Finite groups with very few character values

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ABSTRACT
Finite groups with very few character values are characterized. The following is the main result of this article: A finite non-abelian group has precisely four character values if and only if it is the generalized dihedral group of a non-trivial elementary abelian 3-group. The proof involves the analysis of the centralizers of involutions.

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1. Introduction
A character table often contains substantial information about a finite group. Even its partial data affect considerably on the group structure, and character degrees has been drawn special attention over the years. Surprisingly, however, the author cannot find any study on how the set of character values

$$cv(G) = \{ \chi(g) \mid \chi \in \text{Irr}(G), \ g \in G \}$$

affects the structure of a finite group $G$, where $\text{Irr}(G)$ denotes the set of irreducible characters of $G$.

Question. What information about a finite group can be obtained from its character values?

This article presents the characterizations of finite groups with very few character values. Finite groups with less than four character values are abelian, and their characterizations are rather easy (Proposition 2.3). The first non-trivial example is, as in many cases, the symmetric group $S_3$ of degree three. (See the character table.) It is non-abelian and has precisely four character values. This is the case we settle in this article.

|    | (1) | (2) | (3) | (1, 2) | (3) | (1, 2, 3) |
|----|-----|-----|-----|--------|-----|-----------|
| $\chi^{(3)}$ | 1   |     |     | 1      |     |           |
| $\chi^{(1,2)}$ | 1   |   -1 |     | 1      |     |           |
| $\chi^{(1,2,3)}$ | 2   |     | 0   | -1     |     |           |
Theorem. A finite non-abelian group has precisely four character values if and only if it is the generalized dihedral group of a non-trivial elementary abelian 3-group.

The generalized dihedral group $\text{Dih} A$ of $A$ is defined to be the relative holomorph of $A$ by the inversion $A \rightarrow A, a \mapsto a^{-1}$, where $A$ is an abelian group that is not an elementary abelian 2-group. In particular, the generalized dihedral group $\text{Dih} C_3^t$ of a non-trivial elementary abelian 3-group $C_3^t$ has a presentation

$$\langle a_1, \ldots, a_r, t \mid a_i^3 = [a_i, a_j] = t^2 = 1, \ a_i^t = a_i^{-1} \rangle.$$ 

Its centralizers of involutions have order two; this is the key to identify the isomorphism classes of finite non-abelian groups with precisely four character values.

Remark. It is natural to wonder what happens if a finite group has more character values. One can easily check that $C_3^*, C_3^* \times \text{Dih} C_3^t$, and $S_4$ have precisely five character values where $*$ indicates some positive integer. But there are also many finite 2-groups with precisely five character values, including the dihedral group $D_8$ and the quaternion group $Q_8$ of order eight. This makes it hard to prove or even guess the next case. Having said that, the character values seem to impose severe restrictions on the group structure. For instance, it appears that finite groups with less than eight character values are solvable.

We refer the readers to the book by Huppert [1] for standard terminology and notation in the character theory of finite groups.

2. Character values

The preliminary results on character values are summarized in this section. The largest character degree of a finite group $G$ is denoted by $b(G)$.

Lemma 2.1. A finite group is non-abelian if and only if it has zero as a character value. In particular, every finite group with less than four character values is abelian.

Proof. See [1, Theorem 6.13 (Burnside)] for the first part. For the second part, we shall prove the contraposition. Let $G$ be a finite non-abelian group. We have $\text{cv}(G) \supseteq \{0, 1, b(G)\}$ by the first part. From the column orthogonality, there must be a character value with a negative real part. Hence $|\text{cv}(G)| \geq 4$. \hfill $\square$

Lemma 2.2. The number of character values of a finite abelian group equals the maximum order of elements.

Proposition 2.3. Let $G$ be a finite group.

(i) $|\text{cv}(G)| = 1$ if and only if $G$ is trivial.
(ii) $|\text{cv}(G)| = 2$ if and only if $G$ is a non-trivial elementary abelian 2-group.
(iii) $|\text{cv}(G)| = 3$ if and only if $G$ is a non-trivial elementary abelian 3-group.
(iv) Assume $G$ is abelian. Then $|\text{cv}(G)| = 4$ if and only if $G$ has exponent 4.

Proof. The assertions follow from Lemmas 2.1 and 2.2. \hfill $\square$

3. Generalized dihedral groups

The easy part of Theorem is proved in this section. For a finite group $G$, the principal character of $G$ is denoted by $1_G$, the set of linear characters of $G$ is denoted by $\hat{G}$, and the largest normal subgroup of $G$ of odd order is denoted by $O(G)$. 
Proposition 3.1. Let $A$ be a non-trivial finite abelian group of odd order, $G$ the generalized dihedral group of $A$, and $t$ an involution of $G$. Then $G$ has a unique non-principal linear character $\sigma$ and the character table of $G$ is given by the following.

|     | 1   | $t$ | $a$ |
|-----|-----|-----|-----|
| $1_0$ | 1   | 1   | 1   |
| $\sigma$ | 1   | $-1$ | 1   |
| $\theta^a$ | 2   | 0   | $2\text{Re}(a)$ |

Here, representatives of $A \setminus \{1\}$ under the equivalence relation generated by $a \sim a'$ are indicated by $a$ and representatives of $A \setminus \{1_A\}$ under the equivalence relation generated by $\theta \sim \theta'$ are indicated by $\theta$. In particular, $G$ has $(|A| + 3)/2$ conjugacy classes, $G' = O(G)$, and $C_G(t) = \langle t \rangle$.

Proof. Because $A$ is a non-trivial group of odd order, $G$ is a Frobenius group with the Frobenius kernel $A$ and a Frobenius complement $\langle t \rangle$. Then, the assertions follow from [1, Theorem 18.7]. □

Corollary 3.2. $\text{Dih}C_3$ has precisely four character values for every $r \geq 1$.

4. Proof of theorem

The difficult part of Theorem is proved in this section. For a finite group $G$, the set of character degrees of $G$ is denoted by $\text{cd}(G)$.

Lemma 4.1. Let $G$ be a finite non-abelian group with precisely four character values.

(i) $\text{cd}(G) = \{1, b(G)\}$.

(ii) $b(G)$ divides $|G : G'|$.

(iii) $\text{cv}(G) = \{-1, 0, 1, b(G)\}$.

Proof. (i) As $G$ is non-abelian, $b(G) \neq 1$. By Lemma 2.1, we have $\text{cv}(G) \supseteq \{0, 1, b(G)\}$. Suppose that $\psi(t) \in \text{cv}(G)$ is the remaining character value. From the row orthogonality $0 = \langle \psi, 1_G \rangle = \frac{1}{|G|} \sum_{g \in G} \psi(g)$, the remaining character value $\psi(t)$ must be a negative rational number. Hence, $\text{cd}(G) = \{1, b(G)\}$.

(ii) As $b(G)$ divides $|G| = \sum_{\chi \in \text{Irr}(G)} \chi(1)^2 = |G : G'| + b(G)^2 + \cdots + b(G)^2$, it follows that $b(G)$ divides $|G : G'|$.

(iii) As $|G : G'| \neq 1$ by the part (ii), there is a non-principal linear character $\sigma$. Since values of $\sigma$ are complex numbers of modulus one, we get $\psi(t) = -1$. Hence, $\text{cv}(G) = \{-1, 0, 1, b(G)\}$. □

Lemma 4.2. Let $G$ be a finite non-abelian group with precisely four character values.

(i) $G/G'$ is a non-trivial elementary abelian 2-group.

(ii) $|C_G(t)| = |G : G'|$ for every involution $t$ of $G$.

Proof. (i) It follows from Lemma 4.1 and Proposition 2.3.

(ii) Let $t$ be an involution of $G$. We claim that

$$\chi(t) = 0 \quad (\dagger)$$

for every nonlinear character $\chi$ of $G$. Suppose that $\chi'$ is a representation of $G$ affording $\chi$. Because $\chi'(t)$ is involutory, its eigenvalues are $\pm 1$. As $b(G)$ is even, $\chi(t)$ equals 0 or $b(G)$ by Lemma 4.1. From the column orthogonality,
\[
0 = \sum_{\psi \in \text{Irr}(G)} \psi(t)\psi(1) = \sum_{\sigma \in G} \sigma(t) + b(G) \sum_{\psi \in \text{Irr}(G) \setminus \hat{G}} \psi(t) \geq \sum_{\sigma \in G} \sigma(t).
\]

(Note that \(\psi(t)\) equals 0 or \(b(G)\) for \(\psi \in \text{Irr}(G) \setminus \hat{G}\).) Hence, \(\sigma(t) = -1\) for some non-principal linear character \(\sigma\). Thus, if \(\chi(t) \neq 0\), then \(-b(G) = \sigma \otimes \chi(t) \in \text{cv}(G)\), a contradiction. Therefore, by the claim \((\dagger)\) and the column orthogonality,

\[
|C_G(t)| = \sum_{\chi \in \text{Irr}(G)} |\chi(t)|^2 = \sum_{\sigma \in G} |\sigma(t)|^2 = |G : G'|.
\]

**Lemma 4.3.** Let \(G\) be a finite non-abelian group with precisely four character values.

(i) \(G' = O(G)\).

(ii) Every involution of \(G\) acts on \(O(G)\) as the inversion.

**Proof.** (i) By Lemma 4.2(i), it suffices to prove that \(G'\) has odd order. Let \(P\) be a Sylow 2-subgroup of \(G\) and suppose that \(G'\) has even order. Then, \(G' \cap Z(P)\) is non-trivial because \(G' \cap P\) is a non-trivial normal subgroup of \(P\). Hence, there exists an involution \(t \in G' \cap Z(P)\). Since \(G/G'\) is a 2-group, there is a transversal \(S\) of \(G'\) in \(G\) that is contained in \(P\). Then, we have \(|C_G(t)| = |S||C_G(t)| > |S| = |G : G'|\), but this contradicts to Lemma 4.2(ii).

(ii) Let \(t\) be an involution of \(G\). As \(C_G(t)\) is a 2-group by Lemma 4.2, it acts fixed-point-freely on \(O(G)\). From [1, Proposition 16.9e], it follows that every element of \(O(G)\) is inverted by \(t\).

**Proof of Theorem.** As the ‘if’ part is already done in Corollary 3.2, we shall prove the ‘only if’ part. Let \(G\) be a finite non-abelian group with \(|\text{cv}(G)| = 4\) and a Sylow 2-subgroup \(P\). By Lemma 4.2(i) and Lemma 4.3(i), it follows that \(P\) is elementary abelian. Let \(s\) and \(t\) be non-trivial elements of \(P\). By Lemma 4.3(ii), every element of \(O(G)\) is inverted by \(s\) and \(t\). Then, \(O(G)\) is abelian and \(O(G) \leq C_G(st)\). It follows that \(C_G(st)\) cannot be a 2-group by Lemma 4.3(i) and thus \(st = 1\) by Lemma 4.2. Hence, \(P\) has order two and \(G\) is the generalized dihedral group of \(O(G)\). By Proposition 3.1, \(O(G)\) must be a non-trivial elementary abelian 3-group.

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**References**

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