Surface Layers in General Relativity and Their Relation to Surface Tensions

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Abstract

For a thin shell, the intrinsic 3-pressure will be shown to be analogous to $-A$, where $A$ is the classical surface tension: First, interior and exterior Schwarzschild solutions will be matched together such that the surface layer generated at the common boundary has no gravitational mass; then its intrinsic 3-pressure represents a surface tension fulfilling Kelvin’s relation between mean curvature and pressure difference in the Newtonian limit. Second, after a suitable definition of mean curvature, the general relativistic analogue to Kelvin’s relation will be proven to be contained in the equation of motion of the surface layer.

1 Introduction

In general relativity, an energy-momentum tensor concentrated on a time-like hypersurface is called a surface layer. Via Einstein’s equations it is related to nonspurious jumps of the Christoffel affinities or equivalently to jumps of the second fundamental tensor. In [1-6] there have been given algorithms for
their calculating, and in [1], [4], and [7-13] the spherically symmetric case was of a special interest. The discussion of disklike layers as models for accretion disks was initiated in [14] and [15], and [21] contains surface layers as sources of the Kerr geometry.

A surface layer has no component in the normal direction, otherwise the delta-like character would be destroyed. Hence, an ideal fluid with nonvanishing pressure cannot be concentrated on an arbitrary thin region and therefore in most cases, the layer is considered to be composed of dust. Additionally, an intrinsic 3-pressure is taken into account in [16] (cf. [17]), where for a spherically symmetric configuration each shell $r = \text{const}$ is thought to be composed of identical particles moving on circular orbits without a preferred direction, and the tangential pressure (= intrinsic 3-pressure) is due to particle collisions, whereas a radial pressure does not appear. (See also [10], [13], and [26], where the equation of motion for a spherically symmetric layer has been discussed.)

Such an intrinsic 3-pressure as well as surface tensions are both of the physical dimension “force per unit length.” There the question arises whether an intrinsic 3-pressure of a surface layer may be related to a surface tension, and the present paper deals with just this question. Then a general relativistic formulation of thermodynamics (cf. [18] or [19] for this and [20] for nonrelativistic surface tensions) can be completed by equations for surface tensions to answer, e.g., the question how long a drop, say, a liquid comet, remains connected while falling towards a compact object. The only paper concerned with such questions seems to be [22]. There the influence of surface tensions on the propagation of gravitational waves has been calculated by perturbation methods, yielding a possibly measurable effect.

The paper proceeds as follows: Section 2 contains Kelvin’s relation for nonrelativistic surface tensions. Section 3 discusses the matching of the interior to the exterior Schwarzschild solution such that a surface tension ap-
pears (calculations are found in the appendices) and compares with Kelvin’s relation in the Newtonian limit. Sections 4 and 5 are devoted to the nonspherically symmetric case. Section 4 contains a suitable definition of mean curvature and Section 5 deduces Kelvin’s relation from the equation of motion of the surface layer without any weak field assumptions.

2 Nonrelativistic Surface Tensions

Imagine a drop of some liquid moving in vacuo. Its equilibrium configuration is a spherical one, and Kelvin’s relation [23] between surface tension $A$, pressure difference (outer minus inner pressure) $\Delta P$, and mean curvature $H$ ($H = 1/R$ for a sphere of radius $R$) reads

$$\Delta P = -2HA.$$  

$A$ is a material-dependent constant. Equation (1) means, an energy $A \cdot \Delta F$ is needed to increase the surface area by $\Delta F$. This supports our description of $-A$ as a kind of intrinsic pressure. (But of course, it is a quite different physical process: Pressure, say, of an ideal gas, can be explained by collisions of freely moving particles, whereas the microphysical explanation of surface tensions requires the determination of the intermolecular potential, which looks like

$$\Phi(r) = -\mu r^{-6} + Ne^{-r/\rho}$$

with certain constants $\mu$, $N$, and $\rho$; see [20]. In this context the surface has a thickness of about $10^{-7}$ cm but in most cases this thickness may be neglected.)

In addition, for a nonspherically symmetric surface, the mean curvature $H$ in equation (1) may be obtained from the principal curvature radii $R_1, R_2$ by means of the relation

$$H = \frac{1}{2R_1} + \frac{1}{2R_2},$$  

(2)
3 Spherical Symmetry

Now the drop shall be composed of an incompressible liquid. For a general relativistic description we have to take the interior Schwarzschild solution

\[ ds^2 = -\left[ \frac{3}{2}(1 - r_g/r_0)^{1/2} - \frac{1}{2}(1 - r_g r^2/r_0^3)^{1/2} \right]^2 dt^2 + \frac{dr^2}{1 - r_g r^2/r_0^3} + r^2 d\Omega^2 \]

\[ d\Omega^2 = d\psi^2 + \sin^2 \psi d\varphi^2 \]

whose energy-momentum tensor represents ideal fluid with energy density \( \mu = 3r_g/\kappa r_0^3 \) and pressure

\[ p(r) = \mu \cdot \frac{(1 - r_g r^2/r_0^3)^{1/2} - (1 - r_g/r_0)^{1/2}}{3(1 - r_g/r_0)^{1/2} - (1 - r_g r^2/r_0^3)^{1/2}}. \]

One has \( p(r_0) = 0 \), and therefore usually \( 0 \leq r \leq r_0 \) is considered. But now we require only a nonvanishing pressure at the inner surface and take (3) for values \( r \) with \( 0 \leq r \leq R \) and a fixed \( R < r_0 \) only. The gravitational mass of the inner region equals

\[ M = \mu \cdot 4\pi R^3/3 = r_g (R/r_0)^3/2 \]

where \( G = c = 1 \). The outer region shall be empty and therefore we have to insert the Schwarzschild solution for \( r \geq R \). Neglecting the gravitational mass\(^1\) of the boundary \( \Sigma \subset V_4 \) which is defined by \( r - R = 0 \), just \( M \) of equation (5) has to be used as the mass parameter of the exterior Schwarzschild metric. Then only delta-like tensions appear at \( \Sigma \), and \( \Delta P = -p(R) < 0 \).

On \( \Sigma \), the energy-momentum tensor is

\[ T_i^k = \tau_i^k \cdot \delta_\Sigma \]

\(^1\) The vanishing of the gravitational mass is required to single out the properties of an intrinsic 3-pressure. In general, a surface layer is composed of both parts, not at least to ensure the validity of the energy condition \( T_{00} \geq |T_{ik}| \) which holds for all known types of matter.
the nonvanishing components of which are

\[ \tau_2^2 = \tau_3^3 = -p(R)R/2(1 - 2M/R)^{1/2} \] (7)

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cf. Appendix A.

The \( \delta_\Sigma \) distribution is defined such that for all smooth scalar functions \( f \) the invariant integrals

\[ \int_\Sigma f \, d^{(3)}x \quad \text{and} \quad \int_{V_4} f \cdot \delta_\Sigma d^{(4)}x \]

coincide. For a more detailed discussion of distribution-valued tensors in curved spacetime, cf. [24].

The fact that all components \( \tau_0^\alpha \) in (6) vanish reflects the nonexistence of a delta-like gravitational mass on \( \Sigma\).

Now consider the Newtonian limit \( M/R \ll 1; \) then the mean curvature becomes again \( H = 1/R \), and together with (7) Kelvin’s relation (1) is just equivalent to

\[ \tau_2^2 = \tau_3^3 = -A[1 + O(M/R)]. \] (8)

Therefore: At least for static spherically symmetric configurations and weak fields a delta-like negative tangential pressure coincides with the classical surface tension.

In the next two sections we investigate to what extent these presumptions are necessary.

4 Mean Curvature in Curved Space-Time

To obtain a general relativistic analogue to Kelvin's relation (1) we have to define the mean curvature \( H \) of the time-like hypersurface \( \Sigma \) contained in a space-time \( V_4 \) such that for weak fields just the usual mean curvature arises.

To get the configuration we have in mind, we make the ansatz

\[ \tau_{\alpha\beta} = -A(g_{\alpha\beta} + u_\alpha u_\beta) \quad u_\alpha u^\alpha = -1 \] (9)
with $A > 0$. Thereby again the gravitational mass of $\Sigma$ will be neglected. Now the mean curvature shall be defined. But there is a problem: In general, there does not exist a surface $S \subset \Sigma$ which can serve as “boundary at a fixed moment” for which we are to determine the mean curvature. To circumvent this problem we start considering the special case

$$u_{\alpha\beta} = u_{\beta\alpha} \cdot$$  \hspace{1cm} (10)

Then there exists a scalar $t$ on $\Sigma$ such that $u_{\alpha} = t_{\alpha}$, and the surface $S \subset \Sigma$ defined by $t = 0$ may be called “boundary at a fixed moment.” Now $S$ has to be embedded into a “space at a fixed moment”: We take intervals of geodesics starting from points of $S$ in the normal direction $n_{i}$ and the opposite one. The union $V_{3}$ of these geodetic segments will be called “space at a fixed moment,” and $S \subset V_{3}$ is simply a two-surface in a three-dimensional positive definite Riemannian manifold, for which mean curvature has a definite sense: Let $v^{\alpha}$, $w^{\alpha}$ be the principal curvature directions inside $S$ and $R_{1}$, $R_{2}$ the corresponding principal curvature radii, then equation (2) applies to obtain $H$.

Of course, $v^{\alpha}w_{\alpha} = 0$ holds, and $v^{\alpha}v_{\alpha} = w^{\alpha}w_{\alpha} = 1$ shall be attained. Then, inserting the second fundamental tensor (cf. Appendix B), this becomes equivalent to

$$H = \frac{1}{2}(v^{\alpha}v_{\alpha} + w^{\alpha}w_{\alpha})k_{\alpha\beta} = \frac{1}{2}(g^{\alpha\beta} + u^{\alpha}u_{\beta})k_{\alpha\beta} \cdot$$  \hspace{1cm} (11)

But this latter relation makes sense without any reference to condition (10). Therefore, we define (11) to be the general relativistic analogue to the mean curvature of a surface. For the case $H^{+} \neq H^{-}$ we take their arithmetic mean$^{2}$

$$H = \frac{(H^{+} + H^{-})}{2},$$

$^{2}$This choice can be accepted noting that in the Newtonian limit

$$|H^{+} - H^{-}| \ll |H^{+} + H^{-}|$$

anyhow.
and then we obtain from (9) and (11)

\[-2HA = \frac{1}{2} \left( k^+_{\alpha\beta} + k^-_{\alpha\beta} \right) \tau^{\alpha\beta}. \tag{12}\]

To compare this with Kelvin’s relation we have to relate the right-hand side of equation (12) to the pressure difference \(\Delta P\) at \(\Sigma\). To this end we investigate the equation of motion for the surface layer.

5 Equation of Motion for the Surface Layer

The equation of motion, \(T^{k}_{i;k} = 0\), contains products of \(\delta\) distributions and \(\theta\)-step functions at points where \(\Gamma^i_{jk}\) has a jump discontinuity. These products require special care; cf. [25] for a discussion of his point. But defining \(\theta \cdot \delta = \frac{1}{2}\delta\) we obtain (cf. Appendix C)

\[
\Delta P \equiv \Delta n_i n^k T^i_k = \frac{1}{2} \left( k^+_{\alpha\beta} + k^-_{\alpha\beta} \right) \tau^{\alpha\beta} \quad \text{and} \quad \Delta T^1_\alpha \equiv \Delta n_i \epsilon^k_\alpha T^i_k = -\tau^\beta_{\alpha\parallel \beta}. \tag{13}\]

From equation (14) we see the following: The equation \(\tau^\beta_{\alpha\parallel \beta} = 0\) holds only under the additional presumption that the regular (i.e., not delta-like) part of \(T^1_\alpha\) has no jump on \(\Sigma\). This condition is fulfilled e.g., presuming \(\Sigma\) to be such a boundary that the regular energy flow does not cross it and the four-velocity is parallel to \(\Sigma\) in both \(V_+\) and \(V_-\). This we will presume in the following. Then \(\Delta P\) is indeed the difference of the pressures on both sides, and together with equations (12) and (13) we obtain exactly Kelvin’s relation (1). That means, it is the definition of mean curvature used here that enables us to generalize Kelvin’s formula to general relativity. Furthermore, \(O(M/R)\) of equation (8) vanishes.

Finally we want to discuss the equation \(\tau^\beta_{\alpha\parallel \beta} = 0\). Transvection with \(u^\alpha\) and \(\delta^\alpha_\gamma + u^\alpha u_\gamma\) yields

\[
u^\alpha_{\parallel \alpha} = 0 \quad \text{and} \quad u^\alpha u_{\gamma\parallel \alpha} + (\ln A)_{\parallel \alpha} (\delta^\alpha_\gamma + u^\alpha u_\gamma) = 0 \tag{15}\]

\(\theta(x) = 1\) for \(x \geq 0\), and \(\theta(x) = 0\) else.
respectively. But $A$ is a constant here, and therefore $u^\alpha$ is an expansion-free geodesic vector field in $\Sigma$.\footnote{But observe that the $u^\alpha$ lines are geodesics in $V_4$ under additional presumptions only.}

Here, we have only considered a phenomenological theory of surface tensions, and, of course, a more detailed theory has to include intermolecular forces. But on that phenomenological level equations (9) and (15) together with Kelvin’s relation (1) (which has been shown to follow from the equation of motion) and $A = \text{const}$ as a (solely temperature-dependent) equation of state complete the usual general relativistic Cauchy problem for a thermodynamical system by including surface tensions.

**Appendix A**

To deduce equation (7) we take proper time $\xi^0$ and angular coordinates $\psi = \xi^2$ and $\varphi = \xi^3$. Then, inside $\Sigma$,

$$ds^2 = -\left(d\xi^0\right)^2 + R^2 d\Omega^2.$$ Using equation (20) and the exterior Schwarzschild solution one obtains $k^+_{\alpha\beta}$ the nonvanishing components of which are

$$k^+_{00} = -M/R^2(l - 2M/R)^{1/2}, \quad k^+_{22} = R(l - 2M/R)^{1/2}$$

and

$$k^+_{33} = k^+_{22} \sin^2 \psi$$

because of spherical symmetry. To avoid long calculations with the metric (3) one can proceed as follows. By construction, $\tau_{00} = 0$, and together with equation (21) and the spherical symmetry $k^-_{22} = k^+_{22}$, $k^-_{33} = k^+_{33}$ follows. From equation (24), equation (7) follows then immediately without the necessity of determining the actual value of $k^-_{00}$.
And to be independent of the discussions connected with equation (23),
we deduce equation (24) another way. First, (independent of surface layers),
for an arbitrary timelike hypersurface and a coordinate system such that (22)
holds, we have
\[ \kappa T_{11} = \frac{1}{2} \left( (3)R + k^2 - k_{\alpha \beta}k^{\alpha \beta} \right) \tag{16} \]
where \( (3)R \) is the curvature scalar within that surface. Now turn to a surface
layer with \( k^+_{\alpha \beta} \neq k^-_{\alpha \beta} \). Then equation (16) splits into a “+” and a “−”
equation, having in common solely \( (3)R \). Inserting all this into equation
(21), one obtains
\[ \frac{1}{2} \left( k^+_{\alpha \beta} + k^-_{\alpha \beta} \right) \tau^{\alpha \beta} = \frac{1}{2\kappa} \left( k^+_{\alpha \beta} + k^-_{\alpha \beta} \right) \left( g^{\alpha \beta} \Delta k - \Delta k^{\alpha \beta} \right) \]
\[ = \frac{1}{2\kappa} \left[ (k^+)^2 - (k^-)^2 - k^+_{\alpha \beta}k^{+\alpha \beta} + k^-_{\alpha \beta}k^{-\alpha \beta} \right] = T^+_i - T^-_i \]
i.e., just equation (24) below.

**Appendix B**

To make the paper more readable, some conventions and formulas shall be
given. Let \( \xi^\alpha, \alpha = 0, 2, 3, \) be coordinates in \( \Sigma \) and \( x^i, i = 0, 1, 2, 3, \) those
for \( V_4 \). The embedding \( \Sigma \subset V_4 \) is performed by functions \( x^i(\xi^\alpha) \) whose
derivatives
\[ e^i_\alpha = \partial x^i/\partial \xi^\alpha \equiv x^i_{;\alpha} \tag{17} \]
form a triad field in \( \Sigma \). \( \Sigma \) divides, at least locally, \( V_4 \) into two connected
components, \( V_+ \) and \( V_- \), and the normal \( n_i \), defined by
\[ n_i n^i = 1, \quad n_i e^i_\alpha = 0 \tag{18} \]
is chosen into the \( V_+ \) direction (which can be thought being the outer region).
Possibly \( V_+ \) and \( V_- \) are endowed with different coordinates \( x^i_+, x^i_- \) and metrics \( g_{ik+} \) and \( g_{ik-} \), respectively. For this case all subsequent formulas had to
be indexed with \(+/-\), and only the inner metric of \(\Sigma\), its first fundamental tensor

\[ g_{\alpha\beta} = e^i_{\alpha} e^k_{\beta} g_{ik} \]  \hspace{1cm} (19)

has to be the same in both cases. As usual, we require \(g_{ik}\) to be \(C^2\)-differentiable except for jumps of \(g_{ij,k}\) at \(\Sigma\). Covariant derivatives within \(V_4\) and \(\Sigma\) will be denoted by \(;\) and \(\parallel\) respectively.

The second fundamental tensor \(k^\pm_{\alpha\beta}\) on both sides of \(\Sigma\) is defined by

\[ k^\pm_{\alpha\beta} = (e^i_{\alpha} e^k_{\beta} n_{i;k})^\pm = (e^i_{\alpha;k} e^k_{\beta} n_i)^\pm \]  \hspace{1cm} (20)

and the difference, \(\Delta k_{\alpha\beta} = k^+_{\alpha\beta} - k^-_{\alpha\beta}\), \(\Delta k = g^{\alpha\beta} \Delta k_{\alpha\beta}\), enters the energy-momentum tensor via equation (6) and the relation

\[ \tau^{ik} \equiv e^i_{\alpha} e^k_{\beta} \tau_{\alpha\beta}, \hspace{1cm} \text{where} \hspace{1cm} \kappa \tau_{\alpha\beta} = g^{\alpha\beta} \Delta k - \Delta k_{\alpha\beta} \]  \hspace{1cm} (21)

cf. e.g., [4]. From equation (18) and the Lanczos equation (21) one obtains \(n^i \tau_{ik} = 0\), i.e., indeed the absence of a delta-like energy flow in the normal direction.

**Appendix C**

Now take a special coordinate system: \(x^\alpha = \xi^\alpha\), and the \(x^1\) lines are geodesics starting from \(\Sigma\) into \(n_i\) direction with natural parameter \(x^1\). Then the line element of \(V_4\) reads

\[ ds^2 = - \left(dx^1\right)^2 + g_{\alpha\beta} dx^\alpha dx^\beta \]  \hspace{1cm} (22)

and the only jumps of \(\Gamma^i_{jk}\) are

\[ \Gamma^\pm_{\alpha\beta} = -k^\pm_{\alpha\beta} = -\frac{1}{2} g^\pm_{\alpha\beta,1}. \]

The most natural definition of \(\theta \cdot \delta\) is \(\frac{1}{2} \delta\) being equivalent to the choice

\[ \Gamma^i_{jk} = \frac{1}{2} \left( \Gamma^+_{jk} + \Gamma^-_{jk} \right) \hspace{1cm} \text{on} \hspace{1cm} \Sigma. \]  \hspace{1cm} (23)
But cf. [3] for another choice of $\Gamma_{jk}^i$ with the consequence that $T_{i;k}^k \neq 0$ at $\Sigma$. Now the $\delta$ part of the equation $T_{i;k}^k = 0$ reads

$$\Delta T^1_i \equiv T^1_{i+1} - T^1_{i-1} = \frac{1}{2} \left( k_{\alpha\beta}^+ + k_{\alpha\beta}^- \right) \tau^{\alpha\beta}. \quad (24)$$

Analogously one obtains for the other components

$$\Delta T^1_{\alpha} = -\tau^{\alpha \beta}_{\alpha \beta}. \quad (25)$$

Reintroducing the original coordinate system, the left-hand sides of equations (24) and (25) have to be replaced by $\Delta P_i = \Delta n_i n^k T^i_k$ and $\Delta P_{\alpha} = \Delta n_{\alpha} e^k_{\alpha} T^i_k$, respectively; cf. [26]. Thereby, $\Delta P_i$ is the difference of the energy flows on both sides of $\Sigma$.

**Acknowledgment**

Discussions with members of our relativity group, especially with S. Gottlöber and U. Kasper, are gratefully acknowledged.

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Received February 14, 1983

In this reprint we removed only obvious misprints of the original, which was published in General Relativity and Gravitation, Gen. Rel. Grav. 16 (1984) Nr. 11, pages 1053 - 1061; Author’s address that time: Zentralinstitut für Astrophysik der Akademie der Wissenschaften der DDR, 1502 Potsdam–Babelsberg, R.-Luxemburg-Str. 17a.