Quantum coherence in mutually unbiased bases

Yao-Kun Wang,1, 2 Li-Zhu Ge,3 and Yuan-Hong Tao4

1College of Mathematics, Tonghua Normal University, Tonghua, Jilin 134001, China
2Research Center for Mathematics, College of Mathematics, Tonghua Normal University, Tonghua, Jilin 134001, China
3The Branch Campus of Tonghua Normal University, Tonghua, Jilin 134001, China
4Department of Mathematics College of Sciences, Yanbian University, Yanji 133002, China

We investigate the $l_1$ norm of coherence of quantum states in mutually unbiased bases. We find that the sum of squared $l_1$ norm of coherence of the mixed state single qubit is less than two. We derive the $l_1$ norm of coherence of three classes of $X$ states in nontrivial mutually unbiased bases for 4-dimensional Hilbert space is equal. We propose “autotensor of mutually unbiased basis(AMUB)” by the tensor of mutually unbiased bases, and depict the level surface of constant the sum of the $l_1$ norm of coherence of Bell-diagonal states in AMUB. We find the $l_1$ norm of coherence of Werner states and isotropic states in AMUB is equal respectively.

I. INTRODUCTION

Quantum coherence is a special feature of quantum mechanic like entanglement and other quantum correlations. Quantum coherence is an essential factor in quantum information processing, quantum optics, quantum metrology, low-temperature thermodynamics and quantum biology. Recently, a structure to quantify coherence has been proposed, and various quantum coherence measures, such as the $l_1$ norm of coherence, trace norm of coherence, relative entropy of coherence, Tsallis relative $\alpha$ entropies and Relative Rényi $\alpha$ monotones, have been defined. With the help of the coherence measures, a variety of properties of quantum coherence, such as the relations between quantum correlations and quantum coherence, the freezing phenomenon of coherence, have been studied.

Mutually unbiased bases are used in detection of quantum entanglement, quantum state reconstruction, quantum error correction, and the mean kings problem. Many features of mutually unbiased bases are reviewed in reference. When $d$ is power of a prime number, maximal sets of $d+1$ mutually unbiased bases have been built for the case. Maximal sets of MUBs are an open problem, when the dimensionality is another composite number. Entropic uncertainty relations for $d+1$ mutually unbiased bases in $d$-dimensional Hilbert space were obtained in references. The fine-grained uncertainty relation for mutually unbiased bases is derived in. The relation between mutually unbiased bases and unextendible maximally entangled is investigated in.

In this article, we investigate the $l_1$ norm of coherence of quantum states in mutually unbiased bases. We evaluate analytically the sum of squared $l_1$ norm of coherence of the mixed state single qubit. We derive the relation of the $l_1$ norm of coherence of three classes of $X$ states in nontrivial mutually unbiased bases for 4-dimensional Hilbert space. We propose “autotensor of mutually unbiased basis(AMUB)” by the tensor of mutually unbiased bases, and depict the level surface of constant the sum of the $l_1$ norm of coherence of Bell-diagonal states in AMUB. We obtain the relations of the $l_1$ norm of coherence of Werner states and isotropic states in AMUB respectively.
II. THE $l_1$ NORM OF COHERENCE OF QUANTUM STATES IN 2 DIMENSION MUTUALLY UNBIASED BASES

Under fixed reference basis, the $l_1$ norm of coherence of state $\rho$ is defined by

$$C_{l_1}(\rho) = \sum_{i \neq j} |\rho_{i,j}|,$$

and the relative entropy of coherence is given by

$$C_r(\rho) = S(\rho_{\text{diag}}) - S(\rho),$$

where $S(\rho) = -Tr \rho \log \rho$ is von Neumann entropy.

A set of orthonormal bases $\{B_k\}$ for a Hilbert space $H = C^d$ where $\{B_k\} = \{|0_k\}, \cdots, |d-1_k\}$ is called mutually unbiased (MU) iff

$$|\langle i_k | j_l \rangle|^2 = \frac{1}{d}, \forall i, j \in \{0, \cdots, d-1\},$$

holds for all basis vectors $|i_k\rangle$ and $|j_l\rangle$ that belong to different bases, i.e. $\forall k \neq l$.

In dimension $d = 2$, a set of three mutually unbiased bases is readily obtained from the eigenvectors of the three Pauli matrices $\sigma_z$, $\sigma_x$ and $\sigma_y$:

$$\begin{align*}
\alpha_1 &= \{\alpha_{11}, \alpha_{12}\} = \{|0\}, |1\}, \\
\alpha_2 &= \{\alpha_{21}, \alpha_{22}\} = \{\frac{1}{\sqrt{2}}(|0|+|1|), \frac{1}{\sqrt{2}}(|0| - |1|)\}, \\
\alpha_3 &= \{\alpha_{31}, \alpha_{32}\} = \{\frac{1}{\sqrt{2}}(|0| + i|1|), \frac{1}{\sqrt{2}}(|0| - i|1|)\}.
\end{align*}$$

In dimension $d = 3$, there are four mutually unbiased bases as follows:

$$\begin{align*}
\beta_1 &= \{\beta_{11}, \beta_{12}, \beta_{13}\} = \{|0\}, |1\}, |2\}, \\
\beta_2 &= \{\beta_{21}, \beta_{22}, \beta_{23}\} = \{\frac{1}{\sqrt{3}}(|0| + |1| + |2|), \frac{1}{\sqrt{3}}(|0| + \omega|1| + \omega^2|2|), \frac{1}{\sqrt{3}}(|0| + \omega^2|1| + \omega|2|)\}, \\
\beta_3 &= \{\alpha_{31}, \alpha_{32}, \alpha_{33}\} = \{\frac{1}{\sqrt{3}}(|0| + |1| + \omega|2|), \frac{1}{\sqrt{3}}(|0| + \omega|1| + |2|), \frac{1}{\sqrt{3}}(|0| + \omega^2|1| + \omega|2|)\}, \\
\beta_4 &= \{\alpha_{41}, \alpha_{42}, \alpha_{43}\} = \{\frac{1}{\sqrt{3}}(|0| + |1| + \omega^2|2|), \frac{1}{\sqrt{3}}(|0| + \omega|1| + |2|), \frac{1}{\sqrt{3}}(|0| + \omega^2|1| + \omega^2|2|)\},
\end{align*}$$

where $\omega = e^{i\frac{2\pi}{3}}$.

An arbitrary density matrix for a mixed state single qubit may be written as

$$\rho_s = \frac{I + \overrightarrow{r} \cdot \overrightarrow{\sigma}}{2},$$

where $\overrightarrow{r} = (x, y, z)$ is a real three-dimensional vector such that $x^2 + y^2 + z^2 \leq 1$, and $\overrightarrow{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. In particular, $\rho$ is pure if and only if $x^2 + y^2 + z^2 = 1$.

Next, we will consider the relation of the $l_1$ norm of coherence among $\rho_s$ in three mutually unbiased bases $\alpha_1, \alpha_2, \alpha_3$. 
The density matrix of mixed state single qubit $\rho_s$ in base $\alpha_1 = \{\alpha_{11}, \alpha_{12}\} = \{|0\rangle, |1\rangle \}$ is

$$\rho_s = \frac{1}{2} \left( \begin{array}{cc} 1 + z & x - iy \\ x + iy & 1 - z \end{array} \right)$$

$$= \frac{1}{2} (1 + z)|0\rangle\langle 0| + \frac{1}{2} (x - iy)|0\rangle\langle 1| + \frac{1}{2} (x + iy)|1\rangle\langle 0| + \frac{1}{2} (1 - z)|1\rangle\langle 1|,

(4)$$

Using Eq. (1) directly, the $l_1$ norm of coherence of state $\rho_s$ in base $\alpha_1$ is

$$C_{l_1}(\rho_s)_{\alpha_1} = \left| \frac{1}{2} (x - iy) \right| + \left| \frac{1}{2} (x + iy) \right| = \sqrt{x^2 + y^2}. \quad (5)$$

The density matrix of $\rho_s$ in base $\alpha_2 = \{\alpha_{21}, \alpha_{22}\}$ is

$$\rho_s = \left( \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right)$$

$$= a_{11} \alpha_{21} \alpha_{21}^\dagger + a_{12} \alpha_{21} \alpha_{22}^\dagger + a_{21} \alpha_{22} \alpha_{22}^\dagger + a_{22} \alpha_{22} \alpha_{22}^\dagger$$

$$= \frac{1}{2} (a_{11} + a_{12} + a_{21} + a_{22}) |0\rangle\langle 0| + \frac{1}{2} (a_{11} - a_{12} + a_{21} - a_{22}) |0\rangle\langle 1|$$

$$+ \frac{1}{2} (a_{11} + a_{12} - a_{21} - a_{22}) |1\rangle\langle 0| + \frac{1}{2} (a_{11} - a_{12} - a_{21} + a_{22}) |1\rangle\langle 1|. \quad (6)$$

As $\rho_s$ in Eq. (1) and Eq. (6) is the same, using the method of undetermined coefficients, we obtain

$$\begin{align*}
\frac{1}{2} (a_{11} + a_{12} + a_{21} + a_{22}) &= \frac{1}{2} (1 + z) \\
\frac{1}{2} (a_{11} - a_{12} + a_{21} - a_{22}) &= \frac{1}{2} (x - iy) \\
\frac{1}{2} (a_{11} + a_{12} - a_{21} - a_{22}) &= \frac{1}{2} (x + iy) \\
\frac{1}{2} (a_{11} - a_{12} - a_{21} + a_{22}) &= \frac{1}{2} (1 - z)
\end{align*}$$

The solution of the equation is

$$\begin{align*}
a_{11} &= \frac{1 + z}{2} \\
a_{12} &= \frac{z + iy}{2} \\
a_{21} &= \frac{z - iy}{2} \\
a_{22} &= \frac{1 - z}{2}
\end{align*} \quad (7)$$

The $l_1$ norm of coherence of state $\rho_s$ in base $\alpha_2$ is

$$C_{l_1}(\rho_s)_{\alpha_2} = \left| \frac{1}{2} (z + iy) \right| + \left| \frac{1}{2} (z - iy) \right| = \sqrt{z^2 + y^2}. \quad (8)$$

The density matrix of $\rho_s$ in base $\alpha_3 = \{\alpha_{31}, \alpha_{32}\}$ by the above method is

$$\rho_s = \frac{1}{2} \left( \begin{array}{cc} 1 + y & z - ix \\ z + ix & 1 - y \end{array} \right) \quad (9)$$

The $l_1$ norm of coherence of state $\rho_s$ in base $\alpha_3$ is

$$C_{l_1}(\rho_s)_{\alpha_3} = \left| \frac{1}{2} (z - ix) \right| + \left| \frac{1}{2} (z + ix) \right| = \sqrt{z^2 + x^2}. \quad (10)$$

As $x^2 + y^2 + z^2 \leq 1$, $|C_{l_1}(\rho_s)_{\alpha_1}|^2 + |C_{l_1}(\rho_s)_{\alpha_2}|^2 + |C_{l_1}(\rho_s)_{\alpha_3}|^2 \leq 2.$
III. **The \( l_1 \) Norm of Coherence of \( X \) States in the Tensor of 3 Dimension Mutually Unbiased Bases**

For the three classes of \( X \) states in base \( \beta_1 = \{ \beta_{11}, \beta_{12}, \beta_{13} \} = \{ |0\rangle, |1\rangle, |2\rangle \} \)

\[
\rho_X = \begin{pmatrix} x & 0 & z \\ 0 & 1 - x - y & 0 \\ z & 0 & y \end{pmatrix},
\]

(11)

where \( x, y, z \) are all real number, we will consider the \( l_1 \) norm of coherence of \( \rho_X \) in the 3 dimension mutually unbiased bases \( \beta_2, \beta_3, \beta_4 \).

Let the density matrix of \( \rho_X \) in base \( \beta_2 = \{ \beta_{21}, \beta_{22}, \beta_{23} \} \) be

\[
\rho_X = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix},
\]

(12)

and \( \rho_X = b_{11}\beta_{21}\beta_{21}^\dagger + b_{12}\beta_{21}\beta_{22}^\dagger + b_{13}\beta_{21}\beta_{23}^\dagger + b_{21}\beta_{22}\beta_{21}^\dagger + b_{22}\beta_{22}\beta_{22}^\dagger + b_{23}\beta_{22}\beta_{23}^\dagger + b_{31}\beta_{23}\beta_{21}^\dagger + b_{32}\beta_{23}\beta_{22}^\dagger + b_{33}\beta_{23}\beta_{23}^\dagger \). As \( \rho_X \) in Eq. (11) and Eq. (12) is the same, using the method of undetermined coefficients, we obtain

\[
\frac{1}{3}(b_{11} + b_{12} + b_{13} + b_{21} + b_{22} + b_{23} + b_{31} + b_{32} + b_{33}) = x
\]
\[
\frac{1}{3}(b_{11} + \omega^2 b_{12} + \omega b_{13} + b_{21} + \omega^2 b_{22} + \omega b_{23} + b_{31} + \omega^2 b_{32} + \omega b_{33}) = 0
\]
\[
\frac{1}{3}(b_{11} + \omega b_{12} + \omega^2 b_{13} + b_{21} + \omega b_{22} + \omega^2 b_{23} + b_{31} + \omega b_{32} + \omega^2 b_{33}) = z
\]
\[
\frac{1}{3}(b_{11} + b_{12} + b_{13} + \omega b_{21} + \omega b_{22} + \omega b_{23} + \omega^2 b_{31} + \omega^2 b_{32} + \omega^2 b_{33}) = 0
\]
\[
\frac{1}{3}(b_{11} + \omega^2 b_{12} + \omega b_{13} + b_{21} + \omega^2 b_{22} + \omega b_{23} + \omega^2 b_{31} + \omega b_{32} + \omega b_{33}) = 1 - x - y
\]

(13)

The solution of the equation is

\[
\begin{aligned}
b_{11} &= \frac{1 + \sqrt{2} + \sqrt{3}}{3}, b_{12} = \frac{3 \sqrt{x + z} - 1 - \sqrt{3} (x + 2y + z - 1)}{6}, b_{13} = \frac{3 \sqrt{x + z} + \sqrt{3} (x + 2y + z - 1)}{6}, \\
b_{21} &= \frac{1 - \sqrt{2} + \sqrt{3}}{3}, b_{22} = \frac{3 \sqrt{x - 2z} - 1 + \sqrt{3} (x + 2y - 2z - 1)}{6}, b_{23} = \frac{3 \sqrt{x - 2z} + \sqrt{3} (x + 2y - 2z - 1)}{6}, \\
b_{31} &= \frac{1 - \sqrt{2} - \sqrt{3}}{3}, b_{32} = \frac{3 \sqrt{x - z} - 1 + \sqrt{3} (x - 2y - z - 1)}{6}, b_{33} = \frac{3 \sqrt{x - z} - \sqrt{3} (x - 2y - z - 1)}{6}.
\end{aligned}
\]

(14)

The \( l_1 \) norm of coherence of state \( \rho_X \) in base \( \beta_2 \) is

\[
C_{l_1}(\rho_X)_{\beta_2} = 2(|b_{12}| + |b_{13}| + |b_{23}|).
\]

(15)

Similarly, the density matrix of \( \rho_X \) in base \( \beta_3 \) is

\[
\rho_X = \begin{pmatrix} b_{22} & b_{12} & b_{23} \\ b_{12} & b_{11} & b_{13} \\ b_{23} & b_{13} & b_{33} \end{pmatrix},
\]

(16)

The \( l_1 \) norm of coherence of state \( \rho_X \) in base \( \beta_3 \) is

\[
C_{l_1}(\rho_X)_{\beta_3} = 2(|b_{12}| + |b_{13}| + |b_{23}|).
\]

(17)

The density matrix of \( \rho_X \) in base \( \beta_4 \) is

\[
\rho_X = \begin{pmatrix} b_{22} & b_{12} & b_{23} \\ b_{12} & b_{11} & b_{13} \\ b_{23} & b_{13} & b_{33} \end{pmatrix},
\]

(18)
The $l_1$ norm of coherence of state $\rho_X$ in base $\beta_4$ is

$$C_{l_1}(\rho_X)_{\beta_4} = 2(|b_{12}| + |b_{13}| + |b_{23}|).$$ (19)

At last, we find that the $l_1$ norm of coherence of state $\rho_X$ in base $\beta_2, \beta_3, \beta_4$ is equal, i.e

$$C_{l_1}(\rho_X)_{\beta_2} = C_{l_1}(\rho_X)_{\beta_3} = C_{l_1}(\rho_X)_{\beta_4}. \quad (20)$$

Furthermore, let

$$\rho_\Delta = \begin{pmatrix} 1 - x - y & 0 & 0 \\ 0 & x & z \\ 0 & z & y \end{pmatrix},$$ \quad (21)

and

$$\rho_\circ = \begin{pmatrix} x & z & 0 \\ z & y & 0 \\ 0 & 0 & 1 - x - y \end{pmatrix}, \quad (22)$$

where $x, y, z$ are all real number, using above method, we can find that the $l_1$ norm of coherence of state $\rho_\Delta$ and $\rho_\circ$ in base $\beta_2, \beta_3, \beta_4$ is also equal respectively.

**IV. THE $l_1$ NORM OF COHERENCE OF BELL-DIAGONAL STATES IN THE TENSOR OF 2 DIMENSION MUTUALLY UNBIASED BASES**

In this section, we extend the concept of mutually unbiased basis by the tensor.

*Definition.* For the set of mutually unbiased bases $\{B_k\}$ for a Hilbert space $H = C^d$ where $\{B_k\} = \{\{0_k\}, \cdots, \{d - 1_k\}\}$, we call the set $\{\gamma_k\} = \{|i\rangle_k \otimes |j\rangle_k\forall i, j \in \{0, \cdots, d - 1\}\}$ autotensor of mutually unbiased basis (AMUB) if

$$|\langle i|_k \otimes \langle j|_k(|m\rangle_1 \otimes |n\rangle_1)| = \frac{1}{d},$$ \quad (23)

where $k \neq l$. Furthermore, we can construct a set of AMUB by $d = 2$ dimension mutually unbiased bases.

For example, let

$$\gamma_1 = \{\gamma_{11}, \gamma_{12}, \gamma_{13}, \gamma_{14}\} = \{\alpha_{11} \otimes \alpha_{11}, \alpha_{11} \otimes \alpha_{12}, \alpha_{12} \otimes \alpha_{11}, \alpha_{12} \otimes \alpha_{12}\},$$

$$\gamma_2 = \{\gamma_{21}, \gamma_{22}, \gamma_{23}, \gamma_{24}\} = \{\alpha_{21} \otimes \alpha_{21}, \alpha_{21} \otimes \alpha_{22}, \alpha_{22} \otimes \alpha_{21}, \alpha_{22} \otimes \alpha_{22}\},$$

$$\gamma_3 = \{\gamma_{31}, \gamma_{32}, \gamma_{33}, \gamma_{34}\} = \{\alpha_{31} \otimes \alpha_{31}, \alpha_{31} \otimes \alpha_{32}, \alpha_{32} \otimes \alpha_{31}, \alpha_{32} \otimes \alpha_{32}\}.\,$$

Next, we will consider the relation of the coherence of quantum states in above AMUB.

A two-qubit Bell-diagonal states can be written as

$$\rho_B = \frac{1}{4}(I \otimes I + \sum_{i=1}^{3} c_i \sigma_i \otimes \sigma_i),$$ \quad (24)

where $\{\sigma_i\}_{i=1}^{3}$ are the Pauli matrices, and $c_1, c_2, c_3 \in [-1, 1]$. The density matrix of $\rho_B$ in base $\gamma_1 = \{\gamma_{11}, \gamma_{12}, \gamma_{13}, \gamma_{14}\} = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ is:

$$\rho_B = \frac{1}{4} \begin{pmatrix} 1 + c_3 & 0 & 0 & c_1 - c_2 \\ 0 & 1 - c_3 & c_1 + c_2 & 0 \\ 0 & c_1 + c_2 & 1 - c_3 & 0 \\ c_1 - c_2 & 0 & 0 & 1 + c_3 \end{pmatrix}, \quad (25)$$
The $l_1$ norm of coherence of state $\rho_B$ in base $\gamma_1$ is

$$C_{l_1}(\rho_B)_{\gamma_1} = 2\left(\frac{1}{4}(c_1 - c_2) + \frac{1}{4}(c_1 + c_2)\right) = \frac{1}{2}(|(c_1 - c_2)| + |(c_1 + c_2)|).$$  \hspace{1cm} (26)

Let the density matrix of $\rho_B$ in base $\gamma_2 = \{\gamma_{21}, \gamma_{22}, \gamma_{23}, \gamma_{24}\}$ is

$$\rho_B = \begin{pmatrix}
    d_{11} & d_{12} & d_{13} & d_{14} \\
    d_{21} & d_{22} & d_{23} & d_{24} \\
    d_{31} & d_{32} & d_{33} & d_{34} \\
    d_{41} & d_{42} & d_{43} & d_{44}
\end{pmatrix},$$  \hspace{1cm} (27)

and $\rho_B = d_{11}\gamma_{21}\gamma_{21}^\dagger + d_{12}\gamma_{21}\gamma_{22}^\dagger + d_{13}\gamma_{21}\gamma_{23}^\dagger + d_{14}\gamma_{21}\gamma_{24}^\dagger + d_{21}\gamma_{22}\gamma_{21}^\dagger + d_{22}\gamma_{22}\gamma_{22}^\dagger + d_{23}\gamma_{22}\gamma_{23}^\dagger + d_{24}\gamma_{22}\gamma_{24}^\dagger + d_{31}\gamma_{23}\gamma_{21}^\dagger + d_{32}\gamma_{23}\gamma_{22}^\dagger + d_{33}\gamma_{23}\gamma_{23}^\dagger + d_{34}\gamma_{23}\gamma_{24}^\dagger + d_{41}\gamma_{24}\gamma_{21}^\dagger + d_{42}\gamma_{24}\gamma_{22}^\dagger + d_{43}\gamma_{24}\gamma_{23}^\dagger + d_{44}\gamma_{24}\gamma_{24}^\dagger$. As $\rho_B$ in Eq. (25) and Eq. (27) is the same, using the method of undetermined coefficients, we obtain

$$\begin{align*}
    d_{11} + d_{12} + d_{13} + d_{14} + d_{21} + d_{22} + d_{23} + d_{24} + d_{31} + d_{32} + d_{33} + d_{34} + d_{41} + d_{42} + d_{43} + d_{44} &= 1 + c_3 \\
    d_{11} - d_{12} + d_{13} - d_{14} + d_{21} - d_{22} + d_{23} - d_{24} + d_{31} - d_{32} + d_{33} - d_{34} + d_{41} - d_{42} + d_{43} - d_{44} &= 0 \\
    d_{11} + d_{12} - d_{13} - d_{14} + d_{21} + d_{22} - d_{23} + d_{24} + d_{31} + d_{32} - d_{33} - d_{34} + d_{41} + d_{42} - d_{43} - d_{44} &= 0 \\
    d_{11} - d_{12} - d_{13} + d_{14} + d_{21} - d_{22} + d_{23} + d_{24} + d_{31} - d_{32} - d_{33} + d_{34} - d_{41} + d_{42} + d_{43} + d_{44} &= c_1 - c_2 \\
    d_{11} + d_{12} + d_{13} + d_{14} - d_{21} - d_{22} - d_{23} + d_{24} + d_{31} + d_{32} - d_{33} - d_{34} + d_{41} = c_3 + c_2 \\
    d_{11} + d_{12} + d_{13} + d_{14} - d_{21} - d_{22} - d_{23} - d_{24} + d_{31} + d_{32} - d_{33} + d_{34} - d_{41} + d_{42} - d_{43} - d_{44} &= 0
\end{align*}$$

\hspace{1cm} (28)

The solution of the equation is

$$\begin{align*}
    d_{11} &= \frac{1+c_1}{4},
    d_{12} &= 0,
    d_{13} &= 0,
    d_{14} &= \frac{c_3 - c_2}{4},
    d_{21} &= 0,
    d_{22} &= \frac{1-c_1}{4},
    d_{23} &= \frac{-c_3 + c_2}{4},
    d_{24} &= 0,
    d_{31} &= 0,
    d_{32} &= \frac{c_3 - c_2}{4},
    d_{33} &= \frac{1+c_1}{4},
    d_{34} &= 0,
    d_{41} &= \frac{c_3 - c_2}{4},
    d_{42} &= 0,
    d_{43} &= 0,
    d_{44} &= \frac{1+c_1}{4}.
\end{align*}$$

\hspace{1cm} (29)

So, the density matrix of $\rho_B$ in base $\gamma_2 = \{\gamma_{21}, \gamma_{22}, \gamma_{23}, \gamma_{24}\}$ is

$$\rho_B = \frac{1}{4} \begin{pmatrix}
    1 + c_1 & 0 & 0 & c_3 - c_2 \\
    0 & 1 - c_1 & c_3 + c_2 & 0 \\
    0 & c_3 + c_2 & 1 - c_1 & 0 \\
    c_3 - c_2 & 0 & 0 & 1 + c_1
\end{pmatrix}.$$  \hspace{1cm} (30)

The $l_1$ norm of coherence of state $\rho_B$ in base $\gamma_2$ is

$$C_{l_1}(\rho_B)_{\gamma_2} = 2\left(\frac{1}{4}(c_3 - c_2) + \frac{1}{4}(c_3 + c_2)\right) = \frac{1}{2}(|(c_3 - c_2)| + |(c_3 + c_2)|).$$  \hspace{1cm} (31)

Similarly, the density matrix of $\rho_B$ in base $\gamma_3 = \{\gamma_{31}, \gamma_{32}, \gamma_{33}, \gamma_{34}\}$ is

$$\rho_B = \frac{1}{4} \begin{pmatrix}
    1 + c_2 & 0 & 0 & c_3 - c_1 \\
    0 & 1 - c_2 & c_3 + c_1 & 0 \\
    0 & c_3 + c_1 & 1 - c_2 & 0 \\
    c_3 - c_1 & 0 & 0 & 1 + c_2
\end{pmatrix}.$$  \hspace{1cm} (32)
The $l_1$ norm of coherence of state $\rho_B$ in base $\gamma_3$ is

$$C_{l_1}(\rho_B)_{\gamma_3} = 2\left(\left|\frac{1}{4}(c_3 - c_1)\right| + \left|\frac{1}{4}(c_3 + c_1)\right|\right) = \frac{1}{2}\left(\left|(c_3 - c_1)\right| + \left|(c_3 + c_1)\right|\right).$$  \hspace{1cm} (33)
In Eq. (25), let $c_1 = c_2 = c_3 = \frac{4p}{3} - 1$, where $0 \leq p \leq 1$, Bell-diagonal states $\rho_B$ turn into Werner state

$$\rho_W = \begin{pmatrix}
\frac{E}{3} & 0 & 0 & 0 \\
0 & \frac{1}{3} - \frac{E}{3} & 0 & \frac{2E}{3} - \frac{1}{6} \\
0 & 0 & \frac{1}{3} - \frac{E}{3} & 0 \\
0 & \frac{2E}{3} - \frac{1}{6} & \frac{1}{3} - \frac{E}{3} & \frac{1}{3} + \frac{1}{6}
\end{pmatrix}. \quad (35)$$

We denoted the $l_1$ norm of coherence of Werner states $\rho_W$ in bases $\gamma_1, \gamma_2, \gamma_3$ by $C_{l_1}(\rho_W)_{\gamma_1}, C_{l_1}(\rho_W)_{\gamma_2}, C_{l_1}(\rho_W)_{\gamma_3}$ respectively. By Eqs. (26), (31), (33), we find that $C_{l_1}(\rho_W)_{\gamma_1} = C_{l_1}(\rho_W)_{\gamma_2} = C_{l_1}(\rho_W)_{\gamma_3} = |\frac{4p}{3} - 1|$.

In Eq. (25), let $c_1 = 4\frac{F-1}{3}$, $c_2 = -4\frac{F-1}{3}$, $c_3 = 4\frac{F-1}{3}$, where $0 \leq F \leq 1$, Bell-diagonal states $\rho_B$ turn into isotropic state

$$\rho_{iso} = \begin{pmatrix}
\frac{E}{3} + \frac{1}{6} & 0 & 0 & 0 \\
0 & \frac{1}{3} - \frac{E}{3} & 0 & \frac{2E}{3} - \frac{1}{6} \\
0 & 0 & \frac{1}{3} - \frac{E}{3} & 0 \\
0 & \frac{2E}{3} - \frac{1}{6} & \frac{1}{3} - \frac{E}{3} & \frac{1}{3} + \frac{1}{6}
\end{pmatrix}. \quad (36)$$

We denoted the $l_1$ norm of coherence of isotropic states $\rho_{iso}$ in bases $\gamma_1, \gamma_2, \gamma_3$ by $C_{l_1}(\rho_{iso})_{\gamma_1}, C_{l_1}(\rho_{iso})_{\gamma_2}, C_{l_1}(\rho_{iso})_{\gamma_3}$ respectively. By Eqs. (26), (31), (33), we find that $C_{l_1}(\rho_{iso})_{\gamma_1} = C_{l_1}(\rho_{iso})_{\gamma_2} = C_{l_1}(\rho_{iso})_{\gamma_3} = |\frac{4F-1}{3}|$.

V. SUMMARY

In this work, we studied the $l_1$ norm of coherence of quantum states in mutually unbiased bases. We have found the sum of squared $l_1$ norm of coherence of the mixed state single qubit is less than two. We have obtained the $l_1$ norm of coherence of three classes of $X$ states in nontrivial mutually unbiased bases for 4-dimensional Hilbert space is equal. We have proposed “autotensor of mutually unbiased basis (AMUB)” by the tensor of mutually unbiased bases, and given the level surface of constant the sum of the $l_1$ norm of coherence of Bell-diagonal states in AMUB. We have found the $l_1$ norm of coherence of Werner states and isotropic states in AMUB is equal respectively.

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