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Quantum hyper-CPHASE gates with polarisation and orbital angular momentum degrees of freedom and generalisation to arbitrary hyper-conditional gates

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Abstract
In this paper, a controlled-phase (CPHASE) gate using the polarisation and orbital angular momentum degrees of freedom for single-photon two-qubit quantum logic is proposed. This is critical to the realisation of quantum cluster states and graph networks using transverse degrees of freedom. A generalisation of the proposed scheme to arbitrary number and kinds of degrees of freedom, for optical systems, as well as arbitrary operations to be conditionally performed is proposed.

Keywords Entanglement · Quantum operation · Cluster states

1 Introduction
Universal quantum information processing can be performed with single-qubit rotations and the two-qubit controlled-NOT operations, among other possible sets of universal generators of quantum gates [1,2]. Quantum information processing in optical systems, using photonic qubits and qumodes, has been realised for gate-based as well as measurement-based quantum computation and communication [3–7]. In optical systems, single-qubit operations as well as controlled logic operations over two or more qubits can be realised using linear optical elements such as beam-splitters and waveplates [8–11]. Quantum information processing can be done over multiple transverse degrees of freedom in optical systems, using hyper-entanglement and hybrid-entanglement [12,13]. This allows efficient computation and communication using multiple qubits over a lesser number of photons. Measurement-based quantum computation has been realised in optical systems [14,15], with Yokoyama et al. [16] multiplexing generating and characterised a continuous-variable cluster state containing more than 10,000 entangled modes, and Larsen et al. [17] recently proposed a

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scheme to generate more than 30,000 entangled modes in a two-dimensional cluster state.

In cluster-state quantum computing, the role of the controlled-PHASE (CPHASE), as an entangler, is primary [18,19]. It can be used as a stand-alone two-qubit operation as well as an entangling gate, as is the case in the generation of entanglement in cluster states. The CPHASE operation has the form

\[
U_{\text{CPHASE}} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]

In optical systems with multiple degrees of freedom, the realisation of a CPHASE gate has previously been undertaken, between polarisation and momentum degrees of freedom, thereby helping create a high-fidelity four-qubit linear cluster state [20]. Cluster states have been created using the simultaneous entanglement of photons in three degrees of freedom, in a hybrid approach to one-way quantum computing [13], while recently, an 18-qubit Greenberger–Horne–Zeilinger (GHZ) entangled state was experimentally demonstrated by simultaneously exploiting three different degrees of freedom (polarisation, path and orbital angular momentum) of six photons [21].

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With regard to a single degree of freedom, Zou et al. [22] proposed a linear optical scheme for direct implementation of a non-destructive N-qubit controlled phase (CPHASE) gate on the polarisation degree of freedom. Lemr et al. [23] experimentally realised an optimal linear-optical controlled phase shift gate for arbitrary phases. Controlled phase shift and CPHASE gates have been implemented using quantum dots and optical cavities [24–27].

In this work, a scheme for a conditional phase (CPHASE) gate using the polarisation and orbital angular momentum degrees of freedom is proposed. The scheme utilises linear optical elements and is a fundamental step towards the realisation of one-time measurement-based quantum computation using optical systems. A generalisation of this scheme to arbitrary number and kinds of degrees of freedom as well as arbitrary operation to be conditionally performed is proposed, for optical systems.

2 Realisation of hyper-CPHASE gate between polarisation and OAM degrees of freedom

The scheme utilises the state discrimination for the different degrees of freedom (DOF) performed by a DOF-specific beam splitter, to selectively introduce a phase-flip operation in the other degree of freedom. In the fundamental unit of the hyper-CPHASE using polarisation and orbital angular momentum degrees of freedom, polarisation beam splitters (PBS) for polarisation control and OAM-based beam splitters formed of Mach–Zehnder interferometer with two Dove prisms for OAM-control are used.

Let us briefly look at the optical elements being used in the circuit before moving ahead with the realisation of the polarisation-controlled and OAM-controlled CPHASE operations. In the OAM-controlled CPHASE operation, we utilise a half-
wave plate, which is basically a wave retarder with its function being to delay the phase of the polarisation component that lies in the direction of its slow axes by a phase of $\pi$ relative to the phase of the perpendicular component. The Jones matrix for a half-wave plate is given by

$$J_{\text{HWP}} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

with angle $\theta$ being the orientation of the fast axis, relative to the horizontal [28,29]. Birefringent wave plates only affect the polarisation degree of freedom, and therefore, the effect of the wave plate can be written as

$$|P', O'\rangle = J_{\text{HWP}} \times |P, O\rangle$$

where the initial and final states—$|P, O\rangle$ and $|P', O'\rangle$, respectively, are described in terms of the polarisation ($P$ and $P'$) and orbital angular momentum ($O$ and $O'$) degrees of freedom. Single-qubit gates, such as the Hadamard and Pauli gates, for qubits encoded in the polarisation degree of freedom, can be realised using appropriately oriented half-wave plates. The Z-gate can be realised using a half-wave plate with its fast axes parallel to the horizontal plane, while an X-gate can be realised using a half-wave plate with its fast axes oriented at an angle of $\pi/4$ with respect to the horizontal plane.

Besides the birefringent wave plates, the Dove prism is a primary building block in the proposed realisation of both: polarisation-controlled hyper-CPHASE and OAM-controlled hyper-CPHASE operations. A Dove prism flips the sign of the orbital angular momentum of light; for instance, it can convert an OAM mode from $l = 1$ to $-1$. It tends to provide reflection as well as refraction to a beam of light and thereby affects both the polarisation and OAM degrees of freedom. We consider the M-shaped Dove prisms for the purposes of this work, for which the Jones matrix is

$$J_{\text{DP}} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

The M-shaped Dove prism was invented for the purposes of being a quarter wave retarder and therefore acts as a quarter wave plate on the polarisation degree of freedom [30]. The composite action of such a Dove prism is therefore given by

$$|P', O'\rangle = J_{\text{DP}} \times \sigma_x |P, O\rangle$$

Therefore, one usually uses a quarter-wave plate before or after a Dove prism to compensate for the effect on the polarisation degree of freedom, if we only want to invert the OAM state of a light beam. For ease of this study, we shall be assuming this correction while talking of the Dove prisms with the compensating birefringent wave plates (Fig. 1).

Considering a Laguerre–Gaussian beam, its complex amplitude has a phase-dependent term of the form $e^{il\phi}$, where $l$ is the orbital angular momentum of the beam...
[31]. If we rotate the light beam by an angle $\alpha$, the phase-dependent terms becomes $e^{il(\phi+\alpha)}$, thereby contributing a phase shift of $l\alpha$ [32,33]. Dove prisms can be oriented suitably to rotate optical beams. A non-rotated Dove prism can act as an effective X-gate for qubits encoded in the orbital angular momentum degree of freedom, since they change the OAM state of the beam to the opposite state, for instance, going from $|l\rangle$ to $|-l\rangle$. A Dove prism, when rotated by an angle of $\pi/4$ with respect to the vertical plane, transforms a qubit encoded in the orbital angular momentum degree of freedom as follows:

$$
|0\rangle_{OAM}^{DP_{\pi/4}} \rightarrow e^{-i\pi/2} |1\rangle_{OAM} \quad \text{and} \quad |1\rangle_{OAM} \rightarrow e^{i\pi/2} |0\rangle_{OAM}.
$$

This effectively implements the Pauli-Y operation. Since $i\sigma_z = \sigma_x \sigma_y$, we can implement the Pauli-Z operation for qubits encoded in the orbital angular momentum degree of freedom by using two Dove prisms, with 0 and $\pi/4$ rotation, with respect to the vertical plane. We can also construct an interferometer circuit, using Dove prisms in two arms, as shown in Fig. 2. This helps create what can be described as an OAM sorter [32,35,36], in which one output port has the odd OAM states while the other port has the even OAM states.

### 2.1 Generation of input states

In this paper, we consider the most general case

$$
|\psi_{in}\rangle = |\psi_P\rangle |\psi_{OAM}\rangle = (\alpha |0\rangle_P + \beta |1\rangle_P)(\gamma |0\rangle_{OAM} + \delta |1\rangle_{OAM})
$$

with subscripts defining the degree of freedom, and $\sqrt{\alpha^2 + \beta^2 + \gamma^2 + \delta^2} = 1$. Here, the polarisation qubits are: $|0\rangle_P = |H\rangle$ and $|1\rangle_P = |V\rangle$, while the OAM qubits are: $|0\rangle_{OAM} = |l = -1\rangle$ and $|1\rangle_{OAM} = |l = +1\rangle$, where $l$ defines the orbital angular momentum $\ell \hbar$.

The generation of a photon pulse is carried out using a single Rubidium (Rb) atom, trapped in an optical cavity by a two-dimensional optical lattice: a strong retro-reflected dipole laser confines the atom in a direction perpendicular to a cavity axis while a weak laser, known as the cavity-stabilisation laser, confines the atom along the cavity axis [37]. The cavity is excited by a sequence of trigger-laser pulses that induce a Raman transition, causing emission of a single photon pulse. The initial arbitrary OAM state: $|\psi_{OAM}\rangle = \gamma |0\rangle_{OAM} + \delta |1\rangle$ can be prepared using a number
of methods: using forked gratings and computer generated holograms [38], using cylindrical lenses [39], using spiral phase plates [40] and using q-plates [41]. Recently, the use of versatile metasurfaces has been used to generate arbitrary OAM states, using the concept conventional frequency-selective surfaces (FSSs), which are composed of sub-wavelength scatterers with varied orientations and geometries [42-46]. The can be used to construct arbitrary OAM states. Similarly, the arbitrary initial polarisation state $|\psi_P\rangle = \alpha|0\rangle_P + \beta|1\rangle_P$ can be prepared using polarisers [47] with birefringent plates as well as metasurfaces [48], and liquid crystal waveplates [49].

### 2.2 Polarisation-controlled hyper-CPHASE

For the realisation of polarisation-controlled hyper-CPHASE, we consider an initial arbitrary quantum state written in the form $|\psi_{in}\rangle = A|0\rangle|E_{(0)}\rangle + B|1\rangle|E_{(1)}\rangle$, where $\sqrt{A^2 + B^2}$, $|i\rangle$ ($i = 0, 1$) are the polarisation states, while $|E_{(i)}\rangle$ ($i = 0, 1$) are the orbital angular momentum states in the input quantum state that are associated with corresponding $|i\rangle$ polarisation states. Here, the states $|E_{(0)}\rangle$ and $|E_{(1)}\rangle$ simply denote the OAM states corresponding to the polarisation vector states $|0\rangle$ and $|1\rangle$, and since, in principle, $|E_{(0)}\rangle = |E_{(1)}\rangle$ is possible at the beginning of this hyper-CPHASE operation, the polarisation and OAM degrees of freedom need not be entangled at this point.

In Fig. 2, the two branches emergent from PBS$_1$ have orthogonal polarisation states. We consider the transmitted $|V\rangle$ or $|1\rangle$ state in the upper branch and reflected $|H\rangle$ or $|0\rangle$ state in the lower branch. We implement a OAM-Phase Flip gate using a realisation of

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**Fig. 2** Polarisation-controlled hyper-CPHASE scheme: here PBS—polarising beam splitter, DP—Dove prism. The light beam passes through two Dove prisms DP$_1$ and DP$_2$ in the lower branch. The first Dove prism has no relative angle with respect to the vertical plane, while the second Dove prism has angle $\pi/4$ with respect to the vertical plane. BP represented the compensating birefringent plate that corrects for polarisation changes by the Dove prism. The figure is a two-dimensional representation with the Dove-prisms being shown in three dimensions to highlight the arbitrary angles they could take with respect to the vertical plane.

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Fig. 3 OAM-controlled hyper-CPHASE scheme: here DP—Dove prism, BS—beam splitter and BP—compensating birefringent plates. We sort the even and odd OAM states using the interferometer designed using the two Dove prisms, with the first prism at no relative angle to the vertical plane but the second prism with relative angle $\pi/2$. The half-wave plate in the circuit has its fast access parallel to the horizontal plane. The optical path-length matching in the two branches of the OAM-sorter can be improved using optical elements with matched refractive index or elongation of path-length in the lower branch using mirrors. The figure is a two-dimensional representation with the Dove-prisms being shown in three dimensions to highlight the arbitrary angles they could take with respect to the vertical plane.

A Pauli-Z gate for qubits encoded in the orbital angular momentum degree of freedom, using two Dove prisms (oriented at angles of 0 and $\pi/4$, with respect to the vertical plane):

$$|\psi_{in}\rangle \xrightarrow{\text{PBS}_1+\text{DP}_1+\text{DP}_2} A|0\rangle|E(0)\rangle + B|1\rangle\sigma_z^{(\text{OAM})}|E(1)\rangle$$

(7)

This implements a phase-flip selectively for those states where the control-qubit is in the state $|1\rangle$ (in the polarisation degree of freedom) (Fig. 3).

2.3 OAM-controlled hyper-CPHASE

The realisation of OAM-controlled hyper-CPHASE gate is proposed using the OAM-sorter circuit and selective polarisation manipulation. Instead of using the decomposition of a general OAM state in terms of $|L\rangle$ and $|R\rangle$, one can express such a state in terms of a decomposition over odd/even states

$$|\psi_{in}\rangle = \frac{1}{\sqrt{2k}} \sum_k (c_{2k}|2k\rangle|P_{\text{even}}\rangle + c_{2k+1}|2k+1\rangle|P_{\text{odd}}\rangle)$$

(8)

where $|P_{\text{even}}\rangle$ and $|P_{\text{odd}}\rangle$ are the polarisation states corresponding to OAM even and odd states, respectively.
The OAM-sorter separates the odd OAM states and even OAM states and makes them pass through different output ports to the sorter. The Dove prisms introduce an OAM-dependent phase $e^{i\alpha}$ for the OAM state $|l\rangle$. When $\alpha = \pi$, the even states acquire a phase $e^{i2k\pi}$, while the odd states acquire a phase $e^{i(2k+1)\pi}$. Therefore, the state $|\psi_{\text{down}}\rangle$ that has passed through the Dove prism (with $\alpha = \pi$) in the lower branch of the OAM-sorter will be of the form

$$
|\psi_{\text{down}}\rangle = \frac{i}{\sqrt{2k}} \sum_k (c_{2k} |2k\rangle + c_{2k+1} |2k + 1\rangle)
$$

where $c_i$ define the relative phases for each superposition term $|i\rangle$ and the factor $i$ is due to the reflection at the first beam splitter. For the upper branch of the OAM-sorter, since the relative phase is 0,

$$
|\psi_{\text{up}}\rangle = \frac{1}{\sqrt{2k}} \sum_k (c_{2k} |2k\rangle + c_{2k+1} |2k + 1\rangle)
$$

where $|\psi_{\text{up}}\rangle$ is the state of the beam after passing through the Dove prism in the upper branch. After passing through the second beam splitter $\text{BS}_2$, we have the transmitted state (having considered the requisite phases due to reflection and transmission) with respect to the beam that has transversed the Dove prism $\text{DP}_2$,

$$
|\psi_1\rangle = |\psi_{\text{down}}\rangle + i |\psi_{\text{up}}\rangle = \frac{i}{\sqrt{2k}} \sum_k (c_{2k} |2k\rangle + c_{2k+1} |2k + 1\rangle)
$$

Similarly, we can find the expression for the other output port,

$$
|\psi_2\rangle = \frac{1}{\sqrt{2k}} \sum_k (c_{2k} |2k\rangle + c_{2k+1} |2k + 1\rangle) + \frac{1}{\sqrt{2k}} \sum_k (-c_{2k} |2k\rangle + c_{2k+1} |2k + 1\rangle)
$$

In the channel emerging out of the output port carrying $|\psi_2\rangle$, we place a half wave plate (HWP) with an angle $\pi/4$ with respect to the horizontal plane.
In this scheme, we will be considering the OAM states in the even/odd basis. If we consider the initial input state in the proposed scheme to be of the form $|\psi_{in}\rangle = \alpha|P_{\text{even}}\rangle|O_{\text{even}}\rangle + \beta|P_{\text{odd}}\rangle|O_{\text{odd}}\rangle$, for any general OAM states $|O_{\text{even}}\rangle$ and $|O_{\text{odd}}\rangle$ and polarisation vectors $|P_{\text{even}}\rangle$ and $|P_{\text{odd}}\rangle$ which are simply the polarisation states of those photons that have OAM states $|O_{\text{even}}\rangle$ and $|O_{\text{odd}}\rangle$, respectively, we obtain the result:

$$|\psi_{in}\rangle \xrightarrow{\text{OAM Sorter + HWP}} \alpha|P_{\text{even}}\rangle|O_{\text{even}}\rangle + \beta\sigma_{z}^{P}|P_{\text{odd}}\rangle|O_{\text{odd}}\rangle,$$

where $\sigma_{z}^{P}$ is the Pauli-Z operation on qubits encoded in the polarisation degree of freedom.

Since the aforementioned state differentiation is for odd–even OAM states primarily, we can model the circuit also for L/R OAM states, as shown in Fig. 4. In this we use Pancharatnam–Berry optical phase elements (PBOE). The fundamental idea behind this is to unwrap the beam’s azimuthal phase variations into variations in the Cartesian basis [50]. This unwrapped form is constituted by tilted wavefronts with their tilt being proportional to the orbital angular momentum value of the original beam. When these tilted waves are made to pass through a lens, they focus on different positions based on the orbital angular momentum value of the original beam. The unwrapping is performed using a conformal mapping between Cartesian and log-polar coordinates ($u = -a \ln(\sqrt{x^2 + y^2})$, $v = a \arctan(y/x)$ with $a$ and $b$ being scaling parameters), which can be realised using two confocal phase elements: unwrapper ($\phi_1$) and corrector ($\phi_2$), given by Hossack et al. [51]:

$$\phi_1(x, y) = \frac{2\pi a}{\lambda f} [y \arctan \frac{y}{x} - x \ln \frac{\sqrt{x^2 + y^2}}{b} + x] \quad (11)$$

$$\phi_2(u, v) = -\frac{2\pi ab}{\lambda f} e^{-\frac{u}{a}} \cos \frac{v}{a} \quad (12)$$

where $f$ is the focal length of the lenses used (both Lens$_1$ and Lens$_2$ as shown in Fig. 4) and $\lambda$ is the wavelength of the beam. The corrector element provides final phase corrections to the light beam, to complete the mapping. A point to note here is that PBOEs introduce a polarisation-dependent transformation onto light beams, which is corrected by using the corrector element [51,52]. Since the output waves have different spatial output dependent on the OAM-value(s) in the original beam, we can collect these OAM-states, using optical fibre collectors, and transmit them to the next section of the circuit that introduces the controlled Pauli-Z operation on the polarisation degree of freedom.

2.4 Illustrative example

We can understand the working of this scheme by considering an illustrative example: let us begin with the initial state

$$|\psi_{in}\rangle = \frac{1}{2}(|0\rangle + |1\rangle)_{P}(|0\rangle + |1\rangle)_{OAM}.$$
where subscripts ‘$P$’ and ‘OAM’ stand for the polarisation and OAM degrees of freedom, respectively. Upon using the circuit for the polarisation-controlled hyper-CPHASE scheme, we shall have the final state

$$|\psi\rangle_{out} = \frac{1}{2} (|0\rangle_P |0\rangle_{OAM} + |0\rangle_P |1\rangle_{OAM} + e^{i3\pi} |1\rangle_P |1\rangle_{OAM})$$

$$= \frac{1}{2} (|0\rangle_P |0\rangle_{OAM} + |0\rangle_P |1\rangle_{OAM} + |1\rangle_P |0\rangle_{OAM} - |1\rangle_P |1\rangle_{OAM}).$$

We can similarly implement the OAM-controlled hyper-CPHASE operation.

### 2.5 Accounting for errors and phase drifts using non-destructive measurements

The proposed scheme is primarily for the ideal case, wherein we do not account for errors in either degree of freedom or phase drifts due to the environment. The correction for either degree of freedom relies on the non-destructive measurement of the same, followed by compensatory operations to account for the error or phase drift. In the case of polarisation, this can be attained in a non-deterministic manner by using a measurement induced non-linearity, with the reported quantum non-demolition (QND) fidelity—$F_{QND} > 0.99$ for such a scheme [53]. Thereafter, error correction in the polarisation degree of freedom can be performed using linear optical elements [54]. Similarly, non-demolition measurement of the orbital angular momentum degree of freedom and subsequent error correction can help reduce errors and phase drifts in this degree of freedom as well [55–57].

In classical interferometer systems, continuous stabilisation and fine-level phase control of time-bin interferometers can be achieved [58] using passive stabilisation techniques such as ‘plug and play’ techniques [59,60] and active stabilisation techniques such as active thermal management of photonic integrated circuits [61] and adding single-photon detectors and ports to interferometer outputs for tomography [62]. Most of these active stabilisation techniques involve the injection of a reference signal that usually has a wavelength that differs from the quantum signal, which is monitored and used to cancel any relative path length drifts in the interferometers [59,63,64].
Fig. 5 Scheme for an arbitrary hyper-conditional gate. Here, we have two degrees of freedom: DOF$_1$ and DOF$_2$, with the first as control and second as target. DOF$_i$BS$_i$ represents the $i$th beam splitter that discriminates states based on the states of the first degree of freedom. DOF$_2$U is the operation being performed on the second qubit.

In this way, the hyper-CPHASE gate can be implemented, with the polarisation degree of freedom as control and the OAM degree of freedom as target, in noisy environments that may introduce errors and phase drifts.

3 Generalisation of hyper-conditional gates

The implementation of the hyper-CPHASE gives us an idea about how to implement a general hyper-conditional quantum gate of the form $|0⟩⟨0| + U|1⟩⟨1|$ (Fig. 5).

For a hyper-controlled gate, we have three relevant degrees of freedom: the path-variable ($S$), the first degree of freedom that acts as control ($D_1$) and the second degree of freedom that acts as control ($D_2$). If both the degrees of freedom are two level systems with $D_i = \begin{pmatrix} d_{1}^{(i)} \\ d_{2}^{(i)} \end{pmatrix}$, then we can define a general state as

$$|SD_1D_2⟩ = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \otimes \begin{pmatrix} d_{1}^{(1)} \\ d_{2}^{(1)} \end{pmatrix} \otimes \begin{pmatrix} d_{1}^{(2)} \\ d_{2}^{(2)} \end{pmatrix} = \begin{pmatrix} s_1d_{1}^{(1)}d_{1}^{(2)} \\ s_1d_{1}^{(1)}d_{2}^{(2)} \\ s_1d_{2}^{(1)}d_{1}^{(2)} \\ s_1d_{2}^{(1)}d_{2}^{(2)} \\ s_2d_{1}^{(1)}d_{1}^{(2)} \\ s_2d_{1}^{(1)}d_{2}^{(2)} \\ s_2d_{2}^{(1)}d_{1}^{(2)} \\ s_2d_{2}^{(1)}d_{2}^{(2)} \end{pmatrix}$$

Analysing the state emerging from the DOF-based beam splitter that splits the light beam based on the control degree of freedom, we can write the composite beam-splitter action as

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\[ C = \begin{pmatrix}
  t_1 & 0 & 0 & 0 & t_1 & 0 & 0 & 0 \\
  0 & t_1 & 0 & 0 & 0 & t_1 & 0 & 0 \\
  0 & 0 & t_2 & 0 & 0 & 0 & t_2 & 0 \\
  0 & 0 & 0 & t_2 & 0 & 0 & 0 & t_2 \\
  r_1 & 0 & 0 & 0 & r_1 & 0 & 0 & 0 \\
  0 & r_1 & 0 & 0 & 0 & r_1 & 0 & 0 \\
  0 & 0 & r_2 & 0 & 0 & 0 & r_2 & 0 \\
  0 & 0 & 0 & r_2 & 0 & 0 & 0 & r_2
\end{pmatrix} \]  

(14)

where \( t_i \) and \( r_i \) are the transmittance and reflectance, respectively, for the \( i \)th state of the control (DOF) qubit. The beam splitter action for the second beam splitter \( (C') \) is the same operation with \( t_i \leftrightarrow r_i \). In the controlled two-qubit operation, the beam-splitter action constitutes what can be called the \textit{control operation}.

Just like the CPHASE operation is given by \( \text{CPHASE} = I_{2 \times 2} \oplus \sigma_z \), The \textit{Target Operation}—\( T \) that conditionally applies an operation \( U = \begin{pmatrix}
  U_{11} & U_{12} \\
  U_{21} & U_{22}
\end{pmatrix} \), is defined as

\[ T = \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & U_{11} & U_{12} \\
  0 & 0 & U_{21} & U_{22}
\end{pmatrix} \]  

(15)

where \( U_{ij}, i, j \in [1, 2] \) are elements of the operation \( U \) that is being performed as part of the conditional two qubit operation: \( |0\rangle\langle 0| + U|1\rangle\langle 1| \).

Cumulatively, the operations in the circuit can be represented as:

\[ O = C \cdot (I \otimes (I \oplus U)) \cdot C' \]  

(16)

whose matrix form is

\[ O = \begin{pmatrix}
  0 & t_1 - r_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
  t_1 - r_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & t_2 - r_2 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & t_2 - r_2 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & t_1 - r_1 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & t_1 - r_1 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \]  

(17)

If we consider the perfect DOF1-based beam splitter is perfect with \( t_1 = i r_2 = 1 \) and \( t_2 = i r_1 = 0 \) and the operator for DOF2 to have \( U_{11} = 1, U_{12} = 0, U_{21} = 0 \) and \( U_{22} = -1 \), giving us \( \sigma_x \otimes \text{CPHASE} \). This effectively implements the CPHASE gate between the two degrees of freedom DOF1 and DOF2. While the polarising beam splitter is well known, beam splitters based on other degrees of freedom, such as orbital angular momentum, have been analysed and used recently [32]. The proposed scheme...
A DOF-based beam splitter (DOF-BS) can be replaced by a general beam splitter (BS) followed by DOF-based filters (DOS-F). The transmission and reflection coefficients need to be fixed consistently:

\[
\begin{align*}
t_{\text{DOF-BS}} &= t_{\text{BS}} + t_{\text{DOS-F}}^2, \\
r_{\text{DOF-BS}} &= r_{\text{BS}} + r_{\text{DOS-F}}^1
\end{align*}
\]

Table 1

| Degree of freedom | DOF-BS          | Gates          |
|-------------------|-----------------|----------------|
| Polarisation      | Polarising beam splitter | Universal [a] |
| OAM               | OAM-sorter      | Universal [b]  |
| Frequency         | Dichroic mirror | Universal [c]  |
| Time-bin          | Optical switch [d] | Non-universal |

[a] [8], [b] [75], [c] [76], [d] [77]

works for any degree of freedom wherein experimental resources for mode sorting and mode shifting are available.

Instead of a DOF-based beam splitter, we can also use a probabilistic version of this scheme with a beam splitter followed by DOF-filters that project the states in the relevant DOF to either \( |0\rangle \) or \( |1\rangle \), as shown in Fig. 6. The coefficient matching condition, with reference to Fig. 6, is:

\[
\begin{align*}
t_{\text{DOF-BS}} &= t_{\text{BS}} + t_{\text{DOS-F}}^2, \\
r_{\text{DOF-BS}} &= r_{\text{BS}} + r_{\text{DOS-F}}^1
\end{align*}
\]

where \( t_i \) and \( r_i \) stand for the transmission coefficient and reflection coefficient of element \( i \), respectively. These filters are DOF-specific: we have polarising filters to filter out specific polarisation values [65,66], dichroic filters for frequency/wavelength filtering [67,68], an OAM-sorter using metasurfaces and holograms, time-division multiplexing and Mach–Zehnder Interferometer-based schemes for filtering out OAM values [69–71], and apertures/path-filters and non-destructive measurement(s) of photons using optical resonators to filter out path-specific values for a beam of light [72,73].

While a generalised physical model for all DOF-based sorters is not present at the moment, we can look at the relevant optical elements for an exhaustive set of all degrees of freedom of a photon [74]—polarization, spatial (OAM) mode, frequency and time (bin) (Table 1).
4 Conclusion

In this paper, a hyper-CPHASE gate using the polarisation and orbital angular momentum degrees of freedom for single-photon two-qubit quantum logic was proposed. The cluster state formalism for quantum computation [18] proposed the realisation of one-way measurement-based quantum computation on cluster states, with the CPHASE gate being the entangling operation. A natural extension of this formalism to multiple degrees of freedom requires the realisation of a hyper-CPHASE operation. The proposed realisation of the hyper-CPHASE gate in this paper therefore gives us a way to implement this fundamental building block for any cluster-state using multiple degrees of freedom. As a stand-alone two-qubit conditional gate in multiple degrees of freedom, the proposed scheme (along with the proposed generalisation to any arbitrary conditional gate) can be used to generate hyper-entanglement across these multiple degrees of freedom, which can be used as a resource for tasks such as hyper-teleportation [12, 78, 79], hyper-QKD [78, 80, 81] and hyper-superdense coding [82–84]. The hyper-CPHASE gate can also be useful for any gate-based quantum computation task where the CPHASE need be applied for qubits encoded in multiple degrees of freedom.

References

1. DiVincenzo, D.P.: Two-bit gates are universal for quantum computation. Phys. Rev. A 51, 1015 (1995)
2. Gottesman, D., Chuang, I.L.: Demonstrating the viability of universal quantum computation using teleportation and single-qubit operations. Nature 402, 390 (1999)
3. O’Brien, J.L., Furusawa, A., Vučković, J.: Photonic quantum technologies. Nat. Photonics 3, 687 (2009)
4. Duan, L.-M., Kimble, H.: Scalable photonic quantum computation through cavity-assisted interactions. Phys. Rev. Lett. 92, 127902 (2004)
5. Azuma, K., Tamaki, K., Lo, H.-K.: All-photonic quantum repeaters. Nat. Commun. 6, 6787 (2015)
6. Chen, K., Li, C.-M., Zhang, Q., Chen, Y.-A., Goebel, A., Chen, S., Mair, A., Pan, J.-W.: Experimental realization of one-way quantum computing with two-photon four-qubit cluster states. Phys. Rev. Lett. 99, 120503 (2007)
7. Prevedel, R., Walther, P., Tiefenbacher, F., Böhi, P., Kaltenbaek, R., Jennewein, T., Zeilinger, A.: High-speed linear optics quantum computing using active feed-forward. Nature 445, 65 (2007)
8. Knill, E., Laflamme, R., Milburn, G.J.: A scheme for efficient quantum computation with linear optics. Nature 409, 46 (2001)
9. Ralph, T.C., Langford, N.K., Bell, T., White, A.: Linear optical controlled-NOT gate in the coincidence basis. Phys. Rev. A 65, 062324 (2002)
10. O’Brien, J.L., Pryde, G.J., White, A.G., Ralph, T.C., Branning, D.: Demonstration of an all-optical quantum controlled-NOT gate. Nature 426, 264 (2003)
11. Gasparoni, S., Pan, J.-W., Walther, P., Rudolph, T., Zeilinger, A.: Realization of a photonic controlled-NOT gate sufficient for quantum computation. Phys. Rev. Lett. 93, 020504 (2004)
12. Deng, F.-G., Ren, B.-C., Li, X.-H.: Quantum hyperentanglement and its applications in quantum information processing. Sci. Bull. 62, 46 (2017)
13. Vallone, G., Donati, G., Ceccarelli, R., Mataloni, P.: Six-qubit two-photon hyperentangled cluster states: characterization and application to quantum computation. Phys. Rev. A 81, 052301 (2010)
14. Schwartz, I., Cogan, D., Schmidgall, E.R., Don, Y., Gantz, L., Kenneth, O., Lindner, N.H., Gershoni, D.: Deterministic generation of a cluster state of entangled photons. Science 354, 434 (2016)
15. Kues, M., Reimer, C., Sciara, S., Roztocki, P., Islam, M., Cortés, L.R., Zhang, Y., Fischer, B., Loranger, S., Kashyap, R. et al.: High-dimensional one-way quantum processing enabled by optical d-level cluster states. In: Quantum Information and Measurement, pp. S2C–3. Optical Society of America (2019)
16. Yokoyama, S., Ukai, R., Armstrong, S.C., Somphiphatphong, C., Kaji, T., Suzuki, S., Yoshikawa, J.-I., Yonezawa, H., Menicucci, N.C., Furusawa, A.: Ultra-large-scale continuous-variable cluster states multiplexed in the time domain. Nat. Photonics 7, 982 (2013)
17. Larsen, M.V., Guo, X., Breum, C.R., Neergaard-Nielsen, J.S., Andersen, U.L.: Deterministic generation of a two-dimensional cluster state. Science 366, 369 (2019)
18. Raussendorf, R., Briegel, H.J.: A one-way quantum computer. Phys. Rev. Lett. 86, 5188 (2001)
19. Raussendorf, R., Harrington, J., Goyal, K.: A fault-tolerant one-way quantum computer. Ann. Phys. 321, 2242 (2006)
20. Vallone, G., Pomarico, E., Mataloni, P., De Martini, F., Berardi, V.: Realization and characterization of a two-photon four-qubit linear cluster state. Phys. Rev. Lett. 98, 180502 (2007)
21. Wang, X.-L., Luo, Y.-H., Huang, H.-L., Chen, M.-C., Su, Z.-E., Liu, C., Chen, C., Li, W., Fang, Y.-Q., Jiang, X., et al.: 18-qubit entanglement with six photons’ three degrees of freedom. Phys. Rev. Lett. 120, 260502 (2018)
22. Zou, X., Li, K., Guo, G.: Linear optical scheme for direct implementation of a nondestructive N-qubit controlled phase gate. Phys. Rev. A 74, 044305 (2006)
23. Lemr, K., Černoch, A., Soubusta, J., Kieling, K., Eisert, M.: Experimental implementation of the optimal linear-optical controlled phase gate. Phys. Rev. Lett. 106, 013602 (2011)
24. Das, S., Grankin, A., JakouPOv, I., Brion, E., Borregaard, J., Boddema, R., Usman, I., Ourjoumtsev, A., Grangier, P., Sorensen, A.S.: Photonic controlled-phase gates through Rydberg blockade in optical cavities. Phys. Rev. A 93, 040303 (2016)
25. Xiao, Y.-F., Zou, X.-B., Guo, G.-C.: One-step implementation of an N-qubit controlled-phase gate with neutral atoms trapped in an optical cavity. Phys. Rev. A 75, 054303 (2007)
26. Fushman, I., Englund, D., Faroq, A., Stoltz, N., Petroff, P., Vučković, J.: Controlled phase shifts with a single quantum dot. Science 320, 769 (2008)
27. Chen, L.-B., Yang, W.: All-optical controlled phase gate in quantum dot molecules. Laser Phys. Lett. 11, 105201 (2014)
28. Fowles, G.R.: Introduction to Modern Optics. Courier Corporation, United States (1989)
29. Hecht, E., Zajac, A.: Optics. Addison-Wesley Publishing Company, Boston (1974)
30. Bennett, J.: A critical evaluation of rhomb-type quarterwave retarders. Appl. Opt. 9, 2123 (1970)
31. Allen, L., Beijersbergen, M.W., Spreeuw, R., Woerdman, J.: Orbital angular momentum of light and the transformation of Laguerre–Gaussian laser modes. Phys. Rev. A 45, 8185 (1992)
32. Leach, J., Padgett, M.J., Barnett, S.M., Franke-Arnold, S., Courtial, J.: Measuring the orbital angular momentum of a single photon. Phys. Rev. Lett. 88, 257901 (2002)
33. Courtial, J., Robertson, D., Dholakia, K., Allen, L., Padgett, M.: Rotational frequency shift of a light beam. Phys. Rev. Lett. 81, 4828 (1998)
34. Nicolas, A., Veissier, L., Giacobino, E., Maxein, D., Laurat, J.: Quantum state tomography of orbital angular momentum photonic qubits via a projection-based technique. New J. Phys. 17, 033037 (2015)
35. Leach, J., Courtial, J., Skeldon, K., Barnett, S.M., Franke-Arnold, S., Padgett, M.J.: Interferometric methods to measure orbital and spin, or the total angular momentum of a single photon. Phys. Rev. Lett. 92, 013601 (2004)
36. Rodenburg, B., Magaña-Loaiza, O.S., Mirhosseini, M., Taherirostami, P., Chen, C., Boyd, R.W.: Multiplexing free-space channels using twisted light. J. Opt. 18, 050415 (2016)
37. Predojević, A., Zhai, Z., Caballero, J.M., Mitchell, M.W.: Rubidium resonant squeezed light from a diode-pumped optical-parametric oscillator. Phys. Rev. A 78, 063820 (2018)
38. Bazhenov, V., Vasnetsov, M., Koskin, M.: Laser beams with screw dislocations in their wavefronts. JETP Lett. 52, 429 (1990)
39. Beijersbergen, M.W., Allen, L., Van der Veen, H., Woerdman, J.: Astigmatic laser mode converters and transfer of orbital angular momentum. Opt. Commun. 96, 123 (1993)
40. Beijersbergen, M., Coerwinkel, R., Kristensen, M., Woerdman, J.: Helical-wavefront laser beams produced with a spiral phaseplate. Opt. Commun. 112, 321 (1994)
41. Marrucci, L., Manzo, C., Paparo, D.: Optical spin-to-orbital angular momentum conversion in inhomogeneous anisotropic media. Phys. Rev. Lett. 96, 163905 (2006)
42. Yu, N., Capasso, F.: Flat optics with designer metasurfaces. Nat. Mater. 13, 139 (2014)
43. Yu, N., Genevet, P., Kats, M.A., Aieta, F., Tetienne, J.-P., Capasso, F., Gaburro, Z.: Light propagation with phase discontinuities: generalized laws of reaction and refraction. Science 334, 333 (2011)
44. Genevet, P., Yu, N., Aieta, F., Lin, J., Kats, M.A., Blanchard, R., Scully, M.O., Gaburro, Z., Capasso, F.: Ultra-thin plasmonic optical vortex plate based on phase discontinuities. Appl. Phys. Lett. 100, 013101 (2012)
45. Yu, N., Genevet, P., Aieta, F., Kats, M.A., Blanchard, R., Aoust, G., Tetienne, J.-P., Gaburro, Z., Capasso, F.: Flat optics: controlling wavefronts with optical antenna metasurfaces. IEEE J. Sel. Top. Quantum Electron. 19, 4700423 (2013)
46. Munk, B.: Frequency Selective Surface: Design and Theory. Wiley, New York (2000)
47. Peters, N.A., Barreiro, J.T., Goggin, M.E., Wei, T.-C., Kwiat, P.G.: Remote state preparation: arbitrary remote control of photon polarization. Phys. Rev. Lett. 94, 150502 (2005)
48. Mueller, J.B., Rubin, N.A., Devlin, R.C., Groever, B., Capasso, F.: Metasurface polarization optics: independent phase control of arbitrary orthogonal states of polarization. Phys. Rev. Lett. 118, 113901 (2017)
49. Dupont, L., Sansoni, T., et al.: Endless smectic A* liquid crystal polarization controller. Optics Communications 209, 101 (2002)
50. Berkhour, G.C., Lavery, M.P., Courtial, J., Beijersbergen, M.W., Padgett, M.J.: Efficient sorting of orbital angular momentum states of light. Phys. Rev. Lett. 105, 153601 (2010)
51. Hossack, W., Darling, A., Dahdouh, A.: Coordinate transformations with multiple computer-generated optical elements. J. Mod. Opt. 34, 1235 (1987)
52. Walsh, G.F.: Pancharatnam–Berry optical element sorter of full angular momentum eigen-state. Opt. Express 24, 6689 (2016)
53. Pryde, G., O’Brien, J., White, A., Bartlett, S., Ralph, T.: Measuring a photonic qubit without destroying it. Phys. Rev. Lett. 92, 190404 (2004)
54. Do Nascimento, J.C., Mendonça, F.A., Ramos, R.V.: Linear optical setups for active and passive quantum error correction in polarization encoded qubits. J. Mod. Opt. 54, 1467 (2007)
55. Wei, D., Wang, Y., Liu, D., Zhu, Y., Zhong, W., Fang, X., Zhang, Y., Xiao, M.: Simple and nondestructive on-chip detection of optical orbital angular momentum through a single plasmonic nanohole. ACS Photonics 4, 996 (2017)
56. Wang, F.-X., Chen, W., Yin, Z.-Q., Wang, S., Guo, G.-C., Han, Z.-F.: Scalable orbital-angular-momentum sorting without destroying photon states. Phys. Rev. A 94, 033847 (2016)
57. Alonso, J.R.G., Brun, T.A.: Protecting orbital-angular-momentum photons from decoherence in a turbulent atmosphere. Phys. Rev. A 88, 022326 (2013)
58. Toliver, P., Dailey, J.M., Agarwal, A., Peters, N.A.: Continuously active interferometer stabilization and control for time-bin entanglement distribution. Opt. Express 23, 4135–4143 (2015)
59. Sticki, D., Gisin, N., Guinnard, O., Ribordy, R., Zbinden, H.: Quantum key distribution over 67 km with a plug & play system. New J. Phys. 4, 411–418 (2002)
60. Chapman, J.C., Graham, T.M., Zeitler, C.K., Bernstein, H.J., Kwiat, P.G.: Time-bin and polarization superdense teleportation for space applications. Phys. Rev. Appl. 14(1), 014044 (2020)
61. Takesue, H., Inoue, K.: Generation of 1.5 μm band time-bin entanglement using spontaneous fiber four-wave mixing and planar light-wave circuit interferometers. Phys. Rev. A 72(4), 041804 (2005)
62. Wang, S.X., Chan, C., Moraw, P., Reilly, D.R., Altpeter, J.B., Kanter, G.S.: High-speed tomography of time-bin-entangled photons using a single-measurement setting. Phys. Rev. A 86(4), 042122 (2012)
63. Marcikic, I., de Riedmatten, H., Tittel, W., Zbinden, H., Legr, M., Gisin, N.: Distribution of time-bin entangled qubits over 50 km of optical fiber. Phys. Rev. Lett. 93(18), 180502 (2004)
64. Xavier, G.B., von der Weid, J.P.: Stable single-photon interference in a 1 km fiber-optic Mach–Zehnder interferometer with continuous phase adjustment. Opt. Lett. 36(10), 1764–1766 (2011)
65. Kitaura, K., Kigoshi, S., Hisaki, H.: Polarizer for visible light. US Patent 5,087,985 (1992)
66. Saxe, R.L.: Light polarizing materials and suspensions thereof. US Patent 5,002,701 (1991)
67. Ohkamoto, M.: Dichroic filter. US Patent App. 12/224,134 (2009)
68. Trost, D., Baumeister, P., Fischer, D.: Dichroic optical filter. US Patent 5,341,238 (1994)
69. Yu, S., Li, L., Kou, N.: Generation, reception and separation of mixed-state orbital angular momentum vortex beams using metasurfaces. Opt. Mater. Express 7, 3312 (2017)
70. Chen, D.-X., Zhang, P., Liu, R.-F., Li, H.-R., Gao, H., Li, F.-L.: Orbital angular momentum filter of photon based on spin-orbital angular momentum coupling. Phys. Lett. A 379, 2530 (2015)
71. Karimi, E., Marrucci, L., de Lizio, C., Santamato, E.: Time-division multiplexing of the orbital angular momentum of light. Opt. Lett. 37, 127 (2012)
72. Bailey, T.B.: Absorption path controlled filter. US Patent 4,502,758 (1985)
73. Reiserer, A., Ritter, S., Rempe, G.: Nondestructive detection of an optical photon. Science 342, 1349 (2013)
74. Barreiro, J.T., Langford, N.K., Peters, N.A., Kwiat, P.G.: Generation of hyperentangled photon pairs. Phys. Rev. Lett. 95, 260501 (2005)
75. García-Escartín, J.C., Chamorro-Posada, P.: Universal quantum computation with the orbital angular momentum of a single photon. J. Opt. 13, 064022 (2011)
76. Lu, H.-H., Lukens, J.M., Williams, B.P., Imany, P., Peters, N.A., Weiner, A.M., Lougovski, P.: A controlled-NOT gate for frequency-bin qubits. NPJ Quantum Inf. 5, 1 (2019)
77. Lo, H.-P., Ikuta, T., Matsuda, N., Honjo, T., Takesue, H.: Entanglement generation using a controlled-phase gate for time-bin qubits. Appl. Phys. Express 11, 092801 (2018)
78. Perumangatt, C., Rahim, A.A., Salla, G.R., Prabhakar, S., Samanta, G.K., Paul, G., Singh, R.P.: Three-particle hyper-entanglement: teleportation and quantum key distribution. Quantum Inf. Process. 14, 3813 (2015)
79. Shi, J., Ma, P.-C., Chen, G.-B.: Schemes for bidirectional quantum teleportation via a hyper-entangled state. Int. J. Theor. Phys. 58, 372 (2019)
80. Djordjevic, I.B.: Multidimensional QKD based on combined orbital and spin angular momenta of photon. IEEE Photonics J. 5, 7600112 (2013)
81. Smith III, J.F.: Superdense coding facilitated by hyper-entanglement and quantum networks. In: Ultrafast Bandgap Photonics II, vol. 10193, p. 1019316. International Society for Optics and Photonics (2017)
82. Zheng, C., Gu, Y., Li, W., Wang, Z., Zhang, J.: Complete distributed hyper-entangled-bell-state analysis and quantum super dense coding. Int. J. Theor. Phys. 55, 1019 (2016)
83. Rui-Tong, Z., Qi, G., Li, C., Hong-Fu, W., Shou, Z.: Quantum superdense coding based on hyperentanglement. Chin. Phys. B 21, 080303 (2012)

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