THE $n$ LINEAR EMBEDDING THEOREM

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Abstract. Let $\sigma_i$, $i = 1, \ldots, n$, denote positive Borel measures on $\mathbb{R}^d$, let $\mathcal{D}$ denote the usual collection of dyadic cubes in $\mathbb{R}^d$ and let $K : \mathcal{D} \to [0, \infty)$ be a map. In this paper we give a characterization of the $n$ linear embedding theorem. That is, we give a characterization of the inequality

$$\sum_{Q \in \mathcal{D}} K(Q) \prod_{i=1}^n \left| \int_Q f_i \, d\sigma_i \right| \leq C \prod_{i=1}^n \| f_i \|_{L^{p_i}(d\sigma_i)}$$

in terms of multilinear Sawyer’s checking condition and discrete multinonlinear Wolff’s potential, when $1 < p_i < \infty$.

1. Introduction

The purpose of this paper is to investigate the $n$ linear embedding theorem. We first fix some notations. We will denote by $\mathcal{D}$ the family of all dyadic cubes $Q = 2^{-k}(m + [0, 1)^d)$, $k \in \mathbb{Z}$, $m \in \mathbb{Z}^d$. Let $K : \mathcal{D} \to [0, \infty)$ be a map and let $\sigma_i$, $i = 1, \ldots, n$, be positive Borel measures on $\mathbb{R}^d$. In this paper we give a necessary and sufficient condition for which the inequality

$$(1.1) \quad \sum_{Q \in \mathcal{D}} K(Q) \prod_{i=1}^n \left| \int_Q f_i \, d\sigma_i \right| \leq C \prod_{i=1}^n \| f_i \|_{L^{p_i}(d\sigma_i)},$$

to hold when $1 < p_i < \infty$.

For the bilinear embedding theorem, in the case $\frac{1}{p_1} + \frac{1}{p_2} \geq 1$, Sergei Treil gives a simple proof of the following.

Proposition 1.1 ([19, Theorem 2.1]). Let $K : \mathcal{D} \to [0, \infty)$ be a map and let $\sigma_i$, $i = 1, 2$, be positive Borel measures on $\mathbb{R}^d$. Let $1 < p_i < \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} \geq 1$. The following statements are equivalent:

(a) The following bilinear embedding theorem holds:

$$\sum_{Q \in \mathcal{D}} K(Q) \prod_{i=1}^2 \left| \int_Q f_i \, d\sigma_i \right| \leq C \prod_{i=1}^2 \| f_i \|_{L^{p_i}(d\sigma_i)} < \infty;$$

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Proposition 1.2

The author prove the following. The discrete Wolff’s potential in \([5, 6]\) called “the Sawyer type checking condition”, since this was first introduced by Eric T. Sawyer.

Moreover, the least possible \(c_1\) and \(c_2\) are equivalent.

Here, for each \(1 < p < \infty\), \(p'\) denote the dual exponent of \(p\), i.e., \(p' = \frac{p}{p-1}\), and \(1_E\) stands for the characteristic function of the set \(E\).

Proposition 1.1 was first proved for \(p_1 = p_2 = 2\) in [4] by the Bellman function method. Later in [3], this was proved in full generality. The checking condition in Proposition 1.4 is called “the Sawyer type checking condition”, since this was first introduced by Eric T. Sawyer in [5, 6].

To describe the case \(\frac{1}{p_1} + \frac{1}{p_2} < 1\), we need discrete Wolff’s potential.

Let \(\mu\) and \(\nu\) be positive Borel measures on \(\mathbb{R}^d\) and let \(K : \mathcal{D} \rightarrow [0, \infty)\) be a map. For \(p > 1\), the discrete Wolff’s potential \(W_{K,\mu}^p[\nu](x)\) of the measure \(\nu\) is defined by

\[
W_{K,\mu}^p[\nu](x) := \sum_{Q \in \mathcal{D}} K(Q)\mu(Q) \left( \frac{1}{\mu(Q)} \sum_{Q' < Q} K(Q')\mu(Q')\nu(Q') \right)^{p-1} 1_Q(x), \quad x \in \mathbb{R}^d.
\]

The author prove the following.

Proposition 1.2 ([7, Theorem 1.3]). Let \(K : \mathcal{D} \rightarrow [0, \infty)\) be a map and let \(\sigma_i, \ i = 1, 2\), be positive Borel measures on \(\mathbb{R}^d\). Let \(1 < p_i < \infty\) and \(\frac{1}{p_1} + \frac{1}{p_2} < 1\). The following statements are equivalent:

(a) The following bilinear embedding theorem holds:

\[
\sum_{Q \in \mathcal{D}} K(Q) \prod_{i=1}^2 \int_Q f_i \, d\sigma_i \leq c_1 \prod_{i=1}^2 \|f_i\|_{L^{p_i}(d\sigma_i)} < \infty;
\]

(b) For \(\frac{1}{p} + \frac{1}{p_1} + \frac{1}{p_2} = 1\),

\[
\begin{align*}
\|W_{K,\sigma_1}^{p_1}[\sigma_1]^{1/p_1}\|_{L^p(d\sigma_1)} & \leq c_2 < \infty, \\
\|W_{K,\sigma_1}^{p_2}[\sigma_2]^{1/p_2}\|_{L^p(d\sigma_2)} & \leq c_2 < \infty.
\end{align*}
\]

Moreover, the least possible \(c_1\) and \(c_2\) are equivalent.

In his exercent survey of the \(A_2\) theorem [2], Tuomas P. Hytönen introduces another proof of Proposition 1.4 which uses the “parallel corona” decomposition. In this paper, following Hytönen’s arguments in [2], we shall establish the following theorems (Theorems 1.3 and 1.4).

Theorem 1.3. Let \(K : \mathcal{D} \rightarrow [0, \infty)\) be a map and let \(\sigma_i, \ i = 1, \ldots, n\), be positive Borel measures on \(\mathbb{R}^d\). Let \(1 < p_i < \infty\) and \(\sum_{i=1}^n \frac{1}{p_i} \geq 1\). The following statements are equivalent:
(a) The following \( n \) linear embedding theorem holds:
\[
\sum_{Q \in \mathcal{D}} K(Q) \prod_{i=1}^{n} \int_{Q} f_{i} \, d\sigma_{i} \leq c_{1} \prod_{i=1}^{n} \|f_{i}\|_{L^{p_{i}}(d\sigma_{i})} < \infty;
\]
(b) For all \( j = 1, \ldots, n \) and for all \( Q \in \mathcal{D} \),
\[
\sum_{Q' \subset Q} K(Q') \sigma_{j}(Q') \prod_{i=1}^{n} \int_{Q'} f_{i} \, d\sigma_{i} \leq c_{2} \sigma_{j}(Q)^{1/p_{j}} \prod_{i=1}^{n} \|f_{i}\|_{L^{p_{i}}(d\sigma_{i})} < \infty.
\]
Moreover, the least possible \( c_{1} \) and \( c_{2} \) are equivalent.

Let the symmetric group \( S_{n} \) be the set of all permutations of the set \( \{1, \ldots, n\} \), that is, the set of all bijections from the set \( \{1, \ldots, n\} \) to itself. Let \( K : \mathcal{D} \to [0, \infty) \) be a map and let \( \sigma_{i} \), \( i = 1, \ldots, n \), be positive Borel measures on \( \mathbb{R}^{d} \). Let \( 1 < p_{i} < \infty \) and \( \sum_{i=1}^{n} \frac{1}{p_{i}} < 1 \).

Let \( \phi \in S_{n} \). Set
\[
\frac{1}{r_{1}^{\phi}} + \frac{1}{p_{\phi(1)}} = 1,
\frac{1}{r_{2}^{\phi}} + \frac{1}{p_{\phi(1)}} + \frac{1}{p_{\phi(2)}} = 1,
\vdots
\frac{1}{r_{n-1}^{\phi}} + \sum_{i=1}^{n-1} \frac{1}{p_{\phi(i)}} = 1,
\frac{1}{r} + \sum_{i=1}^{n} \frac{1}{p_{\phi(i)}} = 1.
\]
Let, for \( Q \in \mathcal{D} \),
\[
K_{1}^{\phi}(Q) := K(Q) \sigma_{\phi(1)}(Q) \left( \frac{1}{\sigma_{\phi(1)}(Q)} \sum_{Q' \subset Q} K(Q') \prod_{i=1}^{n} \sigma_{\phi(i)}(Q') \right)^{r_{1}^{\phi} - 1},
\]
let
\[
K_{2}^{\phi}(Q) := K_{1}^{\phi}(Q) \sigma_{\phi(2)}(Q) \left( \frac{1}{\sigma_{\phi(2)}(Q)} \sum_{Q' \subset Q} K_{1}^{\phi}(Q') \prod_{i=2}^{n} \sigma_{\phi(i)}(Q') \right)^{r_{2}^{\phi} / r_{1}^{\phi} - 1}
\]
and, inductively, for \( j = 3, \ldots, n - 1 \), let
\[
K_{j}^{\phi}(Q) := K_{j-1}^{\phi}(Q) \sigma_{\phi(j)}(Q) \left( \frac{1}{\sigma_{\phi(j)}(Q)} \sum_{Q' \subset Q} K_{j-1}^{\phi}(Q') \prod_{i=j}^{n} \sigma_{\phi(i)}(Q') \right)^{r_{j}^{\phi} / r_{j-1}^{\phi} - 1}.
\]

**Theorem 1.4.** With the notation above, the following statements are equivalent:

(a) The following \( n \) linear embedding theorem holds:
\[
\sum_{Q \in \mathcal{D}} K(Q) \prod_{i=1}^{n} \int_{Q} f_{i} \, d\sigma_{i} \leq c_{1} \prod_{i=1}^{n} \|f_{i}\|_{L^{p_{i}}(d\sigma_{i})} < \infty;
\]
For all \( \phi \in S_n \),
\[
\left\| \left( \sum_{Q \in \mathcal{D}} K_{n-1}^\phi(Q)1_Q \right)^{1/r_{n-1}} \right\|_{L^r(d\sigma_{\phi(n)})} \leq c_2 < \infty.
\]

Moreover, the least possible \( c_1 \) and \( c_2 \) are equivalent.

Even though Theorems 1.3 and 1.4 both characterize the same \( n \) linear embedding theorem, it seems that the characterizations are very different. In very recent paper [1], Timo S. Hänninen, Tuomas P. Hytönen and Kangwei Li give a unified approach saying “sequential testing” characterization, when \( n = 2, 3 \). Especially, our Theorem 1.4 with \( n = 3 \) is obtained in [1, Theorem 1.16]. (An alternative form of another unified characterization has been simultaneously obtained by Vuorinen [10].) In [8], the author gives a characterization of the trilinear embedding theorem in terms of Theorem 1.3 and Propositions 1.1 and 1.2.

The letter \( C \) will be used for constants that may change from one occurrence to another.

\[2. \text{Proof of the necessity} \]

In what follows we shall prove the necessity of theorems. The necessity of Theorem 1.3, that is, (b) follows from (a) at once if we substitute the test function \( f_j = 1_Q \). So, we shall verify the necessity of Theorem 1.4. We need a lemma (cf. Lemma 2.1 in [7]).

**Lemma 2.1.** Let \( \sigma \) be a positive Borel measure on \( \mathbb{R}^d \). Let \( 1 < s < \infty \) and \( \{\alpha_Q\}_{Q \in \mathcal{D}} \subset [0, \infty) \).

Define, for \( Q_0 \in \mathcal{D} \),
\[
A_1 := \int_{Q_0} \left( \sum_{Q \subset Q_0} \frac{\alpha_Q}{\sigma(Q)}1_Q \right)^s d\sigma,
\]
\[
A_2 := \sum_{Q \subset Q_0} \alpha_Q \left( \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \alpha_{Q'} \right)^{s-1},
\]
\[
A_3 := \int_{Q_0} \sup_{Q \subset Q_0} \left( \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \alpha_{Q'} \right)^s d\sigma(x).
\]

Then
\[A_1 \leq c(s)A_2, \quad A_2 \leq c(s)\frac{1}{s}A_3 \quad \text{and} \quad A_3 \leq (s')^s A_1.\]

Here,
\[c(s) := \begin{cases} 
  s, & 1 < s \leq 2, \\
  (s(s-1) \cdots (s-k))^{\frac{1}{s-k-1}}, & 2 < s < \infty,
\end{cases}\]

where \( k = \lceil s - 2 \rceil \) is the smallest integer greater than \( s - 2 \).

We will use \( \frac{1}{\sigma(Q)} \int_Q f \, d\sigma \) to denote the integral average \( \sigma(Q)^{-1} \int_Q f \, d\sigma \). The dyadic maximal operator \( M_{\mathcal{D}}^s \) is defined by
\[
M_{\mathcal{D}}^s f(x) := \sup_{Q \in \mathcal{D}} \frac{1}{\sigma(Q)} \int_Q |f(y)| \, d\sigma(y).
\]
Suppose that (a) of Theorem 1.4. Then, for \( \phi \in S_n \),

\[
(2.1) \quad \sum_{Q \in \mathcal{D}} K(Q) \prod_{i=1}^{n} \left| \int_Q f_{\phi(i)} \, d\sigma_{\phi(i)} \right| \leq c_1 \prod_{i=2}^{n} \| f_{\phi(i)} \|_{L^{p_{\phi(i)}}(d\sigma_{\phi(i)})}.
\]

Recall that \( \frac{1}{r_1} + \frac{1}{p_{\phi(i)}} = 1 \). By duality, we see that

\[
\int_{\mathbb{R}^d} \left( \sum_{Q \in \mathcal{D}} K(Q) \prod_{i=2}^{n} \left| \int_Q f_{\phi(i)} \, d\sigma_{\phi(i)} \right| \right)^{r_1^\phi} \, d\sigma_{\phi(1)} \leq c_1 \prod_{i=2}^{n} \| f_{\phi(i)} \|_{L^{p_{\phi(i)}}(d\sigma_{\phi(i)})}^{r_1^\phi},
\]

which implies by Lemma 2.1

\[
\sum_{Q \in \mathcal{D}} K(Q)\sigma_{\phi(1)}(Q) \prod_{i=2}^{n} \left| \int_Q f_{\phi(i)} \, d\sigma_{\phi(i)} \right| \times \left[ \frac{1}{\sigma_{\phi(1)}(Q)} \sum_{Q' \subset Q} K(Q')\sigma_{\phi(1)}(Q') \prod_{i=2}^{n} \left| \int_{Q'} f_{\phi(i)} \, d\sigma_{\phi(i)} \right| \right]^{r_1^\phi-1}
\]

\[
\leq C c_1 \prod_{i=2}^{n} \| f_{\phi(i)} \|_{L^{p_{\phi(i)}}(d\sigma_{\phi(i)})}^{r_1^\phi}.
\]

It follows from this inequality that

\[
\sum_{Q \in \mathcal{D}} K_0^\phi(Q) \prod_{i=2}^{n} \left| \int_Q g_{\phi(i)} \, d\sigma_{\phi(i)} \right|
\]

\[
= \sum_{Q \in \mathcal{D}} K(Q)\sigma_{\phi(1)}(Q) \prod_{i=2}^{n} \left| \int_Q g_{\phi(i)} \, d\sigma_{\phi(i)} \right| \left[ \frac{1}{\sigma_{\phi(1)}(Q)} \sum_{Q' \subset Q} K(Q') \prod_{i=2}^{n} \sigma_{\phi(i)}(Q') \right]^{1/r_1^\phi}
\]

\[
= \sum_{Q \in \mathcal{D}} K(Q)\sigma_{\phi(1)}(Q) \prod_{i=2}^{n} \sigma_{\phi(i)}(Q) \left| \int_Q g_{\phi(i)} \, d\sigma_{\phi(i)} \right|^{1/r_1^\phi} \times \left[ \frac{1}{\sigma_{\phi(1)}(Q)} \sum_{Q' \subset Q} K(Q') \sigma_{\phi(1)}(Q') \prod_{i=2}^{n} \sigma_{\phi(i)}(Q') \left| \int_Q g_{\phi(i)} \, d\sigma_{\phi(i)} \right| \right]^{r_1^\phi-1}
\]

\[
\leq \sum_{Q \in \mathcal{D}} K(Q)\sigma_{\phi(1)}(Q) \prod_{i=2}^{n} \int_Q (M_\phi^\sigma g_{\phi(i)})^{1/r_1^\phi} \, d\sigma_{\phi(i)}
\]

\[
\times \left[ \frac{1}{\sigma_{\phi(1)}(Q)} \sum_{Q' \subset Q} K(Q') \sigma_{\phi(1)}(Q') \prod_{i=2}^{n} \int_{Q'} (M_\phi^\sigma g_{\phi(i)})^{1/r_1^\phi} \, d\sigma_{\phi(i)} \right]^{r_1^\phi-1}
\]

\[
\leq C c_1 \prod_{i=2}^{n} \left\| M_\phi^\sigma g_{\phi(i)} \right\|_{L^{p_{\phi(i)}/r_1^\phi}(d\sigma_{\phi(i)})}^{r_1^\phi} \prod_{i=2}^{n} \left\| g_{\phi(i)} \right\|_{L^{p_{\phi(i)}/r_1^\phi}(d\sigma_{\phi(i)})}^{r_1^\phi},
\]

where we have used the boundedness of dyadic maximal operators. Thus, we obtain

\[
(2.2) \quad \sum_{Q \in \mathcal{D}} K_1^\phi(Q) \prod_{i=2}^{n} \left| \int_Q f_{\phi(i)} \, d\sigma_{\phi(i)} \right| \leq C c_1 \prod_{i=2}^{n} \| f_{\phi(i)} \|_{L^{p_{\phi(i)}}(d\sigma_{\phi(i)})}.
\]
Notice that
\[
\begin{align*}
\frac{r_{i-1}^\phi}{r_i^\phi} + \frac{r_{i-1}^\phi}{p_{\phi(i)}} &= 1, \quad i = 2, \ldots, n - 1, \\
\frac{r_{n-1}^\phi}{r} + \frac{r_{n-1}^\phi}{p_{\phi(n)}} &= 1.
\end{align*}
\]
(2.3)

By the same manner as the above but starting from (2.2), instead of (2.1), and using (2.3) with \(i = 2\), we obtain
\[
\sum_{Q \in D} K_2^\phi(Q) \prod_{i=3}^n \left| \int_Q f_{\phi(i)} d\sigma_{\phi(i)} \right| \leq C c_1 r_2^\phi \prod_{i=3}^n \|f_{\phi(i)}\|_{L^{p_{\phi(i)}/r^\phi_i}\left(d\sigma_{\phi(i)}\right)}.
\]

By being continued inductively until the \(n - 1\) step, we obtain
\[
\sum_{Q \in D} K_{n-1}^\phi(Q) \left| \int_Q f_{\phi(n)} d\sigma_{\phi(n)} \right| \leq C c_1 r_{n-1}^\phi \|f_{\phi(n)}\|_{L^{p_{\phi(n)}/r_{n-1}^\phi(d\sigma_{\phi(n)})}}.
\]

Notice that the last equation of (2.3). Then by duality
\[
\left\| \sum_{Q \in D} K_{n-1}^\phi(Q) 1_Q \right\|_{L^{r/r_{n-1}^\phi(d\sigma_{\phi(n)})}} \leq C c_1 r_{n-1}^\phi
\]
and, hence,
\[
\left\| \left( \sum_{Q \in D} K_{n-1}^\phi(Q) 1_Q \right)^{1/r_{n-1}^\phi} \right\|_{L^r(d\sigma_{\phi(n)})} \leq C c_1,
\]
which completes the necessity of Theorem 1.4.

3. Proof of the sufficiency

In what follows we shall prove the sufficiency of theorems.

Let \(Q_0 \in D\) be taken large enough and be fixed. We shall estimate the quantity
\[
(3.1) \quad \sum_{Q \subset Q_0} K(Q) \prod_{i=1}^n \left( \int_Q f_i d\sigma_i \right),
\]
where \(f_i \in L^p(d\sigma_i)\) is nonnegative and is supported in \(Q_0\). We define the collection of principal cubes \(\mathcal{F}_i\) for the pair \((f_i, \sigma_i), i = 1, \ldots, n\). Namely,
\[
\mathcal{F}_i := \bigcup_{k=0}^\infty \mathcal{F}_i^k,
\]
where \(\mathcal{F}_i^0 := \{Q_0\}\),
\[
\mathcal{F}_i^{k+1} := \bigcup_{F \in \mathcal{F}_i^k} \text{ch}_{\mathcal{F}_i}(F)
\]
and \(\text{ch}_{\mathcal{F}_i}(F)\) is defined by the set of all “maximal” dyadic cubes \(Q \subset F\) such that
\[
\int_Q f_i d\sigma_i > 2 \int_F f_i d\sigma_i.
\]
Observe that
\[ \sum_{F' \in ch_{\mathcal{F}_i}(F)} \sigma_i(F') \leq \left( 2 \int_F f_i \, d\sigma_i \right)^{-1} \sum_{F' \in ch_{\mathcal{F}_i}(F)} \int_{F'} f_i \, d\sigma_i \leq \left( 2 \int_F f_i \, d\sigma_i \right)^{-1} \int_F f_i \, d\sigma_i = \frac{\sigma_i(F)}{2}, \]
which implies
\[ \sigma_i(E_{\mathcal{F}_i}(F)) := \sigma_i \left( F \setminus \bigcup_{F' \in ch_{\mathcal{F}_i}(F)} F' \right) \geq \frac{\sigma_i(F)}{2}, \]
where the sets \( E_{\mathcal{F}_i}(F), F \in \mathcal{F}_i, \) are pairwise disjoint. We further define the stopping parents, for \( Q \in \mathbb{D}, \)
\[ \begin{aligned}
\pi_{\mathcal{F}_i}(Q) &:= \min \{ F \supset Q : F \in \mathcal{F}_i \}, \\
\pi(Q) &:= (\pi_{\mathcal{F}_1}(Q), \ldots, \pi_{\mathcal{F}_n}(Q)).
\end{aligned} \]
Then we can rewrite the series in \( (3.1) \) as follows:
\[ \sum_{Q \subset Q_0} = \sum_{(F_1, \ldots, F_n) \in (\mathcal{F}_1, \ldots, \mathcal{F}_n)} \sum_{\pi(Q) \in (F_1, \ldots, F_n)} \cdot \]
We notice the elementary fact that, if \( P, R \in \mathbb{D}, \) then \( P \cap R \in \{ P, R, \emptyset \}. \) This fact implies, if \( \pi(Q) = (F_1, \ldots, F_n), \) then
\[ Q \subset F_{\phi(1)} \subset \cdots \subset F_{\phi(n)} \quad \text{for some} \quad \phi \in S_n. \]
Thus, for fixed \( \phi \in S_n, \) we shall estimate
\[ \sum_{F_{\phi(1)} \subset \cdots \subset F_{\phi(n)}} K(Q) \prod_{i=1}^{n} \left( \int_Q f_{\phi(i)} \, d\sigma_{\phi(i)} \right). \]

**Proof of (a) of Theorem 1.23** It follows that, for fixed \( F_{\phi(n)} \in \mathcal{F}_{\phi(n)}, \)
\[ \sum_{F_{\phi(1)} \subset \cdots \subset F_{\phi(n)}} \sum_{Q: \pi(Q) = (F_{\phi(i)})} K(Q) \prod_{i=1}^{n} \left( \int_Q f_{\phi(i)} \, d\sigma_{\phi(i)} \right) \leq \left( 2 \int_{F_{\phi(n)}} f_{\phi(n)} \, d\sigma_{\phi(n)} \right)^{-1} \sum_{F_{\phi(1)} \subset \cdots \subset F_{\phi(n)}} \sum_{Q: \pi(Q) = (F_{\phi(i)})} K(Q) \sigma_{\phi(n)}(Q) \prod_{i=1}^{n-1} \left( \int_Q f_{\phi(i)} \, d\sigma_{\phi(i)} \right). \]

We need two observations. Suppose that \( F_{\phi(1)} \subset \cdots \subset F_{\phi(n)} \) and \( \pi(Q) = (F_{\phi(i)}). \) Let \( i = 1, \ldots, n - 1. \) If \( F' \in ch_{\mathcal{F}_{\phi(n)}}(F_{\phi(n)}) \) satisfies \( F' \subset Q. \) Then
\[ \pi_{\mathcal{F}_{\phi(n)}}(\pi_{\mathcal{F}_{\phi(n)}}(F')) = \begin{cases} F_{\phi(n)}, & \text{when } F' \notin \mathcal{F}_{\phi(i)}; \\
F', & \text{when } F' \in \mathcal{F}_{\phi(i)}. \end{cases} \]
By this observation, we define
\[ ch_{\mathcal{F}_{\phi(n)}}(F_{\phi(n)}) := \{ F' \in ch_{\mathcal{F}_{\phi(n)}}(F_{\phi(n)}) : F' \text{ satisfies } (3.4) \}. \]
We further observe that, when $F' \in \mathcal{H}^{\phi(i)}(F_{\phi(n)})$, we can regard $f_{\phi(i)}$ as a constant on $F'$ in the above integrals, that is, we can replace $f_{\phi(i)}$ by $f_{\phi(i)}^{F_{\phi(n)}}$ in the above integrals, where

$$
  f_{\phi(i)}^{F_{\phi(n)}} := f_{\phi(i)}1_{E_{\phi(n)}}(F_{\phi(n)}) + \sum_{F' \in ch^{\phi(i)}(F_{\phi(n)})} \left( \int_{F'} f_{\phi(i)} d\sigma_{\phi(i)} \right) 1_{F'}.
$$

It follows from Hölder’s inequality with exponents $(1)$ that

$$
  \sum_{F_{\phi(n)} \subseteq \cdots \subseteq F_{\phi(n)}} \prod_{i=1}^{n-1} K(Q)\sigma_{\phi(n)}(Q) \prod_{i=1}^{n-1} \left( \int_{Q} f_{\phi(i)}^{F_{\phi(n)}} d\sigma_{\phi(i)} \right) \leq c_2 \sigma_{\phi(n)}(F_{\phi(n)})^{1/p_{\phi(n)}} \prod_{i=1}^{n-1} \| f_{\phi(n)}^{F_{\phi(n)}} \|_{L^{p_{\phi(i)}}(d\sigma_{\phi(i)})} \left( \int_{F_{\phi(n)}} f_{\phi(n)} d\sigma_{\phi(n)} \right) \sigma_{\phi(n)}(F_{\phi(n)})^{1/p_{\phi(n)}}.
$$

Thus, we obtain

$$
  \sum_{i=1}^{n} \frac{1}{s_{\phi(i)}} + \frac{1}{p_{\phi(n)}} = 1 \quad \text{and} \quad 1 < p_{\phi(i)} \leq s_{\phi(i)} < \infty.
$$

It follows from Hölder’s inequality with exponents $s_{\phi(1)}, \ldots, s_{\phi(n-1)}, p_{\phi(n)}$ that

$$
  \frac{1}{s_{\phi(i)}} \leq C_2 \prod_{i=1}^{n-1} \left[ \sum_{F_{\phi(n)} \subseteq \cdots \subseteq F_{\phi(n)}} \| f_{\phi(n)}^{F_{\phi(n)}} \|_{L^{p_{\phi(i)}}(d\sigma_{\phi(i)})} \right]^{1/s_{\phi(i)}} \times \left[ \sum_{F_{\phi(n)} \subseteq \cdots \subseteq F_{\phi(n)}} \left( \int_{F_{\phi(n)}} f_{\phi(n)} d\sigma_{\phi(n)} \right)^{p_{\phi(n)}} \sigma_{\phi(n)}(F_{\phi(n)}) \right]^{1/p_{\phi(n)}} \leq C_2 \prod_{i=1}^{n-1} \left[ \sum_{F_{\phi(n)} \subseteq \cdots \subseteq F_{\phi(n)}} \| f_{\phi(n)}^{F_{\phi(n)}} \|_{L^{p_{\phi(i)}}(d\sigma_{\phi(i)})} \right]^{1/p_{\phi(i)}} \times \left[ \sum_{F_{\phi(n)} \subseteq \cdots \subseteq F_{\phi(n)}} \left( \int_{F_{\phi(n)}} f_{\phi(n)} d\sigma_{\phi(n)} \right)^{p_{\phi(n)}} \sigma_{\phi(n)}(F_{\phi(n)}) \right]^{1/p_{\phi(n)}} =: C_2 (I_1) \times \cdots \times (I_n),
$$

where we have used $\| \cdot \|_{L^{p_{\phi(i)}}} \geq \| \cdot \|_{L^{s_{\phi(i)}}}$.

For $(I_n)$, using $\sigma_{\phi(n)}(F_{\phi(n)}) \leq 2\sigma_{\phi(n)}(E_{\phi(n)}(F_{\phi(n)}))$ (see (3.22)), the fact that

$$
  \int_{F_{\phi(n)}} f_{\phi(n)} d\sigma_{\phi(n)} \leq \inf_{y \in F_{\phi(n)}} M_{D}^{\sigma_{\phi(n)}} f_{\phi(n)}(y)
$$
and the disjointness of the sets $E_{\mathcal{F},(n)}(F_{\phi(n)})$, we have

$$(I_n) \leq C \left[ \sum_{F_{\phi(n)} \in F_{\phi(n)}} \int_{E_{\mathcal{F},(n)}(F_{\phi(n)})} (M^p_{D} f_{\phi(n)})^{p_{\phi(n)}} d\sigma_{\phi(n)} \right]^{1/p_{\phi(n)}}$$

$$\leq C \left[ \int_{Q_0} (M^p_{D} f_{\phi(n)})^{p_{\phi(n)}} d\sigma_{\phi(n)} \right]^{1/p_{\phi(n)}} \leq C \| f_{\phi(n)} \|_{L^p_{\phi(n)}(d\sigma_{\phi(n)})}.$$ 

It remains to estimate $(I_i)$, $i = 1, \ldots, n - 1$. It follows that

$$(I_i)_{p_{\phi(i)}} = \sum_{F_{\phi(n)} \in F_{\phi(n)}} \int_{E_{\mathcal{F},(n)}(F_{\phi(n)})} f_{\phi(i)}^{p_{\phi(i)}} d\sigma_{\phi(i)}$$

$$+ \sum_{F_{\phi(n)} \in F_{\phi(n)}} \sum_{F' \in ch_{\mathcal{F},(n)}(F_{\phi(n)})} \left( \int_{F'} f_{\phi(i)}^{p_{\phi(i)}} d\sigma_{\phi(i)} \right)^{p_{\phi(i)}} \sigma_{\phi(i)}(F').$$

By the pairwise disjointness of the sets $E_{\mathcal{F},(n)}(F_{\phi(n)})$, it is immediate that

$$\sum_{F_{\phi(n)} \in F_{\phi(n)}} \int_{E_{\mathcal{F},(n)}(F_{\phi(n)})} f_{\phi(i)}^{p_{\phi(i)}} d\sigma_{\phi(i)} \leq \| f_{\phi(i)} \|_{L^{p_{\phi(i)}}(d\sigma_{\phi(i)})}.$$ 

For the remaining double sum, there holds by the uniqueness of the parent

$$\sum_{F_{\phi(n)} \in F_{\phi(n)}} \sum_{F' \in ch_{\mathcal{F},(n)}(F_{\phi(n)}): F' \text{ satisfies } (3.3)} \left( \int_{F'} f_{\phi(i)}^{p_{\phi(i)}} d\sigma_{\phi(i)} \right)^{p_{\phi(i)}} \sigma_{\phi(i)}(F')$$

$$\leq 2 \sum_{F \in \mathcal{F}_{\phi(i)}} \sum_{F' \in ch_{\mathcal{F}_{\phi(i)}(F_{\phi(n)})}: \pi_{\mathcal{F}_{\phi(i)}(F')} = F_{\phi(n)}} \left( \int_{F'} f_{\phi(i)}^{p_{\phi(i)}} d\sigma_{\phi(i)} \right)^{p_{\phi(i)}} \sigma_{\phi(i)}(F')$$

$$\leq 2 \sum_{F \in \mathcal{F}_{\phi(i)}} \left( 2 \int_{F} f_{\phi(i)}^{p_{\phi(i)}} d\sigma_{\phi(i)} \right)^{p_{\phi(i)}} \sigma_{\phi(i)}(F)$$

$$\leq C \| M^p_{D} f_{\phi(i)} \|_{L^{p_{\phi(i)}}(d\sigma_{\phi(i)})} \leq C \| f_{\phi(i)} \|_{L^{p_{\phi(i)}}(d\sigma_{\phi(i)})}.$$ 

Altogether, we obtain

$$(3.3) \leq C C_2 \prod_{i=1}^{n} \| f_{\phi(i)} \|_{L^{p_{\phi(i)}}(d\sigma_{\phi(i)})}. $$

This yields (a) of Theorem 1.3.

**Proof of (a) of Theorem 1.3.** We shall estimate (3.3) by use of multilinear Wolff’s potential. We first observe that if $F_{\phi(i)} \in \mathcal{F}_{\phi(i)}$, $i = 1, \ldots, n$, satisfy $F_{\phi(1)} \subset \cdots \subset F_{\phi(n)}$ and, for some $Q \in D$, $\pi(Q) = (F_{\phi(i)})$, then

$$\pi_{\mathcal{F}_{\phi(j)}(F_{\phi(i)})} = F_{\phi(j)} \text{ for all } 1 \leq i < j \leq n.$$ 

Fix $F_{\phi(i)} \in \mathcal{F}_{\phi(i)}$, $i = 1, \ldots, n$, that satisfy (3.3). Then

$$C \left( \sum_{\pi(Q) = (F_{\phi(i)})} K(Q) \prod_{i=1}^{n} \left( \int_{Q} f_{\phi(i)}^{\pi_{\phi(i)}} d\sigma_{\phi(i)} \right) \right) \leq \prod_{i=1}^{n} \left( 2 \int_{F_{\phi(i)}} f_{\phi(i)}^{\pi_{\phi(i)}} d\sigma_{\phi(i)} \right) \sum_{\pi(Q) = (F_{\phi(i)})} K(Q) \prod_{i=1}^{n} \sigma_{\phi(i)}(Q).$$
Recall that

\[
\begin{aligned}
\frac{1}{r^j} + \sum_{i=1}^{j} \frac{1}{p_{\phi(i)}} &= 1, \quad j = 1, \ldots, n - 1, \\
\frac{1}{r} + \sum_{i=1}^{n} \frac{1}{p_{\phi(i)}} &= 1.
\end{aligned}
\]

(3.6)

In the following estimates, \( \sum_{F_{\phi(i)}} \) runs over all \( F_{\phi(1)} \in \mathcal{F}_{\phi(1)} \) that satisfy (3.5) for fixed \( F_{\phi(i)} \in \mathcal{F}_{\phi(i)}, i = 2, \ldots, n \).

\[
\begin{aligned}
\sum_{F_{\phi(1)}} & \left( \int_{F_{\phi(1)}} f_{\phi(1)} d\sigma_{\phi(1)} \right) \sum_{Q: \pi(Q) = (F_{\phi(1)})} K(Q) \prod_{i=1}^{n} \sigma_{\phi(i)}(Q) \\
& \leq \sum_{F_{\phi(1)}} \left( \int_{F_{\phi(1)}} f_{\phi(1)} d\sigma_{\phi(1)} \right) \sum_{Q \subset F_{\phi(1)}} K(Q) \prod_{i=1}^{n} \sigma_{\phi(i)}(Q) \\
& = \sum_{F_{\phi(1)}} \left( \int_{F_{\phi(1)}} \left( \sum_{Q \subset F_{\phi(1)}} K(Q) \prod_{i=2}^{n} \sigma_{\phi(i)}(Q) \right) d\sigma_{\phi(1)} \right) \frac{\sigma_{\phi(1)}(F_{\phi(1)})}{\|\phi(1)\|_{\rho}} \\
& \quad \times \left( \sum_{F_{\phi(1)}} \left( \int_{F_{\phi(1)}} \left( \sum_{Q \subset F_{\phi(1)}} K(Q) \prod_{i=2}^{n} \sigma_{\phi(i)}(Q) \right) d\sigma_{\phi(1)} \right) \frac{\sigma_{\phi(1)}(F_{\phi(1)})}{\|\phi(1)\|_{\rho}} \right)^{-1/r^\rho_{i}},
\end{aligned}
\]

where we have used (3.6) with \( j = 1 \). By Hölder’s inequality, we have further that

\[
\begin{aligned}
& \leq \left[ \sum_{F_{\phi(1)}} \left( \int_{F_{\phi(1)}} f_{\phi(1)} d\sigma_{\phi(1)} \right)^{p_{\phi(1)}} \sigma_{\phi(1)}(F_{\phi(1)}) \right]^{1/p_{\phi(1)}} \\
& \quad \times \left[ \sum_{F_{\phi(1)}} \left( \int_{F_{\phi(1)}} \left( \sum_{Q \subset F_{\phi(1)}} K(Q) \prod_{i=2}^{n} \sigma_{\phi(i)}(Q) \right) d\sigma_{\phi(1)} \right)^{r^\rho_{i}} \sigma_{\phi(1)}(F_{\phi(1)}) \right]^{1/r^\rho_{i}}.
\end{aligned}
\]

By the same way as the estimate of \( (I_n) \), we see that the last term is majorized by

\[
C \left( \int_{F_{\phi(2)}} \left( \sum_{Q \subset F_{\phi(2)}} K(Q) \prod_{i=2}^{n} \sigma_{\phi(i)}(Q) \right)^{r^\rho_{i}} d\sigma_{\phi(1)} \right) \frac{1}{r^\rho_{i}}.
\]

By Lemma 2.1 we have further that

\[
\leq C \left( \sum_{Q \subset F_{\phi(2)}} K^\rho_{1}(Q) \prod_{i=2}^{n} \sigma_{\phi(i)}(Q) \right)^{1/r^\rho_{i}}.
\]

By (2.3), we notice that

\[
\frac{1}{r^\rho_{i}} + \frac{1}{p_{\phi(i)}} = \frac{1}{r^\rho_{i-1}}, \quad i = 2, \ldots, n - 1.
\]

In the following estimates, \( \sum_{F_{\phi(2)}} \) runs over all \( F_{\phi(2)} \in \mathcal{F}_{\phi(2)} \) that satisfy, for fixed \( F_{\phi(i)} \in \mathcal{F}_{\phi(i)}, i = 3, \ldots, n, \)

\[
\pi_{\mathcal{F}_{\phi(i)}}(F_{\phi(i)}) = F_{\phi(j)} \quad \text{for all} \quad 2 \leq i < j \leq n.
\]

(3.8)
There holds
\[
\sum_{F_{\phi(2)}} \left( \int_{F_{\phi(2)}} f_{\phi(2)} \, d\sigma_{\phi(2)} \right) \times \left( \sum_{Q \subset F_{\phi(2)}} K_{1}^{\phi}(Q) \prod_{i=2}^{n} \sigma_{\phi(i)}(Q) \right)^{1/r_{1}^{\phi}}
\]
\[
\times \left( \sum_{F_{\phi(1)}} \left( \int_{F_{\phi(1)}} f_{\phi(1)} \, d\sigma_{\phi(1)} \right)^{p_{\phi(1)}} \sigma_{\phi(1)}(F_{\phi(1)}) \right)^{1/p_{\phi(1)}}
\]
\[
= \sum_{F_{\phi(2)}} \left( \int_{F_{\phi(2)}} f_{\phi(2)} \, d\sigma_{\phi(2)} \right) \sigma_{\phi(2)}(F_{\phi(2)})^{1/p_{\phi(2)}}
\]
\[
\times \left( \int_{F_{\phi(2)}} \left( \sum_{Q \subset F_{\phi(2)}} K_{1}^{\phi}(Q) \prod_{i=3}^{n} \sigma_{\phi(i)}(Q) \right) \, d\sigma_{\phi(2)} \right)^{1/r_{1}^{\phi}} \sigma_{\phi(2)}(F_{\phi(2)})^{1/r_{2}^{\phi}}
\]
\[
\times \left( \sum_{F_{\phi(1)}} \left( \int_{F_{\phi(1)}} f_{\phi(1)} \, d\sigma_{\phi(1)} \right)^{p_{\phi(1)}} \sigma_{\phi(1)}(F_{\phi(1)}) \right)^{1/p_{\phi(1)}}
\],

where we have used (3.7) with \(i = 2\). Recall that (3.6) with \(j = 2\). Then Hölder’s inequality gives

\[
\leq \left[ \sum_{F_{\phi(2)}} \left( \int_{F_{\phi(2)}} f_{\phi(2)} \, d\sigma_{\phi(2)} \right)^{p_{\phi(2)}} \sigma_{\phi(2)}(F_{\phi(2)}) \right]^{1/p_{\phi(2)}}
\]
\[
\times \left[ \sum_{F_{\phi(2)}} \sum_{F_{\phi(1)}} \left( \int_{F_{\phi(1)}} f_{\phi(1)} \, d\sigma_{\phi(1)} \right)^{p_{\phi(1)}} \sigma_{\phi(1)}(F_{\phi(1)}) \right]^{1/p_{\phi(1)}}
\]
\[
\times \left[ \sum_{F_{\phi(2)}} \left( \int_{F_{\phi(2)}} \left( \sum_{Q \subset F_{\phi(2)}} K_{1}^{\phi}(Q) \prod_{i=3}^{n} \sigma_{\phi(i)}(Q) \right) \, d\sigma_{\phi(2)} \right)^{1/r_{1}^{\phi}} \sigma_{\phi(2)}(F_{\phi(2)}) \right]^{1/r_{2}^{\phi}}
\].

The last term is majorized by

\[
C \left( \int_{F_{\phi(3)}} \left( \sum_{Q \subset F_{\phi(3)}} K_{1}^{\phi}(Q) \prod_{i=3}^{n} \sigma_{\phi(i)}(Q) \right) \, d\sigma_{\phi(2)} \right)^{1/r_{2}^{\phi}}
\]
\[
\leq C \left( \sum_{Q \subset F_{\phi(3)}} K_{2}^{\phi}(Q) \prod_{i=3}^{n} \sigma_{\phi(i)}(Q) \right)^{1/r_{2}^{\phi}}
\].

By Lemma 2.1 we have further that

\[
\leq C \left( \sum_{Q \subset F_{\phi(3)}} K_{2}^{\phi}(Q) \prod_{i=3}^{n} \sigma_{\phi(i)}(Q) \right)^{1/r_{2}^{\phi}}
\].
By being continued inductively until the \( n - 1 \) step, we obtain
\[
\begin{align*}
(3.3) & \leq C \left[ \sum_{F_{\phi(n)}} \left( \int_{F_{\phi(n)}} f_{\phi(n)} d\sigma_{\phi(n)} \right)^{P_{\phi(n)}} \sigma_{\phi(n)}(F_{\phi(n)}) \right]^{1/P_{\phi(n)}} \\
& \times \left[ \sum_{F_{\phi(n)} \in F_{\phi(n-1)}} \sum_{F_{\phi(1)}} \left( \int_{F_{\phi(1)}} f_{\phi(1)} d\sigma_{\phi(1)} \right)^{P_{\phi(1)}} \sigma_{\phi(1)}(F_{\phi(1)}) \right]^{1/P_{\phi(1)}} \\
& \times \left[ \sum_{F_{\phi(n)} \in F_{\phi(n-1)}} \left( \sum_{Q \subset F_{\phi(n)}} K_{n-1}^{\phi}(Q) \right) d\sigma(n) \sigma_{\phi(n)}(F_{\phi(n)}) \right]^{1/r},
\end{align*}
\]
where \( \sum_{F_{\phi(n)}} \) runs over all \( F_{\phi(n)} \in F_{\phi(n)} \) and \( \sum_{F_{\phi(k)}} \), \( k = 3, \ldots, n-1 \), runs over all \( F_{\phi(k)} \in F_{\phi(k)} \) that satisfy, for fixed \( F_{\phi(i)} \), \( i = k+1, \ldots, n \),
\[
\pi_{\phi(i)}(F_{\phi(j)}) = F_{\phi(j)} \quad \text{for all} \quad k \leq i < j \leq n.
\]
The last term is majorized by
\[
C \left( \int_{Q_0} \left( \sum_{Q \subset Q_0} K_{n-1}^{\phi}(Q) \right)^{r/r_n^{\phi}} d\sigma_{\phi(n)} \right)^{1/r} \leq c_2.
\]
It follows from \((3.2), (3.3), (3.4)\) and the uniqueness of the parents that
\[
\begin{align*}
& \left[ \sum_{F_{\phi(n)} \in F_{\phi(n-1)}} \cdots \sum_{F_{\phi(1)}} \left( \int_{F_{\phi(1)}} f_{\phi(1)} d\sigma_{\phi(1)} \right)^{P_{\phi(1)}} \sigma_{\phi(1)}(F_{\phi(1)}) \right]^{1/P_{\phi(1)}} \\
& \leq \left[ \sum_{F_{\phi(n)} \in F_{\phi(n)}} \sum_{\pi_{\phi(n)}(F_{\phi(n)}) = F_{\phi(n)}} \left( \int_{F_{\phi(n)}} f_{\phi(n)} d\sigma_{\phi(n)} \right)^{P_{\phi(n)}} \sigma_{\phi(n)}(F_{\phi(n)}) \right]^{1/P_{\phi(n)}} \\
& = \left[ \sum_{F_{\phi(n)} \in F_{\phi(n)}} \left( \int_{F_{\phi(n)}} f_{\phi(n)} d\sigma_{\phi(n)} \right)^{P_{\phi(n)}} \sigma_{\phi(n)}(F_{\phi(n)}) \right]^{1/P_{\phi(n)}} \\
& \leq C \| f_{\phi(i)} \|_{L^{P_{\phi(i)}}(d\sigma_{\phi(i)})},
\end{align*}
\]
Altogether, we obtain
\[
(3.3) \leq C c_2 \prod_{i=1}^{n} \| f_{\phi(i)} \|_{L^{P_{\phi(i)}}(d\sigma_{\phi(i)})}.
\]
This yields (a) of Theorem 1.4.
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