Coefficient Systems and Jacquet modules

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August 20, 2014

Abstract

Let $F$ be a locally compact non-archimedean field and $G$ denote the group of $F$-rational points of a reductive group $G$ assumed to be defined over $F$, semisimple, simply connected and of $F$-rank 1. Fix a maximal $F$-split torus $T$ in $G$ and write $L$ for the centralizer of $T$ in $G$. Let $P$ be a minimal parabolic subgroup of $G$ with Levi decomposition $P = LU$ and let $A$ be the apartment of the Bruhat-Tits building $X$ of $G$ attached to the torus $T$. Let $\mathcal{F}$ be a $G$-equivariant coefficient system on $X$. We show that we can attach to $\mathcal{F}$ a $L$-equivariant coefficient system on $X$. We show that we can attach to $\mathcal{F}$ a $L$-equivariant coefficient system $\mathcal{G}$ on $A$, such that the Jacquet module $H_0(X, \mathcal{F})_U$ is naturally isomorphic to $H_0(A, \mathcal{G})$ as $L$-module, where $H_0$ denote the 0-th homology module of a coefficient system.

As an application of this result, we prove the following. Let $\pi$ be an irreducible supercuspidal representation of $G$. Then there exist a maximal compact subgroup $K$ of $G$, as well as an irreducible smooth representation $\lambda$ of $K$ such that $\pi |_K$ contains $\lambda$, and such that the compactly induced representation $c$-ind$_K^G \lambda$ decomposes as a finite sum of irreducible supercuspidal representations.

Introduction

Let $F$ be a non-archimedean locally compact field of residue characteristic $p$ and $G$ denote the group of $F$-rational points of a reductive group $G$ defined over $F$. We assume that $G$ is semisimple, simply connected and of $F$-rank 1. Let $T$ be the group of rational points of a maximal split torus in $G$ and $L$ be its centralizer. Let $P$ be a minimal parabolic subgroup of $G$ with Levi decomposition $P = LU$. We denote by $A$ the apartment of the Bruhat-Tits building $X$ of $G$ attached to the torus $T$. Let $C$ be a field of characteristic $l \neq p$. Let $\mathcal{F}$ be a $G$-equivariant coefficient system of $C$-vector spaces on $X$ in the sense of Schneider and Stuhler [SS].
The homology module \( H_0(X, \mathcal{F}) \) is a smooth representation of \( G \). Our first result is the following (see Theorem (4.3)): there exists an \( L \)-equivariant coefficient system \( \mathcal{G} \) on \( \mathcal{A} \), naturally attached to \( \mathcal{F} \), such that the Jacquet module \( H_0(X, \mathcal{F})_U \) is isomorphic to \( H_0(\mathcal{A}, \mathcal{G}) \) as an \( L \)-module. The section spaces of \( \mathcal{G} \) are defined as follows. For any simplex \( \sigma \) of \( T \), the space \( \mathcal{F}_\sigma \) is a \( G_\sigma \)-module in a natural way, where \( G_\sigma \) is the stabilizer of \( \sigma \) in \( G \); we then define \( \mathcal{G}_\sigma \) as the space of \( G_\sigma \cap U \)-fixed vectors in \( \mathcal{F}_\sigma \).

This result may be viewed as a generalization of a result of Bushnell and Kutzko which in certain cases gives the Jacquet module of a compactly induced representation as a compactly induced representation ([BK] Lemma (10.3)).

As an application, we prove the following result.

**Theorem 0.1.** Let \( \pi \) be a complex irreducible supercuspidal representation of \( G \). Then there exist a maximal compact subgroup of \( G \), as well as an irreducible representation \( \lambda \) of \( K \), such that:

1. the compactly induced representation \( \text{c-ind}^G_K \lambda \) decomposes as a finite sum \( \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_s \) of irreducible supercuspidal representations of \( G \),
2. \( \pi \) contains the pair \( (K, \lambda) \) by restriction to \( K \), i.e. \( \pi \) is isomorphic to \( \pi_i \) for some \( i = 1, \ldots, s \).

The proof of this result relies on two ingredients. First we use the fact due to Schneider and Stuhler (cf. [SS]) that if \( G \) is the group of \( F \)-rational points of a reductive group defined over \( F \), then any irreducible smooth representation of \( G \) is isomorphic to \( H_0(X, \mathcal{F}) \), for some \( G \)-equivariant coefficient system on the building \( X \) of \( G \). The second ingredient (Theorem 5.1.2) is the fact that if \( \lambda \) is a finite dimensional representation of a maximal compact subgroup \( K \) of \( G \) such that \( \lambda^{U \cap K} = 0 \), then the induced representation \( \text{c-ind}^G_K \lambda \) is a finite direct sum of supercuspidal representations of \( G \).

The idea of the proof is then the following. We start with an irreducible supercuspidal representation \( \pi \) of \( G \) that we write \( H_0(X, \mathcal{F}) \) for some \( G \)-equivariant coefficient system \( \mathcal{F} \) on \( X \). By our first result, we have \( \pi_U = 0 = H_0(X, \mathcal{F})_U \simeq H_0(\mathcal{A}, \mathcal{G}) \). As a consequence the boundary map in the chain complex computing \( H_0(X, \mathcal{G}) \) is surjective. From this we prove a *technical lemma* asserting that, in the chain complex

\[
\partial : C_1(X, \mathcal{F}) \longrightarrow C_0(X, \mathcal{F})
\]

computing \( H_0(X, \mathcal{F}) \), certain special elements of \( C_0(X, \mathcal{F}) \) actually lies in the image of the boundary map \( \partial \). Now, by definition \( \pi \) is a quotient of \( C_0(X, \mathcal{F}) \), which may be written as a sum of certain compactly induced representations.
Certain of these induced representations are good in the sense that they are direct sums of a finite number of supercuspidal representations, the other which do not have this property are the bad ones. Our technical lemma allows then to prove that the bad parts of those compactly induced representations are killed in the quotient. From this it is not difficult to conclude that our Theorem holds.

It is a folklore conjecture that any irreducible supercuspidal representation of a $p$-adic reductive group should have a type in the sense of [BK]§4. The following result taken from [BK] gives a characterization of types for supercuspidal representations.

**Proposition 0.2** ([BK], Proposition (5.2), page 602.) Let $G$ be the group of $F$-rational points of a connected reductive group $\mathbf{G}$ defined over $F$, and let $\pi$ be an irreducible supercuspidal complex smooth representation of $G$. Let $Z$ denote the center of $\mathbf{G}$, $Z$ the group of its $F$-rational points and $^0Z$ the unique maximal compact subgroup of $Z$. Finally let $K$ be a compact open subgroup of $G$ containing $^0Z$ and $\rho$ be an irreducible smooth representation of $K$ such that the restriction of $\rho$ to $^0Z$ is a multiple of the central character of $\pi$.

Let $\tilde{\rho}$ denote some extension of $\rho$ to $ZK$. Then $(K, \rho)$ is a type for $\pi$ if and only if there exist unramified quasicharacters $\chi_1, \chi_2, \ldots, \chi_r$ of $G$ such that

$$c\text{-ind}^{G}_{ZK}\tilde{\rho} \simeq \bigoplus_{i=1}^{r} \pi \otimes \chi_i.$$ 

It follows that in our Theorem 0.1, the pair $(K, \rho)$ is a type for $\pi$ if and only if the representation $\pi_i$, $i = 1, \ldots, s$, are unramified twists of a single representation. Our result, in our framework of semisimple, simply connected, of $F$-rank 1 groups, may be seen as a first step in proving that irreducible supercuspidal representations have types.

In the litterature, when it is known that a given supercuspidal representation has a type, a stronger result is actually proved: the representation is compactly induced (i.e. $s = 1$ in our Theorem 0.1). If our result is less precise it avoids using the deep arithmetic structure of the reductive group $G$ and give a uniform proof for the class of groups that we consider. Moreover our result seems to be genuinely new for the group $\text{SL}(2, D)$, where $D$ is a central division $F$-algebra of degree $d > 1$, and for some non-split forms of $\text{Sp}_4$, $\text{Sp}_6$, $\text{Spin}_6$, $\text{Spin}_8$ and $\text{Spin}_{10}$, which are in the isogeny class of special unitary groups over quaternion algebras with involution.
This paper is organized as follows. Section 1 consists of some notation and of basic facts on the action of a rank 1 reductive group on its Bruhat-Tits building. In section 2 we review the notion of an equivariant coefficient system on $X$ or $\mathcal{A}$ and define the $L$-equivariant coefficient system $\mathcal{G}$. In section 3 we recall Bushnell and Kutzko’s result on the Jacquet module of a compactly induced representation and prove a slight generalization. Section 4 is devoted to the proof of a commutative diagram (Theorem 4.1) that is then applied to the proof of our first result (Theorem 4.3). The proof of Theorem (0.1) is given in section 5. In §5.1 we review the structure of compactly induced representations and we state the general theorem of Schneider and Stuhler. Our technical lemma is proved in §5.2 and the end of the proof of Theorem (0.1) is given in §5.3.

I would like to thank François Courtès and Peter Schneider for their comments, and Guy Henniart for his encouragement to tackle the rank 1 case (a first version of this work was restricted to the case of $\text{SL}(2, F)$). Finally, I thank Maarten Solleveld whose remarks helped me to improve a former version of this work.

1 Reductive groups of relative rank 1

For the relative theory of reductive algebraic groups we refer to Chapter V of [Bo]. Basic facts on the Bruhat-Tits building of a $p$-adic reductive group may be found in [Ti] and [BT]. The aim of this section is to fix the notation and gather together a few lemmas on the action of a rank 1 reductive group on its Bruhat-Tits building that we shall need later in the paper.

We fix an non archimedean locally compact field $F$. We fix a reductive $F$-algebraic group $G$ that we suppose semisimple, simply connected and of $F$-rank 1. If $H$ is an algebraic group defined over $F$, we shall denote by $H$ the group of its $F$-rational points.

We fix a maximal $F$-split torus $T$ in $G$ so that $T$ is isomorphic to the multiplicative group $\mathbb{G}_m$ over $F$. We denote by $L = Z_G(T)$ the centralizer of $T$ in $G$ and $N = N_G(T)$ its normalizer. The spherical Weyl group $W = N/L$ is a group with 2 elements. The group $L$ is the Levi component of a minimal parabolic subgroup $P$ defined over $F$ and one has $P = L.U$ (semidirect product), where $U = R_u(P)$ is the unipotent radical of $P$.

The locally compact group $L = L(F)$ is compact mod center and we denote by $L^0$ its unique maximal compact subgroup.

We denote by $X$ the Bruhat-Tits building of $G$ in the sense of [Ti]. It is a simplicial complex of dimension 1 (indeed a tree) on which $G$ acts
by simplicial automorphisms. For \( i = 0, 1 \), we denote by \( X_i \) the set of \( i \)-dimensional simplices of \( X \). Let \( \mathcal{A} \) be the apartment attached to \( \mathbf{T} \). As a simplicial complex it is isomorphic to the line \( \mathbb{Z} \), whose vertex set is the set of integers \( \mathbb{Z} \) and edge set is

\[
\{\{i, i + 1\}; \ i \in \mathbb{Z}\}
\]

We fix such an isomorphism in such a way that any \( u \in U \) fixes the vertex number \( k \) for \( k \) large enough. For \( i \in \mathbb{Z} \), we denote by \( s_i \) the vertex corresponding to \( i \), and by \( a_i \) the edge corresponding to \( \{i, i + 1\} \).

If \( \sigma \) is a simplex of \( X \), we denote by \( G_\sigma \) the \( G \)-stabilizer of \( \sigma \). Moreover we sometimes abbreviate \( G_0 = G_{s_0}, G_1 = G_{s_1} \) and \( I = G_{a_0} \). Since \( G \) is semisimple and simply connected, we have the following facts:

- \( I = G_0 \cap G_1 \) is an Iwahori subgroup of \( G \),
- the \( G_i, i = 0, 1 \) are maximal compact subgroups of \( G \),
- the action of \( G \) on the vertices of \( X \) has two orbits: \( G.s_0 \) and \( G.s_1 \).

The apartment \( \mathcal{A} \) is the Coxeter complex of an affine Weyl group of rank \( 1 \). If, for \( i = 0, 1 \), we denote by \( r_i \) the reflection relative to the wall \( \{s_i\} \), the affine Weyl group of \( \mathcal{A} \) is the dihedral group generated by \( r_0 \) and \( r_1 \). Recall that \( N \) acts on \( \mathcal{A} \) via affine isomorphisms and that the image of the corresponding morphism \( N \rightarrow \text{Aff}(\mathcal{A}) \) is the affine Weyl group of \( \mathcal{A} \). For \( i = 0, 1 \), we fix an element \( w_i \in N \) which induced the reflection \( r_i \) on \( \mathcal{A} \). The element \( w_1w_0 \) induces the translation \( t = r_1r_0 \). Let us notice that \( s_k = t^{k/2} s_0 \), if \( k \) is an even integer, and \( s_k = t^{(k-1)/2} s_1 \), if \( k \) is odd.

**Lemma 1.1** We have \( G = G_0P = G_1P \). Moreover \( G_0 \cap L = G_1 \cap L = L^0 \) the maximal compact subgroup of \( L \).

**Proof.** Since \( \mathcal{A} \) is a Coxeter complex of rank \( 1 \), all its vertices are automatically special. In particular we have \( G = G_0P = G_1P \). The compact group \( L^0 \subset N \) fixes \( \mathcal{A} \) pointwise, so that \( L^0 \subset G_i \cap L \), for \( i = 0, 1 \). On the other hand, for \( i = 0, 1 \), the intersection \( G_i \cap L \) is compact. So by maximality of \( L^0 \), we have the equality \( G_i \cap L = L^0 \).

**Lemma 1.2** Let \( k \in \mathbb{Z} \). We have the equalities:

\[ G_{sk} \cap P = L^0.(G_{sk} \cap U) \]

**Proof.** The map \( f : P \rightarrow L \), defined by: \( f(p) \) is the unique element of \( L \) such that there exists \( u \in U \) satisfying \( p = f(p)u \) is well defined and continuous. In particular \( f(P \cap G_{sk}) \) is a compact subset of \( L \). Since \( L \) normalizes \( U \), this is a subgroup of \( L \). On the other hand, we certainly have \( L^0 \subset f(P \cap G_{sk}) \), whence by maximality of \( L^0 \), we have \( f(P \cap G_{a_0}) = L^0 \). The lemma follows easily.
Lemma 1.3 Let $k$ be a fixed integer.

(i) We have $G_{s_k} \cap U = G_{a_k} \cap U$.

(ii) The group $G_k \cap U$ acts transitively on the set of edges $a$ of $X$ such that $a$ contains $s_k$ and $a \neq a_k$.

Proof. (i) We clearly have $G_{a_k} \cap U \subset G_{s_k} \cap U$. On the other hand if $u \in G_{a_k} \cap U$, then $u$ fixes $s_l$ for $l$ large enough, whence fixes the segment $[s_k, s_l]$ pointwise for $l$ large enough. In particular $u$ fixes $a_k = [s_k, s_{k+1}]$, that is $u \in G_{a_k}$.

(ii) By [BT] Corollaire (2.2.6), page 36, the stabilizer $G_{a_k}$ of $a_k$ acts transitively on the set of apartments containing $a_k$. So it acts transitively on the set of edges $a$ containing $s_k$ and different from $a_k$. Moreover by [BT] Proposition (5.2.11), page 100, we have the Iwahori decomposition $G_{a_k} = U_{a_k}L^0\bar{U}_{a_k}$, where $\bar{U}$ is the unipotent radical of the parabolic subgroup opposed to $P$ relative to $T$ and $\bar{U}_{a_k} = G_{a_k} \cap \bar{U}$. It follows that

$$\{a \in X_1; a \ni s_k, a \neq a_k\} = \{ka_{k-1}; k \in U_{a_k}L^0\bar{U}_{a_k}\} = \{ka_{k-1}; k \in U_{a_k}\}$$

since the subgroup $L^0\bar{U}_{a_k}$ fixes $a_{k-1}$.

Lemma 1.4 a) We have

$$G = I.N.P$$

b) More precisely, we have

$$G = I.L.U \cup I.Lw_0.U$$

and the natural map

$$\Psi : L^0 \setminus L \coprod L^0 \setminus Lw_0 \rightarrow I \setminus G/U,$$

given by $L^0l \mapsto llU$, $L^0lw_0 \mapsto llw_0U$, is a bijection.

Proof. a) This is a classical property of double Tits system: it is given by [BT] Théorème (5.1.3)(vi).

b) Moreover since $L$ acts on $A$ via translations, we have $w_0 \in N \setminus L$, so that $N/L \simeq \mathbb{Z}/2\mathbb{Z}$ is generated by the image of $w_0$ in $N/L$. This proves that $G = I.LU \cup ILw_0U$ and that $\Psi$ is surjective.

For the injectivity of $\Psi$, we have to prove that for $l_1, l_2 \in L, i \in I, u \in U$, we have:

(i) $il_1 = l_2u$ implies $l_1l_2^{-1} \in L^0$,
(ii) $il_1w_0 = l_2w_0u$ implies $l_1l_2^{-1} \in L^0$
(iii) $il_1 \neq l_2w_0u$. 

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If \( il_1 = l_2u \), then, since \( L \) normalizes \( U \), \( il_1 = u'l_2 \), for some \( u' \in U \), so that \( i = u'l_2l_1^{-1} \). In particular \( i \in G_0 \cap P = (U \cap G_0)(L \cap G_0) = (U \cap G_0)L_0 \), by lemma 1.2. Hence \( l_2l_1^{-1} \in L^0 \).

If \( il_1w_0 = l_2w_0u \), then \( il_1 = l_2(w_0uw_0^{-1}) \), where \( w_0uw_0^{-1} \in \tilde{U} \) the unipotent radical of the parabolic subgroup \( \tilde{P} \) opposed to \( P \) with respect to the torus \( T \). So (ii) follows from (i) by replacing \( P \) by \( \tilde{P} \).

To prove (iii), assume for a contradiction that \( il_1 = l_2w_0u \). There exists \( k_0 \in \mathbb{Z} \), such that \( u \) fixes \( s_k \) for \( k \geq k_0 \). Then for \( k \geq k_0 \), we have \( il_1s_k = l_2w_0s_k \). Two points of \( A \) conjugate by \( I \) must be equal, so that \( l_1s_k = l_2w_0s_k \), for \( k \geq k_0 \), that is \( l_1^{-1}l_2w_0s_k = s_k, k \geq k_0 \). Since the element \( l_1^{-1}l_2w_0 \) fixes a half-apartment it lies in \( L^0 \). This implies that \( w_0 \in l_2l_1^{-1}L^0 \subset L \), a contradiction.

**Lemma 1.5** The set of simplices of the apartment \( A \) is a fondamental domain for the action of \( U \) on the simplices of \( X \).

**Proof.** We must prove that for any simplex \( \sigma \) of \( X \) there exist \( u \in U \) and a unique \( \sigma_0 \subset A \) such that \( \sigma = u\sigma_0 \). If \( \sigma \) is a vertex (resp. an edge), the existence of \( \sigma_0 \) is a consequence of the decompositions \( G = ULG_{a_0} \), \( G = ULG_{s_1} \) (resp. of the decomposition \( G = ULNG_{a_0} \)) and of the fact that any vertex of \( X \) is conjugate to \( s_0 \) or to \( s_1 \) (resp. that any edge is conjugate to \( a_0 \)).

For the uniqueness, assume that \( \sigma_0 = u\sigma_0 \) for some simplex \( \sigma_0 \) of \( A \), and some \( u \in U \). Since \( \sigma_0 \) and \( u\sigma_0 \) are two conjugate simplices of \( A \), there exists \( l \in L \) such that \( \sigma_0 = lu\sigma_0 \). So \( lu \in G_{\sigma_0} \). If \( \sigma \) is a vertex, we apply Lemma 1.2 to obtain \( u \in G_{\sigma_0} \), so that \( u\sigma_0 = \sigma_0 \) as required. If \( \sigma_0 = \{s,t\} \) is an edge, we have \( lu \in G_s \cap G_t \) and we conclude by applying Lemma 1.2 twice.

## 2 Equivariant coefficient systems

We fix a commutative field \( C \). We denote by \( S(G) \) the category of smooth representations of \( G \) in \( C \)-vector spaces. In this section we make no assumption on the characteristic of \( C \) and we do not assume that it is algebraically closed.

### 2.1 \( G \)-equivariant coefficient systems on \( X \)

Following Schneider and Stuhler [SS], we define a coefficient system \( F \) (of \( C \)-vector spaces) on \( X \) to be a collection \((F_\sigma)_{\sigma}, (r^\sigma_r)_{r \subset \sigma})\), where:
- for each simplex \( \sigma \) of \( X \), \( F_\sigma \) is a \( C \)-vector space,
– for each pair of simplices $\tau \subset \sigma$, $r^\sigma_\tau : F_\sigma \rightarrow F_\tau$ is a linear map,
– we have $r^\sigma_\sigma = \text{id}_{F_\sigma}$, for any simplex $\sigma$,
– we have $r^\theta_\sigma \circ r^\sigma_\tau = r^\theta_\tau$, for any triple of simplices $\theta \subset \tau \subset \sigma$.

A $G$-equivariant coefficient system on $X$ is a collection $\mathcal{F} = (F_\sigma, (r^\sigma_\tau)_{\tau \subset \sigma}, \varphi_{\sigma,g})$, where

– $(F_\sigma, (r^\sigma_\tau)_{\tau \subset \sigma})$ is a coefficient system on $X$,
– for any $g \in G$ and any $\sigma$ simplex of $X$, $\varphi_{\sigma,g} : F_\sigma \rightarrow F_{g.\sigma}$ is an isomorphism of $\mathbb{C}$-vector spaces,
– for all $g, h \in G$, any $\sigma$ simplex of $X$, we have

$$\varphi_{g.\sigma,h} \circ \varphi_{\sigma,g} = \varphi_{\sigma,gh},$$
– for any $g \in G$, for any pair of simplices $\tau \subset \sigma$ of $X$, we have

$$\varphi_{\tau,g} \circ r^\sigma_\tau = r^{g.\sigma}_{g.\tau} \circ \varphi_{\sigma,g},$$
– for any simplex $\sigma$, the $G_\sigma$-module $F_\sigma$ is smooth, where the action of $G_\sigma$ is given by

$$g.v := \varphi_{\sigma,g}(v), \quad g \in G, \quad v \in F_\sigma.$$ 

If $\mathcal{F}$ is a $G$-equivariant coefficient system on $X$, for any simplex $\sigma$, we shall denote by $\rho_\sigma$ the natural representation of $G_\sigma$ in $F_\sigma$.

We fix incidence coefficients on $X$ in the following way. If $a$ is an edge and $s$ a vertex, we set $[a : s] = 0$, if $s \not\in a$, $[a : s] = 1$ is $s \in a$ and $s \in G.s_0$, and $[a : s] = -1$ if $s \in a$ and $s \in G.s_1$. Since $G$ is simply connected, its action on $X$ preserves incidence coefficients:

$$[g.a : g.s] = [a : s], \quad g \in G, \quad a \in X_1, \quad s \in X_0.$$

Let $\mathcal{F}$ be a fixed $G$-equivariant coefficient system on $X$. For $i = 1, 2$, the space $C_i(X, \mathcal{F})$ of $i$-chains of $X$ with coefficients in $\mathcal{F}$ is the space of functions $f : X_i \rightarrow \bigcup_{\text{dim } \sigma = i} F_\sigma$, such that $f$ has finite support and for any $i$-dimensional simplex $\sigma$, $f(\sigma) \in F_\sigma$. We have a boundary operator $\partial : C_1(X, \mathcal{F}) \rightarrow C_0(X, \mathcal{F})$ given by

$$\partial(\omega)(s) = \sum_{a \in X_1, \ a \ni s} [a : s] \omega(s).$$

The chain complex

$$0 \rightarrow C_1(X, \mathcal{F}) \rightarrow C_0(X, \mathcal{F}) \rightarrow 0$$
is by definition the chain complex of $X$ with coefficients in $F$. It is a complex in the category $S(G)$ and its homology spaces $H_i(X, F) \in S(G)$, $i = 0, 1$, are by definition the homology spaces of $X$ with coefficients in $F$. They are smooth representations of $G$.

**Remark.** The notion of an equivariant coefficient complex extends naturally to the case where $X$ is any simplicial complex endowed with an action of a locally profinite group $G$ via simplicial automorphisms. If moreover $X$ possesses an orientation with $G$-invariant incidence numbers, we may in the same way define the chain complex of $X$ with coefficients in any $G$-equivariant coefficient system $F$, as well as the homology groups $H_*(X, F)$. In particular we have the notion of a $T$-equivariant coefficient system on $A$.

We now describe the chain complex in terms of induced representations. Let us be given the compactly induced representation $\text{c-ind}^G_{G_{a_0}}(\rho_{a_0}, F_{a_0}) = \text{c-ind}^G_{G_{a_0}} F_{a_0}$ as the space of compactly supported functions $f : G \rightarrow F_{a_0}$ satisfying $f(kg) = \rho_{a_0}(k)f(g)$, $k \in G_{a_0}, g \in G$, where $G$ acts on functions by right translation.

**Lemma 2.1.1** The map $C_1(X, F) \rightarrow \text{c-ind}^G_{G_{a_0}} F_{a_0}$, $\omega \mapsto f_\omega$, where

$$f_\omega(g) = \varphi_{a_0, g^{-1}}^{-1} \omega(g^{-1}a_0), \ g \in G$$

is an isomorphism of $G$-modules. Its inverse is given by $f \mapsto \omega_f$, where

$$\omega_f(ga_0) = \varphi_{a_0, g} f(g^{-1}).$$

**Proof.** Straightforward.

Similarly, one may consider the two induced representations $\text{c-ind}^G_{G_{s_i}}(\rho_{s_i}, F_{s_i}) = \text{c-ind}^G_{G_{s_i}} F_{s_i}$, $i = 0, 1$, given in their standard models.

**Lemma 2.1.2** The map

$$C_0(X, F) \rightarrow \text{c-ind}^G_{G_{s_0}} F_{s_0} \oplus \text{c-ind}^G_{G_{s_1}} F_{s_1}, \ \alpha \mapsto (f_\alpha^0, f_\alpha^1)$$

given by

$$f_\alpha^0(g) = \varphi_{s_0, g^{-1}}^{-1} \alpha(g^{-1}s_0) \text{ and } f_\alpha^1(g) = \varphi_{s_1, g^{-1}}^{-1} \alpha(g^{-1}s_1), \ g \in G$$

is an isomorphism of $G$-modules, whose inverse is given by $(f^0, f^1) \mapsto \alpha$, where

$$\alpha(gs_i) = \varphi_{s_i, g} f^i(g^{-1}), \ i = 0, 1, \ g \in G.$$ 

**Proof.** Straightforward.
2.2 \( L \)-equivariant coefficient systems on \( A \)

Now let \( \mathcal{G} = ((G_\sigma), (R^\sigma_\tau), (\psi_{l,\sigma})) \) be an \( L \)-equivariant coefficient system on \( A \) and let \( (C_\ast(A, \mathcal{G}), \partial) \) denote the chain complex of \( A \) with coefficients in \( \mathcal{G} \). For any simplex \( \sigma \) of \( A \), we denote by \( \rho^\sigma_\sigma \) the natural representation of the stabilizer \( L_\sigma \) of \( \sigma \) in \( L \) (indeed \( L_\sigma = L^0 \)). Contrary to the case of \( X \), which admits a single \( G \)-orbit of edges, \( A \) has two \( L \)-orbits of edges: that of \( a_0 \) and that of \( a_{-1} = \{s_{-1}, s_0\} \).

The following lemmas whose proofs are straightforward will be useful.

**Lemma 2.2.1**

a) The map \( C_0(A, \mathcal{G}) \longrightarrow \text{c-ind}_{L_{a_0}}^L G_{a_0} \oplus \text{c-ind}_{L_{a_1}}^L G_{a_1}, \ \alpha \mapsto (F^0_\alpha, F^1_\alpha) \), given by

\[
F^i_\alpha(l) = \psi^{-1}_{s_{-i}, l-1} \alpha(l^{-1}.s_i), \ l \in L, \ i = 0, 1
\]

is an isomorphism of \( L \)-modules whose inverse is given by

\[
\alpha(l.s_i) = \psi_{s_{-i}, l} F^i_\alpha(l^{-1}), \ l \in L, \ i = 0, 1.
\]

b) The map \( C_1(A, \mathcal{G}) \longrightarrow \text{c-ind}_{L_{a_0}}^L G_{a_0} \oplus \text{c-ind}_{L_{a_{-1}}}^L G_{a_{-1}}, \ \omega \mapsto (F^0_\omega, F^1_\omega) \), given by

\[
F^i_\omega(l) = \psi^{-1}_{a_{-i}, l-1} \omega(l^{-1}.a_i), \ l \in L, \ i = 0, -1
\]

is an isomorphism of \( L \)-modules whose inverse is given by

\[
\omega(l.a_i) = \psi_{a_{-i}, l} F^i_\omega(l^{-1}), \ l \in L, \ i = 0, -1.
\]

Let us be given a \( G \)-equivariant coefficient system \( \mathcal{F} = ((\mathcal{F}_\sigma)_\sigma, (r^\sigma_\tau)_{\tau \subseteq \sigma}, (\varphi_{\sigma, \rho})) \) on \( X \). We shall attach to \( \mathcal{F} \) an \( L \)-equivariant coefficient system on \( A \) in a natural way.

For any simplex \( \sigma \) of \( X \), we set \( U_\sigma = U \cap G_\sigma \). We define a collection \( \mathcal{G} = ((G_\sigma)_\sigma, (R^\sigma_\tau)_\tau \subseteq \sigma, (\psi_{l,\sigma})) \) in the following way:

- for any simplex \( \sigma \) of \( A \), we set \( G_\sigma = \mathcal{F}^{U_\sigma}_\sigma \), the set of vectors fixed by \( \rho_\sigma(U_\sigma) \) in \( \mathcal{F}_\sigma \);

- for any pair of simplices \( \tau \subseteq \sigma \) in \( A \), we define a linear map \( R^\sigma_\tau : G_\sigma \longrightarrow G_\tau \) by

\[
R^\sigma_\tau(v) = \sum_{u \in U_\tau/U_\sigma} \rho_\tau(u) r^\sigma_\tau(v), \ v \in G_\sigma.
\]

- for \( l \in L \) and \( \sigma \) a simplex of \( A \), we define \( \psi_{l,\sigma} : G_\sigma \longrightarrow G_{l,\sigma} \) as the linear map induced by \( \varphi_{l,\sigma} \).

**Lemma 2.2.2** The collection \( \mathcal{G} = ((G_\sigma)_\sigma, (R^\sigma_\tau)_{\tau \subseteq \sigma}, (\psi_{l,\sigma})) \) is well defined and is an \( L \)-equivariant coefficient system on \( A \).

**Proof.** Easy and follows mainly from the equality \( U_{l,\sigma} = lU_\sigma l^{-1} \) due to the fact that \( L \) normalizes \( U \).
3 Compact induction and Jacquet modules

The aim of this section is to describe the Jacquet module of a compactly induced representation as a (sum of) compactly induced representation(s). This was done by Bushnell and Kutzko in [BK] (Lemma 10.3 of §10, page 628). We shall in fact need a slightly more general version of their lemma. In this section we assume that the characteristics $l$ of the field $C$ is different from the residue characteristics $p$ of $F$.

In this section only, $G$ denotes the group of $F$-rational points of a connected reductive $F$-algebraic group. We fix a parabolic subgroup $P$ with Levi decomposition $P = MU$, as well as a compact open subgroup $K$ of $G$. We make the following assumption:

(A) $K_M := K \cap M$ is a maximal compact subgroup of $M$.

Lemma 3.1 We have the equalities:

$$K \cap P = K_M (K \cap U) \text{ and } K_M = (KU) \cap M.$$  

Proof. The map $f : P \rightarrow M$, defined by: $f(p)$ is the unique element of $M$ such that there exists $u \in U$ satisfying $p = f(p)u$ is well defined and continuous. In particular $f(P \cap K)$ is a compact subset of $M$. Since $M$ normalizes $U$, this is a subgroup of $M$. On the other hand, we certainly have $K_M \subset f(P \cap K)$, whence by maximality of $K_M$, we have $f(P \cap K) = K_M$. The equalities follow then easily.

If $(\pi, \mathcal{V})$ is a smooth representation of $G$ in a $C$-vector space $\mathcal{V}$, we define its Jacquet module $(\pi_U, \mathcal{V}_U)$ as the natural representation of $M$ in $\mathcal{V}/\mathcal{V}[U]$, where $\mathcal{V}[U]$ is the subspace of $\mathcal{V}$ generated by the vectors of the form $\pi(u).v - v$, $u \in U$, $v \in \mathcal{V}$.

Since $U$ is a pro-$p$-group, we may and do fix a $C$-valued Haar measure $\mu$ on $U$.

Now, with the notation of loc. cit., take a smooth representation $(\rho, W)$ of $K$. Let $\rho_M$ be the representation of $K_M = (K \cap P)/(K \cap U)$ on the space $W^K_{K \cap U}$ of $\rho(K \cap U)$-fixed vectors in $W$.

Let us realize the representation c-ind$_K^G \rho$ in the usual model $\mathcal{V}$ of certain functions on $G$ with values in $W$. Following Bushnell and Kuztko, for $f \in \mathcal{V}$, we define $\Phi f : M \rightarrow W$ by

$$\Phi f(x) = \int_U f(ux) \, d\mu(u), \ x \in M$$  \hspace{1cm} (1)
Then one easily checks that $\Phi f$ belongs to the space of $c\text{-}\text{ind}_{K_{\text{M}}}^{M}\rho_{\text{M}}$ and that $\Phi$ defines an $M$-intertwining operator $\mathcal{V} \longrightarrow c\text{-}\text{ind}_{K_{\text{M}}}^{M}\rho_{\text{M}}$. Moreover it vanishes on $\mathcal{V}[U]$ and defines an element of $\text{Hom}_{M}(\mathcal{V}_{U}, c\text{-}\text{ind}_{K_{\text{M}}}^{M}\rho_{\text{M}})$ that we still denote by $\Phi$.

**Lemma 3.2** The intertwining operator

$$\Phi : \mathcal{V}_{U} \longrightarrow c\text{-}\text{ind}_{K_{\text{M}}}^{M}\rho_{\text{M}}$$

is surjective. Moreover, under the assumption that $G = KP$, it is injective.

**Proof.** The bijectivity of $\Phi$ when $G = KP$ is [BK], Lemma 10.3. We just have to observe that Bushnell and Kutzko do not use the assumption $G = KP$ to prove the surjectivity of $\Phi$.

**Remarks**

1. In their Lemma 10.3 of [BK], Bushnell and Kutzko assume that $W$ is finite dimensional. However they never use this assumption in their proof.

2. Bushnell and Kutzko normalize $\mu$ in such a way that $\mu(K \cap U) = 1$. The effect of another normalization multiplies $\Phi$ by a non-zero constant.

We now specialize to the case of our $F$-rank 1 group $G$. Here we take $P = LU$, the minimal parabolic subgroup introduced in §1. We consider the two compact open subgroups $I = G_{a_{0}}$ and $w_{0}^{-1}I = w_{0}^{-1}Iw_{0} = G_{a_{-1}}$. These Iwahori subgroups are not special, but they do satisfy assumption (A). For $i = 0, -1$, we set $U_{a_{i}} = U \cap G_{a_{i}}$. We fix a smooth representation $(\rho, W)$ of $I$ and write $\rho_{w_{0}^{-1}}^{w_{0}}$ for the representation of $w_{0}^{-1}I$ given by $\rho_{w_{0}^{-1}}^{w_{0}}(w_{0}^{-1}kw_{0}) = \rho(k)$.

The representations $c\text{-}\text{ind}_{I}^{G}\rho$ and $c\text{-}\text{ind}_{w_{0}^{-1}I}^{G}\rho_{w_{0}^{-1}}$ are isomorphic and a specific intertwining operator is $f \mapsto f_{w_{0}}^{w_{0}}$, where $f_{w_{0}}^{w_{0}}$ is defined by $f_{w_{0}}^{w_{0}}(g) = f(w_{0}g)$.

With the notation as above, we have $K_{M} = L^{0}$, if $K = I$ or $w_{0}^{-1}I$. Let $\rho_{L}$ denote the natural representation of $L^{0}$ in the space of $\rho(U_{a_{0}})$-fixed vectors in $\rho$, and similarly let $\rho_{T}^{w_{0}^{-1}}$ be the natural representation of $L^{0}$ in the space of $\rho_{w_{0}^{-1}}^{w_{0}^{-1}}(U_{a_{-1}})$-fixed vectors in $\rho_{w_{0}^{-1}}^{w_{0}^{-1}}$. For $f \in c\text{-}\text{ind}_{I}^{G}\rho$ (resp. $f \in c\text{-}\text{ind}_{I}^{G}\rho_{T}^{w_{0}^{-1}}$), we define $\Phi_{0}(f) \in c\text{-}\text{ind}_{L^{0}}^{L}\rho_{L}$ (resp. $\Phi_{-1}(f) \in c\text{-}\text{ind}_{L^{0}}^{L}\rho_{L}^{w_{0}^{-1}}$) by

$$\Phi_{0}f(x) = \int_{U} f(ux) \, d\mu(u), \quad x \in L$$

(resp. $\Phi_{-1}f(x) = \int_{U} f(ux) \, d\mu(u), \quad x \in L$).
Finally we consider the following map:

$$\Phi : \text{c-ind}_I^L \rho \longrightarrow \text{c-ind}_{I_0}^L \rho_L \oplus \text{c-ind}_{I_0}^{L_1} \rho_L^{w_0^{-1}}$$

$$f \mapsto (\Phi_0(f), \Phi_{-1}(f^{w_0}))$$

**Proposition 3.3**. The map $\Phi$ induces a bijective isomorphism of $L$-modules:

$$(\text{c-ind}_I^L \rho)_U \longrightarrow \text{c-ind}_{I_0}^L \rho_L \oplus \text{c-ind}_{I_0}^{L_1} \rho_L^{w_0^{-1}}$$

**Proof.** We closely follow the proof of [BK] Lemma (10.3), by replacing the assumption “$G = KP$”, which does not hold in our case, by Lemma 1.4.

Let us first prove the surjectivity of $\Phi$. Write $(\pi, V)$ for the induced representation $\text{c-ind}_I^L \rho$. Then $V$ is generated as a $C$-vector space by the $f_{g,v}$, $g \in G$, $v \in W$, where $f_{g,v}$ is the function with support $Ig$ given by $f_{g,v}(kg) = \rho(g).v$, $k \in I$. Since for $u \in U$ and $f \in V$, we have $\Phi(\pi(u)f) = \Phi(f)$, and using the identity $\pi(u)f_{g,v} = f_{g_{u,v}}$, we deduce that the image of $\Phi$ is generated as a $C$-vector space by the functions $\Phi(f_{g,v})$, where $u \in U$ and $g$ runs over a system of representatives of $I \backslash G$. Moreover by the previous lemma, we may make $g$ run over a system of representatives of $L^0 \backslash L \cup (L^0 \backslash L)w_0$. Hence we first calculate the images of functions of the form $f_{l,v}, f_{l_0,v_0}, l \in L, v \in W$.

Write $\rho_0$ (resp. $\rho_{-1}$) for the canonical $U_{\rho^{-1}}$-projection $W \longrightarrow W^\rho_0(U_{\rho^{-1}})$ (resp for the canonical $U_{\rho^{-1}}$-projection $W \longrightarrow W^\rho_{-1}(U_{\rho^{-1}})$). For $l \in L$, $v \in W^\rho_0(U_{\rho^{-1}})$, $v' \in W^\rho_{-1}(U_{\rho^{-1}})$, we define $F_{l,v} \in \text{c-ind}_{L_0}^L \rho_L$ and $H_{l,v'} \in \text{c-ind}_{L_0}^L \rho_L^{w_0^{-1}}$, with support $L^0l$, by the formulas:

$$F_{l,v}(l^0) = \rho_L(l^0).v, \quad H_{l,v'}(l^0) = \rho_L^{w_0^{-1}}(l^0).v', \quad l^0 \in T^0$$

Then the $F_{l,v}$ (resp. the $H_{l,v'}$) generate $\text{c-ind}_{L_0}^L \rho_L$ (resp. $\text{c-ind}_{L_0}^L \rho_L^{w_0^{-1}}$).

**Lemma 3.4**. For $l \in L$ and $v \in W$ we have:

a) $\Phi_0(f_{l,v}) = \mu(I \cap U)F_{l,p_0(v)}$, and $\Phi_{-1}(f_{l,v}^{w_0}) = 0$,

b) $\Phi_0(f_{l_0,v_0}) = 0$ and $\Phi_{-1}(f_{l_0,v_0}^{w_0}) = \mu(w_0^{-1}I \cap U)H_{l_0,v_0}^{w_0^{-1}}$.

**Proof.** For $x \in L$, we first compute

$$\Phi_0 f(x) = \int_N f(ux) d\mu(u) = \delta_P(x) \int_U f(xu) d\mu(u)$$

where $f = f_{l,v}$ and $\delta_P$ denote the modulus character of $P$. For $u \in U$, if $f(ux) \neq 0$ then $u \in Ilx^{-1}$, whence $1 \in Ilx^{-1}U$. By lemma 1.4 this implies
$l^{-1} \in L^0$. Therefore the support of $\Phi_0 f$ lies in $L^0 l = ll^0$. Write $x = l^0 l$, $l^0 \in L^0$. We have:

$$\Phi_0(l^0 l) = \int_N f(u l^0 l) \, d\mu(n) = \rho(l^0) \int_U f(u l) \, d\mu(u)$$

where we used the change of variable $u \mapsto (l^0)^{-1} u l^0$ and the fact that $\delta_{F}(l^0) = 1$. Now we have $f(u l) \neq 0$ if and only if $u \in I \cap U$ so that

$$\Phi_0 f(l^0 l) = \rho(l^0) \int_{I \cap U} f(u l) \, d\mu(u)
= \rho(l^0) \int_{I \cap U} \rho(u) v \, d\mu(u)
= \rho(l^0) \mu(I \cap U) p_0(v)$$

as required.

On the other hand, for $x \in L$, since $U x \cap Iw_0 = x U \cap Iw_0 = \emptyset$ (lemma 1.4), the support of $\Phi_0(f_{i w_0, v})$ is empty and therefore $\Phi_0(f_{i w_0, v}) = 0$. Similarly, we prove that $\Phi_{-1}(f_{i w_0}) = 0$.

Let $x \in L$ and $f := f_{i w_0, v}$. For $u \in U$, if $f^u(ux) \neq 0$, we have $w_0 u x \in ll^0 w_0$ so that $w_0 U x \cap I Iw_0 = w_0 x U \cap I Iw_0 \neq \emptyset$ and $x \in w_0^{-1} lw L^0$. It follows that the support of $\Phi_{-1}(f_{i w_0})$ is contained in $w_0^{-1} lw L^0$. Write $x = l^0 w_0^{-1} lw$. We have

$$\Phi_{-1}(f_{i w_0})(x) = \int_U f_{i w_0, v}(w_0 u l^0 w_0^{-1} lw_0) \, d\mu(u)
= \rho(w_0 l^0 w_0^{-1}) \int_U f_{i w_0, v}(w_0 u w_0^{-1} lw_0) \, d\mu(u)$$

Moreover $f_{i w_0, v}(w_0 u w_0^{-1} lw_0) \neq 0$ is and only if $w_0 u w_0^{-1} lw_0 \in I w_0$, that is $u \in w_0^{-1} I w_0$. Therefore

$$\Phi_{-1}(f_{i w_0})(x) = \rho_{w_0}^{-1}(l^0) \int_{U \cap w_0^{-1} l w_0} f_{i w_0, v}(w_0 u w_0^{-1} lw_0) \, d\mu(u)
= \rho_{w_0}^{-1}(l^0) \int_{U \cap w_0^{-1} l w_0} \rho_{w_0}^{-1}(u) v \, d\mu(u)
= \rho_{w_0}^{-1}(l^0) \mu(U \cap w_0^{-1} l w_0) p_0(v)$$

as required.

It follows from the lemma that $\Phi$ is onto. Indeed $c\text{-ind}_{l L^0}^L \rho_L \oplus c\text{-ind}_{l L^0}^L \rho_L^{-1}$ is generated as a $C$-vector space by the $(F_{l, v_0, H_{m, v_0}})$, $l, m \in L$, $v_0 \in W^0(U_{w_0})$,
\[v_{-1} \in W^{\rho^{-1}_{w_0}(U_{a_{-1}})},\text{ and}\]

\[
(F_{l,v_0}, H_{m,v_{-1}}) = \Phi\left(\frac{1}{\mu(I \cap U)} f_{l,v_0} + \frac{1}{\mu(w_0^{-1}I \cap U)} f_{w_0m_0w_0^{-1},v_{-1}}\right).
\]

For the injectivity of \(\Phi\), let \(\bar{f} = f \mod \mathcal{V}[U]\) be an element of \(\mathcal{V}/\mathcal{V}[U]\) such that \(\Phi(\bar{f}) = \Phi(f) = 0\). We must prove that \(f \in \mathcal{V}[U]\). One may write \(f\) as a linear combination

\[f = \sum_{i=1}^{n} x_i f_{g_i,v_i},\ g_i \in G, \ x_i \in \mathbb{C}, \ v_i \in W.\]

Moreover changing \(f\) by an element of \(\mathcal{V}[U]\), we may assume that \(f\) writes

\[f = \sum_{i} \lambda_i f_{l_i,v_i} + \sum_{j} \mu_j f_{m_j,w_0,v'_j}, \ \lambda_i, \ \mu_j \in \mathbb{C}, \ v_i, \ v'_j \in W\]

and where the \(l_i\) (resp. the \(m_j\)) are pairwise distinct elements in a set of representatives of \(L^0\setminus L\). Then

\[
\Phi(f) = \left(\sum_{i} \mu(I \cap U) F_{l_i,p_0(v_i)}, \sum_{j} \mu(w_0^{-1}I \cap U) H_{w_0^{-1}m_jw_0,p_{-1}(v'_j)}\right)
\]

so that for all \(i\), \(F_{l_i,p_0(v_i)} = 0\), and for all \(j\), \(H_{w_0^{-1}m_jw_0,p_{-1}(v'_j)} = 0\). So we may as well as assume that \(f\) has the form \(f_{l,v}\), or \(f_{l,w_0,v}\), for some \(l \in L\) and some \(v \in W\).

In the first case, we get \(F_{l,p_0(v)} = 0\), whence \(p_0(v) = 0\). It follows that \(v\) writes \(\sum_{i} (\rho(u_i)v'_i - v'_i)\), for some \(u_i \in U \cap I\) and \(v'_i \in W\). Using the identity

\[\pi(x^{-1}nx)f_{x,w} = f_{x,\rho(n),w},\ x \in L, \ n \in I \cap U, \ w \in W,\]

we obtain:

\[f_{l,v} = \sum_{i} \left(\pi(l^{-1}u_i l)f_{l_i,v'_i} - f_{l_i,v'_i}\right) \in \mathcal{V}[U]\]

as required. The second case is similar. This achieves the proof of the proposition.

### 4 Coefficient systems and Jacquet modules

In this section we assume that the characteristic \(l\) of \(\mathbb{C}\) is not \(p\) and we fix a \(\mathbb{C}\)-valued Haar measure \(\mu\) on \(U\).
We fix a $G$-equivariant coefficient system $F = ((F_\sigma), (r_\sigma^\sigma), (\varphi_{\sigma,\varphi}))$ on $X$.

For any simplex $\sigma$ of $X$, we set $U_\sigma = U \cap G_\sigma$, and we denote by $\rho_{\sigma}^U$ the natural representation of $L^0 = L \cap G_\sigma$ in the space $F_{\sigma}^{U_\sigma}$ of $\rho_\sigma(U_\sigma)$-invariant vectors in $F_\sigma$.

By Lemmas 1.1 and 3.2, we have an isomorphism of $L$-modules:

$$\Phi_{s_0} \oplus \Phi_{s_1} : (c\text{-}\text{ind}_{G_{a_0}}^G F_{s_0})_U \oplus (c\text{-}\text{ind}_{G_{s_1}}^G F_{s_1})_U \rightarrow c\text{-}\text{ind}_{L^0\rho_{s_0}}^L U_{a_0} \oplus c\text{-}\text{ind}_{L^0\rho_{s_1}}^L U_{s_1}$$

where, for $i = 0, 1$ and $\Phi \in (c\text{-}\text{ind}_{G_{s_1}}^G F_{s_1})_U$, we have

$$\Phi_{s_i}(\Phi)(x) = \int_U f(ux) \, d\mu(x), \ x \in L.$$

By Proposition 3.3, we have an isomorphism of $L$-modules:

$$\Phi : (c\text{-}\text{ind}_{G_{a_0}}^G F_{a_0})_U \rightarrow c\text{-}\text{ind}_{L^0\rho_{a_0}}^L U_{a_0} \oplus c\text{-}\text{ind}_{L^0\rho_{a_0}^w}^L U_{a_0}^{-1}$$

given by $\Phi \mapsto (\Phi_0(f), \Phi_{a_1}(f_{a_0}^{-1}))$, where for $i = 0, -1$, we have

$$\Phi_{a_i}(f) = \int_U f(ux) \, d\mu(x).$$

We denote by $\Psi_1 : C_1(X, F)_U \rightarrow (c\text{-}\text{ind}_{G_{a_0}}^G F_{a_0})_U$ the $L$-isomorphism induced by the $G$-isomorphism $C_1(X, F) \rightarrow c\text{-}\text{ind}_{G_{a_0}}^G F_{a_0}$ of Lemma 2.1.1. Similarly, we denote by

$$\Psi_0 : C_0(X, F)_U \rightarrow (c\text{-}\text{ind}_{G_{a_0}}^G F_{a_0})_U \oplus (c\text{-}\text{ind}_{G_{s_1}}^G F_{s_1})_U$$

the $L$-isomorphism induced by the $G$-isomorphism of Lemma 2.1.2.

One easily check that the map $f \mapsto \varphi_{a_0, w_0^{-1}} \circ f$ induces an isomorphism of $L$-modules:

$$c\text{-}\text{ind}_{L^0\rho_{a_0}}^L (\rho_{a_0}^{-1}) U_{a_0} \rightarrow c\text{-}\text{ind}_{L^0\rho_{a_0}^{-1}}^L U_{a_0}^{-1}.$$

It follows that we have an isomorphism of $L$-modules:

$$\Psi : c\text{-}\text{ind}_{L^0\rho_{a_0}^w}^L U_{a_0} \oplus c\text{-}\text{ind}_{L^0\rho_{a_0}^w}^L U_{a_0}^{-1} \rightarrow c\text{-}\text{ind}_{L^0\rho_{a_0}}^L U_{a_0} \oplus c\text{-}\text{ind}_{L^0\rho_{a_0}^{-1}}^L U_{a_0}^{-1}$$

given by $(f_0, f_{-1}) \mapsto (f_0, \varphi_{a_0, w_0^{-1}} \circ f_{-1})$.

Let $G$ be the $L$-equivariant coefficient system on $A$ attached to $F$ as in Lemma 2.2.2. Using Lemma 2.2.1, we have natural $L$-isomorphisms:

$$\Psi_0 : c\text{-}\text{ind}_{L^0\rho_{s_0}}^L U_{s_0} \oplus c\text{-}\text{ind}_{L^0\rho_{s_1}}^L U_{s_1} \rightarrow C_0(A, G).$$
and

$$\Psi'_1 : \text{c-ind}_{L_0}^L U_{\rho_0} \oplus \text{c-ind}_{L_0}^L U_{\rho_{-1}} \rightarrow C_0(\mathcal{A}, \mathcal{G})$$

Now define $L$-isomorphisms

$$\varphi_0 : C_0(X, \mathcal{F})_U \rightarrow C_0(\mathcal{A}, \mathcal{G}), \quad \varphi_1 : C_1(X, \mathcal{F})_U \rightarrow C_1(\mathcal{A}, \mathcal{G})$$

by

$$\varphi_0 = \Psi'_0 \circ (\Phi_{s_0} \oplus \Phi_{s_1}) \circ \Psi_0, \quad \varphi_1 = \Psi'_1 \circ \Upsilon \circ \Phi \circ \Psi_1.$$

**Theorem 4.1** Write $\partial_A$ for the boundary map $C_1(\mathcal{A}, \mathcal{G}) \rightarrow C_0(\mathcal{A}, \mathcal{G})$. Then the following diagram is commutative:

$$
\begin{array}{ccc}
C_1(X, \mathcal{F})_U & \xrightarrow{\partial_U} & C_0(X, \mathcal{F})_U \\
\downarrow \varphi_1 & & \downarrow \varphi_0 \\
C_1(\mathcal{A}, \mathcal{G}) & \xrightarrow{\partial_A} & C_0(\mathcal{A}, \mathcal{G})
\end{array}
$$

where $\partial_U$ is the map induced by the boundary map $\partial : C_1(X, \mathcal{F}) \rightarrow C_0(X, \mathcal{F})$ on Jacquet modules.

**Proof.** We start by giving much simpler formulas for the maps $\varphi_0$ and $\varphi_1$. For $i = 0, 1$, let us write $\pi_i$ for the natural smooth representation of $G$ in $C_i(X, \mathcal{F})$. For $\omega \in C_1(X, \mathcal{F})$, $\alpha \in C_0(X, \mathcal{F})$, $g \in G$, $a \in X_1$ and $s \in X_0$ we have

$$(\pi_1(g)\omega)(a) = \varphi_{g^{-1}a,g}\omega(g^{-1}a), \quad (\pi_0(g)\alpha)(s) = \varphi_{g^{-1}s,g}\alpha(g^{-1}s).$$

**Lemma 4.2** Let $\bar{\omega} \in C_1(X, \mathcal{F})_U$ and $\bar{\alpha} \in C_0(X, \mathcal{F})$. Then we have

$$\varphi_1(\bar{\omega})(la_i) = \delta_P(l) \int_U (\pi_1(u)\omega)(la_i) \, d\mu(u), \quad l \in L, \quad i = -1, 0,$$

and

$$\varphi_0(\bar{\alpha})(ls_i) = \delta_P(l) \int_U (\pi_0(u)\alpha)(ls_i) \, d\mu(u), \quad l \in L, \quad i = 0, 1,$$

where $\delta_P$ is the modulus character of $L$ corresponding to the change of variable $d\mu(u) = \delta_P(l) d\mu(lul^{-1})$, $u \in U$, $l \in L$.

**Proof of the Lemma.** Let $\alpha \in C_0(X, \mathcal{F})$. Then $\Psi_0(\alpha) = (f^0_\alpha, f^1_\alpha)$, where for $i = 0, 1$ and $g \in G$, $f^i_\alpha(g) = \varphi_{s_i,g^{-1}}\alpha(g^{-1}s_i)$. Then for $i = 0, 1$, we have

$$\Phi_{s_i}(f^i_\alpha)(x) = \int_U \varphi_{s_i,ux}^{-1}\alpha((ux)^{-1}s_i) \, d\mu(u), \quad x \in L.$$
It follows that for $i = 0, 1$ and $l \in L$, we have
\[
\varphi_{0}(\bar{\alpha})(ls_i) = \varphi_{s_i,l}\Phi_{s_i}(f_{\alpha}^{-1})(l^{-1})
\]
\[
= \varphi_{s_i,l} \int_{U} \varphi_{s_i,lu^{-1}} \alpha(lu^{-1}s_i) \, d\mu(u)
\]
\[
= \int_{U} \varphi_{s_i,l} \circ \varphi_{lu^{-1}s_i,ul^{-1}} \alpha(lu^{-1}s_i) \, d\mu(u)
\]
\[
= \int_{U} \varphi((lu^{-1}l^{-1})ls_i,ul^{-1}) \alpha((lu^{-1}l^{-1})ls_i) \, d\mu(u)
\]
\[
= \delta_{P}(l) \int_{U} \varphi_{v^{-1}(ls_i),v} \alpha(v^{-1}(ls_i)) \, d\mu(v)
\]
\[
= \delta_{P}(l) \int_{U} (\pi_{0}(v) \alpha)(ls_i) \, d\mu(v)
\]
as required.

Now let $\omega \in C_{1}(X, F)$. For $g \in G$, we write $\Psi_{1}(\omega)(g) = \varphi_{a_0,g^{-1}}(\omega(g^{-1}a_0))$, and $\Phi(\Psi_{1}(\omega)) = (F_{0}, F_{-1})$, where
\[
F_{0}(x) = \int_{U} \varphi_{a_0,(ux)^{-1}}^{-1} \omega((ux)^{-1}a_0) \, d\mu(u)
\]
\[
F_{-1}(x) = \int_{U} \varphi_{a_0,(w_{0}ux)^{-1}}^{-1} \omega((w_{0}ux)^{-1}a_0) \, d\mu(u), \quad x \in L.
\]

Moreover we have
\[
\Upsilon \circ \Phi(\Psi_{1}(\omega)) = (F_{0}, \varphi_{a_0,w_{0}^{-1}} F_{-1})
\]
Let $l \in L$. The proof of the equality
\[
\varphi_{1}(\bar{\omega})(la_0) = \delta_{P}(l) \int_{U} (\pi_{1}(u) \omega)(la_0) \, d\mu(u)
\]
is similar to the case of $\alpha$. So we compute:
\[
\varphi_{1}(\bar{\omega})(la_{-1}) = \varphi_{a_{-1},l} \varphi_{a_0,w_{0}^{-1}} \int_{U} \varphi_{a_0,(w_{0}ul^{-1})^{-1}}^{-1} \omega((w_{0}ul^{-1})^{-1}a_0) \, d\mu(u)
\]
\[
= \int_{U} \varphi_{a_{-1},l} \varphi_{a_0,w_{0}^{-1}} \varphi_{lu^{-1}(w_{0}^{-1}a_0),ul^{-1}} \omega(lu^{-1}(w_{0}^{-1}a_0)) \, d\mu(u)
\]
\[
= \int_{U} \varphi((lu^{-1}l^{-1})(la_{-1}),ul^{-1}) \omega((lu^{-1}l^{-1})(la_{-1})) \, d\mu(u)
\]
\[
= \delta_{P}(l) \int_{U} (\pi_{1}(v) \omega)(la_{-1}) \, d\mu(v)
\]
as required.

Let us go back to proof of the Theorem. We must prove that for \( \bar{\omega} \in C_1(X, \mathcal{F})_U, l \in L \) and \( i = 0, 1 \), we have

\[
\phi_0(\partial_U(\bar{\omega}))(l s_i) = \partial_A(\phi_1(\bar{\omega}))(l s_i).
\]

We give a proof for \( i = 0 \), the other case being similar.

For any \( \omega' \in C_1(A, \mathcal{F}) \) and \( l \in L \), we have:

\[
\partial_A(\omega')(l s_0) = R^{l a_0}_{l s_0} \omega'(l a_0) + R^{l a_{-1}}_{l s_0} \omega'(l a_{-1}).
\]

Therefore we have

\[
\partial_A(\phi_1(\bar{\omega}))(l s_0) = \sum_{i=-1,0} \delta_P(l) R^{l a_i}_{l s_0} \int_U \phi_{u^{-1}la_i,u}(u^{-1}la_i) \, d\mu(u)
\]

\[
= \sum_{i=-1,0} \delta_P(l) \phi_{s_0,l} \circ R^{a_i}_{s_0} \circ \phi_{a_i,l}^{-1} \int_U \phi_{u^{-1}la_i,u}(u^{-1}la_i) \, d\mu(u)
\]

\[
= \delta_P(l) \phi_{s_0,l} R^{a_i}_{s_0} \int_U \phi_{u^{-1}la_i,l^{-1}u}(u^{-1}la_i) \, d\mu(u)
\]

\[
= \phi_{s_0,l} \sum_{i=-1,0} R^{a_i}_{s_0} \int_U \phi_{l v^{-1}a_i,l v^{-1}a_i}(u^{-1}la_i) \, d\mu(u)
\]

by the change of variable \( v = l^{-1}u l \).

On the other hand, for \( s \in X_0 \), we have

\[
\partial(\omega)(s) = \sum_{a \ni s} [a : s] \omega(a)
\]
and

$$\varphi_0(\partial(\bar{w}))(l_{s_0}) = \delta_p(l) \int_{U} \varphi_{u^{-1}l_{s_0}u} \partial_U(\omega)(u^{-1}l_{s_0}) \, d\mu(u)$$

$$= \delta_p(l) \int_{U} \sum_{a \in u^{-1}l_{s_0}} [a : u^{-1}l_{s_0}] r_{u^{-1}l_{s_0}}^a (a) \, d\mu(u)$$

$$= \delta_p(l) \int_{U} \varphi_{u^{-1}l_{s_0}u} \sum_{a \in s_0} [u^{-1}la : u^{-1}l_{s_0}] r_{u^{-1}l_{s_0}}^a \omega(u^{-1}la) \, d\mu(u)$$

$$= \delta_p(l) \int_{U} \varphi_{u^{-1}l_{s_0}u} \sum_{a \in s_0} r_{u^{-1}l_{s_0}}^a \omega(u^{-1}la) \, d\mu(u)$$

$$= \delta_p(l) \int_{U} \sum_{a \in s_0} \varphi_{u^{-1}l_{s_0}u} \varphi_{s_0,u^{-1}l} \circ r_{s_0}^a \circ \varphi_{a,u^{-1}l}^{-1} \omega(ul^{-1}a) \, d\mu(u)$$

$$= \delta_p(l) \varphi_{s_0,l} \int_{U} \sum_{a \in s_0} r_{s_0}^a \varphi_{u^{-1}la,l^{-1}u} \omega(ul^{-1}a) \, d\mu(u)$$

$$= \varphi_{s_0,l} \int_{U} \sum_{a \in s_0} \varphi_{ul^{-1}a,vl^{-1}a} \omega(ul^{-1}a) \, d\mu(v)$$

by the change of variable $v = l^{-1}u_l$.

By Lemma (1.2)(ii), the edges of $X$ containing $s_0$ are $a_0$ and the $ua_{-1}$, where $u$ runs over a set of representatives of $G_{s_0} \cap U$ or $a_{-1}$. Moreover by Lemma (1.2)(i), we have $r_{s_0}^{-1} = R_{s_0}^{\alpha_0}$. So we obtain

$$\varphi_0(\partial(\bar{w}))(l_{s_0}) = \varphi_{s_0,l} R_{s_0}^{\alpha_0} \int_{U} \varphi_{ul^{-1}a_0,vl^{-1}a} \omega(ul^{-1}a_0) \, d\mu(v) + \Sigma$$

where the term $\Sigma$ is given by

$$\Sigma = \varphi_{s_0,l} \sum_{u \in U_{s_0}/U_{a_{-1}}} r_{s_0}^{u a_{-1}} \int_{U} \varphi_{ul^{-1}u a_{-1},vl^{-1}u} \omega(ul^{-1}u a_{-1}) \, d\mu(v)$$

$$= \varphi_{s_0,l} \sum_{u \in U_{s_0}/U_{a_{-1}}} \varphi_{s_0,u} \varphi_{a_{-1},u}^{-1} \int_{U} \varphi_{ul^{-1}u a_{-1},vl^{-1}u}^{-1} \omega(ul^{-1}u a_{-1}) \, d\mu(v)$$

$$= \varphi_{s_0,l} \sum_{u \in U_{s_0}/U_{a_{-1}}} \varphi_{s_0,u} \varphi_{a_{-1},u}^{-1} \int_{U} \varphi_{ul^{-1}a_{-1},vl^{-1}u} \omega(ul^{-1}a_{-1}) \, d\mu(v)$$

$$= \varphi_{s_0,l} \sum_{u \in U_{s_0}/U_{a_{-1}}} \varphi_{s_0,u} \varphi_{a_{-1},u}^{-1} \int_{U} \varphi_{ul^{-1}a_{-1},vl^{-1}u} \omega(ul^{-1}a_{-1}) \, d\mu(v)$$

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by the change of variable $w = u^{-1}v$. By definition of the restriction map $R_{s_0}^{a_{-1}}$, we finally obtain

$$
\Sigma = \varphi_{s_0} \cdot R_{s_0}^{a_{-1}} \int_{U} \varphi_{lw^{-1}a_{-1},wl^{-1}} \omega(lw^{-1}a_{-1}) d\mu(w)
$$

and this proves the commutativity of the diagram.

**Theorem 4.3** Let $\mathcal{F}$ be a $G$-equivariant coefficient system on $X$. Let $\mathcal{G}$ be the $L$-equivariant coefficient system on $A$ attached to $\mathcal{F}$ as in §2.2. Then we have an isomorphism of $L$-modules:

$$
H_0(X, \mathcal{F})_U \simeq H_0(A, \mathcal{G})
$$

**Proof.** This is a direct consequence of Theorem 4.1 and of the exactness of the Jacquet module functor.

## 5 Application to supercuspidal representations

In this section the field $\mathbb{C}$ is assumed to be algebraically closed and of characteristic 0.

We now apply Theorem 4.3 to supercuspidal representations. More precisely, we shall prove the following result.

**Theorem 5.1** Let $\pi$ be an irreducible supercuspidal representation of $G$. Then there exists a maximal compact subgroup $K \in \{G_{s_0}, G_{s_1}\}$, an irreducible smooth representation $\lambda$ of $K$ such that:

i) $c\text{-}\text{ind}_K^G \lambda$ is a finite sum $\pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_s$ of irreducible supercuspidal representations.

ii) there exists $i \in \{1, \ldots, s\}$ such that $\pi$ is isomorphic to $\pi_i$.

### 5.1 Some useful general results

The proof of Theorem (5.1) relies on two results that we review in this section. These results are quite general and hold for any $p$-adic reductive groups.

We first need to review the Bernstein decomposition of the category $\mathcal{S}(G)$ of smooth representations of $G$. This decomposition is actually available for any connected reductive group over $F$. For more details the reader may refer to [BK]§1, or to [Be]§2. Recall that a *cuspidal pair* in $G$ is a couple $(M, \sigma)$ formed of a Levi subgroup $M$ of $G$ and of an irreducible supercuspidal representation $\sigma$ of $M$. An unramified character of $M$ is a character of the
form \( g \mapsto |\chi(g)|_F^s \), where \( \chi : M \rightarrow G_m \) is an \( F \)-rational character of \( M \), \( s \) a complex number, and where \( | - |_F \) denotes the absolute value of \( F \) (normalized in such a way that e.g. \( |\varpi_F|_F = 1 \), for any uniformizer \( \varpi_F \) of \( F \)). Let us notice that any unramified character of \( G \) is trivial. Two cuspidal pairs \( (M_i, \sigma_i), i = 1, 2 \), are called inertially equivalent if there exist \( g \in G \) and an unramified character \( \phi \) of \( L_2 \) such that \( M_2 = gM_1g^{-1} \) and \( \sigma_2 \simeq \sigma_1^g \otimes \phi \).

We let \( \mathcal{B}(G) \) denote the set of equivalence classes \( [M, \sigma] \) of cuspidal pairs.

For \( s = [M, \sigma] \in \mathcal{B}(G) \), we denote by \( S_s(G) \) the full subcategory of \( S(G) \) whose objects are the smooth representations \( \pi \) of \( G \) satisfying: any irreducible subquotient \( \pi' \) of \( \pi \) is a subquotient of a parabolically induced representation \( \text{ind}^G_P\sigma \otimes \phi \), for some parabolic subgroup \( P \) of \( G \) with Levi component \( M \), and some unramified character \( \phi \) of \( M \).

Then the category \( S(G) \) decomposes as the direct sum:

\[
S(G) = \prod_{s \in \mathcal{B}(G)} S_s(G). \tag{2}
\]

We shall need the following finiteness result.

**Proposition 5.1.1** Let \( K \) be an open compact subgroup of \( G \). Let \( (\pi, \mathcal{V}) \) be an object of \( S(G) \) such that \( \mathcal{V} \) is generated by \( \mathcal{V}^K \) as a \( G \)-module. Let

\[
\mathcal{V} = \bigoplus_{s \in \mathcal{B}(G)} \mathcal{V}_s
\]

be the decomposition of \( \pi \) with respect to (2). Then for all but a finite number of \( s \), we have \( V_s \neq 0 \).

**Proof.** This is a direct consequence of [Be] Corollaire (3.9)(i), page 29.

**Theorem 5.1.2** Let \( K = G_{s_i} \) for \( i = 0 \) or \( 1 \). Let \( (\lambda, W) \) be a finite dimension smooth representation of \( K \). Assume that \( W \) has no non-zero fixed vectors under \( \rho(U \cap K) \). Then \( (\pi, \mathcal{V}) = \text{c-ind}^G_K \lambda \) is a finite sum of irreducible supercuspidal representations.

**Proof.** By the Bernstein decomposition of \( S(G) \), we may write \( \mathcal{V} = \mathcal{V}_{\text{cusp}} \oplus \mathcal{V}_\infty \), where \( \mathcal{V}_{\text{cusp}} \) is supercuspidal, and where no irreducible subquotient of \( \mathcal{V}_\infty \) is supercuspidal. By Lemma (3.2) we have \( \pi_U = 0 \). Hence the representation \( \mathcal{V} \) is supercuspidal so that \( \mathcal{V}_\infty = 0 \). If \( K \) is a good open compact subgroup of \( G \) contained in the kernel of \( \lambda \), we have that \( \mathcal{V} \) is generated by \( \mathcal{V}^K \).

By applying Proposition 5.1.1, there exists a finite number \( \sigma_1, ..., \sigma_s \) of supercuspidal representations of \( G \) such that

\[
\pi = \bigoplus_{i=1,...,s} \pi_i
\]

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with \( \pi_i \in S_{[G,\sigma_i]} \), \( i = 1, ..., s \). Since \( \pi \) is finitely generated, so is each \( \pi_i \). It follows that the representations \( \pi_i \) are finitely generated and cuspidal, whence admissible by [BZ] Corollay (2.41). Since any irreducible subquotient of \( \pi_i \) is isomorphic to \( \sigma_i \), it follows that \( \pi_i \) is of finite length. The connected center of \( G \) being trivial, any supercuspidal irreducible representation of \( G \) is a projective object of \( S(G) \) ([Cas] Theorem 5.4.1). Now a basic inductive argument shows that each \( \pi_i \) is a direct sum of a finite number of irreducible representations isomorphic to \( \sigma_i \). The theorem follows.

**Theorem 5.1.3** (Schneider and Stuhler) Let \( G \) be the group of \( F \)-rational points of a reductive group defined over \( F \). Let \( X \) be it semisimple Bruhat-Tits building. Then for all irreducible smooth representation \((\pi, V)\) of \( G \), there exists an equivariant coefficient system \( F \) on \( X \) such that \( \pi \cong H_0(X, F) \) and such that for all polysimplex \( \sigma \) of \( X \), the space of sections \( F_\sigma \) is finite dimensional.

**Proof.** The existence of a \( G \)-equivariant coefficient system \( F \) such that \( \pi \cong H_0(X, F) \) is given by Theorem II.3.1 of [SS]. In fact Schneider and Stuhler do not construct a single coefficient system \( F \) but a family depending on an integer \( e \) chosen large enough.

Moreover for any polysimplex \( \sigma \), the space \( F_\sigma \) is the set of vectors in \( V \) fixed by a certain congruence subgroups of the parahoric subgroup fixing \( \sigma \). Since this congruence subgroup is open and since \( \pi \) is admissible, \( F_\sigma \) is indeed finite dimensional.

Note that the coefficient system \( F \) of the theorem is far from being unique.

### 5.2 A technical lemma

Let \( F = ((F_\sigma)_\sigma, (r_\tau^\sigma)_{\tau \subset \sigma}, (\varphi_{\sigma,g})_{\sigma,g}) \) be a \( G \)-equivariant coefficient system on \( X \). Let \( G = ((\mathcal{G}_\sigma)_\sigma, (R_\tau^\sigma)_{\tau \subset \sigma}, (\psi_{\sigma,g})_{\sigma,g}) \) be the \( T \)-equivariant coefficient system on \( \mathcal{A} \) attached to \( F \) as in section (2.2).

For \( i \in \{0, 1\} \) and \( v \in F^{U_{\alpha_i}} \), let \( \alpha = \alpha_{i,v} \) be the 0-chain of \( F \) with support \( \{s_i\} \) and defined by \( \alpha(v) = v \).

**Lemma 5.2.1** With the notation as above, if \( H_0(X, F)_U = 0 \), then there exists \( \omega \in C_1(X, F) \) such that \( \alpha_{i,v} = \partial(\omega) \).

**Proof.** Since \( H_0(X, F)_U = 0 \), we have \( H_0(\mathcal{A}, \mathcal{G}) = 0 \) by the isomorphism of Theorem 4.3. It follows that the boundary map \( \partial_\mathcal{A} : C_1(\mathcal{A}, \mathcal{G}) \rightarrow C_0(\mathcal{A}, \mathcal{G}) \) is surjective. We may view \( \alpha \) as a 0-chain of \( \mathcal{G} \), and there exists \( \omega_T \in C_1(\mathcal{A}, \mathcal{G}) \) such that \( \alpha = \partial_T(\omega_T) \).
The support \( \text{Supp}(\omega_T) \) of \( \omega_T \) consists of a finite number of vertices. Hence there exist two integers \( k < l \) such that

\[
\text{Supp}(\omega_T) \subset [s_k, s_l] := \{s_k, s_{k+1}, \ldots, s_{l-1}, s_l\}.
\]

We are going to define a 1-chain \( \omega \in C_1(X, F) \). For this we have to give a value to \( \omega(a) \) for all edges \( a \in X_1 \).

**Case 1. Assume first that** \( a \in A \). We set \( \omega(a) = 0 \) is \( a \) does not lie on the geodesic segments \([s_k, s_l] \). If \( a = [s_{i-1}, s_i], i \in \{k+1, k+2, \ldots, l\} \), we set \( \omega(a) = \omega_T(a) \).

**Case 2. Assume that** \( a \notin A \). Let \( m \) be the middle of the edge \( a \), and \( s_j \), for some \( j \in \mathbb{Z} \), be the projection of \( m \) on the apartment \( A \) (i.e. the unique vertex \( s \) of \( A \) which makes the distance \( d(m, s) \) minimal). If \( j > l \), we set \( \omega(a) = 0 \). If \( j \leq l \), there exist \( u \in U \) and a unique \( i \leq l \) such that \( a = u[s_{i-1}, s_i] \). Here we used the fact that the apartment is a fundamental domain for the action of \( U \) on the simplices of \( X \) (Lemma 1.5). We then set \( \omega(a) = \varphi_{a_T, u_1} \omega_T(a) \), where \( a_T = [s_{i-1}, s_i] \), and \( \omega(a_T) \) is defined as in case 1.

The chain \( \omega \) is well defined. Indeed, with the notation of Case 2, assume that we have \( a = u_1 a_T = u_2 a_T \), then \( u_2^{-1} u_1 \in G_{a_T} \cap U = U_{a_T} \). So

\[
\varphi_{a_T, u_1} \omega_T(a_T) = \varphi_{a_T, u_2 u_1^{-1} u_1} \omega_T(a_T) \\
= \varphi_{a_T, u_2} \circ \varphi_{u_1^{-1} u_1 a_T, u_2} \omega_T(a_T) \\
= \varphi_{a_T, u_2} \omega_T(a_T)
\]

since \( \omega_T(a_T) \) is fixed by the action of \( U_{a_T} \).

Moreover \( \omega \) has finite support. Indeed if \( \omega(a) \neq 0 \), then with the notation of the definition, the projection of \( m \) onto \( A \) is \( s_m \), for some \( m \leq l \), and \( a \) may be written \( a = u a_T \), with \( u \in U \) and \( a_T \) belonging to the finite set of edges

\[
\{[s_k, s_{k+1}], [s_{k+1}, s_{k+2}], \ldots, [s_{l-1}, s_l]\}
\]

Moreover any \( u \) such that \( a = u a_T \) must fix \( s_k \) and therefore lies in the compact group \( U_{s_k} \). From this we deduce that \( \text{Supp}(\omega) \) is finite.

It remains to prove that \( \partial(\omega) = \alpha \), that is:

\[
\sum_{a \in s} [a : s] r_s^a \omega(a) = \alpha(s), \ s \in X_0.
\]

(3)

We split the proof of this equality in three cases.
Case 1. Assume that the projection of $s$ onto $A$ is $s_i$ for some $i > l$. Then $\alpha(s) = 0$ and by construction $\omega(a) = 0$ for all $\omega$ containing $s$, so that the equality trivially holds.

Case 2. Assume that $s = s_i$, for some $i \leq l$. By assumption we have
\[
\sum_{a \in A_1, \ a \ni s} [a : s] R^a_s \omega_T(a) = \alpha(s) .
\] (4)
In the set $\{a \in A_1 ; \ a \ni s\}$, there is one edge $a^+ = [s_i, s_{i+1}]$ such that $U_{a^+} = U_{s_i}$, and an edge $a^- = [s_{i-1}, s_i]$ such that $U_{a^-} \neq U_{s_i}$. By definition of the restriction maps $R$, Equality (23) writes:
\[
\sum_{n \in U_{s_i} / U_{a^-}} [a^- : s] \varphi_{s,n} r^{a^-}_{s} \omega_T(a^-) + r^{a^+}_{s} \omega_T(a^+) = \alpha(s) .
\] (5)
that is
\[
\sum_{n \in U_{s_i} / U_{a^-}} [n a^- : s] r^{a^-}_{s} \varphi_{a^- , n} \omega_T(a^-) + r^{a^+}_{s} \omega_T(a^+) = \alpha(s) .
\] (6)
The set $\{a^+, u.a^- ; \ u \in U_{s_i} / U_{a^-}\}$ is precisely the set of edges of $X$ containing $s$. Using the definition of $\omega$, we obtain:
\[
\sum_{a \in X_1, \ a \ni s} [a : s] r^a_s \omega(a) = \alpha(a)
\] (7)
as required.

Case 3. Assume that $s \not\in A$ and that the projection of $s$ onto $A$ write $s_j$, for some $j \leq l$. Write $s = u.s_i$ with $i \leq l$, $u \in U$, so that
\[
\{a \in X_1; \ a \ni s\} = u. \{b \in X_1; \ b \ni s_i\} .
\]
We have:
\[
\sum_{a \ni s} [a : s] r^a_s \omega(a) = \sum_{b \ni s} [u.b : u.s_i] r^{u.b}_{u.s_i} \omega(u.b)
\] (8)
\[
= \sum_{b \ni s} [b : s] r^{u.b}_{u.s_i} \varphi_{b,u} \omega(b)
\] (9)
\[
= \sum_{b \ni s} [b : s] \varphi_{s,u} r^b_s \omega(b)
\] (10)
\[
= \varphi_{s,u} \left( \sum_{b \ni s} [b : s] r^b_s \omega(b) \right)
\] (11)
\[
= 0
\] (12)
the sum in Equality (11) being trivial using case (2), and we are done.
5.3 Proof of Theorem (5.1)

Let $\pi$ be an irreducible supercuspidal representation of $G$. Using Theorem (5.1.3), fix a $G$-equivariant coefficient system $\mathcal{F}$ on $X$, with finite dimensional section spaces, such that $\pi \simeq H_0(X, \mathcal{F})$ as $G$-modules.

For $i = 0, 1$ we shall identify the $G_{s_i}$-module $\mathcal{F}_{s_i}$ as the $G_{s_i}$-submodule $C_0(X, \mathcal{F})_{s_i}$ of $C_0(X, \mathcal{F})$ formed of those chains whose support is contained in $\{s_i\}$. A natural isomorphism is given by

$$C_0(X, \mathcal{F})_{s_i} \longrightarrow \mathcal{F}_{s_i}, \quad \alpha \mapsto \alpha(s_i).$$

In this way, we identify $C_0(X, \mathcal{F})$ with the direct sum of the two induced representations $c\text{-}\text{ind}_{G_{s_i}}^G\mathcal{F}_{s_i} = \sum_{g \in G/G_{s_i}} g.\mathcal{F}_{s_i}, i = 0, 1$. Moreover for $i = 0, 1$, we abbreviate $G_{s_i} = G_i$ and $\mathcal{F}_{s_i} = \lambda_i$.

For $i = 0, 1$, we decompose the $G_i$-module $\lambda_i$ as

$$\lambda_i = \lambda_i^{\text{cusp}} \oplus \bigoplus_{j=1,\ldots,n_i} \lambda_i^j$$

where the subrepresentation $\lambda_i^{\text{cusp}}$ has no non-zero fixed vector by $U_{s_i}$, and where the $\lambda_i^j$, $j = 1, \ldots, n_j$ are irreducible subrepresentations satisfying $(\lambda_i^j)^{U_{s_i}} \neq 0$.

**Lemma 5.3.1** Let $i = 0, 1$ and $j \in \{1,\ldots,n_j\}$. Then the representation $\lambda_i^j$ lies in the image of $\partial : C_1(X, \mathcal{F}) \longrightarrow C_0(X, \mathcal{F})$.

**Proof.** Pick a non-zero vector $v \in (\lambda_i^j)^{U_{s_i}} \subset \mathcal{F}_{s_i}^{U_{s_i}}$. Under our identifications, it corresponds to the 0-chain $\alpha_{i,v}$ with support $\{s_i\}$ and satisfying $\alpha_{i,v}(s_i) = v$. Since $\pi$ is supercuspidal, we have $H_0(X, \mathcal{F})_U \simeq \pi_U = 0$. Thus we may apply Lemma (5.2.1): this cochain $\alpha_{i,v}$ lies in the image of $\partial$. Since $v$ generates $\lambda_i^j$ as a $G_{s_i}$-module and since $\partial$ is $G_{s_i}$-equivariant, we deduce that $\lambda_i^j$ lies in the image of $\partial$.

**Lemma 5.3.2** The representation $\pi$ is a quotient of $c\text{-}\text{ind}_{G_0}^G \lambda_0^{\text{cusp}} \oplus c\text{-}\text{ind}_{G_1}^G \lambda_1^{\text{cusp}}$.

**Proof.** By assumption we have $\pi \simeq C_0(X, \mathcal{F})/\text{Im} \partial$. Moreover the $G$-module $C_0(X, \mathcal{F})$ decomposes as

$$c\text{-}\text{ind}_{G_0}^G \lambda_0^{\text{cusp}} \oplus c\text{-}\text{ind}_{G_1}^G \lambda_1^{\text{cusp}} \oplus \bigoplus_{i=0,1} \bigoplus_{j=1,\ldots,n_i} c\text{-}\text{ind}_{G_i}^G \lambda_i^j.$$ 

For $i = 0, 1$, $j = 1, \ldots, n_i$, we have that $\lambda_i^j \subset \text{Im} (\partial)$. Since $c\text{-}\text{ind}_{G_i}^G \lambda_i^j$ is generated by $\lambda_i^j$ as a $G$-module, and since $\partial$ is $G$-equivariant, we have $c\text{-}\text{ind}_{G_i}^G \lambda_i^j \subset \text{Im} (\partial)$. The lemma follows.
By Theorem 5.1.2, for $i = 0, 1$, we may write

$$c\text{-ind}_{G_i}^{G} \lambda_i^{\text{cusp}} = \bigoplus_{j=1,...,m_i} \pi_i^j$$

where $m_i$ is some positive integer and the $\pi_i^j$, $j = 1,...,m_i$ are irreducible supercuspidal representations. So there exist $i \in \{0,1\}$ and $j \in \{1,...,m_i\}$ such that $\pi$ is isomorphic to $\pi_i^j$. Fix such an integer $i \in \{0,1\}$ and decomposes $\lambda_i^{\text{cusp}}$ as a direct sum of irreducible $G_i$-modules:

$$\lambda_i^{\text{cusp}} = \bigoplus_{k=1,...,l_i} \sigma_i^k.$$ 

By Frobenius reciprocity, there exists $k \in \{1,...,l_i\}$ such that $\pi$ is a quotient of $c\text{-ind}_{G_i}^{G} \sigma_i^k$. The irreducible representation $(G_i, \sigma_i^k)$ satisfies the assumption of Corollary (5.1.3) so that $c\text{-ind}_{G_i}^{G} \sigma_i^k$ is a finite direct sum of irreducible supercuspidal representations. Of course $\pi$ is one of those representations. This finishes the proof of Theorem (5.1).

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