Hinojosa, Pedro A.
Surfaces of Constant Mean Curvature in Euclidean 3-space Orthogonal to a Plane along its Boundary
Anais da Academia Brasileira de Ciências, vol. 74, núm. 1, março, 2002, pp. 33-35
Academia Brasileira de Ciências
Rio de Janeiro, Brasil

Available in: http://www.redalyc.org/articulo.oa?id=32774104
Surfaces of Constant Mean Curvature in Euclidean 3-space
Orthogonal to a Plane along its Boundary

PEDRO A. HINOJOSA

Centro de Ciências Exatas e da Natureza, Departamento de Matemática
Universidade Federal da Paraíba, João Pessoa, Brasil, and
Universidade Federal do Ceará, Centro de Ciências, Pós Graduação em Matemática
Campus do Pici, Bloco 914 – 60455-760 Fortaleza, Ce, Brasil

Manuscript received on October 25, 2001; accepted for publication on November 28, 2001;
presented by J. Lucas Barbosa

ABSTRACT

We consider compact surfaces with constant nonzero mean curvature whose boundary is a convex
planar Jordan curve. We prove that if such a surface is orthogonal to the plane of the boundary,
then it is a hemisphere.

Key words: surfaces with boundary, constant mean curvature, elliptic partial differential equation.

Let $M$ be a compact surface immersed in $\mathbb{R}^3$ with constant mean curvature $H$ whose boundary
$\partial M = \Gamma$ is a planar Jordan curve of length $L$. Let $D$ be a planar region enclosed by $\Gamma$ and let $A$ be
the area of $D$. Let us consider the cycle $M \cup D$ oriented in such a way that its orientation, along
$M$, coincides with the one defined by the mean curvature vector. Let $Y$ be a Killing vector field in
$\mathbb{R}^3$ and $n_D$ be a unitary vector field normal to $D$ in the orientation of $M \cup D$. Let $\nu$ be the unitary
co-normal vector field along $\partial M = \Gamma$ pointing inwards $M$. By the flux formula it is known that
$|H| \leq \frac{L}{2A}$ where equality holds if and only if $\nu = n_D$. That is, if and only if $\nu$ is constant and
orthogonal to $D$ along $\Gamma$.

In this work we consider the case $|H| \leq \frac{L}{2A}$ and we show that, in the above conditions, if $M$
is embedded and $\Gamma$ is convex, then $M$ is a hemisphere. Explicitly we prove that:

THEOREM 1. Let $M$ be a compact embedded surface in $\mathbb{R}^3$ with constant mean curvature $H \neq 0$
whose boundary $\partial M$ is a Jordan curve $\Gamma$ in a plane $\mathbb{P} \subset \mathbb{R}^3$. Suppose that $\Gamma$ is convex and $M$ is
perpendicular to the plane $\mathbb{P}$ along its boundary. Then $M$ is a hemisphere of radius $\frac{1}{|H|}$.

Correspondence to: Universidade Federal do Ceará
e-mail: hinojosa@mat.ufpb.br
This theorem generalizes a result obtained by Brito and Earp (Brito and Earp 1991). We succeed in discarding their assumption that $\partial M$ should be a circle of radius $\frac{1}{|H|}$.

A sketch of the proof of the theorem is as follows.

First, under the hypothesis of the theorem, $M$ must be totally contained in one of the halfplanes determined by $P$ (see Brito et al. 1991, for example). Now let $M^*$ be the reflection of $M$ with respect to the plane $P$. Since $M$ is orthogonal to $P$ along $\Gamma$, we have that $\tilde{M} := M \cup M^*$ is a compact surface without boundary, embedded in $\mathbb{R}^3$. Note that a priori $\tilde{M}$ is only of class $C^1$. In this way we are able to use a classical result due to Alexandrov (see Hopf 1983, for example) in order to establish that $\tilde{M}$ is a sphere and therefore $M$ is a hemisphere.

The regularity of $\tilde{M}$ along $\Gamma$ is achieved by means of the theory of elliptic partial differential equations. Let $p$ be any point in $\Gamma \subset \tilde{M}$ and $\Omega$ be an open neighborhood of 0 in $T_p \tilde{M}$ chosen in such a way that locally around $p$, $\tilde{M}$ may be described as the graph of a function $u : \Omega \to \mathbb{R}$. For our purposes, it is sufficient to consider $\Omega$ of class $C^{1,1}$.

It is clear that $u \in C^1(\Omega)$. So, $\nabla u$ is well-defined and continuous. Since $\Omega$ is bounded we have that $u \in W^{1,2}(\Omega)$.

Let us denote the linear space of $k$-times weakly differentiable functions by $W^k(\Omega)$. For $p \geq 1$ and $k$ a non-negative integer, we let $W^{k,p}(\Omega) = \{u \in W^k(\Omega), D^\sigma u \in L^p(\Omega) \text{ for all } |\sigma| \leq k\}$.

The Hölder spaces $C^{k,\alpha}(\Omega)$ are defined as the subspaces of $C^k(\Omega)$ consisting of functions whose $k$-th order partial derivatives are locally Hölder continuous with exponent $\alpha$ in $\Omega$.

We define on $\Omega$ the following linear operators:

$$L_1 v := D_i (a^{ij} D_j v), \quad v \in W^{1,2}(\Omega) \quad i, j = 1, 2$$

and

$$L_2 v := A^{ij} D_{ij} v, \quad v \in W^{2,2}(\Omega) \quad i, j = 1, 2,$$

where the coefficients $a^{ij}$ are given by

$$a^{11} = a^{22} = \frac{1}{1 + |\nabla u|^2}, \quad a^{12} = a^{21} = 0$$

and the coefficients $A^{ij}$ are defined by $A^{11} = 1 + u_x^2$, $A^{12} = A^{21} = -u_x u_y$, $A^{22} = 1 + u_y^2$. Finally, the symbols $D_i, D_{ij}, i, j = 1, 2$ stand for partial differentiation.

We prove that $u$ is a weak solution to the equation $L_1 u = 2H$. By the Corollary 8.36 (Gilbarg and Trudinger 1983) we have $u \in C^{1,\alpha}(\Omega)$. Moreover, by the Lebesgue’s dominated convergence theorem and Lemma 7.24 (Gilbarg and Trudinger 1983) we can conclude that $u \in W^{2,p}(\Omega^\prime)$ for any subdomain $\Omega^\prime \subset \subset \Omega$. Fixed $\Omega^\prime \subset \subset \Omega$, we consider the equation

$$L_2 v = 2H \left(1 + |\nabla u|^2\right)^2.$$  (1)
We observe that \( u \in W^{2,2}(\Omega') \). Thus, \( L_2 u \) is well-defined. Moreover, we have that \( L_2 u = 2H (1 + |\nabla u|^2)^{\frac{1}{2}} \) in \( \Omega' \). It means that \( u \in W^{2,2}(\Omega') \) is a solution to the equation (1) just above. Now using the Theorem 9.19 (Gilbarg and Trudinger 1983) we obtain \( u \in C^{2,\alpha}(\Omega') \). Repeating the same procedure we conclude that \( u \in C^\infty(\Omega') \). Thus, \( \tilde{M} \) is \( C^\infty \). So, \( \tilde{M} \) is a regular compact closed surface embedded in \( \mathbb{R}^3 \) with constant mean curvature. By the Theorem 5.2 (Chapter V, (Hopf 1983)) we conclude that \( \tilde{M} \) is a (round) sphere and therefore \( M \) is a hemisphere.

ACKNOWLEDGMENTS

This work is part of my Doctoral Thesis at the Universidade Federal do Ceará – UFC. I want to thank my advisor, Professor J. Lucas Barbosa, and also Professor A. G. Colares, by the encouragement and many helpful conversations. The research was supported by a scholarship of CAPES.

REFERENCES

Brito F and Earp RS. 1991. Geometric Configurations of Constant Mean Curvature Surfaces with Planar Boundary. An Acad Bras Ci 63: 5-19.

Brito F, Earp RS, Meeks W and Rosenberg H. 1991. Structure Theorems for Constant Mean Curvature Surfaces Bounded by a Planar Curve. Indiana Univ Math J 40(1): 333-343.

Gilbarg D and Trudinger NS. 1983. Elliptic Partial Differential Equations of Second Order. 2nd edition, Springer-Verlag, Berlin.

Hopf H. 1983. Differential Geometry in the Large. Lectures Notes in Mathematics, 1000, Springer-Verlag, Berlin.

RESUMO

Consideramos superfícies compactas com curvatura média constante e não nula as quais têm como bordo uma curva de Jordan plana convexa. Provamos que, se uma tal superfície é ortogonal ao plano do bordo então é um hemisfério.

Palavras-chave: superfície com bordo, curvatura média constante, equações diferenciais parciais elípticas.