Casimir effect in de Sitter spacetime with compactified dimension

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Abstract

We investigate the Hadamard function, the vacuum expectation values of the field square and the energy-momentum tensor of a scalar field with general curvature coupling parameter in de Sitter spacetime compactified along one of spatial dimensions. By using the Abel-Plana summation formula, we have explicitly extracted from the vacuum expectation values the part due to the compactness of the spatial dimension. The topological part in the vacuum energy-momentum tensor violates the local de Sitter symmetry and dominates in the early stages of the cosmological evolution. At late times the corresponding vacuum stresses are isotropic and the topological part corresponds to an effective gravitational source with barotropic equation of state.

1 Introduction

De Sitter (dS) spacetime is the maximally symmetric solution of Einstein’s equation with a positive cosmological constant. Recent astronomical observations of supernovae and cosmic microwave background \cite{1} indicate that the universe is accelerating and can be well approximated by a world with a positive cosmological constant. If the universe would accelerate indefinitely, the standard cosmology leads to an asymptotic dS universe. De Sitter spacetime plays an important role in the inflationary scenario, where an exponentially expanding approximately dS spacetime is employed to solve a number of problems in standard cosmology \cite{2}. In this regard in recent years, many string theorists have devoted to understand and shed light on the cosmological constant or dark energy within the string framework. The famous Kachru-Kallosh-Linde-Trivedi (KKLT) model \cite{3} is a typical example, which tries to construct metastable de Sitter vacua in the light of type IIB string theory. The quantum field theory on dS spacetime is also of considerable interest. In particular, the inhomogeneities generated by fluctuations of a quantum field during inflation provide an attractive mechanism for the structure formation in the universe. Another motivation for investigations of dS based quantum theories is related to the recently proposed holographic duality between quantum gravity on dS spacetime and a quantum field theory living on boundary identified with the timelike infinity of dS spacetime \cite{4}.

In quantum field theory on curved backgrounds among the important quantities describing the local properties of a quantum field and quantum back-reaction effects are the expectation

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values of the field square and the energy-momentum tensor for a given quantum state. In particular, the vacuum expectation values of these quantities are of special interest. For a scalar field on background of dS spacetime the renormalized vacuum expectation values of the field square and the energy-momentum tensor are investigated in Refs. [5, 6, 7, 8] by using various regularization schemes. The corresponding effects upon phase transitions in an expanding universe are discussed in [9]. Recently it was argued that there is no reason to believe that the version of dS spacetime which may emerge from string theory, will necessarily be the most familiar version with symmetry group $O(1,4)$ and there are many different topological spaces which can accept the dS metric locally (see [3, 10] and references therein). There are many reasons to expect that in string theory the most natural topology for the universe is that of a flat compact three-manifold [10]. From an inflationary point of view universes with compact spatial dimensions, under certain conditions, should be considered a rule rather than an exception [11] (for observational bounds on the size of compactified dimensions see Refs. [12]).

The compactification of spatial dimensions leads to the modification of the spectrum of vacuum fluctuations and, as a result, to Casimir-type contributions to the vacuum expectation values of physical observables (topological Casimir effect, see [13]). In the present paper we investigate the effect of the compactification of one of spatial dimensions in dS spacetime on the properties of quantum vacuum for a scalar field with general curvature coupling parameter (for quantum effects in braneworld models with dS spaces see, for instance, [14]).

The paper is organized as follows. In the next section we consider the Hadamard function. By using the Abel-Plana summation formula, we decompose this function in two parts: the first one is the corresponding function for the uncompactified dS spacetime and the second one is induced by the compactness of the spatial dimension. In Section 3 we use the Hadamard function for the evaluation of the vacuum expectation values of the field square and the energy-momentum tensor. As the parts corresponding to the uncompactified dS spacetime are well-investigated in literature, we are mainly concerned with the topological part. The asymptotic behavior of the latter is investigated in detail in early and late stages of the cosmological evolution. The main results of the paper are summarized in Section 4.

2 Hadamard function in de Sitter spacetime with a compact spatial dimension

We consider a quantum scalar field with curvature coupling parameter $\xi$ on background of the $(D+1)$-dimensional de Sitter spacetime. We will write the corresponding line element in the form most appropriate for cosmological applications:

$$ ds^2 = dt^2 - e^{2t/\alpha} \sum_{i=1}^{D} (dz^i)^2. $$

The parameter $\alpha$ is related to the Ricci scalar and the corresponding cosmological constant by the formulæ

$$ R = D(D+1)/\alpha^2, \quad \Lambda = D(D-1)/2\alpha^2. $$

For the further discussion, in addition to the synchronous time coordinate $t$ it is convenient to introduce the conformal time in accordance with

$$ \eta = \alpha e^{-t/\alpha}, \quad 0 < \eta < \infty. $$
In terms of this coordinate the line element takes conformally flat form:

\[ ds^2 = \alpha^2 \eta^{-2} [d\eta^2 - \sum_{i=1}^{D} (dz^i)^2]. \]

We will assume that the spatial coordinate \( z^D \) is compactified to \( S^1: 0 \leq z^D \leq L \). This geometry is a limiting case of toroidally compactified dS spacetime discussed in \( [11] \) and can be used to describe two types of models. For the first one \( D = 4 \) and it corresponds to the universe with Kaluza-Klein type single extra dimension. As it will be shown the presence of extra dimension generates an additional gravitational source in the cosmological equations which is of barotropic type at late stages of the cosmological evolution. For the second model \( D = 3 \) and the results given below describe how the properties of the universe with dS geometry are changed by one-loop quantum effects induced by the compactness of a single spatial dimension.

The field equation has the form

\[ \left( \nabla_I \nabla^I + m^2 + \xi R \right) \varphi = 0, \]

where \( \nabla_I \) is the covariant derivative operator associated with line element \( (4) \). The values of the curvature coupling parameter \( \xi = 0 \) and \( \xi = \xi_D \equiv (D-1)/4D \) correspond to the most important special cases of minimally and conformally coupled fields. For a scalar field with periodic boundary condition one has \( \varphi(\eta, z^D + L) = \varphi(\eta, z^D) \), where \( z = (z^1, \ldots, z^{D-1}) \) is the set of uncompactified dimensions. In this paper we are interested in the effects of non-trivial topology on the VEVs of the field square and the energy-momentum tensor. This VEVs are obtained from the corresponding Hadamard function in the coincidence limit of the arguments.

To evaluate the Hadamard function we employ the mode-sum formula

\[ G^{(1)}(x, x') = \langle 0|\varphi(x)\varphi(x') + \varphi(x')\varphi(x)|0\rangle = \sum_{\sigma} \left[ \varphi_{\sigma}(x)\varphi_{\sigma}^*(x') + \varphi_{\sigma}(x')\varphi_{\sigma}^*(x) \right], \]

where \( \{\varphi_{\sigma}(x), \varphi_{\sigma}^*(x)\} \) is a complete set of positive and negative frequency solutions to the classical field equations and satisfying the periodicity condition along the \( z^D \)-direction. The collective index \( \sigma \) specifies these solutions. For the problem under consideration the eigenfunctions have the form

\[ \varphi_{\sigma}(x) = C_{\sigma} \eta^{D/2} H^{(2)}(k_n \eta) e^{i k_\perp x + 2 i \pi n z^D/L}, \quad n = 0, \pm 1, \pm 2, \ldots, \]

with the notations \( k = (k_1, \ldots, k_{D-1}) \), \( k = |k| \), and

\[ k_n = \sqrt{k^2 + (2\pi n/L)^2}, \quad \nu = \left[ D^2/4 - D(D + 1)\xi - m^2\alpha^2 \right]^{1/2}. \]

In \( (7) \) \( H_{\nu}(x) \) is the Hankel function. The coefficient \( C_{\sigma} \) with \( \sigma = (k, n) \) is found from the orthonormalization condition

\[ -i \int d^D x \sqrt{|g|} g^{00} [\varphi_{\sigma}(x) \partial_0 \varphi_{\sigma}^*(x) - \varphi_{\sigma}^*(x) \partial_0 \varphi_{\sigma}(x)] = \delta_{\sigma \sigma'}, \]

where the integration goes over the spatial hypersurface \( \eta = \text{const} \), and \( \delta_{\sigma \sigma'} \) is understood as the Kronecker delta for discrete indices and as the Dirac delta-function for continuous ones. This leads to the result

\[ C_{\sigma}^2 = \frac{\alpha^{1-D} e^{-i(\nu - \nu^*)\pi/2}}{2^{D+1} \pi^{D-2} L}. \]
Substituting the eigenfunctions (7) with the normalization coefficient (10) into the mode-sum formula for the Hadamard function, one finds
\[
G^{(1)}(x, x') = \frac{\alpha^{1-D}(\eta \eta')^{D/2}}{2D\pi^{D-2}L} \int dk e^{ik\Delta z} \sum_{n=0}^{\infty} \cos(2\pi n \Delta z D/L) \times \left[ H^{(2)}(k_n \eta) H^{(1)}(k_n \eta') + H^{(2)}(k_n \eta') H^{(1)}(k_n \eta) \right],
\]
where \(\Delta z D = z^D - z'^D\). For the further evaluation we apply to the series over \(n\) the Abel-Plana summation formula [13, 15]
\[
\sum_{n=0}^{\infty} f(n) = \int_0^{\infty} dx f(x) + i \int_0^{\infty} dx \frac{f(ix) - f(-ix)}{e^{2\pi x} - 1},
\]
where the prime on the summation sign means that the term \(n = 0\) should be halved. This enables us to present the Hadamard function in the decomposed form
\[
G^{(1)}(x, x') = G^{(1)}_0(x, x') + G^{(1)}_c(x, x'),
\]
with
\[
G^{(1)}_0(x, x') = \frac{(\eta \eta')^{D/2}}{2D+2\pi D-1\alpha^{D-1}} \int dk_D e^{ik_D \cdot \Delta z D} \times \left[ H^{(2)}(k_D \eta) H^{(1)}(k_D \eta') + H^{(2)}(k_D \eta') H^{(1)}(k_D \eta) \right],
\]
being the Hadamard function for the uncompactified dS spacetime. In formula (13), the second term on the right is induced by the compactness of the \(z^D\)-direction and is given by the formula
\[
G^{(1)}_c(x, x') = \frac{2(\eta \eta')^{D/2}}{(2\pi)^D \alpha^{D-1}} \int dk e^{ik \Delta z} \int_0^{\infty} dx \frac{x \cosh(\sqrt{x^2 + k^2 \Delta z D})}{\sqrt{x^2 + k^2 (e^{L\sqrt{x^2 + k^2}} - 1)}} \times \left\{ [I_{-\nu}(x \eta') + I_{\nu}(x \eta')] K_{\nu}(x \eta) + [I_{-\nu}(x \eta) + I_{\nu}(x \eta)] K_{\nu}(x \eta') \right\}.
\]
Note that in this formula the integration with respect to the angular part of \(k\) can be done explicitly. Two-point function in the uncompactified dS spacetime given by formula (14) can be expressed in terms of the hypergeometric function and is investigated in [3, 6, 7, 8] (see also [16]).

3 Vacuum expectation values of the field square and the energy-momentum tensor

3.1 Field square

The VEV of the field square is obtained from the two-point function \(G^{(1)}(x, x')\) taking the coincidence limit of the arguments. In this limit the Hadamard function diverges and some renormalization procedure is necessary. The important point here is that the divergences are contained in the part corresponding to the uncompactified dS spacetime and the topological part is finite. As we have already extracted the first part, the renormalization procedure is reduced to the renormalization of the uncompactified dS part which is already done in literature. As a result the renormalized VEV of the field square is presented in the form
\[
\langle \varphi^2 \rangle_{\text{ren}} = \langle \varphi^2 \rangle_{\text{0, ren}} + \langle \varphi^2 \rangle_{\text{c}},
\]
(16)
where in the case $D = 3$ the renormalized VEV for the uncompactified dS space is given by the formula [5, 7, 8]

$$
\langle \varphi^2 \rangle_{0,\text{ren}} = \frac{1}{8\pi^2\alpha^2} \left\{ (m^2\alpha^2/2 + 6\xi - 1) \left[ \psi\left(\frac{3}{2} + \nu\right) + \psi\left(\frac{3}{2} - \nu\right) - \ln\left(m^2\alpha^2\right) \right] 
- \frac{(6\xi - 1)^2}{m^2\alpha^2} + \frac{1}{30m^2\alpha^2} - 6\xi + \frac{2}{3} \right\},
$$

(17)

where $\psi(x)$ is the logarithmic derivative of the gamma-function. Due to the maximal symmetry of the dS spacetime this VEV does not depend on the spacetime point.

The second term on the right of Eq. (16) is the part due to the compactness of the $z^D$-direction. This part is directly obtained from (15) taking the coincidence limit of the arguments:

$$
\langle \varphi^2 \rangle_c = \frac{\alpha^{1-D} \eta^D}{2^{D-2} \pi^{D/2} \Gamma\left(D/2\right) \int_0^\infty dk D^{D-2} \times \int_0^\infty dx \frac{x[I_{-\nu}(x\eta) + I_{\nu}(x\eta)]}{\sqrt{x^2 + k^2 (e^{Lx/\eta} + k^2 - 1)}} K_\nu(x\eta).
$$

(18)

Here $I_\nu(x)$ and $K_\nu(x)$ are the modified Bessel functions. Introducing a new integration variable $y = \sqrt{x^2 + k^2}$ and expanding $(e^{Lx/\eta} - 1)^{-1}$, the integral over $y$ is easily evaluated and one finds

$$
\langle \varphi^2 \rangle_c = \frac{\alpha^{1-D}}{2^{D/2-1} \pi^{D/2+1}} \sum_{n=1}^\infty \int_0^\infty dx \frac{x}{D^{D-1}} \left[I_{-\nu}(x) + I_{\nu}(x)\right] K_\nu(x) \frac{K_{D/2-1}(nLx/\eta)}{(nLx/\eta)^{D/2-1}}.
$$

(19)

The integral in this formula can be expressed in terms of the hypergeometric function $\,_4F_3$. By taking into account the relation between the conformal and synchronous time coordinates, we see that the VEV of the field square is a function of the combination $L/\eta = L_e^{t/\alpha}\alpha$. For a conformally coupled massless scalar field one has $\nu = 1/2$ and $[I_{-\nu}(x) + I_{\nu}(x)] K_\nu(x) = 1/x$. The corresponding integral in (19) is explicitly evaluated and we find

$$
\langle \varphi^2 \rangle_c = \zeta_R(D-1) \left(\frac{\eta}{\alpha L}\right)^{D-1} \frac{D-1}{\Gamma\left(D/2\right)}, \quad \xi = \xi_D, \quad m = 0,
$$

(20)

where $\zeta_R(x)$ is the Riemann zeta function. We could obtain this result directly from the corresponding result in the Minkowski spacetime compactified along the direction $z^D$, by using the fact that two problems are conformally related: $\langle \varphi^2 \rangle_c = (\eta/\alpha)^{D-1}\langle \varphi^2 \rangle_c^{(\text{M})}$.

Let us consider the behavior of the topological part $\langle \varphi^2 \rangle_c$ in the VEV of the field square in the asymptotic regions of the ratio $L/\eta$. For small values of this ratio, $L/\eta \ll 1$, we introduce a new integration variable $y = Lx/\eta$. By taking into account that for large values $x$ one has $[I_{-\nu}(x) + I_{\nu}(x)] K_\nu(x) \approx 1/x$, we find that to the leading order $\langle \varphi^2 \rangle_c$ coincides with the corresponding result for a conformally coupled massless field, given by (20):

$$
\langle \varphi^2 \rangle_c \approx \zeta_R(D-1) \left(\frac{\eta}{\alpha L}\right)^{D-1} \frac{D-1}{\Gamma\left(D/2\right)}, \quad L/\eta \ll 1.
$$

(21)

In the opposite limit of large values for the ratio $L/\eta$, again we introduce a new integration variable $y = Lx/\eta$ and expand the integrand. For real values $\nu$ to the leading order we find

$$
\langle \varphi^2 \rangle_c \approx \frac{\eta/L}{}^{D-2\nu} \Gamma(\nu) \zeta_R(D-2\nu) \Gamma(D/2 - \nu), \quad \eta/L \ll 1.
$$

(22)
For pure imaginary values $\nu$ by a similar way we can see that

$$
\langle \varphi^2 \rangle_c \approx \frac{\alpha^{1-D}(\eta/L)^D}{\pi^{D/2} \sinh(\nu/\pi)} \Im \left[ \frac{\Gamma(D/2 - i|\nu|)}{\Gamma(1 - i|\nu|)} \zeta_R(D - 2i|\nu|)(L/\eta)^{2|\nu|} \right], \ \eta/L \ll 1. \quad (23)
$$

By taking into account relation (3) between synchronous and conformal time coordinates, the corresponding formula can also be written in the form

$$
\langle \varphi^2 \rangle_c \approx \frac{\alpha a_D(|\nu|)e^{-Dt/\alpha}}{\pi^{D/2} \sinh(|\nu|\pi)\alpha^D} \sin [2|\nu|t/\alpha + 2|\nu|\ln(L/\alpha) + \phi_0], \ t \gg \alpha. \quad (24)
$$

where $a_D(|\nu|)$ and $\phi_0$ are defined by the relation

$$
\frac{\Gamma(D/2 - i|\nu|)}{\Gamma(1 - i|\nu|)} \zeta_R(D - 2i|\nu|) = a_D(|\nu|)e^{i\phi_0}. \quad (25)
$$

As we see, unlike to the case of real $\nu$, here the damping of the VEV has an oscillatory nature.

In figure 1 we have plotted $\langle \varphi^2 \rangle_c$ as a function of the ratio $L/\eta$ for conformally (left panel) and minimally (right panel) coupled fields in $D = 3$. The numbers near the curves correspond to the value of the parameter $m\alpha$. Note that in the case $m\alpha = 1$ for a conformally coupled scalar the parameter $\nu$ is pure imaginary and the corresponding oscillatory behavior is seen in the region $L/\eta > 7$ (the first zero is at $L/\eta \approx 7.28$). Similarly, for a minimally coupled scalar field the parameter $\nu$ is pure imaginary in the case $m\alpha = 2$ and the first zero of $\langle \varphi^2 \rangle_c$ is at $L/\eta \approx 4.12$.

![Figure 1: The topological part in the VEV of the field square for $D = 3$ conformally (left panel) and minimally (right panel) coupled fields, $\alpha^{D-1}\langle \varphi^2 \rangle_c$, as a function of the ratio $L/\eta$. The numbers near the curves correspond to the values of the parameter $m\alpha$.](image)

3.2 Energy-momentum tensor

Now we turn to the investigation of the VEV for the energy-momentum tensor. Having the Hadamard function we can evaluate this VEV by making use of the formula

$$
\langle 0|T_{ik}|0 \rangle = \frac{1}{2} \lim_{x' \rightarrow x} \partial_i \partial_k G^{(1)}(x, x') + \left[ \left( \xi - \frac{1}{4} \right) g_{ik} \nabla_l \nabla^l - \xi \nabla_i \nabla_k - \xi R_{ik} \right] \langle 0|\varphi^2|0 \rangle, \quad (26)
$$
where $R_{ik} = Dg_{ik}/\alpha^2$ is the Ricci tensor for the dS spacetime. Similar to the case of the field square, the renormalized VEV of the energy-momentum tensor is presented as the sum

$$\langle T^k_l \rangle_{\text{ren}} = \langle T^k_l \rangle_{0,\text{ren}} + \langle T^k_l \rangle_c,$$

where $\langle T^k_l \rangle_{0,\text{ren}}$ is the part corresponding to the uncompactified dS spacetime and $\langle T^k_l \rangle_c$ is induced by the compactness along the $z^D$ - direction. For $D = 3$ the standard dS part is given by the formula [3, 7, 8] (see also [16])

$$\langle T^k_l \rangle_{0,\text{ren}} = \frac{\delta^k_l}{32\pi^2 \alpha^4} \left\{ \frac{m^2 \alpha^2}{2} \left( \frac{m^2 \alpha^2}{2} + 6 \xi - 1 \right) \left[ \psi \left( \frac{3}{2} + \nu \right) + \psi \left( \frac{3}{2} - \nu \right) - \ln \left( m^2 \alpha^2 \right) \right] 
- (6\xi - 1)^2 + 1/30 + (2/3 - 6\xi) m^2 \alpha^2 \right\},$$

(28)

By using the asymptotic expansion of the function $\psi(x)$ for large values of the argument it can be seen that for large values of the parameter $m\alpha$ from (28) one has

$$\langle T^k_l \rangle_{0,\text{ren}} \approx \frac{\delta^k_l}{32\pi^2 m^2 \alpha^6} \left( \frac{7}{12} \frac{58\xi}{5} + 72\xi^2 - 144\xi^3 \right), \quad m\alpha \gg 1.$$

(29)

For a conformally coupled scalar field the coefficient in braces is equal $-1/60$. The energy-momentum tensor (28) is a gravitational source of the cosmological constant type. Due to the problem symmetry this will be the case for general values $D$. As a result, in combination with the initial cosmological constant $\Lambda$ given by (2), the one-loop effects lead to the effective cosmological constant (for a recent discussion of the one-loop topological contributions to the cosmological constant see [17, 18])

$$\Lambda_{\text{eff}} = D(D - 1)/2\alpha^2 + 8\pi G \langle T^0_0 \rangle_{0,\text{ren}},$$

(30)

where $G$ is the Newton gravitational constant. In figure [2] we have plotted the renormalized vacuum energy density in the uncompactified dS spacetime as a function of the parameter $m\alpha$ for $D = 3$ minimally and conformally coupled scalar fields. As it is seen from the plots, the one-loop correction to the cosmological constant for a minimally coupled scalar field is always positive, whereas for a conformally coupled scalar it can be also negative. Below we will see that the energy density induced by the compactness of a spatial dimension is not of cosmological constant type and corresponds to an additional source in the cosmological equations which is of barotropic type at late stages of the cosmological evolution.

The topological part in the VEV of the energy-momentum tensor is obtained substituting the function [15] into formula (26). After lengthy but straightforward calculations this leads to the result

$$\langle T^k_l \rangle_c = \frac{2\delta^k_l}{(2\pi)^{D/2+1} \alpha^{D+1}} \sum_{n=1}^{\infty} \int_0^\infty dx x^{D-1} \frac{K_{D/2-1}(nLx/\eta)}{(nLx/\eta)^{D/2-1}} F^{(l)}(x)$$

$$+ \frac{4F^{(l)}_0}{(2\pi)^{D/2+1} \alpha^{D+1}} \sum_{n=1}^{\infty} \int_0^\infty dx x^{D+1} [I_{-\nu}(x) + I_{\nu}(x)] K_{\nu}(x) \frac{K_D(nLx/\eta)}{(nLx/\eta)^{D/2}}.$$  

(31)

where we have used the notations

$$F^{(0)}_0 = 0, \quad F^{(\beta)}_0 = -1, \quad F^{(D)}_0 = (D - 1), \quad \beta = 1, \ldots, D - 1,$$

(32)
are obtained from (31) after the integration by parts:

An alternative expressions for the topological part of the VEV of the energy-momentum tensor with the notations

Figure 2: Renormalized vacuum energy density in uncompactified dS spacetime, $\eta \alpha^{D+1}(T_0^{\text{ren}})$ as a function of $m\alpha$ for minimally and conformally coupled scalar fields in $D = 3$. The scaling coefficient $\eta = 10^3(10^4)$ for minimally (conformally) coupled scalar fields.

and

$$F^{(0)}(x) = (I_{-\nu}(x) + I_{\nu}(x)) K_{\nu}(x) (\nu^2 + 2m^2\alpha^2 - x^2) + x^2 [I'_{-\nu}(x) + I'_{\nu}(x)] K_{\nu}'(x)$$

$$+ D(1/2 - 2\xi)x [(I_{-\nu}(x) + I_{\nu}(x))K_{\nu}(x)]',$$  \hspace{1cm} (33)

$$F^{(\beta)}(x) = (4\xi - 1) [I_{-\nu}(x) + I_{\nu}(x)] K_{\nu}(x) (x^2 + \nu^2) + (4\xi - 1) x^2 K_{\nu}'(x),$$

$$\times [I'_{-\nu}(x) + I'_{\nu}(x)] + [2(D + 1)\xi - D/2] x [(I_{-\nu}(x) + I_{\nu}(x))K_{\nu}(x)]',$$  \hspace{1cm} (34)

$$F^{(D)}(x) = 2x^2 K_{\nu}(x) [I_{-\nu}(x) + I_{\nu}(x)] + F^{(1)}(x).$$  \hspace{1cm} (35)

An alternative expressions for the topological part of the VEV of the energy-momentum tensor are obtained from (31) after the integration by parts:

$$\langle T^k_l \rangle_c = \frac{4\alpha^{-D} \delta^k_l}{(2\pi)^{D/2+1}} \sum_{n=1}^\infty \int_0^\infty dx \ x^{D-1} \ [I_{-\nu}(x) + I_{\nu}(x)] K_{\nu}(x) f^{(l)}(x, nLx/\eta),$$  \hspace{1cm} (36)

with the notations

$$f^{(0)}(x, u) = \frac{K_{D/2+1}(u)}{4u^{D/2-3}} - \left(D\xi + \frac{D + 2}{4}\right) \frac{K_{D/2}(u)}{u^{D/2-2}}$$

$$+ \left( m^2\alpha^2 - x^2 + D\xi \right) \frac{K_{D/2-1}(u)}{u^{D/2-1}},$$  \hspace{1cm} (37)

$$f^{(\beta)}(x, u) = \left(\xi - 1/4\right) \frac{K_{D/2+1}(u)}{u^{D/2-3}} - x^2 \frac{K_{D/2}(u)}{u^{D/2}} - \xi D \frac{K_{D/2-1}(u)}{u^{D/2-1}}$$

$$- \left[ (D + 1)\xi - \frac{D + 2}{4}\right] \frac{K_{D/2}(u)}{u^{D/2-2}}, \ \beta = 1, \ldots, D - 1,$$  \hspace{1cm} (38)

$$f^{(D)}(x, u) = (D - 1) x^2 \frac{K_{D/2}(u)}{u^{D/2}} + (x^2 - \xi D) \frac{K_{D/2-1}(u)}{u^{D/2-1}} + (\xi - 1/4) \frac{K_{D/2+1}(u)}{u^{D/2-3}}$$

$$- \left[ (D + 1)\xi - \frac{D + 2}{4}\right] \frac{K_{D/2}(u)}{u^{D/2-2}}.$$  \hspace{1cm} (39)
Note that the equivalent form \([36]\) for the VEV of the energy-momentum tensor can also be directly obtained from formula \((26)\) introducing a new integration variable \(y = Lx/\eta\) in \([19]\) before applying the differential operator. As it is seen from the obtained formulae, the topological parts in the VEVs are time-dependent and, hence, the local dS symmetry is broken by them.

It can be checked that the topological terms in the VEVs satisfy the trace relation

\[
\langle T^l_i \rangle_c = D(\xi - \xi_D)\nabla^l \langle \varphi^2 \rangle_c + m^2 \langle \varphi^2 \rangle_c.
\]  

In particular, from here it follows that the topological part is traceless for a conformally coupled massless scalar field. The trace anomaly is contained in the uncompactified dS part only. Of course, we could expect this result, as the trace anomaly is determined by the local geometry and the local geometry is not changed by the compactification. For a conformally coupled massless scalar field, similar to the case of the field square we find (no summation over \(l\))

\[
\langle T^l_i \rangle_c = -\frac{\zeta_R(D + 1)}{\pi^{(D+1)/2}} \left( \frac{\eta}{\alpha L} \right)^{D+1} \Gamma \left( \frac{D + 1}{2} \right), \ l = 0, 1, \ldots, D - 1,
\]

\[
\langle T^l_D \rangle_c = -D\langle T^0_0 \rangle_c, \ \xi = \xi_D, \ m = 0.
\]

Again, this result can be directly obtained by using the conformal relation between the problem under consideration and the corresponding problem in the Minkowski spacetime.

Now we turn to the investigation of the topological part in the asymptotic regions of the parameters. For small values of the ratio \(L/\eta\) we can see that to the leading order \(\langle T^k_i \rangle_c\), coincides with the corresponding result for a conformally coupled massless field given by \([31], [42]\):

\[
\langle T^k_i \rangle_c \approx \langle T^k_i \rangle_c(\xi = \xi_D, m = 0), \ L/\eta \ll 1.
\]

In terms of the synchronous time coordinate this limit corresponds to \(L e^{\nu/\alpha} \ll \alpha\). Now we see that in this limit the topological part dominates the effective cosmological constant given by \([30]\) and the back-reaction of one-loop quantum effects is important. From formulae \((41), (42)\) it follows that \(\langle T^0_0 \rangle_c < 0, \langle T^0_0 \rangle_c - \langle T^D_D \rangle_c < 0, \langle T^0_0 \rangle_c - \sum_{i=1}^{D} \langle T^i_i \rangle_c < 0,\) and this part violates all (weak, null, strong, dominant) energy conditions.

For large values of the ratio \(L/\eta\) and in the case of real \(\nu\) the part \(\langle T^k_i \rangle_c\) has the leading behavior given by the formula (no summation over \(l\))

\[
\langle T^l_i \rangle_c \approx \frac{\alpha^{-1-D}}{2\pi^{D/2+1}} \left( \frac{\eta}{L} \right)^{D-2}\zeta_R(D - 2\nu)\Gamma(\nu)\Gamma(D/2 - \nu) f_0^{(l)}(\nu), \ \eta/L \ll 1,
\]

with the notations

\[
f_0^{(0)}(\nu) = 2\nu D \left( \xi - \frac{D - 2\nu}{4D} \right) + m^2 \alpha^2,
\]

\[
f_0^{(l)}(\nu) = -2\nu(D + 1 - 2\nu) \left[ \xi - \frac{D - 2\nu}{4(D + 1 - 2\nu)} \right], \ l = 1, 2, \ldots, D.
\]

In this limit the topological part corresponds to an effective gravitational source with barotropic equation of state. In the same limit and for imaginary \(\nu\), to the leading order we have the following asymptotic:

\[
\langle T^l_i \rangle_c \approx \frac{\alpha^{-1-D}(\eta/L)^D}{\pi^{D/2} \sinh(\nu/\pi)} \text{Im} \left[ \frac{\Gamma(D/2 - i|\nu|)}{\Gamma(1 - i|\nu|)} \zeta_R(D - 2i|\nu|)(\eta/\nu)^{2|\nu|} f_0^{(l)}(i|\nu|) \right], \ \eta/L \ll 1.
\]
As we see in both cases the leading terms in the vacuum stresses are isotropic. The latter relation can also be written in terms of the synchronous time coordinate:

\[
\langle T^t_t \rangle_c \approx \frac{a^{(l)}_D(|\nu|) e^{-Dt/\alpha}}{\pi^{D/2} \sinh(|\nu|\pi)\alpha L^D} \sin \left[ 2|\nu| t/\alpha + 2|\nu| \ln(L/\alpha) + \phi_0^{(l)} \right], \ t \gg \alpha,
\]  

(48)

where \(a^{(l)}_D\) and \(\phi_0^{(l)}\) are defined by the formula

\[
\frac{\Gamma(D/2 - i|\nu|)}{\Gamma(1 - i|\nu|)} \zeta_R(D - 2i|\nu|) f_0^{(l)}(i|\nu|) = a^{(l)}_D(|\nu|) e^{i\phi_0^{(l)}}.
\]  

(49)

In figure 3 we have plotted \(\langle T^0_0 \rangle_c\) as a function of the ratio \(L/\eta\) for conformally (left panel) and minimally (right panel) coupled fields in \(D = 3\). The numbers near the curves correspond to the value of the parameter \(m\alpha\). The oscillatory behavior of the energy density in the case \(m\alpha = 1\) for a conformally coupled scalar and in the case \(m\alpha = 2\) for a minimally coupled scalar can be seen for larger values of \(L/\eta\).

![Figure 3: The part in the VEV of the energy density for conformally (left panel) and minimally (right panel) coupled fields in \(D = 3\) induced by the compactness of a spatial dimension, \(\alpha^{D+1}\langle T^0_0 \rangle_c\), as a function of the ratio \(L/\eta\). The numbers near the curves correspond to the values of the parameter \(m\alpha\).](image)

4 Conclusion

Compactified spatial dimensions appear in various physical models including Kaluza-Klein type theories, supergravity, string theory and cosmology. Motivated by the fact that dS spacetime plays an important role in all these fields, in this paper we investigate the quantum vacuum effects in dS spacetime induced by non-trivial topology of one of spatial dimensions. We consider a scalar field with general curvature coupling parameter satisfying the periodic boundary condition along the compactified dimension. Among the most important characteristics of the vacuum are the VEVs of the field square and the energy-momentum tensor. Though the corresponding operators are local, due to the global nature of the vacuum these VEVs carry an important information on the global structure of the background spacetime.
In order to derive formulae for the VEVs of the square of the field operator and the energy-momentum tensor, we first construct the Hadamard function. The application of the Abel-Plana summation formula enabled us to extract from this function the part corresponding to the Hadamard function for the uncompactified de Sitter spacetime. As the topological part is finite in the coincidence limit, by this way the renormalization procedure is reduced to that for the standard dS case. The latter was already realized in literature. As a result the VEVs of the field square and the energy-momentum tensor are decomposed into dS and topological parts. Due to the maximal symmetry of dS spacetime the first one does not depend on the spacetime point and is given by formula (17) in the case of the field square and by formula (28) for the energy-momentum tensor. This is not the case for topological parts which are given by formulae (19) and (36) for the field square and the energy-momentum tensor respectively. In addition, the corresponding vacuum stresses along uncompactified and compactified dimensions are anisotropic. As a result, though the local geometry is not changed by the compactification, here we deal with the topologically induced anisotropy in the properties of the vacuum. Of course, this effect is well-known from the corresponding flat spacetime examples. The topological terms take a simple form, given by formulae (19), (20) for special case of a conformally coupled massless scalar field. These formulae can also be obtained from the corresponding flat spacetime results by using the conformal relation between the geometries.

The topological parts in the VEVs of the field square and the energy-momentum tensor are functions of the ratio \( L/\eta \). For general values of the curvature coupling parameter the corresponding formulae are simplified in the asymptotic regions of small and large values of this parameter. In the first case the leading terms in the VEVs are the same as those for a conformally coupled field and the topological parts behave like \( \exp[-(D - 1)t/\alpha] \) for the field square and as \( \exp[-(D + 1)t/\alpha] \) for the energy-momentum tensor. In this limit the topological part dominates the effective cosmological constant and the back-reaction of one-loop quantum effects is important. The corresponding energy-momentum tensor violates the energy conditions and can essentially change the cosmological dynamics. For large values of the ratio \( L/\eta \) the behavior of the topological parts is different for real and pure imaginary values of the parameter \( \nu \). In the first case these parts behave like \( \exp[-(D - 2\nu)t/\alpha] \), whereas in the second case the decay has an oscillatory nature \( e^{-Dt/\alpha} \sin (2|\nu| t/\alpha + \phi_1) \). In this limit the vacuum stresses are isotropic and the topological part corresponds to an effective gravitational source with barotropic equation of state.

As it was argued in [11], the models of a compact universe with nontrivial topology may play an important role in inflationary cosmology by providing proper initial conditions for inflation. In particular, it was shown that in this class of cosmological models the probability of inflation is not exponentially suppressed. The quantum creation of the universe having compact spatial topology is discussed in [13] and in [20] within the framework of various supergravity theories on non-simply connected backgrounds. Note that on manifolds with non-trivial topologies or non-zero curvature, in the evaluation of the vacuum energy no term-by-term cancellation occurs between the bosonic and fermionic degrees of freedom (see, for instance, [20] for a flat spacetime with non-trivial topology and [21] for the anti de Sitter bulk). In view of this it will be interesting to generalize the results of the present paper to other classes of the compactification, in particular, for the toroidal compactification, including higher spin fields. We will consider these topics in our future work.

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