Abstract

We study the effect of feature noise (measurement error) on the discrepancy between losses across two groups (e.g., men and women) in the context of linear regression. Our main finding is that adding even the same amount of noise on all individuals impacts groups differently. We characterize several forms of loss discrepancy in terms of the amount of noise and difference between moments of the two groups, for estimators that either do or do not use group membership information. We then study how long it takes for an estimator to adapt to a shift in the population that make the groups have the same mean. We finally validate our results on three real-world datasets.

1 Introduction

Standard learning procedures such as empirical risk minimization have been shown to result in models that perform well on average but whose performance differ widely across groups (Angwin et al., 2016; Barocas and Selbst, 2016). This loss discrepancy across groups is especially problematic in critical applications that impact people’s lives (Berk, 2012; Chouldechova, 2017). Despite the vast literature on removing loss discrepancy (Hardt et al., 2016; Khani et al., 2019; Agarwal et al., 2018; Zafar et al., 2017), the direct removal of loss discrepancy might introduce other problems such as intra-group loss discrepancy (Lipton et al., 2018) and adverse long-term impacts (Liu et al., 2018). Therefore, it is important to understand the source of loss discrepancy.

Why do such loss discrepancies exist? The literature generally studies sources of loss discrepancy due to an “information deficiency” of one group—that is, one group has, for example, more noise (Corbett-Davies et al., 2017), less training data (Chouldechova and Roth, 2018; Chen et al., 2018), biased prediction targets (Madras et al., 2019), or less-predictive features (Chen et al., 2018). Some work also state that groups have different risk distributions, and thus making hard (binary) decisions on such distributions causes loss discrepancy (Corbett-Davies and Goel, 2018; Canetti et al., 2019). In this work, we show that even under very favorable conditions—i.e., no bias in the prediction targets, infinite data, perfect predictive features for both groups, and no hard decisions (in regression)—adding the same amount of feature noise to all individuals leads to loss discrepancy.

There are two common flavors of loss discrepancies: (i) statistical, which measures the difference between the expected losses of groups (Hardt et al., 2016; Agarwal et al., 2018; Woodworth et al., 2017; Pleiss et al., 2017; Khani et al., 2019); and (ii) counterfactual, which measures the difference between the loss of an individual and a “counterfactual” individual with the same characteristics but from another group (Kusner et al., 2017; Chiappa, 2019; Lofthus et al., 2018; Nabi and Shpitser, 2018; Kilbertus et al., 2017).

In order to study the effect of feature noise on loss discrepancy, we consider the following linear regression setup. We assume each individual belongs to a group $g \in \{0, 1\}$ and has latent features $z \in \mathbb{R}^d$ which causes the target $y = \beta^T z + \alpha$. However, we only observe a noisy version of $z$ (which includes omitting features as a special case) through one of the following observation functions:

$$o_{-g}(z, g, u) = [z + u],$$  \hspace{1cm} (1)

$$o_{+g}(z, g, u) = [z + u, g],$$  \hspace{1cm} (2)

where $u \in \mathbb{R}^d$ is mean-zero noise independent of the rest of the variables, and the group membership $g$ can be either included or not. We study the discrepancy of both the residual $(y - \hat{y})$, which measures the amount of underestimation and the squared error $(y - \hat{y})^2$, which measures the overall performance. Abusing terminology, we call both losses.

First, we show that statistical loss discrepancy is determined by four factors: the amount of feature noise and the difference between means, variances, and sizes of the
groups. In particular, the loss discrepancy based on residual is proportional to the difference between means, and the loss discrepancy based on squared error is proportional to the difference between variances. Second, we show a trade-off between statistical and counterfactual notions of loss discrepancy. We show that the least squares estimator with no group information \((o_{-g})\) has high statistical loss discrepancy but low counterfactual loss discrepancy (Proposition 1). With group information \((o_{+g})\), it has low statistical loss discrepancy but high counterfactual loss discrepancy (Proposition 2).

We then study the role of historical inequalities on present day estimators. We decompose the incurred loss discrepancy by linear regression into two terms, where the first term is related to the moments of training distribution, and the second term is related to the moments of the test distribution. We show that the high statistical loss discrepancy of \(o_{-g}\) is mainly due to differences in the test distribution, while the high loss discrepancy of \(o_{+g}\) is mainly due to differences in the training distribution. We then study how long it takes a linear model using \(o_{+g}\) to adapt to a shift where the difference in means between groups vanishes. In particular, we show that if in the training distribution, the shifted distribution is \(K\) times more likely than the initial distribution, then the loss discrepancy (according to the shifted distribution) converges to zero with the rate of \(\frac{1}{K+1}\) (Proposition 3).

We validate our results on three real-world datasets for predicting the final grade of secondary school students, final GPA of law students, and crime rates in the US communities, where the group \(g\) is either race or gender. We consider two types of feature noise: (i) adding the same amount of noise to every feature and (ii) removing features. We show that in the Communities and Crime and Students datasets where groups have different means, variances, and sizes, noise leads to high loss discrepancy. On the other hand, in the Law School dataset, where groups have similar means and variances, noise does not affect the loss discrepancy. Finally, for the datasets with high loss discrepancy, we consider a shift to a reweighted dataset where groups have similar means and show loss discrepancy vanishes with a similar rate studied in Proposition 3.

## 2 Setup

We consider the following regression setup. We assume each individual belongs to a group \(g \in \{0, 1\}\), e.g., men and women. Each individual has latent features, \(z \in \mathbb{R}^d\) which cause the prediction target \(y \in \mathbb{R}\). However, for each individual, we observe \(x = o(z, g, u)\) through an observation function \(o\), where \(u \sim \mathbb{R}^d\) is a random vector which is the source of randomness in the observation function. As an example, the latent feature \((z)\) can be education background, and we only observe a noisy version of it \((x)\) through some observation functions such as exam performance.

Let \(h : \mathbb{R}^d \to \mathbb{R}\) be a predictor, and \(\hat{y} = h(o(z, g, u))\) be the predicted value for individual \(z\). We measure the impact of the predictor for an individual through a loss function \(\ell(\hat{y}, y)\) (e.g., for squared error, \(\ell(\hat{y}, y) = (\hat{y} - y)^2\)), abbreviated as \(\ell\) when clear from the context. Finally, note that in the entire paper, we analyze only the population setting since the effect of feature noise does not vanish even with a larger amount of data.

### 2.1 Loss Discrepancy Notions

There are mainly two flavors of loss discrepancies: the counterfactual loss discrepancy between similar individuals (same \(z\), not necessarily the same \(x\)) who belong to different groups \((g)\), and the statistical loss discrepancy between different groups.

**Counterfactual.** Introduced by Kusner et al. (2017), counterfactual fairness measures how much two similar individuals (in our setup, same \(z\)) are treated differently because of their group membership. In the original work, they state that an algorithm is counterfactually fair if its prediction is the same between two similar individuals but different demographic group \(g\). In this work, we compare the expected losses instead of predictions:

**Definition 1. (Counterfactual Loss Discrepancy (CLD))**

For a predictor \(h\), observation function \(o\), and loss function \(\ell\), counterfactual loss discrepancy is the expected difference between the loss of an individual and its counterfactual counterpart:

\[
\text{CLD}(h, o, \ell) = \mathbb{E}[|L_0 - L_1|],
\]

where \(L_{g'} = \mathbb{E}[\ell(h(o(z, g', u), y)|z]\).

Of course, one does not have access to \(z\), and it is hard to compute CLD; however, we can still study it under idealized settings.

**Statistical.** Statistical notions of fairness have been studied in economics and machine learning under the names of disparate impact, equal opportunity, classification parity, etc. (Arrow, 1973; Phelps, 1972; Hardt et al., 2016; Corbett-Davies and Goel, 2018). Here, we define statistical loss discrepancy similar to mentioned work, but looking at a general loss function.

**Definition 2 (Statistical Loss Discrepancy (SLD)).** For a predictor \(h\), observation function \(o\), and loss function \(\ell\), statistical loss discrepancy is the difference between the expected loss between two groups:

\[
\text{SLD}(h, o, \ell) = |\mathbb{E}[\ell | g = 1] - \mathbb{E}[\ell | g = 0]|.
\]

1 For comparison to other work related to counterfactual loss discrepancy see Section 7.
Unlike CLD, SLD operates at the group level and no counterfactual reasoning is needed to compute it. However, it does not provide any guarantee at the individual level.

If an observation function \( o \) maps \( z \) to different \( x \) for different groups \( (o(z,0,u) \neq o(z,1,u)) \), then individuals and groups can be adversely affected—for example, when \( o \) adds noise to the features only when \( g = 0 \).

If an observation function \( o \) maps \( z \) to the same \( x \) for both groups \( (o(z,0,u) = o(z,1,u)) \), then CLD = 0; however, SLD of a model can be large if \( z \) and \( g \) are correlated \( (z \notin g) \); for example, when \( o \) removes features which are highly informative only for one group.

Finally, note that CLD and SLD are not comparable. A model can have the same loss for similar individuals (CLD = 0), but since groups have different distributions over individuals, it can have higher loss for one group (SLD \( \neq 0 \)). Conversely, a model can treat similar individuals differently due to their group membership (CLD \( \neq 0 \)), but when averaged over the groups, it can result in similar expected losses for both groups (SLD = 0).

## 3 Linear Regression Background

For an unknown random variable \( y \) and a known random vector \( x \), the least squares estimator (in the population setting) is,

\[
\alpha, \beta = \arg \min_{\alpha', \beta'} \mathbb{E} \left[ (y - \beta'x - \alpha')^2 \right].
\]

(5)

For any two random vectors \( u \) and \( v \), let \( E_u = \mathbb{E}[u] \) denote the average of \( u \) and \( \Sigma_{uv} = \mathbb{E}[(u - \mu_u)(v - \mu_v)^\top] \) denote the covariance matrix between \( u \) and \( v \), and we write \( \Sigma_u \) for \( \Sigma_{uv} \). Assuming \( \Sigma_x \) is invertible, then the least squares estimator (5) is given in closed form:

\[
\beta = \Sigma_x^{-1}\Sigma_{xy}, \quad \alpha = \mu_y - \beta^\top\mu_x.
\]

(6)

### 3.1 Feature noise in Linear Regression

Feature noise (also known as measurement error or error-in-variable) is the difference between the observed/measured quantity and the true variables. When the noise on each feature is additive, independent of the rest of variables \( (y \) and \( z \)), and has mean zero, then we can analyze the estimated parameters via least squares (Carroll et al., 2006; Fuller, 2009; Frisch, 1934) (Note that the actual estimator only has access to \( x \), but our analysis is in terms of \( z \) and \( u \)). Recalling that \( y = \beta^\top z + \alpha \) and \( x = z + u \), using (6) we have:

\[
\hat{\beta} = \Sigma_x^{-1}\Sigma_{xy} = (\Sigma_z + \Sigma_u)^{-1}\Sigma_z\beta
\]

(7)

\[
\hat{\alpha} = (\beta - \hat{\beta})^\top\mathbb{E}[z] + \alpha.
\]

(8)

Figure 1: In the presence of feature noise, least squares estimator is not consistent; and the estimated slope (red line) is smaller than the true slope (black line). Here the true feature is \( z \sim \mathcal{N}(1,1) \), the observed features is \( x \sim \mathcal{N}(z,1) \), and the prediction target \( y = z \).

To simplify notation, let \( \Lambda \) def \( = (\Sigma_z + \Sigma_u)^{-1}\Sigma_u \), then \( \hat{\beta} = (I - \Lambda)\beta \) and \( \hat{\alpha} = (\Lambda\beta)^\top\mathbb{E}[z] + \alpha \). Finally, for these estimated parameters the squared error is:

\[
(\Lambda\beta)^\top\Sigma_z\Lambda\beta + ((I - \Lambda)\beta)^\top\Sigma_u(I - \Lambda)\beta
\]

(9)

If all variables are one-dimensional, then \( \sigma_y^2 \leq 1 \) denotes the relative size of the true signal and it is known as attenuation bias (note that in this case \( \hat{\beta} \leq \beta \)). Figure 1 shows the estimated line (which predicts \( y \) from \( x \)) in comparison to the true line (which predicts \( y \) from \( z \)). Here \( y = z, z \sim \mathcal{N}(1,1) \), and \( u \sim \mathcal{N}(0,1) \).

Next, we will show that feature noise affects different groups differently and leads to high loss discrepancy.

## 4 CLD and SLD for Linear Regression

We now show how feature noise affect CLD (Definition 1) and SLD (Definition 2) for linear regression. Let \( \beta, \alpha \) denote the true parameters such that for each individual, \( y = \beta^\top z + \alpha \). We focus on two loss functions when computing CLD and SLD:

1. Residual: measures the amount of underestimation.

\[
\ell_{\text{res}}(\hat{y}, y) = y - \hat{y}.
\]

(10)

2. Squared error: measures overall performance.

\[
\ell_{\text{sq}}(\hat{y}, y) = (y - \hat{y})^2.
\]

(11)

In the next two sections, we consider two different observation functions. For each of them, we are interested in four metrics: (i) CLD based on residual, (ii) CLD based on squared error, (iii) SLD based on residual, and (iv) SLD based on squared error.

### 4.1 Independent noise without group information

We first show observing a noisy version of \( z \) without any information on the group membership, lead to high statistical loss discrepancy. Formally, let \( u \) be a mean-zero noise
Figure 2: An illustration of feature noise and its effect on CLD and SLD. There are two groups (green and orange) distributed normally centered in 1 and 4. The true function (dashed black line) is \( y = z \). Predicting \( y \) from \( x = o_{-g}(z, g, u) = z + \mathcal{N}(0,1) \) is the blue line which underestimates the target values for the orange group and thus have high SLD(\( o_{-g}, \ell_{res} \)); however, since the prediction is independent of the group membership, it has CLD = 0. Predicting \( y \) from \( o_{+g}(z, g, u) = [z + \mathcal{N}(0,1), g] \) is the red lines, which has high CLD since groups are treated differently according to their group membership but low SLD.

and independent of the rest of the variables \((z, g, y)\).

\[
o_{-g}(z, g, u) = z + u
\]

In this case, group information is not encoded in the observation function \( o_{-g}(z, 0, u) = o_{+g}(z, 1, u) \); therefore, CLD = 0. However, as discussed in Section 2, SLD depends on the distribution of the groups over \( z \).

Let’s first consider a simple one-dimensional case to identify the important factors in loss discrepancy between groups. Figure 2 shows two groups, where \( z \sim \mathcal{N}(1,0.5) \) for the first group (green), and \( z \sim \mathcal{N}(4,1) \) for the second group (orange), also the first group is twice as likely as the second group (\( \mathbb{P}[g = 1] = 2 \mathbb{P}[g = 0] \)). The prediction target here is \( y = z \). Let the noise be Gaussian \( u \sim \mathcal{N}(0,1) \), and we observe a noisy version of the true features, \( x = o_{-g}(z, g, u) = z + u \). Concretely, we are interested in the loss discrepancy between groups for the least squares estimator, which predicts \( y \) using \( x \).

As shown in Section 3, having noise in the features causes attenuation bias. In particular, in this example we have \( \text{Var}[z] = E[\text{Var}[z | g]] + \text{Var}[E[z | g]] = \frac{8}{3} \). Therefore \( \hat{\beta} = \frac{\sigma^2_z}{\sigma^2_z + \sigma^2_u} = \frac{3}{11} \), and \( \hat{\alpha} = \frac{\sigma^2_u}{\sigma^2_z + \sigma^2_u} = \frac{8}{11} \) (see blue line in Figure 2).

Let’s see how this attenuation bias affects different groups. As shown in the Figure 2, the prediction target for the orange group is underestimated. Intuitively if the mean of a group deviates from the mean of the population, then the average of residual for that group is large. Therefore, the difference between the mean of the groups is a factor in loss discrepancy based on residual \( \ell_{res} \).

\[
\Delta \mu_z \overset{\text{def}}{=} E[z | g = 1] - E[z | g = 0] \quad (13)
\]

Secondly, since the green group is in majority (\( \mathbb{P}[g = 1] > \mathbb{P}[g = 0] \)), the line has to less bias for the green group; As a result the difference between size of the groups also play an important role in loss discrepancy:

\[
\mathbb{P}[g = 1] - \mathbb{P}[g = 0] \quad (14)
\]

Thirdly, as shown in [5] the squared error is related to variance of data points; therefore, the difference in variance is also a main factor in loss discrepancy.

\[
\Delta \Sigma_z \overset{\text{def}}{=} \text{Var}[z | g = 1] - \text{Var}[z | g = 0] \quad (15)
\]

Finally (and most importantly), as noise increases, the attenuation bias increases, thus the estimated line deviates more from the true line, leading to a higher loss discrepancy. The following proposition formalizes how SLD depends on the four factors above:

**Proposition 1.** Let \( \Lambda \overset{\text{def}}{=} (\Sigma_z + \Sigma_u)^{-1} \Sigma_u \). The least squares estimator [12] based on \( o_{-g} \) results in:

\[
\begin{align*}
\text{CLD}(o_{-g}, \ell_{res}) &= \text{CLD}(o_{-g}, \ell_{sq}) = 0 \\
\text{SLD}(o_{-g}, \ell_{res}) &= |(\Lambda \beta)^\top \Delta \mu_z| \\
\text{SLD}(o_{-g}, \ell_{sq}) &= |(\Lambda \beta)^\top \Delta \Sigma_z (\Lambda \beta)| \\
&= - (\mathbb{P}[g = 1] - \mathbb{P}[g = 0])(|\Lambda \beta|^\top \Delta \mu_z)^2.
\end{align*}
\]

where \( \Delta \mu_z \) and \( \Delta \Sigma_z \) are as defined in (13) and (15).

**Proof sketch.** Using (7) and (8), the expected \( \ell_{res} \) for the first group (\( g = 0 \)) is:

\[
E[y - \hat{y}] = -(1 - \mathbb{P}[g = 0])(\Lambda \beta)^\top \Delta \mu_z
\]

Note that the first group has lower expected \( \ell_r \) if its size is large or \( \Delta \mu_z \) (projected on \( \beta^\top \Lambda^\top \)) is small. Similarly for the second group, we have:

\[
E[y - \hat{y}] = (1 - \mathbb{P}[g = 1])(\Lambda \beta)^\top \Delta \mu_z
\]

Computing the difference results in: \( \text{SLD}(o_{-g}, \ell_{res}) = |(\Lambda \beta)^\top \Delta \mu_i|. \) For calculating SLD(\( o_{-g}, \ell_{sq} \)), first note that the squared error can be decomposed into squared of bias and variance,

\[
E[(\hat{y} - y)^2] = E[(\hat{y} - \hat{y})]^2 + \text{Var}[(\hat{y} - y)].
\]

Using the above equations, the difference between squared of bias of the groups is \( (\mathbb{P}[g = 0] - \mathbb{P}[g = 1])(|\Lambda \beta|^\top \Delta \mu)^2. \) For the difference between variances, we have:

\[
\text{Var}[(\Lambda \beta)^\top z + \hat{\beta}^\top u | g = 1] - \text{Var}[(\Lambda \beta)^\top z + \hat{\beta}^\top u | g = 0].
\]

Since \( u \) is independent of \( z \) and \( g \) the difference is only related to the difference between variances. \( \square \)
### 4.2 Independent Noise with Group Information

We now show containing group information in observation function alleviates the statistical loss discrepancy (SLD) discussed in the previous section but leads to high counterfactual loss discrepancy. Formally, we define a new observation function as follows:

\[ o_{+g}(z, g) = [z + u, g] \]  \hspace{1cm} (20)

Let’s go back to the discussed example in Figure 2, where there are two groups with different means, variances, and sizes. The goal is to predict \( y \) (where in this example, we simply have \( y = z \)). The red lines indicate the estimated line that least squares estimator predicts for \( y \) given a noisy version of \( z \) and the group membership \( (x = o_{+g}(z, g, u) = [z + u, g]) \). In this case, having \( g \) as an additional feature enables the model to have different intercepts for each group. As a result, the average residual for each group is zero; therefore \( \text{SLD}(o_{+g}, \ell_{res}) = 0 \). However, this benefit comes at the expense of treating individuals with the same \( z \) differently.

It turns out that for the squared error (\( \ell_{sq} \)) since each group has its own intercept, the squared error is only related to the difference between variance of the groups, and is no longer related to difference is sizes or means. The following proposition computes CLD and SLD:

**Proposition 2.** Consider the observation function \( o_{+g}(z, g) \) \hspace{1cm} (20). Let \( \Sigma_{z|g} = \mathbb{E}[\text{Var}[z \mid g]] \), and \( \Lambda' = (\Sigma_{z|g} + \Sigma_u)^{-1} \Sigma_u \). The estimated parameters using \( \hat{o}_{+g} \) are:

\[ \hat{\beta} = \left( I - \Lambda' \right) \beta \]

\[ \hat{\alpha} = \left( \Lambda' \right)^\top \mathbb{E}[z \mid g = 0] + \alpha. \]  \hspace{1cm} (21) \hspace{1cm} (22)

The loss discrepancies are as follows:

\[ \text{CLD}(o_{+g}, \ell_{res}) = \left| (\Lambda' \right)^\top \Delta \mu_z \]

\[ \text{CLD}(o_{+g}, \ell_{sq}) = \left| (\Lambda' \right)^\top \Delta \mu_z \right| \mathbb{E} \left[ \left( \Lambda' \right)^\top (2z - \mu_1 - \mu_0) \right] \]

\[ \text{SLD}(o_{+g}, \ell_{res}) = 0 \]

\[ \text{SLD}(o_{+g}, \ell_{sq}) = \left| (\Lambda' \right)^\top \Delta \Sigma_{z}(\Lambda' \beta) \right| , \]

where \( \Delta \mu_z = \mu_1 \mathbb{E}[z \mid g = 1] \) and \( \Delta \mu_0 = \mathbb{E}[z \mid g = 0] \).  

**Proof sketch.** Having \( g \) as an additional feature enables the model to have different intercepts for each group. As a result the bias for each groups is zero; therefore \( \text{SLD}(o_{+g}, \ell_{res}) = 0 \). As shown in \( \text{18} \), squared error can be decomposed to squared of bias and variance. As explained, squared of bias is zero. Similar to \( \text{19} \), we can see that loss discrepancy based on square loss is only related to the difference between variance of the groups.

According to the estimated parameters, the difference between loss of any two similar individuals (same \( z \)) who belong to different groups is the coefficient of \( \hat{\beta} \) for \( g \); therefore, \( \text{CLD}(o_{+g}, \ell_{res}) = \left| (\Lambda' \right)^\top \Delta \mu \right| \). When computing \( \text{CLD}(o_{+g}, \ell_{sq}) \) this difference (coefficient of \( \hat{\beta} \) for \( g \)) has more significance if data points deviate more from the center of two groups.

By the one-dimensional case \( \Lambda' \leq \Lambda \). Therefore the attenuation factor is more significant, and the predicted line is more tilted in comparison to regression using \( o_{-g} \).

In the previously discussed example shown in Figure 2 using \( o_{+g} \) the estimated line is the red lines. In this example, \( \beta_{+g}^{2} = \mathbb{E}[\text{Var}[z \mid g]] = \frac{2}{3} \); therefore \( \Lambda' = \frac{3}{2} \) and consequently \( \hat{\beta} = \frac{2}{7}, \hat{\alpha} = \frac{2}{7}, \hat{\beta}_y = \frac{9}{7} \). The red lines have better performance for each group, both in terms of residuals and squared error. However, this benefit comes at the expense of having high CLD (in this example, \( \text{CLD}(o_{+g}, \ell_{res}) = \frac{9}{7} \)).

Table 1 presents a summary of the computed 8 metrics in this section. We also study general noise in Appendix C and infinite noise in Appendix D.

### 5 Persistence of Loss Discrepancy

So far, we assumed the training distribution used to estimate parameters is the same as the test distribution that we are interested measuring loss discrepancy with respect to. But what if the train and test distributions are different? As stated in Section 3.1 the estimated parameters of linear regression in the presence of noise depend on the mean and variance of training distribution. So it is possible that the
model have poor performance when mean and variance of test distribution is different (see Figure 3).

Our formulation presented in Proposition 1 and 2 can be rewritten in terms of train and test distribution as follows:

\[
\begin{align*}
\text{CLD}(o_{+g}, \ell_{\text{res}}) &= \left| (A'_{\text{train}}\beta)^\top \Delta \mu_{z(\text{train})} \right| \\
\text{SLD}(o_{+g}, \ell_{\text{res}}) &= \left| (A'_{\text{train}}\beta)^\top (\Delta \mu_{z(\text{train})} - \Delta \mu_{z(\text{test})}) \right| \\
\text{CLD}(o_{-g}, \ell_{\text{res}}) &= 0 \\
\text{SLD}(o_{-g}, \ell_{\text{res}}) &= \left| (A'_{\text{train}}\beta)^\top \Delta \mu_{z(\text{test})} \right|,
\end{align*}
\]

where the subscript denotes whether the statistics are computed on the training or test distribution.

For the sake of space, we only focus on residual loss function \(\ell_{\text{res}}\). We consider one specific type of covariate shift, which we call demographic shift, where the mean of one group \((E[z | g = 0])\) shifts toward the other group \((E[z | g = 1])\). How long does it take for a linear predictor to adapt itself to the demographic shift? For simplicity, we consider the following two distributions (the same analysis work if groups have different variances or sizes).

- Initial distribution: The mean of \(z\) for group \(g = 1\) is \(-\mu\), and for group \(g = 0\) is \(\mu\), the variance of \(z\) for both groups is \(\Sigma\).
- Shifted distribution: The mean of \(z\) for both groups is \(\mu\) and its variance is \(\Sigma\).

The following proposition states that SLD and CLD converges to 0 as we see more examples from the shifted distribution.

\textbf{Proposition 3.} For \(0 \leq t \leq 1\), let training distribution be a mixture of initial distribution with probability \(t\) and the shifted distribution with probability \(1-t\). Let \(c_1 = \left( (\Sigma + \Sigma_u)^{-1} \Sigma_u \beta \right)^\top (2\mu)\), \(c_2 = \left( (\Sigma + \Sigma_u)^{-1} \Sigma_u \beta \right)^\top (2\mu)\). For a linear predictor which is trained on the above distribution and test on the shifted distribution, we have:

\[
\begin{align*}
t (c_1 - c_2) &\leq \text{SLD}(o_{+g}, \ell_{\text{res}}) = \text{CLD}(o_{+g}, \ell_{\text{res}}) \\
&\leq t (c_1 + |c_2|) \\
\text{SLD}(o_{-g}, \ell_{\text{res}}) &= \text{CLD}(o_{-g}, \ell_{\text{res}}) = 0.
\end{align*}
\]

\textbf{Proof sketch.} In shifted distribution, the means of the groups are the same; therefore using (23), \(\text{CLD}(o_{-g}, \ell_{\text{res}}) = \text{SLD}(o_{-g}, \ell_{\text{res}}) = 0\) and \(\text{CLD}(o_{+g}, \ell_{\text{res}})\) and \(\text{CLD}(o_{+g}, \ell_{\text{res}})\) are equal.

As shown in Proposition 2, \(\text{CLD}(o_{+g}, \ell_{\text{res}}) = (\Lambda'\beta)^\top \Delta \mu_{z(\text{train})}\). The mean difference \(\Delta \mu_{z(\text{train})} = 2t\mu\), and converges to zero as \(t\) decreases. The challenge is to bound the mean difference \(\Delta \mu_{z(\text{train})}\).

Using this result and noting that inverse of covariance matrix is positive definite, we can compute the above bound.

\textbf{Proposition 3} states that when group membership is not used \((o_{-g})\), then linear predictor will have no loss discrepancy if the demographic shift occurs. However, this is not the case for \(o_{+g}\). For example, let’s assume we have training data of \(K + 1\) batches such that the first batch is from the initial distribution which has loss discrepancy \(\text{SLD}_{\text{initial}}\) (this is equal to \(c_1\) in the above proposition) and \(K\) batches from the shifted distribution, the loss discrepancy on the shifted distribution is \(\text{SLD}_{\text{new}} \approx \frac{1}{K+1} \text{SLD}_{\text{initial}}\) and converges to zero with rate of \(\frac{1}{K+1}\).

\section{Experiments}

We now study three real world datasets and show that in the presence of feature noise, the datasets with groups of different means, variances, and sizes are more susceptible to loss discrepancy. We then study the adaptation of the least squares estimator to demographic shift (as explained in Section 5), and observe the same rate of adaptation as shown in Proposition 3.

Since we do not have access to the true features \((z)\) and only have access to the observed features \((x)\), we cannot calculate CLD, so we only report SLD. We also simulate feature noise by treating \(x\) as \(z\) and adding noise to obtain \(x\). Although our assumptions do not hold anymore, we still observe that the difference between moments of the groups are relevant factors governing loss discrepancy.

\textbf{Datasets.} We consider three real-world datasets which are common in fairness literature. A summary of the datasets are shown in Table 2, for more information see Appendix E

\textbf{Setup.} We standardize all features and the target in all datasets (except the group membership feature) to have mean and variance of 0 and 1, respectively. We run each experiment 100 times, each time randomly performing a 80–20 train-test split of the data, and reporting the average on the test set. We train two least squares estimator on two observation functions: \(o_{-g}\) which only have access to non-group features, and \(o_{+g}\) which have access to all features. We consider two types of noise:

![Figure 3: In the presence of noise, the estimated model (red line) has poor performance on a shifted distribution.](image-url)
of mean and variance of the prediction target between groups, respectively.

In C&C, we see that as we increase the noise, the squared error is similar for both groups, or the noise cause it to systematically over/underestimate some group features and simulate dropping out features by adding normal noise with a very high variance ($u \sim N(0, 10000)$) to them sequentially.

Loss discrepancy based on squared error. Expectedly, increases in the amount of noise results in larger squared errors (SE). We see a smoother increase in Figure 4a as opposed to the large jumps in Figure 4b related to the “importance” of the feature. For instance, for the Students dataset, removing the 1st and 2nd period grade makes the prediction of the 3rd period grade (prediction target) more difficult and results in a large jump in average squared error.

We are interested to see whether the observed increase in the squared error is similar for both groups, or the noise affected the groups differently (i.e., inducing high loss discrepancy). In C&C, we see that as we increase the noise, SLD increases (blue lines in Figure 4a), meaning that one group is incurring higher loss in comparison to the other group. In Law (race) dataset, the sizes of the groups are very different (as shown in Table 2) whites represent 86% of the population. Recalling from Proposition 1 when group membership is not used ($o_{-g}$), the minority group gets higher loss; this fact reflects in the observation that SLD($o_{-g}$, $\ell_{sq}$) (dotted blue line) increases as we add more noise. On the other hand, once group membership is used, the group size does not influence the loss discrepancy; therefore, we do not observe an increase in SLD($o_{+g}$, $\ell_{sq}$) (solid blue line).

Loss discrepancy based on residual. As discussed in Section 3, when we estimate the parameters of linear regression, the bias (average residual) is always zero. Therefore, as we increase the noise, the average residual remains zero.

Is the average residual also 0 for both groups or does adding noise cause it to systematically over/underestimate some groups (i.e., inducing loss discrepancy based on residual)? As discussed in Section 4.2, when group membership is used, then the average residual for each group is always zero (SLD($o_{+g}$, $\ell_{res}$) = 0). As shown in Figure 4, the solid green line is very close to zero. However, when we do not use the group membership ($o_{-g}$), as discussed in Section 4.1 if groups have different means then there will be a difference between the average residuals of the groups. In all datasets, without any noise, the residual loss discrepancy (SLD($o_{-g}$, $\ell_{res}$)) is very small. However, as we increase the noise in Figure 4a or as the informative features are dropped in Figure 4b, we see that SLD($o_{-g}$, $\ell_{res}$) increases. Therefore, noise affects different groups differently and causes the prediction target to be over/underestimated for different groups. The only dataset in which the loss discrepancy for residuals does not increase is law (sex), in which the considered groups have similar means.

As shown in Proposition 2 and 3 there is a close relationship between CLD($o_{+g}$, $\ell_{res}$) and SLD($o_{-g}$, $\ell_{res}$). In these datasets we cannot calculate CLD($o_{+g}$, $\ell_{res}$); however, as shown in Figure 4, the weight associated with the group membership feature ($\beta_{group}$) is very close to the residual loss discrepancy (SLD($o_{-g}$, $\ell_{res}$)).

Adaptation of least squares estimator to demographic shift. We now study the adaptation of linear regression to demographic shift (as introduced in Section 5). To simulate demographic shift, we consider the original distribution (uniform distribution over all data points), and a re-weighted distribution where weights are chosen such that the mean of the features (except for the group membership) and the mean of the prediction target are the same between both groups. We compute such a re-weighting using linear programming. For different values of $K$, we train two least square estimators (with and without group membership) on a batch of size $n = 1000$ from the original distribution and a batch of size $Kn$ from the re-weighted distribution. We then calculate loss discrepancy and average squared error (SE) of both models on the re-weighted distribution.

As shown in Proposition 3, if the mean of the groups are similar at the test time, then the estimator without group membership achieves zero residual loss discrepancy (see dotted green line in Figure 5). On the other hand, due to the demographic shift the average loss discrepancy when considering group membership is no longer zero, and in accordance with Proposition 5 it converges to zero with rate of $O(1/K+1)$ (see solid green lines).

Table 2: Statistics of the used datasets. Size of the first group is denoted by $E[g = 1]$ and $\Delta \mu_y$ and $\Delta \sigma_y^2$ denote the difference of mean and variance of the prediction target between groups, respectively.

| name  | #records | #features | prediction target | features example | group $E[g = 1]$ | $\Delta \mu_y$ | $\Delta \sigma_y^2$ |
|-------|----------|-----------|-------------------|------------------|-----------------|----------------|-------------------|
| C&C   | 1994     | 91        | crime rate        | #homeless, average income, . . . | race            | 0.50           | 1.10              |
| law   | 20798    | 25        | final GPA         | undergraduate GPA, LSAT, . . . race | sex             | 0.86           | 0.87              | 0.01              |
| students | 649     | 33        | final grade       | study time, #absences, . . . sex | sex             | 0.56           | 0.01              | 0.04              |
Noise Induces Loss Discrepancy Across Groups for Linear Regression

![Graphs showing statistical loss discrepancy (SLD) and squared error (SE) with different noise levels and features](image)

(a) Adding noise increases squared error (SE) in all datasets; however, noise induces different loss discrepancy across the datasets.

(b) Depending on the dropped feature, abrupt steps may be observed on SE and SLD.

Figure 4: Statistical loss discrepancy (SLD) and squared error (SE) when (a) independent normal noise \((u \in \mathcal{N}(0, \sigma^2_u))\), is added to each feature (except for the group membership) (b) normal noise with high variance is added to the features sequentially (except for the group membership).

![Graphs showing loss discrepancy for the predictor learned on \(o^-\)g (dotted green line) and \(o^+\)g (solid green line) per noisy features](image)

Figure 5: Loss discrepancy for the predictor learned on \(o^-\) (dotted green line) is always zero. Loss discrepancy for the predictor learned on \(o^+\) (solid green line) converges to 0 with rate of approximately \(\frac{1}{K+T}\).

7 Related work

While many papers focus on measuring loss discrepancy (Kusner et al., 2017; Hardt et al., 2016; Pierson et al., 2017; Simoiu et al., 2017; Khani et al., 2019) and mitigating loss discrepancy (Calmon et al., 2017; Hardt et al., 2016; Zafar et al., 2017), there are relatively few that study how loss discrepancy arises in machine learning models.

Chen et al. (2018) decompose the loss discrepancy into three components—bias, variance, and noise. They mainly focus on the bias and variance, and also consider scenarios in which available features are not equally predictive for both groups. There are also lines of work which assume the loss discrepancy of the model is because of biased target values (e.g., Madras et al., 2019). Some work states that high loss discrepancy is due to lack of data for minority groups (Chouldechova and Roth, 2018). Some assume different groups have different functions (sometime in conflict with each other) (Dwork et al., 2018), and therefore, fitting the same model for both groups is suboptimal. In this work, we showed even when the prediction target is correct (not biased), with infinite data, the same function for both groups, equal noise for both groups, there is still loss discrepancy.

Recently, there is some work showing that enforcing fairness constraints without accurately understanding how they change the predictor results in worse outcomes for both groups. Corbett-Davies and Goel (2018) look at different group fairness notions and show how they can lead to worse results if groups have different risk distributions. Liu et al. (2018) show that enforcing some fairness notions hurts the minorities in the long term. Lipton et al. (2018) show that removing disparate treatment and disparate impact simultaneously causes suboptimal solutions and in-group discrimination. Our result that simple feature noise leads to loss discrepancy even under otherwise favorable conditions points at a more fundamental problem in the lack of information about individuals.

Finally, in this work, we assumed that we have access to the true features \(z\), and the observation function \(o\); therefore, we did not need to infer them from data. We are agnos-
tic to the causal graph and assumed either $z$ are resolving features as defined by Kilbertus et al. (2017), or there is an unobserved variable $v$, which affects both $z$ and $g$. There is a rich line of work on checking the fairness of a model when true features and observation function need to be inferred from data (Kusner et al. 2017; Nabi and Shpitser 2018; Kilbertus et al. 2017).

References

Agarwal, A., Beygelzimer, A., Dudík, M., Langford, J., and Wallach, H. (2018). A reductions approach to fair classification. arXiv preprint arXiv:1803.02453.

Angwin, J., Larson, J., Mattu, S., and Kirchner, L. (2016). Machine bias: There’s software used across the country to predict future criminals. and it’s biased against blacks. ProPublica, 23.

Arrow, K. (1973). The theory of discrimination. Discrimination in labor markets, 3(10):3–33.

Barocas, S. and Selbst, A. D. (2016). Big data’s disparate impact. 104 California Law Review, 5:671–732.

Bechavod, Y. and Ligett, K. (2017). Penalizing unfairness in binary classification. arXiv preprint arXiv:1707.00044.

Berk, R. (2012). Criminal justice forecasts of risk: A machine learning approach. Springer Science & Business Media.

Calmon, F., Wei, D., Vizamuri, B., Ramamurthy, K. N., and Varshney, K. R. (2017). Optimized pre-processing for discrimination prevention. In Advances in Neural Information Processing Systems (NeurIPS), pages 3992–4001.

Canetti, R., Cohen, A., Dikkala, N., Ramnarayan, G., Scheffler, S., and Smith, A. (2019). From soft classifiers to hard decisions: How fair can we be? In Proceedings of the Conference on Fairness, Accountability, and Transparency, pages 309–318.

Carroll, R. J., Ruppert, D., Stefanski, L. A., and Crainiceanu, C. M. (2006). Measurement error in nonlinear models: a modern perspective. Chapman and Hall/CRC.

Chen, I., Johansson, F. D., and Sontag, D. (2018). Why is my classifier discriminatory? In Advances in Neural Information Processing Systems (NeurIPS), pages 3539–3550.

Chiappa, S. (2019). Path-specific counterfactual fairness. In Association for the Advancement of Artificial Intelligence (AAAI), volume 33, pages 7801–7808.

Chouldechova, A. (2017). A study of bias in recidivism prediction instruments. Big Data, pages 153–163.

Chouldechova, A. and Roth, A. (2018). The frontiers of fairness in machine learning. arXiv preprint arXiv:1810.08810.
Nabi, R. and Shpitser, I. (2018). Fair inference on outcomes. In Association for the Advancement of Artificial Intelligence (AAAI).

Phelps, E. S. (1972). The statistical theory of racism and sexism. The American Economic Review, 62(4):659–661.

Pierson, E., Corbett-Davies, S., and Goel, S. (2017). Fast threshold tests for detecting discrimination. arXiv preprint arXiv:1702.08536.

Pleiss, G., Raghavan, M., Wu, F., Kleinberg, J., and Weinberger, K. Q. (2017). On fairness and calibration. In Advances in Neural Information Processing Systems (NeurIPS), pages 5684–5693.

Redmond, M. and Baveja, A. (2002). A data-driven software tool for enabling cooperative information sharing among police departments. European Journal of Operational Research, 141(3):660–678.

Sherman, J. and Morrison, W. J. (1950). Adjustment of an inverse matrix corresponding to a change in one element of a given matrix. The Annals of Mathematical Statistics, 21(1):124–127.

Simoiu, C., Corbett-Davies, S., Goel, S., et al. (2017). The problem of infra-marginality in outcome tests for discrimination. The Annals of Applied Statistics, 11(3):1193–1216.

Wightman, L. F. and Ramsey, H. (1998). LSAC national longitudinal bar passage study. Law School Admission Council.

Woodworth, B., Gunasekar, S., Ohannessian, M. I., and Srebro, N. (2017). Learning non-discriminatory predictors. In Conference on Learning Theory (COLT), pages 1920–1953.

Zafar, M. B., Valera, I., Rodriguez, M. G., and Gummadi, K. P. (2017). Fairness beyond disparate treatment & disparate impact: Learning classification without disparate mistreatment. In World Wide Web (WWW), pages 1171–1180.
A Proposition

**Proposition 1.** Let $A \overset{\text{def}}{=} (\Sigma_z + \Sigma_u)^{-1} \Sigma_u$. The least squares estimator \[^{[6]}\] based on $o_{-g}$ \[^{[12]}\] results in:

- $CLD(0, \ell_{res}) = CLD(0, \ell_{sq}) = 0$
- $SLD(0, \ell_{res}) = |(\Lambda \beta)^T \Delta \mu_z|$
- $SLD(0, \ell_{sq}) = |(\Lambda \beta)^T \Delta \Sigma_z (\Lambda \beta) - (\mathbb{P}[g = 1] - \mathbb{P}[g = 0])((\Lambda \beta)^T \Delta \mu_z)^2|$

where $\Delta \mu_z$ and $\Delta \Sigma_z$ are as defined in \[^{[13]}\] and \[^{[15]}\].

**Proof.** As stated in \[^{[6]}\] :

\[
\hat{\beta} = \Sigma_x^{-1} \Sigma_{xy} = (\Sigma_z + \Sigma_u + \Sigma_{zu} + \Sigma_{uy} + \Sigma_{uz} + \Sigma_{uu})^{-1} (\Sigma_{zy} + \Sigma_{uy}) \tag{26}
\]

Due to assumptions ($u$ is independent of other variables), $\Sigma_{zu} = 0$ and $\Sigma_{uy} = 0$, and from \[^{[6]}\], $\Sigma_{zy} = \Sigma_z \beta$.

\[
\hat{\beta} = (\Sigma_z + \Sigma_u)^{-1} \Sigma_z \beta = (I - (\Sigma_z + \Sigma_u)^{-1} \Sigma_u) \beta = (I - \Lambda) \beta \tag{27}
\]

The intercept formulation is as follows:

\[
\hat{\alpha} = \beta^T \mathbb{E}[z] + \alpha - \hat{\beta}^T (z + u) - \hat{\alpha} \mid g = 0 = \mathbb{E}[\beta^T z + \alpha - \hat{\beta}^T (z + u) - \hat{\alpha} \mid g = 0] \tag{28}
\]

Since $o(z, 0, u) = o(z, 1, u)$, we have $CLD(0, \ell_{res}) = CLD(0, \ell_{sq}) = 0$.

For computing SLD based on residual, we first compute the expected $\ell_{res}$ for the first group:

\[
\mathbb{E}[y - \hat{y} \mid g = 0] = \mathbb{E}[\beta^T z + \alpha - \hat{\beta}^T (z + u) - \hat{\alpha} \mid g = 0]
\]

\[
= \mathbb{E}[(\Lambda \beta)^T z - (\Lambda \beta)^T \mathbb{E}[z] - \hat{\beta}^T u \mid g = 0] \tag{29}
\]

Using $\mathbb{E}[u] = 0$ and $\mathbb{E}[z] = \mathbb{P}[g = 0] \mathbb{E}[z \mid g = 0] + \mathbb{P}[g = 1] \mathbb{E}[z \mid g = 1]$, we have:

\[
\mathbb{E}[y - \hat{y} \mid g = 0] = (\Lambda \beta)^T (\mathbb{E}[z \mid g = 0] - \mathbb{E}[z]) \tag{30}
\]

\[
= (\Lambda \beta)^T (\mathbb{E}[z \mid g = 0] - \mathbb{P}[g = 0] \mathbb{E}[z \mid g = 0] - \mathbb{P}[g = 1] \mathbb{E}[z \mid g = 1]) \tag{31}
\]

\[
= (\Lambda \beta)^T (\mathbb{E}[z \mid g = 0] - \mathbb{P}[g = 0]) \mathbb{E}[z \mid g = 1] - \mathbb{E}[z \mid g = 0]) \tag{32}
\]

\[
= (\Lambda \beta)^T (\mathbb{E}[z \mid g = 0] - \mathbb{P}[g = 0]) \mathbb{E}[z \mid g = 1] - \mathbb{E}[z \mid g = 0]) \tag{33}
\]

\[
= (\Lambda \beta)^T \Delta \mu_z. \tag{34}
\]

Note that the first group ($g = 0$) have lower expected $\ell_{r}$ if its size is large or $\Delta \mu_z$ (projected on $\beta^T \Lambda^T$) is small. Similarly for the second group ($g = 1$) we have:

\[
\mathbb{E}[y - \hat{y} \mid g = 1] = (1 - \mathbb{P}[g = 1]) (\Lambda \beta)^T \Delta \mu_z. \tag{35}
\]

Computing the difference between the expected residuals of the groups we have: $SLD(0, \ell_{res}) = |(\Lambda \beta)^T \Delta \mu|$. For computing $SLD(0, \ell_{sq})$, first note that the squared error can be decomposed to squared of bias and variance,

\[
\mathbb{E}[(\hat{y} - y)^2] = \mathbb{E}[(\hat{y} - y)^2] + \text{Var}[(\hat{y} - y)]. \tag{36}
\]

Using this decomposition, we have:

\[
SLD(0, \ell_{sq}) = |\mathbb{E}[\ell_{sq} \mid g = 1] - \mathbb{E}[\ell_{sq} \mid g = 0]| \tag{37}
\]

\[
= |\mathbb{E}[\ell_{res} \mid g = 1] - \mathbb{E}[\ell_{res} \mid g = 0]| + \text{Var}[\ell_{res} \mid g = 1] - \text{Var}[\ell_{res} \mid g = 0] \tag{38}
\]
Using (34) and (35), we have:
\[
\mathbb{E}[\ell_{\text{res}} \mid g = 1]^2 - \mathbb{E}[\ell_{\text{res}} \mid g = 0]^2 = \mathbb{P}[g = 0]^2((\Lambda\beta)^T \Delta \mu_z)^2 - \mathbb{P}[g = 1]^2((\Lambda\beta)^T \Delta \mu_z)^2
\]
\[= - (\mathbb{P}[g = 1] - \mathbb{P}[g = 0])((\Lambda\beta)^T \Delta \mu_z)^2.
\]
(39)

For the difference between variances, we have:
\[
\text{Var}[\ell_{\text{res}} \mid g = 1] - \text{Var}[\ell_{\text{res}} \mid g = 0] = \text{Var}\left[ (\Lambda\beta)^T z + \hat{\beta}^T u \mid g = 1 \right] - \text{Var}\left[ (\Lambda\beta)^T z + \hat{\beta}^T u \mid g = 0 \right]
\]
\[= (\Lambda\beta)^T \Delta \Sigma_z (\Lambda\beta)
\]
(41)

Since \(u\) is independent of \(z\) and \(g\) the difference is only related to the difference between variances.
\[
\text{Var}[\ell_{\text{res}} \mid g = 1] - \text{Var}[\ell_{\text{res}} \mid g = 0] = \text{Var}\left[ (\Lambda\beta)^T z \mid g = 1 \right] - \text{Var}\left[ (\Lambda\beta)^T z \mid g = 0 \right]
\]
\[= (\Lambda\beta)^T \Delta \Sigma_z (\Lambda\beta)
\]
(43)

Combining (40) and (44) completes the proof.

\[\square\]

**B Proposition 2**

**Proposition 2.** Consider the observation function \(o_{+g} \) [20]. Let \(\Sigma_{z\mid g} = \mathbb{E}[z \mid g]\), and \(\Lambda' = (\Sigma_{z\mid g} + \Sigma_u)^{-1}\Sigma_u\). The estimated parameters using (8) are:
\[
\hat{\beta} = \begin{bmatrix}(I - \Lambda')\beta \\ (\Lambda'\beta)^T \Delta \mu_z\end{bmatrix}
\]
\[\hat{\alpha} = (\Lambda'\beta)^T \mathbb{E}[z \mid g = 0] + \alpha.
\]
(21)

The loss discrepancies are as follows:
\[
\text{CLD}(o_{+g}, \ell_{\text{res}}) = \left| (\Lambda'\beta)^T \Delta \mu_z \right|
\]
\[
\text{CLD}(o_{+g}, \ell_{sq}) = \left| (\Lambda'\beta)^T \Delta \mu_z \right| \mathbb{E}\left[ ((\Lambda'\beta)^T (2z - \mu_1 - \mu_0)) \right]
\]
\[
\text{SLD}(o_{+g}, \ell_{res}) = 0
\]
\[
\text{SLD}(o_{+g}, \ell_{sq}) = \left| (\Lambda'\beta)^T \Delta \Sigma_z (\Lambda'\beta) \right|
\]
\[\text{where } \Delta \mu_z \text{ and } \Delta \Sigma_z \text{ are as defined in (13) and (15), and } \mu_1 \overset{\text{def}}{=} \mathbb{E}[z \mid g = 1] \text{ and } \mu_0 \overset{\text{def}}{=} \mathbb{E}[z \mid g = 0].
\]

**Proof.** For simplicity, let \(z' \overset{\text{def}}{=} z + u\). We are interested in finding the best linear estimator for \(y\), given \(z'\) and \(g\). According to our assumptions we have:
\[
y = \beta^T z + \alpha
\]
(45)
\[
\mathbb{E}[y \mid z', g] = \beta^T \mathbb{E}[z \mid z', g] + \alpha
\]
(46)

First note that, we can represent \(\mathbb{E}[z' \mid g]\) according to \(g\) linearly as follows:
\[
\mathbb{E}[z' \mid g] = \mathbb{E}[z' \mid g = 0] + (\mathbb{E}[z' \mid g = 1] - \mathbb{E}[z' \mid g = 0])g
\]
(47)

We now write a linear predictor for \(\mathbb{E}[z \mid z', g]\) given \(z'\) and \(g\) with some re-parametrization for simplicity. Define \(v \overset{\text{def}}{=} z' - \mathbb{E}[z' \mid g]\) and \(w \overset{\text{def}}{=} g - \mathbb{E}[g]\), we have:
\[
\gamma_0 + \gamma_1(v - \mathbb{E}[v \mid g]) + \gamma_2(w - \mathbb{E}[w])
\]
(48)

If \(n\) denotes the dimension of \(z\) then \(\gamma_0 = n \times 1\), \(\gamma_1 = n \times n\) and finally \(\gamma_2 = n \times 1\).
First note that $E[v] = E[z'] - E[E[z' | g]] = 0$. Therefore, we have:

$$\text{Cov}(v, w) = E[vw]$$

$$= E[z'g] - E[z']E[g] - E[E[z' | g]g] + E[E[z' | g]]E[g]$$

$$= E[z'g] - E[E[z' | g]g] = 0$$

(49) (50) (51) (52)

As a result, due to orthogonality, we can compute the parameters as follows:

$$\gamma_1 = \Sigma_v^{-1}\Sigma_{vz}$$

$$\Sigma_v = E \left[ \left( z' - E[z' | g] \right) \left( z' - E[z' | g] \right)^\top | g \right] = E[\text{Var}[z' | g]]$$

(53) (54)

For the covariance between $v$ and $z$ we have:

$$\Sigma_{vz} = E[vz^\top] - E[v]E[z]^\top$$

$$= E \left[ \left( z' - E[z' | g] \right)z^\top \right]$$

$$= E[z'z^\top] - E[E[z' | g]z^\top]$$

$$= E[zz^\top] + E[uz^\top] - E[E[z + u | g]z^\top]$$

$$= E[zz^\top] - E[E[z | g]z^\top]$$

$$= E \left[ \left( z - E[z | g] \right) \left( z - E[z | g] \right)^\top | g \right]$$

$$= E[\text{Var}[z | g]]$$

(55)

Combining the above equations and presenting $\hat{\beta}$ to two parts ($\hat{\beta}_z$ for coefficient of $z'$ and $\hat{\beta}_g$ for the coefficient of $g$), we have:

$$\hat{\beta}_z = \gamma_1 = E[\text{Var}(z' | g)]^{-1}E[\text{Var}[z | g]].$$

(56)

Using (47) and noting $E[z' | g] = E[z + u | g] = E[z | g]$, we have $\gamma_2 = \Delta \mu$; therefore, $\hat{\beta}_g = \gamma_2 - \gamma_1 \Delta \mu$

Now define $\Lambda'$ to be:

$$\Lambda' = I - (E[\text{Var}(z' | g)]^{-1}E[\text{Var}[z | g]])$$

(57)

Then the estimated parameters are:

$$\hat{\beta}_z = (I - \Lambda')\beta, \quad \hat{\beta}_g = (\Lambda')^\top \Delta \mu,$$

(58)

For the intercept we have:

$$\hat{\alpha} - \alpha = \beta^\top E[z] - \hat{\beta}_z E[z] - \hat{\beta}_g E[g]$$

$$= (I - (I - \Lambda'))\beta^\top E[z] - (\Lambda')^\top \Delta \mu E[z] - \hat{\beta}_g E[g]$$

$$= (\Lambda')^\top \left( E[z] - (E[z | g = 1] - E[z | g = 0])E[g] \right)$$

$$= (\Lambda')^\top \left( E[z] - (E[z | g = 1] - E[z | g = 0])E[g] \right)$$

$$= (\Lambda')^\top \left( E[z | g = 1] - E[z | g = 0] \right).$$

(59)

Now that we calculated the estimated parameters, we compute loss discrepancies. SLD based on residuals is zero, since $E[\ell_{\text{res}} | g = 0] = E[\ell_{\text{res}} | g = 1] = 0$. We can calculate SLD based on the squared error using the same techniques as Proposition 1

$$\text{SLD}(o + g, \ell_{\text{sq}}) = (\Lambda')^\top \Delta \Sigma_z (\Lambda')^\top.$$

(60)
We now compute CLD$(o_{+g}, \ell_{\text{res}})$. Recall $L_{g'} \overset{\text{def}}{=} \mathbb{E}[\ell(y, h(o(z, g, u)) \mid z]$.

\[
\text{CLD}(o_{+g}, \ell_{\text{res}}) = \mathbb{E}[\|L_1 - L_0\|]
= \mathbb{E}\left[\mathbb{E}[y - \hat{\beta}_z^T z - \hat{\beta}_g - \hat{\alpha} \mid z] - \mathbb{E}[y - \beta_z^T z - \beta_g u - \alpha \mid z]\right]
= |\hat{\beta}_g|
= |(\Lambda' \beta)^\top \Delta \mu_z|.
\]

For calculating CLD$(o_{+g}, \ell_{\text{sq}})$, let $\mu_1 = \mathbb{E}[z \mid g = 1]$ and $\mu_0 = \mathbb{E}[z \mid g = 0]$, then we have:

\[
\text{CLD}(o_{+g}, \ell_{\text{sq}}) = \mathbb{E}[\|L_1 - L_0\|]
= \mathbb{E}\left[\left(y - \hat{\beta}_z^T (z + u) - \hat{\beta}_g - \hat{\alpha}\right)^2 \mid z\right] - \mathbb{E}\left[\left(y - \beta_z^T (z + u) - \alpha\right)^2 \mid z\right]
= \mathbb{E}\left[\|((\Lambda' \beta)^\top (z - \mu_1))^2 - ((\Lambda' \beta)^\top (z - \mu_0))^2\|\right]
= \mathbb{E}\left[((\Lambda' \beta)^\top (z - \mu_1)(z - \mu_1)^\top - (z - \mu_0)(z - \mu_0)^\top)(\Lambda' \beta)\right]
= \mathbb{E}\left[((\Lambda' \beta)^\top (\mu_1 - \mu_2)(2z - \mu_1 - \mu_2)^\top)(\Lambda' \beta)\right]
= |((\Lambda' \beta)^\top \Delta \mu_z)|^2 \mathbb{E}\left[\left((\Lambda' \beta)^\top (z - \frac{\mu_1 + \mu_0}{2})\right)^2\right]
\]

\[
\text{Proof.} \quad \text{Due to assumption, } \Delta \mu_{z,\text{test}} = 0; \text{ therefore, } \text{CLD}(o_{-g}, \ell_{\text{res}}) = \text{SLD}(o_{-g}, \ell_{\text{res}}) = 0 \text{ and } \text{CLD}(o_{+g}, \ell_{\text{res}}) \text{ are equal.}
\]

As shown in Proposition 2, SLD$(o_{+g}, \ell_{\text{res}}) = (\Lambda' \beta)^\top \Delta \mu_{z,\text{train}}$. The mean difference $\Delta \mu_{z,\text{train}} = 2t\mu$, and converges to zero as $t$ decreases. The challenge is to bound $\Lambda' = (\Sigma + 2t(1 - t)\mu\mu^\top + \Sigma_u)^{-1}\Sigma_u$. Sherman and Morrison (1950) shows that if $A$ is nonsingular and $u, v$ are column vectors then

\[
(A + uv^\top)^{-1} = A^{-1} - \frac{1}{1 + v^\top A^{-1} u} A^{-1}uv^\top A^{-1}
\]

We can simplify $\Lambda'$ using the above equation.

\[
((\Sigma + \Sigma_u) + 2t(1 - t)\mu\mu^\top)^{-1} = (\Sigma + \Sigma_u)^{-1} - \frac{2t(1 - t)}{1 + 2t(1 - t)\mu^\top (\Sigma + \Sigma_u)^{-1}\mu}(\Sigma + \Sigma_u)^{-1}\mu\mu^\top (\Sigma + \Sigma_u)^{-1}
\]

First note that since we assumed the inverse of covariance matrix $(\Sigma + \Sigma_u)$ exists, therefore, it should be positive definite. As a result $\mu^\top (\Sigma + \Sigma_u)^{-1}\mu \geq 0$. Also note that $0 \leq 2t(1 - t) \leq 1$; therefore, we can have the following bound:

\[
0 \leq \frac{2t(1 - t)}{1 + 2t(1 - t)\mu^\top A^{-1} \mu} \leq 1
\]
For simplicity, let \( r = \frac{2t(1-t)}{1+2t(1-t)\mu - \mu} \), using the above bound we can bound loss discrepancy when \( o_{+g} \) is used:

\[
CLD(\o_{+g}, \ell_{res}) = (\Delta_{\text{train}}) \Delta_{\mu_{\text{train}}} = t \left( (\Sigma + \Sigma_u)^{-1}\Sigma_u \beta \right)^{T} (2\mu) - r \left( (\Sigma + \Sigma_u)^{-1}\mu \mu^{T} (\Sigma + \Sigma_u)^{-1}\Sigma_u \beta \right)^{T} (2\mu) \tag{74}
\]

(75)

Defining \( c_1 \) and \( c_2 \) we have:

\[
t (c_1 - |c_2|) \leq SLD(o_{+g}, \ell_{res}) = CLD(o_{+g}, \ell_{res}) \leq t (c_1 + |c_2|) \tag{76}
\]

D Infinite noise

When noise is infinite, the predictor simply predicts \( \mu_y = \mathbb{E}[y] \) for all data points when group membership is not available \((o_{-g})\), and it predicts the average of each group for the members of that group when group membership is available \((o_{+g})\). In this case, statistical loss discrepancy is only related to the moments of groups on \( y \), see Table 3.

| CLD | SLD | average performance |
|-----|-----|---------------------|
| \( \ell_{res} \) | \( o_{-g} : \Delta \mu_y \) | \( o_{-g} : \mathbb{E}[\ell_{res}] = 0 \) |
| \( o_{+g} : \Delta \mu_y \) | \( o_{+g} : \mathbb{E}[\ell_{res}] = 0 \) |
| \( \ell_{sq} \) | \( o_{-g} : 0 \) | \( o_{-g} : \mathbb{E}[\ell_{sq}] = \sigma^2_y \) |
| \( o_{+g} : 2\Delta \mu_y \mathbb{E}[y - \mu_y - (\frac{1}{2} - \mathbb{E}[g])\Delta \mu_y] \) | \( o_{+g} : \mathbb{E}[\ell_{sq}] = \sigma^2_{gg} \) |

Table 3: A summery of metrics in the presence of infinite noise. Here, \( \Delta \mu_y \) and \( \Delta \sigma^2_y \) are defined as \( \mathbb{E}[y \mid g = 1] - \mathbb{E}[y \mid g = 0] \) and \( \mathbb{V}[y \mid g = 1] - \mathbb{V}[y \mid g = 0] \), respectively.

E General noise

The independence assumption on the noise in Section 4.1 and 4.2 enables us to have a closed-form for CLD and SLD. Without any assumptions on the noise, we cannot specify anything about the estimated parameters more than (8). However, we can still analyze the form of SLD given the estimated parameters.

**Proposition 4.** Fix \( o(z, g, u) \) as an arbitrary observation function, such that for \( x = o(z, g, u) \) the covariance matrix \( \Sigma_x \) is invertible. Let \( \hat{\beta} \) and \( \hat{o} \) be the estimated parameters as computed in (6). Equalize dimensions of \( \hat{\beta}, \beta, z, \) and \( x \) by adding extra \( 0 \)'s at the end; and let \( u = x - z \) denote the add-on error to the value of the true feature. The statistical loss discrepancies are as follows:

\[
SLD(o, \ell_{res}) = |\hat{\beta}^{T} \Delta \mu_x - \Delta \mu_y| \tag{77}
\]

\[
= |(\hat{\beta}^{T} - \beta^{T}) \Delta \mu_x + \hat{\beta}^{T} \Delta \mu_u| \tag{78}
\]

where for any random variable \( t \), \( \Delta \mu_t = \mathbb{E}[t \mid g = 1] - \mathbb{E}[t \mid g = 0] \). For SLD based on \( \ell_{sq} \), we have:

\[
SLD(o, \ell_{sq}) = |\Delta \Sigma_y + \hat{\beta}^{T} \Delta \Sigma_x \hat{\beta} - 2\hat{\beta}^{T} \Delta \Sigma_{xy}| \tag{79}
\]

\[
= |(\beta - \hat{\beta})^{T} \Delta \Sigma_y (\beta - \hat{\beta}) + \hat{\beta}^{T} \Delta \Sigma_u \hat{\beta} - 2(\beta - \hat{\beta})^{T} \Delta \Sigma_{zu} \hat{\beta}| \tag{80}
\]

where for any two random variable \( s, t \) we define \( \Delta \Sigma_{st} \) to be:

\[
\Sigma_{st} \overset{\text{def}}{=} \mathbb{E}[(s - \mu_s)(t - \mu_t)^{T} \mid g = 1] - \mathbb{E}[(s - \mu_s)^{T}(t - \mu_t) \mid g = 0] \tag{81}
\]
Proof.

\[
\text{SLD}(o, \ell_{\text{res}}) = |E[\ell_{\text{res}} | g = 1] - E[\ell_{\text{res}} | g = 0]| \\
= |E[\beta^T z + \alpha - \hat{\beta}^T (z + u) - \hat{\alpha} | g = 1] - E[\beta^T z + \alpha - \hat{\beta}^T (z + u) - \hat{\alpha} | g = 0]| \\
= |(\beta - \hat{\beta})^T (E[z | g = 1] - E[z | g = 0]) - \hat{\beta}^T (E[u | g = 1] - E[u | g = 0])| \\
= |(\beta - \hat{\beta})^T \Delta \mu_z - \hat{\beta} \Delta \mu_u |
\]  

(82)

This formulation implies that if \( \Delta \mu_x = 0 \) and \( \Delta \mu_y = 0 \) then for any arbitrary linear predictor \( \text{SLD}(o, \ell_{\text{res}}) \) is always zero.

For computing the statistical loss discrepancy based on squared error, first note that using (6), we have: \( \hat{\alpha} = \beta \mu_z + \alpha - \hat{\beta} \mu_x \Rightarrow \alpha - \hat{\alpha} = -(\beta - \hat{\beta})^T \mu_z + \beta \mu_u \). We can now compute the expected squared error for the first group, as follows:

\[
E[\ell_{\text{sq}} | g = 0] = E\left[ \left( \beta^T z + \alpha - \hat{\beta}^T (z + u) - \hat{\alpha} \right)^2 | g = 0 \right] \\
= E\left[ \left( (\beta - \hat{\beta})^T (z - \mu_z) - \hat{\beta}^T (u - \mu_u) \right)^2 | g = 0 \right] \\
= E\left[ (\beta - \hat{\beta})^T (z - \mu_z)(z - \mu_z)^T (\beta - \hat{\beta}) + \hat{\beta}^T (u - \mu_u)(u - \mu_u)^T \hat{\beta} - 2(\beta - \hat{\beta})^T (z - \mu_z)(u - \mu_u)^T \hat{\beta} | g = 0 \right]
\]  

(85)

The expected squared error for \( g = 1 \) is similar, computing the difference we have:

\[
\text{SLD}(o, \ell_{\text{sq}}) = |E[\ell_{\text{sq}} | g = 1] - E[\ell_{\text{sq}} | g = 0]| \\
= |(\beta - \hat{\beta})^T \Delta \Sigma_z (\beta - \hat{\beta}) + \hat{\beta}^T \Delta \Sigma_u \hat{\beta} - 2(\beta - \hat{\beta})^T (z - \mu_z)(u - \mu_u)^T \hat{\beta} | g = 0 \]
\]  

(87)

Same as (83), we can compute SLD based on squared error in another form as well:

\[
E[\ell_{\text{sq}} | g = 0] = E\left[ \left( y - \hat{\beta}^T x - \hat{\alpha} \right)^2 | g = 0 \right] \\
= E\left[ \left( y - \hat{\beta}^T x - \mu_y + \hat{\beta}^T \mu_x \right)^2 | g = 0 \right] \\
= E\left[ \left( (y - \mu_y) - \hat{\beta}^T (x - \mu_x) \right)^2 | g = 0 \right]
\]  

(88)

\[
\text{SLD}(o, \ell_{\text{sq}}) = |E[\ell_{\text{sq}} | g = 1] - E[\ell_{\text{sq}} | g = 0]| = |\Delta \Sigma_y + \hat{\beta}^T \Delta \Sigma_x \hat{\beta} - 2\hat{\beta}^T \Delta \Sigma_{xy} |
\]  

(89)

Future work could explore the impact of different error patterns across groups on the SLD. Equation (77) states that if the average over features and the target values are the same between groups then \( \text{SLD}(o_g, \ell_{\text{res}}) = 0 \) for any linear predictor. Equation (78) states that if we cannot estimate the true parameters \( \hat{\beta} \neq \beta \) then in order to have \( \text{SLD}(o_g, \ell_{\text{res}}) = 0 \) we should enforce different average add-on errors for the groups (\( \Delta \mu_u \neq 0 \)).
F Datasets

**Students** ([Cortez and Silva](https://archive.ics.uci.edu/ml/datasets/student+performance) 2008) This dataset represents student achievements in secondary education of one Portuguese school in mathematics subject. The data features ($x$) include student grades, demographic, social, and school-related features and it was collected by using school reports and questionnaires. The target ($y$) is the final year grade, and we set the groups ($g$) to be males and females. Students dataset contains the students’ 1st and 2nd period grades, which are strongly correlated with the prediction target (the final grade issued at the 3rd period).

**Law School Admissions Council’s National Longitudinal Bar Passage Study** ([Wightman and Ramsey](https://github.com/jjgold012/lab-project-fairness) 1998) This dataset consists of the records of graduate students in law major. We set the target ($y$) to be the final Grade Point Average. The features include student grades and school-related features. We consider two versions of this data, one where $g$ is race and the other where $g$ is sex. The Law school dataset contains features such as first-year GPA; which are strongly correlated with the prediction target.

**Communities and Crime** ([Redmond and Baveja](http://archive.ics.uci.edu/ml/datasets/communities+and+crime) 2002) This dataset represents communities within the United States, each data point represents a community, and the goal is to predict the per capital violent crimes ($y$) given features such as average income in that community. This dataset contains eight continuous features related to the race, specifying the number and percentage of different races within that community. We replaced these features with a single binary feature; which is 1 if a community is in top 50% of communities with a majority of whites, and 0 if otherwise. We set this binary feature to be the indicate the groups.