T) \ll \varepsilon T^{\frac{593}{1824}}, \quad 593/1824 = 0.32510 \ldots .
\]

If the conjectural \( \alpha = \frac{1}{2} \) held (\( \alpha < \frac{1}{2} \) is impossible by (1.14)), then we would obtain from (3.15)

\[
\zeta\left(\frac{1}{2} + it\right) \ll \varepsilon |t|^\frac{1}{8}, \quad E^*(T) \ll \varepsilon T^{\frac{1}{4}},
\]

which is out of reach by present day methods. See (4.5) for the best known bound for \( \zeta\left(\frac{1}{2} + it\right) \); the best known exponent for \( E^*(T) \) is \( \frac{131}{416} = 0.31490 \ldots \). This exponent was proved for \( E(T) \) by N. Watt [20], but since the same exponent holds for \( \Delta(x) \) and \( \Delta^*(x) \), it holds for \( E^*(T) \) as well. Thus, although the bounds in (3.17) are non-trivial, they are not the best ones known at present.

We return now to our proof of (2.2). Suppose now that \(|E^*(t)| \geq V\) on a set of points \( \{t_r\}_{r=1}^R \) lying in \([T, T + H]\) and spaced at least \( CG \) apart. We take \( G = \delta V/L \) \((< H)\) for sufficiently small \( \delta > 0 \). Then from (3.15) we have, for a representative set of the \( t_r \)'s such that the intervals \((t_r - G, t_r + G)\) are disjoint,

\[
\mathcal{R} V^2 L^{-2} \ll \sum_{r=1}^R \int_{t_r - G}^{t_r + G} |R(u)| \, du
\]
\[
\ll \sum_{r=1}^R \left( \int_{t_r - G}^{t_r + G} |R(u)|^k \, du \right)^{\frac{1}{k}} G^{1 - \frac{1}{k}}
\]
\[
\ll \left( \sum_{r=1}^R \int_{t_r - G}^{t_r + G} |R(u)|^k \, du \right)^{\frac{1}{k}} (RG)^{1 - \frac{1}{k}},
\]
on applying Hölder’s inequality for integrals. Since the intervals \((t_r - G, t_r + G)\) are disjoint, and their union is contained in \([T - H, T + 2H]\), the preceding bound gives us

\[
\mathcal{R} \ll \int_{T - H}^{T + 2H} |R(u)|^k \, du \cdot V^{-3k} L^{2k} G^{k-1},
\]

which simplifies to

\[
(3.18) \quad \mathcal{R} \ll \int_{T - H}^{T + 2H} |R(u)|^k \, du \cdot V^{-1-2k} L^{k+1}.
\]

Splitting \( \int_{T - H}^{T + H} |E^*(t)|^{2k} \, dt \) into \( O(\log T) \) integrals \( I_V \) where

\[
V \leq |E^*(t)| \leq 2V,
\]
we estimate each of these integrals by (3.18), keeping in mind that $V \leq T^{1/3} \ll H$. The bound in (2.2) follows at once.

An obvious corollary of Theorem 1 is that
\[(3.19)\]
\[\int_T^{T+H} |\zeta(\frac{1}{2} + it)|^{4k+2} dt \ll (\log T)^{3k+4} \int_{T-2H}^{T+4H} |R(t)|^k dt \quad (T^{1/3} \ll H \ll T).\]

From (1.17) and (3.19) with $k = 2$ we obtain
\[(3.20)\]
\[\int_{T-H}^{T+H} |\zeta(\frac{1}{2} + it)|^{10} dt \ll \varepsilon T^\varepsilon (HT + T^{5/3}) \quad (T^{1/3} \ll H \ll T).\]

It seems that this bound is new in the range when $H$ is close to $T^{1/3}$. It gives, by (3.1), the classical bound $|\zeta(\frac{1}{2} + it)| \ll |t|^{1/6 + \varepsilon}$.

We shall now pass to the proof of Theorem 2. To obtain (2.4) we use (3.5) of Lemma 1 with $G \equiv H$. This gives, for fixed $k \in \mathbb{N}$, $T^\varepsilon \leq H = H(T) \leq T$,
\[(3.21)\]
\[\int_T^{2T} \left( \int_{t-H}^{t+H} |\zeta(\frac{1}{2} + i\nu)|^2 d\nu \right)^k dt \ll H^{-k} \int_T^{2T} \left( \int_{-HL}^{HL} |E^*(t + x)| e^{-(x/H)^2} dx \right)^k dt + TH^k L^k.\]

Hölder’s inequality for integrals shows that the integral on the right-hand side of (3.21) is
\[(3.22)\]
\[\ll \int_T^{2T} \int_{-HL}^{HL} |E^*(t + x)|^k e^{-(x/H)^2} dx \cdot \left( \int_{-HL}^{HL} e^{-(x/H)^2} dx \right)^{k-1} dt \ll H^{k-1} \int_{-HL}^{HL} e^{-(x/H)^2} \left( \int_T^{2T+HL} |E^*(t + x)|^k dt \right) dx.\]

From (3.21) and (3.22) we obtain (2.4) if we take into account (2.3). Note that the constant $A(k)$ in (2.3) actually must satisfy $A(k) \geq 1 + k/6$ for any $k \geq 1$, and not necessarily when $k$ is an integer. If $k \geq 2$, then by Hölder’s inequality for integrals
\[\int_T^{2T} |E^*(t)|^2 dt \leq \left( \int_T^{2T} |E^*(t)|^k dt \right)^{2/k} T^{1-2/k},\]
and the desired bound for $A(k)$ follows from the mean square formula (1.5). If $1 \leq k \leq 2$ then it follows in a similar fashion from (1.5) and (1.7). We remark that if $A(k) = 1 + k/6$ holds for some $k$, then (2.1) and (3.1) yield the bound

$$\zeta\left(\frac{1}{2} + it\right) \ll \varepsilon \frac{t^{|\frac{k+6}{12(k+1)} + \varepsilon|}}{t},$$

and this improves the exponent $32/205 = 0, 15609 \ldots$ (see (4.5)) for $k \geq 5$, since for $k = 5$ it gives $11/72 = 0, 152777 \ldots$.

It remains to prove Theorem 3. We begin by noting that the author in [11] proved the following result, which improves on an earlier result of M. Jutila [16]: If $1 \ll U = U(T) \leq \frac{1}{2} \sqrt{T}$, then we have ($c_3 = 8 \pi^{-2}$)

$$\int_T^{2T} \left(\Delta(x + U) - \Delta(x)\right)^2 \, dx = TU \sum_{j=0}^{3} c_j \log^j \left(\frac{\sqrt{T}}{U}\right) + O_\varepsilon(T^{1/2 + \varepsilon} U^2) + O_\varepsilon(T^{1+\varepsilon} U^{1/2}),$$

a similar result being true if $\Delta(x + U) - \Delta(x)$ is replaced by $E(x + U) - E(x)$, with different constants $c_j$ ($c_3 > 0$). But the integral in (2.7) can be reduced to the evaluation of the mean square of $E(t + h) - E(t - h)$, since by (1.2) one has

$$\int_{t-H}^{t+H} |\zeta(\frac{1}{2} + it)|^2 \, dt = E(t+H) - E(t-H) + 2H \left(\log \left(\frac{t}{2\pi}\right) + 2\gamma\right) + O \left(\frac{H^2}{T}\right).$$

Therefore

$$\int_T^{2T} \left(\int_{t-H}^{t+H} |\zeta(\frac{1}{2} + it)|^2 \, dt\right)^2 \, dx = I_1 + 2I_2 + I_3,$$

say, where

$$I_1 := \int_T^{2T} (E(t + H) - E(t - H))^2 \, dt = \int_{T+H}^{2T+2H} (E(x + 2H) - E(x))^2 \, dx,$$

$$I_2 := \int_T^{2T} 2H \left(\log \frac{t}{2\pi} + 2\gamma + O \left(\frac{H}{T}\right)\right) (E(t + H) - E(t - H)) \, dt,$$

$$I_3 := \int_T^{2T} 4H^2 \left(\log \frac{t}{2\pi} + 2\gamma + O \left(\frac{H}{T}\right)\right)^2 \, dx.$$

To evaluate $I_1$ we write

$$I_1 = \int_{T+H}^{2T+2H} = \int_{T}^{2T} + \int_{2T}^{2T+2H} - \int_{T}^{T+H} = J_1 + J_2 - J_3,$$
say. By trivial estimation, in view of \( E(t) \ll t^{1/3} \) (see e.g., [4, Ch. 15]), it follows that
\[
J_2 - J_3 \ll HT^{2/3}.
\]
To evaluate \( J_1 \) we use the analogue of (3.23) (with \( U = 2H \)) for \( E(x + U) - E(x) \). This gives, with suitable constants \( d_j \) \((d_3 > 0)\) and \( 1 \ll H \ll \sqrt{T} \),
\[
J_1 = TH \sum_{j=0}^{3} d_j \log \left( \frac{\sqrt{T}}{2H} \right) + O_\varepsilon(T^{1/2+\varepsilon}H^2) + O_\varepsilon(T^{1+\varepsilon}H^{1/2}).
\]
One can evaluate \( I_3 \) in a straightforward way to obtain
\[
I_3 = 4H^2 \int_T^{2T} \left( \log^2 \left( \frac{t}{2\pi} \right) + 4\gamma^2 + 4\gamma + \log \left( \frac{t}{2\pi} \right) + O \left( \frac{H \log T}{T} \right) \right) \, dt
= H^2 T (4 \log^2 T + e_1 \log T + e_0) + O(H^3 \log T)
\]
with suitable constants \( e_0 \) and \( e_1 \).

Finally to bound \( I_2 \) we invoke the result of J.L. Hafner and the author [2], namely
\[
(3.26) \quad E_1(T) := \int_T^{T} E(u) \, du = \pi T + O(G(T)), \quad G(T) = O(T^{3/4}) \quad (T > 2).
\]
Actually in [2] an explicit expression is given for \( G(T) \) (from which one can deduce that \( G(T) = \Omega_\pm(T^{3/4}) \)). Thus from (3.25), (3.26) we obtain, on integrating by parts,
\[
I_2 = 2H \left\{ \left( E_1(t + H) - E_1(t - H) \right) \left( \log \frac{t}{2\pi} + 2\gamma \right) \right\} \bigg|_{t=T}^{2T}
- 2H \int_T^{2T} \left( E_1(t + H) - E_1(t - H) \right) \frac{dt}{t} + O(H^2 T^{1/3})
= O(H^2 \log T) + O(HT^{3/4} \log T) + O(H^2 T^{1/3}) = O(HT^{3/4} \log T)
\]
in view of the range for \( H \), namely \( T^\varepsilon \leq H = H(T) \leq T^{1/2-\varepsilon} \).

Combining the expressions for \( I_1, I_2 \) and \( I_3 \) we obtain (2.7), which in the range \( T^\varepsilon \leq H = H(T) \leq T^{1/2-\varepsilon} \) provides an asymptotic formula for the integral in question. Note that in this range \( HT^{3/4}L \ll T^{1+\varepsilon}H^{1/2} \), so only the error terms in (2.7) remain. For \( T^{1/2-\varepsilon} \leq H \leq T \) the upper bound in (2.6) follows easily from (2.4) and \( A(2) \leq 4/3 \), see (4.1).
It remains yet to prove (2.5). Note that, by (3.24), the integral in question is easily seen to be equal to

\[
(3.27) \quad 2H \int_T^{2T} \left( \log \frac{t}{2\pi} + O\left(\frac{H}{T}\right) \right) dt + \int_T^{2T} \left( E(t + H) - E(t - H) \right) dt.
\]

But by using (3.26) again it is seen that (3.27) reduces to

\[
2H \left( T \log \left( \frac{2T}{e\pi} \right) \right) + O(H^2) + O(T^{3/4}).
\]

Hence, for \( T^\varepsilon \leq H = H(T) \leq T \),

\[
\int_T^{2T} \left( \int_{t-H}^{t+H} |\zeta(\frac{1}{2} + iu)|^2 du \right) dt = 2H \left( T \log \left( \frac{4T}{e} \right) \right) + O(H^2) + O(T^{3/4}),
\]

as asserted by (2.5).

3. Some corollaries and remarks

If \( A(k) \) is defined by (2.3), then from (1.7)–(1.9) we have

\[
(4.1) \quad A(2) \leq \frac{4}{3}, \quad A(3) \leq \frac{3}{2}, \quad A(4) \leq \frac{7}{4}, \quad A(5) \leq 2.
\]

We also have \( A(1) \leq 7/6 \) by \( A(2) \leq 4/3 \) and the Cauchy-Schwarz inequality. Then (with \( H = T \)) (2.1) of Theorem 1 yields

\[
\int_0^T |\zeta(\frac{1}{2} + it)|^8 dt \ll_{\varepsilon} T^{3/2+\varepsilon},
\]

\[
(4.2) \quad \int_0^T |\zeta(\frac{1}{2} + it)|^{10} dt \ll_{\varepsilon} T^{7/4+\varepsilon},
\]

\[
\int_0^T |\zeta(\frac{1}{2} + it)|^{12} dt \ll_{\varepsilon} T^{2+\varepsilon},
\]

with \( k = 3, 4, 5 \), respectively. The bounds in (4.2) (up to \( T^\varepsilon \), which can be replaced by a log-factor) are the sharpest known bounds for the moments in question (see e.g., [4, Chapter 8]).

On the other hand, by using (1.4), we also have from (2.1)

\[
(4.3) \quad \int_T^{T+H} |\zeta(\frac{1}{2} + it)|^6 dt \ll_{\varepsilon} H T^{1/3} \log^{12} T + T^{1+\varepsilon} \quad (T^{1/3} \leq H \leq T).
\]
Although this is not trivial, it can be improved if one uses the bound of H. Iwaniec \[13\]
\[
\int_T^{T+H} |\zeta(\frac{1}{2} + it)|^4 \, dt \ll_\varepsilon T^\varepsilon (H + TH^{-1/2}) \quad (T^\varepsilon \leq H \leq T).
\]
The bound in (4.4) was obtained by sophisticated methods from the spectral theory of the non-Euclidean Laplacian, and if coupled with the best known bound of M.N. Huxley \[3\] for $|\zeta(\frac{1}{2} + it)|$, namely
\[
\zeta(\frac{1}{2} + it) \ll_\varepsilon |t|^{32/205+\varepsilon}, \quad 32/205 = 0.15609 \ldots,
\]
one gets an improvement of (4.3). Note that the famous, yet unsettled Lindelöf conjecture states that, instead of (4.5), one has $\zeta(\frac{1}{2} + it) \ll_\varepsilon |t|^\varepsilon$.

If we combine (1.16) and (2.2) (with $k = 2$), it follows that
\[
\int_T^{T+H} |E^s(t)|^4 \, dt \ll_\varepsilon HT \log^7 T + T^{5/3+\varepsilon} \quad (T^{1/3+\varepsilon} \leq H \leq T).
\]
The bound in (4.6) does not follow from (1.9), as it is better for $T^{2/3} \leq H \leq T^{3/4}$.

As a corollary to Theorem 2, we obtain with (4.1)
\[
\int_T^{2T} \left( \int_{t-H}^{t+H} |\zeta(\frac{1}{2} + it)|^2 \, du \right)^3 dt \ll_\varepsilon T^{3/2+\varepsilon} + TH^3 L^3,
\]
\[
\int_T^{2T} \left( \int_{t-H}^{t+H} |\zeta(\frac{1}{2} + it)|^2 \, du \right)^4 dt \ll_\varepsilon T^{7/4+\varepsilon} + TH^4 L^4,
\]
\[
\int_T^{2T} \left( \int_{t-H}^{t+H} |\zeta(\frac{1}{2} + it)|^2 \, du \right)^5 dt \ll_\varepsilon T^{2+\varepsilon} + TH^5 L^5.
\]
All the bounds in (4.7) are valid for $T^\varepsilon \leq H \leq T$, but as we have (see e.g., K. Ramachandra \[18\])
\[
\int_{t-H}^{t+H} |\zeta(\frac{1}{2} + it)|^{2k} \, dt \gg_k H (\log H)^{k^2} \quad (\log \log T \ll H \leq T, \, k \in \mathbb{N}),
\]
we have the expected upper bounds $T(HL)^m$ ($m = 3, 4, 5$) for the integrals in (4.7). Indeed, we obtain from (4.7)
\[
\int_T^{2T} \left( \int_{t-H}^{t+H} |\zeta(\frac{1}{2} + it)|^2 \, du \right)^3 dt \ll_\varepsilon TH^3 L^3 \quad (H \geq T^{1/6+\varepsilon}),
\]
\[
\int_T^{2T} \left( \int_{t-H}^{t+H} |\zeta(\frac{1}{2} + it)|^2 \, du \right)^4 dt \ll_\varepsilon TH^4 L^4 \quad (H \geq T^{3/16+\varepsilon}),
\]
\[
\int_T^{2T} \left( \int_{t-H}^{t+H} |\zeta(\frac{1}{2} + it)|^2 \, du \right)^5 dt \ll_\varepsilon TH^5 L^5 \quad (H \geq T^{1/5+\varepsilon}).
\]
The bounds in (4.8) seem to be the best unconditional bounds yet.

Note that for the analogous, but less difficult, problem of moments of
\[
J_k(t, G) := \frac{1}{\sqrt{\pi G}} \int_{-\infty}^{\infty} |\zeta(\frac{1}{2} + it + iu)|^{2k} e^{-(u/G)^2} \, du \quad (t \approx T, T^\varepsilon \leq G \ll T),
\]
where \(k\) is a natural number, we refer the reader to the author’s work [7]. Not only do we have
\[
\int_{-T}^{T+G} |\zeta(\frac{1}{2} + it)|^{2k} \, dt = \int_{-G}^{G} |\zeta(\frac{1}{2} + iT + iu)|^{2k} \, du \leq \sqrt{\pi} e G J_k(T, G),
\]
but the presence of the smooth Gaussian exponential factor in \(J_k(T, G)\) facilitates the ensuing estimations. We have (this is [7, Theorem 1])
\[
(4.9) \quad \int_{T}^{2T} J_1^m(t, G) \, dt \ll_{\varepsilon} T^{1+\varepsilon}
\]
for \(T^\varepsilon \leq G \leq T\) if \(m = 1, 2\); for \(T^{1/7+\varepsilon} \leq G \leq T\) if \(m = 3\), and for \(T^{1/5+\varepsilon} \leq G \leq T\) if \(m = 4\); and these bounds were sharpened in [10] to \(T^{7/36} \leq G \leq T\) when \(m = 4\), \(T^{1/5} \leq G \leq T\) when \(m = 5\) and \(T^{2/9} \leq G \leq T\) when \(m = 6\). The bounds in (4.9) can be compared to those in (4.8).

We remark that in [8] the author proved that
\[
(4.10) \quad \int_{T}^{2T} \left( \int_{t-G}^{t+G} |\zeta(\frac{1}{2} + iu)|^4 \, du \right)^2 \, dt \ll_{\varepsilon} G^2 T^{1+\varepsilon}
\]
for \(T^{1/2} \leq G = G(T) \ll T\). In fact, (4.10) is connected with the following, more general result (Theorem 1 of [8]): Let \(T < t_1 < t_2 < \ldots < t_R < 2T\), \(t_{r+1} - t_r \geq G\) for \(r = 1, \ldots, R - 1\). If, for fixed \(m, k \in \mathbb{N}\), we have
\[
(4.11) \quad \int_{T}^{2T} \left( \frac{1}{G} \int_{t-G}^{t+G} |\zeta(\frac{1}{2} + iu)|^{2k} \, du \right)^m \, dt \ll_{\varepsilon} T^{1+\varepsilon}
\]
for \(T^{\alpha_{k,m}} \leq G = G(T) \ll T\) and \(0 \leq \alpha_{k,m} \leq 1\), then
\[
\sum_{r=1}^{R} \int_{t_r-G}^{t_{r+1}+G} |\zeta(\frac{1}{2} + it)|^{2k} \, dt \ll_{\varepsilon} (RG)^{m-1} T^{\frac{1}{m} + \varepsilon}.
\]
In this notation, (4.10) is implied by \(\alpha_{2,2} = \frac{1}{2}\). In fact, if (4.11) holds, then we have
\[
\int_{0}^{T} |\zeta(\frac{1}{2} + it)|^{2km} \, dt \ll_{\varepsilon} T^{1+(m-1)\alpha_{k,m}+\varepsilon}.
\]
Non-trivial bounds of the type (4.11) (with \(0 \leq \alpha_{k,m} \leq 1\)) are hard to obtain when \(m > 2\) or \(k > 2\).
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