Mapping properties of some integral operators associated with generalized Bessel functions

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Abstract
In the present paper, we study the mapping properties of some integral operators on certain classes of harmonic univalent functions associated with generalized Bessel functions of the first kind. To be more precise, we study the mapping properties of Goodman-Rønning-type harmonic univalent functions in the open unit disc \(\mathbb{U}\).

Keywords
Harmonic, univalent functions, generalized Bessel functions.

AMS Subject Classification
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The differential equation (1.2) permits the study of Bessel, modified Bessel, spherical Bessel function and modified spherical Bessel functions all together. Solutions of (1.2) are referred to as the generalized Bessel function of order \(p\). The particular solution given by (1.3) is called the generalized Bessel function of the first kind of order \(p\). Although the series defined above is convergent everywhere, the function \(\omega_{p,b,c}\) is generally not univalent in \(U\). It is worth mentioning that, in particular, when \(b = c = 1\), we reobtain the Bessel function \(\omega_{p,1,1} = I_p\) and for \(c = 1, b = 1\) the function \(\omega_{p,1,-1}\) becomes the modified Bessel function \(I_p\). Now, consider the function \(u_{p,b,c}\) defined by the transformation

\[
u_{p,b,c}(z) = 2^{p} \Gamma\left(p + \frac{b+1}{2}\right) z^{-p/2} \omega_{p,b,c}(z^{1/2}).
\]

By using the well-known Pochhammer (or Appell) symbol, defined in terms of the Euler Gamma function for \(a \neq 0, -1, -2, \ldots\) by

\[
(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & : \text{if } n = 0 \\ a(a+1) \cdots (a+n-1) & : \text{if } n = 1, 2, 3, \cdots \end{cases}
\]

We obtain for the function \(u_{p,b,c}\) by the following representa-
where \( p + (b + 1)/2 \neq 0, -1, -2, \ldots \). This function is analytic on \( C \) and satisfies the second-order linear differential equation

\[
4z^2 u''(z) + 2(2p + b + 1)zu'(z) + czu(z) = 0.
\]

For convenience throughout the sequel, we use the following notations:

\[
u_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c/4)^n}{p + (b + 1)/2} \frac{z^n}{n!}, \quad (1.4)
\]

where \( p + (b + 1)/2 \neq 0, -1, -2, \ldots \). We also let the subclass \( S^0_H \) of \( S_H \) be in the class \( A \). Denote by \( S^0_H \) the subclass of \( S_H \) that are univalent and sense-preserving in \( U \). We also let the subclass \( S^0_H \) of \( S_H \) as

\[
S^0_H = \{ f = h + g \in S_H : g'(0) = B_1 = 0 \}.
\]

The classes \( S^0_H \) and \( S_H \) were first studied in [5]. Also, we let \( K^0_H, S^0_H \) and \( C^0_H \) denote the subclasses of \( S^0_H \) of harmonic functions which are, respectively, convex, starlike, and close-to-convex in \( U \). For definitions and properties of these classes, one may refer to [5] or [6].

A function \( f(z) \) of the form (1.5) is said to be in the class \( N_H(\beta) \), where \( 0 \leq \beta < 1 \), if the following condition is satisfied

\[
\Re \left( \frac{f'(z)}{z} \right) \geq \beta, \quad z = re^{i\theta} \in U.
\]

Further we let \( TN_H(\beta) = N_H(\beta) \cap T \), where \( T \) consists of the functions \( f = h + g \) in \( S_H \) so that \( h \) and \( g \) are of the form

\[
h(z) = z - \sum_{n=2}^{\infty} |A_n| z^n, \quad g(z) = \sum_{n=2}^{\infty} |B_n| z^n. \quad (1.6)
\]

The classes \( N_H(\beta), TN_H(\beta) \) were studied by Ahuja and Jahangiri [2].

In 2001, Rosy et al. [14] introduced the class \( G_H(\gamma) \) consisting of functions of the form (1.5) if it satisfies the following condition

\[
\Re \left\{ (1 + e^{i\alpha}) \frac{zf'(z)}{f(z)} - e^{i\alpha} \right\} \geq \gamma, \quad \alpha \in R, \quad z \in U.
\]

Further we define \( TG_H(\gamma) \equiv G_H(\gamma) \cap T \). The class \( G_H(\gamma) \) is called Goodman-Rønning-type harmonic univalent functions in \( U \).

For complex parameters \( c_1, c_2, k_1, k_2 \) \((\gamma = k_1, k_2, \neq 0, -1, -2, \ldots)\), we introduce the following convolution operator

\[
\Omega = \Omega \left( \begin{array}{c} k_1, c_1 \\ k_2, c_2 \end{array} \right) : H \rightarrow H
\]

defined by

\[
\Omega \left( \begin{array}{c} k_1, c_1 \\ k_2, c_2 \end{array} \right) f = h(z) * \int_{0}^{z} u_{p_1}(t) dt + g(z) * \int_{0}^{z} u_{p_2}(t) dt
\]

for any function \( f = h + g \) in \( H \).

Letting

\[
\Omega \left( \begin{array}{c} k_1, c_1 \\ k_2, c_2 \end{array} \right) f(z) = H(z) + G(z),
\]

where

\[
H(z) = z + \sum_{n=2}^{\infty} \frac{(-c_1/4)^n-1}{(k_1)n-1!} A_n z^n
\]

and

\[
G(z) = \sum_{n=2}^{\infty} \frac{(-c_2/4)^n-1}{(k_2)n-1!} B_n z^n. \quad (1.7)
\]

Similarly we define the Libera type integral operator

\[
L \left( \begin{array}{c} k_1, c_1 \\ k_2, c_2 \end{array} \right) f(z) = h(z) * \int_{0}^{z} u_{p_1}(t) dt + g(z) * \int_{0}^{z} u_{p_2}(t) dt \quad + \frac{1}{2} \sum_{n=2}^{\infty} \int_{0}^{z} u_{p_2}(t) dt, \quad (1.8)
\]

or equivalently

\[
L \left( \begin{array}{c} k_1, c_1 \\ k_2, c_2 \end{array} \right) f(z) = H(z) + G(z),
\]

where

\[
H(z) = z + \sum_{n=2}^{\infty} \frac{2(-c_1/4)^n-1}{(k_1)n-1(n+1)(n-1)!} A_n z^n,
\]

and

\[
G(z) = \sum_{n=2}^{\infty} \frac{2(-c_2/4)^n-1}{(k_2)n-1(n+1)(n-1)!} B_n z^n. \quad (1.9)
\]

Throughout this paper, we will frequently use the notation

\[
\Omega(f) = \Omega \left( \begin{array}{c} k_1, c_1 \\ k_2, c_2 \end{array} \right) f
\]

and

\[
L(f) = L \left( \begin{array}{c} k_1, c_1 \\ k_2, c_2 \end{array} \right) f.
\]
The generalized Bessel function is a recent topic of study in the Geometric Function Theory (e.g., see the work of [3], [7]-[11]). Motivated by results on connections between various subclasses of analytic and harmonic univalent functions by using hypergeometric functions (see [1], [4], [12], [13], [15], and [16]), we establish a number of connections between the classes $G_H(\gamma)$, $K_H^{0}$, $S_H^{0}$, $C_H^{0}$, and $N_H(\beta)$ by applying the convolution operators $\Omega$ and $L$.

### 2. Main Results

In order to establish connections between harmonic convex functions and Goodman-Rønning-type harmonic univalent functions, we shall require the following lemmas.

**Lemma 2.1.** ([5], [6]). If $f = h + \overline{g} \in K_H^0$ where $h$ and $g$ are given by (1.5) with $B_1 = 0$, then

$$|A_n| \leq \frac{n+1}{2}, \quad |B_n| \leq \frac{n-1}{2}.$$  

**Lemma 2.2.** ([14]). Let $f = h + \overline{g}$ be given by (1.5). If $0 \leq \gamma < 1$ and

$$\sum_{n=2}^{\infty} (2n-1-\gamma) |A_n| + \sum_{n=1}^{\infty} (2n+1+\gamma) |B_n| \leq 1 - \gamma,$$  

then $f$ is sense-preserving, Goodman-Rønning-type harmonic univalent functions in $U$ and $f \in G_H(\gamma)$.

**Remark 2.3.** In [14], it is also shown that $f = h + \overline{g}$ given by (1.6) is in the family $T G_H(\gamma)$, if and only if the coefficient condition (2.1) holds. Moreover, if $f \in T G_H(\gamma)$, then

$$|A_n| \leq \frac{1-\gamma}{2n-1-\gamma}, \quad n \geq 2,$$

and

$$|B_n| \leq \frac{1-\gamma}{2n+1+\gamma}, \quad n \geq 1.$$  

**Lemma 2.4.** ([3]). If $b, p, c \in C$ and $k \neq 0, -1, -2, \cdots$ then the function $u_p$ satisfies the recursive relation $4k u_p(z) = -cu_{p+1}(z)$ for all $z \in C$.

**Lemma 2.5.** If $c < 0$ and $k > 1$, then

$$\sum_{n=0}^{\infty} \frac{(c/4)^n}{(k)_n(n+1)!} = \frac{-4(k-1)}{c} [u_{p-1}(1) - 1].$$

**Proof.** We can write

$$\sum_{n=0}^{\infty} \frac{(c/4)^n}{(k)_n(n+1)!} = \frac{(k-1)}{c} \sum_{n=0}^{\infty} \frac{(c/4)^{n+1}}{(k-1)_n(n+1)!}$$

$$= \frac{-4(k-1)}{c} [u_{p-1}(1) - 1].$$

**Theorem 2.6.** Let $c_1, c_2 < 0, k_1, k_2 > 1$. If for some $\gamma(0 \leq \gamma < 1)$ and the inequality

$$2u_{p_1}(1) + (3-\gamma)u_{p_1}(1) - (1+\gamma)$$

$$\left[ -\frac{4(k_1-1)}{c_1} \left[ u_{p_1-1}(1) - 1 \right] \right] + 2u_{p_2}(1) + (1+\gamma)u_{p_2}(1)$$

$$- (1+\gamma) \left[ -\frac{4(k_2-1)}{c_2} \left[ u_{p_2-1}(1) - 1 \right] \right] \leq 4(1-\gamma)$$

is satisfied then $\Omega(K_H^0) \subset G_H(\gamma)$.

**Proof.** Let $f = h + \overline{g} \in K_H^0$ where $h$ and $g$ are of the form (1.5) with $B_1 = 0$. We need to show that $\Omega(f) = H + \overline{T} \in G_H(\gamma)$, where $H$ and $G$ defined by (1.7) with $B_1 = 0$ are analytic functions in $U$.

In view of Lemma 2.2, we need to prove that

$$P_1 \leq 1 - \gamma,$$

where

$$P_1 = \sum_{n=2}^{\infty} (2n-1-\gamma) \left| \frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}n!} A_n \right|$$

$$+ \sum_{n=2}^{\infty} (2n+1+\gamma) \left| \frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}n!} B_n \right|.$$

In view of Lemma 2.1, we have

$$P_1 \leq \frac{1}{2} \left[ \sum_{n=2}^{\infty} (n+1)(2n-1-\gamma) \left( -\frac{c_1/4}{k_1} \right)^{n-1} \frac{1}{n!} \right]$$

$$+ \sum_{n=2}^{\infty} (n+1)(2n+1+\gamma) \left( -\frac{c_2/4}{k_2} \right)^{n-1} \frac{1}{n!}$$

$$= \frac{1}{2} \left[ \sum_{n=2}^{\infty} \left( 2n(n-1) + (3-\gamma)n - (1+\gamma) \right) \right]$$

$$\times \left( -\frac{c_1/4}{k_1} \right)^{n-1} \frac{1}{n!}$$

$$+ \sum_{n=2}^{\infty} \left( (\gamma+1)n - (\gamma+1) \right) \right]$$

$$\times \left( -\frac{c_2/4}{k_2} \right)^{n-1} \frac{1}{n!}$$

$$= \frac{1}{2} \left[ \sum_{n=0}^{\infty} \left( -\frac{c_1/4}{k_1} \right)^{n+1} \frac{1}{(k_1)_{n+1}(n+1)!} \right]$$

$$+ (3-\gamma) \sum_{n=0}^{\infty} \left( -\frac{c_1/4}{k_1} \right)^{n+1} \frac{1}{(k_1)_{n+1}(n+1)!}$$

$$- (1+\gamma) \sum_{n=0}^{\infty} \left( -\frac{c_2/4}{k_2} \right)^{n+1} \frac{1}{(k_2)_{n+1}(n+2)!}$$

$$+ (\gamma+1) \sum_{n=0}^{\infty} \left( -\frac{c_2/4}{k_2} \right)^{n+1} \frac{1}{(k_2)_{n+1}(n+2)!}$$

$$- (\gamma+1) \sum_{n=0}^{\infty} \left( -\frac{c_2/4}{k_2} \right)^{n+1} \frac{1}{(k_2)_{n+1}(n+2)!}.$$
\[ \frac{1}{2} \left[ 2 \left( -\frac{c_1}{4} \right) \sum_{n=0}^{\infty} \left( -\frac{c_1}{4} \right)^n_{k_1} \sum_{n=0}^{\infty} \frac{(3-\gamma)}{(k_1)_{n+1}(n+2)} \right] \]

\[ \left\{ u_{p_1}(1) - 1 \right\} + (1 + \gamma) \sum_{n=0}^{\infty} \left( -\frac{c_1}{4} \right)^n \frac{1}{(k_1)_{n+1} n!} \]

\[ + 2 \left( -\frac{c_1}{4} \right) \sum_{n=0}^{\infty} \frac{1}{(k_1)_{n+1} (n+2)} \]

\[ \left\{ u_{p_2}(1) - 1 \right\} + (1 + \gamma) \sum_{n=0}^{\infty} \frac{1}{(k_1)_{n+1} (n+2)} \]

Now \( P_1 \leq 1 - \gamma \) follows from the given condition

This completes the proof of Theorem 2.6.

Analogous to Theorem 2.6, we next find conditions of the classes \( \mathcal{S}_H \), \( C_H \) with \( G_H(\gamma) \). However we first need the following result which may be found in [5], [6].

**Lemma 2.7.** If \( f = h + g \in \mathcal{S}_H \) or \( C_H \) where \( h \) and \( g \) are given by (1.5) with \( B_1 = 0 \), then

\[ |A_n| \leq \frac{(2n+1)(n+1)}{n}, \quad |B_n| \leq \frac{2n-1}{n} \]

**Theorem 2.8.** If \( c_1, c_2 < 0, k_1, k_2 > 1 \). If for some \( \gamma(0 \leq \gamma < 1) \) and the inequality

\[ 4u''_{p_1}(1) + (16 - 2\gamma)u'_{p_1}(1) + (7 - 5\gamma)u_{p_1}(1) \]

\[ - (1 + \gamma) \left[ 4 \left( \frac{c_1}{k_1} \right) u_{p_1-1}(1) - 1 \right] \]

\[ + 4u''_{p_1}(1) + (8 + 2\gamma)u'_{p_2}(1) - (1 + \gamma)u_{p_2}(1) \]

\[ + (1 + \gamma) \left[ 4 \left( \frac{c_1}{k_1} \right) u_{p_2-1}(1) - 1 \right] \]

is satisfied, then

\[ \Omega(S_H) \subset G_H(\gamma) \text{ and } \Omega(C_H) \subset G_H(\gamma). \]

**Proof.** Let \( f = h + \overline{g} \in \mathcal{S}_H(\gamma) \) where \( h \) and \( g \) are given by (1.5) with \( B_1 = 0 \). We need to show that \( \Omega(f) = H + \overline{G} \in G_H(\gamma) \), where \( H \) and \( G \) defined by (1.7) with \( B_1 = 0 \) are analytic functions in \( U \). In view of Lemma 2.2, it is enough to show that \( P_1 \leq 1 - \gamma \), where

\[ P_1 = \sum_{n=2}^{\infty} (2n - 1) \left( -\frac{c_1}{4} \right)^{n-1} \frac{|A_n|}{(k_1)_{n-1} n!} \]

\[ + \sum_{n=2}^{\infty} (2n + 1) \left( -\frac{c_1}{4} \right)^{n-1} \frac{|B_n|}{(k_2)_{n-1} n!} \]

In view of Lemma 2.7, we have

\[ P_1 \leq \frac{1}{\gamma} \left[ \sum_{n=2}^{\infty} (2n + 1)(n+1)(2n - 1) \left( -\frac{c_1}{4} \right)^{n-1} \frac{1}{(k_1)_{n-1} n!} \right] \]

\[ + \sum_{n=2}^{\infty} (2n - 1)(n+1)(2n + 1) \left( -\frac{c_1}{4} \right)^{n-1} \frac{1}{(k_2)_{n-1} n!} \]

\[ \leq \frac{1}{\gamma} \left[ \sum_{n=2}^{\infty} 4n(n-1)(n-2) \left( -\frac{c_1}{4} \right)^{n-1} \frac{1}{(k_1)_{n-1} n!} \right] \]

\[ + (16 - 2\gamma)(n-1) + (7 - 5\gamma)n - (\gamma + 1) \]

\[ \sum_{n=2}^{\infty} \left( -\frac{c_1}{4} \right)^{n-1} \frac{1}{(k_1)_{n-1} n!} \]

\[ + \sum_{n=2}^{\infty} 4n(n-1)(n-2) + (8 + 2\gamma) \]

\[ n(n-1) - (1 + \gamma)n + (1 + \gamma) \left( -\frac{c_1}{4} \right)^{n-1} \]

\[ \left( -\frac{c_1}{4} \right)^{n-1} \frac{1}{(k_2)_{n-1} n!} \]

\[ \leq \frac{1}{\gamma} \left[ \sum_{n=2}^{\infty} \frac{(-c_1/4)^{n-1}}{(k_1)_{n-1} n!} + (16 - 2\gamma) \sum_{n=2}^{\infty} \frac{(-c_1/4)^{n-1}}{(k_1)_{n-1} n!} \right] \]

\[ - (1 + \gamma) \sum_{n=0}^{\infty} \frac{(-c_1/4)^{n+1}}{(k_1)_{n+1} (n+1)!} \]

\[ + \left( 4 \sum_{n=0}^{\infty} \frac{(-c_1/4)^{n+1}}{(k_2)_{n+1} (n+2)!} \right) \]

\[ + (8 + 2\gamma) \sum_{n=0}^{\infty} \frac{(-c_1/4)^{n+1}}{(k_2)_{n+1} (n+2)!} \]

\[ - (1 + \gamma) \sum_{n=0}^{\infty} \frac{(-c_1/4)^{n+1}}{(k_2)_{n+1} (n+2)!} \]

\[ + (1 + \gamma) \sum_{n=0}^{\infty} \frac{(-c_1/4)^{n+1}}{(k_2)_{n+1} (n+2)!} \]

\[ + (1 + \gamma) \sum_{n=0}^{\infty} \frac{(-c_1/4)^{n+1}}{(k_2)_{n+1} (n+2)!} \]

Now \( P_1 \leq 1 - \gamma \) follows from the given condition.

**In order to determine connection between \( T \mathcal{N}_H(\beta) \) and \( G_H(\gamma) \), we need the following results in Lemma 2.9.**
Lemma 2.9. (Taylor). Let \( f = h + g \) where \( h \) and \( g \) are given by (1.6) with \( B_1 = 0 \), and suppose that \( 0 \leq \beta < 1 \). Then
\[
f \in TN_H(\beta) \iff \sum_{n=2}^{\infty} n |A_n| + \sum_{n=2}^{\infty} n |B_n| \leq 1 - \beta.
\]

Remark 2.10. If \( f \in TN_H(\beta) \), then \( |A_n| \leq \frac{1-\beta}{n} \) and \( |B_n| \leq \frac{1-\beta}{n^2} \), \( n \geq 2 \).

Theorem 2.11. If \( c_1, c_2 < 0, k_1, k_2 > 1 \) \( (k_1, k_2 \neq 0, -1, -2, \cdots) \). If for some \( \beta(0 \leq \beta < 1) \), \( \gamma(0 \leq \gamma < 1) \) and the inequality
\[
(1 - \beta) \left[ 2 \{ u_{n1} (1) - 1 \} + (1 + \gamma) \frac{d(k_{11}-1)}{c_1} u_{n2} (1) - 1 \right] + 2u_{n2} (1) - (1 + \gamma) \frac{d(k_{11}-1)}{c_2} u_{n2} (1) - 1 \leq 1 - \gamma
\]
is satisfied then
\[
\Omega(TN_H(\beta)) \subseteq G_H(\gamma).
\]

Proof. Let \( f = h + g \in TN_H(\beta) \) where \( h \) and \( g \) are given by (1.5). In view of Lemma 2.2, it is enough to show that
\[
P_2 \leq 1 - \gamma,
\]
where
\[
P_2 = \sum_{n=2}^{\infty} (2n - 1 - \gamma) \frac{d(k_{11}-1)}{c_1} |A_n| + \sum_{n=1}^{\infty} (2n + 1 + \gamma) \frac{d(k_{11}-1)}{c_2} |B_n|.
\]

Using Remark 2.10, we have
\[
P_2 \leq (1 - \beta) \left[ \sum_{n=2}^{\infty} \left( 2 - \frac{(1 + \gamma) d(k_{11}-1)}{c_1} \right) \frac{d(k_{11}-1)}{c_1} n! |A_n| + \sum_{n=1}^{\infty} \left( 2 + \frac{(1 + \gamma) d(k_{11}-1)}{c_2} \right) \frac{d(k_{11}-1)}{c_2} n! |B_n| \right] + 2u_{n2} (1) - (1 + \gamma) \frac{d(k_{11}-1)}{c_2} u_{n2} (1) - 1 \leq 1 - \gamma
\]
by the given hypothesis.

In next theorem, we establish connections between \( TG_H(\gamma) \) and \( G_H(\gamma) \).

Theorem 2.12. Let \( c_1, c_2 < 0, k_1, k_2 > 1 \). If for some \( \gamma(0 \leq \gamma < 1) \) the inequality
\[
\frac{d(k_{11}-1)}{c_1} u_{n1} (1) - 1 + \frac{d(k_{11}-1)}{c_2} u_{n2} (1) - 1 \geq -2 \frac{d(k_{11}-1)}{c_2} u_{n2} (1) - 1 \leq 1 - \gamma
\]
is satisfied, then \( \Omega(TG_H(\gamma)) \subseteq G_H(\gamma) \).

Proof. Let \( f = h + g \in TG_H(\gamma) \) where \( h \) and \( g \) are given by (1.6). In view of Lemma 2.2, it is enough to show that
\[
P_2 \leq 1 - \gamma,
\]
where
\[
P_2 = \sum_{n=2}^{\infty} (2n - 1 - \gamma) \frac{d(k_{11}-1)}{c_1} |A_n| + \sum_{n=1}^{\infty} (2n + 1 + \gamma) \frac{d(k_{11}-1)}{c_2} |B_n|.
\]

Using Remark 2.3, it follows that
\[
P_2 \leq (1 - \beta) \left[ \sum_{n=2}^{\infty} \left( 2 - \frac{(1 + \gamma) d(k_{11}-1)}{c_1} \right) \frac{d(k_{11}-1)}{c_1} n! |A_n| + \sum_{n=1}^{\infty} \left( 2 + \frac{(1 + \gamma) d(k_{11}-1)}{c_2} \right) \frac{d(k_{11}-1)}{c_2} n! |B_n| \right] + 2u_{n2} (1) - (1 + \gamma) \frac{d(k_{11}-1)}{c_2} u_{n2} (1) - 1 \leq 1 - \gamma
\]
by the given condition and this completes the proof.
In next theorem, we present conditions on the parameters $k_1, k_2, c_1, c_2$ and obtain a characterization for operator $\Omega$ which maps $TG_H(\gamma)$ onto itself.

**Theorem 2.13.** If $c_1, c_2 < 0$, $k_1, k_2 > 1$ and $\gamma(0 \leq \gamma < 1)$. Then

\[
\Omega(TG_H(\gamma)) \subset TG_H(\gamma),
\]

if and only if the inequality \ref{2.4} is satisfied.

**Proof.** The proof of above theorem is similar to that of Theorem 2.4. Therefore we omit the details involved. \quad \Box

**Theorem 2.14.** Let $c_1, c_2 < 0$, $k_1 > 0, k_2 > 2$. If for some $\gamma(0 \leq \gamma < 1)$ and the inequality

\[
2u'_{p_1}(1) + (1 - \gamma) \left(u_{p_1}(1) - 1\right) + 2u'_{p_2}(1) + (1 - \gamma)
\]

\[
(u_{p_2}(1) - 1) + 2(1 - \gamma) \left[\frac{-4(k_2-1)}{c_2}(u_{p_2-1}(1) - 1)\right] - 2(1 - \gamma)
\]

\[
\left[\frac{16(k_2-2)(k_2-1)}{c_2^2}u_{p_2-2}(1) - 1\right] + \frac{c_2/4}{k_2-2} - \frac{(c_2/4)^2}{2(k_2-2)(k_2-1)}\]

is satisfied then $L(K_H^p) \subset G_H(\gamma)$.

**Proof.** Let $f = h + \overline{g} \in K_H^0$ where $h$ and $g$ are of the form \ref{1.5} with $B_1 = 0$. We need to show that $L(f) = H + \overline{G} \in G_H(\gamma)$, where $H$ and $G$ defined by \ref{1.9} with $B_1 = 0$ are analytic functions in $U$.

In view of Lemma 2.2, we need to prove that

\[
P_{3} \leq 1 - \gamma,
\]

where

\[
P_{3} = \sum_{n=2}^{\infty} \left(2n - 1 - \gamma\right) \left[\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(\alpha-1)!} + \frac{2k_1}{n+1}\right] + \sum_{n=2}^{\infty} \left(2n + 1 + \gamma\right) \left[\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(\beta-1)!}\right].
\]

In view of Lemma 2.1, we have

\[
P_{3} \leq \left[\sum_{n=2}^{\infty} \left(2n - 1 - \gamma\right) \left[\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(\alpha-1)!}\right] + \sum_{n=2}^{\infty} \left(2n + 1 + \gamma\right)n(n-1) \left[\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(\beta-1)!}\right]\right]
\]

\[
= \left[\sum_{n=2}^{\infty} \left(2n - 1 + (1 - \gamma)\right) \left[\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(\alpha-1)!}\right] + \sum_{n=2}^{\infty} \left(2(n+1)n(n-1) + (\gamma - 1)(n+1)n + 2(1 - \gamma)(n+1) + 2(\gamma - 1)\right) \left[\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(\beta-1)!}\right]\right]
\]

performing the similar calculation as in Theorem 2.6 we obtain the required condition.

This completes the proof. \quad \Box

**Theorem 2.15.** Let $c_1, c_2 < 0$, $k_1, k_2 > 2$, $0 \leq \beta < 1$. If for some $\gamma(0 \leq \gamma < 1)$ and the inequality

\[
2(1 - \beta) \left[\frac{-4(k_1-1)}{c_1} \left(u_{p_1-1}(1) - 1\right)\right] - (3 + \gamma) \left[\frac{16(k_2-2)(k_2-1)}{c_2^2} \left(u_{p_2-1}(1) - 1\right) + \frac{c_2}{4(k_2-2)}\right]
\]

is satisfied then $L(TN_H(\beta)) \subset G_H(\gamma)$.

**Proof.** The proof of above theorem is similar to that of Theorem 2.14, therefore we omit the details involved. \quad \Box

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**References**

[1] O.P. Ahuja, Connections between various subclasses of planar harmonic mappings involving hypergeometric functions, *Appl. Math. Comput.*, 198 (1)(2008), 305–316.

[2] O.P. Ahuja and J.M. Jahangiri, Noshiro-type harmonic univalent functions, *Sci. Math. Jpn.*, 6(2)(2002), 253–259.

[3] A. Baricz, *Generalized Bessel Functions of the First Kind*, Lecture Notes in Math., Vol. 1994 , Springer-Verlag, 2010.

[4] B.C. Carleson and D.B. Shaffer, Starlike and prestarlike hypergeometric functions, *SIAM J. Math. Anal.*, 15 (1984), 737–745.

[5] J. Clunie and T. Sheil-Small, Harmonic univalent functions, *Ann. Acad. Sci. Fen. Series A.I. Math.*, 9 (1984), 3–25.

[6] P. Duren, *Harmonic Mappings in the Plane*, Cambridge University Press, 2004.

[7] Saurabh Porwal, Mapping properties of generalized Bessel functions on some subclasses of univalent functions, *Anal. Univ. Oradea Fasc. Matematica*, XX(2)(2013), 51–60.

[8] Saurabh Porwal, Harmonic starlikeness and convexity of integral operators generated by generalized Bessel functions, *Acta Math. Vietnamica*, 39 (2014), 337–346.
Saurabh Porwal, Connections between various subclasses of planar harmonic mappings involving generalized Bessel functions, *Thai J. Math.*, 13(1) (2015), 33–42.

Saurabh Porwal and Moin Ahmad, Some sufficient condition for generalized Bessel functions associated with conic regions, *Vietnam J. Math.*, 43(2015), 163–172.

Saurabh Porwal and K.K. Dixit, An application of generalized Bessel functions on certain analytic functions, *Acta Universitatis Matthiae Belii, series Mathematics*, 21 (2013), 55–61.

Saurabh Porwal and K.K. Dixit, An application of certain convolution operator involving hypergeometric functions, *J. Rajasthan Acad. Physical Sci.*, 9 (2)(2010), 173–186.

Saurabh Porwal and K.K. Dixit, An application of hypergeometric functions on harmonic univalent functions, *Bull. Math. Anal. Appl.*, 2(4)(2010), 97–105.

T. Rosy, B.A. Stephen, K.G. Subramanian and J.M. Jahangiri, Goodman-Rønning-type harmonic univalent functions, *Kyungpook Math. J.*, 41 (1)(2001), 45–54.

S. Ruscheweyh and V. Singh, On the order of starlikeness of hypergeometric functions, *J. Math. Anal. Appl.*, 113 (1986), 1–11.

A.K. Sharma, Saurabh Porwal and K.K. Dixit, Class mappings properties of convolutions involving certain univalent functions associated with hypergeometric functions, *Electronic J. Math. Anal. Appl.*, 1(2)(2013), 326–333.