Diagrammatic Exponentiation for Products of Wilson Lines

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Abstract

We provide a recursive diagrammatic prescription for the exponentiation of gauge theory amplitudes involving products of Wilson lines and loops. This construction generalizes the concept of webs, originally developed for eikonal form factors and cross sections with two eikonal lines, to general soft functions in QCD and related gauge theories. Our coordinate space arguments apply to arbitrary paths for the lines.
1 Introduction

Path-ordered exponentials of gauge fields over space-time curves are an essential ingredient in many descriptions of gauge theory dynamics. A generic form for these functions is

$$\Phi_C^{(f)}(y, x) = P \exp \left[ \int_0^\infty d\tau \xi_C(\tau) \cdot A^{(f)}(\xi_C(\tau)) \right],$$

where as $\tau$ varies, the coordinate $\xi_C^\mu(\tau)$ traces curve $C$ from point $x = \xi(0)$ to point $y = \xi(\infty)$, which may be open or closed ($y = x$). The gauge field may be in any representation $f$ of the group. Such ordered exponentials are generally referred to as Wilson loops and lines. Matrix elements of products of Wilson lines have become a familiar feature of factorized amplitudes and resummed cross sections in QCD and related gauge theories. Applications to physical process with an electroweak hard scattering are generally based on products of two Wilson lines that meet at a point in a color singlet configuration. This vertex defines a composite operator that requires renormalization, and is often termed the “cusp” vertex. The description of QCD hard scatterings generally requires several lines to meet at what we will call a “multi-eikonal vertex”, of which the simplest example is the cusp. Providing our Wilson lines with color indices, a four-line multi-eikonal vertex, for example, can be represented by a constant matrix, $c_I$ in color space that links the indices $3, 4$,

$$W^{[f]}_{I \{r_k\}} = \sum_{\{d_i\}} \langle 0 | \Phi^{[f_3]}_{v_3}(\infty, 0)_{r_4, d_4} \Phi^{[f_2]}_{v_2}(0, -\infty)_{d_2, r_2} \Phi^{[f_1]}_{v_1}(0, \infty)_{r_3, d_3} | 0 \rangle.$$

For the eikonal Wilson lines of this expression, constant velocities $v_i$ label the curves, which we can choose to be $\xi_j(\tau_j) = v_j \tau_j$ (outgoing) or $\xi_i(\tau_i) = -v_i/\tau_i$ (incoming). The curves meet at the origin, either from infinity in the past or to infinity in the future.

Amplitudes of multi-eikonal vertices contain the bulk of information necessary to reconstruct the full infrared structure of multiparton amplitudes in QCD. An even more direct connection can be found in \( N = 4 \) supersymmetric Yang-Mills theory (SYM), for which certain closed loops of Wilson lines with sequential cusp vertices have the remarkable property of being dual descriptions to full partonic scattering amplitudes, a feature with suggestive connections to the strong-coupling limit.

Many of the useful features of the examples mentioned above result from their exponentiation properties. These may be formulated quite generally in terms of the anomalous

\footnote{In particular, “soft functions”, which organize non-collinear soft radiation, are also a key part of factorizations and resummations based on direct analyses in perturbative QCD and soft-collinear effective theory.}

\footnote{Generally, the term ‘eikonal’ refers to a source with constant velocity. From the point of view of renormalization, however, it is only necessary that the lines meeting at a point have a well-defined local velocity, or tangent vector. We will therefore use this terminology even though our considerations below apply to lines whose local velocities are otherwise arbitrary.}
dimensions of multi-eikonal vertices \[3,10\]. For the cusp anomalous dimension, however, there is a more detailed exponential form, in which the singlet product of Wilson lines is written as the exponential of a sum of two-eikonal irreducible diagrams with modified color factors, the so-called “webs” [15]. The renormalization of the cusp anomalous dimension can be reformulated simply in terms of the webs, and many regularities at nonleading order flow from the underlying exponentiation. The original proofs [15] of diagrammatic exponentiation were formulated in momentum space, specifically for the cusp vertex and for Wilson lines of fixed velocity. These considerations were revisited and extended recently in Ref. [16]. In the present paper, we will develop a diagrammatic exponentiation, not only for multi-eikonal scattering, but also for arbitrary products of Wilson lines of arbitrary length, which may or may not meet at cusp or multi-eikonal vertices.

A potential application of our results is to the systematic calculation of anomalous dimensions for multi-eikonal vertices. The cusp anomalous dimension, for example, can be determined directly from the webs, which can streamline calculations \[17,18\]. More general composite vertices, involving multiple Wilson lines require matrices of anomalous dimensions. By now, these products are understood to two loops for massless \[19\] as well as massive lines \[20–22\], and interesting progress has been made at higher orders of the massless case \[23–25\]. In what follows, we will show that as for the cusp, diagrammatic exponentiation can simplify, and certainly clarifies, the calculation of anomalous dimensions beyond one loop. As in Ref. \[22\], we will find it useful to work in a coordinate space representation.

Graphical exponentiation also has potential applications in the phenomenological treatment of cross sections. Because the underlying structure is fundamentally nonperturbative, webs have been used to organize the structure of power corrections due to soft radiation for Drell-Yan and related cross sections \[26\]. We anticipate analogous applications to QCD hard scattering processes.

We present our diagrammatic construction in Sec. 2 with a discussion of general diagrams for vacuum expectation values of products of Wilson lines. We give a simple combinatorial identity for products of Wilson line amplitudes considered directly in the coordinate space that contains their defining curves. From this identity, we derive an iterative construction for the logarithm of the amplitude, based purely on considerations of counting. We lose the simple condition of diagrammatic irreducibility characteristic of webs for the cusp, but in the general case the color factors of the subdiagrams that make up the logarithm are intertwined. As a result, we will continue to use the term “web” below to describe the result of this procedure. The mixing of color structure gives the general product a more complex form than the familiar case based on the cusp vertex. In this special case, however, the classic web formula is readily derived. In Sec. 3 we analyze the renormalization of multi-eikonal vertices as they occur in the diagrams discussed in Sec. 2, and describe some features of renormalized webs for multiple lines. We go on in Sec. 4 to conclude with a brief discussion the extension of the formalism to cross sections and to the massless case.
Figure 1: Representation of the eikonal amplitude $A$ discussed in the text. Each gluon line represents an arbitrary number of connections to each Wilson line, indicated by $e_a$ for line $a$, with $a = 1 \ldots L$ for $L$ Wilson lines.

2 Coordinate Identities and Diagrammatic Exponentiation

We will be interested in the sums of diagrams of the sort illustrated in Fig. 1, in which an arbitrary number of Wilson lines (four in the figure) are connected by gluon attachments in all possible ways. The attached gluons may interact in an arbitrary fashion, as indicated by the bubble $S$ in the figure. For the purposes of this argument, we denote the sum of all such diagrams for a set of Wilson lines over smooth paths $C_i$, $i = 1 \ldots L$, as

$$A[C_i] = \sum_{N\geq0} A^{(N)}[C_i], \quad (3)$$

where the superscript $N$ denotes the order in $\alpha_s$. For convenience, we absorb $(\alpha_s/\pi)^N$ into $A^{(N)}$ rather than to show it explicitly. We also suppress color indices, but generally the amplitude in Eq. (2) is a vector in the space of color tensors [3]. We will assume that the curves $C_i$ may meet at one or more multi-eikonal vertices, although they need not do so. We shall assume that they are otherwise nonintersecting. The curves need not be of infinite extent, although they may be.

In this section, we will treat the Wilson lines as they appear in perturbation theory, including in principle renormalization of all terms in the Lagrange density, but not for the composite multi-eikonal vertices. Thus, we assume that the diagrams are regularized for both ultraviolet and infrared divergences, and treat them as convergent integrals. We return
to the renormalization of the composite vertices in the next section.

Of course, we can always write the amplitude formally as an exponential,

\[ A[C_i] = \exp \left( w[C_i] \right), \quad w[C_i] = \sum_{i \geq 1} w^{(i)}[C_i], \tag{4} \]

for some matrix \( w[C_i] \), the logarithm of \( A[C_i] \), which can also be expanded in powers of \( \alpha_s \), beginning at order \( \alpha_s \) as indicated. Both \( w \) and \( A \) are functionals of the curves \( C_i \), as shown, but we shall suppress this dependence as well below.

Our goal is to characterize the matrix \( w \) order-by-order in perturbation theory for a general set of Wilson lines (that is, curves \( C_i \)). We will refer to diagrams that contribute to \( w^{(i)} \) as “webs” by analogy to the form factor [15], although the diagrammatic structure that will emerge from the reasoning below generalizes the case of the cusp vertex.

We actually know the webs at lowest order, \( w^{(1)} \), clearly given by the sum of all possible single-gluon exchanges between the curves \( C_i \), including the case of self-energies. It is therefore natural to pose our construction in recursive terms.

Suppose then, that we know the \( w^{(i)} \) up to some fixed order \( N \). This knowledge is enough to construct \( A \) up to \( N \)-loop order. That is, we can write

\[ \sum_{j=1}^{N} A^{(j)} = \exp \left( \sum_{i=1}^{N} w^{(i)} \right) + \mathcal{O}(\alpha_s^{N+1}), \tag{5} \]

with corrections at the next order. To relate the \( N \)th order expression in Eq. (5) to \( w^{(N+1)} \), we next expand the exponential, remembering that the \( w \)’s are matrices,

\[ \exp \left( \sum_{i=1}^{N} w^{(i)} \right) = \sum_{m=1}^{\infty} \frac{1}{m!} \left( \sum_{i=1}^{N} w^{(i)} \right)^{m} = \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{i_1=1}^{N} \cdots \sum_{i_m=1}^{N} w^{(i_m)} w^{(i_{m-1})} \cdots w^{(i_1)}. \tag{6} \]

Of course, some of these \( w \)’s may be identical. We will denote the number of copies of \( w^{(i_c)} \) as \( m_c \) where \( m_c \) can take any integer value, including zero.

Now consider the \( N + 1 \)st order, which we write in two ways. The first is as the sum of all \( N + 1 \)st order diagrams, denoted by \( D^{(N+1)} \),

\[ A^{(N+1)} = \sum_{D^{(N+1)}} D^{(N+1)}. \tag{7} \]

\[ \]
The second expression for \( A^{(N+1)} \) is as the \( N + 1 \)st order term from the exponential of \( W \) up to order \( N + 1 \),

\[
A^{(N+1)} = \left( \exp \left[ \sum_{i=1}^{N+1} w^{(i)} \right] \right)^{(N+1)}.
\]  

(8)

We can now solve for \( w^{(N+1)} \) in terms of the sum of \( N + 1 \)-order diagrams; it is only necessary to subtract from the sum over \( D^{(N+1)} \) the expansion of the exponential of the sum of \( w^{(m)} \)'s up to \( m = N + 1 \). Using Eq. (6) for the expansion of the exponential, and observing that \( w^{(N+1)} \) can appear only by itself in the sum, we find

\[
w^{(N+1)} = \sum_{D^{(N+1)}} D^{(N+1)} - \left[ \sum_{m=2}^{N+1} \frac{1}{m!} \sum_{i_m=1}^{N} \cdots \sum_{i_1=1}^{N} w^{(i_m)} w^{(i_{m-1})} \cdots w^{(i_1)} \right]^{(N+1)}.
\]  

(9)

Thus, the \( N + 1 \)st order contribution to the exponent is just what is left over from the sum of all diagrams at that order when we subtract the \( N + 1 \)st order result of the exponentiation of lower orders. In the following, we will learn how to interpret these products of the \( w \)'s, which will allow us to write a form that is more informative than Eq. (9).

The \( w^{(i)} \) are sums of diagrams with different attachments of gluons to each of the Wilson lines, and we will find it convenient to group together those diagrams at each order with definite numbers of gluons, \( e_a \), attached to the \( a \)th Wilson line, \( a = 1 \ldots L \). Clearly, the values of \( e_a \) are limited by the total loop order,

\[
2 \leq \sum_{a=1}^{L} e_a \leq 2N,
\]  

(10)

at \( N \)th order. To label the possible connections at each order explicitly, we write

\[
w^{(i)} = \sum_{E} w^{(i)}_{E},
\]  

(11)

where each \( E = \{e_1 \ldots e_L\} \) is a member of the set of possible assignments of the \( e_a \), subject to (10). In fact, the information contained in the subscript \( E \) is all we will need to know about the webs. To see this, we consider their general form as integrals over Wilson lines.

Each term \( w^{(i)}_{E} \) is itself an integral over the positions of each of its external gluons along the Wilson line to which it attaches. By construction, \( w^{(i)}_{E} \) includes a sum over all web diagrams with \( e_a \) gluons attached to the \( a \)th line, so that by Bose symmetry it is symmetric under permutations of the gluons attached to each line. Thus, without loss of generality we order the parameters \( \tau^{(a)}_j, j = 1 \ldots e_a \) for each of the Wilson lines, defined as in Eq. (11), and write

\[
w^{(i)}_{E} = \prod_{a=1}^{L} \prod_{j=1}^{e_a} \int_{\tau^{(a)}_j}^{\infty} d\tau^{(a)}_j W^{(i)}_{E} \left( \{\tau^{(a)}_j\} \right) \equiv \mathcal{I}_E \left[ W^{(i)}_{E} \right],
\]  

(12)
where in the first equality we define $\tau_0^{(a)} \equiv 0$. The second form represents the integrals as a functional $I_E$, acting on the “internal web function” $W_E^{(i)}$ corresponding to $w_E^{(i)}$. $W_E^{(i)}$ is a function of all the $\tau_{j_s}^{(a)}$s, and includes all color and velocity dependence associated with the gluons, including the vectors $\xi^\mu(\tau_{j_s}^{(a)})$ that are contracted with the gluon propagators.

Notice that $I_E$ depends only on the assignment of gluon connections, $E$, and is otherwise independent of the internal function $W_E^{(i)}$, including its order, $i$. We now use this property of the $I$’s to derive an identity that will serve as a lemma for our main result.

Let us consider the product of functionals, $I_{E_{s}}$, $s = 1 \ldots m$, with each factor defined by (12). For a given choice of Wilson line $a$, the integrals within each factor of the product are ordered as in (12) above, but they are not otherwise mutually ordered between different products. We can, however, write the product as a sum of terms, in which all the integration parameters $\tau_{j_s}^{(a)}$ from every factor $I_{E_{s}}$, $s = 1 \ldots m$ are ordered with respect to the integrals along every line from every other factor, while maintaining the original ordering within each factor. The sum is effectively over all possible interleaving of the integrals with each other. We label each such ordering by $E_\pi(\cup_{s=1}^{m}E_{s})$, with $\pi$ an element of the set $\Pi(\{E_{s}\})$ of the permutations of all the parameters $\tau_{j_s}^{(a)}$, which preserve the original ordering internal to each $I_{E_{s}}$

$$
\prod_{s=1}^{m} I_{E_{s}} = \sum_{\pi \in \Pi(\{E_{s}\})} I_{E_{\pi(\cup_{s=1}^{m}E_{s})}},
$$

(13)

This identity holds for any sets of Wilson lines, which need not be straight, or of infinite length. We note that at this stage, every term on the right-hand side of (13) is different, because the integrals within each $I_{E_{s}}$ will act on different functions. We will come back to this point shortly. Figure 2 illustrates Eq. (13), where the sum in the figure represents the sum over all interleavings of gluons connecting the two $w$’s to the lines. As the figure
shows, on the right-hand side of Eq. \[(13)\] we have a set of terms whose integrals can be identified with those of \(N + 1\)st order diagrams. Because they act on color-dependent web factors, however, which do not correspond to the resulting diagrams in general, we will end up combining diagrams with nonstandard color factors.

It may also be helpful to write a simple example of Eq. \[(13)\], for one line, involving a single integral with two mutually ordered integrals, written in the notation introduced above,

\[
\int_0^\infty d\tau_{j_1} \int_0^\infty d\tau_{k_1} \int_0^\infty d\tau_{j_2} = \int_0^\infty d\tau_{j_1} \int_0^\infty d\tau_{k_1} \int_0^\infty d\tau_{j_2} + \int_0^\infty d\tau_{k_1} \int_0^\infty d\tau_{j_1} \int_0^\infty d\tau_{k_2} + \int_0^\infty d\tau_{k_1} \int_0^\infty d\tau_{j_2} \int_0^\infty d\tau_{j_1} \tag{14}
\]

The relation \[(13)\] is simply a generalization of this trivial rewriting.

We are now ready to go back to the expression, Eq. \[(9)\] for \(w^{(N+1)}\). We expand each factor, \(w^{(i)}\) as a sum over \(E\), to which we apply the representation \[(12)\],

\[
\sum_{D^{(N+1)}} D^{(N+1)} \left[ \sum_{m=2}^{N+1} \frac{1}{m!} \prod_{j=1}^{m} \left( \sum_{i_j=1}^{N} \sum_{E_j \in E[w^{(i)}]} I_{E_m} \left[ W_{E_m}^{(i_m)} \right] \ldots I_{E_1} \left[ W_{E_1}^{(i_1)} \right] \right) \right] \tag{15}
\]

This enables us to use the integration identity \[(13)\] to combine the \(I_s\), which now act on the product of the \(W_s\),

\[
\sum_{D^{(N+1)}} D^{(N+1)} \left[ \sum_{m=2}^{N+1} \frac{1}{m!} \prod_{j=1}^{m} \left( \sum_{i_j=1}^{N} \sum_{E_j \in E[w^{(i)}]} I_{E_s(\cup_{j=1}^{m} E_s)} \left[ W_{E_m}^{(i_m)} \right] \ldots W_{E_1}^{(i_1)} \right) \right] \tag{16}
\]

The combination of sums in this expression over the orders \(i_j\) and over choices of connections of gluons, \(E_j\) to the Wilson lines for each of the \(W_s\) is equivalent to the sum over all diagrams that can be formed by combing the \(W_s\). As above, we let \(m_c\) be the number of identical factors \(W_{E_s}^{(i_s)}\) in Eq. \[(15)\]. The \(m_c!\) sets of diagrams found by permuting the roles of the integrals and internal factors \(W^{(i)}\) in these classes of identical \(W\)s are also identical. On the other hand, sets of diagrams found by permuting distinguishable internal factors are also distinguishable. We may therefore replace the product of sums over orders and gluon connections of the internal factors in Eq. \[(16)\] with a sum over all distinguishable
permutations of the integration factors in $\Pi(\{E_i\})$ found by combining the $I$s, taking into account the distinguishability properties of the $W$s. We need only weight this sum by the factor $\prod_c m_c!$.

Once the sum over permutations $\pi$ in (16) is over distinguishable combinations of the integrals and $W$s, each choice of $\pi$ uniquely determines a set of diagrams, found by summing over each of the internal factors $W_{E_j}^{(ij)}$ while leaving the orderings of its connections to the Wilson lines fixed. We can therefore replace the sum over $\pi$ with a sum over distinguishable $N+1$st-order diagrams in (16), found from the lower-order internal functions $W_{E}^{(i)}$, combined with a sum over all possible ways of forming that diagram from a product of internal functions $W$ of lower order.

To be specific, we write $D_{E}^{(N+1)} = \sum_E D_{E}^{(N+1)}$, where the sum is over all possible connections $E$ to the Wilson lines for $N+1$st order diagrams $D_{E}^{(N+1)}$. Let $\Omega_m(D_{E}^{(N+1)})$ be the set of all combinations of $m$ $W$s that give diagrams that are topologically equivalent to $D_{E}^{(N+1)}$ in this manner. That is, only those combinations of $E_{ij}$ and $i_{j}$ that reproduce the integrals of $D_{E}^{(N+1)}$ are included in the sum over $\Omega_m(D_{E}^{(N+1)})$. Their color factors, of course, remain defined by the original product. Then Eq. (16) may be rewritten as

$$w^{(N+1)} = \sum_E \sum_{D_{E}^{(N+1)}} \left( D_{E}^{(N+1)} - \mathcal{I}_E \left[ \sum_{m=2}^{N+1} \sum_{\Omega_m(D_{E}^{(N+1)})} \frac{\prod_c m_c!}{m!} \sum_{\text{sym}} W_{E_m}^{(i_m)} \ldots W_{E_1}^{(i_1)} \right] \right),$$

where $\mathcal{I}_E$ represents the integrals along the Wilson lines for diagram $D_{E}^{(N+1)}$, and $\sum_{\text{sym}}$ indicates the sum over all permutations of the factors $W_{E_j}^{(ij)}$ in set $\Omega_m(E)$. The symmetric sum is over distinguishable permutations only, so that if two $W$s are identical, there is only a single term. Equation (17) is our basic recursive result: from each diagram at order $N+1$ we subtract a specific set of diagrams with the same integrals over Wilson lines, but times a color-symmetrized product of lower order internal web factors.

The result we have just derived generalizes the web construction to arbitrary products of Wilson lines. It is apparently more complex than the original construction for the cusp vertex, because of the non-commutativity of the lower order webs. For the cusp vertex (15), the simple condition on web diagrams is that they be irreducible under cuts of two Wilson lines. This criterion does not extend to the general case. In particular, the diagrams in Fig. 3 whose color factors do not commute survive in two-loop generalized webs.

If the color factors commute in Eq. (17), however, we recover the familiar web formulas. This is the case for an abelian theory or for a special case in a nonabelian theory, such as Wilson lines coupled at (singlet) two- or three-eikonal vertices. More generally, there is a cancellation for all diagrams that can be decomposed into commuting subdiagrams. Let us see how this comes about.
Figure 3: Double exchange diagrams discussed in connection with Eq. (28). The shaded circle represents a multi-eikonal vertex.

Figure 4: Examples of two-web diagrams for the form factor.

For commuting internal functions $W$ in (17), we can combine all terms in the symmetric sum over color structures that give the same result, remembering that terms related by permutations of identical factors $W$ are counted only once. We then find

$$w^{(N+1)}_{\text{commuting}} = \sum_{E} \sum_{D_{E}^{(N+1)}} \left( D_{E}^{(N+1)} - \mathcal{I}_{E} \left[ \sum_{m=2}^{N+1} \sum_{\Omega_{m}(D^{(N+1)})} \prod_{j} \mathcal{W}_{E_{j}}^{(i_{j})} \right] \right). \quad (18)$$

This result is formally equivalent to the usual web formula for the form factor and related cross sections [15,16]. We should check, however, that the expression vanishes for diagrams $D^{(N+1)}$ in Eq. (18) that do not appear in the web formulation. In the case of the cusp vertex, webs are defined by irreducibility under cuts of the two Wilson lines. Two examples are shown in Fig. 4. Neither Fig. 4a nor b is a web, because in both cases we can cut the diagram between the two exchanged gluons in a and between the inner gluon and the crossed ladder in b. We will use these diagrams to sketch a demonstration of the result at all orders.

For the two Wilson line case, we assume that up to $N$ loops all webs are two Wilson line irreducible, and we consider what happens when $D^{(N+1)}$ is not such a diagram. For definiteness, let $D^{(N+1)}$ consist of a ladder with two rungs, as in Fig. 4.

Suppose first that the two rungs have no web subdiagrams. In this case the color factor
for diagram $D^{(N+1)}$ is the product of the (two) web color factors, and is clearly cancelled by the product of web color factors in Eq. (18). This is the case for Fig. 4a.

Next suppose that the rungs themselves are decomposable into sets of webs. An example is Fig. 4b, because the outer, crossed ladder, although a web, is formed of two single-gluon exchanges, each of which can play the role of a first-order web. In this case, the color factors of the $W$s are not the same as in the original diagram $D^{(N+1)}$, but have additional subtractions, corresponding to each decomposition into webs. For each such decomposition, however, there is also a corresponding term in the original sum over web decompositions of $D^{(N+1)}$. For our example, this is the unique three-web decomposition of Fig. 4b into three webs. Thus, here also, this diagram does not contribute to $w^{(N+1)}$. This pattern clearly continues for more and more complex ladders for the form factor.

3 Renormalization of Multi-eikonal Vertices

In the following we clarify the systematics of the all-order renormalization of multi-eikonal vertices. As we demonstrate, in the presence of non-commuting color tensors, renormalization becomes more complicated, yet remains tractable. Our results generalize to all orders the non-trivial two-loop contribution to the anomalous dimension matrix for massive partons found recently [21,22] with the help of a direct calculation. We recall that at two loops in the purely massless case no such non-trivial contributions involving counterterms appear [19,22]. Clearly, it will be very interesting to understand if the apparent simplicity observed in the massless case both in QCD and related theories is accidental or, if found to persist through higher orders, is due to some deeper reason.

Our starting point is the all-order multiplicative renormalization property of the effective multi-eikonal vertex

$$A_{\text{ren}} = A Z^{-1} \equiv \exp[w] \exp[\zeta].$$

Note that the “un-renormalized” vertex $A$ contains UV renormalization for sub-divergences not related to the effective vertex (i.e. coupling renormalization). Moreover, we observe that $w$ and $\zeta$ are, in general, color matrices and do not commute with each other. Combining the two exponentials appearing in Eq. (19), and introducing the perturbative expansions

$$w = \sum_{i \geq 1} w^{(i)}, \quad \zeta = \sum_{i \geq 1} \zeta^{(i)},$$

we can define all-order “renormalized webs” by

$$A_{\text{ren}} = \exp \left\{ w + \zeta + H(w, \zeta) \right\} = \exp \left\{ \sum_{i \geq 1} [w^{(i)} + \zeta^{(i)}] + \sum_{j \geq 2} H^{(j)}(w, \zeta) \right\},$$

**We note that for eikonal Wilson lines, beyond zeroth order $A$ is defined by scaleless integrals, and formally vanishes in dimensional regularization. This is not the case, however, for more general Wilson loops.**
where the matrix $H(w, \zeta)$ and its perturbative expansion $H^{(j)}(w, \zeta)$ follow from the usual Baker-Campbell-Hausdorff series,

$$H(w, \zeta) = \frac{1}{2} [w, \zeta] + \frac{1}{12} [w, [w, \zeta]] - \frac{1}{12} [\zeta, [w, \zeta]] + \ldots \quad (22)$$

Therefore, for any theory, the amplitude can be written as the exponential of “renormalized” web functions, defined at $n$th order by

$$w^{(n)}_{\text{ren}} = w^{(n)} + \zeta^{(n)} + H^{(n)}(w, \zeta). \quad (23)$$

The commutators $H$ are absent in any amplitude where the webs commute in their color content. In gauge theories these are the singlet products of one and two webs (singlet form factors). Therefore, in such special cases, the terms $\zeta^{(n)}$ act as local counterterms directly. This is the case for the cusp vertex, and in this and similar cases, renormalization proceeds by simple addition in the exponent. Strikingly, a similar feature holds in large-$N$ QCD, for which the commutator terms are non-leading in the number of colors $N$. In this case, a multi-eikonal vertex breaks up into sums of cusps, and the complete web is the sum of all cusp exponents, each of which is renormalized additively.

Turning to the general case, we need to clarify how the $\zeta$’s determine the anomalous dimensions. For any function $f(\alpha_s(\mu))$, we define the derivative with respect to the scale $\mu$ as

$$f'(\alpha_s) \equiv \mu \frac{d}{d\mu} f(\alpha_s(\mu)) = [-2\varepsilon \alpha_s + \beta(\alpha_s)] \frac{\partial}{\partial \alpha_s} f(\alpha_s(\mu)), \quad (24)$$

with the second form for a dimensionally-regulated theory in $D = 4 - 2\varepsilon$ dimensions, and $\beta(\alpha_s) = -\alpha_s^2/(2\pi) \sum_{n=0}^{\infty} \beta_n (\alpha_s/\pi)^n$ with $\beta_0 = 11C_A/3 - 2n_f/3$.

The anomalous dimensions can be found from the relation (see also Ref. [19])

$$\mu \frac{d}{d\mu} A_{\text{ren}} = -A_{\text{ren}} \Gamma = e^w \mu \frac{d}{d\mu} e^{\zeta} = e^w e^{\zeta} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} C_k(\zeta, \zeta'), \quad (25)$$

with $\Gamma = Z^{-1} \mu dZ/d\mu$ and with the matrices $C_k$ defined recursively as nested commutators,

$$C_0(\zeta, \zeta') = \zeta', \quad C_{k+1} = [\zeta, C_k(\zeta, \zeta')]. \quad (26)$$

Thus, to any order, the anomalous dimension matrix is found from the single pole of the $n$th-order counterterm, plus nested commutators of lower order counterterms,

$$\Gamma^{(n)} = - (\zeta')^{(n)} - \left( \sum_{k=1}^{\infty} \frac{(-1)^k}{(k+1)!} C_k(\zeta, \zeta') \right)^{(n)}. \quad (27)$$
For the cusp, and related cases where there is no mixing between color structures, the nested commutators vanish, and the anomalous dimension is determined directly from $\zeta' = C_0$.

It is instructive to see low-order examples. At two and three loops we have

$$w_{\text{ren}}^{(2)} = w^{(2)} + \zeta^{(2)} + \frac{1}{2} \left[ w^{(1)}, \zeta^{(1)} \right] ,$$

$$w_{\text{ren}}^{(3)} = w^{(3)} + \zeta^{(3)} + \frac{1}{2} \left[ w^{(1)}, \zeta^{(2)} \right] + \frac{1}{2} \left[ w^{(2)}, \zeta^{(1)} \right] + \frac{1}{12} \left[ w^{(1)} - \zeta^{(1)}, [w^{(1)}, \zeta^{(1)}] \right] ,$$

(28)

corresponding, with the help of Eq. (27), to anomalous dimensions

$$\Gamma^{(2)} = - (\zeta')^{(2)} ,$$

$$\Gamma^{(3)} = - (\zeta')^{(3)} + \frac{1}{2} \left[ \zeta^{(1)} , (\zeta')^{(2)} \right] + \frac{1}{2} \left[ \zeta^{(2)} , (\zeta')^{(1)} \right] .$$

(29)

The structure of the two-loop web $w_{\text{ren}}^{(2)}$ is very transparent in light of the results of Ref. [22]. In terms of the notation given here, it was found that (in Feynman gauge) $w^{(2)}$ gets no contributions from the double exchange diagrams illustrated by Fig. 3, but that the full two-loop counterterm, $\zeta^{(2)}$, and therefore $\Gamma^{(2)}$ requires diagrams in which the inner loops of Fig. 3 are replaced by one-loop counterterms. In Eqs. (28) and (29), these terms are generated by the commutator between the derivative of the one-loop counterterm $\zeta^{(1)}$ and the one-loop web $w^{(1)}$. Because the poles of the one loop web are proportional to the poles of $\zeta^{(1)}$, there are no double poles in the commutator. We postpone the detailed analysis of the three-loop web $w_{\text{ren}}^{(3)}$ for future work.

In applying the procedure sketched above at fixed order, the “practical” approach in deriving renormalized webs, counterterms and anomalous dimension matrices is, first to expand the all-order exponent Eq. (21) and to compare the $n$th order of this expansion to the usual diagrammatic prescription for the calculation of the corresponding amplitude at the same order. This is analogous to the use of Eq. (9) for unrenormalized webs. Then, requiring that $w_{\text{ren}}^{(n)}$ be finite fixes the counterterm $\zeta^{(n)}$ in terms of the webs of the same order and a combination of webs and counterterms of lower orders. Finally, one can determine the anomalous dimension (matrix) $\Gamma^{(n)}$ in terms of the $n$th order counterterm $\zeta^{(n)}$ through Eq. (27).

4 Extensions and Conclusions

In this brief account, we have shown that a large class of products of Wilson lines and loops share a straightforward diagrammatic construction that results in an exponentiated form. This construction of an exponent generalizes the webs of color-singlet form factors...
and related cross sections [15]. For general multi-eikonal vertices, the diagrams necessary to compute the exponent do not share quite the simple rule of irreducibility under cuts of Wilson lines. In fact, this feature was already clear on the basis of two-loop renormalization for vertices connecting several massive eikonal lines [21, 22].

The construction is based on counting, and is carried out in coordinate space. The underlying identity, Eq. (13), between products of integrals along the paths of ordered exponentials, reduces to the Fourier transform of the momentum space eikonal identity [15] in the limit of straight, semi-infinite paths, but is much more general. It applies as well to closed paths, with or without sequential cusp singularities, so that the diagrammatic construction described here applies in those cases as well.

We have seen in Sec. 3 that diagrammatic exponentiation is most direct before the renormalization of the multi-eikonal vertices. The renormalization of such a vertex leads to a fairly complex, but highly structured, formalism for determining anomalous dimensions. We anticipate that the investigation of generalized webs at higher orders will shed further light on matrices of anomalous dimensions that appear in many resummed cross sections, and perhaps on the dynamics of soft interactions in gauge theory more generally.

The extension of these rules to eikonal cross sections involving the scattering of Wilson lines with arbitrary colors is possible, because in squared amplitudes involving products of ordered exponentials, we can follow the path of a line that extends from a multi-eikonal vertex into the final state to infinite time in the amplitude, and then back again to the vertex in the complex conjugate amplitude. Because the arguments given above apply to any path, and can accommodate any form of propagators and interactions between gluons, they extend to QCD hard scattering in eikonal approximation, just as the web formalism extends to two-jet cross sections, to DIS, and to Drell-Yan annihilation processes.

In conclusion, we touch on the zero-mass limit for the generalized web construction. For soft functions in resummed cross sections, the soft function is conveniently described as a ratio of eikonal amplitudes or cross sections divided by eikonal form factors [11] or jet functions [27], respectively. In such ratios, double poles and logarithms associated with collinear behavior cancel. Because the form factors and jet functions exponentiate according to the original web construction, the complexities associated with color structure remain in the soft function, which exponentiates single logarithms in matrix form. If the conjecture that the anomalous dimensions of massless Wilson lines reduce to a dipole structure only holds, in this limit the general construction will simplify to a sum of exponentiated webs [23-24]. The general pattern of higher order counterterms described here may help in the investigation of this possibility.

Note added: During the completion of this project, we learned that E. Gardi, E. Laenen, G. Stavenga and C. White were completing a related study on the generalization of webs [28].

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