ON A COMPLETE TOPOLOGICAL INVERSE POLYCYCLIC MONOID

We give sufficient conditions when a topological inverse λ-polycyclic monoid $P_\lambda$ is absolutely $H$-closed in the class of topological inverse semigroups. For every infinite cardinal $\lambda$ we construct the coarsest semigroup inverse topology $\tau_{mi}$ on $P_\lambda$ and give an example of a topological inverse monoid $S$ which contains the polycyclic monoid $P_2$ as a dense discrete subsemigroup.

Key words and phrases: inverse semigroup, bicyclic monoid, polycyclic monoid, free monoid, semigroup of matrix units, topological semigroup, topological inverse semigroup, minimal topology.

Ivan Franko National University, 1 Universytetska str., 79 000, Lviv, Ukraine
E-mail: sbardyla@yahoo.com (Bardyla S.O.), o_gutik@franko.lviv.ua, ovgutik@yahoo.com (Gutik O.V.)

In this paper all topological spaces will be assumed to be Hausdorff. We shall follow the terminology of [10, 12, 16, 31]. If $A$ is a subset of a topological space $X$, then we denote the closure of the set $A$ in $X$ by $\text{cl}_X(A)$. By $\mathbb{N}$ we denote the set of all positive integers and by $\omega$ the first infinite cardinal.

A semigroup $S$ is called an inverse semigroup if every $a$ in $S$ possesses a unique inverse, i.e. if there exists a unique element $a^{-1}$ in $S$ such that

$$aa^{-1}a = a$$

and

$$a^{-1}aa^{-1} = a^{-1}.$$

A map that associates to any element of an inverse semigroup its inverse is called the inversion.

A band is a semigroup of idempotents. If $S$ is a semigroup, then we shall denote the subset of idempotents in $S$ by $E(S)$. If $S$ is an inverse semigroup, then $E(S)$ is closed under multiplication. The semigroup operation on $S$ determines the following partial order $\leq$ on $E(S)$: $e \leq f$ if and only if $ef = fe = e$. This order is called the natural partial order on $E(S)$. A semilattice is a commutative semigroup of idempotents. A semilattice $E$ is called linearly ordered or a chain if its natural order is a linear order. A maximal chain of a semilattice $E$ is a chain which is properly contained in no other chain of $E$. The Axiom of Choice implies the existence of maximal chains in any partially ordered set. According to [35, Definition II.5.12] a chain $L$ is called $\omega$-chain if $L$ is order isomorphic to $\{0, -1, -2, -3, \ldots\}$ with the usual order $\leq$. Let $E$ be a semilattice and $e \in E$. We denote $\downarrow e = \{f \in E \mid f \leq e\}$ and $\uparrow e = \{f \in E \mid e \leq f\}$.

If $S$ is a semigroup, then we shall denote by $R$, $L$, $D$ and $H$ the Green relations on $S$ (see [17] or [12, Section 2.1]):

$$aRb \text{ if and only if } aS^1 = bS^1; \quad aLb \text{ if and only if } S^1a = S^1b;$$

$$D = L \circ R = R \circ L; \quad H = L \cap R.$$

The $R$-class (resp., $L$, $H$, or $D$-class) of the semigroup $S$ which contains an element $a$ of $S$ will be denoted by $R_a$ (resp., $L_a$, $H_a$, or $D_a$).

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The bicyclic monoid \( \mathscr{C}(p, q) \) is the semigroup with the identity 1 generated by two elements \( p \) and \( q \) subjected only to the condition \( pq = 1 \). The semigroup operation on \( \mathscr{C}(p, q) \) is determined as follows:

\[
q^k p^l \cdot q^m p^n = q^{k+m-\min\{l,m\}} p^{l+n-\min\{l,m\}}.
\]

It is well known that the bicyclic monoid \( \mathscr{C}(p, q) \) is a bisimple (and hence simple) combinatorial \( E \)-unitary inverse semigroup and every non-trivial congruence on \( \mathscr{C}(p, q) \) is a group congruence [12]. Also the well known Andersen Theorem states that an idempotent is completely simple if and only if \( S \) does not contain an isomorphic copy of the bicyclic semigroup (see [2] and [12, Theorem 2.54]).

Let \( \lambda \) be a non-zero cardinal. On the set \( B_\lambda = (\lambda \times \lambda) \cup \{0\} \), where \( 0 \notin \lambda \times \lambda \), we define the semigroup operation "\( \cdot \)" as follows

\[
(a, b) \cdot (c, d) = \begin{cases} (a, d), & \text{if } b = c; \\ 0, & \text{if } b \neq c, \end{cases}
\]

and \((a, b) \cdot 0 = 0 \cdot (a, b) = 0 \cdot 0 = 0 \) for \( a, b, c, d \in \lambda \). The semigroup \( B_\lambda \) is called the semigroup of \( \lambda \times \lambda \)-matrix units (see [12]).

In 1970 Nivat and Perrot proposed the following generalization of the bicyclic monoid (see [34] and [31, Section 9.3]). For a non-zero cardinal \( \lambda \), the polycyclic monoid on \( \lambda \) generators \( P_\lambda \) is the semigroup with zero given by

\[
P_\lambda = \langle \{p_i\}_{i \in \lambda}, \{p_i^{-1}\}_{i \in \lambda} \mid p_i p_i^{-1} = 1, p_i p_j^{-1} = 0 \text{ for } i \neq j \rangle.
\]

If \( \lambda = 1 \) the semigroup \( P_1 \) is isomorphic to the bicyclic semigroup with adjoined zero. For every finite non-zero cardinal \( \lambda = n \) the polycyclic monoid \( P_n \) is congruence free, combinatorial, 0-bisimple, 0-\( E \)-unitary inverse semigroup (see [31, Section 9.3]).

A topological (inverse) semigroup is a Hausdorff topological space together with a continuous semigroup operation (and an inversion, respectively). Obviously, the inversion defined on a topological inverse semigroup is a homeomorphism. If \( S \) is a semigroup (an inverse semigroup) and \( \tau \) is a topology on \( S \) such that \((S, \tau)\) is a topological (inverse) semigroup, then we shall call \( \tau \) an (inverse) semigroup topology on \( S \). A semitopological semigroup is a Hausdorff topological space endowed with a separately continuous semigroup operation.

Let \( \mathcal{STSG}_\lambda \) be a class of topological semigroups. A semigroup \( S \in \mathcal{STSG}_\lambda \) is called \( H \)-closed in \( \mathcal{STSG}_\lambda \), if \( S \) is a closed subsemigroup of any topological semigroup \( T \in \mathcal{STSG}_\lambda \) which contains \( S \) both as a subsemigroup and as a topological space. The \( H \)-closed topological semigroups were introduced by Stepp in [39], and there they were called maximal semigroups. A topological semigroup \( S \in \mathcal{STSG}_\lambda \) is called absolutely \( H \)-closed in the class \( \mathcal{STSG}_\lambda \), if any continuous homomorphic image of \( S \) into \( T \in \mathcal{STSG}_\lambda \) is \( H \)-closed in \( \mathcal{STSG}_\lambda \). Absolutely \( H \)-closed topological semigroups were introduced by Stepp in [40], and there they were called absolutely maximal.

Recall [1], a topological group \( G \) is called absolutely closed if \( G \) is a closed subgroup of any topological group which contains \( G \) as a subgroup. In our terminology such topological groups are called \( H \)-closed in the class of topological groups. In [36] Raikov proved that a topological group \( G \) is absolutely closed if and only if it is Raikov complete, i.e., \( G \) is complete with respect to the two-sided uniformity. A topological group \( G \) is called \( h \)-complete if for every
continuous homomorphism \( h: G \to H \) the subgroup \( f(G) \) of \( H \) is closed [13]. In our terminology such topological groups are called absolutely \( H \)-closed in the class of topological groups. The \( h \)-completeness is preserved under taking products and closed central subgroups [13]. \( H \)-closed paratopological and topological groups in the class of paratopological groups were studied in [37]. The paper [7] contains a sufficient condition for a quasitopological group to be \( H \)-closed, which allowed us to solve a problem by Arhangel’skii and Choban [3] and show that a topological group \( G \) is \( H \)-closed in the class of quasitopological groups if and only if \( G \) is Raikov-complete. In [18] it is proved that a topological group \( G \) is \( H \)-closed in the class of semitopological inverse semigroups with continuous inversion if and only if \( G \) is compact.

In [40] Stepp studied \( H \)-closed topological semilattices in the class of topological semigroups. He proved that an algebraic semilattice \( E \) is algebraically \( h \)-complete in the class of topological semilattices if and only if every chain in \( E \) is finite. In [27] Gutik and Repovš studied the closure of a linearly ordered topological semilattice in a topological semilattice. They obtained a characterization of \( H \)-closed linearly ordered topological semilattices in the class of topological semilattices and showed that every \( H \)-closed linear topological semilattice is absolutely \( H \)-closed in the class of topological semilattices. Such semilattices were studied also in [11,20]. In [5] the closures of the discrete semilattices \((\mathbb{N}, \min)\) and \((\mathbb{N}, \max)\) were described. In that paper the authors constructed an example of an \( H \)-closed topological semilattice in the class of topological semilattices, which is not absolutely \( H \)-closed in the class of topological semilattices. The constructed example gives a negative answer to Question 17 from [40]. \( H \)-closed and absolutely \( H \)-closed (semi)topological semigroups and their extensions in different classes of topological and semitopological semigroups were studied in [8,18,19,21–26]

In [6] we showed that the \( \lambda \)-polycyclic monoid for an infinite cardinal \( \lambda \geq 2 \) has similar algebraic properties to that of the polycyclic monoid \( P_n \) with finitely many \( n \geq 2 \) generators. In particular we proved that for every infinite cardinal \( \lambda \) the polycyclic monoid \( P_\lambda \) is congruence-free, combinatorial, 0-bisimple, 0-\( E \)-unitary, inverse semigroup. Also we showed that every non-zero element \( x \in P_\lambda \) is an isolated point in \((P_\lambda, \tau)\) for every Hausdorff topology on \( P_\lambda \), such that \( P_\lambda \) is a semitopological semigroup; moreover, every locally compact Hausdorff semigroup topology on \( P_\lambda \) is discrete. The last statement extends results of the paper [32] treating topological inverse graph semigroups. We described all feebly compact topologies \( \tau \) on \( P_\lambda \) such that \((P_\lambda, \tau)\) is a semitopological semigroup. Also in [6] we proved that for every cardinal \( \lambda \geq 2 \) any continuous homomorphism from a topological semigroup \( P_\lambda \) into an arbitrary countably compact topological semigroup is annihilating and there exists no Hausdorff feebly compact topological semigroup containing \( P_\lambda \) as a dense subsemigroup.

This paper is a continuation of [6]. In this paper we give sufficient conditions on a topological inverse \( \lambda \)-polycyclic monoid \( P_\lambda \) to be absolutely \( H \)-closed in the class of topological inverse semigroups. For every infinite cardinal \( \lambda \) we construct the coarsest semigroup inverse topology \( \tau_{mi} \) on \( P_\lambda \) and give an example of a topological inverse monoid \( S \) which contains the polycyclic monoid \( P_2 \) as a dense discrete subsemigroup.

It is well known that for an arbitrary topological inverse semigroup \( S \) and every element \( x \in S \) the continuity of the semigroup operation and the inversion in \( S \) implies that any \( \mathcal{L} \)-class \( L_x \) and any \( \mathcal{R} \)-class \( R_x \) which contain the element \( x \) are closed subsets in \( S \). Indeed, the Wagner–Preston Theorem (see Theorem 1.17 from [12]) implies that \( L_x = L_{x^{-1}} \) and \( R_x = R_{xx^{-1}} \) for arbitrary \( x \in S \) and since the maps \( \varphi: S \to E(S) \) and \( \psi: S \to E(S) \) defined by the formulae

\[
(x)\varphi = xx^{-1} \quad \text{and} \quad (x)\psi = x^{-1}x
\]
are continuous, we get that \( L_x = (x^{-1}x)\psi^{-1} \) and \( R_x = (xx^{-1})\phi^{-1} \) are closed subsets of the topological semigroup \( S \). This implies that for any idempotents \( e \) and \( f \) of a topological inverse semigroup \( S \) the following \( \mathcal{H} \)-classes of \( S \):

\[
H_e = R_e \cap L_e \quad \text{and} \quad H_{e,f} = R_e \cap L_f
\]

are closed subsets of the topological inverse semigroup \( S \) too. Moreover, the relations \( \mathcal{L}, \mathcal{R} \) and \( \mathcal{H} \) are closed subsets in \( S \times S \), but \( \mathcal{D} \) and \( \mathcal{J} \) are not necessary closed subsets in \( S \times S \) for an arbitrary topological inverse semigroup \( S \) (see [15, Section II]).

The following proposition describes \( \mathcal{D} \)-equivalent \( \mathcal{H} \)-classes in an arbitrary topological inverse semigroup.

**Proposition 1.** Let \( S \) be a Hausdorff topological inverse semigroup and \( a, c \) be \( \mathcal{D} \)-equivalent elements of \( S \). Then there exists \( b \in S \) such that \( aRb \) and \( b\mathcal{L}c \) in \( S \), and hence \( as = b, bs' = a, \)

\[\begin{align*}
tb &= c, & t'c &= b, \quad \text{for some } s, s', t, t' \in S.
\end{align*}\]

The mappings \( f_{a,c}: H_a \to H_c: x \mapsto txs \) and \( f_{c,a}: H_c \to H_a: x \mapsto t'xs' \) are continuous and mutually inverse, and hence are homeomorphisms of closed subspaces \( H_a \) and \( H_c \) of the topological space \( S \). Moreover, if \( H_a \) and \( H_c \) are subgroups of \( S \) then \( H_a \) and \( H_c \) are topologically isomorphic closed topological subgroups in the topological inverse semigroup \( S \).

**Proof.** The above arguments imply that \( H_a \) and \( H_c \) are closed subspaces of \( S \). Also, the algebraic part of the statement of our theorem follows from Theorem 2.3 of [12] and Theorem 1.2.7 from [28]. The continuity of the semigroup operation in \( S \) implies that the maps \( f_{a,c}: H_a \to H_c \) and \( f_{c,a}: H_c \to H_a \) are continuous and hence are homeomorphisms. Now, the proof of Theorem 1.2.7 from [28] implies that in the case when \( H_a \) and \( H_c \) are subgroups of \( S \), then there exist \( u, u' \in S \) such that the maps \( f_{a,c}: H_a \to H_c: x \mapsto uxu' \) and \( f_{c,a}: H_c \to H_a: x \mapsto u'xu \) are mutually inverse isomorphisms and the continuity of the semigroup operation in \( S \) implies that so defined maps are topological isomorphisms. \( \square \)

**Remark 1.** The proof of Proposition 1 implies that any two \( \mathcal{D} \)-equivalent \( \mathcal{H} \)-classes of a Hausdorff semitopological semigroup \( S \) are homeomorphic subspaces in \( S \), but they are not necessary closed subspaces in \( S \), and a similar statement holds for maximal subgroups in \( S \) (see [18]).

**Lemma 1.** Let \( T \) and \( S \) be a Hausdorff topological inverse semigroup such that \( S \) is an inverse subsemigroup of \( T \). Let \( G \) be an \( \mathcal{H} \)-class in \( S \) which is a closed subset of the topological inverse semigroup \( T \) and \( D_G \) be a \( \mathcal{D} \)-class of the semigroup \( S \) which contains the set \( G \). Then every \( \mathcal{H} \)-class \( H \subseteq D_G \) of the semigroup \( S \) is a closed subset of the topological space \( T \).

**Proof.** First we consider the case when \( G \) has an idempotent, i.e., \( G \) is a maximal subgroup of the semigroup \( S \) (see Theorem 2.16 of [12]).

In the case when the \( \mathcal{H} \)-class \( H \) contains an idempotent, Theorem 2.16 in [12] implies that \( H \) is a maximal subgroup of \( S \) and hence \( H \) is a subgroup of topological inverse semigroup \( T \). We put \( e \) and \( f \) are unit elements of the groups \( G \) and \( H \), respectively. Since the idempotents \( e \) and \( f \) are \( \mathcal{D} \)-equivalent in \( S \), Proposition 3.2.5 of [31] implies that there exists \( a \in S \) such that \( aa^{-1} = e \) and \( a^{-1}a = f \). Now by Proposition 3.2.11(5) of [31] the idempotents \( e \) and \( f \) are \( \mathcal{D} \)-equivalent in the semigroup \( T \). Put \( H_e^T \) and \( H_f^T \) be the \( \mathcal{H} \)-classes of idempotents \( e \) and \( f \) in the semigroup \( T \), respectively. We define the maps \( f_{e,f}: T \to T \) and \( f_{f,e}: T \to T \) by the formulae
(x)f_{e,f} = a^{-1}xa and (x)f_{f,e} = axa^{-1}, respectively. Then for any s ∈ H_f^T and t ∈ H_f^T we get the equalities

\[(s)f_{e,f}(s)f_{e,f})^{-1} = a^{-1}sa(a^{-1}sa)^{-1} = a^{-1}sa^{-1}sa^{-1}a = a^{-1}ses^{-1}a = a^{-1}ss^{-1}a = a^{-1}ea = a^{-1}a = f,\]
\[\ ((s)f_{e,f})^{-1}(s)f_{e,f} = (a^{-1}sa)^{-1}a^{-1}sa = a^{-1}s^{-1}aa^{-1}sa = a^{-1}s^{-1}esa = a^{-1}s^{-1}sa = a^{-1}ea = a^{-1}a = f,\]
\[\ (f)f_{e,f}((t)f_{e,f})^{-1} = ata^{-1}(ata^{-1})^{-1} = ata^{-1}at^{-1}a^{-1} = atft^{-1}a^{-1} = att^{-1}a^{-1} = af^{-1}a^{-1} = aa^{-1} = e,\]
\[\ ((t)f_{e,f})^{-1}(t)f_{e,f} = (ata^{-1})^{-1}ata^{-1} = at^{-1}a^{-1}ata^{-1} = at^{-1}fa^{-1} = at^{-1}ta^{-1} = af^{-1}a^{-1} = aa^{-1} = e,\]
\[\ ((s)f_{e,f})f_{e,f} = aa^{-1}sa^{-1} = ese = s,\]
\[\ ((t)f_{e,f})f_{e,f} = a^{-1}ata^{-1}a = ftf = t,\]

because \(aa^{-1} = ss^{-1} = s^{-1}s = e, ea = a, af = a\) and \(a^{-1}a = tt^{-1} = t^{-1} = f\). Similarly, for arbitrary \(s, v ∈ H_f^T\) and \(t, u ∈ H_f^T\) we have that

\[\ (s)f_{e,f}(v)f_{e,f} = a^{-1}sa^{-1}va = a^{-1}seva = a^{-1}sva = (sv)f_{e,f}\]

and

\[\ (t)f_{e,f}(u)f_{e,f} = ata^{-1}uva^{-1} = atfua^{-1} = atua^{-1} = (tu)f_{e,f}.\]

Hence the restrictions \(f_{e,f}|_{H_f^T}: H_f^T → H_f^T\) and \(f_{f,e}|_{H_f^T}: H_f^T → H_f^T\) are mutually invertible group isomorphisms. Also, since \(a ∈ S\) we get that the restrictions \(f_{e,f}|_G: G → H\) and \(f_{f,e}|_H: H → G\) are mutually invertible group isomorphisms too. This and the continuity of left and right translations in \(T\) imply that \(H\) is a closed subgroup of the topological inverse semigroup \(T\).

Next we consider the case when the \(H\)-class \(H\) contains no idempotents. Then there exists distinct idempotents \(e, f ∈ S\) such that \(ss^{-1} = e\) and \(s^{-1}s = f\) for all \(s ∈ H\). Suppose to the contrary that \(H\) is not a closed subset of the topological inverse semigroup \(T\). Then there exists an accumulation point \(x ∈ T \setminus H\) of the set \(H\) in the topological space \(T\). Since every \(H\)-class of a topological inverse semigroup \(T\) is a closed subset of \(T\) we get that \(H\) and \(x\) are contained in a same \(H\)-class \(H_x\) of the semigroup \(T\). Then \(xx^{-1} = e\) and \(x^{-1}x = f\). Now the \(H\)-class \(H_x^T\) in \(T\) which contains the idempotent \(e ∈ S\) is a topological subgroup of the topological inverse semigroup \(T\) and by Proposition 1 the subspace \(H_x^T\) of the topological space \(T\) is homeomorphic to the subspace \(H_x\) of \(T\). Moreover, Theorem 1.2.7 from [28] implies that there exists a homeomorphism \(f: H_x → H_x^T\) such that the image \((H)f\) is a topological subgroup of the topological inverse semigroup \(T\) and \((H)f\) is topologically isomorphic to the topological group \(G\). Then \((H)f\) is not a closed subgroup of \(T\) which contradicts our above part of the proof.

Assume that \(G\) has no idempotents. By the previous part of the proof it suffices to show that there exists a maximal subgroup \(H_e\) with an idempotent \(e\) in the \(H\)-class \(D_G\) such that \(H_e\) is a closed subgroup of topological semigroup \(T\). Suppose to the contrary that every maximal subgroup in the \(H\)-class \(D_G\) is not a closed in \(T\). Fix and arbitrary subgroup \(H_e\) with an idempotent \(e\) in the \(H\)-class \(D_G\) such that \(xx^{-1} = e\) for all \(x ∈ G\). Then Proposition 3.2.11(3) of [31] implies
that there exist $\mathcal{H}$-classes $H^T_G$ and $H^T_e$ in the semigroup $T$ which contain the set $G$ and group $H_e$. Since in the topological semigroup $T$ every $\mathcal{H}$-class is a closed subset in $T$, we have that $G$ is a closed subset of the space $H^T_G$ and $H_e$ is not a closed subgroup of the topological group $H^T_e$. Then Proposition 3.2.11 of [31] and Proposition 1 imply that there exist $s, s', t, t^\prime \in S$ such that the maps $f_e: H^T_G \to H^T_G$: $x \mapsto txs$ and $f_G: H^T_G \to H^T_G$: $x \mapsto t'xs'$ are mutually invertible homeomorphisms of the topological spaces $H^T_G$ and $H^T_e$ such that the restrictions $f_e|_{H_e}: H^T_G \to G$ and $f_G|_{G}: G \to H_e$ are mutually invertible homeomorphisms. This is a contradiction, because $H_e$ is not a closed subset of $H^T_e$. This completes proof of the lemma.

Lemma 1 implies the following corollary.

**Corollary 1.** Let $T$ and $S$ be a Hausdorff topological inverse semigroup such that $S$ is an inverse subsemigroup of $T$. Let $G$ be a maximal subgroup in $S$ which is $H$-closed in the class of topological inverse semigroups and $D_G$ be a $D$-class of the semigroup $S$ which contains the group $G$. Then every $\mathcal{H}$-class $H \subseteq D_G$ of the semigroup $S$ is a closed subset of the topological space $T$.

**Lemma 2.** Let $S$ be a Hausdorff topological inverse semigroup such following conditions hold:

(i) every maximal subgroup of the semigroup $S$ is $H$-closed in the class topological groups;

(ii) all non-minimal elements of the semilattice $E(S)$ are isolated points in $E(S)$.

If there exists a topological inverse semigroup $T$ such that $S$ is a dense subsemigroup of $T$ and $T \setminus S \neq \emptyset$ then for every $x \in T \setminus S$ at least one of the points $x \cdot x^{-1}$ or $x^{-1} \cdot x$ belongs to $T \setminus S$.

**Proof.** First we consider the case when the topological semilattice $E(S)$ does not have the smallest element. Then the space $E(S)$ is discrete and Theorem 3.3.9 of [16] implies that $E(S)$ is an open subset of the topological space $E(T)$ and hence every point of the set $E(S)$ is isolated in $E(T)$. Also by Proposition II.3 [15] we have that $\text{cl}_T(E(S)) = \text{cl}_{E(T)}(E(S))$ and hence the points of the set $E(T) \setminus E(S)$ are not isolated in the space $E(T)$.

Fix an arbitrary point $x \in T \setminus S$. By Corollary 1 every $\mathcal{H}$-class is a closed subset of the topological inverse semigroup $T$. Since $x$ is an accumulation point of the set $S$ in the topological space $T$ we have that every open neighbourhood $U(x)$ of the point $x$ in $T$ intersects infinitely many $\mathcal{H}$-classes of the semigroup $S$. By Proposition II.1 of [15] the inversion on $T$ is a homeomorphism of the topological space $T$ and hence $(U(x))^{-1}$ is an open neighbourhood of the point $x^{-1}$ in $T$ which intersects infinitely many $\mathcal{H}$-classes of the semigroup $S$. Then the continuity of the semigroup operations and the inversion in $T$ implies that at least one of the sets $\left(U(x)(U(x))^{-1}\right) \cap E(T)$ or $\left((U(x))^{-1}U(x)\right) \cap E(T)$ is infinite for every open neighbourhood $U(x)$ of the point $x$ in the topological semigroup $T$. This implies that at least one of $\min x \cdot x^{-1}$ or $x^{-1} \cdot x$ is a non-isolated point in the topological space $E(T)$.

In the case when the semilattice $E(S)$ has a minimal idempotent the presented above arguments imply that for arbitrary point $x \in T \setminus S$ and every open neighbourhood $U(x)$ of the point $x$ in $T$ one of the sets $\left(U(x)(U(x))^{-1}\right) \cap E(T)$ or $\left((U(x))^{-1}U(x)\right) \cap E(T)$ is infinite for every open neighbourhood $U(x)$ of the point $x$ in the topological semigroup $T$. Since $H_e$ is a minimal ideal of $S$ and it is a Ratkó complete topological group. Then there exists an open neighborhood $U(x)$ of $x$ in $T$, such that $U(x) \cap H_e = \emptyset$. If $xx^{-1} = e$ or $x^{-1}x = e$ then $x = xx^{-1}x \in H_e$, which contradicts that $x \in T \setminus S$. Hence $xx^{-1} \in T \setminus S$ or $x^{-1}x \in T \setminus S$.  

\[ \Box \]
Lemma 2 implies the following two corollaries.

**Corollary 2.** Let $S$ be a Hausdorff topological inverse semigroup satisfying the following conditions:

(i) every maximal subgroup of the semigroup $S$ and the semilattice $E(S)$ are $H$-closed in the class of topological inverse semigroups;

(ii) all non-minimal elements of the semilattice $E(S)$ are isolated points in $E(S)$.

Then $S$ is $H$-closed in the class of topological inverse semigroups.

**Corollary 3.** Let $\lambda \geq 2$ and let $P_\lambda$ be a proper dense subsemigroup of a topological inverse semigroup $S$. Then either $xx^{-1} \in S \setminus P_\lambda$ or $x^{-1}x \in S \setminus P_\lambda$ for every $x \in S \setminus P_\lambda$.

The following theorem gives sufficient condition when a topological inverse $\lambda$-polycyclic monoid $P_\lambda$ is absolutely $H$-closed in the class of topological inverse semigroups.

**Theorem 1.** Let $\lambda$ be a cardinal $\geq 2$ and $\tau$ be a Hausdorff inverse semigroup topology on $P_\lambda$ such that $U(0) \cap L$ is an infinite set for every open neighborhood $U(0)$ of zero 0 in $(P_\lambda, \tau)$ and every maximal chain $L$ of the semilattice $E(P_\lambda)$. Then $(P_\lambda, \tau)$ is absolutely $H$-closed in the class of topological inverse semigroups.

**Proof.** First we observe that the definition of the $\lambda$-polycyclic monoid $P_\lambda$ implies that for every maximal chain $L$ in $E(P_\lambda)$ the set $L \setminus \{0\}$ is an $\omega$-chain. Then Theorem 2 of [5] implies that every maximal chain $L$ in $E(P_\lambda)$ with the induced topology from $(P_\lambda, \tau)$ is an absolutely $H$-closed topological semilattice. Suppose that $E(P_\lambda)$ with the induced topology from $(P_\lambda, \tau)$ is not an $H$-closed topological semilattice. Then there exists a topological semilattice $S$ which contains $E(P_\lambda)$ as a dense proper subsemilattice. Also the continuity of the semilattice operation in $S$ implies that zero 0 of $E(P_\lambda)$ is zero in $S$. Fix an arbitrary element $x \in S \setminus E(P_\lambda)$. Then for an arbitrary open neighbourhood $U(x)$ of the point $x$ in $S$ such that $0 \notin U(x)$ the continuity of the semilattice operation in $S$ implies that there exists an open neighbourhood $V(x) \subseteq U(x)$ of $x$ in $S$ such that $V(x) \cdot V(x) \subseteq U(x)$. Now, the neighbourhood $V(x)$ intersects infinitely many maximal chains of the semilattice $E(P_\lambda)$, because all maximal chains in $E(P_\lambda)$ with the induced topology from $(P_\lambda, \tau)$ are absolutely $H$-closed topological semilattices. Then the semigroup operation of $P_\lambda$ implies that $0 \in V(x) \cdot V(x) \subseteq U(x)$, which contradicts the choice of the neighbourhood $U(0)$. Therefore, $E(P_\lambda)$ with the induced topology from $(P_\lambda, \tau)$ is an $H$-closed topological semilattice.

Now, by Corollary 2 the topological inverse semigroup $(P_\lambda, \tau)$ is $H$-closed in the class of topological inverse semigroups. Since the $\lambda$-polycyclic monoid $P_\lambda$ is congruence free, every continuous homomorphic image of $(P_\lambda, \tau)$ is $H$-closed in the class of topological inverse semigroups. Indeed, if $h : (P_\lambda, \tau) \to T$ is a continuous (algebraic) homomorphism from $(P_\lambda, \tau)$ into a topological inverse semigroup $T$, then the set $U(h(0)) \cap h(L)$ is infinite for every open neighbourhood $U(h(0))$ of the image zero $h(0)$ in $T$. Then the previous part of the proof implies that $h(P_\lambda)$ is a closed subsemigroup of $T$. \qed

**Remark 2.** By Remark 2.6 from [30] (also see [30, p. 453], [29, Section 2.1] and [31, Proposition 9.3.1]) for every positive integer $n \geq 2$ any non-zero element $x$ of the polycyclic monoid...
$P_n$ has the form $u^{-1}v$, where $u$ and $v$ are elements of the free monoid $\mathcal{M}_n$, and the semigroup operation on $P_n$ in this representation is defined in the following way:

$$a^{-1}b \cdot c^{-1}d = \begin{cases} (c_1a)^{-1}d, & \text{if } c = c_1b \text{ for some } c_1 \in \mathcal{M}_n; \\
-1b_1d, & \text{if } b = b_1c \text{ for some } b_1 \in \mathcal{M}_n; \\
0, & \text{otherwise} \\
\end{cases}$$

(1)

Then Lemma 2.4 of [6] implies that every any non-zero element $x$ of the polycyclic monoid $P_\lambda$ has the form $u^{-1}v$, where $u$ and $v$ are elements of the free monoid $\mathcal{M}_\lambda$, and the semigroup operation on $P_\lambda$ in this representation is defined by formula (1).

Now we shall construct a topology $\tau_{\lambda}$ of the $\lambda$-polycyclic monoid $P_\lambda$ such that $(P_\lambda, \tau_{\lambda})$ is absolutely $H$-closed in the class of topological inverse semigroups.

**Example 1.** We define a topology $\tau_{\lambda}$ on the polycyclic monoid $P_\lambda$ in the following way. All non-zero elements of $P_\lambda$ are isolated point in $(P_\lambda, \tau_{\lambda})$. For an arbitrary finite subset $A$ of $\mathcal{M}_\lambda$ put

$$U_A(0) = \{ a^{-1}b : a, b \in M_\lambda \setminus A \}.$$  

We put $\mathcal{B}_{\lambda} = \{ U_A(0) : A \text{ is a finite subset of } \mathcal{M}_\lambda \}$ to be a base of the topology $\tau_{\lambda}$ at zero $0 \in P_\lambda$.

We observe that $\tau_{\lambda}$ is a Hausdorff topology on $P_\lambda$ because $U_{\{a,b\}}(0) \not\ni a^{-1}b$ for every non-zero element $a^{-1}b \in P_\lambda$. Also, since $(U_A(0))^{-1} = U_A(0)$ for any $A \subseteq \mathcal{B}_{\lambda}$, the inversion is continuous in $(P_\lambda, \tau_{\lambda})$. Fix an arbitrary $a^{-1}b \in P_\lambda$ and any basic neighbourhood $U_A(0)$ of zero in $(P_\lambda, \tau_{\lambda})$. Let $S_b$ be a set of all suffixes of the word $b$. Put $B = P_\lambda \cup \{ kb \in \mathcal{M}_\lambda : ka \in A \}$. It is obvious that the set $B$ is finite and hence formula (1) implies that $a^{-1}b \cdot B(0) \subseteq U_A(0)$. Let $S_a$ be a set of all suffixes of the word $a$. Put $D = S_a \cup \{ ta \in \mathcal{M}_\lambda : tb \in A \}$. It is obvious that the set $D$ is finite and hence formula (1) implies that $D(0) \cdot a^{-1}b \subseteq U_A(0)$. Also $U_T(0) \cdot U_T(0) \subseteq U_A(0)$ for $T = A \cup \{ b \in \mathcal{M}_\lambda : b$ is a suffix of some $a \in A \}$. Therefore $(P_\lambda, \tau_{\lambda})$ is a topological inverse semigroup.

Theorem 1 and Example 1 implies the following corollary.

**Corollary 4.** The topological inverse semigroup $(P_\lambda, \tau_{\lambda})$ is absolutely $H$-closed in the class of topological inverse semigroups.

**Definition 1.** (23). A Hausdorff topological (inverse) semigroup $(S, \tau)$ is said to be minimal if no Hausdorff semigroup (inverse) topology on $S$ is strictly contained in $\tau$. If $(S, \tau)$ is minimal topological (inverse) semigroup, then $\tau$ is called a minimal (inverse) semigroup topology.

Minimal topological groups were introduced independently in the early 1970’s by Doîtchinov [14] and Stephenson [38]. Both authors were motivated by the theory of minimal topological spaces, which was well understood at that time (cf. [9]). More than 20 years earlier L. Nachbin [33] had studied minimality in the context of division rings, and B. Banaschewski [4] investigated minimality in the more general setting of topological algebras. In [23] on the infinite semigroup of $\lambda \times \lambda$-matrix units $B_\lambda$ the minimal semigroup and the minimal semigroup inverse topologies were constructed.
Theorem 2. For any infinite cardinal \( \lambda \), \( \tau_{mi} \) is the coarsest inverse semigroup topology on \( P_\lambda \), and hence \( (P_\lambda, \tau_{mi}) \) is a minimal topological inverse semigroup.

Proof. The definition of the topology \( \tau_{mi} \) on \( P_\lambda \) implies that the subsemigroup of idempotents \( E(P_\lambda) \) of the semigroup \( P_\lambda \) is a compact subset of the space \( (P_\lambda, \tau_{mi}) \). By Proposition 3.1 of [6] every non-zero-element of a semitopological monoid \( (P_\lambda, \tau) \) is an isolated point in the space \( (P_\lambda, \tau) \). This and above arguments imply that the topology \( \tau_{mi} \) on \( P_\lambda \) induces the coarsest semigroup topology on the semilattice \( E(P_\lambda) \). Also by Remark 2.6 from [30] (also see [30, p. 453], [29, Section 2.1] and [31, Proposition 9.3.1]) we have that every non-zero element of the semilattice \( E(P_\lambda) \) can be represented in the form \( a^{-1}a \) where \( a \) are elements of the free monoid \( \mathcal{M}_\lambda \), and the semigroup operation on \( E(P_\lambda) \) in this representation is defined by formula (1).

Also, we observe that for any topological inverse semigroup \( S \) the following maps \( \varphi: S \to E(S) \) and \( \psi: S \to E(S) \) defined by the formulae \( \varphi(x) = xx^{-1} \) and \( \psi(x) = x^{-1}x \), respectively, are continuous. Since the inverse element of \( u^{-1}v \) in \( P_\lambda \) is equal to \( v^{-1}u \), we have that \( U_A = P_\lambda \setminus (\varphi^{-1}(A) \cup \psi^{-1}(A)) \), for any finite subset \( A \) of the free monoid \( \mathcal{M}_\lambda \). This implies that \( U_A(A) \in \tau \) for every inverse semigroup topology \( \tau \) on \( P_\lambda \), where \( A \) is an arbitrary finite subset of \( \mathcal{M}_\lambda \). Thus, \( \tau_{mi} \) is the coarsest inverse semigroup topology on the \( \lambda \)-polycyclic monoid \( P_\lambda \).

In the next example we construct a topological inverse monoid \( S \) which contains the polycyclic monoid \( P_2 = \langle p_1, p_2 \mid p_1p_1^{-1} = p_2p_2^{-1} = 1, p_1p_2^{-1} = p_2p_1^{-1} = 0 \rangle \) as a dense discrete subsemigroup, i.e., the polycyclic monoid \( P_2 \) with the discrete topology is not \( H \)-closed in the class of topological inverse semigroups. Also, later we assume that the free monoid \( \mathcal{M}_2 \) is generated by two element \( p_1 \) and \( p_2 \).

Example 2. Let \( \mathcal{F} \) be the filter on the bicyclic semigroup \( \mathcal{C}(p_1, p_1^{-1}) = \langle p_1, p_1^{-1} \mid p_1p_1^{-1} = 1 \rangle \), generated by the base \( \mathcal{B} = \{ U_n : n \in \mathbb{N} \} \), where \( U_n = \{ p_1^{-k}p_1^j : k, j > n \} \). We denote \( A = \{ a^{-1}b \in P_2 : a \neq p_1a_1 \text{ and } b \neq p_1b_1 \text{ for any } a_1, b_1 \in \mathcal{M}_2 \} \).

For any element \( a^{-1}b \) of the set \( A \) let \( \mathcal{F}_{a^{-1}b} \) be the filter on \( P_2 \) generated by the base \( \mathcal{B}_{a^{-1}b} = \{ V_n : n \in \mathbb{N} \} \), where \( V_n = a^{-1}U_n b = \{ (p_1^{-k}a)^{-1}p_1^mb : k, m > n \} \). It is obvious that \( \mathcal{F} = \mathcal{F}_{1} \), where 1 is the unit element of the free monoid \( \mathcal{M}_2 \).

We extend the binary operation from \( P_2 \) onto \( S = P_2 \cup \{ \mathcal{F}_{a^{-1}b} : a^{-1}b \in A \} \) by the following formulae:

(I) \( a^{-1}b \cdot \mathcal{F}_{c^{-1}d} = \begin{cases} \mathcal{F}_{(ea)^{-1}d} & \text{if } c = e b; \\ \mathcal{F}_{(c^{-1}d)^{-1}e} & \text{if } b = p_1^n c \text{ for some } n \in \mathbb{N}, \text{ where } e \text{ is the longest suffix of } c \text{ such that } e \neq p_1 f \text{ for some } f \in M_2; \\ 0 & \text{otherwise}; \end{cases} \)

(II) \( \mathcal{F}_{c^{-1}d} \cdot a^{-1}b = \begin{cases} \mathcal{F}_{c^{-1}eb} & \text{if } d = ea; \\ \mathcal{F}_{c^{-1}e} & \text{if } a = p_1^m d \text{ for some } n \in \mathbb{N}, \text{ where } e \text{ is the longest suffix of } b \text{ such that } e \neq p_1 f \text{ for some } f \in M_2; \\ 0 & \text{otherwise}; \end{cases} \)

(III) \( \mathcal{F}_{a^{-1}b} \cdot \mathcal{F}_{c^{-1}d} = \begin{cases} \mathcal{F}_{a^{-1}d} & \text{if } b = c; \\ 0 & \text{otherwise}. \end{cases} \)
It is obvious that the subset $T = S \setminus P_2 \cup \{0\}$ with the induced binary operation from $S$ is isomorphic to the semigroup of $\omega \times \omega$-matrix units $B_\omega$ and moreover we have that $(\mathcal{F}_{a^{-1}b})^{-1} = \mathcal{F}_{b^{-1}a}$ in $T$.

We determine a topology $\tau$ on the set $S$ in the following way: assume that the elements of the semigroup $P_2$ are isolated points in $(S, \tau)$ and the family

$$\mathcal{B}(\mathcal{F}_{a^{-1}b}) = \{U_n(\mathcal{F}_{a^{-1}b}) : U_n \in \mathcal{F}_{a^{-1}b}\}$$

of the set $U_n(\mathcal{F}_{a^{-1}b}) = U_n \cup \{\mathcal{F}_{a^{-1}b}\}$ is a neighborhood base of the topology $\tau$ at the point $\mathcal{F}_{a^{-1}b} \in S$.

Now we show that so defined binary operation on $(S, \tau)$ is continuous.

In case (I) we consider three cases.

If $a^{-1}b \cdot \mathcal{F}_{c^{-1}d} = 0$ then we have that $a^{-1}b \cdot U_n(\mathcal{F}_{c^{-1}d}) = \{0\}$ for any positive integer $n$.

If $a^{-1}b \cdot \mathcal{F}_{c^{-1}d} = \mathcal{F}_{(ea)^{-1}d}$ then $c = eb$. We claim that $a^{-1}b \cdot U_n(\mathcal{F}_{c^{-1}d}) \subseteq U_n(\mathcal{F}_{(ea)^{-1}d})$ for any open basic neighbourhood $U_n(\mathcal{F}_{(ea)^{-1}d})$ of the point $\mathcal{F}_{(ea)^{-1}d}$ in $(S, \tau)$. Indeed, if $x \in U_n(\mathcal{F}_{c^{-1}d})$ then $x = (p_1^m c)^{-1} p_1^d$ for some positive integers $m, k > n$, and hence we have that

$$a^{-1}b \cdot (p_1^m c)^{-1} p_1^d = a^{-1}b \cdot (p_1^m eb)^{-1} p_1^d = (p_1^m ea)^{-1} p_1^d = U_n(\mathcal{F}_{(ea)^{-1}d}).$$

If $a^{-1}b \cdot \mathcal{F}_{c^{-1}d} = \mathcal{F}_{e^{-1}d}$, then $e$ is the longest suffix of the word $a$ in $\mathcal{M}_2$ which is not equal to the word $p_1 f$ for some $f \in \mathcal{M}_2$. This holds when $b = p_1^t c$ for some positive integer $t$. We claim that $a^{-1}b \cdot U_{n+t}(\mathcal{F}_{c^{-1}d}) \subseteq U_n(\mathcal{F}_{c^{-1}d})$ for any open basic neighbourhood $U_n(\mathcal{F}_{c^{-1}d})$ of the point $\mathcal{F}_{c^{-1}d}$ in $(S, \tau)$. Indeed, if $x \in U_{n+t}(\mathcal{F}_{c^{-1}d})$, then $x = (p_1^{m+t} c)^{-1} p_1^{k+t} d$ for some positive integers $m, k > n$, and hence we have that

$$a^{-1}b \cdot (p_1^{m+t} c)^{-1} p_1^{k+t} d = e^{-1} p_1^{m+t} c \cdot (p_1^{m+t} c)^{-1} p_1^{k+t} d = (p_1^{m+t} c)^{-1} p_1^{k+t} d = U_n(\mathcal{F}_{c^{-1}d}).$$

In case (II) the proof of the continuity of binary operation in $(S, \tau)$ is similar to case (I).

Now we consider case (III).

If $\mathcal{F}_{a^{-1}b} \cdot \mathcal{F}_{c^{-1}d} = 0$ then $U_n(\mathcal{F}_{a^{-1}b}) \cdot U_n(\mathcal{F}_{c^{-1}d}) \subseteq \{0\}$, for any open basic neighbourhoods $U_n(\mathcal{F}_{a^{-1}b})$ and $U_n(\mathcal{F}_{c^{-1}d})$ of the points $\mathcal{F}_{a^{-1}b}$ and $\mathcal{F}_{c^{-1}d}$ in $(S, \tau)$, respectively.

If $\mathcal{F}_{a^{-1}b} \cdot \mathcal{F}_{c^{-1}d} = \mathcal{F}_{a^{-1}d}$ then $b = c$ and for every any open basic neighbourhood $U_n(\mathcal{F}_{a^{-1}d})$ of the point $\mathcal{F}_{a^{-1}d}$ in $(S, \tau)$ we have that $U_n(\mathcal{F}_{a^{-1}b}) \cdot U_n(\mathcal{F}_{b^{-1}d}) \subseteq U_n(\mathcal{F}_{a^{-1}d})$. Indeed if $(p_1^k a)^{-1} p_1^b \in U_n(\mathcal{F}_{a^{-1}b})$ and $(p_1^l b)^{-1} p_1^m d \in U_n(\mathcal{F}_{b^{-1}d})$ then

$$(p_1^k a)^{-1} p_1^b \cdot (p_1^l b)^{-1} p_1^m d = (p_1^k a)^{-1} p_1^b \cdot (b \cdot b^{-1}) p_1^l p_1^m d = (p_1^s a)^{-1} p_1^d,$$

for some positive integers $s, z > n$, and hence $(p_1^s a)^{-1} p_1^d \in U_n(\mathcal{F}_{a^{-1}d})$.

Thus, we proved that the binary operation on $(S, \tau)$ is continuous. Taking into account that $P_2$ is a dense subsemigroup of $(S, \tau)$ we conclude that $(S, \tau)$ is a topological semigroup.

Also, since $T = S \setminus P_2 \cup \{0\}$ with the induced binary operation from $S$ is isomorphic to the semigroup of $\omega \times \omega$-matrix units $B_\omega$ we have that idempotents in $S$ commute and moreover $\mathcal{F}_{a^{-1}b} \cdot \mathcal{F}_{b^{-1}a} \cdot \mathcal{F}_{a^{-1}b} = \mathcal{F}_{b^{-1}a}$. This implies that $S$ is an inverse semigroup. Also, for every open basic neighbourhood $U_n(\mathcal{F}_{a^{-1}b})$ of the point $\mathcal{F}_{a^{-1}b}$ in $(S, \tau)$ we have that $(U_n(\mathcal{F}_{a^{-1}b}))^{-1} = U_n(\mathcal{F}_{b^{-1}a})$ for all $n \in \mathbb{N}$ and hence the inversion in $(S, \tau)$ is continuous.

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Вказано достатні умови, за яких топологічний інверсний λ-поліциклічний моноїд $P_λ$ є абсолютно $H$-замкненим в класі топологічних інверсних напівгруп. Для довільного нескінченного кардиналу $λ$ побудовано найслабшу напівгрупову інверсну топологію $τ_{mi}$ на $P_λ$ та наведено приклад топологічного інверсного моноїда $S$, що містить поліциклічний моноїд $P_2$ як щільну дискретну піднапівгрупу.

Ключові слова і фрази: інверсна напівгрупа, біциклічний моноїд, поліциклічний, вільний моноїд, напівгрупа матричних одиниць, топологічна напівгрупа, топологічна інверсна напівгрупа, мінімальна топологія.