A NOTE ON RATIONAL CURVES ON GENERAL FANO HYPERSURFACES

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Abstract. We show the space of smooth rational curves of degree at most roughly \( \frac{2 - \sqrt{2}}{2} n \) on a general hypersurface \( X \subset \mathbb{P}^n \) of degree \( n - 1 \) is irreducible and of the expected dimension. This proves more cases of a conjecture of Coskun, Harris, and Starr.

1. Introduction

Our investigation is motivated by the following conjecture by Coskun, Harris, and Starr.

**Conjecture 1.1** ([4, Conjectures 1.3 and 1.4]). Let \( X \subset \mathbb{P}^n \) be a general hypersurface of degree \( d \leq n \) and dimension at least 3. Then, the open locus \( R_e(X) \) in the Hilbert scheme of \( X \) parameterizing smooth rational curves of degree \( e \) is irreducible of dimension \( e(n + 1 - d) + n - 4 \). Furthermore, if \( d \leq n - 1 \), then the evaluation map \( \overline{M}_{0,1}(X,e) \to X \) is flat.

There has been progress on Conjecture [1.1] by using induction on \( e \) via bend and break [9, 11]. Most recently, Riedl and Yang showed that Conjecture [1.1] holds when \( d \leq n - 2 \) [11]. For work on rational curves on arbitrary smooth hypersurfaces, see [4, 3].

In this note, we will look at rational curves of low degree on general hypersurfaces of degree \( n - 1 \). Specifically, we will show

**Theorem 1.2.** If \( X \subset \mathbb{P}^n \) is a general hypersurface of degree \( n - 1 \) for \( n \geq 4 \), then Conjecture [1.1] holds if \( e \leq n \) and

\[
\frac{n^2}{2} - (1 + 4e)n + 2e(e + 1) + 8 > 0.
\]

In particular, this holds if \( e \leq \frac{n}{4} \) or if \( e < \frac{2 - \sqrt{2}}{2} n \) and \( n >> 0 \).

Another method that has been successful in controlling the dimensions of \( R_e(X) \) has been to consider the incidence correspondence between hypersurfaces and rational curves in projective space. In the setting of Conjecture [1.1], \( R_e(X) \) is smooth and of the expected dimension for \( e \leq d + 2 \) by work of Gruson, Lazarsfeld and Peskine [7]. Furukawa made this connection explicit and gave a weaker bound that also works in positive characteristic [6].

Therefore, the content of Theorem [1.2] is the connectedness of \( R_e(X) \). As a product of the bend and break methods, we will also see that the space of rational curves through each point of the general hypersurface is also the expected dimension, which is equivalent to flatness of the evaluation map. To show Theorem [1.2] we will show

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Theorem 1.3. Under the assumptions of Theorem 1.2, the Kontsevich space $\overline{M}_{0,0}(X,e)$ is a local complete intersection stack of dimension $2e+n-4$ and has two components. One of the components consists of $e$ to $1$ covers of a line and the other component consists generically of smooth rational curves.

In addition, the evaluation map $\overline{M}_{0,1}(X,e) \to X$ is flat.

In order to apply bend and break in families as in [11], we will need apply a couple of results from the author [14] to control the locus of hypersurfaces with more lines through a point than expected and the locus of hypersurfaces with positive dimensional singular loci. Also, the space of lines through a point is expected to be finite, and so in particular is not irreducible. This will present a technical obstacle in showing irreducibility of the main component in Theorem 1.3. To deal with this, we will show that the space of conics through a general point is irreducible, and then use an argument that is similar in spirit to the irreducibility argument in [9] but will require us to specialize further.

1.1. Outline. The argument will have two parts. In Section 4, we will show the fibers of the evaluation map $\overline{M}_{0,1}(X,e) \to X$ have the expected dimension in Theorem 4.1. We will look at the irreducible components of the general fiber in Theorem 5.7 in Section 5. Theorem 1.3 will follow from Theorems 4.1 and 5.7.

In Section 4, we are mostly interested in dimension, so it suffices to work with the coarse moduli space of $\overline{M}_{0,1}(X,e)$, but we will need look at smoothness in Section 5, so we will need to work with $\overline{M}_{0,1}(X,e)$ as a stack. In general, the fact that the Kontsevich space $\overline{M}_{0,1}(X,e)$ and, more generally, the Behrend-Manin stacks [2], are Deligne Mumford stacks in characteristic zero will allow us to avoid technical difficulties with stacks by passing to an étale cover.

2. Acknowledgements

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3. Conventions

Throughout the paper, we will work over an algebraically closed field of characteristic zero. We will let

1. $X \subset \mathbb{P}^n$ be a hypersurface of degree $2 \leq d \leq n - 1$
2. $N = \left(\frac{d+n}{n}\right) - 1$ so $\mathbb{P}^N$ parameterizes hypersurfaces of degree $d$ in $\mathbb{P}^n$
3. $\mathcal{X} \to \mathbb{P}^N$ be the universal hypersurface $\mathcal{X} = \{(p,X)|p \in X\} \subset \mathbb{P}^n \times \mathbb{P}^N$
4. $\overline{M}_{0,r}(X,e)$ is the Kontsevich space parameterizing stable maps $C \to X$ of degree $e$, where $C$ is a genus $0$ with $r$ marked points
5. $\overline{M}_{0,r}(\mathcal{X}/\mathbb{P}^N,e)$ is the relative Kontsevich space parameterizing stable maps mapping into the fibers of $\mathcal{X} \to \mathbb{P}^N$.

We are interested in when $d = n - 1$, but will state results more generally when there is no drawback.
4. Flatness

The goal of the first half of the note is to prove

**Theorem 4.1.** If \( n^2 - (1 + 4e)n + 2e(e + 1) + 8 > 0 \) and \( e \leq n \), then for a general hypersurface \( X \) of degree \( d = n - 1 \) in \( \mathbb{P}^n \), the evaluation map

\[ \overline{M}_{0,1}(X,e) \to X \]

is flat and \( \overline{M}_{0,1}(X,e) \) is a local complete intersection stack.

4.1. Definitions. We recall the notion of \( e \)-level \([11]\), with some slight modifications.

**Definition 4.1.** Given a point \( p \in X \subset \mathbb{P}^n \) of a hypersurface of degree \( d \), \( p \) is called \( e \)-level if either

1. \( p \) is a smooth point and the space of degree \( e \) rational curves through \( p \) is dimension at most \( e(n - d + 1) - 2 \)
2. \( p \) is a singular point and the space of degree \( e \) rational curves through \( p \) is at most \( e(n - d + 1) - 1 \).

By the space of rational curves through \( p \), we mean the fiber of the evaluation map \( \overline{M}_{0,1}(X,e) \to X \) over \( p \in X \). When \( p \) fails to be \( e \)-level, there is a bigger dimensional family of rational curves through \( p \) than expected.

Note that because the definition of \( e \)-level treats singular and nonsingular points separately, the locus of level points is not closed or open in the universal hypersurfaces. However, it is open if we restrict to either the locus of smooth points or the locus of singular points. To see why we need to treat singular points separately, it suffices to consider the space of lines through \( p \), where being singular at \( p \) means the linear term in the Taylor expansion of the form defining \( X \) at \( p \) vanishes and hence doesn’t impose a condition on the space of lines through \( p \).

**Definition 4.2.** A hypersurface \( X \) is \( e \)-level if,

1. the singular locus is finite, and
2. every point of \( X \) is \( k \) level for all \( k \leq e \).

**Remark 1.** Instead of requiring only finitely many singular points, what is actually being used in the dimension counts of [4 Proposition 2.5] \([11]\) Proposition 5.5] is that there is no rational curve \( C \subset X \) of degree less than \( e \), for which the space of rational curves of degree \( k < e \) through every point \( p \in C \) exceeds \( k(n - d + 1) - 2 \). For example, the original definition of \( e \)-level \([11]\) replaced the condition of finitely many singular points with the condition that there is no rational curve of degree at most \( e \) in the singular locus.

4.2. Basic lemmas. We collect here some crucial facts needed to run the argument.

**Proposition 4.2.** For any map \( \phi : T \to \overline{M}_{0,a}(X,e) \) from an irreducible scheme \( T \), the pullback \( \phi^{-1}(\partial \overline{M}_{0,a}(X,e)) \) is codimension at most 1.

**Proof.** This follows from the fact that \( \partial \overline{M}(\mathbb{P}^n,e) \) is a divisor in \( \overline{M}(\mathbb{P}^n,e) \). Explicitly, consider an etale cover \( \pi : Y \to \overline{M}(\mathbb{P}^n,e) \) by a scheme and note that \( \partial \overline{M}(\mathbb{P}^n,e) \) pulls back to a divisor on the scheme \( T \times \overline{M}(\mathbb{P}^n,e) \) \( Y \) in the fiber diagrams below.
\[ T \times \overline{\mathcal{M}}(\mathbb{P}^n, e) \xrightarrow{\phi} M_{0,1}(X, e) \xrightarrow{\pi^{-1}} \overline{\mathcal{M}}(\mathbb{P}^n, e) \xrightarrow{\pi} Y \]

We will also need a version of bend and break.

**Lemma 4.3.** ([11, Corollary 3.3]) If \( T \) is a closed locus in \( \overline{\mathcal{M}}_{0,1}(\mathbb{P}^n, e) \) of dimension at least \( 2n - 1 \), then \( T \) contains maps with reducible domains.

### 4.3. Codimension of the locus of hypersurfaces that are not 1-level.

As described in [11], the idea of the argument is to borrow rational curves from nearby hypersurfaces to apply bend and break. To run the argument, we need to know the locus of hypersurfaces that are not 1-level has high codimension. For the rest of this section, let \( \mathbb{P}^N \) be the space of all hypersurfaces of degree \( d \) in \( \mathbb{P}^n \). We are primarily interested in the case where \( d = n - 1 \).

**Theorem 4.4** ([14]). Let \( U \subset \mathbb{P}^N \) be the open locus of smooth hypersurfaces. For \( 4 \leq d \leq n - 1 \), a largest component of the closed locus \( Z \subset U \) of hypersurfaces that are not 1-level consists of hypersurfaces containing a 2-plane.

We won’t need this but the largest component is unique when \( n - 1 = d > 4 \).

We also need to consider hypersurfaces with lines through a singular point than expected.

**Proposition 4.5.** Let \( U \subset \mathbb{P}^N \) be the open locus of hypersurfaces with at most finitely many singular points. Let \( Z \subset \mathbb{P}^N \) be the locus of hypersurfaces \( X \) for which there exists a singular point \( p \) containing a \( n - d + 1 \)-dimensional family of lines in \( X \). For \( d \leq n - 1 \), the codimension of \( Z \) in \( \mathbb{P}^N \) is at least \( \binom{n+1}{2} \).

**Proof.** This is proven in [11, Proposition 5.10]. Even though they assume \( d \leq n - 2 \), the analysis for the case of a singular point goes through. The main obstruction to extending [11, Proposition 5.10] to the case where \( d = n - 1 \) (case of smooth points) is covered by Theorem 4.4. □

To prove a hypersurface is \( e \)-level, we need to rule out a positive dimensional singular locus.

**Theorem 4.6** ([14]). For \( d \geq 7 \), the unique largest component of the closed locus \( Z \subset \mathbb{P}^N \) of hypersurfaces with positive dimensional singular locus consists of hypersurfaces singular along a line.

Theorem 4.6 is expected to hold for all degrees \( d \) [12], but as it stands we will need to consider the case \( d \leq 6 \) separately, as we cannot control the locus of hypersurfaces with positive dimensional singular loci. Since Theorem 1.3 only applies when \( e \leq 2 \) in this case, this will not be a technical obstacle. Conjecture 1.1 is proven for \( n - 1 = d \leq 5 \) and all \( e \) [4, Theorem 1.6], with the small caveat that when \( (d, n) = (5, 6) \), the evaluation map is only flat away from finitely many points. Therefore, we will only mention the case \( d \leq 6 \) briefly.

**Corollary 4.7.** Suppose \( 7 \leq d = n - 1 \). Let \( Z \subset \mathbb{P}^N \) be the locus of hypersurfaces that are not 1-level. Then, the codimension of \( Z \) is \( \binom{n+1}{2} - 3(n - 2) \).
Proof. We need to compare the contributions of hypersurfaces

1) with more lines than expected through a smooth point,
2) with more lines than expected through a singular point,
3) singular along a line.

The first case is codimension \((\binom{n+1}{2} - 3(n-2))\) by Theorem 4.4. The dimension of hypersurfaces singular along a line is \(dn - 2n + 3 = n^2 - 3n + 3\) [12, Lemma 5.1]. The third case is covered by Proposition 4.5. Comparing these bounds yields the result. □

4.4. Bend and break in families. Let \(\mathbb{P}^N\) be the space of degree \(d\) hypersurfaces in \(\mathbb{P}^n\), and \(S_e \subset \mathbb{P}^N\) denote the closure of the hypersurfaces that are not \(e\)-level. We have a chain of inclusions

\[ S_1 \subset S_2 \subset S_3 \subset \cdots \subset \mathbb{P}^N. \]

We will use the argument in [11, Theorem 6.2] to bound the codimension of each inclusion \(S_e \subset S_{e+1}\).

Theorem 4.8. The codimension of \(S_{e-1} \subset S_e\) is at most \(2n - (n - d + 1)e\).

Proof. Let \(a = e(n-d+1) - 2\) be the expected dimension of a fiber \(ev : \overline{M}_{0,1}(\mathcal{X}/\mathbb{P}^N, e) \to \mathcal{X}\).

\[
\begin{array}{ccc}
\overline{M}_{0,1}(\mathcal{X}/\mathbb{P}^N, e) & \longrightarrow & \overline{M}_{0,1}(\mathbb{P}^n, e) \\
\downarrow & & \downarrow \\
\mathcal{X} & \longrightarrow & \overline{M}_{0,0}(\mathbb{P}^n, e)
\end{array}
\]

Suppose \(A \subset \overline{M}_{0,1}(\mathcal{X}/\mathbb{P}^N, e)\) is an irreducible component of

\([C] \in \overline{M}_{0,1}(\mathcal{X}/\mathbb{P}^N, e) \mid ev([C])\) is a singular point, dimension of fiber at \([C] \geq a + 2\).

If \(A\) consists of covers of lines, then \(A \to \mathbb{P}^N\) maps into \(S_1\). If \(A\) contains a map that isn’t a cover of a line, then the image of \(A \to \overline{M}_{0,0}(\mathbb{P}^n, e)\) has dimension at least \(3n - 3\) from \(PGL_{n+1}\)-invariance, as we can interpolate a curve from \(A\) through 3 general points by taking the \(PGL\)-translates of a single curve.

Now, we apply Lemma 4.9 to the image of \(A\) under the forgetful map

\(A \subset \overline{M}_{0,1}(\mathcal{X}/\mathbb{P}^N, e) \to \overline{M}_{0,0}(\mathcal{X}/\mathbb{P}^N, e) \subset \mathbb{P}^N \times \overline{M}_{0,0}(\mathbb{P}^n, e)\).

to cut \(A\) by hyperplane sections in \(\mathbb{P}^N\) to produce \(A' \subset A\) such that \(A' \to \overline{M}_{0,0}(\mathbb{P}^n, e)\) has image dimension \(2n - 1\) and \(\text{im}(A' \to \overline{M}_{0,0}(\mathcal{X}/\mathbb{P}^N, e)) \to \overline{M}_{0,0}(\mathbb{P}^n, e)\) is generically finite onto its image, so \(\text{im}(A' \to \overline{M}_{0,0}(\mathcal{X}/\mathbb{P}^N, e))\) is also dimension \(2n - 1\). By Lemma 4.3 \(A'\) contains reducibles. The image of \(A' \to \overline{M}_{0,0}(\mathcal{X}/\mathbb{P}^N, e) \to \mathbb{P}^N\) has dimension at most \(2n - 1 - (a + 2)\).

Finally, we assume for the sake of contradiction that the codimension of \(S_{e-1}\) in the image of \(A \to \mathbb{P}^N\) is at least \(2n - (n - d + 1)e + 1 = 2n - (a + 2) + 1\). Then, by construction, \(A' \to \mathbb{P}^N\)
If a singular point and is dimension at most $S\rightarrow P$.

Similarly, to finish we need to consider the case where a smooth point is not level. We let $\mathcal{A}$ be an irreducible component of

$$\{[C] \in \overline{M}_{0,1}(\mathcal{X}/\mathbb{P}^N, e) | \operatorname{ev}([C]) \text{ is a smooth point, dimension of fiber at } [C] \geq a + 1\}.$$ 

If $\mathcal{A}$ consists of covers of lines, then $\mathcal{A} \rightarrow \mathbb{P}^N$ maps into $S_1$. Otherwise, as before, $\mathcal{A} \rightarrow \overline{M}_{0,0}(\mathbb{P}^n, e)$ has dimension at least $3n - 3$. Let $\mathcal{A}$ be hyperplane sections in $\mathbb{P}^n$ to produce $\mathcal{A}' \subset \mathcal{A}$ such that $\mathcal{A}' \rightarrow \overline{M}_{0,0}(\mathbb{P}^n, e)$ has image dimension $2n - 1$ and the image of $\mathcal{A}' \rightarrow \mathbb{P}^N$ is at most $2n - 1 - (a + 1)$. As before, Lemma 4.3 shows $\mathcal{A}'$ contains reducibles.

If we assume for the sake of contradiction that the codimension of $S_{e-1}$ in the image of $\mathcal{A} \rightarrow \mathbb{P}^N$ is at least $2n - (n - d + 1)e + 1 = 2n - (a + 1) + 1$, then $\mathcal{A}' \rightarrow \mathbb{P}^N$ misses the locus $S_{e-1}$ completely. Applying [11] Proposition 5.5 shows the locus of reducible curves in $\mathcal{A}'$ is dimension at most $e(n - d + 1) - 2 = a$ in each fiber of the evaluation map $\mathcal{A}' \rightarrow \mathcal{X}$ over a singular point and is dimension at most $a - 1$ in each fiber over a smooth point. Since the general point in the image of $\mathcal{A}' \rightarrow \mathcal{X}$ is smooth, again we have the locus of reducibles in $\mathcal{A}'$ is codimension at least 2, contradicting Proposition 4.2.

**Lemma 4.9.** Suppose $A \subset B \times C$ with projections $\pi_1 : A \rightarrow B$ and $\pi_2 : A \rightarrow C$. If $C$ is a projective scheme and $H \subset C$ is a general linear section. Then, $\dim(\pi_1(\pi_2^{-1}(H))) = \dim(\pi_1(A))$ if $\pi_1 : A \rightarrow B$ has positive-dimensional fibers and $\dim(\pi_1(\pi_2^{-1}(H))) = \dim(\pi_1(A)) - 1$ if $\pi_1 : A \rightarrow B$ is generically finite onto its image.

**Proof.** If $A \rightarrow B$ has positive dimensional fibers, choose $H$ so that it cuts down the dimension of a general fiber of $\pi_1$ by 1. □

4.5. **Conclusion of argument.**

**Proof.** (of Theorem 4.1) To prove Theorem 4.1, it suffices to show that the fibers of evaluation map are of the expected dimension [9, Lemma 4.5].

First suppose $d \geq 7$. We apply Theorem 4.8 to show $S_e$ is not all of $\mathbb{P}^N$. First, by Corollary 4.7, we have $S_1$ is codimension at least $(\frac{n+1}{2}) - 3(n-2) - \sum_{e'=2}^{e} (2n-2e')$ in $\mathbb{P}^N$. Theorem 4.8 implies then that $S_e$ is codimension at least

$$\left(\frac{n+1}{2}\right) - 3(n-2) - \sum_{e'=2}^{e} (2n-2e') = \left(\frac{n+1}{2}\right) - 3(n-2) - 2n(e-1) + e(e+1) - 2$$

$$= \frac{1}{2}(n^2 - n - 4ne + 2e^2 + 2e + 8).$$

If $d < 7$, then $e \leq 2$. First, we can assume $d \in \{5, 6\}$ [4, Theorem 1.6]. As mentioned in Remark 1, we chose to restrict to hypersurfaces with finitely many singular points to simplify the argument for $d \geq 7$. When $e = 2$, we can replace $S_1$ with the locus of hypersurfaces for which all the points are 1-level and the singular locus does not contain a line. Hypersurfaces singular along a line is codimension $n^2 - 3n + 3$ [12, Lemma 5.1], which is at least the codimension of hypersurfaces containing a 2-plane for $n \geq 3$. Then, we can run the same argument in Proposition 4.8 to see flatness in Theorem 4.1 as [11] Proposition 5.5 still applies. □
5. Irreducible components

5.1. Conics through a point. We will work with the Hilbert scheme of conics instead of the Kontsevich space to focus on degree 2 maps that are not covers of a line.

Definition 5.1. We let

1. \( \text{Hilb}_{2t+1}(\mathbb{P}^n) \) denote the Hilbert scheme of conics in \( \mathbb{P}^n \)
2. \( \text{Hilb}_{2t+1}(X) \) denote the Hilbert scheme of conics in \( X \)
3. \( \text{Hilb}_{2t+1}(X/\mathbb{P}^N) \) denote the relative Hilbert scheme of conics in the fibers of \( X \to \mathbb{P}^N \)
4. \( \mathcal{C} \to \text{Hilb}_{2t+1}(X/\mathbb{P}^N) \) be the universal curve.

Note that \( \text{Hilb}_{2t+1}(\mathbb{P}^n) \) is smooth, as a \( \mathbb{P}^5 \) bundle over \( G(2,n) \), \( \text{Hilb}_{2t+1}(X/\mathbb{P}^N) \) is smooth as \( \text{Hilb}_{2t+1}(X/\mathbb{P}^N) \to \text{Hilb}_{2t+1}(\mathbb{P}^n) \) is a \( \mathbb{P}^N-5 \)-bundle, and \( \mathcal{C} \) is smooth as \( \mathcal{C} \to \mathbb{P}^n \) is a \( G(1,n-1) \times \mathbb{P}^4 \times \mathbb{P}^N-5 \) bundle.

The goal of this section is on conics is to prove

Proposition 5.1. The general fiber of \( \mathcal{C} \to \mathcal{X} \) is smooth and connected.

Proof. Since \( \mathcal{C} \to \mathcal{X} \) is a surjective map between smooth varieties, the general fiber is smooth by generic smoothness. To see connectedness, let \( \mathcal{C}' \to \mathcal{X} \) be the Stein factorization of \( \mathcal{C} \to \mathcal{X} \). Let \( D \subset \mathcal{C}' \to \mathcal{X} \) be the singular locus of \( \mathcal{C}' \to \mathcal{X} \). The inverse image of \( D \) in \( \mathcal{C} \) is contained in the singular locus of the map \( \mathcal{C} \to \mathcal{X} \). By Proposition 5.2, \( D \) is codimension at least 2 when pulled back to \( \mathcal{C} \) hence codimension at least 2 in \( \mathcal{C}' \). By the Purity theorem [13, Tag 0BMB], \( \mathcal{C}' \to \mathcal{X} \) is étale.

To finish, it suffices to see that \( \mathcal{X} \) is étale simply-connected. This follows from the fact that \( \mathcal{X} \to \mathbb{P}^n \) is a \( \mathbb{P}^{N}-1 \) bundle and the homotopy exact sequence for étale fundamental groups [13, Tag 0BUM]. □

Proposition 5.2. For \( n \geq 4 \), the singular locus of \( \mathcal{C} \to \mathcal{X} \) is codimension at least 2 in \( \mathcal{C} \).

Proof. By Proposition 5.3, we know Proposition 5.2 is true when we restrict \( \mathcal{C} \) to the locus where \( \mathcal{C} \to \text{Hilb}_{2t+1}(\mathcal{X}/\mathbb{P}^N) \) is smooth. Now, consider the locally closed subset \( \mathcal{Y} \subset \mathcal{C} \) consisting of pointed curves \( (\mathcal{C},p) \), where \( \mathcal{C} \) is a union of two distinct lines \( L_1, L_2, p \in L_1 \setminus L_2 \). The locus \( \mathcal{Y} \) is irreducible, as we can parameterize \( \mathcal{Y} \) by a smooth, irreducible variety \( \mathcal{Z} \to \mathcal{Y} \) by specifying \( (\mathcal{C},p) \) by first choosing \( p \in \mathbb{P}^n \), the two plane \( P \) containing \( C \), a line \( L_1 \) containing \( p \) and contained in \( P \), a second line \( L_2 \) contained in \( P \), and finally a hypersurface \( X \) containing \( L_1 \cup L_2 \). Since the complement of the union of \( \mathcal{Y} \) and the smooth locus of \( \mathcal{C} \to \text{Hilb}_{2t+1}(\mathcal{X}/\mathbb{P}^N) \) is codimension 2, it suffices to show the singular locus of \( \mathcal{C} \to \mathcal{X} \) is codimension 1 in \( \mathcal{Y} \).

A parameter count involving the normal bundle of a line in a hypersurface similar to [10, Proposition 4.3.9] shows the singular locus of the map from the universal line \( \overline{M}_{0,1}(\mathcal{X}/\mathbb{P}^N,1) \to \mathcal{X} \) is singular in codimension at least \( \min\{n-2, n-d\} = n-d \). A more complicated version of this parameter count is given in detail in the proof of Proposition 5.3 below. This means the map \( \mathcal{Z} \to \mathcal{X} \) is smooth at a general point. Since the image \( \mathcal{Y} \) of \( \mathcal{Z} \) is codimension 1 and \( \mathcal{Z} \to \mathcal{Y} \) has finite reduced fibers, this means \( \mathcal{C} \to \mathcal{X} \) is smooth at a general point of \( \mathcal{Y} \). □
Proposition 5.3. Let \( \text{Hilb}_{2t+1}(\mathcal{X}/\mathbb{P}^N) \subset \text{Hilb}_{2t+1}(\mathcal{X}/\mathbb{P}^N) \) denote the open locus of smooth conics. The singular locus of \( C \to \mathcal{X} \) is codimension at least \( \min\{n - 2, 2n - 2d + 1\} \) in \( \mathcal{C}|_{\text{Hilb}_{2t+1}(\mathcal{X})} \) for \( d \geq 4 \). If \( d = 3, n = 4 \) then the codimension is at least 2.

Proof. Since all smooth conics are projectively equivalent, we can fix the smooth conic \( C \subset \mathbb{P}^n \) with ideal \((Q(X_0, X_1, X_2), X_3, \ldots, X_n)\) and parameterization \( \mathbb{P}^1 \to \mathbb{P}^n \) given by \((s, t) \to (s^2, st, t^2, 0, \ldots, 0)\). It suffices to show that, in the \( \mathbb{P}^{N-5} \) hypersurfaces \( X \) that contain \( C \), the locus of hypersurfaces \( X \) with \( h^1(\mathcal{N}_{C/X}(-p)) \neq 0 \) is codimension at least 2 for \( p \in C \).

Let \( F(X_0, \ldots, X_n) \) cut out a hypersurface \( X \) containing \( C \). Then, \( F \) can be written as

\[
F = Q(X_0, X_1, X_2)G(X_0, X_1, X_2) + X_3F_3(X_0, \ldots, X_n) + \cdots + X_nF_n(X_0, \ldots, X_n),
\]

where \( G \) is degree \( d - 2 \) and each \( F_i \) is degree \( d - 1 \). The polynomials \( G \) and \( F_i \) can be chosen independently. For \( p \in C \),

\[
\partial_i F(p) = \begin{cases} 
G(p)\partial_i Q(p) & \text{if } 0 \leq i \leq 2 \\
F_i(p) & \text{if } 3 \leq i \leq n,
\end{cases}
\]

so \( X \) is smooth along \( C \) if and only if \( G, F_3, \ldots, F_n \) do not have a common zero on \( C \), in particular the locus of hypersurfaces singular at a point of \( C \) is codimension at least \( n - 2 \) in \( \mathbb{P}^{N-5} \). Therefore, it suffices to restrict to the open locus in \( \mathbb{P}^{N-5} \) of hypersurfaces smooth along \( C \). Consider the short exact sequence

\[
0 \longrightarrow N_{C/X} \longrightarrow N_{C/\mathbb{P}^n} \longrightarrow N_{X/\mathbb{P}^n}|_C \longrightarrow 0
\]

\[
\mathcal{O}_C(2H) \oplus \mathcal{O}_C(H)^{n-2} \longrightarrow \mathcal{O}_C(dH)
\]

Given \( N_{C/\mathbb{P}^n}^\vee = (Q, X_3, \ldots, X_n)/(Q, X_3, \ldots, X_n)^2 \) and \( N_{X/\mathbb{P}^n}|_C^\vee = (F)/(F)(Q, X_3, \ldots, X_n) \) and the map \( N_{X/\mathbb{P}^n}|_C^\vee \to N_{C/\mathbb{P}^n}^\vee \) is induced by inclusion \((F) \subset (Q, X_3, \ldots, X_n)\), the map \( \mathcal{O}_C(2H) \oplus \mathcal{O}_C(H)^{n-2} \to \mathcal{O}_C(dH) \) is given by multiplication by the vector \((G, F_3, \ldots, F_n)\). The long exact sequence in cohomology implies \( H^1(N_{C/X}(-p)) = 0 \) if and only if \( H^0(\mathcal{O}_C(2H - p) \oplus \mathcal{O}_C(H)^{n-2}) \to H^0(\mathcal{O}_C(dH)) \) is surjective.

By pulling back via the parameterization \( \mathbb{P}^1 \to \mathbb{P}^n \), we can reduce ourselves to the following problem. Let \( W_a = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a)) \) be the degree \( a \) forms in 2 variables. Consider the locus \( U \subset W_{2d-4} \times W_{2d-2}^{n-2} \) of tuples \((G', F'_3, \ldots, F'_n)\) that have no common zero on \( \mathbb{P}^1 \). We want to show the locus \( Z \subset U \) of forms \((G', F'_3, \ldots, F'_n)\) for which the map \( W_3 \times W_1^{n-2} \to W_{2d-1} \) given by \((b, b_3, \ldots, b_n) \to bG' + b_3F'_3 + \cdots + b_nF'_n \) is not surjective is codimension at least 2.

Let \( V \subset W_{2n-1} \) be a hyperplane. We want to bound the loci of forms \((G', F'_3, \ldots, F'_n)\) for which the image \( W_3 \times W_1^{n-2} \to V \) is contained in \( V \). Since the forms \((G', F'_3, \ldots, F'_n)\) have no common zero, we can assume \( V \) is not of the form \( \{A \in W_{2d-1} | A(p) = 0\} \) for \( p \in \mathbb{P}^1 \).

By [10] Lemma 4.3.1.1, the locus of \( F'_i \in W_{2d-2} \) for which the map \( W_1 \to W_{2d-1} \) given by multiplication by \( F'_i \) is contained in \( V \) is codimension 2.

To finish, we have to consider the locus of \( G' \in W_{2d-4} \) for which the map \( W_3 \to W_{2d-1} \) is given by multiplication by \( G' \) is contained in \( V \). Let the general element of \( W_{2d-4} \) be written as \( a_0s^{2d-4} + a_1s^{2d-5}t + \cdots + a_{2d-3}t^{2d-4} \), the general element of \( W_{2d-1} \) be written as \( b_0s^{2d-1} + b_1s^{2d-2}t + \cdots + b_{2d}t^{2d-1} \), and \( V \) be given by \( c_0b_0 + \cdots + c_{2d}b_{2d} = 0 \). Then, the
Consider the map contains an open dense subset contained in $B$. Let $G$ be a chain of inclusions
$$\begin{align*}
\{ \mathcal{C} \in \mathcal{M}_{0,1}(\mathcal{X}/\mathbb{P}^n, e) \mid \text{a component of ev}^{-1}(\text{ev}([\mathcal{C}])) \text{ containing } [\mathcal{C}] \text{ has no reducibles} \}.
\end{align*}$$

Note that $\mathcal{B}$ is constructible by Proposition 5.5, so in particular every component of $\mathcal{B}$ contains an open dense subset contained in $\mathcal{B}$.

Consider the map $\mathcal{B} \to \mathbb{P}^N$, consider the closed locus $\mathcal{B}' \subset \mathcal{B}$ where the fiber of $\text{im}(\mathcal{B} \to \mathcal{X}) \to \mathbb{P}^N$ is dimension $n - 1$ (e.g. the entire hypersurface). We want to control the image

The space of hyperplanes $V \subset W_{2d-1}$ is parameterized by $\mathbb{P}W_{2d-1}^*$. From the description above, the locus in $\mathbb{P}W_{2d-1}^*$ of all $V$ that impose $1 \leq i \leq 3$ conditions on the $G' \in W_{2d-4}$ for the multiplication by $G'$ map $W_3 \to W_{2d-1}$ to have image in $V$ is the $i^{th}$ secant variety to the rational normal curve [8, Proposition 9.7]. Let $S^1 \subset S^2 \subset S^3 \in \mathbb{P}W_{2d-1}^*$ be the first, second and third secant varieties to the rational normal curve. Since we have restricted ourselves to the open locus $U$ of forms with no common zero, we do not have to consider the case where $V$ is in $S^1$. We have to consider $S^2 \setminus S^1$, $S^3 \setminus S^2$, $S^3 \setminus \mathbb{P}W_{2d-1}^*$.

If $d \geq 4$, then $S^3 \in \mathbb{P}W_{2d-1}^*$ is a proper subvariety and combining these cases yields the locus $Z \subset U$ is codimension at least $4 + 2(n - 2) - (2d - 1) = 2n - 2d + 1$. If $d = 3$ and $n = 4$, then $S^3 = \mathbb{P}W_{2d-1}^*$, so the codimension is at least $3 + 2(n - 2) - (2d - 1) = 2n - 2d = 2$. \square

5.2. Layeredness. Let $X \subset \mathbb{P}^n$ be a hypersurface of degree $d$, where $d \leq n - 1$.

**Definition 5.2.** A point $p \in X$ is called $e$-layered if it is 1-level and for every $1 < k \leq e$, every irreducible component parameterizing degree $k$ rational curves through $p$ contains reducibles.

**Definition 5.3.** A hypersurface $X$ is $e$-layered if it is $e$-level and a general point is $e$-layered.

**Remark 2.** We have corresponded with the authors of [11] and it’s not clear how their argument as written shows $e$-layeredness at all points as claimed, but they brought us to the attention that $e$-levelness at all points and $e$-layeredness at a general point suffices to prove their main theorem.

5.3. Existence of $e$-layered hypersurfaces. Let $\mathbb{P}^N$ be the space of degree $d$ hypersurfaces in $\mathbb{P}^n$, and $T_e \subset \mathbb{P}^N$ denote the closure of the hypersurfaces that are not $e$-layered. We have a chain of inclusions
$$T_1 \subset T_2 \subset T_3 \subset \cdots \subset \mathbb{P}^N.$$ 

**Theorem 5.4.** The codimension of $T_{e-1} \subset T_e$ is at most $2n - (n - d + 1)e$.

**Proof.** From Theorem 4.8, we know the codimension of $T_{e-1} \subset T_{e-1} \cup S_e$ is at most $2n - (n - d + 1)e$. Using the same setup as the proof of Theorem 4.8, let $\mathcal{B}$ be

$$\{ [\mathcal{C}] \in \mathcal{M}_{0,1}(\mathcal{X}/\mathbb{P}^N, e) \mid \text{a component of ev}^{-1}(\text{ev}([\mathcal{C}])) \text{ containing } [\mathcal{C}] \text{ has no reducibles} \}.$$

Note that $\mathcal{B}$ is constructible by Proposition 5.5, so in particular every component of $\mathcal{B}$ contains an open dense subset contained in $\mathcal{B}$. We want to control the image
$B' \to \mathbb{P}^N$. Let $\mathcal{A} \subset B'$ be an irreducible component. If $\mathcal{A}$ contains a map that isn’t a cover of a line, then $\mathcal{A} \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$ has dimension at least $3n - 3$. Applying Lemma 4.9 as before allows us to cut $\mathcal{A}$ by hyperplane sections in $\mathbb{P}^N$ to obtain $\mathcal{A}' \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$ has image dimension $2n - 1$ and the image of the generic fiber of $\mathcal{A}' \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$ in $\mathbb{P}^N$ is finite. Lemma 4.3 shows $\mathcal{A}'$ contains reducibles.

This means $\text{im}(\mathcal{A}' \to \overline{\mathcal{M}}_{0,0}(\mathcal{X}/\mathbb{P}^N, e))$ is also dimension $2n - 1$. Since the fiber dimension of $\text{im}(\mathcal{A}' \to \mathcal{X}) \to \mathbb{P}^N$ is $n - 1$, the fiber dimension of $\text{im}(\mathcal{A}' \to \overline{\mathcal{M}}_{0,0}(\mathcal{X}/\mathbb{P}^N, e))) \to \mathbb{P}^N$ is at least $(n - 1) + (a - 1)$. The image of $\mathcal{A}' \to \mathbb{P}^N$ is dimension at most $2n - 1 - (n + a - 2)$, where $a = (n - d + 1)e - 2$ as in the proof of Theorem 4.8.

Assume for the sake of contradiction that the codimension of $T_{e-1} \subset T_e$ is at least $2n - (n - d + 1)e + 1$. Then, since $n \geq 3$ the image $\mathcal{A}' \to \mathbb{P}^N$ misses the locus $S_{e-1}$, which contradicts [11] Proposition 5.5 and Proposition 4.2. Here, the point is [11] Proposition 5.5 shows the reducible curves is codimension at least 1 in each fiber of $\mathcal{A}' \to \mathcal{X}$, the general fiber of $\mathcal{A}' \to \mathcal{X}$ contains no reducible curves, and singular points occur in the image of $\mathcal{A}' \to \mathcal{X}$ in codimension at least $n - 1$ by levelness. This means reducibles occur in $\mathcal{A}'$ in codimension at least 2, which contradicts Proposition 4.2. □

**Proposition 5.5.** If $f: X \to Y$ is a map between Noetherian schemes and $Z \subset X$ a closed subset. Then, the set $A \subset X$ of all $x \in X$ such that a geometric component of the fiber $f^{-1}(f(x))$ containing $x$ is disjoint from $Z$ is constructible.

**Proof.** By restricting to each component of $X$, we can assume $X$ is irreducible. By Noetherian induction on $X$, it suffices to prove Proposition 5.5 after restriction to some open subset of $Y$. Let $\eta \in Y$ be the generic point and $X_\eta$ be the fiber over the generic point. After an étale base change of $Y$, we can assume all the irreducible components of $X_\eta$ are geometrically irreducible [13 Tag 0551]. Let $X_{1,\eta}, \ldots, X_{r,\eta}$ denote the irreducible components of $X_\eta$ and $X_1, \ldots, X_r$ closures of $X_{1,\eta}, \ldots, X_{r,\eta}$ in $X$ respectively. We can replace $Y$ by an open subset so that $X$ is the set-theoretically the union of $X_1, \ldots, X_r$ [13 Tag 054Y].

By generic flatness [13 Tag 052A] and the fact that flat morphisms are open [13 Tag 01UA], we can replace $Y$ by an open subset so that each $X_i \to Y$ is flat and surjective. If $Z$ does not intersect $X_{i,\eta}$, then $Z \cap X_i$ maps to a constructible subset of $Y$ not containing $\eta$, so we can replace $Y$ by an open subset so that $Z \cap X_i$ is empty. Similarly, if $Z$ does intersect $X_{i,\eta}$, then $Z \cap X_i$ maps to a constructible subset of $Y$ that does contain $\eta$, so we can replace $Y$ by an open subset so that $Z \cap X_i$ surjects onto $Y$. Therefore, we can assume for each $i$ that $Z \cap X_i$ is empty or $Z \cap X_i$ surjects onto $Y$.

By replacing $Y$ with an open subset, we can assume that $X_i \cap X_j$ is nowhere dense in each fiber of $X_i \to Y$ [13 Tag 054X]. By replacing $Y$ with an open subset, we can assume that each geometric fiber of $X_i \to Y$ is irreducible [13 Tag 0559]. Finally, if we let $S \subset \{1, \ldots, r\}$ be the subset of indices $i$ such that $X_i$ does not intersect $Z$, then $A$ is the union $\bigcup_{i \in S} X_i$. □

5.3.1. **Case of degree $d \leq 6.$** When $d \leq 6$, it suffices to consider the case $e \leq 2$, in which case Proposition 5.1 suffices.

5.4. **Behrend-Manin stacks.** Let $X$ be a smooth hypersurface. In order to keep track of the combinatorics of the components of reducible rational curves, we will use Behrend-Manin stacks. We refer the reader to [2 Definitions 1.6, 3.13] for the precise definitions of a stable
A-graph $\tau$ and the associated Behrend-Manin stack $\overline{M}(X, \tau)$. Also see [9] for a shorter account that suffices for our purposes.

Roughly, a stable A-graph keeps track of the combinatorics of the irreducible components of a stable map $C \to X$, including the dual graph of how they intersect, the marked points on each component, the degree of the map restricted to each component, and the genus of each component. Since we are dealing with rational curves, all the stable A-graphs we consider will have genus zero, meaning the genus of each vertex is zero and the underlying graph is a tree.

Associated to a stable A-graph $\tau$, there is a set of vertices $\text{Vertex}(\tau)$, a set of edges $\text{Edge}(\tau)$ connecting them, and a set of tails $\text{Tail}(\tau)$, which can be thought of as half edges attached to vertices. There is also a map $\beta : \text{Vertex}(\tau) \to \mathbb{Z}_{\geq 0}$, assigning a degree to each vertex. We also let $\beta(\tau) = \sum_{v \in \text{Vertex}(\tau)} \beta(v)$ and the expected dimension

$$\dim(X, \tau) := (n + 1 - d)\beta(\tau) + \#\text{Tail}(\tau) - \#\text{Edge}(\tau) + \dim(X) - 3$$

[9] Definition 3.4]. Finally, there is a set of flags $\text{Flag}(\tau)$, where we have two flags corresponding to each edge in $\text{Edge}(\tau)$, corresponding to the two endpoints, and one flag for each tail. In particular, $\#\text{Flag}(\tau) = 2\#\text{Edge}(\tau) + \#\text{Flag}(\tau)$.

The Behrend-Manin stack $\overline{M}(X, \tau)$ parameterizes stable maps $C \to X$, where the curve $C$ consists of prestable curves $C_v$ [2] Definition 2.1, one for each vertex of $\tau$, that glue together and map to $X$ according to the data in $\tau$. The open locus of $\overline{M}(X, \tau)$ of strict maps is quicker to define. See [9] Definition 3.7] for more details.

Definition 5.4. A stable map $C \to X$ in $\overline{M}(X, \tau)$ is a strict map if $C_v \cong \mathbb{P}^1$ for each $v \in \text{Vertex}(\tau)$. The locus of strict maps is an open substack $\mathcal{M}(X, \tau) \subset \overline{M}(X, \tau)$.

A point in $\overline{M}(X, \tau) \subset \overline{M}(X, \tau)$ can be specified by the data $((C_v)_{v \in \text{Vertex}(\tau)}, (h_v : C_v \to X)_{v \in \text{Vertex}(\tau)}, (q_f)_{f \in \text{Flag}(\tau)})$ such that $q_f \in C_v$, where $v$ is the vertex to which $q_f$ is attached. Each map $h_v : C_v \to X$ is specified to have degree $\beta(v)$.

Since we want to think of $\text{Tail}(\tau)$ as parameterizing marked points, for each $f \in \text{Tail}(\tau)$, we have an evaluation map

$$\text{ev}_f : \overline{M}(X, \tau) \to X$$

[9] Definition 3.11]. Similarly, we have $\text{ev}_f$ for all $f \in \text{Flag}(\tau)$, as the remaining flags correspond to the points of intersection between different prestable curves $C_v$ that piece together to give the domain of a stable map $C \to X$, and we can ask for the image of such an intersection point.

5.5. A criterion for smoothness.

Definition 5.5. Let $\tau_r(e)$ be the stable A-graph that has one vertex $v$, no edges, $r$ tails such that $\beta(\tau) = \beta(v) = e$. By definition, $\overline{M}(X, \tau_r(e))$ is the Kontsevich space $\overline{M}_{0,r}(X, e)$. Given any A-stable graph with $\beta(\tau) = e$ and $\#\text{Tail}(\tau) = r$, there is a morphism $\overline{M}(X, \tau) \to \overline{M}_{0,r}(X, e)$ canonical up to relabeling the tails.

If we repeatedly specialize a rational curve $C \to X$ so that it breaks up into more and more components, then we eventually end up with a tree of lines. Since we will care about rational curves through a general point $p \in X$ given by a tree of lines, we make the following definition. By abuse of notation, it is different than the one given in [9] Definition 5.8].
Definition 5.6. Let a stable $A$-graph $\tau$ be called a basic $A$-graph if $\beta(v) \in \{0, 1\}$ for all $v \in \text{Vertex}(\tau)$ and $\#\text{Tail}(\tau) = 1$.

Definition 5.7. Let a basic $A$-graph be called nondegenerate if $\beta(v) = 1$ for all $v \in \text{Vertex}(\tau)$.

The argument in [12 Proposition 6.6] applied in our case gives

Proposition 5.6. Let $\tau$ be a basic $A$-graph and $X \subset \mathbb{P}^n$ be a smooth $e$-level hypersurface of degree $d = n - 1$ with an irreducible Fano scheme of lines. Then, the morphism $\overline{M}(X, \tau) \to \overline{M}_{0,1}(X, e)$ maps a general point of $\overline{M}(X, \tau)$ to a point in the smooth locus of $\text{ev} : \overline{M}_{0,1}(X, e) \to X$.

Proof. Applying the argument in [12 Proposition 6.6], reduces the question of checking whether a point $(h : C \to X) \in \overline{M}(X, \tau)$ is a smooth point of $\overline{M}_{0,1}(X, e) \to X$ to checking whether $H^1(C, h^*T_X(-p)) = 0$, where $p \in C$ corresponds to the unique tail in $\text{Tail}(\tau)$.

The tangent space to a fiber of $\overline{M}_{0,1}(X, 1) \to X$ at a pair $(\ell, p)$, where $p \in \ell \subset X$ and $\ell$ is a line is $H^0(N_{\ell/X}(-p))$, and this is of the expected dimension if and only if $H^1(N_{\ell/X}(-p)) = 0$. By generic smoothness, this holds for a general pair $(\ell, p)$. From the short exact sequence, $0 \to T\ell \to TX|_{\ell} \to N_{\ell/X} \to 0$, $H^1(TX|_{\ell}(-p)) = 0$ for a general point $(\ell, p) \in \overline{M}_{0,1}(X, 1)$. Applying [12 Lemma 6.2] allows us to conclude.

Remark 3. Instead of arguing via the smoothness of the nonseparated Artin stack of prestable curves as in [12 Proposition 6.6] in the beginning of the proof of Proposition 5.6, an equivalent way is to add $c$ marked points to $C \to X$ so the prestable curve $C$ is actually stable. Then, it suffices to show smoothness of $\overline{M}_{0,1+c}(X, e) \to \overline{M}_{0,1+c} \times X$ at $C \to X$ as this implies the map $\overline{M}_{0,1+c}(X, e) \to X$ is smooth at $C \to X$.

Note that $e$-levelness guarantees flatness of the evaluation map [9 Lemma 4.5], so smoothness at a point is equivalent to being smooth in its fiber.

The condition on the Fano scheme is automatically satisfied in our case since the Fano scheme of lines is smooth and connected for a general hypersurface [10 Theorem 4.3] if the degree $d$ of $X \subset \mathbb{P}^n$ is at most $2n - 4$ and $X$ is not a quadric surface.

5.6. Rational curves through a point.

Theorem 5.7. Let $e \geq 2$ and $d = n - 1$. If there exist $e$-layered hypersurfaces, then for a general hypersurface $X$, the fiber $F_p$ of the evaluation map

$$\overline{M}_{0,1}(X, e) \to X$$

over a general point $p \in X$ has only one component that contains curves $C \to X$ that are not multiple covers of a line.

Proof. The case $e = 2$ is Proposition 5.1, so suppose $e > 2$. We will use strong induction on $e$. By $e$-levelness, each component of $F_p$ has the same dimension. If $C \to X$ is a rational curve in $X$ through $p$, we can use $e$-layeredness to specialize $C \to X$ to $C_0 \to X$, so that $C_0 \to X$ lies in $\mathcal{M}(X, \tau)$, where $\tau$ is a nondegenerate basic $A$-graph. By Proposition 5.6, we can assume $C_0 \to X$ is a smooth point of $F_p$. 


Each component of the fiber of $\overline{M}(X, \tau) \to X$ over $p$ lies in a unique component of $F_p$. What we need to show is that as we vary over all nondegenerate basic $A$-graphs $\tau$ we only get one component of $F_p$ that contains curves that are not covers of a line. To do this, we will reduce ourselves to looking at “combs” of lines, where the backbone gets collapsed. One can get this by specializing a tree of lines to a “broom”, where all the lines pass through $p$. For clarity, we will instead first reduce to the case of chains of lines and then specialize the chain of lines to a comb.

As before, let $C_0 \to X$ be in $\mathcal{M}(X, \tau)$, where $\tau$ is a nondegenerate basic $A$-graph. Let $(C_v)_{v \in \text{Vertex}(\tau)}$ be the components of $C_0$. Note that each $C_v \cong \mathbb{P}^1$. Let $v_0$ be the vertex to which the unique tail of $\tau$ is attached. By abuse of notation, we call the marked point in $C_{v_0}$ that maps to $p$ under $C \to X$ also as $p \in C_{v_0}$. Let $q_{a_1}, \ldots, q_{a_r} \in C_{v_0}$ correspond to the edges attached to $v_0$ in $\tau$, or the points of attachment of the other components of $C$ to $C_{v_0}$.

Now, we specialize the points $q_{a_1}, \ldots, q_{a_r}$ one by one to a fixed general point $q_a \in C_{v_0}$. Let the resulting curve be $C'_0 \to X$, given by gluing together the prestable curves $(C''_v)_{v \in \text{Vertex}(\tau)}$. From the argument in [5, Proposition 6], if we want to understand what happens to $C_{v_0}$ in the limit as we specialize, it suffices to understand what happens to the map $(C_{v_0}, q_{a_1}, \ldots, q_{a_r}) \to X$ from the pointed curve $(C_{v_0}, q_{a_1}, \ldots, q_{a_r})$ as we specialize the points $q_{a_1}, \ldots, q_{a_r}$.

Then, $C'_{v_0}$ gets replaced with the prestable curve $C''_{v_0}$ that is $C_{v_0}$ with a chain of rational curves attached at $q_a$. Proposition 5.6 tells us that $C'_0 \to X$ is a smooth point of $F_p$. By induction, the space of degree $e - 1$ curves through $q_a$ contains only one component with curves that do not cover a line.

This means $C'_0 \to X$ is in the same component of $F_p$ as the curve we get when we take $C''_{v_0} \cong \mathbb{P}^1$ and attach a general chain of lines to $q_a$ in the same component of degree $e - 1$ curves through $q_a$. This chain of lines may be a cover of a line. In this way, we have reduced to the case where $\tau$ is a chain.

Now, let $v_0, \ldots, v_{e-1}$ be $\text{Vertex}(\tau)$, where each $v_i$ is connected to $v_{i+1}$ for $0 \leq i \leq e - 2$. Let the unique tail of $\tau$ be attached to $v_0$ and $h : C \to X$ given by $(h_i : C_{v_i} \to X)$ be a point of $\overline{\mathcal{M}}(X, \tau) \cap F_p$, general in its component. As before, each $h_i : C_{v_i} \cong \mathbb{P}^1 \to X$ is an embedding of a line. Now, we want to specialize the points of attachment.

Let $p_0 = p$ and $p_i$ be the point of $C_{v_i}$ that is attached to $C_{v_{i-1}}$ under $h : C \to X$ for $1 \leq i \leq e - 1$. For $0 \leq i \leq e - 2$, let $q_i \in C_{v_i}$ be the point that is attached to $C_{v_{i+1}}$. Now,
we take a 1-dimensional family that specializes $q_0$ to $p_0$. Then, we specialize $q_1$ to $p_1$. Let $C' \to X$ given by $(h'_i : C'_{v_i} \to X)$ be the result after specializing $q_i$ to $p_i$ for all $0 \leq i \leq n - 2$. Then, each $C''_{v_i}$ for $0 \leq i \leq e - 2$ becomes a union of two rational curves, where the one containing $q_i$ and $p_i$ is collapsed under the map to $X$.

Proposition 5.6 tells us that $C' \to X$ is a smooth point of $F_p$. However, we note that $C' \to X$ is also a specialization of a strict map $D \to X$ lying in $\mathcal{M}(X, \tau_{\text{comb}}) \cap F_p$, where $\text{Vertex}(\tau_{\text{comb}}) = \{v_{\text{center}}, v_0, \ldots, v_{e-1}\}$ with $\beta(v_{\text{center}}) = 0$ and $\beta(v_i) = 1$, and the edges of $\tau_{\text{comb}}$ connect $v_{\text{center}}$ to each of $v_0, \ldots, v_{e-1}$. The unique tail of $\tau_{\text{comb}}$ is attached to $v_{\text{center}}$. Proposition 5.6 says that a general choice of a strict map $D$ given by $(g_v : D_v \to X)_{v \in \text{Vertex}(\tau_{\text{comb}})}$ is a smooth point of $F_p$. In fact, since the choice of the maps $D_{v_i} \to X$ is discrete, every choice of a strict map $D \to X$ is smooth.

The backbone $D_{v_{\text{center}}}$ gets collapsed to $p$ under the map to $X$ and each $D_{v_i}$ gets mapped to a line through $p$. To finish, we need to show that all strict maps $D \to X$, where not all of the $D_{v_i}$ get mapped to the same line, are in the same component of $F_p$.

Let $S \subset \{v_0, \ldots, v_{e-1}\}$ be a strict subset, with $\# S \geq 2$ and such that the maps $D_v \to X$ for $v \in S$ don’t all embed as the same line. Then, we can specialize $D \to X$ to $D' \to X$, where $D'_{v_{\text{center}}}$ is now a rational curve with two components, one that is attached to $D'_v$ for all $v \in S$ and the other that contains $p$ and is attached to $D'_v$ for $v \notin S$. The maps $D_{v_i} \cong D'_{v_i} \to X$ are unchanged. Then, using the induction hypothesis, this map $D' \to X$ is in the same component of $F_p$ as the map we would get by modifying the lines $D'_v \to X$ for $v \in S$, so long as $D'_v \to X$ for $v \in S$ don’t all embed as the same line.

The end result is that $D \to V$ is in the same component as the map we would get by modifying $D_v \to X$ for $v \in S$, so long as we maintain $D_v \to X$ for $v \in S$ don’t all embed as the same line. In this way, we see that all strict maps $D \to X$, where not all of the $D_{v_i}$ get mapped to the same line, are in the same component of $F_p$. \qed
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