Labelled $\lambda$-calculi with Explicit Copy and Erase

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We present two rewriting systems that define labelled explicit substitution $\lambda$-calculi. Our work is motivated by the close correspondence between Lévy’s labelled $\lambda$-calculus and paths in proof-nets, which played an important role in the understanding of the Geometry of Interaction. The structure of the labels in Lévy’s labelled $\lambda$-calculus relates to the multiplicative information of paths; the novelty of our work is that we design labelled explicit substitution calculi that also keep track of exponential information present in call-by-value and call-by-name translations of the $\lambda$-calculus into linear logic proof-nets.

1 Introduction

Labelled $\lambda$-calculi have been used for a variety of applications, for instance, as a technology to keep track of residuals of redexes [6], and in the context of optimal reduction, using Lévy’s labels [14]. In Lévy’s work, labels give information about the history of redex creation, which allows the identification and classification of copied $\beta$-redexes. A different point of view is proposed in [4], where it is shown that a label is actually encoding a path in the syntax tree of a $\lambda$-term. This established a tight correspondence between the labelled $\lambda$-calculus and the Geometry of Interaction interpretation of cut elimination in linear logic proof-nets [13].

Inspired by Lévy’s labelled $\lambda$-calculus, we define labelled $\lambda$-caluli where the labels attached to terms capture reduction traces. However, in contrast with Lévy’s work, our aim is to use the dynamics of substitution to include information in the labels about the use of resources, which corresponds to the exponentials in proof-nets. Exponential structure in proof-nets involves box-structures and connectives that deal with their management, for instance, copying and erasing of boxes. Different translations of the $\lambda$-calculus into proof-nets place boxes at different positions; such choices are also reflected in the paths of the nets. In Lévy’s calculus substitution is a meta-operation: substitutions are propagated exhaustively and in an uncontrolled way. We would like to exploit the fact that substitutions copy labelled terms and hence paths, but it is difficult to tell with a definition such as $(MN)^\alpha[P/x] = (M[P/x]N[P/x])^\alpha$, whether the labels in $P$ are actually copied or not: $P$ may substitute one or several occurrences of a variable, or it may simply get discarded.

In order to track substitutions we use calculi of explicit substitutions, where substitution is defined at the same level as $\beta$-reduction. Over the last years a whole range of explicit substitution calculi have been proposed, starting with the work of de Bruijn [9] and the $\lambda\sigma$-calculus [11]. Since we need to track copy and erasing of substitutions, we will use a calculus where not only substitutions are explicit, but also copy and erase operations are part of the syntax. Specifically, in this paper we use explicit substitution calculi that implement closed reduction strategies [10, 11]. This may be thought of as a more powerful form of combinatory reduction [8] in the sense that $\beta$-redexes may be contracted when the argument part or the function part of the redex is closed. This essentially allows more reductions to take place under abstractions. The different possibilities of placing restrictions on the $\beta$-rule give rise to different closed reduction strategies, corresponding to different translations of the $\lambda$-calculus into proof-nets (a survey of
available translations can be found in [16]). Closed reduction strategies date back to the late 1980’s, in fact, such a strategy was used in the proof of soundness of the Geometry of Interaction [13].

Labelled \( \lambda \)-calculi are a useful tool to understand the structure of paths in the Geometry of Interaction: Lévy’s labels were used to devise optimisations in GoI abstract machines, defining new strategies of evaluation and techniques for the analysis of \( \lambda \)-calculus programs [5, 3]. The labels in our calculi of explicit substitutions contain, in addition to the multiplicative information contained in Lévy’s labels, also information about the exponential part of paths in proof-nets. In other words, our labels relate a static concept—a path—with a dynamic one: copying and erasing operations. This is demonstrated in this paper using two different labelled calculi. In the first system, the \( \beta \)-rule applies only if the function part of the redex is closed. We relate this labelled system with proof-nets using the so-called call-by-value translation. We then define a second labelled \( \lambda \)-calculus where the \( \beta \)-rule applies only if the argument part of the redex is closed; thus, all the substitutions in this system are closed. We show that there is a tight relationship between labels in this system and paths in proof-nets, using the so-called call-by-name translation.

The rest of the paper is organised as follows. In Section 2 we review the syntax of the calculus of explicit substitutions (\( \lambda_c \)-terms) and introduce basic terminology regarding linear logic proof-nets. In Section 3 we introduce labelled versions of \( \lambda_c \)-terms. Section 4 presents the labelled calculus of closed functions (\( \lambda_{cf} \)) that we relate to paths in proof-nets coming from the call-by-value translation. Similarly, we relate in Section 5 the labelled version of the calculus of closed arguments \( \lambda_{ca} \) to closed cut-elimination of nets obtained from the call-by-name translation. We conclude in Section 6.

## 2 Background

We assume some basic knowledge of the \( \lambda \)-calculus [6], linear logic [12], and the Geometry of Interaction [13]. In this section we recall the main notions and notations that we will use in the rest of the paper. **Labels.** There is a well known connection between labels and paths: the label associated to the normal form of a term in Lévy’s labelled \( \lambda \)-calculus describes a path in the graph of the term [4]. The set of labels is generated by the grammar: \( \alpha, \beta ::= a | \alpha \beta | \overline{\alpha} | \alpha \), where \( a \) is an atomic label. Labelled terms are terms of the \( \lambda \)-calculus where each sub-term \( T \) has a label attached on it: \( T_\alpha \). Labelled \( \beta \)-reduction is given by \( (\lambda x. M)_\alpha N \beta \rightarrow \beta \alpha \cdot M[\alpha \cdot N / x] \), where \( \cdot \) concatenates labels: \( \beta \cdot T^\alpha = T^{\beta \alpha} \). Substitution assumes the variable name convention [6]:

\[
\begin{align*}
  x^\alpha[N/x] &= \alpha \cdot N \\
  y^\alpha[N/x] &= y^\alpha
\end{align*}
\]

\[
(\lambda y. M)^\alpha[N/x] = (\lambda y. M[N/x])^\alpha
\]

\[
(MN)^\alpha[P/x] = (M[P/x]N[P/x])^\alpha
\]

For example, \( III \), where \( I = \lambda x. x \), can be labelled, and then reduced as follows:

\[
\begin{align*}
  h &\quad e \\
  \lambda x &\quad \alpha \times d \\
  a &\quad c \\
  b &\quad \lambda x \\
  x &\quad x \\
  f &\quad x
\end{align*}
\]

The final label generated describes a path in the tree representation of the initial term, if we reverse the
underlines. Following this path will lead to the sub-term which corresponds to the normal form, without performing β-reduction. This is just one perspective on Girard’s Geometry of Interaction, initially set up to explain cut-elimination in linear logic. Here we are interested in the λ-calculus, but we can use the Geometry of Interaction through a translation into proof-nets. These paths are precisely the ones that the GoI Machine follows [17]. In this example, the structure of the labels (overlining and underlining) tells us about the multiplicative information, and does not directly offer any information about the exponentials. To add explicitly the exponential information we would need to choose one of the known translations of the λ-calculus into proof-nets, and it would be different in each case. Further, to maintain this information, we would need to monitor the progress of substitutions, so we need to define a notion of labelled λ-calculus for explicit substitutions. The main contribution of this paper is to show how this can be done.

**Explicit substitutions and resource management.** Explicit substitution calculi give first class citizenship to the otherwise meta-level substitution operation. Since we need to track copy and erasing of substitutions, in this paper we will use a calculus where not only substitutions are explicit, but also copy and erase operations are part of the syntax. The explicit substitution calculi defined in [10, 11] are well-adapted for this work: besides having explicit constructs for substitutions, terms include constructs for copying (δ) and erasing (ε) of substitutions. The motivation of such constructs can be traced back to linear logic, where the structural rules of weakening and contraction become first class logical rules, and to Abramsky’s work [2] on proof expressions.

The table below defines the syntax of λc-terms, together with the variable constraints that ensure that variables occur linearly in a term. We use fv(·) to denote the set of free variables of a term. We refer the reader to [11] for a compilation from λc-terms to λc-terms.

| Term | Variable Constraint | Free variables |
|------|---------------------|----------------|
| x    | ~                   | {x}            |
| λx.M | x ∈ fv(M)           | fv(M) – {x}    |
| MN   | fv(M) ∩ fv(N) = {}  | fv(M) ∪ fv(N)  |
| εx.M | x ∉ fv(M)           | fv(M) ∪ {x}   |
| δx^y.z.M | x ∉ fv(M), y ≠ z; {y, z} ⊆ fv(M) | (fv(M) – {y, z}) ∪ {x} |
| M[N/x] | x ∈ fv(M), (fv(M) – {x}) ∩ fv(N) = {} | (fv(M) – {x}) ∪ fv(N) |

For example, the compilation of λx.λy.x is λx.λy.εy.x, and the compilation of (λx.xx)(λx.xz) is (λx.δx^y.z.M) (λx.xz). We remark that both yield terms that satisfy the variable constraints.

Using λc-terms, two explicit substitution calculi were defined in [11]: λcf, the calculus of closed functions, and λcu, the calculus of closed arguments. In the former, the β-rule requires the function part of the redex to be closed, whereas in the latter, the argument part must be closed to trigger a β-reduction. In the rest of the paper we define labelled versions of these calculi and relate labels to proof-net paths.

**Proof-nets and the Geometry of Interaction.** The canonical syntax of a linear logic proof is a graphical one: a proof-net. Nets are built using the following set of nodes: *Axiom (●)* and *cut (○)* nodes (such nodes are dummy-nodes in our work and simply represent “linking” information); *Multiplicative nodes:* ⊗ and ⊕; *Exponential nodes:* contraction (fan), of-course (!), why-not (?), dereliction (D) and weakening (W). A sub-net may be enclosed in a box built with one of course-node and n ≥ 0 why-not-nodes, which we call auxiliary doors of a box. The original presentation of a net is oriented so that edges designated as conclusions of a node point downwardly: we abuse the natural orientation and indicate with an arrow-head the conclusion of the node (see Figure 1); all other edges are premises. Such structures may be
obtained by one of the standard translations, call-by-name or call-by-value \([16, 12]\), of \(\lambda\)-terms. For instance, the net in Figure 1a is obtained by the call-by-value translation of the \(\lambda\)-term \((\lambda x.y)(\lambda x.x)\).

We assume a weak form of cut elimination—closed cut elimination—denoted by \(\Rightarrow\). This is ordinary multiplicative and exponential proof-net reduction with the restriction that every exponential step can handle only boxes with no auxiliary doors. We refer the reader to \([13, 16]\) for a specification of this cut-elimination strategy, which is also used in the context of Interactions Nets \([15]\).

In this work we obtain labelled (weighted) versions of nets via inductive translations of \(\lambda_c\)-terms, which we define later on.

**Weighted nets.** To each edge of a net, we associate a *weight* \((w)\) built from terms of the dynamic algebra \(L^\ast\): constants \(p, q\) (multiplicative), \(r, s, t, d\) (exponential), 0 and 1; an associative composition operator \(\cdot\) with unit 1 and absorbing element 0; an involution \((\cdot)^\ast\) and a unary morphism \(!\). We use meta-variables \(\alpha, \beta, \ldots\) for terms and we shall write \(!^n(\cdot)\) for \(n \geq 0\) applications of the morphism. Intuitively, weights are used to identify paths, and the algebra is used to pick out the paths that survive reduction. We omit the definition of the correct labelling (with weights) of a net: this will again be obtained by translations of \(\lambda_c\)-terms into labelled proof-nets.

We use metavariables \(\phi, \chi \ldots\) to range over paths. Paths are assumed to be a) *non-twisting*, that is, paths are not over different premises of the same node and b) *non-bouncing*, that is, paths do not bounce off nodes. We call such paths *straight*; these traverse a weighted edge \(e\) forwardly when moving towards the premise of the incident node (resp. backwardly \(e^r\) when moving towards the conclusion) such that direction changes happen only at cut and axiom links. For instance, Figure 1b shows a bouncing, a twisting, and a straight path respectively. The weight of a path is 1 if it traverses no edge — this weight is the identity for composition which we usually omit; if \(\phi = e \cdot \psi\) is a path then its weight \(w(\phi)\) is defined to be \(w(e) \cdot w(\psi)\) and we have \(w(e^r) = w(e)^\ast\). We are mainly interested in the statics of the algebra and omit the equations that terms satisfy. We refer the reader to \([7, 3, 5]\) for a more detailed treatment.

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1 Traditionally, weights are composed antimorphically but in this work we read paths and compose weights from left to right.
3 Labelled terms

We attach labels to $\lambda_c$-terms in order to capture information not only about $\beta$-reductions but also about propagation of substitutions. We adopt the same language for labels as in [20], where a confluent system $(\lambda_{lcf})$ is defined and informally related to traces in a call-by-value proof-net translation. To establish the required correspondence, in this paper we translate (in Sections 4 and 5) our labels into terms of $L^*$ (defined in the previous section), which is the de-facto language for labels in proof-nets.

**Definition 1.** Labels are defined by the following grammar; $a$ is an atomic label taken from a denumerable set $\{a, b, \ldots\}$, and all labels in $C$ are atomic.

\[
\alpha, \beta := a \mid \alpha \cdot \beta \mid \overline{\alpha} \mid \underline{\alpha} \mid a; \quad C := \overline{E} \mid \underline{E}; \quad E := D \mid ! \mid ? \mid R \mid S \mid W.
\]

These labels are similar to Lévy’s labels except that we have (atomic) markers to describe exponentials, motivated by the constants in the algebra $L^*$. Atomic labels from $\{a, b, \ldots\}$ correspond to 1 while $W$ corresponds to 0; the marker $?$ corresponds to $t$ and the marker $\{d, r, s\}$ are easily recognised in our labels. Multiplicative constants $\{p, q\}$ are treated implicitly via over-lining and underlining. One may recover the multiplicative information, which is simply a bracketing, using the function $\lambda$ which is the identity transformation on all labels except for $f \overline{\alpha} = \overline{\alpha} = f \alpha$ and $f \underline{\alpha} = \underline{\alpha} = f \alpha$. The marker $!$ deserves more attention since the straightforward analogue in the algebra is the morphism $!(\cdot)$. The purpose of the marker is to delimit regions in the label, that is, paths that traverse edges entirely contained in a box.

**Initialisation:** To each term in $\lambda_c$, except for $\delta, \varepsilon$ and substitution-terms, we associate a unique and pairwise distinct label from $\{a, b, \cdots\}$. Notice that we do not place any exponential markers on initialised terms: it is the action of the substitution that yields their correct placement. From now on, we assume that all $\lambda_c$-terms come from the compilation of a $\lambda$-term and receive an initial labelling. Free and bound variables of labelled $\lambda_c$-terms are defined in the usual way.

4 The labelled calculus of closed functions

In this section we define the labelled calculus $\lambda_{lcf}$, that yields traces for the call-by-value translation of $\lambda_c$-terms into linear logic proof nets.

**Definition 2** (Labelled Reduction in $\lambda_{lcf}$). The Beta-rule of the labelled calculus $\lambda_{lcf}$ is defined by

\[
((\lambda x.M)^\alpha N)^\beta \rightarrow_{lcf} \beta \overline{D} \alpha^\top \bullet M[\overline{D} \alpha^\top]^\gamma \bullet N/x \quad \text{if } \nu((\lambda x.M)^\alpha) = \emptyset
\]

The operator $\bullet$ and the function $(\cdot)^\gamma$ on labels are defined in Table 1. We place substitution rules $(\sigma)$ at the same level as the Beta-rule. These are given in Table 2. We write $\rightarrow_{lcf}^*$ for the transitive reflexive closure of $\rightarrow_{lcf}$ and we may omit the name of the relation when it is clear from the context. Reduction is allowed to take place under any context satisfying the conditions.

\[
\begin{align*}
(a)^r &= a & (\beta \cdot x^\alpha)^r &= \chi^\beta \alpha \\
(\alpha \beta)^r &= (\beta)^r \cdot (\alpha)^r & (\beta \cdot (\lambda x.M)^\alpha)^r &= (\lambda x.M)^\beta \alpha \\
(\overline{\alpha})^r &= (\alpha)^r & (\beta \cdot (MN)^\alpha)^r &= (MN)^\beta \alpha \\
(\underline{\alpha})^r &= (\alpha)^r & \alpha \cdot (\delta^{y,z} \cdot M)^r &= (\delta^{y,z} \cdot \alpha \cdot M) \\
(\overline{E})^r &= \overline{E} & \alpha \cdot (\varepsilon_x \cdot M)^r &= (\varepsilon_x \cdot \alpha \cdot M) \\
(\underline{E})^r &= \underline{E} & \alpha \cdot (M[N/x])^r &= (\alpha \cdot M)[N/x]
\end{align*}
\]

Table 1: Operation of $(\cdot)^r$ and $\cdot$. 

\[
\begin{align*}
(\alpha)^r &= a & (\beta \cdot x^\alpha)^r &= \chi^\beta \alpha \\
(\alpha \beta)^r &= (\beta)^r \cdot (\alpha)^r & (\beta \cdot (\lambda x.M)^\alpha)^r &= (\lambda x.M)^\beta \alpha \\
(\overline{\alpha})^r &= (\alpha)^r & (\beta \cdot (MN)^\alpha)^r &= (MN)^\beta \alpha \\
(\underline{\alpha})^r &= (\alpha)^r & \alpha \cdot (\delta^{y,z} \cdot M)^r &= (\delta^{y,z} \cdot \alpha \cdot M) \\
(\overline{E})^r &= \overline{E} & \alpha \cdot (\varepsilon_x \cdot M)^r &= (\varepsilon_x \cdot \alpha \cdot M) \\
(\underline{E})^r &= \underline{E} & \alpha \cdot (M[N/x])^r &= (\alpha \cdot M)[N/x]
\end{align*}
\]
which is initially empty. Formally, this rewriting system is working on pairs lead to unneeded arguments. We omit explicit labels on copying (not only the arguments that get discarded but also their labels, i.e. the paths starting from variables that function application gives rise to traversals of paths that lead to terms that belong in a strategy where such sub-paths may be traversed indeed. For instance, evaluating arguments before action of reduction (they are killed off by the dynamics of the algebra), however, one could impose the weak reduction is weak, the system does not restrict reduction under abstraction altogether as do theories of the weak \( \lambda \)-calculus.

Moreover, the labelled calculus has the following useful properties:

- **Propagation of substitutions**. The propagation of open substitutions through abstractions is \( \alpha \)-conversion free: the propagation of open substitutions through abstractions is

- **Confluence**. \( \lambda_{lcf} \) reductions are confluent: if \( M \xrightarrow{\alpha \cdot N} P \) and \( N_1 \xrightarrow{\alpha \cdot N} P \) then there exists a term \( P \) such that \( N_1 \xrightarrow{\alpha \cdot N} P \) and \( N_2 \xrightarrow{\alpha \cdot N} P \).

The proof of termination of \( \sigma \) is based on the observation that rules push the substitution down the term (which can be formalised using the standard interpretation method). Since the propagation rules are defined for closed substitutions, it is easy to see that closed substitutions do not block. The proof of confluence is more delicate. We derive confluence in three steps: First, we show confluence of the

| Rule   | Reduction                                      | Condition               |
|--------|------------------------------------------------|-------------------------|
| Lam    | \((\lambda x. M)^\alpha[N/x]\) \xrightarrow{lcf} (\lambda x. M[N/x])^\alpha\] | \(fv(N) = \emptyset\) |
| App1   | \((MN)^\alpha[P/x]\) \xrightarrow{lcf} (M[P/x]N)^\alpha\]       | \(x \in fv(M)\)       |
| App2   | \((MN)^\alpha[P/x]\) \xrightarrow{lcf} (MN[P/x])^\alpha\]       | \(x \in fv(N)\)       |
| Cpy1   | \((\delta x z.M)[N/x]\) \xrightarrow{lcf} M[R \bullet N/y][S \bullet N/z]\] | \(fv(N) = \emptyset\) |
| Cpy2   | \((\delta x z.M)[N/x]\) \xrightarrow{lcf} (\delta x z.M[N/x])\] | ~                      |
| Ers1   | \((\epsilon x.M)[N/x]\) \xrightarrow{lcf} M, \{\{W \bullet N\} \cup B\] | \(fv(N) = \emptyset\) |
| Ers2   | \((\epsilon x.M)[N/x]\) \xrightarrow{lcf} (\epsilon x.M[N/x])\] | ~                      |
| Var    | \(x^\alpha[N/x]\) \xrightarrow{lcf} \alpha \bullet N\]            | ~                      |
| Cmp    | \(M[P/y][N/x]\) \xrightarrow{lcf} M[P[N/x]/y]\] | \(x \in fv(P)\)       |

Table 2: Labelled substitution (\( \sigma \)) rules in \( \lambda_{lcf} \)

The calculus defined above is a labelled version of \( \lambda_{lcf} \), the calculus of closed functions [11]. The conditions on the rules may not allow a substitution to fully propagate (i.e., a normal form may contain substitutions) but the calculus is adequate for evaluation to weak head normal form [11]. Although reduction is weak, the system does not restrict reduction under abstraction altogether as do theories of the weak \( \lambda \)-calculus.

Intuitively, the purpose of the **Beta**-rule is to capture two paths, one leading into the body and one to the argument of the proof-net representation of the function application. The controlled copying and erasing (Cpy1 and Ers1) of substitutions allows the identification of paths that start from contraction nodes and weakening nodes respectively. Rule Ers1 has a side effect; erased paths are kept in a set \( B \), which is initially empty. Formally, this rewriting system is working on pairs \((M, B)\) of a \( \lambda \)-term and a set of labels. The set \( B \) deserves more regards: there exist paths in the GoI that do not survive the action of reduction (they are killed off by the dynamics of the algebra), however, one could impose a strategy where such sub-paths may be traversed indeed. For instance, evaluating arguments before function application gives rise to traversals of paths that lead to terms that belong in \( B \). In this sense, it is not only the arguments that get discarded but also their labels, i.e. the paths starting from variables that lead to unneeded arguments. We omit explicit labels on copying (\( \delta \)) and erasing (\( \epsilon \)) constructs: these are used just to guide the substitutions. The composition rule Cmp is vital in the calculus because we may create open substitutions in the **Beta**-rule.

The calculus is \( \alpha \)-conversion free: the propagation of open substitutions through abstractions is the source of variable capture, which is here avoided due to the conditions imposed on the Lam-rule. Moreover, the labelled calculus has the following useful properties:

**Property.** 1. Strong normalisation of substitution rules. The reduction relation generated by the \( \sigma \)-rules is terminating.

2. Propagation of substitutions. Let \( T = M[N/x] \). If \( fv(N) = \emptyset \) then \( T \) is not a normal form. As a corollary closed substitutions can be fully propagated.

3. Confluence. \( \lambda_{lcf} \) reductions are confluent: if \( M \xrightarrow{\alpha \cdot N} N_1 \) and \( M \xrightarrow{\alpha \cdot N} N_2 \) then there exists a term \( P \) such that \( N_1 \xrightarrow{\alpha \cdot N} P \) and \( N_2 \xrightarrow{\alpha \cdot N} P \).
\[ G_n(x) = \ldots \]

\[ G_n(MN) = \ldots \]

\[ G_n(\lambda x.M) = \ldots \]

\[ G_n(\epsilon_x.M) = \ldots \]

\[ G_n(\delta^{y,z}_x.M) = \ldots \]

\[ G_n(M[N/x]) = \ldots \]

Figure 2: Translation of \( \lambda_c \)-terms into weighted call-by-value proof-nets

\( \sigma \)-rules (local confluence suffices, by Newman’s lemma [18], since the rules are terminating). Then we show that \( \beta \) alone is confluent, and finally we use Rosen’s lemma [19], showing the commutation of the \( \beta \) and \( \sigma \) reduction relations. For detailed proofs we refer the reader to [21].

**Labels in \( \lambda_{lcf} \) and paths in the call-by-value translation**

There is a correspondence between our labels and the paths in weighted proof nets. The aim of the remainder of the section is to justify the way in which this calculus records paths in proof-nets. This will provide us with a closer look at the operational behaviour of the calculus, especially at the level of propagation of substitutions, and will highlight the relationship, but also the differences, between term reduction and proof-net reduction.

We first define a call-by-value translation from labelled \( \lambda_c \)-terms to proof-nets labelled with weights taken from algebra \( L^* \), and then we show that the set of labels generated at each rewrite step in the calculus coincides with the set of weights in the corresponding nets. For simplicity, we first consider the translation of unlabelled terms; then we extend the translation to labelled terms.

**Definition 3.** In Figure 2 we give the translation function \( G_n(\cdot) \) from unlabelled \( \lambda_c \)-terms to call-by-value proof-nets. We use a parameter \( n \) \((n \geq 0)\) to record a current-box level, which indicates the box-nesting in which the translation works. The translation of a term \( M \) is obtained with \( G_0(M) \), indicating the absence of box-nesting in initial terms. In our translation we omit the weights 1 of cuts and axioms. In general, the translation of a term \( M \) at level \( n \) is a proof-net:

\[ \framebox{G_n(M)} \]

where the crossed wire at the bottom represents a set of edges corresponding to the free variables in \( M \).

At the syntactical level, the correspondence of a \( \delta \)-term to a duplicator (fan) node is evident as well as the correspondence of \( \varepsilon \)-term to weakening (W) nodes. In the former case, we assume that the variable \( y \) (resp. \( z \)) corresponds to the link labelled with \( r \) (resp. \( s \)). The free variable \( x \) corresponds to the wire at the conclusion of the fan. The case for the erasing term is similar, the free variable \( x \) corresponds to
Atomic labels

| Label   | Level |
|---------|-------|
| \( lw[\alpha]_n \) | \((1, n)\) |
| \( lw[\alpha\beta]_n \) | \((1, n)\) |
| \( lw[\alpha\beta']_n \) | \((1, n+1)\) |
| \( lw[\alpha\beta^*]_n \) | \((1, n-1)\) |
| \( lw[?\alpha]_n \) | \((1, n)\) |
| \( lw[?\beta]_n \) | \((1, n)\) |

Composite labels

| Label   | Level |
|---------|-------|
| \( lw[\overline{\alpha}]_n \) | \(((w^{\overline{\alpha}}) \cdot w \cdot w^{\overline{\alpha}}(q^*), n')\) |
| \( lw[\overline{\alpha}\beta]_n \) | \((w \cdot w', n'')\) |

Table 3: Translation of labels into weights

the conclusion of the weakening node. The encoding of a substitution term is the most interesting: a substitution redex corresponds to a cut in proof-nets.

We next give the translation of labelled terms, which is similar except that now we must translate labels of the calculus into terms of the algebra. This is a two-step process: first we must consider how labels correspond to weights in proof-nets, and then we must place the weight on an edge of the graph.

**Definition 4.** The weight of a label is obtained by the function 

\[ lw : \text{label} \rightarrow \text{level} \rightarrow (\text{weight}, \text{level}) \]

where \( \text{level} \in \mathbb{N} \) is a level (or box depth) number. Given a label and the level number of the first label, the function defined in Table 3 yields a weight together with the level number of the last label. We assume that the input level number always is an appropriate one.

Before we give the translation, we introduce a convention that will help us to reason about input and output levels: instead of projecting the weight from the tuple in the previous definition, we place the tuple itself on a wire like this:

\[ (a, o) \quad !^o(\beta) \]

where \( \alpha \) is the weight of the translated label, \( o \) is a level number. Now, after projecting the weight from the tuple one may compose with existing weights on the wire \((!^o(\beta))\). This simply introduces a delay in our construction that helps us to maintain the levels.

The translation of a labelled term is obtained using the function \( G_i(M) \), which is the same as in Definition 3 with the difference that now when we call \( G_i(M) \), the parameter \( i \) depends on the translation of the label. Specifically, for each term that has a label on its root, we first translate the label using Definition 4 and place the obtained output onto the root of \( G \). In Figure 3 we show the translation of labelled application and substitution terms and the remaining cases can be easily reconstructed from the translation of the unlabelled terms. Thus, the only difference is that when a term has a label on the root, the translation must use the output level to propagate to subterms.

To extract the external label of a term we use the function:

\[
\begin{align*}
\text{label } x^\alpha & = \alpha \\
\text{label } (\lambda x. M)^\alpha & = \alpha \\
\text{label } (\varepsilon_\alpha^x.M) & = \text{label } M \\
\end{align*}
\]
Similarly, we can get the label of a free variable in a term. This is just a search and we omit the definition. Thanks to the linearity of terms, there is exactly one wire for each free variable in the term.

**Main result.** Before proving the main result of this section (Theorem 1), which states the correspondence between labels and paths, we need a few general properties.

**Proposition 1** (First and last atomic labels). Let $k$ be the external label of an initialised term $T$, and assume $T \rightarrow^* T'$.

1. If $(\text{label } T') = l_1 \ldots l_n, n \geq 1$, then $l_1$ is $k$.

2. Let $N$ be an application, abstraction or variable subterm of $T'$ with $(\text{label } N) = l_1 \ldots l_n, n \geq 1$.

   The atomic label $l_n$ identifies an application (resp. abstraction, resp. variable) term in $T$ iff $N$ is an application (resp. abstraction, resp. variable) term.

This property captures the idea that we cannot lose the original root of a reduction and terms never forget about their initial label. Notice that we consider terms that do receive a label by the initialisation. Thus the last atomic label of a string on a term-construct is the label the construct has obtained by initialisation. This is because labels get prefixed by the actions of the calculus. Additionally, notice that in $(\lambda x.M)^\alpha N^\beta \rightarrow \beta\alpha \sigma \cdot M_\alpha N_\beta / z$, where we forget about the markers, we know that the last label of $\beta$ must be the one of the application node in which $M$ was the functional part. But we also know the first label of $\alpha$: if it is atomic, then it is the label of this $\lambda$ in the initial term. Otherwise it is the label of the functional edge of the application node identified by the last label of $\beta$. One argues similarly for the argument.

**Lemma 1.** If $T = M[N/x]$ then there is a decomposition $(\text{label } N) = \omega \alpha \sigma$ such that $\omega$ is a prefix built with exponential markers having the shape $E_1 \ldots E_n$ with $n \geq 0$.

**Proof.** A simple inspection of the rewrite rules shows that we always prefix the external label of $N$ with some exponential marker as long as the substitution propagates with label sensitive rules. The moment where $\omega$ itself gets prefixed is during a variable substitution which stops the process.

The previous statement allows us to point out the distinguishing pattern of labels on substitution terms where we shall see that label sensitive propagation of substitutions corresponds to building an exponential path in a proof-net.

**Corollary 1.** The atomic label that stands on an initialised variable can be followed only by $\omega \alpha \sigma$.

**Proof.** This is a consequence of the last-label property stated above and the action of the rewrite rule $\text{Var}$.  

---

**Figure 3**: Translation of labelled application and substitution terms
Next we show that the labels of the calculus adequately trace paths in proof-nets. In particular, we show that the set of labels generated in the calculus coincides with weights of straight paths in proof-nets in the following sense:

**Theorem 1.** Let $W_G = \{w(\phi) \mid \phi \in \text{straight paths of } G\}$ denote the set of weights of straight paths observable in a graph $G$ and let $T$ be a labelled term. If $T \rightarrow T'$ then $W_{G(T)} = W_{G(T')}$. 

*Proof.* The proof is by induction on $T$ and is given in the appendix. 

\[ \square \]

5 The labelled calculus of closed arguments

In this section we define a labelled calculus, called $\lambda_{lca}$, that yields traces for the call-by-name translation of $\lambda$-terms into linear logic proof-nets.

**Definition 5** (Labelled reduction in $\lambda_{lca}$). The new Beta-rule is defined by:

\[ ((\lambda x.M)^{\alpha}\ N)^{\beta} \rightarrow_{lca} \beta \bullet \alpha M[\alpha/x] \quad \text{if } \text{fv}(N) = \emptyset \]

where the operator $\bullet$ is given in Definition 2. The substitution rules for this system are presented below:

| Rule | Reduction | Condition |
|------|-----------|-----------|
| Lam  | $(\lambda y.M)^{\alpha}[N/x] \rightarrow_{lca} (\lambda y.M[N/x])^{\alpha}$ | $\sim$ |
| App1 | $(MN)^{\alpha}[P/x] \rightarrow_{lca} (M[P/x]N)^{\alpha}$ | $x \in \text{fv}(M)$ |
| App2 | $(MN)^{\alpha}[P/x] \rightarrow_{lca} (MN[\overrightarrow{P/x}]^\alpha$ | $x \in \text{fv}(N)$ |
| Cpy1 | $(\delta^{\overrightarrow{y,z}}M)[N/x] \rightarrow_{lca} M[\overrightarrow{N/x}]$ | $\sim$ |
| Cpy2 | $(\delta^{\overrightarrow{y,z}}M)[N/x'] \rightarrow_{lca} (\delta^{\overrightarrow{y,z}}M[N/x'])$ | $\sim$ |
| Ers1 | $(\varepsilon_x.M)[N/x] \rightarrow_{lca} M, \{\overrightarrow{W \bullet N}\} \cup B$ | $\sim$ |
| Ers2 | $(\varepsilon_x.M)[N/x'] \rightarrow_{lca} (\varepsilon_x.M[N/x'])$ | $\sim$ |
| Var  | $x^{\alpha}[N/x] \rightarrow_{lca} \alpha \overrightarrow{D \bullet N}$ | $\sim$ |

This is the labelled version of the calculus of closed arguments in [11] and we refer the reader to [21] for a proof of confluence for the labelled version.

Labels in $\lambda_{lca}$ and paths in the call-by-name translation

The particularities of the calculus are best understood via the correspondence to the call-by-name translation. Thus, let us move directly to the translation of terms into call-by-name proof-nets. We provide a simplified presentation where instead of placing a translated label on a wire, we simply place the label itself. As before, multiplicative information is kept implicit via overlining and underlining. Hence, the general form of the translation takes a labelled term and places the label of the term (when it has one) at the root of the graph: We give the translation in Figure 4. Remark that the translation of an argument (and the substitution) always involves a box structure. We do not repeat translations for $\delta$ and $\varepsilon$-terms since these translate in the same way as before.

The following theorem is the main result of this section. It establishes a correspondence between labels and paths in the call-by-name proof-net translation.

**Theorem 2.** 1. If $T \rightarrow_{lca} T'$ then $W_{G(T)} = W_{G(T')}$, where $W_G$ is defined as in Theorem 1.

2. Suppose $T$ is a term obtained by erasing the labels and $\rightarrow_{ca}$ is the system generated by the rules for $\lambda_{lca}$ by removing the labels. If $T \rightarrow_{ca} T'$ then $G(T) \Rightarrow^* G(T')$ using closed cut elimination, where $G$ is the call-by-name translation.
This theorem is stronger than Theorem 1: there is a correspondence between labels in \( \lambda_{lca} \) and paths in the call-by-name proof-net translation, and between the dynamics of the calculus and the proof-net dynamics of closed cut elimination \( \Rightarrow \) (due to space constraints we omit the definition of \( \Rightarrow \) and refer to [21]). Rules \( \text{App1, Lam, Cpy2 and Ers2} \) correspond to identities while the remaining rules correspond to single step graph rewriting. To obtain a similar result for \( \lambda_{lc} \) with the call-by-value translation, we need to impose more conditions on the substitution rules (requiring closed values instead of simply closed terms).

6 Conclusions and future work

We have investigated labelled \( \lambda \)-calculi with explicit substitutions. The labels give insight into how the dynamics of a \( \lambda \)-calculus corresponds to building paths in proof-nets, and they also allow us to understand better the underlying calculi. For instance, label insensitive substitution rules witness that some actions in the calculi capture “less essential” computations, that is, an additional price for bureaucracy of syntax is paid in relation to the corresponding proof-net dynamics. From this point of view, it is interesting to ask about whether strategies for these calculi exist such that each propagation of substitution corresponds to stretching a path in a corresponding proof-net. On the other hand, investigation of new labelled versions of known strategies defined for the underlying calculi could help in understanding and establishing requirements for new proof-net reduction strategies.

The use of closed reduction in the current work simplifies the computation of the labels, since only closed substitutions are copied/erased. The methodology can be extended to systems that copy terms with free variables, but one would need to use global functions to update the labels; instead, using closed reduction, label computations are local, in the spirit of the Geometry of Interaction. Notice that duplication (resp. erasing) of free variables causes further copying (resp. erasing), which in our case would require on the fly instantiation of additional \( \delta \)-terms (resp. \( \varepsilon \)-terms). Without implying that such calculi need to be confluent, we remark that the system without the closed conditions introduces non joinable critical pairs in the reduction rules resulting in non-confluent systems. Notice that path computation in the GoI has only been shown sound for nets that do not contain auxiliary doors.

The main results of this paper establish that the labels are adequate enough for the representation of paths in proof-nets. This makes these calculi appealing for intermediate representation of implementations of programming languages where target compilation structures are linear logic proof-nets.

For the study of shared reductions in proof-nets, and in the calculus itself, a few additions would be useful: it would be certainly interesting to track not only cuts that correspond to Beta-redexes but also exponential cuts corresponding to substitutions. For this, we should allow copy, erase and substitution terms to bear labels. These additions could also be useful towards obtaining a standardisation result for closed reduction calculi.
References

[1] M. Abadi, L. Cardelli, and P. L. Curien. Explicit substitutions. *Journal of Functional Programming*, 1:31–46, 1991.

[2] S. Abramsky. Computational Interpretations of Linear Logic. *Theoretical Computer Science*, 111:3–57, 1993.

[3] A. Asperti, V. Danos, C. Laneve, and L. Regnier. Paths in the lambda-calculus. In *Logic in Computer Science*, pages 426–436, 1994.

[4] A. Asperti and C. Laneve. Paths, computations and labels in the lambda-calculus. In *RTA-93: Selected papers of the fifth international conference on Rewriting techniques and applications*, pages 277–297, Amsterdam, The Netherlands, The Netherlands, 1995. Elsevier Science Publishers B. V.

[5] A. Asperti and S. Guerrini. *The optimal implementation of functional programming languages*. Cambridge University Press, 1998.

[6] H. P. Barendregt. *The Lambda Calculus – Its Syntax and Semantics*, volume 103 of *Studies in Logic and the Foundations of Mathematics*. North-Holland, 1984.

[7] V. Danos. *La Logique Linéaire appliquée a l’étude de divers processus de normalisation (et principalement du λ-calcul)*. PhD thesis, Universite Paris VII, 1990.

[8] N. Çağman and J. R. Hindley. Combinatory weak reduction in lambda calculus. *Theor. Comput. Sci.*, 198(1-2):239–247, 1998.

[9] N. G. de Bruijn. A namefree lambda calculus with facilities for internal definition of expressions and segments. Department of Mathematics, Eindhoven University of Technology. T.H.-Report 78-WSK-03, 1978.

[10] M. Fernández and I. Mackie. Closed reductions in the lambda-calculus. In *CSL ’99: Proceedings of the 13th International Workshop and 8th Annual Conference of the EACSL on Computer Science Logic*, pages 220–234, London, UK, 1999. Springer-Verlag.

[11] M. Fernández, I. Mackie, and F.-R. Sinot. Closed reduction: explicit substitutions without α-conversion. *Mathematical Structures in Comp. Sci.*, 15(2):343–381, 2005.

[12] J.-Y. Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.

[13] J.-Y. Girard. Geometry of Interaction I: Interpretation of system F. In C. Bonotto, R. Ferro, S. Valentini, and A. Zanardo, editors, *Logic Colloquium ’88*, pages 221–260. North-Holland, 1989.

[14] J.-J. Levy. *Reductions correctes et optimales dans le lambda-calcul*. PhD thesis, These de doctorat d’etat, Universite Paris VII, 1978.

[15] I. Mackie. Interaction nets for linear logic. *Theor. Comput. Sci.*, 247(1-2):83–140, 2000.

[16] I. Mackie. *The Geometry of Implementation*. PhD thesis, Department of Computing, Imperial College of Science, Technology and Medicine, September 1994.

[17] I. Mackie. The geometry of interaction machine. In *POPL ’95: Proceedings of the 22nd ACM SIGPLAN-SIGACT symposium on Principles of programming languages*, pages 198–208, New York, NY, USA, 1995. ACM.

[18] M. Newman. On theories with a combinatorial definition of “equivalence”. *Annals of Mathematics*, 43(2):223–243, 1942.

[19] B. Rosen. Tree-manipulating systems and Church-Rosser theorems. *Journal of the ACM*, 20(1):160–187, 1973.

[20] N. Siafakas. A fully labelled lambda calculus: Towards closed reduction in the Geometry of Interaction machine. In *Proceedings of Developments in Computation Models 2006*. *Electron. Notes in Theor. Comput. Sci.*, 171(3):111–126, 2007.

[21] N. Siafakas. A framework for path-based computation in functional programming languages. PhD thesis, King’s College London, University of London, 2008. Available from [http://www.dcs.kcl.ac.uk/pg/siafakas/thesis.pdf](http://www.dcs.kcl.ac.uk/pg/siafakas/thesis.pdf)
APPENDIX

Proof of Theorem A-1

Theorem A-1. Let $W_G = \{w(\phi) \mid \phi \in \text{straight paths of } G\}$ denote the set of weights of straight paths observable in a graph $G$ and let $T$ be a labelled term. If $T \rightarrow T'$ then $W_{G(T)} = W_{G(T')}$. 

Proof. By induction on $T$. The only interesting case is when the reduction takes place at the root position. We show the property by cases on the rule applied.

Case Beta: We give the graphical representation of the left- and right-hand sides of the Beta-rule below:

Notice that there are no auxiliary doors since the function must be closed. We must show that weights of straight paths in the left hand side are found in the right hand side, that is, the weights of

1. $\phi = \rho - (\beta, o) - !^o(q) - !^o(d) - (\alpha, o') - !^{o' + 1}(q^*) - \mu$ where $\rho$ is a path ending at the root of the left hand side and $\mu$ is a path starting in $M$;
2. $\psi = \mu' - !^{o' + 1}(p) - (a^*, o') - !^o(d^*) - !^o(p^*) - \nu$ with $\mu'$ ending in the (translation of) free variable of $M$ and $\nu$ starting in $N$; and
3. $\psi', \phi'$ are found in the right hand side of the figure. Notice that in the right hand side, the translation of a substitution term does not place any label at the immediate root of the graph and there seems to be a kind of mismatch with the level numbers in which the subgraphs are called, however this is correct. Since we do not have labels on substitution terms, the translation will respect the $i$'s later on, that is, we can apply the induction hypothesis. We show the property by checking that the external label of $M$ fixes the level number yielding the weight we are after.

- We have $l = \text{label } M = \beta \text{ over } \bar{D}\alpha \Downarrow \beta'$, where $\beta'$ is some suffix. The weight of $l$ is given by
  \[ lw[\beta \text{ over } \bar{D}\alpha \Downarrow \beta'] = w((\beta, o) - !^o(q) - !^o(d) - (\alpha, o') - !^{o' + 1}(q^*)) \cdot lw[\beta']^{o' + 1} \]
  which completes the first case.

- We work in a similar fashion with the second case where we first translate $\text{label } N = (\bar{D}\alpha \Downarrow \gamma)'$. Under the assumption that the wire connecting the free variable $x$ is at level $o + 1$ we have
  \[ lw[(\bar{D}\alpha \Downarrow \gamma)'] = w(!^{o' + 1}(p) - (a^*, o') - !^o(d^*) - !^o(p^*)) \cdot lw[\gamma]^o \]
The translation calls with a suitable \( i \) big enough to cover the open scopes at the first label and returns the open scopes at the last label. Now in \( \psi \), reading \( o \) in reverse means that the indicated scope number is the one for the first label and hence decreases.

One argues in the same way for the reverse cases of \( \phi \) and \( \psi \).

**Case Var:** This is where two paths meet and get glued via an axiom-link. The translation of the left- and right-hand sides do not tell us anything useful since we have just wires (identities) and thus the weights of paths remain the same.

**Case Lam:** The translations of each side are:

\[
\begin{align*}
G_i(LHS) &= G(M)_{\alpha+1} G(N)_{o} \\
G_i(RHS) &= G(M)_{\alpha+1} G(N)_{o} \cup \{ G(N) \} \cup B
\end{align*}
\]

Recall that the external label of \( N \) may give us a prefix of exponential markers followed by an underline or just an underline. The interesting point is that the right hand side suggests that the translation is called with \( o + 1 \) for \( N \) and looks like a source of a mismatch. But the external label on \( N \) in the rhs must start with an exponential marker \( ? \) and this decreases the \( o \) upon which \( N \) is translated. Thus the weights of paths remain the same and this completes the case.

**Case Cpy1:** We have \( (\delta^{\{x\}}.M)[N/x] \rightarrow_{lcf} M[R \bullet N/y][S \bullet N/z] \) with the corresponding translations

\[
\begin{align*}
G_i(LHS) &= G(M)_{(r,s)} G(N) \\
G_i(RHS) &= G(M)_{(r,s)} G(N) \\
\end{align*}
\]

We argue as before and there is nothing to say about the levels. The case is similar for erasing.

**Case Ers1:** The rule behaves as follows

\[
\begin{align*}
G_i(LHS) &= G(M)_{(r,s)} G(N) \\
G_i(RHS) &= \{ G(N) \} \cup B
\end{align*}
\]
There are killed paths in the left hand side and we have the same on the right. Note that erased paths do not survive reduction but one may walk these with a strategy. This is the reason for keeping the set \( B \), which is initially empty.

The translation of rules App1, App2, Cpy2, Ers2 and Cmp correspond to

\[ \text{Proof of Theorem 2} \]

**Lemma A-1.** If \( T = M[N/x] \) then there is a decomposition (label \( N \)) = \( \omega \alpha \xrightarrow{\nabla} \sigma \) such that \( \omega \) is a prefix built with exponential markers having the shape \( \overrightarrow{E_1 \ldots E_n} \) where \( n \geq 0 \) and \( E \) is not a \( D \) marker.

**Proof.** Since the term \( N \) belongs to a substitution term, its external label must have been generated by a \( \text{Beta} \)-rule generating the sub-label \( \alpha \xrightarrow{\nabla} \sigma \). By using the rules in \( \sigma \), we can generate only an \( \omega \) prefix. It cannot contain a dereliction marker since this can come from the \( \text{Var} \)-rule which stops the process; that is, the root of \( T \) is not a substitution term anymore.

The consequence of the lemma is that derelictions are followed directly by exponentials. Notice that this is different form the previous system.

**Theorem A-2.** 1. If \( T \rightarrow_{lca} T' \) then \( W_{G(T)} = W_{G(T')} \).

2. Suppose \( T \) is a term obtained by erasing the labels and \( \rightarrow_{ca} \) is the system generated by the rules for \( \lambda_{lca} \) by removing the labels. If \( T \rightarrow_{ca} T' \) then \( G(T) \Rightarrow G(T') \) using closed cut elimination, where \( G \) is the call-by-name translation.

**Proof.** We proceed by cases and argue about both properties.

**Case Beta:** \( ((\lambda x.M)^\alpha N)^\beta \rightarrow_{lca} \beta \bullet \overline{\alpha M}[\overline{\alpha} \nabla \overline{\beta} \bullet N/x] \) if \( \text{fv}(N) = \emptyset \) where \( \text{fv}(N) = \emptyset \). The situation is the following:

Two paths are of interest: one entering from the root, moving along the cut and ending at the root of \( G(M) \) and one coming from the free variable, travelling along the cut and ending up at the root of \( G(N) \). Both weights are preserved in the right hand side. The second point of our claim is satisfied since the closed cut elimination sequence corresponds to one multiplicative cut.

**Case App2:** \( (MN)^\alpha [P/x] \rightarrow_{lca} (MN[\overrightarrow{\nabla} \bullet P/x])^\alpha \) where \( x \in \text{fv}(N) \) and the translation of both sides become
For the first point in our claim, weights are preserved by recording the auxiliary marker and for the second point, the graph rewrite corresponds to a closed commutative cut. Notice that there are no closedness conditions on the $\sigma$-rules, but since $Beta$ is the only rule that can generate a substitution, it must be a closed one.

**Case Var:** $x^\alpha[N/x] \longrightarrow^{lca} [\alpha \overrightarrow{D} \bullet N]$ and the situation becomes

There seems to be a mismatch with the graph rewriting rule and paths that we record since we removed the box in the right hand side. However, by Lemma A-1, the external label of the argument must have the shape $(label \ N) = \omega^\alpha \overrightarrow{\bullet} \sigma$ such that $\omega$ is built with exponential markers and the trailing exponential box marker restores our level information. Regarding our second point, this simply corresponds to a closed dereliction cut.

**Case Cpy1, Ers1:** With respect to the paths, both cases are similar as before. Regarding the second point of the claim, the translations correspond to a closed contraction and weakening cut respectively.

For the remaining cases, the left and right hand side translations are identical.