Irreducible symplectic 4-folds numerically equivalent to \((K3)^2\)

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Abstract

First steps towards a classification of irreducible symplectic 4-folds whose integral 2-cohomology with 4-tuple cup product is isomorphic to that of \((K3)^2\). We prove that any such 4-fold deforms to an irreducible symplectic 4-fold of Type A or Type B. A 4-fold of Type A is a double cover of a (singular) sextic hypersurface and a 4-fold of Type B is birational to a hypersurface of degree at most 12. We conjecture that Type B 4-folds do not exist.

1 Introduction

Kodaira [15] proved that any two \(K3\) surfaces are deformation equivalent. A \(K3\) surface is the same as an irreducible symplectic 2-fold - recall that a compact Kähler manifold is irreducible symplectic if it is simply connected and it carries a holomorphic symplectic form spanning \(H^{2,0}\) (see [1, 12]). A classification of higher-dimensional irreducible symplectic manifolds up to deformation equivalence appears to be out of reach at the moment (see [1, 12]). We will take the first steps towards a solution of the classification problem for numerical \((K3)^2\)’s. We explain our terminology: two irreducible symplectic manifolds \(M_1, M_2\) of dimension \(2n\) are numerically equivalent if there exists an isomorphism of abelian groups \(\psi: H^2(M_1; \mathbb{Z}) \rightarrow H^2(M_2; \mathbb{Z})\) such that \(\int_{M_1} \alpha^{2n} = \int_{M_2} \psi(\alpha)^{2n}\) for all \(\alpha \in H^2(M_1; \mathbb{Z})\). Recall [1] that if \(S\) is a \(K3\) then \(S^{[n]}\) - the Douady space parametrizing length-\(n\) analytic subsets of \(S\) - is an irreducible symplectic manifold of dimension \(2n\). A numerical \((K3)^2\) is an irreducible symplectic 4-fold numerically equivalent to \(S^{[2]}\) where \(S\) is a \(K3\).

Theorem 1.1. Let \(M\) be a numerical \((K3)^2\). Then \(M\) is deformation equivalent to one of the following:

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(1) An irreducible symplectic 4-fold $X$ carrying an anti-symplectic involution $\phi: X \to X$ such that the quotient $X/\langle \phi \rangle$ is isomorphic to a sextic hypersurface $Y \subset \mathbb{P}^5$. Let $f: X \to Y$ be the quotient map and $H := f^*O_Y(1)$; the fixed locus of $\phi$ is a smooth irreducible Lagrangian surface $F$ such that

$$c_2(F) = 192, \quad O_F(2K_F) \cong O_F(6H), \quad c_1(F)^2 = 360. \quad (1.0.1)$$

(2) An irreducible symplectic 4-fold $X$ admitting a rational map $f: X \cdots > \mathbb{P}^5$ which is birational onto its image $Y$, with $6 \leq \deg Y \leq 12$.

We give a brief outline of the proof of the theorem. By applying surjectivity of the period map and Huybrechts’ projectivity criterion \[12\] \[13\] we will be able to deform $M$ to an irreducible symplectic 4-fold $X$ such that Items (1) through (6) of Proposition (4.4) hold. The first item gives (via Hirzebruch-Riemann-Roch and Kodaira Vanishing) that there is an ample divisor $H$ on $X$ such that

$$\int_X c_1(H)^4 = 12, \quad h^0(O_X(H)) = 6. \quad (1.0.2)$$

Let $h := c_1(H)$; Items (2), (3) and (4) state that $h$ generates $H^{2,1}(X)$ and that $H^4(X)$ has no rational Hodge substructures other than those forced by $h$ and the Beauville quadratic form. Items (5)-(6) imply, via Proposition (4.1), the following Irreducibility property of $|H|$: if $D_1, D_2 \in |H|$ are distinct then $D_1 \cap D_2$ is a reduced and irreducible surface in $X$. Next we will study the rational map $f: X \cdots > |H|^\vee \cong \mathbb{P}^5$. A straightforward argument based on ampleness of $H$ and the Irreducibility property of $|H|$ will show that either Item (1) or Item (2) of Theorem (1.1) holds or $Y := Im(f)$ is one of the following

(a) a 3-fold of degree at most 6,

(b) a 4-fold of degree at most 4.

We will prove that (a) or (b) cannot hold arguing by contradiction: assuming that (a) or (b) holds we will get that either $H^4(X)$ has a non-existant Hodge substructure or the Irreducibility property of $|H|$ does not hold - with the exception of $Y$ a normal quartic 4-fold, this case will require an ad hoc argument. Thus we will need to analyze 3-folds and 4-folds in $\mathbb{P}^5$ of low degree. In particular we will prove some results on cubic 4-folds $Y \subset \mathbb{P}^5$ which might be of independent interest. First we will show that if $\dim(singY) \geq 1$ then $Y$ contains a plane. Secondly we will prove that if $Y$ is singular with isolated singularities and it does not contain planes then $Gr^W_1 H^4(Y)$ contains a Hodge substructure isomorphic to the transcendental part of the $H^2$ of a $K3$ surface (shifted by (1,1)), namely the minimal desingularization of the set of lines in $Y$ through any of its singular points. This result should be equivalent to a statement about degenerations of the variety $F(Y)$ parametrizing lines on a cubic 4-folds $Y \subset \mathbb{P}^5$ - recall that if $Y$ is smooth then $F(Y)$ is a deformation of $(K3)^2$ (see [2]) and if $Y$ is singular then $F(Y)$ is singular [11]. The relevant statement is the following. Let $U$ be the parameter space for cubic 4-folds $Y \subset \mathbb{P}^5$ not containing a plane; there exists a finite cover $\tilde{U} \to U$ such that the pull-back to $\tilde{U}$ of the family over $U$ with fiber $F(Y)$ at $[Y]$ has a simultaneous resolution of singularities. The proof of Theorem (1.1) should be compared to that given in [23] of Kodaira’s theorem on deformation equivalence of $K3$ surfaces. The general strategies are
the same however we have to work harder and the result is not as conclusive as Kodaira’s\(^1\).

**Conjecture 1.2.** Suppose that \(X\) is a numerical \((K3)^2\) and that Items (1) through (6) of Proposition (3.2) hold. Then Item (1) of Theorem (1.1) holds.

We notice that if \(X\) satisfies Item (1) of Theorem (1.1) then any small deformation of \(X\) that keeps \(c_1(H)\) of type \((1, 1)\) is a variety which again satisfies Item (1) and the hyperplane class on the deformed variety is the deformation of the hyperplane class on \(X\) - see Proposition (4.6). In other words the conjecture is stable for small deformations. If the above conjecture is true then any numerical \((K3)^2\) is deformation equivalent to an \(X\) as in Item (1) of Theorem (1.1). In another paper we prove that the quotient \(Y\) belongs to the set of sextic hypersurfaces described by Eisenbud-Popescu-Walter in Example (9.3) of [7]. We will also show that if \(Z\) is a generic EPW-sextic and \(W \to Z\) is the natural double cover then \(W\) is deformation of \((K3)^2\); this will imply that if Conjecture (1.2) holds then any numerical \((K3)^2\) is a deformation of \((K3)^2\).

**Notation:** If \(X\) is a topological space then \(H^*(X)\) denotes cohomology with complex coefficients.

Topology of algebraic varieties (or analytic spaces) will be either the classical topology or the Zariski topology: in general it will be clear from the context in which topology we are working.

Let \(X\) be a smooth projective variety. If \(W\) is a closed subscheme of \(X\) of pure dimension \(d\) we let

\[ [W] \in Z_d(X) \tag{1.0.3} \]

be the fundamental cycle associated to \(W\) as in [2], p. 15.

Let \(P(V)\) be a projective space. If \(A \subset P(V)\) we let \(\text{span}(A) \subset P(V)\) be the span of \(A\), i.e. the intersection of all linear subspaces containing \(A\). If \(A, B \subset P(V)\) we let

\[ J(A, B) := \bigcup_{p \in A, q \in B} \text{span}(p, q). \tag{1.0.4} \]

If \(A, B\) are closed and \(A \cap B = \emptyset\) then \(J(A, B)\) is closed - in general \(J(A, B)\) is not closed.

Let \(X\) be a scheme and \(x \in X\) a (closed) point; we let \(\Theta_x X\) be the Zariski tangent space to \(X\) at \(x\). Now assume that \(X\) is a subscheme of a projective space \(P(V)\). Then \(\Theta_x X \subset \Theta_x P(V)\): the projective tangent space to \(X\) at \(x\) is the unique linear subspace

\[ T_x X \subset P(V) \tag{1.0.5} \]

containing \(x\) whose Zariski tangent space at \(x\) is equal to \(\Theta_x X\).

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\(^1\)We do have a proof that \(f\) cannot be birational onto its image with \(\deg(\text{Im}(f)) \leq 8\).
2 Preliminaries

2.1 Beauville’s form and Fujiki’s constant

Let $M$ be an irreducible symplectic manifold of dimension $2n$. By Beauville and Fujiki (see [1] and Thm. (4.7) of [2]) there exist a rational positive number $c_M$ and an integral indivisible symmetric bilinear form $(,)_M$ on $H^2(M)$ characterized by the following properties. First $(,)_M$ is positive definite on the span of $\{\sigma + \sigma^\perp\}_{\sigma \in H^{2,0}(M)}$ and an arbitrarily chosen Kähler class. Secondly we have the equality

$$\int_M \alpha^{2n} = c_M(\alpha, \alpha)_M^n, \quad \alpha \in H^2(M).$$

(2.1.1)

Thus $H^2(M; \mathbb{Z})$ has a canonical structure of lattice. If two irreducible symplectic manifolds of the same dimension have the same Beauville form and Fujiki constant then by (2.1.1) they are numerically equivalent. The converse is “almost true”. In fact let $\omega \in H^{1,1}(M; \mathbb{R})$ be a Kähler class; by (2.1.1) the primitive (with respect to $\omega$) cohomology $H^2(M)_{prim}$ is equal to $\omega^4$ (orthogonality is with respect to $(,)_M$) and hence by the Hodge index Theorem the signature of $(,)_M$ is $(3, b_2(M) - 3)$. It follows that if two irreducible symplectic manifolds $M_1, M_2$ of dimension $2n$ are numerically equivalent then they have the same Beauville form and Fujiki constant unless $n$ is even and $b_2(M_1) = b_2(M_2) = 6$: in this case numerical equivalence implies that $(,)_M = \pm (,)_M$ and $c_{M_1} = c_{M_2}$. Let $\Lambda$ be the lattice given by

$$\Lambda := U^{\oplus 3} \oplus (-E_8)^{\oplus 2} \oplus (-2),$$

(2.1.2)

where $U$ is the standard hyperbolic plane. Let $S$ be a K3 surface; the Beauville form and Fujiki constant of $S^{[2]}$ are given (see [1]) by

$$H^2(S^{[2]}; \mathbb{Z}) \cong \Lambda, \quad c_{S^{[2]}} = 3.$$  

(2.1.3)

Thus a numerical $(K3)^{[2]}$ is an irreducible symplectic 4-fold $M$ such that

$$H^2(M; \mathbb{Z}) \cong \Lambda, \quad c_M = 3.$$  

(2.1.4)

In particular $b_2(M) = 23$; as is well-known - see [10, 21] - this implies that

$$H^3(M; \mathbb{Q}) = 0, \quad Sym^2 H^2(M; \mathbb{Q}) \cong H^4(M; \mathbb{Q}),$$

(2.1.5)

where the second isomorphism is given by cup-product. The equations of (2.1.5) will be crucial for what follows.

2.2 Quadratic forms on $V$ and $S^2V$

Let $A$ be a ring and $V$ be an $A$-module. Let $(V \otimes V)^+, (V \otimes V)^- \subset V \otimes V$ be the submodules of tensors which are invariant, respectively anti-invariant, for the involution of $V \otimes V$ interchanging the factors. We let $Sym^2 V := (V \otimes V)^+$ and $Sym^2 V := V \otimes V/(V \otimes V)^-$. Assume that $(,)$ is a symmetric bilinear form on $V$; we let $(,)$ be the unique symmetric bilinear form on $S^2V$ such that

$$\langle \alpha_1 \alpha_2, \alpha_3 \alpha_4 \rangle = (\alpha_1, \alpha_2)(\alpha_3, \alpha_4) + (\alpha_1, \alpha_3)(\alpha_2, \alpha_4) + (\alpha_1, \alpha_4)(\alpha_2, \alpha_3)$$

(2.2.1)

for $\alpha_1, \ldots, \alpha_4 \in V$. Using (2.1.1) and the second equality of (2.1.4) we get the following.
Remark 2.1. Let $M$ be a numerical $(K3)^{[2]}$. The intersection form on
\[ \text{Sym}^2 H^2(M) \cong H^4(M) \]  
(2.2.2)
is the bilinear form constructed as above from $V := H^2(M)$ and $(,) := (,)_M$.

3 The deformation

Let $M$ be a numerical $(K3)^{[2]}$. We will show that $M$ can be deformed to a projective irreducible symplectic 4-fold $X$ such that $H^*(X)$ has few integral Hodge substructure. First we introduce the tautological rational Hodge substructures of $H^*(X)$ for $X$ a numerical $(K3)^{[2]}$ with an $h \in H^{1,1}(X)$ such that
\[ \int_X h^4 \neq 0. \]  
(3.0.1)
To simplify notation we let $(,)$ be the Beauville form of $X$; thus \( (,)^{[2]} \) is equivalent by \( \epsilon_{[2]} \) to $(h,h) \neq 0$. We have an orthogonal direct sum decomposition
\[ H^2(X; \mathbb{C}) = Ch \oplus 
⊥ h \]  
(3.0.2)
into Hodge substructures of levels 0 and 2 respectively. By \( \epsilon_{[2]} \) we have a direct sum decomposition
\[ H^4(X; \mathbb{C}) = Ch^2 \oplus (Ch \otimes h^\perp) \oplus \text{Sym}^2(h^\perp) \]  
(3.0.3)
into Hodge substructures of levels 0, 2 and 4 respectively. There is a refinement of Decomposition \( \epsilon_{[2]} \): to explain this we need to introduce the dual of Beauville’s form. Let $q \in \text{Sym}_2(H^2(X)^\vee)$ be Beauville’s symmetric bilinear form; it is non-degenerate \( \Pi_1 \) and hence it defines an isomorphism $L_q: H^2(X) \cong H^2(X)^\vee$. Let $\Pi_2: \text{Sym}_2 H^2(X) \to \text{Sym}^2 H^2(X)$ be the composition of the inclusion $\text{Sym}^2 H^2(X) \hookrightarrow H^2(X) \otimes H^2(X)$ and the projection map $H^2(X) \otimes H^2(X) \to \text{Sym}^2 H^2(X)$. Let
\[ q^{\vee} := \Pi_2 \circ \text{Sym}_2(L_q^{-1})(q) \in \text{Sym}^2 H^2(X). \]  
(3.0.4)
Explicitly: let \( \{\alpha_1, \ldots, \alpha_{23}\} \) be a basis of $H^2(X)$ and \( \{\alpha_1^{\vee}, \ldots, \alpha_{23}^{\vee}\} \) be the dual basis. Thus
\[ q = \sum_{ij} g_{ij} \alpha_i^{\vee} \otimes \alpha_j^{\vee} \]  
(3.0.5)
where $(g_{ij})$ is a symmetric matrix. Then
\[ q^{\vee} = \sum_{ij} m_{ij} \alpha_i \alpha_j, \quad (m_{ij}) = (g_{ij})^{-1}. \]  
(3.0.6)
We know that $q$ is integral and that $(\alpha_1, \alpha_2) = 0$ if $\alpha_i \in H^{r_1,2-r_2}(X)$ with $r_1 + r_2 \neq 2$; this implies that
\[ q^{\vee} \in H^{2,2}_{\mathbb{Q}}(X). \]  
(3.0.7)
In terms of Decomposition \( \epsilon_{[2]} \) we have $q^{\vee} \subseteq Ch^2 \oplus \text{Sym}^2(h^\perp)$. More precisely let $q_h := q|_{h^\perp}$ and let $q_h^{\vee} \in \text{Sym}^2(h^\perp)$ be its dual (this makes sense because $(h,h) \neq 0$ and hence $q_h$ is non-degenerate); then
\[ q^{\vee} = (h,h)^{-1} h^2 + q_h^{\vee}. \]  
(3.0.8)
Let $\langle , \rangle$ be the intersection form on $H^4(X)$ - the notation is consistent with that of Subsection 2.2, see Remark 2.1 - and let

$$W(h) := (q^\vee)^\perp \cap \text{Sym}^2(h^\perp), \quad (3.0.9)$$

where the first orthogonality is with respect to $\langle , \rangle$ and the second is with respect to $(,)$.

**Claim 3.1.** Keeping notation as above, $W(h)$ is a codimension-1 rational sub Hodge structure of $\text{Sym}^2(h^\perp)$, and we have a direct sum decomposition

$$\text{Ch}^2 \oplus \text{Sym}^2(h^\perp) = \mathbb{C}h^2 \oplus Cq^\vee \oplus W(h). \quad (3.0.10)$$

**Proof.** $W(h)$ is a sub Hodge structure because $q^\vee$ is rational of type $(2,2)$; let’s show that $\text{Sym}^2(h^\perp) \not\subset (q^\vee)^\perp$. (3.0.11)

From Remark 2.1 one gets that $\langle q^\vee, \alpha \beta \rangle = 25(\alpha, \beta), \quad \alpha, \beta \in H^2(X)$. (3.0.12)

From this we get immediately (3.0.11) and thus $W(h)$ has codimension 1. Now let’s prove that we have (3.0.10). By (3.0.8) $h^2$ and $q^\vee$ are linearly independent and hence it suffices to show that

$$\left( \text{Ch}^2 \oplus Cq^\vee \right) \cap W(h) = \{0\}. \quad (3.0.13)$$

It follows from (3.0.12) that

$$\langle q^\vee, q^\vee \rangle = 25 \cdot 23, \quad (3.0.14)$$

and hence

$$\left( \text{Ch}^2 \oplus Cq^\vee \right) \cap (q^\vee)^\perp = \mathbb{C}(23h^2 - (h, h)q^\vee). \quad (3.0.15)$$

On the other hand by (3.0.8) we have

$$\left( \text{Ch}^2 \oplus Cq^\vee \right) \cap \text{Sym}^2(h^\perp) = \mathbb{C}(h^2 - (h, h)q^\vee). \quad (3.0.16)$$

Equation (3.0.13) follows immediately from (3.0.15)-(3.0.16). \qed

By the above claim we have a decomposition

$$H^4(X; \mathbb{C}) = \left( \text{Ch}^2 \oplus Cq^\vee \right) \oplus \left( \text{Ch} \otimes h^\perp \right) \oplus W(h) \quad (3.0.17)$$

into sub-H.S.’s of levels 0, 2 and 4 respectively. The following is the main result of this section.

**Proposition 3.2.** Keep notation as above. Let $M$ be a numerical $(K3)^2$. There exists an irreducible symplectic manifold $X$ deformation equivalent to $M$ such that:

1. $X$ has an ample divisor $H$ with $(h, h) = 2$, where $h := c_1(H)$,
2. $H^{1,1}_Z(X) = \mathbb{Z}h$,
3. Let $\Sigma \in Z_1(X)$ be an integral algebraic 1-cycle on $X$ and $\text{cl}(\Sigma) \in H^{3,3}_Q(X)$ be its Poincaré dual. Then $\text{cl}(\Sigma) = mh^3/6$ for some $m \in \mathbb{Z}$.  

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an irreducible symplectic manifold deformation equivalent to $M$ period domain $Q$
that $H_K$ the context of numerical ([2, 3]) - in $X$ and ($M$ to moduli space of marked irreducible symplectic manifolds deformation equivalent
$H_K \otimes$ its linear extension $\Lambda$
see [1]). Let $P$ where ($, ,$ $H_K$)
$\Lambda$ is the symmetric bilinear form on $\Lambda$. The period map is given by
$
\mathcal{M} \xrightarrow{P} Q, \psi^{-1}(H_{2,0}^2(X)).$
Here and in the following $\psi$ denotes both the isometry $\Lambda \sim H^2(X; \mathbb{Z})$ and $\Lambda \otimes \mathbb{C} \to H^2(X; \mathbb{C})$. The map $P$ is locally an isomorphism, see [1]. Let $\mathcal{M}^0$ be a connected component of $\mathcal{M}$. Huybrechts’ Theorem on surjectivity of the global period map (Thm. (8.1) of [12]) states that the restriction of $P$ to $\mathcal{M}^0$ is surjective. Given $\alpha \in \Lambda$ we let
$\mathcal{M}_0^\alpha := \{t \in \mathcal{M}^0 | \psi_t(\alpha) \in H_{Z}^{1,1}(X_t)\}. \tag{3.0.19}$

**Lemma 3.3.** Let $M$ be a numerical $(K3)^2$ and $\mathcal{M}$ be the moduli space of marked irreducible symplectic manifolds deformation equivalent to $M$. Let $\alpha \in \Lambda$ with $(\alpha, \alpha) \neq 0$. For $t \in \mathcal{M}_0^\alpha$ outside of a countable union of proper analytic subsets we have:

1. $H_{Q}^{1,1}(X_t) = Q(\psi_t(\alpha))$
2. any rational sub Hodge structure of $W(\psi_t(\alpha))$ is trivial.

*Proof.* Let
$\mathcal{L}_\alpha := \{[\sigma] \in Q | (\sigma, \alpha)_\Lambda = 0\}. \tag{3.20}$
As is easily checked $\mathcal{L}_\alpha$ is a non-empty codimension 1 subvariety of $Q$ and furthermore
$\mathcal{M}_0^\alpha = P^{-1}(\mathcal{L}_\alpha). \tag{3.21}$
By surjectivity of the period map $\mathcal{M}_0^\alpha$ is non-empty of dimension 20. It is well-known that the set of $t \in \mathcal{M}_0^\alpha$ for which (1) does not hold is a countable union of proper analytic subsets of $\mathcal{M}_0^\alpha$ (see [12]). Next we show that the set of $t \in \mathcal{M}_0^\alpha$
for which (2) does not hold is also a countable union of proper analytic subsets of $M_{\alpha}^0$. Let
\[ W(\alpha) := Sym^2(\alpha^\perp) \cap (q^\perp_\alpha \cap Sym^2(\Lambda \otimes \mathbb{C}) \]
where $q_\Lambda$ is the quadratic form on $\Lambda$. For a linear subspace $V \subset W(\alpha)$ defined over $\mathbb{Q}$ let
\[ M_{\alpha}^0(V) := \{ t \in M_{\alpha}^0 | Sym^2(\psi_t)(V) \text{ is a sub-H.S. of } W(\psi_t(\alpha)) \}. \]
(3.0.22)
Since the set of subspaces $V \subset W(\alpha)$ defined over $\mathbb{Q}$ is countable it suffices to prove that $M_{\alpha}^0(V)$ is a proper analytic subset of $M_{\alpha}^0$ whenever $V \neq 0$ or $V \neq W(\alpha)$. It is well-known that $M_{\alpha}^0(V)$ is an analytic subset of $M_{\alpha}^0$. Assume that $M_{\alpha}^0(V)$ contains a non-empty open subset $U \subset M_{\alpha}^0$, we will show that either $V = 0$ or $V = W(\alpha)$. We have
(a) $Sym^2(\psi_t)(V) \cap H^{4,0}(X_t) \neq \{0\}$ for all $t \in U$, or
(b) $Sym^2(\psi_t)(V) \cap H^{4,0}(X_t) = \{0\}$ for all $t \in U$.
Assume that (a) holds. Then
\[ Sym^2(\psi_t)(V) \supset H^{4,0}(X_t) = H^{2,0}(X_t) \wedge H^{2,0}(X_t) \]
for all $t \in U$ and hence
\[ V \supset \{ \sigma^2 | [\sigma] \in P(U) \}, \]
(3.0.23)
where $P$ is the period map. Let $V \subset L_\alpha$ be open and non-empty: as is easily verified
\[ \text{span}\{[\sigma^2] | [\sigma] \in V\} = \mathbb{P}(W(\alpha)). \]
(3.0.24)
Since $P(U)$ is an open non-empty subset of $L_\alpha$ we get by (3.0.23) that $V = W(\alpha)$. Now assume that (b) holds. Then
\[ \langle Sym^2(\psi_t)(V), H^{0,4}(X_t) \rangle = 0 \]
for all $t \in U$ and hence $V \perp \{ [\tau^2] | [\tau] \in P(U) \}$. Arguing as above we get that $V = \{0\}$.

We will apply Lemma (3.3) with a particular choice of $\alpha$. First we prove two preliminary results.

**Lemma 3.4.** The vectors $\alpha \in \Lambda$ with
\[ (\alpha, \alpha)_\Lambda = 2 \]
belong to a single $O(\Lambda)$-orbit.

**Proof.** Let
\[ \Lambda := U^{\otimes 3} \oplus (\Lambda) \oplus U. \]
Choose an embedding $\Lambda \subset \Lambda$ such that $\Lambda^\perp = Z_\gamma$ where $\gamma \in U$ is a vector with $(\gamma, \gamma)_\Lambda = 2$. Given $\alpha_1, \alpha_2 \in \Lambda$ with $(\alpha_1, \alpha_i)_\Lambda = 2$ the lattices $Z_\gamma \oplus Z\alpha_1$ and $Z_\gamma \oplus Z\alpha_2$ are saturated and isometric. By a standard result on lattices (see Theorem 1, p. 578 of [20]) there exists $g \in O(\Lambda)$ with $g(\gamma) = \gamma$ and $g(\alpha_1) = \alpha_2$. Since $g$ sends $\Lambda = \gamma^\perp$ to itself it restricts to an isometry of $\Lambda$ taking $\alpha_1$ to $\alpha_2$. \[ \square \]
Lemma 3.5. Let $M$ be a numerical $(K3)^2$. Let $\mathcal{M}$ be the moduli space of marked irreducible symplectic manifolds deformation equivalent to $M$ and let $\mathcal{M}^0$ be a connected component of $\mathcal{M}$. Suppose that $\alpha_1, \alpha_2 \in \Lambda$ satisfy

$$(\alpha_1, \alpha_1)_\Lambda = (\alpha_2, \alpha_2)_\Lambda = 2, \quad (\alpha_1, \alpha_2)_\Lambda = 1 \mod 2. \quad (3.0.26)$$

There exists $1 \leq i \leq 2$ such that for every $t \in \mathcal{M}^0$ the class of $\psi_t(\alpha_i)^2$ in $H^4(X_t; \mathbb{Z})/\text{Tors}$ is indivisible.

Proof. First notice that it suffices to show that for one $t_0 \in \mathcal{M}^0$ there exists $1 \leq i \leq 2$ such that $\psi_{t_0}(\alpha_i)^2$ is indivisible; in fact for any other $t \in \mathcal{M}^0$ there exists a diffeomorphism $f : X_{t_0} \cong X_t$ such that $H^2(f) \circ \psi_t = \psi_{t_0}$ and hence if $\psi_t(\alpha_i)^2$ is divisible then $\psi_{t_0}(\alpha_i)^2$ is divisible too. Next we claim that for $i = 1, 2$ the class of $\psi_t(\alpha_i)^2$ is divisible at most by $2$. First notice that there exists $\beta_i \in \Lambda$ with

$$(\alpha_i, \beta_i)_\Lambda = 1, \quad (\beta_i, \beta_i)_\Lambda = 0. \quad (3.0.27)$$

In fact by Lemma (3.3) it suffices to exhibit $\alpha', \beta' \in \Lambda$ such that

$$(\alpha', \alpha')_\Lambda = 2, \quad (\alpha', \beta')_\Lambda = 1, \quad (\beta', \beta')_\Lambda = 0, \quad (3.0.28)$$

and this is a trivial exercise. Now let $\beta_i$ be as above. Then

$$\langle \psi_t(\alpha_i)^2, \psi_t(\beta_i)^2 \rangle = 2 \quad (3.0.29)$$

and this proves that $\psi_t(\alpha_i)^2$ is divisible at most by $2$. Now we prove the lemma arguing by contradiction. Assume that $\psi_t(\alpha_i)^2$ is divisible by $2$ (modulo torsion) for $i = 1$ and $i = 2$; thus

$$\psi_t(\alpha_i)^2 = 2\gamma_i + \xi_i \quad (3.0.30)$$

where $\gamma_i \in H^4(M; \mathbb{Z})$ and $\xi_i \in Tors(H^4(M; \mathbb{Z}))$. By Remark (2.1) we have

$$\langle \psi_t(\alpha_1)^2, \psi_t(\alpha_2)^2 \rangle = 6. \quad (3.0.31)$$

On the other hand by (3.0.30) the left-hand side is equal to $4\langle \gamma_1, \gamma_2 \rangle$, contradiction. \hfill \Box

Proof of Proposition (3.2). Let $\mathcal{M}$ be the moduli space of marked irreducible symplectic manifolds deformation equivalent to $M$ and let $\mathcal{M}^0$ be a connected component of $\mathcal{M}$. By Lemma 3.5 there exists $\alpha \in \Lambda$ with $(\alpha, \alpha) = 2$ such that for every $t \in \mathcal{M}^0$ the class of $\psi_t(\alpha)^2$ in $H^4(X_t; \mathbb{Z})/\text{Tors}$ is indivisible. Let $t \in \mathcal{M}^0$ satisfying Items (1)-(2) of Lemma (3.3). Set $X := X_t$. Since $\psi_t(\alpha) \in H^4_{cl}(X)$ and $(\psi_t(\alpha), \psi_t(\alpha)) = 2$ we know that $X$ is projective by Huybrechts’ projectivity criterion [12]; since $H^1_{cl}(X) = H^2_{cl}(X)$ either $\psi_t(\alpha)$ or $-\psi_t(\alpha)$ is the class of an ample divisor. Let $h := \psi_t(\alpha)$ in the former case and $h := -\psi_t(\alpha)$ in the latter case. We let $H$ be a divisor with $c_1(H) = h$. Let’s prove that Items (1)-(5) of Proposition 3.2 hold for $(X, H)$. Of course $X$ is a deformation of $M$ by definition. (1)-(2): They hold by construction. (3): By Item (2) and Hard Lefschetz we have $H^1_{cl}(X) = Q^{h^3}$ and hence $cl(\Gamma) = x h^3$ for some $x \in \mathbb{Q}$. There exists $e \in H^2(X; \mathbb{Z})$ with $(e, h) = 1$, see (3.0.27), and hence

$$\int_X e = \langle x h^3, e \rangle = 3x(h, h)(h, e) = 6x.$$
(4): Follows from Item (2) of Lemma \[85.5\], from the fact that \( \mathbb{C} h \otimes h^\perp \) has no non-trivial sub-H.S.’s and an easy argument based on the observation that the three summands of Decomposition \[\text{3.0.14}\] have pairwise distinct levels. (5): Holds by our choice of \( \alpha \), thanks to Lemma \[\text{3.0.35}\]. (6): First we show that
\[
c_2(X) = 6q^\vee/5 \quad \text{in } H^4(X; \mathbb{Q}).
\] (3.0.32)

It is well-known that any \( \theta \in \text{Sym}^2 H^2(X; \mathbb{Q}) \cap H^{2,2}(X) \) which stays of type \((2,2)\) for all deformations of \( X \) is a multiple of \( q^\vee \): to prove it let \( u \in M \) and use \( L_{q^\vee} \colon H^2(X_u) \overset{\sim}{\longrightarrow} H^2(X_u)^\vee \) to produce from \( \theta \) a \( \theta'_u \in \text{Sym}^2 H^2(X_u)^\vee \) with \( \theta_u(\sigma_u, \sigma_u) = 0 \) for \( \sigma_u \in H^{2,0}(X_u) \). Applying this to \( \theta = c_2(X) \) we get that \( c_2(X) = aq^\vee \) for some \( a \in \mathbb{Q} \). Applying Hirzebruch-Riemann-Roch and keeping in mind that all odd Chern classes of \( X \) vanish we get that
\[
3 = \chi(O_X) = \frac{1}{240} \left(c_2(X)^2 - \frac{1}{3} c_4(X)\right),
\] (3.0.33)

By \[\text{2.2.5.}\] we know that
\[
c_4(X) = 324
\] (3.0.34)

and hence it follows that
\[
c_2(X)^2 = 828.
\] (3.0.35)

Applying Formula \[\text{3.0.14}\] we get that \( a = \pm 6/5 \). On the other hand Theorem (1.1) of \[18\] together with \[\text{3.0.12}\] gives that
\[
0 \leq \langle c_2(x), h^2 \rangle = \langle aq^\vee, h^2 \rangle = 50a.
\] (3.0.36)

This proves \[\text{3.0.14}\]. Since \( 2q^\vee \in \text{Sym}^2 H^2(X; \mathbb{Z}) \) Formula \[\text{3.0.2}\] gives that
\[
H^4(X; \mathbb{Z})/\text{Tors} \ni (2c_2(X) - 2q^\vee) = 2q^\vee/5 = c_2(X)/3.
\] (3.0.37)

In particular
\[
\Omega(h) := Z h^2 \oplus Z (2q^\vee/5) \subset (H^{2,2}(X)/\text{Tors})
\] (3.0.38)

By Item (4) of the proposition we know that \( h^{2,2}_{\mathbb{Q}} = 2 \) and hence \( \Omega(h) \) is of finite index in \( H^{2,2}(X)/\text{Tors} \). A straightforward computation (use \[\text{3.0.14}\] and \[\text{3.0.12}\]) shows that
\[
\text{discr} \left( (\cdot)|_{\Omega(h)} \right) = 2^6 \cdot 11,
\] (3.0.39)

and hence
\[
[H^{2,2}_{\mathbb{Z}}(X)/\text{Tors} : \Omega(h)] \leq 8.
\] (3.0.40)

Now let \( xh^2 + y(2q^\vee/5) \in H^{2,2}_{\mathbb{Z}}(X)/\text{Tors} \): we must show that \( 2x \in \mathbb{Z} \) and \( 2y \in \mathbb{Z} \). Let \( \beta \in H^2(X; \mathbb{Z}) \) with \( \langle h, \beta \rangle = 1 \) and \( \langle \beta, \beta \rangle = 0 \): such a \( \beta \) exists, see the proof of Lemma \[\text{3.0.4}\]. Using \[\text{3.0.12}\] we get that
\[
\mathbb{Z} \ni \langle xh^2 + y(2q^\vee/5), \beta^2 \rangle = 2x.
\]

Next let \( \gamma, \delta \in H^2(X; \mathbb{Z}) \) with \( \langle \gamma, \delta \rangle = 1 \). Then
\[
\mathbb{Z} \ni \langle xh^2 + y(2q^\vee/5), \gamma \delta \rangle = 2x(1 + (h, \gamma)(h, \delta)) + 10y.
\]

Since \( 2x \in \mathbb{Z} \) we get that \( 10y \in \mathbb{Z} \). By \[\text{3.0.10}\] we know that \( 8y \in \mathbb{Z} \) and hence \( 2y \in \mathbb{Z} \). This finishes the proof of Proposition \[\text{3.2}\].
Remark 3.6. In the proof of Proposition 8.29 we appealed to Huybrechts’ Global Surjectivity Theorem. It is plausible that local surjectivity is sufficient.

If $M$ is a numerical $(K3)^{[2]}$ we cannot exclude the existence of a $\gamma \in H^2(M;\mathbb{Z})$ such that $(\gamma, \gamma) = 2$ and the image of $\gamma^2$ in $H^4(M;\mathbb{Z})/\text{Tors}$ is divisible by 2; we only proved that it is impossible that $\gamma^2$ is divisible by 2 for all $\gamma$ with $(\gamma, \gamma) = 2$. If $M$ is a deformation of $(K3)^{[2]}$ the picture is simpler.

Proposition 3.7. Let $M$ be a deformation of $(K3)^{[2]}$ and $\gamma \in H^2(M;\mathbb{Z})$ such that $(\gamma, \gamma) = 2$. The image of $\gamma^2$ in $H^4(M;\mathbb{Z})/\text{Tors}$ is not divisible.

Proof. Let $S$ be a $K3$ surface. We may assume that $\gamma \in H^2(S^{[2]};\mathbb{Z})$. Let $\Delta \subset S^{[2]}$ be the codimension-1 locus parametrizing non-reduced subschemes of $S$. There exists $\xi \in H^2(S^{[2]};\mathbb{Z})$ such that $2\xi = c_1(\Delta)$. There is an orthogonal direct sum decomposition (see Prop. 6, p. 768 and pp. 777-778 of [1])

$$H^2(S^{[2]};\mathbb{Z}) = \mu(H^2(S;\mathbb{Z})) \oplus \mathbb{Z}\xi$$

(3.0.41)

where $\mu : H^2(S;\mathbb{Z}) \to H^2(S^{[2]};\mathbb{Z})$ is the symmetrization map (Donaldson map). If $C \subset S$ is an algebraic curve a representative of $\mu(C)$ is the divisor

$$\Sigma_C := \{[Z] \in S^{[2]} | Z \cap C \neq \emptyset\}. \quad (3.0.42)$$

By 3.0.41 we have $\gamma = \mu(\alpha') - x\xi$. We know that $\gamma$ is at most divisible by 2, see the proof of Lemma 3.5, and hence we may add to $\gamma$ arbitrary elements of $2H^2(S^{[2]};\mathbb{Z})$. Thus we may assume that $\gamma = \mu(\pm\alpha) - \xi$ where $(\alpha, \alpha) = 4$. We can deform the complex structure of $S$ so that either $\alpha$ or $-\alpha$ is represented by a very ample divisor on $S$ giving an embedding $S \subset \mathbb{P}^3$. We can furthermore assume that $S$ contains a conic $C$. Now consider the map

$$S^{[2]} \overset{g}{\rightarrow} \text{Gr}(1,\mathbb{P}^3) \quad (3.0.43)$$

where $\langle Z \rangle$ is the line spanned by $Z$. Let $p : \text{Gr}(1,\mathbb{P}^3) \overset{p}{\rightarrow} \mathbb{P}^5$ be the Plücker embedding. Then $c_1((pg)^*\mathcal{O}_{\mathbb{P}^5}(1)) = \gamma$, see Formula (4.1.9) of [19]. Now consider the surface $C^{(2)} \subset S^{[2]}$. Since $g$ maps $C^{(2)}$ isomorphically onto a linear $\mathbb{P}^2$ in $\mathbb{P}^5$ we get that

$$\int_{C^{(2)}} \gamma^2 = 1. \quad (3.0.44)$$

Thus the image of $\gamma^2$ in $H^4(S^{[2]};\mathbb{Z})/\text{Tors}$ is not divisible. \hfill \Box

4 The linear system $|H|

Let $X, H$ be as in Proposition 3.2. In this section we will prove some basic properties of the complete linear system $|H|$. A key result is the following.

Proposition 4.1. Keep notation as above.

(1) If $D_1, D_2 \in |H|$ are distinct then $D_1 \cap D_2$ is a reduced irreducible surface.

(2) If $D_1, D_2, D_3 \in |H|$ are linearly independent the subscheme $D_1 \cap D_2 \cap D_3$ has pure dimension 1 and the Poincaré dual of the fundamental cycle $[D_1 \cap D_2 \cap D_3]$ is equal to $h^3$. 11
Proof. (1): Assume that $\Gamma \in \mathbb{Z}^2(X)$ is an effective non-zero algebraic cycle of pure codimension 2. Assume that

$$cl(\Gamma) = (sh^2 + t(2q^2/5)) \in H^4(X; \mathbb{Z})/Tors,$$

where $cl(\Gamma)$ is the image of the Poicaré dual of the homology class represented by $\Gamma$, and $h := c_1(H)$. Let $\sigma \in \Gamma(\Omega^2_X)$ be a symplectic form. Then

$$0 < \langle cl(\Gamma), h^2 \rangle = \langle sh^2 + t(2q^2/5), h^2 \rangle = 12s + 20t,$$

$$0 \leq \langle cl(\Gamma), (\sigma + \sigma^2) \rangle = \langle sh^2 + t(2q^2/5), (\sigma + \sigma^2) \rangle = (2s + 10t)(\sigma + \sigma^2).$$

Since $(\sigma + \sigma^2, \sigma + \sigma^2) > 0$ we get that

$$3s + 5t > 0, \quad s + 5t \geq 0. \quad (4.0.1)$$

Now let $D_1, D_2 \in |H|$ be distinct. By Item (2) of Proposition 3.2 we know that $D_1 \cap D_2$ is a subscheme of $X$ of pure codimension 2 representing $h^2$. Assume that $D_1 \cap D_2$ is not reduced and irreducible: then we have an equality of cycles $[D_1 \cap D_2] = A + B$ with $A, B$ effective non-zero. By Item (5) of Proposition 3.2 we have

$$cl(A) = xh^2 + y(2q^2/5), \quad cl(B) = (1 - x)h^2 - y(2q^2/5)$$

with $2x, 2y \in \mathbb{Z}$. Applying (4.0.1) we get that

$$0 < 3x + 5y < 3, \quad 0 \leq x + 5y \leq 1.$$

"Eliminating $x$" we get that

$$-3/5 < 2y < 3/5.$$ 

Since $2y \in \mathbb{Z}$ we get that $y = 0$ and hence $cl(A) = xh^2$ with $0 < x < 1$. This contradicts Item (4) of Proposition 3.2 and proves Item (1). Item (2) follows immediately from Item (1). $\Box$

Let $B$ be the base-scheme of $|H|$, i.e.

$$B := \bigcap_{D \in |H|} D. \quad (4.0.2)$$

Item (2) of Proposition 4.1 gives that

$$\dim B \leq 1. \quad (4.0.3)$$

We claim that

$$h^0(\mathcal{O}_X(nH)) = \frac{1}{2}n^4 + \frac{5}{2}n^2 + 3, \quad n \in \mathbb{N}_+ \quad (4.0.4)$$

In fact applying H.-R.-R. and keeping in mind that all odd Chern classes of $X$ vanish we get that for any $n \in \mathbb{Z}$ we have

$$\chi(\mathcal{O}_X(nH)) = \frac{1}{24} \left( \int_X h^4 \right) n^4 + \frac{1}{24} \left( \int_X c_2(X)h^2 \right) n^2 + \chi(\mathcal{O}_X). \quad (4.0.5)$$
By using (3.0.32) and (3.0.12) we get that
\[ \chi(O_X(nH)) = \frac{1}{2} n^4 + \frac{5}{2} n^2 + 3, \quad n \in \mathbb{Z}. \] (4.0.6)

Since \( K_X \cong O_X \) Kodaira vanishing gives that for \( n > 0 \) we have \( h^0(O_X(nH)) = \chi(O_X(nH)) \). Thus (4.0.4) follows from (4.0.6). In particular we have \( \chi(O_X(H)) = 6 \). We choose once and for all an isomorphism
\[ |H| \sim \mathbb{P}^5 \] (4.0.7)
and we let
\[ f : X \rightarrow \mathbb{P}^5 \] (4.0.8)
be the rational map given by the composition \( X \rightarrow |H| \sim \mathbb{P}^5 \). Let
\[ \tilde{X} := Bl_B(X), \quad E \in \text{Div}(\tilde{X}) \] (4.0.9)
be the blow-up of the scheme \( B \) and the corresponding exceptional divisor respectively. Let
\[ \tilde{f} : \tilde{X} \rightarrow \mathbb{P}^5 \] (4.0.10)
be the regular map which resolves the indeterminacies of \( f \). Let \( Y := \text{Im}(\tilde{f}) \); thus \( Y \subset \mathbb{P}^5 \) is closed and we have (abusing notation) a dominant map
\[ f : X \rightarrow Y. \] (4.0.11)

We let \( \text{deg} f \) be the degree of the map above. Let \( Y_0 \) be the interior of \( \tilde{f}(X \setminus B) \) (we may view \( (X \setminus B) \) as an open subset of \( \tilde{X} \)); thus \( Y_0 \subset Y \) is open and dense. Let \( X_0 := (X \setminus B) \cap \tilde{f}^{-1}(Y_0) \); thus \( X_0 \subset X \) is open and dense. The restriction of \( \tilde{f} \) to \( X_0 \) defines a regular surjective map
\[ f_0 : X_0 \rightarrow Y_0. \] (4.0.12)

**Proposition 4.2.** Keep notation as above. Let \( L \subset \mathbb{P}^5 \) be a linear subspace of codimension at most 2. Then \( L \cap Y_0 \) is reduced and irreducible and, if non-empty, it has pure codimension equal to \( \text{cod}(L, \mathbb{P}^5) \).

**Proof.** If \( L = \mathbb{P}^5 \) there is nothing to prove. Assume that \( \text{cod}(L, \mathbb{P}^5) = 1 \). Let \( D \in |H| \) be the divisor corresponding to \( L \) via (4.0.7). Then \( D \cap X_0 = f_0^*L \); since \( X_0 \) is open dense in \( X \) and \( f_0 \) is surjective the result follows from Item (2) of Proposition (3.2). Assume that \( \text{cod}(L, \mathbb{P}^5) = 2 \) and write \( L = L_1 \cap L_2 \) where \( L_1, L_2 \subset \mathbb{P}^5 \) are hyperplanes. Let \( D_1, D_2 \in |H| \) be the divisors corresponding to \( L_1, L_2 \) via (4.0.7). Then \( D_1 \cap D_2 \cap X_0 = f_0^*L \); since \( X_0 \) is open dense in \( X \) and \( f_0 \) is surjective the result follows from Item (1) of Proposition (3.2). \( \square \)

The following result is the first step towards the proof that the manifold \( X \) satisfies (1) or (2) of Theorem (1.1).

**Proposition 4.3.** Keep notation as above. One of the following holds:

(1) \( \dim Y = 3 \) and \( 3 \leq \text{deg} Y \leq 6 \). If \( \dim Y = 3 \) and \( \text{deg} Y = 6 \) then \( B \) is 0-dimensional.
(2) \( \dim Y = 4, \deg Y = 2. \)

(3) \( \dim Y = 4, \deg Y = 3 \) and \( \deg f = 3. \)

(4) \( \dim Y = 4, \deg Y = 3, \deg f = 4 \) and \( B = \emptyset. \)

(5) \( \dim Y = 4, \deg Y = 4, \deg f = 3 \) and \( B = \emptyset. \)

(6) There exists a regular anti-symplectic involution \( \phi: X \to X \) such that \( Y \cong X/\langle \phi \rangle \) and the quotient map \( X \to X/\langle \phi \rangle \) is identified with \( f: X \to Y. \) The \((\pm 1)\)-eigenspaces of \( H^2(\phi) \) are \( CH \) and \( h^\perp \) respectively. The fixed locus of \( \phi \) is a smooth irreducible Lagrangian surface \( F \) such that

\[
    c_2(F) = 192, \quad O_F(2K_F) \cong O_F(6H), \quad c_1(F)^2 = 360. \tag{4.0.13}
\]

(7) \( \dim Y = 4, \ f: X \cdots > Y \) is birational and \( 6 \leq \deg Y \leq 12. \)

The rest of this section is devoted to the proof of the above proposition. We let \( d := \deg Y. \)

**Claim 4.4.** Keeping notation as above, we have \( \dim Y \geq 3. \)

**Proof.** This is a straightforward consequence of Proposition (4.12). Suppose that \( \dim Y = 1. \) Since \( Y \) is an irreducible non-degenerate curve in \( \mathbb{P}^5 \) we have \( d \geq 5. \)

Let \( L \subset \mathbb{P}^5 \) be a generic hyperplane; since \( Y_0 \) is open dense in \( Y \) the intersection \( Y_0 \cap L \) consists of \( d \) points, contradicting Proposition (4.2). Now suppose that \( \dim Y = 2; \) since \( Y \) is an irreducible non-degenerate surface in \( \mathbb{P}^5 \) we have \( d \geq 4. \)

Let \( L \subset \mathbb{P}^5 \) be a generic linear subspace of codimension 2; since \( Y_0 \) is open dense in \( Y \) the intersection \( Y_0 \cap L \) consists of \( d \) points, contradicting Proposition (4.2). \( \square \)

**The case** \( \dim Y = 3. \) We will show that (1) holds. Since \( Y \) is an irreducible non-degenerate 3-fold in \( \mathbb{P}^5 \) we have \( 3 \leq \deg Y. \) Let’s prove that \( \deg Y \leq 6. \)

Let \( L, L', L'' \subset \mathbb{P}^5 \) be generic linearly independent hyperplanes. Then the intersection \( Y \cap L \cap L' \cap L'' \) is transverse and it consists of \( d \) points \( p_1, \ldots, p_d \in Y_0. \)

Let \( D, D', D'' \in [H] \) correspond to \( L, L', L'' \) via (4.0.13). By Item (2) of Proposition (4.11) the scheme \( D \cap D' \cap D'' \) has pure dimension 1. Let \( \Gamma_{0,i} := f_0^{-1}(p_i) \) and \( \Gamma_i \) be its closure in \( X. \) We have

\[
    [D \cap D' \cap D''] = \Gamma_1 + \cdots + \Gamma_d + \Sigma \tag{4.0.14}
\]

where \( \Sigma \) is an effective 1-cycle with \( \text{supp} \Sigma \subset \text{supp} B. \) (See (4.0.13) for the notation \([D \cap D' \cap D'']\).) Of course \( B \neq \emptyset \) because \( \dim Y < \dim X, \) and

\[
    \dim B = 0 \text{ if and only if } \Sigma = 0. \tag{4.0.15}
\]

By (3) of Proposition (4.12)

\[
    cl(\Gamma_i) = m_i h^3/6, \quad m_i \in \mathbb{N}_+. \tag{4.0.16}
\]

By Item (2) of Proposition (4.11) the 1-cycle \([D \cap D' \cap D'']\) represents \( h^3 \) and hence (4.0.14) gives that

\[
    12 = \langle h, \Gamma_1 + \cdots + \Gamma_d + \Sigma \rangle = 2 \sum_{i=1}^d m_i + \langle h, \Sigma \rangle \geq 2d + \langle h, \Sigma \rangle. \tag{4.0.17}
\]

\[
    12 = \langle h, \Gamma_1 + \cdots + \Gamma_d + \Sigma \rangle = 2 \sum_{i=1}^d m_i + \langle h, \Sigma \rangle \geq 2d + \langle h, \Sigma \rangle. \tag{4.0.17}
\]
Since $h$ is ample and $\Sigma$ is effective we get that $d \leq 6$. Furthermore if $d = 6$ then $(h, \Sigma) = 0$ and hence $\Sigma = 0$; by (4.0.10) we get that $\dim B = 0$.

**The case dim $Y = 4$: elementary considerations.** Let $D, D', D'', D''' \in |H|$ be linearly independent divisors. We will make some elementary considerations on the relation between the intersection number $\int_X h^4$ and the intersection $D \cap \cdots \cap D'''$. These facts will also be useful later on. Let $L, L', L'', L''' \subset \mathbb{P}^5$ be the hyperplanes corresponding to $D, D', D'', D'''$ via (4.0.7). Let $\bar{f}$ and $E$ be as in (4.0.10) and (4.0.9) respectively; we can and will assume that

$$\dim(L \cap \cdots \cap L''' \cap Y) = 0, \quad L \cap \cdots \cap L''' \cap \bar{f}(\text{supp} E) = \emptyset. \quad (4.0.18)$$

By Item (2) of Proposition (4.1) the intersection $D' \cap D'' \cap D'''$ is of pure dimension 1. There is a unique decomposition

$$[D' \cap D'' \cap D'''] = \Gamma + \Sigma \quad (4.0.19)$$

with $\Gamma, \Sigma$ effective 1-cycles and

$$\dim(\text{supp}(\Gamma) \cap \text{supp}(B)) \leq 0, \quad \text{supp} \Sigma \subset \text{supp} B. \quad (4.0.20)$$

From (4.0.19) we get that

$$12 = \int_X h^4 = \deg(H \cdot (\Gamma + \Sigma)) = \deg(D \cdot \Gamma) + \int_\Sigma h. \quad (4.0.21)$$

By (4.0.18) the divisor $D$ intersects $\Gamma$ in $d \cdot \deg f$ points (counting multiplicities) outside $\text{supp} B$ and hence we have

$$\deg(D \cdot \Gamma) = d \cdot \deg f + \sum_{p \in \text{supp} B} \text{mult}_p(D \cdot \Gamma). \quad (4.0.22)$$

(The sum on the right is finite because of (4.0.20).)

**Lemma 4.5.** Keep notation as above. Assume that $\dim Y = 4$. Then

$$\deg Y \cdot \deg f \leq 12 \quad (4.0.23)$$

with equality if and only if $B = \emptyset$.

**Proof.** Since $H$ is ample the integral appearing in (4.0.21) is non-negative and hence (4.0.23) follows from (4.0.21) and (4.0.22). It is clear that if $B = \emptyset$ then (4.0.23) is an equality, we must prove the converse. Assume that (4.0.23) is an equality. By (4.0.21) and (4.0.22) we have $\Sigma = 0$ and hence Equation (4.0.19) gives that $\text{supp} \Gamma \supset \text{supp} B$. Since $\text{supp} D \supset \text{supp} B$ every $p \in \text{supp} B$ is contained in $D \cap \Gamma$. By (4.0.22) we get that $B = \emptyset$. \qed

**The case dim $Y = 4$ and $\deg f = 1$.** We must show that (7) holds. From Lemma 4.5 we get that $d \leq 12$. One gets the lower bound $6 \leq d$ by adjunction. Explicitly, let $\bar{Y} \subset \mathbb{P}^5$ be an embedded resolution of $Y \subset \mathbb{P}^5$: then

$$h^0(K_{\bar{Y}}) = 1 \quad (4.0.24)$$
because $\overline{Y}$ is birational to $X$. On the other hand by adjunction and vanishing of the Hodge numbers $h^{5,1}(\mathbb{P}^5), h^{5,0}(\mathbb{P}^5), h^{4,0}(\mathbb{P}^5)$ we get an isomorphism

$$H^0(K_{\overline{Y}}) = H^0(I_Z(d-6)),$$  \hspace{1cm} (4.0.25)

where $Z \subset \mathbb{P}^5$ is a subscheme supported on $\text{sing}Y$. From \textbf{(4.0.24)} we get that $6 \leq d$. We have proved that if $\deg f = 1$ then (7) holds.

**The case $\dim Y = 4$ and $\deg f = 2$.** Since $f: X \cdots \to Y$ is generically a double cover it defines a birational involution $\phi: X \cdots \to X$. We claim that $\phi$ is regular: since $K_X \sim 0$ there exist closed subsets $I_1, I_2 \subset X$ of codimension at least 2 such that $\phi$ restricts to a regular map $(X \setminus I_1) \to (X \setminus I_2)$ and since $H^{1,1}(X) = \mathbb{Z}h$ we have $\phi^*H \sim H$: it follows by a well-known argument (see [12]) that $\phi$ is regular. The map $f: X \cdots \to Y$ factors as

$$X \xrightarrow{\rho} X/\langle \phi \rangle \cdots \to Y$$  \hspace{1cm} (4.0.26)

where $\rho$ is the quotient map. Since $\deg f = 2$ we have $\deg \overline{f} = 1$, i.e. $\overline{f}$ is birational. We claim that

$$d = 6, \quad \overline{f} \text{ is regular, } \dim(\text{sing}Y) \leq 2.$$  \hspace{1cm} (4.0.27)

Let $\sigma$ be a symplectic form on $X$: since $H^0(\Omega^2_X) = \mathbb{C}\sigma$ and since $\phi$ is an involution we have $\phi^*\sigma = \pm \sigma$ and hence $\phi^*(\sigma \wedge \sigma) = \sigma \wedge \sigma$. Thus if $W$ is any desingularization of $X/\langle \phi \rangle$ we have $H^0(K_W) \neq 0$. Since $\overline{f}$ is birational we get that $H^0(K_{\overline{Y}}) \neq 0$ for any desingularization $\overline{Y} \to Y$. By \textbf{(4.0.26)} we get that $d \geq 6$, and hence Lemma \textbf{(4.0.25)} gives that $d = 6$ and that $B = \emptyset$. Since $B = \emptyset$ the map $\overline{f}$ is regular. Since $d = 6$ we get that $\dim(\text{sing}Y) \leq 2$ - if $\dim(\text{sing}Y) = 3$ then $\text{sing}Y$ certainly “imposes conditions on adjoints”. We have proved \textbf{(4.0.27)}. Let’s show that $\overline{f}$ is an isomorphism. The fibers of $\overline{f}$ are finite because $\overline{f}^*\mathcal{O}_Y(1)$ is ample, $Y$ is normal because it is a hypersurface smooth in codimension 1: this implies that the birational map $\overline{f}$ is an isomorphism. Let $H^2_\pm(X) \subset H^2(X)$ be the $(\pm 1)$-eigenspace of $H^2(\phi)$ respectively. Then $h^2_+(X)$ is equal to $h^2(Y)$, which is 1 by Lefschetz’ Hyperplane Section Theorem: since $h$ belongs to $H^2(\phi)_+$ we get that

$$H^2(\phi)_+ = \mathbb{C}h.$$  \hspace{1cm} (4.0.28)

Since $\phi$ preserves Beauville’s form $(,) we get that

$$H^2(\phi)_- = h^\perp.$$  \hspace{1cm} (4.0.29)

In particular $\phi$ is anti-symplectic. Let’s prove that the fixed locus $F$ has the stated properties. Since $F$ is the fixed locus of an involution on a smooth manifold it is smooth. Since $\phi$ is anti-symplectic $F$ has pure dimension equal to $\dim X/2 = 2$, and $F$ is Lagrangian. Let’s prove that $F$ is irreducible. Let $F = \bigcup_{i \in I} F_i$ be the decomposition into irreducible components. For $i \in I$ let $\text{cl}(F_i) \in H^{2,2}_Q(X)$ be the Poincaré dual of $F_i$; we claim that

$$\text{cl}(F_i) = k_i(15h^2 - c_2(X)), \quad k_i \in \mathbb{Q}.$$  \hspace{1cm} (4.0.30)
In fact since $F_i$ is effective and Lagrangian we have
\[
\int_X (\text{cl}(F_i) \wedge h^2) > 0, \quad \int_X (\text{cl}(F_i) \wedge \sigma \wedge \bar{\sigma}) = 0. \tag{4.0.31}
\]
By Item (6) of Proposition \ref{prop6} and by \ref{eq3.0.32} we have
\[
\text{cl}(F_i) = (x_i h^2 + y_i c_2(X)), \quad x_i, y_i \in \mathbb{Q}. \tag{4.0.32}
\]
Substituting the above expression for $\text{cl}(F_i)$ in \ref{eq4.0.31} and applying \ref{eq2.1} and \ref{eq3.0.12} we get \ref{eq4.0.30}. Now suppose that there exist two distinct irreducible components $F_i, F_j$ of $F$. Then $F_i \cap F_j = \emptyset$ because $F$ is smooth and hence by \ref{eq4.0.30} we get that
\[
0 = \int_X (\text{cl}(F_i) \wedge \text{cl}(F_i)) = k_i k_j \int_X (15h^2 - c_2(X))^2. \tag{4.0.33}
\]
Thus $\int_X (15h^2 - c_2(X))^2 = 0$. On the other hand using \ref{eq2.1} and \ref{eq3.0.12} we get that
\[
\int_X (15h^2 - c_2(X))^2 = 1728, \tag{4.0.34}
\]
contradiction. This shows that $F$ is irreducible. Let’s prove that
\[
c_2(F) = 192. \tag{4.0.35}
\]
First we compute the Euler characteristic of $Y$. We have $b_i(Y) = \dim H^i(\phi)_+$. Thus $b_i(Y) = 0$ for odd $i$ and $b_2(Y) = 1$ by \ref{eq4.0.28}. By \ref{eq2.1} and \ref{eq4.0.28} we get that $H^4(\phi)_+ = \mathbb{C}(h \wedge h) \oplus \text{Sym}^2(h^\perp)$ and hence $b_4(Y) = 254$. Thus
\[
\chi(Y) = 258. \tag{4.0.36}
\]
On the other hand the decompositions $X = (X \setminus F) \bigsqcup F$ and $Y = (Y \setminus \rho(F)) \bigsqcup \rho(F)$ give that
\[
324 = \chi(X) = 2\chi(Y \setminus \rho(F)) + \chi(F). \tag{4.0.37}
\]
By \ref{eq4.0.36} we have $258 = (\chi(Y \setminus \rho(F)) + \chi(F))$; together with \ref{eq4.0.35} this gives $\chi(F) = 192$, i.e. \ref{eq4.0.35}. Before proving the stated properties of $K_F$ we show that
\[
\text{cl}(F) = 5 h^2 - \frac{1}{3} c_2(X). \tag{4.0.38}
\]
We have
\[
\int \text{cl}(F) \wedge \text{cl}(F) = \int_F c_2(N_{F/X}) = \int_F c_2(\Omega^1_F) = 192, \tag{4.0.39}
\]
where the second equality holds because $F$ is Lagrangian and the third equality is given by \ref{eq4.0.33}; replacing $\text{cl}(F)$ by the right-hand side of \ref{eq4.0.33} and using \ref{eq4.0.34} one gets \ref{eq4.0.38}. Now let’s prove that
\[
\mathcal{O}_F(2K_F) \cong \mathcal{O}_F(6H). \tag{4.0.40}
\]
Let $F' := \rho(F)$; thus $\rho: F \rightarrow F'$ is an isomorphism. The embedding of $Y \cong (X/\langle \phi \rangle)$ into $\mathbb{P}^5$ defines by pull-back an isomorphism
\[
\rho^* N_{F'/\mathbb{P}^5} \cong \text{Sym}^2(N_{Y/X}). \tag{4.0.41}
\]
Since \( F \) is Lagrangian in \( X \) we have \( N_{F/X}^* \cong \Theta_F \); substituting in (4.0.41) and taking determinants we get an isomorphism
\[
\rho^* \det(N_{F/P^5}) \cong \mathcal{O}_F(3K_F).
\] (4.0.42)

On the other hand the normal sequence for the embedding \( F' \hookrightarrow \mathbb{P}^5 \) gives
\[
\det(N_{F'/\mathbb{P}^5}) \cong \mathcal{O}_{F'}(6) \otimes \mathcal{O}_{F'}(K_{F'}). \] (4.0.43)

Since \( \rho \) is an isomorphism and \( \rho^* \mathcal{O}_{F'}(1) \cong \mathcal{O}_F(H) \) we get that
\[
\rho^* \det(N_{F'/\mathbb{P}^5}) \cong \mathcal{O}_F(6H) \otimes \mathcal{O}_F(K_F). \] (4.0.44)

The above isomorphism together with (4.0.42) gives (4.0.40). Finally to get \( \phi \) we remark that we have the following stability result for the \( X \) satisfying (6) of Proposition 4.6.

**Proposition 4.6.** Let \( X \) be a numerical \((K3)^2 \) and suppose that there exist an anti-symplectic involution \( \phi: X \to X \) with quotient map \( f: X \to Y \) and an embedding \( j: Y \to \mathbb{P}^5 \) with \( j(Y) \) a sextic hypersurface. Let \( H \sim \mathcal{O}_Y(1) \). Let \( X' \) be a small deformation of \( X \) for which \( H \) remains of type \((1,1)\). There is an involution \( \phi': X' \to X' \) which is a deformation of \( \phi \) and letting \( f': X' \to Y' \) be the quotient map there is an embedding \( Y' \to \mathbb{P}^5 \) which deforms \( Y \to \mathbb{P}^5 \). Furthermore \( (f')^* \mathcal{O}_{Y'}(1) \) is the divisor-class deformation of \( H \).

**Proof.** Let \( h := c_1(H) \). Since \( j(Y) \) is a sextic and \( \deg f = 2 \) we have \( \int_X h^4 = 12 \). By Remark (2.1) and Equation (2.2.1) we get that \( (h,h) = 2 \). The invariant subspace \( H^2(X)_+ \subset H^2(X) \) for the action of \( H^2(\phi) \) contains \( h \) and has rank 1 because \( H^2(Y) \) has rank 1; thus \( H^2(X)_+ = \mathbb{C}h \). It follows that \( H^2(\phi) = R_h \) the reflection in the span of \( h \). The result then follows from Proposition (3.3) of [13]. (Notice that in that proposition we have \( 0 \in \mathcal{V} \).)

The case \( \dim Y = 4 \) and \( \deg f \geq 3 \). By Lemma (4.3) we get that one of (2), (3), (4), (5) holds.

We have proved Proposition 4.6.

## 5 Proof of Theorem (1.1)

It suffices to prove that (1)-(5) of Proposition 4.3 cannot hold. We assume that \( f: X \to Y \) satisfies on of (1), (2), ... (5) of Proposition 4.3 and we reach a contradiction. If \( f: X \to Y \) satisfies one of (1),..., (4) we show that there exists a linear subspace \( L \subset \mathbb{P}^5 \) of codimension 2 such that \( L \cap Y_0 \) is not reduced and irreducible of pure codimension 2, contradicting Proposition 4.2, or the pull-back \( f^*: H^4(Y) \to H^4(X) \) gives a rational Hodge substructure of \( H^4(X) \) which does not exist by Proposition 4.2. If \( f: X \to Y \) satisfies (5) of Proposition 4.3 and \( \dim(\text{sing}(Y)) = 3 \) then the first argument given above works. If \( f: X \to Y \) satisfies (5) of Proposition 4.3 and \( \dim(\text{sing}(Y)) \leq 2 \) then we show that the ramification divisor of \( f \) is the pull-back of a divisor on \( X \); since the ramification divisor is non-empty this is absurd.
5.1 (1) of Proposition (4.3) does not hold

We will prove the following result.

**Proposition 5.1.** Let $Y \subset \mathbb{P}^5$ be an irreducible non-degenerate linearly normal 3-dimensional subvariety of degree at most 6.

1. If $\deg Y \leq 5$ then given an arbitrary non-empty subset $U \subset Y$ there exists a 3-dimensional linear subspace $L \subset \mathbb{P}^5$ such that $L \cap U$ is reducible.

2. If $\deg Y = 6$ then there exists a 3-dimensional linear subspace $L \subset \mathbb{P}^5$ such that $L \cap Y$ is not reduced or not irreducible.

Granting the above proposition let's show that (1) of Proposition (4.3) does not hold. The proof is by contradiction. First assume that (1) of Proposition (4.3) holds with $\deg Y \leq 5$. Clearly $Y$ is irreducible non-degenerate and linearly normal and hence Proposition (5.1) applies with $U := Y^0$; thus there exists a 3-dimensional linear subspace $L \subset \mathbb{P}^5$ such that $L \cap Y^0$ is reducible; this contradicts Proposition (4.2). This proves that we cannot have $\deg Y \leq 5$.

**Claim 5.2.** Suppose that (1) of Proposition (4.3) holds with $\deg Y = 6$. Then $Y^0 = Y$.

**Proof.** By Item (1) of Proposition (4.3) we know that $\dim B = 0$. Let $n$ be such that $nH$ is very ample and let $D \in |nH|$ be generic; in particular since $\dim B = 0$ we have $D \subset (X \setminus B) = X^0$. It suffices to show that

$$f_0(D) = Y. \quad (5.1.1)$$

Since $\dim Y = 3$ the generic fiber of $f_0: X^0 \to Y$ is 1-dimensional and hence its intersection with $D$ consists of a finite set of points. Thus $f_0(D)$ is 3-dimensional. Since $f_0(D)$ is closed in $Y$ and $Y$ is irreducible of dimension 3 we get (5.1.1). \[\square\]

Now assume that (1) of Proposition (4.3) holds with $\deg Y = 6$; we will get to a contradiction. By Claim (5.1.1) we have $Y^0 = Y$. Since $Y$ is irreducible non-degenerate and linearly normal Proposition (5.1) applies and we get that there exists a 3-dimensional linear subspace $L \subset \mathbb{P}^5$ such that $L \cap Y^0$ is not reduced or not irreducible, contradicting Proposition (4.2).

**Proof of Proposition (5.1).** First consider the case of $Y$ a cone; thus $Y = J(p, Y)$ where $\overline{Y}$ is a surface with $\dim(\text{span}(\overline{Y})) = 4$. (See (1.0.4) for the notation $J(\cdot, \cdot)$.) Let $L \subset \text{span}(\overline{Y})$ be a generic linear subspace of dimension 2. Then $L := J(p, L)$ is a 3-dimensional linear subspace of $\mathbb{P}^5$ and

$$L \cap Y = J(p, \overline{L} \cap \overline{Y}). \quad (5.1.2)$$

Thus $L \cap U$ has $\deg \overline{Y}$ irreducible components - they are open dense subsets of lines through $p$. Since $\deg \overline{Y} = \deg Y \geq 3$ we get that $L \cap Y$ is reducible. This proves the proposition for $Y$ a cone. Now assume $Y$ is not a cone. We prove Item (1). Assume first that $Y$ is singular. Let $p \in \text{sing}(Y)$ and let $m$ be its multiplicity. Let $A \subset \mathbb{P}^5$ be a hyperplane not containing $p$ and let

$$\rho: (Y \setminus \{p\}) \to A \quad (5.1.3)$$
be projection from $p$. Let $Z := \text{Im}(\rho)$ and let $\overline{Z}$ be its closure. Since $Y$ is not a cone, $\overline{Z}$ is a hypersurface with $\deg \overline{Z} = (\deg Y - m)$. Thus $\overline{Z}$ is a hypersurface in $A \cong \mathbb{P}^4$ of degree at most 3 and hence it is covered by lines. The image $\rho(U \setminus \{p\}) \subset \overline{Z}$ contains an open dense $V \subset Z$. Let $\ell \subset \overline{Z}$ be a generic line: then $\ell \cap V$ is dense in $\ell$. Let $q \in (V \setminus \ell)$ be generic and let $\mathcal{L} := J(q, \ell)$. Thus $\mathcal{L} \subset A$ is a plane and
\[
\mathcal{L} \cap V = (\ell \cap V) \cup C
\] (5.1.4)
where $C$ is an open dense subset of a line or of a conic. (Notice that $\mathcal{L} \not\subset \overline{Z}$ because $\ell$ and $q$ are generic in $\overline{Z}$.) Let $L := J(p, \mathcal{L})$; this is a 3-dimensional linear subspace of $\mathbb{P}^5$. We have
\[
L \cap (\rho^{-1}V) = \rho^{-1}(\mathcal{L} \cap V)
\] (5.1.5)
and hence $L \cap (\rho^{-1}V)$ is reducible because of (5.1.4). Since $\rho^{-1}V$ is an open subset of $U$ we get that $L \cap U$ is reducible. Finally assume that $\rho^{-1}V$ is smooth with $\deg Y \leq 5$. All smooth non-degenerate linearly normal 3-folds in $Y \subset \mathbb{P}^5$ of degree at most 5 have been classified, see [14]: $Y$ is the Segre 3-fold i.e. $\mathbb{P}^1 \times \mathbb{P}^2$ embedded by $\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1)$, or a complete intersection of two quadric hypersurfaces, or a quadric fibration, i.e. it fibers over $\mathbb{P}^1$ with fibers which are embedded quadric surfaces. In each case $Y$ is covered by lines; it follows immediately that Item (1) of Proposition 5.1 holds for $Y$. Now we prove Item (2). First assume that $\dim(\text{sing}Y) = 2$. Let $V \subset \text{sing}Y$ be a 2-dimensional component. We claim that
\[
\deg V \leq 4.
\] (5.1.6)
In fact let $\Sigma \subset \mathbb{P}^5$ be a generic 3-dimensional linear subspace: then
\[
\text{sing}(\Sigma \cap Y) = \Sigma \cap \text{sing}Y \supset \Sigma \cap V
\] (5.1.7)
and $|\Sigma \cap V| = \deg V$. Now $\Sigma \cap Y$ is an irreducible non-degenerate curve in $\Sigma$, and hence it has at most 4 singular points. Thus (5.1.6) follows from (5.1.7). A straightforward argument shows that any surface $V$ of degree at most 4 contains a plane curve. Explicitly: If $\dim(\text{span}V) = 2$ there is nothing to prove. If $\dim(\text{span}V) \leq 3$ intersect $V$ with a plane contained in $\text{span}(V)$. If $\dim(\text{span}V) \geq 4$ and $V$ is singular the projection of $V$ from $q \in (\text{sing}V)$ is a quadric surface $Q$; if $\ell \subset Q$ is a line the intersection $J(q, \ell) \cap V$ has dimension 1. If $\dim(\text{span}V) \geq 4$ and $V$ is smooth then (see [11]) $V$ is a rational scroll, a complete intersection of quadric hypersurfaces in a hyperplane of $\mathbb{P}^5$ or the Veronese surface. In the first two cases $V$ contains lines, in the third case it contains conics. Thus we verified that $V$ contains a plane curve $C$. Let $L \subset \mathbb{P}^5$ be the generic 3-dimensional linear space containing $C$: then $L \cap Y$ is a reducible curve. This proves that Item (2) holds if $\dim(\text{sing}Y) = 2$. Now assume that $\dim(\text{sing}Y) = 1$. Let $W \subset (\text{sing}Y)$ be a 1-dimensional component. If $\dim(\text{span}W) \leq 2$ then $V$ contains a plane curve and we are done. Assume that $\dim(\text{span}W) \geq 4$. Then $\dim((\text{span}W) \cap Y) \geq 2$ and hence there exists $p \in ((\text{span}W) \cap (Y \setminus W))$. Since curves are never defective (see [11]) there exists a 3-secant plane of $W$ containing $p$, call it $\Omega$. We claim that $\dim(\Omega \cap Y) \geq 1$. In fact if this is not the case then $\dim(\Omega \cap Y) = 0$ and hence the multiplicity of the intersection $\Omega \cap Y$ is equal to $\deg Y = 6$: but the points in $\Omega \cap W$ give a contribution of at least 6 because $\Omega$ is 3-secant to $W$ and $W \subset (\text{sing}Y)$, and we
have a contribution of at least 1 from $p$, for a total of at least 7, contradiction. Thus $Y$ contains a plane curve and we are done. We are left with the case $\dim(\text{span}W) = 3$. If $\dim((\text{span}W) \cap Y) = 2$ then $Y$ contains plane curves and we are done. If $\dim((\text{span}W) \cap Y) = 1$ let $L := \text{span}W$; since $Y$ is singular along $W$ the intersection $L \cap Y$ is not reduced along $W$. We have proved that Item (2) holds if $\dim(\text{Sing}Y) \geq 1$. Now assume that $\dim(\text{Sing}Y) \leq 0$. Let $\Lambda \subset \mathbb{P}^5$ be a generic hyperplane; thus $S := \Lambda \cap Y$ is a smooth non-degenerate (in $\Lambda$!) surface of degree 6. Since $\deg(S) \neq 4$ we know that $S$ is linearly normal (Severi) and we may apply the known classification of such surfaces (see [14]): $S$ is the complete intersection of a quadric and a cubic or it is a Bordiga surface i.e. the blow up of $\mathbb{P}^2$ at 10 points embedded by the linear system of plane quartics through the 10 points. If $S$ is a Bordiga surface it contains lines; if $\ell \subset S$ is a line and $L \subset \mathbb{P}^5$ a generic 3-dimensional linear subspace containing $\ell$ the intersection $L \cap Y$ is reducible. If $S$ is the complete intersection of a quadric and a cubic then since $Y$ is linearly normal the quadric hypersurface in $\Lambda$ containing $S$ lifts to a quadric hypersurface $Q \subset \mathbb{P}^5$ containing $Y$. There exist 3-dimensional linear spaces $L \subset \mathbb{P}^5$ such that $L \cap Q$ is the union of 2 planes; if $L$ is a generic such space then $L \cap Y$ is reducible. This finishes the proof of Proposition (5.1).

5.1.1 Comments

One may ask the following: does there exist a numerical $(K3)^{[2]}$ with an ample $H$ with $(c_1(H), c_1(H)) = 2$ and $Y := \text{Im}(f : X \cdots > |H|)$ of dimension strictly smaller than 4? We do not know of any such example however we do have examples with $H$ big and nef such that $\dim Y < \dim X$. (The case of big and nef divisors will be needed in order to construct complete moduli spaces.) An explicit example is the following. Let $\pi : S \to \mathbb{P}^2$ be a double cover ramified over a smooth sextic; thus $S$ is a $K3$ surface. Let $H_S := \pi^* O_{\mathbb{P}^2}(1)$ and let $X := M(0, H_S, 0)$ be the Moduli space of $H_S$-semistable rank-0 pure sheaves $G$ on $S$ with $c_1(G) = c_1(H_S)$ and $\chi(G) = 0$: a typical $G$ is given by $\iota_* \xi$ where $\iota : C \hookrightarrow S$ is the inclusion of a curve $C \subset |H_S|$ and $\xi$ is a degree-1 line-bundle on $C$. It is known that $X$ is a deformation of $(K3)^{[2]}$ - see [22]. There is a Lagrangian fibration $\rho : X \to |H_S|$ mapping $[G] \in M(0, H_S, 0)$ to its support; the fiber over $C \in |H_S|$ is $\text{Jac}^1(C)$ (suitably defined if $C$ is singular). Thus on $X$ we have the divisor class $F := \rho^* \mathcal{O}_{|H_S|}(1)$. We also have a unique effective divisor $A$ on $X$ whose restriction to any Lagrangian fiber $\rho^{-1}([C]) \cong \text{Jac}^1(C)$ is the canonical $\Theta$-divisor. Let $H := A + 2F$; a straightforward argument shows that $(c_1(H), c_1(H)) = 2$ - use (2.1.3). One can also show that $H$ is nef; since $\int_X c_1(H)^4 = 12$ we get that $H$ is big. The image $Y = \text{Im}(f : X \cdots > |H|)$ is the Veronese surface in $\mathbb{P}^5$.

5.2 (2) of Proposition (4.3) does not hold

We assume that $Y \subset \mathbb{P}^5$ is an irreducible quadric hypersurface and we will get to a contradiction. Since $Y_0 \subset Y$ is open dense in a quadric 4-fold there exists a 3-dimensional linear subspace $L \subset \mathbb{P}^5$ such that $L \cap Y_0$ is reducible; this contradicts Proposition (4.2).
5.2.1 Comments

There exist examples \((X, H)\) with \(X\) a deformation of \((K3)^{[2]}\) and \(H\) an ample divisors with \((c_1(H), c_1(H)) = 2\) such that \(Y = Im(f : X \cdot > |H|)\) is a quadric hypersurface - see (4.1) of [19].

5.3 (3) of Proposition (4.3) does not hold

We will use the following elementary result.

Proposition 5.3. Let \(Y \subset \mathbb{P}^5\) be a cubic hypersurface containing a 3-dimensional linear space \(\Omega\). There exists a hyperplane \(Z \subset \mathbb{P}^5\) containing \(\Omega\) such that \(Z \cap Y\) is swept out by planes, i.e. either \(Z \subset Y\) or \(Z \cdot Y = \Omega + Q\) where \(Q \subset Z\) is a singular quadric hypersurface.

Proof. Let \(I \subset \mathbb{G}(3, \mathbb{P}^5) \times |\mathcal{O}_{\mathbb{P}^5}(3)| \times (\mathbb{P}^5)^V\) be the set of triples \((\Omega, Y, Z)\) where \(\Omega \subset Y\) and \(\Omega \subset Z\), let \(J \subset \mathbb{G}(3, \mathbb{P}^5) \times |\mathcal{O}_{\mathbb{P}^5}(3)|\) be the set of couples \((\Omega, Y)\) where \(\Omega \subset Y\) and let

\[
I \quad \rho \quad J
\]

be the forgetful map. Let \(I^0 \subset I\) be the subset of triples \((\Omega, Y, Z)\) such that \(Z \cdot Y = \Omega + Q\) with \(Q \subset Z\) a smooth quadric hypersurface. We must show that \(\rho(I \setminus I^0) = J\). The map \(\rho\) is proper with 1-dimensional fibers, \(J\) is irreducible and \((I \setminus I^0)\) is closed of codimension at most 1 at every point; thus it suffices to exhibit one couple \((\Omega, Y) \in J\) such that

\[
\rho^{-1}(\Omega, Y) \cap (I \setminus I^0) \neq \emptyset, \rho^{-1}(\Omega, Y). \tag{5.3.2}
\]

Let \([X_0, \ldots, X_5]\) be homogeneous coordinates on \(\mathbb{P}^5\). Let \(\Omega = V(X_4, X_5)\) and \(Y = V(F \cdot X_4 + G \cdot X_5)\) where \(F, G \in \mathbb{C}[X_0, \ldots, X_5]\) are homogeneous of degree 2 with \(F(X_0, \ldots, X_4, 0)\) and \(G(X_0, \ldots, X_5, 0, X_5)\) quadratic forms of rank 4 and 5 respectively. Then \((\Omega, Y)\) is a couple satisfying (5.3.2).

Now suppose that (3) of Proposition (4.3) holds i.e. that \(f : X \cdot > Y\) is a map of degree 3 and that \(Y\) is a cubic hypersurface; we will arrive at a contradiction. First we notice the following corollary of Proposition (5.3).

Corollary 5.4. Suppose that \(f : X \cdot > Y\) is a map of degree 3 and that \(Y\) is a cubic hypersurface. Then \(Y\) does not contain a 3-dimensional linear space.

Proof. By contradiction. Let \(\Omega \subset Y\) be a 3-dimensional linear space. By Proposition 5.3 there exists a hyperplane \(Z \subset \mathbb{P}^5\) containing \(\Omega\) such that \(Z \cap Y\) is swept out by planes. We claim that \(Z \cap Y_0 \neq \emptyset\). In fact let \(\bar{f}\) and \(E\) be as in (1.10.10) and (1.10.9) respectively; if \(Z \cap Y_0 = \emptyset\) then \(supp(f^*Z) \subset supp(E)\), absurd. Let \(y \in Z \cap Y_0\); by Proposition 5.3 there exists a plane \(\Lambda \subset (Z \cap Y)\) with \(y \in \Lambda\). Now let \(y' \in (Y_0 \setminus Z)\) and let \(L \subset \mathbb{P}^5\) be the 3-dimensional linear space \(L := J(y', \Lambda)\). Then

(a) either \(L \subset Y\), or

(b) \(L \cap Y = \Lambda \cup \Gamma\) where \(\dim \Gamma = 2\) and \(\Gamma \ni y'\).
If Item (a) holds then \( L \cap Y_0 \) is non-empty 3-dimensional (notice: we do not know whether our “original” 3-dimensional linear space \( \Omega \subset Y \) intersects \( Y_0 \)) and if Item (b) holds then \( L \cap Y_0 \) is reducible. In either case we contradict Proposition 4.2.

Let \( B \) be the base-scheme of \(|H|\). We know by (4.0.3) that \( \dim B \leq 1 \): we consider separately the two cases \( \dim B = 0 \) and \( \dim B = 1 \). Suppose first that \( \dim B = 0 \). Then the 1-cycle \( \Sigma \) appearing in (4.0.19) is zero and hence (4.0.21) gives that
\[
\sum_{p \in \text{supp}B} \text{mult}_p(D \cap D' \cap D'' \cap D''') = 3.
\]
(5.3.3)

This implies that \( B \) is the disjoint union of 0-dimensional schemes \( B_i \) each of which is curvilinear (contained in a smooth curve) and supported on a single point. Let \( \ell_i \) be the length of \( B_i \); a straightforward computation shows that \( E = \sum_i \ell_i E_i \) with \( E_i \) a prime divisor such that \( \pi(E_i) = \text{supp}B_i \). (Recall that \( \pi: \tilde{X} \to X \) is the blow-up of \( B \).) Furthermore each \( E_i \) is isomorphic to \( \mathbb{P}^3 \) and \( \tilde{f}^* \mathcal{O}_Y(1) \cong \mathcal{O}_{E_i}(1) \). Thus \( \tilde{f}(E_i) \subset Y \) is a 3-dimensional linear space; this is absurd by Corollary 5.4. Thus we are left with the case \( \dim B = 1 \).

**Proposition 5.5.** Keep notation as above and assume that \( \dim B = 1 \). Then \( B \) is a reduced, irreducible, local complete intersection of pure dimension 1. Furthermore the following hold:

(a) Let \( \Sigma \) be the 1-cycle appearing in (4.0.19). Then \( \Sigma = [B] \), hence we may identify \( \Sigma \) with \( B \).

(b) Let \( \Gamma \) be the 1-cycle appearing in (4.0.19). Then \( \text{supp}(\Gamma) \) intersects \( \Sigma \) in a single point \( p \), \( \text{supp}(\Gamma) \) is smooth at \( p \) with tangent direction not contained in \( \Theta_p \Sigma \), and the unique component of \( \text{supp}(\Gamma) \) through \( p \) appears with multiplicity 1 in the cycle \( \Gamma \).

(c) As \( D, \ldots, D''' \) vary among divisors such that (4.0.18) holds the point of intersection \( \text{supp}(\Gamma) \cap \Sigma \) varies in \( \Sigma \), i.e. it is not constant.

**Proof.** By (4.0.21) and (4.0.22) we get that
\[
\sum_{p \in \text{supp}B} \text{mult}_p(D \cap \Gamma) + \int_{\Sigma} h = 3.
\]
(5.3.4)

Using Item (3) of Proposition 3.2 we get that \( cl(\Sigma) = mh^3/6 \) for some positive integer \( m \); in fact \( m \) is strictly positive or else \( \dim B = 0 \). Thus \( \int_{\Sigma} h = 2m \) and by (5.3.4) we get that
\[
\int_{\Sigma} h = 2, \quad cl(\Sigma) = h^3/6.
\]
(5.3.5)

Furthermore, again by Item (3) of Proposition 3.2, we get that \( \text{supp}(\Sigma) \) is irreducible and that the multiplicity of \( \Sigma \) equals 1, i.e. \( \Sigma \) is equal to the reduced irreducible curve \( \text{supp}(\Sigma) \). Since \( H \) is ample the scheme \( D' \cap D'' \cap D''' \) is
connected and hence \( \text{supp}(\Gamma) \cap \Sigma \neq \emptyset \); let \( p \in \text{supp}(\Gamma) \cap \Sigma \). Since \( \Sigma \subset B \) we have \( p \in \text{supp}B \); thus gives that
\[
\text{mult}_p(D \cdot \Gamma) = 1, \quad \text{supp}(B) \cap \text{supp}(\Gamma) = \{p\}. \tag{5.3.6}
\]
This proves Item (b) because \( \Sigma \subset \text{supp}(B) \). Furthermore since \( \text{supp}(B) \subset \text{supp}(\Gamma) \cup \Sigma \) we get that \( \text{supp}(B) = \text{supp}(\Sigma) \). On the other hand \( B \) is a subscheme of \( D \cap \cdots \cap D'' \) and away from \( \text{supp}(\Gamma) \) the latter scheme coincides with the reduced and irreducible l.c.i. \( \Sigma \); thus \( B \) is a reduced and irreducible l.c.i. away from \( p \). This shows that if Item (c) holds then also Item (a) holds. We prove Item (c) arguing by contradiction. If Item (c) is false then \( \text{supp}(\Gamma) \cap \Sigma = \{p\} \) for a fixed \( p \in \Sigma \). Since all \( p \in \Sigma \) whenever \( \text{mult}_{\Sigma} \geq 1 \) holds. Let \( \Lambda \subset |H| \) be the \( \mathbb{P}^1 \) spanned by \( D, \ldots, D'' \) and let
\[
\Lambda_p := \{ Z \in \Lambda \mid \text{mult}_p Z \geq 2 \}. \tag{5.3.7}
\]
Since all \( Z \in \Lambda \) contain \( \Sigma \) we get that \( \Lambda_p \) is a linear subspace of \( \Lambda \) with \( \text{cod}(\Lambda_p, \Lambda) \leq 3 \) and hence \( \Lambda_p \) is not empty because \( \dim \Lambda = 3 \). Renaming \( D, \ldots, D'' \) we may assume that \( D \in \Lambda_p \). Since \( p \in \text{supp}(\Gamma) \cap \text{supp}(B) \) Equation \( \text{(5.3.7)} \) gives that
\[
4 \leq \sum_{p \in \text{supp}B} \text{mult}_p(D \cap \Gamma) + \int_{\Sigma} h = 3, \tag{5.3.8}
\]
absurd.
\[\Box\]

Let’s show that if \( \text{dim} B = 1 \) then we get a contradiction. By Proposition \( \text{(5.3)} \) the exceptional divisor is a \( \mathbb{P}^2 \)-fibration \( E \to B \) and \( \tilde{f} \) embeds each fiber \( \pi^{-1}(p) \) over \( p \in B \) as a plane in \( \mathbb{P}^5 \). We claim that \( \tilde{f}(E) \) is a 3-dimensional linear subspace of \( \mathbb{P}^5 \). In fact let \( L', L'', L''' \subset \mathbb{P}^5 \) be the hyperplanes corresponding to \( D', D'', D''' \) via \( \text{(4.0.7)} \). By Proposition \( \text{(5.5)} \) the divisors \( \tilde{f}^*(L'), \tilde{f}^{-1}(L''), \tilde{f}^*(L''') \) intersect transversely in a single point. Thus \( \dim \tilde{f}(E) = 3 \), because if we had \( \dim \tilde{f}(E) = 2 \) then the intersection would be either empty or of dimension 1. Furthermore since \( L', L'', L''' \) are generic hyperplanes we get that \( \tilde{f}(E) \) has degree 1, i.e. it is 3-dimensional linear space. Since \( \tilde{f}(E) \subset Y \) this contradicts Proposition \( \text{(5.4)} \). This completes the proof that Item (3) of Proposition \( \text{(4.3)} \) does not hold.

### 5.4 (4) of Proposition (4.3) does not hold

We will prove that our map \( f : X \to \mathbb{P}^5 \) cannot be a degree-4 regular map onto an irreducible cubic hypersurface \( Y \subset \mathbb{P}^5 \). The proof is by contradiction. We assume that we have \( f : X \to Y \) a finite regular map of degree 4 onto a cubic 4-fold \( Y \subset \mathbb{P}^5 \) and we reach a contradiction. If \( Y \) is smooth a straightforward argument shows that \( f^* H^4(Y) \) is a non-existant Hodge substructure of \( H^4(X) \) - see Subsubsection \( \text{(5.4.1)} \). The proof that \( Y \) cannot be a singular cubic 4-fold is more involved: it will follow from some results on singular cubic 4-folds which should be of independent interest. Let \( Y \subset \mathbb{P}^5 \) be an arbitrary singular cubic hypersurface: for \( p \in \text{sing}(Y) \) we let
\[
S_p := \{ \ell \in \text{Gr}(1, \mathbb{P}^5) \mid p \in \ell \subset Y \}. \tag{5.4.1}
\]
The definition above is set-theoretic but of course \( S_p \) has a natural structure as subscheme of \( \text{Gr}(1, \mathbb{P}^5) \). We will prove the following result.
Proposition 5.6. Let \( Y \subset \mathbb{P}^5 \) be a singular cubic hypersurface and let \( p \in \text{sing} Y \). Then

(1) either \( Y \) contains a plane or

(2) \( Y \) has isolated quadratic singularities, the scheme \( S_p \) is a reduced, normal surface not containing lines nor conics, with Du Val singularities\(^2\). The minimal desingularization of \( S_p \) is a K3 surface \( \tilde{S}_p \).

The proof of Proposition 5.6 goes as follows. If \( Y \) is reducible then \( Y \) satisfies Item (1) trivially - and of course it does not satisfy Item (2). If \( Y \) is a cone then \( Y \) certainly does not satisfy Item (2), and it satisfies Item (1) by the following elementary result.

Lemma 5.7. Let \( Y \subset \mathbb{P}^5 \) be a cubic hypersurface which is a cone. Then \( Y \) contains a plane.

Proof. We have \( Y = J(p, \bar{Y}) \) where \( \bar{Y} \) is a cubic hypersurface in \( \mathbb{P}^4 \). Thus \( \bar{Y} \) contains a line \( \ell \) and hence \( Y \) contains the plane \( J(p, \ell) \).

We are left with the case of \( Y \) an irreducible singular cubic 4-fold which is not a cone, i.e. every singular point of \( Y \) is quadratic. In Subsubsections (5.4.3)-(5.4.4)-(5.4.5) we will prove that if \( \dim(\text{sing} Y) \) is equal to 3, 2, 1 respectively then \( Y \) contains a plane; the basic (well-known) observation is that the line joining two distinct singular points of \( Y \) is contained in \( Y \). Thus if \( \dim(Y) \geq 1 \) then Item (1) of Proposition 5.6 holds - and Item (2) does not hold by hypothesis. In Subsubsection (5.4.6) we prove that if \( Y \) has isolated quadratic singularities then either Item (1) or Item (2) of Proposition 5.6 holds. It is elementary that (1) and (2) cannot both hold; the hard part is to show that if (1) does not hold then \( S_p \) has Du Val singularities - the remaining statements of (2) are straightforward with the exception of the assertion about the minimal desingularization of \( S_p \), this follows from the fact that the singularities are Du Val. First we prove by explicit computation that the singularities of \( S_p \) which are not lines joining \( p \) to another singular point of \( Y \) are Du Val. Then by analyzing the relation between \( S_p \) and \( S_{p'} \) for \( p' \neq p \) we are able to get that \( S_p \) is Du Val also at the points span\((p, p')\) for \( p' \in \text{sing} Y \). This will complete the proof of Proposition 5.6. In order to prove that (4) of Proposition 4.3 does not hold we will need a result on the (mixed) Hodge structure of a cubic 4-fold \( Y \) satisfying Item (2) of Proposition 5.6. Let

\[ \ldots \subset W_3 H^4(Y) \subset W_4 H^4(Y) = H^4(Y) \] (5.4.2)

be Deligne’s weight filtration \[.\] In particular

\[ W_3 H^4(Y) = \ker(H^4(Y) \xrightarrow{H^4(\zeta)} H^4(\bar{Y})) \] (5.4.3)

where \( \zeta: \bar{Y} \to Y \) is any desingularization, see Proposition (8.5.2) of \[.\] Thus \( W_3 H^4(Y) \) is in the kernel of the intersection form on \( H^4(Y) \) and hence the intersection form is well-defined on \( Gr^W H^4(Y) := H^4(Y)/W_3 H^4(Y) \). Let \( p \in \text{sing} Y \); since we are assuming that \( Y \) satisfies Item (2) of Proposition 5.6 we

\[^2\text{See Ch.4 of } \text{(3)} \text{ for definition and properties of Du Val singularities.}\]
know that $\tilde{S}_p$ is a $K3$ surface. Let $T(\tilde{S}_p) \subset H^2(\tilde{S}_p; \mathbb{Z})$ be the transcendental lattice of $\tilde{S}_p$ i.e.

$$T(\tilde{S}_p) := \{ \alpha \in H^2(\tilde{S}_p; \mathbb{Z}) | \alpha \perp H^1_\mathbb{Z}(\tilde{S}_p) \}. \tag{5.4.4}$$

Then

$$T(\tilde{S}_p)_C := T(\tilde{S}_p) \otimes \mathbb{C} \subset H^2(\tilde{S}_p) \tag{5.4.5}$$

is a sub-Hodge structure of level 2 with

$$h^{2,0}(T(\tilde{S}_p)_C) = h^{0,2}(T(\tilde{S}_p)_C) = 1, \quad 1 \leq h^{1,1}(T(\tilde{S}_p)_C) \leq 19. \tag{5.4.6}$$

The following result will be proved in Subsubsection (5.4.7).

**Proposition 5.8.** Suppose that a cubic hypersurface $Y \subset \mathbb{P}^5$ satisfies Item (2) of Proposition (5.6). Then there is a morphism of type $(1, 1)$ of Hodge structures $\gamma: T(\tilde{S}_p)_C \to \text{Gr}^W_W H^4(Y). \tag{5.4.7}$

If $\eta, \theta \in T(\tilde{S}_p)_C$ then

$$\int_Y \gamma(\eta) \wedge \gamma(\theta) = -\int_{\tilde{S}_p} \eta \wedge \theta. \tag{5.4.8}$$

Granting Propositions (5.6)-(5.8) let’s prove that it is impossible to have $f: X \to Y$ finite of degree 4 onto a singular cubic 4-fold. Assume that such an $f$ exists; since $f$ is regular $Y = Y_0$ and hence $Y$ does not contain planes by Proposition (4.2). Let $p \in \text{sing}Y$. By Propositions (5.6)-(5.8) we have the morphism of type $(1, 1)$ of Hodge structures $\gamma$ of (5.4.7). Composing $\gamma$ with $f^*$ we get a morphism of type $(1, 1)$ of Hodge structures

$$T(\tilde{S}_p)_C \xrightarrow{f^* \gamma} H^4(X). \tag{5.4.9}$$

Let $\eta, \theta \in T(\tilde{S}_p)_C$; by (5.4.8) we have

$$\int_X f^* \gamma(\eta) \wedge f^* \gamma(\theta) = -4 \int_{\tilde{S}_p} \eta \wedge \theta. \tag{5.4.10}$$

Since the restriction to $T(\tilde{S}_p)_C$ of the intersection form on $H^2(\tilde{S}_p)$ is non-degenerate we get that $f^* \gamma$ is injective. Thus $\text{Im}(f^* \gamma)$ is a rational Hodge substructure of $H^4(X)$ with Hodge numbers $h^{p, q} = h^{p-1, q-1}(T(\tilde{S}_p)_C)$. By (5.4.6) this contradicts Item (4) of Proposition (3.2).

In the last subsusbsection we comment on the possibility that $f: X \to Y$ is of degree 4 onto a cubic when one drops one of the hypotheses of Proposition (4.3).

5.4.1 (4) of Proposition (4.3) with $Y$ smooth does not hold

We assume that $f: X \to Y$ with $Y \subset \mathbb{P}^5$ a smooth cubic hypersurface, $f$ finite of degree 4 and we get to a contradiction. Since $\deg f = 4$ we have

$$\langle f^* \alpha, f^* \beta \rangle_X = 4 \langle \alpha, \beta \rangle_Y, \quad \alpha, \beta \in H^4(Y) \tag{5.4.11}$$
where \((\cdot)_X\) and \((\cdot)_Y\) are the intersection forms on \(H^4(X)\) and \(H^4(Y)\) respectively. Thus \(f^*: H^4(Y) \to H^4(X)\) is an injection of rational Hodge structures. Let

\[ H^4(Y)_{prim} := \{ \alpha \in H^4(Y) | \alpha \wedge c_4(O_Y(1)) = 0 \} \]

be the primitive cohomology of \(Y\): this a rational sub Hodge structure of \(H^4(Y)\). Since \(\dim H^4(Y)_{prim} = 22\) Item (4) of Proposition (3.2) gives that \(f^* H^4(Y)_{prim} = \mathbb{C} h \otimes h^\perp\). Thus

\[ f^* H^4(Y; \mathbb{Q})_{prim} = \mathbb{Q}h \otimes h^\perp_{\mathbb{Q}} \]  

(5.4.12)

where \(h^\perp := h^\perp \cap H^2(X; \mathbb{Q})\). Let \(\mathcal{B} = \{ \alpha_1, \ldots, \alpha_{22} \}\) be a \(\mathbb{Z}\)-basis of \(H^4(Y; \mathbb{Z})_{prim}\). Let \(Q_B\) be the matrix of the restriction of \((\cdot)_Y\) to \(H^4(Y; \mathbb{Z})_{prim}\) in the basis \(\mathcal{B}\).

Since \((\cdot)_Y\) is unimodular and \(\deg Y = 3\) we have

\[ |\det(Q_B)| = 3. \]  

(5.4.13)

Let \(B' := \{ f^* \alpha_1, \ldots, f^* \alpha_{22} \}\); by (5.4.12) we know that \(B'\) is a \(\mathbb{Q}\)-basis of \(Qh \otimes h^\perp_{\mathbb{Q}}\).

Let \(Q_{B'}\) be the matrix of the restriction of \((\cdot)_X\) to \(Qh \otimes h^\perp_{\mathbb{Q}}\) in the basis \(B'\); by (5.4.13)-(5.4.14) we have

\[ |\det(Q_{B'})| = 3 \cdot 2^{44}. \]  

(5.4.14)

Now let \(\{ \beta_1, \ldots, \beta_{22} \}\) be a \(\mathbb{Z}\)-basis of \(h^\perp := H^2(X; \mathbb{Z}) \cap h^\perp\); then \(B' := \{ h\beta_1, \ldots, h\beta_{22} \}\) is a \(\mathbb{Q}\)-basis of \(Qh \otimes h^\perp_{\mathbb{Q}}\). Let \(Q_{B''}\) be the matrix of the restriction of \((\cdot)_X\) to \(Qh \otimes h^\perp_{\mathbb{Q}}\) in the basis \(B''\). By Remark (2.1) one gets (use also Lemma (5.4)) that

\[ |\det(Q_{B''})| = 2^{24}. \]  

(5.4.15)

Since both \(B'\) and \(B''\) are \(\mathbb{Q}\)-bases of \(Qh \otimes h^\perp_{\mathbb{Q}}\) the determinants appearing in Equations (5.4.13)- (5.4.15) must represent the same class in \(\mathbb{Q}^*/(\mathbb{Q}^*)^2\). This is visibly false, contradiction.

### 5.4.2 \(Y\) a singular cubic 4-fold: elementary considerations

Let \(Y \subset \mathbb{P}^5\) be a singular cubic hypersurface. Suppose that \(p, q \in \text{sing} Y\) are distinct points: \(\text{span}(p, q)\) and \(Y\) intersect with multiplicity at least 2 at \(p\) and at \(q\) hence by Bézout we get that \(\text{span}(p, q) \subset Y\). Thus for a subset \(W \subset \text{sing} Y\) we have

\[ \text{chord}(W) \subset Y \]  

(5.4.16)

where \(\text{chord}(W) \subset \mathbb{P}^5\) is the subvariety swept out by the chords of \(W\) i.e.

\[ \text{chord}(W) := \text{closure of } \{ \text{span}(p, q) | p, q \in W, \ p \neq q \}. \]  

(5.4.17)

Now assume that \(Y\) is irreducible and reduced, \(p \in \text{sing} Y\) and \(Y\) is not a cone with vertex \(p\). Choose homogeneous coordinates \([X_0, \ldots, X_4, Z]\) on \(\mathbb{P}^5\) such that \(p = [0, \ldots, 0, 1]\). We have

\[ Y = V(F(X_0, \ldots, X_4)Z + G(X_0, \ldots, X_4)) \]  

(5.4.18)

where \(F, G\) are homogeneous non-zero of degrees 2 and 3 respectively. We have

\[ \mathbb{P}(C_p Y) = V(F(X_0, \ldots, X_4)) \subset \mathbb{P}^4_{[X_0, \ldots, X_4]} = \mathbb{P}(\Theta_p Y) = \mathbb{P}(\Theta_p \mathbb{P}^5). \]  

(5.4.19)
Let
\[ \psi_p : Y \rightarrow \mathbb{P}(\Theta_p \mathbb{P}^5) \]  
be projection from \( p \). The map \( \psi_p \) is birational: letting \( X := X_0, \ldots, X_4 \) the inverse of \( \psi_p \) is given by
\[
\mathbb{P}(\Theta_p \mathbb{P}^5) \stackrel{\psi_p^{-1}}{\rightarrow} Y
\]
\[
[X] \mapsto [F(X)X_0, \ldots, F(X)X_4, -G(X)]]
\]

The indeterminacy locus of \( \psi_p^{-1} \) is clearly the set of lines through \( p \) contained in \( Y \) (see (5.4.1)). Using the coordinates introduced above we see that the natural inclusion \( S_p \subset \mathbb{P}(\Theta_p \mathbb{P}^5) \) is given by
\[
S_p = V(F,G) \subset \mathbb{P}[X] = \mathbb{P}(\Theta_p \mathbb{P}^5).
\]
Notice that since \( Y \) is irreducible, reduced and not a cone with vertex \( p \) the polynomials \( F,G \) have no common factors and hence
\[
S_p \text{ is a complete intersection of } \mathbb{P}(C_p Y) \text{ and a cubic hypersurface}. \quad (5.4.23)
\]

Formula (5.4.21) says that \( \psi_p^{-1} \) is defined by the linear system \(|I_{S_p}(3)|\) on \( \mathbb{P}(\Theta_p \mathbb{P}^5) \). Since \( I_{S_p}(3) \) is globally generated we get that the resolution of indeterminacies of \( \psi_p \) defines an isomorphism
\[
\tilde{\psi}_p : \text{Bl}_p Y \xrightarrow{\sim} \text{Bl}_{S_p} \mathbb{P}(\Theta_p \mathbb{P}^5).
\]
We will need to relate properties of \( Y \) and of \( S_p \). A first observation: if \( y \in \text{sing}(Y \setminus \{p\}) \) then \( \text{span}(p,y) \subset Y \) by (5.4.19) and hence
\[
\psi_p(\text{sing}(Y \setminus \{p\})) \subset S_p \subset \mathbb{P}(C_p Y).
\]

**Proposition 5.9.** Suppose that \( Y \subset \mathbb{P}^5 \) is a singular reduced and irreducible cubic hypersurface, that \( p \in \text{sing} \ Y \) and that \( Y \) is not a cone with vertex \( p \).

1. If \( y \in \text{sing}(Y \setminus \{p\}) \) then \( s := \psi_p(y) \in \text{sing}(S_p) \). If \( \text{span}(p,y) \subset \text{sing}(Y) \)
   then \( \dim \Theta_s(S_p) = 4 \), in particular \( \mathbb{P}(C_p Y) \) is singular at \( s \). If \( \text{span}(p,y) \not\subset \text{sing}(Y) \)
   then \( \mathbb{P}(C_p Y) \) is smooth at \( s \).

2. Let \( s \in \text{sing}(S_p) \) and assume that \( \dim \Theta_s(S_p) = 4 \). Then \( Y \) is singular at all points of the line corresponding to \( s \).

3. Let \( s \in \text{sing}(S_p) \) and assume that \( \dim \Theta_s(S_p) = 3 \). If \( \mathbb{P}(C_p Y) \) is smooth at \( s \) there exists a unique \( y \in \text{sing}(Y \setminus \{p\}) \) such that \( \psi_p(y) = s \). If \( \mathbb{P}(C_p Y) \)
   is singular at \( s \) there is no \( y \in \text{sing}(Y \setminus \{p\}) \) such that \( \psi_p(y) = s \).

4. \( Y \) contains a plane if and only if \( S_p \) contains a line or a conic.

**Proof.** Let \( [X_0, \ldots, X_4, Z] \) be homogeneous coordinates on \( \mathbb{P}^5 \) with \( p = [0, \ldots, 0, 1] \); thus we have (5.4.18) (5.4.22). Let \( y = [a_0, \ldots, a_4, b] \in \mathbb{P}^5 \setminus \{p\} \); thus
\[
\psi_p(y) = [a_0, \ldots, a_4] = [a].
\]
Differentiating the defining equation of $Y$ we get that $y \in \text{sing}(Y \setminus \{p\})$ if and only if
\begin{equation}
  b \frac{\partial F}{\partial x_i}(\mathbf{a}) + \frac{\partial G}{\partial x_i}(\mathbf{a}) = 0 \quad i = 0, \ldots, 4, \quad \text{and} \quad F(\mathbf{a}) = 0.
\end{equation}

(1): From the two equations above we get that $G(\mathbf{a}) = 0$ (we already noticed this), and hence the first equation shows that $s \in \text{sing}(S_p)$. Assume that for a fixed $\mathbf{a} \neq (0, \ldots, 0)$ the first equation holds with an arbitrary choice of $b$: then both $\mathcal{V}(F)$ and $\mathcal{V}(G)$ are singular at $s$ and this proves the second statement. Assume that for a fixed $\mathbf{a} \neq (0, \ldots, 0)$ the first equation holds for some but not for all choices of $b$: then $\mathcal{V}(F)$ is smooth at $s$ and this proves the third statement.

Items (2)-(3) are proved by similar elementary considerations. Now let’s prove Item (4). Assume that $Y$ contains a plane. If $p \in L$ then $\psi_p(L \setminus \{p\})$ is a line contained in $S_p$. If $p \notin L$ then $\Lambda := \psi_p(L)$ is a plane in $\mathbb{P}^4_{[x]}$. The restriction of $\psi^{-1}_p$ to $\Lambda$ is the linear system $|I_{\Lambda \cap S_p}(3)|$. Since $\psi^{-1}_p(\Lambda) = L$ is a plane we get that necessarily $\Lambda \cap S_p$ is a cone in $\Lambda$; thus $S_p$ contains a cone. The proof of the converse is similar. \hfill \Box

### 5.4.3 Proof of Proposition (5.6) for $Y$ with $\dim(\text{sing}(Y)) = 3$

As shown in the introduction to the subsection we may assume that $Y$ is reduced, irreducible and not a cone. Let $Y \subset \mathbb{P}^5$ be a reduced and irreducible cubic hypersurface with $\dim(\text{sing}(Y)) = 3$. The intersection of $Y$ and a generic plane is a singular reduced and irreducible cubic curve and hence it has exactly one singular point. Thus $\text{sing}(Y)$ has exactly one 3-dimensional irreducible component, call it $V$, and $V$ is a linear space. Thus $Y$ contains (many) planes.

### 5.4.4 Proof of Proposition (5.6) for $Y$ with $\dim(\text{sing}(Y)) = 2$

$Y$ is necessarily reduced and irreducible. We may also assume that $Y$ is not a cone by Lemma (2.7). Assume that there exists a 2-dimensional irreducible component $V$ of $\text{sing}(Y)$ with $\dim(\text{span}(V)) \leq 4$. Then $\text{chord}(V) = \text{span}(V)$ and hence by (2.4.4) $Y$ contains a linear subspace of dimension at least 2. Now assume that every 2-dimensional irreducible component $V$ of $\text{sing}(Y)$ is non-degenerate. By (2.4.10) we get that $\dim(\text{chord}(V)) \leq 4$, i.e. the non-degenerate surface $V \subset \mathbb{P}^5$ is defective: a classical result of Severi (see [3]) states that $V$ is either a cone over a degree-4 rational normal curve or the Veronese surface. One verifies easily that in both cases $\text{chord}(V)$ is a cubic hypersurface in $\mathbb{P}^5$ and hence $Y = \text{chord}(V)$. If $V$ is a cone over a degree-4 rational normal curve then $\text{chord}(V)$ is itself a cone, excluded by hypothesis. If $V$ is a Veronese surface let $\psi: \mathbb{P}^2 \cong V$ be an isomorphism with $\psi^*\mathcal{O}_V(1) \cong \mathcal{O}_{\mathbb{P}^2}(2)$; if $\ell \subset \mathbb{P}^2$ is a line then $\psi(\ell)$ is a conic spanning a plane contained in $\text{chord}(V)$. Thus $\text{chord}(V) = Y$ contains a plane.

### 5.4.5 Proof of Proposition (5.6) for $Y$ with $\dim(\text{sing}(Y)) = 1$

$Y$ is necessarily reduced and irreducible. We may also assume that $Y$ is not a cone by Lemma (2.7). Let $(\text{sing}(Y))^1$ be the union of 1-dimensional irreducible components of $\text{sing}(Y)$. Choose $p \in (\text{sing}(Y))^1$ such that $(\text{sing}(Y))^1$ is smooth at $p$. \hfill (5.4.28)
Let $S_p$ be the set of lines in $Y$ through $p$ - see (5.4.1). Assume first that $S_p$ is not reduced or that it is reducible. By (5.4.23) we get that there exists a surface $T \subset S_p$ of degree at most 3 and hence $S_p$ contains a line $\ell$. The lines in $\mathbb{P}^5$ parametrized by points of $\ell$ sweep out a plane contained in $Y$, and we are done. Now assume that $S_p$ is reduced and irreducible: let’s prove that

$$\deg(sing Y)^1 \leq 5. \quad (5.4.29)$$

Let $\psi_p: Y \to \mathbb{P}(\Theta_p \mathbb{P}^5)$ be projection from $p$. Let $(sing S_p)^1$ be the union of 1-dimensional irreducible components of $sing(S_p)$ - notice that $sing(S_p)$ has dimension at most 1 because $S_p$ is a reduced surface. By Proposition (5.9) the closure of $\psi_p(sing Y \setminus \{p\})$ is an irreducible component of $(sing S_p)^1$ and hence

$$deg(\psi_p(sing Y \setminus \{p\})) \leq deg(sing S_p)^1. \quad (5.4.30)$$

We claim that

$$deg(sing S_p)^1 \leq 4. \quad (5.4.31)$$

In fact let $\Lambda \subset \mathbb{P}(\Theta_p \mathbb{P}^5)$ be a generic 3-dimensional linear space; thus $S_p \cap \Lambda$ is irreducible. By (5.4.22) $S_p \cap \Lambda$ is a complete intersection of a quadric and a cubic in $\Lambda \cong \mathbb{P}^3$ and hence it has arithmetic genus 4; since it is irreducible we get that it has at most 4 singular points. Inequality (5.4.31) follows because $sing(S_p \cap \Lambda) = sing(S_p)^1 \cap \Lambda$. By Assumption (5.4.28) we have

$$deg(\psi_p((sing Y)^1)) = deg(sing Y)^1 - 1. \quad (5.4.32)$$

Inequality (5.4.29) follows from (5.4.32), (5.4.30) to (5.4.31). Thus if $S_p$ is reduced and irreducible one of the following holds:

(I) $(sing Y)^1$ contains a line.

(II) There is an irreducible component $\Gamma$ of $(sing Y)^1$ with $2 \leq \dim(span(\Gamma)) \leq 3$.

(III) There is an irreducible component $\Gamma$ of $(sing Y)^1$ with $\dim(span(\Gamma)) = 4$ and $4 \leq \deg(\Gamma) \leq 5$.

(IV) $(sing Y)^1$ is the rational normal curve of degree 5 in $\mathbb{P}^5$.

We will examine (I) through (IV) separately and we will show in each case that $Y$ contains a plane. (I): Let $\ell \subset sing Y$ be a line. We will prove that there exists a plane $\Lambda \subset Y$ containing $\ell$. Let $[X_0, \ldots, X_3]$ be homogeneous coordinates on $\mathbb{P}^3$ such that $\ell = V(X_0, \ldots, X_3)$. Since $Y$ is singular along $\ell$ we have

$$Y = V(A \cdot X_4 + B \cdot X_5 + C)$$

where $A, B, C \in \mathbb{C}[X_0, \ldots, X_3]$ are homogeneous with $\deg A = \deg B = 2$ and $\deg C = 3$. There exists a point

$$[a_0, \ldots, a_3] \in V(A, B, C) \subset \mathbb{P}^3_{[X_0, \ldots, X_3]}.$$

The plane

$$\Lambda := \{[\lambda a_0, \ldots, \lambda a_3, \mu, \theta] \mid [\lambda, \mu, \theta] \in \mathbb{P}^2 \}$$

is contained in $Y$. (II): Since $\dim(span(\Gamma)) \leq 3$ we have $chord(\Gamma) = span(\Gamma)$. By (5.4.10) we know that $Y \supset span(\Gamma)$. Since by hypothesis $\dim(span(\Gamma)) \geq 2$ we get that $Y$ contains a plane. (III): First we prove the following.
Lemma 5.10. Let $Y \subset \mathbb{P}^5$ be a reduced and irreducible cubic hypersurface such that $\text{sing} Y$ contains an irreducible curve $\Gamma$ with $\dim(\text{span}(\Gamma)) = 4$ and $4 \leq \deg(\Gamma) \leq 5$. Then $\Gamma$ is a degree-4 rational normal curve and $Y \cap (\text{span}(\Gamma))$ is the cubic 3-fold $\text{chord}(\Gamma)$.

Proof. By (5.4.16) $\text{chord}(\Gamma) \subset Y$. The intersection $Y \cap (\text{span}(\Gamma))$ is a hypersurface because $Y$ is reduced and irreducible. Since $\text{chord}(\Gamma)$ is a hypersurface in $\text{span}(\Gamma)$ we get that

$$3 = \deg(Y \cap \text{span}(\Gamma)) \geq \deg(\text{chord}(\Gamma)), \quad (5.4.33)$$

with equality only if $(Y \cap \text{span}(\Gamma)) = (\text{chord}(\Gamma))$. From our hypotheses we get that either $\Gamma$ is a degree-4 rational normal curve in $\text{span}(\Gamma)$ or it has degree 5 and arithmetic genus at most 1. A straightforward computation shows that

$$\deg(\text{chord}(\Gamma)) = \begin{cases} 3 & \text{if } \deg \Gamma = 4, \\ 6 & \text{if } \deg \Gamma = 5 \text{ and } p_a(\Gamma) = 0, \\ 5 & \text{if } \deg \Gamma = 5 \text{ and } p_a(\Gamma) = 1. \end{cases}$$

The result follows from the above formulae and (5.4.33).

Now fix a degree-4 rational normal curve $\Gamma \subset \mathbb{P}^5$. If it were true that $\text{chord}(\Gamma)$ contains a plane we would be done; unfortunately this is not the case. Let $\mathcal{I}_\Gamma \subset \mathcal{O}_{\mathbb{P}^5}$ be the ideal sheaf of $\Gamma$; thus $|\mathcal{I}_\Gamma^2(3)|$ is the linear system of cubic hypersurfaces $Y \subset \mathbb{P}^5$ which are singular at each point of $\Gamma$. Before formulating the next result we remark that the 3-fold $\text{chord}(\Gamma)$ contains lines which are not chords of $\Gamma$.

Proposition 5.11. Keep notation as above, and let $Y \in |\mathcal{I}_\Gamma^2(3)|$. Then $Y$ contains a 1-dimensional family of planes $\Lambda$ such that $\Lambda \cap \text{span}(\Gamma)$ is a chord of $\Gamma$.

Proof. Let $Z \subset \Gamma^{(2)} \times \mathbb{G}r(2, \mathbb{P}^5)$ be the subset defined by

$$Z := \{(p + q, \Lambda) \mid \Lambda \supset \overline{p,q}\},$$

where $\overline{p,q} = \text{span}(p, q)$ if $p \neq q$ and $\overline{p,p} = T_p \Gamma$. Projecting $Z$ to the first factor we get that $Z$ is smooth irreducible and

$$\dim Z = 5. \quad (5.4.34)$$

Let $(p + q, \Lambda) \in Z$: we let $\mathcal{I}_{p+q,\Lambda} \subset \mathcal{O}_\Lambda$ be the ideal sheaf of the subscheme $(p, q)$ (reduced structure) if $p \neq q$ and of the length-2 subscheme supported at $p$ with tangent direction $\Theta_p \Gamma \subset \Theta_p \Lambda$ if $p = q$. Let $F \to Z$ be a vector-bundle with fiber $H^0(\mathcal{I}_{p+q,\Lambda}(2))$ over $(p + q, \Lambda)$. Of course $F$ is only defined modulo tensorization by a line-bundle on $Z$: any choice of $F$ is good for our argument. We have

$$\text{rk} F = 4. \quad (5.4.35)$$

Let $Y = [P] \in |\mathcal{I}_\Gamma^2(3)|$ where $P \in \mathbb{C}[X_0, \ldots, X_5]$ is homogeneous of degree 3. Let $z = (p + q, \Lambda) \in Z$ and $\tau_z \in H^0(\mathcal{O}_\Lambda(1))$ be an equation of the line $\overline{p,q} \subset \Lambda$. We have

$$P|_\Lambda = \tau_z \otimes \sigma_{z,p}, \quad \sigma_{z,p} \in H^0(\mathcal{I}_{p+q,\Lambda}(2)). \quad (5.4.36)$$
Letting $\pi: Z \times |I^2_\Gamma(3)| \to Z$ be the projection the above equation gives that there is a section $\sigma \in H^0(\pi^*F \otimes \mathcal{L})$, where $\mathcal{L} \to Z \times |I^2_\Gamma(3)|$ is a suitable line-bundle, such that
\[ \sigma(z, [P]) = c \cdot \sigma_{z, P}, \quad c \in \mathbb{C}^*. \quad (5.4.37) \]
Let $W := \{ \sigma \}$ be the locus of zeroes of $\sigma$. Letting $\rho: Z \times |I^2_\Gamma(3)| \to |I^2_\Gamma(3)|$ be the projection we have
\[ \rho(W) = \{ Y \in |I^2_\Gamma(3)| \mid \exists \Lambda \subset Y \text{ a plane with } \Lambda \cap \text{span}(\Gamma) \text{ a chord of } \Gamma. \} \]

Claim 5.12. Keep notation as above. There exist $(z_0, Y_0) \in W$ and an open $U \subset Z \times |I^2_\Gamma(3)|$ containing $(z_0, Y_0)$ such that $U \cap W \cap \rho^{-1}(Y_0)$ is purely 1-dimensional.

Proof. As is easily checked there exists a smooth $Q \in |I^\Gamma(2)|$. Since $\Gamma$ is cut out by quadrics we may assume that $Q \not\supset \text{chord}(\Gamma)$.
\[ (5.4.38) \]
Let $Y_0 := Q + \text{span}(\Gamma)$; clearly $Y_0 \in |I^2_\Gamma(3)|$. Before choosing $z_0$ we notice that $\Sigma_Q := \{ p + q \in \Gamma(2) \mid p, q \subset Q \}$ is 1-dimensional because of (5.4.38). Let $p_0 + q_0 \in \Sigma_Q$. There exist two planes $\Lambda \subset Q$ which contain $p_0, q_0$, let $\Lambda_0$ be one of them: we set $z_0 := (p_0 + q_0, \Lambda_0)$.

We let $U \subset Z \times |I^2_\Gamma(3)|$ be the open subset given by
\[ U := \{ (p + q, \Lambda, Y) \mid \Lambda \not\subset \text{span}(\Gamma) \}. \]
One easily checks that with these choices the claim holds. \( \square \)

Let’s finish the proof of the proposition. By (5.4.35) we get that $\text{cod}(W, Z \times |I^2_\Gamma(3)|) \leq 4$, and thus by (5.4.34)
\[ \dim W \geq 1 + \dim |I^2_\Gamma(3)|. \quad (5.4.39) \]
By Claim (5.12) the fibers of $\rho$ restricted to $W \cap U$ have dimension at most 1 in a neighborhood of $Y_0$ and hence
\[ \dim \rho(W) = \dim |I^2_\Gamma(3)|. \]
Since $\rho$ is proper and $|I^2_\Gamma(3)|$ is irreducible we get that $\rho(W) = |I^2_\Gamma(3)|$, i.e. every $Y \in |I^2_\Gamma(3)|$ contains a plane intersecting $\text{span}(\Gamma)$ in a chord of $\Gamma$. Furthermore the set of such planes has dimension at least 1 because every fiber of $\rho|_W$ has dimension at least 1 by (5.4.39) and because every plane in $\mathbb{P}^5$ intersects $\Gamma$ in a finite set of points. \( \square \)

(IV): Let $\Gamma \subset \mathbb{P}^5$ be a rational normal curve of degree 5. We will explicitly construct cubic hypersurfaces $Y \subset \mathbb{P}^5$ with $\Gamma \subset \text{sing}(Y)$; by construction these cubics are ruled by planes. Then we will prove that every $Y \in |I^2_\Gamma(3)|$ is one of the cubics that we constructed; thus every cubic satisfying (IV) contains a plane - actually a 2-dimensional family. Let $L \to \Gamma$ be “the”degree-1 line-bundle. Given a degree-3 linear system $G$ of dimension 2 on $\Gamma$ i.e. $G \in |L^{[3]}|$, we let
\[ Y_G := \bigcup_{p_1, p_2, p_3 \in G} p_1 + p_2 + p_3 \quad (5.4.40) \]
be the variety swept out by the planes spanned by divisors parametrized by \( G \) - of course if \( p_1 = p_2 = p \) and \( p_3 \neq p \) then \( \overline{p_1, p_2, p_3} := J(T_p \Gamma, p_3) \) and if \( p_1 = p_2 = p_3 = p \) then \( \overline{p_1, p_2, p_3} \) is the projective osculating plane to \( \Gamma \) at \( p \).

One easily checks that \( Y_G \) is a hypersurfaces and that \( \text{sing}(Y_G) = \Gamma \). Furthermore \( Y_G \) is a cone with vertex \( p \) if and only if \( p \in \Gamma \) and \( G = p + |L^{\otimes 2}| \); if this is the case then \( Y_G = (p, \text{chord}(\Gamma_p)) \) where \( \Gamma_p \subset \mathbb{P}(\Theta_p(\mathbb{P}^3)) \) is the projection of \( \Gamma \) from \( p \). Since \( \Gamma_p \) is a degree-4 rational normal curve \( \text{chord}(\Gamma_p) \) is a cubic 3-fold and hence we get that \( \deg(Y_G) = 3 \) whenever \( G \) has a base point. Since \( \deg(Y_G) \) is independent of \( G \) we get that \( Y_G \) is a cubic hypersurface for all \( G \in |L^{\otimes 3}|^\vee \). Thus we have defined an injection

\[
|L^{\otimes 3}|^\vee \hookrightarrow |T_3^2(3)|
\]

\[ \tag{5.4.41} \]

**Proposition 5.13.** Keep notation as above. The map \(5.4.41\) is an isomorphism.

**Proof.** Let \( p \in \Gamma \) and let \( \Sigma_p \subset |T_3^2(3)| \) be the linear subspace of cubics which are cones with vertex \( p \). Let \( G_p := (p + |L^{\otimes 2}|) \in |L^{\otimes 3}|^\vee \); a straightforward argument shows that

\[
\Sigma_p = \{ Y_{G_p} \}.
\]

(5.4.42)

Now let’s prove that

\[
\text{cod}(\Sigma_p, |T_3^2(3)|) \leq 3.
\]

(5.4.43)

Let \( U \ni p \) be an open affine space containing \( p \); associating to \( Y \in |T_3^2(3)| \) an affine cubic equation of \( Y \cap U \) we may identify \( H^0(T_3^2(3)) \) with a subvector-space \( A \subset \mathbb{C}[U] \). If \( Y \in |T_3^2(3)| \) then \( Y \) is singular at \( p \); thus \( p \) is a critical point of \( \phi \) for all \( \phi \in A \). Associating to \( \phi \in A \) its Hessian at \( p \) we get a linear map

\[
A \xrightarrow{\text{Hessian}} \text{Sym}_2(\Omega_1^1(p^5))
\]

(5.4.44)

Since \( \Sigma_p = \mathbb{P}(\ker \mathcal{H}) \) it suffices to prove that

\[
\dim(\text{Im} \mathcal{H}) \leq 3.
\]

(5.4.45)

Let \( Q \in \mathbb{P}(\text{Im} \mathcal{H}) \); we may view \( Q \) as a quadric hypersurface in \( \mathbb{P}^5 \) with vertex at \( p \). Since cubics in \( |T_3^2(3)| \) are singular at all points of \( \Gamma \) the quadric \( Q \) is singular at all points of \( T_p \Gamma \). Moreover \( Q \) contains all the lines \( \overline{p, q} \) for \( q \in \Gamma \) because such lines are contained in any \( Y \in |T_3^2(3)| \). Hence projecting \( Q \) from the line \( T_p \Gamma \) we get a quadric \( \widetilde{Q} \subset \mathbb{P}(N_{T_p \Gamma, p^5}) \) containing the degree-3 rational normal curve \( \Gamma \) obtained projecting \( \Gamma \) from \( T_p \Gamma \). The linear system of quadrics in \( \mathbb{P}(N_{T_p \Gamma, p^5}) \cong \mathbb{P}^3 \) containing \( \Gamma \) has (projective) dimension 2 and hence we get \(5.4.45\). This proves \(5.4.43\). By \(5.4.42\) we get that \( \dim |T_3^2(3)| \leq 3 \). Since the map of \(5.4.41\) is injective and since \( \dim |L^{\otimes 3}|^\vee = 3 \) we get the proposition.

**5.4.6 Proof of Proposition (5.6) for \( Y \) with \( \dim(\text{sing} Y) = 0 \)**

We assume that \( Y \subset \mathbb{P}^5 \) is a singular cubic hypersurface with isolated singularities and that \( p \in \text{sing} Y \). First let’s show that Items (1) and (2) of Proposition 5.6 are mutually exclusive. Assume that \( Y \) contains a plane. We may
assume that $Y$ is reduced and irreducible because otherwise $Y$ does not have isolated singularities. If $Y$ is a cone with vertex $p$ then $\dim(S_p) = 3$ and hence Item (2) of Proposition 5.6 does not hold. If $Y$ is not a cone with vertex $p$ then by Item (4) of Proposition 5.9 we know that $S_p$ contains a line or a conic and hence Item (2) of Proposition 5.6 does not hold. This shows that Items (1) and (2) of Proposition 5.6 can not both hold. If $Y$ is a cone with vertex $p$ then $\dim(S_p) = 3$ and hence Item (2) of Proposition 5.6 does not hold. By Lemma 5.7 we know that $Y$ is not a cone, i.e. it has quadratic singularities. By 5.4.23 we know that $S_p$ is a surface. We say that a surface is Du Val if it is reduced, normal with Du Val singularities. We notice the following

\[(S_p \text{ is Du Val}) \implies \text{Item (2) of Proposition 5.6 holds.}\]  

In fact by 5.4.23 we know that $S_p$ is an intersection of a quadric and a cubic in $\mathbb{P}^4$ and hence by simultaneous resolution of Du Val singularities it follows that the minimal desingularization $\tilde{S}_p$ is a deformation of a smooth intersection of a quadric and a cubic in $\mathbb{P}^4$. Since a smooth intersection of a quadric and a cubic in $\mathbb{P}^4$ is a $K3$ surface we get that $\tilde{S}_p$ is a $K3$. Thus it remains to show that $S_p$ is Du Val.

Claim 5.14. Let $Y \subset \mathbb{P}^5$ be a singular cubic hypersurface with isolated quadratic singularities. Assume that $Y$ does not contain any plane. Let $q \in \text{sing}(Y)$. Then:

1. $\dim(\text{sing}\mathbb{P}(C_q Y)) \leq 1$.
2. $S_q \subset \mathbb{P}(\Theta_q \mathbb{P}^5)$ is a reduced and irreducible normal complete intersection of $\mathbb{P}(C_q Y)$ and a cubic hypersurface. $S_q$ has hypersurface singularities (embedding dimension 3).

Proof. (1): Suppose that $\dim(\text{sing}\mathbb{P}(C_q Y)) \geq 2$. Then $\mathbb{P}(C_q Y)$ is the union of two hyperplanes in $\mathbb{P}(\Theta_q \mathbb{P}^5) \cong \mathbb{P}^4$ or a double hyperplane, and hence $S_q$ is the union of two cubic surfaces or a double cubic surface. In either case $S_q$ contains a line, contradicting Item (4) of Proposition 5.6. (2): By Item (4) of Proposition 5.9 $S_q$ contains no lines and hence by Item (1) we get that $S_q \cap \text{sing}\mathbb{P}(C_q Y)$ is empty or finite. This fact together with Items (1)-(2)-(3) of Proposition 5.9 gives that $S_q$ is reduced normal and that the embedding dimension of $S_q$ is equal to 3 at every singular point. Furthermore 5.4.23 gives that $S_q$ is a complete intersection as stated. $S_q$ is connected because it is a complete intersection: since $S_q$ is normal we get that it is irreducible.

Keep notation as above. By the above claim $S_p$ is a reduced and normal surface with locally trivial dualizing sheaf $\omega_{S_p}$ (actually $\omega_{S_p}$ is globally trivial by adjunction). It remains to prove that the singularities of $S_p$ are Du Val, i.e. that given any $s \in \text{sing}(S_p)$ there exists a desingularization of $s$, call it $\epsilon_s: T_s \rightarrow S_p$, such that $\omega_{T_s} \cong \epsilon_s^* \omega_{S_p}$. Let

\[|\text{sing}(Y)| = k + 1.\]  

Let $q \in \text{sing}(Y)$: we write $\text{sing}(Y) = \{q, q_1, \ldots, q_k\}$. The line $\text{span}(q, q_i)$ for $1 \leq i \leq k$ is contained in $Y$ and hence it is parametrized by a point $|\text{span}(q, q_i)| \in S_q$. Let

\[U_q := S_q \setminus \{[\text{span}(q, q_1)], \ldots, [\text{span}(q, q_k)]\}.\]  

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Proposition 5.15. Let $Y \subset \mathbb{P}^5$ be a cubic with isolated quadratic singularities and assume that $Y$ does not contain any plane. Let $q \in \text{sing}(Y)$. The singularities of $U_q$ are Du Val.

Proof. By Proposition 5.9 we know that $U_q$ is smooth away from $\text{sing}(\mathbb{P}(C_q Y)) \cap \mathbb{P}(Y_q)$. Thus we must prove that $S_q$ has a Du Val singularity at all $s \in \text{sing}(\mathbb{P}(C_q Y)) \cap \mathbb{P}(Y_q)$. Choose such an $s$. By Claim 5.14 we get that $S_q$ is the complete intersection of $\mathbb{P}(C_q Y)$ and a cubic $\Xi \subset \mathbb{P}(\Theta_q \mathbb{P}^5)$ which is smooth at $s$. From $\dim(\text{sing}(\mathbb{P}(C_q Y))) \leq 1$ one easily gets that $\text{mult}_s(S_q) = 2$. Let $\pi : \tilde{S}_q \to S_q$ be the blow-up of $s$. Since $S_q$ has a hypersurface singularity of multiplicity 2 at $s$ we have $\pi^* \omega_{S_q} \cong \omega_{\tilde{S}_q}$. Thus it suffices to prove that

$$\tilde{S}_q \text{ has Du Val singularities along } \pi^{-1}(s). \quad (5.4.49)$$

Since $\mathbb{P}(C_q Y)$ is singular at $s$ it is the join $J(s, Q)$ where $Q \subset \mathbb{P}(\Theta_q \mathbb{P}^5)$ is a quadric surface not containing $s$. By Item (1) of Claim 5.14 we know that $Q$ is either smooth or the cone over a smooth conic. By Item (4) of Proposition 5.9 we know that $S_q$ contains no lines and hence projection from $s$ defines a regular finite map $\psi : \tilde{S}_q \to Q$ of degree 2. We describe explicitly $\psi$. Let $X := X_0, \ldots, X_3$. Choose projective coordinates $[X, Z]$ on $\mathbb{P}(\Theta_q \mathbb{P}^5)$ so that $s = [0, \ldots, 0, 1]$ and $\text{span}(Q) = V(Z)$; thus $[X]$ are projective coordinates on $\text{span}(Q)$. We have

$$\mathbb{P}(C_q Y) = V(F), \quad \Xi = V(AZ^2 + BZ + C) \quad (5.4.50)$$

where $F, A, B, C \in \mathbb{C}[X]$ are homogeneous of degrees 2, 1, 2 and 3 respectively. Since $S_q$ contains no lines we have

$$\mathbb{P}^3[X] \supset V(F, A, B, C) = \emptyset. \quad (5.4.51)$$

Since $\tilde{S}_q$ is normal the branch divisor of $\psi : \tilde{S}_q \to Q$ is the reduced effective divisor $D(\psi) \subset \text{Div}(Q)$ defined by

$$D(\psi) = V(F, B^2 - 4A \cdot C) \subset Q = V(F) \subset \mathbb{P}^3[X]. \quad (5.4.52)$$

In general suppose that $V$ is a smooth surface, $W$ is a normal surface and $\varphi : W \to V$ is a double cover branched over the effective reduced divisor $D \subset \text{Div}(V)$. One can get a desingularization $\tilde{W}$ of $W$ by constructing an embedded resolution $\tilde{D}$ of $D$ in a suitable blow-up $\tilde{V}$ of $V$ and taking a double cover $\tilde{W} \to \tilde{V}$ branched over $\tilde{D}$ and a suitable sum of components of the exceptional divisors: from this construction one easily gets the following criterion.

Condition 5.16. Keep notation as above. Let $w \in W$ and $v := \varphi(w)$. Suppose that $\text{mult}_v(D) \leq 3$ and moreover that if $\text{mult}_v(D) = 3$ the strict transform of $D$ in $Bl_v(V)$ intersects the exceptional divisor in at least two distinct points. Then $W$ has a Du Val singularity at $w$.

Now let’s prove 5.4.49. Let $t \in \pi^{-1}(s)$ and let $[\epsilon] = \psi(t)$. We have

$$[\epsilon] \in \psi(\pi^{-1}(s)) = V(F, A) \subset \mathbb{P}^3[X]. \quad (5.4.53)$$

If $B(\epsilon) \neq 0$ then by 5.4.52 and 5.4.54 we get that $[\epsilon] \notin D(\psi)$. Thus a neighborhood of $t$ in $\tilde{S}_q$ is isomorphic to a neighborhood of $[\epsilon]$ in $Q$. Since $Q$
has Du Val singularities we get that $\tilde{S}_q$ is Du Val at $t$. Thus we may assume from now on that
\[ B(\epsilon) = 0. \] (5.4.54)

By (5.4.52) we have
\[ C(\epsilon) \neq 0. \] (5.4.55)

We treat separately the two cases:

1. $Q$ is smooth at $[\epsilon]$.
2. $Q$ is singular at $[\epsilon]$.

(1): If $V(A)$ is transverse to $Q = V(F)$ at $[\epsilon]$ then by (5.4.55) we get that $D(\psi)$ is smooth at $[\epsilon]$ and hence $\tilde{S}_q$ is Du Val at $t$ - actually smooth. If $V(A)$ is tangent to $Q$ at $[\epsilon]$ we distinguish two cases: $Q$ smooth and $Q$ singular. If $Q$ is smooth then $V(A,F)$ is the union of two distinct lines through $[\epsilon]$ and we get from (5.4.55) and (5.4.52) that $D(\psi)$ has a quadratic singularity at $[\epsilon]$; thus $\tilde{S}_q$ is Du Val at $t$ by Criterion (5.10). If $Q$ is singular then $V(A,F)$ is a “double line” supported on $\ell := \text{span}([\epsilon], \text{sing}Q)$. If $V(B)$ is singular at $[\epsilon]$ or if it is smooth at $[\epsilon]$ and transverse to $\ell$ then $D(\psi)$ has a quadratic singularity at $[\epsilon]$; thus $\tilde{S}_q$ is Du Val at $t$ by Criterion (5.10). Finally assume that $V(B)$ is smooth at $[\epsilon]$ and that $\ell$ is tangent to $V(B)$ at $[\epsilon]$. We notice that
\[ (\ell \cdot V(B))[\epsilon] = 2. \] (5.4.56)

In fact if this does not hold then $\ell \subset V(B)$ because $V(B)$ is a quadric and hence $\ell \cap V(C) \subset V(F,A,B,C)$; this contradicts (5.4.51). Let $0 \leq i \leq 3$ be such that $e_i \neq 0$ and let $a,b,c \in \mathbb{C}[\mathbb{P}^3 \setminus V(X_i)]$ be the regular functions $a := A/X_i, b := B/X_i^2, c := C/X_1^3$. From (5.4.58) we get that there exists an open (in the classical topology) $U \subset Q$ containing $[\epsilon]$ and analytic coordinates $(x,y)$ on $U$ centered at $[\epsilon]$ such that
\[ b|_U = y + x^2, \quad I(\ell \cap U) = (y). \] (5.4.57)

Then $a|_U = \lambda y^2$ and $c|_U = \mu$ with $\lambda, \mu \in \mathbb{C}\{x,y\}$ units. Let $\lambda \cdot \mu = \sum_{i,j} f_{i,j} x^i y^j$, where $f_{i,j} \in \mathbb{C}$. Then
\[ (b^2 - 4a \cdot c)|_U \equiv (1 - 4f_{0,0})y^2 + 2y(x^2 - 2f_{1,0}xy - 2f_{0,1}y^2) \mod (x,y)^4. \] (5.4.58)

If $4f_{0,0} \neq 1$ then $D(\psi)$ has a quadratic singularity at $[\epsilon]$ and hence $\tilde{S}_q$ is Du Val at $t$ by Criterion (5.10). On the other hand if $4f_{0,0} = 1$ then the multiplicity of $D(\psi)$ at $[\epsilon]$ is 3 and the strict transform of $D(\psi)$ under the blow-up of $Q$ at $[\epsilon]$ intersects the exceptional divisor in at least 2 distinct points; thus Criterion (5.10) applies again and we get that $\tilde{S}_q$ is Du Val at $t$. This finishes the proof that if Item (1) above holds then $\tilde{S}_q$ is Du Val at $t$. Now we assume that Item (2) holds, i.e. that $Q$ is a cone with vertex $[\epsilon]$ over a smooth conic. Let $\rho: \tilde{Q} \to Q$ be the blow-up of $[\epsilon]$ and $R$ be the exceptional divisor of $\rho$. Let $\tilde{D}(\psi) \subset \tilde{Q}$ be the strict transform of $D(\psi)$. Since $0 = A([\epsilon]) = B([\epsilon])$ and $C([\epsilon]) \neq 0$ we get that
\[ \rho^* D(\psi) = \tilde{D}(\psi) + R. \] (5.4.59)
Thus \( \rho^*D(\psi) \) is reduced, and there is a unique square-root of \( \mathcal{O}_{\hat{Q}}(\rho^*D(\psi)) \), namely \( \rho^*\mathcal{O}_{\hat{Q}}(2) \); let \( \varphi: W \to \hat{Q} \) be the corresponding normal double cover with branch divisor \( \rho^*D(\psi) \). We have a natural map \( \zeta: W \to \hat{S}_q \) which is an isomorphism outside \( t \) and such that \( \zeta^{-1}(t) = \varphi^{-1}(R) \). Furthermore the dualizing sheaf \( \omega_W \) is locally-free because \( W \) has hypersurface singularities and we have

\[
\omega_W \cong \zeta^*\omega_{\hat{S}_q}. \tag{5.4.60}
\]

Thus it suffices to prove that \( W \) has Du Val singularities at all points of \( \varphi^{-1}(R) \). Since \( \rho^*D(\psi) \) is smooth at all points of \( R \setminus (\text{supp}(\hat{D}(\psi))) \) we get that \( W \) is smooth at points of \( (\varphi^{-1}(R) \setminus \varphi^{-1}(\text{supp}(\hat{D}(\psi)))) \). Let \( \tilde{V}(A,F) \subset \hat{Q} \) be the strict transform of \( V(A,F) \subset Q \); we have

\[
R \cap (\text{supp}(\hat{D}(\psi))) = R \cap \tilde{V}(A,F). \tag{5.4.61}
\]

Either \( V(A,F) \) consists of two lines \( \ell_1, \ell_2 \) or it is a “double line” supported on the line \( \ell \). In the first case \( R \cap V(A,F) \) consists of two points \( r_1, r_2 \). One easily checks that \( \rho^*D(\psi) \) has a quadratic singularity at \( r_1 \) and at \( r_2 \); thus \( W \) is Du Val at \( \varphi^{-1}(r_1), \varphi^{-1}(r_2) \) by Criterion \ref{criterion:du_val}. In the second case \( R \cap V(A,F) \) consists of a single point \( r \); one easily checks that the multiplicity of \( \rho^*D(\psi) \) at \( r \) is at most 3 and that if it is equal to 3 then the strict transform of \( \rho^*D(\psi) \) under the blow-up of \( r \) intersects the exceptional divisor in 2 distinct points; thus \( W \) is Du Val at \( \varphi^{-1}(r) \) by Criterion \ref{criterion:du_val}. \( \square \)

Now we prove that \( S_p \) has Du Val singularities. Let \( k \) be as in \ref{claim:curve} and write \( \text{sing}(Y) = \{p, p_1, \ldots, p_k\} \). If \( k = 0 \) then \( U_p = S_p \) and hence by the above proposition \( S_p \) has Du Val singularities. In order to prove that \( S_p \) has Du Val singularities when \( k > 0 \) we study the relation between \( S_p \) and \( S_{p_i} \). Let \( r_i := \text{span}(p, p_i) \); thus \( r_i \subset Y \). Let \( \Sigma(r_i) \subset \mathbb{P}r(2, \mathbb{P}^5) \) be the subset parametrizing planes containing \( r_i \). If \( [\Lambda] \in \Sigma(r_i) \) then \( Y|_{\Lambda} \) is an effective divisor because \( Y \) does not contain planes and we have

\[
Y|_{\Lambda} = r_i + c, \quad c \in |\mathcal{O}_\Lambda(2)|. \tag{5.4.62}
\]

Let \( \Gamma_0^i \subset \Sigma(r_i) \) be the subset parametrizing planes \( \Lambda \) such that the conic \( c \) of \ref{claim:curve} is reducible and \( r_i \not\subset \text{supp}(c) \). Let \( \Gamma_i \subset \mathbb{P}r(2, \mathbb{P}^5) \) be the closure of \( \Gamma_0^i \). Let \( [\Lambda] \in \Gamma_0^i \); since \( Y \) is singular at \( p \) and at \( p_i \) we must have \( p, p_i \in \text{supp}(c) \) and hence there is a unique decomposition \( c = \ell + \ell' \) with \( p \in \ell \) and \( p_i \in \ell' \). Thus we have regular maps

\[
\begin{array}{ccc}
\Gamma_i^0 & \overset{\tau_i^0}{\rightarrow} & S_{p_i} \\
\{\Lambda\} & \\ \\
\rightarrow & \rightarrow & \\
|\ell| & |\ell'| \\
\end{array} \tag{5.4.63}
\]

As is easily verified the above maps extend to regular maps

\[
\pi_i: \Gamma_i \to S_p, \quad \tau_i: \Gamma_i \to S_{p_i}. \tag{5.4.64}
\]

The fiber of \( \pi_i \) over a point of \( S_p \setminus \{[r_i]\} \) consists of a single point, and the same holds for the fiber of \( \tau_i \) over a point of \( S_{p_i} \setminus \{[r_i]\} \). By Item (2) of Claim \ref{claim:curve} we know that \( S_p \) and \( S_{p_i} \) are normal and hence \( \pi_i \) and \( \tau_i \) define isomorphisms

\[
(\Gamma_i \setminus \pi_i^{-1}(\{r_i\})) \overset{\sim}{\rightarrow} S_p \setminus \{[r_i]\}, \quad (\Gamma_i \setminus \tau_i^{-1}(\{r_i\})) \overset{\sim}{\rightarrow} S_{p_i} \setminus \{[r_i]\}. \tag{5.4.65}
\]

In particular \( \pi_i \) and \( \tau_i \) are birational maps and hence \( S_p \) is birational to \( S_{p_i} \).
Proposition 5.17. Keep assumptions and notation as above. The embedding \( \Gamma_i \hookrightarrow \Sigma(r_i) \cong \mathbb{P}^3 \) realizes \( \Gamma_i \) as a quartic surface. Furthermore \( \pi_i^{-1} (U_p \cup \{ [r_i] \}) \) is an open subset of \( \Gamma_i \) with Du Val singularities.

Proof. Over \( \Sigma(r_i) \) we have a tautological family of conics: the conic over \( [\Lambda] \) is given by the divisor \( c \) appearing in (5.4.62). Thus we have a discriminant divisor \( \Delta_i \subset \Sigma(r_i) \) locally defined by the determinant of a symmetric matrix defining the family of conics. We have \( \Gamma_i \subset \text{supp}(\Delta_i) \), however \( \Gamma_i \neq \text{supp}(\Delta_i) \). In fact let

\[
\Omega_i := \{ [\Lambda] \in \Sigma(r_i) \mid Y|_\Lambda = 2r_i + \ell, \quad \ell \in |\mathcal{O}_\Lambda(1)| \} \quad (5.4.66)
\]

Clearly \( \Omega_i \subset \text{supp}(\Delta_i) \) and \( \text{supp}(\Delta_i) = \Gamma_i \cup \Omega_i \) (5.4.67).

A plane \( \Lambda \) is parametrized by a point of \( \Sigma(r_i) \) if and only if it is tangent to \( Y \) at each point of \( r_i \). Let

\[
L_i := \bigcap_{y \in r_i} \Theta_y Y. \quad (5.4.68)
\]

Since \( Y \) is singular at \( p \) and \( p_i \) but \( r_i \not\subset \text{sing}(Y) \) the linear space \( L_i \) is a hyperplane. Thus \( \Omega_i \) is a plane and hence it is an irreducible component of \( \text{supp}(\Delta_i) \). Now we write out explicit equations for \( \Omega_i, \Gamma_i \), etc. Let

\[
\underline{X} := X_0, \ldots, X_3, \quad \underline{Z} := Z_0, Z_1. \quad (5.4.69)
\]

Choose projective coordinates \( [X, Z] \) on \( \mathbb{P}^5 \) so that

\[
p = [0, \ldots, 0, 1, 0], \quad p_i = [0, \ldots, 0, 1]. \quad (5.4.70)
\]

Thus \( r_i = V(\underline{X}) \) and we have an obvious identification \( \Sigma(r_i) \cong \mathbb{P}^3_{[\underline{X}]} \). Since \( r_i \subset Y \) we have \( Y = V(\sum_j A_j X_j) \) where \( A_j \in \mathbb{C}[\underline{X}, \underline{Z}] \) is homogeneous of degree 2. Since \( Y \) is singular at \( p \) and \( p_i \) we have \( 0 = A_j(0, \ldots, 0, 1, 0) = A_j(0, \ldots, 0, 1) \). Thus

\[
A_j = B_j + C_j Z_0 + D_j Z_1 + F_j Z_0 Z_1 \quad (5.4.71)
\]

where \( B_j, C_j, D_j, F_j \in \mathbb{C}[\underline{X}] \) are homogeneous of degrees 2, 1, 1 and 0 respectively. An easy computation gives that

\[
\Omega_i = V(\sum_j F_j X_j). \quad (5.4.72)
\]

Let \( [\underline{X}] \) correspond to the plane \( \Lambda \); a straightforward computation gives that the conic \( c \) appearing in (5.4.62) is defined by the \( 3 \times 3 \) symmetric matrix

\[
M_i := \begin{pmatrix}
\sum_j B_j X_j & \sum_j C_j X_j & \sum_j D_j X_j \\
\sum_j C_j X_j & 0 & \sum_j F_j X_j \\
\sum_j D_j X_j & \sum_j F_j X_j & 0
\end{pmatrix} \quad (5.4.73)
\]

computed at \( \underline{X} \). In particular we get that

\[
V \left( \sum_j B_j X_j, \sum_j C_j X_j, \sum_j D_j X_j, \sum_j F_j X_j \right) = \emptyset. \quad (5.4.74)
\]
The divisor $\Delta_i$ is defined by

$$\det M_i = \left( \sum_j F_j X_j \right) \cdot \left( \sum_{j,h} (2C_j X_j D_h X_h - B_j X_j F_h X_h) \right). \quad (5.4.75)$$

Let $P_i \in \mathbb{C}[X]$ be the second factor appearing in the right-hand side of $5.4.75$.

It follows from $5.4.74$ and $5.4.74$ that $P_i$ does not vanish identically on $\Omega_i$; thus by Equality $5.4.67$ the zero-set of $P_i$ is equal to $\Gamma_i$. By Item (2) of Claim $5.14$ we know that $\Gamma_i$ is irreducible and hence we get that

$$(P_i) = m_i \Gamma_i \quad (5.4.76)$$

for some positive integer $m_i$. Let

$$[\varepsilon] \in V(\sum_j F_j X_j, \sum_j C_j X_j, \sum_j D_j X_j). \quad (5.4.77)$$

Then

$$P_i(\varepsilon) = 0, \quad \frac{\partial P_i}{\partial X_s}(\varepsilon) = -F_s \sum_j B_j(\varepsilon)e_j. \quad (5.4.78)$$

Since $F_s \neq 0$ for some $0 \leq s \leq 3$ and since $\sum_j B_j(\varepsilon)e_j \neq 0$ by $5.4.74$ we get that

if $5.4.77$ holds then $P_i(\varepsilon) = 0$ and $\nabla P_i(\varepsilon) \neq 0 \quad (5.4.79)$

This proves that the $m_i$ appearing in $5.4.76$ is equal to 1; since deg $P_i = 4$ we get that $\Gamma_i$ is a quartic, defined by the vanishing of $P_i$. Let’s show that $\pi_i^{-1}(U_p \cup \{|r_i|\})$ is an open subset of $\Gamma_i$ with Du Val singularities. The subset $(U_p \cup \{|r_i|\}) \subset S_p$ is open, see $5.4.48$, and hence $\pi_i^{-1}(U_p \cup \{|r_i|\})$ is open.

Next we notice that if $[\Lambda] \in \Gamma_i$ and $\pi_i([\Lambda]) = [r_u]$ with $u \neq i$ then

$$\tau_i([\Lambda]) = [\text{span}(p_i, p_u)]. \quad (5.4.80)$$

In fact $Y|_\Lambda = r_i + r_u + \ell$ and since $Y$ is singular at $p_i$ and at $p_u$ we get that $\ell = \text{span}(p_i, p_u)$. From $5.4.80$ we get that

$$\tau_i(\pi_i^{-1}(U_p \cup \{|r_i|\})) = U_{p_i} \cup \{|r_i|\}. \quad (5.4.81)$$

Let $[\Lambda] \in \pi_i^{-1}(U_p \cup \{|r_i|\})$. By $5.4.81$ one of the following holds:

1. $\pi_i([\Lambda]) \in U_p$.
2. $\pi_i([\Lambda]) \in U_{p_i}$.
3. $[\Lambda] \in \pi_i^{-1}(\{r_i\}) \cap \tau_i^{-1}(\{r_i\})$.

Suppose that (1) holds. By $5.4.85$ the map $\pi_i$ is a local isomorphism onto $S_p$ in a neighborhood of $[\Lambda]$. Applying Proposition $5.4.15$ with $q = p$ we get that $\Gamma_i$ is Du Val at $[\Lambda]$. If (2) holds a similar proof works: we apply Proposition $5.4.15$ with $q = p_i$. Finally suppose that (3) holds. We claim that

$$\pi_i^{-1}(\{r_i\}) = V(\sum_j F_j X_j, \sum_j D_j X_j), \quad (5.4.82)$$

$$\tau_i^{-1}(\{r_i\}) = V(\sum_j F_j X_j, \sum_j C_j X_j). \quad (5.4.83)$$
In fact let $[\Lambda] \in \pi_i^{-1}([r_i])$ and let $[X]$ be its projective coordinates. Since $\Lambda \cap Y = 2r_i + \ell$ where $p_i \in \ell$ we have $[\Lambda] \in \Omega$, and $\text{span}(p_i, [X, 0, 0]) \subset \mathbb{P}(C_pY)$. This gives that $\pi_i^{-1}([r_i])$ consists of those points of the right-hand side of (5.4.82) which are contained in $\Gamma_i$. Since $\Gamma_i$ is the zero-locus of $P_i$ we get that the right-hand side of (5.4.82) is contained in $\Gamma_i$; this proves (5.4.82). Exchanging the roles of $p$ and $p_i$ we get Equation (5.4.83). From (5.4.82) and (5.4.83) we get that

$$\pi_i^{-1}([r_i]) \cap \tau_i^{-1}([r_i]) = V(\sum_j F_jX_j, \sum_j C_jX_j, \sum_j D_jX_j).$$

(5.4.84)

By (5.4.79) we get that $\Gamma_i$ is smooth at every point of $\pi_i^{-1}([r_i]) \cap \tau_i^{-1}([r_i])$. □

Suppose that $k > 0$ where $k$ is given by (5.4.47): we prove that $S_p$ has Du Val singularities. By Proposition 5.15 we know that $U_p$ has Du Val singularities. It remains to show that $S_p$ has a Du Val singularity at each $[r_i]$, where $1 \leq i \leq k$.

Let $X, Z$ be as in (5.4.69) and assume that (5.4.71) holds. Projection of $S_p$ from $[r_i]$ defines an embedding $Bl_{[r_i]}(S_p) \to \mathbb{P}(X)$; the image of this embedding is $\Gamma_i$ and it gives an identification of $\tau_i: \Gamma_i \to S_p$ with the blow-up of $[r_i]$. In particular since $\text{deg} S_p = 6$ and $\text{deg} \Gamma_i = 4$ we get that $\text{mult}_{[r_i]} S_p = 2$. On the other hand $S_p$ has embedding dimension 3 at $[r_i]$ by Claim (5.14) and hence we get that

$$\omega_{\Gamma_i} = \pi_i^*(\omega_{S_p}).$$

(5.4.85)

Let $\rho_i: \tilde{\Gamma}_i \to \Gamma_i$ be the minimal desingularization of the singularities belonging to $\pi_i^{-1}([r_i])$. By Proposition (5.14) we know that $\Gamma_i$ has Du Val singularities along $\pi_i^{-1}([r_i])$ and hence

$$\omega_{\Gamma_i} = \rho_i^*(\omega_{\Gamma_i}).$$

(5.4.86)

The regular map $\pi_i \circ \rho_i: \tilde{\Gamma}_i \to S_p$ gives a desingularization of the singular point $[r_i]$ and by (5.4.85) and (5.4.86) we have

$$\omega_{\Gamma_i} = (\pi_i \circ \rho_i)^*(\omega_{S_p}).$$

(5.4.87)

This proves that $S_p$ has a Du Val singularity at $[r_i]$.

5.4.7 Proof of Proposition (5.8)

Let $S_{p_{sm}} \subset S_p$ be the smooth locus of $S_p$. We have a cylinder map

$$\text{cyl}: H_2(S_{p_{sm}}; \mathbb{Z}) \to H^4(Bl_{S_{p_{sm}}}(\Theta_p \mathbb{P}^5); \mathbb{Z})$$

(5.4.88)

defined as follows. Let

$$\pi: Bl_{S_p}(\Theta_p \mathbb{P}^5) \to \mathbb{P}(\Theta_p \mathbb{P}^5)$$

(5.4.89)

be the blow-down map. Given a homology class $\alpha \in H_2(S_{p_{sm}}; \mathbb{Z})$ represented by an oriented closed smooth real surface $\Sigma \subset S_{p_{sm}}$ the oriented smooth real 4-fold $\pi^{-1}\Sigma$ is in the smooth locus of $Bl_{S_p}(\Theta_p \mathbb{P}^5)$, hence $\pi^{-1}\Sigma$ has a well-defined Poincaré dual class $PD(\pi^{-1}\Sigma) \in H^4(Bl_{S_p}(\Theta_p \mathbb{P}^5); \mathbb{Z})$ independent of the choice of representative $\Sigma$: we set $\text{cyl}(\alpha) := PD(\pi^{-1}\Sigma)$. Now let $\ldots, R_i, \ldots$ be the irreducible components of the desingularization map $\tilde{S}_p \to S_p$; thus we have

$$j: S_{p_{sm}} \to \tilde{S}_p, \quad j(S_{p_{sm}}) = \left(\tilde{S}_p \setminus \bigcup_i R_i\right).$$

(5.4.90)
Since \( S_p \) has du Val singularities the map \( H_2(j) \) is injective and it gives an identification

\[
H_2(S_p^{sm}) = \{ \alpha \in H_2(\tilde{S}_p; \mathbb{Z}) | \langle \alpha, R_i \rangle = 0 \ \forall R_i \},
\]

where \( \langle \cdot, \cdot \rangle \) is the intersection pairing on \( H_2(\tilde{S}_p; \mathbb{Z}) \). If \( \alpha \in H_2(\tilde{S}_p; \mathbb{Z}) \) is Poincaré dual to a class in \( T(\tilde{S}_p) \) then \( \alpha \) belongs to the right-hand side of (5.4.91). Thus via Poincaré duality we get an injection

\[
T(\tilde{S}_p) \hookrightarrow H_2(S_p^{sm}; \mathbb{Z}).
\]

Composing the above inclusion with the cylinder map and tensoring with \( \mathbb{C} \) we get a map

\[
\tilde{\gamma} : T(\tilde{S}_p)_\mathbb{C} \rightarrow H^4(Bl_{p}\mathbb{P}(\Theta_p^{\mathbb{P}^5})).
\]

A moment’s thought will convince the reader that the map above is a morphism of type (1, 1) of Hodge structures. Furthermore for \( \alpha, \beta \in T(\tilde{S}_p)_\mathbb{C} \) we have

\[
\int_{Bl_{p}\mathbb{P}(\Theta_p^{\mathbb{P}^5})} \tilde{\gamma}(\alpha) \wedge \tilde{\gamma}(\beta) = -\int_{\tilde{S}_p} \alpha \wedge \beta.
\]

In fact this follows from a standard computation based on the fact that the normal bundle of the exceptional divisor of (5.4.98) has degree -1 on a fiber of the \( \mathbb{P}^1 \)-bundle \( \pi^{-1}(S_p) \rightarrow S_p \). By Isomorphism (5.4.24) we may replace the right-hand side of (5.4.93) by \( H^4(Bl_p Y) \); thus \( \tilde{\gamma} \) defines a morphism (we do not change its name) of type (1, 1)

\[
\tilde{\gamma} : T(\tilde{S}_p)_\mathbb{C} \rightarrow H^4(Bl_{p}Y).
\]

Let

\[
\rho : Bl_{p}Y \rightarrow Y
\]

be the blow-down map. The exceptional divisor of \( \rho \) is the projectivized normal cone \( \mathbb{P}(C_p Y) \). Composing the map of (5.4.95) with the restriction map \( H^4(Bl_{p}Y) \rightarrow H^4(\mathbb{P}(C_p Y)) \) we get

\[
T(\tilde{S}_p)_\mathbb{C} \rightarrow H^4(\mathbb{P}(C_p Y)).
\]

We claim that the above map is zero. It suffices to prove triviality of the map

\[
T(\tilde{S}_p)_\mathbb{C} \rightarrow H^4(\mathbb{P}(C_p Y))/W_3H^4(\mathbb{P}(C_p Y))
\]

obtained by composing (5.4.97) with the quotient map. The right-hand side of (5.4.98) is a sub Hodge structure of \( H^4 \) of any desinglarization of \( \mathbb{P}(C_p Y) \); since \( \mathbb{P}(C_p Y) \) is a quadric we get that the right-hand side of (5.4.98) is of pure type (2, 2). By (5.4.96) we get that \( (5.4.98) \) has a non-zero kernel, and since \( T(\tilde{S}_p)_\mathbb{C} \) has no non-trivial rational sub-Hodge structure we get that the kernel of (5.4.98) is all of \( T(\tilde{S}_p)_\mathbb{C} \). Thus (5.4.97) is zero and \( Im(\tilde{\gamma}) \subset ImH^4(\rho) \) where \( \rho \) is the blow-down map (5.4.96). Hence there exists a morphism of type (1, 1) of Hodge structures

\[
\tilde{\gamma} : T(\tilde{S}_p)_\mathbb{C} \rightarrow H^4(Y)/\ker(\rho^*)
\]

such that \( \tilde{\gamma} = H^4(\rho) \circ \tilde{\gamma} \). Clearly \( \ker(\rho^*) \subset W_3H^4(Y) \); we let \( \gamma \) be the composition of \( \tilde{\gamma} \) with the quotient map \( H^4(Y)/\ker(\rho^*) \rightarrow Gr^W_3 H^4(Y) \). This defines the morphism of Hodge structures (5.4.7). Equation (5.4.8) follows from Equation (5.4.99).
5.4.8 Comments

Following is an example of $X$ a numerical $(K3)^{[3]}$ and $H$ a big and nef divisor on $X$ with $(c_1(H), c_1(H)) = 2$ such that $f: X \to |H|'$ is a regular double covering of a cubic hypersurface - we do not know of any such example with $H$ ample. Let $V$ be a 3-dimensional complex vector space and $\pi: S \to \mathbb{P}(V)$ be a double covering ramified over a smooth sextic curve; thus $S$ is a $K3$ surface. Let $X := S^{[2]}$ and let $f$ be the composition

$$S^{[2]} \to S^{(2)} \to \mathbb{P}(V)^{(2)} \to \mathbb{P}(Sym^2 V) \cong \mathbb{P}^5.$$  \hfill (5.4.100)

The image of $\mathbb{P}(V)^{(2)} \to \mathbb{P}(Sym^2 V)$ is the discriminant cubic hypersurface; since $f$ has degree 4 onto its image we get that $\int_X c_1(H)^4 = 12$ and hence $(c_1(H), c_1(H)) = 2$ by (2.1.3). The divisor $H$ is big and nef and $f$ can be identified with the natural map $f: X \to |H|'$: thus $f$ has the stated properties.

5.5 (5) of Proposition (4.3) does not hold

In Subsubsection (5.5.1) we will prove the following result.

Proposition 5.18. Let $Y \subset \mathbb{P}^5$ be a quartic hypersurface such that $\dim(\text{sing} Y) \geq 3$. Then $Y$ contains a plane.

Granting the above proposition let’s prove that Item (5) of Proposition (4.3) does not hold. We argue by contradiction. Assume that we have $f: X \to Y$ regular of degree 3 onto a quartic hypersurface $Y \subset \mathbb{P}^5$. By Propositions (4.2) and (5.18) we get that $\dim(\text{sing} Y) \leq 2$. Let $R \in \text{Div}(X)$ be the ramification divisor of $f$. Applying the adjunction formula to $Y^{sm} := (Y \setminus \text{sing} Y)$ and Hurwitz’ formula to $f^{-1}(Y^{sm}) \to Y^{sm}$ we get that

$$R \in |\mathcal{O}_X(2H)|.$$  \hfill (5.5.1)

By applying (4.0.4) we get that

$$h^0(\mathcal{O}_X(2H)) = 21 = h^0(\mathcal{O}_Y(2)).$$  \hfill (5.5.2)

Thus the pull-back map $f^*: H^0(\mathcal{O}_Y(2)) \to H^0(\mathcal{O}_X(2H))$ is an isomorphism and from (5.5.1) we get that there exists an effective Cartier divisor $D \in \text{Div}(Y)$ such that $f^*D = R$. Comparing the orders of vanishing of $f^*D$ and $R$ at a prime component of $R$ we get a contradiction.

5.5.1 Proof of Proposition (5.18)

If $Y$ is not reduced or not irreducible then there is an irreducible component of $Y$ of degree at most 2 and the result follows immediately. Thus we may assume that $Y$ is irreducible and reduced. Let $V$ be an irreducible component of $\text{sing} Y$; intersecting $Y$ with a generic plane we get that $\deg V \leq 3$. If $\deg V = 1$ there is nothing to prove. Assume that $\deg V = 2$. If $V$ is singular then $V$ contains planes and we are done. Thus we may assume that $V$ is smooth. Let $L := \text{span}(V)$. Then $L \cong \mathbb{P}^4$ and $V$ is a quadric hypersurface in $L$. Since $Y$ is irreducible of degree 4 we have the cycle-theoretic intersection

$$Y \cdot L = 2V.$$  \hfill (5.5.3)
We claim that there exists a complete intersection of two quadrics
\[ \tilde{Y} = Q_1 \cap Q_2 \subset \mathbb{P}^6 \] (5.5.4)
such that \( Y \) is isomorphic to the projection of \( \tilde{Y} \) from a point outside \( \tilde{Y} \). In fact let \( \mathcal{I}_V \subset \mathcal{O}_{\mathbb{P}^5} \) be the ideal sheaf of (the reduced) \( V \). The linear system \( |\mathcal{I}_V(2)| \) has dimension 6. The rational map
\[ \varphi: \mathbb{P}^5 \to |\mathcal{I}_V(2)|^\vee \cong \mathbb{P}^6 \] (5.5.5)
is the composition of the blow-up of \( V \) and contraction of the strict transform of \( L \) to a point, call it \( p \). The image of \( \varphi \) is a smooth quadric \( Q_1 \subset \mathbb{P}^6 \). The inverse of \( \mathbb{P}^5 \to Q_1 \) is projection from \( p \). The image (strict transform) of \( Y \) under \( \varphi \) is a codimension-1 subset \( \tilde{Y} \subset Q_1 \) which does not intersect \( p \). Use (5.5.3) to get this last statement. Thus \( \deg \tilde{Y} = \deg Y = 4 \) and hence there exists a quadric \( Q_2 \subset \mathbb{P}^6 \) such that (5.5.4) holds. By a theorem of Debarre-Manivel [5] we get that \( \tilde{Y} \) contains a plane \( \Lambda \). Since projection from \( p \) will map \( \Lambda \) to a plane in \( Y \) we are done. Finally assume that \( \deg V = 3 \). The variety is non-degenerate; in fact if \( \dim(\text{span}(V)) = 4 \) then \( \text{span}(V) \subset \tilde{Y} \) contradiction. Since \( V \) is non-degenerate of degree 3 we get that \( V \) is smooth and linearly normal; as is well-known [14] it follows that \( V \) is the Segre 3-fold i.e. \( \mathbb{P}^1 \times \mathbb{P}^2 \) embedded by \( \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1) \). Since the Segre 3-fold contains planes we are done.

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