1. Introduction

Whereas usual Hodge theory concerns mainly the usual or abelian cohomology of an algebraic variety—or eventually the rational homotopy theory or nilpotent completion of $\pi_1$ which are in some sense obtained by extensions—nonabelian Hodge theory concerns the cohomology of a variety with nonabelian coefficients. Because of the basic fact that homotopy groups in higher dimensions are abelian, and since cohomology theories can generally be interpreted as spaces of maps into classifying (or Eilenberg-MacLane) spaces, nonabelian cohomology occurs essentially only in degree 1. There are certainly some degree 2 aspects which are as of yet totally untouched; and the same goes for the degree 1 case with twisted coefficient systems. (See however [70] for a direction of development combining the nonabelian coefficients in degree 1 with abelian coefficients in higher degrees). If we leave these aside, we are left with the case of $H^1(X, G)$ for $G$ a nonabelian group. It is most natural to interpret this cohomology as a groupoid, or, when $G$ is a group-scheme, to interpret $H^1(X, G)$ as a stack. It is the stack of flat principal $G$-bundles on $X$. Recall from the usual abelian case that in order to obtain a Hodge structure, we must consider cohomology with complex coefficients. The analogue in the nonabelian case is that we must take as coefficient group a group-scheme $G$ over the complex numbers (and in fact it should be affine too). This then is the domain of application of the work that has been done in nonabelian Hodge theory: the study of properties and additional structure on the moduli stack $\mathcal{M}(X, G) := H^1(X, G)$ which are the analogues in an appropriate sense of the main structures or properties of abelian cohomology.

By its nature, the first nonabelian cohomology is an invariant of the fundamental group $\pi_1(X)$. The study of nonabelian Hodge theory may thus be thought of as the study of fundamental groups of algebraic varieties or compact Kähler manifolds. It is important to note, specially in light of Toledo’s examples of $\pi_1(X)$ not residually finite, that the study of $\pi_1(X)$ via its nonabelian cohomology, i.e. via the spaces of homomorphisms $\pi_1(X) \to G$, will only “see” a certain part of $\pi_1(X)$ and in particular will not at all see the intersection of subgroups of finite index. It is an interesting question to try to understand what Hodge-theoretic methods could say about this more mysterious part of the fundamental group.

We start in §2 by reviewing Corlette’s nonabelian Hodge theorem [10] (cf also [18] and [17]) which is actually a generalization of the theorem of Eells and Sampson [19]. This
theorem allows us to choose a preferred metric on any flat bundle. By a Bochner technique on Kähler manifolds ([74], [77], [10]) a harmonic metric is in fact pluriharmonic, and we recover in this way the holomorphic data of a Higgs bundle. The correspondence in the other direction characterizes exactly which Higgs bundles arise in this way ([33], [63]).

We then mention (referring to [57], [69] for proofs) the existence of moduli spaces for all of the objects in question. The space $M_{DR}(X, G)$ is the moduli scheme for principal holomorphic $G$-bundles with integrable connection. It is a coarse moduli space for the moduli stack $\mathcal{M}_{DR}(X, G)$; it is this moduli stack which should be thought of as the nonabelian de Rham cohomology, and the moduli space is a convenient scheme-theoretic version. The space $M_{Dol}(X, G)$ is the moduli scheme for semistable principal Higgs bundles with vanishing rational Chern classes; again it is a coarse moduli space for $\mathcal{M}_{Dol}(X, G)$, the moduli stack which is what should be thought of as the nonabelian Dolbeault cohomology. The harmonic metric construction and the Bochner technique (together with the inverse construction) give a homeomorphism $M_{DR}(X, G) \cong M_{Dol}(X, G)$ which is $C^\infty$ on the smooth points.

The above work is the result of a long series of generalizations of the original work of Narasimhan and Seshadri. Without being exhaustive, I should at least mention the names of Mumford, Gieseker, Maruyama, Mehta and Ramanathan for the constructions of moduli spaces; and Donaldson, Uhlenbeck, Yau, Deligne and Beilinson for the inverse construction to the harmonic map construction. See the introductions and references of [65], [69] for more detailed historical references.

\[ \star \]

\[ \star \]

\[ \star \]

These constructions (or their predecessors in work of Eells and Sampson [13], [12] and Siu [71]) are the starting point for much of the work which has been done in nonabelian Hodge theory in the past several years. As many aspects are covered by other lectures (and their corresponding articles), and in any case many of the papers on the subject contain survey-like introductions, I will not try to survey all of these topics in the main part of the paper but will just mention some of them here in the introduction.

Hitchin was interested from the beginning in the completely integrable holomorphic Hamiltonian system given by the moduli space of Higgs bundles [32]. This direction of research has branched off toward the Verlinde formula, quantization and so forth. I won’t try to give references as this gets away from our principal concern of Hodge theory.

One of the principal applications of Hodge theory has always been to give restrictions on the topological type of varieties and their subvarieties. The nonabelian version presents this same feature. The existence of all of the various structures described above on the nonabelian cohomology $H^1(X, G)$ and various related considerations give restrictions on which groups can be fundamental groups of compact Kähler manifolds, or more generally on which homotopy types can arise. Some of these results such as [1], [5], [12], [71] pre-date
the general Hodge-theoretic point of view given above, being based on harmonic map considerations à la Eells-Sampson and Siu. Others come directly from the full correspondence between Higgs bundles and local systems and the subsidiary fact that Higgs bundles invariant under the natural action of $G_m = \mathbb{C}^*$ correspond to variations of Hodge structure \cite{65}. The restrictions on higher homotopy types are generally of two sorts: either one uses information about a harmonic map to obtain additional information on the harmonic map (such as its rank) and then concludes that such a harmonic map cannot exist (\cite{71} \cite{7} \cite{11}); or one uses various notions about the cohomology of local systems to rule out higher homotopy types (these are the restrictions coming from work of Green-Lazarsfeld \cite{23}, Beauville \cite{3}, Arapura \cite{1}, recently Hironaka \cite{31}, also \cite{66}—the idea of using these results to get restrictions will be explored in §3 below since many of these papers don’t explicitly mention the aspect “restrictions on homotopy types” which comes out of their results).

Gromov has an $L^2$-Hodge theoretic argument to rule out free (and certain amalgamated) products of groups \cite{24}. This is particularly interesting in relation to the theory we sketch here, because it allows one to “see” the whole fundamental group (for example, the amalgamated product of two groups with no subgroups of finite index is ruled out, which would evidently be impossible to do by looking at representations into linear groups). Gromov and Schoen \cite{25} have also developed a generalization of the harmonic map theory to cover harmonic maps into negatively curved Euclidean buildings. Coupled with a Lefschetz technique \cite{67} this gives results about fundamental groups and in particular seems to give an alternative proof of the result about amalgamated products. This technique is generalized in \cite{38} \cite{40}.

With all of these restrictions on fundamental groups coming from Hodge theory, one might well wonder if there are any interesting fundamental groups at all. Toledo’s example of a non residually finite fundamental group (\cite{73}—there have since been several other generalizations) shows that the family of groups which can occur is, on the contrary, quite complicated. We can still ask whether the part “seen” by nonabelian Hodge theory with linear groups as coefficients can be nontrivial, for example, are there nontrivial ways of obtaining positive dimensional moduli spaces (other than easily known ways using curves and abelian varieties)? The answer here is affirmative too, and in fact the Higgs bundle picture is essential for calculating what happens to obtain examples \cite{68}.

One of the main types of results has been the factorization theorem. This type of result relates the fundamental group and the geometry of $X$. The typical type of statement is that if $\rho : \pi_1(X) \to \Gamma$ is a certain type of representation then it must factor: there is a morphism of varieties $X \to Y$ such that $\rho$ factors through $\pi_1(Y)$. Perhaps the original result of this type is that of Siu \cite{71} (cf also Beauville’s appendix to \cite{3}) stating that when $\Gamma$ is the fundamental group of a Riemann surface, then any $\rho$ must factor through a Riemann surface $Y$. Gromov’s result of \cite{24} also passes through a similar type
of statement: certain $L^2$ cohomology classes on the universal cover of $X$ must factor through maps to a Riemann surface. One of the first statements involving factorization through a higher dimensional variety is that of Zuo [77]; and we now have a fairly complete picture of this type of result (cf [38] [39] [40] [43] [76] [77]): any nonrigid representation to a linear group $G$ must factor through a variety $Y$ of dimension less than or equal to the rank of $G$. Note that the example of [68] shows that we do not always get factorization through a curve...but apart from this we do not know for sure if the bound rank of $G$ is sharp. See also [32] and various generalizations for additional geometric information on the factorization variety $Y$.

Using the theory of harmonic maps to buildings mentioned above, one can extend these factorization theorems to the case of representations into linear groups over $p$-adic fields not going into a maximal compact subgroup [38] [40] [77].

These factorization techniques obviously have a certain application to the Shafarevich conjecture [16]. This has been pursued by Katzarkov, Lasell, Napier, Ramachandran ([12] [15] [19] [53] [57]), see also Zuo [77]. The main problem is that only the part of the fundamental group seen by a given linear representation can be treated. They obtain a full proof of the Shafarevich conjecture for surfaces whose fundamental group injects into a linear group.

Another recent development worth mentioning is Reznikov’s proof of the Bloch conjecture that the Chern-Simons classes of flat bundles on Kähler manifolds are torsion [59].

The last principal area of work I would like to mention, one where there is still a fair amount to be done, is the noncompact (quasiprojective) case. The problem is to do the analogues of everything which we discuss in the compact Kähler (or projective algebraic) case, in the case of a quasiprojective variety. This problem becomes much more difficult in dimension $\geq 2$. Aside from the dimension distinction, the problem can be divided into several parts.

The first part is to obtain the analogue of Corlette’s theorem. The main difficulty here is to get a starting point for a heat equation minimization process, that is to say an equivariant map of finite energy. If the eigenvalues of the monodromy at infinity are not of norm one, this becomes impossible and the problem is more difficult. Modulo this difficulty, the problem has been solved by Corlette [12] and Jost and Zuo [38].

The next problem is to obtain the analogues of the Bochner results, yielding a Higgs bundle. This is discussed to some extent in [38]. There may be a problem with Chern classes in general. The other aspect of this problem is that the appropriate Higgs bundle notion must include some data at infinity, namely a parabolic structure. The problem of associating a parabolic Higgs bundle with nice properties, to a harmonic bundle, is treated in Biquard [5] in the case when the divisor at infinity is smooth.
Biquard also provides the converse construction: given a parabolic Higgs bundle satisfying appropriate conditions, he gets back a Yang-Mills connection and hence a representation. This provides a relatively complete generalization in the case of smooth divisor at infinity. What is left open for the moment is to treat the case where the divisor at infinity has normal crossings.

A fourth aspect of the problem is to construct moduli spaces. This is now well understood for Higgs bundles and the like, due to work of Yokogawa and Maruyama [51] [74]. I think there is still a little work (probably not too hard) left to be done on the side of filtered local systems, which are the general representation-like objects which correspond to parabolic Higgs bundles.

It remains to be seen how all of these aspects fit together, and then to proceed with the generalizations of all of the further structures inherited by the moduli spaces in the compact case (i.e. the structures we will discuss in the present paper).

Rather than going into further detail on all of these applications and developments, I would like in the body of the paper to concentrate on a fundamental aspect—the non-abelian Hodge filtration [64]. We will discuss many of the basic subjects surrounding the Hodge filtration, such as the quaternionic structure and twistor spaces. And we give proofs of the results announced in [64], in particular the compactification of $M_{DR}$ which is a consequence of (and practically equivalent to) the Hodge filtration. The goal will be in the last section to introduce an open problem, that of studying degenerations of nonabelian Hodge structure coming from a degenerating family of varieties.

After our discussion of Corlette’s theorem, its converse and the moduli spaces in §§1-2, we turn in §3 to a discussion of Hitchin’s quaternionic structure on the moduli space for representations [33] [21]. We give an application to cohomology jump-loci retrieving the results of Green and Lazarsfeld via an argument of Deligne and along the way see how these results give restrictions on the higher homotopy type of non-simply connected Kähler manifolds.

In §4 we look at Deligne’s complex analytic construction of the twistor space corresponding to the quaternionic structure [13]. He obtains the twistor space by glueing two copies of a family $M_{Hod}$ deforming between the moduli space $M_{DR}$ of vector bundles with integrable connection and the moduli space $M_{Dol}$ of Higgs bundles. This deformation (parametrized by $A^1$) is the moduli space of vector bundles with $\lambda$-connections; over $\lambda = 0$ a $\lambda$-connection is just a Higgs field, whereas over $\lambda \neq 0$ a $\lambda$-connection is $\lambda$ times an integrable connection.

In §5 we explain the analogy with Rees modules which allows us to interpret the space $M_{Hod}$ as the Hodge filtration on $M_{DR}$. 
As justification we establish in §6 the relationship with the Morgan-Hain Hodge filtration on the nilpotent completion of the fundamental group \([53][27]\).

We then proceed with certain results about the Hodge filtration in the nonabelian case, notably Griffiths transversality for its variation in a family, and regularity of the Gauss-Manin connection at singular points of a family (§8). In order to do this, we first introduce in §7 the notion of formal groupoid \([4][36]\). This provides a general framework for looking at connections, Higgs fields and so forth, and in particular allows us to actions of these types of things on schemes rather than just vector bundles.

In a detour §9 we investigate Goldman-Millson theory \([22]\) for the local structure of \(M_{Hod}\). The isosingularity principle which says that the singularities of \(M_{Dol}\) are the same as those of \(M_{DR}\), generalizes to give a trivialization of \(M_{Hod}\) formally along preferred sections. This allows us to conclude, for example, that \(M_{Hod}\) is flat over \(A^1\).

Then we come to a properness property of \(M_{Hod}\); the limits of \(G_m\)-orbits always exist (§10). This is the analogue of the classical property of the Hodge filtration, that \(F^0\) is the whole space.

This weight property allows us to obtain a compactification of \(M_{DR}\) in §11, by taking the quotient of an open set in \(M_{Hod}\) by the action of \(G_m\). This compactification was announced without proofs in \([54]\) so we take this opportunity to provide a complete version of the argument. (Drinfeld recently informed me that some of his students have obtained a compactification of the moduli space of logarithmic connections on \(P^1\) with singularities at a finite set of points, also using the method of \(\lambda\)-connections but independently of \([54]\).)

At the end of the section we revisit Griffiths transversality in terms of this compactification: it says that the Gauss-Manin connection on \(M_{DR}\) has poles of order 1 at infinity in the compactification. This gives a picture of a compact space \(\overline{M_{DR}(X/S,G)}\) with a lift of a frame vector field on \(S\) to a vector field having a simple pole at infinity. A similar interpretation holds for the regularity of the Gauss-Manin connection.

In the penultimate section 12 we define the nonabelian Noether-Lefschetz locus \(NL(X/S, GL(n))\). If \(X \to S\) is a family then this is essentially the locus of \(s \in S\) where \(X_s\) supports an integral variation of Hodge structure. It is the nonabelian analogue of the classical Noether-Lefschetz locus of Hodge cycles. If \(S\) is projective then we can see that \(NL(X/S, GL(n))\) is algebraic (as would be a consequence of the Hodge-type conjecture that one could formulate, that integral variations of Hodge structure are motivic). We conjecture that this is true even for \(S\) quasiprojective, which would be a nonabelian version of the result of \([7]\).

In the last section we present an open problem, the problem of understanding the degeneration of all of our structures (Hodge filtration, Gauss-Manin connection, quaternionic structure, etc.) near degenerations of a family \(X/S\). This problem is motivated by the problem of proving that \(NL(X/S)\) is algebraic when \(S\) is quasiprojective—the nonabelian analogue of the work of Cattani, Deligne and Kaplan for the classical Noether-Lefschetz
locus of Hodge cycles \[\mathfrak{z}\].

I would like to thank P. Deligne for sharing with me his ideas on how to construct Hitchin’s twistor space. This construction provided the starting point for everything (new) done below.

Everything is over the field \(\mathbb{C}\) of complex numbers.

2. The nonabelian Hodge theorem

Suppose \(X\) is a Riemannian manifold with basepoint \(x \in X\), and suppose \(G\) is a reductive algebraic group. A representation \(\rho : \pi_1(X, x) \to G\) corresponds to a flat principal left \(G\)-bundle over \(X\) (in other words a locally constant sheaf of principal homogeneous spaces for \(G\) over \(X\)), or equally well to a \(\mathcal{C}^\infty\) principal \(G\)-bundle \(P\) with an integrable connection \(D\). We think of a connection on a principal bundle as a \(G\)-invariant operator \(\nabla\) from functions on \(P\) to sections of \(T^*(X)|_P\) (satisfying a Leibniz rule with respect to functions pulled back from the base). Such an operator then has a square \(\nabla^2\) from functions on \(P\) to sections of \(\Lambda^2 T^*(X)|_P\), and the integrability condition is \(\nabla^2 = 0\). The flat principle bundle is the sheaf of \(\nabla\)-horizontal sections of \(P\).

Fix a maximal compact subgroup \(K \subset G\). A \(K\)-reduction for a principal bundle \(P\) is a \(\mathcal{C}^\infty\) principal \(K\)-subbundle \(P_K \subset P\) giving \(P = P_K \times^K G\).

A flat principal \(G\)-bundle \((P, \nabla)\) gives rise to a flat family of homogeneous spaces over \(X\) which we can write as \(P \times^G (G/K)\). If \(P_K \subset P\) is a \(K\)-reduction for \(P\) then the image of \(P_K \times (eK)\) in \(P_K \times^K G\) is a smooth section of the bundle \(P \times^G (G/K)\). This smooth section can also be thought of as a \(\rho\)-equivariant map \(\phi : \tilde{X} \to G/K\). We define the energy of the \(K\)-reduction or of its associated equivariant map by

\[
    \mathcal{E}(\phi) := \int_X |d\phi|^2,
\]

where the integral is taken with respect to the volume form on \(X\), and the norm of the differential \(d\phi\) is measured with respect to an invariant metric on \(G/K\) (note that one has to fix a \(K\)-invariant metric on the complement \(p\) to \(k \subset g\) when making this discussion—if \(G\) is semisimple then we can fix the Killing form as a canonical choice).

An equivariant map or \(K\)-reduction \(P_K\) is called harmonic if it is a critical point of \(\mathcal{E}(\phi)\). The Euler-Lagrange equation for for a harmonic equivariant map is \(d^*d\phi = 0\). This is a nonlinear equation with the Laplacian as its principal term.

The main theorem in this subject is the following generalisation of the theorem of Eells and Sampson [19]:

**Theorem 2.1** (Corlette [10]) If \(\rho\) is a representation such that \(\rho(\pi_1(X, x))\) is Zariski-dense in \(G\) (or such that the Zariski closure is itself reductive), then there exists a harmonic equivariant map \(\phi\).
Proof: We indicate here a variant of Corlette’s proof which might be useful for people with an algebraic geometry background. Assume that the Zariski closure $G$ is semisimple (the general reductive case may then be obtained by using the linear theory of harmonic forms for $C^*$ representations).

Eells and Sampson [19] prove the existence of harmonic maps from $X$ to a compact negatively curved manifold $M$ by a heat-equation minimization technique. We can start off with an equivariant map $\phi_0$ and apply the same heat equation to obtain a family of maps $\phi_t$. We get the same local estimates. In particular, for $x, y \in \tilde{X}$ the distance from $\phi_t(x)$ to $\phi_t(y)$ is a bounded function of $t$. Furthermore, if for any one point $x$ we can show the “$C^0$-estimate” that $\phi_t(x)$ stays in a compact subset of $G/K$ then the estimates of [19] will allow us to show that the $\phi_t$ converge to a smooth harmonic map $\phi$. Note, for example, that the $C^0$-estimate is not true if the Zariski closure of $\rho(\pi_1(X, x))$ is not reductive (in fact, the existence of an equivariant harmonic map implies reductivity of the Zariski closure). In the original case of [19] the target was a compact manifold so this problem was avoided.

Now apply a little bit of geometric invariant theory to get the $C^0$-estimate. Choose elements $g_t \in G$ bringing us back to the basepoint: $g_t\phi_t(x) = eK$. Fix a set of generators $\gamma_i$ for $\pi_1(X, x)$. From [19] we know that the distance from $\phi_t(x)$ to $\phi_t(\gamma_ix)$ remains bounded. As the distance on $G/K$ is $G$-invariant, we have

$$d(g_t\phi_t(x), g_t\phi_t(\gamma_ix)) \leq C.$$  

On the other hand the equivariance of $\phi_t$ gives $\phi_t(\gamma x) = \rho(\gamma)\phi_t(x)$ so (also plugging in $g_t\phi_t(x) = eK$) we get

$$d(eK, g_t\rho(\gamma_i)g_t^{-1}eK) \leq C.$$  

Since $K$ is compact the map $G \rightarrow G/K$ is proper so the $g_t\rho(\gamma_i)g_t^{-1}$ remain in a compact subset of $G$.

The representation variety $R := Hom(\pi_1(X, x), G)$ embeds in a product $G \times \ldots \times G$ by $\rho \mapsto (\ldots, \rho(\gamma_i), \ldots)$. The group $G$ acts on $R$ by the adjoint action $Ad(g)(\rho)(\gamma) := g\rho(\gamma)g^{-1}$, and this is compatible with the above embedding via the adjoint action in each variable of $G \times \ldots \times G$. The previous paragraph tells us that, in our situation, $Ad(g_t)\rho$ remain in a compact subset of $R$.

The basic information from the geometric invariant theory of spaces of representations of finitely generated groups, is that the hypothesis that $\rho(\pi_1(X, x))$ is Zariski-dense in $G$ implies that the $Ad(G)$-orbit of $\rho$ is closed in $R$. This is well known [50] but we discuss it anyway in the next two paragraphs.

If $G = GL(n, C)$ then $\rho$ corresponds to an irreducible $n$-dimensional representation which we denote $V_\rho$. If $V_0$ is a representation in the closure of the orbit of $\rho$ then we have a family of representations $\{V_t\}$ parametrized by $t$ in a smooth curve with $V_0$ being the
value at a point 0 and $V_t \cong V_\rho$ for $t \neq 0$. By semicontinuity there is a nontrivial morphism of representations from $V_\rho$ to $V_\circ$, but since $V_\rho$ is irreducible this must be an isomorphism and we get that $V_0$ is in the orbit of $V_\rho$.

To prove this for a semisimple group $G$ note that $G$ admits a faithful irreducible representation $V$; this gives a composed representation $V_\rho$ of $\pi_1(X, x)$. Since $\rho$ is Zariski-dense, $V_\rho$ is irreducible. Suppose we have a family $\rho_t$ of representations (parametrized by an affine curve) with $\rho_t \sim \rho$ for $t \neq 0$ and $\rho_0$ different from $\rho$. This gives a family of linear representations $V_t$ of $\pi_1(X, x)$; as above, semicontinuity and irreducibility of $V_\rho$ imply that $V_0 \cong V_\rho$. This then implies that $\rho$ and $\rho_0$ are conjugate by an automorphism of $G$ where furthermore this automorphism is a limit of inner automorphisms. The group of outer automorphisms being finite (hence discrete) we conclude that $\rho_0$ and $\rho$ are conjugate by an inner automorphism, that is $\rho_0$ is in the $Ad(G)$-orbit of $\rho$.

Now we complete the proof. The $Ad(g_t)\rho$ remain in a compact subset of the orbit $Ad(G)\rho$ because of the fact that the orbit is closed. Since $\rho(\pi_1(X, x))$ is Zariski-dense in $G$, the stabilizer of $\rho$ is just the center of $G$, which is finite (since we have assumed that $G$ is reductive). In particular the map $G \to Ad(G)\rho$ given by the action on $\rho$ is proper, so the $g_t$ themselves remain in a compact subset of $G$. Finally this implies that the $\phi(x) = g_t^{-1}eK$ remain in a compact subset of $G/K$, which is the $C^0$-estimate we need. □

Remark: Donaldson proved this theorem for rank 2 representations independently of Corlette [18]. It was also proved by Diederich and Ohsawa for representations into $SL(2, \mathbb{R})$ [17]. On the other hand there have since been several generalizations to the noncompact case, for example by Corlette [12] and Jost and Zuo [38] [39].

The Kähler case

Assume now that $X$ is a compact Kähler manifold. Let $\omega$ denote the Kähler form (of a Kähler metric which we choose); let $\Lambda$ denote the adjoint of wedging with $\omega$; and let $\partial$ and $\overline{\partial}$ denote the operators coming from the complex structure. A holomorphic principal bundle $P$ may be considered as a $C^\infty$ principal bundle together with a $G$-invariant operator $\overline{\partial}$ from functions on $P$ to sections of $\Omega^{0,1}_X|_P$ satisfying the appropriate Leibniz rule and $\overline{\partial}^2 = 0$.

We say that a section, map or whatever is pluriharmonic if it is harmonic when restricted to any locally defined smooth complex subvariety. This condition is independent of the choice of metric. The classical Bochner formula states that harmonic forms are pluriharmonic. The fundamental result about equivariant harmonic maps on Kähler manifolds is just the analogue:

Proposition 2.2 If $\phi$ is a harmonic equivariant map from $\tilde{X}$ to $G/K$ then $\phi$ is pluriharmonic.
Proof: See [71] [37] and [10].

Suppose $P$ is a flat principal $G$-bundle with flat connection denoted by $d = d' + d''$. Suppose $P_K$ is a $K$-reduction corresponding to equivariant harmonic map $\phi$. We can decompose the connection into a component $d^+$ preserving $P_K$ and a component $a$ orthogonal to $P_K$. Then decompose according to type, $d^+ = \partial + \overline{\partial}$ and $a = \theta' + \theta''$. We obtain the decompositions

$$d' = \partial + \theta'$$

and

$$d'' = \overline{\partial} + \theta''.$$  

The components orthogonal to $P_K$ operate on functions only via the restrictions of the functions to the fiber, which is to say that they operate on functions via the Lie algebra $ad(P) = P \times^G g$ of $G$-invariant vector fields on $P$. Thus these components are sections $\theta'$ of $ad(P) \otimes \Omega^0_X$ and $\theta''$ of $ad(P) \otimes \Omega^1_X$. The pluriharmonic map equations translate into:

$$\overline{\partial}^2 = 0;$$

$$\overline{\partial} \theta' + \theta' \overline{\partial} = 0 \text{ (which we write } \overline{\partial}(\theta') = 0);$$

and

$$[\theta', \theta'] = 0.$$

In the last equation the form coefficients are wedged and the Lie algebra coefficients bracketed.

We also obtain of course the complex-conjugate equations for $\partial$ and $\theta''$. These are complex conjugates in view of the hermitian or antihermitian properties of $\partial + \overline{\partial}$ and $\theta' + \theta''$ respectively.

The first equation says that $(P, \overline{\partial})$ has a structure of holomorphic principal bundle. This is in general different from the structure of holomorphic principal bundle $(P, d'')$ which comes from the flat structure. The second equation says that $\theta'$ corresponds to a holomorphic section which we now denote simply by $\theta \in \mathcal{H}^0(X, ad(P) \otimes \Omega^1_X)$; and the third equation says $[\theta, \theta] = 0$.

We define a principal Higgs bundle to be a holomorphic principal $G$-bundle $P$ together with $\theta \in \mathcal{H}^0(X, ad(P) \otimes \Omega^1_X)$ such that $[\theta, \theta] = 0$. From the previous results, a flat principal $G$-bundle $P$ with harmonic $K$-reduction $P_K$ gives a principal Higgs bundle $(P, \theta)$.

If $(P, \theta)$ is a principal Higgs bundle and $V$ is a linear representation of $G$ then we obtain an associated Higgs bundle $(E, \theta_E)$ where $E = P \times^G V$ and $\theta_E \in \mathcal{H}^0(X, End(E) \otimes \Omega^1_X)$ is the associated form associated to $\theta$.

Recall that a Higgs bundle $(E, \theta)$ is stable if for any subsheaf $F \subset E$ preserved by $\theta$ we have $deg(F)/r(F) < deg(E)/r(E)$ (the notion of degree depends on choice of Kähler
class). Say that \( E \) is **polystable** if it is a direct sum of stable Higgs bundles of the same slope (degree over rank). Say that a principal Higgs bundle \( P \) is **polystable** if for every representation \( V \), the associated Higgs bundle is polystable. If the generalized first Chern classes of \( P \) (corresponding to all degree one invariant polynomials) vanish then it is enough to check this for one faithful representation \( V \) (cf [65] p. 86).

**Theorem 2.3** Suppose \( \rho : \pi_1(X) \to G \) with \( G \) reductive, and suppose that the Zariski closure of the image of \( \rho \) is reductive. Let \( P \) be the associated flat bundle and \( P_K \) be a pluriharmonic reduction. The structure of principal Higgs bundle \((P, \theta)\) obtained above doesn’t depend on choice of pluriharmonic reduction \( P_K \). The principal Higgs bundle has vanishing rational Chern classes and is polystable. Furthermore, any polystable principal Higgs bundle with vanishing rational Chern classes arises from a unique representation \( \rho \) in this way.

**Proof:** See [33] [63] [65]. \(\square\)

**Moduli spaces**

Fix a reductive complex algebraic group \( G \) and a smooth projective variety \( X \).

Let \( R_{DR}(X, x, G) \) denote the moduli scheme of principal \( G \)-bundles with integrable connection and frame at \( x \in X \) constructed in [69]; similarly let \( R_{Dol}(X, x, G) \) denote the moduli scheme of semistable principal Higgs bundles with vanishing rational Chern classes with a frame at \( x \in X \); and finally let \( R_B(X, x, G) := \text{Hom}(\pi_1(X, x), G) \) denote the space of representations of the fundamental group in \( G \). In all three cases these schemes represent the appropriate functors. We call these spaces the de Rham, Dolbeault and Betti **representation spaces**.

The group \( G \) acts on each of the representation spaces. In all three cases, all points are semistable for an appropriate linearized line bundle, so by [54] the universal categorical quotients

\[
M_{DR}(X, G) := R_{DR}(X, x, G)//G
\]

\[
M_{Dol}(X, G) := R_{Dol}(X, x, G)//G
\]

\[
M_B(X, G) := R_B(X, x, G)//G
\]

exist [50] [52] [57]. They are independent of the choice of basepoint. The points of these quotients parametrize the closed orbits in the representation spaces. The closed orbit in the closure of an orbit corresponding to a given representation is the **semisimplification** of the representation. Two points in a representation space map to the same point in the moduli space if and only if their semisimplifications coincide.
For some purposes it is useful to think about the moduli stacks instead. These are the stack-theoretic quotients

\[ M_{DR}(X, G) := R_{DR}(X, x, G)/G \]
\[ M_{Dol}(X, G) := R_{Dol}(X, x, G)/G \]
\[ M_B(X, G) := R_B(X, x, G)/G. \]

Properly speaking, it is the moduli stacks which should be thought of as the first non-abelian cohomology stacks. The moduli spaces are the hausdorffifications or associated coarse moduli spaces for the stacks (they universally co-represent the functors \( \pi_0 \) of the stacks).

We have a complex analytic isomorphism \( R_{DR}(X, x, G)^{an} \cong R_{B}(X, x, G)^{an} \) compatible with the action of \( G \) coming from the Riemann-Hilbert correspondence between holomorphic systems of ODE’s and their monodromy representations. This projects to the universal categorical quotients ([69] §5) giving \( M_{DR}(X, G)^{an} \cong M_{B}(X, G)^{an} \) as well as to the stack quotients giving \( M_{DR}(X, G)^{an} \cong M_{B}(X, G)^{an} \).

The correspondence of Theorem 2.3 gives an isomorphism between the underlying sets of points of \( M_{DR}(X, G) \) and \( M_{Dol}(X, G) \), because the points of these spaces correspond exactly to representations which have reductive Zariski closure (really the corresponding de Rham or Dolbeault analogues of this notion defined using the Tannakian formalism). This isomorphism is a homeomorphism of underlying topological spaces ([69] which we thus write

\[ M_{DR}(X, G)^{top} \cong M_{Dol}(X, G)^{top}. \]

Hitchin’s original point of view ([33] was slightly different, in that he constructed a single moduli space for all objects, and noted that it had several different complex structures. This amounts to the same thing if one ignores the algebraic structures (and in fact it is difficult to say anything concrete about the relationship between the algebraic structures).

The homeomorphism between the moduli spaces does not lift to a homeomorphism between the representation spaces (cf the counterexample of ([69] II, pp 38-39). I don’t know what happens when we look at the stacks, so I’ll give that as a question for future research.

**Question:** Does there exist a natural homeomorphism \( M_{DR}(X, G)^{top} \cong M_{Dol}(X, G)^{top} \) inducing the previous one on moduli spaces?

In order to attack this question one must first define the notion of the ”underlying topological space” of a stack.

3. The quaternionic structure on the moduli space
Let $M_{\text{sm}}^{\text{DR}}(X, G)$ (resp. $M_{\text{sm}}^{\text{Dol}}(X, G)$, $M_{\text{sm}}^{B}(X, G)$) denote the open subset of smooth points of $M_{\text{sm}}^{\text{DR}}(X, G)$ (resp. $M_{\text{sm}}^{\text{Dol}}(X, G)$, $M_{\text{sm}}^{B}(X, G)$) parametrizing Zariski-dense representations (the notion of Zariski denseness makes sense for the de Rham or Dolbeault spaces using the Tannakian point of view). Then the isomorphism

$$M_{\text{sm}}^{\text{DR}}(X, G)^{\text{top}} \cong M_{\text{sm}}^{\text{Dol}}(X, G)^{\text{top}}$$

is $C^\infty$ (and even real analytic) \[33\] \[21\]. Denoting by $M_{\text{sm}}(X, G)$ the differentiable manifold underlying these isomorphic spaces, we obtain two complex structures $I$ and $J$ on $M_{\text{sm}}(X, G)$ coming respectively from $M_{\text{sm}}^{\text{Dol}}(X, G)$ and $M_{\text{sm}}^{\text{DR}}(X, G)$.

The tangent space to $M_{\text{sm}}^{\text{DR}}(X, G)$ (resp. $M_{\text{sm}}^{\text{Dol}}(X, G)$, $M_{\text{sm}}^{B}(X, G)$) at a point corresponding to a principal bundle $P$ is $H^1_{\text{DR}}(X, \text{ad}(P))$ (resp. $H^1_{\text{Dol}}(X, \text{ad}(P))$, $H^1_{B}(X, \text{ad}(P))$). These tangent spaces have natural $L^2$ metrics coming from the interpretation of classes as harmonic forms.

**Theorem 3.1** (Hitchin \[33\]) Put $K = IJ$. Then the triple $(I, J, K)$ is a quaternionic structure for the manifold $M_{\text{sm}}(X, G)$. Furthermore if $g$ denotes the natural Riemannian metric on $M_{\text{sm}}(X, G)$ obtained from the $L^2$ metric on the tangent space induced by the harmonic metric (these are the same up to a constant for all structures) then $g$ is a Kähler metric for each of the structures $(I, J, K)$, in other words $M_{\text{sm}}(X, G)$ becomes a hyperkähler manifold.

**Proof:** See \[33\] for the result when $X$ is a curve. The theorem for any $X$ follows from the corresponding theorem for a curve, using the embedding $M(X, G) \subset M(C, G)$ for a curve $C$ which is a complete intersection of hyperplane sections in $X$. On the other hand Fujiki proves this for a general Kähler manifold $X$ \[21\] (where the method of taking hyperplane sections is no longer available).

We will see how to calculate that $I, J, K$ form a quaternionic structure in the proof of Theorem 4.2 below. \[\square\]

The embedding $M(X, G) \subset M(C, G)$ obtained from a hyperplane section is complex analytic for all structures, along any naturally defined subvariety such as the Whitney strata of the singular locus. Consequently the smooth points of the underlying reduced scheme structure of these strata inherit quaternionic (and even hyperkähler) structures.

**The twistor space**

The notion of twistor space of a quaternionic manifold $N$ is explained, for example, in \[34\]. Suppose $N$ is a manifold with three integrable complex structures $(I, J, K)$ defining a
quaternionic structure on each tangent space. We identify \( \mathbf{P}^1 \) with the sphere \( x^2 + y^2 + z^2 = 1 \) by the stereographic projection; this gives

\[
\lambda = u + iv \leftrightarrow (x = \frac{1 - |\lambda|^2}{1 + |\lambda|^2}, y = \frac{2u}{1 + |\lambda|^2}, z = \frac{2v}{1 + |\lambda|^2}).
\]

The twistor space \( TW(N) \) is a complex manifold with a \( C^\infty \) trivialisation

\[
TW(N) \cong N \times \mathbf{P}^1
\]
such that the projection \( TW(N) \to \mathbf{P}^1 \) is holomorphic; such that for any \( n \in N \) the section \( \{n\} \times \mathbf{P}^1 \subset TW(N) \) is holomorphic; and such that for any \( \lambda \in \mathbf{P}^1 \) the complex structure on \( M \times \{\lambda\} \) is \( xI + yJ + zK \) where \((x, y, z)\) corresponds to \( t \) via the stereographic projection defined above (note that \((xI + yJ + zK)^2 = -1\)).

If we denote by \( I_{u+iv} \) the complex structure corresponding to \( \lambda = u + iv \in \mathbf{A}^1 \) then one can see using the above definitions that the formula

\[
I_{u+iv} = (1 - uK + vJ)^{-1}I(1 - uK + vJ)
\]

holds (we’ll need this in the proof of Theorem 4.2 below).

This definition serves to determine an almost complex structure on \( TW(N) = N \times \mathbf{P}^1 \). The almost complex structure is integrable \([34, 4, 33, 31]\): in our case we will indicate below an explicit construction which is integrable so we can avoid using the general integrability result.

The twistor space has various other structures, notably an antilinear involution \( \sigma \) covering the antipodal involution \( \sigma_{\mathbf{P}^1} \) of \( \mathbf{P}^1 \). Define \( \sigma(n, t) := (n, \sigma_{\mathbf{P}^1}(t)) \). This is antilinear because it is antilinear in the horizontal directions (along preferred sections) and in vertical directions because \(-xI - yJ - zK\) is the complex conjugate complex structure to \( xI + yJ + zK \).

The preferred sections are by definition \( \sigma \)-invariant.

If \( N \) is a quaternionic vector space of quaternionic rank \( r \) then the twistor space may be constructed by hand. It the direct sum bundle \( \mathcal{O}_{\mathbf{P}^1}(1)^{2r} \) over \( \mathbf{P}^1 \). In this case one can check that the preferred sections are the only \( \sigma \)-invariant sections.

**Application: subvarieties of \( M(X, \mathbf{G}_m) \) defined by cohomological conditions**

The quaternionic structure gives a nice way of looking at the results of Green-Lazarsfeld \([23]\) on subvarieties defined by cohomological conditions. In \([23]\) they look at the subvarieties \( \Sigma^i_k(Pic^0) \subset Pic^0(X) \) of line bundles \( \mathcal{L} \) with \( h^i(\mathcal{L}) \geq k \). They show that these are unions of translates of subtori of \( Pic^0(X) \). We look at the subvarieties \( \Sigma^i_k(M) \subset M(X, GL(n)) \) consisting of those local systems \( V \) such that \( h^i(X, V) \geq k \). We have not
specified whether we look at $M_B$, $M_{DR}$ or $M_{Dol}$ because the same locus is defined in all three cases, and they correspond under the homeomorphisms $M_B \cong M_{DR} \cong M_{Dol}$. This is due to the fact that if $\rho$ is a representation corresponding to vector bundle $V$ with integrable connection and corresponding to Higgs bundle $E$ then the interpretation of cohomology classes as harmonic forms and the Kähler identities between the laplacians [65] gives isomorphisms

$$H^i(X, \rho) \cong H^i_{DR}(X, V) \cong H^i_{Dol}(X, E)$$

(see [65] Lemma 2.2). Now $\Sigma^i_k(M_{Dol})$ is a complex analytic subvariety of $M_{Dol}$ whereas $\Sigma^i_k(M_{DR})$ is a complex analytic subvariety of $M_{DR}$. At any smooth point of the reduced subvariety, the tangent space of $\Sigma^i_k(M)$ is preserved by both complex structures $I$ and $J$ of the quaternionic structure; thus at smooth points $\Sigma^i_k$ is a quaternionic submanifold of $M^{sm}$. This puts a big restriction on the possibilities for $\Sigma^i_k$ (for example, it must have real dimension divisible by 4 i.e. even complex dimension; the same is true for any stratum in its Whitney stratification, for intersections of various $\Sigma^i_k$, etc.).

Deligne pointed out [13] that we can use this compatibility with the quaternionic structure to recover the results of [23]. This method (which I described as an alternative in [66]) adds to the many various points of view on [23] that are now available ([1] [3] [8] [66]). It seems worthwhile to mention the quaternionic point of view here, since similar considerations may come into play for local systems of higher rank.

Deligne makes use of the following observation which is probably classical.

**Lemma 3.2** Any locally defined smooth quaternionic subvariety of a quaternionic vector space is flat (i.e. a linear subspace).

**Proof:** If it were not flat, the second fundamental form would be a quaternionic quadratic form, but an easy calculation shows that this cannot exist. \qed

**Corollary 3.3** If $G = G_m$ then the $\Sigma^i_k(M) \subset M(X, G_m)$ are unions of translates of subtori.

**Proof:** The universal covering of $M(X, G)$ is just $H^1(X, \mathbb{C})$ and the quaternionic structure here is linear. Thus the previous lemma (which is a local statement) applies to show that $\Sigma^i_k(M)$ is flat at smooth points of reduced irreducible components. A standard argument (such as in [23]) gives the conclusion. \qed

This property and its generalizations (one can show that the translates of subtori are translations by torsion points [3] [8] [1] [66]) have implications for the topology of $X$. One can easily fabricate examples of homotopy types such that the corresponding jump loci are not translates of subtori.
We give a crude version here involving additions of 2- and 3-cells to a torus (one can analyze in a similar way examples made by adding cells of any dimensions). Put $\Gamma := \mathbb{Z}^a$ with $a$ even, and put $U_1 = K(\Gamma, 1)$ (which we can take as a real torus). Note that $\mathbb{C}_\Gamma \cong \mathbb{C}[t_1, t_1^{-1}, \ldots, t_a, t_a^{-1}]$ is the Laurent polynomial ring in $a$ variables, and

$$M_B(U_1, \mathbb{G}_m) = \text{Hom}(\Gamma, \mathbb{G}_m) \cong \mathbb{G}_m^a.$$ In fact one can canonically identify $M_B(U_1, \mathbb{G}_m) \cong \text{Spec}(\mathbb{C}_\Gamma)$ (be careful that this reasoning only works well for $\mathbb{G} = \mathbb{G}_m$). Let $u \in U_1$ be the basepoint and let $U_2$ be obtained from $U_1$ by attaching $m$ 2-spheres at $u$. Finally let $U_3$ be obtained by attaching $\ell$ 3-cells to $U_2$ with attaching maps $\alpha_i \in \pi_2(U_2, u)$ for $i = 1, \ldots, \ell$. We calculate the cohomology jump loci $\Sigma^k_i(U_3) \subset M_B(U_3, \mathbb{G}_m)$ for $i = 2, 3$. Note that $U_1 \hookrightarrow U_3$ induces an isomorphism $\pi_1$, so $M_B(U_3, \mathbb{G}_m) = \text{Spec}(\mathbb{C}_\Gamma)$ too. If $L$ is a rank one local system corresponding to a nontrivial representation $\rho : \Gamma \to \mathbb{G}_m$ then $H^2(U_2, L) \cong L_\mathbb{C}^\ell$ and Mayer-Vietoris gives an exact sequence

$$0 \to H^2(U_3, L) \to L_\mathbb{C}^\ell \xrightarrow{A(\rho)} L_\mathbb{C}^\ell \to H^3(U_3, L) \to 0.$$ The matrix $A(\rho)$ comes from the attaching maps: we have $\pi_2(U_2) \cong (\mathbb{C}_\Gamma)^m$ so the collection $\{\alpha_i\}$ can be considered as an $\ell \times m$ matrix $A$ with coefficients in $\mathbb{C}_\Gamma$. The matrix $A(\rho)$ is obtained by evaluating $A$ at the algebra homomorphism $\mathbb{C}_\Gamma \to \mathbb{C}$. In this case the jump loci $\Sigma^k(U_3) = \Sigma^k_{\ell+m}(U_3)$ are the sets of $\rho \in M_B(U_3, \Gamma)$ where $A(\rho)$ has rank $\leq m - k$. In particular they are defined by the ideals of $m - k$ by $m - k$ minors of $A$. Since our choice of $\alpha_i$ and hence of $A$ is arbitrary (except that the matrix must actually have coefficients in $\mathbb{Z}_\Gamma$), we can get our jump loci to be any subscheme of $\mathbb{G}_m^a$ defined by equations in $\mathbb{Z}_\Gamma$, that is any subscheme defined over $\mathbb{Z}$.

We can, for example, get the jump loci to be subschemes which are not of even complex dimension and in any case not unions of translates of subtori—this gives constructions of many homotopy types which cannot be the homotopy types of complex Kähler manifolds. We can arrange that the jump loci do not go through the identity representation (or even any torsion point), in particular this characteristic of the homotopy type will not be seen by rational homotopy theory (we can insure that the cohomology with constant coefficients and even the rational homotopy type are those of the real torus e.g. an abelian variety of dimension $a/2$, or equally well those of any complex subvariety with the same $\pi_1$).

Getting back to the result of Corollary 3.3, we can recover the results of Green and Lazarsfeld by looking at the Dolbeault realization. There is a natural embedding
Pic⁰(X) ⊂ M_Dol(X, G_m) sending a line bundle L to the Higgs bundle (L, 0). The Dolbeault cohomology of (L, 0) is just the direct sum of the \( H^{i-k}(X, \mathcal{L} \otimes \Omega^k_X) \). It is a consequence of semicontinuity that the jump loci for a direct sum must contain as irreducible components the jump loci for each of the factors. Thus the irreducible components of \( \Sigma^i_k(\text{Pic}^0) \) are among the irreducible components of \( \Sigma^i_k(M_{Dol}) \), and the conclusion of the corollary implies the result of \[23\].

4. Deligne’s construction of \( TW(M^{\text{sm}}) \)

In \[13\] Deligne indicated a complex analytic construction of the complex manifold \( TW(M^{\text{sm}}) \) (the idea is based on some properties that Hitchin established). This is interesting because the construction given above of the quaternionic structure on \( M^{\text{sm}}(X, G) \) comes from the homeomorphism \( M_{DR}(X, G) \cong M_{Dol}(X, G) \) which itself comes from the non-complex analytic harmonic metric construction. Of course the trivialization \( TW(M^{\text{sm}}(X, G)) \cong M^{\text{sm}}(X, G) \times \mathbb{P}^1 \) depends on the harmonic metric construction.

It turns out that Deligne’s construction of \( TW(M^{\text{sm}}(X, G)) \) is useful for two other things that were not mentioned in \[13\]: (1) it gives an approach to defining the nonabelian analogue of the Hodge filtration on \( M_{DR}(X, G) \); and (2) it gives a way of compactifying \( M_{DR}(X, G) \). Both of these were announced in \[64\] but with only brief sketches of proofs. In this paper we will fill in the details about these two things and give some natural extensions of these ideas. Before doing that, we review Deligne’s construction, since it has not otherwise appeared in print (to my knowledge).

An antilinear morphism \( T \rightarrow T' \) between two complex analytic spaces is a morphism of ringed spaces such that the composition \( C \rightarrow \mathcal{O}_T \rightarrow \mathcal{O}_T \) is the complex conjugate of the structural morphism \( C \rightarrow \mathcal{O}_T \). An antilinear involution of \( T \) is an antilinear morphism \( \sigma : T \rightarrow T \) with \( \sigma^2 = 1 \).

Let \( \sigma_{\mathbb{P}^1} \) denote the antilinear antipodal involution of \( \mathbb{P}^1 \). If \( z \) is the standard linear coordinate on \( \mathbb{A}^1 \) then \( \sigma(z) = -z^{-1} \). Let \( \sigma_{G_m} \) denote the restriction of \( \sigma \) to \( G_m \subset \mathbb{P}^1 \).

The data of a morphism of complex analytic spaces \( T \rightarrow \mathbb{P}^1 \) together with an antilinear involution \( \sigma \) covering \( \sigma_{\mathbb{P}^1} \) is equivalent to the data of a morphism \( T \rightarrow \mathbb{A}^1 \) and an antilinear involution \( \sigma' \) of \( T_{G_m} := T' \times_{\mathbb{A}^1} G_m \). Given \( T' \) and \( \sigma' \), let \( T' \) denote the complex conjugate analytic space (that is the same ringed space but with structural morphism \( C \rightarrow \mathcal{O}_{T'} \) the complex conjugate of the structural morphism for \( T' \)). The involution \( \sigma' \) becomes a complex linear isomorphism

\[
T'_m \cong T'_m,
\]

which we can use to glue \( T' \) to \( T' \) to obtain \( T \). By construction \( T \) comes with an antilinear involution \( \sigma \) (it comes from the tautological antilinear morphism \( T' \rightarrow T' \) and its inverse).
One can see that $\mathbb{P}^1$ is obtained from $\mathbb{A}^1$ and the involution $\sigma_{\mathbb{G}_m}$ by the same construction, so we obtain our map $T \to \mathbb{P}^1$ compatible with involutions.

To give a holomorphic $\sigma$-invariant section $\eta : \mathbb{P}^1 \to T$ it suffices to give a holomorphic section $\eta' : \mathbb{A}^1 \to T'$ such that $\eta'|_{\mathbb{G}_m}$ is $\sigma'$-invariant.

Hitchin noticed that the twistor space $TW(M_{\text{sm}})$ comes equipped with an action of $\mathbb{G}_m$ identifying the fibers over all different $\lambda \in \mathbb{G}_m \subset \mathbb{P}^1$ [33] (note however that the twistor space of a general hyperkähler manifold doesn’t come equipped with such an action). Deligne’s idea is to use this and the remark of the previous paragraph to obtain a direct construction of $TW(M_{\text{sm}})$.

For simplicity we treat the case $G = \text{GL}(n, \mathbb{C})$ but the case of a general reductive group can be treated directly by working with principal bundles, or indirectly using the Tannakian formalism such as in [69] (note that for the constructions of moduli spaces the indirect Tannakian method is the only one I know of).

Deligne makes the following definition. Suppose $\lambda : S \to \mathbb{A}^1$ is a morphism. A $\lambda$-connection on a vector bundle $E$ over $X \times S$ consists of an operator

$$\nabla : E \to E \otimes \Omega^1_{X \times S/S}$$

such that $\nabla(\lambda e) = \lambda e \otimes d(a) + a\nabla(e)$ (Leibniz rule multiplied by $\lambda$) and such that $\nabla^2 = 0$ as defined in the usual way (integrability).

Note that if $\lambda = 1$ then this is the same as the usual notion of connection, whereas if $\lambda = 0$ then this is the same as the notion of Higgs field making $(E, \nabla)$ into a Higgs bundle [33] [65].

**Proposition 4.1** Fix $x \in X$. The functor which to $\lambda : S \to \mathbb{A}^1$ associates the set of triples $(E, \nabla, \beta)$ where $E$ is a vector bundle on $X \times S$, $\nabla$ is a $\lambda$-connection on $E$ (such that the resulting Higgs bundles over $\lambda = 0$ are semistable with vanishing rational Chern classes), and $\beta : E|_{\{x\} \times S} \cong \mathcal{O}_S^n$ is a frame, is representable by a scheme $R_{\text{Hod}}(X, x, \text{GL}(n)) \to \mathbb{A}^1$. The group $\text{GL}(n)$ acts on $R_{\text{Hod}}(X, x, \text{GL}(n))$ by change of frame and all points are semistable for this action (with respect to the an appropriate linearized bundle). The geometric-invariant theory quotient $M_{\text{Hod}}(X, \text{GL}(n)) \to \mathbb{A}^1$ is a universal categorical quotient. In particular the fibers of $M_{\text{Hod}}$ over $\lambda = 0$ and $\lambda = 1$ are $M_{\text{Dol}}(X, \text{GL}(n))$ and $M_{\text{DR}}(X, \text{GL}(n))$ respectively.

**Proof:** This follows by applying the results of [69] to the ring $\Lambda^R$ defined in [32] p. 87. □

**Remark:** We can define the notion of $\lambda$-connection on a principal bundle, and obtain the corresponding statement for principal $G$-bundles. We obtain schemes $R_{\text{Hod}}(X, x, G)$ and $M_{\text{Hod}}(X, G)$. The construction is done by applying the Tannakian considerations of [33] §9.
Concerning the terminology $R_{\text{Hod}}$ and $M_{\text{Hod}}$: this reflects the fact that, as we shall see below, these spaces incarnate the Hodge filtrations on $R_{\text{DR}}$ and $M_{\text{DR}}$.

Let $\mathcal{M}_{\text{Hod}}(X, GL(n))$ (or $\mathcal{M}_{\text{Hod}}(X, G)$) denote the stack-theoretic quotient of $R_{\text{Hod}}(X, x, GL(n))$ by $GL(n)$ (or $R_{\text{Hod}}(X, x, G)$ by $G$).

The group $G_m$ acts on the functor $\{(E, \nabla, \beta)\}$ over its action on $A^1$: if $t \in G_m(S)$ and $(E, \nabla, \beta)$ is a $\lambda$-connection then $(E, t\nabla, \beta)$ is a $t\lambda$ connection. Since $R_{\text{Hod}}(X, x, GL(n))$ represents the functor, we get an action of $G_m$ on $R_{\text{Hod}}(X, x, GL(n))$ covering its action on $A^1$. Since $\mathcal{M}_{\text{Hod}}(X, GL(n))$ is a universal categorical quotient, this descends to an action on $\mathcal{M}_{\text{Hod}}(X, GL(n))$. This action serves to identify the fibers over any $\lambda, \lambda' \neq 0$ in $A^1$—they are all isomorphic to $M_{\text{DR}}(X, GL(n))$.

The space $M_{\text{Hod}}(X, GL(n))$ will play the role of the space $T'$ in constructing the twistor space $T$. According to our general discussion, in order to obtain $T$ by glueing, it suffices to have an antilinear involution $\sigma'$ of $\mathcal{M}_{\text{Hod}}(X, GL(n))|_{G_m}$. As we have seen above, the action of $G_m$ gives an isomorphism

$$M_{\text{Hod}}(X, GL(n))|_{G_m} \cong M_{\text{DR}}(X, GL(n)) \times G_m.$$  

On the other hand we have an antilinear involution $\tau$ of $M_B(X, GL(n))$ obtained by setting $\tau(\rho)$ equal to the dual of the complex conjugate representation (where complex conjugation is taken with respect to the real structure $GL(n, \mathbb{R})$; the dual of the complex conjugate is also the complex conjugate with respect to the compact real form). To be totally explicit, for $\gamma \in \pi_1(X, x)$ we set $\tau(\rho)(\gamma) := \overline{\rho(\gamma)}^{-1}$. The complex analytic isomorphism $M_B(X, GL(n))^{an} \cong M_{\text{DR}}(X, GL(n))^{an}$ given by the Riemann-Hilbert correspondence allows us to interpret $\tau$ as an antilinear involution of $M_{\text{DR}}(X, GL(n))$. Finally we define the involution $\sigma'$ of $M_{\text{DR}}(X, GL(n)) \times G_m$ by the formula $\sigma'(u, \lambda) = (\tau(u), -\overline{\lambda}^{-1})$. Using $\sigma'$ and $T' = \mathcal{M}_{\text{Hod}}(X, GL(n))$ in the general recipe given above, we obtain a space $T$ which we denote $M_{\text{Del}}(X, GL(n)) \to \mathbb{P}^1$.

The complex conjugate scheme $\overline{T'} = \mathcal{M}_{\text{Hod}}(\overline{X}, GL(n))$ which appears above can be identified with $\mathcal{M}_{\text{Hod}}(\overline{X}, GL(n))$.

Exercise: write down the glueing isomorphism between $\mathcal{M}_{\text{Hod}}(X, GL(n))$ and $\mathcal{M}_{\text{Hod}}(\overline{X}, GL(n))$ over $G_m \subset A^1$. Note that it will be analytic but not algebraic (depending on the Riemann-Hilbert correspondence).

We note rapidly some properties of $M_{\text{Del}}(X, GL(n))$ which are immediate consequences of the construction. The fiber of $M_{\text{Del}}(X, GL(n)) \to \mathbb{P}^1$ over a point $\lambda \in \mathbb{P}^1$ (which is denoted using the coordinate system of the first embedding of $A^1$ which corresponds to the part concerning $X$) is equal to $M_{\text{Del}}(X, GL(n))^{an}$ if $\lambda = 0$; the fiber is isomorphic to $M_{\text{DR}}(X, GL(n))^{an} \cong M_B(X, GL(n))^{an} \cong M_{\text{DR}}(\overline{X}, GL(n))^{an}$ if $\lambda \neq 0, \infty$; and the fiber is equal to $M_{\text{Del}}(\overline{X}, GL(n))^{an}$ if $\lambda = \infty$. There is an analytic action of $G_m$ covering the standard action on $\mathbb{P}^1$ (this action is constructed by glueing the
natural action over the first open set \( M_{Hod}(X, GL(n))^{an} \) with the composition with \( i \) of the natural action on the second open set \( M_{Hod}(\overline{X}, GL(n))^{an} \). The antilinear involution \( \sigma \) of \( M_{Del}(X, GL(n)) \) comes from the first version of the construction discussed above.

**Remark:** Let \( R_{Hod}(X, x, GL(n)) \) denote the representation space constructed with reference to the basepoint \( x \in X \). We can construct, exactly as above, an involution \( \sigma \) which can be thought of as an isomorphism between the inverse images of \( G_m \subset A^1 \) in \( R_{Hod}(X, x, GL(n)) \) and \( R_{Hod}(\overline{X}, \overline{x}, GL(n)) \). We obtain \( R_{Del}(X, x, GL(n)) \rightarrow P^1 \) by glueing \( R_{Hod}(X, x, GL(n)) \) to \( R_{Hod}(\overline{X}, \overline{x}, GL(n)) \) using this isomorphism. The group \( GL(n, C) \) acts analytically and on each open subset the associated moduli space \( M_{Hod}(X, GL(n)) \) is a universal categorical quotient in the analytic category (\[69\] §5). The glueing (along invariant open sets which are pullbacks of open sets in the quotients) preserves this property, so

\[
R_{Del}(X, x, GL(n)) \rightarrow M_{Del}(X, GL(n))
\]
is a universal categorical quotient by the action of \( GL(n, C) \) in the analytical category. There is again an action of \( G_m \) on \( R_{Del}(X, x, GL(n)) \) and the fibers are again respectively \( R_{Del}(X, x, GL(n))^{an} \), \( R_{B}(X, x, GL(n))^{an} \) and \( R_{Del}(\overline{X}, \overline{x}, GL(n)) \) over \( \lambda = 0, \lambda \neq 0, \infty \), and \( \lambda = \infty \) in \( P^1 \).

Denote by \( \mathcal{M}_{Del}(X, GL(n)) \) the stack-theoretic quotient of \( R_{Del}(X, x, GL(n)) \) by \( GL(n); \) it is an analytic stack with a morphism to \( P^1 \).

We now show how a harmonic bundle defines a section \( P^1 \rightarrow M_{Del}(X, GL(n)) \) which we refer to as a preferred section. As mentioned before, it suffices to obtain a \( \sigma' \)-invariant section \( \mathbf{A}^1 \rightarrow M_{Hod}(X, GL(n)) \).

Suppose \( P \) is a flat principal \( GL(n) \)-bundle. Choose a pluriharmonic \( K \)-reduction \( P_K \) and consider the decomposition defined previously

\[
d' = \partial + \theta',
\]
\[
d'' = \overline{\partial} + \theta''.
\]

For \( \lambda \in \mathbf{A}^1 \) we define a holomorphic structure

\[
\overline{\partial}_\lambda := \overline{\partial} + \lambda \theta'',
\]
and an operator

\[
\nabla_\lambda := \lambda \partial + \theta'.
\]

We claim that \( \overline{\partial}_\lambda \) is an integrable holomorphic structure and \( \nabla_\lambda \) an integrable holomorphic \( \lambda \)-connection on \( (P, \overline{\partial}_\lambda) \). The equations \( \overline{\partial}^2 = 0 \), \( [\theta'', \theta'''] = 0 \) and \( (d'')^2 = 0 \) imply that \( \overline{\partial}(\theta'') = 0 \) and hence \( \overline{\partial}_\lambda = 0 \). Similarly, \( \overline{\partial}(\theta') = 0 \) and \( \partial(\theta'') = 0 \) and furthermore we get

\[
\overline{\partial} \partial + \theta' \theta'' + \theta'' \theta' = 0
\]
which gives \([\overline{\partial}_\lambda, \nabla_\lambda] = 0\); finally by the same argument as previously \(\partial(\theta') = 0\) so \(\nabla_\lambda^2 = 0\). This gives the claim. Note that at \(\lambda = 0\) we recover the Higgs bundle structure \((\overline{\partial}, \theta')\) which we know to be polystable with vanishing Chern classes. This construction thus gives a section \(A^1 \to M_{Hod}(X, GL(n))\). It is holomorphic in \(\lambda\) (since \(\lambda\) appears linearly in the equations).

We have to check that our section is \(\sigma'\)-invariant over \(G_m \subset A^1\). This is a bit technical so feel free to skip it! A point of \(M_{Hod}(X, GL(n))\) can be represented as a quadruple \((E, \delta', \delta'', \lambda)\) where \(E\) is a \(C^\infty\) bundle, \(\lambda \in C\), \(\delta\) is an operator satisfying Leibniz’ rule for \(\lambda \partial\), and \(\delta''\) is an operator satisfying Leibniz’ rule for \(\overline{\partial}\), such that \((\delta')^2 = 0\), \((\delta'')^2 = 0\), and \(\delta' \delta'' + \delta'' \delta' = 0\). If \(\lambda \neq 0\) this corresponds to a flat bundle \((E, \lambda^{-1} \delta' + \delta'')\). The dual complex conjugate flat bundle (corresponding to the dual of the complex conjugate representation on \(X\)) is \((E^*, \delta''^* + \overline{\lambda} \delta')\) (the superscript * on the operators means the induced operators on the dual). If we take the point obtained by multiplying this complex conjugate flat bundle by \(-\lambda^{-1}\), we obtain the point

\[
\sigma(E, \delta', \delta'', \lambda) = (E^*, -\overline{\lambda}^{-1} \delta''^*, \overline{\lambda}^{-1} \delta', -\overline{\lambda}^{-1}).
\]

We have to check that this operation preserves our preserved section, which is the collection of points of the form \((E, \lambda \partial + \theta', \overline{\partial} + \lambda \theta'', \lambda)\). We have an isomorphism \(E^* \cong E\) given by the harmonic metric, and via this isomorphism the dual complex conjugation operation has the following effect on operators:

\[
\partial \leftrightarrow \overline{\partial},
\]

\[
\theta' \leftrightarrow -\theta'
\]

(this is from the definition of \(\overline{\partial} + \partial\) and \(\theta' + \theta''\) as the components parallel to and perpendicular to the unitary structure). In view of these formulae, when we apply the operation \(\sigma\) to such a point we get

\[
\sigma(E, \lambda \partial + \theta', \overline{\partial} + \lambda \theta'', \lambda) = (E^* \cong E, -\overline{\lambda}^{-1} \partial + \theta', \overline{\partial} - \overline{\lambda}^{-1} \theta'', -\overline{\lambda}^{-1})
\]

which is indeed a point on our preferred section.

It is clear from the definition that through any point of \(M_{Del}(X, GL(n))\) passes exactly one preferred section.

The set of preferred sections gives a set-theoretic trivialization

\[
M_{Del}(X, GL(n)) \cong M_B(X, GL(n)) \times \mathbb{P}^1.
\]

This trivialisation is in fact a homeomorphism, as can be seen by using the techniques of [69] which are used in proving that \(M_{DR}(X, GL(n))^{\text{top}} \cong M_{Dol}(X, GL(n))^{\text{top}}\). This is also verified in [21].
Let $M_{\text{sm}}^{\text{Del}}(X, GL(n))$ denote the open subset of $M_{\text{Del}}(X, GL(n))$ where the projection to $\mathbb{P}^1$ is smooth. By the etale local triviality of $M_{\text{Hod}}$ (explained in §9 below) a point in $M_{\text{Del}}(X, GL(n))$ lies in $M_{\text{sm}}^{\text{Del}}(X, GL(n))$ if and only if it is a smooth point of the fiber $M_{\text{Dol}}(X, GL(n))$, $M_{\text{DR}}(X, GL(n))$ or $M_{\text{Dol}}(\overline{X}, GL(n))$.

The trivialisation via preferred sections gives

$$M_{\text{sm}}^{\text{Del}}(X, GL(n))^{\text{top}} \cong M_{\text{sm}}^{\text{Del}}(X, GL(n))^{\text{top}} \times \mathbb{P}^1.$$ 

This is in fact a $C^\infty$ isomorphism, as follows from the construction of the preferred sections and the fact that the harmonic maps or metrics vary smoothly with parameters (since they are solutions of the appropriate kind of nonlinear elliptic equation).

**Theorem 4.2** (Deligne) The space $M_{\text{sm}}^{\text{Del}}(X, GL(n))$ with all of its structures is analytically isomorphic to the twistor space $TW(M_{\text{sm}})$; via this isomorphism, the preferred section trivialisations of $M_{\text{sm}}^{\text{Del}}(X, GL(n))$ and $TW(M_{\text{sm}})$ coincide.

**Proof:** This is actually a consequence of the properties obtained by Hitchin for his twistor space in [33]. For intrepid readers, we indicate a self-contained calculation—partly because this also serves to show that $(I, J, K)$ defined a quaternionic structure in the first place. Both the twistor space and $M_{\text{sm}}^{\text{Del}}(X, GL(n))$ are $C^\infty$ isomorphic to the product $M_{\text{sm}}^{\text{Del}} \times \mathbb{P}^1$. This gives the isomorphism between the two. Furthermore we know in both cases that the horizontal sections $\{x\} \times \mathbb{P}^1$ are holomorphic, so the isomorphism is analytic in the horizontal direction. We have to check that this isomorphism is compatible with the complex structures in the vertical direction. Choose a tangent direction to $M_{\text{sm}}^{\text{Del}}$ which we will look at first in the Dolbeault realization. The tangent direction can be thought of as a change of operator $\overline{\partial} + \theta' \mapsto \overline{\partial} + \theta' + \alpha$ where $\alpha$ is an endomorphism-valued form representing the cohomology class of the tangent vector. We may (by gauging back if necessary) assume that the associated harmonic metric remains fixed; the infinitesimal change $\alpha$ then induces a change of operator $\partial + \theta'' \mapsto \partial + \theta'' + \beta$. Write $\alpha = \alpha' + \alpha''$ and $\beta = \beta' + \beta''$ according to type. In terms of the isomorphism $E \cong \overline{E}$ the condition that the fixed metric still relates our new operators is

$$\beta' = \overline{\alpha''},$$

$$\beta'' = -\overline{\alpha'}.$$ 

One can see that if $\alpha$ is harmonic then the form $\beta$ defined by these formulas is also harmonic. If we denote by $B(\alpha)$ the form $\beta$ defined by these formulas then $B$ becomes an endomorphism of the space of harmonic forms. It is antilinear (that is $Bi = -iB$), and $B^2 = -1$. Thus $B$ is another complex structure which forms part of a quaternionic triple with $i$. 

22
The complex structure $I$ on $M_{Dol}(X, GL(n))$ corresponds to multiplication of $\alpha$ by $i$ (because $\alpha$ is the representative of our tangent vector in the Dolbeault realization). The complex structure on $M_{DR}(X, GL(n))$ is the operator on $\alpha$ which causes $\alpha + \beta = \alpha + B(\alpha)$ to be multiplied by $i$. Thus we have the formula

$$I(1 + B)\alpha = (1 + B)J\alpha.$$  

From whence $J = IB$. This now shows that the pair $(I, J)$ form a part of a quaternionic triple, for which $B = -K$ (Theorem 3.1).

For $\lambda \in A^1$ the change of associated $\lambda$-connection is

$$\lambda \partial + \theta' + \bar{\theta} + \lambda \theta'' \mapsto \lambda \partial + \theta' + \bar{\theta} + \lambda \theta'' + \lambda \beta' + \alpha' + \alpha'' + \lambda \beta''.$$  

Thus if $I_\lambda$ denotes the complex structure on the fiber of $M_{Del}(X, GL(n))$ over $\lambda \in A^1$ then we get the formula

$$I(1 + \lambda B) = (1 + \lambda B)I_\lambda.$$  

One has to be careful about what $\lambda B$ means: if $\lambda = u + iv$ then $\lambda B = uB + vIB$. We obtain (replacing $B$ by $-K$):

$$I_{u+iv} = (1 - uK + vJ)^{-1}I(1 - uK + vJ).$$  

This coincides with the formula given in §3. □

**Question:** Are the preferred sections the only sections which are preserved by the involution $\sigma$?

This is certainly locally true, since the normal bundle to a preferred section is a direct sum of $\mathcal{O}_{P^1}(1)$. In fact, locally the morphism from the space of all sections to the product of any two distinct fibers is an isomorphism. If we take two antipodal fibers then $\sigma$ gives an antilinear involution of the product of the two fibers, and the preferred sections correspond to the fixed points.

Hitchin’s discussion in [34] (Theorem 1) is actually a bit unclear on this point: as written the converse in Theorem 1 would imply that the answer is yes in general, but one can easily imagine that he meant only to look at the real sections in the given family of sections. A glance at [35] didn’t resolve the problem, so I think that the answer to the above question is not known.

An affirmative answer would mean that $(M_{Del}(X, GL(n)), \sigma)$ determines the twistor space structure and in particular the isomorphism $M_{DR} \cong M_{Dol}$, an interesting point since the construction of $(M_{Del}(X, GL(n)), \sigma)$ is entirely complex analytic, so we could bypass the nonlinear elliptic theory necessary to define the harmonic metrics—conceptually speaking at least.

The answer to this question is ‘yes’ for the twistor space of a quaternionic vector space. As a consequence we obtain this property for the moduli space of rank one representations:
Theorem 4.3 Suppose \( G = G_m \). Then the preferred sections are the only \( \sigma \)-invariant sections of \( TW(M^{\text{sm}}) \rightarrow \mathbb{P}^1 \).

Proof: The moduli space is a quotient \( M = H^1(X, \mathbb{C})/H^1(X, \mathbb{Z}) \) (as can be seen by a flat version of the exponential exact sequence). The quaternionic structure is the quotient by the lattice of a linear quaternionic structure on \( H^1(X, \mathbb{C}) \). Thus the twistor space is the quotient

\[ TW(M) = TW(H^1(X, \mathbb{C}))/H^1(X, \mathbb{Z}). \]

Since \( \mathbb{P}^1 \) is simply connected, the sections from \( \mathbb{P}^1 \) to \( TW(M) \) are just projections of sections from \( \mathbb{P}^1 \) to \( TW(H^1(X, \mathbb{C})) \). The involution \( \sigma \) acts compatibly on everything. From the theory of the twistor space for quaternionic vector spaces (which is just a bundle which is a direct sum of \( \mathbb{O}_{\mathbb{P}^1}(1) \)) we see that through any point of \( TW(H^1(X, \mathbb{C})) \) there is a unique \( \sigma \)-invariant section; this gives the same result on \( TW(M) \) which implies the theorem. \( \square \)

5. The Hodge filtration

In [33] Hitchin introduced an \( S^1 \) action on the moduli space of representations. This was taken up again in [65] as a \( \mathbb{C}^* \)-action. This action is defined via the isomorphism \( M_{\text{top}}^B \cong M_{\text{top}}^{\text{Dol}} \), \( t \in \mathbb{C}^* \) sends the Higgs bundle \((E, \theta)\) to \((E, t\theta)\).

The \( \mathbb{C}^* \) or \( S^1 \) actions are the analogue in nonabelian Hodge theory of the Hodge decomposition of cohomology coming from harmonic forms. In the usual case, the Hodge decomposition does not vary holomorphically with parameters, because it includes complex conjugate information. Similarly, if the variety is defined over a small field, there is no particular reason for the Hodge decomposition to be defined over a small field. In order to obtain something which comes from algebraic geometry and thus has the properties of holomorphic variation, and compatibility with fields of definition, one looks at the Hodge filtration of the algebraic de Rham cohomology. We will define and investigate the analogue for nonabelian cohomology.

Begin with the following observation. Suppose \( V \) is a vector space with complete decreasing filtration \( F^{-} \) (complete means that the filtration starts with \( V \) and ends with \( \{0\} \)). Define a locally free sheaf \( \xi(V, F) \) over \( \mathbb{A}^1 \) with action of \( G_m \) as follows. Let \( j : G_m \rightarrow \mathbb{A}^1 \) denote the inclusion. Then \( \xi(V, F) \) is the subsheaf of \( j_* (V \otimes \mathcal{O}_{G_m}) \) generated by the sections of the form \( z^{-p} v_p \) for \( v_p \in F^p V \) (where \( z \) denotes the coordinate on \( \mathbb{A}^1 \)). Conversely if \( W \) is a locally free sheaf on \( \mathbb{A}^1 \) with action of \( G_m \) then we obtain a decreasing filtration \( F \) on the fiber \( W_1 \) of \( W \) over \( 1 \in \mathbb{A}^1 \) by looking at orders of poles of \( G_m \)-invariant sections. These constructions are inverses.

The locally free sheaf \( \xi(V, F) \) is the tilde of the Rees module of \( (V, F) \).
If \((V, F)\) is a filtered vector space then the fiber \(\xi(V, F)_0\) over \(0 \in \mathbb{A}^1\) is naturally identified with the associated-graded \(\bigoplus F^p/F^{p+1}\).

Let \(\Sigma\) be a sheaf of sets on the big etale site \(\mathcal{X}'\). We define a filtration \(\mathcal{F}\) of \(\Sigma\) to be a sheaf of sets with morphism \(\Sigma_{\mathcal{F}} \to \mathbb{A}^1\) together with action of \(\mathbb{G}_m\) (here an action means a morphism \(\Sigma_{\mathcal{F}} \times \mathbb{G}_m \to \Sigma_{\mathcal{F}}\) satisfying the usual axioms) and an isomorphism \(\Sigma_{\mathcal{F}} \times \mathbb{A}^1 \{1\} \cong \Sigma\). Note that \(\Sigma_{\mathcal{F}}\) may be interpreted as a sheaf on \(\mathcal{X}/\mathbb{A}^1\), and using this interpretation we can make a similar definition for sheaves of objects of any appropriate category. We obtain a similar definition for stacks (or homotopy-sheaves of spaces, or even \(n\)-stacks or \(\infty\)-stacks once those are defined).

Normally we will be interested in the case where \(\Sigma\) is represented by a scheme or eventually an algebraic stack, in this case we expect \(\Sigma_{\mathcal{F}}\) to be a scheme or at least an algebraic stack.

**Caution:** As we will see in one of our main examples in the section on formal categories, the notion of filtration of a sheaf of sets in the context of stacks is different from the notion of filtration in the context of sets, in other words we might have \(\Sigma_{\mathcal{F}}\) a stack whereas \(\Sigma\) is a set.

Now, getting back to our main discussion, in terms of this definition the space \(M_{\text{Hod}} \to \mathbb{A}^1\) with action of \(\mathbb{G}_m\) is a filtration on \(M_{\text{DR}}\). We call this the Hodge filtration on \(M_{\text{DR}}\). Similarly \(R_{\text{Hod}}\) is the Hodge filtration on \(R_{\text{DR}}\). And most properly speaking, it is \(M_{\text{Hod}}\) which provides the Hodge filtration on the nonabelian cohomology stack \(M_{\text{DR}}\).

In the next section we will see how this filtration is compatible with the usual Hodge filtration on the nilpotent completion of the fundamental group.

The idea of interpreting the Hodge filtration in this way is very closely related to the interpretation of Deninger [16]. Essentially he looks at a derivation expressing the infinitesimal action instead of the full action of \(\mathbb{G}_m\). In turn he refers to Fontaine [20] (1979) for a reworking of Hodge theory from this point of view (which is what led Fontaine to all of his rings such as \(B_{\text{cris}}\) . . . I guess . . . ).

A word about purity. If \((V, F, \overline{F})\) is a vector space with two filtrations (which can be complex conjugates with respect to a real structure, for example) then \(\xi(V, F, \overline{F})\) is a vector bundle over \(\mathbb{P}^1\) obtained by glueing \(\xi(V, F)\) to \(\xi(V, \overline{F})\) much as in \S 4. The two filtrations define a Hodge decomposition pure of weight \(w\) if and only if the vector bundle \(\xi(V, F, \overline{F})\) is a direct sum of copies of \(O_{\mathbb{P}^1}(w)\). The construction \(M_{\text{Del}}\) is in effect the nonabelian analogue of the construction \(\xi(V, F, \overline{F})\) where \(F\) is replaced by the “filtration” \(M_{\text{Hod}}\). The fact that this construction gives the twistor space for a quaternionic structure is equivalent to the statement that the normal bundle along any preferred section is a direct sum of \(O_{\mathbb{P}^1}(1)\) (cf [34], [35]). This can be interpreted as purity of weight one. I don’t know how far one can go toward making this analogy more precise than it is.

6. The nilpotent completion of \(\pi_1\) and representations near the identity
We will justify our definition of $M_{Hod}$ as the Hodge filtration by making the connection with the usual Hodge filtration on the nilpotent completion of the fundamental group \cite{b}. For simplicity we work with the group algebra $\mathbb{C}\pi_1^\wedge$ (completed at the augmentation ideal). The mixed Hodge structure on $\pi_1$ is usually defined via the mixed Hodge structure on $\mathbb{C}\pi_1^\wedge$ and the inclusion of the Lie algebra corresponding to $\pi_1$ into this group algebra.

We first consider the relationship between the completed group algebra and the completions of the spaces of representations at the identity.

Suppose $A$ is an augmented $\mathbb{C}$-algebra which is complete with respect to the augmentation ideal $J_A$. Let $R(A,n)$ denote the functor of artinian local $\mathbb{C}$-algebras $B$ defined by setting

$$R(A,n)(B) := Hom_{\text{aug}}(A, M_n(B))$$

where $Hom_{\text{aug}}$ denotes the set of algebra homomorphisms sending $J_A$ to the ideal $M_n(m_B)$ ($m_B$ denotes the maximal ideal of $B$).

If $A = \mathbb{C}\Gamma^\wedge$ is the completion of the group algebra of a finitely presented group $\Gamma$ then $R(A,n)$ is pro-representable by the completion at the identity representation $R(\Gamma, GL(n))^\wedge$ of the space of representations of $\Gamma$ in $GL(n)$ (there are probably more abstract conditions on $A$ which could be used to insure representability but we don’t need those here).

Conversely let $\mathcal{C}$ denote the category of algebras which are direct products of algebras of the form $M_n(\mathbb{C})$. Suppose $\Upsilon : \mathcal{C} \to \text{ForSch}$ is a functor from $\mathcal{C}$ to the category of formal schemes, compatible with products (so we can think of $\Upsilon$ as a collection of formal schemes $\Upsilon_n$ together with morphisms of functoriality corresponding to morphisms of products of algebras in $\mathcal{C}$). Then we define $\mathcal{A}(\Upsilon)$ to be the algebra of natural transformations $\Upsilon \to 1_{\mathcal{C}}$. The elements of $\mathcal{A}(\Upsilon)$ are functions $a$ which for each $n$ associate a section $a_n : \Upsilon_n \to M_n(\mathbb{C})$ with the $a_n$ compatible with morphisms of products of objects of $\mathcal{C}$.

**Lemma 6.1** Suppose $A = \mathbb{C}\Gamma^\wedge$ is the completion of the group algebra of a finitely presented group. Then the $R(A,n)$ give a functor $R(A,\cdot) : \mathcal{C} \to \text{ForSch}$ and we can recover $A$ by the construction of the previous paragraph:

$$A = \mathcal{A}(R(A,\cdot)).$$

□

**Remark:** The morphisms of functoriality defining $R(A,\cdot)$ can be obtained from the morphisms of functoriality of $R(\Gamma, GL(n))$ for morphisms between products of groups $GL(n)$.

Now we investigate the Hodge filtrations. Suppose $X$ is a smooth projective variety. We obtain a family of formal completions $R_{Hod}(X,x,GL(n))^\wedge \to \mathbb{A}^1$, with an action of
The technique of Goldman and Millson used in (§10) to give an isomorphism $R_{Dol}(X, x, GL(n))^\wedge \cong R_{DR}(X, x, GL(n))^\wedge$ actually gives a trivialization 

$$R_{Hod}(X, x, GL(n))^\wedge \cong R_{Dol}(X, x, GL(n))^\wedge \times^\wedge \mathbb{A}^1,$$

with the action of $\mathbb{G}_m$ coming from the action defined in [65] on $R_{Dol}(X, x, GL(n))^\wedge$ and the standard action on $\mathbb{A}^1$. We discuss this further in §9 below.

We obtain a functor $R_{Hod}(X, x, \cdot) : C \to \text{ForSch}/\mathbb{A}^1$ which we think of as a family of functors parametrized by $\mathbb{A}^1$ (with action of $\mathbb{G}_m$). Because of the trivialization we can apply the previous lemma. This family of functors gives rise to a completed algebra $\mathcal{A}$ over $\mathbb{A}^1$, by a relative version of the construction of Lemma 6.1 (which poses no problem since everything is a product). Conversely starting from $\mathcal{A}$ we get back the $R_{Hod}(X, x, GL(n))^\wedge$.

Finally, the fiber $\mathcal{A}_1$ over $1 \in \mathbb{A}^1$ is isomorphic to the completed group algebra $C\pi_1(X, x)^\wedge$ again by the above discussion. This family of algebras together with $\mathbb{G}_m$-action (which as we have seen is equivalent to the data of the $R_{Hod}$ functorially in $n$) corresponds to a filtration of $C\pi_1(X, x)^\wedge$. We claim that this filtration is the Hodge filtration of Morgan-Hain [53] [27].

To see this, note a consequence of the trivializations and $\mathbb{G}_m$-actions in the above discussion, that the filtration on $C\pi_1(X, x)^\wedge$ corresponding to $\mathcal{A}$ is just the filtration associated to the grading given by the $\mathbb{G}_m$-action. This $\mathbb{G}_m$-action is that which was defined in [65]. Finally, in §§5-6 of [65] it was verified that this $\mathbb{G}_m$-action gives rise to the Hodge filtration of Morgan-Hain.

To sum up, starting with the completed group algebra $C\pi_1(X, x)^\wedge$ and the Morgan-Hain Hodge filtration $F$ we can form the family of algebras $\mathcal{A} = \xi(C\pi_1(X, x)^\wedge, F)$ over $\mathbb{A}^1$ with $\mathbb{G}_m$-action; then the family of completed representation spaces associated to this family of algebras is isomorphic (together with $\mathbb{G}_m$-action) to the completion $R_{Hod}(X, x, GL(n))^\wedge$ along the identity section. In the other direction, the data of the completions $R_{Hod}(X, x, GL(n))^\wedge$ functorially in $GL(n)$ serve to define (via Lemma 6.1) a family of completed algebras $\mathcal{A}$ over $\mathbb{A}^1$, again with $\mathbb{G}_m$-action and isomorphism $\mathcal{A}_1 \cong C\pi_1(X, x)^\wedge$, and this family yields the Morgan-Hain Hodge filtration on $C\pi_1(X, x)^\wedge$ by reversing the construction $\xi$.

Thus the completion of our Hodge filtration at the identity representation corresponds to the Hodge filtration on the nilpotent completion of the fundamental group. The whole Hodge filtration $R_{Hod}(X, x, G)$ or $M_{Hod}(X, G)$ should be thought of as an analytic continuation of the Hodge filtration on the nilpotent completion.

It might be interesting to try to express the existence of this analytic continuation in terms of estimates on the mixed Hodge structure on $C\pi_1(X, x)^\wedge$ in the spirit of Hadamard’s technique [20].

**Writing formulas**

27
We can combine what we know so far to sketch a method which should allow, in principle, to write down the local equations for the correspondence $M_{Dol} \cong M_{DR}$ (and hence for the quaternionic structure) near a complex variation of Hodge structure in $M^{sm}$.

The method sketched above should work equally well along any preferred section coming from a complex variation of Hodge structure $\rho$. The Hodge filtration on the relative Malcev completion $[28]$ should give a Hodge filtration $F$ on the complete local ring $\hat{O}_{M_{DR,\rho}}$; then taking $\xi(\hat{O}_{M_{DR,\rho}}, F)$ we get a family of complete local rings over $A^1$, and taking a formal spectrum we get a formal scheme along $A^1$. The real structure or at least the invariant indefinite hermitian form underlying the variation $\rho$ should give an involution allowing us to glue the formal scheme with itself to get a formal scheme along $P^1$. We should get back in this way the formal completion of $M_{Del}$ along the preferred section.

The normal bundle of $M_{Del}$ along a preferred section is a direct sum of $O_{P^1}(1)$’s, so the space of sections near the given section (which has a structure of formal scheme in this case) will map isomorphically to the product of any two fibers. Taking two antipodal fibers (for example the fibers over 0 and $\infty$) we obtain explicitly the involution on the space of sections and the preferred sections are those which are invariant. Finally looking at the isomorphism from the space of sections to the product of the formal completions of $M_{Dol}$ and $M_{DR}$, the space of invariant sections gives the graph of the real analytic isomorphism $M_{Dol} \cong M_{DR}$.

One can imagine following out this entire construction explicitly to obtain the Taylor series for the isomorphism $M_{Dol} \cong M_{DR}$ near the point $\rho$. The only ingredients are the Hodge filtration and the real structure (and of course an analysis of the space of sections of our formal scheme, but this is an algebraic question).

One can see just from the existence of this method that the algebraically closed field generated by the coefficients of the Hodge filtration on the relative Malcev completion (and their complex conjugates) will contain the coefficients of the power series for the isomorphism $M_{Dol} \cong M_{DR}$ and hence for the power series of the quaternionic structure.

I have not checked the details of this construction any more than what is written above.

7. Formal categories

One of the main properties of the Hodge filtration on usual abelian cohomology is Griffiths transversality. This is a property of the variation of the Hodge filtration with respect to the Gauss-Manin connection which arises from a smooth family of varieties. We would like to obtain a similar property for our nonabelian Hodge filtration. Let’s first look at how to interpret the usual Griffiths transversality in terms of the construction $\xi$.

Suppose $S$ is a smooth variety and $V$ is a vector bundle with integrable connection $\nabla$. Suppose $F^\cdot$ is a decreasing filtration of $V$ by subbundles. Then $F^\cdot$ satisfies the Griffiths
transversality condition $\nabla F^p \subset F^{p-1} \otimes \Omega^1_S$ if and only if the action of $T(S)$ on $V$ extends to a $G_m$-invariant action of the sheaf $T(S \times \mathbb{A}^1/\mathbb{A}^1)(-S \times \{0\})$ (of relative tangent vector fields vanishing to order one along $S \times \{0\}$) on $\xi(V, F^\cdot)$. This can be seen by calculating directly with the definition of $\xi(V, F^\cdot)$ (cf Lemma 7.2 below).

In the nonabelian context suppose $X \rightarrow S$ is a smooth projective morphism. We have a family $M_{\text{DR}}(X/S, G)$ of moduli spaces over $S$ and we would like our “Griffiths transversality” to say that the lifting of vector fields on $S$ to vector fields on $M_{\text{DR}}(X/S, G)$ given by the Gauss-Manin connection, extends to a $G_m$-invariant lifting of sections of $T(S \times \mathbb{A}^1/\mathbb{A}^1)(-S \times \{0\})$ to vector fields on $M_{\text{Hod}}(X/S, G)$.

The difficulty in making this precise is that the Gauss-Manin connection can no longer be interpreted in terms of vector fields if $M_{\text{DR}}(X/S, G)$ is not smooth—so the above interpretation makes sense only on the smooth points. The calculations in terms of vector fields are also difficult to follow through. To remedy these problems we introduce the point of view of formal categories.

Recall that the Gauss-Manin connection on $M_{\text{DR}}(X/S, G)$ is an isomorphism

$$p_1^* M_{\text{DR}}(X/S, G)|_{(S \times S)} \cong p_2^* M_{\text{DR}}(X/S, G)|_{(S \times S)}$$

where $p_1, p_2 : S \times S \rightarrow S$ are the projections and $(S \times S)^\wedge$ is the formal completion of the diagonal. If we set $N := (S \times S)^\wedge$ then the pair $(S, N)$ has a structure of category in the category of formal schemes. The notion of formal category is a generalization of this example. It provides a general framework for operations on families of things over $S$.

A formal category is a pair $(X, N)$ consisting of a scheme $X$ and a formal scheme $M$ mapping to $X \times X$, together with a structure of category, that is morphisms $N \times_X N \rightarrow N$ giving composition and $X \rightarrow N$ giving the identity, subject to the usual axioms for a category. A formal category gives in a natural way a presheaf of categories on $\text{Sch}/C$. A formal groupoid is a formal category such that the values of the associated presheaf are groupoids. We say that a formal category is of smooth type if $X$ is smooth, the underlying scheme of $N$ is the scheme $X$ (via the identity morphism), and $N$ is formally smooth.

Let $X_N$ denote the stack over $\text{Sch}/\mathbb{C}$ associated to the presheaf of groupoids given by $(X, N)$. We have a morphism $p : X \rightarrow X_N$. Note that $N$ represents the functor $X \times_{X_N} X$, so we can recover $(X, N)$ from the stack $X_N$ with its morphism $X \rightarrow X_N$. In practice we confuse the notions (and notations) of formal groupoid $(X, N)$ and associated stack $X_N$.

Suppose $(X, N)$ is a formal groupoid of smooth type. Note that the structure sheaf $\mathcal{O}$ of $\text{Scsh}/\mathbb{C}$ restricts to a sheaf of rings which we also denote by $\mathcal{O}$ on $\text{Sch}/X_N$. There is a complex of $\mathcal{O}$-modules over $X_N$ which we denote by $p_* \Omega^\cdot_{X/X_N}$ with differential denoted $d$, giving a resolution

$$\mathcal{O} \rightarrow p_* \Omega^0_{X/X_N} \rightarrow p_* \Omega^1_{X/X_N} \rightarrow \cdots \rightarrow p_* \Omega^r_{X/X_N} \rightarrow 0$$
(here \( n = \text{dim}(N/X) \) is the relative (formal) dimension of the formally smooth scheme \( N \) over \( X \) via either of the projections).

The notation \( p_* \Omega^i_{X/N} \) is justified by the fact that each component of this resolution is actually the direct image of a locally free sheaf \( \Omega^i_{X/N} \) on \( X \). This locally free sheaf comes from \( \Omega^i_{N/X} \) by descent from \( N = X \times_X N \) to \( X \). Note that the differential is in general a differential operator (of first order) between these component sheaves, so it becomes a morphism only over \( X_N \).

A local system \( V \) on \( X_N \) is a sheaf on \( \text{Sch}/X_N \) which locally isomorphic to \( \mathcal{O}^a \). This is equivalent to a vector bundle \( V_X \) on \( X \) together with a connection \( \varphi : V_X \to V_X \otimes_{\mathcal{O}_X} \Omega^1_{X/X_N} \) satisfying an integrability condition \( \varphi^2 = 0 \).

We obtain the resolution \( p_* \Omega^i_{X/N} \otimes_{\mathcal{O}} V \) of \( V \), which we can use to calculate the cohomology of \( V \) over \( X_N \). Note that the morphism \( p \) is cohomologically trivial for coherent sheaves on \( X \). So the cohomology of \( V \) may be calculated as the hypercohomology of the complex \( \Omega^i_{X/X_N} \otimes_{\mathcal{O}} V \) of Zariski (or etale) sheaves on \( X \); in particular there is a spectral sequence

\[
H^i(X, \Omega^j_{X/X_N} \otimes_{\mathcal{O}} V) \Rightarrow H^{i+j}(X_N, V).
\]

The cohomology groups are finite dimensional vector spaces if \( X \) is proper.

To a formal category \((X, N)\) of smooth type we can associate a split almost polynomial sheaf of rings of differential operators \( \Lambda_N \). It is the sheaf of rings associated to the differentials in the above complex. Note that \( p_*(\mathcal{O}_N) \) is naturally a projective limit of locally free sheaves on \( X \). We can construct \( \Lambda_N \) as the continuous dual, which is a union of locally free sheaves. The ring structure is dual to the cogebra structure \( p_*(\mathcal{O}_N) \to p_*(\mathcal{O}_N) \otimes_{\mathcal{O}_X} p_*(\mathcal{O}_N) \) which itself comes from the composition morphism \( N \times_X N \to N \). A local system \( V \) on \( X_N \) is the same thing as a \( \Lambda_N \)-module; the underlying \( \mathcal{O}_X \)-module is \( V_X \) and the \( \Lambda_N \)-module structure is given by \( \varphi \).

A principal \( G \)-bundle on \( X_N \) means a \( G \)-torsor on the stack \( X_N \). This is the same thing as a principal \( G \)-bundle \( P \) on \( X \) together with an isomorphism \( \varphi : s^*(P) \cong b^*(P) \) on \( N \) (where \( s, b : N \to X \) are the two tautological morphisms), such that \( \varphi \) satisfies the appropriate cocycle condition on \( N \times_X N \).

There is a notion of semistability for local systems over \( X_N \) which is analogous to the usual notion: we can define a notion of coherent sheaf over \( X_N \) (a coherent sheaf on \( X \) with descent data to \( X_N \)) and a bundle over \( X_N \) is semistable if for every \( X_N \)-subsheaf, the normalized Hilbert polynomial is less than or equal to that of the original object. Again as usual we can define the notion of semistability of a principal \( G \)-bundle on \( X_N \). Lacking a direct proof of the conservation of semistability by tensor product in the case of local systems over a general formal category (this is a good question for further research),
we put in the definition here that the local systems associated to all representations of $G$ should be semistable.

When considering a relative situation $X_N \to S$, semistability means semistability on each fiber $(X_N)_s$ (it is an open condition on the base $S$).

Because, in all of the examples we interested in in this paper, it is necessary to include a condition of vanishing rational Chern classes when defining the moduli spaces, we put this directly into the definition in the formal category setting. Of course, for formal groupoids different from our examples, this condition may not necessarily be a sensible one; and even in our examples, it may also be interesting to consider other components of the moduli spaces. One could make the same definitions and obtain moduli spaces without this condition, but we include the condition here for simplicity of notation.

Suppose $(X, N) \to S$ is a morphism from a formal groupoid of smooth type to a base scheme $S$, such that $X$ is smooth and proper over $S$ and $N$ is formally smooth over $S$. Suppose $x : S \to X$ is a section. Define the functor $R(\mathcal{X}_N/S, x, G)$ which to an $S$-scheme $S'$ associates $\{(P, \varphi, \beta)\}$ where $(P, \varphi)$ is a principal $G$-bundle over $X_N \times_S S'$, semistable and with vanishing rational Chern classes relative to $S'$; and $\beta : x^*(P) \cong G \times S'$ is a framing along the section $x$.

**Theorem 7.1** The functor $R(\mathcal{X}_N/S, x, G)$ is representable by a scheme which we denote by $R(\mathcal{X}_N/S, x, G) \to S$. Furthermore all points are semistable for the action of $G$ so a universal categorical quotient $M(\mathcal{X}_N/S, G) = R(\mathcal{X}_N/S, x, G)/\!/G$ exists.

This theorem follows from the interpretation of local systems over $X_N$ as $\Lambda_N$-modules \cite{69}; the construction of the moduli and representation schemes for $\Lambda_N$-modules; and from the tannakian point of view used in \cite{69} §9.

We denote the stack quotient by

$$\mathcal{M}(\mathcal{X}_N/S, G) := R(\mathcal{X}_N/S, x, G)/G.$$  

This is the first relative nonabelian cohomology of $\mathcal{X}_N/S$ with coefficients in $G$. It is an algebraic stack.

**The basic examples**

The main example (which we used to introduce this section and which you meet in most treatments of the subject—cf \cite{4} for example) is the formal groupoid obtained by setting $N := (X \times X)^\wedge$ (completion along the diagonal). We denote this formal groupoid (or the associated stack) by $X_{DR}$. In this stack there is at most one morphism between any pair of objects, so the stack is equivalent to a sheaf of sets which we also denote $X_{DR}$. The sheaf of sets is the quotient of $X$ by the equivalence relation $N$: heuristically we identify

31
any two points which are infinitesimally close together, and it is this infinitesimal glue which makes it so that $X_{DR}$ actually reflects the topology of the underlying usual space. More precisely if $S$ is any scheme over $\mathbb{C}$ then the $S$-valued points of $X_{DR}$ are the $S$-valued points of $X$ modulo the relation that two points are equivalent if their restrictions to the underlying reduced scheme $S^{\text{red}}$ are the same; except that we have to divide by this equivalence relation and then sheafify. Since $X$ is smooth, any $S^{\text{red}}$-valued point extends, locally on $S$, to an $S$-valued point. Thus after taking the quotient and sheafifying the result is simply that $X_{DR}(S) = X(S^{\text{red}})$.

The sheaf of rings of differential operators associated to the formal groupoid $X_{DR}$ is just the full ring $\Lambda_{DR}$ of differential operators on $X$ [29]. A principal bundle on $X_{DR}$ is just a principal bundle on $X$ with integrable connection, and a vector bundle or local system over $X_{DR}$ is just a vector bundle on $X$ with integrable connection. We recover

$$M(X_{DR}, G) = M_{DR}(X, G)$$

and similarly for the representation spaces and stacks

$$R(X_{DR}, x, G) = R_{DR}(X, x, G), \quad \mathcal{M}(X_{DR}, G) = \mathcal{M}_{DR}(X, G).$$

The cohomology of $X_{DR}$ with coefficients in a local system is just the algebraic de Rham cohomology of $X$ with coefficients in the corresponding vector bundle with integrable connection.

We now define a formal groupoid $X_{Dol}$ which gives rise to the Dolbeault theory in the same way as $X_{DR}$ gave rise to the de Rham theory. In this formal groupoid the object object is $X$ and the morphism object is the formal completion of the zero section in the tangent bundle of $X$, lying over the diagonal in $X \times X$. A principal bundle on $X_{Dol}$ is just a principal Higgs bundle; a local system is a Higgs bundle; the cohomology of a local system is the Dolbeault cohomology; and the associated sheaf of rings of differential operators is the ring $\Lambda_{Dol}$ defined in [29]. We recover

$$M(X_{Dol}, G) = M_{Dol}(X, G)$$

and similarly for the representation spaces and stacks.

Now we define a formal groupoid $X_{Hod} \to \mathbb{A}^1$ which serves as a deformation from $X_{DR}$ to $X_{Dol}$ and from which we can recover the moduli spaces $M_{Hod}(X, G)$. It is the stack associated to the realisation of the nerve of the presheaf of groupoids given by a formal groupoid which we denote by $\tilde{X}_{Hod} \to \mathbb{A}^1$. The object object is

$$\text{Ob}(\tilde{X}_{Hod}) := X \times \mathbb{A}^1.$$

Let $Y$ be the complement of the strict transform of $X \times X \times \{0\}$ in the blow-up of $X \times X \times \mathbb{A}^1$ along $\Delta(X) \times \{0\}$. (Here $\Delta(X)$ is the diagonal.) There is a unique composition

$$Y \times_{X \times \mathbb{A}^1} Y \to Y$$

32
compatible with the trivial composition

$$(X \times X \times A^1) \times_{X \times A^1} (X \times X \times A^1)$$

via the morphism $Y \rightarrow X \times X \times A^1$.

There is a morphism $\Delta' : X \times A^1 \rightarrow Y$ covering the inclusion $\Delta : X \rightarrow X \times X \times A^1$. This section provides an identity for the above composition—in particular, setting the morphism object equal to $Y$ would define a category. As with the case of $X_{DR}$ itself, we take the formal completion of this morphism object: the morphism object $\text{Mor}(\tilde{X}_{Hod})$ is defined to be the formal completion of $Y$ along $\Delta'(X \times A^1)$.

The formal groupoid defined in this way is a groupoid; it maps to $A^1$; it has fiber over $\{0\}$ equal to the formal groupoid defining $X_{Dol}$; and its fiber over $\{t\}$ for any $t \neq 0$ is equal to the formal groupoid defining $X_{DR}$.

The general construction of Theorem 7.1 gives back for $X_{Hod}$ over $A^1$ the result of Proposition 4.1:

$$M(X_{Hod}/A^1, G) = M_{Hod}(X, G)$$

and similarly for the representation space and moduli stack

$$R(X_{Hod}/A^1, x, G) = R_{Hod}(X, x, G), \quad \mathcal{M}(X_{Hod}/A^1, G) = \mathcal{M}_{Hod}(X, G).$$

**Lemma 7.2** In the case $G = GL(n)$ a section of the moduli stack $A^1 \rightarrow \mathcal{M}(X_{Hod}/A^1, GL(n))$ preserved by $G_m$ (or more precisely with action of $G_m$ specified) corresponds to a vector bundle with filtration satisfying Griffiths transversality. The relative cohomology of such a family of local systems is just the sheaf over $A^1$ with action of $G_m$ corresponding to the induced filtration on the cohomology of the local system.

**Proof:** A section of the moduli stack with action of $G_m$ is just a vector bundle on $X_{Hod}$ with action of $G_m$. In particular we have a vector bundle on the underlying scheme $X \times A^1$ together with action of $G_m$; by a relative version of the inverse of $\xi$ this corresponds to a bundle $V$ on $X$ with filtration by subbundles $F^p$ such that the associated-graded is a bundle. The descent data to $X_{Hod}$ are determined by the descent data over $G_m \subset A^1$, and by the $G_m$-invariance of our section, these are determined by the descent data over $1 \in A^1$, which is to say an integrable connection on $V$. The original bundle over $X \times A^1$ is the locally free sheaf $\xi(V, F) = \sum t^{-p} F^p$. The statement that the connection extends to descent data for this bundle down to $X_{Hod}$ is equivalent to the condition

$$t\nabla(\sum t^{-p} F^p) \subset (\sum t^{-p} F^p) \otimes \Omega^1_X,$$

which translates to $\nabla F^p \subset F^{p-1} \otimes \Omega^1_X$—the Griffiths transversality condition. \qed
8. The Gauss-Manin connection

Suppose $X \to S$ is a smooth projective morphism. Then we obtain

$$M_{DR}(X/S, G) \to S,$$

a family whose fiber over $s \in S$ is $M_{DR}(X_s, G)$. This family has an algebraic integrable connection \[64\]. We can interpret the connection in terms of formal categories in the following way (this is a simple variant of the crystalline interpretation of \[69\]). We have a morphism $X_{DR} \to S_{DR}$ and the fiber product $X_{DR} \times_{S_{DR}} S$ has a structure of smooth formal groupoid over $S$, which we call $X_{DR/S}$. The morphism space is the formal completion of the diagonal in $X \times_S X$. The moduli stack $\mathcal{M}_{DR}(X/S)$ is just the nonabelian cohomology of $X_{DR/S}$ relative to $S$ with coefficients in $G$, or in our previous notations

$$\mathcal{M}_{DR}(X/S, G) = \mathcal{M}(X_{DR/S}/S, G).$$

The same holds for the moduli spaces and representation spaces. But we could equally well take the nonabelian cohomology of $X_{DR}$ relative to $S_{DR}$. We obtain a stack over $S_{DR}$ which, when pulled back to $S$, gives $\mathcal{M}_{DR}(X/S, G)$. To put this another way, we get descent data for $\mathcal{M}_{DR}(X/S, G)$ from $S$ down to $S_{DR}$. This is exactly the data of an integrable connection which is the nonabelian version of the Gauss-Manin connection.

One can see that the associated analytic connection on the analytic family is the same as that induced by the fact that (locally over the base) all of the fibers of $\mathcal{M}_{DR}(X/S)$ are of the form $\mathcal{M}_B(\Gamma)$ for $\Gamma$ the fundamental group of the fiber \[69\], Theorem 8.6).

In abelian Hodge theory there are two principal results about the Gauss-Manin connection: Griffiths transversality with respect to the Hodge filtration, and regular singularities at the singular points of a family. We obtain their analogues for nonabelian cohomology by using the theory of formal categories and following the above description of the connection.

These properties are easily obtained by using a variant of our construction of the connection. Suppose $X_N \to S_K$ is a morphism of formal categories of smooth type such that the fiber product $X_N \times_{S_K} S$ is a formal groupoid of smooth type on $X/S$. Then the schemes $\mathcal{M}(X_N \times_{S_K} S/S, G)$ and stacks $\mathcal{M}(X_N \times_{S_K} S/S, G)$ have descent data down to $S_K$, that is they are pullbacks of sheaves or stacks on $\text{Sch}/S_K$. The same is true for the $\mathcal{R}(X_N \times_{S_K} S/S, x, G)$ if $x : S_K \to X_N$ is a section.

Griffiths transversality

Suppose $X \to S$ is a smooth family of projective varieties (and we now ask that the base be smooth, although we don’t need it to be projective). Then we obtain a morphism of formal categories $X_{Hod} \to S_{Hod}$ over $\mathbb{A}^1$. Put

$$X_{Hod/S} := X_{Hod} \times_{S_{Hod}} (S \times \mathbb{A}^1).$$

34
It is given by a smooth formal groupoid on $X \times \mathbb{A}^1$ relative to $S \times \mathbb{A}^1$. The relative nonabelian cohomology is

$$\mathcal{M}_{Hod}(X/S, G) := \mathcal{M}(X_{Hod}/S, G) \to S \times \mathbb{A}^1.$$  

This morphism is provided with an action of the formal groupoid $S_{Hod}$ (i.e. $\mathcal{M}_{Hod}(X/S, G)$ is the pullback of a stack over $S_{Hod}$). In particular over $\lambda \neq 0$ we get back the action of $S_{DR}$, that is to say the Gauss-Manin connection. There is a $\mathbb{G}_m$-action compatible with everything. This whole situation is the nonabelian analogue of Griffiths transversality, an interpretation which, comparing with the abelian case, is justified by Lemma VBonXHod.

After we discuss the compactification below we will give another interpretation in terms of poles of the Gauss-Manin connection at infinity.

**Regularity of the Gauss-Manin connection**

Suppose $S' \subset S$ is an open subset whose complement is a divisor $D$ with normal crossings. Define $S_{DR}(\log D)$ to be the formal groupoid which is associated to the bundle of vector fields tangent to $D$. To construct it explicitly we first treat the case where $S$ is a smooth curve and $D$ a point. Then $S \times S$ is a smooth surface. Blow up at the point $(D, D)$, and take the formal completion along strict transform of the diagonal, as morphism scheme. The maps of a smooth scheme into the blow up minus the transform of $D \times S$ are the same as maps into $S \times S$ which send the intersection with the divisor $D \times S$ to the point $(D, D)$. From this description we obtain the composition. The formal completion then becomes a formal groupoid. Now, for any $(S, D)$, glue this construction in along each component of the divisor (or more precisely along an $i$-fold intersection of the divisor in an $n$-dimensional $S$, glue the product of $i$ copies of this construction with $n - i$ copies of a smooth curve with the usual de Rham construction.

A vector bundle or local system over $S_{DR}(\log D)$ is just a vector bundle on $S$ with integrable connection with logarithmic singularities along $D$. In particular, the condition that a vector bundle on $S'_{DR}$ (corresponding to a vector bundle with integrable connection on $S'$) extends to a bundle on $S_{DR}(\log D)$ is equivalent to the condition that the connection have regular singularities.

Now suppose that we have a projective morphism of smooth varieties $f : S \to S$ and a divisor $D \subset S$ which has normal crossings, such that $Y := f^{-1}(D)$ has normal crossings, and such that $f : X' \to S'$ is smooth where $S' = S - D$ and $X' = f^{-1}(S') = X - Y$. There is a morphism of formal groupoids $S_{DR}(\log D) \to S_{DR}$. Let

$$X_{DR}(\log D) := X_{DR} \times_{S_{DR}} S_{DR}(\log D).$$

Again we have a morphism of formal groupoids

$$X_{DR}(\log D) \to S_{DR}(\log D),$$
and the fiber product

\[ X_{DR/S}(\log D) := X_{DR}(\log D) \times_{S_{DR}(\log D)} S \]

is a formal groupoid on \( X \) over \( S \). Even though \( X/S \) is not smooth, this formal groupoid corresponds to a split almost polynomial sheaf of rings of differential operators \( \Lambda_{DR/S}(\log D) \) on \( X \) over \( S \). Thus the moduli problems are solved by \[69\] so the first relative nonabelian cohomology stack \( M(X_{DR/S}(\log D)/S, G) \) is an algebraic stack; the associated representation functor for objects provided with a frame along a fixed section (not passing through the singular points) is representable by a scheme, and the universal categorical quotient scheme \( M(X_{DR/S}(\log D)/S, G) \) exists.

Finally, since \( X_{DR/S}(\log D) \) comes from fiber product as in our general situation, these moduli spaces, moduli stacks and representation spaces descend to \( S_{DR}(\log D) \).

This statement is the regularity of the Gauss-Manin connection.

If \( G = GL(n) \) then we can do slightly better. Recall that \( M(X_{DR/S}(\log D)/S, GL(n)) \) parametrizes bundles with descent data to the formal groupoid \( X_{DR/S}(\log D) \), which are semistable with vanishing rational Chern classes on the fibers \( X_s \). But over the singular fibers, where the extra structure is a connection away from the singularities but simply a logarithmic connection near the singularities, it is also natural to consider objects which are no longer bundles but torsion-free sheaves. This moduli space, which we can denote by \( M^{tf}(X_{DR/S}(\log D)/S, GL(n)) \), will have the advantage that, when combined with the techniques used below for compactifying \( M_{DR}(X_s, GL(n)) \), will give a compact total space over projective \( S \).

**Caution:** One might be tempted to try the above argument with \( X_{DR/S} \). In this case the fiber product \( X_{DR} \times_{S_{DR}} S \) no longer has the required smoothness properties, and the relative moduli stack is no longer an algebraic stack—otherwise we would have an extension of the Gauss-Manin connection over the singularities! In the case \( G = G_a \) for example this would give an extension of the Gauss-Manin connection for ordinary first cohomology, easily seen as impossible in most examples.

**Exercise:** Explain what goes wrong in an example (such as a family of curves acquiring a node) if we try to do the previous construction with \( X_{DR/S} \). This can be done in the context of abelian cohomology.

9. **Etale local triviality of \( M_{Hod} \)**

**Goldman-Millson theory**

A deformation problem is often controlled by a differential graded Lie algebra (dgla) \( L \) over the base field which we are assuming is \( \mathbb{C} \). As explained in \[22\] this means that the
deformations with values in an artin local ring $A$ (with maximal ideal $m$) correspond to elements $\eta \in L^1 \otimes \mathbb{C} m$ such that

$$d(\eta) + \frac{1}{2} [\eta, \eta] = 0.$$  

The isomorphisms between deformations $\eta$ and $\eta'$ correspond to elements $s \in L^0 \otimes \mathbb{C} m$ with

$$\eta' = d(s) + e^{-s} \eta e^s.$$  

If $P$ is a principal $G$-bundle, let $A(ad P)$ denote the graded Lie algebra of $\mathcal{C}^\infty$ forms on $X$ with coefficients in the adjoint bundle $P \times^G \mathfrak{g}$ (with Lie bracket combining the wedge of forms and the Lie bracket of $\mathfrak{g}$). If $(P, \theta)$ is a principal Higgs bundle structure then we obtain a dgla $(A(ad P), \bar{\partial} + \theta)$ which gives the deformation theory of $(P, \theta)$. If $(P, \nabla)$ is a principal bundle with integrable connection then we obtain a dgla $(A(ad P), \nabla)$ which gives the deformation theory of $(P, \nabla)$ [22].

More generally, suppose $(X, N)$ is a formal groupoid structure for $X$; then we have the algebra of differentials $\Omega^\cdot_{X/X_N}$ (cf §7 above). Let

$$A^i_N := \bigoplus_{p+q=i} A^{0,q}(\Omega^p_{X/X_N})$$

This is a differential graded algebra with differential equal to $\bar{\partial} + \delta$ where $\delta$ is the first order differential operator corresponding to the differential of $p_* \Omega_{X/X_N}$. If $P$ is a principal $G$-bundle over $X_N$ then we obtain a dgla

$$A_N(ad P) := A_N \otimes \mathcal{O} (P \times^G \mathfrak{g}).$$

The differential comes from that of $A_N$. This dgla controls the deformation theory of $P$ as a principal bundle on $X_N$ (we leave the proof as an exercise following [22] and [69] §10).

If $X_N = X_{DR}$ then we get $A_{DR}(ad P) := (A(ad P), \nabla)$, whereas if $X_N = X_{Dol}$ then we get $A_{Dol}(ad P) := (A(ad P), \bar{\partial} + \theta)$.

If $L'$ is a dgla then let $H(L')$ denote the dgla of cohomology with differential equal to zero. Recall that we say that $L'$ is formal if there is a quasiisomorphism between $L'$ and $H(L')$. According to the theory of Goldman and Millson [22], a quasiisomorphism induces an equivalence of deformation theories, and on the other hand the deformation theory of a dgla with zero differential is quadratic, in other words the universal deformation space is the quadratic cone in $H^1$ which is defined as the zero scheme of the map $H^1 \to H^2$; $\eta \mapsto [\eta, \eta]$. It follows that the deformation theory of a formal dgla is quadratic.

We need a relative version of this theory for $X \times \mathbb{A}^1/\mathbb{A}^1$ near a preferred section. It is actually an interesting question (which I don’t think has yet been addressed) to
develop a relative version of Goldman-Millson theory in all generality. In our case we are
helped by the fact that the total space is a product. Also we will restrict to deformations
over artinian base (whereas in a better version one should consider arbitrary base with
nilpotent ideal).

Suppose $P$ is a flat principal $G$-bundle with harmonic $K$-reduction $P_K$ and associated
operators $d' = \partial + \theta'$ and $d'' = \overline{\partial} + \theta''$. We define the differential graded Lie algebra over
$\mathbb{C}[\lambda]$
\[ A_{Hod} := \left( A(\text{ad } P) \otimes \mathbb{C}[\lambda], \lambda \partial + \theta' + \overline{\partial} + \lambda \theta'' \right). \]
Note that the differential has square zero and, as varying with parameters, gives a defor-
mation from the de Rham to the Dolbeault differentials. Notice also that the components
of $A_{Hod}$ are flat $\mathbb{C}[\lambda]$-modules. For any dgla $L$ over $\mathbb{C}[\lambda]$ where the components are flat,
we define a stack of groupoids over the category of artinian local $\mathbb{C}[\lambda]$-algebras $(B, m)$ in
the same way as above (except that tensor products are taken over $\mathbb{C}[\lambda]$): the objects are
elements $\eta \in L^1 \otimes_{\mathbb{C}[\lambda]} m$ such that
\[ d(\eta) + \frac{1}{2}[\eta, \eta] = 0. \]
The isomorphisms between deformations $\eta$ and $\eta'$ correspond to elements $s \in L^0 \otimes_{\mathbb{C}[\lambda]} m$
with
\[ \eta' = d(s) + e^{-s} \eta e^s. \]
Again we have the same theorem that quasiisomorphisms between flat dgla’s induce induc-
ences of deformation groupoids. And, on the other hand, the dgla $A_{Hod}$ gives the defor-
mation theory of $\mathcal{M}_{Hod}$ near the preferred section corresponding to $P$. More pre-
cisely the groupoid defined above for $(B, m)$ is equivalent to the groupoid of morphisms
$\text{Spec}(B) \to \mathcal{M}_{Hod}$ with isomorphism between the morphism $\text{Spec}(B/m) \to \mathcal{M}_{Hod}$ and the composed morphism $\text{Spec}(B/m) \to A^1 \to \mathcal{M}_{Hod}$ (where the first morphism is the projection to $A^1 = \text{Spec}(\mathbb{C}[\lambda])$ and the second morphism is the preferred section correspond-
ing to $P$). The proofs are left as exercises.

Finally, the dgla $A_{Hod}$ defined above is formal over $\mathbb{C}[\lambda]$. Let $\ker(\partial + \theta'')[\lambda]$ denote the
subcomplex of $A_{Hod}$ consisting of forms $\alpha$ with $\partial \alpha + \theta'' \alpha = 0$. Since this operator doesn’t
depend on $\lambda$, the subcomplex is just a tensor product of the usual complex $\ker(\partial + \theta'')$ with
$\mathbb{C}[\lambda]$. Furthermore the subcomplex is a sub-dgla; and finally note that the differential of the sub-complex is $\overline{\partial} + \theta'$ which also doesn’t depend on $\lambda$. The “principle of two types”
(cf Lemma 2.2 of [37]) implies that the morphisms
\[ H'(A_{Hod}) \leftarrow \ker(\partial + \theta'')[\lambda] \to A_{Hod} \]
are quasiisomorphisms. Finally, let $\mathcal{H}$ denote the $\mathbb{C}$-vector space of harmonic forms in
$A(\text{ad } P)$ (which is the same for the flat connection or the Dolbeault operator $\overline{\partial} + \theta'$ or,
for that matter, anything in between). Then the morphism

$$\mathcal{H} \otimes C[\lambda] \to H(\mathcal{A}_{Hod})$$

is an isomorphism of graded vector spaces (the space of harmonic forms does not \textit{a priori} have a product structure). This shows that $H(\mathcal{A}_{Hod})$ is flat over $C[\lambda]$ so we can apply the quasiisomorphism result to conclude that the deformation theory of $\mathcal{A}_{Hod}$ is the same as that of the graded Lie algebra $H(\mathcal{A}_{Hod})$. Finally, in order to obtain local triviality we have to show that there is a product structure on $\mathcal{H}$ such that the isomorphism $H(\mathcal{A}_{Hod}) \cong \mathcal{H} \otimes C[\lambda]$ becomes an isomorphism with product structure. This can be seen by identifying $\mathcal{H}$ with the cohomology of the complex $\ker(\partial + \theta'')$ and noting that this latter has a product structure. With this result, the deformation theory along the preferred section becomes a product. We have shown this result for deformations over artinian base, so we get the result on the level of formal completions. Artin approximation then gives that locally in the etale topology at any point of a preferred section, $\mathcal{M}_{Hod}$ is a product. Any point (even non-semisimple) is isomorphic to a point in a neighborhood of a semisimple point, so we obtain local triviality at any point in the union $\mathcal{M}_{Hod}(X, G)$ of components corresponding to bundles with vanishing rational Chern classes.

The formal trivialization along the preferred section is the total-space version of the \textit{isosingularity principle} stated in the introduction and §10 of [19].

We can sum up in the following theorem.

\textbf{Theorem 9.1} Suppose $X$ is a smooth projective variety. Let $M_{Hod}(X, G) \to A^1$ denote the union of components corresponding to objects with vanishing rational Chern classes. Then etale locally (above) $M_{Hod}(X, G)$ is a product, in other words any point $P \in M_{Hod}(X, G)$ over $\lambda \in A^1$ admits an etale neighborhood $P \in U \to M_{Hod}(X, G)$ with an etale morphism $U \to M_{Hod}(X, G)_\lambda \times A^1$. The same holds for $R_{Hod}(X, x, G)$ and the moduli stack $\mathcal{M}_{Hod}(X, G)$.

\hfill \Box

\textbf{Corollary 9.2} The morphism $M_{Hod}(X, G, 0) \to A^1$ is flat (and similarly for $R_{Hod}(X, x, G)$ and the moduli stack $\mathcal{M}_{Hod}(X, G)$).

\hfill \Box

On the other hand, if we have a family of varieties $X \to S$ then the Gauss-Manin connection guarantees that the family $M_{DR}(X/S, G) \to S$ is etale locally a product. It seems reasonable to guess that these two results combine into the following.
**Conjecture 9.3** Suppose $X \to S$ is a smooth projective family. Let $M_{Hod}(X/S, G) \to S \times \mathbb{A}^1$ denote the relative $M_{Hod}$ space. Then etale locally (above) $M_{Hod}(X/S, G)$ is a product, in other words any point $P \in M_{Hod}(X/S, G)$ over $(s, \lambda) \in S \times \mathbb{A}^1$ admits an etale neighborhood $P \in U \to M_{Hod}(X/S, G)$ with an etale morphism $U \to M_{Hod}(X/S, G)_{(s,\lambda)} \times S \times \mathbb{A}^1$. The same holds for $R_{Hod}(X/S, x, G)$ and the moduli stack $M_{Hod}(X/S, G)$.

To prove this one would have to analyze much more closely the deformation theory. In particular, the fact that the total space is a product, which helped a lot in the previous argument, is no longer there to help us here.

We do give an argument showing this conjecture over smooth points (i.e. showing that the morphism to $S$ is smooth at smooth points of the fibers) in the subsection “Griffiths transversality revisited” of §11 below.

A consequence of this conjecture would be that the deformation class of $M_{Hod}$ over a point $(s, 0)$ (we take $\lambda = 0$ because the theory over $\lambda \neq 0$ is trivial due to the Gauss-Manin connection) is determined by a class $\zeta \in H^1(M_{Dol}(X_s), \Theta_{M_{Dol}}) \otimes (TS_s \oplus \mathbb{C})$, where $\Theta_{M_{Dol}}$ is the etale sheaf of infinitesimal automorphisms of $M_{Dol}$.

10. **A weight property for the Hodge filtration**

From its very definition, the usual Hodge filtration on cohomology has the property that $F^0 V = V$. In terms of the Rees bundle this translates to the statement that if $v \in V$ then the $\mathbb{G}_m$-orbit $\mathbb{G}_m v$ has a limit point, i.e. it extends to a section $\mathbb{A}^1 \to \xi(V, F)$. We will establish a similar property for $M_{Hod}$ and $M_{Hod}$.

We don’t explicitly review the notion of sheaf of rings of differential operators $\Lambda$ on $X/S$, for $X \to S$ a projective morphism, from 

Recall that for any formal groupoid of smooth type we obtain an almost-polynomial sheaf of rings of differential operators as described in 

and in §7 above. The $\Lambda$-modules are just the coherent sheaves on $X$ with descent data down to $X_M$.

On the other hand, in the case which interests us (the formal groupoid $X_{Hod}$ on $X \times \mathbb{A}^1$ over $\mathbb{A}^1$) we can explicitly describe the sheaf of rings $\Lambda$ (it is the sheaf of rings denoted $\Lambda^R$ in 

Note first of all that $\Lambda_{DR}$ (corresponding to $X_{DR}$) is just the sheaf of rings of differential operators. It has a filtration $\Lambda^i_{DR}$ being the differential operators of order $\leq i$. Define a decreasing filtration by indexing negatively, $F^{-i} = \Lambda^i_{DR}$. This filtration is compatible with the ring structure so the construction $\xi$ gives a sheaf of rings on $X \times \mathbb{A}^1$ over $\mathbb{A}^1$,

$$\Lambda_{Hod} = \xi(\Lambda_{DR}, F).$$

It is the sheaf of rings associated to the formal groupoid $X_{Hod}$. The relative moduli theory for $\Lambda_{Hod}$-modules on $X \times \mathbb{A}^1/\mathbb{A}^1$ yields the moduli space $M_{Hod}$ and representation space $R_{Hod}$. The stack theoretic quotient gives the moduli stack $M_{Hod}$. 40
Langton theory

We recall the notations and terminology of [69]. In particular, $p$-semistability and $p$-stability refer to Gieseker’s definition involving Hilbert polynomials. We will work with $G = GL(n)$ at the start.

The following theorem is the generalisation of Langton’s theory [47] of properness of moduli spaces, in the case of sheaves of $\Lambda$-modules (M. Maruyama pointed out to me that Langton’s theory carries over in this type of general context).

**Theorem 10.1** Suppose $S = \text{Spec}(A)$ where $A$ is a discrete valuation ring with fraction field $K$ and residue field $A/m = \mathbb{C}$. Let $\eta$ denote the generic point and $s$ the closed point of $S$. Suppose $X \to S$ is a projective morphism of schemes with relatively very ample $O(1)$ on $X$. Suppose $\Lambda$ is a split almost polynomial sheaf of rings of differential operators on $X/S$ as in [69]. Suppose $F$ is a sheaf of $\Lambda$-modules on $X$ which is relatively of pure dimension $d$, flat over $S$, and such that the generic fiber $F_\eta$ is $p$-semistable. Then there exists a sheaf of $\Lambda$-modules $F'$ on $X$ which is relatively of pure dimension $d$, flat over $S$, and such that $F'_\eta \cong F_\eta$ and also $F'_s$ is $p$-semistable.

**Proof:** Langton’s proof carries over into our situation. We give a brief sketch for compatibility with our notations. Let $p_X(\cdot)$ denote the absolute normalized Hilbert polynomial of a sheaf on $X$ with proper support, and let $p_{X/S}(\cdot)$ denote the relative normalized Hilbert polynomial of a sheaf flat over $S$. Let $F_n$ denote the sheaf of $\Lambda$-modules

$$F_n := F \otimes_A A/m^{n+1}.$$ 

It is of pure dimension $d$ on $X$. Let

$$F_n \to G_n$$

denote the destabilizing quotient, that is the quotient with the minimal normalized Hilbert polynomial. (Note that if $G_0 = F_0$ then $F_0$ is $p$-semistable and we’re done—so we assume that this is not the case). We make the following claims:

1. Let $p_0 = p_X(G_0)$. Then for all $n$, $p_X(G_n) = p_0$.
2. There are morphisms $G_n \to G_{n-1}$ compatible with the morphisms $F_n \to F_{n-1}$.
3. There there is $q$ such that for $n \geq q$ we have $G_n \cong G_q$.

**Proof of 1:** Assume it is known for $n - 1$. We have an exact sequence of $\Lambda$-modules

$$0 \to F_0 \to F_n \to F_{n-1} \to 0.$$ 

From this, we obtain a quotient $F_n \to G_{n-1}$, with $p_X(G_{n-1}) = p_0$. This shows that the normalized Hilbert polynomial of the destabilizing quotient $G_n$ is $\leq p_0$. On the other hand,

$$p_X(\text{im}(F_0 \to G_n)) \leq p_X(G_n)$$

41
(since \( G_n \) is \( p \)-semistable)—unless this morphism is zero in which case \( G_n = G_{n-1} \) and we’re done anyway. Therefore (by the definition of \( p_0 \)) we have that \( p_0 \leq p_X(G_n) \). This proves claim (1).

**Proof of 2:** Since \( G_{n-1} \) is a quotient of \( F_n \) with \( p_X(G_{n-1}) = p_0 \), it factors through the destabilizing quotient giving the morphism \( G_n \to G_{n-1} \).

**Proof of 3:** Suppose not. We may assume that \( A \) is complete. Let \( G := \lim_{\to} G_n \). This gives a quotient of \( F \) destabilizing \( F \) over the generic point.

Now we proceed with the construction of \( F' \). Starting with \( F \) (and assuming that \( F_0 \) is not \( p \)-semistable), we construct the quotient \( G_q \) as above. By statement (3), \( G_q \) is the maximal quotient of \( F \) which has normalized Hilbert polynomial \( \leq p_0 \) and which is supported over some \( \text{Spec}(A/m^n) \).

Let \( F^{(1)} \) be the kernel of the map \( F \to G_q \). Let \( p_1 \) be the normalized Hilbert polynomial of the destabilizing quotient \( Q \) of \( F^{(1)} \otimes_A A/m \). We claim that \( p_1 > p_0 \). To see this, suppose to the contrary that \( p_1 \leq p_0 \). Let \( K \) be the kernel of the map \( F^{(1)} \to Q \), and let \( G' = F/K \). Then \( G' \) is a quotient of \( F \) which is an extension of \( G_q \) by \( Q \); in particular its normalized Hilbert polynomial is \( \leq p_0 \). Furthermore \( G' \) is supported over \( \text{Spec}(A/m^{n+2}) \). This contradicts maximality of \( G_q \), showing that \( p_1 > p_0 \).

Now start with \( F^{(1)} \) and repeat the same construction to obtain \( F^{(2)} \) etc.; and for each \( i \) let \( p_i \) be the normalized Hilbert polynomial of the destabilizing quotient of \( F^{(i)} \otimes_A A/m \) (we stop if \( F^{(i)} \otimes_A A/m \) is \( p \)-semistable). We have \( p_0 < p_1 < p_2 < \ldots \). Since all of these sheaves are flat over \( S \) (they are subsheaves of \( F \) and hence without \( A \)-torsion), the Hilbert polynomials of \( F^{(i)} \otimes_A A/m \) are all equal to the Hilbert polynomial of \( F \otimes_A K \). Finally, the set of \( \Lambda \)-modules with a given Hilbert polynomial and with destabilizing quotient having normalized Hilbert polynomial \( \geq p_0 \), is bounded. Thus the set of possible normalized Hilbert polynomials of the destabilizing quotients is finite. This shows that the process must stop. At the stopping point \( F^{(i)} \otimes_A A/m \) is \( p \)-semistable, and we take \( F' := F^{(i)} \).

**Application to \( \mathcal{M}_{\text{Hod}} \)**

We apply Langton theory to limits of \( G_m \)-orbits in \( \mathcal{M}_{\text{Hod}} \). In fact this applies equally well to the moduli stack \( \mathcal{M}_{\text{Hod}} \). Suppose \( p \in \mathcal{M}_{\text{Hod}}(X_s, GL(n)) \). The \( G_m \)-orbit of \( p \) is a morphism \( G_m \to \mathcal{M}_{\text{Hod}} \) which corresponds to a \( \lambda \)-connection \((F', \nabla')\) on \( X_s \times G_m \) (where \( \lambda : G_m \to A^1 \) is the projection of the orbit).

**Corollary 10.2** With the notations of the above paragraph, there is an extension \((F, \nabla)\) of \((F', \nabla')\) to a \( \lambda \)-connection on \( X_s \times A^1 \), such that \( F|_{X_s \times \{0\}} \) is a bundle, is semistable and has vanishing rational Chern classes.
Proof: First of all note that there exists an extension \((\mathcal{F}_1, \nabla_1)\). For this note that \(p\) corresponds to a \(\lambda(1)\)-connection \((\mathcal{E}, \varphi)\) on \(X_s\), and \(\mathcal{F}' = p_1^* (\mathcal{E})\) on \(X \times \mathbb{G}_m\) with \(\nabla = t \varphi\) (here \(t\) denotes the coordinate on \(\mathbb{G}_m\)). We can simply put \(\mathcal{F}_1 = p_1^* (\mathcal{E})\) and \(\nabla_1 = t \varphi\) on \(X \times \mathbb{A}^1\).

Theorem [10.1] now implies that there exists an extension \((\mathcal{F}, \nabla)\) which is semistable over \(X_s \times \{0\}\). Note that \(\lambda(0) = 0\) so the restriction to \(X_s \times \{0\}\) is a Higgs sheaf. By flatness of \(\mathcal{F}\) over \(\mathbb{A}^1\), the restriction has vanishing rational Chern classes. By ([52] Theorem 2 p. 39), our restriction is actually a bundle. 

Corollary 10.3 Suppose now that \(G\) is any reductive group. If \(p \in M_{\text{Hod}}(X/S, G)\) then the limit \(\lim_{t \to 0} t \cdot p\) exists in \(M_{\text{Hod}}(X/S, G)\).

Proof: We can choose an injection \(G \hookrightarrow GL(n)\). By a variant of [52] Corollary 9.15 concerning \(M_{\text{Hod}}\) (we can get this by using the topological trivialization \(M_{\text{Hod}} \cong M_{\text{DR}} \times \mathbb{A}^1\) which is functorial in \(G\)) the induced map \(M_{\text{Hod}}(X_s, G) \to M_{\text{Hod}}(X_s, GL(n))\) is finite. Since the limit exists in \(M_{\text{Hod}}(X_s, GL(n))\) by the previous corollary, it exists in \(M_{\text{Hod}}(X_s, G)\). 

Question: What happens for non-reductive groups? If \(G = \mathbb{G}_a\) then the limits again exist (this is exactly the weight property of the Hodge filtration refered to at the start of the section), so it seems likely that this will be true in general.

Lemma 10.4 Suppose \(G\) is a reductive group. Let \(V \subset M_{\text{Hod}}(X, G)\) be the fixed point set of the \(\mathbb{G}_m\)-action (note that \(V\) is concentrated over the origin so in fact \(V \subset M_{\text{Dol}}(X, G)\)). Then \(V\) is proper over \(S\).

Proof: The fixed point set lies over the origin in \(\mathbb{A}^1\), so it is just the fixed point set of the \(\mathbb{G}_m\)-action on the moduli space of Higgs bundles. For \(G = GL(n)\) this fixed point set is proper by ([33], [57], [59] Theorem 6.11). For any \(G\), argue as in the previous corollary. Alternatively one can obtain properness using Langton theory as above. 

11. Compactification of \(M_{\text{DR}}\)

The space \(M_{\text{Hod}}(X/S, G) \to \mathbb{A}^1\) together with the action of \(\mathbb{G}_m\) and the isomorphism between \(M_{\text{DR}}(X/S, G)\) and the fiber over \(\lambda = 1\), allow us to compactify \(M_{\text{DR}}(X/S, G)\) relative to \(S\). This depends on the properness results of the previous section.

Structure theory for \(\mathbb{G}_m\)-orbits and construction of some quotients
Suppose \( X \to S \) is a projective morphism with an action of \( \mathbb{G}_m \) covering the trivial action on \( S \). Choose a relatively very ample line bundle \( \mathcal{L} \) and a compatible action of \( \mathbb{G}_m \). Let \( V_i \) denote the connected components of the fixed point set \( V \). For each \( i \) there is an integer \( \alpha_i \) such that \( t \in \mathbb{G}_m \) acts by \( t^{\alpha_i} \) on \( \mathcal{L}|_{V_i} \).

Define a partial ordering \( \preceq \) on \( V \), by saying that \( u \preceq v \) if there is a sequence of points \( x_1, \ldots, x_m \in X \) with
\[
\lim_{t \to 0} x_1 = u \\
\lim_{t \to 0} x_k = \lim_{t \to \infty} x_{k+1} \\
\lim_{t \to \infty} x_m = v.
\]
Notice that if \( u \preceq v \) and \( u \in V_i \), \( v \in V_j \) then \( \alpha_i \geq \alpha_j \) (and if \( \alpha_i = \alpha_j \) then \( u = v \)).

Suppose \( V = V_+ \cup V_- \) is a decomposition of the fixed point set into two disjoint closed subsets (which are consequently unions of connected components), with the properties that
\[
v \in V_+, \ u \in V, \ u \preceq v \Rightarrow u \in V_+. \]
and
\[
v \in V_-, \ u \in V, \ u \succeq v \Rightarrow u \in V_. \]
Put
\[
Y_+ = \{ y \in X : \lim_{\lambda \to \infty} \lambda \cdot y \in V_+ \}
\]
and
\[
Y_- = \{ y \in X : \lim_{\lambda \to 0} \lambda \cdot y \in V_- \}. \]
These are disjoint closed subsets. They are closed by an argument similar to the proof of properness in Theorem 11.1 below. They are disjoint because if there existed \( y \in Y_+ \cap Y_- \) then
\[
\lim_{\lambda \to 0} \lambda \cdot y \leq \lim_{\lambda \to \infty} \lambda \cdot y,
\]
so we would obtain two points \( u, v \) with \( u \in V_- \) and \( v \in V_+ \) but \( u \preceq v \); whence \( u \in V_+ \) and \( v \in V_- \) (by the conditions on \( V_- \) and \( V_+ \) contradicting the disjointness of \( V_+ \) and \( V_- \). Finally, note that \( Y_+ \) and \( Y_- \) are, by the nature of their definitions, \( \mathbb{G}_m \)-invariant.

Let \( U := X - Y_+ - Y_- \). This is a \( \mathbb{G}_m \)-invariant open set in \( X \).

**Theorem 11.1** With the above notations, a universal geometric quotient \( U/\mathbb{G}_m \) exists. It is separated and proper over \( S \).

**Remark:** When the subsets \( V_+ \) and \( V_- \) are defined by choosing \( a \in \mathbb{Q} - \mathbb{Z} \) and setting \( V_+ = \bigcup_{\alpha_i > a} V_i \) and \( V_- = \bigcup_{\alpha_i < a} V_i \) then the quotient defined above is just the geometric invariant theory quotient of the set of semistable points (for the linearized action obtained
when the linearization is translated by $a$). In particular, in this case the quotient is projective. I don’t know if the quotient given by this theorem will be projective in general, nor if it is projective in our example (the compactification of $M_{DR}$) below.

**Proof:** Let $X^{(pre)}$ denote the set of pre-stable points \([54]\). In our case it is easy to see that $X^{(pre)} = X - V$ is just the complement of the fixed point set. Mumford constructs a universal geometric quotient $\phi : X^{(pre)} \to X^{(pre)}/\mathbb{G}_m$. This morphism is submersive so $\phi(U)$ is open, and by the universality we obtain a universal geometric quotient $\phi : U \to U/\mathbb{G}_m$. The only problem is to prove that $U/\mathbb{G}_m$ is separated and proper over $S$.

Suppose $R$ the henselian local ring of $\mathbb{C}[z]$ at the origin $P$, with maximal ideal $m$ and residue field $R/m = \mathbb{C}$. Let $K$ be the fraction field of $R$, and let $z \in R$ denote a uniformizing parameter. Let $\tilde{K}$ denote the algebraic closure of $K$ and let $\tilde{R}$ denote the normalization of $R$ in $\tilde{K}$. The extension $\tilde{K}$ is obtained from $K$, as $\tilde{R}$ is obtained from $R$, by adjoining the elements $z^{1/n}$. Let $\tilde{m}$ denote the maximal ideal of $\tilde{R}$. Note that $\tilde{R}$ is a valuation ring with $\mathbb{Q}$ as value group, $\tilde{m}$ is the valuation ideal, and $\tilde{R}/\tilde{m} = \mathbb{C}$.

Any finite extension of $K$ is isomorphic to $K$ (by changing the parameter).

Suppose $\eta : Spec(K) \to U$ is a point. We have to show that there is $\varphi \in \mathbb{G}_m(\tilde{K})$ such that $\varphi \eta$ extends to a point $Spec(\tilde{R}) \to U$, and furthermore that $\varphi$ is unique up to $\mathbb{G}_m(\tilde{R})$.

The action of $\mathbb{G}_m$ on the point $\eta$ gives a morphism $Spec(K) \times \mathbb{G}_m \to X$ which completes to $Spec(K) \times P^1 \to X$. Let

$$\eta_0 := \lim_{t \to 0} t \cdot \eta$$

$$\eta_\infty := \lim_{t \to \infty} t \cdot \eta$$

as points $Spec(K) \to X$. These complete to points $Spec(R) \to X$. There is a scheme $W$ (the closure of the graph of the previous morphism in $Spec(R) \times X$) with a diagram

$$
\begin{array}{ccc}
Spec(K) \times P^1 & \hookrightarrow & W \\
\downarrow & & \downarrow \\
Spec(K) & \hookrightarrow & Spec(R) \to S
\end{array}
$$

where the vertical arrows are proper, and where $\mathbb{G}_m$ acts compatibly on everything in the top row. The fiber of $W$ over the closed point of $Spec(R)$ decomposes as a string of $P^1$’s meeting at fixed points for the action. Let $y_1, \ldots, y_r$ denote the images in $X$ of the fixed points in the string of $P^1$’s over the origin. Since $W$ was taken as the closure of the graph, these points are distinct. We can order them so that $y_1 = \eta_0(P)$, $y_r = \eta_\infty(P)$, and $y_i$ is joined to $y_{i+1}$ by a $P^1$ in the fiber. In this case for a general point $x$ on the $P^1$ joining $y_i$ to $y_{i+1}$ we have (\*)

$$\lim_{t \to 0} t \cdot x = y_i, \quad \lim_{t \to \infty} t \cdot x = y_{i+1}.$$
In particular, refering to our partial ordering above we have
\[ \eta_0(P) = y_1 < y_2 < \ldots < y_r = \eta_\infty(P). \]

Note that since \( \eta \in U(K) \) we have \( \eta_0 \in V_+(K) \) and \( \eta_\infty \in V_-(K) \). Thus \( y_1 \in V_+ \) and \( y_r \in V_- \) (as \( V_+ \) and \( V_- \) are closed). By the conditions on \( V_+ \) and \( V_- \) there is \( k \) such that \( y_1, \ldots, y_k \in V_+ \) and \( y_k+1, \ldots, y_r \in V_- \). From the definitions of \( Y_+ \) and \( Y_- \) as well as the the property (\( \ast \)) we find that the \( \mathbb{P}^1 \) joining \( y_i \) to \( y_{i+1} \) lies in \( Y_+ \) if \( i < k \) and in \( Y_- \) if \( i > k \), whereas it meets \( U \) if \( i = k \). The uniqueness of the \( \mathbb{G}_m \)-orbit meeting \( U \) in the closed fiber gives the separatedness. Choose \( \varphi \in \mathbb{G}_m(\bar{K}) \) so that \( \varphi \eta : \text{Spec}(\bar{K}) \to W \) completes to a point \( \text{Spec}(\bar{R}) \to W \) with \( P \) mapping to a general point on the \( \mathbb{P}^1 \) joining \( y_k \) to \( y_{k+1} \). This gives the desired \( \varphi \) for properness. \( \square \)

We obtain the following theorem as a corollary.

**Theorem 11.2** Suppose \( Z \to S \) is an \( S \)-scheme on which \( \mathbb{G}_m \) acts (acting trivially on \( S \)). Suppose that the fixed point set \( W \subset Z \) is proper over \( S \), and that for any \( z \in Z \) the limit \( \lim_{t \to 0} t \cdot z \) exists in \( W \). Let \( U \subset Z \) be the subset of points \( z \) such that the limit \( \lim_{t \to \infty} t \cdot z \) does not exist in \( Z \). Then \( U \) is open and there exists a geometric quotient \( Q = U/\mathbb{G}_m \) by the action of \( \mathbb{G}_m \). This geometric quotient is separated and proper over \( S \).

**Proof:** Chose a \( \mathbb{G}_m \)-linearized very ample line bundle \( \mathcal{L} \) on \( Z \) (this exists by [54]). Then \( \mathbb{G}_m \) acts in a locally finite way on \( H^0(Z, \mathcal{L}) \) so we may choose a fixed subspace which gives a projective embedding of \( Z \). Thus we may assume that \( Z \subset \mathbb{P}^N \) as a locally closed subscheme, and that \( \mathbb{G}_m \) acts linearly on \( \mathbb{P}^N \) preserving \( Z \) and inducing the given action there. Taking the graph we can consider this as an embedding \( Z \subset \mathbb{P}^N \times S \). Let \( X \) be the subscheme closure of \( Z \) in \( \mathbb{P}^N \times S \) (that is, the subscheme defined by the homogeneous ideal of forms which vanish on \( Z \)). Note that \( X \) is projective over \( S \) and that \( \mathbb{G}_m \) acts on \( X \) preserving the open set \( Z \). Let \( V \) denote the fixed point set in \( X \), and let \( V_+ := W \) be the fixed point set in \( Z \). Let \( V_- := V \cap (X - Z) \) denote the fixed point set in the complement. Note that the complement \( X - Z \) is closed, hence proper over \( S \), so \( V_- \) is proper over \( S \). By hypothesis \( V_+ \) is proper over \( S \). We obtain a decomposition \( V = V_+ \cup V_- \) as a disjoint union of two closed subsets.

Suppose \( u, v \in V \) with \( v \leq u \). This means that there is a sequence of points \( v_0 = v, \ldots, v_n = u \) such that \( v_i \) is joined to \( v_{i+1} \) by a \( \mathbb{G}_m \)-orbit (i.e. there is an orbit whose limits are \( v_i \) at \( \lambda \to 0 \) and \( v_{i+1} \) at \( \lambda \to \infty \)). Suppose \( v_i \in V_- \). Then the orbit corresponds to a point \( x \in X \) with \( \lim_{\lambda \to 0} \lambda \cdot x = v_i \). But if \( x \in Z \) then our hypothesis would give \( v_i \in V_+ \), so \( x \) must be in \( X - Z \). Since \( X - Z \) is closed, the other limit \( v_{i+1} \) must be in \( X - Z \) also. We thus show by induction that if \( v = v_0 \) is in \( V_- \) then so is \( u = v_n \). The contrapositive says that if \( u \) is in \( V_+ \) then so is \( v \). We have shown on the one hand that
if \( v \in V_- \) and \( u \in V \) with \( v \leq u \) then \( u \in V_- \); and on the other hand that if \( u \in V_+ \) and \( v \in V \) with \( v \leq u \) then \( v \in V_+ \).

We are now ready to apply the general construction above. Define the subsets \( Y_+ \) and \( Y_- \) as before, and let \( U' = X - Y_+ - Y_- \). We claim that \( Y_+ \) is the set of points \( x \in Z \) such that \( \lim_{\lambda \to \infty} \lambda \cdot x \in Z \). Recall that \( Y_+ := \{ x \in X : \lim_{\lambda \to \infty} \lambda \cdot x \in V_+ \} \). But if \( x \in Z \) with \( \lim_{\lambda \to \infty} \lambda \cdot x \in Z \) then this limit is in \( V \cap Z = V_+ \).

On the other hand, if \( x \in X \) with \( \lim_{\lambda \to \infty} \lambda \cdot y \in V_+ \) then \( x \not\in (X - Z) \) because \( X - Z \) is closed and \( V_+ \cap (X - Z) = \emptyset \). This shows the claim. We next show that \( Y_- = X - Z \). To see this recall that \( Y_- := \{ x \in X : \lim_{\lambda \to 0_+} \lambda \cdot x \in V_- \} \). If \( x \in Z \) then by hypothesis \( \lim_{\lambda \to 0_+} \lambda \cdot x \in Z \) and \( V_- \cap Z = \emptyset \), so this shows that \( Y_- \subset X - Z \). On the other hand if \( x \in X - Z \) then since \( X - Z \) is closed, \( \lim_{\lambda \to 0_+} \lambda \cdot x \in X - Z \) and hence this limit is in \( Y_- \), which shows that \( Y_- = X - Z \).

With the two statements of the previous paragraph we obtain that the complement \( U' \) of \( Y_+ \) and \( Y_- \) is equal to the subset of points of \( Z \) whose limits at \( \lambda \to \infty \) do not exist in \( Z \), that is to say that \( U' \) is the same as the subset \( U \) described in the statement of the theorem. The result of Theorem 11.1 now gives the universal geometric quotient \( U/G_m \) which is separated and proper over \( S \).

\[
\text{Relative compactification of } M_{DR}(X/S, G)
\]

Suppose \( G \) is a reductive group and \( X \to S \) a smooth projective morphism. Apply Theorem 11.2 to \( Z = M_{Hod}(X/S, G) \). The hypotheses on \( Z \) are given by Corollary 10.3 and Lemma 10.4. Note that the open set \( U \) certainly contains the open set

\[
M_{Hod}(X/S, G) \times^1_A G_m \cong M_{DR}(X/S, G) \times G_m
\]

as a \( G_m \)-invariant open set. Since the quotient \( U \to Q \) is a geometric quotient, the image of the open set \( M_{DR}(X/S, G) \times G_m \) is a geometric quotient of \( M_{DR}(X/S, G) \times G_m \), but we already know the geometric quotient here, it is just \( M_{DR}(X/S, G) \). Thus our quotient \( Q \) contains \( M_{DR}(X/S, G) \) as an open set, and \( Q \) is proper over \( S \). We have proved the following theorem.

**Theorem 11.3** If \( G \) is a reductive group and \( X \to S \) a smooth projective morphism, then there exists a natural relative compactification \( \overline{M}_{DR}(X/S, G) \) proper over \( S \) and containing \( M_{DR}(X/S, G) \) as an open subset.

\[
\text{Proof: Take } \overline{M}_{DR}(X/S, G) \text{ to be the quotient } Q \text{ of the previous paragraph.} \quad \square
\]

**Remark:** There is a natural stack-theoretic compactification \( \overline{M}_{DR}(X/S, GL(n)) \) containing \( M_{DR}(X/S, GL(n)) \) as an open subset and satisfying the valuative criterion of
properness over $S$. The valuative criterion comes from Corollary 10.2. We state this only in the case $G = GL(n)$ because the finiteness result (E Corollary 9.15) used to pass to any group $G$ in the proof of Corollary 10.3 is only available for the moduli spaces, not for the moduli stacks.

There is a natural Cartier divisor on $M_{Hod}(X/S, GL(n))$ given as the pullback of the divisor $\{0\} \subset \mathbb{A}^1$. This divisor is $G_m$-invariant, so it projects to a Cartier divisor in the stack-theoretic quotient $\overline{M}_{DR}(X/S, GL(n))$; and the open set $M_{DR}(X/S, GL(n))$ is just the complement of this divisor. Thus the “divisor at infinity” exists as a natural Cartier divisor.

In the moduli space compactification $\overline{M}_{DR}(X/S, G)$ the divisor at infinity is only a Weil divisor. We can define an intermediate orbifold compactification by taking the quotient $M_{Hod}(X/S, G)/G_m$ in the sense of stacks. Here the divisor at infinity is again a Cartier divisor. In the orbifold compactification there may be orbifold points corresponding to fixed points of finite subgroups of $G_m$. In the scheme-theoretic compactification these project to certain quotient singularities. They correspond to objects which are like systems of Hodge bundles (or variations of Hodge structure) except that the Hodge bundles are only indexed by a cyclic group instead of $\mathbb{Z}$ so the Kodaira-Spencer map $\theta$ can go “around and around” to no longer be nilpotent.

The orbifold structure of the divisor at infinity is that of the quotient $U \times_{\mathbb{A}^1} \{0\}/G_m$. But this is the quotient of the open subset of $M_{Dol}(X/S, G)$ corresponding to Higgs bundles with non-nilpotent Higgs field, by the action of $G_m$.

**Interpretation of the Hodge filtration in terms of the compactification**

Let $\overline{M}_{DR}$ denote the orbifold compactification described above. There is a principal $G_m$-bundle over this space, it is just the total space from before taking the quotient. This principal bundle corresponds to a line bundle $L$. One can see that $L = \mathcal{O}_{\overline{M}_{DR}}(-D)$ where $D$ is the divisor at infinity. Conversely, from the data of $\overline{M}_{DR}$ and the divisor at infinity $D$ (which has multiplicity one) we obtain a line bundle and hence a principal $G_m$-bundle with a section defined over $M_{DR}$. There is only one function on this total space which is constant on the multiples of our copy of $M_{DR}$ so this fixes the morphism to $\mathbb{A}^1$. The total space is $M_{Hod}$, the open subset which is the complement of the locus of Higgs bundles with nilpotent Higgs field. Thus we recover most but not all of the Hodge filtration $M_{Hod}$ from our compactification with its divisor at infinity.

**Griffiths transversality revisited**

In the relative case we have obtained a family of compactifications

$\overline{M}_{DR}(X/S) \to S$.  

48
On the other hand, recall that $M_{DR}(X/S)$ has the Gauss-Manin connection which, analytically, translates the fact that $M_{DR}(X/S)^{an}$ is locally over $S^{an}$ a product of the form $S^{an} \times M_B$ where $M_B$ is the moduli space of representations of the fundamental group of the fiber $X_s$. The Griffiths transversality condition basically says that the Gauss-Manin connection has poles of order 1 along the divisor at infinity.

In order to make this precise we restrict ourselves to a case where the moduli space is smooth. Fix a family $X \to S$ over a base $S$, and suppose $S$ is a smooth curve. Suppose for simplicity that $G$ and Chern class data $c$ are fixed so that the corresponding unions of components $M_{DR}(X_s, G)_c$ is smooth and equidimensional (for example $X/S$ a family of curves, $G = PGL(n)$ and $c$ means we look at bundles of degree $d$ prime to $n$). We obtain

$$M_{Hod}(X/S, G)_c \to S \times \mathbb{A}^1.$$  

We claim that this map is smooth. This is a special case of Conjecture 9.3, and requires some care. Apply the criterion of ([30] Chapter III Lemma 10.3.A—for which Hartshorne refers to Bourbaki and Altman and Kleiman) where $t$ is the coordinate on $\mathbb{A}^1$. We have to show that $t$ is not a zero divisor upstairs, and that $M_{Dol}(X/S, G)_c \to S$ is flat. Since all of the fibers of our map are smooth, the only way $t$ could be a zero divisor is if there were an irreducible component lying over $0 \in \mathbb{A}^1$. But Theorem 9.1 shows that this is not the case. To show that $M_{Dol}(X/S, G)_c \to S$ is flat it suffices (again in view of the smoothness of the fibers) to show that no irreducible component of $M_{Dol}(X/S, G)_c$ lies over a point in the curve $S$. Any component of $M_{Dol}(X_s, G)_c$ is contained in the closure of $M_{DR}(X_s, G)_c \times G_m$, so any component of $M_{Dol}(X/S, G)_c$ is contained in the closure of $M_{DR}(X/S, G)_c \times G_m$. If $n$ denotes the dimension of any components of $M_{DR}(X_s, G)_c$ (by hypothesis these dimensions are all the same) then the dimension of any component of $M_{Hod}$ is at least $n + 2$ and (since $M_{Dol}$ is defined by one equation $t = 0$ inside $M_{Hod}$) the dimension of any component of $M_{Dol}(X/S, G)_c$ must be at least $n + 1$. But the fibers $M_{Dol}(X_s, G)_c$ are all of dimension $n$, so they cannot contain irreducible components of $M_{Dol}(X/S, G)_c$. This proves that our map is flat. As the fibers are smooth, the map is smooth.

Now we can get back to the thread of our discussion. Let $U \subset M_{Hod}(X/S, G)_c$ denote the open set used in defining the compactification. Let $\overline{M}_{DR}(X/S, G)$ denote the orbifold compactification of $M_{DR}(X/S, G)$ obtained by taking the quotient $U/G_m$ in the sense of stacks. This can introduce orbifold points at places where the stabilizer is a nontrivial finite subgroup of $G_m$ (it has to be the $m$-th roots of unity). These orbifold points would be replaced by the corresponding cyclic quotient singularities in the usual compactification defined previously. The advantage here is that $\overline{M}_{DR}(X/S, G)$ is smooth over $S$.

Let $D \subset \overline{M}_{DR}(X/S, G)$ denote the divisor at infinity. It is reduced (since $M_{Hod}$ is smooth over $\mathbb{A}^1$). The Gauss-Manin connection can be interpreted as a lifting of vector fields on $S$ to vector fields on $M_{DR}(X/S, G)$. If $p$ denotes the projection to $S$, we obtain
a vector field with coefficients in the line bundle $p^*(\Omega^1_S)$ which we denote as
\[ \eta \in H^0(M_{DR}(X/S, G), T(M_{DR}(X/S, G)) \otimes p^*\Omega^1_S).\]

Note that it projects to the identity section of $T(S) \otimes \Omega^1_S = O_S$. This means that flowing along $\eta$ takes us from one fiber of $p$ to another.

**Theorem 11.4** The Griffiths transversality property says that the vector field $\eta$ has simple poles along $D$, and the residue is tangent to $D$. More precisely let
\[ F := \frac{T(M_{DR}(X/S, G)) \otimes p^*\Omega^1_S \otimes O_D(D)}{T(D) \otimes p^*\Omega^1_S \otimes O_D(D)} \]
(which is supported on $D$), then
\[ \ker(H^0(M_{DR}(X/S, G), T(M_{DR}(X/S, G)) \otimes p^*\Omega^1_S \otimes O(D)) \to H^0(D, F). \]

**Proof:** We leave this to the reader. \qed

It should be interesting to study the behavior of the dynamical system given by this vector field with poles. The transport between fibers $M_{DR}(X_s, G)$ and $M_{DR}(X_t, G)$ has the effect of composing the analytic isomorphisms
\[ M_{DR}(X_s, G)^{an} \cong M_B(X_s, G)^{an} = M_B(X_t, G)^{an} \cong M_{DR}(X_t, G)^{an} \]
where the left and right isomorphisms are the Riemann-Hilbert correspondence and the middle equality comes from the isomorphism of fundamental groups (which depends on the path we choose from $s$ to $t$).

**Exercise:** Interpret the regularity of the Gauss-Manin connection in way similar to the above interpretation of Griffiths transversality. On the moduli space $M(X_{DR/S}(\log D)/S, G)$ the lifts of vector fields given by the Gauss-Manin connection will have simple poles along inverse image of the singular set in $S$.

**Compactifications of spaces of $\Lambda$-modules**

By a technique similar to our construction of the relative compactification of $M_{DR}$ we have the following general theorem.

**Theorem 11.5** Suppose $X \to S$ is a projective flat morphism, and $\Lambda$ is a split almost polynomial sheaf of rings of differential operators on $X/S$. Let $M(\Lambda, P) \to S$ denote the moduli space of semistable $\Lambda$-modules with Hilbert polynomial $P$ on $X/S$. Then there exists a relative compactification, a scheme $\overline{M}(\Lambda) \to S$ containing $M(\Lambda)$ as an open set and which is proper over $S$.  

50
Proof: The proof is the same as the previous one, with the following changes. We replace the ring \( \Lambda_{Hod} \) by the ring \( \xi(\Lambda, F) \) on \( X \times \mathbb{A}^1 \) for the filtration \( F \) of \( \Lambda \) by degree. Even if not admitted in the definition of \( M(\Lambda, P) \), we must now admit torsion-free objects in the space \( M(\xi(\Lambda, F), P) \) used to get the compactification. \( \square \)

We could even obtain the same result with \( \Lambda \)-modules which are of pure dimension \( d < \dim(X/S) \)—this is an interpretation of the statement for \( \deg(P) = d \). The proof is again exactly the same.

A total space compactification of \( M_{DR}(X/S, GL(n)) \)

Suppose \( S \) is smooth, projective, with \( S' = S - D \) the complement of a divisor with normal crossings. Suppose \( X \to S \) is smooth over \( S' \) (we denote \( X' := X \times_S S' \)) and has inverse image of \( D \) being a divisor with normal crossings. In this situation We can get a compactification for the total space \( M_{DR}(X'/S', GL(n)) \). Let \( \Lambda = \Lambda_{DR/S}(log D) \) be the split almost polynomial sheaf of rings of differential operators corresponding to the formal groupoid \( X_{DR/S}(log D) \) defined in §8. Apply the construction of Theorem 11.5 to obtain a compactification. We can describe this more precisely. There is a formal groupoid \( X_{Hod/S}(log D) \) combining all of the constructions of §8, with underlying scheme \( X \times \mathbb{A}^1 \). Let \( Mtf(X_{Hod/S}(log D), GL(n)) \) denote the moduli space for torsion free semistable objects on \( X_{Hod/S}(log D) \) (with vanishing rational Chern classes). Our total space compactification is

\[
Mtf(X_{DR/S}(log D), GL(n)) := U/G_m
\]

where \( U \subset Mtf(X_{Hod/S}(log D), GL(n)) \) is the open set of points \( p \) such that \( \lim_{t \to \infty} tp \) does not exist. It is proper over \( S \) and since \( S \) itself is proper, it is compact.

The necessity to include torsion-free sheaves over the singular fibers is why we must assume here that the structure group is \( GL(n) \).

Combining the interpretations of Griffiths transversality and regularity of Gauss-Manin (exercise), we get that the lifts of vector fields on \( S \) given by the Gauss-Manin connection on \( M_{DR}(X/S, GL(n)) \), have simple poles at all components of infinity in the total space compactification of \( M_{DR}(X/S, GL(n)) \).

12. The nonabelian Noether-Lefschetz locus

A. Beilinson made a comment to the effect that one could get the moduli for \( \mathbb{Z} \)-variations of Hodge structure as an intersection between the integral representations, and the filtered local systems. Of course for \( X \) fixed the moduli space is just a finite set of points, but this becomes interesting when we let \( X \) vary in a family.

For this section we suppose \( G = GL(n) \).
Suppose $X \rightarrow S$ is a smooth projective morphism. Let $V \subset M_{\text{Dol}}(X/S, GL(n))$ denote the fixed point set of the $G_m$-action; it is also the moduli space for systems of Hodge bundles or equivalently for complex variations of Hodge structure \[3\]. Let $V_{DR}$ denote the image in $M_{DR}(X/S, GL(n))$ (note that this is not a complex analytic subset). On the other hand, let $M_B(X_s, GL(n, \mathbb{Z})) \subset M_B(X_s, GL(n))$ denote the image of $\text{Hom}(\pi_1(X_s, x), GL(n, \mathbb{Z}))$ (it is the subset of integral representations), and let $M_{DR}(X/S, GL(n, \mathbb{Z}))$ denote the subset of points of $M_{DR}(X/S, GL(n))$ which over each fiber $X_s$ correspond to elements of $M_B(X_s, GL(n, \mathbb{Z}))$. Note that $M_{DR}(X/S, GL(n, \mathbb{Z}))$ is a complex analytic subset of $M_{DR}(X/S, GL(n))$. Finally put

$$NL(X/S, GL(n)) := V_{DR} \cap M_{DR}(X/S, GL(n, \mathbb{Z})).$$

There is a morphism $NL(X/S, GL(n)) \rightarrow S$.

**Theorem 12.1** For each $s \in S$, the fiber $NL(X/S, GL(n))_s$ is the set of isomorphism classes of integral representations $\rho$ such that $\rho \oplus \rho$ underlies an integral variation of Hodge structure. The morphism $NL(X/S, GL(n)) \rightarrow S$ is proper, and $NL(X/S, GL(n))$ has a unique structure of normal analytic variety such that the inclusions $NL(X/S, GL(n)) \rightarrow M_{DR}(X/S, GL(n))$ and $NL(X/S, GL(n)) \rightarrow M_{\text{Dol}}(X/S, GL(n))$ are complex analytic.

**Remark:** In this first treatment of the subject, we are ignoring a possibly more natural non-reduced or non-normal structure of complex analytic space on $NL(X/S, GL(n))$.

**Proof:** If $\rho \in NL(X/S, GL(n))_s$ then $\rho$ is a complex variation of Hodge structure on $X_s$, and $\rho$ is integral. It is easy to see that $\rho \oplus \rho$ has a structure of integral variation of Hodge structure (see for example the arguments in \[14\], \[15\], \[65\]). Conversely if $\rho \oplus \rho$ has a structure of variation of Hodge structure then $\rho$ is fixed by the $G_m$ action, so $\rho$ lies in $V$.

To see that $NL(X/S, GL(n)) \rightarrow S$ is proper, it suffices to work locally near a point $s_0 \in S$. By the main result of \[14\] (done with small variations in the parameters, which still works) there is a finite subset of points of $M_B(X_{s_0}, GL(2n, \mathbb{Z}))$ which correspond to representations are doubles of $n$-dimensional representations and which could possibly be $\mathbb{Z}$-variations of Hodge structure on $X_s$ with $s$ in a given relatively compact neighborhood of $s_0$. Over this neighborhood, $NL(X/S, GL(n))$ is the intersection of $V_{DR}$ (which is proper over $S$ \[59\]) with this finite set of sections; thus $NL(X/S, GL(n))$ is proper over $S$.

We show the complex analyticity of $NL(X/S, GL(n))$ inside $M_{DR}(X/S, GL(n))$. The question is local over $S$, so we can fix a neighborhood $U$ of $s \in S$ and a section $\sigma : U \rightarrow M_{DR}(X/S, GL(n, \mathbb{Z}))$ corresponding to an integral representation of $\pi_1(X_s)$; we have to show that $\sigma^{-1}(V_{DR})$ is an analytic subset of $U$. Our representation $\rho$ corresponds to a local system $W$ on $X_s$. Let $W_i$ denote the complex irreducible factors of $W$. Then
$\sigma^{-1}(V_{DR})$ is the intersection of the subsets $N_i$ of points in $u$ where $W_i$ admits a complex variation of Hodge structure. It suffices to show that $N_i$ are complex analytic. If $W_i$ does not admit a flat hermitian form then $N_i$ is empty, so we can assume that it does admit such a form. This form $\langle \ , \ \rangle$ is uniquely determined up to a scalar. Fix an integer $w$, and data of ranks $r_i$ and degrees $d_i$. Let
\[ HF(X/S, W_i, r_i, d_i) \rightarrow U \]
denote the parameter scheme for filtrations $F^\cdot$ of the local systems $W_i(s)$ (considered as vector bundles with integrable connection on the fibers $X_s$) with $r(F^\cdot) = r_i$ and $deg(F^\cdot) = d_i$, and satisfying the Griffiths transversality condition. It is an analytic variety over $U$, in fact the pullback of a quasiprojective variety over $M_{DR}(X/S, GL(n))$ by the section $\sigma$ (the parameter variety is the closed subscheme of the Hilbert scheme of filtrations defined by the Griffiths transversality condition). Let
\[ HF(X/S, W_i, \langle \ , \ \rangle, w, r_i, d_i) \subset HF(X/S, W_i, r_i, d_i) \]
denote the open subset of filtrations such that $F^\cdot$ and $F^\cdot\perp$ together determine a Hodge structure of weight $w$ on the fiber over every $x \in X$. The morphism
\[ HF(X/S, W_i, \langle \ , \ \rangle, w, r_i, d_i) \rightarrow U \]
is injective (because there is at most one structure of complex variation of Hodge structure on the irreducible representation $W_i$ up to translations, but the translations are fixed by specifying the ranks $r_i$), so the image is an analytic subset. There are only a finite number of possible sets of degrees and ranks which can occur (and it suffices to consider $w = 0$ for example) so the union of this finite number of analytic subsets is $N_i$. This shows that $N_i$ is analytic, hence that the intersection is analytic. Thus (taking the union over the finite number of sections we have to consider) $NL(X/S, GL(n))$ is an analytic subset of $M_{DR}(X/S, GL(n))$.

From the above construction one gets that over any component of $NL(X/S, GL(n))$ the Hodge filtration of the resulting variation of Hodge structure varies analytically. Since the associated Higgs bundle is the associated graded of the Hodge filtration (with $\theta$ as the projection of the connection, which also varies analytically), the associated Higgs bundle varies analytically with the point in $NL(X/S, GL(n))$, that is to say that $NL(X/S, GL(n))$ is an analytic subset of $M_{Dol}(X/S, GL(n))$.

\[ \square \]

**Corollary 12.2** If $S$ is projective, then $NL(X/S, GL(n))$ has a structure of normal projective variety such that the morphisms $NL(X/S, GL(n)) \rightarrow M_{DR}(X/S, GL(n))$ and $NL(X/S, GL(n)) \rightarrow M_{Dol}(X/S, GL(n))$ are algebraic morphisms.

53
Proof: We can think of $NL(X/S, GL(n))$ as the normalization of an analytic subvariety of $M_{DR}(X/S, GL(n))$ which is proper over $S$. In particular it is a closed subvariety of $\overline{M}_{DR}(X/S, GL(n))$ so we can apply GAGA to say that it is algebraic. Again by GAGA the morphism to $M_{Dol}(X/S, GL(n))$ is algebraic.

After the relatively surprising results of the theorem and this corollary, the reader might well be asking if $NL(X/S, GL(n))$ isn’t just a finite collection of points. In fact, $NL(X/S, GL(n))$ can have positive dimensional components. For example if $Z$ is a surface with a $\mathbf{Z}$-variation of Hodge structure and then if $X/S$ is a pencil of curves on $Z$, the family of restrictions of the variation on $Z$ gives a component of $NL(X/S, GL(n))$ dominating $S$. More generally if $X/S$ contains a pencil of curves on $Z$ as a subfamily, then we get a component of $NL(X/S, GL(n))$ dominating this subfamily. One might ask whether all positive dimensional components of $NL(X/S, GL(n))$ come from such a construction.

We can also ask for an extension of the result of Corollary 12.2 to the quasiprojective case.

**Conjecture 12.3** If $S$ is a quasiprojective variety then $NL(X/S, GL(n))$ is an algebraic variety and the morphisms to $M_{DR}(X/S, GL(n))$ and $M_{Dol}(X/S, GL(n))$ are algebraic.

This conjecture would be the nonabelian analogue of the result of [9]. It is a globalized version of a problem Deligne posed to me some time ago, of obtaining a generalization of the finiteness results of [14] uniformly near singularities. In [9] it is explained how their result would be a consequence of the Hodge conjecture. In a similar way, Conjecture 12.3 would be a consequence of the following nonabelian version of the Hodge conjecture.

**Conjecture 12.4** The points of $NL(X_\mathbf{s}, GL(n))$ (i.e. the $\mathbf{Z}$-variations of Hodge structure) are motivic representations on $X_\mathbf{s}$.

I don’t know who first thought of this conjecture but that must have been a long time ago. Of course there is little chance of making any progress on this—I have presented it only for the light it sheds on Conjecture 12.3.

The main problem in proving Conjecture 12.3 is to make a local analysis around the singularities of a good completion of the family $X/S$ to $\overline{X}/\overline{S}$. We hope that the total space compactification constructed above will be useful for studying this algebraicity question locally at the singularities of the family.

13. An open problem: degeneration of nonabelian Hodge structure

I would like to end this paper by proposing an open problem for further research. This is the problem (motivated at the end of the previous section) of studying the degeneration
of nonabelian Hodge structure which arises from a degenerating family of varieties $X \rightarrow S$. It is already complicated enough to study the case of a degenerating family of curves, so we can suppose that $X \rightarrow S$ is a family of curves. Let $0 \in S$ denote a point where the fiber $X_0$ has a simple singularity (node). Let $U = S - \{0\}$ and suppose $X_U$ is smooth over $U$. We assume $G = GL(n)$ for reasons we will see in a minute. Then the relative moduli space $M_{DR}(X_U/U, GL(n))$ over $U$ is provided with the Gauss-Manin connection; with a family of hyperkähler structures; with a family of Hodge filtrations satisfying Griffiths transversality; and with a Noether-Lefschetz locus $NL(X_U/U, GL(n)) \subset M_{DR}(X_U/U, GL(n))$ for which we would like to prove algebraicity. The problem, then, is to study the degeneration of all of these structures as one approaches $0 \in S$.

The first step in studying degenerations of variations of Hodge structure in the abelian case was to have a canonical extension of the underlying holomorphic bundle, and a regular singularity theorem for the flat connection. We have obtained the analogues of these things for the nonabelian case. Note that in defining a canonical extension, it is important to get something which is proper over $S$. This was perhaps not apparent in the abelian case, where nobody cares about the fact that a vector space is noncompact! But in the nonabelian case, a noncompact extension would leave room for some asymptotic behavior as one goes off to infinity, difficult to see. To get a compactification of $M_{DR}(X_U/U, GL(n))$ we take the total space compactification $M^{\text{tf}}_{DR}(X_{DR/S}(\log 0)/S, GL(n))$ which comes from looking at the moduli of torsion free sheaves with $\lambda$-connection logarithmic at the singularities, and applying the construction of §§10-11. After doing all of this we obtain a compactification of $M_{DR}(X_U/U, GL(n))$ which is the analogue of the canonical extension (together with the Hodge filtration which corresponds to the fiberwise compactification). The regularity of the Gauss-Manin connection, coupled with Griffiths transversality, give that the Gauss-Manin connection on $M_{DR}(X_U/U, GL(n))$ over $U$ has poles of order one at infinity (both along the fiberwise infinity and at the singular point $0 \in S$).

We need a good local description of this compactified moduli space near all points at infinity, specially torsion-free sheaves on the singular fibers but also at infinity in the fiberwise direction.

One approach to the analysis of the degeneration might be to directly analyze the lifted vector field giving the Gauss-Manin connection with its simple poles, and try to deduce asymptotic properties of the transport along this vector field.

Another option would be to try to analyze the degeneration of the hyperkähler structure, thinking of it as a family of quaternionic structures on the fixed variety $M_{B}(X_s, GL(n))$.

Among the goals of this study should be: to prove algebraicity of the Noether-Lefschetz locus, in a nonabelian version of the work of Cattani-Deligne-Kaplan; to obtain a nonabelian version of the Clemens-Schmid exact sequence; to obtain estimates and asymptotic expansions for everything in the spirit of the $SL_2$ and nilpotent orbit theorems; and finally to be able to apply all of this to obtain a devissage principle for nonabelian cohomology...
(even with coefficients in higher homotopy types [20]), i.e. a version of the Leray spectral sequence.

References

[1] D. Arapura. Higgs line bundles, Green-Lazarsfeld sets and maps of Kähler manifolds to curves. *Bull. A.M.S.* **26** (1992), 310-314.

[2] M. Atiyah, N. Hitchin, I. Singer. Self duality in four dimensional Riemannian geometry. *Proc. Royal Soc. Lond.* **A362** (1978), 425-461.

[3] A. Beauville. Annulation du $H^1$ et systèmes paracanoniques sur les surfaces. *Crelle Journ.* **388** (1988), 149-157.

[4] P. Berthelot. *Cohomologie Cristalline des Schémas de Caractéristique $p > 0*, Springer L.N.M. **407** (1974).

[5] O. Biquard. Fibrés de Higgs et connexions intégrables: le cas logarithmique (diviseur lisse). Preprint.

[6] O. Biquard. Fibrés paraboliques stables et connexions singulières plates. *Bull. S.M.F.* **119** (1991), 231-257.

[7] J. Carlson and D. Toledo. Harmonic mappings of Kähler manifolds to locally symmetric spaces. *Publ. Math. I.H.E.S.* **69** (1989), 173-201.

[8] F. Catanese. Moduli and classification of irregular Kähler manifolds (and algebraic varieties) with Albanese general type fibrations. *Invent. Math.* **104** (1991), 263-289.

[9] E. Cattani, P. Deligne, A. Kaplan. On the locus of Hodge classes. *Jour. A.M.S.* **8** (1995), 483-506.

[10] K. Corlette. Flat $G$-bundles with canonical metrics. *J. Diff. Geom.* **28** (1988), 361-382.

[11] K. Corlette. Rigid representations of Kählerian fundamental groups. *J. Diff. Geom.* **33** (1991), 239-252.

[12] K. Corlette. Nonabelian Hodge theory and square integrability. *Actes du colloque “Géométrie et Topologie des Variétés Projectives” Toulouse, 1992* (preprint).

[13] P. Deligne. Various letters to the author.
[14] P. Deligne. Un théorème de finitude pour la monodromie. *Discrete Groups in Geometry and Analysis*, Birkhauser (1987), 1-19.

[15] P. Deligne. Théorie de Hodge II. *Publ. Math. I.H.E.S.* 40 (1971), 5-58.

[16] C. Deninger. On the $\Gamma$-factors attached to motives. *Invent. Math.* 104 (1991), 245-263.

[17] K. Diederich and T. Ohsawa. Harmonic mappings and disc bundles over compact Kähler manifolds. *Publ. R.I.M.S.* 21 (1985), 819-833.

[18] S. Donaldson. Twisted harmonic maps and self-duality equations. *Proc. London Math. Soc.* 55 (1987), 127-131.

[19] J. Eells and J. Sampson. Harmonic mappings of Riemannian manifolds. *Amer. J. of Math.* 86 (1964), 109-160.

[20] J.M. Fontaine. Modules galoisiens, modules filtrés et anneaux de Barsotti-Tate, *Journée de Géométrie Algébrique de Rennes, Astérisque* 65 (1979), 3-80.

[21] A. Fujiki. Hyperkähler structure on the moduli space of flat bundles. Preprint (Kyoto University).

[22] W. Goldman and J. Millson. The deformation theory of representations of fundamental groups of compact Kähler manifolds. *Publ. Math. I.H.E.S.* 67 (1988), 43-96.

[23] M. Green and R. Lazarsfeld. Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville. *Invent. Math.* 90 (1987), 389-407.

[24] M. Gromov. Sur le groupe fondamentale d’une variété kählérienne. *C.R. Acad. Sci. Paris* Ser. I Math. 308 (1989), 67-70.

[25] M. Gromov and R. Schoen. Harmonic maps into singular spaces and $p$-adic superrigidity for lattices in groups of rank one. *Publ. Math. I.H.E.S.* 76 (1992), 165-246.

[26] J. Hadamard. Essai sur l’étude des fonctions données par leurs développements de Taylor. *J. Math. Pures Appl.* 8 (1892), 101-186.

[27] R. Hain. The de Rham homotopy theory of complex algebraic varieties I. *K-theory* 1 (1987), 271-324.

[28] R. Hain. Completions of mapping class groups and the cycle $C - C^-$. Preprint.
[29] R. Hain. Torelli groups and geometry of moduli spaces of curves. Preprint alg-geom 9403015 (plus later versions).

[30] R. Hartshorne. Algebraic Geometry. Springer, New York (1977).

[31] E. Hironaka. Alexander stratifications of character varieties. Preprint alg-geom 9602004.

[32] N. Hitchin. Stable bundles and integrable systems. Duke Math. J. 54 (1987), 91-114.

[33] N. Hitchin. The self-duality equations on a Riemann surface. Proc. London Math. Soc. 55 (1987), 59-126.

[34] N. Hitchin. Hyperkähler manifolds. Exposé 748, Seminaire Bourbaki, Astérisque 206 (1992), 137-166.

[35] N. Hitchin, A. Karlhede, U. Lindstrom, M. Roćcek. Hyperkähler metrics and supersymmetry. Comm. Math. Phys. 108 (1987), 535-559.

[36] L. Illusie. Complexe cotangent et déformations I, Springer L.N.M. 239 (1971); II, Springer L.N.M. 283.

[37] J. Jost and S.T. Yau. Harmonic maps and Kähler manifolds. Math. Ann. 262 (1983), 145-166.

[38] J. Jost and K. Zuo. Harmonic maps into Tits buildings, factorization of nonintegral $p$-adic and nonrigid representations of $\pi_1$ of algebraic varieties. Preprint.

[39] J. Jost and K. Zuo. Harmonic maps and $Sl(r, \mathbb{C})$-representations of fundamental groups of quasiprojective manifolds. Preprint.

[40] L. Katzarkov. Factorization theorems for the representations of the fundamental groups of quasiprojective varieties and some applications. Preprint alg-geom 9402012.

[41] L. Katzarkov. Nilpotent groups and universal coverings of smooth projective varieties. Preprint alg-geom 9510009.

[42] L. Katzarkov. On the Shafarevich maps. Preprint.

[43] L. Katzarkov and T. Pantev. Representations of fundamental groups whose Higgs bundles are pullbacks. J. Diff. Geo. 39 (1994), 103-121.
[44] L. Katzarkov and T. Pantev. Stable $G_2$ bundles and algebraically completely integrable systems. *Compositio Math.* **92** (1994), 43-60.

[45] L. Katzarkov and M. Ramachandran. On the Shafarevich conjecture for surfaces. Preprint (1994).

[46] J. Kollar. Shafarevich maps and plurigenera of algebraic varieties. *Invent. Math.* **113** (1993), 177-215.

[47] S. Langton. Valuative criteria for families of vector bundles on algebraic varieties. *Ann. of Math.* **101** (1975), 88-110.

[48] B. Lasell. Complex local systems and morphisms of varieties. *Compositio Math.* **98** (1995), 141-166.

[49] B. Lasell and M. Ramachandran. Local systems on Kähler manifolds and harmonic maps. Preprint (1994).

[50] A. Lubotzky and A. Magid. Varieties of representations of finitely generated groups. *Memoirs A.M.S.* **336** (1985).

[51] M. Maruyama and K. Yokogawa. Moduli of parabolic stable sheaves. *Math. Ann.* **293** (1992), 77-99.

[52] N. Mok. Factorization of semisimple discrete representations of Kähler groups. *Invent. Math.* **110** (1992), 557-614.

[53] J. Morgan. The algebraic topology of smooth algebraic varieties. *Publ. Math. I.H.E.S.* **48** (1978), 137-204.

[54] D. Mumford. *Geometric Invariant Theory*, Springer, New York (1965).

[55] T. Napier. Convexity properties of coverings of smooth projective varieties. *Math. Ann. 286* (1990), 433-480.

[56] T. Napier and M. Ramachandran. Structure theorems for complete Kähler manifolds and applications to a Lefschetz type of theorem. To appear.

[57] N. Nitsure. Moduli spaces of semistable pairs on a curve. *Proc. London Math. Soc.* **62** (1991), 275-300.

[58] N. Nitsure. Moduli of semistable logarithmic connections. *Jour. A. M. S.* **6** (1993), 597-610.
[59] A. Reznikov. All regulators of flat bundles are torsion. *Ann. of Math.* **141** (1995), 373-386.

[60] A. Reznikov. Harmonic maps, hyperbolic cohomology and higher Milnor inequalities. *Topology* **32** (1993), 899-907.

[61] S. Salamon. Differential geometry of quaternionic manifolds. *Ann. Sci. E.N.S.* **19** (1986), 31-55.

[62] J. Sampson. Applications of harmonic maps to Kähler geometry. *Contemp. Math.* **49** (1986), 125-133.

[63] C. Simpson. Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization. *Journ. A. M. S.* **1** (1988), 867-918.

[64] C. Simpson. Nonabelian Hodge theory. *Proceedings, ICM-90, Kyoto* Springer, Tokyo (1991), 198-230.

[65] C. Simpson. Higgs bundles and local systems. *Publ. Math. I.H.E.S.* **75** (1992), 5-95.

[66] C. Simpson. Subspaces of moduli spaces of rank one local systems. *Ann. Sci. Ec. Norm. Sup.* **26** (1993), 361-401.

[67] C. Simpson. Lefschetz theorems for the integral leaves of a holomorphic one form. *Compositio Math.* **87** (1993), 99-113.

[68] C. Simpson. Some families of local systems over smooth projective varieties. *Annals of Math.* **138** (1993), 337-425.

[69] C. Simpson. Moduli of representations of the fundamental group of a smooth projective variety, I: *Publ. Math. I.H.E.S.* **79** (1994), 47-129; II: *Publ. Math. I.H.E.S.* **80** (1994), 5-79.

[70] C. Simpson. Homotopy over the complex numbers and de Rham cohomology. To appear, proceedings of the Taniguchi symposium on vector bundles, Kyoto 1994.

[71] Y.T. Siu. Complex analyticity of harmonic maps and strong rigidity of complex Kähler manifolds. *Ann. of Math.* **112** (1980), 73-110.

[72] Y. T. Siu. Complex analyticity of harmonic maps, vanishing and Lefschetz theorems. *J. Diff. Geo.* **17** (1982), 55-138.

[73] D. Toledo. Projective varieties with non-residually finite fundamental group. *Publ. Math. I.H.E.S.* **77** (1993), 103-119.
[74] K. Yokogawa. Compactification of moduli of parabolic sheaves and moduli of parabolic Higgs sheaves. *J. Math. Kyoto Univ.* **33** (1993), 451-504.

[75] K. Yokogawa. Infinitesimal deformations of parabolic Higgs sheaves. *Internat. J. Math.* **6** (1995), 125-148.

[76] K. Zuo. Some structure theorems for semisimple representations of $\pi_1$ of algebraic manifolds. *Math. Ann.* **295** (1993), 365-382.

[77] K. Zuo. *Representations of Fundamental Groups of Algebraic Varieties*, Habilitation, Universität Kaiserslautern (1995), to appear as a book.