A two step algorithm for learning from unspecific reinforcement

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Abstract

We study a simple learning model based on the Hebb rule to cope with “delayed”, unspecific reinforcement. In spite of the unspecific nature of the information-feedback, convergence to asymptotically perfect generalization is observed, with a rate depending, however, in a non-universal way on learning parameters. Asymptotic convergence can be as fast as that of Hebbian learning, but may be slower. Moreover, for a certain range of parameter settings, it depends on initial conditions whether the system can reach the regime of asymptotically perfect generalization, or rather approaches a stationary state of poor generalization.

1 Introduction

Introducing biologically motivated features in models for learning has usually a double role: testing hypotheses for natural learning and finding hints for artificial learning. These problems can be stated at various sophistication levels. Here we do not take the more ambitious point of view of describing the complexity of the former or of finding optimal algorithms for the latter. On the contrary, our motivation is to investigate which are the capabilities of very elementary mechanisms.

One urgent problem with which a system, either natural or artificial, may be confronted when trying to improve its performance is to learn only from the final success/failure of series of consecutive decisions. The typical situation we may consider is that of an “agent” which let free in a complicated “landscape” tries many “paths” to reach a “goal” and has to optimize its path (a local problem) knowing only the “time” (or cost) it needs to reach the goal (global information). Here “goal” may be a survival interest or the solution of a problem, “path” a series of moves or of partial solution steps in a complex geographical or mathematical “landscape” etc. The problem we want to approach here is to find out whether there are elementary features characterizing learning under such unspecific reinforcement conditions. From the point of view of reinforcement learning our problem may be seen under the “class III” problems in the classification of Hertz et
However, we stress that our attitude is not that of finding good algorithms for tackling special problems, like movement, control or games – see, e.g., [2]. For this reason we do not consider evolved algorithms from the class of Q-learning [3], of TD learning [4], agent and critic [5], etc but restrict to most primitive algorithms which we may think of having a chance to have developed under natural conditions. On the other hand, if such algorithms will prove capable of tackling the problem they may well give further insights.

In the case of neural network systems the normal situation is already that of lacking detailed control over the synapses and learning is achieved by confronting the “pupil” system with the correct answer after each presentation of a pattern. For perceptrons both the unsupervised Hebb rule and the supervised perceptron algorithm are known to lead to asymptotically perfect generalization, although with different asymptotic laws. In our problem setting, however, the pupil never knows the right answer to each question, but only the average error it makes over many tests. In previous work concerned with this problem [7] (see also [8]) we presented an analysis of a 2-step algorithm based on the Hebb rule for perceptrons and used computer simulations and a rough approximation to estimate the convergence conditions. In the present work we undertake a detailed study of this learning algorithm which we call for simplicity “association-reinforcement(AR)-Hebb-rule”. This algorithm introduces two learning parameters and we find that its generalization behaviour is highly nontrivial: in the pre-asymptotic region and depending on the network parameters fixed points of the learning dynamics may appear. This leads either to asymptotically perfect generalization with non-universal power laws depending on the (ratio of the) learning parameters, or to stationary states of very poor generalization, according to the network parameters and initial conditions.

That this AR-Hebb-algorithm may be of a more general interest is suggested by applying it to a concrete problem of optimizing paths in a landscape with obstacles and traps, in a neural network recasting of [6]; this study will be presented elsewhere (partial results have been given in [7]).

In the next section we shall introduce the problem and the algorithm, and in section 3 we shall present results from numerical simulations. In section 4 we shall study a coarse grained approximation which is appropriate for large networks (“thermodynamic limit”). Section 5 is reserved for conclusions.

2 Learning rule for perceptrons under unspecific reinforcement

We consider perceptrons with Ising units $s, s_i = \pm 1$ and real weights (synapses) $J_i$:

$$s = \text{sign} \left( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} J_i s_i \right) = \text{sign} \left( \frac{1}{\sqrt{N}} J \cdot s \right)$$

(1)

Here $N$ is the number of input nodes, and we put no explicit thresholds. The network (pupil) is presented with a series of patterns $\xi^{(q,l)}_i$, $q \in \mathbb{N}$, $l = 1, ..., L$ to which it answers with $s^{(q,l)}$. A training period consists of the successive presentation of $L$ patterns. The answers are compared

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1 An illustration of the problem was provided in an early paper [6] dealing with these questions in the simulation of a device moving on a board.
with the corresponding answers \( t^{(q,l)} \) of a teacher with pre-given weights \( B_i \) and the average error made by the pupil over one training period is calculated:

\[
e_q = \frac{1}{2L} \sum_{l=1}^{L} |t^{(q,l)} - s^{(q,l)}|.
\]  

(2)

The training algorithm consists of two parts:

I. - a “blind” Hebb-type association at each presentation of a pattern:

\[
J^{(q,l+1)} = J^{(q,l)} + \frac{a_1}{\sqrt{N}} s^{(q,l)} \xi^{(q,l)};
\]  

(3)

II. - an “unspecific” but graded reinforcement proportional to the average error \( e_q \) introduced in (2), also Hebbian, at the end of each training period,

\[
J^{(q+1,1)} = J^{(q,L+1)} - \frac{a_2}{\sqrt{N}} e_q \sum_{l=1}^{L} s^{(q,l)} \xi^{(q,l)}.
\]  

(4)

Because of these 2 steps we call this algorithm “association/reinforcement(AR)-Hebb-rule” (or “2-Hebb-rule”, [7]). We are interested in the behavior with the number of iterations \( q \) of the generalization error \( \epsilon_g(q) \):

\[
\epsilon_g(q) = \frac{1}{\pi} \arccos \left( \frac{J \cdot B}{|J||B|} \right)
\]  

(5)

The training patterns \( \xi^{(q,l)} \) are generated randomly, and are taken to be unbiased in the present paper. The case of structured patterns is more complicated, and will be dealt with in a separate publication [9]. We shall test whether the behavior of \( \epsilon_g(q) \) follows a power law at large \( q \):

\[
\epsilon_g(q) \simeq \text{const} \ q^{-p}
\]  

(6)

Notice the following features:

a) During training the pupil only uses its own associations \( \xi^{(q,l)} \leftrightarrow s^{(q,l)} \) and the average error \( e_q \) which does not refer specifically to the particular steps \( l \).

b) Since the answers \( s^{(q,l)} \) are made on the basis of the instantaneous weight values \( J^{(q,l)} \) which change at each step according to eq. (3), the series of answers form a correlated sequence with each step depending on the previous one. Therefore \( e_q \) measures in fact the performance of a “path”, an interdependent set of decisions.

c) For \( L = 1 \) the algorithm reproduces the usual “perceptron rule” (for \( a_1 = 0 \)) or to the usual “unsupervised Hebb rule” (for \( a_2 = 2a_1 \)) for on-line learning, for which the corresponding asymptotic behavior is known [10], [11].
3 Numerical results

In a preliminary analysis \[7\] we have tested various combinations of \( L = 1, 5, 10, 15 \) and \( N = 50, 100, 200, 300 \). We went with \( q \) up to \( 4.10^5 \). We found the convergence of the learning procedure to depend on the ratio \( a_1/a_2 \), in particular no convergence was found for \( L \) of 5 and higher if this ratio was decreased significantly below 0.2. For fixed \( a_1, a_2 \) the asymptotic behavior with \( q \) appeared well reproduced by a power law and the exponent was found to depend on \( L \). For \( L = 1 \) varying \( a_1/a_2 \) between 0 and 1/2 interpolates between perceptron and Hebbian learning, for ratios larger than 1 new asymptotic behavior can be expected to show up (see sect. 4) – we did not perform a systematic numerical analysis for \( L = 1 \), however.

![Figure 1: Generalization error \( \epsilon_g \) vs. \( \alpha \) for \( N = 100 \), \( L = 5 \) and \( L = 10 \) (upper plots) and \( N = 300 \), \( L = 5 \) and \( L = 10 \) (lower plots), for the algorithms (a - d) of eqs. (9-12). The lines indicate the expected asymptotic behaviour as suggested by the coarse grained approximation discussed in section 4 for the corresponding \( a_1/a_2 \) ratio, as well as 2 further power laws for illustration.](image)

In the present, more precise analysis we use \( L = 5, 10 \) and \( N = 100, 300 \), going up to \( 8.10^5 \) iterations. We introduce:
\[
\alpha = qL/N. \tag{7}
\]

We present here results for the following choices of parameters:

\[
a_2 = 0.012, \quad \text{and} \quad \begin{align*}
(a) \quad & a_1 = a_2/20; \\
(b) \quad & a_1 = a_2/5; \\
(c) \quad & a_1 = a_2/5 \text{ for } \alpha < 100L, \\
& a_1 = a_2/(2L) \text{ for } \alpha \geq 100L; \\
(d) \quad & a_1 = a_2/5 \text{ for } \alpha < 100L, \\
& a_1 = 0. \text{ for } \alpha \geq 100L. \tag{8}
\end{align*}
\]

We use random initial conditions with the same normalization for the teacher and pupil weights, \(B^2/N = J^2/N = 1\). The results are shown in Fig. 1. In agreement with the preliminary results of [7] we find no convergence in the case (a) and convergence in the case (b). If a certain threshold in \(\epsilon_g\) is achieved, switching to a smaller ratio \(a_1/a_2\) is seen to accelerate the asymptotic convergence – case (c) –, but even then \(a_1\) cannot be set to zero – case (d). Similar behaviour is observed for other \(N\) and \(L \geq 5\).

This intriguing behaviour incited us to try to obtain analytic understanding by using the coarse grained analysis discussed in the next section.

4 Coarse grained analysis

We combine blind association (3) during a learning period of \(L\) elementary steps and the graded unspecific reinforcement (4) at the end of each learning period into one coarse grained step

\[
J^{(q+1,1)} = J^{(q,1)} + \frac{1}{\sqrt{N}} (a_1 - a_2 \epsilon_g) \sum_{l=1}^{L} \text{sign}(J^{(q,l)} \cdot \xi^{(q,l)}) \xi^{(q,l)}, \tag{13}
\]

\[
\epsilon_g = \frac{1}{2L} \sum_{l=1}^{L} |\text{sign}(B \cdot \xi^{(q,l)}) - \text{sign}(J^{(q,l)} \cdot \xi^{(q,l)})|. \tag{14}
\]

We introduce the notations:

\[
\hat{R}(q,l) = \frac{1}{N} B \cdot J^{(q,l)}, \quad \hat{Q}(q,l) = \frac{1}{N} [J^{(q,l)}]^2 \tag{15}
\]

and we normalize the teacher weights to 1, i.e. \(B^2/N = 1\). In the “thermodynamic limit” \(L/N \to 0\) one can treat \(\alpha\) as a continuous variable. We shall follow standard procedures [1], [11], [12], [8] and obtain the following expressions for the changes of \(\hat{R}\) and \(\hat{Q}\) over a coarse grained step:
\[ L \frac{d}{d\alpha} \hat{R} = \frac{1}{\sqrt{N}} (a_1 - a_2 e^q) \sum_{l=1}^{L} \text{sign}(J^{(q,l)} \cdot \xi^{(q,l)}) (B \cdot \xi^{(q,l)}), \quad (16) \]

\[ L \frac{d}{d\alpha} \hat{Q} = \frac{2}{\sqrt{N}} (a_1 - a_2 e^q) \sum_{l=1}^{L} \text{sign}(J^{(q,l)} \cdot \xi^{(q,l)}) (J^{(q,l)} \cdot \xi^{(q,l)}) \]

\[ + \frac{1}{N} (a_1 - a_2 e^q)^2 \left( \sum_{l=1}^{L} \text{sign}(J^{(q,l)} \cdot \xi^{(q,l)}) \xi^{(q,l)} \right)^2. \quad (17) \]

In the following we shall consider unbiased random input–patterns with

\[ \langle \xi_i^{(l,q)} \xi_j^{(k,r)} \rangle = \delta_{ij} \delta_{lk} \delta_{qr}. \quad (18) \]

The local fields:

\[ h_j^{(q,l)} = \frac{1}{\sqrt{N}} J^{(q,l)} \cdot \xi^{(q,l)}, \quad h_B^{(q,l)} = \frac{1}{\sqrt{N}} B \cdot \xi^{(q,l)}, \quad (19) \]

are then normally distributed with second moments

\[ \langle (h_j^{(q,l)})^2 \rangle = \langle \hat{Q}(q,l) \rangle = Q, \quad \langle (h_B^{(q,l)})^2 \rangle = 1, \quad \langle (h_j^{(q,l)} h_B^{(q,l)}) \rangle = \langle \hat{R}(q,l) \rangle = R. \quad (20) \]

Their joint probability density is thus given by

\[ p(h_J, h_B) = \frac{1}{2\pi\sqrt{\Delta}} \exp \left( -\frac{1}{2\Delta} (Qh_B^2 - 2Rh_Jh_B + h_J^2) \right), \quad (21) \]

with

\[ \Delta = Q - R^2. \quad (22) \]

In the thermodynamic limit \( N \to \infty \), the self-overlap of the learner \( \hat{Q} \) and its overlap \( \hat{R} \) with the teacher are self–averaging, so that their evolution equations (16), (17) can be directly rewritten in terms of evolution equations for their averages. Moreover, these averages \( Q \) and \( R \) become smooth functions on the \( \alpha \)–scale, so that we can neglect the dependence of \( R \) and \( Q \) on \( l \) in (21) when used to perform averages on the right hand sides of (16) and (17), as it would only produce \( \mathcal{O}(1/N) \) corrections to the evolution equations, which become negligible as \( N \to \infty \). One thus obtains

\[ \frac{dR}{d\alpha} = \sqrt{2} \left[ a_1 \frac{R}{\sqrt{Q}} - a_2 \left( \frac{R}{\sqrt{Q}} - \frac{1}{L} - (1 - \frac{1}{L})P \cdot \frac{R}{\sqrt{Q}} \right) \right], \quad (23) \]

\[ \frac{dQ}{d\alpha} = 2 \sqrt{2} \left[ a_1 \sqrt{Q} - a_2 \left( \sqrt{Q} - \frac{R}{L} - (1 - \frac{1}{L})P \sqrt{Q} \right) \right] \]

\[ + \left[ a_1^2 - a_1 a_2 (1 - P) + a_2^2 \left( 1 - 2P + \frac{1}{L} + (1 - \frac{1}{L})P^2 \right) \right], \quad (24) \]
where

\[
P = -\frac{1}{\pi \sqrt{\Delta}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} x^2} \text{sign}(x) \int_{R_x}^{\infty} dy e^{-\frac{1}{2} \Delta y^2} = 1 - \frac{2}{\pi} \arccos \left( \frac{R}{\sqrt{Q}} \right).
\]  

(25)

The generalization error is:

\[
\epsilon_g = \frac{1}{\pi} \arccos \left( \frac{R}{\sqrt{Q}} \right).
\]  

(26)

We may formally eliminate one of the learning parameters by rescaling our quantities by the parameter \(a_2\):

\[
R = Ra_2, \quad Q = Q a_2^2, \quad \lambda = \frac{a_1}{a_2},
\]  

(27)

We then obtain:

\[
\frac{d\epsilon_g}{d\alpha} = -\frac{1}{\sqrt{2\pi \pi L \sqrt{Q}}} \sin(\pi \epsilon_g)
\]

\[
+ \frac{1}{2\pi Q} \cotg(\pi \epsilon_g) \left( \lambda^2 - (2\lambda - \frac{1}{L})\epsilon_g + (1 - \frac{1}{L})\epsilon_g^2 \right),
\]  

(28)

\[
\frac{d\sqrt{Q}}{d\alpha} = \sqrt{\frac{2}{\pi}} \left( \lambda - (1 - \frac{1}{L})\epsilon_g - \frac{1}{2L} \left( 1 - \cos(\pi \epsilon_g) \right) \right)
\]

\[
+ \frac{1}{2\sqrt{Q}} \left( \lambda^2 - (2\lambda - \frac{1}{L})\epsilon_g + (1 - \frac{1}{L})\epsilon_g^2 \right).
\]  

(29)

To establish the asymptotic behaviour we look for solutions of the equations (28,29) in the limit of small \(\epsilon_g\), large \(Q\). To leading order (for \(\lambda > 0\)), these equations become:

\[
\frac{d\epsilon_g}{d\alpha} \simeq -\frac{\epsilon_g}{\sqrt{2\pi L \sqrt{Q}}} + \frac{\lambda^2}{2\pi^2 Q \epsilon_g},
\]  

(30)

\[
\frac{d\sqrt{Q}}{d\alpha} \simeq \sqrt{\frac{2}{\pi}} \lambda,
\]  

(31)

which can be solved exactly to give:

\[
\epsilon_g^2 \simeq \frac{\lambda}{\sqrt{2\pi \pi} (\frac{1}{4L} - 1)} Q^{-1/2} + c_1 Q^{-2/2\pi L} \quad \text{for } \lambda \neq \frac{1}{L},
\]  

(32)

\[
\epsilon_g^2 \simeq \left( \frac{1}{\pi \sqrt{2\pi L}} \ln Q^{-1/2} + c_2 \right) Q^{-1/2} \quad \text{for } \lambda = \frac{1}{L},
\]  

(33)

i.e. explicitly
\[ \epsilon_g^2 \simeq \frac{1}{2\pi \left( \frac{1}{\lambda L} - 1 \right)} \alpha^{-1} + \tilde{c}_1 \alpha^{-\frac{1}{2\lambda L}} \quad \text{for } \lambda \neq \frac{1}{L}, \]  

\[ \epsilon_g^2 \simeq \left( \frac{1}{2\pi \ln \alpha + \tilde{c}_2} \right) \alpha^{-1} \quad \text{for } \lambda = \frac{1}{L}, \]  

\[ Q \simeq \frac{2}{\pi} \lambda^2 \alpha^2 \]  

asymptotically at large \( \alpha \).

We see that for \( \lambda < \frac{1}{L} \) we obtain asymptotically perfect generalization, the dominant term exhibiting the usual power -1/2 (and, for \( L = 1, \lambda = 0.5 \), also the usual coefficient [1]), while for \( \lambda > \frac{1}{L} \) the second term in (32,34) dominates and ensures again perfect generalization but with a different power law, \(-1/(2\lambda L)\). For \( \lambda = \frac{1}{L} \) we obtain logarithmic corrections – see eq. (33,35). Notice that these results hold also for \( L = 1 \).

In the case \( \lambda = 0 \) one can see from (28,29) that starting with any finite \( Q \) one cannot have perfect generalization for \( L > 1 \). For \( L = 1 \) one reobtains the asymptotic behavior found in [10].

There is, however, a nontrivial pre-asymptotic region, which turns out to be dominated by two stationarity conditions, one for the self-overlap, \( dQ/d\alpha = 0 \), and one for the overlap with the teacher-configuration, \( dR/d\alpha = 0 \) or, alternatively, that for the generalization error \( d\epsilon_g/d\alpha = 0 \). For suitable values of the network parameters, the two stationarity conditions may simultaneously be satisfied, leading to fixed points of the learning dynamics, one of these fully stable and with poor generalization, the other partially stable.

To this pre-asymptotic region we shall now turn our attention and thereby also obtain further specifications for the parameters. In Fig. 2 we show the evolution of \( \epsilon_g \) and \( Q \) according to

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**Figure 2:** Evolution of the generalization error \( \epsilon_g \) (vertical axis) and of \( \sqrt{Q} \) (horizontal axis) at \( L = 10 \) for various \( \lambda \) and starting points \( Q(0) \).
Figure 3: Flow in the plane $\epsilon_g, Q$. The dot-dashed line corresponds to stationarity condition $d\epsilon_g/da = 0$, the dashed line to $dQ/da = 0$. Left: $L = 10, \lambda = 0.075$; two fixed points (one all stable, one partially stable) clearly show up, the full lines represent the stable and the unstable manifolds of the partially stable fixed point. Right: $L = 5, \lambda = 0.2$, a parameter setting for which there are no fixed points. For every starting point we have convergence to perfect generalization. Compare also with Fig. 2.

eqs. (28, 29), starting from $\epsilon_g(0) = 0.5$ and various $Q(0) = Q_0$. The various trajectories are parameterized by $\lambda$. In all cases there is a critical value $\lambda_c(Q_0)$ which separates flows toward a stationary state of poor generalization from flows toward perfect asymptotic generalization. The fixed point in the $Q, \epsilon_g$ plane (with a location parameterized by $\lambda$) which is responsible for this behaviour has an attractive and a repulsive direction. For a given initial condition $Q_0$, the critical value $\lambda_c(Q_0)$ is defined as that value for which the attractive manifold connects the initial condition to the partially stable fixed point; for smaller values of $\lambda$ the flow always is from the initial condition to the fully stable fixed point with poor generalization, for slightly larger values of $\lambda$ the flow is towards asymptotically perfect generalization. At still larger values of $\lambda$ the two fixed points eventually coalesce and disappear altogether. Then we always have asymptotically perfect generalization. Some values for $\lambda_c(Q_0)$ are given in Table 1.

| $\sqrt{Q_0}$ | $1$ | $10$ | $100$ | $1000$ | $10000$ |
|--------------|-----|------|-------|--------|--------|
| $\lambda_c(Q_0)$ | $0.2545(5)$ | $0.2185(5)$ | $0.1385(5)$ | $0.0875(5)$ | $0.0485(5)$ |

Table 1: Critical value of $\lambda$ for $L = 10$ and various initial conditions.

Notice that due (27) the dependence on the initial conditions $Q_0$ may be translated into a dependence on the learning rate for the initial network: for a fixed ratio $\lambda$ of learning rates, and given values of the original overlaps $Q$ and $R$, finer updating (smaller $a_1$ and $a_2$) is equivalent to larger values of rescaled overlaps, hence a larger value of $Q_0$. 
In Fig. 3 we describe the flow in this plane for a given \( \lambda \), this should be compared with the \( \alpha \)-trajectories in the \( Q, \epsilon_g \) plane for various \( \lambda \) with different starting points \( Q_0 \), Fig. 2. In Fig. 4 we plot directly \( \epsilon_g(\alpha) \). As can be seen from all these figures, for \( \lambda < \lambda_c \) the training leads to an initial improvement which is however limited and followed by a very rapid deterioration toward confusion. For \( \lambda > \lambda_c \), on the contrary, the learning stabilizes and leads to asymptotically perfect generalization with a \( \lambda \)-dependent power law in agreement with eqs. (34,35).

These analytic results compare very well with the numerical results given in the previous section, both in the pre-asymptotic and in the asymptotic region (cf. Fig. 4).

Figure 4: Generalization error \( \epsilon_g \) vs. \( \alpha \) at \( L = 10 \) for various \( \lambda \) and for starting point \( Q(0) = 1000 \). The straight lines on the left plot show the dominant asymptotic behaviour for the corresponding \( \lambda \) from (34,35) (notice that for \( \lambda \leq 1/L \) the normalization is fixed; for \( \lambda \leq 1/L \) we give also a fit using the subdominant terms in (34,35)). Right: Amplified view at the pre-asymptotic region.

5 Summary and Discussion

In the present paper we have investigated a two-phase learning algorithm for perceptrons, named AR-Hebb-algorithm. Its first phase consists of a series of Hebb-type synaptic modifications, correlating, however, input and self-computed output (blind association) rather than input and clamped teacher output. This first phase is followed by an unspecific but graded reinforcement-type learning step which leads to a partial reversal of the previous series of Hebb-type synaptic modifications, depending on current average success rates.

Our main motivation has been biological, attempting to honour the observation that a learner’s control over its neurons and synapses might be less specific and direct than ordinary
supervised learning algorithms usually presume, while basically adhering to the Hebbian learning paradigm.

Our central results can be stated as follows:

a) Despite the fact that feedback on the learner’s performance enters its learning dynamics only in an unspecific way in that it cannot be associated with single identifiable correct or incorrect associations, convergence of the AR-Hebb-algorithm in the sense of asymptotically perfect generalization is observed.

b) For given initial conditions, this convergence depends on the parameters of the algorithm; in particular none of these parameters can be set to 0. Alternatively, at fixed $L$ and ratio of the algorithm parameters convergence may depend on initial conditions.

In the details the dynamics of this algorithm was found to be unexpectedly complex. Depending on the parameters, fixed points in the dynamic flow may emerge — one stable, the other only partially stable. The attracting manifold of the latter constitutes a separatrix dividing initial states into two sets, one for which the algorithm converges, and another for which it doesn’t in which case the flow is driven to the all–stable fixed point with poor generalization. Seen from a different point of view, a given initial condition (given updating speed) may be found to belong to the asymptotically converging lot, or to end up in a state of poor generalization, depending on network parameters.

On the other hand, parameter settings may be varied in such a way that the two fixed points eventually coalesce and disappear, rendering convergence of the algorithm independent of initial conditions. The pre-asymptotic regime of the learning process is then still influenced by the lines in the $\epsilon_g$–$Q$ plane along which either $d\epsilon_g/d\alpha$ or $dQ/d\alpha$ (but not both) vanish.

Much to our surprise, the convergence–rate of the algorithm was found to depend in a non-universal manner on the ratio of learning parameters. In spite of the non-specific nature of the information–feedback on the learning dynamics, convergence can be as fast as that of Hebbian learning, $\epsilon_g \sim \alpha^{-1/2}$, if $\lambda L < 1$, whereas it is slower and exhibits a non-universal parameter dependent rate, $\epsilon_g \sim \alpha^{-1/2}\lambda L$, if $\lambda L > 1$. Logarithmic corrections appear in the marginal case $\lambda L = 1$.

One may ask oneself, why there is no generalization for a perceptron–type algorithm $\lambda = 0$ (i.e., $a_1 = 0$). We can offer a simple observation which may be of heuristic value: since for $L = 1$ $e_q$ can only be 0 or 1 $a_1 = 0$ means penalty for failure, no change for success, i.e. the usual perceptron learning rule known to converge. However, for $L > 1$ $e_q$ can take fractional values in the interval $[0,1]$. In this case $a_1 = 0$ means penalty for all answers which are short of perfect, i.e. even if the pupil is successful in far above 50% of the cases. This procedure can turn out to be destructive.

To put our findings into a broader perspective, it is perhaps appropriate to note that a similar kind of unspecific information feedback as in our setup occurs in committee–machine learning. While in our case, information feedback is unspecific in time (with respect to the pattern labels within a longer series on which the learner may have been in error), unspecificity in the committee–machine refers to space, i.e., the label(s) of the node(s) which may have contributed to a wrong output upon presentation of a single pattern. In the details, though, the way in which unspecific feedback is utilized in the dynamics is different in the two setups, leading to different asymptotic laws, and to different behaviour in the preasymptotic regime. Although plateaus in the learning dynamics occur in both setups, this similarity is superficial. Whereas
in the committee-machine, the appearance of plateaus is related to a permutation symmetry of
the nodes and escape therefrom to its breaking (a transition to specialization), there is strictly
speaking no time-translation symmetry within a coarse-grained step, and no breaking thereof, as
each coarse-grained step constitutes a whole correlated path of events during which the learner
already evolves in response to the patterns presented. Quantitatively the difference manifests
itself in the fact that plateaus in our setup have a much higher generalization error than those
in the committee-machines, and that the AR-Hebb rule may converge to a state of poor gener-
alization even if its initial performance is almost perfect (as can be seen in Fig. 3a). Still,
it may be interesting to enquire whether techniques akin to those invented in order to decrease
the extent of plateaus in committee-machine learning (see [14] for a recent reference) might be
utilized to improve the present setup.

We have not up to now addressed issues related to optimal parameter settings or optimal
online-control of parameters (the latter issue would in some sense run against our original bio-
logically minded starting point), nor did we so far investigate the performance of the algorithm
in multi-layer architectures. Clearly these may be interesting topics to pursue in future research,
as may be more detailed investigations of the algorithm as an intricate dynamical system per se.

Note added in proof: We should like to add the following interesting observation. A
variant of the present algorithm which introduces an additional biologically motivated element
of indeterminism by including patterns in the second (reinforcement) phase of a session only with
probability \( p < 1 \) (the student does not remember everything it did in the first phase) shows
qualitatively the same behaviour as the algorithm studied in the present paper. A rough first
quantitative characterization of this modification would be that it leads to an effective rescaling of
the parameter \( a_2 \) of the algorithm by approximately a factor \( p \), entailing corresponding rescalings
of the parameter \( \lambda \) and the scaled self-overlap \( Q \), viz. \( \lambda \rightarrow \lambda/p \) and \( Q \rightarrow Q/p^2 \). Asymptotic
convergence will then be slower, with an exponent computed from the rescaled value \( \lambda/p \) rather
than from the bare \( \lambda \). It also leads to a corresponding reduction of critical \( \lambda \)'s for given initial
condition \( Q_0 \) or, alternatively, to a reduction of the minimum \( Q_0 \) required for convergence at a
given bare \( \lambda \). These results are well corroborated by numerical simulations.

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