THE CATEGORIES $\mathcal{T}^c$ AND $\mathcal{T}^b$ DETERMINE EACH OTHER

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Abstract. The main result is very general—it works in the abstract setting of weakly approximable triangulated categories. But it has the following concrete, immediate corollaries.

1. Suppose $X$ is a separated, noetherian scheme. There is a recipe which, out of the category $\mathbf{D}^{\text{perf}}(X)$, constructs $\mathbf{D}^{\text{coh}}(X)$ as a triangulated category.
2. There is a recipe which, out of the category $\mathbf{D}^{\text{coh}}(X)$, constructs $\mathbf{D}^{\text{perf}}(X)$ as a triangulated category.
3. Let $R$ be any ring, possibly noncommutative. The recipe takes the triangulated category $\mathbf{D}^b(R^{\text{-proj}})$, that is the category whose objects are bounded complexes of finitely generated projective modules, and out of it constructs the triangulated category $\mathbf{D}^{-b}(R^{\text{-proj}})$—that is the category of bounded-above cochain complexes of finitely generated projective modules, with bounded cohomology.
4. Now assume $R$ is left-coherent. Starting with $\mathbf{D}^b(R^{\text{-mod}})$ we construct $\mathbf{D}^b(R^{\text{-proj}})$.
5. Out of the homotopy category of finite spectra we construct the homotopy category of spectra with finitely many nonzero stable homotopy groups, all of them finitely generated.
6. Out of the homotopy category of spectra with finitely many nonzero stable homotopy groups, all of them finitely generated, we construct the homotopy category of finite spectra.

More abstractly: given a triangulated category $\mathcal{S}$ it is possible to put a metric on it—the definition is given in the paper. We may complete any essentially small triangulated category $\mathcal{S}$ with respect to any metric, obtaining a category $\mathcal{L}(\mathcal{S})$ which isn’t usually triangulated. But inside $\mathcal{L}(\mathcal{S})$ there is a subcategory $\mathcal{G}(\mathcal{S})$, of objects compactly supported with respect to the metric. And the first main theorem tells us that $\mathcal{G}(\mathcal{S})$ is always triangulated. The second main theorem gives a practical method that can help in computing $\mathcal{G}(\mathcal{S})$.

In the numbered examples above, the metric on each of the triangulated categories can be described intrinsically—it isn’t added structure. There are recipes that start with essentially small triangulated categories and, under some weak hypotheses, cook up metrics.

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0. Introduction

This article began with a question from Krause. In Oberwolfach, in March 2018, Krause told me he had reread Rickard’s old paper [10] on derived Morita equivalence, and didn’t believe some of the proofs. Specifically: Rickard [10, Theorem 6.4] asserts that, if \( R \) and \( S \) are coherent rings, then
\[
D^b(R\text{-proj}) \cong D^b(S\text{-proj}) \iff D^b(R\text{-mod}) \cong D^b(S\text{-mod})
\]
Krause asked if I could find a counterexample. More specifically: he was wondering if the direction \( \iff \) is true. Instead of a counterexample I discovered a proof, based on the ideas of approximability—it’s easily modified to be symmetric enough to do both directions. This article is about a vast generalization, but for the sake of clarity it seems best to start with the simple Oberwolfach argument. But first

Reminder 0.1. Suppose \( S \) is a triangulated category, \( G \in S \) is an object and \( A \leq B \) are integers. Then \( \langle G \rangle^{[A,B]} \) is the smallest full subcategory of \( S \), containing \( \Sigma^{-i}G \) for \( A \leq i \leq B \), and closed under direct summands and extensions.

In the above we allow \( A \) and/or \( B \) to be infinite. For example: the case when \( A = -\infty \) and \( B = \infty \) gives a subcategory \( \langle G \rangle^{(-\infty,\infty)} \), containing all suspensions of \( G \). It is usually abbreviated \( \langle G \rangle \), and is the smallest thick subcategory containing \( G \).

An object \( G \in S \) is called a classical generator if \( S = \langle G \rangle \).

Suppose \( S \) is essentially small. The category \( \text{Mod-}S \) is the category of all additive functors \( H : S^{\text{op}} \rightarrow \text{Ab} \). The Yoneda functor \( Y : S \rightarrow \text{Mod-}S \), taking \( A \in S \) to \( Y(A) = \text{Hom}(\text{-}, A) \in \text{Mod-}S \), is a fully faithful embedding.

If \( \mathcal{C} \) is any pointed category and \( \mathcal{P} \subset \mathcal{C} \) is a subcategory, then \( \mathcal{P}^\perp \) is the full subcategory of all objects \( c \in \mathcal{C} \) with \( \text{Hom}(\mathcal{P}, c) = 0 \). And \( ^\perp \mathcal{P} \) is the full subcategory of all \( c \in \mathcal{C} \) with \( \text{Hom}(c, \mathcal{P}) = 0 \). We will allow ourselves to take perpendiculars in both \( S \) and in \( \text{Mod-}S \). To avoid confusion, when we give a subcategory \( P \subset S \) we will write \( P\perp \) for its perpendicular in \( S \), and \( Y(P)\perp \) for its perpendicular in \( \text{Mod-}S \).

Now we come to something new:

Definition 0.2. Let \( S \) be a triangulated category, and suppose we are given a sequence of subcategories \( \{ P_i \subset S, i \in \mathbb{N} \} \) with \( \Sigma^{-1} P_i \cup P_i \cup \Sigma P_i \subset P_{i+1} \). A sequence in \( S \) of the form
\( E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow \cdots \) is declared to be Cauchy with respect to \( \{ P_i \subset S, n \in \mathbb{N} \} \) if, for every \( i \in \mathbb{N} \), there exists an integer \( N > 0 \) such that \( \text{Hom}(P_i, -) \) takes \( E_n \rightarrow E_{n+1} \) to an isomorphism for all \( n \geq N \) and all \( P \in P_i \).

**Remark 0.3.** There is an obvious notion of equivalence—two sequences of subcategories are declared equivalent if they yield the same Cauchy sequences. For example: if \( S \) has a classical generator \( G \) we can define \( P_i(G) = \langle G \rangle_{[-i, \infty]} \), and the resulting Cauchy sequences don’t depend on the choice of \( G \). The Cauchy sequences in this particular “metric” are intrinsic, they depend only on \( S \).

**Example 0.4.** If \( S = \text{D}^b(\text{R–proj}) \), with \( R \) a ring, then the object \( R \) is a classical generator. Remark 0.3 gives an intrinsic notion of Cauchy sequences—to compute what they are let us put \( P_i = P_i(R) = \langle R \rangle_{[-i, \infty]} \) as in Remark 0.3. The reader can check that sequence \( E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow \cdots \) is Cauchy precisely if, for every integer \( i > 0 \), there exists an integer \( N > 0 \) such that \( H^j(E_n) \rightarrow H^j(E_{n+1}) \) is an isomorphism whenever \( n \geq N \) and \( j \geq -i \).

**Definition 0.5.** Let \( S \) be an essentially small triangulated category. Let the notation be as in Definition 0.2: that is \( \{ P_i \subset S, i \in \mathbb{N} \} \) is a sequence of subcategories satisfying the hypotheses, with corresponding Cauchy sequences. We let \( \mathcal{L}(S) \) be the full subcategory of \( \text{Mod-}S \) whose objects are the colimits of Cauchy sequences in \( S \). And we declare

\[
\mathcal{G}(S) = \mathcal{L}(S) \cap \left[ \bigcup_{i \in \mathbb{N}} Y_{i} \left( \Sigma^n P_i \right)^{\perp} \right]
\]

**Example 0.6.** It is a small exercise to check that, in the case where \( S = \text{D}^b(\text{R–proj}) \) and the Cauchy sequences are as in Example 0.4, the category \( \mathcal{L}(S) \) comes down to the image in \( \text{Mod-}S \) of \( \text{D}^-(\text{R–proj}) \), and if \( R \) is coherent the category \( \mathcal{G}(S) \) is nothing other than \( \text{D}^b(\text{R–mod}) \). We have found an intrinsic way to construct \( \text{D}^b(\text{R–mod}) \) out of \( \text{D}^b(\text{R–proj}) \).

It is interesting to go in the other direction. Our problem becomes to intrinsically define Cauchy sequences in \( \text{D}^b(\text{R–mod}) \). Note that, in general, I have no idea when \( \text{D}^b(\text{R–mod}) \) has a classical generator—there are some theorems, for example Rouquier [11, Theorem 7.38], but here we’re working in the generality of any, possibly noncommutative, noetherian rings.

In the absence of a classical generator the recipe of Remark 0.3 isn’t much use, we need an alternative.

**Definition 0.7.** Let \( S \) be a triangulated category. We define a partial order on its subcategories: we declare \( P \leq Q \) if there exists an integer \( n \) with \( \Sigma^n P \subset Q \).

For any object \( G \in S \) consider the subcategory \( \langle G \rangle_{(-\infty,0]} \). If there is a minimal one, with respect to the partial order above, we call it \( \Omega(S) \). It is well-defined up to equivalence with respect to the partial order.
Example 0.8. If $R$ is noetherian and $S = \mathcal{D}^b(R\text{-mod})$ then there is a minimal $[(G)^{(-\infty,0]}\perp$. After all: any object $G \in \mathcal{D}^b(R\text{-mod})$ is contained in $\mathcal{D}^b(R\text{-mod})^{\leq n}$ for some $n$, therefore $[(G)^{(-\infty,0]}\perp$ contains $\mathcal{D}^b(R\text{-mod})^{\geq n+1}$. But if we take $G = R$ then $[(G)^{(-\infty,0]}\perp = \mathcal{D}^b(R\text{-mod})^{\geq 1}$ and must therefore be minimal.

Definition 0.9. Assume $S$ is an essentially small triangulated category, and assume that a minimal $Q(S)$ as in Definition 0.7 exists. Let the increasing sequence of subcategories $\{P_i \subset S, i \in \mathbb{N}\}$ be $P_i = \Sigma^i Q(S)$.

Example 0.10. Now apply the construction of Definition 0.5 to $S = [\mathcal{D}^b(R\text{-mod})^{\text{op}}$ and to the sequence of categories $\{P_i^{\text{op}} \subset S, i \in \mathbb{N}\}$, with $P_i = \Sigma^i Q(S)$ as in Definition 0.9. A Cauchy sequence turns out to be an inverse system $\cdots \to E_3 \to E_2 \to E_1$ in $\mathcal{D}^b(R\text{-mod})$, such that for every $i > 0$ there exists an $N > 0$ with $H^j(E_{n+1}) \to H^j(E_n)$ an isomorphism whenever $n \geq N$ and $j \geq -i$. The category $\mathcal{L}(S)$ comes down to the image in Mod-$S$ of the category $[\mathcal{D}^-(R\text{-mod})]^{\text{op}}$, and the category $\mathcal{S}(S)$ is nothing other than $[\mathcal{D}^b(R\text{-proj})]^{\text{op}}$. We have found a recipe that goes back.

Remark 0.11. For the direction of passing from $\mathcal{D}^b(R\text{-proj})$ to $\mathcal{D}^b(R\text{-mod})$ Krause [5] has a different argument—to put it succinctly he works with different Cauchy sequences. We did discuss the two approaches in Oberwolfach, and by email in the months since then. The current manuscript sticks to my Oberwolfach Cauchy sequences.

We have explained the simple idea that led to this article. Now it’s time to elaborate on how we expand the ideas—it’s time to tell the reader what else she can expect to find in the article, beyond the simple argument of the last couple of pages.

Let $S$ be an essentially small triangulated category. We will define the notion of a metric on $S$, and with respect to any metric there will be Cauchy sequences—this involves a slight generalization of what we have already seen. As in Definition 0.5 we will define, in the category Mod-$S$, two subcategories $\mathcal{L}(S)$ and $\mathcal{S}(S)$. And the first theorem will be

Theorem 0.12. For any essentially small $S$, and any metric on $S$, the category $\mathcal{S}(S)$ has a triangulated structure which can be defined purely in terms of $S$ and the metric.

We need hardly tell the reader how remarkable this is—there are not many known recipes that start with a triangulated category $S$, and out of it cook up another. The conventional wisdom is that this can only be done in the presence of some enhancement. Maybe a minimal enhancement—like Keller’s towers in Krause [5]. See Keller’s appendix to [5], as well as the original exposition in Keller [4]. But, in defiance of conventional wisdom, in this article there is no enhancement.

It becomes interesting to compute $\mathcal{S}(S)$ in examples. For this it turns out to be helpful to study the following situation.

Notation 0.13. Let $S$ be an essentially small triangulated category with a metric. Suppose we are given a fully faithful triangulated functor $F: S \to \mathcal{T}$; we consider also the
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functor \( \mathcal{Y} : \mathcal{T} \rightarrow \text{Mod-} \mathcal{S} \), which takes an object \( A \in \mathcal{T} \) to the functor \( \text{Hom}(F(\_), A) \). The functor \( F \) is called a good extension with respect to the metric if \( \mathcal{T} \) has coproducts, and for every Cauchy sequence \( E_* \) in \( \mathcal{S} \) the natural map \( \text{colim} Y(E_*) \rightarrow \mathcal{Y} (\text{Hocolim} F(E_*)) \) is an isomorphism.

For any good extension \( F : \mathcal{S} \rightarrow \mathcal{T} \) we proceed to define the full subcategory \( \mathcal{L}'(\mathcal{S}) \subset \mathcal{T} \) to have for objects all the homotopy colimits of Cauchy sequences, and inside \( \mathcal{L}'(\mathcal{S}) \) we define a full subcategory \( \mathcal{L}'(\mathcal{S}) \cap \mathcal{Y}^{-1}(\mathcal{C}(\mathcal{S})) \) — we ask the reader for patience, the definition will come in the body of the paper. The next result is

**Theorem 0.14.** The category \( \mathcal{L}'(\mathcal{S}) \cap \mathcal{Y}^{-1}(\mathcal{C}(\mathcal{S})) \) is a triangulated subcategory of \( \mathcal{T} \), and the functor \( \mathcal{Y} : \mathcal{T} \rightarrow \text{Mod-} \mathcal{S} \) restricts to a triangulated equivalence

\[
\mathcal{Y} : \mathcal{L}'(\mathcal{S}) \cap \mathcal{Y}^{-1}(\mathcal{C}(\mathcal{S})) \rightarrow \mathcal{S}(\mathcal{S})
\]

In the presence of a good extension \( F : \mathcal{S} \rightarrow \mathcal{T} \) this allows us to compute \( \mathcal{S}(\mathcal{T}^c) \), up to triangulated equivalence, as the triangulated subcategory \( \mathcal{L}'(\mathcal{S}) \cap \mathcal{Y}^{-1}(\mathcal{C}(\mathcal{S})) \) of \( \mathcal{T} \).

Next suppose \( \mathcal{T} \) is a triangulated category with coproducts, and assume it has a compact generator \( H \) with \( \text{Hom}(H, \Sigma^i H) = 0 \) for \( i \gg 0 \). In this case the theory introduced in [8, 9] kicks in: there is a preferred equivalence class of \( t \)-structures on \( \mathcal{T} \), and it is possible to define, intrinsically, thick subcategories \( \mathcal{T}^b \subset \mathcal{T}^c \). In terms of the preferred \( t \)-structures it is possible to endow \( \mathcal{T}^c \) and \( [\mathcal{T}^b]^{\text{op}} \) with metrics, we will define them in Example 1.5. It turns out that the embedding \( \mathcal{T}^c \rightarrow \mathcal{T} \) is always a good extension, while the embedding \( [\mathcal{T}^b]^{\text{op}} \rightarrow \mathcal{T}^{\text{op}} \) is a good extension provided \( \mathcal{T} \) is weakly approximable. And the next result says

**Proposition 0.15.** For the metrics above we have

(i) \( \mathcal{S}(\mathcal{T}^c) = \mathcal{T}^b \).

(ii) If \( \mathcal{T} \) is noetherian and weakly approximable then \( \mathcal{S}([\mathcal{T}^b]^{\text{op}}) = [\mathcal{T}^c]^{\text{op}} \).

The notion of a noetherian triangulated category, in Proposition 0.15(ii), is new. It will be defined in Section 5. It is a hypothesis that guarantees there are enough nonzero objects in \( \mathcal{T}^c \), after all there is no a priori reason to expect any.

From the perspective of the Oberwolfach discussion this is still unsatisfactory: in the special case where \( \mathcal{T} = \mathcal{D}(R) \), the derived category of a noetherian ring, the recipe tells us how to pass from

\[
\mathcal{T}^c = \mathcal{D}^b(\text{R-proj}), \text{ together with its metric} \quad \Rightarrow \quad \mathcal{D}^b(\text{R-mod}) = \mathcal{T}^b \]
\[
[\mathcal{T}^b]^{\text{op}} = \mathcal{D}^b(\text{R-mod})^{\text{op}}, \text{ together with its metric} \quad \Rightarrow \quad \mathcal{D}^b(\text{R-mod})^{\text{op}} = [\mathcal{T}^b]^{\text{op}}
\]

But the metrics are defined in terms of the preferred equivalence class of \( t \)-structures on \( \mathcal{T} \). As presented, the metrics depend on the embedding into \( \mathcal{T} \).

Hence it becomes interesting to see when we can construct the metrics intrinsically, without reference to \( \mathcal{T} \). It turns out we can always do this. More precisely: the recipe
of Definitions 0.7 and 0.9 works in general, to give the metric on \([\mathcal{T}_c^b]^\text{op}\) in Proposition 0.15(ii). There is also an intrinsic description of the metric on \(\mathcal{T}^c\) used in Proposition 0.15(i), we will see it in Definition 4.5(i) and Remark 4.7, but it isn’t the recipe given in Remark 0.3. For the metric of Remark 0.3 to agree with the metric in Proposition 0.15(i) we will need to assume \(\mathcal{T}\) weakly approximable, see Proposition 4.8.

Remark 0.16. In Remark 0.11 we mentioned that the development in Krause [5] is different. One way to say it is that the completion of the triangulated category \(\mathcal{S}\) depends on a choice of metric—Krause prefers to work with a metric different from mine.

The general theory developed here applies to Krause’s metric. I have only fully worked out what happens for \(\mathcal{S} = D^b(R\text{-proj})\), with \(R\) a noetherian ring. In this case the triangulated category \(\mathcal{S}(\mathcal{S})\) turns out to be \(D^b(R\text{-mod})\), the category of bounded complexes of injective \(R\)-modules whose cohomology modules are finite. If \(R\) has a dualizing complex this category is equivalent to \(D^b(R\text{-proj})\). For more detail see Examples 3.4 and 3.10 as well as Remark 4.9.

Finally we should say something about the structure of the article. The first two sections work with a triangulated category \(\mathcal{S}\) and its metric—there is no mention of good extensions, the sections are devoted to the proof of Theorem 0.12 and are self-contained. Section 3 is where we prove Theorem 0.14—if the reader ignores the examples, Section 3 is also self-contained. But the later sections, which work out the general theory in the examples \(\mathcal{T}^c\) and \([\mathcal{T}_c^b]^\text{op}\), assume familiarity with approximability.

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1. The basic definitions

Reminder 1.1. Let \(\mathcal{S}\) be a triangulated category and let \(\mathcal{A}, \mathcal{C}\) be subcategories. As in [2, 1.3.9] we define the full subcategory \(\mathcal{A} \ast \mathcal{C} \subset \mathcal{S}\) to have for objects those \(b \in \mathcal{S}\) for which there exists, in \(\mathcal{S}\), a triangle \(a \to b \to c \to a \) with \(a \in \mathcal{A}\) and \(c \in \mathcal{C}\).

Definition 1.2. Let \(\mathcal{S}\) be a triangulated category. A metric on \(\mathcal{S}\) will be a sequence of additive subcategories \(\{M_i \in \mathcal{S} \mid i \in \mathbb{N}\}\) such that

(i) \(M_{i+1} \subset M_i\) for every \(i\).
(ii) \(M_i \ast M_i = M_i\).

A metric \(\{M_i\}\) is declared to be finer than the metric \(\{N_i\}\) if, for every integer \(i > 0\), there exists an integer \(j > 0\) with \(M_j \subset N_i\); we denote this partial order by \(\{M_i\} \preceq \{N_i\}\).

The metrics \(\{M_i\}\), \(\{N_i\}\) are equivalent if \(\{M_i\} \preceq \{N_i\} \preceq \{M_i\}\).

Example 1.3. The dumb example is to let \(M_i = \mathcal{S}\) for every \(i\).
Reminder 1.4. For the next example we remind the reader of some constructions from \[8\]. Let \(T\) be a triangulated category. In \[8\] Definition 0.10 we declared two \(t\)-structures \((T_1^{<0}, T_2^{>0})\) and \((T_2^{<0}, T_2^{>0})\) to be equivalent if there exists an integer \(A > 0\) with \(T_1^{<0} \subset T_2^{>0} \subset T_2^{<0} \subset T_1^{>0}\). Now assume \(T\) has coproducts and a compact generator \(G\). Then \[8\] Definition 0.14] defines a preferred equivalence class of \(t\)-structures, namely the one containing the \(t\)-structure generated by \(G\) in the sense of Alonso, Jeremias and Souto \[1\]. And \[8\] Definition 0.16] allows one to construct two subcategories \(T_c \subset T_c\) and \(T_c \subset T_c\). If the compact generator \(G \in T\) is such that \(\text{Hom}(G, \Sigma^i G) = 0\) for \(i \gg 0\), then \(T_c \subset T_c\) are both thick subcategories of \(T\), see \[8\] Proposition 2.10].

Example 1.5. Suppose \(T\) is a triangulated category with coproducts, and assume \(T\) has a compact generator \(G\) with \(\text{Hom}(G, \Sigma^i G) = 0\) for \(i \gg 0\). With the notation as in Reminder \[\text{1.4}\], let \((T^{<0}, T^{>0})\) be a \(t\)-structure in the preferred equivalence class. Out of this data we can construct two examples of \(S\)'s with metrics:

(i) Let \(S\) be the subcategory \(T_c \subset T\), and put \(M_i = T_c \cap T^{<0}\).

(ii) Let \(S\) be the subcategory \([T_c]\)\(^{op}\), and put \(M_i^{op} = T_c \cap T^{<0}\).

It's obvious that equivalent \(t\)-structures define equivalent metrics. Thus up to equivalence we have a canonical metric on \(T_c\) and a canonical metric on \([T_c]\)\(^{op}\). But the definition depends on the embedding into \(T\), which is the category with the \(t\)-structure.

Definition 1.6. Let \(S\) be a triangulated category with a metric \(\{M_i\}\). A Cauchy sequence in \(S\) is a sequence of \(E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow \cdots\) so that, for every pair of integers \(i > 0\), \(j \in \mathbb{Z}\), there exists an integer \(M > 0\) such that, in any triangle \(E_m \rightarrow E_{m'} \rightarrow D_{m,m'}\) with \(M \leq m < m'\), the object \(\Sigma^j D_{m,m'}\) lies in \(M_i\).

Remark 1.7. Note that the Cauchy sequences depend only on the equivalence class of the metric.

The following observation is useful for constructing Cauchy sequences.

Lemma 1.8. Suppose we are given in \(S\) a sequence of \(E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow \cdots\) so that, for every integer \(i > 0\), \(j \in \mathbb{Z}\) there exists an integer \(M > 0\) such that, in any triangle \(E_m \rightarrow E_{m+1} \rightarrow D_m\) with \(M \leq m\), the object \(\Sigma^j D_m\) lies in \(M_i\).

Then for all integers \(M \leq m < m'\), the triangles \(E_m \rightarrow E_{m'} \rightarrow D_{m,m'}\) have \(\Sigma^j D_{m,m'}\) \(\in M_i\). In particular: the sequence \(E_*\) is Cauchy.

Proof. We prove the assertion by induction on \(m' - m\), the case \(m' - m = 1\) being the hypothesis. Suppose the result is true for \(m' - m \leq k\), and construct an octahedron on the composable maps \(E_m \rightarrow E_{m+k} \rightarrow E_{m+k+1}\); we obtain a triangle \(D_{m,m+k} \rightarrow D_{m,m+k+1} \rightarrow D_{m+k}\), and induction gives that \(\Sigma^j D_{m,m+k}, \Sigma^j D_{m+k+1} \in M_i\). It follows that \(\Sigma^j D_{m,m+k+1} \in M_i * M_i = M_i\). \(\square\)

Remark 1.9. Given an essentially small triangulated category \(S\) it is customary to consider the Yoneda functor on it, we wish to explore the functor \(Y : S \rightarrow \text{Mod-S}\). To
recall the notation: \( \text{Mod-S} \) is the category of additive functors \( S^{\text{op}} \rightarrow \mathbb{Z}-\text{Mod} \), and the functor \( Y \) takes the object \( A \in S \) to the additive functor \( Y(A) = \text{Hom}(\cdot, A) \).

**Definition 1.10.** Suppose \( S \) is an essentially small triangulated category with a metric \( \{ M_i \} \). We define three full subcategories \( \mathcal{L}(S) \), \( \mathcal{C}(S) \) and \( \mathcal{S}(S) \) of the category \( \text{Mod-S} \) as follows:

(i) The objects of \( \mathcal{L}(S) \) are the functors \( S^{\text{op}} \rightarrow \mathbb{Z}-\text{Mod} \) which can be expressed as \( \text{colim} \rightarrow Y(E_i) \), where \( E_* \) is a Cauchy sequence in \( S \).

(ii) The objects of \( \mathcal{C}(S) \) are the compactly supported functors with respect to the metric. Concretely, they are given by the formula

\[
\mathcal{C}(S) = \left\{ A \in \text{Mod-S} \left| \begin{array}{c}
\text{For every integer } j \in \mathbb{Z} \text{ there exists an integer } i > 0 \text{ with } \text{Hom}(Y(\Sigma^j M_i), A) = 0
\end{array} \right. \right\}.
\]

(iii) Finally \( \mathcal{S}(S) \) is defined by \( \mathcal{S}(S) = \mathcal{L}(C) \cap \mathcal{C}(S) \).

**Remark 1.11.** First of all: it’s obvious that the categories \( \mathcal{L}(S) \), \( \mathcal{C}(S) \) and \( \mathcal{S}(S) \) depend only on the equivalence class of the metric.

Next note that the Yoneda functor \( Y : S \rightarrow \text{Mod-S} \) factors through the subcategory \( \mathcal{L}(S) \), after all the constant sequence \( E \overset{\text{id}}{\rightarrow} E \overset{\text{id}}{\rightarrow} E \overset{\text{id}}{\rightarrow} \cdots \) is Cauchy for any metric, and the colimit in \( \text{Mod-S} \) of the image of this sequence under Yoneda is \( Y(E) \).

Finally observe that all the objects of \( \mathcal{L}(S) \) are homological functors \( S^{\text{op}} \rightarrow \mathbb{Z}-\text{Mod} \). After all they are filtered colimits of the homological functors \( Y(E_i) \).

**Example 1.12.** In the special case where the metric is the dumb one in Example 1.3, that is \( M_i = S \) for every \( i \), every sequence is Cauchy and \( \mathcal{L}(S) \) is the Ind-completion of \( S \). The category \( \mathcal{C}(S) \) and \( \mathcal{S}(S) \) are both equal to \( \{ 0 \} \). The theory doesn’t produce much.

2. The category \( \mathcal{S}(S) \) is triangulated

**Notation 2.1.** Throughout this section we will fix the triangulated category \( S \) together with its metric \( \{ M_i \} \). The only categories we will study in the section are full subcategories of \( \text{Mod-S} \): the subcategories \( \mathcal{L}(S) \), \( \mathcal{C}(S) \) and \( \mathcal{S}(S) \) of Definition 1.10, as well as the subcategory \( S \subset \mathcal{L}(S) \). Note that we view \( S \) as embedded in \( \mathcal{L}(S) \subset \text{Mod-S} \) through the fully faithful functor \( Y \). And most of the time we will freely confuse \( S \) with its image under \( Y : S \rightarrow \text{Mod-S} \).

It’s only in the statements—not proofs—of results that we plan to appeal to in later sections that we try to be careful with the notation. The reason is that in later sections we will allow ourselves to embed \( S \) into other categories \( \mathcal{T} \), and confusion could arise.

**Discussion 2.2.** We define an invertible automorphism \( \Sigma : \mathbb{S}-\text{Mod} \rightarrow \mathbb{S}-\text{Mod} \) by the rule

(i) If \( A \) is an object of \( \text{Mod-S} \), meaning a functor \( A : S^{\text{op}} \rightarrow \mathbb{Z}-\text{Mod} \), and \( s \) is an object of \( S \), then \( [\Sigma A](s) = A(\Sigma^{-1}s) \).
The Yoneda isomorphism \( A(s) \cong \text{Hom}_{\text{Mod-}\mathcal{S}}(Y(s), A) \) permits us, in our sloppy conventions established in Notation 2.1, to rewrite (i) as \( \text{Hom}(s, \Sigma A) = \text{Hom}(\Sigma^{-1}s, A) \). The homological functors \( A : S^{op} \to \text{Mod-}\mathcal{Z} \) are precisely the objects \( A \in \text{Mod-}\mathcal{S} \) such that \( \text{Hom}_{\text{Mod-}\mathcal{S}}(-, A) \) restricts to a homological functor on \( S^{op} \). In particular: in Remark 1.11 we noted

(ii) If \( A \) belongs to \( \mathcal{L}(S) \), then the restriction to \( S^{op} \) of the functor \( \text{Hom}_{\text{Mod-}\mathcal{S}}(-, A) \) is homological.

Finally, a sequence \( A \to B \to C \) in \( \text{Mod-}\mathcal{S} \) is exact if it is exact when evaluated at each \( s \in \mathcal{S} \). Our notation translates this into

(iii) The sequence \( A \to B \to C \) in \( \text{Mod-}\mathcal{S} \) is exact if and only if, for every object \( s \in \mathcal{S} \), the functor \( \text{Hom}(s, -) \) takes it to an exact sequence.

Observation 2.3. Since the formula will be cited in future sections our notation is careful, we write

\[
\mathcal{C}(\mathcal{S}) = \bigcap_{j \in \mathbb{Z}} \bigcup_{i \in \mathbb{N}} [Y(\Sigma^j M_i)]^\perp
\]

Reverting to the sloppy conventions of Notation 2.1 and Discussion 2.2 for the explanation: for \( A \in \text{Mod-}\mathcal{S} \) the condition \( \text{Hom}_{\text{Mod-}\mathcal{S}}(\Sigma^j M_i, A) = 0 \) rewrites as \( A \notin \left[ \Sigma^j M_i \right]^\perp \), and the displayed formula above just codifies the quantifiers on \( i, j \) in Definition 1.10(ii).

The above makes it clear that

(i) \( \mathcal{C}(\mathcal{S}) \) is closed in \( \text{Mod-}\mathcal{S} \) under direct summands.

(ii) \( \Sigma \mathcal{C}(\mathcal{S}) = \mathcal{C}(\mathcal{S}) \).

(iii) Given an exact sequence \( A \to B \to C \) in \( \text{Mod-}\mathcal{S} \), we have

\[
A, C \in \mathcal{C}(\mathcal{S}) \implies B \in \mathcal{C}(\mathcal{S}) .
\]

Perhaps we should explain (iii). Since \( M_i, M_j \) both contain \( M_{i+j} \) we have

\[
A \in M_i^\perp \quad \text{and} \quad C \in M_j^\perp \implies B \in M_{i+j}^\perp,
\]

and hence

\[
A \in \bigcup_i M_i^\perp \quad \text{and} \quad C \in \bigcup_i M_i^\perp \implies B \in \bigcup_i M_i^\perp .
\]

Now (iii) follows by applying the above to the exact sequences \( \Sigma^j A \to \Sigma^j B \to \Sigma^j C \) for all \( j \in \mathbb{Z} \).

Definition 2.4. Let \( S \subset \text{Mod-}\mathcal{S} \) be as in Notation 2.1. A pre-triangle is a diagram \( A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \) in \( \text{Mod-}\mathcal{S} \) such that \( A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \xrightarrow{\Sigma f} \Sigma B \) is an exact sequence.

Remark 2.5. From Observation 2.3(iii) we learn that, if \( A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \) is a pre-triangle in \( \text{Mod-}\mathcal{S} \) and if two of \( A, B, C \) lie in \( \mathcal{C}(\mathcal{S}) \), then so does the third.

Having made the definition, it becomes interesting to construct examples.
Lemma 2.6. Let $f : A \rightarrow B$ be a morphism in $\mathfrak{L}(S)$. Then we may complete it to a pre-triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ in the category $\mathfrak{L}(S)$, which is the colimit in Mod-$S$ of the image under $Y$ of a Cauchy sequence of triangles $a_* \xrightarrow{f_*} b_* \xrightarrow{g_*} c_* \xrightarrow{h_*} \Sigma a_*$ in the category $S$.

Moreover: we may choose the Cauchy sequence $a_*$ such that $A = \text{colim Y}(a_*)$ in advance. And if we are given a Cauchy sequence $b'_* \in S$ with $B = \text{colim Y}(b'_*)$, we may choose $b_*$ to be a subsequence of $b'_*$. We may even specify in advance the choice of the first triangle $a_1 \xrightarrow{f_1} b_1 \xrightarrow{g_1} c_1 \xrightarrow{h_1} \Sigma a_1$, provided the square

$$
\begin{array}{ccc}
Y(a_1) & \xrightarrow{Y(f_1)} & Y(b_1) \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & B
\end{array}
$$

commutes.

Finally: if we are given an integer $\ell > 0$, and assume that the given Cauchy sequences $a_*$ and $b'_*$ of the paragraph above are such that, for all positive integers $m < m'$, the triangles $a_m \rightarrow a_{m'} \rightarrow d_{m,m'}$ and $b'_m \rightarrow b'_{m'} \rightarrow d_{m,m'}$ are such that the objects $\Sigma d_{m,m'}, \Sigma^{-1} d_{m,m'}, d_{m,m'}$ lie in $M_\ell$, then in the triangles $c_m \rightarrow c_{m'} \rightarrow \widehat{d}_{m,m'}$ we can guarantee that $\Sigma^{-1} \widehat{d}_{m,m'}, \widehat{d}_{m,m'}$ will also lie in $M_\ell$.

Proof. Because $A, B$ belong to $\mathfrak{L}(S)$ we can find Cauchy sequences converging to them—if the sequences are given we work with those. But in any case we may choose Cauchy sequences $a_*, b'_* \in S$ with $A \cong \text{colim} a_*$ and $B \cong \text{colim} b'_*$. We are given in $\mathfrak{L}(S)$ the composite $a_1 \rightarrow A \xrightarrow{f} B$, which is an element in $\text{colim} \text{Hom}(a_1, b'_*)$. We may choose a preimage in some $\text{Hom}(a_1, b'_*)$, constructing a commutative square

$$
\begin{array}{ccc}
a_1 & \xrightarrow{f_1} & b'_1 \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & B
\end{array}
$$

If we are given $f_1$ we begin with it.
And then we continue inductively: if we have carried out the construction as far as the integer \( i \), then we have a commutative diagram

\[
\begin{array}{ccc}
  a_i & \xrightarrow{f_i} & b'_i \\
  \downarrow & & \downarrow \\
  a_{i+1} & \xrightarrow{f_{i+1}} & b'_{i+1} \\
  \downarrow & & \downarrow \\
  A & \xrightarrow{f} & B \\
\end{array}
\]

which we may, by choosing \( \ell_{i+1} \) large enough, complete to a commutative diagram

\[
\begin{array}{ccc}
  a_i & \xrightarrow{f_i} & b'_i \\
  \downarrow & & \downarrow \\
  a_{i+1} & \xrightarrow{f_{i+1}} & b'_{i+1} \\
  \downarrow & & \downarrow \\
  A & \xrightarrow{f} & B \\
\end{array}
\]

constructing the subsequence \( b_* \) of \( b'_* \) and the map \( f_* : a_* \to b_* \).

Next complete each \( f_i : a_i \to b_i \) to a triangle \( a_i \xrightarrow{f_i} b_i \xrightarrow{g_i} c_i \xrightarrow{h_i} \Sigma a_i \); if the triangle \( a_1 \xrightarrow{f_1} b_1 \xrightarrow{g_1} c_1 \xrightarrow{h_1} \Sigma a_1 \), is already given we work with it. In the paragraph above we constructed commutative squares

\[
\begin{array}{ccc}
  a_i & \xrightarrow{f_i} & b_i \\
  \downarrow \alpha_i & & \downarrow \beta_i \\
  a_{i+1} & \xrightarrow{f_{i+1}} & b_{i+1} \\
\end{array}
\]

which we complete to a 3 \times 3 diagram of triangles

\[
\begin{array}{ccc}
  a_i & \xrightarrow{f_i} & b_i \xrightarrow{g_i} c_i \xrightarrow{h_i} \Sigma a_i \\
  \downarrow \alpha_i & \downarrow \beta_i & \downarrow \gamma_i \\
  a_{i+1} & \xrightarrow{f_{i+1}} & b_{i+1} \xrightarrow{g_{i+1}} c_{i+1} \xrightarrow{h_{i+1}} \Sigma a_{i+1} \\
  \downarrow d_i & \downarrow \hat{d}_i & \downarrow \Sigma d_i \\
  \Sigma d_i & & \Sigma d_i \\
\end{array}
\]

We note that, so far

(i) We have extended the sequence of maps \( f_* : a_* \to b_* \) to a sequence of triangles

\[
a_* \xrightarrow{f_*} b_* \xrightarrow{g_*} c_* \xrightarrow{h_*} \Sigma a_* .
\]
Suppose we are given a pair of integers \( j \in \mathbb{Z} \) and \( n \in \mathbb{N} \). We may choose an integer \( M > 0 \) so that, for all \( i \geq M \), we have \( \Sigma^j d_i, \Sigma^{j+1} d_i \in \mathcal{M}_n \). But then \( \Sigma^j \hat{d}_i \in \mathcal{M}_n + \mathcal{M}_n = \mathcal{M}_n \), and since this is true for all \( i \geq M \), Lemma 1.8 permits us to conclude

(ii) In the sequence of triangles of (ii), the sequence \( c_* \) is Cauchy.

Moreover the “finally” part of the Lemma holds by construction: if the sequences \( a_* \) and \( b'_* \) that we began with satisfy the hypotheses then so do the sequences \( a_* \) and \( b_* \)—passing to a subsequence is harmless. Thus in the \( 3 \times 3 \) diagram above the objects \( \overline{d}_i, \Sigma \overline{d}_i, \Sigma^{-1} \overline{d}, \overline{d} \) lie in \( \mathcal{M}_\ell \) for every integer \( i \geq 0 \), hence so do \( \Sigma^{-1} \overline{d}_i, \overline{d}_i \), and the “finally” assertion comes from Lemma 1.8.

\[ \square \]

Remark 2.7. Lemma 2.6 produces examples of pre-triangles. Another source is mapping cones: given a morphism of pre-triangles

\[
\begin{array}{c}
A \\ \downarrow u
\end{array} \longrightarrow
\begin{array}{c}
B \\ \downarrow v
\end{array} \longrightarrow
\begin{array}{c}
C \\ \downarrow w
\end{array} \longrightarrow
\begin{array}{c}
\Sigma A \\ \downarrow \Sigma u
\end{array}
\]

\[
\begin{array}{c}
A' \\ \downarrow f'
\end{array} \longrightarrow
\begin{array}{c}
B' \\ \downarrow g'
\end{array} \longrightarrow
\begin{array}{c}
C' \\ \downarrow h'
\end{array} \longrightarrow
\begin{array}{c}
\Sigma A' \\ \downarrow \Sigma A'
\end{array}
\]

then the mapping cone is also a pre-triangle

\[
\begin{array}{c}
A' \oplus B \\ \downarrow \begin{pmatrix} f' & v \\ 0 & -g \end{pmatrix}
\end{array} \longrightarrow
\begin{array}{c}
B' \oplus C \\ \downarrow \begin{pmatrix} g' & w \\ 0 & -h \end{pmatrix}
\end{array} \longrightarrow
\begin{array}{c}
C' \oplus \Sigma A \\ \downarrow \begin{pmatrix} h' & \Sigma u \\ 0 & -\Sigma f \end{pmatrix}
\end{array} \longrightarrow
\begin{array}{c}
\Sigma A' \oplus \Sigma B \\ \downarrow \Sigma A' \oplus \Sigma B
\end{array}
\]

In view of Remark 2.7 our next project is to learn how to construct morphisms of pre-triangles. For this the next little lemma is helpful.

Lemma 2.8. Suppose we are given:

(i) A homological object \( B \in \mathcal{C}(\mathcal{S}) \). We remind the reader: the fact that \( B \) is homological means that \( \text{Hom}_{\mathcal{S}_{\text{Mod-S}}}(-, B) \) restricts to a homological functor on \( \mathcal{S}_{\text{op}} \subset [\mathcal{S}_{\text{Mod-S}}]_{\text{op}} \).

(ii) An object \( A \in \mathcal{L}(\mathcal{S}) \). Assume also that we are given a Cauchy sequence \( a_* \) in \( \mathcal{S} \) with colimit \( A \).

Then there exists an integer \( n > 0 \) so that, for any integer \( i \geq n \), any map \( a_i \to B \) factors uniquely as \( a_i \to A \to B \). More precisely: if we choose an integer \( j > 0 \) with \( B \in \mathcal{M}_j \), then just choose \( n \) to be an integer such that, for all \( i \geq n \), the triangles \( a_i \to a_{i+1} \to d_i \) have \( \Sigma^{-1} d_i, d_i \in \mathcal{M}_j \).

Proof. Apply the homological functor \( \text{Hom}(-, B) \) to the triangles \( \Sigma^{-1} d_i \to a_i \to a_{i+1} \to d_i \). The hypotheses guarantee that \( \text{Hom}(a_{i+1}, B) \to \text{Hom}(a_i, B) \) is an isomorphism whenever \( i \geq n \), allowing us to extend any map \( a_i \to B \), uniquely, to a map from \( a_* \) to \( B \).

\[ \square \]

Corollary 2.9. Assume we are given
(i) A pre-triangle as constructed in Lemma 2.6. That is: in the category $\mathcal{S}(S)$ we have a diagram $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$, which is the colimit of the image under $Y$ of some given Cauchy sequence of triangles $a_* \xrightarrow{f_*} b_* \xrightarrow{g_*} c_* \xrightarrow{h_*} \Sigma a_*$ in the category $S$.

(ii) Composable morphisms $A' \xrightarrow{f'} B' \xrightarrow{g'} C' \xrightarrow{h'} \Sigma A'$ in the category $\mathcal{E}(S)$, where the objects $A', B'$ and $C'$ are all homological. Then there exists an integer $n > 0$ such that, for any integer $i \geq n$, any commutative diagram

\[
\begin{array}{cccccc}
Y(a_i) & \xrightarrow{Y(f_i)} & Y(b_i) & \xrightarrow{Y(g_i)} & Y(c_i) & \xrightarrow{Y(h_i)} & Y(\Sigma a_i) \\
\downarrow u_i & & \downarrow v_i & & \downarrow w_i & & \downarrow \Sigma u_i \\
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & \Sigma A'
\end{array}
\]

factors uniquely through a map

\[
\begin{array}{cccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\
\downarrow u & & \downarrow v & & \downarrow w & & \downarrow \Sigma u \\
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & \Sigma A'
\end{array}
\]

More precisely: if we choose $j > 0$ so that $A', B', C', \Sigma A'$ all belong to $Y(M_j)^\perp$, it suffices to choose the integer $n > 0$ large enough so that, for any $i \geq n$, in the triangles

\[
a_i \xrightarrow{a_i} a_{i+1} \rightarrow \overline{d}_i, \quad b_i \xrightarrow{\beta_i} b_{i+1} \rightarrow d_i, \quad c_i \xrightarrow{\gamma_i} c_{i+1} \rightarrow \tilde{d}_i
\]

we have $\Sigma^{-1}\overline{d}_i, \overline{a}_i, \Sigma^{-1}d_i, d_i, \Sigma^{-1}\tilde{d}_i, \tilde{d}_i$ belonging to $M_j$.

**Proof.** Lemma 2.8 says that, with our choice of integer $n$, if we are given an integer $i \geq n$ and a map from any of $a_i$, $b_i$ or $c_i$ to any of $A'$, $B'$, $C'$ or $\Sigma A'$, then the map factors uniquely through the respective $a_i \rightarrow A$, $b_i \rightarrow B$ or $c_i \rightarrow C$. Applying this to the maps $u_i : a_i \rightarrow A'$, $v_i : b_i \rightarrow B'$ and $w_i : c_i \rightarrow C'$ we factor them uniquely as

\[
a_i \xrightarrow{\tilde{a}_i} A \xrightarrow{u} A', \quad b_i \xrightarrow{\tilde{\beta}_i} B \xrightarrow{v} B', \quad c_i \xrightarrow{\tilde{\gamma}_i} C \xrightarrow{u} C',
\]

producing a diagram

\[
\begin{array}{cccccc}
a_i & \xrightarrow{f_i} & b_i & \xrightarrow{g_i} & c_i & \xrightarrow{h_i} & \Sigma a_i \\
\tilde{a}_i & & \tilde{\beta}_i & & \tilde{\gamma}_i & & \Sigma \tilde{a}_i \\
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\
\downarrow u & & \downarrow v & & \downarrow w & & \downarrow \Sigma u \\
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & \Sigma A'
\end{array}
\]

By construction we know that, when we delete the middle row of (2), we are left with diagram (1) in the statement of the Corollary—this diagram commutes by hypothesis.
Deleting the bottom row of (2) leaves us a commutative diagram, given by the map from the triangle \( a_i \to b_i \to c_i \to \Sigma a_i \) to the colimit of \( a_s \to b_s \to c_s \to \Sigma a_s \). Thus the composites in each of the squares

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow u & & \downarrow v \\
A' & \xrightarrow{f'} & B'
\end{array}
\quad
\begin{array}{ccc}
B & \xrightarrow{g} & C \\
\downarrow v & & \downarrow w \\
B' & \xrightarrow{g'} & C'
\end{array}
\quad
\begin{array}{ccc}
C & \xrightarrow{h} & \Sigma A \\
\downarrow w & & \downarrow \Sigma u \\
C' & \xrightarrow{h'} & \Sigma A'
\end{array}
\]

give a pair of maps rendering equal the composites

\[
a_i \xrightarrow{\tilde{\alpha}_i} A \xrightarrow{=} B' \quad b_i \xrightarrow{\tilde{\beta}_i} B \xrightarrow{=} C' \quad c_i \xrightarrow{\tilde{\gamma}_i} C \xrightarrow{=} \Sigma A'
\]

The uniqueness assertion of Lemma 2.8 gives that the three squares must commute. □

**Definition 2.10.** In the category \( \mathfrak{G}(S) = \mathfrak{L}(S) \cap \mathfrak{C}(S) \), we declare a sequence \( A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \) to be a distinguished triangle if it is isomorphic to the colimit of the image under \( Y \) of some Cauchy sequence of triangles \( a^* \xrightarrow{f^*} b^* \xrightarrow{g^*} c^* \xrightarrow{h^*} \Sigma a^* \) in the category \( S \).

**Theorem 2.11.** With the distinguished triangles as in Definition 2.10, the category \( \mathfrak{G}(S) \) is triangulated.

**Proof.** We need to show that the axioms of triangulated categories are satisfied. We begin with the obvious: for any object \( A \in \mathfrak{G}(S) \) the sequence \( A \xrightarrow{id} A \to 0 \to \Sigma A \) is clearly a triangle, just choose a Cauchy sequence \( a_s \) with colimit \( A \) and consider the Cauchy sequence of triangles \( a_s \xrightarrow{id} a_s \to 0 \to \Sigma a_s \). Any isomorph of a triangle is a triangle by definition.

Suppose we are given a morphism \( A \to B \) in \( \mathfrak{G}(S) \). Lemma 2.6 permits us to extend this to a pre-triangle \( A \to B \to C \to \Sigma A \) in the category \( \mathfrak{L}(S) \), which is a colimit of a Cauchy sequence of triangles \( a^* \xrightarrow{f^*} b^* \xrightarrow{g^*} c^* \xrightarrow{h^*} \Sigma a^* \) in the category \( S \). Remark 2.5 permits us to conclude that \( C \in \mathfrak{C}(S) \), hence \( C \) belongs to \( \mathfrak{G}(S) = \mathfrak{L}(S) \cap \mathfrak{C}(S) \).

This completes the proof of [TR1]. The axiom [TR2] is obvious, the rotations of triangles in \( \mathfrak{G}(S) \) are triangles.

It remains to prove [TR3] and [TR4] as in [7] Definitions 1.1.2 and 1.3.13; we need to show that, given a commutative diagram in the category \( \mathfrak{G}(S) \) where the rows are triangles

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow u & & \downarrow v \\
A' & \xrightarrow{f'} & B'
\end{array}
\quad
\begin{array}{ccc}
B & \xrightarrow{g} & C \\
\downarrow v & & \downarrow w \\
B' & \xrightarrow{g'} & C'
\end{array}
\quad
\begin{array}{ccc}
C & \xrightarrow{h} & \Sigma A \\
\downarrow w & & \downarrow \Sigma u \\
C' & \xrightarrow{h'} & \Sigma A'
\end{array}
\]

we may complete it to a morphism of triangles, and even do so in such a way that the mapping cone is a triangle.
OK: because the diagram lies in $\mathcal{S}(\mathcal{S}) \subset \mathcal{C}(\mathcal{S})$ we may choose an integer $j > 0$ so that the objects $A, B, C, \Sigma A, \Sigma B, A', B', C', \Sigma A', \Sigma B'$ all lie in $M_j^\perp$. Next: because the rows are triangles we may choose Cauchy sequences of triangles $a_\ast \xrightarrow{f_\ast} b_\ast \xrightarrow{g_\ast} c_\ast \xrightarrow{h_\ast} \Sigma a_\ast$ and $a'_\ast \xrightarrow{f'_\ast} b'_\ast \xrightarrow{g'_\ast} c'_\ast \xrightarrow{h'_\ast} \Sigma a'_\ast$ whose colimits are, respectively, $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ and $A' \xrightarrow{f'} B' \xrightarrow{g'} C' \xrightarrow{h'} \Sigma A'$. And since all these sequences are Cauchy we can, by ingoring the early parts of the Cauchy sequences, assume that, with $x_\ast$ standing for any of $a_\ast, b_\ast, c_\ast, d_\ast, b'_\ast$ or $c'_\ast$, in any triangle $x_m \rightarrow x_m' \rightarrow d$ the objects $\Sigma^{-1} d, d$ and $\Sigma d$ lie in $M_j$.

We are given the commutative diagram

\[
\begin{array}{ccc}
a_1 & \xrightarrow{f_1} & b_1 \\
\downarrow{\tilde{\alpha}_1} & & \downarrow{\tilde{\beta}_1} \\
A & \xrightarrow{f} & B \\
\downarrow{u} & & \downarrow{v} \\
A' & \xrightarrow{f'} & B'
\end{array}
\]

and, because $A' = \text{colim } a'_\ast$ and $B' = \text{colim } b'_\ast$, we may factor the composite through some commutative diagram

\[
\begin{array}{ccc}
a_1 & \xrightarrow{f_1} & b_1 \\
\downarrow{u_\ell} & & \downarrow{v_\ell} \\
da'_\ell & \xrightarrow{f'_\ell} & b'_\ell \\
\downarrow{\tilde{\alpha}'_\ell} & & \downarrow{\tilde{\beta}'_\ell} \\
A' & \xrightarrow{f'} & B'
\end{array}
\]

In the triangulated category $\mathcal{S}$ we may complete the commutative diagram whose rows are triangles

\[
\begin{array}{ccc}
a_1 & \xrightarrow{f_1} & b_1 \\
\downarrow{u_\ell} & & \downarrow{v_\ell} \\
da'_\ell & \xrightarrow{f'_\ell} & b'_\ell \\
\downarrow{w_\ell} & & \downarrow{h'_\ell} \\
A' & \xrightarrow{f'} & B'
\end{array}
\]

\[
\Sigma\end{array}
\]

to a morphism of triangles

\[
\begin{array}{ccc}
a_1 & \xrightarrow{f_1} & b_1 \\
\downarrow{u_\ell} & & \downarrow{v_\ell} \\
da'_\ell & \xrightarrow{f'_\ell} & b'_\ell \\
\downarrow{w_\ell} & & \downarrow{h'_\ell} \\
A' & \xrightarrow{f'} & B'
\end{array}
\]

\[
\Sigma\end{array}
\]
and even do so in such a way that the mapping cone is a triangle. Now consider the composite

![Diagram](attachment:image.png)

Corollary 2.9 applies. Actually, we use the “more precisely” refinement, with \( n = 1 \). The Corollary allows us to factor the composite, uniquely, through

![Diagram](attachment:image2.png)

This already establishes [TR3], but we assert further that the mapping cone is a triangle in \( \mathcal{S}(S) \).

To simplify the notation let us write the mapping cone of (2) as \( X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \). By Remark 2.7 we know this to be a pre-triangle. Note that, by our construction, the objects in this pre-triangle, that is \( X = A' \oplus B, Y = B' \oplus C, Z = C' \oplus \Sigma A \) and \( \Sigma X = \Sigma A' \oplus \Sigma B \), all lie \( M_j^+ \). We can furthermore express \( X = A' \oplus B \) as \( X \cong \text{colim} (a'_s \oplus b_s) \) and \( Y = B' \oplus C \) as \( Y \cong \text{colim} (b'_s \oplus c_s) \), that is we have explicit Cauchy sequences converging to \( X \) and \( Y \). If we chop off the sequences \( a'_s \) and \( b'_s \), deleting all the terms \( a'_i \) and \( b'_i \) with \( i < \ell \), we can even express \( X = \text{colim} x_s \) and \( Y = \text{colim} y'_s \), so that \( x_1 = a'_\ell \oplus b_1 \) and \( y'_1 = b'_\ell \oplus c_1 \).

And the sequences \( x_s, y'_s \) are such that, in the triangles \( x_m \rightarrow x_{m'} \rightarrow \Sigma x_{m'} \) and \( y'_m \rightarrow y'_{m'} \rightarrow d_{m,m'} \), we have \( \Sigma^{-1} d_{m,m'}, \Sigma d_{m,m'}, \Sigma^{-1} d_{m,m'}, \Sigma d_{m,m'} \) all belonging to \( M_j \). And finally the morphism from the mapping cone of (1) to the mapping cone of (2) rewrites as

![Diagram](attachment:image3.png)

Where the top row is a distinguished triangle in \( S \), while the bottom row is a pre-triangle in \( \mathcal{S}(S) \).

Now we apply Lemma 2.6 to the morphism \( \tilde{f} : X \rightarrow Y \) in \( \mathcal{S}(S) \). Actually: we apply the “moreover” part. We are given in \( S \) a triangle \( x_1 \xrightarrow{f_1} y'_1 \xrightarrow{g_1} z_1 \xrightarrow{h_1} \Sigma x_1 \), as well as
a commutative square
\[
\begin{array}{ccc}
x_1 & \xrightarrow{f_1} & y_1' \\
\downarrow{\alpha_1'} & & \downarrow{\beta_1'} \\
X & \xrightarrow{f} & Y
\end{array}
\]
and Cauchy sequences \(x_s, y_s'\) with \(X = \text{colim} x_s\) and \(Y = \text{colim} y_s'\). We may construct in \(\mathcal{S}\) a Cauchy sequence of triangles \(x_s \xrightarrow{\tilde{f}_s} y_s \xrightarrow{\tilde{g}_s} z_s \xrightarrow{\tilde{h}_s} \Sigma x_s\), extending the given triangle \(x_1 \xrightarrow{\bar{f}_1} y_1' \xrightarrow{\bar{g}_1} z_1 \xrightarrow{\bar{h}_1} \Sigma x_1\), with \(x_s\) the given sequence and \(y_s\) a subsequence of \(y_s'\) such that \(y_1 = y_1'\), and so that, in the Cauchy sequence \(z_s\), we have that the triangles \(z_m \rightarrow z_{m'} \rightarrow \hat{d}_{m,m'}\) have \(\Sigma^{-1}\hat{d}_{m,m'}\) both in \(\mathcal{M}_f\). Let the colimit of \(x_s \xrightarrow{\tilde{f}_s} y_s \xrightarrow{\tilde{g}_s} z_s \xrightarrow{\tilde{h}_s} \Sigma x_s\) be written \(\xrightarrow{\Sigma f} X \xrightarrow{\Sigma Y} \Sigma X\).

Now we apply Corollary 2.9 to the diagram in (3) and the Cauchy sequence of triangles \(x_s \xrightarrow{\tilde{f}_s} y_s \xrightarrow{\tilde{g}_s} z_s \xrightarrow{\tilde{h}_s} \Sigma x_s\). Actually: we apply the “more precisely” part with \(n = 1\), to factor (3) uniquely through a morphism

\[
\begin{array}{ccc}
X & \xrightarrow{\tilde{f}} & Y \\
\downarrow{\bar{f}} & & \downarrow{\bar{g}} \\
X & \xrightarrow{\Sigma f} & \Sigma Y
\end{array}
\]

The top row is a pre-triangle in \(\mathcal{L}(\mathcal{S})\), while the bottom row is a pre-triangle in \(\mathcal{E}(\mathcal{S})\).

The 5-lemma, coupled with the exactness of the rows, tells us that the map \(\varphi\) is an isomorphism. Therefore \(\hat{Z} \cong Z\) lies in \(\mathcal{G}(\mathcal{S}) = \mathcal{L}(\mathcal{S}) \cap \mathcal{E}(\mathcal{S})\), the top row is a triangle in \(\mathcal{G}(\mathcal{S})\), and the bottom row, which is the mapping cone on the morphism in (2), is isomorphic to the triangle in the top row.

\[\square\]

3. IN THE PRESENCE OF AN GOOD EXTENSION \(\mathcal{S} \rightarrow \mathcal{T}\)

In Sections \(\mathcal{S}\) and \(\mathcal{T}\) we fixed a triangulated category \(\mathcal{S}\) with a metric, and out of it constructed and studied several subcategories of \(\text{Mod-}\mathcal{S}\). But it turns out to be useful to embed \(\mathcal{S}\) into other triangulated categories. In this section we will assume given a fully faithful, triangulated functor \(F: \mathcal{S} \rightarrow \mathcal{T}\). Let us set up the conventions.

**Notation 3.1.** With \(F: \mathcal{S} \rightarrow \mathcal{T}\) a fully faithful, triangulated functor we let \(\mathcal{Y}: \mathcal{T} \rightarrow \text{Mod-}\mathcal{S}\) be the functor taking an object \(A \in \mathcal{T}\) to the functor \(\mathcal{Y}(A) = \text{Hom}(F(-), A)\). Clearly \(\mathcal{Y} \circ F = Y\), with \(Y: \mathcal{S} \rightarrow \text{Mod-}\mathcal{S}\) as in the previous sections. Because in this section we will be considering both the embedding \(Y: \mathcal{S} \rightarrow \text{Mod-}\mathcal{S}\) and the embedding \(F: \mathcal{S} \rightarrow \mathcal{T}\), we will try to be careful and not confuse \(s \in \mathcal{S}\) with its image under either of these embeddings.

We begin with
Observation 3.2. With $F : \mathcal{S} \to \mathcal{T}$ as above, we have the formula

$$y^{-1}(\mathcal{C}(\mathcal{S})) = \bigcap_{j \in \mathbb{Z}} \bigcup_{i \in \mathbb{N}} \left[ F(\Sigma^j \mathcal{M}_i) \right]^\perp$$

To see this observe that, for every $s \in \mathcal{M}_i \subset \mathcal{S}$, Yoneda gives that $\text{Hom}(Y(s), y(t)) = \text{Hom}(F(s), t)$. Hence $y^{-1}[Y(\Sigma^i \mathcal{M}_i)]^\perp = [F(\Sigma^i \mathcal{M}_i)]^\perp$. The formula above now follows from Observation 2.3 and the fact that $y^{-1}$ respects unions and intersections.

From Observation 2.3 (i), (ii) and (iii) we furthermore deduce that $y^{-1}(\mathcal{C}(\mathcal{S}))$ is a thick subcategory of $\mathcal{T}$.

Example 3.3. Let us specialize to the situation of Example 1.5(i): the category $\mathcal{T}$ has coproducts, there is a compact generator $H$ with $\text{Hom}(H, \Sigma^i H) = 0$ for $i \gg 0$, and we are given a $t$-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ in the preferred equivalence class. As in Example 1.5(i) we set $\mathcal{S} = \mathcal{T}^c$ and the metric is given by $M_i = \mathcal{T}^c \cap \mathcal{T}^{\leq -i}$.

With $F : \mathcal{T}^c \to \mathcal{T}$ the natural embedding, the inclusion $F(M_i) \subset \mathcal{T}^{\leq -i}$ gives $\mathcal{T}^{\geq -i+1} = [\mathcal{T}^{\leq -i}]^\perp \subset [F(M_i)]^\perp$. We want to prove an inclusion in the other direction. Note that, because $H$ is compact and the $t$-structure is in the preferred equivalence class, there is an integer $n > 0$ with $\Sigma^n H \in \mathcal{T}^{\leq 0}$, and hence $[H]\langle -\infty, -n-i \rangle \subset \mathcal{T}^c \cap \mathcal{T}^{\leq -i} = M_i$. Therefore $[F(M_i)]^\perp \subset [H]\langle -\infty, -n-i \rangle^\perp = \langle [H]\langle -\infty, -n-i \rangle \rangle^\perp$. The definition of the $t$-structure $(\mathcal{T}^{\leq 0}_H, \mathcal{T}^{\geq 0}_H)$ generated by $H$ is that $\mathcal{T}^{\leq 0}_H = \langle [H]\langle -\infty, 0 \rangle \rangle^\perp$, and as the $t$-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is in the preferred equivalence class it is equivalent to $(\mathcal{T}^{\leq 0}_H, \mathcal{T}^{\geq 0}_H)$, and there is an integer $n' > 0$ with $\langle [H]\langle -\infty, -n-i \rangle \rangle^\perp = \mathcal{T}^{\leq -n-i} \supset \mathcal{T}^{\leq -n-n'-i}$. Taking perpendiculars, and combining with the earlier inclusions, we deduce

$$\mathcal{T}^{\geq -i+1} \subset [F(M_i)]^\perp \subset \langle [H]\langle -\infty, -n-i \rangle \rangle^\perp \subset [\mathcal{T}^{\leq -n-n'-i}]^\perp = \mathcal{T}^{\geq -n-n'-i+1}.$$  

For any integer $j \in \mathbb{Z}$ this gives $\mathcal{T}^{\geq -i-j+1} \subset [F(\Sigma^j \mathcal{M}_i)]^\perp \subset \mathcal{T}^{\geq -n-n'-i-j+1}$, and taking the union over $i \in \mathbb{N}$ we discover $\cup_{i \in \mathbb{N}} [F(\Sigma^j \mathcal{M}_i)]^\perp = \mathcal{T}^+$. Intersecting over $j \in \mathbb{Z}$, and appealing to the formula for $y^{-1}(\mathcal{C}(\mathcal{S}))$ given in Observation 3.2 this combines to

$$y^{-1}(\mathcal{C}(\mathcal{S})) = \mathcal{T}^+.$$  

Example 3.4. Still with $\mathcal{T}$ being a triangulated category with coproducts and with $\mathcal{S} = \mathcal{T}^c$, we can consider a different metric. More explicitly: with $H$ still a compact generator for $\mathcal{T}$ we can let $M_i = \langle \langle H \rangle^{[-i,i]} \rangle^\perp$.

First we note that this is—up to equivalence—the metric studied in Krause [5]. Krause doesn’t state his theory in these terms, hence let us sketch the translation. If a sequence $E_*$ is Cauchy with respect to the metric above then in the triangle $E_n \to E_{n+1} \to D_n$ we have $\Sigma^{-1} D_n, D_n \in \mathcal{M}_i$ for $n \gg 0$. This implies that, for all objects $G \in \langle H \rangle^{[-i,i]}$, the functor $\text{Hom}(G, -)$ takes the maps $E_n \to E_{n+1}$ to isomorphisms whenever $n \gg 0$. Since every $G \in \mathcal{T}^c = \langle H \rangle = \cup_{i \in \mathbb{Z}} \langle H \rangle^{[-i,i]}$ belongs to some $\langle H \rangle^{[-i,i]}$, it follows that a Cauchy sequence with respect to the metric is Cauchy in Krause’s sense.
Conversely: if the sequence is Cauchy in Krause’s sense then, with \( G = \bigoplus_{\ell = -1}^i \Sigma^\ell H \) we have that \( \text{Hom}(G, -) \) takes \( E_n \to E_{n+1} \) to isomorphisms for all \( n \gg 0 \). Put \( G = \bigoplus_{\ell = -1}^i \Sigma^\ell H \); since both \( \Sigma^{-1}G \) and \( G \) are direct summands of \( G \), this, \( \text{Hom}(\Sigma^{-1}G, -) \) take \( E_n \to E_{n+1} \) to isomorphisms, or to rephrase this, \( \text{Hom}(G, -) \) takes both \( E_n \to E_{n+1} \) and \( \Sigma E_n \to \Sigma E_{n+1} \) to isomorphisms. We conclude that, in the triangle \( E_n \to E_{n+1} \to D_n \), we have \( D_n \in G^\perp \) for \( n \gg 0 \). As \( G^\perp = [\bigoplus_{\ell = -1}^i \Sigma^\ell H]^\perp = \left[ \langle H \rangle^{[1, i]} \right]^\perp = M_i \) the sequence is Cauchy in the metric.

We can, of course, try to compute what the theory developed here yields when applied to the metric that underlies Krause’s construction. The only case I have computed in detail is when \( \mathcal{T} = D(R) \) with \( R \) a noetherian ring. We can choose the compact generator to be \( R \in D(R) \), and then the categories \( \mathcal{M}_i \) come down to

\[
\mathcal{M}_i = \{ X \in D^b(R\text{-proj}) \mid H^\ell(X) = 0 \text{ whenever } -i \leq \ell \leq i \}.
\]

Because \( \Sigma^j R \in \mathcal{M}_i \) whenever \( |j| > i \), we have that any object \( X \in \left[ F(\mathcal{M}_i) \right]^\perp \) must have \( H^j(X) = 0 \) if \( |j| > i \), in other words \( X \in D(R)^{\geq -i} \cap D(R)^{\leq i} \). But the category \( \mathcal{M}_i \) also contains good approximations for every object of the form \( \Sigma^{-i-1}M \), where \( M \in R\text{-mod} \).

Precisely: choose a resolution for \( \Sigma^{-i-1}M \) by finitely generated, projective \( R \)-modules, that is a complex

\[
\cdots \to P^{-1} \to P^0 \to P^1 \to 0 \to \cdots
\]

whose only cohomology is \( M \) in degree \( i+1 \). Then form the brutal truncation, deleting everything in degree \( < -i - 1 \). We obtain an object in \( P^* \in D^b(R\text{-proj}) \) with only two nonvanishing cohomology groups, \( H^{i+1}(P^*) = M \) and \( H^{-i-1}(P^*) \).

Hence \( P^* \in \mathcal{M}_i \). The triangle \( (P^*)^{\leq -i-1} \to P^* \to \Sigma^{-i-1}M \to \Sigma(P^*)^{\leq -i-1} \) tells us that, for \( X \in \left[ F(\mathcal{M}_i) \right]^\perp \subseteq D(R)^{\geq -i} \) the map \( \text{Hom}(\Sigma^{-i-1}M, X) \to \text{Hom}(P^*, X) = 0 \) must be an isomorphism. But \( X \in \left[ F(\mathcal{M}_i) \right]^\perp \) also belongs to \( D(R)^{\leq i} \), and the vanishing of \( \text{Hom}(\Sigma^{-i-1}M, X) \) for every finite \( R \)-module \( M \) guarantees that \( X \) must be isomorphic to a bounded complex of injective \( R \)-modules, vanishing outside degrees \( -i \leq j \leq i \).

Now use the formula of Observation \( \text{[32]} \) for \( \mathcal{Y}^{-1}(\mathcal{C}(\mathcal{S})) \). It tells us that, for Krause’s metric on \( \mathcal{S} = D^b(R\text{-proj}) \), the category \( \mathcal{Y}^{-1}(\mathcal{C}(\mathcal{S})) \subseteq D(R) \) turns out to be the category of bounded complexes of injectives.

So far we have computed a couple of examples, we have worked out what the category \( \mathcal{Y}^{-1}(\mathcal{C}(\mathcal{S})) \) comes down to in the special cases of Example \( \text{[33]}(\text{i}) \) and of Krause’s metric on \( D^b(R\text{-proj}) \). We would like to also say something about \( \mathcal{L}(\mathcal{S}) \), and for this it helps to restrict the class of embeddings \( \mathcal{S} \to \mathcal{T} \) we consider. This leads us to

**Definition 3.5.** Let \( F : \mathcal{S} \to \mathcal{T} \) be a fully faithful triangulated functor between triangulated categories. Assume \( \mathcal{T} \) has coproducts and \( \mathcal{S} \) is given a metric \( \{ M_i \} \). We say that \( \mathcal{T} \) is a good extension with respect to the metric if, for any Cauchy sequence \( E_\ast \) in \( \mathcal{S} \), the natural map \( \text{colim} Y(E_i) \to \mathcal{Y}(\text{Hocolim} F(E_i)) \) is an isomorphism.
Example 3.6. If $\mathcal{T}$ is a triangulated category with coproducts, and $\mathcal{S} = \mathcal{T}^c \subset \mathcal{T}$ is the subcategory of compact objects, then the embedding $F : \mathcal{S} \rightarrow \mathcal{T}$ is a good extension for any metric on $\mathcal{S}$. This follows from [9, Lemma 2.8], which tells us that for any sequence $E_*$ in $\mathcal{T}$ the map $\liminj \gamma(E_*) \rightarrow \gamma(\text{Hocolim} \ E_*)$ is an isomorphism.

Example 3.7. Now let $\mathcal{T}$ be a weakly approximable triangulated category, and choose a $t$–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ in the preferred equivalence class. As in Example 1.5(ii) let $\mathcal{S} = [\mathcal{T}^c_0]^{\text{op}}$, and the metric is given by $M^{\text{op}}_i = \mathcal{T}^c_i \cap \mathcal{T}^{\leq -i}$.

I assert that the embedding $F : [\mathcal{T}^c_0]^{\text{op}} \rightarrow \mathcal{T}^{\text{op}}$ is a good extension—the reader can find the proof in [9, Lemmas 2.11 and 3.1].

The definition of $\mathcal{L}(\mathcal{S})$ was to take colimits in the category $\text{Mod-}\mathcal{S}$, but since $\mathcal{T}$ has coproducts we could consider an alternative, there is nothing to stop us from taking homotopy colimits in $\mathcal{T}$. We define

Definition 3.8. The full subcategory $\mathcal{L}'(\mathcal{S}) \subset \mathcal{T}$ has for objects all the isomorphs in $\mathcal{T}$ of homotopy colimits of the images under $F$ of Cauchy sequences in $\mathcal{S}$.

Observation 3.9. If $F : \mathcal{S} \rightarrow \mathcal{T}$ is a good extension, then the functor $\gamma : \mathcal{T} \rightarrow \text{Mod-}\mathcal{S}$ restricts, on objects, to an essential surjection $\text{Ob}(\mathcal{L}'(\mathcal{S})) \rightarrow \text{Ob}(\mathcal{L}(\mathcal{S}))$. After all: by Definition 3.8 the functor $\gamma$ takes homotopy colimits of Cauchy sequences to colimits.

Example 3.10. We return to Example 1.5(i), where $\mathcal{S} = \mathcal{T}^c$ and the metric is given by $M_i = \mathcal{T}^c_i \cap \mathcal{T}^{\leq -i}$. We assert that in this case the category $\mathcal{L}'(\mathcal{S})$ turns out to be $\mathcal{T}^c$. The reader can find this in [8, Lemma 7.5].

Example 3.11. With $\mathcal{T}$ a triangulated category with coproducts and a single compact generator $H$, we can let $\mathcal{S} = \mathcal{T}^c$ as above, but endow $\mathcal{S}$ with Krause’s metric—see Example 3.4. By Example 3.6 the pair $\mathcal{S} \subset \mathcal{T}$ is a good extension. We can form the category $\mathcal{L}'(\mathcal{S})$, but I have only computed it when $\mathcal{T} = \text{D}(R)$ for a noetherian ring $R$.

Every object $E \in \mathcal{S} = \text{D}^b(R-\text{proj})$ has bounded cohomology, with $H^i(E)$ a finite $R$–module for every $i$. In any Cauchy sequence, with respect to Krause’s metric, the cohomology eventually stabilizes. Therefore for any $X \in \mathcal{L}'(\mathcal{S})$ and any $j \in \mathbb{Z}$ we have that $H^j(X)$ is a finite $R$–module. In symbols: $\mathcal{L}'(\mathcal{S}) \subset \text{D}_{R-\text{mod}}(R)$, the category of all complexes of $R$–modules with finite cohomology modules.

I assert that this inclusion is an equality. Suppose $X$ belongs to $\text{D}_{R-\text{mod}}(R)$, I want to exhibit $X$ as the homotopy colimit of a Cauchy sequence. To this end pick an integer $i > 0$ and consider the map $X^{\leq i} \rightarrow X$ from the truncation with respect to the standard $t$–structure on $\text{D}(R)$. The object $X^{\leq i}$ is bounded above and has finite cohomology modules, hence admits a resolution by finitely generated projectives—there is in $\text{D}(R)$ an isomorphism $P \rightarrow X^{\leq i}$, with $P \in \text{D}^-(R-\text{proj})$. Now take the brutal truncation, killing everything in degree $< -i - 1$ to obtain a map $E_i \rightarrow X^{\leq i} \rightarrow X$ with $E_i \in \text{D}^b(R-\text{proj})$. The functor $H^j(\cdot)$ takes this map to an isomorphism whenever $-i \leq j \leq i$, and these assemble to a Cauchy sequence with homotopy colimit $X$. 
Now that we have seen a few examples, let us turn to general results that hold under the assumptions introduced.

**Lemma 3.12.** Let $F : S \to \mathcal{T}$ be a good extension, as in Definition 3.5. Suppose $E$ belongs to the category $\mathcal{C}(S)$ of Definition 3.8 and let $X \in \mathcal{T}$ be arbitrary. Then the natural map $\text{Hom}(E, X) \to \text{Hom}(\mathcal{Y}(E), \mathcal{Y}(X))$ extends to a short exact sequence

$$0 \to K(E, X) \to \text{Hom}(E, X) \to \text{Hom}(\mathcal{Y}(E), \mathcal{Y}(X)) \to 0.$$ 

Moreover: if $E_*$ is a Cauchy sequence with $E \cong \text{Hocolim} F(E_*)$, then there is an isomorphism $K(E, X) \cong \lim_{\leftarrow} \text{Hom}(\Sigma F(E_i), X)$.

**Proof.** Choose a Cauchy sequence $E_*$ in $S$ with $E = \text{Hocolim} F(E_*)$, and consider the commutative square

$$\begin{array}{ccc}
\text{Hom}(E, X) & \to & \lim \text{Hom}(F(E_i), X) \\
\downarrow & & \downarrow \\
\text{Hom}(\mathcal{Y}(E), \mathcal{Y}(X)) & \cong & \lim \text{Hom}(\mathcal{Y}(E_i), \mathcal{Y}(X)).
\end{array}$$

The vertical map on the right is an isomorphism by Yoneda, and the bottom horizontal map is an isomorphism because

$$\text{Hom}(\mathcal{Y}(E), \mathcal{Y}(X)) = \text{Hom}(\mathcal{Y}(\text{Hocolim} F(E_i)), \mathcal{Y}(X))$$
$$= \text{Hom}(\text{colim} \mathcal{Y}(E_i), \mathcal{Y}(X))$$
$$= \lim_{\leftarrow} \text{Hom}(\mathcal{Y}(E_i), \mathcal{Y}(X)).$$

where the second isomorphism is because we’re dealing with a good extension. We are therefore reduced to showing that the top horizontal map is a surjection, and computing its kernel.

For this recall the definition of homotopy colimits: the homotopy colimit $E = \text{Hocolim} F(E_*)$ sits in a triangle

$$\prod_{i>0} F(E_i) \xrightarrow{1\text{-shift}} \prod_{i>0} F(E_i) \to E \to \prod_{i>0} \Sigma F(E_i)$$

and, applying the functor $\text{Hom}(-, X)$, we obtain a short exact sequence

$$0 \to \lim_{\leftarrow} \text{Hom}(\Sigma F(E_i), X) \to \text{Hom}(E, X) \to \lim_{\leftarrow} \text{Hom}(F(E_i), X) \to 0.$$ 

This completes the proof of the Lemma. $\square$

The next result is

**Lemma 3.13.** If $E$ is an object of $\mathcal{C}(S)$ and $X \in \mathcal{T}$ is an object with $\mathcal{Y}(X) \in \mathcal{C}(S)$, then the natural map $\text{Hom}(E, X) \to \text{Hom}(\mathcal{Y}(E), \mathcal{Y}(X))$ is an isomorphism.
Proof. Since $X \in \mathcal{I}$ is such that $y(X) \in \mathcal{C}(\mathcal{S})$, Observation 3.2 allows us to choose an integer $n$ with $\text{Hom}_\mathcal{I}(F(\Sigma M_n), X) = 0 = \text{Hom}_\mathcal{I}(F(M_n), X)$. Because the sequence $E_m$ is Cauchy there exists an integer $M > 0$ so that, for all $M \leq m < m'$, the triangle $D_{m,m'} \to \Sigma E_m \to \Sigma E_{m'} \to D_{m,m'}$ in the category $\mathcal{S}$ has $D_{m,m'} \in M_n$. Applying the exact functor $\text{Hom}(F(-), X)$ to this triangle we have that the map $\text{Hom}(F(\Sigma E_{m'}), X) \to \text{Hom}(F(\Sigma E_m), X)$ is an isomorphism whenever $M \leq m < m'$, and hence $\varprojlim \text{Hom}(F(\Sigma E_i), X) = 0$. The current Lemma now follows Lemma 3.12. 

**Corollary 3.14.** The restriction of $y : \mathcal{I} \to \text{Mod}-\mathcal{S}$, to the subcategory $\mathcal{L}'(\mathcal{S}) \cap y^{-1}(\mathcal{C}(\mathcal{S})) \subset \mathcal{I}$, induces an equivalence with the category $\mathcal{G}(\mathcal{S})$ of Definition 1.10(iii).

**Proof.** In Observation 3.9 we noted that the functor $y$ yields an essential surjection $\text{Ob}(\mathcal{L}'(\mathcal{S})) \to \text{Ob}(\mathcal{L}(\mathcal{S}))$, and restricting to the inverse image of $\mathcal{G}(\mathcal{S}) = \mathcal{L}(\mathcal{S}) \cap \mathcal{C}(\mathcal{S})$ will yield an essential surjection $\text{Ob}[\mathcal{L}'(\mathcal{S}) \cap y^{-1}(\mathcal{C}(\mathcal{S}))] \to \text{Ob}(\mathcal{G}(\mathcal{S}))$. So on objects the functor is essentially surjective.

On the other hand Lemma 3.10 tells us that, for $E, X$ in $\mathcal{L}'(\mathcal{S}) \cap y^{-1}(\mathcal{C}(\mathcal{S}))$, the map $\text{Hom}(E, X) \to \text{Hom}(y(E), y(X))$ is an isomorphism. Thus $y$ is fully faithful on the subcategory.

And now we come to the point.

**Theorem 3.15.** Let $\mathcal{S}$ be a triangulated category with a metric $\{M_i\}$, and let $F : \mathcal{S} \to \mathcal{I}$ be a good extension. Then the category $\mathcal{L}'(\mathcal{S}) \cap y^{-1}(\mathcal{C}(\mathcal{S}))$ is a triangulated subcategory of $\mathcal{I}$, and the natural map $y : \mathcal{L}'(\mathcal{S}) \cap y^{-1}(\mathcal{C}(\mathcal{S})) \to \mathcal{G}(\mathcal{S})$ is a triangulated equivalence, where the category on the left has the triangulated structure in inherits from being a triangulated subcategory of $\mathcal{I}$, and the category $\mathcal{G}(\mathcal{S})$ has the triangulated structure of Definition 2.10.

**Proof.** The fact that $y$ is an equivalence of categories was proved in Corollary 3.14 we only have to worry about the triangulated structure. Let $A, B$ be objects in $\mathcal{L}'(\mathcal{S}) \cap y^{-1}(\mathcal{C}(\mathcal{S}))$, and suppose $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ is a triangle in $\mathcal{I}$. We need to prove that $C$ belongs to $\mathcal{L}'(\mathcal{S}) \cap y^{-1}(\mathcal{C}(\mathcal{S}))$, and that $y(A) \xrightarrow{y(f)} y(B) \xrightarrow{y(g)} y(C) \xrightarrow{y(h)} y(\Sigma A)$ is a triangle in $\mathcal{G}(\mathcal{S})$.

Remark 2.2 guarantees that, in the pre-triangle $y(A) \xrightarrow{y(f)} y(B) \xrightarrow{y(g)} y(C) \xrightarrow{y(h)} y(\Sigma A)$, the object $y(C)$ must belong to $\mathcal{C}(\mathcal{S})$. As the objects $y(A), y(B), y(C)$ and $y(\Sigma A)$ all belong to $\mathcal{C}(\mathcal{S})$, we choose and fix an integer $n$ so that all four objects belong to $[Y(M_n)]$. Because the objects $y(A)$ and $y(B)$ belong to $\mathcal{L}(\mathcal{S})$ we may choose Cauchy sequences converging to them: that is we pick a Cauchy sequence $a_s$ with $y(A) = \text{colim} \ Y(a_s)$ and a Cauchy sequence $b'_s$ with $y(B) = \text{colim} \ Y(b'_s)$. Passing to subsequences if necessary, we make sure that for all integers $0 < m < m'$, in the triangles $a_m \to a_{m'} \to \bar{a}_{m,m'}$ and...
factor this map, uniquely, through $\Sigma^{-1}d_{m,m'},\Sigma d_{m,m'},\Sigma^{-1}d_{m,m'}, d_{m,m'}$ all belong to $M_n$.

In the category $\mathcal{E}(S)$ we have the map $\mathcal{y}(f) : \mathcal{y}(A) \rightarrow \mathcal{y}(B)$, to which we may apply Lemma 2.6—or rather the “more precisely” version: we may choose a subsequence $b_s$ of $b'_s$, a sequence of triangles $a_s \xymatrix{ f_s \ar[r] & b_s \ar[r] & c_s \ar[r] & a_s }$ in $S$, do it in such a way that in the triangles $c_m \rightarrow c_{m'} \rightarrow \tilde{d}_{m,m'}$ we have $\Sigma^{-1}\tilde{d}_{m,m'}, \tilde{d}_{m,m'} \in M_n$, and ensure that the colimit of $\mathcal{y}(a_s) \xymatrix{ Y(f_s) \ar[r] & Y(b_s) \ar[r] & Y(c_s) \ar[r] & Y(a_s) }$ is a pre-triangle $\mathcal{y}(A) \xymatrix{ \mathcal{y}(f) \ar[r] & \mathcal{y}(B) \ar[r] & \tilde{C} \ar[r] & \mathcal{y}(\Sigma A) }$.

In particular the construction gives us a commutative square in $\text{Mod-}S$

\[
\begin{array}{ccc}
Y(a_1) & \xymatrix{ \ar[r]^{Y(f_1)} & } & Y(b_1) \\
\mathcal{y}(A) & \xymatrix{ \ar[r]^{\mathcal{y}(f)} & } & \mathcal{y}(B)
\end{array}
\]

which must be the image under $\mathcal{y}$ of a commutative square in $\mathcal{T}$

\[
\begin{array}{ccc}
F(a_1) & \xymatrix{ \ar[r]^{F(f_1)} & } & F(b_1) \\
A & \xymatrix{ \ar[r]^{f} & } & B
\end{array}
\]

This last square may be extended to a morphism of triangles in $\mathcal{T}$

\[
\begin{array}{ccc}
F(a_1) & \xymatrix{ \ar[r]^{F(f_1)} & } & F(b_1) & \xymatrix{ \ar[r]^{F(g_1)} & } & F(c_1) & \xymatrix{ \ar[r]^{F(h_1)} & } & F(\Sigma a_1) \\
A & \xymatrix{ \ar[r]^{f} & } & B & \xymatrix{ \ar[r]^{g} & } & C & \xymatrix{ \ar[r]^{h} & } & \Sigma A
\end{array}
\]

and applying the functor $\mathcal{y}$ we deduce a commutative diagram in $\text{Mod-}S$

\[
\begin{array}{ccc}
Y(a_1) & \xymatrix{ \ar[r]^{Y(f_1)} & } & Y(b_1) & \xymatrix{ \ar[r]^{Y(g_1)} & } & Y(c_1) & \xymartr{ \ar[r]^{Y(h_1)} & } & Y(\Sigma a_1) \\
\mathcal{y}(A) & \xymartr{ \ar[r]^{\mathcal{y}(f)} & } & \mathcal{y}(B) & \xymartr{ \ar[r]^{\mathcal{y}(g)} & } & \mathcal{y}(C) & \xymartr{ \ar[r]^{\mathcal{y}(h)} & } & \mathcal{y}(\Sigma A)
\end{array}
\]

And now we apply Corollary 2.9 or rather the “more precisely” version with $n = 1$, to factor this map, uniquely, through

\[
\begin{array}{c}
\mathcal{y}(A) \xymat{ \mathcal{y}(f) & } \xymat{ \mathcal{y}(g) & } \xymatr{ \mathcal{y}(\Sigma f) & } \xymatr{ \mathcal{y}(\Sigma B) & } \\
\mathcal{y}(C) \xymatr{ \mathcal{y}(f) & } \xymatr{ \mathcal{y}(g) & } \xymatr{ \mathcal{y}(\Sigma f) & } \xymatr{ \mathcal{y}(\Sigma B) & }
\end{array}
\]
The 5-lemma forces $\varphi$ to be an isomorphism. The top row is a triangle in $\mathcal{S}(\mathcal{S})$ by construction, and the isomorphism tells us that so is the bottom row.

It remains to prove that $\mathcal{L}'(\mathcal{S}) \cap \mathcal{Y}^{-1}(\mathcal{C}(\mathcal{S}))$ is a triangulated subcategory of $\mathcal{T}$, concretely we still need to check that $C$ belongs to $\mathcal{L}'(\mathcal{S}) \cap \mathcal{Y}^{-1}(\mathcal{C}(\mathcal{S}))$. What we know so far is that $\mathcal{Y}(C) \in \mathcal{C}(\mathcal{S})$, or equivalently that $C \in \mathcal{Y}^{-1}(\mathcal{C}(\mathcal{S}))$; it remains to prove that $C \in \mathcal{L}'(\mathcal{S})$. Because $F : \mathcal{S} \rightarrow \mathcal{T}$ is a good extension we have isomorphisms $\bar{C} = \underset{\longrightarrow}{\text{colim}} \mathcal{Y}(c_\ast) \cong \mathcal{Y}(\text{Hocolim} c_\ast)$. This makes $\varphi$ a morphism $\varphi : \mathcal{Y}(\text{Hocolim} c_\ast) \rightarrow \mathcal{Y}(C)$ with $\text{Hocolim} c_\ast \in \mathcal{L}'(\mathcal{S})$ and $\mathcal{Y}(C) \in \mathcal{C}(\mathcal{S})$. Lemma \ref{lem:5-lemma} allows us to lift the isomorphism $\varphi$ to a (unique) morphism $\rho : \text{Hocolim} c_\ast \rightarrow C$ in the category $\mathcal{T}$. And since $\mathcal{Y}(\rho) = \varphi$ is an isomorphism, it follows that in the triangle $D \rightarrow \text{Hocolim} c_\ast \rightarrow C$ we have $\mathcal{Y}(D) = 0$, or to rephrase we have $D \in \mathcal{S}^\perp$. Hence $D \in \text{Loc}(\mathcal{S})^\perp$, where $\text{Loc}(\mathcal{S})$ is the smallest localizing subcategory of $\mathcal{T}$ containing $\mathcal{S}$.

On the other hand we have that $A, B$ and $\text{Hocolim} c_\ast$ belong to $\mathcal{L}'(\mathcal{S}) \subset \text{Loc}(\mathcal{S})$, the triangle $A \rightarrow B \rightarrow C \rightarrow \Sigma A$ informs us that $C \in \text{Loc}(\mathcal{S})$, and from the triangle $D \rightarrow \text{Hocolim} c_\ast \rightarrow C$ we learn that $D \in \text{Loc}(\mathcal{S})$. The identity map $\text{id} : D \rightarrow D$ is a morphism from $D \in \text{Loc}(\mathcal{S})$ to $D \in \text{Loc}(\mathcal{S})^\perp$ and must vanish. Hence $D = 0$, the map $\rho : \text{Hocolim} c_\ast \rightarrow C$ is an isomorphism, and this exhibits $C$ as isomorphic to $\text{Hocolim} c_\ast \in \mathcal{L}'(\mathcal{S})$. \hfill \qed

4. THE EXAMPLE OF $\mathcal{T}^c$ DETERMINING $\mathcal{T}^b_c$

It’s time to see what the generality of Sections \ref{sec:general} and \ref{sec:examples} reduces to in some concrete examples. But first a few global conventions for this section.

**Notation 4.1.** Throughout this section $\mathcal{T}$ will be a triangulated category with coproducts, we will assume there is a compact generator $H \in \mathcal{T}$ with $\text{Hom}(H, \Sigma^i H) = 0$ for $i \gg 0$, and we will suppose given a $t$-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ in the preferred equivalence class.

**Example 4.2.** Let the conventions be as in Notation \ref{not:1.1}. In Example \ref{ex:1.5}(i) we studied the following: we put $\mathcal{S} = \mathcal{T}^c$, and let the metric be given by $\{M_i = \mathcal{T}^c \cap \mathcal{T}^{\leq -i}\}$. In Example \ref{ex:3.6} we learned that the embedding $F : \mathcal{T}^c \rightarrow \mathcal{T}$ is a good extension, Example \ref{ex:3.3} teaches us that the category $\mathcal{Y}^{-1}(\mathcal{C}(\mathcal{S}))$ turns out to be $\mathcal{T}^+$, while in Example \ref{ex:3.10} we saw that $\mathcal{L}'(\mathcal{S}) = \mathcal{T}^-$. This means that
\[
\mathcal{L}'(\mathcal{S}) \cap \mathcal{Y}^{-1}(\mathcal{C}(\mathcal{S})) = \mathcal{T}^- \cap \mathcal{T}^+ = \mathcal{T}^b_c.
\]

Theorem \ref{thm:5-lemma} tells us that the functor $\mathcal{Y}$ induces a a triangulated equivalence of $\mathcal{T}^b_c = \mathcal{L}'(\mathcal{S}) \cap \mathcal{Y}^{-1}(\mathcal{C}(\mathcal{S}))$ with the category $\mathcal{S}(\mathcal{S})$. This computes for us what $\mathcal{S}(\mathcal{S})$ turns out to be, as a triangulated category, as long as the metric is as in Example \ref{ex:1.5}(i).

From Example \ref{ex:1.2} we learn that, in the generality given in Notation \ref{not:1.1}, the triangulated category $\mathcal{T}^b_c$ is fully determined by the category $\mathcal{T}^c$ together with its metric. The way we defined the metric was to use the embedding into $\mathcal{T}$; our definition was $M_i = \mathcal{T}^c \cap \mathcal{T}^{\leq -i}$.
While it’s true that, up to equivalence of metrics, the $t$–structure doesn’t matter much—equivalent $t$–structures induce equivalent metrics, and the $t$–structures in the preferred equivalence class are all equivalent—the preferred equivalence class of $t$–structures is defined on $\mathcal{T}$, not $\mathcal{T}^c$.

Hence the reader might wonder if there is some way to define the metric on $\mathcal{T}^c$ without referring to the embedding into $\mathcal{T}$. We begin with

Reminder 4.3. A classical generator of a triangulated category $S$ is an object $G \in S$ with $S = \langle G \rangle$. From [S] Lemma 0.9(iii) it follows that, given two classical generators $G$ and $H$, there exists an integer $A$ with $H \in \langle G \rangle_{[-A,A]}$ and $G \in \langle H \rangle_{[-A,A]}$.

Lemma 4.4. Suppose $\mathcal{T}$ is a compactly generated triangulated category. Then any classical generator for $\mathcal{T}^c$ is a compact generator for $\mathcal{T}$.

Proof. Let $G \in \mathcal{T}^c$ be a classical generator. Then $\mathcal{T}^c = \langle G \rangle$, and $\mathcal{T} = \text{Loc}(\mathcal{T}^c) = \overline{\langle G \rangle}$. □

Definition 4.5. Let $S$ be a triangulated category, and assume $G \in S$ is a classical generator. We define two metrics $\{\mathcal{L}_i\}, \{\mathcal{N}_i\}$ by the formulas

(i) $\mathcal{L}_i = \langle G \rangle_{(-\infty,-i]}$.
(ii) $\mathcal{N}_i = \left(\langle G \rangle_{[-i,\infty]}\right)^\perp$.

Remark 4.6. The Cauchy sequences with respect to the metric $\{\mathcal{N}_i\}$ manifestly agree with those of Remark 4.3.

In this generality all that’s clear is that, up to equivalence, the metrics $\{\mathcal{L}_i\}$ and $\{\mathcal{N}_i\}$ don’t depend on the choice of classical generator. This follows immediately from Reminder 4.3 more specifically from the fact that, given two classical generators $G$ and $H$, we can find an integer $A > 0$ with $H \in \langle G \rangle_{[-A,A]}$ and $G \in \langle H \rangle_{[-A,A]}$.

Remark 4.7. Now return to the situation of Notation 4.1. Put $S = \mathcal{T}^c$ and consider the metrics $\{\mathcal{L}_i\}, \{\mathcal{N}_i\}$ of Definition 4.5 as well as the metric $\{\mathcal{M}_i\}$ of Example 1.5(i). The classical generator $G \in S = \mathcal{T}^c$ of Definition 4.5 is a compact object in $\mathcal{T}$, and the $t$–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ of Notation 4.1 belongs to the preferred equivalence class. By [S] Observation 0.12 and Lemma 2.8 there exists an integer $A$ with $\Sigma^A G \in \mathcal{T}^{\leq 0}$ and with $\text{Hom}(\Sigma^{-A} G, \mathcal{T}^{\leq 0}) = 0$. We deduce that

(i) $\mathcal{L}_{i+A} = \langle G \rangle_{(-\infty,-i-A]} \subset \mathcal{T}^c \cap \mathcal{T}^{\leq i} = \mathcal{M}_i$
(ii) $\mathcal{M}_{i+A} = \mathcal{T}^c \cap \mathcal{T}^{\geq i-A} \subset \mathcal{T}^c \cap \left(\langle G \rangle_{[-i,\infty]}\right)^\perp = \mathcal{N}_i$

By Lemma 4.3 the classical generator $G \in \mathcal{T}^c$ is a compact generator of $\mathcal{T}$. But then the fact that the $t$–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is in the preferred equivalence class says that $t$–structures $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ and $(\mathcal{T}^{\leq 0}_G, \mathcal{T}^{\geq 0}_G)$ must be equivalent, which gives an integer $B > 0$ for which the inclusion $\mathcal{T}^{\leq -i-B} \subset \mathcal{T}^{\leq -i}_G = \overline{\langle G \rangle}_{-\infty,-i}$ holds. Now [S] Proposition 1.8 gives the second last equality in

$\mathcal{M}_{i+B} = \mathcal{T}^c \cap \mathcal{T}^{\leq -i-B} \subset \mathcal{T}^c \cap \overline{\langle G \rangle}_{-\infty,-i} = \langle G \rangle_{(-\infty,-i]} = \mathcal{L}_i$.

That is: without assuming any approximability we have $\{\mathcal{L}_i\} \cong \{\mathcal{M}_i\}$ and $\{\mathcal{M}_i\} \preceq \{\mathcal{N}_i\}$.
If we assume weak approximability we can do better.

**Proposition 4.8.** Assume \( \mathcal{T} \) is a weakly approximable triangulated category, and let \( S = \mathcal{T}^c \). Then the metric \( \{N_i\} \) on \( S \) given in Definition 4.5 is equivalent to the metric \( \{M_i\} \) of Example 4.3(i).

Proof. In Remark 4.7 we noted the inequalities \( \{M_i\} \leq \{N_i\} \); what needs proof is the reverse inequality. By Lemma 4.4 the classical generator \( G \in \mathcal{T}^c \) is a compact generator of \( \mathcal{T} \). By [3, Lemma 2.9] there is an integer \( B > 0 \) with \( [\langle G \rangle^{[-i, \infty]}] \perp \mathcal{T}^{\leq -i + B} \), hence

\[
N_{i + B} = \mathcal{T}^c \cap [\langle G \rangle^{[-i - B, \infty]}] \perp \mathcal{T}^c \cap \mathcal{T}^{\leq -i} = M_i.
\]

□

**Remark 4.9.** Let us remain with the conventions of this section, that is \( S = \mathcal{T}^c \) where \( \mathcal{T} \) satisfies the assumptions of Notation 4.1; it’s natural to ask what happens with Krause’s metric. As I have said before, I have only worked out completely what happens in the case \( \mathcal{T} = \mathbf{D}(R) \) with \( R \) a noetherian ring.

In Example 3.4 we saw that, with respect to the good extension \( F : \mathbf{D}^b(\mathcal{R} \text{-proj}) \to \mathbf{D}(\mathcal{R}) \), the subcategory \( \mathcal{Y}^{-1}(\mathbf{C}(S)) \) turns out to be \( \mathbf{D}^b(\mathcal{R} \text{-Inj}) \), the category of bounded complexes of injective modules. In Example 3.11 we learned that the category \( \mathbf{L}'(S) \) is \( \mathbf{D}_{R \text{-mod}}(R) \), the full subcategory of \( \mathbf{D}(\mathcal{R}) \) of complexes whose cohomology modules are finite. This makes the category \( \mathbf{L}'(S) \cap \mathcal{Y}^{-1}(\mathbf{C}(S)) \) equal to \( \mathbf{D}^b_{R \text{-mod}}(R \text{-Inj}) \), meaning the objects are the bounded complexes of injectives whose cohomology modules are finite.

Assume there is a dualizing complex \( C \in \mathbf{D}^b(R \text{-mod}) \), meaning a complex with a bounded injective resolution, and such that \( \text{Hom}(\cdot, C) \) induces an equivalence \( \mathbf{D}^b(R \text{-mod})^{\text{op}} \to \mathbf{D}^b(R \text{-mod}) \). Then \( \text{Hom}(\cdot, C) \) takes an object of \( \mathbf{L}'(S) \cap \mathcal{Y}^{-1}(\mathbf{C}(S)) \) to a complex in \( \mathbf{D}^b(R \text{-mod}) \) with a bounded flat resolution, hence a bounded projective resolution—in other words to an object of \( \mathbf{D}^b(R \text{-proj}) \). Thus the category \( \mathbf{G}(S) \cong \mathbf{L}'(S) \cap \mathcal{Y}^{-1}(\mathbf{C}(S)) \) is equivalent to \( \mathbf{D}^b(R \text{-proj})^{\text{op}} \cong \mathbf{D}^b(R \text{-proj}) \).

5. **Noetherian approximable categories, and passing from \( \mathcal{T}^b_c \) back to \( \mathcal{T}^c \)**

The reader may have noticed that in the treatment so far we have been strangely reticent about Example 1.5(ii). Recall: under the hypotheses of Notation 4.1 we can look at the category \( S = [\mathcal{T}^b_c]^{\text{op}} \), endow it with the metric \( \mathcal{M}^{\text{op}} = \mathcal{T}^b_c \cap \mathcal{T}^{\leq -1} \), and consider the embedding \( F : S = [\mathcal{T}^b_c]^{\text{op}} \to \mathcal{Y}^{\text{op}}. \) In Example 3.7 we mentioned that, as long as \( \mathcal{T} \) is weakly approximable, this is a good extension with respect to the metric. But since then there has been silence—no mention of the example. The reason is that without further hypotheses there isn’t much to say, what we can prove is that \( \mathbf{L}'(S) \) is contained in \( [\mathcal{T}^b_c]^{\text{op}} \); this follows from [9, Lemma 3.3].

The reason we can’t say much more without hypotheses is simple: without some noetherianness I see no reason for the category \( \mathcal{T}^b_c \) to be nonzero—we have proved that \( \mathcal{L}'(S) \subset \mathcal{T}^b_c \), but to expect an inclusion in the other direction there better be some nonzero
objects in $\mathcal{T}_c^b$. And by way of cautionary example: if $R$ is a DGA, and $H^i(R) = 0$ for $i > 0$, then \cite{section3.3} teaches us that the category $\mathcal{T} = \mathcal{H}^0(R\text{-Mod})$ is an approximable triangulated category. But unless $R$ is noetherian or $H^i(R) = 0$ for $i \leq 0$, I don’t know any nonzero objects in $\mathcal{T}_c^b$. In order to proceed we need to impose some hypothesis that guarantees the existence of enough objects in $\mathcal{T}_c^b$.

**Definition 5.1.** Suppose $\mathcal{T}$ is a triangulated category with coproducts, and assume it has a compact generator $H$ with $\text{Hom}(H, \Sigma^i H) = 0$ for $i > 0$. We declare $\mathcal{T}$ to be noetherian if there exists an integer $N > 0$, and a $t$-structure $(\mathcal{T}_{\leq 0}, \mathcal{T}_{\geq 0})$ in the preferred equivalence class, such that: for every object $X \in \mathcal{T}_c^-$ there exists a triangle $A \to X \to B$ with $A \in \mathcal{T}_c^- \cap \mathcal{T}_{\leq 0}$ and $B \in \mathcal{T}_c^- \cap \mathcal{T}_{\geq -N} = \mathcal{T}_c^b \cap \mathcal{T}_{\geq -N}$.

**Remark 5.2.** The definition is clearly robust. If one $t$-structure has an integer $N$ as above, then so does any equivalent $t$-structure. The integer will of course depend on the $t$-structure.

**Example 5.3.** Let $X$ be a separated, noetherian scheme. Then $\mathcal{T} = \mathcal{D}_{\text{qc}}(X)$ is approximable and noetherian. After all: from \cite{section3.6} we learn that $\mathcal{D}_{\text{qc}}(X)$ is approximable, that the standard $t$-structure belongs to the preferred equivalence class, and that the subcategory $\mathcal{T}_c^- \subset \mathcal{T}$ turns out to be the subcategory $\mathcal{D}_{\text{coh}}^-(X) \subset \mathcal{D}_{\text{qc}}(X)$ of complexes with bounded-above, coherent cohomology. And the standard $t$-structure on $\mathcal{D}_{\text{qc}}(X)$ respects the subcategory $\mathcal{D}_{\text{coh}}^-(X)$; given any object $F \in \mathcal{D}_{\text{coh}}^-(X)$ we may form the triangle $F_{\leq 0} \to F \to F_{\geq 1}$ in $\mathcal{T}$, and for the $t$-structure at hand we have $F_{\leq 0}, F_{\geq 1} \in \mathcal{D}_{\text{coh}}^-(X)$. Thus the category $\mathcal{T} = \mathcal{D}_{\text{qc}}(X)$ satisfies the condition of Definition 5.1; it is noetherian. For the standard $t$-structure we may even set the integer $N$ to be $N = -1$; of course any larger integer also works.

**Remark 5.4.** The argument of Example 5.3 works whenever there exists, in the preferred equivalence class, a $t$-structure which respects $\mathcal{T}_c^-$. This happens, for example, with the standard $t$-structure on the category $\mathcal{D}(R)$ as long as $R$ is a coherent ring. It also happens with the standard $t$-structure on the homotopy category of spectra.

**Lemma 5.5.** Let $\mathcal{T}$ be a noetherian, weakly approximable triangulated category, and choose a $t$-structure $(\mathcal{T}_{\leq 0}, \mathcal{T}_{\geq 0})$ in the preferred equivalence class. For any object $X \in \mathcal{T}_c^- \cap \mathcal{T}_{\leq 0}$ there exists a Cauchy sequence $B_i$ in $\mathcal{T}_c^b \cap \mathcal{T}_{\leq 0}$ with $X = \text{Holim} B_i$.

**Proof.** Let $N$ be the integer whose existence is guaranteed by Definition 5.1, and suppose we are given an object $X \in \mathcal{T}_c^- \cap \mathcal{T}_{\leq 0}$. Definition 5.1 permits us to produce, for every integer $i > 0$, a triangle $A_i \to X \to B_i$ with $A_i \in \mathcal{T}_c^- \cap \mathcal{T}_{\leq -i(N+1)}$ and $B_i \in \mathcal{T}_c^- \cap \mathcal{T}_{\geq -i(N+1) - N}$. Therefore the composite $A_{i+1} \to X \to B_i$ is a morphism from $A_{i+1} \in \mathcal{T}_{\leq -i(N+1) - N}$ to $B_i \in \mathcal{T}_{\geq -i(N+1) - N}$ and must vanish. We can factor $X \to B_i$ through $X \to B_{i+1} \to B_i$, creating an inverse system in $\mathcal{T}_c^b$. Because $A_i \in \mathcal{T}_{\leq -i(N+1)}$ the functor $(-)_{\geq -i(N+1) + 1}$ takes the maps $X \to B_i$ to an isomorphism, showing that $B_i \in \mathcal{T}_c^b \cap \mathcal{T}_{\geq -i}$ and that the sequence $B_i$ is Cauchy. And \cite{section3.2} shows that $X \to \text{Holim} B_i$ is an isomorphism. \hfill $\Box$
Proposition 5.6. Let $\mathcal{T}$ be a noetherian, weakly approximable triangulated category, and choose a $t$–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ in the preferred equivalence class. With $S = [\mathcal{T}_b]^{\text{op}}$, with the metric $M_i^{\text{op}} = \mathcal{T}_c^b \cap \mathcal{T}^{\leq -i}$, and with the good extension $F : S = [\mathcal{T}_c^b]^{\text{op}} \to \mathcal{T}^{\text{op}}$, we compute

(i) $\mathcal{L}'(S) = [\mathcal{T}_c^b]^{\text{op}}$.

(ii) $\mathcal{L}'(S) \cap \text{Y}^{-1}(\mathcal{C}(S)) = [\mathcal{T}^c]^{\text{op}}$.

Proof. The inclusion $\mathcal{L}'(S) \subset [\mathcal{T}_c^b]^{\text{op}}$ is contained in [9, Lemma 3.3], and the inclusion $[\mathcal{T}_c^b]^{\text{op}} \subset \mathcal{L}'(S)$ follows from Lemma 5.5. This proves (i).

Now for (ii). Suppose $X \in \mathcal{T}_c^b$ belongs to $[\text{F}(\mathcal{M}_i)]^{-1}$, that is $\text{Hom}(X, B) = 0$ for all $B \in M_i^{\text{op}} = \mathcal{T}_c^b \cap \mathcal{T}^{\leq -i}$. Choose any object $Y \in \mathcal{T}_c^b \cap \mathcal{T}^{\leq -i-1}$. By Lemma 5.5 we can express $Y$ as $Y = \text{Holim} B_n$, with $B_n \in \mathcal{T}_c^b \cap \mathcal{T}^{\leq -i-1}$. Hence $Y$ sits in a triangle

$$
\prod_{j=n}^{\infty} \Sigma^{-1} B_n \longrightarrow Y \longrightarrow \prod_{j=n}^{\infty} B_n
$$

Since $\Sigma^{-1} B_n, B_n$ belong to $\mathcal{T}_c^b \cap \mathcal{T}^{\leq -i} = M_i^{\text{op}}$ for every $n$, the functor $\text{Hom}(X, -)$ annihilates both outside terms in the triangle above. Therefore $\text{Hom}(X, Y) = 0$. We learn that, if $X \in \mathcal{T}_c^b$ belongs to $[\text{F}(\mathcal{M}_i)]^{-1}$, then $\text{Hom}(X, Y) = 0$ for all $Y \in \mathcal{T}_c^b \cap \mathcal{T}^{\leq -i-1}$. But the fact that $X \in \mathcal{T}_c^b$ means that there must exist a triangle $E \to X \to Y$ with $E \in \mathcal{T}$ and $Y \in \mathcal{T}_c^b \cap \mathcal{T}^{\leq -i-1}$. The vanishing of the map $X \to Y$ forces $X$ to be a direct summand of $E \in \mathcal{T}_c$, proving (ii). 

6. The category $\mathcal{T}_c^b$ determines $\mathcal{T}^c$

In Proposition 5.6 we saw that, if $\mathcal{T}$ is noetherian and weakly approximable, then the category $\mathcal{T}_c^b$ together with its metric determines $\mathcal{T}^c$. We want to find a recipe to produce the metric just from the triangulated category $\mathcal{T}_c^b$.

Notation 6.1. In this section it is quite easy to become confused by perpendiculars. The convention will be: if $P \subset \mathcal{T}$ is any subset, then $P^{\perp}$ means the perpendicular of $P$ in $\mathcal{T}$. If it so happens that $P \subset \mathcal{T}_c^b \subset \mathcal{T}$, our notation for the perpendicular of $P$ in $\mathcal{T}_c^b$ will be $\mathcal{T}_c^b \cap P^{\perp}$.

Lemma 6.2. Let $\mathcal{T}$ be a noetherian triangulated category, and choose a $t$–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ in the preferred equivalence class. For any object $H \in \mathcal{T}_c^b$ and any integer $m \geq 0$ we can find an object $G \in \mathcal{T}_c^b$ with $H^{\perp} \supset \mathcal{T}^{\leq -m} \cap G^{\perp}$.

Proof. The noetherian hypothesis permits us to construct a triangle $A \to H \to G$, with $A \in \mathcal{T}^{\leq -m-1}$ and $G \in \mathcal{T}_c^b \cap \mathcal{T}^{\leq -m-1-N} \subset \mathcal{T}_c^b$. The result now follows because $H^{\perp} \supset A^{\perp} \cap G^{\perp} \supset \mathcal{T}^{\leq -m} \cap G^{\perp}$.

Lemma 6.3. Let $\mathcal{T}$ be a noetherian triangulated category. In the partial order on the subcategories of $\mathcal{T}_c^b$, introduced in Definition 0.3, the subcategories $\mathcal{T}_c^b \cap [(G)^{(-\infty, 0)}]^{\perp}$ with
Lemma 6.4. Let $G \in T_c^b$ have a minimal member $Q(T_c^b)$. The subcategory $Q(T_c^b)$ is equivalent, in the partial order, to $T_c^b \cap T^0$, where $(T^{\leq 0}, T^{\geq 0})$ is a t-structure in the preferred equivalence class.

Proof. To produce a $G$ which gives a minimal $T_c^b \cap (G)^{(\leq 0)}_+$ we apply Lemma 6.2 to a compact generator $H \in T$, to the t-structure $(T^{\leq 0}, T^{\geq 0})$, and to the integer $m = 0$. We learn that we may find an object $G \in T_c^b$ with $H^{-1} \supset T_c^b \cap G^\perp$. Suspending $i$ times, with $i \geq 0$, gives

$$(\Sigma^i H)^\perp \ni T_c^b \cap (\Sigma^i G)^\perp \ni T_c^b \cap (\Sigma^i G)^\perp$$

Intersecting over $i \geq 0$ we obtain

$$\bigcap_{i=0}^{\infty} (\Sigma^i H)^\perp \ni T_c^b \cap \left[ \bigcap_{i=0}^{\infty} (\Sigma^i G)^\perp \right]$$

which rewrites as $T_c^b \supset T_c^b \cap (G)^{(\leq 0)}_+$. Suspending $i + 1$, times, with $i \geq 0$, gives $T_c^b \supset T_c^b \cap (G)^{(\leq 0)}_+ \ni (G)^{(\leq 0)}_+$, and induction allows us to prove, for any $i \geq 0$, $T_c^b \supset T_c^b \cap (G)^{(\leq 0)}_+ \ni (G)^{(\leq 0)}_+$. After all it’s true for $i = 0$, and the inductive step follows from

$$T_c^b \supset T_c^b \cap (G)^{(\leq 0)}_+ \ni (G)^{(\leq 0)}_+ \ni (G)^{(\leq 0)}_+ \ni (G)^{(\leq 0)}_+$$

Now taking the union over $i > 0$ gives $T_c^b \supset T_c^b \cap (G)^{(\leq 0)}_+$. Intersecting with $T_c^b$ we have $T_c^b \cap T_c^b \supset T_c^b \cap (G)^{(\leq 0)}_+$. This gives us that, in the partial order of Definition 0.7 we have $T_c^b \cap (G)^{(\leq 0)}_+ \leq T_c^b \cap T_c^b$. On the other hand for any object $\tilde{G} \in T_c^b$ we have that $\tilde{G}$ belongs to $T_c^b$ for some $n$, hence $(\tilde{G})^{(\leq 0)} \subset T_n^c$, and $(\tilde{G})^{(\leq 0)} \subset T_c^b$. Thus for any $\tilde{G} \in T_c^b$ we have $T_c^b \cap (\tilde{G})^{(\leq 0)} \supset T_c^b \cap T^{\geq n+1}_c$; in other words the inequality $T_c^b \cap T^{\geq n+1}_c \leq T_c^b \cap (\tilde{G})^{(\leq 0)}_+$ is cheap.

\[\square\]

Lemma 6.4. Let $T$ be a noetherian triangulated category. With $Q(T_c^b)$ a minimal $T_c^b \cap (G)^{(\leq 0)}_+$ as in Definition 6.2 and Lemma 6.3, the subcategory $\frac{1}{2} Q(T_c^b)_c$ is equivalent, in the partial order, to $T_c^b \cap T^{\geq 0}$, where $(T^{\leq 0}, T^{\geq 0})$ is a t-structure in the preferred equivalence class.

Proof. By Lemma 6.3 the subcategory $Q(T_c^b)$ is equivalent to $T_c^b \cap T^{\geq 0}$, hence $\frac{1}{2} Q(T_c^b)_c$ is equivalent to $\frac{1}{2} (T_c^b \cap T^{\geq 0}) \cap T_c^b$. Clearly $T_c^b \cap T^{\leq -1} \cap T_c^b$, that is $T_c^b \cap T^{\leq -1}$, is a subcategory of $\frac{1}{2} (T_c^b \cap T^{\geq 0}) \cap T_c^b$. We need to prove the reverse inequality in the partial order.

Let $N > 0$ be the integer of Definition 5.1 that is any object $X \in T_c^b$ admits a triangle $A \to X \to B$ with $A \in T_c^b \cap T^{\geq N}$ and $B \in T_c^b \cap T^{\geq 0}$. I assert that, with this choice
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of $N$, we have $\perp [T^b_c \cap T^{\geq 0}] \cap T^b_c \subset T^b_c \cap T^{\leq N}$. To prove this take $X \in \perp [T^b_c \cap T^{\geq 0}] \cap T^b_c$ and form the triangle $A \rightarrow X \rightarrow B$ above. As $X$ belongs to $\perp [T^b_c \cap T^{\geq 0}] \cap T^b_c$ and $B$ belongs to $T^b_c \cap T^{\geq 0}$ the map $X \rightarrow B$ must vanish, making $X$ an direct summand of $A \in T^b_c \cap T^{\leq N}$. □

Summarizing the lemmas in this section we have:

**Proposition 6.5.** Let $T$ be a noetherian triangulated category. For this Proposition we will work entirely in the subcategory $T^b_c$; that is perpendiculars are understood in $T^b_c$.

Then with the partial order of Definition 0.7 there is a minimal subcategory $\mathcal{Q}(T^b_c)$ among the $\langle G \rangle^{-\infty,0} \perp \subset T^b_c$. If we define $\mathcal{L}_i$ by the formula $\mathcal{L}_i^{op} = \perp [\Sigma' \mathcal{Q}(T^b_c)]$, then the subcategories $\{\mathcal{L}_i, i \in \mathbb{N}\}$ form a metric equivalent to the $\{M_i\}$ defined on $[T^b_c]^{op}$ in Example 1.5(ii).

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