Quantum Dynamics of the Driven and Dissipative Rabi Model

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The Rabi model considers a two-level system (or spin-1/2) coupled to a quantized harmonic oscillator and describes the simplest interaction between matter and light. The recent experimental progress in solid-state circuit quantum electrodynamics has engendered theoretical efforts to quantitatively describe the mathematical and physical aspects of the light-matter interaction beyond the rotating wave approximation. We develop a stochastic non-perturbative Schrödinger equation approach which enables us to access the strong-coupling limit of the Rabi model and study the effects of dissipation and AC drive in an exact manner. We consider the high-Q cavity limit and include the effect of ohmic noise on the non-Markovian spin dynamics resulting in Kondo-type correlations. We compute the time evolution of spin variables in various conditions. As a scope, we discuss the possibility to reach a steady state with one polariton in realistic experimental conditions.

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Introduction. — Originally, the Rabi model had been introduced to describe the effect of a weak and rapidly rotating magnetic field on an atom possessing a nuclear spin [1, 2]. Nowadays, this model is applied to a variety of quantum systems, from quantum optics to condensed matter physics. A few examples include microwave and optical cavity quantum electrodynamics (QED) [3, 4], ion traps [5], quantum dots, and superconducting qubits in circuit QED [6, 7]. Recent on-chip experiments, by using artificial two-level systems made of superconducting qubits, allow a high control on the coupling between the system and the light field [8, 9]. Effective photon-photon interactions and photon blockade effects may also be engineered [10, 11]. Such spin-boson systems are of importance for applications in quantum computing [12, 13].

The application of the rotating wave approximation (RWA) is justified in the weak coupling limit and results in the Jaynes-Cummings (JC) model [14], which is exactly solvable. Analytical solutions of the quantum Rabi model beyond the RWA have been recently explored based on the underlying discrete $Z_2$ (parity) symmetry [15, 16]. Moreover, some dynamical properties of the model have been addressed [17] and other theoretical efforts in the strong-coupling limit are achieved [18, 19]. In this Letter, we study the Rabi model in a wide regime of parameters, from the weak to the strong coupling, and account for external driving and non-Markovian dissipation effects on the two-level system from the environment. The latter is modeled by a bath of harmonic oscillators which gives rise to ohmic dissipation and entanglement phenomena. At low temperatures, this engenders a renormalized (many-body) Lamb frequency for the two-level system and non-trivial damping processes which can be measured in cold atom, ion trap, mesoscopic, and photon systems [20]. By introducing two stochastic fields, we extend the non-perturbative Schrödinger equation method of Refs. [21, 22]. We show the applicability of this stochastic method by focusing on the spin dynamics in various conditions. We complement our results via physical and other analytical arguments. We also discuss non-trivial dynamical final states with one polariton. By increasing the drive amplitude we decrease the characteristic time to reach a pure state with one polariton. This may find applications to realize a driven Mott state of polaritons [23, 24], in the weak-coupling limit between light and matter [25].

Model. — The Hamiltonian describing the driven and dissipative quantum Rabi model reads

\[
H = \frac{\Delta}{2} \sigma^z + \omega_0 \left( a^\dagger a + \frac{1}{2} \right) + \frac{g}{2} \sigma^x (a + a^\dagger) + V(t)(a + a^\dagger) + \sum_k \left[ \omega_k b_k^\dagger b_k + \lambda_k (b_k + b_k^\dagger) \sigma^x \right],
\]

where $a^\dagger$ and $a$ are creation and annihilation operators for the quantized harmonic oscillator with frequency $\omega_0$, $\sigma^i$ ($i = x, y, z$) are the Pauli spin operators for the spin-1/2, $\Delta$ is the resonant frequency between the two levels, and $g$ denotes the interaction strength (we set $\hbar = 1$). The first line of the Hamiltonian represents the Rabi model, while the terms containing $b_k^\dagger b_k$ and $\lambda_k$ describe the microscopic interaction of the two-level system with the environment, which we assume to be of ohmic type. The Jaynes-Cummings weak-coupling limit of the Rabi model is reproduced when neglecting the counter-rotating terms, $(\sigma^+ a^\dagger + h.c.)$ where $\sigma^+ = (\sigma^x \pm i \sigma^y)/2$, which ensures a continuous $U(1)$ symmetry and an associated conserved quantity, the polariton number $N = (a^\dagger a + \sigma^+ \sigma^-)$.

The combined effect of the cavity and of the ohmic bath is encapsulated through the spectral function $J(\omega) = \pi g^2 \delta(\omega - \omega_0) + 2\pi \alpha \omega \exp(-\omega/\omega_c)$. Here,
\( \alpha \) determines the effective (dimensionless) coupling between the spin and the bath, while \( \omega_c \) is a high-frequency cutoff. We consider the high-Q cavity limit, where dissipation effects are more important in the two-level system, and the regime \( 0 < \alpha \lesssim 0.1 \) of the spin-boson model [55, 59]. Note, the case of very strong dissipation could also be addressed both theoretically and experimentally [50, 63–79]. Furthermore, in the absence of the cavity, the physics of the model (1) is already quite rich as it describes, e.g., the ohmic spin-boson model, Kondo physics, and long-range Ising models [55, 59, 81, 83].

A typical example of experimental setup is a Cooper pair box system at resonance [82, 83] where, within our notations, the operator \( \sigma^z \) represents the presence or absence of excess Cooper pairs in the island. The transverse field \( \sigma^x \) can be realized by coupling the Cooper pair box to a macroscopic superconductor via the Josephson effect. We assume that the Cooper pair box is capacitively coupled to the electromagnetic cavity and that ohmic dissipation embodies resistive effects stemming from the mesoscopic circuit [82, 83]. Other superconducting circuits known as flux [85] or phase qubits [59] provide equivalent systems. Superconducting systems [57–59] yield a long decoherence time which corresponds to very small values of \( \alpha \). A similar Hamiltonian could be derived in the case of a dissipative flux qubit [90].

**Method.** — We assume without lack of generality that the spin and bath are uncoupled at the initial time \( t_0 \) when they are brought into contact, and therefore the total density matrix can be factorized [55]: \( \rho_{tot}(t_0) = \rho_B(t_0) \otimes \rho_S(t_0) \). Here, \( \rho_B \) and \( \rho_S \) are respectively the bosonic and spin reduced density matrices. Suppose that we are first interested in computing the dynamics of \( \langle \sigma^z(t) \rangle \) [91]. Then we consider that the spin is initially in a pure state along the \( x \)-axis \( \rho_S(t_0) = |+_x\rangle\langle +_x| \) and we parametrize the spin path according to its value along the \( x \)-axis. We reintegrate out the bosonic degrees of freedom which are described by a quadratic action through the Feynman-Vernon influence functional [92], and then we adopt the “blip-sojourn approach” of Leggett et al. [55]. Following Refs. [69, 59], we introduce a matrix formalism in the four-dimensional vector space of states \( \{|+_+\rangle, |-_+\rangle, |_+-\rangle, |-_+\rangle\} \). The states \( |+_+\rangle \) and \( |+_+\rangle \) associated with the \( x \)-direction correspond to the diagonal elements of the spin density matrix (or sojourns) while \( |_-\rangle \) and \( |+\rangle \) correspond to the off-diagonal elements (or blips). To describe the spin dynamics, we extend the methodology of Ref. [59] and introduce two coupled stochastic fields. We briefly summarize the main steps of the procedure. Other technical details about the method can be found in the Supplemental Material [91].

We express the probability for the spin to come back in the state \( |+_x\rangle \) at time \( t \geq t_0 \), exactly as

\[
p(t) = \frac{1 + \langle \sigma^z(t) \rangle}{2} = \langle \Phi_f | T e^{-i \int_{t_0}^t dW(s)} | \Phi_i \rangle,
\]

where \( |\Phi_i\rangle = (e^{i\eta(t_0)}, 0, 0, 0)^T \), \( |\Phi_f\rangle = (e^{-i\eta(t)}, 0, 0, 0) \), and the effective spin Hamiltonian reads

\[
W(t) = \frac{\Delta}{2} \begin{pmatrix}
0 & e^{-i\xi} & e^i\xi & 0 \\
e^{-i\xi} & 0 & 0 & e^{-i\xi} \\
e^i\xi & 0 & 0 & e^i\xi \\
0 & e^i\xi & e^{-i\xi} & 0
\end{pmatrix}.
\]

Here, \( h_\xi \) and \( h_\eta \) are the two complex Gaussian stochastic functions which verify:

\[
\begin{align*}
&h_\xi(t)h_\xi(s) = \frac{1}{\pi} Q_2(t - s) + k_1, \\
&h_\eta(t)h_\eta(s) = k_2, \\
&h_\xi(t)h_\eta(s) = \frac{i}{\pi} Q_1(t - s)\theta(t - s) + k_3,
\end{align*}
\]

where \( k_1, k_2, \) and \( k_3 \) are arbitrary complex numbers. The functions \( Q_1 \) and \( Q_2 \), which describe the feedback of the electromagnetic field and of the dissipative environment, are directly obtained from the spectral function \( J(\omega) \). At zero temperature, they read [91]:

\[
\begin{align*}
Q_1(t) &= \pi \left[ \frac{g^2}{\omega_0^2} \sin \omega_0 t + 2\alpha \tan^{-1}(\omega_0 t) \right], \\
Q_2(t) &= \pi \left[ \frac{g^2}{\omega_0^2} (1 - \cos \omega_0 t) + \alpha \log(1 + \omega_0^2 t^2) \right].
\end{align*}
\]

The log-function in \( Q_2 \) reflects the non-Markovian features of the ohmic bath [60]. Resorting to Eq. (2), we rewrite \( p(t) \) as a stochastic average \( \langle \Phi_f | \Phi(t) \rangle \), where \( |\Phi(t)\rangle \) is the solution of the Schödinger equation

\[
i\partial_t \Psi(t) = W(t) |\Psi(t)\rangle.
\]
with the initial condition $|\Phi_i\rangle$. Note that other stochastic approaches were developed [93-95]. It is relevant to mention that we can also access the spin observable $\langle \sigma^z(t) \rangle$ in a similar manner using the boundary conditions $|\Phi'_i\rangle = |\Phi_i\rangle$ and $|\Phi'_j\rangle = (0, e^{-h_\xi(t)}, 0, 0)$ [91]. In the case of driving the system, the only change is the appearance of an additional term to the field $h_\xi$ [91]. This makes connections with the Landau-Zener-Majorana-Stückelberg oscillations [96, 97]. Indeed, by integrating out the cavity field, this produces a field term $-gV(t)\sigma^x/(2\delta)$, where $\delta = \omega_0 - \Delta$ is the detuning.

**Results without drive and dissipation.** — First, we check that the results for the free Rabi model reproduce the dynamics of the JC model in the weak coupling limit $g/\omega_0 \ll 1$ with weak detuning $\delta/\omega_0 \ll 1$. Since the RWA holds we can easily diagonalize the free undamped JC Hamiltonian in the so-called dressed basis. The ground state $|g\rangle$ of the system consists of the two-level system in its lower state and vacuum for the photons, while the excited eigenstates $|N\pm\rangle$ are pairs of combined light-matter excitations (polaritons) described in terms of the polariton number operator $N$ which commutes with the Hamiltonian leading to the well-known structure of the anharmonic JC ladder (Fig. 1(a)).

If one prepares the two-level system in its upper state $|+\rangle$ in vacuum, the dynamics shows coherent oscillations between the two polariton states $|1-\rangle$ and $|1+\rangle$, also known as Rabi oscillations. We obtain consistent results, and the first corrections to the JC limit when the coupling $g$ is increased, are in agreement with (second order) perturbation theory (Fig. 1(b)). The presence of the counter-rotating terms in the quantum Rabi model gives rise to a shift of the resonance frequency between the atom and photon, leading to an additional negative detuning $\delta = -g^2/[2(\omega_0 + \Delta)]$ when $\Delta < \omega_0$. This Bloch-Siegert shift [98] has been observed in circuit QED [14]. Moreover, the dynamics towards the deep strong coupling regime with $g \approx \omega_0$ reveals other features [32, 99] (Fig. 1(c)) which are also reproduced within our approach.

The regime of the Rabi model corresponding to a highly detuned system with $\Delta/\omega_0 \ll 1$ is known as the adiabatic limit [100]. One can visualize such a system as a set of two displaced oscillator wells (characterized by the value of $\sigma^x$), whose degeneracy is lifted by the field along the $z$-direction. The dynamics of the two-level system, initially prepared in a displaced Fock state of one well, should undergo coherent and complete oscillations between this state and its symmetric counterpart in the other well. Such an initial state can be prepared by applying a strong bias field along $x$-direction for negative times, letting the system relax towards its shifted equilibrium position before the release of the constraint at time $t = 0$. The frequency of oscillations only depends on the overlap between these two states, and one can show that this frequency is $\Omega = \Delta e^{-g^2/2\omega_0^2}$ [101] (Fig. 2).

The convergence of the numerical evaluation in our procedure is ensured from weak coupling to ratios $g/\omega_0$ of the order of 1, allowing us to reach the ultra-strong coupling regime. Below, we go one step further and consider dissipation and drive effects non-perturbatively.

**Dissipation.** — Decoherence effects are characterized by a prominent suppression of the off-diagonal elements of the spin reduced density matrix at equilibrium [40, 108] as well as a damping (disappearance) of the Rabi oscillations [38, 39, 56, 109, 115] (see Fig. 3). More precisely, the effects of the ohmic bath at low coupling are both a damping of the Rabi oscillations and a
dephasing due to a renormalization of the tunneling element $\Delta$ to $\Delta_r = \Delta(\Delta/\omega_c)^{1/2}$ \[38, 39\]. At small $\alpha$, these features can be reproduced through a master equation \[102\]. The latter phenomenon engenders an effective detuning $\delta_r = \omega_c - \Delta_r$ between the two-level system and light, which can be seen explicitly via a change in the frequency of Rabi oscillations in the dynamics of $\langle \sigma^x \rangle$ and $\langle \sigma^z \rangle$. The numerical estimation of this effective detuning matches the theoretical expectation $\delta_r$ for coupling strengths $a/g \ll 1$. Note that $\Delta_r$ can also be identified as the effective Kondo energy scale in the ohmic spin-boson model \[38, 39\]. When the dissipation strength increases the net field along the $x$-axis progressively becomes zero, since $\langle a + a^\dagger \rangle \approx 0$ at $g/\alpha \ll 1$. Then, the system relaxes to a final state with $\langle \sigma^x \rangle = 0$, and $\langle \sigma^z \rangle$ (within our notations) can be evaluated through Bethe Ansatz calculations \[40, 103, 106\].

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4}
\caption{Dynamics of $\langle \sigma^x \rangle$ with the system initially in its ground state; the parameters are $g/\omega_0 = 0.02$, $\Delta/\omega_0 = 0.9$ and $V_0/\omega_0 = 0.1$. A driving AC field is applied until the system has reached the first lower polariton; $t_s$ refers to the time at which we switch off the drive. The blue curve is the ideal dissipationless case ($\alpha = 0$); the red curve is the same result in the dressed state basis. The cyan and the green curves are for $\alpha = 10^{-5}$ and $10^{-4}$; $\omega_c = 100\omega_0$ (see also inset).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5}
\caption{Dynamics of the mean number of polaritons $\langle N(t) \rangle$ in the weak-coupling $g$ limit (and essentially $\alpha \lesssim 10^{-5}$). We have chosen parameters in accordance with Fig. 4. From the ground state, the system is brought into a non-trivial polaritonic final state by driving the cavity. The black dotted line refers to the moment when the AC coherent drive is switched off. Inset: Standard deviation with the same parameters.}
\end{figure}

Drive and One Polariton State. — Now, we consider the weak-coupling limit $g/\omega_0 \ll 1$ which allows us to realize a pure state with a single polariton. Applying a coherent semi-classical drive to this anharmonic system can account for several non linear effects which have been explored recently \[69, 116\]. We focus on the case of a system initially prepared in its ground state and probed via an AC drive with a frequency $\omega_d$ which is chosen to match the transition towards the first polariton $|1^-\rangle$.

In the limit of infinitely small drive $V_0/g \ll 1$ the dynamics shows complete semi-classical Bloch oscillations of frequency $\alpha_1 V_0/2$ between these two levels; $\alpha_1 = [(A - \delta_r)/2A]^{1/2}$ and $A = \sqrt{g^2 + \delta_r^2}$. However, the switch-off time $t_s$ necessary to bring the system into the state $|1^-\rangle$ is typically longer than the decoherence time. In the general case, we can express the driving term in the dressed states basis \[117\] and we can compute the occupancies of all the levels associated with the JC ladder (see Fig. 1). The price to pay for an increase of the drive strength is the subtle interplay of the upper levels. But, the anharmonicity of the JC ladder makes it possible to quantitatively reach the first polariton beyond the linear response limit, in typical times $t_s$ which are much smaller and therefore enable us to minimize the effect of dissipation. The dynamics of $\langle \sigma^z \rangle$ is shown in Fig. 4, by applying both the stochastic approach and a numerical procedure in the dressed state basis for $\alpha = 0$. The mean number of photons $\langle a^\dagger a \rangle$ is also evaluated using a straightforward numerical integration of the Schrödinger equation in the dressed state basis keeping the 91 lowest eigenstates, which enables us to evaluate the mean number of polaritons $\langle N \rangle = \langle a^\dagger a \rangle + (\langle \sigma^z \rangle + 1)/2$.

This analysis may have further implications in the realization of a driven polariton Mott state in arrays of electromagnetic resonators \[61\]. Let $\kappa$ denote the (capacitive) coupling between cavities. If the time $t_\kappa \approx 1/\kappa$ is much greater than the switch-off time $t_s$ we can reasonably treat the drive term individually on each site (cavity). This suggests that if the transition from the ground state to the first polariton is performed in a fast manner and if dissipation effects are weak \[118\] ($\alpha \lesssim 10^{-3} g/\omega_0$), we reach a state with one polariton on each cavity (Fig. 5) due to the polariton blockade \[62\].

Conclusion. — We have addressed the dynamics of the driven and dissipative quantum Rabi model. We have made quantitative predictions for the spin dynamics
which can be tested experimentally. We have also shown the possibility to reach a steady state with one polaron. This constitutes a step towards the realization of a driven Mott state of polaritons in realistic conditions [118]. The stochastic approach described in the present work could be generalized to (other) hybrid systems [119–126], photon lattices [61, 127–137] with artificial gauge fields [138–146], and fermion systems subject to time-dependent fields (potentials) [147–165]. Other avenues could be the study of the matter-phonon coupling, dissipative spin and Kondo models.

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Supplemental Material: “Quantum Dynamics of the Driven and Dissipative Rabi Model”

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This Supplemental material is provided to bring additional technical details concerning the procedure applied to compute the spin dynamics in various conditions. We develop the derivation of the Schrödinger equation with two stochastic fields.

I. PATH INTEGRAL APPROACH

Let $\rho_{\text{tot}}$ be the density matrix of the whole system and $\rho_S$ and $\rho_B$ the density matrices of the two-level and the bosonic (cavity+dissipative ohmic bath) subsystems. The full Hilbert space is the tensor product of the spin space and the bosonic space $\epsilon = \epsilon_S \otimes \epsilon_B$; basis of these spaces are denoted $\{u_{n,S}\}$ and $\{u_{n,B}\}$. The reduced density matrix is given by the partial trace of the total density matrix $\rho_S = \tr_B(\rho_{\text{tot}}) = \sum_B \langle u_{n,B} | \rho_{\text{tot}} | u_{n,B} \rangle$, whose evolution can be expressed with the unitary time-evolution operator of the whole system $U(t,t_0)$: $\rho_{\text{tot}}(t) = U(t,t_0) \rho_{\text{tot}}(t_0) U^\dagger(t,t_0)$.

We parametrize the spin path according to its value along the $x$-axis (corresponding to the direction of the coupling with the bosonic bath). We therefore choose notations in which the density matrix corresponding to a pure state along the $x$-axis is:

$$\rho_{(+x)}(+x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{1}$$

We moreover assume without lack of generality that the density matrix can be factorized at the initial time $t_0$ as $\rho_{\text{tot}}(t = t_0) = \rho_B(t_0) \otimes \rho_S(t_0)$, when spin and bath are brought into contact. We consider that the spin is initially in a pure state along the $x$-axis $\rho_S(t_0) = \langle +x | \rho_S(t) | +x \rangle$. We will first introduce the Feynman-Vernon (FV) influence functional [1] and follow the seminal work by Leggett et al. [2]. We are interested in computing the diagonal element of the spin density matrix describing the probability $p(t) = \langle +x | \rho_S(t) | +x \rangle$ to find the system back in the state $| +x \rangle$ at time $t \geq t_0$. Elements of the reduced density matrix can be expressed as:

$$\langle \sigma_f | \rho_S(t) | \sigma'_f \rangle = \sum_{\sigma_0, \sigma'_0} \langle \sigma_0 | \rho_S(t_0) | \sigma'_0 \rangle \int D\sigma D\sigma' A[\sigma] A[\sigma']^* F[\sigma, \sigma']. \tag{2}$$

$D\sigma$ and $D\sigma'$ denote here integration over all real-time spin paths $\sigma(.)$ and $\sigma'(.)$ with fixed initial conditions $\sigma_0$ and $\sigma'_0$ and final conditions $\sigma_f$ and $\sigma'_f$. The terms $A[\sigma]$ and $A[\sigma']$ denote the free amplitude for the spin to follow a given path. The influence of the bosonic bath (photons + bosonic modes describing the dissipation) is fully contained in the FV influence functional $F[\sigma, \sigma']$ which reads [1]:

$$F[\sigma, \sigma'] = \exp \left\{ -\frac{1}{\pi} \int_{t_0}^t ds \int_{t_0}^s ds' \left[ -i L_1(s-s') \eta(s) \xi(s') + L_2(s-s') \xi(s) \eta(s') \right] \right\}, \tag{3}$$

where $\eta$ and $\xi$ are the symmetric and antisymmetric classical spin paths: $\eta(s) = \frac{1}{2} [\sigma(s) + \sigma'(s)]$, $\xi(s) = \frac{1}{2} [\sigma(s) - \sigma'(s)]$. Here $\sigma$ and $\sigma'$ are classical spin paths taking value $\pm 1$, and the two functions $L_1$ and $L_2$ characterize the interaction with the bath:

$$L_1(t) = \int_0^\infty d\omega J(\omega) \sin \omega t,$$

$$L_2(t) = \int_0^\infty d\omega J(\omega) \cos \omega t \coth \frac{\beta \omega}{2}. \tag{4}$$

In the following we will focus on the quantum problem at zero temperature, and Eqs. (4) become:
The original spin path has been mapped onto a classical spin path, and the price to pay is to introduce another dimension. The double path integral along \( \sigma() \) and \( \sigma'(t) \) in Eq. (2) can be viewed as a single path that visits the four states \( A = |++\rangle, B = |+-\rangle, C = |-+\rangle \) and \( D = |--\rangle \). States A and D correspond to the diagonal elements of the density matrix (also named 'sojourn' states) whereas B and C correspond to the off-diagonal ones (also called 'blip' states), see Fig. 1 [2–4].

The double spin path is initially constrained to the diagonal state A because of the initial condition \( \rho_S(t_0) = |+\rangle\langle+x| \). Since we intend to compute the diagonal element of the density matrix describing the probability to find back the system in the state \( |+\rangle \) at time \( t \), we consider spin paths that end in the sojourn state A. One path of this type makes \( 2n \) transitions at times \( t_i \), \( i \in \{1, 2, \ldots, 2n\} \) such that \( t_0 < t_1 < t_2 < \ldots < t_{2n} \) along the way. We can write this spin path as \( \xi(t) = \sum_{j=1}^{2n} \Xi_j \theta(t - t_j) \) and \( \eta(t) = \sum_{j=0}^{2n} \Upsilon_j \theta(t - t_j) \) where the variables \( \Xi_i \) and \( \Upsilon_i \) take values in \( \{-1, 1\} \). Such a path is visualized in Fig. 2. The variables \( \Xi \) (in blue) describe the blip parts, and the variables \( \Upsilon \) (in red) on the other hand characterize the sojourn parts.

The diagonal element of the density matrix is given by a series in the tunneling coupling \( \Delta^2 \) [2, 3, 5]:

\[
p(t) = \frac{\langle \sigma_1^z(t) \rangle + 1}{2} = 1 + \sum_{n=1}^{\infty} \left( \frac{i\Delta}{2} \right)^{2n} \int_{t_0}^{t} dt_2 \ldots \int_{t_0}^{t_2} dt_1 \sum_{\{\Xi_j\}} \sum_{\{\Upsilon_j\}'} \mathcal{F}_n[\{\Xi_j\}, \{\Upsilon_j\}', \{t_j\}]. \tag{6}
\]

The prime in \( \{\Upsilon_j\}' \) in Eq. (6) indicates that the initial and final sojourn states are fixed according to the initial and final conditions. More precisely we have \( \Upsilon_0 = \Upsilon_{2n} = 1 \). Therefore:
\[ F_n[[\Xi_j], \{\Upsilon_j\}', \{t_j\}] = Q_1 Q_2, \] (7)

\[ Q_1 = \exp \left[ \frac{i}{\pi} \sum_{k=0}^{2n-1} \sum_{j=k+1}^{2n} \Xi_j \Upsilon_k Q_1(t_j - t_k) \right], \] (8)

\[ Q_2 = \exp \left[ \frac{1}{\pi} \sum_{k=1}^{2n-1} \sum_{j=k+1}^{2n} \Xi_j \Xi_k Q_2(t_j - t_k) \right]. \] (9)

The two coupling functions \( Q_1 \) and \( Q_2 \) are:

\[ Q_1(t) = \int_0^\infty d\omega \frac{J(\omega)}{\omega^2} \sin \omega t = \pi \left[ \frac{g^2}{\omega_0^2} \sin \omega_0 t + 2 \alpha \tan^{-1}(\omega, t) \right] \]

\[ Q_2(t) = \int_0^\infty d\omega \frac{J(\omega)}{\omega^2} (1 - \cos \omega t) = \pi \left[ \frac{g^2}{\omega_0^2} (1 - \cos \omega_0 t) + \alpha \log(1 + \omega_0^2 t^2) \right]. \] (10)

It is important to notice that blips and sojourns do not have symmetric effects. \( Q_1 \) describes the coupling between the blips and all previous sojourns, and \( Q_2 \) contains the interaction between all blips (including self interaction). The index for the \( \Upsilon \) variables starts at 0 and ends at \( 2n - 1 \) whereas the index for the \( \Xi \) variables starts at 1 and ends at \( 2n \). Note that the last sojourn does not contribute, the last coupling period is the blip which lasts from \( t_{2n-1} \) to \( t_{2n} \).

Let now \( h_\xi \) and \( h_\eta \) be two complex gaussian random fields, which verify:

\[ \overline{h_\xi(t) h_\xi(s)} = \frac{1}{\pi} Q_2(t - s) + k_1 \]

\[ \overline{h_\eta(t) h_\eta(s)} = k_2 \]

\[ \overline{h_\xi(t) h_\eta(s)} = \frac{i}{\pi} Q_1(t - s) \theta(t - s) + k_3. \] (11)

The overline denotes statistical average, and the \( k_p \) are arbitrary complex constants. Making use of the identity \( \overline{\exp(X)} = \exp(\overline{X^2}/2) \), Eq. (7) can then be re-expressed as:

\[ F_n[[\Xi_j], \{\Upsilon_j\}', \{t_j\}] = \exp \left\{ \frac{1}{2} \sum_{j,l=1}^{2n} \Xi_j \Xi_l \Xi_l(t_j) h_\xi(t_l) + 2 \Xi_j \Upsilon_{l-1} h_\xi(t_j) h_\eta(t_{l-1}) + \Upsilon_j \Upsilon_{l-1} \Upsilon_{l-1} h_\eta(t_{j-1}) h_\eta(t_{l-1}) \right\} \]

\[ = \prod_{j=1}^{2n} \exp [h_\xi(t_j) \Xi_j + h_\eta(t_{j-1}) \Upsilon_{j-1}]. \] (12)

The complex constants \( k_p \) do not contribute because \( \sum_{k=0}^{2n-1} \Upsilon_k = \sum_{j=1}^{2n} \Xi_j = 0 \). Practically such fields can be sampled by Fourier series decomposition (see subsection B for more details).

II. STOCHASTIC SCHRODINGER EQUATION

From Eqs. (6) and (12), the resulting formula for \( p(t) \) can be expressed as:

\[ p(t) = \sum_{n=0}^{\infty} (-1)^n \left( \frac{\Delta}{2} \right)^{2n} \int_{t_0}^{t} dt_{2n} \int_{t_0}^{t} dt_2 \sum_{\{\Xi_j\}} \sum_{\{\Upsilon_j\}} \prod_{j=1}^{2n} \exp [h_\xi(t_j) \Xi_j + h_\eta(t_{j-1}) \Upsilon_{j-1}]. \] (13)

As shown in Ref. [4], the summation over blip and sojourns variables \( \{\Xi_j, \Upsilon_j\} \) can be captured by introducing a matrix formalism in the four-dimensional vector space of states \( \{|+, -\rangle, |+-, -\rangle, |-, -\rangle\} \), which leads to:
\[ p(t) = \sum_{n=0}^{\infty} (-1)^n \int_{t_0}^{t} dt_{2n} \ldots \int_{t_0}^{t_1} dt_1 \langle \Phi_f | W(t_2n) \ldots W(t_2) W(t_1) | \Phi_i \rangle, \quad (14) \]

where the effective Hamiltonian for the spin is

\[ W(t) = \frac{\Delta}{2} \begin{pmatrix} 0 & e^{h_\xi(t)} & 0 & 0 \\ -e^{-h_\xi(t)} & 0 & 0 & 0 \\ 0 & 0 & e^{-h_\eta(t)} & 0 \\ 0 & e^{h_\eta(t)} & 0 & 0 \end{pmatrix}. \quad (15) \]

Here the vector \( |\Phi(t)\rangle \) represents the double spin state.

We have \( |\Phi_f\rangle = (e^{h_\eta(t_0)}, 0, 0, 0)^T \) and \( |\Phi_f\rangle = (e^{-h_\eta(t_2n)}, 0, 0, 0) \): these choices account for the asymmetry between blips and sojourns. The contribution from the first sojourn is encoded in \( |\Phi_i\rangle \), and we artificially suppress the contribution of the last sojourn via \( |\Phi_f\rangle \). This final vector depends on an intermediate time, but we can notice that replacing \( (e^{-h_\eta(t_2n)}, 0, 0, 0) \) by \( (e^{-h_\eta(t)}, 0, 0, 0) \) does not add any contribution on average because:

\[ \sum_{j=1}^{2n} h_\xi(t_j) \Xi_j + h_\eta(t_{j-1}) \Upsilon_{j-1} + \Upsilon_{2n} (h_\eta(t_{2n}) - h_\eta(t)) \]

\[ + \sum_{j=1}^{2n} \Upsilon_{2n} (h_\eta(t_{2n}) - h_\eta(t))^2 + 2 \sum_{j=1}^{2n} \Upsilon_{2n} \Xi_j (h_\eta(t_{2n}) - h_\eta(t)) h_\xi(t_j) + 2 \sum_{j=1}^{2n} \Upsilon_{2n} \Xi_j (h_\eta(t_{2n}) - h_\eta(t)) h_\eta(t_{j-1}) \]  \[ (15) \]

The three last terms vanish because \( (h_\eta(t_{2n}) - h_\eta(t))^2 = 0, \sum_{j=1}^{2n} \Xi_j = 0 \) and \( \sum_{j=0}^{2n-1} \Upsilon_j = 0 \). Then we can write \( p(t) \) as a time-ordered product:

\[ p(t) = \langle \Phi_f | Te^{-i \int_{t_0}^{t} ds W(s)} | \Phi_i \rangle, \quad (17) \]

where \( T \) is the time-ordering operator. This probability is then given by the stochastic average \( \langle \Phi_f | \Phi(t) \rangle \) where \( |\Phi(t)\rangle \) is the solution of the stochastic Schödinger equation (18) with initial condition \( |\Phi_i\rangle \).

\[ i \partial_t |\Psi\rangle = W|\Psi\rangle, \quad (18) \]

Starting from an initial state characterized by the density matrix \( \rho_S(t_0) = |+_z\rangle \langle +_x | \) we have therefore shown that it is possible to compute numerically \( \langle \sigma^z(t) \rangle \) in an exact manner.

In general, the fields \( h_\xi \) and \( h_\eta \) have both a real and imaginary parts, which could lead to numerical convergence issues. The real part of these fields leads to a convergence speed which decays exponentially with time. We focus on these convergence issues in Sec. VI. We now turn to the computation of \( \langle \sigma^z(t) \rangle \), which corresponds to the off-diagonal terms of the density matrix.

### III. COMPUTATION OF THE OFF-DIAGONAL ELEMENTS OF THE DENSITY MATRIX

Following the work by Weiss [3], it is also possible to compute an off-diagonal term of the density matrix in terms of a series expansion in \( \Delta \), considering spin paths that end in a blip state. An example of such a path can be seen in Fig. 3.

Such paths make now \( 2n - 1 \) transitions and we have:

\[ \langle + | \rho_S(t) | - \rangle = \sum_{n=1}^{\infty} \left( \frac{i \Delta}{2} \right)^{2n-1} \int_{t_0}^{t} dt_{2n-1} \ldots \int_{t_0}^{t_1} dt_1 \sum_{\Xi_j} \sum_{\Upsilon_j} \mathcal{F}_n[\{\Xi_j\}', \{\Upsilon_j\}', \{t_j\}]. \quad (19) \]

Here the initial sojourn state is fixed, as well as the final blip state.
All blips are coupled to all previous sojourns and blips (see Eqs. (8) and (9)). In Section I and II, we have considered paths that ended in a sojourn state. For a given path the last coupling period was from $t_{2n-1}$ to $t_{2n}$ as can be seen in Eq. (6). The situation is different here because paths end up in a blip state. For a given path the final coupling period then lasts from $t_{2n-1}$ to the final time $t$. Providing that we formally set $t_{2n} = t$ and $\Xi_{2n} = -\sum_{j=1}^{2n-1} \Xi_j$, we have:

$$\mathcal{F}_n[\{\Xi_j\}', \{\Upsilon_j\}', \{t_j\}] = \exp \left[ \frac{i}{\pi} \sum_{k=0}^{2n-1} \sum_{j=k+1}^{2n} \Upsilon_j \Upsilon_k Q_1(t_j - t_k) \right] \exp \left[ \frac{1}{\pi} \sum_{k=1}^{2n-1} \sum_{j=k+1}^{2n} \Xi_j \Xi_k Q_2(t_j - t_k) \right],$$

(20)

and

$$\mathcal{F}_n[\{\Xi_j\}', \{\Upsilon_j\}', \{t_j\}] = \prod_{j=1}^{2n} \exp \left[ h_\xi(t_j) \Xi_j + h_\eta(t_{j-1}) \Upsilon_j - 1 \right].$$

(21)

We can conclude that $\langle \sigma^z(t) \rangle$ is given by the average $2Re\langle \Phi'_f | \Phi'(t) \rangle$ where $|\Phi'(t)\rangle$ is the solution of the stochastic Schrödinger equation with the initial condition $|\Phi'_f\rangle = |\Phi_0\rangle$ and $\langle \Phi'_f | = (0, e^{-hc(t)}, 0, 0)$. These boundary conditions are important because they account again for the differences of indices in Eq. (21).

IV. INITIAL CONDITION

It is also possible to consider a protocol in which the spin is initially prepared in an eigenstate along $z$-direction: $\rho_S(t_0) = \langle +z | +z \rangle$. Eq. (2) is linear, so that we can evolve the four initial components of the density matrix separately. The treatment of the evolution of a diagonal element of the density matrix (the case of a path begining in the $D = |\downarrow\downarrow\rangle$ state can be deduced considering $\Upsilon_0 = -1$) have already been done in A and C. We focus then on the evolution of a path which is initially in a blip state, as in Fig. 4. Off-diagonal elements have also been computed for the Ohmic spin-boson model, using a slightly different procedure [5].

We have:

$$\langle +z | \rho_S(t) | -z \rangle = \sum_{n=0}^{\infty} \left( \frac{i\Delta}{2} \right)^{2n} \int_{t_0}^{t} dt_{2n} \sum_{\{\Xi_j\}', \{\Upsilon_j\}', \{t_j\}} \mathcal{F}_n[\{\Xi_j\}', \{\Upsilon_j\}', \{t_j\}],$$

(22)

where

$$\mathcal{F}_n[\{\Xi_j\}', \{\Upsilon_j\}', \{t_j\}] = \exp \left[ \frac{i}{2} \sum_{k=1}^{2n-1} \sum_{j=k+1}^{2n+1} \Xi_j \Xi_k Q_1(t_j - t_k) \right] \exp \left[ \frac{1}{2} \sum_{k=1}^{2n-1} \sum_{j=k+1}^{2n+1} \Xi_j \Xi_k Q_2(t_j - t_k) \right].$$

(23)

Here initial and final blip states are constrained. We formally set $t_{2n+1} = t$ again and considering for example a path that starts in state B, we find:

$$\langle \sigma^z(t) \rangle = 2 Re \sum_{n=0}^{\infty} \int_{t_0}^{t} dt_{2n} \int_{t_0}^{t} dt_{2n+1} \sum_{\{\Xi_j\}', \{\Upsilon_j\}', \{t_j\}} \mathcal{F}_n[\{\Xi_j\}', \{\Upsilon_j\}', \{t_j\}],$$

(24)
FIG. 4. Spin path- \( \eta(t) = \sum_{j=1}^{2n} \Upsilon_j \theta(t-t_j) \) in red; \( \xi(t) = \sum_{j=0}^{2n} \Xi_j \theta(t-t_j) \) in blue. Here the spin path starts and ends in the blip state \( B = |+\rangle \).

where \( |\Phi_f''\rangle = (0,e^{i\xi(t_0)},0,0)^T, \langle \Phi_f'| = \langle \Phi_f| \).

V. INCLUSION OF THE EXTERNAL DRIVE IN OUR STOCHASTIC METHOD

Applying a coherent semi-classical drive can be taken into account by adding to the Hamiltonian a new term of the form \( V(t)(a + a\dagger) \); see main text. The precise treatment of the drive adds a new contribution which can be treated exactly within our formalism by substituting \( \sigma^2(t) \) by \( (\sigma^2(t) + V(t)) \) in the path integral approach. This is simply reflected in Eq. (3) by the appearance of a new coupling term. Assuming \( V(t) \) to be of the form \( V_0 \cos \omega_d t \) and beginning the procedure at time \( t_0 \), the functional \( F[\sigma, \sigma'] \) is changed into \( F_d[\sigma, \sigma'] \) which reads, for \( t \geq t_0 \):

\[
F_d[\sigma, \sigma'] = e^{2iV_0g \int_{t_0}^{t} ds \int_{s}^t ds' \sin \omega_0 (s-s') \xi(s) \cos \omega_d s'} F[\sigma, \sigma'].
\]

(25)

We consider for example a path that starts in a sojourn state and ends in a blip state. The integration leads to:

\[
F_d[\sigma, \sigma'] = \exp \left\{ 2iV_0g \frac{\omega_0}{\omega_d^2 - \omega_0^2} \sum_{j=1}^{2n-1} \Xi_j \left[ \sin \omega_dt - \sin \omega dt_j - \sin(\omega_0 t + (\omega_0 + \omega_d) t_j) - \sin(\omega_0 t + (\omega_0 + \omega_d) t_0) \right] \right\} F[\sigma, \sigma']
\]

(26)

\[
= \exp \left\{ 2iV_0g \frac{\omega_0}{\omega_d^2 - \omega_0^2} \sum_{j=1}^{2n} \Xi_j \left[ \frac{\sin \omega_0 t_j + (\omega_0 + \omega_d) t_0}{\omega_0} - \frac{\sin \omega_0 t_j}{\omega_0} \right] \right\} F[\sigma, \sigma'].
\]

The new contribution in Eq. (26) can be taken into account into one height field. Let us call \( h_\xi^d \) the stochastic field coupling blips in the presence of the drive. It reads:

\[
h_\xi^d(t) = h_\xi(t) + 2iV_0g \frac{\omega_0}{\omega_d^2 - \omega_0^2} \left[ \sin \omega_dt + \sin(\omega_0 t + (\omega_0 + \omega_d) t_0) \right].
\]

(27)

Let us underline that it is possible to consider the drive term with a RWA-type approximation \( V_0/2 \left( a e^{i\omega_d t} + a\dagger e^{-i\omega_d t} \right) \), which results in:

\[
F_d[\sigma, \sigma'] = e^{iV_0g \int_{t_0}^{t} ds \int_{s}^t ds' \xi(s)(\sin \omega_0 (s-s') \cos \omega_d s' + \cos \omega_0 (s-s') \sin \omega_d s')} F[\sigma, \sigma']
\]

(28)

\[
= \exp \left\{ iV_0g \frac{2n}{\omega_0 - \omega_d} \sum_{j=1}^{2n} \Xi_j \left[ \sin \omega_0 t_j + (\omega_0 + \omega_d) t_0 \right] \right\} F[\sigma, \sigma'].
\]

Eq. (27) still holds provided that we replace \( 2V_0g\omega_0/(\omega_d^2 - \omega_0^2) \) by \( V_0g/(\omega_d - \omega_0) \). The two expressions give the same result for small drive strength \( V_0/\omega_0 \ll 1 \).
VI. SAMPLING OF THE FIELDS $h_\xi$ AND $h_\eta$

Below, we describe how one can sample the two stochastic variables with the properties given by Eq. (11). We introduce variables $\tau_k = t_k/t_f$ with $t_f$ being the final time of the experiment/simulation. Hence $Q_2(\tau)$ and $Q_1(\tau)\theta(\tau)$ are defined on $[-1,1]$. We extend their definitions by making them 2-periodic functions and it is then possible to expand them into Fourier series [5]. For the Rabi problem, we have:

$$Q_1(t) = \frac{g^2}{\omega_0^2} \sin \omega_0 t$$

$$Q_2(t) = \frac{g^2}{\omega_0^2} (1 - \cos \omega_0 t).$$

We define:

$$h_\xi(t_j) = i \frac{g}{\omega_0} (s_1 \cos \omega_0 t_j + s_2 \sin \omega_0 t_j)$$

$$+ \frac{g}{2\sqrt{2\omega_0}} \left\{ v_1 \phi(t_j) + iv_2 \phi(t_j) + v_3 \phi^*(t_j) + iv_4 \phi^*(t_j) \right\}$$

$$+ \sum_{m=1}^{\infty} \phi_m(\tau_j) \left( \frac{if_m}{4} \right)^{1/2} (u_{1,m} + iu_{2,m}) + \phi_m^*(\tau_j) \left( \frac{if_m}{4} \right)^{1/2} (u_{3,m} + iu_{4,m}) \right],$$

$$h_\eta(t_j) = \frac{g}{2\sqrt{2\omega_0}} \left\{ v_1 \phi^*(t_j) - iv_2 \phi^*(t_j) - v_3 \phi(t_j) + iv_4 \phi(t_j) \right\}$$

$$+ \sum_{m=1}^{\infty} \phi_m(\tau_j) \left( \frac{if_m}{4} \right)^{1/2} (u_{1,m} - iu_{2,m}) + \phi_m^*(\tau_j) \left( \frac{if_m}{4} \right)^{1/2} (u_{3,m} - iu_{4,m}) \right],$$

where $\phi(\tau) = \exp(i\omega_0 \tau t_f)$, and $\phi_m(\tau) = \exp(im\pi\tau)$. The choice of a Fourier basis is particularly well-suited for the Rabi model where coupling functions are trigonometric functions. We have an analytical expression for the Fourier coefficients $\{f^*_m = (g^2/\omega_0^2) \int_{-1}^{1} d\theta(\tau) \sin \omega_0 t_f \tau \cos m\pi\tau = (g^2t_f/\omega_0) \left[ 1 - (-1)^m \cos \omega_0 t_f \right] \left/ (\omega_0^2 t_f^2 - m^2\pi^2) \right\}$. The convergence is then controlled, and we are limited to ratios $g/\omega_0$ around unity for experiment/simulation times of the order of $2\pi/\omega_0$.

The contribution from the ohmic spin-boson model follows directly from the procedure of Ref. [5].

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