AUTOMORPHISMS OF K3 SURFACES, SIGNATURES, AND ISOMETRIES OF LATTICES

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Abstract. Let $\alpha$ be a Salem number of degree $d$ with $4 \leq d \leq 18$. We show that if $d \equiv 0, 4$ or $6 \pmod{8}$, then $\alpha$ is the dynamical degree of an automorphism of a complex (non-projective) $K3$ surface. We define a notion of signature of an automorphism, and use it to give a criterion for Salem numbers of degree 10 and 18 to be realized as the dynamical degree of such an automorphism. The first part of the paper contains results on isometries of lattices.

0. General introduction

A Salem polynomial is a monic, irreducible polynomial $S \in \mathbb{Z}[X]$ such that $S(X) = X^{\deg(S)}S(X^{-1})$ and that $S$ has exactly two roots outside the unit circle, both positive real numbers; hence $\deg(S)$ is even, and $\geq 2$; the unique real root $> 1$ of a Salem polynomial is called a Salem number. The degree of the Salem number $\alpha$ is by definition the degree of the Salem polynomial $S$.

Let $X$ be a complex analytic $K3$ surface, and let $T : X \to X$ be an automorphism; it induces an isomorphism $T^* : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$. The characteristic polynomial of $T^*$ is a product of at most one Salem polynomial, and of a product of finitely many cyclotomic polynomials (cf. [McM 02], Theorem 3.2. and Corollary 3.3). The dynamical degree of $T$ is by definition the spectral radius of $T^*$, i.e. the maximum of the absolute values of the eigenvalues of $T^*$; hence it is either 1 or a Salem number of degree $\leq 22$. We denote by $\lambda(T)$ the dynamical degree of $T$.

McMullen raised the following question (see [McM 02], [McM 11], [McM 16])

Question 1. Which Salem numbers occur as dynamical degrees of automorphisms of complex $K3$ surfaces?

We say that a Salem number is $K3$-realizable (or realizable, for short) if it is the dynamical degree of an automorphism of a complex $K3$ surface. McMullen (see [McM 02], [McM 11], [McM 16]) gave several examples of such Salem numbers (see also Oguiso [O 10], Brandhorst-Gonzalez [BGA 16], Iwasaki-Takada [IT 21]), as well as an infinite family of realizable degree 6 Salem numbers; these are obtained by automorphisms of Kummer surfaces (see [McM 02], §4). Other infinite families of realizable Salem numbers are given by Gross-McMullen (in degree 22, see [GM 02], Theorems 1.7 and 1.6), by Reschke (in degree 14, see [R 12], Theorem 1.2), and Brandhorst proved that some power of every Salem number of degree $\leq 20$ is realizable (cf. [Br 20], Theorem 1.1). The case of Salem numbers of degree 22 was treated in [BT 20].

We now consider a more precise question, as follows. Let

$$H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$$

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be the Hodge decomposition of $H^2(\mathcal{X}, \mathbb{C})$. Since the subspace $H^{2,0}(\mathcal{X})$ is one-dimensional, the isomorphism $T^*$ acts on it by multiplication by a scalar, denoted by $\omega(T)$; we have $|\omega(T)| = 1$.

The value of $\omega(T)$ depends on the geometry of the surface: if $\mathcal{X}$ is projective, then $\omega(T)$ is a root of unity (see McM 02, Theorem 3.5).

**Definition 1.** If $\alpha$ is a Salem number, and if $\delta \in \mathbb{C}^\times$ is such that $|\delta| = 1$, we say that $(\alpha, \delta)$ is realizable if there there exists an automorphism $T$ of a $K3$ surface such that $\lambda(T) = \alpha$ and $\omega(T) = \delta$.

Since the characteristic polynomial of $T$ is of the form $SC$, where $S$ is a Salem polynomial and $C$ a product of cyclotomic polynomials, $\omega(T)$ is either a root of $S$, or a root of unity.

Suppose first that $\omega(T)$ is not a root of unity, i.e. $\lambda(T)$ and $\omega(T)$ are roots of the same Salem polynomial; this implies that the surface is not projective, it is of algebraic dimension 0 (see Proposition 20.1). Note that this also implies that $\deg(S) \geq 4$.

**Theorem 1.** Let $S$ be a Salem polynomial of degree $d$ with $4 \leq d \leq 22$, and let $\alpha$ be the corresponding Salem number. The following are equivalent:

1. There exists a root $\delta$ of $S$ with $|\delta| = 1$ such that $(\alpha, \delta)$ is realizable.
2. $(\alpha, \delta)$ is realizable for all roots $\delta$ of $S$ with $|\delta| = 1$.

This is proved in 21.8. For Salem polynomials of degree 22, we have the following result (see Theorem 21.9): 

**Theorem 2.** Let $\alpha$ be a Salem number of degree 22 with Salem polynomial $S$, and let $\delta$ be a root of $S$ such that $|\delta| = 1$. Then $(\alpha, \delta)$ is realizable if and only if $|S(1)|$ and $|S(-1)|$ are squares.

Theorem 2 can also be deduced from the main result in BT 20 and Theorem 1. Indeed, it is proved in BT 20 that if $|S(1)|$ and $|S(-1)|$ are squares, then $(\alpha, \delta)$ is realizable for some root $\delta$ of $S$ such that $|\delta| = 1$, hence by Theorem 1 the pair $(\alpha, \delta)$ is realizable for all roots $\delta$ of $S$ with $|\delta| = 1$.

We now consider Salem polynomials $S$ of degree $d$ with $4 \leq d \leq 20$.

**Theorem 3.** Let $\alpha$ be a Salem number of degree $d$ with Salem polynomial $S$; assume that $4 \leq d \leq 20$ and $d \equiv 0, 4$ or 6 (mod 8). Let $\delta$ be a root of $S$ such that $|\delta| = 1$. Then $(\alpha, \delta)$ is realizable.

For $d = 20$, this is Theorem 1.3 in Takada’s paper 122, for the other values of $d$, it is a consequence of Theorem 21.3.

If $d = 10$ or 18, we give a condition for $(\alpha, \delta)$ to be realizable, in terms of signature maps (see Theorem 21.10). This condition does not always hold when $d = 18$, leading to non-realizable pairs (see Example 22.1). On the other hand, the case $d = 10$ remains open; numerical evidence seems to indicate that all pairs $(\alpha, \delta)$ might be realizable.

In a different direction, we show that if $\alpha$ is a Salem number of degree $d$ with $4 \leq d \leq 22$ and Salem polynomial $S$, and $\delta$ a root of $S$ such that $|\delta| = 1$, then there exist only finitely many $K3$ surfaces having an automorphism $T$ with $\lambda(T) = \alpha$ and $\omega(T) = \delta$ (see Proposition 25.1). This implies that, if $\alpha$ is a Salem number, there exists only finitely many $K3$ surfaces of algebraic dimension 0 having an automorphism of dynamical degree $\alpha$. 

The last sections of the paper contain some remarks on automorphisms of projective K3 surfaces.

The paper is divided into three parts, each preceded by an introduction. The first part concerns isometries of lattices: these are needed for the above mentioned results on automorphisms of K3 surfaces. Indeed, the intersection form
\[ H^2(\mathcal{X}, \mathbb{Z}) \times H^2(\mathcal{X}, \mathbb{Z}) \to \mathbb{Z} \]
of a K3 surface \( \mathcal{X} \) is an even unimodular lattice of signature \((3, 19)\). It is well-known that such a lattice is unique up to isomorphism; we denote it by \( \Lambda_{3,19} \). Automorphisms of \( \mathcal{X} \) induce isometries of the intersection form, hence of the lattice \( \Lambda_{3,19} \).

Part III concerns automorphisms of K3 surfaces. A criterion of McMullen (see [McM 11], Theorem 6.2) plays an important role in the proofs of Theorems 1-3; he gives explicit conditions for isometries to be induced by automorphisms of K3 surfaces.

Part II is the bridge between the arithmetic results of Part I and the geometric applications of Part III. Starting with an isometry of \( \Lambda_{3,19} \), one may have to modify it to be able to apply McMullen’s criterion; in this process, the characteristic polynomial can change: we keep the same Salem factor, but the cyclotomic factor can be different.

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Part I: Isometries of even unimodular lattices

A lattice is a pair \((L, q)\), where \(L\) is a free \(\mathbb{Z}\)-module of finite rank, and \(q: L\times L \to \mathbb{Z}\) is a symmetric bilinear form; it is unimodular if \(\det(1) = \pm 1\), and even if \(q(x, x)\) is an even integer for all \(x \in L\). Let \(r, s \geq 0\) be integers such that \(r \equiv s \mod 8\); this congruence condition is equivalent to the existence of an even unimodular lattice with signature \((r, s)\). When \(r, s \geq 1\), such a lattice is unique up to isomorphism (see for instance [S 77], chap. V); we denote it by \(\Lambda_{r,s}\). In [GM 02], Gross and McMullen raise the following question (see [GM 02], Question 1.1):

**Question 2.** What are the possibilities for the characteristic polynomial \(F(X) = \det(X - t)\) of an isometry \(t \in \text{SO}(\Lambda_{r,s})\)?

The condition \(t \in \text{SO}(\Lambda_{r,s})\) implies that \(F(X) = X^{\deg(F)}F(X^{-1})\), hence \(F\) is a symmetric polynomial (cf. [11]). Let \(2n = \deg(F)\), and let \(m(F)\) be the number of roots \(z\) of \(F\) such that \(|z| > 1\). As shown in [GM 02], we have the further necessary conditions:

\[(C 1) \ |F(1)|, |F(-1)| \text{ and } (-1)^nF(1)F(-1) \text{ are squares.}\]
(C 2) \( r \geq m(F), \ s \geq m(F), \) and if moreover \( F(1)F(-1) \neq 0, \) then \( m(F) \equiv r \equiv s \pmod{2}. \)

Gross and McMullen prove that if \( F \in \mathbb{Z}[X] \) is an irreducible, symmetric and monic polynomial satisfying condition (C 2) and such that \( |F(1)F(-1)| = 1 \), then there exists \( t \in \text{SO}(\Lambda_{r,s}) \) with characteristic polynomial \( F \) (see \cite{GM02}, Theorem 1.2). They speculate that conditions (C 1) and (C 2) are sufficient for a monic irreducible polynomial to be realized as the characteristic polynomial of an isometry of \( \Lambda_{r,s} \); this is proved in \cite{BT20}, Theorem A. More generally, Theorem A of \cite{BT20} implies that if a monic, irreducible and symmetric polynomial \( F \) satisfies conditions (C 1) and (C 2), then there exists an even unimodular lattice of signature \( (r, s) \) having an isometry with characteristic polynomial \( F \). This is also the point of view of the present paper - we treat the definite and indefinite cases simultaneously.

The first result is that condition (C 1) is sufficient locally at the finite places. If \( p \) is a prime number, we say that a \( \mathbb{Z}_p \)-lattice \( (L, q) \) is even if \( q(x, x) \in 2\mathbb{Z}_p \) for all \( x \in L \); note that if \( p \neq 2 \), then every lattice is even, since \( 2 \) is a unit in \( \mathbb{Z}_p \). The following is proved in Theorem 2.1.

Let \( F \in \mathbb{Z}[X] \) be a monic, symmetric polynomial of even degree.

**Theorem 4.** Condition (C 1) holds if and only if for all prime numbers \( p \), there exists an even unimodular \( \mathbb{Z}_p \)-lattice having a semi-simple isometry with characteristic polynomial \( F \).

If \( F \) is irreducible, then Theorem A of \cite{BT20} implies that we have a local-global principle; however, this does not extend to the case where \( F \) has distinct irreducible factors. We define an obstruction to the Hasse principle in terms of an equivalence relation on the irreducible factors of \( F \); this leads to an obstruction group, see \S7.

Let us write \( F(X) = F_1(X)(X - 1)^{n^+}(X + 1)^{n^-} \) with \( F_1(1)F_1(-1) \neq 0 \), and suppose for simplicity that \( n^+ \neq 2 \) and \( n^- \neq 2 \) (the case \( n^+ = n^- = 2 \) is also treated in the paper, but the results are more complicated to state). In this case, the obstruction group only depends on \( F \), and we denote it by \( G_F \). In \S12 we show the following:

**Theorem 5.** Let \( r, s \geq 0 \) be integers such that \( r \equiv s \pmod{8} \). Suppose that Conditions (C 1) and (C 2) hold and that \( G_F = 0 \). Let \( t \in \text{SO}_{r,s}(\mathbb{R}) \) be a semi-simple isometry with characteristic polynomial \( F \). Then \( t \) preserves an even unimodular lattice.

If \( G_F \neq 0 \), then we need additional conditions for a semi-simple isometry to stabilize an even unimodular lattice; finding such conditions is the subject matter of sections \S5 - \S12, leading to Theorem 6 below:

Let \( t \in \text{SO}_{r,s}(\mathbb{R}) \) be a semi-simple isometry with characteristic polynomial \( F \), and suppose that Condition (C 1) holds. In \S12 we define a homomorphism

\[ \epsilon_t : G_F \rightarrow \mathbb{Z}/2\mathbb{Z}, \]

and we show (cf. Theorem \S12,2):

**Theorem 6.** The isometry \( t \) preserves an even unimodular lattice if and only if \( \epsilon_t = 0 \).

A more convenient way of expressing Theorem 6 is given in terms of signature maps of isometries, see \S3.
1. Definitions, notation and basic facts

We start by recalling some notions and results from [M 69], [B 15], [BT 20] and [B 21]. Let $K$ be a field.

**Isometries**

Let $V$ be a finite dimensional $K$-vector space, and let $q : V \times V \to K$ be a non-degenerate symmetric bilinear form. An isometry $t : V \to V$ such that $q(tx, ty) = q(x, y)$ for all $x, y \in V$.

If $f \in K[X]$ is a monic polynomial such that $f(0) \neq 0$, set

$$f^*(X) = f(0)^{-1}X^{\deg(f)}f(X^{-1});$$

we say that $f$ is symmetric if $f^* = f$. It is well-known that the characteristic polynomial of an isometry is symmetric (see for instance [B 15], Proposition 1.1).

**Equivariant Witt groups and residue maps**

Let $\Gamma$ be the infinite cyclic group denoted multiplicatively, and let $K[\Gamma]$ be the associated group ring. Sending $\gamma \in \Gamma$ to $\gamma^{-1}$ induces a $K$-linear involution of $K[\Gamma]$; let $W_\Gamma(K)$ be the Witt ring of symmetric $K[\Gamma]$-bilinear forms (see [BT 20], §3). If $M$ is a simple $K[\Gamma]$-module, we denote by $W_\Gamma(K, M)$ the subgroup of $W_\Gamma(K)$ generated by the classes of $K[\Gamma]$-bilinear forms $(M, q)$.

Let $V$ be a finite dimensional $K$-vector space, let $q : V \times V \to K$ be a non-degenerate symmetric bilinear form, and let $t : V \to V$ be an isometry of $q$. Let $\gamma$ be a generator of $\Gamma$. Let us define a $K[\Gamma]$-module structure on $V$ by setting $\gamma.x = t(x)$ for all $x \in V$. This gives rise to a $K[\Gamma]$-bilinear form $(V, q)$, and we obtain an element $[V] = [V, q]$ of $W_\Gamma(K)$.

If $K$ is a local field with residue field $\kappa$, we have a residue map $W_\Gamma(K) \to W_\Gamma(\kappa)$, see [BT 20], §4; a $K[\Gamma]$-bilinear form contains a unimodular lattice if and only if it is defined on a bounded module, and its class in $W_\Gamma(K)$ is in the kernel of this map (cf. [BT 20], Theorem 4.3 (iv)).

**Symmetric polynomials over the integers**

The following definitions and notation are used throughout the paper.

**Definition 1.1.** Let $f \in \mathbb{Z}[X]$ be a monic, symmetric polynomial. We say that $f$ is of

- **type** 0 if $f$ is a product of powers of $X - 1$ and of $X + 1$;
- **type** 1 if $f$ is a product of powers of monic, symmetric, irreducible polynomials in $\mathbb{Z}[X]$ of even degree;
- **type** 2 if $f$ is a product of polynomials of the form $gg^*$, where $g \in \mathbb{Z}[X]$ is monic, irreducible, and $g \neq \pm g^*$.

It is well-known that every monic, symmetric polynomial is a product of polynomials of type 0, 1 and 2.

If $F \in \mathbb{Z}[X]$ is a monic, symmetric polynomial, we write $F = F_0F_1F_2$, where $F_i$ is the product of the irreducible factors of type $i$ of $F$. We have $F_0(X) = (X - 1)^{n^+}(X + 1)^{n^-}$ for some integers $n^+, n^- \geq 0$, and $F_1 = \prod_{f \in I_1} f^{n_1}$, where $I_1$ is the set of irreducible factors of type 1 of $F$.

Finally, we need the following notation:
Notation 1.2. Let $E_0$ be an étale $K$-algebra of finite rank, and let $E$ be an étale $E_0$-algebra which is free of rank 2 over $E_0$. Let $\sigma : E \to E$ be the involution fixing $E_0$.

(a) If $\lambda \in E_0^\times$, we denote by $b_\lambda$ the symmetric bilinear form $b_\lambda : E \times E \to K$ such that $b_\lambda(x, y) = \Tr_{E/K}(\lambda x\sigma(y))$.

(b) Set $T(E, \sigma) = E_0^\times / \mathcal{N}_{E_0}(E^\times)$.

Note that the isomorphism class of $b_\lambda$ only depends on the class of $\lambda$ in $T(E, \sigma)$.

2. Local results

Let $F \in \mathbb{Z}[X]$ be a monic, symmetric polynomial of even degree such that $F(0) = 1$, and let $\deg(F) = 2n$. Recall from the introduction to Part I that the following condition is necessary for the existence of an even, unimodular lattice having an isometry with characteristic polynomial $F$:

$$(C\, 1) \mid F(1), \mid F(-1) \text{ and } (-1)^n F(1)F(-1) \text{ are squares.}$$

The aim of this section is to show that this condition is necessary and sufficient for the existence of such a lattice at all the finite places:

Theorem 2.1. Condition $(C\, 1)$ holds if and only if for all prime numbers $p$, there exists an even unimodular $\mathbb{Z}_p$-lattice having a semi-simple isometry with characteristic polynomial $F$.

Let us write $F = F_0F_1F_2$ as in §1 with $F_0(X) = (X - 1)^{n_+}(X + 1)^{n_-}$. Since the hyperbolic lattice of rank $\deg(F_2)$ has a semi-simple isometry with characteristic polynomial $F_2$, we assume that $F = F_0F_1$. Let $\deg(F_1) = 2n_1$.

If $p$ is a prime number, we denote by $v_p$ the $p$-adic valuation.

Notation 2.2. If $q$ is a non-degenerate quadratic form over $\mathbb{Q}_p$, we denote by $w_2(q) \in \Br_2(\mathbb{Q}_p)$ its Hasse-Witt invariant; we identify $\Br_2(\mathbb{Q}_p)$ with $\{0, 1\}$.

Notation 2.3. If $f \in \mathbb{Z}[X]$ is an irreducible, symmetric polynomial of even degree, set $E_f = \mathbb{Q}[X]/(f)$, let $\sigma_f : E_f \to E_f$ be the involution induced by $X \mapsto X^{-1}$, and let $(E_f)_0$ be the fixed field of $\sigma$ in $E_f$. Let $\alpha \in E_f$ be the image of $X$.

Definition 2.4. Let $f \in \mathbb{Z}[X]$ be an irreducible, symmetric polynomial of even degree, and let $p$ be a prime number. We say that $f$ is ramified at $p$ if there exists a place $w$ of $(E_f)_0$ above $p$ that is ramified in $E_f$; otherwise, we say that $f$ is unramified at $p$. We denote by $\Pi_f^+$ the set of prime numbers $p$ such that $f$ is ramified at $p$.

Let $\pi_f$ be an odd prime number. If $w$ is a place of $(E_f)_0$ above $p$ that is ramified in $E_f$, we denote by $\kappa_w$ the residue field of $w$, and by $\overline{\alpha}$ be the image of $\alpha$ in $\kappa_w$; we denote by $S_+$ the set of places $w$ above $p$ such that $\overline{\alpha} = 1$, and by $S_-$ the set of places $w$ above $p$ such that $\overline{\alpha} = -1$. We denote by $\Pi_f^{+/-}$ the set of prime numbers $p$ such that there exists a place $w$ above $p$ with $w \in S_+$, and by $\Pi_f^{-/-}$ the set of prime numbers $p$ such that there exists a place $w$ above $p$ with $w \in S_-$.

Definition 2.5. Let $F \in \mathbb{Z}[X]$ be a monic, symmetric polynomial, and let $I_1$ be the set of monic, symmetric, irreducible factors of $F$ of even degree. If $p$ is a prime number, we say that $F$ is ramified at $p$ if there exists $f \in I_1$ such that $f$ is ramified at $p$; otherwise, we say that $F$ is unramified at $p$. We say that $F$ is unramified if $F$ is unramified at every prime number $p$; otherwise, we say that $F$ is ramified.
If $|F(1)F(1)| = 1$, then $F$ is unramified (see [GM 02], Proposition 3.1; note that in [GM 02] an irreducible, symmetric polynomial $S$ is said to be unramified if $|S(1)S(-1)| = 1$.

**Proposition 2.6.** Let $p$ be a prime number. There exists an even, unimodular $\mathbb{Z}_p$-lattice having a semi-simple isometry with characteristic polynomial $F$ if and only if the following conditions hold:

(a) If $v_p(F_1(1)) \equiv 1 \pmod{2}$, then $n^+ \geq 1$;

(b) If $v_p(F_1(-1)) \equiv 1 \pmod{2}$, then $n^- \geq 1$;

(c) If moreover $p = 2$ and $n^+ = n^- = 0$, then the class of $(-1)^n F_1(1) F_1(-1)$ in $\mathbb{Q}_2^\times/\mathbb{Q}_2^{\times 2}$ lies in $\{1, -3\}$.

**Proof.** We start by showing that if conditions (a), (b) and (c) hold, then there exists an even, unimodular $\mathbb{Z}_p$-lattice having a semi-simple isometry with characteristic polynomial $F$.

We use the notation of §1 in particular, $F_1 = \prod_{f \in I_1} f^{n_f}$. For all $f \in I_1$, we consider the étale $\mathbb{Q}$-algebra $E_f = \mathbb{Q}[X]/(f)$; since $f$ is irreducible, $E_f$ is a finite field extension of $\mathbb{Q}$. We denote by $\sigma : E_f \to E_f$ the involution induced by $X \mapsto X^{-1}$, and by $(E_f)_0$ the fixed field of $\sigma_f$ in $E_f$. Let $L_f$ be an extension of degree $n_f$ of $(E_f)_0$, linearly disjoint from $E_f$ over $(E_f)_0$; set $\tilde{E}_f = E_f \otimes_{(E_f)_0} L_f$, and let $\tilde{\sigma}_f : \tilde{E}_f \to \tilde{E}_f$ be the involution induced by $\sigma_f$. Let $\alpha_f$ be a root of $f$ in $E_f$.

Set $E = \prod_{f \in I_1} \tilde{E}_f$, $E_0 = \prod_{f \in I_1} L_f$, and let $\sigma = (\tilde{\sigma}_f)$ be the involution $\sigma : E \to E$ induced by the involutions $\tilde{\sigma}_f : \tilde{E}_f \to \tilde{E}_f$; we have $E_0 = \{e \in E \mid \sigma(e) = e\}$. Let $\alpha = (\alpha_f)$; the characteristic polynomial of the multiplication by $\alpha$ on $E$ is equal to $F_1$.

Let $V^+$ and $V^-$ be $\mathbb{Q}_p$-vector spaces with $\text{dim}(V^+) = n^+$, $\text{dim}(V^-) = n^-$, and set $V = V^+ \oplus V^-$. Let $\epsilon = (\epsilon^+, \epsilon^-) : V \to V$ be the isomorphism given by $\epsilon^+ : V^+ \to V^+$, $\epsilon^- : V^- \to V^-$, such that $\epsilon^+ = \text{id}$ and $\epsilon^- = -\text{id}$.

We denote by $\partial : W_1(\mathbb{Q}_p) \to W_1(\mathbb{F}_p)$ the residue map (see §1). If $\lambda \in E_0^\times$, we obtain a non-degenerate quadratic form $b_\lambda : E \times E \to \mathbb{Q}_p$, cf. notation [1.2].

The rest of the proof is somewhat different for $p \neq 2$ and $p = 2$. We first consider the case where $p \neq 2$:

**Claim 1:** Assume that $p \neq 2$. Let $u_+, u_- \in \mathbb{Z}_p^\times$. There exists $\lambda \in E_0^\times$ and non-degenerate quadratic forms $q^+ : V^+ \times V^+ \to \mathbb{Q}_p$, $q^- : V^- \times V^- \to \mathbb{Q}_p$ such that for $q = q^+ \oplus q^-$, we have

$$\partial[E \oplus V, b_\lambda \oplus q, \alpha \oplus \epsilon] = 0,$$

and, if moreover $n^+ \neq 0$, then $\det(q^+) = u_+ F_1(1)$, and if $n^- \neq 0$, then $\det(q^-) = u_- F_1(-1)$.

**Proof of Claim 1:** If $v_p(F_1(1)) \equiv 0 \pmod{2}$ and $v_p(F_1(-1)) \equiv 0 \pmod{2}$, then this is an immediate consequence of [BT 20], Proposition 7.1. The algebra $E_0$ decomposes as a product of fields $E_0 = \prod_{v \in S} E_{0,v}$, where $S$ is the set of places above $p$. For all $v \in S$, set $E_v = E \otimes_{E_0} E_{0,v}$. The algebra $E_v$ is of one of the following types:

- (sp) $E_v = E_{0,v} \times E_{0,v}$;
- (un) $E_v$ is an unramified extension of $E_{0,v}$;
(+) \( E_v \) is a ramified extension of \( E_{0,v} \), and the image \( \overline{\alpha} \) of \( \alpha \) in the residue field \( \kappa_v \) of \( E_v \) is 1;

(-) \( E_v \) is a ramified extension of \( E_{0,v} \), and the image \( \overline{\alpha} \) of \( \alpha \) in the residue field \( \kappa_v \) of \( E_v \) is \(-1\).

This gives a partition \( S = S_{sp} \cup S_{un} \cup S_+ \cup S_- \).

Let \( \gamma \) be a generator of \( \Gamma \), and let \( \chi_\pm : \Gamma \to \{ \pm 1 \} \) be the character sending \( \gamma \) to \( \pm 1 \).

Let us choose \( \lambda = (\lambda_v)_{v \in S} \) in \( \prod_{v \in S} E_0^\times \) such that for every \( v \in S_{un} \), we have \( \partial[E_v, b_{\lambda_v}, \alpha] = 0 \) in \( W_\Gamma(F_p) \); this is possible by [BT 20], Proposition 6.4. For all \( u \in \mathbb{Z}_p^\times \), let \( \overline{u} \) be the image of \( u \) in \( F_p \).

Assume first that \( v_p(F_1(1)) \equiv 1 \pmod{2} \) and \( v_p(F_1(-1)) \equiv 0 \pmod{2} \). For \( v \in S_- \) we choose \( \lambda_v \) such that
\[
\sum_{v \in S_-} \partial[E_v, b_{\lambda_v}, \alpha] = 0 \quad \text{in} \quad W(F_p) = W(F_p, \chi_-) \subset W_\Gamma(F_p). \]
This is possible by [BT 20], Proposition 6.6; indeed, by [BT 20], Lemma 6.8 (ii) we have
\[
\sum_{v \in S_-} [\kappa_v : F_p] \equiv v_p(F_1(-1)) \pmod{2}. \]

We now come to the places in \( S_+ \). Recall that by [BT 20], Lemma 6.8 (i) we have
\[
\sum_{v \in S_+} [\kappa_v : F_p] \equiv v_p(F_1(1)) \pmod{2}. \]
Since \( v_p(F_1(1)) \equiv 1 \pmod{2} \) by hypothesis, this implies that \( \sum_{v \in S_+} [\kappa_v : F_p] \equiv 1 \pmod{2} \). Therefore there exists \( w \in S_+ \) such that \( [\kappa_w : F_p] \) is odd. By [BT 20], Proposition 6.6, we can choose \( \lambda_w \) such that \( \partial[E_w, b_{\lambda_w}, \alpha] \) is either one of the two classes of \( W(F_p) = W(F_p, \chi_+) \subset W_\Gamma(F_p) \) represented by a one-dimensional quadratic form over \( F_p \). Let us choose the class of determinant \( -\overline{1} \), and set \( \partial[E_w, b_{\lambda_w}, \alpha] = \delta \). Since \( v_p(F_1(1)) \equiv 1 \pmod{2} \), by hypothesis we have \( n^+ \geq 1 \). Let \( (V^+, q^+) \) be a non-degenerate quadratic form over \( Q_p \) such that \( \det(q^+) = u_+ F_1(1) \), and that \( \partial[V^+, q^+, id] = -\delta \) in \( W(F_p) = W(F_p, \chi_+) \subset W_\Gamma(F_p) \).

Let \( S'_+ = S_+ - \{ w \} \); we have \( \sum_{v \in S'_+} [\kappa_v : F_p] \equiv 0 \pmod{2} \), hence by [BT 20], Proposition 6.6, for all \( v \in S'_+ \) there exists \( \lambda_v \in E_{0,v}^\times \) such that \( \sum_{v \in S'_+} \partial[E_v, b_{\lambda_v}, \alpha] = 0 \) in \( W(F_p) = W(F_p, \chi_+) \subset W_\Gamma(F_p) \). We have
\[
\partial[E \oplus V^+, b_\lambda \oplus q^+, (\alpha, id)] = 0 \quad \text{in} \quad W_\Gamma(F_p). \]
Taking for \( (V^-, q^-) \) a quadratic form over \( Q_p \) of determinant \( u_- F_1(-1) \) and setting \( q = q^+ \oplus q^- \), we get \( \partial[E \oplus V, b \oplus q, (\alpha, \epsilon)] = 0 \) in \( W_\Gamma(F_p) \). This completes the proof when \( v_p(F_1(1)) \equiv 1 \pmod{2} \) and \( v_p(F_1(-1)) \equiv 0 \pmod{2} \). The proof is the same when \( v_p(F_1(1)) \equiv 0 \pmod{2} \) and \( v_p(F_1(-1)) \equiv 1 \pmod{2} \), exchanging the roles of \( S_+ \) and \( S_- \).

Assume now that \( v_p(F_1(1)) \equiv 1 \pmod{2} \) and \( v_p(F_1(-1)) \equiv 1 \pmod{2} \). By [BT 20], Lemma 6.8 (i) and (ii), we have
\[
\sum_{v \in S} [\kappa_v : F_p] \equiv v_p(F_1(1)) \pmod{2}, \quad \text{and} \quad \sum_{v \in S_-} [\kappa_v : F_p] \equiv v_p(F_1(-1)) \pmod{2}. \]
Therefore \( \sum_{v \in S_+} [\kappa_v : F_p] \equiv \sum_{v \in S_-} [\kappa_v : F_p] \equiv 1 \pmod{2} \). Hence there exist
$w_\pm \in S_\pm$ such that $[\kappa_{w_+} : F_p]$ and $[\kappa_{w_-} : F_p]$ are odd. By [BT 20], Proposition 6.6, we can choose $\lambda_{w_+}$ such that $\partial[E_{w_\pm}, b_{\lambda_{w_\pm}}, \alpha]$ is either one of the two classes of $\gamma \in W(F_p) = W(k, \chi_\pm) \subset W_T(F_p)$ with $\dim(\gamma) = 1$. Let us choose $\lambda_{w_+}$ such that $\partial[E_{w_\pm}, b_{\lambda_{w_\pm}}, \alpha]$ is represented by a form of dimension 1 and determinant $\mu_\pm$, and set $\delta_\pm = \partial[E_{w_\pm}, b_{\lambda_{w_\pm}}, \alpha]$. By hypothesis, we have $n^+ \geq 1$ and $n^- \geq 1$. Let $(V^\pm, q^\pm)$ be non-degenerate quadratic forms over $Q_p$ such that $\det(q^\pm) = u_\pm f(e \pm 1)$ and that

$$\partial[V^\pm, q^\pm, e^\pm] = -\delta_\pm \in W(F_p) = W_T(F_p, \chi_\pm) \subset W_T(F_p).$$

Let $S'_+ = S_+ - \{w_\pm\}$; we have $\sum_{v \in S'_+} [\kappa_v : F_p] \equiv 0 \pmod{2}$, hence by [BT 20], Proposition 6.6, for all $v \in S'_+$ there exists $\lambda_v \in E_{0,v}^\times$ such that $\sum_{v \in S'_+} \partial[E_v, b_{\lambda_v}, \alpha] = 0$ in $W(F_p) = W(F_p, \chi_+) \subset W_T(F_p)$. Similarly, set $S'_- = S_+ - \{w_\pm\}$; we have $\sum_{v \in S'_-} [\kappa_v : F_p] \equiv 0 \pmod{2}$, hence by [BT 20], Proposition 6.6, for all $v \in S'_-$ there exists $\lambda_v \in E_{0,v}^\times$ such that $\sum_{v \in S'_-} \partial[E_v, b_{\lambda_v}, \alpha] = 0$ in $W(F_p) = W_T(F_p, \chi_-)$. Let $q = q^+ \oplus q^-$, and note that $\partial[E \oplus V, b_\lambda \oplus q, (\alpha, e)] = 0$ in $W_T(F_p)$. This completes the proof of Claim 1. We now treat the case where $p = 2$.

**Claim 2:** Let $u_+, u_- \in Z_2^\times$ be such that $u_+ u_- = (-1)^n$. Then there exists $\lambda \in E_0^\times$ and non-degenerate quadratic forms $q^+ : V^+ \times V^+ \to Q_2$, $q^- : V^- \times V^- \to Q_2$ such that for $q = q^+ \oplus q^-$, we have

$$\partial[E \oplus V, b_\lambda \oplus q, (\alpha, e)] = 0,$$

that $(E \oplus V, b_\lambda \oplus q)$ contains an even unimodular $Z_2$-lattice, and, if moreover $n^+ \neq 0$ then $\det(q^+) = u_+ F_1(1)$, and if $n^- \neq 0$ then $\det(q^-) = u_- F_1(-1)$.

**Proof of Claim 2:** The algebra $E_0$ decomposes as a product of fields $E_0 = \prod_{v \in S} E_{0,v}$. For all $v \in S$, set $E_v = E \otimes E_0, E_{0,v}$. The algebra $E_v$ is of one of the following types:

- (sp) $E_v = E_{0,v} \times E_{0,v}$;
- (un) $E_v$ is an unramified extension of $E_{0,v}$;
- (r) $E_v$ is a ramified extension of $E_{0,v}$.

This gives a partition $S = S_{sp} \cup S_{un} \cup S_r$. Assume first that $v_2(F_1(1)) + v_2(F_1(-1)) \equiv 0 \pmod{2}$. By [BT 20], Lemma 6.8 and Proposition 6.7, we can choose $\lambda_v$ such that

$$\sum_{v \in S} \partial[E_v, b_{\lambda_v}, \alpha] = 0 \text{ in } W(F_2) = W_T(F_2, 1) \subset W_T(F_2).$$

Therefore $\partial[E, b_\lambda, \alpha] = 0$ in $W_T(F_2)$. Suppose that $v_2(F_1(1)) \equiv v_2(F_1(-1)) \equiv 0 \pmod{2}$. We have $\det(E, b_\lambda) = F_1(1) F_1(-1)$ in $Q_2^\times/Q_2^{\times 2}$ (see for instance [B 15], Corollary 5.2). If $n^+ = n^- = 0$, then by condition (c) the class of $(-1)^n F_1(1) F_1(-1)$ in $Q_2^\times/Q_2^{\times 2}$ lies in $\{1, -3\}$, and this implies that the discriminant of the form $(E, b_\lambda)$ in $Q_2^\times/Q_2^{\times 2}$ belongs to $\{1, -3\}$. Let us choose $\lambda$ such that the quadratic form $(E, b_\lambda)$ contains an even unimodular $Z_2$-lattice. If $S_r = \emptyset$, this is automatic; indeed, in that case every $Z_2$-lattice in $(E, b_\lambda)$ is even. If not, by [BT 20] Propositions 8.4 and 5.4 we can choose $\lambda$ having this additional property. If $n^+ \neq 0$ and $n^- = 0$, we choose $q^+$ such that $\det(q^+) = (-1)^n F_1(1) F_1(-1)$; since $\det(b_\lambda) = F_1(1) F_1(-1)$, this implies
that \( \det(E \oplus V, b_\lambda \oplus q^+) = (-1)^n \). Moreover, let us choose the Hasse-Witt invariant of \( b_\lambda \oplus q^+ \) in such a way that the quadratic form \((E \oplus V, b_\lambda \oplus q)\) contains an even unimodular \( \mathbb{Z}_2 \)-lattice; this is possible by [BT 20] Corollary 8.4. Note that since \( v_p(\det(q^+)) = 0 \), we have \( \partial[V, q^+] = 0 \) in \( W(V_2) \), hence \( \partial[E \oplus V, b_\lambda \oplus q, \alpha \oplus \epsilon] = 0 \) in \( W_\Gamma(F_2) \). The same argument applies if \( n^+ = 0 \) and \( n^- \neq 0 \). Suppose that \( n^+ \neq 0 \) and \( n^- \neq 0 \). Let us choose \( q^+ \) such that \( \det(q^+) = u_+F_1(1) \) and \( q^- \) such that \( \det(q^-) = u_-F_1(-1) \). Since \( u_+u_- = (-1)^n \) and \( \det(b_\lambda) = F_1(1)F_1(-1) \), this implies that \( \det(E \oplus V, b_\lambda \oplus q^+ \oplus q^-) = (-1)^n \). As in the previous cases, we can choose \( \lambda, q^+ \) and \( q^- \) such that \( (E \oplus V, b_\lambda \oplus q) \) contains an even unimodular \( \mathbb{Z}_2 \)-lattice, and that \( \partial[E \oplus V, b_\lambda \oplus q, \alpha \oplus \epsilon] = 0 \) in \( W_\Gamma(F_2) \).

Suppose that \( v_2(F_1(1)) \equiv v_2(F_1(-1)) \equiv 1 \pmod{2} \); note that the hypothesis implies that \( n^+, n^- \geq 1 \). With our previous choice of \( \lambda \), we have \( \partial[E, b_\lambda, \alpha] = 0 \) in \( W_\Gamma(F_2) \). Let us choose \( q^+ \) and \( q^- \) such that \( \det(q^+) = u_+F_1(1) \), and note that this implies that \( \det(E \oplus V, b_\lambda \oplus q) = (-1)^n \), and that \( v_2(\det(q^-)) = v_2(F_1(-1)) \). Moreover, choose the Hasse-Witt invariants of \( b_\lambda, q^+ \) and \( q^- \) so that \( (E \oplus V, b_\lambda \oplus q^+ \oplus q^-) \) contains an even unimodular \( \mathbb{Z}_2 \)-lattice; this is possible by [BT 20] Corollary 8.4. We have \( \partial[V, q, \epsilon] = 0 \) in \( W_\Gamma(F_2) \), hence \( \partial[E \oplus V, b_\lambda \oplus q, \alpha \oplus \epsilon] = 0 \) in \( W_\Gamma(F_2) \).

Assume now that \( v_2(F_1(1)) \equiv 1 \pmod{2} \), and \( v_2(F_1(-1)) \equiv 0 \pmod{2} \). By hypothesis, this implies that \( n^+ \geq 1 \). If \( n^- \neq 0 \), then choose \( q^- \) such that \( \det(q^-) = u_-F_1(-1) \), and note that this implies that \( v_2(\det(q^-)) \equiv v_2(F_1(-1)) \equiv 0 \pmod{2} \); choose \( q^+ \) such that \( \det(q^+) = u_+F_1(1) \). Since \( v_2(F_1(1)) \equiv 1 \pmod{2} \), this implies that \( \partial[V^+, q^+, id] \) is the unique non-trivial element of \( W(F_2) = W_\Gamma(F_2, 1) \subset W_\Gamma(F_2) \). Note that \( \partial[E \oplus V, b_\lambda \oplus q^+ \oplus q^-] = (-1)^n \) in \( \mathbb{Q}_2^\times / \mathbb{Q}^\times \). Let us choose \( \lambda, q^+ \) and \( q^- \) such that the quadratic form \( (E \oplus V, b_\lambda \oplus q^+ \oplus q^-) \) contains an even unimodular \( \mathbb{Z}_2 \)-lattice; this is possible by [BT 20] Corollary 8.4. Note that \( \partial[E, b_\lambda, \alpha] \) and \( \partial[V^+, q^+, id] \) are both equal to the unique non-trivial element of \( W(F_2) = W_\Gamma(F_2, 1) \), which is a group of order 2. Therefore \( \partial[E \oplus V, b_\lambda \oplus q, \alpha \oplus \epsilon] = 0 \) in \( W_\Gamma(F_2) \). If \( v_2(F_1(1)) \equiv 0 \pmod{2} \) and \( v_2(F_1(-1)) \equiv 1 \pmod{2} \), the same argument gives the desired result. This completes the proof of Claim 2.

Let us show that conditions (a), (b) and (c) imply the existence of an even unimodular lattice having a semi-simple isometry with characteristic polynomial \( F \). If \( p \neq 2 \), then this is an immediate consequence of Claim 1. If \( p = 2 \), we apply Claim 2 and [BT 20], Theorem 8.1; note that \( v_2(\det(q^-)) = v_2(F_1(-1)) \), hence Theorem 8.5 of [BT 20] implies that condition (iii) of this theorem is satisfied.

Conversely, we now show that if there exists an even unimodular \( \mathbb{Z}_p \)-lattice having a semi-simple isometry with characteristic polynomial \( F \), then conditions (a), (b) and (c) hold. Assume that such a lattice exists, and let \( (V, q) = (V^0, q^0) \oplus (V^1, q^1) \oplus (V^2, q^2) = V^0 \oplus V^1 \oplus V^2 \) the corresponding orthogonal decomposition of \( \mathbb{Q}_p[\Gamma] \)-bilinear forms. Since \( \partial[V^2] = 0 \), we have \( \partial[V^0 \oplus V^1] = 0 \). We have the further orthogonal decomposition \( V^1 = \oplus V_f \), and \( V_f \) is the orthogonal sum of quadratic forms of the type \( b_\lambda \). By [BT 20] Lemma 6.8 and Proposition 6.6, the component of \( \partial(V^1) \) in \( W_\Gamma(F_p, \chi_\pm) \) is represented by a quadratic form over \( F_p \) of dimension \( v_p(F_1(1)) \) if \( p \neq 2 \), and by a form of dimension \( v_2(F_1(1)) + v_2(F(-1)) \) if \( p = 2 \).

Suppose that \( p \neq 2 \), and let us show that (a) and (b) hold. Assume that \( n^+ = 0 \). Then the component of \( \partial[V^0] \) in \( W_\Gamma(F_p, \chi_\pm) \) is trivial, and therefore \( v_p(F_1(1)) \equiv 0 \pmod{2} \), hence (a) holds. By the same argument, (b) holds as well.
Let \( p = 2 \). We have \( v_2(F_1(-1)) \equiv v_2(\det(q^-)) \pmod{2} \) by [BT 20], Proposition 8.6 and Theorem 8.5. If \( n^+ = n^- = 0 \), then \( v_2(F(-1)) \equiv 0 \pmod{2} \), and the above argument shows that \( v_2(F_1(1)) + v_2(F_1(-1)) \equiv 0 \pmod{2} \). This implies (a) and (b); property (c) also holds, since the discriminant of the lattice is \((-1)^n F_1(1) F_1(-1)\).

If \( n^+ \neq 0 \) or \( n^- \neq 0 \), then \( v_2(F_1(1)) + v_2(F_1(-1)) \equiv 1 \pmod{2} \); in both cases, we see that the congruence \( v_2(F_1(-1)) \equiv v_2(\det(q^-)) \pmod{2} \) implies (a) and (b).

This completes the proof of the proposition.

Proof of Theorem 2.1. Let \( p \) be a prime number. By Proposition 2.6, the existence of an even, unimodular lattice over \( \mathbb{Z}_p \) having a semi-simple isometry with characteristic polynomial \( F \) implies that either \( F(1) = 0 \), or \( v_p(F(1)) \) is even; similarly, either \( F(-1) = 0 \), or \( v_p(F(-1)) \) is even. This implies that \( |F(1)| \) and \( |F(-1)| \) are squares. If \( F(1)F(-1) = 0 \), we are done. If not, we apply condition (c): the class of \((-1)^n F_1(1) F_1(-1)\) in \( \mathbb{Q}_p^2 / \mathbb{Q}_p^{x^2} \) lies in \( \{1, -3\} \). Since \( |F(1)F(-1)| \) is a square of an integer, this implies that \((-1)^n F_1(1) F_1(-1)\) is a square, and hence condition (C 1) holds. The converse is an immediate consequence of Proposition 2.6.

The following results and notions will be useful in the next sections:

Corollary 2.7. If there exists an even unimodular lattice having an isometry with characteristic polynomial \( F \), then the \( F \) satisfies Condition (C 1).

Proof. This is an immediate consequence of 2.1.

Corollary 2.8. Condition (C 1) holds if and only if for all prime numbers \( p \), there exists an even \( \mathbb{Z}_p \)-lattice of determinant \((-1)^n\) having a semi-simple isometry with characteristic polynomial \( F \).

Proof. This follows from Theorem 2.1 noting that if \((L, q)\) is an even, unimodular \( \mathbb{Z}_p \)-lattice having a semi-simple isometry with characteristic polynomial \( F \), then there exists such a lattice of determinant \((-1)^n\). If \( F(1)F(-1) \neq 0 \), then \( |F(1)| |F(-1)| \) are squares. If \( F(1)F(-1) = 0 \), we are done. If not, we apply condition (C 1) we have \( \det(q) = (-1)^n \) in \( \mathbb{Q}_p^2 / \mathbb{Q}_p^{x^2} \). Since \( \det(q) \) is a unit of \( \mathbb{Z}_p \) for all \( p \), this implies that \( \det(q) = (-1)^n \). If \( F(1)F(-1) = 0 \), we apply Claim 1 or Claim 2 of the proof of Theorem 2.1 to modify the determinant of \( q^+ \) or \( q^- \), if necessary.

Proposition 2.9. Let \( f \in \mathbb{Z}[X] \) be an irreducible, monic, symmetric polynomial of even degree, and let \( p \) be an odd prime number. If \( v_p(f(1)) \equiv 1 \pmod{2} \), then \( p \in \Pi_p^+ \); if \( v_p(f(-1)) \equiv 1 \pmod{2} \), then \( p \in \Pi_p^- \).

Proof. With the notation of the proof of Proposition 2.6, Claim 1, we have

\[
\sum_{v \in S_+} [\kappa_v : \mathbb{F}_p] \equiv v_p(f(1)) \pmod{2}, \quad \text{and} \quad \sum_{v \in S_-} [\kappa_v : \mathbb{F}_p] \equiv v_p(f(-1)) \pmod{2},
\]

see [BT 20], Lemma 6.8. This implies the proposition.

Proposition 2.10. Let \( f \in \mathbb{Z}[X] \) be an irreducible, monic, symmetric polynomial of even degree, and suppose that \( f \) is unramified at the prime number 2. Then \( v_2(f(1)) \) and \( v_2(f(-1)) \) are even, and the class of \((-1)^n f(1)f(-1)\) in \( \mathbb{Q}_2^2 / \mathbb{Q}_2^{x^2} \) lies in \( \{1, -3\} \).

Proof. Set \( E = E_f \otimes \mathbb{Q} \mathbb{Q}_2 \) and \( E_0 = (E_f)_0 \otimes \mathbb{Q} \mathbb{Q}_2 \); let \( \alpha \) be the image of \( X \) in \( E_f \). \( O_E \) be the ring of integers of \( E \), and \( O_{E_0} \) the ring of integers of \( E_0 \). By hypothesis, the polynomial \( f \) is unramified at 2, therefore all the places of \( E_0 \) are unramified in \( E \). The algebra \( E_0 \) decomposes as a product of fields \( \prod_{v \in S} E_{0,v} \), and
Lemma 3.2. Let \( f : V \rightarrow V \) be a non-degenerate quadratic form over \( \mathbb{R} \); let \( t_1 \) be an isometry of \( q_1 \), and \( t_2 \) an isometry of \( q_2 \). If \((V,q)\) is the orthogonal sum of \((V_1,q_1)\) and \((V_2,q_2)\) and \( t = t_1 \oplus t_2 \), then \( \text{sign}_t = \text{sign}_{t_1} + \text{sign}_{t_2} \).

Proof. If \( p \) is an odd prime number, then Proposition 2.9 implies that \( v_p(f(1)) \equiv 0 \) (mod 2) and \( v_p(f(-1)) \equiv 0 \) (mod 2). On the other hand, Proposition 2.10 implies that \( v_2(f(1)) \equiv 0 \) (mod 2) and \( v_2(f(-1)) \equiv 0 \) (mod 2), and that the class of \((-1)^n f(1)f(-1)\) in \( \mathbb{Q}_2^x/\mathbb{Q}_2^{x^2} \) lies in \( \{1,-3\} \). But \( f(1) \) and \( f(-1) \) are integers, hence this implies that \((-1)^n f(1)f(-1)\) is a square, and that \(|f(1)|\) and \(|f(-1)|\) are squares. Therefore Condition (C 1) holds for \( f \).

3. The signature map of an isometry

The aim of this section is to define a notion of signature for an isometry; we start with reminders concerning signatures of quadratic forms.

Let \( V \) be a finite dimensional vector space over \( \mathbb{R} \), and let \( q : V \times V \rightarrow \mathbb{R} \) be a quadratic form; it is the orthogonal sum of a non-degenerate quadratic form and the zero form; its signature is by definition the signature of the non-degenerate form. The signature of \( q \) is an element of \( \mathbb{N} \times \mathbb{N} \), denoted by \( \text{sign}(q) \). Let

\[
\text{sum} : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}
\]

be defined by \((n,m) \mapsto n+m\), and

\[
\text{diff} : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{Z}
\]

by \((n,m) \mapsto n-m\). Note that \( \text{sum} \circ \text{sign}(q) \) is the dimension of the non-degenerate part of \( q \), and that \( \text{diff} \circ \text{sign}(q) \) is the index of \( q \). In particular, the quadratic form \( q \) is non-degenerate if and only if \( \text{sum} \circ \text{sign}(q) = \dim(V) \).

Definition 3.1. Let \( t : V \rightarrow V \) be an isometry of \( q \). If \( f \in \mathbb{R}[X] \), set \( V_f = \text{Ker}(f(t)) \) and let \( q_f \) be the restriction of \( q \) to \( V_f \). Let

\[
\text{sign}_t : \mathbb{R}[X] \rightarrow \mathbb{N} \times \mathbb{N}
\]

be the map sending \( f \in \mathbb{R}[X] \) to the signature of \((V_f,q_f)\); it is called the signature map of the isometry \( t \).

Lemma 3.2. Let \((V_1,q_1)\) and \((V_2,q_2)\) be two non-degenerate quadratic forms over \( \mathbb{R} \); let \( t_1 \) be an isometry of \( q_1 \), and \( t_2 \) an isometry of \( q_2 \). If \((V,q)\) is the orthogonal sum of \((V_1,q_1)\) and \((V_2,q_2)\) and \( t = t_1 \oplus t_2 \), then \( \text{sign}_t = \text{sign}_{t_1} + \text{sign}_{t_2} \).
Proof. For all \( f \in \mathbb{R}[X] \), we have \( V_f = (V_1)_f \oplus (V_2)_f \) and \( q_f = (q_1)_f \oplus (q_2)_f \), hence \( \text{sign}_t(f) = \text{sign}_{t_1}(f) + \text{sign}_{t_2}(f) \).

The aim of this section is to characterize signature maps of semi-simple isometries; we start with some notation and definitions:

**Definition 3.3.** Let \( \text{Sym}(\mathbb{R}[X]) \) be the set of symmetric polynomials in \( \mathbb{R}[X] \). The symmetric radical of \( f \in \mathbb{R}[X] \) is by definition the monic, symmetric divisor of highest degree of \( f \); we denote it by \( \text{rad}(f) \in \mathbb{R}[X] \).

**Notation 3.4.** If \( F \in \text{Sym}(\mathbb{R}[X]) \) and if \( f \in \text{Sym}(\mathbb{R}[X]) \) is irreducible, we denote by \( n_f(F) \) the largest integer \( \geq 0 \) such that \( f^{n_f(F)} \) divides \( F \).

**Notation 3.5.** For \( i = 1, 2 \) let \( \text{proj}_i : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, (n_1, n_2) \mapsto n_i \) be the projection on the \( i \)-th component.

Let \( \text{proj}_1 : \mathbb{N} \times \mathbb{N} \) be the projection of the first factor, \( \text{proj}_1(n, m) = n \), and let \( \text{proj}_2 : \mathbb{N} \times \mathbb{N} \) be the projection of the second factor, \( \text{proj}_1(n, m) = m \).

**Definition 3.6.** We say that a map \( \tau : \mathbb{R}[X] \to \mathbb{N} \times \mathbb{N} \) is bounded if there exists \( (r, s) \in \text{Im}(\tau) \) such that \( \text{proj}_1(\tau(f)) \leq r \) and \( \text{proj}_2(\tau(f)) \leq s \) for all \( f \in \mathbb{R}[X] \), and \( (r, s) \) is called the maximum of the bounded map \( \tau \).

**Proposition 3.7.** Let \( \tau : \mathbb{R}[X] \to \mathbb{N} \times \mathbb{N} \) be a bounded map with maximum \( (r, s) \) with the following properties:

(a) There exists a unique monic polynomial \( F \in \text{Sym}(\mathbb{R}[X]) \) of degree \( r + s \) such that \( \tau(F) = (r, s) \), and such that if \( f \in \mathbb{R}[X] \) is irreducible with \( \tau(f) \neq (0, 0) \), then \( f \) divides \( F \).

(b) If \( f, g \in \text{Sym}(\mathbb{R}[X]) \) are relatively prime polynomials, then
\[
\tau(fg) = \tau(f) + \tau(g).
\]

(c) \( \tau(f) = \tau(\text{rad}(f)) \) for all \( f \in \mathbb{R}[X] \) and \( \tau(f) = 0 \) if \( f \) is constant.

(d) If \( f \in \text{Sym}(\mathbb{R}[X]) \) is irreducible, then \( \tau(f) \in \text{deg}(f)\mathbb{N} \times \text{deg}(f)\mathbb{N} \),
\[
\text{sum} \circ \tau(f) = \text{deg}(f)n_f(F)
\]
and
\[
\tau(f^k) = \tau(f)
\]
for all integers \( k \geq 1 \).

(e) If \( f(X) = (X - a)(X - a^{-1}) \) with \( a \neq \pm 1 \), then
\[
\tau(f) = (n_{X-a}(F), n_{X-a}(F)),
\]
and \( \tau(f^k) = \tau(f) \) for all integers \( k \geq 1 \).

Then there exists a non-degenerate quadratic form \( q \) over \( \mathbb{R} \) of signature \( (r, s) \) and a semi-simple isometry \( t \) of \( q \) such that \( \text{sign}_t = \tau \).

Conversely, if \( t \) is a semi-simple isometry of a non-degenerate quadratic form \( q \) of signature \( (r, s) \), then \( \tau = \text{sign}_t \) is bounded of maximum \( (r, s) \) and satisfies conditions (a)-(e).
Lemma 3.1, we have an orthogonal decomposition of the signature of this quadratic form is (\(2r, 2s\)). Set \(W_f = E^*\), let \(h_f : W_f \times W_f \to E_f\) be the hermitian form of signature \((r, s)\), and let \(q_f : W_f \times W_f \to \mathbb{R}\) be defined by \(q_f(x, y) = \text{Tr}_{E_f/\mathbb{R}}(h_f(x, y))\); the signature of this quadratic form is \((2r, 2s)\), and it has a semi-simple isometry with characteristic polynomial \(f\). Let \(q_1 = \bigoplus q_f\). Let \(r_0, s_0 \geq 0\) be integers such that \(\tau(X - 1) + \tau(X + 1) = (r_0, s_0)\), and let \(q_0\) be the non-degenerate quadratic form over \(\mathbb{R}\) with signature \((r_0, s_0)\). Finally, let \(q_2\) be the hyperbolic quadratic form over \(\mathbb{R}\) of dimension \(\text{deg}(F) - \dim(q_1) - \dim(q_0)\), and set \(q = q_0 \oplus q_1 \oplus q_2\); the quadratic form has a semi-simple isometry with signature map \(\tau\).

The converse follows from Milnor’s classification of isometries in [M 69], §3. Indeed, let \((V, q)\) be a non-degenerate quadratic form over \(\mathbb{R}\) of signature \((r, s)\), let \(t : V \to V\) be a semi-simple isometry of \(q\), and let \(F \in \mathbb{R}[X]\) be the characteristic polynomial of \(t\). Set \(\tau = \text{sign}_q\).

Let \(I_1\) be the set of irreducible factors of degree 2 of \(F\), and let us write \(F = F_0F_1F_2\), where \(F_0(X) = (X - 1)^{n_0}(X + 1)^{n_0}\), \(F_1 = \prod_{f \in I_1} f^{n_f(X)}\), and \(F_2\) is a product of polynomials of the form \((X - a)(X - a^{-1})\) with \(a \in \mathbb{R}\) and \(a \neq \pm 1\). By [M 69], Lemma 3.1, we have an orthogonal decomposition \(V = V_0 \oplus V_1 \oplus V_2\); moreover, the restriction of \(q\) to \(V_2\) is hyperbolic (see [M 69], page 94, Case 3). This implies that \(\tau\) has properties (a), (b), (c) and (e). Property (d) follows from [M 69], Theorems 3.3 and 3.4, combined with [M 69], §1.

Definition 3.8. A bounded map \(\tau : \mathbb{R}[X] \to \mathbb{N} \times \mathbb{N}\) satisfying conditions (a)-(e) is called a (semi-simple) signature map. The polynomial of condition (a) is by definition the polynomial associated to \(\tau\).

Since we only consider semi-simple isometries in this paper, we use the terminology of signature map without the adjective “semi-simple”.

Finally, we note that if \(\tau\) is a signature map with polynomial \(F\) and maximum \((r, s)\), then condition (C 2) of Gross and McMullen (see the introduction to Part I) holds for \(F\) and \((r, s)\).

4. Signature maps of automorphisms of K3 surfaces

Let \(T : \mathcal{X} \to \mathcal{X}\) be an automorphism of a K3 surface. The intersection form of \(\mathcal{X}\) induces a non-degenerate quadratic form

\[H^2(\mathcal{X}, \mathbb{R}) \times H^2(\mathcal{X}, \mathbb{R}) \to \mathbb{R}\]

of signature (3,19). The isomorphism \(T^*\) is an isometry of the intersection form, and the signature map of the automorphism \(T : \mathcal{X} \to \mathcal{X}\) is by definition the signature map of the isometry \(T^*\).

The signature map of \(T\) determines the characteristic polynomial of \(T^*\), hence also the dynamical degree \(\lambda(T)\). Note that either \(\lambda(T) = 1\), or \(\lambda(T)\) is the unique real number \(\alpha > 1\) such that

\[\text{sign}_{T^*}((X - \alpha)(X - \alpha^{-1})) = (1,1).\]
In addition, the signature map of $T$ also determines $\omega(T)$, up to complex conjugation. Indeed, let $\delta$ be a complex number, let $\overline{\delta}$ be its complex conjugate, and let $f_\delta \in \mathbb{R}[X]$ be the minimal polynomial of $\delta$ over $\mathbb{R}$.

Recall that $\text{proj}_1 : \mathbb{N} \times \mathbb{N}$ denotes the projection of the first factor, $\text{proj}_1(n, m) = n$. We have

$$\omega(T) = \delta \text{ or } \overline{\delta} \iff \text{proj}_1 \circ \text{sign}_{T^*}(f_\delta) = 2.$$  

Let $F$ be the characteristic polynomial of $T^*$; we have $F = SC$, where $S$ is a Salem polynomial and $C$ is a product of cyclotomic polynomials; set $d = \deg(S)$. We have either $\text{sign}_{T^*}(S) = (3, d-3)$ and, or, $\text{sign}_{T^*}(S) = (1, d-1)$ and $\text{sign}_{T^*}(C) = (2, 20-d)$.

Let us assume that $\lambda(T) > 1$ and that $\omega(T)$ is not a root of unity; this implies that $\lambda(T)$ and $\omega(T)$ are roots of the Salem polynomial $S$. Then we are in the first case, $\text{sign}_{T^*}(S) = (3, d-3)$, and the characterization of $\lambda(T)$ and $\omega(T)$ in terms of the signature map takes a very simple form. Let $\alpha \in \mathbb{R}$, $\delta \in \mathbb{C}^*$ be such that $\alpha > 1$ and $|\delta| = 1$. We have

$$\lambda(T) = \alpha \iff \text{sign}_{T^*}((X - \alpha)(X - \alpha^{-1})) = (1, 1).$$

and

$$\omega(T) = \delta \text{ or } \overline{\delta} \iff \text{sign}_{T^*}((X - \delta)(X - \delta^{-1})) = (2, 0).$$

Conversely, note that under this hypothesis, the signature map $\text{sign}_{T^*}$ is determined by the characteristic polynomial $F = SC$, and by the choice of $\delta \in \mathbb{C}^*$ such that $\text{sign}_{T^*}((X - \delta)(X - \delta^{-1})) = (2, 0)$.

5. Local data

To explain the results of this section, we start with a special case. As in the introduction, let $S$ be a Salem polynomial and let $C$ be a product of cyclotomic polynomials. Let $\tau$ be a signature map with associated polynomial $SC$.

We consider the following questions - the first one being the central topic of Part I, the second of Part II:

(a) Does there exist an even unimodular lattice having an isometry with signature map $\tau$?

Let $t : L \to L$ be an isometry of an even unimodular lattice $L$ with signature map $\tau$; set $L_S = \text{Ker}(S(t))$ and $L_C = \text{Ker}(C(t))$. The properties of these lattices determine whether or not $t : L \to L$ is induced by an automorphism of a $K3$ surface; this follows from results of McMullen (see [McM 11], Theorem 6.2). This shows that it is not enough to answer question (a); we also need to understand the flexibility we have in the choice of the isometry in case of a positive answer.

More precisely, we ask:

(b) Does there exist an even unimodular lattice $L$ having an isometry $t$ with signature map $\tau$, such that the restriction of $t$ to $L_C$ satisfies McMullen’s criterion?

Let $V = L \otimes \mathbb{Z} Q$, $V_S = L_S \otimes \mathbb{Z} Q$ and $V_C = L_C \otimes \mathbb{Z} Q$; we have $V = V_S \oplus V_C$. As we will see, the “building blocks” $V_S$ and $V_C$ play an important role in both questions (a) and (b). The first step is to describe, for all prime numbers $p$, the possibilities for the quadratic spaces with isometry $V_S \otimes Q_p$ and $V_C \otimes Q_p$; this will give rise to combinatorial data, as indicated in the introduction to Part I.
We now return to the general case. Let $\tau$ be a signature map with associated polynomial $F$, and assume that $F \in \mathbb{Z}[X]$. Let $p$ be a prime number, let $V$ be a finite dimensional $\mathbb{Q}_p$-vector space, let $q : V \times V \to \mathbb{Q}_p$ a non-degenerate quadratic form, and let $t : V \to V$ a semi-simple isometry of the quadratic form $q$ with characteristic polynomial $F$. The aim of this section is to associate combinatorial data to this isometry.

This turns out to be especially simple in an important special case: when at least one of $F(1)$ and $F(-1)$ is non-zero. Indeed, let $F = F_0F_1$, where $F_i$ is the product of the irreducible factors $f \in \mathbb{Z}[X]$ of type $i$ of $F$, with $F_1 = \prod_{f \in I_1} f^{n_f}$ and $F_0(X) = (X - 1)^{n^+}(X + 1)^{n^-}$ for some integers $n^+, n^- \geq 0$.

Set $V_{F_i} = \text{Ker}(F_i(t))$, we have an orthogonal decomposition of $\mathbb{Q}_p[\Gamma]$-quadratic forms
\[(V, q) = (V_{F_0}, q_0) \oplus (V_{F_1}, q_1).\]

Note that $\det(q) = \det(q_0)\det(q_1)$ in $\mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2}$ and that $\det(q_1) = F_1(1)F_1(-1)$. Set $V_f = \text{Ker}(f(t))$; we have the further orthogonal decomposition of $\mathbb{Q}_p[\Gamma]$-quadratic forms $(V_{F_1}, q_1) = \bigoplus_{f \in I_1} (V_f, q_f)$, and $\det(q_f) = [f(1)f(-1)]^{n_f}$. The dimensions and determinants of the quadratic forms $(V_f, q_f)$ are determined by the polynomial $f$; these forms can however have different Hasse-Witt invariants, and this is what will determine the combinatorial data.

The dimension of $q_0$ is $n_0 = n^+ + n^-$, and its determinant is
\[\det(q_0) = \det(q)\det(q_1) = \det(q)F_1(1)F_1(-1) \text{ in } \mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2},\]

hence $\det(q_0)$ only depends on the polynomial $F$ and the determinant of the quadratic form $q$.

In the special case where at least one of $F(1)$ or $F(-1)$ is non-zero, then we have $n_0 = n^+$ or $n_0 = n^-$, and we can define the combinatorial data as a function of $F$ and $\det(q)$; however, if $F(1) = F(-1) = 0$, then we have a non-trivial orthogonal decomposition of $\mathbb{Q}_p[\Gamma]$-quadratic forms $(V_0, q_0) = (V_+, q_+) \oplus (V_-, q_-)$, and we only know $\det(q_0) = \det(q_+)\det(q_-)$ as a function of $q$ and $F$, but not the factors $\det(q_+)$ and $\det(q_-)$.

We deal with this question in \[\text{[3]}\]. For now, we assume that at least one of $F(1)$ or $F(-1)$ is non-zero, and hence $n_0 = n^+$ or $n_0 = n^-$. In the remainder of this section, we define the local data in this case: first, we deal with the polynomials $f \in I_1$, the hypothesis is not needed for these, and then we pass to $f \in I_0$, under the above assumption.

We start by introducing some notation.

**Notation 5.1.** For all $f \in I_1$, let $Q_f : (E_f)^{n_f} \times (E_f)^{n_f} \to \mathbb{Q}$ be the orthogonal sum of $n_f$ copies of the quadratic form $E_f \times E_f \to \mathbb{Q}$ defined by $(x, y) \mapsto \text{Tr}_{E_f/\mathbb{Q}}(x\sigma_f(y))$.

Set $E_f^0 = E_f \otimes \mathbb{Q}$ and $(E_f)^0_f = (E_f)^0_f \otimes \mathbb{Q}$; there exists a unique hermitian form $(V_f, h_f^0)$ over $(E_f^0_f, \sigma_f)$ such that $q_f(x, y) = \text{Tr}_{E_f^0_f/\mathbb{Q}_p}(h_f^0(x, y))$ for all $x, y \in V_f$; see for instance \[\text{[M 69]}, \text{Lemma 1.1 or [B 15]}, \text{Proposition 3.6}.\]

Set $\lambda_f^0 = \det(h_f^0) \in (E_f^0_f)^{\times}/N_{E_f^0_f/(E_f^0_f)}(E_f^0_f)^{\times}$. Note that the hermitian form $h_f^0$ is isomorphic to the $n_f$-dimensional diagonal hermitian form $(\lambda_f^0, 1, \ldots, 1)$ over $E_f^0_f$; hence $q_f$ is determined by $\lambda_f^0$. 
Notation 5.2. With the notation above, set
\[ \partial_p(\lambda_f^p) = \partial_p[q_f^p] \in W_\Gamma(F_p). \]
If \( \lambda = (\lambda_f^p)_{f \in I_1} \), set \( \partial_p(\lambda) = \oplus_{f \in I_1} \partial_p(\lambda_f^p) = \partial_p[\oplus_{f \in I_1} q_f^p] \in W_\Gamma(F_p). \)

Notation 5.3. If \( q \) is a non-degenerate quadratic form over \( \mathbb{Q}_p \), we denote by \( w_2(q) \in \text{Br}(\mathbb{Q}_p) \) its Hasse-Witt invariant; we identify \( \text{Br}(\mathbb{Q}_p) \) with \( \{0,1\} \).

For all \( f \in I_1 \), let \( d_f \in (E_f)_0^\times \) be such that \( E_f = (E_f)_0(\sqrt{d_f}) \).

Proposition 5.4. We have \( \dim(q_f^p) = n_f \deg(f) = \dim(Q_f) \), \( \det(q_f^p) = [f(1)f(-1)]^{n_f} = \det(Q_f) \) in \( \mathbb{Q}_p^\times /\mathbb{Q}_p^{\times 2} \), and
\[ w_2(q_f^p) + w_2(Q_f) = \text{cor}_{(E_f)_0^p/\mathbb{Q}_p}(\lambda_f^p, d_f) \]
in \( \text{Br}(\mathbb{Q}_p) \), where \( \text{cor} \) denotes the corestriction map.

Proof. The assertion concerning the dimension is clear, the one on the determinant follows from [B 15], Corollary 5.2, and the property of the Hasse-Witt invariants from [B 21], Proposition 12.8.

Notation 5.5. For \( f \in I_1 \), set \( a^p(f) = \text{cor}_{(E_f)_0^p/\mathbb{Q}_p}(\lambda_f^p, d_f) \) in \( \text{Br}(\mathbb{Q}_p) \).

By Proposition 5.4 we have \( a^p(f) = w_2(q_f^p) + w_2(Q_f) \).

We now assume that \( F(1) \neq 0 \) or \( F(-1) \neq 0 \).

If \( f \in I_0 \), then either \( f(X) = X - 1 \) or \( f(X) = X + 1 \). Set \( n_0 = n_f \) and note that \( F_0(X) = (X - 1)^{n_0} \) in the first case, and \( F_0(X) = (X + 1)^{n_0} \) in the second one.

We have \( \dim(V_{F_0}^p) = n_0 \) and \( \det(q_0^p) = \det(q)F_1(1)F_1(-1) \) in \( \mathbb{Q}_p^\times /\mathbb{Q}_p^{\times 2} \).

Set \( D_0 = \det(q)F_1(1)F_1(-1) \), and let \( Q_0 \) be the diagonal quadratic form of dimension \( n_0 \) over \( \mathbb{Q} \) defined by \( Q_0 = \langle D_0, 1, \ldots, 1 \rangle \).

Notation 5.6. With \( f \in I_0 \) as above, set \( a^p(f) = w_2(q_0^p) + w_2(Q_0) \) in \( \text{Br}(\mathbb{Q}_p) \).

Set \( I = I_0 \cup I_1 \). If \( f \in I_0 \), \( a^p(f) \) is defined as in Notation 5.6 and if \( f \in I_1 \), then \( a^p(f) \) is as in Notation 5.5. Let us denote by \( C(I) \) the set of maps \( I \to \mathbb{Z}/2\mathbb{Z} \); we obtain \( a^p \in C(I) \). In summary, we have

Definition 5.7. Assume that \( F(1) \neq 0 \) or \( F(-1) \neq 0 \), and let \( \lambda = (\lambda_f^p)_{f \in I_1} \) and \( q_0^p \) be as above. We define
\[ a^p = a^p[\lambda^p, q_0] \in C(I) \]
by setting \( a^p(f) = w_2(q_0^p) + w_2(Q_0) \) if \( f \in I_0 \), and \( a^p(f) = w_2(q_f^p) + w_2(Q_f) = \text{cor}_{(E_f)_0^p/\mathbb{Q}_p}(\lambda_f^p, d_f) \) if \( f \in I_1 \).

Notation 5.8. Let \( \delta \in W_\Gamma(F_p) \), and let \( C^p \) be the set of \( a^p = a^p[\lambda^p, q_0] \in C(I) \) as in definition 5.7 such that \( \partial_p(\lambda^p) \oplus \partial_p[q_0] = \delta \).
6. Local data - continued

We keep the notation of §5 in particular, \( F \in \mathbb{Z}[X] \) is a monic, symmetric polynomial, \( p \) is a prime number, \((V, q)\) is a non-degenerate quadratic form over \( \mathbb{Q}_p \), and \( t : V \to V \) is a semi-simple isometry of \( q \) with characteristic polynomial \( F \). Recall that \( F = F_0 F_1 \), with

\[
F_1 = \prod_{f \in I_1} f^{|f|} \quad \text{and} \quad F_0(X) = (X - 1)^{n^+} (X + 1)^{n^-}
\]

for some integers \( n^+, n^- \geq 0 \).

In §5 we defined a set \( \mathcal{C}_p \) under the hypothesis that \( n^+ = 0 \) or \( n^- = 0 \); we now do this in general. Recall that \( D_0 = \det(q) F_1(1) F_1(-1) \), and that we have the orthogonal decompositions of \( \mathbb{Q}_p[\Gamma] \)-quadratic forms

\[
(V, q) = (V F_0, q_0) \oplus (V F_1, q_1)
\]

and \((V_0, q_0) = (V_+, q_+) \oplus (V_-, q_-)\). We have \( \det(q_0) = D_0 \), but \( \det(q_+) \) and \( \det(q_-) \) are not determined by \( F \) and \( q \).

Let \( D_+, D_- \in \mathbb{Q}_p^2/\mathbb{Q}_p \) be such that \( D_0 = D_+ D_- \); the set \( \mathcal{C}_p \) we now define also depends on the choice of \( D_+ \) and \( D_- \).

Let \( Q_+ \) be the diagonal quadratic form of dimension \( n_+ \) over \( \mathbb{Q} \) defined by

\[
Q_+ = (D_+, 1, \ldots, 1)
\]

and similarly, let \( Q_- \) be the diagonal quadratic form of dimension \( n_- \) over \( \mathbb{Q} \) defined by

\[
Q_- = (D_-, 1, \ldots, 1)
\]

Notation 6.1. Set \( a^p(X - 1) = w_2(q_+) + w_2(Q_+) \) in \( \text{Br}_2(\mathbb{Q}_p) \), and \( a^p(X + 1) = w_2(q_-) + w_2(Q_-) \) in \( \text{Br}_2(\mathbb{Q}_p) \).

This defines \( a^p(f) \) for \( f \in I_0 \), and if \( f \in I_1 \), then \( a^p(f) \) is as in Notation 5.5. We identify \( \text{Br}_2(\mathbb{Q}_p) \) with \{0, 1\}, and obtain \( a^p \in C(I) \). In summary, we have

Definition 6.2. Let \( \lambda^p = (\lambda_f^p)_{f \in I_1} \) be as in §5. We define

\[
a^p = a^p[\lambda^p, q_\pm] \in C(I)
\]

by setting

\[
a^p(X - 1) = w_2(q_+) + w_2(Q_+), \quad a^p(X + 1) = w_2(q_-) + w_2(Q_-), \quad \text{and}
\]

\[
a^p(f) = w_2(q_f^p) + w_2(Q_f) = \text{cor}_{(E_f)_{/p}_0}^{(\mathbb{Q}_p)}(\lambda_f^p, d_f) \quad \text{if} \ f \in I_1.
\]

Notation 6.3. Let \( \delta \in W_I(\mathbb{F}_p) \), and let \( \mathcal{C}_p^\delta \) be the set of \( a^p = a^p[\lambda^p, q_\pm] \in C(I) \) as in definition 6.2 such that \( \partial_\delta(\lambda^p) \oplus \partial_\delta[q_+] \oplus \partial_\delta[q_-] = \delta \).

7. Combinatorial data - obstruction group and equivalence relations

The aim of this section is to define some combinatorial objects that we need in the next sections. We keep the notation of §5: we fix monic, symmetric polynomial \( F \in \mathbb{Z}[X] \) of even degree. Let \( I \) be the set of irreducible, symmetric factors of \( F \), and we write \( F \) as \( F = F_0 F_1 \), with

\[
F_1 = \prod_{f \in I_1} f^{|f|} \quad \text{and} \quad F_0(X) = (X - 1)^{n^+} (X + 1)^{n^-}
\]

for some integers \( n^+, n^- \geq 0 \).

If \( f \in \mathbb{Z}[X] \) is a symmetric, irreducible polynomial of even degree, recall that we denote by \( \Pi_f \) the set of prime numbers \( p \) such that \( f \) is ramified at \( p \) (see Definition 2.4).
**Notation 7.1.** If \( f, g \in \mathbb{Z}[X] \) are monic, irreducible, symmetric polynomials of even degree, we denote by \( \Pi_{f,g} \) the set of prime numbers \( p \) such that the following conditions holds:

The polynomial \( f \) has a symmetric, irreducible factor \( f' \in \mathbb{Z}_p[X] \), the polynomial \( g \) has a symmetric, irreducible factor \( g' \in \mathbb{Z}_p[X] \), such that \( f' \pmod{p} \) and \( g' \pmod{p} \) have a common irreducible, symmetric factor in \( \mathbb{F}_p[X] \).

For all prime numbers \( p \), let \( D^p_+ = \mathbb{Q}_p^\times / \mathbb{Q}_p^2 \).

**Notation 7.2.** If \( f \in I_1 \), let \( \Pi_{f,X-1} \) be the set of prime numbers \( p \) such that \( p \in \Pi_f \), that \( f \pmod{p} \) is divisible by \( X - 1 \) in \( \mathbb{F}_p[X] \), and that if \( n^+ = 2 \), then \( D^p_+ \neq -1 \).

Let \( \Pi_{f,X+1} \) be the set of prime numbers \( p \) such that \( p \in \Pi_f \), and that \( f \pmod{p} \) is divisible by \( X + 1 \) in \( \mathbb{F}_p[X] \), and that if \( n^- = 2 \), then \( D^p_- \neq -1 \).

Let \( \Pi_{X-1,X+1} = \{2\} \) if the following conditions hold: \( n^+ \neq 0 \), \( n^- \neq 0 \), and if \( n^+ = 2 \), then \( D^2_+ \neq -1 \); if \( n^- = 2 \), then \( D^2_- \neq -1 \). Otherwise, set \( \Pi_{X-1,X+1} = \emptyset \).

We denote by \( C(I) \) the set of maps \( I \to \mathbb{Z}/2\mathbb{Z} \).

**Notation 7.3.** If \( f, g \in I \), let \( c_{f,g} \in C(I) \) be such that

\[
 c_{f,g}(f) = c_{f,g}(g) = 1, \quad c_{f,g}(h) = 0 \quad \text{if} \quad h \neq f, g.
\]

Let \((f, g) : C(I) \to C(I) \) be the map sending \( c \) to \( c + c_{f,g} \).

**Notation 7.4.** Let \( C_0(I) \) be the set of \( c \in C(I) \) such that

\[
 c(f) = c(g) \quad \text{if} \quad \Pi_{f,g} \neq \emptyset,
\]

and we denote by \( G_F(D_+, D_-) \) the quotient of the group \( C_0(I) \) by the subgroup of constant maps.

In general, the group depends on \( D_+_p = (D^p_+)^\times \) and \( D_- = (D^p_-)^\times \). If \( n^+ \neq 2 \) and \( n^- \neq 2 \), then \( G_F(D_+, D_-) \) only depends on \( F \), and we denote it by \( G_F \); similarly, we use the notation \( G_F(D_-) \) if \( n^+ \neq 2 \), and \( G_F(D_+) \) if \( n^- \neq 2 \).

The following proposition is useful in some of the examples.

**Proposition 7.5.** Let \( f, g \in \mathbb{Z}[X] \) be monic, irreducible, symmetric polynomials of even degree, and \( p \) be a prime number not dividing \((fg)(1)(fg)(-1)\). If \( f \pmod{p} \) and \( g \pmod{p} \) have a common irreducible, symmetric factor of even degree in \( \mathbb{F}_p[X] \), then \( p \in \Pi_{f,g} \).

The proposition is a consequence of the following lemma:

**Lemma 7.6.** Let \( p \) be a prime number, and let \( f_0 \in \mathbb{Z}_p[X] \) be a monic, irreducible polynomial of degree \( n \). Set \( f(X) = X^n f_0(X + X^{-1}) \), and assume that \( p \) does not divide \( f(1)f(-1) \). Let \( \overline{f} \) be the image of \( f \) in \( \mathbb{F}_p[X] \). If \( \overline{f} \) has an irreducible, symmetric factor in \( \mathbb{F}_p[X] \), then \( f \in \mathbb{Z}_p[X] \) is irreducible.

**Proof.** Set \( E = \mathbb{Q}_p[X]/(f) \), and let \( \alpha \) be the image of \( X \) in \( E \); set \( E_0 = \mathbb{Q}_p[X]/(f_0) \). If \( f \) is irreducible, then \( E/E_0 \) is a quadratic field extension, otherwise \( E = \mathbb{Q}_p[X]/(h) \times \mathbb{Q}_p[X]/(h^*) \) for some irreducible polynomial \( h \in \mathbb{Q}_p[X] \) such that \( h \neq h^* \), and \( f = hh^* \). Set \( \delta = (\alpha - \alpha^{-1})^2 \). Since \( p \) does not divide \( f(1)f(-1) \) by hypothesis, the elements \( \alpha + 1 \) and \( \alpha - 1 \) are units, and hence \( \delta \) is a unit of \( E_0 \). Let \( O_0 \) be the ring of integers of \( E_0 \), let \( m_0 \) be its maximal ideal, and set \( \kappa_0 = O_0/m_0 \). Suppose that \( f = hh^* \) with \( h \in \mathbb{Q}_p[X] \) such that \( h \neq h^* \); then \( \delta \) is a square in \( O_0 \),
hence its image $\overline{\delta}$ is a square in $\kappa_0$. This implies that the irreducible factors of $\overline{f}$ in $\mathbb{F}_p[X]$ are not symmetric, and this concludes the proof of the lemma.

**Proof of Proposition 7.5.** Let $h \in \mathbb{F}_p[X]$ be a common irreducible, symmetric factor of $f \pmod{p}$ and $g \pmod{p}$. By Lemma 7.6 there exist symmetric, irreducible polynomials $f', g' \in \mathbb{Z}_p[X]$ such that $f'$ divides $f$, $g'$ divides $g$, and that $h$ is a common factor of $f' \pmod{p}$ and $g' \pmod{p}$. This implies that $p \in \Pi_{f,g}$.

In the case where $F$ has no linear factors, the group $G_F$ was defined in [21], and several examples are given in [21], §25 and §31. Here are some more examples:

**Example 7.7.** Let $a$ be an integer $\geq 0$, set

$$S_a(X) = X^6 - aX^5 - X^4 + (2a - 1)X^3 - X^2 - aX + 1,$$

and $R_a(X) = (X^2 - 4)(X - a) - 1$; we have $S_a(X) = X^3R_a(X + X^{-1})$.

If $m$ is an integer $\geq 3$, recall that $\Phi_m$ is the $m$-th cyclotomic polynomial, and that $\text{Res}(S_a, \Phi_m)$ is the resultant of $S_a$ and $\Phi_m$.

(i) We have $\Pi_{S_a, \Phi_3} \neq \emptyset$. Indeed,

$$\text{Res}(S_a, \Phi_3) = N_{\mathbb{Q}(\zeta_3)/\mathbb{Q}}(S(\zeta_3)) = N_{\mathbb{Q}(\zeta_3)/\mathbb{Q}}(R(-1)) = 3(a + 1) - 1.$$ 

Since $a \geq 0$, we have $3(a + 1) - 1 \equiv 2 \pmod{3}$, therefore $\text{Res}(S_a, \Phi_3)$ is divisible by a prime number $p$ such that $p \equiv 2 \pmod{3}$. The polynomial $\Phi_3$ is irreducible modulo $p$, hence $p \in \Pi_{S_a, \Phi_3}$.

Set $F_a(X) = S_a(X)\Phi_3^2(X)(X - 1)^{12}$; note that with the above notation we have $n^+ = 12$ and $n^- = 0$, hence the obstruction group only depends on $F_a$. We have $\Phi_3(1) = 3$ and $3 \in \Pi_{\Phi_3}$, hence $3 \in \Pi_{\Phi_3(X)X^{-1}}$, and therefore $G_{F_a} = 0$.

(ii) Set $P_a(X) = S_a(X)\Phi_4^2(X)(X - 1)^{12}$. We have $2 \in \Pi_{\Phi_4(X)X^{-1}}$, and

$$\text{Res}(S_a, \Phi_4) = N_{\mathbb{Q}(i)/\mathbb{Q}}(R(0)) = 4a - 1.$$ 

If $a \geq 1$, then $\Pi_{S_a, \Phi_4} \neq \emptyset$, and $G_{P_a} = 0$. On the other hand, $\Pi_{S_0, \Phi_4} = \emptyset$, and $G_{P_a}$ is of order 2.

**Example 7.8.** With the notation of Example 7.7 set $a = 3$,

$$S(X) = S_3(X) = X^6 - 3X^5 - X^4 + 5X^3 - X^2 - 3X + 1.$$ 

(i) Set $F = S\Phi_{10}^4$. Then $G_F$ is of order 2. Indeed, the resultant of $S$ and $\Phi_{10}$ is 121, and these two polynomials have the irreducible common factors $X + 3$ and $X + 4$ in $\mathbb{F}_{11}[X]$. These polynomials are not symmetric, therefore $\Pi_{S, \Phi_{10}} = \emptyset$; hence $G_F \simeq \mathbb{Z}/2\mathbb{Z}$.

(ii) Set $F(X) = S(X)\Phi_{10}(X)^2(X - 1)^8$; in this case, we have $G_F \simeq (\mathbb{Z}/2\mathbb{Z})^2$. Indeed, we have seen in (i) that $\Pi_{S, \Phi_{10}} = \emptyset$, and it is clear that $\Pi_{S(X),X^{-1}} = \emptyset$, $\Pi_{\Phi_{10}(X),X^{-1}} = \emptyset$.

**Definition 7.9.** If $p$ is a prime number, we consider the equivalence relation on $C(I)$ generated by the elementary equivalence

$$a \sim b \iff b = (f, g) \circ a \text{ with } p \in \Pi_{f,g},$$

and we denote by $\sim_p$ the equivalence relation on $C(I)$ generated by this elementary equivalence.
Lemma 7.10. Let \( c \in C_0(I) \), let \( p \) be prime number, and let \( a, b \in C(I) \) such that \( a \sim_p b \). Then
\[
\sum_{h \in I} c(h)a(h) = \sum_{h \in I} c(h)b(h).
\]

Proof. We can assume that \( b = (f, g) \circ a \) with \( p \in \Pi_{f,g} \). By definition, we have \( b(h) = a(h) \) if \( h \neq f, g \), \( b(f) = a(f) + 1 \) and \( b(g) = a(g) + 1 \), hence
\[
\sum_{h \in I} c(h)b(h) = \sum_{h \in I} c(h)a(h) + c(f) + c(g).
\]
Since \( c \in C_0(I) \) and \( \Pi_{f,g} \neq \emptyset \) we have \( c(f) = c(g) \), and this proves the claim.

8. Local data and equivalence relation

We now combine the results and notions of §5, §6 and §7. We keep the notation of these sections : \( F \in \mathbb{Z}[X] \) is a monic, symmetric polynomial, \( p \) is a prime number, \( (V, q) \) is a non-degenerate quadratic form over \( \mathbb{Q}_p \), and \( t : V \to V \) is a semi-simple isometry of \( q \) with characteristic polynomial \( F \). Recall that \( F = F_0F_1 \), with
\[
F_1 = \prod_{f \in I_1} f^{n_f} \quad \text{and} \quad F_0(X) = (X - 1)^{n_+}(X + 1)^{n_-}
\]
for some integers \( n_+, n_- \geq 0 \).

Recall that \( D_0 = \text{det}(q)F_1(1)F_1(-1) \), and let \( D_+, D_- \in \mathbb{Q}_p^X/\mathbb{Q}_p^{X^2} \) be such that \( D_0 = D_+D_- \).

We fix \( \delta \in W_1(\mathbb{F}_p) \), and set \( C_p = C^p_\delta \); see Notation 6.3 for the notation \( C_p \), and Definition 7.3 for the definition of the equivalence relation \( \sim_p \).

Proposition 8.1. The set \( C_p \) is a \( \sim_p \)-equivalence class of \( C(I) \).

Proof. Set \( A^p = w_2(q) + w_2(Q) \in \text{Br}_2(\mathbb{Q}_p) = \mathbb{Z}/2\mathbb{Z} \), and note that for all \( a^p \in C_p \), we have
\[
\sum_{f \in I} a^p(f) = A^p.
\]
We start by proving that the set \( C_p \) is stabilized by the maps \( (f, g) \) for \( p \in \Pi_{f,g} \). Let \( a^p[\lambda^p, q^p_\pm] \in C_p \), let \( f, g \in I \) be such that \( p \in \Pi_{f,g} \), and let us show that \( (f, g)(a^p[\lambda^p, q^p_\pm]) \in C_p \). Note that if \( f \in I_1 \), then \( p \in \Pi_{f,g} \) implies that \( (E^p_{f})_0^{\infty}/(E^p_{f^0})_0(E^p_{f}) \neq 0 \). Assume first that \( f, g \in I_1 \). There exist \( \mu_f, \mu_g \in (E^p_{f})_0^{\infty}/(E^p_{f^0})_0(E^p_{f}) \) such that \( \text{cor}_{(E^p_{f})_0^{\infty}/Q_p}(\mu_f, d_f) \neq \text{cor}_{(E^p_{f})_0^{\infty}/Q_p}(\lambda_f, d_f) \) and \( \text{cor}_{(E^p_{f})_0^{\infty}/Q_p}(\mu_g, d_g) \neq \text{cor}_{(E^p_{f})_0^{\infty}/Q_p}(\lambda_g, d_g) \). Let \( \mu^p \in E^p_0 \) be obtained by replacing \( \lambda^p_f \) by \( \mu^p_f \), \( \lambda^p_g \) by \( \mu^p_g \), and leaving the other components unchanged. We have
\[
a^p[\mu^p, q^p_\pm] = (f, g)(a^p[\lambda^p, q^p_\pm]).
\]
Using the arguments of §22, Propositions 16.5 and 22.1 we see that \( a^p[\mu^p, q^p_\pm] \in C_p \). Assume now that \( f \in I_1 \) and \( g(X) = X - 1 \). In this case, the hypothesis \( p \in \Pi_{f,g} \) implies that there exists a place \( w \) of \( (E^p)_{f^0} \) above \( p \) that ramifies in \( E_f \), and such that the \( w \)-component \( \lambda^w \) of \( \lambda^p \) is such that with the notation of [B 21], §22, \( \partial_\nu(\lambda^w) \) is in \( W_1(\mathbb{F}_p, N_+) \). We modify the \( w \)-component of \( \lambda^p \) to obtain \( \mu^p \in (E^p_{f})_0^{\infty}/(E^p_{f^0})_0(E^p_{f}) \) such that \( \text{cor}_{(E^p_{f})_0^{\infty}/Q_p}(\mu_f, d_f) \neq \text{cor}_{(E^p_{f})_0^{\infty}/Q_p}(\lambda_f, d_f) \), and let \( b^p \) be a quadratic form over \( \mathbb{Q}_p \) with \( \text{dim}(b^p) = \text{dim}(q^p_+) \), \( \text{det}(b^p) = \text{det}(q^p_+) \), and \( w_2(b^p) = w_2(q^p_+) + 1 \). We have \( a^p[\mu^p, b^p, q^p_\pm] = (f, g)(a^p[\lambda^p, q^p_\pm]) \). The arguments of [B 21], Propositions 16.5 and 22.1 show that \( a^p[\mu^p, b^p, q^p_\pm] \in C_p \). The proof is the same when \( f \in I_1 \) and \( g(X) = X + 1 \).
Conversely, let us show that if \( a^p[\lambda^p, q^p_{\pm}] \) and \( a^p[\mu^p, b^p_{\pm}] \) are in \( C^p \), then \( a^p[\lambda^p, q^p_{\pm}] \sim_p a^p[\mu^p, b^p_{\pm}] \). Let \( I' \) be the set of \( f \in I \) such that \( a^p[\lambda^p, q^p_{\pm}](f) \neq a^p[\mu^p, b^p_{\pm}](f) \). Since \( \sum_{h \in I} a^p(h) = A^p \) for all \( a^p \in C^p \), the set \( I' \) has an even number of elements.

Assume first that \( p \neq 2 \). This implies that for all \( f \in I' \), we have \( \partial_p(\lambda^p_f) \neq \partial_p(\mu^p_f) \) and that if \( f(X) = X \pm 1 \), then \( \partial_p(\lambda^p_f) \neq \partial_p(\mu^p_f) \). Hence there exist \( f, g \in I' \) with \( f \neq g \) such that \( \partial_p(W_1(Q_p, M^p_f)) \) and \( \partial_p(W_1(Q_p, M^p_g)) \) have a non-zero intersection. This implies that \( p \in \Pi_{f,g} \). The element \( (f,g)(a^p[\lambda^p, q^p_{\pm}]) \) differs from \( a^p[\mu^p, b^p_{\pm}] \) in less elements than \( a^p[\lambda^p, q^p_{\pm}] \). Since \( I' \) is a finite set, continuing this way we see that \( a^p[\lambda^p, q^p_{\pm}] \sim_p a^p[\mu^p, b^p_{\pm}] \).

Suppose now that \( p = 2 \). Let \( I'' \) be the set of \( f \in I' \) such that \( \partial_2(\lambda^p_f) \neq \partial_2(\mu^p_f) \), and note that \( I' \) has an even number of elements. The same argument as in the case \( p \neq 2 \) shows that applying maps \( (f, g) \), we can assume that \( I'' = \emptyset \). If \( f \in I' \) and \( f \notin I'' \), then \( \partial_2(\lambda^p_f) \) belongs to \( W_1(F_2, 1) \subset W_1(F_2) \). Therefore \( f, g \in I' \) and \( f, g \notin I'' \), then \( 2 \in \Pi_{f,g} \). The number of these elements is also even, hence after a finite number of elementary equivalences we see that \( a^p[\lambda^p, q^p_{\pm}] \sim_p a^p[\mu^p, b^p_{\pm}] \). This completes the proof of the proposition.

**Notation 8.2.** Let \( a^p \in C^p \), and let \( c \in C(I) \). Set

\[
\epsilon_{a^p}(c) = \sum_{f \in I} c(f) a^p(f).
\]

**Lemma 8.3.** Let \( a^p, b^p \) be two elements of \( C^p \), and let \( c \in C_0(I) \). Then

\[
\epsilon_{a^p}(c) = \epsilon_{b^p}(c).
\]

**Proof.** By Proposition 8.1, we have \( a^p \sim_p b^p \); therefore Lemma 7.10 implies the desired result.

Since \( \epsilon_{a^p}(c) \) does not depend on the choice of \( a^p \in C^p \), we set \( e^p(c) = \epsilon_{a^p}(c) \) for some \( a^p \in C^p \), and obtain a map

\[
e^p : C_0(I) \rightarrow \mathbb{Z}/2\mathbb{Z}.
\]

9. A Hasse principle

The aim of this section is to decide whether the “building blocks” of \( \mathfrak{S}_5 \) and \( \mathfrak{S}_6 \) give rise to a global isometry. We do this in a more general context than needed in Part I; this is done to prepare Part II, by including some results that will be useful there.

Let \( \tau \) be a signature map with associated polynomial \( F \), and assume that \( F \in \mathbb{Z}[X] \); we write \( F = F_0 F_1 \) as in the previous sections. Let \( V \) be a finite dimensional \( \mathbb{Q} \)-vector space, let \( q : V \times V \rightarrow \mathbb{Q} \) be a non-degenerate quadratic form, and let \( D_+, D_- \in \mathbb{Q}^\times/\mathbb{Q}^{\times 2} \) be such that \( D_+ D_- = \text{det}(q) F_1(1) F_1(-1) \). Let \( \delta = (\delta_p) \) with \( \delta_p \in W_1(F_p) \) be such that \( \delta_p = 0 \) for almost all prime numbers \( p \).

Suppose that for all prime numbers \( p \), there exists a semi-simple isometry \( t : V \otimes \mathbb{Q} Q_p \rightarrow V \otimes \mathbb{Q} Q_p \) of \( (V, q) \otimes \mathbb{Q} Q_p \) with characteristic polynomial \( F \) such that the associated \( Q_p[\Gamma] \)-bilinear form \( (V, q) \otimes \mathbb{Q} Q_p \) satisfies

\[
\partial_p((V, q) \otimes \mathbb{Q} Q_p) = \delta_p.
\]
Proposition 9.2. \( \tau \partial W f \)

Let \( C \) (the Hasse-Witt invariant of \( q \) from \([B 21]\), Proposition 12.8. follows from \([B 15]\), Corollary 5.2, and the property of the Hasse-Witt invariants the unique infinite place. We have an orthogonal decomposition

\[ q \in \mathbb{Q} \rightarrow H \in \text{Br}(\mathbb{Q}) \]

Proposition 9.1. For almost all prime numbers \( p \), the zero map belongs to the set \( C_p \).

Proof. Let \( S \) be the set of prime numbers such that \( p \) is ramified in the extension \( E_f/\mathbb{Q} \) for some \( f \in I_1 \), or \( w_2(q) \neq w_2(Q) \) in \( \text{Br}_2(\mathbb{Q}) \); this is a finite set. We claim that if \( p \not\in S \), then the zero map belongs to \( C_p \). Indeed, set \( q_f^p = Q_f^p \) for all \( f \in I_1 \), and \( q_{f+}^p = Q_{f+}^p \). We have \( \det(q) = \det(Q) \) in \( \mathbb{Q}^\times/\mathbb{Q}^{\times 2} \), and if \( p \not\in S \) we have \( w_2(q) = w_2(Q) \) in \( \text{Br}_2(\mathbb{Q}) \), therefore, for \( p \not\in S \), we have

\[ \det(V, q) \otimes_R Q_p = \bigoplus_{f \in I_1} (V_f^p, q_f^p) \bigoplus (V_{f+}^p, q_{f+}^p) \bigoplus (V_f^p, q_f^p). \]

If \( p \) is unramified in all the extensions \( E_f/\mathbb{Q} \) for \( f \in I_1 \), by \([B 21]\), Lemma 11.2 we have \( \partial_p[V_f, q_f^p] = 0 \) in \( W_{\Gamma}(\mathbb{F}_p) \); moreover, \( v_p(D_{f+}) = 0 \), hence \( \partial[V_f^p, q_{f+}^p] = 0 \) in \( W_{\Gamma}(\mathbb{F}_p) \).

The above arguments show that if \( p \not\in S \), then the choice of \( q_f^p = Q_f^p \) for all \( f \in I_1 \) and \( q_{f+}^p = Q_{f+}^p \) gives rise to the element \( a^p = 0 \) of \( C \), as claimed. This completes the proof of the proposition.

Let us denote by \( \mathcal{V} \) the set of places of \( \mathbb{Q} \), by \( \mathcal{V}' \) the set of finite places, and by \( \infty \) the unique infinite place. We have an orthogonal decomposition

\[ (V, q) \otimes_R \mathbb{Q} = (V_{f_0}, q_{f_0}^\infty) \oplus (V_{f_1}, q_{f_1}^\infty) \]

of \( \mathbb{R}[\Gamma] \)-quadratic forms determined by the signature map \( \tau \), with \( (V_{f_1}, q_{f_1}^\infty) = \bigoplus_{f \in I_1} (V_f^\infty, q_f^\infty) \). If moreover \( n_+ > 0 \) and \( n_- > 0 \), we then have the further orthogonal decomposition of \( \mathbb{R}[\Gamma] \)-quadratic forms \( (V_{f_0}, q_{f_0}^\infty) \simeq (V_{f_+}^\infty, q_{f+}^\infty) \oplus (V_{f_0}^\infty, q_{f_0}^\infty) \). The \( \mathbb{R}[\Gamma] \)-quadratic forms \( (V_{f_0}^\infty, q_{f_0}^\infty) \) and \( (V_{f_+}^\infty, q_{f+}^\infty) \) are also determined by the signature map \( \tau \).

Proposition 9.2. We have \( \dim(q_f^\infty) = \deg(f) n_f \), \( \det(q_f^\infty) = [f(1)f(-1)]^{n_f} \), and the Hasse-Witt invariant of \( q_f^\infty \) satisfies

\[ w_2(q_f^\infty) + w_2(Q_f) = \text{cor}(E_f^\infty)/\mathbb{R} \det(h_f^\infty), d_f \]

in \( \text{Br}_2(\mathbb{R}) \).

Proof. The assertion concerning the dimension is clear, the one on the determinant follows from \([B 15]\), Corollary 5.2, and the property of the Hasse-Witt invariants from \([B 21]\), Proposition 12.8.

For \( f \in I_1 \), set \( E_f^\infty = E_f \otimes_R \mathbb{Q} \) and \( (E_f)_0^\infty = (E_f)_0 \otimes_R \mathbb{Q} \). There exists a unique non-degenerate hermitian form \( (M_f^\infty, h_f) \) over \( (E_f^\infty, \sigma_f) \) such that

\[ q_f^\infty(x, y) = \text{Tr}_{E_f^\infty/\mathbb{R}}(h_f^\infty(x, y)), \]

see for instance \([M 69]\), Lemma 1.1 or \([B 15]\), Proposition 3.6. Set

\[ \lambda_f^\infty = \det(h_f^\infty) \in (E_f^\infty)_0^\infty/N_{E_f^\infty/(E_f)_0^\infty}. \]
Set \( a^\infty(f) = w_2(q^\infty_f) + w_2(Q_f) \) if \( f \in I_1 \), and note that by Proposition 9.2 we have \( a^\infty(\tau) = \text{cor}(E_f)^\infty_{\mathbb{R}}(\lambda^\infty f, df) \).

Set \( a^\infty(\tau)(X \pm 1) = w_2(q^\infty_{\pm}) + w_2(Q_{\pm}). \)

We record the following observation for later use:

**Proposition 9.3.** If \( \tau \) and \( \tau' \) are two signature maps with the same maximum and associated polynomial, and if \( f \in I \) is such that \( \text{proj}_2(\tau(f)) \equiv \text{proj}_2(\tau'(f)) \) (mod 4), then \( a^\infty(\tau) = a^\infty(\tau'). \)

**Proof.** Let \( f \in I \), let \( \tau(f) = (r_f, s_f) \). Note that \( \text{proj}_2(\tau(f)) = s_f \), and that \( q^\infty_f \) is the diagonal quadratic form over \( \mathbb{R} \) with \( r_f \) entries 1 and \( s_f \) entries -1. The Hasse-Witt invariant \( w_2(q^\infty_f) \) is determined by the class of \( s_f \) (mod 4), and this implies the proposition.

We obtain a map

\[ \epsilon^\infty_{\tau} : C(I) \to \mathbb{Z}/2\mathbb{Z} \]

by setting

\[ \epsilon^\infty_{\tau}(c) = \sum_{f \in I} c(f) a^\infty_{\tau}(f). \]

For \( v \in \mathcal{V} \), set \( \epsilon^v = \epsilon^p \) if \( v = v_p \), and \( \epsilon^v = \epsilon^\infty \) if \( v = v_\infty \). Set

\[ \epsilon_{\tau}(c) = \sum_{v \in \mathcal{V}} \epsilon^v(c). \]

Since \( \epsilon^v = 0 \) for almost all \( v \in \mathcal{V} \) (cf. Proposition 9.1), this is a finite sum.

Set \( \epsilon^\text{finite}_\tau = \sum_{v \in \mathcal{V}} \epsilon^v \), and note that if \( n^+ = 0 \) or \( n^- = 0 \), then \( \epsilon^\text{finite}_\tau \) does not depend on \( D_+ \) and \( D_- \). By definition, we have \( \epsilon_{\tau} = \epsilon^\text{finite}_\tau + \epsilon^\infty \). We obtain a homomorphism

\[ \epsilon_{\tau} : C_0(I) \to \mathbb{Z}/2\mathbb{Z}. \]

**Proposition 9.4.** The homomorphism \( \epsilon_{\tau} : C_0(I) \to \mathbb{Z}/2\mathbb{Z} \) induces a homomorphism

\[ \epsilon_{\tau} : G_F(D_+, D_-) \to \mathbb{Z}/2\mathbb{Z}. \]

**Proof.** It suffices to show that if \( c(f) = 1 \) for all \( f \in J \), then \( \epsilon_{\tau}(c) = 0 \). For all \( v \in \mathcal{V} \), set \( A^v = w_2(q) + w_2(Q) \) in \( \text{Br}_2(\mathbb{Q}_v) = \mathbb{Z}/2\mathbb{Z} \), where \( \mathbb{Q}_v \) is either \( \mathbb{R} \) or \( \mathbb{Q}_p \), for a prime number \( p \). Note that \( A^v = 0 \) for almost all \( v \in \mathcal{V} \), and that \( \sum_{v \in \mathcal{V}} A^v = 0. \)

Moreover, for all \( a^v \in C^v \), we have by definition \( \sum_{f \in J} a^v(f) = A^v \).

Let \( c \in C(I) \) be such that \( c(f) = 1 \) for all \( f \in J \). We have

\[ \epsilon_{\tau}(c) = \sum_{v \in \mathcal{V}} \sum_{f \in J} c(f) a^v(f) = \sum_{v \in \mathcal{V}} \sum_{f \in J} a^v(f) = \sum_{v \in \mathcal{V}} A^v = 0. \]

**Notation 9.5.** We denote by \( \mathcal{C} \) the set of \( (a^v) \) with \( a^v \in C^v \) and \( a^v = 0 \) for almost all \( v \).

By Proposition 9.1 the set \( \mathcal{C} \) is not empty.
Theorem 9.6. Let \((a^\nu) \in \mathcal{C}\) be such that \(\sum_{v \in \mathcal{V}} a^\nu(f) = 0\) for all \(f \in I\). Then there exists a semi-simple isometry \(t : V \to V\) of \((V, q)\) with signature map \(\tau\) and local data \((a^\nu)\).

Proof. If \(f \in I_1\), we have \(a^\nu(f) = a[\lambda^p, q^p_{+}, q^p_{-}](f) = \text{cor}_{(E_f)^0/\mathbb{Q}_p}(\lambda^p_f, df)\), and \(a^{\nu_\infty}(f) = a[\lambda^{\infty}, q^{\infty}_{+}, q^{\infty}_{-}](f) = \text{cor}_{(E_f)^\infty/\mathbb{R}}(\lambda^{\infty}_f, df)\) (cf. Propositions 5.4 and 9.2). Since \(\sum_{v \in \mathcal{V}} a^\nu(f) = 0\), this implies that

\[
\sum_{v \in \mathcal{V}} \text{cor}_{(E_f)^0/\mathbb{Q}_v}(\lambda^v_f, df) = 0,
\]

where \(\mathcal{W}\) is the set of primes of \((E_f)^0\). This implies that there exists \(\lambda_f \in (E_f)^0/\mathcal{N}_{E_f}(E_f)^0\) mapping to \(\lambda^w_f\) for all \(w \in \mathcal{W}\) (see for instance [B 21], Theorem 10.1). In particular, we have \((\lambda_f, df) = (\lambda^w_f, df)\) in \(\text{Br}_2((E_f)^0)\) for all \(w \in \mathcal{W}\).

Note that \(\text{diff} \circ \tau(f)\) is an even integer. Let \(h_f : V_f \times V_f \to E_f\) be a hermitian form of determinant \(\lambda_f\) and index \(\frac{1}{2}\text{diff} \circ \tau(f)\); such a hermitian form exists (see for instance [Sch 85], 10.6.9). Let us define \(q_f : V_f \times V_f \to \mathbb{Q}\) by \(q_f(x, y) = \text{Tr}_{E_f/\mathbb{Q}}(h_f(x, y))\).

Let \(f = X \pm 1\). We have \(\sum_{v \in \mathcal{V}} a^\nu(f) = 0\), hence by the Brauer-Hasse-Noether theorem there exists \(a(\pm) \in \text{Br}_2(\mathbb{Q})\) mapping to \(a^\nu(f)\) in \(\text{Br}_2(\mathbb{Q}_v)\) for all \(v \in \mathcal{V}\). Let \(q_{\pm}\) be a quadratic form over \(\mathbb{Q}\) of dimension \(n_{\pm}\), determinant \(D_{\pm}\), Hasse-Witt invariant \(w_2(q_{\pm}) = a(\pm) + w_2(Q_{\pm})\) and signature \(\tau(X \pm 1) = (r_{\pm}, s_{\pm})\). Such a quadratic form exists; see for instance [S 77], Proposition 7.

Let \(q' : V \times V \to \mathbb{Q}\) be the quadratic form given by

\[
(V, q') = \bigoplus_{f \in I_1} (V_f, q_f) \oplus \bigoplus_{f \in I_0} (V_f, q_f).
\]

By construction, \((V, q')\) has the same dimension, determinant, Hasse-Witt invariant and signature as \((V, q)\), hence the quadratic forms \((V, q')\) and \((V, q)\) are isomorphic.

Let \(t : V \to V\) be defined by \(t(x) = \gamma x\), where \(\gamma\) is a generator of \(\Gamma\). By construction, \(t\) is an isometry of \((V, q')\) and it is semi-simple with signature map \(\tau\) and local data \((a^\nu)\).

Theorem 9.7. Suppose that \(\epsilon_\tau = 0\). Then there exists \((a^\nu) \in \mathcal{C}\) such that \(\sum_{v \in \mathcal{V}} a^\nu(f) = 0\) for all \(f \in I\).

Proof. This follows from [B 21], Theorem 13.5.

Corollary 9.8. The quadratic form \((V, q)\) has a semi-simple isometry with signature map \(\tau\) such that the associated \(\mathbb{Q}\Gamma\) quadratic form \((V, q)\) satisfies \(\partial_\delta[V, q] = \delta_p\) for all prime numbers \(p\) if and only if \(\epsilon_\tau = 0\).

Proof. Assume that \((V, q)\) has a semi-simple isometry as above, and let \((a^\nu)\) be the associated local date. Recall that the element \(a^p \in C^p\) given by \(a^p(f) = w_2(q_f^p) + w_2(Q_f)\) if \(f \in I_1\), by \(a^p(X \pm 1) = w_2(q_{+}^p) + w_2(Q_{\pm})\), and \(a^p(f) = 0\) if \(f \in J\) with \(f \not\in I, f \neq X \pm 1\) (see [S 5]). Similarly, we have the element \(a^{\infty} \in C(I)\)
given by \( a^\infty(f) = w_2(q_f^\infty) + w_2(Q_f) \) if \( f \in I_1 \), \( a^\infty(X \pm 1) = w_2(q_f^\infty) + w_2(Q_{\pm}) \), and \( a^\infty(f) = 0 \) if \( f \in J \) with \( f \notin I, f \neq X \pm 1 \). Since \( q_f^p = q_f \otimes_{\mathbb{Q}} \mathbb{Q}_p \) and \( q_f^\infty = q_f \otimes_{\mathbb{Q}} \mathbb{R} \), we have

\[
\sum_{v \in \mathcal{V}} a^v(f) = 0 \text{ for all } f \in J.
\]

This implies that \( \epsilon_r = 0 \). The converse follows from Theorems 9.6 and 9.7.

10. Necessary conditions

Our aim is to determine the signature maps of semi-simple isometries with determinant 1 of even unimodular lattices; in this section, we collect some necessary conditions for this.

We start by recalling a well-known result:

**Lemma 10.1.** Let \((L, q)\) be an even unimodular lattice. Then

(a) For all prime numbers \( p \), we have \( w_2(q) = w_2(h) \) in \( \text{Br}_2(\mathbb{Q}_p) \), where \( h \) is the hyperbolic form over \( \mathbb{Q}_p \) of dimension equal to the rank of \( L \).

(b) Let \((r, s)\) be the signature of \((L, q)\). Then \( r \equiv s \pmod{8} \).

**Proof.** (a) The reduction mod 2 of \((L, q)\) is a non-degenerate alternating symmetric bilinear form over \( \mathbb{F}_2 \), hence it is even dimensional; its dimension is equal to the rank of \( L \), therefore this rank is even. Set \( \text{rank}(L) = 2n \). The determinant of \( q \) is \( \pm 1 \); by [BC 05], Proposition 5.2 this implies that \( q \otimes_{\mathbb{Z}} \mathbb{Z}_2 \) is isomorphic to \( h \otimes_{\mathbb{Z}} \mathbb{Z}_2 \), and hence \( w_2(q) = w_2(h) \) in \( \text{Br}_2(\mathbb{Q}_2) \). If \( p \) is a prime number with \( p \neq 2 \), then \( w_2(q) = w_2(h) = 0 \); this proves (a).

(b) This is proved for instance in [S 77], Chap. V, Corollaire 1. Here is another proof: since \( q \) is a global form and \( w_2(q) = 0 \) in \( \text{Br}_2(\mathbb{Q}_p) \) for \( p \neq 2 \), we have \( w_2(q) = 0 \) in \( \text{Br}_2(\mathbb{Q}_2) \) if and only if \( w_2(q) = 0 \) in \( \text{Br}_2(\mathbb{R}) \). Moreover, \( w_2(q) = 0 \) in \( \text{Br}_2(\mathbb{R}) \) if and only if \( n \equiv 0 \) or 1 (mod 4). On the other hand, by (a) we know that \( w_2(q) = w_2(h) \) in \( \text{Br}_2(\mathbb{Q}_2) \), therefore \( w_2(q) = 0 \) in \( \text{Br}_2(\mathbb{Q}_2) \) if and only if \( n \equiv 0 \) or 1 (mod 4). This implies that \( n \equiv s \pmod{4} \). We have \( 2n = r + s \), therefore \( r - s = 2(n - s) \), hence \( r \equiv s \pmod{8} \), as claimed.

This motivates the following definition. Let \( F \in \mathbb{R}[X] \) and let \( r, s \geq 0 \) be two integers.

**Definition 10.2.** We say that condition (C 0) holds for \( F \) and \((r, s)\) if

(C 0) \( F \in \mathbb{Z}[X] \) with \( F(0) = 1 \), \( \deg(F) = r + s \), and \( r \equiv s \pmod{8} \).

We say that condition (C 0) holds for a signature map \( \tau \) if it holds for the maximum \((r, s)\) and the associated polynomial \( F \) of \( \tau \).

**Lemma 10.3.** Suppose that \( \tau \) is the signature map of a semi-simple isometry with determinant 1 of an even unimodular lattice. Then condition (C 0) holds.

**Proof.** It is clear that \( F \in \mathbb{Z}[X] \) and that \( F(0) = 1 \), we have \( \deg(F) = r + s \) by definition, and \( r \equiv s \pmod{8} \) follows from Lemma 10.1.

Recall from §2 that condition (C 1) is said to hold for a polynomial \( F \in \mathbb{Z}[X] \) with \( 2n = \deg(F) \) if

(C 1) \( |F(1)|, |F(-1)| \) and \( (-1)^n F(1) F(-1) \) are squares.
Definition 10.4. We say that condition (C1) holds for a signature map $\tau$ if it holds for the polynomial $F \in \mathbb{Z}[X]$ associated to $\tau$.

Lemma 10.5. Suppose that $\tau$ is the signature map of a semi-simple isometry with determinant 1 of an even unimodular lattice. Then condition (C1) holds.

Proof. This follows from Theorem 2.1 (see also [GM 02], Theorem 6.1).

Notation 10.6. If $r, s \geq 0$ are two integers, we denote by $(V_{r,s}, q_{r,s})$ the diagonal quadratic form over $\mathbb{Q}$ with $r$ diagonal entries 1 and $s$ diagonal entries $-1$.

Lemma 10.7. Suppose that $r \equiv s (\text{mod } 8)$. Then for all prime numbers $p$, we have $w_2(q_{r,s}) = w_2(h)$ in $\text{Br}_2(\mathbb{Q}_p)$, where $h$ is the hyperbolic form over $\mathbb{Q}_p$ of dimension $r + s$.

Proof. This follows from an easy computation.

Lemma 10.8. Suppose that $\tau$ is the signature map of a semi-simple isometry with determinant 1 of an even unimodular lattice $(L, q)$, and let $(r, s)$ be the maximum of $\tau$. Then $(L, q) \otimes \mathbb{Q} \simeq (V_{r,s}, q_{r,s})$.

Proof. It is clear that $q$ and $q_{r,s}$ have the same dimension, determinant and signature. They have the same Hasse-Witt invariant by Lemma 10.1 (a) and Lemma 10.7 hence they are isomorphic.

Let $\tau$ be a signature map, let $F \in \mathbb{Z}[X]$ be the associated polynomial, and assume that $F = F_0 F_1$ as in §5 with

$$F_0(X) = (X - 1)^{n^+} (X + 1)^{n^-}$$

for some integers $n^+, n^- \geq 0$. Let $(V, q) = (V_{r,s}, q_{r,s})$.

Notation 10.9. Let $p$ be a prime number, and let $t : V \otimes \mathbb{Q} \mathbb{Q}_p \rightarrow V \otimes \mathbb{Q} \mathbb{Q}_p$ be a semi-simple isometry with characteristic polynomial $F$. Let

$$(V, q) \otimes \mathbb{Q} \mathbb{Q}_p = (V_{F_0}, q_0^p) \oplus (V_{F_1}, q_1^p)$$

be the orthogonal decomposition of $\mathbb{Q}_p[\Gamma]$-quadratic forms of §5. If moreover $n_+ > 0$ and $n_- > 0$, we then have the further orthogonal decomposition of $\mathbb{Q}_p[\Gamma]$-quadratic forms $(V_{F_0}, q_0^p) \simeq (V_+, q_+^p) \oplus (V_-, q_-^p)$, as in §5.

Proposition 10.10. Assume that $n^+ > 0$ and $n^- > 0$, and let $p$ be a prime number. With the above notation, we have

$$\det(q_0^p) = u_+ F_1(1), \det(q_0^p) = u_- F_1(-1) \text{ in } \mathbb{Q}_p^\times / \mathbb{Q}_p^{\times 2}$$

for some $u_+, u_- \in \mathbb{Q}_p^\times$ such that $u_+ u_- = (-1)^n$.

Proof. This follows from [BT 20], Lemma 6.8 and proposition 6.6 if $p \neq 2$, from [BT 20], Lemma 6.8, Proposition 6.7 and the fact that $v_2(\det(q_-)) \equiv v_2(F_1(-1)) (\text{mod } 2)$ since the lattice is even if $p = 2$.

We also need the following observation:

Lemma 10.11. Let $\tau(X - 1) = (r_+, s_+)$ and $\tau(X + 1) = (r_-, s_-)$. Let

$$(V, q) = (V_{F_0}, q_0^\infty) \oplus (V_{F_1}, q_1^\infty)$$

be the orthogonal decomposition of $\mathbb{R}[\Gamma]$-quadratic forms of corresponding to the signature map $\tau$. If moreover $n^+ > 0$ and $n^- > 0$, we then have the further
orthogonal decomposition of $\mathbb{R}[\Gamma]$-quadratic forms $(V_{F_0}, q_0^\infty) \simeq (V_+, q_+^\infty) \oplus (V_-, q_-^\infty)$. Then we have

$$\det(q_+^\infty) = (-1)^{s_+}, \quad \det(q_-^\infty) = (-1)^{s_-} \text{ in } \mathbb{R}^\times/\mathbb{R}^2.$$ 

Proof. This is clear.

11. Local data - isometries of even unimodular lattices

Let $\tau$ be a signature map with maximum $(r, s)$ and associated polynomial $F$. We assume that conditions (C 0) and (C 1) hold for $\tau$; let $n \geq 1$ be an integer such that $\deg(F) = 2n$. Suppose that $F = F_0F_1$, with the notation of the previous section.

In the next section, we give a necessary and sufficient condition for $\tau$ to be the signature map of a semi-simple isometry with determinant 1 of an even unimodular lattice. In order to apply the results of §9 we need to fix a non-degenerate quadratic form $q : V \times V \to \mathbb{Q}$, and $D_+, D_- \in \mathbb{Q}^\times/\mathbb{Q}^\times 2$ such that $D_+D_- = \det(q)F_1(1)F_1(-1)$.

The results of the previous section imply that the only possible choice is $(V, q) = (V_{r,s}, q_{r,s})$ (see Lemma [10.7]). We have $\det(q) = (-1)^n$, hence the condition for $D_+$ and $D_-$ becomes $D_+D_- = (-1)^nF_1(1)F_1(-1)$.

We now determine the values of $D_+$ and $D_-$, also using the results of the previous section. Let $\tau(X - 1) = (r_+, s_+)$ and $\tau(X + 1) = (r_-, s_-)$. With the notation of Lemma [10.11] we have $\det(q) = (-1)^n = F_1(1)F_1(-1)(-1)^{s_+}(-1)^{s_-}$ in $\mathbb{R}^\times/\mathbb{R}^\times 2$. Set $\epsilon_+ = \frac{F_1(1)}{|F_1(1)|}$ and $\epsilon_- = \frac{F_1(-1)}{|F_1(-1)|}$; then we have

$$(-1)^n = \epsilon_+(-1)^{s_+}\epsilon_-(-1)^{s_-}.$$ 

Set $u_+ = \epsilon_+(-1)^{s_+}$ and $u_- = \epsilon_-(-1)^{s_-}$. Then Lemma [10.11] and Lemma [10.10] show that the only possible choice for $D_+$ and $D_-$ is

$$D_+ = u_+F_1(1) = (-1)^{s_+}|F_1(1)|, \quad D_- = u_-F_1(-1) = (-1)^{s_-}|F_1(-1)|.$$ 

Note that $D_+$ and $D_-$ are determined by the signature map $\tau$. We now show that this choice gives rise to non-empty local data.

Proposition 11.1. Let $u_+, u_- \in \{\pm 1\}$ be such that $u_+u_- = (-1)^n$, and let $(V, q) = (V_{r,s}, q_{r,s})$. Then for every prime number $p$, the quadratic form $(V, q) \otimes \mathbb{Q} \mathbb{Q}_p$ has a semi-simple isometry with characteristic polynomial $F$ that stabilizes an even unimodular $\mathbb{Z}_p$-lattice such that if $n_+ > 0$ and $n_- > 0$, then $\det(q_+^p) = u_+F_1(1), \det(q_-^p) = u_-F_1(-1)$ in $\mathbb{Q}_p^\times/\mathbb{Q}_p^\times 2$.

Proof. Condition (C 1) implies that for every prime number $p$, there exists an even $\mathbb{Z}_p$-lattice of determinant $(-1)^n$ having a semi-simple isometry with characteristic polynomial $F$ (cf. Corollary [2.8]); the statement concerning $\det(q_+^p)$ and $\det(q_-^p)$ follows from Claim 1 and Claim 2 (see [2] proof of Theorem [2.1]). It remains to show that for every prime number $p$, the extension to $\mathbb{Q}_p$ of such a lattice is isomorphic to $(V, q) \otimes \mathbb{Q} \mathbb{Q}_p$. Note that the conditions $2n = r + s$ and $r \equiv s \pmod{8}$ imply that $n \equiv s \pmod{4}$, therefore $(-1)^s = (-1)^n$; this implies that $\det(q) = (-1)^n$.

Moreover, for all prime numbers $p$ we have $w_2(q) = w_2(h)$ in $\text{Br}_p(\mathbb{Q}_p)$, where $h$ is the hyperbolic lattice over $\mathbb{Q}_p$ of rank $2n$. This coincides with the Hasse-Witt invariants of an even unimodular $\mathbb{Z}_p$-lattice of rank $2n$ (this is clear for $p \neq 2$, and follows from [BG 05], Proposition 5.2 for $p = 2$). Hence the extension to $\mathbb{Q}_p$ of an even $\mathbb{Z}_p$-lattice of rank $2n$ and determinant $(-1)^n$ is isomorphic to $(V, q) \otimes \mathbb{Q} \mathbb{Q}_p$. 
12. A necessary and sufficient condition

Let $\tau$ be a signature map. The aim of this section is to apply the results of §11 to decide whether $\tau$ is the signature map of a semi-simple isometry of an even unimodular lattice. Let $F$ be the polynomial associated to $\tau$, let $(r, s)$ be the maximum of $\tau$, and set $(V, q) = (V_{r,s}, q_{r,s})$.

Recall that condition (C 0) holds for $F$ and $(r, s)$ if

(C 0) $F \in \mathbb{Z}[X]$ with $F(0) = 1$, $\deg(F) = r + s$, and $r \equiv s \pmod{8}$.

and that condition (C 0) holds for $\tau$ if it holds for $(r, s)$ and $F$.

Recall that condition (C 1) is said to hold for a polynomial $F \in \mathbb{Z}[X]$ with $2n = \deg(F)$ if

(C 1) $|F(1)|$, $|F(-1)|$ and $(-1)^n F(1) F(-1)$ are squares.

and that condition (C 1) holds for $\tau$ if it holds for $F$.

In the following, we assume that conditions (C 0) and (C 1) hold for $\tau$.

Let us write $F = F_0 F_1$ as in the previous sections. With the notation of §11 set

$$D_+ = u_+ F_1(1) = (-1)^{s_+}|F_1(1)|, \ D_- = u_- F_1(-1) = (-1)^{s_-}|F_1(-1)|.$$ 

Note that the group $G_F(D_+, D_-) = G_F((-1)^{s_+}|F_1(1)|, (-1)^{s_-}|F_1(-1)|)$ only depends on the polynomial $F$ and the integers $s_+, s_1$.

Let $\delta_p = 0$ for all prime numbers $p$.

For every prime number $p$, the quadratic form $(V, q) \otimes_{\mathbb{Q}} \mathbb{Q}_p$ has a semi-simple isometry with characteristic polynomial $F$ that stabilizes an even unimodular $\mathbb{Z}_p$-lattice (cf. Proposition 1.1). To this isometry, we associate a homomorphism $\epsilon^p : C_0(I) \to \mathbb{Z}/2\mathbb{Z}$ as in §10 with $\epsilon^p = 0$ for almost all $p$, and we obtain a homomorphism (cf. §11)

$$\epsilon^\infty_\tau : C_0(I) \to \mathbb{Z}/2\mathbb{Z}.$$ 

Note that if $F(1) \neq 0$ or $F(-1) \neq 0$, then $\epsilon^\infty_\tau$ only depends only on the polynomial $F$, and in this case, is denoted by

$$\epsilon^\infty_F : C_0(I) \to \mathbb{Z}/2\mathbb{Z}.$$ 

We associate to $\tau$ a homomorphism $\epsilon_\tau : C_0(I) \to \mathbb{Z}/2\mathbb{Z}$, and set $\epsilon_\tau = \epsilon^\infty_\tau + \epsilon^\infty_F$, obtaining a homomorphism

$$\epsilon_\tau : G_F(D_+, D_-) \to \mathbb{Z}/2\mathbb{Z}$$

(see Proposition 11.4).

**Theorem 12.1.** There exists an even unimodular lattice having a semi-simple isometry with signature map $\tau$ if and only if $\epsilon_\tau = 0$.

**Proof.** Assume that $\epsilon_\tau = 0$. By Corollary 9.8 the quadratic form $(V, q)$ has a semi-simple isometry with signature map $\tau$ such that for all prime numbers $p$, the quadratic form $(V, q) \otimes_{\mathbb{Q}} \mathbb{Q}_p$ contains a unimodular $\mathbb{Z}_p$-lattice $L_p$ that is stabilized by $t$; for $p = 2$, the $\mathbb{Z}_2$-lattice $L_2$ can be chosen to be even (this follows from [BT 20], Theorem 8.1, in which conditions (i) and (ii) are clearly satisfied, and condition (iii) follows from [BT 20], Theorem 8.4 and Claim 2). We can also suppose that for almost all prime numbers $p$, the lattice $L_p$ is equal to $L' \otimes_{\mathbb{Z}} \mathbb{Z}_p$ for some $\mathbb{Z}$-lattice $L'$ of $(V, q)$ stabilized by $t$. Set $L = \cap_p L_p$, where the intersection is taken over all prime
numbers $p$; the lattice $L$ is even unimodular and stabilized by $t$. The converse is an immediate consequence of Corollary 9.8.

Note that if $n^+ \neq 2$ and $n^- \neq 2$, then the group $G_\tau$ only depends on the polynomial $F$; in this case, we denote it by $G_F$. Similarly, if $n^+ \neq 2$, we use the notation $G_F(D_-)$, and if $n^- \neq 2$, the notation $G_F(D_+)$. We now derive some consequences of Theorem 12.1: in particular, we prove the results announced in the introduction to Part I.

Let $t \in \text{SO}_{r,s}(\mathbb{R})$ be a semi-simple isometry with characteristic polynomial $F$, and suppose that $n^+ \neq 2$ and $n^- \neq 2$.

The following result is Theorem 6 of the introduction:

**Theorem 12.2.** The isometry $t$ preserves an even unimodular lattice if and only if $\epsilon_{\text{sign}} t = 0$.

Theorem 12.1 also implies the following corollaries:

**Corollary 12.3.** If $G_F = 0$, then there exists an even unimodular lattice having an isometry with signature map $\tau$.

**Corollary 12.4.** If $G_F = 0$, then there exists an even unimodular lattice of signature $(r,s)$ having an isometry with characteristic polynomial $F$.

If $S$ is a Salem polynomial of degree 22, then Condition (C 1) holds for $S$ if and only if $|S(1)|$ and $|S(-1)|$ are squares; since $S$ is irreducible, we have $G_S = 0$. Therefore the following result is a consequence of Corollary 12.3:

**Corollary 12.5.** Let $S$ be a Salem polynomial of degree 22 such that $|S(1)|$ and $|S(-1)|$ are squares, and let $\tau$ be a signature map with maximum $(3,19)$ and polynomial $S$. Then $\Lambda_{3,19}$ has an isometry with signature map $\tau$.

In particular, the lattice $\Lambda_{3,19}$ has an isometry with characteristic polynomial $S$; this was proved in [BT 20].

The following sections contain some special cases and examples that are of interest for the applications to $K3$ surfaces; the first of these consists of reminders on Salem numbers and Salem polynomials.

13. Salem polynomials

The aim of this section is to record some basic properties of Salem polynomials, and more generally, of symmetric polynomials.

**Lemma 13.1.** Let $n \geq 1$ be an integer, and let $F \in \mathbb{Z}[X]$ be a symmetric polynomial of degree $2n$. Then $F(1) \equiv (-1)^n F(-1) \pmod{4}$.

**Proof.** Let $I = \{1, \ldots, n-1\}$ and let $a_1, \ldots, a_n$ be integers such that

$$F(X) = X^{2n} + \sum_{i \in I} a_i (X^i + X^{2n-i}) + a_n X^n + 1.$$ 

Let $I_{\text{even}}$ be the set of $i \in I$ such that $i$ is even, and let $I_{\text{odd}}$ be the set of $i \in I$ such that $i$ is odd. If $n$ is even, then we have $F(1) - F(-1) = 4 \sum_{i \in I_{\text{odd}}} a_i$; if $n$ is odd, then $F(1) + F(-1) = 4 + 4 \sum_{i \in I_{\text{even}}} a_i$. 


Lemma 13.2. Let \( S \) be a Salem polynomial. Then \( S(1) < 0 \) and \( S(-1) > 0 \).

Proof. We have \( S(X) = (X - \alpha)(X - \alpha^{-1}) \prod_{\rho \in P} (X - \rho)(X - \overline{\rho}) \), where \( \alpha > 1 \) is a real number. Note that for all \( \rho \in P \) we have \((1 - \rho)(1 - \overline{\rho}) > 0 \) and \((1 + \rho)(1 + \overline{\rho}) > 0 \). On the other hand, we have \((1 - \alpha)(1 - \alpha^{-1}) < 0 \) and \((1 + \alpha)(1 + \alpha^{-1}) < 0 \). Therefore we have \( S(1) < 0 \) and \( S(-1) > 0 \), as claimed.

Recall that a monic polynomial \( F \in \mathbb{Z}[X] \) of degree \( 2n \) satisfies condition (C 1) if \(|F(1)|, |F(-1)|\) and \((-1)^n F(1)F(-1)\) are squares.

Proposition 13.3. Let \( n \geq 1 \) be an integer, and let \( S \) be a Salem polynomial of degree \( 2n \). Then we have

(a) If \( S \) satisfies condition (C 1), then \( n \) is odd.

(b) If \(|S(1)S(-1)| = 1\), then \( n \) is odd.

Proof. (a) The condition that \((-1)^n S(1)S(-1)\) is a square implies that \( n \) is odd; indeed, by Lemma 13.2 we have \( S(1) < 0 \) and \( S(-1) > 0 \), therefore \( S(1)S(-1) < 0 \).

(b) Suppose that \( n \) is even; then by Lemma 13.1 we have \( S(1) \equiv S(-1) \mod 4 \). On the other hand, Lemma 13.2 implies that \( S(1) < 0 \) and \( S(-1) > 0 \), therefore \(|S(1)S(-1)| > 1\).

Note that part (b) of the lemma was proved by Gross and McMullen, see [GM 02], Proposition 3.3.

Proposition 13.4. If \( \alpha \) is a Salem number of degree \( d \), then so is \( \alpha^n \), for all integers \( n \) with \( n \geq 1 \).

Proof. For \( d \geq 4 \), this is a result of Salem, see [Sa 45], page 169; see also [Sm 15], Lemma 2, and it is well-known if \( d = 2 \).

Proposition 13.5. If \( K = \mathbb{Q}(\alpha) \) for some Salem number \( \alpha \), then there exists a Salem number \( \beta \) such that the set of Salem numbers in \( K \) consists of the powers of \( \beta \).

Proof. This is well-known if \( d = 2 \); for \( d \geq 4 \), see [Sm 15], Proposition 3, (iii) (and [Sa 45], page 169, for a slightly less general statement).

We refer to the survey articles [GH 01] and [Sm 15] for more information about Salem polynomials and Salem numbers.

14. Examples

The aim of this section and the next one is to illustrate the results of §12 by some examples; in view of applications to automorphisms of K3 surfaces, we look for isometries of the lattice \( \Lambda_{3,19} \) with characteristic polynomial \( SC \), where \( S \) is a Salem polynomial, and \( C \) a product of cyclotomic polynomials. We start with the case where \( C \) is a power of a cyclotomic polynomial.

Example 14.1. Let \( F = SC \), where \( S \) is a Salem polynomial of degree \( d \) with \( 4 \leq d \leq 20 \) and \( C \) is a power of a cyclotomic polynomial; assume that \( \deg(F) = 22 \), and that \( C(X) \neq (X - 1)^2 \), \( C(X) \neq (X + 1)^2 \). If \( z \) is a root of \( F \) with \(|z| = 1\), let \( f_z \in \mathbb{R}[X] \) be the minimal polynomial of \( z \) over \( \mathbb{R} \), and let \( \tau_z \) be the signature map with maximum \((3,19)\) and polynomial \( F \) such that \( \text{proj}_1 \circ \tau(f_z) = 2 \), where \( \text{proj}_1 : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) is the projection on the first factor.
We say that $\tau_z$ is \textit{realizable} if it is the signature map of a semi-simple isometry of the lattice $\Lambda_{3,19}$.

(a) Suppose that $F$ satisfies condition (C 1), and that $G_F = 0$; then $\tau_z$ is realizable for all $z$.

(b) Suppose that $F$ satisfies condition (C 1), and that $G_F \neq 0$. We have $I = \{ S, \Phi \}$ where $\Phi$ is cyclotomic and $C$ is a power of $\Phi$. Let $c : I \rightarrow \mathbb{Z}/2\mathbb{Z}$ be such that $c(S) = 1$ and $c(\Phi) = 0$, and let $c' : I \rightarrow \mathbb{Z}/2\mathbb{Z}$ be such that $c'(S) = 0$ and $c'(\Phi) = 1$.

The map $\epsilon_F^{\text{finite}} : C(I) \rightarrow \mathbb{Z}/2\mathbb{Z}$ only depends on the polynomial $F$. On the other hand, the map $\epsilon^\infty : C(I) \rightarrow \mathbb{Z}/2\mathbb{Z}$ also depends on the choice of $z$, as follows:

- Assume that $z$ is a root of $S$. Then we have:
  \[ d \equiv 0 \pmod{4} \implies a^\infty(S) = 0 ~ \text{and} ~ a^\infty(\Phi) = 1. \]
  \[ d \equiv 2 \pmod{4} \implies a^\infty(S) = 1 ~ \text{and} ~ a^\infty(\Phi) = 0. \]

  With $c$ and $c'$ as above, we have:
  \[ d \equiv 0 \pmod{4} \implies \epsilon_{\tau_z}^\infty(c) = 0 ~ \text{and} ~ \epsilon_{\tau_z}^\infty(c') = 1. \]
  \[ d \equiv 2 \pmod{4} \implies \epsilon_{\tau_z}^\infty(c) = 1 ~ \text{and} ~ \epsilon_{\tau_z}^\infty(c') = 0. \]

  This implies that $\tau_z$ is realizable if and only if either
  \[ d \equiv 0 \pmod{4} \text{ and } \epsilon_F^{\text{finite}}(c) = 0, \epsilon_F^{\text{finite}}(c') = 1, \text{ or} \]
  \[ d \equiv 2 \pmod{4} \text{ and } \epsilon_F^{\text{finite}}(c) = 1, \epsilon_F^{\text{finite}}(c') = 0. \]

- Assume that $z$ is a root of $\Phi$. Then we have:
  \[ d \equiv 0 \pmod{4} \implies a^\infty(S) = 1 ~ \text{and} ~ a^\infty(\Phi) = 0. \]
  \[ d \equiv 2 \pmod{4} \implies a^\infty(S) = 0 ~ \text{and} ~ a^\infty(\Phi) = 1. \]

  With $c$ and $c'$ as above, we have:
  \[ d \equiv 0 \pmod{4} \implies \epsilon_{\tau_z}^\infty(c) = 1 ~ \text{and} ~ \epsilon_{\tau_z}^\infty(c') = 0. \]
  \[ d \equiv 2 \pmod{4} \implies \epsilon_{\tau_z}^\infty(c) = 0 ~ \text{and} ~ \epsilon_{\tau_z}^\infty(c') = 1. \]

  This implies that $\tau_z$ is realizable if and only if either
  \[ d \equiv 0 \pmod{4} \text{ and } \epsilon_F^{\text{finite}}(c) = 1, \epsilon_F^{\text{finite}}(c') = 0, \text{ or} \]
  \[ d \equiv 2 \pmod{4} \text{ and } \epsilon_F^{\text{finite}}(c) = 0, \epsilon_F^{\text{finite}}(c') = 1. \]

In summary, this implies that whether or not $\tau_z$ is realizable depends on $\epsilon_F^{\text{finite}}$, and whether $z$ is a root of $S$ or of $\Phi$. In particular,

- if $z$ and $z'$ are roots of the same irreducible polynomial, then
  \[ \tau_z \text{ is realizable } \iff \tau_{z'} \text{ is realizable.} \]

On the other hand,

- if $z$ and $z'$ are roots of different irreducible polynomials, then
  \[ \tau_z \text{ realizable } \implies \tau_{z'} \text{ is not realizable.} \]

Moreover, by Proposition 9.4, we have $\epsilon_{\tau_z}(c) = \epsilon_{\tau_z}(c')$, and this implies the stronger statement

- if $z$ and $z'$ are roots of different irreducible polynomials, then
\( \tau_z \) realizable \iff \( \tau_{z'} \) is not realizable.

Therefore, either \( \tau_z \) is realizable for all roots \( z \) of \( S \) with \( |z| = 1 \), or \( \tau_z \) is realizable for all roots \( z \) of \( \Phi \) – but not both. In particular, the lattice \( \Lambda_{3,19} \) has a semi-simple isometry with characteristic polynomial \( SC \).

(c) Assume that both \( S \) and \( C \) satisfy condition (C 1). In this case, one gets a more precise result. Suppose that \( G_F \neq 0 \) – otherwise, every signature map is realizable by (a). We keep the notation of (b).

Since \( S \) is irreducible, and \( C \) is a power of an irreducible polynomial, we have \( G_S = G_C = 0 \). By hypothesis, \( S \) satisfies condition (C 1), and this implies that \( d \equiv \pm 2 \pmod{8} \).

Assume first that \( z \) is a root of \( S \) with \( |z| = 1 \); then \( \tau_z(S) = (3, d - 3) \). Condition (C 0) holds for \( S \) and \( (3, d - 3) \) if and only if \( d \equiv -2 \pmod{8} \). If this is the case, then Condition (C 0) also holds for \( C \) and \( (0, 22 - d) \), and by Theorem 12.1 the signature map \( \tau_z \) is realizable.

Note that by (b), this implies that if \( d \equiv -2 \pmod{8} \) and \( z' \) is a root of \( \Phi \), then \( \tau_{z'} \) is not realizable.

Assume that \( z \) is a root of \( \Phi \). In this case, we have \( \tau_z(S) = (1, d - 1) \) and \( \tau_z(C) = (2, 20 - d) \). Condition (C 0) holds for \( S \) and \( (1, d - 1) \) if and only if it holds for \( C \) and \( (2, 20 - d) \), and this happens if and only if \( d \equiv 2 \pmod{8} \). In this case, by Theorem 12.1 the signature map \( \tau_z \) is realizable.

By (b), this implies that if \( d \equiv 2 \pmod{8} \) and \( z' \) is a root of \( S \), then \( \tau_{z'} \) is not realizable.

The following examples are numerical illustrations of Example 14.1.

**Example 14.2.** Let \( S(X) = X^6 - 3X^5 - X^4 + 5X^3 - X^2 - 3X + 1 \); this is a Salem polynomial (see for instance [McM 02], §4, or [GM 02], §7, Example 1 with \( a = 3 \)), let \( C = \Phi_{10} \) and \( F = SC \). We keep the notation of Example 14.1. We are in case (c) : the polynomials \( S \) and \( C \) both satisfy condition (C 1), and Example 14.3 shows that \( G_F \simeq \mathbb{Z}/2\mathbb{Z} \).

If \( z \) is a root of \( S \) with \( |z| = 1 \); then Example 14.1 (c) shows that \( \tau_z \) is realizable. On the other hand, if \( z' \) is a root of \( \Phi_{10} \), then \( \tau_{z'} \) is not realizable.

**Example 14.3.** Let \( S(X) = X^{14} - X^{11} - X^{10} + X^7 - X^4 - X^3 + 1 \), let \( C = \Phi_{20} \), and set \( F = SC \). Note that \( S \) is a Salem polynomial, associated to the third smallest known Salem number (see for instance [Bo 77]). We have \( G_F \simeq \mathbb{Z}/2\mathbb{Z} \). Indeed, the resultant of \( S \) and \( \Phi_{20} \) is equal to \( 41^2 \), and the common factors of \( S \) and \( C \) modulo \( 41 \) are \( X + 5 \) and \( X + 33 \); these polynomials are not symmetric, hence \( \Pi_{S,C} = \emptyset \).

We are in case (c) : the polynomials \( S \) and \( C \) both satisfy Condition (C 1), and \( G_F \simeq \mathbb{Z}/2\mathbb{Z} \). Example 14.1 shows that \( \tau_z \) is realizable if \( z \) is a root of \( S \) with \( |z| = 1 \), and not realizable if \( z \) is a root of \( C \).

The next examples are slight variations of the same principle.

**Example 14.4.** Let \( S(X) = X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1 \), the minimal polynomial of the smallest known Salem number (see for instance [Bo 77]), and let \( C(X) = \Phi_{22}(X)(X + 1)^2 \); set \( F = SC \), and note that \( G_F \simeq \mathbb{Z}/2\mathbb{Z} \). Indeed, \( \text{Res}(S, \Phi_{22}) = 23^2 \), and the common irreducible factors of \( S \) and \( \Phi_{22} \) in \( F_{23} \) are \( X + 4 \) and \( X + 6 \); these polynomials are not symmetric, hence \( \Pi_{S,\Phi_{22}} = \emptyset \). We also
have $\Pi_{S(X),X+1} = \varnothing$, since $S(-1) = 1$. On the other hand, $\Phi_{22}(-1) = 11$, hence $\Pi_{\Phi_{22}(X),X+1} = \{1\}$, and $G_F \simeq \mathbb{Z}/2\mathbb{Z}$.

The polynomial $F$ satisfies condition (C 1); with the notation of Example 14.1 and reasoning as in part (b) of this example, we see that the signature map $\tau_z$ is realizable either for all roots $z$ of $S$ with $|z| = 1$, or for all roots of $C$, but not both. It is easy to see that $\tau_z$ is realizable for all roots of $C$. Indeed, both polynomials $S$ and $C$ satisfy condition (C 1), and we have $G_S = G_C = 0$. If $z$ is a root of $C$, then condition (C 0) also holds, and hence $\tau_z$ is realizable. On the other hand, $\tau_z$ is not realizable if $z$ is a root of $S$.

**Example 14.5.** Let $a$ be an integer $\geq 0$, and set

$$S_a(X) = X^6 - aX^5 - X^4 + (2a - 1)X^3 - X^2 - aX + 1.$$  

The polynomial $S_a$ is a Salem polynomial (see for instance [McM 02], §4, or [GM 02], §7, Example 1). Set

$$F_a(X) = S_a(X)(X^2 + 1)^2(X - 1)^{12}.$$  

By Example 14.1, we have $G_{F_a} = 0$ if $a \geq 1$, and $G_{F_0}$ is of order 2. Therefore if $a \geq 1$, then every signature map of maximum (3, 19) and polynomial $F_a$ is realizable. On the other hand, if $a = 0$, then the method of the previous examples shows that $\tau$ is realizable if and only if $\tau = \tau_z$ where $z$ is a root of $S$ with $|z| = 1$; equivalently, if we have $\tau(S) = (3, 3)$.

**Part II : Salem signature maps and surgery on lattices**

This part is the bridge between the arithmetic results of Part I, and the applications to $K3$ surfaces of Part III. The strategy is the following:

Let $t$ be a semi-simple isometry of an even unimodular lattice $(L, q)$ of signature $(3, 19)$ with characteristic polynomial $SC$, where $S$ is a Salem polynomial of degree $d$ with $4 \leq d \leq 20$, and $C$ is a product of cyclotomic polynomial. Let $\alpha$ be the Salem number corresponding to $S$. The criterion of McMullen [McM 11], Theorem 6.2 (see also [McM 16], Theorem 6.1) determines whether $t$ is induced by an automorphism of a $K3$ surface.

Set $L_S = \text{Ker}(S(t))$ and $L_C = \text{Ker}(C(t))$; then $(L_C, q)$ is either negative definite, or of signature $(2, \text{deg}(C) - 2)$. As in the introduction, we assume that $(L_C, q)$ is negative definite. In this case, McMullen’s criterion concerns the restriction of $t$ to $L_C$ : namely, it is satisfied if and only if $t$ stabilizes a chamber of the root system of $L_C$ (see [McM 11], §5, §6 and [McM 16], §2, for details; see also Theorem 21.4). If this criterion is fulfilled, then we are done : $\alpha$ is realizable. If not, we replace $t$ by an isometry $t'$ satisfying McMullen’s criterion, of characteristic polynomial $SC'$, where $C'$ is a product of cyclotomic polynomials.

**15. Salem signature maps and realizable isometries**

We start with some definitions and notation.

**Definition 15.1.** A signature map $\tau$ is called a Salem signature map if its associated polynomial is of the form $F = SC$, where $S$ is a Salem polynomial of degree $d$ with $4 \leq d \leq 22$ and $C$ is a product of cyclotomic polynomial, and moreover

- $\tau(S) = (3, d - 3)$ and $\tau(C) = (0, \text{deg}(C))$.
- $\tau$ satisfies conditions (C 0) and (C 1).
The Salem polynomial $S$ is called the Salem polynomial of $\tau$, and the Salem number of the polynomial $S$ is called the Salem number of $\tau$.

**Definition 15.2.** Let $S$ be a Salem polynomial. Two signature maps $\tau$ and $\tau'$ are said to be equivalent at $S$ if 

$$\tau(f) = \tau'(f)$$

for all factors $f \in \mathbb{R}[X]$ of $S$.

**Notation 15.3.** Let $S$ be a Salem polynomial of degree $d$ with $4 \leq d \leq 22$, and let $z \in \mathbb{C}$ be a root of $S$ with $|z| = 1$. Set $f_z(X) = (X - z)(X - z^{-1})$. We denote by $\tau_{S,z}$ the signature map of maximum $(3, d - 3)$ and polynomial $S$ characterized by $\tau_{S,z}(f_z) = (2, 0)$.

We next define the notion of realizable isometry.

**Definition 15.4.** If $(N, q)$ is an even, negative definite lattice, let 

$$\Delta(N) = \{x \in N \mid q(x, x) = -2\}$$

be the set of roots of $N$; it is a root system (see for instance [E 13], Theorem 1.2). Set $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. For all $x \in \Delta(N)$, let $H_x$ be the hyperplane of $N_{\mathbb{R}}$ orthogonal to $x$, and let $H$ be the union of the hyperplanes $H_x$ for $x \in \Delta(L)$. The complement of $H$ in $N_{\mathbb{R}}$ is a disjoint union of connected open subsets, called the chambers of $N_{\mathbb{R}}$ with respect to $q$. An isometry $t : N \rightarrow N$ is said to be of type (P) if $t$ stabilizes a chamber of $N_{\mathbb{R}}$ with respect to $q$.

**Definition 15.5.** Let $t$ be a semi-simple isometry of an even unimodular lattice $(L, q)$ with characteristic polynomial $SC$, where $S$ is a Salem polynomial of degree $d$ with $4 \leq d \leq 20$, and $C$ is a product of cyclotomic polynomials. Set $L_S = \text{Ker}(S(t))$ and $L_C = \text{Ker}(C(t))$; suppose that $(L_C, q)$ is negative definite. We say that $t$ is realizable if the restriction of $t$ to $L_C$ is of type (P).

Let $S$ be a Salem polynomial of degree $d$ with $4 \leq d \leq 22$, and let $z \in \mathbb{C}$ be a root of $S$ with $|z| = 1$.

**Theorem 15.6.** Suppose that $d \leq 16$ and $d \equiv 0, 4$ or $6 \pmod{8}$. Then $\Lambda_{3,19}$ has a realizable isometry with signature map equivalent to $\tau_{S,z}$.

We start with some preliminary results.

**Lemma 15.7.** Let $(L, q)$ be an even negative definite lattice of rank 2, and set $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$. The lattice $(L, q)$ has an isometry $t : L \rightarrow L$ of type (P) such that 

$$\partial_v[V, q, t] = \partial_v[V, q, -id]$$

for all $v \in \mathcal{V}'$.

**Proof.** There exist integers $D \geq 1$ and $f \geq 1$ such that $\text{det}(L, q) = f^2 D$, where $-D$ is the discriminant of an imaginary quadratic field. If $(L, q)$ contains no roots, then we are done: we can take $t = -id$. Otherwise, the lattice $(L, q)$ is isomorphic to $-q'$, where $q'$ is a quadratic form associated to an order $O$ of the imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$ (see for instance [C 89], Theorem 7.7 (iii)). Complex conjugation induces an isometry of the quadratic form $(O, q')$ with characteristic polynomial $X^2 - 1$. If $D = 3$ and $f = 1$, then $(O, q')$ is isomorphic to the root lattice $A_2$, and complex conjugation is an isometry of type (P) of $(O, q')$ (see [McM 11], §5, Example); if $D = 4$ and $f = 1$, then $(O, q')$ is isomorphic to $A_1 \oplus A_1$, and exchanging the two copies of $A_1$ gives rise to an isometry of type (P) of $(O, q')$; otherwise, $(O, q')$ contains only two roots, fixed by complex conjugation, hence we obtain an isometry $t$ of type
follows from Proposition 15.8 (i).

If $S \equiv Q$ Salem polynomial, we have $L$ for \( \tau \) and by Proposition 2.10, \( t \) is realizable. Assume now that $d \not\equiv 0 \pmod{4}$; then $n$ is even, and hence both $S(1)$ and $S(-1)$ are not squares, and the result follows from Proposition 15.8 (i).

Assume that $|S(1)|$ and $S(-1)$ are both squares, and set $n = d/2$. Since $S$ is a Salem polynomial, we have $S(1) < 0$ and $S(-1) > 0$.

Suppose first that $d \equiv 0 \pmod{4}$; then $n$ is even, and hence $S(1)^n S(1) S(-1) < 0$. If $S$ is unramified at 2, then by Proposition 2.10 the class of $-1^n S(1) S(-1)$ in $\mathbb{Q}^\times / \mathbb{Q}^{2 \times}$ lies in $\{1, -3\}$, and this leads to a contradiction. Therefore $S$ is ramified at 2, and Proposition 15.8 (ii) gives the desired result.

Assume now that $d \equiv 6 \pmod{8}$. Then $n$ is odd, and hence $S(1)^n S(1) S(-1) > 0$. By hypothesis, $|S(1)|$ and $S(-1)$ are both squares, therefore condition (C 1) holds for $S$.

Set $(r, s) = (3, d - 3)$; note that the hypothesis $d \equiv 6 \pmod{8}$ implies that $r \equiv s \pmod{8}$. Since $S$ is irreducible, we have $G_S = 0$; therefore by Corollary 12.3 the lattice $\Lambda_{3, d - 3}$ has an isometry with signature map $\tau_{S,z}$. The identity is a semi-simple
isometry of the lattice $-E_8$; since $\Lambda_{3,19} = \Lambda_{3,3} \oplus (-E_8) \oplus (-E_8) = \Lambda_{3,11} \oplus (-E_8)$, we obtain a realizable isometry, as claimed.

**Proposition 15.9.** Suppose that $d = 22$ and that $|S(1)|$ and $S(-1)$ are both squares. Then $\Lambda_{3,19}$ has a realizable isometry with signature map equivalent to $\tau_{S,z}$.

**Proof.** This is an immediate consequence of Corollary 12.5.

If $d = 10$ or 18 and if $|S(1)|$ and $S(-1)$ are not both squares, then $\Lambda_{3,19}$ has a realizable isometry with signature map equivalent to $\tau_{S,z}$ by Proposition 15.8. The aim of the next sections is to give a necessary and sufficient condition in the case where $|S(1)S(-1)| = 1$, and more generally, when $S$ is unramified. We start with a definition.

Recall that if $\tau$ is a signature map, we use the notation $\tau(X - 1) = (r_+, s_+)$ and $\tau(X + 1) = (r_-, s_-)$. Note that if $\tau$ is a Salem signature map, then $s_+$ and $s_-$ are both even, and hence $D_+ = |F_1(1)|$, $D_- = |F_1(-1)|$.

**Definition 15.10.** Let $\tau$ be a Salem signature map with polynomial $F$. We say that $\tau$ is a Salem signature map with trivial obstruction if $G_F(D_+, D_-) = 0$.

**Theorem 15.11.** Suppose that $d = 10$ or 18 and that $S$ is unramified. Suppose that there exists a Salem signature map with trivial obstruction of maximum $(3,19)$ or $(3,11)$ and Salem factor $S$. Then the lattice $\Lambda_{3,19}$ has a realizable isometry with signature map equivalent to $\tau_{S,z}$.

This result will be proved in the next sections.

16. Preparing for surgery

Let $S$ be a Salem polynomial of degree $d$ with $4 \leq d \leq 18$ such that $d \equiv 2 \pmod{4}$. We assume that $S$ is unramified; for instance, this is the case if $|S(1)S(-1)| = 1$. The aim of this section is to prove a preliminary result that will be useful in the proof of Theorem 15.11 in the next sections.

Let $\Phi$ be a cyclotomic polynomial of even degree, let $k \geq 1$ be an integer, and set $C = \Phi^k$. Assume that $\deg(C) \equiv 4 \pmod{8}$, and that condition (C 1) holds for $C$. Recall that if $t : L \rightarrow L$ is a semi-simple isometry with characteristic polynomial $SC$, we set $L_S = \ker(S(t))$ and $L_C = \ker(\Phi(t))$, $V = L \otimes \mathbb{Z} Q$, $V_S = L_S \otimes \mathbb{Z} Q$ and $V_C = L_C \otimes \mathbb{Z} Q$. Recall that $V$ is the set of places of $Q$ and that $V'$ is the set of finite places.

**Proposition 16.1.** Let $\tau$ be a Salem signature map with trivial obstruction and polynomial $SC$, and let $p \in \Pi_S,\Phi$. Then there exists an even unimodular lattice $L$ having a semi-simple isometry with characteristic polynomial $F$ and signature map $\sigma$ such that $\partial_v([V_S]) = \partial_v([V_C]) = 0$ for all $v \in V'$ with $v \neq v_p$.

**Proof.** Let $L$ be an even unimodular lattice and let $t : L \rightarrow L$ be a semi-simple isometry with signature map $\tau$; these exist by Theorem 12.1. Let $V = L \otimes \mathbb{Z} Q$, and note that $\partial_v([V]) = 0$ for all $v \in V'$. We have $V = V_S \oplus V_C$, therefore for all $v \in V'$, we have

$$\partial_v([V_S]) = 0 \iff \partial_v([V_C]) = 0.$$

Let $U = U(L,t)$ be the set of $v \in V'$ such that $\partial_v([V_S]) \neq 0$. Let $v \in U$; this implies that $v = v_q$ with $q \in \Pi_S,\Phi$. Since $S$ is unramified, the common, irreducible, symmetric factors of $S$ and $\Phi \pmod{q}$ are of even degree; hence $\partial_v([V_S])$ is contained in a product of Witt groups of the type $W(\kappa, \sigma)$, where $\kappa$ is an even degree extension.
of $\mathbb{F}_q$ and $\sigma$ is a non-trivial involution of $\kappa$. Since these Witt groups are of order 2, the (non-trivial) element $\partial_v([V_S])$ is of order 2; moreover, $\partial_v([V_S]) = \partial_v([V_C])$.

Note that $I = \{S, \Phi\}$, and recall that the map $(S, \Phi) : C(I) \to C(I)$ was defined in §7.3, cf. Notation 7.3. Set $E_S = Q[X]/(S)$, $E_\Phi = Q[X]/(\Phi)$ and $E = E_S \times E_\Phi$. Let $(\lambda^v)$ be the local data associated to $V$, and for all $v \in V$, let $a^v = a(\lambda^v) \in \mathcal{C}$. Since $V_S$ and $V_C$ are global forms, we have $\sum_{w \in V} a^v(f) = 0$ for all $f \in I$.

Let $N$ be the cardinality of $\mathcal{U}$. Assume first that $N \geq 2$; we claim that in this case there exists an even unimodular lattice $L'$ and a semi-simple isometry $t : L' \to L'$ with signature map $\tau$ such that the cardinal of $\mathcal{U}(L', t')$ is $N - 2$.

Let $v$ and $v'$ be two distinct elements of $\mathcal{U}$. Set $b^v = (S, \Phi)(a^v)$ and $b^{v'} = (S, \Phi)(a^{v'})$; we have $b^v \in \mathcal{C}$ and $b^{v'} \in \mathcal{C}$; note that $b^v(f) + b^{v'}(f) = a^v(f) + a^{v'}(f)$ for all $f \in I$. Let $\mu^v, \mu^{v'} \in T(E^v, \sigma)$ be such that $b^v = a(\mu^v)$ and $b^{v'} = a(\mu^{v'})$; let us chose $\mu^v = (\mu^v_S, \mu^v_\Phi)$ and $\mu^{v'} = (\mu^{v'}_S, \mu^{v'}_\Phi)$ such that $\partial_v(\mu^v_S) = 0$ and $\partial_v(\mu^{v'}_S) = 0$ (hence also $\partial_v(\mu^v_\Phi) = 0$ and $\partial_v(\mu^{v'}_\Phi) = 0$). For all $w \in V$ with $w \neq v, v'$, set $\mu^w = \lambda^w$, and let $b^w = a(\mu^w) \in \mathcal{C}$; we have $\sum_{w \in V} b^w(f) = \sum_{w \in V} a^w(f)$ for all $f \in I$. This implies that $\sum_{w \in V} b^w(f) = 0$ for all $f \in I$. By Theorem 9.6, there exists an even unimodular lattice $L'$ and a semi-simple isometry $t : L' \to L'$ with signature map $\tau$ and local data $(\mu^w)$. Set $V' = L' \otimes \mathbb{Z} Q$. We have $\partial_v([V'_S]) = \partial_v(\mu^v_S)$, $\partial_v([V'_S]) = \partial_v(\mu^v_\Phi)$, hence $\partial_v([V'_S]) = \partial_v([V'_S]) = 0$. Since $\partial_w([V'_S]) = \partial_w([V'_S])$ if $w \neq v, v'$, this implies that the cardinal of $\mathcal{U}(L', t')$ is $N - 2$, as claimed.

Therefore we can assume that either $N = 0$ or $N = 1$.

Note that the signature of $V_S$ is $(3, d - 3)$, and that $V_C$ is negative definite. Hence $V_S$ and $V_C$ do not contain even unimodular lattices. This implies that there exists at least one $v \in V'$ such that $\partial_v([V'_S]) \neq 0$ and $\partial_v([V'_S]) \neq 0$, and this implies that $N \neq 0$.

Assume that $N = 1$, and let $\mathcal{U} = \{v\}$; if $v = v_p$, we are finished. Suppose that $v \neq v_p$. Set $b^v = (S, \Phi)(a^v)$ and $b^{v_p} = (S, \Phi)(a^{v_p})$; we have $b^v \in \mathcal{C}$ and $b^{v_p} \in \mathcal{C}$; note that $b^v(f) + b^{v_p}(f) = a^v(f) + a^{v_p}(f)$ for all $f \in I$. Let $\mu^v, \mu^{v_p} \in T(E^v, \sigma)$ be such that $b^v = a(\mu^v)$ and $b^{v_p} = a(\mu^{v_p})$; choose $\mu^v$, so that $\partial_v(\mu^v_S) = 0$. For all $v' \in V$ with $v' \neq v, v_p$, set $\mu^{v'} = \lambda^{v'}$, and let $b^{v'} = a(\mu^{v'}) \in \mathcal{C}$; we have $\sum_{w \in V} b^{v'}(f) = \sum_{w \in V} a^{v'}(f)$ for all $f \in I$. This implies that $\sum_{w \in V} b^{v'}(f) = 0$ for all $f \in I$. By Theorem 9.6, there exists an even unimodular lattice $L'$ and a semi-simple isometry $t : L' \to L'$ with signature map $\tau$ and local data $(\mu^w)$. Set $V' = L' \otimes \mathbb{Z} Q$. We have $\partial_v([V'_S]) = \partial_v(\mu^v_S)$, hence $\partial_v([V'_S]) = \partial_v([V'_S]) = 0$. Since $\partial_w([V'_S]) = \partial_w([V'_S])$ if $w \neq v, v_p$, this implies that $\partial_w([V'_S]) = 0$ for all $w \in V'$ with $w \neq v_p$. This completes the proof of the proposition.

17. Unramified Salem polynomials of degree 10 or 18

Let $S$ be an unramified Salem polynomial of degree $d$, with $d = 10$ or 18.

Theorem 17.1. Let $\tau$ be a Salem signature map with trivial obstruction of maximum $(3, d + 1)$ and Salem factor $S$. Then the lattice $\Lambda_{3,19}$ has a realizable isometry with signature map equivalent to $\tau$. Moreover, if $d = 10$, then this signature map can be chosen so that its polynomial is divisible by $(X - 1)^8$. 
Proof. If \( m \geq 1 \) is an integer, we denote by \( \Phi_m \) the \( m \)-th cyclotomic polynomial. Let \( F \) be the polynomial associated to \( \tau \). We have \( G_F = 0 \) by definition; since \( |S(1)S(-1)| = 1 \), this implies that the polynomial \( F \) is divisible by a cyclotomic polynomial \( \Phi_m \) with \( m = 3, 4, 6 \) or 12 such that \( \Pi_S, \Phi_m \neq \emptyset \). Let \( \Phi_m \) be such a polynomial, and let \( p \) be a prime number such that \( p \in \Pi_S, \Phi_m \). Let \( f \in \mathbb{F}_p[X] \) be an irreducible, symmetric common factor of \( S \) and \( \Phi_m \) (mod \( p \)); the polynomial \( f \) has even degree. Let \( M = \mathbb{F}_p[X]/(f) \), considered as a simple \( \mathbb{F}_p[\Gamma] \)-module, and let \( [M] \) denote the only non-trivial element of \( W\Gamma(\mathbb{F}_p, M) \).

Set \( C = \Phi^2_n \) if \( m = 3, 4 \) or 6 and \( C = \Phi_m \) if \( m = 12 \), and let \( \tau' \) be the Salem signature map of maximum \((3, d + 1)\) and associated polynomial \( SC \) such that \( \tau'(g) = \tau(g) \) for all factors \( g \in \mathbb{R}[X] \) of \( S \). Let \( t : L \to L \) be a semi-simple isometry with signature map \( \tau' \) of an even unimodular lattice \( L \) such that that \( \partial_v([V_S]) = \partial_v([V_C]) = 0 \) for all \( v \in \mathcal{V}' \) with \( v \neq v_p \); these exist by Proposition 16.1. Note that \( \tau' \) is equivalent to \( \tau \).

Suppose first that \( m = 4 \), and note that with the notation above, the polynomial \( f \) is the image of \( \Phi_4 \) in \( \mathbb{F}_p[X] \), and that \( p \equiv 3 \) (mod \( 4 \)). We now construct a negative definite lattice \( N \) and a semi-simple isometry \( t_N : N \to N \) of type \((P)\) such that

\[
\partial_v([N \otimes \mathbb{Z} Q]) = \partial_v([V_C])
\]

for all \( v \in \mathcal{V}' \).

Let \( \mathcal{O} \) be the ring of integers of the quadratic field \( \mathbb{Q}(\sqrt{-p}) \), and let \( q_\mathcal{O} : \mathcal{O} \times \mathcal{O} \to \mathbb{Z} \) be the associated binary quadratic form; \( q_N \) is even, positive definite, and has determinant \( p \). Let \( N = \mathcal{O} \oplus \mathcal{O} \) and let \( q_N = (\mathcal{O} \mathcal{O}) \oplus (\mathcal{O} \mathcal{O}) \). For all \( x \in \mathcal{O} \), we denote by \( x \mapsto \overline{x} \) the complex conjugation. Let \( t_N : N \to N \) be the isometry of \( q_N \) given by \( t_N(x, y) = (y, \overline{x}) \) for all \( x, y \in \mathcal{O} \). The characteristic polynomial of \( t_N \) is \( X^4 - 1 = (X^2 - 1)(X^2 + 1) \), and \( t_N \) is of type \((P)\). We have \( \partial_v([N \otimes \mathbb{Z} Q]) = \partial_v([V_C]) \) for all \( v \in \mathcal{V}' \), as claimed; hence \( \partial_v([N \otimes \mathbb{Z} Q]) = \partial_v([V_S]) \) for all \( v \in \mathcal{V}' \).

Set \( V' = V_S \oplus (N \otimes \mathbb{Z} Q) \), and let \( t' : V' \to V' \) be the isometry such that the restriction of \( t' \) to \( V_S \) is \( t \), and the restriction to \( N \) is \( t_N \). The lattice \( L_S \oplus N \) is stabilized by \( t' \). Note that \( (L_S \oplus N) \otimes \mathbb{Z} \mathbb{Z}_q \) is unimodular if \( q \neq p \), and it is even for \( q = 2 \).

Since \( \partial_{v_p}([N \otimes \mathbb{Z} Q]) = \partial_{v_p}([V_S]) \), we have \( \partial_{v_p}[V'] = 0 \), therefore \( V' \otimes \mathbb{Q} \mathbb{Q}_p \) contains a unimodular \( \mathbb{Z}_p \)-lattice stabilized by \( t' \); let \( L'_p \) be such a lattice. If \( q \) is a prime number \( q \neq p \), set \( L'_q = (L_S \oplus N) \otimes \mathbb{Z} \mathbb{Z}_q \), and let \( L' = \bigcap_{q} L'_q \), where \( q \) runs over all the prime numbers. The lattice \( L' \) is even, unimodular, and the signature map of \( t' \) is equivalent to \( \tau' \).

If \( d = 18 \), then \( L' \) is of rank 22, isomorphic to \( \Lambda_{3,19} \), hence the theorem is proved in this case. Assume that \( d = 10 \); in this case, \( L' \) is of rank 14. Set \( L'' = L' \oplus (-E_8) \), and let \( t'' : L'' \to L'' \) be the isometry such that the restriction of \( t'' \) to \( L' \) is equal to \( t' \), and that \( t'' \) is the identity on \(-E_8 \). The lattice \( L'' \) is isomorphic to \( \Lambda_{3,19} \), the isometry \( t'' \) is realizable, and its signature map is equivalent to \( \tau' \), hence the theorem is proved in this case as well.

Suppose that \( m = 3 \). Since \( S \) is unramified, the image of the polynomial \( \Phi_3 \) in \( \mathbb{F}_p[X] \) is the common irreducible factor of \( S \) and \( \Phi_3 \) (mod \( p \)). With the notation above, we have \( \partial_{v_p}([V_S]) = \partial_{v_p}([V_C]) = [M] \), and \( \partial_{v_p}([V_S]) = \partial_{v_p}([V_C]) = 0 \) if \( v \neq v_p \). If \( L_C \) has no roots, then the restriction of \( t \) to \( L_C \) is of type \((P)\); suppose that \( L_C \) has roots.
Assume first that $p = 2$; in this case, we have $L_C \simeq -D_4$, where $D_4$ is the lattice associated to the root system $D_4$ (see for instance [13]). Let $t' : L_C \to L_C$ be the trilateral isometry of order 3 of $D_4$, and note that $t'$ is of type (P). Let $V$ denote the $\mathbb{Q}[\Gamma]$-bilinear form on $L_C \otimes \mathbb{Z} \mathbb{Q}$ obtained by letting a generator $\gamma$ of $\Gamma$ act via $t'$; we have $\partial_{\gamma_2}(V) = \{M = \partial_{\gamma_2}([V_C]),$ and hence $\partial_{\gamma_2}([V]) = \partial_{\gamma_2}([V_S])$. Let $t' : V_S \oplus V \to V_S \oplus V$ be defined by $t'(x, y) = (t(x), t'(y))$ for all $x \in V_S$ and $y \in V$. Since $\partial_{\gamma_2}([V]) = \partial_{\gamma_2}([V_S])$, we have $t'(L) = L$. The isometry $t' : L \to L$ is realizable, and its signature map is similar to $\tau'$.

Suppose now that $p \neq 2$, and note that $p \neq 3$. Still assuming that $L_C$ has roots, this implies that $L_C$ contains a sublattice of rank 2 isomorphic to the root lattice $A_2$. Let $L_0 \subset L_C$ be this sublattice, and let $L_1 \subset L_C$ be its orthogonal complement in $L_C$; we have $t(L_1) = L_1$. Set $V_0 = L_0 \otimes \mathbb{Z} \mathbb{Q}$ and $V_1 = L_1 \otimes \mathbb{Z} \mathbb{Q}$. We have $\partial_{\gamma_p}[V_0] = 0$ and $\partial_{\gamma_3}[V_0] = ([\mathbb{F}_3, -1, id])$. This implies that $\partial_{\gamma}[V_1] = 0$ for $v \neq \gamma_3, \gamma_p$, that $\partial_{\gamma_3}[V_1] = ([\mathbb{F}_3, 1, id])$, and $\partial_{\gamma_p}[V_1] = \partial_{\gamma_p}[V_C]$. Let $t' : L_C \to L_C$ be determined by $t'|L_0 = id$ and $t'|L_1 = t$. The isometry $t'$ is of type (P), and determines an isometry $t' : L \to L$ of signature map $\tau'$.

The argument is the same for $m = 6$. In both cases, if $d = 18$, then $L$ is isomorphic to $A_{3,19}$, hence the theorem is proved. If $d = 10$, then the same argument as in the case $m = 4$ provides a realizable isometry of $A_{3,19}$ with signature map equivalent to $\tau'$.

Assume now that $m = 12$. If $L_C$ has roots, then either $p = 2$ and $L_C$ is isomorphic to $-D_4$, or $p = 3$, and $L_C$ is isomorphic to $-A_2 \oplus -A_2$. In the first case, let $t' : L_C \to L_C$ be obtained from the trilateral isometry of order 3 of $D_4$; this isometry is of type (P). We have $\partial_{\gamma}[V_C, t'] = \partial_{\gamma}[V_C, t]$ for all $v \in \mathcal{V'}$, therefore the lattice $L$ has a realizable isometry $t' : L \to L$ with signature map equivalent to $\tau'$. If $p = 3$ and $L_C \simeq -A_2 \oplus -A_2$, then we have $\partial_{\gamma_3}[V_C] = ([\mathbb{F}_3^2, q, \overline{7}])$, the characteristic polynomial of $\overline{7}$ being the image of $\Phi_4$ in $\mathbb{F}_3[X]$. We identify the lattice $A_2$ to $\mathbb{Z}[\zeta_3] \times \mathbb{Z}[\zeta_3] \to \mathbb{Z}$, sending $(x, y)$ to $Tr_{\mathbb{Q}(\zeta_3)}(x\overline{y})$, where $x \mapsto \overline{x}$ is the complex conjugation of the cyclotomic field $\mathbb{Q}(\zeta_3)$. Let $t' : L_C \to L_C$ be given by $t'(x, y) = (y, \overline{x})$ for $x, y \in A_2$; we have $\partial_{\gamma_3}[V_C, t'] = \partial_{\gamma_3}[V_C, t]$. The lattice $L$ has a realizable isometry with signature map equivalent to $\tau'$. If $d = 18$, then $L$ is isomorphic to $A_{3,19}$, hence the theorem is proved. If $d = 10$, then the same argument as in the case $m = 4$ gives us a realizable isometry of $A_{3,19}$ with signature map equivalent to $\tau'$.

18. Isometries of root lattices

In the following section, we need more information on isometries of root lattices of characteristic polynomial a power of a cyclotomic polynomial. This question was studied in several papers, in particular [BM 94], théorème A.2. and [B 99], §3 (see also [BB 92], [B 84]).

**Proposition 18.1.** Let $\ell$ be an odd prime number. There exists a root lattice $\Lambda$ having an isometry with minimal polynomial $\Phi_\ell$ or $\Phi_{2\ell}$ if and only if the irreducible components of $\Lambda$ are

(a) of type $A_{\ell-1}$,

(b) of type $A_4$ or $E_8$ if $\ell = 5$,

(c) $A_2, D_4, E_6$ or $E_8$ if $\ell = 3$. 
Proof. See [BM 94], théorème A.2.

Proposition 18.2. Let \( \ell \) be an odd prime, and let \( \Lambda \) be a root lattice of type \( A_{\ell-1} \) having an isometry with characteristic polynomial \( \Phi_\ell \) (resp. \( \Phi_{2\ell} \)). Set \( V = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q} \) and let \( [V] \in W_{\Gamma}(\mathbb{Q}) \) be the Witt class of the \( \mathbb{Q}[\Gamma] \)-form \( V \). Then \( \partial_p[V] = 0 \) if \( p \) is a prime number with \( p \neq \ell \), and

\[
\partial_\ell[V] = ([F_\ell, (1), id]),
\]
resp.

\[
\partial_\ell[V] = ([F_\ell, (1), -id]).
\]

Proof. This is an easy computation.

The following observation will be used several times:

Proposition 18.3. Let \( L \) be a negative definite lattice with an isometry \( t : L \to L \), \( V = L \otimes_{\mathbb{Z}} \mathbb{Q} \). Let \( \ell \) be an odd prime number. Let \( \Lambda \subset L \) be a lattice stabilized by \( t \), assume that \( \Lambda \) is isometric to an orthogonal sum of lattices \( -A_{\ell-1} \), and that the minimal polynomial of the restriction of \( t \) to \( \Lambda \) is \( \Phi_\ell \) or \( \Phi_{2\ell} \). Let \( V' \) be the orthogonal of \( \Lambda \otimes_{\mathbb{Z}} \mathbb{Q} \) in \( V_L \). Then

\[
\partial_p[V] = \partial_p[V']
\]
for all prime numbers \( p \neq \ell \).

Proof. Set \( V_\Lambda = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q} \). Let \( p \) be a prime number with \( p \neq \ell \). We have

\[
\partial_p[V] = \partial_p[V_\Lambda] + \partial_p[V'].
\]
Since \( p \neq \ell \), by Proposition [IS 2] we have \( \partial_p[V_\Lambda] = 0 \), hence \( \partial_p[V] = \partial_p[V'] \), as claimed.

We need a similar result for the lattice \( E_6 \).

Proposition 18.4. (a) Let \( \Lambda \) be a root lattice of type \( E_6 \) having an isometry with characteristic polynomial \( \Phi_9 \) (resp. \( \Phi_{18} \)). Set \( V = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q} \) and let \( [V] \in W_{\Gamma}(\mathbb{Q}) \) be the Witt class of the \( \mathbb{Q}[\Gamma] \)-form \( V \). Then \( \partial_p[V] = 0 \) if \( p \) is a prime number with \( p \neq 3 \).

(b) Let \( L \) be a negative definite lattice with an isometry \( t : L \to L \), \( V = L \otimes_{\mathbb{Z}} \mathbb{Q} \). Let \( \Lambda \subset L \) be a lattice stabilized by \( t \), assume that \( \Lambda \) is isometric to an orthogonal sum of lattices \( -E_6 \), and that the minimal polynomial of the restriction of \( t \) to \( \Lambda \) is \( \Phi_9 \) or \( \Phi_{18} \). Let \( V' \) be the orthogonal of \( \Lambda \otimes_{\mathbb{Z}} \mathbb{Q} \) in \( V_L \). Then

\[
\partial_p[V] = \partial_p[V']
\]
for all prime numbers \( p \neq 3 \).

Proof. Part (a) is an immediate consequence of the definition of the root lattice \( E_6 \), and the proof of (b) is the same argument as the one of the proof of Proposition [IS 3]
19. Unramified Salem polynomials of degree 10

Let $S$ be an unramified Salem polynomial of degree 10. The case of Salem signature maps of Salem factor $S$ and maximum $(3,11)$ was treated in Section 17; we now consider those of maximum $(3,19)$.

**Theorem 19.1.** Let $\tau$ be a Salem signature map with trivial obstruction of Salem factor $S$ and maximum $(3,19)$. Then the lattice $\Lambda_{3,19}$ has a realizable isometry with signature map equivalent to $\tau$.

**Proof.** Let $F$ be the polynomial associated to $\tau$; we have $G_F = 0$ by definition. Since $S$ is unramified, this implies that the polynomial $F$ is divisible by a cyclotomic polynomial $\Phi_m$ with $m \geq 3$ such that $\Pi_{S,\Phi_m} \neq \emptyset$. Let $\Phi_m$ be such a polynomial, and let $p$ be a prime number such that $p \in \Pi_{S,\Phi_m}$. Let $f \in \mathbb{F}_p[X]$ be an irreducible, symmetric common factor of $S$ and $\Phi_m \pmod{p}$, and let $M = \mathbb{F}_p[X]/(f)$, considered as a simple $\mathbb{F}_p[\Gamma]$-module. Note that the polynomial $f$ has even degree, and let $[M]$ denote the only non-trivial element of $W_T(\mathbb{F}_p,M)$.

The case where $m = 3,4,6$ or 12 was already treated in Theorem 17.1. In the remaining cases, we construct a polynomial $C$ of degree 12, product of cyclotomic polynomials and divisible by $\Phi_m$, such that $SC$ satisfies conditions (C 0) and (C 1) and that $G_{SC} = 0$. We then consider the Salem signature map $\tau_C$ of maximum $(3,19)$ and associated polynomial $SC$ such that $\tau_C(g) = \tau(g)$ for all factors $g \in \mathbb{R}[X]$ of $S$. The signature map $\tau_C$ is equivalent to $\tau$. Let $t : L \to L$ be a semi-simple isometry with signature map $\tau_C$ of an even unimodular lattice $L$ such that $\partial_v([V_S]) = \partial_v([V_C]) = 0$ for all $v \in \mathcal{V}'$ with $v \neq v_p$; these exist by Proposition 16.1. Note that $\tau_C$ is equivalent to $\tau$. If $L_C$ has no roots, then $t$ is realizable, and we are done. Assume that $L_C$ contains roots; we then construct a realizable isometry with signature map equivalent to $\tau_C$. This strategy is the same in all cases, but the polynomial $C$ and the construction of the realizable isometry vary slightly from case to case.

**Case 1:** Suppose that $m = 5,7,8,9,11,14,18$ or 22. Set $C = \Phi_m^2$ if $m = 7,9,14$ or 18, $C(X) = \Phi_m(X)(X-1)^2$ if $m = 5$ or 8, $C(X) = \Phi_m(X)(X-1)^2$ if $m = 11$ and $C(X) = \Phi_m(X)(X+1)^2$ if $m = 22$.

Suppose that $L_C$ contains a root lattice $L_0$ with minimal polynomial $\Phi_m$, and let $L_1$ be the orthogonal of $L_0$ in $L_C$. Set $V_1 = L_1 \otimes \mathbb{Q}$. Then by Propositions 18.3 and 18.4 we have $\partial_{v_p}[V_C] = \partial_{v_p}[V_1]$; note that if $m = 8$, then $\partial_v[V_C] = \partial_v[V_1]$ for all $v \in \mathcal{V}'$. If $m = 11$ or $m = 22$, then this leads to a contradiction, hence $t$ is realizable. Otherwise, let $t' : L_C \to L_C$ be such that $t'|L_0 = id$ and $t'|L_1 = t|L_1$. The isometry $t'$ is of type (P), and determines a realizable isometry $t : L \to L$ with signature map equivalent to $\tau_C$.

**Case 2:** Suppose that $m = 10$, and set $C(X) = \Phi_{10}(X)(X-1)^6(X+1)^2$; let $L'$ be the rank 2 sublattice of $L_C$ given by $L' = \text{Ker}(t+1)$, and note that the restriction of $t$ to $L'$ is $-id$. By Lemma 15.7 the lattice $L'$ has an isometry $t'$ of type (P) such that $\partial_v[L_1 \otimes \mathbb{Q}, t_1] = \partial_v[L_1 \otimes \mathbb{Q}, -id]$. We now continue as in Case 1: the isometry $t'$ obtained with this method is of type (P).

**Case 3:** Suppose that $m = 15,16,20,24$ or 30. Set $C = \Phi_m \Phi_2$ if $m = 15$ or 16, $C(X) = \Phi_m(X)(X-1)^2(X+1)^2$ if $m = 20$ and $C = \Phi_m \Phi_4$ if $m = 30$. Suppose that $L_C$ contains a root lattice $L_0$ isomorphic to $-E_8$ with minimal polynomial $\Phi_m$, and let $L_1$ be the orthogonal of $L_0$ in $L_C$. Set $V_1 = L_1 \otimes \mathbb{Q}$. Since $E_8$ is unimodular, we have $\partial_v[V_C] = \partial_v[V_1]$ for all $v \in \mathcal{V}'$. This
implies that \( \Pi S, \Phi \neq \emptyset \) with \( n = 3, 4 \) or 6, hence we obtain a realizable isometry by Theorem 17.1. If \( m = 16 \) and \( L_C \) contains a root lattice isomorphic to \(-D_8\), then \( \partial_{\nu}[V_C] = \partial_{\nu}[V_1] \) for all \( v \in V' \), and this leads to a contradiction. Otherwise, \( L_C \) contains a root lattice isomorphic to \( A_{\ell-1} \) for \( \ell = 3 \) or 5, and we conclude as in Case 1.

**Case 4 :** Assume that \( m = 21, 28, 36 \) or 42, and set \( C = \Phi_m \). Suppose that \( L_C \) contains a root lattice; this lattice is an orthogonal sum of root lattices of type \( A_{\ell-1} \) for \( \ell = 3 \), 5 or 7, and this implies that \( \partial_{\nu}[V_C] \) is non-trivial. Hence \( \partial_{\nu}[V_C] \) is also non-trivial, and therefore \( \Pi S, \Phi \neq \emptyset \) for some cyclotomic polynomial \( \Phi \) of degree at most 6; we conclude by applying Theorem 17.1, or Cases 1 or 2.

**Part III : Automorphisms of \( K_3 \) surfaces**

This part combines the arithmetic results of the first two parts to obtain applications to automorphisms of \( K_3 \) surfaces. We start by recalling some definitions and results (see §20); in particular, the results of McMullen in \([\text{McM 11}]\) needed in the following sections, and some consequences of fundamental results on \( K_3 \) surfaces, such as the strong Torelli theorem and the surjectivity of the period map.

The following sections concern realization of Salem numbers; we prove the results announced in the introduction, and §25 gives some finiteness results.

The last two sections contain some remarks on automorphisms of projective \( K_3 \) surfaces: examples of Salem numbers that cannot be realized by automorphisms of projective \( K_3 \) surfaces in §26 and an alternative approach to some results of Kondo \([\text{K 92}]\) and Vorontsov \([\text{V 83}]\) in §27.

### 20. \( K_3 \) surfaces

The aim of this section is to give some basic facts on \( K_3 \) surfaces and their automorphisms in a more general framework than needed in this paper; we refer to \([\text{H 16}], [\text{K 20}], [\text{Ca 14}], [\text{Ca 99}], [\text{Ca 01}], [\text{O 07}], [\text{O 08}]\) for details.

A \( K_3 \) surface \( \mathcal{X} \) is a simply-connected compact complex surface with trivial canonical bundle. The dimension of the complex vector space \( H^2(\mathcal{X}, \mathbb{C}) \) is 22, and we have the Hodge decomposition

\[
H^2(\mathcal{X}, \mathbb{C}) = H^{2,0}(\mathcal{X}) \oplus H^{1,1}(\mathcal{X}) \oplus H^{0,2}(\mathcal{X})
\]

with \( \dim H^{2,0} = \dim H^{0,2} = 1 \). The Picard group of \( \mathcal{X} \) is by definition

\[
\text{Pic}(\mathcal{X}) = H^2(\mathcal{X}, \mathbb{Z}) \cap H^{1,1}(\mathcal{X}).
\]

The intersection form \( H^2(\mathcal{X}, \mathbb{Z}) \times H^2(\mathcal{X}, \mathbb{Z}) \to \mathbb{Z} \) of \( \mathcal{X} \) is an even unimodular lattice of signature \((3, 19)\); the signature of its restriction to \( \text{Pic}(\mathcal{X}) \) depends on the geometry of the surface. Let \( a(\mathcal{X}) \) be the algebraic dimension of \( \mathcal{X} \) (see for instance \([\text{K 20}], 3.2\)). The following result is well-known (see \([\text{H 16}], \text{Chapter 17, Proposition 1.3}, \text{or [K 20], Proposition 4.11}; \text{see also [N 79], 3.2} \)).

**Proposition 20.1.** Let \( \rho(\mathcal{X}) \) be the rank of \( \text{Pic}(\mathcal{X}) \).

(a) If \( a(\mathcal{X}) = 2 \), then \( \text{Pic}(\mathcal{X}) \) is non-degenerate and its signature is

\[
(1, \rho(\mathcal{X}) - 1).
\]

(b) If \( a(\mathcal{X}) = 1 \), then \( \text{Pic}(\mathcal{X}) \) has a one-dimensional kernel, and the quotient by the kernel is negative definite.
(c) If \( a(\mathcal{X}) = 0 \), then \( \text{Pic}(\mathcal{X}) \) is negative definite.

The global Torelli theorem implies that \( K3 \) surfaces are classified in terms of their Hodge structures and intersection pairing (see for instance [H 16, Theorem 2.4]):

**Theorem 20.2.** Two \( K3 \) surfaces \( \mathcal{X} \) and \( \mathcal{X}' \) are isomorphic if and only if there exists an isomorphism \( H^2(\mathcal{X}, \mathbb{Z}) \to H^2(\mathcal{X}', \mathbb{Z}) \) respecting the Hodge decomposition and the intersection form.

An automorphism of a \( K3 \) surface \( \mathcal{X} \) is a biholomorphic map \( T : \mathcal{X} \to \mathcal{X} \); such a map induces an isomorphism \( T^* : H^2(\mathcal{X}, \mathbb{Z}) \to H^2(\mathcal{X}, \mathbb{Z}) \) preserving the intersection form, the Hodge decomposition, as well as the \( \text{Kähler cone} \) \( C_X \), that is, the subset of \( H^{1,1}(\mathcal{X}) \) consisting of the \( \text{Kähler classes} \) of \( \mathcal{X} \) (see for instance [H 16, Chapter 8, §5]). The Torelli theorem implies that the map sending \( T \) to \( T^* \) induces an isomorphism

\[
\text{Aut}(\mathcal{X}) \to \text{Aut}(H^2(\mathcal{X}), H^{2,0}(\mathcal{X}), C_X)
\]

(see for instance [H 16, Chapter 15, Theorem 2.3]).

Recall from the introduction that we denote by \( \lambda(T) \) the dynamical degree of \( T \), and by \( \omega(T) \) the complex number such that \( T^* : H^{2,0}(\mathcal{X}) \to H^{2,0}(\mathcal{X}) \) is multiplication by \( \omega(T) \).

Following McMullen (cf. [McM 11, §6]), we introduce the notion of \( K3 \) structure of a lattice; if an isometry preserves this structure, then it is induced by an automorphism of a \( K3 \) surface.

**Definition 20.3.** Let \( (L, q) \) be an even unimodular lattice of signature \((3, 19)\). A \( K3 \) structure on \( L \) consists of the following data:

1. A Hodge decomposition

\[
L \otimes_{\mathbb{Z}} \mathbb{C} = L_{2,0} \oplus L_{1,1} \oplus L_{0,2}
\]

of complex vector spaces such that \( L_{i,j} = L_{j,i} \), and that the hermitian spaces \( L_{1,1} \) and \( L_{2,0} \oplus L_{0,2} \) have signatures \((1, 19)\) and \((2, 0)\), respectively.

2. A positive cone \( C \subset L_{1,1} \cap (L \otimes_{\mathbb{Z}} \mathbb{R}) \), consisting of one of the two components of the locus \( q(x, x) > 0 \).

3. A set of positive roots \( \Psi^+ \subset \Psi = \{ x \in L \cap L_{1,1} \mid q(x, x) = -2 \} \) such that \( \Psi = \Psi^+ \cup (-\Psi^+) \), and that the set (called the Kähler cone)

\[
C(L) = \{ x \in C \mid q(x, y) > 0 \text{ for all } y \in \Psi^+ \}
\]

is not empty.

We say that a \( K3 \) structure is realized by a \( K3 \) surface \( \mathcal{X} \) if there exists an isomorphism \( \iota : L \to H^2(\mathcal{X}, \mathbb{Z}) \) sending \( L_{i,j} \) to \( H^{i,j}(\mathcal{X}) \), and sending \( C(L) \) to the Kähler cone \( C_X \); an isometry \( t : L \to L \) is realized by an automorphism \( T : \mathcal{X} \to \mathcal{X} \) if \( t \) can be chosen so that \( \iota \circ T^* = t \circ \iota \).

Note that if a \( K3 \) structure on \( L \) is realized by \( \mathcal{X} \), then \( \text{Pic}(\mathcal{X}) = \iota(L \cap L_{1,1}) \). The strong Torelli theorem and the surjectivity of the period map imply the following (see [McM 11, Theorem 6.1]):

**Theorem 20.4.** Let \( L \) be an even unimodular lattice of signature \((3, 19)\). Then every \( K3 \) structure on \( L \) is realized by a unique \( K3 \) surface \( \mathcal{X} \), and every isometry of \( L \) preserving the \( K3 \) structure is realized by a unique automorphism of \( \mathcal{X} \).
Recall that the transcendental lattice of a K3 surface $\mathcal{X}$ is the primitive sublattice $\text{Trans}(\mathcal{X})$ of minimal rank of $H^2(\mathcal{X}, \mathbb{Z})$ such that the vector space $\text{Trans}(\mathcal{X}) \otimes \mathbb{C}$ contains $H^{2,0}(\mathcal{X}) \oplus H^{0,2}(\mathcal{X})$. The sublattices $\text{Pic}(\mathcal{X})$ and $\text{Trans}(\mathcal{X})$ of $H^2(\mathcal{X}, \mathbb{Z})$ are orthogonal to each other; moreover, if $a(\mathcal{X}) = 0$ or $2$, then $\text{Pic}(\mathcal{X}) \oplus \text{Trans}(\mathcal{X})$ is a sublattice finite index in $H^2(\mathcal{X}, \mathbb{Z})$.

The following result is due to Oguiso, [Ogu 08], Theorem 2.4 (1):

**Lemma 20.5.** Let $T : \mathcal{X} \to \mathcal{X}$ be an automorphism of a K3 surface. The minimal polynomial of the restriction of $T^*$ to $\text{Trans}(\mathcal{X})$ is irreducible.

**Proof.** This is [Ogu 08], Theorem 2.4 (1); we give a proof for the convenience of the reader. Let $f \in \mathbb{Z}[X]$ be an irreducible factor of the minimal polynomial of the restriction of $T^*$ to $\text{Trans}(\mathcal{X})$, and suppose that $\omega(T)$ is a root of $f$. Let $N = \text{Ker}(f(T^*))$; then $N \otimes \mathbb{C}$ contains $H^{2,0}(X) \oplus H^{0,2}(X)$. By the minimality property of $\text{Trans}(\mathcal{X})$, this implies that $N = \text{Trans}(\mathcal{X})$.

The existence of an automorphism of dynamical degree $> 1$ imposes some restrictions on the K3 surface; in particular, its algebraic dimension is 0 or 2 (see Corollary 20.7).

**Proposition 20.6.** Let $T : \mathcal{X} \to \mathcal{X}$ be an automorphism with $\lambda(\mathcal{X}) > 1$, and let $F$ be the characteristic polynomial of $T$. Then we have

(a) $F = SC$, where $S$ is a Salem polynomial, and $C$ is a product of cyclotomic polynomials.

(b) Suppose that $\omega(T)$ is a root of $S$. Then the signature of $\text{Pic}(\mathcal{X})$ is $(0, \rho(\mathcal{X}))$, and $\text{Pic}(\mathcal{X}) = \text{Ker}(C(T^*))$; the signature of $\text{Trans}(\mathcal{X})$ is $(3, 19 - \rho(\mathcal{X}))$ and $\text{Trans}(\mathcal{X}) = \text{Ker}(S(T^*))$.

(c) If $\omega(T)$ is a root of unity, then the signature of $\text{Pic}(\mathcal{X})$ is $(1, \rho(\mathcal{X}) - 1)$ and the signature of $\text{Trans}(\mathcal{X})$ is $(2, 20 - \rho(\mathcal{X}))$.

(d) $T^*$ is semi-simple.

**Proof.** (a) Recall that by [McM 02], Corollary 3.3, the polynomial $F$ is a product of at most one Salem polynomial and of cyclotomic polynomials. Since we are assuming that $\lambda(T) > 1$, this implies that $F$ is divisible by a Salem polynomial.

Let $L = H^2(\mathcal{X}, \mathbb{Z})$, $L_S = \text{Ker}(S(T^*))$ and $L_C = \text{Ker}(C(T^*))$. The lattices $L_S$ and $L_C$ are orthogonal to each other, and $L_S \oplus L_C$ is of finite index in $L$. Set $d = \deg(S)$.

(b) $L_S \otimes \mathbb{C}$ contains $H^{2,0} \oplus H^{0,2}$, since $\omega(T)$ is a root of $S$, therefore the signature of $L_S$ is $(3, d - 3)$; we have $\text{Trans}(\mathcal{X}) = L_S$ by the minimality of the lattice $\text{Trans}(\mathcal{X})$. This implies that $\text{Pic}(\mathcal{X}) = L_C$, and the signature of $\text{Pic}(\mathcal{X})$ is $(0, \rho(\mathcal{X}))$.

(c) Since $\omega(T)$ is a root of unity, the lattice $L_S$ is orthogonal to $H^{2,0} \oplus H^{0,2}$, and therefore $L_S$ is a sublattice of $\text{Pic}(\mathcal{X})$. This implies that the signature of $\text{Pic}(\mathcal{X})$ is $(1, \rho(\mathcal{X}) - 1)$, and the signature of $\text{Trans}(\mathcal{X})$ is $(2, 20 - \rho(\mathcal{X}))$.

(d) If $\omega(T)$ is a root of $S$, the lattice $L_C$ is negative definite, therefore the restriction of $T^*$ to $L_C$ is semi-simple; the restriction of $T^*$ to $L_S$ is also semi-simple, because $S$ is irreducible. Suppose that $\omega(T)$ is a root of unity. Then $S$ divides the minimal polynomial of $T^*|\text{Pic}(\mathcal{X})$, and the signature of $\text{Pic}(\mathcal{X})$ is $(1, \rho(\mathcal{X}) - 1)$; this implies that the restriction of $T^*$ to $\text{Pic}(\mathcal{X})$ is semi-simple. The restriction of $T^*$ to $\text{Trans}(\mathcal{X})$ is semi-simple by Lemma 20.5.
Corollary 20.7. If \( X \) has an automorphism \( T \) with \( \lambda(T) > 1 \), then \( a(X) = 0 \) or \( a(X) = 2 \).

Proof. This follows from Theorem 20.1 combined with Proposition 20.6 (b) and (c).

21. Realizable Salem numbers

We now return to the main topic of this paper; following McMullen ([McM 02, McM 11, McM 16]), we investigate the following question:

Question 21.1. Which Salem numbers occur as dynamical degrees of automorphisms of complex K3 surfaces?

More precisely, which pairs \( (\lambda(T), \omega(T)) \) occur? This leads to the definition:

Definition 21.2. Let \( \alpha \) be a Salem number, and let \( \delta \in \mathbb{C}^\times \) be such that \( |\delta| = 1 \). We say that the pair \( (\alpha, \delta) \) is realizable if there exists an automorphism \( T \) of a K3 surface with \( \lambda(T) = \alpha \) and \( \omega(T) = \delta \).

The following implies Theorem 3 of the introduction:

Theorem 21.3. Let \( S \) be Salem polynomial of degree \( d \) with \( 4 \leq d \leq 18 \), and let \( \delta \in \mathbb{C}^\times \) be such that \( |\delta| = 1 \). Assume that one of the following holds

(a) \( d \equiv 0, 4 \) or \( 6 \) (mod 8).

(b) \( d \equiv 2 \) (mod 8) and \( |S(1)|, S(−1) \) are not both squares.

Then \( (\alpha, \delta) \) is realizable.

The proof uses the results of Parts I and II, as well as a criterion of McMullen; the following result is a reformulation of this criterion in the case we need here. See Definition 15.5 for the notion of a realizable isometry.

Theorem 21.4. Let \( L \) be an even unimodular lattice of signature \( (3, 19) \), and let \( t : L \to L \) be a realizable isometry with signature map equivalent to \( \tau_{S, \delta} \) and characteristic polynomial \( SC \), where \( C \) is a product of a finite number of cyclotomic polynomials such that \( \deg(C) = 22 - d \). Set \( L_C = \text{Ker}(C(t)) \).

Then there exists a K3 surface automorphism \( T : \mathcal{X} \to \mathcal{X} \) with \( t = T^* \). Moreover:

(a) We have \( \lambda(T) = \alpha \) and \( \omega(T) = \delta \).

(b) The Picard lattice of the K3 surface \( \mathcal{X} \) is isomorphic to \( L_C \).

Proof. Let us check that the hypotheses of [McM 11], Theorem 6.2 are fulfilled. We have \( \alpha > 1 \) by definition, and \( \alpha \) is an eigenvalue of \( t \), since \( S \) divides the characteristic polynomial of \( t \). Let \( \nu = \delta + \delta^{-1} \); then \( E_\nu = \text{Ker}(t + t^{-1} - \nu) \) has signature \( (2, 0) \). Set \( M = L \cap E_\nu^\perp \). Then \( M \) is negative definite, and \( M = \text{Ker}(C(t)) \). Since \( t \) is a realizable isometry, the restriction of \( t \) to \( M \) is of type (P). With the terminology of [McM 11], this implies that the restriction of \( t \) to \( M(−1) \) is positive. Therefore we can apply Theorem 6.2 of [McM 11], and conclude that \( t \) is realizable by a K3 surface automorphism \( T : \mathcal{X} \to \mathcal{X} \). We have \( \lambda(T) = \alpha \) and \( \omega(T) = \delta \) by construction, and Pic(\( \mathcal{X} \)) is isomorphic to \( M = \text{Ker}(C(t)) \) by the above remark, hence (a) and (b) hold. This completes the proof of the theorem.

With the terminology of §15 this implies the following:
Corollary 21.5. If the lattice $\Lambda_{3,19}$ has a realizable isometry with signature map equivalent to $\tau_{S,\delta}$, then $(\alpha, \delta)$ is realizable.

Proof of Theorem 21.3. Part (a) follows from Theorem 15.6 and Corollary 21.3 and Part (b) from Proposition 15.8 combined with Corollary 21.5.

Example 21.6. Let $\alpha = \lambda_{16} = 1,2363179318...$ be the smallest degree 16 Salem number (see for instance the table in Boyd [Bo 77]). The corresponding Salem polynomial is

$$S(X) = X^{16} - X^{15} - X^8 - X + 1.$$ 

Theorem 21.3 (a) implies that $(\alpha, \delta)$ is realizable for every root $\delta$ of $S$ with $|\delta| = 1$. On the other hand, McMullen proved that $\alpha$ is not realizable by an automorphism of a projective $K3$ surface (see [Mcm 16], §9).

Example 21.7. Let $\alpha = 1,2527759374...$ be the Salem number with polynomial

$$S(X) = X^{18} - X^{12} - X^{11} - X^{10} - X^8 - X^7 - X^6 + 1.$$ 

We have $S(1) = -5$, hence Theorem 21.3 (b) implies that $(\alpha, \delta)$ is realizable for all roots $\delta$ of $S$ with $|\delta| = 1$.

We now prove Theorem 1 of the introduction:

Theorem 21.8. Let $S$ be a Salem polynomial of degree $d$ with $4 \leq d \leq 22$, and let $\alpha$ be the corresponding Salem number. The following are equivalent:

1. There exists a root $\delta$ of $S$ with $|\delta| = 1$ such that $(\alpha, \delta)$ is realizable.
2. $(\alpha, \delta)$ is realizable for all roots $\delta$ of $S$ with $|\delta| = 1$.

Proof. It is clear that (2) $\implies$ (1); let us prove that (1) $\implies$ (2). Since $(\alpha, \delta)$ is realizable, there exists an even unimodular lattice $L$ of signature $(3,19)$ and a realizable isometry $t : L \to L$ with signature map equivalent to $\tau_{S,\delta}$; let us denote by $\tau$ this signature map. Let $F$ be the characteristic polynomial of $t$; we have $F = SC$, where $C$ is a product of cyclotomic polynomials. Set $L_S = \text{Ker}(S(t))$, $L_C = \text{Ker}(C(t))$, and let $V_S = L_S \otimes \mathbb{Z} Q$, $V_C = L_C \otimes \mathbb{Z} Q$; both $V_S$ and $V_C$ are stabilized by $\tau$. Let $V = L \otimes Q$, and let $q : V \times V \to Q$ be the quadratic form given by the lattice $L$; we have $V = V_S \oplus V_C$, and this decomposition is orthogonal with respect to the form $q$. Let $[V_S], [V_C] \in W_T(Q)$ be the corresponding Witt classes. For all prime numbers $p$, let $\delta_p^S = \partial_p[V_S]$, $\delta_p^C = \partial_p[V_C]$ be their images in $W_T(F_p)$; since $L$ is unimodular, we have $\delta_p^S \oplus \delta_p^C = 0$.

For all prime numbers $p$, let $a^p(S) \in C_{\delta_p^S}$ and $a_{\tau_{S,\delta}}^p(S)$ be as in §5. Notation 5.5 and §9. Let $t'$ be a root of $S$ with $|\delta| = 1$. By Proposition 9.3 we have $a_{\tau_{S,t'}}^p(S) = a_{\tau_{S,\delta}}^p(S)$. Theorem 9.6 implies that there exists a semi-simple isometry $t' : V_S \to V_S$ with signature map $\tau_{S,t'}$ and local data $(a_{\delta_p^{t'}}^p)$; let $[V_S'] \in W_T(Q)$ be the corresponding Witt class. For all prime numbers $p$, let $N^p$ be a maximal $Z_p$-lattice in $V_S \otimes Q Z_p$ such that the class of $((N^p)^{\otimes p}, t')$ in $W_T(F_p)$ is equal to $\partial_p[V_S']$; note that for almost all $p$, we have $N^p = L_S \otimes Z_p$. Let $N$ be the intersection of the lattices $N^p$ in $V_S$; then $N$ is stabilized by the isometry $t'$. Let $t' : V \to V$ be such that the restriction of $t'$ to $V_S$ is $t$, and the restriction of $t'$ to $V_C$ is $t$. Since $\partial_p[V_S'] = \delta_p^S$ for all prime numbers $p$, the isometry $t' : V \to V$ stabilizes an even unimodular lattice $L'$ such that $\text{Ker}(C(t')) = L_C$. The isometry $t'$ is realizable, since the restriction of $t'$ to $V_C$ is equal to $t$. The signature map of $t'$ is equivalent to $\tau_{S,t'}$, hence by Corollary 21.5 $(\alpha, \delta')$ is realizable.
This concludes the proof of the theorem.

The following is Theorem 2 of the introduction:

**Theorem 21.9.** Suppose that \( d = 22 \) and that \( |S(1)| \) and \( S(-1) \) are squares. Then \((\alpha, \delta)\) is realizable.

**Proof.** This follows from Theorem 15.9 and Corollary 21.5.

In the case of unramified Salem numbers of degree 10 or 18, we obtain a necessary and sufficient criterion in terms of Salem signature maps.

**Theorem 21.10.** Assume that \( d = 10 \) or 18 and that \( S \) is unramified. Then \((\alpha, \delta)\) is realizable if and only if there exists a Salem signature map of maximum \((3, 19)\) with trivial obstruction equivalent to \( \tau_{S, \delta} \).

**Proof.** Let \( \tau \) be a Salem signature map of maximum \((3, 19)\) with trivial obstruction equivalent to \( \tau_{S, \delta} \); then by Theorem 17.1 the lattice \( \Lambda_{3,19} \) has a realizable isometry with signature map equivalent to \( \tau \). By 21.5, we conclude that \((\alpha, \delta)\) is realizable.

Conversely, suppose that \((\alpha, \delta)\) is realizable, and let \( t : L \to L \) be a semi-simple isometry \( t \) of an even unimodular lattice \( L \) with signature map \( \tau \) equivalent to \( \tau_{S, \delta} \). Let \( F \) be the polynomial associated to \( \tau \), i.e. the characteristic polynomial of \( t \); we have \( F = SC \), where \( C \) is a product of cyclotomic polynomials. The polynomial \( F \) satisfies condition (C 1). Set \( L_S = \text{Ker}(S(t)) \), \( L_C = \text{Ker}(C(t)) \) and let \( V_S = L_S \otimes \mathbb{Z} Q \) and \( V_C = L_C \otimes \mathbb{Z} Q \). We are assuming that \( S \) is unramified, hence \( S \) satisfies condition (C 1); this implies that \( C \) also satisfies condition (C 1), and therefore we have \( \partial_v[V_S] = \partial_v[V_C] = 0 \) for all \( v \in V' \). On the other hand, since \( \delta \) is a root of \( S \), the signature of \( L_S \) is \( (3, d-3) \) and \( L_C \) is negative definite; hence \( L_S \) and \( L_C \) cannot be unimodular. This implies that \( C \) has an irreducible factor \( \Phi \) such that \( \Pi_{S, \delta} \neq \varnothing \). The hypothesis \( S \) unramified implies that \( \Phi(X) \neq X - 1 \) or \( X + 1 \), hence \( \Phi \) is a cyclotomic polynomial of even degree. There exists a product of cyclotomic polynomials \( C' \) of degree \( 22 - d \) divisible by \( \Phi \) satisfying conditions (C 0) and (C 1) such that \( G_{SC'} = 0 \). Indeed, if \( \text{deg}(\Phi) = 22 - d \), then by Condition (C 1) we have \( \Phi(1) = \Phi(-1) = 1 \); in this case, set \( C' = \Phi \). Otherwise, \( \text{deg}(\Phi) \leq 20 - d \). Using the properties of cyclotomic polynomials and the fact that \( 22 - d \) is divisible by 4, we check that such a polynomial \( C' \) exists. Let \( \tau \) be the signature map of maximum \((3, 19)\) with polynomial \( SC' \) equivalent to \( \tau_{S, \delta} \); this is a Salem signature map, hence the theorem is proved.

Recall that if \( S \) is unramified and if \( m \) is an integer \( \geq 3 \), then \( \Pi_{S, \Phi_m} \) is the set of prime numbers \( p \) such that \( S \) (mod \( p \)) and \( \Phi_m \) (mod \( p \)) have a common irreducible symmetric factor of even degree in \( \mathbb{F}_p[X] \).

**Theorem 21.11.** Assume that \( d = 10 \) or 18 and that \( S \) is unramified. If we have \( \Pi_{S, \Phi_m} \neq \varnothing \) for some \( m = 3, 4, 6 \) or 12, then

(a) \((\alpha, \delta)\) is realizable.

(b) If moreover \( d = 10 \), then the lattice \( \Lambda_{3,19} \) has a realizable isometry with signature map equivalent to \( \tau_{S, \delta} \) and polynomial divisible by \((X - 1)^8\).

**Proof.** (a) Set \( C = \Phi_m^2 \) if \( m = 3, 4 \) or 6, and \( C = \Phi_m \) if \( m = 12 \). Then the polynomial \( SC \) satisfies conditions (C 0) and (C 1), and \( \Pi_{S, \Phi_m} \neq \varnothing \) implies that \( G_{SC} = 0 \). Let \( \tau \) be the signature map of maximum \((3, d + 4)\) and associated polynomial \( SC \) be such that \( \tau(f) = \tau_{S, \delta}(f) \) for all factors \( f \in \mathbb{R}[X] \) of \( S \), and
note that $\tau$ is a Salem signature map; hence Theorem 21.10 implies that $(\alpha, \delta)$ is realizable. Part (b) follows from Theorem 17.1.

22. Salem numbers of degree 18

We keep the notation of the previous section. Suppose that $d = 18$ and $S$ is unramified (for instance, $|S(1)S(-1)| = 1$). In this case, the necessary and sufficient condition of Theorem 21.10 can be reformulated as follows:

**Theorem 22.1.** Suppose that $S$ is unramified. Then $(\alpha, \delta)$ is realizable if and only if $\Pi_{S, \Phi_m} \neq \emptyset$ for some $m = 3, 4, 6$ or 12.

**Proof.** We already know that if $\Pi_{S, \Phi_m} \neq \emptyset$ for $m = 3, 4, 6$ or 12, then $(\alpha, \delta)$ is realizable (cf. Theorem 21.11). The converse is a consequence of Theorem 21.10 combined with some properties of cyclotomic polynomials.

**Example 22.2.** Let $\alpha = 1, 21972085590$ be the second smallest degree 18 Salem number, with Salem polynomial

$$S(X) = X^{18} - X^{17} - X^{10} + X^9 - X^8 - X + 1$$

(see the table in [Bo.77]). We have $\Pi_{S, \Phi_3} = \{5\}$, therefore by Theorem 22.1 the pair $(\alpha, \delta)$ is realizable for every root $\delta$ of $S$ with $|\delta| = 1$.

On the other hand, we now show that there exist nonrealizable pairs $(\alpha, \delta)$. If $f \in \mathbb{Z}[X]$ is a monic polynomial, we denote by $\text{Res}(S,f)$ the resultant of the polynomials $S$ and $f$.

**Proposition 22.3.** Assume that $|S(1)S(-1)| = 1$, and that $\text{Res}(S, \Phi_m) = 1$ for $m = 3, 4$ and 6. If $(\alpha, \delta)$ is realizable, then $\Pi_{S, \Phi_{12}} \neq \emptyset$.

**Proof.** Suppose that $(\alpha, \delta)$ is realizable. This implies that $\Lambda_{3,19}$ has a semi-simple isometry with signature map $\tau$ of maximum $(3, 19)$ and polynomial $SC$, where $C$ is a product of cyclotomic polynomials. Assume that all the factors of $C$ belong to the set $\{\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_6\}$. Then $S$ and $C$ are relatively prime over $\mathbb{Z}$. If $\Lambda_{3,19}$ has an isometry with characteristic polynomial $F$, then $\Lambda_{3,19} = L_1 \oplus L_2$, where $L_1$ and $L_2$ are even unimodular lattices such that $L_1$ has an isometry with characteristic polynomial $S$ and signature map $\tau_{S,\delta}$, and $L_2$ has an isometry with characteristic polynomial $C$. This implies that the signature of $L_1$ is $(3, 15)$ and that the signature of $L_2$ is $(0, 4)$, and this is impossible. Therefore the only possibility is $C = \Phi_{12}$.

Let us show that if $\Pi_{S, \Phi_{12}} = \emptyset$, then $(\alpha, \delta)$ is not realizable. Note that $\Pi_{S, \Phi_{12}} = \emptyset$ implies that $G_{S, \Phi_{12}} \neq 0$. We apply Example 14.1 (c); since $S$ and $\Phi_{12}$ both satisfy condition (C 1), the group $G_{S, \Phi_{12}}$ is $\neq 0$, and $\delta$ is a root of $S$, the signature map $\tau$ is not realizable, and this implies that $(\alpha, \delta)$ is not realizable.

**Example 22.4.** Let $\lambda_{18} = 1.1883681475...$, the smallest degree 18 Salem number (it is also the second smallest known Salem number). Let $S$ be the corresponding Salem polynomial, and let $\delta$ be a root of $S$ with $|\delta| = 1$; then $(\lambda_{18}, \delta)$ is not realizable. Indeed, the polynomial $S$ satisfies the conditions of Proposition 22.3; we have $\text{Res}(S,f) = 1$ for all $f \in \{\Phi_3, \Phi_4, \Phi_6\}$. Therefore by Proposition 22.3 if $(\lambda_{18}, \delta)$ is realizable, then $\Pi_{S, \Phi_{12}} = \emptyset$.

We have $\text{Res}(S, \Phi_{12}) = 169$, and the common factors modulo 13 of $S$ and $\Phi_{12}$ in $\mathbb{F}_{13}[X]$ are $X + 6, X + 11 \in \mathbb{F}_{13}[X]$. These polynomials are not symmetric. Therefore $\Pi_{S, \Phi_{12}} = \emptyset$; hence Proposition 22.3 implies that $(\lambda_{18}, \delta)$ is not realizable.
On the other hand, McMullen proved that \( \lambda_{18} \) is realized by an automorphism of a projective \( K3 \) surface (see [McM 16], Theorem 8.1).

**Example 22.5.** With the notation of Example \( 22.4 \) set \( \alpha_2 = \lambda_{18}^2 \) and \( \alpha_3 = \lambda_{18}^3 \); let \( S_2 \) and \( S_3 \) be the associated Salem polynomials. The Salem numbers \( \alpha_2 \) and \( \alpha_3 \) are realizable: indeed, we have \( \Pi_{S_2, \Phi_6} = \{13\} \) and \( \Pi_{S_3, \Phi_3} = \{17\} \).

### 23. Salem numbers of degree 10

We keep the notation of the previous sections: \( S \) is a Salem polynomial of degree \( d \) with Salem number \( \alpha \), and \( \delta \) is a root of \( S \) such that \( |\delta| = 1 \). Suppose that \( d = 10 \), and that \( S \) is *unramified*; for instance, this is the case when \( |S(1)S(-1)| = 1 \). Recall that if \( m \geq 3 \) is an integer, then \( \Pi_{S, \Phi_m} \) is the set of prime numbers \( p \) such that \( S \pmod{p} \) and \( \Phi_m \pmod{p} \) have a common irreducible symmetric factor of even degree in \( \mathbf{F}_p[X] \).

**Theorem 23.1.** Assume that there exists an integer \( m \geq 3 \) with \( \varphi(m) \leq 12 \) and \( m \neq 13 \) or 26 such that \( \Pi_{S, \Phi_m} \neq \emptyset \). Then \( (\alpha, \delta) \) is realizable.

**Proof.** The hypothesis implies that there exists a product of cyclotomic polynomials \( C \) of degree 12 such that the polynomial \( F = SC \) satisfies conditions (C 0) and (C 1), and \( G_F = 0 \). Let \( \tau \) be the signature map of maximum (3,19) and associated polynomial \( SC \) be such that \( \tau(f) = \tau_{S, \delta}(f) \) for all factors \( f \in \mathbf{R}[X] \) of \( S \), and note that \( \tau \) is a Salem signature map; hence Theorem 21.10 implies that \( (\alpha, \delta) \) is realizable.

Examples 23.2 and 23.3 give infinite families of realizable degree 10 Salem numbers:

**Example 23.2.** Let \( a \geq 1 \) be an integer, let

\[
R(X) = (X + 1)^2(X^2 - 4)(X - a) - 1, \quad \text{and} \quad S(X) = X^5 R(X + X^{-1}).
\]

Then \( S \) is a Salem polynomial (see [GM 02], §7); let \( \alpha_a \) be the corresponding Salem number. We have \( S(i) = R(0) = 4a - 1 \). Since \( a \geq 1 \), this is congruent to 3 modulo 4, hence it is divisible by a prime number \( p \equiv 3 \pmod{4} \). Note that the resultant of \( S \) and \( \Phi_4 \) is \( \mathbf{N}\mathbf{Q}(i)/\mathbf{Q}(S(i)) \), and that \( \Phi_4 \) is irreducible mod \( p \). Therefore we have \( p \in \Pi_{S, \Phi_4} \), and Theorem 23.1 implies that \( (\alpha, \delta) \) is realizable for all roots \( \delta \) of \( S \) with \( |\delta| = 1 \).

**Example 23.3.** Let \( a, b, c \in \mathbf{Z} \), set

\[
R(X) = (X^2 - 4)(X^3 + aX^2 + (b - 1)X + c) - 1,
\]

and

\[
S(X) = X^5 R(X + X^{-1}).
\]

Suppose that \( c \geq 0 \) and that \( a + c < -|b| \). Then we have \( R(0) < 0 \) and \( R(-1), R(1) > 0 \). Since \( R(-2), R(2) < 0 \), the polynomial \( R \) has 4 roots between \( -2 \) and \( 2 \). Moreover, \( R \) is irreducible (otherwise, one of the factors of \( R \) would be a "cyclotomic trace polynomial"). Therefore \( S \) is a Salem polynomial.

We have \( S(\zeta_3) = R(-1) = -3(a - b + c) - 1 \); since \( a + c < -|b| \) by hypothesis, this number is divisible by a prime number \( p \) congruent to 2 modulo 3, and this implies that \( p \in \Pi_{S, \Phi_3} \). Let \( \alpha \) be the Salem number corresponding to \( S \). By Theorem 23.1 the pair \( (\alpha, \delta) \) is realizable for all roots \( \delta \) of \( S \) with \( |\delta| = 1 \).
Example 23.4. Let \( \alpha = \lambda_{10} \) be the smallest known Salem number (also called “Lehmer number”), and let \( S \) be the corresponding Salem polynomial. The polynomial \( S \) belongs to the family of Example 5.7 for \( a = 1 \); we have \( 3 \in \Pi_{S, \Phi_{14}} \), hence \((\alpha, \delta)\) is realizable for all roots \( \delta \) of \( S \) with \( |\delta| = 1 \). An explicit construction (for \( \delta + \overline{\delta} = -1.886... \) ) is given by McMullen in [McM 11], Theorem 7.1.

We also have \( \Pi_{S, \Phi_{12}} = \{3\} \), \( \Pi_{S, \Phi_{14}} = \{13\} \), \( \Pi_{S, \Phi_{15}} = \{29\} \) and \( \Pi_{S, \Phi_{36}} = \{3\} \), also leading to realizations of \((\alpha, \delta)\) for all roots \( \delta \) of \( S \) with \( |\delta| = 1 \).

The Lehmer number is also the dynamical degree of automorphisms of projective \( K3 \) surfaces, as shown by McMullen (see [McM 16], Theorems 7.1 and 7.2.)

These results and examples suggest that all degree 10 Salem numbers might be realizable, suggesting the question:

**Question 23.5.** Is every Salem number of degree 10 realizable?

In particular, I do not know the answer to the following question:

**Question 23.6.** Let \( S \) be a Salem polynomial of degree 10 with \( |S(1)S(-1)| = 1 \). Does there exist an integer \( m \geq 3 \) with \( \varphi(m) \leq 12 \) and \( m \neq 13 \) or 26 such that \( \Pi_{S, \varphi_m} \neq \emptyset \) ?

24. Salem numbers of degree 20

The aim of this section is to give some partial results and examples concerning Salem numbers of degree 20. More recently, Takada proved that every Salem number of degree 20 is realizable (see [T 22], Theorem 1.3).

Let \( \alpha \) be a Salem number of degree 20, let \( S \) be the corresponding Salem polynomial, and let \( \delta \) be a root of \( S \) with \( |\delta| = 1 \). Let \( F(X) = S(X)(X^2 - 1) \), and let \( \tau_3 \), \( \tau_{X^2 - 1} \) be the signature maps of maximum (3, 19) and polynomial \( F \) such that \( \tau_3 \) is equivalent to \( \tau_{S, \delta} \), and that \( \tau_{X^2 - 1}(X - 1) = \tau_{X^2 - 1}(X + 1) = 1 \).

**Theorem 24.1.** Suppose that \( S(1) \) and \( S(-1) \) are odd, square-free, and relatively prime. Then

(a) The lattice \( \Lambda_{3,19} \) has a realizable isometry with signature map \( \tau_3 \).

(b) The lattice \( \Lambda_{3,19} \) has an isometry with signature map \( \tau_{X^2 - 1} \).

**Proof.** We prove (a) and (b) simultaneously. Set \( D = S(1)S(-1) \), and let \( K = \mathbb{Q}(\sqrt{D}) \); since \( S(-1) < 0 \) and \( S(1) > 0 \), the field \( K \) is an imaginary quadratic field. For all prime numbers \( p \) dividing \( D \), we denote by \( P \) the ramified ideal of \( K \) above \( p \); let \( I \) be the product of the ideals \( P \) for \( p \) dividing \( S(1) \). Let \( q_I : K \times K \to \mathbb{Q} \) be the quadratic form defined by \( q_I(x, y) = \frac{1}{N(I)} \text{Tr}_{K/\mathbb{Q}}(xc(y)) \), where \( x \mapsto c(x) \) denotes complex conjugation; the form \( q_I \) is positive definite. We have \( q_I(x, y) \in \mathbb{Z} \) for all \( x, y \in I \), and \( q_I((x, x)) \in 2\mathbb{Z} \) for all \( x \in I \). The complex conjugation \( c : I \to I \) is an isometry of \( q_I \) with characteristic polynomial \( X^2 - 1 \).

In case (a), set \( e_p = \partial_p[K, -q_I, c] \) in \( W(\mathbb{F}_p, id) \) if \( p \) divides \( S(1) \), and in \( W(\mathbb{F}_p, -id) \) if \( p \) divides \( S(-1) \).

In case (b), set \( e_p = \partial_p[K, q_I, c] \) in \( W(\mathbb{F}_p, id) \) if \( p \) divides \( S(1) \), and in \( W(\mathbb{F}_p, -id) \) if \( p \) divides \( S(-1) \).

Let \( E = \mathbb{Q}[X]/(S) \), let \( \alpha \) be the image of \( X \) in \( E \), and let \( \sigma : E \to E \) be the involution induced by \( X \mapsto X^{-1} \). Let \( E_0 \) be the fixed field of \( \sigma \) in \( E \), and let \( d \in (E_0)^\times \) be such that \( E = E_0(\sqrt{d}) \). If \( p \) is a prime number, set \( E_p = E \otimes \mathbb{Q} \mathbb{Q}_p \).
For all prime numbers $p$ dividing $S(1)S(-1)$, there exists $\lambda_p \in T(E_p, \sigma)$ such that $\partial_p[E_p, b_{\lambda_p}, \alpha] = -\epsilon_p$; if $p$ does not divide $S(1)S(-1)$, we take $\lambda_p \in T(E_p, \sigma)$ such that $(E_p, b_{\lambda_p}, \alpha)$ contains an even unimodular lattice; this is possible by [K1-20], §6. Set $V = E \oplus K$ and $V_p = V \otimes \mathbb{Q}_p$.

In case (a), set $q_p = b_{\lambda_p} - q_I$; and in case (b), set $q_p = b_{\lambda_p} + q_I$. In both cases, we have $\partial_p[V_p, q_p, \alpha \oplus e] = 0$ for all prime numbers $p$.

In case (a), let us choose $\lambda_\infty$ such that the quadratic form $b_{\lambda_\infty}$ has signature $(3, 17)$ and that the signature map of $(b_{\lambda_\infty}, \alpha)$ is equal to $\tau_{S, \delta}$. In case (b), we choose $\lambda_\infty$ such that the quadratic form $b_{\lambda_\infty}$ has signature $(1, 19)$.

Let us determine the invariants of the quadratic forms $(V, q_p)$ and $(E_p, b_{\lambda_p})$. We have $\det(q_p) = -S(1)^2S(-1)^2 = -1$ in $\mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2}$. The Hasse-Witt invariant $w(q_p)$ at $p$ is $0$ if $p \neq 2$, and it is $1$ for $p = 2$. We have $\det(b_{\lambda}) = -\det(q_I) = -\det(-q_I)$, therefore we have $w(b_{\lambda} \oplus -q_I) = w(b_{\lambda}) + w(-q_I)$ and $w(b_{\lambda} \oplus q_I) = w(b_{\lambda}) + w(q_I)$; this implies that the Hasse-Witt invariants of $(E_p, b_{\lambda_p})$ and of $q_I$ are equal at $p \neq 2$, and different at $2$ and at infinity; since $q_I$ is a global form, the sum of its Hasse-Witt invariants is $0$, and therefore the same thing is true for the Hasse-Witt invariants of $(E_p, b_{\lambda_p})$.

We have $w(b_{\lambda_p}) = w(b_1) + \text{cor}_{E_p/\mathbb{Q}_p}(\lambda_p, d)$ for all $p$ (see Proposition 5.4); since $b_1$ is a global form, the above argument shows that $\sum_{v \in V} \text{cor}_{E_v/\mathbb{Q}_v}(\lambda_v, d) = 0$. By Theorem 9.3 this implies that there exists $\lambda \in E_0^\times$ such that $(E, b_{\lambda}) \otimes \mathbb{Q}_v \simeq (E_v, b_{\lambda_v})$ for all $v$, and that in case (a), $(E, b_{\lambda})$ has signature map $\tau_{S, \delta}$; in case (b), $(E, b_{\lambda})$ has signature $(1, 19)$. Set $(V, q, t) = (E, b_{\lambda}, \alpha) \oplus (K, -q_I, c)$ in case (a), and $(V, q, t) = (E, b_{\lambda}, \alpha) \oplus (K, q_I, c)$ in case (b).

The quadratic form $(V, q)$ contains an even unimodular lattice stabilized by $t$ everywhere locally, and the intersection of these lattices is an even unimodular lattice stabilized by the isometry $t$. In case (a), the isometry $t$ is realizable by construction.

**Corollary 24.2.** Suppose that $S(1)$ and $S(-1)$ are odd, square-free, and relatively prime. Then $(\alpha, \delta)$ is realizable.

**Proof.** This follows from Theorem 24.1 and Corollary 24.5.

**Example 24.3.** Let

$$S(X) = X^{20} - X^{19} - X^{15} + X^{14} - X^{11} + X^{10} - X^9 + X^6 - X^5 - X + 1.$$  

The corresponding Salem number is $\alpha = \lambda_{20} = 1, 2326135486...$, the smallest degree 20 Salem number (cf. the table in [De 77]). We have $S(1) = -1$ and $S(-1) = 11$, hence Corollary 24.2 implies that $(\alpha, \delta)$ is realizable for all roots $\delta$ of $S$ with $|\delta| = 1$; McMullen proved that $\lambda_{20}$ is not realized by an automorphism of a projective K3 surface (see [McM 16], §9).

**Example 24.4.** Let

$$S(X) = X^{20} - X^{17} - X^{16} - X^{15} - X^{11} - X^{10} - X^9 - X^5 - X^4 - X^3 + 1.$$  

The corresponding Salem number is $\alpha = 1, 37892866...$. We have $S(1) = -7$ and $S(-1) = 5$; by Corollary 24.2 the pair $(\alpha, \delta)$ is realizable for all roots $\delta$ of $S$ with $|\delta| = 1$.

**Remark 24.5.** The method of Theorem 24.1 and Corollary 24.2 can be used to prove that if $S(1)$ and $S(-1)$ are odd, then $(\alpha, \delta)$ is realizable.
25. Finiteness results

McMullen proved that up to isomorphism, there are only countably many pairs $(X, T)$ where $T : X \to X$ is an automorphism of a K3 surface and $\omega(T)$ is not a root of unity (see [McM 02], Theorem 3.6); we record here some related finiteness results.

Let $\alpha$ be a Salem number with Salem polynomial $S$, let $d = \deg(S)$, and let $\delta$ be a root of $S$ such that $|\delta| = 1$.

**Proposition 25.1.** There exist only finitely many isomorphism classes of K3 surfaces having an automorphism $T$ with $\lambda(T) = \alpha$ and $\omega(T) = \delta$.

We start with a definition:

**Definition 25.2.** Let $(L, q)$ be a lattice, and let $t : L \to L$ be an isometry of $L = (L, q)$; we say that $(L, t) = (L, q, t)$ is an isometric structure. Two isometric structures $(L, q, t)$ and $(L', q', t')$ are isomorphic if there exists an isomorphism $f : L \to L'$ such that $q'(f(x), f(y)) = q(x, y)$ and $f \circ t = t' \circ f$. An isometric structure $(L, q, t)$ is said to be realizable if $t : L \to L$ is a realizable isometry of $(L, q)$.

**Example 25.3.** Let $T : X \to X$ be an automorphism of a K3 surface; then $(H^2(X, \mathbb{Z}), T^*)$ is a realizable isometric structure.

**Proposition 25.4.** There exist only finitely many isomorphism classes of isometric structures $(H^2(X, \mathbb{Z}), T^*)$, where $T : X \to X$ is an automorphism of the K3 surface $X$ with $\lambda(T) = \alpha$ and $\omega(T) = \delta$.

**Proof.** If $T : X \to X$ is an automorphism of a K3 surface with $\lambda(T) = \alpha$ and $\omega(T) = \delta$, then the characteristic polynomial of $T^*$ is equal to $SC$, where $C$ is a product of cyclotomic polynomials with $\deg(C) = 22 - d$. There are only finitely many cyclotomic polynomials of a given degree; therefore it suffices to show the finiteness of the number of isometric structures $(H^2(X, \mathbb{Z}), T^*)$ with fixed characteristic polynomial. Since $\omega(T)$ is a root of $S$, the isometry $T^*$ is semi-simple; therefore Proposition 4 of [BM 79] implies the desired finiteness result.

**Proposition 25.5.** Let $T : X \to X$ and $T' : X' \to X'$ be two automorphisms of K3 surfaces with $\omega(T) = \omega(T') = \delta$. If the isometric structures $(H^2(X, \mathbb{Z}), T^*)$ and $(H^2(X', \mathbb{Z}), (T')^*)$ are isomorphic, then the K3 surfaces $X$ and $X'$ are isomorphic.

**Proof.** Let $L = H^2(X, \mathbb{Z})$, $L' = H^2(X', \mathbb{Z})$, $t = T^*$ and $t' = (T')^*$, and let $f : L \to L'$ be an isomorphism of the isometric structures $(L, t)$ and $(L', t')$.

Note that $H^{2,0}(X)$ is the eigenspace of $\delta$ in $L \otimes \mathbb{C}$; this follows from the fact that $\delta$ is a simple root of $S$. The isomorphism $f$ respects the eigenspaces, hence $f(H^{2,0}(X)) = H^{2,0}(X')$; this implies that $f$ respects the Hodge decompositions. Therefore by Theorem 20.2, the K3 surfaces $X$ and $X'$ are isomorphic.

**Proof of Proposition 25.1**. This follows from Proposition 25.4 and Proposition 25.5.

**Corollary 25.6.** Let $\alpha$ be a Salem number. Then there exist at most finitely many isomorphism classes of K3 surfaces of algebraic dimension 0 having an automorphism of dynamical degree $\alpha$.

**Proof of Corollary 25.6**. This follows from Proposition 25.1 since $\lambda(T)$ and $\omega(T)$ are roots of the same Salem polynomial if $T$ is an automorphism of a K3 surface of algebraic dimension 0.
26. Non-realizable Salem numbers

This section and the next one contain some remarks on automorphisms of projective $K3$ surfaces.

If $T : \mathcal{X} \to \mathcal{X}$ is an automorphism of a $K3$ surface, recall that we denote by $\lambda(T)$ the dynamical degree of $T$, and by $\omega(T)$ the complex number such that $T^*: H^2(\mathcal{X}) \to H^2(\mathcal{X})$ is multiplication by $\omega(T)$; a pair $(\alpha, \delta)$ is said to be realizable if there exists an automorphism $T$ of a $K3$ surface with $\lambda(T) = \alpha$ and $\omega(T) = \delta$ (see Definition 21.2); we say that $\alpha$ is realizable if there exists a complex number $\delta$ with $|\delta| = 1$ such that $(\alpha, \delta)$ is realizable. The previous sections were concerned with realizability of pairs $(\alpha, \delta)$, such that $\alpha$ and $\delta$ are roots of the same Salem polynomial; this implies that the $K3$ surface has algebraic dimension 0.

This section takes up this problem for projective $K3$ surfaces. Recall that if $T : \mathcal{X} \to \mathcal{X}$ is an automorphism of a projective $K3$ surface, then $\omega(T)$ is a root of unity (see [McM 02], Theorem 3.5). Moreover, the characteristic polynomial of the restriction of $T^*$ to $\text{Trans}(\mathcal{X})$ is a power of a cyclotomic polynomial (see for instance Lemma 20.5).

Degree 18 Salem numbers

The first result concerns degree 18 Salem numbers: we give an example of a Salem number $\alpha$ of degree 18 such that $\alpha$ is not realizable, in other words, there does not exist any automorphism of a $K3$ surface, projective or not, with dynamical degree $\alpha$. We start with a lemma.

Lemma 26.1. Let $S \in \mathbb{Z}[X]$ be a monic polynomial of degree 18 with $|S(1)S(-1)| = 1$, and such that $\text{Res}(S, \Phi_m) = 1$ for $m = 3, 4, 6$ and 12. Let $C$ be a product of cyclotomic polynomials with $\deg(C) = 4$, let $L$ be an even, unimodular lattice and let $t : L \to L$ be an isometry with characteristic polynomial $SC$. Then $L \simeq L_1 \oplus L_2$, where $L_1$ and $L_2$ are even, unimodular lattices stables by $t$, and the characteristic polynomial of the restriction of $t$ to $L_1$ is $S$, the the characteristic polynomial of the restriction of $t$ to $L_2$ is $C$.

Proof. Set $F = SC$; the polynomial $F$ satisfies condition (C 1), hence $|F(1)|$ and $F(-1)$ are squares. Since $|S(1)S(-1)| = 1$, this implies that $C(1)$ and $C(-1)$ are squares; hence $C$ cannot be equal to $\Phi_8$. Therefore all the factors of $C$ are equal to $\Phi_m$, for $m = 1, 2, 3, 4, 6$ or 12. All these polynomials are relatively prime to $S$ over $\mathbb{Z}$, therefore $L = L_1 \oplus L_2$, where $L_1$ and $L_2$ are even, unimodular lattices stables by $t$, and the restriction of $t$ to $L_1$ has characteristic polynomial $S$, the characteristic polynomial of the restriction of $t$ to $L_2$ is equal to $C$.

Proposition 26.2. Let $S$ be a Salem polynomial of degree 18, and let $\alpha$ be the corresponding Salem number. Assume that $|S(1)S(-1)| = 1$, and that $\text{Res}(S, \Phi_m) = 1$ for $m = 3, 4, 6$ and 12. Then $\alpha$ is not realizable.

Proof. By Proposition 22.3 we already know that $(\alpha, \delta)$ is not realizable for any root $\delta$ of $S$. Suppose that $(\alpha, \zeta)$ is realizable for some root of unity $\zeta$. Let $T : \mathcal{X} \to \mathcal{X}$ be an automorphism with $\lambda(T) = \alpha$ and $\omega(T) = \zeta$; note that $\mathcal{X}$ is then a projective $K3$ surface. Set $L = H^2(\mathcal{X}, \mathbb{Z})$, and $t = T^*$. Let $F$ be the characteristic polynomial of $t$; we have $F = SC$, where $C$ is a power of a cyclotomic polynomial with $C(\zeta) = 0$. By Lemma 26.1 we have $L \simeq L_1 \oplus L_2$, where $L_1$ and $L_2$ are even, unimodular lattices stables by $t$, and the characteristic polynomial of the restriction of $t$ to $L_1$ is $S$, the the characteristic polynomial of the restriction of $t$ to $L_2$ is $C$. Since $\omega(T) = \zeta$, the
lattice $L_2$ is isomorphic to $\text{Trans}(\mathcal{X})$, and its signature is $(2, 2)$; hence the lattice $L_1$ is isomorphic to $\text{Pic}(\mathcal{X})$, and its signature is $(1, 17)$. Therefore $\text{Pic}(\mathcal{X})$ is isomorphic to $\Lambda_{1,17}$, hence $\text{Aut}(\mathcal{X})$ is finite (see [PS 71, §7, Example 2]). This implies that all the automorphisms of $\mathcal{X}$ have dynamical degree 1, contradicting the assumption that $\lambda(T) > 1$; hence $(\alpha, \zeta)$ is not realizable, as claimed.

**Example 26.3.** I thank Chris Smyth for this example. Let $a$ be an integer, let $R_a(X) = X^2(X^2 - 4)(X^2 - 3)(X^2 - (X - a) - 1)$, and set $S_a(X) = X^9R_a(X + X^{-1})$. For $a = 3$, the polynomial $S = S_3$ is a Salem polynomial, and the corresponding Salem number is $2,618575...$ We have $|S(1)S(-1)| = 1$ and $\text{Res}(S, \Phi_m) = 1$ for $m = 3, 4, 6$ and 12, hence by Proposition 26.2 the Salem number $\alpha = 2,618575...$ is not realizable.

**Degree 20 Salem numbers**

We now give examples of degree 20 Salem numbers that are not realized by automorphisms of projective K3 surfaces.

**Proposition 26.4.** Let $S$ be a Salem polynomial of degree 20, and let $\alpha$ be the corresponding Salem number. Let $T : \mathcal{X} \to \mathcal{X}$ is an automorphism of a projective K3 surface such that $\lambda(T) = \alpha$. Then one of the following holds:

(i) $S(-1)$ is a square.

(ii) $|S(1)|$ is a square.

(iii) There exist integers $m$ and $n$ such that $|S(1)| = 2m^2$ and $S(-1) = 2n^2$.

**Proof.** Let $F$ be the characteristic polynomial of $T^*$; then $F = SC$, where $C$ is a power of a cyclotomic polynomial. Therefore we have either $C(X) = (X - 1)^2$, $C(X) = (X + 1)^2$, or $C = \Phi_m$, for $m = 3, 4$ or 6. The polynomial $F$ satisfies condition (C 1), hence (i) holds if $C(X) = (X - 1)^2$ or $C = \Phi_3$, (ii) holds if $C(X) = (X + 1)^2$ or $C = \Phi_6$, and (iii) holds if $C = \Phi_4$.

**Corollary 26.5.** Let $p$ and $q$ be two distinct prime numbers, let $S$ be a Salem polynomial of degree 20 such that $S(1) = -p$ and $S(-1) = q$, and let $\alpha$ be the corresponding Salem number. Then $\alpha$ is not realizable by any automorphism of a projective K3 surface.

**Proof.** This follows from Proposition 26.4 since $S$ does not satisfy any of the conditions (i), (ii) or (iii).

**Example 26.6.** It is easy to find Salem numbers satisfying the hypothesis of Corollary 26.5: for instance, $\alpha = 1,378928...$ (with $S(1) = -7$, $S(-1) = 5$), $\alpha = 1,394338...$ (with $S(1) = -5$, $S(-1) = 3$), $\alpha = 1,464501...$ (with $S(1) = -13$, $S(-1) = 3$), $\alpha = 1,464843...$ (with $S(1) = -7$, $S(-1) = 29$),... By Corollary 26.5 these Salem numbers are not realized by any automorphism of a projective K3 surface; note that they are realized by automorphisms of non-projective K3 surfaces (see Corollary 24.2 or Takada [T 22, Theorem 1.3]).

**Remark 26.7.** Let $\alpha = \lambda_{20} = 1,2326135486...$ be the smallest degree 20 Salem number, and let $S$ be the corresponding Salem polynomial. We have $S(1) = -1$ and $S(-1) = 11$, hence $S$ satisfies condition (ii) of Proposition 26.4. Nevertheless, $\alpha$ is not realized by any automorphism of a projective K3 surface, as shown by McMullen in [McM 16, §9].
27. Automorphisms which act trivially on Picard groups

In this section, we consider automorphisms of projective $K3$ surfaces that induce the identity on their Picard groups - in particular, they have dynamical degree one. The aim is to give an alternative approach to some results of Vorontsov [V 83], Kondo [K 92], Machida-Oguiso [MO 98] and Oguiso-Zhang [OZ 00]; see also Brandhorst [Br 19] and the references therein for more recent results.

Let $C$ be a power of a cyclotomic polynomial of even degree, i.e $C = \Phi_m^\ast$, where $m \geq 3$ and $n \geq 1$ are integers, and assume that $2 \leq \deg(C) \leq 20$.

**Lemma 27.1.** Let $T : \mathcal{X} \to \mathcal{X}$ be an automorphism of a projective $K3$ surface such that $T^\ast|\text{Pic}(\mathcal{X})$ is the identity, and that the characteristic polynomial of $T^\ast|\text{Trans}(\mathcal{X})$ is equal to $C$. Then the following conditions hold

(i) $C(-1)$ is a square.

(ii) If $C(1) = 1$, then $\deg(C) \equiv 4 \pmod{8}$.

Moreover, if $C(1) = 1$, then the lattice $\text{Trans}(\mathcal{X})$ is unimodular.

**Proof.** Set $F(X) = C(X)(X - 1)^{22 - \deg(C)}$; the polynomial $F$ is the characteristic polynomial of $T^\ast$. Since $T^\ast$ is an isometry of a unimodular lattice, the polynomial $F$ satisfies condition (C 1) (see §2), and hence $F(-1)$ is a square; this implies that $C(-1)$ is a square; therefore condition (i) holds.

Set $L = H^2(\mathcal{X}, \mathbb{Z})$, set $L_1 = \text{Ker}(C(T^\ast))$, and let $L_2$ be the sublattice of fixed points of $T^\ast$. If $C(1) = 1$, then the polynomials $X - 1$ and $C$ are relatively prime over $\mathbb{Z}$; this implies that $L = L_1 \oplus L_2$, hence $L_1$ and $L_2$ are both unimodular. We have $L_2 = \text{Pic}(\mathcal{X})$ and $L_1 = \text{Trans}(\mathcal{X})$; this shows that the lattice $\text{Trans}(\mathcal{X})$ is unimodular. The signature of the lattice $L_1$ is $(2, \deg(C) - 2)$; this lattice is unimodular, therefore $\deg(C) - 4 \equiv 0 \pmod{8}$, and this implies that $\deg(C) \equiv 4 \pmod{8}$, hence (ii) holds.

**Proposition 27.2.** Suppose that $C$ is a cyclotomic polynomial. There exists an automorphism $T : \mathcal{X} \to \mathcal{X}$ of a projective $K3$ surface such that $T^\ast|\text{Pic}(\mathcal{X})$ is the identity and that the characteristic polynomial of $T^\ast|\text{Trans}(\mathcal{X})$ is equal to $C$ if and only if the following conditions hold:

(i) $C(-1) = 1$.

(ii) If $C(1) = 1$, then $\deg(C) \equiv 4 \pmod{8}$.

The $K3$ surface $\mathcal{X}$ with this property is unique up to isomorphism. Moreover, the lattice $\text{Trans}(\mathcal{X})$ is unimodular if and only if $C(1) = 1$.

**Proof.** Let $T : \mathcal{X} \to \mathcal{X}$ be an automorphism of a projective $K3$ surface such that $T^\ast|\text{Pic}(\mathcal{X})$ is the identity, and that the characteristic polynomial of $T^\ast|\text{Trans}(\mathcal{X})$ is equal to $C$. Condition (ii) follows from part (ii) of Lemma 27.1. Since $C$ is a cyclotomic polynomial, $C(-1)$ is a square if and only if $C(-1) = 1$, therefore part (i) of Lemma 27.1 implies (i).

Conversely, assume that conditions (i) and (ii) hold, and let us prove the existence of $T : \mathcal{X} \to \mathcal{X}$ with the required properties. Set $F(X) = C(X)(X - 1)^{22 - \deg(C)}$; by (i), the polynomial $F$ satisfies condition (C 1).

If $C(1) > 1$ and $\deg(C) < 20$, then $G_F = 0$; if $\deg(C) = 20$ and $C(1)$ is not a square, then $G_F(D_\perp) = 0$ (see §11 and §12). By Corollary 12.3 there exists an even, unimodular lattice $L$ of signature $(3, 19)$ and an isometry $t : L \to L$ with...
characteristic polynomial $F$ and signature map $\tau$ satisfying $\tau(C) = (2, \deg(C) - 2)$. Let $L_1 = \ker(C(t))$ and let $L_2$ be the sublattice of $L$ of fixed points by $t$.

Suppose now that $C(1)$ is a square; since $C$ is a cyclotomic polynomial, this implies that $C(1) = 1$; condition (ii) implies that $\deg(C) \equiv 4 \mod 8$. Hence the polynomial $C$ satisfies Condition (C1), and by [BT20, Theorem A (or Corollary 12.3) there exists an even, unimodular lattice $L_1$ of signature $(2, \deg(C) - 2)$ having an isometry $t_1 : L_1 \rightarrow L_1$ of characteristic polynomial $C$. Let $L_2$ be an even, unimodular lattice of signature $(1, 22 - \deg(C))$. Then $L = L_1 \oplus L_2$ is isomorphic to $\Lambda_{3,19}$. Let $t : L \rightarrow L$ be the isometry that is equal to $t_1$ on $L_1$, and is the identity on $L_2$.

The isometry $t : L \rightarrow L$ is realizable (see McMullen, [McM16, Theorem 6.1), and gives rise to an automorphism $T : X \rightarrow X$ of a projective $K3$ surface such that $\text{Pic}(X)$ is isomorphic to $L_2$, and $\text{Trans}(X)$ is isomorphic to $L_1$.

If $C(1) = 1$, then the lattice $\text{Trans}(X)$ is unimodular by Lemma 27.1. Conversely, suppose that $\text{Trans}(X)$ is unimodular. Its signature is $(2, \deg(C) - 2)$, and this implies that $\deg(C) \equiv 4 \mod 8$. Moreover, $C$ satisfies condition (C1), hence $C(1)$ and $C(-1)$ are squares; since $C$ is cyclotomic, this implies that $C(1) = C(-1) = 1$.

The uniqueness of the $K3$ surface up to isomorphism follows from a result of Brandhorst, [Br19], Theorem 1.2.

We now show that Proposition 27.2 implies some results of Vorontsov, Kondo, Machida-Oguiso and Oguiso-Zhang. If $X$ is a $K3$ surface, we denote by $H_X$ the kernel of the map $\text{Aut}(X) \rightarrow \text{O} \left( \text{Pic}(X) \right)$; it is a cyclic group of finite order (see [N79, Corollary 3.3]). Let $m_X$ be the order of $H_X$. Set $\Sigma = \{12, 28, 36, 42, 44, 66\}$ and $\Omega = \{3, 9, 27, 5, 25, 7, 11, 13, 17, 19\}$. The following corollaries are a combination of results of [V83], [K92], [MO98] and [OZ00]:

**Corollary 27.3.** Let $X$ be a projective $K3$ surface, and suppose that the rank of $\text{Trans}(X)$ is equal to $\varphi(m_X)$. Assume that the lattice $\text{Trans}(X)$ is unimodular. Then $m_X \in \Sigma$. Conversely, for each $m \in \Sigma$, there exists a unique (up to isomorphism) projective $K3$ surface with $m_X = m$ and $\text{rank}(\text{Trans}(X)) = \varphi(m)$.

**Corollary 27.4.** Let $X$ be a projective $K3$ surface, and suppose that the rank of $\text{Trans}(X)$ is equal to $\varphi(m_X)$. Assume that the lattice $\text{Trans}(X)$ is not unimodular. Then $m_X \in \Omega$. Conversely, for each $m \in \Omega$, there exists a unique (up to isomorphism) projective $K3$ surface with $m_X = m$ and $\text{rank}(\text{Trans}(X)) = \varphi(m)$.

**Proof of Corollaries 27.3 and 27.4.** Let $T$ be a generator of the cyclic group $H_X$, and note that $m_X = m$ if and only if the characteristic polynomial of the restriction of $T^*$ to $\text{Trans}(X)$ is equal to $\Phi_m$.

Suppose that the characteristic polynomial of the restriction of $T^*$ to $\text{Trans}(X)$ is equal to $\Phi_m$. Since the rank of $\text{Trans}(X)$ is at most 20, we have $\deg(\Phi_m) \leq 20$. By Proposition 27.2, $C = \Phi_m$ satisfies conditions (i) and (ii), and this implies that $m \in \Sigma$ if $\Phi_m(1) = 1$, and $m \in \Omega$ otherwise; therefore $m_X \in \Sigma$ if $\text{Trans}(X)$ is unimodular, and $m_X \in \Omega$ otherwise.

Conversely, if $m \in \Sigma$, then $\Phi_m(1) = \Phi_m(-1) = 1$ and $\deg(\Phi_m) \equiv 4 \mod 8$; if $m \in \Omega$, then $\Phi_m(-1) = 1$ and $\Phi_m(1) > 1$, hence Proposition 27.2 implies the desired result.
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