PATTERSON-SULLIVAN MEASURES AND GROWTH OF RELATIVELY HYPERBOLIC GROUPS

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Abstract. We prove that for a relatively hyperbolic group $G$ there is a sequence of relatively hyperbolic proper quotients such that their growth rates converge to the growth rate of $G$. Under natural assumptions, the same conclusion holds for the critical exponent of a cusp-uniform action of $G$ on a hyperbolic metric space. As a corollary, we obtain that the critical exponent of a torsion-free geometrically finite Kleinian group can be arbitrarily approximated by those of quotient groups. This resolves a question of Dal’bo-Peigné-Picaud-Sambusetti.

Our approach is based on the study of Patterson-Sullivan measures on Bowditch boundary of a relatively hyperbolic group. The uniqueness and ergodicity of Patterson-Sullivan measures are proved when the group is divergent. We prove a variant of the Sullivan Shadow Lemma, called Partial Shadow Lemma. This tool allows us to prove several results on growth functions of balls and cones. One central result is the existence of a sequence of geodesic trees with growth rates sufficiently close to the growth rate of $G$, and transition points uniformly spaced in trees. By taking small cancellation over hyperbolic elements of high powers, we can embed these trees into properly constructed quotients of $G$. This proves the results in the previous paragraph.

Contents

1. Introduction 1
2. Preliminaries 7
3. Visual and quasi-conformal densities 15
4. Patterson-Sullivan measures on Bowditch boundary: I 18
5. Applications towards growth of cones and partial cones 25
6. Lift paths in nerve graphs 32
7. Patterson-Sullivan measures on Bowditch boundary: II 36
8. Small cancellation in relatively hyperbolic groups 46
9. Proofs of Theorems 1.2 & 1.4 52
References 53

1. Introduction

In recent years, there has been an increasing interest in the study of relatively hyperbolic groups. The notion of relative hyperbolicity originated in seminal work...
of Gromov in [29], and was later re-formulated and elaborated on by many authors, cf. [19], [3], [36], [17], [22], to name a few. The class of relatively hyperbolic groups includes many naturally occurred groups, such as word hyperbolic groups, fundamental groups of non-uniform lattices with negative curvature [3], limit groups [13], and CAT(0) groups with isolated flats [31].

It is a popular theme in geometric group theory to study a group via a proper action on a model space with nice geometric properties. The asymptotic geometry of the group action then encodes information from the group. For example, one could simply count the number of orbit points in a growing ball. A celebrated theorem of Gromov in [28] says that the polynomial growth in a Cayley graph implies the group being virtually nilpotent.

A group is elementary if it is finite or a finite extension of a cyclic group. Recall that non-elementary relatively hyperbolic groups have exponential growth, to which one can associate a real number called growth rate to measure the speed. The goal of the present paper is then to investigate growth rate for quotients of a relatively hyperbolic group which admits a proper action on various geometric spaces. In what follows, we describe our results in detail.

1.1. Growth rates of relatively hyperbolic groups. We first prepare some general setup. Let $G$ be a group acting properly on a geodesic metric space $(X,d)$. Fix a basepoint $o \in X$. Denote $N(o,n) = \{go : g \in G, d(o,go) \leq n\}$ for $n \in \mathbb{N}$. The growth rate $\delta_A$ of a subset $A \subset G$ relative to $d$ is thus defined as

$$\delta_A = \limsup_{n \to \infty} n^{-1} \ln \#(N(o,n) \cap A o).$$

Note that $\delta_A$ does not depend on the choice of $o$.

Let $\Gamma$ be a normal subgroup in $G$. Consider $\bar{G} = G/\Gamma \sim X/\Gamma$ by $\Gamma y \cdot (\Gamma x) \rightarrow \Gamma yx$. Equip $\bar{X}$ with the quotient metric $\bar{d}$. The $\bar{G}$ is called a proper quotient if $\Gamma$ is infinite. Denote by $\delta_{\bar{G}}$ the growth rate of $\bar{G}$ relative to $\bar{d}$.

**Definition 1.1.** The action $G \acts (X,d)$ is called growth tight if $\delta_G > \delta_{\bar{G}}$ for any proper quotient $\bar{G}$ of $G$.

We consider here proper quotients, because otherwise $\delta_G = \delta_{\bar{G}}$ always holds.

**Remark.** Growth tightness was introduced by Grigorchuk and de la Harpe in [27] for word metric. The study of growth tightness for general metrics was proposed by Sambusetti in [41]. See [1], [40], [42], [39], [16] for further references about growth tightness.

First of all, we consider the model space arising from Cayley graphs of $G$. Let $1 \notin S$ be a finite generating set with $S = S^{-1}$. Then $G$ acts on the Cayley graph $\mathcal{G}(G,S)$ endowed with word metric $d$. Here $\delta_G$ is the usual growth rate of $G$ with respect to $S$. In [48], we proved that any (non-elementary) relatively hyperbolic group is growth tight with respect to word metric. Hence, a natural question arises as follows: is there a gap between $\delta_G$ and $\sup \{\delta_{\bar{G}}\}$ over all proper quotients $\bar{G}$? The first main result is to answer this question negatively by proving the following.

**Theorem 1.2.** Suppose that $G$ is a relatively hyperbolic group with a finite generating set $S$. Then there exists a sequence $\bar{G}_n$ of relatively hyperbolic proper quotients of $G$ such that

$$\lim_{n \to \infty} \delta_{\bar{G}_n} = \delta_G.$$
We compare Theorem 1.2 with some related works. In a hyperbolic group $G$ without torsion, Coulon recently showed in [12] that the growth rate of the periodic quotient $G/G^n$ converges to $\delta_G$ as $n$ odd approaches $\infty$. In Theorem 1.2, we construct $\bar{G}_n$ as small cancellation quotients $G/\langle \langle h^n \rangle \rangle$ for $n > 0$ over any hyperbolic element $h \in G$.

In [27], Grigorchuk and de la Harpe asked under which conditions the growth rate of a sequence of one-relator groups tends to that of a free group. In [43], Shukhov showed that this is true for one-relator small cancellation groups, when the length of this relator goes to infinity. In [18, Corollary 2], Erschler established a generalization for any small cancellation group $G$: the growth rate of small cancellation quotient of $G$ by adjoining a new relator $r$ goes to $\delta_G$ as $\ell(r) \to \infty$. Hence Theorem 1.2 represents a further generalization in this direction.

We next consider the model space which comes from the definition of a relatively hyperbolic group. Relative hyperbolicity of a group admits many equivalent formulations [19], [5], [17], [36], [30], [22], among which is the following stated in [30], generalizing the definition of a geometrically finite Kleinian group in [33].

**Definition 1.3.** Suppose $G$ admits a proper and isometric action on a proper hyperbolic space $(X, d)$, and $\mathcal{P}$ the collection of maximal parabolic subgroups in $G$. Assume that there is a $G$-invariant system of (open) horoballs $U$ centered at parabolic points of $G$ such that the action $G \actson X \setminus U, U := \bigcup U \in U U$ is co-compact. Then the pair $(G, \mathcal{P})$ is said to be relatively hyperbolic, and the action $G \actson (X, d)$ is called cusp-uniform.

In this setting, the growth rate $\delta_G$ for the action $G \actson X$ is usually called critical exponent, which has been studied for a long time in fields of Kleinian groups, dynamical systems, and so on. It is known that parabolic subgroups are quasi-isometrically embedded in word metric, cf [36]. By contrast, the asymptotic behavior of $G \actson X$ even module a compact part can be still quite complicated due to exponential distortion of parabolic subgroups. In [15], Dal’bo-Otal-Peigné introduced a parabolic gap property to resolve this issue. By definition, a proper action $G \actson X$ has a parabolic gap property if $\delta_G > \delta_P$ for every maximal parabolic subgroup $P$.

In [16], Dal’bo-Peigné-Picaud-Sambusetti considered growth tightness of a geometrically finite Kleinian group $G$ with pinched negative curvature. They generalized the cocompact case in [42] and proved that $G$ is growth tight, provided that $G$ satisfies parabolic gap property. Thus, we can consider an analogue of growth gap problem as to hyperbolic metric. Parallel to Theorem 1.2, the following theorem asserts the same result for growth relative to hyperbolic metric.

**Theorem 1.4.** Suppose that $G \actson (X, d)$ is a cusp-uniform action with $\delta_G < \infty$. Assume that $G$ satisfies parabolic gap property. Then there exists a sequence $\bar{G}_n$ of relatively hyperbolic proper quotients of $G$ such that

$$\lim_{n \to \infty} \delta_{\bar{G}_n} = \delta_G.$$

**Remark.** It is possible that $\delta_G = \infty$, see Example 1 in [21, Section 3.4]. We do not need the the growth tightness of $G$ with respect to $d$, although it appears to the author that the proof for growth tightness of Kleinian groups in [16] is valid under the general assumption of Theorem 1.4.
Recall that growth rate is not a quasi-isometric invariant. Hence, Theorem 1.4 cannot be deduced, even in hyperbolic case, from Theorem 1.2. The overall strategy in proving Theorem 1.4 shares much similarity with that in Theorem 1.2. On the other hand, the parabolic gap property plays a crucial role in technical part. This will be well commented in next subsection.

It is well-known that a geometrically finite Kleinian group with constant curvature has parabolic gap property. Hence we obtain the following corollary, which settles a question of Dal’bo-Peigné-Picaud-Sambusetti in [16] asking whether there is a gap between $\delta_G$ and $\sup \{\delta_{\bar{\Gamma}}\}$ over all proper quotients $\bar{\Gamma}$.

**Corollary 1.5.** Let $G$ be a torsion-free geometrically finite Kleinian group with constant curvature. Then

$$\inf \{\delta_G - \delta_{\bar{\Gamma}}\} = 0$$

over all quotients $\bar{\Gamma}$ of $G$.

### 1.2. Patterson-Sullivan measures.

We now describe the approach in proving Theorems 1.2 and 1.4. In [12] and [18], the typical approach consists of two components: automata theory of a hyperbolic group (or equivalently, Cannon’s theory about finiteness of cone types in [9]), and small cancellation theory. Unfortunately, there is no automata that can recognize the geodesic language of any relatively hyperbolic group.

The tool that we find to replace the automata theory turns out to be the theory of Patterson-Sullivan measures (or PS-measures for shorthand). In fact, the idea of using PS-measures to study growth problems dates back to works of Patterson [37] and Sullivan [45], see [45, Theorem 9] for example.

In [10], Coornaert has established the theory of PS-measures on the limit set of a discrete group acting on a $\delta$-hyperbolic space. In the setting of Kleinian groups (with variable negative curvature), Dal’bo-Otal-Peigné used the parabolic gap property in [15] to deduce the divergence of $G$, and demonstrated that this property can be very helpful to obtain a satisfactory theory of PS-measures. A significant part of this study is to further develop their idea in a general setting.

It is well-known that the construction of PS-measures can be performed in a general geodesic metric space with a proper group action, via the horofunction boundary, cf. [7]. For a relatively hyperbolic group $G$, we can consider PS-measures on the Bowditch boundary $\partial G$, which is defined to be the (topological) Gromov boundary $\partial X$ of $X$ in Definition 1.3. In [3], Bowditch observed that $\partial G$ is independent of the choice of $X$. In [22], Gerasimov proved that $\partial G$ can also be obtained as quotients of Floyd boundary which compactifies the Cayley graph of $G$. These give two natural settings to construct PS-measures on $\partial G$.

In what follows, we shall mainly discuss PS-measures $\partial G$ constructed through $G \sim X$. Under the condition that $G \sim X$ is divergent, or has parabolic gap property, all results below hold with no (or minor) changes for PS-measures obtained from a cusp-uniform action $G \sim X$.

Let $G$ be a relatively hyperbolic group with a finite generating set $X$. Compactify the Cayley graph $\mathcal{G}(G, S)$ by $\partial G$. One difficulty in proving quasiconformal density of PS-measures is the lack of a well-defined notion of horofunctions at all boundary points in $\partial G$. We get around this difficulty by defining horofunctions only at conical points. We summarize the results about PS-measures as follows. See Section 4 for detail and precise definitions.
Theorem 1.6 (=Proposition [4.15]). Let μ be a PS-measure on ∂G constructed through the action \( G \sim \mathcal{G}(G,S) \). Then μ is a quasiconformal density without atoms. Moreover, μ is unique and ergodic.

Remark. Under the assumption that \( G \sim X \) is divergent, we obtain the same conclusion for PS-measures on ∂G constructed through the cusp-uniform action \( G \sim X \). See Proposition [7.3]

In the theory of PS-measures, a key tool is the Sullivan Shadow Lemma, which connects the geometry inside and the analytic on boundary. In our setup of group completion \( \mathcal{G}(G,S) \cup \partial G \) the Sullivan Shadow Lemma is proved. In fact, we prove a variant of Shadow Lemma that holds for partial shadows. Before stating our result, we need go into some technical parts of this paper.

Denote \( \mathbb{P} = \{gP : g \in G, P \in \mathcal{P}\} \), where \( \mathcal{P} \) is a complete set of conjugacy representatives in \( \mathcal{P} \). For a path \( p \) in \( \mathcal{G}(G,S) \), a point \( v \in p \) is called an \((\epsilon, R)\)-transition point for \( \epsilon, R \geq 0 \) if the \( R \)-neighborhood \( P \cap B(v, R) \) around \( v \) in \( p \) is not contained in the \( \epsilon \)-neighborhood of any \( gP \in \mathbb{P} \). This notion of transition points was due to Hruska in [30], and further elaborated on by Gerasimov-Potyagailo in [24].

Hence, the partial shadow \( \Pi_{r, \epsilon, R}(g) \) at \( r \) for \( r \geq 0 \) is the set of boundary points \( x \in \partial G \) such that some geodesic \([1, x]\) intersects non-trivially in \( B(g, r) \) and contains an \((\epsilon, R)\)-transition point \( v \) in \( B(g, 2R) \). We prove the following variant of Shadow Lemma, of which an analogous version in the setting of \( G \sim X \) is Lemma [5.19]

Lemma 1.7 (=Partial Shadow Lemma [5.18]). Let \( \mu \) be the PS-measure on ∂G. There are constants \( r_0, \epsilon, R \geq 0 \) such that the following holds

\[
\begin{align*}
\exp(-\delta_G d(1,g)) &< \mu(\Pi_{r, \epsilon, R}(g)) < \exp(-\delta_G d(1,g)),
\end{align*}
\]

for any \( g \in G \) and \( r \geq r_0 \).

By Partial Shadow Lemma, we prove a series of results about the growth functions of balls and cones.

The cone \( \Psi_r(g) \) at \( g \) for \( r \geq 0 \) in \( \mathcal{G}(G,S) \) is the set of elements \( h \in G \) such that some geodesic \([1, h]\) intersects non-trivially in \( B(g, r) \). A notion of partial cone \( \Psi_{r, \epsilon, R}(g) \) at \( g \) is defined similarly, by demanding the existence of \((\epsilon, R)\)-transition points on \([1, h]\) \( 2R \)-close to \( g \). As an analogue to Cannon’s result, it is proven in Lemma [6.14] that there are finitely many partial cone types \( \Psi_0, \epsilon, R(g) \) for fixed \( \epsilon, R \).

Theorem 1.8 (=Lemmas [6.10] and Corollary [5.6]). Let \( r, \epsilon, R \geq 0 \) be given by Lemma [1.7]. Then the following hold.

1. \( \# N(1, n) \asymp \exp(n\delta_G) \) for any \( n \geq 0 \).
2. \( \#(\Psi_{r, \epsilon, R}(g) \cap N(g, n)) \asymp \exp(n\delta_G) \) for any \( g \in G \).
3. In particular, \( \#(\Psi_r(g) \cap N(g, n)) \asymp \exp(n\delta_G) \) for any \( g \in G \).

Remark. The statements (1) and (3) in hyperbolic case were proved in [10] Théorème 7.2 and [11] Lemma 4) respectively. Under the assumption that \( G \) has parabolic gap property, we obtain analogous versions in the setting of \( G \sim X \). See Lemma [7.8] and Corollary [7.9].

Let’s describe the strategy in proofs of Theorems [1.2] and [1.4]. The central result in proving Theorem [1.2] is the existence of a sequence of sufficiently large trees with uniformly spaced transitions points. We also prove an analogue, Theorem [7.20] for the cusp-uniform action \( G \sim X \) under the assumption that \( G \sim X \) has parabolic gap property.
Theorem 1.9 (=Theorem 5.13). Suppose that $G$ is a relatively hyperbolic group with a finite generating set $S$. Then there exist $\epsilon, R > 0$ such that the following holds.

For any $0 < \sigma < \delta_G$, there exist $r > 0$ and a geodesic tree $T$ in $\mathcal{G}(G, S)$ with the following properties.

1. Let $\gamma$ be a geodesic in $T$. For any $x \in \gamma$, there exists an $(\epsilon, R)$-transition point $v$ in $\gamma$ such that $d(x, v) < r$.
2. Let $\delta_T$ be the growth rate of $T$. Then $\delta_T > \sigma$.

Remark. This result is new even in hyperbolic case. We remark that if $\sigma$ is closer to $\delta_G$, then $r$ is bigger. Consequently, transition points are sparser and geodesics in $T$ thus get wilder and less controlled.

The other ingredient in proving Theorem 1.2 is small cancellation theory. The idea is to embed sufficiently large trees given by Theorem 1.9 into small cancellation quotients constructed as follows.

Fix an arbitrary hyperbolic element $h \in G$, and denote by $E(h)$ the maximal elementary subgroup in $G$ containing $h$. Endowing $S$ with a total order, we consider a language $L$ over $S$ consisting of words $w \in L$ that are lexi-minimal in representing elements in $G$.

For $\epsilon > 0, W \subset L$ the notation $L(W, \epsilon)$ denotes the set of words $w$ in $L$ that $w \epsilon$-contains some word in $W$. See Section 8 for precise definitions. Using small cancellation theory via rotating family developed in [14], we prove the following.

Theorem 1.10 (=Lemma 8.14). There exist $\epsilon = \epsilon(h), N = N(h)$ and $L = L(h, n)$ for $n > N$. Let $W$ be the set of words in $L$ representing elements of $E(h)$ with length at least $L$. Then the following hold for $n > N$.

1. $L(n, h) \to \infty$ as $n \to \infty$.
2. $G \to \overline{G} = G/\langle \langle h^n \rangle \rangle$ is injective on $L \setminus L(W, \epsilon)$.

To prove Theorem 1.2 it suffices to observe that sufficient large trees $T$ can be embedded into $L \setminus L(W, \epsilon)$. The proof of Theorem 1.4 is similar, but with more complicated arguments. The complete proofs can be found in Section 9.

We conclude with a brief discussion about the parallel development of PS-measures on $G \cup \partial G$ and $X \cup \partial G$. As remarked previously, many results using PS-measures on $G \cup \partial G$ have an analogous version for PS-measures on $X \cup \partial G$ under the assumption that $G \curvearrowright X$ is divergent. Recall that $G \curvearrowright \mathcal{G}(G, S)$ is always divergent. It may be conceivable that the assumption on divergence of $G$ is the common explanation. However, contrary to its analogue Lemma 5.5, Lemma 7.8 uses crucially the parabolic gap property, or a weaker exponential decay property, cf. Lemma 7.13. It is not clear that whether Lemma 7.8 is true with the only assumption that $G \curvearrowright X$ is divergent. Hence it would be interesting to have an unified theory for both cases.

1.3. Organization of paper. In Section 2, we introduce contracting systems and discuss the notion of transitions points relative to a contracting system. This provides an uniform background for further concrete studies. We also define the notion of horofunctions at conical points in Bowditch boundary. This allows to construct PS-measures on Bowditch boundary in Section 4.

Section 3 gives an abstract formulation of quasiconformal density and describes the construction of PS-measures. The notions of partial shadows and cones are
introduced in a general setting. These serve as a protocol which will be implemented in Sections 4, 5 & 7.
Sections 4, 5 focus on the PS-measures on the compactification $\mathcal{G}(G, S) \cup \partial G$ and use them to derive results about growth of balls and cones in Cayley graphs. Theorems 1.6, 1.8 and 1.9 are proved.

The aim of Section 6 is to link the action of $G \sim \mathcal{G}(G, S)$ to the action of $G \sim X$, via lift operations. Then Section 7, parallel to Sections 4, 5, is to prove analogous versions of Theorems 1.6, 1.8 and 1.9 in the setting of the action $G \sim X$.

Section 8 studies the small cancellation over hyperbolic elements of high power. We make use of rotating families to prove Theorem 1.10. In Section 9, Theorems 1.9 and 1.10 are combined to prove Theorems 1.12 and 1.14.

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2. Preliminaries

2.1. Notations and Conventions. Let $(X, d)$ be a geodesic metric space. We collect some notations and conventions used globally in the paper.

1. $B_d(x, r) := \{ y : d(x, y) \leq r \}$. The metric $d$ will be omitted, if $d$ is understood.
2. $N_r(A) := \{ y \in X : d(y, A) \leq r \}$ for a subset $A$ in $X$.
3. $\| A \|$ denotes the diameter of $A$ with respect to $d$.
4. A subset $A$ is $C$-separated for $C > 0$ if $d(a, a') > C$ for any two distinct $a, a' \in A$.
5. Let $p$ be a path in $X$. Denote by $p_-, p_+$ the initial and terminal endpoints of $p$ respectively, and by $\ell(p)$ the length of $p$. Given two points $x, y \in p$, denote by $[x, y]_p$ the subpath of $p$ going from $x$ to $y$.
6. A path $p$ going from $p_-$ to $p_+$ induces a first-last order as follows. Given a property (P), a point $z$ on $p$ is called the first point satisfying (P) if $z$ is among the points $w$ on $p$ with the property (P) such that $\ell([p_-, w]_p)$ is minimal. The last point satisfying (P) is defined in a similar way.
7. For $x, y \in X$, denote by $[x, y]$ a geodesic $p$ in $X$ with $p_- = x, p_+ = y$. Note that the geodesic between two points is usually not unique. But the ambiguity of $[x, y]$ is usually made clear or does not matter in the context.
8. Let $p$ a path and $Y$ be a closed subset in $X$ such that $p \cap Y \neq \emptyset$. So the entry and exit points of $p$ in $Y$ are defined to be the first and last points $z$ in $p$ respectively such that $z$ lies in $Y$.
9. In a triangle, two points $x, y$ in sides $a, b$ respectively are called congruent if $d(x, o) = d(y, o)$ where $o$ is the common endpoint of $x$ and $y$.
10. Let $f, g$ be two real-valued functions with domain understood in the context. Then $f \prec_{c_1, c_2, \ldots, c_n} g$ means that there is a constant $C > 0$ depending on parameters $c_i$ such that $f < Cy$. And $f \succ_{c_1, c_2, \ldots, c_n} g$, $f \sim_{c_1, c_2, \ldots, c_n} g$ are used in a similar way. For simplicity we omit $c_i$ if they are uniform.
11. The notation $n \gg 0$ means that all sufficiently large integers.
Lemma 2.1. Let \((X,d)\) be a \(\delta\)-hyperbolic space for \(\delta \geq 0\). Then any geodesic triangle is \(\delta\)-thin. Any two (possibly infinite) geodesic with same endpoints stay in \(\delta\)-neighborhoods of each other.

2.2. Contracting systems. Some materials are borrowed from [47]. The terminology of contracting sets is due to Bestvina-Fujiwara in [2]. We also refer to [44] for alternative formulation of contracting sets.

Given a subset \(Y\) in \(X\), the projection \(\text{Pr}_Y(x)\) of a point \(x\) to \(Y\) is the set of nearest points in the closure of \(Y\) to \(x\). Then for \(A \subset X\) define \(\text{Pr}_Y(A) = \bigcup_{a \in A} \text{Pr}_Y(x)\).

Definition 2.2. Let \(\tau, D > 0\). A subset \(Y\) is called \((\tau, D)\)-contracting in \(X\) if the following holds

\[ \|\text{Pr}_Y(\gamma)\| < D \]

for any geodesic \(\gamma\) in \(X\) with \(N_\tau(Y) \cap \gamma = \emptyset\).

A collection of \((\tau, D)\)-contracting subsets is referred to as a \((\tau, D)\)-contracting system. The constants \(\tau, D\) will be often omitted, if understood.

Lemma 2.3. [45] Let \(Y\) be a contracting set in \(X\). Then the following holds.

1. For any \(r \geq 0\), if \(\gamma\) is a geodesic in \(X\) such that \(d(\gamma_-, Y), d(\gamma_+, Y) < r\), then there exists \(\epsilon = \epsilon(r)\) such that \(\gamma \subset N_\epsilon(Y)\).

2. There exists a constant \(M > 0\) such that for any geodesic \(\gamma\) with \(\gamma_- \in Y\), we have \(d(\text{Pr}_Y(\gamma_-), \gamma) \leq M\).

3. There exists a constant \(D > 0\) such that \(\|\text{Pr}_Y(\alpha)\| \leq \ell(\gamma) + D\) for any geodesic \(\gamma\) in \(X\).

Lemma 2.4. [45] Let \(\mathcal{Y}\) be a contracting system. Then the following two properties are equivalent.

1. (bounded intersection property) If for any \(\epsilon > 0\) there exists \(R = R(\epsilon) > 0\) such that

\[ \|N_\epsilon(Y) \cap N_\epsilon(Y')\| < R \]

for any two distinct \(Y, Y' \in \mathcal{Y}\).

2. (bounded projection property) If there exists a finite number \(D > 0\) such that

\[ \|\text{Pr}_Y(Y')\| < D \]

for any two distinct \(Y, Y' \in \mathcal{Y}\).

We describe a typical setting which gives rise to a contracting system with bounded intersection (and projection) property. All examples in this paper are of this sort. The proof is straightforward.

Lemma 2.5. Suppose that a group \(G\) acts properly and isometrically on a geodesic metric space \(X\). Let \(\mathcal{Y}\) be a \(G\)-invariant contracting system such that \(\|\mathcal{Y}/G\| < \infty\) and for each \(Y \in \mathcal{Y}\) the stabilizer \(G_Y\) of \(Y\) in \(G\) acts cocompactly on \(Y\). Assume that \(\|N_\epsilon(Y) \cap N_\epsilon(Y')\| < \infty\) for any \(r > 0\) and all \(Y \neq Y' \in \mathcal{Y}\). Then \(\mathcal{Y}\) has bounded projection property.

Remark. Suppose \(X\) is a hyperbolic space or the Cayley graph of a relatively hyperbolic group \(G\). The condition \("\|N_\epsilon(Y) \cap N_\epsilon(Y')\| < \infty\"\) is satisfied if the limit set of \(Y\) is disjoint with that of \(Y'\).
In the following definition, we put into an abstract context a notion of transition points which was introduced in [30] by Hruska in the setting of relatively hyperbolic groups.

**Definition 2.6.** Let \( \mathcal{Y} \) be a contracting system with bounded intersection in \( X \). Fix \( \epsilon, R > 0 \). Let \( \gamma \) be a path in \( X \) and \( v \in \gamma \) a vertex such that \( \gamma_, \gamma, \notin B(v, R) \). Given \( Y \in \mathcal{Y} \), we say that \( v \) is \((\epsilon, R)\)-deep in \( Y \) if it holds that \( \gamma \cap B(v, R) \subset N_\epsilon(Y) \). If \( v \) is not \((\epsilon, R)\)-deep in any \( Y \in \mathcal{Y} \), then \( v \) is called an \((\epsilon, R)\)-transition point of \( \gamma \).

**Example 2.7.** In the sequel we mainly consider transition points in the following two setups.

1. Let \((G, \mathcal{P})\) be a relatively hyperbolic group with a finite generating set \( S \). Then the collection \( \mathcal{P} \) of all peripheral cosets is a contracting system with bounded intersection in \( \mathcal{Y}(G, S) \), cf. [17].
2. Let \( G \) acts properly on a hyperbolic space \( X \). Assume that there exists a collection of horoballs \( U \) in \( X \) such that \( G \curvearrowright (X \setminus \cup_{U \in \mathcal{P}}) \) is co-compact. Then \( \mathcal{U} \) is a contracting system with bounded intersection.

Even in this abstract setting we can still obtain the following non-trivial facts about transitions points.

**Lemma 2.8.** There exists \( \epsilon_0 > 0 \) such that for any \( \epsilon \geq \epsilon_0, R > 0 \), there exists \( L = L(\epsilon, R) > 0 \) with the following property.

Let \( Y \in \mathcal{Y} \) and \( \gamma = [x, y] \) a geodesic in \( X \) such that \( d(x, y), d(y, \gamma) < \epsilon \). Take \( z \in \gamma \) such that \( d(x, z), d(\gamma, z) > L \). Then \( z \) is \((\epsilon_0, R)\)-deep in \( Y \).

**Proof.** Assume that \( \mathcal{Y} \) is a \((\tau, D)\)-contracting system for \( \tau, D > 0 \). Let \( \epsilon_0 = \epsilon(\tau) \) be given by Lemma 2.3 for \( \epsilon > \epsilon_0 \), set \( L = 2\epsilon + \tau + D + R + 1 \).

Observe that \( \gamma \cap N_{\epsilon}(Y) \neq \emptyset \). Indeed, if not, by projecting \( \gamma \) to \( Y \) we see that \( d(x, y) < D + 2\epsilon \). This contradicts the choice of \( L \). Denote by \( x', y' \in \gamma \) the entry and exit points of \( \gamma \) in \( N_{\epsilon}(Y) \) respectively. By projection on sees that \( d(x, x'), d(y, y') \leq D + \epsilon + \tau \). Hence \( d(z, x'), d(z, y') > R + 1 \).

Note that \([x', y'] \subset N_{\epsilon_0}(Y)\) by Lemma 2.3. Consequently, we have that \( z \) is \((\epsilon_0, R)\)-deep in \( Y \).

**Lemma 2.9.** Let \( \epsilon_0 > 0 \) given by Lemma 2.3. For any \( \epsilon \geq \epsilon_0 \), there exists \( R = R(\epsilon) > 0 \) with the following property.

Let \( \gamma \) be a geodesic in \( X \). Assume that \( Y \in \mathcal{Y} \) and \( \gamma = (\epsilon, R')\)-deep in some \( Y \in \mathcal{Y} \) for \( R' \geq R \). Denote by \( x, y \) the entry and exit points of \( \gamma \) in \( N_{\epsilon}(Y) \) respectively. If \( d(x, \gamma_1) > R \) (resp. \( d(y, \gamma_1) > R \)), then \( x \) (resp. \( y \)) is an \((\epsilon, R)\)-transition point in \( \gamma \).

**Proof.** By Lemma 2.3, there exists \( \sigma \geq \epsilon \) such that \([x, y] \subset N_\sigma(Y)\). Set \( R = R(\sigma) + 1 \), where \( R(\sigma) \) is given by Lemma 2.4 for \( \mathcal{Y} \).

As \( d(x, v) \geq R' \geq R \), we have \([\gamma \cap N_\sigma(Y)] \geq R \). Clearly, one sees that \( x \) is not \((\epsilon, R)\)-deep in \( Y \). Assume that \( x \) is \((\epsilon, R)\)-deep in \( Y' \neq Y \in \mathcal{Y} \). Then \( \gamma \cap B(x, R) \subset N_\sigma(Y') \). This gives a contraction, as \( R > R(\sigma) \) and \( Y \neq Y' \). Hence it is shown that \( x \) is \((\epsilon, R)\)-transitional.

**2.3 Relatively hyperbolic groups.** We always consider a non-elementary relatively hyperbolic group \((G, \mathcal{P})\) given by Definition 1.3. The elements in \((G, \mathcal{P})\) are classified into elliptic, hyperbolic and parabolic types according to their actions on \( X \), see [29] for more detail.
Recall that $\mathcal{P}$ contains only finitely many $G$-conjugacy classes, cf. [46]. Take a choice $\mathcal{P}$ of a complete set of conjugacy representatives in $\mathcal{P}$. Define $\mathcal{P} = \{ gp : g \in G, P \in \mathcal{P} \}$, in which each member shall be referred to as a peripheral coset. It is worth noting that $\mathcal{P}$ depends on the choice of $\mathcal{P}$.

Denote by $\bar{U}$ a complete set of $G$-representatives in $U$. As a consequence, $\| \bar{U} \| < \infty$. It is a useful fact that $G_U$ acts cocompactly on the (topological) boundary $\partial U$.

**Definition 2.10.** Let $T$ be a compact metrizable space on which a group $G$ acts by homeomorphisms. We recall the following definitions, cf [4].

1. $G \actson T$ is a convergence group action if the induced group action of $G$ on the space of distinct triples over $T$ is proper.
2. The limit set $\Lambda(\Gamma)$ of a subgroup $\Gamma \subset G$ is the set of accumulation points of every $\Gamma$-orbit in $T$.
3. A point $\xi \in T$ is called conical if there are a sequence of elements $g_n \in G$ and a pair of distinct points $a, b \in \partial G$ such that the following holds

\[ g_n(\xi, \xi) \rightarrow (a, b), \]

for any $\zeta \in \partial G \setminus \xi$.

4. A point $\xi \in \partial G$ is called bound parabolic if the stabilizer $G_\xi$ in $G$ of $\xi$ is infinite, and acts properly and cocompactly on $T \setminus \xi$.

5. A convergence group $G \actson T$ is called geometrically finite if every point $\xi \in T$ is either a conical point or a bounded parabolic point.

Let $\partial X$ be the Gromov boundary of $X$ endowed with visual metrics, see [26]. It is well-known that $G$ acts geometrically finitely on $\partial X$. Bowditch showed that these metrizable compact spaces $\partial X$ are topologically the same for every choice of $X$ in the definition [13]. So we can associate to $G$ (with respect to $\mathcal{P}$) a topological boundary called Bowditch boundary.

Note that the boundary at infinity of a horoball $U$ in $X \cup \partial X$ is a single point, which is fixed by $G_U$ and is a bounded parabolic point. This gives the following observation which will be invoked implicitly many times.

**Observation 2.11.** Let $\xi$ be a conical point in $\partial X$. Then for any $\epsilon > 0$ any geodesic ending at $\xi$ exits the $\epsilon$-neighborhood of any horoball $U \in \bar{U}$, which the geodesic enters into.

2.4. **Floyd boundary.** Recall that any relatively hyperbolic group admits a so-called Bowditch boundary (relative to $\mathcal{P}$). In this subsection we shall describe a kind of 'absolute' boundary, due to Floyd, for any finitely generated group.

Let $G$ be a group with a finite generating set $S$. Assume that $1 \in S$ and $S = S^{-1}$. Recall that the Cayley graph $\mathcal{G}(G, S)$ of $G$ relative to $S$ is a directed edge-labeled graph with the vertex set $V(\mathcal{G}(G, S)) = G$ and the edge set $E(\mathcal{G}(G, S)) = G \times S$. An edge $e = (g, s)$ goes from the origin $e_- = g$ to the terminus $e_+ = gs$, and has the label $\text{Lab}(e) = s$. By definition, we set $s^{-1} := [ga, s^{-1}]$. Denote by $d_{S'}$ (or simply by $d$ if no ambiguity) the combinatorial metric on $\mathcal{G}(G, S)$.

Let $f(n) = \lambda^{-n} (\lambda > 1)$. For any point $a \in G$, we define a Floyd metric $\rho_a$ as follows. The Floyd length of an edge $e$ in $\mathcal{G}(G, S)$ is set to be $f(n)$, where $n = d(a, e)$. This naturally gives a length metric $\rho_a$ on $\mathcal{G}(G, S)$. Let $\overline{G}_\lambda$ be the Cauchy completion of $G$ with respect to $\rho_a$. The complement $\partial_\Lambda G$ of $G$ in $\overline{G}_\lambda$ is called Floyd boundary of $G$. The $\partial_\Lambda G$ is called non-trivial if $\| \partial_\Lambda G \| > 2$. We refer the reader to [20], [32] for more detail.
It is easy to see that \( \rho_0(x, y) = \rho_{g_0}(gx, gy) \) for any \( g \in G \). Floyd metrics relative to different basepoints are linked by the following formulae,

\[
\lambda^{-d(o, o')} \leq \frac{\rho_0(x, y)}{\rho_{o'}(x, y)} \leq \lambda^{d(o, o')}
\]

This implies that the action \( G \sim \mathcal{G}(G, S) \) extends continuously to \( \partial_\lambda G \). In \([22]\), Karlsson showed that \( G \sim \partial_\lambda G \) is a convergence group action. The key lemma that he used to prove his result is also very helpful in this paper. His lemma holds, in fact, for any quasi-geodesic, but we only need the following simplified version.

**Lemma 2.12 (Visibility Lemma).** \([32]\) There is a function \( \varphi : \mathbb{R} \to \mathbb{R} \) such that for any \( v \in G \) and any geodesic \( \gamma \), the following holds

\[
\rho_v(\gamma_-, \gamma_+) \leq \varphi(r),
\]

where \( r = d(v, \gamma) \).

**Remark (Inverse of \( \varphi \)).** Given \( \epsilon > 0 \), we define \( \varphi^{-1}(\epsilon) \) to be the maximal \( r \) such that \( \varphi(r) \geq \epsilon \). Let \( a, b \in \partial_\lambda G \) be any two points and \( \gamma \) a geodesic between them. It follows that if \( \rho_v(a, b) > \epsilon \), then \( \gamma \cap B(v, r) \neq \emptyset \), where \( r = \varphi^{-1}(\epsilon) \).

The relevance of Floyd boundary to relative hyperbolicity is the following theorem by Gerasimov.

**Theorem 2.13.** \([23]\) Suppose \((G, \mathcal{P})\) is relatively hyperbolic with the Bowditch boundary \( \partial G \). Then there exists \( \lambda_0 > 1 \) such that there exists a continuous \( G \)-equivariant surjective map

\[
\partial_\lambda G \to \partial G
\]

for any \( \lambda > \lambda_0 \).

Let \( \bar{\rho}_v \) be the maximal pseudo-metric on \( \partial_\lambda G \) such that \( \bar{\rho}_v(\cdot, \cdot) \leq \rho_v(\cdot, \cdot) \) and \( \bar{\rho}_v \) vanishes on any pair of points in the preimages of any point \( \xi \in \partial G \). Pushing forward \( \bar{\rho}_v \), we can obtain a true metric on \( \partial G \) (cf. \([24]\)), which is called shortcut metric, still denoted by \( \rho_v \). Obviously, Karlsson Visibility Lemma \(2.12\) holds for shortcut metrics.

Let \( x \in G, U \subset G \). The projection \( \text{Pr}_U(x) \) of \( x \) to \( U \) is the set of points \( y \in U \) such that \( d(y, x) = d(U, x) \). Let \( \bar{U} = U \cup \partial U \) be the closure of \( U \) in \( G \cup \partial G \).

We need consider a projection of boundary points introduced in \([24]\). The definition replies on the following observation which is a consequence of Lemma \(2.12\).

**Lemma 2.14.** \([24]\) Proposition 3.3.2] There exists a non-increasing function \( \psi : ]0, \infty[ \to ]0, \infty[ \) such that for any subsets \( U, V \subset G \) such that \( \bar{U} \cap \partial V = \emptyset \), we have

\[
\| \text{Pr}_V(U) \| < \psi(\epsilon),
\]

where \( \epsilon = \max\{\rho_v(\bar{U}, \partial V) : o \in V\} \).

**Remark.** Consider a hyperbolic space \( X \) instead of \( G \), \( \rho_v \) the visual metric at \( v \) on Gromov boundary \( \partial X \). Then the same conclusion holds for any subsets \( U, V \subset X \) such that \( \bar{U} \cap \partial V = \emptyset \). This is because that the visual metric also satisfies the conclusion of Lemma \(2.12\).

Let \( \xi \in \partial_\lambda G \) and \( V \subset G \) a subset. Then by Lemma \(2.14\) there exists an open neighborhood \( U \) of \( \xi \) such that for any \( U' \subset U \cap (G \cup \partial_\lambda G) \) we have \( \text{Pr}_V(U') = \text{Pr}_V(U) \). We define \( \text{Pr}_V(\xi) = \text{Pr}_V(U) \).
2.5. Characterization of conical points. In this subsection we shall give a description of conical points in Floyd boundary or Bowditch boundary. For simplicity we keep the same notation \( \partial G \) for both of them.

Lemma 2.15. Let \( \xi \in \partial G \) be a conical point. Then there exist a constant \( r > 0 \) and a sequence of elements \( g_n \in G \) such that for any geodesic \( \gamma \) ending at \( \xi \), we have

\[
\gamma \cap B(g_n, r) \neq \emptyset
\]

for all but finitely many \( g_n \).

Proof. Let \( a, b \in \partial G, g_n \in G \) be given by definition for the conical point \( \xi \). Consider a geodesic \( \gamma \) ending at \( \xi \). We do allow \( \zeta = \gamma^- \) to be on \( \partial G \).

Take \( (\epsilon/4) \)-neighborhoods \( U, V \) of \( a, b \) respectively, where \( \epsilon = \rho_o(a, b) \). By Lemma 2.12 it follows that \( g_n \gamma \cap B(1, r) \neq \emptyset \), for all but finitely many \( g_n \). Hence \( \{g_n^{-1}\} \) is the sequence we are looking for. This completes the proof. □

The converse of Lemma 2.15 is also true, under the following assumption.

Corollary 2.16. Suppose that \( G \) acts geometrically finitely on \( \partial G \). Then the converse of Lemma 2.15 is also true.

Proof. Recall that in [25] a quasi-geodesic \( \gamma \) in \( \mathcal{G}(G, S) \) is called a horocycle at \( \xi \in \partial G \) if the following holds

\[
\lim_{t \to \infty} \gamma(t) = \lim_{t \to \infty} \gamma(-t) = \xi.
\]

Consider \( \xi \in \partial G \) satisfying the conclusion of Lemma 2.15. Since \( G \sim \partial G \) is geometrically finite, \( \xi \) is either bounded parabolic or conical. At parabolic points there always exists a quasi-geodesic horocycle. Observe that Lemma 2.15 holds for a quasi-geodesic \( \gamma \), as Lemma 2.12 applies to quasi-geodesics. Thus, these quasi-geodesic horocycles violate Lemma 2.15. So \( \xi \) could only be conical points. □

Recall that \( G \) acts 2-cocompactly on \( \partial G \) if the action \( G \) on the space of pairs of distinct points in \( \partial G \) is co-compact. In [22], Gerasimov proved that a 2-cocompact action is equivalent to a geometrically finite action.

Lemma 2.17. Suppose that \( G \) acts 2-cocompactly on \( \partial G \). Then there is a constant \( r > 0 \) such that the conclusion of Lemma 2.15 holds for any conical point \( \xi \in \partial G \).

Proof. The same proof as in Lemma 2.15 works, with the additional observation by 2-cocompactness of \( G \sim \partial G \) that there is an uniform constant \( \epsilon > 0 \) such that \( \rho_o(a, b) > \epsilon \) for every \( a, b \) given by definition of conical points. □

2.6. Transition points. From now on we shall always assume, unless explicitly stated, that \( G \) is hyperbolic relative to a collection of subgroups \( \mathcal{P} \). Fix a finite generating set \( S \). Denote by \( \partial G \) the Bowditch boundary of \( G \) equipped with a family of shortcut metrics \( \{\rho_v\}_{v \in G} \).

Recall that \( \mathcal{P} \) is the collection of all peripheral cosets in \( G \). Then \( \mathcal{P} \) is a contracting system in \( \mathcal{G}(G, S) \). Hence the notion of transition points relative to \( \mathcal{P} \) applies in \( \mathcal{G}(G, S) \). In fact, this notion was introduced by Hruska in [30]. It is further generalized and elaborated on via Floyd metrics by Gerasimov-Potyagailo in [24]. This viewpoint will be adopted in this paper.

Following Gerasimov-Potyagailo, we explain the role of transition points via Floyd metrics in the following manner.
Lemma 2.18. [24] Proposition 5.2.3 There exists $\epsilon_0 > 0$ with the following property.

For any $\epsilon > \epsilon_0, R > 0$, there exists $\kappa = \kappa(\epsilon, R) > 0$ such that for any geodesic $\gamma$ and an $(\epsilon, R)$-transition point $v \in \gamma$ we have $\rho_v(\gamma_-, \gamma_+) > \kappa$.

In what follows, we make ourselves conform to the following convention.

Convention 2.19 (about $\epsilon, R$). When talking about $(\epsilon, R)$-transition points in $\mathcal{G}(G, S)$ we always assume that $\epsilon \geq \epsilon_0$, where $\epsilon_0$ are given by Lemmas 2.8, 2.9 and 2.18. In addition, assume that $R > R(\epsilon)$, where $R(\epsilon)$ is given by Lemma 2.4.

The following lemma is proven in [30] by Hruska in the case that $r = 0$, with an alternative proof that can be found in [24] by Gerasimov-Potyagailo. The proof below is inspired by the one of Proposition 6.2.1 in [24].

Lemma 2.20. Let $\epsilon, R$ be chosen as in convention (2.19). There exists $D = D(\epsilon, R) > 0$ such that for any $r \geq 0$, there exists $L = L(r, \epsilon, R)$ with the following property.

Let $\gamma, \gamma'$ be two geodesics in $\mathcal{G}(G, S)$ such that $\gamma_- = \gamma'_-$ and $d(\gamma_+, \gamma'_+) < r$. Let $v \in \gamma$ be an $(\epsilon, R)$-transition point such that $d(v, \gamma_+) > L$. Then there exists an $(\epsilon, R)$-transition point $w \in \gamma'$ such that $d(v, w) < D$.

Proof. We first define some constants. Set $R_0 = R(\epsilon)$ and $r_0 = \varphi^{-1}(\kappa/2)$, where $R(\epsilon)$, $\kappa > 0$ are given by Lemmas 2.4, 2.8, and 2.18 respectively. Denote $\kappa' = \lambda^{r_0} \kappa/2$. By Proposition 4.1.2 in [24] there exists $D_0 = D(\epsilon, \kappa') > 0$ such that the following holds for any $V \in \mathbb{P}$

\[(2) \quad N_r(V) \setminus (B_{r_0} + (q, \kappa'/6) \cap G) \subseteq B(w, D_0),\]

where $q$ is the one-singleton limit set of $V$ in $\partial G$.

Assume that $V$ is $(\tau, D_1)$-contracting for $\tau, D_1 > 0$. Let $L_0 = L(R_0 + \epsilon + \tau, R)$ given by Lemma 2.8. Set

\[L_1 = L_0 + D_0 + R_0 + r_0 + r_1 + D_1 + 2\epsilon + \tau,\]

where $r_1 = \varphi^{-1}(\kappa'/3)$. Note that $L_1$ does not depend on $r$. Without loss of generality, assume that $d(v, \gamma_-) > L_1$.

Choose $L > L_1 + r_1 + 1$ large enough, which depends on $r$, such that $\rho_v(\gamma_+, \gamma'_+) < \kappa/2$. As $\rho_v(\gamma_-, \gamma'_+) > \kappa$, we have $\rho_v(\gamma_+, \gamma'_+) > \kappa/2$. By Lemma 2.12 there exists $w \in \gamma'$ such that $d(v, w) < r_0$. Note that $d(v, \gamma_-) > L_1 > R_0 + r_0$. Thus $d(w, \gamma'_+) > R_0$. Choose further $L > r_0 + r_1 + 1$. Then $d(w, \gamma'_+) > R_0$.

Assume that $w$ is $(\epsilon, R_0)$-deep in some $V \in \mathbb{P}$, otherwise we are done. Let $x, y$ be the entry and exit points of $\gamma'$ in $N_r(V)$ respectively. We claim that at least one of $x, y$ is far from the endpoints.

Claim. $\max\{d(x, \gamma'_+), d(y, \gamma'_+)\} > R_0$.

Proof of Claim. Suppose to the contrary that $d(x, \gamma'_+), d(y, \gamma'_+) \leq R_0$. Then $d(\gamma_+, V) < R_0 + r + \epsilon$ and $d(v, V) \leq r_0 + \epsilon$.

By projecting $[v, \gamma_+]$ to $V$, it is easy to see that $[v, \gamma_+] \cap N_r(V) = \emptyset$. Denote by $v'$ the exit point of $[v, \gamma_+]$ in $N_r(V)$. By projection, we see that $d(v', \gamma_+) < \tau + D_0 + R_0 + r + \epsilon$. As $L > L_1 + r + 1$, it follows that $d(v, v') > L_0$.

Note that $d(v, \gamma_-), d(v, v') > L_0$ and $d(\gamma_-, V), d(v', V) \leq R_0 + \epsilon + \tau$. By Lemma 2.8, we have $v$ is $(\epsilon, R_0)$-deep in $V$. This is a contradiction. Thus the claim is proved.

By Lemma 2.31 at least one of $\{x, y\}$ is an $(\epsilon, R)$-transition point in $\gamma'$. We now prove the following bound on the distance of $w$ to $x$ and $y$ respectively.
Claim. $\min\{d(w,x), d(w,y)\} < D_0 + r_1.$

Proof of Claim. By the property \[\text{(1)}\] it follows that $\rho_w(\gamma'_-, \gamma'_+) > \kappa'$. If $\rho_w(x, y) > \kappa'/3$, then by the \[\text{(2)}\] we have $\{x, y\} \cap B(w, D_0) \neq \emptyset$. As $d(v, \gamma_-) > L_1 > D_0 + r_0$, we have $d(y, w) < D_0$.

Otherwise, we have $\max\{\rho_w(\gamma'_-, x), \rho_w(y, \gamma'_+)\} > \kappa'/3$. It follows by Lemma \[\text{(2.12)}\] that $\min\{d(w, y), d(w, x)\} < \varphi^{-1}(\kappa'/3) = r_1$.

By the choice of $L_1$, note that $d(w, \gamma'_-), d(w, \gamma'_+) > R_0 + D_0 + r_1$. By the two claims above, if $d(w, x) < D_0 + r_1$, then $x$ is an $(\epsilon, R)$-transition point; or if $d(w, y) < D_0 + r_1$, then $y$ is an $(\epsilon, R)$-transition point. Setting $D = D_0 + r_0 + r_1$ completes the proof. \[\square\]

2.7. Horofunctions. In this subsection, we shall define horofunctions cocycle at conical points. We start with a horofunction associated to a geodesic ray.

Definition 2.21. Let $\gamma$ be an infinite geodesic ray (with length parametrization) in $\mathcal{F}(G, S)$. A horofunction $b : G \to \mathbb{R}$ associated to $\gamma$ is defined as follows:

$$b_\gamma(x) = \lim_{t \to \infty} (d(x, \gamma(t)) - t),$$

for any $x \in G$.

Let $\kappa > 0$ given by Lemma \[\text{2.18}\] $r > 0$ given by Lemma \[\text{2.17}\] and $\varphi$ given by Lemma \[\text{2.12}\]. We fix $C = \max\{2\varphi^{-1}(\kappa/2), 4r\}$ until the end of this subsection.

Lemma 2.22. Let $\xi \in \partial G$ be a conical point at which terminate two geodesics $\gamma, \gamma'$. Then the following holds

$$|(b_\gamma(x) - b_\gamma(y)) - (b_{\gamma'}(x) - b_{\gamma'}(y))| < C,$$

for any $x, y \in G$.

Proof. Note that $\xi$ is a conical point. By Lemma \[\text{2.15}\] there exists a sequence of points $z_n \in G$ such that $d(z_n, \gamma), d(z_n, \gamma') < r$. Then we obtain two sequences of points $v_n \in \gamma, v'_n \in \gamma'$ such that $d(v_n, v'_n) \leq 2r$. Thus it follows that

$$|b_\gamma(x) - b_\gamma(y))| - (b_{\gamma'}(x) - b_{\gamma'}(y)|,$$

$$= \lim_{n \to \infty} |(d(v_n, x) - d(v_n, y)) - (d(v'_n, x) - d(v'_n, y))|,$$

$$\leq 4r.$$

The conclusion follows. \[\square\]

We are now going to define a notion of horofunction cocycle, up to some uniform constant, independent of the choice of geodesic rays.

Let $\xi \in \partial G$ be a conical point. Let $B_\xi(\cdot, \cdot) : G \times G \to \mathbb{R}$ be defined as follows:

$$\forall x, y \in G : B_\xi(x, y) = \sup_{\gamma, \gamma' \neq \xi} \{\phi_\gamma(x) - \phi_{\gamma'}(y)\}.$$  

For any $z \in G$, let $\forall x, y \in G : B_z(x, y) = d(z, x) - d(z, y)$. By Lemma \[\text{2.22}\] we see that $B_\xi(x, y)$ differs from

$$\forall \gamma, \gamma' = \xi : b_\gamma(x) - b_{\gamma'}(y)$$

an uniform constant depending only on $G$. In what follows, we usually view $b_\gamma(x) - b_{\gamma'}(y)$ as a representative of $B_\xi(x, y)$ and deal with it directly. In this case, we suppress the index $\gamma$ for the convenience.

The following lemma is crucial in establishing quasi-conformal density on Bowditch boundary.
Lemma 2.23. Let $\xi \in \partial G$ be a conical point. For any $x, y \in G$, there is a neighborhood $V$ of $\xi$ such that the following property holds:

$$|B_\xi(x, y) - B_z(x, y)| < 4C, \forall z \in V \cap G.$$ 

Proof. Let $\gamma$ be a geodesic between $x$ and $\xi$. Since $b(x)$ is the limit of an increasing function, for any $\epsilon > 0$ there exists a number $L > 0$ such that

$$|B_\xi(x, y) - (d(x, v) - d(y, v))| \leq \epsilon$$

for any $v \in \gamma$ with $d(x, v) > L$. Fix $\epsilon = C$ in the remainder of the proof.

Let $\epsilon, R$ be chosen as in convention 2.19. We claim that there exists a sequence \(\gamma, R\)-transition points in $\gamma$. In fact, let $u, v$ be the entry and exit points of $\gamma$ in $N_\epsilon(V)$ respectively. If the middle point $z$ of $[u, v]_\gamma$ is $(\epsilon, R)$-deep in $V$, then $u, v$ are both $(\epsilon, R)$-transition points in $\gamma$ by Lemma 2.9. Otherwise, either $z$ is $(\epsilon, R)$-transitional or $z$ is $(\epsilon, R)$-deep in some $V' \neq V \in \mathbb{P}$. The first case already gives an $(\epsilon, R)$-transition point $z$. In the second case, the entry and exit point of $\gamma$ in $N_\epsilon(V')$ is $(\epsilon, R)$-transition points in $\gamma$. Note that $\gamma$ exits the $\epsilon$-neighborhood of peripheral cosets $V \in \mathbb{P}$ which it enters into. This proves our claim.

Choose a $(\epsilon, R)$-transition point $v$ of $\gamma$ such that $d(x, v) > L$. So $\rho_\epsilon(x, \xi) \geq \kappa$ by Lemma 2.18. By taking $L$ large enough, we can assume by Lemma 2.12 that

$$\rho_\epsilon(x, y) \leq \kappa/4.$$

Let $V = B_{\rho_\epsilon}(\xi, \lambda^{-d(\epsilon, v)}\kappa/4)$, the metric ball at $\xi$ radius $\lambda^{-d(\epsilon, v)}\kappa/4$ with respect to the metric $\rho_\epsilon$. We shall show that $V$ is the desired neighborhood.

Take any $z \in V \cap G$. Then the inequality (1) implies that $\rho_\epsilon(z, \xi) < \kappa/4$. Hence the (4) yields

$$\rho_\epsilon(y, z) \geq \kappa/2.$$ 

Let $\gamma' = [y, z]$. By Lemma 2.12, there exists $w \in \gamma'$ such that $d(v, w) \leq \varphi^{-1}(\kappa/2)$. As $\rho_\epsilon(x, z) > \kappa/2$, we have $d(v, [x, z]) < \varphi^{-1}(\kappa/2)$. Then we obtain the following

$$d(x, z) \geq d(x, v) + d(v, z) - 2\varphi^{-1}(\kappa/2)$$

$$\geq d(x, v) + d(w, z) - 3\varphi^{-1}(\kappa/2),$$

for $\forall z \in V \cap G$, which proves

$$|d(x, z) - (d(x, v) + d(w, z))| \leq 3\varphi^{-1}(\kappa/2), \forall z \in V \cap G.$$ 

Hence by (3) and (6) it follows that

$$|B_\xi(x, y) - (d(x, z) - d(y, z))|$$

$$\leq C + |(d(x, v) - d(y, v)) - (d(x, z) - d(y, z))|$$

$$\leq 2C + |(d(x, v) - d(w, z) - d(x, z)) + (d(y, w) - d(y, v))|$$

$$\leq 2C + 3\varphi^{-1}(\kappa/2) \leq 4C.$$ 

for $\forall z \in V \cap G$. This finishes the proof. 

\[
\]

3. Visual and quasi-conformal densities

3.1. Protocol. Suppose that $G$ acts properly and isometrically on a proper geodesic metric space $(X, d)$. Assume, in addition, that $X$ admits a compactification denoted by $\partial G$ such that $X \cup \partial G$ is a compact metrizable space, and the action $G \curvearrowright X$ extends to a minimal convergence group action $G \curvearrowright \partial G$.

Fix a basepoint $o \in X$. Given $z \in \text{Go}$, let $B_z(x, y) = d(x, z) - d(y, z)$ for $x, y \in X$. We denote by $\partial^G G$ the set of conical limit points in $\partial G$. 

\[
\]
Assumption A. There exists a constant $C > 0$ and a family of functions
\[ \{B_{\xi}(\cdot, \cdot) : G \times G \to \mathbb{R}\}_{\xi \in \partial G} \]
such that the following holds.

For any $x, y \in X$, there is a neighborhood $V$ of $\xi \in \partial G$ in $X \cup \partial G$ such that the following property holds:
\[ |B_{\xi}(x, y) - B_{\xi}(x, y)| < 4C, \forall z \in V \cap G. \]

A Borel measure $\mu$ on a topological space $T$ is regular if $\mu(A) = \inf\{\mu(U) : A \subset U, U$ is open\} for any Borel set $A$ in $T$. The $\mu$ is called tight if $\mu(\{K : K \subset A, K$ is compact\}) for any Borel set $A$ in $T$.

Recall that Radon measures on a topological space $T$ are finite, regular, tight and Borel measures. It is well-known that all finite Borel measures on compact metric spaces are Radon. Denote by $\mathcal{M}(\partial G)$ the set of finite positive Radon measures on $\partial G$. Then $G$ possesses an action on $\mathcal{M}(\partial G)$ given by $g_*\mu(A) = \mu(g^{-1}A)$ for any Borel set $A$ in $\partial G$.

Endow $\mathcal{M}(\partial G)$ with the weak-convergence topology. Write $\mu (f) = \int f d\mu$ for a continuous function $f \in C^1(\partial G)$. Then $\mu_n \to \mu$ for $\mu_n \in \mathcal{M}(\partial G)$ iff $\mu_n(f) \to \mu(f)$ for any $f \in C^1(\partial G)$, equivalently $\lim_{n \to \infty} \inf \mu_n(U) \geq \mu(U)$ for any open set $U \subset \partial G$.

**Definition 3.1.** Let $\sigma \in [0, \infty[$. A $G$-equivariant map
\[ \mu : G \to \mathcal{M}(\partial G), \; g \to \mu_g \]
is called a $\sigma$-dimensional visual density on $\partial G$ if $\mu_g$ are absolutely continuous with respect to each other and their Radon-Nikodym derivatives satisfy
\[ \exp(-\sigma d(go, ho)) < \frac{d\mu_g}{d\mu_h}(\xi) < \exp(\sigma d(go, ho)), \]
for any $g, h \in G$ and $\mu_h$-a.e. point $\xi \in \partial G$.

Here $\mu$ is called $G$-equivariant if the following holds $\mu_{h^g}(A) = h_*\mu_g(A)$ for any Borel set $A \subset \partial G$.

**Remark.** The terminology ”visual density” is due to Paulin in [38] to generalize Hausdorff measures of visual metrics on Gromov boundary. Since $G$ has no global fixed point on $\partial G$. Then $\mu_g$ is not an atom measure.

The reader should keep in mind that the upper bound in (7) is usually not applicable in practice. On improving the Radon-Nikodym derivative we propose.

**Definition 3.2.** Let $\sigma \in [0, \infty[$. A $\sigma$-dimensional visual density
\[ \mu : G \to \mathcal{M}(\partial G), \; g \to \mu_g \]
is a $\sigma$-dimensional quasiconformal density if for any $g, h \in G$ the following holds
\[ \frac{d\mu_g}{d\mu_h}(\xi) \asymp \exp(-\sigma B_{\xi}(go, ho)), \]
for $\mu_h$-a.e. conical points $\xi \in \partial G$.

**Remark.** Fix $o \in G$ and let $\nu = \mu_o$. Define $g^*\nu(A) = \nu(gA)$. By the equivariant property of $\mu$ we obtain the following
\[ C^{-1} \exp(-\sigma B_{\xi}(g^{-1}o, o)) \leq \frac{dg^*\nu}{d\nu}(\xi) \leq C \exp(-\sigma B_{\xi}(g^{-1}o, o)), \]
for any $g, h \in G$ and $\nu$-a.e. point $\xi \in \partial G$. 

\[ ∂G \]
for $\mu_h$-a.e. conical points $x, \xi \in \partial G$. This is used by some authors instead of the quasiconformal density, by assuming for $\mu_h$-a.e. $x \in \partial G$.

**Lemma 3.3.** Let $\mu$ be a $\sigma$-dimensional quasiconformal density on $\partial G$. Then the support of $\mu_o$ is $\partial G$.

We now give a protocol of the (partial) shadow and (partial) cone which will be examined in detail in further sections. In order to do so, we make the following assumption.

**Assumption B.** For any $x \in \partial G$ and $x \in G_o$, there exists at least one geodesic ray $[x, \xi]$ between $x, \xi$.

**Definition 3.4 (Shadow).** Let $r \geq 0$ and $g \in G$. The shadow $\Pi_r(g_o)$ at $g_o$ is the set of points $x \in \partial G$ such that for ANY geodesic $[o, x]$ we have $[o, x] \cap B(g_o, r) \neq \emptyset$.

The weak shadow $\Pi_r(g_o)$ at $g_o$ is the set of points $x \in \partial G$ such that there exists SOME geodesic $[o, x]$ intersecting nontrivially in $B(g_o, r)$.

Inside $X$ the shadowed region motivates the notion of a cone.

**Definition 3.5 (Cone).** Let $g \in G$ and $r \geq 0$. The cone $\Phi_r(g_o)$ at $g_o$ is the set of elements $h \in G$ such that there exists SOME geodesic $[o, ho]$ in $X$ such that $[o, ho] \cap B(g_o, r) \neq \emptyset$.

The strong cone $\Psi_r(g_o)$ at $g_o$ is the set of elements $h \in G$ such that for ANY geodesic $[o, ho]$ in $X$ we have $[o, ho] \cap B(g_o, r) \neq \emptyset$.

In case $r = 0$, we omit the index $r$ and write $\Phi(g_o), \Psi(g_o)$ for simplicity.

The key notion in this paper is the following variant of shadow and cone.

**Definition 3.6 (Partial Shadow).** Let $r, \epsilon, R > 0$ and $g \in G$. The partial shadow $\Pi_{r, \epsilon, R}(g_o)$ at $g_o$ is the set of points $x \in \partial G$ such that SOME geodesic $[o, x]$ contains an $(\epsilon, R)$-transition point $v \in [o, x] \cap B(g_o, 2R)$.

Similarly the weak partial shadow $\Pi_{r, \epsilon, R}(g_o)$ is the set of points $x \in \Pi_{r, \epsilon, R}(g_o)$ such that SOME geodesic $[o, x]$ contains an $(\epsilon, R)$-transition point $v \in [o, x] \cap B(g_o, 2R)$.

The notion of a partial cone is closely related to that of a partial shadow.

**Definition 3.7 (Partial Cone).** Let $g \in G$ and $\epsilon, R \geq 0$. The partial cone $\Phi_{r, \epsilon, R}(g_o)$ at $g_o$ is the set of elements $h \in \Phi(g_o)$ such that one of the two following statements holds.

1. $d(o, ho) \leq d(o, g_o) + 2R$.
2. There exists SOME geodesic $\gamma = [o, ho]$ and an $(\epsilon, R)$-transition point $v \in \gamma$ such that $d(v, g_o) \leq 2R$.

For $r > 0$, define $\Phi_{r, \epsilon, R}(g_o) = \cup_{h \in B(g_o, r)} \Phi_{r, \epsilon, R}(h_o)$.

**Remark.** By definition the partial cone $\Phi_{r, \epsilon, R}(g_o)$ is a subset of the weak cone $\Phi_r(g_o)$. The set $\Phi_{r, \epsilon, R}(g_o)$ grows, as $R$ increases and $\epsilon$ decreases.

For $\Delta \geq 0, n \geq 0$, define

$$A(g_o, n, \Delta) = \{h \in G : n - \Delta \leq d(o, ho) - d(o, g_o) < n + \Delta\},$$

for any $g \in G$. For simplicity we write $A(n, \Delta) := A(o, n, \Delta)$.
3.2. **Patterson-Sullivan measures.** The aim of this subsection is to recall a construction of Patterson and show that a Patterson-Sullivan measure is a quasiconformal density.

Choose a basepoint \(o \in X\). We associate the Poincaré series to \(G_o\),

\[
P_G(s, o) = \sum_{g \in G} \exp(-sd(o, go)), \quad s \geq 0.
\]

Note that the convergence radius of \(P_G(s, o)\) is given by

\[
\delta_G = \limsup_{R \to \infty} \frac{\log \#N(o, R)}{R}.
\]

The action \(G \acts X\) is called divergent (resp. convergent) if \(P_G(s, o)\) is divergent (resp. convergent) at \(s = \delta_G\).

We can construct a family of measures supported on \(G_o\). First assume that \(P_G(s, o)\) is divergent at \(s = \delta_G\). Set

\[
\mu^s_v = \frac{1}{P_G(s, o)} \sum_{g \in G} \exp(-sd(v, go)) \cdot \text{Dirac}(go),
\]

where \(s > \delta_G\) and \(v \in Go\). Note that \(\mu^s_v\) is a probability measure.

If \(P_G(s, o)\) is convergent at \(s = \delta_G\), Patterson introduced in [37] a monotonically increasing function \(H\) with the following property:

\[
\forall \epsilon > 0, \exists t_\epsilon, \forall t, \forall a > 0 : H(a + t) \leq \exp(at)H(t).
\]

such that the series \(\sum_{g \in G} H(d(v, go)) \exp(-sd(v, go))\) is divergent for \(s \leq \sigma\) and convergent for \(s > \sigma\). Then define measures as follows:

\[
\mu^s_v = \frac{1}{P_G(s, o)} \sum_{g \in G} \exp(-sd(v, go)) H(d(v, go)) \cdot \text{Dirac}(go),
\]

where \(s > \delta_G\) and \(v \in Go\).

Choose \(s_i \to \delta_G\) such that \(\mu^s_{v_i}\) are convergent in \(\mathcal{M}(\partial G)\). Let \(\mu_v = \lim \mu^s_{v_i}\) be the limit measures, which are so called **Patterson-Sullivan measures** at \(v\). Note that \(\mu_o(\partial G) = 1\).

In the sequel, we write PS-measures as shorthand for Patterson-Sullivan measures.

**Lemma 3.8.** **PS-measures are \(\delta_G\)-dimensional quasiconformal densities.**

**Proof.** With Assumption A, the proof goes exactly as that of Théorème 5.4 in [10]. \(\square\)

4. **Patterson-Sullivan measures on Bowditch boundary: I**

In this section, we consider the PS-measures constructed on the completion \(\mathcal{G}(G, S) \cup \partial G\), where \(\partial G\) is the Bowditch boundary of \(G\) with shortcuts metrics.

The generalities of Section 2 shall get specialized and simplified in this context with aid of the following two facts:

1. The action \(G \acts \mathcal{G}(G, S)\) is divergent for the word metric \(d\).
2. The Karlsson Visibility Lemma [212] holds for the compactification \(G \cup \partial G\).
4.1. **Shadow Lemma.** The key observation in the theory of PS-measures is the Sullivan Shadow Lemma, linking the analytic properties on boundary and geometric properties inside.

We begin with some weak forms of Shadow Lemma (under weak assumption), with proofs following closely the presentation of Coornaert in [10].

**Lemma 4.1 (Shadow Lemma I).** Let \( \mu \) be a \( \sigma \)-dimensional visual density on \( \partial G \). Then there exists \( r > 0 \) such that the following holds

\[
\exp(-\sigma d(o, go)) \leq \mu_o(\Pi_r(go)) \leq \mu_o(\Pi_r(go)),
\]

for any \( g \in G \).

**Proof.** Let \( m_0 \) be the maximal value over atoms of \( \mu_o \). Since \( \mu_o \) is not an atom measure, it follows that \( m_0 < \mu_o(\partial G) \).

Let \( m_0 < m < \mu_o(\partial G) \). Then there is a constant \( \epsilon > 0 \) such that any subset of diameter at most \( \epsilon \) has a measure at most \( m \). Indeed, if not, then there is a sequence of positive numbers \( \epsilon_n \to 0 \) and subsets \( X_n \) of diameter at most \( \epsilon_n \) such that \( \mu(x)(X_n) > m \). Then up to passage of a subsequence, \( \{ X_n \} \) converges to a point \( p \in \partial G \). Since \( \mu_o \) is regular, we have \( \mu_o(p) = \inf(\mu_o(U)) \), where the infimum is taken over all open sets \( p \in U \). It follows that \( \lim \mu(x)(X_n) = \mu_o(p) \). This contradicts the choice of \( m \).

The following fact is a consequence of Visibility Lemma 2.12.

**Claim.** Given any \( \epsilon > 0 \), there is a constant \( r_0 > 0 \) such that the following holds

\[
[\partial G \setminus g^{-1} \Pi_r(go)] < \epsilon
\]

for all but finitely many \( g \in G \) and \( r > r_0 \).

**Proof of Claim.** Given \( \epsilon > 0 \), let \( r_0 = \varphi^{-1}(\epsilon/2) \) be the constant provided by Lemma 2.12. Assume that \( d(go, o) > r > r_0 \).

Observe that \( g^{-1} \Pi_r(go) \) is the set of boundary points \( \xi \in \partial G \) such that any geodesic \( [g^{-1}o, \xi] \) passes through the closed ball \( B(o, r) \). Let \( \xi \in \partial G \setminus g^{-1} \Pi_r(go) \). Thus some geodesic \( [g^{-1}o, \xi] \) misses the ball \( B(o, r) \). Hence \( \rho_g(g^{-1}o, \xi) \leq \epsilon/2 \). This implies that \( [\partial G \setminus g^{-1} \Pi_r(go)] < \epsilon \).

**Remark.** The same claim holds for \( \partial G \setminus g^{-1} \Pi_r(go) \) by the same argument.

Hence for \( \epsilon = \mu_o(\partial G) - m \), there is a constant \( r_0 > 0 \) given by the Claim such that the following holds

\[
\mu_o(\partial G) - m < \mu_o(g^{-1} \Pi_r(go)) < \mu_o(\partial G), \quad \forall r > r_0,
\]

for all but finitely many \( g \in G \).

Since \( \mu \) is a \( \sigma \)-dimensional visual density, there is a constant \( C > 0 \) such that the following holds

\[
C^{-1} \exp(-\sigma d(o, go)) \leq \frac{\mu_{g^{-1}}(\Pi_r(go))}{\mu_o(g^{-1} \Pi_r(go))} < C \exp(\sigma d(o, go)).
\]

By the equivariant property of \( \mu \), it follows that

\[
\mu_{g^{-1}}(\Pi_r(go)) = \mu_o(\Pi_r(go)).
\]

This implies that

\[
(\mu_o(\partial G) - m)C^{-1} \exp(-\sigma d(o, go)) \leq \mu_o(\Pi_r(go)),
\]

for all but finitely many \( g \in G \). This clearly concludes the proof.
Lemma 4.2 (Shadow Lemma II). Let $\mu$ be a $\sigma$-dimensional quasiconformal density on $\partial G$. Then there exists $r_0 > 0$ such that the following holds

$$\mu_o(\Pi_r^c(go)) \leq \mu_o(\Pi_r^c(go)) \exp(-\sigma d(o, go)) \cdot \exp(2\sigma r),$$

for any $r > r_0$.

**Proof.** Let $\xi \in g^{-1}\Pi_r^c(go)$. Then there is a geodesic $\gamma$ between $g^{-1}o$ and $\xi$ such that $\gamma \cap B(o, r) \neq \emptyset$. Hence the following holds

$$|d(z, g^{-1}o) - d(z, o) - d(g^{-1}o, o)| < 2r,$$

for any $z \in \gamma$.

By definition of a horofunction cocycle, we have

$$B_\xi(g^{-1}o, o) = \lim_{t \to \infty} (d(\gamma(t), g^{-1}o) - d(\gamma(t), o)).$$

Since $\mu$ is a $\sigma$-dimensional quasiconformal density, there is a constant $C > 0$ such that the following holds

$$C^{-1} \exp(-\sigma B_\xi(g^{-1}o, o)) \leq \frac{d\mu_{g^{-1}o}(\xi)}{d\mu_o} \exp(-\sigma B_\xi(g^{-1}o, o))$$

for $\mu_o$-a.e. conical points $\xi \in \partial^c G$.

By the equivariant property of $\mu$, it follows that

$$\mu_{g^{-1}o}(g^{-1}\Pi_r(go)) = \mu_o(\Pi_r(go)).$$

Observe that the shadow set $\Pi_r(go)$ is closed and thus $\Pi_r^c(go)$ is Borel. We use Radon-Nikodym derivative to derive the following

$$\mu_o(\Pi_r^c(go)) \leq \int_{g^{-1}\Pi_r^c(go)} C \exp(-\sigma B_\xi(g^{-1}o, o)) d\mu_o(\xi)$$

$$< C \exp(-\sigma d(g^{-1}o, o) + 2\sigma) \mu_o(g^{-1}\Pi_r^c(go)),$$

which finishes the proof by the fact $\mu_o(g^{-1}\Pi_r(go)) < \mu_o(\partial G) = 1$. \(\square\)

Combining Shadow Lemma (I) \(\ref{shadowlemmanonat} \) and (II) \(\ref{shadowlemmanonat2} \) we obtain the full strength of Shadow Lemma for a $\sigma$-dimensional quasiconformal density **without atoms at parabolic points.**

**Lemma 4.3 (Shadow Lemma).** Let $\mu$ be a $\sigma$-dimensional quasiconformal density **without atoms at parabolic points.** Then there is $r_0 > 0$ such that the following holds

$$\exp(-\delta_C d(o, go)) < \mu_o(\Pi_r(go)) \leq \mu_o(\Pi_r(go)) \exp(-\delta_C d(o, go))$$

for any $r > r_0$.

**4.2. First applications to growth functions.** In this subsection we shall see how the growth rate of a relatively hyperbolic group is related to the dimension of visual and conformal density.

**Proposition 4.4.** Let $\mu$ be a $\sigma$-dimensional visual density on $\partial G$. Then the following holds

$$\|N(o, n) < \exp(\sigma n),$$

for any $n \geq 0$. 
Proof. Let \( r_0, C_1 \) be constants given by Weak Shadow Lemma (I). Let \( r = r_0 + 1 \). Denote by \( S_k \) the elements \( g \) in \( G \) with \( d(go,o) = k \). Then we have
\[
\mu_o(\Pi_r(go)) > C_1 \exp(-\sigma d(o,go)) > C_1 \exp(-\sigma k)
\]
for any \( g \in S_k \).

Observe that each point of \( \partial G \) is contained, if at all, in at most \( C_2 \) sets of form \( \Pi_r(go) \), where \( C_2 \) depends only on \( r \). Indeed, given \( \xi \in \partial G \), let \( g \) be such that \( \xi \in \Pi_r(go) \). Fix a geodesic \( \gamma = [o,\xi] \) and a point \( x \in \gamma \) such that \( d(o,x) = k \). By the definition of shadow, we have \( d(go,\gamma) < r \). Then \( go \in B(x,2r) \). Hence \( C_2 = \|B(x,2r)\| \) gives the desired number.

Hence for each \( k > 0 \), we have \( \cup_{g \in S_k} \mu_o(\Pi_r(go)) < C_2 \mu_o(\partial G) \). Then we obtain
\[
\| S_k \| < C_3 \exp(\sigma k),
\]
where \( C_3 = C_1^{-1} C_2 \mu_o(\partial G) \). Consequently, the following holds
\[
\| N(o,n) \| = \sum_{k=0}^{n-1} \| S_k \| < C \exp(\sigma n),
\]
for a constant \( C \geq 1 \).

Remark. In the proof, the (strong) shadow \( \Pi_r(go) \) is necessary to obtain the uniform constant \( C_2 \).

Corollary 4.5. Let \( \mu \) be a \( \sigma \)-dimensional visual density on \( \partial G \). Then \( \sigma \geq \delta_G \).

Corollary 4.6. The following holds
\[
\| N(o,n) \| < \exp(\delta_G n),
\]
for any \( n \geq 0 \).

Proof. By Lemma 4.5, PS-measure is a \( \delta_G \)-dimensional visual density.

Let’s list \( G = \{g_1, \ldots, g_i, \ldots\} \). By Lemma 2.15, it follows that
\[
\partial^c G \subset \bigcup_{r=1}^{\infty} \bigcap_{n=1}^{\infty} \Pi_r(g_i).
\]

If, in addition, \( G \) acts geometrically finitely on \( \partial G \), then by Lemma 2.17, we have that
\[
\partial^c G \subset \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} \Pi_r(g_i) = \limsup_{n \to \infty} \Pi_r(g_n),
\]
for any \( r \gg 0 \). In other words, every conical point is shadowed infinitely many times by the orbit \( Go \).

The observation (11) can be used to impose an upper bound on the dimension of an arbitrary quasiconformal density on \( \partial G \) by the growth rate of \( G \).

Lemma 4.7. Let \( \mu \) be a \( \sigma \)-dimensional quasiconformal density on \( \partial G \). If \( \mu \) gives positive measure to the set of conical points, then \( \sigma \leq \delta_G \).

Proof. List \( G = \{g_1, \ldots, g_i, \ldots\} \) with elements \( g_i \) of increasing lengths. As \( \mu_o(\partial^c G) > 0 \), there exists \( r > 0 \) such that \( \mu_o(A_r) > 0 \), where
\[
A_r = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} \Pi_r(g_i,o).
\]
Recall that
\[ P_G(s, o) = \sum_{g \in G} \exp(-sd(o, go)) = \sum_{k=0}^{\infty} \frac{1}{k!} s^k \exp(-sk). \]

We claim that \( P_G(s, o) \) is divergent at \( s = \sigma \). Indeed, by Lemma 4.2 we see that for sufficiently large \( n > 0 \), the following holds
\[ \sum_{k=n}^{\infty} \frac{1}{k!} s^k \exp(-\sigma k) \geq \exp(-2\sigma r) \sum_{d(go, o) > n} \mu_o(\Pi_r^e(go)) \geq \exp(-2\sigma r) \mu_o\left( \sum_{d(go, o) > n} \Pi_r^e(go) \right). \]

Observe that \( \sum_{d(go, o) > n} \Pi_r^e(go) \ni A_n \). It follows that
\[ \sum_{k=n}^{\infty} \frac{1}{k!} s^k \exp(-\sigma k) > \exp(-2\sigma r) \mu_o(A_r) > 0, \]
for \( n >> 0 \). Hence, \( P_G(s, o) \) is divergent at \( s = \sigma \). \( \square \)

**Corollary 4.8.** If \( \mu \) is a \( \sigma \)-dimensional quasiconformal density for \( G \) giving positive measure on conical points, then \( \sigma = \delta_G \).

**4.3. PS-measures are quasiconformal densities without atoms.** Let \( \mu \) be a PS-measure constructed on \( \mathcal{G}(G, S) \cup \partial G \). We shall show that \( \mu \) has no atoms.

We first show that \( \mu \) have no atoms at parabolic points, following an argument of Dal’bo-Otal-Peigné in [13, Propositions 1 & 2].

**Lemma 4.9.** Let \( H \) be a subgroup in \( G \) such that \( \Lambda(H) \) is properly contained in \( \partial G \). Then \( P_H(s, o) \) is convergent at \( s = e_{\lambda} \). In particular, if \( H \) is divergent, then \( \nu_H < \nu_G \).

**Proof.** Since \( H \) acts properly on \( \partial G \setminus \Lambda(H) \). Let \( K \) be a Borel fundamental domain for this action. Then we obtain the following
\[ \mu_o(\partial G) = \sum_{h \in H} \mu_o(hK) + \mu_o(\Lambda(H)). \]

Observe that \( \mu_o(K) > 0 \). Indeed, if not, we see that \( \mu_o \) is supported on \( \Lambda(H) \). This gives a contradiction, as \( \mu \) is supported on \( \partial G \). Thus \( \mu_o(K) > 0 \).

Note that \( \mu \) is a \( \delta_G \)-dimensional visual density. It follows that we have
\[ \mu_o(h^{-1}K) = \mu_o(K) \geq \exp(-\delta_G d(o, ho)) \mu_o(K) \]
for any \( h \in H \). Hence
\[ \mu_o(\partial G) \geq \sum_{h \in H} \mu_o(hK) \geq \sum_{h \in H} \exp(-\delta_G d(o, ho)) \mu_o(K), \]

implying that \( \sum \exp(-\delta_G d(o, ho)) \) is finite. This concludes the proof. \( \square \)

**Lemma 4.10.** The PS-measure \( \mu \) has no atoms at bounded parabolic points.

**Proof.** Let \( q \in \partial G \) be a bounded parabolic point, the stabilizer \( P \) in \( G \) of \( q \). Choose a compact fundamental domain \( K \) for the action of \( P \sim (\partial G \cup G) \setminus q \). Enlarge and keep \( K \) be compact in \( G \cup \partial G \) such that \( q \notin K \) and the boundary of \( K \) is \( \mu_o \)-null for some (hence any) \( v \in G \).

By Lemma 2.13 one sees that \( \| \Pr_P(K) \| \) is uniformly bounded. Choose \( o \in \Pr_P(K) \). By Lemma 2.13 there is a constant \( D > 0 \) such that the following holds
\[ d(z, o) + D > d(z, po) + d(o, po) > d(o, z), \]
for any $z \in pK$.

Define $V_n = \bigcup_{d(o,po) > n} pK$. Then $V_n \cup q$ is a decreasing sequence of open neighborhoods of $q$. Note that the boundary of $V_n$ is $\mu_o$-null. It follows that $\mu_o^s(V_n) \to \mu_o(V_n)$ for each $V_n$, as $s \to \delta_G$. Then we have

$$\mu_o^s(V_n) = \mu_o^s\left(\bigcup_{d(o,po) > n} pK\right) \leq \sum_{d(o,po) > n} \mu_o^s(pK)$$

$$\leq \frac{1}{P_G(s,o)} \sum_{d(o,po) > n} \sum_{go \in pK} \exp(-sd(o,go)) \cdot \text{Dirac}(go)$$

yielding

$$\mu_o^s(V_n) \leq \frac{\exp(sD)}{P_G(s,o)} \sum_{d(o,po) > n} \exp(-sd(o,po)) \cdot \text{Dirac}(go)$$

$$\leq \exp(sD) \sum_{d(o,po) > n} \exp(-sd(o,po)) \mu_o^s(K).$$

Note that $P_H(s,o)$ is convergent at $s = \delta_G$. Let $s \to \delta_G$ and then $n \to \infty$. We obtain that $\mu_o(q) = 0$. Therefore, PS-measures have no atom at parabolic points.

We next consider conical points. In what follows until the end of this subsection, we fix $r_0 = \varphi^{-1}(\kappa/2)$, where $\kappa > 0$ is given by Lemma 2.12 and $\varphi$ given by Lemma 2.12.

**Lemma 4.11.** Let $\xi \in \partial G$ be a conical limit point and $\gamma$ be a geodesic between $o$ and $\xi$. Let $v$ be a transition point on $\gamma$. If $d(v,o) > r_0$, then

$$B_{\rho_o}(\xi, \frac{\lambda^{d(v,o)} \kappa}{2}) \subset \Pi_{r_0}(v).$$

**Proof.** Let $\eta \in B_{\rho_o}(\xi, \frac{\lambda^{d(v,o)} \kappa}{2})$. Then $\rho_o(\eta, \xi) \leq \frac{\lambda^{d(v,o)} \kappa}{2}$. By the property (1) it follows that $\rho_o(\eta, \xi) \leq \lambda^{-\frac{d(v,o)}{2}} \rho_o(\eta, \xi) \leq \kappa/2$. Hence any geodesic between $o, \eta$ has to pass through the ball $B(v, r_0)$.

We are able to estimate PS-measures of a sequence of shrinking $\rho_o$-metric disks around a conical point.

**Lemma 4.12.** Let $\mu_o$ be a $\delta_G$-dimensional quasiconformal density without atoms at parabolic points on $\partial G$. Denote $\sigma = -\delta_G/\log \lambda$. Then there exist constants $C_1, C_2 > 0$ with the following property.

Let $\xi \in \partial G$ be a conical limit point and $\gamma$ a geodesic between $o$ and $\xi$. Then there exists a sequence of decreasing numbers $\{r_i > 0 : r_i \to 0\}$ such that the following holds

$$C_1 r_i^\sigma \leq \mu_o(B_{\rho_o}(\xi, r_i)) \leq C_2 r_i^\sigma,$$

for all $i > 0$.

**Proof.** By Lemma 4.10 all parabolic points are not atoms. Then by Lemma 4.12 and Lemma 4.2 we obtain

$$\exp(-\delta_G d(o,v)) \leq \mu_o(B_{\rho_o}(\xi, \frac{\lambda^{d(v,o)} \kappa}{4})) \leq \exp(-\delta_G d(o,v)) \exp(2\delta_G r_0).$$

Let $r = \frac{\lambda^{d(v,o)} \kappa}{4}$. Then we have $\exp(-\delta_G d(o,v)) = (4r/\kappa)^\sigma$, where $\sigma = -\delta_G/\log \lambda$. 


Observe that there are infinitely many transitional points on $\gamma$. So $d(v,o) \to \infty$. This gives the sequence $\{r_i > 0\}$. The proof is complete. \hfill \square

As a corollary, we obtain the following.

**Lemma 4.13.** Conical points are not atoms of PS-measures.

As $G \sim \partial G$ is geometrically finite, there exist only bounded parabolic points and conical points in $\partial G$. Hence, Lemmas 4.10 and 4.13 together prove the following.

**Proposition 4.14.** Any PS-measure constructed on $\mathcal{H}(G,S) \cup \partial G$ is a $\delta_G$-dimensional quasiconformal density without atoms.

### 4.4. Uniqueness and Ergodicity of PS-measures

This subsection is devoted to the proof for the uniqueness and ergodicity of PS-measures (in fact, for any quasiconformal density without atoms). The results proven here will not be used anymore in this paper.

Let $B$ be a metric ball of radius $\text{rad}(B)$ in a metric space $X$. For $t > 0$ define $tB$ to be the union of all balls of radius $t \cdot \text{rad}(B)$ intersecting non-trivially in $B$. Recall the following covering result.

**Lemma 4.15.** [34 Theorem 2.1] Let $X$ be a proper metric space and $B$ a family of balls in $X$ with uniformly bounded radii. Then there exists a subfamily $B' \subset B$ of pairwise disjoint balls such that the following holds

$$\bigcup_{B \in B} B \subset \bigcup_{B \in B'} 5B.$$ 

Let $B = \{B_{\rho_\xi}(\xi, r_\xi) : \xi \in \partial^o G\}$ be a collection of balls centered at conical points $\xi$, with radii $r_\xi$ provided by Lemma 4.12. Note that $B$ covers $\partial^o G$.

We consider a Hausdorff-Caratheodory measure $H_\sigma(\cdot, B)$ relative to $B$ and the gauge function $r^\delta$. The Hausdorff measure in the usual sense is taking $B$ as the set of all metric balls in $\partial G$.

**Lemma 4.16.** Let $\mu$ be a $\delta_G$-dimensional quasiconformal density without atoms on $\partial G$. Then we have

$$H_{\delta_G}(A, B) \asymp \mu_\circ(A)$$

for any subset $A \subset \partial^o G$. In particular, $\mu$ is unique in the sense of if $\mu, \mu'$ are two PS-measures, then the Radon-Nikodym derivative $d\mu/d\mu'$ is bounded from up and below.

**Proof.** Let $A$ be any subset in $\partial^o G$. Let $B' \subset B$ be an $\epsilon$-covering of $A$. Then $\mu_\circ(A) < \sum_{B \in B'} \mu_\circ(B)$. Let $\epsilon \to 0$. By Lemma 4.12 we obtain that $\mu_\circ(A) \leq C \cdot H_{\delta_G}(A, B)$.

To establish the other inequality, we use an adaption of an argument of McMullen, cf. [5, Remark 2.5.11]. Note that $\mu_\circ, H_\sigma$ are Radon measures. Then for any $\tau > 0$ there exists a compact set $K$ and an open set $U$ such that $K \subset A \subset U$ and $H_\sigma(U \setminus K) < \tau, \mu_\circ(U \setminus K) < \tau$.

Set $\epsilon_0 := \rho_\circ(K, \partial G \setminus U) > 0$. For any $0 < \epsilon < \epsilon_0$, let $B_1 \subset B$ be an $\epsilon$-covering of $K$. Then there exists a sub-collection $B_2$ of $B_1$ given by Lemma 4.12 such that

$$H_\sigma(K, B) < \sum_{B \in B_1} \text{rad}(B) \epsilon < \sum_{B \in B_2} (5\text{rad}(B)) \epsilon < C \cdot \mu_\circ(U),$$

for some constant $C > 0$ provided by Lemma 4.12. Letting $\tau \to 0$ yields $H_\sigma(A, B) < C\mu_\circ(A)$. \hfill \square
Proposition 4.17. Let \( \mu \) be a \( \delta_G \)-dimensional quasiconformal density without atoms on \( \partial G \). Then \( \mu \) is ergodic with respect to \( G \rightharpoonup \partial G \).

Proof. Let \( A \) be a \( G \)-invariant Borel subset in \( \partial G \) such that \( \mu_o(A) > 0 \). Restricting \( \mu \) on \( A \) gives rise to a \( \delta_G \)-dimensional conformal density without atoms on \( A \).

By Lemma 4.16, there exists \( C > 0 \) such that for any subset \( X \subset \partial G \), we have \( \mu_o(X) < CH_\sigma(A \cap X, \mathcal{B}). \) Thus \( \mu_o(\partial G \setminus A) = 0. \)

Summarizing the results obtained previously, we have the following.

Proposition 4.18. The PS-measures \( \mu \) are \( \delta_G \)-dimensional quasiconformal density without atoms. Moreover, \( \mu \) is unique and ergodic.

5. Applications towards growth of cones and partial cones

This section begins with the study of the growth of cones. Let’s fix some notations. For \( r, \epsilon, R, \Delta > 0 \), define

\[
\Phi_r(go,n,\Delta) = \Phi_r(go) \cap A(go,n,\Delta)
\]

and

\[
\Phi_{r,\epsilon,R}(go,n,\Delta) = \Phi_{r,\epsilon,R}(go) \cap A(go,n,\Delta),
\]

for any \( g \in G, n \geq 0 \). The set \( \Psi_r(go,n,\Delta) \) is defined similarly.

5.1. Growth of Cones. We first consider the growth of (weak) cones. The proof of Lemma 5.1 is simple, but illustrates the basic logic theme of proofs for more complicated Lemmas 5.5 and 7.8.

Lemma 5.1. There are constants \( r, \Delta, \kappa > 0 \) such that the following holds

\[
\sharp \Phi_r(go,n,\Delta) \geq \kappa \cdot \exp(\delta_G n)
\]

for any \( g \in G \) and \( n \geq 0 \).

Proof. Let \( r_0 \) be the constant given by Shadow Lemma 4.3. We first note the following observation, roughly saying that the shadow of a ball at \( go \) can be covered by those of all elements at level \( n \) in the cone at \( go \).

Claim. Let \( \Delta > 0 \). Then the following holds

\[
\mu_o(\Pi_r(go)) \leq \sum_{h \in \Phi_r(go,n,\Delta)} \mu_o(\Pi_r(ho)),
\]

for any \( r > r_0 \).

Proof of Claim. Let \( \xi \in \Pi_r(go) \). Then there is SOME geodesic \( \gamma = [o,\xi] \) such that \( \gamma \cap B(go,r) \neq \varnothing \). Take \( ho \in \gamma \) such that \( d(ho,o) = n+d(o,go) \). Then \( ho \in \Phi_r(go,n,0) \). Hence the conclusion follows.

By Shadow Lemma 4.3 it follows that

\[
\exp(-n\delta_G) \cdot \mu_o(\Pi_r(ho)) \asymp_r,\Delta \mu_o(\Pi_r(go)),
\]

for any \( h \in \Phi_r(go,n,\Delta) \). Hence by the Claim above, we obtain that \( \sharp \Phi_r(g,n,\Delta) \asymp_r,\Delta \exp(n\delta_G) \).
5.2. Partial Shadow Lemma. To study the growth of partial cones, the key tool is a variant of Shadow Lemma for Partial Shadows, which we call Partial Shadow Lemma.

The following observation will be used in the proof of Partial Shadow Lemma.

**Lemma 5.2.** For any \( \epsilon, r > 0 \), there exists \( R = R(\epsilon, r) > 0 \) such that the following holds
\[
d_{(z, o)} > R \sum_{z \in gP} \exp(-\delta_G d(o, z)) < \epsilon,
\]
for all \( gP \in \mathbb{P} \) and \( o \in N_r(gP) \).

**Proof.** By Lemma 4.9, we have that \( P(s, o) \) is convergent at \( s = \delta_G \) for each \( P \in \mathbb{P} \). The conclusion follows.

We can now state the main result in this subsection. Also see Lemma 7.3 the analogue for the PS-measure obtained via a cusp uniform \( G \sim X \).

**Lemma 5.3 (Partial Shadow Lemma).** Let \( \epsilon \) be as in convention (2.19). There exist constants \( r, R > 0 \) such the following holds
\[
\exp(-\delta_G d(o, go)) < \mu_o(\Pi_{r, r, R}(go)) \leq \mu_o(\Pi_{r, r, R}(go)),
\]
for any \( g \in G \).

**Proof.** Let \( r_0, C > 0 \) be given by Shadow Lemma 4.3 such that the following holds
\[
\mu_o(\Pi_r(go)) \geq C \exp(-\delta_G d(o, go)),
\]
for any \( r \geq r_0 \). Fix \( r = r_0 \).

Denote by \( F \) the set of peripheral cosets \( hP \in \mathbb{P} \) such that \( hP \cap B(go, r + \epsilon) \neq \emptyset \). Then \( F \) is a uniform number depending only on \( G \).

Let \( R_1 = R(C(2 \| F \|)^{-1}, r + \epsilon) \) be the constant given by Lemma 5.2 such that
\[
\sum_{z \in hP} \exp(-\delta_G d(go, z)) < C/\| F \|.
\]

Let \( R_0 > 0 \) be given by Lemma 2.9. Choose \( R = r + \max\{R_0, R_1\} \). In the reminder of proof, we show that \( r, R \) are the desired constants.

Let \( \xi \in \Xi = \Pi_r(go) \setminus \Pi_{r, r, R}(go) \). Then any geodesic \( [o, \xi] \) does not contain an \( (\epsilon, R) \)-transition point in the ball \( B(go, 2R) \).

Choose \( g'o \in B(go, r) \cap [o, \xi] \). Then \( d(go, g'o) < r \). Thus \( [o, \xi] \) does not contain an \( (\epsilon, R) \)-transition point in \( B(go, 2R - r) \). As \( R > r \), \( g'o \) is \( (\epsilon, R) \)-deep in some \( hP \in \mathbb{P} \). Since \( d(go, hP) < r + \epsilon \), we have \( hP \in F \).

Let \( z \) be the exit point of \( [o, \xi] \) in \( N_r(hP) \). Then by Lemma 2.9, \( z \) is an \( (\epsilon, R) \)-transition point in \( [o, \xi] \). Hence \( d(z, g'o) > 2R - r \).

Note that \( d(o, z) > d(o, go) + d(go, z) + 2r \). We estimate \( \mu_o(\Xi) \) as follows:
\[
\mu_o(\Xi) \leq \sum_{hP \in F} \sum_{z \in hP} \mu_o(\Pi_r(z)) \leq \sum_{hP \in F} \sum_{z \in hP} \exp(-\delta_G d(o, z)) \exp(2\delta_G r) \leq \exp(-\delta_G d(o, go)) \cdot \sum_{hP \in F} \sum_{z \in hP} \exp(-\delta_G d(go, z)) \leq \exp(-\delta_G d(o, go)) \cdot C/2.
\]
The last inequality in (15) uses the inequality (14).

Notice that \( \mu_o(\Xi) + \mu_o(\Pi_{\epsilon,R}(go)) = \mu_o(\Pi_{\epsilon}(go)) \geq C \exp(-\delta_{\epsilon} d(o,go)) \). So the inequalities (13) and (15) yield
\[
\mu_o(\Pi_{\epsilon,R}(go)) \geq (C/2) \exp(-\delta_{\epsilon} d(o,go)).
\]

The proof is complete. \( \Box \)

5.3. Growth of partial cones. We are in a position to prove the exponential growth of the annulus of a partial cone. Let’s first note the following relation between the partial cone and strong cone.

Lemma 5.4. Let \( \epsilon, R \) be as in convention (2.19). For any \( r > 0 \), there exists \( r' = r'(r, \epsilon, R) > 0 \) such that \( \Phi_{r,\epsilon,R}(go) \subset \Psi_r(go) \).

Proof. Let \( h \in \Phi_{r,\epsilon,R}(go) \). Then there exists some geodesic \( \gamma = [o, ho] \) such that \( \gamma \) contains an \( (\epsilon, R) \)-transition point \( v \) in \( B(go, 2R) \).

Let \( \alpha = [o, ho] \) be any geodesic. By Lemma 2.20 there exists \( D = D(\epsilon, R) \) such that \( d(o, v) < D \). Hence \( d(go, \alpha) < D + 2R \). This implies that \( h \in \Psi_r(go) \), where \( r' = 2R + D \).

Lemma 5.5. Let \( r, \epsilon, R \) be as in Lemma 5.3. There are constants \( \Delta, \kappa > 0 \) such that the following holds
\[
\| \Phi_{r,\epsilon,R}(go, n, \Delta) \| \geq \kappa \cdot \exp(n\delta_G)
\]
for any \( g \in G \) and \( n \geq 0 \).

Proof. We first note the following observation, roughly saying that the partial shadow \( \Pi_{\epsilon,R}(go) \) can be covered by those of all elements at level \( n \) in \( \Phi_{r,\epsilon,R}(go, n) \).

Claim. There exists a constant \( D = D(\epsilon, R) > 0 \) such that the following holds
\[
\mu_o(\Pi_{\epsilon,R}(go)) \leq \sum_{h \in \Phi_{r,\epsilon,R}(go, n, \Delta)} \mu_o(\Pi_r(ho)),
\]
for any \( n \geq D + R + r, \Delta \geq 0 \).

Proof of Claim. Let \( \xi \in \Pi_{\epsilon,R}(go) \). Then there is SOME geodesic \( \gamma = [o, \xi] \) containing an \( (\epsilon, R) \)-transition point \( v \) in \( B(go, 2R) \).

Since \( \xi \in \Pi_{\epsilon}(go) \), there exists a geodesic \( \alpha = [o, \xi] \) (possibly \( \alpha \neq \gamma \)) such that \( \alpha \cap B(go, r) \neq \emptyset \). Let \( D = D(\epsilon, R) \) be the constant given Lemma 2.20. Then there exists an \( (\epsilon, R) \)-transition point \( w \in \alpha \) such that \( d(v, w) < D \).

Let \( ho \in \alpha \) such that \( d(ho, o) = n + d(o, go) \). Since \( n > D + R + r \), it follows that \( d(go, [0, ho]_o) < r \) and \( w \in [0, ho]_o \).

Notice that \( w \) is also an \( (\epsilon, R + D) \)-transition point in \( \alpha \). As \( d(w, go) < 2R + D \), it follows that \( h \in \Phi_{r,\epsilon,R+D}(go, n, 0) \). Hence the claim is proved. \( \Box \)

By Shadow Lemmas 4.3 and 5.3 it follows that
\[
\exp(-n\delta_G) \cdot \mu_o(\Pi_r(ho)) \geq_{\epsilon,\epsilon, R, \Delta} \mu_o(\Pi_{r,\epsilon,R}(go)),
\]
for any \( h \in \Phi_{r,\epsilon,R+D}(go, n, \Delta) \). By the Claim above we obtain that
\[
\| \Phi_{r,\epsilon,R+D}(go, n, \Delta) \| >_{r,\epsilon, R, \Delta} \exp(n\delta_G)
\]
for \( n > D + R + r \). The proof is complete. \( \Box \)

By Lemma 5.4 we have the exponential growth of the strong cones.
Corollary 5.6. There are constants \( r, \Delta, \kappa > 0 \) such that the following holds

\[
\Psi_r(go, n, \Delta) \geq \kappa \cdot \exp(n\delta_G)
\]

for any \( g \in G \) and \( n \geq 0 \).

Recall that an element \( g \) in \( \mathcal{G}(G, S) \) is called a dead end if \( d(1, h) \leq d(1, g) \) for all \( h \in G \). cf. \cite{27} and references therein. For example, such elements exist in the Cayley graph of \( G \times Z_2 \) with respect to a particular generating set. However, one could remedy this situation by the following result, saying that for any \( g \in G \) there exists a companion element near \( g \) with a very large cone.

Proposition 5.7 (Companion Cone). Let \( r, \epsilon, R \) be as in Lemma 5.5. There are constants \( \Delta, \kappa > 0 \) such that the following holds.

For any \( g \in G \), there exists \( g' \in G \) such that \( d(g, g') < r \) and the following holds

\[
\Phi_{\epsilon, R}(g'o, n, \Delta) \geq \kappa \cdot \exp(n\delta_G)
\]

for any \( n \geq 0 \).

Proof. By Lemma 5.5 this is proven by using a pigeonhole principle on \( B(go, r) \).

5.4. Large Transitional Trees. In this subsection, we shall construct a tree \( T \) in \( \mathcal{G}(G, S) \) such that the growth rate of \( T \) is sufficiently close to \( \delta_G \) and transition points are uniformly spaced in \( T \). Our construction is inspired by a nice argument of Coulon in a hyperbolic group.

Endow \( S \triangleleft G \) with a total order. Let’s consider the lexicographic order over the free monoid \( \mathcal{F}(S) \) on \( S \) as follows. For any two words \( w, w' \in \mathcal{F}(S) \), we say \( w \leq w' \) if in the first letter that they differ, the corresponding letter of \( w \) is smaller than that of \( w' \).

Fix the basepoint at 1 in \( G \). For \( g \in G \), denote by \( \omega_g \) the lexi-geodesic among all geodesics between 1 and \( g \) such that the word labeling \( \omega_g \) is minimal with respect to “\( \leq \)”. Define \( \mathcal{L} = \{ \omega_g : g \in G \} \).

Definition 5.8 (Partial Lexi-Cone). For any \( g \in G \), the lexi-cone of \( \mathcal{L}(g) \) at \( g \) is the set of \( h \in G \) such that \( g \in \omega_h \). For \( \epsilon, R \geq 0 \), the partial lexi-cone \( \mathcal{L}_{\epsilon, R}(g) \) at \( g \) is the set of elements \( h \in \mathcal{L}(g) \) such that one of the two following holds.

1. \( d(1, h) \leq d(1, g) + 2R \),
2. \( \omega_h \) contains an \((\epsilon, R)\)-transition point \( v \) such that \( d(v, g) \leq 2R \).

For \( \epsilon, R, \Delta, n \geq 0 \), define

\[
\mathcal{L}_{\epsilon, R}(g, n, \Delta) = \mathcal{L}_{\epsilon, R}(g) \cap A(g, n, \Delta).
\]

Proposition 5.9 (Companion Lexi-cone). Let \( r, \epsilon, R \) be as in Lemma 5.5. There are constants \( \Delta, \kappa > 0 \) with the following property.

For any \( g \in G \), there exists \( g' \in G \) such that \( d(g, g') < r \) and

\[
\mathcal{L}_{\epsilon, R}(g', n, \Delta) \geq \kappa \cdot \exp(n\delta_G)
\]

for any \( n \geq 0 \).

Proof. Let \( h \in \Phi_{\epsilon, R}(g, n, \Delta) \). By definition, there exists \( \gamma = [1, h] \) containing an \((\epsilon, R)\)-transition point \( v \) such that \( d(v, g) < 2R \). Assume that \( \omega_h \neq \gamma \). By Lemma \cite{22} there exist \( D = D(\epsilon, R) > 0 \) and an \((\epsilon, R)\)-transition point \( w \in \omega_h \) such that
\begin{align*}
d(v, w) < D. \text{ Using a pigeonhole principle on the finite set } B(g, 2R + D), \text{ there exist } g' \in B(g, 2R + D) \text{ and } \kappa > 0 \text{ such that the following holds}
\| \mathcal{L}_{c, R}(g', n, \Delta) \| \geq \kappa \cdot \exp(n\delta_G),
\end{align*}
which completes the proof. \hfill \Box

The following lemma is useful in choosing a sufficiently separated set.

**Lemma 5.10.** [Lemma 5.1] For any \( M > 0 \) there exists a constant \( \theta = \theta(M) > 0 \) depending on \( G \) and \( S \) with the following property.

Let \( Y \) be any finite set in \( G \). Then there exists an \( M \)-separated subset \( Z \subset Y \) such that \( \| Z \| > \theta \cdot \| X \| \).

We now prepare the constants used in the proof of Lemma 5.11.

Take \( r, \epsilon, R, \Delta, \kappa > 0 \) given by Proposition 5.9. Define the auxiliary set
\begin{equation}
E = \{ g \in G : \| \mathcal{L}_{c, R}(g, n, \Delta) \| > \kappa \cdot \exp(n\delta_G), \forall n \geq 0 \}.
\end{equation}
Note that \( N_r(E) = G \).

Let \( D = D(\epsilon, R), L = L(\epsilon, R, C + r) > 0 \) be given by Lemma 2.20, where
\[ C := D + 2R. \]

Let \( \theta = \theta(2C + 2r) > 0 \) be given by Lemma 5.10.

Let \( \kappa' = \kappa / (\theta \cdot \| B(1, C) \|) \) and \( R' = 2C \).

Fix \( n > 0 \) and let \( \Delta' = \Delta + 2C + r \). We make use of a construction of Coulon in [12], defining a sequence of decreasing sets \( G_{i,n} \) for \( i \geq 0 \) inductively. Let \( G_{0,n} = G \).

Then set
\[ G_{i+1,n} = \{ g \in G_{i,n} : \| \mathcal{L}_{c, R'}(g, n, \Delta') \cap G_{i,n} \| > \kappa' \cdot \exp(n\delta_G) \} . \]

Define \( n_0 = L + 2R + \Delta \).

**Lemma 5.11.** Let \( n > n_0 \). Then the following holds
\[ B(g, C) \cap G_{i,n} \neq \emptyset \]
for each \( g \in E \) and \( i \geq 0 \).

**Proof.** We proceed by induction on \( i \). For \( i = 0 \), it is automatically true. Assume that \( B(g, C) \cap G_{j,n} \neq \emptyset \) for each \( j \leq i, g \in E \). We shall show that
\[ B(g, C) \cap G_{i+1,n} \neq \emptyset \]
for each \( g \in E, n > n_0 \).

By definition of the set \( E \) in (16), \( \| \mathcal{L}_{c, R}(g, n, \Delta) \| \geq \kappa \cdot \exp(\delta_G n) \) for any \( g \in E \).

By Lemma 5.10 there exist \( \theta = \theta(2r + 2C) > 0 \) and a \( (2r + 2C) \)-separated set \( Z \subset \mathcal{L}_{c, R}(g, n, \Delta) \) such that
\begin{equation}
\| Z \| > \kappa / \theta \cdot \exp(\delta_G n).
\end{equation}

Let \( x \in Z \). By \( N_r(E) = G \) and the induction assumption on \( i \), there exists \( h_x \in G_{i,n} \) such that \( d(x, h_x) < C + r \).

Observe that \( h_x \neq h_y \) for any \( x \neq y \in Z \). Indeed, if \( h_x = h_y \), then \( d(x, y) < 2C + 2r \) for \( x, y \in Z \). This contradicts the choice of \( X \) as a \( (2C + 2r) \)-separated set in \( \mathcal{L}_{c, R}(g, n, 0) \). Hence the (17) yields
\begin{equation}
\| \{ h_x : x \in Z \} \| > \kappa / \tau \cdot \exp(\delta_G n).
\end{equation}

Recall that \( \kappa' = \kappa / (\theta \cdot \| B(1, C) \|) \). We next show the following claim.
Claim. There exists an element \( g' \in B(g, C) \) such that
\[
\| (\mathcal{L}_{c,R'}(g', n, \Delta') \cap G_{i,n}) \| > \kappa' \cdot \exp(n\delta_G).
\]

Proof of Claim. Since \( x \in Z \subset \mathcal{L}_{c,R}(g, n, \Delta) \), the lexi-geodesic \( \omega_x \) contains an \((\epsilon, R)\)-transition point \( v \in \omega_x \cap B(g, 2R) \).

Since \( d(o,go) > n - \Delta \) and \( n > n_0 \geq L + 2R + \Delta \). Then \( d(v, x) > L \). Note that \( d(x, h_x) < C + r \) and apply Lemma \([\ref{2.20}]\) to the pair of geodesics \( \omega_x, \omega_{h_x} \). Then there exists an \((\epsilon, R)\)-transition point \( w \in \omega_{h_x} \) such that \( d(v, w) < D \). Hence it follows that
\[
d(g, \omega_{h_x}) < D + 2R \leq C,
\]
for any \( h_x \) where \( x \in X \).

Recall that \( \Delta' = \Delta + C + r \). By using a pigeonhole principle on \( B(g, C) \), there exist \( g' \in B(g, C) \) and a subset
\[
Y \subset \{ h_x : x \in Z \} \subset \mathcal{L}(g', n, \Delta')
\]
such that the following holds
\[
(19) \quad \| (Y \cap G_{i,n}) \| > \kappa(\theta \cdot \| B(1, C) \|) \cdot \exp(n\delta_G) \geq \kappa' \cdot \exp(n\delta_G).
\]

For each \( h_x \in Y \) the lexi-geodesic \( \omega_{h_x} \) contains an \((\epsilon, R)\)-transition point \( w \) satisfying \( d(w, g') < 2C \). Note that \( R' = 2C > R \). Hence, \( Y \subset \mathcal{L}_{c,R'}(g', n, \Delta') \) and thus \([\ref{19}]\) yields
\[
\| (\mathcal{L}_{c,R'}(g', n, \Delta') \cap G_{i,n}) \| > \kappa' \cdot \exp(n\delta_G).
\]
This proves the claim. \( \square \)

Note that \( G_{i,n} \subset G_{j,n} \) for \( j \leq i \). Then by the Claim above we have
\[
\| (\mathcal{L}_{c,R}(g, n, \Delta') \cap G_{i,n}) \| > \kappa' \cdot \exp(n\delta_G)
\]
for any \( j \leq i \). By definition, \( g' \in G_{i,n} \) and thus \( g' \in G_{i+1,n} \) by the Claim. Therefore, \( B(g, C) \cap G_{i+1,n} \neq \emptyset \). \( \square \)

Lemma 5.12. There exist \( \epsilon, R, \Delta, \kappa, n_0 > 0 \) with the following property.

For any \( n > n_0 \) there exists a subset \( 1 \in \mathcal{G} \) in \( G \) with the following property
\[
\| (\mathcal{L}_{c,R}(g, n, \Delta) \cap \mathcal{G}) \| > \kappa \cdot \exp(n\delta_G)
\]
for any \( g \in \mathcal{G} \).

Proof. Denote \( R = R', \Delta = \Delta', \kappa = \kappa' \), where \( R', \Delta', \kappa' \) are calculated as above.
Define \( \tilde{G} = \bigcap_{l \geq 0} G_{i,n} \). By Lemma \([\ref{5.11}]\) it suffices to show that \( 1 \in \tilde{G} \). By the special role of 1 which acts as the basepoint of partial lexi-cone, it is readily seen that the same reasoning as in Lemma \([\ref{5.11}]\) applies and shows that \( 1 \in G_{i,n} \). \( \square \)

Theorem 5.13 (Large Trees). Let \( \epsilon, R \) be given by Lemma \([\ref{7.12}]\), satisfying also convention \([\ref{2.14}]\).

For any \( 0 < \sigma < \delta_G \), there exist \( r > 0 \) and a geodesic tree \( T \in G(S,G) \) with the following properties.

1. Let \( \gamma \) be a geodesic in \( T \) and \( x \in \gamma \). Then there exists an \((\epsilon, R)\)-transition point \( v \in \gamma \) such that \( d(x, v) < r \).
2. Let \( \delta_T \) be the growth rate of \( T \) in \( G \). Then \( \delta_G > \delta_T > \sigma \).
By Lemma 2.20, there exist \( D \) geodesic. By definition of the partial cone \( \Phi \)

Proof. \( \delta \) which yields the following holds

\[
\text{Finiteness of Partial Cone Types.} \quad \text{This subsection is complementary to our discussion. Recall that finiteness of cone types is established by Cannon in 5.5.}
\]

Lemma 5.14 (Finiteness of Partial Cone Types). There exist \( \epsilon, R > 0 \) such that for any \( R > R_0 \), there are at most \( N = N(\epsilon, R) \) types of all \( (\epsilon, R) \)-partial cones \( \{ \Phi_{\epsilon, R}(g) : \forall g \in G \} \).

Proof. Let \( \epsilon, R_0 > 0 \) be given by Lemma 5.14. For any \( R > R_0 \), let \( \Phi_{\epsilon, R}(g) \) be a partial cone and \( h \) an element in \( \Phi_{\epsilon, R}(g) \). We first make the following claim.

Claim. Assume that \( d(h, g) \geq 2R + 1 \). Then there exists a constant \( \kappa > 0 \) such that the following holds

\[
\rho_{go}(1, h) > \kappa.
\]

Proof. By definition of the partial cone \( \Phi_{\epsilon, R}(g) \) in which lies \( h \), there exists some geodesic \( \gamma = [1, h] \) such that \( \gamma \) contains an \( (\epsilon, R) \)-transition point \( v \) in \( B(g, 2R) \). As \( h \in \Phi(g) \), there exists a geodesic \( \alpha = [o, ho] \) (possibly \( \alpha \neq \gamma \)) such that \( go \in \alpha \). By Lemma 5.18 there exist \( D = D(\epsilon, R) \) and an \( (\epsilon, R) \)-transition point \( w \) in \( \alpha \) such that \( d(w, v) < D \). Hence \( d(go, w) < D + 2R \).

Since \( w \) is an \( (\epsilon, R) \)-transition point in \( \alpha \), there exists \( \kappa = \kappa(\epsilon, R) \) given by Lemma 5.18 such that \( \rho_w(o, ho) > \kappa \). Note that \( d(w, go) < D + 2R \). By the property \( \{1\} \), we can assume that \( \rho_{go}(o, ho) > \kappa \), up to a decrease of \( \kappa \) by a constant depending on \( D + 2R \).

Define \( C = \varphi^{-1}(\kappa) \), where \( \varphi \) is given by Lemma 5.14. Take \( F_g \subset B(g, 2C + 1) \) such that \( d(1, gf) \leq d(1, g) \) for \( f \in F_g \). Then we have the following.

Claim. The sets \( F_g \) and \( B(g, 2R) \cap \Phi(g) \) together determine the type of the partial cone \( \Phi_{\epsilon, R}(g) \).
Proof of Claim. Let \( g,g' \in G \) such that \( F_g = F_{g'} \) and \( B(g,2R) \cap \Phi(g) = B(g',2R) \cap \Phi(g') \). It suffices to show that for any \( h \in \Phi_{e,R}(g) \) with \( d(g,h) \geq 2R + 1 \), we have \( h \in \Phi_{e,R}(g') \). We proceed by induction.

Let \( w = g^{-1}h \). By induction, assume that \( gw \in \Phi_{e,R}(g), g'w \in \Phi_{e,R}(g') \) were proven. Let \( s \in S \) be a generator such that \( (21) \)

\[
d(1, gw_s) = d(1, g) + d(1, w) + 1.
\]

We shall show that \( g'ws \in \Phi_{e,R}(g'o) \). Suppose not. Then we have \( (22) \)

\[
d(1, g'ws) < d(1, g') + d(1, w) + 1.
\]

Consider a geodesic \( \gamma \) between 1 and \( g'ws \). The previous Claim gives \( d(g'o, \gamma) \leq C \).

Thus there exists \( u \in \gamma \) such that \( d(g'o, u) < 2C + 1 \) and \( d(o, u) = d(o, g'o) \). By \( (22) \), we see that \( d(u, \gamma) \leq d(1, w) \).

Thus \( f := g^{-1}u \in F \). Since \( f \text{Lab}([u, \gamma]) = ws \), we obtain \( gws = g \text{Lab}([u, \gamma]) \).

Note that \( d(1, gf) \leq d(1, g) \). Hence we have

\[
d(1, g \text{Lab}([u, \gamma])) \leq d(1, gf) + d(u, \gamma) \leq d(1, g) + d(1, w).
\]

This contradicts with \( (21) \). Hence \( g'ws \in \Phi_{e,R}(g') \).

The finiteness of partial cones types thus follows from the second claim.

6. Lift paths in nerve graphs

Recall that \( G, X, U, \hat{U}, \mathcal{U} \) are as in Definition 1.3.

Fix a finite generating set \( S \) for \( G \). The purpose of this section is to make precise a relation between the graph \( \mathcal{G}(G, S) \) and the structure of orbits of \( G \sim X \).

In [5], Bowditch introduced a nerve graph over a collection of horoballs. We specify his construction in a way such that the nerve graph displays explicit information from \( \mathcal{G}(G, S) \).

6.1. Nerve graph. Fix a basepoint \( o \in X - \mathcal{U} \), and choose a projection point \( o_U \) of \( o \) to each \( U \in \hat{U} \). The nerve graph \( \mathcal{G} \) over \( G \) is constructed as follows.

Let \( V(\mathcal{G}) = Go \cup \hat{U} \) be the union of two types of vertices: orbit vertices \( Go \) and horoball vertices \( \hat{U} \). Connect \( o \) to \( o_U \) by edge \( (o, o_U) \) for each \( U \in \hat{U} \), and \( o \) to \( so \) by edge \( (o, so) \) for each \( s \in S \). This incident relation extends over \( V(\mathcal{G}) \) in a \( G \)-equivariant manner. There are no edges between two horoball vertices. This defines the nerve graph \( \mathcal{G} \). It is obvious that \( \mathcal{G} \) is connected, on which \( G \) acts cocompactly but not properly in general.

We shall consider two natural length metrics on \( \mathcal{G} \). Assigning each edge length 1 induces a combinatorial metric \( d \) on \( \mathcal{G} \). Then the nerve graph is the same as the cone-off Cayley graph \( \mathcal{G}(G, S \cup P) \) in the sense of Farb [19]. With respect to \( d \) the graph \( \mathcal{G} \) is a geodesic metric space, where geodesics are usually called relative geodesics.

By construction, we can set the length of (translated) edges \( (o, o_U) \) (resp. \( (o, so) \)) to be \( d(o, o_U) \) (resp. \( d(o, so) \)). This gives the other length metric \( d_{\mathcal{G}} \) on \( \mathcal{G} \).

Hence \( (\mathcal{G}, d) \) is \( (M-)b\)i-Lipschitz to \( (\mathcal{G}, d_{\mathcal{G}}) \), where \( M \) is defined as follows

\[
M = \max\{d(o, o_U), d(o, so) : U \in \hat{U}; s \in S\}.
\]

Consider a pseudo-metric space \( (X, \tilde{d}) \), the space \( X \) endowed with the maximal pseudometric \( \tilde{d} \) such that \( \tilde{d}(\cdot, \cdot) \leq d(\cdot, \cdot) \) and \( \tilde{d}(\cdot, \cdot) \) vanishes on every \( U \in \hat{U} \). The proof of the following lemma is elementary.
Lemma 6.1. The map

\[ \iota : (\mathcal{G}, \hat{d}) \to (\hat{X}, \hat{d}), v \mapsto \iota(v) = v \]

is a bi-Lipschitz map between \((\mathcal{G}, \hat{d})\) and \((\hat{X}, \hat{d})\).

Note that \((\mathcal{G}, \hat{d})\) is identical to \(\hat{\mathcal{G}}(G, S \cup P)\). The following result is due to Hruska, cf. [30, Proposition 8.13], plus the fact that \(G \simeq X\) is proper.

Lemma 6.2. There exists a constant \(D > 0\) such that the following holds. Assume that \(p\) is a relative geodesic in \((\mathcal{G}, \hat{d})\) and \(q = [1, g]\) a geodesic in \(\hat{\mathcal{G}}(G, S)\). For any orbit vertex \(ho \in p\), there exists \(f \in q\) such that \(d(ho, \hat{f}o) < D\).

6.2. Lift paths. Let \(p\) be a path in \(\mathcal{G}\). We shall construct a lift path \(\hat{p}\) in \(X\). Fix a (choice of) geodesic \([o, o_U]\) for each \(U \in \hat{U}\), and a geodesic \([o, so]\) for each \(s \in S\).

Note that if \(p\) contains no horoball vertices, then \(p\) can be naturally seen as a concatenation path in \(X\), which consists of (translated) geodesic segments \([o, so]\) for edges labeled by \(s\) in \(p\).

In order to obtain the lift path in general case, it suffices to modify pairs of edges adjacent to a horoball vertex.

Definition 6.3 (Lift path in \(\mathcal{G}\)). Let \(p\) be a path in \(\mathcal{G}\) with at least one orbit vertex. The lift of \(p\) is a path \(\hat{p}\) in \(X\) obtained as follows.

Let \(U \in \hat{U}\) be a horoball vertex in \(p\). Denote by \(U_-, U_+\), if exist, the previous and next vertices respectively of \(U\) in \(p\). Choose projection points \(u_-, u_+\), if exist, of \(U_-, U_+\) on the horoball \(U\) respectively.

If \(U = p_-(\text{resp. } U = p_+),\) we replace the edge \((U, U)(\text{resp. } (U_-, U))\) of \(p\) by a geodesic \([u_-, U_+](\text{resp. } [U_-, u_+])\) in \(X\).

Assume that \(U \neq p_-, p_+\). Then \(U_-, U_+ \in \hat{G}o\). We replace the subpath \([U_-, U_+]\) of \(p\) by the concatenated path

\[ [U_-, u_-][u_-, u_+][u_+, U_+] \]

in \(X\).

We do the above procedure for every horoball vertex \(U\) to get the lift path \(\hat{p}\).

Remark. With notions in definition 6.3 the lift path \(\hat{p}\) can be divided as

\[ \hat{p} = \hat{q}_1 \cdot [(u_1)-, (u_1)_+] \cdot \hat{q}_2 \cdot \ldots \cdot \hat{q}_i \cdot [(u_i)-, (u_i)_+] \cdot \hat{q}_{i+1} \ldots, \]

where \(q_i\) are (consecutive) maximal subsegments of \(p\) with horoball vertices at endpoints, and \(U\) are the common horoball vertex with \(U = (q_i)_+ = (q_{i+1})-\).

The main result in this subsection is the following.

Lemma 6.4. The lift of a quasi-geodesic in \((\mathcal{G}, d_{\mathcal{G}})\) is a quasi-geodesic in \((X, d)\).

Proof. For simplicity of estimates, we shall show that the lift of a geodesic is quasi-geodesic. The general case is completely analogous.

We first prove the bounded projection of certain segments of \(p\) to each horoball vertex \(U\).

Claim. There exists a uniform constant \(D > 0\) with the following property. Let \(q\) be any maximal segment of \(p\) containing horoball vertices only at endpoints, and \(\hat{q}\) the lift path of \(p\). Then

\[ \|Pr_U(\hat{q})\| < D, \]

for any horoball vertex \(U\) of \(p\).
Proof. By Lemma 6.1 \( \iota : (\mathcal{G}, d_\mathcal{G}) \to (\hat{X}, \hat{d}) \) is an \((L, 0)\)-quasi-isometry for \( L > 0 \). Let \( q' \) be the maximal segment of \( q \) without horoball vertices. Thus \( q' \) is a \((L, 0)\) quasi-geodesic in \( X \).

Let \( V_-, V_+ \) be the corresponding horoball vertices adjacent to \( q'_-, q'_+ \) respectively. Choose projection points \( u, v \) of \( q'_-, q'_+ \) to \( V_-, V_+ \) respectively. So we have

\[
d(u, q'_-) \cdot d(q'_+, v) < M,
\]

where \( M \) is defined in (23). Then the lift path \( \hat{q} = [u, q'_-] \cdot q'_- \cdot [q'_+, v] \) is a \((L, 2M)\)-quasi-geodesic.

It is well-known that quasi-convex subsets in a hyperbolic space is contracting for quasi-geodesics. Let \( \tau, D_0 > 0 \) such that any \((L, 2M)\)-quasi-geodesic outside \( N_\tau(U) \) has the projection to \( U \) of diameter at most \( D_0 \).

By Lemma 2.3, we increase \( D_0 \) such that a geodesic segment of length \( M \) has the projection to \( U \) of diameter at most \( D_0 \).

Let \( z, w \) be the entry and exit points, if exist, of \( q' \) in \( N_\tau(U) \). Then

\[
\hat{d}(z, U), \hat{d}(w, U) < \tau
\]

and thus

\[
d_\mathcal{G}(z, U), d_\mathcal{G}(w, U) < L\tau.
\]

Hence, there are at most 2\( L\tau \) vertices in \( [z, w]_p \). Projecting \( [z, w]_p \) to \( U \), we see that

\[
\|\text{Pr}_U(\hat{q})\| < \|\text{Pr}_U([u, z]_\hat{q})\| + \|\text{Pr}_U([z, w]_p)\| + \|\text{Pr}_U([w, v]_\hat{q})\| < 2D_0 + 2D_0 L\tau.
\]

This completes the proof. \( \square \)

We sketch an argument of Bowditch in [5, Lemma 7.1] to show that \( \hat{p} \) is a quadratic path in \( X \).

Let \( \hat{p} \) be decomposed as in Remark 6.2 and connect endpoints of \( \hat{p} \) by a geodesic \( \gamma \). For each horoball vertex \( U \) of \( p \), we project to \( U \) the other horoball vertices of \( p \), paths \( \gamma \) and \( \hat{q}_i \). Note that \( d_\mathcal{G}(\hat{p}_-, \hat{p}_+) \leq L \cdot d(\hat{p}_-, \hat{p}_+) \). By the Claim above, one obtains that \( d(u_-, u_+) \) is bounded by a linear function of \( d(\hat{p}_-, \hat{p}_+) \). Consequently, \( \ell(\hat{p}) \) is bounded by a quadratic function of \( d(\hat{p}_-, \hat{p}_+) \). By the exponential divergence of geodesics in hyperbolic spaces, \( \ell(\hat{p}) \) is in fact linearly bounded. This proves the lemma. \( \square \)

6.3. Transitions points revisited. In this subsection, we discuss the notion of transition points in \( X \) with respect to the horoball system \( \mathcal{U} \).

Lemma 6.5. There exist \( \epsilon, D > 0 \) such that the following holds.

For any \( g \in G \), consider a geodesic \( \gamma = [o, go] \) in \( X \) and a relative geodesic \( p = [o, go] \) in \( (\mathcal{G}, d) \). Let \( x \in \gamma \) such that \( x \notin N_\tau(U) \) for any horoball vertex \( U \) of \( p \). Then there exists an orbit vertex \( y \in p \) such that \( d(x, y) < \epsilon \).

Proof. By Lemma 6.4, the lift path \( \hat{p} \) is a \((\lambda, c)\)-quasi-geodesic in \( X \) for some \( \lambda, c > 0 \). Recall that for a horoball vertex \( U \) of \( p \), we denote by \( u_-, u_+ \) projection points to \( U \) of previous and next orbit vertices in \( p \) of \( U \). We first claim that \( \gamma \) travels closely every horoball vertices \( U \) of \( p \) with sufficiently large \( d(u_-, u_+) \).
Claim. There are constants $D_0, R > 0$ with the following property.

Let $\{U_1, U_2, \ldots, U_n\}$ be the set of horoball vertices $U$ of $p$ such that $d(u_-, u_+) > D_0$. Then there exists a sequence of consecutive pairs $(z_i, w_i)$ on $\gamma$ (i.e. $w_i \in [z_i, \gamma_+]$, $z_{i+1} \in [w_i, \gamma_+]$) such that

$$d(z_i, U_i), d(w_i, U_i) < R.$$ 

**Sketch of proof of claim.** We use a result in [48]. Let $R = R(\lambda, c), D_0 = D(\lambda, c)$ be given by Proposition 2.16 in [48]. Note that $p$ can be divided at horoball vertices $U$ with $d(u_-, u_+) > D$ such that $p$ is a $(D_0, \lambda, c)$-admissible path. Then the claim is just the conclusion of Proposition 2.16 in [48].

Set $\epsilon = R + 8\delta$. Let $x \in \gamma$ such that $x \notin N_\epsilon(U)$ for any horoball vertex $U$ of $p$.

By the choice of $\epsilon$ and the Claim above, $x$ lies in $[w_i, z_{i+1}]$, for some $1 \leq i \leq n$. Let $w'_i, z'_{i+1}$ be projection points of $w_i, z_{i+1}$ to $U_i, U_{i+1}$ respectively. Thus we have

$$d(w'_i, w'_i), d(z'_{i+1}, z'_{i+1}) < R.$$ 

Let $q = [U_i, U_{i+1}]_p$ be the subsegment of $p$ between $U_i$ and $U_{i+1}$. Consider the geodesic pentagon formed by

$$[w_i, w'_i] : [w'_i, (\hat{q})_-] : [(\hat{q})_-, \gamma_+] : [(\gamma_+, \gamma_+)], [z'_{i+1}, z_{i+1}] : [z_{i+1}, w_i],$$

which is $6\delta$-thin. Since every horoball is $2\delta$-quasi-convex, we have

$$[w'_i, (\hat{q})_-] \subset N_{2\delta}(U_i), [(\hat{q})_+, z'_{i+1}] \subset N_{2\delta}(U_{i+1}).$$

As $x \notin N_\epsilon(U_i \cup U_{i+1})$, there exists $y \in [(\hat{q})_-, (\hat{q})_+]$ such that $d(x, z) < 6\delta$.

As $\hat{q}$ is a $(\lambda, c)$-quasi-geodesic, there exists $D_1 = D(\lambda, c) > 0$ such that $[(\hat{q})_-, (\hat{q})_+] \subset N_{D_1}(q)$. By the Claim above, $d(u_-, u_+) < D_0$ for every horoball vertex $U$ in $q$. Then there exists an orbit vertex $y$ in $p$ such that $d(y, z) < D_0 + D_1 + M$, where $M$ is defined in (23). Hence, $d(x, y) < D_0 + D_1 + M + 6\delta$, proving the lemma.

**Convention 6.6 (about $\epsilon, R$).** When talking about $(\epsilon, R)$-transition points in $X$ we always assume that $\epsilon > \epsilon_0$, where $\epsilon_0$ are given by Lemmas 2.8, 2.9, and 6.5. In addition, assume that $R > R(\epsilon)$, where $R(\epsilon)$ is given by Lemma 2.7.

The following lemma says that the image of a geodesic in $\mathcal{G}(G, S)$ follows closely transition points of any geodesic with same endpoints in $X$.

**Lemma 6.7.** Let $\epsilon > 0$ be in convention (6.6). For any $R > 0$, there exists $D = D(\epsilon, R)$ with the following property.

For any $g \in G$, consider a geodesic $\gamma = [o, go]$ in $X$ and a word geodesic $\alpha = [1, g]$ in $\mathcal{G}(G, S)$. Then for any $(\epsilon, R)$-transition point $x \in \gamma$, there exists a vertex $h \in \alpha$ such that $d(ho, x) < D$.

**Proof.** As $x$ is not $(\epsilon, R)$-deep in any horoball $U \in U$, there exists a point $y \in \gamma$ such that $d(x, y) \leq R$ and $y \notin N_\epsilon(U)$ for any horoball vertex $U$ of $p$.

Let $D_1, D_2 > 0$ given by Lemmas 6.2 and 6.5 respectively. Thus there exists $h \in \alpha$ such that $d(ho, y) < D_1 + D_2$. This proves the lemma.

The main result is the following analogue of Lemma 2.20.

**Lemma 6.8.** Let $\epsilon, R > 0$ be in convention (6.6). For any $r > 0$, there exist $D = D(\epsilon, R), L = L(\epsilon, R, r) > 0$ with the following property.

Let $\alpha, \gamma$ be two geodesics in $X$ such that $\alpha_- = \gamma_-, d(\alpha_+, \gamma_+) < r$. Take an $(\epsilon, R)$-transition point $v$ in $\alpha$ such that $d(v, \alpha_-) > L$. Then there exists an $(\epsilon, R)$-transition point $w$ in $\gamma$ such that $d(v, w) < D$. 
Remark. Since $G$ acts cocompactly on $X \setminus U$, we can always find $h \in G$ such that $d(ho, x) < D$. The strengthening of this Lemma lies in the statement that $h$ can be chosen to lie on the geodesic $[1, g]$ in $\mathcal{G}(G, S)$.

Proof. Let $L = R_0 + L_0 + r + 3\delta$, where $L_0 = L(R, \delta + \epsilon)$ is given by Lemma 2.8. Without loss of generality, assume that $d(v, \alpha_+) > L_0 + R_0$, as $L_0$ does not depend on $r$.

Let $w \in \gamma$ be a congruent point of $v$. Since $d(v, \alpha_+) > r$, we have $d(v, w) < \delta$ by Lemma 2.1. Assume that $w$ is $(\epsilon, R)$-deep in a horoball $U \in \mathcal{U}$. Otherwise, $w$ is an $(\epsilon, R)$-transition point and it is done.

We prove the following two claims as in proof of Lemma 2.20. Let $x, y$ be the entry and exit points respectively of $\gamma$ in $N_\epsilon(U)$.

Claim. $\max\{d(\gamma_-, x), d(\gamma_+, y)\} > R_0$.

Proof. Assume that $d(\gamma_-, x), d(\gamma_+, y) \leq R_0$. Choose $y'' \in \gamma$ such that $d(y'', \gamma_+) = r + R_0$. Let $x', y' \in \alpha$ be congruent points of $x, y$ respectively. Then $d(x', U), d(y', U) < \delta + \epsilon$. As $d(v, \alpha_-) > L_0 + R_0$, we have $d(v, x') > L_0$. And $d(v, \alpha_+) > L_0 + R_0 + r$ implies $d(v, y') > L_0$. By Lemma 2.8 $v$ is $(\epsilon, R)$-deep in $U$. This is a contradiction.

Claim. $\min\{d(w, x), d(w, y)\} \leq L_0 + 2\delta$.

Proof. Assume that $d(w, x), d(w, y) > L_0 + 2\delta$. If $d(y, \gamma_+) < r$, choose $y'' \in \gamma$ such that $d(y'', \gamma_+) = r$. As $L > L_0 + r + \delta$, we have $d(v, y'') > L_0 + \delta$. Let $x', y' \in \alpha$ be congruent points of $x, y''$ respectively. Thus $d(x', U), d(y', U) < \delta + \epsilon$. As $d(v, w) < \delta$, we have $d(x', v), d(y', v) > L_0$. By Lemma 2.8 $v$ is $(\epsilon, R)$-deep in $U$. This is a contradiction.

Note that $d(v, \alpha_-), d(v, \alpha_+) > L_0 + R_0 + 3\delta$. The proof completes exactly as in Lemma 2.20.

7. Patterson-Sullivan measures on Bowditch boundary: II

Recall that $(G, \mathcal{P})$ is relatively hyperbolic, and $X, U, U, \bar{U}, \mathcal{P}$ are as in Definition 1.3. Analogous to the development of PS-measures on $\partial G$ constructed through $G \acts\mathcal{G}(G, S)$ in Section 4, this section is devoted to the study of PS-measures obtained via the cusp-uniform $G \acts X$.

7.1. PS-measures have no atoms. Apply Patterson’s construction for the hyperbolic space $X$ on which $G$ acts as a cusp-uniform action. Let $\mu$ be a so-constructed PS-measure on $\partial X$. In [10], Coornaert proved the quasi-conformal density of $\mu$.

Theorem 7.1. Let $\mu$ be a PS-measure on $\partial X$ constructed through $G \acts X$. Then $\mu$ is a $\delta_G$-dimensional quasi-conformal density.

In the sequel we shall show that PS-measures have no atoms, under the assumption that $G$ is a divergent group or $G$ has parabolic gap property. The proof below follows closely an argument of Dal’bo-Otal-Peigné in [15 Propositions 1 & 2].

Lemma 7.2. Assume that $G \acts X$ is divergent or has the parabolic gap property. Then $\mu$ has no atoms at bounded parabolic points.
Proof. Let \( q \in \partial X \) be a bounded parabolic point, with the stabilizer \( P \) of \( q \). Choose a compact set \( K \) in \( X \cup \partial X \) in the same way as in the proof of Proposition 4.10 such that \( q \notin K \) and the boundary of \( K \) is \( \mu_r \)-null for some (hence any) \( v \in Go \).

Let \( V_n = \cup_{d(o,po) > n} pK \). By the Remark after Lemma 2.14 one sees that \( \| Pr_{\mu_v}(K) \| \) is uniformly bounded. Choose \( \theta' \in Pr_{\mu_v}(K) \). Without loss of generality, we assume that \( \theta' = o \). Then there is a constant \( D > 0 \) such that the following holds
\[
(24) \quad d(z, o) + D > d(z, po) + d(o, po) > d(o, z),
\]
for any \( z \in pK o \).

In the case that \( G \) is divergent, the proof of Lemma 4.10 goes without changes in the current setting. So below, we assume that \( G \) has parabolic gap property.

Let \( \epsilon = (\delta_G - \delta_P)/2 \). Then there exists \( \epsilon > 0 \) such that (10) holds for a monotonically increasing function \( H \). Deleting finitely many points, assume that \( d(z, o) > \epsilon \) for each \( z \in K \). Thus
\[
\mu^o_n(V_n) \leq \sum_{d(o,po)>n} \mu^o_n(pK) \leq \sum_{d(o,po)>n} \mu^o_n(pK) \leq \sum_{d(o,po)>n} H(d(o,go)) \exp(-sd(o,go)) \text{Dirac}(go)
\]
\[
\leq \sum_{d(o,po)>n} H(d(po,go)) \exp(\epsilon d(o,po)) \exp(-sd(po,go) - sd(o,po)) \text{Dirac}(go)
\]
Hence, up to a translation, we obtain
\[
\mu^o_n(V_n) \leq \sum_{d(o,po)>n} \exp(-s \epsilon d(o,po)) \sum_{d(o,po)>n} H(d(o,go)) \exp(-sd(o,go)) \text{Dirac}(go)
\]
\[
\leq \sum_{d(o,po)>n} \exp(-s \epsilon d(o,po)) \mu^o_n(K).
\]

By Lemma 4.9 \( P_H(s,a) \) is convergent for \( s > \delta_H \). Let \( s \rightarrow \delta_G \) and then \( n \rightarrow \infty \). We obtain that \( \lim_{n \rightarrow \infty} \mu^o_n(V_n) = 0 \). The proof is complete.

7.3. Assume that \( G \) is divergent or has the parabolic gap property. Then PS-measures \( \mu \) are \( \delta_G \)-dimensional quasiconformal density without atoms. Moreover, \( \mu \) is unique and ergodic.

Proof. It is well-known that under appropriate choices of parameters, Floyd metric is bi-Lipschitz to visual metric. cf. [29]. Hence the proof of Lemma 4.12 shows that \( \mu \) has no atoms at conical points. The proof is now completed by Propositions 4.10, 4.17.

7.2. Partial Shadow Lemma. We shall prove an analogue of Partial Shadow Lemma in the setting of \( G \searrow X \). The proof will be completely analogous to that of Lemma 5.3, with the ingredients proven below. We first recall two results of Coornaert in [10].

Lemma 7.4 (Shadow Lemma). There exists \( r_0 > 0 \) such that the following holds,
\[
\exp(-\delta_G d(go, o)) < \mu_o(\Pi_r(go)) < \exp(-\delta_G d(go, o)),
\]
for any \( g \in G \) and \( r > r_0 \).

Lemma 7.5. There exists \( \Delta_0 > 0 \) such that the following holds
\[
\| A(n, \Delta) \|_\Delta \exp(n \delta_G),
\]
for any \( n \geq 0 \) and \( \Delta > \Delta_0 \).

The following observation is straightforward from Lemma 4.9.
Lemma 7.6. For any $\epsilon, r > 0$, there exists $R = R(\epsilon, r) > 0$ such that the following holds
\[
d(z, hw) > R \sum_{h \in G_U} \exp(-\delta_G d(z, hw)) < \epsilon,
\]
for any $U \in \U$ and any $z, w \in N_r(\partial U)$.

Before stating Partial Shadow Lemma, we want to emphasize the flexibility of a contracting system in definition of partial cones. We shall consider another relative hyperbolic structure of $G$, which is obtained by adjoining into $P$ a collection of subgroups $\mathcal{E}$. This can happen in the following way. Let $h \in G$ be a hyperbolic element. Denote by $E(h)$ the stabilizer in $G$ of the fixed points of $h$ in $\partial X$. Then $E = \{gE(h) : g \in G\}$ gives such an example. See [35] for more detail.

Let $C(E)$ denote the convex hull in $X$ of $\Lambda(E)$ for each $E \in \mathcal{E}$. Then $U \cup \mathcal{E}$ is a contracting system with bounded intersection. Observe that a transition point relative to $U \cup \mathcal{E}$ is also a transition point relative to $U$.

In what follows (in particular, in proof of Theorem 4.3), we are interested in transition points relative to the enlarged $U \cup \mathcal{E}$. Below the partial shadow $\Pi_{r, \epsilon, R}$ is defined using transition points relative to $U \cup \mathcal{E}$. Analogous to Lemma 5.3, the following version of Partial Shadow Lemma holds in this setting.

**Lemma 7.7 (Partial Shadow Lemma).** There are constants $r, \epsilon, R \geq 0$ such that the following holds
\[
\exp(-\delta_G d(o, go)) < \mu_o(\Pi_{r, \epsilon, R}(go)) \leq \mu_o(\Pi_{r, \epsilon, R}(go)),
\]
for any $g \in G$.

**Proof.** The proof Lemma 5.3 works without changes in this setting, with Lemma 5.2 replaced by Lemma 7.6. □

### 7.3. Growth of the orbit in partial cones

This section is aiming to prove the exponential growth of partial cones.

**Lemma 7.8.** Suppose that $G$ has parabolic gap property. Then there are constants $r, \epsilon, R, \Delta, \kappa > 0$ such that the following holds
\[
\# \Phi_{r, \epsilon, R}(go, n, \Delta) \geq \kappa \cdot \exp(n\delta_G)
\]
for any $g \in G, n \geq 0$.

Applying Lemma 7.8 to $g = 1$, we obtain the following, improving [16, Lemma 4.3(i)].

**Corollary 7.9.** Suppose that $G$ has parabolic gap property. There exist $\Delta, \kappa > 0$ such that the following holds
\[
\# A(n, \Delta) > \kappa \cdot \exp(n\delta_G),
\]
for any $n \geq 0$.

The rest of this subsection is devoted to the proof of Lemma 7.8. Let’s first prepare some preliminary results. Recall that $o_U \in \partial U$ is a projection point of $o$ to $U$.

**Lemma 7.10.** There exists a constant $M > 0$ with the following property.

For each $U \in \U$, there exists $t_U \in G$ such that $d(t_U o, o_U) \leq M$ and $\partial U \subset N_M((G_U t_U) \cdot o)$.
In what follows, let $t_U \in G$ given by Lemma 7.10 and $M > 0$ be a constant bigger than the diameter of $(X \setminus U)/G$, while satisfying Lemmas 2.3, 7.10.

Before stating the next lemma, we need the following notation.

**Definition 7.11.** Given $r, n \geq 0, g \in G$, the set $U_{r,go,n}$ denotes the collection of horoballs $U \in \mathcal{U}$, where $U$ contains a point $z \in [0, \xi]$ for some $\xi \in \Pi_r(go)$ such that $d(o, z) - d(o, go) = n$.

Recall that $\Pi_{r,\epsilon,R}(go)$ denotes the set of conical limit points in $\Pi_{r,\epsilon,R}(go)$. The constant $\delta$ below is the hyperbolicity constant of $X$. The key step in proving Lemma 7.11 is the following.

**Lemma 7.12.** There exists $\epsilon, r_0 > 0$ such that the following holds:

For any $R, L > 0$, there exist $R' = R(\epsilon, R), \Delta = \Delta(L) > 0$ such that the following holds:

\begin{equation}
\Pi_{r,\epsilon,R}(go) \subset \bigcup_{h \in \Psi_{r,\epsilon,R}(go,n,\Delta)} \Pi_r(ho) \cup \bigcup_{\gamma \in \mathcal{U}_{r,\epsilon,R}(go,n)} \Pi_r(ht_U o),
\end{equation}

for any $n > r_0$ and any $g \in G$, where $h$ in the second union set satisfies

\begin{equation}
h \in \Psi_r(go), \ d(o, ht_U o) - d(o, go) > n + L.
\end{equation}

*Proof.* Fix $L, R > 0$. Set $r_0 = M + \delta$ and $\Delta = L + 2M$. Let $L_1 = L(\epsilon, R, M), D = D(\epsilon, R)$ be given by Lemma 6.8.

For any $\xi \in \Pi_{r,\epsilon,R}(go)$, there exists a geodesic $\gamma = [o, \xi]$ such that $\gamma$ contains an $(\epsilon, R)$-transition point $v$ in $B(go, 2R)$. Take $z \in \gamma$ such that

\begin{equation}
d(o, z) - d(o, go) = n > L_1 + 2R.
\end{equation}

We have two cases to consider.

**Case 1.** There exists $h \in G$ such that $d(ho, z) < M$. By Lemma 2.1, we have

\begin{equation}
\xi \in \Pi_{M+\delta}(ho) = \Pi_r(ho).
\end{equation}

Let $n > M$. Applying Lemma 2.1 gives

\begin{equation}
d(go, [o, ho]) < r + \delta.
\end{equation}

Note that $d(z, v) > L_1$ by (28). By Lemma 6.8 there exists an $(\epsilon, R)$-transition point $w$ in $[o, ho]$ such that $d(v, w) < D$. Then $d(w, go) < 2R + D$. Since $w$ is also $(\epsilon, R)$-transition point, we have

\begin{equation}
h \in \Psi_{r,\epsilon,R}(go, n, \Delta).
\end{equation}

Denote $R' = 2R + D$. Hence by (29) and (30), we see that $\xi$ is in the first term of right-hand set in (26).

**Case 2.** The ball $B(z, M)$ contains no point in $Go$. By the choice of $M$, we see that $z \in U$ for some $U \in \mathcal{U}_{go,n}$.

As $\xi$ is a conical point, any geodesic $[o, \xi]$ leaves every horoball which it enters. Let $x \in \partial U$ be the exit point of $[o, \xi]$ in $U$. Since $G_U$ acts cocompactly on $\partial U$, by Lemma 7.10 there exist $h \in G_U, t_U \in G$ such that $d(ht_U o, x) < M$. So

\begin{equation}
\xi \in \Pi_r(ht_U o).
\end{equation}

To finish the proof, we continue to examine two subcases of Case 2 as follows.

**Subcase 1:** assume that $d(x, z) \leq L + M$. Then $d(z, ht_U o) \leq 2M + L \leq \Delta$. Let $n > 2M + L$. By Lemma 2.1, $d(go, [o, ht_U o]) < r + \delta$. Arguing similarly as in the
Case 1, we see that there exists an \((\epsilon, R)\)-transition point \(w\) in \([o, ht_Uo]\) such that 
\[d(go, w) < 2R + D.\]  This implies that

(32) \[ht_U \in \Psi_{r+\delta, \epsilon, 2R+D}(go, n, \Delta).\]

Hence by (31) and (32), \(\xi\) also goes in the first term of right-hand set in (26).

Subcase 2: assume that \(d(x, z) > L + M\). Then we have

\[d(x, o) > d(o, go) + n + L + M\]

and thus

\[d(ht_Uo, o) - d(o, go) > n + L.\]

By (31), this shows that \(\xi\) is in the second term of right-hand set in (26), thus completing the proof of the lemma. \(\square\)

The observation below, a strengthening version of Lemma 7.6 is also crucial in the proof of Lemma 7.8. We remark that the parabolic gap property would not be necessary in proving exponential growth of cone in groups, cf. Lemma 5.5.

**Lemma 7.13 (Exponential Decay).** Suppose that \((G, d)\) has parabolic gap property. Then there exists \(\kappa > 0\) such that the following holds

\[
\sum_{h \in G_U} \exp(-\delta_G d(o, ho))) < \exp(-\kappa n),
\]

for any \(U \in \mathcal{U}, o \in \partial U\) and \(n \geq 0\).

**Proof.** Denote by \(\delta_U\) the critical exponent of \(G_U < G\). It is assumed that \(\delta_U < \delta_G\).

For any \(i, \Delta > 0\), denote \(a(i, \Delta) = \{h \in G_U : 0 \leq d(ho, o) - i < \Delta\}\). It is clear that

\[
\limsup_{i \to \infty} i^{-1} \log a(i, \Delta) = \delta_U.
\]

Then there exist \(\epsilon, C > 0\) such that \(\delta_U + \epsilon < \delta_G\) and

\[
a(i, \Delta) < C \exp(i(\delta_U + \epsilon))
\]

for any \(i > 0\). For \(n \geq 0\), we estimate

\[
\sum_{h \in G_U} \exp(-\delta_G d(o, ho)) \leq \sum_{i=n}^{\infty} a(n + \Delta(i - n), \Delta) \exp(-\delta_G(n + \Delta(i - n))) \leq C \exp((\delta_U + \epsilon - \delta_G)(n + \Delta(i - n))) \leq C_1 \exp(-\kappa n),
\]

where \(C_1 = C/(1 - \exp(\kappa \Delta))\) and \(\kappa = \delta_G - \delta_U - \epsilon\). This completes the proof. \(\square\)

We need introduce some auxiliary sets to carry out a finer analysis of \(U_{r, go, n}\). Let \(X\) be the set of horoballs \(U \in U_{r, go, n}\) such that \(d(o, o_U) - d(o, go) < 2r\). Consider the annulus set \(U_{r, go, n}(i, \Delta)\) in \(U_{r, go, n} \setminus X\), which is the set of horoballs \(U \in U_{r, go, n} \setminus X\) such that the following holds

(34) \[2r \leq n + 3r - (i + 1)\Delta \leq d(go, o_U) < n + 3r - i\Delta\]

for \(i, \Delta \geq 0\).

**Lemma 7.14.** There exist \(r_0, \Delta > 0\) such that the following hold for \(r > r_0\).

1. \(U_{r, go, n} = X \cup (\cup_{i=0} U_{r, go, n}(i, \Delta))\),
Indeed, if not, assume $y$ for $i$ (36).

Proof. Given $U \in \mathbb{U}_{r,go,n}$, let $x \in \Pi_r(go)$ and $z \in [o, x] \cap U$ with $d(o, z) = d(o, go) + n$. Denote $\gamma = [o, x]$. Choose $x \in \gamma$ such that $d(o, x) = d(o, go)$. As $x \in \Pi_r(go)$, we have $d(go, x) < 2r$.

By Lemma 2.3, $d(o_U, [o, z]) < M$. Let $y \in [o, z]$, such that $d(y, o_U) < M$. As $U \in \mathbb{U}$ are quasi-convex, let $M$ also satisfy that $[y, z], \gamma \subset N_M(U)$. Set $r_0 = M$.

We first prove (1), (2). Let $U \in \mathbb{X}$. Then $d(go, go) > d(o, o_U) - 2r$. We see that $y \in N_{3r}([o, x])$. If not, then $y \in [x, z]_\gamma$ and $d(x, y) > 3r$. As $r > M$, it follows that $d(go, go) = d(o, x) < d(o, y) - 3r < d(o, o_U) + M - 3r$, giving a contradiction. This proves $y \in N_{3r}([o, x])$. As $[y, z], \gamma \subset N_M(U)$, we obtain that $d(go, U) < M + 3r < 4r$.

Let $U \in \mathbb{U}_{r,go,n} \setminus \mathbb{X}$. Then $d(o, o_U) - d(o, go) \geq 2r$. We claim that $y \in [x, z]_\gamma$. Indeed, if not, assume $y \in [o, x]$. Then $d(o, go) > d(o, o_U) - M - r$. Let $r < M$, we have $d(go, go) > d(o, o_U) - 2r$. This gives a contradiction. Thus, $y \in [o, z]_\gamma$ and $d(x, y) \leq n$. Then

$$d(go, o_U) \leq d(x, y) + M + 2r < n + 3r.$$

Let’s now prove (3). For each $U \in \mathbb{U}_{r,go,n}(i, \Delta_0)$, there exists $h_U \in G$ such that $d(h_U, o, o_U) < M$. As $U$ is locally finite, we have

$$\gamma \in \mathbb{U}_{r,go,n}(i, \Delta) \subset \{h_U : U \in \mathbb{U}_{r,go,n}(i, \Delta)\}.$$

Let $\Delta > M$ be given by Lemma 7.5. Thus $A(n, \Delta) \leq \exp(\delta_G n)$. Note that $n + 3r - M - (i + 1)\Delta < d(go, h_U, o) \leq n + 3r + M - i\Delta$, which yields $g^{-1}h_U \in A(n + 3r + M - (i + 1)\Delta, \Delta)$. Therefore,

$$d(go, h_U, o) = d(U, o, go) < \exp((n - i\Delta)\delta_G).$$

for $i \geq 0$.

We first handle the second term in the right-hand union set in (26).

Lemma 7.15. For any $r > M, C > 0$, there exists $L = L(r, C) > 0$ such that the following holds.

$$\gamma \sum_{U \in \mathbb{U}_{r,go,n}} \exp(-\delta_G d(o, h_U, o)) < C \cdot \exp(-\delta_G d(o, go))$$

for any $n > 0, g \in G$.

Proof. Let $h \in G_U$, $U \in \mathbb{U}_{r,go,n}$ satisfy (27). Recall that $d(o_U, t_U, o) < M$. Then $h_U \in \Psi_r(go)$ and

$$d(o, h_U, o) - d(o, go) > n + L.$$

Denote $\gamma = [o, h_U, o]$. Choose $x, z \in \gamma$ such that $d(o, x) = d(o, go)$ and $d(o, z) = d(o, go) + n$. As $h_U \in \Psi_r(go)$ we have that

$$\gamma \delta_G d(x, go) < 2r.$$

By Lemma 2.3, there exists $y \in \gamma$ such that

$$\gamma \delta_G d(o_U, y) < C.$$

Since $\mathbb{U}_{r,go,n} = \mathbb{X} \cup (\bigcup_{i \geq 0} \mathbb{U}_{r,go,n}(i, \Delta))$ by Lemma 7.4, we examine the following two cases.
Case 1. Consider $U \in X$. Then $d(go, U) < 4r$, and thus $X$ is a finite set. Arguing as in proof of Lemma 7.14 we have that $y \in N_{3r}([o, x])$. By the choice of $x$, one sees that

$$
d(hoU, go) \geq d(hoU, x) - 2r > n + L - 2r - \delta.
$$

Choose $L = R_1 + 2M + 2r + \delta$, where $R_1 = R(C/(2\#X), 4r)$ is given by Lemma 7.6. As $d(go, U) < 4r$, we obtain by Lemma 7.4 and 39,

$$
\sum_{U \in X} \exp(-\delta_G d(o, hoU)) \leq \sum_{U \in X} \exp(-\delta_G d(o, go)) \exp(-\delta_G d(go, hoU)) \\
\leq (C/2) \exp(-\delta_G d(o, go)).
$$

Case 2. Consider $U \in U_{r, go, n}(i, \Delta)$ for $i \geq 0$. For any $h$ satisfying 27, we have that

$$
d(o, hoU) > d(o, hoU) - M \\
> d(o, go) + n + d(oU, hoU) - d(oU, z) - M,
$$

which yields

$$
\exp(-\delta_G d(o, hoU)) < \exp(-\delta_G d(o, go) - n\delta_G)) \cdot \exp(-\delta_G d(oU, hoU)) \cdot \exp(\delta_G d(oU, z)).
$$

By the same argument as in Lemma 7.14 we see that $y \in [x, z]$. Note that $r > M$. Then by 37 and 38 we have

$$
n + 4r > d(go, ou) + d(oU, z) > n - 2r.
$$

Consequently by 27, the distance $d(oU, z)$ gets bounded as follows

$$
i\Delta - 5r < d(oU, z) < (i + 1)\Delta + r
$$

which yields

$$
d(oU, hoU) > d(oU, z) + d(z, hoU) - 2M \\
> d(oU, z) + L - 2M > i\Delta + L - 6r,
$$

for $i \geq 0$.

Let $\kappa > 0$ given by Lemma 7.14. Hence by 43, the following holds

$$
\sum_{U \in X} \exp(-\delta_G d(o, hoU)) < \exp(-\kappa(i\Delta + L))
$$

for each $U \in U_{r, go, n}(i, \Delta)$.

By the 35 and 42, we have

$$
\exp(\delta_G d(oU, z)) \cdot \# U_{r, go, n}(i, \Delta) < \exp(n\delta_G).
$$

for any $i$.

Take into account all $U \in U_{r, go, n}(i, \Delta)$ for a fixed $i$. By 11, 14 and 15, we obtain

$$
\sum_{U \in U_{r, go, n}(i, \Delta)} \exp(-\delta_G d(o, hoU)) < \exp(-\kappa(i\Delta + L)).
$$

Clearly by choosing $L$ large enough, one can obtain

$$
\sum_{U \in U_{r, go, n}(i, \Delta), i \geq 0} \exp(-\delta_G d(o, hoU)) \leq (C/2) \exp(-\delta_G d(o, go)).
$$
Finally, as a consequence of **Cases** (1) and (2), the conclusion follows from \((40)\) and \((47)\).

We are ready to give a proof of the main result in this subsection.

**Proof of Lemma 7.8** Let \(r,\epsilon, R\) be given by Lemma \(7.7\) also satisfying Lemma \(7.12\). Apply Shadow Lemmas \(7.7\) and \(7.4\) to the formulae \((26)\). There exists a constant \(C > 0\), and for any \(L > 0\), there exist \(R', \Delta > 0\) such that the following holds

\[
C \exp(-\delta_Gd(o,go)) \leq \sum_{h \in \Phi_{r+\delta, R'}} \exp(-\delta_Gd(o,ho)) \cdot \exp(2r) \\
+ \sum_{h \in \Phi_{r+\delta, U}, U \in U_{r,go,n}} \exp(-\delta_Gd(o,ht_Uo)).
\]

for \(g \in G\) and \(n >> 0\).

For \(r, C/2 > 0\), there exists \(L = L(r, C/2) > 0\) given by Lemma \(7.15\) such that

\[
\sum_{h \in \Phi_{r+\delta, U}, U \in U_{r,go,n}} \exp(-\delta_Gd(o,ht_Uo)) < (C/2) \cdot \exp(-\delta_Gd(o,go)).
\]

Note that \(d(o,ho) > d(o,go) + n - \Delta\) for any \(h \in \Phi_{r+\delta, R'}(go, n, \Delta)\). Hence, we have the following

\[
C/2 < \| \Phi_{r+\delta, R+D}(go, n, \Delta) \cdot \exp(-\delta_Gn) \cdot \exp(2r + \Delta),
\]

yielding

\[
\| \Phi_{r+\delta, R'}(go, n, \Delta) \times_{r, \Delta} \exp(-\delta_Gn),
\]

which completes the proof.

7.4. **Large Quasi-Trees in Hyperbolic Spaces.** This subsection is devoted to an analogue of Theorem 5.13. The difficulty of doing so is that there is no obvious way to "label" geodesics with endpoints in \(Go\) in the space \(X\). The strategy that we take is to map into \(X\) the set of lexi-geodesics between \(1\) and \(g\) in \(\mathcal{G}(G, S)\). The transition points in the partial cone will make these (images of) lexi-geodesics travel uniformly close to the base \(go\) of the cone. This allows us to adapt the methods in proving Theorem 5.13 in this setting.

Below, we always assume that \(G\) has parabolic gap property.

**Definition 7.16 (Partial Lexi-Cone).** For \(\epsilon, R \geq 0\) and \(g \in G\), the partial lexi-cone \(\mathcal{L}_{\epsilon, R}(go)\) at \(go\) is the set of elements \(h \in \mathcal{L}(g)\) such that one of the two following statements holds.

1. \(d(o, ho) \leq d(o, go) + 2R\).
2. There exists some geodesic \([o, ho]\) containing an \((\epsilon, R)\)-transition point \(v\) such that \(d(v, go) \leq 2R\).

Compare with the definition of \(\mathcal{L}_{\epsilon, R}(g)\) in \(\mathcal{G}(G, S)\). For \(n, \Delta \geq 0\), set

\[
\mathcal{L}_{\epsilon, R}(go, n, \Delta) = \mathcal{L}_{\epsilon, R}(go) \cap A(go, n, \Delta).
\]

**Proposition 7.17 (Companion Lexi-cone).** There are constants \(r, \epsilon, R, \Delta, \kappa > 0\) such that the following holds.

For any \(g \in G\), there exists \(g' \in G\) such that \(d(go, g'o) < r\) and the following holds

\[
\| \mathcal{L}_{\epsilon, R}(g'o, n, \Delta) \| \geq \kappa \cdot \exp(n\delta_G)
\]

for any \(n \geq 0\).
Proof. For any \( h \in \Phi_{r,e,R}(go) \), there exists some geodesic \([o,ho]\) containing an \((\epsilon, R)\)-transition point \( v \) in \( B(go, 2R) \). Let \( D = D(\epsilon, R) > 0 \) be given by Lemma 6.7. Then there exists \( f \in \omega_h \) such that \( d(fo, v) < D \). Hence, \( d(fo, go) < D + 2R \). Applying a pigeonhole principle gives
\[
\|L(g') \cap \Phi_{r,e,R}(go,n,\Delta)\| \geq \kappa \cdot \exp(n\delta_g),
\]
for some \( g' \in B(go, D+2R) \). Note that \( L(g') \cap \Phi_{r,e,R}(go,n,\Delta) \subset L_e, D+3R(g'o, n, \Delta) \).
The proof is complete. \( \square \)

The proof of Lemma 7.19 goes similarly as that of Lemma 5.11. We single out only the differences.

Take \( r, \epsilon, R, \Delta, \kappa > 0 \) given by Proposition 7.11. Define the auxiliary set
\[
E = \{ g \in G : \|L_{e,R}(go,n,\Delta)\| > \kappa \cdot \exp(n\delta_g), \forall n \geq 0 \}.
\]
Note that \( N_r(E_0) = Go \).

Let \( D = D(\epsilon, R) \) satisfy Lemmas 6.7, 6.8. \( L = L(C + r) > 0 \) be given by Lemma 6.8 where
\[
C := 2D + 2R.
\]

By Lemma 6.10 and the proper action of \( G \) on \( X \), there exists \( \theta = \theta(2C + 2r) > 0 \) with the following property. For any finite subset \( Y \) of \( G \), there exists a subset \( Z \) of \( Y \) such that \( Z \) is \((2C + 2r)\)-separated in \( X \) and \( \|Z \| > \theta \cdot \|Y\| \).

Let \( \kappa' = \kappa/\theta \cdot \|N(o,C)\| \) and \( R' = 2C \).

Fix \( n > 0 \) and let \( \Delta' = \Delta + 2C + r \). We define a sequence of decreasing sets \( G_{i,n} \) for \( i \geq 0 \) inductively. Let \( G_{0,n} = G \). Then set
\[
G_{i+1,n} = \{ g \in G_{i,n} : \|L_{e,R}(go,n,\Delta')\cap G_{i,n} \| > \kappa' \cdot \exp(n\delta_g) \}.
\]

Define \( n_0 = L + 2R + \Delta \).

Lemma 7.18. Let \( n > n_0 \). Then the following holds
\[
B(go,C) \cap G_{i,n, o} \neq \emptyset
\]
for each \( g \in E \) and \( i \geq 0 \).

Proof. Assume that \( B(go,C) \cap G_{j,n, o} \neq \emptyset \) for each \( j \leq i, g \in E \). Then there exists a set \( Z \subset L_{e,R}(go,n,\Delta) \) such that \( Zo \) is \((2r + 2C)\)-separated and the following holds
\[
\|Z \| > \kappa/\theta \cdot \exp(\delta_g n).
\]
Note that \( N_r(E_0) = Go \). For any \( x \in Z \), there exists \( h_x \in G_{i,n} \) such that \( d(xo, h_x o) < C + r \). Then \( h_x \neq h_y \) for any \( x \neq y \in Z \).

Claim. There exists an element \( g' \in B(go,C) \) such that
\[
\|L_{e,R}(g'o,n,\Delta')\cap G_{i,n} \| > \kappa' \cdot \exp(n\delta_g).
\]

Proof of Claim. Since \( x \in Z \subset L_{e,R}(go,n,\Delta) \), there exists a geodesic \([o, xo]\) containing an \((\epsilon, R)\)-transition point \( v \in B(go, 2R) \).

Since \( d(x, go) > n - \Delta \) and \( n > n_0 \geq L + 2R + \Delta \), we have \( d(v,xo) > L \). As \( d(xo, h_x o) < C + r \), apply Lemma 6.8 to the pair of geodesics \([o, xo], [o, h_x o]\). Then there exists an \((\epsilon, R)\)-transition point \( w \) in \([o, h_x o]\) such that \( d(v,w) < D \). By Lemma 6.7, there exists \( f \in \omega_{h_x} \) such that \( d(fo, w) < D \). Hence, \( d(fo, go) < 2D + 2R \leq C \).

By a pigeonhole principle, there exist \( g'o \in B(go,C) \) and a subset
\[
Y \subset \{ h_x : x \in Z \} \subset L(g',n,\Delta')
\]
such that the following holds
\[(Y \cap G_{i,n}) > \kappa/(\theta \cdot N(o,C)) \cdot \exp(n\delta_C) \geq \kappa' \cdot \exp(n\delta_C).\]

Recall that for each $h_x \in Y$, the geodesic $[o,h_xo]$ contains an $(\epsilon, R)$-transition point $w$ satisfying $d(w, g'w) < 2C$. Hence, $Y \in \mathcal{L}_{c,R}(g', n, \delta')$ and thus \[(50) \quad \| (\mathcal{L}_{c,R}(g', n, \delta') \cap G_{i,n}) > \kappa \cdot \exp(n\delta_C).\]

This proves the claim. \[\square\]

Note that $G_{i,n} \subset G_{j,n}$ for $j < i$. Therefore, $B(go, C) \cap G_{i+1,n}o = \emptyset$. \[\square\]

We have the analogue of Lemma \[5.11\]

Lemma 7.19. There exist $\epsilon, R, \Delta, \kappa, n_0 > 0$ such that for any $n > n_0$ there exists a subset $1 \in G$ in $G$ with the following property
\[\| (\mathcal{L}_{c,R}(g, n, \Delta)) > \kappa \cdot \exp(n\delta_C)\]

for any $g \in G$.

The following is analogous to Theorem \[5.13\]

Theorem 7.20 (Large Trees). There exist $\epsilon, R, D > 0$ such that the following holds. For any $0 < \sigma < \delta_G$, there exist $r > 0$ and a geodesic tree $T$ in $G(G, S)$ with the following properties.

1. For any vertex $g$ in $T$, the image of the geodesic $\omega_g$ has at most a $D$-Hausdorff distance to any geodesic $[o, go]$.

2. Consider a geodesic $\gamma = [o, go]$. For any $x \in \gamma$, there exists an $(\epsilon, R)$-transition point $y \in \gamma$ such that $d(x, y) < r$.

3. Let $\delta_T$ be the growth rate of the set $T_o$ in $X$. Then $\delta_G > \delta_T > \sigma$.

Proof. Let $\epsilon, R, \Delta, \kappa, n_0 > 0$ be given by Lemma \[7.19\] For any $r > n_0 + 6R + \delta + \Delta$, the geodesic tree $T = \lim_{i \to \infty} T_i$ is constructed exactly as in proof of Lemma \[5.13\] cf. \[20\]. By construction, $|d(xo, yo) - r| < \Delta$ for any $y \in T_i(x)$. Following $T$, we construct a quasi-embedded tree in $X$. Let $T'_0 = o$ and $T'_{i+1} = \cup_{x \in T_i}(\cup_{y \in T_{i+1}(x)}[xo, yo])$, where $[xo, yo]$ is a choice of a geodesic between $xo, yo$. Set $T' = \cup_{i \geq 0} T'_i$. The estimate of growth rate of $T_o$ is same as in proof of Lemma \[5.13\] For any $\tau < \delta_G$ we can find $r > 0$ such that (3) holds.

We show (1) by proving the following claim.

Claim. Assume that $r >> 0$. Then there exist $\lambda = \lambda(R), c = c(R)$ such that each path $p$ in $T'$ is a $(\lambda, c)$-quasi-geodesic.

Proof of Claim. Assume that $p = [o, xo][xo, yo][yo, zo] \cdots$, where $x \in T_0(1), y \in T_1(x), z \in T_2(y)$. As $y \in \mathcal{L}_{c,R}(xo)$, it follows that $d(xo, [o, yo]) < 2R$. Hence $[o, yo]p$ is a $(1, 4R)$-quasi-geodesic.

Let $w \in [o, zo]$ be an $(\epsilon, R)$-transition point such that $d(w, yo) < 2R$. Observe that $d(w, [xo, zo]) \leq \delta$. If not, by Lemma \[2.1\] there exists $w' \in [o, xo]$ such that $d(w, w') \leq \delta$. As $[o, yo]p$ is a $(1, 4R)$-quasi-geodesic, we have $\ell([w', yo]p) \leq d(w', yo) + 4R < 6R + \delta$. As $T_i(x) \subset \mathcal{L}_{c,R}(xo, r, \Delta)$, we have $d(xo, yo) > r - \Delta$. Assuming $r > 6R + \Delta + \delta$ would give a contradiction. Hence, $d([o, xo, zo]) \leq \delta$, which yields $d([o, xo, zo]) \leq 2R + \delta$. As a consequence, $[xo, zo]p$ is a $(1, 4R + 2\delta)$-quasi-geodesic.
Inductively, this shows that $p$ is a local quasi-geodesic. By taking $r$ large enough, we see that $p$ becomes a global quasi-geodesic. 

We now prove (2). Let $p$ be the path in $T'$ between $o$ and $g_0$. By the Claim above, there exists $D_1 = D(\lambda, c)$ such that $p \subset N_{D_1}(\gamma)$. Let $ho \in p$ for some $h \in T^0$ such that $d(x, ho) < D_1$. Assume $h \in L_{\epsilon, R}(zo, r, \Delta)$ for some $z \in T^0$. Thus some geodesic $[o, ho]$ contains an $(\epsilon, R)$-transition point $w$ in $B(z, 2R)$.

Let $D_2 = D(\epsilon, R), L = L(\epsilon, R, D_1)$ given by Lemma 6.8. Assume that $r > L$. Then there exists an $(\epsilon, R)$-transition point $y \in \gamma$ such that $d(w, y) < D_2$. Hence $d(x, y) < d(x, ho) + d(ho, zo) + d(zo, w) + d(w, y) < r + D_1 + D_2 + 2R$. This shows the second statement. 

\section{Small cancellation in relatively hyperbolic groups}

We use the theory of a rotating family and the construction of hyperbolic cone-off from [14] and [11] to study small cancellation over hyperbolic elements. The objective is to describe the kernel and find an embedded set of elements in the quotient.

The key tool here is Qualitative Greendlinger Lemma in [14]. However, this lemma lives in a cone-off space, and in order to use it we need a notion of lift paths to transfer information to the original space before the cone-off construction.

\subsection{Rotating family and hyperbolic cone-off.}

Assume that $(\hat{X}, \hat{d})$ is a $\delta$-hyperbolic space and $G$ acts on $\hat{X}$ by isometries.

\textbf{Definition 8.1.} Let $A$ be a $G$-invariant set in $\hat{X}$ and $\{G_a : a \in A\}$ a collection of subgroups in $G$ such that $G_a(a) = a$ and $gG_ag^{-1} = G_{ga}$ for any $a \in A$. We call the pair $A = (A, (G_a)_{a \in A})$ a rotating family.

We are particularly interested in a very rotating family $A$. Informally speaking, this requires $A$ to be sufficiently separated and each $G_a$ rotates $a$ with a very large angle. In [14], Dahmani-Guirardel-Osin proved the following.

\textbf{Lemma 8.2 (Qualitative Greendlinger Lemma).} Let $\hat{X}$ be a $\delta$-hyperbolic geodesic metric space with a very rotating family $A = (A, (G_a)_{a \in A})$. Let $g \in (G_a : a \in A) \setminus 1 < G$. Then for any $o \in \hat{X} \setminus N_{20\delta}(A)$, there exists $a \in A \cap [o, ga]$ and $h \in G_a, q_1, q_2 \in [o, ga]$ satisfying $40\delta < d(q_1, q_2) < 50\delta$ and $\hat{d}(q_1, hq_2) < 8\delta$.

We shall describe a cone-off construction over a hyperbolic space relative to a collection of quasi-convex subsets, which yields the underlying space $\hat{X}$ of a rotating family. We begin with the cone over a metric space.

Let $Y$ be a metric space. For $r_0 \geq 0$, define $C(Y, r_0)$ to the quotient of $Y \times [0, r_0]$ by collapsing $Y \times 0$ as a point. The apex $a(Y)$ and base are the images of $Y \times 0$ and $Y \times r_0$ in $C(Y, r_0)$ respectively. If a group $G$ acts on $Y$ by isometries, then $G$ acts naturally on $Y \times [0, r_0]$ by $g(y, r) = (gy, r)$. This action descends to $C(Y, r_0)$ and fixes $a(Y)$.

We endow $C(Y, r_0)$ with a geodesic metric denoted by $d_Y$, whose precise definition is not relevant here but has the following consequence.

\textbf{Lemma 8.3.} [6] Let $(Y, d)$ be a metric space and $r_0 > 0$. Then the following holds.

1. $d_Y((y, r), (y', r')) \leq \pi \sinh r_0$ for any two $(y, r_0)$ and $(y', r_0)$ in $C(Y, r_0)$.

2. A geodesic between $(y, r_0)$ and $(y', r_0)$ passes the apex in $C(Y, r_0)$ if and only if $d(y, y') \geq \pi \sinh r_0$. 

Let \((X, d)\) be a \(\delta\)-hyperbolic space with a collection \(Q\) of strongly \(10\delta\)-quasi-convex subspaces. For \(r_0 > 0\), the cone-off \(\hat{X}(Q, r_0)\) over \(X\) relative to \(Q\) is the quotient of disjoint union

\[
X \bigsqcup (\bigsqcup_{Q \in \mathcal{C}(Q, r_0)} \mathcal{C}(Q, r_0))
\]

by identifying the base of \(\mathcal{C}(Q, r_0)\) with \(Q\) in \(X\).

We put a length metric on the cone-off \(\hat{X}\), cf. \([11, \text{Section } 3.1]\). Let \(x, y \in \hat{X}\). If there exists \(Q \in \mathcal{Q}\) such that \(x, y \in \mathcal{C}(Q, r_0)\), then \(|x - y| = d_Q(x, y)\). Otherwise if \(x, y \in X\) and there exists no \(Q \in \mathcal{Q}\) with \(x, y \in Q\), set \(|x - y| = d(x, y)\). Define \(|x - y| = \infty\) in all other cases. Endow \(\hat{X}\) with the chain metric:

\[
d(x, y) = \inf \left\{ \sum_{0 \leq i < n} |x_i - x_{i+1}| : x_0 = x, x_n = y, x_i \in \hat{X} \right\}, \forall x, y \in \hat{X}.
\]

If \(X\) and \(Q \in \mathcal{Q}\) are length spaces, then \(d\) is a length metric on \(\hat{X}\). By replacing \(X\) and \(Q\) by (quasi-isometric) graphs, we can assume that \((\hat{X}, d)\) is a geodesic space.

We assume that \(Q\) has bounded intersection property. Then \(\hat{X}\) is a hyperbolic space, cf. \([11, \text{Thm 3.5.2}]\). Let \(\hat{d}\) be the maximal pseudo-metric on \(\hat{X}\) such that \(\hat{d} \leq d\) and \(\hat{d}\) vanishes on every \(Q \in \mathcal{Q}\). By Lemma \(\text{[8.3]}\) we see that the embedding \((X, d) \rightarrow (\hat{X}, \hat{d})\) is a quasi-isometry with constants depending on \(r_0\).

Denote by \(A(Q) = \{a(Q) : Q \in \mathcal{Q}\}\) the set of apices. Usually when \(Q\) carries a group action, i.e. is \(G\)-invariant, the stabilizers of each \(Q \in \mathcal{Q}\) together \(A(Q)\) gives rise to a rotating family. In what follows, we consider a particular example arising from a hyperbolic element \(h \in G\).

The [axe] \(Ax(h)\) of \(h\) is the \(2\delta\)-neighborhood of the union of all geodesics between two fixed points of \(h\) on \(\partial X\). Then \(Ax(h)\) is strongly \(2\delta\)-quasi-convex in \(X\), from which we form a contracting system as follows

\[
(51) \quad \mathcal{Q} = \{gAx(h) : g \in G\}.
\]

Then \(\mathcal{Q}\) has bounded intersection by Lemma \(\text{[24.5]}\). Note that \(Ax(h)\) admits a proper and co-compact action of the group

\[
(52) \quad E(h) = \{g \in G : \exists n \in \mathbb{Z}, gh^n g^{-1} = h^{\pm n}\}.
\]

Since \(\langle h \rangle\) is normal in \(E(h)\), there is a bijection between \(\mathcal{Q}\) and the following

\[
(53) \quad \mathcal{H}_n = \{g(h^n)g^{-1} : g \in G\}.
\]

Lemma \(\text{[8.4]}\). \([14]\) There exists universal real numbers \(\delta_0 > 0, r_0 > 20\delta_0\) such that for any hyperbolic element \(h\) in \(G\), there exist \(N = N(h), \lambda = \lambda(h) > 0\) with the following property.

Consider the cone-off \(\lambda \hat{X}(Q, r_0)\) over the scaled metric space \(\lambda \hat{X} = \langle X, \lambda d \rangle\). Then for any \(n > N\) the pair \((A(Q), \mathcal{H}_n)\) is a very rotating family on the \(\delta_0\)-hyperbolic space \(\lambda \hat{X}(Q, r_0)\).

If \(G\) is a relatively hyperbolic group, then the quotient group \(\hat{G} = G / \langle h^n \rangle\) is relatively hyperbolic for any \(n > N\). See \([14]\) for more detail.

8.2. Lift paths in a cone-off space. We consider an other version of lift paths in a cone-off space, analogous to that of lift paths in Section 6.

Definition 8.5 (Lift path in \(\hat{X}\)). Let \(\gamma\) be a geodesic in \(\hat{X}\) with endpoints in \(X\). The lift path of \(\gamma\) is defined as follows.
Assume that $\gamma \cap C(Q, r_0) \neq \emptyset$ for some $Q \in \mathbb{Q}$, and let $a_-, a_+$ be the corresponding entry and exit points of $\gamma$ in $C(Q, r_0)$. If $\gamma$ contains $a(Q)$, then $\gamma$ intersects exactly in $Q$ two points, between which is a geodesic in $C(Q, r_0)$. We replace $[a_-, a_+]$, by some geodesic in $X$ with same endpoints.

Otherwise, we deduce from Lemma 8.3 that $d(a_-, a_+) < \pi \sinh r_0$. Note that $[a_-, a_+]$ and $[a'_-, a'_+]$, does not cross for two distinct such $Q, Q'$. If there exists no $Q' \in \mathbb{Q}$ such that $a_-, a_+ \in [a'_-, a'_+]$, for $a'_-, a'_+$ constructed as above for $Q'$, then we replace $[a_-, a_+]$, by some geodesic with same endpoints in $X$.

Repeating the above procedure for all $Q \in \mathbb{Q}$ with $\gamma \cap C(Q, r_0) \neq \emptyset$ gives the lift path $\hat{\gamma}$.

As expected, we prove an analogue to Lemma 6.4 that a lift path is a quasi-geodesic.

**Lemma 8.6.** There exist $\lambda = \lambda(r_0), c = c(r_0) \geq 0$ such that the following hold for any geodesic $\gamma$ in $X(Q, r_0)$. Then the lift $\hat{\gamma}$ is a $(\lambda, c)$-quasi-geodesic in $X$.

**Proof.** The lemma can be proven exactly as in proof of Lemma 6.4 We leave the details to the interested reader. $\square$

**Lemma 8.7.** There exist $\epsilon, R, L, D \geq 0$ depending on $r_0$ with the following property.

Let $\gamma$ be a geodesic in $X(Q, r_0)$ with $\gamma_-, \gamma_+ \in X$. For any $Q \in \mathbb{Q}$ with $Q \cap \gamma \neq \emptyset$, denote by $a_-, a_+$ the entry and exit points of $\gamma$ in $C(Q, r_0)$ respectively. Assume that $d(a_-, a_+) > L$. Then there exist $(\epsilon, R)$-transition points $x, y$ in $\beta = [\gamma_-, \gamma_+]$ in $X$ such that $d(x, a_-) < D$ and $d(y, a_+) < D$.

**Proof.** Let $\lambda, c$ be given by Lemma 8.6 Thus, $\hat{\gamma}$ is a $(\lambda, c)$-quasi-geodesic. Assume that for some $\epsilon, D_0 > 0$ is a $(\epsilon, D_0)$-contracting system with $D_0$-bounded projection. Without loss of generality, further assume that any $(\lambda, c)$-quasi-geodesic $q$ and $Q \in \mathbb{Q}$ with $q \cap N_r(Q) = \emptyset$, we have $|P_{rQ}(q)| < D_0$.

Let $Q \in \mathbb{Q}$ be assumed in the lemma. Assume that $L > \pi \sinh r_0$ and then $a(Q) \in \gamma$.

Observe that $\beta \cap N_r(Q) \neq \emptyset$. Indeed, suppose $\beta \cap N_r(Q) = \emptyset$. Then $|P_{rQ}(\beta)| < D_0$. Let $z, w$ be the corresponding entry and exit points of $\hat{\gamma}$ in $N_r(Q)$. Project $z, w$ to $z', w' \in Q$ respectively. By projection, we obtain $d(z', w') < 3D_0$.

Taking into account the possibility that $z, w$ may not be in $\gamma$, we assume that $z$ lies in a geodesic segment of $\hat{\gamma}$ with both endpoints $z_1, z_2$ in some $Q' \in \mathbb{Q}$. By projection, it is easy to see that $d(z_i, a_-) < (\epsilon + 2\pi \sinh r_0)$ for $i = 1, 2$. Thus, $d(z', a_-) < (\epsilon + 2\pi \sinh r_0)D_0$. Similarly, we obtain that $d(w', a_+) < (\epsilon + 2\pi \sinh r_0)D_0$. So $d(a_-, a_+) < (\epsilon + 2\pi \sinh r_0 + 3)D_0$. This is a contradiction as it is assumed that $L > (\epsilon + 2\pi \sinh r_0 + 3)D_0$.

Hence $\beta \cap N_r(Q) \neq \emptyset$.

Let $x, y$ be the entry and exit points of $\beta$ respectively in $N_r(Q)$. By projection, $d(z, x), d(x, y) < 2D_0 + 2\epsilon$. Hence, $d(a_-, x), d(a_+, y) < D$, where $D = 4D_0 + 2\epsilon$.

Let $R = \sup_{Q \in \mathbb{Q}, U \in \mathbb{U}} |N_r(Q) \cap N_r(U)| + 1$. It suffices to show that $x$ is an $(\tau, R)$-transition point in $\gamma$.

Suppose, to the contrary, that $x$ is $(\tau, R)$-deep in a horoball $U \in \mathbb{U}$. As it is assumed that $L > D + R$, we have $d(x, y) > R$. Observe that $[x, y]_z \in N_r(Q) \cap N_r(U)$. This contradicts to the choice of $R$. Therefore, $x$ is a $(\epsilon, R)$-transition point in $\beta$. $\square$
Remark. The condition that \( d(a_-, a_+) > L \) can be dropped (so the constant \( L \) can be deleted also). We won’t do it here as more cases need be considered in proof, and the restricted case as above already suffices in our application.

8.3. Finding embedded subsets in quotients. Endow \( S \subset G \) with a total order. Recall that \( \mathcal{L} = \{ \omega_g : g \in G \} \) is the set of lexi-minimal geodesics \( \omega_g \) between 1 and \( g \in G \). Below, we will not distinguish the word labeling \( \omega_g \), the geodesic \( \omega_g \) and the element \( g \).

The following notion quasifies the relation of a subword being contained in a word, cf. [1]. We put it in a general manner.

Definition 8.8. Suppose \( G \) acts on a geodesic metric space \((X, d)\). Fix \( o \in X, u \in G \) and \( \epsilon \geq 0 \). For \( g \in G \), we say \( \gamma = [o, go] \) \( \epsilon \)-contains a \( u \)-subsegment if there exists \( h \in G \) such that \( d(ho, \gamma) < \epsilon, d(huo, \gamma) < \epsilon \).

If \( X = \mathcal{G}(G, S) \), we say that \( g \) \( \epsilon \)-contains a \( u \)-subword if \( \omega_g \) \( \epsilon \)-contains a \( u \)-subsegment.

Let \((G, \mathcal{P}), X, \mathcal{U}, \mathcal{U} \) be as in Definition 1.3. Fix a basepoint \( o \in X \setminus \mathcal{U} \). Let \( h \) be a hyperbolic element, and \( E(h), \mathcal{Q}, \mathcal{H}_n \) defined in (52), (51), (53) respectively. Recall that \( G \) is also hyperbolic relative to \( \mathcal{P} \cup \mathcal{E} \), where \( \mathcal{E} = \{ gE(h)g^{-1} : g \in G \} \), cf (35). Set \( \mathcal{E} = \{ gE(h) : g \in G \} \). Then \( \mathcal{P} \cup \mathcal{E} \) is a contracting system with bounded intersection in \( \mathcal{G}(G, \mathcal{S}) \).

The following observation, a corollary of Lemma 2.8, will be helpful in proving Theorems 1.2, 1.4. Let \( G \), \( n \), \( \mathcal{G} \) and the restricted case as above already suffices in our application.

Lemma 8.9. For any \( \epsilon, R > 0 \), there exists \( L = L(\epsilon, R) \) with the following property.

Assume that \( g \in G \) \( \epsilon \)-contains a \( u \)-subword for some \( u \in \mathcal{P} \cup \{ E(h) \} \) with \( d(1, u) > L \). Then \( \omega_g \) contains a vertex \( v \) that is \((\epsilon_0, R)\)-deep in some \( \mathcal{P} \)-coset.

Proof. Let \( h \) be given by definition such that \( d(ho, \omega_g) < \epsilon, d(huo, \omega_g) < \epsilon \). Set \( L = 2\epsilon + L_0 \), where \( L_0 = L(\epsilon, R) \) given by Lemma 2.8.

Let \( x, y \in \omega_g \) such that \( d(x, ho), d(y, hu \cdot o) < \epsilon \). As \( d(1, u) > L \), one sees that \( d(x, y) > L_0 \). By Lemma 2.8 there exists a point \( v \) in \( [x, y]_{\omega_g} \) that is \((\epsilon_0, R)\)-deep in \( hP \).

We shall construct a subset of \( G \) which injects in \( G/\langle h^n \rangle \) for \( n > 0 \). This is explained by a series of lemmas.

Let \( \delta_0, r_0 > 20\delta_0, N = N(h), \lambda = \lambda(h) \) be given by Lemma 8.4. Then for any \( n > N, (A(\mathbb{Q}), \mathcal{H}_n) \) is a very rotating family on \( \lambda X(\mathbb{Q}, r_0) \), where \( \lambda X = (X, \lambda d) \) is \( \delta_0 \)-hyperbolic. In what follows, we omit the scaling factor \( \lambda \), as the reasoning for \((X, \lambda d)\) is valid for \((X, \lambda d)\) upon a scaling of the metric \( d \).

Fix \( o \in X \setminus (\mathcal{U} \cup \mathcal{Q}) \), where \( \mathcal{Q} = \cup_{Q \in \mathcal{Q}} Q \). We first consider kernel elements, which shall quasi-contain some high power of the killing relator \( h^n \).

Lemma 8.10. There exist \( \epsilon = \epsilon(h), N = N(h) > 0 \) such that for any \( g \in \langle h^n \rangle - 1 \) with \( n > N \) there exists \( f \in \langle h^n \rangle \) with the following properties.

1. any geodesic \( \gamma = [o, go] \) in \( X \) \( \epsilon \)-contains an \( f \)-subsegment.

2. \( g \) contains an \( f \)-subword.

Proof. For a subgroup \( H \) in \( G \), define \( \text{Inj}(H) = \inf\{ d(hx, x) : x \in X, h \in H \} \).

Let \( \epsilon_1, R_1, L_1, D_1 \) by given by Lemma 8.7 while \( \epsilon_1 \) satisfies convention (0.6). Choose \( N \) sufficiently large such that

\[
\text{Inj}(\langle h^n \rangle) > L_1 + 2r_0 \sinh r_0
\]
for $n > N$.

(1). Let $\beta = [\alpha, \alpha o]$ be a geodesic in $(X, d)$. By Lemma 8.2 there exist $a = a(Q) \in A \cap \beta$ such that there exists $f' \in G_a, q_1, q_2 \in \beta$ with $40\delta_0 < d(q_1, q_2) < 50\delta_0$ and $\dot{d}(q_1, f q_2) < 8\delta_0$. For definiteness assume that $G_a = g_a(h^n)$ for some $g_a$ and

$$f' = g_a h^n a^{-1}$$

for some $i > 0$.

Let $q_1', q_2'$ be the radical projections of $q_1, q_2$ to the base of $C(Q)$ respectively. Thus $a(Q) \notin [q_1, f q_2]_{a}$. Indeed, if $a(Q) \notin [q_1, f q_2]_{a}$, then $\dot{d}(q_1, q_2) = d(q_1, f q_2) = 8\delta_0$. This is a contradiction. Hence, $a(Q) \notin [q_1, f q_2]_{a}$ and then $a(Q) \notin [q_1', f q_2']_{a}$.

By Lemma 8.3 we have

$$d(q_1', f q_2') < 2r_0 \sinh r_0.$$ 

Note that $q_1', q_2'$ are corresponding entry and exit points of $\beta$ in $C(Q, r_0)$. Thus by (53), $d(q_1', f q_2') > L_1 + 2r_0 \sinh r_0$. Hence $d(q_1', q_2') > L_1$. By Lemma 8.7 there are $(\epsilon_1, R_1)$-transition points $x, y$ in $\gamma$ such that $d(q_1', x), d(q_2', y) < D_1$. It follows that

$$d(f q_2', x), d(q_2', y) < 2r_0 \sinh r_0 + D_1.$$ 

Since $E(h)$ acts cocompactly on $Q = Ax(h)$, there exist $M > 0$ and $j > 0$ such that $d(g_j h^j o, q_2') < M$. Hence by (53) and (56),

$$d(g_a h^{ni+j} o, x), d(g_a h^j o, y) < 2r_0 \sinh r_0 + D_1 + M.$$ 

Set $\epsilon = 2r_0 \sinh r_0 + D_1 + M$ and $f = h^{ni}$. Therefore by (57), $\gamma$-contains an $f$-subsegment.

(2). Let $\alpha = [1, g]$ be a word geodesic in $\mathcal{G}(G, S)$. Let $D_2 = D(\epsilon_1, R_1) > 0$ be given by Lemma 8.8. There exist $g_1, g_2 \in \alpha$ such that $d(g_1 o, x), d(g_2 o, y) < D_2$. It follows by (57) that

$$d(g_a h^{ni+j} o, g_1 o), d(g_a h^j o, g_2 o) < \epsilon + D_2.$$ 

Since $G \sim X$ is proper, set $\epsilon' = \max\{d_2(1, g) : d(g_2 o, o) < \epsilon + D_2\} < \infty$. Then $g$ $\epsilon'$-contains an $f$-subsegment for $f = h^{ni}$. This concludes the proof.

Let $\Delta(v_0, v_1, v_2)$ denote (a choice of) a geodesic triangle in $\mathcal{G}(G, S)$ with vertices $v_i$. A relative version of thin-triangle property in 17 is useful later.

**Lemma 8.11.** There exists $\sigma > 0$ such that for any geodesic triangle $\Delta(v_0, v_1, v_2)$, one of the following two cases holds

1. there exists $\alpha \in G$ such that $d(\alpha, [v_i, v_{i+1}]) < \sigma$ for $i \mod 3$,
2. there exists a unique $g \in \mathbb{P}$ such that $d(gP, [v_i, v_{i+1}]) < \sigma$ for $i \mod 3$.

In the second case denote by $(v_i)_-, (v_i)_+$ the entry points of $[v_i, v_{i-1}]$ and $[v_i, v_{i+1}]$ respectively in $N_\sigma(G P)$ for $i \mod 3$. Then $d((v_i)_-, (v_i)_+) < \sigma$.

We need another lemma in the following general form, which is a consequence of contracting property.

**Lemma 8.12.** Let $Y$ be a $(\tau, D)$-contracting subset in $X$. For any $r > 0$ there exists $L = L(r)$ with the following property. Let $p, q$ two geodesics in $X$ with $p_- = q_-$ and $d(p_+, q_+) < r$. If $[p \cap N_\tau(Y)] > L(r)$, then $\|p \cap N_\tau(Y)\| > [q \cap N_\tau(Y)] - 4D - 2r$.

**Proof.** The proof is straightforward using projection. 

□
Lemma 8.11 replaced by thin-triangle property. In this proof we write \(E = E(h)\) for simplicity.

Consider the geodesic triangle \(\Delta(1,g_1,g_2)\) in \(G(\mathcal{G},S)\) with sides \(\gamma_1 = \omega_{g_1}, \gamma = g_1\omega_{g_2}, \gamma_2 = \omega_{g_2}\) such that \((\gamma_1)_+ = (\gamma_2)_-, (\gamma_1)_- = (\gamma_2)_+.\)

Assume that for some \(\tau, D > 0\), \(P \cup E\) is a \((\tau, D)\)-contracting system. Let \(\epsilon > \tau\) be given by Lemma 8.12. Thus, \(\gamma\) \(\epsilon\)-contains an \(f\)'-subsegment for some \(f' \in \langle h^n \rangle\).

That is, \(\|\gamma \cap N_{\epsilon}(g_0E)\| > \text{Inj}(\langle h^n \rangle).\)

Let \(x, y\) be the entry and exit points of \(\gamma\) in \(N_{\epsilon}(g_0E)\) respectively. Assume further that \(N_{\epsilon}(\langle h^n \rangle) = E\). Then

\[
(59) \quad d(x, y) > \text{Inj}(\langle h^n \rangle)/2 - \epsilon.
\]

Let \(\sigma > 0\) be given by Lemma 8.11. By taking the bigger one, assume that \(\sigma \geq \epsilon\).

We examine the following two cases according to Lemma 8.11.

**Case 1.** There exists \(z \in \gamma\) such that \(d(z, \gamma_i) < \sigma\) for \(i = 1, 2\). We consider the subcase that \(z \in [x, y]_\gamma\). The subcase that \(z \notin [x, y]_\gamma\) is obvious by Lemma 8.12.

Assume by (59) that

\[
(60) \quad d(z, x) > \text{Inj}(\langle h^n \rangle)/2 - \epsilon,
\]

for example. The case that \(d(z, y) < \text{Inj}(\langle h^n \rangle)/2 - \epsilon\) is symmetric.

Let \(z' \in \gamma_1\) such that \(d(z, z') < \sigma\). Apply Lemma 8.12 to the pair of geodesics \(p = [\gamma_2, z'], q = [\gamma_1, z]\). We get \(\|\gamma_1 \cap N_{\epsilon}(E)\| > d(z, x) - 4D - 2\sigma\).

Set

\[
L(h, n) = \text{Inj}(\langle h^n \rangle)/2 - 4D - 3\sigma.
\]

Then \(\gamma_1\) \(\epsilon\)-contains an \(f\)-subword, where \(f \in E(h, L)\).

**Case 2.** Let \(gP \in \mathcal{P}\) given by Lemma 8.11 and \(u, v\) the entry and exit points of \(\gamma\) in \(N_{\epsilon}(gP)\) respectively. By Lemma 8.11 \(d(u, \gamma_1), d(v, \gamma_2) < \sigma\).

As \(\mathcal{P} \cup \mathcal{E}\) has bounded intersection, there exists a constant \(C = C(\epsilon_0, \sigma) > 0\) such that \(\|x, y\|, \|u, v\| < C\). Without loss of generality, assume that \(d(y, u) < C\). The case that \(d(x, v) < C\) is symmetric.

Similarly as above in Case 1, we see that \(\gamma_1\) \(\epsilon\)-contains an \(f\)-subword where \(f \in E(h, L)\).

Let \(W\) be a set of words in \(\mathcal{L}\) and \(\epsilon \geq 0\). Denote by \(\mathcal{L}(W, \epsilon)\) the set of words \(w\) in \(\mathcal{L}\) such that \(w\) \(\epsilon\)-contains a \(u\)-subword for some \(u \in W\). Similarly, denote by \(\mathcal{O}(W, \epsilon)\) the set of \(g_0\) in \(X\) such that some \([o, g_0]\) \(\epsilon\)-contains a \(u\)-segment for some \(u \in W\).

**Lemma 8.14.** Let \(\epsilon = \epsilon(h), N = N(h)\) and \(L = L(h, n)\) be as in Lemma 8.13. Set \(W = E(h, L)\). Then
9. Proofs of Theorems 1.2 & 1.4

We are now ready to give proofs of main theorems. Assume that \((G, \mathcal{P})\) is a relatively hyperbolic group. Let \(S\) be a finite generating set.

Fix a hyperbolic element \(h \in G\). The idea is, for any geodesic tree \(T\) given by Theorem 5.13 with growth rate arbitrarily close to \(\delta_G\), to find a sufficiently large \(n\) such that \(T^n\) is contained in the subset \(\mathcal{L} \setminus \mathcal{L}(W, \epsilon)\) given by Lemma 8.14.

A priori, the \(\epsilon\) provided by Theorem 5.13 may be different from the one by Lemma 8.14. This technical difficulty is addressed in Lemma 8.9.

9.1. Proof of Theorem 1.2

Recall that \(\mathcal{P} \cup \mathcal{E}\) is a contracting system with bounded intersection, where \(\mathcal{E} = \{gE(h) : g \in G\}\). We consider the notion of transition points relative to \(\mathcal{P} \cup \mathcal{E}\). Let \(\epsilon, R\) given by Lemma 5.13. Note that \(\epsilon > \epsilon_0\), where \(\epsilon_0\) is given by Lemma 2.8.

Fix \(\sigma > 0\). By Lemma 6.13 there exists a geodesic tree \(\mathcal{T}\) with \(\delta_T > \sigma\) and for any geodesic \(\gamma\) in \(\mathcal{T}\) and \(x \in \gamma\) there exists an \((\epsilon, R)\)-transition point \(y \in \gamma\) such that \(d(x, y) < r\).

Let \(R_0 = r + R + 1, L_0 = L(\epsilon, R_0)\) be given by Lemma 8.14.

Let \(\epsilon' = \epsilon(h), N = N(h)\) given by Lemma 8.14. Choose \(n > N\) large enough such that \(L = L(h, n) > L_0\). Then \(\pi : G \to G/\langle\langle h^n\rangle\rangle\) is injective in \(\mathcal{L} \setminus \mathcal{L}(W, \epsilon')\), where \(W \approx E(h, L)\). We claim that \(\mathcal{T}^n \subset \mathcal{L} \setminus \mathcal{L}(W, \epsilon')\).

Suppose \(g \in \mathcal{T}^n \cap \mathcal{L}(W, \epsilon')\). It is possible that \(\epsilon' > \epsilon\). However, if \(g \epsilon'\)-contains an \(f\)-subword for \(f \in E(h, L)\), then by Lemma 8.14 \(\omega_g\) contains a point \(v\) that is \((\epsilon_0, R_0)\)-deep in some \(E(h)\)-coset, say \(g' E(h)\). That is, \(B(v, R_0) \subset N_{\epsilon_0}(g' E(h))\).

By the property of \(\mathcal{T}\), there exists an \((\epsilon, R)\)-transition point \(w\) such that \(d(v, w) < r\). As \(R_0 > r + R\), we see that \(w\) is \((\epsilon_0, R)\)-deep in \(g' E(h)\). This is a contradiction. Hence \(\mathcal{T}^n \not\subset \mathcal{L} \setminus \mathcal{L}(W, \epsilon')\).

As \(\pi\) is length-decreasing, it follows that \(\delta_T \leq \delta_{\mathcal{L} \setminus \mathcal{L}(W, \epsilon')}\). Hence \(\delta_G > \delta_T > \sigma\).

The proof is complete.

9.2. Proof of Theorem 1.4

The proof is analogous to the previous one.

Assume that \(G\) acts properly on \(X\). Fix a basepoint \(o \in X \setminus \mathcal{U}\). Let \(h\) be a hyperbolic element in \(G\). Then \(\bar{G} = G/\langle\langle h^n\rangle\rangle\) acts on \(\bar{X} = X/\langle\langle h^n\rangle\rangle\) naturally by \(\bar{g} \cdot \bar{x} = \bar{g} \bar{x}\). Equip \(\bar{X}\) with metric \(\bar{d}(\bar{x}, \bar{y}) = \bar{d}(\langle\langle h^n\rangle\rangle x, \langle\langle h^n\rangle\rangle y)\). Then \(\bar{G}\) acts isometrically and properly on \(\bar{X}\).

We consider the notion of transition points relative to \(\mathcal{U} \cup \mathcal{Q}\), where \(\mathcal{Q}\) is the set of \(G\)-translates of axes of \(h\). Let \(\epsilon, R\) given by Lemma 7.20. Note that \(\epsilon > \epsilon_0\), where \(\epsilon_0\) is given by Lemma 2.8.

Fix \(\sigma > 0\). By Lemma 7.20 there exists a geodesic tree \(\mathcal{T}\) with \(\delta_T > \sigma\) with the following property. Let \(g \in \mathcal{T}^0\). Consider any geodesic \(\gamma = [o, go]\) in \(\mathcal{T}\) and \(x \in \gamma\) there exists an \((\epsilon, R)\)-transition point \(y \in \gamma\) such that \(d(x, y) < r\).

Let \(R_0 = r + R + 1, L_0 = L(\epsilon, R_0)\) be given by Lemma 8.9.
Let \( \epsilon' = \epsilon(h), N = N(h) \) given by Lemma 8.14. Choose \( n > N \) large enough such that \( L = L(h,n) > L_0 \). Then \( \pi : X \to X/\langle h^n \rangle \) is injective in \( G_0 \setminus O(W,\epsilon') \), where \( W = E(h,L) \). As in proof of Theorem 1.2, we obtain that \( T_0 \subset G_0 \setminus O(W,\epsilon') \).

As \( \pi \) is length-decreasing, it follows that \( \delta_T \leq \delta_{G_0 \setminus O(W,\epsilon')} \). Hence \( \delta_G > \delta_T > \sigma \). Theorem 1.3 is proved.

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