ON THE NUMBER OF NON-CONGRUENT LATTICE TETRAHEDRA

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ABSTRACT. We prove that there are "many" non-congruent tetrahedra in the truncated lattice $[0, q]^3 \cap \mathbb{Z}^3$. This answers a question from [2].

1. INTRODUCTION

Our goal is to answer the following question asked in the last section of [2].

**Question 1.1.** Let $T_3([0, q]^3 \cap \mathbb{Z}^3)$ denote the collection of all equivalence classes of congruent tetrahedra with vertices in $[0, q]^3 \cap \mathbb{Z}^3$. Is there a $\delta > 0$ and some $C > 0$, both independent of $q$ such that

$$\#T_3([0, q]^3 \cap \mathbb{Z}^3) \leq Cq^{9-\delta}$$

for each $q > 1$?

A positive answer to this question would have implications on producing lower bounds for the Falconer distance-type problem for tetrahedra. We refer to [2] for details on this interesting problem.

Here we give a negative answer to this question. We hope that our approach to answering this question will inspire further constructions which might eventually improve the lower bound.

**Theorem 1.2.** We have for each $\epsilon > 0$ and each $q > 1$

$$\#T_3([0, q]^3 \cap \mathbb{Z}^3) \gtrsim q^{9-\epsilon}.$$

Note the following trivial upper bound, which shows the essential tightness of our result

$$\#T_3([0, q]^3 \cap \mathbb{Z}^3) \leq Cq^9.$$

Indeed, by translation invariance it suffices to fix one vertex at the origin. The upper bound follows since there are $(q+1)^3$ possibilities for each of the remaining three vertices.

2. SYSTEMS OF QUADRATIC EQUATIONS

The proof of Theorem 1.2 will rely on the theory developed in [1] around Siegel’s mass formula. To keep the presentation essentially self-contained, we briefly recall the relevant theorems here and refer the interested reader to [1] for further details.

Let $m = n + 1 \geq 3$ and let $\gamma \in M_{m,m}(\mathbb{Z})$ and $\Lambda \in M_{n,n}(\mathbb{Z})$ be two positive definite matrices with integer entries. Denote by $A(\gamma, \Lambda)$ the number of solutions $\mathcal{L} \in M_{m,n}(\mathbb{Z})$ for

$$\mathcal{L}^* \gamma \mathcal{L} = \Lambda. \quad (1)$$

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For primes $p$ define
\[
\nu_p(\gamma, \Lambda) = \lim_{r \to \infty} \frac{1}{p^{1 - 2m/n + m/n}} \left| \left\{ \mathcal{L} \in M_{m,n}(\mathbb{Z}/p^r) : \mathcal{L}^* \gamma \mathcal{L} \equiv \Lambda \mod p^r \right\} \right|.
\]

The following is an immediate consequence of the Siegel’s mass formula in [3].
\[
A(\gamma, \Lambda) \lesssim_{n, \gamma} \prod_{p \text{ prime}} \nu_p(\gamma, \Lambda).
\]

In our application here $m = 3, n = 2$ and $\gamma$ is the identity matrix $I_3$.

Fix $\Lambda \in M_{n,n}(\mathbb{Z})$, a positive definite matrix, in particular $\det(\Lambda) \neq 0$. In evaluating $\nu_p(I_{n+1}, \Lambda)$ we distinguish two separate cases: $p \nmid \det(\Lambda)$ and $p|\det(\Lambda)$.

**Proposition 2.1.** Assume $p$ is not a factor of $\det(\Lambda)$. Then
\[
\nu_p(I_{n+1}, \Lambda) \leq 1 + \frac{C}{p^2},
\]
where $C$ is independent of $p, \Lambda$.

The contribution of these primes to the product (2) is easily seen to be $O(1)$.

For an $n \times n$ matrix $\Lambda$ and for $A, B \subset \{1, \ldots, n\}$ with $|A| = |B|$ we define
\[
\mu_{A,B} = \det((\Lambda_{i,j})_{i \in A, j \in B}).
\]

Also, for an integer $k$ and a prime number $p$ we denote by $o_p(k)$ the largest positive integer $\alpha$ such that $p^\alpha|k$.

**Proposition 2.2.** Assume $p|\det(\Lambda)$. Then there is $C$ independent of $p, \Lambda$ such that
\[
\nu_p(I_{n+1}, \Lambda) \leq C \sum_{0 \leq l_1, \ldots, l_n \leq o_p(\det(\Lambda))} p^{\beta_2(l_1, \ldots, l_n) + \ldots + \beta_n(l_1, \ldots, l_n)},
\]
where $\beta_i = \beta_i(l_1, \ldots, l_n)$ satisfies
\[
\beta_i = \min \{(i-1)l_i, (i-2)l_i + \min_{|A|=1} o_p(\mu_{\{1\}, A}) - l_1, (i-3)l_i + \min_{|A|=2} o_p(\mu_{\{1,2\}, A}) - l_1 - l_2, \ldots, \min_{|A|=i-1} o_p(\mu_{\{1,2,\ldots,i-1\}, A}) - l_1 - l_2 - \ldots - l_{i-1}\}
\]

3. **Proof of Theorem 1.2**

In the following discussion, unless specified otherwise, $\epsilon$ will denote an arbitrary positive number.

It will be useful to recall the classical facts that circles of radius $r$ contain $O(r^2)$ lattice points while spheres of radius $r$ centered at the origin contain $O(r^{2+\epsilon})$ lattice points.

As mentioned before, we fix one vertex to be the origin $0 = (0, 0, 0)$. A class of congruent tetrahedra in $T_3([0, q]^3 \cap \mathbb{Z}^3)$ can be identified with a matrix $\Lambda \in M_{3,3}(\mathbb{Z})$. Namely, the congruence class of the tetrahedron with vertices $0, x, y, z \in [0, q]^3 \cap \mathbb{Z}^3$ is represented by the matrix
\[
\Lambda = \begin{bmatrix}
\langle x, x \rangle & \langle x, y \rangle & \langle x, z \rangle \\
\langle y, x \rangle & \langle y, y \rangle & \langle y, z \rangle \\
\langle z, x \rangle & \langle z, y \rangle & \langle z, z \rangle
\end{bmatrix}.
\]
A tetrahedron is called non-degenerate if \(x, y, z\) are linearly independent. We will implicitly assume the congruence classes correspond to non-degenerate tetrahedra.

We seek for an upper bound on the number \(N_\Lambda\) of integral solutions \(L = (x, y, z) \in (\mathbb{Z}^3)^3\) to the equation

\[
L^* L = \Lambda.
\]

This will represent the number of congruent tetrahedra with side lengths specified by \(\Lambda\).

In the numerology from the previous section, this corresponds to \(n = m = 3\). To make the theorems in that section applicable we reduce the counting problem to the \(m = 3, n = 2\) case as follows. One can certainly bound \(N_\Lambda\) by \(q^\epsilon N'_\Lambda\), where \(N'_\Lambda\) is the number of integral solutions \(L' = (x, y) \in (\mathbb{Z}^3)^2\) satisfying

\[
(L')^* L' = \Lambda'.
\]

and \(\Lambda'\) is the \(2 \times 2\) minor of \(\Lambda\) obtained from the first two rows and columns of \(\Lambda\). Indeed, if \(x, y\) are fixed, the matrix \(\Lambda\) forces \(z\) to lie on the intersection of the sphere of radius \(\Lambda_{1,3}^{1/2}\) centered at the origin with, say, a sphere centered at \(x\) whose radius is determined only by the entries of \(\Lambda\). These radii are \(O(q)\), so the resulting circle can only have \(O(q^\epsilon)\) points.

Note also that we only care about those \(\Lambda'\) for which there exist \(x, y \in [0, q]^3 \cap \mathbb{Z}^3\) linearly independent, such that

\[
\Lambda' = \begin{bmatrix}
(x, x) & \langle x, y \rangle \\
\langle y, x \rangle & \langle y, y \rangle
\end{bmatrix}.
\]

This in particular forces \(\Lambda'\) to be positive definite.

Apply now Propositions 2.1 and 2.2 combined with (2) to the matrix \(\Lambda'\) \((n = 2)\). For Proposition 2.2 use \(\beta_2 = \min_{|A|=1} o_p(\mu(1, A) - l_1)\), then \(\beta_2 = \min_{|A|=1} o_p(\mu(2, A) - l_1)\) and the fact that there are

\[
O\left(\frac{\log \det(\Lambda)}{\log \log \det(\Lambda)}\right)
\]

primes \(p\) which divide \(\det(\Lambda)\). This will bound \(N'_\Lambda\) by

\[
q^\epsilon \gcd(\Lambda_{i,j} : i, j \neq 3) \leq q^\epsilon \gcd(\Lambda_{1,1}, \Lambda_{2,2}).
\]

Thus

\[
N_\Lambda \lesssim q^\epsilon \gcd(\Lambda_{1,1}, \Lambda_{2,2}),
\]

for each \(\Lambda\) corresponding to a non-degenerate tetrahedron.

Denote by \(M_r\) the number of lattice points on the sphere or radius \(r^{1/2}\) centered at the origin. In our case \(r \leq q^2\) so we know that \(M_r \lesssim q^{1+}\epsilon\). We need to work with spheres that contain many points. Let

\[
A := \{r \leq q^2 : M_r \geq q/2\}.
\]

Since for each \(\epsilon > 0\) we have \(M_r \leq C_{\epsilon} q^{1+\epsilon}\), a double counting argument shows that

\[
q^3 \leq C_{\epsilon} \# A q^{1+\epsilon} + 1/4 q^2 q. \quad \text{Thus} \quad \# A \gtrsim q^{2-}\epsilon.
\]

Note that for each \(r_i \in A\) there are \(\sim M_{r_1} M_{r_2} M_{r_3}\) non-degenerate tetrahedrons with vertices \(x, y, z\) on the spheres centered at the origin and with radii \(r_1^{1/2}, r_2^{1/2}, r_3^{1/2}\) respectively. The congruence class of such a tetrahedron contains

\[
\lesssim q^\epsilon \gcd(r_1, r_2)
\]
elements, according to (4).

We conclude that there are at least
\[
\frac{M_{r_1} M_{r_2} M_{r_3}}{q^\varepsilon \gcd(r_1, r_2)}
\]
congruence classes generated by such non-degenerate tetrahedra. As distinct radii necessarily give rise to distinct congruence classes, we obtain the lower bound
\[
\# T_3([0, q]^3 \cap \mathbb{Z}^3) \gtrsim \varepsilon \sum_{r_1, r_2, r_3 \in A} \frac{1}{\gcd(r_1, r_2)}.
\]

It is immediate that for each integer \(d\) there can be at most \(\frac{q^6}{d^2}\) tuples \((r_1, r_2, r_3) \in [0, q]^3\), hence also in \(A^3\), with \(\gcd(r_1, r_2) = d\). Using this observation and the bound \(#(A^3) \geq C_\varepsilon q^6\), it follows that for each \(\varepsilon > 0\) at least \(\frac{1}{2}\#(A^3)\) among the triples \((r_1, r_2, r_3) \in A^3\) will have \(\gcd(r_1, r_2) \leq \frac{10q^6}{C_\varepsilon}\).

This implies that
\[
\sum_{r_1, r_2, r_3 \in A} \frac{1}{\gcd(r_1, r_2)} \gtrsim q^{6-\varepsilon},
\]
which finishes the proof of the theorem.

REFERENCES

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