THE NERETIN GROUPS

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1. Introduction

The Neretin group $N_q$ was introduced in [5] as an analogue of the diffeomorphism group of the circle. It is a subgroup of the homeomorphism group of the boundary of an infinite $q$-regular tree $T$, consisting of elements which locally act by similarities of the visual metric. We define the Neretin group, endow it with a group topology, and present the proof of its simplicity, following [4].

In Section 2 we discuss the structure of the boundary of a regular tree. Sections 3 and 4 define the Neretin group, and describe its locally compact totally disconnected group topology. In Sections 5 and 6 we define the Higman-Thompson groups $G_{q,r}$, another family of groups related to boundaries of regular trees, and show how they can be embedded into the Neretin group. Then we prove that the Neretin group $N_q$ is generated by any of the embedded copies of the Higman-Thompson group $G_{q,2}$ together with the canonically embedded group of type-preserving automorphisms of the tree $T$. Finally, Section 7 presents the proof of simplicity of the Neretin groups.

2. Preliminaries

A tree $T$ is a nonempty connected undirected simple graph without nontrivial cycles. We will interchangeably treat $T$ as a set of vertices endowed with a binary relation of adjacency, or as a topological space obtained from the set of vertices by gluing in unit intervals corresponding to edges. A fixed basepoint $o \in T$ defines a partial order on $T$, namely $v \leq_o w$ if and only if the path from $v$ to $o$ passes through $w$. The basepoint $o$ is the greatest element in this order. The tree structure on $T$ can be recovered from the poset $(T, \leq_o)$ as follows. Two elements $v, w \in T$ are adjacent if and only if they are comparable, and there are no other elements between them. It follows that tree automorphisms of $T$ fixing $o$ are exactly the order automorphisms of $(T, \leq_o)$.

The distance between vertices $v, w \in T$, i.e. the number of edges on the unique path joining them, will be denoted by $|vw|$. A vertex $v \in T$ of degree 1 is called a leaf. A tree is $q$-regular for some $q \in \mathbb{N}$ if all its vertices have degree $q+1$. A finite $q$-regular tree is a finite tree, whose every vertex is either a leaf, or has degree $q+1$, in which case we call it internal. A rooted tree is a tree with a fixed choice of base vertex $o \in T$, called its root. In case of rooted trees we slightly modify the definition of $q$-regularity by requiring the root to be of degree $q$ instead of $q+1$. In the subsequent sections we will deal only with regular trees, so let us assume

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from now on that $T$ is a rooted or unrooted $q$-regular tree with $q \geq 2$. This will relieve us from considering some special cases, which would otherwise appear in the following discussion.

By a ray in $T$ we understand an infinite path, i.e., a sequence $(v_0, v_1, \ldots)$ of distinct vertices of $T$ such that the consecutive ones are connected by edges. Two rays are said to be asymptotic if, after removing some finite initial subsequences, they become equal. Equivalence classes of rays in $T$ are called the ends of $T$. The set of all ends of $T$ is denoted by $\partial T$ and referred to as the boundary of $T$. Any end $\xi \in \partial T$ has a unique representative $\xi_o$ with a given initial vertex $v \in T$. To see it, one has to pick a representative $(v_0, v_1, \ldots)$ of $\xi$, find a minimal path from $v$ to one of the vertices $v_i$, and replace the initial segment $(v_0, \ldots, v_i)$ by this path. Thus, if we choose a base vertex $o \in T$, we may identify $\partial T$ with the set of rays emanating from $o$.

For $o \in T$ denote by $(\xi, \eta)_o$ the length of the common initial segment of the representatives $\xi_o$ and $\eta_o$ of two ends $\xi, \eta \in \partial T$. It takes values in $\mathbb{N} \cup \{\infty\}$. Together with a choice of $\epsilon > 0$ this allows to define a visual metric $d_o, \epsilon$ on $\partial T$ by

$$d_o, \epsilon(\xi, \eta) = e^{-\epsilon(\xi, \eta)_o}. \tag{1}$$

The change of the basepoint $o$ leads to a bi-Lipschitz equivalent metric, and changing $\epsilon$ still gives the same topology. It is an exercise to check that this unique natural topology on $\partial T$ is compact and second countable, provided that $T$ is locally finite, i.e., every vertex has finite degree.

If $j: T_1 \to T_2$ is an embedding of trees, it sends rays to rays, and preserves asymptoticity. Hence, it induces a map $j_o: \partial T_1 \to \partial T_2$. If we choose basepoints $o_i \in T_i$ in such a way that $j(o_1) = o_2$, then $j_o$ can be seen to be an isometric embedding, so in particular it is continuous. Taking the boundary in fact gives a functor from the category of trees and tree embeddings into the category of topological spaces and continuous embeddings.

From now on we will fix $\epsilon = 1$ and $o \in T$, and suppress them from notation whenever possible. There may exist more natural choices for $\epsilon$, e.g., in the case of regular trees, but they will not be of any use to us. The geometrically obvious inequality $(\xi, \eta) \geq \min\{\langle \xi, \zeta \rangle, \langle \zeta, \eta \rangle\}$ implies that $d$ is in fact an ultrametric, i.e., it satisfies a stronger variant of the triangle condition,

$$d(\xi, \eta) \leq \max\{d(\xi, \zeta), d(\zeta, \eta)\}. \tag{2}$$

As a consequence, two open balls in $(\partial T, d)$ are either disjoint, or one of them is contained in the other. It follows that the covering of $\partial T$ by open balls of fixed radius is in fact a partition into open—and hence also closed—sets, and $\partial T$ is totally disconnected. Additionally, since the metric $d$ takes values in a discrete set, any closed ball is also an open ball with a slightly larger radius, and vice versa.

Ultrametricity implies that any point of a ball in $(\partial T, d)$ is its center. It is however still possible to effectively enumerate the balls in a one-to-one manner. To this end, for $v \in T$ define $T_v$ as the subtree of $T$ spanned by the vertices $\{w \in T : w \leq_o v\}$. It is a rooted $q$-regular tree with root $v$, and its boundary $\partial T_v$ is a subset of $\partial T$. It is in fact a closed ball of radius equals $e^{-|v|}$, and the embedding $(\partial T_v, d_v) \to (\partial T, d_o)$ is a similarity. On the other hand, any ball $B \subseteq \partial T$ is a closed ball of radius $e^{-n}$ for some $n \in \mathbb{N}$, and can be written as $\partial T_v$ where $v$ is the last vertex of the common initial segment of all the rays $\xi_o$ representing points $\xi \in B$. 
The family of non-empty balls in $\partial T$ is therefore in a one-to-one correspondence with vertices of $T$.

As a consequence of the above discussion, the assignment $v \mapsto \partial T_v$ is an order-isomorphism between $(T, \leq_o)$ and the set $B(\partial T, d_o)$ of all balls in $(\partial T, d_o)$ ordered by inclusion. Moreover, if $\phi: T \rightarrow T'$ is a basepoint-preserving isomorphism of trees, then $\phi(T_v) = T_{\phi(v)}$, and

$$\phi_{\ast}(\partial T_v) = \partial \phi(T_v) = \partial T_{\phi(v)}.$$  

(3)

This means that the order-isomorphism between $B(\partial T, d_o)$ and $B(\partial T', d_{\phi(o)})$, induced by $\phi$ is the same as the one induced by $\phi_{\ast}: \partial T \rightarrow \partial T'$. This correspondence can be reversed, namely if $\Phi: \partial T \rightarrow \partial T'$ is a homeomorphism preserving balls, it necessarily preserves their inclusion, and induces an order-isomorphism of $B(\partial T, d_o)$ and $B(\partial T', d_{\phi(o)})$ yielding a basepoint-preserving isomorphism $\phi: T \rightarrow T'$. It satisfies

$$\phi_{\ast}(\partial T_v) = \partial T_{\phi(v)} = \Phi(\partial T_v),$$  

(4)  

but since the balls form a basis of the topology of $\partial T$, this means that $\Phi = \phi_{\ast}$.

Finally, let us introduce the notion of a forest. It is what we obtain if we remove the assumption of connectedness from the definition of a tree. In other words, a forest is a graph $F$ which decomposes into a disjoint union of trees. We may define its boundary $\partial F$ as the disjoint union of the boundaries of its constituent trees. It is again functorial. Most of the discussion above extends to forests.

3. The Neretin groups of spheromorphisms

Let $T$ be a $q$-regular tree. For a nonempty finite $q$-regular subtree $F \subseteq T$, by the difference $T \setminus F$ we will understand the rooted $q$-regular forest obtained by removing from $T$ all the edges and internal vertices of $F$, and designating the leaves of $F$ as the roots; geometrically, this amounts to removing the interior of $F$ from $T$. Clearly, $\partial (T \setminus F) = \partial T$, as every ray in $T$ has a subray disjoint from $F$.

Now, let $F_1, F_2 \subseteq T$ be two finite $q$-regular subtrees, such that there exists an isomorphism of forests $\phi: T \setminus F_1 \rightarrow T \setminus F_2$. It induces a homeomorphism $\phi_{\ast}$ of $\partial T$, called a spheromorphism of $\partial T$. The isomorphism $\phi$ will be referred to as a representative of $\phi_{\ast}$.

Observe that the identity map of $\partial T$ is a spheromorphism. More generally, if $\phi \in \text{Aut}(T)$, then for any subtree $F \subseteq T$ the map $\phi$ restricts to an isomorphism of forests $T \setminus F \rightarrow T \setminus \phi(F)$, and thus the induced homeomorphism $\phi_{\ast}$ is a spheromorphism. Moreover, the inverse of a spheromorphism is also a spheromorphism, and for any pair of spheromorphisms $\phi_{\ast}$ and $\psi_{\ast}$ we may find representatives which are composable, showing that $\psi_{\ast} \circ \phi_{\ast}$ is also a spheromorphism.

**Definition 1.** The Neretin group $N_q$ is the group of all spheromorphisms of the boundary of a $q$-regular tree.

The group $N_q$ has another description, based upon the metric structure of the boundary. We will call a homeomorphism of metric spaces $\Phi: X \rightarrow Y$ a local similarity if for each $x \in X$ there exists an open neighborhood $U$ of $x$ and a constant $\lambda_U > 0$ such that for every $x_1, x_2 \in U$ we have

$$d_Y(\Phi(x_1), \Phi(x_2)) = \lambda_U d_X(x_1, x_2),$$  

(5)
i.e. the restriction $\Phi|_U : U \to \Phi(U)$ is a similarity \cite{3}. The requirement that $\phi$ is a homeomorphism allows to choose $U$ to be a ball $B(x,r)$ centered at $x$, such that $\Phi(B(x,r)) = B(\Phi(x),\lambda x r)$. It is clear that all local similarities of a metric space form a group.

**Proposition 3.1.** For a homeomorphism $\Phi \in \text{Homeo}(\partial T)$ the following conditions are equivalent.

1. $\Phi$ is a spheromorphism,
2. $\Phi$ is a local similarity with respect to any visual metric on $\partial T$,
3. $\Phi$ is a local similarity with respect to some visual metric on $\partial T$.

**Proof.** We begin by showing that (1) implies (2). Fix a basepoint $o \in T$ and the corresponding visual metric $d$. Let $\Phi = \Phi_\phi$ be a spheromorphism represented by $\phi : T \setminus F_1 \to T \setminus F_2$. We may assume that both $F_1$ and $F_2$ contain $o$ as internal vertex. Let $T \setminus F_1 = T_1 \cup \cdots \cup T_k$ be the decomposition into disjoint trees. Then $T \setminus F_2$ decomposes into $\phi(T_1) \cup \cdots \cup \phi(T_k)$. These decompositions induce partitions of $\partial T$ into open balls $\partial T_i$ and $\partial(\phi(T_i)) = \Phi(\partial T_i)$.

Let $v$ be the root of $T_i$. Then the root of $\phi(T_i)$ is $\phi(v)$. For $\xi, \eta \in \partial T_i \subseteq \partial T$ we have

$$
(\phi_*(\xi), \phi_*(\eta))_o = (\phi_*(\xi), \phi_*(\eta))_{\phi(v)} + |o\phi(v)| = (\xi, \eta)_o + |o\phi(v)| - |o\phi(v)|,
$$

which implies that

$$
d(\phi_*(\xi), \phi_*(\eta)) = e^{O(v)-|o\phi(v)|} d(\xi, \eta),
$$

and $\phi_*(\partial T_i) : \partial T_i \to \phi_*(\partial T_i)$ is a similarity.

The other nontrivial implication is from (3) to (1). Let $\Phi \in \text{Homeo}(\partial T)$ be a local similarity of $(\partial T, d_o)$. By compactness, we may cover $\partial T$ by finitely many balls $B$ on which $\Phi$ is a similarity, and $\Phi(B)$ is also a ball. By ultrametricity, we may assume that this covering is disjoint, and contains at least 2 balls. The balls in the covering are of the form $\partial T_v$ with $v$ in some finite set $L \subseteq T$, and $\Phi(\partial T_v) = \partial T_{v'}$ for some $v' \in T$. The restriction $\Phi|_{\partial T_v} : \partial T_v \to \partial T_{v'}$ preserves balls, and therefore is induced by a root-preserving isomorphism $\phi_v : T_v \to T_{v'}$. It now remains to observe that the forests $\bigcup T_v$ and $\bigcup T_{v'}$ are obtained by removing finite regular subtrees from $T$, so that the isomorphisms $\phi_v$ assemble into an isomorphism of these forests, representing a spheromorphism. $\square$

4. **Topology on the Neretin groups**

The Neretin group $N_q$ is a subgroup of the homeomorphism group of $\partial T$, which is a topological group when endowed with the compact-open topology. Since $\partial T$ is compact, this topology is metrizable: for $\Phi, \Psi \in \text{Homeo}(\partial T)$ we have

$$
d(\Phi, \Psi) = \sup_{\xi \in \partial T} d_o(\Phi(\xi), \Psi(\xi)).
$$

A first choice for the group topology on $N_q$ would be to restrict the compact-open topology. Unfortunately, this restriction is not locally compact, as we will now observe, using the following lemma.

**Lemma 4.1.** If a subgroup $H$ of a topological group $G$ is locally compact, then it is closed.
Proof. First, assume that $H$ is dense in $G$. Let $U$ be an open neighborhood of 1 in $G$ such that the closure $K$ of $U \cap H$ in $H$ is compact. We then have $K \cap U = H \cap U$. This set is both closed and dense in $U$, hence it is equal to $U$. Therefore $U \subseteq H$, so the subgroup $H$ is open, and hence closed.

In the general case $H$ is dense, and hence closed in its closure in $G$. This means that it is closed in $G$. □

Now, we can see that if $N_q$ with the compact-open topology was locally compact, it would be a closed subgroup. We will show this is false by constructing a sequence of spheromorphisms converging to a homeomorphism outside $N_q$, using the description of spheromorphisms as local similarities. Let $B_i \in N_q$, which is identity outside $B_i$, and on some ball inside $B_i$ it restricts to a similarity with scale greater than $i$. The sequence $\Psi_k = \Phi_k \circ \cdots \circ \Phi_1$ is Cauchy, as for $k > l$

$$d(\Psi_k, \Psi_l) = d(\Phi_k \circ \cdots \circ \Phi_{l+1}, \text{id}) \leq \max_{l<i \leq k} \text{diam } B_i \to 0.$$  

Therefore, $\Psi_k$ converge to a homeomorphism $\Psi \in \text{Homeo}(\partial T)$. It has the same restriction to $B_i$ as $\Phi_i$, and therefore on some ball it restricts to a similarity of scale at least $i$. But local similarities are Lipschitz, so $\Psi \not\in N_q$.

The issue of endowing $N_q$ with a locally compact group topology can be resolved by observing that it already contains a locally compact group as a subgroup. Indeed, $\text{Aut}(T)$ naturally embeds in $N_q$ (and will be identified with its image) and carries the compact-open topology coming from its action on the set of vertices of $T$. It is locally compact and totally disconnected. We can extend it to $N_q$ by declaring the left cosets of $\text{Aut}(T)$ to be open and homeomorphic to $\text{Aut}(T)$ by the translation maps. If $g\text{Aut}(T) = h\text{Aut}(T)$, then the translation maps induce the same topology, so it is well defined, and clearly locally compact and totally disconnected. What is not so clear is whether this makes $N_q$ a topological group—if it does, then this topology is clearly the unique one making $\text{Aut}(T)$ with its original topology an open subgroup of $N_q$. This issue is addressed by the following two lemmas.

Lemma 4.2. Suppose that an abstract group $G$ contains a topological group $H$ as a subgroup. Then $G$ admits a unique group topology, in which $H$ becomes an open subgroup, provided that for all open subsets $U \subseteq H$ and $g, g' \in G$ the intersection $gUg' \cap H$ is open in $H$.

Proof. The basis for the topology on $G$ is necessarily the family of left translates of open subsets of $H$. This indeed makes $H$ embedded homeomorphically as an open subset. Moreover, right translates of open subsets of $H$ are also open, since for $U \subseteq H$ open and $g' \in G$ the set $U g'$ can be written as

$$U g' = \bigcup_{g \in G} (gH \cap U g') = \bigcup_{g \in G} g (H \cap g^{-1} U g'),$$

which is a union of left translates of open subsets of $H$. As a consequence, left and right translations are homeomorphisms of $G$.

Now we need to ensure that multiplication and inversion are continuous. Let $g_\alpha \to g$ and $g'_\alpha \to g'$ be two convergent nets in $G$. The cosets $gH$ and $Hg'$ are open neighborhoods of $g$ and $g'$ respectively, so without loss of generality we may assume
that $g_\alpha = gh_\alpha$ and $g'_\alpha = h'_\alpha g'$ with $h_\alpha, h'_\alpha \in H$ converging to 1. Then $h_\alpha h'_\alpha \to 1$ in $H$, and therefore $g_\alpha g'_\alpha = gh_\alpha h'_\alpha g' \to gg'$. Similarly $g_\alpha^{-1} = h^{-1}_\alpha g^{-1} \to g^{-1}$. \hfill \qed

In order to show that the topology we put on $N_q$ is indeed a group topology, it remains to show that for every open $U \subseteq \text{Aut}(T)$ and $g,g' \in N_q$ the subset $\text{Aut}(T) \cap gUg'$ is open in $\text{Aut}(T)$. Observe that this property is preserved under unions and finite intersections, so it is enough to show it for $U$ in a certain subbasis of the topology on $\text{Aut}(T)$.

Let $o \in T$ be a base vertex, and denote by $K$ the stabilizer of $o$ in $\text{Aut}(T)$. Then for $g,h \in \text{Aut}(T)$ the set $gKh$ consists exactly of the automorphisms sending $h^{-1}(o)$ to $g(o)$, so the finite intersections of the sets of the form $gKh$ yield the standard basis for the topology of $\text{Aut}(T)$. We are thus left with proving the following.

**Lemma 4.3.** If $K$ is the stabilizer of the base vertex $o \in T$ in the group $\text{Aut}(T)$, then for all $\phi_*, \psi_* \in N_q$ the intersection $\psi_* K \phi_* \cap \text{Aut}(T)$ is open in $\text{Aut}(T)$.

**Proof.** The spheromorphisms $\phi_*$ and $\psi_*$ admit representatives $\phi: T \setminus F_1 \to T \setminus B$ and $\psi: T \setminus B \to T \setminus F_2$, where $B$ is a sufficiently large ball in $T$, centered at $o$. Let $K_B \subseteq K$ denote the pointwise stabilizer of this ball; it is an open subgroup of $\text{Aut}(T)$. We have

$$\psi_* K \phi_* = \bigcup_{k \in K} \psi_* k K_B \phi_* = \bigcup_{k \in K} \psi_* k \phi_*(\phi_*^{-1} K_B \phi_*),$$

where $\phi_*^{-1} K_B \phi_*$ consists of elements $\eta_*$ whose representatives $\eta: T \setminus F_1 \to T \setminus F_1$ leave the trees of the forest $T \setminus F_1$ in place, and thus extend to automorphisms of $T$. Hence, it is an open subgroup of $\text{Aut}(T)$, namely the pointwise stabilizer of $F_1$, and therefore the intersection $\psi_* K \phi_* \cap \text{Aut}(T)$ is open. \hfill \qed

This shows that the topology we defined on $N_q$ is indeed a group topology. We may summarize this as follows.

**Theorem 4.4.** The Neretin group $N_q$ admits a unique group topology such that the natural embedding $\text{Aut}(T) \to N_q$ is continuous and open. With this topology, $N_q$ is a totally disconnected locally compact group.

### 5. The Higman-Thompson groups

A tree $T$ is **planar** if it is rooted and for every $v \in T$ there is a fixed linear order on the set of children of $v$. This corresponds to specifying a way to draw the tree on the plane, so that for every $v \in T$ its children are below it, ordered from left to right. The structure of a planar tree is very rigid—an isomorphism of planar trees, which is required to preserve the roots and orders on the sets of children, is always unique, if it exists.

Let $\mathcal{F}$ be a forest consisting of $r$ planar $q$-regular trees $T_1, \ldots, T_r$. For every $i$ choose rooted (in particular, this implies that $F_i$ and $F'_i$ have the same root as $T_i$) finite regular subtrees $F_i, F'_i \subseteq T_i$ in such a way that the forests $F_1 = \mathcal{F} \setminus \bigcup F_i$ and $F_2 = \mathcal{F} \setminus \bigcup F'_i$ have the same number of trees. The forests $F_1$ and $F_2$ consist of planar $q$-regular trees, and hence for every bijection of the sets of trees in $F_1$ and $F_2$ there exists a unique isomorphism $\phi: F_1 \to F_2$ of planar forests realizing it. It induces a homeomorphism $\phi_* \in \partial \mathcal{F}$, and the subgroup of $\text{Home}(\partial \mathcal{F})$ containing all such homeomorphisms is called the Higman-Thompson group $G_{q,r}$. The group $G_{2,1}$ is known as Thompson group $V$. 
This definition shows some ties between $G_{q,r}$ and the permutation groups $S_n$, which we will now make more explicit. The order of children on each $T_i$ induces a lexicographic order on paths starting from the root, which correspond to vertices. This defines a linear order on the set of vertices of each $T_i$. Moreover, the trees themselves can be ordered from $T_1$ to $T_n$, so we have a linear order on the set of vertices of $F$. This allows to order the trees in $F_1$ and $F_2$ by looking at the order of their roots. An isomorphism $\phi: F_1 \to F_2$ of planar forests is now completely determined by a permutation $\sigma \in S_n$, where $n$ is the number of trees in the forests $F_i$.

We will use this to define a homomorphism $\theta: G_{q,r} \to \mathbb{Z}/2\mathbb{Z}$. If $q$ is even, $\theta$ is just the zero homomorphism. On the other hand, if $q$ is odd, we claim that the sign of the permutation $\sigma$ associated to $\phi$ in the above discussion depends only on the element $\phi_* \in G_{n,r}$, and we put $\theta(\phi_*) = \text{sgn} \, \sigma$. To see this, observe that if $F_1 = \bigcup_{i=1}^n L_i$ and $F_2 = \bigcup_{i=1}^n L'_i$ are decompositions into trees, numbered in accordance with the order, we may modify the representative $\phi$ in an elementary way as follows. Choose one of the trees $L_i$ and remove its root, replacing it with $q$ new trees. Do the same with $\phi(L_i) = L'_i \sigma(i)$. This gives a new representative $\phi'$ of $\phi_*$, obtained by restricting $\phi$. The number of inversions $I(\sigma')$ in the permutation $\sigma'$ associated to $\phi'$ is equal to

$$I(\sigma') = \left| \{(j,k) \in \{1, \ldots, n\}^2 : j \neq k, j < k, \sigma(j) > \sigma(k)\} \right| + q \left| \{k \in \{i+1, \ldots, n\} : \sigma(k) < i\} \right| + q \left| \{j \in \{1, \ldots, i-1\} : \sigma(j) > i\} \right| = I(\sigma) + (q-1)C,$$

where $(q-1)C$ is even. This means that the sign of the permutation does not change when we apply the described elementary modification to a representative of $\phi_*$. It remains to observe that any two representatives of $\phi_*$ can be transformed by a sequence of elementary modifications into the same third representative.

Using the homomorphism $\theta$ defined above, we may now describe the commutator subgroup of $G_{q,r}$. The argument below is based on an idea of Mati Rubin [1].

**Theorem 5.1.** The commutator subgroup $G'_{q,r}$ of the Higman-Thompson group $G_{q,r}$ is equal to the kernel of the homomorphism $\theta: G_{q,r} \to \mathbb{Z}/2\mathbb{Z}$. It is a simple group.

**Proof.** It is clear that $G'_{q,r} \subseteq \ker \theta$, and we need to prove the opposite inclusion. First, we claim that $G'_{q,r}$ is generated by elements with representatives $\phi: F_1 \to F_2$ such that $F_1 = F_2$. Indeed, if $\phi_* \in G_{q,r}$ is represented by $\phi: F_1 \to F_2$, in both $F_1$ and $F_2$ we may find families of $q$ trees whose roots have the same parent in $F$. If we compose $\phi$ with a suitable $\psi: F_2 \to F_2$ such that $\psi \circ \phi$ sends the $q$ fixed trees from $F_1$ to the $q$ fixed trees in $F_2$ in an order preserving way, then $(\psi \circ \phi)_*$ can be represented by a map $\chi: F'_1 \to F'_2$, where $F'_1$ is obtained from $F_1$ by adding the common parent of the fixed $q$ trees, and joining them into a single tree. This process stops after finitely many steps, yielding a decomposition of $\phi_*$ into a product of the claimed generators.

An element of $G_{q,r}$ supported in a proper subset of $\partial F$, represented by $\phi: F_1 \to F_1$ which exchanges two trees of $F_1$ and leaves the rest in place, will be called a transposition. It is now clear that $G'_{q,r}$ is generated by transpositions, and if $q$ is odd, then $\ker \theta$ is generated by products of pairs of transpositions, which can be
further assumed to have disjoint supports not covering the whole boundary $\partial F$ (we will always assume this when speaking about a product of a pair of transpositions). Moreover, if $q$ is even, any transposition can be decomposed into a product of an even number of transpositions with disjoint supports. Hence, $\ker \theta$ is always generated by products of pairs of disjoint transpositions.

Now, observe that any two products of pairs of transpositions are conjugate in $G_{q,r}$. Thus, in order to complete the proof of the inclusion we need to show that the commutator subgroup $G'_{q,r}$ contains a product of two disjoint transpositions, supported in a proper subset of $\partial F$. To this end, we just need to take the commutator of two transpositions, one exchanging two balls in $\partial F$, and the other supported inside one of these balls.

We are now left with observing that the commutator subgroup $G'_{q,r}$ is simple. Let $N$ be a normal subgroup of $G'_{q,r}$ containing a nontrivial element $\phi_*$. There exists an open set $U \subseteq \partial F$ such that $\phi_* (U)$ is disjoint from $U$, and $U \cup \phi_* (U)$ is a proper subset of $\partial F$. If $\psi_*$ is a product of two disjoint transpositions supported in $U$ then the commutator $[\phi_*, \psi_*]$ is a product of four disjoint transpositions, is supported in $U \cup \phi_* (U)$, and belongs to $N$. If $\chi_*$ is a transposition supported outside $U \cup \phi_* (U)$, then $[\phi_*, \psi_*]$ is invariant under conjugation by $\chi_*$. Since any product of two disjoint transpositions, supported in a proper subset of $\partial F$, is conjugate to $[\phi_*, \psi_*]$ by an element of $G_{q,r} = G'_{q,r} \cup \chi_* G'_{q,r}$, the normal subgroup $N$ contains all such products. But from simplicity of the alternating groups it follows that for $n \geq 8$ the alternating group $A_n$ is generated by products of four disjoint transpositions, hence $N$ contains all elements possessing a representative whose associated permutation is even. This means that $N = G_{q,r}$.

6. A CONVENIENT GENERATING SET FOR $N_q$

Fix a 2-coloring of vertices of $T$; in this context one usually refers to colors as types. An automorphism $\phi$ of $T$ is type-preserving if and only if whenever $\phi$ fixes an edge of $T$, then it fixes its endpoints. The subgroup of $\text{Aut}(T)$ consisting of type-preserving automorphisms is denoted by $\text{Aut}^+(T)$.

It is generated by the union of pointwise edge stabilizers in $\text{Aut}(T)$. Indeed, denote by $H$ the subgroup of $\text{Aut}(T)$ generated by the edge stabilizers. It is clearly a subgroup of $\text{Aut}^+(T)$. Any two edges with a common endpoint can be exchanged using an element of the stabilizer of another edge with the same endpoint. It follows that all edges with a common endpoint lie in the same orbit of $H$, and thus $H$ acts transitively on the edges of $T$. Hence, if $\phi \in \text{Aut}^+(T)$ and $e$ is an edge of $T$, then there exists $\psi \in H$ such that $\psi(e) = \phi(e)$. The element $\psi^{-1} \phi \in \text{Aut}^+(T)$ fixes the edge $e$ pointwise, so $\psi^{-1} \phi \in H$.

As an instance of the Tits Simplicity Theorem \[6\], we obtain the following.

**Theorem 6.1.** The group $\text{Aut}^+(T)$ of type-preserving automorphisms is simple.

Now, consider the Higman-Thompson group $G_{q,2}$ acting on the boundary of the planar forest $F$. Pick an edge $e$ of $T$ and an embedding $i : F \to T$ sending $F$ onto $T \setminus e$. It defines an embedding of $G_{q,2}$ into $N_q$ given by $\Phi \mapsto i_* \circ \Phi \circ i_*^{-1}$, whose image we will denote by $G'_{q,2}$. If $j : F \to T$ is another such embedding, with image $T \setminus e'$, it can be written as $j = \eta \circ i \circ e$ where $\eta \in \text{Aut}^+(T)$, and $e$ is either the identity map, or the unique automorphism of $F$ preserving its structure of a planar
forest, and exchanging its two trees. As a consequence, in \( N_q \) the subgroup \( G^j_{q,2} \) is conjugate to \( G^j_{q,2} \) by an element of \( \text{Aut}^+(T) \).

**Lemma 6.2.** The Neretin group \( N_q \) is generated by \( \text{Aut}^+(T) \) and \( G^j_{q,2} \), for any embedding \( i \).

**Proof.** Let \( \phi \in N_q \) be represented by \( \phi: T \setminus F_1 \to T \setminus F_2 \). We may suppose that the subtrees \( F_1 \) and \( F_2 \) share a common edge \( e \). Choose an isomorphism \( j: \mathcal{F} \to T \setminus e; \) it defines the subgroup \( G^j_{q,2} \subseteq \langle \text{Aut}^+(T), G^i_{q,2} \rangle \).

There exists an element \( \psi \in G^j_{q,2} \) with representative \( \psi: T \setminus F_1 \to T \setminus F_2 \) inducing the same bijection of trees as \( \phi \). Then \( \psi^{-1}\phi: T \setminus F_1 \to T \setminus F_1 \) preserves the trees of the forest \( T \setminus F_1 \), and therefore extends to an automorphism of \( T \) fixing pointwise the subtree \( F_1 \), and in particular the edge \( e \subseteq F_1 \). Thus, \( \phi \) is a product of \( \psi \in G^j_{q,2} \) and a type-preserving automorphism. \( \square \)

Since the embedded copies of \( G_{q,2} \) are conjugate by elements of \( \text{Aut}^+(T) \), we may slightly abuse the notation and write \( N_q = \langle \text{Aut}^+(T), G_{q,2} \rangle \).

### 7. The simplicity of the Neretin groups

In this section we present the proof of the simplicity of the Neretin groups following [4] using a method that was introduced by Epstein in [2]. We remark that by using the group topology one can also provide potentially simpler alternative proofs which do not use Epstein’s method. Nevertheless, the following lemmas apply to a large variety of examples and are thus worth recalling.

We begin by two general lemmas about actions on topological spaces. The setting we will consider will consist of a compact Hausdorff topological space \( X \), and a faithful group action by homeomorphisms of a group \( G \) on \( X \).

**Lemma 7.1.** Let \( X \) and \( G \) be as above, let \( \mathcal{U} \) be a basis of \( X \) on which \( G \) acts transitively, and let \( 1 \neq H \triangleleft G \) be a non-trivial normal subgroup of \( G \). Then, for all \( g \in G \) such that \( \text{supp}(g) \subseteq V \in \mathcal{U} \) there exists an element \( \rho \in H \) such that \( \rho|_V = g|_V \).

**Proof.** Let \( 1 \neq \alpha \in H \) be any non-trivial element of \( H \). Let \( x \in X \) be a point for which \( \alpha^{-1}(x) \neq x \). One can find a basis set \( V_0 \in \mathcal{U} \) such that \( \alpha^{-1}(V_0) \cap V_0 = \emptyset \).

Assume first that \( V = V_0 \), and consider \( \rho = [g, \alpha] = g \alpha g^{-1} \alpha^{-1} \). Since \( \alpha \in H \) and \( H \) is normal in \( G \) we get that \( \rho \in H \). Moreover, by our assumption \( \text{supp}(g) \subseteq V \), thus when restricted to \( V \) we see that \( \rho|_V = g|_V \), as required.

More generally, if \( V \neq V_0 \), we may find \( h \in G \) such that \( h V = V_0 \). Now the element \( g' = h g h^{-1} \) satisfies \( \text{supp}(h g h^{-1}) \subseteq V_0 \) and thus by the above we can find \( \rho' \in H \) that satisfies \( \rho'|_{V_0} = g'|_{V_0} \). Since \( H \) is normal \( \rho = h^{-1} \rho h \in H \), and \( \rho \) satisfies \( \rho|_V = g|_V \). \( \square \)

**Lemma 7.2.** Let \( X \) and \( G \) be as above, let \( \mathcal{U} \) be a basis of \( X \) on which \( G \) acts transitively, and let \( 1 \neq H \triangleleft G \) be a non-trivial normal subgroup of \( G \) in which there exists \( \alpha_1, \alpha_2 \in H \) such that for some \( x \in X \) the points \( x, \alpha_1(x), \alpha_2(x) \) are distinct. Then, for all \( g_1, g_2 \in G \) such that \( \text{supp}(g_1), \text{supp}(g_2) \subseteq V \in \mathcal{U} \) there exist element \( \rho_1, \rho_2 \in H \) such that \( [g_1, g_2] = [\rho_1, \rho_2] \).

**Proof.** Let \( x \in X \) and \( \alpha_1, \alpha_2 \in H \) be as assumed. One can find a basis set \( V_0 \in \mathcal{U} \) such that \( V_0, \alpha_1(V_0), \alpha_2(V_0) \) are pairwise disjoint.
As in the proof of the previous lemma we may assume up to conjugation that $V = \{0\}$, and consider $\rho_1 = [g, \alpha_1]$ and $\rho_2 = [g, \alpha_2]$. Again we have $\rho_1, \rho_2 \in H$ as required. Moreover, by our assumption $\text{supp}(g_1), \text{supp}(g_2) \subseteq V$, thus when restricted to $V$ we see that $[\rho_1, \rho_2]|_V = [g_1, g_2]|_V$. Moreover, $\rho_i (i = 1, 2)$ preserves the pairwise disjoint sets $V_0, \alpha_1(V_0), \alpha_2(V_0)$ and is supported on $V_0 \cup \alpha_1(V_0)$. It follows that $[\rho_1, \rho_2]|_X \setminus V = [g_1, g_2]|_X \setminus V$. Thus overall we get $[g_1, g_2] = [\rho_1, \rho_2]$. □

Remark 7.3. Note that in order to find such $\alpha_1, \alpha_2$ as in Lemma 7.2, by Lemma 7.1 it is enough to find two such elements in $G$ that are supported on a basis set.

We now apply the previous lemmas to prove the simplicity of the Neretin group.

**Theorem 7.4.** The Neretin group $N_q$ is simple.

**Proof.** Let $1 \neq H \triangleleft N_q$ be a non-trivial subgroup of the Neretin group. From Lemma 6.2 the Neretin group is generated by $\text{Aut}^+(T_q)$ and $G_{q,2}$. In fact, it is enough to take the commutator subgroup $G'_{q,2}$ of $G_{q,2}$, since $\text{Aut}^+(T_q) \cap (G_{q,2} \setminus G'_{q,2}) \neq \emptyset$ whenever $G_{q,2} \neq G'_{q,2}$. From Theorems 6.1 and 5.1 the subgroups $\text{Aut}^+(T_q)$ and $G'_{q,2}$ are simple. Thus, in order to prove the claim it is enough to show that $H \cap \text{Aut}^+(T_q) \neq 1$ and $H \cap G'_{q,2} \neq 1$.

We observe that the Neretin group acts faithfully by homeomorphisms on the boundary of the tree $T_q$, and acts transitively on the basis of ends of half trees. We complete the proof by Lemma 7.2 and 7.3 after finding two pairs of non-commuting elements in $\text{Aut}^+(T_q)$ and $G_{q,2}$ that are supported in a half-tree. □

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