EXCHANGE RELATION PLANAR ALGEBRAS
OF SMALL RANK

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Abstract. The main purpose of this paper is to classify exchange relation
planar algebras with 4 dimensional 2-boxes. Besides its skein theory, we em-
phasize the positivity of subfactor planar algebras based on the Schur product
theorem. We will discuss the lattice of projections of 2-boxes, specifically the
rank of the projections. From this point, several results about biprojections
are obtained. The key break of the classification is to show the existence of
a biprojection. By this method, we also classify another two families of sub-
factor planar algebras: subfactor planar algebras generated by 2-boxes with
4 dimensional 2-boxes and at most 23 dimensional 3-boxes; subfactor planar
algebras generated by 2-boxes, such that the quotient of 3-boxes by the basic
construction ideal is abelian. They extend the classification of singly generated
planar algebras obtained by Bisch, Jones and the author.

1. Introduction

In [20], Jones classified the indices of subfactors of type II$_1$ as follows:

$$\{4 \cos^2\left(\frac{\pi}{n}\right), n = 3, 4, \cdots \} \cup [4, \infty].$$

One approach to the classification of subfactors is to treat the index. Thus
the simplest subfactors are those of index less than 4 and then those of index
between 4 and 5. An early result is the classification of subfactors of index at most
4; see [14][17][33][39]. This approach has been extremely successful in the hands
of Haagerup [16] and others [1][3][7][17][41]. Recently the classification has been
extended up to index 5, see [18][28][29][31][34].

Below index 4 a deep theorem of Popa’s [37] showed that the standard invariant
is a complete invariant of subfactors of the hyperfinite factor of type II$_1$. Subfactor
planar algebras were introduced by Jones as a diagrammatic axiomatization of the
standard invariant [19]. Other axiomatizations are known as Ocneanu’s paragroups
[33] and Popa’s $\lambda$-lattices [38].

From the planar algebra perspective it seems far more natural to say that the
simplest subfactors are those whose standard invariants are generated by the fewest
elements satisfying the simplest relations. The simplest subfactor planar algebra
is the one generated by the sequence of Jones projections, also well known as the
Temperley-Lieb algebra, denoted by $TL(\delta)$, and $TL$ for short, where $\delta$ is the square
root of the index. The next most complicated planar algebras after Temperley-Lieb
should be those generated by a single element. See [3][30][35][44] for examples.
For a planar algebra $S = \{S_{n,\pm}\}_{n\in\mathbb{N}_{0}}$, an element in $S_{n,\pm}$ is called an $n$-box. Planar algebras generated by 1-boxes were completely analyzed by Jones in [19]. Subfactor planar algebras generated by a non-trivial 2-box were considered by Bisch and Jones, and classified by them for $\dim(S_{3,\pm}) \leq 12$ in [11]; for $\dim(S_{3,\pm}) = 13$ in [12]; by Bisch, Jones and the author for $\dim(S_{3,\pm}) = 14$ in [8]. They are given by the crossed product group subfactor planar algebra $S^{Z_{2}}$, the free product of two TL's, well known as Fuss-Catalan [10]; the crossed product subgroup subfactor planar algebra $S^{Z_{2}} \subset Z_{5} \rtimes Z_{2}$; BMW [4,32,44], precisely one family from quantum $Sp(4,\mathbb{R})$ and one from quantum $O(3,\mathbb{R})$, respectively. The classification for $\dim(S_{3,\pm}) = 15$ is still unclear. In these cases, we always have $\dim(S_{2,\pm}) = 3$, since $\dim(S_{2,\pm})^{2} \leq \dim(S_{3,\pm})$.

In this paper, we hope to classify subfactor planar algebras generated by 4 dimensional 2-boxes. Observe that the free product of the index 2 subfactor planar algebra and a subfactor planar algebra generated by a 2-box with 15 dimensional 3-boxes has 4 dimensional 2-boxes and 24 dimensional 3-boxes. So we can only expect a classification for at most 23 dimensional 3-boxes.

**Theorem 1.1.** Suppose $S$ is a subfactor planar algebra generated by 2-boxes, with $\dim(S_{2,\pm}) = 4$ and $\dim(S_{3,\pm}) \leq 23$. Then $S$ is one of the following:

1. $S^{Z_{4}}$ or $S^{Z_{2} \oplus Z_{2}}$;
2. $A \ast TL$ or $TL \ast A$, where $A$ is generated by a non-trivial 2-box with $\dim(A_{3,\pm}) \leq 13$;
3. $B \ast S^{Z_{2}}$ or $S^{Z_{2}} \ast B$, where $B$ is generated by a non-trivial 2-box with $\dim(A_{3,\pm}) \leq 14$;
4. $S^{Z_{2}} \otimes TL$.

Another approach to the classification of planar algebras is to consider the relations of the generators, instead of the boundary of dimensions. Several kinds of relations of 2-boxes appeared naturally in planar algebras generated by a non-trivial 2-box with at most 15 dimensional 3-boxes. If $S$ is a subfactor planar algebra generated by a non-trivial 2-box with $\dim(S_{3,\pm}) \leq 12$, then $S_{3,\pm} / I_{3,\pm}$ is abelian, where $I_{3,\pm}$ is the basic construction ideal of $S_{3,\pm}$, i.e., the two sided ideal of $S_{3,\pm}$ generated by the Jones projection. Motivated by this condition, we have the following classification.

**Theorem 1.2.** If $S$ is a subfactor planar algebra generated by 2-boxes and $S_{3,\pm} / I_{3,\pm}$ is abelian, then $S$ is either depth-2 or the free product $A_{1} \ast A_{2} \ast \cdots \ast A_{n}$, such that $A_{1}$ is Temperley-Lieb or the dual of $S^{G_{1}}$, for a group $G_{1}$; $A_{n}$ is Temperley-Lieb or $S^{G_{n}}$, for a group $G_{n}$; $A_{m}$, for $1 < m < n$, is Temperley-Lieb or $S^{G_{m}}$, for an abelian group $G_{m}$. The converse statement is also true.

If $S$ is a subfactor planar algebra generated by a non-trivial 2-box with $\dim(S_{3,\pm}) \leq 13$, then $S$ is an exchange relation planar algebra [28]. Motivated by the exchange relation, we have the following classification.

**Theorem 1.3.** Suppose $S$ is an exchange relation planar algebra with $\dim(S_{2,\pm}) = 4$. Then $S$ is one of the following:

1. $S^{Z_{4}}$ or $S^{Z_{2} \oplus Z_{2}}$;
2. $A \ast TL$ or $TL \ast A$, where $A$ is generated by a non-trivial 2-box with $\dim(A_{3,\pm}) \leq 13$;
3. $S^{Z_{2}} \otimes TL$;
4. $S^{Z_{2} \subset Z_{7} \times Z_{2}}$.
The three classification results rely on a new approach to the complexity of subfactors, the rank of 2-boxes. We will show that the rank of the coproduct of two 2-box minimal projections is bounded by the number of length-2 paths between the two corresponding vertices in the principal graph; see Lemma 1.35. We can show that a subfactor planar algebra is a free product by looking at its principal graph; see Theorem 1.26.

In section 2, we recall some facts and notation about planar algebras. In section 3, a diagrammatic interpretation of the tensor product is discovered based on the construction of a biunitary in the planar algebra. It is related to the flatness of a planar algebra with respect to two biprojections. If a subfactor planar algebra contains two commuting and co-commuting biprojections, then the planar subalgebra generated by the flat parts with respect to the two biprojections forms a tensor product; see Theorem 3.14.

In section 4, we prove the Schur product theorem for subfactor planar algebras; see Theorem 4.1. Based on it, a new equivalent definition of biprojections in given; see Theorem 4.12. Consequently the support of the pure depth 2 parts of an irreducible subfactor planar algebra is a biprojection; see Theorem 4.16. From a von Neumann algebra perspective, this tells the existence of an intermediate subfactor which is the crossed product of the smaller factor by a Kac algebra. By the new definition of biprojections, we can talk about the biprojection generated by a 2-box; see Definition 4.13. By the Schur product theorem, we show that the norm of the Fourier transform of a positive 2-box is achieved on the Jones projection; see Lemma 4.18. Then we prove that the Fourier transform of the biprojection generated by a positive 2-box is the spectrum projection of the Fourier transform of the 2-box at its maximal spectrum; see Theorem 4.21. This result generalises a well known result in representation theory; see Proposition 4.22 and the remark following.

In section 5 we discuss the construction and the decomposition of exchange relation planar algebras under the free product and the tensor product. We obtain two general constructions of exchange relation planar algebras (see Propositions 5.3 and 5.5) and one family of exchange relation planar algebras (see Theorem 5.6). In section 6 we prove the three main classification results.

2. Preliminaries

We refer the reader to [22] for the definition of subfactor planar algebras. The dual of a planar algebra is given by switching its shading.

2.1. Notation. In a planar tangle, we use a thick string with a number $k$ to indicate $k$ parallel strings. The distinguished intervals of a planar tangle are marked by $'$s (corresponding to '*'s in [22]).

In this paper, the planar algebra $\mathcal{S} = \{\mathcal{S}_{n,\pm}\}_{n \in \mathbb{N} \cup \{0\}}$ is always the standard invariant of an irreducible subfactor, which is automatically spherical. Equivalently we assume that $\dim(\mathcal{S}_{1,\pm}) = 1$. Since we only work with $\mathcal{S}_{n,+}$, we write $\mathcal{S}_n$ for $\mathcal{S}_{n,+}$, and the dollar sign $\$$ of a planar tangle is always in an unshaded region. An element in $\mathcal{S}_n$ is written as a rectangle with the dollar sign on the left, called an $n$-box. The dollar sign and the boundary are omitted, if there is no confusion. For example, we may use $a$ instead of $\begin{array}{c}$\$\end{array}a$, and $\begin{array}{c}$\$\end{array}a$ instead of $\begin{array}{c}$\$\end{array}$. 
The value of a closed circle is \( \delta \); \( id \) means the identity of \( S_2 \); \( e_n = \frac{1}{n} \) is the Jones projection in \( S_{n+1} \); \( e = e_1 \). The (unnormalized) Markov trace on \( S_n \) is denoted by \( tr_n(x) = \sum x \) for all \( x \in S_n \). When \( n = 2 \), we write \( tr(x) \) for short.

For \( a, b \in S_n \), the product of \( a \) and \( b \) is defined as \( ab = \begin{array}{c} b \\ a \end{array} \) if \( a, b \in S_2 \), then we define \( a' = \begin{array}{c} c \\ a \end{array} \) to be the contragredient of \( a \) and \( ab = \begin{array}{c} a \\ b \end{array} \) to be the (1-string) coproduct of \( a \) and \( b \). Furthermore for \( a_i \in S_2, i = 1, 2, \ldots, k \), we write \( *_{i=1}^k a_i \) for \( a_1 * a_2 * \cdots * a_k \), and \( a^{*k} \) for \( *_{i=1}^k a \).

Note that \( S_2 \) is embedded in \( S_3 \) by adding one string to the right. Thus a 2-box \( a \) can be viewed as an element in \( S_3 \), still written as \( a \).

The Fourier transform, i.e., the one click rotation, is an isometry from \( S_2 \) to \( S_{2,-} \), and \( S_{2,-} \) is identified as a subspace of \( S_3 \) by adding one string to the left. Let us define \( 1 \square a \) to be the element \( \begin{array}{c} a \end{array} \) of \( S_3 \), and \( S_{1,3} \) to be \( \{ 1 \square z | z \in S_2 \} \). Then \( S_{1,3} \) is isomorphic to \( S_{2,-} \) as an algebra. It is easy to check that \( (1 \square a)(1 \square b) = 1 \square (a + b) \), \( e_2 = \frac{1}{2} (1 \square id) \) and \( 1 \square a' = \) the adjoint of \( 1 \square a^* \), where \( a^* \) is the adjoint of \( a \).

**Definition 2.1.** For two self-adjoint operators \( x \) and \( y \), we say \( x \) is weaker (resp. stronger) than \( y \) if the support of \( x \) (resp. \( y \)) is a subprojection of the support of \( y \) (resp. \( x \)), written as \( x \preceq y \) (resp. \( y \succeq x \)). If \( x \preceq y \) and \( y \preceq x \), then they have the same support, written as \( x \sim y \).

For a self-adjoint operator \( x \) and a projection \( p \), \( x \preceq p \) is equivalent to \( x = p xp \).

**Notation 2.2.** The support of a two sided ideal of a finite dimensional \( C^* \)-algebra is the maximal projection in the ideal.

### 2.2. Principle graphs, depth-2 subfactors and subgroup subfactors

We refer the reader to [25] for the definition of the (dual) principal graph of a subfactor. It is also defined for a subfactor planar algebra, since it does not depend on the presumed subfactor [6].

The principal graph and the dual principal graph are parts of the Ocneanu 4-partite principal graph [23][33].

Suppose \( S \) is the planar algebra of \( N \subset M \). Let us define \( \mathcal{I}_{n+1} \) to be the two sided ideal of \( S_{n+1} \) generated by the Jones projection \( e_n \), called the basic construction ideal; then \( \mathcal{I}_{n+1} = S_{n+1} e_n S_{n+1} = S_n e_n S_n \). Let us define \( S_n/\mathcal{I}_n \) to be the orthogonal complement of \( \mathcal{I}_n \) in \( S_n \); there is a bijection between the equivalent classes of minimal projections of \( S_n/\mathcal{I}_n \) and vertices in the principal graph whose distance from the marked point is \( n \).

**Definition 2.3.** In the principal graph, a vertex is said to be depth-\( n \) if its distance from the marked vertex is \( n \). Its multiplicity is the number of length-\( n \) paths from the marked vertex to it. The depth of a principal graph is defined to be the maximal depth of its vertices.
Remark 2.4. If a depth- \( n \) vertex has multiplicity \( m \), then the vertex corresponds to an \( M_m(\mathbb{C}) \) component in \( S_{n}/I_{n} \).

Notation 2.5. If the principal graph of a subfactor planar algebra is depth-2, equivalently \( S_3 = I_3 \), then we call it a depth-2 subfactor planar algebra.

There is a one to one correspondence between depth-2 subfactor planar algebras and finite dimensional Kac algebras, or finite dimensional C* Hopf algebras [26,40,42]. Precisely, for any depth-2 subfactor planar algebra \( S \), \( S_2 \) forms a Kac algebra. On the other hand for any finite dimensional Kac algebra \( K \) there is an outer action of \( K \) on the hyperfinite factor \( R \) of type II\(_1\). Then \( R' \cap R \rtimes K = \mathbb{C} \). Thus we obtain an irreducible subfactor planar algebra as the standard invariant of \( R \subset R \rtimes K \), denoted by \( S^K \). Then \( S^K \) is depth-2, and \( (S^K)_2 \) is isomorphic to the dual of \( K \) as a Kac algebra. Specially when \( G \) is a finite group, we obtain a subfactor planar algebra \( S^G \) of \( R \subset R \rtimes G \). If \( H \) is a subgroup of \( G \), then \( R \rtimes H \) is subfactor of \( R \rtimes G \). Thus we obtain a subfactor planar algebra of \( R \rtimes H \subset R \rtimes G \).

Definition 2.6. Let us define \( S^G \) to be the planar algebra of the crossed product group subfactor \( R \subset R \rtimes G \) and \( S^{H \subset G} \) to be the planar algebra of the crossed product subgroup subfactor \( R \rtimes H \subset R \rtimes G \).

The principal graph of a subgroup subfactor is described in [25,27].

2.3. Wenzl’s formula. Suppose \( S \) is a subfactor planar algebra, \( I_{n+1} \) is the basic construction ideal of \( S_{n+1} \), and \( S_{n+1}/I_{n+1} \) is its orthogonal complement in \( S_{n+1} \). Then \( S_{n+1} = I_{n+1} \oplus S_{n+1}/I_{n+1} \). Let \( s_{n+1} \) be the support of \( S_{n+1}/I_{n+1} \).

If \( S \) is Temperley-Lieb (with \( \delta^2 \geq 4 \)), then \( s_{n} \) is the \( n_{th} \) Jones-Wenzl projection. The following relation is called Wenzl’s formula [43]:

\[
\begin{align*}
\begin{array}{c}
\text{n} \\
\downarrow
\end{array}
\begin{array}{c}
\text{S_n} \\
\downarrow
\end{array}
\quad = \quad \begin{array}{c}
\text{tr}_{n-1}(s_{n-1}) \\
\downarrow
\end{array}
\begin{array}{c}
\text{S_n} \\
\downarrow
\end{array} \quad + \quad \begin{array}{c}
\text{tr}_{n}(s_{n}) \\
\downarrow
\end{array}
\begin{array}{c}
\text{S_n} \\
\downarrow
\end{array}
\end{align*}
\]

It tells how a minimal projection is decomposed after adding one string to the right.

In general, suppose \( P \) is a minimal projection in \( S_{n}/I_{n} \). Note that \( S_{n+1} = I_{n+1} \oplus S_{n+1}/I_{n+1} \). When \( P \) is included in \( S_{n+1} \), it is decomposed as two projections \( P = P_{\text{old}} + P_{\text{new}} \), such that \( P_{\text{old}} \in I_{n+1} \) and \( P_{\text{new}} \in S_{n+1}/I_{n+1} \). By the definition of \( s_{n+1} \), we have \( P_{\text{new}} = s_{n+1}P \). Now let us construct \( P_{\text{old}} \). Let \( v \) be the depth-\( n \) vertex in the principal graph corresponding to \( P \), and let \( V \) be the central support of \( P \). Suppose \( v_i, 1 \leq i \leq m, \) are the depth-(\( n - 1 \)) vertices adjacent to \( v \), the multiplicity of the edge between \( v_i \) and \( v \) is \( m(i) \), and \( Q_i \) is a minimal projection in \( S_{n-1} \) corresponding to \( v_i \). For each \( i \), take partial isometries \( \{U_{ij}\}_{j=1}^{m(i)} \) in \( S_{n} \), such that

\[
U_{ij}U_{ij}^* = P, \forall 1 \leq j \leq m(i); \quad \sum_{j=1}^{m(i)} U_{ij}U_{ij}^* = Q_iV.
\]

It is easy to check that \( \frac{\text{tr}_{n-1}(Q_i)}{\text{tr}_{n}(P)} \) is a subprojection of \( P \), and they are mutually orthogonal for all \( i, j \). By Frobenius reciprocity, their sum is \( P_{\text{old}} \). Then
the general Wenzl’s formula \( P_{new} = P - P_{odd} \) is given as

\[
s_{n+1} = P - \left( \sum_{i=1}^{m} \sum_{j=1}^{n(i)} \frac{tr_{n-1}(Q_i)}{tr_n(P)} \right) U_{ij}.
\]

Now we give an alternative proof of the general Wenzl formula without applying Frobenius reciprocity. This proof is useful when the planar algebra is constructed by generators and relations. Based on this proof, we may derive the Bratteli diagram inductively without assuming it is a subfactor planar algebra.

**Proposition 2.7.** Take \( P_+ = P - \left( \sum_{i=1}^{m} \sum_{j=1}^{n(i)} \frac{tr_{n-1}(Q_i)}{tr_n(P)} \right) U_{ij} \). For any \( x \in \mathcal{I}_{n+1} \), we have \( xP_+ = \sum_{i=1}^{m} \sum_{j=1}^{n(i)} x_{i,j} e_{n}U_{ij} \), for some \( x_{i,j} \in \mathcal{S}_n \). So \( xP_+ = (xP)P_+ = 0 \). Similarly \( P_+x = 0 \). □

2.4. **Tensor products, free products and biprojections.** Let us recall some facts about tensor products, free products and biprojections due to Bisch and Jones; see [2,5,9,10] for more details.

**Notation 2.8 ([19]).** Suppose \( A \) and \( B \) are planar algebras and their tensor product is a planar algebra, denoted by \( A \otimes B \).

The tensor product \( A \otimes B \) is a planar algebra for which \( (A \otimes B)_n = A_n \otimes B_n, n \geq 0 \), and the action of an unlabeled tangle \( T \) from \( \bigotimes_{i=1}^{k}(A \otimes B)_{n_i} \) to \( (A \otimes B)_m \) is defined as a linear extension of the map

\[
T(\bigotimes_{i=1}^{k}(x_i \otimes y_i)) = T(\bigotimes_{i=1}^{k}x_i) \otimes T(\bigotimes_{i=1}^{k}y_i), \quad \forall x_i \otimes y_i \in A_{n_i} \otimes B_{n_i}, \, 1 \leq i \leq m.
\]

If both \( A \) and \( B \) admit the adjoint operation *, then we define \( (x \otimes y)^* \) to be \( x^* \otimes y^* \).

**Proposition 2.9 ([19]).** Suppose \( A \) and \( B \) are subfactor planar algebras. Then \( A \otimes B \) is a subfactor planar algebra.

**Notation 2.10 ([2,5,10]).** Suppose \( A \) and \( B \) are planar algebras. Their free product is a planar algebra, denoted by \( A * B \).
Each element in the free product $A \ast B$ is a linear sum of $A,B$-colour diagrams which consist of non-intersecting $A,B$-colour strings, labels of $A$ which only connect with $A$-coloured strings, and labels of $B$ which only connect with $B$-coloured strings. The colour of its boundary points are ordered by $ABBA ABBA \cdots ABBA$. For an action of an unlabeled tangle $T$, we substitute each string of $T$ by a pair of parallel $A,B$-colour strings and then glue the boundaries.

There is an equivalent definition of the free product. We say an element $x \otimes y \in (A \ast B)^n$ is separated by a Temperley-Lieb $n$-tangle $T_n$ if $x$ can be written as a diagram in unshaded regions of $T_n$ and $y$ can be written a diagram in shaded regions of $T_n$. Then $A \ast B$ is the planar subalgebra of $A \otimes B$ consisting of all separated elements. For example, the diagram \[
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}\begin{array}{c}
\begin{array}{c}
B
\end{array}
\end{array}\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}\begin{array}{c}
\begin{array}{c}
B
\end{array}
\end{array}\]
\[
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}\begin{array}{c}
\begin{array}{c}
B
\end{array}
\end{array}\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}\begin{array}{c}
\begin{array}{c}
B
\end{array}
\end{array}\end{array}
\]
\[\text{is separated by the tangle} \delta_B \text{id} \otimes e, \text{ where } \delta_B \text{ is the value of a circle of } B. \text{ Moreover,}\]
\[
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}\begin{array}{c}
\begin{array}{c}
B
\end{array}
\end{array}\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}\begin{array}{c}
\begin{array}{c}
B
\end{array}
\end{array}\end{array}
\begin{array}{c}
\begin{array}{c}
a
\end{array}
\end{array}= \delta_B a \otimes e;\]
\[
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}\begin{array}{c}
\begin{array}{c}
B
\end{array}
\end{array}\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}\begin{array}{c}
\begin{array}{c}
B
\end{array}
\end{array}\end{array}
\begin{array}{c}
\begin{array}{c}
b
\end{array}
\end{array}= \text{id} \otimes b. \text{ Consequently we have the following result.}\]

\textbf{Proposition 2.11} (\cite{2,9}). Suppose $A$ and $B$ are subfactor planar algebras. Then $A \ast B$ is a subfactor planar algebra.

\textbf{Notation 2.12.} The free product is associative but not commutative. The free product of $A$ and $B$ may be denoted as either $A \ast B$ or $B \ast A$.

\textbf{Proposition 2.13.} The dual of $A \ast B$ is the free product of the dual of $B$ with the dual of $A$.

The free product of Temperley-Lieb subfactor planar algebras is called a \textit{Fuss-Catalan} subfactor planar algebra \cite{10}.

\textbf{Definition 2.14.} Suppose $Q$ is a projection in $S_2$. If $1 \square Q$ is a multiple of a projection in $S_{1,3}$, then we call $Q$ a biprojection. In this case, \[
\frac{\delta}{tr(Q)}1 \square Q \text{ is the projection.}\]

\textbf{Notation 2.15.} There are two trivial biprojections, $e$ and $\text{id}$, in $S_2$. A biprojection is said to be non-trivial if it is neither $e$ nor $\text{id}$.

In the free product $A \ast B$, there is a non-trivial biprojection $\delta_B^{-1}$. \[
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}\begin{array}{c}
\begin{array}{c}
B
\end{array}
\end{array}\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}\begin{array}{c}
\begin{array}{c}
B
\end{array}
\end{array}\end{array}
\begin{array}{c}
\begin{array}{c}
AB
\end{array}
\end{array}= \text{id} \otimes b. \text{ Consequently we have the following result.}\]

\textbf{Proposition 2.16} (\cite{5}). Suppose $Q$ is a biprojection in a subfactor planar algebra. Then $Q$ satisfies \[
\begin{array}{c}
\begin{array}{c}
\delta
\end{array}
\end{array}\begin{array}{c}
\begin{array}{c}
\delta
\end{array}
\end{array}= \text{id} \otimes b. \text{ Called the exchange relation of the biprojection } Q.\]

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Proof. Let \( x = \begin{array}{c}
\includegraphics{a.png}
\end{array} - \begin{array}{c}
\includegraphics{b.png}
\end{array} \). Then it is easy to check that \( \text{tr}_3(x^*x) = 0 \). By the positivity of the trace, we have \( x = 0 \). That means \( \begin{array}{c}
\includegraphics{a.png}
\end{array} = \begin{array}{c}
\includegraphics{b.png}
\end{array} \). \qed

It was known to Bisch and Jones and also appeared in \cite{2} that the planar algebra of \( \mathcal{N} \subset \mathcal{Q} \) can be realised as a \( \mathcal{Q} \) cut down on shaded intervals of diagrams in \( \mathcal{S} \), denoted by \( \mathcal{S}_Q \). That means \( (\mathcal{S}_Q)_n = \Psi_Q(\mathcal{S}_n) \), where \( \Psi_Q \) is the annular action

\[
\begin{array}{c}
\includegraphics{c.png}
\end{array}
\]

and the action of an unlabeled tangle \( T \) on \( \mathcal{S}_Q \) is defined to be

\[
(\delta \sqrt{\text{tr}(Q)}/fudge(T))\Psi_Q \circ T,
\]

where \( fudge(T) = n - m \), \( n \) is number of shaded intervals of the outside boundary of \( T \) and \( m \) is the number of closed circles after adding a cap at each shaded interval of the (outside and inside) boundary of \( T \). Considering the duality, the planar algebra of \( \mathcal{Q} \subset \mathcal{M} \) is realised as a \( \mathcal{Q} \) cut down on unshaded intervals of diagrams in \( \mathcal{S} \), denoted by \( \mathcal{S}^Q \).

The meaning of the fudge factor is explained in the following proposition \cite{9}.

**Proposition 2.17.** Let \( A \) and \( B \) be subfactor planar algebras with circle parameters \( \delta_A \) and \( \delta_B \) respectively. Form the free product \( A \ast B \). Let \( Q \) be the biprojection \( \begin{array}{c}
\includegraphics{d.png}
\end{array} \) and \( v_n \in B \) be \( \begin{array}{c}
\includegraphics{e.png}
\end{array} \) or \( \begin{array}{c}
\includegraphics{f.png}
\end{array} \), when \( n \) is odd or even respectively. Then the map

\[
\alpha_Q : A \rightarrow (A \ast B)_Q, \quad \alpha_Q(x) = x \otimes v_n, \quad \forall x \in A_n,
\]

is a planar algebra isomorphism.

The following result was first known by Bisch and Jones \cite{9}; see also \cite{2}.

**Theorem 2.18 (\cite{2,9}).** Let \( \mathcal{S} \) be a subfactor planar algebra containing a biprojection \( Q \). Then the planar subalgebra \( \mathcal{S}_Q \vee \mathcal{S}^Q \) of \( \mathcal{S} \) generated by the vector spaces \( \mathcal{S}_Q \) and \( \mathcal{S}^Q \) is naturally the free product \( \mathcal{S}_Q \ast \mathcal{S}^Q \).

**Corollary 2.19.** Suppose \( A \) and \( B \) are subfactor planar algebras generated by 2-boxes. Then \( A \ast B \) is generated by 2-boxes.

**Theorem 2.20.** Suppose \( \mathcal{S} \) is a subfactor planar algebra. If \( Q \) is a biprojection in \( \mathcal{S}_2 \), and \( \mathcal{S} \) is generated by \( \{x \in \mathcal{S}_2 | QxQ = x \text{ or } QxQ = (\frac{\text{tr}(Q)}{\delta})^2 x\} \) as a planar algebra, then \( \mathcal{S} = \mathcal{S}_Q \ast \mathcal{S}^Q \), and both \( \mathcal{S}_Q \) and \( \mathcal{S}^Q \) are generated by 2-boxes. In this case, \( \mathcal{S} \) is said to be separated by the biprojection \( Q \) as a free product.

**Proof.** Suppose \( A \) is the planar subalgebra of \( \mathcal{S}_Q \) generated by 2-boxes and \( B \) is the planar subalgebra of \( \mathcal{S}^Q \) generated by 2-boxes. Then \( A \ast B \subseteq \mathcal{S}_Q \ast \mathcal{S}^Q \subseteq \mathcal{S} \).
On the other hand, if \( x \in S_2 \) satisfies \( QxQ = Q \), then \( x = \alpha_Q^{-1}(x) \otimes \delta_B e \), and if \( y \in S_2 \) satisfies \( QyQ = (\frac{\text{tr}(Q)}{\delta})y \), then \( y = \text{id} \otimes \beta_Q^{-1}(y) \). If \( S \) is generated by these 2-boxes, then \( S \subseteq A* B \). So \( A* B = Q* S^Q = S \). Note that the generating function of a free product is Voiculescu’s free product of generating functions \([9]\). By counting the dimension of \( n \)-boxes, we obtain \( A = S_Q \) and \( B = S^Q \).

**Corollary 2.21.** Suppose \( A \) and \( B \) are subfactor planar algebras. If \( A* B \) is generated by 2-boxes, then both \( A \) and \( B \) are generated by 2-boxes.

**Proof.** Suppose \( Q \) is the central biprojection \( \text{id} \otimes e \); then \( (A* B)_2 = \{ x \in S_2 | QxQ = Q \text{ or } QxQ = (\frac{\text{tr}(Q)}{\delta})x \} \). The statement follows from Theorem 2.20. \( \square \)

The following result should be known to experts. We give two proofs based on the Schur product theorem; see Theorem 4.1.

**Theorem 2.22.** Suppose \( A \) and \( B \) are subfactor planar algebras. Then for any biprojection \( Q \) in \( (A* B)_2 \), either \( Q \geq \text{id} \otimes e \) or \( Q \leq \text{id} \otimes e \).

**Proof.** Note that \( Q = a \otimes e + \text{id} \otimes b \) for some \( a \in A_2, b \in B_2 \). Furthermore we may assume \( be = 0 \), otherwise \( a, b \) are replaced by \( \lambda \text{id} + a, b - \lambda e \), when \( be = \lambda e \). Then this decomposition is unique since \( a \otimes e = Q(id \otimes e) \). By assumption \( Q \) is a projection, thus \( a \otimes e \) and \( \text{id} \otimes b \) are projections. Then \( a \) and \( b \) are projections. Moreover \( Q = Q' \), thus both \( a \otimes e \) and \( \text{id} \otimes b \) are self-contragredient, and so \( a = a' \) and \( b = b' \). Furthermore \( \frac{\text{tr}(Q)}{\delta} Q = Q * Q \), so

\[
a \otimes e + \text{id} \otimes b = a*a \otimes e + a* \text{id} \otimes e * b + \text{id} \otimes a \otimes b * e + \text{id} \otimes \text{id} \otimes b * b
\]

\[
= \frac{1}{\delta_1} a*a \otimes e + \frac{2\text{tr}(a)}{\delta_1 \delta_2} \text{id} \otimes b + \delta_1 \text{id} \otimes b * b
\]

\[
= \frac{1}{\delta_1} a*a \otimes e + \delta_1 \frac{\text{tr}(b)}{\delta_2} \text{id} \otimes e + \frac{2\text{tr}(a)}{\delta_1 \delta_2} \text{id} \otimes b + \delta_1 \text{id} \otimes (b * b - \frac{\text{tr}(b)}{\delta_2} e).
\]

Both \( b \) and \( b * b - \frac{\text{tr}(b)}{\delta_2} e \) are orthogonal to \( e \), so

\[
\frac{\text{tr}(Q)}{\delta} a \otimes e = \frac{1}{\delta_1} a*a \otimes e + \delta_1 \frac{\text{tr}(b)}{\delta_2} \text{id} \otimes e.
\]

If \( \text{tr}(b) = 0 \), then \( b = 0 \) because \( b \) is a projection. Thus \( P = a \otimes e \leq \text{id} \otimes e \). Otherwise \( \text{tr}(b) > 0 \). By Theorem 4.1 we have that \( a*a \otimes e \) is positive. So \( \text{id} \otimes e \leq a \otimes e \). Recall that \( a \) is a projection, so \( a = \text{id} \). Then \( P \geq \text{id} \otimes e \). \( \square \)

**A second proof.** Suppose \( Q \) is a biprojection. Note that \( \text{id} \otimes e \) is central. If \( Q \) is not a subprojection of \( \text{id} \otimes e \), then \( Q \) has a subprojection \( \text{id} \otimes b \). By Theorem 4.1 we have

\[
\text{id} \otimes e \leq (\text{id} \otimes \text{id}) \otimes (p' \otimes p) = (\text{id} \otimes p)' \otimes (\text{id} \otimes p) \leq Q' * Q \sim Q.
\]

So \( \text{id} \otimes e \leq Q \). \( \square \)
2.5. Skein theory. Comparing to group theory, a subfactor planar algebra could be constructed by generators and relations \[19\]. While trying to construct a subfactor planar algebra \( S = \{S_{n,\pm}\}_{n \in \mathbb{N}_0} \), we will encounter four problems:

1. Is \( S \) finite dimensional, i.e., is \( S_{n,\pm} \) finite dimensional for each \( n \)?
2. Is \( S \) evaluable, i.e., is \( S_{0,\pm} \) 1 dimensional?
3. Is \( S \) the zero planar algebra?
4. Is the Markov trace positive definite?

If \( S \) is the planar algebra of an irreducible subfactor, then \( \dim(S_{0,\pm}) = \dim(S_{1,\pm}) = 1 \). We shall consider a planar algebra generated by a finite subset of 2-boxes. Then each generator can be viewed as a crossing with a label at the intersection, and each element in \( S_{n,\pm} \) can be viewed as a linear combination of diagrams with \( 2n \) boundary points which consist of finitely many crossings and finitely many strings. For example, as an element in \( S_{3,+} \), is a diagram with 6 boundary points and 2 crossings. What kind of relations should be endowed? One type of relation, termed an exchange relation, is discussed by Landau \[28\]. It was motivated by the exchange relation of a biprojection discovered by Bisch \[5\]; see Proposition \[2.16\]. The planar algebra \( S \) has an exchange relation means that the diagram can be replaced by a finite sum of the diagrams and , and the diagram can be replaced by a multiple of a string. Note that a closed string contributes a scalar \( \delta \). By these three operations, a face of a diagram can be removed without increasing the number of crossings. Given the number of boundary points, up to isotopy, there are only finitely many diagrams without faces and closed strings. Thus problem (1) is solved. Furthermore if such a diagram has no boundary points, then it has to be the empty diagram. Thus problem (2) is solved. Given generators and an exchange relation, to solve problem (3) is equivalent to checking a finite system of equations. However it is hard to solve these equations directly. What’s worse, it is much harder to solve problem (4). In this paper, we focus on classifying exchange relation planar algebras. The ones that appeared could be constructed by other methods. So we will not deal with problems (3) and (4) directly.

**Definition 2.23.** Suppose \( S \) is an irreducible subfactor planar algebra. If \( S \) is generated by \( S_2 \) with the relations

\[
(1 \boxdot a)b = \sum_i c_i (1 \boxdot d_i) + f_i (1 \boxdot id)g_i,
\]

for any \( a, b \in S_2 \) and for finitely many \( c_i, d_i, f_i, g_i \in S_2 \), then \( S \) is called an exchange relation planar algebra.

It is easy to check that this definition is equivalent to Landau’s definition in \[28\]. By definition, Temperley-Lieb subfactor planar algebras and depth-2 subfactor planar algebras are exchange relation planar algebras.
Proposition 2.24. Suppose $S$ is an exchange relation planar algebra. Then
\[ \dim(S_{n+1}) \leq \dim(S_n)^2 + (\dim(S_2) - 1)^n \]
and $P_3$ is generated by $S_2$ and $S_{1,3}$ as an algebra. Specifically,
\[ \dim(S_3) \leq \dim(S_2)^2 + (\dim(S_2) - 1)^2. \]

Proof. We view 2-boxes as crossings, the labels at the intersection as points, and
the strings as edges. Then using exchange relations, we may replace
\[ \text{ by } \]
by \[ \text{ and } \]. By this operation, the number of edges of one
face will decrease without adding faces. Combining with the relation \[ \text{ we only need to consider diagrams without faces.}
If a diagram with $2n + 2$ boundary points, $n + 1$ on the top and $n + 1$ on the
bottom, has no faces, then either it is in the ideal $\mathcal{I}_{n+1}$ generated by the Jones
projection $e_n$, or it has $n + 1$ through strings. The dimension of the ideal of $\mathcal{I}_{n+1}$ is
at most $\dim(S_n)^2$. In $S_{n+1}/\mathcal{I}_{n+1}$, applying the exchange relation, we only need to
consider one diagram, \[ \text{ If the label at an intersection is the Jones projection}
$e$, then this diagram is in the ideal $\mathcal{I}_{n+1}$. Thus the dimension of $S_{n+1}/\mathcal{I}_{n+1}$ is at
most $(\dim(S_2) - 1)^n$. Then
\[ \dim(S_{n+1}) \leq \dim(S_n)^2 + (\dim(S_2) - 1)^n. \]

If a diagram with 6 boundary points has no faces and it has two crossings or
more, then it has to be one of the following three diagrams: \[ \text{ Thus } P_3 \text{ is generated by } S_2 \text{ and } S_{1,3} \text{ as an algebra.} \]

If a closed diagram has no faces, then it is the empty diagram. So each closed
diagram is evaluable based on the exchange relation. Furthermore the exchange
relation is determined by the algebraic structure of 2-boxes in the following sense.

Definition 2.25. The structure of 2-boxes of a subfactor planar algebra consists
of the data of adjoints, contragredients, products and coproducts of 2-boxes.

The following data is also derived from the structure of 2-boxes: the identity $id$
is identified as the unique unit of 2-boxes under the product; the value of a closed
circle $\delta$ is determined by the coproduct of two identities; $\delta e$ is identified as the
unique unit of 2-boxes under the coproduct; the trace of a 2-box is determined by
its coproduct with the identity $id$. If the planar algebra is irreducible, then capping
a 2-box is also determined.
The following result is known to experts.

**Theorem 2.26.** Suppose $S$ is an exchange relation planar algebra and $A$ is a subfactor planar algebra generated by 2-boxes. If a linear map $\phi : S_2 \rightarrow A_2$ is surjective and it preserves the structure of 2-boxes, i.e., adjoints, contragredients, products and coproducts, then $\phi$ extends to a planar algebra isomorphism from $S$ to $A$.

**Proof.** We extend $\phi$ to the universal planar algebra generated by 2-boxes of $S$. If

$$(1 \boxdot a)b - \left( \sum_i c_i(1 \boxdot d_i) + f_i(1 \boxdot id)g_i \right)$$

is a relation, denoted by $y$, then $tr(y^*y) = 0$. The computation of $tr(\phi(y)^*\phi(y))$ only depends on the structure of 2-boxes, which are preserved by $\phi$, so

$$tr(\phi(y)^*\phi(y)) = tr(y^*y) = 0.$$

Then $\phi(y) = 0$ by the positivity of the trace. So $\phi$ induces a planar algebra homomorphism from the quotient $S$ to $A$. By assumption $\phi$ is surjective on 2-boxes, and $A$ is generated by 2-boxes, so $\phi$ is a planar algebra isomorphism. \qed

**Remark 2.27.** A planar algebra homomorphism of subfactor planar algebras induces a homomorphism on the 0-box space $\mathbb{C}$, so it is either zero or injective.

The following classification is given by Bisch and Jones [11][12].

**Theorem 2.28** ([11][12]). Suppose $S$ is a subfactor planar algebra generated by a non-trivial 2-box with $\dim(P_2) \leq 13$. Then $S$ is one of the following: (1) $S^{Z_2}$; (2) $TL \ast TL$; (3) $S^{Z_2 \times Z_2}$.

Suppose $S$ is a subfactor planar algebra generated by $S_2$ with $\dim(S_2) = 3$. Assume $id$, $e$ and $r$ form a basis of $S_2$. In $S_2$, there are 5 Temperley-Lieb diagrams, 6 diagrams with one $r$, the orbits or $r, 1 \boxdot r$ under the rotation, and 3 diagrams with two $r$, $r(1 \boxdot id)r$, $r(1 \boxdot r)$ and $(1 \boxdot r)r$. If $\dim(S_3) \leq 13$, then those 14 elements are linear dependent. Without loss of generality, we may assume one diagram with two $r$ is a linear sum of the other 13 diagrams. Up to a 2-click rotation, it can be chosen as $(1 \boxdot r)r$. This implies $S$ is an exchange relation planar algebra. Conversely if $S$ is an exchange relation planar algebra with $\dim(S_2) = 3$, then $\dim(S_3) \leq 2^2 + 3^2 = 13$. Thus Theorem 2.28 is equivalent to

**Theorem 2.29.** Suppose $S$ is an exchange relation planar algebra with $\dim(P_2) = 3$. Then $S$ is one of the following: (1) $S^{Z_2}$; (2) $TL \ast TL$; (3) $S^{Z_2 \times Z_2}$.

### 3. Tensor Products

In this section, sometimes we draw a diagram with all the boundary points on the top. For example, $v_n$ is the Temperley-Lieb $n$-tangle $\left\{ \begin{array}{c} \ \ \ $\bigcup \bigcup \cdots \bigcup \ $\end{array} \right. \text{.}$ Recall that $A \ast B \subset A \otimes B$ and it contains a biprojection $id \otimes e$. By Proposition 2.17 there is a planar algebra isomorphism $\alpha_1 : A \rightarrow (A \ast B)_{id \otimes e}$, $\alpha_1(a) = a \otimes v_n$, $\forall a \in A$. By the definition of the tensor product, we have $(A \otimes B)_{id \otimes e} = \alpha_1(A)$. So $(A \otimes B)_{id \otimes e} = (A \ast B)_{id \otimes e}$. Then we have a planar algebra isomorphism

$\alpha_1 : A \rightarrow (A \otimes B)_{id \otimes e}$, $\alpha_1(a) = a \otimes v_n$, $\forall a \in A$. 
Similarly we have a planar algebra isomorphism
\[ \alpha_2 : B \rightarrow (A \otimes B)_{e \otimes id}, \quad \alpha(b) = v_n \otimes b, \quad \forall \ b \in B. \]

The tensor product of planar algebras is defined via simple tensors of vectors. We hope to interpret the simple tensor \( a \otimes b \) as a diagram in terms of \( \alpha_1(a) \) and \( \alpha_2(b) \).

**Theorem 3.1.** \( A \otimes B \) is generated by the two vector spaces \( (A \otimes B)_{id \otimes e} \) and \( (A \otimes B)_{e \otimes id} \) as a planar algebra.

**Proof.** For any \( a \in A_n \) and \( b \in B_n \), let us construct an element \( T \) in \( (A \otimes B)_n \) as

\[
T_A = \lambda_n a
\]

Replacing id, e, \( v_n \) in the above tangle by Temperley-Lieb diagrams, we have
\[ T_A = \lambda_n a, \quad \text{for some} \ \lambda_n > 0. \]

Similarly the second component is \( T_B = \lambda_n b \). So \( T = \lambda_n a \otimes \lambda_n b \). Note that \( a \otimes v_n \in (A \otimes B)_{id \otimes e}, \ v_n \otimes b \in (A \otimes B)_{e \otimes id} \), and \( A \otimes B \) is generated by all \( a \otimes b \)'s.

Thus \( A \otimes B \) is generated by \( (A \otimes B)_{id \otimes e} \) and \( (A \otimes B)_{e \otimes id} \) as a planar algebra. \( \square \)

**Corollary 3.2.** If both \( A \) and \( B \) are generated by 2-boxes, then \( A \otimes B \) is generated by 2-boxes.

**Lemma 3.3.** Suppose \( S \) is a subfactor planar algebra and \( A, B \) are two biprojections such that \( AB = e \) and \( A \ast B \) is a multiple of id. Then \( A \ast B = \frac{1}{\delta} id \) and \( \text{tr}(A)\text{tr}(B) = \delta^2 \). Moreover

\[
\frac{1}{\delta} = \begin{array}{cc}
A & B \\
A & B
\end{array}
\]
Proof. By assumptions, we have
\[ AB = (AB)^* = B^* A^* = BA; \]
\[ A * B = (A * B)' = B' * A' = B * A. \]

By computing the trace of \( A * B \), we have \( A * B = \frac{tr(A)tr(B)}{\delta^3} \) id. Then \( tr((A * B)e) = \frac{tr(A)tr(B)}{\delta^3} \). On the other hand, we view \( tr((A * B)e) \) as a diagram; then \( tr((A * B)e) = \frac{1}{\delta} tr(AB) = \frac{1}{\delta} \). Thus \( tr(A)tr(B) = \delta^2 \) and \( A * B = \frac{1}{\delta} \) id. Using the exchange relation of biprojections (Proposition 2.16), we have

\[
(A * B) = \frac{1}{\delta} \]

Notation 3.4. Let \( \delta_A, \delta_B \) be \( \sqrt{tr(A)}, \sqrt{tr(B)} \); then they are the value of a closed circle in \( S_A, S_B \) respectively, and \( \delta = \delta_A \delta_B \).

To show \( S_A \) and \( S_B \) are “independent”, we use two kinds of coloured strings, an A-colour string \(-\cdots-\) connecting with elements in \( S_A \) and a B-colour string \( \cdots\cdots\cdots \) connecting with elements in \( S_B \). A crossing means \( \delta \). Because \( A \) and \( B \) are biprojections, it does not matter where we put the \( \$ \)'s. If a non-closed A-colour string does not intersect with a B-colour string, then the A-colour string is just an ordinary Temperley-Lieb string. By our assumption, an A-colour string \(-\cdots-\cdots-\) only connects with elements in \( S_A \), thus \( \$ = \$A \). Moreover we can view \( \$ \) as \( \$A \) i.e., \( \$A \). Thus \( \$ = tr(A) \). A similar formula holds for \( B \).

Proposition 3.5.

\[
\$ = \$
\]

Proof. By Lemma 3.3 we have

\[
\$ = \delta^2 \]

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Definition 3.6. For $a \in S_A$, we call $a$ $B$-flat if

By definition, the empty diagram $\emptyset$ in $(S_A)_0$ is $B$-flat.

Definition 3.7. Let us define a map $\wedge : S_m \otimes S_n \to S_{m+n}$ as a linear extension of

$$x \wedge y = \begin{array}{l}
\begin{array}{c}
\text{A} \\
\text{A} \\
\vdots \\
\text{A} \\
\text{A} \\
\end{array}
\begin{array}{c}
\text{X} \\
\text{Y} \\
\vdots \\
\text{X} \\
\text{Y} \\
\end{array}
\end{array}, \forall x \in S_m, y \in S_n, m, n \geq 0.
$$

Its restriction on $S_A$ induces a map $\wedge : (S_A)_m \otimes (S_A)_n \to (S_A)_{m+n}$.

Proposition 3.8. Suppose $x, y \in S_A$ are $B$-flat; then $x \wedge y$ is $B$-flat.

Proof.

Thus $x \wedge y$ is $B$-flat. \hfill \Box

Definition 3.9. Let us define the map $\Theta_B$ on $S_A$ given by the following annular tangle:

$$\Theta_B(a) = \begin{array}{l}
\begin{array}{c}
\text{A} \\
\text{A} \\
\vdots \\
\text{A} \\
\text{A} \\
\end{array}
\begin{array}{c}
\text{X} \\
\text{Y} \\
\vdots \\
\text{X} \\
\text{Y} \\
\end{array}
\end{array}, \forall a \in S_A.
$$

Proposition 3.10. For an element $a \in S_A$, $a$ is $B$-flat if and only if $\Theta_B(a) = tr(B)a$. 

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Proof. If \(a\) is \(B\)-flat, then \(\Theta_B(a) = \Theta_B(\emptyset) a = \text{tr}(B) a\).

On the other hand, we assume that \(\Theta_B(a) = \text{tr}(B) a\); then \(\Theta_B(a^*) = (\Theta_B(a))^* = \text{tr}(B) a^*\). Take

\[
x = \begin{array}{c}
\includegraphics{tangle1.png} \\
\end{array} - \begin{array}{c}
\includegraphics{tangle2.png} \\
\end{array}.
\]

Then

\[
\text{tr}(xx^*) = \Theta_B(\emptyset) \text{tr}(aa^*) + \Theta_B(\text{tr}(aa^*))\emptyset - \text{tr}(\Theta_B(a) a^*) - \text{tr}(a\Theta_B(a^*))
\]

\[
= \text{tr}(B) \text{tr}(aa^*) + \text{tr}(B) \text{tr}(aa^*) - \text{tr}(B) \text{tr}(aa^*) - \text{tr}(B) \text{tr}(aa^*) = 0.
\]

Thus \(x = 0\). That means \(a\) is \(B\)-flat.

\[\square\]

**Corollary 3.11.** For any \(a \in S_A\), if \(a\) is \(B\)-flat, then \(a^*\) is \(B\)-flat.

**Theorem 3.12.** All Temperley-Lieb elements of \(S_A\) are \(B\)-flat. Moreover, all \(B\)-flat elements in \(S_A\), denoted by \(\text{Flat}_B(S_A)\), form a non-zero planar subalgebra of \(S_A\).

Proof. We have known that the empty diagram \(\emptyset \in (S_A)_0\) is \(B\)-flat and \(\text{Flat}_B(S_A)\) is invariant under the adjoint action and the map \(\wedge\). Note that the action of a planar tangle can be decomposed as actions of \(\wedge\) and annular tangles \([19]\). Thus it is enough to show that \(\text{Flat}_B(S_A)\) is invariant under the annular Temperley-Lieb action. By Proposition 3.10 it is sufficient to prove that the annular Temperley-Lieb action commutes with \(\Theta_B\). Observe that, each Temperley-Lieb annular action is a composition of the following five operations:

1. a two-string rotation;
2. adding a cap on a shaded boundary;
3. adding a cap on an unshaded boundary;
4. adding a string in a shaded boundary;
5. adding a string in an unshaded boundary.

Thus we only need to prove \(\Theta_B\) commutes with (1)-(5).

\(\Theta_B\) commutes with (1) follows from their definitions.

\(\Theta_B\) commutes with (2) is equivalent to

\[
\begin{array}{c}
\includegraphics{tangle3.png} \\
\end{array} = \begin{array}{c}
\includegraphics{tangle4.png} \\
\end{array}.
\]

That is

\[
\begin{array}{c}
\includegraphics{tangle5.png} \\
\end{array} = \begin{array}{c}
\includegraphics{tangle6.png} \\
\end{array}.
\]
\( \Theta_B \) commutes with (3) is equivalent to 

\[
\begin{array}{ccc}
\includegraphics{diagram1.png} & \cdots & \includegraphics{diagram2.png} \\
\includegraphics{diagram3.png} & \cdots & \includegraphics{diagram4.png}
\end{array}
\]

That is

\[
\begin{array}{ccc}
\includegraphics{diagram5.png} & \cdots & \includegraphics{diagram6.png} \\
\includegraphics{diagram7.png} & \cdots & \includegraphics{diagram8.png}
\end{array}
\]

Observe that \( \{ (S_A)_n \} \) with the Markov trace forms a sequence of Hilbert spaces, and \( \Theta_B \) is self-adjoint on them due to the spherical property. Moreover (4) (resp. (5)) is a multiple of the adjoint of (3) (resp. (2)); thus \( \Theta_B \) commutes with (4) (resp. (5)).

**Notation 3.13.** Similarly let us define \( \text{Flat}_A(S_B) \) to be the planar subalgebra of \( B \) consisting of all \( A \)-flat elements.

**Theorem 3.14.** The planar subalgebra of \( S \) generated by the two vector spaces \( \text{Flat}_B(S_A) \) and \( \text{Flat}_A(S_B) \), denoted by \( \text{Flat}_B(S_A) \lor \text{Flat}_A(S_B) \), is naturally isomorphic to the tensor product \( \text{Flat}_B(S_A) \otimes \text{Flat}_A(S_B) \).

**Proof.** Let us define a map \( \Phi : \text{Flat}_B(S_A) \otimes \text{Flat}_A(S_B) \to S \) as a linear extension of

\[
\Phi(a \otimes b) =
\]

for any \( a \otimes b \in (\text{Flat}_B(S_A) \otimes (\text{Flat}_A(S_B)))_n, n > 0 \), and define \( \Phi(\emptyset \otimes \emptyset) = \emptyset \), where \( \emptyset \) is the unshaded empty diagram.

To show it is well defined, we prove that \( \Phi \) preserves the inner product. By equation (3.1) and Proposition 3.5, for any \( a_1 \otimes b_1, a_2 \otimes b_2 \in (\text{Flat}_B(S_A) \otimes (\text{Flat}_A(S_B)))_n \), we have

\[
\langle \phi(a_1 \otimes b_1), \phi(a_2 \otimes b_2) \rangle_S = tr_n(\phi(a_1 \otimes b_1)^* \phi(a_2 \otimes b_2)) = \delta^{-n} tr_n(a_1^* a_2) tr_n(b_1^* b_2).
\]

While computing the inner product in \( S_A \) and \( S_B \), the fudge factor will be involved, and we have

\[
\langle a_1, a_2 \rangle_{S_A} = \delta^{-n} tr(a_1^* a_2); \quad \langle b_1, b_2 \rangle_{S_B} = \delta^{-n} tr(b_1^* b_2).
\]

So

\[
\langle a_1 \otimes b_1, a_2 \otimes b_2 \rangle_{S_A \otimes S_B} = \langle a_1, a_2 \rangle_{S_A} \langle b_1, b_2 \rangle_{S_B} = \delta^{-n} tr(a_1^* a_2) \delta^{-n} tr(b_1^* b_2)
\]

\[
= \delta^{-n} tr_n(a_1^* a_2) tr_n(b_1^* b_2)
\]

\[
= \langle \phi(a_1 \otimes b_1), \phi(a_2 \otimes b_2) \rangle_S.
\]
To prove $\Phi$ is a planar algebra isomorphism, we need to check that $\phi$ commutes with the following seven operations:

1. the 2-click rotation;
2. adding a cap on a shaded boundary;
3. adding a cap on an unshaded boundary;
4. adding a string in a shaded boundary;
5. adding a string in an unshaded boundary;
6. the adjoint operation;
7. the $\wedge$ operation.

(1) follows from the flatness of $a$ and $b$, and the fudge factor is 0 under the 2-click rotation.
(7) follows from the flatness of $a$ and $b$, and the fudge factor is 0 under the $\wedge$ operation.

(6) follows from $\begin{array}{c} B \\ A \end{array} = \delta \begin{array}{c} B \\ A \end{array} = \begin{array}{c} A \\ B \end{array}$, and the flatness of $a$ and $b$.

(2) follows from $\begin{array}{c} A \\ B \end{array} = \frac{1}{\delta} \begin{array}{c} A \\ B \end{array}$, and the flatness of Temperley-Lieb elements. The factor $\delta^{-1} = \delta^{-1}_A \delta^{-1}_B$ contributes $-1$ to the fudge factor of adding a cap on a shaded boundary.

(3) follows from $\begin{array}{c} B \\ A \end{array} = \begin{array}{c} A \\ B \end{array}$, and the flatness of Temperley-Lieb elements. The fudge factor is 0 while adding a cap on an unshaded boundary.

(4) follows from $\begin{array}{c} A \\ B \end{array} = \delta^2 \begin{array}{c} A \\ B \end{array} = \delta \begin{array}{c} A \\ B \end{array}$, and the flatness of Temperley-Lieb elements. The factor $\delta = \delta_A \delta_B$ contributes $+1$ to the fudge factor of adding a string in a shaded boundary.

(5) follows from $\begin{array}{c} A \\ B \end{array} = \begin{array}{c} A \\ B \end{array}$, and the flatness of Temperley-Lieb elements. The fudge factor is 0 while adding a string in an unshaded boundary. □

**Theorem 3.15.** Suppose $S$ is a subfactor planar algebra with two biprojections $A, B$ such that

\[
AB = e, \quad A \star B \text{ is a multiple of } \text{id},
\]

\[
a \star B = B \star a, \quad \forall \ a \preceq A, \quad A \star b = b \star A, \quad \forall \ b \preceq B,
\]

and $S$ is generated by $\{ x \in S_2 \mid x \preceq A \text{ or } x \preceq B \}$ as a planar algebra. Then the planar subalgebra of $S$ generated by the two vector spaces $S_A$ and $S_B$, denoted by $S_A \lor S_B$, is naturally isomorphic to the tensor product $S_A \otimes S_B$, $S_A$ and $S_B$ are...
generated by 2-boxes, and \( S = S_A \vee S_B \). In this case, \( S \) is said to be separated by
the biprojections \( A \) and \( B \) as a tensor product.

\[
\Theta_B(a) = \delta^2 B = \delta^2 B = \delta^2 B = \delta^2 B
\]

by the exchange relation of \( A \)

Thus \( a \in \text{Flat}_B(S_A) \) by Proposition 3.10. Any operator in \((S_A)_2\) is a linear sum of
four positive operators, so \((S_A)_2 \subseteq \text{Flat}_B(S_A)\). Take \( A \) to be the planar subalgebra
of \( \text{Flat}_B(S_A) \) generated by 2-boxes. Symmetrically \((S_B)_2 \subseteq \text{Flat}_A(S_B)\). Take \( B \)
to be the planar subalgebra of \( \text{Flat}_A(S_B) \) generated by 2-boxes. Then \( A \vee B \subseteq
\text{Flat}_B(S_A) \vee \text{Flat}_A(S_B) \subseteq S \). On the other hand \( x \leq A \) implies \( x \in (S_A)_2 \), and
\( y \leq B \) implies \( y \in (S_B)_2 \). By assumption, we have \( S \subseteq A \vee B \). So \( A \vee B =
\text{Flat}_B(S_A) \vee \text{Flat}_A(S_B) = S \). Then \( \text{Flat}_B(S_A) = A \) and \( \text{Flat}_A(S_B) = B \) by
counting the dimensions. So they are generated by 2-boxes. By Theorem 3.14,
\( S = \text{Flat}_B(S_A) \vee \text{Flat}_A(S_B) \) is naturally isomorphic to \( \text{Flat}_B(S_A) \otimes \text{Flat}_A(S_B) \).

\[ \square \]

Corollary 3.16. Suppose \( A, B \) are subfactor planar algebras. If \( A \otimes B \) is generated
by 2-boxes, then both \( A \) and \( B \) are generated by 2-boxes.

\[ \square \]

4. Biprojections

In this section, we assume that \( S \) is a subfactor planar algebra.

For crossed product group subfactor planar algebra, all 2-box minimal projections are indexed by group elements, and their coproduct behaves like the group multiplication. Motivated by this fact, we will discuss the coproduct of 2-box positive operators of \( S \), specifically the lattice of the supports of those positive operators, based on the Schur product theorem. For a positive operator \( x \), and \( e \leq x \), the support of \( x^k \) will be increasing. The limit is a projection \( P \) such that
$P \ast P \sim P$. We shall expect $P$ to be a biprojection, viewed as the biprojection generated by $x$.

If we assume $P$ is a projection, $e \preceq p$, $P = P'$ and $P \ast P \sim P$, then by GJS construction [15], we obtain a factor $\mathcal{N} = \text{Gr}_0$ consisting of the linear span of diagrams with all boundary points on the left, and a factor $\mathcal{M} = \text{Gr}_1$ consisting of the linear span of diagrams with all boundary points on the left except one on the top and one on the bottom. Let us construct $\mathcal{P}$ as a cut down of $\mathcal{M}$ by an action of the projection $P$ on the right. By our assumptions, we can show that $P$ is an intermediate subfactor of $\mathcal{N} \subset \mathcal{M}$, and $P$ is a biprojection.

Furthermore we can drop the assumptions $e \preceq p$, $P = P'$, because the support of $P \ast k$ will contain $e$ and $P'$ when $k$ is large enough. This phenomenon is similar to the fact that any group element $g$ of a finite group generates identity and $g^{-1}$. We will not expect this phenomenon without the finite-index condition. A well known result of representation theory of finite groups reduces to this phenomenon and the Stone-Weierstrass theorem. See Theorems 4.12 and 4.21, Proposition 4.22 and the remark after that.

Let $Q$ be a projection. Take the maximal projection $P$ subject to the condition $P \ast Q \preceq Q$; then $P \ast P \sim P$. By our new result, $P$ is a biprojection. If $Q$ behaves like a normalizer under the coproduct, then the planar algebra $S$ is a free product; see Theorem 4.26. Moreover we can find such a normalizer $Q$ by looking at the principal graph; see Lemma 4.5. The combination of these two results will be the key break in our main classification results.

**Theorem 4.1** (Schur product theorem). If $A, B$ are positive operators in $S_2$, then $A \ast B$ is a positive operator.

**Proof.** Set $A = a^*a, B = b^*b$. Then $A \ast B = \delta \Phi((a(1 \boxtimes b))^*(a(1 \boxtimes b)))$, where $\Phi$ is the conditional expectation from $S_3$ to $S_2$. Since $(a(1 \boxtimes b))^*(a(1 \boxtimes b)) \geq 0$ in $S_3$, we have $A \ast B \geq 0$ in $S_2$. While $tr(A \ast B) > 0$, it follows that $A \ast B$ is positive. □

**Remark 4.2.** Diagrammatically

\[
\begin{array}{ccc}
\text{a} & \text{b} \\
\text{a}^* & \text{b}^* \\
\end{array}
\]

\[
A \ast B = \begin{array}{c}
\text{a} \\
\text{a}^* \\
\end{array}
\begin{array}{c}
\text{b} \\
\text{b}^* \\
\end{array}
\geq 0.
\]

**Remark 4.3.** If we consider the subfactor planar algebra as a planar subalgebra of its graph planar algebra \cite{21,24}, then this Schur product theorem reduces to the classical one.

**Definition 4.4.** Suppose $X \in S_2$ is a positive operator, and $X = \sum_{i=1}^{k} C_i P_i$ for some mutually orthogonal minimal projections $P_i \in S_2$ and $C_i > 0$, for $1 \leq i \leq k$. Let us define the rank of $X$ to be $k$, denoted by $r(X)$.

It is easy to see that $r(X)$, the rank of $X$ in $S_2$, is independent of the decomposition. Also, $r(X) = 1$ means $X$ is a multiple of a minimal projection.

**Lemma 4.5.** Suppose $P$ and $Q$ are projections in $S_2$. Then

\[r(P' \ast Q) \leq \dim(QS_3 P),\]

where $QS_3 P = \{QxP | x \in S_3\}$. 

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Proof. Suppose \( P' \ast Q = \sum_{i=1}^{k} C_i R_i, C_i > 0, \) for some mutually orthogonal minimal projections \( R_i \in S_2 \). Take \( v_i = Q(1 \boxdot R_i') P, \) for \( 1 \leq i \leq k \). Then \( v_i \in QS_3 P \). It is easy to check that
\[
tr_3(v_i^* v_i) = tr((P' \ast Q) R_i) > 0, \quad \forall 1 \leq i \leq k;
\]
\[
tr_3(v_i^* v_j) = 0, \quad \forall 1 \leq i, j \leq k, \ i \neq j.
\]
So \( \{v_i\} \) are mutually orthogonal non-zero vectors. Then
\[
r(P' \ast Q) = k \leq \dim(Q(1 \boxdot R_i') P).
\]

If \( P, Q \) are two minimal projections in \( S_2 \), then \( P, Q \) correspond to two vertices in the principal graph, and \( \dim(QS_3 P) \) is the number of length-2 paths between the two vertices. If \( \dim(S_3) \) is small, then for most depth-2 vertices in the principal graph, the number of length-2 paths between them is small. So the rank of their coproduct is small.

Definition 4.6. Suppose \( S \) is a subfactor planar algebra generated by 2-boxes. Let us define the rank of \( S_2 \) to be the maximal number of length-2 paths between a pair of depth-2 vertices in the principal graph, for all pairs of depth-2 vertices.

It was first shown in \[36\] that a trace-one 2-box minimal projection induces a normalizer of the subfactor. Consequently we have the following result.

Proposition 4.7. Suppose \( S \) is a subfactor planar algebra and the rank of \( S_2 \) is 1; then it is a group subfactor planar algebra \( S^G \), for some group \( G \). The converse statement is also true.

The following three lemmas are basic facts about positive operators and planar algebras. They will be used frequently.

Lemma 4.8. Suppose \( A, B, C, D \) are positive operators in \( S_2 \). If \( A \preceq C, B \preceq D \), then \( A \ast B \preceq C \ast D \).

Proof. Because \( S_2 \) is finite dimensional, the spectrum of a positive operator is finite. If \( A \preceq C, B \preceq D \), then \( A \prec \lambda C \) and \( B \prec \lambda D \), for some \( \lambda \) large enough. Thus \( A \ast B \preceq \lambda^2 C \ast D \) by Theorem [4.1]. Then \( A \ast B \preceq C \ast D \). \( \Box \)

Lemma 4.9. Suppose \( C \) and \( D \) are two positive operators. If \( tr(CD) = 0 \), then \( CD = 0 \). Furthermore if \( E \) is a positive operator and \( E \preceq C \), then \( ED = 0 \).

Proof. If \( tr(CD) = 0 \), then \( tr(C^{\frac{1}{2}} D C^{\frac{1}{2}}) = 0 \). Thus \( C^{\frac{1}{2}} D^{\frac{1}{2}} = 0 \). Then \( CD = 0 \). Furthermore if \( E \preceq C \), then \( E \prec \lambda C \) for some \( \lambda > 0 \). Thus \( 0 \leq tr(ED) \leq \lambda tr(CD) = 0 \). Thus \( ED = 0 \). \( \Box \)
Lemma 4.10. If $C, D, E \in S_2$, then
\[ \text{tr}((C \ast D)E') = \text{tr}((D \ast E)C') = \text{tr}((E \ast C)D') \]
\[ = \text{tr}(E'(C \ast D)) = \text{tr}(C'(D \ast E)) = \text{tr}(D'(E \ast C')) \]
\[ = \text{tr}(C(E' \ast D')) = \text{tr}(D(C' \ast E')) = \text{tr}(E(D' \ast C')) \]
\[ = \text{tr}((E' \ast D')C) = \text{tr}((C' \ast E')D) = \text{tr}((D' \ast C')E). \]

Recall that $a'$ is the contragredient of $a$.

Proof. Follows directly from the isotopy and the spherical property of a planar algebra. \(\square\)

Suppose $x$ is a positive operator in $S_2$ and $e \preceq x$. Then the support of $x^{*k}$ is an increasing sequence bounded by $\text{id}$, when $k$ approaches to infinity. We will prove that their union is a biprojection.

Before that, let us prove a lemma. For convenience, we mark the boundary points of a 3-box by 1, 2, 3, 4, 5, 6 from the dollar sign clockwise as:

\[ \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array} \]

Lemma 4.11. Suppose $P, Q, R$ are projections in $S_2$. Then the following are equivalent:

1. $P(1 \Box Q)R = 0$;
2. $(P \ast Q)R = 0$;
3. $P(R \ast Q') = 0$;
4. $\text{tr}(P \ast Q)R = 0$.

Proof. (1) $\implies$ (2) (resp. (2')) follows from adding a cap on the 3-box $P(1 \Box Q)R$ which connects boundary points 4, 5 (resp. 2, 3).

(2), (2') $\implies$ (3) are obvious.

(3) $\implies$ (1) follows from the positivity of the trace and $\text{tr}((P(1 \Box Q)R)^*P(1 \Box Q)R) = \text{tr}(P(1 \Box Q)R) = 0$. \(\square\)

Theorem 4.12. Suppose $P$ is a projection in $S_2$. If $P \ast P \preceq P$, equivalently $(P \ast P)P = P \ast P$, then $P$ is a biprojection.

Proof. Set $Q = \text{id} - P$. The condition $(P \ast P)P = P \ast P$ implies $(P \ast P)Q = 0$. By Lemma 4.11, we have $P(1 \Box P)Q = 0$. Then $P(1 \Box P) = P(1 \Box P)P$. Adding a cap which connects boundary points 2, 3, we obtain

\[ \text{tr}(P) \delta P = P(P \ast P'). \]

Computing the trace, we have \[ \frac{\text{tr}(P)^2}{\delta} = \text{tr}(P(P \ast P')). \] Note that $P \ast P'$ is positive by Theorem 4.11. By Hölder’s inequality, we have

\[ \text{tr}(P(P \ast P')) \leq \|P\|\text{tr}(P \ast P') = \frac{\text{tr}(P)^2}{\delta}. \]
The equality of the inequality implies \( P \ast P' \leq P \). By equation (4.11), we have \( \frac{tr(P)}{\delta}P = P \ast P' \). Note that \((P \ast P')' = P \ast P'\), so \( P = P' \) and \( \frac{tr(P)}{\delta}P = P \ast P \). Then \( P \) is a biprojection. 

In a group, some elements and their inverses will generate a subgroup under group multiplication. Thinking of the coproduct of minimal projections as the group multiplication of group elements, we can define the biprojection generated by an element in \( S_2 \).

For an element \( y \in S_2 \), take the positive operator \( x = e + y^*y + yy^* \). Let \( P \) be the union of the support of \( x^{*k} \), for \( k = 1, 2, \ldots \). Then \( P \ast P \leq P \). So \( P \) is a biprojection. Moreover \( PyP = y \). If \( Q \) is a biprojection such that \( QyQ = y \), then \( QxQ = Q \). So \( x^{*k} \leq Q \). Then \( P \leq Q \). That means \( P \) is the smallest biprojection “containing” \( y \).

**Definition 4.13.** Suppose \( y \) is an element in \( S_2 \) and \( P \) is the smallest biprojection satisfying \( PyP = y \). Then we call \( P \) the biprojection generated by \( y \).

**Lemma 4.14.** Suppose \( A \in S_2 \) is a positive operator and \( P \) is the biprojection generated by \( A \). Then \( P \sim \sum_{i=1}^{k} A^{*i} \) for \( k \) large enough.

**Proof.** Let \( X_k \) be the support of \( \sum_{i=1}^{k} A^{*i} \), for \( k = 1, 2, \ldots \). By Theorem 4.1, \( A^{*i} \) is positive, so \( X_k \leq X_{k+1} \). Since \( S_2 \) is finite dimensional, there is an \( m \) such that \( X_m = X_{m+1} \). Then \( X_m \ast A \preceq X_m \). Thus \( X_m \ast X_m \preceq X_m \). By Theorem 4.12, \( X_m \) is a biprojection. Moreover \( A \preceq P \) implies \( X_m \preceq P \). Thus \( X_m = P \). Then \( P \sim \sum_{i=1}^{k} A^{*i} \), whenever \( k \geq m \). \( \square \)

**Remark 4.15.** Note that the generator of the group \( \mathbb{Z} \) will not generate the whole group under group multiplication. We can only expect the above result for finite index subfactors.

Let \( \iota : S_2 \rightarrow S_3 \) be the inclusion by adding one string to the right, and let \( \mathcal{I}_3 \) be the basic construction ideal of \( S_3 \). Then \( S_2 \cap \mathcal{I}_3 = \{ x \in S_2 | \iota(x) \in \mathcal{I}_3 \} \) is a two sided ideal of \( S_2 \). Thus the support of \( S_2 \cap \mathcal{I}_3 \) is a central projection. A minimal projection of \( S_2 \) belongs to \( S_2 \cap \mathcal{I}_3 \) if and only if the corresponding vertex in the principal graph is not adjacent to a depth-3 vertex.

**Theorem 4.16.** Let \( P \) be the support of \( S_2 \cap \mathcal{I}_3 \). Then \( P \) is a central biprojection and \( S_P \) is depth-2.

**Proof.** By definition \( \iota(P) \in S_3 \), we have \( \iota(P) = \sum_{i} c_i(1 \boxdot id)d_i \), for finitely many \( c_i, d_i \in S_2 \). Then

\[
\iota(P \ast P) = \sum_{i} \left[ \begin{array}{cc} \text{di} & \text{ci} \\ \text{Pi} & \text{ci} \end{array} \right] = \sum_{i, i'} \left[ \begin{array}{cc} \text{di'} & \text{ci'} \\ \text{ci} & \text{ci} \end{array} \right].
\]

Thus \( P \ast P \in S_2 \cap \mathcal{I}_3 \), Then \((P \ast P)P = P \ast P \). By Theorem 4.12 we have that \( P \) is a biprojection. So \( P = P' \).
Let $\iota_P : (S_P)_2 \to (S_P)_3$ be the inclusion by adding one string to the right. Then for any $x \in (S_P)_2$, we have $\iota_P(x)$ is a multiple of $P_i(x)$. Note that $\iota(x) \in I_3$, so

$$\iota_P(x) = \sum f_i(1 \boxtimes id)g_iP,$$

for finitely many $f_i, g_i \in S_2$. Note that $P$ is central, so

$$\iota_P(x) = \sum f_iP(1 \boxtimes id)Pg_i = \sum f_iP(1 \boxtimes P)Pg_i.$$

The last equation follows from the exchange relation of the biprojection $P$. Observe that $f_iP, Pg_i \in (S_P)_2$ and $P(1 \boxtimes P)$ is a multiple of the Jones projection in $(S_P)_3$. Thus $\iota_P(x)$ is in the basic construction ideal of $(S_P)_3$. Therefore $S_P$ is depth-2.  

Corollary 4.17. If $A \in S_2 \cap I_3$, then $A' \in S_2 \cap I_3$.

Proof. If $S \in S_2 \cap I_3$, then $A = PA$. So $A' = A'P$. Then $A' \in S_2 \cap I_3$.  

If $S$ is the planar algebra of an irreducible subfactor $N \subset M$, then the support $P$ of $S_2 \cap I_3$ as a central biprojection corresponds to an intermediate subfactor $P$ of $N \subset M$. Note that the planar algebra of $N \subset P$ is $S_P$. So $P = N \times K$, where $K$ is the dual of $(S_P)_2$ as a Kac algebra.

For a positive operator $A$ in $S_2$, let $P$ be the biprojection generated by $A$. We will show that $\frac{\delta}{tr(P)} 1 \boxtimes P$ is the spectral projection $E_{\{||1 \boxtimes (A + A')||\}}(1 \boxtimes (A + A'))$, where $|| \cdot ||$ is the operator norm. To prove this, we need some lemmas.

Lemma 4.18. Suppose $A$ is a positive operator in $S_2$. Then

$$||1 \boxtimes A|| = \frac{tr(A)}{\delta}.$$  

Proof. Note that $(1 \boxtimes A)e_2 = \frac{tr(A)}{\delta} e_2$, so $||1 \boxtimes A|| \geq \frac{tr(A)}{\delta}$. Recall that $(1 \boxtimes A)^* = 1 \boxtimes A'$, so $(1 \boxtimes A)(1 \boxtimes A')$ is positive. Then

$$||((1 \boxtimes A)(1 \boxtimes A'))|| = \lim_{k \to \infty} tr(((1 \boxtimes A)(1 \boxtimes A'))^k)^{\frac{1}{k}}.$$  

By an isotopy, we have

$$tr(((1 \boxtimes A)(1 \boxtimes A'))^k) = \delta tr((A * A')^k(A * A')) \leq \delta ||A * A'|| tr((A * A')^k) = \delta^2 ||A * A'|| \left(\frac{tr(A)}{\delta}\right)^{2k}.$$  

Thus $||((1 \boxtimes A)(1 \boxtimes A'))|| \leq \left(\frac{tr(A)}{\delta}\right)^2$. Then $||1 \boxtimes A|| \leq \frac{tr(A)}{\delta}$. Therefore $||1 \boxtimes A|| = \frac{tr(A)}{\delta}$.  

Lemma 4.19. Suppose $A$ is a positive operator in $S_2$ and $1 \boxtimes Q$ is a minimal projection in $S_{1,3}$. If $Q * A * Q = \frac{tr(A)}{\delta} Q$, then for any self-adjoint operator $B \in S_2$, $B \preceq A$, we have $Q * B * Q = \frac{tr(B)}{\delta} Q$.

Proof. For a positive operator $C < A$, by Lemma 4.18, we have

$$||1 \boxtimes C|| = \frac{tr(C)}{\delta}; \quad ||1 \boxtimes (A - C)|| = \frac{tr(A - C)}{\delta}. $$
By assumption $1 \square Q$ is minimal; thus
\[
(1 \square Q)(1 \square C)(1 \square Q) = \lambda 1 \square Q, \quad \text{for some } |\lambda| \leq \frac{\text{tr}(C)}{\delta}; \\
(1 \square Q)(1 \square (A - C))(1 \square Q) = \mu 1 \square Q, \quad \text{for some } |\mu| \leq \frac{\text{tr}(A - C)}{\delta}.
\]

If $Q \ast A \ast Q = \frac{\text{tr}(A)}{\delta}Q$, then $(1 \square Q)(1 \square A)(1 \square Q) = \frac{\text{tr}(A)}{\delta}1 \square Q$. Therefore $\frac{\text{tr}(A)}{\delta} = \lambda + \mu$. Then $\lambda = \frac{\text{tr}(C)}{\delta}$ and $\mu = \frac{\text{tr}(A - C)}{\delta}$. Thus $Q \ast C \ast Q = \frac{\text{tr}(C)}{\delta}Q$.

In general, if $B$ is a self-adjoint operator and $B \preceq A$, then $B$ is a linear sum of the $C_i$'s, such that $0 < C_i < A$. So $Q \ast B \ast Q = \frac{\text{tr}(B)}{\delta}Q$. \hfill \square

**Lemma 4.20.** Suppose $X$ is an operator and $P$ is a projection acting on a finite dimensional Hilbert space. If $X$ has an eigenvalue $\|X\|$ and $E_{\|X\|}(X)$ is the corresponding spectral projection, then $P \leq E_{\|X\|}(X)$ if and only if $PX P = \|X\|P$.

**Proof.** If $P \leq E_{\|X\|}(X)$, then obviously $PX P = \|X\|P$.

If $PX P = \|X\|P$, then for any vector $v$ in the Hilbert space such that $Pv = v$, we have $PX v = \|X\|v$. Then
\[
\|X\|^2 \|v\|^2_2 = \|PX v\|^2_2 \leq \|PX v\|^2_2 + \| (I - P) Xv\|^2_2 = \|Xv\|^2_2 \leq \|X\|^2 \|v\|^2_2.
\]

So $(I - P) Xv = 0$. Then $Xv = PX v = \|X\|v$. Therefore $P \leq E_{\|X\|}(X)$. \hfill \square

**Theorem 4.21.** Suppose $A$ is a positive operator in $S_2$, $P$ is the biprojection generated by $A$ and $1 \square Q$ is a minimal projection in $S_{1,3}$. Then the following are equivalent:

1. $Q \ast P \ast Q = \frac{\text{tr}(P)}{\delta}Q$;
2. $Q \ast A \ast Q = \frac{\text{tr}(A)}{\delta}Q$;
3. $Q \ast A = A \ast Q = \frac{\text{tr}(A)}{\delta}Q$;
4. $Q \ast (A + A') \ast Q = \frac{\text{tr}(A + A')}{\delta}Q$.

Consequently $\frac{\delta}{\text{tr}(P)}1 \square P$ is the spectral projection $E_{\{\|1 \square A + A'\|\}}(1 \square (A + A'))$.

**Proof.** (1) $\implies$ (4) $\implies$ (2) follows from Lemma 4.19 and the fact $A \preceq A + A' \preceq P$.

(2) $\implies$ (3): If $Q \ast A \ast Q = \frac{\text{tr}(A)}{\delta}Q$, then $(1 \square Q)(1 \square A)(1 \square Q) = \frac{\text{tr}(A)}{\delta}(1 \square Q)$.

By Lemma 4.18 we have $\|1 \square A\| = \frac{\text{tr}(A)}{\delta}$. By Lemma 4.20 we have $1 \square Q \leq E_{\{\|1 \square A\|\}}(1 \square A)$. Thus $(1 \square Q)(1 \square A) = (1 \square A)(1 \square Q) = \frac{\text{tr}(A)}{\delta}(1 \square Q)$. Then $Q \ast A = A \ast Q = \frac{\text{tr}(A)}{\delta}Q$. 

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By assumption 1 \( \sqsubset Q \) is a projection, thus \( Q \ast Q = Q \). If \( Q \ast A = A \ast Q = \frac{\text{tr}(A)}{\delta} Q \), then \( Q \ast (\sum_{i=1}^{k} A^{ri}) \ast Q = \delta^{-1} \text{tr}(\sum_{i=1}^{k} A^{ri}) Q \). By Lemma 4.14, we have \( P \preceq \sum_{i=1}^{k} A^{ri} \), for \( k \) large enough. So \( Q \ast P \ast Q = \frac{\text{tr}(P)}{\delta} Q \) by Lemma 4.19.

Note that \( \delta \frac{\text{tr}(P)}{\delta} 1 \sqsubset P \) is a projection. Moreover
\[
1 \square Q \leq \frac{\delta}{\text{tr}(P)} 1 \square P \iff \frac{\delta}{\text{tr}(P)} (1 \square Q)(1 \square P)(1 \square Q) = 1 \square Q \\
\iff Q \ast (A + A') \ast Q = \frac{\text{tr}(A + A')}{\delta} Q.
\]

By Lemma 4.20, we have
\[
1 \square Q \leq \frac{\delta}{\text{tr}(P)} 1 \square P \iff 1 \square Q \leq E_{\{1 \square (A + A')\}}(1 \square (A + A')).
\]

Thus \( 1 \square Q \leq \frac{\delta}{\text{tr}(P)} 1 \square P \iff 1 \square Q \leq E_{\{1 \square (A + A')\}}(1 \square (A + A')). \)

So \( \frac{\delta}{\text{tr}(P)} 1 \square P = E_{\{1 \square (A + A')\}}(1 \square (A + A')). \)

**Proposition 4.22.** For a finite group \( G \), take \( S = S^G \) to be a group subfactor planar algebra and \( A \) to be the (minimal) central projection corresponding to an (irreducible) representation \( V \) of \( G \). Then the following are equivalent:

1. the representation \( V \) is faithful;
2. \( E_{\{1 \square A\}}(1 \square A) = e_2 \);
3. \( E_{\{1 \square (A + A')\}}(1 \square (A + A')) = e_2 \);
4. every irreducible representation of \( G \) is contained in some tensor power of \( V \).

**Proof.** \( 1 \iff 2 \) and \( 3 \iff 4 \) are analogies between subfactors and representation theory. \( 2 \iff 2' \) follows from Lemmas 4.18 and 4.20. \( 2' \iff 3 \) follows from Theorem 4.21.

**Remark 4.23.** Note that \( 1 \iff 4 \) is a well known result in representation theory [13]. Observe that \( V \) is faithful is equivalent to the characteristic function of \( V \) separates the identity and other group elements. The characteristic function of the regular representation of the group is a multiple of the Dirac function of the identity. Moreover, the characteristic function of the contragradient of \( V \) is the complex conjugate of the characteristic function of \( V \). Therefore if the condition of \( 4 \) is replaced by the tensor power of \( V \) and its contragredient, then \( 1 \implies 4 \) reduces to the Stone-Weierstrass theorem and the Peter-Weyl theorem. \( 4 \implies 1 \) is because the regular representation is faithful.
Recall that a trace-one minimal projection in \( S_2 \) induces a normalizer \[36\]. Consequently we have the following proposition.

**Proposition 4.24.** Suppose \( P \in S_2 \) is a minimal projection and \( \operatorname{tr}(P) = 1 \). Then \( \delta P * (\cdot) \) is a *-isomorphism of \( S_2 \) as a \( C^*-\)algebra. Consequently if \( Q \in S_2 \) is a minimal projection, then \( \delta P * Q \) is a minimal projection.

We give a direct proof here.

**Proof.** Suppose \( P, x, y \in S_2 \) and \( P \) is a trace-one minimal projection. By Wenzl’s formula, we have

\[
\delta (\cdot) = \delta^2.
\]

Therefore \( \delta P * (\cdot) \) is a *-homomorphism of \( S_2 \) as a \( C^*-\)algebra. Note that \( \delta e * (\cdot) \) is the identity map and \( \delta P' * \delta P = \delta e. \) So \( \delta P' * (\cdot) \) is the inverse of \( \delta P * (\cdot) \). Thus \( \delta P * (\cdot) \) is a *-isomorphism. Consequently if \( Q \in S_2 \) is a minimal projection, then \( \delta P * Q \) is a minimal projection. \( \square \)

**Definition 4.25.** If \( P \) is a central minimal projection in \( S_2 \), such that \( \operatorname{tr}(P) > 1 \) and \( r(P * Q) = 1 \) (resp. \( r(Q * P) = 1 \)), for any minimal projection \( Q \) in \( S_2 \), \( Q \neq P' \), then we call \( P \) a left (resp. right) virtual normalizer. If \( P \) is a left and right virtual normalizer, then we call it a virtual normalizer.

Obviously \( P \) is a left virtual normalizer if and only if \( P' \) is a right virtual normalizer. If the coproduct is commutative, then a left virtual normalizer is a virtual normalizer.

**Theorem 4.26.** Suppose \( S \) is a subfactor planar algebra generated by \( S_2 \). If \( S_2 \) contains a left (or right) virtual normalizer, then either \( S \) is Temperley-Lieb or \( S \) is separated by a non-trivial biprojection as a free product.

**Proof.** When \( \dim(S_2) = 2 \), we have that \( S \) is Temperley-Lieb.

When \( \dim(S_2) \geq 3 \), we assume that \( P \) is a left virtual normalizer, so \( P' \) is central.

By Lemma 4.11, we have \( \operatorname{tr}(e(P' * P)) = \operatorname{tr}((P * e)P) = \frac{\operatorname{tr}(P)}{\delta} \). So \( e \leq P' * P \), and the coefficient of \( e \) in \( P' * P \) is \( \frac{\operatorname{tr}(P)}{\delta} \).

**Case 1.** If \( P' * P \preceq e + P' \), then \( P' * P \sim e + P' \); otherwise \( P' * P \sim e \) implies \( \operatorname{tr}(P) = 1 \), which contradicts the assumption \( \operatorname{tr}(P) > 1 \). Note that \( P' * P \) is self-contragredient, so is \( e + P' \). Then \( P = P' \) and \( e + P \) is a biprojection by Theorem 4.12. Computing the trace, we have \( P * P = \frac{\operatorname{tr}(P)}{\delta} e + \frac{\operatorname{tr}(P) - 1}{\delta} P \). For any minimal projection \( Q \) orthogonal to \( (e + P) \), we have \( r(P * Q) = 1 \) since \( P \) is a left virtual normalizer. Moreover \( P * Q \sim P \), because \( \operatorname{tr}((P * Q)P) = \operatorname{tr}(Q(P * P)) = 0 \) by Lemma 4.10. Then \( r(P * (P * Q)) = 1 \) since \( P \) is a left virtual normalizer. Note that

\[
P * (P * Q) = (P * P) * Q = \frac{\operatorname{tr}(P)}{\delta} e * Q + \frac{\operatorname{tr}(P) - 1}{\delta} P * Q;
\]

thus \( P * Q \sim Q \). Then \( (e + P) * Q \sim Q \). Recall that \( P = P' \), so \( Q' * (e + P) \sim Q' \).

By Theorem 4.20, \( S \) is separated by \( e + P \) as a free product.
Case 2. Otherwise \((id - e - P')(P' * P) \neq 0\), since \(e\) and \(P'\) are central. Moreover \((id - e - P')(P' * P)\) is positive by Theorem 4.1. Let \(S\) be the support of \((id - P')(P' * P)\); we have \(S < id\). Recall that \(e \preceq P' * P\) and \(P' \neq e\), so \(e < S\). We will show that \(S\) is a central biprojection which separates \(B_2\) as a free product.

Note that \(S_2\) is a direct sum of matrix algebras. For each \(M_n(\mathbb{C})\) summand of \(\mathcal{P}_2\), the space \(\mathcal{CP}^n\) of minimal projections is path connected in the norm topology. To show the projection \(S\) is central, it is enough to show that \(SQ_1 = 0 \iff SQ_2 = 0\) when two minimal projections \(Q_1\) and \(Q_2\) are close enough. For a minimal projection \(Q\), \(Q \neq P'\), we have

\[
tr((P' * P)Q) > 0 
\iff tr((P * Q)P) > 0 \quad \text{by Lemma 4.10}
\]

\[
\iff P * Q = \frac{tr(Q)}{\delta} P
\]

since \(P\) is a left virtual normalizer

\[
\iff tr((P * Q)P) = \frac{tr(P)tr(Q)}{\delta}
\]

\[
\iff tr((P' * P)Q) = \frac{tr(P)tr(Q)}{\delta} \quad \text{by Lemma 4.10}
\]

If \(Q_1, Q_2\) are two minimal projections such that \(\|Q_1 - Q_2\| < \frac{tr(P)tr(Q)}{\delta\|P' * P\|}\) and \(Q_i \neq P'\), for \(i = 1, 2\), then

\[
tr((P' * P)Q_1) > 0 \implies tr((P' * P)Q_1) = \frac{tr(P)tr(Q_1)}{\delta}
\]

\[
\implies tr((P' * P)Q_2) > 0.
\]

By Theorem 4.11 we have \(tr((P' * P)Q_i) \geq 0\), for \(i = 1, 2\). Thus

\[
tr((P' * P)Q_2) = 0 \implies tr((P' * P)Q_1) = 0.
\]

Combining with Lemma 4.9 we have

\[
SQ_2 = 0 \implies (S + P')Q_2 = 0
\]

\[
\implies (P' * P)Q_2 = 0
\]

\[
\implies tr((P' * P)Q_2) = 0
\]

\[
\implies tr((P' * P)Q_1) = 0
\]

\[
\implies SQ_1 = 0.
\]

Thus \(S\) is central.

Recall that \(P'\) is central and \(S\) is the support of \((id - P')(P' * P)\), so for any minimal projection \(R, R \leq S\), we have \(R \neq P'\) and \(R \preceq P' * P\). By Lemma 4.10 we have

\[
tr((P * R)P) = tr(R(P' * P)) > 0.
\]

Recall that \(P\) is a left virtual normalizer, so \(P * R \sim P\). Then \(P * S \sim P\). Therefore

\[
P * (S * S) = (P * S) * S \sim P.
\]

Then for any minimal projection \(U, U \preceq S * S\), we have \(P * U \sim P\) by Lemma 4.8. By Lemma 4.10 we have

\[
tr(U(P' * P)) = tr((P * U)P) > 0.
\]
If $U = P'$, then $e \preceq P * P' = P * U \sim P$. This is a contradiction. Otherwise $U \neq P'$; then $UP' = 0$ since $P'$ is central. Then
\[ \text{tr}(U(id - P')(P' * P)) = \text{tr}(U(P' * P)) > 0. \]
Note that $S \sim (id - P')(P' * P)$; by Lemma 4.9, we have $\text{tr}(US) > 0$. Recall that $S$ is central, so $U \leq S$. Thus $S * S \not\leq S$. By Theorem 4.12, $S$ is a biprojection.

To show $S$ is separated by $S$ as a free product, we need the following lemma.

**Lemma 4.27.** For any $y \in \mathcal{P}_2$, $yS = 0$, we have $S * y = y$.

For any element $x \in \mathcal{P}_2$, we have $x = x_1 + x_2$ such that $x_1S = x_1, x_2S = 0$. By Lemma 4.27, $S * x = x_2$. Note that $S$ is central and $S = S'$, so $x_2S = 0$. Then $S * x_2 = x_2'$ by Lemma 4.27. So $x_2 * S = x_2$. Then $Sx_1S = x_1$ and $S * x_2 * S = x_2$. By Theorem 2.20, $S_2$ is separated by $S$ as a free product.

To prove Lemma 4.27, we need the following lemma.

**Lemma 4.28.** Suppose $R_1, R_2 \in S_2$ are two minimal projections orthogonal to $S + P'$. If $P * R_1 \sim P * R_2$, then $R_1 = R_2$.

**Proof of Lemma 4.27.** Recall that $P * S \sim P$, so $S * P' \sim P'$. Suppose $R_1$ is a minimal projection orthogonal to the central projection $S + P'$. By Lemma 4.10, we have
\[ \text{tr}((S * R_1)(S + P')) = \text{tr}(R_1(S * (S + P'))) = 0. \]
Thus $S * R_1$ is orthogonal to $S + P'$. Moreover
\[ P * (S * R_1) = (P * S) * R_1 \sim P * R_1. \]
Suppose $R_2$ is a minimal projection such that $R_2 \preceq S * R_1$. Then $R_2$ is orthogonal to $S + P'$ and
\[ P * R_2 \preceq P * (S * R_1) \sim P * R_1. \]
Recall that $P$ is a left virtual normalizer, so $P * R_1 \sim P * R_2$. By Lemma 4.28, we have $R_2 = R_1$. So $S * R_1 \sim R_1$. Note that for any $y \in S_2$ with $yS = 0$, we have $y$ is a linear sum of such $R_1$’s and $P'$. So $S * y = y$. □

Finally let us prove Lemma 4.28.

**Proof of Lemma 4.28.** Suppose $R_1, R_2 \in S_2$ are two minimal projections orthogonal to $S + P'$. If $P * R_1 \sim P * R_2$, then $P^{*n} * R_1 \sim P^{*n} * R_2$, for $n = 1, 2, \ldots$. Recall that $P$ is a left virtual normalizer. If $P^{*n} * R_1 \sim P'$ and $r(P^{*n} * R_1) = 1$, then $r(P^{*(n+1)} * R_1) = 1$.

**Case 1.** If $P^{*n} * R_1 \sim P'$, $\forall n > 0$, then $r(P^{*n} * R_1) = 1, \forall n > 0$. By Lemma 4.14, $e \preceq P^{*m}$ for some $m > 0$; thus
\[ R_1 \sim P^{*m} * R_1 \sim P^{*m} * R_2 \sim R_2. \]

Then $R_1 = R_2$.

**Case 2.** If $P^{*n} * R_1 \sim P'$, for some $n > 0$, assuming this $n$ is the minimal one, then $r(P^{*j} * R_1) = 1, \forall 1 \leq j \leq n$.

**Subcase 2.1.** If $P' \preceq P^{*k}$, for some $1 \leq k \leq n - 1$, then $e \preceq P * P' \preceq P^{*(k+1)}$ by Lemma 4.8. Thus
\[ R_1 \sim P^{*(k+1)} * R_1 \sim P^{*(k+1)} * R_2 \sim R_2. \]

Then $R_1 = R_2$. 
Subcase 2.2. Otherwise \( P' \sim P^{*j} \) and \( r(P^{*(j+1)}) = 1, \forall 1 \leq j \leq n-1 \). If \( P' \sim P^{*n} \), then \( P \ast P' = P' \ast P \) and \( P' \ast R_1 \sim P^{*n} \ast R_1 \sim P' \). By Lemma [4.10] we have
\[
tr(R_1(P' \ast P)) = tr(R_1(P \ast P')) = tr((P' \ast R_1)P') > 0.
\]
On the other hand, we have \( R_1(S + P') = 0 \) and \( P' \ast P \leq S + P' \), so \( R_1(P' \ast P) = 0 \) by Lemma [4.9]. This is a contradiction.

Subcase 2.3. Otherwise \( P' \sim P^{*j} \) and \( r(P^{*(j+1)}) = 1, \forall 1 \leq j \leq n \). First we will show that \((P')^{*l}\) is central by induction, for \( 1 \leq l \leq n+1 \). The virtually normalizer \( P \) is central, so \( P' \) is central. For \( 1 \leq l \leq n \), suppose \((P')^{*l}\) is central. Take a minimal projection \( V \) such that \( tr(((P')^{*(l+1)})V) > 0 \). By Lemma [4.10] we have
\[
tr(((P')^{*(l+1)})V) = tr(((P')^{*(l+1)})V) > 0.
\]
If \( V = P' \), then \((P')^{*(l+1)} = P'\), and it is central. If \( V \neq P' \), then \( r(P \ast V) = 1 \) since \( P \) is a left virtual normalizer. Recall that \( r((P')^{*l}) = 1 \) and \((P')^{*l}\) is central by induction, so \( P \ast V \sim (P')^{*l} \). Computing the trace, we have
\[
tr(((P')^{*(l+1)})V) = tr(((P')^{*l}(P \ast V)) = C(P)tr(V),
\]
for a positive constant \( C(P) \) only depending on \( P \). Now we have
\[
tr(((P')^{*(l+1)})V) > 0 \implies tr(((P')^{*(l+1)})V) = C(P)tr(V).
\]
Note that the space of minimal projections in the central support of \( V \) is path connected in norm topology. When the minimal projection \( V \) moves continuously, the assumption \( tr(((P')^{*(l+1)})V) > 0 \) always holds. Recall that \( r((P')^{*(l+1)}) = 1 \), so \((P')^{*(l+1)}\) is central. By induction, \((P')^{*(n+1)}\) is central. Recall that \( P^{*n} \ast R_1 \sim P' \); by Lemma [4.10] we have
\[
tr(R_1((P')^{*(n+1)})) = tr((P^{*n} \ast R_1)P') > 0.
\]
Recall that \( r((P')^{*(n+1)}) = 1 \) and \((P')^{*(n+1)}\) is central, so \( R_1 \sim (P')^{*(n+1)} \). Similarly \( R_2 \sim (P')^{*(n+1)} \). So \( R_1 \sim R_2 \). Then \( R_1 = R_2 \). □

Remark 4.29. By Corollary [2.21] \( S^S \) is generated by 2-boxes. In \( S^S \), either \( tr(P) = 1 \) or \( P \) is a virtual normalizer. In the latter case, \( e + P \) is a biprojection and the planar algebra \( S^S \) is separated by \( e + P \) as a free product. Furthermore \((S^S)_{e+P}\) is Temperley-Lieb, and \( P \) is the second Jones-Wenzl projection. After decomposing \( S \) as a free product, a virtual normalizer becomes either a trace-one projection or the second Jones-Wenzl projection of a Temperley-Lieb component in the free product. We leave the details to the reader.

Corollary 4.30. Suppose \( S \) is a subfactor planar algebra with \( \text{dim}(S_2) \geq 3 \) which contains a left (or right) virtual normalizer. Then \( S_2 \) contains a non-trivial biprojection corresponding to an intermediate subfactor.

Proof. Let \( P \subseteq S \) be the subfactor planar algebra generated by \( S_2 \). Then \( S_2 = P_2 \). By Theorem [4.26] \( P_2 \) contains a non-trivial biprojection. □
5. Constructions and decompositions

5.1. Exchange relation planar algebras. In general, it is not easy to show that a subfactor planar algebra is an exchange relation planar algebra. In this section we will give two general constructions of exchange relation planar algebras by the free product and the tensor product. Moreover we will show that the subgroup subfactor planar algebra $S^C_2 \times \mathbb{Z}_p \rtimes \mathbb{Z}_2$ is an exchange relation planar algebra, for an odd prime number $p$. For the classification, we will show how an exchange relation planar algebra decomposes as a free product or a tensor product.

Proposition 5.1. If $S$ is an exchange relation planar algebra, then its dual is an exchange relation planar algebra.

Proof. Recall that the dual of a subfactor planar algebra is given by switching its shading. Thus the dual of an exchange relation planar algebra is still generated by 2-boxes, and its exchange relation is given by the 180° rotation of the adjoint of the original exchange relation. □

Proposition 5.2. If $A \ast B$ is an exchange relation planar algebra, then both $A$ and $B$ are exchange relation planar algebras.

Proof. By Theorem 2.20 both $A$ and $B$ are generated by 2-boxes. Suppose $Q = id \otimes e$, the central biprojection separating $A \ast B$ as a free product. Then $A$ is isomorphic to $(A \ast B)_Q$. For any $x, y \leq Q$, we have

$$(1 \boxtimes x)y = \sum_i c_i (1 \boxtimes d_i) + f_i (1 \boxtimes id)g_i,$$

for finitely many two boxes $c_i, d_i, f_i, g_i$. Then

$$Q(1 \boxtimes Qx)yQ = \sum_i Qc_i (1 \boxtimes Qd_i)Q + Qf_i (1 \boxtimes Q)g_iQ.$$ 

Note that $Qc_iQ, d_iQ, Qf_i, g_iQ \leq Q$, so that is the exchange relation of $(A \ast B)_Q$. Thus $A$ is an exchange relation planar algebra. Considering the duality of exchange relation planar algebras, we have that $B$ is an exchange relation planar algebra. □

Proposition 5.3. If $A$ and $B$ are exchange relation planar algebras, then $A \ast B$ is an exchange relation planar algebra.

Proof. By Corollary 2.19, $A \ast B$ is generated by 2-boxes. Suppose $Q = id \otimes e$ is the central biprojection which separates $A \ast B$ as a free product. Then any 2-box in $A \ast B$ is of the form $x \otimes e + id \otimes y$, for some $x \in A_2$, $y \in B_2$. We need to check the exchange relation for four cases. For any $x_1, x_2 \in A_2$, $y_1, y_2 \in B_2$:

1. the exchange relation of $(1 \boxtimes (x_1 \otimes e))(x_2 \otimes e)$ follows from the exchange relation of $A$;
2. the exchange relation of $(1 \boxtimes (id \otimes y_1))(id \otimes y_2)$ follows from the exchange relation of $B$;
3. $(1 \boxtimes (x_1 \otimes e))(id \otimes y_1) = (id \otimes y_1)(1 \boxtimes (x_1 \otimes e));$
4. $(1 \boxtimes (id \otimes y_1))(x_1 \otimes e) = (id \otimes y_1')(1 \boxtimes id)(x_1 \otimes e).$

Therefore $A \ast B$ is an exchange relation planar algebra. □

Proposition 5.4. If $A \otimes B$ is an exchange relation planar algebra, then both $A$ and $B$ are exchange relation planar algebras.
Proof. By Corollary 3.16 both $A$ and $B$ are generated by 2-boxes. The rest is the same as the proof of Proposition 5.2.

Its converse is not true. The tensor product of two Temperley-Lieb subfactor planar algebras may not be an exchange relation planar algebra. It is a corollary of Theorem 6.5. However a weak version is true.

**Proposition 5.5.** If $A$ is a depth-2 subfactor planar algebra and $B$ is an exchange relation planar algebra, then $A \otimes B$ is an exchange relation planar algebra.

Proof. By Corollary 3.2, $A \otimes B$ is generated by 2-boxes. Take $A = \text{id} \otimes e$ and $B = e \otimes \text{id}$. Note that any 2-box in $S_A \otimes S_B$ is a finite sum of $x^*y$’s for which $x \preceq A, y \preceq B$, so we only need to check the exchange relation for $(1 \otimes (x_1 * y_1))(x_2 * y_2)$, for any $x_1, x_2 \preceq A, y_1, y_2 \preceq B$. Since $B$ is an exchange relation planar algebra, and $(A \otimes B)_B$ is isomorphic to $B$, we have

$$B(1 \Box By_1)y_2B = \sum_i Bc_i(1 \Box Bd_i)B + Bf_i(1 \Box B)g_iB,$$

for finitely many $c_i, d_i, f_i, g_i \preceq B$. Then

$$(1 \Box y_1)y_2 = \sum_i c_i(1 \Box d_i) + f_i(1 \Box id)g_i,$$

by the exchange relation of the biprojection $B$. On the other hand,

$$x_1 = \sum_j s_j(1 \Box id)t_j; \quad x_2 = \sum_k u_k(1 \Box id)v_k,$$

for finitely many $s_j, t_j, u_k, v_k \preceq A$, because $A$ is a depth-2 subfactor planar algebra. Then

Thus $S_A \otimes S_B$ is an exchange relation planar algebra.

**Theorem 5.6.** For an odd prime number $p$, the subgroup subfactor planar algebra $S^\mathbb{Z}_2 \subset \mathbb{Z}_p \rtimes \mathbb{Z}_2$ is an exchange relation planar algebra, where $\mathbb{Z}_p \rtimes \mathbb{Z}_2 = \{a, t | a^p = 1, t^2 = 1, tat = a^{-1}\}$.

Proof. Note that the principal graph of $S^\mathbb{Z}_2 \subset \mathbb{Z}_p \rtimes \mathbb{Z}_2$ is $\xymatrix{ & x_2 \ar[dl]_{y_2} \ar[dr]^{x_1} & \\
& x_1 &}$, with $\frac{p-1}{2}$ depth-2 vertices. So

$$\dim((S^\mathbb{Z}_2 \subset \mathbb{Z}_p \rtimes \mathbb{Z}_2)_2) = \frac{p+1}{2}; \quad \dim((S^\mathbb{Z}_2 \subset \mathbb{Z}_p \rtimes \mathbb{Z}_2)_3) = \left(\frac{p+1}{2}\right)^2 + \left(\frac{p-1}{2}\right)^2.$$
Considering $\mathcal{S}^{\mathbb{Z}_2 \subset \mathbb{Z}_p \times \mathbb{Z}_2}$ as a biprojection cutdown of $\mathcal{S}^{\mathbb{Z}_p \times \mathbb{Z}_2}$, it is easy to check that the minimal projections $e, g_1, \cdots, g_{\frac{p-1}{2}}$ of $(\mathcal{S}^{\mathbb{Z}_2 \subset \mathbb{Z}_p \times \mathbb{Z}_2})_2$ satisfy the relation
\[
\delta g_m * g_n = g_{m+n} + g_{m-n}, \quad \forall \ 0 \leq m, n \leq \frac{p-1}{2},
\]
where $g_0 = 2e$, $g_{m+n} = g_{n-m}$ when $m + n > \frac{p-1}{2}$, and $g_{m-n} = g_{n-m}$ when $m-n < 0$. Note that $(\mathcal{S}^{\mathbb{Z}_2 \subset \mathbb{Z}_p \times \mathbb{Z}_2})_{1,3}$ is isomorphic to $(\mathcal{S}^{\mathbb{Z}_2 \subset \mathbb{Z}_p \times \mathbb{Z}_2})_{2,-}$ as an algebra. Take
\[
\chi_k = \frac{\delta}{p}(1 \boxtimes 2e + \sum_{m=1}^{\frac{p-1}{2}} (\omega^{mk} + \omega^{-mk})(1 \boxtimes g_m)),
\]
where $\omega = e^{\frac{2\pi i}{p}}$, for $k = 0, 1, \cdots, \frac{p-1}{2}$. Then $\chi_0 = 2e_2$ and $\{\chi_k\} = \{\chi_k\}_{k=1}^{\frac{p-1}{2}}$ is the set of minimal projections of $(\mathcal{S}^{\mathbb{Z}_2 \subset \mathbb{Z}_p \times \mathbb{Z}_2})_{1,3}$.

Suppose $\mathcal{S}$ is the planar subalgebra of $\mathcal{S}^{\mathbb{Z}_2 \subset \mathbb{Z}_p \times \mathbb{Z}_2}$ generated by 2-boxes, and $\mathcal{I}_3$ is the basic construction ideal of $\mathcal{S}_3$. Then $\mathcal{S}_3 = \mathcal{I}_3 \oplus \mathcal{S}_3/\mathcal{I}_3$. Let us define $s_3$ to be the support of $\mathcal{S}_3/\mathcal{I}_3$; we have Wenzl’s formula for $g_m$,
\[
s_3g_m = g_m - \frac{\delta}{tr(g_m)(1 \boxtimes id)}g_m.
\]
By a direct computation, we have $tr(s_3g_m \chi_k) \neq 0$, for $1 \leq m, k \leq \frac{p-1}{2}$, and $\{s_3g_m \chi_k\}_{1 \leq m, k \leq \frac{p-1}{2}}$ is a set of pairwise orthogonal vectors. By Proposition 2223, we have
\[
\dim(\mathcal{S}_3/\mathcal{I}_3) \leq (\dim(\mathcal{S}_2) - 1)^2 = \left(\frac{p-1}{2}\right)^2.
\]
Thus $\{s_3g_m \chi_k\}_{1 \leq m, k \leq \frac{p-1}{2}}$ forms a basis of $\mathcal{S}_3/\mathcal{I}_3$, and $\dim(\mathcal{S}_3/\mathcal{I}_3) = \left(\frac{p-1}{2}\right)^2$. That means $\mathcal{S}$ is an exchange relation planar algebra. Moreover, $\mathcal{S}_3 = \mathcal{S}^{\mathbb{Z}_2 \subset \mathbb{Z}_p \times \mathbb{Z}_2}_3$. So they have the same principal graph up to depth-3. Then their principal graphs have to be the same by the restriction of the index. So $\mathcal{S} = \mathcal{S}^{\mathbb{Z}_2 \subset \mathbb{Z}_p \times \mathbb{Z}_2}_3$. \hfill \Box

5.2. Commute relation planar algebras. To classify subfactor planar algebras generated by 2-boxes subject to the condition that the quotient of 3-boxes by the basic construction ideal is abelian, let us prove two lemmas for the construction and the decomposition of such planar algebras via the free product. For convenience we use following notation.

Notation 5.7. A subfactor planar algebra $\mathcal{S}$ is called a commute relation planar algebra if it is generated by 2-boxes and $\mathcal{S}_3/\mathcal{I}_3$ is abelian. Moreover it is of type AN if $\mathcal{S}_2$ is abelian, of type NA if $\mathcal{S}_{1,3}$ is abelian, and of type AA if both $\mathcal{S}_2$ and $\mathcal{S}_{1,3}$ are abelian.

A Temperley-Lieb subfactor planar algebra is a commute relation planar algebra of type AA. A depth-2 subfactor planar algebra is a commute relation planar algebra. Furthermore it is of type AN if and only if it is $\mathcal{S}^G$, for a group $G$; it is of type NA if and only if it is the dual of $\mathcal{S}^G$, for a group $G$; and it is of type AA if and only if it is $\mathcal{S}^G$, for an abelian group $G$.

Proposition 5.8. Suppose $\mathcal{S}$ is a commute relation planar algebra. Then it is an exchange relation planar algebra. Consequently $\mathcal{S}_3$ is generated by $\mathcal{S}_2$ and $\mathcal{S}_{1,3}$ as an algebra.
Proof. If \( S_3/I_3 \) is abelian, then for any \( a, b \in S_2 \), we have \( (1 \square a)b - b(1 \square a) \in I_3 \). Thus

\[
(1 \square a)b = b(1 \square a) + \sum_i f_i (1 \square id)g_i,
\]

for finitely many \( f_i, g_i \in S_2 \).

\[ \square \]

Lemma 5.9. Suppose \( \mathcal{A}, \mathcal{B} \) are commute relation planar algebras of type NA and AN respectively. Then \( \mathcal{S} = \mathcal{A} \ast \mathcal{B} \) is a commute relation planar algebra. Furthermore if \( \mathcal{A} \) is of type AA, then \( \mathcal{S} \) is of type AN, and if \( \mathcal{B} \) is of type AA, then \( \mathcal{S} \) is of type NA.

Proof. Suppose \( \mathcal{A}, \mathcal{B} \) are commute relation planar algebras of types NA and AN respectively. Then by Propositions 5.3 and 5.8, we have that \( \mathcal{S} = \mathcal{A} \ast \mathcal{B} \) is an exchange relation planar algebra. Therefore \( \mathcal{S} \) is generated by \( S_2 \) and \( S_{1,3} \) as an algebra. To prove \( S_3/I_3 \) is abelian, it is enough to prove it for the generating sets \( S_2 \) and \( S_{1,3} \). That is, for any \( x, y \in S_2 \), we need to show:

1. \( (1 \square x)y - y(1 \square x) \in I_3 \);
2. \( xy - yx \in I_3 \);
3. \( (1 \square x)(1 \square y) - (1 \square y)(1 \square x) \in I_3 \).

Note that \( x = a \otimes e + id \otimes b, y = c \otimes e + id \otimes d \), for some \( a, c \in A_2 \) and \( b, d \in B_2 \). Each case splits into four subcases:

1. \( x = a \otimes e, y = c \otimes e \);
2. \( x = id \otimes b, y = id \otimes d \);
3. \( x = a \otimes e, y = id \otimes d \);
4. \( x = id \otimes b, y = c \otimes e \).

Now we use the A,B-colour diagrams to express elements in the free product. For convenience, we use dotted lines to express \( \mathcal{B} \) strings. We omit the labels of the boundary of a diagram, which should be ordered as \( ABBA ABBA \cdots ABBA \).

(1.1) By assumption \( \mathcal{A} \) is a commute relation planar algebra, so

\[
(1 \square a)c - c(1 \square a) = \sum_i f_i (1 \square id)g_i,
\]

for finitely many \( f_i, g_i \in A_2 \). Then

Thus

\[
(1 \square (a \otimes e))(c \otimes e) - (c \otimes e)(1 \square (a \otimes e)) \in I_3.
\]

(1.2) By assumption \( \mathcal{B} \) is a commute relation planar algebra. Similarly we have

\[
(1 \square (id \otimes b))(id \otimes d) - (id \otimes d)(1 \square (id \otimes b)) \in I_3.
\]

(1.3) \( (1 \square (a \otimes e))(id \otimes d) - (id \otimes d)(1 \square (a \otimes e)) = 0. \)

(1.4) Note that \( c \otimes e \in I_3 \); thus

\[
(1 \square (id \otimes b))(c \otimes e) - (c \otimes e)(1 \square (id \otimes b)) \in I_3.
\]
(2.1) By assumption $\mathcal{A}$ is a commute relation planar algebra, so

$$ac - ca = \sum_j f_j (1 \boxtimes id) g_j,$$

for finitely many $f_j, g_j \in A_2$. Then

$$-(1 \boxtimes e)(c \boxtimes e) - (c \boxtimes e)(1 \boxtimes e) \in I_3.$$

(2.2) By assumption $\mathcal{B}$ is of type AN, so $bd - db = 0$. Thus $(id \otimes b)(id \otimes d) - (id \otimes b)(id \otimes d) = 0.$

(2.3) $(a \otimes e)(id \otimes d) - (id \otimes d)(a \otimes e) = a \otimes cd - a \otimes de = 0.$

(2.4) $(id \otimes b)(c \otimes e) - (c \otimes e)(id \otimes b) = c \otimes be - c \otimes eb = 0.$

Considering the $180^\circ$ rotation, the proof of (3) is the same as that of (2) while assuming $\mathcal{A}$ is of type NA.

Therefore $\mathcal{S}$ is a commute relation planar algebra.

Furthermore, if $\mathcal{A}$ is of type AA, then $A_2$ is abelian. Recall that if $\mathcal{B}$ is of type AN, then $A_2$ is abelian. So $\mathcal{A} \ast B_2$ is abelian and $A \ast B$ is of type AN. If $\mathcal{B}$ is of type AA, similarly $A \ast B$ is of type NA by duality.

**Lemma 5.10.** Suppose $\mathcal{A}, \mathcal{B}$ are subfactor planar algebras with circle parameters greater than 1. If $\mathcal{S} = \mathcal{A} \ast \mathcal{B}$ is a commute relation planar algebra, then $\mathcal{A}, \mathcal{B}$ are commute relation planar algebras of type NA and AN respectively. Furthermore if $\mathcal{S}$ is of type AN, then $\mathcal{A}$ is of type AA, and if $\mathcal{S}$ is of type NA, then $\mathcal{B}$ is of type AA.

**Proof.** Because $\mathcal{S}_2$ is a commute relation planar algebra, for any $a, c$ in $A_2$, we have

$$-(1 \boxtimes a)c - c(1 \boxtimes a) = \sum_i f_i (1 \boxtimes id) g_i.$$
Similarly
\[ \begin{array}{c|c|c|c|c}
 & a & c & & \\
| & & & & \\
| & & & & \\
\end{array} \begin{array}{c|c|c|c|c}
 & a & c & & \\
| & & & & \\
| & & & & \\
\end{array} = \sum_j \begin{array}{c|c|c|c|c}
 & x_j & & & \\
| & & & & \\
| & & & & \\
\end{array} \begin{array}{c|c|c|c|c}
 & y_j & & & \\
| & & & & \\
| & & & & \\
\end{array} \]
for finitely many \( x_j, y_j \in S_2 \). Moreover
\[(id \otimes e) \begin{array}{c|c|c|c|c}
 & a & c & & \\
| & & & & \\
| & & & & \\
\end{array} \begin{array}{c|c|c|c|c}
 & a & c & & \\
| & & & & \\
| & & & & \\
\end{array} (id \otimes e) = \begin{array}{c|c|c|c|c}
 & a & c & & \\
| & & & & \\
| & & & & \\
\end{array} \]
so we may assume that \( x_j, y_j \preceq id \otimes e \). Then
\[ \begin{array}{c|c|c|c|c}
 & a & c & & \\
| & & & & \\
| & & & & \\
\end{array} \begin{array}{c|c|c|c|c}
 & a & c & & \\
| & & & & \\
| & & & & \\
\end{array} = \sum_j \begin{array}{c|c|c|c|c}
 & f_j & & & \\
| & & & & \\
| & & & & \\
\end{array} \begin{array}{c|c|c|c|c}
 & g_j & & & \\
| & & & & \\
| & & & & \\
\end{array} , \]
for finitely many \( f_j, g_j \in A_2 \). Therefore
\[ ac - ca = \sum_j f_j (1 \square \text{id}) g_j. \]

If \( A_{1,3} \) is not abelian, then there is a system of matrix units \( \{ u_{11}, u_{12}, u_{21}, u_{22} \} \) in \( A_{1,3} \). By assumption the index of \( B \) is greater than 1, so there is a projection \( p \in B_2 \) orthogonal to the Jones projection. Then \( \{ \begin{array}{c|c|c|c|c}
 & P & & & \\
| & & & & \\
| & & & & \\
\end{array} \begin{array}{c|c|c|c|c}
 & U & & & \\
| & & & & \\
| & & & & \\
\end{array} \}_{1 \leq i,j \leq 2} \) forms a system of matrix units in \( S_3 \), and they are in the orthogonal complement of \( I_3 \). So \( S_3/I_3 \) is not abelian. This is a contradiction.

Therefore \( A \) is a commute relation planar algebra of type NA. Furthermore if \( S \) is of type AN, then \( S_2 \) is abelian. Note that \( A \) is isomorphic to \( S_{id \otimes e} \), so \( A_2 \) is abelian. Then \( A \) is of type AA.

Considering the duality, we have that \( B \) is a commute relation planar algebra of type AN. Furthermore if \( S \) is of type NA, then \( B \) is of type AA.

### 6. Classifications

Recall that the classification of a subfactor planar algebra generated by a non-trivial 2-box are given by: \( S^{Z_2} \), \( TL \ast TL \), for at most 12 dimensional 3-boxes \[11\]; \( S^{Z_2 \times Z_2} \), for 13 dimensional 3-boxes \[12\]; BMW’s, precisely one family from quantum \( Sp(4, \mathbb{R}) \) and one from quantum \( O(3, \mathbb{R}) \), for 14 dimensional 3-boxes \[8\].

Now let us prove the main classification results.

**Lemma 6.1.** Suppose \( S \) is a subfactor planar algebra generated by 2-boxes with \( \dim(S_2) = 4 \). If \( S \) has a biprojection \( P \) and \( r(P) = 3 \), then \( S \) is a free product.

**Proof.** If \( S \) has a biprojection \( P \) and \( r(P) = 3 \), then \( \dim((S_P)_2) = 3 \). Note that \( \dim((S_P)_2) \geq 2 \), so
\[ \dim((S_P \ast S^P)_2) = \dim((S_P)_2) + \dim((S^P)_2) - 1 \geq 4. \]
By the assumption \( \dim(S_2) = 4 \), we have \( S_2 = (S_P \ast S^P)_2 \). Note that \( S \) is generated by 2-boxes, so \( S = S_P \ast S^P \). \( \square \)
Lemma 6.1. Suppose $S$ is a subfactor planar algebra generated by 2-boxes with $\dim(S_2) = 4$. If $S$ has two non-trivial biprojections, then $S$ is either $TL \otimes TL$ or $TL \otimes TL$.

Proof. Suppose $S$ has two non-trivial biprojections $P, Q$. If $r(P) = 3$, then by Lemma 6.1, $S$ is separated by $P$ as a free product. By Theorem 2.22, $Q$ is a subprojection of $P$. So $S = TL \otimes TL \otimes TL$. If $S \neq TL \otimes TL \otimes TL$, then $r(P) = 2$. Similarly $r(Q) = 2$. So $PQ = e$. Applying the same argument to the dual of $S$, we have that $P \otimes Q$ is a multiple of $id$. Note that the 2-box space of the dual of $S$ is 4 dimensional, so it is abelian. That means the coproduct of $S_2$ is commutative. By Theorem 4.15 we have $S = TL \otimes TL$. □

Lemma 6.3. Suppose $S$ is a subfactor planar algebra with $\dim(S_2) = 4$. If $S_3/I_3$ is abelian, then $S$ is either depth-2, or a free product.

Proof. If $S_3/I_3$ is abelian, then in the principal graph, each depth-3 vertex only connects with one depth-2 vertex. That means $\dim(S_2) = 4$, so $\dim(S_2)$ is abelian. That means there is only one edge between each depth-2 vertex and the depth-1 vertex. So there is only one length-2 path between any two different depth-2 vertices. Let $e$, $P_1$, $P_2$, $P_3$ be the minimal projections of $S_2$; then $\dim(P_1S_3P_2) = 1$, whenever $1 \leq i, j \leq 3$ and $i \neq j$. By Lemma 4.5, we have $r(P_i' \ast P_j) = 1$. Thus either $\text{tr}(P_i) = 1$, or $P_i'$ is a virtual normalizer. If $\text{tr}(P_i) = 1$, for $i = 1, 2, 3$, then $S_2$ is depth-2. Otherwise one of them is a virtual normalizer. By Theorem 4.26, $S$ is a free product. □

Lemma 6.4. Suppose $S$ is a subfactor planar algebra with $\dim(S_2) = 4$. If $S_3/I_3$ is a direct sum of $M_{2 \times 2}$ and the C’s, then $S$ is either a free product or $S^\mathbb{Z}_2 \otimes TL$.

Proof. Suppose $e, P_1, P_2, P_3$ are minimal projections of $S_2$. Let $a_i, i = 1, 2, 3$, be the depth-2 vertex in the principal graph corresponding to $P_i$. If $S_3/I_3$ is a direct sum of $M_{2 \times 2}$ and the C’s, then there is one depth-3 vertex $v$ with multiplicity 2 in $e$ and $P_3$ be the minimal projections of $S_2$; then $\dim(P_1S_3P_2) = 1$. By Lemma 4.5, we have $r(P_i' \ast P_j) = 1$. Therefore either $P_i'$ is a left virtual normalizer or $\text{tr}(P_i) = 1$. If $P_i'$ is a left virtual normalizer, then $S$ is a free product by Theorem 4.26. Otherwise $\text{tr}(P_i) = 1$. By Theorem 4.16, the support $T$ of $S_2 \cap I_3$ is a biprojection. If $r(T) = 3$, then $S$ is a free product by Lemma 6.1. Otherwise $T = e + P_1$.

To sum up, if $S$ is not a free product, then it has a trace-2 biprojection. Applying the same argument to the dual of $S$, we also have a trace-2 biprojection in the dual. That gives a biprojection $Q$ in $S$, and $\text{tr}(Q) = \frac{\delta^2}{2}$. If $P = Q$, then $\delta^2 = 4$. This is a contradiction. Otherwise $S = TL \otimes TL$ by Theorem 6.2. Furthermore $\text{tr}(e + P_1) = 2$, so $S = S^\mathbb{Z}_2 \otimes TL$. □

Theorem 6.5. Suppose $S$ is an exchange relation planar algebra with $\dim(S_2) = 4$. Then $S$ is one of the following:

1. $S^\mathbb{Z}_2$, or $S^\mathbb{Z}_2 \otimes \mathbb{Z}_2$;
2. $A^*TL$ or $TL^*A$, for an exchange relation planar algebra $A$ with $\dim(A_2) = 3$;
3. $S^\mathbb{Z}_2 \otimes TL$;
4. $S^\mathbb{Z}_2 \otimes \mathbb{Z}_2 \times \mathbb{Z}_2$. 

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Recall that an exchange relation planar algebra $\mathcal{A}$ with $\dim(\mathcal{A}_2) = 3$ is the same as a subfactor planar algebra $\mathcal{A}$ generated by a non-trivial 2-box with $\dim(\mathcal{A}_3) \leq 13$; see the arguments at the end of section 2.5.

**Proof.** Suppose $\mathcal{S}$ is an exchange relation planar algebra with $\dim(\mathcal{S}_2) = 4$.

1. If $\mathcal{S}$ is depth-2, then it is $S^{Z_4}$ or $S^{Z_2 \oplus Z_2}$. Conversely a depth-2 subfactor planar algebra is an exchange relation planar algebra. Therefore we obtain class (1), $S^{Z_4}$, and $S^{Z_2 \oplus Z_2}$.
2. If $\mathcal{S}$ is separated by a non-trivial biprojection $Q$ as a free product, then by Proposition 5.2 both $S^Q$ and $S_Q$ are exchange relation planar algebras. Note that

$$\dim(S^Q)_2 + \dim(S_Q)_2 = \dim(S_2) + 1 = 5,$$

so one of them is $TL$ and the other is an exchange relation planar algebra $\mathcal{A}$ with $\dim(\mathcal{A}_2) = 3$. Conversely by Proposition 5.3 the free product of an exchange relation planar algebra $\mathcal{A}$ with $\dim(\mathcal{A}_2) = 3$ and $TL$ is an exchange relation planar algebra. Thus we obtain class (2).

3. By Proposition 5.5, the tensor product of $S^{Z_2}$ and $TL$ is an exchange relation planar algebra $\mathcal{S}$ with $\dim(S_2) = 2 \times 2 = 4$.

4. By Theorem 5.6 the subgroup subfactor planar algebra $S^{Z_2} \subset Z_2 \times Z_2$ is an exchange relation planar algebra $\mathcal{S}$ with $\dim(S_2) = \frac{7 + 1}{2} = 4$.

By Proposition 2.24 we have $\dim(S_3/I_3) \leq 9$. We need to consider the following four cases:

(a) $S_3/I_3$ is abelian;
(b) $S_3/I_3$ is a direct sum of $M_{2 \times 2}$ and the $\mathbb{C}$’s.
(c) $S_3/I_3$ contains $M_{2 \times 2} \oplus M_{2 \times 2}$;
(d) $S_3/I_3 = M_{3 \times 3}$;

Case (a): If $S_3/I_3$ is abelian, then $\mathcal{S}$ is in class (1) or (2) by Lemma 6.3.

Case (b): If $S_3/I_3$ is a direct sum of $M_{2 \times 2}$ and the $\mathbb{C}$’s, then $\mathcal{S}$ is in class (2) or (3) by Lemma 6.4.

To prove cases (c) and (d), let us prove some general results for exchange relation planar algebras. Note that $\dim(S_2) = 4$, so $S_2$ and $S_{1,3}$ are abelian. Suppose $e_i, P_1, P_2, P_3$ are mutually orthogonal minimal projections of $S_2$, and $e_2, 1 \square Q_1, 1 \square Q_2$, and $1 \square Q_3$ are mutually orthogonal minimal projections of $S_{1,3}$. Then

$$(1 \square P_i)(1 \square Q_j) = \lambda_{i,j} 1 \square Q_j, \text{ for some } \lambda_{i,j} \in \mathbb{C}.$$

So $P_i \ast Q_j = \lambda_{i,j} Q_j$. Let $I_3$ be the basic construction ideal of $S_3$; then $S_3 = I_3 \oplus S_3/I_3$. Let us define $s_3$ to be the support of $S_3/I_3$. Then $\{s_3(1 \square Q_j)P_1\}_{1 \leq i, j \leq 3}$ are pairwise orthogonal. By assumption $\mathcal{S}$ is an exchange relation planar algebra, so $S_3/I_3$ is generated by $\{s_3P_i(1 \square Q_j)\}_{1 \leq i, j \leq 3}$ as a linear space. Then we have $\dim(s_3(1 \square Q_j)S_3P_i) \leq 1$ and

$$\dim(s_3(1 \square Q_j)S_3P_i) = 1 \iff s_3P_i(1 \square Q_j) \neq 0.$$
It is easy to check that
\[ s_3P_i(1 \square Q_j) = 0 \iff tr(s_3P_i(1 \square Q_j)) = 0 \]
\[ \iff tr((P_i - \frac{\delta}{tr(P_i)}P_i(1 \square id)P_i)(1 \square Q_j)) = 0 \]
\[ \iff |\lambda_{i,j}| = \frac{tr(P_i)}{\delta}. \]

Note that \( P_i \) corresponds to a depth-2 vertex in the principal graph, denoted by \( a_i \), \( 1 \square Q_j \) corresponds to a depth-2 vertex in the dual principal graph, denoted by \( b_j \), and \( \dim(s_3(1 \square Q_j)S_3P_i)) \) is the number of length-2 paths from \( a_i \) to \( b_j \) passing through a depth-3 vertex of the principal graph in the 4-partite principal graph. Thus \( \dim(s_3(1 \square Q_j)S_3P_i)) \leq 1 \) implies that the number of edges connecting a depth-3 vertex of the principal graph with \( a_i \) (or \( b_j \)) is at most 1.

Case (c): If \( S_3/T_3 \) contains \( M_{2 \times 2} \oplus M_{2 \times 2} \), then there are two depth-3 vertices with multiplicity 2 in the principal graph. Thus there is a vertex \( a_i \) which connects with both of them. Moreover there is a vertex \( b_j \) which connects with both of them in the 4-partite principal graph. Then there are two length-2 paths from \( a_i \) to \( b_j \) passing through a depth-3 vertex of the principal graph. This is a contradiction.

Case (d): If \( S_3/T_3 = M_{3 \times 3} \), then \( \dim(S_3/T_3) = 9 \). Thus \( s_3P_i(1 \square Q_j) \neq 0 \), for any \( 1 \leq i, j \leq 3 \). Then \( |\lambda_{i,j}| \neq \frac{tr(P_i)}{\delta} \). By Theorem 4.10, the biprojection generated by \( P_i \) is \( id \). In the principal graph, there is only one depth-3 vertex, and it connects with each depth-2 vertex with one edge. Thus
\[ tr(P_i) = tr(P_2) = tr(P_3) > 1; \]
\[ \dim(P_1S_3P_2) = 2, \forall 1 \leq i, j \leq 3. \]

Take \( c = tr(P_1) \). By Lemma 4.5 we have
\[ r(P_iP_j) \leq 2, \forall 1 \leq i, j \leq 3. \]

Note that among the three projections, at least one is self-contragredient; we assume that \( P_1 = P_1' \). By Lemma 4.10, we have
\[ tr((P_1 \ast P_1)e) = tr((e \ast P_1)P_1) = \frac{c}{\delta}. \]
So the coefficient of \( e \) in \( P_1 \ast P_1 \) is \( \frac{c}{\delta} \). Recall that \( r(P_1 \ast P_1) \leq 2 \). Computing the trace, we have
\[ P_1 \ast P_1 = \frac{c}{\delta} e + \frac{c - 1}{\delta} P_k, \text{ for some } 1 \leq k \leq 3. \]

By Theorem 4.12, we have \( k \neq 1 \); otherwise the biprojection generated by \( P_1 \) is \( e + P_1 \). Without loss of generality, we assume that \( k = 2 \). Then \( P_2 = P_2' \). By Lemma 4.10,
\[ tr((P_1 \ast P_2)P_1) = tr(P_2(P_1 \ast P_1)) = \frac{c(c - 1)}{\delta}. \]
So the coefficient of \( P_1 \) in \( P_1 \ast P_2 \) is \( \frac{c - 1}{\delta} \). Recall that \( r(P_1 \ast P_2) \leq 2 \). Computing the trace, we have
\[ P_1 \ast P_2 = \frac{c - 1}{\delta} P_1 + \frac{1}{\delta} P_l, \text{ for some } 2 \leq l \leq 3. \]
By Theorem 4.12, we have $l \neq 2$; otherwise the biprojection generated by $P_1$ is $e + P_1 + P_2$. So

$$ P_1 * P_2 = \frac{c - 1}{\delta} P_1 + \frac{1}{\delta} P_3. $$

Applying Lemma 4.10 and computing the trace again, we have

$$ P_1 * P_3 = \frac{1}{\delta} P_2 + \frac{c - 1}{\delta} P_3. $$

By Lemma 4.10

$$ tr((P_2 * P_2)P_1) = tr((P_1 * P_2)P_2) = 0. $$

So $P_1 \geq P_2 * P_2$. Recall that $r(P_2 * P_3) \leq 2$ and $e \leq P_2 * P_2$. If $P_2 * P_2 \cong e + P_2$, then $e + P_2$ is a biprojection. This is a contradiction. So $P_2 * P_2 \sim e + P_3$. Computing the trace, we have

$$ P_2 * P_2 = \frac{c}{\delta} e + \frac{c - 1}{\delta} P_3. $$

By a similar argument, we have

$$ P_2 * P_3 = \frac{1}{\delta} P_1 + \frac{c - 1}{\delta} P_3; $$

$$ P_3 * P_3 = \frac{c}{\delta} e + \frac{c - 1}{\delta} P_1. $$

Therefore

$$ P_1 * (P_1 * P_2) = \frac{c(c - 1)}{\delta^2} e + \frac{(c - 1)^2}{\delta^2} P_2 + \frac{1}{\delta^2} P_2 + \frac{c - 1}{\delta^2} P_3; $$

$$ (P_1 * P_1) * P_2 = \frac{c}{\delta^2} P_2 + \frac{c(c - 1)}{\delta^2} e + \frac{(c - 1)^2}{\delta^2} P_3. $$

Comparing the coefficient of $P_3$, we obtain $c = 2$. By Theorem 2.26, we have that $S$ is isomorphic to $S_{Z_2 \subset Z_7 \times Z_2}$.

**Theorem 6.6.** Suppose $S$ is a subfactor planar algebra generated by 2-boxes with $\dim(S_{2, \pm}) = 4$ and $\dim(S_{3, \pm}) \leq 23$. Then $S$ is one of the following:

1. $S^{Z_4}$ or $S^{Z_2 \oplus Z_2}$;
2. $A * TL$ or $TL * A$, where $A$ is generated by a non-trivial 2-box with $\dim(A_{3, \pm}) \leq 13$;
3. $B * S^{Z_2}$ or $S^{Z_2} * B$, where $B$ is generated by a non-trivial 2-box with $\dim(A_{3, \pm}) \leq 14$;
4. $S^{Z_2} \otimes TL$.

**Proof.** Suppose $S$ is a subfactor planar algebra generated by 2-boxes with $\dim(S_{2, \pm}) = 4$ and $\dim(S_{3, \pm}) \leq 23$, and $I_3$ is the basic construction ideal of $S_3$. Then $\dim(S_3/I_3) \leq 23 - 4^2 = 7$. Thus either $S_3/I_3$ is abelian or $S_3/I_3$ is a direct sum of $M_{2x2}$ and the $C$’s. By Lemmas 6.3 and 6.4, either $S$ is one of $S^{Z_4}$, $S^{Z_2 \oplus Z_2}$ and $S^{Z_2} \otimes TL$, corresponding to classes (1) and (3) in the statement, or $S$ is separated by a non-trivial biprojection $Q$ as a free product. In the latter case, both $S^Q$ and $S_Q$ are generated by 2-boxes. Counting the dimensions of 2-boxes, we have

$$ \dim((S^Q)_2) - 1 + \dim((S_Q)_2) - 1 = \dim(S_2) - 1 = 3. $$

Thus one of them is $TL$, and the other is a subfactor planar algebra generated by 2-boxes with 3 dimensional 2-boxes. Furthermore counting the dimensions of 3-boxes, we have

$$ \dim((S^Q)_3) + \dim((S_Q)_3) = 3^2 + 2^2 - (3 - 1)(2 - 1) + \dim(S_2) - 4^2 \leq 18. $$
Thus either the Templey-Lieb one is \( S^3 \) and the other has at most 14 dimensional 3-boxes, or the other one has at most 13 dimensional 3-boxes. They correspond to classes (2a) and (2b) in the statement.

**Theorem 6.7.** Suppose \( S \) is a subfactor planar algebra generated by 2-boxes such that \( S_3/I_3 \), \( S_2 \) and \( S_{1,3} \) are abelian. Then it is a free product of Templey-Lieb subfactor planar algebras and group subfactor planar algebras for abelian groups. The converse statement is also true.

**Proof.** By assumption \( S_3/I_3 \) is abelian, so the multiplicity of a depth-3 vertex in the principal graph is 1. Furthermore \( S_2 \) is abelian, so the multiplicity of a depth-2 vertex in the principal graph is 1. Then for two distinct depth-2 vertices, there is only one length-2 path between them. Thus \( \dim(P_iS_3P_j) = 1 \) in \( S_3 \), for any two distinct minimal projections \( P_i, P_j \) of \( P_2 \). By Lemma 4.5, we have \( r(P_j^*P_i) = 1 \). Thus either \( r(P_i) = 1 \) or \( P_i^* \) is a virtual normalizer. If \( r(P_i) = 1 \), for any minimal projection \( P_i \) in \( S_2 \), then \( S \) is depth-2. By assumption it is of type AA, so it is a group subfactor planar algebra for some abelian group. Otherwise \( S \) contains a virtual normalizer. By Theorem 4.26, either \( S \) is Templey-Lieb or \( S \) is separated by a non-trivial biprojection as a free product \( A \ast B \). In the latter case, both \( A \) and \( B \) have smaller index. By Lemma 5.10, both \( A \) and \( B \) are commute relation planar algebras of type AA. Then we may decompose them again until they are either depth-2 or Templey-Lieb. Therefore \( S \) is a free product of Templey-Lieb subfactor planar algebras and depth-2 subfactor planar algebras, and each of them is of type AA.

Conversely both Templey-Lieb subfactor planar algebras and group subfactor planar algebras for abelian groups are commute relation planar algebras of type AA. By Lemma 5.9, their free product is a commute relation planar algebra of type AA.

**Theorem 6.8.** Suppose \( S \) is a subfactor planar algebra generated by 2-boxes and \( S_3/I_3 \) is abelian. Then: \( S \) is either depth-2 or the free product \( A_1 \ast A_2 \ast \cdots \ast A_n \), such that \( A_1 \) is Templey-Lieb or the dual of \( S^{G_1} \), for a group \( G_1 \); \( A_n \) is Templey-Lieb or \( S^{G_n} \), for a group \( G_n \); \( A_m \), for \( 1 < m < n \), is Templey-Lieb or \( S^{G_m} \), for an abelian group \( G_m \). The converse statement is also true.

In this general case, we still want to show the existence of a virtual normalizer in a commute relation planar algebra, whenever it is not depth-2. Then we may decompose a commute relation planar algebra as a free product of commute relation planar algebras, until they are either Templey-Lieb or depth-2.

**Notation 6.9.** Recall that \( S_2 \cap I_3 \) is a two sided ideal of \( S_2 \). Let us define \( S_2/I_3 \) to be the orthogonal complement of \( S_2 \cap I_3 \) in \( S_2^* \). Then \( S_2 = (S_2 \cap I_3) \oplus S_2/I_3 \).

**Lemma 6.10.** Suppose \( S \) is a commute relation planar algebra. If \( S \) is not depth-2, then each minimal projection \( P_i \) in \( S_2/I_3 \) is a virtual normalizer.

Based on Lemma 6.10 the proof of Theorem 6.8 is similar to the proof of Theorem 6.7.

**Proof of Theorem 6.8** Suppose \( S \) is a commute relation planar algebra. If it is not depth-2, then by Lemma 6.10 it contains a virtual normalizer. If \( S \) is not Templey-Lieb, then by Theorem 4.26 and Lemma 5.10 \( S \) is a free product of two commute relation planar algebras with smaller index. Repeating this process, we
have \( S = A_1 \ast A_2 \ast \cdots \ast A_n \) such that each \( A_i \) is either Temperley-Lieb or depth-2. By Lemma 5.10, \( A_1 \) is of type NA and \( A_n \) is of type AN; the others are of type AA.

Conversely by Lemma 5.9, their free product is a commute relation planar algebra.

To prove Lemma 6.10 let us prove some basic results first.

**Lemma 6.11.** Suppose \( S \) is a commute relations planar algebra. If \( P_i, P_j \) are distinct minimal projections in \( S_2/I_3 \), then \( r(P_i \ast P_j) = 1 \).

**Proof.** Suppose \( P_i, P_j \) are distinct minimal projections in \( S_2/I_3 \) and \( v_i, v_j \) are the corresponding depth-2 vertices in the principal graph. By assumption \( S_3/I_3 \) is abelian, so \( P_i, P_j \) are central in \( S_2 \). Then the multiplicity of \( v_i, v_j \) is 1. Note that the multiplicity of a depth-3 vertex is 1, so there is only one length-2 path between \( v_i \) and \( v_j \). By Lemma 4.5 we have \( r(P_i \ast P_j) = 1 \). \( \square \)

We want to show that \( r(P_i \ast P_j) = 1 \) whenever \( P_i \) is a minimal projection in \( S_2/I_3 \) and \( P_j \) is a minimal projection in \( S_2 \cap I_3 \). If \( tr(P_j) = 1 \), then \( r(P_i \ast P_j) = 1 \) is a minimal. If \( tr(P_j) > 1 \), we will see \( P_i \ast P_j \sim P_i \).

**Lemma 6.12.** Suppose \( S \) is a commute relations planar algebra, \( P_i, P_j, P_k \) are minimal projections of \( S_2 \), \( P_i \in S_2/I_3 \), and \( P_j \in S_2 \cap I_3 \). If \( P_k \preceq P_i \ast P_j \), then \( P_k \in S_2/I_3 \).

**Proof.** Suppose \( S \) is a commute relations planar algebra, \( P_k \in S_2 \cap I_3 \), and \( P_k \preceq P_i \ast P_j \). Then by Lemma 4.10

\[
tr(P_k (P_i \ast P_j)) = tr((P_i \ast P_j)P_k) > 0.
\]

However \( P_j, P_k \in S_2 \cap I_3 \); by Theorem 4.16 we have \( P_k \ast P_j' \in S_2 \cap I_3 \). So \( P_i (P_k \ast P_j') = 0 \). This is a contradiction. \( \square \)

**Lemma 6.13.** Suppose \( S \) is a commute relations planar algebra, \( P_i, P_j, P_k \) are minimal projections of \( S_2 \), \( P_i, P_k \in S_2/I_3 \), and \( P_i \neq P_k \). If \( tr(P_j (P_i \ast P_k)) > 0 \), then \( tr(P_j (P_i \ast P_k)) = \frac{tr(P_i)tr(P_k)}{\delta} \). Furthermore, \( P_j \in S_2/I_3 \) or \( tr(P_j) = 1 \).

**Proof.** If \( tr(P_j (P_i \ast P_k)) > 0 \), then \( tr((P_i \ast P_j)P_k) > 0 \) by Lemma 4.10. Thus \( (P_i \ast P_j)P_k = \lambda P_k \), for some \( \lambda > 0 \). Note that \( I_3 \) is the basic construction ideal of \( S_3 \). Let us take \( s_3 \) to be the support of \( S_3 \). Then \( s_3 P_i \) is central in \( S_3 \), since \( S_3/I_3 \) is abelian. So

\[
(s_3 P_i)(1 \square P_j)P_k = (1 \square P_j)(s_3 P_i)P_k = 0.
\]

By Wenzl’s formula, we have

\[
s_3 P_i = P_i - \frac{\delta}{tr(P_i)}P_i (1 \square id)P_i.
\]

So

\[
P_i (1 \square P_j)P_k = \frac{\delta}{tr(P_i)}P_i (1 \square id)P_i (1 \square P_j)P_k = \frac{\delta}{tr(P_i)}P_i (1 \square id)(P_i \ast P_j)P_k \text{ by an isotopy} = \frac{\delta}{tr(P_i)} \lambda P_i (1 \square id)P_k.
\]
i.e.,

\[ (6.1) \]

Then \( P_i(1 \sqcup P_j)P_k \neq 0 \). Multiplying \( P_j \) on the right side (the 3, 4 position), we have

\[ P_i(1 \sqcup P_j)P_k = \frac{\delta}{\text{tr}(P_j)} \lambda P_i(1 \sqcup P_j)P_k. \]

Thus \( \lambda = \frac{\text{tr}(P_j)}{\delta} \). Adding a cap on the right bottom (the 4, 5 position), we have

\[ \text{tr}(P_i * P_j)P_k) = \frac{\text{tr}(P_i)\text{tr}(P_k)}{\delta}. \]

By Lemma 4.10 \( \text{tr}(P_j(P_i' * P_k)) = \frac{\text{tr}(P_i)\text{tr}(P_k)}{\delta} \). Therefore

\[ \text{tr}(P_j(P_i' * P_k)) > 0 \implies \text{tr}(P_j(P_i' * P_k)) = \frac{\text{tr}(P_i)\text{tr}(P_k)}{\delta}. \]

If \( P_i \) is a minimal projection in \( S_2 \), such that \( \|P_i - P_j\| < 1 \), then \( \text{tr}(P_i(P_i' * P_k)) > 0 \). Thus \( \text{tr}(P_i(P_i' * P_k)) = \frac{\text{tr}(P_i)\text{tr}(P_k)}{\delta} \). Note that the space of projections in the central support of \( P_j \) is path connected in norm topology. When \( P_j \) moves continuously, \( \text{tr}(P_j(P_i' * P_k)) > 0 \) always holds. On the other hand, \( r(P_i' * P_k) = 1 \) by Lemma 6.11. So \( P_j \) is central. Then \( P_j \in S_2/I_3 \) or \( \text{tr}(P_j) = 1 \). □

**Proof of Lemma 6.10** Suppose \( S \) is not depth-2, \( P_i \) is a minimal projection in \( S_2/I_3 \), \( P_j \) is a minimal projection in \( S_2 \), and \( P_j \neq P_i' \).

If \( P_j \in S_2/I_3 \), then \( r(P_i * P_j) = 1 \) by Lemma 6.11.

If \( \text{tr}(P_j) = 1 \), then \( r(P_i * P_j) = 1 \).

Otherwise \( P_j \in S_2 \cap I_3 \) and \( \text{tr}(P_j) > 1 \). By Lemma 6.12 if \( P_k \leq P_i * P_j \), then

\[ P_k \in S_2/I_3 \text{ and } \text{tr}(P_j(P_i' * P_k)) = \text{tr}(P_i * P_j)P_k > 0. \]

Furthermore by Lemma 6.13 if \( P_k \neq P_i \), then \( P_j \in S_2/I_3 \) or \( \text{tr}(P_j) = 1 \). This is a contradiction. So \( P_k = P_i \). Then \( P_i * P_j \sim P_i \).

Therefore \( P_i \) is a left virtual normalizer. By Theorem 4.16 \( P_i' \) is a minimal projection in \( S_2/I_3 \). So \( P_i' \) is a left virtual normalizer. Then \( P_i \) is a virtual normalizer. □

**Definition 6.14.** A subfactor (or a subfactor planar algebra) is said to be \( k \)-supertransitive if its principal graph is the Dynkin diagram \( A_{k+1} \) up to depth \( k \).

From a subfactor perspective, we have the following weak version of Theorem 6.8.

**Corollary 6.15.** Suppose \( N \subseteq M \) is a finite index irreducible subfactor such that the quotient of \( N' \cap M_2 \) by the basic construction ideal \( (N' \cap M_2)e_2(N' \cap M_2) \) is abelian, where \( N \subseteq M \subseteq M_1 \subseteq M_2 \subseteq \cdots \) is the Jones tower and \( e_2 \) is the Jones projection onto \( L^2(M) \). Then either \( N \subseteq M \) is depth-2 or there exists a sequence of intermediate subfactors \( N \subseteq R_1 \subseteq \cdots \subseteq R_n \subseteq M \) such that:

1. either \( N \subseteq R_1 \) is 2-supertransitive or \( R_1 = N \rtimes G \) for an outer action of a group \( G \);
(2) either $\mathcal{R}_n \subset \mathcal{M}$ is 2-supertransitive or $\mathcal{R}_n = \mathcal{M}^H$ for an outer action of a group $H$, where $\mathcal{M}^H$ is the fixed point algebra under the action of $H$;
(3) either $\mathcal{R}_i \subset \mathcal{R}_{i+1}$ is 2-supertransitive or $\mathcal{R}_{i+1} = \mathcal{R}_i \rtimes A$ for an outer action of an abelian group $A$, for $1 \leq i \leq n-1$.

Furthermore any intermediate subfactor of $\mathcal{N} \subset \mathcal{M}$ is either one of the sequence or an intermediate subfactor of some adjacent pair of the sequence.

Proof. Suppose $\mathcal{F}$ is the planar algebra of $\mathcal{N} \subset \mathcal{M}$, and $\mathcal{S}$ is the planar subalgebra of $\mathcal{F}$ generated by 2-boxes. Since $\mathcal{S}_2 = \mathcal{F}_2$, the basic construction ideal $\mathcal{I}_3$ of $\mathcal{F}$ is also the basic construction ideal of $\mathcal{S}_3$. Note that $(\mathcal{N}' \cap \mathcal{M}_2)/((\mathcal{N}' \cap \mathcal{M}_2)\mathcal{e}_2(\mathcal{N}' \cap \mathcal{M}_2))$ is abelian means $\mathcal{F}_3/\mathcal{I}_3$ is abelian. Then $\mathcal{S}_3/\mathcal{I}_3$ is abelian. So $\mathcal{S}$ is a commut rate planar algebra. Recall that intermediate subfactors correspond to biprojections in $\mathcal{F}_2 = \mathcal{S}_2$. By Theorem 6.8, $\mathcal{S}$ is either depth-2 or a free product of Temperley-Lieb subfactor planar algebras and depth-2 subfactor planar algebras. Thus either $\mathcal{N} \subset \mathcal{M}$ is depth-2 or there exists a sequence of intermediate subfactors $\mathcal{N} \subset \mathcal{R}_1 \subset \cdots \subset \mathcal{R}_n \subset \mathcal{M}$ corresponding to the sequence of biprojections $P_1, P_2, \cdots, P_n$ which separate $\mathcal{S}$ as a free product such that $\mathcal{S}_{P_{i+1}}^{P_i}$ is either Temperley-Lieb or depth-2, for $1 \leq i \leq n-1$. Note that $\mathcal{S}_{P_{i+1}}^{P_i} = \mathcal{S}_{P_{i+1}}^{P_i}$, so $\mathcal{S}_{P_{i+1}}^{P_i}$ is either 2-supertransitive or depth-2. Moreover if $\mathcal{N} \subset \mathcal{R}_1$ is depth-2, then its planar algebra is of type NA; thus $\mathcal{R}_1 = \mathcal{N} \rtimes G$, for an outer action of a group $G$. If $\mathcal{R}_n \subset \mathcal{M}$ is depth-2, then its planar algebra is of type AN; thus $\mathcal{R}_n = \mathcal{M}^H$, for an outer action of a finite group $H$. If $\mathcal{R}_i \subset \mathcal{R}_{i+1}$ is depth-2, for some $1 \leq i \leq n-1$, then its planar algebra is of type AA; thus $\mathcal{R}_{i+1} = \mathcal{R}_i \rtimes A$, for an outer action of an abelian group $A$.

Furthermore, by Theorem 2.22 any intermediate subfactor of $\mathcal{N} \subset \mathcal{M}$ is either one of the sequence or an intermediate subfactor of some adjacent pair of the sequence. □

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REFERENCES

[1] M. Asaeda and U. Haagerup, Exotic subfactors of finite depth with Jones indices $(5+\sqrt{13})/2$ and $(5+\sqrt{11})/2$, Comm. Math. Phys. 202 (1999), no. 1, 1–63, DOI 10.1007/s002200050574. MR1685551(2000c:46120)
[2] B. Bhattacharyya and Z. Landau, Intermediate standard invariants and intermediate planar algebras, submitted to Journal of Functional Analysis.
[3] Stephen Bigelow, Emily Peters, Scott Morrison, and Noah Snyder, Constructing the extended Haagerup planar algebra, Acta Math. 209 (2012), no. 1, 29–82, DOI 10.1007/s11511-012-0081-7. MR2979509
[4] Joan S. Birman and Hans Wenzl, Braids, link polynomials and a new algebra, Trans. Amer. Math. Soc. 313 (1989), no. 1, 249–273, DOI 10.2307/2001074. MR992598(90g:57004)
[5] Dietmar Bisch, A note on intermediate subfactors, Pacific J. Math. 163 (1994), no. 2, 201–216. MR1262294(95c:46105)
[6] Dietmar Bisch, Bimodules, higher relative commutants and the fusion algebra associated to a subfactor, Operator algebras and their applications (Waterloo, ON, 1994/1995), Fields Inst. Commun., vol. 13, Amer. Math. Soc., Providence, RI, 1997, pp. 13–63. MR1424954(97f:46109)
[7] Dietmar Bisch, Principal graphs of subfactors with small Jones index, Math. Ann. 311 (1998), no. 2, 223–231, DOI 10.1007/s002080050185. MR1625762(2000k:46087)
[8] D. Bisch, V. Jones, and Z. Liu, Singly generated planar algebras of small dimension, part III.
9. D. Bisch and V. F. R. Jones, The free product of planar algebras, and subfactors, unpublished.
10. Dietmar Bisch and Vaughan Jones, Algebras associated to intermediate subfactors, Invent. Math. 128 (1997), no. 1, 89–157, DOI 10.1007/s002220050137. MR1437496 (99c:46072)
11. Dietmar Bisch and Vaughan Jones, Singly generated planar algebras of small dimension, Duke Math. J. 101 (2000), no. 1, 41–75, DOI 10.1215/S0012-7094-00-10112-3. MR1733737
12. Dietmar Bisch and Vaughan Jones, Singly generated planar algebras of small dimension. II, Adv. Math. 175 (2003), no. 2, 297–318, DOI 10.1016/S0001-8708(02)00060-9. MR1972635 (2004d:46073)
13. William Fulton and Joe Harris, Representation theory: A first course, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991. MR1153249 (93a:20069)
14. Frederick M. Goodman, Pierre de la Harpe, and Vaughan F. R. Jones, Coxeter graphs and towers of algebras, Mathematical Sciences Research Institute Publications, vol. 14, Springer-Verlag, New York, 1989. MR999799 (91c:46082)
15. A. Guionnet, V. F. R. Jones, and D. Shlyakhtenko, Random matrices, free probability, planar algebras and subfactors, Quanta of maths, Clay Math. Proc., vol. 11, Amer. Math. Soc., Providence, RI, 2010, pp. 201–239. MR2732052 (2012g:46094)
16. Uffe Haagerup, Principal graphs of subfactors in the index range $4 < [M : N] < 3 + \sqrt{2}$, Subfactors (Kyuzeso, 1993), World Sci. Publ., River Edge, NJ, 1994, pp. 1–38. MR1317352 (96d:46081)
17. Masaki Izumi, Application of fusion rules to classification of subfactors, Publ. Res. Inst. Math. Sci. 27 (1991), no. 6, 953–994, DOI 10.2977/prims/1195169007. MR1145672 (93b:46121)
18. Masaki Izumi, Vaughan F. R. Jones, Scott Morrison, and Noah Snyder, Subfactors of index less than 5, Part 3: Quadruple points, Comm. Math. Phys. 316 (2012), no. 2, 531–554, DOI 10.1007/s00220-012-1472-5. MR2993924
19. V. F. R. Jones, Planar algebras, I, arXiv:math.QA/9909027.
20. V. F. R. Jones, Index for subfactors, Invent. Math. 72 (1983), no. 1, 1–25, DOI 10.1007/BF01389127. MR696688 (84d:46097)
21. Vaughan F. R. Jones, The planar algebra of a bipartite graph, Knots in Hellas '98 (Delphi), Ser. Knots Everything, vol. 24, World Sci. Publ., River Edge, NJ, 2000, pp. 94–117, DOI 10.1142/9789812792679.0008. MR1865703 (2003c:57003)
22. Vaughan F. R. Jones, Quadratic tangles in planar algebras, Duke Math. J. 161 (2012), no. 12, 2257–2295, DOI 10.1215/00127094-1723608. MR2972458
23. Vaughan F. R. Jones, Scott Morrison, and Noah Snyder, The classification of subfactors of index at most 5, Bull. Amer. Math. Soc. (N.S.) 51 (2014), no. 2, 277–327, DOI 10.1090/S0273-0979-2013-01442-3. MR3166042
24. Vaughan F. R. Jones and David Penneys, The embedding theorem for finite depth subfactor planar algebras, Quantum Topol. 2 (2011), no. 3, 301–337, DOI 10.4171/QT/23. MR2812459 (2012f:46128)
25. V. Jones and V. S. Sunder, Introduction to subfactors, London Mathematical Society Lecture Note Series, vol. 234, Cambridge University Press, Cambridge, 1997. MR1473221 (98k:46067)
26. Vijay Kodiyalam, Zeph Landau, and V. S. Sunder, The planar algebra associated to a Kac algebra, Functional analysis (Kolkata, 2001), Proc. Indian Acad. Sci. Math. Sci. 113 (2003), no. 1, 15–51, DOI 10.1007/BF02829677. MR1971553 (2004d:46075)
27. Hideki Kosaki and Shigeru Yamagami, Irreducible bimodules associated with crossed product algebras, Internat. J. Math. 3 (1992), no. 5, 661–676, DOI 10.1142/S0129167X9200031X. MR1189679 (94f:46087)
28. Zeph A. Landau, Exchange relation planar algebras, Proceedings of the Conference on Geometric and Combinatorial Group Theory, Part II (Haifa, 2000), Geom. Dedicata 95 (2002), 183–214, DOI 10.1023/A:1021296230310. MR1950890 (2003k:46091)
29. Scott Morrison, David Penneys, Emily Peters, and Noah Snyder, Subfactors of index less than 5, Part 2: Triple points, Internat. J. Math. 23 (2012), no. 3, 1250016, 33, DOI 10.1142/S0129167X11007586. MR2902285
30. Scott Morrison, Emily Peters, and Noah Snyder, Skein theory for the $D_{2n}$ planar algebras, J. Pure Appl. Algebra 214 (2010), no. 2, 117–139, DOI 10.1016/j.jpaa.2009.04.010. MR2559686 (2011c:46131)
[31] Scott Morrison and Noah Snyder, Subfactors of index less than 5, Part 1: The principal graph odometer, Comm. Math. Phys. 312 (2012), no. 1, 1–35, DOI 10.1007/s00220-012-1426-y. MR1291405

[32] Jun Murakami, The Kauffman polynomial of links and representation theory, Osaka J. Math. 24 (1987), no. 4, 745–758. MR1297059 (89c:57007)

[33] Adrian Ocneanu, Quantized groups, string algebras and Galois theory for algebras, Operator algebras and applications, Vol. 2, London Math. Soc. Lecture Note Ser., vol. 136, Cambridge Univ. Press, Cambridge, 1988, pp. 119–172. MR996454 (91k:46068)

[34] David Penneys and James E. Tener, Subfactors of index less than 5, Part 4: Vines, Internat. J. Math. 23 (2012), no. 3, 1250017, 18, DOI 10.1142/S0129167X11006741. MR2902286

[35] Emily Peters, A planar algebra construction of the Haagerup subfactor, Internat. J. Math. 21 (2010), no. 8, 987–1045, DOI 10.1142/S0129167X10006380. MR2679382 (2011i:46077)

[36] Mihai Pimsner and Sorin Popa, Entropy and index for subfactors, Ann. Sci. École Norm. Sup. (4) 19 (1986), no. 1, 57–106. MR860811 (87m:46040)

[37] S. Popa, Classification of subfactors: the reduction to commuting squares, Invent. Math. 101 (1990), no. 1, 19–43, DOI 10.1007/BF01231494. MR1055708 (91h:46109)

[38] Sorin Popa, An axiomatization of the lattice of higher relative commutants of a subfactor, Invent. Math. 120 (1995), no. 3, 427–445, DOI 10.1007/BF01241137. MR1334479 (96g:46051)

[39] S. Popa, Classification of amenable subfactors of type II1 factors, J. Reine Angew. Math. 442 (1993), 131–170. MR1237144 (94k:46074)

[40] V. S. Sunder and A. K. Vijayarajan, On the nonoccurrence of the Coxeter graphs $\beta_{2n+1}$, $D_{2n+1}$ and $E_7$ as the principal graph of an inclusion of $II_1$ factors, Pacific J. Math. 161 (1993), no. 1, 185–200. MR1237144 (94k:46067)

[41] Wojciech Szymański, Finite index subfactors and Hopf algebra crossed products, Proc. Amer. Math. Soc. 120 (1994), no. 2, 519–528, DOI 10.2307/2159890. MR1136139 (94e:46061)

[42] Hans Wenzl, On sequences of projections, C. R. Math. Rep. Acad. Sci. Canada 9 (1987), no. 1, 5–9. MR873400 (88k:46070)

[43] Hans Wenzl, Quantum groups and subfactors of type B, C, and D, Comm. Math. Phys. 133 (1990), no. 2, 383–432. MR1090432 (92k:17032)

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