Deformations of Hyperelliptic and Generalized Hyperelliptic Polarized Varieties

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Abstract. The purpose of this article is twofold. Firstly, we address and completely solve the following question: Let \((X,L)\) be a smooth, hyperelliptic polarized variety and let \(\varphi : X \to Y \subset \mathbb{P}^N\) be the morphism induced by \(|L|\); when does \(\varphi\) deform to a birational map? Secondly, we introduce the notion of “generalized hyperelliptic varieties” and carry out a study of their deformations. Regarding the first topic, we settle the non trivial, open cases of \((X,L)\) being Fano-K3 and of \((X,L)\) having dimension \(m \geq 2\), sectional genus \(g\) and \(L^m = 2g\). This was not addressed by Fujita in his study of hyperelliptic polarized varieties and requires the introduction of new methods and techniques to handle it. In the Fano-K3 case, all deformations of \((X,L)\) are again hyperelliptic except if \(Y\) is a hyperquadric. By contrast, in the \(L^m = 2g\) case, with one exception, a general deformation of \(\varphi\) is a finite birational morphism. This is especially interesting and unexpected because, in the light of earlier results, \(\varphi\) rarely deforms to a birational morphism when \(Y\) is a rational variety, as is our case. The Fano-K3 case contrasts with canonical morphisms of hyperelliptic curves and with hyperelliptic K3 surfaces of genus \(g \geq 3\). Regarding the second topic, we completely answer the question for generalized hyperelliptic polarized Fano and Calabi–Yau varieties. For generalized hyperelliptic varieties of general type we do this in even greater generality, since our result holds for \(Y\) toric. Standard methods in deformation theory do not work in the present setting. Thus, to settle these long standing open questions, we bring in new ideas and techniques building on those introduced by the authors concerning deformations of finite morphisms and the existence and smoothings of certain multiple structures. We also prove a new general result on unobstructedness of morphisms that factor through a double cover and apply it to the case of generalized hyperelliptic varieties.

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Introduction

In this article we study the deformations of morphisms \( \varphi \) to projective space, which are finite and of degree 2 onto their image. Two things may happen: either all deformations of \( \varphi \) are of degree 2 onto their image; or there exist deformations of \( \varphi \) for which the degree changes. One would like to know under what circumstances each situation occurs. In this article we completely settle this question for those smooth polarized hyperelliptic varieties \((X, L)\) of dimension \( m \) such that the image \( Y \) of the morphism \( \varphi \) induced by \(|L|\) is also smooth. Indeed, for these varieties we find out exactly in what cases the degree of \( \varphi \) remains constant under deformation and in what cases the degree changes; in those cases, the general deformation of \( \varphi \) is a finite, birational morphism onto its image. To do this, we completely settle the non trivial, open cases \( L^m = 2g - 2 \) (see Theorems 2.9 and 2.10) and \( L^m = 2g \) (see Theorems 3.6, 3.8 and 3.10), where \( g \) is the sectional genus of \((X, L)\). The situations where a finite, non birational morphism can be deformed to a morphism of degree 1 are important, because they provide a way to construct new subvarieties in projective space with interesting properties and invariants, with further applications to moduli spaces, among other things. Indeed, there have been several works in that direction, especially concerning deformations of canonical double covers of surfaces of general type. In this regard, we recall the works of Horikawa, Catanese, Beauville, Schreyer, Ciliberto–Pardini–Tovena, Ashikaga–Konno and the authors, to mention just a few (see [2,6–9,14,15,22]).

In this article we characterize all smooth hyperelliptic varieties with smooth \( Y \) for which \( \varphi \) can be deformed to a degree 1 morphism. Furthermore, we extend the concept of hyperelliptic varieties and define so-called generalized hyperelliptic varieties (see Definition 0.4). Then we settle the question for those Fano varieties (see Theorem 2.6), Calabi–Yau varieties (see Theorem 4.5) and varieties of general type (see Theorem 5.6) which fit in this new, more general notion of hyperellipticity. For general type we even settle the case of deformations of generalized hyperelliptic covers of smooth toric varieties. Except for one exception, for all of these generalized hyperelliptic varieties of dimension \( m \geq 3 \), the morphism \( \varphi \) always deforms to a morphism of degree 2. This is very interesting, for it contrasts with the situation in dimension 2 (indeed, K3 surfaces behave differently).

The role played by \( H^1(\mathcal{F} \otimes \mathcal{E}) \) in the deformations of varieties \( X \) and of finite, degree 2 morphisms \( \pi \) from \( X \) to \( Y \) is well known and is described in detail in the literature by Fujita, Konno, Seiler, Wavrik or Wheler among others (see [12,24,26,28] or [29]). Precisely, the vanishing of \( H^1(\mathcal{F} \otimes \mathcal{E}) \)
implies that any deformation of $X$ carries a deformation of $\pi$ (see [29, Corollary 1.11]). This criterion has its counterpart for morphisms to projective space (see Theorem 1.7, that follows from [29, Proposition 1.10], and [14, Theorem 2.6]). All this does not address the case $H^1(\mathcal{F} \otimes \mathcal{E}) \neq 0$ or, more generally, the question of under what conditions the degree of $\varphi$ varies under deformation. Thus, to tackle this we need to carry out a study of the deformations of $\varphi$ that does not follow from methods of standard deformation theory. Rather, it fits in a recent paradigm, introduced by the authors in [14], about deformations of morphisms and existence and deformations of multiple structures. Applying this paradigm in the cases dealt with in this article is neither trivial nor straight-forward. The lack of this paradigm in the past might be one of the reasons why the problem of deformations of hyperelliptic polarized varieties has not been settled before. Another ingredient and by-product of our arguments is the proof the unobstructedness of $\varphi$, which we achieve except in one case. In fact, we prove a general, new result on unobstructedness (see Theorem 1.6) of morphisms that factor through a double cover.

The classic situation in which the degree of $\varphi$ changes when we deform $\varphi$ is the canonical morphism of a hyperelliptic curve of genus $g$, $g > 2$. Thus, from the point of view of deformations, moduli and related issues, it is natural to look for a generalization of the notion of hyperellipticity to higher dimensions. There will naturally be more than one analogue of this notion and we study such analogues from the perspective of deformations and moduli in this article. Before introducing a more general notion (see Definition 0.4) of hyperellipticity, we will first deal with a more classic notion, that carries over directly from canonically polarized hyperelliptic curves (compare with [12, Definition 1.1]):

**Definition 0.1.** Let $X$ be a smooth variety and let $L$ be a polarization (i.e., an ample line bundle) on $X$. We say that the polarized variety $(X, L)$ is hyperelliptic if $L$ is base-point-free and the complete linear series $|L|$ induces a morphism of degree 2 onto its image, which is a variety $Y$ embedded in projective space as a variety of minimal degree.

Consider now a deformation $(X_t, L_t)$ of a hyperelliptic polarized variety $(X, L)$. Having in account Definition 0.1 and the fact that ampleness and base-point-freeness are open conditions, we may rephrase the question on deformations of hyperelliptic morphisms as a question on deformations of polarized hyperelliptic varieties:

**Question 0.2.** Given a polarized hyperelliptic variety $(X, L)$ and considering deformations $(X', L')$ of $(X, L) = (X_0, L_0)$ with $h^0(L_t) = h^0(L)$, are all of them hyperelliptic or are there nonhyperelliptic deformations, i.e., such that the morphism induced by $|L_t|$ is of degree 1 for $t \neq 0$?

We will denote by $g$ the **sectional genus** of a polarized variety $(X, L)$ of dimension $m$, i.e., $g$ is the genus of the intersection of $m - 1$ general members of $|L|$. If $L^m = 2g - 2, g > 2$ or $L^m = 2g, g > 1$, then the answer to Question 0.2 was unknown and non trivial. We do give this answer in this
Before proceeding to these non-trivial cases, we recall the previously known, easy-to-answer cases:

**Remark 0.3.** Let \((X, L)\) be a polarized variety of dimension \(m, m > 1\). Except for case (3), the following follows essentially from looking at the restriction of the polarization to a general curve section of \((X, L)\):

1. If \((X, L)\) is hyperelliptic and \(L^m < 2g - 2\), all deformations \((X', L')\) with \(h^0(L')\) constant are hyperelliptic (see e.g. [12, (7.5)]). This follows essentially from Clifford’s theorem.

2. If \((X, L)\) is hyperelliptic, \(L^m = 2g - 2\) and \(g = 2\), then all deformations of \((X, L)\) are hyperelliptic. This is because the morphism induced by \([L]\) is two-to-one onto \(\mathbb{P}^m\), so \(H^1(L) = 0\), hence \(h^0(L') = h^0(L)\) by [20, Chapitre III, §7] (see also [25, Chapter 0, §5]). Since \(L_t^m = L^m = 2\), the claim follows.

3. If \((X, L)\) is hyperelliptic, \(L^m = 2g - 2, g > 2\) and \(m = 2\), then a general deformation of \((X, L)\) is nonhyperelliptic. This is well-known, because in this case \(X\) is a K3 surface.

4. If \((X, L)\) is hyperelliptic, \(L^m = 2g\) and \(g = 1\), then, by Riemann–Roch all deformations of \((X, L)\) are hyperelliptic.

5. If \(L^m > 2g\) and \(X\) is regular, again by Riemann–Roch, \((X, L)\) is nonhyperelliptic.

Now we proceed with the open cases, which are non-trivial. In these cases we can expect the deformations of a hyperelliptic polarized variety \((X, L)\) to be nonhyperelliptic; they are the case \(L^m = 2g - 2, m > 2\) and \(g > 2\) and the case \(L^m = 2g, m > 1\) and \(g > 1\). The case \(L^m = 2g - 2\), which was partially addressed by Fujita, is settled completely in this article when \(Y\) is smooth. The case \(L^m = 2g\) was not studied previously and we also settle it completely when \(Y\) is smooth.

The deformations of (higher dimensional) hyperelliptic Fano-K3 varieties \((X, L)\) (see Definition 2.7) were partially studied in [12, Remark 7.7, Corollary 7.14], where the existence of a hyperelliptic deformation of \((X, L)\) is shown. In Sect. 2 we settle completely the question on the deformations of hyperelliptic Fano-K3 varieties when \(Y\) is smooth. Precisely, we prove that any deformation of a Fano-K3 variety \((X, L)\) is hyperelliptic except if \(Y\) is embedded as a hyperquadric (see Theorems 2.9 and 2.10). This was not previously known. Our result contrasts with the results for surfaces (see Remark 0.3 (3)) and curves which are general sections of \((X, L)\) (those sections are, respectively, polarized hyperelliptic K3 surfaces and canonically polarized hyperelliptic curves).

The case \(L^m = 2g\) was not previously known either and, in Sect. 3, we deal with it completely when \(Y\) is smooth. We prove that, except for one case, (see Theorems 3.6, 3.8, and 3.10) \((X, L)\) deforms to a nonhyperelliptic polarized variety, in clear contrast with Fano-K3 varieties. This provides families of varieties and morphisms \(\varphi\) where the degree of \(\varphi\) varies under deformation. In the view of earlier results, this is unusual when the target of the morphism \(\varphi\) is a rational variety, as is the case here; this makes this case quite relevant (when the target of \(\varphi\) is not rational, the situation differs,
see [3]). If $Y$ is as in (2e) of Proposition 3.2 we overcome an added difficulty regarding the possible obstructedness of $\varphi$ (see the end of the last paragraph of this introduction).

The upshot of all the above is that, in the context of hyperelliptic varieties, the degree of $\varphi$ varying under deformation is rather rare. This compels us to investigate a deeper structural reason for this to happen, by looking at the behaviour under deformation of the double covers of those smooth abstract varieties that can be embedded as varieties of minimal degree. This leads us to introduce another, broader generalization of hyperellipticity:

**Definition 0.4.** Let $(X, L)$ be a smooth polarized variety with $L$ base-point-free. Let the morphism from $X$ to $\mathbb{P}^N$, induced by $|L|$ be of degree 2 onto its image $i(Y)$ ($i$ is the embedding of $Y$ in $\mathbb{P}^N$). Let the variety $Y$ be smooth and isomorphic to any of this:

1. projective space;
2. a hyperquadric;
3. a projective bundle over $\mathbb{P}^1$.

The variety $Y$ is *not necessarily* embedded by $i$ as a variety of minimal degree in $\mathbb{P}^N$. Then we say that $(X, L)$ is a generalized hyperelliptic polarized variety.

In Sect. 5 we extend, for varieties of general type, the concept of generalized hyperelliptic polarized variety even further, as in this case we allow $Y$ to be toric as well. Thus, we also study the deformations of morphisms $\varphi : X \to \mathbb{P}^N$, when $X$ is a (minimal) variety of minimal degree and $\varphi$ is finite of degree 2 onto its image $i(Y)$, which is a smooth toric variety. In Sects. 2, 4 and 5 we show that, but for one exception, the deformations of the morphism $\varphi$ associated to generalized hyperelliptic Fano, Calabi–Yau and general type polarized varieties are morphisms of degree 2 (see Theorems 2.6, 4.5 and 5.6; the last one holds also for $Y$ toric). In particular, the results on Calabi–Yau varieties are in sharp contrast to the results for their lower dimensional analogue, namely, K3 surfaces.

We study the deformations of $\varphi$ by proving the existence or non existence of certain double structures called *ribbons* (see Definition 1.9), mapped to $\mathbb{P}^N$ onto possibly non locally Cohen-Macaulay double structures, and by deforming them. In those instances when $\varphi$ can be deformed to a degree 1 morphism, the existence of these ribbons is crucial in proving so (see Corollary 2.4 and Remark 3.5 (1)). Actually, in our arguments, being able to deform $\varphi$ to a degree 1 morphism amounts to being able to deform those double structures in $\mathbb{P}^N$ to non reduced ones. On the contrary, those cases in which any deformation of $\varphi$ is of degree 2 are the ones in which ribbons do not exist (see Corollaries 2.4, 4.4 and 5.4 and Remark 3.5 (2)). The contrast mentioned above between morphism deformation results for Calabi–Yau varieties and K3 surfaces can also be traced to the question of the existence of double structures. Indeed, in [17] and [4] it is shown the existence of K3 carpets (i.e., ribbons with the same invariants as smooth K3 surfaces), supported on rational normal scrolls and, more generally, on general embeddings of Hirzeburch
surfaces and other rational surfaces. Contrarily, here we prove that higher dimensional analogues of K3 carpets do not exist. Precisely, we show (see Corollary 4.4) that there are no non split ribbons with Calabi–Yau invariants supported on varieties \( Y \) as in Definition 0.4. Similarly, in Corollaries 2.4 and 5.4 we prove (with one exception, see Example 2.11), that Fano and general type non split ribbons of dimension bigger than 1, supported on varieties \( Y \) as in Definition 0.4, do not exist. We also prove the non existence of ribbons of general type supported on smooth toric varieties.

In Sect. 1, we prove a new general result on unobstructedness (see Theorem 1.6) of morphisms that factor through a double cover. To handle situations when \( \varphi \) can be deformed to a birational morphism, one usually requires \( \varphi \) to be unobstructed. We prove it is so in all the cases but one. In that special case (see Proposition 3.3) it is not clear whether the obstruction space \( H^1(\mathcal{N}_\varphi) \) vanishes, nor if \( \varphi \) is unobstructed (see Question 3.9). Despite the ensuing difficulty, we find a way to overcome this obstacle and succeed in studying the deformations of \( \varphi \) (see the proof of Theorem 3.8).

1. General Results on Deformations of Morphisms

In this section we give some general results on deformations of morphisms, that we will apply in the remaining of the article. First we make clear what we will mean when we talk of a deformation of a morphism:

**Definition 1.1.** Let \( X \) be an algebraic, projective variety and let

\[
\varphi : X \longrightarrow \mathbb{P}^N
\]

be a morphism. By a deformation of \( \varphi \) we mean a flat family of morphisms

\[
\Phi : \mathcal{X} \longrightarrow \mathbb{P}^N_Z
\]

over a smooth, irreducible algebraic variety \( Z \) (i.e., \( \Phi \) is a \( Z \)-morphism for which \( \mathcal{X} \longrightarrow Z \) is proper, flat and surjective) with a distinguished point \( 0 \in Z \) such that

1. \( \mathcal{X} \) is irreducible and reduced;
2. \( \mathcal{X}_0 = X \) and \( \Phi_0 = \varphi \).

Unless otherwise specified, when we say that certain property (*) is satisfied by a deformation \( \Phi \), we will mean that there exists an open neighborhood \( U \) of 0 such that, for all \( z \in U \), property (*) is satisfied by the fibers

\[
\Phi_z : \mathcal{X}_z \longrightarrow \mathbb{P}^N
\]

of \( \Phi \).

That certain property (*) is satisfied by a general deformation will mean that there exists an open set \( \tilde{U} \) of the base of an algebraic formally semiuniversal deformation \( \Phi \) of \( \varphi \) such that, for all \( z \in \tilde{U} \), the morphism \( \Phi_z \) satisfies (*).

We will use analogous definitions for a deformation of a variety \( X \) or for a deformation of a polarized variety \( (X, L) \).
1.2 Notation and Conventions

Throughout this article, unless otherwise stated, we will use the following notation and conventions:

(1) We will work over an algebraically closed field $k$ of characteristic 0.
(2) $X$ and $Y$ will denote smooth, irreducible, algebraic projective varieties.
(3) $\omega_X$ and $\omega_Y$ denote respectively the canonical bundle of $X$ and $Y$; $K_X$ and $K_Y$ denote respectively the canonical divisor of $X$ and $Y$. We use the same notation for all the other varieties that appear.
(4) $\pi$ will denote a finite morphism $\pi : X \rightarrow Y$ of degree $n = 2$; in this case, $E$ will denote the trace-zero module of $\pi$ ($E$ is a line bundle on $Y$) and $B$ will be the branch divisor of $\pi$ (then $E - 2 = O_Y(B)$).
(5) $i$ will denote a projective embedding $i : Y \hookrightarrow P^N$. In this case, $I$ will denote the ideal sheaf of $i(Y) \subseteq P^N$. We will often abridge $i_* O_{P^N}(1)$ as $O_Y(1)$, $E \otimes i_* O_{P^N}(1)$ as $E(1)$, etc.
(6) $\varphi$ will denote a projective morphism $\varphi : X \rightarrow P^N$ such that $\varphi = i \circ \pi$.

Remark 1.3. The morphism $\varphi$ is induced by a complete linear series if and only if $H^0(E(1)) = 0$.

Remark 1.4. It is well known (see e.g. [14, (2.3)]) that the normal sheaf $N_{\pi}$ of $\pi$ is isomorphic to $O_B$.

We introduce a homomorphism defined in [19, Proposition 3.7]:

Proposition 1.5. Let $N_{\pi}$ and $N_{\varphi}$ be respectively the normal bundles of $\pi$ and $\varphi$ and let $N_{i(Y), P^N}$ be the normal bundle of $i(Y)$ in $P^N$. There exists a homomorphism

$$H^0(N_{\varphi}) \xrightarrow{\Psi} \text{Hom}(\pi^*(\mathcal{I}/\mathcal{I}^2), \mathcal{O}_X),$$

that appears when taking cohomology on the exact sequence

$$0 \rightarrow N_{\pi} \rightarrow N_{\varphi} \rightarrow \pi^* N_{i(Y), P^N} \rightarrow 0,$$

(1.1)

that arises from commutative diagram [19, (3.3.2)]. Since

$$\text{Hom}(\pi^*(\mathcal{I}/\mathcal{I}^2), \mathcal{O}_X) = \text{Hom}(\mathcal{I}/\mathcal{I}^2, \pi_* \mathcal{O}_X)$$

$$= \text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y) \oplus \text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{E}),$$

the homomorphism $\Psi$ has two components

$$H^0(N_{\varphi}) \xrightarrow{\Psi_1} \text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y)$$

$$H^0(N_{\varphi}) \xrightarrow{\Psi_2} \text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{E}).$$

Taking cohomology on (1.1), the homomorphism $\Psi$ fits in this long exact sequence of cohomology:

$$H^0(N_{\varphi}) \xrightarrow{\Psi} \text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y) \oplus \text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{E}) \xrightarrow{\epsilon}$$

$$H^1(N_{\pi}) \xrightarrow{\eta} H^1(N_{\varphi}) \rightarrow \text{Ext}^1(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y) \oplus \text{Ext}^1(\mathcal{I}/\mathcal{I}^2, \mathcal{E}).$$

(1.2)

The following two results give sufficient conditions for $\varphi$ to be unobstructed and for the deformations of $\varphi$ to be morphisms of degree 2 onto their image:
Theorem 1.6. With the notation of 1.2, assume furthermore that

1. the line bundle \( \mathcal{E} \) can be lifted to a line bundle on any infinitesimal deformation of \( Y \) (or, equivalently, the map \( H^1(\mathcal{O}_Y) \to H^2(\mathcal{O}_Y) \) induced by the cohomology class \( c_1(\mathcal{E}) \in H^1(\Omega_Y) \) via cup product and duality is zero; \( \Omega_Y \) is the cotangent bundle of \( Y \), see [27, Theorem 3.3.11 (iii)];
2. \( H^1(\mathcal{E}^{-2}) = 0 \);
3. \( H^1(\mathcal{O}_Y) = 0 \);
4. the subvariety \( i(Y) \) is unobstructed in \( \mathbb{P}^N \); and
5. \( \Psi_2 = 0 \).

Then \( \varphi \) is unobstructed.

Proof. Let \( \text{Def}_{(Y,B)} \) be the functor of deformations of the pair \( (Y \subset \mathbb{P}^N, B \in |\mathcal{E}^{-2}|) \).

Since \( H^1(\mathcal{O}_Y) = 0 \), for any embedded infinitesimal deformation \( Y \subset \tilde{Y} \subset \tilde{Y} \) we see that \( \text{Pic}(\tilde{Y}) \hookrightarrow \text{Pic}(\tilde{Y}) \hookrightarrow \text{Pic}(Y) \). Then, from (1) we see that for any \( Y \subset \tilde{Y} \) there is a unique lifting \( (\tilde{Y}, \tilde{\mathcal{E}}) \) of \( (Y, \mathcal{E}) \). Then, by uniqueness, for the liftings \( (\tilde{Y}, \tilde{\mathcal{E}}) \) and \( (\tilde{Y}, \tilde{\mathcal{E}}') \) of \( (Y, \mathcal{E}) \) we have \( \tilde{\mathcal{E}}_Y = \tilde{\mathcal{E}}'. \) Thus we see that the local Hilbert functor of deformations of \( B \in Y \) is a subfunctor of \( \text{Def}_{(Y,B)} \).

Let \( H^P_Y \) denote the local Hilbert functor of \( Y \) in \( \mathbb{P}^N \). Then for any small extension \( \tilde{A} \to A \) and elements \( \tilde{Y} \in H^P_Y(\tilde{A}), (\tilde{Y}, \tilde{B}) \in \text{Def}_{(Y,B)}(A) \) such that \( \tilde{Y} \times \text{Spec} A = \tilde{Y} \), we have an exact sequence

\[
0 \to \mathcal{E}^{-2} \to \tilde{\mathcal{E}}^{-2} \to \tilde{\mathcal{E}}^{-2} \to 0
\]

and then an exact sequence

\[
H^0(\tilde{\mathcal{E}}^{-2}) \to H^0(\tilde{\mathcal{E}}^{-2}) \to H^1(\tilde{\mathcal{E}}^{-2}) = 0.
\]

Therefore \( \tilde{B} \) can be lifted to \( \tilde{B} \) such that \( (\tilde{Y}, \tilde{B}) \in \text{Def}_{(Y,B)}(\tilde{A}) \).

Since \( Y \subset \mathbb{P}^N \) is unobstructed, the map \( H^P_Y(\tilde{A}) \to H^P_Y(\tilde{A}) \) is surjective. Therefore the argument above shows that the map

\[
\text{Def}_{(Y,B)}(\tilde{A}) \to \text{Def}_{(Y,B)}(A)
\]

is surjective, which shows that the functor \( \text{Def}_{(Y,B)} \) and the forgetful map \( \text{Def}_{(Y,B)} \to H^P_Y \) are smooth. Therefore we have an exact sequence on tangent spaces

\[
0 \to H^0(\mathcal{O}_B(B)) \to \text{Def}_{(Y,B)}(k[\mathcal{E}]) \to H^0(\mathcal{N}_{Y,P^N}) \to 0. \quad (1.3)
\]

One can show that \( \text{Def}_{(Y,B)} \) has a semiuniversal formal element by checking Schlessinger’s conditions (see e.g. [27, Theorem 2.3.2]).

There is a map

\[
\text{Def}_{(Y,B)} \xrightarrow{F} \text{Def}_\varphi \quad (1.4)
\]

defined as follows. Let \( (\tilde{Y}, \tilde{B}) \) be an element in \( \text{Def}_{(Y,B)}(A) \), where \( \tilde{B} = (\tilde{r})_0 \), with \( \tilde{r} \in H^0(\tilde{\mathcal{E}}^{-2}) \) lifting \( r \in H^0(\mathcal{E}^{-2}) \) such that \( B = (r)_0 \). Then, on the
total space of $\bar{\mathcal{E}}^{-1}$, there is a tautological section $\bar{t}$ lifting the tautological section on the total space of $\mathcal{E}^{-1}$. Then $(\bar{X}, \bar{\varphi}) \in \text{Def}_\varphi(A)$ is given by

$$(\bar{t}^2 - \bar{r})_0 = \bar{X} \xrightarrow{\bar{\pi}} \bar{\mathcal{E}}^{-1} \xrightarrow{\bar{\varphi}} Y,$$

and

$$\bar{\varphi} : \bar{X} \xrightarrow{\bar{\pi}} \bar{Y} \hookrightarrow \mathbb{P}_A^N.$$ Then $\bar{X}$ is flat over $A$ and $(\bar{X}, \bar{\varphi})$ is a lifting of $(X, \varphi)$.

Recall that $\text{Def}_\varphi(k[\epsilon]) = H^0(\mathcal{N}_\varphi)$ and the long exact sequence of cohomology (1.2). Since the restriction of $\pi$ becomes an isomorphism between the ramification divisor and the branch locus $B$, this isomorphism identifies $\mathcal{O}_B(B) = \mathcal{N}_\pi$.

Let $(\bar{Y}, \bar{B}) \in \text{Def}_{Y,B}(k[\epsilon])$ be a first order deformation of $(Y \subset \mathbb{P}^N, B \in |\mathcal{E}^{-2}|)$ and $(\bar{X}, \bar{\varphi}) \in H^0(\mathcal{N}_\varphi)$ be the first order deformation of $(X, \varphi)$ associated to $(\bar{Y}, \bar{B})$ by $d\bar{F}$. From the construction we made for $\bar{F}$, we see that $\bar{X} \xrightarrow{\bar{\varphi}} \mathbb{P}_k^N$ factors through $\bar{Y} \hookrightarrow \mathbb{P}_k^N$. Therefore $\text{im} \bar{\varphi} = \bar{Y}$. Then, using [19, Theorem 3.8 (2) and Propositions 3.11, 3.12], we see that there is a commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & H^0(\mathcal{O}_B(B)) \xrightarrow{\sim} H^0(\mathcal{N}_\pi) \\
\downarrow & & \downarrow \\
\text{Def}_{Y,B}(k[\epsilon]) & \xrightarrow{d\bar{F}} & H^0(\mathcal{N}_\varphi) \\
\downarrow_{d\nu} & & \downarrow_{\Psi_1 \oplus \Psi_2} \\
0 & \longrightarrow & H^0(\mathcal{N}_{Y/\mathbb{P}^N}) \longrightarrow H^0(\mathcal{N}_{Y/\mathbb{P}^N} \oplus H^0(\mathcal{N}_{Y/\mathbb{P}^N} \otimes \mathcal{E}^*) \\
\downarrow & & \downarrow \\
0 & & 0 \\
\end{array}
$$

(1.5)

We see, from diagram (1.5), that if $\Psi_2 = 0$, then $d\bar{F}$ is an isomorphism.

Since $\text{Def}_{Y,B}$ and $\text{Def}_\varphi$ have a semiuniversal formal element, $\text{Def}_{Y,B}$ is smooth and $d\bar{F}$ is an isomorphism it follows that $\text{Def}_\varphi$ and $\bar{F}$ are smooth. This completes the proof. \hfill $\square$

Next result follows from [29, Proposition 1.10]. The details of the proof are in [3, Section 2] and we include them here for the sake of completeness.
Theorem 1.7. With the notation of 1.2, if Hom(\mathcal{I}/\mathcal{I}^2, \mathcal{E}) = 0, then the deformations of \phi are morphisms of degree 2 onto their image.

Proof. We apply [29, Proposition 1.10] to this commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & Y \\
\downarrow{\phi} & & \downarrow{\iota} \\
P^N & & 
\end{array}
\]

The maps \beta_1 and \beta_2 of [29, Proposition 1.10] become \[\beta_1 : H^0(\mathcal{I}_i(Y), \mathcal{P}^N) \rightarrow H^0(\pi^*\mathcal{I}_i(Y), \mathcal{P}^N), \quad \text{and} \quad \beta_2 : H^1(\mathcal{I}_i(Y), \mathcal{P}^N) \rightarrow H^1(\pi^*\mathcal{I}_i(Y), \mathcal{P}^N).\]

The assertion follows since the map \beta_2 is always injective and \beta_1 is surjective if \[H^0(\mathcal{I}_i(Y), \mathcal{P}^N \otimes \mathcal{E}) = 0.\] □

Proposition 1.8. Let \mathcal{X} be a smooth, algebraic projective variety, let \phi : \mathcal{X} \rightarrow P^N be a morphism and let L be a polarization on \mathcal{X}. If \[H^2(\mathcal{O}_\mathcal{X}) = 0,\] then \phi and (\mathcal{X}, L) have an algebraic formally semiuniversal deformation.

Proof. Since \mathcal{X} is a projective variety, it has a formal versal deformation (see e.g. [11] or [30]). Since \[H^2(\mathcal{O}_\mathcal{X}) = 0,\] then, by Grothendieck’s existence theorem (see [27, Theorem 2.5.13]), this formal versal deformation of \mathcal{X} is effective. It follows from general deformation theory the existence of an algebraic formally versal (even semiuniversal) deformation of \phi (see [27, Theorem 3.4.8]). In this case, the formal semiuniversal deformation of \phi is also effective, so it is algebraizable by Artin’s algebraization theorem (see [1]).

By [27, Theorem 3.3.11.(i)], the functor Def_{(\mathcal{X}, L)} has a semiuniversal formal element. Then arguing as above, this element is algebraizable, so there exists an algebraic formally semiuniversal deformation for (\mathcal{X}, L). □

We end this section by giving the definitions of ribbon and split ribbon (for more details, see [5, §1]):

Definition 1.9. Let \mathcal{Y} be as in Notation 1.2 and let \widetilde{\mathcal{E}} be a line bundle on \mathcal{Y}.

(1) A ribbon supported on \mathcal{Y} with conormal bundle \widetilde{\mathcal{E}} is a scheme \widetilde{\mathcal{Y}} with \[\widetilde{\mathcal{Y}}_{\text{red}} = \mathcal{Y} \text{ such that}\]

(i) \[\mathcal{J}_{\mathcal{Y}, \widetilde{\mathcal{Y}}}^2 = 0,\]

(ii) \[\mathcal{J}_{\mathcal{Y}, \widetilde{\mathcal{Y}}} \simeq \widetilde{\mathcal{E}} \text{ as } \mathcal{O}_\mathcal{Y}\text{-modules}.\]

(2) A ribbon \widetilde{\mathcal{Y}} as above is split if the inclusion \mathcal{Y} \hookrightarrow \widetilde{\mathcal{Y}} has a retraction \[\widetilde{\mathcal{Y}} \longrightarrow \mathcal{Y}.\]

2. Deformations of Hyperelliptic Fano-K3 Varieties and Generalized Hyperelliptic Polarized Fano Varieties

Notation 2.1. Unless otherwise stated, in the remaining of this article the variety \mathcal{Y} has dimension \(m, m \geq 2,\) and is isomorphic to a variety of one of these three types:
(i) \( P^m \).

(ii) A smooth hyperquadric of dimension \( m \geq 3 \); in this case, let \( \mathcal{E} \) be the restriction of the hyperplane section of \( P^{m+1} \) to \( Y \).

(iii) A (smooth) projective bundle on \( P^1 \).

In order to study the deformations of hyperelliptic Fano-K3 varieties (see Definition 2.7) we need first to carry out certain cohomology computations. We do them in the broader setting of generalized hyperelliptic polarized Fano varieties.

**Proposition 2.2.** Let \( X \), \( Y \) and \( \mathcal{E} \) be as in Notations 1.2 and 2.1, let \( \mathcal{T}_Y \) be the tangent bundle of \( Y \) and assume \( X \) is Fano (i.e., \( -K_X \) is ample). If \( Y \) is a hyperquadric, let \( \mathcal{E} \neq \mathcal{O}_Y(-\mathfrak{h}) \) and \( \mathcal{E} \neq \mathcal{O}_Y(-2\mathfrak{h}) \). Then \( h^1(\mathcal{T}_Y \otimes \mathcal{E}) = 0 \).

**Proof.** Case 1: \( Y = P^m \). If \( m > 2 \), then the vanishing of \( H^1(\mathcal{T}_Y \otimes \mathcal{E}) \) follows from suitably twisting and taking cohomology on the Euler sequence of the tangent bundle of \( P^m \) and by the vanishing of the intermediate cohomology of line bundles on \( P^m \). If \( m = 2 \), we also need to check that \( H^2(\mathcal{E}) = 0 \). Recall \( \omega_X = \pi^*(\omega_Y \otimes \mathcal{E}^{-1}) \). Since \( -K_X \) is ample, so is \( \pi_Y^{-1} \otimes \mathcal{E} \). Then the degree of \( \mathcal{E} \) is greater than or equal to \(-2 \), so \( H^2(\mathcal{E}) \) indeed vanishes.

Case 2: \( Y \) is a hyperquadric. By Lefschetz hyperplane theorem the Picard group of \( Y \) is generated by \( \mathcal{O}_Y(\mathfrak{h}) \). Then \( \mathcal{E} = \mathcal{O}_Y(\delta \mathfrak{h}) \), with \( \delta \) a negative integer, for \( X \) is connected. Then, by hypothesis \( \delta \leq -3 \). Consider the exact sequence

\[
0 \longrightarrow \mathcal{T}_Y \otimes \mathcal{E} \longrightarrow \mathcal{T}_{P^{m+1}}|_Y \otimes \mathcal{E} \longrightarrow \mathcal{O}_Y(2\mathfrak{h}) \otimes \mathcal{E} \longrightarrow 0, \quad (2.1)
\]

obtained by tensoring the normal sequence of \( Y \) in \( P^{m+1} \) by \( \mathcal{E} \). We want to see

\[
H^1(\mathcal{T}_{P^{m+1}}|_Y \otimes \mathcal{E}) = 0. \quad (2.2)
\]

Consider the exact sequence

\[
0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_Y(\mathfrak{h})^{m+2} \otimes \mathcal{E} \longrightarrow \mathcal{T}_{P^{m+1}}|_Y \otimes \mathcal{E} \longrightarrow 0, \quad (2.3)
\]

which is obtained by restricting to \( Y \) the Euler sequence of the tangent bundle of \( P^{m+1} \) and tensoring with \( \mathcal{E} \). Taking cohomology on (2.3) we get

\[
H^1(\mathcal{O}_Y((\delta + 1)\mathfrak{h}))^{m+2} \longrightarrow H^1(\mathcal{T}_{P^{m+1}}|_Y \otimes \mathcal{E}) \longrightarrow H^2(\mathcal{O}_Y(\delta \mathfrak{h})).
\]

Then we need \( H^1(\mathcal{O}_Y((\delta + 1)\mathfrak{h})) = 0 \) and

\[
H^2(\mathcal{O}_Y(\delta \mathfrak{h})) = 0. \quad (2.4)
\]

Both vanishings follow from suitably twisting and taking cohomology on the exact sequence

\[
0 \longrightarrow \mathcal{O}_{P^{m+1}}(-2) \longrightarrow \mathcal{O}_{P^{m+1}} \longrightarrow \mathcal{O}_Y \longrightarrow 0, \quad (2.5)
\]

because of the vanishing of the intermediate cohomology of line bundles on \( P^{m+1} \) (recall \( m + 1 \geq 4 \)).

Now we study \( H^0(\mathcal{E} \otimes \mathcal{O}_Y(2\mathfrak{h})) = H^0(\mathcal{O}_Y((\delta + 2)\mathfrak{h})) \). If \( \delta \leq -3 \), then \( H^0(\mathcal{O}_Y((\delta + 2)\mathfrak{h})) = 0 \). This together with (2.1) and (2.2) yields \( H^1(\mathcal{T}_Y \otimes \mathcal{E}) = 0 \) if \( Y \) is a hyperquadric.

Case 3: \( Y \) is a projective bundle on \( P^1 \). We use the following notation:
(i) Let $Y = \mathbf{P}(E_0)$ with $E_0$ normalized, and let
\[ E_0 = \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(-e_{m-1}), \] (2.6)
with $0 \leq e_1 \leq \cdots \leq e_{m-1}$.

(ii) Let $H_0$ be the divisor on $Y$ such that $\mathcal{O}_Y(H_0) = \mathcal{O}_{\mathbf{P}(E_0)}(1)$.

(iii) Let $p$ be the structure morphism from $Y$ to $\mathbf{P}^1$ and let $F$ be a fiber of $p$.

(iv) Let $B = 2\alpha H_0 + 2\beta F$.

Recall $K_Y = -mH_0 - (e_1 + \cdots + e_{m-1} + 2)F$. Recall $\omega_X = \pi^*(\omega_Y \otimes \mathcal{O}^{-1})$.

Since $\omega_X^1$ is ample, so is $\omega_Y^1 \otimes \mathcal{O}$, therefore $\alpha < m$.

Let $\mathcal{F}_Y\mathcal{P}_1$ be the relative tangent bundle to $p$ and consider the exact sequence
\[ 0 \rightarrow \mathcal{F}_Y\mathcal{P}_1 \rightarrow \mathcal{F}_Y \rightarrow p^* \mathcal{F}_P \rightarrow 0 \] (2.7)
and the relative Euler sequence
\[ 0 \rightarrow \mathcal{O}_Y \rightarrow p^*E^\prime_0 \otimes \mathcal{O}_Y(H_0) \rightarrow \mathcal{F}_Y\mathcal{P}_1 \rightarrow 0. \] (2.8)

First we argue for $\alpha > 1$. We see that $H^1(p^* \mathcal{F}_P \otimes \mathcal{O}) = H^1(\mathcal{O}_Y(\alpha H_0 + (2 - \beta)F)) = 0$.
Indeed,
\[ H^1(\mathcal{O}_{\mathbf{P}^1}(-\alpha)) = \cdots = H^{m-2}(\mathcal{O}_{\mathbf{P}^1}(-\alpha)) = H^{m-1}(\mathcal{O}_{\mathbf{P}^1}(-\alpha) = 0, \]

because of the vanishing of the intermediate cohomology of $\mathbf{P}^1$ and, for the topmost cohomology, because $\alpha \leq m - 1$. Then $R^0p_* \mathcal{O}(\alpha H_0 + (2 - \beta)F)) = 0$ for all $i > 1$, so, by the Leray’s spectral sequence, $H^1(\mathcal{O}_Y(\alpha H_0 + (2 - \beta)F)) = H^1(p_* \mathcal{O}_Y(\alpha H_0 + (2 - \beta)F)) = 0$.

On the other hand, $H^0(\mathcal{O}_{\mathbf{P}^1}(-\alpha)) = 0$ for $\alpha > 0$, so $p_* \mathcal{O}_Y(\alpha H_0 + (2 - \beta)F) = 0$, so $H^1(p_* \mathcal{O}_Y(\alpha H_0 + (2 - \beta)F) = 0$.

We see that $H^1(p^*E_0^\prime \otimes \mathcal{O}_Y(H_0) \otimes \mathcal{O}) = 0$. Indeed, $p^*E_0^\prime \otimes \mathcal{O}_Y(H_0) \otimes \mathcal{O}$ is a direct sum of $m$ line bundles on $Y$, each of them of the form $\mathcal{O}_Y((1 - \alpha)H_0 + \delta_j F)$, for certain $\delta_j \in \mathbb{Z}$, $1 \leq j \leq m$. Since $\alpha > 1$, we can argue as above. We see that $H^2(\mathcal{O}) = H_2(\mathcal{O}_Y(\alpha H_0 - \beta F)) = 0$ if $\alpha > 0$ using the same arguments. Then exact sequences (2.7) and (2.8) and the vanishing of $H^1(p^* \mathcal{F}_P \otimes \mathcal{O})$, $H^1(p^*E_0^\prime \otimes \mathcal{O}_Y(H_0) \otimes \mathcal{O})$ and $H^2(\mathcal{O})$ imply $H^1(\mathcal{F}_Y \otimes \mathcal{O}) = 0$.

Now we argue for $\alpha = 1$. Note that we have already proved $H^1(p^* \mathcal{F}_P \otimes \mathcal{O}) = H^2(\mathcal{O}) = 0$ if $\alpha > 0$. Thus we only need to compute $H^1(p^*E_0^\prime \otimes \mathcal{O}_Y(H_0) \otimes \mathcal{O})$. Recall
\[ \omega_Y^1 \otimes \mathcal{O} = \mathcal{O}_Y((m - 1)H_0 + (e_1 + \cdots + e_{m-1} + 2 - \beta)F). \]

is ample. This is equivalent to $e_1 + \cdots + e_{m-1} + 2 - \beta > (m - 1)e_{m-1}$. Then
\[(m - 1)e_{m-1} + 2 - \beta > (m - 1)e_{m-1}, \text{ so } \beta < 2. \]

On the other hand
\[ H^1(p^*E_0^\prime \otimes \mathcal{O}_Y(H_0) \otimes \mathcal{O}) = H^1(\mathcal{O}_Y(\beta F)) \oplus H^1(\mathcal{O}_Y((e_1 - \beta)F)) \]
\[ \oplus \cdots \oplus H^1(\mathcal{O}_Y((e_{m-1} - \beta)F)) \]
\[ = H^1(\mathcal{O}_{\mathbf{P}^1}(-\beta)) \oplus H^1(\mathcal{O}_{\mathbf{P}^1}(e_1 - \beta)) \oplus \cdots \oplus H^1(\mathcal{O}_{\mathbf{P}^1}(e_{m-1} - \beta)) = 0 \]
if $\beta < 2$. Then exact sequences (2.7) and (2.8) and the vanishing of $H^1(p^* \mathcal{F}_P \otimes \mathcal{O})$, $H^1(p^*E_0^\prime \otimes \mathcal{O}_Y(H_0) \otimes \mathcal{O})$ and $H^2(\mathcal{O})$ imply $H^1(\mathcal{F}_Y \otimes \mathcal{O}) = 0$. 

Finally we argue for $\alpha = 0$. Now
\[ \omega_{Y}^{-1} \otimes \mathcal{E} = \mathcal{O}_{Y}(mH_{0} + (e_{1} + \cdots + e_{m-1} + 2 - \beta)F) \]
is ample. This is equivalent to $e_{1} + \cdots + e_{m-1} + 2 - \beta > me_{m-1}$. Then $(m-1)e_{m-1} + 2 - \beta > me_{m-1}$, so $\beta < 2 - e_{m-1}$. In addition, $\beta \neq 0$, otherwise $X$ would be disconnected. Then $1 \leq \beta < 2 - e_{m-1}$, so $e_{m-1} = 0$ and $\beta = 1$. Then $H^{1}(p^{*}\mathcal{F}_{P^{1}} \otimes \mathcal{E}) = H^{1}(F) = 0$, $H^{2}(\mathcal{E}) = H^{2}(-F) = 0$ and $H^{1}(p^{*}E^{0}_{\mathcal{Y}} \otimes \mathcal{O}_{Y}(H_{0}) \otimes \mathcal{E}) = H^{1}(\mathcal{O}_{Y}(H_{0} - F))^{m} = H^{1}(\mathcal{O}_{P^{1}}(-1))^{m} = 0$. Then exact sequences (2.7) and (2.8) and the vanishing of $H^{1}(p^{*}\mathcal{F}_{P^{1}} \otimes \mathcal{E})$, $H^{1}(p^{*}E^{0}_{\mathcal{Y}} \otimes \mathcal{O}_{Y}(H_{0}) \otimes \mathcal{E})$ and $H^{2}(\mathcal{E})$ imply $H^{1}(\mathcal{Y} \otimes \mathcal{E}) = 0$. \(\square\)

We now define Fano ribbons and study the existence or nonexistence of Fano ribbons which are not split, on varieties $Y$ as in Notation 2.1.

**Definition 2.3.** Let $\tilde{Y}$ be a ribbon as in Definition 1.9. We say that $\tilde{Y}$ is a Fano ribbon if $h^{0}(\mathcal{O}_{Y}) = 1$ and the dual of its dualizing sheaf is ample.

**Corollary 2.4.** Let $Y$ be as in Notation 2.1. There are no nonsplit Fano ribbons $\tilde{Y}$ on $Y$ except if $Y$ is a hyperquadric embedded in $P^{m+1}$ and $\tilde{Y}$ is the unique ribbon embedded in $P^{m+1}$ and supported on $Y$.

**Proof.** Let $\tilde{Y}$ be a Fano ribbon supported on $Y$ and let $\tilde{\mathcal{E}}$ be the conormal bundle of $Y$ in $\tilde{Y}$. Since $\omega_{Y}^{-1}$ is ample, by [13, Lemma 1.4], so is $\omega_{Y}^{-1} \otimes \tilde{\mathcal{E}}$, so $\tilde{\mathcal{E}}$ can be thought as the trace-zero module of a cover $\pi : X \to Y$ as in Proposition 2.2, except for the fact that $|\tilde{\mathcal{E}}|$ might not contain a smooth divisor. Since we do not use this property of $\mathcal{E}$ in the proof of Proposition 2.2 and since $h^{0}(\mathcal{O}_{Y}) = 1$ translates into $X$ being connected, it follows from Proposition 2.2 that $\text{Ext}^{1}(\Omega_{Y}, \tilde{\mathcal{E}}) = 0$ except maybe if $Y$ is a hyperquadric and $\tilde{\mathcal{E}} = \mathcal{O}_{Y}(-2\mathfrak{h})$ or $\tilde{\mathcal{E}} = \mathcal{O}_{Y}(-\mathfrak{h})$. In the former case, it follows from suitably twisting and taking cohomology on (2.3) and (2.5) that
\[ H^{0}(\mathcal{F}_{P^{m+1}}|_{Y} \otimes \tilde{\mathcal{E}}) = 0. \tag{2.9} \]
Since $h^{0}(\mathcal{O}_{Y}(2\mathfrak{h}) \otimes \tilde{\mathcal{E}}) = h^{0}(\mathcal{O}_{Y}) = 1$, it follows from (2.1), (2.2) (which holds also for $\delta = -2$) and (2.9) that $h^{1}(\mathcal{Y} \otimes \tilde{\mathcal{E}}) = 1$ in this case. In the latter case, $H^{0}(\mathcal{F}_{P^{m+1}} \otimes \tilde{\mathcal{E}}) = H^{0}(\mathcal{O}_{Y})^{[m+2]}$ and (2.2) also holds if $\delta = -1$. Thus, taking cohomology on (2.1), we see that $H^{1}(\mathcal{Y} \otimes \tilde{\mathcal{E}})$ is the cokernel of the map
\[ H^{0}(\mathcal{O}_{Y}(\mathfrak{h})) \otimes H^{0}(\mathcal{O}_{Y}) \to H^{0}(\mathcal{O}_{Y}(\mathfrak{h})) \]
of multiplication of global sections, which is, trivially, an isomorphism. Thus $\text{Ext}^{1}(\Omega_{Y}, \tilde{\mathcal{E}}) = 0$ if $Y$ is a hyperquadric and $\tilde{\mathcal{E}} = \mathcal{O}_{Y}(-\mathfrak{h})$. Then the result follows from [5, Corollary 1.4]. \(\square\)

**Corollary 2.5.** Let $X$, $Y$, $\varphi$, $\mathcal{E}$ and $\mathcal{I}$ be as in Notations 1.2 and 2.1. Assume $X$ is Fano and $\varphi$ is induced by a complete linear series. Then $\text{Hom}(\mathcal{I} / \mathcal{I}^{2}, \mathcal{E}) = 0$, except if $Y$ is a hyperquadric and $\mathcal{E} = \mathcal{O}_{Y}(-2\mathfrak{h})$, in which case $\text{Hom}(\mathcal{I} / \mathcal{I}^{2}, \mathcal{E})$ has dimension 1.
Proof. Taking cohomology on the conormal sequence of \( i(Y) \) in \( P^N \) we get
\[
\operatorname{Hom}(\Omega_{P^N | i(Y)}, \mathcal{E}) \longrightarrow \operatorname{Hom}(\mathcal{I} / \mathcal{I}^2, \mathcal{E}) \longrightarrow \operatorname{Ext}^1(\Omega_Y, \mathcal{E}) \longrightarrow \operatorname{Ext}^1(\Omega_{P^N | i(Y)}, \mathcal{E}).
\]
(2.10)

We are going to see \( H^0(\mathcal{I}) \) is a hyperquadric, then the result follows from (2.10) and Proposition 2.2. If \( Y \) is a hyperquadric, since \( H^0(\mathcal{E}(1)) = 0 \), then \( \mathcal{E} = \mathcal{O}_Y(\mathcal{E}) \) with \( \delta \leq -2 \). If \( \delta \leq -3 \), then the result follows from (2.10) and Proposition 2.2.

If \( Y \) is a hyperquadric and \( \mathcal{E} = \mathcal{O}_Y(-2\mathcal{E}) \), then \( H^0(\mathcal{E}(1)) = 0 \) implies that \( \mathcal{E} \) is induced by \( |\pi^* \mathcal{O}_Y(\mathcal{E})| \) so \( i \) is the embedding of \( Y \) as a hyperquadric in \( P^{n+1} \). Then \( \mathcal{I} / \mathcal{I}^2 \simeq \mathcal{O}_Y(-2\mathcal{E}) \), so \( \operatorname{Hom}(\mathcal{I} / \mathcal{I}^2, \mathcal{E}) \) is isomorphic to \( H^0(\mathcal{E}) \) in this case. \( \square \)

Theorem 2.6. Let \( X, Y \) and \( \varphi \) be as in Notations 1.2 and 2.1. Let \( X \) be a Fano variety and let the morphism \( \varphi \) from \( X \) to \( P^N \) be induced by a complete linear series. If \( Y \) is a hyperquadric, assume \( \mathcal{E} \neq \mathcal{O}_Y(-2\mathcal{E}) \).

(1) Then \( \varphi \) has an algebraic formally semiuniversal deformation, which is a finite morphism of degree 2 onto its image, which is a deformation of \( i(Y) \) in \( P^N \).

(2) Assume furthermore that \( B \) is base-point-free in case \( Y \) is a projective bundle over \( P^1 \). Then \( \varphi \) is unobstructed.

Proof. Part (1) follows from Theorem 1.7, Proposition 1.8 and Corollary 2.5. Part (2) will follow from Theorem 1.6. We check now that the hypotheses of Theorem 1.6 are satisfied. If \( Y \) is as in Notation 2.1, then it is well-known that \( H^1(\mathcal{O}_Y), H^2(\mathcal{O}_Y) \) and \( H^1(\mathcal{K}(Y), P^N) \) vanish. Thus hypotheses (1), (3) and (4) of Theorem 1.6 are satisfied. If \( Y \) is a projective bundle, then \( H^1(\mathcal{E}(-2)) = 0 \) because of the vanishing of cohomology in projective space. If \( Y \) is a projective bundle over \( P^1 \), then \( B \) being base-point-free is equivalent to \( \beta \geq \alpha e_{m-1} \), with \( \alpha, \beta \) and \( e_{m-1} \) as in (2.6). Then condition (2) implies \( H^1(\mathcal{E}(-2)) = 0 \). Thus hypothesis (2) of Theorem 1.6 is also satisfied. Finally, hypothesis (5) of Theorem 1.6 follows from Corollary 2.5. \( \square \)

We give the definition of Fano-K3 polarized variety, which is equivalent to [12, Definition 1.5] (note that the condition of the ring \( R(L) = \oplus_{n=0}^\infty H^0(L^{\otimes n}) \) being Cohen-Macaulay, which is required by [12, Definition 1.5], is deduced from Definition 2.7, Kodaira vanishing theorem and Serre duality).

In essence, Fano-K3 polarized varieties are the higher dimensional generalization, in terms of adjunction, of polarized canonical curves and polarized K3 surfaces. By this we mean that, when we consider subsequent general hyperplane sections of a Fano-K3 polarized variety, we eventually end with a (smooth) polarized K3 surface and a (smooth) canonically polarized curve:

Definition 2.7. We say that a polarized variety \((X, L)\) of dimension \( m, m \geq 3 \), is Fano-K3 if it is a Fano polarized variety of index \( m-2 \), i.e., if \( \omega_X^{-1} = L^{\otimes m-2} \).
We give now the classification, done by Fujita, of those hyperelliptic Fano-K3 varieties $(X, L)$ of dimension $m$ such that the image $Y$ of the morphism induced by $|L|$ is smooth. Recall that, since $(X, L)$ is hyperelliptic, $i(Y)$ (see Notation 1.2) is either $\mathbb{P}^m$, a smooth hyperquadric or a smooth rational normal scroll, hence $Y$ is as in Notation 2.1. When $i(Y)$ is a hyperquadric we denote the generator of the Picard group of $Y$ as $\mathfrak{h}$ as in Notation 2.1 and, when $i(Y)$ is a rational normal scroll, we denote the generators of the Picard group of $Y$ as $H_0$ and $F$, as in (2.6). Although, after looking at curves and K3 surfaces, one might expect a priori the existence of many different hyperelliptic Fano-K3 polarized varieties and their sectional genus to be unbounded, this is not the case. Note also that if $i(Y)$ is a rational normal scroll, $m$ is bounded ($m \leq 4$).

**Proposition 2.8.** Let $X, Y, \varepsilon, i$ and $\varphi$ be as in Notation 1.2, 2.1 and (2.6), and assume $(X, L)$ is a hyperelliptic Fano-K3 variety of dimension $m \geq 3$ and sectional genus $g$ and that $\varphi$ is induced by $|L|$. Then $i(Y)$ is a variety of minimal degree and $Y, \varepsilon^{-2}$ and $B$ are as follows:

1. If $Y = \mathbb{P}^m$, then $\varepsilon^{-2} = \mathcal{O}_{\mathbb{P}^m}(6)$ (in this case, $g = 2$).
2. If $Y$ is a hyperquadric, then $B \sim 4\mathfrak{h}$ (in this case, $g = 3$).
3. If $Y$ is a projective bundle over $\mathbb{P}^1$, then
   a. $m = 4$, $e_1 = e_2 = e_3 = 0$, $L \sim H_0 + F$ and $B \sim 4H_0$ (in this case, $g = 5$);
   b. $m = 3$, $e_1 = e_2 = 0$ and $L \sim H_0 + 2F$ and $B \sim 4H_0$ (in this case, $g = 7$);
   c. $m = 3$, $e_1 = e_2 = 0$ and $L \sim H_0 + F$ and $B \sim 4H_0 + 2F$ (in this case, $g = 4$);
   d. $m = 3$, $e_1 = e_2 = -1$, $L \sim H_0 + 2F$ and $B \sim 4H_0$ (in this case, $g = 5$).

**Proof.** If $(X, L)$ is a Fano-K3 variety of dimension $m$, then by adjunction $L^m = 2g - 2$. Now the result follows from [12, Proposition 5.18 (3), (6.7)].

As a consequence of Theorem 2.6 we obtain this:

**Theorem 2.9.** Let $X$ and $Y$ be as in Notations 1.2 (2). Let $(X, L)$ be a hyperelliptic Fano-K3 variety of dimension $m \geq 3$. If $Y$ is $\mathbb{P}^m$ or a rational normal scroll, then any deformation of $(X, L)$ is hyperelliptic.

**Proof.** If $Y = \mathbb{P}^m$, the result follows from Remark 0.3 (2). Let now $Y$ be a projective bundle over $\mathbb{P}^1$ and let $\varphi$ be the morphism induced by $|L|$. Let $Z$ be a smooth, algebraic variety with a distinguished point $0 \in Z$. Let $(\mathcal{X}, \mathcal{L})$ be a flat family over $Z$ such that the fiber $(\mathcal{X}_0, \mathcal{L}_0)$ over 0 is isomorphic to $(X, L)$. Recall that $H^1(L) = H^1(\pi_*L)$ and that, by the projection formula, the latter equals $H^1(\mathcal{O}_Y(1)) \oplus H^1(\mathcal{O}_{\mathcal{L}}(1))$. We have $H^1(\mathcal{O}_Y(1)) = 0$; $H^1(\mathcal{O}_{\mathcal{L}}(1))$ also vanishes because of the ampleness of $-K_X$, hence $H^1(L) = 0$. Then, shrinking $Z$ if necessary, by semicontinuity $H^1(\mathcal{L}_z) = 0$ for all $z \in Z$. Then, shrinking $Z$ again if necessary, the push-out of $\mathcal{L}$ to $Z$ is a free sheaf on $Z$ and the formation of the push-out commutes with base extension (see [20,
3.3.11 (ii)], if \( H \) is such that \( \Phi \) satisfies condition (2) of Theorem 2.6. Then Theorem 2.6 implies that, for all \( z \in Z \), the morphism \( \Phi_z \) has degree 2 onto its image, which is a deformation of \( i(Y) \) in \( P^N \). Since any deformation of \( i(Y) \) is a variety of minimal degree, the morphism \( \Phi_z \) is induced by the complete linear series \( H^0(\mathcal{L}_z) \) and \( \mathcal{L}_z \) is a hyperelliptic polarized variety.

**Theorem 2.10.** Let \( X \) and \( Y \) be as in Notations 1.2 (2). Let \( (X, L) \) be a hyperelliptic Fano-K3 variety of dimension \( m \geq 3 \) and let \( \varphi \) be the morphism induced by the complete linear series \( |L| \). If \( Y \) is a hyperquadric, then we have:

1. The morphism \( \varphi \) and the polarized variety \( (X, L) \) are unobstructed.
2. A general deformation of \( \varphi \) is an embedding. Likewise, a general deformation of \( (X, L) \) is nonhyperelliptic but its complete linear series induces an embedding. The images of these embeddings are quartic hypersurfaces of \( P^{m+1} \).

**Proof.** If \( Y \) is a hyperquadric, then \( \mathcal{E} = \mathcal{O}_Y(-2h) \) by Proposition 2.8 (2) and \( \mathcal{O}_Y(1) = \mathcal{O}_Y(h) \). Since \( H^2(\mathcal{O}_Y) = 0 \) and \( H^2(\mathcal{E}) = 0 \) by (2.4), then \( H^2(\mathcal{E}_X) = 0, \) so, by Proposition 1.8, there exist algebraic formally semiuniversal deformations of \( \varphi \) and \( (X, L) \).

If follows from Remark 1.4 that \( H^1(\mathcal{N}_x) \) is isomorphic to \( H^1(\mathcal{O}_B(B)) \). To compute \( H^1(\mathcal{O}_B(B)) \) we consider the sequence

\[
H^1(\mathcal{O}_Y(B)) \to H^1(\mathcal{O}_B(B)) \to H^2(\mathcal{O}_Y).
\]  

(2.11)

Note that, by the Kodaira vanishing theorem, \( H^1(\mathcal{O}_Y(B)) = H^1(\mathcal{O}_Y(4h)) = 0 \) and \( H^2(\mathcal{O}_Y) = 0, \) so \( H^1(\mathcal{O}_B(B)) \) and \( H^1(\mathcal{N}_x) \) vanish. From (1.2) and from the vanishing of \( H^1(\mathcal{O}_Y) \) and \( H^1(\mathcal{O}_Y(2h)) \), it follows that \( H^1(\mathcal{N}_x) = 0, \) so \( \varphi \) is unobstructed. Now let \( \Sigma_L \) be the sheaf of first order differential operators in \( L \) and let

\[
0 \to \mathcal{E}_X \to \Sigma_L \to \mathcal{T}_X \to 0
\]

(2.12)

be the Atiyah extension of \( L \) (see [21, p. 96] or [27, (3.30)]). By [27, Theorem 3.3.11 (ii)], if \( H^2(\Sigma_L) = 0, \) then \( (X, L) \) is unobstructed. To prove the vanishing of \( H^2(\Sigma_L) \), by exact sequence (2.12) and the vanishing of \( H^2(\mathcal{E}_X) \), we only need to show \( H^2(\mathcal{T}_X) = 0. \) For that we take cohomology on the exact sequence

\[
0 \to \mathcal{T}_X \to \varphi^* \mathcal{T}_P \to \mathcal{N}_\varphi \to 0.
\]

(2.13)

By the vanishing of \( H^1(\mathcal{N}_x) \), the projection formula and the Leray’s spectral sequence, it is enough to show the vanishing of \( H^2(\mathcal{T}_P|_{i(Y)}) \) and \( H^2(\mathcal{T}_P|_{i(Y)} \otimes \mathcal{E}) \). For that we use the restriction to \( i(Y) \) of the Euler sequence of the tangent bundle of \( P^N \), so the wanted vanishings will follow from
the vanishing of $H^2(\mathcal{O}_Y(h))$, $H^3(\mathcal{O}_Y)$, $H^2(\mathcal{O}_Y(-h))$ and $H^3(\mathcal{O}_Y(-2h))$. All those vanishings follow by Kodaira vanishing (recall that $m \geq 3$). Therefore $H^2(\mathcal{J}_X) = 0$ and so $H^2(\Sigma_L) = 0$ and $(X, L)$ is unobstructed. This completes the proof of (1).

To prove (2) we are going to use [15, Theorem 1.5]. By Corollary 2.5, $\text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{E})$ has dimension 1. In fact, since $\mathcal{I}/\mathcal{I}^2 = \mathcal{E}$, any nonzero element of $\text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{E})$ is an isomorphism. Recall the homomorphism $\Psi_2$, introduced in Proposition 1.5, and cohomology sequence (1.2). Since $H^1(\mathcal{N}_\varphi)$ vanishes, the homomorphism $\Psi_2$ is surjective. Then, given a nonzero $\mu \in \text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{E})$, there exists $\nu$ in $H^0(\mathcal{N}_\varphi)$ such that $\Psi_2(\nu) = \mu$. Since there is an algebraic formally semiuniversal deformation of $\varphi$ with base $Z$ and $\varphi$ is unobstructed, we see that all the hypotheses of [15, Theorem 1.5] are satisfied, so [15, Theorem 1.5] and its proof imply that there exists a smooth, algebraic curve $T$ in $Z$, passing through 0 and tangent to the tangent vector $\nu$ of $Z$ corresponding to $\nu$ (recall that $H^0(\mathcal{N}_\varphi)$ is isomorphic to the tangent space of $Z$ at 0) and a deformation $\Phi_T$ of $\varphi$ over $T$ such that $\Phi_0 = \varphi$ and $\Phi_t$ is an embedding for all $t \in T, t \neq 0$. Taking the pullback by $\Phi_T$ of $\mathcal{O}_{\mathbb{P}^N}^1(1)$ gives us a deformation $(\mathcal{X}_T, \mathcal{L}_T)$ of $(X, L)$. Since for all $t \in T \setminus \{0\}$, $\Phi_t$ is an embedding, then $(\mathcal{X}_t, \mathcal{L}_t)$ is nonhyperelliptic. Let $Z'$ be the base of an algebraic formally semiuniversal deformation $(\mathcal{X}, \mathcal{L})$ of $(X, L)$. Then, after shrinking $T$ if necessary, $(\mathcal{X}_T, \mathcal{L}_T)$ is obtained from $(\mathcal{X}, \mathcal{L})$ by étale base change, so there is a point $z' \in Z'$ such that $(\mathcal{X}_t', \mathcal{L}_t')$ is nonhyperelliptic but $\mathcal{L}_z'$ is very ample. Since very ampleness is an open condition, this finishes the proof of (2).

Deformations of finite morphisms of degree 2 to embeddings are linked to the existence of smoothable embedded ribbons:

**Example 2.11.** Let $m \geq 3$. Given a hyperquadric $i(Y)$ in $\mathbb{P}^{m+1}$, the dualizing sheaf of the (unique) ribbon structure $\tilde{Y}$ on $i(Y)$ embedded in $\mathbb{P}^{m+1}$ is $\omega_{\tilde{Y}} = \mathcal{O}_{\tilde{Y}}(-m + 2)$. In addition, since $\tilde{Y}$ is a divisor in $\mathbb{P}^{m+1}$, it is arithmetically Cohen-Macaulay. Thus, we can say that $\tilde{Y}$ is a Fano-K3 ribbon.

The ribbon $\tilde{Y}$ can be smooth inside $\mathbb{P}^{m+1}$, i.e., there exists an embedded deformation of $\tilde{Y}$ whose general fiber is a smooth (Fano-K3) variety in $\mathbb{P}^{m+1}$. This follows from the general theory developed in [15, §1] but, in this case, can be achieved in an ad-hoc, more straight forward fashion, by deforming $\tilde{Y}$ in the linear system of degree 4 divisors on $\mathbb{P}^{m+1}$.

**Remark 2.12.** In [23, (7.2)] Iskovskikh classified hyperelliptic $(X, L)$ Fano-K3 threefolds. They are of five types and their sectional genera are $g = 2, 3, 4, 5$ and 7 (compare with Proposition 2.8). Iskovskikh also proved (see [10, Theorem 3]) that, if $X$ is a prime Fano threefold of genus $g$ and $g \geq 4$, then $-K_X$ is very ample. Proposition 2.8 and Theorem 2.9 say that hyperelliptic Fano-K3 threefolds with $g \geq 4$ deform only to hyperelliptic Fano-K3 threefolds, so for each genus 4, 5 and 7 there are at least two kinds of Fano-K3 threefolds, hyperelliptic and anticanonically embedded, that do not deform to each other.
3. Deformations of Hyperelliptic Polarized Varieties with $L^m = 2g$

In this section we will study the deformations of those hyperelliptic polarized varieties $(X, L)$ with $L^m = 2g$ when the image of the morphism induced by $|L|$ is smooth. We will start by classifying them, but, before doing so, we will introduce the notation we will use for the case in which $i(Y)$ is a (smooth) rational normal scroll (note that this notation is different from (2.6)):

**Notation 3.1.** If $Y$ is a projective bundle embedded by $i$ as a rational normal scroll, then

(i) $E$ is the very ample vector bundle such that $Y = \mathbb{P}(E)$ and $i$ is induced by $|\mathcal{O}_{\mathbb{P}(E)}(1)|$;

(ii) $E = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)$ with $a_1 \geq \cdots \geq a_n > 0$ and denote $a = a_1 + \cdots + a_n$ (thus $i(Y) = S(a_1, \ldots, a_n)$ and $N = a + n - 1$);

(iii) $H$ will be the divisor on $Y$ such that $\mathcal{O}_Y(H) = \mathcal{O}_{\mathbb{P}(E)}(1)$;

(iv) $p$ will be the structure morphism from $Y$ to $\mathbb{P}^1$ and $F$ will be the fiber of $p$.

As in the case of Fano-K3 varieties, the classification of hyperelliptic polarized varieties $(X, L)$ with $L^m = 2g$ such that the image of the morphism induced by $|L|$ is smooth follows from the work of Fujita. Unlike in the Fano-K3 case, now the sectional genus is unbounded and, if $i(Y)$ is a rational normal scroll, the dimension $m$ is also unbounded. In the latter case though, there are a few sporadic types with bounded sectional genus $g$ and bounded dimension $m$ ($m \leq 3$):

**Proposition 3.2.** Let $X, Y, \mathcal{E}, i$ and $\varphi$ be as in Notation 1.2, 2.1, (2.6) and 3.1. Assume $(X, L)$ is a hyperelliptic variety of dimension $m \geq 2$ with $L^m = 2g$ and $\varphi$ is induced by $|L|$. Then $i(Y)$ is a variety of minimal degree and $Y$, $\mathcal{E}^{2g-2}$ and $B$ are as follows:

1. If $Y = \mathbb{P}^m$, then $\mathcal{E}^{-2} = \mathcal{O}_{\mathbb{P}^m}(4)$ ($g = 1$).
2. If $Y$ is a projective bundle over $\mathbb{P}^1$, then
   (a) $B \sim 2H + 2F$; or
   (b) $m = 3$, $a_1 = a_2 = a_3 = 1$ and $B \sim 4H - 4F$ (i.e., $B \sim 4H_0$); or
   (c) $m = 2$, $a_1 = a_2$ and $B \sim 4H - (4a_1 - 2)F$ (i.e., $B \sim 4H_0 + 2F$); or
   (d) $m = 2$, $a_1 = a_2 + 1$ and $B \sim 4H - 4a_2F$ (i.e., $B \sim 4H_0 + 4F$); or
   (e) $m = 2$, $a_1 = a_2 + 2$ and $B \sim 4H - (4a_2 + 2)F$ (i.e., $B \sim 4H_0 + 6F$).

**Proof.** The result follows from [12, Proposition 5.18 (2), (6.7)].

We need to carry out several cohomological computations on the varieties $X$ and $Y$ of Proposition 3.2:

**Proposition 3.3.** Let $X, Y, \mathcal{E}, i, \pi$ and $\varphi$ be as in Notation 1.2, 2.1 and 3.1. Let $L = \varphi^*\mathcal{O}_Y(1)$. Assume furthermore that $Y$ and $B$ are as in Proposition 3.2 (2a), (2c), (2d) or (2e). Then $\varphi$ is induced by the complete linear series $|L|$, the groups $\text{Hom}((\mathcal{I}/\mathcal{I}^2, \mathcal{E})$ and $\text{Ext}^1(\Omega_Y, \mathcal{E})$ are isomorphic, and the following holds:
(1) If \( Y \) and \( B \) are as in Proposition 3.2 (2a), then the dimension of \( \text{Hom} (\mathcal{I} / \mathcal{I}^2, \mathcal{E}) \) is \( g \); otherwise, the dimension of \( \text{Hom} (\mathcal{I} / \mathcal{I}^2, \mathcal{E}) \) is 2.

(2) \( h^2(\mathcal{O}_Y) = 0 \).

(3) \( h^1(\mathcal{N}_\varphi) = 0 \), except if \( Y \) and \( B \) are as in Proposition 3.2 (2e); in this case, \( h^1(\mathcal{N}_\varphi) = 1 \).

(4) \( h^1(\mathcal{N}_\varphi) = 0 \); except, maybe, if \( Y \) and \( B \) are as in Proposition 3.2 (2e); in this case, \( h^1(\mathcal{N}_\varphi) = 0 \) or \( h^1(\mathcal{N}_\varphi) = 1 \).

**Proof.** We first see that \( \varphi \) is induced by the complete linear series \(|L|\). By Remark 1.3, this is equivalent to

\[
H^0(\mathcal{E}(1)) = 0, \tag{3.1}
\]

so we check this in each case of Proposition 3.2 (2a), (2c), (2d) or (2e). Indeed, if \( Y \) and \( B \) are as in Proposition 3.2 (2a), (2c), (2d) or (2e), then \( \mathcal{E}(1) \) equals \( \mathcal{O}_Y(-F), \mathcal{O}_Y(-H + (2a_1 - 1)F), \mathcal{O}_Y(-H + 2a_2F) \) and \( \mathcal{O}_Y(-H + (2a_2 + 1)F) \) respectively, and none of these line bundles have global sections.

Now we prove \( \text{Hom}(\mathcal{I} / \mathcal{I}^2, \mathcal{E}) \) and \( \text{Ext}^1(\Omega_Y, \mathcal{E}) \) are isomorphic and then we will compute the dimension of \( \text{Ext}^1(\Omega_Y, \mathcal{E}) \), i.e., \( h^1(\mathcal{N}_Y \otimes \mathcal{E}) \). For the former, we will prove that the connecting homomorphism \( \gamma \) of (2.10) is an isomorphism. To prove that \( \gamma \) is an isomorphism, we will show

\[
\text{Hom}(\Omega_{\mathbf{P}^N}|_{i(Y)}, \mathcal{E}) = \text{Ext}^1(\Omega_{\mathbf{P}^N}|_{i(Y)}, \mathcal{E}) = 0. \tag{3.3}
\]

For that, we consider the restriction to \( i(Y) \) of the Euler sequence of \( \mathbf{P}^N \) and get

\[
\text{Hom}(\mathcal{O}_{i(Y)}^{N+1}(-1), \mathcal{E}) \longrightarrow \text{Hom}(\Omega_{\mathbf{P}^N}|_{i(Y)}, \mathcal{E}) \longrightarrow \text{Ext}^1(\mathcal{O}_Y, \mathcal{E})
\]

and

\[
\text{Ext}^1(\mathcal{O}_{i(Y)}^{N+1}(-1), \mathcal{E}) \longrightarrow \text{Ext}^1(\Omega_{\mathbf{P}^N}|_{i(Y)}, \mathcal{E}) \longrightarrow \text{Ext}^2(\mathcal{O}_Y, \mathcal{E}).
\]

It suffices to see that \( \text{Hom}(\mathcal{O}_{i(Y)}^{N+1}(-1), \mathcal{E}), \text{Ext}^1(\mathcal{O}_{i(Y)}^{N+1}(-1), \mathcal{E}), \text{Ext}^1(\mathcal{O}_Y, \mathcal{E}) \) and \( \text{Ext}^2(\mathcal{O}_Y, \mathcal{E}) \) all vanish. On the one hand, \( \text{Hom}(\mathcal{O}_{i(Y)}^{N+1}(-1), \mathcal{E}) \) vanishes because of (3.1) and \( \text{Ext}^1(\mathcal{O}_{i(Y)}^{N+1}(-1), \mathcal{E}) \) vanishes because none of the line bundles of (3.2) has higher cohomology. On the other hand, if \( Y \) and \( B \) are as in Proposition 3.2 (2a), then \( \text{Ext}^1(\mathcal{O}_Y, \mathcal{E}) \) and \( \text{Ext}^2(\mathcal{O}_Y, \mathcal{E}) \) are isomorphic to \( H^1(\mathcal{O}_Y(-H - F)) \) and \( H^2(\mathcal{O}_Y(-H - F)) \) and both also vanish. If \( Y \) is as in the other three cases, \( \text{Ext}^1(\mathcal{O}_Y, \mathcal{E}) \) and \( \text{Ext}^2(\mathcal{O}_Y, \mathcal{E}) \) are, respectively, Serre dual of \( H^1(\mathcal{O}_Y(-F)) \) and \( H^0(\mathcal{O}_Y(-F)) \) and both cohomology groups vanish. This completes the proof of \( \gamma \) being an isomorphism.

Now we prove (1). We compute the dimension \( \text{Ext}^1(\Omega_Y, \mathcal{E}) \), which is isomorphic to \( H^1(\mathcal{N}_Y \otimes \mathcal{E}) \). In order to compute the dimension \( H^1(\mathcal{N}_Y \otimes \mathcal{E}) \), we compute the exact sequence (2.7). If \( Y \) and \( B \) are as in Proposition 3.2 (2a), then \( p^* \mathcal{N}_{\mathbf{P}^1} \otimes \mathcal{E} = \mathcal{O}_Y(-H + F) \), so both \( H^0(p^* \mathcal{N}_{\mathbf{P}^1} \otimes \mathcal{E}) \) and \( H^1(p^* \mathcal{N}_{\mathbf{P}^1} \otimes \mathcal{E}) \) vanish. Then \( H^1(\mathcal{N}_Y \otimes \mathcal{E}) \) is isomorphic to \( H^1(\mathcal{N}_Y/\mathbf{P}^1 \otimes \mathcal{E}) \). In order to compute \( H^1(\mathcal{N}_Y/\mathbf{P}^1 \otimes \mathcal{E}) \), we consider the dual of the relative Euler sequence

\[
0 \longrightarrow \mathcal{O}_Y \longrightarrow p^* E^\vee \otimes \mathcal{O}_Y(H) \longrightarrow \mathcal{N}_Y/\mathbf{P}^1 \longrightarrow 0. \tag{3.4}
\]
Since $H^1(\mathcal{O}_Y(-H - F)) = H^2(\mathcal{O}_Y(-H - F)) = 0$, $H^1(\mathcal{T}_Y \otimes \mathcal{E})$ is isomorphic to $H^1(p^* E^\vee \otimes \mathcal{O}_Y(H) \otimes \mathcal{E})$, which is the same as

$$H^1(\mathcal{O}_Y(-(a_1 + 1)F)) \oplus \cdots \oplus H^1(\mathcal{O}_Y(-(a_n + 1)F)),$$

which has dimension $a_1 + \cdots + a_n = a = \frac{1}{2} L^m = g$. Then, if follows from all the above that the dimension of $\text{Hom}(\mathcal{S}, \mathcal{F}^2, \mathcal{E})$ is $g$ if $Y$ and $B$ are as in Proposition 3.2 (2a).

Now we compute $h^1(\mathcal{T}_Y \otimes \mathcal{E})$ when $Y$ and $B$ are as in Proposition 3.2 (2c), (2d) or (2e). Since in this case $Y$ has dimension 2, with notation (2.6), exact sequence (2.7) becomes

$$0 \longrightarrow \mathcal{O}_Y(2H_0 + eF) \longrightarrow \mathcal{T}_Y \longrightarrow \mathcal{O}_Y(2F) \longrightarrow 0, \quad (3.5)$$

where $e = a_1 - a_2$, i.e., $e = 0$ in case (2c), $e = 1$ in case (2d), $e = 2$ in case (2e). Tensoring with $\mathcal{E}$, (3.5) becomes

$$0 \longrightarrow \mathcal{O}_Y(-F) \longrightarrow \mathcal{T}_Y \otimes \mathcal{E} \longrightarrow \mathcal{O}_Y(-2H_0 + (1 - e)F) \longrightarrow 0. \quad (3.6)$$

Note that $H^1(\mathcal{O}_Y(-F)) = H^2(\mathcal{O}_Y(-F)) = h^1(\mathcal{O}_Y(-2H_0 + (1 - e)F))$. By Serre duality, $h^1(\mathcal{O}_Y(-2H_0 + (1 - e)F)) = h^1(\mathcal{O}_Y(-3F)) = 2$, so $h^1(\mathcal{T}_Y \otimes \mathcal{E})$ and hence, the dimension of $\text{Hom}(\mathcal{S}/\mathcal{F}^2, \mathcal{E})$, is 2 if $Y$ and $B$ are as in Proposition 3.2 (2c), (2d) or (2e).

Now we prove (2). By the projection formula and by the Leray spectral sequence,

$$H^2(\mathcal{O}_X) = H^2(\mathcal{O}_Y) \oplus H^2(\mathcal{E}).$$

It is well known $H^2(\mathcal{O}_Y) = 0$. If $Y$ and $B$ are as in Proposition 3.2 (2a), then

$$H^2(\mathcal{E}) = H^2(\mathcal{O}_Y(-H - F)) = 0.$$

If $Y$ and $B$ are as in Proposition 3.2 (2c), (2d) or (2e), then

$$H^2(\mathcal{E}) = H^2(\mathcal{O}_Y(-2H_0 - (e + 1)F)),$$

which, by Serre duality is isomorphic to $H^0(\mathcal{O}_Y(-F))^\vee$, which vanishes. Hence $H^2(\mathcal{O}_X) = 0$ in all cases.

Now we prove (3). If follows from Remark 1.4 that $H^1(\mathcal{N}_z)$ is isomorphic to $H^1(\mathcal{O}_B(B))$. To compute $H^1(\mathcal{O}_B(B))$ we consider the exact sequence (2.11). Since $H^1(\mathcal{O}_Y) = H^2(\mathcal{O}_Y) = 0$, the groups $H^1(\mathcal{O}_B(B))$ and $H^1(\mathcal{O}_B(B))$ are isomorphic. Using the projection formula and the Leray spectral sequence, we see that $h^1(\mathcal{O}_Y(B))$ is the sum of the dimensions of the first cohomology group of certain line bundles on $\mathbb{P}^1$. If $Y$ and $B$ are as in Proposition 3.2 (2a), then the smallest among the degrees of those line bundles is $2a_n + 2$, which is greater than or equal to 4; if $Y$ and $B$ are as in Proposition 3.2 (2c), then the smallest among the degrees of those line bundles is 2; and if $Y$ and $B$ are as in Proposition 3.2 (2d), then the smallest among the degrees of those line bundles is 0. In all these three cases the first cohomology group of the line bundles vanishes, so $H^1(\mathcal{O}_Y(B))$ vanishes and so do $H^1(\mathcal{O}_B(B))$ and $H^1(\mathcal{N}_z)$. If $Y$ and $B$ are as in Proposition 3.2 (2e), then

$$h^1(\mathcal{O}_Y(B)) = h^1(\mathcal{O}_{\mathbb{P}^1}(6)) \oplus h^1(\mathcal{O}_{\mathbb{P}^1}(4)) \oplus h^1(\mathcal{O}_{\mathbb{P}^1}(2)) \oplus h^1(\mathcal{O}_{\mathbb{P}^1}(-2)) = 1,$$

so $h^1(\mathcal{N}_z) = h^1(\mathcal{O}_B(B)) = h^1(\mathcal{O}_Y(B)) = 1$. 
Now we prove (4). We use cohomology sequence (1.2). First we will prove the vanishing of \( H^1(\mathcal{N}_i(Y), P_N) \oplus H^1(\mathcal{N}_i(Y), P_N \otimes \mathcal{E}) \). The vanishing of \( H^1(\mathcal{N}_i(Y), P_N) \) is well known. To prove the vanishing of \( H^1(\mathcal{N}_i(Y), P_N \otimes \mathcal{E}) \), consider the normal sequence

\[
0 \longrightarrow \mathcal{T}_Y \longrightarrow \mathcal{T}_P N|_{i(Y)} \longrightarrow \mathcal{N}_i(Y), P_N \longrightarrow 0.
\]

Then, for \( H^1(\mathcal{N}_i(Y), P_N \otimes \mathcal{E}) \) to vanish, it suffices to have the vanishings of \( H^1(\mathcal{T}_P N|_{i(Y)} \otimes \mathcal{E}) \) and \( H^2(\mathcal{T}_Y \otimes \mathcal{E}) \). The vanishing of \( H^1(\mathcal{T}_P N|_{i(Y)} \otimes \mathcal{E}) \) has been already proved (see (3.3)). For the vanishing of \( H^2(\mathcal{T}_Y \otimes \mathcal{E}) \) we use (2.7), (3.4) and (3.6). We argue first for \( Y \) and \( B \) as in Proposition 3.2 (2a). It suffices to prove the vanishings of \( H^2(p^* \mathcal{T}_P 1 \otimes \mathcal{E}), H^2(p^* E Y \otimes \mathcal{O}_Y (H) \otimes \mathcal{E}) \) and \( H^3(\mathcal{E}) \). The first cohomology group is isomorphic to \( H^2(\mathcal{O}_Y (H + F)) \) and this group vanishes. For the vanishings of the second cohomology group it suffices to show the vanishings of \( H^2(\mathcal{O}_Y (-a_1 + 1 F)) \) for all \( i = 1, \ldots, n \); all of these vanishings occur. Finally \( H^3(\mathcal{E}) \) is isomorphic to \( H^3(\mathcal{O}_Y (H - F)) \), which also vanishes. Now we argue for \( Y \) and \( B \) as in Proposition 3.2 (2c), (2d) or (2e). To prove the vanishing of \( H^2(\mathcal{T}_Y \otimes \mathcal{E}) \) in these cases we look at the exact sequence (3.6). Note that \( H^2(\mathcal{O}_Y (-F)) = 0 \). On the other hand, by Serre duality \( h^2(\mathcal{O}_Y (2H_0 + (1 - e) F)) = h^0(\mathcal{O}_Y (-3F)) = 0 \). Then \( H^2(\mathcal{T}_Y \otimes \mathcal{E}) = 0 \) as wished.

Because \( H^1(\pi^* \mathcal{N}_i(Y), P_N) \) vanishes, \( H^1(\mathcal{N}_\pi) \) surjects onto \( H^1(\mathcal{N}_\pi) \). If \( Y \) and \( B \) are as in Proposition 3.2 (2a), (2c) or (2d), then \( H^1(\mathcal{N}_\pi) = 0 \) because \( H^1(\mathcal{N}_\pi) = 0 \). If \( Y \) and \( B \) are as in Proposition 3.2 (2e), then \( h^1(\mathcal{N}_\pi) \) is 0 or 1 because \( h^1(\mathcal{N}_\pi) = 1 \). □

**Proposition 3.4.** Let \( X, Y, \mathcal{E}, i, \pi \) and \( \varphi \) be as in Notation 1.2, 2.1 and 3.1. Let \( L = \varphi^* \mathcal{O}_Y (1) \). If \( Y \) and \( B \) are as in Proposition 3.2 (2b), then \( \varphi \) is induced by the complete linear series \(|L|\) and \( \text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{E}) = \text{Ext}^1(\Omega_Y, \mathcal{E}) = 0 \).

**Proof.** Since \( B \sim 4H - 4F \), we have \( H^0(\mathcal{E}, 1) = H^0(\mathcal{O}_Y (-H + 2F)) = 0 \), so \( \varphi \) is induced by the complete linear series \(|L|\).

In view of (2.10), in order to prove the vanishing of \( \text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{E}) \) it will suffice to prove the vanishing of \( H^0(\mathcal{T}_P N|_{i(Y)} \otimes \mathcal{E}) \) and of \( H^1(\mathcal{T}_Y \otimes \mathcal{E}) \). Arguing as in the proof of Proposition 3.3, in order to show the vanishing of \( H^0(\mathcal{T}_P N|_{i(Y)} \otimes \mathcal{E}) \) it suffices to prove that \( H^0(\mathcal{E}, 1) \) and \( H^1(\mathcal{E}) \) vanish. We have already seen \( H^0(\mathcal{E}, 1) = 0 \). For \( H^1(\mathcal{E}) = 0 \), since \( B \sim 4H - 4F \), we have

\[
H^1(\mathcal{E}) = H^1(\mathcal{O}_Y (-2H + 2F)) = 0.
\]

Now we prove \( H^1(\mathcal{T}_Y \otimes \mathcal{E}) = 0 \). We use exact sequences (2.7) and (3.4). Then it suffices to prove the vanishings of \( H^1(p^* \mathcal{T}_P 1 \otimes \mathcal{E}), H^1(p^* E Y \otimes \mathcal{O}_Y (H) \otimes \mathcal{E}) \) and \( H^2(\mathcal{E}) \). For the first one, note \( H^1(p^* \mathcal{T}_P 1 \otimes \mathcal{E}) = H^1(\mathcal{O}_Y (-2H + 4F)) = 0 \). The vanishing of the second one follows from \( H^1(\mathcal{O}_Y (-H + F)) = 0 \). The vanishing of the third one follows because

\[
H^2(\mathcal{E}) = H^2(\mathcal{O}_Y (-2H + 2F)) = 0.
\]

□
Remark 3.5. (1) It follows from Proposition 3.3 and [5, Corollary 1.4] that there exist nonsplit ribbons supported on $Y$ and with conormal bundle $\mathcal{E}$, where $Y$ and $\mathcal{E}$ are as in Proposition 3.2 (2a), (2c), (2d) or (2e).

(2) It follows from Proposition 3.4 and [5, Corollary 1.4] that there do not exist nonsplit ribbons supported on $Y$ and with conormal bundle $\mathcal{E}$, where $Y$ and $\mathcal{E}$ are as in Proposition 3.2 (2b).

Theorem 3.6. Let $X$ and $Y$ be as in Notations 1.2 (2). Let $(X,L)$ be a hyperelliptic variety such that $L^m = 2g$ and let $\varphi$ be the morphism induced by the complete linear series $|L|$. If $Y$ and $B$ are as in (2a), (2c) or (2d) of Proposition 3.2, then we have:

1. The morphism $\varphi$ and the polarized variety $(X,L)$ are unobstructed.
2. A general deformation of $\varphi$ is a finite morphism of degree 1 onto its image. Likewise, a general deformation of $(X,L)$ is nonhyperelliptic but its complete linear series induces a finite morphism of degree 1 onto its image.

Proof. By Proposition 3.3 (2), $H^2(\mathcal{E}_X) = 0$. Thus, by Proposition 1.8, there exist algebraic formally semiuniversal deformations of $\varphi$ and $(X,L)$. It follows from Proposition 3.3 (4) that $\varphi$ is unobstructed. To prove that $(X,L)$ is also unobstructed, we see that $H^2(\Sigma_L) = 0$. For this we use exact sequence (2.12).

Since $H^2(\mathcal{E}_X) = 0$, we only need to see that $H^2(\mathcal{F}_X)$ vanishes. Since $H^1(\mathcal{N}_{\varphi})$ vanishes, taking cohomology on exact sequence (2.13), we see that, by the projection formula and the Leray’s spectral sequence, it is enough to prove the vanishing of $H^2(\mathcal{P}_N|Y)$ and $H^2(\mathcal{P}_N|Y \otimes \mathcal{E})$. Taking cohomology on the restriction to $Y$ of the Euler sequence of the tangent bundle of $\mathcal{P}_N$, since $H^2(\mathcal{E}_Y(1))$, $H^2(\mathcal{E}(1))$, $H^3(\mathcal{E}_Y)$ and $H^3(\mathcal{E})$ vanish, we obtain the desired vanishings. Then $H^2(\Sigma_L) = 0$ and by [27, Theorem 3.3.11 (ii)], the polarized variety $(X,L)$ is unobstructed. This completes the proof of (1).

To prove (2) we are going to use [14, Theorem 1.4]. By Proposition 3.3, $\text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{E}) \neq 0$. If $\mu \in \text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{E})$, $\mu \neq 0$, then $\mu$ is a homomorphism of rank 1 because $\mathcal{E}$ is a line bundle. By Proposition 3.3, $H^1(\mathcal{N}_{\varphi}) = 0$, so it follows from the long exact sequence of cohomology (1.2) that the map $\Psi_2$ of Proposition 1.5 is surjective. Then, given a nonzero $\mu \in \text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{E})$, there exists $\nu$ in $H^0(\mathcal{N}_{\varphi})$ such that $\Psi_2(\nu) = \mu$. Since there is an algebraic formally semiuniversal deformation of $\varphi$ with base $Z$ and $\varphi$ is unobstructed, we see that all the hypotheses of [14, Theorem 1.4] are satisfied, so [14, Theorem 1.4] and its proof imply that there exists a smooth, algebraic curve $T$ in $Z$, passing through 0 and tangent to the tangent vector $v$ of $Z$ corresponding to $\nu$ and a deformation $\Phi_T$ of $\varphi$ over $T$ such that $\Phi_0 = \varphi$ and $\Phi_t$ is a finite morphism of degree 1 onto its image, for all $t \in T, t \neq 0$. Taking the pullback by $\Phi_T$ of $\mathcal{O}_{\mathcal{P}_N}(1)$ gives us a deformation $(\mathcal{I}_T, \mathcal{L}_T)$ of $(X,L)$, such that for all $t \in T \setminus \{0\}$, $(\mathcal{I}_t, \mathcal{L}_t)$ is nonhyperelliptic. Let $Z'$ be the base of an algebraic formally semiuniversal deformation $(\mathcal{I}', \mathcal{L}')$ of $(X,L)$. Then, after shrinking $T$ if necessary, $(\mathcal{I}'_T, \mathcal{L}'_T$) is obtained from $(\mathcal{I}, \mathcal{L})$ by etale base change, so there is a point $z' \in Z'$ such that $(\mathcal{I}'_{z'}, \mathcal{L}'_{z'})$ is nonhyperelliptic but $|\mathcal{L}'_{z'}|$ induces a finite morphism of degree 1 onto its image. Since this is an open condition, this finishes the proof of (2). \qed
Remark 3.7. The proof of Theorem 3.6 yields a more precise statement for the deformations of $\varphi$. Indeed, note that the element $\nu$ in the proof is a general element of $H^0(\mathcal{N}_\varphi)$. In fact, $\nu$ belongs to the complement $\mathcal{U}$ of a linear subspace of $H^0(\mathcal{N}_\varphi)$, namely, the kernel of $\Psi_2$, which has codimension $h^0(\mathcal{N}_{i(Y)}, \mathcal{P}_N \otimes \mathcal{E})$ in $H^0(\mathcal{N}_\varphi)$ (for the value of $h^0(\mathcal{N}_{i(Y)}, \mathcal{P}_N \otimes \mathcal{E}$), see Proposition 3.3 (1)). Then, for any $\nu \in \mathcal{U}$, there exists a smooth, algebraic curve $T$ in $Z$, passing through 0 and tangent to $\nu$ and a deformation $\Phi_T$ of $\varphi$ over $T$ such that $\Phi_0 = \varphi$ and, for all $t \in T, t \neq 0$, the morphism $\Phi_t$, we have is finite of degree 1 onto its image.

We can also give a similar, more precise statement for the deformations of $(X, L)$. Recall that the space $H^1(\Sigma L)$ parameterizes first-order infinitesimal deformations of the pair $(X, L)$ up to isomorphism (see [21, p. 96], [27, Theorem 3.3.11 (ii)] or [30, pp. 126–128]). Taking cohomology in the commutative diagram [27, (3.39)] one obtains the exact sequence

$$H^0(L)^* \otimes H^0(L) \longrightarrow H^0(\mathcal{N}_\varphi) \xrightarrow{\zeta} H^1(\Sigma_L) \longrightarrow H^0(L)^* \otimes H^1(L).\quad (3.7)$$

By the projection formula and the Leray’s spectral sequence, $H^1(L) = 0$, so $\zeta$ is surjective.

Consider the commutative diagram

$$
\begin{array}{ccc}
H^1(\Sigma_L) & & \Phi_2 \\
\zeta \downarrow & & \zeta' \\
H^0(\mathcal{N}_\varphi) & & H^1(\mathcal{F}_X) \\
\Phi_2 \downarrow & & \gamma \\
H^0(\mathcal{N}_{i(Y)}, \mathcal{P}_N \otimes \mathcal{E}) & & H^1(\mathcal{F}_Y \otimes \mathcal{E})
\end{array}
$$

(3.8)

where the square arises from [19, Proposition 3.7 (1)]), the homomorphism $\zeta$ is the one in (3.7), the homomorphism $\zeta'$ arises when taking cohomology in exact sequence (2.12) and $\zeta'$ and $\zeta''$ are the homomorphisms that forget, in the obvious way, part of the information of the first order infinitesimal deformations of $\varphi$ and $(X, L)$.

From the commutativity of (3.8) and the injectivity of $\gamma$ (in fact, $\gamma$ is an isomorphism, see the proof of Proposition 3.3), it follows that $\zeta^{-1}(\zeta(\ker \Psi_2)) = \ker \Psi_2$.

Then, since there exist elements in $H^0(\mathcal{N}_\varphi) \setminus \ker \Psi_2$, we have that $\zeta(\ker \Psi_2)$ is a proper vector subspace of $H^1(\Sigma_L)$. Since $\zeta$ is surjective, $\mathcal{U}' = H^1(\Sigma_L) \setminus \zeta(\ker \Psi_2)$ is a (non empty) open set of $H^1(\Sigma_L)$. Then, for any $\varpi \in \mathcal{U}'$, there exists a smooth, algebraic curve $T'$ in $Z'$, passing through 0 and tangent to $\varpi$ and a deformation $(\mathcal{F}_{T'}, \mathcal{L}_{T'})$ of $(X, L)$ over $T'$ such that $(\mathcal{F}_0, \mathcal{L}_0) = (X, L)$ and, for all $t' \in T', t' \neq 0$, we have that $[\mathcal{L}_{t'}]$ induces a finite of degree 1 onto its image.

An analogous remark can be made in relation to Theorem 2.10.

**Theorem 3.8.** Let $X$ and $Y$ be as in Notations 1.2 (2). Let $(X, L)$ be a hyperelliptic variety such that $L^m = 2g$ and let $\varphi$ be the morphism induced by the
complete linear series $|L|$. If $Y$ and $B$ are as in (2e) of Proposition 3.2, then a general deformation of $\varphi$ is a finite morphism of degree 1 onto its image and a general deformation of $(X, L)$ is nonhyperelliptic but its complete linear series induces a finite morphism of degree 1 onto its image.

**Proof.** By Proposition 3.3 (2), $H^2(\mathcal{O}_X) = 0$, so, by Proposition 1.8, there exist algebraic formally semiuniversal deformations of $\varphi$ and $(X, L)$. We look now at long exact sequence of cohomology (1.2). Recall that in this case $h^1(\mathcal{M}_\varphi) = 1$ and that we saw in the proof of Proposition 3.3 that $\eta$ is surjective, so that $h^1(\mathcal{M}_\varphi) = 0$ or 1. Let $Z$ be the base of an algebraic formally semiuniversal deformation of $\varphi$. We have (see [27, Corollary 2.2.11])

$$h^0(\mathcal{M}_\varphi) - h^1(\mathcal{M}_\varphi) \leq \dim Z \leq h^0(\mathcal{M}_\varphi).$$

We argue first for $h^1(\mathcal{M}_\varphi) = 1$. Then $\eta$ is an isomorphism, so $\Psi_2$ is surjective. If $\dim Z = h^0(\mathcal{M}_\varphi)$, then $\varphi$ is unobstructed, so we can argue as in the proof of Theorem 3.6 to find deformations $\Phi_T$ and $(\mathcal{H}_T, \mathcal{L}_T)$ over a smooth algebraic curve $T$ in $Z$.

Now, if $h^1(\mathcal{M}_\varphi) = 1$ and $\dim Z = h^0(\mathcal{M}_\varphi) - 1$, then the tangent cone of $Z$ at 0 has codimension 1 in $H^0(\mathcal{M}_\varphi)$. By Proposition 3.3, the dimension of $\text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{E}) = 2$, so $\ker \Psi_2$ has codimension 2 in $H^0(\mathcal{M}_\varphi)$. Thus the tangent cone of $Z$ is not contained in $\ker \Psi_2$ and there exists an element $\nu$ in the tangent cone of $Z$ such that $\Psi_2(\nu) = \mu \neq 0$. Since $\mathcal{E}$ is a line bundle, $\mu$ is a homomorphism of rank 1. Since $\nu$ is in the tangent cone of $Z$, there exists an algebraic curve $\hat{T}$, $0 \in \hat{T}$ such that $\nu$ is tangent to $\hat{T}$. Desingularizing $\hat{T}$ if necessary we obtain a flat family of morphisms satisfying the hypotheses of [14, Proposition 1.3], so there exists a deformation $\Phi_T$ of $\varphi$ over a smooth algebraic curve $T$ such that the fiber $\Phi_t$ over any $t \in T \setminus \{0\}$ is a morphism to $\mathbb{P}^N$, which is finite and of degree 1 onto its image. Now, taking the pullback by $\Phi_T$ of $\mathcal{O}_{\mathbb{P}^N}(1)$ gives us a deformation $(\mathcal{H}_T, \mathcal{L}_T)$ of $(X, L)$. Since for all $t \in T \setminus \{0\}$ is of degree 1, $(\mathcal{H}_t, \mathcal{L}_t)$ is non hyperelliptic. Let $Z'$ be the base of an algebraic formally semiuniversal deformation of $(X, L)$. Since $\Phi_T$ and $(\mathcal{H}_T, \mathcal{L}_T)$ are obtained, by étale base change, from the algebraic formally semiuniversal deformations over, respectively, $Z$ and $Z'$, we may conclude that there are $z \in Z$ and $z' \in Z'$ such that $\Phi_z$ and the morphism induced by $|\mathcal{L}_{z'}|$ are finite and of degree 1 onto its image.

We argue now for $h^1(\mathcal{M}_\varphi) = 0$. In this case $\varphi$ is unobstructed. The kernel of the homomorphism $\epsilon$ of $(1.2)$ has codimension 1 in $\text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y) \oplus \text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{E})$. By Proposition 3.3, the linear subspace

$$W = \text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y) \times \{0\}$$

has codimension 2 in $\text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y) \oplus \text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{E})$. Then the kernel of $\epsilon$ is not contained in $W$. Thus there exists $\nu \in H^0(\mathcal{M}_\varphi)$ such that $\Psi_2(\nu) \neq 0$. Then we can argue as in the proof of Theorem 3.6 to find deformations $\Phi_T$ and $(\mathcal{H}_T, \mathcal{L}_T)$ over a smooth algebraic curve $T$ in $Z$.

In all cases, there are $z \in Z$ and $z' \in Z'$ such that $\Phi_z$ and the morphism induced by $|\mathcal{L}_{z'}|$ are finite and of degree 1 onto its image. Thus we may conclude the proof of (2) using that being a finite morphism of degree 1 is an open condition. $\square$
As seen in the proof of Theorem 3.8, since we do not know if \( h^1(\mathcal{N}_\varphi) = 0 \) (see Proposition 3.3), it is not clear whether \( \varphi \) is unobstructed or not. It would be interesting to settle the question one way or the other.

**Theorem 3.10.** Let \( X \) and \( Y \) be as in Notations 1.2 (2). Let \((X, L)\) be a hyperelliptic variety such that \( L^m = 2g \) and let \( \varphi \) be the morphism induced by \(|L|\). If \( Y \) and \( B \) are as in Proposition 3.2 (1) or (2b), then \( \varphi \) is unobstructed and the algebraic formally semiuniversal deformation of \((X, L)\) is hyperelliptic.

**Proof.** Let first \( Y = \mathbb{P}^m \). We know by Remark 0.3 (4) that any deformation of \((X, L)\) is hyperelliptic. Alternatively, this follows from Theorem 1.7. The rest of the result, including the unobstructedness of \( \varphi \), follows from Theorem 1.6 and Proposition 1.8.

Let now \( Y \) and \( B \) be as in Proposition 3.2 (2b). Let the flat family \((\mathcal{X}, \mathcal{L})\) over be \( Z \) the algebraic formally semiuniversal deformation of \((X, L)\), which exists by Proposition 1.8. Since \( H^1(\mathcal{O}_Y(1)) = 0 \) and \( H^1(\mathcal{E}(1)) = 0 \), because \( H^1(\mathcal{E}(1)) = H^1(\mathcal{O}_Y(-H + 2F)) \), we have \( H^1(L) = H^1(\pi_*L) = 0 \), so, arguing as in the proof of Theorem 2.10 and shrinking \( Z \) if necessary, we obtain from \( \Phi_*L \) a \( Z \)-morphism
\[
\Phi : \mathcal{X} \longrightarrow \mathbb{P}^N
\]
such that \( \Phi_0 = \varphi \), i.e., \( \Phi \) is a deformation of \( \varphi \).

We apply Theorems 1.6 and 1.7 to \( \Phi \). Proposition 3.4 tells \( \text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{E}) = 0 \) so, \( \Psi_2 = 0 \). As already mentioned, \( H^1(\mathcal{O}_Y) = 0 \), and \( H^2(\mathcal{O}_Y) = 0 \) also. In addition,
\[
H^1(\mathcal{E}^{-2}) = H^1(\mathcal{O}_Y(B)) = H^1(\mathcal{O}_Y(4H_0)) = H^1(\mathcal{O}_\mathbb{P}^1)^{\oplus 15} = 0
\]
and it is well known that \( i(Y) \) is unobstructed in projective space. Then all the hypotheses of Theorems 1.6 and 1.7 are satisfied so \( \varphi \) is unobstructed and, for all \( z \in Z \), the morphism \( \Phi_z \) has degree 2 onto its image, which is a deformation of \( i(Y) \) in \( \mathbb{P}^N \). Since any deformation of \( i(Y) \) is variety of minimal degree, the morphism \( \Phi_z \) is induced by the complete linear series \( H^0(\mathcal{L}_z) \) and \( \mathcal{L}_z^m = L^m \), we conclude that \((\mathcal{X}_z, \mathcal{L}_z)\) is a hyperelliptic polarized variety.

\[\square\]

4. Deformations of Generalized Hyperelliptic Polarized Varieties of Calabi–Yau

In the last two sections we continue the study of deformations of certain generalized hyperelliptic polarized varieties, looking this time at Calabi–Yau and general type varieties. If \((X, L)\) is either a hyperelliptic polarized Calabi–Yau variety of dimension \( m \), \( m \geq 3 \), or a hyperelliptic variety of general type, canonically polarized, of dimension \( m \), \( m \geq 2 \), then, by adjunction, \( L^m < 2g - 2 \) (\( g \) is the sectional genus of \((X, L)\)) and \( h^0(\mathcal{L}_i) \) is constant for any deformation of \((X, L)\) (because \( H^1(L) = 0 \), by the Kodaira vanishing theorem, if \( X \) is Calabi–Yau and because the invariance by deformation of the geometric genus otherwise); thus by Remark 0.3 (1), all deformations of \((X, L)\) are hyperelliptic. Then in this context it is interesting to see if
there are further reasons for this phenomenon, namely, all deformations of a hyperelliptic polarized variety are hyperelliptic, to happen. That is the case, since all the deformations of generalized hyperelliptic Calabi–Yau and general type varieties are generalized hyperelliptic, as we will see in Theorems 4.5 and 5.6. First, we recall the definition of Calabi–Yau variety:

**Definition 4.1.** Let $\mathcal{X}$ be a smooth variety of dimension $m$, $m \geq 3$. We say that $\mathcal{X}$ is a Calabi–Yau variety if

1. $\omega_X = \mathcal{O}_X$; and
2. $H^i(\mathcal{O}_X) = 0$, for all $1 \leq i \leq m - 1$.

**Lemma 4.2.** Let $X, Y, \pi$ and $\mathcal{E}$ be as in Notations 1.2 and 2.1 with $m \geq 3$. If $\omega_X = \mathcal{O}_X$, then $\mathcal{E} = \omega_Y$ and $X$ is Calabi–Yau.

**Proof.** Since $\omega_X = \mathcal{O}_X$, $\pi^*(\omega_Y \otimes \mathcal{E}^{-1}) = \mathcal{O}_X$, so $\omega_Y \otimes \mathcal{E}^{-1}$ is numerically trivial. Since $Y$ is either a projective space or a hyperquadric or a projective bundle over $P^1$, $\omega_Y \otimes \mathcal{E}^{-1}$ is in fact trivial, so $\mathcal{E} = \omega_Y$.

By the projection formula and the Leray’s spectral sequence, $H^i(\mathcal{O}_X) = H^i(\mathcal{O}_Y) \oplus H^i(\mathcal{E})$. If $1 \leq i \leq m - 1$, then $H^i(\mathcal{O}_Y) = 0$ and, by Serre duality $h^i(\mathcal{E}) = h^i(\omega_Y) = h^{m-i}(\mathcal{O}_Y) = 0$. □

**Proposition 4.3.** Let $X, Y, \pi$ and $\mathcal{E}$ be as in Notations 1.2 and 2.1 and assume $X$ is Calabi–Yau (of dimension $m$, $m \geq 3$). Then $H^1(\mathcal{T}_Y \otimes \mathcal{E}) = 0$.

**Proof.** If $Y = P^m$, then $H^1(\mathcal{E}(1)) = H^2(\mathcal{E}) = 0$ by the vanishing of the intermediate cohomology of line bundles in projective space. In view of the Euler sequence for the tangent bundle of $P^m$, this implies the vanishing of $H^1(\mathcal{T}_Y \otimes \mathcal{E})$.

If $Y$ is a hyperquadric, then it follows from Lemma 4.2 that $\mathcal{E} = \mathcal{O}_Y(-mh)$. In view of the exact sequence for the normal bundle of $Y$ in $P^{m+1}$, we need to check the vanishings of $H^0(\mathcal{O}_Y((-m+2)h))$ and $H^1(\mathcal{T}_P^{m+1} \otimes \mathcal{E})$. The first happens because $m \geq 3$. The second follows from the Euler sequence for the tangent bundle of $P^{m+1}$ restricted to $Y$, since $H^1(\mathcal{E}(1)) = 0$ by the Kodaira vanishing theorem and $h^2(\mathcal{E}) = h^{m-2}(\mathcal{O}_Y) = 0$.

If $Y$ is a projective bundle over $P^1$, we use the notation in (2.6) and exact sequences (2.7) and (2.8). Then it is enough to prove the vanishings of $H^1(p^* \mathcal{T}_{P^1} \otimes \mathcal{E})$, $H^1(\mathcal{E} \otimes \mathcal{O}_Y(H_0))$, $H^1(\mathcal{E} \otimes \mathcal{O}_Y(H_0 + e_1F))$, $H^1(\mathcal{E} \otimes \mathcal{O}_Y(H_0 + e_mF))$ and $H^2(\mathcal{E})$. Then $h^1(p^* \mathcal{T}_{P^1} \otimes \mathcal{E}) = h^{m-1}(\mathcal{O}_Y(-2F)) = 0$ by the projection formula and the Leray spectral sequence (recall $m \geq 3$). In addition, $h^1(\mathcal{E} \otimes \mathcal{O}_Y(H_0)) = h^{m-1}(\mathcal{O}_Y(-H_0)) = 0$, also by the projection formula and the Leray spectral sequence. The vanishings of $H^1(\mathcal{E} \otimes \mathcal{O}_Y(H_0 + e_1F))$, $H^1(\mathcal{E} \otimes \mathcal{O}_Y(H_0 + e_mF))$ are argued analogously. Finally, $h^2(\mathcal{E}) = h^{m-2}(\mathcal{O}_Y) = 0$. □

In analogy with Definition 4.1, by a Calabi–Yau ribbon we mean a ribbon of dimension bigger than 2, with trivial dualizing sheaf and such that the intermediate cohomology of its structure sheaf vanishes. We now deduce from Proposition 4.3 the non existence of nonsplit Calabi–Yau ribbons:

**Corollary 4.4.** Let $Y$ be as in Notations 1.2 (2) and 2.1. There are no nonsplit Calabi–Yau ribbons on $Y$. 
Proof. Let \( \tilde{Y} \) be a ribbon supported on \( Y \) and let \( \tilde{E} \) be the conormal bundle of \( Y \) in \( \tilde{Y} \). The same argument of the proof of [13, Proposition 1.5] implies that \( \tilde{Y} \) is Calabi–Yau if and only if \( \tilde{E} = \omega_Y \). Then the result follows from Proposition 4.3 and [5, Corollary 1.4]. \( \square \)

**Theorem 4.5.** Let \( X, Y, \pi, \) and \( \varphi \) be as in Notations 1.2 and 2.1. Let \( X \) be a Calabi–Yau variety of dimension \( m \), \( m \geq 3 \), and let the morphism \( \varphi \) be induced by a complete linear series. Then the algebraic formally semiuniversal deformation of \( \varphi \) is a finite morphism of degree 2 onto its image, which is a deformation of \( i(Y) \) in \( \mathbb{P}^N \).

**Proof.** The result follows from (2.10), Remark 1.3, Lemma 4.2 (for the vanishing of \( H^1(\mathcal{E}) \)), Theorem 1.7 and Propositions 1.8 and 4.3. \( \square \)

**Remark 4.6.** Assume \( X, Y \) and \( \varphi \) are as in Theorem 4.5 and that, if \( Y \) is a projective bundle over \( \mathbb{P}^1 \), then \( -K_Y \) is base-point-free (i.e., \( e_1 + \cdots + e_{m-2} + (1-m)e_{m-1} \geq -2 \), with \( e_1, \ldots, e_{m-1} \) as in the notation in (2.6)). Then we can argue as in the proof of Theorem 2.6 and, using Theorem 1.6, conclude that \( \varphi \) is unobstructed.

Although we already observed at the beginning of the section that next result follows from Remark 0.3 (1), it is interesting to see how it is deduced from the broader setting of Theorem 4.5:

**Corollary 4.7.** Let \( X \) be a Calabi–Yau variety (of dimension \( m \), \( m \geq 3 \)) and let \( L \) be a polarization on \( X \). If \((X, L)\) is hyperelliptic and the image of \( X \) by the morphism induced by \( |L| \) is smooth, then any deformation of \( (X, L) \) is hyperelliptic.

**Example 4.8.** If \((X, L)\) is a polarized Calabi–Yau threefold with \( L^3 = 8 \) and \( h^0(L) = 7 \), then, by adjunction and Clifford’s theorem, \((X, L)\) is hyperelliptic. Thus, if the image \( i(Y) \) of the morphism induced by \( |L| \) is smooth, then \( i(Y) \) is a rational normal scroll \( S(1,1,2) \) of \( \mathbb{P}^6 \). This implies that polarized Calabi–Yau threefolds \((X, L)\) with \( L^3 = 8 \) and \( h^0(L) = 7 \) such that the image of the morphism induced by \( |L| \) is smooth are parameterized by an irreducible variety whose general points correspond to hyperelliptic polarized Calabi–Yau threefolds.

Corollary 4.7 shows that a hyperelliptic polarized Calabi–Yau threefold \((X, L)\) only deforms to hyperelliptic polarized Calabi–Yau threefolds. If the image of the morphism induced by \( |L| \) is a smooth rational normal scroll, then \( X \) is fibered by \( K3 \) surfaces (see [18, Proposition 1.6]); thus, in this case, any deformation of \( X \) carries also the \( K3 \) fibration. This motivates the following question.

**Question 4.9.** Let \( X \) be a Calabi–Yau threefold. If \( X \) carries a \( K3 \) fibration, does any deformation of \( X \) carry the \( K3 \) fibration? It is tempting but, probably, too optimistic, to expect that a Calabi–Yau threefold \( X \) carries a \( K3 \) fibration if and only if there exists a polarization \( L \) on \( X \) so that \((X, L)\) is hyperelliptic and the image of the morphism induced by \( |L| \) is a smooth rational normal scroll. If this expectation were true then one would easily show that the answer to our question is affirmative.
5. Deformation of Generalized Hyperelliptic Polarized
Varieties of General Type

For varieties of general type, we extend our notion of generalized hyperelliptic polarized varieties to the case of $Y$ being a toric variety (note that if $Y$ is as in (i) and (iii) of Notation 2.1, then $Y$ is toric):

**Notation 5.1.** In this section, unless otherwise specified, $Y$ will be as in Notation 2.1 (ii) or a toric variety of dimension $m \geq 2$.

Now we study the deformations of generalized hyperelliptic polarized varieties of general type when $Y$ is as in Notation 5.1:

**Proposition 5.2.** Let $X, Y, \pi$ and $\mathcal{E}$ be as in Notations 1.2 and 5.1 and assume $X$ is a minimal variety of general type (i.e., with $K_X$ ample) of dimension $m$, $m \geq 2$. Then $H^1(T_Y \otimes \mathcal{E}) = 0$.

**Proof.** Let $\mathcal{E}' = \omega_Y \otimes \mathcal{E}^{-1}$. Recall that $\omega_X = \pi^* \mathcal{E}'$, so $\mathcal{E}'$ is ample. By Serre duality $h^1(T_Y \otimes \mathcal{E}) = h^{m-1}(\Omega_Y \otimes \mathcal{E}')$, that vanishes if $Y$ is toric by the Bott vanishing. If $Y$ is a hyperquadric, since $\mathcal{E}'$ is ample, then $\mathcal{E}'$ is very ample. Then $\mathcal{E} = \omega_Y \otimes \mathcal{E}'^{-1}$, so the result follows from $H^1(T_Y \otimes \omega_Y \otimes \mathcal{E}'^{-1}) = 0$, which was showed in the proof of [16, Proposition 1.6].

**Remark 5.3.** Under the hypotheses of Proposition 5.2, the variety $X$ is regular. Indeed, $\pi_* (\mathcal{E}_X) = \mathcal{O}_Y \oplus \mathcal{E}$. We have $H^1(\mathcal{O}_Y) = 0$. We see that

$$H^1(\mathcal{E}) = 0$$

also. This follows from Serre duality and Bott vanishing if $Y$ is toric. If $Y$ is a (smooth) hyperquadric and $m \geq 3$, the vanishing of $H^1(\mathcal{E})$ follows from sequence (2.5) and the vanishing of the intermediate cohomology of $\mathbb{P}^N$.

We say that a ribbon of dimension greater than or equal to 2 is a minimal ribbon of general type if its dualizing sheaf is ample. From Proposition 5.2 we deduce the non existence of nonsplit minimal ribbons of general type supported on $Y$ as in Notation 5.1:

**Corollary 5.4.** Let $Y$ be as in Notation 5.1. There are no nonsplit minimal ribbons of general type on $Y$.

**Proof.** Let $\tilde{Y}$ be a minimal ribbon of general type supported on $Y$ and let $\tilde{\mathcal{E}}$ be the conormal bundle of $Y$ in $\tilde{Y}$. Since $\omega_\varphi$ is ample, by [13, Lemma 1.4], so is $\omega_Y \otimes \tilde{\mathcal{E}}^{-1}$. Then $\text{Ext}^1(\Omega_Y, \tilde{\mathcal{E}}) = H^1(T_Y \otimes \tilde{\mathcal{E}}) = 0$ as in the proof of Proposition 5.2, so the result follows from [5, Corollary 1.4].

**Corollary 5.5.** Let $X, Y, \varphi, \mathcal{E}$ and $\mathcal{I}$ be as in Notations 1.2 and 5.1. Assume $X$ is a minimal variety of general type (i.e., with $K_X$ ample) of dimension $m$, $m \geq 2$ and $\varphi$ is induced by a complete linear series. Then $\text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{E}) = 0$.

**Proof.** The result follows from exact sequence (2.10), the vanishing of $H^0(\mathcal{E}(1))$ (this is because of Remark 1.3), (5.1) and Proposition 5.2.
Theorem 5.6. Let $X, Y$ and $\varphi$ be as in Notations 1.2 and 5.1. Assume $X$ is a minimal variety of general type of dimension $m$, $m \geq 2$ and $\varphi$ is induced by a complete linear series. Then the algebraic formally semiuniversal deformation of $\varphi$ is a finite morphism of degree 2 onto its image, which is a deformation of $i(Y)$ in $\mathbb{P}^N$.

Proof. The result follows from Theorem 1.7 and Corollary 5.5. □

Remark 5.7. Let $X, Y$ and $\varphi$ be as in Notations 1.2 and 2.1, let $X$ be a minimal variety of general type of dimension $m$, $m \geq 2$ and let $\varphi$ be induced by a complete linear series. If $Y$ is a projective bundle over $\mathbb{P}^1$, assume $B$ is base-point-free. Then $\varphi$ is unobstructed, as argued in Theorem 2.6 and Remark 4.6.

The divisor $B$ being base-point-free is not a very restrictive condition. Indeed, $\mathcal{E}^{-2} = \omega_Y^{-2} \otimes \mathcal{E'}^\otimes 2$, where $\mathcal{E'}$ is ample. If $Y$ is “balanced”, e.g. if $e_1 = \cdots = e_{m-1} = 0$, then $\omega_Y^{-2}$ is ample and free, and so is $\mathcal{E}^{-2}$.

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