Time-dependent Pattern Speeds in Barred Galaxies

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Abstract

Based on a high-quality N-body simulation of a double-bar galaxy model, we investigate the evolution of the bar properties, including their size, strength, and instantaneous pattern speed derived by using three distinct methods: the Fourier, Jacobi integral, and moment of inertia methods. The interaction of the two bars, which rotate at distinct speeds, primarily affects the size, strength, and pattern speed of the inner bar. When the two bars are perpendicular to each other, the size and the pattern speed of the inner bar decrease and its strength increases. The emergence of a strong Fourier $m=1$ mode increases the oscillation amplitude of the size, strength, and pattern speed of the inner bar. On the other hand, the characteristics of the outer bar are substantially influenced by its adjacent spiral structure. When the spiral structure disappears, the size of the outer bar increases and its strength and pattern speed decrease. Consequently, the ratio of the pattern speed of the outer bar with respect to the inner bar is not constant and increases with time. Overall, the double-bar and disk system displays substantial high-frequency semichaotic fluctuations of the pattern strengths and speeds in both space and time, superposed on the slow secular evolution, which invalidates the assumption that the actions of individual stars should be well conserved in barred galaxies, such as the Milky Way.

Key words: galaxies: evolution – galaxies: kinematics and dynamics – galaxies: spiral – Galaxy: kinematics and dynamics – methods: numerical

1. Introduction

Single or double stellar bars are common structures existing in most disk galaxies, playing an important role in the secular evolution of galaxies driven by internal processes. More than two-thirds of nearby disk galaxies observed in the near-infrared are considered to have bars, although this fraction is likely higher since bars viewed edge-on are difficult to detect, as illustrated by the Milky Way, the bar of which was established observationally only in the 1990s. The fraction of galaxies with bars is also known to depend on the specific properties of galaxies, such as Hubble type (Marinova & Jogee 2007; Menéndez-Delmestre et al. 2007). Physically, the presence of bars significantly affects the evolution of a galaxy by redistributing the angular momentum between its different components, for example, between the disk and halo (Weinberg 1985; Villa-Vargas et al. 2009).

In addition, the interaction of two bars can also affect the stellar kinematics of the galaxies, especially for the central region, which is closely related to the inner bar. For example, the $\sigma$-humps (Emsellem et al. 2001), which measure the velocity dispersion, have two local maxima along the minor axis of the inner bar, are generated by the existence of vertically thin bars (Wozniak et al. 2003; Du et al. 2017b), and can oscillate in strength according to the relative angle between the two bars (Du et al. 2016).

Furthermore, the bars affect not only the stars but also the gas in the galaxies. Bars are expected to induce gas inflows, enhancing episodic star formation activity (Aguerri 1999) and/or feeding supermassive black holes in the central regions of galaxies (Shlosman et al. 1989; Friedli & Martinet 1993). However, the existence of the black hole with mass $\sim 0.05\%$–$0.2\%$ of the total stellar mass can destroy the inner bar of a double-bar galaxy (Du et al. 2017a), as well as in single-barred galaxies (Hasan & Norman 1990; Pfenniger & Norman 1990; Friedli & Pfenniger 1991; Hasan et al. 1993; Norman et al. 1996). Therefore, the gas inflows that are driven by the inner bar may be terminated when the black hole grows to $\sim 0.1\%$ of the total stellar mass.

Although bars are expected to affect the evolution of galaxies profoundly, the details of their formation, evolution, and characteristics are still unclear and need to be investigated in more detail. In particular, the evolutions of bar sizes, strength, and pattern speeds, as well as the mutual perturbations of double bars and adjacent spirals, remain to be better investigated.

Since the 1960s, stellar bars have been assumed as rigidly and steadily rotating structures perturbing a background fixed gravitational potential (e.g., de Vaucouleurs et al. 1968; Weinberg 1985). This unchecked assumption is implicit and is used in most studies on the topic (e.g., Hernquist & Weinberg 1992; Binney & Tremaine 2008, Section 3.3.2). However, Sellwood & Sparke (1988) first showed that a single bar and its adjacent spiral arms rotate on average at distinct speeds. When adjacent bar or spiral structures rotate with different speeds, their mutual torques cause the intermediate region to be strongly time dependent in any rotating frame. This point was investigated quantitatively by Wu et al. (2016), who demonstrated that bars and spiral arms in a double-barred galaxy model are actually flexible structures, especially in the vicinity of the corotation region. This results from the fact that the equilibrium points within the corotation region are time dependent in any rotating frame owing to the mutual interaction with their adjacent structure (bar–bar or bar–spiral structures). One of the important consequences of the time-dependent dynamics is that the bars in their entirety do not rotate at a constant pattern speed, especially near their ends, where the
parameter values in Table 1.}

| Parameter | Bulge | Disk | Halo |
|-----------|--------|------|------|
| Mass M(10^10 M_\odot) | 1.3496 | 8.6504 | 15.0 |
| Scale length (a + b) (kpc) | 0.50 | 4.50 | 15.0 |
| Scale height h (kpc) | 0.15 | 0.45 | 15.0 |
| Number of particles | 1,079,680 | 6,920,320 | 12,000,000 |

pattern speed varies both spatially and temporally. This implies that we need to be cognizant of the fact that the instantaneous pattern speed near the corotation region may differ from the instantaneous pattern speed in the other parts of the bars. From this view, the meaning of the corotation resonance is also challenged since it lies midway between two patterns of similar strength, but with differing pattern speeds. At best we could use the term *time-dependent resonance* for its description. The significance of the bar outer Lindblad resonance is even more challenged since it lies in the spiral region (in the Milky Way close to the Sun), where the bar torque may be weaker than the local spiral torque.

The existing methods to observationally determine pattern speeds in the Milky Way and in external galaxies (e.g., Tremaine & Weinberg 1984) can, to a certain degree, measure instantaneous pattern speeds, whereas the pattern speeds determined in theoretical studies using N-body simulations (e.g., Sellwood & Athanassoula 1986) are usually space and time averaged. To ensure a more meaningful comparison between theory and observations, it is necessary to develop new methods to measure the local and instantaneous pattern speeds in numerical simulations, which have been advanced in a companion paper (D. Pfenniger et al. 2018, in preparation). In the present paper, three of these methods (explained in Appendices A–C) are used for analyzing a high-quality N-body simulation of a double-bar disk galaxy model whose potential depends on time in any coordinates. Following the results in Wu et al. (2016), it is essential to quantify the different time-dependent characteristics of bars.

The purpose of evolving and analyzing N-body models in this paper is not to describe a fully realistic evolution of a galaxy, but to check the degree of time dependence of different patterns in a collisionless self-gravitating rotating disk similar to a galaxy. Three main properties are studied: the size, strength, and instantaneous mode/pattern speeds of the bars.

Our paper is organized as follows. In Section 2, the galaxy model is described. The comparison of three methods for determining the mode/pattern speed and the spatial and temporal evolution of the entire galaxy over 7.8 Gyr is shown in Section 3. In Sections 4 and 5, we focus on the evolution of the inner and outer bar and their interplay. Finally, we conclude in Section 6.

2. Galaxy Model

2.1. The Parameters

Our initial galaxy model consists of three concentric axisymmetric Miyamoto–Nagai (hereafter MN) components (Miyamoto & Nagai 1975), which could be associated with a bulge, a disk, and a halo. Table 1 shows the parameters of these three components, which are the same as in Wu et al. (2016). To generate double-barred galaxies, our equilibrium initial models should have (i) a double-peaked rotation curve with equal maxima, one peak corresponding to the disk component and the other to the bulge component, (ii) the smaller component characterized by colder kinematics (Q ∼ 1), and (iii) the medium component described by hotter kinematics (Q ∼ 1.5), similar to Du et al. (2015). In addition, to maintain a high velocity in the rotation curve in the spiral region, a spherical component (i.e., a Plummer model) is adopted for the largest MN component. For this setup, a long-lived double-barred galaxy surrounded by a spiral region ensues.

The initial 3D N-body model is constructed using the GaLiC code (Yurin & Springel 2014), modified to generate MN components. It is evolved using the pure stellar dynamical code gyrfalcON (Dehnen 2000), using the fast multipole method, which has better momentum conservation than the Barnes–Hut Treecode method. We adopt a total number of equal-mass particles N = 2 × 10^7, which corresponds to a gravitational softening length of ∼0.03 kpc, in order to reduce the particle noise and to allow investigation of the inner bar, which has a half-length of ∼0.6 kpc. In our model, all components are live, so that the disk particles are allowed to interact with the halo and bulge particles. The dynamics is then more realistic than in models that have a fixed halo (e.g., Du et al. 2015), thereby suppressing the growth of odd modes.

Our model galaxy is simulated for about 8 Gyr, which is sufficiently long to study the overall evolution of the two nested bars and the surrounding spiral arms. The time evolution of the projected density of the double-barred system will be shown in the next subsection.

2.2. Time Evolution of the Galaxy

Figure 1 presents the projected density of the double-barred system in log scale at selected times. The inset of each panel shows the projected density near the galaxy center, clearly revealing the inner bar.

Starting from the equilibrium initial condition with the bulge, disk, and halo components, the inner bar and transient inner spirals form first owing to the colder kinematics near the galaxy center, as shown in panel (a). The inner spirals recur several times until t ∼ 0.5 Gyr. Subsequently, the transient outer spirals and the outer bar form. It is worth noticing that in this scenario, forming the inner bar first and the outer bar later is not necessarily the only scenario for forming a double-barred galaxy. A plausible scenario (Shlosman et al. 1989; Friedli & Martinet 1993) is that the inner bar forms after the outer bar when sufficient gas has accumulated as a nuclear disk, and star formation from this disk injects stars with cold kinematics therein, triggering a bar instability.

Because the pattern speeds of the inner bar and outer bar differ, the phase angle between the two bars varies with time. Panel (b) shows an example at a time when the inner bar and the outer bar are perpendicular to each other. Panel (c) illustrates the time when the two bars are aligned. We note that after t ∼ 0.7 Gyr, the transient outer spirals recur several times, instead of forming steady spirals, until the end of the simulation. For example, as shown in panels (d) and (e), the outer strong spirals appear at t = 3.128 Gyr and then disappear in about 155 Myr.

After t ∼ 3.8 Gyr, the center of the inner bar starts to clearly shift around the center of the galaxy, representing motion associated with a Fourier m = 1 mode, an example of which can be seen in the inset of panel (f). After t ∼ 4.8 Gyr, the outer
Figure 1. Time evolution of the projected density in log scale at selected times. The inset of each panel shows the projected density near the galaxy center.
Figure 2. Time evolution of the angular momentum change \( L_z - L_z(0) \), where \( L_z \) is the \( z \)-component of the angular momentum and \( L_z(0) \) is its initial value. The black, red, and blue lines correspond to the halo, disk, and bulge components, respectively. The green line shows the change of the total angular momentum. The black, red, and blue lines correspond to the halo, disk, and bulge components, respectively. The green line shows the change of the total angular momentum. The bar becomes oval in shape and cannot be identified well, as shown in panel (g). However, at the late stage of the simulation, a longer outer bar with a size of \( \sim 6 \) kpc and transient spirals recur, as shown in panel (h), lasting until the end of the simulation.

Figure 2 shows the angular momentum change \( L_z - L_z(0) \) as a function of time, where \( L_z \) is the \( z \)-component of the angular momentum and \( L_z(0) \) is its initial value. The black, red, and blue lines correspond to the halo, disk, and bulge components, respectively. The green line shows the total \( L_z \), which is well conserved over the full simulation time. For \( t \sim 0 \)–0.7 Gyr, when the inner bar starts to develop, angular momentum is mainly exchanged between the bulge and disk components, as the red and blue curves mirror each other. After \( t \sim 2 \) Gyr, the red and blue lines show that the angular momentum is exchanged primarily between the disk and halo. The angular momentum of the halo component increases as the bulge and disk components lose their angular momentum, as suggested by Debattista & Sellwood (2000) and Athanassoula (2003). Finally, the exchange of angular momentum between the disk and the halo is inefficient for \( t \sim 4.5 \)–6.0 Gyr as the outer bar becomes weaker.

3. The Mode/Pattern Speeds and Strengths

Here three methods are used to evaluate the rotation speeds of the inner and outer bars: (1) the Fourier method for the mode pattern speed of different modes, (2) the Jacobi integral method for the rotational angular speed of the gravitational potential, and (3) the moment of inertia method for the bisymmetric mass moment angular speed. An advantage in using these methods is that the instantaneous speed is determined rather than the time-averaged speed frequently used in the \( N \)-body spiral/bar model literature. Hence, the short-timescale variability of the system can be probed.

A brief description of the three methods is given in Appendices A–C, with each method applied to a few snapshots of the \( N \)-body simulation to illustrate their adequacy. Note that the different methods measure different characteristics of the structure, giving slightly different results because the structure is actually not time independent. No discrepancies occur when the structure is rigid and constantly rotating, which has been tested by imposing a solid-body rotation to the particles and by calculating the rotation speed with the described methods. Using three different methods allows us to better grasp the weak or strong points of each method and to better understand the double-barred system. In the following, we compare the results using the different methods in Section 3.1 and show the time evolution of the mode/pattern speed and the bar strength over the full simulation time in Section 3.2.

3.1. Comparison of the Different Methods

Figure 3 shows the radial profile of the mode/pattern speed and the strength of the inner and outer bars at selected times when the two bars are (1) aligned \( (t = 2.385 \) Gyr), (2) misaligned by \( 45^\circ \) \( (t = 2.393 \) Gyr), and (3) perpendicular to each other \( (t = 2.401 \) Gyr). The different colors represent the results of the three different methods. The magenta, blue, and black lines correspond to the Fourier method \( (m = 2 \) mode), Jacobi integral method, and moment of inertia method, respectively. The mode speed of the Fourier \( m = 2 \) mode and the pattern speeds \( \Omega_J \) and \( \Omega_M \) are obtained using Equations (4), (9), and (12), respectively. The strength of the Fourier \( m = 2 \) mode, \( \eta_2 \), and the strength \( \eta_M \), which is determined by the moment of inertia method, are calculated using Equations (3) and (13). The inset within each left panel reveals the mode/pattern speed near the outer bar and the spiral region with an expanded scale on the \( y \)-axis in order to clearly show the variation of the pattern speed in the radial direction.

It is evident from the left panels of Figure 3 that the mode/pattern speed of the two bars varies somewhat with radius, \( R \). Specifically, the degree of variation of the mode/pattern speed of the inner bar \( (R \sim 0.3 \)–0.7 kpc) is greater than that of the outer bar \( (R \sim 3 \)–5 kpc), especially when two bars are perpendicular to each other, as shown in Figure 3(e). We note that the mode/pattern speed cannot be well determined near the galaxy center \( (R < 0.2 \) kpc), especially for the Fourier method (magenta line) and the moment of inertia method (black line). This is due to the fact that these two methods require a bisymmetric component to calculate the mode/pattern speed. Finally, none of these three methods find similar speeds in the region between the bars, which is the expected consequence of the absence of a well-defined steady pattern there. For example, all three methods yield a negative pattern speed at \( R = 1.4 \)–2 kpc in Figure 3(e), which results from the
interaction of two different pattern speeds of the two bars rather than only one single pattern. Overall, the Jacobi integral method (blue line) reveals a smoother radial pattern speed profile than the other two methods mainly because it probes the gravitational potential, which is relatively less reactive to the spatial density fluctuations used in the Fourier and moment of inertia methods.

The right panels of Figure 3 compare the strength of the Fourier $m = 2$ mode $\eta_2$ (magenta line) and the strength $\eta_M$ (black line) at selected times. The strength $\eta_M$ appears slightly smaller than the strength $\eta_2$, but their variations are consistent. The regions of the inner bar and the outer bar can be easily identified corresponding to the two peaks near $R = 0.3$ and 5 kpc, respectively. In addition, the strength corresponding to $\eta_2$ and $\eta_M$ in the transition region lying between the inner and outer bars ($R \sim 1\text{–}2$ kpc) is small owing to the lack of a bisymmetric component.

### 3.2. Time Evolution of the Mode/Pattern Speeds and Strengths

In Section 3.1, the radial profiles of the mode/pattern speed and bar strength at selected times are displayed. In this subsection, the temporal variation of the mode/pattern speed and the strength over the full duration of the simulation are presented.

Specifically, the time evolution of the mode/pattern speed is illustrated in Figure 4. The color map indicates the pattern speed in units of $\text{km s}^{-1}\text{kpc}^{-1}$ for the asinh function argument. The regions in white indicate the radii where the pattern speed cannot be well determined owing to a weak strength or strongly negative values. The criterion of the strength is 0.1 for panels (a), (c), and (d), and 0.02 for panel (b). Panels (a) and (b) illustrate the mode speed of the Fourier $m = 2$ and 4 modes, respectively. Panels (c) and (d) show the pattern speed obtained using the Jacobi integral method and the moment of inertia method, respectively. The radial bin size is 0.1 kpc, and the time interval between two snapshots is 1.96 Myr.

As shown in Figure 4, it is apparent that the mode or pattern speeds near the galaxy center ($R < 0.2$ kpc) and in the region between the inner and outer bars are not well determined, as mentioned in Section 3.1. Furthermore, the color variations show that the mode/pattern speed varies not only in radius but also, more importantly, in time.

The time evolution of the strength is shown in Figure 5. Panels (a)–(c) show the strength of the Fourier $m = 1$, 2, and 4 modes. The strength of the Fourier $m = 3$ mode is not shown here since it is similar to the $m = 1$ mode but weaker. Panel (d) illustrates the strength that is determined from the moment of inertia method. The radial bin size and time interval between two snapshots are the same as in Figure 4. As can be seen from panels (b) and (d), the strength of the Fourier $m = 2$ mode is similar to the strength derived from the moment of inertia.
method. As for the pattern speed, the color variations show that the strength varies both in radius and in time.

For example, during half of the rotation period that corresponds to the relative phase angle between the two bars changing from 0° to 90°, such as from $t = 2.385$ to $2.401$ Gyr, the pattern speed $\Omega_J$ varies $\sim 4\%$ at $R = 0.5$–$0.6$ kpc, which is located within the inner bar region, and the pattern speeds $\Omega_{F2}$ and $\Omega_M$ change by $\sim 18\%$ and $\sim 37\%$, respectively. In the same interval, the strengths $\eta_2$ and $\eta_M$ at $R = 0.5$–$0.6$ kpc change by $\sim 8\%$.

Spatial variations of the pattern speed in the radial direction, for example, in the region $R = 0.3$–$0.8$ kpc with the radial bin size 0.1 kpc, when two bars are perpendicular to each other at $t = 2.401$ Gyr, are about $16\%$ and $12\%$ for $\Omega_J$ and $\Omega_{F2}$, respectively. However, the variation of $\Omega_M$ can be much larger and up to $75\%$. The strength is also seen to vary, with $\eta_2$ and $\eta_M$ varying by $\sim 30\%$. Hence, it is clear from Figures 4 and 5 that both the mode/pattern speed and the strength vary substantially with respect to the radius $R$ and time.

In comparing the different modes, the $m = 2$ mode dominates the whole system for most of the time, as shown in Figures 5(a)–(c). However, the strength of the Fourier $m = 1$ mode increases near the galaxy center and is comparable to the strength of the Fourier $m = 2$ mode at later times such as $t = 4.4$–$4.5$ Gyr. The strength of the Fourier $m = 4$ mode also increases prior to $t = 3.2$ Gyr but weakens thereafter. Further

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**Figure 4.** Time evolution of the mode/pattern speed. The color map illustrates the mode/pattern speed in asinh scale in units of km s$^{-1}$ kpc$^{-1}$. The radial bin size and the time interval between two snapshots are 0.1 kpc and 1.96 Myr, respectively. Panels (a) and (b) present the mode speeds of the Fourier $m = 2$ and 4 modes, respectively. Panels (c) and (d) show the pattern speeds that are obtained from the Jacobi integral method and the moment of inertia method, respectively.
Dynamically the strength and rotation speed variations described here are significantly strong and fast, placing in question the often-adopted assumption of individual star action conservation in galaxies (e.g., Sanders & Binney 2016), at least in barred galaxies, and thus in the Milky Way. The overall dynamics of our simulated system is not characterized by an adiabatic slow evolution with low space and time frequencies, required for action conservation, but rather by high-frequency spacetime semichaotic fluctuations superposed on a slow evolution. For individual stars the consequence is that they may diffuse through phase space with characteristic timescales commensurable with the rotational period. Such a behavior has already been documented several times in $N$-body simulation of barred galaxies (e.g., Brunetti et al. 2011).

### 4. Time Evolution of the Two Bars

In Section 3, we have shown that the mode/pattern speed and the strength in our model galaxy vary with radius $R$, as well as with time. In the following, we quantitatively discuss not only the temporal variation of the pattern speeds and strengths of the inner and outer bars but also their sizes.

#### 4.1. Methods for Determining the Sizes of the Two Bars

To determine the size of a bar, we need first to define an explicit procedure to calculate it in a reproducible way. It was
found that the procedures required are distinct for the inner and outer bars as their adjacent perturbing patterns are different.

4.1.1. The Inner Bar

To determine the size of the inner bar, we consider the strength and the phase parameters from the $m = 2$ Fourier mode and adopt the following procedure. For illustration, Figure 6 shows two examples for determining the size of the inner bar. Panels (a) and (b) show the strength profile at $t = 2.385$ and 3.944 Gyr, respectively. The selected particles are located within the $|z| < \text{half of the disk scale height}$, and the radial bin size is 0.1 kpc. Panels (c) and (d) present their corresponding phase profile $\phi_2$.

The quantity, $\eta_{0.8}$, corresponding to a strength that is 80% of its maximum value, $\eta_{\text{max}}$, in the inner bar region, is used to define the boundary of the bar. The 80% strength allows us to exclude the regions mentioned in Section 3, where the pattern speed is ill determined, and to cover the main portion of the bar. The blue dashed line in Figures 6(a) and (b) indicates $\eta_{0.8}$ at $t = 2.385$ and 3.944 Gyr. Since the strength $\eta_2$ continuously decreases with respect to the radial distance away from the point corresponding to $\eta_{\text{max}}$, the radius corresponding to $\eta_{0.8}$ is derived from the linear interpolation of the two closest points to $\eta_{0.8}$. For example, as shown in Figure 6(a), points “a” and “b” are selected to interpolate for the inner radius corresponding to $\eta_{0.8}$. The same method is used to determine the outer radius corresponding to $\eta_{0.8}$ using points “c” and “d.”

A phase difference criterion is also used to determine the size of the inner bar. The phase difference between $\phi_{\text{max}}$, which is the phase at the radius corresponding to $\eta_{\text{max}}$, and the points within the inner bar is required to be less than $5^\circ$. In the first example, as shown in Figure 6(c), the phase difference between $\phi_{\text{max}}$ and at all points between $R = R_a$ and $R_j$ is less than $5^\circ$; hence, the boundary of the inner bar is defined at $R_1$ and $R_2$. On the other hand, in the second example, Figure 6(d) shows that the boundary of the inner radius $R_1$ is located at point “b” because the phase difference between $\phi_{\text{max}}$ and point “a” is greater than $5^\circ$. Therefore, using the strength and the phase difference criteria together, the boundaries of the inner bar $R_1$ and $R_2$, marked by the red crosses in Figures 6(a) and (b), can be determined, and the size of the inner bar is defined by the outer boundary, that is, $R_{2}$. The radii $R_{1}$ and $R_{2}$ are used for determining the strength and the mode/pattern speed of the inner bar, as described in Section 4.2.
4.1.2. The Outer Bar

A different method is adopted to determine the boundaries of the outer bar because the presence of the inner bar and the spiral structure complicates the radial profiles of the strength and phase in the outer bar region. For example, the strength of the spiral can be comparable to or even stronger than the outer and phase in the outer bar region. As can be seen from Figure 7, during the initial development of the spiral structure complicates the radial propagation.

To determine the radial boundaries of the outer bar, we adopt the following procedure. We require that the outer bar has a nearly constant phase and constant pattern speed. In addition, we require that the strength within the outer bar boundaries is not too weak since we seek to identify a strong outer bar. Similar to the case for the inner bar, the phase, strength, and mode speed of the Fourier $m = 2$ mode are considered in determining the boundaries.

To reduce the fluctuations in the phase, strength, and pattern speed, these properties are averaged using centered finite differencing over two time intervals of $\pm 1.96$ Myr before and after the evaluated time. We restrict the radial region to be larger than 1 kpc and require the simultaneous satisfaction of the following five criteria:

1. The phase difference within the outer bar boundaries is less than $\pm 5^\circ$;
2. The slope of the phase, that is, $(\phi_2(R_{i+1}) - \phi_2(R_i))/(R_{i+1} - R_i)$, is less than $\pm 1.5^\circ$/kpc.
3. The pattern speed difference within the outer bar boundaries is less than $\pm 3$ km s$^{-1}$ kpc$^{-1}$.
4. The slope of the pattern speed, that is, $(\Omega_p(R_{i+1}) - \Omega_p(R_i))/(R_{i+1} - R_i)$, is less than $\pm 1$ km s$^{-1}$ kpc$^{-2}$.
5. The strength within the outer bar boundaries exceeds the minimal strength $\eta_k = 0.12$.

With the above criteria, the boundaries of the strong outer bar, namely, $R_{o1}$ and $R_{o2}$, can be determined over large time intervals. The outer boundary $R_{o2}$ is also used to determine the size of the outer bar. If the above criteria are not satisfied, we consider the outer bar to be absent. In the following, only the strong outer bar as defined above is discussed.

4.2. The Properties of the Two Bars

4.2.1. The Bar Size

Adopting the above procedure, Figures 7 and 8 show the time evolution of the inner boundary (the black dots) and the outer boundary (the red dots) of the two bars once they start to develop. As can be seen from Figure 7, during the initial development of the inner bar ($t \sim 30$–70 Myr), the size of the inner bar increases rapidly from $\sim 0.55$ to $\sim 0.8$ kpc. After $t = 570$ Myr, the outer boundary $R_2$ (the red dots) starts to oscillate with the period $\sim 35$ Myr, while the inner boundary $R_1$ (the black dots) does not significantly change. In addition, $R_1$ and $R_2$ oscillate with greater amplitude at certain times, such as at $t \sim 4.74$–4.87 Gyr.

As shown in Figure 8, the outer bar starts to develop at $t \sim 1.44$ Gyr but only lasts for $\sim 10$ Myr because the phase in the outer bar region changes quickly in the radial direction, which causes the phase variation to not satisfy our criterion. At $t \sim 1.66$ Gyr, the outer bar can be defined again and lasts about 3.37 Gyr, existing until $t \sim 5.03$ Gyr. During this stage, the size of the outer bar is about 5 kpc. After $t \sim 5.03$ Gyr, the outer bar cannot be well defined owing to its weak strength for about 1.4 Gyr. At $t \sim 6.40$ Gyr, another outer bar starts to develop at $R = 7$–9 kpc and can be well determined most of the time after $t = 6.83$ until the simulation ends. At some times the outer boundary $R_{o2}$ (red dots) suddenly increases, such as at $t = 2.21$, 2.55, and 7.52 Gyr, due to the disappearance of spiral structure (see Section 5.3).

4.2.2. The Bar Strength

Figures 9 and 10 show the strength of the inner and outer bar, respectively. The strength of the bar is calculated from the particles located between the inner and outer boundary in the $R$-direction and $|z| < 0.225$ kpc (half of the disk scale height). The yellow, magenta, and green dots illustrate the strength of the Fourier mode $m = 1$, 2, and 4. The black dots represent the strength $\eta_m$. The strength $\eta_m$ is consistent with the strength $\eta_2$ differing by only about 0.03 and 0.01 for the inner and outer bar, respectively. The strength $\eta_2$ is not shown here because its time evolution is similar to the strength $\eta_1$ with about half of its value.

It can be seen from Figure 9 that the strength $\eta_2$ of the inner bar increases rapidly from $\sim 0.05$ to $\sim 0.5$ during the initial development of the inner bar ($t \sim 30$–70 Myr). Subsequently, it oscillates and decreases, reaching a relatively stable stage at $t \sim 570$ Myr. At $t \sim 2.55$ Gyr, the strength $\eta_2$ starts to increase and oscillate significantly. For example, it is about half of the strength $\eta_2$ mode at $t = 2.6$ Gyr, becoming comparable to the strength $\eta_2$ at certain times, such as at $t \sim 4.8$ Gyr. Compared with the strength $\eta_1$ and the strength $\eta_2$, it can be seen that $\eta_1$ is about five times smaller than $\eta_2$ on average.

In the case of the outer bar, as shown in Figure 10, the $m = 2$ mode dominates during the whole simulation. The strength $\eta_2$ is about 2.5–3 times smaller than the strength $\eta_2$. In contrast to the inner bar, the strength $\eta_1$ is very weak and is never comparable to the strength $\eta_2$. The strength $\eta_1$ is even weaker than the $\eta_1$ and not shown here. Furthermore, it is evident that the strength of the outer bar oscillates in time, especially for $\eta_2$ and $\eta_M$. For example, the strength $\eta_2$ decreases at $t = 2.21$, 2.55, 2.90, and 3.28 Gyr.

4.2.3. The Mode/Pattern Speed

Figures 11 and 12 illustrate the time evolution of the mode/pattern speed of the inner bar and outer bar, respectively. As for calculating the strength of the bars, their mode/pattern speed is calculated only for the particles located between the inner and outer boundary in the $R$-direction and $|z| < 0.225$ kpc. The magenta dots represent the mode speed of the Fourier $m = 2$ mode. The blue and black dots represent the pattern speed, which is determined by the Jacobi integral method and the moment of inertia method, respectively.

As shown in Figure 11, it is apparent that the mode speed of the $m = 2$ mode of the inner bar is $\sim 50$ km s$^{-1}$ kpc$^{-1}$ greater than the pattern speeds, which are derived from the other two methods, but has a similar evolutionary trend. At the beginning, the mode/pattern speed of the inner bar decreases rapidly. After the outer bar-like structure forms ($t \sim 570$ Myr), this speed decreases gradually.

Similar to the oscillation of the boundary $R_2$ as shown in Figure 7, the mode/pattern speed also oscillates with a higher frequency (corresponding to a time interval of $\sim 35$ Myr). In addition, the pattern speed $\Omega_2$ decreases by about $50$ km s$^{-1}$ kpc$^{-1}$ and oscillates with a lower frequency.
(corresponding to a time interval of $\sim 400$ Myr), while the oscillation amplitude for the pattern speed $\Omega_M$ increases.

For the outer bar, as shown in Figure 12, the mode speed $\Omega_{F2}$ is about 35 km s$^{-1}$ kpc$^{-1}$ and the pattern speeds $\Omega_J$ and $\Omega_M$ are about 28 km s$^{-1}$ kpc$^{-1}$ on average. After $t = 6.40$ Gyr, a new longer outer bar forms, as mentioned in Section 4.2.1. Its mode/pattern speed is about 10 km s$^{-1}$ kpc$^{-1}$ lower than the shorter outer bar, which exists before $t \sim 5.03$ Gyr.

We note that the time evolution of the mode speed $\Omega_{F2}$ has a similar oscillation trend to the pattern speed $\Omega_M$. For example, both the mode speed $\Omega_{F2}$ and the pattern speed $\Omega_M$ decrease by about 5 km s$^{-1}$ kpc$^{-1}$ at $t = 2.99$ and 3.28 Gyr. This oscillation is consistent with the results in Figures 4(a) and (d). Similarly, the pattern speed $\Omega_J$ shows a high-frequency ($\sim 35$ Myr) oscillation, but with a smaller pattern speed variation ($\sim 3$ km s$^{-1}$ kpc$^{-1}$).

5. The Kinematic Effects in the Double-barred Galaxy

The description for the time evolution of the size, the strength, and the mode/pattern speed of the two bars has been presented in Section 4.2. Here, we describe the effect of the inner bar–outer bar
interaction, the \( m = 1 \) mode structure, and the outer bar–spiral interaction on the properties of the two bars.

5.1. The Interaction of the Two Bars

Once the outer bar (like) structure forms (\( t = 570 \) Myr), the two bars regularly interact, resulting in the oscillation in the size, strength, and mode/pattern speed of the inner bar. The period of the oscillation is related to the phase difference between the inner and outer bars and can be calculated as follows: \( \pi / (\Omega_i - \Omega_o) \), where \( \Omega_i \) and \( \Omega_o \) are the pattern speeds of the inner and outer bar, respectively. For example, at \( t \sim 2.8 \) Myr, the oscillation period is \( \sim 35 \) Myr because the difference of the pattern speed between the two bars is about 85 km s\(^{-1}\) kpc\(^{-1}\).

The size of the inner bar, i.e., the radius \( R_{i,2} \), is smaller when the two bars are perpendicular to each other. This follows from the fact that the strength of the inner bar decreases more rapidly from \( \eta_{\text{max}} \) to the region between two bars (\( R \sim 1–2 \) kpc) when the two bars are perpendicular to each other, and it reaches \( \eta_{0.8} \) at smaller radius, as demonstrated in Figure 3. This result is in good agreement with Maciejewski & Sparke (2000).

![Figure 8](image.png)

Figure 8. Time evolution of the boundaries of the outer bar. The black and the red dots represent the radii \( R_{o,1} \) and \( R_{o,2} \), respectively.
Similar to the oscillation of the inner bar size, as shown in Figure 11, the strength and the pattern speed of the inner bar also oscillate corresponding to the phase angle between the inner bar and the outer bar when the $m = 2$ mode dominates the whole system. For example, when the two bars are aligned, such as at $t \sim 2.385$ Gyr, the strength of the inner bar is at its local minimum and both the mode speed $\Omega_{F2}$ and the pattern speed $\Omega_{J}$ are at their local maximum. On the other hand, when two bars are perpendicular with respect to each other, such as at $t \sim 2.401$ Gyr, the strength of the inner bar is at its local maximum and $\Omega_{F2}$ and $\Omega_{J}$ are at their local minimum. This behavior is consistent with the results in Maciejewski & Sparke (2000), Debattista & Shen (2007), and Du et al. (2015).

Due to their mutual gravitational interaction, the pattern speed of the outer bar is also affected. As can be seen in Figure 12, the pattern speed $\Omega_{J}$ of the outer bar shows a high-frequency oscillation with a smaller pattern speed variation ($\sim 3 \text{ km s}^{-1} \text{ kpc}^{-1}$). When the two bars are aligned/perpendicular,
is higher/lower. This result also agrees with those found by Debattista & Shen (2007) and Du et al. (2015). However, in our simulation, the strength of the outer bar is affected more by the spiral structure, which is beyond the scope of their paper, and will be mentioned in Section 5.3.

5.2. The $m = 1$ Mode

As described previously, an $m = 1$ mode of the inner bar develops during the evolution. Off-centered bars have long been described in observations (e.g., de Vaucouleurs & Freeman 1970), indicating that an $m = 1$ mode can occur as naturally as an $m = 2$ mode. In our simulation, the $m = 1$ mode strength increases and oscillates significantly after $t \sim 2.55$ Gyr, as mentioned in Section 4.2.2. At certain times, such as at $t \sim 3.90$ and 4.47 Gyr, it is even comparable to the strength of the $m = 2$ mode. When the strength $\eta_1$ becomes large, such as during $t \sim 2.73$–2.87 Gyr, the strength of the $m = 2$ and 4 modes and the size of the inner bar $R_{i2}$ start to oscillate with greater amplitude, as shown in Figures 7 and 9.

Figure 10. Time evolution of the strength of the outer bar. The dots have the same meaning as in Figure 9.
This situation recurs repeatedly with the period about 300–400 Myr before $t \sim 5.85$ Gyr.

Furthermore, the existence of the strong $m = 1$ mode also affects the pattern speed of the inner bar. $\Omega_J$ decreases and oscillates with a lower frequency (corresponding to a time interval of $\sim 400$ Myr), and $\Omega_M$ oscillates with greater amplitude, as can be seen in Figure 11 at $t \sim 2.73–2.87$.

As a result, the existence of the strong $m = 1$ mode affects not only the strength of other modes of the inner bar but also the size and the pattern speed of the inner bar. Among the three different mode/pattern speeds, the pattern speed $\Omega_J$ is more sensitive to the existence of the Fourier $m = 1$ mode because it probes the gravitational potential, which is more reactive to the movement of the system.

The spontaneous occurrence of odd modes here is a reminder that it is important in $N$-body simulations not to impose a rigid and fixed halo potential, as it prevents the evolution of odd modes and violates Newton’s third law.
5.3. The Spiral

In the evolution of our galaxy model, spiral structure is found to develop in the outer parts of the galaxy. No steady-state pattern is found to develop, as the local spiral pattern speed decreases with \( R \) and is actually close to the local circular rotation frequency (Figure 3). As a consequence, the spiral arms are transient especially after \( t \sim 0.7 \text{ Gyr} \), as mentioned in Section 2.2. Given its location, the appearance of the spiral structure and the size, strength, and mode/pattern speed of the outer bar are related. In particular, the size of the outer bar \( R_{o2} \) increases upon the disappearance of the spirals, such as at \( t = 2.21 \) and 2.55 Gyr, or at similar phases, such as during \( t = 7.52–7.56 \text{ Gyr} \). Under these two circumstances, the phase near the tips of the outer bar can remain constant within \( \pm 5^\circ \), the first criterion for determining the outer bar as mentioned in Section 4.1.2, up to a larger radius.

Figure 13 shows at two times the torque \( z \)-component when the transient spirals appear (panel (a)) and disappear (panel (b)). It is apparent that in the bar rotating frame, the torque changes more in the spiral region than in the bar region. In other words, the time modulation of the torque by the spirals on
the bar is less noticeable than the torque by the bar on the spirals. Furthermore, the torque varies more in time than the morphology, as it is the driver of the morphology changes. However, knowing the torque is not sufficient for predicting the pattern speed change because a bar is not a solid body but a nonlinear density wave with internal streaming.

As mentioned in the Introduction, the cause of the time dependence studied here could have been expected long ago from the work of Sellwood & Sparke (1988), who first showed that a single bar and its adjacent spiral arms rotate on average at distinct speeds. Thus, in the case that an adjacent bar or spiral structures rotate with different speeds, their mutual interactions cause the intermediate region to be strongly time dependent in any rotating frame. The torque modulation timescale is given by $\tau_{\text{mod}} = 2\pi/|m_X \Omega_X - m_X \Omega_X|$, where $m_X$ denotes the number of arms and $\Omega_X$ the pattern speed of structure $X$. For a typical bar–bar or a bar—four-arm spiral pattern, $\tau_{\text{mod}}$ is comparable (by a factor of $\sim 2$) to the rotational period evaluated at a radius between the two structures, $\tau_{\text{mid}} = 4\pi/(\Omega_X + \Omega_X)$, meaning that the modulation is dynamically fast. Since the cause of time dependence is identified for a single bar and spiral system, the description detailed here for a double-bar system should be even more valid owing to the existence of additional but weaker time-dependent torque, such as the modulation of the spiral by the inner bar, and vice versa.

6. Conclusions

To further elaborate on the short-timescale variations of the barred structure in a disk galaxy model reported in Wu et al. (2016), three methods are used to measure the instantaneous and local mode/pattern speed rather than the time-averaged mode/pattern speed as often done in other works. Using the Fourier mode method (which measures a mode speed), the Jacobi integral method (which measures the potential rotation speed), and the moment of inertia method (which measures the rotation speed of the second-order mass moment tensor), we have investigated the instantaneous variations of the bar size, strength, and pattern speed of the bars.

Since the $N$-body bars and spiral arms in our double-barred galaxy model are found to be flexible at different locations, the different methods measure different characteristics of the structure, allowing one to quantify the time dependence of the structures at different locations. The discrepancies between the methods are large when the time dependence is strong and small when the pattern speed is well defined and nearly constant. For example, we have shown that in our galaxy model the $m = 2$ Fourier mode speed of the inner/outer bar is larger than the pattern speed determined by the Jacobi integral method and the moment of inertia method. In addition, the pattern speed of the inner bar determined by the Jacobi integral method can vary by $50$ km s$^{-1}$ kpc$^{-1}$ owing to the appearance of the $m = 1$ Fourier mode.

Although different methods yield slightly different results, all of them demonstrate that the inner and outer bars are flexible and time dependent in our model. As shown in Figure 3, the pattern speed and the strength vary with radius. The discrepancy between the methods is especially large when the hypothesis of a single pattern at constant speed is false, for example, between the inner and outer bar regions. The used methods allow us to localize the regions with an approximate single pattern and quantify their time dependence. It is found that the regions where the patterns are well defined and constant are smaller than commonly assumed.

To verify the changes of the bar structure, we quantify and investigate the variations of the bar characteristics in time. The results in Section 4 confirm that both the inner and outer bars are time-dependent structures. For the inner bar, the size, strength, and mode/pattern speed are affected more by the interaction with the outer bar. For example, when the two bars are aligned, the size and mode speed of the inner bar increase, but the strength decreases slightly. In the case of the outer bar, its characteristics are affected not only by the inner bar but also by the adjacent spiral structure. When the spiral structure is absent, the size of the outer bar increases, but the strength and mode/pattern speed decrease. Taken together, these results
confirm that the inner and outer bars are strongly time-
dependent structures and do not have a constant mode/pattern
speed. The characteristic timescales of substantial pattern speed
variations can be comparable to the rotational timescale or
longer, similar to the timescales of substantial pattern strength
variations. These results invalidate the assumption that the
actions of individual stars are well conserved, since the
morphology changes of the potential contain space and time
high-frequency semichaotic components superposed on a
slowly varying component.

In comparing the outer bar with the inner bar, the ratios of
their size, strength, and mode/pattern speed are also time
dependent. The size ratio between the outer and inner bars
pulsates and increases from 7 to 10 owing to the increment of
the outer bar size. Because the strength of the outer bar
decreases substantially with respect to time, the strength ratio
between the outer and inner bars decreases from about 1 to 0.5
before the shorter outer bar disappears. Finally, the mode/
pattern speed ratio between the outer and inner bars rises from
\( \sim 0.23 \) to 0.3 owing to the decrement of mode/pattern speed
of the inner bar. These results strengthen the view that the
evolution of two bars is time dependent and does not settle into
a specific resonant state.

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Appendix A
Fourier Method

A.1. The Mode Speed and Strength

The Fourier method measures the mode speed of the Fourier
mode \( m \) in the galactic plane. The inverse Fourier transform
of the particle distribution can be obtained by

\[ F_m = \sum_j m_j \exp(i m\theta_j) = \sum_j m_j (\cos(m\theta_j) + i \sin(m\theta_j)), \]

where \( \theta_j = \arctan(y_j, x_j) \) is the azimuthal angle of the particle \( j \)
given its Cartesian coordinates \( x_j, y_j \) and \( m_j \) is the particle mass.
Therefore, the phase of any mode \( m \) is

\[ \phi_m = \arctan(\Im(F_m), \Re(F_m)), \]

where \( \Re(F_m) = \sum_j m_j \cos(m\theta_j) \) and \( \Im(F_m) = \sum_j m_j \sin(m\theta_j) \),
and the strength of any mode \( m \) is defined as

\[ \eta_m = \frac{\sqrt{\Im^2(F_m) + \Re^2(F_m)}}{N}, \]

where \( N \) is the total number of selected particles. By
differentiating the phase \( \phi_m \) with respect to time, we obtain
the phase speed \( \dot{\phi}_m \), which is \( m \) times the mode speed \( \Omega_{Fm} \) in
ordinary space. In compact form it reads

\[ \Omega_{Fm} = \frac{\dot{\phi}_m}{m} = \frac{C + S}{C^2 + S^2}, \]

where the terms \( C, S, C_1, \) and \( S_1 \) are

\[ C = \sum_j m_j \cos(m\theta_j), \quad S = \sum_j m_j \sin(m\theta_j), \]

\[ C_1 = \sum_j m_j \cos(m\theta_j)\dot{\theta}_j, \quad S_1 = \sum_j m_j \sin(m\theta_j)\dot{\theta}_j, \]

and \( \dot{\theta}_j = (x_j v_{x,j} - y_j v_{y,j})/(x_j^2 + y_j^2). \) Thus, given the particle
positions \( x_j, y_j \) and velocities \( v_{x,j}, v_{y,j} \), the instantaneous mode speed \( \Omega_{Fm} \) can be calculated without time averaging.

A.2. Examples

In this subsection, we demonstrate the utility of the Fourier
method using the selected snapshots in our double-barred
system. Figure 14 shows examples of the radial mode speed
profile (left panels) and the strength profile (right panels) at
three different selected times when the relative phase angle
between the two bars is \( 0^\circ \) (\( t = 2.385 \) Gyr), \( 45^\circ \) (\( t = 2.393 \) Gyr), and \( 90^\circ \) (\( t = 2.401 \) Gyr). To calculate the mode speed
\( \Omega_{Fm} \) and the corresponding strength \( \eta_m \) at different radii, the
particles are selected in an annular bin with radial and vertical
sizes of \( 0.1 \) kpc and \( |z| < 0.225 \) kpc, respectively. The vertical
bin size is half the disk scale height. The black line and the red
line in Figure 14 denote the mode speed of the

The red and black lines in the right panels of Figure 14
illustrate the strength of the \( m = 2 \) and 4 modes. The two peaks
of the \( m = 2 \) strength correspond to the regions of the inner bar
and the outer bar, respectively. Note that the \( m = 4 \) mode
usually appears with the \( m = 2 \) mode. The strength of the
\( m = 2 \) mode is about five times larger than the \( m = 4 \) mode
for the inner bar. On the other hand, the relative strength in the
outer bar is smaller, with the strength of the \( m = 2 \) mode about
2–3 times larger than the \( m = 4 \) mode.
perpendicular to each other at $t = 2.385$ Gyr. The inner bar and the outer bar are aligned at $t = 2.385$ Gyr (panels (a) and (b)), 45° misalignment at $t = 2.393$ Gyr (panels (c) and (d)), and perpendicular to each other at $t = 2.401$ Gyr (panels (e) and (f)). The black line and the red line are for modes $m = 2$ and $4$, respectively. The subplots in the left panels show the radial mode speed near the outer bar with different scale.

Figure 14. Radial mode speed profile (left panels) and the strength profile (right panels) that are obtained by the Fourier method at the selected times $t = 2.385, 2.393$, and 2.401 Gyr. The inner bar and the outer bar are aligned at $t = 2.385$ Gyr (panels (a) and (b)), 45° misalignment at $t = 2.393$ Gyr (panels (c) and (d)), and perpendicular to each other at $t = 2.401$ Gyr (panels (e) and (f)). The black line and the red line are for modes $m = 2$ and $4$, respectively. The subplots in the left panels show the radial mode speed near the outer bar with different scale.

mentioned above, at $R \sim 1\sim 2$ kpc, which is the transition region between the inner and outer bar, the strength of the $m = 2$ mode is small, especially when two bars are perpendicular to each other, as shown in Figure 14(f).

The relative angle between two bars affects the system since Figures 14(a), (c), and (e) show that the mode speed profile changes according to the phase angle between two bars, especially near the inner bar region. This might be due to the change of the bar morphology, which causes the change of the rotation profile and the pattern speed along the major axis of the inner bar.

In summary, the Fourier method allows a determination of the instantaneous phase, strength, and mode speed of different modes. We find that the $m = 2$ and 4 modes usually exist together with different strengths and different mode speeds for both the inner and outer bar. Furthermore, the determination of the strength of the $m = 2$ mode can be used to determine the boundary of the bar as described in Section 4.

Appendix B
Jacobi Integral Method

B.1. The Pattern Speed

Here we describe a method for evaluating the pattern speed of the global gravitational potential given the positions, velocities, and accelerations of a set of particles, under the assumptions that (1) the Hamiltonian $H = E - \Omega_L \cdot L$, the so-called Jacobi integral, is conserved for a test particle in a constantly rotating potential $\Phi(x(t))$ (so $H = 0 = E - \Omega_L L$) and (2) the rotation axis of $\Omega_L$ is along the $z$-axis. As shown by D. Pfenniger et al. (2018, in preparation), the pattern speed $\Omega_f$ can be determined by

$$\Omega_f(t) = \frac{E}{L_z} = \frac{v(t) \cdot a(t) + \Phi(x(t))}{(x(t) \times a(t))_z},$$

(7)

where $x(t), v(t), a(t)$ are the position, velocity, and acceleration of a particle in the inertial frame, respectively, $E = \frac{1}{2}v^2(t) + \Phi(x(t))$, and $L_z = (x \times a)_z$ is the $z$-component of the local torque. For our analysis, the time derivative of the potential, $\dot{\Phi}(x(t))$, is calculated by taking a finite difference over the time interval of $\pm 0.12$ Myr.

In regions with a vanishing or small gravitational torque (the denominator of Equation (7)), the pattern speed is sensitive to noise. A more robust approach is to solve by linear least squares the following system:

$$\begin{pmatrix} x_1(t) \times a_1(t) \\ x_2(t) \times a_2(t) \\ \vdots \\ x_n(t) \times a_n(t) \end{pmatrix}_z \Omega_f(t) \approx \begin{pmatrix} v_1(t) \cdot a_1(t) + \Phi(x_1(t)) \\ v_2(t) \cdot a_2(t) + \Phi(x_2(t)) \\ \vdots \\ v_n(t) \cdot a_n(t) + \Phi(x_n(t)) \end{pmatrix},$$

(8)
simultaneously for \( n \) particles belonging to a given spatial bin. In such a system, the particles with vanishing torque do not weight the solution; thus, the result is more robust. Indeed, the analytical least-squares solution reads

\[
\Omega_j(t) = \frac{\sum_i \dot{E}_i \dot{L}_{ij}}{\sum_i \dot{L}_{ij}^2},
\]

which shows that the least-squares solution is an arithmetic mean weighted by the torque, so small torque particles have small weight in the mean, in contrast to solving directly Equation (7) for several particles and taking an average. A simple extension of this method described in D. Pfenniger et al. (2018, in preparation) allows one to find the full local pattern speed vector by adding to the left of Equation (8) the \( x \) and \( y \) components of the torque and solving simultaneously the least-squares system for the three components of \( \Omega_j \).

### B.2. Examples

The subsection below reveals the utility of the Jacobi integral method using the same selected snapshots as in Appendix A. Figure 15 illustrates the pattern speed \( \Omega_j \) in the \((x-y)\)-plane obtained using Equation (9) for the same dimensions as in Figure 14 (0.1 kpc square cells in the \((x-y)\)-plane, and 0.45 kpc height). The black curves represent the contours of the projected surface density. Panels (a)–(c) and panels (d)–(f) display the outer and inner bars at 15 and 2 kpc scales, respectively.

In the top panels, the pattern speed of the outer bar appears relatively constant \((\approx 25 \text{ km s}^{-1} \text{ kpc}^{-1})\) in contrast to the inner bar, except for the regions along the semimajor and semiminor axes of the outer bar. The pattern speed \( \Omega_j \) is somewhat noisy along the semimajor and semiminor axes of the outer bar because the torque \( L_z \) nearly vanishes in the swastika-shaped regions.

The bottom panels of Figure 15 illustrate the pattern speed of the inner bar. Similar to the top panels, the pattern speed is not well determined along the semimajor and semiminor axes of the inner bar, where the torque is close to zero. The morphology of the pattern speed in the \((x-y)\)-plane changes according to the relative angle between the inner and the outer bars. Specifically, the ill-defined region (dark-blue and gray regions) of the pattern speed is wider and circular when two bars are perpendicular to each other, as shown in panel (f).

Figure 16 presents the radial distribution of the pattern speed \( \Omega_j \) of the particles that are located within the \(|z| < 0.5 \text{ kpc}\) half of the disk scale height, i.e., 0.225 kpc, at the selected times, as in Figures 14 and 15. The radial bin size is 0.1 kpc, and the color represents the number of particles in a log scale. The white dots represent \( \Omega_j \) obtained by using Equation (9) in each radial bin, and the connected white line shows the radial profile of the pattern speed \( \Omega_j \).

From Figure 16, it is apparent that the pattern speed \( \Omega_j \) of most of the particles is about \( 120 \text{ km s}^{-1} \text{ kpc}^{-1} \) near the galaxy center \((R \approx 0.5 \text{ kpc})\) and about \( 25 \text{ km s}^{-1} \text{ kpc}^{-1} \) near the outer bar region \((R \approx 5 \text{ kpc})\). This is consistent with the results in Figure 15. We find that the pattern speed as determined from Equation (9) can be defined well near the inner bar and the outer bar region, as indicated by the white line. However, it is difficult to define the pattern speed close to the region between the two bars \((R \approx 1–2 \text{ kpc})\), as shown in panel (c), since the torque \( L_z \) is close to zero and the pattern speed is not well defined in this region, as shown in Figure 15(f).
From the radial profile of the pattern speed (white line) in Figure 16, it is evident that the pattern speed of the inner bar \( R \sim 0.3–0.7 \) kpc is not constant, varying by about \( \pm 10 \) km s\(^{-1}\) kpc\(^{-1}\). However, the pattern speed of the outer bar \( R \sim 3–5 \) kpc is relatively constant. In summary, the Jacobi integral method can reveal the pattern speed structure in two dimensions, as shown in Figure 15, and provides another measure of the pattern speed of the inner and outer bars, as shown in Figure 16.

**Appendix C**

**Moment of Inertia Method**

**C.1. The Pattern Speed and Strength**

In this subsection, we describe a method for evaluating the pattern speed of the mass-weighted second moment of the particle positions, the moment of inertia, knowing their positions and velocities. We restrict the discussion to the \( x-y \)-plane here, but the method can be extended to 3D for obtaining the full pattern speed vector as described in D. Pfenniger et al. (2018, in preparation).

The largest eigenvector of the tensor \( I \) of a set of particles,

\[
I \equiv \begin{pmatrix} I_{xx} & I_{xy} \\ I_{xy} & I_{yy} \end{pmatrix} = \begin{pmatrix} \sum_i m_i x_i^2 & \sum_i m_i x_i y_i \\ \sum_i m_i x_i y_i & \sum_i m_i y_i^2 \end{pmatrix},
\]

describes the principal axis of the selected particles and allows the azimuthal angle \( \phi_M \) of the principal axis to be calculated. Therefore, if the selected particles of mass \( m_i \) are located within the bar region, on the \((x-y)\)-plane of Cartesian coordinates \( x_i, y_i \) whose center of mass is at the origin, the azimuthal angle of the semimajor axis of the bar reads

\[
\phi_M = \frac{1}{2} \arctan(2L_{xy}, I_{xx} - I_{yy}),
\]

as described in Wu et al. (2016). The time derivative of the angle \( \phi_M \) yields the angular speed of the principal axis,

\[
\Omega_M = \frac{\dot{\phi}_M(t)}{- \frac{1}{2} \left( D_{xx}I_{yy} - D_{yy}I_{xx} \right)},
\]

where \( D_{xx} = \frac{1}{2}(I_{xx} - I_{yy}) \) and \( D_{yy} = \sum_i m_i(x_i y_i - y_i x_i) \).

In addition, the strength of the bar \( \eta_M \) can be estimated by the eigenvalues \( \lambda_\pm \) of the tensor \( I \). Defining \( S = \frac{1}{2}(I_{xx} + I_{yy}) \),

\[
D = \frac{1}{2}(I_{xx} - I_{yy}), \quad \text{and} \quad P = \sqrt{D^2 + I_{xy}^2},
\]

the strength of the bar parameter reads

\[
\eta_M = 1 - \frac{\lambda_-}{\lambda_+},
\]

where \( \lambda_- = S - P \) and \( \lambda_+ = S + P \). Further details can be found in D. Pfenniger et al. (2018, in preparation).

Figure 16. Radial pattern speed \( \Omega_p \) profile at the selected times \( t = 2.385, 2.393, \) and 2.401 Gyr, the same as in Figure 15. The color representation illustrates the number of particles in log scale. The radial and the pattern speed bin size are 0.1 kpc and 5 km s\(^{-1}\) kpc\(^{-1}\), respectively. The white dots denote the pattern speed \( \Omega_p \), which is obtained by solving Equation (9) in each radial bin, and the connected white line shows the radial pattern speed profile.
C.2. Examples

The utility of the moment of inertia method will be demonstrated using three selected snapshots, the same as in Appendices A and B. To obtain the radial profile of the pattern speed and the strength, the particles located within the \(|z| < \frac{1}{2}\) of the disk scale height are binned in the \(R\)-direction with the bin size \(\Delta R = 0.1\) kpc, the same as in the Fourier method and the Jacobi integral method. Then, the pattern speed \(\Omega_M\) and the strength \(\eta_M\) are calculated for each annulus.

The left panels of Figure 17 show the radial profile of the pattern speed \(\Omega_M\) at selected times \(t = 2.385, 2.393,\) and 2.401 Gyr. The radial bin size is 0.1 kpc. The subplot in each left panel shows the radial pattern speed \(\Omega_M\) near the outer bar and spiral region with a different scale in the \(y\)-axis.

The right panels of Figure 17 illustrate the radial profile of the strength \(\eta_M\) at selected times. Similar to the strength of the Fourier \(m = 2\) mode, two peaks around \(R = 0.3\) and 5 kpc show the regions of the inner bar and the outer bar, respectively. In addition, at the transition region between the inner and outer bars \((R \sim 1-2\) kpc\), the strength \(\eta_M\) is small, which corresponds to the region having ill-determined pattern speed.

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