On the quantum mechanics of $M(atrix)$ theory.

Jan Plefka\footnote{e-mail: plefka@nikhef.nl} and Andrew Waldron\footnote{e-mail: waldron@nikhef.nl}

NIKHEF, P.O. Box 41882, 1009 DB Amsterdam,
The Netherlands

Abstract

We present a study of $M(atrix)$ theory from a purely canonical viewpoint. In particular, we identify free particle asymptotic states of the model corresponding to the supergraviton multiplet of eleven dimensional supergravity. These states have a natural interpretation as excitations in the flat directions of the matrix model potential. Furthermore, we provide the split of the matrix model Hamiltonian into a free part describing the free propagation of these particle states along with the interaction Hamiltonian describing their interactions. Elementary quantum mechanical perturbation theory then yields an effective potential for these particles as an expansion in their inverse separation. Remarkably we find that the leading velocity independent terms of the effective potential cancel in agreement with the fact that there is no force between stationary $D0$ branes. The scheme we present provides a framework in which one can perturbatively compute the $M(atrix)$ theory result for the eleven dimensional supergraviton $S$ matrix.
1 Introduction.

$M$ (atrix) theory is the conjectured description of $M$ theory in terms of a quantum mechanical supersymmetric $U(N)$ matrix model and as such has recently been the subject of intense study. At low energies and large distances $M$ theory, by definition, reduces to eleven dimensional supergravity. Thus a principal test of the conjecture is the computation of graviton scattering amplitudes in $M$ (atrix) theory followed by a comparison to the supergravity result. According to Susskind, equivalence is expected to hold even for finite $N$, where $M$ (atrix) theory is conjectured to provide the discretized light cone quantization of $M$ theory.

To date all computations have been based on path integral quantization and semiclassical expansions thereof, yielding several rather spectacular consistency tests. However, although dominant in many quantum field theoretical contexts, path integrals have played a subdominant role in the development of quantum mechanics for which canonical methods reign. In this paper we study $M$ (atrix) theory from a purely canonical point of view, concentrating on the two body sector of the theory with each particle carrying one unit of quantized light cone momentum. Let us stress that our ultimate goal is the computation of $M$ (atrix) theory supergraviton $S$ matrix elements, rather than to test the well known equivalence between path integral and canonical methods.

The calculation of scattering amplitudes in gauge theories typically splits into two steps. The first involves the computation of Greens functions subject to Ward identities expressing the gauge invariance of the theory. These Greens functions are neatly encapsulated in the effective action or generating functional of one particle irreducible diagrams. The perturbative calculation of the effective action can be performed efficiently using path integral methods in which the gauge is fixed via the Faddeev Popov procedure. In particular the background field formalism provides an elegant framework for such calculations. The above techniques have already been rather successfully applied to $M$ (atrix) theory and the connection between semiclassical path integral quantization around particular classical backgrounds and the eikonal approximation has been exploited to extract physical results.

However, typically one is interested in $S$ matrix elements for which a second step is required. Namely, the asymptotic analysis of Lehmann, Syzmannik and Zimmermann (LSZ) in which one identifies the asymptotic states of the theory and derives reduction formulae describing the connection between off-shell Greens functions and physical matrix elements. A central theme of this work is the “LSZ formalism” for $M$ (atrix) theory. The bounty is obvious, since such a formalism allows the (perturbative) computation of the $M$ (atrix) theory

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3It is to be emphasised that $M$ (atrix) theory is an on-shell theory as regards eleven dimensional energy momentum, but we are interested in the connection between path integral $M$ (atrix) theory effective actions and the eleven dimensional supergraviton $S$ matrix.
supergraviton $S$ matrix.

Our primary result is the identification of the $M$(atrix) theory asymptotic supergraviton particle states. Furthermore, the $M$(atrix) theory is a first quantized quantum mechanical theory so that knowledge of these asymptotic states allows the formulation of "reduction formulae" relating $S$ matrix elements to covariant gauge path integrals. These formulae may be found in our concluding remarks and their derivation, being relatively standard, is given in the appendix.

Since the model is quantum mechanical, rather than a quantum field theory, one may also directly compute the $S$ matrix by applying the techniques of quantum mechanical scattering theory to our asymptotic states. This computation amounts to studying $M$(atrix) theory in the temporal gauge and one suspects that a covariant gauge path integral computation along with the reduction of off-shell Greens functions to physical amplitudes via an LSZ type analysis involving our asymptotic states ultimately provides the most efficient computational framework. Nonetheless, the first results from elementary quantum mechanical perturbation theory are very encouraging. In particular we find the correct cancellation of velocity independent terms in the effective potential at the leading order of perturbation theory.

Quantum mechanical scattering theory is described by identifying the free part of the Hamiltonian $H_0$ whose eigenstates may be interpreted as asymptotic free particle states with interactions governed by the interaction Hamiltonian $H_{\text{int}} = H - H_0$. For the $M$(atrix) theory the asymptotic free particle states should correspond to the eleven dimensional supergraviton multiplet. Although classical backgrounds corresponding to free particle motion are known and form the basis of the existing semiclassical path integral quantization of $M$(atrix) theory, wavefunctions corresponding to asymptotic particle states have not yet appeared. In particular the description of the polarisations of asymptotic particles and the dependence of scattering amplitudes upon these polarisations remains unclear in the conventional background field approach, whereas it becomes transparent in the canonical treatment as we shall show.

The construction of such asymptotic wavefunctions is rendered non-trivial by the presence of constraints expressing the super Yang-Mills gauge invariance of the $M$(atrix) theory along with the interpretation of spacetime as an asymptotic limit of the theory. Our work provides the resolution of both these issues.

The main results of this paper are the construction of wavefunctions describing pairs of free supergravitons along with the free and interaction Hamiltonians describing their free propagation and interactions. We also calculate the Born amplitude along with the leading contribution from second order perturbation theory. A supersymmetric cancellation between the coefficients of the $1/r^2$ contribution to the effective potential is found in agreement with existing two loop path integral calculations [7]. It is shown that spin interactions give no contribution to the Born amplitude and that the systematic computation of spin effects is made possible by our framework.
1.1 Results and Outline.

Some aspects of our work are rather technical, therefore we have found it useful to include in the introduction a summary of our findings and explanations of how they were obtained. Conventions and detailed derivations can be found in the following sections.

The model.

The Hamiltonian of the $M$(atrix) theory is that of 10d super Yang-Mills dimensionally reduced to $0 + 1$ dimensions and arises in a remarkable way from two rather disparate viewpoints. On the one hand, it emerges as a regulating theory of the eleven dimensional supermembrane in light cone gauge quantization [12] and on the other hand it is the effective Hamiltonian describing the short distance properties of $D0$ branes [13, 14, 15]. Employing the conjecture of [4], the finite $N$ model is to be identified with the compactification of a null direction of $M$ theory (henceforth called the $-\alpha$ direction). The quantized total momentum of the $U(N)$ system in this direction is then given by $P_{-} = N/R$, where $R$ denotes the compactification radius.

We shall be primarily interested in the $U(2)$ theory, studying the Hilbert space of two supergravitons with momentum $P_{-} = 1/R$ each. The coordinates and Majorana spinors of the transverse nine dimensional space then take values in the adjoint representation of $U(2)$, i.e.

$$X_{\mu} = X_{\mu}^{0} i 1 + X_{\mu}^{A} i \sigma^{A} \quad \mu = 1, \ldots, 9$$

$$\theta_{\alpha} = \theta_{\alpha}^{0} i 1 + \theta_{\alpha}^{A} i \sigma^{A} \quad \alpha = 1, \ldots, 16$$

where $\sigma^{A}$ are the Pauli matrices. We shall often employ a vector notation for the $SU(2)$ part in which $\vec{X}_{\mu} = (X_{1\mu}, X_{2\mu}, X_{3\mu}) \equiv (X_{\mu}^{A})$ and similarly for $\vec{\theta}$.

The Hamiltonian is then given by

$$H = H_{CoM} + R \left( \frac{1}{2} \vec{P}_{\mu} \cdot \vec{P}_{\mu} + M^6 \frac{1}{4} (\vec{X}_{\mu} \times \vec{X}_{\mu})^2 + M^3 \frac{i}{2} \vec{X}_{\mu} \cdot \vec{\theta} \gamma_{\mu} \times \vec{\theta} \right)$$

where $H_{CoM} = \frac{1}{2} R P_{\mu}^{0} P_{\mu}^{0}$ is the $U(1)$ centre of mass Hamiltonian. These two contributions to $H$ are completely independent and may both be written as a square of a supersymmetry charge [14]. $M$ denotes the eleven dimensional Planck mass, which we from now on set to unity. Due to the linear dependence of the Hamiltonian on $R$ this quantity may also be dropped. The explicit $M$ and $R$ dependence can be reinstated at any stage by a dimensional analysis. Note that we are using a real, symmetric representation of the $SO(9)$ Dirac matrices in which the nine dimensional charge conjugation matrix is equal to unity.

The Hamiltonian (1.3) is augmented by the Gauss law constraint

$$\vec{L} = \vec{X}_{\mu} \times \vec{P}_{\mu} - \frac{i}{2} \vec{\theta} \times \vec{\theta}, \quad [L^{A}, L^{B}] = i \epsilon^{ABC} L^{C}$$

(1.4)
whose action is required to vanish on physical states.

The task is now to identify the free asymptotic two-particle states of the Hamiltonian (1.3) which describe the on-shell supergraviton multiplet of eleven dimensional supergravity. This problem manifestly factorises into a $U(1)$ centre of mass state and an $SU(2)$ invariant state describing the relative dynamics of the particles.

**The centre of mass theory.**

The eigenstates of the free $U(1)$ centre of mass matrix theory are

$$|k_\mu; h_{\mu\nu}, B_{\mu\nu\rho}, h_{\mu\hat{\alpha}}\rangle_0 = e^{ik_\mu X^0_\mu} |h_{\mu\nu}, B_{\mu\nu\rho}, h_{\mu\hat{\alpha}}\rangle_0$$

and possess $SO(9)$ momentum $k_\mu$ and on-shell $SO(9)$ supergraviton polarisations $h_{\mu\nu}$, $B_{\mu\nu\rho}$ and $h_{\mu\hat{\alpha}}$ (graviton, antisymmetric tensor and gravitino, respectively).

The state $|h_{\mu\nu}, B_{\mu\nu\rho}, h_{\mu\hat{\alpha}}\rangle_0$ is the $44 \oplus 84 \oplus 128$ representation of the centre of mass spinor degrees of freedom. The construction of this state is carried out in section 2 and allows explicit calculations of the spin dependence of $M$ matrix theory supergraviton amplitudes to be carried out.

**Asymptotic states.**

Relative motions are described in the $M$ matrix theory by the constrained $SU(2)$ quantum mechanical matrix theory defined above. However, spacetime is only an asymptotic concept in this theory. In particular diagonal matrix configurations, i.e., those corresponding to Cartan generators of $SU(N)$, span flat directions in the matrix model potential and describe spacetime configurations [1]. Transverse directions are described by supersymmetric harmonic oscillator degrees of freedom.

Due to the gauge constraint (1.4) quantum mechanical wavefunctions must be invariant under $SU(2)$ rotations so that there is no preferred Cartan direction. This is not a contradiction with the identification of spacetime spatial degrees of freedom with the diagonal Cartan degrees of freedom of the $M$ (matrix) model since we only require that the concept of spacetime emerges in an asymptotic limit.

To find asymptotic states corresponding to supergraviton (i.e., spacetime) excitations in a gauge invariant way we proceed as follows. Let us suppose we wish to study states describing particles widely separated in the (say) ninth spatial direction, then we may simply declare the $SU(2)$ vector $\vec{X}_9$ to be large. The limit $|\vec{X}_9| = \sqrt{\vec{X}_9 \cdot \vec{X}_9} \to \infty$ is $SU(2)$ rotation (and therefore gauge) invariant. We search for asymptotic particle-like solutions in this limit. The solution of this

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\[4\] Note that the polarisation tensors $h_{\mu\nu}$, $B_{\mu\nu\rho}$ and $h_{\mu\hat{\alpha}}$ correspond to physical polarisations. The $M$ (matrix) theory does away with unphysical timelike and longitudinal polarisations at the price of manifest eleven dimensional Lorentz invariance.
problem along with the identification of the free and interaction Hamiltonians pertaining to their propagation and interactions is the subject of section 3.

To this end it is convenient to employ the (partial) gauge choice in which one chooses a frame where $\vec{X}_9$ lies along the $z$-axis, 

$$X_9^1 = 0 = X_9^2.$$  \hspace{1cm} (1.6)

Calling $X_9 = (0, 0, x)$ and $\vec{X}_a = (Y_a^1, Y_a^2, x_a)$ (with $a = 1, \ldots, 8$) the Hamiltonian in this frame includes the terms:

$$H_V = -\frac{1}{2x} (\partial_x)^2 x - \frac{1}{2} (\partial x_a)^2$$ \hspace{1cm} (1.7)

$$H_B = -\frac{1}{2} (\frac{\partial}{\partial Y_a^I})^2 + \frac{1}{2} r^2 Y_a^I Y_a^I$$ \hspace{1cm} (1.8)

$$H_F = r \tilde{\theta}^I \gamma_9 \theta.$$ \hspace{1cm} (1.9)

The sum of the Hamiltonians $H_B$ and $H_F$ is that of a supersymmetric harmonic oscillator with frequency $r$ and describes excitations transverse to the flat directions. Particle motions in the flat directions correspond to the Hamiltonian $H_V$ whereby we interpret the Cartan variables $x_\mu = (x_a, x)$ asymptotically as the $SO(9)$ space coordinates.

The Hilbert space may be treated as a “product” of transverse superoscillator degrees of freedom and Cartan wavefunctions depending on $x_\mu$ and the third component of $\vec{\theta}$ via the identity

$$H = \sum_{m,n} |m\rangle \langle m| H |n\rangle \langle n|.$$ \hspace{1cm} (1.10)

where $\{|n\rangle\}$ denote the complete set of eigenstates of $H_B$ and $H_F$. Since the frequency $r$ of the superoscillators is coordinate dependent, operators $\partial/\partial x_\mu$ do not commute with $|n\rangle$ so that this “product” is not direct. This construction allows us to study an “effective” Hamiltonian $H_{mn} = \langle m| H |n\rangle$ for the Cartan degrees of freedom pertaining to asymptotic spacetime. Similarly, note that $H_{mn}$ is a differential operator in the Cartan variables. In particular the free Hamiltonian is given by the diagonal terms:

$$H_0 = \sum_n |n\rangle \langle n| \left( H_V + H_B + H_F - \frac{c_n}{r^2} \right) |n\rangle \langle n|. $$ \hspace{1cm} (1.11)

Since supersymmetric harmonic oscillator zero point energies vanish, eigenstates of $H_0$ are

$$|k_{\mu}, h_{\mu\nu}, B_{\mu\nu\rho}, h_{\mu\hat{\alpha}}\rangle = \frac{1}{x} e^{ik_{\mu}x_\mu} |h_{\mu\nu}, B_{\mu\nu\rho}, h_{\mu\hat{\alpha}}\rangle \otimes |0_B, 0_F\rangle.$$  \hspace{1cm} (1.12)

5The spinors $\tilde{\theta}$ are built from $\theta^1$ and $\theta^2$ by complexification and a spin(9) rotation (see equations (3.8) and (3.9)). Note that $r^2 \equiv x_a x_a + x^2$.

6We subtract terms $c_n/r^2$ to ensure the correct asymptotic behaviour of the interaction Hamiltonian. A detailed explanation of this point may be found in section 3.
where \( |0_B, 0_F\rangle \) is the supersymmetric harmonic oscillator vacuum. These states satisfy the correct free particle dispersion relation

\[
H_0 |k_\mu; h_{\mu\nu}, B_{\mu\nu\rho}, h_{\mu\hat{\alpha}}\rangle = \frac{1}{2} k_\mu k_\mu |k_\mu; h_{\mu\nu}, B_{\mu\nu\rho}, h_{\mu\hat{\alpha}}\rangle .
\]  

(1.13)

Here, the supergraviton polarisation multiplet \( |h_{\mu\nu}, B_{\mu\nu\rho}, h_{\mu\hat{\alpha}}\rangle \) is built from the \( 44 \oplus 84 \oplus 128 \) representation of \( \theta^3 \).

What are the properties of these states? Firstly, it is to be stressed that \( k_\mu \) and \( x_\mu \) are not \( SO(9) \) vectors since manifestly the gauge choice (1.6) breaks \( SO(9) \) covariance (the same statement holds for supersymmetry). However, the solutions (1.12) may easily be written in the original variables by undoing the gauge fixing procedure. In the limit \( |\vec{X}_9| \to \infty \) one finds that \( k_\mu, h_{\mu\nu}, B_{\mu\nu\rho} \) and \( h_{\mu\hat{\alpha}} \) transform correctly under both \( SO(9) \) and supersymmetry. I.e., we have found states which asymptotically have the required quantum numbers to describe supergravitons.

Furthermore in the limit \( |\vec{X}_9| \to \infty \), we find \( H \to H_0 \) and that our particle states have eigenvalues \( k_\mu \) of the momentum operator \( |\vec{X}_9|^{-1} \vec{X}_9 \cdot \vec{P}_\mu \).

Therefore, upon taking the direct product of an asymptotic state (1.12) with a centre of mass eigenstate (1.3), one obtains a state describing a pair of supergravitons widely separated in the ninth spatial direction whose interactions are governed by the interaction Hamiltonian \( H_{\text{int}} = H - H_{\text{CoM}} - H_0 \).

**Born amplitude.**

The leading contribution to the quantum mechanical scattering amplitude is the Born amplitude

\[
\langle k'_{\mu}; h'_{\mu\nu}, B'_{\mu\nu\rho}, h'_{\mu\hat{\alpha}} | H_{\text{int}} | k_\mu; h_{\mu\nu}, B_{\mu\nu\rho}, h_{\mu\hat{\alpha}}\rangle
\]  

(1.14)

which yields a nine dimensional Fourier transform of an effective potential \( V(x_\mu, \partial_\mu) \) in which the superoscillator degrees are “integrated out”. In section 4 we show that the result for \( V \) is

\[
V = \frac{16}{r^2} + \frac{(r^2 - x^2)^2}{2 x^2 r} - \frac{r^2 - x^2}{2 x^2 r^2} + \frac{19 r^2 + x^2}{2 x^2 r^5} + \frac{x^a}{2 x^2 r^3} \partial x_a - \frac{1}{2 x^2 r} (\partial x_a)^2 .
\]  

(1.15)

The first term was also obtained in [16] in a different setting. The result (1.13) is encouraging, firstly, since the leading \( 1/r^2 \) term is \( SO(9) \) invariant in accordance with our argument that \( SO(9) \) invariance should be recovered in the large \( x = |\vec{X}_9| \) limit. Secondly, even though we have not yet recovered the reversed \( v^4/r^7 \) potential for \( D0 \) particles, the \( 1/r^2 \) term is, in fact, the leading term allowed.

\footnote{Actually a tree level contribution proportional to \( r \) is also allowed by dimensionality but is clearly absent.}
on dimensional grounds in a loop expansion of the effective action of the original super Yang-Mills model. However, in explicit calculations such a velocity independent $1/r^2$ term has been shown to be absent. Fortunately, it is not hard to see that second (but not higher) order perturbation theory can also yield a $1/r^2$ contribution and in section we show that the result is $-16/r^2$. This supersymmetric cancellation yields a strong test of our proposal.

2 The centre of mass $U(1)$ matrix theory.

The $U(N)$ matrix theory may be decomposed into a free $U(1)$ supersymmetric matrix theory representing the centre of mass motion of the system and a $SU(2)$ theory describing relative motions. The two systems are independent and in this section we give the complete description of the $U(1)$ part.

The $U(1)$ Hamiltonian is given by

$$H_{\text{CoM}} = \frac{1}{2}P^0_{\mu}P^0_{\mu}, \quad \mu = 1, \ldots, 9$$

(2.1)

and there is no constraint since the structure constants of the $U(1)$ gauge group vanish. The Hamiltonian acts in a phase space spanned by the real variables $(X^0_{\mu}, P^0_{\mu})$ and the real sixteen component $SO(9)$ spinors $\theta^0$ where

$$[P^0_{\mu}, X^0_{\nu}] = -i\delta^0_{\mu\nu}$$

(2.2)

$$\{\theta^0_{\hat{\alpha}}, \theta^0_{\hat{\beta}}\} = \delta_{\hat{\alpha}\hat{\beta}}, \quad \hat{\alpha}, \hat{\beta} = 1, \ldots, 16.$$ 

(2.3)

The Hamiltonian (2.1) is the square of the real, sixteen component supersymmetry generator

$$Q_{\hat{\alpha}} = P^0_{\mu}(\gamma^0_{\mu}\theta)_{\hat{\alpha}}$$

(2.4)

$$\{Q_{\hat{\alpha}}, Q_{\hat{\beta}}\} = 2H_{\text{CoM}}\delta_{\hat{\alpha}\hat{\beta}}.$$ 

(2.5)

The $SO(9)$ Dirac matrices $\gamma^0_{\mu}$ are symmetric, satisfy $\{\gamma^0_{\mu}, \gamma^0_{\nu}\} = 2\delta^0_{\mu\nu}$ and the nine dimensional charge conjugation matrix is taken to be unity.

The eigenstates of (2.1) are simply

$$|k_{\mu}; h_{\mu\nu}, B_{\mu\nu\rho}, h_{\mu\hat{\alpha}}\rangle_0 = e^{ik_{\mu}X^0_{\mu}}|h_{\mu\nu}, B_{\mu\nu\rho}, h_{\mu\hat{\alpha}}\rangle_0$$

(2.6)

and are parametrised by the $SO(9)$ momentum vector $k_{\mu}$ and the on-shell $SO(9)$ graviton, antisymmetric tensor and gravitino polarisation tensors $(h_{\mu\nu}, B_{\mu\nu\rho}, h_{\mu\hat{\alpha}}$, respectively) of the eleven dimensional supergraviton multiplet. It is important to note that the tensors $h_{\mu\nu}$ and $h_{\mu\hat{\alpha}}$ are traceless and gamma-traceless, respectively. The state $|h_{\mu\nu}, B_{\mu\nu\rho}, h_{\mu\hat{\alpha}}\rangle_0$ is the $44 \oplus 84 \oplus 128$ representation of the algebra (2.3) and the rest of this section is devoted to the explicit construction of this representation along with explicit realizations of the supersymmetry and $SO(9)$ Lorentz transformations of these states.
In order to define the fermionic vacuum and creation and annihilation operators we perform a decomposition of the $SO(9)$ Lorentz algebra with respect to an $SO(7) \otimes U(1)$ subgroup \cite{12}. This is done as follows. Firstly split vector indices $\mu = (1, \ldots, 9)$ as $(m = 1, \ldots, 7; 8, 9)$ so that an $SO(9)$ vector $V_\mu$ may be rewritten as $(V_m, V, V^*)$ where $V = V_8 + i V_9$ and $V^* = V_8 - i V_9$. The parameters $\Lambda_{\mu\nu}$ of an $SO(9)$ Lorentz transformation decompose with respect to $SO(7) \otimes U(1)$ into $\Lambda_{mn}$ and $\Lambda_{89}$ corresponding to $SO(7)$ and $U(1)$ transformations, respectively, and the remaining parameters may be written in the $SO(7) \otimes U(1)$ covariant form $l^m = \Lambda_m 8 + i \Lambda_m 9$ and $l^{*m} = \Lambda_m 8 - i \Lambda_m 9$. The $SO(9)$ transformation of a vector is then given by

\begin{align}
V_m &\rightarrow V_m + \frac{1}{2}(l_m V^* + l^{*m} V) \\
V &\rightarrow V - i \Lambda_{89} V - l_m V_m \\
V^* &\rightarrow V^* + i \Lambda_{89} V^* - l^{*m} V^* 
\end{align}

(2.7) \hspace{1cm} (2.8) \hspace{1cm} (2.9)

For an $SO(9)$ spinor the same decomposition is made by complexifying, in particular, for the canonical spinor variables we have

$$\lambda = \frac{\theta^0 + i \theta^0}{\sqrt{2}}, \quad \lambda^\dagger = \frac{\theta^0 - i \theta^0}{\sqrt{2}},$$

(2.10)

where the subscript $\pm$ denotes projection by $(1 \pm \gamma_9)/2$. The $SO(9)$ transformations then read

$$\lambda \rightarrow \lambda - \frac{1}{4} \Lambda_{mn} \gamma_{mn} \lambda + \frac{i}{2} \Lambda_{89} \lambda + \frac{1}{2} l^{*m} \gamma_m \lambda^\dagger,$$

(2.12)

$$\lambda^\dagger \rightarrow \lambda^\dagger - \frac{1}{4} \Lambda_{mn} \gamma_{mn} \lambda^\dagger - \frac{i}{2} \Lambda_{89} \lambda^\dagger + \frac{1}{2} l^m \gamma_m \lambda,$$

(2.13)

The canonical anticommutation relations are now

$$\{\lambda_\alpha, \lambda^\dagger_\beta\} = \delta_\alpha\beta, \quad \alpha, \beta = 1, \ldots, 8.$$ 

(2.14)

and we define the fermionic vacuum $|\rangle$ by

$$\lambda |\rangle = 0.$$ 

(2.15)

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8Our index conventions are as follows. Nine dimensional vector indices are given by $\mu, \nu, \ldots$. Vector indices $a, b, \ldots$ stand for $1, \ldots, 8$ whereas $m, n, \ldots$ denote the values $1, \ldots, 7$. For spinor indices, sixteen dimensional $SO(9)$ indices are denoted by $\hat{\alpha}, \hat{\beta}, \ldots$ and eight dimensional $SO(7)$ indices are given by $\alpha, \beta, \ldots$.

9We employ the following representation for the $SO(9)$ Dirac matrices

$$\gamma_\mu = \left\{ \begin{pmatrix} 0 & \gamma_m \\ -\gamma_m & 0 \end{pmatrix}, \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \right\}$$

(2.11)

where the real, antisymmetric $SO(7)$ Dirac matrices satisfy $\{\gamma_m, \gamma_n\} = -2 \delta_{mn}$. 

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We denote the completely filled state by \( |+\rangle = \lambda_1^\dagger \cdots \lambda_8^\dagger |\rangle \).

The most general state is then some linear combination of 256 states of the form

\[
\sum_{i=0}^{8} H_{\alpha_1 \cdots \alpha_i} \lambda_{\alpha_1}^\dagger \cdots \lambda_{\alpha_i}^\dagger |\rangle.
\]  

(2.16)

It is now a simple exercise to extend the \( SO(7) \otimes U(1) \) decomposition to the polarisation tensors \( h_{\mu\nu}, B_{\mu\nu\rho} \) and \( h_{\mu\alpha} \) and then identify combinations of states in (2.16) with the same \( SO(7) \otimes U(1) \) transformation properties. One finds then the following expansion for the supergraviton polarisation state

\[
|h_{\mu\nu}, B_{\mu\nu\rho}, h_{\mu\alpha}\rangle_0 = h|\rangle - \frac{1}{4} h_m|\rangle_m + \frac{1}{16} h_{mn}|\pm\rangle_{mn} + \frac{1}{4} h_m^*|\rangle_m + h^*|\rangle

- \frac{\sqrt{3}}{8} i \left( B_{mn}|\rangle_m + \frac{i}{6} B_m|\rangle_m + \frac{1}{6} B_{mnp}|\pm\rangle_{mnp} - B_{mn}^*|\rangle_{mn} \right)

+ \frac{i}{\sqrt{2}} \left( h_\alpha|\rangle_{\alpha} - \frac{1}{2} h_{\alpha\alpha}|\rangle_m - \frac{1}{2} h_{\alpha\alpha}^*|\rangle_m - h_{\alpha\alpha}^*|\rangle_{\alpha} \right).
\]  

(2.17)

The states in (2.17) are defined in table 1. Note that we have, without loss of generality, assigned the vacuum the \( U(1) \) weight 2 so that \( |\rangle - \rightarrow |\rangle - + 2i\Lambda_{89}|\rangle \).

The less obvious \( SO(7) \otimes U(1) \) decompositions of the supergraviton polarisation tensors are defined below

\[
h_m = h_{m8} + ih_{m9}
\]

(2.18)
\[
h = \frac{h_{88} - h_{99}}{2} + ih_{89}
\]

(2.19)
\[
B_{mn} = B_{m8n} + iB_{mn9}
\]

(2.20)
\[
B_m = B_{m89}
\]

(2.21)
\[
h_\alpha = \frac{(h_{8+} + ih_{9+} - i[h_{8-} + ih_{9-}])_{\alpha}}{\sqrt{2}}
\]

(2.22)
\[
h_{\alpha\alpha} = \frac{(h_{m+} - ih_{m-})_{\alpha}}{\sqrt{2}}.
\]

(2.23)

The relative coefficient of each term in (2.17) is not fixed by \( SO(7) \otimes U(1) \) invariance alone. Firstly note\(^{10}\) that (2.17) is real with respect to complex con-

\(^{10}\)To this end one needs to use the equalities of the states \( |\pm\rangle_{mn} = |\mp\rangle_{mn}, |\pm\rangle_{mnp} = -|\mp\rangle_{mnp} \) and \( |\pm\rangle_m = |\mp\rangle_m \) which follow, respectively, from the identities

\[
(\gamma_m)_{[\alpha\beta}(\gamma_n)_{\gamma\delta]} = \frac{1}{4!} \epsilon_{\alpha\beta\gamma\delta\alpha'\beta'\gamma'\delta'} (\gamma_m)_{\alpha'\beta'}(\gamma_n)_{\gamma'\delta'}
\]

\[
(\gamma_{mn})_{[\alpha\beta}(\gamma_{p})_{\gamma\delta]} = -\frac{1}{4!} \epsilon_{\alpha\beta\gamma\delta\alpha'\beta'\gamma'\delta'} (\gamma_{mn})_{\alpha'\beta'}(\gamma_{p})_{\gamma'\delta'}
\]

\[
(\gamma_{nm})_{[\alpha\beta}(\gamma_{n})_{\gamma\delta]} = \frac{1}{4!} \epsilon_{\alpha\beta\gamma\delta\alpha'\beta'\gamma'\delta'} (\gamma_{nm})_{\alpha'\beta'}(\gamma_{n})_{\gamma'\delta'}
\]

We (anti)symmetrise with unit weight.
| State | $U(1)$ weight | $SO(7)$     |
|-------|---------------|-------------|
| $|\rightarrow\rangle$ | 2 | scalar |
| $|\rightarrow\rangle_\alpha = \lambda^\dagger_\alpha |\rightarrow\rangle$ | 3/2 | spinor |
| $|\rightarrow\rangle_m = (\lambda^\dagger_\gamma m \lambda^\dagger) |\rightarrow\rangle$ | 1 | vector |
| $|\rightarrow\rangle_{mn} = (\lambda^\dagger_\gamma mn \lambda^\dagger) |\rightarrow\rangle$ | 1 | a/symmetric tensor |
| $|\rightarrow\rangle_{m\alpha} = \lambda^\dagger_\alpha (\lambda^\dagger_\gamma m \lambda^\dagger) |\rightarrow\rangle$ | 1/2 | vector-spinor |
| $|\uparrow\rangle_{mn} = (\lambda^\dagger_\gamma m \lambda^\dagger)(\lambda^\dagger_\gamma n \lambda^\dagger) |\rightarrow\rangle$ | 0 | symmetric tensor |
| $|\uparrow\rangle_{mn} = (\lambda^\dagger_\gamma m \lambda^\dagger)(\lambda^\dagger_\gamma n \lambda^\dagger) |\rightarrow\rangle$ | 0 | symmetric tensor |
| $|\uparrow\rangle_{mnp} = (\lambda^\dagger_\gamma [mn \lambda^\dagger](\lambda^\dagger_\gamma p \lambda^\dagger) |\rightarrow\rangle$ | 0 | a/symmetric tensor |
| $|\uparrow\rangle_{mnp} = (\lambda^\dagger_\gamma [mn \lambda^\dagger](\lambda^\dagger_\gamma p \lambda^\dagger) |\rightarrow\rangle$ | 0 | a/symmetric tensor |
| $|\uparrow\rangle_m = (\lambda^\dagger_\gamma mn \lambda^\dagger)(\lambda^\dagger_\gamma n \lambda^\dagger) |\rightarrow\rangle$ | 0 | vector |
| $|\uparrow\rangle_m = (\lambda^\dagger_\gamma mn \lambda^\dagger)(\lambda^\dagger_\gamma n \lambda^\dagger) |\rightarrow\rangle$ | 0 | vector |
| $|\uparrow\rangle_{m\alpha} = \lambda_\alpha (\lambda^\gamma m \lambda^\dagger) |\rightarrow\rangle$ | -1/2 | vector-spinor |
| $|\uparrow\rangle_{mn} = (\lambda^\gamma mn \lambda^\dagger) |\rightarrow\rangle$ | -1 | a/symmetric tensor |
| $|\uparrow\rangle_m = (\lambda^\gamma m \lambda^\dagger) |\rightarrow\rangle$ | -1 | vector |
| $|\uparrow\rangle_\alpha = \lambda_\alpha |\rightarrow\rangle$ | -3/2 | spinor |
| $|\uparrow\rangle$ | -2 | scalar |

Table 1: States transforming covariantly with respect to $SO(7) \otimes U(1)$. Note that $(\lambda^\dagger_\gamma m \lambda^\dagger) \equiv \lambda^\dagger_\alpha (\gamma_m)_{\alpha\beta} \lambda^\dagger_\beta$. 
jugation where $|−\rangle^* = |+\rangle$. Having fixed reality, one must employ covariance with respect to the remaining $SO(9)$ transformations (those with parameters $l_m$ and $l_m^*$) and supersymmetry to fix the coefficients as given in (2.17). From the definition (2.13) and the transformation law (2.12), one deduces that the vacuum is not inert under the leftover $SO(9)$ transformations but rather

$$|−\rangle \rightarrow |−\rangle - \frac{1}{4} l_m^* |−\rangle_m$$

(2.24)

and similarly

$$|+\rangle \rightarrow |+\rangle - \frac{1}{4} l_m |+\rangle_m.$$  

(2.25)

Then using the transformation laws (2.12), (2.13), (2.24) and (2.25) the transformation laws of the states in table 1 may be calculated. So long as one remembers the tracelessness conditions on the graviton and gravitino it is not hard to check that the transformation laws of the states induce the correct $SO(9)$ transformation laws for the supergraviton polarisation tensors.

The supersymmetry transformations of the states in table 1 are obtained by acting with the explicit supersymmetry charge (2.4) which may be written in $SO(7) \otimes U(1)$ covariant language as

$$i \Lambda_\alpha Q_\alpha = k_m (l^* \gamma_m \lambda - l \gamma_m \lambda^*) + kl \lambda - k^* l^* \lambda^*,$$

(2.26)

where we have replaced the operator $P_\mu^0$ by its eigenvalue $k_\mu = (k_m, k, k^*)$. Note that $l$ and $l^*$ are defined (in terms of $\Lambda$) analogously to $\lambda$ and $\lambda^*$, respectively, in (2.10). It is easy to check that the algebra is, as required

$$[i \Lambda_1 Q_\alpha, i \Lambda_2 Q_\beta] = \frac{1}{2} k_\mu k_\nu (\Lambda_1)_\alpha (\Lambda_2)_\beta (2 \delta_\alpha^\beta).$$

(2.27)

Again it is a simple exercise to verify that the action of the Hermitean generator $i \Lambda_\alpha Q_\alpha$ in (2.26) on the state (2.17) induces the correct supersymmetry transformations of the supergraviton polarisation tensors

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \frac{i}{\sqrt{2}} \Lambda_\alpha k_\rho \left( \gamma_\rho \gamma_{(\mu} h_{\nu)} \right)_\alpha$$

(2.28)

$$B_{\mu\nu\rho} \rightarrow B_{\mu\nu\rho} + i \sqrt{\frac{3}{8}} \Lambda_\alpha k_\sigma \left( \gamma_\sigma \gamma_{[\mu\nu} h_{\rho]} \right)_\alpha$$

(2.29)

$$h_{\mu\dot{\alpha}} \rightarrow h_{\mu\dot{\alpha}} + \frac{i}{\sqrt{2}} k_\rho h_{\sigma\mu} \left( \gamma_\sigma \gamma_{\rho\mu} \right)_\dot{\alpha} + \frac{i}{6 \sqrt{6}} B_{\rho\sigma\eta} k_\kappa \left( \gamma_\mu \gamma_{\rho\sigma\eta} \gamma_{\kappa} \right)_\dot{\alpha}$$

$$- i \sqrt{\frac{3}{8}} B_{\mu\rho\sigma} k_\eta \left( \gamma_{\rho\sigma} \gamma_{\eta} \right)_\dot{\alpha}.$$  

(2.30)

One may check that the right hand sides of equations (2.28) and (2.30) are traceless and gamma-traceless, respectively. Furthermore, the commutator of two supersymmetry transformations, as given in (2.28)-(2.30), satisfy the algebra (2.27).
(this result, of course, is ensured since we have an explicit operator representation (2.26) of the algebra (2.27)).

Finally, for the sake of physics, we are interested in inner products of these supergraviton polarisation states. Defining \( \langle -| - \rangle = \{ - \rangle \rangle^\dagger | - \rangle = 1 \) where \( \langle -| \lambda_\alpha = 0, \) we find

\[
\langle h'_\mu \nu, B'_\mu \nu \rho, h'_{\mu \hat{\alpha}} | h_\mu \nu, B_\mu \nu \rho, h_{\mu \hat{\alpha}} \rangle = h'_{\mu \nu} h_\mu \nu + B'_{\mu \nu \rho} B_\mu \nu \rho + h'_{\mu \hat{\alpha}} h_{\mu \hat{\alpha}}.
\]

(2.31)

The SO(9) and supersymmetric invariance\(^{11}\) of this result provides a simple check of our computations.

3 Construction of asymptotic states.

We now turn to the constrained SU(2) sector of the theory describing the relative motion of the two particles. Similar to its U(1) counterpart the state to be constructed must be parametrised in terms of SO(9) momentum and polarisation tensors. Its construction, however, is complicated by the constraint condition \((1.4)\). Incidentally, this condition even forbids momentum eigenstates with nonzero eigenvalue. This follows from the eigenstate equation \( P_A^\mu |p_\mu\rangle = p_\mu |p_\mu\rangle \) (for some fixed \( A \)). Acting twice with appropriate components of \( \vec{L} \) and taking into account that \( \vec{L} |p_\mu\rangle = 0 \), along with the constraint algebra \((1.4)\), implies that \( p_\mu \) vanishes. How can we then hope to find free particle states of definite momentum? The solution lies in the fact that the states to be constructed are not exact momentum eigenstates. Rather, they need only be eigenstates in the limit that the separation between the two particles becomes large. Similarly they will only transform correctly under the transverse Lorentz group SO(9) in this asymptotic sense. These features fit in nicely with the advocated principle of \( M(atrix) \) theory \([1]\) that the concept of (commuting) spacetime emerges only in an asymptotic limit of the theory.

To find these asymptotic states let us suppose that the particles are separated in (say) the ninth spatial direction. We have found it useful to partially gauge fix the SU(2) rotational invariance of the state by choosing a frame in which \( \vec{X}_9 \) lies along the z-axis, i.e. \( X_9^1 = X_9^2 = 0 \). We emphasize, however, that this is only a technical manoeuvre and all our results may be re-expressed in terms of gauge invariant wavefunctions of the original variables. We define

\[
\vec{X}_9 = (0, 0, x) \\
\vec{X}_a = (Y_a^1, Y_a^2, x_a) \quad \text{where} \quad a = 1, \ldots, 8.
\]

(3.1)

\(^{11}\)Of course this is manifest when one inserts \( 1 = e^{-\Lambda Q} e^{\Lambda Q} \) into the left hand side of (2.34), however this induces supersymmetry transformations of the primed polarisation tensors whose signs require some care.
The Cartan subalgebra variables $x_\mu = (x_a, x)$ will acquire the asymptotic interpretation of the nine dimensional spatial coordinates, whereas the $Y^I_a$ (where $I = 1, 2$) describe the oscillatory “off diagonal” modes. Similarly the fermions split up into $\theta^I$ taking care of the polarisation structure of our state, whereas the $\theta^I$ serve as the fermionic partners to the bosonic oscillators $Y^I_a$. The following identification of supersymmetric harmonic oscillator contributions to the $M$(atrix) theory Hamiltonian has already been employed in the beautiful work of de Wit, Lüscher and Nicolai [17] to prove the continuity of the supermembrane spectrum.

To rewrite the Hamiltonian in the frame $Y^I \equiv X^I_9 = 0$, derivatives with respect to $Y^I$ acting on gauge invariant wavefunctions must be reexpressed as

$$
\begin{align*}
[\partial_{Y^I}]_{Y^I=0} &= -i \frac{\epsilon^{IJ}}{x} \hat{L}^J \\
[\partial^2_x + \partial_{Y^I}^2]_{Y^I=0} &= \frac{1}{x} \partial^2_x x - \frac{1}{x^2} \left[ (\hat{L}^1)^2 + (\hat{L}^2)^2 \right]
\end{align*}
$$

(3.2)

where

$$
\hat{L} = \bar{L} - \hat{X}_9 \times \hat{P}_9
$$

(3.3)

do not depend on $x$ or $Y^I$. Note however that (3.1) does not completely fix the gauge, further $U(1)$ rotations about the $z$-axis generated by the Cartan generator $L^3$ are still possible.

The Hamiltonian in this gauge then reads [17]

$$
\begin{align*}
H_V &= -\frac{1}{2x} (\partial_x^2 x - \frac{1}{2} (\partial_x a)^2) \\
H_B &= -\frac{1}{2} \left( \frac{\partial}{\partial Y^I_a} \right)^2 + \frac{1}{2} r^2 Y^I_a Y^I_a \\
H_F &= r \bar{\theta}^I \gamma_9 \theta \\
H_4 &= \frac{1}{4} \epsilon^{IJ} Y^I_a Y^J_b \epsilon^{KL} Y^K_a Y^L_b - \frac{1}{2} x_a x_b Y^I_a Y^I_b \\
&\quad + \frac{1}{x^2} \left( x_a \partial_{x_a} + \frac{1}{2} Y^I_a \partial_{Y^I_a} \right) - \frac{1}{2x^2} x_a x_b \partial_{Y^I_a} \partial_{Y^I_b} \\
&\quad - \frac{1}{2x^2} Y^I_a Y^I_b \partial_{x_a} \partial_{x_b} + \frac{1}{x^2} x_b Y^I_a \partial_{x_a} \partial_{y^I_b} + i \epsilon^{IJ} Y^I_a \theta^I \gamma_a \theta^3 \\
&\quad - \frac{1}{2x^2} (\theta^I \theta^3) (\theta^I \theta^3) - \frac{1}{x^2} (\theta^I \theta^3) [Y^I_a \partial_{x_a} - x_a \partial_{Y^I_a}]
\end{align*}
$$

(3.7)

where $r^2 = x^2 + x_a x_a$ and the spinors $\theta^I$ enter $H_F$ through the complexified and spin(9) rotated combination

$$
\begin{align*}
\theta &= \frac{1}{\sqrt{2}} (\theta^1 + i \theta^2) \\
\tilde{\theta} &= \frac{r + \gamma_9 \gamma_\mu x^\mu}{\sqrt{2r(r + x)}} \theta.
\end{align*}
$$

(3.8)

(3.9)
These complex eight component spinors obey the canonical anti-commutation relations
\[ \{ \theta_\alpha, \theta^\dagger_\beta \} = \{ \tilde{\theta}_\alpha, \tilde{\theta}^\dagger_\beta \} = \delta_{\alpha\beta}, \] (3.10)
although the \( \tilde{\theta}_\alpha \) no longer commute with the bosonic momentum operators. Observe that the Hamiltonian (3.4)-(3.7) commutes with the generator of residual \( U(1) \) Cartan rotations
\[ L^3 = -i \epsilon^{ij} Y^I a \frac{\partial}{\partial Y^J a} - \frac{i}{2} \epsilon^{ij} \theta^I_{\tilde{\alpha}} \theta^J_{\tilde{\alpha}}. \] (3.11)
Physical wavefunctions must, of course, be annihilated by \( L^3 \).

The sum of \( H_B \) and \( H_F \) in (3.4) and (3.6), respectively, is the Hamiltonian of a superharmonic oscillator in the transverse coordinates \( Y^I a \) and \( \theta^I_{\tilde{\alpha}} \) with frequency \( r \). The bosonic groundstate \( |0_B\rangle \) of (3.4) is
\[ |0_B\rangle = (r/\pi)^4 e^{-\frac{r^2}{2} Y^I a Y^I a} \] (3.12)
\[ H_B |0_B\rangle = 8r |0_B\rangle. \] (3.13)
In the fermionic sector we introduce the chiral spinors
\[ \gamma_9 \tilde{\theta}_\pm = \pm \tilde{\theta}_\pm \] (3.14)
to rewrite \( H_F \) as
\[ H_F = r [ (\tilde{\theta}^\dagger_+)_{\alpha} (\tilde{\theta}_+)_\alpha + (\tilde{\theta}^\dagger_-)_{\alpha} (\tilde{\theta}_-)_{\alpha} - 8]. \] (3.15)
The fermionic groundstate is
\[ |0_F\rangle = \prod_{\alpha=1}^8 (\tilde{\theta}^\dagger_-)_{\alpha} |0_{F(\text{ock})}\rangle \] (3.16)
\[ H_F |0_F\rangle = -8r |0_F\rangle. \] (3.17)
where the canonical fermion vacuum \( |0_{F(\text{ock})}\rangle \) is defined by \( \tilde{\theta}_\alpha |0_{F(\text{ock})}\rangle = 0 \). The zero point energies of the combined system \( H_B + H_F \) cancel identically, as expected by supersymmetry. One may check [17] that the groundstates (3.12) and (3.16) are annihilated by the residual constraint (3.11). We have normalised the states \( |0_B\rangle \) and \( |0_F\rangle \) to unity.

Let us now perform the split of the total Hamiltonian into a free and interacting part. At first sight one might naively think that the free Hamiltonian \( H_0 \) is given by \( H_V + H_B + H_F \). However, \( H_V \) is not diagonal in the superoscillator space, as the frequency of the oscillator states depends on the Cartan variables \( x_\mu = (x_a, x) \).

We overcome this difficulty as follows. We would like to factor the Hilbert space into a product of transverse oscillatory degrees of freedom with the space of
particle-like excitations in the valley/flat directions. To this end one may envisage the SU(2) Hilbert space of our problem as a product (albeit non-direct) between the space $H_{S}$ spanned by all possible superoscillator states (denoted symbolically by $\{|n\rangle\}$) and the Cartan Hilbert space $H_{C}$ of wavefunctions depending on $x_{\mu} = (x_{a}, x)$ and $\theta_{3}^{\alpha}$. More concretely we write the total Hamiltonian as

$$H = \sum_{n,m} |n\rangle \langle n| H |m\rangle \langle m|$$

where $\langle n| H |m\rangle : H_{C} \rightarrow H_{C}$, (3.18)

via the identity $1_{H_{SU(2)}} = \sum_{n} |n\rangle \langle n| \times 1_{H_{C}}$ with $1_{H_{C}} = 1$. Let us stress once more that the differential operator $H_{nm} = \langle n| H |m\rangle$ does not commute with the states $|m\rangle$. It represents an “effective” Hamiltonian of the Cartan degrees of freedom pertaining to asymptotic spacetime.

We may now identify the free Hamiltonian as the diagonal piece

$$H_{0} = \sum_{n} |n\rangle \langle n| \left( H_{V} + H_{B} + H_{F} - \frac{c_{n}}{r^{2}} \right) |n\rangle \langle n|.$$ (3.19)

The term $-\frac{c_{n}}{r^{2}}$ on the right hand side deserves some explanation. Our scheme is that all terms in the Hamiltonian of order $1/x$ should be relegated to the interaction part of the Hamiltonian. However, when one computes the expectations of the kinetic terms $H_{V}$ between states $\langle n| H_{C}$ and $|n\rangle$ one finds terms proportional to $1/r^{2}$ which we subtract off order by order as indicated. Note that $c_{0} = 9$. The “effective” Hamiltonian of $H_{0}$ acting in $H_{C}$ then takes the form

$$(H_{0})_{nn} = -\frac{1}{2x} (\partial_{x})^{2} x - \frac{1}{2} (\partial_{x_{a}})^{2} + d_{n} r.$$ (3.20)

Crucially, $d_{0} = 0$ as demonstrated above. Therefore we now have an obvious candidate for the relative asymptotic supergraviton states

$$|k_{\mu}; h_{\mu\nu}, B_{\mu\nu\rho}, h_{\mu\hat{\alpha}}\rangle = \frac{1}{x} e^{ik_{\mu}x_{\mu}} |h_{\mu\nu}, B_{\mu\nu\rho}, h_{\mu\hat{\alpha}}\rangle \otimes |0_{B}, 0_{F}\rangle.$$ (3.21)

Here, the supergraviton polarisation multiplet $|h_{\mu\nu}, B_{\mu\nu\rho}, h_{\mu\hat{\alpha}}\rangle$ is built from the $44 \oplus 84 \oplus 128$ representation of $\theta_{3}^{\alpha}$ in complete analogy to the $U(1)$ sector discussed in the previous section. Indeed, in accordance with the $M$atrix theory picture in which the diagonal blocks describe individual particle degrees of freedom, one should perform this construction in terms of the variables $\theta^{0} \pm \theta^{3}$ in order to obtain polarisations corresponding to individual particle states.

The states (3.21) are true eigenstates of the free Hamiltonian satisfying the correct free particle dispersion relation

$$H_{0} |k_{\mu}; h_{\mu\nu}, B_{\mu\nu\rho}, h_{\mu\hat{\alpha}}\rangle = \frac{1}{2} k_{\mu} k_{\mu} |k_{\mu}; h_{\mu\nu}, B_{\mu\nu\rho}, h_{\mu\hat{\alpha}}\rangle.$$ (3.22)
They are invariant under the residual gauge transformations (3.1) as \( L^3 \) only acts on the \( U(1) \) invariant superharmonic oscillator vacuum \( |0_B, 0_F \rangle \). Furthermore they are plane wave normalisable\(^{12}\).

It follows from the outlined construction that the interaction Hamiltonian \( H_{\text{int}} = H - H_0 \) reads

\[
H_{\text{int}} = \sum_{n \neq m} |n \rangle \langle n| H_V |m \rangle \langle m| + \sum_{n,m} |n \rangle \langle n| \left( H_4 + \frac{c_n}{r^2} \right) |m \rangle \langle m|.
\]

(3.23)

In the limit of large particle separation, \( x \to \infty \), the interaction Hamiltonian scales as

\[
\lim_{x \to \infty} H_{\text{int}} = \mathcal{O}(1/\sqrt{x}).
\]

(3.24)

Hence the interactions die off and the proposed supergraviton states (3.21) are seen to be free in this limit. Let us explain this important observation in detail. Expectation values of the operators \( \theta^I \) between superoscillator states \( \langle m| \) and \( |n \rangle \) do not scale with \( x \). However, it is well known that harmonic oscillator expectations of \( Y^I_a \) scale as the inverse square root of the frequency. I.e., \( \langle m|Y^p|n \rangle \sim r^{-p/2} \), which one can easily see writing the \( Y^I_a \) in terms of creation and annihilation operators (see (4.9)).

Similarly, our states are asymptotic momentum eigenstates, that is, defining the spacetime momentum operator by

\[
P_\mu \equiv -i(\partial_\mu, \partial_{xa}) = -i|\vec{X}_9|^{-1} \vec{X}_9 \cdot \partial/\partial \vec{X}_\mu,
\]

(3.25)

we have

\[
P_\mu |k_\mu; h_{\mu \nu}, B_{\mu \nu \rho}, h_{\mu \alpha} \rangle = k_\mu |k_\mu; h_{\mu \nu}, B_{\mu \nu \rho}, h_{\mu \alpha} \rangle + \mathcal{O}(1/x)
\]

(3.26)

as promised in the beginning of this section.

Let us remark at this point that the free Hamiltonian \( H_0 \) may also possess normalisable eigenstates built from excited superoscillators of the symbolic form \( \psi(x_\mu, \theta^3) |n \rangle \). However, their energy will rise linearly with particle separation \( x \) (due to the third term on the right hand side of (3.20)) and hence they do not qualify as scattering states.

What are the transformation properties of the states (3.21) under SO(9)? The SO(8) subgroup is manifest in the index \( a \), but the gauge fixing \( Y^I = 0 \) of (3.1) breaks SO(9). To gain some insight, let us study the state (3.21) in gauge unfixed language. (This is a familiar problem, given some expression corresponding to some \( SU(2) \) invariant quantity evaluated in a special frame, find the original manifestly invariant expression from which it came). For example, the exponential term in (3.21) becomes

\[
(\vec{X}_9 \cdot \vec{X}_9)^{-1/2} \exp \left[ i(k \sqrt{\vec{X}_9 \cdot \vec{X}_9} + k_a \frac{\vec{X}_a \cdot \vec{X}_9}{\sqrt{\vec{X}_9 \cdot \vec{X}_9}}) \right]
\]

(3.27)

\(^{12}\)Note that the measure for the \( x \) coordinate is \( 4\pi x^2 \) as is evident from the gauge fixing procedure.
which is manifestly gauge invariant as it is built only from dot products of $SU(2)$ 3-vectors. The above expression reverts to $x^{-1} \exp i (k x + k_a x_a)$ in the $Y_I = 0$ gauge. Similarly the term $r Y^I_a Y^I_a$ appearing in the exponential of $|0_B \rangle$ of (3.12) may be written gauge invariantly as
\[
\left( \vec{X}_9 \cdot \vec{X}_9 + \frac{(\vec{X}_a \cdot \vec{X}_9)^2}{\vec{X}_9 \cdot \vec{X}_9} \right)^{1/2} \frac{(\vec{X}_9 \times \vec{X}_a)^2}{\vec{X}_9 \cdot \vec{X}_9}.
\] (3.28)

One may make analogous calculations for the fermions. In this gauge unfixed language the $SO(9)$ Lorentz transformations are obvious, e.g. an infinitesimal rotation $\epsilon$ in the 8-9 plane reads
\[
\vec{X}_9 \rightarrow \vec{X}_9 + \epsilon \vec{X}_8 \quad (3.29) \\
\vec{X}_8 \rightarrow \vec{X}_8 - \epsilon \vec{X}_9 \quad (3.30)
\]
under which (3.27) and (3.28) transform non-covariantly. However, in the limit $|\vec{X}_9| = \sqrt{\vec{X}_9 \cdot \vec{X}_9} \rightarrow \infty$ all contributions violating $SO(9)$ covariance are suppressed. A similar observation holds for supersymmetry. This blends nicely with the advocated principle of $M$atrix theory in which the spacetime interpretation arises only in the asymptotic regime of widely separated particles. Hence we see that our states (3.21) possess all the requirements for free particle states in the limit of large separation, i.e. $SO(9)$ covariance, the correct dispersion relations, definite momenta and polarisations.

4 The Born approximation (and beyond).

The leading term in conventional quantum mechanical scattering theory is the Born amplitude and its computation is the subject of this section. However, we shall find a rather different type of expansion to that encountered in a loopwise diagrammatical expansion of the supersymmetric Yang-Mills theory [7, 8, 18]. To see this let us reinstate the factors $M$ and $R$ as in (1.3) and make the following rescalings $\vec{X}_\mu \rightarrow M^{-3} \vec{X}_\mu$, $\vec{P}_\mu \rightarrow M^{-3} \vec{P}_\mu$ and $\vec{\theta} \rightarrow M^{-3} \vec{\theta}$. One then finds
\[
H = \frac{R}{M^6} \mathcal{H}
\] (4.1)
where
\[
\mathcal{H} = \frac{1}{2} \vec{P}_\mu \cdot \vec{P}_\mu + \frac{1}{4} (\vec{X}_\mu \times \vec{X}_\mu)^2 + \frac{i}{2} \vec{X}_\mu \cdot \vec{\theta} \gamma_\mu \times \vec{\theta}
\]
and the canonical commutation relations now read
\[
[X^A_\mu, P^B_\nu] = i M^6 \delta^B_\mu \delta^{AB}
\] (4.2)
\[
\{\theta^A_\alpha, \theta^B_\beta\} = M^6 \delta^B_\alpha \delta^{AB}.
\] (4.3)
We see that $M^6 \equiv \hbar$ plays the rôle of $\hbar$ and henceforth will be denoted as such. A loopwise expansion leads, of course, to an expansion in $\hbar$, but this will not be the case for our quantum mechanical expansion as we shall see from the following developments.

Employing the results of the previous sections the computation of a $2 \to 2$ supergraviton amplitude in $M(atrix)$ theory may now be performed via elementary quantum mechanical scattering theory. The centre of mass part is trivial and the relative piece of the Born amplitude, using (3.23), reads

$$\langle k'_\mu; h_{\mu\nu}, B'_{\mu\nu}, h'_{\mu\nu}\rangle|H_{\text{Int}}|k_{\mu}; h_{\mu\nu}, B_{\mu\nu}, h_{\mu\nu}\rangle = \int 4\pi x^2 d^3 x \frac{e^{ik'_a x_a}}{x} \langle h'_{\mu\nu}, B'_{\mu\nu}, h'_{\mu\nu}|H_{\text{Eff}}(x_{\mu}, \partial_{\mu}, \theta^3)|h_{\mu\nu}, B_{\mu\nu}, h_{\mu\nu}\rangle \frac{e^{ik_a x_a}}{x}$$  \hspace{1cm} (4.4)

where

$$H_{\text{Eff}}(x_{\mu}, \partial_{\mu}, \theta^3) \equiv \langle 0_B, 0_F|H_{\text{Int}}|0_B, 0_F\rangle.$$  \hspace{1cm} (4.5)

Therefore the $M(atrix)$ theory Born amplitude takes the form of a conventional nine-dimensional Born amplitude, namely the nine dimensional Fourier transform of some interaction Hamiltonian $H_{\text{Eff}}$. The new ingredient is that the “effective” interaction Hamiltonian $H_{\text{Eff}}$ is obtained as the superoscillator vacuum expectation value of the full interaction Hamiltonian $H_{\text{Int}}$. Let us now turn our attention to the calculation of this vacuum expectation.

In rescaled variables, the object we wish to calculate is

$$\langle 0_B, 0_F|(H_4 + \frac{c_0}{r^2})|0_B, 0_F\rangle,$$  \hspace{1cm} (4.6)

where

$$H_4 = \frac{1}{4} \epsilon^{IJKL} Y^I_a Y^J_b \epsilon^{KLM} Y^K_a Y^L_b - \frac{1}{2} x_a x_b [Y^I_a Y^J_b - \frac{1}{2} \frac{\hbar^2}{x^2} x_a x_b \partial_{Y^I_a} \partial_{Y^J_b},$$

$$+ \frac{\hbar^2}{x^2} (x_a \partial_{x_a} + \frac{1}{2} Y^I_a \partial_{Y^I_a}) - \frac{\hbar^2}{2 x^2} x_a x_b \partial_{Y^I_a} \partial_{Y^J_b}$$

$$\frac{\hbar^2}{x^2} x_a x_b Y^I_a \partial_{x_a} \partial_{x_b} + \frac{\hbar^2}{x^2} x_a x_b [Y^I_a \partial_{x_a} \partial_{Y^J_b} + i \epsilon^{IJK} Y^J_a \theta^J \gamma_a \theta^3$$

and, as explained in section 3 the constant $c_0$ is fixed such that

$$\langle 0_B, 0_F||\left(-\frac{\hbar^2}{2x} \partial_{\mu}^2 x - \frac{c_0}{r^2}\right)|0_B, 0_F\rangle = -\frac{\hbar^2}{2x} \partial_{\mu}^2 x.$$  \hspace{1cm} (4.8)

To handle the non-commutativity of derivatives $\partial_{x_\mu}$ with superoscillator states we define mode operators

$$Y^I_a = \frac{a^I_a + a^{I\dagger}_a}{\sqrt{2r/\hbar}}, \quad \partial Y^I_a = \frac{a^I_a - a^{I\dagger}_a}{\sqrt{2\hbar/r}},$$  \hspace{1cm} (4.9)

$$[a^I_a, a^{I\dagger}_b] = \delta_{ab} \delta^{IJ},$$  \hspace{1cm} (4.10)
whose derivatives read

\[ \partial_{x^\mu} a^I_a = \frac{1}{2} \frac{x^\mu}{r^2} (a^I_a)^\dagger, \quad \partial_{x^\mu} (a^I_a)^\dagger = \frac{1}{2} \frac{x^\mu}{r^2} a^I_a. \]  

(4.11)

Derivatives with respect to \( x_\mu \) on bosonic oscillator wavefunctions can be equivalently expressed in terms of combinations of these mode operators acting on states. The \( x_\mu \) dependence of the spin(9) rotated spinor variables \( \tilde{\theta} \) is given by (3.9) so that the fermion oscillator vacuum state is also \( x_\mu \) dependent. Using these results one finds the following for derivatives on the superoscillator ground state

\[ \left[ \partial_{x^\mu}, |0_B,0_F\rangle \right] = \left( -\frac{x^\mu}{4 r^2} a^I_a \right)^\dagger + \frac{1}{2} \left( (\delta_{a9} + \frac{(1 - \delta_{a9})x_\mu}{r + x}) \frac{x_a (\tilde{\theta}^\dagger \gamma_\mu \tilde{\theta}) - \frac{1 - \delta_{a9}}{r} (\tilde{\theta}^\dagger \gamma_\mu \tilde{\theta})}{x_a} \right) |0_B,0_F\rangle \]  

(4.12)

where no summation is implied over \( \mu \). Higher derivatives can be computed in a similar fashion, in particular, the vacuum expectation value of the Cartan kinetic operator is

\[ \langle 0_B,0_F | -\frac{\hbar^2}{2x} \partial_x^2 x | 0_B,0_F \rangle = -\frac{\hbar^2}{2x} \partial_x^2 x + \frac{9\hbar^2}{r^2}. \]  

(4.13)

yielding, as promised, \( c_0 = 9\hbar^2 \).

Using the definitions of the mode operators (1.19) and the above formulae for derivatives on the superoscillator groundstate plus analogous formulas for the fermions, the computation of (4.6) is reduced to some simple mode operator algebra and we find

\[ \langle 0_B,0_F | (H_4 + \frac{c_0}{r^2}) | 0_B,0_F \rangle = \frac{16\hbar^2}{r^2} + \frac{\hbar (r^2 - x^2)^2}{2x^2 r} - \frac{\hbar^2 (r^2 - x^2)}{2x^2 r^2} + \frac{\hbar^3 (19r^2 + x^2)}{2x^2 r^5} + \frac{\hbar^3 x_a}{2x^2 r^3} \partial_{x_a} - \frac{\hbar^3}{2x^2 r} (\partial_{x_a})^2. \]  

(4.14)

In terms of an expansion in large \( x \), the leading term, \( 16/r^2 \), is \( SO(9) \) invariant in accordance with our previous arguments that \( SO(9) \) symmetry should hold to leading order in a large \( x \) expansion. Furthermore there is no explicit \( \theta^3 \) dependence which matches the suspicion that spin effects should be found only at higher orders. However, the quantum mechanical perturbation theory in which transverse oscillator modes are effectively "integrated out" is not organised directly as an expansion in \( \hbar \).

The perturbative expansion we are performing is an expansion in \( 1/x \) of the effective potential. However, as the Born approximation (4.14) shows, this expansion does not scale uniformly in \( 1/x \) for every given order, rather there will be
an overlap of contributions to a specific $1/x^n$ term from a finite number of orders in quantum mechanical perturbation theory. In particular the leading $16\hbar/r^2$ term of (4.14) also receives contributions in second order perturbation theory, but there will be no further contributions at higher orders as a simple scaling analysis of the interaction terms in (4.7) shows. We hence proceed to evaluate this contribution.

Here fortune smiles upon us since the sole contribution to order $r^{-2}$ in the effective potential at second order perturbation theory stems from

$$V_2 = \langle 0_B, 0_F| (i\epsilon^{KL} Y^K_b \theta^L \gamma_b \theta^3) \frac{1}{E - H_0 + i\epsilon} (i\epsilon^{IJ} Y^I_a \theta^J \gamma_a \theta^3)|0_B, 0_F\rangle .$$

To see this observe that $Y^I_a$ scales as $r^{-1/2}$ from (4.9) and the leading contribution to $H_0$ is linear in $r$ as (3.20) shows. As a matter of fact this linear contribution to $H_0$ is the only one entering the $r^{-2}$ part of $V_2$ and hence

$$V_2 = \sum_n \langle 0_B, 0_F| (i\epsilon^{KL} Y^K_b \theta^3 \gamma_b \theta^L)|n\rangle \langle n| \frac{1}{d_n \hbar r}|n\rangle \langle n| (i\epsilon^{IJ} Y^I_a \theta^3 \gamma_a \theta^J)|0_B, 0_F\rangle + O(1/x^3)$$

Now reading this amplitude from the right hand side, we see that the interaction term homogeneously excites one bosonic and one fermionic oscillator mode. Thus the sum over oscillator projectors collapses to the $n = 2$ sector with $d_2 = 2$ and we are left with the computation of a norm. Therefore, dropping the $O(1/x^3)$ terms, we have

$$V_2 = -\frac{1}{2 \hbar r} \left| i\epsilon^{IJ} Y^I_a \theta^J \gamma_a \theta^3|0_B, 0_F\rangle \right|^2 = -\frac{16 \hbar^2}{r^2}$$

which one shows using (4.9), $\langle 0_B, 0_F| \theta^I_a \theta^J_b|0_B, 0_F\rangle = \delta_{a\beta}$ and $(\theta^I_a \theta^3_a)^2 = 8$ due to the canonical anti-commutation relations.

But this contribution cancels exactly with the one obtained in the Born amplitude (4.14). Therefore the overall $1/r^2$ dependence of the effective potential for two widely separated supergravitons vanishes! This result agrees with the effective action found in a path integral background field calculation [7] where a possible $\hbar/r^2$ contribution arises at two loops but also vanishes. A priori the connection between this field theoretic effective action and our canonical effective Hamiltonian for the Cartan degrees of freedom is not at all obvious. Nonetheless, this supersymmetric cancellation is a strong confirmation of our proposal. Furthermore, the understanding of the precise connection between the two approaches that we provide, yields an effective and transparent means of computing the $\mathcal{M}$atrix theory $S$ matrix for eleven dimensional supergravitons.

As demonstrated in [8] the loopwise expansion of the path integral background field effective action is an expansion in $\hbar \equiv M^0$. However our quantum mechanical
perturbation theory is manifestly not organised as an expansion in $\hbar$, e.g., the Born amplitude (4.14) is comprised of terms arising from one, two and three loop level from the point of view of [8]. In fact every order in quantum mechanical perturbation theory will give contributions to a given order of $\hbar$. One may see this by studying the $\hbar$ dependence of the $m$-th order perturbation theory amplitude

$$\langle 0_B, 0_F | H_{\text{Int}} \frac{1}{E - H_0 + i\epsilon} H_{\text{Int}} \frac{1}{E - H_0 + i\epsilon} H_{\text{Int}} \cdots \frac{1}{E - H_0 + i\epsilon} H_{\text{Int}} | 0_B, 0_F \rangle$$

(4.18)

Now as $H_{\text{Int}} \approx (1 + \hbar + \hbar^2)$ and $E - H_0 \approx (1 + \hbar)$ we see that at any order of $\hbar$ all orders of quantum mechanical perturbation theory contribute. Let us stress again, that this conclusion applies only to the $\hbar$ dependence of the effective Hamiltonian. The expansion in $1/x$ is well under control in the sense that for a given order in $1/x$ we will only have to compute a finite number of orders in perturbation theory, just as was the case for the $1/r^2$ term in the above.

5 Conclusion.

Although we have not yet carried out the goal of computing the $M(atrix)$ theory supergraviton $S$ matrix it is clear that our proposal provides a transparent framework for such a calculation. We have also reached the point at which one can pause and consider how this calculation can be most efficiently undertaken. Looking back however, it is pleasing to see that the $M(atrix)$ theory really admits asymptotic quantum mechanical wavefunctions with the correct interpretation as supergraviton particle states possessing the appropriate polarisations, momenta and dispersion relations.

Certainly it is possible to carry out second and higher order quantum mechanical perturbation theory in the manner indicated in section 4. Such a calculation, despite its somewhat untidy appearance, is not inordinately difficult and yields directly $M(atrix)$ theory $S$ matrix elements. This is to be contrasted with the existing background path integral approaches relevant to eikonal scattering from which one extracts phase shifts and in turn the effective potential for a D0 brane probe moving under the influence of a D0 brane source. Ultimately, our results should be directly comparable with $S$ matrix amplitudes of eleven dimensional supergravity, rather than simply the effective potential for a probe moving in a classical eleven dimensional gravitational background. Furthermore, the manifest inclusion of spin degrees of freedom through the physical polarisation tensors of the eleven dimensional supergraviton multiplet allows spin effects to be directly calculated\[13]\.

Note that our calculation of the $M(atrix)$ theory Born amplitude considered the case in which the large ingoing and outgoing separations of the supergraviton

\[13\] The first $M(atrix)$ theory calculations of the spin-orbit interactions of a spinning D0 brane probe have recently been carried out in [19].
particle pairs were both in the nine direction which is the kinematical regime described by the eikonal approximation. There is however, in principle, no impediment to considering more general kinematical situations (see the appendix for further discussion of this point).

Nonetheless, efficient perturbative calculations should also be possible by a loopwise computation of the generating functional of one particle irreducible Greens functions (the effective action), which is readily obtainable from existing background field computations evaluated at arbitrary values of the background fields. To this end one needs reduction formulae connecting these Greens functions with the scattering amplitudes of our asymptotic states. However, for a first quantized quantum mechanical model such as the \( M \)\( (\text{atrix} \) theory the solution to this problem is well known (for completeness, a brief derivation of the following formulae is presented in the appendix). The result for the relevant \( S \) matrix elements is given by

\[
S_{fi} = \int d^9 \vec{X}' \, d^9 \vec{\theta}' \, \Phi_f(\vec{X}_\mu, \vec{\theta}_\alpha) \langle \vec{X}'_\mu, \vec{\theta}'_\alpha | U(\infty, -\infty) | \vec{X}_\mu, \vec{\theta}_\alpha \rangle \Phi_i(\vec{X}_\mu, \vec{\theta}_\alpha)
\]

where we have rewritten our ingoing and outgoing asymptotic supergraviton states as wavefunctions \( \Phi_i(\vec{X}_\mu, \vec{\theta}_\alpha) \) and \( \Phi_f(\vec{X}'_\mu, \vec{\theta}'_\alpha) \) respectively. The quantum mechanical transition element \( \langle \vec{X}'_\mu, \vec{\theta}'_\alpha | U(\infty, -\infty) | \vec{X}_\mu, \vec{\theta}_\alpha \rangle \) may be expressed as a gauge fixed BRST symmetric path integral in the standard way

\[
\langle \vec{X}'_\mu, \vec{\theta}'_\alpha | U(\infty, -\infty) | \vec{X}_\mu, \vec{\theta}_\alpha \rangle = \lim_{T \to \infty} \int_{\vec{X}_\mu(-T/2)=\vec{X}_\mu}^{\vec{X}_\mu(T/2)=\vec{X}_\mu} D^9 \vec{X} D^9 \vec{\theta} D\vec{A} D\vec{b} D\vec{c} e^{iS_{\text{BRST}}}
\]

The boundary conditions of this path integral can be rendered standard by the introduction of appropriate background fields. The BRST action is given (in covariant gauge) by

\[
S_{\text{BRST}} = S_{\text{SYM}} + \int_{-T/2}^{T/2} dt \left( \frac{1}{2} \dot{\vec{A}}^2 + \vec{b} \cdot \frac{D(\vec{A})}{Dt} \right)
\]

where \( S_{\text{SYM}} \) is the usual one dimensional super Yang-Mills action of the \( M \)\( (\text{atrix} \) theory. One is now in the realm of the existing path integral calculations that one finds in the literature \[3, 4, 5, 6\] and the connection between these calculations and supergraviton \( S \) matrix elements is now clear.

Penultimately we mention that we have only considered asymptotic states of the \( U(2) \) \( M \)\( (\text{atrix} \) theory describing a pair of widely separated particles. Three body graviton scattering, for example, would require knowledge of the \( SU(2) \) zero energy ground state wavefunction, since a single outgoing supergraviton state
is described by the product of this ground state with the centre of mass $U(1)$ supergraviton wave function outlined above. Similar ground state considerations are necessary in the study of pairs of particles in the $U(N)$ $M$atrix theory for $N > 2$ where one could even describe an exchange of momentum in the $x_-$ direction. Existence properties of such ground states are discussed in \cite{10, 21} and progress towards their construction has recently been made in \cite{22}.

Our final remark concerns the construction of $n$-body asymptotic states for $n > 2$. It is obvious how our scheme can be generalised to the limited kinematical regime in which the particles are collinear. However, more general kinematical situations require further input. Nonetheless, such a generalisation should certainly draw its main ingredients from our asymptotic state construction.

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Appendix

A. Reduction formulae for $M$(atrix theory).

In this appendix we outline the $M$(atrix) theory analogues of the LSZ reduction formulae relating Greens functions and $S$ matrix elements. Detailed discussions of the below considerations in a more general context may be found in the review and books listed in [20].

We wish to compute $S$ matrix elements of the form

$$S_{fi} = \langle k'_{\mu}; h'_{\mu\nu}, B'_{\mu\nu\rho}, h'_{\mu\hat{\alpha}}; \tilde{X}'_{\theta} | U(\infty, -\infty) | k_{\mu}; h_{\mu\nu}, B_{\mu\nu\rho}, h_{\mu\hat{\alpha}}; \tilde{X}_{\theta} \rangle$$  \hspace{1cm} (A.1)

with $U(\infty, -\infty) = \lim_{T \to \infty} \exp(-iHT)$. The in and out states in (A.1) are our asymptotic supergraviton states written in a general, ungauged frame in variables $(\tilde{X}_\mu, \tilde{\theta}_\hat{\alpha})$. The additional labels $\tilde{X}_9$ and $\tilde{X}'_{9'}$, respectively, indicate the direction in which the asymptotic ingoing and outgoing particles are widely separated. Our earlier calculation of the Born amplitude considered the case in which the large ingoing and outgoing separations of the supergraviton particle pairs were both in the nine direction which is the kinematical regime described by the eikonal approximation. However, here we take the more general kinematical situation in which $\tilde{X}_9 \neq \tilde{X}'_{9'}$.

Introducing twice the resolution of unity in the form \(^{14}\)

$$1 = \int d^9 \tilde{X} d^{16} \tilde{\theta} | \tilde{X}_\mu, \tilde{\theta}_\hat{\alpha} \rangle \langle \tilde{X}_\mu, \tilde{\theta}_\hat{\alpha} |$$  \hspace{1cm} (A.2)

and denoting our asymptotic in and out states in wavefunction form as $\Phi_i(\tilde{X}_\mu, \tilde{\theta}_\hat{\alpha})$ and $\Phi_f(\tilde{X}_\mu, \tilde{\theta}_\hat{\alpha})$, respectively, we have

$$S_{fi} = \int d^9 \tilde{X}' d^{16} \tilde{\theta}' d^9 \tilde{X} d^{16} \tilde{\theta} \Phi_f(\tilde{X}'_\mu, \tilde{\theta}'_\hat{\alpha}) \langle \tilde{X}'_\mu, \tilde{\theta}'_\hat{\alpha} | U(\infty, -\infty) | \tilde{X}_\mu, \tilde{\theta}_\hat{\alpha} \rangle \Phi_i(\tilde{X}_\mu, \tilde{\theta}_\hat{\alpha})$$  \hspace{1cm} (A.3)

The quantum mechanical transition element $\langle \tilde{X}'_\mu, \tilde{\theta}'_\hat{\alpha} | U(\infty, -\infty) | \tilde{X}_\mu, \tilde{\theta}_\hat{\alpha} \rangle$ may be expressed as a path integral in the usual way. If one is interested in (say) covariant gauges it is necessary to introduce the Hermitian and nilpotent BRST charge

$$Q = \bar{c} \cdot \bar{L} + \bar{b} \cdot \bar{p}_A + \frac{1}{2} (\bar{c} \times \bar{c}) \cdot \bar{p}_c$$  \hspace{1cm} (A.4)

along with the ghosts $\bar{c}$ and antighosts $\bar{b}$ and their respective canonical momenta $\bar{p}_c$ and $\bar{p}_b$. Furthermore, we reinstate the Lagrange multiplier $\bar{A}$ of the original super Yang-Mills theory and its canonical momentum $\bar{p}_A$. Our gauge invariant

\(^{14}\)Here we are somewhat schematic. For the sixteen dimensional real spinors $\tilde{\theta}_\hat{\alpha}$ one should really complexify and build coherent states. Technical problems of this type have been considered in the context of quantum mechanical path integration in [23].
asymptotic states are clearly physical and therefore annihilated by $Q$ without being exact and so, according to the Batalin Vilkovisky theorem, we are free to add a piece to the Hamiltonian $\{Q, \Psi\}$ for some gauge fixing fermion $\Psi$. In particular Feynman gauge is reached via the choice

$$\Psi = i\vec{p}_c \cdot \vec{A} + \frac{i}{2} \vec{b} \cdot \vec{p}_A.$$  

(A.5)

After integrating out appropriate canonical momenta via their algebraic field equations the gauge fixed path integral result for the transition element is

$$\langle \vec{X}'_{\mu}, \vec{\theta}'_{\dot{\alpha}} | U(\infty, -\infty) | \vec{X}_\mu, \vec{\theta}_{\dot{\alpha}} \rangle = \lim_{T \to \infty} \int_{\vec{X}_\mu(-T/2) = \vec{X}_\mu}^{\vec{X}_\mu(T/2) = \vec{X}_\mu} D^9 \vec{X} D^{16} \vec{\theta} D \vec{A} D \vec{b} D \vec{c} e^{i S_{\text{BRST}}}$$

(A.6)

$$\equiv e^{\Gamma(\vec{X}', \vec{\theta}'; \vec{X}, \vec{\theta})}$$  

(A.7)

where the BRST action is the sum of the dimensionally reduced one dimensional super Yang-Mills action and gauge fixing terms $S_{\text{BRST}} = S_{\text{SYM}} + S_{\text{Fix}}$ with

$$S_{\text{Fix}} = \int_{-T/2}^{T/2} dt \left( \frac{1}{2} \dot{\vec{A}}^2 + \dot{\vec{b}} \cdot \frac{D(\vec{A})c}{Dt} \right).$$  

(A.8)

The perturbative computation of $\Gamma(\vec{X}', \vec{\theta}'; \vec{X}, \vec{\theta})$ may be carried out by a straightforward loopwise expansion of the path integral (A.6).

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15In principal one must be somewhat careful here, but for technical details we refer the reader to the texts [20].
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