On Quillen’s calculation of graded $K$-theory

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Abstract We adapt Quillen’s calculation of graded $K$-groups of $\mathbb{Z}$-graded rings with support in $\mathbb{N}$ to graded $K$-theory, allowing gradings in a product $\mathbb{Z} \times G$ with $G$ an arbitrary group. This in turn allows us to use induction and calculate graded $K$-theory of $\mathbb{Z}^m$-graded rings.

Keywords Graded ring · Graded $K$-theory

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1 Introduction

Let $G$ be a group, written additively, and let $A$ be a $G$-graded ring. We denote the category of finitely generated $G$-graded projective right $A$-modules by $\mathcal{P}_G^G(A)$. This is an exact category with the usual notion of (split) short exact sequence, and we denote...
Theorem 1

Let \( G \) be a group, and let \( A \) be a \( \mathbb{Z} \)-graded ring. Then there is a group isomorphism \( K_i^G(\mathbb{Z}[G]) \). The group \( G \) acts on the category \( \mathbb{Z}[G] \) from the right via \( (P, g) \mapsto P(g) \), where \( P(g)_h = P_{g+h} \). By functoriality of \( K \)-groups this equips \( K_i^G(A) \) with the structure of a right \( \mathbb{Z}[G] \)-module.

If \( A \) is strongly graded, i.e., if \( 1 \in A_g A_{-g} \) for all \( g \in G \), then by Dade’s Theorem ([3, Theorem 3.1.1]) the functor \( (-)_0 : \mathbb{Z}[G] \to \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \) so that it induces an equivalence of categories. This implies there is a group isomorphism \( K_i^G(A) \cong K_i(A_0) \), for \( i \geq 0 \).

The relation between graded \( K \)-groups and non-graded \( K \)-groups is not always apparent. For example consider the \( \mathbb{Z} \)-graded matrix ring \( A = \mathbb{M}_5(F)(0, 1, 2, 2, 3) \), where \( F \) is a field (see [3, §9.2] for details). Using graded Morita theory one can show that \( K_0(A_0) \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \). Note also that \( K_0(A) \cong \mathbb{Z} \).

For a \( \mathbb{Z} \)-graded ring \( A \) with support in \( \mathbb{N} \) the graded \( K \)-theory of \( A \) was determined by Quillen [4, Proposition, p. 107]: The functor \( P \mapsto P \otimes_{A_0} A \) induces an isomorphism of \( \mathbb{Z}[x, x^{-1}] \)-modules

\[
K_i(A_0) \otimes_{\mathbb{Z}[x, x^{-1}]} \mathbb{Z}[x, x^{-1}] \cong K_i^\mathbb{Z}(A).
\]

(1)

Contrary to other central theorems in the subject, such as fundamental theorem of \( K \)-theory (i.e., \( K_i(R[x, x^{-1}]) = K_i(R) \times K_{i-1}(R) \), for \( R \) a regular ring), one cannot use an easy induction on (1) to write a similar statement for “multi-variables” rings. For example, it appears that there is no obvious inductive approach to generalise (1) to \( \mathbb{Z}^m \times G \)-graded rings. However, by generalising Quillen’s argument to take gradings into account on both sides of the isomorphism, such a procedure becomes feasible.

We will prove the following statement:

**Theorem 1** Let \( G \) be a group, and let \( A \) be a \( \mathbb{Z} \times G \)-graded ring with support in \( \mathbb{N} \times G \). Then there is a \( \mathbb{Z}[\mathbb{Z} \times G] \)-module isomorphism

\[
K_i^G(A_{(0,-)}) \otimes_{\mathbb{Z}[\mathbb{Z} \times G]} \mathbb{Z}[\mathbb{Z} \times G] \cong K_i^{\mathbb{Z} \times G}(A),
\]

where \( A_{(0,-)} = \bigoplus_{g \in G} A_{(0,g)} \).

By a straightforward induction this now implies:

**Corollary 2** For a \( \mathbb{Z}^m \times G \)-graded ring \( A \) with support in \( \mathbb{N}^m \times G \) there is a \( \mathbb{Z}[x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_m, x_m^{-1}] \)-module isomorphism

\[
K_i^G(A_{(0,-)}) \otimes_{\mathbb{Z}} \mathbb{Z}[x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_m, x_m^{-1}] \cong K_i^{\mathbb{Z}^m \times G}(A).
\]

For a trivial group \( G \) this is a direct generalisation of Quillen’s theorem to \( \mathbb{Z}^m \)-graded rings.

2 Swan’s theorem

As in Quillen’s calculation the proof of the Theorem is based on a version of Swan’s Theorem, modified to the present situation: it provides a correspondence between
isomorphism classes of \( \mathbb{Z} \times G \)-finitely generated graded projective \( A \)-modules and of \( G \)-finitely generated graded projective \( A_{(0, \cdot)} \)-modules.

**Proposition 3** Let \( \Gamma \) be a (possibly non-abelian) group. Let \( A \) be a \( \Gamma \)-graded ring, \( A_0 \) a graded subring of \( A \) and \( \pi : A \to A_0 \) a graded ring homomorphism such that \( \pi|_{A_0} = 1 \). (In other words, \( A_0 \) is a retract of \( A \) in the category of \( \Gamma \)-graded rings.) We denote the kernel of \( \pi \) by \( A_+ \).

Suppose that for any finitely generated graded right \( A \)-module \( M \) the condition \( MA_+ = M \) implies \( M = 0 \). Then the natural functor

\[
S = - \otimes_{A_0} A : \mathcal{P}^\Gamma(A_0) \to \mathcal{P}^\Gamma(A)
\]

induces a bijective correspondence between the isomorphism classes of finitely generated graded projective \( A_0 \)-modules and of finitely generated graded projective \( A \)-modules. An inverse of the bijection is given by the functor

\[
T = - \otimes_A A_0 : \mathcal{P}^\Gamma(A) \to \mathcal{P}^\Gamma(A_0).
\]

There is a natural isomorphism \( T \circ S \cong \text{id} \), and for each \( P \in \mathcal{P}^\Gamma(A) \) a non-canonical isomorphism \( S \circ T(P) \cong P \). The latter is given by

\[
T(P) \otimes_{A_0} A_0 \to P, \quad x \otimes a \mapsto g(x) \cdot a,
\]

where \( g \) is an \( A_0 \)-linear section of the epimorphism \( P \to T(P) \).

**Proof** For any finitely generated graded projective \( A_0 \)-module \( Q \) we have a natural isomorphism \( T S(Q) \cong Q \) given by

\[
\nu_Q : T S(Q) = Q \otimes_{A_0} A \otimes_A A_0 \to Q, \quad q \otimes a \otimes a_0 \mapsto q \pi(a) a_0.
\]

We will show that for a graded projective \( A \)-module \( P \) there is a non-canonical graded isomorphism \( ST(P) \cong_{\text{gr}} P \). The lemma then follows.

Consider the natural graded \( A \)-module epimorphism

\[
f : P \to T(P) = P \otimes_A A_0, \quad p \mapsto p \otimes 1.
\]

Here \( T(P) \) is considered as an \( A \)-module via the map \( \pi \). Since \( T(P) \) is a graded projective \( A_0 \)-module, the map \( f \) has a graded \( A_0 \)-linear section \( g : T(P) \to P \). This section determines an \( A \)-linear graded map

\[
\psi : ST(P) = P \otimes_A A_0 \otimes A_0 \to P, \quad p \otimes a_0 \otimes a \mapsto g(p \otimes a_0) \cdot a,
\]

and we will show that \( \psi \) is an isomorphism. First note that the map

\[
T(f) : T(P) \to TT(P), \quad p \otimes a_0 \mapsto f(p) \otimes a_0 = p \otimes 1 \otimes a_0
\]

is an isomorphism (consider \( T(P) \) as an \( A \)-module via \( \pi \) here). In fact the inverse is given by the isomorphism \( TT(P) = P \otimes_A A_0 \otimes A_0 \to P \otimes_A A_0 \) which
maps $p \otimes a_0 \otimes b_0$ to $p \otimes (a_0 b_0)$. Tracing the definitions now shows that both composites

$$TST(P) \xrightarrow{T(\psi)} T(P) \xrightarrow{T(f)} TT(P)$$

map $p \otimes a_0 \otimes a \otimes b_0 \in P \otimes_A A_0 \otimes_A A_0 \otimes A_0 = TST(P)$ to the element $f(g(p \otimes a_0) \cdot a) \otimes b_0 = p \otimes (a_0 \pi(a) b_0) \otimes 1 \in TT(P)$. This implies that $T(\psi) = \nu_{T(P)}$, which is an isomorphism.

The exact sequence

$$0 \to \ker \psi \to ST(P) \xrightarrow{\psi} P \to \coker \psi \to 0 \quad (4)$$

gives rise, upon application of the right exact functor $T$, to an exact sequence

$$TST(P) \xrightarrow{T(\psi)} T(P) \to T(\coker \psi) \to 0.$$ 

Since $T(\psi)$ is an isomorphism we have $T(\coker \psi) = \coker T(\psi) = 0$. Since $\coker \psi$ is finitely generated by (4) this implies $\ker \psi = 0$ (note that $T(M) = M / MA_+$ for every finitely generated module $M$). In other words, $\psi$ is surjective and (4) becomes the short exact sequence $0 \to \ker \psi \to ST(P) \xrightarrow{\psi} P \to \ker \psi = 0$. This latter sequence splits since $P$ is projective; this immediately implies that $\ker \psi$ is finitely generated, and since $T(\psi)$ is injective we also have $T(\ker \psi) = \ker T(\psi) = 0$. The hypotheses guarantee $\ker \psi = 0$ now so that $\psi$ is injective as well as surjective, and thus is an isomorphism as claimed.

3 A lemma on graded $K$-theory

**Lemma 4** Let $G$ and $\Gamma$ be groups, and let $A$ be a $G$-graded ring. Then, considering $A$ as a $\Gamma \times G$-graded ring in a trivial way where necessary, the functorial assignment $(M, \gamma) \mapsto M(\gamma, 0)$ induces a $\mathbb{Z}[\Gamma \times G]$-module isomorphism

$$K_i^{\Gamma \times G}(A) \cong K_i^{G}(A) \otimes_{\mathbb{Z}[G]} \mathbb{Z}[\Gamma \times G].$$

**Proof** Let $P = \bigoplus_{(\gamma, g) \in \Gamma \times G} P_{(\gamma, g)}$ be a $\Gamma \times G$-finitely generated graded projective $A$-module. Since the support of $A$ is $G = 1 \times G$, there is a unique decomposition $P = \bigoplus_{\gamma \in \Gamma} P_{\gamma}$, where the $P_{\gamma} = \bigoplus_{g \in G} P_{(\gamma, g)}$ are finitely generated $G$-graded projective $A$-modules. This gives a natural isomorphism of categories

$$\Psi : \mathcal{P}_{g}^{\Gamma \times G}(A) \xrightarrow{\cong} \bigoplus_{\gamma \in \Gamma} \mathcal{P}_{g}^{G}(A).$$

The natural right action of $\Gamma \times G$ on these categories is described as follows: for a given module $P \in \mathcal{P}_{g}^{\Gamma \times G}(A)$ as above and elements $(\gamma, g) \in \Gamma \times G$ we have

\[ \Psi : \mathcal{P}_{g}^{\Gamma \times G}(A) \xrightarrow{\cong} \bigoplus_{\gamma \in \Gamma} \mathcal{P}_{g}^{G}(A). \]
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so that $\Psi(P(\gamma, g)) = \Psi(P)(\gamma, g)$. Since $K$-groups respect direct sums we thus have a chain of $\mathbb{Z}[(\Gamma \times G)]$-linear isomorphisms

$$K_i^{\Gamma \times G}(A) = K_i(\mathcal{P}_{gr}^{\Gamma \times G}(A)) \cong K_i \left( \bigoplus_{\gamma \in \Gamma} \mathcal{P}_{gr}^G(A) \right) = \bigoplus_{\gamma \in \Gamma} K_i^G(A) \cong K_i^G(A) \otimes_{\mathbb{Z}[\Gamma]} \mathbb{Z}[\Gamma \times G].$$

$\square$

4 Proof of Theorem 1

Let $A$ be a $\mathbb{Z} \times G$-graded ring with support in $\mathbb{N} \times G$. That is, $A$ comes equipped with a decomposition

$$A = \bigoplus_{\omega \in \mathbb{N}} A_{(\omega, -)} \quad \text{where} \quad A_{(\omega, -)} = \bigoplus_{g \in G} A_{(\omega, g)}.$$

The ring $A$ has a $\mathbb{Z} \times G$-graded subring $A_{(0, -)}$ (with trivial grading in $\mathbb{Z}$-direction). The projection map $A \to A_{(0, -)}$ is a $\mathbb{Z} \times G$-graded ring homomorphism; its kernel is denoted $A_+$. Explicitly, $A_+$ is the two-sided ideal

$$A_+ = \bigoplus_{\omega > 0} A_{(\omega, -)}.$$

We identify the quotient ring $A/A_+$ with the subring $A_{(0, -)}$ via the projection.

4.1 Functors

If $P$ is a finitely generated graded projective $A$-module, then $P \otimes_A A_{(0, -)}$ is a finitely generated $\mathbb{Z} \times G$-graded projective $A_{(0, -)}$-module. Similarly, if $Q$ is a finitely generated graded projective $A_{(0, -)}$-module then $Q \otimes_{A_{(0, -)}} A$ is a $\mathbb{Z} \times G$-finitely generated graded projective $A$-module. We can thus define functors

$$T = - \otimes_A A_{(0, -)} : \mathcal{P}_{gr}^{\mathbb{Z} \times G}(A) \longrightarrow \mathcal{P}_{gr}^{\mathbb{Z} \times G}(A_{(0, -)})$$

and

$$S = - \otimes_{A_{(0, -)}} A : \mathcal{P}_{gr}^{\mathbb{Z} \times G}(A_{(0, -)}) \longrightarrow \mathcal{P}_{gr}^{\mathbb{Z} \times G}(A).$$

Since $T(P) = P/PA_+$ we see that the support of $T(P)$ is contained in the support of $P$.

Observe now that if $M$ is a finitely generated $\mathbb{Z} \times G$-graded $A$-module and $MA_+ = M$ then $M = 0$; for if $M \neq 0$ there is a minimal $\omega \in \mathbb{Z}$ such that $M_{(\omega, -)} \neq 0$, but $(MA_+)(\omega, -) = 0$. It follows from Proposition 3 that for each graded finitely generated
projective $A$-module $P$ there is a non-canonical isomorphism $P \cong T(P) \otimes_{A(0,-)} A$ as in (2) which respects the $\mathbb{Z} \times G$-grading. Explicitly, for a given $(\omega, g) \in \mathbb{Z} \times G$ we have an isomorphism of abelian groups

$$P_{(\omega,g)} \cong \bigoplus_{(k,h)} T(P)_{(k,h)} \otimes A(-k+\omega,-h+g); \quad (5)$$

the tensor product $T(P)_{(k,h)} \otimes A(-k+\omega,-h+g)$ denotes, by convention, the abelian subgroup of $T(P) \otimes_{A(0,-)} A$ generated by primitive tensors of the form $x \otimes y$ with homogeneous elements $x \in T(P)$ of degree $(k, h)$ and $y \in A$ of degree $(-k + \omega, -h + g)$.

4.2 Filtration

For a $\mathbb{Z} \times G$-graded $A$-module $P$ write $P = \bigoplus_{\omega \in \mathbb{Z}} P_{(\omega,-)}$, where $P_{(\omega,-)} = \bigoplus_{g \in G} P_{(\omega,g)}$. For $\lambda \in \mathbb{Z}$ let $F^\lambda P$ denote the $A$-submodule of $P$ generated by the elements of $\bigcup_{\omega \leq \lambda} P_{(\omega,-)}$; this is $\mathbb{Z} \times G$-graded again. As an explicit example, we have

$$F^\lambda A(\omega, g) = \begin{cases} A(\omega, g) & \text{if } \lambda \geq -\omega, \\ 0 & \text{else.} \end{cases}$$

Suppose that $P$ is a finitely generated graded projective $A$-module. Since the support of $A$ is contained in $\mathbb{N} \times G$ there exists $n \in \mathbb{Z}$ such that $F^{-n} P = 0$ and $F^n P = P$. Write $\mathcal{P}gr_{\mathbb{Z} \times G}^n (A)$ for the full subcategory of $\mathcal{P}gr_{\mathbb{Z} \times G} (A)$ spanned by those modules $P$ which satisfy $F^{-n} P = 0$ and $F^n P = P$. Then $\mathcal{P}gr_{\mathbb{Z} \times G} (A)$ is the filtered union of the $\mathcal{P}gr_{\mathbb{Z} \times G}^n (A)$.

Let $P \in \mathcal{P}gr_{\mathbb{Z} \times G} (A)$; we want to identify $F^\lambda P$. By definition, the $A$-module $F^\lambda P$ is generated by the elements of $P_{(\omega,g)}$ for $\omega \leq \lambda$, with $P_{(\omega,g)}$ having been identified in (5). We remark that the direct summands in (5) indexed by $\kappa > \omega$ are trivial as $A$ has support in $\mathbb{N} \times G$. On the other hand, for $\omega \geq \kappa$ a given primitive tensor $x \otimes y \in P_{(\omega,g)}$ with $x \in T(P)_{(k,h)}$ and $y \in A(-k+\omega,-h+g)$ can always be re-written, using the right $A$-module structure of $T(P) \otimes_{A(0,-)} A$, as

$$x \otimes y = (x \otimes 1) \cdot y \quad \text{where } x \otimes 1 \in T(P)_{(k,h)} \otimes A(0,0) \subseteq P_{(k,h)}.$$ 

That is, the $A$-module $F^\lambda P$ is generated by those summands of (5) with $\kappa = \omega \leq \lambda$. We claim now that $F^\lambda P$ is isomorphic to

$$M^{(\lambda)} = \bigoplus_{\kappa \leq \lambda} T(P)_{(k,-)} \otimes_{A(0,-)} A(-\kappa, 0), \quad (6)$$
considering $T(P)_{(k,-)}$ as a $\mathbb{Z} \times G$-graded $A_{(0,-)}$-module with support in $[0] \times G$. The homogeneous components of $M^{(k)}$ are given by

$$M^{(k)}_{(\omega,g)} = \bigoplus_{\kappa \leq k} \bigoplus_{h \in G} T(P)_{(\kappa,h)} \otimes A(-\kappa,0)_{(\omega,h+g)}.$$ 

Now elements of the form $x \otimes 1 \in T(P)_{(\kappa,h)} \otimes A(-\kappa,0)_{(\kappa,-h+h)} \subseteq M^{(k)}_{(\kappa,h)}$ clearly form a set of $A$-module generators for $M^{(k)}$ so that, by the argument given above, $F^k P$ and $M^{(k)}$ have the same generators in the same degrees. The claim follows. The module $F^k P$ is finitely generated (viz., by those generators of $P$ that have $\mathbb{Z}$-degree at most $\lambda$). Since $T(P)$ is a finitely generated projective $A_{(0,-)}$-module so is its summand $T(P)_{(k,-)}$; consequently, $P \mapsto F^k P$ is an endofunctor of $\mathbb{G}_q$. It is exact as can be deduced from the (non-canonical) isomorphism in (6), using exactness of tensor products.

4.3 Filtration quotients

From the isomorphism $F^k P \cong M^{(k)}$, cf. (6), we obtain an isomorphism

$$(F^{k+1} P/F^k P \cong T(P)_{(k,-)} \otimes_{A_{(0,-)}} A(-k,0); \ (7))$$

in particular, $F^{k+1} P/F^k P \in \mathbb{G}_q\otimes_{\mathbb{G}_q}(A)$. The isomorphism (7) depends on the isomorphism (6), and thus ultimately on (2). The latter depends on a choice of a section $g$ of $P \rightarrow T(P)$. Given another section $g_0$, the difference $g - g_0$ has image in $\ker(P \rightarrow T(P)) = PA_+$. Since $A_+$ consists of elements of positive $\mathbb{Z}$-degree only, this implies that the isomorphism $F^{k+1} P \cong M^{(k+1)}$ does not depend on $g$ up to elements in $F^k P$; in other words, the quotient $F^{k+1} P/F^k P$ is independent of the choice of $g$. Thus the isomorphism (7) is, in fact, a natural isomorphism of functors.

4.4 $K$-theory

We are now in a position to perform the $K$-theoretical calculations. First define the exact functor

$$\Theta_q : \mathbb{G}_q\otimes_{\mathbb{G}_q}(A_{(0,-)}) \longrightarrow \mathbb{G}_q\otimes_{\mathbb{G}_q}(A)$$

$$P = \bigoplus_{\omega} P_{(\omega,-)} \mapsto \bigoplus_{\omega} P_{(\omega,-)} \otimes_{A_{(0,-)}} A(-\omega,0);$$

here $\mathbb{G}_q\otimes_{\mathbb{G}_q}(A_{(0,-)})$ denotes the full subcategory of $\mathbb{G}_q\otimes_{\mathbb{G}_q}(A_{(0,-)})$ spanned by modules with support in $[-q,q] \times G$, and $P_{(\omega,-)}$ on the right is considered as a $\mathbb{Z} \times G$-graded $A_{(0,-)}$-module with support in $[0] \times G$. 

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Next define the exact functor
\[ \Psi_q : \mathcal{P}_{gr}^{\mathbb{Z} \times G}(A) \rightarrow \mathcal{P}_{gr}^{\mathbb{Z} \times G}(A_{(0,-)}) \]
\[ P \mapsto \bigoplus_{\omega \in \mathbb{Z}} T(P)_{(\omega,-)} ; \]

here \( T(P)_{(\omega,-)} \) is considered as an \( A_{(0,-)} \)-module with support in \( \{\omega\} \times G \).

Now \( \Psi_q \circ \Theta_q \cong \text{id} \); indeed, the composition sends the summand \( P_{(\omega,-)} \) of \( P \) to the \( \kappa \)-indexed direct sum of
\[ T(P_{(\omega,-)} \otimes A_{(0,-)} A(-\omega,0))_{(\kappa,-)} \cong \begin{cases} P_{(\omega,-)} & \text{if } \kappa = \omega , \\ 0 & \text{else.} \end{cases} \]

In particular, \( \Psi_q \circ \Theta_q \) induces the identity on \( K \)-groups. — As for the other composition, we have
\[ \Theta_q \circ \Psi_q(P) = \bigoplus_{\omega} T(P)_{(\omega,-)} \otimes A_{(0,-)} A(-\omega,0) = \bigoplus_{j=-q}^{q-1} F_{j+1} P / F_j P . \]

Since \( F^q = \text{id} \), additivity for characteristic filtrations [4, p. 107, Corollary 2] implies that \( \Theta_q \circ \Psi_q \) induces the identity on \( K \)-groups.

For any \( P \in \mathcal{P}_{gr}^{\mathbb{Z} \times G}(A_{(0,-)}) \) we have
\[ (P \otimes A_{(0,-)} A(-\omega,0))(0, g) = P(0, g) \otimes A_{(0,-)} A(-\omega,0) , \]
by direct calculation. Hence the functor \( \Theta_q \) induces a \( \mathbb{Z}[G] \)-linear isomorphism on \( K \)-groups. Since \( K \)-groups are compatible with direct limits, letting \( q \rightarrow \infty \) yields a \( \mathbb{Z}[G] \)-linear isomorphism \( K_i^{\mathbb{Z} \times G}(A_{(0,-)}) \cong K_i^{\mathbb{Z} \times G}(A) \) and thus, by Lemma 4, a \( \mathbb{Z}[\mathbb{Z} \times G] \)-module isomorphism
\[ K_i^G(A_{(0,-)}) \otimes \mathbb{Z}[G] \mathbb{Z}[\mathbb{Z} \times G] \cong K_i^{\mathbb{Z} \times G}(A) . \]

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