On the rank of a finite group of odd order with an involutory automorphism

Cristina Acciarri · Pavel Shumyatsky

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Abstract
Let $G$ be a finite group of odd order admitting an involutory automorphism $\phi$, and let $G_{-\phi}$ be the set of elements of $G$ transformed by $\phi$ into their inverses. Note that $[G, \phi]$ is precisely the subgroup generated by $G_{-\phi}$. Suppose that each subgroup generated by a subset of $G_{-\phi}$ can be generated by at most $r$ elements. We show that the rank of $[G, \phi]$ is $r$-bounded.

Keywords Finite groups · Automorphisms · Rank of a group

Mathematics Subject Classification 20D45

1 Introduction
Let $G$ be a finite group of odd order admitting an involutory automorphism $\phi$. Here the term “involutory automorphism” means an automorphism $\phi$ such that $\phi^2 = 1$. We let $G_{-\phi}$ stand for the set $\{g \in G | g^\phi = g^{-1}\}$ and $G_{\phi}$ for the centralizer of $\phi$, that is, the subgroup of fixed points of $\phi$. As usual we denote by $[G, \phi]$ the subgroup generated by all elements of $G$ that can be written as $g^{-1}g^\phi$ for a suitable $g \in G$. It is well known that $[G, \phi]$ is normal in $G$ and $\phi$ induces the trivial automorphism of $G/[G, \phi]$. Observe that $[G, \phi]$ is precisely the subgroup generated by $G_{-\phi}$. This is because an automorphism of order at most two of a group of odd order is nontrivial.
if and only if $G_{-\phi} \neq \{1\}$ (cf Lemma 1(i) in the next section). The following theorem was proved in [10, Theorem B].

**Theorem 1** Let $G$ a finite group of odd order admitting an involutory automorphism $\phi$ such that the rank of $G_{\phi}$ is at most $r$. Then the rank of $[G, \phi]'$ is $r$-bounded.

Recall that a rank of a finite group $G$ is the least number $r$ such that each subgroup of $G$ can be generated by at most $r$ elements. Throughout this manuscript we use the term “$(a, b, c \ldots)$-bounded” to mean “bounded from above by some function depending only on the parameters $a, b, c \ldots$.”

Since in a finite group of odd order with an involutory automorphism $\phi$ there is a kind of (very vague) duality between $G_{\phi}$ and $G_{-\phi}$, in this paper we address the question whether a rank condition imposed on the set $G_{-\phi}$ has an impact on the structure of $G$. We emphasize that $G_{-\phi}$ in general is not a subgroup of $G$ and therefore the usual concept of rank does not apply to $G_{-\phi}$. Instead we impose the condition that each subgroup of $G$ generated by a subset of $G_{-\phi}$ can be generated by at most $r$ elements. Our main result is as follows.

**Theorem 2** Let $G$ be a group of odd order admitting an involutory automorphism $\phi$ and suppose that any subgroup generated by a subset of $G_{-\phi}$ can be generated by $r$ elements. Then $[G, \phi]$ has $r$-bounded rank.

It is noteworthy that in the literature there are several papers dealing with finite groups admitting a (not necessarily involutory) automorphism whose fixed-point subgroup is of rank $r$ (see for example [5, 6]). In particular, [5] contains a result similar to the above Theorem 1. Thus, it seems plausible that some analogues of Theorem 2 are valid for the case where the order of $\phi$ is bigger than two.

## 2 Nilpotent groups with involutory automorphisms

We start with a collection of well-known facts about involutory automorphisms of groups of odd order (see for example [3, Lemma 4.1, Chap. 10]).

**Lemma 1** Let $G$ be a finite group of odd order admitting an involutory automorphism $\phi$. The following conditions hold:

(i) $G = G_{\phi}G_{-\phi} = G_{-\phi}G_{\phi}$ and $|G_{-\phi}| = [G : G_{\phi}]$;

(ii) The subgroup generated by $G_{-\phi}$ is exactly $[G, \phi]$;

(iii) If $N$ is any $\phi$-invariant normal subgroup of $G$ we have $(G/N)_{\phi} = G_{\phi}N/N$, and $(G/N)_{-\phi} = \{gN \mid g \in G_{-\phi}\}$;

(iv) If $N$ is any $\phi$-invariant normal subgroup of $G$ such that $N = N_{-\phi}$ or $N = N_{\phi}$, then $[G, \phi]$ centralizes $N$;

(v) The subgroup $G_{\phi}$ normalizes $G_{-\phi}$.

It is well known that a maximal abelian normal subgroup of a nilpotent group coincides with its centralizer. We will require the following related result.

**Lemma 2** Let $G$ be a nilpotent group of odd order, $\phi$ an involutory automorphism of $G$, and $A$ a maximal $\phi$-invariant abelian normal subgroup of $G$. Then $A = C_G(A)$.
Lemma 5

Let $G$ be a group of prime exponent $p$ and rank $r$. The nilpotency of $G/A$ implies that $C/A \cap Z(G/A) \neq 1$.

Let $U$ be the full inverse image of $C/A \cap Z(G/A)$ in $G$. Since $C/A \cap Z(G/A) \neq 1$, the subgroup $A$ is properly contained in $U$. From Lemma 1(i) we know that $U = U_\phi U_{-\phi}$. Thus, either $U_\phi \not\subset A$ or $U_{-\phi} \not\subset A$. In any case we can choose $u \in U \setminus A$ satisfying either $u^\phi = u$ or $u^\phi = u^{-1}$. Take $H = A\langle u \rangle$ and note that $A < H$. Furthermore, $H$ is a $\phi$-invariant abelian normal subgroup of $G$. This yields a contradiction. $\square$

Note that the previous lemma fails if $\phi$ is allowed to be a coprime automorphism of arbitrary order. For example, the quaternion group of order 8 admits an automorphism $\alpha$ of order 3 and the maximal $\alpha$-invariant abelian normal subgroup is central.

Lemma 3

Let $p$ be an odd prime and $G$ a $p$-group admitting an involutory automorphism $\phi$ such that $G = [G, \phi]$. Let $M$ be a $\phi$-invariant normal subgroup of $G$ and assume that $|M_{-\phi}| = p^n$ for some nonnegative integer $n$. Then $M \leq Z_{2n+1}(G)$.

Proof

If $n = 0$, then the result follows from Lemma 1(iv), so assume that $n \geq 1$ and use induction on $n$.

Let $N = M \cap Z_2(G)$. If $N \not\subsetneq Z(G)$, then Lemma 1(iv) implies that $N_{-\phi} \neq 1$, in which case we have $|N/N_{-\phi}| = |M_{-\phi}| = p^n$. By induction $M/N \leq Z_{2n-1}(G/N)$, whence $M \leq Z_{2n+1}(G)$. If $N \subsetneq Z(G)$, then it turns out that $M \cap Z(G) = M \cap Z_i(G)$ for any $i \geq 2$ and so, obviously, $M \leq Z(G)$. This concludes the proof. $\square$

We now fix some notation and hypotheses that will be used throughout this section.

Hypothesis 1

Let $p$ be an odd prime, $r$ a positive integer and $G$ a finite $p$-group with an involutory automorphism $\phi$ such that $G = [G, \phi]$. Assume that any subgroup generated by a subset of $G_{-\phi}$ can be generated by $r$ elements.

Lemma 4

Assume Hypothesis 1 and suppose that $G$ is of exponent $p$. There exists a number $l = l(r)$, depending on $r$ only, such that the rank $r(G)$ of $G$ is at most $l$.

Proof

Let $A$ be a maximal $\phi$-invariant abelian normal subgroup of $G$. The subgroup $\langle A_{-\phi} \rangle$ is an $r$-generated abelian subgroup of exponent $p$ and so $|A_{-\phi}| \leq p^r$. Lemma 3 implies that $A \leq Z_{2r+1}(G)$. Since $\gamma_{2r+1}(G)$ commutes with $Z_{2r+1}(G)$, we deduce that $\gamma_{2r+1}(G)$ centralizes $A$. Furthermore, by Lemma 2, $A = C_G(A)$. Thus $\gamma_{2r+1}(G) \leq A$, that is, the quotient group $G/A$ is nilpotent of class $2r$. We deduce that $G$ has $r$-bounded nilpotency class as well. Since $G = \langle A_{-\phi} \rangle$ is $r$-generated by hypothesis, it follows that the rank $r(G)$ of $G$ is $r$-bounded, as desired. $\square$

The following result from [10, Lemma 2.2] is also useful.

Lemma 5

Let $G$ be a group of prime exponent $p$ and rank $r_0$. Then there exists a number $s = s(r_0)$, depending only on $r_0$, such that $|G| \leq p^s$.

Lemma 6

Let $G$ be a group satisfying Hypothesis 1. There exists a number $\lambda = \lambda(r)$, depending only on $r$, such that $\gamma_{2\lambda+1}(G)$ is powerful.
Proof Let \( s(r_0) \) be as in Lemma 5 and let \( l(r) \) be as in Lemma 4. Take \( N = \gamma_{2\lambda+1}(G) \), where \( \lambda = s(l(r)) \). In order to show that \( N' \leq N^p \), we assume that \( N \) is of exponent \( p \) and prove that \( N \) is abelian.

Note that the subgroup \( \langle N_{-\phi} \rangle \) is of exponent \( p \). By Lemma 4 the rank of \( \langle N_{-\phi} \rangle \) is at most \( l(r) \). It follows from Lemma 5 that \( |N_{-\phi}| \leq p^{s(l(r))} = p^\lambda \). Now Lemma 3 yields \( N \leq Z_{2\lambda+1}(G) \). By using the well-known fact that \( [\gamma_i(G), Z_i(G)] = 1 \), for any positive integer \( i \) and any group \( G \), we conclude that \( N \) is abelian, as required. \( \square \)

Lemma 7 Assume Hypothesis 1. For any \( i \geq 1 \), there exists a number \( m_i = m_i(i, r) \), depending only on \( i \) and \( r \), such that \( \gamma_i(G) \) is an \( m_i \)-generated group.

Proof Let \( N = \gamma_i(G) \). In view of the Burnside Basis Theorem [9, 5.3.2], we can pass to the quotient \( G/\Phi(N) \) and assume that \( N \) is elementary abelian. Now \( \langle N_{-\phi} \rangle \) is an elementary abelian \( r \)-generated group, so \( |\langle N_{-\phi} \rangle| \leq p^r \). Thus, by Lemma 3, we have \( N \leq Z_{2r+1}(G) \) and deduce that \( G \) has nilpotency class bounded only in terms of \( i \) and \( r \). Since \( G = \langle G_{-\phi} \rangle \) is \( r \)-generated, we conclude that \( r(G) \) is \((i, r)\)-bounded as well. Therefore \( N \) is \( m_i \)-generated for some \((i, r)\)-bounded number \( m_i \). This concludes the proof. \( \square \)

Proposition 1 Under Hypothesis 1 the rank of \( G \) is \( r \)-bounded.

Proof Let \( s(r_0) \) be as in Lemma 5 and \( l(r) \) as in Lemma 4. Take \( N = \gamma_{2\lambda+1}(G) \), where \( \lambda = \lambda(r) = s(l(r)) \). Let \( d \) be the minimal number such that \( N \) is \( d \)-generated. Lemma 7 tells us that \( d \) is an \( r \)-bounded integer and \( N \) is powerful by Lemma 6. It follows from [1, Theorem 2.9] that \( r(N) \leq d \), and so the rank of \( N \) is \( r \)-bounded. Since the nilpotency class of \( G/N \) is \( r \)-bounded (recall that \( \lambda \) depends only on \( r \)) and \( G = \langle G_{-\phi} \rangle \) is \( r \)-generated, we conclude that \( r(G/N) \) is \( r \)-bounded as well. Now \( r(G) \leq r(G/N) + r(N) \) and the result follows. \( \square \)

3 Main results

Throughout this section the Feit–Thompson Theorem [2] is used without explicit references and \( p \) stands for a fixed odd prime. Given a finite soluble group \( G \), we denote by \( r_p(G) \) and \( l_p(G) \) the rank of a Sylow \( p \)-subgroup and the \( p \)-length of \( G \), respectively. Recall that \( l_p(G) \) is by definition the number of \( p \)-factors (that is, factors that are \( p \)-groups) of the lower \( p \)-series of \( G \) given by:

\[
1 \leq O_{p'}(G) \leq O_{p', p}(G) \leq O_{p', p, p'}(G) \leq \cdots .
\]

We aim to establish the following generalisation of Proposition 1.

Theorem 3 Let \( G \) be a group of odd order admitting an involutory automorphism \( \phi \) such that \( G = [G, \phi] \). Let \( r \) be a positive integer and assume that any subgroup generated by a subset of \( G_{-\phi} \) can be generated by \( r \) elements, then \( r_p(G) \) is \( r \)-bounded.

We start with an extension of Lemma 3.
Lemma 8  Let $G$ be a group of odd order admitting an involutory automorphism $\phi$ such that $G = [G, \phi]$. Let $M$ be a $\phi$-invariant normal subgroup of $G$ and assume that $|M\phi| \leq p^n$, for some nonnegative integer $n$. Then $M \leq Z_{2n+1}(O_p(G))$.

**Proof**  The proof can be reproduced word-by-word following that of Lemma 3. We argue by induction on $n$, being Lemma 1(iv) the case $n = 0$. Let $n \geq 1$. If $M \not\subseteq Z(O_p(G))$, then by Lemma 1(iv) we have $N\phi \neq 1$, where $N = M \cap Z_2(O_p(G))$. This implies that $|(M/N)\phi| < |M\phi|$. Thus we can pass to the quotient $G/N$ and use the inductive hypothesis. The result follows. $\square$

For the sake of simplicity we fix the following hypothesis that we will use in the next arguments.

**Hypothesis 2**  Let $r$ be a positive integer and $G$ a group of odd order admitting an involutory automorphism $\phi$ such that $G = [G, \phi]$. Assume that any subgroup generated by a subset of $G\phi$ can be generated by $r$ elements.

As usual, we denote by $F(G)$ the Fitting subgroup of a group $G$. Write $F_0(G) = 1$, $F_1(G) = F(G)$ and let $F_{i+1}(G)$ be the inverse image of $F(G/F_i(G))$. If $G$ is soluble, then the least number $h$ such that $F_h(G) = G$ is called the Fitting height of $G$.

One key step forward to the proof of Theorem 3 consists in showing that there exists an $r$-bounded number $f$ such that the $f$th term of the derived series of $G$ is nilpotent. For our purpose we will require the following result which is an immediate corollary of Hartley-Isaacs Theorem B in [4].

**Proposition 2**  Let $H$ be a finite group of odd order admitting an involutory automorphism $\phi$ such that $H = [H, \phi]$. Let $k$ be a field with characteristic different from 2 and $V$ a simple $k(\phi)H$-module. Suppose that $\dim V \phi = r$. There exists an $r$-bounded number $\delta = \delta(r)$ such that $\dim V \leq \delta$.

In the proof of the next proposition we will use the well-known theorem of Zassenhaus (see [11, Satz 7] or [8, Theorem 3.23]) stating that for any $n \geq 1$ there exists a number $j = j(n)$, depending only on $n$, such that, whenever $k$ is a field, the derived length of any soluble subgroup of $GL(n, k)$ is at most $j$.

**Proposition 3**  Assume Hypothesis 2. There exists a number $f = f(r)$, depending only on $r$, such that the $f$th term $G^{(f)}$ of the derived series of $G$ is nilpotent.

**Proof**  Let $\delta = \delta(r)$ be as in Proposition 2 and $f = j(\delta)$ the number given by the Zassenhaus theorem.

Suppose that the proposition is false and let $G$ be a group of minimal possible order for which Hypothesis 2 holds while $G^{(f)}$ is not nilpotent. Then $G$ has a unique minimal $\phi$-invariant normal subgroup $M$. Indeed, suppose that $G$ has two minimal $\phi$-invariant normal subgroups, say $M_1$ and $M_2$. Then $M_1 \cap M_2 = 1$, being both elementary abelian $p$-groups for some prime $p$. Since $|G/M_1| < |G|$, the minimality of $G$ implies that $(G/M_1)^{(f)}$ is nilpotent. For a symmetric argument $(G/M_2)^{(f)}$ is nilpotent too. This yields a contradiction since $G^{(f)}$ can be embedded into a subgroup of $G/M_1 \times$
$G/M_2$ which is nilpotent, being isomorphic to the direct product of $(G/M_1)^{(f)}$ and $(G/M_2)^{(f)}$.

We claim that $M = C_G(M)$. Since $M$ is a $p$-subgroup, for some prime $p$ and it is unique, the Fitting subgroup $F = F(G)$ is a $p$-subgroup too. If $\Phi(F)$ is nontrivial, then we immediately get a contradiction because $F(\Phi(F)) = F/\Phi(F)$ and, again by the minimality of $G$, we know that $(G/\Phi(F))^{(f)}$ is nilpotent, so in particular $G^{(f)} \leq F$.

Assume now that $\Phi(F) = 1$ and so $F$ is elementary abelian. If $M = F$, then $M = C_G(M)$, since the Fitting subgroup of a soluble group contains its own centralizer (see, for example, [3, Theorem 1.3, Chap. 6]). Thus we can assume that $M < F$. By hypotheses, on one hand, we know that $G^{(f)} \leq F_2(G)$ (to clarify, for the minimality of $G$ the quotient $(G/F)^{(f)}$ is nilpotent, so it is contained in $F(G/F)$) and, on the other hand, that $(G/M)^{(f)}$ is nilpotent (again by the minimality of $G$). Now let $T$ be a $\phi$-invariant Hall $p'$-subgroup of $G^{(f)}$. It follows that both $FT$ and $MT$ are $\phi$-invariant normal subgroups of $G$. Indeed, $FT/F$ is normal in $G/F$, since $(G/F)^{(f)}$ is nilpotent and, similarly, $MT/M$ is normal in $G/M$, being $(G/M)^{(f)}$ nilpotent as well.

Suppose first that $C_F(T) \neq 1$. Note that $C_F(T) = Z(FT)$, since $F$ is abelian. Thus $C_F(T)$ is a $\phi$-invariant normal subgroup of $G$, because $FT$ is normal and $\phi$-invariant. Hence $M \leq C_F(T)$. This implies that $T$ centralizes $M$ and so $MT = T \times M$. Recall that $T \leq F_2(G)$ and $T \cap F = 1$. It follows that $T$ is nilpotent. Then $T \times M$ is normal nilpotent and $T \leq F$, a contradiction.

Thus, $C_F(T) = 1$. On the other hand, we see that $[F, T] \leq M$, since the nilpotent $p'$-subgroup $MT/M$ and the $p$-subgroup $F/M$ are both contained in $F(G/M)$ and commute, being $F(G/M)$ nilpotent. Now we have $M < F$ and $F = [F, T] \times C_F(T)$, so it should be $C_F(T) \neq 1$, a contradiction. Thus $M = C_G(M)$, as claimed above.

Then $G/M$ acts faithfully and irreducibly on $M$. Moreover $\langle M_{-\phi} \rangle$ is $r$-generated and elementary abelian, so $|\langle M_{-\phi} \rangle| \leq p^r$. Now we can view $M$ as a $G/M\langle \phi \rangle$-module over the field with $p$ elements. By Proposition 2 we have $\dim(M) \leq \delta(r)$. Applying the theorem of Zassenhaus the derived length of $G/M$ is at most $f = j(\delta(r))$. Then $G^{(f)} \leq F$, which concludes the proof.

As a by-product of the previous result we obtain a bound for the $p$-length of $G$.

Corollary 1 Assume Hypothesis 2. Then $l_p(G)$ is $r$-bounded, for any $p \in \pi(G)$.

Proof By Proposition 3 we know that $G^{(f)}$ is nilpotent for some $r$-bounded number $f$. This implies that the Fitting height $h(G) \leq f$. The result easily follows since it can be shown, by induction on the Fitting height $h(K)$, that $l_p(K) \leq h(K)$ for any finite soluble group $K$ and for any prime $p \in \pi(K)$.

The next result will be useful for a reduction argument inside the proof of Theorem 3.

Lemma 9 Let $G$ be a group of odd order admitting an involutory automorphism $\phi$. Assume that $G = PB$, where $P$ is a $\phi$-invariant normal elementary abelian $p$-subgroup and $B$ is a cyclic subgroup such that $B = B_{-\phi}$. If $r(P_{-\phi}) = r$, then the rank of $[P, B]$ is at most $2r$. 
Proof Let \( B = \langle b \rangle \), where \( b \) is a generator of \( B \). Let \( C = P_{\phi} \) and \( C_0 = C \cap C^b \). Then it follows from Lemma 1(i) that

\[
[P : C_0] \leq [P : C][P : C^b] \leq p^{2r},
\]

since \( r(P_{-\phi}) = r \). We claim that \( C_0 \leq C_G(b) \). Indeed, choose \( x \in C \) such that \( x^b \in C \). Then, we have \( x^b = (x^b)^\phi = x^{b^{-1}} \) and so \( x \) commutes with \( b^2 \). Since \( b \) has odd order, it follows that \( C_0 \leq C_G(b) \), as claimed. Thus \( C_0 \leq Z(G) \). Choose now \( a_1, \ldots, a_{2r} \) elements that generate \( P \) modulo \( C_0 \). By using linearity in \( P \) and the fact that \( C_0 \) is central in \( G \), we deduce that \( [P, b] \) is generated by \( [a_1, b], \ldots, [a_{2r}, b] \). Hence the result.

We are ready to embark on the proof of Theorem 3.

Proof of Theorem 3 Recall that \( G \) is a group satisfying Hypothesis 2 and we want to show that \( r_p(G) \) is \( r \)-bounded for any fixed prime \( p \in \pi(G) \).

First, we show that \( G \) is generated by \( r \)-boundedly many elements from \( G_{-\phi} \). If \( G \) is a \( p \)-group, then the claim follows from the Burnside Basis Theorem since \( G = \langle G_{-\phi} \rangle \) is \( r \)-generated. In the case where \( G \) is nilpotent, we have \( [G, \phi] = [P_1, \phi] \times \cdots \times [P_s, \phi] \), where \( \{P_1, \ldots, P_s\} \) are the Sylow subgroups of \( G \), so the result easily follows from the case of \( p \)-groups. Assume now that \( G \) is not nilpotent. Let \( h = h(G) \geq 2 \). Since we know from the proof of Corollary 1 that \( h \) is \( r \)-bounded, it is sufficient to show that \( G \) is generated by \( (h, r) \)-boundedly many elements from \( G_{-\phi} \). We argue by induction on \( h \). Let \( F = F(G) \). By induction there are boundedly many elements \( a_1, \ldots, a_d \in G_{-\phi} \) such that \( G = F\langle a_1, \ldots, a_d \rangle \). Let \( D = \langle F_{-\phi}, a_1, \ldots, a_d \rangle \). Note that \( D \) has an \( r \)-bounded number of generators from \( G_{-\phi} \). Let \( N \) be the normal closure of \( \langle F_{-\phi} \rangle \) in \( G \). Then \( N \) is precisely \( \langle F_{-\phi} \rangle^D \) because \( F \) normalizes \( \langle F_{-\phi} \rangle \) by Lemma 1(v). Thus \( N \leq D \). Recall that by Lemma 1(i) we have \( F = F_0F_{-\phi} \). Hence the image of \( F \) in \( G/N \) is contained in \( (G/N)_{-\phi} \). Hence, it is central by Lemma 1(iv). Since \( G = FD \), it follows that \( D/N \) becomes normal in \( G/N \) and, therefore, \( D \) is normal in \( G \) (because \( N \leq D \)). Now \( \phi \) acts trivially on the quotient \( G/D \), that is \( \langle G, \phi \rangle \leq D \). Since \( G = \langle G, \phi \rangle \), we have \( G = D \). This concludes the proof that \( G \) can be generated by \( r \)-boundedly many elements from \( G_{-\phi} \).

If \( G \) is a \( p \)-group, then the theorem follows immediately from Proposition 1. Assume that \( G \) is not a \( p \)-group and use induction on \( l = l_p(G) \) that is \( r \)-bounded by Corollary 1. So it is sufficient to show that \( r_p(G) \) is \( (l, r) \)-bounded. By induction assume that there exists \( r_1 \), depending only on \( l \) and \( r \), such that \( r_p(K) \leq r_1 \) for any \( \phi \)-invariant quotient \( K \) of \( G \) having \( l_p(K) \) at most \( l - 1 \).

Since \( l = l_p(G/O_{p'}(G)) \), we can assume that \( O_{p'}(G) = 1 \). Take \( P = O_p(G) \).

Note that

\[
r_p(G) \leq r(P) + r_P(G/P).
\]

Since \( l_p(G/P, G) \leq l - 1 \), by induction the rank \( r_p(G/[P, G]) \leq r_1 \). Then it is sufficient to bound the rank of \( P \).

Let us show first that \( P \) has an \( r \)-bounded number of generators. Passing to the quotient \( G/\Phi(P) \), we can assume that \( P \) is elementary abelian. As showed above, we
know that $G$ can be generated by $t = t(r)$ elements from $G_{-\phi}$, say $d_1, \ldots, d_t$. Note that $[P, G] = [P, d_1][P, d_2] \cdots [P, d_t]$. In view of Lemma 9 each $[P, d_i]$ has rank at most $2r$. Therefore the rank of the image of $P$ in $G/\Phi(P)$ is at most $2rt$ and by induction on $l$, $r_p(G/P)$ is $r$-bounded, so $P$ has an $r$-bounded number of generators, as claimed.

Next, we claim that for any $i \geq 2$ there exists a number $m_i = m_i(i, r)$, depending only on $i$ and $r$, such that $V = \gamma_i(P)$ has $m_i$-bounded number of generators. We can pass to the quotient $G/\Phi(V)$ and assume that $V$ is elementary abelian. Now $\langle V_{-\phi} \rangle$ is an elementary abelian $r$-generated group, so $\{(V_{-\phi}) \leq p^r$. Thus, by Lemma 8, we have $V \leq Z_{2r+1}(P)$ and deduce that the nilpotency class of $P/\Phi(V)$ is bounded only in terms of $i$ and $r$. Since $P$ has an $r$-bounded number of generators, we conclude that $r(P/\Phi(V))$ is $(i, r)$-bounded as well. Therefore $V$ is $m_i$-generated for some $(i, r)$-bounded number $m_i$, as claimed.

Let $s(r_0)$ be as in Lemma 5 and let $l(r)$ be as in Lemma 4. Take $M = \gamma_{2\lambda+1}(P)$, where $\lambda = s(l(r))$. We want to prove that $M$ is powerful. In order to show that $M' \leq M^p$, we assume that $M$ is of exponent $p$ and prove that $M$ is abelian. Note that the subgroup $\langle M_{-\phi} \rangle$ is of exponent $p$. By Lemma 4 the rank of $\langle M_{-\phi} \rangle$ is at most $l(r)$. It follows from Lemma 5 that $|M_{-\phi}| \leq p^{s(l(r))} = p^b$. Now Lemma 8 yields that $M \leq Z_{2\lambda+1}(P)$. Since $[\gamma_i(P), Z_i(P)] = 1$, for any positive integer $i$, we conclude that $M$ is abelian, as required.

Let now $d_0$ be the minimal number such that $M$ is $d_0$-generated. It was shown above that $d_0$ is an $r$-bounded integer. Since $M$ is powerful, it follows from [1, Theorem 2.9] that $r(M) \leq d_0$, and so the rank of $M$ is $r$-bounded. Since the nilpotency class of $P/M$ is $r$-bounded and $P$ has an $r$-bounded number of generators, we conclude that $r(P/M)$ is $r$-bounded as well. Now $r(P) \leq r(P/M) + r(M)$ and the result follows.

It is now easy to give the proof of our main result, Theorem 2, which states that if $G$ is a group satisfying Hypothesis 2, then the rank of $G$ is $r$-bounded.

**Proof of Theorem 2** Without loss of generality we can assume that $G = [G, \phi]$. By a result of Kovács [7] for any soluble group $H$ we have $r(H) \leq \max\{r_p(H) \mid p \in \pi(H)\} + 1$. Therefore it is enough to check that $r_p(G)$ is bounded in terms of $r$ only for any $p \in \pi(G)$. This is immediate from Theorem 3.

**References**

1. Dixon, J.D., du Sautoy, M.P.F., Mann, A., Segal, D.: Analytic Pro-$p$ Groups. Cambridge University Press, Cambridge (1991)
2. Feit, W., Thompson, J.: Solvability of groups of odd order. Pac. J. Math. **13**, 773–1029 (1963)
3. Gorenstein, D.: Finite Groups. Chelsea Publishing Company, New York (1980)
4. Hartley, B., Isaacs, I.M.: On characters and fixed points of coprime operator groups. J. Algebra **131**, 342–358 (1990)
5. Khukhro, E.I.: Groups with an automorphism of prime order that is almost regular in the sense of rank. J. Lond. Math. Soc. **77**(2), 130–148 (2008)
6. Khukhro, E.I., Mazurov, V.D.: Finite groups with an automorphism of prime order whose centralizer has small rank. J. Algebra **301**, 474–492 (2006)
7. Kóvacs, L.G.: On finite soluble groups. Math. Z. **103**, 37–39 (1967)
8. Robinson, D.J.S.: Finiteness Conditions and Generalized Soluble Groups, Part 1. Springer-Verlag, Berlin (1972)
9. Robinson, D.J.S.: A Course in the Theory of Groups. Springer-Verlag, New York (1996)
10. Shumyatsky, P.: Involutory automorphisms of finite groups and their centralizers. Arch. Math. 71, 425–432 (1998)
11. Zassenhaus, H.: Beweis eines Satzes über diskrete Gruppen. Abh. Math. Sem. Univ. Hamburg 12, 289–312 (1938)

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