Primordial Gravitational Waves Predictions for GW170817-compatible Einstein-Gauss-Bonnet Theory

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In this work we shall calculate in detail the effect of a GW170817-compatible Einstein-Gauss-Bonnet theory on the energy spectrum of the primordial gravitational waves. The spectrum is affected by two characteristics, the overall amplification/damping factor caused by the GW170817-compatible Einstein-Gauss-Bonnet theory and by the tensor spectral index and the tensor-to-scalar ratio. We shall present the formalism for studying the inflationary dynamics and post-inflationary dynamics of GW170817-compatible Einstein-Gauss-Bonnet theories for all redshifts starting from the radiation era up to the dark energy era. We exemplify our formalism by using two characteristic models, which produce viable inflationary and dark energy eras. As we demonstrate, remarkably the overall damping/amplification factor is of the order of unity, thus the GW170817-compatible Einstein-Gauss-Bonnet models affect the primordial gravitational waves energy spectrum only via their tensor spectral index and the tensor-to-scalar ratio. Both models have a blue tilted tensor spectrum, and therefore the predicted energy spectrum of the primordial gravity waves can be detectable by most of the future gravitational waves experiments, for various reheating temperatures.

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I. INTRODUCTION

Undoubtedly General Relativity (GR) is the most correct description for astrophysical phenomena and objects, however even at the astrophysical level, already appear examples that cast doubt on its full validity at astrophysical levels, see for example [1]. Furthermore, GR fails to describe in a consistent way persisting large scale phenomena in the Universe, such as dark energy. The GR description of dark energy is problematic, since if dark energy is phantom, phantom scalar fields must be used to describe it, and phantom fields are not appealing. On the other hand, if one sticks with GR descriptions of the primordial era of the Universe, and specifically the inflationary era [2–5], inevitably the scalar field description is the only option. The inflaton description however is deemed problematic, due to the tremendous fine tuning required for the inflaton interactions with other particles, its mass and its very own self interaction scalar potential. Although scalar fields frequently occur in string theory, which is the most correct theoretical description of a unified theory of everything, it seems that a simple GR description might not be the complete answer for inflation. Conceptually, single scalar fields are expected to exist in the post-Planck era, as remnants of the UV-complete theory. However, scalar fields are not the only constituents of the post-Planck inflationary Lagrangian, since higher order curvature terms might also be present in the effective inflationary Lagrangian. In fact, these quantum terms are either higher powers of the Ricci scalar $R$, square or cubic powers, and also Gauss-Bonnet terms. Thus, modified gravity [6–11] might eventually be the correct description of nature primordially in the post-Planck era. The so-called Einstein-Gauss-Bonnet theories thus offer another viable description of the inflationary era, combining string motivated quantum corrections and scalar fields [12–57]. However, Einstein-Gauss-Bonnet theories have a serious flaw related with the propagation speed of the primordial tensor perturbations, which is not equal to that of light’s. The GW170817 event indicated that the propagation speed of gravitational waves is almost identical to that of light’s, therefore since there is no fundamental particle physics reason for the graviton to change its mass in the post-inflationary era, the graviton must be massless primordially. Ordinary Einstein-Gauss-Bonnet theories have this flaw related to the propagation speed of the tensor modes. However, in our recent works [55, 56] we provided a theoretical remedy for the problem of massive primordial gravitons. Specifically we showed that if the scalar potential and scalar coupling function are related in a specific way, the speed of the tensor perturbations can be equal to unity in natural units, for Einstein-Gauss-Bonnet theories.

Now apart from the theoretical problems of the GR description of inflation using solely scalar fields, there might be another problem for it in the near future. The future gravitational wave experiments [58, 59] will seek for stochastic gravitational waves in the Universe. If a signal is detected by the future experiments, then single scalar field descriptions will be in a very bad position, since the current predictions of those theories lead to undetectable signals.

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A possible signal will either mean that some modified gravity is the underlying theory that produces the signal, or that some abnormal reheating era takes place. Even if the latter scenario occurs, the GR description produces a low scalar energy spectrum of primordial gravitational waves. On the other hand, theories such as the Einstein-Gauss-Bonnet theories lead to detectable signals from future gravitational wave experiments. Thus it is vital to thoroughly investigate all the aspects of Einstein-Gauss-Bonnet theories.

In this work we shall consider the GW170817-compatible Einstein-Gauss-Bonnet theories developed in Ref. [56], and we shall calculate the predicted energy spectrum of the primordial gravitational waves. We shall consider inflationary aspects of several well chosen models and also we shall focus on models which also produce a viable late-time era. Overall we shall provide a unified description of the early and late-time accelerations with the same models. However the major issue we shall address in this work is that we shall calculate numerically the overall amplification or damping factor of the primordial gravitational wave signal, starting from zero redshift and up to the radiation domination era. This is the first time that such a calculation is performed for viable Einstein-Gauss-Bonnet theories. In this way, we shall have a complete picture for these inflationary theories.

This paper is organized as follows: In section II we shall present the essential features of the inflationary dynamics for GW170817-compatible Einstein-Gauss-Bonnet theories. In the same section, the formalism for describing the post-inflationary and late-time dynamics is described in detail. In section III two characteristic Einstein-Gauss-Bonnet models are studied in detail, and specifically their late-time and inflationary behavior is studied. In section IV, the formalism for extracting the energy spectrum of the primordial gravitational waves for Einstein-Gauss-Bonnet is described in detail. We present the exact formulas needed for the calculation of the overall amplification/damping factor caused by the Einstein-Gauss-Bonnet theory and we present the predicted energy spectrum of the primordial gravitational waves for the models developed in section IV. Finally the conclusions follow at the end of the paper.

II. ESSENTIAL FEATURES OF GW170817-COMPATIBLE EINSTEIN-GAUSS-BONNET INFLATION AND THE DARK ENERGY ERA

A. Inflationary Phenomenology Theoretical Framework of GW170817-compatible Einstein-Gauss-Bonnet Theory

In this section we shall present the essential features of an GW170817-compatible Einstein-Gauss-Bonnet gravity, which was developed in [56], see also [55] for an alternative approach. First we consider the Einstein-Gauss-Bonnet action,

$$S = \int d^4x \sqrt{-g} \left( \frac{R}{2\kappa^2} - \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - V(\phi) - \frac{1}{2} \xi(\phi) \mathcal{G} \right),$$  

with $R$ denoting as usual the Ricci scalar, $\kappa = \frac{1}{M_p}$ with $M_p$ being the reduced Planck mass. Also $\mathcal{G}$ denotes the Gauss-Bonnet invariant in four dimensions, which is $\mathcal{G} = R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ with $R_{\alpha\beta}$ and $R_{\alpha\beta\gamma\delta}$ denoting the Ricci and Riemann tensor. For the whole analysis that follows we shall assume that the geometric background is described by a flat Friedmann-Robertson-Walker (FRW) metric, with line element,

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1}^{3} (dx^i)^2,$$

where $a(t)$ is the scale factor. Assuming that the scalar field is solely time-dependent and by varying the gravitational action with respect to the metric tensor and the scalar field, we obtain the following equations of motion,

$$\frac{3H^2}{\kappa^2} = \frac{1}{2} \dot{\phi}^2 + V + 12\xi H^3,$$

$$\frac{2\dot{H}}{\kappa^2} = -\ddot{\phi}^2 + 4\ddot{\xi} H^2 + 8\dot{\xi} H \dot{H} - 4\dot{\xi} H^3,$$

$$\ddot{\phi} + 3H \dot{\phi} + V' + 12\dot{\xi} H^2 (H + H^2) = 0.$$

For the whole analysis of the inflationary era, we shall assume that the slow-roll conditions are valid,

$$\dot{H} \ll H^2, \quad \frac{\dot{\phi}^2}{2} \ll V, \quad \ddot{\phi} \ll 3H \dot{\phi}.$$
Moreover, the speed of the primordial tensor modes for the Einstein-Gauss-Bonnet theory has the following functional form,

\[ c_T^2 = 1 - \frac{Q_f}{2Q_t}, \]  

(7)

where \( Q_f = 8(\ddot{\xi} - H\dot{\xi}) \), \( Q_t = F + \frac{Q_b}{2} \), \( F = \frac{1}{\kappa^2} \) and \( Q_b = -8\dot{\xi}H \). In order for the Einstein-Gauss-Bonnet theory to be compatible with the GW170817 event, we must require \( c_T^2 = 1 \) in natural units, therefore this can be achieved if \( Q_f = 0 \) which in turn results to the following differential equation \( \ddot{\xi} = H\dot{\xi} \). Clearly this differential equation constrains the scalar coupling function \( \xi(\phi) \). Expressed in terms of the scalar field, the constraint equation reads,

\[ \xi''\dot{\phi}^2 + \xi'\ddot{\phi} = H\xi'\dot{\phi}, \]  

(8)

where the “prime” indicates differentiation with respect to the scalar field. Motivated the slow-roll conditions of the scalar field, by assuming that,

\[ \xi'\ddot{\phi} \ll \xi''\dot{\phi}^2, \]  

(9)

the constraint (8) is rewritten as,

\[ \dot{\phi} \simeq \frac{H\xi'}{\xi''}, \]  

(10)

hence by combining Eqs. (5) and (10) we get,

\[ \frac{\xi'}{\xi''} \simeq -\frac{1}{3H^2}(V' + 12\xi'H^4). \]  

(11)

Furthermore we shall assume that the following conditions hold true for the theory at hand,

\[ \kappa\frac{\xi'}{\xi''} \ll 1, \]  

(12)

\[ 12\dot{\xi}H^3 = 12\frac{\xi'^2H^4}{\xi''} \ll V. \]  

(13)

In view of Eqs. (6), (10) and (13), the equations of motion take the following form,

\[ H^2 \simeq \frac{\kappa^2 V}{3}, \]  

(14)

\[ \dot{H} \simeq -\frac{1}{2}\kappa^2\dot{\phi}^2, \]  

(15)

\[ \dot{\phi} \simeq \frac{H\xi'}{\xi''}, \]  

(16)

while the condition (13) becomes,

\[ \frac{4\kappa^4\xi'^2V}{3\xi''} \ll 1. \]  

(17)

Also the constraint differential equation (11) which relates the functional forms of the scalar coupling function and of the scalar potential, becomes,

\[ \frac{V'}{V^2} + \frac{4\kappa^4\xi'}{3} \approx 0. \]  

(18)
In view of the above simplifications, the slow-roll indices for the Einstein-Gauss-Bonnet inflationary framework at hand take the following simplified form [56],

\begin{align}
\epsilon_1 &\simeq \frac{\kappa^2}{2} \left( \frac{\xi'}{\xi''} \right)^2, \\
\epsilon_2 &\simeq 1 - \epsilon_1 - \frac{\xi'\xi'''}{\xi''^2}, \\
\epsilon_3 &= 0, \\
\epsilon_4 &\simeq \frac{\xi'}{2\xi''} \frac{E'}{E}, \\
\epsilon_5 &\simeq -\frac{\epsilon_1}{\lambda}, \\
\epsilon_6 &\simeq \epsilon_5 (1 - \epsilon_1),
\end{align}

where \( E = E(\phi) \) and \( \lambda = \lambda(\phi) \) are defined as follows,

\[ E(\phi) = \frac{1}{\kappa^2} \left( 1 + 72 \frac{\epsilon_1^2}{\lambda^2} \right), \quad \lambda(\phi) = \frac{3}{4\xi''\kappa^2 V}. \]  

(25)

Having the slow-roll indices available, one can easily calculate the observational indices of inflation. We shall mainly be interested in the spectral index of the primordial scalar curvature perturbations \( n_S \), the spectral index of the tensor perturbations \( n_T \) and the tensor-to-scalar ratio \( r \), which in terms of the slow-roll indices are expressed as follows,

\begin{align}
n_S &= 1 - 4\epsilon_1 - 2\epsilon_2 - 2\epsilon_4, \\
n_T &= -2(\epsilon_1 + \epsilon_6), \\
r &= 16 \left| \left( \frac{\kappa^2 Q_e}{4H} - \epsilon_1 \right) \frac{2c_A^3}{2 + \kappa^2 Q_b} \right|, 
\end{align}

(26) \hspace{1cm} (27) \hspace{1cm} (28)

with \( c_A \) being the sound speed,

\[ c_A^2 = 1 + \frac{Q_a Q_e}{3Q_a^2 + \phi^2(\frac{2}{\kappa^2} + Q_b)}, \]

(29)

with,

\[ Q_a = -4\dot{\xi}H^2, \quad Q_b = -8\dot{\xi}H, \quad Q_t = F + \frac{Q_b}{2}, \quad Q_c = 0, \quad Q_e = -16\dot{\xi}\dot{H}. \]

(30)

As it was shown in Ref. [56], by using the previous simplifications, the tensor-to-scalar ratio takes the following simplified form,

\[ r \simeq 16\epsilon_1, \]  

(31)

while the spectral index of the primordial tensor perturbations takes the following simplified form,

\[ n_T \simeq -2\epsilon_1 \left( 1 - \frac{1}{\lambda} + \frac{\epsilon_1}{\lambda} \right), \]

(32)
with $\lambda$ being defined in Eq. (25). Also, the $e$-foldings number can be evaluated in terms of the scalar coupling function $\xi(\phi)$ as follows,

$$N = \int_{t_i}^{t_f} H dt = \int_{\phi_i}^{\phi_f} \frac{\xi''}{\xi'} d\phi,$$  \hspace{1cm} (33)$$

where $\phi_f$ and $\phi_i$ are the values of the scalar field at the end of inflation and at the first horizon crossing during the beginning of inflation, respectively. Having the theoretical framework we just presented at hand, one can easily investigate if a specific Einstein-Gauss-Bonnet model can be viable when compared to the Planck 2018 data.

Before closing this section, an important remark is in order. Specifically, in this work we considered mainly linear effects governing the dynamics of Einstein-Gauss-Bonnet inflation and the corresponding gravitational wave speed. So the constraint equation (8) is valid only at a linear level, and indeed this linear level would persist for gravitational waves with wavelengths larger than 10 Mpc. However, for these modes which are mainly probed by CMB experiments and are basically CMB modes, the linear theory applies. Below 10 Mpc, one expects non-linear effects to take place, in the form of higher order operators and quantum corrections, see the comprehensive analysis in Refs. [66, 67]. Indeed, for wavelengths corresponding to gravitational waves corresponding to the GW170817 event, one should certainly include in the study of cosmological gravity waves, the non-linear effects imposed by higher order corrections and quantum corrections. The stability of the solutions and the effects of these corrections on the background evolution should be addressed carefully, taking also into account the quantum and higher order corrections. Hence the constraint (8) is explicitly background dependent and only works up to the linear level. Therefore, any small deviation coming from any non-linear effects will effectively spoil this constraint relation. Hence, although the GW170817 measurement is particularly important since it directly probes the gravitational wave speed over cosmological distances, one should take into account that the gravitational wave speed might be locally reduced in high energy environments. Also, although beyond 10 Mpcs, screening effects are believed not to persist and affect the gravitational wave speed, for low energy measurements at distances 10 000 km, the effective field theory of dark energy or modified gravity must be used in a compelling way [66]. Hence the higher order corrections and quantum corrections terms originating from the underlying effective field theory of any origin, must be used in a formal way. In this work though we need to clarify that these effects are not taken into account. However, such an extension of the present paper to include the effects of these operators is compelling, since all the relevant modes to future gravitational waves experiments have wavelengths smaller that 10 Mpc, thus all the quantum and higher order effects apply. We aim to formally address this important issue in a future work.

### B. From Reheating to Dark Energy Phenomenology of GW170817-compatible Einstein-Gauss-Bonnet Theory

In the previous subsection we presented the inflationary theoretical framework for a general GW170817-compatible Einstein-Gauss-Bonnet gravity. A crucial assumption for this theory was that the tensor perturbations propagate with a speed equal to that of light’s. Since there is no fundamental reason for the gravitons to propagate differently compared to light during the post-inflationary era, in this section we shall develop a formalism for studying the dark energy and the previous eras up to the inflationary era, for GW170817-compatible Einstein-Gauss-Bonnet theories. The reason for studying the post-inflationary eras up to the dark energy era, is mainly the fact that in a later section we will need to quantify the effect of the Einstein-Gauss-Bonnet gravity on the energy spectrum of the primordial gravitational waves from reheating up to the dark energy era.

The condition $\ddot{\xi} = H\dot{\xi}$ which ensures that primordially $c_T^2 = 1$, reads,

$$\dot{\xi} = C e^{\int H dt},$$  \hspace{1cm} (34)$$

with $C$ being an integration constant. Since, $\frac{dz}{dt} = -H(1 + z)$, the above equation yields,

$$\dot{\xi} = a(t) C = \frac{C}{1 + z}.$$  \hspace{1cm} (35)$$

The above relation greatly simplifies the study of the dark energy era, since it specifies the functional form of $\dot{\xi}$ which enters the gravitational equations of motion. In view of Eq. (35), the field equations read,

$$\frac{3H^2}{\kappa^2} = \rho_m + \frac{1}{2} \dot{\phi}^2 + V + 24 \frac{C}{1 + z} H^3,$$  \hspace{1cm} (36)$$
\[ -\frac{2H}{\kappa^2} = \rho_m + P_m + \dot{\phi}^2 - 16\frac{C}{1+z} H\dot{H}, \quad (37) \]

\[ V_\phi + \dot{\phi} + 3H\dot{\phi} + \frac{C}{1+z} \frac{G}{\phi} = 0. \quad (38) \]

The parameter \( C \) is a free parameter of the theory, and it proves that when it takes reasonable values, not fine-tuned to be larger than \( \sim \mathcal{O}(10^6) \), it does not crucially affect the dark energy era. In order to study the post-inflationary evolution of the Einstein-Gauss-Bonnet theory, we shall express the field equations with respect to the redshift. Hence we shall use the following,

\[ \dot{H} = -H(1+z)H', \quad (39) \]

\[ \dot{\phi} = -H(1+z)\phi', \quad (40) \]

\[ \ddot{\phi} = H^2(1+z)^2\phi'' + H^2(1+z)\phi' + HH'(1+z)^2\phi', \quad (41) \]

\[ \dot{R} = 6H(1+z)^2 \left( HH'' + (H')^2 - \frac{3HH'}{1+z} \right), \quad (42) \]

in order to transform the field equations in terms of the redshift. Also we shall introduce the statefinder quantity \( y_H(z) \) \cite{[68, 69]},

\[ y_H = \frac{\rho_{DE}}{\rho_{d0}}, \quad (43) \]

where \( \rho_{DE} \) denotes the energy density of dark energy, which is in the case at hand,

\[ \rho_{DE} = \frac{1}{2} \dot{\phi}^2 + V + 24\xi H^3, \quad (44) \]

while \( \rho_{d0} \) is the value of density for non-relativistic matter at present day. The pressure for the dark energy fluid is,

\[ P_{DE} = -V - 24\xi H^3 - 8\xi H\dot{H}, \quad (45) \]

with the dark energy fluid satisfying,

\[ \dot{\rho}_{DE} + 3H(\rho_{DE} + P_{DE}) = 0. \quad (46) \]

Therefore, equations (3) and (4) can be written in the Friedmann equation-like form of standard Einstein-Hilbert gravity,

\[ \frac{3H^2}{\kappa^2} = \rho(m) + \rho_{DE}, \quad (47) \]

\[ -\frac{2\dot{H}}{\kappa^2} = \rho(m) + P(m) + \rho_{DE} + P_{DE}, \quad (48) \]

The Hubble rate and its derivatives entering in the field equations, can be expressed in terms of the statefinder quantity \( y_H(z) \) as follows,

\[ H^2 = m_s^2 \left(y_H(z) + \frac{\dot{\rho}(m)}{\rho_{d0}} \right), \quad (49) \]

\[ HH' = m_s^2 \left(y_H' + \frac{\rho'(m)}{\rho_{d0}} \right), \quad (50) \]
\[ H'^2 + H H'' = \frac{m^2_s}{2} \left( y''_H + \frac{\rho''(m)}{\rho_{d0}} \right), \tag{51} \]

with \( m^2_s = \kappa^2 \frac{\rho_{d0}}{3} = 1.87101 \cdot 10^{-67} \). In the next section we shall numerically solve the differential equations (11) and (14) with respect to the scalar field and the statefinder quantity \( y_H \), for the redshift range from zero up to several millions. In this way, we will find the behavior of the model from the dark energy era up to the reheating era. For our study we shall consider several important quantities characterizing the geometrical contribution of the Einstein-Gauss-Bonnet gravity to the post-inflationary evolution and to the dark energy era. We shall consider the equation of state parameter \( \omega_{DE} \) and the dark energy density parameter \( \Omega_{DE} \), which expressed as functions of the statefinder quantity \( y_H(z) \) are defined as follows \([68, 69]\),

\[ \omega_{DE} = -1 + \frac{1 + z}{3} \frac{d \ln y_H}{dz}, \quad \Omega_{DE} = \frac{y_H}{y_H + \frac{\rho_{d0}}{\rho_{d0}}}. \tag{52} \]

Also for the late-time study, we shall consider another important statefinder quantity, the deceleration parameter, defined as,

\[ q = -1 - \frac{\dot{H}}{H^2}. \tag{53} \]

In the next sections we shall study the inflationary and post-inflationary evolution of two Einstein-Gauss-Bonnet models and we shall set the stage for the study of the predicted energy spectrum of the primordial gravitational waves for these models.

### III. STUDY OF THE PHENOMENOLOGY FOR SPECIFIC EINSTEIN-GAUSS-BONNET MODELS: FROM INFLATION TO DARK ENERGY

In this section we shall consider the evolution and phenomenology of two distinct Einstein-Gauss-Bonnet models, starting from inflation until the dark energy era. For the inflationary era, we shall be interested in calculating the spectral index of the primordial scalar curvature perturbations and the tensor-to-scalar ratio of both models, which will be essential for the calculation of the primordial gravitational wave energy spectrum. Recall that the Planck 2018 collaboration \([70]\) constrains the spectral index and the tensor-to-scalar ratio as follows,

\[ n_S = 0.9649 \pm 0.0042, \quad r < 0.064. \tag{54} \]

Accordingly, we shall numerically solve the equations of motion for the post-inflationary era, which will enable us to measure the effect of the Einstein-Gauss-Bonnet theory on the energy spectrum of the primordial gravitational waves.

#### A. Model I

Let us start with an exponential functional form for the scalar coupling function \( \xi(\phi) \), which has the following form,

\[ \xi(\phi) = \beta \exp \left( \frac{\phi}{M} \right)^2, \tag{55} \]

with \( \beta \) being a dimensionless parameter, and \( M \) is a parameter with mass dimensions \([m]\)^1. Combining (55) and Eq. (18) and by solving the resulting differential equation, the scalar potential is obtained, which is,

\[ V(\phi) = \frac{3}{3\gamma \kappa^4 + 4\beta \kappa^4 \epsilon \phi^2}, \tag{56} \]

with \( \gamma \) being a dimensionless integration constant. Accordingly, the slow-roll indices (19)-(24) become,

\[ \epsilon_1 \simeq \frac{\kappa^2 M^4 \phi^2}{2 (M^2 + 2\phi^2)^2}, \tag{57} \]
while the tensor spectral index reads, 

\[ \epsilon_2 \simeq \frac{M^4 \left( 2 - \kappa^2 \phi^2 \right) - 4M^2 \phi^2}{2 \left( M^2 + 2 \phi^2 \right)^2}, \]  

\[ \epsilon_3 = 0, \]  

\[ \epsilon_5 \simeq -\frac{4\beta \phi^2 e^{\frac{\phi}{M}}} \left( M^2 + 2 \phi^2 \right) \left( 3\gamma + 4\beta e^{\frac{\phi}{M}} \right), \]  

\[ \epsilon_6 \simeq -\frac{2\beta \phi^2 e^{\frac{\phi}{M}} \left( M^4 \left( 2 - \kappa^2 \phi^2 \right) + 8M^2 \phi^2 + 8\phi^4 \right)} \left( M^2 + 2 \phi^2 \right)^3 \left( 3\gamma + 4\beta e^{\frac{\phi}{M}} \right), \]  

and we omitted the slow-roll index \( \epsilon_4 \) the functional form of which was quite lengthy. Accordingly, by solving the equation \( \epsilon_1 \simeq \mathcal{O}(1) \) yields the value of the scalar field when inflation ends \( \phi_f \simeq \frac{1}{2} \sqrt{\kappa^2 M^4 + \kappa M^3 \sqrt{\kappa^2 M^2 - 16} - 8M^2} \). Moreover, in order to calculate the value of the scalar field at first horizon crossing, we shall use the e-foldings number Eq. (33), and by solving the resulting algebraic equation with respect to \( \phi \) we get \( \phi_i = \frac{1}{2} M \sqrt{\kappa^2 M^2 + \kappa M \sqrt{\kappa^2 M^2 - 16} - 16} - 8 \). Moreover, the scalar spectral index as a function of the scalar field reads, 

\[ n_s \simeq -1 - \frac{\kappa^2 M^4 \phi^2}{(M^2 + 2 \phi^2)^2} + \frac{4\phi^2 \left( 3M^2 + 2 \phi^2 \right)}{(M^2 + 2 \phi^2)^2} \]  

\[ + \frac{4608\beta^2 \phi^2 e^{\frac{\phi}{M}} \left( 6\gamma \phi^2 + 16\beta e^{\frac{\phi}{M}} \left( M^2 + \phi^2 \right) + 9\gamma M^2 \right)}{(M^2 + 2 \phi^2)^4 \left( 3\gamma + 4\beta e^{\frac{\phi}{M}} \right)^3}, \]  

while the tensor spectral index reads, 

\[ n_T \simeq \frac{\phi^2 \left( -4\beta e^{\frac{\phi}{M}} \left( M^4 \left( 3\kappa^2 \phi^2 - 2 \right) + \kappa^2 M^6 - 8M^2 \phi^2 - 8\phi^4 \right) - 3\gamma \kappa^2 M^4 \left( M^2 + 2 \phi^2 \right) \right)}{(M^2 + 2 \phi^2)^3 \left( 3\gamma + 4\beta e^{\frac{\phi}{M}} \right)^3}. \]  

and the tensor-to-scalar ratio is, 

\[ r \simeq \frac{8\kappa^2 M^4 \phi^2}{(M^2 + 2 \phi^2)^2}. \]  

The observational indices must be evaluated at the first horizon crossing, so for \( \phi = \phi_i \) and also the parameter \( M \) is assumed to be of the form \( M = \mu / \kappa \) where \( \mu \) is a dimensionless free parameter. The model at hand is compatible with the Planck data for a wide range of the free parameters. In Fig. 4 we present the marginalized curves of Planck 2018 and the predictions of Model I, for the free parameters taking values \( \mu = [22.09147657871, 22.09147657877], \) \( \beta = -1.5, \gamma = 2, \) for \( N = 60 \) e-foldings. As it can be seen in Fig. 4 the model is fitted quite well within the Planck 2018 data. Also for the above range of values for the free parameters, the tensor spectral index takes values in the range \( n_T = [0.378856, 0.379088] \), hence it is positive. This is of particular importance and shall play an important role for energy spectrum of the primordial gravitational waves. Let us now consider the evolution of Model I during the post-inflationary era up to present day. For the calculation of the impact of the Einstein-Gauss-Bonnet theory on the energy spectrum of the primordial gravitational waves it is essential to study the evolution of the model from present day, hence for \( z = 0 \) up to high redshifts which stretch up to the reheating era, hence for redshifts \( z \sim 10^6 \) or higher. For the numerical analysis of the differential equations, we shall impose the following initial conditions, 

\[ y_H(z = z_{fin}) = \frac{\Delta z}{a_H} \left| \frac{dz}{dz} \right|_{z_{fin}} = 0, \phi(z = z_{fin}) = 10^1.9 M_p, \frac{d\phi}{dz} \right|_{z = z_{fin}} = -10^{-6} M_p, \text{ where } z_{fin} = 10^6. \]  

The results of our numerical analysis for Model I can be found in Fig. 2 where we plot the total equation of state (EoS) parameter \( w_{eff} \) (left plot) and the deceleration parameter (right plot) as functions of the redshift \( z \). As it can be seen in Fig. 2 the total EoS parameter approaches the value \( w_{eff} \sim 1/3 \), hence the Model I for redshifts \( z \sim 10^6 \) has entered deeply in the radiation domination era. Also the Model I mimics the Λ-Cold-Dark-Matter (ΛCDM) model at late times, as it can be seen by looking the behavior of the deceleration parameter. In Table 4 we have gathered the values of the dark energy EoS parameter \( \omega_{DE} \), the dark energy density parameter \( \Omega_{DE} \) and of the deceleration parameter \( q \) at present day. Overall, Model I offers a viable description regarding it’s dark energy predictions.
FIG. 1: Marginalized curves of the Planck 2018 data for the Model I (red curve).

FIG. 2: The total EoS parameter $w_{eff}$ (left plot) and the deceleration parameter $q$ (right plot), as functions of the redshift.

| Parameter | Model I | $\Lambda$CDM Value or Planck 2018 Constraints |
|-----------|---------|---------------------------------------------|
| $q(z=0)$  | -0.51626| -0.535                                      |
| $\Omega_{DE}(0)$ | 0.67754 | 0.6847±0.0073                               |
| $\omega_{DE}(0)$ | -1      | -1.018±0.031                                |

**B. Model II**

Let us now consider a power law functional form for the scalar coupling function $\xi(\phi)$, which has the following form,

$$\xi(\phi) = \beta \left( \frac{\phi}{M} \right)^{\nu},$$

(65)

where $\beta$ is a dimensionless parameter, and $M$ is a parameter with mass dimensions $[m]^1$. Upon combining (65) and Eq. (18) and by solving the resulting differential equation, the scalar potential is obtained, which in this case is,

$$V(\phi) = \frac{3}{3\gamma\kappa^4 + 4\beta\kappa^4 \left( \frac{\phi}{M} \right)^{\nu}},$$

(66)

with $\gamma$ being a dimensionless integration constant. Accordingly, the slow-roll indices (19)-(24) become in this case,

$$\epsilon_1 \simeq \frac{\kappa^2 \phi^2}{2(\nu - 1)^2},$$

(67)
\[ \epsilon_2 \simeq -\frac{\kappa^2 \dot{\phi}^2 - 2\nu + 2}{2(\nu - 1)^2}, \]
\[ \epsilon_3 = 0, \]
\[ \epsilon_5 \simeq \frac{2\beta \nu \left( \frac{\phi}{\phi_0} \right)^{\nu}}{(\nu - 1) \left( 3\gamma + 4\beta \left( \frac{\phi}{\phi_0} \right) \right)^\nu}, \]
\[ \epsilon_6 \simeq -\frac{\beta \nu \left( -\kappa^2 \dot{\phi}^2 + 2\nu^2 - 4\nu + 2 \right) \left( \frac{\phi}{\phi_0} \right)^\nu}{(\nu - 1)^3 \left( 3\gamma + 4\beta \left( \frac{\phi}{\phi_0} \right) \right)^\nu}, \]

and we omitted the slow-roll index \( \epsilon_4 \) in this model too, because its functional form is quite lengthy. Accordingly, by solving the equation \( \epsilon_1 \simeq 0 \) yields the value of the scalar field when inflation ends \( \phi_f \simeq \sqrt{2/(\nu - 20)} \). Also by using the e-foldings number Eq. (33), and by solving the resulting algebraic equation with respect to \( \phi_i \) we get 
\[ \phi_i = \sqrt[3]{2/(\nu - 20)}e^{\nu/\nu_f}. \]
Moreover, the scalar spectral index as a function of the scalar field reads,

\[ n_S \simeq \frac{-\kappa^2 \dot{\phi}^2 + 2(\nu - 2)}{(\nu - 1)^2} + 1 \]
\[ = \frac{192\beta \nu^4 \phi_0^4 \left( \phi_0^2 \right)^{\nu - 1} \left( \frac{\phi}{\phi_0} \right)^{\nu - 3}}{M^3(3\gamma + 4\beta \left( \phi_0^2 \right))}, \]

while the tensor spectral index reads,

\[ n_T \simeq \frac{2\beta (\nu - 2 - 3\kappa^2 \phi^2) + 2\kappa^2 \phi^2 + 2\nu^2 - 4\nu^2 \left( \frac{\phi}{\phi_0} \right)^{\nu - 3} (\nu - 1)\phi^2}{(\nu - 1)^3 \left( 3\gamma + 4\beta \left( \frac{\phi}{\phi_0} \right) \right)^\nu}. \]

and the tensor-to-scalar ratio is,

\[ r \simeq \frac{8\kappa^2 \phi^2}{(\nu - 1)^2}. \]

The observational indices must be evaluated at the first horizon crossing, so for \( \phi = \phi_i \) and also the parameter \( M \) is assumed in this case too, to be of the form \( M = \mu/\kappa \) where \( \mu \) is a dimensionless free parameter. The model at hand is compatible with the Planck data for a wide range of the free parameters. In Fig. 3 we present the marginalized curves of Planck 2018 and the predictions of Model II, for the free parameters taking values \( \mu = 2.09 \times 10^{-22} \times \kappa \), \( \beta = [0.000025, 1.69055] \), \( \gamma = 10^{20} \), for \( N = [50, 63] \) e-foldings. As it can be seen in Fig. 1, the model is fitted quite well within the Planck 2018 data. Also for this model too, the tensor spectrum is blue tilted and the corresponding tensor spectral index takes values in the range \( n_T = [0.00679, 1.10833] \). Let us now consider the evolution of Model II during the post-inflationary era up to present day. For the numerical analysis of the differential equations, we shall impose the \( y_{H}(z = z_{fin}) = \frac{\lambda}{3m^2}, \frac{d\mu}{dz} \bigg|_{z_{fin}} = 0, \phi(z = z_{fin}) = 10^{-20}M_p, \frac{d\phi}{dz} \bigg|_{z = z_{fin}} = -10^{-10}M_p, \) where \( z_{fin} = 10^6 \).

The results of our numerical analysis for Model I can be found in Fig. 4 where we plot the total equation of state (EoS) parameter \( w_{eff} \) (left plot) and the deceleration parameter (right plot) as functions of the redshift \( z \). As it can be seen in Fig. 2, the total EoS parameter approaches in this case too the value \( w_{eff} \sim 1/3 \), hence the Model II for redshifts \( z \sim 10^6 \) has entered deeply in the radiation domination era. Also the Model II mimics in this case too the \( \Lambda \)CDM model at late times, by looking the behavior of the deceleration parameter. In Table II we have gathered the values of the dark energy EoS parameter \( \omega_{DE} \), the dark energy density parameter \( \Omega_{DE} \) and of the deceleration parameter \( q \) at present day. Overall, Model II, as Model I, offers a viable description regarding its dark energy predictions.

What now remains is to calculate the effect of the two Einstein-Gauss-Bonnet models presented in this section, on the energy spectrum of the primordial gravitational waves. This issue is addressed in the next section.
FIG. 3: Marginalized curves of the Planck 2018 data for the Model II (red curve).

FIG. 4: The total EoS parameter $w_{eff}$ (left plot) and the deceleration parameter $q$ (right plot), as functions of the redshift.

| Parameter | Model II | $\Lambda$CDM Value or Planck 2018 Constraints |
|-----------|----------|---------------------------------------------|
| $q(z=0)$  | -0.518953 | -0.535                                      |
| $\Omega_{DE}(0)$ | 0.679335 | $0.6847\pm0.0073$ |
| $\omega_{DE}(0)$ | -1 | $-1.018\pm0.031$ |

IV. PRIMORDIAL GRAVITATIONAL WAVE ENERGY SPECTRUM FOR EINSTEIN-GAUSS-BONNET THEORIES

In this section we shall calculate in detail the effect of the GW170817-compatible Einstein-Gauss-Bonnet models we developed in the previous sections on the energy spectrum of the primordial gravitational wave energy spectrum. In the literature there exist many works which consider theoretical predictions on primordial gravitational waves, for a mainstream of articles see for example Refs. [71-113] and references therein. The results of our study can be compared with the results of Ref. [74] where also the Einstein-Gauss-Bonnet primordial gravitational waves are considered. The new development that our article brings in the field is the detailed calculation of the effects that the Einstein-Gauss-Bonnet term brings along to the energy spectrum of the primordial gravitational waves, from present time back to the radiation domination era. As we show shortly, the results are quite interesting and surprisingly simple. Let us present at this point how to quantify the effects of the Einstein-Gauss-Bonnet gravity on the primordial gravitational waveform. The evolution of the Fourier transformation of the tensor perturbation of a flat FRW background has the following form [12, 100],

$$\frac{1}{a^3 Q_i} \frac{d}{d\tau} \left( a^3 Q_i \hat{h}(k) \right) + \frac{k^2}{a^2} \hat{h}(k) = 0,$$  \hspace{1cm} (75)
FIG. 5: The \( h^2 \)-scaled gravitational wave energy spectrum for the Einstein-Gauss-Bonnet Model I gravity. The Einstein-Gauss-Bonnet Model I curves correspond to three different reheating temperatures, the blue curve to \( T_R = 10^{2} \) GeV, the red curve to \( T_R = 10^{7} \) GeV and the purple curve to \( T_R = 10^{12} \) GeV.

where \( Q_t \) for Einstein-Gauss-Bonnet gravity is defined below Eq. (7), and we also quote its explicit form here for convenience, \( Q_t = F + \frac{Q_b}{2} \), \( F = \frac{1}{\kappa^2} \) and \( Q_b = -8\xi H \). The evolution equation can be written as,

\[
\ddot{h}(k) + (3 + \alpha_M) H \dot{h}(k) + \frac{k^2}{a^2} h(k) = 0 ,
\]

where the parameter \( \alpha_M \) is defined to be equal to,

\[
\alpha_M = \frac{Q_t}{Q_t H} .
\]

In the case of Einstein-Gauss-Bonnet gravity, the exact form of the parameter \( \alpha_M \) can easily be calculated by using the explicit form of \( Q_t \),

\[
\alpha_M = \frac{-4\ddot{\xi} H - 4\dot{\xi} H}{H(\frac{1}{\kappa^2} - 4\xi H)} .
\]

Basically, the parameter \( \alpha_M \) quantifies the deviation of the Einstein-Gauss-Bonnet primordial gravitational wave waveform from that of standard GR. This can easily be measured by using a WKB approach introduced in [90, 91]. According to Refs. [90, 91], by using a WKB approach, the solution of the differential equation (76) can be written in the following way,

\[
h = e^{-D} h_{GR} ,
\]

where \( h_{GR} \) denotes the GR waveform which corresponds to the case \( \alpha_M = 0 \). The quantity \( D \) is defined as,

\[
D = \frac{1}{2} \int^\tau a_M \mathcal{H} d\tau_1 = \frac{1}{2} \int_0^z \frac{a_M}{1 + z'} dz' .
\]

So in order to find the actual effect of the Einstein-Gauss-Bonnet theory on the energy spectrum of the primordial gravitational waves, one has to calculate the quantity \( D \) from present day up to the reheating era, so for redshifts \( z \sim 10^6 \). This can easily be done by using the numerical outcomes of the previous section regarding the behavior of the Einstein-Gauss-Bonnet theory up to redshifts \( z \sim 10^6 \). In addition to this, in the section where we calculated the
inflationary characteristics of the model, we also evaluated the tensor spectral index and the tensor-to-scalar ratio, which is also needed for the calculation of the energy spectrum of the primordial gravitational wave. Let us recall in brief the form of the energy spectrum of the primordial gravitational waves in the context of GR, which is,

\[ \Omega_{gw}(f) = \frac{k^2}{12H_0^2} \Delta_h^2(k), \]  

with \( \Delta_h^2(k) \) being [71 [84 [SS 90] 92] 100].

\[ \Delta_h^2(k) = \Delta_h^{(p)}(k)^2 \left( \frac{\Omega_m}{\Omega_\Lambda} \right)^2 \left( \frac{g_*(T_{in})}{g_{*0}} \right) \left( \frac{g_{*0}}{g_{*s}(T_{in})} \right)^{4/3} \left( \frac{3j_1(k\tau_0)}{k\tau_0} \right)^2 T_{in}^2(x_{eq}) T_S^2(x_R), \]

and the quantity \( \Delta_h^{(p)}(k)^2 \) denotes the inflationary tensor power spectrum, which is,

\[ \Delta_h^{(p)}(k)^2 = A_T(k_{ref}) \left( \frac{k}{k_{ref}} \right)^{n_T}. \]  

The inflationary tensor power spectrum has to be evaluated at the Cosmic Microwave Background pivot scale \( k_{ref} = 0.002 \, \text{Mpc}^{-1} \) and also recall that \( n_T \) stands for the spectral index of the tensor perturbations, while \( A_T(k_{ref}) \) denotes the amplitude of the tensor perturbations, the actual form of which is,

\[ A_T(k_{ref}) = r P_\zeta(k_{ref}). \]

Finally \( r \) is the tensor-to-scalar ratio and \( P_\zeta(k_{ref}) \) stands for the amplitude of the primordial scalar perturbations. In effect we have,

\[ \Delta_h^{(p)}(k)^2 = r P_\zeta(k_{ref}) \left( \frac{k}{k_{ref}} \right)^{n_T}, \]

therefore the energy spectrum of the primordial gravitational waves for Einstein-Gauss-Bonnet gravity takes the final form,

\[ \Omega_{gw}(f) = e^{-2D} \times \frac{k^2}{12H_0^2} r P_\zeta(k_{ref}) \left( \frac{k}{k_{ref}} \right)^{n_T} \left( \frac{\Omega_m}{\Omega_\Lambda} \right)^2 \left( \frac{g_*(T_{in})}{g_{*0}} \right) \left( \frac{g_{*0}}{g_{*s}(T_{in})} \right)^{4/3} \left( \frac{3j_1(k\tau_0)}{k\tau_0} \right)^2 T_{in}^2(x_{eq}) T_S^2(x_R), \]

with the quantity \( D \) be defined in Eq. [80]. In effect, the impact of the Einstein-Gauss-Bonnet theory on the GR energy spectrum of primordial gravitational waves is mainly contained on the quantity \( e^{-2D} \) and also to the inflationary observational indices \( n_T \) and \( r \). Now we proceed to the predictions of the energy power spectrum of the primordial gravitational waves for the Einstein-Gauss-Bonnet models I and II we developed in the previous sections. For these models we calculated numerically the parameter \( D \) for the redshift interval \( z = [0, 10^6] \) for both the models I and II. The result is astonishing as it reveals a pattern, and for the Model I we found that \( D = 6.9564 \times 10^{-6} \) while for the Model II we found that, \( D = 9.0102 \times 10^{-6} \). The remarkable result is that these are tiny for both models, thus the amplification/damping factor is equal to unity for both models I and II. Therefore, the overall amplification effect of the Einstein-Gauss-Bonnet models which are compatible with the GW170817 event is null, and this result of ours also validates the approach developed in Ref. [72], where the overall WKB factor \( e^{-2D} \) was not taken into account. As we showed, this is a correct approach since the overall amplification is null. Now let us proceed in examining the predictions of Model I and Model II regarding the energy spectrum of the primordial gravitational waves. Our results are presented in Fig. 5 for Model I while in Fig. 6 we present the results for Model II, regarding the \( h^2 \)-scaled primordial gravitational wave energy spectrum. In both models we considered three different reheating temperatures, depicted in three distinct colors namely, the blue curve to \( T_R = 10^2 \, \text{GeV} \), the red curve to \( T_R = 10^7 \, \text{GeV} \) and the purple curve to \( T_R = 10^{12} \, \text{GeV} \). In all cases, for large reheating temperatures, the signal from the Einstein-Gauss-Bonnet models is detectable by the future experiments due to the fact that for both models, the tensor spectral index is blue tilted and significantly large. However, however for Model I, a low reheating temperature will keep the gravitational wave signal undetectable. Furthermore, the signal for Model II violates the BBN bound constraints, for high frequencies, larger than 0.01 Hz. Thus the Einstein-Gauss-Bonnet models, which are compatible with the GW170817 event, serve as a potential viable candidate theory for the description of the early Universe, if a stochastic gravitational wave is detected by future experiments.
FIG. 6: The $h^2$-scaled gravitational wave energy spectrum for the Einstein-Gauss-Bonnet Model II gravity. The Einstein-Gauss-Bonnet Model II curves correspond to three different reheating temperatures, the blue curve to $T_R = 10^2\text{GeV}$, the red curve to $T_R = 10^7\text{GeV}$ and the purple curve to $T_R = 10^{12}\text{GeV}$.

V. CONCLUSIONS

In this paper we calculated for the first time the overall amplification/damping factor for the energy spectrum of the primordial gravitational waves for Einstein-Gauss-Bonnet theories. We focused on GW170817-compatible Einstein-Gauss-Bonnet theories, and we presented the inflationary dynamics formalism for these theories. In addition, we developed a formalism for the post-inflationary evolution of GW170817-compatible Einstein-Gauss-Bonnet theories, using the redshift as a dynamical variable. In this way by numerically solving the field equations using physically motivated initial conditions, we were able to know in detail the dynamics of the Einstein-Gauss-Bonnet theories at late times up to the reheating era. We applied the formalism to two well chosen Einstein-Gauss-Bonnet models, and we demonstrated that a viable inflationary era and a viable dark energy era can be obtained by those models. In this way we provided a way toward a unified description of early and late-time acceleration eras in the context of Einstein-Gauss-Bonnet theories. Notably, both the models we studied resulted to a blue tilted tensor spectral index, and this has significant effects on the production of stochastic primordial gravitational waves. Finally, we calculated the overall effect of the Einstein-Gauss-Bonnet gravity on the amplification/damping factor of the energy spectrum of the primordial gravitational wave. As we showed, the amplification is of the order of unity, and this validates the work of [74] who also considered similar Einstein-Gauss-Bonnet theories, without the GW170817 constraint. A natural extension of the present study is to include more quantum corrections to the single scalar field Lagrangian, except from the Gauss-Bonnet term only. For example one could consider power-law curvature corrected Einstein-Gauss-Bonnet theories like in Ref. [114]. Work is in progress toward this research line.

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