Discrete mean field games

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Abstract

In this paper we study a mean field model for discrete time, finite number of states, dynamic games. These models arise in situations that involve a very large number of agents moving from state to state according to certain optimality criteria. The mean field approach for optimal control and differential games (continuous state and time) was introduced by Lasry and Lions [LL06a, LL06b, LL07]. The discrete time, finite state space setting is motivated both by its independent interest as well as by numerical analysis questions which appear in the discretization of the problems introduced by Lasry and Lions.

Our setting is the following: we assume that there is a very large number of identical agents which can be in a finite number of states. Because the number of agents is very large, we assume the mean field hypothesis, that is, that the only relevant information for the global evolution is the fraction \( \pi_n^i \) of players in each state \( i \) at time \( n \). The agents look for minimizing a running cost, which depends on \( \pi \), plus a terminal cost \( V_N \). In contrast with optimal control, where usually only the terminal cost \( V_N \) is necessary to solve the problem, in mean-field games both the initial distribution of agents \( \pi^0 \) and the terminal cost \( V_N \) are necessary to determine the solutions, that is, the distribution of players \( \pi^n \) and value function \( V^n \), for \( 0 \leq n \leq N \). Because both initial and terminal data needs to be specified, we call this problem the initial-terminal value problem. Existence of solutions is non-trivial. Nevertheless, following the ideas of Lasry and Lions, we were able to establish existence and uniqueness, both for the stationary and for the initial-terminal value problems. We discuss in some detail a particular model, the entropy penalized problem. In the last part of the paper we prove the main result of the paper, namely the exponential convergence to a stationary solution of \((\pi^0, V^0)\), as \( N \to \infty \), for the initial-terminal value problem with (fixed) data \( \pi^{-N} \) and \( V^N \).

1 Introduction

In this paper we study a mean field model for discrete time, finite number of states, dynamic games. These models arise in situations that involve a very large number of agents moving from state to state according to certain optimality criteria. The mean field approach for optimal control and differential games (continuous state and time) was introduced by Lasry and Lions [LL06a, LL06b, LL07]. In the continuous state and time setting, mean field problems gives rise to Hamilton-Jacobi equations coupled with transport equations. The discrete time, finite state space setting is motivated both by its independent interest as well as by numerical analysis questions which appear in the discretization of the problems introduced by Lasry and Lions. The discretization of these models has been studied by I. Capuzzo-Dolcetta and Y. Achdou.

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Our setting is the following: we assume that there is a very large number of identical agents which can be in a finite number of states. Each agent behaves individually and rationally, moving from state to state according to certain optimality criteria. Furthermore, its decisions are based solely on the following information, which is known to every agent, the current state, and the fraction of agents in each state. As in non-cooperative games, there may be interactions between the players in different states, as we will explain in more detail below. Because the number of agents is very large, we assume the mean field hypothesis, that is, that only the fraction $\pi^n_i$ of players in each state $i$ at time $n$ is the relevant information for the global evolution. The mathematical justification of mean field models has been investigated extensively by Lions and Lasry in yet to be published papers and we do not address these issues in this work.

Let $d > 1$ and $N \geq 1$ be natural numbers, representing, respectively, the number of possible states in which the agents can be at any given time, and the total duration of the process. Let $\pi^0$ and $V^N$ be given $d$-dimensional vectors. We suppose that $\pi^0$ is a probability vector, the initial probability distribution of agents among states, and that $V^N$, the terminal cost, is an arbitrary vector. A solution to the mean field game is a sequence of pairs of $d$-dimensional vectors

$$\{(\pi^n, V^n) : 0 \leq n \leq N\},$$

where $\pi^n_i$ is the probability distribution of agents among states at time $n$ and $V_j^n$ is the expected minimum total cost for an agent at state $j$, at time $n$. These pairs must satisfy certain optimality conditions that we describe in what follows: at every time step, the agents in state $i$ choose a transition probability, $P_{ij}$, from state $i$ to state $j$. Given the transition probabilities $P_{ij}^n$ at time $0 \leq n < N$, the distribution of agents at time $n + 1$ is simply

$$\pi_j^{n+1} = \sum_i \pi_i^n P_{ij}^n.$$

Associated to this choice there is a transition cost $c_{ij}(\pi, P)$. In the special case in which $c_{ij}$ only depends on $\pi$ and on the $i$th line of $P$ we use the simplified notation $c_{ij}(\pi, P_i)$. This last case arises when the choices of players in states $j \neq i$ do not influence the transition cost to an agent in state $i$. Let $e_i(\pi, P, V)$ be the average cost that agents which are in state $i$ incur when matrix $P$ is chosen, given the current distribution $\pi$ and the cost vector $V$ at the subsequent instant. We assume that

$$e_i(\pi, P, V) = \sum_j c_{ij}(\pi, P) P_{ij} + V_j P_{ij}.$$

Define the probability simplex $S = \{(q_1, \ldots, q_d) : q_j \geq 0 \forall j, \sum_{j=1}^d q_j = 1\}$. The set of $d \times d$ stochastic matrices is identified with $S^d$. Given a stochastic matrix $P \in S^d$ and a probability vector $q \in S$, we define $P(P, q, i)$ to be the $d \times d$ stochastic matrix obtained from $P$ by replacing the $i$-th row by $q_i$ and leaving all others unchanged. 

**Definition 1.** Fix a probability vector $\pi \in S$ and a cost vector $V \in \mathbb{R}^d$. A stochastic matrix $P \in S^d$ is a Nash minimizer of $e(\pi, \cdot, V)$ if for each $i \in \{1, \ldots, d\}$ and any $q \in S$

$$e_i(\pi, P, V) \leq e_i(\pi, P(P, q, i), V).$$

**Definition 2.** Suppose that for each $\pi \in S$ and $V \in \mathbb{R}^d$ there exists a Nash minimizer $P \in S^d$ of $e(\pi, \cdot, V)$. Let $N \geq 1$, $\pi^0 \in S$ (the initial distribution of states), and $V^N \in \mathbb{R}^d$ (the terminal cost). A sequence of pairs of $d$-dimensional vectors

$$\{(\pi^n, V^n) : 0 \leq n \leq N\}$$

is a solution of the mean field game if for every $0 \leq n \leq N - 1$

$$\begin{align}
V_j^n &= \sum_j c_{ij}(\pi^n, P^n) P_{ij}^n + V_j^{n+1} P_{ij}^n \\
\pi_j^{n+1} &= \sum_i \pi_i^n P_{ij}^n,
\end{align}$$

(1)
for some Nash minimizer \( P^n \in \mathbb{S}^d \) of \( e(\pi^n, \cdot, V^{n+1}) \).

Until the end of this section we will assume that for all \((\pi, V) \in \mathbb{S} \times \mathbb{R}^d\) there exists a unique Nash minimizer \( \bar{P} \) of \( e(\pi, P, V) \). Conditions which guarantee the uniqueness of a Nash minimizer will be studied in §3.2. Under the uniqueness of a Nash minimizer for \( e \), we can define the (backwards) evolution operator for the value function

\[
G_\pi(V) = e(\pi, \bar{P}, V),
\]

as well as the (forward) evolution operator for \( \pi \)

\[
K_V(\pi) = \pi \bar{P}.
\]

Since the operator \( G_\pi \) commutes with addition with constants, it can be regarded as a map from \( \mathbb{R}^d / \mathbb{R} \) to \( \mathbb{R}^d / \mathbb{R} \). Here \( \mathbb{R}^d / \mathbb{R} \) is the set of equivalence classes of vectors in \( \mathbb{R}^d \) whose components differ by the same constant. In \( \mathbb{R}^d / \mathbb{R} \) we define the norm

\[
\|\psi\|_\# := \inf_{\lambda \in \mathbb{R}} \|\psi + \lambda\|
\]

In this paper we will regard \( G_\pi \), depending on what is convenient, as both a map in \( \mathbb{R}^d \) as well as a map in \( \mathbb{R}^d / \mathbb{R} \).

We have the compact equivalent form for (1)

\[
\begin{cases}
V^n = G_\pi^n(V^{n+1}) \\
\pi^{n+1} = K_{\pi^{n+1}}(\pi^n).
\end{cases}
\]

In this paper we will consider solutions to (2) which satisfy initial-terminal value conditions, \( \pi^0 \) (or \( \pi^{-N} \)) and \( V^N \), as well as stationary solutions, that we discuss in what follows.

**Definition 3.** A pair of vectors \((\bar{\pi}, \bar{V})\) is a stationary solution to the mean field game if there exists a constant \( \bar{\lambda} \), called critical value, such that

\[
\begin{cases}
\bar{V} = G_\pi(\bar{V}) + \bar{\lambda} \\
\bar{\pi} = K_V(\bar{\pi}).
\end{cases}
\]

We should remark that the first equation in (3) can be written in \( \mathbb{R}^d / \mathbb{R} \) as \( G_\pi(\bar{V}) = \bar{V} \). Therefore, solutions to (3) can be regarded as fixed points of \((G_\pi, K_V)\) in \( \mathbb{R}^d / \mathbb{R} \times \mathbb{S} \).

The structure of the paper is as follows: we first start, in §2 by listing our main assumptions, as well as explaining where they are needed in the paper. In §3 we address the issue of existence, theorem [1], and uniqueness, theorem [2] of Nash-minimizing transition matrices. Some general properties of the operator \( G \) are studied in §4. In §5 we establish several results concerning the existence, theorem [3], and uniqueness, propositions [4] of stationary solutions to mean field games. The initial-terminal value problem is studied in §6. We show also existence, theorem [5], and uniqueness, theorem [6], of solutions for this problem. Both in the stationary and initial-terminal value problem the uniqueness proofs use a version of the monotonicity argument of Lasry and Lions in [LL06a, LL06b, LL07]. In the last section we address the convergence to equilibrium and establish one of the main results of the paper, theorem [7], which states that, as we take the initial and terminal conditions far apart \((n = \pm N)\), the solutions at \( n = 0 \) converge exponentially to a stationary solution. Throughout the paper we discuss with detail the entropy penalized problem For this model we give an independent proof of existence of stationary solutions, proposition [8] and study the large entropy limit, proposition [9]. Another important example is the optimal stationary solutions, discussed in §5.7 which give a variational interpretation of a solution of certain mean field games in terms of non-linear programming problems, proposition [10].
2 Main Assumptions

In this section, for the convenience of the reader, we list the main assumptions that will be needed in the text. We list here only the assumptions that will be used in the main results or the ones which are repeatedly used in the text. A few other assumptions will be introduced later in the text and will only be used "locally" in the section they are stated.

The first two assumptions will be used in theorem \[1\], §3.1, to establish existence of a Nash minimizer of \(e\).

**Assumption 1.** For each \(\pi \in S, V \in \mathbb{R}^d, P \in S^d\), and each index \(1 \leq i \leq d\), the mapping \(q \mapsto e_i(\pi, P(q, i), V)\), defined for \(q \in S\), and taking values on \(\mathbb{R}\), is convex.

**Assumption 2.** The map \(P \mapsto e_i(\pi, P, V)\) is continuous for all \(i\).

Concerning the uniqueness of Nash minimizers, addressed in §3.2, we need the following definition (see [HK] for the motivation of this assumption):

**Definition 4.** A function \(g: \mathbb{R}^{d \times d} \times \mathbb{R}^{d \times d} \to \mathbb{R}^{d \times d}\) is diagonally convex if for all \(P_1, P_2 \in \mathbb{R}^{d \times d}\), \(P_1 \neq P_2\), we have

\[
\sum_{ij} (P_{ij}^1 - P_{ij}^2)(g_{ij}(P^1) - g_{ij}(P^2)) > 0.
\]

With this definition we can state the next assumption:

**Assumption 3.** Let

\[
g_{ij}(P) = \frac{\partial e_i(\pi, P, V)}{\partial P_{ij}}.
\]

Then \(g\) is diagonally convex.

Since diagonal convexity may not be the unique way to ensure uniqueness of Nash minimizers, it is convenient to add uniqueness as an assumption, which obviously holds under assumptions \[1-3\] but which can also hold under other alternative hypothesis.

**Assumption 4.** For each \((\pi, V)\) there exists a unique Nash minimizer \(P(\pi, V)\) of \(e(\pi, \cdot, V)\).

The uniqueness of \(P(\pi, V)\) makes the operators \(K_V\) and \(G_\pi\) well defined. Therefore, from §3 on we will always suppose that assumption \[4\] holds, even without explicit mention.

To establish continuity of \(P(\pi, V)\), §3.3, proposition \[1\], we need:

**Assumption 5.** For each index \(1 \leq i \leq d\), \(e_i: S \times S^d \times \mathbb{R}^d \to \mathbb{R}\) is a continuous function.

Denote by \(\rho_{i,i'}(P)\) the matrix we obtain from \(P\) by replacing its \(i'\)-th row by its \(i\)-th row, and leaving all other rows (including the \(i\)-th) unchanged.

**Assumption 6.** There exists \(C > 0\) such that for all \(i\) and \(i'\), and any \(\pi \in S, P \in S^d\)

\[
\sum_j |c_{ij}(\pi, P) - c_{i'j}(\pi, \rho_{i,i'}(P))| P_{ij} \leq C. \tag{7}
\]

Note that the previous assumption holds if \(c_{ij}\) is bounded, for instance.

Some of the results of the paper only hold for transition costs which have a special dependence on \(P\). The next assumption will be required frequently:

**Assumption 7.** The cost \(c_{ij}(\pi, P_i)\) depends on \(\pi\) and, for each \(i\), only on the \(i\)-th line of \(P\).

To establish uniqueness of solutions (§5.4, theorem \[7\], §6.2, theorem \[6\]), as well as to obtain exponential convergence to stationary solutions (§7 theorem \[7\]), it is convenient to have the following assumption on the operator \(G\):
Assumption 8. There exists a constant $\gamma > 0$ such that
\[
\tilde{\pi} \cdot (\mathcal{G}_\pi(V) - \mathcal{G}_\tilde{\pi}(\tilde{V})) + \pi \cdot (\mathcal{G}_\pi(\tilde{V}) - G_{\tilde{\pi}}(\tilde{V})) \geq \gamma \|\pi - \tilde{\pi}\|^2,
\]
for any $V, \tilde{V} \in \mathbb{R}^d$ and all $\pi, \tilde{\pi} \in S$.

An example where this last hypothesis is satisfied is the following:
\[
c_{ij}(\pi, P_i) = W_i(\pi) + \tilde{c}_{ij}(P_i),
\]
where $W$ is a monotone function, that is,
\[
(\pi - \tilde{\pi}) \cdot (W(\pi) - W(\tilde{\pi})) \geq \gamma \|\pi - \tilde{\pi}\|^2,
\]
where $\gamma$ is a positive constant. For instance, the gradient of a convex function is a monotone function. In this case we have
\[
\mathcal{G}_\pi(V)(i) = W_i(\pi) + \min_{P_{ij}} \sum_j (\tilde{c}_{ij}(P_i) + V_j)P_{ij}.
\]

The special structure in (8) arises naturally in certain problems, see §5.7.

To establish the uniqueness of the critical value we need the following assumption:

**Assumption 9.** For any $\pi \in S$, the operator $V \mapsto \mathcal{G}_\pi(V)$ satisfies the following property: for all $V^1, V^2$ and any $i \in \text{argmax}(V^1 - V^2)$ we have
\[
\mathcal{G}_\pi(V^1)_i - V^1_i \leq \mathcal{G}_\pi(V^2)_i - V^2_i,
\]
with the opposite inequality if $i \in \text{argmin}(V^1 - V^2)$.

As it will be proved in §4.1, proposition 3, assumption 7 implies assumption 9. However, we leave it explicit to make easier the understanding of what follows.

The following strict concavity of $\mathcal{G}$ is important to establish uniqueness of stationary solutions and the exponential convergence to equilibrium.

**Assumption 10.** For all $\pi \in S$ and all $V^1, V^2 \in \mathbb{R}^d$ we have
\[
\pi \cdot (\mathcal{G}_\pi(V^2) - \mathcal{G}_\pi(V^1)) + K_{V^1}(\pi)(V^1 - V^2) \leq -\gamma \pi \|V^1 - V^2\|^2.
\]

This last assumption is a slightly stronger version of inequality (13), which is a consequence of assumptions 4 and 5.

A final hypothesis allow us to establish certain bounds (lemma 3 in §7.1) which are useful in proving the exponential convergence to equilibrium:

**Assumption 11.** There exists $K > 0$ such that for all $\pi, \tilde{\pi} \in S$, and for any matrix $P \in S^d$
\[
|c_{ij}(\pi, P) - c_{ij}(\tilde{\pi}, P)| \leq K.
\]

Note that the previous assumption holds if $c_{ij}$ is bounded, for instance.

### 3 The Transition Matrix

In this section we discuss the problem of existence, uniqueness and continuity of the Nash equilibrium transition matrix $P$. As we will see, this problem is non-trivial and requires, in the general case, the use of Kakutani’s fixed point theorem, see theorem 1 in §3.1. Once existence is established, uniqueness can be proven for a general class of cost functionals, theorem 2 in §3.2. We finish this section with the discussion of a special case: the entropy penalized model, §3.4.
3.1 Existence in the general case

**Theorem 1.** Suppose that assumptions 1 and 2 hold. Then, for any pair of vectors $\pi$ and $V$ there exists a Nash minimizer $P$ of $e(\pi, \cdot, V)$.

**Proof.** Given a stochastic matrix $P$, define $F_i(P)$ to be the set of vectors in $\mathcal{S}$ given by

$$F_i(P) = \arg\min_{q \in \mathcal{S}} e_i(\pi, \mathcal{P}(P, q, i), V).$$

The set $F_i(P)$ is non-empty and convex. Define $F(P) = F_1(P) \times F_2(P) \times \ldots \times F_d(P)$, where we identify the cartesian product with the the set of all stochastic matrices where the i-th row belongs to $F_i(P)$. Clearly, $F(P)$ is convex for all $P$. Furthermore, as we argue next, the graph $\{(P; F(P)); P \in \mathcal{S}^d\}$ is closed. Indeed, suppose that $P^n \to P^0$ and take $Q^n \in F(P^n)$, if $Q^n \to Q^0$ we want to show that $Q^0 \in F(P^0)$. Fix $i$ and call $q^n_i$ the i-th coordinate of $Q^n$, then by hypothesis $e_i(\pi, \mathcal{P}(P^n, q^n_i, i), V) \leq e_i(\pi, \mathcal{P}(P^n, q'_i, i), V)$ for all $q'_i \in \mathcal{S}$. As $q^n_i \to q^0_i$ and $P^n \to P^0$ we have that $\mathcal{P}(P^n, q^n_i, i) \to \mathcal{P}(P^0, q^0_i, i)$ then we get that $e_i(\pi, \mathcal{P}(P^0, q^0_i, i), V) \leq e_i(\pi, \mathcal{P}(P^0, q'_i, i), V)$ for all $q'_i \in \mathcal{S}$.

Then, because for each $P$, $F(P)$ is a convex set and the graph $(P, F(P))$ is closed, we can apply Kakutani’s fixed point theorem, which implies the existence of a matrix $P$ that belongs to $F(P)$. Thus $P$ is a Nash minimizer of $e(\pi, \cdot, V)$.

\[ \square \]

3.2 Uniqueness for diagonally convex costs

As shown in the previous section, if assumptions 1 and 2 hold, for each $\pi$ and $V$ there exists a transition matrix $P$ which is a Nash minimizer of $e(\pi, \cdot, V)$. In general, such minimizer may fail to be unique. Under the diagonally convex assumption 3 we will show uniqueness.

**Theorem 2.** Suppose assumptions 3 holds. Then there exists a unique transition matrix $P$ which is a Nash minimizer of $e(\pi, \cdot, V)$.

**Proof.** Note that if $P$ is any Nash minimizer of $e(\pi, \cdot, V)$, its i-th line solves the constrained optimization problem

$$\min_{q} e_i(\pi, \mathcal{P}(P, q, i), V),$$

$$q_i \geq 0,$$

$$\sum_i q_i = 1.$$

Thus, if $P^1$ and $P^2$ are two Nash minimizers, they satisfy the KKT conditions [Ped04]

$$\frac{\partial e_i(\pi, \mathcal{P}^k, V)}{\partial P_{ij}} - \nu^k_i - \theta^k_{ij} = 0, \quad k = 1, 2,$$

$$\sum_j P_{ij} - 1 = 0,$$

$$\text{and } \theta_{ij} P_{ij} = 0,$$

where $\nu_i$ is the Lagrange multiplier associated with $\sum_j P_{ij} = 1$ and $\theta_{ij} \geq 0$ corresponds to the constraint $P_{ij} \geq 0$. From the first of the three equations above we conclude that

$$\sum_{ij} (P^1_{ij} - P^2_{ij})(g_{ij}(P^1) - g_{ij}(P^2) - \nu^1_i + \nu^2_i - \theta^1_{ij} + \theta^2_{ij}) = 0.$$

Therefore, using the diagonally convex property, we have

$$\sum_{ij} (P^1_{ij} - P^2_{ij})(-\nu^1_i + \nu^2_i - \theta^1_{ij} + \theta^2_{ij}) < 0,$$
which implies, when we use $\theta_{ij}^k P_{ij}^k = 0$, that
\[
\sum_{ij} \left( -P_{ij}^1 \nu_i^l - P_{ij}^2 \nu_i^l + P_{ij}^1 \nu_i^l + P_{ij}^1 \theta_{ij}^2 + P_{ij}^2 \theta_{ij}^1 \right) < 0.
\]
Now we can use that
\[
\sum_{j} P_{ij}^k \nu_i^l = \nu_i^l, \quad \forall 1 \leq l, k \leq 2,
\]
to get
\[
\sum_{ij} \left( P_{ij}^1 \theta_{ij}^2 + P_{ij}^2 \theta_{ij}^1 \right) < 0.
\]
Since $P_{ij}^1 \theta_{ij}^2, P_{ij}^2 \theta_{ij}^1 \geq 0$, we obtain a contradiction.

### 3.3 Uniqueness and continuity

Suppose assumption 4 holds. Consider the map which associates to each pair $(\pi, V)$ its unique optimizing transition matrix $P(\pi, v)$. Is it natural to ask whether this map is a continuous function. This is addressed in the next proposition.

**Proposition 1.** Suppose assumptions 4-5 hold. Then $P(\pi, V)$ is a continuous function of $\pi$ and $V$.

**Proof.** Consider sequences $\pi_n \to \pi$ and $V_n \to V$. The corresponding sequence of Nash minimizers $P_n = P(\pi_n, V_n)$ converges to a Nash minimizer, by the continuity of $e$, assumption 5. Therefore, by the uniqueness hypothesis (assumption 4) $P(\pi_n, V_n) \to P(\pi, V)$.

### 3.4 The entropy penalized model

Now we consider a special example, the entropy penalized model. We fix a positive constant $\epsilon$, and consider assume that $c_{ij}(\pi, P_i \cdot) = \tilde{c}_{ij}(\pi) + \epsilon \ln(P_{ij})$, where $\tilde{c}_{ij}$ is a continuous function of $\pi$. For simplicity we will drop the $\sim$. We have
\[
e_i(\pi, P, V) = \sum_j P_{ij} \left( c_{ij}(\pi) + \epsilon \ln P_{ij} + V_j \right).
\]
(11)
The term $\epsilon P_{ij} \ln P_{ij}$, with $\epsilon > 0$, is related to entropy and forces the agents to diversify their transition choices by enforcing a penalty if they do not do so.

It is easy to prove that there exists a unique Nash minimizing transition matrix given by
\[
P_{ij}(\pi, V) = \frac{e^{-\tilde{c}_{ij}(\pi) + V_j}}{\sum_k e^{-\tilde{c}_{ik}(\pi) + V_k}}.
\]
(12)
Also, this transition matrix is a continuous function of $\pi$ and $V$.

Now we present a useful formula for the second derivatives of $G_{\pi}(V)$:
\[
\frac{\partial^2 G_{\pi}(V)}{\partial V_k \partial V_l} = \frac{p_k p_l - p_k \delta_{kl}}{\epsilon} \equiv J_{kl},
\]
where
\[
p_k = \frac{e^{-\tilde{c}_{ik}(\pi) + V_k}}{\sum_m e^{-\tilde{c}_{im}(\pi) + V_m}}.
\]
(13)
Because $\sum_k J_{kl} = 0$, the matrix $J_{kl}$ has a zero eigenvalue, which is a reflection of the fact that $G_{\pi}$ commutes with addition of constants. As we show in the next proposition this eigenvalue is simple.
Proposition 2. Suppose $0 < p_k < 1$, $\sum_k p_k = 1$ and let

$$J_{kl} = \frac{p_k p_l - p_k \delta_{kl}}{\epsilon}.$$  

Then 0 has simple multiplicity.

Proof. Observe that

$$J = \frac{1}{\epsilon} D(Q - I),$$

where

$$D = \text{diag}\{p_1, \ldots, p_n\}$$

and

$$Q_{kl} = p_l$$

If 0 is not a simple eigenvalue, it would mean that there exists $w$ and $v$ which are linearly independent eigenvectors corresponding to this eigenvalue. But then $v$ and $w$ are eigenvectors of $Q$ corresponding to the eigenvalue 1. But this contradicts Perron-Frobenius theorem because the eigenvalue 1 is a simple eigenvalue of $Q$.

\[ \square \]

4 Properties of $G$

In this section we discuss the main properties of the operator $G$. In §4.1 we show that assumption 9 is a consequence of assumptions 4 and 7, and in §4.2 we study concavity properties of $G$. A-priori bounds, essential to establishing existence of stationary solutions are considered in §4.3 and, finally, in §4.4 we prove strict concavity of $G$ for the entropy penalized model with two states.

4.1 Assumption 9

Proposition 3. Suppose assumptions 4 and 7 hold. Then assumption 9 holds.

Proof. Let $V^k \in \mathbb{R}^d$, $k = 1, 2$. Let $i \in \arg\max V^1 - V^2$. By adding a constant, we may assume that $V^1_i = V^2_i$, and so because $G$ commutes with the addition of constants, it suffices to check that

$$G_\pi(V^1)_i \leq G_\pi(V^2)_i.$$  

Because $i$ is a maximizer of $V^1_j - V^2_j$, for all $j$ we have $V^1_j - V^2_j \leq 0$, that is $V^1_j \leq V^2_j$. Let $P^2_{ij}$ be such that

$$G_\pi(V^2)_i = \sum_j c_{ij}(\pi, P^2_{ij}) P^2_{ij} + V^2_j P^2_{ij}.$$  

Then

$$G_\pi(V^1)_i \leq \sum_j c_{ij}(\pi, P^1_{ij}) P^1_{ij} + V^1_j P^1_{ij} \leq \sum_j c_{ij}(\pi, P^2_{ij}) P^2_{ij} + V^2_j P^2_{ij} = G_\pi(V^2)_i.$$  

Arguing similarly we obtain the opposite inequality when $i \in \arg\min V^1 - V^2$.  \[ \square \]

4.2 Concavity

If assumption 7 holds, for each fixed index $i$ the mapping

$$V \mapsto G_\pi(V)_i$$

is concave since

$$G_\pi(V)_i = \min_{P_i \in S} \sum_j c_{ij}(\pi, P_i) P_{ij} + V_j P_{ij},$$

(14)
is a pointwise minimum of linear functions of $V$. Furthermore, since $\pi \geq 0$,

$$ V \mapsto \pi \cdot G_\pi(V) $$

is also concave.

Suppose that $P(\pi, V)$, the transition matrix that realizes the minimum in (14), is differentiable with respect to $V$. We will use the notation $P^{\pi,V} = P(\pi, V)$. Then

$$ \frac{\partial G_\pi(V)_i}{\partial V_j} = P^{\pi,V}_{ij}. $$

In general, however, $P^{\pi,V}$ may not be differentiable. Nevertheless, we have the following rigorous statement:

**Proposition 4.** Suppose assumptions 4 and 7 hold. Then

(a) If $V^1$ and $V^2$ are any two given vectors, we have

$$ G_\pi(V^2)_i \leq G_\pi(V^1)_i + \sum_j P^{V^1,V^2}_{ij} \cdot (V^2 - V^1)_j. \tag{15} $$

(b) If $V^2 \geq V^1$, then

$$ 0 \leq G_\pi(V^2)_i - G_\pi(V^1)_i \leq \sum_j P^{V^1,V^2}_{ij} \cdot (V^2 - V^1)_j. $$

**Proof.** (a) Observe that

$$ G_\pi(V^2)_i \leq \sum_j c_{ij}(\pi, P^{V^1,V^2}_i) P^{V^1,V^2}_{ij} + V^2_j P^{V^1,V^2}_{ij} \cdot \pi \cdot (V^2 - V^1)_j. $$

(b) We just need to prove the first inequality. For this, note that

$$ G_\pi(V^2)_i = \sum_j P^{V^2,V^1}_{ij} \cdot (V^2_j + c_{ij}(\pi, P^{V^2,V^1}_i)) \geq \sum_j P^{V^2,V^1}_{ij} \cdot (V^1_j + c_{ij}(\pi, P^{V^2,V^1}_i)) \geq G_\pi(V^1)_i. $$

Note that, by multiplying (15) by $\pi_i$ and adding, we obtain

$$ \pi \cdot (G_\pi(V^2) - G_\pi(V^1)) - (V^2 - V^1) \cdot K^{V^1}(\pi) \leq 0. \tag{16} $$

Since $G$ commutes with the addition of constants it is not possible to establish a strict concavity estimate like

$$ \pi \cdot (G_\pi(V^2) - G_\pi(V^1)) + (V^1 - V^2) \cdot K^{V^1}(\pi) \leq -\gamma \|V^1 - V^2\|^2. $$

However, in certain cases it is possible to obtain the following strict concavity estimate:

$$ \pi \cdot (G_\pi(V^2) - G_\pi(V^1)) + (V^1 - V^2) \cdot K^{V^1}(\pi) \leq -\gamma \|V^1 - V^2\|_\pi^2. $$

In §4.4 we will give an explicit example.
4.3 A-priori bounds

In the next proposition we give some a-priori bounds which are essential to establishing the existence of fixed points (theorem 3).

Proposition 5. Suppose assumption 6 holds. Then for any $(\pi, V) \in \mathbb{S} \times \mathbb{R}^d$ and all indices $i, i'$ we have

$$|G_\pi(V)_{i} - G_\pi(V)_{i'}| \leq C. \quad (17)$$

Proof. Fix $\pi$ and $V$ let $(G_\pi, K_V)$ be as in (4) and (5). Let $P$ be the optimal transition matrix such that

$$G_\pi(V)_{i} = \sum_j c_{ij}(\pi, P)P_{ij} + V_j P_{ij}.$$ 

Since replacing the $i'$-th line by the $i$-th line in $P$ yields a sub-optimal choice, we have

$$G_\pi(V)_{i'} \leq \sum_j c_{i'j}(\pi, \rho_{i',i}(P))P_{ij} + V_j P_{ij}.$$ 

Hence

$$G_\pi(V)_{i'} - G_\pi(V)_{i} \leq \sum_j [-c_{ij}(\pi, P) + c_{i'j}(\pi, \rho_{i,i'}(P))]P_{ij}.$$ 

If we exchange the role of $i$ and $i'$ we have the desired estimate. \qed

4.4 Strict concavity for the entropy penalized model

To show that the entropy penalized model satisfies the strict concavity property of assumption 10, it suffices to show that the restriction of the linear form given by the matrix $D^2_{\nu} G_\pi(V)_{i}$ to the space of vectors $X \in \mathbb{R}^d$ with $\sum_k X_k = 0$ is uniformly definite positive for each $i$. This holds because of proposition 8, which states that the eigenvalue 0 is simple, and the corresponding eigenvector is $Y = (1, \ldots, 1)$. The uniformity follows from the a-priori bounds in the previous section, which allow us to assume that $\|V\|$ is bounded. In fact (uniform) strict concavity only holds for bounded $\|V\|$. However, for the purposes of this paper this is enough because of the a-priori bounds in §4.3. Thus if $c_{ij}$ is bounded and $\|V\|$ is bounded we have $0 < p_k < 1$, and so assumption 10 holds.

5 Stationary Solutions

In this section we study stationary solutions to mean field games. After the characterization of the critical value, in §5.1 as the average cost for the population of agents, we address the question of uniqueness of the critical value $\bar{\lambda}$ for which (4) admits a solution. In §5.2 we give an example where $\bar{\lambda}$ is non-unique. However, after addressing the issue of existence of stationary solutions (in §5.3), we revisit the uniqueness problem in §5.4 giving conditions which imply uniqueness $\bar{\lambda}$, $\bar{\pi}$ and $V$. These conditions are variations of the monotonicity conditions in [LL06a]. The entropy penalized model is revisited in §5.5 and the large entropy limit is considered in §5.6, where we establish uniqueness of stationary solution (proposition 10). This uniqueness proof uses a strong contraction argument and is thus suitable for the numerical approximation of large entropy penalized mean field games. We end this section, in §5.7, with a discussion of optimal stationary solutions, where certain variational problems give rise to mean field games (see [LL07], §2.6, for related problems).

5.1 Representation of the critical value

We will now give a representation formula for the critical value as the average transition cost.
Proposition 6. Suppose assumption 4 holds. Let \((\bar{\pi}, \bar{V})\) be a stationary solution to (6), and \(\bar{\lambda}\) the corresponding critical value. Let \(\bar{P}\) be the optimal transition matrix. Then

\[
\bar{\lambda} = \sum_{ij} \pi_i c_{ij}(\bar{\pi}, \bar{P}) P_{ij}.
\]

Proof. For each \(1 \leq i \leq d\),

\[
\bar{\lambda} = \sum_{i,j} c_{ij}(\bar{\pi}, \bar{P}) \bar{P}_{ij} - \left( \bar{V}_i - \sum_{i,j} \bar{P}_{ij} \bar{V}_j \right).
\]

(18)

Note that \(\sum_{j} c_{ij}(\bar{\pi}, \bar{P}) \bar{P}_{ij}\) can be seen as the expected cost of transition agents that are in state \(i\) will have when moving to other states. If we multiply (18) by \(\bar{\pi}_i\) and add, for \(1 \leq i \leq d\), we get

\[
\bar{\lambda} = \sum_{i,j} c_{ij}(\bar{\pi}, \bar{P}) \bar{\pi}_i \bar{P}_{ij} - \sum_{i,j} (\bar{V}_i - \bar{V}_j) \bar{\pi}_i \bar{P}_{ij}.
\]

Let \(\mu_{\bar{\pi}}\) denote the probability measure on the set \([1, 2, ..., d]^2\) given by \(\mu_{\bar{\pi}}_{ij} = \pi_i \bar{P}_{ij}\). Since \(\bar{\pi} = \bar{\pi} \bar{P}\), we have

\[
\sum_{i} \mu_{\bar{\pi}}_{ij} = \sum_{i} \mu_{\bar{\pi}}_{ji} = \pi_j.
\]

Therefore

\[
\sum_{i,j} (\bar{V}_i - \bar{V}_j) \pi_i \bar{P}_{ij} = \sum_{i,j} (\bar{V}_i - \bar{V}_j) \mu_{\bar{\pi}}_{ij} = 0.
\]

So

\[
\bar{\lambda} = \sum_{i,j} c_{ij}(\bar{\pi}, \bar{P}) \mu_{\bar{\pi}}_{ij}.
\]

5.2 Non-uniqueness of the critical value

In this section we show that the critical value may not be unique. Consider the following example: \(c_{ij}(\pi)\) given by

\[
c_{12} = c_{21} = 100,
\]

and

\[
c_{11}(\pi^\theta) = c_{22}(\pi^\theta) = \theta.
\]

where \(\pi^\theta = (\theta, 1 - \theta)\), for \(0 \leq \theta \leq 1\). Then \(V^\theta = (0, 0)\), \(\lambda^\theta = \theta\), \(\pi^\theta\) and

\[
P^\theta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

is a stationary solution.

5.3 Existence of stationary solutions

Theorem 3. Suppose that assumptions 4-5 hold. Assume further that there exists \(C\) such that for any \((\pi, V) \in S \times \mathbb{R}^d\) and all indices \(i, i'\) we have (13). Then there exists a pair of vectors \((\bar{\pi}, \bar{V})\), a constant \(\bar{\lambda}\) and a transition matrix \(\bar{P}\) such that for all \(i\),

\[
G_{\bar{\pi}}(\bar{V})_i = \sum_{j} c_{ij}(\bar{\pi}, \bar{P}) P_{ij} + \bar{V}_j P_{ij} = \bar{V}_i + \bar{\lambda},
\]

and \(\bar{\pi} = \bar{\pi} \bar{P}\).
Note that (17) will hold, by proposition 3, if we assume additionally assumption 4.

**Proof.** Since (4,5) hold, by proposition 1, the optimal transition matrix \( P(\pi, V) \) is a continuous function. Therefore the operator \( G_{\pi}(\cdot) : \mathbb{R}^d/\mathbb{R} \to \mathbb{R}^d/\mathbb{R} \) is continuous. Furthermore, by estimate (17), \( \|G_{\pi}(V)\|_\varnothing \) is uniformly bounded for any \( V \in \mathbb{R}^d \).

Consider the mapping \( (G_{\pi}(V), K_V(\pi)) : (\mathbb{R}^d/\mathbb{R}) \times S \to (\mathbb{R}^d/\mathbb{R}) \times S \). By Brouwer’s fixed point theorem this mapping has a fixed point, which is a stationary solution since

\[
\begin{align*}
G_{\pi}(\bar{V}) &= \bar{\lambda} + \bar{V} \\
K_V(\bar{\pi}) &= \bar{\pi}.
\end{align*}
\]

\( \square \)

### 5.4 Uniqueness of stationary solutions

Now we address the problem of uniqueness of stationary solutions. The results in this section use the monotonicity methods introduced in [L06a, L06b, L07] - in this discrete setting, different versions of the hypothesis will yield several uniqueness results. Under assumption 3 we will prove, in proposition 4, the uniqueness of stationary distribution \( \pi \). The uniqueness of critical value is established in proposition 5, using assumption 4, and finally under assumption 6 we obtain the uniqueness of \( \pi, V \) and \( \lambda \) in theorem 4.

**Proposition 7.** Suppose assumptions 4, 7 and 3 hold. Let \( (\pi^k, V^k), k = 1, 2 \), be stationary solutions:

\[
K_{V^k}(\pi^k) = \pi^k, \quad G_{\pi^k}(V^k) = \lambda^k + V^k
\]

where \( \lambda^k \) are constants. Then \( \pi^1 = \pi^2 \).

**Proof.** From the hypothesis we have

\[
0 = (V^1 - V^2) \cdot (K_{V^1}(\pi^1) - \pi^1 - K_{V^2}(\pi^2) + \pi^2) + (\pi^1 - \pi^2)((G_{\pi^2}(V^2) - V^2) - (G_{\pi^1}(V^1) - V^1)) + (\lambda^1 - \lambda^2) \sum_i (\pi^1_i - \pi^2_i).
\]

Note that the last term vanishes since \( \sum_i (\pi^1_i - \pi^2_i) = 0 \). Rewriting we have

\[
0 = \pi^1 \cdot (G_{\pi^2}(V^2) - G_{\pi^1}(V^1)) + (V^1 - V^2) \cdot K_{V^1}(\pi^1) + \\
\quad + \pi^2 \cdot (G_{\pi^2}(V^1) - G_{\pi^1}(V^2)) + (V^2 - V^1) \cdot K_{V^2}(\pi^2) + \\
\quad + \pi^1 \cdot (G_{\pi^2}(V^2) - G_{\pi^1}(V^2)) + \pi^2 \cdot (G_{\pi^1}(V^1) - G_{\pi^2}(V^1)).
\]

By proposition 4 and (16), the first term above satisfies

\[
\pi^1 \cdot (G_{\pi^2}(V^2) - G_{\pi^1}(V^1)) + (V^1 - V^2) \cdot K_{V^1}(\pi^1) \leq 0,
\]

and similarly for the second term.

To analyze the third term, observe that, by assumption 4, we have

\[
\pi^1 \cdot (G_{\pi^2}(V^2) - G_{\pi^1}(V^2)) + \pi^2 \cdot (G_{\pi^1}(V^1) - G_{\pi^2}(V^1)) \leq -\gamma \|\pi^1 - \pi^2\|^2.
\]

Therefore, the estimates above imply \( \gamma \|\pi^1 - \pi^2\|^2 \leq 0 \).

Now we establish the uniqueness of the critical value:

**Proposition 8.** Suppose assumptions 4, 7 and 3 hold. Let \( \pi \in S, \lambda^k \in \mathbb{R} \) and \( V^k \in \mathbb{R}^d, k = 1, 2 \) be solutions of

\[
G_{\pi}(V^k) = \lambda^k + V^k.
\]

Then \( \lambda^1 = \lambda^2 \).
Proof. Choose \( i \in \text{argmax} V^1 - V^2 \). Then
\[
\lambda^1 = G_\pi(V^1)_i - V^1_i \leq G_\pi(V^2)_i - V^2_i = \lambda^2.
\]

By choosing \( i \in \text{argmin} V^1 - V^2 \) we obtain the opposite inequality, which then implies \( \lambda^1 = \lambda^2 \).

Therefore, under assumptions 8 and 9, we have both uniqueness of the stationary distribution \( \pi \) and critical value \( \lambda \). We now address the uniqueness of the stationary value function \( V \).

**Theorem 4.** Suppose assumptions 4, 7, 8 and 10 hold. Let \( (\pi^k, V^k) \), \( k = 1, 2 \), be stationary solutions:
\[
K_{V^k}(\pi^k) = \pi^k, \quad G_{\pi^k}(V^k) = \lambda^k + V^k
\]
where \( \lambda^k \) are constants. Then
(a) \( \pi^1 = \pi^2 \),
(b) \( V^2 = V^1 + k \), where \( k \) is a constant vector,
(c) \( \lambda_1 = \lambda_2 \).

Proof. If we follow the proof of Proposition 7, we can use assumption 10 to get
\[
0 = -\gamma_1 \| V^1 - V^2 \|_2^2 - \gamma_2 \| V^1 - V^2 \|_2^2 - \gamma \| \pi^1 - \pi^2 \|_2^2.
\]
This implies items (a) and (b). To get item (c), we observe that
\[
V^2 + \lambda^2 = G_\pi(V^2) = G_\pi(V^1 + k) = G_\pi(V^1) + k = V^1 + \lambda^1 + k = V^2 + \lambda^1,
\]
where \( \pi = \pi_1 = \pi_2 \), in the first and fourth equalities we used \( G_\pi(V^k) = V^k + \lambda^k \), in the second and fifth we used item (b), and in the third we used the fact that \( G_\pi \) commutes with constants.

### 5.5 Entropy penalized stationary solutions

Now we consider the entropy penalized model. We will present a simple proof of existence of solutions that relies on the special structure of the problem. A simple computation yields
\[
G_\pi(V)_i = -\epsilon \ln \left[ \sum_k e^{-\frac{c_{ik}(\pi) + V_k}{\epsilon}} \right].
\]
We will suppose further:

**Assumption 12.** The function \( c_{ij}(\pi) \) is continuous.

**Proposition 9.** Suppose assumption 12 holds. Consider the entropy penalized model (11). Then there exists a pair of vectors \( (\bar{\pi}, \bar{V}) \in S \times \mathbb{R}^d \), a constant \( \bar{\lambda} \in \mathbb{R} \) such that
\[
G_{\bar{\pi}}(\bar{V}) = \bar{\lambda} + \bar{V}
\]
and \( \bar{\pi} = \bar{\pi}P = K_{\bar{V}}(\bar{\pi}) \).

Proof. Define the strictly positive linear operator that associates to each vector \( \psi \in \mathbb{R}^d \) the vector
\[
L_\psi(\psi)_i = \sum_k e^{-\frac{c_{ik}(\pi) + V_k}{\epsilon}} \psi_k.
\]
Let \( e^{-\frac{\bar{\lambda}}{\epsilon}} \) be the largest eigenvalue of the operator \( L_\psi(\psi) \) and \( \psi^{\pi} \) the unique normalized eigenvector associated to \( e^{-\frac{\bar{\lambda}}{\epsilon}} \), i.e.,
\[
L_\psi(\psi^{\pi}) = e^{-\frac{\bar{\lambda}}{\epsilon}} \psi^{\pi}.
\]
By Perron-Frobenius Theorem, $\psi_\pi$ is a strictly positive vector which is a continuous function of $\pi$. We can define $V^\pi$ as $\psi_k^\pi = e^{-\frac{\lambda^\pi_k}{\epsilon}}$. Let $E[\phi]_j = e^{-\frac{\phi_j}{\epsilon}}$ be the exponential transformation. These operators are related by

$$\mathcal{L}_\pi \circ \mathcal{E} = \mathcal{E} \circ \mathcal{G}_\pi.$$ 

Hence

$$\mathcal{G}_\pi(V_\pi) = V_\pi + \lambda_\pi.$$ \hspace{1cm} (19)

Define a new probability vector $K(\pi)_j = \sum_i \pi_i P_{ij}(\pi, V^\pi) = \sum_i \pi_i e^{-\frac{c_{ij}(\pi) + V^\pi_j}{\epsilon}}$. Let $\hat{E}[\phi]_j = e^{-\frac{\phi_j}{\epsilon}}$ be the exponential transformation. These operators are related by

$$\mathcal{L}_\pi \circ \mathcal{E} = \mathcal{E} \circ \mathcal{G}_\pi.$$ 

Hence

$$\mathcal{G}_\pi(V_\pi) = V_\pi + \lambda_\pi.$$ \hspace{1cm} (19)

Thus we have defined a operator $K : S \rightarrow S$ which is continuous. By Brower’s fixed point theorem, $K$ has a fixed point $\bar{\pi}$. Define $\bar{V}_\pi$ and $\bar{\lambda}_\pi$ as above, and $\bar{P}_{ij} = e^{-\frac{c_{ij}(\pi) + V^\pi_j}{\epsilon}}$, then (19) holds.

5.6 Stationary Solutions with large entropy

The construction of fixed points for the entropy penalized model in the last section depends on Brower’s fixed point theorem. In the case of large entropy we can use a contraction argument to establish the existence of a stationary solution. Before proving and stating this result we need an elementary lemma

**Lemma 1.** Let $T : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^n$ be a $C^1$ mapping. Suppose

$$DT = \begin{bmatrix} E_1 & M \\ E_2 & E_3 \end{bmatrix},$$

where $E_k, M$ are $n \times n$ matrices. If $\|M\|$ is bounded and $\|E_k\|$ is sufficiently small then $T^2$ is a strong contraction in $\mathbb{S}^d \times \mathbb{R}^d / \mathbb{R}$.

**Proof.** It suffices to observe that

$$D(T^2)(x) = DT(T(x)x)DT(x)$$

can be written as

$$D(T^2) = \begin{bmatrix} E_1 & M \\ E_2 & E_3 \end{bmatrix} \begin{bmatrix} E_1 & M \\ E_2 & E_3 \end{bmatrix} = \begin{bmatrix} E_1 E_1 + ME_2 & E_1 M + ME_3 \\ E_2 E_1 + E_3 E_2 & E_2 M + E_3 E_3 \end{bmatrix},$$

and therefore $\|D(T^2)\| < 1$. \hspace{1cm} \(\square\)

To establish the main result in this section we need to replace assumption \[2\] by

**Assumption 13.** The function $c_{ij}(\pi)$ is a $C^1$ function.

**Proposition 10.** Suppose assumption \[3\] holds. Then, for large $\epsilon$, there is a unique stationary solution. Additionally, let $T(\pi, V) = (\mathcal{K}_{V,\epsilon}(\pi), \mathcal{G}_\pi(V))$, then $T^2$ is a strong contraction.

**Proof.** From proposition \[3\] we have $\|\mathcal{G}_\pi(V)\|_\#$ is uniformly bounded. Therefore it suffices to show that the operator $(\mathcal{K}, \mathcal{G})$ is a strong contraction for $V$ in a compact set (with respect to the norm $\| \cdot \|_\#$). Also, we can replace $\mathcal{G}$ by

$$\hat{\mathcal{G}}_\pi(V)_i = \mathcal{G}_\pi(V)_i - \frac{1}{d} \sum_k \mathcal{G}_\pi(V)_k,$$
and $K_V(\pi)$ by

$$\hat{K}_V(\pi)_i = K_V(\pi)_i - \frac{1}{d} \left( \sum_j K_V(\pi)_j - 1 \right).$$

Since any fixed point to $(\hat{G}, \hat{K})$ is a stationary solution. In this way $\|G_\pi(V)\|_\# = \|\hat{G}_\pi(V)\|$, and for any $\pi$ (not necessarily a probability measure), $\hat{K}_V(\pi)$ is a probability measure, and agrees with $K_V(\pi)$ if $\pi$ is a probability measure.

We will show that, for $\epsilon$ sufficiently large, the pair $(\hat{G}, \hat{K})$ satisfies the hypothesis of lemma 1. To do so, we first compute the matrix

$$\begin{bmatrix}
\frac{\partial \hat{G}}{\partial \pi} & \frac{\partial \hat{G}}{\partial \pi} \\
\frac{\partial \hat{K}}{\partial \pi} & \frac{\partial \hat{K}}{\partial \pi}
\end{bmatrix}.$$

We will show that, when $\epsilon \to \infty$, this matrix converges to

$$\begin{bmatrix}
[1/d] & [1/d \sum_k \frac{\partial c_{ki}}{\partial \pi_j}]
\end{bmatrix},$$

where we denote by $[a]$ the $d \times d$ matrix whose entries are all identical to $a$.

Since $G_{\pi, \epsilon}(V)_i = -\epsilon \ln \left( \sum_k e^{-\frac{c_{ik}(\pi) + V_j}{\epsilon}} \right)$, we have:

$$\left( \frac{\partial G_{\pi}(V)}{\partial V_j} \right)_i = \frac{e^{-\frac{c_{ij}(\pi) + V_j}{\epsilon}}}{\sum_k e^{-\frac{c_{ik}(\pi) + V_j}{\epsilon}}} \to 1/d,$$

when $\epsilon \to \infty$. We also have

$$\left( \frac{\partial G_{\pi}(V)}{\partial \pi_j} \right)_i = \frac{\sum_k \partial c_{ik}(\pi) e^{-\frac{c_{ij}(\pi) + V_j}{\epsilon}}}{\sum_k e^{-\frac{c_{ik}(\pi) + V_j}{\epsilon}}} \to \frac{1}{d} \sum_k \partial c_{ik}(\pi),$$

when $\epsilon \to \infty$.

Now we consider $K_V(\pi)_i = \sum_k \pi_k P_{ki}$, where

$$P_{ki} = \frac{e^{-\frac{c_{ik}(\pi) + V_j}{\epsilon}}}{\sum_l e^{-\frac{c_{il}(\pi) + V_j}{\epsilon}}}.$$

We have

$$\left( \frac{\partial K_V(\pi)}{\partial \pi_j} \right)_i = P_{ji} + \sum_k \pi_k \sum_l e^{-\frac{c_{ik}(\pi) + V_j}{\epsilon}} e^{-\frac{c_{lj}(\pi) + V_j}{\epsilon}} \left( \frac{\partial c_{kj}(\pi)}{\partial \pi_j} - \frac{\partial c_{ik}(\pi)}{\partial \pi_j} \right) \epsilon \left( \sum_l e^{-\frac{c_{il}(\pi) + V_j}{\epsilon}} \right)^2.$$

If we take $\epsilon \to \infty$, the second term tends to zero while $P_{ji} \to \frac{1}{d}$. Thus

$$\left( \frac{\partial K_{\pi, \epsilon}(\pi)}{\partial \pi_j} \right)_i \to \frac{1}{d}.$$
when $\epsilon \to \infty$.

Finally, observe that
\[
\begin{bmatrix}
\frac{\partial \hat{G}}{\partial K} & \frac{\partial \hat{G}}{\partial \pi}
\end{bmatrix} = \begin{bmatrix}
I - [1/d] & 0
\end{bmatrix} \begin{bmatrix}
\frac{\partial G}{\partial K} & \frac{\partial G}{\partial \pi}
\end{bmatrix}.
\]

This shows that, in the limit $\epsilon \to \infty$, we have
\[
\begin{bmatrix}
\frac{\partial \hat{G}}{\partial V} & \frac{\partial \hat{G}}{\partial \pi}
\end{bmatrix} \to \begin{bmatrix}
0 & M
\end{bmatrix} = \begin{bmatrix}
I - [1/d] & \frac{1}{d} \sum_k \frac{\partial c_{ik}}{\partial \pi_j}
\end{bmatrix}.
\]

Thus, for $\epsilon$ large enough lemma \[\] yields the strong contraction property of $T^2$. \[\]

### 5.7 Optimal Stationary Solutions

Given a probability measure $\eta_{ij}, 1 \leq i, j \leq d$, define $\pi^n_i = \sum_j \eta_{ij}$ and let $P^n_{ij}$ be a stochastic matrix such that $\eta_{ij} = \pi^n_i P^n_{ij}$. If $\pi^n$ never vanishes then $P^n$ is uniquely defined by $P^n_{ij} = \frac{\eta_{ij}}{\pi^n_i}$. A probability measure $\eta_{ij}$ is stationary if
\[
\sum_j \eta_{ij} = \sum_j \eta_{ji}.
\] (20)

For our purposes, in this section it is convenient to consider the following two auxiliary assumptions:

**Assumption 14.** For each $1 \leq i \leq d$ the mapping $P_i \mapsto \sum_j c_{ij}(P_i)P_{ij}$ is convex.

and

**Assumption 15.** The mapping $\eta \mapsto \sum_{i,j} \pi^n_i c_{ij}(P^n_{ij})$ is strictly convex.

Consider a $C^1$ convex function function $f : \mathbb{R}^d \to \mathbb{R}$. Consider the problem
\[
\min_{\eta} \sum_{i,j} \pi^n_i c_{ij}(P^n_{ij})P^n_{ij} + f(\pi^n),
\] (21)

where the minimum is taken over all probability measures $\eta$ satisfying (21).

**Proposition 11.** Let $\eta > 0$ be a solution of (21). Let $V^n \in \mathbb{R}^d$ be the Lagrange multiplier associated to the constraint (21), and $\lambda^n$ the Lagrange multiplier corresponding to $\sum_{i,j} \eta_{ij} = 1$. Then $(\pi^n, V^n)$ is a stationary solution of
\[
V^n + \lambda^n = \mathcal{G}_{\pi^n}(V^n),
\]

where
\[
\mathcal{G}_\pi(V)_i = \frac{\partial f}{\partial \pi_i}(\pi) + \min_{P_i} \sum_j c_{ij}(P_i)P_{ij} + P_{ij}V_j.
\]

**Proof.** Let $\eta$ be as in the statement. As before write $\eta_{ij} = \pi^n_i P^n_{ij}$. Then both $\pi^n$ and $P^n_{ij}$ are critical points of the functional
\[
f(\pi) + \sum_i \pi_i \left( \sum_j c_{ij}(P)P_{ij} + V^n_j P_{ij} \right) - V^n_i - \lambda^n.
\] (22)
Consequently, for each $i$, $P_i$ is a critical point of
\[
\sum_j c_{ij}(P)P_{ij} + V^n_j P_{ij},
\]
which by the convexity hypothesis, assumption 14 is a minimizer. Furthermore, by differentiating (22) with respect to $\pi_i$, we obtain
\[V_i + \lambda^0 = \mathcal{G}_{\pi}(V^n)_i.\]
Finally, we can write (20) as
\[\pi^n = \mathcal{K}_{V^n}(\pi^n),\]
which ends the proof.

**Proposition 12.** Suppose assumptions 7, and 15 hold. Then there exists at most one stationary solution with $\pi, P > 0$.

**Proof.** This proposition follows from the well known fact (see for instance [Ped04]) that for strictly convex objective functions under linear constraints the KKT conditions are not only necessary but also sufficient.

Note that if $f$ is a strictly convex function, then the previous proposition gives us another proof of the existence and uniqueness of the stationary solution in the case $c_{ij}(\pi, P) = \tilde{c}_{ij}(P_i) + W(\pi)$, with $W(\pi) = \frac{\partial f}{\partial \pi}(\pi)$.

## 6 Solutions to the Mean Field Game initial-terminal value problem

In this section we prove the existence (§6.1) and uniqueness (§6.2) of solutions to the initial-terminal value problem.

### 6.1 Existence of Solutions

**Theorem 5.** Suppose assumptions 3 and 4 hold. Then for any initial probability vector $\tilde{\pi} \in \pi$ and terminal cost $\tilde{V}$ there exists a solution
\[
\{(\pi^n, V^n) : 0 \leq n \leq N\}
\]
for the initial-terminal value problem for the mean field game with $\pi^0 = \tilde{\pi}$ and $V^N = \tilde{V}$.

**Proof.** Suppose we are given a sequence $\pi^{n,0} \in \pi^{N+1}$ of probability vectors, with $\pi^{0,0} = \tilde{\pi}$. Define, for $0 \leq n \leq N$
\[V^{n,0} = \mathcal{G}_{\pi^{n,0}}(V^{n+1,0}),\]
with $V^{N,0} = \tilde{V}$. Then let
\[\pi^{n+1,1} = \mathcal{K}_{V^{n,0}}(\pi^{n,1}),\]
with $\pi^{0,1} = \tilde{\pi}$. This procedure defines a continuous mapping from $\pi^{N+1}$ into itself that associates to the sequence $\pi^{n,0}$ the new sequence of probability vectors $\pi^{n,1}$. Therefore, by Brouwer’s fixed point theorem, it has a fixed point, which corresponds to a solution to the problem. \qed
6.2 Uniqueness

As for stationary solution we adapt Lasry and Lions monotonicity arguments to obtain uniqueness of solutions.

**Theorem 6.** Suppose assumptions 4, 7 and 10 hold. Let \( \{ (\pi^n, V^n) : 0 \leq n \leq N \} \) and \( \{ (\tilde{\pi}^n, \tilde{V}^n) : 0 \leq n \leq N \} \) be solutions of the the mean field game with \( \pi^0 = \tilde{\pi}^0 \) and \( V^N = \tilde{V}^N \). Then \( \pi^n = \tilde{\pi}^n \), and \( V^n = \tilde{V}^n \), for all \( 0 \leq n \leq N \).

**Proof.** We have

\[
G_{\pi^n}(V^{n+1}) = V^n, \quad K_{V^{n+1}}(\pi^n) = \pi^{n+1},
\]

and

\[
G_{\tilde{\pi}^n}(\tilde{V}^{n+1}) = \tilde{V}^n, \quad K_{\tilde{V}^{n+1}}(\tilde{\pi}^n) = \tilde{\pi}^{n+1}.
\]

Then

\[
0 = \sum_{n=0}^{N-1} (V^{n+1} - \tilde{V}^{n+1}) : \left[ (K_{V^{n+1}}(\pi^n) - \pi^{n+1}) - (K_{\tilde{V}^{n+1}}(\tilde{\pi}^n) - \tilde{\pi}^{n+1}) \right]
+ \sum_{n=0}^{N-1} (\pi^n - \tilde{\pi}^n) : \left[ (G_{\pi^n}(V^{n+1}) - \pi^n) - (G_{\tilde{\pi}^n}(\tilde{V}^{n+1}) - \tilde{\pi}^n) \right].
\]

Note that \( (V^{N} - \tilde{V}^{N}) \cdot (\tilde{\pi}^{N} - \pi^{N}) = 0 \) and \( (\pi^0 - \tilde{\pi}^0) \cdot (V^0 - \tilde{V}^0) = 0 \). Thus rewriting the identity above we have

\[
0 = \sum_{n=0}^{N-1} \pi^n \cdot (G_{\pi^n}(\tilde{V}^{n+1}) - G_{\pi^n}(V^{n+1})) + K_{V^{n+1}}(\pi^n) \cdot (V^{n+1} - \tilde{V}^{n+1}) +
+ \sum_{n=0}^{N-1} \tilde{\pi}^n \cdot (G_{\tilde{\pi}^n}(V^{n+1}) - G_{\tilde{\pi}^n}(\tilde{V}^{n+1})) + K_{\tilde{V}^{n+1}}(\tilde{\pi}^n) \cdot (\tilde{V}^{n+1} - V^{n+1}) +
+ \sum_{n=0}^{N-1} \pi^n \cdot (G_{\pi^n}(\tilde{V}^{n+1}) - G_{\pi^n}(\tilde{V}^{n+1})) + \tilde{\pi}^n \cdot (G_{\tilde{\pi}^n}(V^{n+1}) - G_{\tilde{\pi}^n}(V^{n+1})).
\]

Now, using 16, we have, for each \( 0 \leq n \leq N - 1 \),

\[
\pi^n \cdot (G_{\pi^n}(\tilde{V}^{n+1}) - G_{\pi^n}(V^{n+1})) + K_{V^{n+1}}(\pi^n) \cdot (V^{n+1} - \tilde{V}^{n+1}) \leq 0
\]

and similarly for the terms of the second line. In the third line we have

\[
\pi^n \cdot (G_{\pi^n}(\tilde{V}^{n+1}) - G_{\pi^n}(\tilde{V}^{n+1})) + \tilde{\pi}^n \cdot (G_{\tilde{\pi}^n}(V^{n+1}) - G_{\tilde{\pi}^n}(V^{n+1})) \leq -\gamma \| \pi^n - \tilde{\pi}^n \|^2.
\]

Hence

\[
\sum_{n=0}^{N-1} \gamma \| \pi^n - \tilde{\pi}^n \|^2 \leq 0.
\]

This implies \( \pi^n = \tilde{\pi}^n \) for all \( 0 \leq n \leq N \).

To obtain \( V^n = \tilde{V}^n \) for all \( 0 \leq n \leq N \), we just have to use \( V^N = \tilde{V}^N \) and apply iteratively the operator \( G_{\pi^n}(V^{n+1}) = V^n \) and \( G_{\tilde{\pi}^n}(\tilde{V}^{n+1}) = \tilde{V}^n \), from \( n = N - 1 \) to \( n = 0 \).

7 Convergence to equilibrium

In this last section we discuss the main contribution of this paper, namely the exponential convergence to equilibrium for the initial-terminal value problem. Our setting is the following: consider a initial-terminal value problem with initial data \( \pi^{-N} \) and terminal data \( V^N \). We will now study conditions under which \( \pi^0 \rightarrow \bar{\pi} \) and \( V^0 \rightarrow \bar{V} \) where \( (\bar{\pi}, \bar{V}) \) are stationary solutions, as \( N \rightarrow \infty \). In fact we will show this is true if assumptions 4, 7, 10 and 11 hold.
7.1 A-priori bounds

We start by establishing some useful a-priori bounds.

**Lemma 2.** Suppose assumption 4, 5 and 11 holds. Let \( \{(\pi^n, V^n) : -N \leq n \leq N\} \) and \( \{\tilde{\pi}^n, \tilde{V}^n\) : \( -N \leq n \leq N\) be two solutions of the mean field game. Then we have

\[
\|\tilde{V}^N - V^{-N}\| \leq \|\tilde{V}^N - V^N\| + N2K.
\]  

**Proof.** Applying proposition 3 we have that

\[
G_{\pi^{-N}, \pi^{-N}}(\tilde{V}^N)_i - G_{\pi^{-N}, \pi^{-N}}(V^N)_i \leq \sum_j P^{\gamma_{\tilde{\pi}^{-N}, \pi^{-N}}}_i (\tilde{V}^N_j - V^N_j).
\]

Then

\[
G_{\pi^{-N}, \pi^{-N}}(\tilde{V}^N)_i - G_{\pi^{-N}, \pi^{-N}}(V^N)_i + \tilde{V}^{N-1}_i - V^{N-1}_i \leq \|\tilde{V}^N - V^N\|
\]

also we have

\[
G_{\pi^{-N}, \pi^{-N}}(\tilde{V}^N)_i - G_{\pi^{-N}, \pi^{-N}}(\tilde{V}^N)_i \leq \sum_j [c_{ij}(\pi^{-N-1}, P^{\gamma_{\tilde{\pi}^{-N}, \tilde{\pi}^{-N}}}_i) - c_{ij}(\pi^{-N-1}, P^{\gamma_{\pi^{-N}, \pi^{-N}}}_i)] P^{\gamma_{\tilde{\pi}^{-N}, \pi^{-N}}}_i \leq K.
\]

Hence,

\[
\tilde{V}^{N-1}_i - V^{N-1}_i \leq \|\tilde{V}^N - V^N\| + K
\]

Exchanging the roles of \( \pi^{-N-1}, V^N \) and \( \tilde{\pi}^{-N-1}, \tilde{V}^N \) we get

\[
V^{N-1}_i - \tilde{V}^{N-1}_i \leq \|\tilde{V}^N - V^N\| + K,
\]

thus

\[
\|\tilde{V}^{N-1} - V^{N-1}\| \leq \|\tilde{V}^N - V^N\| + K
\]

Reasoning by induction we obtain that

\[
\|\tilde{V}^N - V^N\| \leq \|\tilde{V}^N - V^N\| + N2K.
\]

\[\square\]

7.2 Exponential convergence

We recover the proof of theorem 3 to obtain an important estimate:

**Proposition 13.** Suppose assumptions 4, 5, 8 and 11 hold. Let \( \{(\pi^n, V^n) : -N \leq n \leq N\} \) and \( \{\tilde{\pi}^n, \tilde{V}^n\) : \( -N \leq n \leq N\) be solutions of the mean field game. Let \( C = 1/\gamma \). Then

\[
\sum_{n=-N}^{N-1} \|\pi^n - \tilde{\pi}^n\|^2 + \|V^n - \tilde{V}^n\|^2 \leq C \left( \|\pi^N - \tilde{\pi}^N\|^2 + \|V^N - \tilde{V}^N\|^2 + \|\pi^{-N} - \tilde{\pi}^{-N}\|^2 + \|V^{-N} - \tilde{V}^{-N}\|^2 \right)
\]

**Proof.** As before, observe that

\[
0 = \sum_{n=-N}^{N-1} (V^{n+1} - \tilde{V}^{n+1}) \cdot \left[ (\mathcal{K}_{\pi^{n+1}}(\pi^n) - \pi^{n+1}) - (\mathcal{K}_{\tilde{\pi}^{n+1}}(\pi^n) - \tilde{\pi}^{n+1}) \right] + \sum_{n=-N}^{N-1} (\pi^n - \tilde{\pi}^n) \cdot \left[ (\mathcal{G}_{\pi^n}(\tilde{V}^{n+1}) - \tilde{V}^n) - (\mathcal{G}_{\tilde{\pi}^n}(V^{n+1}) - V^n) \right].
\]
Then rewriting the equation above we have
\[
\sum_{n=-N}^{N-1} (V^{n+1} - \tilde{V}^{n+1}) \cdot (\pi^{n+1} - \tilde{\pi}^{n+1}) - (V^n - \tilde{V}^n) \cdot (\pi^n - \tilde{\pi}^n)
\]
\[
= \sum_{n=-N}^{N-1} \pi^n \cdot (\mathcal{G}_{\pi^n}(\tilde{V}^{n+1}) - \mathcal{G}_{\pi^n}(V^{n+1})) + K_{\mathcal{V}_{n+1}}(\pi^n) \cdot (V^{n+1} - \tilde{V}^{n+1}) + \\
+ \sum_{n=N}^{N-1} \tilde{\pi}^n \cdot (\mathcal{G}_{\tilde{\pi}^n}(V^{n+1}) - \mathcal{G}_{\tilde{\pi}^n}(\tilde{V}^{n+1})) + K_{\mathcal{V}_{n+1}}(\tilde{\pi}^n) \cdot (\tilde{V}^{n+1} - V^{n+1}) + \\
+ \sum_{n=-N}^{N-1} \pi^n \cdot (\mathcal{G}_{\pi^n}(\tilde{V}^{n+1}) - \mathcal{G}_{\pi^n}(V^{n+1})) + \tilde{\pi}^n \cdot (\mathcal{G}_{\pi^n}(V^{n+1}) - \mathcal{G}_{\tilde{\pi}^n}(\tilde{V}^{n+1})).
\]

Now, for each $-N \leq n \leq N - 1$, we have that
\[
\pi^n \cdot (\mathcal{G}_{\pi^n}(\tilde{V}^{n+1}) - \mathcal{G}_{\pi^n}(V^{n+1})) + K_{\mathcal{V}_{n+1}}(\pi^n) \cdot (V^{n+1} - \tilde{V}^{n+1}) \leq -\gamma \|V^{n+1} - \tilde{V}^{n+1}\|_\#^2,
\]
and similarly for the terms of the second line. In the third line we have
\[
\pi^n \cdot (\mathcal{G}_{\tilde{\pi}^n}(V^{n+1}) - \mathcal{G}_{\pi^n}(\tilde{V}^{n+1}))) + \tilde{\pi}^n \cdot (\mathcal{G}_{\pi^n}(V^{n+1}) - \mathcal{G}_{\tilde{\pi}^n}(\tilde{V}^{n+1})) \leq -\gamma \|\pi^n - \tilde{\pi}^n\|^2.
\]
Consequently
\[
\sum_{n=-N}^{N-1} \|\pi^n - \tilde{\pi}^n\|^2 + 2\|V^{n+1} - \tilde{V}^{n+1}\|_\#^2
\]
\[
\leq \frac{1}{\gamma} \left( (\pi^N - \tilde{\pi}^N) \cdot (\tilde{V}^N - V^N) + (\pi^{-N} - \tilde{\pi}^{-N}) \cdot (V^{-N} - \tilde{V}^{-N}) \right).
\]

Note that, if $c$ is the constant vector then $(\pi^k - \tilde{\pi}^k) \cdot \mu = 0$, where $k = N, -N$. Also, there exists $\mu_k$ such that $\|V^k - \tilde{V}^k\| = |V^k - V^k + \mu_k|$. Hence
\[
\sum_{n=-N+1}^{N-1} \|\pi^n - \tilde{\pi}^n\|^2 + \|V^n - \tilde{V}^n\|_\#^2
\]
\[
\leq C \left( \|\pi^N - \tilde{\pi}^N\|^2 + \|V^N - \tilde{V}^N\|_\#^2 + \|\pi^{-N} - \tilde{\pi}^{-N}\|^2 + \|V^{-N} - \tilde{V}^{-N}\|_\#^2 \right),
\]
if we denote $C = 1/\gamma$.

Define $f_0 = \|\pi^0 - \tilde{\pi}^0\|^2 + \|V^0 - \tilde{V}^0\|_\#^2$, and, for $n > 0$
\[
f_n = \|\pi^n - \tilde{\pi}^n\|^2 + \|V^n - \tilde{V}^n\|_\#^2 + \|\pi^{-n} - \tilde{\pi}^{-n}\|^2 + \|V^{-n} - \tilde{V}^{-n}\|_\#^2.
\]
The previous proposition implies
\[
\sum_{n=0}^{N-1} f_n \leq C f_N.
\]

Note that the previous proposition and lemma \[\] imply
\[
f_N \leq \|\pi^N - \tilde{\pi}^N\|^2 + \|\pi^{-N} - \tilde{\pi}^{-N}\|^2 + \|V^N - \tilde{V}^N\|_\#^2 + (\|V^N - \tilde{V}^N\| + N^2 K)^2.
\]
The next lemma is the only missing tool to get exponential decay:
Lemma 3. Suppose \( f_n \geq 0 \) and that
\[
\sum_{n=0}^{N-1} f_n \leq Cf_N.
\]
Then
\[
f_0 \leq C \left( \frac{C}{C+1} \right)^{N-1} f_N.
\]

Proof. The proof follows by induction. The case \( N = 1 \) is simply a particular case of (26):
\[
f_0 \leq Cf_1.
\]
Now, observe that
\[
Cf_{N+1} \geq \sum_{n=0}^{N} f_n \geq f_0 + \sum_{n=1}^{N} \frac{1}{C} \left( \frac{C+1}{C} \right)^{n-1} f_0
\]
\[
= f_0 \left[ 1 + \frac{1}{C} \left( \frac{C+1}{C} \right)^{N-1} - 1 \right] = f_0 \left[ 1 + \left( \frac{C+1}{C} \right)^{N} - 1 \right] = f_0 \left( \frac{C+1}{C} \right)^{N},
\]

which ends the proof.

\( \square \)

Theorem 7. Suppose assumptions \( \ref{assumption:1} \), \( \ref{assumption:2} \), \( \ref{assumption:3} \), \( \ref{assumption:4} \) and \( \ref{assumption:5} \) hold. Fix \( \tilde{V}, \tilde{\pi} \). Given \( N > 0 \), denote by \( (\pi_0^N, V_0^N) \) the solution of the mean field game at time 0 that has initial distribution \( \pi^{-N} = \tilde{\pi} \) and terminal cost \( V^N = \tilde{V} \).

Then, as \( N \to \infty \)
\[
V_0^N \to \tilde{V} \ (\text{in } \mathbb{R}^d/\mathbb{R}), \quad \pi_0^N \to \tilde{\pi}
\]
where \( \tilde{V} \) and \( \tilde{\pi} \) is the unique stationary solution.

Proof. By theorem 6, we can define, for each \( N \), a map \( \Xi_N : S \times \mathbb{R}^d/\mathbb{R} \to S \times \mathbb{R}^d/\mathbb{R} \), that associates to each pair \( (\tilde{\pi}, \tilde{V}) \) the pair \( (\pi_0^N, V_0^N) \). Here \( S \times \mathbb{R}^d/\mathbb{R} \) is given the product topology where in \( \mathbb{R}^d/\mathbb{R} \) we consider the norm \( \|U\| = \inf_{k \in \mathbb{R}} \|U + k\| \).

Now, lemma 3 and equation (25) show that, for any two pairs \( (V, \pi) \) and \( (\tilde{V}, \tilde{\pi}) \), we have
\[
\Xi_N(V, \pi) - \Xi_N(\tilde{V}, \tilde{\pi}) \to (0,0)
\]
as \( N \to \infty \). Thus,
\[
\Xi_N(V, \pi) - \Xi_N(\tilde{V}, \tilde{\pi}) \to (0,0)
\]
as \( N \to \infty \).

\( \square \)

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