PARTIAL REGULARITY AND LIOUVille THEOREMS FOR STABLE SOLUTIONS OF ANISOTROPIC ELLIPTIC EQUATIONS

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Abstract. We study the quasilinear elliptic equation
\[-Qu = e^u \quad \text{in} \quad \Omega \subset \mathbb{R}^N,\]
where the operator $Q$, known as the Finsler-Laplacian (or anisotropic Laplacian) operator, is defined by
\[Qu := \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(F(\nabla u)F_{\xi_i}(\nabla u)).\]
Here $F_{\xi_i} = \frac{\partial F}{\partial \xi_i}$ and $F : \mathbb{R}^N \to [0, +\infty)$ is a convex function of $C^2(\mathbb{R}^N \setminus \{0\})$ that satisfies certain assumptions. For a bounded domain $\Omega$ and for a stable weak solution of the above equation, we prove that the Hausdorff dimension of the singular set does not exceed $N - 10$. For the case of entire space, we apply Moser iteration arguments, established by Dancer-Farina and Crandall-Rabinowitz in the context, to prove Liouville theorems for stable solutions and for finite Morse index solutions in dimensions $N < 10$ and $2 < N < 10$, respectively. We also provide an explicit solution that is stable outside a compact set in two dimensions $N = 2$. In addition, we present similar Liouville theorems for the related equations with power-type nonlinearities.

1. Introduction and main results. We examine stable weak solutions of the quasilinear Finsler-Liouville equation
\[-Qu = e^u \quad \text{in} \quad \Omega,\]
where $\Omega$ is a subset of $\mathbb{R}^N$ and the operator $Q$ is defined by
\[Qu := \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(F(\nabla u)F_{\xi_i}(\nabla u)),\]

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where \( F_\xi := \frac{\partial F}{\partial \xi} \) and \( F : \mathbb{R}^N \to [0, +\infty) \) is a convex function of \( C^2(\mathbb{R}^N \setminus \{0\}) \) such that \( F(t\xi) = |t|F(\xi) \) for any \( t \in \mathbb{R} \) and \( \xi \in \mathbb{R}^N \). The above equation is a particular case of the quasilinear equation with nonlinearity \( f \in C^1(\mathbb{R}) \),

\[
-Qu = f(u) \quad \text{in} \quad \Omega.
\]

We assume that \( F(\xi) > 0 \) for any \( \xi \neq 0 \) and for such a function \( F \). We also assume that there exist constants \( 0 < a \leq b < \infty \) and \( 0 < \lambda \leq \Lambda < \infty \) such that

\[ a|\xi| \leq F(\xi) \leq b|\xi|, \quad (1.3) \]

and

\[ \lambda^2|V|^2 \leq F_\xi(\xi) V_i V_j \leq \Lambda|V|^2, \quad (1.4) \]

for any \( \xi \in \mathbb{R}^N \) and \( V \in \xi^\perp \) where \( \xi^\perp := \{V \in \mathbb{R}^N : \langle V, \xi \rangle = 0\} \). The operator \( \Omega \) is known as the anisotropic Laplacian or the Finsler-Laplacian operator in the literature. When \( F(\xi) = |\xi| \), that is the isotropic case, the operator \( \Omega \) becomes the classical Laplacian operator. As an anisotropic Laplacian, such operators have been studied vastly in the literature. In early twentieth century, Wulff [36] used such operators to study crystal shapes and minimization of anisotropic surface tensions. The operator \( \Omega \) is closely connected with a smooth, convex hypersurface in \( \mathbb{R}^N \), called the Wulff shape (or equilibrium crystal shape) of \( F \). The Wulff shape was introduced and studied by Wulff in [36]. In order to provide a few references in this context, Wang and Xia in [33] extended the classical result of Brezis and Merle [4] to equation (1.1) in two dimensions. See also [32] where the authors study an overdetermined problem for anisotropic equations. Caffarelli et al. in [5] established gradient estimates and monotonicity formulae for quasilinear equations in order to study entire solutions, see also [20]. Cozzi et al. in [8, 9] proved such estimates and formulae for singular, degenerate, anisotropic equations, see also [19, 28]. In addition, we refer interested readers to [6, 22, 31] and references therein in the context of quasilinear equations.

The definitions of weak and stable solutions of (1.2) are as follows.

**Definition 1.1.** We say that \( u \) is a weak solution of (1.2), if \( u \in H^1_{\text{loc}}(\Omega) \), \( f(u) \in L^1_{\text{loc}}(\Omega) \) and the following holds

\[
\int_\Omega F(\nabla u) F_\xi(\nabla u) \cdot \nabla \phi dx = \int_\Omega f(u) \phi dx,
\]

for all \( \phi \in C^\infty_c(\Omega) \).

**Definition 1.2.** We say that a weak solution of equation (1.2) is stable, if \( f'(u) \in L^1_{\text{loc}}(\Omega) \) and

\[
\int_\Omega F_\xi(\nabla u) F_{\xi j}(\nabla u) \phi_{x_i} \phi_{x_j} + F(\nabla u) F_\xi(\nabla u) \phi_{x_i} \phi_{x_j} - f'(u) \phi^2 dx \geq 0, \quad (1.5)
\]

for all \( \phi \in C^\infty_c(\Omega) \).

Here, we provide some properties of the operator \( \Omega \). Let \( F^0 \) be the support function of \( K := \{x \in \mathbb{R}^N : F(x) < 1\} \) which is defined by

\[ F^0(x) := \sup_{\xi \in K} \langle x, \xi \rangle. \]

Let \( B_r(x_0) := \{x \in \mathbb{R}^N : F^0(x - x_0) < r\} \) be the Wulff ball of radius \( r \) with center at \( x_0 \). Set \( \kappa_0 := |B_1(x_0)| \), where \( |B_1(x_0)| \) is the Lebesgue measure of \( B_1(x_0) \). By the assumptions on \( F \), one can see that the following properties hold. Some of these are discussed in detail in [22, 32].
Theorem A. We have the following properties:
1. \(|F(x) - F(y)| \leq F(x + y) \leq F(x) + F(y)|;
2. \(\|\nabla F(x)\| \leq C\) for any \(x \neq 0\);
3. \(\langle \xi, \nabla F(\xi) \rangle = F(\xi), \langle x, \nabla F_0(x) \rangle = F_0(x)\) for any \(x \neq 0, \xi \neq 0\);
4. \(\sum_{j=1}^{N} F_{\xi_j}(\xi) \xi_j = 0\), for any \(i = 1, 2, \ldots, N;\)
5. \(F(\nabla F_0(x)) = 1, F_0(\nabla F(x)) = 1;\)
6. \(F_0(t\xi) = \text{sgn}(t)F_0(\xi);\)
7. \(F_0(x)F_0(\nabla F_0(x)) = x.\)

Here, we provide definitions of the Hausdorff dimension and the singular set, see [26].

Definition 1.3. Let \(A\) be a subset of \(\mathbb{R}^N, 0 \leq s \leq \infty\) and \(0 \leq \delta \leq \infty\). Set
\[
H^s_\delta := \inf \left\{ \sum_{j=1}^{\infty} \alpha(s) \left( \frac{\text{diam}C_j}{2} \right)^s \left| A \subset \bigcup_{j=1}^{\infty} C_j, \text{diam}C_j \leq \delta \right. \right\},
\]
where \(\alpha(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2}+1)}, 0 \leq s < \infty\) and \(\Gamma(s)\) is the \(\Gamma\)-function. Let \(H^s\) be the \(s\)-dimensional Hausdorff measure that is defined as
\[
H^s(A) := \lim_{\delta \to 0} H^s_\delta(A) = \sup_{\delta > 0} H^s_\delta(A).
\]
The Hausdorff dimension of a set \(A \subset \mathbb{R}^N\) is defined as
\[
H_{\text{dim}}(A) := \inf \{ 0 \leq s < \infty | H^s(A) = 0 \}.
\]

Here is the definition of the singular set \(S\), see [34].

Definition 1.4. The singular set \(S\) of a solution \(u\) contain those point where in any neighborhood of this point \(u\) is not bounded, its complement is the regular set of \(u\).

Our main result addressing partial regularity of solutions of (1.1) is as follows.

Theorem 1.1. Let \(\Omega \subset \mathbb{R}^N\) be a bounded domain. Assume that for all \(x, y \in \mathbb{R}^N,\)
\[
\langle F_0(x), F_0(y) \rangle = \frac{\langle x, y \rangle}{F(x)F_0(y)}, \quad (1.6)
\]
If \(u\) is a stable weak solution of (1.1), then the Hausdorff dimension of the singular set \(S\) does not exceed \(N - 10\).

In order to provide an example for the above assumption (1.6), consider
\[
F(\xi) = \left[ \sum_{i=1}^{N} (\beta_i^2 \xi_i^2) \right]^\frac{1}{2} \quad \text{and} \quad F_0(\eta) = \left[ \sum_{i=1}^{N} (\eta_i^2 / \beta_i^2) \right]^\frac{1}{2},
\]
where \(\beta = (\beta_1, \ldots, \beta_N) \in \mathbb{R}^N\) be a constant vector.

When \(F(\xi) = |\xi|\), for the Laplacian operator, Da Lio [11] proved that the Hausdorff dimension of singular set of stationary solutions is at most 1 in dimension \(N = 3\). Wang [34, 35] showed that the Hausdorff dimension of singular set for stable weak solutions does not exceed \(N - 10\).

We now consider \(\Omega\) to be the entire space \(\mathbb{R}^N\). Here, we list our main results for such domains. The first result is a Liouville theorem for stable solutions.

Theorem 1.2. If \(N < 10\), then equation (1.1) does not admit any stable solution.
The following is a Liouville theorem for finite Morse index solutions.

**Theorem 1.3.** For $3 \leq N \leq 9$, under the assumption of (1.6), then equation (1.1) does not admit any stable outside a compact set of solution. If $N = 2$, then

$$u(x) = -2 \log(1 + \frac{1}{8} \lambda^2 F^0(x - x_0)^2) + 2 \log \lambda,$$

for some $\lambda > 0$ and $x_0 \in \mathbb{R}^2$, is stable outside a compact set of solution of (1.1).

When $F(\xi) = |\xi|$, Farina in [18] proved an analogues of Theorem 1.2 and Dancer with Farina in [12, 13] proved a counterpart of Theorem 1.3. The methods applied in here are the Moser iteration arguments developed in this context by Crandall and Rabinowitz [10].

For power-type nonlinearities, we prove the following Liouville theorem for stable solutions of (1.2). This is a counterpart of Theorem 1.2.

**Theorem 1.4.** The equation (1.2) does not admit positive stable weak solution if

(i) $f(u) = e^u$ for $p > 3$ and $N < \frac{6p\sqrt{p^2 - p - 2}}{p - 1}$.

(ii) $f(u) = -u^{-p}$ for $p > \frac{1}{3}$ and $N < \frac{6p\sqrt{p^2 + p + 2}}{p + 1}$.

When $F(\xi) = |\xi|$, the above result is given by Farina in [17] and Esposito et al. in [15, 16] for Part (i) and Part (ii), respectively.

This article is organized as follows. In Section 2, we first recall a well-known sharp anisotropic Hardy’s inequality. Then, we prove certain integral estimates using Moser iteration arguments. All of these inequalities are essential tools in next sections. In Section 3, we prove the partial regularity result, i.e., Theorem 1.1. In Section 4, we prove Liouville theorems for stable solutions and for finite Moser index solutions, i.e., Theorem 1.2, Theorem 1.3 and Theorem 1.4. In addition, we also show the existence of the finite Moser index solutions. In Section 5, we discuss monotonicity formulas which are of independent interests in this context.

2. **Integral estimates.** In this section, we provide some essential elliptic estimates and inequalities needed to establish our main results. We start with the following sharp anisotropic Hardy inequality, given in [27].

**Proposition 2.1.** Let $\Omega$ be a domain in $\mathbb{R}^N$ and assume $1 \leq s < N$ or $s > N$. Then, the following inequality

$$\left| \frac{N - s}{s} \int_\Omega (F(x))^s dx \right|^s \leq \frac{1}{\left( \frac{s}{N} \right)^s} \int_\Omega \left\| \nabla \varphi \right\|^s dx,$$

holds for any $\varphi \in C_0^\infty(\Omega)$ if $1 \leq s < N$, and for any $\varphi \in C_0^\infty(\Omega \setminus \{0\})$ if $s > N$.

We now prove some integral estimates for stable solutions. The methods and ideas are inspired by Moser iteration arguments given in [10, 17, 18] and references therein.

**Proposition 2.2.** Assume that $N \geq 2$ and $\Omega$ is a domain (possibly unbounded) of $\mathbb{R}^N$. Let $u$ be a stable weak solution of (1.2).

(i) If $f(u) = e^u$ then for any integer $m \geq 10$ and any $\alpha \in (0, 4)$, we have

$$\int_\Omega e^{(\alpha + 1)u^2} dx \leq C \int_\Omega (|\nabla \varphi|^2 + |\nabla \varphi|^4)^{\alpha + 1} dx.$$  

(2.1)
(ii) If \( f(u) = u^p \) with \( p > 3 \), then for any integer \( m \geq \frac{4\sqrt{p^2 - p + 6p - 2}}{p - 3} \) and \( p - \sqrt{p^2 - p} < \alpha < p + \sqrt{p^2 - p} \), we have
\[
\int_{\Omega} u^{p+2\alpha-1}\psi^{2m} \, dx \leq C \int_{\Omega} \left( |\nabla \psi|^{\frac{p}{p-\alpha}} + |\nabla \psi|^{\frac{p}{p+\alpha}} \right)^{2\alpha - 1} \, dx. \tag{2.2}
\]

(iii) If \( f(u) = -u^{-p} \) with \( p > \frac{1}{2} \), then for any integer \( m \geq \max\{ \frac{2p+2}{p+1}, \frac{6p+4}{p+2} \} \) and \( 1 < \alpha < p + \sqrt{p^2 + p}, \) we have
\[
\int_{\Omega} u^{-2\alpha - p - 1}\psi^{2m} \, dx \leq C \int_{\Omega} \left( |\nabla \psi|^{\frac{p}{p-\alpha}} + |\nabla \psi|^{\frac{p}{p+\alpha}} \right)^{2\alpha + p + 1} \, dx. \tag{2.3}
\]

Here, \( \psi \in C^1_c(\Omega) \) is a test function satisfying \( 0 \leq \psi \leq 1 \) in \( \Omega \).

Proof. (i) If \( f(u) = e^u \), for any \( \alpha \in (0, 4) \) and any \( k > 0 \), we set
\[
a_k(t) = \begin{cases} 
  e^{\frac{\alpha}{2} t}, & \text{if } t < k, \\
  |\alpha| (t - k) + 1)e^{\frac{\alpha}{2} t}, & \text{if } t \geq k,
\end{cases}
\]
and
\[
b_k(t) = \begin{cases} 
  e^{\alpha t}, & \text{if } t < k, \\
  |\alpha| (t - k) + 1)e^{\alpha t}, & \text{if } t \geq k.
\end{cases}
\]

Straightforward calculations yield
\[
a_k^2(t) \geq b_k(t), \quad (a_k'(t))^2 = \frac{\alpha}{4} b_k'(t), \tag{2.4}
\]
and
\[
(a_k'(t))^2 \leq c_2 e^{\alpha t}, \quad (a_k(t))^2 \leq e^{\alpha t}, \quad (b_k(t))^{-1} b_k(t) \leq c_2 e^{\alpha t}, \tag{2.5}
\]
for some positive constant \( c_1 \) and \( c_2 \) which depends only on \( \alpha \). For any \( \phi \in C^1_c(\Omega) \), multiply (1.2) with test function \( b_k(u)\phi^2 \) and integrate by parts. It follows from properties (2) and (3) of Theorem A that
\[
\int_{\Omega} F(\nabla u) F_{\xi_i}(\nabla u) \partial_{x_i} (b_k(u) \phi^2) \, dx \\
= \int_{\Omega} F(\nabla u) F_{\xi_i}(\nabla u) b_k(u) u_x \phi + F(\nabla u) F_{\xi_i}(\nabla u) b_k(u) 2\phi \phi_x, \, dx \\
= \int_{\Omega} F^2(\nabla u) b_k(u) \phi^2 + F(\nabla u) F_{\xi_i}(\nabla u) b_k(u) 2\phi \phi_x, \, dx \\
= \int_{\Omega} e^{\alpha} b_k(u) \phi^2 \, dx.
\]

It follows that
\[
\int_{\Omega} F^2(\nabla u) b_k(u) \phi^2 \, dx \leq 2C \int_{\Omega} F(\nabla u) b_k(u) |\phi||\nabla \phi| \, dx + \int_{\Omega} e^{\alpha} b_k(u) \phi^2 \, dx.
\]

For the Cauchy inequality, we have
\[
\int_{\Omega} F^2(\nabla u) b_k(u) \phi^2 \, dx \leq \frac{2C}{(1 - 2Ce^\alpha)^2} \int_{\Omega} (b_k(u))^{-1} b_k^2(u) |\nabla \phi|^2 \, dx + \frac{1}{1 - 2Ce^\alpha} \int_{\Omega} e^{\alpha} b_k(u) \phi^2 \, dx. \tag{2.6}
\]

Since \( u \) is stable solution of equation (1.2), for any \( \varphi \in C^1_c(\Omega) \), we have
\[
\int_{\Omega} F_{\xi_i}(\nabla u) F_{\xi_j}(\nabla u) \phi_{x_i} \phi_{x_j} + F(\nabla u) F_{\xi_i \xi_j}(\nabla u) \phi_{x_i} \phi_{x_j} - e^{\alpha} \phi^2 \, dx \geq 0. \tag{2.7}
\]
Set \( \varphi = a_k(u) \phi \), in the light of \( \varphi_x = a_k'(u)u_x, \phi + a_k(u)\phi_x \). Applying (1.4) and (2) and (3) of Theorem A with Cauchy inequality, we deduce that

\[
\int_\Omega F(\nabla u)F(\nabla u)(\nabla u)_{x_i} \varphi_x \, dx
\]

and

\[
\int_\Omega (1 + 2C\varepsilon_1)F^2(\nabla u)(a_k'(u))^2 \phi^2 + (C^2 + \frac{2C}{\varepsilon_1})(a_k(u))^2 |\nabla \phi|^2 \, dx.
\]

Hence, we have

\[
\int_\Omega (1 + 2C\varepsilon_1)F^2(\nabla u)(a_k'(u))^2 \phi^2 + (C^2 + \frac{2C}{\varepsilon_1})(a_k(u))^2 |\nabla \phi|^2 \, dx,
\]

and

\[
\int_\Omega F(\nabla u)F(\nabla u)(\nabla u)_{x_i} \varphi_x \, dx
\]

and

\[
\int_\Omega (1 + 2C\varepsilon_1)F^2(\nabla u)(a_k'(u))^2 \phi^2 + (C^2 + \frac{2C}{\varepsilon_1})(a_k(u))^2 |\nabla \phi|^2 \, dx.
\]

Combining (2.4)-(2.9) to obtain

\[
\int_\Omega e^u(a_k(u))^2 \phi^2 \, dx \leq \frac{\alpha(1 + 2C\varepsilon_1 + \Lambda\varepsilon_2)}{4(1 - 2C\varepsilon)} \int_\Omega e^u(a_k(u))^2 \phi^2 \, dx + \frac{\Lambda}{\varepsilon_2} \int_\Omega e^u|\nabla \phi|^4 \phi^2 \, dx
\]

where \( C_1 \) and \( C_2 \) are positive constants and independent of \( k \). Then, letting \( k \to +\infty \), by Fatou’s lemma, we get

\[
\int_\Omega e^{(\alpha+1)u} \phi^2 \, dx \leq C_1 \int_\Omega e^{\alpha u} |\nabla \phi|^4 \phi^2 \, dx + C_2 \int_\Omega e^{\alpha u} |\nabla \phi|^2 \, dx.
\]

Let \( \phi = \psi^m \) and \( 0 \leq \psi \leq 1 \), by young’s inequality, we obtain

\[
\int_\Omega e^{(\alpha+1)u} \phi^2 \, dx = \int_\Omega e^{(\alpha+1)u} \psi^{2m} \, dx
\]

\[
\leq \tilde{C}_1 \varepsilon \int_\Omega e^{(\alpha+1)u} \psi^{2m} \, dx + \tilde{C}_1 \varepsilon \int_\Omega (|\psi|^{2m - 2 - 2m - 2m - 2m - 4 + 2m + 2m - 4 + 2m - 4} \phi^2)^{\alpha+1} \, dx
\]

\[
+ \tilde{C}_2 \varepsilon \int_\Omega e^{(\alpha+1)u} \psi^{2m} \, dx + \tilde{C}_2 \varepsilon \int_\Omega (|\psi|^{2m - 2 - 2m - 2m - 2m - 4 + 2m + 2m - 4} \phi^2)^{\alpha+1} \, dx.
\]

Since \( m \geq 10 \), we have \( 2m - 4 - 2m - 4 + 2m - 4 + 2m - 4 \geq 0 \) and one can choose \( \varepsilon \) small enough such that

\[
\int_\Omega e^{(\alpha+1)u} \psi^{2m} \, dx \leq C \int_\Omega (|\nabla \psi|^2 + |\nabla \psi|^4)^{\alpha+1} \, dx.
\]
This completes the proof of (2.1).

(ii) Let \( f(u) = u^p \). For any \( \alpha \in (p - \sqrt{p^2 - p}, p + \sqrt{p^2 - p}) \) and any \( k > 0 \), set

\[
a_k(t) = \begin{cases} 
\frac{\alpha^2}{(2\alpha - 1)k}(t - k) + 1 & \text{if } t < k, \\
t^\alpha & \text{if } t \geq k,
\end{cases}
\]

and

\[
b_k(t) = \begin{cases} 
\frac{\alpha^2}{(2\alpha - 1)k}(t - k) + 1 & \text{if } t < k, \\
t^{2\alpha - 1} & \text{if } t \geq k.
\end{cases}
\]

Notice that

\[
(a'_k(t))^2 = \frac{\alpha^2}{2\alpha - 1} b'_k(t) \quad \text{and} \quad (a_k(t))^2 \geq tb_k(t),
\]

and

\[
(a'_k(t))^{-2}(a_k(t))^4 \leq C_3 t^{2\alpha + 2}, \quad (a_k(t))^2 \leq C_4 t^{2\alpha} \quad \text{and} \quad (b'_k(t))^{-1}(b_k(t))^2 \leq C_5 t^{2\alpha},
\]

where \( C_3, C_4 \) and \( C_5 \) are positive constant and independent of \( k \). For any \( \phi \in C^1_c(\Omega) \), multiple \((1.2)\) with test function \( b_k(u)\phi^2 \) and integrate by parts. It follows from properties (2) and (3) in Theorem A that

\[
\int_{\Omega} F^2(\nabla u)b'_k(u)\phi^2 dx 
\leq \frac{2C}{(1 - 2C\varepsilon)} \int_{\Omega} (b'_k(u))^{-1}(b_k(u))^2|\nabla \phi|^2 dx + \frac{1}{1 - 2C\varepsilon} \int_{\Omega} w^p b_k(u)\phi^2 dx. \tag{2.12}
\]

Since \( u \) is a stable solution of equation \((1.2)\), for any \( \varphi \in C^1_c(\Omega) \), we have

\[
\int_{\Omega} F_{\xi_i}(\nabla u)F_{\xi_j}(\nabla u)\varphi_{x_i}\varphi_{x_j} + F(\nabla u)F_{\xi_i,\xi_j}(\nabla u)\varphi_{x_i}\varphi_{x_j} - pu^p \varphi^2 dx \geq 0. \tag{2.13}
\]

Set \( \varphi = a_k(u)\phi \), in the light of \( \varphi_{x_i} = a'_k(u)u_{x_i}\phi + a_k(u)\phi_{x_i} \). Applying \((1.4)\) and \((2)\) and \((3)\) in Theorem A and Cauchy inequality, we derive

\[
p \int_{\Omega} w^{-1}(a_k(u))^2\phi^2 dx 
\leq (1 + 2C\varepsilon_1 + \Lambda\varepsilon_2) \int_{\Omega} F^2(\nabla u)(a'_k(u))^2\phi^2 dx 
+ (C^2 + \frac{2C}{\varepsilon_1}) \int_{\Omega} (a_k(u))^2|\nabla \phi|^2 dx + \frac{\Lambda}{\varepsilon_2} \int_{\Omega} (a'_k(u))^{-2}(a_k(u))^4 \frac{|\nabla \phi|^4}{\phi^2} dx. \tag{2.14}
\]

It follows from \((2.10)-(2.14)\) that

\[
p \int_{\Omega} w^{-1}(a_k(u))^2\phi^2 dx 
\leq \frac{\alpha^2(1 + 2C\varepsilon_1 + \Lambda\varepsilon_2)}{(2\alpha - 1)(1 - 2C\varepsilon)} \int_{\Omega} w^{-1}(a_k(u))^2\phi^2 dx + \frac{\Lambda}{\varepsilon_2} C_3 \int_{\Omega} w^{2\alpha + 2}|\nabla \phi|^4 \phi^2 dx 
+ \left[ \frac{2C\alpha(1 + 2C\varepsilon_1 + \Lambda\varepsilon_2)}{(2\alpha - 1)(1 - 2C\varepsilon)} C_5 + (C^2 + \frac{2C}{\varepsilon_1}) C_4 \right] \int_{\Omega} w^{2\alpha}|\nabla \phi|^2 dx.
\]
Since \( p - \sqrt{p^2 - p} < \alpha < p + \sqrt{p^2 - p} \), we can choose \( \varepsilon, \varepsilon_1 \) and \( \varepsilon_2 \) small enough such that

\[
p > \alpha^2 \left( 1 + 2C\varepsilon_1 + \Lambda\varepsilon_2 \right) \overline{(2\alpha - 1)(1 - 2\varepsilon)}.
\]

It follows that

\[
\int_{\Omega} u^{p-1}(a_k(u))^2 \phi^2 \,dx \leq C_6 \int_{\Omega} u^{2\alpha} |\nabla \phi|^2 \,dx + C_7 \int_{\Omega} u^{2\alpha + 2} \frac{|\nabla \phi|^4}{\phi^2} \,dx,
\]

where \( C_5, C_6 \) are positive constant and independent of \( k \), let \( k \to +\infty \), by Fatou’s lemma we have

\[
\int_{\Omega} u^{2\alpha + p-1} \phi^2 \,dx \leq C_6 \int_{\Omega} u^{2\alpha} |\nabla \phi|^2 \,dx + C_7 \int_{\Omega} u^{2\alpha + 2} \frac{|\nabla \phi|^4}{\phi^2} \,dx.
\]

Since \( p > 3 \), let \( \phi = \psi^m \) and \( 0 \leq \psi \leq 1 \), by Young’s inequality we obtain

\[
\int_{\Omega} u^{2\alpha + p-1} \phi^2 \,dx = \int_{\Omega} u^{2\alpha + p-1} \psi^{2m} \,dx
\]

\[
\leq \bar{C}_6 \varepsilon \int_{\Omega} u^{2\alpha + p-1} \psi^{2m} \,dx + \overline{C_6} \varepsilon \int_{\Omega} \left[ |\nabla \psi|^2 \psi^{2m - 2 - 2m} \frac{2n + 2}{p - 1} \right]^{2n + p - 1} \,dx
\]

\[
+ \bar{C}_7 \varepsilon \int_{\Omega} u^{2\alpha + p-1} \psi^{2m} \,dx + \overline{C_7} \varepsilon \int_{\Omega} \left[ |\nabla \psi|^4 \psi^{2m - 4 - 2m} \frac{2n + 2}{p - 1} \right]^{2n + p - 1} \,dx.
\]

Since \( m \geq \frac{6p + 4\sqrt{p^2 - p} - 2}{p - 3} \), we have

\[
2m - 4 - 2m \frac{2\alpha + 2}{2\alpha + p - 1} > 0.
\]

This finishes the proof of (2.2).

(iii) Let \( f(u) = -u^{-p} \). For any \( \alpha \in (1, p + \sqrt{p^2 + p}) \) and for any \( k > 0 \), we set

\[
a_k(t) = \begin{cases} 
\frac{\alpha^2}{(2\alpha + 1)k} (k - t) + 1 \quad & \text{if } \frac{1}{\sqrt{t}} \leq \frac{1}{k}, \\
\frac{\alpha^2}{(2\alpha + 1)k} (k - t) + 1 \quad & \text{if } \frac{1}{\sqrt{t}} > \frac{1}{k},
\end{cases}
\]

and

\[
b_k(t) = \begin{cases} 
t^{-(2\alpha + 1)} \quad & \text{if } \frac{1}{\sqrt{t}} \leq \frac{1}{k}, \\
\frac{\alpha^2}{(2\alpha + 1)k} (k - t) + 1 \quad & \text{if } \frac{1}{\sqrt{t}} > \frac{1}{k}.
\end{cases}
\]

Notice that

\[
(a_k(t))^2 = \frac{\alpha^2}{2\alpha + 1} |b_k(t)| \quad \text{and} \quad (a_k(t))^2 \geq tb_k(t),
\]

and

\[
|a_k(t)|^{-2} (a_k(t))^4 \leq c_3 t^{-2\alpha + 2}, \quad (a_k(t))^2 \leq c_4 t^{-2\alpha} \quad \text{and} \quad |b_k(t)|^{-1} (b_k(t))^2 \leq c_5 t^{-2\alpha},
\]

where \( c_3, c_4 \) and \( c_5 \) are positive constant and independent of \( k \). For any \( \phi \in C^1_c(\Omega) \), multiply (1.2) with test function \( b_k(u) \phi^2 \) and integrate by parts. It follows from
properties (2) and (3) of Theorem A that
\[
\int_{\Omega} F^2(\nabla u)|b_k'(u)|\phi^2 dx \\
\leq \frac{2C}{(1-2C\varepsilon)} \int_{\Omega} |b_k'(u)|^{-1}(b_k(u))^2 |\nabla \phi|^2 dx + \int_{\Omega} \frac{1}{1-2C\varepsilon} \int_{\Omega} u^{-p}b_k(u)\phi^2 dx.
\] (2.17)

Since \( u \) is a stable solution of equation (1.2), for any \( \varphi \in C^1_c(\Omega) \), we have
\[
\int_{\Omega} F_{\xi}(\nabla u)F_{\xi}(\nabla u)\varphi_x, \varphi_x \phi_x + F(\nabla u)F_{\xi}(\nabla u)\varphi_x \phi_x - p u^{-p-1}\varphi^2 dx \geq 0.
\] (2.18)

Set \( \varphi = a_k(u)\phi \), in the light of \( \varphi_x = a_k'(u)u_x \phi + a_k(u)\phi_x \). Applying (1.4) and (2) and (3) of the Theorem A and Cauchy inequality, we deduce that
\[
p \int_{\Omega} u^{-p-1}(a_k(u))^2 \phi^2 dx \\
\leq (1 + 2C\varepsilon_1 + \Lambda\varepsilon_2) \int_{\Omega} F^2(\nabla u)(a_k(u))^2 \phi^2 dx \\
+ (C^2 + \frac{2C}{\varepsilon_1}) \int_{\Omega} (a_k(u))^2 |\nabla \phi|^2 dx + \Lambda \frac{1}{\varepsilon_2} \int_{\Omega} (a_k(u))^{-2}(a_k(u))^4 |\nabla \phi|^4 \phi^2 dx.
\] (2.19)

It follows from (2.15)-(2.19) that
\[
p \int_{\Omega} u^{-p-1}(a_k(u))^2 \phi^2 dx \\
\leq \left[ \frac{2C^2(1 + 2C\varepsilon_1 + \Lambda\varepsilon_2)}{(2\alpha + 1)(1-2C\varepsilon)} + (C^2 + \frac{2C}{\varepsilon_1})c_4 \right] \int_{\Omega} u^{-2\alpha}|\nabla \phi|^2 dx \\
+ \frac{\alpha^2(1 + 2C\varepsilon_1 + \Lambda\varepsilon_2)}{(2\alpha + 1)(1-2C\varepsilon)} \int_{\Omega} u^{-p-1}(a_k(u))^2 \phi^2 dx + \Lambda \frac{1}{\varepsilon_2} c_3 \int_{\Omega} u^{-2\alpha+2} |\nabla \phi|^4 \phi^2 dx.
\]

Since \( \alpha \in (1, p + \sqrt{p^2 + p}) \), so we can choose \( \varepsilon, \varepsilon_1 \) and \( \varepsilon_2 \) small enough such that
\[
p > \frac{\alpha^2(1 + 2C\varepsilon_1 + \Lambda\varepsilon_2)}{(2\alpha + 1)(1-2C\varepsilon)}.
\]

Hence, we have
\[
\int_{\Omega} u^{-p-1}(a_k(u))^2 \phi^2 dx \leq c_6 \int_{\Omega} u^{-2\alpha}|\nabla \phi|^2 dx + c_7 \int_{\Omega} u^{-2\alpha+2} |\nabla \phi|^4 \phi^2 dx,
\]
where \( c_6, c_7 \) are positive constants and independent of \( k \). Let \( k \to +\infty \), by Fatou’s lemma we get
\[
\int_{\Omega} u^{-2\alpha-p-1} \phi^2 dx \leq c_6 \int_{\Omega} u^{-2\alpha}|\nabla \phi|^2 dx + c_7 \int_{\Omega} u^{-2\alpha+2} |\nabla \phi|^4 \phi^2 dx.
\]

Let \( \phi = \psi^m \) and \( 0 \leq \psi \leq 1 \), by Young’s inequality, we conclude
\[
\int_{\Omega} u^{-2\alpha-p-1} \psi^{2m} dx \leq c_6 \int_{\Omega} (\psi^{2m-2-2m+\frac{2\alpha+2}{p+1}} |\nabla \psi|^2)^{\frac{p+1+2\alpha}{p+1}} dx \\
+ c_7 \int_{\Omega} (\psi^{2m-4-2m+\frac{2\alpha-2}{p+1+2\alpha}} |\nabla \psi|^4)^{\frac{p+1+2\alpha}{p+1}} dx.
\]
Since \( m \geq \max\{\frac{3p + 2\sqrt{p^2 + p + 1}}{p + 1}, \frac{6p + 4\sqrt{p^2 + p + 2}}{p + 1}\} \), we have
\[
2m - 2 - 2m \frac{2\alpha}{p + 1 + 2\alpha} \geq 0,
\]
and
\[
2m - 4 - 2m \frac{2\alpha - 2}{p + 1 + 2\alpha} \geq 0.
\]
This completes the proof of (2.3).

3. The partial regularity result. To prove our regularity theorem, the level set method plays an important role, see [30, 33]. In order to use this method, let us first recall the important tools: the co-area formula and isoperimetric inequality for anisotropic version. We define the total variation of \( u \in BV(\Omega) \) with respect to \( F \) by
\[
\int_{\Omega} |\nabla u|_F := \sup \left\{ \int_{\Omega} u \text{div} \sigma dx : \sigma \in C^1_c(\Omega; \mathbb{R}^N), F^0(\sigma) \leq 1 \right\}.
\]
From this definition, the perimeter of \( E \subset \Omega \) is defined as
\[
P_F(E) := \int_{\Omega} |\nabla \chi_E|_F,
\]
where \( \chi_E \) is the characteristic function of \( E \). Then the co-area formula
\[
\int_{\Omega} |\nabla u|_F = \int_0^\infty P_F(\{|u| > t\}) dt,
\]
and the isoperimetric inequality
\[
P_F(E) \geq N\kappa_0^{1\over p} |E|^{1 - \frac{1}{p}},
\]
hold, and the equality holds if and only if \( E \) is a Wulff ball, for the proof we refer to [1, 23]. Moreover, in [2], we know that if \( u \in W^{1,1}(\Omega) \), then
\[
\int_{\Omega} |\nabla u|_F = \int_{\Omega} F(\nabla u) dx,
\]
and the co-area formula becomes
\[
-\frac{d}{dt} \int_{\{u > t\}} F(\nabla u) dx = P_F(\{u > t\}),
\]
for almost every \( t \).

Here, we recall the definition of the Morrey space \( M^p(\Omega) \), see [24].

**Definition 3.1.** A function \( f \in L^1(\Omega) \) is said to belong to \( M^p(\Omega) \) when \( 1 \leq p \leq \infty \), if there exists a constant \( K \) such that
\[
\int_{\Omega \cap B_r} |f| \leq Kr^{N(1 - 1 \over p)}
\]
for all \( B_r \subset \mathbb{R}^N \) with the norm
\[
\| f \|_{M^p(\Omega)} = \inf\{K \int_{\Omega \cap B_r} |f| \leq Kr^{N(1 - 1 \over p)}\}.
\]

In order to prove the main result, we need the following decay estimate for solutions of (1.1). Without loss of generality, we always assume \( \Omega = B_2(0) \) in (1.1).
Lemma 3.1. Under the assumption of (1.6), there exist \( \varepsilon_0 > 0, \ r \in (0, \frac{1}{2}) \), which depend only on the dimension \( N \), such that for a stable solution \( u \) of (1.1), if

\[
2^{2-N} \int_{B_r(0)} e^u dx \leq \varepsilon,
\]

where \( \varepsilon \leq \varepsilon_0 \), then

\[
r^{2-N} \int_{B_r(0)} e^{v_0} dx \leq \frac{1}{2} \varepsilon.
\] (3.1)

Proof. Apply Proposition 2.2, when \( \alpha = 1, \ \Omega = B_2(0) \) and \( \psi = 1 \) in \( B_1(0) \), to get

\[
\int_{B_1(0)} e^{2v} dx \leq \int_{B_2(0)} e^{2v} \psi^{2m} dx \leq C \int_{B_2(0)} e^u (|\nabla \psi|^2 + |\nabla \psi|^4) dx.
\]

Hence \( \| e^u \|_{L^2(B_1(0))} \leq C \varepsilon^{\frac{1}{2}} \). Set the decomposition \( u = v + w \) in \( B_1(0) \), where

\[
\begin{aligned}
&-Qw = 0 \quad \text{in} \ B_1(0), \\
&w = u \quad \text{on} \ \partial B_1(0),
\end{aligned}
\] (3.2)

and

\[
\begin{aligned}
&-\tilde{Q}v := -(Qu - Qw) = e^u \quad \text{in} \ B_1(0), \\
v = 0 \quad \text{on} \ \partial B_1(0).
\end{aligned}
\] (3.3)

Set \( \Omega = B_1(0), \ \Omega_t = \{ x \in \Omega | v > t \} \) and \( \mu(t) = |\Omega_t| \), and deduce that

\[
\int_{\Omega_t} e^u dx = \int_{\Omega_t} -(Qu - Qw) dx
\]

\[
= \int_{\partial \Omega_t} \langle F(\nabla u)F_\xi(\nabla u) - F(\nabla w)F_\xi(\nabla w), \frac{\nabla(u - w)}{|\nabla(u - w)|} \rangle dS
\]

\[
\geq d_0 \int_{\partial \Omega_t} \frac{F^2(\nabla(u - w))}{|\nabla(u - w)|} dS = d_0 \int_{\partial \Omega_t} \frac{F^2(\nabla v)}{|\nabla v|} dS,
\]

where

\[
d_0 = \inf \left\{ d_{X,Y} \big| X,Y \in \mathbb{R}^N, X \neq 0, Y \neq 0, X \neq Y \right\},
\]

with

\[
d_{X,Y} := \frac{\langle F(X)F_\xi(X) - F(Y)F_\xi(Y), X - Y \rangle}{F^2(X - Y)}.
\]

It is straightforward to check \( \min \left\{ \frac{\lambda_1}{2}, 1 \right\} \leq d_0 \leq 1 \), where \( \lambda_1 \) is the smallest eigenvalue of \( Hess(F^2) \). By the isoperimetric inequality, the co-area formula and the Hölder inequality, we obtain

\[
N^\frac{1}{2} \mu(t)^{1-\frac{1}{2}} \leq \frac{d}{dt} \int_{\Omega_t} F(\nabla v) dx = \int_{\partial \Omega_t} \frac{F(\nabla v)}{|\nabla v|} dS
\]

\[
\leq \left( \int_{\partial \Omega_t} \frac{F^2(\nabla v)}{|\nabla v|} dS \right)^{\frac{1}{2}} \left( \int_{\partial \Omega_t} \frac{1}{|\nabla v|} dS \right)^{\frac{1}{2}}
\]

\[
\leq \left( \frac{1}{d_0} \int_{\Omega_t} e^u dx \right)^{\frac{1}{2}} \left( -\mu'(t) \right)^{\frac{1}{2}}.
\]

It follows that

\[
-\mu'(t) \geq \frac{d_0 N^2 \kappa N \mu(t)^{2-\frac{1}{2}}}{\int_{\Omega_t} e^u dx},
\]
and hence
\[ -\frac{dt}{d\mu} \leq \frac{\int_{\Omega} e^u dx}{d_0 N^2 K_{\Omega}^2 \mu(t)^{2-\frac{4}{N}}} \leq C \frac{\int_{\Omega} e^u dx}{\mu(t)^{2-\frac{4}{N}}} \]

Integrating the above inequality over \((\mu, [\Omega])\), we have
\[ t(\mu) \leq C \| e^u \|_{L^1(\Omega)} \int_{\mu}^{[\Omega]} \frac{1}{s^{\frac{4}{N}}} ds \]
\[ \leq C \| e^u \|_{L^1(\Omega)} \left( \frac{1}{\mu^{1-\frac{4}{N}}} - \frac{1}{[\Omega]^{1-\frac{4}{N}}} \right). \]

Using the co-area formula again, we derive
\[ \int_{\Omega} v dx = \int_0^\infty t \cdot (-\mu'(t)) dt \]
\[ \leq \int_0^{[\Omega]} C \| e^u \|_{L^1(\Omega)} \left( \frac{1}{\mu^{1-\frac{4}{N}}} - \frac{1}{[\Omega]^{1-\frac{4}{N}}} \right) d\mu \]
\[ \leq C [\Omega]^{\frac{2}{N}} \| e^u \|_{L^1(\Omega)} \leq C \varepsilon. \]

Hence, we have \( \| v \|_{L^1(B_t(0))} \leq C \varepsilon \) and \( \| e^u \|_{L^2(B_t(0))} \leq C \varepsilon^{\frac{3}{2}} \). By the elliptic estimate, we have \( \| v \|_{W^{2,2}(B_t(0))} \leq C \varepsilon^{\frac{3}{2}} \). Then it follows from the Sobolev embedding theorem, we get \( \| v \|_{L^{\frac{2N}{N-2}}(B_t(0))} \leq C \varepsilon^\frac{3}{2}. \) By an interpolation inequality between \( L^q \) spaces, we derive
\[ \| v \|_{L^2(B_t(0))} = \| v \|_{L^2(B_t(0))} \| v \|_{L^{\frac{2N}{N-2}}(B_t(0))} \leq C \varepsilon^\alpha, \]
where \( \alpha = \frac{N+8}{2N+8} > \frac{1}{2} \). Then by interpolation inequality between Sobolev space, we get
\[ \| \nabla v \|_{L^2(B_t(0))} \leq C \left( \varepsilon^{\frac{1}{2} (\alpha - 1)} \| \nabla v \|_{L^2(B_t(0))} + \varepsilon^{-\frac{1}{2} (\alpha - 1)} \| v \|_{L^2(B_t(0))} \right) \leq C \varepsilon^\beta, \]
where \( \beta > \frac{1}{2} \) depends only on \( N \). It follows from the above inequality that
\[ \int_{B_t(0)} v e^u dx = \int_{B_t(0)} -v \tilde{Q} v dx \leq d_0 \int_{B_t(0)} F^2 (\nabla v) dx \]
\[ \leq d_0 b^2 \int_{B_t(0)} |\nabla v|^2 dx \leq C \varepsilon^{2\beta}. \]

We decompose the estimate of \( r^{2-N} \int_{B_r(0)} e^u dx \) into two parts: \( \{ v \leq \varepsilon^\gamma \} \) and \( \{ v > \varepsilon^\gamma \} \), where \( \gamma = \frac{1}{2} (2\beta - 1) > 0 \). Since \( Qw = 0 \), we have
\[ Q(e^w) = e^w F^2 (\nabla w) \geq 0. \]

Under the assumption of (1.6), we have the mean-value inequality,
\[ e^{w(y)} \leq \frac{1}{\kappa_0 r^N} \int_{B_r(y)} e^{w(x)} dx, \]
for all \( B_r(y) \subset B_1(0) \), see [22]. For \( r \in (0, \frac{1}{2}) \) and for \( x \in B_r(0) \) we have \( B_{\frac{1}{2}}(x) \subset B_1(0) \). Hence,
\[ r^{-N} \int_{B_r(0)} e^u dx \leq 2^N \int_{B_1(0)} e^u dx \leq 2^N \int_{B_1(0)} e^u dx. \]
It follows that
\[-r^2 \int_{B_r(0) \cap \{v \geq \varepsilon \gamma \}} e^u \, dx \leq r^2 e^{\gamma} \int_{B_r(0) \cap \{v \leq \varepsilon \}} e^w \, dx \leq 2N r^2 e^{\gamma} \int_{B_1(0)} e^u \, dx \leq Cr^2 \varepsilon.\]

For the second part, we conclude
\[-r^2 \int_{B_r(0) \cap \{v \geq \varepsilon \gamma \}} e^u \, dx \leq \varepsilon - \gamma r^2 \int_{B_r(0) \cap \{v \leq \varepsilon \gamma \}} e^w \, dx \leq C r^2 \varepsilon + C r^{2-N} \varepsilon^{2\beta-\gamma}.\]

Therefore,
\[r^2 \int_{B_r(0)} e^u \, dx \leq C r^2 \varepsilon + C r^{2-N} \varepsilon^{2\beta-\gamma}.\]

Note that \(2\beta - \gamma > 1\). So, one can choose \(r\) and \(\varepsilon_0\) small enough such that for any \(\varepsilon \leq \varepsilon_0\),
\[C r^2 \varepsilon + C r^{2-N} \varepsilon^{2\beta-\gamma} \leq \frac{1}{2} \varepsilon.\]

This completes the proof.

From the above decay estimate, Lemma 3.1, and the priori estimate in Morrey space, we have the following \(\varepsilon\)-regularity theorem.

**Lemma 3.2.** Assume that (1.6) hold. Suppose that \(u\) is a stable weak solution of (1.1). If there exists \(\varepsilon_0 > 0\) such that
\[\int_{B_{1/2}(0)} e^u \, dx \leq \varepsilon,\]
where \(\varepsilon \leq \varepsilon_0\), then
\[\sup_{B_{1/4}(0)} u < \infty.\]

**Proof.** We choose \(\varepsilon_0\) small enough such that for any \(y \in B_{1/2}(0)\)
\[2^{N-2} \int_{B_{1/2}(y)} e^u \leq \varepsilon.\]

Applying Lemma 3.1, there exist \(\delta > 0\) and \(r < \frac{1}{2}\) such that for \(y \in B_{1/2}(0)\) by a standard induction,
\[\int_{B_{r}(y)} e^u \, dx \leq C r^{N-2+\delta}.\]

This implies \(e^u \in M^{\frac{N}{N-\delta}}\). Take the decomposition \(u = v + w\), where
\[\begin{cases} -Qw = 0 & \text{in } B_{1/2}(0) \\ w = u & \text{on } \partial B_{1/2}(0), \end{cases} \quad (3.4)\]
and
\[
\begin{aligned}
-\tilde{Q}v &:= -(Qu - Qw) = e^u & \text{in } B_{\frac{1}{2}}(0) \\
v & = 0 & \text{on } \partial B_{\frac{1}{2}}(0).
\end{aligned}
\tag{3.5}
\]

From the elliptic estimate, we know that \( w \) is bounded in \( B_{\frac{1}{2}}(0) \). Next, to estimate \( v \), we also use the level set method. Denote \( \Omega = B_{\frac{1}{2}}(0) \), set \( \Omega_t = \{ x \in \Omega | v > t \} \) and \( \mu(t) = |\Omega_t| \). According to the property of (3) in the Theorem A, it implies that
\[
\int_{\Omega_t} e^u dx = \int_{\Omega_t} -Qv dx = \int_{\partial \Omega_t} F(\nabla v)F(\nabla v) |\nabla v| dS = \int_{\partial \Omega_t} F^2(\nabla v) |\nabla v| dS.
\]
By the isoperimetric inequality, the co-area formula and Holder’s inequality, we have
\[
N \kappa_0^{1/N} \mu(t)^{1-1/N} \leq P_F(\Omega_t) = -\frac{d}{dt} \int_{\Omega_t} F(\nabla v) dx
\]
\[
= \int_{\partial \Omega_t} \frac{F(\nabla v)}{|\nabla v|} dS \leq \left( \int_{\partial \Omega_t} \frac{F^2(\nabla v)}{|\nabla v|} dS \right)^{1/2} \left( \int_{\partial \Omega_t} \frac{1}{|\nabla v|} dS \right)^{1/2}
\]
\[
= \left( \int_{\Omega_t} e^u dx \right)^{1/2} (-\mu'(t))^{1/2}.
\]
It follows that
\[
-\mu'(t) \geq \frac{N \kappa_0^{2/N} \mu^{2-2/N}}{\int_{\Omega_t} e^u dx}.
\]

Hence
\[
-\frac{dt}{d\mu} \leq \frac{\int_{\Omega_t} e^u dx}{N \kappa_0^{2/N} \mu(t)^{2-2/N}} \leq C \frac{\mu^{1 - \frac{1}{2p}}}{\mu^{2-2/N}} \leq C \frac{\mu^{1 - \frac{1}{2p}}}{\mu^{2-2/N}} \leq C \frac{\mu^{1 - \frac{1}{2p}}}{\mu^{2-2/N}}.
\]
Integrating the above inequality over \( (\mu, |\Omega|) \), we have
\[
t(\mu) \leq \int_{\mu}^{\mu(t)} C \frac{\mu^{1 - \frac{1}{2p}}}{s^{1 - \frac{1}{2p}}} ds
\]
\[
\leq C \frac{\mu^{1 - \frac{1}{2p}}}{s^{1 - \frac{1}{2p}}} (|\Omega|^{1 - \frac{1}{2p}} - \mu^{1 - \frac{1}{2p}}) < \infty.
\]
This inequality implies that \( \|v\|_{L^\infty(\Omega)} < \infty \). Thus we get \( u \) is bounded in \( B_{\frac{1}{2}}(0) \).

Now, we prove our main partial regularity result.

**Proof of Theorem 1.1.** The equation (1.1) is invariant under the rescaling
\[
u' (x) = u(rx) + 2 \log r.
\]
From (2.1), we know that for \( p \in (1, 5) \), there exist \( C > 0 \) such that
\[
\int_{B_r(x)} e^{pu} \leq Cr^{N-2p}.
\]
Hence, if
\[
\int_{B_r(x)} e^{pu} \leq \varepsilon,
\]
by Hölder’s inequality, we deduce that
\[ \int_{B_1(x)} e^{u(r)}dy \leq \varepsilon. \]
Therefore, it follows from Lemma 3.2 that \( u(r) \) is bounded in \( B_{1}(x) \), which implies that \( u \) is bounded in \( B_{\frac{1}{4}}(x) \). Thus, for any \( x \in S \) and \( r > 0 \), we have
\[ r^{2p-N} \int_{B_r(x)} e^{pu} > \varepsilon. \]
From the Besicovitch covering Lemma, see [26], we obtain
\[ H^{N-2p}(S) = 0. \]
Since \( p \) is arbitrary in (1.5). This completes the proof. □

4. The Liouville theorems. In this section, we are mainly devoted to the proof of Liouville theorems for stable solutions and finite Morse index solutions.

**Proof of Theorem 1.2.** By contradiction, suppose that \( u \) is a stable solution of equation (1.1). Notice that (2.1) implies that we can fix an integer \( m \geq 10 \), and choose \( \alpha \in (0, 4) \) such that \( N - 2(\alpha + 1) < 0 \). Consider the function \( \phi_R(x) = \phi \left( \frac{F^0(x)}{R} \right) \), where \( \phi \in C^1_c(\mathbb{R}) \), \( 0 \leq \phi \leq 1 \) for \( x \in \mathbb{R}^N \) and
\[ \phi(t) = \begin{cases} 1, & \text{if } |t| \leq 1, \\ 0, & \text{if } |t| \geq 2. \end{cases} \]
For every \( R > 0 \), we get
\[ \int_{B_R(0)} e^{(\alpha+1)u}dx \leq \tilde{C} R^{N-2(\alpha+1)}, \]
where \( \tilde{C} \) is a positive constant independent on \( R \). Letting \( R \to +\infty \), we obtain \( \int_{\mathbb{R}^{N}} e^{(\alpha+1)u}dx = 0 \). This is a contradiction. □

**Proof of Theorem 1.4.** Similar to the proof of Theorem 1.2, for Part (i) and Part (ii) we conclude
\[ \int_{\mathbb{R}^{N}} u^{p+2\alpha-1}dx = 0, \]
and
\[ \int_{\mathbb{R}^{N}} u^{-p-2\alpha-1}dx = 0, \]
respectively. These contradict the assumptions. □

**Proof of Theorem 1.3.** When \( N = 2 \), without loss of generality, take \( x_0 = 0 \). We observe that there exists \( R = R(\lambda) > 1 \) such that
\[ e^{u(x)} \leq \frac{1}{4F^0(x)^2 \ln^2(F^0(x))}, \]
for \( F^0(x) > R \). It is straightforward to see that \( v(x) = \ln^\frac{1}{2}(F^0(x)) \) solves quasilinear equation
\[ -Qv = \frac{1}{4F^0(x)^2 \ln^2(F^0(x))}v. \]
For any \( \phi \in C_c^\infty(\mathbb{R}^2 \setminus B_R) \), we have
\[
\int_{\mathbb{R}^2 \setminus B_R} |F_x(\nabla v) \cdot \nabla \phi|^2 + \frac{1}{4F^0(x)^2 \ln^2(F^0(x))} \phi^2 \, dx \geq 0.
\]
From the properties of \( F \), we can see that \( F_x(\nabla v) = -F_x(\nabla u) \). So, we deduce that
\[
\int_{\mathbb{R}^2 \setminus B_R} |F_x(\nabla v) \cdot \nabla \phi|^2 + \sum_{i,j=1}^2 F(\nabla u) F_{x_i}(\nabla u) \phi_{x_i} \phi_{x_j} - c^u \phi^2 \, dx \geq 0,
\]
where we used the definition of stable solutions.

We now prove nonexistence of stable outside a compact set solutions of \( \mathbb{R}^N \) when \( 3 \leq N \leq 9 \). By contradiction, we assume that \( u \) is a solution of (1.1) which is stable outside a compact set of \( \mathbb{R}^N \). In order to get the contradiction, we split the proof into four steps.

**Step 1.** There exists \( R_0 = R_0(u) > 0 \) such that
(a) for any \( \alpha \in (0, 4) \) and \( r > R_0 + 3 \) there exist positive constants \( A \) and \( B \) depending on \( \alpha, N \) and \( R_0 \) but not \( r \), holds
\[
\int_{B_r \setminus B_{R_0 + 3}} e^{(\alpha+1)u} \, dx \leq A + Br^{-2(\alpha+1)}, \tag{4.1}
\]
(b) For any \( y \in \mathbb{R}^N \) such that \( B_{2R}(y) \subset \{ x \in \mathbb{R}^N : F^0(x) > R_0 \} \) and for any \( \alpha \in (0, 4) \), we have
\[
\int_{B_{2R}(y)} e^{(\alpha+1)u} \, dx \leq CR^{-2(\alpha+1)}, \tag{4.2}
\]
where \( C \) is a positive constant depending on \( \alpha, N \) and \( R_0 \) but not on \( R \) and \( y \).

Since \( u \) is stable outside a compact set of \( \mathbb{R}^N \), there exist \( R_0 > 0 \) such that (2.1) holds with \( \Omega := \mathbb{R}^N \setminus \overline{B_{R_0}}(0) \). Let \( m = 10 \), and for every \( r > R_0 + 3 \), consider the following test function \( \xi_r \in C_c^1(\mathbb{R}^N) \)
\[
\xi_r(x) = \begin{cases} 
\theta_{R_0}(F^0(x)), & \text{if } x \in B_{R_0+3}, \\
\phi \left( \frac{F^0(x)}{r} \right), & \text{if } x \in \mathbb{R}^N \setminus B_{R_0+3},
\end{cases}
\]
where \( \phi \) is defined in the proof of Theorem 1.2 and for \( s > 0 \), \( \theta_s \) belongs to \( C_c^1(\mathbb{R}) \), \( 0 \leq \theta_s \leq 1 \) everywhere on \( \mathbb{R} \) and
\[
\theta_s(t) = \begin{cases} 
0, & \text{if } |t| \leq s + 1, \\
1, & \text{if } |t| \geq s + 2.
\end{cases}
\]
It follows from Proposition 2.2 that
\[
\int_{B_r \setminus B_{R_0 + 3}} e^{(\alpha+1)u} \, dx \leq \int_{\Omega} e^{(\alpha+1)u} \, dx
\]
\[
\leq C \int_{\Omega} (|\nabla \xi_r|^2 + |\nabla \xi_r|^4)^{\alpha+1} \, dx
\]
\[
\leq C_1(\alpha, N, \theta_{R_0}) + C_2(\alpha, N, \phi)r^{N-2(\alpha+1)},
\]
and hence the inequality (4.1) holds.
The integral estimate (4.2) is obtained in the same way by using the test function \( \psi_{R,y}(x) = \phi\left(\frac{F^0(x-y)}{R}\right) \) in Proposition 2.2.

Step 2. There exist \( \eta > 0 \) and \( R_1 = R_1(N, \eta, u) > R_0 \) such that

\[
\int_{B_{R}} e^{\frac{N}{2}u} \, dx \leq \eta^{\frac{N}{2}}.
\] (4.3)

Let \( \alpha_1 := \frac{N-2}{2} \in (0, 4) \). For \( r > R_0 + 3 \), from (4.1) we have

\[
\int_{B_r \setminus B_{R_0+2}} e^{\frac{N}{2}u} \, dx \leq \int_{B_r \setminus B_{R_0+2}} e^{(\alpha_1+1)u} \, dx \leq A + B_{r}^{N-2(\alpha_1+1)}.
\]

Sending \( r \to \infty \), completes the arguments.

Step 3. The following asymptotic limit holds,

\[
\lim_{F^0(x) \to \infty} F^0(x)^2 e^{u(x)} = 0.
\]

Set \( \epsilon = \frac{1}{10} \), we observe that \( \frac{N}{2-\alpha} \in (1, 5) \). Since \( 3 \leq N \leq 9 \), there exists \( \alpha_2 = \alpha_2(N) \in (0, 4) \) such that \( \alpha_2 + 1 = \frac{N}{\epsilon} \). Next we fix \( \eta > 0 \) and observe that \( w = e^u \) satisfies

\[-Qw - e^u w \leq 0 \quad \text{in} \quad B_{2R}(y).\]

Applying the Harnack’s inequality, for positive solutions of the quasilinear equation

\[-Qw = e^u w,
\]

for any \( t > 1 \), we have

\[
\| w \|_{L^\infty(B_{R}(y))} \leq CR^{-\frac{N}{2}} \| w \|_{L^t(B_{2R}(y))},
\]

where \( C \) is a positive constant depending on \( N \) and \( R^\epsilon \| e^u \|_{L^\infty(B_{2R}(y))} \), see [31]. In order to apply the above result, we consider some point \( y \in \mathbb{R}^N \) such that \( F^0(y) > 10R_1 \). Set \( R = \frac{F^0(y)}{4} \) and \( t = \frac{N}{2} > 1 \). Since \( R_1 > R_0 \), where \( R_1 \) is defined by step 2, we conclude

\[
B_{2R}(y) \subset \{ x \in \mathbb{R}^N : F^0(x) > R_0 \},
\]

\[
\int_{\{ F^0(x) > R_1 \}} e^{\frac{N}{2}u} \, dx < \eta^{\frac{N}{2}},
\]

and

\[
R^\epsilon \| e^u \|_{L^\infty(B_{2R}(y))} = R^\epsilon \left( \int_{B_{2R}(y)} e^{(\alpha_2+1)u} \, dx \right)^{\frac{2-\alpha}{2}} \leq R^\epsilon \left[ CR^{-2(\alpha_2+1)} \right]^{\frac{2-\alpha}{2}} \leq C_1.
\]

Step 4. In this step, we complete the proof. Let \( v(r) = \frac{1}{N\kappa_0 r^{N-1}} \int_{\partial B_r} udS \), then

\[
v'(r) = \frac{1}{N\kappa_0 r^{N-1}} \int_{\partial B_r} (\nabla u, x) dS.
\]

It is straightforward to see that \( F^0(x) = r \) and \( \nu = F^0_\xi(x) \) on \( \partial B_r \). By the assumption (1.6), we have \( \langle \nabla u, x \rangle = F(\nabla u) \langle F_\xi(\nabla u), F^0_\xi(x) \rangle F^0(x) \). Therefore,

\[
v'(r) = \frac{1}{N\kappa_0 r^{N-1}} \int_{\partial B_r} \sum_{i=1}^{N} F(\nabla u) F_{\xi_i}(\nabla u) \nu_i dS = \frac{1}{N\kappa_0 r^{N-1}} \int_{B_r} Qudx.
\]

Since

\[
\lim_{F^0(x) \to \infty} (F^0(x))^2 e^{u(x)} = 0,
\]
there exist constants $0 < \delta < 1$ and $r_0 > 0$ such that $e^{u(x)} \leq \delta (F^0(x))^{-2}$ if $F^0(x) > r_0$. For $r$ large enough, we have

$$-u'(r) = \frac{1}{N\kappa_0 r^{N-1}} \int_{B_r} -Qu dx = \frac{1}{N\kappa_0 r^{N-1}} \int_{B_r} e^u dx$$

$$= \frac{1}{N\kappa_0 r^{N-1}} \left( \int_{B_{r_0}} e^u dx + \int_{B_r \setminus B_{r_0}} e^u dx \right) \leq \frac{\delta}{r}.$$ 

Therefore,

$$r^2 e^{u(r)} \geq Cr.$$ 

From the Jensen’s inequality, we have

$$\max_{\partial B_r} (F^0(x))^2 e^{u(x)} = r^2 \max_{\partial B_r} e^{u(x)} \geq \frac{r^2}{N\kappa_0 r^{N-1}} \int_{\partial B_r} e^u \partial S \geq r^2 e^{u(r)} \geq Cr.$$ 

This is a contradiction. 

Here, we classify stable outside a compact set solutions if $f(u) = u^p$ with $p = \frac{N+2}{N-2}$. When $F(\xi) = |\xi|$, for the Laplacian operator, such a classification is established by Farina in [17]. For the quasilinear setting, Ciraklo-Figalli-Roncoroni in [7] studied (1.2) for $f(u) = u^p$ with the critical exponent.

**Theorem 4.1.** If $f(u) = u^p$ with the critical exponent $p = \frac{N+2}{N-2}$, then $u$ is a stable outside a compact set solution of (1.2) in $\mathbb{R}^N$ if and only if

$$u(x) = \left( \frac{\lambda \sqrt{N(N-2)}}{\lambda^2 + F^0(x-x_0)^2} \right)^{\frac{N-2}{2}},$$ 

for some $\lambda > 0$ and $x_0 \in \mathbb{R}^N$.

**Proof.** It is discussed in [7] that any positive weak solution of equation (1.2) is radial and is of the form

$$u_\lambda(x) = \left( \frac{\lambda \sqrt{N(N-2)}}{\lambda^2 + F^0(x-x_0)^2} \right)^{\frac{N-2}{2}},$$ 

for some $\lambda > 0$ and $x_0 \in \mathbb{R}^N$. Next, we claim that $u_\lambda(x)$ is stable outside a compact set. Assume that $x_0 = 0$ and observe that $p|u_\lambda(x)|^{p-1} = O((F^0(x))^{-4})$ as $F^0(x) \to \infty$. Therefore, one can find $R_0 > 0$ such that for any $F^0(x) > R_0$,

$$p|u_\lambda(x)|^{p-1} \leq \frac{(N-2)^2}{4} (F^0(x))^{-2}.$$ 

From the property (7) in Theorem A, we have $F_{\xi_i}(\nabla u_\lambda) = \frac{x}{F^0(x)}$. Hence

$$\int_{\Omega} F_{\xi_i}(\nabla u_\lambda) F_{\xi_j}(\nabla u_\lambda) \phi_x \phi_{x_j} + F(\nabla u_\lambda) F_{\xi_i}(\nabla u_\lambda) \phi_x \phi_{x_j} - p_u^{p-1} \phi^2 dx$$

$$\geq \int_{\Omega} F_{\xi_i}(\nabla u_\lambda) F_{\xi_j}(\nabla u_\lambda) \phi_x \phi_{x_j} - p_u^{p-1} \phi^2 dx$$

$$= \int_{\Omega} \left| \frac{x}{F^0(x)} \cdot \nabla \phi \right|^2 - p_u^{p-1} \phi^2 dx$$

$$\geq \int_{\Omega} \left| \frac{x}{F^0(x)} \cdot \nabla \phi \right|^2 - \frac{(N-2)^2}{4} \frac{\phi^2}{F^0(x)^2} dx \geq 0.$$
Hence, the last inequality follows from Proposition 2.1 with $s = 2$. The desired result is proved. 

5. Monotonicity formulas. In this section, we state monotonicity formulas for the equation (1.2) when $f(u)$ is an exponential-type or a power-type nonlinearity. For the isotropic case, the following result (i) is given in [3], and (ii) and (iii) are given by [29] and [25], respectively.

**Theorem 5.1.** Let $u \in H^1_{\text{loc}}(\mathbb{R}^N)$ be a weak solution of (1.2), for $x_0 \in \mathbb{R}^N$ and $\lambda > 0$, under the assumption of (1.6).

(i) Let $f(u) = e^u$ and $e^u \in L^1_{\text{loc}}(\mathbb{R}^N)$. Define

$$E(u, x_0, \lambda) := \lambda^{2-n} \int_{B_\lambda(x_0)} \frac{1}{2} F(\nabla u)^2 - e^u dx + 2\lambda^{1-n} \int_{\partial B_\lambda(x_0)} (u + 2 \log \lambda) dS.$$  
(5.1)

Then $E(u, x_0, \lambda)$ is a nondecreasing function of $\lambda$.

(ii) Let $f(u) = u^p$ and $u \in L^p_{\text{loc}}(\mathbb{R}^N)$. Define

$$E_1(u, x_0, \lambda) := \lambda^{\frac{2-p}{p+1}} - \int_{B_\lambda(x_0)} \frac{1}{2} F(\nabla u)^2 - \frac{1}{p+1} u^{p+1} dx + \frac{1}{p-1} \lambda^{\frac{p+1}{p-1}} - \int_{\partial B_\lambda(x_0)} u^2(x) dS.$$  
(5.2)

Then $E_1(u, x_0, \lambda)$ is a nondecreasing function of $\lambda$.

(iii) Let $f(u) = -u^{-p}$ and $u^{-p} \in L^1_{\text{loc}}(\mathbb{R}^N)$. Define

$$E_2(u, x_0, \lambda) := \lambda^{\frac{2-p}{p+1}} - \int_{B_\lambda(x_0)} \frac{1}{2} F(\nabla u)^2 + \frac{1}{1-p} u^{-p} dx - \frac{1}{p+1} \lambda^{\frac{p+1}{p-1}} - \int_{\partial B_\lambda(x_0)} u^2(x) dS.$$  
(5.3)

Then $E_2(u, x_0, \lambda)$ is a nondecreasing function of $\lambda$.

**Proof.** (i) For $f(u) = e^u$, define

$$E(\lambda) := \lambda^{2-n} \int_{B_\lambda(x_0)} \frac{1}{2} F(\nabla u)^2 - e^u dx.$$  

Set $u^\lambda(x) = u(\lambda x) + 2 \log \lambda$ to get

$$E(\lambda) = \lambda^{2-n} \int_{B_\lambda(x_0)} \frac{1}{2} F(\nabla u)^2 - e^u dx$$
$$= \lambda^{2-n} \int_{B_1(x_0)} \frac{1}{2} F(\nabla u^\lambda)^2 - e^{u^\lambda}) \lambda^{n-2} dy$$
$$= \int_{B_1(x_0)} \frac{1}{2} F(\nabla u^\lambda)^2 - e^{u^\lambda}) dy.$$  

It follows from assumption (1.6) that

$$\frac{d}{d\lambda} E(\lambda) = \int_{B_1(x_0)} F(\nabla u^\lambda) F_\xi(\nabla u^\lambda) \frac{du^\lambda}{d\lambda} \frac{\partial u^\lambda}{\partial x_i} - e^{u^\lambda} \frac{d u^\lambda}{d\lambda} dy$$
$$= \int_{\partial B_1(x_0)} F(\nabla u^\lambda) F_\xi(\nabla u^\lambda) \frac{d u^\lambda}{d\lambda} \nu_i dS$$
$$= \int_{\partial B_1(x_0)} (\nabla u^\lambda, y) \frac{d u^\lambda}{d\lambda} dS.$$
\[ E(u, x_0, \lambda) := \lambda^{2-n} \int_{B_\lambda(x_0)} \left( \frac{1}{2} F(\nabla u)^2 - e^u dx + 2\lambda^{1-n} \int_{\partial B_\lambda(x_0)} (u + 2 \log \lambda) dS. \right) \]

Therefore,
\[ \frac{d}{d\lambda} E(u, x_0, \lambda) \geq 0. \]

(ii) Let \( f(u) = u^p \). Then
\[ E_1(\lambda) := \lambda^{\frac{2p+2}{p+1} - N} \int_{B_\lambda(x_0)} \left( \frac{1}{2} F^2(\nabla u) - \frac{1}{p+1} u^{p+1} dx. \right) \]

Set \( u_\lambda(x) = \lambda^{\frac{2}{p+1}} u(\lambda x) \) to conclude
\[ E_1(\lambda) = \int_{B_1(x_0)} \frac{1}{2} F^2(\nabla u_\lambda(y)) - \frac{1}{p+1} u_\lambda(y)^{p+1} dy. \]

It follows from assumption (1.6) that
\[ \frac{d}{d\lambda} E_1(\lambda) = \int_{B_1(x_0)} F(\nabla u_\lambda(y)) F_{\xi_i}(\nabla u_\lambda(y)) \frac{d\partial u_\lambda(y)}{\partial y_i} - u_\lambda(y)^p \frac{du_\lambda(y)}{d\lambda} dy \]
\[ = \int_{\partial B_1(x_0)} \nabla u_\lambda(y), y \frac{du_\lambda(y)}{d\lambda} dS \]
\[ = \int_{\partial B_1(x_0)} [\lambda \frac{du_\lambda(y)}{d\lambda} - \frac{2}{p+1} u_\lambda(y) \frac{du_\lambda(y)}{d\lambda}] dS \]
\[ = \int_{B_1(x_0)} \lambda (\frac{du_\lambda(y)}{d\lambda})^2 - \frac{1}{p+1} \frac{du_\lambda(y)}{d\lambda} dS. \]

Define
\[ E_1(u, x_0, \lambda) := \int_{B_1(x_0)} \frac{1}{2} F^2(\nabla u_\lambda(y)) - \frac{1}{p+1} u_\lambda(y)^{p+1} dy + \int_{\partial B_1(x_0)} \frac{1}{p+1} u_\lambda(y)^{p+1} dS \]
\[ = \lambda^{\frac{2p+2}{p+1} - N} \int_{B_\lambda(x_0)} \frac{1}{2} F^2(\nabla u) - \frac{1}{p+1} u^{p+1} dx + \frac{1}{p+1} \lambda^{\frac{p+3}{p+1} - N} \int_{\partial B_\lambda(x_0)} u^2(x) dS. \]

Therefore, we derive
\[ \frac{d}{d\lambda} E_1(u, x_0, \lambda) \geq 0. \]

(iii) Let \( f(u) = -u^{-p} \). Then
\[ E_2(\lambda) := \lambda^{\frac{2p+2}{p+1} - N} \int_{B_\lambda(x_0)} \frac{1}{2} F^2(\nabla u) + \frac{1}{1-p} u^{1-p} dx. \]

Set \( u_\lambda(x) = \lambda^{\frac{2}{p+1}} u(\lambda x) \) to conclude
\[ E_2(\lambda) = \int_{B_1(x_0)} \frac{1}{2} F^2(\nabla u_\lambda(y)) + \frac{1}{1-p} u_\lambda(y)^{1-p} dy. \]
It follows from assumption (1.6) that
\[
\frac{d}{d\lambda} E_2(\lambda) = \int_{B_1(x_0)} F(\nabla u_\lambda(y)) F_\xi(\nabla u_\lambda(y)) \frac{d\partial u_\lambda(y)}{d\lambda} y_i + u_\lambda(y)^{1-p} \frac{du_\lambda(y)}{d\lambda} dy
\]
\[
= \int_{\partial B_1(x_0)} F(\nabla u_\lambda(y)) F_\xi(\nabla u_\lambda(y)) \frac{du_\lambda(y)}{d\lambda} \nu_i dS
\]
\[
= \int_{\partial B_1(x_0)} \langle \nabla u_\lambda(y), y \rangle \frac{du_\lambda(y)}{d\lambda} dS
\]
\[
= \int_{\partial B_1(x_0)} \left[ \lambda \frac{du_\lambda(y)}{d\lambda} + \frac{2}{p+1} u_\lambda(y) \right] \frac{du_\lambda(y)}{d\lambda} dS
\]
\[
= \int_{\partial B_1(x_0)} \lambda \left( \frac{du_\lambda(y)}{d\lambda} \right)^2 + \frac{1}{p+1} \frac{du_\lambda^2(y)}{d\lambda} dS.
\]

Define
\[
E_2(u, x_0, \lambda) := \int_{B_1(x_0)} \frac{1}{2} p^2 (\nabla u_\lambda(y)) + \frac{1}{1-p} u_\lambda(y)^{1-p} dy - \int_{\partial B_1(x_0)} \frac{1}{p+1} u_\lambda^2(y) dS
\]
\[
= \frac{2p^2}{p+1} \frac{N}{2} \int_{B_0(x_0)} \frac{1}{2} p^2 (\nabla u) + \frac{1}{1-p} u^{1-p} dx - \frac{1}{p+1} \lambda^{1-N} \int_{\partial B_0(x_0)} u^2(x) dS.
\]

Therefore,
\[
\frac{d}{d\lambda} E_2(u, x_0, \lambda) \geq 0.
\]

This completes the proof. \( \square \)

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