LEBESGUE POINTS VIA THE POINCARÉ INEQUALITY

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Abstract. In this article, we show that in a $Q$-doubling space $(X, d, \mu)$, $Q > 1$, which satisfies a chain condition, if we have a $Q$-Poincaré inequality for a pair of functions $(u, g)$ where $g \in L^Q(X)$, then $u$ has Lebesgue points $H^h$-a.e. for $h(t) = \log^{1-Q-\epsilon} (1/t)$. We also discuss how the existence of Lebesgue points follows for $u \in W^{1,Q}(X)$ where $(X, d, \mu)$ is a complete $Q$-doubling space supporting a $Q$-Poincaré inequality for $Q > 1$.

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1. Introduction

The usual argument for obtaining the existence of Lebesgue points outside a small set for a Sobolev function $u \in W$ goes as follows. First of all, Lebesgue points exist except for a set of $W$-capacity zero [HKM06], [MZ97]; this is proven by approximating $u$ by continuous functions. Secondly, each set of positive Hausdorff $h$-measure, for a suitable $h$, is of positive $W$-capacity, see Theorem 7.1 in [KM72] or Theorem 5.1.13 in [AH96].

For the usual euclidean Sobolev space $W^{1,n}(\mathbb{R}^n)$, this argument shows that, given $\epsilon > 0$, a function $u \in W^{1,n}(\mathbb{R}^n)$ satisfies

$$u(x) = \lim_{r \to 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} u(y) \, dy$$

outside a set $E_\epsilon$ with $H^h(E_\epsilon) = 0$, where $h(t) = \log^{1-n-\epsilon} (1/t)$. In fact, any non-decreasing non-negative gauge function $h$ that satisfies

$$\int_0^1 h(t)^{1/(n-1)} \frac{dt}{t} < \infty$$

can be used. To be precise, a function $u \in W^{1,n}(\mathbb{R}^n)$ is a priori only defined almost everywhere with respect to the $n$-dimensional measure. The meaning of (1.1) is that the limit of integral averages of $u$ exists $H^h$-a.e. and after replacing $u$ with this limit, we

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obtain a representative of \( u \) for which (1.1) holds outside \( E_\epsilon \).

The above argument is very general. Let us consider a doubling metric space \((X, d, \mu)\). Then a simple iteration argument shows that there is an exponent \( Q > 0 \) and a constant \( C \geq 1 \) so that

\[
(1.3) \quad \left( \frac{s}{r} \right)^Q \leq C \frac{\mu(B(x, s))}{\mu(B(a, r))}
\]

holds whenever \( a \in X, x \in B(a, r) \) and \( 0 < s \leq r \). We say that \((X, d, \mu)\) is \( Q\)-doubling if \((X, d, \mu)\) is a doubling metric measure space and (1.3) holds with the given \( Q \). Towards defining our Sobolev space, we recall that a Borel-measurable function \( g \geq 0 \) is an upper gradient of a measurable function \( u \) provided

\[
(1.4) \quad |u(\gamma(a)) - u(\gamma(b))| \leq \int_\gamma g \, ds
\]

for every rectifiable curve \( \gamma : [a, b] \to X \). We define \( W^{1,p}(X) \), \( 1 \leq p < \infty \), to be the collection of all \( u \in L^p(X) \) that have an upper gradient that also belongs to \( L^p(X) \), see [Sha00]. In order to obtain lower bounds for the capacity associated to \( W^{1,p}(X) \), it suffices to assume a suitable Poincaré inequality. We say that \((X, d, \mu)\) supports a \( p\)-Poincaré inequality if there exist constants \( C \) and \( \lambda \) such that

\[
(1.5) \quad \int_B |u - u_B| \, d\mu \leq C \text{diam}(B) \left( \int_{\lambda B} g^p \, d\mu \right)^{1/p}
\]

for every open ball \( B \) in \( X \), for every function \( u : X \to \mathbb{R} \) that is integrable on balls, and for every upper gradient \( g \) of \( u \) in \( X \). For simplicity, we will from now on only consider the case of a \( Q\)-doubling space and we will assume that \( p = Q \).

Relying on [KL02], [BO05], and [KZ08] one obtains the following conclusion.

**Theorem A.** Let \( \epsilon > 0 \). Let \((X, d, \mu)\) be a complete \( Q\)-doubling space with \( Q > 1 \) that supports a \( Q\)-Poincaré inequality. If \( u \in W^{1,Q}(X) \), then

\[
(1.6) \quad u(x) = \lim_{r \to 0} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} u(y) \, d\mu(y)
\]

outside a set \( E_\epsilon \) with \( H^h(E_\epsilon) = 0 \), where \( h(t) = \log^{1-Q-\epsilon}(1/t) \).

**Theorem A** is not explicitly stated in literature and thus let us describe how it follows from the indicated references. First of all, [KL02] together with [KZ08] gives the existence
of Lebesgue points capacity almost everywhere. Next, [BO05] gives the desired relation between capacity and Hausdorff measure, but under the assumption that the space supports a 1-Poincaré inequality. However, an examination of the corresponding proof in [BO05] shows that it actually suffices that the Poincaré inequality (1.5) holds for each \( u \in W^{1,Q}(X) \) with \( p = 1 \) for some function \( g \in L^Q(X) \), whose \( Q \)-norm is at most a fixed constant times the infimum of \( Q \)-norms of all upper gradients of \( u \). This requirement holds by the self-improving property of Poincaré inequalities [KZ08], see Section 4.

The argument in the previous paragraph requires that \((X, d)\) be complete: the self-improving property from [KZ08] may fail in the non-complete setting, see [Kos99]. Moreover, even in the complete case, the self-improvement may fail unless we require a \( Q \)-Poincaré inequality for all \( u \in W^{1,Q}(X) \). It is then natural to inquire if these two conditions are necessary for the conclusion of Theorem A.

Our result gives a rather optimal conclusion.

**Theorem B.** Let \( \epsilon > 0 \). Suppose that \((X, \mu)\) is a \( Q \)-doubling space for some \( Q > 1 \). Assume that \( X \) satisfies a chain condition (see definition 3.1) and that the \( p \)-Poincaré inequality (1.5) holds for a pair of functions \((u, g)\) with \( p = Q \) where \( g \in L^Q \) and \( u \) is integrable on balls. Then

\[
(1.7) \quad u(x) = \lim_{r \to 0} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} u(y) \, d\mu(y)
\]

outside a set \( E_\epsilon \) with \( H^h(E_\epsilon) = 0 \), where \( h(t) = \log^{1-Q-\epsilon}(1/t) \).

As in the classical setting, the meaning of (1.7) is that the limit exists outside \( E_\epsilon \) and defines a representative for which (1.7) holds outside \( E_\epsilon \).

Since the integral in (1.2) diverges for \( h(t) = \log^{1-Q}(1/t) \), the conclusion of Theorem B is rather optimal. We do not know if one could obtain the same conclusion as in the classical euclidean setting in this generality; under the assumptions of Theorem A one actually has a full analogue. Theorem B can be viewed as a refined version of a result in [Giu69] on the existence of Lebesgue points that also avoids the use of capacities.
A doubling space that supports a $p$-Poincaré inequality is necessarily connected and even bi-Lipschitz equivalent to a geodesic space, if it is complete [Che99]. Since each geodesic space satisfies a chain condition, the assumption of chain condition in Theorem B is natural. One can actually obtain the existence of a limit in (1.7) outside a larger exceptional set even without a chain condition, see Section 3 below. This leads to gauge functions of the type $h(t) = \log^{-Q-\epsilon}(1/t)$.

This paper is organized as follows. We explain our notation and state a couple of preliminary results in Section 2. The proof of Theorem B is given in Section 3 and the proof of Theorem A in the appendix.

2. Notation and preliminaries

We assume throughout that $X = (X, d, \mu)$ is a metric measure space equipped with a metric $d$ and a Borel regular outer measure $\mu$. We call such a $\mu$ as a measure. The Borel-regularity of the measure $\mu$ means that all Borel sets are $\mu$-measurable and that for every set $A \subset X$ there is a Borel set $D$ such that $A \subset D$ and $\mu(A) = \mu(D)$.

We denote open balls in $X$ with center $x \in X$ and radius $0 < r < \infty$ by $B(x, r) = \{y \in X : d(y, x) < r\}$.

If $B = B(x, r)$ is a ball, with center and radius understood, and $\lambda > 0$, we write $\lambda B = B(x, \lambda r)$.

With small abuse of notation we write $\text{rad}(B)$ for the radius of a ball $B$ and we always have

$$\text{diam}(B) \leq 2\text{rad}(B),$$

and the inequality can be strict.

A Borel regular measure $\mu$ on a metric space $(X, d)$ is called a doubling measure if every ball in $X$ has positive and finite measure and there exist a constant $C_\mu \geq 1$ such that

$$\mu(B(x, 2r)) \leq C_\mu \mu(B(x, r))$$
for each \( x \in X \) and \( r > 0 \). We call a triple \((X, d, \mu)\) a \textit{doubling metric measure space} if \( \mu \) is a doubling measure on \( X \).

If \( A \subset X \) is a \( \mu \)-measurable set with finite and positive measure, then the \textit{mean value} of a function \( u \in L^1(A) \) over \( A \) is

\[
u_A = \int_A u \, d\mu \quad \text{or} \quad \frac{1}{\mu(A)} \int_A u \, d\mu.
\]

A metric space is said to be \textit{geodesic} if every pair of points in the space can be joined by a curve whose length is equal to the distance between the points.

We recall that the \textit{generalized Hausdorff h-measure} is defined by

\[
H^h(E) = \limsup_{\delta \to 0} H^h_\delta(E),
\]

where

\[
H^h_\delta(E) = \inf \left\{ \sum h(\text{diam}(B_i)) : E \subset \bigcup B_i, \text{diam}(B_i) \leq \delta \right\},
\]

where the dimension gauge function \( h \) is required to be continuous and increasing with \( h(0) = 0 \). In particular, if \( h(t) = t^\alpha \) with some \( \alpha > 0 \), then \( H^h \) is the usual \( \alpha \)-dimensional \textit{Hausdorff measure}, denoted also by \( H^\alpha \). See \cite{Rog98} for more information on the generalized Hausdorff measure.

For the convenience of reader we state here a fundamental covering lemma (for a proof see \cite[2.8.4-6]{Fed69} or \cite[Theorem 1.3.1]{Zie89}).

\textbf{Lemma 2.1} (5B-covering lemma). Every family \( \mathcal{F} \) of balls of uniformly bounded diameter in a metric space \( X \) contains a pairwise disjoint subfamily \( \mathcal{G} \) such that for every \( B \in \mathcal{F} \) there exists \( B' \in \mathcal{G} \) with \( B \cap B' \neq \emptyset \) and \( \text{diam}(B) < 2 \text{diam}(B') \). In particular, we have that

\[
\bigcup_{B \in \mathcal{F}} B \subset \bigcup_{B \in \mathcal{G}} 5B.
\]

The following lemma will be essential for the proof of Theorem B.
Lemma 2.2. Suppose that \( \{a_j\}_{j=0}^\infty \) is a sequence of non-negative real numbers such that \( \sum_{j\geq 0} a_j < \infty \). Then
\[
\sum_{j\geq 0} \frac{a_j}{(\sum_{i\geq j} a_i)^{1-\delta}} < \infty \quad \text{for any } \ 0 < \delta < 1.
\]

Proof. For any \( n \geq 1 \), we use summation by parts (Newton series) and Bernoulli’s inequality to obtain
\[
\sum_{j=0}^{n} \frac{a_j}{(\sum_{i\geq j} a_i)^{1-\delta}} = \frac{1}{(\sum_{i\geq 0} a_i)^{1-\delta}} \sum_{j=0}^{n} a_j + \sum_{j=0}^{n-1} \frac{1}{(\sum_{i\geq j+1} a_i)^{1-\delta}} - \frac{1}{(\sum_{i\geq j} a_i)^{1-\delta}} \sum_{k=j+1}^{n} a_k.
\]
Now, if we let \( n \to \infty \), we get
\[
\sum_{j\geq 0} \frac{a_j}{(\sum_{i\geq j} a_i)^{1-\delta}} \leq \left( \sum_{j\geq 0} a_j \right)^{\delta} + (1-\delta) \sum_{j\geq 0} \frac{a_j}{(\sum_{i\geq j} a_i)^{1-\delta}}
\]
and hence
\[
\sum_{j\geq 0} \frac{a_j}{(\sum_{i\geq j} a_i)^{1-\delta}} \leq \frac{1}{\delta} \left( \sum_{j\geq 0} a_j \right)^{\delta} < \infty.
\]

\[\square\]

3. Proof of Theorem B

In this section, we give the proof of Theorem B. Let us begin with a weaker statement that does not require a chain condition. Thus assume only that \((X, \mu)\) is \(Q\)-doubling and that \((u, g)\) satisfies \(Q\)-Poincaré. Given \( \epsilon > 0 \), we wish to find \( E_\epsilon \subset X \) with \( H^h(E_\epsilon) = 0 \) for \( h(t) = \log^{-Q-\epsilon}(1/t) \) and so that the limit
\[
\lim_{r \to 0} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} u(y) \, d\mu(y)
\]
exists for $x$ outside $E_\varepsilon$.

Towards this end, it suffices to show that the sequence $\left(u_{B_j(x)}\right)_j$ of the integral averages of $u$ over the balls $B(x, 2^{-j})$ is a Cauchy sequence outside such a set $E_\varepsilon$. Indeed, given $2^{-j-1} < r < 2^{-j}$,

$$|u_{B(x,r)} - u_{B(x,2^{-j})}| \leq \int_{B(x,r)} |u - u_{B_j(x)}| \leq C \int_{B_j(x)} |u - u_{B_j(x)}| \leq C \left( \int_{\lambda B_j(x)} g^Q \, d\mu \right)^{\frac{1}{Q}}$$

by $Q$-doubling and $Q$-Poincaré. Similarly, for $l < m$,

$$|u_{B_l(x)} - u_{B_m(x)}| \leq C \sum_{j=l}^{m-1} \left( \int_{\lambda B_j(x)} g^Q \right)^{\frac{1}{Q}}.$$

Hence, $\left(u_{B_j(x)}\right)_j$ is Cauchy provided $\int_{B(x,r)} g^Q \, d\mu \leq C \log^{-Q-\varepsilon}(1/r)$ for all suffices small $r > 0$. By usual covering theorems, this holds outside a desired set.

Towards the proof of Theorem B, we give a definition of a chain condition, a version of which is already introduced in [HK00].

**Definition 3.1.** We say that a space $X$ satisfies a chain condition if for every $\lambda \geq 1$ there are constants $M \geq 1$, $0 < m \leq 1$ such that for each $x \in X$ and all $0 < r < \text{diam}(X)/8$ there is a sequence of balls $B_0, B_1, B_2, \ldots$ with

1. $B_0 \subset X \setminus B(x, r)$,
2. $M^{-1} \text{diam}(B_i) \leq \text{dist}(x, B_i) \leq M \text{diam}(B_i)$,
3. $\text{dist}(x, B_i) \leq Mr^{2^{-mi}}$,
4. there is a ball $D_i \subset B_i \cap B_{i+1}$, such that $B_i \cup B_{i+1} \subset MD_i$,
5. no point of $X$ belongs to more than $M$ balls $\lambda B_i$.

The sequence $B_i$ will be called a chain associated with $x, r$.

The existence of a doubling measure on $X$ does not guarantee a chain condition. In fact, such a space can be badly disconnected, whereas a space with a chain condition cannot have “large gaps”. For example, the standard 1/3-Cantor set satisfies a chain condition only for $\lambda < 2$. Here we show that a large number of spaces satisfy our chain condition.
Let \( X \) be a metric space. For \( 0 < r < R \) and \( x \in X \), we define the annulus \( A(x, r, R) \) to be the set \( \overline{B}(x, R) \setminus B(x, r) \).

**Definition 3.2.** A metric space \( X \) is said to be \( L \)-annularly connected if whenever \( y, z \in A(x, r, 2r) \) for some \( x \in X \) and \( r > 0 \), then there exists a curve joining \( y \) and \( z \) in \( A(x, r/L, 2rL) \).

Annular connectivity holds, for example, for complete doubling metric spaces that support a suitable Poincaré inequality [HK00], [Kor07].

**Lemma 3.3.** Suppose that \((X, d, \mu)\) is a doubling metric measure space, \((X, d)\) is connected and \( L \)-annularly connected. Then \((X, d)\) satisfies a chain condition.

**Proof.** Let \( x \in X \) and \( 0 < r < \text{diam}(X)/8 \). Then \( X \setminus \overline{B}(x, 2r) \neq \emptyset \). By connectivity, for each \( j \geq 0 \) there is \( y_j \in X \) with \( d(x, y_j) = 2^{-j+1}r \). Fix \( 0 < \epsilon < 1 \). As \( \mu \) is doubling, we can cover each annulus \( A_j(x) = A(x, 2^{-j}r, 2^{-j+1}r) \) by at most \( N \) balls of radii equal to \( \epsilon 2^{-j}r \) and the annulus \( A(x, 2r, 2rL) \) by at most \( N \) balls of radii equal to \( \epsilon r \) with \( N \) independent of \( x, j \). When \( \epsilon \) is sufficiently small, depending only on \( \lambda \), the balls \( 2\lambda B \) with \( B \) corresponding to \( A_j(x) \) and \( 2\lambda B' \) with \( B' \) corresponding to \( A_i(x) \) do not intersect provided \( |i - j| \geq 2 \). Since \((X, d)\) is annularly connected, we can connect the points \( y_j, y_{j+1}, j \geq 0 \), by a curve in a wider annulus from definition 3.2. Collect all those balls from the collection above which intersect the curve joining \( y_j \) and \( y_{j+1}, j \geq 0 \). Consider the new collection of balls, where each ball chosen above gets replaced by the double of it, i.e., we replace \( B(y, s) \) by \( B(y, 2s) \). Beginning with \( y_0 \), we order our balls into a chain along the curves joining the points \( y_j \) and \( y_{j+1} \). The desired properties follow, with \( m = 1/N \) for condition number 3. \( \square \)

Annular connectivity is not necessary for our chain condition. For example, the real line satisfies a chain condition, and so do geodesic spaces.

**Lemma 3.4.** If \((X, d)\) is a geodesic space, then \((X, d)\) satisfies a chain condition.

Lemma 3.4 follows from the proof of Lemma 8.1.6 in [HKST].
By using the chain condition, the following lemma yields us the condition that we want for the proof of Theorem B.

**Lemma 3.5.** Suppose that \( X \) satisfies a chain condition and let the sequence \( B_i \) be a chain associated with \( x, R_2 \) for \( x \in X \) and \( 0 < R_2 < \text{diam}(X)/8 \). Let \( 0 < R_1 < R_2 \). Then we can find balls \( B_{iR_2}, B_{iR_2+1}, \ldots, B_{iR_1} \) from the above collection such that

\[
\frac{R_2}{M(1 + M)^2} \leq \text{diam}(B_{iR_2}) \leq MR_2, \tag{3.1}
\]

\[
\frac{R_1}{M(1 + M)^2} \leq \text{diam}(B_{iR_1}) \leq MR_1 \tag{3.2}
\]

hold and \( B_{iR_2} \subset B(x, R_2), B_{iR_1} \subset B(x, R_1) \) and also the balls \( B_{iR_2}, B_{iR_2+1}, \ldots, B_{iR_1} \) form a chain.

**Proof.** Let \( i_{R_2} = \min \{i \geq 0 : B_i \subset B(x, R_2)\} \). Hence we have \( \text{dist}(x, B_{iR_2}) \leq R_2 \), which implies that \( \text{diam}(B_{iR_2}) \leq MR_2 \), using the second condition of the above definition. Again \( B_{iR_2-1} \cap (X \setminus B(x, R_2)) \neq \emptyset \). Using the triangle inequality, we obtain \( \text{dist}(x, B_{iR_2-1}) + \text{diam}(B_{iR_2-1}) \geq R_2 \) and hence we have

\[
\text{diam}(B_{iR_2-1}) \geq \frac{R_2}{1 + M}.
\]

Since \( B_{iR_2} \cap B_{iR_2-1} \neq \emptyset \), we write \( \text{dist}(x, B_{iR_2}) + \text{diam}(B_{iR_2}) \geq \text{dist}(x, B_{iR_2-1}) \) and hence we obtain

\[
\text{diam}(B_{iR_2}) \geq \frac{R_2}{M(1 + M)^2}.
\]

Once \( B_{iR_2} \) is chosen, we can choose \( B_{iR_2+1}, B_{iR_2+2}, \ldots, B_{iR_1} \) from the above collection, where \( i_{R_1} = \min \{i \geq 0 : B_i \subset B(x, R_1)\} \). Then obtain the above estimates for \( B_{iR_1} \) in a similar way and clearly the new collection of balls form a chain. \( \square \)

Our next lemma shows that we have an upper bound for the volume quotient in (1.3) under the chain condition.

**Lemma 3.6.** Suppose that a doubling metric measure space \((X, d, \mu)\) satisfies a chain condition. Then there is an exponent \( \hat{Q} > 0 \) and a constant \( C_0 \geq 1 \) so that

\[
\frac{\mu(B(x, s))}{\mu(B(a, r))} \leq C_0 \left( \frac{s}{r} \right)^{\hat{Q}} \tag{3.3}
\]

holds whenever \( a \in X, x \in B(a, r) \) and \( 0 < s \leq r \).
Proof. Let $B$ be an arbitrary ball in $X$. We choose $\tau < 1/2$ such that we get a ball $\tilde{B} \subset B$ disjoint from $\tau B$ using the chain condition and hence using the doubling property we obtain

$$
\mu(B) \geq \mu(\tau B) + \mu(\tilde{B}) \geq \mu(\tau B) + C_{\mu} \mu(B),
$$

which means that we have the “reverse” doubling condition

$$
\mu(\tau B) \leq (1 - C_{\mu}) \mu(B).
$$

Then a simple iteration argument gives us the required condition. \hfill \Box

It immediately follows from Lemma 3.6 that $H^h(E_0) = 0$ implies, in the setting of Theorem B, that $\mu(E_0) = 0$. Hence the conclusion of Theorem B has content.

**Proof of Theorem B.** Let $x \in X$. For given $0 < r < 1$, we can always find $j \in \mathbb{N}$ such that $2^{-(j+1)} < r < 2^{-j}$. It is enough to consider the balls $B(x, 2^{-j})$ instead of $B(x, r)$ as we have, using the doubling property and the Poincaré inequality,

$$
|u_{B(x,r)} - u_{B(x,2^{-j})}| \leq \int_{B(x,r)} |u - u_{B(x,2^{-j})}| \, d\mu
\leq c \int_{B(x,2^{-j})} |u - u_{B(x,2^{-j})}| \, d\mu
\leq c \left( \int_{B(x,2^{-j})} g^Q \, d\mu \right)^{\frac{1}{Q}} \to 0 \text{ as } j \to \infty.
$$

Our aim is to show that the sequence $u_{B(x,2^{-j})} = \int_{B(x,2^{-j})} u(y) \, d\mu(y), j \in \mathbb{N}$ is a Cauchy sequence. Towards this end, for $m, l \in \mathbb{N}, m > l$, let us consider the difference

$$
|u_{B(x,2^{-m})} - u_{B(x,2^{-l})}| \leq |u_{B(x,2^{-m})} - u_{B_{2^{-l}}}| + |u_{B(x,2^{-m})} - u_{B_{2^{-l}}}| + |u_{B_{2^{-l}}} - u_{B_{2^{-m}}}|,
$$
where the balls $B_{i}, B_{i+1}, \ldots, B_{im}$ are obtained from Lemma 3.5 for $R_{1} = 2^{-m}$, $R_{2} = 2^{-l}$. Using the doubling property, Poincaré inequality and Lemma 3.5, we obtain

$$|u_{B_{i}} - u_{B_{i+2^{-l}}}| \leq \int_{B_{i}} |u - u_{B_{i+2^{-l}}}| \, d\mu$$

$$\leq c \int_{B_{i+2^{-l}}} |u - u_{B_{i}}| \, d\mu$$

$$\leq c \left( \int_{B_{i+2^{-l}}} g^{Q} \, d\mu \right)^{\frac{1}{Q}} \to 0 \text{ as } l \to \infty$$

and similarly we get $|u_{B_{i+2^{-m}}} - u_{B_{im}}| \to 0$ as $m \to \infty$. So, it is enough to prove that $|u_{B_{i}} - u_{B_{im}}| \to 0$ when both $m, l$ tend to infinity.

Fix $\epsilon > 0$ and write $h_{1}(t) = \log^{1-\frac{Q}{Q-1}+\epsilon}(1/t)$. Let $\tilde{\epsilon} > 0$, which is to be chosen later. We use a telescopic argument for the balls $B_{i}, B_{i+1}, \ldots, B_{im}$ and also use chain conditions, relative lower volume decay (1.3) and Poincaré inequality (1.5) to estimate

$$|u_{B_{im}} - u_{B_{i}}| \leq \sum_{n=i}^{im-1} |u_{B_{n}} - u_{B_{n+1}}|$$

$$\leq \sum_{n=i}^{im-1} \left( |u_{B_{n}} - u_{D_{n}}| + |u_{B_{n+1}} - u_{D_{n}}| \right)$$

$$\leq \sum_{n=i}^{im-1} \left( \int_{D_{n}} |u - u_{B_{n}}| \, d\mu + \int_{D_{n}} |u - u_{B_{n+1}}| \, d\mu \right)$$

$$\leq c \sum_{n=i}^{im-1} \int_{B_{n}} |u - u_{B_{n}}| \, d\mu$$

$$\leq c \sum_{n=i}^{im-1} \text{diam}(B_{n}) \left( \int_{\lambda B_{n}} g^{Q} \, d\mu \right)^{\frac{1}{Q}}$$

$$\leq c \sum_{n \geq i} \left( \frac{\text{diam}(B_{n})^{Q}}{\mu(B_{n})} \int_{\lambda B_{n}} g^{Q} \, d\mu \right)^{\frac{1}{Q}} n^{-\frac{Q-1+\epsilon}{Q}} n^{-\frac{Q-1+\epsilon}{Q}}$$

$$\leq c \left( \sum_{n \geq i} \frac{\text{diam}(B_{n})^{Q}}{\mu(B_{n})} n^{Q-1+\epsilon} \int_{\lambda B_{n}} g^{Q} \, d\mu \right)^{\frac{1}{Q}} \left( \sum_{n \geq i} n^{-\frac{Q-1+\epsilon}{Q}} \right)^{\frac{Q-1}{Q}}$$

$$\leq \frac{c l^{-\frac{Q}{Q-1}}}{\mu(B(x, 1))} \left( \sum_{n \geq i} n^{Q-1+\epsilon} \int_{\lambda B_{n}} g^{Q} \, d\mu \right)^{\frac{1}{Q}}.$$
Now we consider the convergence of the sum $\sum_{n \geq n_i} n^{Q-1+i} \int_{\lambda B_n} g^Q \, d\mu$ when $l$ is large. If we have

$$\int_{B(x,r)} g^Q \, d\mu \leq \frac{c}{\log^{Q-1+\varepsilon/2} (\frac{1}{r})}$$

for all sufficiently small $0 < r < 1/5$, then

$$\sum_{n \geq n_i} \int_{\lambda B_n} g^Q \, d\mu \leq \frac{c}{n^{Q-1+\varepsilon/2}}$$

for all $n \geq n_i$, provided $l$ is sufficiently large. Then we choose $\tilde{\varepsilon} = \frac{\varepsilon}{2} - \delta (Q - 1 + \frac{\varepsilon}{2})$ for some $0 < \delta < 1$ (we can choose $\delta$ as small as we want to make $\tilde{\varepsilon}$ positive) and use Lemma 2.2 to obtain

$$\sum_{n \geq n_i} n^{Q-1+i} \int_{\lambda B_n} g^Q \, d\mu < \infty.$$ 

Hence we get $|u_{B_{im}} - u_{B_{il}}| \to 0$ when both $l, m$ tend to infinity.

On the other hand, let us consider the set

$$E_\varepsilon = \left\{ x \in X : \text{there exists arbitrarily small } 0 < r_x < \frac{1}{5} \text{ such that} \int_{B(x,r_x)} g^Q \, d\mu \geq \frac{c}{\log^{Q-1+\varepsilon/2} (\frac{1}{r_x})} \right\}.$$ 

Let $0 < \delta_1 < 1/5$. Then we get a pairwise disjoint family $\mathcal{G}$, by the using 5B-covering lemma, such that

$$E_\varepsilon \subset \bigcup_{B \in \mathcal{G}} 5B,$$

where $\text{diam}(B) < 2\delta_1$ for $B \in \mathcal{G}$. Then we estimate

$$\mathcal{H}^{h_1}_{10\delta_1}(E_\varepsilon) \leq \sum_{B \in \mathcal{G}} \log^{1-Q-\frac{\varepsilon}{2}} \left( \frac{1}{5 \text{rad}(B)} \right)$$

$$\leq c \sum_{B \in \mathcal{G}} \log^{1-Q-\frac{\varepsilon}{2}} \left( \frac{1}{\text{rad}(B)} \right)$$

$$\leq c \sum_{B \in \mathcal{G}} \int_B g^Q \, d\mu$$

$$\leq c \int_{\bigcup_{B \in \mathcal{G}}} g^Q \, d\mu < \infty.$$ 

It follows that $\mathcal{H}^{h_1}(E_\varepsilon) < \infty$ and hence we have that $\mathcal{H}^{h}(E_\varepsilon) = 0$ (see [Rog98, Theorem 40]), which gives us the existence of $\lim_{i \to \infty} \int_{B_i} u(y) \, d\mu(y)$ for $\mathcal{H}^h$-a.e. $x \in X$. Since $u$ is
locally integrable, $\mu$-almost every $x$ is a Lebesgue point, and hence (1.7) extends to hold $H^b$-a.e. for a representative of $u$.

\[\square\]

**Remark 3.7.** The proof of Theorem B actually only requires a chain condition for the value of $\lambda$ given in our assumption (1.5) on the pair $(u,g)$.

4. **Appendix**

In this section, we complete the proof of Theorem A. First we recall the definition of maximal functions and a version of the well-known maximal theorem of Hardy, Littlewood and Wiener.

Let $(X, d, \mu)$ be a metric measure space. The *Hardy-Littlewood maximal function* $Mf$ of a locally integrable function $f$ is the function defined by

\[
Mf(x) := \sup_{r>0} \int_{B(x,r)} |f(y)| \, d\mu(y)
\]

and the *restricted maximal function* is defined by

\[
M_Rf(x) := \sup_{0<r<R} \int_{B(x,r)} |f(y)| \, d\mu(y)
\]

for $R > 0$ fixed.

Here we only state the Maximal theorem, for a proof see [Smi56], [Rau56] or [Hei01].

**Theorem 4.1** (Maximal theorem). *Let $X$ be a doubling metric measure space. There exist constants $C_p$, depending only on $p$ and on the doubling constant of $\mu$, such that*

\[
\mu(\{x : Mf(x) > t\}) \leq \frac{C_1}{t} \|f\|_{L^1(X)}
\]

*for all $t > 0$ and that*

\[
\|Mf\|_{L^p(X)} \leq C_p \|f\|_{L^p(X)}
\]

*for all $1 < p \leq \infty$ and for all measurable functions $f$."

We also recall here the Hajlasz-Sobolev space $M^{1,p}(X)$ defined by Hajlasz in [Haj96]. A measurable function $u : X \to \mathbb{R}$ belongs to the Hajlasz-Sobolev space $M^{1,p}(X)$ if and only if $u \in L^p(X)$ and there exists a nonnegative function $g \in L^p(X)$ such that the inequality

\[|u(x) - u(y)| \leq d(x,y)(g(x) + g(y))\]
holds for all \( x, y \in X \setminus E \), where \( \mu(E) = 0 \).

The following theorem completes the sketch of the proof of Theorem A from our introduction.

**Theorem 4.2.** Suppose that \((X, d, \mu)\) is a complete and doubling space that supports a \( Q \)-Poincaré inequality. Let \( u \in W^{1,Q}(X) \) and \( g \) be its upper gradient. Then there exists a function \( h \in L^Q(X) \) such that the inequality

\[
\int_B |u - u_B| \leq Cr \int_B h \, d\mu
\]

holds on every ball \( B \) of radius \( r \) and that \( \|h\|_Q \leq c\|g\|_Q \).

**Proof.** By [KZ08], we know that there exists \( \epsilon > 0 \) such that a \((Q - \epsilon)\)-Poincaré inequality holds for the pair \((u, g)\). Then Theorem 3.1 of [HK00] shows that \( u \in M^{1,Q}(X) \) and in particular, Theorem 3.2 of [HK00] gives us the pointwise inequality

\[
|u(x) - u(y)| \leq Cd(x, y) \left(h(x) + h(y)\right)
\]

for almost every \( x, y \in X \), where \( h(x) = (M_{2\lambda d(x,y)}g^{Q-\epsilon}(x))^{\frac{1}{Q-\epsilon}} \). Now integrating inequality (4.6) over a ball \( B \) with respect to \( x \) and \( y \), we obtain inequality (4.5) and using Maximal Theorem 4.1, we obtain

\[
\|h\|_Q = \|Mg^{Q-\epsilon}\|_Q^{\frac{1}{Q-\epsilon}} \leq c\|g^{Q-\epsilon}\|_Q^{\frac{1}{Q-\epsilon}} = c\|g\|_Q.
\]

Note that we have used the fact that \( g^{Q-\epsilon} \in L^{\frac{Q}{Q-\epsilon}}, \frac{Q}{Q-\epsilon} > 1 \) and that

\[
(M_{2\lambda d(x,y)}g^{Q-\epsilon}(x))^{\frac{1}{Q-\epsilon}} \leq (Mg^{Q-\epsilon}(x))^{\frac{1}{Q-\epsilon}}.
\]

\[ \Box \]

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