Coverage Optimal Empirical Likelihood Inference for Regression Discontinuity Design

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Abstract

This paper proposes an empirical likelihood inference method for a general framework that covers various types of treatment effect parameters in regression discontinuity designs (RDD). Our method can be applied for standard sharp and fuzzy RDDs, RDDs with categorical outcomes, augmented sharp and fuzzy RDDs with covariates and testing problems that involve multiple RDD treatment effect parameters. Our method is based on the first-order conditions from local polynomial fitting and avoids explicit asymptotic variance estimation. We investigate both first-order and second-order asymptotic properties and derive the coverage optimal bandwidth which minimizes the leading term in the coverage error expansion. In some cases, the coverage optimal bandwidth has a simple explicit form, which the Wald-type inference method usually lacks. We also find that Bartlett corrected empirical likelihood inference further improves the coverage accuracy. Easily implementable coverage optimal bandwidth selector and Bartlett correction are proposed for practical use. We conduct Monte Carlo simulations to assess finite-sample performance of our method and also apply it to two real datasets to illustrate its usefulness.

1 Introduction

The regression discontinuity design (RDD) has become one of the most popular methods for causal inference in applied economics and other related fields, as its identification strategy resembles an randomized experiment conducted near the cut-off of the forcing variable and its identified treatment effect parameter is easy to interpret. Exploiting the discontinuous variation in the probability of treatment around the cut-off of the forcing variable, RDD identifies, estimates and makes inference on the local treatment effect by comparing the outcomes of subjects (usually the conditional means) around the cut-off. Its validity is based on mild continuity assumptions on the counterfactuals. Reviews of the theory and practical guidelines of RDD can be found in Imbens and Lemieux (2008) and Lee and Lemieux (2010). Recent developments and extensions are documented in Cattaneo and

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Escanciano (2017). In a recent study, Hyytinen et al. (2018) confirmed that RDD, when implemented with the bias correction and robust inference in Calonico et al. (2014), produces estimates that are in line with the results from a comparable experiment. Meanwhile, they also showed that the results of RDD may be sensitive to the details of implementation such as bandwidth choice even when the number of observations is large. Gelman and Imbens (2019) argued that local polynomial fitting for inference on the RDD parameter has better performance than controlling for global high-order polynomials. Non-parametric estimation and inference for RDD motivate the studies on bandwidths that minimize the asymptotic mean squared error (AMSE) of the RDD estimator (Imbens and Kalyanaraman, 2011; Arai and Ichimura, 2018, 2016), bias-corrected t-statistic that validates use of AMSE-optimal bandwidth for inference (Calonico et al., 2014), the bandwidth that minimizes the coverage error of the confidence interval (Calonico et al., 2018, 2020) and the uniform-in-bandwidth confidence interval (Armstrong and Kolesár, 2018).

In this paper, we consider a more general framework. The parameter of interest is approximately identified in a set of moment conditions from population-level local linear (polynomial) fitting. We show that important special cases are treatment effects identified in various types of RDD, including standard sharp RDD, standard fuzzy RDD, RDD with multiple outcome variables, augmented sharp RDD with covariates and augmented fuzzy RDD with covariates. The approach of local polynomial fitting we consider is slightly different from the conventional one, from which the standard local polynomial regression method (see, e.g., Fan and Gijbels, 1996) can be derived. Both approaches can be derived from the minimum contrast problem in Bickel and Doksum (2015). See Section 2 ahead, Bickel and Doksum (2015, Chapter 11.3) and Jiang and Doksum (2003) for more discussions. The main advantage of the local polynomial fitting we use is that the conditional mean at a boundary point can be approximately identified by just one simple moment condition. Such a parameter can be also identified in moment conditions from more conventional local polynomial fitting at the cost of nuisance parameters that are not easily removed. Such simplification makes it much easier to derive the second-order properties of the proposed method. We also show that it is easy to augment the proposed moment conditions to include covariates in the form of linear projection, which was studied recently in Calonico et al. (2019). This is not easily achieved if moment conditions from more conventional local polynomial fitting are used.

In this paper, we propose empirical likelihood (EL) inference and study its asymptotic properties. Differently from Wald-type inference, EL inference avoids explicit asymptotic variance estimation (studentization) and no separate calculation of standard error is needed. In the literature, EL methods were proposed to make inference on the parameter identified by unconditional or conditional moment conditions. See, e.g., Kitamura (2006) for a comprehensive review. EL inference was also proposed in the context of non-parametric curves. See, e.g., Chen and Qin (2000), Otsu et al. (2013), Otsu et al. (2015) and Ma et al. (2019). It was shown in the literature that EL has favorable properties. See, e.g., Chen and Cui (2007), Kitamura (2001), Matsushita and Otsu (2013), Newey and Smith (2004), Otsu (2010) and Ma (2017) among many others.

The coverage error for inference is the discrepancy between nominal coverage probability and
finite-sample coverage probability. The coverage probability is of the form \((1 - \alpha) + r(n, h)\), where \(1 - \alpha\) is the nominal coverage probability, \(n\) is the sample size, \(h\) is the bandwidth and \(r(n, h) \downarrow 0\), as \(n \uparrow \infty\) and \(h \downarrow 0\). We characterize the coverage error of our method and derive the coverage optimal bandwidth, which is defined to be the minimizer of this leading term in the expansion of the coverage error. It has a simple expression in some cases. We further propose plug-in estimation of the coverage optimal bandwidth. We also show that in the general context our EL ratio statistic admits Bartlett correction, i.e., a simple rescaling device to further improve the coverage accuracy.

This paper is related to several lines of the literature. Inference on the RDD parameters has received much attention in the recent literature. Calonico et al. (2014) focused on Wald-type inference with the standard local polynomial regression. They proposed bias correction and new standard errors and their method is robust to large bandwidths. Calonico et al. (2018, 2020) derived coverage optimal bandwidth for the standard Wald-type inference. Compared with Calonico et al. (2014, 2018, 2020)’s method, ours has the following advantages. First, our EL inference avoids explicit studentization. Second, differently from Calonico et al. (2018, 2020)’s coverage optimal bandwidth, the coverage optimal bandwidth for our EL inference is independent of the nominal coverage probability. Third, our coverage optimal bandwidth has a simple explicit form in some interesting cases and its calculation does not require solving a minimization problem numerically. Fourth, our EL ratio statistic is Bartlett correctable and therefore, achieves a faster coverage error decay rate.

Lastly, we derive coverage optimal bandwidths in a more general context where covariates can be incorporated. Incorporating covariates for inference of the RDD parameters is important and practically very useful. See, e.g., Imbens and Lemieux (2008, Section 4.3) for discussion. It received much attention in recent econometric literature. See Calonico et al. (2019) and Frölich and Huber (2019) for two different approaches. Calonico et al. (2019)’s approach is based on linear projection and does not require smoothing over covariates. In this paper, we follow Calonico et al. (2019)’s approach and propose EL inference of the treatment effect in such an augmented RDD. Our method has advantages over Calonico et al. (2019)’s Wald-type inference including implicit studentization, coverage optimal bandwidth which is independent of the nominal coverage probability and Bartlett correction.\(^1\)

We also consider EL-based joint inference for RDD where multiple RDD treatment effect parameters arise. E.g., the researcher may be interested in the effect of a conditional cash transfer program on household’s consumption on both food and non-food.\(^2\) Moreover, when the outcome variable is categorical, a standard practice is to generate mutually exclusive dummy variables (one for each category) as the outcome variables. In this case, the RDD treatment effect for the categorical outcome is characterized by comparing multiple pairs of conditional probabilities at the cut-off, which naturally calls for joint inference. Xu (2017) proposed local likelihood estimator and Wald-type inference

\(^1\)To the best of our knowledge, the coverage optimal bandwidth for Calonico et al. (2019)’s Wald-type inference has not been derived in the literature.

\(^2\)The empirical exercise of Calonico et al. (2014) serves as an example for the Progresa/Oportunidades program in Mexico.
in this context. Like Xu (2017)’s approach, our EL approach can be applied for joint inference on
treatment effects across categories. Our method has several advantages. First, separate calculation
of standard error is not needed. Second, our method admits a simple coverage optimal bandwidth
that has an explicit form and is independent from the nominal coverage probability. Third, the
shape of our confidence region is data-driven. Fourth, it can also be applied in fuzzy RDD. See
Remark 18 ahead. Fifth, it is also easy to incorporate covariates when our EL approach is adopted.
Lastly, our method is Bartlett correctable.

Joint inference in RDD is also useful when the researcher wants to check the validity of the RDD
by testing for the continuity of a group of pre-treatment covariates at the cut-off. The covariate
balance test has been a common practice in the enterprise of RDD (see, e.g., Imbens and Lemieux,
2008; Lee, 2008; Lee and Lemieux, 2010). While most empirical works conduct the balance test
separately for each covariate, some researchers have noted that the problem of multiple testing may
generate statistical imbalance of some covariates by chance. See, e.g., Eggers et al. (2015); Hyytinen
et al. (2018). In this regard, our EL-based joint test for the smoothness of multiple covariates at the
cut-off complements the current practice of RDD validity check.

Otsu et al. (2015) proposed a different EL-based inference method in the RDD context. Their
method was based on first-order conditions from standard local linear regression while our method
is based on moment conditions derived from different local linear fitting. In this paper, we focus on
a more general context where the need for joint inference on multiple RDD treatment effects and
incorporating covariates can be accommodated. We also derive the coverage optimal bandwidth in
each of the scenarios. In some cases, it has a simple explicit form. For fuzzy RDD, our method
makes use of a different set of moment conditions inspired by Noack and Rothe (2019). Our method
for fuzzy RDD involves fewer nuisance parameters and its coverage optimal bandwidth is as simple
as that for the sharp RDD. Moreover, covariates can be easily incorporated. See Remark 19 ahead.
In another related paper, Ma et al. (2019) studied EL inference for the parameter of interest in the
density discontinuity design (see Jales and Yu, 2016) and derived the coverage optimal bandwidth.
The scope of this paper is different from Ma et al. (2019)’s but the moment conditions in both papers
are from similar population-level local linear fitting, which originated from the minimum contrast
problem in Bickel and Doksum (2015).

This paper is organized as follows. Section 2 introduces the moment conditions from population-
level local linear fitting. Section 3 introduces EL inference in a general framework and discusses
interesting special cases. Section 4 provides first and second order asymptotic properties. Section 5
focuses on interesting special cases including various RDD parameters and provides formulae for the
coverage optimal bandwidths. Section 6 discusses implementation. Section 7 reports results from
Monte Carlo simulations. Section 8 provides two empirical applications. Section 9 concludes. All
proofs are collected in a supplemental appendix.

Notation. $a^T$ denotes the transpose of a vector (or matrix) $a$. For a real sequence $\{a_n\}_{n=1}^{\infty}$, we
denote $b \propto a_n$ if $b = c \cdot a_n$ for some constant $c > 0$. Let $\mathbb{1}(\cdot)$ denote the indicator function.
Let $\phi_d$ be the $d$-dimensional standard multivariate normal density. Let $F_{\chi_d^2}$ and $f_{\chi_d^2}$ denote the cumulative distribution function and density function of a $\chi_d^2$ random variable respectively. Let $q_{\chi_d^2, 1-\alpha} = F_{\chi_d^2}^{-1}(1-\alpha)$ be the $(1-\alpha)$ quantile of a $\chi_d^2$ random variable.

For any $k$-times differentiable univariate function $f$, let $f^{(k)}$ denote the $k$-th order derivative. Let $0_{J \times K}$ denote the $J \times K$ matrix in which all elements are zeros. Let $I_K$ denote the $K \times K$ identity matrix. Let $e_{K, s}$ denote the $s$-th unit vector in $\mathbb{R}^K$. For a $J \times K$ matrix $A$, let $A^{[jk]}$ denote the element on the $j$-th row and $k$-th column. For a vector $Z \in \mathbb{R}^J$, let $Z[^j]$ denote its $j$-th coordinate. $x^{\otimes k}$ denotes the $k$-th Kronecker power of $x$. $\| \cdot \|$ denotes the Euclidean norm. "$a := b$" means that $a$ is defined by $b$ and "$a := b$" means that $b$ is defined by $a$.

\section{Local Linear Fitting}

This section reviews (population-level) local linear fitting. We show that by using a particular type of local linear fitting that is different from the conventional local linear regression, the conditional mean at a boundary point can be approximately identified by just one simple moment condition. This is the building block of our empirical likelihood inference method for RDD. Moreover, the simple form of the moment condition will greatly facilitate our investigation of the second-order asymptotic properties and the derivation of the coverage optimal bandwidth.

Let the forcing variable $X$ be a continuous random variable supported on $[x, \bar{x}]$. Let $f$ denote its density function. We assume that $f$ admits continuous high-order derivatives in the interior of the support $[x, \bar{x}]$. Denote $\varphi := f(c)$ and $\varphi^{(k)} := f^{(k)}(c)$ for simplicity. For a random vector (or matrix) $V$, denote $g_V(x) := E[V \vert X = x]$. Let $c \in (x, \bar{x})$ be some cut-off point. In this paper, we denote $g_{V, -}(x) := g_V(x)$ for $x < c$ and $g_{V, +}(x) := g_V(x)$ for $x \geq c$. Also denote $\mu_{V, -} := \lim_{x \downarrow c} g_{V, -}(x)$, $\mu_{V, +} := \lim_{x \uparrow c} g_{V, +}(x)$, $\mu^{(k)}_{V, -} := \lim_{x \downarrow c} g^{(k)}_{V, -}(x)$ and $\mu^{(k)}_{V, +} := \lim_{x \uparrow c} g^{(k)}_{V, +}(x)$.

Let $K$ denote a symmetric kernel function supported on $[-1, 1]$ and let $h$ denote the bandwidth. Suppose $Y$ is the dependent variable and we observe i.i.d. copies $(Y_i, X_i)_{i=1}^n$ of $(Y, X)$. Suppose that we are interested in estimating $\mu_{Y, -}$. A widely-used approach is the local linear regression:

\begin{equation}
\min_{a_-, b_-} \sum_{i=1}^n \{ Y_i - (a_- + b_-(X_i - c)) \}^2 \mathbb{1}(X_i < c) \frac{1}{h} K\left( \frac{X_i - c}{h} \right).
\end{equation}

Let $(\widehat{a}_-, \widehat{b}_-)$ denote the minimizer that solve (1).

Jiang and Doksum (2003) noted that the standard local linear regression can be derived from the following (population-level) local linear fitting which is also referred to as the minimum contrast problem in Bickel and Doksum (2015):

\begin{equation}
\min_{a_-, b_-} \int_{-\infty}^c \{ g_Y(x) - (a_- + b_-(x - c)) \}^2 \frac{1}{h} K\left( \frac{x - c}{h} \right) f(x) \, dx.
\end{equation}
Let \((a_{*-}, b_{*-})\) denote the minimizers that solve (2). The first-order conditions can be written as

\[
\int \int y \left( \frac{1}{x-c} \right) \mathbb{1} (x < c) \frac{1}{h} K \left( \frac{x-c}{h} \right) f_{Y,X} (y, x) \, dx \, dy
\]

\[
= \left\{ \int \left[ \frac{1}{x-c} \frac{x-c}{(x-c)^2} \right] \mathbb{1} (x < c) \frac{1}{h} K \left( \frac{x-c}{h} \right) f (x) \, dx \right\} \left( \begin{array}{c} a_{*-} \\ b_{*-} \end{array} \right). \tag{3}
\]

By comparing the first-order conditions of (1) and (3), we find that the local linear estimator \((\hat{a}_-, \hat{b}_-)\) is the sample analogue of \((a_{*-}, b_{*-})\), which converge to \((\mu_{Y_-}, \mu_{Y_-}^{(1)})\) as \(h \downarrow 0\).

Jiang and Doksum (2003) also showed that an alternative local linear estimator can be derived from a slightly different local linear fitting (minimum contrast) problem:

\[
\min_{a_- \in \mathbb{R}, b_- \in \mathbb{R}} \int_{c}^{\infty} \left\{ g_Y (x) f (x) - (a_- + b_- (x-c)) \right\}^2 \frac{1}{h} K \left( \frac{x-c}{h} \right) \, dx.
\tag{4}
\]

See Remark 3 ahead for its local polynomial generalization. Let \((a_{*-}, b_{*-})\) denote the minimizers that solve (4). The first-order conditions can be written as

\[
\int_{c}^{\infty} g_Y (x) f (x) \left( \frac{1}{x-c} \right) \frac{1}{h} K \left( \frac{x-c}{h} \right) \, dx
\]

\[
= \left\{ \int \left[ \frac{1}{x-c} \frac{x-c}{(x-c)^2} \right] \frac{1}{h} K \left( \frac{x-c}{h} \right) f (x) \, dx \right\} \left( \begin{array}{c} a_{*-} \\ b_{*-} \end{array} \right). \tag{5}
\]

Note that differently from (3), the coefficients on the right hand side of (5) are fixed constants that do not depend on any unknown feature of the population. Denote

\[
m_{j,-} := \int_{-1}^{0} u^j K (u) \, du, \quad m_{j,+} := \int_{0}^{1} u^j K (u) \, du,
\]

\[
\gamma_j := \int_{-1}^{0} \left\{ \frac{m_{2,-}}{m_{0,-} m_{2,-} - m_{1,-}^2} - \frac{m_{1,-}}{m_{0,-} m_{2,-} - m_{1,-}^2} t \right\}^j K (t) \, dt
\]

and

\[
\omega := \int_{-1}^{0} \left( \frac{m_{2,-} - m_{1,-} t}{m_{0,-} m_{2,-} - m_{1,-}^2} \right) t^2 K (t) \, dt = \frac{m_{2,-} - m_{1,-} m_{3,-}}{m_{0,-} m_{2,-} - m_{1,-}^2}.
\]

When the kernel \(K\) is given, these are known constants.\(^3\) Solving the linear equations (5) for \(a_{*-},\)

\(^3\)It is easy to check that since \(K\) is assumed to be symmetric, we have

\[
\gamma_j = \int_{0}^{1} \left\{ \frac{m_{2,+}}{m_{0,+} m_{2,+} - m_{1,+}^2} - \frac{m_{1,+}}{m_{0,+} m_{2,+} - m_{1,+}^2} t \right\}^j K (t) \, dt
\]

\[
\omega = \int_{0}^{1} \left( \frac{m_{2,+} - m_{1,+} t}{m_{0,+} m_{2,+} - m_{1,+}^2} \right) t^2 K (t) \, dt = \frac{m_{2,+} - m_{1,+} m_{3,+}}{m_{0,+} m_{2,+} - m_{1,+}^2}.
\]
we find that the minimizer is
\[ a_{*,-} = \int_{c}^{x} \left\{ \frac{m_{2,-}}{m_{0,-}m_{2,-} - m_{1,-}^2} - \frac{m_{1,-}}{m_{0,-}m_{2,-} - m_{1,-}^2} \cdot \frac{x-c}{h} \right\} \frac{1}{h} K \left( \frac{x-c}{h} \right) g_Y (x) f (x) \, dx. \]

It can be shown that \( a_{*,-} = \mu_{Y,-} - \varphi + O \left( h^2 \right) \). See the supplemental appendix.

Denote
\[ W_{-i} := \left\{ \frac{m_{2,-}}{m_{0,-}m_{2,-} - m_{1,-}^2} - \frac{m_{1,-}}{m_{0,-}m_{2,-} - m_{1,-}^2} \cdot \frac{X_i-c}{h} \right\} \mathbb{1} (X_i < c) K \left( \frac{X_i-c}{h} \right) \]
for \( i = 1, ..., n \). Note that
\[ a_{*,-} = E \left[ \frac{1}{h} W_- g_Y (X) \right] = E \left[ \frac{1}{h} W_- Y \right] \]
follows from law of iterated expectations. Note that in a special case when \( Y \equiv 1 \) and \( g_Y \equiv 1 \),
\[ a_{*,-} = \varphi + O \left( h^2 \right) \]. Therefore,
\[ \frac{E \left[ h^{-1} W_- Y \right]}{E \left[ h^{-1} W_- \right]} = \mu_{Y,-} + O \left( h^2 \right). \tag{6} \]
And the parameter of interest \( \mu_{Y,-} \) is approximately identified by just one moment condition:
\[ E \left[ \frac{1}{h} W_- (Y - \mu_{Y,-}) \right] \approx 0, \text{ when } h \text{ is small.} \tag{7} \]

Basing EL inference of \( \mu_{Y,-} \) on this moment condition rather than ones derived from (3) has the advantage of “no nuisance parameter”. Moreover, we show that EL inference based on (7) admits a simple coverage optimal bandwidth which minimizes the leading term in the expansion of the coverage error.

For estimation of \( \mu_{Y,+} \), we solve the minimum contrast problem on the other side:
\[ \min_{a_+, b_+} \int_{c}^{x} \left\{ g_Y (x) f (x) - (a_+ + b_+ (x-c)) \right\}^2 \frac{1}{h} K \left( \frac{x-c}{h} \right) \, dx \]
and find that the minimizer is
\[ a_{*,+} = \int_{c}^{x} \left\{ \frac{m_{2,+}}{m_{0,+}m_{2,+} - m_{1,+}^2} - \frac{m_{1,+}}{m_{0,+}m_{2,+} - m_{1,+}^2} \cdot \frac{x-c}{h} \right\} \frac{1}{h} K \left( \frac{x-c}{h} \right) g_Y (x) f (x) \, dx. \]

Let
\[ W_{i,+} := \left\{ \frac{m_{2,+}}{m_{0,+}m_{2,+} - m_{1,+}^2} - \frac{m_{1,+}}{m_{0,+}m_{2,+} - m_{1,+}^2} \cdot \frac{X_i-c}{h} \right\} \mathbb{1} (X_i \geq c) K \left( \frac{X_i-c}{h} \right). \]
Then it is clear that $\mu_{Y,+}$ is identified by one moment condition: $E\left[h^{-1}W_+(Y - \mu_{Y,+})\right] \approx 0$ approximately.

Note that in this paper, we restrict the bandwidths for smoothing the forcing variables on the left and the right of the cut-off point to be the same. It is possible to extend all of the theorems in this paper to accommodate different bandwidths on different sides. However, the optimal pair of bandwidths that minimizes the absolute value of the leading coverage error term is not well-defined. Simultaneous selection of a pair of bandwidths for coverage optimality is more involved than the simultaneous bandwidth selection for minimizing AMSE (Arai and Ichimura, 2018). See Ma et al. (2019, Theorem 2 and Remark 4). This challenging research question is left for future investigation.

In the next section, we consider a more general context where various parameters of interest in RDDs can be identified by moment conditions from local linear fitting.

### 3 A General Framework for RDD

In this section, we first present a general framework for RDD and then specialize it to various scenarios through Examples 1 to 8. Suppose our observations

$$\{(V_{1,i}, V_{2,i}, V_{3,i}, X_i, G_{1,i}, G_{2,i}, G_{3,i})\}_{i=1}^{n}$$

are i.i.d. copies of $(V_1, V_2, V_3, X, G_1, G_2, G_3)$. $V_{j,i} \in \mathbb{R}^{d_j}$, $G_{j,i} \in \mathbb{R}^{d_j \times d}$, $j \in \{1, 2, 3\}$ and $d := d_1 + d_2 + d_3$. Later $V$’s and $G$’s will be specified in different RDD scenarios. Let $b_* \in \mathbb{R}^d$ be the solution to the following linear moment conditions:

$$0_{d_1 \times 1} = E\left[\frac{1}{h}W_+ (V_1 - G_1 b_*)\right]$$

$$0_{d_2 \times 1} = E\left[\frac{1}{h}W_- (V_2 - G_2 b_*)\right]$$

$$0_{d_3 \times 1} = E\left[\frac{1}{h}(W_+ + W_-) (V_3 - G_3 b_*)\right]. \quad (8)$$

In RDD, The parameter $b_*$ corresponds to a vector of conditional means at the cut-off. Since $b_*$ can be written as

$$b_* = \left(\begin{array}{cc}
E\left[h^{-1}W_+ G_1\right] & -1 \\
E\left[h^{-1}W_- G_2\right] & E\left[h^{-1}W_+ V_1\right] \\
E\left[h^{-1} (W_+ + W_-) G_3\right] & E\left[h^{-1} (W_+ + W_-) V_3\right]
\end{array}\right)^{-1} \left(\begin{array}{c}
E\left[h^{-1}W_+ V_1\right] \\
E\left[h^{-1}W_- V_2\right] \\
E\left[h^{-1} (W_+ + W_-) V_3\right]
\end{array}\right). \quad (9)$$
The moment conditions (8) approximately identifies

\[ \beta := \begin{pmatrix} \mu_{G_1,+} \\ \mu_{G_2,-} \\ \mu_{G_3,+} + \mu_{G_3,-} \end{pmatrix}^{-1} \begin{pmatrix} \mu_{V_1,+} \\ \mu_{V_2,-} \\ \mu_{V_3,+} + \mu_{V_3,-} \end{pmatrix} \]

since it can be shown that \( b^* = \beta + O(h^2) \). See the supplemental appendix. The parameter of interest is \( \tau^* := \rho(\beta) \) for some nonlinear function \( \rho : \mathbb{R}^d \to \mathbb{R}^{d_\rho} \) with \( d_\rho < d \).

We define the EL criterion function:

\[ \ell(\theta) := \min_{p_1, \ldots, p_n} -2 \sum_{i=1}^{n} \log (n \cdot p_i) \]

subject to

\[ \sum_{i=1}^{n} p_i \begin{pmatrix} W_{+,i} (V_{1,i} - G_{1,i} \theta) \\ W_{-,i} (V_{2,i} - G_{2,i} \theta) \\ (W_{+,i} + W_{-,i}) (V_{3,i} - G_{3,i} \theta) \end{pmatrix} = 0_{d \times 1}, \]

\[ \sum_{i=1}^{n} p_i = 1, p_i \geq 0, i = 1, \ldots, n \]  

(10)

and the EL ratio statistic:

\[ LR(\tau) = \inf_{\rho(\theta) = \tau} \ell(\theta). \]  

(11)

Note that the dual form of the EL criterion function is

\[ \ell(\theta) = 2 \sup_{\lambda \in \mathcal{L}_n(\theta)} \sum_{i=1}^{n} \log \left( 1 + \lambda^T \begin{pmatrix} W_{+,i} (V_{1,i} - G_{1,i} \theta) \\ W_{-,i} (V_{2,i} - G_{2,i} \theta) \\ (W_{+,i} + W_{-,i}) (V_{3,i} - G_{3,i} \theta) \end{pmatrix} \right), \]

(12)

where

\[ \mathcal{L}_n(\theta) := \left\{ \lambda \in \mathbb{R}^d : \lambda^T \begin{pmatrix} W_{+,i} (V_{1,i} - G_{1,i} \theta) \\ W_{-,i} (V_{2,i} - G_{2,i} \theta) \\ (W_{+,i} + W_{-,i}) (V_{3,i} - G_{3,i} \theta) \end{pmatrix} > -1, \text{ for } i = 1, \ldots, n \right\}. \]

(13)

A standard EL-based confidence region with nominal coverage probability \( 1 - \alpha \) is

\[ \left\{ \tau : LR(\tau) \leq q_{\chi^2_{d_\rho}, 1-\alpha} \right\}. \]

(14)

In this paper, we justify its validity and study the problem of optimal bandwidth selection to minimize its probabilistic coverage error.

**Example 1 (Sharp RDD).** Suppose that \( X \) is the forcing variable and \( Y \) is the outcome variable.
Treatment is assigned if $X \geq c$. The sharp RDD parameter is

$$
\tau^S_\star := \mu_{Y,+} - \mu_{Y,-}.
$$

See Section 5.1 ahead for more details. Let $(g_{*,+}, g_{*,-})$ be the solution to the moment conditions

\begin{align*}
0 &= E \left[ \frac{1}{h} W_+ (Y - g_{*,+}) \right] \\
0 &= E \left[ \frac{1}{h} W_- (Y - g_{*,-}) \right].
\end{align*}

(15)

According to (6), $(g_{*,+}, g_{*,-}) = (\mu_{Y,+}, \mu_{Y,-}) + O(h^2)$. $(\mu_{Y,+}, \mu_{Y,-})$ is approximately identified by the moment conditions. Identification of the sharp RDD parameter is a special case of the general framework we considered since (8) is equivalent to (15) if we do not have the third set of moment conditions ($d_3 = 0$) and take $V_1 = V_2 = Y$, $G_1 = \left[ \begin{array}{c} 1 \\ 0 \end{array} \right]$ and $G_2 = \left[ \begin{array}{cc} 0 & 1 \end{array} \right]$. 

**Example 2 (Sharp RDD with Categorical Outcomes).** In a sharp RDD with categorical outcomes, the outcome variable $\tilde{Y}$ takes value in $\{0, 1, ..., J\}$. The difference between realized values has no meaning. The categories can be ordered, in which case the order between values is meaningful but the difference is not meaningful. Or they can be unordered, in which case the categories have no natural ordering and the values in $\{0, 1, ..., J\}$ are simply numerical codes that represent the categories. The outcome variable $\tilde{Y}$ can be represented by mutually exclusive dummy variables. Let $Y[j] := \mathbb{1} (\tilde{Y} = j)$ and $Y := (Y[1], ..., Y[J])^T$. $j = 0$ is viewed as the base category. The sharp RDD parameters are

$$
\tau^S_{\star,j} := \lim_{x \uparrow c} \Pr \left[ \tilde{Y} = j | X = x \right] - \lim_{x \downarrow c} \Pr \left[ \tilde{Y} = j | X = x \right] = \mu_{Y[j],+} - \mu_{Y[j],-}, \quad j = 1, ..., J.
$$

Joint inference on treatment effects across categories $(\tau^S_{\star,1}, ..., \tau^S_{\star,J})$ is of economic interest. Let $(g_{*,+}, g_{*,-})$ be the solution to

\begin{align*}
0_{J \times 1} &= E \left[ \frac{1}{h} W_+ (Y - g_{*,+}) \right] \\
0_{J \times 1} &= E \left[ \frac{1}{h} W_- (Y - g_{*,-}) \right].
\end{align*}

Since $(g_{*,+}, g_{*,-}) = (\mu_{Y,+}, \mu_{Y,-}) + O(h^2)$, $(\mu_{Y,+}, \mu_{Y,-})$ is approximately identified by the moment conditions. Clearly, identification by these moment conditions can be viewed as a special case of the general framework.

**Example 3 (Testing Covariate Balance).** To check the validity of RDD, one common practice is to test for the continuity of pre-treatment covariates $Z \in \mathbb{R}^d_Z$ at the cut-off. See, e.g., Eggers et al. (2015) for discussion. In many applications, the number of covariates $d_Z > 1$. E.g., in
the electoral RDD of Lee (2008), those covariates include previous democrat vote share, previous democrat victories, democrat political and electoral experience, among others. Therefore, it is useful to have a joint test for balance of all covariates. More specifically, we are interested in testing $d_Z$ restrictions:

$$H_0 : \mu_{Z,+} = \mu_{Z,-} \text{ versus } H_1 : \mu_{Z,+} \neq \mu_{Z,-}.$$ 

Let $(g_{*,+}, g_{*,-})$ be the solution to

$$0_{d_Z \times 1} = E \left[ \frac{1}{h} W_+ (Z - g_{*,+}) \right]$$

$$0_{d_Z \times 1} = E \left[ \frac{1}{h} W_- (Z - g_{*,-}) \right].$$

We have $(g_{*,+}, g_{*,-}) = (\mu_{Z,+}, \mu_{Z,-}) + O(h^2)$. The set of moment conditions is similar to those considered in the previous example.

**Example 4 (Fuzzy RDD).** Suppose $X$ is the forcing variable, $Y$ is the outcome variable and $D \in \{0, 1\}$ is the treatment assignment. The fuzzy RDD parameter is

$$\tau^F_* := \frac{\mu_{Y,+} - \mu_{Y,-}}{\mu_{D,+} - \mu_{D,-}}.$$ 

See Section 5.4 ahead for more details. Consider the moment conditions

$$0_{2 \times 1} = E \left[ \frac{1}{h} W_+ \left( Y - g_{*,Y,+} \right) \right]$$

$$0_{2 \times 1} = E \left[ \frac{1}{h} W_- \left( Y - g_{*,Y,-} \right) \right].$$

Since the solution satisfies $(g_{*,Y,+}, g_{*,Y,-}) = (\mu_{Y,+}, \mu_{Y,-}) + O(h^2)$ and $(g_{*,D,+}, g_{*,D,-}) = (\mu_{D,+}, \mu_{D,-}) + O(h^2)$, $(\mu_{Y,+}, \mu_{Y,-}, \mu_{D,+}, \mu_{D,-})$ is approximately identified by the moment conditions. Identification of the fuzzy RDD parameter is a special case of the general framework we considered since (8) is equivalent to (16) if we do not have the third set of moment conditions ($d_3 = 0$) and take $V_1 = V_2 = (Y,D)^T$, $G_1 = \left( I_2 \ 0 \right)$ and $G_2 = \left( 0 \ I_2 \right)$.

**Example 5 (Alternative Approach to Fuzzy RDD).** Noack and Rothe (2019) noted that

$$\lim_{x \downarrow c} E \left[ Y - \tau \cdot D \mid X = x \right] = \lim_{x \uparrow c} E \left[ Y - \tau \cdot D \mid X = x \right]$$

if and only if $\tau = \tau^F_*$ and a confidence interval for $\tau^F_*$ could be the set of values for $\tau$ under which the null hypothesis $\mu_{Y,+} - \tau \cdot \mu_{D,+} = \mu_{Y,-} - \tau \cdot \mu_{D,-}$ is not rejected. Noack and Rothe (2019)’s idea motivates a set of moment conditions

$$0 = E \left[ \frac{1}{h} W_+ (Y - g_{*,1} D - g_{*,0}) \right].$$
which approximately identifies $\tau^F_*$ in the standard fuzzy RDD. It is easy to check that $g_{s,1} = \tau^F_* + O\left(h^2\right)$ and $g_{s,0} = \mu^F_* + O\left(h^2\right)$, where $\mu^F_* := \mu_{Y,+} - \tau^F_* \cdot \mu_{D,+} = \mu_{Y,-} - \tau^F_* \cdot \mu_{D,-}$. Identification of the fuzzy RDD parameter is a special case since (8) is equivalent to (16) if we do not have the third set of moment conditions ($d_3 = 0$) and take $V_1 = V_2 = Y$ and $G_1 = G_2 = \begin{bmatrix} D & 1 \end{bmatrix}$.

Compared with (16), (17) has only two moment conditions and one nuisance parameter. This alternative approach has less computational burden. Therefore, we recommend taking this approach rather than the more conventional one based on (16).\(^4\) In Section 5.4 ahead, we provide the coverage optimal bandwidth for EL inference using (17) rather than (16).

When $Y$ is the categorical outcome variable introduced in Example 2, the moment conditions

\[
0_{J \times 1} = \mathbb{E} \left[ \frac{1}{h} W_+ (Y - g_{s,1} D - g_{s,0}) \right]
\]

\[
0_{J \times 1} = \mathbb{E} \left[ \frac{1}{h} W_- (Y - g_{s,1} D - g_{s,0}) \right]
\]

identify $g_{s,1} = (\tau^F_{*,1}, \ldots, \tau^F_{*,J})^T + O\left(h^2\right)$ and $g_{s,0} = (\mu^F_{s,1}, \ldots, \mu^F_{s,J})^T + O\left(h^2\right)$, where

\[
\tau^F_{*,j} := \frac{\mu_{Y,[j],+} - \mu_{Y,[j],-}}{\mu_{D,+} - \mu_{D,-}}
\]

is the fuzzy RDD parameter and $\mu^F_{*,j} := \mu_{Y,[j],+} - \tau^F_{*,j} \cdot \mu_{D,+} = \mu_{Y,[j],-} - \tau^F_{*,j} \cdot \mu_{D,-}$. Joint inference on $\left(\tau^F_{*,1}, \ldots, \tau^F_{*,J}\right)$ can be accommodated.

**Example 6 (Sharp RDD with Covariates).** Suppose that we also observe a vector consisting of pre-treatment covariates $Z \in \mathbb{R}^{d_Z}$ in addition to $(Y, X)$ in a sharp RDD. The augmented sharp RD parameter introduced by Calonico et al. (2019) is

\[
\tau^SC_* := \left( \mu_{Y,+} - \mu_{Z,+}^{T} \gamma_* \right) - \left( \mu_{Y,-} - \mu_{Z,-}^{T} \gamma_* \right)
\]

\[
= \frac{\mu^SC_{Y,+} - \mu^SC_{Y,-}}{\mu^SC_{Y,+} - \mu^SC_{Y,-}}
\]

\[
= \tau^S_*, \text{ if } \mu^SC_{Z,+} = \mu^SC_{Z,-},
\]

where

\[
\gamma_* := \left\{ \lim_{x \to c} \text{Var} \left[ Z | X = x \right] + \lim_{x \to c} \text{Var} \left[ Z | X = x \right] \right\}^{-1}
\]

\[
\times \left\{ \lim_{x \to c} \text{Cov} \left[ Z, Y | X = x \right] + \lim_{x \to c} \text{Cov} \left[ Z, Y | X = x \right] \right\}.
\]

\(^4\)It is known that the actual coverage of Wald-type inference based on a plug-in estimator of $\tau^F_*$ deviates substantially from the nominal coverage probability when the jump in the treatment probabilities $\mu_{D,+} - \mu_{D,-}$ is small. See, e.g., Feir et al. (2016). Noack and Rothe (2019)’s approach uses the auxiliary quantity $Y - \tau \cdot D$ and avoids direct estimation of a possibly small denominator.
Consider the following moment conditions

\[ 0 = E \left[ \frac{1}{h} W_{+} (Y - g_{*,Y,+} - Z^T g_{*,Z}) \right] \]
\[ 0 = E \left[ \frac{1}{h} W_{-} (Y - g_{*,Y,-} - Z^T g_{*,Z}) \right] \]
\[ 0_{d_Z \times 1} = E \left[ \frac{1}{h} (W_{+} + W_{-}) Z (Y - g_{*,Y,+} 1 (X \geq c) - g_{*,Y,-} 1 (X < c) - Z^T g_{*,Z}) \right]. \] (20)

Solving the linear equations, we have

\[ g_{*,Z} = \left\{ \frac{E \left[ h^{-1} W_{+} ZZ^T \right] + E \left[ h^{-1} W_{-} ZZ^T \right]}{E \left[ h^{-1} W_{+} Z \right] E \left[ h^{-1} W_{+} Z^T \right]} \right\}^{-1} \]
\times \left\{ \frac{E \left[ h^{-1} W_{+} ZY \right] + E \left[ h^{-1} W_{-} ZY \right]}{E \left[ h^{-1} W_{+} Y \right] E \left[ h^{-1} W_{+} Y \right]} \right\}, \]

which satisfies

\[ g_{*,Z} = \left\{ \mu_{Z,+} - \mu_{Z,+} \mu_{Z,+}^T + \mu_{Z,-} - \mu_{Z,-} \mu_{Z,-}^T \right\}^{-1} \]
\times \left\{ \mu_{Z,+} - \mu_{Z,+} \mu_{Y,+} + \mu_{Z,-} - \mu_{Z,-} \mu_{Y,-} \right\} + O \left( h^2 \right)
= \gamma_\ast + O \left( h^2 \right). \]

Then the first two moment conditions give

\[ g_{*,Y,r} = E \left[ h^{-1} W_{r} Y \right] - E \left[ h^{-1} W_{r} Z^T \right] \gamma_\ast \]
\[ = \mu_{Y,r}^{SC} + O \left( h^2 \right), \quad r \in \{-, +\}. \]

Therefore \( (\mu_{Y,+}^{SC}, \mu_{Y,-}^{SC}, \gamma_\ast) \) is approximately identified by the moment conditions (20). Identification of the covariates-augmented sharp RDD parameter is a special case since (8) is equivalent to (20) if we take \( V_1 = V_2 = Y, \quad V_3 = ZY \) and

\[ G_1 = \begin{pmatrix} 1 & 0 & Z^T \end{pmatrix} \]
\[ G_2 = \begin{pmatrix} 0 & 1 & Z^T \end{pmatrix} \]
\[ G_3 = \begin{pmatrix} Z 1 (X \geq c) & Z 1 (X < c) & ZZ^T \end{pmatrix}. \]

**Example 7 (Fuzzy RDD with Covariates).** Suppose that we also observe covariates \( Z \in \mathbb{R}^{d_Z} \) in addition to \((Y, X, D)\) in a fuzzy RDD. Calonico et al. (2019) proposed an estimator which is the
ratio of local linear estimators of two covariates-augmented sharp RDD parameters. Let
\[
\mu_{Y,r}^{SC} := \mu_{Y,r} - \mu_{Z,r}^T \gamma_{s,Y}, \quad r \in \{-, +\}
\]
\[
\mu_{D,r}^{SC} := \mu_{D,r} - \mu_{Z,r}^T \gamma_{s,D}, \quad r \in \{-, +\},
\]
where \( \gamma_{s,Y} \) is defined by the right hand side of (19) and similarly \( \gamma_{s,D} \) is defined by (19) with \( Y \) being replaced by \( D \). The probabilistic limit of Calonico et al. (2019)’s estimator is the covariates-augmented fuzzy RDD parameter
\[
\tilde{\tau}_s^{FC} := \frac{\mu_{Y,+}^{SC} - \mu_{Y,-}^{SC}}{\mu_{D,+}^{SC} - \mu_{D,-}^{SC}} = \hat{\tau}_s^F, \text{ if } \mu_{Z,+} = \mu_{Z,-}.
\]

We can easily extend the approach introduced in Example 4 and consider the following moment conditions:
\[
\begin{align*}
0_{2 \times 1} &= E \left[ \frac{1}{h} W_+ \begin{pmatrix} Y - g_{s,Y,+} - Z^T g_{s,Y}^{FC} \\ D - g_{s,D,+} - Z^T g_{s,D}^{FC} \end{pmatrix} \right] \\
0_{2 \times 1} &= E \left[ \frac{1}{h} W_- \begin{pmatrix} Y - g_{s,Y,-} - Z^T g_{s,Y}^{FC} \\ D - g_{s,D,-} - Z^T g_{s,D}^{FC} \end{pmatrix} \right] \\
0_{2d \times 1} &= E \left[ \frac{1}{h} (W_+ + W_-) \left\{ \begin{array}{c} Z \begin{pmatrix} Y - g_{s,Y,+} \mathbb{1}(X \geq c) - g_{s,Y,-} \mathbb{1}(X < c) - Z^T g_{s,Y}^{FC} \\ D - g_{s,D,+} \mathbb{1}(X \geq c) - g_{s,D,-} \mathbb{1}(X < c) - Z^T g_{s,D}^{FC} \end{pmatrix} \end{array} \right\} \right]
\end{align*}
\]
(21)

Note that \( (\mu_{Y,+}^{SC}, \mu_{Y,-}^{SC}, \mu_{D,+}^{SC}, \mu_{D,-}^{SC}, \gamma_{s,Y}, \gamma_{s,D}) \) is approximately identified by (21). Identification of \( \tilde{\tau}_s^{FC} \) is a special case since (8) is equivalent to (21) if we take \( V_1 = V_2 = (Y, D)^T, V_3 = (YZ^T, DZ^T)^T \) and
\[
G_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & Z^T & 0_{1 \times d_Z} \\
0 & 0 & 1 & 0 & 0_{1 \times d_Z} & Z^T
\end{pmatrix},
\]
\[
G_2 = \begin{pmatrix}
0 & 1 & 0 & 0 & Z^T & 0_{1 \times d_Z} \\
0 & 0 & 0 & 1 & 0_{1 \times d_Z} & Z^T
\end{pmatrix},
\]
\[
G_3 = \begin{pmatrix}
Z \mathbb{1}(X \geq c) & Z \mathbb{1}(X < c) & 0 & 0 & ZZ^T & 0_{d_Z \times d_Z} \\
0 & 0 & Z \mathbb{1}(X \geq c) & Z \mathbb{1}(X < c) & 0_{d_Z \times d_Z} & ZZ^T
\end{pmatrix}.
\]

Example 8 (Alternative Approach to Fuzzy RDD with Covariates). Combining the approaches of Noack and Rothe (2019) and Calonico et al. (2019), we consider the auxiliary variable \( \hat{Y}(\tau) := Y - \tau \cdot D \) and make inference on the associated covariates-augmented sharp RD parameter. Let \( (\mu_{\hat{Y},+}(\tau), \mu_{\hat{Y},-}(\tau), \gamma_{\hat{Y},+}(\tau)) \) be the parameters approximately identified by the moment conditions (20) with \( Y \) being replaced by \( \hat{Y}(\tau) \). Under the assumption that \( \mu_{Z,+} = \mu_{Z,-}, \tau = \hat{\tau}_s^F \) if and only if \( \mu_{\hat{Y},+}(\tau) = \mu_{\hat{Y},-}(\tau) \). The set of values for \( \tau \) under which the test of the null hypothesis
\[\mu_{Y,+}(\tau) = \mu_{Y,-}(\tau)\] using our EL ratio statistic is not rejected is

\[\left\{ \tau : \inf_{g_Y, g_Z} \ell (\tau, g_Y, g_Z) \leq q_{x_{1.1,1-\alpha}} \right\}, \tag{22}\]

where the EL criterion function is given by

\[\ell (\tau, g_Y, g_Z) = 2 \cdot \sup_\lambda \sum_{i=1}^n \log \left( 1 + \lambda^T \begin{pmatrix} W_{+,i} (Y_i - \tau \cdot D_i - g_Y - Z_i^T g_Z) \\ W_{-,i} (Y_i - \tau \cdot D_i - g_Y - Z_i^T g_Z) \\ (W_{+,i} + W_{-,i}) Z_i (Y_i - \tau \cdot D_i - g_Y - Z_i^T g_Z) \end{pmatrix} \right).\]

Our main result implies that this is an asymptotically valid confidence region for \(\tau^F_s\). It is clear from Example 6 that inference on \(\tau^F_s\) using the auxiliary variable \(\hat{Y}(\tau)\) is a special case of our general framework.

Denote \(\gamma_s^F := \gamma_s,Y - \tau_{s,D}^{FC} \cdot \gamma_s,D\) and \(\mu_s^{FC} := \mu_{Y,+}^{SC} - \tau_{s}^{SC} \cdot \mu_{D,+}^{SC} = \mu_{Y,-}^{SC} - \tau_{s}^{SC} \cdot \mu_{D,-}^{SC}\). It is easy to check that the the solution \((g_{s,1}, g_{s,0}, g_{s,Z})\) to the moment conditions

\[
0 = E \left[ \frac{1}{h} W_+ (Y - g_{s,1} D - g_{s,0} - Z^T g_{s,Z}) \right] \\
0 = E \left[ \frac{1}{h} W_- (Y - g_{s,1} D - g_{s,0} - Z^T g_{s,Z}) \right] \\
0_{d_Z \times 1} = E \left[ \frac{1}{h} (W_+ + W_-) Z (Y - g_{s,1} D - g_{s,0} - Z^T g_{s,Z}) \right] \tag{23}
\]

satisfies \(g_{s,Z} = \gamma_s^F + O (h^2), g_{s,0} = \mu_s^{FC} + O (h^2)\) and \(g_{s,1} = \tau_{s}^{FC} + O (h^2)\). \((22)\) can also be motivated by the observation that the covariates-augmented fuzzy RDD parameter \(\tau_s^{FC}\) of Calonico et al. (2019) is approximately identified by \((23)\). Compared with inference using \((21)\), the advantage of this alternative approach is that much fewer moment conditions and nuisance parameters are needed, especially when the number of covariates is relatively large. We recommend basing EL inference on \((23)\) instead of \((21)\) in applications.

## 4 Asymptotic Properties

This section provides the asymptotic properties. We make the following regularity assumptions. Let

\[K_{s,r}(t) := \left\{ \frac{m_{2,r}}{m_{0,r} m_{2,r} - m_{1,r}^2} - \frac{m_{1,r} t}{m_{0,r} m_{2,r} - m_{1,r}^2} \right\} K(t), \text{ for } r \in \{-, +\}.\]

Note that since \(K\) is symmetric, we have \(K_{s,+}(t) = K_{s,-}(-t)\) for all \(t \in [-1, 1]\). Also denote

\[E_{jk} := (V_j - G_j \beta) (V_k - G_k \beta)^T, \; (j,k) \in \{1, 2, 3\} \times \{1, 2, 3\}\]
and

\[ \Omega_s := \begin{bmatrix} \mu_{E11,+} & 0_{d_1 \times d_2} & \mu_{E13,+} \\ 0_{d_2 \times d_1} & \mu_{E22,-} & \mu_{E23,-} \\ \mu_{E31,+} & \mu_{E32,-} & \mu_{E33,+} + \mu_{E33,-} \end{bmatrix} \gamma^2 \varphi. \]

**Assumption 1 (Kernel).** (a) \( K : \mathbb{R} \to \mathbb{R} \) is a symmetric and continuous probability density function that is supported on \([-1, 1]\);

(b) There exists a partition \( 0 = u_1 < \cdots < u_J = 1 \), such that \( K_{s,+}' \) is bounded and either strictly positive or strictly negative on \((u_j, u_{j+1})\), for \( j = 1, \ldots, J - 1 \).

Assumption 1(b) is a mild condition imposed on the kernel and is used when establishing validity of the Edgeworth expansion in the proof of Theorem 2 ahead. It is satisfied by most commonly-used kernels.\(^5\)

**Assumption 2 (Bandwidth).** For some \( \tau > 0 \), the bandwidth \( h \propto n^{-\tau} \) satisfies \( h \downarrow 0 \), \( nh^5 \downarrow 0 \) and \((nh)^{-1} \downarrow 0\) as \( n \uparrow \infty \).

Assumption 2 requires that the bandwidth satisfies the “undersmoothing” condition \( nh^5 \downarrow 0 \) as \( n \uparrow \infty \). In the context of effective point estimation of various non-parametric curves, optimal bandwidths minimize the asymptotic mean square errors (AMSE). These bandwidths usually obey \( h \propto n^{-1/5} \) and violate this assumption. In the literature, “undersmoothing” always means choosing a bandwidth which vanishes at a rate that is faster than the AMSE-minimizing bandwidths. See Li and Racine (2007) for many examples and Imbens and Kalyanaraman (2011) for local linear estimation of the RDD parameters. If \( h \propto n^{-\tau} \), the assumption is satisfied if \( \tau \in (1/5, 1) \).

**Assumption 3 (Data Generating Process).** Let \( \bar{\delta} > 0 \) denote some positive constant. Let

\[ \eta_1 := \left( V_1^T, \text{vec}(G_1)^T, V_3^T, \text{vec}(G_3)^T \right)^T \quad \text{and} \quad \eta_2 := \left( V_2^T, \text{vec}(G_2)^T, V_3^T, \text{vec}(G_3)^T \right)^T. \]

(a) On the neighborhood \((c, c + \bar{\delta})\), \( g_{\eta_1,+} \) has Lipschitz continuous derivatives up to the fourth order and an analogous condition holds for \( g_{\eta_2,-} \) with the neighborhood \((c - \bar{\delta}, c)\);

(b) On the neighborhood \((c, c + \bar{\delta})\), \( g_{\eta_2,k}^{\otimes,+} \) is bounded and Lipschitz continuous, \( k = 2, 3, 4 \) and an analogous condition holds for \( g_{\eta_2,k}^{\otimes,-} \) with the neighborhood \((c - \bar{\delta}, c)\);

(c) \( g_{\|\eta_1\|^2_0}(X) \) and \( g_{\|\eta_2\|^2_0}(X) \) are bounded;

(d) \( f \) has a Lipschitz third-order derivative on \((c - \bar{\delta}, c + \bar{\delta})\);

(e) The \( d \times d \) matrix \( \left[ \mu_{G1,+}^T, \mu_{G2,-}^T, \mu_{G3,+}^T + \mu_{G3,-}^T \right]^T \) is invertible;

(f) \( \Omega_s \) is positive definite.

Assumption 3 and the continuous extension theorem guarantee the existence of \( \left( \mu_{\eta_1,+}^{(l)}, \mu_{\eta_2,-}^{(l)} \right) \) \( (l = 0, 1, 2, 3) \) and \( \left( \mu_{\eta_2,k}^{\otimes,+}, \mu_{\eta_2,k}^{\otimes,-} \right) \) for \( k = 2, 3, 4 \). Characterization of the order of the remainder

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\(^5\)The restriction of \( K_{s,+} \) to \([0, 1]\) coincides with that of the “equivalent kernel” of local linear regression. See, e.g., Armstrong and Kolesár (2018, Section S2.1). E.g., if \( K \) is the triangular kernel, Assumption 1 is satisfied with \( J = 3 \) and \( u_2 = 3/4 \).
terms in the statements of the theorems in Section 4.2 requires existence of third-order (one-sided) derivatives and boundedness of conditional expectations of $\|\eta_1\|^2_0$ and $\|\eta_2\|^2_0$.

It is convenient to transform the constrained minimization of the EL criterion function (11) to an unconstrained one. Let $\theta_\dagger$ denote the first $d_\rho$ coordinates of the vector $\theta$ and $\theta_\dagger \in \mathbb{R}^{d-d_\rho}$ be the rest of the $d$ coordinates: $\theta = (\theta_\dagger^T, \theta_\dagger^T)^T$. Similarly, let $\beta_\dagger$ denote the first $d_\rho$ coordinates of the vector $\beta$ and $\beta_\dagger \in \mathbb{R}^{d-d_\rho}$ be the vector satisfying $\beta = (\beta_\dagger^T, \beta_\dagger^T)^T$. Since $\rho(\beta) = \tau_*$, under the assumptions that $\rho$ has high-order continuous derivatives and $\partial \rho(\theta) / \partial \theta_\dagger^T$ is invertible for all $\theta \in H$, by the implicit function theorem, there exists some open set $N \subseteq H$ around $\beta_\dagger$, some open set $U$ around $\beta_\dagger$ and some differentiable function $\psi: U \to \mathbb{R}^{d_\rho}$ such that the parameters satisfying the constraint $\rho(\theta) = \tau_*$ that also lie in the neighborhood $N$ of $\beta$, i.e., $\{\theta \in N : \rho(\theta) = \tau_*\}$, is the graph of $\psi$, i.e., $\{(\psi(\theta_\dagger^T), \theta_\dagger^T)^T : \theta_\dagger \in U\}$. In this paper, we assume that $\psi$ is affine. In all the examples introduced in the previous section, this assumption is satisfied.

Assumption 4 (Parameter). (a). The parameter space $H \subseteq \mathbb{R}^d$ is compact and $\beta$ is an interior point of $H$;

(b). The constrained parameter space $\{\theta \in H : \rho(\theta) = \tau_*\}$ is locally equivalent to the graph of an affine function: there exists some open set $N \subseteq H$ around $\beta$, some open set $U$ around $\beta_\dagger$ and coefficients $(\psi_0, \Psi_\dagger) \in \mathbb{R}^{d_\rho} \times \mathbb{R}^{d_\rho \times (d-d_\rho)}$ such that $\{\theta \in N : \rho(\theta) = \tau_*\}$ is equal to $\{(\psi(\theta_\dagger^T), \theta_\dagger^T)^T : \theta_\dagger \in U\}$, where $\psi(\theta_\dagger) = \psi_0 + \Psi_\dagger \theta_\dagger$.

4.1 First-Order Asymptotic Properties

The following theorem shows that $LR(\tau_*)$ is asymptotically $\chi^2_{d_\rho}$, i.e., the Wilks’ phenomenon holds.

Theorem 1. Suppose Assumptions 1 - 4 are satisfied. Then,

$$LR(\tau_*) \to_d \chi^2_{d_\rho}.$$ 

Remark 1. Theorem 1 justifies the first-order validity of the EL confidence region (14):

$$\Pr\left[\tau_* \in \left\{\tau : LR(\tau) \leq q_{\chi^2_{d_\rho}, 1-\alpha}\right\}\right] = \Pr\left[LR(\tau_*) \leq q_{\chi^2_{d_\rho}, 1-\alpha}\right] \to 1 - \alpha,$$

as $n \uparrow \infty$. An asymptotically valid size-$\alpha$ (two-sided) test of the nonlinear restriction

$$H_0 : \rho(\beta) = \tau_0 \text{ versus } H_1 : \rho(\beta) \neq \tau_0$$

can be based on the test statistic $LR(\tau_0)$ and the critical region $\left(q_{\chi^2_{d_\rho}, 1-\alpha}, \infty\right)$. 

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Remark 2 (Undersmoothing). Consider the sample analogue of (9),

\[
\tilde{\beta} = \left( \begin{array}{c}
\sum_{i=1}^{n} h^{-1} W_{+i} G_{1,i} \\
\sum_{i=1}^{n} h^{-1} W_{-i} G_{2,i} \\
\sum_{i=1}^{n} h^{-1} (W_{+i} + W_{-i}) G_{3,i}
\end{array} \right)^{-1} \left( \begin{array}{c}
\sum_{i=1}^{n} h^{-1} W_{+i} V_{1,i} \\
\sum_{i=1}^{n} h^{-1} W_{-i} V_{2,i} \\
\sum_{i=1}^{n} h^{-1} (W_{+i} + W_{-i}) V_{3,i}
\end{array} \right),
\]

which is a local linear estimator of \( \beta \). The bandwidth that minimizes the leading term of the AMSE of \( \rho (\tilde{\beta}) \) satisfies \( h \propto n^{-1/5} \). Such a bandwidth is “too large” for inference since it incurs non-negligible bias. Various papers in the RDD literature recommended shrinking the AMSE-optimal bandwidths by an ad hoc amount. See, e.g., Otsu et al. (2015, Section 5). We may modify the AMSE-optimal bandwidth by multiplying it by \( n^{-\epsilon} \) where \( \epsilon > 0 \) is a small number, e.g., \( \epsilon = 0.1 \). However, we note that this approach lacks theoretical justification and therefore an optimal bandwidth selection rule that is effective for inference is desirable.

Remark 3 (Local Polynomial Fitting). Denote \( r_p (u) := (1, u, \ldots, u^p)^T \). We consider the following \( p \)-th order local polynomial fitting:

\[
\min_{b_{-,i} \in \mathbb{R}^{p+1}} \int_{-c}^{c} \left\{ g_Y (x) f (x) - b_{-,i}^T r_p (x - c) \right\}^2 \frac{1}{h} K \left( \frac{x-c}{h} \right) \, dx.
\]

The minimizer denoted by \( b_{s,-} \) satisfies the moment conditions:

\[
\int_{-c}^{c} g_Y (x) f (x) r_p (x - c) \frac{1}{h} K \left( \frac{x-c}{h} \right) \, dx = \left\{ \int_{-c}^{c} r_p (x - c) r_p (x - c) \frac{1}{h} K \left( \frac{x-c}{h} \right) \, dx \right\} b_{s,-}.
\]

Similarly, we can define \( b_{s,+} \). Note that the right hand side of (24) are known constants. The minimizer \( e_{p+1,1}^T b_{s,r} \) is an approximation for \( \mu_{Y,r} \), \( r \in \{-, +\} \). Let \( M_{p,r} \) be the \( (p+1) \times (p+1) \) matrix with \( m_{k+i-2,r} \) being on its \( k \)-th row and \( l \)-th column, for \( r \in \{-, +\} \). Let \( B_h \) be a \((p+1) \times (p+1)\) diagonal matrix with \( \{1, h, \ldots, h^p\} \) being on its diagonal. Solving the linear equations (24), we have

\[
e_{p+1,1}^T b_{s,-} = \int_{-c}^{c} g_Y (x) f (x) e_{p+1,1}^T (M_{p,-} - B_h)^{-1} r_p \left( \frac{x-c}{h} \right) \frac{1}{h} K \left( \frac{x-c}{h} \right) \, dx.
\]

Let

\[
W_{-,i}^{(p)} := 1 (X_i < c) e_{p+1,1}^T (M_{p,-} - B_h)^{-1} r_p \left( \frac{X_i-c}{h} \right) \frac{1}{h} K \left( \frac{X_i-c}{h} \right)
\]

and

\[
W_{+,i}^{(p)} := 1 (X_i \geq c) e_{p+1,1}^T (M_{p,-} - B_h)^{-1} r_p \left( \frac{X_i-c}{h} \right) \frac{1}{h} K \left( \frac{X_i-c}{h} \right).
\]

Now clearly we have \( e_{p+1,1}^T b_{s,-} = E \left[ h^{-1} W_{-,i}^{(p)} Y \right] \). The \( p \)-th order local polynomial EL criterion function \( \ell^{(p)} \) in its dual form is given by (12) and (13) with \((W_{+,i}, W_{-,i})\) being replaced by \((W_{+,i}^{(p)}, W_{-,i}^{(p)})\).

The \( p \)-th order EL ratio \( LR^{(p)} \) is defined by (11) with \( \ell \) being replaced by \( \ell^{(p)} \).
In the literature, it is argued that in the context of estimation and inference for RDD, the preferred choice of the order of the local polynomial is \( p = 1 \), since the finite-sample performance of higher-order polynomial approximation at or near boundary points is often poor. See, e.g., Gelman and Imbens (2019) and Calonico et al. (2020) for discussion.

**Remark 4 (Robust Inference).** For Wald-type inference on the RDD parameters using the standard local linear regression, “robust inference” proposed by Calonico et al. (2014) uses bias correction and standard errors which are based on linearization that takes into account estimation of the bias. This method is robust to large bandwidths that obey \( h \propto n^{-1/5} \). Calonico et al. (2014, Remark 7) noted that explicit bias estimation and their bias-estimation-aware standard errors for standard local polynomial estimation for the RDD parameter is equivalent to using higher-order local polynomials. Using higher-order local polynomial fitting for robustness against large bandwidths (e.g., the AMSE-optimal ones that are effective for point estimation with local linear fitting) is popular in recent literature. See, e.g., Chiang et al. (2019). The same approach can be taken in our context.

It is straightforward to adapt the proof of Theorem 1 to show that for local quadratic EL \( LR^{(2)}(\tau^*) \rightarrow_d \chi^2_d \rho \) it suffices that \( nh^7 \downarrow 0 \) (instead of \( nh^5 \downarrow 0 \) ) as \( n \uparrow \infty \). This justifies the practical implementation that we can choose the bandwidth to be the AMSE-minimizing bandwidth discussed in Remark 2 since the bias is removed internally by using higher order polynomial approximation.

### 4.2 Second-Order Asymptotic Properties

Now we provide the second order properties of the EL inference. Denote

\[
U_{1,i} := W_{+i}(V_{1,i} - G_{1,i} \beta), \\
U_{2,i} := W_{-i}(V_{2,i} - G_{2,i} \beta), \\
U_{3,i} := (W_{+i} + W_{-i})(V_{3,i} - G_{3,i} \beta)
\]

and \( U_i := (U_{1,i}^T, U_{2,i}^T, U_{3,i}^T)^T \). \( (U_1, U_2, U_3, U) \) are defined similarly. Denote the bias \( \xi := E[h^{-1}U] \).

If local linear fitting is applied, it is shown in the appendix that \( ||\xi|| = O(h^2) \). In the appendix, we find the stochastic expansion (approximation) \( LR_* \) to \( LR(\tau^*) \) satisfying

\[
LR(\tau^*) = LR_* + O_p \left( (nh)^{-3/2} + n^{-1} + \frac{||\xi||}{(nh)^{1/2}} \right),
\]

where the expression of \( LR_* \) can be found in the appendix.\(^6\) The stochastic approximation is of the form \( LR_* = (nh)(R + R_0)^T(R + R_0) \), where \( R \) is the stochastic part and \( R_0 \) is the deterministic bias part, which is a quadratic function of the bias \( \xi \).

---

\(^6\)Applying the “delta method” (see, e.g., Hall, 1992, Section 2.7), we can show that the difference between the cumulative distribution function of \( LR(\tau^*) \) and that of \( LR_* \) is of order that is the same as that of the order of the error term in (25), up to logarithmic factors.
Denote

\[
\Omega := E\left[ \frac{1}{h} UU^T \right] = \begin{bmatrix}
E \left[ h^{-1} U_1 U_1^T \right] & 0_{d_1 \times d_2} & E \left[ h^{-1} U_1 U_3^T \right] \\
0_{d_2 \times d_1} & E \left[ h^{-1} U_2 U_2^T \right] & E \left[ h^{-1} U_2 U_3^T \right] \\
E \left[ h^{-1} U_3 U_1^T \right] & E \left[ h^{-1} U_3 U_2^T \right] & E \left[ h^{-1} U_3 U_3^T \right]
\end{bmatrix},
\]

\[
\Delta := \left( \frac{\partial \psi(\theta_i)}{\partial \theta_i} \right)_{\theta_i = \beta_i} = \left( \Psi_i \right) \text{ and } P_i := \left( \begin{array}{c}
W_{+i} G_{1,i} \Delta \\
W_{-i} G_{2,i} \Delta \\
(W_{+i} + W_{-i}) G_{3,i} \Delta
\end{array} \right).
\]

Let

\[
\Pi := E \left[ \frac{1}{h} P \right], \quad Q := \Omega^{-1} - \Pi (\Pi^T \Omega^{-1} \Pi)^{-1} \Pi^T \Omega^{-1},
\]

\[
O := (\Pi^T \Omega^{-1} \Pi)^{-1}, \quad N := \Omega^{-1} \Pi (\Pi^T \Omega^{-1} \Pi)^{-1}
\]

and

\[
\gamma[jk] := E \left[ \frac{1}{h} U[j] U[k] \right], \quad \gamma[jkl] := E \left[ \frac{1}{h} U[j] U[k] U[l] \right], \quad \gamma[jklm] := E \left[ \frac{1}{h} U[j] U[k] U[l] U[m] \right],
\]

\[
\Gamma[j,s] := E \left[ \frac{1}{h} P[js] \right], \quad \Gamma[k;j,s] := E \left[ \frac{1}{h} U[k] P[js] \right], \quad \Gamma[k;j,s,t] := E \left[ \frac{1}{h} U[k] P[js] P[kt] \right].
\]

The following result is the main distributional expansion theorem of this paper.

**Theorem 2.** Suppose that Assumptions 1 - 4 hold. Let

\[
B_1 := (nh) \left\{ \sum_{j,k} Q[jk] \xi[j] \xi[k] \right\}
\]

\[
B_2 := -2 \cdot \sum_{j,k,l} \sum_s \Gamma[k;j,s] N[k] Q[jl] \xi[l]
\]

\[
B_3 := \frac{1}{nh} \left\{ \sum_{j,k,l,m} \sum_s \Gamma[k;j,s] Q[km] Q[jl] O[st] \Gamma[m;l,t] - 2 \sum_{k,l,m,n,o,v} \sum_s \Gamma[k;l,s] \frac{Q[km]}{Q[kv]} N[os] \gamma[wo] O[nw] \right\}
\]

\[
+ \frac{1}{3} \sum_{n,o,v,w} \Gamma[nos] Q[no] Q[os] - \frac{1}{3} \sum_{n,o,v,v',w} \Gamma[no] Q[no'] Q[o'] Q[vv'] \gamma[no'o'] v' + \frac{1}{3} \sum_{n,o,v,w} \Gamma[nos] Q[no] Q[os] - \frac{1}{3} \sum_{n,o,v,v',w} \Gamma[nos] Q[no'] Q[o'] Q[vv'] \gamma[no'o'] v' + \frac{1}{3} \sum_{n,o,v,w} \Gamma[nos] Q[no] Q[os] - \frac{1}{3} \sum_{n,o,v,v',w} \Gamma[nos] Q[no'] Q[o'] Q[vv'] \gamma[no'o'] v'
\]

\[
- \sum_{j,k,l,m} \sum_s \Gamma[j;k,s]; N[jl] Q[km] N[ls] \Gamma[l;m,t] + \sum_{j,k,l,m} \sum_s \Gamma[j;k,s]; N[j ls] Q[km] \Gamma[l;m,t] N[lt]
\]

\[
- \sum_{j,k} \sum_s \Gamma[j;s;k,t] Q[jk] O[st] + 2 \sum_{j,k} \sum_s \Gamma[j;k;l,s] N[js] Q[kl]
\]
where the ranges are $j,k,l,m,n,o,v,w = 1,2,\ldots,d$ and $s,t = 1,2,\ldots,d-d_{\rho}$, and

\[ B_{c} := \frac{B_{1} + B_{2} + B_{3}}{d_{\rho}}. \]

Then,

\[
\Pr \left[ LR_{s} \leq q_{\chi_{d_{\rho},1-\alpha}}^{2} \right] = (1 - \alpha) - B_{c} \cdot q_{\chi_{d_{\rho},1-\alpha}}^{2} f_{\chi_{d_{\rho},1-\alpha}} \left( q_{\chi_{d_{\rho},1-\alpha}}^{2} \right) + O \left( \frac{\|\xi\|}{(nh)^{1/2}} + n^{-1} + (nh)^{-3/2} + (nh)^{2} \|\xi\|^{4} + (nh) \|\xi\|^{3} \right).
\]

Moreover,

\[
\Pr \left[ (1 + B_{c})^{-1} LR_{s} \leq q_{\chi_{d_{\rho},1-\alpha}}^{2} \right] = (1 - \alpha) + O \left( \frac{\|\xi\|}{(nh)^{1/2}} + n^{-1} + (nh)^{-3/2} + (nh)^{2} \|\xi\|^{4} + (nh) \|\xi\|^{3} \right).
\]

**Remark 5.** In the proof of Theorem 2, it is shown that the density of $(nh)^{1/2} R$ admits an approximation of the form:

\[ x \mapsto \left\{ 1 + \pi_{1}(x) + \pi_{2,1}(x) + \pi_{2,2}(x) \right\} \phi_{d_{\rho}}(x), \]

where terms of smaller order of magnitude are omitted. $\pi_{1}$ is a linear function whose coefficients are of order $O \left( (nh)^{-1/2} \right)$. $\pi_{2,1}$ is a quadratic function whose coefficients are linear functions of the bias $\xi$. $\pi_{2,1}$ is a quadratic function whose coefficients are of order $O \left( (nh)^{-1} \right)$. Then

\[
\Pr \left[ LR_{s} \leq q_{\chi_{1,1-\alpha}}^{2} \right] = \Pr \left[ \| (nh)^{1/2} (R + R_{0}) \| \leq \sqrt{q_{\chi_{d_{\rho},1-\alpha}}^{2}} \right]
\]

is equal to the sum of

\[
\int \| x + (nh)^{1/2} R_{0} \| \leq \sqrt{q_{\chi_{d_{\rho},1-\alpha}}^{2}} \phi_{d_{\rho}}(x) \, dx = (1 - \alpha) - B_{1} \cdot \frac{q_{\chi_{d_{\rho},1-\alpha}}^{2}}{d_{\rho}} f_{\chi_{d_{\rho},1-\alpha}} \left( q_{\chi_{d_{\rho},1-\alpha}}^{2} \right),
\]

\[
\int \| x + (nh)^{1/2} R_{0} \| \leq \sqrt{q_{\chi_{d_{\rho},1-\alpha}}^{2}} \left\{ \pi_{1}(x) + \pi_{2,1}(x) \right\} \phi_{d_{\rho}}(x) \, dx = -B_{2} \cdot \frac{q_{\chi_{d_{\rho},1-\alpha}}^{2}}{d_{\rho}} f_{\chi_{d_{\rho},1-\alpha}} \left( q_{\chi_{d_{\rho},1-\alpha}}^{2} \right),
\]

\[
\int \| x + (nh)^{1/2} R_{0} \| \leq \sqrt{q_{\chi_{d_{\rho},1-\alpha}}^{2}} \pi_{2,2}(x) \phi_{d_{\rho}}(x) \, dx = -B_{3} \cdot \frac{q_{\chi_{d_{\rho},1-\alpha}}^{2}}{d_{\rho}} f_{\chi_{d_{\rho},1-\alpha}} \left( q_{\chi_{d_{\rho},1-\alpha}}^{2} \right),
\]

where terms of smaller order of magnitude are omitted.

**Remark 6 (Coverage Accuracy).** If local linear fitting is applied, it can be shown that $B_{1} = O \left( nh^{5} \right)$, $B_{2} = O \left( h^{2} \right)$ and $B_{3} = O \left( (nh)^{-1} \right)$, since $\|\xi\| = O \left( h^{2} \right)$ and all other quantities are $O \left( 1 \right)$. Therefore the leading coverage error term is $O \left( nh^{5} + h^{2} + (nh)^{-1} \right)$. This coincides with those of the leading coverage error terms of EL-type or Wald-type inference methods for non-parametric
curves in many contexts. See, e.g., Calonico et al. (2018), Otsu et al. (2015) and Ma et al. (2019) among others. It is clear that \( h \propto n^{-1/3} \) yields the smallest possible order of magnitude for the leading coverage error, i.e., \( O \left( n^{-2/3} \right) \).

For local quadratic fitting, Theorem 2 implies
\[
\Pr \left[ LR^{(2)}(\tau) \leq q_{\chi_{d_p,1-\alpha}^2} \right] = (1 - \alpha) + O \left( nh^7 + h^3 + (nh)^{-1} \right),
\]
under stronger assumptions on smoothness than Assumption 3. Applying CCT’s robustifying strategy discussed in Remark 4 would require \( h \propto n^{-1/5} \). It is clear from (26) that the bandwidths that give the fastest coverage error decay rate should obey \( h \propto n^{-1/4} \), under which the fastest possible decay rate of the coverage error is \( O \left( n^{-3/4} \right) \). The rate \( h \propto n^{-1/5} \) required by robust inference is sub-optimal.\(^{7}\) If \( h \propto n^{-1/5} \), the remainder term on the right hand side of (26) is \( O \left( n^{-2/5} \right) \). In contrast, the coverage error decay rate corresponding to the local linear fitting using the optimal rate \( h \propto n^{-1/3} \) is \( O \left( n^{-2/3} \right) \), which is faster than \( O \left( n^{-2/5} \right) \).

Remark 7 (Bartlett Correction). Theorem 2 holds for any bandwidth \( h \) satisfying Assumption 2. It implies that the error of \( \chi^2 \) approximation to the distribution of the rescaled statistic \((1 + B_c)^{-1} LR_x \) is of smaller order of magnitude, since the rescaling eliminates the leading term. It can be shown that \( E[LR_x] \) is equal to the sum of \( 1 + B_c \) and terms of smaller order of magnitude. Rescaling \( LR_x \) by its mean is known as Bartlett correction in the literature. The second part of Theorem 2 shows that Bartlett-corrected EL confidence set \( \left\{ \tau : (1 + B_c)^{-1} LR(\tau) \leq q_{\chi_{d_p,1-\alpha}^2} \right\} \) has more accurate coverage than that of the uncorrected one. Bartlett correction is one of the desired properties of EL-based inference in many contexts. Note that the leading coverage error term, with \( f_{\chi_{d_p}^2} \left( q_{\chi_{d_p,1-\alpha}^2} \right) \) ignored, can be viewed as a linear function of \( q_{\chi_{d_p,1-\alpha}^2} \). The proof of the second part of Theorem 2 hinges on this fact. For Wald-type inference in the context of inference for non-parametric curves, its counterpart is typically an odd polynomial of degree five. See, e.g., Ma et al. (2019, Theorem 3).

In practice, the unknown quantities \((\Upsilon, \Gamma, Q, O, N)\) in \( B_c \) can be estimated by sample analogues. It is shown in the appendix that for some vector \((\zeta_1^T, \zeta_2^T, \zeta_3^T)^T \in \mathbb{R}^d\) that depend on \((\mu_{\eta_1,+}^{(k)}, \mu_{\eta_2,-}^{(k)})\) and \( \varphi^{(k)}, k = 0, 1, 2, \)
\[
\xi = \frac{1}{2} \varphi \left( \zeta_1^T, \zeta_2^T, \zeta_3^T \right)^T h^2 + O \left( h^3 \right).
\]
The bias \( \xi \) can be estimated by plug-in estimation of \((\zeta_1^T, \zeta_2^T, \zeta_3^T)^T \). See Section 6 ahead.

\(^{7}\)A similar result for standard local polynomial regression was discussed in Calonico et al. (2018, Section 3).
5 Regression Discontinuity Design

5.1 Sharp Regression Discontinuity Design

In a sharp RDD, assignment of treatment is determined by the forcing variable $X$ and a cut-off $c$. Treatment is assigned if $X \geq c$. Let $D := \mathbb{1}(X \geq c)$ denote the treatment assignment. Let $Y_1$ be the potential outcome with treatment and let $Y_0$ be the potential outcome without treatment. For any individual, only one of the potential outcomes is observed. The observed outcome $Y$ is determined by treatment assignment ($D$) and potential outcomes ($Y_1$, $Y_0$):

$$Y = D Y_1 + (1 - D) Y_0.$$

We take the widely applied electoral RDD (see Lee, 2008; Eggers et al., 2015; Hyytinen et al., 2018) as an example. The forcing variable $X$, treatment $D$ and the outcome variable $Y$ correspond to the vote share margin in the last election, results of the last election (win or lose) and this election.

Suppose that we observed a random sample $\{(Y_i, X_i)\}_{i=1}^n$ which are i.i.d. copies of $(Y, X)$. We are interested in making inference on the treatment effect. In particular, in a sharp RDD, the local (conditional) average treatment effect $E[Y_1 - Y_0 | X = c]$ is identified under very weak assumptions. If $g_{Y_1}$ and $g_{Y_0}$ are both continuous at $c$, then it is easy to show that

$$\tau^*_S := \mu_{Y,+} - \mu_{Y,-} = E[Y_1 - Y_0 | X = c],$$

where $\mu_{Y,+} - \mu_{Y,-}$ is a feature of the population of the data, on which inference can be made by using our method. See Example 1. An EL confidence set with nominal coverage probability $1 - \alpha$ for the RDD treatment effect $\tau^*_S$ is

$$\{ \tau : LR(\tau) \leq q_{\chi^2, 1 - \alpha} \}$$

where

$$LR(\tau) = \inf_{(g_+, g_-) : g_+ - g_- = \tau} \ell(g_+, g_-),$$

and

$$\ell(g_+, g_-) = 2 \cdot \sup_{\lambda} \sum_{i=1}^n \log \left( 1 + \lambda^T \begin{pmatrix} W_{+,i} (Y_i - g_+) \\ W_{-,i} (Y_i - g_-) \end{pmatrix} \right).$$

Denote

$$\overline{B}(\kappa_{2,+}, \kappa_{2,-}, \kappa_{3,+}, \kappa_{3,-}, \kappa_{4,+}, \kappa_{4,-}) := \frac{1}{2} \frac{\gamma_4}{\gamma_2} \frac{\kappa_{4,+}}{\kappa_{2,+}} + \frac{1}{3} \frac{\gamma_3^2}{\gamma_2^2} \frac{(\kappa_{3,+} - \kappa_{3,-})^2}{(\kappa_{2,+} + \kappa_{2,-})^2} + (4\gamma_3 - 2\gamma_2) \cdot \frac{\kappa_{2,+} \kappa_{2,-}}{\kappa_{2,+} + \kappa_{2,-}},$$

$$\kappa_{j,+}^S := \lim_{x \downarrow c} E \left[ (Y - \mu_{Y,+})^j | X = x \right],$$

$$\kappa_{j,-}^S := \lim_{x \uparrow c} E \left[ (Y - \mu_{Y,-})^j | X = x \right].$$
and

\[ \zeta^g := \mu^{(2)}_{\gamma^g} + 2\mu^{(1)}_{\gamma^g}, \text{ for } r \in \{-, +\}. \]

The following theorem is established as a corollary to Theorem 2. For EL inference on the sharp RDD parameter, the following result provides the leading term in the coverage error expansion.

**Theorem 3 (Sharp RD).** In the special case described in Example 1 (Sharp RDD), suppose that Assumptions 1 - 4 hold. Then,

\[
\Pr \left[ LR_* \leq q_{\chi^2_{1, 1-\alpha}} \right] = (1 - \alpha) - B^S_c \cdot q_{\chi^2_{1-\alpha}} \cdot f_{\chi^2_{1-\alpha}}(q_{\chi^2_{1-\alpha}}) + O \left( \frac{h^2}{(nh)^{1/2}} + n^{-1} + (nh)^{-3/2} + n^2 h^{10} + n h^6 + h^3 \right),
\]

where \( B^S_c := B^S_1 + B^S_3, \)

\[
B^S_1 := nh^5 \cdot \frac{1}{4} \frac{\omega^2 (\zeta^S_+ - \zeta^S_-)^2}{\gamma_2 \varphi (\kappa^S_{2,+} + \kappa^S_{2,-})}, \quad B^S_3 := \frac{1}{n h} \overline{B} \left( \kappa^S_{2,+}, \kappa^S_{2,-}, \kappa^S_{3,+}, \kappa^S_{3,-}, \kappa^S_{4,+}, \kappa^S_{4,-} \right) \frac{\omega^2 (\kappa^S_{2,+} + \kappa^S_{2,-})}{\gamma_2 \varphi (\kappa^S_{2,+} + \kappa^S_{2,-})}.
\]

Moreover,

\[
\Pr \left[ (1 + B^S_c)^{-1} LR_* \leq q_{\chi^2_{1, 1-\alpha}} \right] = (1 - \alpha) + O \left( \frac{h^2}{(nh)^{1/2}} + n^{-1} + (nh)^{-3/2} + n^2 h^{10} + n h^6 + h^3 \right).
\]

**Remark 8.** We specialize Theorem 2 to the case of moment conditions (15) in the sharp RDD. Then we can show that \( B_1 = B^S_1 + O(n h^6), B_2 = O(h^3) \) and \( B_3 = B^S_3 + O(n^{-1}) \). It is interesting to observe that in this special case, \( B_2 \) is of smaller order of magnitude than \( B_1 \) and \( B_3 \), if the bandwidth obeys \( h \propto n^{-1/3} \), which is optimal for coverage error decay rate of the EL confidence set \( \{ \tau : LR(\tau) \leq q_{\chi^2_{1, 1-\alpha}} \} \). The coverage optimal bandwidth is determined only by \( B^S_1 \) and \( B^S_3 \) and an explicit solution of it is possible as a consequence. Note that in the expressions of \( B^S_1 \) and \( B^S_3 \), \( \gamma \)'s and \( \omega \) are known constants that depend only on the kernel.

**Remark 9 (Coverage Optimal Bandwidth).** First, a bandwidth that gives the best coverage should obey \( h \propto n^{-1/3} \) and we restrict our attention to bandwidths that satisfy this rate restriction. Denote

\[
v_S := \overline{B} \left( \kappa^S_{2,+}, \kappa^S_{2,-}, \kappa^S_{3,+}, \kappa^S_{3,-}, \kappa^S_{4,+}, \kappa^S_{4,-} \right) \text{ and } \iota_S := \frac{1}{2} \omega (\zeta^S_+ - \zeta^S_-).
\]

Now if \( h = H \cdot n^{-1/3} \), the leading coverage error term can be written as

\[
- n^{-2/3} \left\{ \frac{\iota^2_S H^5 + v_S H^{-1}}{\gamma_2 \varphi (\kappa^S_{2,+} + \kappa^S_{2,-})} \right\} q_{\chi^2_{1, 1-\alpha}} f_{\chi^2_{1-\alpha}}(q_{\chi^2_{1, 1-\alpha}}).
\]
It follows that the coverage optimal bandwidth is \( h_{\text{opt}}^* = H_{\text{opt}}^* \cdot n^{-1/3} \), where

\[
H_{\text{opt}}^* = \arg\min_{H>0} \left| \iota_2^2 H^5 + v_{\text{opt}} H^{-1} \right| .
\] (30)

If \( v_{\text{opt}} > 0 \), the leading coverage error (29) must be negative and \( \iota_2^2 H^5 + v_{\text{opt}} H^{-1} > 0 \) for \( H > 0 \). Solving the first order condition, we easily find the expression of the minimizer: \( H_{\text{opt}}^* = \left( v_{\text{opt}} / (5 \iota_2^2) \right)^{1/6} \). If \( v_{\text{opt}} < 0 \), the leading coverage error can be positive or negative at different values for \( H \).

In this case, following Calonico et al. (2018), we minimize its absolute value numerically and no explicit solution is available.\(^8\) Note that in either case, the minimization problem (30) is well-defined since the objective is a rational function and tends to \( \infty \) as \( H \downarrow 0 \) or \( H \uparrow \infty \). For practical implementation, we estimate \( H_{\text{opt}}^* \) by replacing the unknown quantities in the expressions of \( \iota_{\text{opt}} \) and \( v_{\text{opt}} \) with their consistent estimators. See Section 6 ahead.

**Remark 10.** Unlike the AMSE-minimizing bandwidths which obey \( h \propto n^{-1/5} \), the coverage optimal bandwidth can be thought of being balancing a term \( B_{3}^{S} \) that is of the same order as the asymptotic variance \( O \left( (nh)^{-1} \right) \) against the asymptotic bias \( O \left( h^2 \right) \), instead of the square of the bias. \( B_{3}^{S} \) reflects first-order contribution of the bias to the leading coverage error. Suppose that different bandwidths \( h_{+} \) and \( h_{-} \) \( (h_{+}, h_{-}) \propto n^{-1/3} \) on the right and the left of the cut-off point are used. It is possible to prove an extension of Theorem 1 which assumes different bandwidths on different sides. Then we wish to simultaneously select the bandwidths and minimize the leading coverage error term with respect to \( (h_{+}, h_{-}) \). The same argument as that invoked in Ma et al. (2019, Remark 4) can be used to show that this minimization problem is not well-defined.\(^10\) It can be shown that if \( \zeta_{S}^{+}, \zeta_{S}^{-} > 0 \) and the bandwidths are chosen to be proportional to each other: \( h_{+}/h_{-} = (\zeta_{S}^{+}/\zeta_{S}^{-})^{1/2} \), the first-order contribution from the bias to the leading coverage error vanishes. Then it can be easily shown that leading coverage error can be made to shrink to zero if \( (h_{+}, h_{-}) \uparrow \infty \).

**Remark 11 (Bartlett Correction).** When the bandwidth is selected to be the coverage optimal bandwidth \( h_{\text{opt}}^* \), the leading coverage error term is (29) with \( H = H_{\text{opt}}^* \). By Theorem 3, we can rescale the EL ratio statistic by the Bartlett correction factor

\[
1 + B_{c}^{S} = 1 + n^{-2/3} \frac{\iota_2^2 (H_{\text{opt}}^*)^5 + v_{\text{opt}} (H_{\text{opt}}^*)^{-1}}{\gamma_2 \varphi \left( \kappa_{2,+}^S + \kappa_{2,-}^S \right)}
\] (31)

to eliminate the leading term in the expansion. It is clear from the second part of Theorem 3 that for Bartlett-corrected EL inference, \( h_{\text{opt}}^* \propto n^{-1/3} \) leads to a faster coverage error decay rate \( O \left( n^{-1} \right) \) for the Bartlett-corrected EL confidence set \( \{ \tau : (1 + B_{c}^{S})^{-1} \left( LR(\tau) \leq q_{\chi^2_{1,1-\alpha}} \right) \} \), which is the coverage error decay rate of standard two-sided confidence intervals for most regular parameters that could be estimated at the standard parametric rate. In practice we can combine the coverage optimal band-

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\(^8\)In all of the Monte Carlo simulation setups with which we experiment, \( v_{\text{opt}} \) is strictly positive.

\(^9\)In practice, we recommend squaring the objective function when numerically solving the minimization problem (30).

\(^10\)Also see Arai and Ichimura (2018, Section 2.1).
width selector and Bartlett correction. In practical implementation, first we consistently estimate $H^*_S$ by replacing the unknown quantities in $\iota_S$ and $\upsilon_S$ with their consistent estimators. Then we fix the bandwidth to be the estimated coverage optimal bandwidth to compute the EL ratio statistic and we estimate the Bartlett correction factor by replacing $\iota_S$, $\upsilon_S$ and $H^*_S$ on the right hand side of (31) with their consistent estimators. This gives a feasible Bartlett-corrected EL ratio statistic for inference.

Remark 12 (Comparison with Existing Methods). Compared with the existing coverage optimal bandwidth selection method for Wald-type inference with standard local linear regression (Calonico et al., 2018, 2020), our EL inference method has at least four theoretical advantages. First, studentization is achieved implicitly and no separate standard error calculation is needed. Second, very often, our coverage optimal bandwidth has an explicit solution and there is no need for solving an optimization problem. Third, our Bartlett-corrected EL inference has better theoretical coverage error decay rate ($O\left(\frac{1}{n}\right)$) compared with that of Calonico et al. (2020) ($O\left(\frac{1}{n^{3/4}}\right)$). See Calonico et al. (2020, Theorem 3.2) with $p = 1$ (local linear regression), which was recommended to be used in applications by the econometric literature. Lastly, it is interesting to note that our coverage optimal bandwidth is independent of the nominal coverage level $1 - \alpha$. In contrast, the coverage optimal bandwidths for Wald-type inference methods usually depend on $1 - \alpha$. The reason is that their leading coverage error terms (with $\phi_1\left(q_{x_1^2,1-\alpha}^{1/2}\right)$ ignored) are usually linear functions of $q_{x_1^2,1-\alpha}^{1/2}$, $q_{x_1^2,1-\alpha}^{3/2}$ and $q_{x_1^2,1-\alpha}^{5/2}$, where coefficients depend on the bandwidths.\textsuperscript{11} See Calonico et al. (2018, 2020) and also Ma et al. (2019, Theorem 3). Therefore the minimizer depends on $1 - \alpha$. In contrast, for EL inference, the leading coverage error term is proportional to $q_{x_1^2,1-\alpha}^{1/2}$ and the minimizer of the absolute value of (29) is the same as that of (30), which does not depend on $1 - \alpha$. In practice, practitioners calculate confidence sets under multiple nominal coverage levels (e.g., 90%, 95% and 99%). For our EL inference, the coverage optimal bandwidth is the same regardless of which nominal coverage level is chosen. For hypothesis testing problems, practitioners very often need to calculate and report the $p$-value. Using the elementary definition of the $p$-value (see, e.g., Wasserman, 2013, Definition 10.11) can be problematic for practitioners if Wald-type test statistic with its coverage optimal bandwidth is used, since in this case, the Wald-type statistic is dependent on the nominal significance level. Our EL method does not suffer from this problem.

5.2 Joint Inference with Multiple Outcome Variables

We consider the extensions proposed in Examples 2 and 3. In a sharp RDD, the multi-dimensional outcome $Y \in \mathbb{R}^{d_Y}$ is determined by treatment assignment and potential outcomes. We are interested in joint inference on the vector $\tau^{S}_\nu := \mu_{Y, +} - \mu_{Y, -}$. In the case of categorical outcome, the vector $Y$ corresponds to the set of dummy variables indicating each category (excluding the base category).

\textsuperscript{11}Note that $q_{x_1^2,1-\alpha}^{1/2} q_{x_1^2,1-\alpha} \left( q_{x_1^2,1-\alpha}^{1/2} \right) = q_{x_1^2,1-\alpha}^{1/2} \phi_1 \left( q_{x_1^2,1-\alpha}^{1/2} \right)$. 

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In the context of (joint) covariate balance test, the vector $Y$ is simply replaced by the the vector of pre-treatment covariate $Z \in \mathbb{R}^{d_Z}$ and the null hypothesis becomes $\mu_{Z,+} = \mu_{Z,-}$. Such a condition is also necessary for the validity of the method of incorporating covariates proposed by Imbens and Lemieux (2008) and Calonico et al. (2019). The joint confidence region for the multiple RDD treatment effects $\tau^* = \{ \tau : LR(\tau) \leq q_{\chi^2_{d_Y},1-\alpha} \}$, where $LR(\tau)$ is a straightforward extension of (27) to multi-dimensional outcome:

$$LR(\tau) = 2 \inf_{(g_+g_-):g_+g_-=\tau} \sup_{x \in \mathcal{X}_c} \sum_{i=1}^n \log \left( 1 + \lambda^T \left( \begin{array}{c} W_{+,i} (Y_i - g_+) \\ W_{-,i} (Y_i - g_-) \end{array} \right) \right).$$

(32)

Note that the shape of such a EL-based confidence region is not pre-determined like a Wald-type confidence region but driven by the data.

In the case of multi-dimensional outcome, it is convenient to use matrix notations to write the expressions in the leading coverage error term. Let

$$D_{+} := \lim_{x \downarrow c} E \left[ (Y - \mu_{Y,+}) (Y - \mu_{Y,+})^T | X = x \right]$$

$$D_{-} := \lim_{x \uparrow c} E \left[ (Y - \mu_{Y,-}) (Y - \mu_{Y,-})^T | X = x \right],$$

(33)

$$D_{[k],+} := \lim_{x \downarrow c} E \left[ (Y^{[k]} - \mu_{Y,+}^{[k]}) (Y - \mu_{Y,+}) (Y - \mu_{Y,+})^T | X = x \right]$$

$$D_{[k],-} := \lim_{x \uparrow c} E \left[ (Y^{[k]} - \mu_{Y,-}^{[k]}) (Y - \mu_{Y,-}) (Y - \mu_{Y,-})^T | X = x \right]$$

(34)

for $k = 1, 2, ..., d_Y$ and

$$D_{[k,l],+} := \lim_{x \downarrow c} E \left[ (Y^{[k]} - \mu_{Y,+}^{[k]}) (Y^{[l]} - \mu_{Y,+}^{[l]}) (Y - \mu_{Y,+}) (Y - \mu_{Y,+})^T | X = x \right]$$

$$D_{[k,l],-} := \lim_{x \uparrow c} E \left[ (Y^{[k]} - \mu_{Y,-}^{[k]}) (Y^{[l]} - \mu_{Y,-}^{[l]}) (Y - \mu_{Y,-}) (Y - \mu_{Y,-})^T | X = x \right]$$

(35)

for $(k,l) \in \{1, 2, ..., d_Y\}^2$. Also define

$$K := D_{+}^{-1} (D_{+}^{-1} + D_{-}^{-1})^{-1} D_{+}^{-1}$$

$$\zeta^r := \mu_{Y,+}^{(2)} \varphi + 2 \mu_{Y,+}^{(1)} \varphi^{(1)}, \text{ for } r \in \{-, +\}.$$  

(36)

The following theorem is an extension of Theorem 3. It accommodates both of Examples 2 and 3, where multiple outcome variables arise naturally.

**Theorem 4.** *In the special cases described in Examples 2 and 3 (sharp RDD with multiple outcome variables), suppose that Assumptions 1 - 4 hold. Then,*

$$\Pr \left[ LR* \leq q_{\chi^2_{d_Y},1-\alpha} \right] = (1 - \alpha) - B^m_c \cdot q_{\chi^2_{d_Y},1-\alpha} f_{\chi^2_{d_Y}} \left( q_{\chi^2_{d_Y},1-\alpha} \right)$$

(36)
\begin{align*}
+ O \left( \frac{h^2}{(nh)^{1/2}} + n^{-1} + (nh)^{-3/2} + n^2 h^{10} + nh^6 + h^3 \right),
\end{align*}

where $B^M_c := (B_1^M + B_3^M) / d_Y$,

\begin{align*}
B_1^M &= n h^5 \cdot \frac{1}{4} \cdot \frac{\vartheta^2}{\gamma_2 \varphi} (\zeta^M_+ - \zeta^M_\cdot)^T K (\zeta^M_+ - \zeta^M_\cdot) \\
B_3^M &= \frac{1}{nh} (\gamma_2 \varphi)^{-1} (\theta_1^M + \theta_2^M + \theta_3^M)
\end{align*}

and

\begin{align*}
\theta_1^M &= \frac{1}{2} \cdot \frac{\gamma_4}{\gamma_2} \cdot \sum_{k=1}^{d_Y} \sum_{l=1}^{d_Y} K^{[k,l]} \left( \text{tr} \left( KD_{[k,l]}^+ \right) + \text{tr} \left( KD_{[k,l]}^- \right) \right) \\
\theta_2^M &= -\frac{1}{3} \cdot \frac{\gamma_2}{\gamma_2} \cdot \left\{ \sum_{k=1}^{d_Y} \sum_{l=1}^{d_Y} K^{[k,l]} \left( \text{tr} \left( KD_{[k,l]}^+ KD_{[l,k]}^- \right) - \text{tr} \left( KD_{[k,l]}^- KD_{[l,k]}^+ \right) \right) \\
&\quad - \text{tr} \left( KD_{[k,l]}^+ KD_{[l,k]}^- \right) + \text{tr} \left( KD_{[k,l]}^- KD_{[l,k]}^+ \right) \right\} \\
\theta_3^M &= (4\gamma_3 - 2\gamma_2) \cdot \text{tr} \left( K (D_+^{-1} + D_-^{-1})^{-1} \right).
\end{align*}

Moreover,

\[ \Pr \left[ (1 + B^M_c)^{-1} LR_\ast \leq q_{\chi^2_{d_Y-1},1-\alpha} \right] = (1 - \alpha) + O \left( \frac{h^2}{(nh)^{1/2}} + n^{-1} + (nh)^{-3/2} + n^2 h^{10} + nh^6 + h^3 \right). \]

**Remark 13.** Theorem 4 is an extension of Theorem 3 to accommodate multiple outcomes. Similarly, we define the coverage optimal bandwidth. Denote

\[ \nu_\ast := \theta_1^M + \theta_2^M + \theta_3^M \text{ and } \tau := \frac{1}{2} \vartheta (\zeta^M_+ - \zeta^M_\cdot). \]

When $h = H \cdot n^{-1/3}$, the leading coverage error term is

\[ -n^{-2/3} \{ (\gamma_2 \varphi)^{-1} \left( (\nu_\ast^T K \nu_\ast) H^5 + \nu_\ast H^{-1} \right) \} \cdot \frac{q_{\chi^2_{d_Y-1},1-\alpha}}{d_Y} f_{\chi^2_{d_Y}} \left( q_{\chi^2_{d_Y-1},1-\alpha}^2 \right), \]

and the optimal bandwidth is defined to be what minimizes the absolute value of the leading coverage error (Calonico et al., 2018): $h^*_\ast = H^*_\ast \cdot n^{-1/3}$, where

\[ H^*_\ast = \underset{H > 0}{\text{argmin}} \left| (\nu_\ast^T K \nu_\ast) H^5 + \nu_\ast H^{-1} \right|. \]

If $\nu_\ast > 0$, $(\nu_\ast^T K \nu_\ast) H^5 + \nu_\ast H^{-1} > 0$ for all $H > 0$ and an explicit solution is available from solving the first order condition: $H^*_\ast = \left\{ \nu_\ast / 5 \nu_\ast^T K \nu_\ast \right\}^{1/6}$. $H^*_\ast$ is independent of the nominal coverage probability.
1 - \alpha. Practical implementation is straightforward. The Bartlett correction factor is
\[
1 + B_c^M = 1 + n^{-2/3} \left( \frac{\mu^T K \mu}{d_Y(\gamma_2^2 \varphi)} \right) (H^*_K)^5 + \nu_H (H^*_K)^{-1},
\]
The Bartlett-corrected EL confidence region \( \left\{ \tau : \left(1 + B_c^M\right)^{-1} LR(\tau) \leq q_{\chi^2_{1-\alpha} \ldots} \right\} \) has better coverage accuracy with error rate \( O(n^{-1}) \).

To test covariate balance \( \mu_{Z,+} = \mu_{Z,-} \), we take \( Y_i = Z_i \) (pre-treatment covariates) in (32) and use the test statistic \( LR(0_{d_Z \times 1}) \) with rejection region \( \left( q_{\chi^2_{d_Z \cdot 1-\alpha}}, \infty \right) \). The EL joint test for covariate balance avoids the problem of size distortion from multiple testing and we can choose the bandwidth to be \( h^*_K \) to further minimize size distortion. The bandwidth \( h^*_K \) is independent of the nominal size \( \alpha \). Therefore, the \( p \)-value can be calculated easily in a standard way.

\textbf{Remark 14 (Categorical Outcome).} When the outcome variables \( Y = (Y^{[1]}, \ldots, Y^{[J]})^T \) are mutually exclusive dummy variables generated by a categorical outcome, the expressions of the unknown quantities (33), (34) and (35) can be simplified by using mutual exclusion. For example,
\[
\lim_{x \to c} \text{E} \left[ (Y^{[k]} - \mu_{Y^{[k]},+}) (Y^{[l]} - \mu_{Y^{[l]},+}) | X = x \right] = \begin{cases} 
\mu_{Y^{[k]},+} - \mu_{Y^{[k]},+}^2 & \text{if } k = l \\
- \mu_{Y^{[k]},+} \mu_{Y^{[l]},+} & \text{if } k \neq l.
\end{cases}
\]
Then it is easy to see that the unknown quantities (33), (34) and (35) depend only on the first moments \( (\mu_{Y,+}, \mu_{Y,-}) \). The probabilities \( (\mu_{Y,+}, \mu_{Y,-}) \) can be consistently estimated by the local maximum likelihood estimator of Xu (2017) or conventional local linear regression. See Section 6 ahead.

Compared with Xu (2017), our method admits a simple coverage optimal bandwidth and the shape of our EL confidence region is data-driven. Studentization is done implicitly. Moreover, inference for fuzzy RDD and incorporating covariates are straightforward extensions. See Remark 18 ahead. Our method is Bartlett correctable. When solving (32), we can easily impose the natural ranges for \( (g_+, g_-) : g_r^{[j]} \in (0, 1) \) for all \( j = 1, 2, \ldots, J \) and \( \sum_{j=1}^J g_r^{[j]} < 1 \), for \( r \in \{-, +\}.^{12} \)

### 5.3 Sharp Regression Discontinuity Design with Covariates

Next, we consider a very important and greatly useful extension of Theorem 3. In an RDD, econometricians always have access to observations on some pre-treatment covariates. For example, in the empirical illustration of electoral RDD in Section 8.1, we consider three covariates: candidates’ age, gender, the incumbency status, all determined prior to the election considered. Following Example

\[^{12}\text{In practical implementation, we impose the ranges by using the one-to-one transformation:} \]
\[
(p^{[1]}, \ldots, p^{[J]}) \mapsto \left( \exp(p^{[1]}) / \left(1 + \sum_{j=1}^J \exp(p^{[j]}) \right), \ldots, \exp(p^{[J]}) / \left(1 + \sum_{j=1}^J \exp(p^{[j]}) \right) \right).
\]
6, we use \( Z \in \mathbb{R}^{d_Z} \) to denote a vector consisting of such variables. In such an augmented RDD, we observe a random sample \( \{(Y_i, X_i, Z_i)\}_{i=1}^n \), which are i.i.d. copies of \((Y, X, Z)\). Note that the local linear estimator for the sharp RDD parameter, which is the difference of the two estimated intercepts in (1), can be obtained from running a single localized regression. See, e.g., Imbens and Lemieux (2008, Section 4.2). In practical applications, a common practice is to augment such a localized regression to incorporate the pre-treatment covariates \( Z_i \):

\[
\min_{a_0, b_0, a_1, b_1, d} \sum_{i=1}^n K \left( \frac{X_i - c}{h} \right) \{ Y_i - a_0 - b_0 (X_i - c) - a_1 D_i - b_1 (X_i - c) D_i - Z_i^T d \}^2.
\]

See Imbens and Lemieux (2008, Section 4.3). The covariates \( Z_i \) enters linearly. Note that this approach does not require kernel smoothing over the covariates \( Z_i \) and therefore avoids selecting additional bandwidths. The estimated coefficient of \( D_i \) is used as an estimator of the sharp RDD parameter in practical applications.

Calonico et al. (2019) studied the probabilistic limit of such an estimator. It was shown that the estimated coefficients of \( Z_i \) in the localized regression (37) converges in probability to \( \gamma_s \) defined by (19) and the estimated coefficient of \( D_i \) converges in probability to \( \tau_s^{SC} \) defined by (18). Note that \( \tau_s^{SC} = \tau_s^g \), which is equal to the local (conditional) treatment effect identified in a sharp RDD, under the weak covariate balance assumption \( \mu_{Z,+} = \mu_{Z,-} \).\(^{13}\)

It was also shown by Calonico et al. (2019) that under a stronger assumption, the estimator from the localized regression (37) is more accurate than the standard local linear estimator without covariates. See Calonico et al. (2019, Section 4.2) for discussion.

It is shown in Example 6 that the augmented sharp RDD parameter \( \tau_s^{SC} \) is identified approximately by the moment conditions (20). Therefore in an augmented sharp RDD, our EL-based method can be applied to make inference on the parameter of interest. The EL confidence set is \( \{ \tau : LR(\tau) \leq q_{\chi^2_{1,1-\alpha}} \} \) where

\[
LR(\tau) = \inf_{(g_{Y,+}, g_{Y,-}, g_Z) : g_{Y,+} - g_{Y,-} = \tau} \ell (g_{Y,+}, g_{Y,-}, g_Z)
\]

and

\[
\ell (g_{Y,+}, g_{Y,-}, g_Z) = 2 \cdot \sup_{\lambda} \sum_{i=1}^n \log \left( 1 + \lambda^T \begin{pmatrix} W_{+,i} (Y_i - g_{Y,+} - Z_i^T g_Z) \\ W_{-,i} (Y_i - g_{Y,-} - Z_i^T g_Z) \\ W_{+,i} Z_i (Y_i - g_{Y,+} - Z_i^T g_Z) + W_{-,i} Z_i (Y_i - g_{Y,-} - Z_i^T g_Z) \end{pmatrix} \right).
\]

Due to the presence of the covariates, the expressions in the leading coverage error term are not

\[^{13}\mu_{Z,+} = \mu_{Z,-} \] is satisfied if the conditional distribution of \( Z \) given \( X = x \) is continuous at \( x = c \).
easy to simplify. We use matrix notations for expressions in this case. Define

\[ \epsilon_{r, \nu}^{sc} := Y - \mu_{Y, \nu}^{sc} - Z^T \gamma_s, \ r \in \{-, +\}. \]

Denote \( T := (\mathbb{1}(X_i \geq c), \mathbb{1}(X_i < c))^T, \ Z := (1, Z^T)^T \) and \( S := (T^T, Z^T)^T \). Let

\[
C_{k;[s,t],+} := \lim_{x \downarrow c} \mathbb{E} \left[ (\epsilon_{r}^{sc})^k Z^{[s]} Z^{[t]} SS^T | X = x \right]
\]

\[
C_{k;[s,t],-} := \lim_{x \downarrow c} \mathbb{E} \left[ (\epsilon_{r}^{sc})^k Z^{[s]} Z^{[t]} SS^T | X = x \right]
\]

\[
C_{k;[s,t]} := C_{k;[s,t],+} + C_{k;[s,t],-}
\]

(38)

and \( \Omega_{SC} := C_{2;[1,1]} \). Let

\[
\Pi_{SC} := \begin{pmatrix}
1 & \mu_{Z,+}^T \\
1 & \mu_{Z,-}^T \\
\mu_{Z,+} + \mu_{Z,-} & \mu_{ZT,+} + \mu_{ZT,-}
\end{pmatrix}
\]

and

\[
\Omega_{SC} := \Omega_{sc} - \Omega_{sc} \Pi_{SC} \left( \Pi_{SC}^T \Omega_{sc}^{-1} \Pi_{SC} \right)^{-1} \Pi_{SC}^T \Omega_{sc}^{-1}
\]

\[
O_{SC} := \left( \Pi_{SC}^T \Omega_{sc}^{-1} \Pi_{SC} \right)^{-1}
\]

\[
N_{SC} := \Omega_{sc}^{-1} \Pi_{SC} \left( \Pi_{SC}^T \Omega_{sc}^{-1} \Pi_{SC} \right)^{-1}.
\]

Let

\[
J_{(l)} := \begin{cases}
C_{3;[1,1],+} & \text{if } l = 1 \\
C_{3;[1,1],-} & \text{if } l = 2 \\
C_{3;[1,l-1]} & \text{if } l \in \{3, \ldots, 2 + dZ\}
\end{cases}
\]

and

\[
J_{(l,m)} := \begin{cases}
C_{4;[1,1],+} & \text{if } l = 1, m = 1 \\
C_{4;[1,1],-} & \text{if } l = 2, m = 2 \\
C_{4;[1,m-1],+} & \text{if } l = 1, m \in \{3, \ldots, 2 + dZ\} \\
C_{4;[1,m-1],-} & \text{if } l = 2, m \in \{3, \ldots, 2 + dZ\} \\
C_{4;[l-1,m-1]} & \text{if } (l, m) \in \{3, \ldots, 2 + dZ\}^2,
\end{cases}
\]

\( J_{(l,m)} = 0_{(2+dZ) \times (2+dZ)} \), if \( (l, m) \in \{(1, 2), (2, 1)\} \) and when \( (l, m) \in \{3, 4, \ldots, 2 + dZ\} \times \{1, 2\}, \ J_{(l,m)} \)
is defined to be $J_{(m,t)}$. Also define

$$
\mathbf{L}(l,s) := \begin{cases} 
\mathbf{C}_2;[1,s],+ & \text{if } l = 1, s \in \{1,...,1+dZ\} \\
\mathbf{C}_2;[1,s],- & \text{if } l = 2, s \in \{1,...,1+dZ\} \\
\mathbf{C}_2;[l-1,s] & \text{if } (l, s) \in \{3,...,2+dZ\} \times \{1,...,1+dZ\}.
\end{cases}
$$

Denote

$$
\zeta^SC_+ := 2\varphi^{(1)}\mu^{(1)}_{Y,+} - 2\varphi^{(1)}\mu^{(1)}_{Z,T,+} + 2\varphi\mu^{(2)}_{Y,+} - \varphi\mu^{(2)}_{Z,T,+} \gamma_*
$$

$$
\zeta^SC_- := 2\varphi^{(1)}\mu^{(1)}_{Y,-} - 2\varphi^{(1)}\mu^{(1)}_{Z,T,-} + 2\varphi\mu^{(2)}_{Y,-} - \varphi\mu^{(2)}_{Z,T,-} \gamma_*
$$

and

$$
\zeta^SC_Z := 2\varphi^{(1)} \{ \mu^{(1)}_{Z,+} - \mu^{(1)}_{Z,+} + \mu^{(1)}_{Z,+} - \mu^{(1)}_{Z,-} \} + 2\varphi^{(1)} \{ \mu^{(1)}_{Z,+} - \mu^{(1)}_{Z,+} + \mu^{(1)}_{Z,+} - \mu^{(1)}_{Z,-} \} \gamma_*
$$

$$
+ \varphi \{ \mu^{(2)}_{Z,+} - \mu^{(2)}_{Z,+} + \mu^{(2)}_{Z,+} - \mu^{(2)}_{Z,-} \} + \varphi \{ \mu^{(2)}_{Z,+} - \mu^{(2)}_{Z,+} + \mu^{(2)}_{Z,+} - \mu^{(2)}_{Z,-} \} \gamma_*.
$$

Let $\zeta^SC := (\zeta^SC_+, \zeta^SC_-, \zeta^SC_Z)^T$. The following theorem is an extension of Theorem 3. It provides the expansion of the coverage probability of the EL confidence set $\{ \tau : LR(\tau) \leq q_{X^T,1-\alpha} \}$.

**Theorem 5 (Sharp RDD with Covariates).** In the special case described in Example 6 (Sharp RDD with covariates), suppose that Assumptions 1 - 4 hold. Then,

$$
\Pr \left[ LR_* \leq q_{X^T,1-\alpha} \right] = (1 - \alpha) - B^SC_c \cdot q_{X^T,1-\alpha}^{1/2} \varphi \left( q_{X^T,1-\alpha}^{1/2} \right)
$$

$$
+ O \left( \frac{h^2}{(nh)^{1/2}} + n^{-1} + (nh)^{-3/2} + n^2 h^{10} + nh^6 \right),
$$

where

$$
B^SC_c := -h^2 \varphi^1 + (nh)^5 \varphi^2 + \frac{1}{nh} \left( \sum_{j=3}^{10} \varphi^j \right),
$$

$$
\varphi^1 := \varphi \cdot \sum_{s=1}^{1+dZ} e_{1+dZ,s} T \mathbf{N}_{SC} \mathbf{C}_{1,[1,s]} \mathbf{Q}_{SC} \zeta^SC
$$

$$
\varphi^2 := \frac{1}{4} \cdot \varphi^2 \gamma_2 \cdot (\zeta^SC)^T \mathbf{Q}_{SC} \zeta^SC
$$

$$
\varphi^3 := \varphi^{-1} \gamma_2 \cdot \sum_{s=1}^{1+dZ} \sum_{t=1}^{1+dZ} O_{SC}^{[s]} \operatorname{tr} \left( \mathbf{Q}_{SC} \mathbf{C}_{1,[1,s]}^T \mathbf{Q}_{SC} \mathbf{C}_{1,[1,t]} \right)
$$

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\begin{align*}
\varphi_4^{sc} & := -2 \cdot \gamma_3 (\varphi_3)^{-1} \cdot \sum_{j=1}^{2+d_h} \sum_{s=1}^{1+d_g} N^{[js]}_s \text{tr} \left( J_{(j)} Q_{sc} C_{1;[1,s]} Q_{sc} \right) \\
\varphi_5^{sc} & := \frac{1}{2} \cdot \gamma_4 \varphi_3^{-2} \varphi_3^{-1} \cdot \sum_{j=1}^{2+d_h} \sum_{k=1}^{2+d_g} Q^{[jk]} \text{tr} \left( Q_{sc} J_{(j,k)} \right)
\end{align*}
and
\begin{align*}
\varphi_6^{sc} & := -\varphi_1 \gamma_2 \cdot \sum_{s=1}^{1+d_g} \sum_{t=1}^{1+d_h} e_{1+d_h,s}^T N_{sc}^T C_{1;[1,s]} Q_{sc} C_{1;[1,t]}^T N_{sc} e_{1+d_h,t} \\
\varphi_7^{sc} & := \varphi_1 \gamma_2 \cdot \sum_{s=1}^{1+d_g} \sum_{t=1}^{1+d_h} e_{1+d_h,s}^T N_{sc}^T C_{1;[1,s]} Q_{sc} C_{1;[1,t]}^T N_{sc} e_{1+d_h,t} \\
\varphi_8^{sc} & := -\varphi_1 \gamma_2 \cdot \sum_{s=1}^{1+d_g} \sum_{t=1}^{1+d_h} O_{sc}^{[st]} \text{tr} \left( Q_{sc} C_{0;[s,t]} \right) \\
\varphi_9^{sc} & := 2 \cdot \gamma_3 \varphi_3^{-1} \cdot \varphi_3^{-1} \sum_{s=1}^{1+d_g} \sum_{t=1}^{1+d_h} e_{1+d_h,s}^T N_{sc}^T L_{(1,s)} Q_{sc} e_{2+d_g,t}.
\end{align*}

Moreover, with $B^sc_c$ being replaced by $B^{sc}_c$, (28) holds.

**Remark 15.** Theorem 5 is an extension of Theorem 3 to incorporate pre-treatment covariates. The proof of Theorem 5 is similar to that of Theorem 3. We specialize Theorem 2 to the case of moment conditions (20) in the sharp RDD with covariates. Then we can show that $B_1 = (nh)^5 \varphi_3^{sc} + O (nh^6)$, $B_2 = -h^2 \varphi_1^{sc} + O (h^6)$ and $B_3 = (nh)^{-1} \left( \sum_{j=3}^{10} \varphi_j^{sc} \right) + O \left( n^{-1} \right)$. The difference is that unlike inference for sharp RDD without covariates, in this case, $B_2$ is of the same order as $B_1$ and $B_3$, if the bandwidth obeys $h \sim n^{-1/3}$. As a consequence, the coverage optimal bandwidth does not have an explicit form.

**Remark 16.** The coverage optimal bandwidth that minimizes the absolute value of the leading coverage error term in the statement of Theorem 5 is $h^{sc}_0 := H^{sc}_0 \cdot n^{-1/3}$ where

\begin{align*}
H^{sc}_0 & := \text{argmin}_{H > 0} \left| \varphi_1^{sc} H^2 - \varphi_3^{sc} H^5 - \left( \sum_{j=3}^{10} \varphi_j^{sc} \right) H^{-1} \right|.
\end{align*}

Note that in this case we need to solve a minimization problem. This minimization problem is well-defined since the objective function is a rational function and tends to $\infty$ as $H \downarrow 0$ or $H \uparrow \infty$. In practical implementation, the bandwidth we use can be based on a plug-in estimator of $H^{sc}_0$ which is obtained by replacing the unknown parameters in the objective function with their consistent estimators. In spite of absence of an explicit form, the coverage optimal bandwidth $h^{sc}_0$ in this case retains the favorable property of independence from the nominal coverage probabil-
ity $1 - \alpha$. The second part of Theorem 5 implies that the Bartlett-corrected EL confidence set 
$$\{ \tau : \left(1 + B_c^{SC}\right)^{-1} LR(\tau) \leq q_{\chi^2_{1,1-\alpha}} \},$$
where
$$1 + B_c^{SC} = 1 + n^{-2/3} \left\{ -\psi_1^{SC}(H_{SC})^2 + \psi_2^{SC}(H_{SC})^5 + \left( \sum_{j=3}^{10} \psi_j^{SC} \right) (H_{SC})^{-1} \right\}$$
when the bandwidth is equal to the coverage optimal bandwidth $h_{SC}^*$, gives better coverage accuracy with error of order $O\left(n^{-1}\right)$.

Compared with Calonico et al. (2019), our method avoids separate calculation of standard error, admits a simple coverage optimal bandwidth which is independent from $1 - \alpha$ and has good coverage accuracy through Bartlett correction. Extension of this method to accommodate multiple outcome variables is straightforward.

5.4 Fuzzy Regression Discontinuity Design

In a fuzzy RDD, incentive is assigned if $X \geq c$, but due to limited compliance, the treatment assignment $D \in \{0, 1\}$ is not equal to $\mathbbm{1}(X \geq c)$ but $D$ is highly correlated with $\mathbbm{1}(X \geq c)$. Let $(Y_1, Y_0)$ be the potential outcomes with treatment and without treatment. Some individuals with $X \geq c$ do not comply and $Y_0$'s are observed and some with $X < c$ receive the treatment and $Y_1$'s are observed. The observed outcome $Y$ is still given by $Y = DY_1 + (1 - D) Y_0$. In a fuzzy RDD, it is assumed that the conditional probability of receiving the treatment $x \mapsto \Pr[D = 1 | X = x]$ has a jump at $x = c$: $\mu_{D,+} \neq \mu_{D,-}$.

Let $D(x)$ denote the potential treatment assignment if the realization of the forcing variable $X$ is $x \in [x_l, x_r]$.\footnote{This is true if $D$ is generated by $D = \mathbbm{1}(X \geq c) \delta_+(X, V) + \mathbbm{1}(X < c) \delta_-(X, V)$, where $V$ is some unobservable random variable supported on $[x_l, x_r]$ and $\delta_+, \delta_- : [x_l, x_r] \times [x_l, x_r] \to \{0, 1\}$. Then $D(x) := \mathbbm{1}(x \geq c) \delta_+(x, V) + \mathbbm{1}(x < c) \delta_-(x, V)$ with $x \in [x_l, x_r]$ is the potential treatment.} We assume that the monotonicity (no defier) assumption hold: $\lim_{x \uparrow c} D(x) \leq \lim_{x \downarrow c} D(x)$. In the literature (see Hahn et al., 2001 and Dong, 2018), it was shown that

$$\tau^F_*= \frac{\mu_{Y,+} - \mu_{Y,-}}{\mu_{D,+} - \mu_{D,-}} = E\left[ Y_1 - Y_0 | X = c, \lim_{x \uparrow c} D(x) = 1, \lim_{x \downarrow c} D(x) = 0 \right],$$

where the right hand side of the second equality is the local treatment effect for the complier group ($\lim_{x \uparrow c} D(x) = 1$ and $\lim_{x \downarrow c} D(x) = 0$) and the left hand side is a feature of the population of the data. Therefore, in a fuzzy RDD, the local treatment effect is non-parametrically identified. We take the approach introduced in Example 5 and use the moment conditions (17) and our EL method to make inference on $\tau^F_*$. The EL confidence set for $\tau^F_*$ is 
$$\{ \tau : LR(\tau) \leq q_{\chi^2_{1,1-\alpha}} \}$$
where $LR(\tau) = \inf_{g_0} \ell(\tau, g_0)$.
and the EL criterion function is given by

\[ \ell (g_1, g_0) = 2 \cdot \sup_{\lambda} \sum_{i=1}^{n} \log \left( 1 + X^T \left( \begin{array}{c} W_{+,i} (Y_i - g_1 D - g_0) \\ W_{-,i} (Y_i - g_1 D - g_0) \end{array} \right) \right). \]

Denote

\[ \kappa^F_{j,+} := \lim_{x \downarrow c} \mathbb{E} \left[ (Y - \tau^F D - \mu_s^F)^j \mid X = x \right] \]

\[ \kappa^F_{j,-} := \lim_{x \uparrow c} \mathbb{E} \left[ (Y - \tau^F D - \mu_s^F)^j \mid X = x \right] \]

and

\[ \zeta^F_r := (\mu^{(2)}_{Y,r} \varphi + 2 \mu^{(1)}_{Y,r} \varphi^{(1)}) - \tau^F_r (\mu^{(2)}_{D,r} \varphi + 2 \mu^{(1)}_{D,r} \varphi^{(1)}) \], for \( r \in \{-, +\}. \]

The following theorem is analogous to Theorem 3.

**Theorem 6 (Fuzzy RDD).** In the special case described in Example 5 (Fuzzy RDD), suppose that Assumptions 1 - 4 hold. Then,

\[ \Pr \left[ LR_s \leq q \chi^2_{3,1-\alpha} \right] = (1 - \alpha) - B^F_c \cdot q^{1/2} (q^{1/2}) \frac{1}{\kappa_{2,+} + \kappa_{2,-}} \]

\[ + O \left( \frac{h^2}{(nh)^{1/2}} + (nh)^{-3/2} + (nh)^{10} + nh^6 + h^3 \right), \]

where \( B^F_c := B^F_1 + B^F_3 \),

\[ B^F_1 := nh^5 \cdot \frac{1}{4} \cdot \frac{\varphi (\zeta^{F_+} - \zeta^{F_-})^2}{\gamma_2 \varphi (\kappa_{2,+} + \kappa_{2,-})} \]

\[ B^F_3 := \frac{1}{nh} \cdot \frac{\overline{B} (\kappa_{2,+}, \kappa_{2,-}, \kappa_{3,+}, \kappa_{3,-}, \kappa_{4,+}, \kappa_{4,-})}{\gamma_2 \varphi (\kappa_{2,+} + \kappa_{2,-})}. \]

Moreover, with \( B^S_c \) being replaced by \( B^F_c \), (28) holds.

**Remark 17.** The proof of Theorem 6 is very similar to that of Theorem 3 and therefore omitted. We specialize Theorem 2 to the case of moment conditions (16). Then we can show that \( B_1 = B^F_1 + O (nh^6) \), \( B_2 = O (h^3) \) and \( B_3 = B^F_3 + O (n^{-1}) \). The order of magnitude of \( B_2 \) is smaller. Denote

\[ \nu_F := \overline{B} (\kappa_{2,+}, \kappa_{2,-}, \kappa_{3,+}, \kappa_{3,-}, \kappa_{4,+}, \kappa_{4,-}) \text{ and } \nu_F := \frac{1}{2} \varphi (\zeta^{F_+} - \zeta^{F_-}) \].

Similarly to Theorem 3, if \( h = H \cdot n^{-1/3} \), the leading coverage error term is

\[ -n^{-2/3} \left( \frac{\nu^2_F H^5 + \nu_F H^{-1}}{\gamma_2 \varphi (\kappa_{2,+} + \kappa_{2,-})} \right) q \chi^2_{3,1-\alpha} f (q \chi^2_{3,1-\alpha}). \]
The coverage optimal bandwidth which minimizes its absolute value is \( h^*_F = H^*_F \cdot n^{-1/3} \), where

\[
H^*_F = \text{argmin}_{H > 0} \left| \frac{\tilde{H}}{2} H^5 + v_F H^{-1} \right|
\]

If \( v_F > 0 \), solving the first order condition, we find \( H^*_F = \left\{ v_F / \left(5\tilde{H}^2\right) \right\}^{1/6} \). Otherwise, we numerically solve this minimization problem. In practical implementation, we non-parametrically estimate \( H^*_F \). See Section 6 ahead. Similarly, it is easy to see that the Bartlett corrected EL confidence set \( \hat{\tau} : (1 + B^*_c)^{-1} L(\hat{\tau}) \leq q_{\chi^2_{1,1-\alpha}} \), where

\[
1 + B^*_c = 1 + n^{-2/3} \frac{\tilde{H}^2}{\gamma_2 \varphi} \left( \kappa^F_{2,+} + \kappa^F_{2,-} \right),
\]

has good coverage accuracy with error of order \( O(n^{-1}) \).

Compared with standard Wald-type inference for fuzzy RDD (e.g., Calonico et al., 2014, 2018, 2020), our method avoids calculation of standard error, admits a simple coverage optimal bandwidth which is independent from the nominal coverage probability \( 1 - \alpha \) and has better theoretical coverage accuracy through Bartlett correction. Compared with the EL-type method of Otsu et al. (2015), our method uses fewer moment conditions and nuisance parameters (Noack and Rothe, 2019) and therefore has less computational burden and admits a simple coverage optimal bandwidth. Moreover, extension of our method to accommodate multiple outcome variables and incorporate information from covariates is straightforward.

**Remark 18 (Categorical Outcome).** In a fuzzy RDD, the outcome can be categorical and corresponding mutually exclusive dummy variables \( Y = (Y[1], ..., Y[J])^T \) are used to represent the outcome. See Example 5. Denote \( \tau^*_F := \left( \tau^*_{F,1}, ..., \tau^*_{F,J} \right)^T \) and \( \mu^*_F := \left( \mu^*_{F,1}, ..., \mu^*_{F,J} \right)^T \), where

\[
\tau^*_{F,j} := \frac{\mu_{Y[j],+} - \mu_{Y[j],-}}{\mu_{D,+} - \mu_{D,-}} \quad \text{and} \quad \mu^*_{F,j} := \mu_{Y[j],+} - \tau^*_{F,j} \cdot \mu_{D,+} = \mu_{Y[j],-} - \tau^*_{F,j} \cdot \mu_{D,-},
\]

for all \( j = 1, 2, ..., J \). The EL joint confidence region for \( \tau^*_F \) is \( \left\{ \tau : \text{LR} (\tau) \leq q_{\chi^2_{J,1-\alpha}} \right\} \) where

\[
\text{LR} (\tau) = 2 \cdot \inf_{g_0} \sup_{\lambda} \sum_{i=1}^n \log \left( 1 + \lambda^T \left( W_{+i} (Y_i - \tau \cdot D - g_0) \right) W_{-i} (Y_i - \tau \cdot D - g_0) \right).
\]

Let \( \hat{Y} := Y - \tau^*_{F} \cdot D \) and

\[
D^*_{F,+,j} := \lim_{x \to c} E \left[ \left( \hat{Y} - \mu^*_F \right) \left( \hat{Y} - \mu^*_F \right)^T | X = x \right]
\]

\[
D^*_{F,[k],+} := \lim_{x \to c} E \left[ \left( \hat{Y} - \mu^*_{F,k} \right) \left( \hat{Y} - \mu^*_{F,k} \right)^T | X = x \right]
\]

\[\text{In all of the simulation setups we considered, } v_F \text{ is strictly positive.}\]
\[
D_{[k],+}^{FM} := \lim_{x \downarrow c} \mathbb{E} \left[ \left( \hat{Y}^{[k]} - \hat{\mu}_s^{Y,k} \right) \left( \hat{Y}^{[l]} - \hat{\mu}_s^{Y,l} \right) \left( \hat{Y} - \hat{\mu}_s^Y \right) \left( \hat{Y} - \hat{\mu}_s^Y \right)^T | X = x \right].
\]

\[
(D_{-}^{FM}, D_{[k],-}^{FM}, D_{[k],-}^{FM}) \text{ are defined similarly. Let}
\]
\[
\zeta_r^{FM} := \left( \mu_{Y,r}^{(2)} \varphi + 2 \mu_{Y,r}^{(1)} \varphi^{(1)} \right) - \tau_r^{FM} \left( \mu_{D,r}^{(2)} \varphi + 2 \mu_{D,r}^{(1)} \varphi^{(1)} \right), \text{ for } r \in \{-, +\}.
\]

With \((D_r, D_{[k],r}, D_{[k],r})\) and \(\zeta_r^{FM}\) being replaced by \((D_{r}^{FM}, D_{[k],+}^{FM}, D_{[k],-}^{FM})\) and \(\zeta_r^{FM}, r \in \{-, +\}\), the statement of Theorem 4 holds true. Then it is easy to adapt the argument in Remark 13 to derive the coverage optimal bandwidth that minimizes the leading coverage error of \(\{\tau : LR(\tau) \leq q_{k,j,1-\alpha}\}\) and is independent from the nominal coverage probability \(1 - \alpha\). Bartlett corrected EL inference can be implemented analogously.

**Remark 19 (Incorporating Covariates).** To incorporate the covariates, we take the approach introduced in Example 8. Let \(\hat{Y} := Y - \tau_s^{FC} . D\) and
\[
\epsilon^{FC} := \hat{Y} - \mu_s^{FC} - Z^T \gamma_s^F.
\]

Let
\[
C_{k:[s,l],+}^{FC} := \lim_{x \downarrow c} \mathbb{E} \left[ (\epsilon^{FC})^k \hat{Z}^s \hat{Z}^l \mathbf{S} \mathbf{S}^T | X = x \right]
\]
and let \(C_{k:[s,l],-}^{FC}\) be defined similarly. Let \(\zeta^{FC} := (\zeta^{FC}_+, \zeta^{FC}_-, (\zeta^{FC}_Z)^T)\), where \((\zeta^{FC}_+, \zeta^{FC}_-, \zeta^{FC}_Z)\) are defined by (39) and (40), with \(Y\) and \(\gamma_s\) being replaced by \(\hat{Y}\) and \(\gamma_s^F\). We can show that with \((C_{k:[s,l],+}, C_{k:[s,l],-})\) and \(\zeta^{FC}\) being replaced by \((C_{k:[s,l],+}^{FC}, C_{k:[s,l],-}^{FC})\) and \(\zeta^{FC}\), the statement of Theorem 5 also holds in the context of inference for covariates-augmented fuzzy RDD (Example 8). Then we easily adapt the argument in Remark 16 to derive the coverage optimal bandwidth that minimizes the leading coverage error of the EL confidence set (22) and is independent from the nominal coverage probability \(1 - \alpha\).

## 6 Practical Implementation

The constant part of the coverage optimal bandwidth in Section 5.1 \(H_s^*\) contains unknown quantities \(\varphi\) and \(\varphi^{(1)}\) (density of \(X\) and its derivative at \(c\)), \(\mu_{Y,r}^{(k)}\), \((r,k) \in \{-, +\} \times \{1, 2\}\) (the first and second order derivatives of conditional means at both sides of \(c\)) and \(\kappa_{j,r}^{S}\), \((r,j) \in \{-, +\} \times \{1, 2, 3, 4\}\) (the centered moments of \(Y\) at both sides of \(c\)). We follow Imbens and Kalyanaraman (2011) and Arai and Ichimura (2018) to use a plug-in estimator of \(H_s^*\) in practical implementation. We first obtain non-parametric estimators of these quantities \((\varphi, \varphi^{(1)}, \mu_{Y,r}^{(k)}\) and \(\kappa_{j,r}^{S}\)) and then plug them into the expression of \(H_s^*\). We calculate and use the AMSE-minimizing bandwidths for these non-parametric estimators. Similarly, we can easily calculate plug-in estimates of \(H_s^*, H_s^{SC}\) and \(H_s^\phi\).

To illustrate our implementation, let \(V\) denote a random variable and we non-parametrically
estimate $\mu_{V,+}^{(k)}$ with the AMSE-minimizing bandwidth. In the first step, we consistently estimate $\sigma_{V,+}^2$, $\sigma_{V,-}^2$, and $\varphi$. These are used as pilot estimates of the unknown quantities in the expressions of AMSE-optimal bandwidths. Let $\tilde{\sigma}_V^2 := (n - 1)^{-1} \sum_{i=1}^n (X_i - \bar{X})^2$. Calculate Silverman’s rule-of-thumb bandwidth $h_0 := 1.84 \cdot \tilde{\sigma}_X \cdot n^{-1/5}$. Denote $n_{-,h_0} := \sum_{i=1}^n \mathbb{1}(c - h_0 \leq X_i < c)$ and $\nabla_{-,h_0} := n_{-,h_0}^{-1} \sum_{i=1}^n \mathbb{1}(c - h_0 \leq X_i < c) V_i$. $n_{+,h_0}$ and $\nabla_{+,h_0}$ are defined similarly. A consistent pilot estimator of $\varphi$ is

$$\bar{\varphi} := \frac{n_{-,h_0} + n_{+,h_0}}{2 \cdot n \cdot h_0}$$

and consistent pilot estimators of $\sigma_{V,+}^2$ and $\sigma_{V,-}^2$ are

$$\tilde{\sigma}_{V,-}^2 := \frac{1}{n_{-,h_0} - 1} \sum_{i=1}^n \mathbb{1}(c - h_0 \leq X_i < c) (V_i - \nabla_{-,h_0})^2$$

$$\tilde{\sigma}_{V,+}^2 := \frac{1}{n_{+,h_0} - 1} \sum_{i=1}^n \mathbb{1}(c \leq X_i \leq c + h_0) (V_i - \nabla_{+,h_0})^2.$$

Let $\tilde{\mu}_{V,-}^{(k)}(h)$ be the local polynomial estimator with order $p = k + 1$:

$$\tilde{\mu}_{V,-}^{(k)}(h) := k! \cdot e_{p+1,k+1}^T \arg\min_{b_-} \sum_{i=1}^n \left\{ V_i - b_-^T r_p(X_i - c) \right\}^2 \mathbb{1}(X_i < c) \frac{1}{h} K \left( \frac{X_i - c}{h} \right).$$

The AMSE of $\tilde{\mu}_{V,-}^{(k)}(h)$ is

$$\left\{ \frac{k! \mu_{V,-}^{(p+1)} h^{p+1-k}}{(p + 1)!} \cdot e_{p+1,k+1}^T \Sigma_{p,-}^{-1} e_{p+1,k+1} \right\}^2 + \frac{(k!)^2 \sigma_{V,-}^2}{n h^{2k+1} \varphi} \cdot e_{p+1,k+1}^T \Sigma_{p,-}^{-1} V_{p,-} M_{p,-}^{-1} e_{p+1,k+1}. \quad (41)$$

For estimating the AMSE-minimizing bandwidths, we replace $\sigma_{V,-}^2$ and $\varphi$ by their pilot estimators $\tilde{\sigma}_{V,-}^2$ and $\bar{\varphi}$. Following the procedure of Arai and Ichimura (2018), we replace the derivative $\mu_{V,-}^{(p+1)}$ by an inconsistent but easily implementable pilot estimator from a global regression. We fit a global polynomial regression for observations for which $X_i < c$. We regress $Y_i$ on $r_{p+1}(X_i - c)$ to obtain the OLS coefficients, denoted by $\tilde{\beta}_-$. The pilot estimator for $\tilde{\mu}_{V,-}^{(p+1)}$ is $\tilde{\mu}_{V,-}^{(p+1)} := (p + 1) e_{p+2,p+2}^T \tilde{\beta}_-$. A plug-in estimator of the bandwidth which minimizes the AMSE (41) is

$$\tilde{h}_{V,-}^{(k)} := n^{-\frac{1}{2p+3}} \left( \frac{\tilde{\sigma}_{V,-}^2 \{(p + 1)!\}^2 (2k + 1) \left\{ e_{p+1,k+1}^T \Sigma_{p,-}^{-1} V_{p,-} M_{p,-}^{-1} e_{p+1,k+1} \right\}}{2 (p + 1) \bar{\varphi} \left\{ \tilde{\mu}_{V,-}^{(p+1)} \right\}^2 \left\{ e_{p+1,k+1}^T \Sigma_{p,-}^{-1} V_{p,-} M_{p,-}^{-1} e_{p+1,k+1} \right\}^2} \right)^{\frac{1}{2p+3}}.$$

Similarly, we can calculate $\tilde{h}_{V,+}^{(k)}$.

Let $\hat{\varphi}(h)$ and $\hat{\varphi}^{(1)}(h)$ be standard kernel density estimators with kernel $K$ and bandwidth $h$. Plug-in estimation of AMSE-optimal bandwidths require pilot estimators of $\varphi$ and higher-order derivatives. Easily implementable estimators of the derivatives can be based on the idea of Cattaneo
et al. (2019). We write \( F(z) = \mathbb{E}[1(X_2 \leq X_1) | X_1 = z] \), where \( F \) is the distribution function of \( X \) and \((X_1, X_2)\) are independent copies of \( X \). We generate \( F_i := (n - 1)^{-1} \sum_{j \neq i} 1(X_j \leq X_i) \) and regress \( F_i \) on \( r_i(X_i - c) \) to obtain the OLS coefficients, denoted by \( \tilde{\beta}_i \). The pilot estimators of \( \varphi(2) \) and \( \varphi(3) \) are given by \( \tilde{\varphi}(2) := 3!e_5^T \tilde{\beta}_i \) and \( \tilde{\varphi}(3) := 4!e_5^T \tilde{\beta}_i \). The estimated AMSE-optimal bandwidths are

\[
\hat{h}_\varphi := n^{-1/5} \left( \frac{\hat{\varphi}}{\{\hat{\varphi}(2)\}^2} \cdot \frac{\int K(u)^2 \, du}{\int K(u) \, du} \right)^{1/5} \quad \text{and} \quad \hat{h}_\varphi^{(1)} := n^{-1/7} \left( \frac{3\hat{\varphi}}{\{\hat{\varphi}(3)\}^2} \cdot \frac{\int K'(u)^2 \, du}{\int K(u) \, du} \right)^{1/7}.
\]

For estimation of \( \iota_8 \), we use \( \hat{\varphi}(\hat{h}_\varphi) \) and \( \hat{\varphi}(\hat{h}_\varphi^{(1)}) \) as non-parametric estimators of \( \varphi \) and \( \varphi^{(1)} \) and use \( \hat{\mu}_{V,r}^{(k)}(\hat{h}_{V,r}) \) as a non-parametric estimator of \( \mu_{V,r}^{(k)} \), for \( V = Y, (k, r) \in \{1, 2\} \times \{-, +\} \). For estimation of \( v_8 \), we use \( \hat{\mu}_{V,r}^{(k)}(\hat{h}_{V,r}) \) as a non-parametric estimator of \( \mu_{V,r}^{(k)} \), for \( V = (Y - \mu_{Y,r})_+, (j, r) \in \{2, 3, 4\} \times \{-, +\} \), \( r \in \{-, +\} \) and \( k = 0 \).

## 7 Monte Carlo Simulations

### 7.1 Basic RDDs: Sharp, Fuzzy and Covariate Adjustment

We conduct simulations to evaluate the finite sample performance of the proposed EL inference for the sharp, fuzzy and covariate-adjusted sharp RDDs. For all three designs, The forcing variable \( X_i \sim 2 \cdot B(2, 4) - 1 \) has a Beta distribution with shape parameters \( \alpha = 2, \beta = 4 \) and re-scaled to the support \([-1, 1]\), which follows Calonico et al. (2014). The cut-off \( c = 0 \). We consider sample sizes \( n = 1, 000, 2, 000 \) and 5, 000. The number of simulation replications is 10, 000. For the sharp RDD, random samples of \((Y_i, X_i)\) are generated from the outcome equation \( Y_i = g(X_i) + \varepsilon_i \), where the functional form of \( g \) corresponds to Models I of Calonico et al. (2014), i.e.,

\[
g(x) = \begin{cases} 
0.48 + 1.27x + 7.18x^2 + 20.21x^3 + 21.54x^4 + 7.33x^5 & \text{if } x < 0 \\
0.52 + 0.84x - 3.00x^2 + 7.99x^3 - 9.01x^4 + 3.56x^5 & \text{if } x \geq 0.
\end{cases}
\]

The additive error term \( \varepsilon_i \sim N(0, \sigma_\varepsilon^2) \), with \( \sigma_\varepsilon = 0.5 \). For the fuzzy RDD, random samples of \((Y_i, X_i, D_i)\) are generated by \( Y_i = D_i Y_{i1} + (1 - D_i) Y_{i0} \), where the counterfactual \( Y_{di} = g_d(X_i) + \varepsilon_{di} \) for \( d \in \{1, 0\} \), and

\[
g_d(x) = \begin{cases} 
\alpha_d + 1.27x + 7.18x^2 + 20.21x^3 + 21.54x^4 + 7.33x^5 & \text{if } x < 0 \\
\alpha_d + 0.84x - 3.00x^2 + 7.99x^3 - 9.01x^4 + 3.56x^5 & \text{if } x \geq 0,
\end{cases}
\]
with \((\alpha_1, \alpha_0) = (0.52, 0.48)\). The error terms \((\varepsilon_{0i}, \varepsilon_{1i})\) are independent and have the same marginal distribution \(N(0, \sigma^2_\varepsilon)\), with \(\sigma_\varepsilon = 0.5\). The treatment status \(D_i\) is determined by

\[
D_i = \begin{cases} 
1 \{X_i - a \geq \nu_i\} & \text{if } X_i < 0 \\
1 \{X_i + a \geq \nu_i\} & \text{if } X_i \geq 0,
\end{cases}
\]

where \(a = 0.84, \nu_i \sim N(0, 1)\) and is independent from \((X_i, \varepsilon_i)\). Our covariate-adjusted sharp RDD incorporates a univariate pre-treatment variable \(Z\). Following Model 2 of Calonico et al. (2019), the random samples of \((Y_i, Z_i, X_i)\) are generated by

\[
Y_i = \mu_y (X_i, Z_i) + \varepsilon_{y,i} \quad \text{and} \quad Z_i = \mu_z (X_i) + \varepsilon_{z,i},
\]

where error terms \((\varepsilon_{y,i}, \varepsilon_{z,i})\) are bivariate normal with the means being both equal to 0, the standard deviation being equal to 0.5 and the correlation coefficient being equal to 0.269.

Tables 1, 2 and 3 present the coverage rates of the EL confidence intervals for the RD parameters in the aforementioned sharp, fuzzy and covariate-adjusted sharp RDDs. Throughout Sections 7, EL CO\(_{tr}\) and EL CO represent the EL inference using the true coverage optimal bandwidth \(h^*_S\) (or \(h^*_F\), \(h^*_SC\)) derived in Section 5 and its estimated version described in Section 6. ELB CO\(_{tr}\) and ELB CO represent the Bartlett-corrected EL inference (Remark 7), using the true and estimated coverage optimal bandwidths, respectively. Note that Bartlett correction does not alter the coverage optimal bandwidth. As a comparison, Column “CCT MSE” and “CCT CO” report the coverage rates of the robust bias-corrected Wald-type confidence intervals that use AMSE-optimal bandwidth (Calonico et al., 2014) and the coverage optimal bandwidth (Calonico et al., 2016), computed using the R package \(rdrobust\) (Calonico et al., 2015).\(^{16}\) Tables 1 to 3 show that the coverage rates of EL CO\(_{tr}\) are close to the nominal coverage probabilities and the Bartlett-corrected version ELB CO\(_{tr}\) further improves on them. The feasible EL confidence interval EL CO exhibits a slight under-coverage but its Bartlett-corrected version, ELB CO, restores the coverage rates to a level very close to the nominal ones. Therefore, we recommend the Bartlett-corrected EL ratio inference method (ELB CO) to practitioners. When compared to the Wald-type inference in the last two columns, ELB CO outperforms CCT MSE in most cases as the coverage optimal bandwidth aims at a more precise coverage rate. Both CCT CO and the proposed ELB CO perform fairly well in our simulations: for the sharp and fuzzy RDDs (Tables 1 and 2), their performances are comparable. For covariate-

\(^{16}\)The implementation imposes no restrictions on the optimal bandwidth \(h\) for RD estimate and optimal bandwidth \(b\) for the bias correction. Using the terminology of CCT, \(\rho = h/b\) is not restricted. See Calonico et al. (2014, 2015).
adjusted sharp design (Table 3), ELB CO performs slightly better than CCT CO.

7.2 Joint Inference with Multiple Outcome Variables

We also conduct simulations for the EL-based joint inference described in Section 5.2, which can be used for testing the hypothesis involving multiple RD treatment parameters, or for testing the balance of multiple pre-treatment covariates. Our experiment considers the first scenario. Let three outcomes are generated by $Y_{ji} = g_j (X_i) + \varepsilon_{ji}$ for $j = 1, 2, 3$, where $g_1 = g$ in Section 7.1, $g_2$ and $g_3$ take the following forms:

$$g_2 (x) = \begin{cases} 0.48 + 1.27x - 0.5 \times 7.18x^2 + 0.7 \times 20.21x^3 + 1.1 \times 21.54x^4 + 1.5 \times 7.33x^5 \quad \text{if } x < 0 \\ 0.52 + 0.84x - 0.1 \times 3.00x^2 - 0.3 \times 7.99x^3 - 0.1 \times 9.01x^4 + 3.56x^5 \quad \text{if } x \geq 0 \end{cases}$$

and

$$g_3 (x) = \begin{cases} 0.03 - 2.26x - 13.14x^2 - 30.89x^3 - 31.89x^4 + 12.1x^5 \quad \text{if } x < 0 \\ 0.09 + 5.76x - 42.56x^2 + 120.90x^3 - 139.71x^4 + 55.59x^5 \quad \text{if } x \geq 0. \end{cases}$$

The forms of $g_2$ and $g_3$ correspond to Model 3 of Calonico et al. (2014) and Design 4 of Arai and Ichimura (2018), respectively. The error terms $\varepsilon_{ji}, j = 1, 2, 3$ have mean 0, standard deviation 0.5 and the pairwise covariance 0.2. The forcing variable $X_i$ follows the one in Section 7.1. We examine two cases of joint inference: $J = 2$ which focuses on two parameters $\mu_{Y_{j,+}} - \mu_{Y_{j,+}}, j = 1, 2$, and $J = 3$ which focuses on three parameters $\mu_{Y_{2,+}} - \mu_{Y_{2,+}}, \mu_{Y_{3,+}}$, $j = 1, 2, 3$. The coverage rates of EL joint inference are summarized in Table 4, for sample sizes $n = 2,000, 4,000$ and $8,000$. The number of simulation replications is 5,000. As one can see, the proposed EL joint inference with the coverage optimal bandwidth performs well in the case $J = 2$ for all sample sizes considered. In the case $J = 3$, the coverage rates of ELB CO are quite close to the nominal ones for $n = 4,000$ and $8,000$. Once again, Bartlett correction improves coverage performance.

7.3 Sharp RDD with Categorical Outcome

In this section, we evaluate the finite sample performance of our EL inference for the categorical outcome (Example 2 and Remark 14). The experiment design follows DGP 1 of Xu (2017), which is a multi-nominal logit model with three categories in the outcome (see Xu, 2017, Page 9 for details). As Example 2 describes, after setting the base category, we focus on two RD treatment parameters $\mu_{Y_{3,+}} - \mu_{Y_{3,+}}, j = 1, 2$, regarding the remaining two categories. We consider EL inference for each category separately and joint inference for both categories. The coverage rates for sample sizes $n = 4,000, 8,000$ and $12,000$ are summarized in Table 5. The number of simulation replications is 5,000. As Panel A shows, EL inference for the individual treatment parameters, especially ELB CO,
performs very well for all sample sizes considered. For the joint inference, Panel B shows that ELB CO generates precise coverage rates when \( n = 8,000 \) and 12,000. As a comparison, for the same experiment design with \( n = 8,000 \), the robust inference (with normal coverage probability 0.90) in Xu (2017) leads to individual coverage rates 0.885 and 0.851 for two categories respectively and a joint coverage rate 0.852, which are comparable to our outputs: 0.8772 and 0.8860 for the individual categories and 0.8778 for joint inference.

Overall, our simulation results demonstrate that our inference method based on EL using its coverage optimal bandwidth works well for various types of problems in the RDD framework. Combining EL with Bartlett correction, and using the corresponding coverage optimal bandwidth, our ELB CO can be a complementary inference method for applied researchers.

8 Empirical Illustrations

8.1 Finnish Municipal Election

We first apply our coverage optimal EL inference method to analyze the individual incumbent advantage in Finnish Municipal elections, which was first studied by Hyytinen et al. (2018). The dataset has two appealing features: first, it includes 1351 candidates “for whom the (previous) electoral outcome was determined via random seat assignment due to ties in vote counts” (Hyytinen et al., 2018, Page 1020), which constitutes a experiment benchmark to evaluate the credibility of the RD treatment effect estimated from the non-experimental data (candidates with previous electoral ties are excluded from the RD sample). Second, the sample size is very large (\( n = 154,543 \)) and thus provides a good platform for evaluating and comparing non-parametric inference approaches. We also incorporate pre-treatment covariates to the sharp RDD following Example 6. The main results are presented in Table 6. The “No Covariates” part refers to the sharp RDD in which the binary outcome variable \( Y \) indicates whether the candidate is elected in the next election, and the forcing variable \( X \) is the vote share margin in the previous election. The “Using Covariates” part incorporates three covariates: candidates’ age, gender, the incumbency status. The last two rows of Table 6 contains the EL-based joint balance test for the continuity of these three covariates at the cut-off. Consistent with the individual balance tests in Hyytinen et al. (2018, Table D1), we do not find enough evidence to reject their continuity at the cut-off (\( p \)-value = 0.45). The “Experiment benchmark” row restates the experiment estimate in Hyytinen et al. (2018), which finds zero treatment effect (see their Table 2, Column 4, the \( p \)-value is imputed by us). We consider different inference approaches and bandwidth selectors similar to the simulation section: EL CO refers to the EL inference method using the coverage optimal bandwidth, and ELB CO refers to its Bartlett-corrected version. CCT MSE and CCT CO refer to the bias-corrected and robust inference proposed by Calonico et al. (2014) using AMSE-optimal bandwidth and coverage optimal bandwidth, respectively. See Calonico et al. (2020) for a thorough analysis of using coverage optimal bandwidth in CCT. All CCT methods are implemented using the R package \texttt{rdrobust}, in which \( h \) refers to
the bandwidth used for estimating the RD treatment effect and $b$ refers to the bandwidth used for estimating the bias correction term. The columns of Table 6 present the estimates of RD treatment effect $\tau$, $p$-values for testing $H_0 : \tau = 0$, the 95% confidence intervals and the selected bandwidths.\footnote{For our EL inference, the point estimate for $\tau$ is computed via the EL estimation using moment conditions (15) or (20). For CCT methods, the point estimates are bias-corrected estimates. The bandwidth column reports the value of $h$ when it comes to CCT methods.}

We make the following observations regarding the empirical results. First, for such a large sample, the EL CO and ELB CO (the Bartlett-corrected version) give almost the same results. Second, inference methods that use the coverage optimal bandwidth, such as our EL/ELB CO and CCT CO, deliver results closer to the experiment benchmark than CCT MSE that uses the AMSE-optimal bandwidth. In particular, the inference results in CCT MSE with unrestricted $h/b$ rejects the null hypothesis of zero treatment effect, which is at odds with the experiment estimate.\footnote{Note that the CCT MSE rows in Table 6 (no covariates) reproduces columns (2) and (8) in Table 4 of Hyytinen et al. (2018). The numbers are slightly different from theirs because we use the upgraded version of package rdrobust, see Hyytinen et al. (2018, Page 1044) for the software update regarding the estimates of AMSE-optimal bandwidth.} The superior performance of the coverage optimal bandwidth is not surprising, as it enjoys an improved coverage error decay rate compared to the AMSE-optimal bandwidth. Third, the $p$-values of EL CO and ELB CO, nearly 0.8, are larger than the CCT methods. In other words, our EL inference method finds less evidence against the null hypothesis which is likely to be true according to the experiment benchmark. The CCT CO method, which is more robust than the CCT MSE in terms of not rejecting the null hypothesis, remains sensitive to the choice of $h/b$. When bandwidths $h$ and $b$ are optimized respectively (i.e., $h/b$ unrestricted), the $p$-value is around 0.15 with and without covariates, which can still be regarded as a considerable body of evidence against the null hypothesis. Fourth, one natural concern of the larger $p$-values given by EL CO and ELB CO is the possible efficiency loss. However, the comparison in the length of confidence interval between the EL/ELB CO and the CCT CO with $h/b = 1$ (which has the $p$-value closest to EL/ELB CO among all CCT methods) reveals that the former is about 19% (or 5%) shorter than the latter in the case with (or without) covariates. As the last column shows, the coverage optimal bandwidths for the EL inference are much narrower than those for the Wald-type CCT method, which is expected to make the EL/ELB CO method less biased. Meanwhile, EL/ELB CO method remains to yield shorter confidence interval than CCT CO with $h/b = 1$ (the best version of CCT methods considered, in terms of reproducing the experiment estimate) in spite of the smaller effective sample size (due to the narrower bandwidth). We conjecture that this reflects another merit of the likelihood-based method. Overall, this example confirms that the theoretical benefit of using the coverage optimal bandwidth materializes in a real dataset. It also shows that the EL inference method, coupled with a coverage optimal bandwidth, can be a viable tool for applied researchers.

### 8.2 Impacts of Academic Probation

We then apply the categorical outcome version (Example 2) of the coverage optimal EL inference method to a Canadian University data on academic performances, which was used by Lindo et al.
A sharp RDD framework is used to evaluate the short and long run impacts of being placed on academic probation after the first year. The forcing variable is the distance between a student’s first year GPA and the probationary cut-off. Two outcome variables are considered: (1) the decision to permanently leave the university at the end of the first year, which measures an immediate response to academic probation; and (2) the graduation rate (graduated within 4 years, graduated in the 5th year, graduated in the 6th year, or graduated after more than 6 years/dropout at some point) which reflects the long-run effect of being placed on academic probation. The first outcome (call it dropout in the following) is a binary variable and the second outcome (graduation rate) has four categories, both fitting the categorical outcome framework described in Example 2.

The available sample consists of 44,362 students for the dropout outcome and 18,983 students for the graduate rates. Table 7 presents the point estimate $\hat{\tau}$, $p$-value, 90% confidence interval and the coverage optimal bandwidth for the immediate dropout effect of all students and the subsamples by gender. The results tell the difference in the response between male and female students to academic probation. While academic probation significantly increased male students’ dropout rate (after the first year) by 4.8 percentage points, its effect on female students was insignificant. Our results are similar to those in Xu (2017, Table 9). The last two rows of Table 7 report the $p$-value of the EL-based joint test for the continuity of eight covariates at the cutoff: students’ high school grades, credits registered in the first year, age at entry, gender, birthplace, native language and two campus dummies. We hardly find any evidence to reject the joint continuity of all these covariates ($p$-value = 0.99) and regard this as a confirmation of the validity of the RDD. Our joint test complements the individual balance tests for each covariate reported by Lindo et al. (2010, Table 2).

Table 8 summarizes the estimated long-run impacts of academic probation on the graduate rates. In terms of the point estimates, being placed on the academic probation after the first year decreased the probability of graduation within 4 years (Category #1) and graduation in the 5th year (Category #2) by 5.0 and 6.3 percentage points respectively, while increased probability of graduation in the 6th year (Category #3) and more than 6 years/dropout at some point (base category) by 1.1 and 10.2 percentage points respectively. All the above estimated effects are not significant at a customary level, but the effect for the graduation rate within 4 years (Category #1) has the $p$-value = 0.151, leaning towards significance. The joint test for all three categories yields a $p$-value = 0.11, marginally above the 10% significance level. As a comparison, Table 10 of Xu (2017) finds a significant effect for Category #1 ($p$-value = 0.04), insignificant effects for other categories and an insignificant joint effect ($p$-value = 0.23). For both methods the effect for Category #1 is the most significant one among all categories. In sum, this example illustrates the applicability of our EL inference method with coverage optimal bandwidth to the case where a categorical outcome is taken into account.
9 Conclusion

This paper proposes an empirical likelihood inference method for various RDDs, including standard sharp and fuzzy designs, designs with multiple outcome variables and augmented sharp and fuzzy designs with covariates as special cases. We show Edgeworth approximation for the coverage probability and derive the optimal bandwidth which minimizes the absolute value of the leading coverage error term in the expansion, in each of the cases. Our method is easy to implement in practice and complements the fast-growing literature of econometric methods for RDD. Compared with existing methods, our method has certain theoretical advantages, including Bartlett correctability, implicit studentization, automatic bandwidth selection that is optimal for inference, and data-driven shape of confidence sets. Unlike bandwidths that are optimal for Wald-type inference, coverage optimal bandwidths for our method are independent from the nominal coverage probability in all cases and have explicit forms in some cases. Our method also performs well in Monte Carlo experiments.

Following the suggestion of Imbens and Lemieux (2008), researchers often calculate confidence sets at multiple bandwidths for sensitivity analysis in applications. The true coverage optimal bandwidth for our method is unknown and has to be estimated. In a complementary paper, we provide the uniform-in-bandwidth theory of our method and justify an adjustment to the critical value in the spirit of Armstrong and Kolesár (2018). In practical applications, we recommend calculating our EL confidence sets at three bandwidths \( \left( \hat{h}^*/2, \hat{h}^*, 2\hat{h}^* \right) \) (\( \hat{h}^* \) denotes the estimated coverage optimal bandwidth) with a larger critical value in place of \( q_{X_{1,1-\alpha}} \). The new critical value takes into account the need for “bandwidth snooping”. However, the proof of such a result is involved and it cannot be established as an extension of the main theorem of Armstrong and Kolesár (2018).

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Table 1: Sharp RDD: Empirical likelihood Inference and Wald-type Inference Using Optimal Bandwidths, $1 - \alpha = \text{Nominal coverage probability}$

| $1 - \alpha$ | $n$  | EL $\text{CO}_{tr}$ | EL CO | ELB $\text{CO}_{tr}$ | ELB CO | CCT MSE | CCT CO |
|--------------|------|----------------------|-------|-----------------------|--------|---------|--------|
| .99          | 1000 | .9859                | .9823 | .9882                 | .9863  | .9848   | .9846  |
|              | 2000 | .9885                | .9856 | .9903                 | .9878  | .9874   | .9878  |
|              | 5000 | .9895                | .9850 | .9903                 | .9855  | .9861   | .9889  |
| .95          | 1000 | .9408                | .9336 | .9437                 | .9432  | .9353   | .9380  |
|              | 2000 | .9448                | .9377 | .9502                 | .9425  | .9405   | .9472  |
|              | 5000 | .9479                | .9392 | .9501                 | .9413  | .9352   | .9462  |
| .90          | 1000 | .8850                | .8737 | .8964                 | .8883  | .8796   | .8839  |
|              | 2000 | .8943                | .8834 | .9018                 | .8898  | .8811   | .8914  |
|              | 5000 | .8958                | .8818 | .8980                 | .8860  | .8852   | .8915  |

Table 2: Fuzzy RDD: Empirical likelihood Inference and Wald-type Inference Using Optimal Bandwidths, $1 - \alpha = \text{Nominal coverage probability}$

| $1 - \alpha$ | $n$  | EL $\text{CO}_{tr}$ | EL CO | ELB $\text{CO}_{tr}$ | ELB CO | CCT MSE | CCT CO |
|--------------|------|----------------------|-------|-----------------------|--------|---------|--------|
| .99          | 1000 | .9872                | .9829 | .9897                 | .9861  | .9943   | .9956  |
|              | 2000 | .9886                | .9857 | .9899                 | .9877  | .9909   | .9934  |
|              | 5000 | .9893                | .9879 | .9906                 | .9883  | .9858   | .9907  |
| .95          | 1000 | .9443                | .9343 | .9507                 | .9411  | .9566   | .9639  |
|              | 2000 | .9424                | .9388 | .9469                 | .9438  | .9433   | .9552  |
|              | 5000 | .9486                | .9408 | .9511                 | .9430  | .9354   | .9470  |
| .90          | 1000 | .8904                | .8820 | .9019                 | .8941  | .9063   | .9159  |
|              | 2000 | .8915                | .8859 | .8985                 | .8936  | .8929   | .9030  |
|              | 5000 | .8971                | .8857 | .9010                 | .8891  | .8741   | .8919  |

Table 3: Sharp RDD with a Covariate: Empirical likelihood Inference and Wald-type Inference Using Optimal Bandwidths, $1 - \alpha = \text{Nominal coverage probability}$

| $\alpha$ | $n$  | EL $\text{CO}_{tr}$ | EL CO | ELB $\text{CO}_{tr}$ | ELB CO | CCT MSE | CCT CO |
|----------|------|----------------------|-------|-----------------------|--------|---------|--------|
| .99      | 1000 | .9824                | .9815 | .9862                 | .9864  | .9811   | .9817  |
|          | 2000 | .9887                | .9868 | .9904                 | .9897  | .9855   | .9873  |
|          | 5000 | .9886                | .9877 | .9893                 | .9887  | .9853   | .9880  |
| .95      | 1000 | .9377                | .9353 | .9469                 | .9459  | .9351   | .9371  |
|          | 2000 | .9423                | .9362 | .9486                 | .9456  | .9372   | .9431  |
|          | 5000 | .9456                | .9451 | .9487                 | .9484  | .9367   | .9449  |
| .90      | 1000 | .8868                | .8812 | .9013                 | .9011  | .8841   | .8890  |
|          | 2000 | .8893                | .8841 | .8968                 | .8955  | .8820   | .8899  |
|          | 5000 | .8962                | .8911 | .9010                 | .8964  | .8844   | .8940  |
### Table 4: Joint Empirical likelihood Inference Multiple RDD Parameters, $J = \text{Number of parameters}, \ 1 - \alpha = \text{Nominal coverage probability}$

|       | $J = 2$       | $J = 3$       |
|-------|---------------|---------------|
|       | EL CO$_{tr}$  | EL CO         | ELB CO$_{tr}$ | ELB CO | EL CO$_{tr}$ | EL CO | ELB CO$_{tr}$ | ELB CO |
| .99   | 2000          | .9812         | .9780         | .9846  | .9818         | .9752 | .9368         | .9812  |
|       |               |               |               |        |               |       |               |        |
|       | 4000          | .9866         | .9852         | .9892  | .9874         | .9834 | .9728         | .9884  |
|       |               |               |               |        |               |       |               |        |
|       | 8000          | .9884         | .9874         | .9890  | .9886         | .9868 | .9848         | .9886  |
| .95   | 2000          | .9344         | .9260         | .9416  | .9364         | .9194 | .8600         | .9360  |
|       |               |               |               |        |               |       |               |        |
|       | 4000          | .9384         | .9346         | .9444  | .9418         | .9336 | .9162         | .9466  |
|       |               |               |               |        |               |       |               |        |
|       | 8000          | .9514         | .9498         | .9556  | .9552         | .9382 | .9360         | .9444  |
| .90   | 2000          | .8752         | .8684         | .8906  | .8890         | .8598 | .7966         | .8874  |
|       |               |               |               |        |               |       |               |        |
|       | 4000          | .8828         | .8752         | .8902  | .8864         | .8768 | .8592         | .8932  |
|       |               |               |               |        |               |       |               |        |
|       | 8000          | .8988         | .8936         | .9056  | .9016         | .8896 | .8812         | .8990  |

### Table 5: Sharp RDD with a Categorical Outcome (Three Categories), $1 - \alpha = \text{Nominal coverage probability}$

|       | Individual: Category #1 | Individual: Category #2 |
|-------|-------------------------|-------------------------|
|       | EL CO$_{tr}$ | EL CO | ELB CO$_{tr}$ | ELB CO | EL CO$_{tr}$ | EL CO | ELB CO$_{tr}$ | ELB CO |
| .99   | 4000          | .9738 | .9270         | .9784  | .9352         | .9676 | .9506         | .9704  |
|       |               |       |               |        |               |       |               |        |
|       | 8000          | .9792 | .9684         | .9820  | .9730         | .9844 | .9860         | .9782  |
|       |               |       |               |        |               |       |               |        |
|       | 12000         | .9888 | .9822         | .9900  | .9842         | .9864 | .9792         | .9872  |
| .95   | 4000          | .9202 | .8682         | .9310  | .8848         | .9190 | .8998         | .9270  |
|       |               |       |               |        |               |       |               |        |
|       | 8000          | .9342 | .9182         | .9406  | .9282         | .9384 | .9276         | .9462  |
|       |               |       |               |        |               |       |               |        |
|       | 12000         | .9474 | .9404         | .9526  | .9484         | .9434 | .9342         | .9470  |
| .90   | 4000          | .8656 | .8176         | .8802  | .8352         | .8718 | .8514         | .8828  |
|       |               |       |               |        |               |       |               |        |
|       | 8000          | .8852 | .8670         | .8948  | .8772         | .8848 | .8744         | .8938  |
|       |               |       |               |        |               |       |               |        |
|       | 12000         | .8960 | .8874         | .9022  | .8978         | .8942 | .9014         | .8812  |

|       | Joint         |
|-------|---------------|
|       | EL CO$_{tr}$ | EL CO | ELB CO$_{tr}$ | ELB CO |
| .99   | 4000          | .9460 | .8970         | .9518  | .9040         |
|       |               |       |               |        |               |
|       | 8000          | .9750 | .9664         | .9778  | .9710         |
|       |               |       |               |        |               |
|       | 12000         | .9842 | .9804         | .9866  | .9830         |
| .95   | 4000          | .8884 | .8394         | .9026  | .8620         |
|       |               |       |               |        |               |
|       | 8000          | .9250 | .9146         | .9316  | .9272         |
|       |               |       |               |        |               |
|       | 12000         | .9422 | .9386         | .9466  | .9472         |
| .90   | 4000          | .8288 | .7808         | .8496  | .8106         |
|       |               |       |               |        |               |
|       | 8000          | .8728 | .8576         | .8840  | .8778         |
|       |               |       |               |        |               |
|       | 12000         | .8910 | .8868         | .9022  | .9012         |

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Table 6: Individual Incumbency Advantage in Finnish Municipal Election $\tau = \text{RD Treatment Effect}$

| Method          | $\tau$  | $p$-value | 95% CI       | bandwidth |
|-----------------|---------|-----------|--------------|-----------|
| **No Covariates** |         |           |              |           |
| $n = 154,543$   |         |           |              |           |
| EL CO           | .0059   | .7990     | [-.039, .050]| .233      |
| ELB CO          | .0059   | .7991     | [-.039, .050]| .233      |
| CCT MSE         | .0450   | .0004     | [.020, .070] | .720      |
| (h/b unrestricted) |       |           |              |           |
| CCT MSE         | .0203   | .2896     | [-.017, .058]| .720      |
| (h/b = 1)       |         |           |              |           |
| CCT CO          | .0258   | .1383     | [.020, .070] | .720      |
| (h/b unrestricted) |     |           |              |           |
| CCT CO          | -.0119  | .6752     | [-.067, .044]| .396      |
| (h/b = 1)       |         |           |              |           |
| **With Covariates** |       |           |              |           |
| $n = 154,543$   |         |           |              |           |
| EL CO           | .0048   | .8611     | [-.058, .048]| .186      |
| ELB CO          | .0048   | .8613     | [-.058, .048]| .186      |
| CCT MSE         | .0437   | .0007     | [.018, .069] | .698      |
| (h/b unrestricted) |     |           |              |           |
| CCT MSE         | .0185   | .3377     | [.019, .056] | .689      |
| (h/b = 1)       |         |           |              |           |
| CCT CO          | .0248   | .1567     | [.010, .059] | .384      |
| (h/b unrestricted) |     |           |              |           |
| CCT CO          | -.0115  | .6868     | [.068, .044] | .384      |
| (h/b = 1)       |         |           |              |           |
| **Experiment benchmark** |       |           |              |           |
| (Hyytinen et al., 2018) |     |           |              |           |
| $n = 1,351$     |         |           |              |           |
| EL CO           | -0.010  | .5157     | [-.060, .040]|           |
| **Joint Test for** |       |           |              |           |
| **Covariate Balance** |       |           |              |           |
| $n = 190,600, J = 3$ |      |           |              |           |
| EL CO           | .4475   | .257      |              |           |
| ELB CO          | .4481   | .257      |              |           |
Table 7: Immediate Permanent Dropout Effect of Academic Probation, $\tau = \text{RD Treatment Effect}$

| Method | $\hat{\tau}$ | p-value | 90% CI          | bandwidth |
|--------|---------------|---------|-----------------|-----------|
| All    |               |         |                 |           |
| $n = 44,362$ | EL CO | -.0104 | .4641 | [-.0338, .0130] | .262 |
|         | ELB CO | -.0104 | .4661 | [-.0339, .0131] | .262 |
| Male   |               |         |                 |           |
| $n = 16,981$ | EL CO | -.0477 | .0004 | [-.0705, -.0257] | .717 |
|         | ELB CO | -.0477 | .0004 | [-.0705, -.0257] | .717 |
| Female |               |         |                 |           |
| $n = 27,381$ | EL CO | .0106  | .5701 | [-.0203, .0421]  | .241 |
|         | ELB CO | .0106  | .5724 | [-.0205, .0422]  | .241 |
| Joint Test for Covariate Balance |         |         |                 |           |
| $n = 44,362, J = 8$ | EL CO | .994   |         | .149        |         |
|         | ELB CO | .994   |         | .149        |         |

Table 8: Graduate Effect of Academic Probation, $\tau = \text{RD Treatment Effect}, n = 18,983$

| Method | $\hat{\tau}$ | p-value | 90% CI          | bandwidth |
|--------|---------------|---------|-----------------|-----------|
| Individual Effect For Each Category |         |         |                 |           |
| Within 4 years | EL CO | .0504 | .1505 | [-.0072, .1080] | .358 |
|         | ELB CO | .0504 | .1510 | [-.0073, .1080] | .358 |
| In 5th year | EL CO | .0629 | .2403 | [-.0252, .1506] | .186 |
|         | ELB CO | .0629 | .2414 | [-.0254, .1508] | .186 |
| In 6th year | EL CO | -.0134 | .4022 | [-.0396, .0129] | .696 |
|         | ELB CO | -.0134 | .4030 | [-.0396, .0130] | .696 |
| Joint Test For All Categories |         |         |                 |           |
| EL CO | .1122 |         |         | .237        |         |
| ELB CO | .1140 |         |         | .237        |         |