Characteristics of solutions of nonlinear neutral integro-differential equation via Chandrasekhar integral

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Abstract
In this paper, we shall study the existence of at least one continuous solution for a nonlinear neutral differential equation via Chandrasekhar integral. Next, continuous dependence of the solution of that equation on the delay functions will be studied. Also, we use Kransnoselskii theorem to prove the existence of solutions and estimate upper and lower bounds for solutions defined in unbounded interval. Some particular cases and remarks are presented to illustrate our results.

Keywords: Neutral differential equation, Kransnoselskii theorem, continuous dependence, estimate upper and lower bounds for solutions.

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1. Introduction
Neutral differential equations arise in many areas of applied mathematics and for this reason these equations have been investigated extensively in the last decades.

The dynamical systems which depend on present and past states are often described by neutral delay differential equations. Practical examples of neutral delay differential systems include biological models of single species growth [14], processes including steam or water pipes, heat exchanges [11], population ecology [12] and other engineering systems [11].

The study of neutral differential equations has grown rapidly. This is largely due to the fact that often the qualitative behavior of solutions of neutral differential equations is very different from those of nonneutral equations. For example, [10, 15, 16, 19]. It has been shown in the literature that even when all the characteristic roots of a neutral differential equation have negative real parts, it is still possible for the equation to have unbounded solutions. Such a behavior is impossible for nonneutral equations. Banaś et al. [1] proved the existence and asymptotic behaviour of solutions of the differential equation with a deviating argument of neutral type

\[ y'(t) = f(t, y(y(t)), y'(h(t))), t \in \mathbb{R}_+ = [0, \infty) \]  \hspace{1cm} (1.1)

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These equations can also support the existence of positive solutions approaching zero at infinity. Moreover, their purpose in this paper is to concern with the solution of nonlinear neutral differential equations with unbounded delay using fixed point theory in a Banach space \((X, ||\cdot||)\), \(f(t), t \in J\) is a bounded linear operator. Hussain et al. [9] presented some fixed point and coupled fixed point results in the generalized setting. Meanwhile, their purpose in this paper is to concern with the solution of nonlinear neutral differential equations with bounded below and above by positive functions for the first-order nonlinear neutral differential equations.

\[
\frac{d}{dt} [x(t) - F(t, x_t)] = A \left[ x(t) + \int_0^t f(t - s)x(s)ds \right] + G \left[ t, x_{\sigma(t, x_t)} \right],
\]

\[t \in J = [0, a], t \neq t_i, i = 1, 2, \ldots, n,\]

\[x_0 = \phi \in \mathcal{B}, \Delta x(t_i) = I_i(x_{t_i}), i = 1, 2, \ldots, n,\]

where \(A\) is the infinitesimal generator of a compact, analytic resolvent operator \(R(t), t > 0\) in a Banach space \((X, ||\cdot||)\), \(f(t), t \in J\) is a bounded linear operator.

Chang et al. [4] considered the existence of mild solutions for a class of first-order impulsive neutral integro-differential equations with state-dependent delay such as

\[
\frac{d}{dt} [x(t) - \varphi(t)] = a(t)x(t) + b(t)g(x(t - \tau(t))) + c(t)x'(t - \tau(t))
\]

with unbounded delay using fixed point theory in \(\mathcal{F}\)-metric space, where \(a(t), b(t)\) are continuous, \(c(t)\) is continuously differentiable and \(\tau(t) > 0\) for all \(t \in \mathbb{R}\) and is twice continuously differentiable. The paper [6] is concerned with the existence of a positive solution of the neutral differential equation of the form

\[
\frac{d}{dt} [x(t) - a(t)x(t - \tau)] = p(t)f(x(t - \sigma)), \quad t \geq t_0
\]

where \(\tau > 0, \sigma \geq 0, a \in C([t_0, \infty), (0, \infty)), p \in C(\mathbb{R}, (0, \infty)), f \in C(\mathbb{R}, \mathbb{R}), f\) is nondecreasing function, and \(xf(x) > 0, x \neq 0\). [6] contains some sufficient conditions for the existence of positive solutions which are bounded below and above by positive functions for the first-order nonlinear neutral differential equations. These equations can also support the existence of positive solutions approaching zero at infinity.

Our paper is concerned with the existence of the nonlinear neutral integro-differential equation of the form

\[
\begin{align*}
\left\{ \frac{d}{dt} \left[ x(t) - a_1 \int_{t_0}^t \int_{t_0}^{t+s} g(s, x(\phi_3(s)))ds \right] \right\}_{t=t_0} &= a_1 f_1(t, x(\phi_1(t))), \\
\left[ x(t) - a_2 \int_{t_0}^t \int_{t_0}^{t+s} g(s, x(\phi_3(s)))ds \right]_{t=t_0} &= 0,
\end{align*}
\]

where \(t \in I = [t_0, T], a_1, a_2 \in \mathbb{R}\).

Motivated by this works, we shall prove the existence of at least one positive solution for the nonlinear neutral integro-differential equation via Chandrasekhar integral (1.2) by applying Schauder fixed point theorem, prove the existence of unique solution and prove that this solution is continuously depending on the delay functions. Our next result is concerned with (1.2) on the real half-axis \(\mathbb{R}_+\) by applying Kransnoselskii theorem to prove an existence result which give sufficient conditions for the existence of a positive solution which is bounded by two positive functions.

By a solution of the nonlinear neutral integro-differential equation (1.2) we mean a function \(x \in C[I, \mathbb{R}]\) such that

(i) the function \(t \to x(t) - a_1 \int_{t_0}^t \int_{t_0}^{t+s} g(s, x(\phi_3(s)))ds\) is continuously differentiable on \(I\);
(ii) \(x\) satisfies the equations in (1.2).
**Theorem 1.1** (Schauder fixed point theorem [5]). Let $Q$ be a nonempty, convex, compact subset of a Banach space $X$, and $T : Q \to Q$ be a continuous map. Then $T$ has at least one fixed point in $Q$.

**Theorem 1.2** (Krasnoselskii’s fixed point theorem [6]). Let $X$ be a Banach space, let $\Omega$ be a bounded closed convex subset of $X$ and $S_1, S_2$ be maps of $\Omega$ into $X$ such that $S_1x + S_2y \in \Omega$ for every pair $x, y \in \Omega$. If $S_1$ is contractive and $S_2$ is completely continuous, then the equation $S_1x + S_2x = x$ has a solution in $\Omega$.

### 2. Existence results on bounded interval

(i) The functions $\phi_i : I \to \mathbb{R}, i = 1, 2, 3$ are continuous.

(ii) $f_1, g : I \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfy Carathéodory condition, i.e., $f_1, g$ are measurable functions in $t$ for any $t \in \mathbb{R}_+$ and continuous in $r$ for almost all $t \in I$. There exist two nonnegative constants $b, b_1$ and two functions $t \to \alpha(t), t \to m(t)$, such that

$$|g(t, r)| \leq \alpha(t) + b_1 |r|, \forall (t, r) \in I \times \mathbb{R}_+, \quad \text{and} \quad |f_1(t, r)| \leq m(t) + b |r|, \forall (t, r) \in I \times \mathbb{R}_+,$$

where $\alpha(.)$, $m(.) \in L^1(I)$ and $\sup_{t \in I} |\alpha(t)| = k_1$.

(iii) $f_2 : I \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and there exists three positive constants $l_0, l_1$ and $l_2$ satisfying:

$$|f_2(t_1, r_1, \eta_1) - f_2(t_2, r_2, \eta_2)| \leq l_0 |t_1 - t_2| + l_1 |r_1 - r_2| + l_2 |\eta_1 - \eta_2|,$$

$$\forall (t_1, r_1, \eta_1), (t_2, r_2, \eta_2) \in I \times \mathbb{R}_+ \times \mathbb{R}_+.$$

(iv) $k = \sup_{t \in I} \{f_2(t, 0, 0), f_2(t, 0, 0)\} \int_{l_0}^t m(s)ds \leq M$.

**Theorem 2.1.** Let assumptions (i)-(iv) be satisfied. If $1 > |a_1| bT + |a_2| b_1 l_2 + |a_2| l_1$, then there exists at least one continuous positive solution for the nonlinear neutral integro-differential equation via Chandraseker integral (1.2).

**Proof.** From assumption (iii), we obtain

$$|f_2(t, r, \eta) - f_2(t, 0, 0)| \leq l_1 |r| + l_2 |\eta|,$$

$$|f_2(t, r, \eta)| \leq |f_2(t, 0, 0)| + l_1 |r| + l_2 |\eta| \leq k + l_1 |r| + l_2 |\eta|.$$

Let $C[I, \mathbb{R}_+]$ be the set of all continuous function with the norm $||x|| = \sup_I |x(t)|$. Now define a closed, bounded, and convex subset $\Omega$ of $C[I, \mathbb{R}_+]$ as follows:

$$\Omega = \{x : x \in C[I, \mathbb{R}_+], ||x|| \leq \tau, \tau \geq 0\}.$$

Let $A$ be an operator defined on $\Omega$ by the formula

$$Ax(t) = a_1 \int_{l_0}^t f_1(s, x(\phi_1(s)))ds + a_2 f_2 \left( t, x(\phi_2(t)) \right) \int_{l_0}^t \frac{t - s}{t + s} g(s, x(\phi_3(s)))ds.$$

We shall prove that for each $x \in \Omega \Rightarrow Ax \in \Omega$. First we make an estimate for $r$:

$$|Ax(t)| = \left| a_1 \int_{l_0}^t f_1(s, x(\phi_1(s)))ds + a_2 f_2 \left( t, x(\phi_2(t)) \right) \int_{l_0}^t \frac{t - s}{t + s} g(s, x(\phi_3(s)))ds \right|$$

$$\leq |a_1| \int_{l_0}^T |m(s) + b |x(\phi_1(s))||ds + |a_2| k + |a_2| l_1 |x(\phi_2(t))| + |a_2| l_2 \int_{l_0}^T \frac{t - s}{t + s} g(s, x(\phi_3(s)))ds |$$
\[
\begin{align*}
\leq & \left| a_1 \right| \int_{t_0}^{T} m(s)ds + \left| a_1 \right| b \int_{t_0}^{T} |\varphi(\phi_1(s))| ds + \left| a_2 \right| k + \left| a_2 \right| |t_1| |\varphi(t)| \\
& + \left| a_2 \right| l_2 \alpha(t) + a_2 l_2 b_1 \int_{t_0}^{t_1} |\varphi(s)| ds \\
\leq & \left| a_1 \right| M + \left| a_1 \right| b \int_{t_0}^{T} |\varphi(s)| ds + \left| a_2 \right| k + \left| a_2 \right| l_1 r + \left| a_2 \right| l_2 k_1 + \left| a_2 \right| b_1 l_2 r \int_{t_0}^{1} ds \\
\leq & \left| a_1 \right| M + \left| a_1 \right| b r T + \left| a_2 \right| k + \left| a_2 \right| l_1 r + \left| a_2 \right| l_2 k_1 + \left| a_2 \right| b_1 l_2 r \\
\leq & \left| a_1 \right| M + \left| a_2 \right| k + \left| a_2 \right| l_2 k_1 + a_2 b_1 l_2 r \\
\leq & r.
\end{align*}
\]

From the last estimate we can deduce that

\[
r = \frac{\left| a_1 \right| M + \left| a_2 \right| k + \left| a_2 \right| l_2 k_1}{1 - \left| a_1 \right| b r T - \left| a_2 \right| b_1 l_2 - \left| a_2 \right| l_1}.
\]

This means that \( r \in \Omega \Rightarrow A r \in \Omega \). Now, we shall show that \( A \Omega \) is relatively compact. It is sufficient to show that \( A \Omega \) is uniformly bounded and equicontinuous by the Arzelà-Ascoli theorem. The uniform boundedness follows from the definition of \( \Omega \). For \( r \in \Omega \) and \( t_1, t_2 \in I, t_2 > t_1 \) (without loss of generality) we get

\[
\begin{align*}
|Ar(t_2) - Ar(t_1)| &= \left| a_1 \int_{t_0}^{t_2} f_1(s, \varphi(\phi_1(s))) ds + a_2 f_2 \left( t_2, \varphi(\phi_2(t_2)) \right) \int_{t_0}^{t_2} t_2 g(s, \varphi(\phi_3(s))) ds \right| \\
& \quad - \left| a_1 \int_{t_0}^{t_1} f_1(s, \varphi(\phi_1(s))) ds - a_2 f_2 \left( t_1, \varphi(\phi_2(t_1)) \right) \int_{t_0}^{t_1} t_1 g(s, \varphi(\phi_3(s))) ds \right| \\
& \leq \left| a_1 \int_{t_0}^{t_2} \left[ m(s) + b |\varphi(\phi_1(s))| \right] ds + |a_2| \left[ t_2 |t_2 - t_1| + t_1 |\varphi(\phi_2(t_2)) - \varphi(\phi_2(t_1))| \right] \\
& \quad + t_2 \left[ \int_{t_0}^{t_2} t_2 g(s, \varphi(\phi_3(s))) ds - \int_{t_0}^{t_1} t_1 g(s, \varphi(\phi_3(s))) ds \right] \\
& \leq \left| a_1 \int_{t_0}^{t_2} m(s) ds + |a_1| b |t_2 - t_1| + |a_2| \left[ t_0 |t_2 - t_1| + t_1 |\varphi(\phi_2(t_2)) - \varphi(\phi_2(t_1))| \right] \\
& \quad + t_2 \left[ \int_{t_0}^{t_2} t_2 g(s, \varphi(\phi_3(s))) ds - \int_{t_0}^{t_1} t_1 g(s, \varphi(\phi_3(s))) ds \right] \\
& \leq \left| a_1 \int_{t_0}^{t_2} m(s) ds + |a_1| b |t_2 - t_1| + |a_2| \left[ t_0 |t_2 - t_1| + t_1 |\varphi(\phi_2(t_2)) - \varphi(\phi_2(t_1))| \right] \\
& \quad + t_2 |t_2 - t_1| \int_{t_0}^{t_2} \frac{1}{t_1 + s} |g(s, \varphi(\phi_3(s)))| ds \right| \\
& \leq \left| a_1 \int_{t_0}^{t_2} m(s) ds + |a_1| b |t_2 - t_1| + |a_2| \left[ t_0 |t_2 - t_1| + t_1 |\varphi(\phi_2(t_2)) - \varphi(\phi_2(t_1))| \right] \\
& \quad + t_2 |t_2 - t_1| \int_{t_0}^{t_2} \frac{1}{t_1 + s} \left[ \varphi(s) + b |\varphi(\phi_3(s))| \right] ds \right|
\end{align*}
\]
\[ \leq |a_1| \int_{t_1}^{t_2} m(s)\,ds + |a_1| b r |t_2 - t_1| + |a_2| l_0 |t_2 - t_1| + |a_2| l_2 |r(t_2) - r(t_1)| \]
\[ + |a_2| l_2 \left( \frac{k_1}{t_0} + b r \right) |t_2 - t_1|. \]

Then
\[ \| (A_1)(t_2) - (A_1)(t_1) \| \to 0 \text{ as } t_2 \to t_1. \]

This means that the functions from \( A\Omega \) are equi-continuous on \( I \). Then by Arzela-Ascoli Theorem \([5]\) the closure of \( A\Omega \) is compact. It is clear that the set \( \Omega \) is nonempty, bounded, closed and convex. Assumptions (ii) and (iii) imply that \( A : \Omega \to C[I, \mathbb{R}_+] \) is a continuous operator. Since all conditions of the Schauder fixed-point theorem hold, then \( A \) has a fixed point in \( \Omega \).

**Remark 2.2.** In general, we can study the existence of the nonlinear neutral integro-differential equation of the form

\[
\begin{aligned}
\frac{d}{dt} \left[ r(t) - a_2 f_2 \left( t, r(\phi_2(t)) \right), \int_{t_0}^{t} k(t, s) g(s, r(\phi_3(s)))\,ds \right] &= a_1 f_1 \left( t, r(\phi_1(t)) \right), \\
\left[ r(t) - a_2 f_2 \left( t, \phi_2(t) \right), \int_{t_0}^{t} k(t, s) g(s, r(\phi_3(s)))\,ds \right]_{t=t_0} &= 0,
\end{aligned}
\]

(2.1)

where \( t \in I = [t_0, T] \), \( a_1, a_2 \in \mathbb{R} \). Assume that \( (V) \) \( k : I \times I \to \mathbb{R}_+ \) is a continuous function and the operator \( K \) defined by

\[ (K\eta)(t) = \int_{t_0}^{t} k(t, s)\eta(s)\,ds, \quad t \in I \]

and

\[ \int_{t_0}^{1} |k(t, s)|\alpha(s)\,ds < \Lambda. \]

By a similar way, we can deduce that

\[ r = \frac{|a_1| M + |a_2| k + |a_2| l_2 \Lambda}{1 - |a_1| b T - |a_2| b_1 l_2 \|K\| - |a_2| l_1}. \]

Moreover, we easily can prove the following result.

**Theorem 2.3.** Let assumptions (i)-(iv) and (V) be satisfied. If \( 1 > |a_1| b T + |a_2| b_1 l_2 \|K\| + |a_2| l_1 \), then there exists at least one continuous positive solution for the nonlinear neutral integro-differential equation (2.1).

2.1. Existence of unique solution

Let the functions \( f_1 \) and \( g \) satisfy the following assumption

(\( v \))

\[ |f_1(t, \xi) - f_1(t, \eta)| \leq L |\xi - \eta|, \quad |g(t, \xi) - g(t, \eta)| \leq L' |\xi - \eta|, \quad \forall (t, \xi), (t, \eta) \in I \times \mathbb{R}_+. \]

Let \( \phi(t) \) and \( \bar{\phi}(t) \) be two solutions of the nonlinear neutral integro-differential equations

\[
|\phi(t) - \bar{\phi}(t)| = \left| a_1 \int_{t_0}^{t} f_1(s, \phi(\phi_1(s)))\,ds + a_2 f_2 \left( t, \phi(\phi_2(t)) \right), \int_{t_0}^{t} \frac{1}{t + s} g(s, \phi(\phi_3(s)))\,ds \right|
\]
\[ - a_1 \int_{t_0}^{t} f_1(s, \bar{\phi}(\phi_1(s)))\,ds - a_2 f_2 \left( t, \bar{\phi}(\phi_2(t)) \right), \int_{t_0}^{t} \frac{1}{t + s} g(s, \bar{\phi}(\phi_3(s)))\,ds \right| \]
Let assumptions of Theorem 2.4 be satisfied. If \( \phi \in x \)
then the following result is proved.

Then

\[
\| \tilde{x}(t) - \tilde{y}(t) \| \leq |a_1| TL \| \tilde{x}(t) - \tilde{y}(t) \| + |a_2| |l_1| \| \tilde{x}(t) - \tilde{y}(t) \| + 2L' \| \tilde{x}(t) - \tilde{y}(t) \|.
\]

Since, \(|a_1| TL - |a_2| l_1 - |a_2| l_2 L' < 1\), then

\[
|\tilde{x}(t) - \tilde{y}(t)| = 0 \Rightarrow \tilde{x}(t) = \tilde{y}(t).
\]

Then the following result is proved.

**Theorem 2.4.** Let assumptions (i)-(v) be satisfied. If \(|a_1| TL - |a_2| l_1 - |a_2| l_2 L' < 1\), then there exists unique continuous solution for the nonlinear neutral differential equation via Chandrasekhar integral (1.2).

**2.2. Continuous dependence of solutions on delay functions**

Here, we study the continuous dependence of solutions of the nonlinear neutral integro-differential equation via Chandrasekhar integral (1.2) on the functions \( \phi_i, i = 1, 2, 3 \).

**Definition 2.5.** The solutions \( \tilde{x} \in C(I, R_+) \) of (1.2) are continuously dependent on the function \( \phi_i \), if \( \forall \epsilon > 0, \exists \delta > 0 \) such that \( \| \phi_i - \tilde{\phi}_i \| \leq \epsilon \) implies that \( \| x - \tilde{x} \| \leq \delta \), where

\[
\tilde{x}(t) = a_1 \int_{t_0}^{t} f_1(s, \tilde{x}(\tilde{\phi}_i(s))) \, ds + a_2 \int_{t_0}^{t} f_2(t, \tilde{x}(\tilde{\phi}_i(s)), \tilde{x}(\tilde{\phi}_i(s))) \, ds.
\]

**Theorem 2.6.** Let assumptions of Theorem 2.4 be satisfied. If \(|a_1| LT + |a_2| l_1 + |a_2| l_2 L' < 1\), then the solution \( \tilde{x} \in C(I, R_+) \) of the nonlinear neutral integro-differential equation via Chandrasekhar integral (1.2) depends continuously on \( \phi_i \).

**Proof.** Let \( x \) and \( \tilde{x} \) be two solutions of the nonlinear neutral integro-differential equation via Chandrasekhar integral (1.2),

\[
| x(t) - \tilde{x}(t) | = | a_1 \int_{t_0}^{t} f_1(s, x(\phi_i(s))) \, ds + a_2 \int_{t_0}^{t} f_2(t, x(\phi_i(s)), x(\phi_i(s))) \, ds |
\]
Let assumptions of Theorem 2.4 be satisfied. If \( |a_2|L_1 + |a_2|L_2L' < 1 \), then the solution \( \bar{r} \in C(I, R_+) \) of (1.2) is continuously dependent on the function \( \phi_1 \).

As done above we can prove the continuous dependence of solutions of (1.2) on the functions \( \phi_2 \) and \( \phi_3 \).

**Theorem 2.7.** Let assumptions of Theorem 2.4 be satisfied. If \( |a_2|L_1 + |a_2|L_2L' < 1 \), then the solution \( r \in C(I, R_+) \) of the nonlinear neutral integro-differential equation via Chandrasekhar integral (1.2) depends continuously on \( \phi_2 \).

**Proof.** Let \( r \) and \( \bar{r} \) be two solutions of the nonlinear neutral differential equation via Chandrasekhar integral (1.2), where

\[
- a_1 \int_{t_0}^{t} f_1(s, \bar{r}(\phi_1(s))) \, ds - a_2 f_2 \left( t, \bar{r}(\phi_2(t)), \int_{t_0}^{t} \frac{t}{t + s} g(s, \bar{r}(\phi_3(s))) \, ds \right)
\]

\[
\leq |a_1| \int_{t_0}^{t} |f_1(s, r(\phi_1(s))) - f_1(s, \bar{r}(\phi_1(s)))| \, ds
\]

\[
+ |a_2| f_2 \left( t, \bar{r}(\phi_2(t)), \int_{t_0}^{t} \frac{t}{t + s} g(s, \bar{r}(\phi_3(s))) \, ds \right) - f_2 \left( t, \bar{r}(\phi_2(t)), \int_{t_0}^{t} \frac{t}{t + s} g(s, \bar{r}(\phi_3(s))) \, ds \right)
\]

\[
\leq |a_1| L \int_{t_0}^{t} \bar{r}(\phi_1(s)) - \bar{r}(\phi_1(s)) \, ds + |a_2| L \int_{t_0}^{t} \bar{r}(\phi_3(s)) - \bar{r}(\phi_3(s)) \, ds
\]

\[
+ |a_2| L_1 \int_{t_0}^{t} |x(\phi_2(t)) - \bar{x}(\phi_2(t))| + |a_2| L_2 L' \int_{t_0}^{t} |x(\phi_3(s)) - \bar{x}(\phi_3(s))| \, ds
\]

\[
\leq |a_1| |L| L \int_{t_0}^{t} \bar{r}(\phi_1(s)) - \bar{r}(\phi_1(s)) \, ds + |a_1| |L| L \int_{t_0}^{t} |x(\phi_3(s)) - \bar{x}(\phi_3(s))| \, ds
\]

\[
+ |a_2| L_1 \int_{t_0}^{t} |x(\phi_2(t)) - \bar{x}(\phi_2(t))| + |a_2| L_2 L' \int_{t_0}^{t} |x(\phi_3(s)) - \bar{x}(\phi_3(s))| \, ds
\]

\[
\leq |a_1| |L| L \int_{t_0}^{t} |r(\phi_1(s)) - \bar{r}(\phi_1(s))| + |a_2| L_1 |L| |r(\phi_2(t)) - \bar{r}(\phi_2(t))| + |a_2| L_2 L' |x(\phi_3(s)) - \bar{x}(\phi_3(s))|
\]

\[
\Rightarrow |a_1| |L| L \int_{t_0}^{t} |r(t) - \bar{r}(t)| + |a_2| L_1 |L| |r(t) - \bar{r}(t)| + |a_2| L_2 L' |x(t) - \bar{x}(t)|
\]

Then

\[
(1 - (|a_1| L + |a_2| L_1 + |a_2| L_2 L')) |r(t) - \bar{r}(t)| \leq |a_1| |L| L \epsilon,
\]

\[
||r - \bar{r}|| \leq \frac{|a_1| |L| L \epsilon}{1 - (|a_1| L + |a_2| L_1 + |a_2| L_2 L')} = \delta.
\]

Since \( |a_1| L + |a_2| L_1 + |a_2| L_2 L' < 1 \), then the solution \( r \in C(I, R_+) \) of (1.2) is continuously dependent on the function \( \phi_1 \).

\[\square\]
\[-a_1 \int_{t_0}^{t} f_1(s, \tilde{x}(\phi_1(s))) \, ds - a_2 \tilde{f}_2 \left( t, \tilde{x}(\tilde{\phi}_2(t)), \int_{t_0}^{1} \frac{t}{t+s} g(s, \tilde{x}(\phi_3(s))) \, ds \right) \leq |a_1| \int_{t_0}^{t} \left| f_1(s, \tilde{x}(\phi_1(s))) - f_1(s, \tilde{x}(\phi_1(s))) \right| \, ds
\]
\[+ |a_2| \left| \tilde{f}_2 \left( t, \tilde{x}(\phi_2(t)), \int_{t_0}^{1} \frac{t}{t+s} g(s, \tilde{x}(\phi_3(s))) \, ds \right) - \tilde{f}_2 \left( t, \tilde{x}(\tilde{\phi}_2(t)), \int_{t_0}^{1} \frac{t}{t+s} g(s, \tilde{x}(\phi_3(s))) \, ds \right) \right| \leq |a_1| L \int_{t_0}^{t} \left| \tilde{x}(\phi_1(s)) - \tilde{x}(\phi_1(s)) \right| \, ds
\]
\[+ |a_2| \left| \tilde{x}(\phi_2(t)) - \tilde{x}(\phi_2(t)) \right| + |a_2| L^\prime \int_{t_0}^{1} \left| \tilde{x}(\phi_3(s)) - \tilde{x}(\phi_3(s)) \right| \, ds
\]
\[\leq |a_1| |LT| |x(t) - \tilde{x}(t)| + |a_2| |L_e| + |a_2| |L_e| |x(t) - \tilde{x}(t)| + |a_2| L^\prime |x(t) - \tilde{x}(t)|
\]
\[\leq |a_2| |L_e| + |a_1| |LT| + |a_2| |L_e| |x(t) - \tilde{x}(t)|.\]

Then
\[
(1 - (|a_2| L^\prime + |a_2| |L_e| + |a_1| |LT|)) |x - \tilde{x}| \leq |a_2| |L_e|,
\]
\[|x - \tilde{x}| \leq \frac{|a_2| |L_e|}{1 - (|a_2| L^\prime + |a_2| |L_e| + |a_1| |LT|)} = \delta.
\]

Since \(|a_1| |LT| + |a_2| |L_e| + |a_2| L^\prime < 1\), then the solution \(x \in C(I, R_+)\) of (1.2) is continuously dependent on the function \(\phi_2\).

Similarly, we have the following result.

**Theorem 2.8.** Let assumptions of Theorem 2.4 be satisfied. If \(|a_2| L^\prime + |a_2| |L_e| + |a_1| |LT| < 1\), then the solution \(x \in C(I, R_+)\) of the nonlinear neutral integro-differential equation via Chandrasekhar integral (1.2) depends continuously on \(\phi_3\).

### 3. Estimating upper and lower bounds for solutions

In most literature, the existence of solutions which are bounded by constants is handled. In this section, we shall study the existence of positive solutions of the neutral differential equation (1.2) on the interval \(I = [t_0, \infty)\) and estimate upper and lower bounds for solutions of the neutral integro-differential equation (1.2) by applying Kransnosel’skii, under the following assumptions.

(i) The functions \(\phi_i : R_+ \rightarrow R_+, \ i = 1, 2, 3\) are continuous.

(ii) \(f_1, g : I \times R_+ \rightarrow R_+\) satisfy Carathéodory condition, i.e., \(f_1, g\) are measurable functions in \(t\) for any \(x \in R_+\) and continuous in \(x\) for almost all \(t \in I\). There exist a nonnegative constant \(b\) and a function \(t \rightarrow m(t)\), such that

\[
|g(t, x) - g(t, \eta)| \leq L'|x - \eta|, \quad \forall (t, x), (t, \eta) \in I \times R_+,
\]
\( \Omega \)

To prove

\[ \text{Theorem 3.1.} \]

Let assumptions (i)-(v) be satisfied and suppose that there exist bounded functions \( f \) and two maps \( S \) such that

\[ \forall (t_1, t_2, \eta_1), (t_2, t_2, \eta_2) \in I \times R_+ \times R_+. \]

and

\[ |f(t_1, \eta_1)| \leq m(t) + b |\eta_1|, \forall (t, \eta_1) \in I \times R_+, \]

where \( m(.) \in L^1(t) \).

(iii) \( f_2 : I \times R_+ \times R_+ \rightarrow R_+ \) is continuous and there exists three positive constants \( l_0, l_1, \) and \( l_2 \) satisfying:

\[ |f_2(t_1, t_1, \eta_1) - f_2(t_2, t_2, \eta_2)| \leq l_0 |t_1 - t_2| + l_1 |t_1 - t_2| + l_2 |\eta_1 - \eta_2|, \]

\[ \forall (t_1, t_1, \eta_1), (t_2, t_2, \eta_2) \in I \times R_+ \times R_+. \]

(iv) \( f_2 \) is monotonic nondecreasing function in second and third argument.

(v) \( f_1, g \) is monotonic nondecreasing function in second argument.

By integrating both sides of (1.2) on \( R_+ \), we get

\[ x(t) = a_1 \int_t^\infty f_1(s, \phi_1(s)) \, ds + a_2f_2 \left( t, \phi_2(t), \int_{t_0}^1 \frac{t}{t+s} g(s, \phi_3(s)) \, ds \right). \]

(3.1)

Now, we define a subset \( \Omega \) of \( C([l_0, \infty), R_+) \) as follows:

\[ \Omega = \{ x = x(t) \in C([l_0, \infty), R_+) : u(t) \leq x(t) \leq v(t), t \geq l_0 \}, \]

and two maps \( S_1 \) and \( S_2 : \Omega \rightarrow C([l_0, \infty), R_+) \) as follows:

\[
(S_1x)(t) = \begin{cases} 
    a_2f_2 \left( t, \phi_2(t), \int_{t_0}^1 \frac{t}{t+s} g(s, \phi_3(s)) \, ds \right), & \text{for } t > t_1, \\
    (S_1x)(t_1), & \text{for } t_0 \leq t \leq t_1,
\end{cases}
\]

\[
(S_2x)(t) = \begin{cases} 
    a_1 \int_t^\infty f_1(s, \phi_1(s)) \, ds, & \text{for } t > t_1, \\
    (S_2x)(t_1) + v(t) - v(t_1), & \text{for } t_0 \leq t \leq t_1.
\end{cases}
\]

Then the functional integral equation (3.1) can be written as:

\[ x(t) = (S_1x)(t) + (S_2x)(t). \]

In this section, we shall consider the existence of a positive solution for the equation (1.2). The next theorem gives us the sufficient conditions for existence of a positive solution which is bounded by two positive functions.

**Theorem 3.1.** Let assumptions (i)-(v) be satisfied and suppose that there exist bounded functions \( u, v \in C^1([l_0, \infty), (0, \infty)) \) and \( t_1 \geq l_0 \) such that

\[ u(t) \leq v(t), \quad t \geq l_0, \]

\[ v(t) - v(t_1) - u(t) + u(t_1) \geq 0, \quad l_0 \leq t \leq t_1. \]

If \( |a_2|l_1 + bL' < 1 \), then (1.2) has a positive solution which is bounded by functions \( u, v \).

**Proof.** Let \( C([l_0, \infty), R_+) \) be the set of all continuous functions with the norm \( \|x\| = \sup_{t \geq l_0} |x(t)| \). Clearly, the subset \( \Omega \) of \( C([l_0, \infty), R_+) \) is a closed, bounded, and convex.

We shall show that for any \( \forall y, \eta \in \Omega \) we have \( S_1x + S_2y \in \Omega \). For every \( \forall y, \eta \in \Omega \) and \( t \geq t_1 \), we obtain

\[
(S_1x)(t) + (S_2y)(t) = a_1 \int_t^\infty f_1(s, \phi_1(s)) \, ds + a_2f_2 \left( t, \phi_2(t), \int_{t_0}^1 \frac{t}{t+s} g(s, \phi_3(s)) \, ds \right)
\]

\[ \leq a_1 \int_t^\infty f_1(s, \phi_1(s)) \, ds + a_2f_2 \left( t, \phi_2(t), \int_{t_0}^1 \frac{t}{t+s} g(s, \phi_3(s)) \, ds \right) = v(t). \]
For $t \in [t_0, t_1]$ we have:

$$(S_1\xi)(t) + (S_2\eta)(t) = (S_1\xi)(t_1) + (S_2\eta)(t_1) + v(t) - v(t_1) \leq v(t_1) + v(t) - v(t_1) = v(t).$$

Furthermore for $t \geq t_1$ we get:

$$(S_1\xi)(t) + (S_2\eta)(t) = a_1 \int_{t_1}^{\infty} f_1(s, \eta_1(\xi_1(s))) \, ds + a_2 f_2 \left( t, \eta(\xi_2(t)), \int_{t_0}^{t} \frac{s}{t + s} \eta(s, \xi_3(s)) \, ds \right) \geq a_1 \int_{t_1}^{\infty} f_1(s, \eta(\xi_1(s))) \, ds + a_2 f_2 \left( t, \eta(\xi_2(t)), \int_{t_0}^{t} \frac{s}{t + s} \eta(s, \xi_3(s)) \, ds \right) = u(t).$$

Let $t \in [t_0, t_1]$, we get:

$$v(t) - v(t_1) + u(t_1) \geq u(t), t_0 \leq t \leq t_1.$$

Then for $t \in [t_0, t_1]$ and any $\xi, \eta \in \Omega$ we obtain

$$(S_1\xi)(t) + (S_2\eta)(t) = (S_1\xi)(t_1) + (S_2\eta)(t_1) + v(t) - v(t_1) \geq u(t_1) + v(t) - v(t_1) \geq u(t).$$

Thus we have proved that $S_1\xi + S_2\eta \in \Omega$ for any $\xi, \eta \in \Omega$. We will show that $S_1$ is contraction mapping on $\Omega$ for $\xi, \eta \in \Omega$ and $t \geq t_1$, we have:

$$||S_1\xi(t) - (S_1\eta)(t)|| \leq ||a_2|| ||I_1 + I_2L'|| ||\eta - \eta||.$$

Also, for $t \in [t_0, t_1]$ the previous inequality is valid and $||a_2|| ||I_1 + I_2L'|| < 1$, we conclude that $S_1$ is a contraction mapping on $\Omega$. We now show that $S_2$ is completely continuous. First we will show that $S_2$ is continuous. Let $\xi_k = \xi_k(t) \in \Omega$ be such that $\xi_k(t) \rightarrow \xi(t)$ as $k \rightarrow \infty$. Because $\Omega$ is closed, $\xi = \xi(t) \in \Omega$. For $t \geq t_1$ we have

$$||S_2\xi_k(t) - S_2\xi(t)|| \leq ||a_1|| \int_{t}^{\infty} f_1(s, \xi_k(\eta_1(s))) - f_1(s, \xi(\eta_1(s))) \, ds \leq ||a_1|| \int_{t_1}^{\infty} f_1(s, \xi_k(\eta_1(s))) - f_1(s, \xi(\eta_1(s))) \, ds.$$

Also, we have

$$a_1 \int_{t_1}^{\infty} f_1(s, v(\eta_1(s))) \, ds < \infty.$$
From assumptions (ii) and (iii) we have
\[ |f_1(s, r_k(\phi_1(s)))| \leq m(s) + b|\phi_k(\phi_1(s))| \leq m(s) + b|v(s)| \in L^1(I) \]
and the function \( f_1(s, r_k(\phi_1(s))) \) is continuous in the second argument, i.e.,
\[ f_1(s, r_k(\phi_1(s))) \to f_1(s, \phi_1(s)) \text{ as } k \to \infty. \]
Therefore the sequence \( \{f_1(s, r_k(\phi_1(s)))\} \) satisfies Lebesgue dominated convergence theorem [5]. This means that \( S_2 \) is continuous.

We now show that \( S_2 \Omega \) is relatively compact. It is sufficient to show by the Arzela-Ascoli theorem that the family of functions \( \{S_2\phi : \phi \in \Omega\} \) is uniformly bounded and equicontinuous on \( [t_0, \infty) \). The uniform boundedness follows from the definition of \( \Omega \). For the the equicontinuity we only need to show, according to Levitan’s result [13], that for any given \( \epsilon > 0 \) the interval \( [t_0, \infty) \) can be decomposed into finite subintervals in such a way that on each subinterval all functions of the family have a change of amplitude less than \( \epsilon \), for \( \phi \in \Omega \) and any \( \epsilon > 0 \), we take \( t^* \geq t_1 \) large enough so that
\[ a_1 \int_{t^*}^{\infty} f_1(s, \phi_1(s)) \, ds < \frac{\epsilon}{2}. \]
Then, for \( \phi \in \Omega, t_2 > t_1 \geq t^* \), we have
\[ |(S_2\phi)(T_2) - (S_2\phi)(T_1)| \leq |a_1| \int_{T_2}^{\infty} f_1(s, \phi_1(s)) \, ds + |a_1| \int_{T_1}^{\infty} f_1(s, \phi_1(s)) \, ds \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \]
For \( \phi \in \Omega, t_1 \leq T_1 \leq T_2 \leq t^* \), we have
\[ |(S_2\phi)(T_2) - (S_2\phi)(T_1)| \leq |a_1| \int_{T_1}^{T_2} f_1(s, \phi_1(s)) \, ds - |a_1| \int_{T_1}^{T_2} f_1(s, \phi_1(s)) \, ds \]
\[ \leq |a_1| \int_{T_1}^{T_2} m(s) \, ds + |a_1| b \int_{T_1}^{T_2} |\phi_1(s)| \, ds \leq |a_1| \int_{T_1}^{T_2} m(s) \, ds + |a_1| b \int_{T_1}^{T_2} v(s) \, ds \leq |a_1| \int_{T_1}^{T_2} m(s) \, ds + |a_1| b \max_{t_0 \leq s \leq t^*} |v(s)||T_2 - T_1|. \]
Thus there exists \( \delta_1 > 0 \) such that
\[ |(S_2\phi)(T_2) - (S_2\phi)(T_1)| < \epsilon, \text{ if } 0 < T_2 - T_1 < \delta_1. \]
Finally for any \( \phi \in \Omega, t_0 \leq T_1 < T_2 \leq t_1 \), there exists a \( \delta_2 > 0 \) such that
\[ |(S_2\phi)(T_2) - (S_2\phi)(T_1)| = |v(T_2) - v(T_1)| \leq | \int_{T_1}^{T_2} v'(s) \, ds | \leq \max_{t_0 \leq s \leq t^*} |v'(s)||T_2 - T_1| < \epsilon, \text{ if } 0 < T_2 - T_1 < \delta_2. \]
Then \( \{S_2\phi : \phi \in \Omega\} \) is uniformly bounded and equicontinuous on \( [t_0, \infty) \), and hence \( S_2 \Omega \) is relatively compact subset of \( C([t_0, \infty), R) \). By Theorem 1.2 there is an \( \tau_0 \in \Omega \) such that \( S_1\tau_0 + S_2\tau_0 = \tau_0 \). We conclude that \( \tau_0(t) \) is a positive solution of (3.1). Thus the proof is complete.
4. Some applications and remarks

Observe that the equation (1.2) includes several classes of functional, integral, and functional integral equations considered in many literature. As particular cases of equation (1.2), we obtain the following.

- When \( a_1 = 0, f_2(t, r, η) = q(t) + rηψ(t) \), then we get a quadratic integral equation of Chandrasekhar type

\[
x(t) = c + q(t) + r(\phi_2(t)) \int_0^1 \frac{t}{t + s} \psi(s) x(\phi_3(s)) ds, \quad t \in [0, 1],
\]

where \( c \) is a real constant and \( q(t) \) is continuous function and \( ψ \) is the characteristic function. Moreover, the quadratic functional integral (4.1) reduces to the well-known Chandrasekhar integral equation in radiative transfer [3],

\[
x(t) = 1 + x(t) \int_0^1 \frac{t}{t + s} \psi(s) x(s) ds, \quad t \in [0, 1].
\]

It describes a scattering through a homogeneous semi-infinite plane atmosphere. In particular, solutions for this equations need not to be continuous.

- When \( a_1 = 0 \) and \( f_2(t, r, η) = -p(t)(x(t)) + r(t) \), then we obtain the delay differential equation with a forcing term

\[
x'(t) = -p(t)(x(\phi_2(t))) + r(t), \quad t \geq 0,
\]

where \( p : R_+ \rightarrow R_+ \) and \( f : R \rightarrow R \) are continuous functions with \( f(\phi) > 0 \) for \( \phi \neq 0 \), and \( r : R_+ \rightarrow R \) is a continuous function. Which is more general than the delay differential equation by Qian et al. [18] with \( \phi_2(t) = t - τ, \quad τ \geq 0 \).

- When \( a_1 = 0 \) and \( f_2(t, N, M) = \left( p \frac{K - N(\phi_2(t))}{K + cpN(\phi_2(t))} + r(t) \right) N(t) \), then we obtain

\[
N'(t) = \left( p \frac{K - N(t - τ)}{K + cpN(t - τ)} + r(t) \right) N(t), \quad t > 0,
\]

which is more general than the well-known delay-logistic equation. For more details about this model, see [8],

\[
N'(t) = \left( p \frac{K - N(t - τ)}{K + cpN(t - τ)} + r(t) \right) N(t), \quad t > 0,
\]

and with initial conditions of the form

\[
N(t) = \phi(t) \text{ for } -τ \leq t \leq 0,
\]

where \( φ(t) \in C([-τ, 0], [0, ∞)) \) with \( φ(0) > 0 \). So, the solution on \([0, τ]\) is given by

\[
N(t) = \phi(0)e^\int_0^t \left( p \frac{K - N(\phi(t - τ))}{K + cpN(\phi(t - τ))} + r(t) \right) dt.
\]

Remark 4.1. Also, we can consider the nonlinear neutral retarded differential equations of the form

\[
\left\{ \begin{array}{l}
\frac{d}{dt} \left[ x(t) - a_2 f_2 \left( t, x(t - τ_1), \int_0^1 \frac{t}{t + s} g(s, x(s - τ_2)) ds \right) \right] = a_1 f_1(t, x(t - τ_1)), \quad t > 0, \\
x(t) = φ(t) \text{ for } -τ \leq t \leq 0, \quad τ = \max(τ_i, i = 1, 2, 3),
\end{array} \right.
\]

where \( a_1, a_2 \in R, τ_1 \geq 0, i = 1, 2, 3 \).

By similar way as done before, we can proof the following theorem.

Theorem 4.2. Let assumptions (ii)-(v) be satisfied and suppose that there exist bounded functions \( u, v \in C^1([t_0, ∞), (0, ∞)) \) and \( t_1 \geq t_0 + τ \) such that

\[
u(t) \leq u(t), \quad t \geq t_0, \quad v(t) - v(t_1) - u(t) + u(t_1) \geq 0, \quad t_0 \leq t \leq t_1.
\]

If \( |a_2| |t_1 + t_2L'| < 1 \), then (4.4) has a positive solution which is bounded by functions \( u, v \).
Conclusion 4.3. We have given some existence theorem for a class of neutral integro-differential equations (1.2) which involved many key integral and functional differential equations that appear in applications of nonlinear analysis. The existence of solutions which are bounded by constants has been treated and received much attention in many papers. The existence of positive solutions which are bounded below and above by positive functions for the nonlinear neutral differential equations (1.2) is proved. In aim of estimating lower bound and upper bound of the solutions, we have made a use of Kransnoselskii fixed point theorem and monotonicity conditions.

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