DOMAINS IN COMPLEX SURFACES WITH A NONCOMPACT AUTOMORPHISM GROUP – II

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ABSTRACT. Let $X$ be an arbitrary complex surface and $D \subset X$ a domain that has a noncompact group of holomorphic automorphisms. A characterization of those domains $D$ that admit a smooth real analytic, finite type, boundary orbit accumulation point and whose closures are contained in a complete hyperbolic domain $D' \subset X$ is obtained.

1. INTRODUCTION

Let $D \subset \mathbb{C}^n$, $n \geq 1$ be a bounded domain and let $\text{Aut}(D)$ be the group of holomorphic automorphisms of $D$. There is a natural action of $\text{Aut}(D)$ on $D$ given by

$$(f, z) \mapsto f(z)$$

where $f \in \text{Aut}(D)$ and $z \in D$. Suppose the orbit of some point $p \in D$ under this action accumulates at $p_\infty \in \partial D$ – call such a point a boundary orbit accumulation point. In this situation, it has been shown that (see [1]–[4], [5], [11], [22], [26] and [33] among others) the nature of $\partial D$ near $p_\infty$ provides global information about $D$. The question of investigating this phenomenon when $D$ is a domain in a complex manifold was raised in [7] and [14] and it was shown in the latter article that the Wong-Rosay theorem remains valid when $D$ is a domain in an arbitrary complex manifold with $p_\infty \in \partial D$ a strongly pseudoconvex point. In short, such a domain $D$ is biholomorphic to the unit ball $B^n \subset \mathbb{C}^n$. Motivated by this result, it was shown in [28] that the analogues of [1] and [5] are also valid, with the same conclusion, when $D$ is a domain in an arbitrary complex surface and $p_\infty$ is a smooth weakly pseudoconvex point of finite type. The pseudoconvexity hypothesis near $p_\infty$ was dropped in [4] and a local version of this result for bounded domains in $\mathbb{C}^2$ and with the boundary $\partial D$ near $p_\infty$ being smooth real analytic and of finite type can be found in [31]. The purpose of this article is to generalise the result in [31] by finding all possible model domains when $D$ is a domain in an arbitrary complex surface $X$.

Theorem 1.1. Let $X$ be an arbitrary complex surface and $D \subset X$ a domain. Suppose that $\overline{D}$ is contained in a complete hyperbolic domain $D' \subset X$ and that there exists a point $p \in D$ and a sequence $\{\phi_j\} \in \text{Aut}(D)$ such that $\{\phi_j(p)\}$ converges to $p_\infty \in \partial D$. Assume that the boundary of $D$ is smooth real analytic and of finite type near $p_\infty$. Then exactly one of the following alternatives holds:

(i) If $\dim \text{Aut}(D) = 2$ then either

- $D \simeq D_1 = \{(z_1, z_2) \in \mathbb{C}^2 : 2\Re z_2 + P_1(\Re z_1) < 0\}$ where $P_1$ is a polynomial that depends on $\Re z_1$, or
- $D \simeq D_2 = \{(z_1, z_2) \in \mathbb{C}^2 : 2\Re z_2 + P_2(|z_1|^2) < 0\}$ where $P_2$ is a polynomial that depends on $|z_1|^2$, or

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\[ D = D_3 = \{ (z_1, z_2) \in \mathbb{C}^2 : 2\Re z_2 + P_{2m}(z_1, \bar{z}_1) < 0 \} \] where \( P_{2m} \) is a homogeneous polynomial of degree \( 2m \) without harmonic terms.

(ii) If \( \dim \text{Aut}(D) = 3 \) then \( D = D_4 = \{ (z_1, z_2) \in \mathbb{C}^2 : 2\Re z_2 + (\Re z_1)^{2m} < 0 \} \) for some integer \( m \geq 2 \).

(iii) If \( \dim \text{Aut}(D) = 4 \) then \( D = D_5 = \{ (z_1, z_2) \in \mathbb{C}^2 : 2\Re z_2 + |z_1|^{2m} < 0 \} \) for some integer \( m \geq 2 \).

(vi) If \( \dim \text{Aut}(D) = 8 \) then \( D = D_6 = \mathbb{B}^2 \) the unit ball in \( \mathbb{C}^2 \).

To clarify several points, first note that \( D \) is hyperbolic since it is contained in \( D' \) which is assumed to be complete hyperbolic in the sense of Kobayashi. Therefore \( \text{Aut}(D) \) is a real Lie group endowed with the topology of uniform convergence on compact subsets of \( D \). Moreover, the family \( \phi_j : D \to D \subset D' \) is normal since \( D' \) is complete. By theorem 2.7 in [30] (which generalises Cartan’s theorem – see [25] pp. 78) we see that every possible limit map \( \phi \) is either in \( \text{Aut}(D) \) or satisfies \( \phi(D) \subset \partial D \). Since \( \phi(p) = p_\infty \in \partial D \), it follows that \( \phi(D) \subset \partial D \). Fix a neighbourhood \( U \) of \( p_\infty \) and a biholomorphism \( \psi : U \to \psi(U) \subset \mathbb{C}^2 \) such that \( \psi(p_\infty) = 0 \) and \( \psi(U \cap \partial D) \) is a smooth real analytic hypersurface of finite type – note that the type is a biholomorphic invariant and hence it suffices to work with a fixed, sufficiently small neighbourhood of \( p_\infty \). Let \( W \) be a neighbourhood of \( p \) small enough so that \( \phi(W) \subset U \). If possible, let \( k > 0 \) be the maximal rank of \( \phi \) which is attained on the complement of an analytic set \( A \subset D \). If \( p \in W \setminus A \), then the image of a small neighbourhood of \( p \) that does not intersect \( A \) under \( \phi \) is a germ of a positive dimensional complex manifold contained in \( U \cap \partial D \) and this is a contradiction. On the other hand if \( p \in A \), pick \( q \in W \setminus A \) and repeat the above argument to see that \( k = 0 \) in this case as well. Thus \( \phi(D) \equiv p_\infty \). Since this is true of any limit map, it follows that the entire sequence \( \phi_j \) converges uniformly on compact subsets of \( D \) to the constant map \( \phi(z) \equiv p_\infty \). It follows that \( D \) must be simply connected (see for example [24]) for any loop \( \gamma \subset D \) is contractible if and only if \( \phi_j(\gamma) \) is so for all \( j \). However, for all large \( j \) the loop \( \phi_j(\gamma) \subset U \cap D \) which is simply connected if \( U \) is small enough. Hence \( \phi_j(\gamma) \) is a trivial loop for large \( j \) and hence so is \( \gamma \).

Second, note that \( \psi(p_\infty) \) cannot belong to the envelope of holomorphy of \( \psi(U \cap D) \). Indeed, for if not, then on the one hand we see from the above reasoning that the Jacobian determinant \( \det(\psi \circ \phi_j) \) must tend to zero uniformly on compact subsets of \( D \). On the other hand, all the maps \( \phi_j^{-1} \circ \psi^{-1} : \psi(U \cap D) \to D \subset D' \) extend to a fixed, open neighbourhood of \( \psi(p_\infty) \) by a theorem of Ivashkovich (see [20]) since \( D' \) is complete. Moreover, the extensions of these maps near \( \psi(p_\infty) \) take values in \( D' \). Hence there is an upper bound for \( \det(\phi_j^{-1} \circ \psi^{-1}) \) near \( \psi(p_\infty) \) and this is a contradiction. As a consequence, this observation of Greene-Krantz is also valid in the situation of the main theorem.

Third, recall the stratification of the smooth real analytic finite type hypersurface \( U \cap \partial D \) that was used in [31]. There is a biholomorphically invariant decomposition of \( U \cap \partial D \) as the union of two relatively open sets, namely \( \partial D^+ \) (for brevity, we drop the reference to \( \psi \)) which consists of points near which \( U \cap \partial D \) is pseudoconvex and \( \partial \cap \partial D \) that has those points which are in the envelope of holomorphy of\( U \cap D \), and their closed complement \( \partial' \) which is a locally finite union of smooth real analytic arcs and points. Note that \( \partial' \) is contained in the set of Levi flat points \( \mathcal{L} \) which by the finite type assumption is a codimension one real analytic subset of \( U \cap \partial D \). By the second remark above, \( p_\infty \notin \partial \cap \partial D \). If \( p_\infty \in \partial D^+ \) then by [28] it follows that

\[ D = \{ (z_1, z_2) \in \mathbb{C}^2 : 2\Re z_2 + P_{2m}(z_1, \bar{z}_1) < 0 \} \]
where $P_{2m}(z_1, \overline{z}_1)$ is a homogeneous subharmonic polynomial of degree $2m$ (this being the 1-type of $U \cap \partial D$ near $p_\infty$) without harmonic terms. In this case, the assumption that $D \subset D'$ plays no role for pseudoconvexity of $U \cap \partial D$ near $p_\infty$ (an orbit accumulation point) is enough to guarantee that $D$ is complete hyperbolic – see [5], [13] for example. In particular, in the situation of the main theorem, the Levi form of $U \cap \partial D$ changes sign in every neighbourhood of $p_\infty$. Finally, a word about the assumption that $\overline{D}$ is contained in a complete hyperbolic domain $D' \subset X$. Perhaps the most natural assumption would be to not assume anything except finite type and smooth real analyticity of $U \cap \partial D$ near $p_\infty$. In this situation, the first thing to do would be to show the normality of $O(\Delta, D)$, the family of holomorphic mappings from the unit disc $\Delta$ into $D$. And as in [5] and [13] this should be a consequence of understanding the rate of blow up of the Kobayashi metric on $D$ near $p_\infty$. That the metric can even be localised near $p_\infty$ near which the Levi form changes sign does not seem to be known. Therefore another possibility is to assume that $D$ is locally taut near $p_\infty$, i.e., $V \cap D$ is taut for some fixed neighbourhood $V$ of $p_\infty$. However, working with this also requires knowledge that an analytic disc $f : \Delta \to D$ with $f(0)$ close to $p_\infty$ can be localised. Moreover, if we strengthen the hypothesis on $D$ by assuming that it is complete hyperbolic, then $D$ would be pseudoconvex near $p_\infty$. The model domains in this case have been determined in [28]. With these observations a plausible hypothesis seemed of requiring that $\overline{D} \subset D'$ where $D' \subset X$ is complete – and this, though being global in nature, seemed to complement well the assumption made in [31] that $D \subset \mathbb{C}^2$ is a bounded domain.

The general strategy is the same as in [31]. Note that since $D$ is hyperbolic it follows from [21], [23] that $0 \leq \dim \text{Aut}(D) \leq n^2 + 2n = 8$ as $n = 2$. Furthermore by [21] it is known that if $\dim \text{Aut}(D) \geq 5$, then $D$ is homogeneous and hence there is an orbit that clusters at strongly pseudoconvex points in $U \cap \partial D$. Such points form a non-empty open subset of $U \cap \partial D$ that contains $p_\infty$ in its closure and this follows from the decompositon of $U \cap \partial D$ alluded to above. Consequently by [14], $D \cong \mathbb{B}^2$. Therefore it suffices to treat the case when $0 \leq \dim \text{Aut}(D) \leq 4$. An initial scaling of $D$ using the orbit $\{ \phi_j(p) \}$ as described below shows that $D$ is biholomorphic to a model domain of the form

$$G = \{(z_1, z_2) \in \mathbb{C}^2 : 2\Re z_2 + P(z_1, \overline{z}_1) < 0\}$$

where $P(z_1, \overline{z}_1)$ is a polynomial without harmonic terms. Let $g : G \to D$ be the biholomorphism. $G$ is evidently invariant under the one parameter subgroup of translations in the imaginary $z_2$-direction, i.e., $T_t(z_1, z_2) = (z_1, z_2 + it)$ for $t \in \mathbb{R}$. This shows that $\dim \text{Aut}(D) \geq 1$. If $\dim \text{Aut}(D) = \dim \text{Aut}(G) = 1$, it is possible to explicitly write down what can element of $\text{Aut}(G)$ should look like and this description shows that the orbits in $G$ stay uniformly away from the boundary and accumulate only at the point at infinity in $\partial G$. Using the assumption that $D$ is contained in a taut domain, it can be seen that the Kobayashi metric in $D$ blows up near $p_\infty$. Let $X = g_* (i\partial / \partial z_2)$; note that $i\partial / \partial z_2$ is a holomorphic vector field in $G$ whose real part generates the translations $T_t$. Then $p_\infty$ is seen to be an isolated zero of $X$ on $\partial D$ and the arguments of [4] show that $X$ must be parabolic and this forces $D$ to be equivalent to an ellipsoid whose automorphism group is four dimensional. This is a contradiction. When $\dim \text{Aut}(D) = 2$, two cases arise depending on whether $\text{Aut}(D) \subset \mathbb{C}$, the connected component of the identity in $\text{Aut}(D)$ is abelian or not. In the former case, $\text{Aut}(D) \subset \mathbb{C}$ must be isomorphic to either $\mathbb{R}^2$ or to $\mathbb{R} \times S^1$. These lead to the conclusion that $D \cong D_1$ or $D \cong D_2$. In the non-abelian case $\text{Aut}(D) \subset \mathbb{C}$ is solvable and it can be shown that $D \cong D_3$. A case-by-case analysis is used when $\dim \text{Aut}(D) = 3, 4$ to identify the relevant domain from the classification obtained by A. V. Isaev in [17], [18]. While the argument remains the same in some cases, we take this opportunity to streamline and provide
alternate proofs in some instances – for example, ruling out the possibility that \( \text{dim Aut}(D) = 1 \) and identifying the right model domain when \( \text{dim Aut}(D) = 3 \). There are several possibilities in \([17]\) and here we focus on three interesting classes from that list, as the proof for the others remains the same. Nothing changes when \( \text{dim Aut}(D) = 2, 4 \), i.e., the same proofs from \([31]\) carry over to these cases and we have decided to be brief, the emphasis being not to merely repeat what carries over to this situation from \([31]\), but to identify and focus on the differences instead.

2. THE DIMENSION OF \( \text{Aut}(D) \) IS AT LEAST TWO

To describe the scaling of \( D \) using the base point \( p \) and the sequence \( \{\phi_j\} \in \text{Aut}(D) \), first note that for \( j \) large, there is a unique point \( \tilde{p}_j \in \psi(U \cap \partial D) \) such that

\[
\text{dist}(\psi \circ \phi_j(p), \psi(U \cap \partial D)) = |\tilde{p}_j - \psi \circ \phi_j(p)|.
\]

By a rotation of coordinates, we may assume that the defining function \( \rho(z) \) for \( \psi(U \cap \partial D) \) is of the form

\[
\rho(z) = 2\Re z_2 + \sum_{k,l} c_{kl}(y_2) z_1^k \bar{z}_1^l
\]

where \( c_{00}(y_2) = O(y_2^2) \) and \( c_{10}(y_2) = c_{01}(y_2) = O(y_2) \). Let \( m \) be the type of \( \psi(U \cap \partial D) \) at the origin. By definition, there exist \( k, l \) both at least one and \( k + l = m \) for which \( c_{kl}(0) \neq 0 \) and \( c_{kl}(0) = 0 \) for all other \( k + l < m \). The pure terms, if any, up to order \( m \) in the defining function can be removed by a polynomial automorphism of the form

\[
(z_1, z_2) \mapsto (z_1, z_2 + \sum_{k \leq m} (c_{k0}(0)/2) z_1^k).
\]

These coordinate changes will be absorbed in \( \psi \). Let \( \psi_{p,1}^j(z) = z - \tilde{p}_j \) so that \( \psi_{p,1}^j(\tilde{p}_j) = 0 \). A unitary rotation \( \psi_{p,2}^j(z) \) then ensures that the outer real normal to \( \psi_{p,1}^j \circ \psi(U \cap \partial D) \) at the origin is the real \( z_2 \)-axis. The defining function for \( \psi_{p,2}^j \circ \psi_{p,1}^j \circ \psi(U \cap \partial D) \) near the origin is then of the form

\[
\rho_j(z) = 2\Re z_2 + \sum_{k,l} c_{kl}^j(y_2) z_1^k \bar{z}_1^l
\]

with the same normalisations on the coefficients \( c_{00}^j(y_2) \) and \( c_{10}^j(y_2) \) as described above. Since \( \tilde{p}_j \to 0 \) it follows that both \( \psi_{p,1}^j \) and \( \psi_{p,2}^j \) converge to the identity uniformly on compact subsets of \( \mathbb{C}^2 \). Note that the type of \( \psi_{p,2}^j \circ \psi_{p,1}^j \circ \psi(U \cap \partial D) \) is at most \( m \) for all \( j \) and an automorphism of the form (2.2) can be used to remove all pure terms up to order \( m \) from \( \rho_j(z) \). Denote this by \( \psi_{p,3}^j \). Lastly, note that \( \psi \circ \phi_j(p) \) is on the inner real normal to \( \psi(U \cap \partial D) \) at \( \tilde{p}_j \) and it follows that \( \psi_{p,2}^j \circ \psi_{p,1}^j \circ \psi \circ \phi_j(p) = (0, -\delta_j) \) for some \( \delta_j > 0 \) and the explicit form of (2.2) shows that this is unchanged by \( \psi_{p,3}^j \). Let

\[
\psi_{p,4}^j(z_1, z_2) = (z_1/e_j, z_2/\delta_j)
\]

where \( e_j > 0 \) will be chosen in the next step. The defining function for \( \psi_{p}^j \circ \psi(U \cap \partial D) \) near the origin, where \( \psi_{p}^j = \psi_{p,4}^j \circ \psi_{p,3}^j \circ \psi_{p,2}^j \circ \psi_{p,1}^j \), is given by

\[
\rho_{j,p}(z) = \delta_j^{-1} \rho_j(e_j z_1, \delta_j z_2) = 2\Re z_2 + \sum_{k,l} e_j^{k+l} \delta_j^{-k} c_{kl}^j(\delta_j y_2) z_1^k \bar{z}_1^l.
\]
Observe that $\psi_p^j \circ \psi \circ \phi_j(p) = (0, -1)$ for all $j$. Now choose $\varepsilon_j > 0$ by demanding that
\[
\max \{ |\epsilon_k^{j+l} \delta_j^{-1} c_k^l(0)| : k + l \leq m \} = 1
\]
for all $j$. In particular, note that $\{ \epsilon_j^m \delta_j^{-1} \}$ is bounded and by passing to a subsequence it follows that
\[
\rho_{j,p}(z) \to \rho_p = 2\Re z_2 + P(z_1, \bar{z}_1)
\]
in the $C^\infty$ topology on compact subsets of $C^2$, where $P(z_1, \bar{z}_1)$ is a polynomial of degree at most $m$ without any harmonic terms. Therefore the domains $G_{j,p} = \psi_p^j \circ \psi(\partial U \cap \partial D)$ converge to
\[
G_p = \{ (z_1, z_2) \in C^2 : 2\Re z_2 + P(z_1, \bar{z}_1) < 0 \}
\]
in the Hausdorff sense. Let $K \subset G_p$ be a relatively compact domain containing the base point $(0, -1)$. Then $K \subset \psi_p^j \circ \psi(\partial U \cap \partial D)$ for all large $j$ and therefore the mappings
\[
g_p^j : (\psi_p^j \circ \psi \circ \phi_j)^{-1} : K \to D \subset D'
\]
are well defined and satisfy $g_p^j(0, -1) = p$. The completeness of $D'$ shows that the family $\{ g_p^j \}$ is normal and hence there is a holomorphic limit $g_p : G_p \to \overline{D}$ with $g_p(0, -1) = p$. It remains to show that $g_p$ is a biholomorphism from $G_p$ onto $D$. For this, recall the observation made in [4] that since $P(z_1, \bar{z}_1)$ is not harmonic, the envelope of holomorphy of $G_p$ is either all of $C^2$ or $\partial G_p$ contains a strongly pseudoconvex point. The former situation cannot hold – indeed, by [20] again, the map $g_p$ will extend to $C^2$ taking values in $D'$ and since $D'$ is complete, $g_p(z) \equiv p$. Let $W \subset C^2$ be a bounded domain that intersects infinitely many of the boundaries $\psi_p^j \circ \psi(\partial U \cap \partial D)$ – and hence also $\partial G_p$. Then for each $j$, note that the cluster set of $W \cap \psi_p^j \circ \psi(\partial U \cap \partial D)$ under $g_p^j$ is contained in $\partial D$ since $\phi_j \in \text{Aut}(D)$. Now, if the envelope of $G_p$ were all of $C^2$, it is possible to find a domain $\Omega$ with $\Omega \cap \partial G_p \neq \emptyset$ on which the family $\{ g_p^j \}$ would converge uniformly. In this case, by passing to the limit, we see that $g_p(U \cap \partial G_p) \subset \partial D$ and thus $g_p$ cannot be the constant map. Therefore there must be a strongly pseudoconvex point, say $\zeta$ on $\partial G_p$. Fix $r > 0$ so that all points on $\partial G_p \cap B(\zeta, r)$ are strongly pseudoconvex and since $\rho_{j,p} \to \rho_{\infty,p}$ in the $C^\infty$ topology on $B(\zeta, r)$, it follows that each of the open pieces $\psi_p^j \circ \psi(U \cap \partial D) \cap B(\zeta, r)$ are themselves strongly pseudoconvex for $j \gg 1$. For a complex manifold $M$, let $F_M(z, v)$ denote the Kobayashi metric at $z \in M$ along a tangent vector $v$ at $z$. By the stability of the Kobayashi metric under smooth strongly pseudoconvex perturbations, it follows that for all $q \in B(\zeta, r) \cap G_p$
\[
F_{G_{j,p}}(q, v) \geq c |v|
\]
for some uniform $c > 0$ and by the invariance of the Kobayashi metric we see that
\[
F_{g_p^j(U \cap \partial D)}(g_p^j(q), d g_p^j(q) v) = F_{G_{j,p}}(q, v) \geq c |v|.
\]
implies that the Kobayashi metrics on $\phi_j^{-1}(U \cap D)$ converge to the corresponding metric on $D$ and thus (2.4) shows that

\begin{equation}
(2.5) \quad F_D(g_p(q), dg_p(q)v) \geq c|v|.
\end{equation}

Thus $dg_p(q)$ has full rank. Thus the rank of $dg_p$ can be smaller only on an analytic set $A \subset G_p$ of dimension at most one. Pick $\tilde{q} \in A$ and let $N_1, N_2$ be small neighbourhoods of $\tilde{q}$ and $g_p(\tilde{q})$ respectively such that $g^j_p(N_1) \subset N_2$ for $j \gg 1$. By identifying $N_2$ with an open subset of $\mathbb{C}^2$, Hurwitz’s theorem applied to the Jacobians $\det(dg^j_p)$ shows that either $\det(dg^j_p)$ never vanishes or is identically zero in $N_1$. Since $A$ has strictly smaller dimension it follows that $dg_p$ has full rank everywhere, i.e., $A$ must be empty. Hence $g_p$ is locally biholomorphic in $G_p$ and therefore $g_p(G_p) \subset D$. Injectivity of $g_p$ is now a consequence of the fact that $g^j_p$ are all biholomorphic and they converge uniformly on compact subsets of $G_p$ to $g_p$.

To conclude, we have to show that $D_p = g_p(G_p)$ is all of $D$. If not, pick $\tilde{p} \in \partial D_p \cap D$ and note that since $\phi_j(\tilde{p}) \to p_\infty$, the scaling argument above can be repeated to get a biholomorphism $g_{\tilde{p}}: G_{\tilde{p}} \to g_{\tilde{p}}(G_p) \subset D$. Here $G_{\tilde{p}}$ has the same form as $G_p$ with possibly a different polynomial than $P(z_1, \bar{z}_1)$. Note that $V = D_p \cap \tilde{p}$ is then a nonempty open subset of $D$. Let $f^j_p = (g^j_{\tilde{p}})^{-1}, f_{\tilde{p}} = g^{-1}_p$ and $f_{\tilde{p}} = (g^j_{\tilde{p}})^{-1}, f_{\tilde{p}} = g^{-1}_p$. Observe that both $f_p, f_{\tilde{p}}$ are biholomorphic on $V$, and that both $f^j_p, f^j_{\tilde{p}}$ are defined on a given compact set in $D$ for large $j$. We may write $f_p = A \circ f_{\tilde{p}}$ where $A = g^{-1}_p \circ g_{\tilde{p}}$ is biholomorphic on $f_{\tilde{p}}(V)$. But more can be said about $A$ – indeed, by definition we have

$$g^j_{\tilde{p}} \circ \psi^j_{\tilde{p}} \circ (\psi^j_{\tilde{p}})^{-1} = g^j_{\tilde{p}}$$

where $A_j = \psi^j_{\tilde{p}} \circ (\psi^j_{\tilde{p}})^{-1}$ are polynomial automorphisms of $\mathbb{C}^2$ of bounded degree as their construction shows. Since $g^j_{\tilde{p}}$ and $g^j_{\tilde{p}}$ converge to $g_{\tilde{p}}$ and $g_{\tilde{p}}$ respectively, we may take $A$ as the limit of $A_j$ on $f_{\tilde{p}}(V)$ and conclude that $A$ is also a polynomial automorphism of $\mathbb{C}^2$. Now the functional equation $f_p = A \circ f_{\tilde{p}}$ extends $f_p$ as a biholomorphic mapping from a small neighbourhood $W$ of $\tilde{p}$ onto $W', a$ neighbourhood of $f_{\tilde{p}}(\tilde{p})$. On the other hand, note first that since $g_{\tilde{p}}$ is biholomorphic near $(0, -1)$ and maps it to $\tilde{p}$, it follows that $f^j_p$ form a normal family on $W$, after possibly shrinking it if necessary. As a consequence, the equality

$$f^j_p = A_j \circ f^j_{\tilde{p}}$$

which holds on $W$ for $j$ large, shows that $f^j_p$ converges to $f_p$ on $W$ and hence in the limit we see that $f_p(W) \subset \overline{G_p}$. That is, $f_p(W)$ cannot contain a neighbourhood of $f_{\tilde{p}}(\tilde{p})$ which is a contradiction. Hence $g_p: G_p \to D$ is biholomorphic and since $G_p$ is invariant under the translations $T_i$, it follows that $\dim \Aut(D) = \dim \Aut(G_p) \geq 1$. In the sequel, we will write $g, G$ in place of $g_p, G_p$ respectively.

Recall that $p_\infty$ is not in the envelope of holomorphy of $U \cap D$ where $(U, \phi)$ is the coordinate chart around $p_\infty$ that was fixed earlier. Let $\Delta \subset \mathbb{C}$ be the unit disc. The following estimate on the Kobayashi metric near $p_\infty$ will be useful.

**Lemma 2.1.** For every $r \in (0, 1)$, there is a neighbourhood $V$ of $p_\infty$ compactly contained in $U$ such that every analytic disc $f: \Delta \to D$ with $f(0) \in V$ satisfies $f(r\Delta) \subset U$. As a result, the Kobayashi metric can be localised near $p_\infty$ – there is a constant $C > 0$ such that

$$C \cdot F_{U \cap D}(p, v) \leq F_D(p, v) \leq F_{U \cap D}(p, v)$$
uniformly for all \( p \in V \cap D \) and tangent vectors \( v \) at \( p \). Moreover,
\[
F_D(p,v)/|v| \to \infty
\]
as \( p \to p_\infty \). In particular, for any neighbourhood \( V \) of \( p_\infty \) and \( R < \infty \), there exists another neighbourhood \( W \subset V \) of \( p_\infty \) such that the Kobayashi ball \( B^k_D(p,R) \subset V \) whenever \( p \in W \cap D \).

Proof. Let \( f_\nu : \Delta \to D \subset D' \) be a sequence of holomorphic disks with \( f_\nu(0) = p_\nu \to p_\infty \). The completeness of \( D' \) implies that some subsequence of \( \{f_\nu\} \) converges uniformly on compact subsets of \( \Delta \) to a holomorphic limit \( f : \Delta \to \overline{D} \) and \( f(0) = p_\infty \). Suppose that \( f(z) \neq p_\infty \) on \( \Delta \).

Let \( \eta > 0 \) be such that \( f(\eta \Delta) \subset U \). Since \( U \cap \partial D \) is of finite type, no open subset of \( f(\eta \Delta) \) can be contained in it and hence \( f(\eta \Delta) \cap D \neq \emptyset \). By the strong disk theorem \((\text{[22]})\) it follows that \( p_\infty \) belongs to the envelope of holomorphy of \( U \cap D \) which is a contradiction. Therefore \( f(z) \equiv p_\infty \) and this shows that all limit functions for the given family of holomorphic disks are constant. The first claim follows and the equivalence of the metrics on \( U \cap D \) and \( D \) is then a consequence of the definition of the Kobayashi metric.

If there exists a sequence \( p_\nu \to p_\infty \) and non-zero vectors \( v_\nu \) at \( p_\nu \) and a constant \( C \) such that \( F_D(p_\nu, v_\nu) \leq C|v_\nu| \), then there would exist a uniform \( r > 0 \) and holomorphic disks \( f_\nu \in O(r\Delta, D) \) with \( f_\nu(0) = p_\nu \) and \( d f_\nu(0) = v_\nu \). By the homogeneity of the metric in the vector variable, we may assume that \( |v_\nu| = 1 \) for all \( v_\nu \). The argument above shows that every possible limit function \( f \) of the family \( \{f_\nu\} \) is constant which contradicts \( |d f(0)| = 1 \). Therefore \( F_D(p,v)/|v| \) blows up as \( p \to p_\infty \).

For the claim about the size of \( B^k_D(p,R) \), let us work in local coordinates around \( \phi(p_\infty) = 0 \). For \( a, b \in U \cap D \), let \( d(a,b) \) denote the euclidean distance on \( U \cap D \) induced by \( \phi \). For a given neighbourhood \( V \) of \( p_\infty \) and \( R < \infty \), let \( p_\infty \in N_2 \subset V \) be such that \( F_D(p,v)/|v| \geq 2R \) for all \( p \in N_2 \cap D \) and tangent vectors \( v \) at \( p \). We may assume without loss of generality that \( N_2 \subset U \) and \( N_2 = \phi^{-1}(B(0,2)) \). Let \( N_1 = \phi^{-1}(B(0,1)) \). Fix \( p \in N_1 \cap D \) and \( q \in D \) and let \( \gamma(t) \) be a path in \( D \) parametrised by \([0,1]\) with \( \gamma(0) = p \) and \( \gamma(1) = q \) such that
\[
\int_0^1 F_D(\gamma(t),\gamma'(t)) \, dt \leq d^k_D(p,q) + \varepsilon
\]
where \( \varepsilon > 0 \) is given and \( d^k_D(p,q) \) is the Kobayashi distance between \( p, q \). Suppose that \( q \in N_1 \cap D \); two cases now arise – first, if the entire path \( \gamma \subset N_2 \cap D \), then
\[
2R \, d(p,q) \leq \int_0^1 F_D(\gamma(t),\gamma'(t)) \, dt \leq d^k_D(p,q) + \varepsilon.
\]
Second, if \( \gamma \) does not entirely lie in \( N_2 \cap D \), then there is a connected component of \( \gamma \) that contains \( p \) and a point \( a \in \partial N_2 \cap D \). The length of this connected component is at least \( 2 \geq d(p,q) \). On the other hand, if \( q \in D \setminus N_1 \), then the length of this path can be bounded from below by simply \( 2R \). Thus we get
\[
d^k_D(p,q) \geq 2R \, d(p,q) - \varepsilon
\]
if \( q \in N_1 \cap D \) and \( d^k_D(p,q) \geq 2R \) otherwise. Now if \( p \in N_1 \cap D \) and \( q \in B^k_D(p,R) \), it follows from these comparisons that \( q \in N_1 \cap D \) which completes the proof. \( \square \)

The holomorphic vector field \( X = g_*(i\partial/\partial z_2) \) on \( D \) is such that its real part \( \Re X = (X + \overline{X})/2 \) generates the one parameter subgroup \( L_t = g \circ T_t \circ g^{-1} = \exp(t \Re X) \in \text{Aut}(D) \). Two observations can be made about \( X \) at this stage – first, Proposition 2.3 of \([31]\) shows that \( (L_t) \) induces a local one parameter group of holomorphic automorphisms of a neighbourhood of \( p_\infty \) when
$D \subset \mathbb{C}^2$ is a bounded domain. In particular, $X$ extends as a holomorphic vector field near $p_\infty$. The proof of this relies on a local parametrised version of the reflection principle from \cite{10}, the main tools being the use of Segre varieties and their invariance property under biholomorphisms to construct the desired extension of $(L_t)$ near $p_\infty$ for all $|t| < \eta$ for a fixed $\eta > 0$. The same arguments can be applied in the local coordinates induced on $U$ by $\phi$ to get the same conclusion in the setting of the main theorem as well. Second, consider the pullback of the orbit $\{\phi_j(p)\} \in D$ under the equivalence $g : G \to D$, i.e., let $g^{-1} \circ \phi_j(p) = (a_j, b_j) \in G$ and
\[ 2\epsilon_j = 2\Re b_j + P(a_j, \overline{a}_j). \]
Note that $\epsilon_j < 0$ for all $j$. Proposition 2.5 of \cite{31} shows that if $|\epsilon_j| > c > 0$ for all large $j$, then $X$ vanishes to finite order at $p_\infty$. The proof of this uses the boundedness of $D \subset \mathbb{C}^2$ which in particular implies that a family of holomorphic maps into $D$ is normal. The same argument can be applied in the situation of the main theorem since $D \subset D'$ and $D'$ is assumed to be complete hyperbolic. Thus we have:

**Proposition 2.2.** The group $(L_t)$ induces a local one parameter group of holomorphic automorphisms of a neighbourhood of $p_\infty$ in $X$. In particular, $X$ extends as a holomorphic vector field near $p_\infty$. Moreover, if $|\epsilon_j| > c > 0$ for all large $j$, then $X$ vanishes to finite order at $p_\infty$.

The next step is to describe what the elements of $\text{Aut}(G)$ look like under the assumption that $\dim \text{Aut}(G) = 1$. This calculation was done in Propositions 2.6 and 2.7 of \cite{31} and they remain valid here since they do not involve any features of $D$. The conclusion is that if $g \in \text{Aut}(G)$ then
\[ g(z_1, z_2) = (g_1(z_1, z_2), g_2(z_1, z_2)) = (a z_1 + \beta, \phi(z_1) + a z_2) \]
where $|a| = 1, a = \pm 1, \beta \in \mathbb{C}$ and $\phi(z_1)$ is a holomorphic polynomial. Moreover, if $q = (q_1, q_2) \in G, g \in \text{Aut}(G)$ and
\[ E = 2\Re (g_2(q_1, q_2)) + P(g_1(q_1, q_2), g_1(q_1, q_2)) \]
then $|E| = |2\Re q_2 + P(q_1, \overline{q}_1)|$ as Lemma 2.8 of \cite{31} shows. Hence $|E|$ is independent of $g$.

**Proposition 2.3.** The dimension of $\text{Aut}(D)$ is at least two.

**Proof.** Suppose that $\dim \text{Aut}(D) = \dim \text{Aut}(G) = 1$. Write
\[ (a_j, b_j) = g^{-1} \circ \phi_j(p) = g^{-1} \circ \phi_j \circ g(p) \]
and note that $g^{-1} \circ \phi_j \circ g \in \text{Aut}(G)$ for all $j \geq 1$. Let $g^{-1}(p) = q = (q_1, q_2) \in G$. By the arguments summarized above, it follows that
\[ |2\Re b_j + P(a_j, \overline{a}_j)| = |2\Re q_2 + P(q_1, \overline{q}_1)| > 0 \]
for all $j \geq 1$. This shows that the orbit $\{g^{-1} \circ \phi_j(p)\} \in G$ can only cluster at the point at infinity in $\partial G$. Let
\[ \eta = |2\Re q_2 + P(q_1, \overline{q}_1)| > 0 \]
and for $r > 0$ define
\[ G_r = \{(z_1, z_2) \in \mathbb{C}^2 : 2\Re z_2 + P(z_1, \overline{z}_1) < -r\} \subset G. \]
Observe that the boundaries of $G$ and $G_r$ intersect only at the point at infinity for all $r > 0$. Furthermore, the entire orbit $(a_j, b_j)$ and $q$ are contained in $G_{\eta/2}$ by (2.6). By Proposition 2.2 above it follows that $X(p_\infty) = 0$ and by Lemma 3.5 of \cite{4} the intersection of the zero set of $X$ with $\partial D$ contains $p_\infty$ as an isolated point. Now regard $g$ as a holomorphic mapping from $G_{\eta/2}$ into $D$. The sequence $(a_j, b_j) \in G_{\eta/2}$ converges to the point at infinity in $\partial G_{\eta/2}$ and its image
under $g$, namely $\phi_j(p)$, converges to $p_\infty$. Proposition 2.2 also shows that if the cluster set of the point at infinity in $\partial G_{\eta/2}$ intersects $\partial D$ near $p_\infty$, then the vector field $X$ vanishes at all such points. Since the cluster set of the point at infinity in $\partial G_{\eta/2}$ under $g$ is connected and contains $p_\infty$ as an isolated point, it must equal $p_\infty$.

Thus for a given small neighbourhood $U$ of $p_\infty$ there exists a neighbourhood of the point at infinity in $\partial G_{\eta/2}$ which is mapped by $g$ into $U \cap D$. However, a neighbourhood of infinity in $G_{\eta/2}$ contains $G_M$ for some large $M > 0$. Fix a point $s \in g(G_M) \subset D$ and let $\tilde{s} = (\tilde{s}_1, \tilde{s}_2) \in G_M$ be such that $s = g(\tilde{s})$. Note that

$$L_t(s) = L_t \circ g(\tilde{s}) = g \circ T_t(\tilde{s}_1, \tilde{s}_2) = g(\tilde{s}_1, \tilde{s}_2 + it)$$

which gives $L_t(s) \to p_\infty$ as $|t| \to \infty$. For any compact $K \subset D$ there exists $R > 0$ such that $K$ is contained in the Kobayashi ball $B^D_K(s, R)$. Hence $L_t(K) \subset B^D_t(L_t(s), R)$ for any $t \in \mathbb{R}$. By Lemma 2.1 it follows that $L_t$ moves any point in $D$ in both forward and backward time to $p_\infty$, i.e., the action of $L_t$ on $D$ is parabolic. The arguments of [4] can now be applied to show that

$$D \subset \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2m} < 1\}$$

for some integer $m \geq 1$. Thus $\text{dim} \text{Aut}(D) = 4$ which is a contradiction.\\

In case $\text{dim} \text{Aut}(D) = 2$, note that the calculations done in section 3 of [31] deal with only the defining function of $G$ and hence they apply in this situation as well. Indeed, the following dichotomy holds – here $\text{Aut}(D)^c$ is the connected component of the identity.

(i) If $\text{Aut}(D)^c$ is abelian, then $D$ is biholomorphic to either

$$D_1 = \{(z_1, z_2) \in \mathbb{C}^2 : 2\Re z_2 + P_1(\Re z_1) < 0\}$$

or

$$D_2 = \{(z_1, z_2) \in \mathbb{C}^2 : 2\Re z_2 + P_2(|z_1|^2) < 0\}$$

for some polynomials $P_1, P_2$ that depend only on $\Re z_1$ or $|z_1|^2$ respectively.

(ii) If $\text{Aut}(D)^c$ is non-abelian then $D$ is biholomorphic to

$$D_3 = \{(z_1, z_2) \in \mathbb{C}^2 : 2\Re z_2 + P_{2m}(z_1, \bar{z}_1) < 0\}$$

where $P_{2m}(z_1, \bar{z}_1)$ is a homogeneous polynomial of degree $2m$ without harmonic terms.

3. Model domains when $\text{Aut}(D)$ is three dimensional

3.1. A tube domain and its finite and infinite sheeted covers. For $0 \leq s < t < \infty$ define

$$\mathcal{S}_{s,t} = \{(z_1, z_2) \in \mathbb{C}^2 : s < (\Re z_1)^2 + (\Re z_2)^2 < t\}$$

which is a non-simply connected tube domain over a nonconvex base. Evidently $D$ cannot be biholomorphic to $\mathcal{S}_{s,t}$ since $D$ is simply connected as observed earlier. It is possible to consider finite and infinite sheeted covers of $\mathcal{S}_{s,t}$. To obtain a finite sheeted cover, consider the $n$-sheeted covering self map

$$\Phi_{\chi}^{(n)} : \mathbb{C}^2 \setminus \{\Re z_1 = \Re z_2 = 0\} \to \mathbb{C}^2 \setminus \{\Re z_1 = \Re z_2 = 0\}$$

whose components are given by

$$\tilde{z}_1 = \Re((\Re z_1 + i\Re z_2)^n) + i\Im z_1,$$

$$\tilde{z}_2 = \Im((\Re z_1 + i\Re z_2)^n) + i\Im z_2.$$
Equip $\mathbb{C}^2 \setminus \{\mathbb{R}z_1 = \mathbb{R}z_2 = 0\}$ with the pull-back complex structure using $\Phi^{(n)}_\lambda$ and call the resulting complex surface $M^{(n)}_\lambda$. For $0 \leq s < t < \infty$ and $n \geq 2$ define

$$\mathcal{G}^{(n)}_{s,t} = \{(z_1, z_2) \in M^{(n)}_\lambda : s^{1/n} < (\mathbb{R}z_1)^2 + (\mathbb{R}z_2)^2 < t^{1/n}\}.$$ 

Then $\Phi^{(n)}_\lambda$ is an $n$-sheeted holomorphic covering map from $\mathcal{G}^{(n)}_{s,t}$ onto $\mathcal{G}_{s,t}$. It is clear that the domains $\mathcal{G}^{(n)}_{s,t}$ are not simply connected and hence $D$ cannot be equivalent to any of them. Proposition 4.7 in [31] provides a different proof of this fact which uses ideas that are applicable for other classes of domains as well. This can be adapted in the setting of theorem 1.1 as follows:

**Proposition 3.1.** There cannot exist a proper holomorphic mapping from $D$ onto $\mathcal{G}_{s,t}$ for all $0 \leq s < t < \infty$. In particular, $D$ cannot be equivalent to $\mathcal{G}^{(n)}_{s,t}$ for any $n \geq 2$ and $0 \leq s < t < \infty$.

**Proof.** Let $\pi : D \to \mathcal{G}_{s,t}$ be a proper holomorphic mapping. The case when $0 < s < t < \infty$ will be considered first. The boundary of $\mathcal{G}_{s,t}$ has two components, namely

$$\partial \mathcal{G}^+_{s,t} = \{(z_1, z_2) \in \mathbb{C}^2 : (\mathbb{R}z_1)^2 + (\mathbb{R}z_2)^2 = t\},$$

and

$$\partial \mathcal{G}^-_{s,t} = \{(z_1, z_2) \in \mathbb{C}^2 : (\mathbb{R}z_1)^2 + (\mathbb{R}z_2)^2 = s\}.$$ 

The orientation induced on these pieces by $\mathcal{G}_{s,t}$ makes them strongly pseudoconvex and strongly pseudoconcave respectively. Lemma 2.1 of [31] shows that there is a two dimensional stratum $S \subset \mathcal{L} \cap \hat{D}$ that clusters at $p_{\infty}$ – this is a purely local assertion and hence it remains valid here as well. Pick $a \in S$ near $p_{\infty}$ and let $W$ be a small neighbourhood of $a$ so that $\pi$ extends holomorphically to $W$. Note that $(W \cap \partial D) \setminus S$ consists of points that are either strongly pseudoconvex or strongly pseudoconcave. Let $V_\pi \subset W$ be the branching locus of $\pi : W \to \mathbb{C}^2$. Since $\partial D$ is finite type, it follows that $V_\pi \cap \partial D$ has real dimension at most one. There are two possibilities now – first, if $\pi(a) \in \mathcal{G}^+_{s,t}$, then choose a strongly pseudoconcave point $a' \in (W \cap \partial D) \setminus V_\pi$. Thus $\pi$ maps a neighbourhood of $a'$, which is strongly pseudoconcave, locally biholomorphically onto a neighbourhood of $\pi(a') \in \mathcal{G}^+_{s,t}$ and this is a contradiction. A similar argument can be given when $\pi(a') \in \mathcal{G}^-_{s,t}$. The only possibility then is that there are no pseudoconcave points near $p_{\infty}$, i.e., $\partial D$ is weakly pseudoconvex near $p_{\infty}$. In this case, [23] shows that

$$D \sim \hat{D} = \{(z_1, z_2) \in \mathbb{C}^2 : 2\mathbb{R}z_2 + P_{2m}(z_1, \zeta_1) < 0\}$$

where $P_{2m}(z_1, \zeta_1)$ is a homogeneous subharmonic polynomial of degree $2m$ – this being the 1-type of $\partial D$ at $p_{\infty}$, without harmonic terms. In particular $D$ is globally pseudoconvex and as $\pi$ is proper, it follows that $\mathcal{G}_{s,t}$ is also pseudoconvex. However, this is not the case.

When $0 = s < t < \infty$, the two components of $\partial \mathcal{G}_{0,t}$ are

$$\partial \mathcal{G}^+_{0,t} = \{(z_1, z_2) \in \mathbb{C}^2 : (\mathbb{R}z_1)^2 + (\mathbb{R}z_2)^2 = t\},$$

and

$$\partial \mathcal{G}^-_{0,t} = \{\mathbb{R}z_1 = \mathbb{R}z_2 = 0\}.$$ 

Choose $a \in S$ as above and let $W, W'$ be small neighbourhoods of $a$ and $\pi(a)$ so that $\pi : W \to W'$ is a well defined holomorphic mapping. Suppose that $\pi(a) \in \mathcal{G}^+_{0,t}$. Since $V_a \cap \partial D$ has real dimension at most one, it follows that there is an open piece of $W \cap \partial D$ near $a$ that is mapped locally biholomorphically onto an open piece in $\mathcal{G}^+_{0,t}$ and this is a contradiction. A similar argument shows that $\pi(a) \notin \partial \mathcal{G}^+_{0,t}$ and therefore the only possibility is that $\partial D$ is weakly pseudoconvex.
near \( p_\infty \). By [28] it follows that \( D \preceq \hat{D} \) where \( \hat{D} \) is as in (3.1). Let \( \pi \) still denote the proper mapping

\[
\pi : \hat{D} \to \mathcal{S}_{0,t}.
\]

Let \( \phi \) be a holomorphic function on \( \hat{D} \) that peaks at the point at infinity in \( \partial \hat{D} \). Then \( \psi = \log |\phi - 1| \) is a plurisubharmonic function that is bounded above on \( \hat{D} \) and has the property that \( \psi \to -\infty \) at the point at infinity in \( \partial \hat{D} \). If \( \pi_0^{-1}, \pi_2^{-1}, \ldots, \pi_m^{-1} \) are the local branches of \( \pi^{-1} \), then it is known that

\[
\tilde{\psi} = \max\{ \psi \circ \pi_j^{-1} : 1 \leq j \leq m \}
\]

extends to a plurisubharmonic function on \( \mathcal{S}_{0,t} \). If there is an open piece of \( \partial \mathcal{S}_{0,t} \) on which \( \tilde{\psi} \to \infty \), then the uniqueness theorem shows that \( \tilde{\psi} \equiv -\infty \) and this is a contradiction. Thus there is a point, say \( p \in \partial \hat{D} \) whose cluster set under \( \pi \) intersects \( \partial \mathcal{S}_{0,t} \). Then \( \pi \) extends continuously up to \( \partial \hat{D} \) near \( p \) and this extension is even locally biholomorphic across strongly pseudoconvex points which are known to be dense on \( \partial \hat{D} \). By Webster’s theorem, \( \pi \) is algebraic. Away from a codimension one algebraic variety \( Z \), the inverse \( \pi^{-1} \) defines a correspondence that is locally given by finitely many holomorphic maps. Since \( Z \cap i\mathbb{R}^2 \) has real dimension at most one, it is possible to pick \( p' \in i\mathbb{R}^2 \setminus Z \). The branches of \( \pi^{-1} \) will now map an open piece of \( i\mathbb{R}^2 \) near \( p' \) locally biholomorphically (shift \( p' \) if necessary to achieve this) to an open piece on \( \partial \hat{D} \). This cannot happen as \( \partial \hat{D} \) is not totally real.

To conclude, let \( f : D \to \mathcal{G}_{s,t}^{(n)} \) be biholomorphic. Since \( \mathcal{G}_{s,t}^{(n)} \) inherits the complex structure from \( \mathcal{G}_{s,t} \) via \( \Phi_{\chi}^{(n)} \), it follows that

\[
\pi = \Phi_{\chi}^{(n)} \circ f : D \to \mathcal{G}_{s,t}
\]

is an unbranched, proper holomorphic mapping between domains with the standard complex structure. Such a map cannot exist as shown above. \( \square \)

To construct an infinite sheeted cover of \( \mathcal{G}_{s,t} \), consider the infinite sheeted covering map

\[
\Phi_{\chi}^{(\infty)} : \mathbb{C}^2 \to \mathbb{C}^2 \setminus \{ \Re z_1 = \Re z_2 = 0 \}
\]

whose components are given by

\[
\begin{align*}
\tilde{z}_1 &= \exp(\Re z_1) \cos(\Im z_1) + i \Re z_2, \text{ and} \\
\tilde{z}_2 &= \exp(\Re z_1) \sin(\Im z_1) + i \Im z_2.
\end{align*}
\]

Equip \( \mathbb{C}^2 \) with the pull-back complex structure using \( \Phi_{\chi}^{(\infty)} \) and denote the resulting complex manifold by \( M_{\chi}^{(\infty)} \). For \( 0 \leq s < t < \infty \) define

\[
\mathcal{G}_{s,t}^{(\infty)} = \{ (z_1, z_2) \in M_{\chi}^{(\infty)} : (\ln s) / 2 < \Re z_1 < (\ln t) / 2 \}.
\]

This is seen to be an infinite sheeted covering of \( \mathcal{G}_{s,t} \), the holomorphic covering map being \( \Phi_{\chi}^{(\infty)} \).

**Proposition 3.2.** \( D \) is not biholomorphic to \( \mathcal{G}_{s,t}^{(\infty)} \) for \( 0 \leq s < t < \infty \).

**Proof.** Let \( f : D \to \mathcal{G}_{s,t}^{(\infty)} \) be a biholomorphism. Then

\[
\pi = \Phi_{\chi}^{(\infty)} \circ f : D \to \mathcal{G}_{s,t}
\]
is a holomorphic infinite sheeted covering map between domains equipped with the standard complex structure. Using the explicit description of $\Phi_{x}^{(\infty)}$, we see that it maps the boundary of $\mathcal{S}_{s,t}^{(\infty)}$ into the boundary of $\mathcal{S}_{s,t}$. Hence the cluster set of $\partial D$ under $\pi$ is contained in $\partial \mathcal{S}_{s,t}$. Now if $0 < s < t < \infty$, then by choosing an appropriate point on the two dimensional stratum $S \subset L$ as in the previous proposition, it follows that $\partial D$ must be weakly pseudoconvex near $p_{\infty}$. By [28], $D \simeq \tilde{D}$ where $\tilde{D}$ is as in (3.1). Hence $\tilde{D}$ covers $\mathcal{S}_{s,t}$ and since the Kobayashi metric on $\tilde{D}$ is complete, it follows that the same must hold for $\mathcal{S}_{s,t}$. Completeness then forces $\mathcal{S}_{s,t}$ to be pseudoconvex which it is not. Contradiction.

If $0 = s < t < \infty$, then first note that the conclusion that $\tilde{D}$ covers $\mathcal{S}_{0,t}$ still holds and let $\pi$ still denote this infinite sheeted covering map. By [3], there exists a point on $\partial \tilde{D}$ whose cluster set under $\pi$ intersects $\partial \mathcal{S}_{0,t}$. By standard arguments involving the Kobayashi metric, $\pi$ extends continuously up to $\partial \tilde{D}$ near this point. This extension is even locally biholomorphic near strongly pseudoconvex points that are known to be dense in $\partial \tilde{D}$. By Webster’s theorem, $\pi$ is algebraic and therefore the cardinality of a generic fibre of $\pi$ is finite. This contradicts the fact that $\pi$ is an infinite sheeted cover. □

3.2. A domain in $\mathbb{P}^2$. Let $Q_+ \subset \mathbb{C}^3$ be the smooth complex analytic set given by

$$z_0^2 + z_1^2 + z_2^2 = 1.$$ 

For $1 \leq s < t < \infty$ define

$$E^{(2)}_{s,t} = \{(z_0, z_1, z_2) \in \mathbb{C}^3 : s < |z_0|^2 + |z_1|^2 + |z_2|^2 < t\} \cap Q_+.$$ 

This is a two sheeted covering of

$$E_{s,t} = \{(z_0 : z_1 : z_2) \in \mathbb{P}^2 : s|z_0^2 + z_1^2 + z_2^2| < |z_0|^2 + |z_1|^2 + |z_2|^2 < t |z_0^2 + z_1^2 + z_2^2|\},$$

the covering map being $\psi(z_0, z_1, z_2) = [z_0 : z_1 : z_2]$. Similarly, for $1 < t < \infty$, the map

$$\psi : E^{(2)}_t \rightarrow E_t$$

is a two sheeted covering, where

$$E^{(2)}_t = \{(z_0, z_1, z_2) \in \mathbb{C}^3 : |z_0|^2 + |z_1|^2 + |z_2|^2 < t\} \cap Q_+$$

and

$$E_t = \{(z_0 : z_1 : z_2) \in \mathbb{P}^2 : |z_0|^2 + |z_1|^2 + |z_2|^2 < t|z_0^2 + z_1^2 + z_2^2|\}.$$ 

To construct a four sheeted cover of $E_{s,t}$, consider the map $\Phi_{\mu} : \mathbb{C}^2 \setminus \{0\} \rightarrow Q_+$ whose components are given by

$$\tilde{z}_1 = -i(z_1^2 + z_2^2) + i(z_1 \overline{z}_2 - \overline{z}_1 z_2)/(|z_1|^2 + |z_2|^2),$$

$$\tilde{z}_2 = z_1^2 - z_2^2 - (z_1 \overline{z}_2 + \overline{z}_1 z_2)/(|z_1|^2 + |z_2|^2),$$

and

$$\tilde{z}_3 = 2z_1 z_2 + (|z_1|^2 - |z_2|^2)/(|z_1|^2 + |z_2|^2).$$

Note that $\Phi_{\mu}$ is a two sheeted cover onto $Q_+ \setminus \mathbb{R}^3$. Therefore we may equip the domain of $\Phi_{\mu}$, i.e., $\mathbb{C}^2 \setminus \{0\}$ with the pull back complex structure using $\Phi_{\mu}$ and denote the resulting complex surface by $M^{(4)}_{\mu}$. For $1 \leq s < t < \infty$, the domain

$$E^{(4)}_{s,t} = \{(z_1, z_2) \in M^{(4)}_{\mu} : (s - 1/2)^{1/2} < |z_1|^2 + |z_2|^2 < ((t - 1/2)^{1/2}\}$$

is a four sheeted cover of $E_{s,t}$, the holomorphic covering map being $\psi \circ \Phi_{\mu}$.
Proposition 3.3. There cannot exist a proper holomorphic mapping from \( D \) onto \( E_{s,t} \) for all \( 1 \leq s < t < \infty \). In particular, \( D \) is not equivalent to either \( E_{s,t}^{(2)} \) or \( E_{s,t}^{(4)} \).

Proof. Let \( f : D \to E_{s,t} \) be a proper holomorphic mapping. Consider the case when \( 1 < s < t < \infty \). The boundary \( \partial E_{s,t} \) has two components, namely
\[
\partial E_{s,t}^+ = \{ [z_0 : z_1 : z_2] \in \mathbb{P}^2 : |z_0|^2 + |z_1|^2 + |z_2|^2 = t|z_0^2 + z_1^2 + z_2^2| \}, \quad \text{and}
\partial E_{s,t}^- = \{ [z_0 : z_1 : z_2] \in \mathbb{P}^2 : |z_0|^2 + |z_1|^2 + |z_2|^2 = s|z_0^2 + z_1^2 + z_2^2| \},
\]
which are strongly pseudoconvex and strongly pseudoconcave hypersurfaces respectively. The argument used in proposition 3.1 can be applied here to conclude that \( p_\infty \in \partial D \) must be a weakly pseudoconvex point. By [28] it follows that \( D \cong \hat{D} \) where \( \hat{D} \) is as in (3.1). Thus we have a proper mapping from \( \hat{D} \) onto \( E_{s,t} \), which implies that \( E_{s,t} \) must be holomorphically convex and this is a contradiction.

Now suppose that \( 1 = s < t < \infty \). Then the boundary \( \partial E_{1,t} \) consists of a strongly pseudoconvex piece, namely \( \partial E_{1,t}^+ \), and a maximally totally real piece given by \( \psi(\partial \mathbb{B}^3 \cap Q_+) \). The argument in the preceeding paragraph applies again to show that \( D \cong \hat{D} \) with \( \hat{D} \) as in (3.1). Let \( f \) still denote the proper map from \( \hat{D} \) onto \( E_{1,t} \). Let \( \phi \) be a holomorphic function on \( \hat{D} \) that peaks at the point at infinity in \( \partial \hat{D} \) and denote by \( f_1^{-1}, f_2^{-1}, \ldots, f_t^{-1} \) the locally defined branches of \( f^{-1} \) that exist away from a closed codimension one analytic set in \( E_{1,t} \). Then
\[
\tilde{\phi} = (\phi \circ f_1^{-1}) \cdot (\phi \circ f_2^{-1}) \cdots (\phi \circ f_t^{-1})
\]
is a well defined holomorphic function on \( E_{1,t} \) and satisfies \( |\tilde{\phi}| < 1 \) there. Now \( \tilde{\phi} \) extends across \( \psi(\partial \mathbb{B}^3 \cap Q_+) \), which has real codimension two and is totally real strata, as well. Thus \( \tilde{\phi} \in O(E_t) \) and \( |\tilde{\phi}| \leq 1 \). If \( |\tilde{\phi}(a')| = 1 \) for some \( a' \in \psi(\partial \mathbb{B}^3 \cap Q_+) \), the maximum principle implies that \( |\tilde{\phi}| \equiv 1 \) on \( E_{1,t} \subset E_t \) and this is a contradiction. This argument shows that for every \( a' \in \psi(\partial \mathbb{B}^3 \cap Q_+) \), there is a point \( a \in \partial \hat{D} \) such that the cluster set of \( a \) under \( f \) contains \( a' \). On the other hand, by [3], there are points \( b, b' \) on \( \partial \hat{D}, \partial E_{1,t}^+ \) respectively such that the cluster set of \( b \) contains \( b' \). Thus \( f \) will be algebraic by Webster’s theorem as before. Away from an algebraic variety \( Z \subset \mathbb{P}^2 \), \( f^{-1} \) defines a holomorphic correspondence that locally splits into finitely many holomorphic mappings. Since \( Z \cap \psi(\partial \mathbb{B}^3 \cap Q_+) \) has real dimension at most one, it is possible to choose \( a' \in \psi(\partial \mathbb{B}^3 \cap Q_+) \setminus Z \). Now one of the branches of \( f^{-1} \) will map \( a' \) into \( \partial \hat{D} \) and therefore an open piece of the totally real component \( \psi(\partial \mathbb{B}^3 \cap Q_+) \) will be mapped locally biholomorphically onto an open piece of \( \partial \hat{D} \). Contradiction.

To conclude, if \( D \cong E_{s,t}^{(2)} \) or \( E_{s,t}^{(4)} \), then this would imply the existence of an unbranched proper holomorphic mapping from \( D \) onto \( E_{s,t} \) and this cannot happen by the arguments given above.

Proposition 3.4. There cannot exist a proper holomorphic mapping from \( D \) onto \( E_t \) for all \( 1 < t < \infty \). In particular, \( D \) cannot be equivalent to \( E_t^{(2)} \) for all \( 1 < t < \infty \).

Proof. By working in local coordinates it can be seen that \( E_t \) is described as a sub-level set of a strongly plurisubharmonic function. Hence \( E_t \) must be holomorphically convex and therefore \( D \) is pseudoconvex if there were to exist a proper map \( f : D \to E_t \). By standard arguments involving the Kobayashi metric, this map \( f \) will be continuous up to \( \partial D \) near \( p_\infty \). By [9] it
follows that $p_\infty$ is a weakly spherical point on $\partial D$, i.e., there is a defining function for $\partial D$ near $p_\infty = 0$ of the form

$$\rho(z) = 2\Re z_2 + |z_1|^{2m} + \ldots.$$ 

Since $p_\infty$ is an orbit accumulation point, [28] shows that $D$ is equivalent to the model domain at $p_\infty$, i.e.,

$$D \simeq \{(z_1, z_2) \in \C^2 : 2\Re z_2 + |z_1|^{2m} < 0\}.$$ 

This shows that $\dim \text{Aut}(D) = 4$ which is a contradiction. To conclude, if $D \simeq E_t^{(2)}$, then there would exist an unbranched proper mapping from $D$ onto $E_t$ which is not possible. □

3.3. Domains constructed by using an analogue of Rossi’s map. For $-1 \leq s < t \leq 1$ let

$$\Omega_{s,t} = \{(z_1, z_2) \in \C^2 : s|z_1|^2 + z_2^2 - 1 < |z_1|^2 + |z_2|^2 - 1 < t|z_1|^2 + z_2^2 - 1\}$$

and for $-1 < t < 1$ let

$$\Omega_t = \{(z_1, z_2) \in \C^2 : |z_1|^2 + |z_2|^2 - 1 < t|z_1|^2 + z_2^2 - 1\}.$$ 

It was shown in [17] that $\Omega_t$ has a unique maximally totally real $\text{Aut}(\Omega_t)^c$-orbit, namely

$$O_5 = \{(\Re z_1, \Re z_2) \in \R^2 : (\Re z_1)^2 + (\Re z_2)^2 < t\}$$

for all $t \in (-1, 1)$. Moreover $\Omega_t = \Omega_{-1,t} \cup O_5$ for all $t \in (-1, 1)$.

For $1 \leq s < t \leq \infty$ let

$$D_{s,t} = \{(z_1, z_2) \in \C^2 : s|1 + z_1^2 - z_2^2| < 1 + |z_1|^2 - |z_2|^2 < t|1 + z_1^2 - z_2^2|, \Im(z_1(1 + \overline{z}_2)) > 0\}$$

where it is assumed that the domain $D_{s,\infty}$ does not contain the complex curve

$$O = \{(z_1, z_2) \in \C^2 : 1 + z_1^2 - z_2^2 = 0, \Im(z_1(1 + \overline{z}_2)) > 0\}.$$ 

For $1 \leq s < \infty$ let

$$D_s = \{(z_1, z_2) \in \C^2 : s|1 + z_1^2 - z_2^2| < 1 + |z_1|^2 - |z_2|^2, \Im(z_1(1 + \overline{z}_2)) > 0\}$$

and note that $D_s = D_{s,\infty} \cup O$.

Observe that $D$ cannot be equivalent to $\Omega_{s,t}$ or $D_{s,t}$ as neither is simply connected. It remains to consider whether $D$ can be equivalent to $\Omega_t$ or $D_s$.

**Proposition 3.5.** There cannot exist a proper holomorphic mapping from $D$ onto $\Omega_t$ for $-1 < t < 1$ or to $D_s$ for $1 \leq s < \infty$.

**Proof.** We first consider $\Omega_t$. Let $z_1 = x + iy, z_2 = u + iv$ so that

$$O_5 = \{(x, u) \in \R^2 : x^2 + u^2 < 1\}$$

and its boundary

$$\partial O_5 = \{(x, u) \in \R^2 : x^2 + u^2 = 1\} \subset \partial \Omega_t$$

for all $t \in (-1, 1)$. Note that $\partial \Omega_t \setminus \partial O_5$ is a smooth strongly pseudoconvex hypersurface. Suppose that $f : D \to \Omega_t$ is proper. As in proposition 3.1, it is possible to choose $a \in S \subset \mathcal{L}$ such that $f$ extends holomorphically to a neighbourhood of $a$. By shifting $a \in S$ if necessary we may assume that $f$ is in fact locally biholomorphic near $a$. Note that $f(a) \notin \partial \Omega_t \setminus \partial O_5$, as otherwise there are strongly pseudoconcave points near $a$ that will be mapped to strongly pseudoconvex points. The remaining possibility is that $f(a) \in \partial O_5$ which is totally real. Since $f$ is locally biholomorphic near $a$, $f$ cannot map an open piece of $\partial D$ near $a$ into $\partial O_5$. Again, there are
strongly pseudoconcave points near $a$ that are mapped by $f$ to $\partial \Omega_4 \setminus \partial \Omega_5$ which is strongly pseudoconvex and this is a contradiction.

Hence the boundary $\partial D$ is weakly pseudoconvex near $p_\infty$ and thus $D \simeq \tilde{D}$ by [28] where $\tilde{D}$ is as in (3.1). Let $f : \tilde{D} \to \Omega_t$ still denote the biholomorphism. Observe that the automorphism group of $\tilde{D}$ is at least two dimensional; apart from the translations $T_t$, it is also invariant under the one parameter subgroup $S_t(z_1, z_2) = (\exp(s/2m)z_1, \exp(s)z_2)$, $s \in \mathbb{R}$. The corresponding real vector fields $X = \Re(i\partial/\partial z_2)$ and $Y = \Re((z_1/2m)\partial/\partial z_1 + z_2\partial/\partial z_2)$ satisfy $[X, Y] = X$. By the arguments in the last part of the proof of proposition 4.1 in [31], it follows that $D \simeq D_4$ where

$$D_4 = \{(z_1, z_2) \in \mathbb{C}^2 : 2\Re z_2 + (\Re z_1)^2m < 0\}.$$ 

Let $f : D_4 \to \Omega_t$ still denote the proper map. Choose an arbitrary strongly pseudoconvex point $b' \in \partial \Omega_4 \setminus \partial \Omega_5$. By [8] there exists $b \in \partial D_4$ such that the cluster set of $b$ under $f$ contains $b'$. Then by well known arguments involving the Kobayashi metric on $D_4$ and $\Omega_t$ near $b$ and $b'$ respectively, it follows that $f$ is continuous up to $\partial D_4$ near $b$ and $f(b) = b'$. By [9], it follows that $b \in \partial D_4$ must be a weakly spherical point, i.e., there exists a coordinate system near $b$ in which the defining equation for $\partial D_4$ is of the form

$$\rho(z) = 2\Re z_2 + |z_1|^{2m} + \ldots,$$

the dots indicating terms of higher order. However, the explicit form of $\partial D_4$ shows that no point on it is weakly spherical.

It remains to show that no proper map $f : D \to D_s$ can exist for $1 \leq s < \infty$. Suppose the contrary. Observe that if $s > 1$ then $\partial D_s$ is the disjoint union of three components, namely

$$C^1 = \{1 + |z|^2 - |z_2|^2 = s|1 + z_1^2 - z_2^2|, \Im(z_1(1 + z_2)) > 0\},$$
$$C^2 = \{1 + |z|^2 - |z_2|^2 > s|1 + z_1^2 - z_2^2|, \Im(z_1(1 + z_2)) = 0\},$$
$$C^3 = \{1 + |z|^2 - |z_2|^2 = s|1 + z_1^2 - z_2^2|, \Im(z_1(1 + z_2)) = 0\}.$$

Note that $C^1$ is a strongly pseudoconvex hypersurface and that $\Im(z_1(1 + z_2)) = 0$ has an isolated singularity at $(z_1, z_2) = (0, -1)$ away from which it is smooth Levi flat. Also, $(0, -1) \notin C^2$ as $s > 1$. As above, choose $a \in S \subset L$ near which $f$ extends locally biholomorphically. Since $C^1$ is strongly pseudoconvex, it follows that $f(a) \notin C^1$. Further if $f(a) \in C^2$, then a small open piece of $\partial D$ near $a$ will be mapped locally biholomorphically into the Levi flat piece $\{\Im(z_1(1 + z_2)) = 0\}$ and this is a contradiction as points of $\partial D \setminus S$ near $a$ are Levi non-degenerate. The remaining possibility is that $f(p) \in C^3$. However, an open piece of $\partial D$ near $a$ cannot be mapped by $f$ into $C^3$ as it has real dimension at most 2 near each of its points. Thus there is an open dense set of points near $a$ that are mapped locally biholomorphically into either $C^1$ or $C^2$. Both cannot occur for reasons mentioned above. Thus $\partial D$ must be weakly pseudoconvex near $p_\infty$ and we may now argue as before to get a contradiction.

When $s = 1$, it was noted in [17] that there is a proper mapping $g$ from the bidisc $\Delta^2$ onto $D_1$. If $f : D \to D_1$ is proper, then $F : f^{-1} \circ g : \Delta^2 \to D$ is a proper holomorphic correspondence. Thus $D$ is pseudoconvex and by [28], it follows that $D \simeq \tilde{D}$ where $\tilde{D}$ is as in (3.1). Let $F : \Delta^2 \to \tilde{D}$ still denote the proper correspondence. Using the holomorphic function on $\tilde{D}$ that peaks at the point at infinity in $\partial \tilde{D}$ it can be seen that there is an open dense subset of $\partial \Delta^2$ whose cluster set under
$F$ intersects the finite part of $\partial \Delta^2$—call this subset $\Gamma$. Fix $\zeta_0 \in \Gamma$ and a small neighbourhood $W$ containing it such that $W \cap \partial \Delta^2$ is smooth. Note that $W \cap \partial \Delta^2$ is defined as the zero locus of either $|z_1|^2 - 1$ or $|z_2|^2 - 1$ both of which are plurisubharmonic. Now well known arguments using the branches of $F^{-1}$, these plurisubharmonic defining equations and a suitable version of the Hopf lemma show that

$$\text{dist}(F(z), \partial \bar{D}) \lesssim \text{dist}(z, \partial \Delta^2)$$

whenever $z \in W \cap \Delta^2$—here $F(z)$ denotes any one of the finitely many branches of $F$. By [6] it follows that $F$ extends continuously up to $W \cap \partial \Delta^2$ as a correspondence. The branching locus of $F$ in $\Delta^2$ is therefore defined by a holomorphic function in $\Delta^2$ that extends continuously up to $W \cap \partial \Delta^2$. Let $h \in O(\Delta^2)$ define the branching locus. If $h \equiv 0$ on $W \cap \partial \Delta^2$, the uniqueness theorem shows that $h \equiv 0$ in $\Delta^2$ which cannot happen. By shifting $\zeta_0$ we may assume that $h(\zeta_0) \neq 0$. Therefore near $\zeta_0$ the correspondence $F$ splits into well defined holomorphic functions, say $F_1, F_2, \ldots, F_k$ each of which is holomorphic on $W$ (shrink $W$ if needed) and continuous up to $W \cap \partial \Delta^2$. Since $W \cap \Delta^2$ is a product domain and each point of $\partial \bar{D}$ supports a holomorphic peak function, arguments from [27] show that these branches $F_1, F_2, \ldots, F_k$ must be independent of either $z_1$ or $z_2$. This contradicts the assumption that $F$ is proper. \hfill \square

It is also possible to construct finite and infinite sheeted covers of $D_{s,t}, \Omega_{s,t}$ as explained in [17]. That $D$ cannot be equivalent to any of them follows by similar arguments and we omit the details.

Finally proposition 4.1 of [31] shows that a bounded domain $D \subset \mathbb{C}^2$ that satisfies the hypotheses of the main theorem and admits a Levi flat $\text{Aut}(D)^c$-orbit must be equivalent to

$$D_4 = \{(z_1, z_2) \in \mathbb{C}^2 : 2\Re(z_2) + (\Re(z_1))^{2m} \leq 0\}.$$  

The proof is purely local and can be applied here as well to conclude that a domain $D \subset X$ as in the main theorem with a Levi flat $\text{Aut}(D)^c$-orbit must be equivalent to $D_4$. This is the only possibility that remains after eliminating all others and the conclusion is that if $\dim \text{Aut}(D) = 3$ then $D \simeq D_4$.

4. Model Domains when $\text{Aut}(D)$ is Four Dimensional

Of the 7 isomorphism classes listed in [18] of hyperbolic surfaces with four dimensional automorphism group, the following cannot be equivalent to $D$ for topological reasons.

- The spherical shell $S_r = \{z \in \mathbb{C}^2 : r < |z| < 1\}$ for $0 \leq r < 1$—the automorphism group here is the unitary group $U_2$ which is compact, or the quotient $S_r/\mathbb{Z}_m$ for some $m \in \mathbb{N}$, none of which are simply connected.
- $E_{r,\theta} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, r(1 - |z_1|^2)^\theta < |z_2| < (1 - |z_1|^2)^\theta\}$, where $\theta \geq 0, 0 < r < 1$ or $\theta < 0, r = 0$. This is not simply connected.
- $D_{r,\theta} = \{(z_1, z_2) \in \mathbb{C}^2 : r \exp(\theta |z_1|^2) < |z_2| < \exp(\theta |z_1|^2)\}$, where $\theta = 1, 0 < r < 1$ or $\theta = -1, r = 0$. This is again not simply connected.

The remaining four classes listed below have a common feature that a large part of their boundary, if not the whole, is spherical.

- $\Omega_{r,\theta} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, r(1 - |z_1|^2)^\theta < \exp(\Re(z_2)) < (1 - |z_1|^2)^\theta\}$, where $\theta = 1, 0 \leq r < 1$ or $\theta = -1, r = 0$
- $S = \{(z_1, z_2) \in \mathbb{C}^2 : -1 + |z_1|^2 < \Re(z_2) < |z_1|^2\}$.  

• $T_\theta = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < (1 - |z_1|^2)^\theta\}$, for $\theta < 0$. Here the boundary $\partial T_\theta$ contains a Levi flat piece $L = \{ |z_1| = 1 \} \times \mathbb{C}_{z_2}$. Away from $L$, $\partial T_\theta$ is spherical and strongly pseudoconcave as seen from $T_\theta$

• $E_\theta = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^\theta < 1\}$, where $\theta > 0$ and $\theta \neq 2$.

To see that $D$ cannot be equivalent to $\Omega_{\gamma, \theta}, \Theta$ or to $T_\theta$, suppose the contrary. Let $f : D \to T_\theta$ be biholomorphic. Let $p \in \partial D$ be a strongly pseudoconcave point near $p_\infty$ across which $f$ extends locally biholomorphically. Note that $f(p) \notin L$ as $\partial D$ is of finite type near $p_\infty$. Then $f(p) \in \partial E_\theta$. Let $g$ be a local biholomorphism defined on a open neighbourhood $W$ of $f(p)$ that takes $W \cap \partial E_\theta$ into $\partial \mathbb{B}^2$. Then $g \circ f$ is a biholomorphic germ at $p$ that maps an open piece of $\partial D$ into $\partial \mathbb{B}^2$. By [29], this germ can be analytically continued along all paths in $U \cap \partial D$ that start at $p$. Thus $p_\infty$ must be a weakly pseudoconvex point and by [19], it must be weakly spherical as well. By [28], it follows that $D \simeq E_{2m}$ and so $T_\theta \simeq E_{2m}$ which is a contradiction.

To conclude, it remains to show that if $D \simeq E_\theta$, then $\theta = 2m$ for some integer $m \geq 2$. Proposition 5.1 in [31] remains valid here too and we omit the details. The conclusion is that if $\text{dim} \text{Aut}(D) = 4$ then $D \simeq E_{2m} \simeq D_\delta$.

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