Simple renormalizable flavor symmetry for neutrino oscillations

Y. H. Ahn*, Seungwon Baek†, Paolo Gondolo‡

School of Physics, KIAS, Seoul 130-722, Korea

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Abstract

The recent measurement of a non-zero neutrino mixing angle \( \theta_{13} \) requires a modification of the tri-bimaximal mixing pattern that predicts a zero value for it. We propose a new neutrino mixing pattern based on a spontaneously-broken \( A_4 \) flavor symmetry and a type-I seesaw mechanism. Our model allows for approximate tri-bimaximal mixing and non-zero \( \theta_{13} \), and contains a natural way to implement low and high energy CP violation in neutrino oscillations, and leptogenesis with a renormalizable Lagrangian. Both normal and inverted mass hierarchies are permitted within 3\( \sigma \) experimental bounds, with the prediction of small (large) deviations from maximality in the atmospheric mixing angle for the normal (inverted) case. Interestingly, we show that the inverted case is excluded by the global analysis in 1\( \sigma \) experimental bounds, while the most recent MINOS data seem to favor the inverted case. Our model make predictions for the Dirac CP phase in the normal and inverted hierarchies, which can be tested in near-future neutrino oscillation experiments. Our model also predicts the effective mass \( |m_{ee}| \) measurable in neutrinoless double beta decay to be in the range \( 0.04 \lesssim |m_{ee}| \lesssim 0.15 \) eV for the normal hierarchy and \( 0.06 \lesssim |m_{ee}| \lesssim 0.11 \) eV for the inverted hierarchy, both of which are within the sensitivity of the next generation experiments.

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* Email: yhahn@kias.re.kr
† Email: swbaek@kias.re.kr
‡ Email: paolo.gondolo@utah.edu. On sabbatical leave from University of Utah.
I. INTRODUCTION

The large values of the solar ($\theta_{12} \simeq 35^\circ$) and atmospheric ($\theta_{23} \simeq 45^\circ$) [1] neutrino mixing angles may be telling us about new symmetries in the lepton sector not present in the quark sector, and may provide us with a clue to the nature of the quark-lepton physics beyond the standard model. Theoretically, a great deal of effort has been put into constructing flavor models with high predictive power, especially those giving the tri-bimaximal (TBM) mixing angles [2]:

$$\theta_{13} = 0, \quad \theta_{23} = \frac{\pi}{4} = 45^\circ, \quad \theta_{12} = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right) \simeq 35.3^\circ. \quad (1)$$

However, the Daya Bay and RENO collaborations [3, 4] have reported the first measurements of a non-zero value for the mixing angle $\theta_{13}$:

$$\sin^2 2\theta_{13} = 0.092 \pm 0.016(\text{stat}) \pm 0.005(\text{syst}), \quad (2)$$

and

$$\sin^2 2\theta_{13} = 0.113 \pm 0.013(\text{stat}) \pm 0.019(\text{syst}), \quad (3)$$

respectively, corresponding to an angle $\theta_{13} \approx 9^\circ$. These results are in good agreement with the previous data from the T2K, MINOS and Double Chooz collaborations [5]. A non-zero value of $\theta_{13}$ indicates that the TBM pattern for neutrino mixing should be modified. In addition, at the Neutrino 2012 conference in Kyoto, the MINOS Collaboration has announced a non-maximal value for the atmospheric mixing angle $\theta_{23}$ [6],

$$\sin^2 2\theta_{23} = 0.94^{+0.04}_{-0.05} \pm 0.04, \quad (4)$$

with maximal mixing disfavored at the 88% C.L. This result, which was not included in the global analysis in [7], comes from the analysis of $\nu_\mu$ disappearance in the MINOS accelerator beam, and points to one of two possible values for $\theta_{23}$, namely $\theta_{23} = 38^\circ$ or $\theta_{23} = 52^\circ$. If it holds, this result also calls for a deviation from the tri-bimaximal mixing pattern.

Furthermore, the presence of CP violation in the lepton sector is still unknown. Experimentally, CP violation may become observable in a future generation of neutrino oscillation experiments [T2K, NO$\nu$A] [8]. Theoretically, a flavor symmetry that describes and explains the large reactor mixing angle $\theta_{13} \simeq 9^\circ$ while keeping the TBM values $\theta_{23} \simeq 45^\circ$ and
\( \theta_{12} \cong 35^\circ \) may originate in two ways: (i) a large \( \theta_{13} = \frac{\lambda_C}{\sqrt{2}} \), with \( \lambda_C \) the Cabbibo angle, mainly governed by higher-order corrections in the charged lepton sector \[9\], where the TBM pattern is a good zero-order approximation to reality, or (ii) a large \( \theta_{13} \) from the neutrino sector itself through a new flavor symmetry without resorting to higher-order corrections in the charged lepton sector \[10\].

In this paper, we propose a new and simple model for the lepton sector with \( A_4 \) flavor symmetry in the framework of a type-I seesaw mechanism. It is different from previous works using \( A_4 \) flavor symmetries \[11–14\] in that the Dirac neutrino Yukawa coupling constants do not all have the same magnitude. Our model can naturally explain the TBM large value of \( \theta_{13} \) and can also provide a possibility for low energy CP violation in neutrino oscillations with a renormalizable Lagrangian and small Yukawa coupling parameters, i.e. neutrino masses. The seesaw mechanism, besides explaining of smallness of the measured neutrino masses, has another appealing feature: generating the observed baryon asymmetry in our Universe by means of leptogenesis \[17\]. Since the conventional \( A_4 \) models realized with type-I or -III seesaw and a tree-level Lagrangian lead to an exact TBM and vanishing leptonic CP-asymmetries responsible for leptogenesis (due to the proportionality of the \( Y_\nu^\dagger Y_\nu \) combination of the Dirac neutrino Yukawa matrix \( Y_\nu \) to the unit matrix), authors usually introduce soft-breaking terms or higher-dimensional operators with many parameters, in order to explain the non-zero \( \theta_{13} \) as well as the non-vanishing CP-asymmetries.

Our model is based on a renormalizable \( SU(2)_L \times U(1)_Y \times A_4 \) Lagrangian with minimal Yukawa couplings, and gives rise to a non-degenerate Dirac neutrino Yukawa matrix and a unique CP-violation pattern. This opens the possibility of explaining the non-zero value of \( \theta_{13} \cong 9^\circ \) still maintaining TBM for the other two neutrino mixing angles \( \theta_{23} \cong 45^\circ \) and \( \theta_{12} \cong 35^\circ \); furthermore, this allows an economic way to achieve low energy CP violation in neutrino oscillations as well as high energy CP violation for leptogenesis.

This paper is organized as follows. In the next section, we lay down the particle content and the field representations under the \( A_4 \) flavor symmetry in our model, as well as explain the characteristic points of our model phenomenology at low and high energy. In Sec. III, we present the neutrino mixing angles, and how the low energy CP violation could be generated

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\footnote{E.Ma and G.Rajasekaran \[15\] have introduced for the first time the \( A_4 \) symmetry to avoid the mass degeneracy of \( \mu \) and \( \tau \) under a \( \mu-\tau \) symmetry \[16\].}
in both normal and inverted mass hierarchies, including our predictions for neutrinoless double beta decay. We give our conclusions in Sec. IV, and in Appendix A we outline the minimization of the scalar potential and the vacuum alignments.

II. FLAVOR $A_4$ SYMMETRY FOR NON-ZERO $\theta_{13}$ AND LEPTOGENESIS

In the absence of flavor symmetries, particle masses and mixings are generally undetermined in a gauge theory. Here, to understand the present non-zero $\theta_{13}$ and TBM angles ($\theta_{12}, \theta_{23}$) of the neutrino oscillation data and baryogenesis via leptogenesis, we propose a new discrete symmetry based on an $A_4$ flavor symmetry for leptons in a renormalizable Lagrangian.$^2$

The group $A_4$ is the symmetry group of the tetrahedron, isomorphic to the finite group of the even permutations of four objects. The group $A_4$ has two generators, denoted $S$ and $T$, satisfying the relations $S^2 = T^3 = (ST)^3 = 1$. In the three-dimensional real representation, $S$ and $T$ are given by

$$
S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.
$$

(5)

$A_4$ has four irreducible representations: one triplet $3$ and three singlets $1, 1', 1''$. An $A_4$ triplet $(a_1, a_2, a_3)$ transforms in the unitary representation by multiplication with the $S$ and $T$ matrices in Eq. (5) above,

$$
S \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ -a_2 \\ -a_3 \end{pmatrix}, \quad T \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_2 \\ a_3 \\ a_1 \end{pmatrix}.
$$

(6)

An $A_4$ singlet $a$ is invariant under the action of $S$ ($Sa = a$), while the action of $T$ produces $Ta = a$ for $1$, $Ta = \omega a$ for $1'$, and $Ta = \omega^2 a$ for $1''$, where $\omega = e^{i2\pi/3}$ is a complex cubic-root of unity. Products of two $A_4$ representations decompose into irreducible representations according to the following multiplication rules: $3 \otimes 3 = 3_s \oplus 3_n \oplus 1 \oplus 1' \oplus 1''$, $1' \otimes 1'' = 1$.

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$^2$ To include the quark sector, the symmetry could be promoted to the binary tetrahedral group $T'_4$ [18].
$1' \otimes 1' = 1''$ and $1'' \otimes 1'' = 1'$. Explicitly, if $(a_1, a_2, a_3)$ and $(b_1, b_2, b_3)$ denote two $A_4$ triplets,

$$(a \otimes b)_3' = (a_2 b_3 + a_3 b_2, a_3 b_1 + a_1 b_3, a_1 b_2 + a_2 b_1),$$

$$(a \otimes b)_3'' = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1),$$

$$(a \otimes b)_1' = a_1 b_1 + a_2 b_2 + a_3 b_3,$$

$$(a \otimes b)_1'' = a_1 b_1 + \omega a_2 b_2 + \omega^2 a_3 b_3,$$

$$(a \otimes b)_1''' = a_1 b_1 + \omega^2 a_2 b_2 + \omega a_3 b_3. \quad (7)$$

To make the presentation of our model physically more transparent, we define the $T$-flavor quantum number $T_f$ through the eigenvalues of the operator $T$, for which $T^3 = 1$. In detail, we say that a field $f$ has $T$-flavor $T_f = 0$, $+1$, or $-1$ when it is an eigenfield of the $T$ operator with eigenvalue $1, \omega, \omega^2$, respectively (in short, with eigenvalue $\omega^{T_f}$ for $T$-flavor $T_f$, considering the cyclical properties of the cubic root of unity $\omega$). The $T$-flavor is an additive quantum number modulo 3. We also define the $S$-flavor-parity through the eigenvalues of the operator $S$, which are $+1$ and $-1$ since $S^2 = 1$, and we speak of $S$-flavor-even and $S$-flavor-odd fields. For $A_4$-singlets, which are all $S$-flavor-even, the $1$ representation has no $T$-flavor ($T_f = 0$), the $1'$ representation has $T$-flavor $T_f = +1$, and the $1''$ representation has $T$-flavor $T_f = -1$. Since for $A_4$-triplets, the operators $S$ and $T$ do not commute, $A_4$-triplet fields cannot simultaneously have a definite $T$-flavor and a definite $S$-flavor-parity. While the real representation of $A_4$ in Eqs. (5), in which $S$ is diagonal, is useful in writing the Lagrangian, the physical meaning of our model is more apparent in the $T$-flavor representation in which $T$ is diagonal. This representation is obtained through the unitary transformation

$$A \rightarrow A' = U_\omega A U_\omega^\dagger, \quad (8)$$

where $A$ is any $A_4$ matrix in the real representation and

$$U_\omega = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}. \quad (9)$$

We have

$$S' = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}. \quad (10)$$
Despite the physical advantages of the $S'$, $T'$ representation, for clarity of exposition and to avoid confusion and complications, in this paper we use the real representation $S$, $T$ almost exclusively. For reference, an $A_4$ triplet field with components $(a_1, a_2, a_3)$ in the real representation can be expressed in terms of $T$-flavor eigenfields $(a_e, a_\mu, a_\tau)$ (the notation comes from our lepton assignments below) as

$$a_1 = \frac{a_e + a_\mu + a_\tau}{\sqrt{3}}, \quad a_2 = \frac{a_e + \omega^2 a_\mu + \omega a_\tau}{\sqrt{3}}, \quad a_3 = \frac{a_e + \omega a_\mu + \omega^2 a_\tau}{\sqrt{3}}. \quad (11)$$

Inversely,

$$a_e = \frac{a_1 + a_2 + a_3}{\sqrt{3}}, \quad a_\mu = \frac{a_1 + \omega a_2 + \omega^2 a_3}{\sqrt{3}}, \quad a_\tau = \frac{a_1 + \omega^2 a_2 + \omega a_3}{\sqrt{3}}. \quad (12)$$

We extend the standard model (SM) by the inclusion of an $A_4$-triplet of right-handed $SU(2)_L$-singlet Majorana neutrinos $N_R$, and the introduction of two types of scalar Higgs fields besides the usual SM $SU(2)_L$-doublet Higgs bosons $\Phi$, which we take to be an $A_4$-singlet with no $T$-flavor (1 representation): a second $SU(2)_L$-doublet of Higgs bosons $\eta$, which is distinguished from $\Phi$ by being an $A_4$-triplet, and an $SU(2)_L$-singlet $A_4$-triplet real scalar field $\chi$:

$$\Phi = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix}, \quad \eta_j = \begin{pmatrix} \eta_j^+ \\ \eta_j^0 \end{pmatrix}, \quad \chi_j, \quad j = 1, 2, 3. \quad (13)$$

We assign each flavor of leptons to one of the three $A_4$ singlet representations: the electron-flavor to the $1$ ($T$-flavor 0), the muon flavor to the $1'$ ($T$-flavor +1), and the tau flavor to the $1''$ ($T$-flavor -1). (Note in this respect that our $A_4$ flavor group is not a symmetry under exchange of any two lepton flavors, like $e$ and $\mu$, for example. Our $A_4$ flavor group is implemented as a global symmetry of the Lagrangian, later spontaneously broken, but some fields are not invariant under $A_4$ transformations, much in the same way as the implementation of $SU(2)_L \times U(1)_Y$ in the SM, where left-handed and right-handed fermions are assigned to different representations of the gauge group.) Then we take the usual Higgs boson doublet $\Phi$ to be invariant under $A_4$, that is to be a flavor-singlet $1$ with no $T$-flavor. The other Higgs doublet $\eta$, the Higgs singlet $\chi$, and the singlet neutrinos $N_R$ are assumed to be triplets under $A_4$, and can so be used to introduce lepton-flavor violation in an $A_4$ symmetric Lagrangian.

The field content of our model and the field assignments to $SU(2)_L \times U(1)_Y \times A_4$ representations are summarized in Table I. These representation assignments and the requirement
that the Lagrangian be renormalizable and $A_4$-symmetry forbid the presence of tree-level leptonic flavor-changing charged currents.

The renormalizable Yukawa interactions in the neutrino and charged lepton sectors invariant under $SU(2)_L \times U(1)_Y \times A_4$ are (including a Majorana mass term for the right-handed neutrinos)

$$-\mathcal{L}_{\text{Yuk}} = y_\nu^e \bar{\ell}_e (\bar{\eta} N_R)_1 + y_\nu^\mu \bar{\ell}_\mu (\bar{\eta} N_R)_1' + y_\nu^\tau \bar{\ell}_\tau (\bar{\eta} N_R)_1'' + \frac{1}{2} M (\bar{N}_R^R N_R)_1 + \frac{1}{2} y_R^e (\bar{N}_R^\ell N_R)_3 \chi + y_e \bar{\ell}_e \Phi e_R + y_\mu \bar{\ell}_\mu \Phi \mu_R + y_\tau \bar{\ell}_\tau \Phi \tau_R + \text{h.c.},$$

where $\bar{\eta} \equiv i \tau_2 \eta^*$ and $\tau_2$ is a Pauli matrix. In this Lagrangian, each flavor of neutrinos and each flavor of charged leptons has its own independent Yukawa term, since they belong to different singlet representations $1, 1', 1''$ of $A_4$: the neutrino Yukawa terms involve the $A_4$-triplets $\eta$ and $N_R$, which combine into the appropriate singlet representation; the charged-lepton Yukawa terms involve the $A_4$-singlet $\Phi$ and the $A_4$-singlet right-handed charged-leptons $e_R, \mu_R$, and $\tau_R$. The right-handed neutrinos have an additional Yukawa term that involves the $A_4$-triplet SM-singlet Higgs $\chi$. The mass term $\frac{1}{2} M (\bar{N}_R^R N_R)_1$ for the right-handed neutrinos is necessary to implement the seesaw mechanism by making the right-handed neutrino mass parameter $M$ large.

The Higgs potential of our model contains many terms and is listed in Appendix A, Eqs. (A1)–(A7). We spontaneously break the $A_4$ flavor symmetry by giving non-zero vacuum expectation values to some components of the $A_4$-triplets $\chi$ and $\eta$. As seen in Appendix A, the minimization of our scalar potential gives the following vacuum expectation values

| Field | $L_e, L_\mu, L_\tau$ | $e_R, \mu_R, \tau_R$ | $N_R$ | $\chi$ | $\Phi$ | $\eta$ |
|-------|----------------------|---------------------|-------|-------|-------|-------|
| $A_4$ | 1, 1', 1''           | 1, 1', 1''         | 3     | 3     | 1     | 3     |
| $SU(2)_L \times U(1)_Y$ | $(2, -\frac{1}{2})$ | $(1, -1)$           | $(1, 0)$ | $(1, 0)$ | $(2, \frac{1}{2})$ | $(2, \frac{1}{2})$ |

3 We note that at TeV-scale the higher dimensional operators ($d \geq 5$) driven by $\chi$ and $\eta$ fields are suppressed by a cutoff scale $\Lambda$ which we assume is a very high energy scale, i.e. GUT or Planck scale. And in this paper we neglect the effects of higher dimensional operators.
(VEVs), all real:

\[ \langle \varphi^0 \rangle = \frac{v_\Phi}{\sqrt{2}} \neq 0, \quad \langle \eta_1^0 \rangle = \langle \eta_2^0 \rangle = \langle \eta_3^0 \rangle \equiv \frac{v_\eta}{\sqrt{2}} \neq 0, \quad \langle \chi_1 \rangle \equiv v_\chi \neq 0, \quad \langle \chi_2 \rangle = \langle \chi_3 \rangle = 0. \quad (15) \]

The SM VEV \( v = (\sqrt{2}G_F)^{-1/2} = 246 \text{ GeV} \) results from the combination

\[ v = \sqrt{v_\Phi^2 + 3v_\eta^2}. \]

The non-zero expectation value \( \langle \varphi^0 \rangle = v_\Phi / \sqrt{2} \) does not break the \( A_4 \) symmetry, because the standard model Higgs is \( A_4 \)-flavorless. The non-zero expectation value \( \langle \eta \rangle = (v_\eta, v_\eta, v_\eta) / \sqrt{2} \) breaks the \( S \)-flavor-parity \((\eta_1, \eta_2, \eta_3) \rightarrow (\eta_1, -\eta_2, -\eta_3)\) but leaves the vacuum \( T \)-flavor \( T_f = 0 \). In other words, after \( \eta \) acquires a non-zero VEV, the \( T \)-flavor is still conserved but the \( S \)-flavor-parity is not. Since \( \eta \) appears only in the Higgs sector and in interactions with the light leptons, we say that the light neutrino sector has a residual \( Z_3 \) symmetry expressed by the subgroup \( \{1, T, T^2\} \) that leads to the conservation of \( T \)-flavor in terms involving mixing with the light neutrinos or interactions with the charged leptons. The non-zero expectation value \( \langle \chi \rangle = (v_\chi, 0, 0) \) maintains the \( S \)-flavor-parity of the vacuum (it is \( S \)-flavor-even) but gives the vacuum the symmetric combination of \( T \)-flavors \((a_0 + a_1 + a_{-1}) / \sqrt{3}\). That is, after \( \chi \) acquires a non-zero VEV, the \( S \)-flavor-parity is conserved but the \( T \)-flavor is not. Since \( \chi \) appears only in the Higgs sector and in interactions with the heavy Majorana neutrinos, we say that the heavy neutrino sector has a residual \( Z_2 \) symmetry expressed by the subgroup \( \{1, S\} \) leading to the conservation of \( S \)-flavor-parity in terms involving mixing or interactions with the heavy Majorana neutrinos.

When a non-Abelian discrete symmetry like our \( A_4 \) is considered, it is crucial to check the stability of the vacuum. In the presence of two \( A_4 \)-triplet Higgs scalars \( \chi \) and \( \eta \), Higgs potential terms involving both \( \chi \) and \( \eta \), which would be written as \( V(\chi \eta) \) in Eqs. (A1)–(A7), would be problematic for vacuum stability. Such stability problems can be naturally solved, for instance, in the presence of extra dimensions or in supersymmetric dynamical completions \[13, 19\]. In these cases, \( V(\chi \eta) \) is not allowed or highly suppressed.

The physical Higgs fields are obtained in the usual way. In the Higgs sector we have four Higgs doublets \( \Phi, \eta_1, \eta_2 \) and \( \eta_3 \), and three Higgs singlets \( \chi_1, \chi_2 \), and \( \chi_3 \). They contain in total 16 degrees of freedom: six charged Higgs fields \( \eta_{1,2,3}^\pm \), with \( \eta_j^\pm \equiv (\eta_j^-)^* \), seven neutral Higgs scalars \( h, h_{1,2,3}, \chi_{1,2,3}^0 \), and three Higgs pseudoscalars \( A_{1,2,3} \). We can write, after
electroweak- and $A_4$-symmetry breaking and minimization of the potential,

$$\Phi = \begin{pmatrix} \varphi^+ \\ \frac{1}{\sqrt{2}} (v_\varphi + h + iA_0) \end{pmatrix}, \quad \chi_1 = v_\chi + \chi_1^0, \quad \chi_2 = \chi_2^0, \quad \chi_3 = \chi_3^0,$$

$$\eta_j = \begin{pmatrix} \eta_j^+ \\ \frac{1}{\sqrt{2}} (v_\eta + h_j + iA_j) \end{pmatrix}, \quad j = 1, 2, 3. \quad (16)$$

The action of the residual $Z_2$ generator $S$ on the physical fields is

$$(N_{R1}, N_{R2}, N_{R3}) \rightarrow (N_{R1}, -N_{R2}, -N_{R3}), \quad (17)$$

$$(\chi_1^0, \chi_2^0, \chi_3^0) \rightarrow (\chi_1^0, -\chi_2^0, -\chi_3^0), \quad (18)$$

$$(h_1, h_2, h_3) \rightarrow (h_1, -h_2, -h_3), \quad (19)$$

$$(A_1, A_2, A_3) \rightarrow (A_1, -A_2, -A_3), \quad (20)$$

$$(\eta_1^+, \eta_2^+, \eta_3^+) \rightarrow (\eta_1^+, -\eta_2^+, -\eta_3^+), \quad (21)$$

all other fields are invariant. The action of the residual $Z_3$ generator $T$ on the physical fields is (the triplet fields $a_1, a_2, a_3$ and the triplet fields $a_e, a_\mu, a_\tau$ are linear combinations of each other, see Eqs. (11)–(12))

$$(e, \mu, \tau) \rightarrow (e, \omega \mu, \omega^2 \tau), \quad (22)$$

$$(\nu_e, \nu_\mu, \nu_\tau) \rightarrow (\nu_e, \omega \nu_\mu, \omega^2 \nu_\tau), \quad (23)$$

$$(N_{Re}, N_{R\mu}, N_{R\tau}) \rightarrow (N_{Re}, \omega N_{R\mu}, \omega^2 N_{R\tau}), \quad (24)$$

$$(\chi_e^0, \chi_\mu^0, \chi_\tau^0) \rightarrow (\chi_e^0, \omega \chi_\mu^0, \omega^2 \chi_\tau^0), \quad (25)$$

$$(h_e, h_\mu, h_\tau) \rightarrow (h_e, \omega h_\mu, \omega^2 h_\tau), \quad (26)$$

$$(A_e, A_\mu, A_\tau) \rightarrow (A_e, \omega A_\mu, \omega^2 A_\tau), \quad (27)$$

$$(\eta_e^+, \eta_\mu^+, \eta_\tau^+) \rightarrow (\eta_e^+, \omega \eta_\mu^+, \omega^2 \eta_\tau^+), \quad (28)$$

all other fields are invariant.

After electroweak and $A_4$ symmetry breaking, the neutral Higgs fields acquire vacuum expectation values and give masses to the charged-leptons and neutrinos: the Higgs doublet gives Dirac masses to the charge leptons, the Higgs doublet $\eta$ gives Dirac masses to the three SM neutrinos, and the Higgs singlet $\chi$ gives a Majorana mass to the right-handed neutrino $N_R$. 
The charged lepton mass matrix is automatically diagonal due to the $A_4$-singlet nature of the charged lepton and SM-Higgs fields. The right-handed neutrino mass has the (large) Majorana mass contribution $M$ and a contribution induced by the electroweak-singlet $A_4$-triplet Higgs boson $\chi$ when the $A_4$-symmetry is spontaneously broken.

After the breaking of the flavor and electroweak symmetries, with the VEV alignments as in Eq. (15), the charged lepton, Dirac neutrino and right-handed neutrino mass terms from the Lagrangian (14) result in

$$\mathcal{L}_m = \frac{v_\Phi}{\sqrt{2}} (y_e \bar{e} e_R + y_\mu \bar{\mu} \mu_R + y_\tau \bar{\tau} \tau_R) + \frac{v_\eta}{\sqrt{2}} \left\{ (y_1^e \bar{\nu} e_l + y_2^e \bar{\nu} \mu_R + y_3^e \bar{\nu} \tau_R) N_{R1} 
+ (y_1^\mu \bar{\nu} e_l + y_2^\mu \bar{\nu} \mu_R + y_3^\mu \bar{\nu} \tau_R) N_{R2} 
+ (y_1^\tau \bar{\nu} e_l + y_2^\tau \bar{\nu} \mu_R + y_3^\tau \bar{\nu} \tau_R) N_{R3} \right\} 
+ \frac{M}{2} (N_{R1}^c N_{R1} + N_{R2}^c N_{R2} + N_{R3}^c N_{R3}) + \frac{y_\nu^2}{2} \left( (N_{R2}^c N_{R3} + N_{R3}^c N_{R2}) \right) + \text{h.c. \ (29)}$$

This form shows clearly that the terms in $v_\eta$ break the $S$-flavor-parity symmetry (17)–(21), while the other mass terms preserve it. Passing to the $T$-flavor eigenfields

$$N_{R e} = \frac{N_{R1} + N_{R2} + N_{R3}}{\sqrt{3}}, \quad (30)$$

$$N_{R \mu} = \frac{N_{R1} + \omega N_{R2} + \omega^2 N_{R3}}{\sqrt{3}}, \quad (31)$$

$$N_{R \tau} = \frac{N_{R1} + \omega^2 N_{R2} + \omega N_{R3}}{\sqrt{3}}, \quad (32)$$

with respective $T$-flavor $T_f = 0, +1, -1$, the lepton mass Lagrangian reads

$$\mathcal{L}_m = \frac{v_\Phi}{\sqrt{2}} (y_e \bar{e} e_R + y_\mu \bar{\mu} \mu_R + y_\tau \bar{\tau} \tau_R) 
+ \frac{v_\eta}{\sqrt{3}} \left\{ (y_1^e \bar{\nu} e_l + y_2^e \bar{\nu} \mu_R + y_3^e \bar{\nu} \tau_R) N_{R1} 
+ (y_1^\mu \bar{\nu} e_l + y_2^\mu \bar{\nu} \mu_R + y_3^\mu \bar{\nu} \tau_R) N_{R2} 
+ (y_1^\tau \bar{\nu} e_l + y_2^\tau \bar{\nu} \mu_R + y_3^\tau \bar{\nu} \tau_R) N_{R3} \right\} 
+ \frac{M}{2} (N_{R1}^c N_{R1} + N_{R2}^c N_{R2} + N_{R3}^c N_{R3}) + \frac{y_\nu^2}{2} \left( (N_{R2}^c N_{R3} + N_{R3}^c N_{R2}) \right) + \text{h.c. \ (33)}$$

This form shows clearly that the terms in $v_\chi$ break the $T$-flavor symmetry (22)–(28), while the other mass terms preserve it.

Inspection of the mass terms in Eq. (33) indicates that, with the VEV alignments in Eq. (15), the $A_4$ symmetry is spontaneously broken to a residual $Z_2$ symmetry in the heavy
Majorana neutrino sector (conservation of \(S\)-flavor-parity in terms not involving \(v_\eta\) or \(h_{1,2,3}\)) and a residual \(Z_3\) symmetry in the Dirac neutrino sector (conservation of \(T\)-flavor in terms not involving \(v_\chi\) or \(\chi_1^0\)).

The mass terms in Eq. (29) and the charged gauge interactions in the weak eigenstate basis can be written in (block) matrix form as, using \(\overline{N}^c_R m_D \nu_L = \overline{\nu}_L m^T_D N_R\),

\[
- \mathcal{L}_{mW} = \frac{1}{2} \overline{\nu}_L N_R N_R + \overline{\nu}_L m_D N_R + \overline{\nu}_L m_L \ell_R + \frac{g}{\sqrt{2}} W^- L \gamma^\mu \nu_L + \text{h.c.} \quad (34)
\]

\[
= \frac{1}{2} \left( \frac{\nu_L}{\overline{N}^c_R} \right) \begin{pmatrix} 0 & m_D \\ m^T_D & M_R \end{pmatrix} \begin{pmatrix} \nu^c_L \\ N_R \end{pmatrix} + \overline{\nu}_L m_L \ell_R + \frac{g}{\sqrt{2}} W^- L \gamma^\mu \nu_L + \text{h.c.} \quad (35)
\]

Here \(\ell = (e, \mu, \tau), \nu = (\nu_e, \nu_\mu, \nu_\tau), N_R = (N_{R1}, N_{R2}, N_{R3})\), and

\[
m_\ell = \frac{v_\phi}{\sqrt{2}} \begin{pmatrix} y_e & 0 & 0 \\ 0 & y_\mu & 0 \\ 0 & 0 & y_\tau \end{pmatrix},
\]

\[
m_D = \frac{v_\eta}{\sqrt{2}} Y_\nu = \frac{v_\eta}{\sqrt{2}} \begin{pmatrix} y_1^\nu & y_2^\nu & y_3^\nu \\ y_2^\nu & \omega y_3^\nu & \omega^2 y_2^\nu \\ y_3^\nu & \omega^2 y_2^\nu & \omega y_3^\nu \end{pmatrix},
\]

\[
M_R = \begin{pmatrix} M & 0 & 0 \\ 0 & M & y_\nu^R \nu_\chi \\ 0 & y_\nu^R \nu_\chi & M \end{pmatrix}.
\]

To find the neutrino masses and mixing matrix we are to diagonalize the \(6 \times 6\) matrix

\[
\begin{pmatrix} 0 & m_D \\ m^T_D & M_R \end{pmatrix}.
\]

We start by diagonalizing \(M_R\). For this purpose, we perform a basis rotation \(\widehat{N}_R = U_R^T N_R\), so that the right-handed Majorana mass matrix \(M_R\) becomes a diagonal matrix \(\widehat{M}_R\) with real and positive mass eigenvalues \(M_1 = aM, M_2 = M\) and \(M_3 = bM\),

\[
\widehat{M}_R = U_R^T M_R U_R = M U_R^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \kappa e^{i\xi} \\ 0 & \kappa e^{i\xi} & 1 \end{pmatrix} U_R = \begin{pmatrix} aM & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & bM \end{pmatrix},
\]
where \( \kappa = |y_R^\nu v_\chi/M| \) and \( \xi = \arg(y_R^\nu v_\chi/M) \). We find
\[
a = \sqrt{1 + \kappa^2 + 2\kappa \cos \xi}, \quad b = \sqrt{1 + \kappa^2 - 2\kappa \cos \xi},
\]
and a diagonalizing matrix
\[
U_R = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & \sqrt{2} & 0 & 0 \\
1 & 0 & -1 & 0 \\
1 & 0 & 1 & 0 \\
0 & 0 & 0 & e^{i\frac{\psi_2}{2}}
\end{pmatrix},
\]
with phases
\[
\psi_1 = \tan^{-1}\left(\frac{-\kappa \sin \xi}{1 + \kappa \cos \xi}\right) \quad \text{and} \quad \psi_2 = \tan^{-1}\left(\frac{\kappa \sin \xi}{1 - \kappa \cos \xi}\right).
\]

As the magnitude of \( \kappa \) defined in Eq. (40) decreases, the phases \( \psi_{1,2} \) go to 0 or \( \pi \). At this point,
\[
-\mathcal{L}_{mW} = \frac{1}{2} \begin{pmatrix}
\bar{\nu}_L & \bar{\nu}_R
\end{pmatrix} \begin{pmatrix}
0 & \bar{m}_D \\
\bar{m}_D^T & M_R
\end{pmatrix} \begin{pmatrix}
\nu_L^c \\
\nu_R^c
\end{pmatrix} + \bar{\ell}_L m_\ell \ell_R + \frac{g}{\sqrt{2}} W^\mu \bar{\ell}_L \gamma^\mu \nu_L + \text{h.c.}
\]
with \( \bar{m}_D = m_D U_R \).

Now we take the limit of large \( M \) (seesaw mechanism) and focus on the mass matrix of the light neutrinos \( M_\nu \),
\[
-\mathcal{L}_{mW} = \frac{1}{2} \begin{pmatrix}
\bar{\nu}_L & \bar{\nu}_R
\end{pmatrix} \begin{pmatrix}
\nu_L^c \\
\nu_R^c
\end{pmatrix} + \frac{g}{\sqrt{2}} W^\mu \bar{\ell}_L \gamma^\mu \nu_L + \text{h.c.} + \text{terms in } N_R
\]
with
\[
M_\nu = -\bar{m}_D M_R^{-1} \bar{m}_D^T.
\]

We perform basis rotations from weak to mass eigenstates in the leptonic sector,
\[
\hat{\ell}_L = P_\ell^\dagger \ell_L, \quad \hat{\ell}_R = P_\ell^\dagger \ell_R, \quad \hat{\nu}_L = U_\nu^\dagger P_\nu^\dagger \nu_L,
\]
where \( P_\ell \) and \( P_\nu \) are phase matrices and \( U_\nu \) is a unitary matrix chosen so as the matrix
\[
\hat{m}_\nu = U_\nu^\dagger P_\nu^\dagger M_\nu P_\nu^\dagger U_\nu = -U_\nu^\dagger P_\nu^\dagger m_D U_R M_R^{-1} (U_\nu^\dagger P_\nu^\dagger m_D U_R)^T
\]
is diagonal. Then from the charged current term in Eq. (43) we obtain the lepton mixing matrix \( U_{\text{PMNS}} \) as
\[
U_{\text{PMNS}} = P_\ell^\dagger P_\nu U_\nu.
\]
The matrix $U_{PMNS}$ can be written in terms of three mixing angles and three $CP$-odd phases (one for the Dirac neutrinos and two for the Majorana neutrinos) as

$$U_{PMNS} = \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta_{CP}} \\ -c_{23}s_{12} - s_{23}c_{12}s_{13}e^{i\delta_{CP}} & c_{23}c_{12} - s_{23}c_{12}s_{13}e^{i\delta_{CP}} & s_{23}s_{13} \\ s_{23}s_{12} - c_{23}c_{12}s_{13}e^{i\delta_{CP}} & -s_{23}c_{12} - c_{23}s_{12}s_{13}e^{i\delta_{CP}} & c_{23}s_{13} \end{pmatrix} Q_\nu , \quad (49)$$

where $Q_\nu = \text{Diag}(e^{-i\psi_1/2}, e^{-i\psi_2/2}, 1)$, and $s_{ij} \equiv \sin \theta_{ij}$ and $c_{ij} \equiv \cos \theta_{ij}$.

It is important to notice that the phase matrix $P_\nu$ can be rotated away by choosing the matrix $P_\ell = P_\nu$, i.e. by an appropriate redefinition of the left-handed charged lepton fields, which is always possible. This is an important point because the phase matrix $P_\nu$ accompanies the Dirac-neutrino mass matrix $\tilde{m}_D$ and ultimately the neutrino Yukawa matrix $Y_\nu$ in Eq. (37). This means that complex phases in $Y_\nu$ can always be rotated away by appropriately choosing the phases of left-handed charged lepton fields. Hence without loss of generality the eigenvalues $y_1', y_2'$, and $y_3'$ of $Y_\nu$ can be real and positive. The Yukawa matrix $Y_\nu$ can then be written as

$$Y_\nu = y_3'\sqrt{3} \begin{pmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} U_\nu^\dagger, \quad (50)$$

where $y_1 = |y_1'|/y_3'$, $y_2 = |y_2'|/y_3'$, and $U_\nu$ is given in Eq. (9).

Concerning $CP$ violation, we notice that the $CP$ phases $\psi_1, \psi_2$ coming from $M_R$ only take part in low-energy $CP$ violation, as can be seen in Eqs. (40-50). Any $CP$-violation relevant for leptogenesis is associated with the neutrino Yukawa matrix $\tilde{Y}_\nu = Y_\nu U_R$ and the combination of Dirac neutrino Yukawa matrices, $H \equiv \tilde{Y}_\nu^\dagger \tilde{Y}_\nu = U_R^\dagger Y_\nu^\dagger Y_\nu U_R$, which is

$$H = 3|y_3'|^2 \begin{pmatrix} \frac{1+4y_1'^2+y_2'^2}{2} & \frac{e^{-i\psi_1}}{\sqrt{2}}(2y_1'^2 - y_2'^2 - 1) & \frac{i\sqrt{3}e^{i\psi_1}}{2}(y_2'^2 - 1) \\ \frac{e^{i\psi_1}}{\sqrt{2}}(2y_1'^2 - y_2'^2 - 1) & 1 + y_1'^2 + y_2'^2 & -i\sqrt{\frac{3}{2}}e^{i\psi_2}(y_2'^2 - 1) \\ -i\sqrt{\frac{3}{2}}e^{-i\psi_1}(y_2'^2 - 1) & i\sqrt{\frac{3}{2}}e^{-i\psi_2}(y_2'^2 - 1) & \frac{3}{2}(1 + y_2'^2) \end{pmatrix}, \quad (51)$$

where $\psi_{ij} \equiv \psi_i - \psi_j$. As expected, in the limit $|y_1'| = |y_2'| = |y_3'|$, i.e. $y_{1,2} \to 1$, the off-diagonal entries of $H$ vanish, and there is no $CP$ violation useful for leptogenesis. If the Dirac neutrino Yukawa couplings $y_1', y_2'$, and $y_3'$ differ in magnitude, they can play a role in baryogenesis via leptogenesis and non-zero $\theta_{13} \simeq 9^\circ$ with TBM ($\theta_{23} \simeq 45^\circ, \theta_{12} \simeq 35^\circ$). Therefore, a low energy
CP violation in neutrino oscillation and/or a high energy CP violation in leptogenesis can be 
generated by the non-degeneracy of the Dirac neutrino Yukawa couplings and a non-zero phase $\xi$ coming from $M_R$.

In the following section we investigate the low energy phenomenology, namely the possible values of the light neutrino mixing angles, how the low energy CP violation could be generated in both normal and inverted mass hierarchies, and neutrinoless double beta decay, which is a probe of lepton number violation at low energy.

### III. PHENOMENOLOGY OF LIGHT NEUTRINOS

After seesawing, in a basis where charged lepton and heavy neutrino masses are real and diagonal, the light neutrino mass matrix is given by

$$m_\nu = -\tilde{m}_D M_R^{-1} \tilde{m}_D^T = -\frac{v_\mu^2}{2} Y_\nu U_R M_R^{-1} U_R^T Y_\nu^T$$

$$= m_0 \begin{pmatrix}
(1 + \frac{2e^{i\psi_1}}{a})y_1^2 & (1 - \frac{e^{i\psi_1}}{a})y_1y_2 & (1 - \frac{e^{i\psi_1}}{a})y_1 \\
(1 - \frac{e^{i\psi_1}}{a})y_1y_2 & (1 + \frac{e^{i\psi_1}}{2a} - \frac{3e^{i\psi_2}}{2b})y_2^2 & (1 + \frac{e^{i\psi_1}}{2a} + \frac{3e^{i\psi_2}}{2b})y_2 \\
(1 - \frac{e^{i\psi_1}}{a})y_1 & (1 + \frac{e^{i\psi_1}}{2a} + \frac{3e^{i\psi_2}}{2b})y_2 & (1 + \frac{e^{i\psi_1}}{2a} - \frac{3e^{i\psi_2}}{2b})
\end{pmatrix}, \quad (52)$$

where $\tilde{m}_D = v_\eta \tilde{Y}_\nu / \sqrt{2}$ and we have defined an overall scale $m_0 = \frac{v_\mu^2}{\eta |y_3^2|}/(6 M)$ for the light neutrino masses. The mass matrix $m_\nu$ is diagonalized by the PMNS mixing matrix $U_{\text{PMNS}}$ as described above,

$$m_\nu = U_{\text{PMNS}} \text{Diag}(m_1, m_2, m_3) U_{\text{PMNS}}^T. \quad (53)$$

Here $m_i (i = 1, 2, 3)$ are the light neutrino masses. As is well-known, because of the observed hierarchy $|\Delta m^2_{\text{Atm}}| \equiv |m_3^2 - m_1^2| \gg |\Delta m^2_{\text{Sol}}| \equiv m_2^2 - m_1^2 > 0$, and the requirement of a Mikheyev-Smirnov-Wolfenstein resonance for solar neutrinos, there are two possible neutrino mass spectra: (i) the normal mass hierarchy (NMH) $m_1 < m_2 < m_3$, and (ii) the inverted mass hierarchy (IMH) $m_3 < m_1 < m_2$.

Interestingly, the combination $U_{\omega}^T U_R$ in Eq. (52) reflects an exact TBM:

$$U_{\omega}^T U_R = \begin{pmatrix}
\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}
\end{pmatrix} \begin{pmatrix}
\frac{1}{\sqrt{3}} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^{i\frac{\psi_2}{2}}
\end{pmatrix}. \quad (54)$$

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Therefore Eq. (52) directly indicates that there could be deviations from the exact TBM if the Dirac neutrino Yukawa couplings do not have the same magnitude. In the limit \(|y_2^e| = |y_3^e| (y_2 \to 1)\), the mass matrix in Eq. (52) acquires a \(\mu-\tau\) symmetry that leads to \(\theta_{13} = 0\) and \(\theta_{23} = -\pi/4\). Moreover, in the limit \(|y_1^e| = |y_2^e| = |y_3^e| (y_1, y_2 \to 1)\), the mass matrix (52) gives the TBM angles in Eq. (1) and the corresponding mass eigenvalues

\[
m_1 = \frac{3m_0}{a}, \quad m_2 = 3m_0, \quad m_3 = \frac{3m_0}{b}.
\] (55)

These mass eigenvalues are disconnected from the mixing angles. However, recent neutrino data, i.e. \(\theta_{13} \neq 0\), require deviations of \(y_{1,2}\) from unity, leading to a possibility to search for \(CP\) violation in neutrino oscillation experiments. These deviations generate relations between mixing angles and mass eigenvalues.

To diagonalize the above mass matrix Eq. (52), we consider the hermitian matrix \(m_{\nu}m_{\nu}^\dagger = U_{PMNS} \text{Diag}(m_1^2, m_2^2, m_3^2) U_{PMNS}^\dagger\), from which we obtain the masses and mixing angles. To see how the neutrino mass matrix given by Eq. (52) can lead to deviations of neutrino mixing angles from their TBM values, we first introduce three small quantities \(\epsilon_i\), \((i = 1, 2, 3)\), which are responsible for the deviations of the \(\theta_{jk}\) from their TBM values:

\[
\theta_{23} = -\frac{\pi}{4} + \epsilon_1, \quad \theta_{13} = \epsilon_2, \quad \theta_{12} = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right) + \epsilon_3.
\] (56)

Then the PMNS mixing matrix up to order \(\epsilon_i\) can be written as

\[
U_{PMNS} = \begin{pmatrix}
\frac{\sqrt{2}-\epsilon_2}{\sqrt{6}} & \frac{1+\epsilon_2}{\sqrt{3}} & \epsilon_2 e^{-i\delta_{CP}} \\
-\frac{1+\epsilon_1+\epsilon_2^2}{\sqrt{6}} + \frac{\epsilon_2 e^{i\delta_{CP}}}{\sqrt{3}} & \frac{\sqrt{2}+\epsilon_1}{\sqrt{6}} + \frac{\epsilon_2 e^{i\delta_{CP}}}{\sqrt{3}} & -\frac{1+\epsilon_1}{\sqrt{2}} \\
-\frac{1+\epsilon_1+\epsilon_2^2}{\sqrt{6}} - \frac{\epsilon_2 e^{i\delta_{CP}}}{\sqrt{3}} & \frac{\sqrt{2}+\epsilon_1}{\sqrt{6}} - \frac{\epsilon_2 e^{i\delta_{CP}}}{\sqrt{3}} & \frac{1+\epsilon_1}{\sqrt{2}}
\end{pmatrix} Q_{\nu} + O(\epsilon_i^2).
\] (57)

The small deviation \(\epsilon_1\) from the maximality of the atmospheric mixing angle \(\theta_{23}\) is expressed in terms of the parameters in Eq. (B1) in Appendix B as

\[
\tan \epsilon_1 = \frac{R(1+y_2)-S(y_2-1)}{R(y_2-1)-S(1+y_2)}.
\] (58)

In the limit of \(y_1, y_2 \to 1\), \(\epsilon_1\) goes to zero (or equivalently \(\theta_{23} \to -\pi/4\)) due to \(R, S \to 0\).

The reactor angle \(\theta_{13}\) and the Dirac-CP phase \(\delta_{CP}\) are expressed as

\[
\tan 2\theta_{13} = \frac{y_1 s_{23}(3Q-P)y_2 - c_{23}(3Q+P) - 3i(s_{23}(R-S)y_2 + c_{23}(R+S))}{(F + G + \frac{9K}{4} + \frac{3D}{2})(c_{23}^2 + y_2^2s_{23}^2) + y_2(F + G - \frac{9K}{4})\sin 2\theta_{23} - y_1^2A},
\]

\[
\tan \delta_{CP} = 3\frac{(R-S)^2 + y_2^2(R+S)^2}{(P+Q)(R-S) - y_2^2(P-Q)(R+S)},
\] (59)
where the parameters $P, Q, F, G, K, D$ and $\tilde{A}$ are given in Eq. (B1) in Appendix B. In the limit of $y_1, y_2 \to 1$, the parameters $Q, R, S$ go to zero, which in turn leads to $\theta_{13} \to 0$ and $\delta_{CP} \to 0$ as expected. Finally, the solar mixing angle is given by

$$\tan 2\theta_{12} = 2y_1 \frac{y_2 c_{23}(3Q - P) + s_{23}(3Q + P)}{c_{13}(\Psi_2 - \Psi_1)}.$$  \hfill (60)

Since in the limit $y_1, y_2 \to 1$ the parameters in Eq. (60) behave as $Q \to 0, P \to 6(\frac{1}{a^2} - 1), \Psi_1 \to 3(1 + \frac{1}{a^2})$ and $\Psi_2 \to 6(1 + \frac{1}{a^2})$, it is clear that the mixing angle $\tan 2\theta_{12}$ goes to $2\sqrt{2}$, that is, $\theta_{12} \to \sin^{-1}(1/\sqrt{3})$.

The squared-mass eigenvalues of the three light neutrinos result in

$$m_1^2 = m_0^2 \left\{ s_{12}^2 \Psi_1 + c_{12}^2 \Psi_2 - y_1 \frac{y_2 c_{23}(3Q - P) + s_{23}(3Q + P)}{2c_{13}} \sin 2\theta_{12} \right\},$$

$$m_2^2 = m_0^2 \left\{ c_{12}^2 \Psi_1 + s_{12}^2 \Psi_2 + y_1 \frac{y_2 c_{23}(3Q - P) + s_{23}(3Q + P)}{2c_{13}} \sin 2\theta_{12} \right\},$$

$$m_3^2 = m_0^2 \left[ \left( F + G + \frac{9K}{4} + \frac{3D}{2} \right)(c_{23}^2 + y_2 s_{23}^2) + y_2 \left( F + G - \frac{9K}{4} \right) \sin 2\theta_{23} \right] c_{13}^2 + y_1^2 \bar{A}s_{13}^2 - \frac{y_1 \sin 2\theta_{13}}{2} \left[ c_{23}((3Q + P) \cos \delta_{CP} - 3(R + S) \sin \delta_{CP}) \right. + s_{23} y_2 ((3Q - P) \cos \delta_{CP} + 3(R - S) \sin \delta_{CP}) \left. \right] \right\}.$$ \hfill (61)

We see from Eqs. (60) and (61) that the deviation $\epsilon_3$ from tri-maximality of solar mixing angle $\theta_{12}$ can be expressed as

$$2\sqrt{2} \cos 2\epsilon_3 + \sin 2\epsilon_3 = \frac{3y_1 m_0^2 \{y_2 c_{23}(3Q - P) + s_{23}(3Q + P)\}}{c_{13} \Delta m_{21}^2}.$$  \hfill (62)

Now we perform a numerical analysis using the linear algebra tools in Ref. [20]. The Daya Bay and RENO experiments have accomplished the measurement of three mixing angles $\theta_{12}, \theta_{23}$, and $\theta_{13}$ from three kinds of neutrino oscillation experiments. A combined analysis of the data from the T2K, MINOS, Double Chooz, Daya Bay and RENO experiments shows [7] that, for the normal mass hierarchy (NMH) and inverted mass hierarchy (IMH), respectively,

$$\sin^2 \theta_{13} = 0.026^{+0.003}_{-0.004} (+0.010) \quad \text{NMH}, \quad \left[ 0.027^{+0.003}_{-0.004} (+0.010) \quad \text{IMH} \right]$$ \hfill (63)

or equivalently

$$\theta_{13} = 9.28^{+0.53}_{-0.75} (^{+1.66}_{-2.24}) \quad \text{NMH}, \quad \left[ 9.46^{+0.52}_{-0.73} (^{+1.64}_{-2.19}) \quad \text{IMH} \right]$$ \hfill (64)
at the $1\sigma$ ($3\sigma$) level. The hypothesis $\theta_{13} = 0$ is now rejected at the $8\sigma$ significance level. In addition to the measurement of the mixing angle $\theta_{13}$, the global fit of the neutrino mixing angles and of the mass-squared differences at the $1\sigma$ ($3\sigma$) level is given by

$$
\theta_{12} = 34.45^{+0.92}_{-1.05} (3.50^\circ) \, , \quad \theta_{23} = 44.43^{+4.60}_{-2.87} (8.70^\circ) \, , \quad \text{NMH}, \quad \left[46.72^{+2.89}_{-4.01} (6.41^\circ) \, \text{IMH}\right]
$$

$$
\Delta m_{\odot}^2 [10^{-5}\text{eV}^2] = 7.62^{+0.19}_{-0.19} (0.58) \, , \quad \Delta m_{\text{Atm}}^2 [10^{-3}\text{eV}^2] = \begin{cases}
2.53^{+0.08}_{-0.10} (0.24) \, , & \text{NMH} \\
2.40^{+0.10}_{-0.07} (0.28) \, , & \text{IMH}
\end{cases}
$$

The matrices $m_D$ and $\hat{M}_R$ in Eq. (52) contain seven parameters: $y_3^\nu, M, v_\eta, y_1, y_2, \kappa, \xi$. The first three ($y_3^\nu, M,$ and $v_\eta$) lead to the overall neutrino scale parameter $m_0$. The next four ($y_1, y_2, \kappa, \xi$) give rise to the deviations from TBM as well as the CP phases and corrections to the mass eigenvalues (see Eq. (55)).

In our numerical examples, we take $M = 10$ TeV and $v_\eta = v_\Phi = 123$ GeV, for simplicity, as inputs. Since the neutrino masses are sensitive to the combination $m_0 = v_\eta^2 |y_3^\nu|^2/(6M)$, other choices of $M$ and $v_\eta$ give identical results. Then the parameters $m_0, y_1, y_2, \kappa, \xi$ can be determined from the experimental results of three mixing angles, $\theta_{12}, \theta_{13}, \theta_{23}$, and the two mass squared differences, $\Delta m_{21}^2, \Delta m_{31}^2$. In addition, the CP phases $\delta_{CP}, \varphi_{1,2}$ can be predicted after determining the model parameters. Using the formulae for the neutrino mixing angles and masses and our values of $M, v_\eta, v_\Phi$, we obtain the following allowed regions of the unknown model parameters: for the normal mass hierarchy (NMH) \footnote{When $y_2 = 1$ and around there, there exist other parameter spaces giving very small values of $\theta_{13}$. So, we have neglected them in our numerical result for normal mass hierarchy.}

$$
0.17 \lesssim \kappa \lesssim 0.90 \, , \quad 0.74 \lesssim y_1 \lesssim 1.0 \, , \quad 0.90 \lesssim y_2 \lesssim 1.11 \, , \quad 94^\circ \lesssim \xi \lesssim 119^\circ \, , \quad 240^\circ \lesssim \xi \lesssim 265^\circ \, , \quad 1.8 \lesssim m_0 \times 10^{-2}[\text{eV}] \lesssim 6.0 \, ; \quad (66)
$$

for the inverted mass hierarchy (IMH),

$$
0.31 \lesssim \kappa \lesssim 0.92 \, , \quad 0.84 \lesssim y_1 \lesssim 1.15 \, , \quad 0.65 \lesssim y_2 \lesssim 1.28 \, , \quad 90^\circ \lesssim \xi \lesssim 117^\circ \, , \quad 245^\circ \lesssim \xi \lesssim 265^\circ \, , \quad 1.7 \lesssim m_0 \times 10^{-2}[\text{eV}] \lesssim 4.5 \, . \quad (67)
$$

Note that here we have used the $3\sigma$ experimental bounds on $\theta_{12}, \theta_{23}, \Delta m_{21}^2, \Delta m_{31}^2$ in Eq. (65), except for $\theta_{13} < 12^\circ$ for which we use the values in Eqs. (66, 67). For these parameter regions,
FIG. 1: The reactor mixing angle $\theta_{13}$ versus the ratio of first-to-second generation neutrino Yukawa couplings $y_1'/y_2'$ (left plot) and the parameter $\kappa = |y_R^\nu_{1}\chi /M|$ (right plot). The (red) crosses and (blue) dots represent the results for the normal and the inverted mass hierarchy, respectively. The horizontal solid (dotted) lines in both plots indicate the upper and lower bounds on $\theta_{13}$ for inverted (normal) mass hierarchy given in Eq. (64) at the 3σ level.

FIG. 2: The atmospheric mixing angle $\theta_{23}$ versus the phase $\xi$ of the parameter combination $y_R^\nu_{1}\chi /M$. The (red) crosses and (blue) dots represent the results for the normal and inverted mass hierarchy, respectively.

we investigate how a non-zero $\theta_{13}$ can be determined for the normal and inverted mass hierarchy. In Figs. 1-5, the data points represented by blue dots and red crosses indicate results for the inverted and normal mass hierarchy, respectively. The left-hand-side plot in Fig. 1 shows how the mixing angle $\theta_{13}$ depends on the ratio $y_1/y_2 = y_1'/y_2'$ of the first-
FIG. 3: The Jarlskog invariant $J_{CP}$ versus the reactor angle $\theta_{13}$ (left plot), and the Dirac CP phase $\delta_{CP}$ versus $\theta_{23}$ (right plot). The (red) crosses and (blue) dots represent the results for the normal and inverted mass hierarchy, respectively. The vertical solid (dashed) lines in both plots indicate the upper and lower bounds on $\theta_{13}$ for the inverted (normal) mass hierarchy given in Eq. (64) at the $3\sigma$ level.

and second-generation neutrino Yukawa couplings; the right-hand-side plot shows how $\theta_{13}$ depends on the parameter $\kappa = |y_{\nu R}^{e}v_{\chi}/M|$. We see that the measured value of $\theta_{13}$ from the Daya Bay and RENO experiments can be achieved at $3\sigma$’s for $0.75 < y_{1}/y_{2} < 1$ (NMH), $1.1 < y_{1}/y_{2} < 1.3$ and $y_{1}/y_{2} \sim 0.9$ (IMH), $0.17 \lesssim \kappa \lesssim 0.82$ (NMH) and $0.3 < \kappa \lesssim 0.74$ (IMH). Fig. 2 shows the atmospheric mixing angle $\theta_{23}$ as a function of the phase $\xi$ of $y_{\nu R}^{e}v_{\chi}/M$.

To see how the parameters are correlated with low-energy CP violation observables measurable through neutrino oscillations, we consider the leptonic CP violation parameter defined by the Jarlskog invariant [21] $J_{CP}$:

$$J_{CP} \equiv \text{Im}[U_{e1}U_{\mu2}U_{e2}^{*}U_{\mu1}^{*}] = \frac{1}{8} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \cos \theta_{13} \sin \delta_{CP}. \quad (68)$$

The Jarlskog invariant $J_{CP}$ can be expressed in terms of the elements of the matrix $h = m_{\nu}m_{\nu}^{\dagger}$ [22]:

$$J_{CP} = -\frac{\text{Im}\{h_{12}h_{23}h_{31}\}}{\Delta m_{21}^{2}\Delta m_{31}^{2}\Delta m_{32}^{2}}. \quad (69)$$

The behavior of $J_{CP}$ as a function of $\theta_{13}$ is plotted on the left plot of Fig. 3. We see that the value of $|J_{CP}|$ lies in the range $0 - 0.04$ (NMH) and $0.02 - 0.04$ (IMH) for the measured
value of $\theta_{13}$ at $3\sigma$'s. Also, in our model we have
\[
\text{Im}\{h_{12}h_{23}h_{31}\} = \frac{27m_0^6}{4a^2b^2}y_1^2y_2^2(1 - y_2^2)\sin\psi_2\{....\},
\] (70)
in which $\{....\}$ stands for a complicated lengthy function of $y_1$, $y_2$, $a$, $b$, $\psi_1$ and $\psi_2$. Clearly, Eq. (70) indicates that in the limit of $y_2 \to 1$ or $\sin\psi_2 \to 0$ the leptonic $C$P violation $J_{CP}$ goes to zero. When $y_2 \neq 1$, i.e. for the normal hierarchy case, $J_{CP}$ could go to zero as $\sin\psi_2$ of Eq. (70). In the case of the inverted hierarchy, $J_{CP}$ has nonzero values for the measured range of $\theta_{13}$ while $J_{CP}$ goes to zero for $\theta_{13} \to 0$, which corresponds to $y_2 \to 1$. The right plot of Fig. 3 shows the behavior of the Dirac CP phase $\delta_{CP}$ as a function of $\theta_{23}$, where $\delta_{CP}$ can have discrete values around $50^\circ$, $120^\circ$, $230^\circ$ and $310^\circ$ for the inverted mass hierarchy (for the normal mass hierarchy, $\delta_{CP}$ can vary over a wide range except near $90^\circ$ and $270^\circ$). Future precise measurements of $\theta_{23}$, whether $\theta_{23} > 45^\circ$ or $\theta_{23} < 45^\circ$, will provide more information on $\delta_{CP}$.

Fig. 4 shows how the values of $\theta_{13}$ depend on the mixing angles $\theta_{23}$ and $\theta_{12}$. As can be seen in the left plot of Fig. 4, the behavior of $\theta_{23}$ in terms of the measured values of $\theta_{13}$ at $3\sigma$’s for the normal hierarchy is different than for the inverted hierarchy. For the normal hierarchy we see that the measured values of $\theta_{13}$ can be achieved for $43^\circ < \theta_{23} < 47^\circ$ and $\theta_{23} \neq 45^\circ$, with small deviations from maximality, while for the inverted hierarchy $50^\circ \lesssim \theta_{23} \lesssim 53.1^\circ$ and $38.6^\circ \lesssim \theta_{23} \lesssim 40^\circ$, which are excluded at $1\sigma$ by the experimental bounds as can be seen in Eq. (65).\(^5\) From the right plot of Fig. 4 we see that the predictions for $\theta_{13}$ do not strongly depend on $\theta_{12}$ in the allowed region.

Moreover, we can straightforwardly obtain the effective neutrino mass $|m_{ee}|$ that characterizes the amplitude for neutrinoless double beta decay:
\[
|m_{ee}| \equiv \left| \sum_i (U_{PMNS})_{ei}^2 m_i \right| ,
\] (71)
where $U_{PMNS}$ is given in Eq. (57). The left and right plots in Fig. 5 show the behavior of the effective neutrino mass $|m_{ee}|$ in terms of $\theta_{13}$ and the lightest neutrino mass, respectively. In the left plot of Fig. 5 for the measured values of $\theta_{13}$ at $3\sigma$’s, the effective neutrino mass $|m_{ee}|$ can be in the range $0.04 \lesssim |m_{ee}|[\text{eV}] < 0.15$ (NMH) or $0.06 \lesssim |m_{ee}|[\text{eV}] \lesssim 0.11$ (IMH).

\(^5\) Interestingly, the most recent data of MINOS seem to disfavor the maximal mixing in the atmospheric mixing angle, Eq. (4), indicating that the inverted mass hierarchy may be favored.
FIG. 4: The behaviors of $\theta_{23}$ and $\theta_{12}$ in terms of $\theta_{13}$. The red crosses and the blue dots represent results for the normal mass hierarchy and the inverted mass hierarchy, respectively. The solid (dashed) vertical lines represent the experimental bounds of Eq. (65) at 3$\sigma$’s for the inverted (normal) mass hierarchy. The horizontal dotted lines indicate the 1$\sigma$ experimental bounds in Eq. (65).

FIG. 5: Plots of $|m_{ee}|$ as a function of $\theta_{13}$ and $m_{\text{lightest}}$. The red crosses and the blue dots represent results for the normal and the inverted mass hierarchy, respectively. The vertical solid (dashed) lines show the experimental bounds of Eq. (65) at 3$\sigma$’s for the inverted (normal) mass hierarchy.

The right plot of Fig. 5 shows $|m_{ee}|$ as a function of $m_{\text{lightest}}$, where $m_{\text{lightest}} = m_1$ for the normal mass hierarchy and $m_{\text{lightest}} = m_3$ for the inverted mass hierarchy. Our model predicts that the effective mass $|m_{ee}|$ is within the sensitivity of planned neutrinoless double-beta decay experiments.
IV. CONCLUSIONS

We have suggested a novel and simple scenario to generate neutrino masses and mixings with a discrete $A_4$ symmetry that is spontaneously broken. In particular our model can accommodate in a renormalizable Lagrangian a large value of the mixing angle, $\theta_{13}$, consistent with the recent reactor neutrino experiments Daya Bay and RENO, as well as high energy CP violation interesting for leptogenesis.

In our model we have introduced a right-handed neutrino $N_R$, a real gauge-singlet scalar $\chi$, and an $SU(2)_L$-doublet scalar $\eta$, all of which are $A_4$ triplets. The light neutrino masses are generated by a seesaw mechanism in which we have assumed the right-handed neutrino masses are at the TeV scale (to evade the introduction of higher dimensional operators). Getting VEVs along the direction $\langle \chi \rangle = v_\chi (1, 0, 0)$ and $\langle \eta^0 \rangle = v_\eta (1, 1, 1)$, which break the $A_4$ symmetry down to a $Z_2$ ($S$-flavor-parity) and a $Z_3$ ($T$-flavor) symmetry, respectively, one obtains bimaximal mixing at the right-handed neutrino sector and trimaximal mixing at the light Dirac neutrino sector with non-degenerate Yukawa couplings that deform the exact TBM pattern. The resulting light neutrino mixing matrix is in the form of a deviated TBM generated through unequal neutrino Yukawa couplings, as can be seen in Figure 1. In the limiting case of equal active-neutrino Yukawa couplings, the mixing matrix recovers the exact TBM. In addition, we have shown that unequal neutrino Yukawa couplings can provide a source of high-energy CP violation, perhaps strong enough to be responsible for leptogenesis. The stability of the vacuum alignments we assume are guaranteed, for example, by embedding our model in an extra dimension.

We showed that deviations from the TBM of about 20% are enough to explain $\theta_{13} \sim 9^\circ$. We predicted that the CP violating Dirac phase $\delta_{CP}$ may have discrete values (see Figure 3). Therefore the measurement of the phase $\delta_{CP}$ in the next generation neutrino experiments can rule out or support our model. We have also shown that the inverted mass hierarchy may be excluded by a global analysis using 1σ experimental bounds, while the most recent MINOS data seem to favor it. We also predicted an effective neutrino mass in neutrinoless double-beta decay in the range, $0.04 \lesssim |m_{ee}|[\text{eV}] < 0.15$ (for the normal hierarchy) and $0.06 \lesssim |m_{ee}|[\text{eV}] \lesssim 0.11$ (for the inverted hierarchy), both ranges within reach of near-future neutrinoless double beta decay experiments.
Appendix A: The Higgs potential

In this Appendix we present our Higgs potential and its minimization, as well as our prescription for effecting the stability of the vacuum alignment. We solve the vacuum alignment problem by extending the model into a spatial extra dimension $y$ [12]. We assume that each field lives on a 4D brane either at $y = 0$ or at $y = L$, as shown in Fig. 6. The heavy neutrino masses arise from local operators at $y = 0$, while the charged fermion masses and the neutrino Yukawa interactions are realized by non-local effects involving both branes $A$.

The most general renormalizable scalar potential for the Higgs fields $\Phi, \eta$ and $\chi$, invariant under $SU(2)_L \times U(1)_Y \times A_4$ and obeying the conditions in the previous paragraph, is then given by

$$V = V_{y=0} + V_{y=L},$$  \hspace{1cm} (A1)

where

$$V_{y=0} = V(\Phi) + V(\eta) + V(\eta \Phi),$$  \hspace{1cm} (A2)
$$V_{y=L} = V(\chi),$$  \hspace{1cm} (A3)

and

$$V(\eta) = \mu_2^2 (\eta^\dagger \eta)_1 + \lambda_1^\eta (\eta^\dagger \eta)_1 (\eta^\dagger \eta)_1 + \lambda_2^\eta (\eta^\dagger \eta)_1 (\eta^\dagger \eta)_1 + \lambda_3^\eta (\eta^\dagger \eta)_3 (\eta^\dagger \eta)_3, \hspace{1cm} (A4)$$

$$V(\Phi) = \mu_3^2 (\Phi^\dagger \Phi) + \chi_3 (\Phi^\dagger \Phi)^2,$$

$$V(\chi) = \mu_4^2 (\chi \chi)_1 + \lambda_1^\chi (\chi \chi)_1 (\chi \chi)_1 + \lambda_2^\chi (\chi \chi)_1 (\chi \chi)_1 + \lambda_3^\chi (\chi \chi)_3 (\chi \chi)_3, \hspace{1cm} (A5)$$

$$V(\eta \Phi) = \lambda_1^{\eta \Phi} (\eta \Phi^\dagger \Phi) + \lambda_2^{\eta \Phi} [((\eta \Phi^\dagger \Phi) (\eta \Phi)], + \{ \lambda_3^{\eta \Phi} [((\eta \Phi^\dagger \Phi) (\eta \Phi)], + \text{h.c.} \}$$

Here $\mu_2, \mu_3, \mu_4, \chi_3, \xi_1^\chi$ and $\xi_2^\chi$ have mass dimension-1, while $\lambda_1^{\eta, \ldots, 5}, \lambda_1^{\chi, \ldots, 5}$ and $\lambda_1^{\eta \Phi, \ldots, 5}$ are dimensionless. In $V(\eta \Phi)$ the usual mixing term $\Phi^\dagger \eta$ is forbidden by the $A_4$ symmetry. In the scalar potential $[A1]–[A7]$ we have for simplicity assumed that $CP$ is conserved, and the couplings $\lambda_3^{\eta \Phi}, \lambda_4^{\eta \Phi}$ and $\lambda_5^\eta$ are real.
This VEV breaks $A_4$ down to a residual $Z_3$.

The vanishing of the derivative of $V$ with respect to $\Phi$ reads

$$\sqrt{2} \frac{\partial V}{\partial \phi_0} \bigg|_{(\phi_0)=v_\Phi} = v_\Phi \left\{ v_\Phi^2 \lambda_1^{\Phi} + \mu_\Phi^2 + \frac{1}{2} (\lambda_1^{\eta \Phi} + \lambda_2^{\eta \Phi} + 2 \lambda_3^{\eta \Phi}) (v_{\eta_1}^2 + v_{\eta_2}^2 + v_{\eta_3}^2) \right\} + 3 v_{\eta_1} v_{\eta_2} v_{\eta_3} \lambda_4^{\eta \Phi} = 0 .$$  (A10)
The real-valued solution of Eq. (A10), for real-valued parameters, is

\[
v_\Phi = \frac{-(\frac{3}{2})^{1/3}\tilde{b}}{\left\{-9\tilde{a}^2\tilde{c} + \sqrt{3(4\tilde{a}^3\tilde{b}^3 + 27\tilde{a}^4\tilde{c}^2)}\right\}^{1/3}} + \left\{-9\tilde{a}^2\tilde{c} + \sqrt{3(4\tilde{a}^3\tilde{b}^3 + 27\tilde{a}^4\tilde{c}^2)}\right\}^{1/3}\tilde{a}(18)^{1/3}, \tag{A11}\]

where \(\tilde{a} = \lambda_1^\Phi, \tilde{b} = \mu_2^2 + \frac{2}{3}v_\eta^2(\lambda_1^\nu_1^\Phi + \lambda_2^\nu_\nu + 2\lambda_3^\nu_1^\Phi)\) and \(\tilde{c} = 3v_\eta^2\lambda_1^\nu_1^\Phi\).

Finally, the minimization equations for the vacuum configuration of \(\chi\) are given by

\[
\begin{align*}
\frac{\partial V}{\partial \chi_1} &\big|_{(\chi_1) = v_{\chi_1}} = 2v_{\chi_1}\left(\mu_1^2 + (2\lambda_1^\chi + \lambda_2^\chi + 4\lambda_3^\chi)(v_{\chi_2}^2 + v_{\chi_3}^2) + 2(\lambda_1^\chi + \lambda_2^\chi)v_{\chi_1}^2\right) \\
&\quad + 6\xi_1^\chi v_{\chi_2} v_{\chi_3} = 0, \\
\frac{\partial V}{\partial \chi_2} &\big|_{(\chi_1) = v_{\chi_1}} = 2v_{\chi_2}\left(\mu_2^2 + (2\lambda_1^\chi + \lambda_2^\chi + 4\lambda_3^\chi)(v_{\chi_1}^2 + v_{\chi_3}^2) + 2(\lambda_1^\chi + \lambda_2^\chi)v_{\chi_2}^2\right) \\
&\quad + 6\xi_1^\chi v_{\chi_1} v_{\chi_3} = 0, \\
\frac{\partial V}{\partial \chi_3} &\big|_{(\chi_1) = v_{\chi_1}} = 2v_{\chi_3}\left(\mu_3^2 + (2\lambda_1^\chi + \lambda_2^\chi + 4\lambda_3^\chi)(v_{\chi_1}^2 + v_{\chi_2}^2) + 2(\lambda_1^\chi + \lambda_2^\chi)v_{\chi_3}^2\right) \\
&\quad + 6\xi_1^\chi v_{\chi_1} v_{\chi_2} = 0. \tag{A12}
\end{align*}
\]

From these equations, we obtain the solution\(^6\)

\[
\langle \chi_1 \rangle \equiv v_\chi = \sqrt{-\frac{\mu_1^2}{2(\lambda_1^\chi + \lambda_2^\chi)}} \neq 0, \quad \langle \chi_2 \rangle = \langle \chi_3 \rangle = 0. \tag{A13}
\]

**Appendix B: Parametrization of the neutrino mass matrix**

We parametrize the hermitian matrix \(m_\nu m_\nu^\dagger\) as follows:

\[
m_\nu m_\nu^\dagger = m_0^2 \begin{pmatrix}
\frac{3Q-P}{2} & i\frac{3(R-S)}{2} & y_1 y_2 \left(\frac{3Q-P}{2} - i\frac{3(R-S)}{2}\right)
y_1 y_2 \left(\frac{3Q-P}{2} + i\frac{3(R-S)}{2}\right) & y_2^2 \left(F + G + \frac{9K}{4} + \frac{3Z}{2}\right) & y_2 \left(F + G - \frac{9K}{4} - \frac{3Z}{2}\right)
y_1 \left(-\frac{3Q-P}{2} + i\frac{3(R-S)}{2}\right) & y_2 \left(F + G - \frac{9K}{4} + \frac{3Z}{2}\right) & y_2 \left(F + G + \frac{9K}{4} + \frac{3Z}{2}\right)
\end{pmatrix}.
\]

\(^6\) There exists another nontrivial solution \(\langle \chi \rangle = v_\chi(1, 1, 1)\) with \(v_\chi = \frac{-3\xi_1^\chi + \sqrt{\xi_1^\chi^2 - 8\mu_2^2(3\lambda_1^\nu_1^\Phi + 4\lambda_2^\nu_{\nu})}}{4(3\lambda_1^\nu_1^\Phi + 4\lambda_2^\nu_{\nu})}\). But this solution is not of interest for our purposes.
All parameters appearing here are real, and equal to

\[
\begin{align*}
\tilde{A} &= 1 + y_1^2 + y_2^2 + \frac{1 + 4y_1^2 + y_2^2}{a^2} - \frac{2(1 - 2y_1^2 + y_2^2) \cos \psi_1}{a}, \\
K &= \frac{1 + y_2^2}{b^2}, \\
F &= 1 + y_1^2 + y_2^2 + \frac{1 + 4y_1^2 + y_2^2}{4a^2}, \\
G &= \frac{a}{a}\left(1 - 2y_1^2 + y_2^2\right) \cos \psi_1, \\
D &= (1 - y_2^2) \cos \psi_{12} + 2a \cos \psi_2, \\
Z &= (1 - y_2^2) \sin \psi_{12} - 2a \sin \psi_2, \\
P &= \frac{1 + 4y_1^2 + y_2^2}{a^2} - 2(1 + y_1^2 + y_2^2) + \frac{(1 - 2y_1^2 + y_2^2) \cos \psi_1}{a}, \\
Q &= (1 - y_2^2) \cos \psi_{12} - a \cos \psi_2, \\
S &= (1 - y_2^2) \sin \psi_{12} + a \sin \psi_2, \\
R &= \frac{1 - 2y_1^2 + y_2^2}{a} \sin \psi_1.
\end{align*}
\]  

(B1)

In Eq. (B1) the parameters \(\Psi_1, \Psi_2\) are defined by

\[
\begin{align*}
\Psi_1 &= c^2_{13}y_1^2 \tilde{A} + s^2_{13}\left\{F + G + \frac{9K}{4} + \frac{3D}{2}\right\} (c^2_{23} + y_2^2 s^2_{23}) + y_2 \left( F + G - \frac{9K}{4} \right) \sin 2\theta_{23} \\
&\quad - \frac{y_1}{2} \sin 2\theta_{13} \left\{3c_{23}(R + S) \sin \delta_{CP} - (3Q + P) \cos \delta_{CP} \\
&\quad + y_2 s_{23} (3(R - S) \sin \delta_{CP} + (3Q - P) \cos \delta_{CP})\right\}, \\
\Psi_2 &= \left( F + G + \frac{9K}{4} + \frac{3D}{2}\right)(s^2_{23} + y_2^2 c^2_{23}) - y_2 \left( F + G - \frac{9K}{4} \right) \sin 2\theta_{23}.
\end{align*}
\]  

(B2)

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[1] K. Nakamura et al. (Particle Data Group), J. Phys. G 37, 075021 (2010) and 2011 partial update for the 2012 edition.

[2] P. F. Harrison, D. H. Perkins and W. G. Scott, Phys. Lett. B 530, 167 (2002); Z. Z. Xing, Phys. Lett. B 533, 85 (2002); P. F. Harrison and W. G. Scott, Phys. Lett. B 535, 163 (2002); X. G. He and A. Zee, Phys. Lett. B 560, 87 (2003).

[3] F. P. An et al. [DAYA-BAY Collaboration], Phys. Rev. Lett. 108, 171803 (2012) arXiv:1203.1669 [hep-ex].
[4] J. K. Ahn et al. [RENO Collaboration], Phys. Rev. Lett. 108, 191802 (2012) arXiv:1204.0626 [hep-ex].

[5] K. Abe et al. [T2K Collaboration], Phys. Rev. Lett. 107, 041801 (2011) arXiv:1106.2822 [hep-ex]; P. Adamson et al. [MINOS Collaboration], Phys. Rev. Lett. 107, 181802 (2011) arXiv:1108.0015 [hep-ex]; H. De Kerret et al. [Double Chooz Collaboration], talk presented at the Sixth International Workshop on Low Energy Neutrino Physics, http://workshop.kias.re.kr/lownu11/ Seoul, November 9-11, 2011.

[6] R. Nichol, Plenary talk at the Neutrino 2012 conference, http://neu2012.kek.jp/.

[7] M. Tortola, J. W. F. Valle and D. Vanegas, arXiv:1205.4018 [hep-ph].

[8] H. Nunokawa, S. J. Parke and J. W. F. Valle, Prog. Part. Nucl. Phys. 60, 338 (2008) arXiv:0710.0554 [hep-ph].

[9] Y. H. Ahn, H.-Y. Cheng and S. Oh, Phys. Rev. D 83, 076012 (2011) arXiv:1102.0879 [hep-ph]; Y. H. Ahn, C. S. Kim and S. Oh, arXiv:1103.0657 [hep-ph]; Y. H. Ahn, H.-Y. Cheng and S. Oh, arXiv:1105.4460 [hep-ph]; Y. H. Ahn, H.-Y. Cheng and S. Oh, Phys. Rev. D 84, 113007 (2011) arXiv:1107.4549 [hep-ph]; Y. H. Ahn and H. Okada, Phys. Rev. D 85, 073010 (2012) arXiv:1201.4436 [hep-ph]; S. Zhou, arXiv:1205.0761 [hep-ph]. S. Antusch, C. Gross, V. Maurer and C. Sluka, arXiv:1205.1051 [hep-ph]; G. Altarelli, F. Feruglio and L. Merlo, arXiv:1205.5133 [hep-ph].

[10] Y. H. Ahn and S. K. Kang, arXiv:1203.4185 [hep-ph].

[11] K. S. Babu, E. Ma and J. W. F. Valle, Phys. Lett. B 552, 207 (2003) hep-ph/0206292.

[12] G. Altarelli and F. Feruglio, Nucl. Phys. B 720, 64 (2005) arXiv:hep-ph/0504165.

[13] X. G. He, Y. Y. Keum and R. R. Volkas, JHEP 0604, 039 (2006) arXiv:hep-ph/0601001.

[14] S. Baek and M. C. Oh, Phys. Lett. B 690, 29 (2010) arXiv:0812.2704 [hep-ph];

[15] E. Ma and G. Rajasekaran, Phys. Rev. D 64, 113012 (2001) arXiv:hep-ph/0106291.

[16] T. Fukuyama and H. Nishiura, arXiv:hep-ph/9702253; R. N. Mohapatra and S. Nussinov, Phys. Rev. D 60, 013002 (1999); E. Ma and M. Raidal, Phys. Rev. Lett. 87, 011802 (2001); C. S. Lam, arXiv:hep-ph/0104116; T. Kitabayashi and M. Yasue, Phys.Rev. D 67 015006 (2003); W. Grimus and L. Lavoura, arXiv:hep-ph/0305046 0309050; W. Grimus and L. Lavoura, Phys. Lett. B 572, 189 (2003); Y. Koide, Phys.Rev. D 69, 093001 (2004); A. Ghosal, hep-ph/0304090; W. Grimus and L. Lavoura, J. Phys. G 30, 73 (2004); R. N. Mohapatra and W. Rodejohann, Phys. Rev. D 72, 053001 (2005) hep-ph/0507312; Y. H. Ahn, Sin
Kyu Kang, C. S. Kim, Jake Lee, arXiv:hep-ph/0602160; Y. H. Ahn, S. K. Kang, C. S. Kim and J. Lee, Phys. Rev. D 75, 013012 (2007).

[17] M. Fukugita and T. Yanagida, Phys. Lett. B 174, 45 (1986); G. F. Giudice et al., Nucl. Phys. B 685, 89 (2004) arXiv:hep-ph/0310123; W. Buchmuller, P. Di Bari and M. Plumacher, Annals Phys. 315, 305 (2005) arXiv:hep-ph/0401240; A. Pilaftsis and T. E. J. Underwood, Phys. Rev. D 72, 113001 (2005) arXiv:hep-ph/0506107.

[18] K. M. Case, R. Karplus and C. N. Yang, Phys. Rev. 101, 874 (1956); P. H. Frampton, T. W. Kephart and S. Matsuzaki, Phys. Rev. D 78, 073004 (2008) arXiv:0807.4713 [hep-ph].

[19] G. Altarelli and F. Feruglio, Nucl. Phys. B 741, 215 (2006) arXiv:hep-ph/0512103; I. de Medeiros Varzielas, S. F. King and G. G. Ross, Phys. Lett. B 644, 153 (2007) arXiv:hep-ph/0512313; G. Altarelli, F. Feruglio and Y. Lin, Nucl. Phys. B 775, 31 (2007) arXiv:hep-ph/0610165.

[20] S. Antusch, J. Kersten, M. Lindner, M. Ratz and M. A. Schmidt, JHEP 0503, 024 (2005) hep-ph/0501272.

[21] C. Jarlskog, Phys. Rev. Lett. 55, 1039 (1985); D. D. Wu, Phys. Rev. D 33, 860 (1986).

[22] G. C. Branco, R. Gonzalez Felipe, F. R. Joaquim, I. Masina, M. N. Rebelo and C. A. Savoy, Phys. Rev. D 67, 073025 (2003) arXiv:hep-ph/0211001.