A non-simply laced version for cluster structures on 2-Calabi-Yau categories

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Abstract
We propose a non simply-laced version for cluster structures on additive Krull-Schmidt categories over arbitrary commutative base field. Starting from the work of Buan-Iyama-Reiten-Scott, it turns out that, under the same weaker assumption as in the simply-laced case, the generalized version of cluster structure holds for 2-Calabi-Yau or stably 2-Calabi-Yau categories. As a direct consequence, we can use the so-called cluster maps to realize in a direct way a large class of non simply-laced cluster algebras of geometric type with coefficients associated with rectangular matrices whose principal parts are skew symmetrizable.

Keywords: cluster structure, modulated quivers, Calabi-Yau category, cluster algebra

1. Introduction
The theory of cluster algebras, introduced by Fomin and Zelevinsky in [1], and further developed in a series a papers (for example [2, 3, 4] ...), appears to have interesting connections with many parts of mathematics. In the representation theory of algebras, the philosophy has been to provide a representation or a categorical theoretic interpretation of the main combinatorics, called mutation of seeds, used to define cluster algebras, where a seed in a cluster algebra consists of two non-overlapping sets one of which is called cluster while the other one is called the coefficient set, together with some rectangular matrix of integers whose principal part is skew symmetrizable. The first categorical model of the combinatorics of cluster algebras was given by cluster categories which were first introduced by Buan-Marsh-Reineke-Reiten-Todorov [5] (and in [6] for $A_n$ case by Caldero-Chapoton-Schiffrer), and which are now shown to be triangulated 2-Calabi-Yau categories by Keller [7]. In [8] the module category mod $\Lambda$ of a preprojective algebra $\Lambda$ associated to a simply-laced Dynkin quiver, which turns out to be exact stably 2-Calabi-Yau, is also used for a similar purpose. Both cluster categories and module categories mod $\Lambda$ for preprojective algebras $\Lambda$ associated to (simply-laced) Dynkin quivers contain some special objects called cluster tilting objects which are analogs of clusters. It turns out that, 2-Calabi-Yau categories and related categories appear as a natural framework in which one can model and study many aspects of cluster algebras, with this in mind, cluster structures and substructures were introduced and studied by Buan-Iyama-Reiten-Scott [9] for 2-Calabi-Yau (or stably 2-Calabi-Yau) categories over algebraically closed fields.

However, most of the important connections and results which had been obtained between cluster algebras and representation theory of algebras were restricted to the simply-laced case which corresponds to cluster algebras associated with skew symmetric matrices and to the use of quiver representations. For the general case one should consider skew symmetrizable matrices, and in order to model this case inside the representation theory of algebras, we must consider 2-Calabi-Yau categories over non-necessarily algebraically closed fields, and hence work directly with representations of modulated quivers (or with species) instead of representations of (ordinary) quivers. Some alternative non simply-laced approach to clusters has been the use of folding technique to pass from quivers to valued quivers, see [10, 11, 12].

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In this paper and in some of our undergoing and related work, we are suggesting the use of representation theory of modulated quivers as a non simply-laced approach to further investigate what is still missing for cluster tilting theory inside the framework of Calabi-Yau categories over non algebraically closed base field, as well as previously obtained connections between the theory of cluster algebras and that of representation theory of algebras.

Our main purpose in this paper is rather modest: using modulated quivers we begin the study of a straightforward non simply-laced approach to cluster structure for 2-Calabi-Yau (or stably 2-Calabi-Yau) categories over non necessarily algebraically closed fields. We then follow [9] and establish the existence of a non simply-laced version of cluster structure under the assumption of a weak cluster structure. The main result of this paper (Theorem 5.2 and Theorem 5.3) serves as a starting point to further generalize many simply-laced connections between cluster algebras and representation theory. As a direct consequence of the main result (Theorem 6.1), we generalize to the case of non simply-laced cluster algebras associated with skew symmetric matrices a result of Buan-Iyama-Reiten-Scott realizing simply-laced cluster algebras of geometric type associated with skew symmetric matrices, with coefficients and allowing possibly infinite clusters.

Motivated by the present successful non simply-laced generalization for cluster structures, in some of our undergoing work ([13]) we introduce a notion of potential for modulated quivers and generalize the mutation of quivers with potentials to the mutation of modulated quivers with potentials, and then construct a non simply-laced version of the cluster category associated with a modulated quiver with potential (in the non-acyclic case). The simply-laced version of the cluster category associated to a quiver with potential was obtained by Amiot in [14] and then used by Keller in [15] to completely solve the simply-laced version of the periodicity conjecture. Then it is of great interest to further develop a non-simply laced approach to the cluster tilting theory inside the framework of Calabi-Yau categories over non necessarily algebraically closed base field. As already mentioned above, preprojective algebras associated with simply-laced Dynkin quivers provide good examples of stably 2-Calabi-Yau categories in which some non-acyclic cluster algebras are modeled. As noticed in the appendix to this paper, preprojective algebra associated with modulated graphs are not well understood (even in the Dynkin case), apart from the introductory work of Dlab-Ringel [16] (1980).

We are indebted to Iyama for his comments about 2-Calabi-Yau condition and about trace maps which are the key tools for the non-simply laced version of cluster structure.

2. Some preliminaries

Let $A$ be any additive category and $\text{Ab} = \text{Mod} \mathbb{Z}$ be the category of abelian groups. We write $A(X, Y)$ or $\text{Hom}_A(X, Y)$ for the abelian group of all morphisms in $A$ from $X$ to $Y$, and for any objects $X, Y, Z$ in $A$, the composition of two morphisms $f \in A(X, Y)$ and $g \in A(Y, Z)$ is the element of $A(X, Z)$ written as $g \circ f$ or $gf$. The additive category $A$ is called Krull-Schmidt if it satisfies the Remak-Krull-Schmidt-Azumaya theorem: the endomorphism ring of each indecomposable object in $A$ is local (or equivalently, idempotents split in $A$ is the sense that, any endomorphism $e \in A(X, X)$ such that $e^2 = e$ factorizes as $e = rs$ where $s$ is a section and $r$ is a retraction), and each object in $A$ decomposes as a finite direct sum of indecomposable objects (the decomposition being therefore unique up to isomorphism and up to the order of terms). We always denote by $\text{ind} A$ the isoclass of indecomposable objects in $A$, and we view $\text{ind} A$ as a subset of $\text{obj}(A)$.

The (Jacobson) radical of an additive category $A$ is the sub-bifunctor $J = J_A = \text{rad}_A$ of the bifunctor $\text{Hom} = A(-, -) : A^2 \times A$ defined on each pair of objects $X, Y \in \text{obj}(A)$ as follows:

$$J(X, Y) = \{ f \in A(X, Y) : \forall g \in A(Y, X), 1_X - gf \text{ is invertible.} \}$$

$$= \{ f \in A(X, Y) : \forall g \in A(Y, X), 1_Y - fg \text{ is invertible.} \}$$

We know that $J$ is a two-sided ideal of $A$ and we will refer to elements of $J$ as radical maps. $N$ standing for the set of integers (with zero), the powers $J^n$, with $n$ running over $N$, are the ideals of $A$ defined by induction on $n$ as follows: $J^0 = A(-, -) = \text{Hom}$, $J^1 = J$, and for $n \geq 2$ and for any two objects $X, Y$ in $A$, $J^n(X, Y)$ consists of compositions $\nu \circ u$ with $u \in J^n(X, X')$ and $v \in J(Y', Y)$ for some object $Y'$ in $A$. We need the following easy observations in Krull-Schmidt categories.
Lemma 2.1. If \( A \) is Krull-Schmidt, then the following assertions hold.

(a) For any morphism \( u \) between two indecomposable objects in \( A \), \( u \) is a section if and only if \( u \) is an isomorphism, if and only if \( u \) is a retraction.

(b) Let \( E = \bigoplus_{i \leq n} E_i \) be a biproduct endowed with a family \( (p_i, q_i)_{1 \leq i \leq n} \) of corresponding canonical projections and injections, let \( f = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} : X \rightarrow E \) and \( g = [g_1, \ldots, g_n] : E \rightarrow Y \) be two morphisms. Then \( f \) is a section if and only there is some \( i \) such that \( f_i = p_i f \) is a section. Dually, \( g \) is retraction if and only there is some \( i \) such that \( g_i = q_i g \) is a retraction.

We also need the following easy properties of radical maps in Krull-Schmidt categories.

Lemma 2.2. Suppose \( A \) is Krull-Schmidt.

(a) Let \( f : X \rightarrow E \) be any morphism whose source \( X \) is indecomposable, then the following assertions are equivalent:

(i) \( f \) is not a section.

(ii) \( \text{Im} \text{Hom}_A(f, X) \subseteq J(X, X) \).

(iii) \( f \) is a radical map.

(b) Let \( g : E \rightarrow Y \) be any morphism whose target \( X \) is indecomposable, then the following assertions are equivalent:

(i) \( g \) is not a retraction.

(ii) \( \text{Im} \text{Hom}_A(Y, g) \subseteq J(Y, Y) \).

(iii) \( g \) is a radical map.

For every \( X, Y \in \text{ind} A \), the bimodule of irreducible maps from \( X \) to \( Y \) is \( \text{Irr}(X, Y) := J(X, Y)J^2(X, Y) \). Thus if \( A \) is a Krull-Schmidt \( k \)-category for a field \( k \), then \( k_X = \text{End}(X) \text{radEnd}(X) = A(X, X)J(X, X) \) is a division \( k \)-algebra for each \( X \in \text{ind} A \), and \( \text{Irr}(X, Y) \) is a \( k_X \)-\( k_Y \)-bimodule for each pair \( X, Y \in \text{ind} A \), where \( A^\circ \) is the opposite algebra of an algebra \( A \). If additionally all the bimodules \( \text{Irr}(X, Y) \) are finite dimensional, then the valued quiver \( Q^\circ_A \) of the category \( A \) is described as follows:

- the set of points is given by \( (Q^\circ_A)_0 = \text{ind} A \) and, for each pair of points \( x, y \in (Q^\circ_A)_0 \), the (fully) valued arrow 
  \( \alpha = \alpha_{x,y} \) representing the set \( (Q^\circ_A)_1(x, y) \) of all valued arrows from \( x \) to \( y \) is given by 
  \( \alpha_{x,y} : x d_x, x, \cdots d_x y \),
  with \( x d_y = x d^n, \text{dim}_k(\text{Irr}(x, y)) \) and \( x d_y, y d^n := y d^n = \text{dim}_k(\text{Irr}(x, y)) \).

It is understood that any zero-valued arrow is not counted among the (true) arrows of \( Q \). For a general definition of valued quivers, see Definition 4.4. We can also associated to \( A \) its modulated quiver \( Q_A \) whose underlying valued quiver is \( Q_A \), this will be done after Definition 3.3 about dualizing pairs of bimodules.

3. Minimal approximation sequences, connecting \( \tau \)-sequences

We need some background material about special sequences that we introduce and give some properties.

Definition 3.1. Let \( f : X \rightarrow Y \) be a morphism in some additive category \( A \). Then \( f \) is called right minimal if \( f \) has no zero direct summand of the form \( U \rightarrow 0 \) with \( U \neq 0 \). Dually, \( f \) is called left minimal if \( f \) has no zero direct summand of the form \( 0 \rightarrow U \) with \( U \neq 0 \).
Let’s say that an additive category $A$ has the finite direct sum decomposition property if for every object $X$ in $A$ there exists an integer $m_X > 0$ such that any non-trivial direct sum decomposition of $X$ is finite and bounded by $m_X$. Thus, Krull-Schmidt categories as well as Hom-finite categories have the finite direct sum decomposition property.

**Lemma 3.1.** Assume $A$ is any additive category over some commutative ring with finite direct sum decomposition property. Then for any morphism $f : X \rightarrow Y$ in $A$ there are direct sum decompositions $X = X' \oplus X''$ and $Y = Y_1 \oplus Y_2$ together with a corresponding commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & X' \oplus X'' \\
\downarrow & & \downarrow \\
\oplus & & \downarrow \\
Y' & \xrightarrow{f} & Y
\end{array}
$$

such that $f' : X' \rightarrow Y$ is right minimal and $f_1 : X \rightarrow Y_1$ is left minimal.

**Proof.** Let $f : X \rightarrow Y$ be any morphism in $A$, by assumption on $A$ we let $m_X$ be the maximal length of non-trivial direct sum decompositions of $X$. Now if $f$ is not right minimal, then there must exist a direct sum decomposition $X = X' \oplus X''$ relatively to which $f = [f' \ 0] : X' \oplus X'' \rightarrow Y$. But we must have $m_X < m_X$, thus by induction on $m_X$, there is a decomposition as above in which $f'$ is right minimal. With a similar argument, $f$ is the direct sum of a left minimal map and a zero map as required.

In the next proposition whose proof is not difficult and is omitted for the sake of brevity, the equivalence of (1) and (4) shows that for Krull-Schmidt categories Definition 3.1 is equivalent to the usual stronger version used in module categories.

**Proposition 3.2.** Let $f : X \rightarrow Y$ be a morphism in a Krull-Schmidt category $A$.

(a) The following assertions are equivalent.

1. $f$ is right minimal.
2. Any section $s : X' \rightarrow X$ such that $fs = 0$ must be zero; and for any section $s : X' \rightarrow X$ the map $fs$ is right minimal as well as any direct summand of $f$.
3. Any morphism $h : Z \rightarrow X$ such that $fh = 0$ must be a radical map.
4. Any endomorphism $h : X \rightarrow X$ such that $fh = h$ is an automorphism.

(b) The following assertions are equivalent.

1. $f$ is left minimal.
2. Any retraction $r : Y \rightarrow Y''$ such that $rf = 0$ must be zero; and for any retraction $r : Y \rightarrow Y'$ the map $rf$ is left minimal as well as any direct summand of $f$.
3. Any morphism $h : Y \rightarrow Z$ such that $hf = 0$ must be a radical map.
4. Any endomorphism $h : Y \rightarrow Y$ such that $hf = h$ is an automorphism.

Let $M$ be any full subcategory of a category $A$. An object in $A$. A map $f : E \rightarrow X$ is called a right $M$-approximation if $E \in M$ and for any object $Z$ in $M$ the sequence $A(Z, E) \rightarrow A(Z, X)$ is exact. A right $M$-approximation $f$ is called minimal if additionally $f$ is right minimal. The dual notion of (minimal) left $M$-approximation may also be defined. A sequence $(\xi) : X \rightarrow Y$ is called pseudo-exact if $f$ is a pseudo-kernel of $g$ and $g$ is a pseudo-cokernel of $f$, or equivalently the sequences

$$
A(M, X) \rightarrow A(M, Z) \rightarrow A(M, Y), \ A(Y, M) \rightarrow A(Z, M) \rightarrow A(X, M),
$$
are exact in $\text{Ab}$ for every object $M$ in $\mathcal{A}$. We said that $(\xi)$ is a minimal sequence if it is a pseudo-exact sequence with $f$ left minimal and $g$ right minimal; similarly, $(\xi)$ is called a minimal $\mathcal{M}$-approximation sequence if additionally $f$ is a minimal left $\mathcal{M}$-approximation and $g$ is a minimal right $\mathcal{M}$-approximation. In the same way, if $\mathcal{A}$ is a triangulated category then a minimal triangle (or a minimal $\mathcal{M}$-approximation triangle) is any triangle $(\xi) : X \xrightarrow{f} Z \xrightarrow{g} Y \xrightarrow{e} X[1]$ such that the subsequence $X \xrightarrow{f} Z \xrightarrow{g} Y$ is minimal (or is a minimal $\mathcal{M}$-approximation sequence). We need the following preliminary result whose proof follows directly from five-lemma in abelian categories together with the use of axioms Tr1, Tr2 and Tr3 of triangulated categories [17].

**Lemma 3.3.** The following assertions hold in any triangulated category $\mathcal{A}$.

(a) A triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ splits if and only if $w = 0$, if and only if $v$ is a retraction, if and only if $u$ is a section.

(b) The translation functor (the shift) preserves arbitrary products and coproducts (direct sums). The direct sum as well as the product of any family of triangles are still triangles.

(c) Suppose the diagram below is a morphism of triangles and $H$ is an homological (or a co-homological) functor from $\mathcal{A}$ to $\text{Ab}$.

\[
\begin{array}{cccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} Z \xrightarrow{w} X[1] \\
\downarrow f & & \downarrow g & \downarrow h \\
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} Z' \xrightarrow{w'} X'[1] \\
\end{array}
\]

For each $n \in \mathbb{Z}$, if $H^n f$ and $H^n g$ are isomorphisms then so is $H^n h$. In particular if $Hf$ and $Hg$ are isomorphisms then so is $Hh$.

Consequently, if two of the three morphisms $f$, $g$ and $h$ are isomorphisms then the third one is also an isomorphism.

Now the following lemma shows that from any given triangle one may construct a minimal one; similarly from any pseudo-exact sequence one may construct a minimal one.

**Lemma 3.4.** Suppose that $\mathcal{A}$ is either a triangulated category with finite direct sum decomposition property or any Krull-Schmidt category.

(a) Assume $(\xi) : X \xrightarrow{f} E \xrightarrow{g} Y \xrightarrow{e} X[1]$ is a triangle in $\mathcal{A}$.

(1) Then $(\xi)$ admit two direct sum decompositions into two triangles given by a commutative diagram as shown below, and such that in the triangle $X \xrightarrow{f_1} E_1 \oplus Y_1 \xrightarrow{e_1} X[1]$ the morphism $f_1$ is left minimal, while in the triangle $X' \xrightarrow{f'} E' \xrightarrow{g'} Y \xrightarrow{e'} X'[1]$ the morphism $g'$ is right minimal.

\[
\begin{array}{cccc}
X & \xrightarrow{f_1} & E_1 \oplus Y_2 & \xrightarrow{g_1} Y_1 \oplus Y_2 \xrightarrow{e_1} X[1] \\
\downarrow f & & \downarrow g & \downarrow e \\
X & \xrightarrow{f'} & E' & \xrightarrow{g'} Y \xrightarrow{e'} X'[1] \\
\end{array}
\]

(b) Moreover $(\xi) = (\xi_{\min}) \oplus (\xi_0)$ where $(\xi_{\min}) : X_1 \xrightarrow{f_1} E_1 \xrightarrow{g_1} Y_1 \xrightarrow{e_1} X_1[1]$ is a minimal triangle and $(\xi_0) : \xrightarrow{[\overline{1}]} X_2 \oplus Y_2 \xrightarrow{[1]} Y_2 \xrightarrow{e_1} X_2[1]$ is a split triangle.
(a') In case $A$ is not triangulated, replacing in (a) the term "triangle" by "pseudo-exact sequence" of the form $L \longrightarrow M \longrightarrow N$, statements (1) and (2) still hold.

Proof. For (1), in order to construct the first row in the diagram above, using Lemma 3.1 we may identify $f$ with a map $f = \begin{bmatrix} f_1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}: X \longrightarrow E_1 \oplus Y_2$ such that $f_1$ is left minimal. Now let $X \overset{f_1}{\longrightarrow} E_1 \overset{g_1}{\longrightarrow} Y_1 \overset{\epsilon_1}{\longrightarrow} X[1]$ be the unique triangle with base $f_1$, and consider the trivial triangle $0 \longrightarrow Y_2 \longrightarrow Y_2 \longrightarrow 0$. By Lemma 3.3 the direct sum of the two previous triangles is again a triangle, and since there is up to isomorphism only one triangle with a given base map, we have that the triangle $(\xi) : X \overset{f}{\longrightarrow} E_1 \overset{g}{\longrightarrow} Y \overset{\epsilon}{\longrightarrow} X[1]$ is isomorphic to the direct sum

$$X \overset{f_1}{\longrightarrow} E_1 \oplus Y_2 \overset{\begin{bmatrix} g_1 & 0 \\ 0 & \epsilon_1 \end{bmatrix}}{\longrightarrow} Y_1 \oplus Y_2 \overset{\epsilon_1}{\longrightarrow} X[1].$$

Similarly, the second row in the diagram of Lemma 3.4 is constructed by decomposing $g$ into a direct sum of a right minimal map and a zero map.

Statement (2) is a direct consequence of (1) and of the fact that any direct summand of a left (or a right) minimal map is still left minimal (or respectively right minimal): As above $(\xi)$ is isomorphic to a triangle

$$(\xi_1) : X \overset{f_1}{\longrightarrow} E_1 \oplus Y_2 \overset{\begin{bmatrix} g_1 & 0 \\ 0 & \epsilon_1 \end{bmatrix}}{\longrightarrow} Y_1 \oplus Y_2 \overset{\epsilon_1}{\longrightarrow} X[1]$$

where $f_1'$ is left minimal. Next writing $g_1'$ as a direct sum of right minimal map and a zero map, the last triangle is isomorphic to a triangle of the form

$$X_1 \oplus X_2 \overset{f_1}{\longrightarrow} E_1 \oplus X_2 \overset{g_1}{\longrightarrow} Y_1 \oplus X_2 \overset{\epsilon_1}{\longrightarrow} X[1]$$

in which $(\xi_{\text{min}}) : X_1 \overset{f_1}{\longrightarrow} E_1 \overset{g_1}{\longrightarrow} Y_1 \overset{\epsilon_1}{\longrightarrow} X[1]$ is the desired minimal triangle. \qed

For any object $M$ in an exact or a triangulated category $A$, let $M^{\perp}$ denote the full subcategory of $A$ with objects $Z$ such that $\text{Ext}_A^1(M, Z) = 0$; similarly $^{\perp}M$ is the full subcategory of $A$ with objects $Z$ such that $\text{Ext}_A^1(Z, M) = 0$.

**Corollary 3.5.** Let $(\xi) : X \overset{f}{\longrightarrow} E \overset{g}{\longrightarrow} Y$ be a non-split short exact sequence or a non-split triangle in some Krull-Schmidt exact or triangulated category $A$.

1. If $X$ is indecomposable then $g$ is right minimal. If $Y$ is indecomposable then $f$ is left minimal.

2. If $X$ and $Y$ are indecomposable with $E \subseteq X \cap Y^{\perp}$, then $\langle \xi \rangle$ is the unique minimal $M$-approximation sequence or triangle for every subcategory $M \subseteq X \cap Y^{\perp}$ containing $E$.

**Definition 3.2.** We call a pseudo-exact sequence (or a triangle) $(\xi) : M^+ \overset{f}{\longrightarrow} B \overset{g}{\longrightarrow} M$ a connecting $\tau$-sequence for a pair $(T, T')$ of full subcategories of $A$ if $(\xi)$ satisfies the following conditions:

(i) $M \in T$, $M' \in T'$, $B \in T \cap T'$, $f, g \in I_C$, and the two following sequences are exact.

$$\text{Hom}_T(B, \cdot) \longrightarrow J_T(M', \cdot) \longrightarrow 0, \text{Hom}_T(\cdot, B) \longrightarrow J_{T'}(\cdot, M) \longrightarrow 0$$

(ii) Moreover, $f$ is left minimal while $g$ is right minimal.

A connecting $\tau$-sequence (lying entirely in $T$) turns out to be a $\tau$-sequence in $T$.

Note that for any connecting $\tau$-sequence $(\xi)$ as above, if moreover $M'$ and $M$ are indecomposable, then $f$ is left minimal almost split in $T'$ while $g$ is right minimal almost split in $T$. Observe that strict $\tau$-sequences (or almost split sequences/triangles) in some category are examples of special connecting $\tau$-sequences. $\tau$-categories are introduced by Iyama in [18, §2], they are also studied in [19, §1] with an approach using the concepts of weak (co)kernels (which we have called here as pseudo-(co)kernels).
Lemma 3.6. Let \( T \) and \( T' \) be two full subcategories of a Krull-Schmidt (additive or triangulated) category \( \mathcal{C} \), and let \( \langle \xi \rangle : M' \xrightarrow{f} B \xrightarrow{g} M \rightarrow \) and \( \langle \zeta \rangle : N' \xrightarrow{s} E \xrightarrow{t} N \rightarrow \) be either minimal \( T \cap T' \)-approximations or connecting \( \tau \)-sequences for the pair \( (T, T') \), with \( M, M' \) indecomposable, and with \( B \neq 0 \) in case \( \langle \xi \rangle \) and \( \langle \zeta \rangle \) are not triangles in \( \mathcal{C} \).

(1) The following statements hold.

(i) Any morphism \( a \in \text{Hom}_C(M, N) \) extends to a morphism \( (a', b, a) \in \text{Hom}(\langle \xi \rangle, \langle \zeta \rangle) \) of sequences; moreover, \( a \in \text{Hom}_C(M, N) \) is an isomorphism if and only if it is the case for any extension \( (a', b, a) \in \text{Hom}(\langle \xi \rangle, \langle \zeta \rangle) \) of \( a \).

(ii) Any morphism \( \alpha' \in \text{Hom}_C(M', N') \) extends to a morphism \( (\alpha', \beta, \alpha) \in \text{Hom}(\langle \xi \rangle, \langle \zeta \rangle) \) of sequences; moreover, \( \alpha' \in \text{Hom}_C(M, N) \) is an isomorphism if and only if it is the case for any extension \( (\alpha', \beta, \alpha) \in \text{Hom}(\langle \xi \rangle, \langle \zeta \rangle) \) of \( \alpha' \).

In particular, \( M \cong N \) if and only if \( \langle \xi \rangle \) is isomorphic to \( \langle \zeta \rangle \), if and only if \( M' \cong N' \).

(2) The correspondence given by \( \phi(a) = a' \) for every endomorphisms \( a \in \text{End}_C(M) \) and \( a' \in \text{End}_C(M') \) which extend to an endomorphism of the sequence (or the triangle) \( \langle \xi \rangle \) yields a natural isomorphism \( \text{km}_{M'} \xrightarrow{\sim} \text{km}_M \) of division \( k \)-algebras.

Proof. First observe that when \( \langle \xi \rangle \) and \( \langle \zeta \rangle \) are triangles with zero middle terms, we must have that \( M \cong M'[1] \cong M' \) and \( N \cong N'[1] \cong N' \), so that the result trivially holds. Thus it only remains to prove the lemma when \( \langle \xi \rangle \) and \( \langle \zeta \rangle \) are both pseudo-exact sequences with non-zero middle terms; this implies that \( f, g, s, t \) are all non-zero maps.

We only need to give the proof for (i) since (ii) is the dual of (i). Let’s introduce the following notation for any map \( X \xrightarrow{h} Y \) in \( \mathcal{C} \): \( \text{Hom}^0_C(X, Z) = \text{Hom}_C(Y, Z) h = \{wh : \text{with } Y \xrightarrow{w} Z\} \). Observe that if \( h \in \mathcal{J}_C(X, Y) \) then \( \text{Hom}^0_C(X, Z) \subseteq \mathcal{J}_C(X, Z) \).

By assumption both maps \( g \) and \( t \) are either minimal right \( T \cap T' \)-approximations or right minimal almost split in \( T \), consequently any element in \( \text{Hom}^0_C(B, N) \) factors through the map \( E \xrightarrow{g} N \) and any element in \( \text{Hom}^0_C(E, M) \) factors through the map \( B \xrightarrow{t} M \). Thus also using the fact that the sequences \( \langle \xi \rangle \) and \( \langle \zeta \rangle \) are pseudo-exact, it comes that each morphism \( a \in \text{Hom}_C(M, N) \) must extend to a morphism of sequences \( (a', b, a) \in \text{Hom}(\langle \xi \rangle, \langle \zeta \rangle) \) as shown in the following commutative diagram:

\[
\begin{array}{ccc}
\langle \xi \rangle : & M' \xrightarrow{f} & B \xrightarrow{g} M \\
\langle \zeta \rangle : & N' \xrightarrow{s} & E \xrightarrow{t} N \\
\end{array}
\]

Next we want to prove that in any commutative diagram as above, \( a \in \text{Hom}_C(M, N) \) is an isomorphism if and only if it is the case for \( (a', b, a) \in \text{Hom}(\langle \xi \rangle, \langle \zeta \rangle) \), and for this we only need to show the non-trivial implication: \( a \in \text{Isom}(M, N) \) implies \( (a', b, a) \in \text{Isom}(\langle \xi \rangle, \langle \zeta \rangle) \). Thus assume \( a \in \text{Isom}(M, N) \), then by above "right factorization property" the map \( a^{-1} t \in \text{Hom}^0_C(E, M) \) must factor through \( g \), so that there is a map \( c \in \text{Hom}_C(E, B) \) with \( a^{-1} t = gc \). But then, \( (gc)b = g \) and \( t(bc) = t \), showing by minimality of non-zero maps \( g \) and \( t \) that \( bc \) and \( cb \) are automorphisms, so that \( b \) is both a section and a retraction, thus \( b \) is an isomorphism. Next, using the pseudo-exactness of \( \langle \xi \rangle \) and \( \langle \zeta \rangle \), the morphism \( (b^{-1}, a^{-1}) \in \text{Hom}(t, g) \) induces another map \( a'' \in \text{Hom}_C(N', M') \) such that \( b^{-1}s = fa'' \). This yields that \( s(1 - a'a'') = 0 \) and \( f(1 - a''a') = 0 \). (Remember that for each element \( u \) in a local algebra \( A \), \( u \) or \( 1 - u \) is invertible). Hence since \( \text{End}_C(M') \) and \( \text{End}_C(N') \) are local while \( f \) and \( s \) are non-zero by assumption, it comes that the
endomorphisms $1_{\mathcal{M}} - a\, a''$ and $1_{\mathcal{M}} - a''\, a'$ cannot be invertible, thus $a\, a''$ and $a''\, a'$ are both automorphisms so that $a'$ in above diagram is also an isomorphism. Therefore $(a', b, a)$ is an isomorphism between $(\xi)$ and $(\zeta)$; so that (i) holds as expected.

(2). Let’s show that $(\xi)$ induces a well-defined epimorphism $\text{End}_k(M) \longrightarrow k_{\mathcal{M}} = \text{End}_k(M')/J_{\text{End}_k(M')}$ taking each $a \in \text{End}_k(M)$ to the coset $\bar{a} \in \text{End}_k(M')$ such that there is an extension $(a', b, a) \in \text{End}_k(M')$. For this let $(a', b, a)$ and $(a'', e, a)$ be any extensions of an endomorphism $a \in \text{End}_k(M)$, thus $g = a = g\, e = a'' \neq f = b\, f$ and $f = f = c\, f$. We get $g(b - c) = 0$ while $f(c - a'') = (b - c)\, f$, showing that $(a' - a'') = (b - c)\, f$, so that $(\xi) - (\zeta)$ extends the zero endomorphism of $M$. Thus $J_{\text{End}_k(M)}$. Hence we have a well-defined morphism $\text{End}_k(M) \longrightarrow k_{\mathcal{M}}$: $a \longmapsto \bar{a}$, which moreover is surjective since by (ii) each endomorphism of $M'$ also extends to an endomorphism of the sequence $(\xi)$. Now observe that the kernel of the previous epimorphism is equal to $J_{\text{End}_k(M)}$: indeed by (i), we know that $a$ and $a''$ is not an automorphism, so that $a - a'' \in J_{\text{End}_k(M)}$. Thus the induced map $k_{\mathcal{M}} \cong k_{\mathcal{M}}$: $a \longmapsto \bar{a}$ is the natural isomorphism expected. This ends the proof of the lemma. 

One important concept we want to introduce is that of dualizing pair of bimodules, for two division $k$-algebras $E$ and $F$, $E$-bimodule denotes the category of finite dimensional $E$-$F$-bimodules.

**Definition 3.3.** Let $E$ and $F$ be two division $k$-algebras.

(i) A trace map on a $k$-algebra $E$ is any $k$-linear map $t: E \longrightarrow k$ such that $t(a\, e) = t(e\, a)$ for all $a, e \in E$; it is called non-degenerated if its radical $R_t := \{e \in E : \forall a \in E, t(a\, e) = 0\}$ is zero.

(ii) Let $B$ be any $E$-$F$-bimodule. Then the left dual, the $k$-dual and the right dual bimodule of $B$ are given respectively by $B' = \text{Hom}_k(E, B'E)$, $B^r = \text{Hom}_k(BF, F)$, and $B'' = \text{Hom}_k(FE, E)$, with the actions given as follows: for $e \in E, a \in F, u \in B'$, $\xi \in \text{Hom}_k(E, B')$, and $v \in B''$, with $u \in B'$, we have (a. $e$)($x) = u(x\, a\, e)$ and $e(v\, x) = v\, e(x)$ for every $x \in B$.

A $B$ is called dualizing (or is said to have a dual) if the left dual and the right dual of $B$ are isomorphic.

(iii) A (symmetrizable) dualizing pair of bimodules is the given of a data $\{E, F, B, B'; b, b'\}$ consisting of two dualizing bimodules and two non-degenerated bilinear forms $B_{\mathcal{E}}B' \longrightarrow B \longrightarrow F$, which are symmetrizable over $k$ via two non-degenerated trace maps $t: E \longrightarrow k$ and $t': F \longrightarrow k$ in the sense that $t(b(u \otimes y)) = t'(y(u \otimes x))$ for every $x \in B$ and $y \in B'$. The two bimodules $B \equiv B'$ are therefore called mutually dual and we write $B = B^\ast$ and $B = B^\ast$.

Sometimes a dualizing pair of bimodule $\{B, B', b, b'\}$ is simply denoted by a pair $\{B, B^\ast\}$ in which the associated (symmetrizable) non-degenerated bilinear forms are omitted.

Note that we can form products of symmetrizable dualizing pairs of bimodules to obtain symmetrizable dualizing pairs of bimodules. Indeed, let $k_1$, $k_2$ and $k_3$ be division $k$-algebras and let $\{B_2, 1 \otimes B_1; i_{B_1} \otimes B_1\}$ and $\{B_3, 2B_1^\ast; 1B_1 \otimes B_1\}$ be two symmetrizable dualizing pairs of bimodules with $i_{B_1} \in \text{bimod}_k$ and with $i_{B_3} \otimes B_1, B_3^\ast \longrightarrow \text{bimod}_k$, for $(i, j) = (1, 2), (2, 3)$. Then their product is the symmetrizable dualizing pair of bimodules $\{B_1 \otimes B_2, B_3; i_{B_1} \otimes i_{B_2}, i_{B_3} \otimes B_1\}$ in which the associated bilinear forms are canonically induced. For all $x \in B_1$, $y \in B_2$, $u \in B_3$, and $v \in B_3^\ast$ we have:

$$i_{B_3}((x \otimes y) \otimes (v \otimes u)) = i_{B_1} i_{B_2}((y \otimes v) \otimes u)$$

We point out that the symmetrizable condition we are requiring for our dualizing pairs of bimodules will not be explicitly used here, but such a condition enhance our study of $2$-Calabi-Yau tilted algebras over $k$-algebraically closed fields (Appendix A.1.2). However, since the so-called trace maps play a crucial role for the existence of non simply-laced version of cluster structure here, we keep such a natural condition on bimodules which is justified by the following technical lemma which we can derive from the study of division $k$-algebras (or skew fields).
**Lemma 3.7.** Let \( B \in \text{bimod}_k \) for any two division \( k \)-algebras \( E \) and \( F \). Then there always exist trace maps \( t \in \text{Hom}_k(E,k) \) and \( t' \in \text{Hom}_k(F,k) \) such that \( B \) induces a symmetricizable dualizing pair of bimodules \( \{ B, B^* \} \) in which the associated non-degenerated bilinear forms are natural (functorial); some typical choices for the dual \( B^* \) are given by \( \text{Hom}_k(B,k) \), \( B \) and \( B^\ast \).

**Proof.** By Lemma 5.1 in section 5, we can always construct two non-degenerated trace forms \( t \in \text{Hom}_k(E,k) \) and \( t' \in \text{Hom}_k(F,k) \).

**Claim.** The traces \( t \) and \( t' \) yield natural isomorphisms of bimodules: \( B \xrightarrow{\text{tr}_t} \text{Hom}_k(B,k) \) and \( B^* \xrightarrow{t'} \text{Hom}_k(B,k) \).

To prove our claim, take \( u \in B, a \in E, b \in F \), then for any \( x \in B \) we have: 
\[
\begin{align*}
[t \circ (b \cdot u)](x) &= t((b \cdot u)(a)) = t(u(xb)a) = t\left(u(axb)\right) = [t \circ u](a)(x) = t(b \cdot (t \circ u)a)(x).
\end{align*}
\]
Thus \( t \circ (b \cdot u) = b \cdot (t \circ u)a \), so that \( t \circ - \) is a morphism of bimodules. But then, this morphism is clearly injective: in fact, since \( t \) is non-degenerated, its radical \( R_t := \{ e \in E : \forall a \in E, t(au) = 0 \} \) is zero. Thus if \( t \circ u = 0 \) for some \( u \in B \), then for all \( a \in E \) and every \( x \in B \) we have \( t(a \cdot u)(x) = t(u)(ax) = 0 \) showing that \( u \) is in the radical of \( t \), hence \( u \) is zero. Next, using the fact that \( B \) and \( \text{Hom}_k(B,k) \) are finite-dimensional with the same dimension over \( k \), we conclude that \( t \circ - \) is an isomorphism as claimed. Similarly, \( t' \circ - \) is also an isomorphism. Finally, the naturality of the previous isomorphisms is induced by the associativity of the composition of morphisms as one can easily check. Thus our claim holds.

Now let’s show that for any bimodule \( B^* \) isomorphic for example to the \( k \)-dual \( \text{Hom}_k(B,k) \), we have a dualizing pair of bimodules \( \{ B, B^*, b, b' \} \) with \( b, b' \) symmetricizable over \( k \) via \( t \) and \( t' \). For this, take any isomorphism \( \phi : B^* \longrightarrow \text{Hom}_k(B,k) \) then we consider the two non-degenerated bilinear forms
\[
\begin{align*}
B \otimes_B B^* \xrightarrow{b=(\cdot \otimes (t \circ -)^{-1}(\phi))} E \text{ and } B^* \otimes_B B \xrightarrow{b'=(\otimes t \circ - \phi(\cdot) \otimes -)} F,
\end{align*}
\]
such that on each pair \( x \in B, u \in B^* \), letting \( t \circ u_1 = \phi(u) = t' \circ u_2 \) with \( u_1 \in B \) and \( u_2 \in B^* \), we have: 
\[
\begin{align*}
(b(x \otimes u))(u_1) &= \phi(u)(a) = (x \otimes \phi(u))(u_1) := u_1(x) \text{ and } b(u \otimes x) = \phi(u)(x) := u_2(x) \text{, it is clear that } b \text{ and } b' \text{ are symmetricizable over } k; \text{ indeed, with previous notation we have: } t'(b(x \otimes u)) = t' \circ u_1 = \phi(u)(a) (x) = t' \circ u_2 = t' \circ u_2 = t' \circ (u_2(x)).
\end{align*}
\]
This completes the proof of Lemma 3.7. \( \square \)

Let \( T \) be a full subcategory of a Krull-Schmidt category \( \mathcal{C} \), such that the bifunctor \( J_T \) is finite dimensional over \( k \).

**Definition 3.4.** To \( T \) we can associate a modulated quiver \( Q_T = (Q_T, \mathcal{M}) \) described as follows: the underlying valued quiver of \( Q_T \) is given by the valued quiver \( Q_T \) of \( T \), whose set of points is \( (Q_T)_0 = (Q_T)_0 = \text{ind } T \), while the arrows and the modulations \( \mathcal{M} \) are given as follows:

**\( \bowtie \) the division \( k \)-algebra attached to point \( M \) in \( Q_T \) is the opposite algebra \( k_M^\diamond = \text{End}_M(M)[M](M, M) \).

By fixing a family of non-zero traces \( t_M \in \text{Hom}_k(k_M, k) \) with \( M \in Q_T \), the modulations \( \mathcal{M} \) may be taken to be symmetricizable.

**\( \bowtie \) For each pair of points \( X, Y \in Q \), the valued arrow \( \alpha_{XY} \in (Q_T)_1(X, Y) \) and the dualizing pair of bimodules attached to \( \alpha_{XY} \) are given by \( X \downarrow_{X BY} Y \) where \( \text{X BY} = \text{Irr}_Y(X,Y) = J_T(X,Y)[M]^2(X,Y) \) is the \( k_X \cdot k_Y \)-bimodule of irreducible maps in \( T \) while \( (X BY)^* \) may be chosen to be the \( k \)-dual \( \text{Hom}_k(X, BY) \).

We now prove the following important connection between dualizing pairs of bimodules induced by connecting \( \tau \)-sequences.

**Theorem 3.8.** Let \( T, T' \) be \( \text{Hom} \)-finite full subcategories of a Krull-Schmidt category \( \mathcal{C} \), and let \( (\xi) : M' \xrightarrow{\xi} B \xrightarrow{\xi^*} M \) a connecting \( \tau \)-sequence for the pair \( (T, T') \), with \( M, M' \) indecomposable, and with \( B \neq 0 \) in case \( (\xi) \) is not a triangle. Then there is a natural identification \( k_M \cong k_M' \) of division \( k \)-algebras induced by the endomorphisms of \( \xi \) such that, for any indecomposable object \( X \) in \( T \cap T' \),

\( (\xi) \) naturally induces a non-degenerated bilinear form \( \text{Irr}_{T'}(M', X) \otimes_{k_M} \text{Irr}_{T}(X, M) \xrightarrow{b} \text{Irr}_{T}(M', X) \) form a dualizing pair of bimodules.
Proof. Observe that in case \((\xi)\) is a triangle with \(B = 0\), we must have \(M \cong M'[1]\) so that the result trivially holds. Thus it only remains to prove the theorem when \((\xi)\) is a pseudo-exact sequence with \(B \neq 0\), this implies that \(f\) and \(g\) are non-zero.

By Lemma 3.6-(2), we have a natural isomorphism of division \(k\)-algebras

\[
\phi : k_M := \text{End}_C(M) \otimes_{\text{End}_C(M)} k_M' := \text{End}_C(M') \otimes_{\text{End}_C(M)} k_M', \quad \text{with } \phi(a) = \tilde{a}' \text{ for every endomorphisms } a \in \text{End}_C(M) \text{ and } a' \in \text{End}_C(M') \text{ which extend to an endomorphism of the sequence } (\xi).
\]

Let \(X \in T \cap T'\) be any indecomposable object and write \(1_B_2 = \text{Irr}_T(X, M)\) and \(2B'_1 = \text{Irr}_T(M', X)\). Under the identification of \(k_M'\) with \(k_M\) using the natural isomorphism \(\phi\), the \(k_X\)-\(k_M'\)-bimodule \(2B'_1\) is naturally a \(k_X\)-\(k_M\)-bimodule. Since by Lemma 3.7 any bimodule \(B \in \text{bimod}_F\) (for two division \(k\)-algebras \(E\) and \(F\)) always induces a dualizing pair of bimodules, to show that the sequence \((\xi)\) yields a dualizing pair of bimodules \(\{2B'_1, 2B_1\}\) we only have to check that \((\xi)\) yields as claimed a non-degenerated bilinear form \(2B'_1 \otimes_{k_M} 2B_1 \rightarrow k_X\) (since in this case, \(2B'_1\) will be isomorphic to the right dual bimodule \(2B_1^\ast\)). For any two radical maps \(u' \in \text{Irr}_T(M', X)\) and \(u \in \text{Irr}_T(X, M)\) and for every endomorphism \(a \in \text{End}_C(M)\), the factorization property of the maps \(f\) and \(g\) (and Lemma 3.6) induces two maps \(\tilde{u}'\) and \(\tilde{u}\) and an endomorphism \((a', b, a)\) of the sequence \((\xi)\) together with the two following commutative diagrams.

\[(\xi) : \begin{array}{ccc} M' & \xrightarrow{f} & B \\ \downarrow u & & \downarrow g \\ X & \xrightarrow{u'} & M \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \xrightarrow{\tilde{u}} & u \\ \downarrow u' & & \downarrow \tilde{u}' \\ X & \xrightarrow{\tilde{u}' \circ T} & M' \end{array} \quad \text{(D1)} \]

**Fact 3.1.**

1. With above notations, the morphism \(u\) is in \(J^2_T(X, M)\) if and only if the morphism \(\tilde{u}\) is in \(J^2_T(X, B)\); thus as \(k_M\)-\(k_X\)-bimodules \(T(X, B) \otimes_{\text{End}_C(M)} k_{X'}(X, M) = 1_{B_1}\). Similarly, the morphism \(u'\) is in \(J^2_T(M', X)\) if and only if the morphism \(\tilde{u}'\) is in \(J^2_T(B', X)\), thus as \(k_X\)-\(k_M'\)-bimodules \(T'(B, X) \otimes_{\text{End}_C(M')} k_{M'}(M', X) = 1_{B'_1}\).

2. In particular, \(g\) and \(f\) are irreducible maps, respectively in \(T\) and in \(T'\), and the right \(k_M'\)-module structure of \(T'(B, X) \otimes_{\text{End}_C(M')} k_{M'}(M', X)\) and the left \(k_M\)-module structure of \(T(X, B) \otimes_{\text{End}_C(M)} k_{X'}(X, M)\) are given as follows: let \(\tilde{u}' \in T'(B, X)\), \(\tilde{u} \in T(X, B)\), \(a \in \text{End}_C(M)\) and \((a', b, a)\) any extension of \(a\) to an endomorphism of \((\xi)\) as in (D1), then \(\tilde{a}' \tilde{u} := \tilde{a}' \phi(\tilde{u}) := \tilde{a}' \circ \tilde{u}\) and \(\tilde{a} \tilde{u} := b \circ \tilde{u}\).

For the non-trivial implications of Fact 3.1, let’s assume that \(u \in J^2_T(X, M)\) and write \(u = w \circ v\) for some radical maps \(v \in J_T(X, Z)\) and \(w \in J_T(Z, M)\). Then since by assumption \((\xi)\) is a connecting \(r\)-sequence, the right-factorization property of \(g\) shows that \(w = g\tilde{w}\) for some \(\tilde{w} \in T(Z, B)\), we therefore get: \(g \circ \tilde{u} = u = gw \circ \tilde{v}\), so that \(g \circ (\tilde{u} - \tilde{w} \circ v) = 0\) and by the right minimality of \(g\) it comes that \(\tilde{u} - \tilde{w} \circ v\) is a radical map, so that \(\tilde{u}\) is a radical map as well. In the same way, if \(u' \in J^2_T(M', X)\) then the map \(\tilde{u}'\) is in \(J^2_T(B', X)\). Hence, the composition by the morphism \(g\) yields an isomorphism of bimodules \(T(X, B) \otimes_{\text{End}_C(M)} k_{X'}(X, M) \cong 1_{B_1}\), and the composition by the morphism \(f\) yields an isomorphism of bimodules \(T'(B, X) \otimes_{\text{End}_C(M')} k_{M'}(M', X) \cong 1_{B'_1}\), moreover these isomorphisms clearly show that the right \(k_M\)-module structure of \(T'(B, X) \otimes_{\text{End}_C(M')} k_{M'}(M', X)\) and the left \(k_M\)-module structure of \(T(X, B) \otimes_{\text{End}_C(M)} k_{X'}(X, M)\) are given as in the second part of Fact 3.1. Thus Fact 3.1 is clear.

From Fact 3.1, it is now clear that the composition of maps yields a bilinear form (which is also \(k_X\)-linear)

\[
2B_1' \otimes_{k_M} 2B_1 \cong T'(B, X) \otimes_{\text{End}_C(M')} k_{M'}(M', X) \xrightarrow{b} k_X, \quad \text{with } b(\tilde{w}' \otimes \tilde{u}) = \tilde{w}' \tilde{u}.
\]
Finally we show that b is also non-degenerated. For this, assume \( \bar{u} \neq 0 \) for some \( \bar{u} \in T(X, B) \), thus since \( X \) in indecomposable \( \bar{u} \) must be a section. Hence choosing some left inverse \( B \xrightarrow{g} X \) for \( \bar{u} \), it comes that \( h(\bar{v} \otimes \bar{u}) = 1 \). Similarly for any non-zero \( u \) in \( T'(B, X) \), for some \( \bar{u}' \in T'(B, X) \), the map \( \bar{u}' \) must be a retraction, so that choosing a right inverse \( \bar{v} \in T(B, X) \) for \( \bar{u}' \) we get that \( b(\bar{u} \otimes \bar{v}) = 1 \). Therefore, the bilinear form \( b \) is non-degenerated and this completes the proof of the theorem.

Now let \( T \) be an additive full subcategory of an exact or a triangulated category \( C \) such that \( T \) is rigid \((\Ext^1_C(T, T) = 0)\) and the functor \( J_T \) is finite dimensional over \( k \), let \( M \in \text{ind} T \) and \( T[M] \) be the full subcategory of \( T \) generated by all objects in \( T \) which are not isomorphic to \( M \), assume there exists another indecomposable object \( M' \) not in \( T \) such that the subcategory given by \( T' = (T[M]) \oplus \text{add}(M') \) is also rigid. Then an exchange sequence \((\text{short exact sequence or triangle})\) between \( T \) and \( T' \) is any minimal \((T[M])-\text{approximation sequence} \ M \xrightarrow{f} B \xrightarrow{f} M \). Thus each exchange sequence \((\xi)\) (with \( B \neq 0 \) in case \((\xi) \) is not a triangle) must be unique up to isomorphism. We say that \( T \) has no loop at some point \( X \in \text{ind} T \) if its modulated quiver has no loop at \( X \). Observe that we have another example of connecting \( \tau \)-sequences given by exchange sequences between \( T \) and \( T' \) when \( T \) and \( T' \) have no loops. We need the following technical lemma.

**Lemma 3.9.** Suppose \( C \) is a Krull-Schmidt exact or triangulated category and assume that there is an exchange sequence \((\xi_{ex}) : M^* \xrightarrow{f} B \xrightarrow{f} M \) between two rigid full subcategories \( T \) and \( T' \) \((T = (T[M]) \oplus \text{add}(M'))\). Then \( T \) has no loop at point \( M \) and only if \( \Ext^1_C(M, M^* \cong k_M \cong k_{M'}, \) and if and only if \( T' \) has no loop at point \( M^* \). Moreover, if this is the case then any non-split short exact sequence \((\text{or non-split triangle})\) \((\xi) : M^* \xrightarrow{f} B \xrightarrow{f} M \) in \( C \) must be up to isomorphism the unique exchange sequence \((\xi_{ex})\).

**Proof.** By assumption, \( \Ext^1_C(T, T) = 0 \) and \( \Ext^1_C(T', T') = 0 \) so that \( \Ext^1_C(M, B) = 0 = \Ext^1_C(B, M^*) \), thus applying the two functors \( C(-, M) \) and \( C(-, M^*) \) to the exchange sequence \((\xi_{ex}) : M^* \xrightarrow{f} B \xrightarrow{f} M \), we get the following two exact sequences:

\[
\begin{align*}
C(M, M^*) & \xrightarrow{h} C(M, B) \xrightarrow{g} \Ext^1_C(M, M^*) \xrightarrow{0} \\
C(M, M^*) & \xrightarrow{h'} C(B, M^*) \xrightarrow{g'} \Ext^1_C(M, M^*) \xrightarrow{0}
\end{align*}
\]

Since by definition of an exchange sequence \( f \) and \( f' \) are radical maps, it is clear that \( \text{Im}(h') \subseteq J_C(M, M^*) \) and \( \text{Im}(h) \subseteq J_C(M^*, M^*) \). But by Lemma 3.6-(2) we have a natural isomorphism of division \( k \)-algebras:

\[
k_M = C(M, M) \bigoplus k_{M}, \quad C(M, M^*) \cong k_{M^*} = C(M^*, M^*) \bigoplus k_{M^*}.
\]

The previous observations therefore show that the following equivalences are true: there is no loop at \( M \) in \( T \) if and only if \( \text{Im}(h) = J_C(M, M^*) = \ker(\partial) \), if and only if \( \text{Im}(h') = J_C(M^*, M^*) = \ker(\partial') \), if and only there is no loop at \( M^* \) in \( T' \). Hence the first part of Lemma 3.9 is proved.

For the second part of Lemma 3.9, assume \( T \) has no loop at point \( M \) (or equivalently, \( T' \) has no loop at \( M^* \)), and let \((\xi) : M^* \xrightarrow{g} B \xrightarrow{f} M \) be any non-split short exact sequence or any non-split triangle in \( C \) in particular, the fact that \( M \) and \( M^* \) are indecomposable together with Corollary 3.5 show that \( g^* \) and \( g \) are radical morphisms with \( g^* \) left minimal and \( g \) right minimal. Note that the non-split sequences \((\xi_{ex}) \) and \((\xi)\) correspond to nonzero elements in the extension group \( \Ext^1_C(M, M^*) \). By the discussion of the previous paragraph, applying the functor \( C(M, -) \) to the exchange sequence \((\xi_{ex})\) yields an exact sequence \( C(M, M^*) \xrightarrow{\partial} C(M, B) \xrightarrow{g} \Ext^1_C(M, M^*) \xrightarrow{0} \) in which \( \partial \) is the so-called connecting morphism associated with \((\xi_{ex})\), such that \( \ker(\partial) = J_C(M, M) \) is the radical of the local algebra \( C(M, M) \). Thus, there exists a morphism \( \alpha \in \text{End}_C(M) \) in \( J_C(M, M) \) such that \( \Ext^1_C(\alpha, M^*) := \partial(\alpha) = (\xi) \), so that we have a commutative diagram as follows:

\[
\begin{align*}
(\xi_{ex}) : & \quad M^* \xrightarrow{g} B \xrightarrow{f} M \\
(\xi) : & \quad M^* \xrightarrow{\partial} \Ext^1_C(M, M^*)
\end{align*}
\]

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But $\alpha \notin \ker(\partial) = J_C(M, M)$, hence $\alpha$ is an automorphism of the local algebra $C(M, M)$. Thus by 5-Lemma (or by the fact that $f$ and $g$ are right minimal), it comes that in above diagram $\beta$ is an isomorphism, hence $\langle \xi \rangle$ is isomorphic to the exchange sequence $\langle \xi_{\alpha} \rangle$, completing the proof of Lemma 3.9. 

\[ \square \]

4. Weak cluster structures and cluster structures

Before introducing weak cluster structure, we start by recalling the 2-Calabi-Yau condition and the definition of cluster-tilting subcategories. Here and for all the rest of the paper, we only consider the weak version of Calabi-Yau condition, thus by 2-Calabi-Yau, we always mean weakly 2-Calabi-Yau, see [20]. The reader is also reminded that all our categories are always assumed to be Krull-Schmidt $k$-categories where $k$ is any arbitrary field.

2-Calabi-Yau condition, cluster tilting subcategories. Let $\mathcal{C}$ be a triangulated category with translation functor $[1]$. One usually writes $\text{Ext}^2_{\mathcal{C}}(X, Y) = \mathcal{C}(X, Y[1])$ for every $X, Y \in \text{obj}(\mathcal{C})$. We write $D = \text{Hom}_{\mathcal{C}}(-, k)$ for the standard duality over module categories. Then $\mathcal{C}$ is called 2-Calabi-Yau (weakly 2-Calabi-Yau in [20]) if for every objects $X, Y \in \mathcal{C}$ we have a functorial isomorphism $D \text{Ext}^2_{\mathcal{C}}(X, Y) \longrightarrow \text{Ext}^2_{\mathcal{C}}(Y, X)$, or equivalently, we have a functorial isomorphism $D \text{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \text{Hom}_{\mathcal{C}}(Y, X[2])$. We observe that any 2-Calabi-Yau category is a triangulated $\tau$-category where the translation $\tau$ acts as the Auslander-Reiten translation and moreover $\tau$ coincides with $[2]$. Cluster-categories are typical examples of 2-Calabi-Yau categories, recall from [5, 21] that for an hereditary abelian Ext-finite category $\mathcal{H}$, the cluster category associated to $\mathcal{H}$ is the orbit category $\mathcal{C}_\mathcal{H} = D^{b}\mathcal{H}/\tau^{-1}[1]$, where $D^{b}\mathcal{H}$ is the bounded derived category of $\mathcal{H}$ and $\tau$ is the Auslander-Reiten translation functor on $D^{b}\mathcal{H}$. From [9], also recall that an exact category $\mathcal{C}$ is called stably 2-Calabi-Yau if it is a Frobenius category such that the stable category $\mathcal{C}^s$ which is triangulated according to [22], is Hom-finite 2-Calabi-Yau. Recall that $\mathcal{C}$ is said to have enough projectives if for each object $X \in \mathcal{C}$ there is an exact sequence $0 \longrightarrow X \longrightarrow P \longrightarrow X \longrightarrow 0$ with $P$ projective. Examples of stably 2-Calabi-Yau categories may be obtained as follows. Let $T$ be a triangulated category, then the module category $\text{mod} T$ of all finitely generated functors from $T^\circ$ to $\text{Mod} k$ is an abelian Frobenius category, see [23] and [17]. It follows from [7, §8.5] that if $T$ is $d$-Calabi-Yau, then the stable category $\text{mod} T$ is $(3d - 1)$-Calabi-Yau. Since according to [7, §8.4], the category of projective modules over the preprojective algebra $\Lambda$ associated to a simply-laced Dynkin quiver is 1-Calabi-Yau, we have in particular that the stable category $\text{mod} \Lambda$ is 2-Calabi-Yau. The latter can also be obtained from [8] or from [24, Propositions 3.1 and 1.2].

Following for example [25], recall that a subcategory $B$ of an additive $\mathcal{A}$ is called contravariantly finite if the restriction $\mathcal{A}(\cdot, X)|_B$ of the representable functor defined by $X$ has finite length for each $X \in \mathcal{A}$, or equivalently if for each $X \in \mathcal{A}$ there exists a right $B$-approximation $f : B \longrightarrow X$. The dual notion of covariantly finite subcategory may be defined in a dual way. $B$ is called functorially finite if it is both contravariantly finite and covariantly finite. For (triangulated or exact stably) 2-Calabi-Yau categories, the three previous notions are equivalent.

**Definition 4.1.** [25, 26, 5, 27] A subcategory $T$ of $\mathcal{C}$ is called cluster tilting if it is functorially finite with the following condition: an object $M$ in $\mathcal{C}$ belongs to $T$ if and only if $\text{Ext}^2_{\mathcal{C}}(X, M) = 0$ for every $X \in \mathcal{C}$ (or equivalently, the object $M$ belongs $T$ if and only if $\text{Ext}^2_{\mathcal{C}}(M, X) = 0$ for every $X \in T$). The last condition states that $T$ is maximal rigid or maximal $1$-orthogonal. Similarly, an object $T$ in $\mathcal{C}$ is called cluster tilting if $\text{add} T$ is a cluster tilting subcategory of $\mathcal{C}$.

**Weak cluster structure.** We now recall the notion of weak cluster structure on any additive category.

**Definition 4.2 ([9, §1.1]).** A weak cluster structure on $\mathcal{C}$ consists of the following data:

- The given of a collection $\text{Clst}$ of sets $X \subset \text{ind} \mathcal{C}$ called clusters, while each indecomposable object in each cluster is called a cluster variable. Additionally there is a subset $P \subset \text{ind} \mathcal{C}$ of indecomposable objects which are not clusters, called coefficients. For each cluster $X$ the union $T = X \cup P$ (view as an object or as a subcategory in $\mathcal{C}$) is referred as an extended cluster. These data are required to satisfy the following two conditions.
Remark 4.3 (BMRRT,Z3,KR,IY,BIRSc). If a (stably) 2-Calabi-Yau category $C$ has a cluster tilting subcategory, then the cluster tilting subcategories in $C$ determine a weak cluster structure. Moreover, the weak cluster structure for cluster categories has no loop nor two cycle.

Mutation of valued quivers.

Definition 4.4. A valued graph (without any restriction on the graph) is a labeled graph $\Gamma = (\Gamma_0, \Gamma_1, \Gamma_2)$ whose labeling $d$ called valuation prescribes for each edge $e \in \Gamma_1(i,j)$ two non-zero integers $d^e_{i,j}, d^e_{j,i} \in \mathbb{N}$. Additionally, it is required that the valuation $d$ is right (or left) symmetric, where the (minimal right) symmetricizing map $\Gamma_0 \xrightarrow{d} \Gamma_0$ for $d$ prescribes for each point $i \in \Gamma_0$ a non-zero integer $d_i \in \mathbb{N}$, such that $d^e_{i,j} = d^e_{j,i}$ for all $e \in \Gamma_1(i,j)$. Each valued edge $e \in \Gamma_1(i,j)$ may be depicted by $i \xrightarrow{d^e_{i,j}} j$ or by $i \xrightarrow{d^e_{j,i}} j$ if one should stress on the valuation, (where $d$ may be dropped when no confusion is possible, for example in case $d$ is the only edge between $i$ and $j$).

A valued quiver $Q = (\Gamma, \delta)$ of type $\Gamma$ consists of a valued graph $\Gamma$ endowed with an orientation given by two maps $\delta, \delta_0: \Gamma_1 \xrightarrow{\delta_{0,1}} \Gamma_0$ which prescribe for each edge $e \in \Gamma_1(i,j)$ its source $\delta(e) = i$ and its target $\delta_0(e) = j$, thus turning $e$ to an arrow from $i$ to $j$. If $a$ is oriented from $i$ to $j$ we also write $d(a) = (d^e_{i,j}, d^e_{j,i})$.

With above notations, if moreover $Q$ is locally finite (for each pair of points $i, j \in Q_0$, the set $Q_1(i,j)$ of all valued arrow from $i$ to $j$ is finite), then the set $Q_1(i,j)$ may be represented by one valued arrow $\xrightarrow{\Delta^Q_{i,j}} j$ with $\Delta^Q_{i,j} = \sum_{\gamma \in Q_1(i,j)} d^e_{i,j}$ and $\sum_{\gamma \in Q_1(i,j)} d^e_{j,i}$, in this case, we call $\Delta^Q_{i,j}$ the fully valued arrow from $i$ to $j$ in $Q$. One can also speak of fully valued paths in $Q$.

For all the rest, all valued quivers are always assumed to be locally finite. Let $Q$ be a valued quiver of type $\Gamma$ with valuation $\delta$ over a set $I$, and $k \in I$ a point which is neither on a loop nor on a 2-cycle, in other words the following condition holds:

for every point $i \in I$, either $Q_1(i,k) = \emptyset$ or $Q_1(k,i) = \emptyset$ \hspace{1cm} (1)

For every pair of points $i, j \in I$, we let $Q_2^{(k)}(i,j)$ be the subset of all length-2 valued paths from $i$ to $j$ crossing $k$. Then with condition (1), either $Q_2^{(k)}(i,j) = \emptyset$ or $Q_2^{(k)}(i,j) = \emptyset$.

Definition 4.5. Assume condition (1) is satisfied, then the mutation of $Q$ at point $k$ is the valued quiver over $I$ given by $Q' = \mu_k(Q)$ with valuation $d'$ described as follows:
for each $i \in I$ and for each valued edge $\alpha : i \overset{\beta}{\rightarrow} k$ in $Q$, in $Q'$ there is a corresponding valued edge $\alpha^* : i \overset{\beta}{\rightarrow} k$ having the same valuation as $\alpha$, but the orientation of the arrow $\alpha^*$ in $Q'$ is opposite to the orientation of $\alpha$ in $Q$.

Let $i, j \in I \setminus \{k\}$ such that the set $Q_{12}^k(i,j)$ is empty, then let $i \overset{2 \cdot k - 2 \cdot \beta}{\rightarrow} j$ be the fully valued (or may be $\emptyset$-valued) length-2 path representing the subset $Q_{12}^k(i,j)$, and also let $i \overset{L}{\rightarrow} j$ be the fully valued (or may be $\emptyset$-valued) arrow representing the set $Q_1(i,j)$.

Then $Q_1(i,j) \subset Q_1'(i,j)$ as sets of valued arrows, and any further (fully) valued edge in $Q_1'(i,j) \cup Q_1(j,i)$ is given by $[d_j(\alpha \beta) - d_j(\gamma)]$, with its orientation determined as follows: let $\text{sign}_{\gamma'}(i,j) = \text{sign}(d_j(\alpha \beta) - d_j(\gamma))$ or $\text{sign}(d_j(\alpha \beta) - d_j(\gamma))$, thus if $\text{sign}_{\gamma'}(i,j)$ equals $+1$ then $\gamma'$ is oriented from $i$ to $j$, otherwise $\gamma'$ is oriented from $j$ to $i$.

In particular, if additionally the three points $i$, $j$, and $k$ determine a (full) acyclic valued subquiver in $Q$, then $Q_1'(i,j) = \emptyset$ and $Q_1(i,j)$ may be identified with the disjoint union $Q_1(i,j) \cup Q_{12}^k(i,j)$.

So defined, the above mutation of valued quivers never increase the number of 2-cycles.

**Example 4.6.** In the valued quivers below, $a, b \in \mathbb{N}$ are two (non-zero) natural numbers.

![Diagram](image)

*Cluster structure.* In the following definition, condition cls4 is a non simply-laced generalized version of condition $(d)$ required in [9] for cluster structures. Note that in contrast to the case of simply-laced categories (over algebraically closed fields) where the quiver of a category carries enough information (about irreducible maps for example), here for the general case where the base field need not be algebraically closed the valued quiver alone has very less information about irreducible maps, the corresponding information should be recovered by the modulated quiver. Thus the reader is warned that it is not restrictive to require in condition cls4ii below that cluster structure should respect in some sense the modulation of each extended cluster.

**Definition 4.7.** The category $\mathcal{C}$ is said to have a *cluster structure* if $\mathcal{C}$ has a weak cluster structure as above, with the two following additional conditions.

cls3 There are no loops nor 2-cycles in the following sense: the extended valued quiver of each extended cluster does not have loops and 2-cycles. Equivalently, exchange triangles (or exchange sequences) are connecting $\tau$-sequences and there are no 2-cycles.

cls4 For each extended cluster $T$, $\dim(J_T B_T^\gamma) < \infty$ and the extended modulated quivers of $T$ and of $T^*$ are related by the following semi-modulated mutation:

cls4i $Q_{T^*} = \mu_M(Q_T)$ is the mutation of the valued quiver $Q_T$ at point $M$.

cls4ii Each exchange pair $(M, M^*)$ comes with a natural isomorphism $k_M \sim k_M^*$, compatible with the two exchanges triangles (sequences), which identifies the division algebra $k_M$ with the division algebra $k_M^*$ such that for every cluster $X \in (T^\gamma) M$ the (fully) valued edge between $X$ and $M$ in $Q$ and the (fully) valued edge between $X$ and $M^*$ in $Q'$ are respectively given by

$$\alpha : X \cdot B_M \cdot M B_X = (x B_M)^* M$$ and $$\alpha^* : X \cdot B_M \cdot M B_X = (x B_M)^* M^*.$$
So that in $\alpha^*$ only the orientation and point $M$ change while the associated dualizing pair of bimodules $\{XB_M, MB_X\} = \{XB, MB = \langle MBX \rangle^*\}$ stay unchanged while passing from $Q_T$ to $Q_T^\alpha$.

We observe the following under the assumption of weak cluster structure: looking at condition cls4ii above and taking into account Theorem 3.8, since exchange triangles (or exchange sequences) are minimal approximation sequences, it is true that for any fixed $M$ as above, there always exist two isomorphisms $k_M \cdot k_M$ given by choosing two exchange triangles (or sequences) for the pair $(M, M^*)$, under which for all indecomposable objects $X \in B$ and $X^* \in B'$, $\overline{\text{Irr}_T}(M^*, X) \cong_{k_M} \overline{\text{Irr}_T}(X, M^*)$ while $\overline{\text{Irr}_T}(M, X') \cong_{k_M} \overline{\text{Irr}_T}(X', M^*)$. Thus in view of Theorem 3.8, condition cls4ii says precisely that there should be a "good" trace forms $t = t_M = \phi_M \circ \rho_M$ associated with a nice choice of the two compatible exchange triangles (sequences). Also note that condition cls4ii is always obviously satisfied for simply-laced case, where bimodules are just vectors spaces over an algebraically closed field.

5. 2-Calabi-Yau and stably 2-Calabi-Yau categories with cluster structures

In this section our main result is to prove that in presence of 2-Calabi-Yau property, condition cls4 holds under the assumption that cls3 holds and we have a weak cluster structure. The proof that condition cls4 holds can be derived from the corresponding result for the simply laced case in [9]. However to prove that condition cls4ii holds is a more complicated task which will require the general setup we developed in previous sections as well additional tools regarding skew fields. From now on, we assume that $C$ is either Hom-finite 2-Calabi-Yau or stably 2-Calabi-Yau.

5.1. Main Statements

For the main result of this section we need the following technical but easy lemma giving the existence of the so-called trace maps for finite dimensional local $k$-algebras.

**Lemma 5.1.** Let $\Lambda$ be a finite dimensional local $k$-algebra with radical $J_\Lambda$, then there are non-degenerated $k$-linear trace forms $t$ on $\Lambda$ such that $t \in \text{soc}(\Lambda \text{Hom}_k(\Lambda, k)) \cap \text{soc}(\text{Hom}_k(\Lambda, k)_\Lambda)$: each such trace form is then a non-zero central element in the $\Lambda$-bimodule $D(\Lambda) = \text{Hom}_k(\Lambda, k)$, thus $a \cdot t = t \cdot a$ for every $a \in \Lambda$, and $J_\Lambda t = 0 = tJ_\Lambda$. In particular for every invertible element $a \in \Lambda$, we have $a \cdot t = t \cdot a^{-1} = t$.

**Proof.**

(i). We first prove the following well-known fact: Let $E$ be a centrally finite division ring with center $C$ ($E$ is finite-dimensional over its center $C$), then there is a non-zero $C$-linear trace map $E \xrightarrow{t} C$ (such that $t(ab) = t(ba)$ for every $a, b \in E$). In fact let $[E, E]$ stands for the $C$-subspace of $E$ generated by all commutators $[a, b] = ab - ba$ with $0 \neq a, b \in E$. We claim that $[E, E]$ is a proper subspace of $E$. Indeed, let $C$ be the algebraic closure of $E$, then the algebra $\bar{E} = E_{\text{alg}} \subseteq C$ is still a finite dimensional simple algebra over the algebraically closed "commutative" field $C$ (see for example [28, 15.1]), so that by Wedderburn-Artin Theorem it coincides with a matrix-algebra $M_m(C)$ where $m = \dim_C E \geq 1$. The $C$-subspace $[E, \bar{E}]$ corresponds to matrices with zero trace, thus it is a proper subspace of $E$. It clearly follows from the commutativity of $C$ that $[E, \bar{E}] = [E, E]_{\text{soc}} C$, it comes that $[E, E]$ is also a proper subspace of $E$, hence our claim holds. Now, choosing some non-zero $C$-linear map $[E, E] \xrightarrow{\lambda} C$ and next taking the composition with the projection $E \xrightarrow{\pi} \bar{E}[E, E] = E_{\text{alg}} C$, we obtain that the map $t := \lambda \circ \rho$ is a non-zero $C$-linear trace map on $E$, and since $E$ is a simple algebra the trace form $t$ is automatically non-degenerated in the sense that its radical $R_t = \{a \in E : t(ab) = 0 \text{ for all } b \in E\}$ is zero.

(ii). Now, since $\Lambda$ is a finite-dimensional $k$-algebra, the factor $\hat{\Lambda} = \Lambda \text{rad} \Lambda$ and its center $C := C(\hat{\Lambda})$ are also finite-dimensional division $k$-algebra, in particular $\Lambda$ is also centrally finite. Hence by point (i) above, there is a non-degenerated $C$-linear trace map $\Lambda \text{rad} \Lambda \xrightarrow{t} C$. Let’s choose any non-zero $k$-linear map $C \xrightarrow{\kappa} k$ and let $\Lambda \xrightarrow{\pi = \text{alg} \kappa} \Lambda \text{rad} \Lambda$ be the canonical projection, then the map $\Lambda \xrightarrow{\text{alg} \lambda \circ \kappa} k$ is a non-degenerated $k$-linear
Theorem 5.2. \( f \in \text{trace form on } \Lambda \) and moreover \( t \in \text{soc}(\Lambda \text{Hom}(\Lambda, k)) \cap \text{soc}(\text{Hom}(\Lambda, k)_\Lambda) \). For the latter fact, we use the fact that \( \Lambda \) is an artinian algebra and the bimodule \( D(\Lambda) = \text{Hom}(\Lambda, k) \) is artinian as left-module and as a right module, so that for any element \( f \in D(\Lambda), f \in \text{soc}(\Lambda D(\Lambda)) \) if and only if \( J_\Lambda f = 0 \) and similarly, \( f \in \text{soc}(D(\Lambda)_\Lambda) \) if and only if \( f J_\Lambda = 0 \).

We consider the following condition for a triple \((T, M, T^*)\):

\[
\begin{align*}
\text{\( T \) is a cluster tilting object (subcategory) in } C, \\
\text{\( M \in \text{ind } T \) and, there is no loop at point } M \text{ in } T, \\
\text{\( T^* = (T|M) \oplus M^* \) is the mutation of } T \text{ at point } M. 
\end{align*}
\]

In view of Lemma 3.9, \((T, M, T^*)\) satisfies (2) if and only so does the triple \((T^*, M^*, T)\).

We now have that the non-simply laced version of cluster structure exists under the assumption that we have a weak cluster structure and condition cls3 holds.

Theorem 5.3. Let \( T \) be any triple in a triangulated or exact stably 2-Calabi-Yau category \( C \) such that condition (2) is satisfied. Then the following statements hold.

(1) The two exchange sequences \((\xi) : M^* \xrightarrow{\xi^*} B \xrightarrow{\xi} M \longrightarrow \) and \((\xi') : M \xrightarrow{\xi'} B' \xrightarrow{\xi''} M^* \longrightarrow \), associated with the exchange pair \((M, M^*)\), may be constructed such that each endomorphism \( a \in \text{End}_C(M) \) extends to an endomorphism \((b, a', a, b[1])\) of the sequence \((\xi)\) and to an endomorphism \((a, a'', b, a[1])\) of the sequence \((\xi')\) with a common common \( b \in \text{End}_C(M^*)\).

(2) The correspondence \( a \mapsto b \), induced by (1), yields a natural isomorphism of division \( k\)-algebras \( k_M \xrightarrow{\varphi} k_M^* \), along which, for every indecomposable direct summand \( X \) in \( T|M \) we have the following natural isomorphisms of (symmetrizable) dualizing pairs of bimodules:

\[
\begin{align*}
\{\text{Irr}_{\text{add } T}(X, M), \text{Hom}_k(\text{Irr}_{\text{add } T}(X, M), k)\} &\cong \{\text{Irr}_{\text{add } T^*}(M^*, X), \text{Hom}_k(\text{Irr}_{\text{add } T^*}(M^*, X), k)\}, \\
\{\text{Irr}_{\text{add } T}(M, X), \text{Hom}_k(\text{Irr}_{\text{add } T}(M, X), k)\} &\cong \{\text{Irr}_{\text{add } T^*}(X, M^*), \text{Hom}_k(\text{Irr}_{\text{add } T^*}(X, M^*), k)\}.
\end{align*}
\]

We now have that the non-simply laced version of cluster structure exists under the assumption that we have a weak cluster structure and condition cls3 holds.

Theorem 5.3. Let \( C \) be \( \text{Hom}\)-finite triangulated (exact stably) 2-Calabi-Yau category over an arbitrary commutative field \( k \), with some cluster tilting subcategory and such that there are no loops and no 2-cycle. Then the cluster tilting subcategories determine a (non-simply laced version of) cluster structure for \( C \).

Before proving Theorem 5.2 and Theorem 5.3, we derive the following useful consequence for 2-Calabi-Yau tilted algebras, where the splitting condition in point (iii) of Corollary 5.4 is needed since the base field \( k \) is not assumed to be perfect. Also recall that if an object \( T \in C \) does not have loops and 2-cycles, then between any two points i and j in the modulated quiver \( Q \) of \( \text{End}(T) \), there is only one (full) valued edge \( _{k_i}k_j \) whose symmetrizable dualizing pair of bimodules may be denoted by \( \{B_i, ^ib_j; _ib_k, B_k\} \) in which the symmetrizable non-degenerated bilinear forms \( i_{B_j \otimes k_i}B_i \overset{\varphi}{\longrightarrow} k_i \) and \( i_{B_i \otimes k_j}B_j \overset{\varphi}{\longrightarrow} k_j \) are induced by non-degenerated traces \( t_i \in \text{Hom}_k(k_i, k) \) and \( t_j \in \text{Hom}_k(k_j, k) \) as in Lemma 3.7.

Corollary 5.4. Let \((T, T_k, T^*)\) be a triple in \( C \) as in condition (2) such that \( T \) does have loops and 2-cycles, with \( \text{ind } T = \{T_i : i \in I\} \). Let \( Q = Q_{\text{End}(T)} \) and \( Q' = Q_{\text{End}(T')} \). Then the mutation of \( Q \) may be chosen to be symmetrizable over \( k \) by fixing a family \( (k_i, t_i)_{i \in I} \) of division \( k \)-algebras \( k \), endowed with non-degenerated traces \( t_i \), and such that \( Q \) and \( Q' \) are related by the following semi-modulated mutation.

(i) The underlying valued quiver of \( Q' \) is the mutation \( Q' = \mu_k(Q) \).

(ii) Under a natural identification of the division \( k \)-algebras attached to point \( k \) in \( Q \) and in \( Q' \), we have that the family of division \( k \)-algebras endowed with non-degenerated traces in \( Q' \) is again given by \( (k_i, t_i)_{i \in I} \). And for any two points \( x, y \) with \( x \neq y \), the (full) valued arrows representing the sets \( Q_1(x, k), Q_1(k, y) \) and \( Q_1'(k, x), Q_1'(y, k) \) are respectively given by...
\( \alpha : x_{iB_k} \rightarrow x_{jB_k}, \beta : k_{B_y} \rightarrow x_{B_y} y \) and \( \alpha^* : x_{iB_k} \rightarrow x_{jB_k}, \beta^* : k_{B_y} \rightarrow x_{B_y} y \)

in which \( \alpha \) and \( \alpha^* \) (or \( \beta \) and \( \beta^* \) respectively) have the same symmetrizable dualizing pair of bimodules.

(iii) Suppose \( i, j \in \{ k \} \) are such that the valued subquiver of \( Q \) induced by the three points \( i, j \) and \( k \) is acyclic, let \( i B_j = \text{Irr}_{\text{add}} T(T_j, T_i) \) and \( i B_j' = \text{Irr}_{\text{add}} T(T_j, T_i) \). In case the bimodule \( i B_k \otimes k B_j \) is nonzero we also require the splitting condition: \( J_{\text{add}} T(T_j, T_i) = \text{Irr}_{\text{add}} T(T_j, T_i) \oplus J_{\text{add}} T(T_j, T_i) \) as bimodules. Then as \( k, k \)-bimodules we have \( i B_j' = \text{Irr}_{\text{add}} (T \mathbb{T} k) (T_j, T_i) = i B_j + \text{Irr}_{\text{add}} (T \mathbb{T} k) (T_j, T_i) \), thus the corresponding symmetrizable dualizing pairs of bimodules in \( Q \) and in \( Q' \) are related as follows:

\[
\{i B_j', j B_j'; i B_j, j B_j\} = \{i B_j, j B_j; i B_k, k B_1\} + \{i B_k, k B_j; k B_j, j B_k\}.
\]

**Proof.** (i) is given by Theorem 5.3-clst4i while (ii) is given by Theorem 5.2-(2), thus it only remains to check that (iii) also holds. Let \( Q_2(i, j) := \{ \alpha \beta : \alpha \in Q(i, k), \beta \in Q(k, j) \} \) stands for the set of all length-2 valued paths in \( Q_2(i, j) \) crossing \( k \) (as in Definition 4.5). By definition we have \( T^* = \mu_k(T) = (T \mathbb{T} k) \oplus T_k^* \) with \( T_k^* \neq T_k \). Let \( i B_j = \text{Irr}_{\text{add}} T(T_j, T_i) \), \( i B_j' = \text{Irr}_{\text{add}} T(T_j, T_i) \), \( i B_k = \text{Irr}_{\text{add}} T(T_k, T_i) \) and \( i B_k' = \text{Irr}_{\text{add}} T(T_k, T_i) \); clearly we always have canonical exact sequences of \( k, k \)-bimodules

\[
\xymatrix{i B_k \otimes k B_j \ar[r] & \text{Irr}_{\text{add}} T \mathbb{T} k (T_j, T_i) \ar[r] & B_j \ar[r] & 0 \quad \text{and} \quad \text{Irr}_{\text{add}} T \mathbb{T} k (T_j, T_i) \ar[r] & B_j' \ar[r] & 0.}
\]

Using the splitting condition: \( J_{\text{add}} T(T_j, T_i) = \text{Irr}_{\text{add}} T(T_j, T_i) \oplus J_{\text{add}} T(T_j, T_i) \) in case \( i B_k \otimes k B_j \neq 0 \), we see that \( i B_j \subset \text{Irr}_{\text{add}} T \mathbb{T} k (T_j, T_i) \) and \( \text{dim}_{k^*} (i B'_j) \leq \text{dim}_{k^*} (\text{Irr}_{\text{add}} T \mathbb{T} k (T_j, T_i)) \leq \text{dim}_{k^*} (i B_k \otimes k B_j) + \text{dim}_{k^*} (i B_j) \).

Then to show that \( i B'_j = i B_j \oplus (i B_k \otimes k B_j) = \text{Irr}_{\text{add}} T \mathbb{T} k (T_j, T_i) \), it is enough to check that the bimodules \( i B'_j \) and \( i B_j \oplus (i B_k \otimes k B_j) \) have the same dimension over \( k \). But by hypothesis the full valued subquiver in \( Q \) determined by the three points \( i, j, k \) is acyclic, and the last point of Definition 4.5 shows that for the valued quiver \( Q' := \mu_k(Q) \) of \( T^* \) we have \( Q'_2(i, j) = Q_1(i, j) \cup Q_2^b(i, j) \). It comes that \( \text{dim}_{k^*} (i B'_j) = \text{dim}_{k^*} (i B_j) + \text{dim}_{k^*} (i B_k \otimes k B_j) \), thus yielding our claim that statement (iii) is also true. \( \square \)

5.2. Proofs of Theorem 5.2 and Theorem 5.3

We prove Theorem 5.2 and Theorem 5.3 in the triangulated case, the "exact stably 2-Calabi-Yau" case is obtained similarly by replacing "triangles" by "pseudo-exact sequences".

5.2.1. Proving Theorem 5.2

Since by assumption the triple \( (T, M, T^*) \) satisfies condition (2) so that by Lemma 3.9 the triple \( (T^*, M^*, T) \) also satisfies condition (2), and since moreover the exchange triangles associated with the pair \( (M, M^*) \) are add\((T \cap T^*)\)-approximation triangles, it follows that these exchange triangles are connecting \( \tau \)-sequences. Therefore, point (2) of Theorem 5.2 is a direct consequence of point (1) and of Theorem 3.8-(2). Thus we only need to prove the condition on \( \xi \).

**Constructing exchange triangles.** Fixing one exchange triangle \( (\xi) : M \overset{f}{\longrightarrow} B \overset{g}{\longrightarrow} M^* \overset{\delta}{\longrightarrow} M^*[1] \), we have to construct the second exchange triangle \( (\xi') : M \overset{g}{\longrightarrow} B' \overset{g'}{\longrightarrow} M^* \overset{\delta'}{\longrightarrow} M^*[1] \) such that the condition below is
satisfied.

Each endomorphism $a \in \text{End}_C(M)$ yields a commutative diagram as follows,

$$
\begin{array}{ccc}
M^* & \xrightarrow{f^*} & B \\
\downarrow{f} & & \downarrow{g} \\
M & \xrightarrow{g} & B' \\
\downarrow{g^*} & & \downarrow{\delta'} \\
M^* & \xrightarrow{\delta} & M[1]
\end{array}
$$

where the component morphism $b \in \text{End}_C(M^*)$ is the same for both triangles $(\xi)$ and $(\xi')$.

By lemma 5.1 we start by choosing a non-degenerated trace map $t \in \text{soc}(D \text{Hom}_C(M, M))$, in particular $t$ is a central element in the $\text{End}_C(M)$-bimodule $\text{soc}(D \text{Hom}_C(M, M))$. By 2-Calabi-Yau property we have a functorial isomorphism $D \text{Hom}_C(X, Y) \xrightarrow{\text{Hom}_C(Y, X[2])}$ for every $X, Y \in \text{ind}\ C$, so that $t$ naturally corresponds to a non-zero morphism $\epsilon \in \text{soc}(\text{Hom}_C(M, M[2]))$ which therefore induces an almost-split triangle $(\xi) : M[1] \xrightarrow{\epsilon} E \xrightarrow{\delta} M[2]$ in $C$.

Next, using $(\xi)$ and $(\xi')$ we construct a morphism $\delta' : M^* \xrightarrow{f^*} M[1]$ such that the diagram below commutes.

$$
\begin{array}{ccc}
(\xi) & : & M^* \xrightarrow{f^*} B \\
\downarrow{f} & & \downarrow{g} \\
(\xi') & : & M[1] \xrightarrow{\delta'} \xrightarrow{\delta' [1]} M[2]
\end{array}
$$

In fact by right factorization property of $v$ the radical map $f$ must factors through $v$ by some map $f'$ making the middle square in above diagram to commute; thus the pair $(f', \chi_{\epsilon})$ must extend to a morphism of triangle $(\delta', f', \chi, \delta'[1])$ with $\delta' : M^* \xrightarrow{f'} M[1]$, so that we get a commutative diagram as above in which $\delta' \neq 0$ since $\epsilon$ was chosen to be non-zero.

**Fact 5.1.** The unique non-split triangle $(\xi) : M \xrightarrow{g} B' \xrightarrow{g'} M^* \xrightarrow{\delta} M[1]$ constructed using the morphism $\delta'$ is an exchange triangle.

Since $(\xi')$ does not split, the proof Fact 5.1 is given by Lemma 3.9 showing that the result is always true for a pair of rigid subcategories without loops at $M$ and at $M^*$. However we can also show directly that $B'$ is $T \cap T^*$: indeed let $T = T'M$ applying the functor $C(T, -)$ to $(\xi')$ yields the following exact sequence

$$C(T, M) \xrightarrow{\text{Ext}^1_c(T, -)} C(T, M^*) \xrightarrow{\text{Ext}^1_c(T, M)} C(T, B') \xrightarrow{\text{Ext}^1_c(T, B')} C(T, M^*)$$

Since $T = T \oplus M$ and $T^* = T \oplus M^*$ are rigid, we have $\text{Ext}^1_c(T, M) = 0$ and $\text{Ext}^1_c(T, M^*) = 0$, it follows that $\text{Ext}^1_c(T, B') = 0$. Next, applying $C(-, M)$ to $(\xi')$ and using the fact that there is no loop at point $M$ in $T$, we get (as in the proof of Lemma 3.9) an exact sequence $0 \xrightarrow{\text{Ext}^1_c(M, M)} C(M, M) \xrightarrow{\text{Ext}^1_c(M, M^*)} \text{Ext}^1_c(B', M) \xrightarrow{\text{Ext}^1_c(B', M^*)} \text{Ext}^1_c(M, M) = 0$, showing that $\text{Ext}^1_c(B', M) = 0$ and because of the 2-Calabi-Yau condition $\text{Ext}^1_c(M, B') = 0$ as well. Similarly, applying $C(M^*, -)$ to $(\xi')$ together with the same argument as the previous one, one also obtain that $\text{Ext}^1_c(M^*, B') = 0 = \text{Ext}^1_c(B', M^*)$. We have therefore proved that $\text{Ext}^1_c(T, B') = 0 = \text{Ext}^1_c(T, B')$ and $\text{Ext}^1_c(T, B') = 0 = \text{Ext}^1_c(T, B')$, showing that $B'$ is in $T \cap T^*$ since $T$ and $T^*$ are maximal rigid.

**Fact 5.2.** The two exchange triangles $(\xi)$ and $(\xi')$ are compatible with respect to the modulations associated with $T$ and $T^* = \mu_M(T)$, in the sense that condition (3) is satisfied.
For the proof of Fact 5.2, let \( M \to M \) be any endomorphism of \( M \). Using the first exchange triangle \((\xi)\) (with the right factorization property of \( f \)), it is clear that each endomorphism \( a \) must extend to an endomorphism \((b, a', a, b[1])\) of the triangle \((\xi)\). Now to have a commutative diagram as in (3) we must show that in the diagram below

\[
\begin{array}{c}
(\xi') : \quad M \xrightarrow{g} B' \xrightarrow{g^*} M^* \xrightarrow{\delta'} M[1] \\
\downarrow a & \downarrow b & \downarrow a[1] \\
(\xi') : \quad M \xrightarrow{g} B' \xrightarrow{g^*} M^* \xrightarrow{\delta'} M[1],
\end{array}
\]

the pair \((b, a)\) also extends to an endomorphism \((a, a'', b, a[1])\) of the second exchange triangle \((\xi')\). For this, we only have to show that the right square in the previous diagram commutes, or equivalently that in the following diagram with left commutative square, the right square also commutes:

\[
\begin{array}{c}
M \xrightarrow{\delta} M^*[1] \xrightarrow{\delta'[1]} M[2] \\
\downarrow a & \downarrow b[1] & \downarrow a[2] \\
M \xrightarrow{\delta} M^*[1] \xrightarrow{\delta'[1]} M[2]
\end{array}
\]

Remember from the construction of \( \delta' \) that \( \varepsilon = \delta'[1]; \delta \); but the functoriality of the 2-Calabi-Yau property yields that \( a[2] \varepsilon = \varepsilon a \); to see this let \( M_1 = M_2 = M'_1 = M'_2 = M \) be just others names for \( M \). Then any pair of maps \( M_1 \to M'_1 \) and \( M'_2 \to M_2 \) induces two maps \( \Hom(M_2, M_1[2]) \to \Hom(M'_2, M'_1[2]) \) and \( D \Hom(M_1, M_2) \to D \Hom(M'_1, M'_2) \) with \( \psi(u) = c[2] \cdot u \) and for any \( M'_1 \xrightarrow{\delta} M'_2 \) and \( \alpha \in D \Hom(M'_1, M'_2) \), \( \eta(\alpha)(a') = \alpha(c \cdot d')(a') = (c \cdot \alpha)(a') \), \( \eta(\alpha) = c \cdot \alpha \cdot d' \). Also Recall that under the functorial isomorphism \( \Hom(M_1, M_2) \to \Hom(M'_1, M'_2) \), \( \eta(\alpha)(a') = \alpha(c \cdot d')(a') \), thus \( \eta(\alpha) = c \cdot \alpha \cdot d' \). Hence for \( (c, c') = (1, a) \) we obtain that \( a[2] \varepsilon \) corresponds to \( a \cdot t \) while \( \varepsilon a \) corresponds to \( a \cdot t \). But since \( t \) was chosen to be a central element, we have \( a \cdot t = a \cdot t \), and therefore we also have \( a[2] \varepsilon = \varepsilon a \) as desired.

Now since the left square in the previous diagram already commutes, the equality \( a[2] \varepsilon = \varepsilon a \) implies that \( (a[2] \cdot \delta'[1] - \delta'[1] \cdot [b]) \cdot \delta = 0 \); hence the map \( a[2] \cdot \delta'[1] - \delta'[1] \cdot [b] \) must factor through the term \( B[1] \) of the triangle with base \( \delta \). But we have \( \Hom(B[1], M[2]) = \text{Ext}^1_B(B, M) = 0 \) so that \( a[2] \cdot \delta'[1] - \delta'[1] \cdot [b] = 0 \), yielding the commutativity of the right square in our above diagram. Thus the pair \((b, a)\) extends to an endomorphism of the second exchange triangle \((\xi')\), so that each endomorphism \( a \in \text{End}_C(M) \) extends to an endomorphism \((b, a', a, b[1])\) of \((\xi)\) and to an endomorphism \((a, a'', b, a[1])\) of \((\xi')\) with a common component \( b \in \text{End}_C(M) \). Thus the proof of Theorem 5.2 is now complete. \( \square \)

5.2.2. Proving Theorem 5.3: Condition 3(iiv)

This part of the proof is derived completely from its simply-laced version obtained in [9, Theorem 1.16]. For the sake of completeness, we adapt the proof of [9, Theorem 1.16] using the language of valued quivers instead of (ordinary) quivers, and we give some additional details omitted in [9].

Let \( T \) be a cluster tilting object in \( C \) with \( \text{ind} T = \{ T_i : i \in I \} \), \( k \in I \) a fixed point and \( T^* = T|_{T_k} \oplus T_k^* = \mu_k(T) \); let \( Q = Q_T \) and \( Q' = Q_{T^*} \). We have two exchange triangles \( T_k \xrightarrow{\varepsilon} B_k \xrightarrow{\delta} T_k \xrightarrow{\varepsilon} T_k^{[1]} \) and \( T_k \xrightarrow{\varepsilon} B_k^* \xrightarrow{\delta} T_k^* \xrightarrow{\varepsilon} T_k^{[1]} \), showing clearly that we have \( Q_t(i, k) = \{ \alpha^* : \alpha \in Q_t(k, i) \} \) and \( Q_t(i, k) = \{ \alpha^* : \alpha \in Q_t^*(i, k) \} \) for every \( i \in I \).

Next, we need to consider the situation where we have in \( Q \) a full valued subquiver of the form:

\[
\begin{array}{c}
\bullet \quad \bullet \quad \bullet \\
\downarrow a & \downarrow b & \downarrow c \\
\bullet \quad \bullet \quad \bullet \\
\downarrow a^* & \downarrow b^* & \downarrow c^* \\
\bullet \quad \bullet \quad \bullet
\end{array}
\]

where \( (a, b) = (0, 0) \) or \( (c, d) = (0, 0) \) since we have no 2-cycle by assumption. By Definition 4.5 the mutation of valued quivers yields in \( \mu_k(Q) \) the following valued subquiver:
so that $C$ commutativity of $\omega$ using the octahedral axiom.

It comes that the valued edge between $i$ and $j$ in $\mu_k(Q)$ is given by $\gamma : j \mapsto -c + ms, [d + ms]$ oriented from $i$ to $j$ if $\text{sign}(a + c + ms) = +1$ (or equivalently, if $\text{sign}(b - d + ms) = +1$, since the valuation is always symmetrizable), otherwise it is oriented from $i$ to $j$.

Hence, to prove that the equality $Q' = \mu_k(Q)$ holds we must show that the valued edge corresponding to the pair $i, j$ in $Q' = Q_T$ is given by the previous valued edge $\gamma$ in $\mu_k(Q)$. For this, we start by considering the two following exchange triangles associated with the summand $T_i$ of $T$: $I[0]: T_i \longrightarrow B_j \longrightarrow T_i$ and $T_i \longrightarrow B_j \longrightarrow T_i \longrightarrow T_i[1]$, where $T_k$ is not a direct summand of $B_i$ since there is no arrow from $i$ to $k$ in $Q$ because there are no 2-cycles. The valued arrow $i^{m, m'}$ shows that we may write $B_i = D_i \oplus T_k$ such that $T_k$ is not a direct summand of $D_i$.

We then have the two commutative diagrams Diag1 and Diag2 below, where the third row of Diag1 is a triangle constructed by starting with the three triangles induced by the maps in the upper square, and then using the octahedral axiom $\text{Tr}4$ of triangulated categories.

\begin{align*}
\text{Diag1} & : & (T_k^m)^[1] & \longrightarrow (T_k^m)^[1] & \hspace{1cm} \text{Diag2} & : & (T_k^m)^[1] & \longrightarrow (T_k^m)^[1] \\
T_i[-1] & \longrightarrow & T_i^* & \longrightarrow & D_i \oplus (T_k^m)^[1] & \longrightarrow & T_i & \hspace{1cm} T_i & \longrightarrow & B_i' & \longrightarrow & T_i^* & \longrightarrow & T_i[1] \\
\| & \| & \| & & \| & & \| & \hspace{1cm} \| & \| & \| & \hspace{1cm} \| & \| & \| & \hspace{1cm} \| & \| & \| & \hspace{1cm} \| & \| & \| & \hspace{1cm} \| & \| & \| \\
T_i[-1] & \longrightarrow & (T_k^m)^[1] & \longrightarrow & T_i & \hspace{1cm} (T_k^m)^[1] & \longrightarrow & (T_k^m)^[1]
\end{align*}

Then using again the octahedral axiom, Diag2 is a commutative diagram of triangles in which the second row is an exchange triangle while the third column is the second column of Diag1. Observe that in Diag2 we must have $Y = B_i' \oplus (T_k^m)^[1]$: using the fact that $T_k$ is not in add $B_i'$, we have $C(B_i', (T_k^m)^[1]) = 0$, so that the triangle in the first column of Diag2 splits.

For $T^* = T^* \cap \mathcal{C}$, we claim that the maps $h$ and $h'$ in the two triangles below are right add $T^*$-approximations,

\begin{align*}
(\xi) & : X \longrightarrow D_i \oplus B_k \longrightarrow T_i \longrightarrow X[1]. & (\xi') & : T_i \longrightarrow B_i' \oplus (T_k^m)^[1] \longrightarrow T_i[1].
\end{align*}

In fact, $B_i' \oplus (T_k^m)^[1]$ is clearly in add $T^*$ since $T_k$ is not a direct summand of $B_i'$, but $T_i$ is in $T \cap T^*$ so that $C(T^*, T_i[1]) = 0$, hence $h'$ is a right add $T^*$-approximation. Now, to deal with the case of the map $h$, we start by observing that $D_i \oplus B_k \in \text{add} T^*$. We know that $B_i = D_i \oplus T_k \in \text{add} T$ while the choice of $D_i$, we have $D_i \in \text{add} T$ for $T = T\|k$; and $T_i$ is not a direct summand of $D_i$ because there is no loop at $i$. Further $T_j$ and $T_k$ are not direct summands of $B_k$ since there is no loop at $k$ and there is no arrow from $i$ to $k$ in $Q_T$. Hence $D_i \oplus B_k \in \text{add} T^*$ as required. Next, for each indecomposable direct summand $T_i$ of $T$ not isomorphic to $T_k$, we see that any map $u : T_i \longrightarrow T_i$ factors through $h$ since the second row in Diag1 is an exchange triangle. $u$ factors through the map $g$ so that $u = g \circ u'$ for some $u' \in \text{C}(D_i \oplus T_k, T_i)$. Now using the triangle in the second column of Diag1 above, together with the fact that $C(T_i, (T_k^m)^[1]) = 0$ since $T_i \in \text{ind}(\text{add} T^*)$ with $T_i \not\cong T_k$, it comes that the maps $u'$ must also factor through $w$ so that $u' = w \circ u''$ for some $u'' \in \text{C}(D_i \oplus B_k, D_i \oplus T_k)$. Thus $u = g \circ w \circ u'' = h \circ u''$ by the commutativity of Diag1. Finally, let $v : T_k \longrightarrow T_i$; using the minimal left add $T^*$-approximation $T_k \longrightarrow B_k$, there must exist a map $v' \in \text{C}(B_k, T_i)$ with $v = v' \circ f'$. But by the previous argument $v'$ factors through $h$. 20
since each one of the indecomposable objects $T_i, T_k, T_i^*$, is not a direct summand of $B_k$. Thus $v$ also factors through $h$, and hence $h$ is a right $\overline{T'}$-approximation as claimed.

Note that the triangles $(\xi)$ and $(\xi')$ need not be minimal. But using Lemma 3.4, the triangle $(\xi)$ is isomorphic to a direct sum, $X = T_i^* \oplus W \left[ \begin{array}{c} h_i^* \\ 0 \end{array} \right] T_i \oplus W \left[ \begin{array}{c} |h| \\ 0 \end{array} \right] T_i^* [b] \overline{T_i^*} + W [b]$, in which $(\xi_0) : T_i^* \left[ \begin{array}{c} h_i^* \\ 0 \end{array} \right] U_i \left[ h_i \right] T_i \left[ \begin{array}{c} e \\ 0 \end{array} \right] T_i^* [b] \overline{T_i^*} + W [b]$, is a minimal add $\overline{T'}$-approximation triangle. Hence by Lemma 5.3 $(\xi_0)$ must be the unique exchange triangle associated with $T_i \in T^*$ and $T_i^* \in T^* = \mu(T^*)$. Similarly, since we already have $X = T_i^* \oplus W$, the previous argument applied to the triangle $(\xi')$ shows that $(\xi')$ is isomorphic to a direct sum, $T_i \left[ \begin{array}{c} h_i \\ 0 \end{array} \right] U_i^* \left[ h_i' \right] T_i^* \left[ \begin{array}{c} e' \\ 0 \end{array} \right] T_i^* [b] \overline{T_i^*} + W [b]$, in which $(\xi_0') : T_i \left[ \begin{array}{c} h_i' \\ 0 \end{array} \right] U_i U_i^* \left[ h_i' \right] T_i^* [b] \overline{T_i^*} + W [b]$ is the second exchange triangle associated with the pair $(T_i, T_i^*)$.

Now for $M, U \in C$ with $M$ indecomposable, let $\alpha_M(U) \in \mathbb{N}$ denotes the multiplicity of $M$ as a direct summand of $U$. Since we don’t have 2-cycles in $Q_{T'}$, one of the two natural numbers $\alpha_T(U_i) = \dim_{k_T} \text{Irr}_T(T_j, T_i)$ and $\alpha_T(U'_i) = \dim_{k_T} \text{Irr}_T(T_i, T_j)$ is zero. Thus it comes that the valued edge $\varepsilon_j(i, j)$ between points $i$ and $j$ in $Q_{T'}$, is such that $\varepsilon_j(i, j) = |\alpha_T(U_i) - \alpha_T(U'_i)|$. But since $B_i = D_i \oplus B_i^m$ and $D_i \oplus B_i^m = U_i \oplus W$, while $B_i' \oplus (T_i^*)^m = U_i \oplus W$, we have $\alpha_T(U_i) = \alpha_T(D_i \oplus B_i^m) = \alpha_T(B_i) + m \cdot \alpha_T(B_k) - \alpha_T(W) = a + ms - \alpha_T(B_i) - \alpha_T(W)$, and $\alpha_T(U'_i) = \alpha_T(B'_i \oplus T_i^*) = \alpha_T(B'_i) - \alpha_T(W) = c - \alpha_T(W)$. Hence $\varepsilon_j(i, j) = |\alpha_T(U_i) - \alpha_T(U'_i)| = |a - c + ms|$ as expected. We then conclude that $Q_{T'} = \mu_k(Q_T)$ as desired, and this ends the proof that condition cls4ii holds. $$\square$$

5.3. Substructures

We end this section by pointing out that a notion of substructure for cluster structures is also introduced and widely studied by the authors of [9]. Also, the term "stably 2-Calabi-Yau" is used in a more general context to qualify any category $B$ which is either a Frobenius category such that $B$ is triangulated 2-Calabi-Yau or a factorially finite extension-closed subcategory of a triangulated 2-Calabi-Yau category (for the latter condition, we refer to [9, Thm II.2.1]). Let $C$ be an exact or a triangulated k-category, $B$ a subcategory of $C$ closed under extensions, and such that both $C$ and $B$ have a weak cluster structure. The weak cluster structure of $B$ is called a substructure of $C$ induced by an extended cluster $T$ in $B$ if the following holds. There is a set $\Xi$ of indecomposable objects in $C$ such that for any extended cluster $T'$ in $B$ obtained by a finite number of exchanges from $T$ the object $T' = T' \cup \Xi$ is an extended cluster in $C$.

6. Cluster algebras, subcluster algebras and cluster maps

In this section, we apply the existence of non simply-laced version of cluster structure to realize skew symmetrizable cluster algebras of geometric type [1], allowing as in [9] the possibility of having clusters with countably many elements. We point that the context which is being handled here is more general than [9, III.1], since we deal with the geometric skew-symmetrizable cluster algebras. 2-Calabi-Yau categories have proven to be very useful to construct new examples of cluster algebras and to give a new categorical model for cluster algebras. However most of development is done only for skew symmetric cluster algebras (the simply-laced ones), one can refer for example to [29, 30, 8, 31]. The only non simply-laced example that has been done is given in [12], where using folding and skew group algebras, the author generalizes to the non simply-laced case a result of [8] about the cluster structure of the coordinate ring of maximal unipotent subgroup of simple Lie groups. The approach developed here, based on Buan-Iyama-Reiten-Scott work [9], enables us to directly realize a large class of non simply-laced cluster algebras of geometric type as well as subcluster algebras.
Along the lines of [9, III.1] and [4], we recall the definition of a cluster algebra, allowing countable cluster. Let \( m \geq n \) be non-zero positive integers or countable cardinal numbers. Let \( \mathbb{F} = \mathbb{Q}(u_1, \ldots, u_m) \) be the field of rational functions in \( m \) independent variables over the field \( \mathbb{Q} \) of rational numbers. A *seed* in \( \mathbb{F} \) is a pair \((\tilde{\mathbb{X}}, \tilde{B})\), where \( \tilde{\mathbb{X}} = \{x_1, \ldots, x_n, \ldots, x_m\} = \tilde{\mathbb{X}} \cup \tilde{\mathbb{C}} \) is a transcendence basis of \( \mathbb{F} \) formed by the union of two non-overlapping subsets \( \tilde{\mathbb{X}} = \{x_1, \ldots, x_n\} \) and \( \tilde{\mathbb{C}} = \{x_{n+1}, \ldots, x_m\} \), and \( \tilde{B} = (b_{ij}) \) is a locally finite \( m \times n \)-matrix with integer elements such that its principal part given by the \( n \times n \)-submatrix \( B \) consisting of the first \( n \) rows is *skew symmetrizable* in the sense that there exists a diagonal \( n \times n \)-matrix \( \eta = (1, \ldots, n) \) of non-zero positive integers such that \( b_{ij} \eta_i = -b_{ji} \eta_j \) for all \( 1 \leq i, j \leq n \). The subset \( \tilde{\mathbb{X}} \) is then called the *cluster* for the seed, while \( \tilde{\mathbb{C}} \) is the *coefficient set*. A *cluster algebra* (of geometric type) is a subalgebra \( \Lambda = \Lambda(S) \) of \( \mathbb{F} \) associated with a collection \( S \) of seeds in \( \mathbb{F} \) constructed in the following combinatorial way. For a seed \((\tilde{\mathbb{X}}, \tilde{B})\) in \( \mathbb{F} \) as above, and for any \( k \in \{1, \ldots, n\} \), a *seed mutation* in direction \( k \) produces a new seed \( \mu_k(\tilde{\mathbb{X}}, \tilde{B}) = (\tilde{\mathbb{X}}', \tilde{B}') \). Here \( \tilde{\mathbb{X}}' = (\tilde{\mathbb{X}} \cup \{x_k\}) \cup \{x_k'\} \) with *exchange relation*

\[
x_k'x_k = \prod_{b_{ik}>0} x_{ik}^{b_{ik}} + \prod_{b_{ik}<0} x_{-ik}^{-b_{ik}},
\]

\( \{x_k, x_k'\} \) is then called an *exchange pair*. Let’s define the common sign of each pair of integers \( a, b \in \mathbb{Z} \) by \( \text{sign}(a, b) = \text{sign}(\text{sign}(a) + \text{sign}(b)) \) where \( \text{sign}(0) = 0 \), thus, if \( a, b > 0 \) or \( a, b < 0 \) then \( \text{sign}(a, b) = \text{sign}(a) = \text{sign}(b) \in \{-1, +1\} \), otherwise \( \text{sign}(a, b) = 0 \). The matrix \( \tilde{B}' = (b_{ij}') \) is then given by the following mutation rule:

\[
b_{ij}' = \begin{cases} 
- b_{ij} & \text{if } k \in \{i, j\} \\
 b_{ij} + \text{sign}(b_{ik}, b_{kj}) b_{ik} b_{kj} & \text{otherwise.}
\end{cases}
\]

It easily seen that the seed mutation is involutive and induces an equivalence relation on the set of all seeds in \( \mathbb{F} \). Then let \( S \) be an equivalence class obtained from a fixed initial seed \((\tilde{\mathbb{X}}, \tilde{B})\) by sequences of seed mutations, next choose a subset \( \tilde{\mathbb{C}}_0 \subseteq \tilde{\mathbb{C}} \) and consider the inverted coefficient subset \( \tilde{\mathbb{C}}^{-1}_0 = \{c^{-1} : c \in \tilde{\mathbb{C}}_0\} \). The union \( \mathcal{X} \) of clusters belonging to the seeds in \( S \) is called the set of *cluster variables* for the cluster algebra \( \Lambda = \Lambda(S) \), with coefficients \( \mathcal{C}_0 \) inverted, defined as the \( \mathbb{Z}[\tilde{\mathbb{C}}^{-1}_0] \)-subalgebra of \( \mathbb{F} \) generated by \( \mathcal{X} \).

Now let \( \Lambda \) be a cluster algebra in an ambient field \( \mathbb{F} \), with \( \mathcal{X} \) as the set of cluster variables and with coefficient set \( \mathcal{C} \). Then a *subcluster algebra* \( \Lambda' \) of \( \Lambda \) is a cluster algebra such that there exists a seed \((\tilde{\mathbb{X}}, \tilde{B})\) for \( \Lambda \) and a seed \((\tilde{\mathbb{X}}', \tilde{B}')\) for \( \Lambda' \) such that

**scls1** \( \tilde{\mathbb{X}} = \tilde{\mathbb{X}}' \cup \tilde{\mathbb{C}}' \) for some subsets \( \tilde{\mathbb{X}}' = \{x_{\sigma_1}, \ldots, x_{\sigma_p}\} \subseteq \tilde{\mathbb{X}} \) and \( \tilde{\mathbb{C}}' = \{x_{\sigma_{p+1}}, \ldots, x_{\sigma_q}\} \subseteq \tilde{\mathbb{C}} \) with \( 1 \leq p \leq n \), \( 1 \leq q \leq m \) and \( \sigma : \{1, \ldots, q\} \rightarrow \{1, \ldots, m\} \) is an injective map taking each \( i \) to \( \sigma_i \).

**scls2** For each \( 1 \leq i \leq q \), and for every \( 1 \leq s \leq m \), if \( b_{\sigma_i s} \neq 0 \) then \( s = \sigma_j \) is in the image of \( \sigma \) for some (unique) \( j \in \{1, \ldots, q\} \) and \( b_{ij}' = b_{\sigma_i \sigma_j} \).

**scls3** The invertible coefficients \( \mathcal{C}'_0 \subseteq \mathcal{C}' \) satisfy \( \mathcal{X}_0 \cap \mathcal{C}'_0 \subseteq \mathcal{C}'_0 \).

Note that 2-acyclic valued quivers without loops correspond bijectively to skew symmetrizable matrices with integer coefficients, in such a way that valued quiver mutation and matrix mutation agree. Let \( Q \) be a 2-acyclic valued quiver without loops over a set of points \( I \), the corresponding skew symmetrizable matrix \((b_{ij}), j \in I\) may be defined such that for each pair \( i, j \in I \), the (only fully) valued edge \( \alpha \) between \( i \) and \( j \) is given by \( \alpha : i \overset{b_{ij}}{\rightarrow} j \), with \( \alpha \) oriented from \( i \) to \( j \) if \( \text{sign}(b_{ij}) = +1 \), otherwise \( \alpha \) is oriented from \( j \) to \( i \). Now with \( 1 \leq n \leq m \) as above, assume \( I = \{1, \ldots, m\} \) and suppose \( Q \) is an extended valued quiver with respect to the subset \( I' = \{1, \ldots, n\} \), then we can associated as before an \( m \times n \)-matrix to \( Q \) whose principal part is skew symmetrizable.

For the rest of this section, let \( C \) be a 2-Calabi-Yau or a stably 2-Calabi-Yau category with a cluster structure determined by cluster tilting objects where coefficients are given by projective points. It is assumed that each cluster tilting object (cluster tilting subcategory) has \( n \) cluster variables and \( c \) coefficients, with \( 1 \leq n \leq m \leq \infty \) and \( 0 \leq c \leq \infty \). For each cluster tilting object \( T \), the valued quiver \( Q_{\text{End}_C}(T) \) of the endomorphism algebra of \( T \) is viewed as an extended valued quiver with respect to projective points.
(by dropping all valued arrows between projective points), the corresponding $m \times n$-matrix is denoted by $B_{\text{End}(T)}$.

We recall from [9, §III.1] the following which is a generalization of cluster character in [32]. Let $\mathbb{F} = \mathbb{Q}(x_1, \ldots, x_m)$ and $\Delta$ a connected component of the cluster tilting graph of $\mathcal{C}$. A cluster map (respectively, strong cluster map) for $\Delta$ is a map $\varphi : \mathcal{E} = \text{add}\{T : T \in \Delta\} \rightarrow \mathbb{F}$ (respectively, $\varphi : \mathcal{C} \rightarrow \mathbb{F}$) which is invariant on each isoclass of objects in $\mathcal{C}$ and satisfies the following conditions.

*(clsm1)* For each cluster tilting object $T$ in $\Delta$, $\varphi(T)$ is a transcendence basis for $\mathbb{F}$.

*(clsm2)* For all indecomposable object $M$ and $N$ in $\mathcal{E}$ (respectively, in $\mathcal{C}$) with $\dim_{k_M} \text{Ext}^1_{\mathcal{C}}(M, N) = 1 = \dim_{k_N} \text{Ext}^1_{\mathcal{C}}(M, N)$, we have $\varphi(M) \varphi(N) = \varphi(U) + \varphi(U')$ where $U$ and $U'$ are the middle terms of the non-split triangles or pseudo-exact sequences $N \rightarrow U \rightarrow M$ and $M \rightarrow U' \rightarrow N$.

*(clsm3)* $\varphi(X \otimes X') = \varphi(X) \varphi(X')$ for all $X, X'$ in $\mathcal{E}$ (respectively, $\mathcal{C}$).

Important examples of strong cluster maps appear in [29, 30, 8] and [33, 32]. We point out that the condition (clsm2) is a non-simply version of conditions (M2) - (M2') required in [9, §III.1], here for each pair of indecomposable objects $M, N$ in $\mathcal{C}$ one should consider the dimensions of $\text{Ext}^1_{\mathcal{C}}(M, N)$ over the division $k$-algebras $k_M = \text{End}_{\mathcal{C}}(M) \otimes k$ and $k_N = \text{End}_{\mathcal{C}}(N) \otimes k$ instead of the base field $k$.

Note that from Lemma 3.9 we can derive the following observation also whose proof may also be derived from [5].

**Remark 6.1 ([5, Thm 7.5]).** For any weak cluster structure without loop in $\mathcal{C}$, a pair $(M, N)$ of indecomposable objects in the cluster tilting graph of $\mathcal{C}$ is an exchange pair if and only if $k_M \cong \text{Ext}^1_{\mathcal{C}}(M, N) \cong k_N$.

Note that if $(M, N)$ is an exchange pair, the the conclusion "$k_M \cong \text{Ext}^1_{\mathcal{C}}(M, N) \cong k_N$" may also be driven from Lemma 3.9 where we only assume that the pair of subcategories involved are rigid in some exact or triangulated Krull-Schmidt category which need not be 2-Calabi-Yau.

We then get the following theorem realizing skew symmetrizable cluster algebras of geometric type and whose proof, due to Buan-Iyama-Reiten-Scoot in the simply-laced case, relies totally on the properties of cluster structures.

**Theorem 6.1 ([9] for the simply-laced case).** Let $\varphi : \mathcal{E} = \text{add}\{T : T \in \Delta\} \rightarrow \mathbb{F}$ be a cluster map with previous notations. Then the following statement are true.

(a) Let $A$ be the $\mathbb{Z}$-subalgebra of $\mathbb{F}$ generated by all variables $\varphi(X)$ with $X \in \mathcal{E}$. Then $A$ is a cluster algebra and $\varphi$ sends each cluster-tilting seed $(T, \text{End}_{\mathcal{C}}(T))$ in $\Delta$ to a seed $(\varphi(T), \text{B}_{\text{End}(T)})$ in $A$.

(b) Let $B$ be a subcategory of $\mathcal{C}$ with a substructure, and $\mathcal{E}'$ a subcategory of $\mathcal{B}$ associated with a connected component of the cluster tilting graph of $\mathcal{B}$. Then the variables $\varphi(X)$ with $X \in \mathcal{E}'$ generate a subcluster algebra of $A$.

**Proof.** Statement (a) is a direct consequence of the fact that $\mathcal{C}$ has a (non-simply-laced version of) cluster structure, and cluster tilting mutation in $\mathcal{C}$ agree with seed mutation in $\mathbb{F}$. Statement (b) follows from (a) together with the following argument. Let $T'$ be a cluster tilting object in $\mathcal{B}$ that extends to a cluster tilting object $T$ in $\mathcal{E}$. Then $\varphi(T')$ gives a transcendence basis for a subfield $\mathbb{F}'$ of $\mathbb{F}$, and $(\varphi(T'), \text{B}_{\text{End}(T)})$ is a seed in a subcluster algebra of $A$. $\square$

Let $\varphi : \mathcal{E} \rightarrow \mathbb{F}$ be a cluster map realizing a cluster algebra $A$ as in Theorem 6.1, then any invertible set of coefficients $c_{ij}$ in $A$ yields a cluster algebra $A[c_{ij}^{-1}]$ called a model of the cluster map $\varphi$.

**Appendix A. Some concerns about non-simply laced preprojective algebras**

**Appendix A.1. Some considerations about dualizing pairs of bimodules**

Let $E$ and $F$ be arbitrary division rings (which are not assumed to be $k$-algebras over some field $k$), in this case we do not require the symmetrizable condition for dualizing pairs of bimodules over $E$ and $F$. Let
Let \( M \) be any \( E \)-bimodule, the left dual and the right dual of \( M \) are the \( E \)-bimodules given respectively by \( 'M := \text{Hom}_E(M,E) \) and \( M^* := \text{Hom}_F(M,F) \). If \( \dim_E(M) \) is finite, then for any vector spaces \( V_1, V_2 \) and \( E \), the functorial adjunction of the tensor product yields a natural isomorphism \( \text{Hom}_E(M \otimes V_1, V_2) \cong \text{Hom}_E(V_1, M \otimes V_2) \). Let each morphism \( u : M \otimes X \to Y \) to its (left) adjoint \( \pi : X \to M^* \otimes V \) defined as follows: if \( S = \{m_1, \ldots, m_p\} \) is a left \( E \)-basis of \( M \) and \( \{\tilde{m}_1, \ldots, \tilde{m}_p\} \) is the corresponding dual basis for the left dual \( 'M \), then for every \( x \in X \) we have: 
\[
\pi(x) = \sum_{k=1}^p \tilde{m}_k \otimes u(m_k \otimes x).
\]

Similarly, if \( \dim_F(M) \) is finite, then for any vector spaces \( V_1, V_2 \) and \( F \), under the functorial adjunction \( \text{Hom}_F(V_1, M) \cong \text{Hom}_F(M \otimes V, F) \), the (right) adjoint of a morphism \( v : V \otimes M \to W \) is the morphism \( \tilde{v} : V \to W \otimes M^* \) defined as follows: if \( \{z_1, \ldots, z_q\} \) is a right \( F \)-basis of \( M \) and \( \{\tilde{z}_1, \ldots, \tilde{z}_q\} \) is the corresponding dual basis for the right dual \( M^* \), then for every \( x \in V \) we have: 
\[
\tilde{v}(x) = \sum_{s=1}^q u(x \otimes z_s) \otimes \tilde{z}_s.
\]

Let \( \{M, M^*, b_E, b_F\} \) be a dualizing pair of bimodules with associated non-degenerated bilinear forms \( M \otimes M^* \overset{b_E}{\to} E \) and \( M^* \otimes M \overset{b_F}{\to} F \). Then the left and the right adjoints of these two non-degenerated bilinear forms are isomorphisms of bimodules; for example, the left adjoint and the right adjoint of \( b_E \) are the following isomorphisms: 
\[
\overline{b_E} : M^* \overset{\sim}{\to} M, \quad b_E : M \overset{\sim}{\to} (M^*)^o \quad \text{with} \quad \overline{b_E}(\xi) = b_E(- \otimes \xi) \text{ and } b_E(x) = b_E(x \otimes -)
\]
for every \( \xi \in M^* \) and \( x \in M \). Thus for each left \( E \)-basis \( S = \{m_1, \ldots, m_p\} \) of \( M \), the corresponding dual basis for \( M^* \) is defined by \( S^* = \{m_1^*, \ldots, m_p^*\} \) such that \( b_E(m_1^*) = \tilde{m}_1 \), so that under the left adjoint \( \overline{b_E} \), \( S^* \) corresponds to the dual basis \( \{\tilde{m}_1, \ldots, \tilde{m}_p\} \) of the left dual bimodule \( 'M \); hence \( S^* \) is characterized by the following property: 
\[
b_E(m_k \otimes m_l^*) = \delta_{k,l} \text{ for all } 1 \leq k, l \leq p.
\]
Similarly, for each right \( F \)-basis \( Z = \{z_1, \ldots, z_q\} \) of \( M \), under the right adjoint \( b_F : M^* \overset{\sim}{\to} M \) the dual basis \( \tilde{Z} = \{\tilde{z}_1, \ldots, \tilde{z}_q\} \) of the right dual \( M^* \) of \( M \) corresponds to the dual basis \( Z^* = \{z_1^*, \ldots, z_q^*\} \) of \( M^* \) associated with \( Z \) and defined by the following characterization property: 
\[
b_F(z_s^* \otimes z_t) = \delta_{s,t} \text{ for all } 1 \leq s, t \leq q.
\]

Now suppose \( \{N, N^*, b_E, b_F\} \) is another dualizing pair of bimodules with non-degenerated bilinear forms \( N \otimes N^* \overset{b_E}{\to} E \) and \( N^* \otimes N \overset{b_F}{\to} F \). Thus given any morphism \( f : M \to N \) of \( E \)-bimodules and considering the morphisms 
\[
(- \circ f) := \text{Hom}_E(f, E) : N \to M^* \text{ and } (- \circ f) := \text{Hom}_F(f, F) : N^* \to M \text{,}
\]
we can form the left dual and the right dual of \( f \) (with respect to the two dualizing pairs above), they are morphisms of bimodules \( f^* : N^* \to M^* \) given as follows:
\[
\begin{align*}
\quad f^* & := (\overline{b_E})^{-1} \circ (- \circ f) \circ b_E \quad \text{ and } \quad f^* := (\overline{b_F})^{-1} \circ (- \circ f) \circ b_F, \quad \text{equivalently,} \\
& \quad b_E(- \otimes f(-)) = b_E(f(-) \otimes -) \quad \text{and} \quad b_F(f^*(-) \otimes -) = b_F(f^*(-) \otimes -).
\end{align*}
\]

We note that the left and the right dual of a morphism as above need not coincide, unless we impose additional assumption on dualizing pairs of bimodules, which for some practical computations could simplify the use of modulated quivers, see Appendix A.1.2. We may then say that a morphism \( f : M \to N \), as above, induces a morphism of dualizing pair of bimodules (or is a dualizing morphism) if the left dual of \( f \) coincides with the right dual of \( f \), in this case their common value is denoted by \( f^* : N^* \overset{\sim}{\to} M^* \).

When no precision is needed, we sometimes omit to specify the non-degenerated bilinear forms \( b \) associated with our dualizing pairs of bimodules, and in this case, using only one symbol \( "()" \) we write \((u \otimes v)\) for \( b(u \otimes v)\).

Next, we can form products of dualizing pairs of bimodules to obtain dualizing pairs of bimodules: let \( L \) be another division ring and suppose \( \{M_1, M_1^*\} \) and \( \{M_2, M_2^*\} \) are two dualizing pairs of bimodules (in which the non-degenerated bilinear forms are omitted and will represented by the same symbol \( "()" \)), with \( M_1 \in \text{bimod}_L \) and \( M_2 \in \text{bimod}_F \), then their product defined by \( \{M_1, M_1^*\} \otimes \{M_2, M_2^*\} = \{M_1 \otimes L, M_2 \otimes L, M_1^* \otimes L, M_2^* \otimes L\} \) is again a dualizing pair of bimodules where the associated non-degenerated bilinear forms \( M_1 \otimes_L M_2 \otimes_F M_1^* \otimes_L M_2^* \otimes_F E \) and \( M_2 \otimes_L M_1^* \otimes_E M_1 \otimes_L M_2^* \otimes_F E \) are canonically induced such that for every \( x_1 \in M_1 \) and \( u_1 \in M_1^* \) with \( i = 1, 2 \), we have:
\[
((x_1 \otimes x_2) \otimes (u_2 \otimes u_1)) = (x_1(x_2 \otimes u_2) \otimes u_1) \quad \text{and} \quad ((u_2 \otimes u_1) \otimes (x_1 \otimes x_2)) = (u_2(u_1 \otimes x_1) \otimes x_2).
\]

We now observe that each division ring \( E \) yields a canonical dualizing pair of \( E \)-bimodules \( \{E, E\} \) with non-degenerated bilinear forms given by the ordinary multiplication of \( E \). Moreover, any dualizing pair of
bimodules \( \{ M, M^*, b_E, b_F \} \), with \( M \in \mathcal{E}_{\text{bimod}} \), induces in a canonical way two dualizing pairs of bimodules \( \{ M \otimes M^*, M \otimes M^* \} \) and \( \{ M^* \otimes M, M^* \otimes M \} \) in which \( M \otimes M^* \) and \( M^* \otimes M \) are self-dual bimodules. With previous notations and with respect to the dualizing pairs of bimodules induced by the self-dual bimodules \( E, F, M \otimes M^* \) and \( M^* \otimes M \), let’s point out the following observation.

**Appendix A.1.1.** The two non-degenerated bilinear forms \( M \otimes M^* \xrightarrow{b_E} E \) and \( M^* \otimes M \xrightarrow{b_F} F \) are dualizing: the left dual and the right dual of each of them coincide. Thus the dual morphism \( E \xrightarrow{b_E^*} M \otimes M^* \) of \( b_E \) is the common value of its left and its right dual, \( b_E^* \) takes the identity element of \( E \) to what we call the central element \( \sum_{y \in Y} y \otimes y^* \) of the self-dual bimodule \( M \otimes M^* \) (with respect to \( b_E \) and \( b_F \)) given by choosing any pair \( \{ Y, Y^* \} \) of two mutually dual basis where \( Y \) is a right \( F \)-basis of \( M \) and \( Y^* = \{ y^*: y \in Y \} \) is the corresponding dual basis for \( M^* \). Similarly, the dual morphism \( F \xrightarrow{b_F^*} M^* \otimes M \) of \( b_F \) takes the identity element of \( F \) to the central element \( \sum_{x \in X} x^* \otimes x \) of the self-dual bimodule \( M^* \otimes M \) where \( X \) is a left \( E \)-basis for \( M \) and \( X^* \) is the corresponding dual basis for \( M^* \).

**Appendix A.1.2.** The canonical morphisms \( b_E^* \) and \( b_F^* \) above inspired us to introduce \([13]\) the notion of potential for \( k \)-modulated quivers \( Q \) with symmetrizable dualizing pairs of modules for a commutative base field \( k \). The symmetrizable condition states that the non-degenerated bilinear forms associated with \( Q \) are chosen to be canonically induced by choosing a fixed family \( (t_i)_{i \in I} \) of non-degenerated so-called trace forms \( t_i \in \text{Hom}_k(k_i, k) \) for the division \( k \)-algebras \( k_i, i \in I \). Without such a symmetrizable condition some necessary computations inside the path algebras of the modulated quiver will fail.

**Appendix A.2. The preprojective algebra of a modulated graph**

Keeping the notations of previous subsection, we now recall the definition of the preprojective algebra associated with a modulated graph along the lines of Dlab-Ringel \([34]\). Let \( (\Gamma, \mathfrak{M}) \) be a modulated graph over a set of points \( \Gamma_0 := I \) (and having a normal form and no loop): between two distinct points \( i \) and \( j \) in \( I \) there is only one valued (may be zero-valued) edge, the modulation \( \mathfrak{M} \) prescribes for each point \( i \in I \) a division ring \( k_i \), and for each edge with endpoints \( i \) and \( j \) a dualizing pair of bimodules \( \{ b_j : j \in I \} \in (B_j) \) with \( i B_j \otimes \mathfrak{M} j B_j \xrightarrow{b_j} k_i \) and \( j B_i \otimes \mathfrak{M} i B_i \xrightarrow{b_i} k_j \), so that the unique fully valued edge of \( \Gamma \) representing the set \( \Gamma_1(i,j) = \Gamma_1(j,i) \) may be pictured as follows: \( k_i \xrightarrow{b_i} k_j \). We usually write \( j B_i = i B_j \) and \( i B_j = j B_i \).

Then consider the semisimple ring \( K = \prod_{i \in I} k_i \) in which each division ring \( k_i \) is viewed as a subring of \( K \) with unit \( e_i \), so that the unit of \( K \) is given by \( 1 = \sum_{i \in I} e_i \) while the set \( \{ e_i : i \in I \} \) is a system of primitive orthogonal idempotents for \( K \). Next consider the \( K \)-bimodule \( B_\Gamma = \bigoplus_{i \in I} B_i \) associated with the modulated graph \( \Gamma \), together with the tensor \( \mathbb{N} \)-graded path algebra \( T(\Gamma) := T_K(B_\Gamma) = \bigoplus_{m \geq 0} T(\Gamma)_m \) where \( T(\Gamma)_m = B_\Gamma^m \) is the \( m \)-fold tensor product of the \( K \)-bimodule \( B_\Gamma \) (called the \( K \)-bimodule of length-\( m \) paths in \( T(\Gamma) \)), with \( T(\Gamma)_0 = K, T(\Gamma)_1 = B_\Gamma \) and \( T(\Gamma)_m+1 = B_\Gamma^m \otimes_K B_\Gamma \) for all \( m \geq 1 \). Now let \( b^* := \bigoplus_{i \in I} b_i^* : K \xrightarrow{b_i^*} B_\Gamma \otimes B_\Gamma \) be the morphism of \( K \)-bimodules induced by the canonical morphisms \( b_i^* : k_i \xrightarrow{b_i} B_i \otimes B_i \). Then \( b^* \) may be identified with the corresponding \( K \)-central element \( b^*(1) \) in \( B_\Gamma \otimes B_\Gamma \subset T(\Gamma) \) which is homogeneous of degree 2.

**Appendix A.2.1.** The preprojective algebra associated with \( \Gamma \) is the quadratic algebra defined as the factor algebra \( \Lambda = \Lambda_\Gamma := T(\Gamma)/\langle b^* \rangle \) where \( \langle b^* \rangle := \langle b^*(1) \rangle \) is the principal ideal of \( T(\Gamma) \) generated by the homogeneous element \( b^*(1) \) of degree 2.

We note the following observation. We may prescribe an orientation for \( \Gamma \) to obtain a modulated quiver \( Q = (Q, \mathfrak{M}) \): thus for each non-zero valued edge \( \alpha : \xrightarrow{i,j} \) we write \( i < j \) (or \( \text{sign}(i,j) = +1 \)) to indicate that in \( Q \), \( \alpha \) is oriented from \( i \) to \( j \), of course when \( \Gamma_1(i,j) = \emptyset \) we write \( \text{sign}(i,j) = 0 = \text{sign}(j,i) \). Now
the arrow $K$-bimodule of $Q$ is $B = \bigoplus_i B_i$, we then call the $K$-bimodule $B^* := \bigoplus_{i,j \in I} B_i \otimes_{K} B_j^*$ the dual of $B$. $B^*$ is indeed the arrow bimodule of the dual modulated quiver $Q^*$ of $Q$, over the same set $I$ with the same modulation, but the underlying valued quiver of $Q^*$ is the opposite $Q^*$ of the underlying valued quiver of $Q$. Now the (tensor) path algebra of $Q$ is the $K$-algebra $T(Q) = T_K(B)$. Then since $B \otimes B^* = \bigoplus_{i,j \in I} B_i \otimes B_j^*$, we see that the preprojective algebra of $\Gamma$ (also called the preprojective algebra of $Q$) is given by $\Lambda_{\Gamma} = T(Q \oplus Q^*)(b^*)$, thus the preprojective algebra of a modulated quiver is just the preprojective algebra of the underlying modulated graph and does not depend on the orientation.

As mentioned in section 4, for a simply-laced Dynkin graph $\Delta$, by Keller's approach [7, §8.4], the category of projective modules over the preprojective algebra $\Lambda_{\Delta}$ is 1-Calabi-Yau, so that the stable category $\text{mod}\Lambda_{\Delta}$ is 2-Calabi-Yau [7, §8.5]; the latter conclusion could already be derived from the work of Auslander-Reiten [24, Propositions 3.1 and 1.2] where they used functors categories and a concept of dualizing $R$-variety, or from the work of Brenner-Butler-King [35, Thm 4.8.4.9] where they used functors categories and a concept of dualizing $R$-variety. One therefore notices that simply-laced preprojective algebras are widely studied by many authors and have proven to be useful for example in the cluster theory. In [12] the author generalizes some results of [8] to non simply-laced preprojective algebras obtained as skew group algebras of the preprojective algebras associated with non Dynkin quivers with the focus on the existence of cluster structures and subcategories.

One therefore notices that simply-laced preprojective algebras are widely studied by many authors and have proven to be useful for example in the cluster theory. In [12] the author generalizes some results of [8] to non simply-laced preprojective algebras obtained as skew group algebras of the preprojective algebras associated with non Dynkin quivers (A, D, E). On the other hand it seems that very few information is known about the representation theory of preprojective algebras of (non-simply laced) modulated graphs; until now what is really known about these algebras is given by the introductory work of Dlab-Ringel in [16] (1980), their main result can be stated as follows: for any modulated quiver $Q$ with underlying modulated graph $\Gamma$, $T(Q)$ is a subalgebra of the preprojective algebra $\Lambda = \Lambda_{\Gamma}$ of $\Gamma$ and, as a (right or left) $T(Q)$-module, $\Lambda$ is the direct sum of all indecomposable preprojective $T(Q)$-modules (each one occurring with multiplicity one). From this result it follows that $\Lambda$ is artinian as a ring if and only if the modulated graph $\Gamma$ is a disjoint union of Dynkin modulated graphs.

Suppose $\Gamma$ is now a $k$-modulated graph of type $B_n, C_n, F_4$ or $G_3$, for an arbitrary (necessary non algebraically closed) commutative field $k$ acting centrally on bimodules in $\Gamma$ (with each division ring in $\Gamma$ being now a finite-dimensional $k$-algebra). For the non simply-laced preprojective algebra $\Lambda$ associated the Dynkin $k$-modulated graph $\Gamma$, it would be interesting to have a clear answer for the following elementary questions:

**Question 1.**

1. Is the category $\text{proj}\Lambda_{\Gamma}$ of finite-dimensional $\Lambda$-modules still equivalent to $\text{D}^b(\Lambda)_{\Gamma}$ (as in [7, §8.4] for the simply laced case)? Here $\text{D}^b(\Lambda)$ is the bounded derived category of finite-dimensional $\Lambda$-modules.

2. Is $\Lambda$ still self-injective (as in [35, Thm 4.8.4.9] for the simply laced case)?

Once it is clear that previous questions yield a positive answer, using may be Keller’s approach (or reworking Geiss-Leclerc-Shröer approach which gives a detailed proof of 2-Calabi-Yau property of $\text{mod}\Lambda_{\Delta}$), it would follows immediately (by [7, §8.5]) that $\text{mod}\Lambda$ is stably 2-Calabi-Yau having cluster structure if in addition $\Lambda$ has finite global dimension [9, Prop I.11].

Let us also point out that in [9, II], Buan Iyama Reiten and Scott also study simply-laced preprojective algebras associated with non Dynkin quivers with the focus on the existence of cluster structures and substructures.

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