ON PSEUDO-HERMITIAN EINSTEIN SPACES

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Abstract. We describe and construct here pseudo-Hermitian structures $\theta$ without torsion (i.e. with transversal symmetry) whose Webster-Ricci curvature tensor is a constant multiple of the exterior differential $d\theta$. We call these structures pseudo-Hermitian Einstein and our result states that they all can be derived locally from Kähler-Einstein metrics. Moreover, we discuss the corresponding Fefferman metrics of the pseudo-Hermitian Einstein structures. These Fefferman metrics are never Einstein, but they are locally always conformally Einstein.

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1. Introduction

CR-geometry is a $|2|$-graded parabolic geometry on a smooth manifold $M^n$. Underlying Weyl-structures are the pseudo-Hermitian forms $\theta$. CR-geometry is closely related to conformal geometry via the Fefferman construction. For conformal structures, there is the notion of being conformally Einstein, that means there is a Riemannian metric in the conformal class which is Einstein. In terms of tractor calculus the conformal Einstein condition can be expressed through the existence of a parallel standard tractor (cf. e.g. [Gov04], [Lei05]). The concept of parallel standard tractors works for CR-geometry as well. One can define in this case that a pseudo-Hermitian structure with parallel standard CR-tractor, whose first 'slot' is a constant real function, is Einstein.

However, we do not use here tractor calculus to define the Einstein condition for a pseudo-Hermitian structure. Instead, we say a pseudo-Hermitian structure $\theta$ is Einstein if and only if its torsion vanishes and the Webster-Ricci curvature is a constant multiple of the exterior differential $d\theta$. The two definitions for pseudo-Hermitian Einstein spaces coincide.

As our main result, we will show here a construction principle for pseudo-Hermitian Einstein spaces. In fact, they are closely related with Kähler-Einstein spaces (cf. Theorem 1). And we will explicitly show that the Fefferman metrics...
which belong to pseudo-Hermitian Einstein spaces admit a local Einstein scale (cf. Theorem 1).

We will proceed as follows. In section 2 we introduce the notions that we use here for pseudo-Hermitian geometry, in particular, Webster curvature. In section 3 we consider pseudo-Hermitian structures with transversal symmetry and define the Einstein condition. In section 4 we compare the pseudo-Hermitian geometry of \( \theta \) with the Riemannian geometry of the induced metric \( g_\theta \). In section 5 we derive the natural Riemannian submersion of a transversally symmetric pseudo-Hermitian space. We will see that the Ricci tensor of the base space of the Riemannian submersion determines the Webster-Ricci tensor of the transversally symmetric pseudo-Hermitian space. In section 6 we find the construction principles for pseudo-Hermitian Einstein spaces taking off with a Kähler-Einstein space (cf. Theorem 1). Finally, in section 7 and 8 we recall the Fefferman construction and prove explicitly the conformal Einstein condition for those Fefferman metrics which come from pseudo-Hermitian Einstein structures (cf. Theorem 2).

2. Pseudo-Hermitian structures

We fix here in brief some notations for pseudo-Hermitian structures. Thhereby, we follow mainly the notations of [Bau99]. More material on pseudo-Hermitian geometry can be found e.g. in [Lee86], [Lee88], [CS00] [Cap01] or [CG02].

With a CR-structure on a smooth manifold \( M \) of odd dimension \( n = 2m + 1 \) we mean here a pair \( (H, J) \), which consists of

1. a contact distribution \( H \) in \( TM \) of codimension 1 and
2. a complex structure \( J \) on \( H \), i.e. \( J^2 = -id|_H \), subject to the integrability conditions \( [JX, Y] + [X, JY] \in \Gamma(H) \) and

\[
J([JX, Y] + [X, JY]) - [JX, JY] + [X, Y] = 0
\]

for all \( X, Y \in \Gamma(H) \).

The conditions that the distribution \( H \) is contact and the complex structure \( J \) is integrable ensures that \( (H, J) \) determines a \( 2|-\)graded parabolic geometry on \( M \) (cf. e.g. [CS00]). In particular, the (infinitesimal) automorphism group of \( (M, H, J) \) is finite dimensional.

A nowhere vanishing real 1-form \( \theta \in \Omega(M) \) is called a pseudo-Hermitian structure on the CR-manifold \( (M, H, J) \) if

\[
\theta|_H \equiv 0 .
\]

Then we call the data \( (M, H, J, \theta) \) a pseudo-Hermitian space. Since the distribution \( H \) is contact, the 1-form \( \theta \) is necessarily a contact form. Such a contact form \( \theta \) exist on \( (M, H, J) \) if and only if \( M \) is orientable. Furthermore, two pseudo-Hermitian structures \( \theta \) and \( \tilde{\theta} \) on \( (M, H, J) \) differ only by multiplication with a real nowhere vanishing function \( f \in C^\infty(M) \):

\[
\tilde{\theta} = f \cdot \theta .
\]

We consider now the exterior differential \( d\theta \) of a pseudo-Hermitian structure. This 2-form is non-degenerate on \( H \), i.e.

\[
(d\theta)^m|_H \neq 0
\]

and the 2-tensor

\[
L_\theta(\cdot, \cdot) := d\theta(\cdot, J\cdot)
\]

is symmetric and non-degenerate on \( H \). If \( L_\theta \) is positive definite the pseudo-Hermitian structure \( \theta \) is called strictly pseudoconvex. In general, the 2-tensor \( L_\theta \)
has complex signature \((p, q)\) on \(H\) (resp. real signature \((2p, 2q)\)). The conditions
\[ T \cdot \theta \equiv 1 \quad \text{and} \quad T \cdot d\theta \equiv 0 \]
uniquely determine a vector field \(T\) on \(M\). This \(T\) is called Reeb vector field. For convenience, we set \(J(T) = 0\).

To a pseudo-Hermitian structure \(\theta\) on \((M, H, J)\) (with arbitrary signature for \(L_\theta\)) belongs a canonical covariant derivative
\[ \nabla^W : \Gamma(TM) \to \Gamma(T^*M \otimes TM), \]
which is called the Tanaka-Webster connection. It is uniquely determined by the following conditions:

1. The connection \(\nabla^W\) is metric with respect to the non-degenerate symmetric 2-tensor \(g_\theta := L_\theta + \theta \circ \theta\) on \(M\), i.e.
\[ \nabla^W g_\theta = 0, \]
and
2. its torsion \(\text{T} or^W(X, Y) := \nabla^W_X Y - \nabla^W_Y X - [X, Y]\) satisfies
\[ \text{T} or^W(X, Y) = L_\theta(JX, Y) \quad \text{for all} \ X, Y \in \Gamma(H) \quad \text{and} \]
\[ \text{T} or^W(T, X) = -\frac{1}{2}(\{T, X\} + J[T, JX]) \quad \text{for all} \ X \in \Gamma(H). \]

In addition, for this connection it holds
\[ \nabla^W \theta = 0 \quad \text{and} \quad \nabla^W \circ J = J \circ \nabla^W. \]

The curvature operator of the connection \(\nabla^W\) is defined in the usual manner:
\[ R^W = \nabla^W Y - \nabla^W X - [X, Y] . \]
The (4, 0)-curvature tensor \(R^W\) is given for \(X, Y, Z, V \in TM\) by
\[ R^W(X, Y, Z, V) := g_\theta(R^W(X, Y)Z, V). \]
This curvature tensor has the symmetry properties
\[ R^W(X, Y, Z, V) = -R^W(Y, X, Z, V) = -R^W(X, Y, V, Z), \]
\[ R^W(X, Y, JZ, V) = -R^W(X, Y, Z, JV). \]
We have not listed here the Bianchi type identities. We just note that the Bianchi identities for \(R^W\) do not (formally) look like those for the Riemannian curvature tensor. We will come back to this point later.

There is also a notion of Ricci curvature for pseudo-Hermitian structures. It is called the Webster-Ricci curvature tensor and can be defined as follows. Let
\[ (e_\alpha, J e_\alpha)_{\alpha=1,\ldots,m} \]
be a local orthonormal frame of \(L_\theta\) on \(H\) and \(\varepsilon_\alpha := g_\theta(e_\alpha, e_\alpha)\). Then it is defined
\[ \text{Ric}^W(X, Y) := i \sum_{\alpha=1}^{m} \varepsilon_\alpha R^W(X, Y, e_\alpha, J e_\alpha). \]
The Webster-Ricci curvature is skew-symmetric with values in the purely imaginary numbers \(i\mathbb{R}\). And the Webster scalar curvature is
\[ \text{scal}^W := i \sum_{\alpha=1}^{m} \varepsilon_\alpha \text{Ric}^W(e_\alpha, J e_\alpha). \]
The function \(\text{scal}^W\) on \((M, H, J, \theta)\) is real.
3. Transversal Symmetry

Let \((M, H, J)\) be a CR-manifold. A vector field \(T \neq 0\) is called a transversal symmetry of \((H, J)\) if it is not tangential to the subbundle \(H\) (i.e. it is transversal to \(H\)) and if the flow of \(T\) consists (at least locally for small parameters) of CR-automorphisms, i.e. the distribution \(H\) is preserved and \(\mathcal{L}_T J = 0\), or equivalently

\[
[T, X] + J[T, JX] = 0 \quad \text{for all } X \in \Gamma(H).
\]

Now let \(\theta\) be a non-degenerate pseudo-Hermitian structure on \((M, H, J)\) and let \(T\) be the corresponding Reeb vector field determined by \(\theta(T) \equiv 1\) and \(T \cdot d\theta \equiv 0\).

Obviously, the Reeb vector field to \(\theta\) is a transversal symmetry of \((H, J)\) if and only if the torsion part \(Tor^W(T, X)\) of the Tanaka-Webster connection \(\nabla^W\) vanishes for all vector fields \(X \in \Gamma(H)\). Equivalently, it is right to say that \(T\) is a transversal symmetry if and only if \(T\) is a Killing vector field for the metric \(g_\theta\), i.e.

\[
\mathcal{L}_T g_\theta = 0.
\]

This uses the fact that

\[
\mathcal{L}_T J = 0 \quad \text{and} \quad \mathcal{L}_T \theta = 0
\]

for the case when \(T\) is a transversal symmetry.

The above observations suggest the following notation. We say that a non-degenerate pseudo-Hermitian structure \(\theta\) on a CR-manifold \((M, H, J)\) is transversally symmetric if its Reeb vector field \(T\) is a transversal symmetry of \((H, J)\). In short, we say \(\theta\) is a (TSPH)-structure on \((M, H, J)\).

We extend our notations here further and say that \(\theta\) is a pseudo-Hermitian Einstein structure on \((M, H, J)\) if and only if \(\theta\) is transversally symmetric and the Webster-Ricci curvature \(Ric^W\) is a constant multiple of \(d\theta\), i.e.

\[
Ric^W = -i \frac{\text{scal}^W}{m} \cdot d\theta \quad \text{and} \quad Tor^W(T, X) = 0
\]

for all \(X \in \Gamma(H)\). In this case \((M, H, J, \theta)\) is called a pseudo-Hermitian Einstein space (cf. [Lee88]).

4. Comparison between \(\nabla^W\) and \(\nabla^{g_\theta}\) and their curvature tensors

We determine in this section the endomorphism

\[
D^\theta := \nabla^W - \nabla^{g_\theta},
\]

where \(\nabla^{g_\theta}\) denotes the Levi-Civita connection of \(g_\theta\), and derive comparison formulas for the Riemannian and the Webster curvature tensors. We will restrict this discussion to the transversally symmetric case.

So let \(\theta\) be a (TSPH)-structure on \((M, H, J)\). A straightforward calculation shows that the covariant derivative

\[
\nabla^W - \frac{1}{2} d\theta \cdot T + \frac{1}{2} (\theta \otimes J + J \otimes \theta)
\]

is metric and has no torsion with respect to \(g_\theta\). We conclude that it is the Levi-Civita connection of \(g_\theta\) and we obtain as comparison tensor

\[
D^\theta := \nabla^W - \nabla^{g_\theta} = \frac{1}{2} (d\theta \cdot T - (\theta \otimes J + J \otimes \theta))
\]
Another straightforward calculation shows that for any \( X, Y, Z \in \Gamma(TM) \) it holds
\[
R^\nabla^W (X, Y) Z = R^g (X, Y) Z - \frac{1}{2} \left( \nabla^g_Z d\theta(X, Y) \right) \cdot T - \frac{1}{2} d\theta(X, Y) \cdot J(Z)
+ \frac{1}{4} d\theta(Y, Z) \cdot J(X) - \frac{1}{4} d\theta(X, Z) \cdot J(Y)
+ \frac{1}{4} \theta(Z) \cdot \theta(X) \cdot Y - \frac{1}{4} \theta(Z) \cdot \theta(Y) \cdot X.
\]

This is the comparison of the curvature tensors. The formula immediately proves that (in the transversally symmetric case!) the Webster curvature \( R^\nabla^W \) resp. \( R^W \) satisfies the first Bianchi identity of the style of a Riemannian curvature tensor, i.e. it holds
\[
R^\nabla^W (X, Y) Z + R^\nabla^W (Y, Z) X + R^\nabla^W (Z, X) Y = 0.
\]

This is our main observation here.

**Lemma 1.** Let \( \theta \) be a \((TSPH)\)-structure on \((M, H, J)\). Then the Webster curvature tensor \( R^W \) satisfies
\[
R^W (X, Y, Z, V) + R^W (Y, Z, X, V) + R^W (Z, X, Y, V) = 0
\]
for all \( X, Y, Z, V \in TM \). In particular, it holds
\[
R^W (X, Y, Z, V) = R^W (Z, V, X, Y) \quad \text{and} \quad
R^W (X, JY, JZ, V) = R^W (JX, Y, Z, JV).
\]

Using the derived symmetry properties of the Webster curvature for the particular case of transversal symmetry, we obtain the following comparison between the Riemannian Ricci tensor and the Webster-Ricci tensor. Let \((e_\alpha, Je_\alpha)_{\alpha=1,\ldots,m} = (e_i)_{i=1,\ldots,2m}\) denote a local orthonormal frame of \( H \) in \( TM \). It is
\[
Ric^g (X, Y) = R^g (X, T, T, Y) + \sum_{i=1}^{2m} \varepsilon_i R^g (X, e_i, e_i, Y)
\]
and
\[
Ric^W (X, Y) = i \sum_{\alpha} \varepsilon_{\alpha} R^W (X, Y, e_\alpha, Je_\alpha)
\]
\[
= i \sum_{\alpha} \varepsilon_{\alpha} R^W (Y, e_\alpha, Je_\alpha, X) + i \sum_{\alpha} \varepsilon_{\alpha} R^W (e_\alpha, X, Je_\alpha, Y)
\]
\[
= i \sum_{\alpha} \varepsilon_{\alpha} R^W (JY, Je_\alpha, Je_\alpha, X) + i \sum_{\alpha} \varepsilon_{\alpha} R^W (X, e_\alpha, e_\alpha, JY)
\]
\[
= i \sum_{i=1} \varepsilon_i R^W (X, e_i, e_i, JY)
\]
for all \( X, Y \in TM \). Moreover, by the comparison formula for the curvature tensors \( R^g \) and \( R^W \), we have
\[
\sum_i \varepsilon_i R^W (X, e_i) e_i = Ric^g (X) - R^g (X, T) T + \frac{3}{4} X
\]
for all \( X \in \Gamma(H) \) and
\[
\sum_i \varepsilon_i R^W (T, e_i) e_i = \sum_i \varepsilon_i \left( R^g (T, e_i) e_i - \frac{1}{2} \left( \nabla^g_{e_i} d\theta(T, e_i) \right) \cdot T \right).
\]
These formulas combined with the fact that $R^{g_\theta}(X, T)T = \frac{1}{2}X$ for $X \in H$ result to

$$Ric^{g_\theta}(X, Y) = iRic^W(X, JY) - \frac{1}{2}g_\theta(X, Y),$$

$$Ric^W(T, X) = 0, \quad Ric^W(T, T) = 0$$

and

$$Ric^{g_\theta}(T, X) = 0, \quad Ric^{g_\theta}(T, T) = \frac{m}{2}g_\theta(T, T),$$

whereby $X, Y \in H$.

5. The Natural Riemannian Submersion of a (TSPH)-structure

We assume here that $\theta$ is a (TSPH)-structure on the CR-manifold $(M, H, J)$ of dimension $n = 2m + 1$. This implies that the Reeb vector field $T$ to $\theta$ is Killing for the induced metric $g_\theta$. At least locally, we can factorise through the integral curves of $T$ on $M$ and obtain a semi-Riemannian metric $h$ on a factor space, which has dimension $2m$. We describe this process here in more detail. In particular, we calculate the relation for the Ricci curvatures of the induced metric $g_\theta$ and the factorised metric $h$.

Let $\theta$ be a (TSPH)-structure on $(M, H, J)$ of signature $(p, q)$. To every point $p \in M$ exists a neighbourhood (e.g. some small ball) $U \subset M$ and a map $\phi_U$ such that $\phi_U$ is a diffeomorphism between $U$ and the $\mathbb{R}^n$, and moreover, it holds $d\phi_U(T) = \frac{\partial}{\partial x_1}$, that is the first standard coordinate vector in $\mathbb{R}^n$. This implies that there exists a smooth submersion

$$\pi_U : U \subset M \to N \subset \mathbb{R}^{2m}$$

such that for all $v \in N$ the inverse image $\pi_U^{-1}(v)$ consists of an integral curve of $T$ through some point in $U$ parametrised by an interval in $\mathbb{R}$. Since $T$ is a Killing vector field, the expression

$$h(X, Y) := L_\theta(\pi_U^{-1}X, \pi_U^{-1}Y)$$

is uniquely defined for any $X, Y \in TN$ and gives rise to a smooth metric tensor on $N$ of dimension $2m$ of signature $(2p, 2q)$. Alternatively, we can define

$$h(X, Y) = g_\theta(X^*, Y^*),$$

where $X^*$ denotes the horizontal lift of the vector $X$ to $M$ with respect to $g_\theta$ and the vertical direction $\mathbb{R}T$. In particular, the map

$$\pi_U : (U, g_\theta) \to (N, h)$$

is a smooth Riemannian submersion. The construction is naturally derived from $\theta$ only (and some chosen neighborhood $U$). The distribution $H$ in $TU$ is horizontal for this submersion (i.e. orthogonal to the vertical).

For simplicity, we assume now that

$$\pi : (M, g_\theta) \to (N, h)$$

is globally a smooth Riemannian submersion, whereby the inverse images are the integral curves of the Reeb vector field $T$ to a (TSPH)-structure $\theta$ on $M$ with CR-structure $(H, J)$. Since the complex structure $J$ acts on $H$ and $T$ is an infinitesimal automorphism of $J$, the complex structure can be uniquely projected to a smooth endomorphism on $N$, which we also denote by $J$ and which satisfies $J^2 = -id|N$. Since $J$ is integrable on $H$, the endomorphism $J$ is integrable on $N$ as well, i.e. $J$ is a complex structure on $N$. In fact, $J$ is a Kähler structure on $(N, h)$, i.e.

$$\nabla h J = 0.$$
The latter fact can be seen with the comparison tensor $D^\theta$. It is
\[(\nabla^g_X J)(Y^*) = \nabla^g_X (JY^*) - J\nabla^g_X Y^*
= \nabla^W_X (JY^*) - (J\nabla^W_X Y^*) - \frac{1}{2}d\theta(X^*, J(Y^*)) \cdot T
= -\frac{1}{2}g_\theta(X^*, Y^*) \cdot T\]
and
\[\text{Vert}_\pi \nabla^g_X (J(Y^*)) = -\frac{1}{2}g_\theta(Y^*, X^*) \cdot T.\]
Together with $\nabla^h \circ \pi = \pi \circ \nabla^g$, this implies $\nabla^h J = 0$ on $N$.

Altogether, we know yet that a $(\text{TSPH})$-space $(M, H, J, \theta)$ gives rise (locally) in a natural manner to a $(2m)$-dimensional K"ahler space $(N, h, J)$. We use now the well known formulas for the Ricci tensor of a Riemannian submersion to calculate $Ric^h$ (cf. [ONe66]). The application of the standard formulas shows that
\[Ric^g(T, T) = \frac{m}{2}g_\theta(T, T), \quad Ric^g(T, X^*) = 0 \quad \text{and} \]
\[Ric^h(X, Y) = Ric^g(X^*, Y^*) + \frac{1}{2}g_\theta(X^*, Y^*)\]
for all $X, Y \in T N$.

Using the above result for the Ricci tensor of $g_\theta$ with respect to the Webster-Ricci curvature, we obtain
\[Ric^h(X, Y) = iRic^W(X^*, JY^*)\]
for all $X, Y \in T N$. Basically, this result says that the Ricci-Webster curvature of a $(\text{TSPH})$-structure is the Ricci curvature of the base space of the natural submersion.

6. Description and Construction of Pseudo-Hermitian Einstein Spaces

We explain here an explicit constructions of pseudo-Hermitian Einstein spaces with arbitrary Webster scalar curvature. We also show that locally this construction principle generates all pseudo-Hermitian Einstein structures. So we gain a locally complete description. The ideas in this section suggest that the construction can be extended to conformal K"ahler geometry, in general. We aim to discuss this approach somewhere else.

Let $(M, H, J, \theta)$ be a pseudo-Hermitian Einstein space with arbitrary signature $(p, q)$, i.e. it holds
\[Ric^W = -i\frac{\text{scal}^W}{m} \cdot d\theta \quad \text{and} \quad Tor^W(T, X) = 0\]
for all $X \in \Gamma(H)$. Moreover, we assume for simplicity that $\theta$ generates globally a smooth Riemannian submersion
\[\pi : (M, g_\theta) \to (N, h).\]

With the relation for the Ricci tensors from the end of the last section we obtain
\[\pi^* Ric^h = \frac{\text{scal}^W}{m}d\theta(\cdot, J\cdot) = \frac{\text{scal}^W}{m} \pi^* h.\]
This shows that the base space of the natural submersion to the $(\text{TSPH})$-structure $\theta$ is a K"ahler-Einstein space of scalar curvature
\[\text{scal}^h = 2 \cdot \text{scal}^W.\]
We conclude that a pseudo-Hermitian Einstein space $(M, H, J, \theta)$ of dimension $n = 2m + 1$ determines uniquely (at least locally) a K"ahler-Einstein manifold $(N, h, J)$ of dimension $2m$ and signature $(2p, 2q)$. 
We want to show now that there is a construction which assigns to any Kähler-Einstein metric (with signature \((2p, 2q)\)) a uniquely determined pseudo-Hermitian structure (which is then Einstein). The construction itself is only unique up to gauge transformations. However, it is easy to check that the resulting pseudo-Hermitian structures to any gauge are isomorphic. To start with, let \((N^{2m}, h, J)\) be a Kähler-Einstein space of dimension \(2m\) with \(\text{scal}^h > 0\) and let \(P(N)\) be the U\((n)\)-reduction of the orthonormal frame bundle to \((N, h)\). Then it is
\[
S_{ac}(N) := P(N) \times_{\det} S^1
\]
the principal \(S^1\)-fibre bundle over \(N\), which is associated to the anti-canonical complex line bundle \(\mathcal{O}(-1)\) of the Kähler manifold \((N, h, J)\). The Levi-Civita connection to \(h\) induces a connection form \(\rho_{ac}\) on the anti-canonical \(S^1\)-bundle \(S_{ac}(N)\) with values in \(i\mathbb{R}\). For its curvature we have
\[
\Omega^{\rho_{ac}}(\pi_{S_{ac}(N)}^{-1} X, \pi_{S_{ac}(N)}^{-1} Y) = i \text{Ric}^h(X, JY), \quad X, Y \in TN.
\]
At first, we see from this formula that the horizontal spaces of \((S_{ac}(N), \rho_{ac})\) generate a contact distribution \(H\) of codimension 1 in \(TS_{ac}(N)\) and the horizontal lift of the complex structure \(J\) to \(H\) produces a non-degenerate CR-structure \((H, J)\) on \(S_{ac}(N)\). This CR-structure is integrable as can be seen from the relation
\[
\Omega^{\rho_{ac}}(X^*, JY^*) + \Omega^{\rho_{ac}}(JX^*, Y^*) = 0
\]
for all \(X, Y \in TN\) and the fact that the Nijenhuis tensor \(N(X^*, Y^*)\) is the horizontal lift of \(J([JX, Y] + [X, JY]) - [JX, JY] + [X, Y] = 0\).

Secondly, we see that
\[
\theta := i \frac{2m}{\text{scal}^h} \rho_{ac}
\]
is a pseudo-Hermitian structure on \(M := S_{ac}(N)\) furnished with the CR-structure \((H, J)\). The Reeb vector field \(T\) on the pseudo-Hermitian space
\[(S_{ac}(N), H, J, \theta)\]
is vertical along the fibres (in fact, it is a fundamental vector field generated by the right action) and by construction of \((H, J)\) transversally symmetric. Since \(d\theta = \pi^*_{S_{ac}(N)} h(J\cdot, \cdot)\) on \(H\), the base space of the corresponding submersion is again the Kähler-Einstein space \((N, h, J)\) that we started with. For that reason, we know that the Webster-Ricci curvature to \(\theta\) must be given by
\[
i \text{Ric}^W(X^*, JY^*) = \text{Ric}^h(X, Y), \quad X, Y \in H.
\]
Since \(h\) is Einstein, we can conclude that the pseudo-Hermitian space
\[(S_{ac}(N), H, J, \theta)\]
is Einstein as well with Webster-Ricci curvature
\[\text{Ric}^W = -i \frac{\text{scal}^h}{2m} \cdot d\theta.
\]
As mentioned before, for the inverse construction on the Kähler-Einstein space \((N, h, J)\), the choice of \(\theta = i \frac{2m}{\text{scal}^h} \rho_{ac}\) as pseudo-Hermitian 1-form is not unique. One might replace \(\theta\) by \(\tilde{\theta} := \theta + df\) for some smooth function \(f\) on \(S_{ac}(N)\) with \(df \neq -\theta\). The latter condition ensures that \(\tilde{\theta}\) is ‘transversal’, which makes it possible to lift the complex structure to the kernel of \(\tilde{\theta}\). We obtain again a pseudo-Hermitian Einstein structure to \(S_{ac}(N)\) with induced CR-structure. It is straightforward to see that there is a diffeomorphism (gauge transformation) on \(S_{ac}(N)\), which transforms \(\theta + df\) into \(\theta\), i.e. there is an isomorphism of pseudo-Hermitian structures. Since \((\text{locally})\) \(\theta + df\) is the most general choice of a ‘transversal’ 1-form whose exterior
differential is the lift of $h(J, \cdot)$ on $N$, we know that our gauged construction exhausts locally all pseudo-Hermitian Einstein structures with positive Webster scalar curvature.

For the case of negative Webster scalar curvature note that if $(M, H, J, \theta)$ with signature $(p, q)$ has positive Webster scalar curvature $\text{sc}l^W > 0$ then $(M, H, J, -\theta)$ has negative Webster scalar curvature $-\text{sc}l^W$ and the base space of the natural submersion is $(N, -h, J)$, which is Kähler-Einstein with reversed complex signature $(q, p)$. So if $(N, h, J)$ has $\text{sc}l^h < 0$ then $(N, -h, J)$ has $\text{sc}l^{-h} > 0$ and $\theta = i \frac{2m}{\text{sc}l^{-h}} \rho_{ac}$ has positive Webster scalar curvature. Hence, the pseudo-Hermitian form $i \frac{2m}{\text{sc}l^{-h}} \rho_{ac}$ has negative Webster scalar curvature. We conclude that any pseudo-Hermitian Einstein structure with $\text{sc}l^W \neq 0$ can be realised locally on $(S_{ac}(N), H, J)$ over a Kähler-Einstein space with $\text{sc}l^h \neq 0$ by $\theta = i \frac{2m}{\text{sc}l^{-h}} \rho_{ac}$.

As we have seen above a Webster-Ricci flat pseudo-Hermitian space $(M, H, J, \theta)$ gives rise to a Ricci-flat Kähler space. Again we aim to find an inverse construction. So let $(N, h, J)$ be a Ricci-flat Kähler space furnished with a 1-form $\gamma$ such that $d\gamma = h(\cdot, J \cdot)$, i.e. $\omega := d\gamma$ is the Kähler form. The $S^1$-principal fibre bundle $S_{ac}(N)$ has a Levi-Civita connection form $\rho_{ac}$ with values in $i\mathbb{R}$ which is flat, i.e. $d\rho_{ac} = 0$. We set

$$\theta := i \rho_{ac} - \pi^* \gamma$$

on $S_{ac}(N)$. Obviously, it holds

$$d\theta = -\pi^* \omega,$$

i.e. $\theta$ is a contact form on $S_{ac}(N)$ and the distribution $H$ in $T S_{ac}(N)$, which is given by $\theta|_H \equiv 0$ is contact as well. By definition, the distribution $H$ is transversal to the vertical direction of the fibre. For that reason we can lift $J$ to $H$. Again, the CR-structure $(H, J)$ on $M := S_{ac}(N)$ is integrable. Moreover, $\theta$ is a pseudo-Hermitian structure on $(S_{ac}(N), H, J)$. As the construction is done, it is clear that locally around every point of $(S_{ac}(N), g_0)$ the base of the natural Riemannian submersion is a subset of the Ricci-flat space $(N, h, J)$. We conclude that

$$i \text{Ric}^W(X, Y) = \text{Ric}^h(X, Y) = 0$$

for all $X, Y \in H$, i.e. the pseudo-Hermitian space

$$(S_{ac}(N), H, J, \theta)$$

over a Ricci-flat Kähler space $(N, h, J)$ with Kähler form $d\gamma$, where $\theta = i \rho_{ac} - \pi^* \gamma$, is Webster-Ricci flat.

In the Webster-Ricci flat construction, the pseudo-Hermitian form $\theta$ can be replaced by $\hat{\theta} = i \rho_{ac} - \pi^* \gamma + df$, where $f$ is some smooth function on $S_{ac}(N)$ with $df \neq -i \rho_{ac}$. This is the most general ‘transversal’ 1-form on $S_{ac}(N)$ with $d\hat{\theta} = -\pi^* \omega$. However, again one can see that $\theta$ and $\hat{\theta} = \theta + df$ are gauge equivalent on $S_{ac}(N)$, i.e. they are isomorphic as pseudo-Hermitian structures. We conclude that with our construction of a particular gauge we found (locally) the most general form of a Webster-Ricci flat pseudo-Hermitian space. We summarise our results.

**Theorem 1.** Let $(N, h, J)$ be a Kähler-Einstein space of dimension $2m$ and signature $(2p, 2q)$ with scalar curvature $\text{sc}l^h$.

1. If $\text{sc}l^h \neq 0$ then the anti-canonical $S^1$-principal bundle

$$S_{ac}(N) = P(N) \times_{\det S^1} S^1$$

with canonically induced CR-structure $(H, J)$ and connection 1-form

$$\theta := i \frac{2m}{\text{sc}l^h} \rho_{ac},$$

where
where $\rho_{ac}$ is the Levi-Civita connection to $h$, is a pseudo-Hermitian Einstein space with $\text{scal}^W = \frac{1}{2} \text{scal}^{h} \neq 0$.

(2) If $\text{scal}^{h} = 0$ and the Kähler form is $\omega = d\gamma$ for some 1-form $\gamma$ on $N$ then $(\mathcal{S}_{ac}(N), H, J)$ with pseudo-Hermitian structure $\theta = i\rho_{ac} - \pi^*\gamma$ is Webster-Ricci flat.

Locally, any pseudo-Hermitian Einstein space $(M, H, J, \theta)$ is isomorphic to one of these two models depending on the Webster scalar curvature $\text{scal}^W$.

We remark here that for the case $\text{scal}^{h} \neq 0$ we could have chosen the gauge $\theta = i\rho_{ac} + \pi^*\eta - \pi^*\gamma$, where $d\gamma$ is the Kähler form and $d\eta$ the Ricci form. This would enable us to treat the two cases of Theorem 1 as one case. However, for the following discussion of the corresponding Fefferman spaces we find the chosen gauge of Theorem 1 more convenient.

7. The Fefferman metric to a pseudo-Hermitian structure

We briefly explain here the construction of the Fefferman space which belongs to any pseudo-Hermitian space (cf. [Fef76], [Spa85], [Lee86], [Bau99]).

Let $(M, H, J, \theta)$ be a pseudo-Hermitian space of dimension $n = 2m + 1$ and signature $(p, q)$. We denote by $(F, \pi_M, M)$ the canonical $S^1$-principal fibre bundle of the CR-manifold $(M, H, J)$. The Webster connection on $F$ is denoted by $A^W$. In general, it holds

$$\Omega^W = dA^W = -\pi_M^* \text{Ric}^W.$$ 

Furthermore, we define

$$A_\theta := A^W - \frac{i}{2(m+1)} \text{scal}^W \pi_M^* \theta.$$ 

The Fefferman metric on $F$ belonging to $\theta$ on $(M, H, J)$ is defined as

$$f_\theta := \pi_M^* L_\theta - i \frac{A}{m+2} \pi_M^* \theta \circ A_\theta.$$ 

The signature of this metric is $(2p+1, 2q+1)$. The 1-forms $\pi_M^* \theta$ and $A_\theta$ are both lightlike with respect to the Fefferman metric $f_\theta$. In particular, if $(M, H, J, \theta)$ is strictly pseudoconvex, the signature of $(F, f_\theta)$ is Lorentzian.

Let $P$ denote the fundamental vector field in vertical direction on $F$ generated by the element $\frac{m+2}{2} i \in i \mathbb{R}$, i.e. $A_\theta(P) = \frac{m+2}{2} i$. Moreover, in this section we denote by $X^*$ the horizontal lift with respect to $A_\theta$ of a vector field $X$ in $\Gamma(H)$ on $M$. With $T^*$ we denote the horizontal lift of the Reeb vector field $T$. With our definitions it is

$$f_\theta(P, T^*) = 1.$$ 

We have the following formulas for commutators and covariant derivatives with respect to $f_\theta$ (cf. [Bau99]):

$$[X^*, P] = [T^*, P] = 0,$$

$$\text{Vert}_\pi[X^*, Y^*] = i \frac{2}{m+2} \Omega^A(X^*, Y^*) \cdot P,$$

$$\text{Horiz}_\pi[X^*, Y^*] = [X, Y]^*,$$

$$[T^*, X^*] = [T, X]^* + i \frac{2}{m+2} \Omega^A(T^*, X^*) \cdot P,$$

$$[X^*, Y^*] = \pi_H[X, Y]^* - d\theta(X, Y) \cdot T^* + i \frac{2}{m+2} \Omega^A(X^*, Y^*) \cdot P.$$
Furthermore,
\[ f_\theta(\nabla^f_{X^*}Y^*, Z^*) = L_\theta(\nabla^f_W Y, Z), \]
\[ f_\theta(\nabla^f_{P^*} Y^*, Z^*) = \frac{1}{2} d\theta(Y, Z), \]
\[ f_\theta(\nabla^f_{T^*} Y^*, Z^*) = \frac{1}{2} \left( L_\theta([T, T], Z) - L_\theta([T, Z], Y) - i \frac{2}{m+2} \Omega^A_s(Y^*, Z^*) \right), \]
\[ f_\theta(\nabla^f_{X^*} Y^*, P) = -(\frac{1}{2} d\theta(X, Y), \]
\[ f_\theta(\nabla^f_{T^*} P, T^*) = \left( L_\theta([T, X], Y) + L_\theta([T, Y], X) + i \frac{2}{m+2} \Omega^A_s(X^*, Y^*) \right), \]
\[ f_\theta(\nabla^f_{P^*} P, T^*) = f_\theta(\nabla^f_{T^*} T^*) = f_\theta(\nabla^f_{P^*} P, P) = 0, \]
\[ f_\theta(\nabla^f_{P^*} P, Z^*) = f_\theta(\nabla^f_{P^*} T^*, Z^*) = f_\theta(\nabla^f_{P^*} P, Z^*) = 0 \]
for all \( X, Y, Z \in \Gamma(H) \).

8. Einstein-Fefferman Spaces

We discuss here the Fefferman metric of a pseudo-Hermitian Einstein space. In particular, we give an explicit local construction for an Einstein metric in the conformal class of any Fefferman metric coming from a pseudo-Hermitian Einstein space.

Let \((M, H, J, \theta)\) be a pseudo-Hermitian Einstein space. We know already that every such pseudo-Hermitian Einstein space is constructed at least locally from a Kähler-Einstein space. We assume here for simplicity that \(M = S_{ac}(N)\) is the total space of the anti-canonical \(S^1\)-principal fibre bundle over a Kähler-Einstein space \((N, h, J)\) furnished with the naturally induced CR-structure \((H, J)\) and pseudo-Hermitian form \(\theta\) as described in Theorem 1.

\[ \pi^{ac}_N : (S_{ac}(N), H, J, \theta) \to (N, h, J). \]

Moreover, we denote by
\[ S_c(N) := P(N) \times_{det^{-1}} S^1 \]
the canonical \(S^1\)-principal fibre bundle over \((N, h, J)\) which is furnished with the Levi-Civita connection denoted by \(\rho_c\). Now let \(F\) be the total space of the canonical \(S^1\)-fibre bundle over the CR-manifold \((S_{ac}(N), H, J)\). We denote by \(\pi\) the projection of \(F\) to \(N\):
\[ \pi : F \to N. \]

Obviously, the lift of \(S_c(N)\) along the anti-canonical projection \(\pi^{ac}_N\) is isomorphic to \(F\). This shows that we can understand the total space \(F\) as a torus bundle over \((N, h, J)\):
\[ F = P(N) \times_{(det, det^{-1})} S^1 \times S^1. \]

On \(F\) we already introduced the 1-forms \(\pi^{ac}_N \theta\) and \(A^W\) resp. \(A_\theta\). In our situation here, where the Fefferman construction is based on a Kähler-Einstein space, we can express the two latter 1-forms on \(F\) by:
\[ A^W = \pi^{ac}_N \rho_c \quad \text{and} \quad A_\theta = \pi^{ac}_N \rho_c - \frac{i \cdot scal^W}{2(m+1)} \pi^{ac}_N \theta, \]
where \(\pi_c : F \to S_c(N)\) and \(\pi^{ac}_N : F \to S_{ac}(N)\) are the natural projections.

As it was defined in the previous section, the Fefferman metric \(f_\theta\) to the pseudo-Hermitian Einstein space \((S_{ac}(N), H, J, \theta)\) lives on the total space \(F\). We can express now the Fefferman metric on \(F\) in the pseudo-Hermitian Einstein case by
\[ f_\theta = \pi^* h - \frac{4}{m+2} \pi^{ac}_N \theta \circ \left( \pi^{ac}_N \rho_c - \frac{i \cdot scal^W}{2(m+1)} \pi^{ac}_N \theta \right). \]
Notice that the metric $f_\theta$ is uniquely derived from $(N,h,J)$ if we assume $\theta$ to be given in the gauge as described in Theorem 1. For simplicity we will in the following omit the subscripts of the projections: $\pi = \pi_c$ and $\pi = \pi_{ac}$. It will be clear from the context which projection is meant.

**Definition 1.** Let $(N,h,J)$ be a Kähler-Einstein space and let

$$F = P(N) \times_{(\det, \det^{-1})} S^1 \times S^1$$

be the ‘canonical-anti-canonical’ torus bundle over $N$. Then we denote by $f_h := f_\theta$ (where $\theta$ is the gauge given as in Theorem 1 depending on $\text{scat}^h$) the Fefferman metric on $F$ which belongs to the pseudo-Hermitian Einstein space $(\mathcal{S}_{ac}(N), H, J, \theta)$. We call $f_h$ the Fefferman metric of the Kähler-Einstein space $(N,h,J)$.

In general, it is

$$d(\pi^* \rho_c) + d(\pi^* \rho_{ac}) = \pi^* \text{Ric}^W - \pi^* \text{Ric}^W = 0,$$

i.e. the 1-form $\pi^* \rho_c + \pi^* \rho_{ac}$ is closed on $F$. In fact, we will see below that this 1-form is parallel in the Einstein case. If $\text{scat}^h = 0$ we calculate the Fefferman metric of $N$ from the above expression to

$$f_h = \pi^* h - i \frac{4}{m+2} (i \pi^* \rho_{ac} - \pi^* \gamma) \circ \pi^* \rho_c,$$

where $d\gamma = \omega$ is the Kähler form. In case that $\text{scat}^h \neq 0$ an orthogonal 1-form to $\pi^* \rho_c + \pi^* \rho_{ac}$ is given by $\pi^* \rho_c - \frac{1}{m+1} \pi^* \rho_{ac}$ and we can write the Fefferman metric as

$$f_h = \pi^* h + \frac{4m(m+1)}{(m+2)^2} \cdot \text{scat}^h \cdot \left( (\pi^* \rho_c + \pi^* \rho_{ac})^2 - (\pi^* \rho_c - \frac{1}{m+1} \pi^* \rho_{ac})^2 \right).$$

We want to calculate the Ricci tensor of $f_h$. As before, let $P$ be the vertical vector field on $F$ along the ‘Fefferman’ fibering with

$$A^W(P) = \pi^* \rho_c(P) = \frac{m+2}{2} i \quad \text{and} \quad \pi^* \rho_{ac}(P) = 0$$

and let $T^*$ be the vertical vector field along the anti-canonical fibering with

$$\pi^* \theta(T^*) = 1 \quad \text{and} \quad A^\theta(T^*) = 0.$$

Furthermore, let

$$(e_i)_{i=1,\ldots,2m}$$

denote a local orthonormal basis on $(N,h,J)$ and let $e_i^*$ be the horizontal lifts of $e_i$ to $F$ with respect to $\theta$ and then $A^\theta$, i.e.

$$\pi^*(e_i^*) = e_i \quad \text{and} \quad \pi^* \theta(e_i^*) = \pi^* \rho_c(e_i^*) = 0.$$ 

Then we have

$$[T^*, e_i^*] = [P, e_i^*] = [P, T^*] = 0 \quad \text{for all} \quad i = 1, \ldots, 2m$$

on $F$. We will work in the following always with a local basis on $F$ of the form

$$(e_i^*, T^*, P).$$

Now, since $\theta$ is Einstein, we observe that

$$dA^\theta = \Omega^A^\theta = -\pi^*_{ac} \text{Ric}^W - i \frac{\text{scat}^W}{2(m+2)} \pi^*_{ac} d\theta = i \frac{(m+2) \cdot \text{scat}^h}{4m(m+1)} \cdot \pi^*_{ac} d\theta.$$
We use this and the formulas from the last section (cf. [Bau99]) to obtain the covariant derivatives for a local basis \((e^*_i, T^*, P)\) on \((F, f_h)\). It is

\[
\nabla^f_{e^*_i} e^*_j = (\nabla^W_{e^*_i} e^*_j)^* - \frac{i}{2} \pi^* d\theta(e^*_i, e^*_j) T^* - \frac{1}{m} S^W \pi^* d\theta(e^*_i, e^*_j) P
\]

\[
\nabla^f_{T^*} e^*_i = \nabla^h_{e^*_i} T^* = \frac{1}{2} S^W (J e_i)^*
\]

\[
\nabla^f_{P} e^*_i = \nabla^h_{e^*_i} P = \frac{1}{2} (J e_i)^*
\]

\[
\nabla^f_{P} T^* = \nabla^h_{e^*_i} P = \nabla^h_{P} T^* = \nabla^h_{P} P = 0
\]

whereby we set

\[
S^W := \frac{\text{scal}^h}{2m(m+1)}.
\]

It follows immediately that

\[
f_h(\nabla^h_{A} T^*, B) = -f_h(\nabla^h_{B} T^*, A)
\]

for all \(A, B \in \Gamma(T F)\), i.e. \(T^*\) is a Killing vector field on \((F, f_h)\). (In general, for any pseudo-Hermitian space the horizontal lift of the Reeb vector field \(T\) is a Killing vector on the Fefferman space if and only if \(T\) is a transversal symmetry \(O^A_{\dot{O}}(T^*, \cdot) = 0\).)

From the formulas for the covariant derivative we see that the vertical vector field

\[
T^* - S^W P
\]

is parallel. The dual of this vector field with respect to the Fefferman metric \(f_h\) is equal to \(-\frac{2i}{m+2} (\pi^* \rho_e + \pi^* \rho_u c)\), which is a parallel 1-form. For the Riemannian curvature tensor of \(f_h\) we find

\[
R^h(e^*_i, e^*_j) e^*_j = (R^W(e_i, e_j)e_j)^* + \frac{1}{2} S^W d\theta(e_i, e_j)(J e_j)^*,
\]

\[
R^h(e^*_i, P) T^* = \frac{1}{4} S^W \cdot e^*_i,
\]

\[
R^h(T^*, e^*_j) e^*_j = \frac{1}{4} S^W \cdot (T^* + S^W \cdot P),
\]

\[
R^h(P, e^*_j) e^*_j = \frac{1}{4} (T^* + S^W \cdot P),
\]

\[
R^h(P, T^*) = 0.
\]

Then we obtain for the Ricci tensor

\[
\text{Ric}^h(e^*_i, e^*_j) = i \text{Ric}^W(e_i, J e_j) - S^W f_h(e^*_i, e^*_j),
\]

\[
\text{Ric}^h(T, e^*_i) = \text{Ric}^h(P, e^*_i) = 0,
\]

\[
\text{Ric}^h(T^*, T^*) = \frac{m}{2} (S^W)^2,
\]

\[
\text{Ric}^h(T^*, P) = \frac{m}{2} S^W,
\]

\[
\text{Ric}^h(P, P) = \frac{m}{2},
\]

i.e. the Ricci tensor of \(f_h\) takes the form

\[
\text{Ric}^f_h = i \pi^* \text{Ric}^W(\cdot, J \cdot) - S^W \pi^* f_h
\]

\[
+ \frac{m}{2} \left( (S^W)^2 \pi^* \theta \circ \pi^* \theta - \frac{4}{(m+2)^2} A_\theta \circ A_\theta - i \frac{4}{m+2} S^W A_\theta \circ \pi^* \theta \right)
\]

\[
= \frac{\text{scal}^h}{2(m+1)} f_h - \frac{2m}{(m+2)^2} \left( A_\theta - \frac{i(m+2) \cdot \text{scal}^h}{4m(m+1)} \pi^* \theta \right)^2.
\]
In particular, if $\text{scal}^h = 0$ on $N$ then
\[ \text{Ric}^f_h = -\frac{2m}{(m+2)^2}(\pi^*_c \rho_c)^2 \]
and if $\text{scal}^h \neq 0$ then
\[ \text{Ric}^f_h = \frac{\text{scal}^h}{2(m+1)} f_h - \frac{2m}{(m+2)^2}(\pi^*_c \rho_c + \pi^*_c \rho_{ac})^2. \]

The calculations show that the Fefferman metric $f_h$ to a Kähler-Einstein space $(N, h, J)$ is never Einstein (cf. [Lee86]). For example, a Webster-Ricci flat pseudo-Hermitian space gives rise to a Fefferman metric $f_h$ with totally isotropic Ricci tensor. However, the Einstein condition should not be expected for the Fefferman metric. Instead, we will show now that the Fefferman metric to any pseudo-Hermitian Einstein space (resp. Kähler-Einstein space) is (locally) conformally Einstein, i.e. there is locally a conformally rescaled metric $\tilde{f}_h$ to $f_h$ which is Einstein. For the calculation of the conformal Einstein scale we introduce the coordinate function $t$ on the torus fibre bundle $F$ by
\[ dt = i\pi^* \rho_c \quad \text{when} \quad \text{scal}^h = 0 \quad \text{and} \]
\[ dt = i\pi^* \rho_c + i\pi^* \rho_{ac} \quad \text{when} \quad \text{scal}^h \neq 0. \]

First, we consider the Webster-Ricci flat case. For simplicity we assume a submersion
\[ \pi : (S_{ac}(N), H, J, \theta) \to (N, h, J), \]
where $\theta = i\pi^* \rho - \pi^* \gamma$ and $\omega = d\gamma$ is the Kähler form. Let $f_h$ be the Fefferman metric on $F$ over $N$ and let $\tilde{f}_h = e^{2\phi} f_h$ be a conformally rescaled metric with real function $\phi$ on $F$. For the Ricci tensor of $\tilde{f}_h$ we find by using standard formulas
\[ \text{Ric}^{\tilde{f}_h} - \text{Ric}^f_h = -2m(\text{Hess}(\phi) - d\phi \circ d\phi) + (-\Delta \phi - 2m \|d\phi\|^2)f_h, \]
where $\Delta$ denotes the Laplacian with respect to $f_h$. We denote the correction term on the right hand side of this formula by $C_\phi$. This is a symmetric 2-tensor. Let $\phi(t)$ be a function on $F$ which depends only on the coordinate $t$ in direction of the canonical $S^1$-fibering. Then the only non-trivial component of $C_\phi$ is
\[ C_\phi(P, P) = -2m \left( PP(\phi) - P(\phi)^2 \right). \]
This shows that for any function $\phi(t)$ on $F$, which satisfies the ODE
\[ \partial_t \partial_t \phi - (\partial_t \phi)^2 = \frac{1}{(m+2)^2}, \]
the Ricci tensor $\text{Ric}^{\tilde{f}_h}$ of $\tilde{f}_h = e^{2\phi} f_h$ vanishes. The most general solution of this ODE is
\[ \phi = c_1 - \ln \left( \cos \left( \frac{t}{m+2} + c_2 \right) \right), \]
where $c_1, c_2$ are constants. We choose here $\phi = -\ln(\cos(\frac{t}{m+2}))$, which then gives as conformal rescaling factor
\[ e^{2\phi} = \cos^{-2} \left( \frac{t}{m+2} \right). \]
Then the conformally changed Fefferman metric (in short: conformally Fefferman metric)
\[ \tilde{f}_h = \cos^{-2}(t/(m+2)) \cdot \left( \pi^* h - i \frac{4}{m+2} (i\pi^* \rho_{ac} - \pi^* \gamma) \circ \pi^* \rho_c \right) \]
is Ricci-flat on an open subset in \( F \) around the hypersurface given by \( \{ t = 0 \} \). Obviously, a global conformal Einstein scale for \( f_h \) on \( F \) does not exist. Everywhere locally it does exists.

We assume now that \( (N, h, J) \) is Kähler-Einstein with \( \text{scal}^h \neq 0 \). Then we find with respect to a conformal scaling function \( \phi(t) \), which depends only on the coordinate \( t \) with \( dt = i(\pi^*\rho_c + \pi^*\rho_{ac}) \),

\[
\text{Ric}^{\tilde{f}_h} - \text{Ric}^f = C_{\phi} = 2m(\partial_t \phi - (\partial_t \phi)^2)(d\rho_c + d\rho_{ac})^2 + \frac{(m+2)^2}{4m(m+1)} \text{scal}^h (\partial_t \phi - (\partial_t \phi)^2)f_h.
\]

Again, if we choose \( \phi = -\ln(\cos(\frac{\pi}{m+2})) \), the metric \( \tilde{f}_h = e^{2\phi} f_h \) is Einstein. In fact, we obtain with this function \( \phi \) for the Ricci tensor of the rescaled metric

\[
\text{Ric}^{\tilde{f}_h} = \frac{(2m+1) \cdot \text{scal}^h}{4m(m+1)} f_h
\]

and the scalar curvature is \( \text{scal}^{\tilde{f}_h} = \frac{2m+1}{2m} \cdot \text{scal}^h \).

**Theorem 2.** Let \((N, h, J)\) be a Kähler-Einstein space of dimension \( 2m \) and signature \((2p, 2q)\) with scalar curvature \( \text{scal}^h \).

1. If \( \text{scal}^h = 0 \) and \( \omega = d\gamma \) for some 1-form \( \gamma \) on \( N \) then the metric
   \[
   \tilde{f}_h = \cos^{-2}(t) \cdot (\pi^* h + 4 dt \circ (\pi^* \gamma + ds))
   \]
   on \( N \times \{ (s, t) : -\frac{\pi}{2} < t < \frac{\pi}{2} \} \subset N \times \mathbb{R}^2 \) (with natural projection \( \pi \) onto \( N \)) is conformally Fefferman and Ricci-flat with signature \((2p+1, 2q+1)\).

2. If \( \text{scal}^h \neq 0 \) then the metric
   \[
   \tilde{f}_h = \cos^{-2}(t) \cdot \left( \pi^* h - \frac{4m(m+1)}{\text{scal}^h} \cdot (dt^2 + \frac{\rho_{ac}^2}{(m+1)^2}) \right)
   \]
   on \( S_{ac}(N) \times ]-\frac{\pi}{2}, \frac{\pi}{2}[ \), where \( S_{ac}(N), \pi, N \) is the anti-canonical \( S^1 \)-bundle over \( N \) with Levi-Civita connection \( \rho_{ac} : TS_{ac}(N) \to i\mathbb{R} \), is conformally Fefferman and Einstein with \( \text{scal}^{\tilde{f}_h} = \frac{2m+1}{2m} \cdot \text{scal}^h \) and signature \((2p+1, 2q+1)\).

Every Fefferman metric, which is conformally Einstein, is locally conformally equivalent to a metric of the form \( f_h \) as described here in (1) resp. (2).

In Theorem 2 we simplified the expressions for the Fefferman metrics. In the Ricci-flat case both the Levi-Civita connections \( \rho_c \) and \( \rho_{ac} \) are flat, i.e. the torus bundle is globally a product and we parametrised the vertical directions by the coordinates \( t, s \), where the coordinate \( t \) is rescaled (compared with our notation before) by a factor \((m+1)\).

For the case when \( \text{scal}^h \neq 0 \) we replaced the 1-form \( \pi^* \rho_c - \frac{1}{m+1} \pi^* \rho_{ac} \) on \( F \) by \( -\frac{m+2}{m+1} \cdot \rho_{ac} \) on \( S_{ac}(N) \). This is possible, since locally the canonical bundle \( S_c(N) \) and the anti-canonical bundle \( S_{ac}(N) \) can be identified such that the Levi-Civita connection \( \rho_c \) becomes \( -\rho_{ac} \). It is useful to note here that the Fefferman metric \( f_h = \cos^2(t) \tilde{f}_h \) as presented in Theorem 2 is the product of a real line with the metric

\[
\pi^* h - \frac{4m(m+1)}{(m+1)^2 \cdot \text{scal}^h} \cdot \rho_{ac}^2.
\]

This is the well-known Einstein-Sasaki metric which is constructed from the Kähler-Einstein metric \( h \).
For the proof of Theorem 2 we remark that it does not follow yet from our discussion that any conformally Einstein Fefferman metric comes from a pseudo-Hermitian Einstein space. To see this point in the proof we can use the argument from tractor calculus, which says that the parallel standard tractor on the Einstein-Fefferman space gives rise to a parallel standard tractor on the underlying CR-space. This implies that the CR-space admits a pseudo-Hermitian Einstein structure.

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