THE CLASSIFICATION OF 2-COMPACT GROUPS

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(JOINT WORK WITH KASPER K. S. ANDERSEN)

Abstract. In this talk I'll announce and explain a proof of the classification of 2-compact groups, joint with K. Andersen, hence completing the classification of p-compact groups at all primes p. A p-compact group, as introduced by Dwyer-Wilkerson, is a homotopy theoretic version of a compact Lie group, but with all its structure concentrated at a single prime p. Our classification states that there is a 1-1-correspondence between connected 2-compact groups and root data over the 2-adic integers (which will be defined in the talk). As a consequence we get the conjecture that every connected 2-compact group is isomorphic to a product of the 2-completion of a compact Lie group and copies of the exotic 2-compact group \( DI(4) \), constructed by Dwyer-Wilkerson. The major new input in the proof over the proof at odd primes (due to Andersen-Grodal-Møller-Viruel) is a thorough analysis of the concept of a root datum for 2-compact groups and its relationship with the maximal torus normalizer. With these tools in place we are able to produce a proof which to a large extent avoids case-by-case considerations.

This is a summary of my lecture given at the “Conference on Pure and Applied Topology”, Isle of Skye, Friday June 24, 2005. I announced and sketched a proof of the classification of 2-compact groups, in joint work with Kasper Andersen. This will appear in the papers [2] and [3].

Recall that a p-compact group is a triple \((X, BX, e : X \xrightarrow{\sim} \Omega BX)\) where \( BX \) is a pointed, connected, p-complete space of the homotopy type of a CW-complex, \( X \) satisfies that \( H^*(X; \mathbb{F}_p) \) is finite over \( \mathbb{F}_p \), and \( e \) is a homotopy equivalence. They are homotopy theoretic analogs of compact Lie groups, and were introduced by Dwyer-Wilkerson in [6]. A p-compact group is said to be connected if \( X \) is a connected space.

Our main theorem is the following

**Theorem 1.1** ([3]). Let \((X, BX, e : X \xrightarrow{\sim} \Omega BX)\) be a connected 2-compact group. Then

\[
BX \simeq BG_2 \times BDI(4)^s
\]

where \( BG_2 \) is the 2-completion of a connected compact Lie group \( G \), and \( BDI(4) \) is the classifying space of the exotic 2-compact group \( DI(4) \) constructed by Dwyer-Wilkerson in [3], \( s \geq 0 \).

A corresponding statement for odd primes was proved by the authors together with Møller and Viruel in [4]. Partial results for \( p = 2 \) have been obtained by Dwyer-Miller-Wilkerson, Notbohm, Viruel, Møller, and Vavpetič-Viruel. Our proof is a self-contained induction.

There is a better, more precise, formulation of our theorem which both makes it clear why it is the correct 2-local version of the classification of compact Lie groups and suggests a possible strategy of proof. It is based on the notion of a root datum over the p-adic integers \( \mathbb{Z}_p \), which we now introduce.

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For $R$ a principal ideal domain, an $R$-root datum $D$ is defined to be a triple $(W, L, \{Rb_\sigma\})$, where $L$ is a free $R$-module of finite rank, $W \subseteq \text{Aut}_R(L)$ is a finite subgroup generated by reflections (i.e., elements $\sigma$ such that $1 - \sigma \in \text{End}_R(L)$ has rank one), and $\{Rb_\sigma\}$ is a collection of rank one submodules of $L$, indexed by the reflections $\sigma$ in $W$, satisfying
\[
\text{im}(1 - \sigma) \subseteq Rb_\sigma \subseteq \ker\left(\sum_{i=0}^{\lvert\sigma\rvert-1} \sigma^i\right) \quad \text{and} \quad w(Rb_\sigma) = Rb_{w\sigma w^{-1}} \quad \text{for all } w \in W.
\]

The element $b_\sigma \in L$, called the coroot corresponding to $\sigma$, is determined up to a unit in $R$. Together with $\sigma$ it determines a root $\beta_\sigma : L \to R$ via the formula
\[
\sigma(x) = x + \beta_\sigma(x)b_\sigma.
\]

If $R = \mathbb{Z}$ then there is a 1-1-correspondence between $\mathbb{Z}$-root data and classically defined root data, by to $(W, L, \{\mathbb{Z}b_\sigma\})$ associating $(L, L^*, \{\pm b_\sigma\}, \{\pm \beta_\sigma\})$; see [8] Prop. 2.16. If $R = \mathbb{Z}$ or $\mathbb{Z}_p$, instead of the collection $\{Rb_\sigma\}$ one can equivalently consider their span, the coroot lattice, $L_0 = +_\sigma Rb_\sigma \subseteq L$, and this was the definition given in [4] § 1, under the name “$R$-reflection datum”. Two $R$-root data $D = (W, L, \{Rb_\sigma\})$ and $D' = (W', L', \{Rb'_\sigma\})$ are said to be isomorphic if there exists an isomorphism $\varphi : L \to L'$ such that $\varphi W \varphi^{-1} = W'$ and $\varphi(Rb_\sigma) = Rb'_{\varphi^{-1}\sigma\varphi^{-1}}$. In particular the automorphism group is given by $\text{Aut}(D) = \{\varphi \in N_{\text{Aut}_R(L)}(W) \mid \varphi(Rb_\sigma) = Rb'_{\varphi^{-1}(\sigma)}\}$ and we define the outer automorphism group as $\text{Out}(D) = \text{Aut}(D)/W$.

We now explain how to associate a $\mathbb{Z}_p$-root datum to a $p$-compact group. By a theorem of Dwyer-Wilkerson any $p$-compact group $(X, BX, e)$ has a maximal torus, which is a map $i : BT = (BS^1_p)^r \to BX$ satisfying that the fiber has finite $\mathbb{F}_p$-cohomology and non-trivial Euler characteristic. Replacing $i$ by an equivalent fibration, we define the Weyl space $W_X(T)$ as the topological monoid of self-maps $BT \to BT$ over $i$. The Weyl group is defined as $W_X(T) = \pi_0(W_X(T))$ and the classifying space of the maximal torus normalizer is defined as $BN_X(T) = BT_{hW_X(T)}$. Now, by definition, $W_X$ acts on $L = \pi_2(BT)$ and it is a theorem of Dwyer-Wilkerson that, if $X$ is connected, this gives a faithful representation of $W_X$ in $\text{Aut}_\mathbb{Z}_p(L)$ as a finite $\mathbb{Z}_p$-reflection group. There is also an easy formula for the $b_\sigma$ in terms of the maximal torus normalizer $N_X$, for which we refer to [8], [2], or [3]. This explains how to associate a $\mathbb{Z}_p$-root datum $D_X$ to a connected $p$-compact group $X$. (In the case where $p$ is odd the root datum is in fact completely determined by the finite $\mathbb{Z}_p$-reflection group $(W, L)$, which explains the formulation of our classification theorem with Møller and Viruel in [4].)

We are now ready to state the precise version of our main theorem.

**Theorem 1.2 ([3]).** The assignment which to a connected 2-compact group $X$ associates its $\mathbb{Z}_2$-root datum $D_X$ root datum gives a one-to-one correspondence between connected 2-compact groups and $\mathbb{Z}_2$-root data. Furthermore the map $\Phi : \pi_0(\text{Aut}(BX)) \to \text{Out}(D_X)$ is an isomorphism, and $B\text{Aut}(BX)$ is the unique total space of a split fibration
\[
B^2\mathbb{Z}(D) \to B\text{Aut}(BX) \to B\text{Out}(D_X)
\]

Here $\mathbb{Z}(D)$ is the center of the root datum $D$, defined so as to agree with the formula for the center of a $p$-compact group given in [7], and $B\text{Aut}(BX)$ denotes the classifying space of the topological monoid of self-homotopy equivalences of $BX$.

We remark that since we have completely described connected $p$-compact groups as well as their space of automorphisms, we also get a classification for non-connected 2-compact groups. (The situation here is totally analogous to the case of compact Lie groups.) If $X$
is a connected \( p \)-compact group with a given root datum \( D \) and \( \pi \) is a given \( p \)-group, then the \( p \)-compact groups with connected component isomorphic to \( X \) and component group \( \pi \) are in 1-1-correspondence with the \( \text{Out}(\pi) \)-orbits of the set \([B\pi, B\text{Aut}(BX)]\). Since both \( X \) and \( B\text{Aut}(BX) \) is completely described in terms of \( D_X \) by the main theorem, this gives a classification also in the non-connected case. We refer to \([3]\) for a more detailed description.

The main theorem has a number of corollaries. The most important is perhaps that it gives a proof of the maximal torus conjecture, giving a purely homotopy theoretic characterization of compact Lie groups amongst finite loop spaces.

**Theorem 1.3** (Maximal torus conjecture \([3]\)). The classifying space functor, which to a compact Lie group \( G \) associates the finite loop space \((G, BG, e : G \xrightarrow{\sim} \Omega BG)\) gives a one-to-one correspondence between compact Lie groups and finite loop spaces with a maximal torus. (Moreover, if \( G \) is connected we have a split fibration \( B^2\mathbb{Z}(D_G) \to B\text{Aut}(BG) \to B\text{Out}(D_G) \).)

The fact that the functor “\( B \)” is faithful was already known by work of Notbohm, Møller, and Osse and the statement about the space \( B\text{Aut}(BG) \) follows easily from earlier work of Jackowski-McClure-Oliver and Dwyer-Wilkerson. The new, and a priori quite surprising, result here is the statement that if a finite loop space has a maximal torus, then it has to come from a compact Lie group.

Another application of the classification of \( 2 \)-compact groups is to give an answer to the so-called Steenrod problem for \( p = 2 \) (see \([13]\) and \([12]\)), which asks which graded polynomial algebras can occur as the mod 2 cohomology ring of a space? Steenrod’s problem was solved for \( p \) “large enough” by Adams-Wilkerson \([1]\) and for all odd primes by Notbohm \([11]\) using a partial classification of \( p \)-compact groups, \( p \) odd.

**Theorem 1.4** (Steenrod’s problem for \( p = 2 \) \([3]\)). Suppose that \( P^* \) is a graded polynomial algebra over \( \mathbb{F}_2 \) in finitely many variables. If \( P^* \) occurs as \( H^*(Y; \mathbb{F}_2) \) for some space \( Y \)(no assumptions), then \( P^* \) is isomorphic, as a graded algebra, to

\[ H^*(BG; \mathbb{F}_2) \otimes H^*(BDI(4); \mathbb{F}_2)^{\otimes s} \otimes Q^* \]

where \( G \) is a connected semi-simple Lie group and \( Q^* \) is a polynomial ring with generators in degrees one and two.

In particular if \( P^* \) is assumed to have generators in degree \( \geq 3 \) then \( G \) has to be simply connected and \( P^* \) is a tensor product of the following graded algebras:

\[
\begin{align*}
\mathbb{F}_2[x_4, x_6, \ldots, x_{2n}] & \quad (\text{SU}(n)) \\
\mathbb{F}_2[x_4, x_8, \ldots, x_{4n}] & \quad (\text{Sp}(n)) \\
\mathbb{F}_2[x_4, x_6, x_7, x_8] & \quad (\text{Spin}(7)) \\
\mathbb{F}_2[x_4, x_6, x_7, x_8, x_{8}'] & \quad (\text{Spin}(8)) \\
\mathbb{F}_2[x_4, x_6, x_7, x_8, x_{16}] & \quad (\text{Spin}(9)) \\
\mathbb{F}_2[x_4, x_6, x_7] & \quad (G_2) \\
\mathbb{F}_2[x_4, x_6, x_7, x_{16}, x_{24}] & \quad (F_4) \\
\mathbb{F}_2[x_8, x_{12}, x_{14}, x_{15}] & \quad (DI(4))
\end{align*}
\]

It seems reasonable that one can in fact list all polynomial rings which occur as \( H^*(BG; \mathbb{F}_2) \) for \( G \) semi-simple, although we have not been able to locate such a list in the literature; for \( G \) simple a list can be found in \([10]\).
We finally point out that many classical theorems from Lie theory by Borel, Bott, Demazure, and others also carry over to 2-compact groups via the classification. Applications of this type were already pointed out in [4], to which we refer.

We end this short summary by saying a few words about the proofs. Our strategy is an elaboration of the strategy we pursued with Møller and Viruel in [4], but with several new additions, most importantly a development and utilization of the theory of $p$-adic root data. The sketch below is intended to sum up the main steps without introducing too much notation. We focus on the uniqueness statement, that two connected $p$-compact groups with isomorphic root data are isomorphic.

It is a recent theorem of Dwyer-Wilkerson [8], extending old work of Tits, that the maximal torus normalizer $N_X$ in a 2-compact group $X$ can be recovered from the root datum $D_X$. However, $N_X$ has a larger group of automorphisms than $X$, which means that it is not the right classifying invariant.

This problem can be overcome, by also keeping track of certain “root subgroups”, a machinery which we set up in [2]. In a given maximal torus normalizer $N$, for each reflection $\sigma$ and coroot $b_\sigma$ one can algebraically construct a root subgroup $N_\sigma$ of $N$. In the setting of algebraic groups, this “root subgroup” $N_\sigma$ will be the maximal torus normalizer of $\langle U_\alpha, U_{-\alpha} \rangle$, where $U_\alpha$ is the root subgroup in the sense of algebraic groups corresponding to the root $\alpha$ dual to the coroot $b_\sigma$.

**Theorem 1.5 ([2]).** Let $X$ be a connected 2-compact group. The canonical map $\text{Out}(BX) \rightarrow \text{Out}(D_X)$ factors

$$\Phi : \text{Out}(BX) \rightarrow \text{Out}(BN, \{BN_\sigma\}) \rightarrow \text{Out}(D_X)$$

and we likewise have a canonical map

$$\Phi : B\text{Aut}(BX) \rightarrow B\text{aut}(D_X)$$

where $B\text{aut}(D_X)$ is a certain space defined in [2] refining $B\text{Aut}(BN, \{BN_\sigma\})$.

These maps will be shown to be isomorphisms inductively, as part of proving Theorem 1.2. Since the pair $(N, \{N_\sigma\})$ carries the same structure and automorphisms as $D$, but is closer to $X$, it is more useful than $D$ for classification purposes.

Assume that we have two connected $p$-compact groups $X$ and $X'$ with the same root datum $D$ and hence the same maximal torus normalizer and root subgroups $(N, \{N_\sigma\})$, and assume by induction that Theorem 1.2 is true for all connected $p$-compact groups of smaller cohomological dimension. The first step is to observe that one can reduce to the case where $X$ and $X'$ are simple and center-free, mimicking the proofs in [4] but everywhere keeping track of the root subgroups. So we can assume that we are in the following situation

$$(BN, \{BN_\sigma\})$$

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component of the centralizer of every element $\nu : \mathbb{Z}/p \to BX$ of order $p$ in $X$.

$$\xymatrix{(BC_X(\nu)_1, \{BC_X(\nu)_1\}_\sigma) \ar[rr] & & BC_{X'}(\nu)_1}$$

Using that we have control of the whole space of self-equivalences of $BC_X(\nu)_1$ by Theorem 1.5 and induction, one sees that this map is equivariant with respect to the component group and hence extends uniquely to the whole centralizer $BC_X(\nu)$.

Now for a general elementary abelian subgroup $\nu : E \to X$ of $X$ we can pick an element of order $p$ in $E$, and we get via restriction a map

$$BC_X(\nu) \to BC_X(\mathbb{Z}/p) \to BC_{X'}(\mathbb{Z}/p) \to BX'$$

To make sure that these maps are chosen in a compatible way, one has to show that this map does not depend on the choice of rank one subgroup of $E$. We developed some techniques in [4] for handling this kind of situation. Using easy (one page) case-by-case arguments on the level of 2-adic root data, we are able to verify that the assumptions of [4] are always satisfied.

Since these maps from centralizers are chosen in a compatible way, they combine to form an element

$$\vartheta \in \lim_{\nu \in A(X)}^0 [BC_X(\nu), BX']$$

where $A(X)$ is the Quillen category of $X$. Furthermore the construction of this element allows one to show that $X$ and $X'$ have the same $p$-fusion and in particular the same maximal torus $p$-normalizer and the same $\mathbb{Z}_p$-cohomology. We are left with a rigidification question.

One approach could be to show that the relevant obstruction groups

$$\lim^*_{\vartheta \in A(X)} \pi_* (\text{map}(BC_X(\nu), BX')[\vartheta])$$

vanish, which would show that our diagram rigidifies and produces an equivalence

$$BX \simeq \text{hocolim}_{\nu \in A(X)} BC_X(\nu) \xrightarrow{\sim} BX'$$

It is immediate to see that this is the case for many groups $X$ including $DI(4)$, and it was the approach taken for $p$ odd in [4]. There we for instance also show that it holds for $X = \text{PU}(n)\mathbb{Z}_2$, by the same calculation as for $p$ odd, but it is more difficult to verify e.g., for the exceptional groups.

We however take a different approach: Since $DI(4)$ is easily dealt with, we can assume that $BX = BG_2$ for some simple, center-free Lie group. For each $p$-radical subgroup $P$ of $G$ we have a map $BP \to BC_G(pZ(P))_p \to BX'$. Since $X$ and $X'$ have the same $\mathbb{Z}_p$-homology, and in particular the same fundamental group, we can lift this map to a map $B\hat{P} \to B\hat{X}'$, where $\hat{P}$ is the preimage of $P$ in $\hat{G}$, the universal cover of $G$, and $\hat{X}'$ is the universal cover of $X'$.

One now verifies that these maps assemble to give an element in

$$\lim_{\hat{G}/\hat{P} \in O_p(\hat{G})}^0 [B\hat{P}, B\hat{X'}]$$
where $O_p^r(G)$ is the full subcategory of the $p$-orbit category of $G$ with objects $G/P$ for $P$ a $p$-radical subgroup of $G$. The obstructions to rigidifying this to get a map

$$B\tilde{G} \simeq \hocolim_{G/P \in O_p^r(G)} B\tilde{P} \to B\tilde{X}'$$

can be shown to lie in groups which identify with

$$\lim_{\tilde{G}/\tilde{P} \in O_p^r(G)} *_{\pi_*} (Z(\tilde{P}))$$

These groups have been shown to identically vanish by earlier work of Jackowski-McClure-Oliver [9]. Hence the map exists and it now follows easily by the construction that it is an equivalence. Passing to a quotient we get the wanted equivalence $BG \to BX'$, finishing the proof of the theorem.

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