BOUNDDED SOLUTIONS TO THE AXIALLY SYMMERIC NAVIER STOKES EQUATION IN A CUSP REGION

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ABSTRACT. A domain in $\mathbb{R}^3$ that touches the $x_3$ axis at one point is found with the following property. For any initial value in a $C^2$ class, the axially symmetric Navier Stokes equations with Navier slip boundary condition has a finite energy solution that stays bounded for any given time, i.e. no finite time blow up of the fluid velocity occurs. The result seems to be the first case where the Navier-Stokes regularity problem is solved beyond dimension 2.

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1. INTRODUCTION

The Navier-Stokes equations (NS) describing the motion of viscous incompressible fluids in a domain $D \subset \mathbb{R}^3$ is

\begin{align}
&\mu \Delta v - v \nabla v - \nabla P - \partial_t v = 0, \quad \text{div} \, v = 0, \quad \text{on} \quad D \times (0, \infty)
\end{align}

Here $v$ is the velocity field, $P$ is the pressure, both of which are the unknowns; $\mu > 0$ is the viscosity constant, which will be taken as 1 in this paper. In order to solve the equation, a initial velocity $v_0$ is usually given, together with suitable boundary conditions. One can also add a forcing term on the righthand side, then it becomes a nonhomogeneous problem. Due to Leray \cite{24}, if $D = \mathbb{R}^3$, $v_0 \in L^2(\mathbb{R}^3)$, the Cauchy problem has a solution in the energy space (c.f. (1.8) below). However, in general it is not known if such solutions stay bounded or regular for all $t > 0$. A typical existence theorem for regular solutions always involves a small parameter in the initial condition, as a perturbation of a known regular solution.

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In this paper, we will focus on a special case of (1.1), namely when $v$ and $P$ are independent of the angle in a cylindrical coordinate system $(r, \theta, x_3)$. That is, for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $r = \sqrt{x_1^2 + x_2^2}$, $\theta = \arctan(x_2/x_1)$, and the basis vectors $e_r, e_\theta, e_3$ are:

$$e_r = (x_1/r, x_2/r, 0), \quad e_\theta = (-x_2/r, x_1/r, 0), \quad e_3 = (0, 0, 1).$$

In this case, solutions can be written in the form of

$$v = v_r(r, x_3, t)e_r + v_\theta(r, x_3, t)e_\theta + v_3(r, x_3, t)e_3.$$

Using tensor notations and doing vector calculus under the cylindrical system, one finds

$$\nabla v = \partial_r v \otimes e_r + \frac{1}{r} \partial_\theta v \otimes e_\theta + \partial_{x_3} v \otimes e_3$$

$$= (\partial_r v_r e_r + \partial_r v_\theta e_\theta + \partial_r v_3 e_3) \otimes e_r + \frac{1}{r} (v_r e_\theta - v_\theta e_r) \otimes e_\theta$$

$$+ (\partial_{x_3} v_r e_r + \partial_{x_3} v_\theta e_\theta + \partial_{x_3} v_3 e_3) \otimes e_3.$$ 

It is convenient to denote $e_r \otimes e_r$, $e_\theta \otimes e_\theta$, $e_r \otimes e_\theta$, ..., $e_3 \otimes e_3$ by the nine single-entry matrices in the standard basis for $3 \times 3$ matrices, going as $J^{11}$, $J^{12}$, $J^{13}$, $J^{21}$, ..., $J^{33}$. Then $\nabla v$ is given by the $3 \times 3$ matrix

$$(1.2) \quad \nabla v = \begin{bmatrix} \partial_r v_r & -\frac{1}{r} v_\theta & \partial_{x_3} v_r \\ \partial_r v_\theta & \frac{1}{r} v_r & \partial_{x_3} v_\theta \\ \partial_{x_3} v_r & 0 & \partial_{x_3} v_3 \end{bmatrix},$$

and $v \nabla v$ is given by the matrix multiplication $(\nabla v)v$ with $v$ being regarded as the column vector $(v_r, v_\theta, v_3)^T$. Therefore, $v_r$, $v_3$ and $v_\theta$ satisfy the axially symmetric Navier-Stokes equations

$$(1.3) \quad \begin{cases} (\Delta - \frac{1}{r^2})v_r - (v_r \partial_r + v_3 \partial_{x_3})v_r + \frac{(v_\theta)^2}{r} - \partial_r P - \partial_i v_r = 0, \\
(\Delta - \frac{1}{r^2})v_\theta - (v_r \partial_r + v_3 \partial_{x_3})v_\theta - \frac{v_r v_\theta}{r} - \partial_r v_\theta = 0, \\
\Delta v_3 - (v_r \partial_r + v_3 \partial_{x_3})v_3 - \partial_{x_3} P - \partial_i v_3 = 0, \\
\frac{1}{r} \partial_r (rv_r) + \partial_{x_3} v_3 = 0, \end{cases}$$

which will be abbreviated as ASNS. Although ASNS looks more complicated than the full 3 dimensional equation, a simplification occurs in the 2nd equation where the pressure term drops out.

If the swirl $v_\theta = 0$, then it is known for a while (O. A. Ladyzhenskaya [22], M. R. Uchovskii and B. I. Yudovich [38]), that finite energy solutions to the Cauchy problem of (1.3) in $\mathbb{R}^3$ are smooth for all time $t > 0$. See also the paper by S. Leonardi, J. Malek, J. Necas, and M. Pokorný [28]. By finite energy, we mean the norm (1.8) below is finite, i.e., the solution is a Leray-Hopf solution.

In the presence of swirl, it is still not known in general if finite energy solutions blow up in finite time. By the partial regularity results in [8], possible singularity for suitable weak solutions of ASNS can only occur at the $x_3$ axis. See also [25]. The same statement without the word "suitable" is also true by [4]. Some existence results with a small parameter can be found in [16] and [18]. Critical or sub-critical regularity conditions can be found in [33], [9] and [19].
Despite the difficulty, there has been no lack of research efforts on ASNS. Let us make a brief description of related recent results, starting with the papers by C.-C. Chen, R. M. Strain, T.-P.Tsai, and H.-T. Yau in [12], [13], G. Koch, N. Nadirashvili, G. Seregin, and V. Sverak in [20], which appeared around 2008. See also the work by G. Seregin and V. Sverak [36] for a localized version. These authors proved that if

\[ |v(x,t)| \leq \frac{C}{r}, \]

then finite energy solutions to the Cauchy problem of ASNS are smooth for all time. Here \( C \) is any positive constant.

The proof is based on the fact that the scaling invariant quantity \( \Gamma = rv_\theta \) satisfies the equation

\[ \Delta \Gamma - b\nabla \Gamma - \frac{2}{r}\partial_r \Gamma - \partial_t \Gamma = 0, \]

where \( b = v_r e_r + v_3 e_3 \). The bound (1.4) says that the equation is essentially scaling invariant and the classical linear regularity theory can be applied after some nontrivial modification. The above result can be summarized as: type I solutions of ASNS are regular.

Two years later, in the paper [26] by Lei and the author, it was proven that if \( v_r, v_3 \) are in the space of \( L^\infty((0, \infty), \text{BMO}^{-1}(\mathbb{R}^3)) \) and \( rv_\theta(\cdot, 0) \in L^\infty \), then the solution is regular. Here \( \text{BMO} \) is the space of functions with bounded mean oscillation, and \( \text{BMO}^{-1} \) is the space of tempered distributions which can be written as partial derivatives of \( \text{BMO} \) functions. Well-posedness and other properties of solutions to NS have been studied by Koch-Tataru [21].

Recently Seregin and Zhou [37] have relaxed the \( L^\infty \text{BMO}^{-1} \) assumption further to \( L^\infty \hat{B}^{-1}_{\infty, \infty} \) assumption. Let us recall \( \hat{B}^{-1}_{\infty, \infty} \) is the Besov space consisted of tempered distributions \( f \) such that the norm

\[ \|f\|_{\hat{B}^{-1}_{\infty, \infty}} = \sup_{t>0} t^{1/2} \sup_x \left| \int_{\mathbb{R}^3} G(x,t,y) f(y) dy \right| \]

is finite. Here \( G(x,t,y) = (1/(4\pi t)^{3/2}) \exp(-|x-y|^2/(4t)) \) is the standard heat kernel on \( \mathbb{R}^3 \). In these papers the regularity conditions are critical and hence are scaling invariant under standard scaling. Improvements are at most logarithmic so far. See the paper by X.H. Pan [34]. In contrast, the energy bound scales as \(-1/2\). So even with axial symmetry, there seems to be a finite scaling gap which makes the ASNS supercritical, just like the full equations.

However in a recent paper [27], Lei and the author made the following observation.

The vortex stretching term of the ASNS is critical after a suitable change of dependent variables.

So the aforementioned scaling gap is 0. This observation has the effect of making ASNS looks less formidable than the full 3 dimensional one which has a positive scaling gap. Nevertheless all major open problems for the latter are still open for the former.

The observation is based on the study of the equations for the functions \( \Omega = \omega_\theta/r \) and \( J = \omega_\rho/r \) where \( \omega_\theta \) and \( \omega_r \) are the angular and radial components of the vorticity. The function \( \Omega \) has been around for longtime [38]. But \( J \) was introduced in the recent paper by H. Chen-D.Y.Fang-T. Zhang [7]. By carrying out an energy estimate for these equations, they proved the following result. Let \( v \) be a Leray-Hopf solution to the Cauchy problem of
ASNS with initial data $v_0 \in H^2$ and $\|r(v_0 \cdot e_\theta)\|_{L^\infty} < \infty$. If
\[
|v_\theta(x, t)| \leq C/r^{1-\epsilon},
\]
for all $x$ and $t > 0$, then $v$ is regular everywhere. Here $\epsilon > 0$ and $C$ are positive constants.

The main result in [27] includes the following statement. Let $\delta_0 \in (0, \frac{1}{2})$ and $C_1 > 1$. If
\[
\text{(1.6)} \quad \sup_{0 \leq t < T} |rv_\theta(r, x_3, t)| \leq C_1 |\ln r|^{-2}, \quad r \leq \delta_0,
\]
then above $v$ is regular globally in time. Note that a priori we have $|rv_\theta(r, x_3, t)| \leq C$ by the maximal principle applied on equation (1.5). So there is still a gap of logarithmic nature from regularity.

After [27] was posted on the arxiv, in the paper by Dongyi Wei [39], the power in the log term has been improved to $-3/2$. Namely, if, for some $\delta_0 \in (0, 1/2),$
\[
\text{(1.7)} \quad \sup_{0 \leq t < T} |rv_\theta(r, x_3, t)| \leq C_1 |\ln r|^{-3/2}, \quad r \leq \delta_0,
\]
then the above $v$ is regular.

In this paper, for a special class of bounded domains with Navier slip boundary condition, we manage to solve the regularity problem to ASNS. By finite energy, we mean the solutions are in the energy space $E = L^2_t W^{1,2}_x \cap L^\infty_t L^2_x$. Here and throughout, the norm in $E$ for a function $v$ on $D \times [0, T]$ is taken as
\[
\text{(1.8)} \quad \|v\|^2_E = \int_0^T \int_D |\nabla v|^2 dx dt + \sup_{t \in [0, T]} \int_D |v(x, t)|^2 dx.
\]
The function $v$ can be vector or scalar valued, depending on the context, and $T > 0$.

These domains, which touch the $x_3$ axis at one point, are the union of a sequence of rectangles in the $r - x_3$ plane. The ratio between the height and width of the rectangles decreases slowly to 0 when they approaches the $x_3$ axis. The side view of the domain is a wedge like region with infinitely many terraces. More specifically, the domains are given in

**Definition 1.1.** (see also the figure below.)
\[
\text{(1.9)} \quad D_* = \bigcup_{m=1}^\infty D_m, \quad \text{with} \quad D_m = \bigcup_{j=1}^m S_j,
\]

\[
S_j = \{(r, x_3) \mid 2^{-j} \leq r < 2^{-(j-1)}, \quad 0 < x_3 < 2^{-\beta(j-1)}\}
\]
where $\beta \in (1, 1.1]$ is any fixed number.
Recall that the Navier \cite{32} slip boundary condition reads

\begin{equation}
(\mathbb{S}(v)n)_{\text{tan}} = 0, \quad v \cdot n = 0, \quad \text{on} \quad \partial D_\ast.
\end{equation}

Here the strain tensor \( \mathbb{S}(v) = [\nabla v + (\nabla v)^T]/2 \) which, by \( \text{(1.2)} \), is given by

\begin{equation}
\mathbb{S}(v) = \begin{bmatrix}
\partial_r v_r & \frac{1}{2}(\partial_r v_\theta - \frac{1}{r} v_\theta) & \frac{1}{2}(\partial_{x_3} v_r + \partial_r v_3) \\
\frac{1}{2}(\partial_{x_3} v_r + \partial_r v_3) & \frac{1}{2} \partial_{x_3} v_\theta & \partial_{x_3} v_3 \\
\frac{1}{2}(\partial_{x_3} v_r + \partial_r v_3) & \frac{1}{2} \partial_{x_3} v_\theta & \partial_{x_3} v_3
\end{bmatrix}.
\end{equation}

In the above \( n \) is the unit outward normal on the smooth part of \( \partial D_\ast \) and the \((\mathbb{S}(v)n)_{\text{tan}}\) stands for the tangential component of the vector \( \mathbb{S}(v)n \). Observe that the boundary \( \partial D_\ast \) can be written as the union of horizontal and vertical parts, which are denoted by \( \partial^H D_\ast \) and \( \partial^V D_\ast \) respectively. Namely,

\begin{equation}
\partial D_\ast = \partial^H D_\ast \cup \partial^V D_\ast
\end{equation}

From \( \text{(1.11)} \) and \( \text{(1.10)} \), one sees that the Navier slip boundary condition can be expressed explicitly as

\begin{equation}
\begin{align*}
\partial_{x_3} v_r &= \partial_{x_3} v_\theta = 0, \quad v_3 = 0, \quad \text{on} \quad \partial^H D_\ast; \\
\partial_r v_\theta &= \frac{v_\theta}{r}, \quad \partial_r v_{x_3} = 0, \quad v_r = 0, \quad \text{on} \quad \partial^V D_\ast.
\end{align*}
\end{equation}

Due to the cusp shape of \( D_\ast \) near \( x_3 \) axis and the boundary condition, one needs to define the space for initial values carefully. We pick the space \( C^2_{nb}(D_\ast) \subset C^2(D_\ast) \) for the initial values, which is defined as

\[
C^2_{nb}(D_\ast) = \{ f \in C^2(D_\ast) | f \text{ is the limit under the } C^2 \text{ norm of functions } f_m \in C^2(D_m), \quad \text{div } f_m = 0 \}.
\]

The above convergence means \( \| f \chi_{D_m} - f_m \|_{C^2} \to 0 \) as \( m \to \infty \). The main result states that finite energy solutions of ASNS in \( D_\ast \) are bounded in finite time, namely, no finite time blow up occurs.

**Theorem 1.2.** For any initial value in the \( C^2_{nb}(D_\ast) \) space, the axially symmetric Navier Stokes equations \( \text{(1.3)} \) with Navier slip boundary condition on \( D_\ast \) has a finite energy solution such that the velocity \( v \) stays bounded for any given time.
The result appears to have two implications. First, it seems to be the first case where the regularity problem for finite energy solutions is solved beyond dimension 2. In the literature ASNS are sometimes considered as $2\frac{1}{2}$ dimensional equations. Recall that Ladyzhenskaya [23] (Theorem 6, Chapter 6, Sec.3) already knew that if the domain is bounded away from the symmetric axis, finite energy solutions to ASNS are regular. But no progress was made afterwards if the domain touches the axis. The second pertains recent interesting non-uniqueness results by Buckmaster and Vicol [5] on some very weak (infinite energy) solutions of the Navier Stokes equations using the technique of convex integration of De Lellis and L. Szekelyhidi [14]. In the paper [5] the authors also stated a conjecture on non-uniqueness and hence non-regularity of some finite energy solutions to the Navier Stokes equations. See also [10] by Cheskidov and Luo and [17] by Hou and Huang. Although the above theorem is not exactly a counter example to the conjecture due to the boundary condition and symmetry, it seems to indicate an additional obstacle to the construction of a non-regular solution with finite energy.

Let us briefly describe the proof of the above theorem. We will first construct solutions $v^{(m)}$ of ASNS in the intermediate regions $D_m$ with Navier slip boundary condition. Since $D_m$ is bounded away from the $x_3$ axis, the solutions are bounded. The key step is to prove that these bound is independent of $m$, after which one can extract a convergent subsequence to obtain the claimed bounded solution. The usual path to do so is to find a uniform $L^2_t W^{2,2}_x$ bound for the solutions. However due to the presence of nonconvex corners in $D_m$, such a bound is not expected to be true. See [31] for an example for 2 dimensional Stokes system. Our observation is that a partial $L^2_t W^{2,2}_x$ bound holds, i.e. some components of the $\nabla^2 v^{(m)}$ are in $L^2_t L^2_x$ space. This will allow us to start an energy estimate on the equations for $\omega^{(m)}_\theta / r$ and $\omega^{(m)}_r / r$. Here $\omega^{(m)}_\theta$ and $\omega^{(m)}_r$ are the angular and radial components of the vorticity $\text{curl} v^{(m)}$. The cusp shape of $D_m$ and boundary conditions are used together to show that the vortex stretching terms are dominated by the diffusion near the $x_3$ axis. This will lead to a $L^\infty_t L^2_x$ bounds for $\omega^{(m)}_\theta / r$ and $\omega^{(m)}_r / r$. These bounds are used to prove the $L^\infty_t$ bound for $v^{(m)}$ via the Biot-Savart law. This is easy in the full space case. But the presence of the cusp makes the proof harder. One reason is the standard Sobolev inequality may fail. To proceed, we employ some new observations on the velocity such as the line integral of $v_r$ is 0 in the $x_3$ direction, and interactions between components of $v$ and $\omega$ and boundary values. In the theorem, the parameter $\beta$ can be made a slightly larger than 1.1 but it remains to be seen if it can be chosen as 1. If $\beta = 1$, then the domain $D_*$, after self similar extension to $r = \infty$, resembles the exterior of a cone globally. One can prove, by the same method, that no finite time blowup occurs under an extra condition $|rv_\theta(\cdot,0)| \leq \epsilon$ for some $\epsilon > 0$. Note that no extra assumption is made on $v_r$ or $v_3$.

We end the introduction by listing a number of notations and conventions to be used throughout, which are more or less standard. The velocity field is usually called $v$ and the vorticity $\nabla \times v$ is called $\omega$. We use subscripts to denote their components in coordinates. For example $v_r = v \cdot e_r$, $\omega_\theta = \omega \cdot e_\theta$, $\omega_3 = \omega \cdot e_3$; $\Omega = \omega_\theta / r$ and $J = \omega_r / r$. We write $b = v, e_r + v_3e_3$. $L^p(D), p \geq 1$, denotes the usual Lebesgue space on a domain $D$ which may be a spatial, temporal or space-time domain. Let $X$ be a Banach space defined for functions on $D \subset \mathbb{R}^3$. $L^p(0, T; X)$ is the Banach space of space-time functions $f$ on the space time domain $D \times [0, T]$.
with the norm $\left(\int_0^T \|f(t, \cdot)\|_p^p dt\right)^{1/p}$. If no confusion arises, we will also use $L^pX$ to abbreviate $\mathbb{L}^p(0, T; X)$. Sometimes we will also use $L^pL^q$ or $L^pL^q$ to denote the mixed $p, q$ norm in space time. Let $D \subset \mathbb{R}^3$ be an open domain, then $H^1(D) = W^{1,2}(D) = \{f | f, \nabla f \in L^2(D)\}$ and $H^2(D) = W^{2,2}(D) = \{f | f, \nabla f, \nabla^2 f \in L^2(D)\}$, the standard Sobolev spaces on $D$. Also, interchangeable notations $div v = \nabla \cdot v$, $\nabla v = \sum v \partial_x v = v \cdot \nabla v$ will be used. If there is no confusion, the vertical variable $x_3$ may be replaced with $z$. Also $B(x, r)$ denotes the ball of radius $r$ centered at $x$ in a Euclidean space; and $B_x(f, r)$ denotes the open ball in a normed space $X$, centered at $f \in X$ with radius $r$. If $s$ is a number, then $s^\ast$ means any number which is close but strictly less than $s$. We use $C$ with or without index to denote a generic constant which may change from line to line and $\overline{C}, C_\ast$ etc, $\alpha, \beta$ etc to denote important constants which may depend on relevant functions such as initial values, solutions.

2. Finite energy solutions with partial $L^2_t W^{2,2}_x$ bounds on approximating domains

In this section, we study the existence of finite energy solutions to the following initial boundary value problem on $D_m \times (0, T)$, $T > 0$. We will also prove some extra regularity result, which, although not uniform in $m$, will be needed in the next section.

**Problem 2.1.** Find a function $v = v_r e_r + v_\theta e_\theta + v_3 e_3$ in the energy space $E = L^2_t W^{1,2}_x \cap L^\infty_t L^2_x$ on the domain $D_m \times (0, T)$, such that the following hold.

(a). $v$ satisfies ASNS [1,3] in the weak sense on $D_m \times (0, T)$.

(b). $v$ is subject to the Navier slip boundary condition:

$$\partial_{x_3} v_r = \partial_{x_3} v_\theta = 0, \quad v_3 = 0, \quad \text{on} \quad \partial^H D_m \times [0, T];$$

$$\partial_r v_\theta = \frac{v_\theta}{r}, \quad \partial_r v_3 = 0, \quad v_r = 0, \quad \text{on} \quad \partial^V D_m \times [0, T].$$

(c). The initial value $v(x, 0) = v_0$ is in $W^{2,2}(D_m)$ and it satisfies the Navier slip boundary condition and the divergence free condition.

We refer the reader to Section 2 of [2] for a precise definition of (a) and (b) and the exact function spaces involved. In our case, due to the relative simplicity of the boundary and standard interior and boundary regularity results, the boundary condition (b) can also be understood in the pointwise sense, except at the corners.

Our starting point is the basic energy estimate for velocity $v$. It is known that a solution to Problem 2.1 exists and it satisfies the energy bound

$$\int_{D_m} |v(x, T)|^2 dx + 4 \int_0^T \int_{D_m} |\nabla v(x, t)|^2 dxdt \leq \int_{D_m} |v(x, 0)|^2 dx.$$

Such a result for piecewise smooth domains can be found in the paper [2] by Benes and [3] by Benes and Kucera e.g., where even more complicated mixed boundary conditions are studied. When the angles in the polyhedron is not greater than $\pi$, it is proven in [2] that these solutions are strong in the sense that the velocity $v$ is in $L^2_t W^{2,2}_x$ space. As mentioned, due to the presence of $3\pi/2$ angle in our case, solution $v$ may not enjoy $L^2_t W^{2,2}_x$ regularity in general. This lack of regularity is an obstacle in proving the main result, which will be circumvented by the $L^2_t L^2_x$ bound for $|\nabla \omega|$ and $|\nabla \omega_b|$ in the next proposition. More general
Navier type boundary conditions on smooth domains was treated earlier by several authors. The interested reader is referred to the relatively recent papers and their references therein: [40] by Xiao and Xin, [11] by G.Q. Chen and Z.M. Qian and [30] by Masmoudi and Rousset. See also the paper [11] by Abe and Seregin on ASNS in the exterior of a cylinder containing the $x_3$ axis, subject to the Navier slip condition.

From (2.2) and Korn’s inequality, one can quickly derive a $L^2_t L^2_x$ bound for $|\nabla v|$. However, this bound will depend on the index $m$ for the domain $D_m$.

The next proposition is the main result in this section. It states that Problem 2.1 has a solution in the energy space $E$ of the existence, using a method involving the contraction mapping theorem. Moreover, for the domain $D_m$ in non-smooth domains such as $\Omega$ with $m$ the $3$ axis, subject to the Navier slip condition.

The proof is done by the Galerkin method based on the study of the (linear) Stokes equation.

Proposition 2.2. Given a divergence free initial value $v_0 \in W^{2,2}(D_m)$ which satisfies the Navier slip boundary condition, Problem 2.1 has a solution in the energy space $E$ on the domain $D_m \times (0, \infty)$. Moreover, $|\nabla \omega_0| + |\nabla \omega_r| + |\nabla^2 v| + |\nabla^2 v_3| \in L^2_t L^2_x$ on $D_m \times (0, T)$ for any $T > 0$. In addition, for $\Omega \equiv \frac{\omega_0}{r}$ and $J \equiv \frac{\omega_r}{r}$,

$\int_0^T \int_{D_m} |\nabla \omega_0|^2 dx dt + \frac{1}{2} \int_0^T \int_{D_m} |\Omega(x, T)|^2 dx - \frac{1}{2} \int_0^T \int_{D_m} |\Omega(x, 0)|^2 dx = 2 \int_0^T \int_{D_m} \frac{\partial_r \Omega}{r^2} \Omega dx dt - \int_0^T \int_{D_m} \frac{\partial_r^3 \Omega}{r^3} \Omega dx dt,$

$\int_0^T \int_{D_m} |\nabla J|^2 dx dt + \frac{1}{2} \int_0^T \int_{D_m} |J(x, T)|^2 dx - \frac{1}{2} \int_0^T \int_{D_m} |J(x, 0)|^2 dx = \int_0^T \int_{D_m} \left\{ \frac{2}{r} \partial_r J + v_0 \partial_r \frac{\partial_r v_0}{r} \partial_x J - v_0 \partial_x \frac{\partial_r v_0}{r} \partial_r J \right\} dx dt.$

Proof. The proof is divided into a number of steps.

Step 1. Outline of the method.

As mentioned, the existence of solutions of Problem 2.1 in the energy space is known. The proof is done by the Galerkin method based on the study of the (linear) Stokes equation in non-smooth domains such as $D_m$. So the main task is to prove (2.3). However, since the usual proof of existence does not seem to give (2.3), we will also need an independent proof of the existence, using a method involving the contraction mapping theorem.

Let $v_0 = (v_0)_e e_r + (v_0)_0 e_\theta + (v_0)_3 e_3$ be the given initial value. Let $S$ be the subspace of the energy space $E$ for functions on $D_m \times [0, T]$, defined by

$S = \{ v = v_e e_r + v_0 e_\theta + v_3 e_3 \mid v \in E, \ \text{div} b(\cdot, t) = 0, \ \text{a.e.t.}, \ \omega_0 e_\theta = \text{curl}(v_e e_r + v_3 e_3) \in L^2_t W^{1,2}_x(D_m) \},$

equipped with the norm

$\|v\|^2_S = \|v\|^2_E + \int_0^T \int_{D_m} |\nabla \omega_0|^2 dx dt.$
We will prove that there is a time $T > 0$ such that Problem 2.1 has a solution in the closed unit ball $\overline{B_S(v_0, 2)} \subset E$, which also satisfies (2.3). Afterwards, it will be shown that this solution can be extended to all positive time.

To this end, we pick scalar functions $v_r, v_3$ such that the vector field $b \equiv v_re_r + v_3e_3 \in \overline{B_E(b(0), 1)}$ with $b(0) = (v_0)e_r + (v_0)3e_3$. The strategy is to use $b$ as given data in the equation for $v_0$. This linearized equation, with suitable boundary condition and $(v_0)_\theta$ as the initial value, determines the vector field $v_\theta e_\theta \in \overline{B_E((v_0)_\theta e_\theta, 1)}$, provided that $T$ is sufficiently small. Using these $v_r, v_\theta, v_3$ as given data in the equation for $\Omega = \omega_\theta / r$, with 0 boundary value and $\omega_\theta(0)/r$ as initial value, one finds a vector field $\tilde{\omega}_\theta e_\theta = r\omega$. Then the Biot-Savart law with a suitable boundary condition determines a vector in $\tilde{b} = \tilde{v}_re_r + \tilde{v}_3e_3 \in \overline{B_S(b(0), 1)}$. The correspondence between $b$ and $\tilde{b}$ gives rise to a contraction mapping from $\overline{B_S(b(0), 1)}$ to itself if $T$ is sufficiently small. Let us denote by $b$ the fixed point. Then $v \equiv b + v_\theta e_\theta$ is a solution to Problem 2.1. The following diagram illustrates this process:

$$b \Rightarrow v_\theta \Rightarrow \Omega \Rightarrow \tilde{\omega}_\theta = r\Omega \Rightarrow \tilde{b}.$$ 

In order to prove (2.3), we need to use crucially the property that $\tilde{\omega}_\theta$ has 0 boundary value and some nonstandard uniqueness results for solutions of elliptic and parabolic equations in rectangular domains with non-convex corners, due to the presence of $3\pi/2$ angles on the boundary of $D_m$. In the following steps, detail of the proof is carried out.

Step 2. constructing $v_\theta$.

Recall from (1.3) and the Navier slip boundary condition that $v_\theta$ is determined by the following initial boundary value problem

$$
\begin{cases}
(\Delta - \frac{1}{r^2})v_\theta - (v_re_r + v_3e_3)v_\theta - \frac{v_\theta}{r} - \partial_3v_\theta = 0, & \text{on } D_m \times (0, T];
\partial_3v_\theta = 0 & \text{on } \partial^H D_m \times (0, T],
\partial_3v_\theta = \frac{v_\theta}{r} & \text{on } \partial^D D_m \times (0, T];
\v_\theta(x, 0) = (v_\theta)_0(x), & x \in D_m.
\end{cases}
$$

(2.6)

Notice that the boundary condition for $v_\theta$ is of mixed Neumann and Robin type, which also has been around for many years. Nevertheless, it is more complicated than the Neumann condition. However the function

$$h \equiv v_\theta/r$$

satisfies the standard Neumann boundary condition, since $\partial_3h = \partial_3v_\theta/r = 0$ on $\partial^H D_m$, and $\partial_ih = (r\partial_i v_\theta - v_\theta)/r^2 = 0$ on $\partial^D D_m$. Therefore, we will first look for solutions in $E$ of the following initial boundary value problem

$$
\begin{cases}
(\Delta + \frac{2\gamma}{r^2})h - b\nabla h - \frac{2\gamma}{r}h - \partial_ih = 0, & \text{on } D_m \times (0, T];
\partial_3h = 0 & \text{on } \partial^H D_m \times (0, T];
\partial_3h = \frac{v_\theta}{r} & \text{on } \partial^D D_m \times (0, T];
h(x, 0) = h_0 = (v_\theta)_0(x)/r, & x \in D_m.
\end{cases}
$$

(2.7)

Then $v_\theta = rh$ will give us a solution to (2.6). Here $\partial_ih$ is the derivative of with respect to the exterior normal of $\partial D_m$, except at the corners. Notice that $D_m$ is bounded away from the $x_3$ axis. So the appearance of $1/r$ will not create singularity for fixed $m$.

In the absence of the lower order terms, the existence and uniqueness of solutions in the energy space $E$ for the initial Neumann problem of the heat equation in bounded piecewise smooth domains is a well known classical result. See [29] e.g. In the presence of the lower
order terms within suitable $L^p_tL^q_x$ class, the result is still true with routine modification in the proof. A typical proof usually relies on a priori energy bound and Galerkin method. Since not all lower order terms in the equation of (2.7) are in these suitable $L^p_tL^q_x$ class, we will describe the necessary modification. As mentioned in the introduction, elements in $E$ can either be scalar or vector valued, depending on the context. For (2.7), elements in $E$ are scalar valued.

In (2.7), we treat $-2v_r/r$ as a potential function for the solution $h$. It lies in the space $L^2_tL^\infty_x$ by our choice that $b \in \mathcal{B}_E(b(0), 1) \subset E$. It is well known that in spatial dimension 3, the standard class for potential functions which guaranty the existence and Hölder continuity of solutions in $E$ is $L^p_tL^q_x$ such that $3/p + 2/q < 2$. Therefore $-2v_r/r$ is in the standard class. In contrast, the standard class for the drift term is $L^p_tL^q_x$ with $3/p + 2/q < 1$. Hence the vector field $b$ in the drift term $-b\nabla h$ is not in the standard class. However, since $\text{div} \ b = 0$ in the $L^2_tL^2_x$ sense and $v_r = 0$ on $\partial^V D_m \times (0, T]$ and $v_3 = 0$ on $\partial^H D_m \times (0, T]$ in the sense of boundary trace of functions in $L^2_tW^{1,2}_x(D_m)$, one can prove that the drift term will be integrated out and standard energy estimate can be proven. With this energy estimate in hand, one can prove existence of solution $h$ in $E$ by the standard approximation argument. Namely, one can approximate $b$ and $v_r$ by a sequence of smooth vector fields $b^{(n)}$ and functions $v_r^{(n)}$. Then one solves (2.7) with $b$ replaced by $b^{(n)}$, and $v_r$ replaced by $v_r^{(n)}$ respectively to obtain a sequence of functions $\{h_n\}$ which are uniformly bounded in the norm of $E$. Then a weak limit of a subsequence of $\{h_n\}$, say $h$, is a solution to (2.7). In addition, since $||h_0||_\infty$ is finite due to $v_0 \in W^{2,2}(D_m)$, one can prove that $||h||_\infty$ on $D_m \times [0, T]$ is finite for each $T > 0$. The proof of these results are essentially given in [41]. The only difference is that one needs to replace the local space time argument by a global one on $D_m \times [0, T]$, which makes the matter simpler. The boundary terms in the integrations all vanish due to the boundary condition of $h$ and $b$. Now that we have a solution $h$ to (2.7), it is straight forward to verify that $v_0 \equiv rh$ is a solution to (2.6). In addition, by the above description of energy estimate,

\begin{align}
||v_0 - (v_0)_0||_E & \leq \alpha(T, ||v_0||_\infty, ||v_0||_{W^{1,2}}), \\
||v_0 - (v_0)_0||_\infty & \leq C + \alpha(T, ||v_0||_\infty, ||v_0||_{W^{1,2}}).
\end{align}

(2.8)

on $D_m \times [0, T]$. Here $\alpha$ is a function such that $\alpha(T, \ldots) \to 0$ as $T \to 0$ and $\alpha(T, \ldots) \leq Ce^{a_0 T}$ for $T \geq 1$ and constants $a_0 > 0$ and $C$ which may depend on $m$. Because these inequalities with the exponential bound on $\alpha$ will be used several times, we will give a sketch of the proof in the next paragraph.

Since $r$ is bounded between two positive constants in $D_m$, we can just prove a similar bound for $h$. Using $h$ as a test function in (2.7) and integrating out the drift term, we see that

\begin{align*}
\int_0^T \int_{D_m} |\nabla h|^2 dxdt + \frac{1}{2} \int_{D_m} |h(x, T)|^2 dx \\
= \frac{1}{2} \int_{D_m} |h(x, 0)|^2 dx + 2 \int_0^T \int_{D_m} \frac{\partial \hbar}{r} h dxdt - \int_0^T \int_{D_m} \frac{2v_r}{r} h^2 dxdt.
\end{align*}
As explained, the finiteness of $L^p_t L^2_x$ norm of $v_r / r$ implies $v_r / r$ subcritical as a potential function, hence, the following energy bound holds
\[ \int_0^T \int_{D_m} |\nabla h|^2 dx dt + \int_0^T \int_{D_m} |h(x, T)|^2 dx \leq \int_0^T \int_{D_m} |h(x, 0)|^2 dx + C \frac{L}{r} \int_0^T \int_{D_m} h^2 dx dt \]

Here we have used the energy inequality for $v$, which tells us the $L^p_t L^2_x$ norm of $v_r / r$ is bounded by $\|v_0\|_{L^2(D_m)}$. From here, Gronwall’s inequality infers
\[ \|h\|_E \leq C \|v_0\|_{L^2(D_m)} e^{C r T}, \quad \text{on} \quad D_m \times [0, T]. \]

This and the argument in [41] (mean value inequality with very singular divergence free drift terms) imply the 2nd inequality in (2.8) and the first one when $T \geq 1$. See also the lines around (3.14) below. To prove the first one for small $T$, we notice that $\Delta h_0$ can be regarded as an element in $W^{-1,2}(D_m)$. So the function $h - h_0$ can be regarded as an energy solution to the problem
\[
\begin{aligned}
&\{(\Delta + \frac{2}{r} \partial_r)(h - h_0) - b \nabla (h - h_0) - \frac{2v_r}{r} (h - h_0) - \partial_t (h - h_0) \\
&\quad - \Delta h_0 - \frac{2}{r} \partial_r h_0 + b \nabla h_0 + \frac{2v_r}{r} h_0, \quad \text{on} \ D_m \times (0, T] ; \\
&\partial_r (h - h_0) = 0, \quad \text{on} \ \partial D_m \times (0, T) ; \\
&h(x, 0) - h_0 = 0, \quad x \in D_m.
\end{aligned}
\]

Using $h - h_0$ as a test function on (2.9), we deduce, after integration by parts,
\[
\begin{aligned}
\int_0^T \int_{D_m} |\nabla (h - h_0)|^2 dx dt + \frac{1}{2} \int_0^T \int_{D_m} |(h - h_0)(x, T)|^2 dx \\
= 2 \int_0^T \int_{D_m} \frac{\partial_t (h - h_0)}{r} (h - h_0) dx dt - \int_0^T \int_{D_m} \frac{2v_r}{r} (h - h_0)^2 dx dt - \int_0^T \int_{D_m} \nabla h_0 \nabla (h - h_0) dx dt \\
+ \int_0^T \int_{D_m} [b \nabla (h - h_0)] h_0 dx dt + 2 \int_0^T \int_{D_m} \frac{\partial_r h_0}{r} (h - h_0) dx dt - \int_0^T \int_{D_m} \frac{2v_r}{r} h_0 (h - h_0) dx dt.
\end{aligned}
\]

Applying Cauchy Schwarz to absorb the gradient terms on the right side, we deduce
\[ \int_{D_m} |(h - h_0)(x, T)|^2 dx \leq C(m)T(\|v_0\|^2_{L^2(D_m)} + \|(v_0)\|^2_{L^p(D_m)} + \|(v_0)\|^2_{W^{1,2}(D_m)}), \]

which yields the first inequality of (2.8) when $T$ is small.

**Step 3. constructing an intermediate angular vorticity $\tilde{\omega}_\theta$.**

With the vector field $b$ from Step 1 and $v_\theta$ from Step 2, we will introduce a function
\[
\tilde{\omega}_\theta = r \Omega
\]

where $\Omega$ is the solution in $E$ of the initial boundary value problem:
\[
\begin{aligned}
&\{(\Delta + \frac{2}{r} \partial_r)\Omega - b \nabla \Omega + \frac{2v_\theta}{r^2} \partial_t v_\theta - \partial_t \Omega = 0, \quad \text{on} \ D_m \times (0, T] ; \\
&\Omega = 0, \quad \text{on} \ \partial D_m \times (0, T) ; \\
&\Omega(x, 0) = (\omega_0)_\theta(x)/r, \quad x \in D_m.
\end{aligned}
\]
Here $\omega_0 = \text{curl} \, v_0$. This is an initial Dirichlet problem of the heat equation with a nonhomogeneous term and a divergence free drift term. The advantage over the equation of $\omega_0$ is the absence of $v_r/r$ term, which will be exploited further in the next section. The existence and uniqueness of solution $\Omega$ to (2.11) in $E$ can be proven in a similar manner to that of (2.7). The key is the a priori energy estimate even with a nonstandard drift term. Since we will need to track dependence of $\Omega$ on $v_0$, let us carry out the a priori energy estimate.

If $\Omega$ is a solution of (2.11) in $E$, then it is a legal test function for the equation in (2.11). This gives

$$\int_0^T \int_{D_m} |\nabla \Omega|^2 \, dx dt + \frac{1}{2} \int_{D_m} |\Omega(x, T)|^2 \, dx = \frac{1}{2} \int_{D_m} |\Omega(x, 0)|^2 \, dx + 2 \int_0^T \int_{D_m} \frac{\partial_r \Omega}{r} \Omega \, dx dt - \int_0^T \int_{D_m} \frac{\nu_\theta^2}{r^2} \partial_3 \Omega \, dx dt.$$ 

Here the drift term is integrated out as explained in Step 2. Using the 0 boundary condition of $\Omega$ and integration by parts, we see that the 2nd integral on the right hand side is 0. Then applying Cauchy-Schwarz inequality on the last integral, we deduce

$$\int_0^T \int_{D_m} |\nabla \Omega|^2 \, dx dt + \int_{D_m} |\Omega(x, T)|^2 \, dx \leq \int_{D_m} |\Omega(x, 0)|^2 \, dx + \int_0^T \int_{D_m} \frac{\nu_\theta^4}{r^4} \, dx dt.$$ 

From (2.8), $v_0$ is a bounded function and hence the right hand side of (2.13) is finite, giving us the energy estimate. Note that $r$ is bounded away from 0 on $D_m$ and $\Omega(\cdot, 0) = \omega_0(\cdot, 0)/r \in W^{1,2}(D_m)$ by assumption. As pointed out in Step 2, existence and uniqueness of the solution to (2.11) in $E$ follows. Thus $\tilde{\omega}_\theta$ is well defined, and by direct calculation from (2.11) and (2.10), it satisfies the equation:

$$\begin{cases} 
(\Delta - \frac{1}{r^2})\tilde{\omega}_\theta - (b \nabla) \tilde{\omega}_\theta + 2 \frac{\nu_\theta}{r} \partial_3 v_\theta + \tilde{\omega}_\theta \frac{\nu_\theta}{r^2} - \partial_3 \tilde{\omega}_\theta = 0, \\
\tilde{\omega}_\theta = 0, \quad \text{on} \quad \partial D_m \times (0, T); \\
\tilde{\omega}_\theta(x, 0) = (\omega_0)_\theta(x), \quad x \in D_m.
\end{cases}$$ 

We mention that this function $\tilde{\omega}_\theta$ may not be equal to $\text{curl} \, b$ yet. In the next step, we will use $\tilde{\omega}_\theta$ to construct a vector field $\tilde{b}$ so that $\text{curl} \, \tilde{b} = \tilde{\omega}_\theta$. Eventually we will prove that the map that assigns $b$ to $\tilde{b}$ has a fixed point. For such a fixed point $\tilde{\omega}_\theta = \text{curl} \, b = \text{curl} \, \tilde{b}$.

In addition, we can prove higher integrability for $\Omega$ which will be useful later. Since $\Omega(\cdot, 0) = \omega_0(\cdot, 0)/r \in W^{1,2}(D_m)$ with 0 boundary, we can regard it as $W^{1,2}$ function in a polygon in the $r x_3$ plane, with 0 boundary value. By the two dimensional Sobolev inequality, we know $\Omega(\cdot, 0) \in L^q(D_m)$ for any $q \geq 1$. Using standard energy estimate for $\Omega^q$, using the fact that the drift terms are integrated out, we have

$$\|\Omega(\cdot, T)\|_{L^q(D_m)} \leq \hat{C}(T, q, \|v_0\|_{L^\infty}) \|\Omega(\cdot, 0)\|_{L^q(D_m)},$$

where $\hat{C}$ depends on $T$ in an exponential way: $\hat{C}(T, \ldots) \leq C e^{\alpha_0 T}$ for some constant $\alpha_0 > 0$. Here $\|v_0\|_{L^\infty}$ is on $D_m \times [0, T]$ which has at most exponential growth by (2.8). Actually, by
Moser’s iteration, one can also prove that $\Omega(\cdot, t)$ is bounded as soon as $t > 0$. See also Lemma 3.10 below.

**Step 4. Constructing a contraction map from $\mathcal{B}_S(b(0), 1)$ into itself.**

**Step 4.1. Definition of the map.**

Using the function $\tilde{\omega}_0$ in Step 3, we construct two functions $\tilde{v}_r, \tilde{v}_3 \in \mathbf{E}$ by solving respectively, for each $t \in (0, T]$, the elliptic problems in $W^{1,2}(D_m)$:

\begin{align}
&\begin{cases}
(\Delta - \frac{1}{r^2}) \tilde{v}_r = \partial_{x_3} \tilde{\omega}_0 = \partial_{x_3} (r \Omega), & \text{in } D_m, \\
\partial_{x_3} \tilde{v}_r = 0, & \text{on } \partial^H D_m; \quad \tilde{v}_r = 0, & \text{on } \partial^V D_m,
\end{cases} \\
&\begin{cases}
\Delta \tilde{v}_3 = -\left(\partial_r \tilde{\omega}_0 + \frac{\tilde{\omega}}{r}\right) = -\frac{1}{r} \partial_r (r^2 \Omega), & \text{in } D_m, \\
\tilde{v}_3 = 0, & \text{on } \partial^H D_m; \quad \partial_r \tilde{v}_3 = 0, & \text{on } \partial^V D_m.
\end{cases}
\end{align}

From Step 3, $\tilde{\omega}_0 \in \mathbf{E}$, hence the right hand side of the above two equations are in $L^2(D_m)$ for a.e. $t \in (0, T]$. For these $t$, problems (2.16) and (2.17) can be regarded as 2 dimensional elliptic problems in a polygon in the $r x_3$ plane, with mixed Dirichlet Neumann boundary values. Note that $\Delta = \partial_r^2 + \partial_{x_3}^2 + (1/r) \partial_r$. These problems have been well studied. For example, the existence and uniqueness of these solutions can be found in the Grisvard’s [15] Chapter 4, together with the following properties. For $s = 3^{-}/2$, there exists a positive constant $\hat{C}_m$, depending only on $m$ and $s$, such that, for a.e. $t$,

\begin{align}
&\|\tilde{v}_r\|_{W^{1,2}(D_m)} + \|\tilde{v}_3\|_{W^{1,2}(D_m)} \leq \hat{C}_m \|\tilde{\omega}_0\|_{L^2(D_m)}, \\
&\|\tilde{v}_r\|_{W^{2,1}(D_m)} + \|\tilde{v}_3\|_{W^{2,1}(D_m)} \leq \hat{C}_m \|\tilde{\omega}_0\|_{W^{1,2}(D_m)}.
\end{align}

In the above the arguments for all functions involved are $(\cdot, t)$. It should be pointed out that the due to the presence of nonconvex corners of angle $3\pi/2$, solutions $\tilde{v}_r$ and $\tilde{v}_3$ are not in $W^{2,2}(D_m)$ in general; and the exponent $s = 3^{-}/2$ is determined by this angle. Another remark is that the lower order coefficient $-1/r^2$ in (2.16) has a good sign that does not affect the solvability of the problem. In addition, by (2.15) with say $q = 5$ and Moser’s iteration on the whole domain $D_m$ for the equations (2.16) and (2.17), we deduce

\begin{align}
&\|\tilde{v}_r\|_{L^\infty(D_m)} + \|\tilde{v}_3\|_{L^\infty(D_m)} \leq \hat{C}_m \hat{C} e^{\alpha_1 T}.
\end{align}

Here $\hat{C}$ depends only on the initial value.

Since $\Omega \in \mathbf{E}$ by (2.13), we know $\omega_0 \in \mathbf{E}$ and hence, by (2.18), the vector field

\begin{align}
\tilde{b} &\equiv \tilde{v}_r e_r + \tilde{v}_3 e_3
\end{align}

also lies in $\mathbf{E} \cap L_t^\infty W^{1,2}(D_m)$. Now we define a map $\mathbb{L}$ from $\mathcal{B}_S(b(0), 1)$ into $\mathbf{E}$ by

\begin{align}
\mathbb{L} \tilde{b} = \tilde{b}.
\end{align}

We must first prove $\text{div} \tilde{b} = 0$ a.e. $t$. Recall that $\text{div} \tilde{b} = \partial_r \tilde{v}_r + \tilde{v}_r/r + \partial_{x_3} \tilde{v}_3$. By direct computation from (2.16),

\begin{align}
&\left(\Delta - \frac{2}{r^2}\right) \partial_r \tilde{v}_r + \frac{2}{r^2} \tilde{v}_r = \partial_r \partial_{x_3} \tilde{\omega}_0, \\
&\Delta \left(\frac{\tilde{v}_r}{r}\right) + \frac{2}{r^2} \partial_r \tilde{v}_r - \frac{2}{r^3} \tilde{v}_r = \partial_{x_3} \left(\frac{\tilde{\omega}_0}{r}\right).
\end{align}
Adding these two equations and (2.17), we find, in pointwise sense,

\[
\begin{aligned}
\Delta \text{div} \tilde{b} &= 0, \quad \text{in} \quad D_m, \\
\partial_n \text{div} \tilde{b} &= 0, \quad \text{on} \quad \partial D_m, \quad \text{except at corners},
\end{aligned}
\]

where \( n \) is the unit outward normal of \( \partial D_m \) except at corners. The Neumann boundary condition is a consequence of the boundary condition of \( \tilde{b} \), (2.16) and (2.17). Indeed, on \( \partial^H D_m \) except for corners, we have

\[
\partial_{x_3} \text{div} \tilde{b} = \partial_r \partial_{x_3} \tilde{v}_r + \frac{1}{r} \partial_s \partial_{x_3} \tilde{v}_r + \partial^2_{x_3} \tilde{v}_3 = \partial^2_{x_3} \tilde{v}_3
\]

\[
= -\partial_r^3 \tilde{v}_3 - \frac{1}{r} \partial_r \tilde{v}_3 - \left( \partial_r \omega_0 + \frac{\bar{\omega}_0}{r} \right), \quad \text{by} \quad (2.17),
\]

\[
= 0, \quad \text{by boundary condition of} \quad \tilde{v}_3, \quad \omega_0 = 0.
\]

On \( \partial^H D_m \) except for corners, we have

\[
\partial_r \partial_{x_3} \text{div} \tilde{b} = \partial^2_r \tilde{v}_r + \frac{1}{r} \partial_r \tilde{v}_r - \frac{1}{r^2} \tilde{v}_r + \partial_r \partial_{x_3} \tilde{v}_3
\]

\[
= \partial^2_r \tilde{v}_r + \frac{1}{r} \partial_r \tilde{v}_r - \frac{1}{r^2} \tilde{v}_r + \partial^2_{x_3} \tilde{v}_r \quad \text{by} \quad \tilde{v}_r = 0, \quad \partial_r \tilde{v}_3 = 0,
\]

\[
= \partial_{x_3} \omega_0, \quad \text{by} \quad (2.16),
\]

\[
= 0, \quad \text{by} \quad \bar{\omega}_0 = 0.
\]

Using \( \text{div} b \in W^{1,2}(D_m) \) with \( s = 3^−/2 \) from (2.18), we work on (2.22) to show that \( \text{div} \tilde{b} \) is identically 0. Let us mention that the uniqueness result is not a standard one since \( \text{div} \tilde{b} \) is not known to be in the energy space \( W^{1,2}(D_m) \) at the moment. Actually we are not able to prove the uniqueness of the Neumann problem for \( W^{1,s} \) solutions directly. For \((r, x_3)\) in the interior of \( D_m \), consider the function

\[
\begin{aligned}
f &= f(r, x_3) = \int_0^{x_3} \text{div} \tilde{b}(r, y_3)dy_3 = \int_0^{x_3} \left[ \partial_r \tilde{v}_r + \tilde{v}_r/r + \partial_{y_3} \tilde{v}_3 \right](r, y_3)dy_3 \\
&= \int_0^{x_3} \left[ \partial_r \tilde{v}_r + \tilde{v}_r/r \right](r, y_3)dy_3 + \tilde{v}_3(r, x_3).
\end{aligned}
\]

which is well defined and smooth since the line segments avoids possible singularity. Hence

\[
\partial_r f = \int_0^{x_3} \left[ -\partial^2_{y_3} \tilde{v}_r + \partial_{y_3} \bar{\omega}_0 \right](r, y_3)dy_3 + \partial_r \tilde{v}_3(r, x_3),
\]

which infers, from equation (2.16), that

\[
\begin{aligned}
\partial_r f &= \int_0^{x_3} \left[ -\partial^2_{y_3} \tilde{v}_r + \partial_{y_3} \bar{\omega}_0 \right](r, y_3)dy_3 + \partial_r \tilde{v}_3(r, x_3) = \left( -\partial_{x_3} \tilde{v}_r + \bar{\omega}_0 + \partial_r \tilde{v}_3 \right)(r, x_3).
\end{aligned}
\]

Here we have used the boundary conditions on \( \partial^H D_m \). On the other hand, on interior points of \( D_m \), we can differentiate equations (2.16) and (2.17) suitably to reach

\[
\begin{aligned}
\left\{ \begin{array}{l}
(\Delta - \frac{1}{r^2}) \left( \partial_{x_3} \tilde{v}_r - \partial_r \tilde{v}_3 - \bar{\omega}_0 \right) = 0, \quad \text{in} \quad D_m^0, \\
\partial_{x_3} \tilde{v}_r - \partial_r \tilde{v}_3 - \bar{\omega}_0 = 0 \quad \text{on} \quad \partial D_m,
\end{array} \right.
\end{aligned}
\]
except at corners. Notice that \( \partial_{x_3} \tilde{v}_r - \partial_r \tilde{v}_3 - \tilde{\omega}_\theta \in W^{1,s}(D_m) \) with \( s = 3^-/2 \). By the uniqueness result in Proposition 4.1, we find

\[
\tilde{\omega}_\theta = \partial_{x_3} \tilde{v}_r - \partial_r \tilde{v}_3.
\]

Substituting this to (2.23), we infer in the interior points of \( D_m \),

\[
(2.24) \quad \partial_r f(r, x_3) = 0.
\]

For these points, we also know from (2.22) that

\[
\Delta f = \partial_r^2 f + \frac{1}{r} \partial_r f + \partial_{x_3}^2 f
\]

\[
= \int_0^{x_3} (\partial_r^2 + \frac{1}{r} \partial_r) \text{div} \tilde{b} \, dy_3 + \partial_{x_3} \text{div} \tilde{b}
\]

\[
= - \int_0^{x_3} \partial_{y_3} \text{div} \tilde{b} \, dy_3 + \partial_{x_3} \text{div} \tilde{b}
\]

\[
= 0, \text{ due to } \partial_{x_3} \text{div} \tilde{b}(r, 0) = 0.
\]

Since \( f = f(x_3) \) by (2.24), this shows \( \partial_{x_3}^2 f = 0 \) so that \( f = cx_3 \).

Next we observe that for \( W^{2,s} \) solutions to (2.16), we have

\[
\int_{D_m \cap \{ r = \text{const.} \}} \tilde{v}_r(r, x_3, t) \, dx_3 = 0.
\]

This result, which only depends on the \( W^{2,s} \) property of \( \tilde{v}_r(\cdot, t) \), is stated as Lemma 2.3 below. To avoid interrupting the flow of the presentation, we will prove the lemma till the end of the section. This shows that \( f = 0 \) and hence \( \text{div} \tilde{b} = \partial_{x_3} f = 0 \).

In the next few sub-steps, we will prove that \( \mathbb{L} \) maps \( \overline{B}_8(b(0), 1) \) into itself and it is a contraction, provided \( T \) is sufficiently small.

**Step 4.2.** We prove \( \mathbb{L} \) is a contraction.

Let \( b^{(i)} = v^{(i)}_r e_r + v^{(i)}_3 e_3 \in \overline{B}_{\mathbb{E}}(b(0), 1) \), \( i = 1, 2 \), be two given vector fields with the same initial value \( b_0 \). Then they determine, by Step 2, \( v^{(i)}_\theta \) and henceforth \( \Omega^{(i)} \) by Step 3. Therefore \( \Omega^{(2)} - \Omega^{(1)} \) satisfies

\[
\begin{cases}
(\Delta + \frac{2}{r} \partial_r)(\Omega^{(2)} - \Omega^{(1)}) - b^{(2)} \nabla(\Omega^{(2)} - \Omega^{(1)}) - (b^{(2)} - b^{(1)}) \nabla \Omega^{(1)} \\
+ \frac{1}{r^2} \partial_{x_3} \left((v^{(2)}_\theta)^2 - (v^{(1)}_\theta)^2\right) - \partial_r(\Omega^{(2)} - \Omega^{(1)}) = 0; \\
(\Omega^{(2)} - \Omega^{(1)}) = 0, \text{ on } \partial D_m \times (0, T]; \\
(\Omega^{(2)} - \Omega^{(1)})(x, 0) = 0, \quad x \in D_m.
\end{cases}
\]  

(2.25)
Then along the same way as deriving (2.13), we arrive at (2.26)
\[
\int_0^T \int_{D_m} |\nabla(\Omega^2) - \Omega^{(1)}|^2 \, dxdt + \sup_{t \in [0, T]} \int_{D_m} |\Omega^2 - \Omega^{(1)}|^2(x, t) \, dx \\
\leq \int_0^T \int_{D_m} |(v^{(2)}_\theta)^2 - (v^{(1)}_\theta)^2|^2 \, r^{-4} \, dxdt - 2 \int_0^T \int_{D_m} (b^{(2)} - b^{(1)}) \nabla \Omega^{(1)} (\Omega^2 - \Omega^{(1)}) \, dxdt \\
= \int_0^T \int_{D_m} |(v^{(2)}_\theta)^2 - (v^{(1)}_\theta)^2|^2 \, r^{-4} \, dxdt + 2 \int_0^T \int_{D_m} (b^{(2)} - b^{(1)}) \Omega^{(1)} \nabla(\Omega^2 - \Omega^{(1)}) \, dxdt.
\]

Here we have used the divergence free property of \(b^{(i)}\) and integration by parts. Using Cauchy Schwarz inequality, we find (2.27)
\[
\int_0^T \int_{D_m} |\nabla(\Omega^2) - \Omega^{(1)}|^2 \, dxdt + 2 \sup_{t \in [0, T]} \int_{D_m} |\Omega^2 - \Omega^{(1)}|^2(x, t) \, dx \\
\leq 2 \int_0^T \int_{D_m} |(v^{(2)}_\theta)^2 - (v^{(1)}_\theta)^2|^2 \, r^{-4} \, dxdt + 4 \int_0^T \int_{D_m} (b^{(2)} - b^{(1)})^2 (\Omega^{(1)})^2 \, dxdt \\
\leq 2 \int_0^T \int_{D_m} |(v^{(2)}_\theta)^2 - (v^{(1)}_\theta)^2|^2 \, r^{-4} \, dxdt + 4\|b^{(2)} - b^{(1)}\|^2_{L^1(\Omega)} \|\Omega^{(1)}\|^2_{L^2} \quad \text{(by Hölder inequality)}.
\]

From this, Sobolev inequality and interpolation, we deduce, after using (2.15) with \(q = 5\), (2.28)
\[
\|\Omega^2 - \Omega^{(1)}\|^2_E \leq C \int_0^T \int_{D_m} |(v^{(2)}_\theta)^2 - (v^{(1)}_\theta)^2|^2 \, r^{-4} \, dxdt + C C^2(T, \|v_\theta\|_{L^\infty}) \|\Omega^2 \|_{L^2(D_m)}^2 T^{2/5} \|b^{(2)} - b^{(1)}\|^2_E
\]

Write \(\tilde{\omega}^{(i)} = r\Omega^{(i)}\), and \(\tilde{b}^{(i)} = \tilde{v}^{(i)} e_r + \tilde{v}^{(i)}_3 e_3\), \(i = 1, 2\), where \(\tilde{v}^{(i)}\) and \(\tilde{v}^{(i)}_3\) are determined respectively from (2.16) and (2.17) with \(\tilde{\omega}^{(i)}\) replaced by \(\tilde{\omega}^{(i)}\). Then, just like (2.18), we have
\[
\|\tilde{\omega}^{(2)} - \tilde{\omega}^{(1)}\|_E = \|\tilde{b}^{(2)} - \tilde{b}^{(1)}\|_E \leq C \tilde{m}(1 + T) \sup_{t \in [0, T]} \|\tilde{\omega}^{(2)} - \tilde{\omega}^{(1)}\|_{L^2(D_m)} \\
\leq C \tilde{m}(1 + T) \sup_{t \in [0, T]} \|\Omega^2 - \Omega^{(1)}\|_{L^2(D_m)} \\
\leq C C(T, \|v_{\theta}\|_{L^\infty}) \|\Omega^2 \|_{L^2(D_m)} T^{1/5} \|b^{(2)} - b^{(1)}\|_E + C \tilde{m}(1 + T) \left( \int_0^T \int_{D_m} |(v^{(2)}_{\theta})^2 - (v^{(1)}_{\theta})^2|^2 \, r^{-4} \, dxdt \right)^{1/2}.
\]

Therefore, (2.30)
\[
\|\tilde{b}^{(2)} - \tilde{b}^{(1)}\|_E \leq C C(T, \|v_{\theta}\|_{L^\infty}) \|\Omega^2 \|_{L^2(D_m)} T^{1/5} \|b^{(2)} - b^{(1)}\|_E + C C \tilde{m}(1 + T) \left[ \|v_{\theta}\|_E + o(T, \|v_{\theta}\|_E, \|v_{\theta}\|_{L^{2/2}}) \right] \left( \int_0^T \int_{D_m} |(v^{(2)}_{\theta})^2 - (v^{(1)}_{\theta})^2|^2 \, r^{-4} \, dxdt \right)^{1/2}.
\]

Here we just used (2.28) and (2.8) in the last lines. To bound the last integral, we write \(h_i = v^{(i)}_{\theta}/r\), \(i = 1, 2\). From (2.7), we see that the function \(h_2 - h_1\) satisfies the initial boundary
value problem

\[
(\Delta + \frac{\partial_t}{T}) (h_2 - h_1) - b^{(2)} \nabla (h_2 - h_1) - (b^{(2)} - b^{(1)}) \nabla h_1 - \frac{2\nu_r}{r} (h_2 - h_1) - 2\nu_r (h_2 - h_1) - \nabla h_1 = 0, \quad \text{on} \quad D_m \times (0, T];
\]
\[
\partial_n (h_2 - h_1) = 0, \quad \text{on} \quad \partial D_m \times (0, T];
\]
\[
(h_2 - h_1) (x, 0) = 0, \quad x \in D_m.
\]

Using \( h_2 - h_1 \) as a test function in the preceding equation, we see that

\[
\int_0^T \int_{D_m} |\nabla (h_2 - h_1)|^2 dxdt + \frac{1}{2} \int_{D_m} (h_2 - h_1)^2 (x, T) dx = \int_0^T \int_{D_m} 2 \nu_r (h_2 - h_1)^2 dxdt - \int_0^T \int_{D_m} (b^{(2)} - b^{(1)}) \nabla h_1 (h_2 - h_1) dxdt
\]
\[
- \int_0^T \int_{D_m} \frac{2\nu_r}{r} (h_2 - h_1)^2 dxdt - \int_0^T \int_{D_m} \frac{2\nu_r}{r} h_1 (h_2 - h_1) dxdt
\]
\[
\equiv I_1 + I_2 + I_3 + I_4.
\]

Next we find suitable bounds for \( I_1, ..., I_4 \). For simplicity we denote \( D_m \times [0, T] \) by \( Q_T \) here.

By Cauchy Schwarz inequality

\[
|I_1| \leq (1/4) \| \nabla (h_2 - h_1) \|_{L^2(Q_T)}^2 + C(m) \| h_2 - h_1 \|_{L^2(Q_T)}^2;
\]

Using integration by parts and divergence free property,

\[
|I_2| = \left| \int_0^T \int_{D_m} \frac{1}{2} \partial_r (h_2 - h_1) (h_2 - h_1) \nabla h_1 (h_2 - h_1) dxdt \right|
\]
\[
\leq (1/4) \| \nabla (h_2 - h_1) \|_{L^2(Q_T)}^2 + C \| h_1 \|_{L^{\infty}(Q_T)} T \| b^{(2)} - b^{(1)} \|_{E}^2;
\]

By Hölder inequality, Sobolev inequality and interpolation,

\[
|I_3| \leq C(m) \| \nu_r \|_{L^{10/3}(Q_T)} \left( \int_0^T \int_{D_m} |h_2 - h_1|^{10/3} dxdt \right)^{3/5} (T |D_m|)^{1/10}
\]
\[
\leq CC(m) T^{1/10} \| \nu_r \|_{E} \| h_2 - h_1 \|_{E}^2;
\]

\[
|I_4| \leq C(m) \| h_2 - h_1 \|_{L^2(Q_T)}^2 + CT \| h_1 \|_{L^{\infty}(Q_T)}^2 \| b^{(2)} - b^{(1)} \|_{E}^2.
\]

Substituting (2.33), ..., (2.36) into (2.32), we find, after observing that \( I_3 \) can be absorbed by the right hand side of (2.32) when \( T \) is sufficiently small, that

\[
\int_{D_m} (h_2 - h_1)^2 (x, T) dx
\]
\[
\leq CC(m) \int_0^T \int_{D_m} |h_2 - h_1|^2 dxdt + C \| h_1 \|_{L^{\infty}(Q_T)}^2 T \| b^{(2)} - b^{(1)} \|_{E}^2.
\]
By the usual trick, this inequality actually holds when $T$ is replaced by any time $T' \in [0, T]$. So Gronwall’s inequality then infers, for these $T'$,

\[ (2.37) \quad \int_{D_m} |v_{\theta}^{(2)} - v_{\theta}^{(1)}|^2 (x, T')r^{-2}dx = \int_{D_m} (\bar{h}_2 - \bar{h}_1)^2(x, T')dx \leq \bar{C}(T, m, ||v_0||_{W^{2,2}(D_m)})T||b^{(2)} - b^{(1)}||^2_E. \]

Here the constant $\bar{C}$ is bounded when $T, m$ are bounded. Substituting this to the last term on the right hand side of (2.30), we find that

\[ ||Lb^{(2)} - Lb^{(1)}||_E \leq 0.5||b^{(2)} - b^{(1)}||_E \]

if $T$ is sufficiently small. Using (2.28) and (2.37), we also deduce

\[ ||\dot{\omega}_\theta^{(2)} - \dot{\omega}_\theta^{(1)}||^2_E \leq C\bar{C}_mC^2(T, ||v_0||_{L^\infty})||\Omega(\cdot, 0)||_{L^2(D_m)}^2T^{2/5}||b^{(2)} - b^{(1)}||^2_E. \]

Combining the last two inequalities, we conclude

\[ ||Lb^{(2)} - Lb^{(1)}||_E \leq 0.75||b^{(2)} - b^{(1)}||_E \leq 0.75||b^{(2)} - b^{(1)}||_S. \]

Therefore the map $L$ is a contraction under the norm of $S$ if $T$ is sufficiently small.

**Step 4.3.** We prove $L$ maps $B_S(b(0), 1)$ into itself if $T$ is sufficiently small.

Pick $b \in B_S(b(0), 1)$, then $Lb = \tilde{b} = \tilde{v}_r e_r + \tilde{v}_3 e_3$, where $\tilde{v}_r$ and $\tilde{v}_3$ are given by (2.16) and (2.17) respectively.

Let $\Omega_0 \equiv (\omega_0)/r$ where $\omega_0 = curl v_0$. By (2.11), $\Omega - \Omega_0$ satisfies, in the energy space,

\[ (\Delta + \frac{2}{r} \partial_r)(\Omega - \Omega_0) - b \nabla(\Omega - \Omega_0) + \frac{2\nu}{r^2} \partial_{x_3}v_0 - \partial_t(\Omega - \Omega_0) \]

\[ = -\left( \Delta + \frac{2}{r} \partial_r \right) \Omega_0 + b\nabla\Omega_0, \quad \text{on} \quad D_m \times (0, T]; \]

\[ (\Omega - \Omega_0) = 0, \quad \text{on} \quad \partial D_m \times (0, T]; \]

\[ (\Omega - \Omega_0)(x, 0) = 0, \quad x \in D_m. \]

Here $\Delta \Omega_0$ is regarded as an element in $H_0^{-1}(D_m)$. Using $\Omega - \Omega_0$ as a test function in the above equation, we obtain, in a similar manner as (2.13),

\[ \int_0^T \int_{D_m} |\nabla(\Omega - \Omega_0)|^2 dx dt + \int_{D_m} |(\Omega - \Omega_0)(x, T)|^2 dx \]

\[ \leq \int_0^T \int_{D_m} \frac{v^4}{r^3} dx dt + 2 \int_0^T \int_{D_m} \left[ \left( \Delta + \frac{2}{r} \partial_r \right) \Omega_0 - b \nabla \Omega_0 \right] (\Omega - \Omega_0) dx dt. \]

This implies, after integration by parts, that

\[ \int_0^T \int_{D_m} |\nabla(\Omega - \Omega_0)|^2 dx dt + \int_{D_m} |(\Omega - \Omega_0)(x, T)|^2 dx \]

\[ \leq \int_0^T \int_{D_m} \frac{v^4}{r^3} dx dt - 4 \int_0^T \int_{D_m} \nabla \Omega_0 \nabla (\Omega - \Omega_0) dx dt \]

\[ - 4 \int_0^T \int_{D_m} \frac{\Omega_0}{r} \partial_r(\Omega - \Omega_0) dx dt + 2 \int_0^T \int_{D_m} b\Omega_0 \nabla(\Omega - \Omega_0) dx dt, \]
which infers, via Cauchy Schwarz inequality, that
\[
\int_0^T \int_{D_m} |\nabla (\Omega - \Omega_0)|^2 dx dt + \int_{D_m} (\Omega - \Omega_0)(x, T)^2 dx \\
\leq 64 \int_0^T \int_{D_m} \vartheta^4 dx dt + 64 \int_0^T \int_{D_m} \left( |\nabla \Omega_0|^2 + \Omega_0^2 r^{-2} \right) dx dt + 64 \int_0^T \int_{D_m} |b|^2 \Omega_0^2 dx dt.
\]
By Hölder, Sobolev inequality and interpolation
\[
\int_0^T \int_{D_m} |b|^2 \Omega_0^2 dx dt \leq \|b\|_{L^{10/3}(Q_T)}^2 \left( \int_0^T \int_{D_m} \Omega_0^5 dx dt \right)^{2/5} \leq C_m T^{2/5} \|b\|_E^2 \|v_0\|^2_{W^{2,2}}.
\]
Combining the previous two inequalities and using (2.8) and the initial condition, we deduce
\[
(2.39) \quad \int_0^T \int_{D_m} |\nabla (\Omega - \Omega_0)|^2 dx dt + \sup_{t \in [0, T]} \int_{D_m} (\Omega - \Omega_0)(x, T)^2 dx \leq C_m T^{2/5} \bar{C}(T, \|v_0\|^2_{W^{2,2}}).
\]

According to the definition in (2.21), \( \mathbb{L} b = \tilde{b} = \tilde{v}_r e_r + \tilde{v}_3 e_3 \), where \( \tilde{v}_r \) and \( \tilde{v}_3 \) are given by (2.16) and (2.17) respectively. Consequently \( \tilde{v}_r - (v_0)_r \) and \( \tilde{v}_3 - (v_0)_3 \) satisfy
\[
(2.40) \quad \begin{cases} 
\left( \Delta - \frac{1}{\varrho^2} \right)(\tilde{v}_r - (v_0)_r) = \partial_{x_3}(\tilde{\omega}_\theta - (\omega_0)_\theta) = \partial_{x_3}(r(\Omega - \Omega_0)), & \text{on } D_m, \\
\partial_{x_3}(\tilde{v}_r - (v_0)_r) = 0, & \text{on } \partial^H D_m; \\
\tilde{v}_r - (v_0)_r = 0, & \text{on } \partial^V D_m,
\end{cases}
\]
and
\[
(2.41) \quad \begin{cases} 
\Delta(\tilde{v}_3 - (v_0)_3) = -\left( \partial_r[\tilde{\omega}_\theta - (\omega_0)_\theta] + \frac{\tilde{\omega}_\theta - (\omega_0)_\theta}{r} \right) = -\frac{1}{r} \partial_r(r^2(\Omega - \Omega_0)), & \text{on } D_m, \\
\tilde{v}_3 - (v_0)_3 = 0, & \text{on } \partial^H D_m; \\
\partial_r(\tilde{v}_3 - (v_0)_3) = 0, & \text{on } \partial^V D_m.
\end{cases}
\]
Using (2.18), we obtain, for all \( t \in [0, T] \),
\[
||\tilde{v}_r - (v_0)_r||_{W^{1,2}(D_m)} + ||\tilde{v}_3 - (v_0)_3||_{W^{1,2}(D_m)} \leq \bar{C}_m \|\omega_\theta - (\omega_0)_\theta\|_{L^2(D_m)}.
\]
This and (2.39) yield
\[
||\tilde{b} - b(0)||_8 \leq C_m T^{2/5} \bar{C}(T, \|v_0\|^2_{W^{2,2}}).
\]
This proves \( \mathbb{L} \) maps \( B_8(b(0), 1) \) into itself if \( T \) is sufficiently small.

Now the contraction mapping theorem tells us that the map \( \mathbb{L} \) has a unique fixed point \( b = v_r e_r + v_3 e_3 \in B_8(b(0), 1) \). Namely \( b = \mathbb{L} b = \mathbb{L} b \), i.e.
\[
(2.42) \quad v_r = \tilde{v}_r, \quad v_3 = \tilde{v}_3.
\]
Let \( v_0 \) be given by (2.6). Now we can claim that
\[
v = v_r e_r + v_\theta e_\theta + v_3 e_3
\]
is a solution to Problem (2.1) such that \( |\nabla \omega_\theta| \in L^2(D_m \times [0, T]) \), where \( \omega_\theta = (\text{curl } v)_\theta \). The reason is the following. Let \( \tilde{\omega}_\theta \) be given by (2.14). Then, due to (2.42), \( v_r \) and \( v_3 \) are given by (2.16) and (2.17) respectively. In vector form, these two equations can be written as
\[
\Delta b = -\text{curl } (\tilde{\omega}_\theta e_\theta).
\]
On the other hand, by the above definition of $\omega_\theta$ and cylindrical curl formula, we actually have $\omega_\theta e_\theta = \text{curl} \, b$. Therefore
\[
\Delta b = -\text{curl} \, (\omega_\theta e_\theta).
\]
Subtraction of the last two equations gives us $\text{curl} \, [(\tilde{\omega}_\theta - \omega_\theta) e_\theta] = 0$. Therefore
\[
\Delta [(\tilde{\omega}_\theta - \omega_\theta) e_\theta] = 0.
\]
By our boundary condition $\omega_\theta = \partial_3 v_r - \partial_r v_3 = 0$ on $\partial D_m$ and by construction $\tilde{\omega}_\theta = 0$ on $\partial D_m$. Since $b \in \mathbf{S}$, we know that $\tilde{\omega}_\theta(-,t)$ and $\omega_\theta(-,t)$ are in $W^{1,2}(D_m)$ for a.e. $t$. Hence, by standard uniqueness result of solutions of Laplace equation in $W^{1,2}$ space, we have $\tilde{\omega}_\theta(-,t) = \omega_\theta(-,t)$, a.e. $t$. Now interior regularity of solutions tells us:
\[
(2.43) \quad \omega_\theta = \tilde{\omega}_\theta \quad \text{in} \quad D_m \times [0,T].
\]
By (2.13) and $\tilde{\omega}_\theta = r\Omega$, we know that $|\nabla \omega_\theta| \in L^2(D_m \times [0,T])$. Finally, by (2.14) and (2.43), we see that $\omega_\theta$ satisfies
\[
(\Delta - \frac{1}{r})\omega_\theta - (b\nabla)\omega_\theta + 2\frac{v_\theta}{r} \partial_3 v_\theta + \frac{\omega_\theta}{r} \partial_r - \partial_r \omega_\theta = 0,
\]
(2.44)
\[
(\omega_\theta = 0, \quad \text{on} \quad \partial D_m \times (0,T];
\]
\[
(\omega_\theta(x,0) = (\omega_\theta)_{\theta}x), \quad x \in D_m.
\]
Converting the above equation into the vector equation for $\omega_\theta e_\theta = \text{curl} \, b$, we check, by vector calculus identity in the cylindrical system and (2.44)
\[
(2.45) \quad \text{curl} \, [(\Delta b - b\nabla b + \frac{v_\theta^2}{r} e_r - \partial_r b) = 0, \quad \text{pointwise in} \quad D_m \times (0,T],
\]
where $b\nabla b = (v_r \partial_r v_r + v_3 \partial_3 v_r)e_r + (v_r \partial_r v_r + v_3 \partial_3 v_r)v_3 e_3$. Since $D_m$ can be regarded as a simply connected domain in $r x_3$ plane, the term inside the brackets of (2.45) is a gradient field. Hence we know that $v_r$ and $v_3$ satisfy their respective equations in (1.3). Note also that $v_\theta$ satisfies its corresponding equation in (1.3) by construction, c.f. (2.6). This proves the claim, namely $v$ is a solution to Problem 2.1 in $\mathbf{S}$ on the time interval $(0,T]$. Due to the bounds (2.8), (2.15) and (2.19), this fixed point argument can be continued indefinitely. Indeed, choose any moment $t_0 > 0$ such that a solution already exists. We can repeat the above fixed point argument starting from $t_0$. From the a priori bounds (2.8), (2.15) with $q = 5$, and (2.29) and (2.28) with initial time 0 replaced by $t_0$, we see that the fixed point argument works at least in the interval $[t_0, t_0 + \epsilon e^{-2\alpha_0 t_0}]$. Here $\alpha_0 > 0$ and $\epsilon > 0$ are constants which are independent of $t_0$. Thus we have proven the existence of solutions to Problem 2.1 in $\mathbf{S}$ for all time. Due to the definition of $\mathbf{S}$, we have also proven $|\nabla \omega_\theta| \in L^2_t L^2_x$. It remains to prove $|\nabla \partial_3 v_r| + |\nabla \omega_r| \in L^2_t L^2_x$.

Step 5. $|\nabla \partial_3 v_r| \in L^2_t L^2_x$.

According to (2.18), for a.e. $t$, we have $v_r(-,t) \in W^{2,s}(D_m)$, $\partial_3 \omega_\theta(-,t) \in L^2(D_m)$ with $s = 3^{-}/2$. From here, in this step, we will always work on these time $t$, and for simplicity, will suppress the time variable $t$ unless there is a confusion. By (2.16), $\partial_3 v_r$ can be regarded as a $W^{1,s}$ solution to the problem:
\[
(2.46) \quad \begin{cases}
(\Delta - \frac{1}{r}) \partial_3 v_r = \partial^2_{x_3} \omega_\theta, & \text{on} \quad D_m, \\
\partial_3 v_r = 0, & \text{on} \quad \partial^3 D_m; \\
\partial_3 v_r = 0, & \text{on} \quad \partial^5 D_m.
\end{cases}
\]
Notice that the boundary value for $\partial_{s_3} v_r$ is actually 0. On the other hand, since $\partial_{s_3} \omega_\theta \in L^2(D_m)$, we can find a $W^{1,2}$ solution to the following problem.

\[(2.47) \begin{cases} \left( \Delta - \frac{1}{r^2} \right) f = \partial^2_{s_3} \omega_\theta, & \text{on } D_m, \\ f = 0, & \text{on } \partial D_m. \end{cases} \]

The existence and uniqueness of $W^{1,2}$ solutions to (2.47) is standard if the right hand side of the equation is in $L^2(D_m)$. See Chapter 4 of [15] e.g. In our case, although the right hand side is only in $H^{-1}$, the result still holds and the proof is more or less the same. But for completeness, we will give a quick proof by an approximating procedure. First we construct a sequence of functions $g_j \in W^{1,2}_0(D_m)$ such that $g_j \to \partial_{s_3} \omega_\theta$ in $L^2(D_m)$. Now that $\partial_{s_3} g_j$ is in $L^2(D_m)$, the standard theory mentioned above states that the problem

\[(2.48) \begin{cases} \left( \Delta - \frac{1}{r^2} \right) f_j = \partial_{s_3} g_j, & \text{on } D_m, \\ f_j = 0, & \text{on } \partial D_m. \end{cases} \]

has an unique solution in $W^{1,2}_0(D_m)$. Using $f_j$ as a test function in the above equation, we obtain

$$\int_{D_m} |\nabla f_j|^2 dx + \int_{D_m} r^{-2} f_j^2 dx = \int_{D_m} g_j \partial_{s_3} f_j dx,$$

which yields the uniform energy bound

$$\int_{D_m} |\nabla f_j|^2 dx + \int_{D_m} r^{-2} f_j^2 dx \leq 2 \int_{D_m} g_j^2 \leq 2 \int_{D_m} |\partial_{s_3} \omega_\theta|^2 dx + C.$$

Therefore, a subsequence of $\{f_j\}$ converges weakly in $W^{1,2}$ to a $W^{1,2}_0$ function $f$. Using a $C^1_0(D_m)$ test function, it is easy to see that this $f$ is a solution to (2.47).

So $\partial_{s_3} v - f \in W^{1,2}_0(D_m)$ is a solution to the homogeneous problem

\[(2.49) \begin{cases} \left( \Delta - \frac{1}{r^2} \right) (\partial_{s_3} v - f) = 0, & \text{on } D_m, \\ \partial_{s_3} v - f = 0, & \text{on } \partial D_m. \end{cases} \]

According to Proposition 4.1 below, uniqueness holds for the above problem. Thus we have proven $\partial_{s_3} v_r = f \in W^{1,2}_0(D_m)$, as stated. We comment that, since the solution space $W^{1,4}_0(D_m)$ with $s = 3\pi/2$ is not the standard energy space and $\partial D_m$ has corners with $3\pi/2$ angle, more efforts are needed for the proof of the uniqueness. To avoid interrupting the flow of the main argument, we have moved this to Section 4.

**Step 6.** $|\nabla \partial_{s_3} v_\theta| \in L^2(D) \text{ or } |\nabla J| = | \nabla \partial_{s_3}(v_\theta/r)| \in L^2(D)$.

The proof is based on a weak-strong uniqueness argument. It is convenient to consider the function $J = \frac{\omega_r}{r} = -\frac{\partial_{s_3} \omega_r}{r}$, which satisfies the following equation pointwise:

\[(2.50) \begin{cases} \Delta J - (b \cdot \nabla) J + \frac{2}{r} \partial_r J + (\omega_r \partial_r + \omega_3 \partial_{s_3}) \frac{v_\theta}{r} - \partial_t J = 0, & \text{on } D_m \times [0, T] \\ J = 0, & \text{on } \partial^H D_m \times [0, T]; \\ \partial_t J = 0, & \text{on } \partial^V D_m \times [0, T] \\ J(x, 0) = -\partial_{s_3} v_\theta(x, 0)/r. \end{cases} \]
This equation can be derived from the vorticity equation for $\omega_r$ and will also play an important role in the next section. Note the boundary condition for $J$ is a homogeneous mixed Dirichlet-Neumann condition. The rest of the step is divided into 3 sub-steps.

**Step 6.1.** Here we prove the following assertion: the Laplace transform of $J(x, \cdot)$ is well defined if the real part of the phase variable $\lambda$ is sufficiently large, and it is in $W^{1,2}(D_m)$ with $s = 3^{-2}/2$ if $\lambda$ is real and sufficiently large.

There may be different ways to do it. But we will use the Laplace transform to convert the problem to an elliptic one which was already studied before. In order to do this, we will extend the solution $v$ to all time interval $(0, \infty)$. Due to the bounds (2.8) and (2.19), this is always possible. We need the following component of the energy inequality

\[
2 \int_0^\infty \int_{D_m} |\partial_x v_\theta|^2 \, dx \, dt + 2 \int_0^T \int_{D_m} |\partial_r v_\theta - \frac{1}{r} v_\theta|^2 \, dx \, dt + 2 \int_0^\infty \int_{D_m} |\partial_x v_r + \partial_r v_3|^2 \, dx \, dt
\]

\[
+ 4 \int_0^\infty \int_{D_m} \left( |\partial_r v_r|^2 + \frac{v_r^2}{r^2} \right) \, dx \, dt + 4 \int_0^\infty \int_{D_m} |\partial_x v_3|^2 \, dx \, dt \leq \int_0^\infty |v(x, 0)|^2 \, dx.
\]

which is a consequences of (2.2) and the formula for the strain tensor (1.11). It is helpful to convert the above inequality into one involving $h = \frac{v_\theta}{r}$. Since $r \partial_r h = \partial_r v_\theta - \frac{v_\theta}{r}$, the preceding inequality implies

\[
2 \int_0^\infty \int_{D_m} r^2 |\partial_x h|^2 \, dx \, dt + 2 \int_0^\infty \int_{D_m} r^2 |\partial_r h|^2 \, dx \, dt \leq \int_0^\infty |v(x, 0)|^2 \, dx,
\]

and consequently

\[
(2.51) \quad \int_0^T \int_{D_m} |\nabla h|^2 \, dx \, dt \leq \tilde{C}_m \int_{D_m} |v(x, 0)|^2 \, dx,
\]

for some $\tilde{C}_m$ depending on $m$.

Next we write the equation (2.7) for $h$ as

\[
\Delta h - \partial_t h = -\frac{2}{r} \partial_r h + b \nabla h + \frac{2v_r}{r} h \equiv R,
\]

By (2.51), the standard energy bound for $L^\infty_t L^2_x$ norm of $v_\theta$ and the $L^\infty$ bound for $v_\theta$ (2.8) and $b$ (2.19), we know that $R \in L^2_t L^2_x$ and there is a constant $\beta > 0$ such that

\[
(2.52) \quad \int_0^\infty e^{-\beta t} \|R(\cdot, t)\|^2_{L^2(D_m)} \, dt + \int_0^\infty e^{-\beta t} \|h(\cdot, t)\|^2_{W^{1,2}(D_m)} \, dt < \infty.
\]

For $\lambda$ such that $Re \lambda \geq \beta$, denote by $w = w(x, \lambda)$ the Laplace transform of $h = h(x, t)$ with respect to the $t$ variable i.e.

\[
w = w(x, \lambda) = \int_0^\infty e^{-\lambda t} h(x, t) \, dt \equiv \mathcal{L} h(x, \lambda).
\]

This is well defined since the integral is absolutely continuous and it is a $W^{1,2}(D_m)$ valued, analytic function for $Re \lambda \geq \beta$. By (2.52), $w = w(\cdot, \lambda)$ is a $W^{1,2}$ solution to the elliptic problem

\[
(2.53) \quad \Delta w(x, \lambda) - \lambda w(x, \lambda) = -h(x, 0) + \mathcal{L} R(x, \lambda), \quad \text{in} \quad D_m; \quad \partial_n w = 0 \quad \text{on} \quad \partial D_m.
\]
By (2.52) again, the right hand side of this equation is in $L^2(D_m)$. If $\lambda \geq \beta$ is real, from [15] Chapter 4 again, we know that, for $s = 3/2$,\
\[ \|w(\cdot, \lambda)\|_{w^{2,4}(D_m)} \leq C \left( \|h(\cdot, 0)\|_{L^2(D_m)} + \|LR(\cdot, \lambda)\|_{L^2(D_m)} + (\lambda + 1) \|w(\cdot, \lambda)\|_{L^2(D_m)} \right) \]
Since $\lambda \geq \beta > 0$, we can use $w = w(x, \lambda)$ as a test function in (2.53) to deduce\n\[ \|w(\cdot, \lambda)\|_{L^2(D_m)} \leq 4\lambda^{-1/2} (\|h(\cdot, 0)\|_{L^2(D_m)} + \|LR(\cdot, \lambda)\|_{L^2(D_m)}). \]
A combination of the previous two inequalities yields, for $\lambda \geq \beta$,
\[ (2.54) \quad \|w(\cdot, \lambda)\|_{w^{2,4}(D_m)} \leq C(1 + \lambda^{1/2} + \lambda^{-1/2}) \left( \|h(\cdot, 0)\|_{L^2(D_m)} + \|LR(\cdot, \lambda)\|_{L^2(D_m)} \right) < \infty. \]
Since $J = -\partial_{x_3} h$, this proves the assertion at the beginning of this sub-step.

**Step 6.2.** Our next task is to find a $L^2_t W^{1,2}(D_m)$ solution to the following problem
\[ \begin{cases} 
\Delta g - \partial_t g + F = 0, & \text{on } D_m \times [0, T] \\
g = 0, & \text{on } \partial H D_m \times [0, T]; \quad \partial_n g = 0, & \text{on } \partial V D_m \times [0, T] \\
g(x, 0) = -\partial_{x_3} v_0(x, 0)/r, & \text{where} \\
F \equiv -(b \cdot \nabla) J + \frac{2 \partial_3 J}{r} + (\omega_3 \partial_3 + \omega_3 \partial_3 v_0) \frac{v_r}{r} \\
= -(b \cdot \nabla) J + \frac{2 \partial_3 J}{r} + [-\partial_{x_3} v_0 \partial_3 + \frac{1}{r} \partial_r (r v_0) \partial_3] \frac{v_r}{r}
\end{cases} \]
(2.56) is the lower order terms in equation (2.50). Note also that $g$ and $J$ share the same initial value and boundary condition. Since $J = -\partial_{x_3} v_0/r$ and $v_0$ is only in the energy space $E$, we can only understand $\nabla J$ in pointwise sense except at the corners at this moment. Let $\{v_0^{(k)}\}$ be a sequence of $L^2_t C_0^1(D_m)$ functions such that $v_0^{(k)} \to v_0$ in $L^2_t W^{1,2}_x(D_m)$. Such approximation sequence always exists since $D_m$ is a bounded Lipschitz domain and hence a $W^{1,2}$ Sobolev extension domain. c.f. Calderon [6]. Since $D_m$ can be regarded as a rectangular polygon in the $r x_3$ plane, we can also require
\[ (2.57) \quad \|v_0^{(k)}(\cdot, t)\|_{L^\infty(D_m)} \leq \|v_0(\cdot, t)\|_{L^\infty(D_m)}. \]
The reason is the following. Fix a small positive number $\delta$. For any positive integer $k$, let $D_{mk}$ be the polygon contained in $D_m$ such that the distance between $\partial D_{mk}$ and $\partial D_m$ is $\delta/k$. Choose a one to one linear map $L_k$ that maps $D_m$ onto $D_{mk}$. For example, one can move the origin to a suitable point inside $D_{mk}$ and just shrink the horizontal and vertical variables in the order of $\delta/k$. Then $v_0^{(k)}(r, x_3, t) = v_0(L_k(r, x_3), t)$ will satisfy all the above requirements.

Denote by $J^{(k)} = -\partial_{x_3} v_0^{(k)} / r$ and
\[ F_k = -(b \cdot \nabla) J^{(k)} + \frac{2 \partial_3 J^{(k)}}{r} + [-\partial_{x_3} v_0^{(k)} \partial_3 + \frac{1}{r} \partial_r (r v_0^{(k)}) \partial_3] \frac{v_r}{r}. \]
Since $F_k \in L^2_t L^2_x$, by standard Galerkin method, the problem
\[ \begin{cases} 
\Delta g_k - \partial_t g_k + F_k = 0, & \text{on } D_m \times [0, T], \quad T > 0, \\
g_k = 0, & \text{on } \partial H D_m \times [0, T]; \quad \partial_n g_k = 0, & \text{on } \partial V D_m \times [0, T] \\
g_k(x, 0) = -\partial_{x_3} v_0(x, 0)/r,
\end{cases} \]
(2.58)
has a unique solution in $L^2_t W^{1,2}_v(D_m) \cap E$. Here $W^{1,2}_v(D_m)$ is the closure, under $W^{1,2}$ norm, of the set

$$\{u \in C^1(\bar{D}_m) \mid u(x) = 0, \ x \in \partial^H D_m; \ \partial_n u(x) = 0, \ x \in \partial^V D_m \}.$$  

Using $g_k$ as a test function in (2.58), we obtain (2.59),

$$\int_0^T \int_{D_m} |\nabla g_k|^2 \, dx \, dt + \frac{1}{2} \int_0^T \int_{D_m} |g_k(x, T)|^2 \, dx - \frac{1}{2} \int_0^T \int_{D_m} |g(x, 0)|^2 \, dx = \int_0^T \int_{D_m} F_k g_k \, dx \, dt$$

$$= \int_0^T \int_{D_m} \left\{ -(b \cdot \nabla ) J^{(k)} + \frac{2}{r} \partial_r J^{(k)} + \left[-\partial_x v^{(k)}_\theta \partial_r + \frac{1}{r} \partial_r (r v^{(k)}_\theta) \partial_\theta \right] \frac{V_r}{r} \right\} g_k \, dx \, dt$$

$$= \int_0^T \int_{D_m} \left\{ (b \cdot \nabla ) J^{(k)} + \frac{g_k}{r} \partial_r J^{(k)} + \frac{v^{(k)}_\theta}{r} \partial_\theta \partial_x v^{(k)}_\theta - \frac{v^{(k)}_\theta}{r} \partial_\theta \partial_r g_k \right\} \, dx \, dt.$$  

In the last step, we have used integration by parts except for the 2nd term $Y_2$, which are justified due to the following facts: $\text{div} \ b = 0$, $\nabla \partial_x v_r \in L^2_t L^2_v(D_m)$ from Step 5, $J^{(k)} \in L^2_t C^1(\bar{D}_m)$ by assumption and by the boundary conditions for $v_r, v_2$ and $g_k$. There is also a cancellation of the second order derivatives of $v_r/r: \partial_r \partial_x v_r/r$ which is in $L^2_t L_v^2$ by Step 5. Using Cauchy-Schwarz inequality on the last line of (2.59) except for the 2nd term in the bracelet $Y_2$, we arrive at

$$\int_0^T \int_{D_m} |\nabla g_k|^2 \, dx \, dt + \int_{D_m} |g_k(x, T)|^2 \, dx - \int_{D_m} |g(x, 0)|^2 \, dx$$

$$\leq C \int_0^T \int_{D_m} \left( |v^{(k)}_\theta|^2 |\nabla v_r|^2 + |b|^2 |J^{(k)}|^2 \right) \, dx \, dt + 4 \int_0^T \int_{D_m} \frac{1}{r} \partial_r J^{(k)} \, dx \, dt.$$

The last term requires a slightly different integration by parts since $g_k$ may not be 0 on the vertical boundary $\partial^V D_m$. We can treat it as follows

$$4 \int_0^T \int_{D_m} \frac{2}{r} \partial_r J^{(k)} \, dx \, dt = -4 \int_0^T \int_{D_m} \frac{1}{r} \partial_r \partial_x v^{(k)}_\theta \, dx \, dt = 4 \int_0^T \int_{D_m} \partial_x g_k \frac{1}{r} \partial_r v^{(k)}_\theta \, dx \, dt$$

since $g_k = 0$ on the horizontal boundary $\partial^H D_m$. Substituting this identity into the previous inequality and using Cauchy Schwarz again, we deduce

$$\int_0^T \int_{D_m} |\nabla g_k|^2 \, dx \, dt + 2 \int_{D_m} |g_k(x, T)|^2 \, dx$$

$$\leq C \int_0^T \int_{D_m} \left( |v^{(k)}_\theta|^2 |\nabla v_r|^2 + |b|^2 |J^{(k)}|^2 + \frac{1}{r^2} \left| \partial_r v^{(k)}_\theta \right|^2 \right) \, dx \, dt + 2 \int_{D_m} |g(x, 0)|^2 \, dx$$

$$\leq C C_m \int_0^T \int_{D_m} \left( |v^{(k)}_\theta|^2 + |b(\cdot, t)|^2 \right) \, dx \, dt + \int_{D_m} \left( \left| \nabla v_r \right|^2 + \left| \nabla v^{(k)}_\theta \right|^2 + v_r^2 \right) \, dx \, dt + 2 \|g(\cdot, 0)\|_{L^2(D_m)}^2.$$
where we have used (2.57). From here, using (2.8), (2.19) and the convergence of \( v_\theta^{(k)} \) to \( v_\theta \) in \( L^2 W^{1,2}(D_m) \), we deduce

\[
\int_0^T \int_{D_m} |\nabla g_k|^2 dx dt + 2 \int_{D_m} |g_k(x, T)|^2 dx \\
\leq CC_m^2 \int_0^T e^{\alpha_0 t} \int_{D_m} \left( |\nabla v_\theta|^2 + |\nabla v_\theta|^2 + v_\rho^2 \right) dx dt + 2\|g(\cdot, 0)\|^2_{L^2(D_m)}.
\]

This says that the energy norm of \( g_k \) on \( D_m \times [0, T] \) can be bounded from above by an uniform constant. Hence we can extract a subsequence converging to a function \( g \) in weak \( L^2 W^{1,2}(D_m) \) sense. This function \( g \) is a solution to (2.55) in \( L^2 W^{1,2}(D_m) \). By standard interior and boundary regularity theory of the heat equation, we also know that \( g \) solves (2.55) except possibly at the corners of \( D_m \). Moreover the term \( b\nabla J \) is understood in the weak sense: for test functions \( \phi \in L^2 W^{1,2}(D_m) \), the following holds

\[
\int_0^T \int_{D_m} (b\nabla J) \phi dx dt = - \int_0^T \int_{D_m} (b\nabla \phi) J dx dt.
\]

**Step 6.3.** From (2.60), the solution \( g \) also satisfies the energy estimate

\[
\int_0^T \int_{D_m} |\nabla g|^2 dx dt + 2 \int_{D_m} |g(x, T)|^2 dx \\
\leq CC_m^2 \int_0^T e^{\alpha_0 t} \int_{D_m} \left( |\nabla v_\theta|^2 + |\nabla v_\theta|^2 + v_\rho^2 \right) dx dt + 2\|g(\cdot, 0)\|^2_{L^2(D_m)}.
\]

Now we prove \( J = g \). Once this is done, since \( J = -\partial_x \phi_\theta / r \), the energy bound (2.61) will imply \( |\nabla \partial_x \phi_\theta| \in L^2 L^2 \), completing the proof of Step 6. Since \( J \) satisfies (2.50) and \( g \) satisfies (2.55), the difference \( J - g \) satisfies the following equation pointwise

\[
\begin{aligned}
&\Delta(J - g) - \partial_y (J - g) = 0, \quad \text{on} \quad D_m \times [0, \infty) \\
&g = 0, \quad \text{on} \quad \partial^H D_m \times [0, \infty); \quad \partial_n g = 0, \quad \text{on} \quad \partial^V D_m \times [0, \infty) \\
&(J - g)(x, 0) = 0.
\end{aligned}
\]

From (2.61), the energy bound for \( h = \phi_\theta / r (2.57) \) and standard energy bound for \( v_\rho \), we find that, for all \( T > 0 \), there is a constant \( C_1 \) such that

\[
\int_0^T \int_{D_m} e^{-2\alpha_0 t} (|\nabla g|^2 + g^2) dx dt \leq C_1 C_m^2 \|v_\theta\|^2_{L^2(D_m)}.
\]

This can be seen by taking \( T \) to be positive integers and by splitting the time integral into the sum of integrals on \([j - 1, j]\) with \( j = 1, 2, ..., T \). Inequality (2.63) and the assertion at Step 6.1 infer that the \( W^{1,2}(D_m) \) valued Laplace transform for \( J - g \) is well defined if the phase variable \( \lambda \) satisfies \( Re \lambda \geq \beta \) for some suitable \( \beta \). Here \( s = 3\gamma / 2 \) as before. Therefore the Laplace transform \( \mathcal{L}(J - g)(\cdot, \lambda) \) is a \( W^{1,2}(D_m) \) solution to the problem:

\[
\begin{aligned}
&\Delta(\mathcal{L}(J - g))(x, \lambda) - \lambda \mathcal{L}(J - g)(x, \lambda) = 0, \quad \text{on} \quad D_m \\
&\mathcal{L}(J - g)(x, \lambda) = 0, \quad \text{on} \quad \partial^H D_m; \quad \partial_n \mathcal{L}(J - g)(x, \lambda) = 0, \quad \text{on} \quad \partial^V D_m.
\end{aligned}
\]
According to Proposition 4.4 below, if \( \lambda \) is real and sufficiently large, then \( \mathcal{L}(J - g)(\cdot, \lambda) = 0 \). But the Laplace transform is a \( W^{1,1}(D_m) \) valued analytic function in \( \lambda \) with \( Re \lambda \geq \beta \). This fact is again due to the assertion in Step 6.1 and (2.63). Thus \( \mathcal{L}(J - g)(\cdot, \lambda) = 0 \) for all these \( \lambda \), implying that \( J = g \). As mentioned, this completes the proof of Step 6.

**Step 7.** \(|\nabla^2 v_r| + |\nabla^2 v_3| \in L^2 L^2_t\) and (2.4) and (2.5).

Since it is already proven that \(|\nabla \omega_0|, |\nabla \partial_x v_r| \in L^2 L^2_t\), and \( \omega_0 = \partial_x v_r - \partial_r v_3 \), it is clear that \(|\nabla \partial_r v_3| \in L^2 L^2_t\). Using the divergence free condition \( \partial_r v_r + \frac{1}{r} \partial_r v_3 = 0 \), one sees

\[
\begin{align*}
\partial_r^2 v_r &= -\partial_r(v_r/r) - \partial_r \partial_x v_3 \in L^2 L^2_t, \\
\partial_3^2 v_3 &= -\partial_x(v_r/r) - \partial_x \partial_r v_r \in L^2 L^2_t.
\end{align*}
\]

Hence \(|\nabla^2 v_r| + |\nabla^2 v_3| \in L^2 L^2_t\).

Finally (2.12) is just (2.4). Also, since \( J = g \), by taking \( k \to \infty \) in (2.59), we find

\[
\int_0^T \int_{D_m} |\nabla J|^2 dxdt + \frac{1}{2} \int_0^T \int_D |J(x, T)|^2 dx - \frac{1}{2} \int_0^T \int_D |J(x, 0)|^2 dx = \int_0^T \int_{D_m} \left\{ (b \cdot \nabla J) + J \frac{2}{r} \partial_r J + \nu_\theta \partial_r v_r \partial_x J - \nu_\theta \partial_x v_r \partial_r J \right\} dxdt.
\]

But \( \int_0^T \int_{D_m} (b \cdot \nabla J) J dxdt = 0 \) since \( div b = 0 \) and \( v_3 = 0 \) on \( \partial^H D_m \) and \( v_3 = 0 \) on \( \partial^H D_m \).

This integration by parts is justified because we already proved \( b \in L^\infty \cap L^2 W^{2,2}(D_m) \) and \( J \in L^2 W^{1,2}(D_m) \). Therefore (2.5) follows from (2.64). This completes the proof of Step 7 and the proposition.

Now let us state and prove the lemma that was used in Step 4.1.

**Lemma 2.3.** Fixing \( t \) and \( s = 3^{-}/2 \), let \( \tilde{v}_r = \tilde{v}_r(\cdot, t) \) be a \( W^{2, s}(D_m) \) solution to the problem

\[
\begin{align*}
\left\{ \begin{array}{ll}
\left( \Delta - \frac{1}{r^2} \right) \tilde{v}_r = \partial_x \omega_\theta, & \text{in } D_m, \\
\partial_x \tilde{v}_r = 0, & \text{on } \partial^H D_m; \\
\tilde{v}_r = 0, & \text{on } \partial^V D_m,
\end{array} \right.
\end{align*}
\]

where \( \omega_\theta = \omega_\theta(\cdot, t) \in W^{1,2}_0(D_m) \) and is smooth except at corners.

Then the following holds

\[
\int_{D_{m,r=\text{const.}}} \tilde{v}_r(r, x_3, t) dx_3 = 0.
\]

**Proof.** For simplicity, we drop the \( \tilde{\cdot} \) in the proof. For fixed \( r > 0 \), denote by \( L_r = \{(r, x_3) | (r, x_3) \in D_m\} \) a vertical cross section of \( D_m \), which is regarded as a polygon in the \( r, x_3 \) plane. By standard interior and boundary regularity result, \( v_r \) is smooth on \( L_r \) if no corners of \( D_m \) are contained in \( L_r \). Since \( \omega_\theta = 0 \), \( \partial_x v_r = 0 \) on \( \partial^H D_m \), we know that

\[
\int_{L_r} \partial_x \omega_\theta dx_3 = 0, \quad \int_{L_r} \partial_x^2 v_r dx_3 = 0.
\]

Using these and integrating along \( L_r \), the equation (c.f. (2.65)):

\[
\partial_x^2 v_r + \frac{1}{r} \partial_x v_r + \partial_x^2 v_r - \frac{1}{r^2} v_r = \left( \Delta - \frac{1}{r^2} \right) v_r = \partial_x \omega_\theta,
\]

Thus

\[
\int_{L_r} \partial_x^2 v_r dx_3 = 0, \quad \int_{L_r} \partial_x^2 v_r dx_3 = 0.
\]
we find

\begin{equation}
\partial_r^2 \int_{L_r} v_r dx_3 + \frac{1}{r} \partial_r \int_{L_r} v_r dx_3 - \frac{1}{r^2} \int_{L_r} v_r dx_3 = 0.
\end{equation}

Let us write \( g_i(r) \equiv \int_{L_{r_i}} v_i(r, x_3, t) dx_3 \). Since \( v_i(\cdot, t) \in W^{2, s}(D_m) \) with \( s = 3^{-1}/2 \), by the trace theorem, \( g_i(r) \) and \( g_i'(r) \) are continuous function of \( r \), and \( g_i(r) \) is smooth unless \( L_r \) contains the corners of \( D_m \). For these \( t \), we can solve the following ode from (2.66)

\[ g_i''(r) + \frac{1}{r} g_i'(r) - \frac{1}{r^2} g_i(r) = 0, \]

piecewise. Let \( L_{r_1}, L_{r_2}, ..., L_{r_{m+1}} \) be those vertical segments containing corners of \( D_m \). Then

\[ g_i(r) = c_i(t) r + \bar{c}_i(t) r^{-1}, \quad r_i < r \leq r_{i+1}, \ i = 1, 2, ..., m, \]

where \( c_i(t), \bar{c}_i(t) \) are independent of \( r \). By continuity of \( g_i(r) \) and \( g_i'(r) \), for each \( i = 1, 2, ..., m-1 \), the following hold:

\[ c_i(t) r_i + \bar{c}_i(t) r_i^{-1} = c_{i+1}(t) r_i + \bar{c}_{i+1}(t) r_i^{-1}, \]

\[ c_i(t) - \bar{c}_i(t) r_i^2 = c_{i+1}(t) - \bar{c}_{i+1}(t) r_i^2. \]

Therefore \( c_i(t) = c_{i+1}(t) \) and \( \bar{c}_i(t) = \bar{c}_{i+1}(t) \) so that

\[ g_i(r) = c_1(t) r + \bar{c}_1(t) r^{-1}. \]

According to the Navier boundary condition, \( g_i(r_1) = g_i(r_{m+1}) = 0 \) since \( L_{r_1} \) is the left most vertical boundary of \( D_m \) and \( L_{r_{m+1}} \) is the right most. Hence \( c_1(t) = \bar{c}_1(t) = 0 \) and \( g_i(r) \equiv 0 \), proving the lemma. \( \square \)

3. Proof of the main result

3.1. A priori bounds for \( \|v\|_{L^\infty_t L^\infty_x}, \|\omega_t/r\|_{L^p_t L^q_x}, \|\omega_t/r\|_{L^\infty_t L^2_x}, \|v_t/r\|_{L^\infty_t L^2_x} \). In this section, we prove a number a priori bounds for finite energy solutions to the following initial boundary value problem of ASNS (1.3) on \( D_m \times (0, T], T > 0 \). These a priori bounds include the usual energy bound for \( v \), and \( L^p_t L^q_x \) bounds for \( |\nabla \omega_t| \) and \( |\nabla \omega_t| \). The main point is to show that these bounds are independent of \( m \), which will allow us to prove the main result in the next subsection after letting \( m \to \infty \).
Lemma 3.1. Let \( v \) be a solution to Problem 2.7. Then the following bounds are true.

\( \int_{D_m} |v(x, T)|^2 dx \leq \int_{D_m} |v(x, 0)|^2 dx; \)

\( 2 \int_0^T \int_{D_m} \left| \partial_3 v_0 \right|^2 dx dt + 2 \int_0^T \int_{D_m} \left| \partial_r v_0 - \frac{1}{r} v_0 \right|^2 dx dt + 2 \int_0^T \int_{D_m} \left| \partial_3 v_r + \partial_r v_3 \right|^2 dx dt \)

\( + 4 \int_0^T \int_{D_m} \left( \left| \partial_r v_r \right|^2 + \frac{v_r^2}{r^2} \right) dx dt + 4 \int_0^T \int_{D_m} \left| \partial_3 v_3 \right|^2 dx dt \leq \int_{D_m} |v(x, 0)|^2 dx; \)

if, in addition, \( v \) comes from Proposition 2.2 then

\( 2 \int_0^T \int_{D_m} \left( |\nabla v_r|^2 + \frac{v_r^2}{r^2} + |\nabla v_3|^2 + |r \partial_r v_0|^2 + |\partial_3 v_0|^2 \right) dx dt \leq \int_{D_m} |v(x, 0)|^2 dx. \)

Proof. Bounds (a) and (b) are immediate consequences of (2.2) and the formula for the strain tensor (1.11). So we just need to prove (c). Starting with part (b), we will just need to work out the third term \( L_3 \) on the left hand side of (b), which will be denoted by \( L_3 \).

Let us compute, using integration by parts,

\[
\int_0^T \int_{D_m} \partial_3 v_r \partial_r v_3 dx dt = - \int_0^T \int_{D_m} v_r \partial_r \partial_3 v_3 dx dt
\]

\[
= \int_0^T \int_{D_m} v_r \partial_r (\partial_r v_r + \frac{1}{r} v_r) dx dt = \int_0^T \int_{D_m} v_r \partial_r (\partial_r v_r + \frac{1}{r} v_r) r dr dx_3 dt
\]

\[
= - \int_0^T \int_{D_m} \chi_{D_m} v_r \partial_r (\partial_r v_r + \frac{1}{r} v_r) r dr dx_3 dt - \int_0^T \int_{D_m} v_r \partial_r (\partial_r v_r + \frac{1}{r} v_r) r dr dx_3 dt
\]

\[
= - \int_0^T \int_{D_m} \chi_{D_m} |\partial_r v_r|^2 r dr dx_3 dt - 2 \int_0^T \int_{D_m} \chi_{D_m} v_r \partial_r v_r r dr dx_3 dt
\]

\[
- \int_0^T \int_{D_m} \frac{1}{r} |v_r|^2 r dr dx_3 dt.
\]

We comment that integration by parts is legal due to the property that \( |\nabla^2 v_r|, |\nabla^2 v_3| \in L^2_t L^2_x \) from Section 2 and the boundary condition \( v_r = 0 \) on \( \partial^V D_m \) and \( v_3 = 0 \) on \( \partial^H D_m \) in the trace sense for a.e. \( t \). Doing integration by parts and using \( v_r = 0 \) on \( \partial^V D_m \) again, we see that the 2nd from last term in the previous identity is 0. Hence

\( \int_0^T \int_{D_m} \partial_3 v_r \partial_r v_3 dx dt = - \int_0^T \int_{D_m} |\partial_r v_r|^2 dx dt - \int_0^T \int_{D_m} \frac{1}{r^2} |v_r|^2 dx dt. \)
Similarly, since $\partial_{x_3} v_r = 0$ on $\partial^I D_m$ and $v_3 = 0$ on $\partial^H D_m$ in the trace sense, we deduce, after using the divergence free condition, that

$$
\int_0^T \int_{D_m} \partial_{x_3} v_r \partial_r v_3 dxdt = - \int_0^T \int_{D_m} v_3 \partial_{x_3} (r v_r) \frac{1}{r} dxdt
$$

(3.3)

$$
= \int_0^T \int_{D_m} v_3 \partial_{x_3} \partial_r v_3 dxdt = - \int_0^T \int_{D_m} |\partial_{x_3} v_3|^2 dxdt.
$$

Inequality (c) is derived after substituting (3.2) and (3.3) into $L_3$ in (b) separately (after expanding the square) and adding the resulting two inequalities. □

The next lemma states that the line integral of $v_r$ in $x_3$ variable is 0 for a.e. $t$.

**Lemma 3.2.** Let $v$ be a solution to Problem [2.7] coming from Proposition [2.2] Then for a.e. $t$, the following holds

$$
\int_{D_m \cap \{r = \text{const.}\}} v_r(r, x_3, t) dx_3 = 0.
$$

**Proof.** This is done in Lemma [2.3] in the previous section if $v_r(\cdot, t) \in W^{2,\infty}(D_m)$. Now that $v_r(\cdot, t) \in W^{2,2}(D_m)$ for a.e. $t$, the conclusion follows. □

The following lemma states that solutions to Problem [2.1] has the property that $v_\theta$ is uniformly bounded near the right boundary of $D_m$ for all $m \geq 2$. Eventually, by the end of the section, we will have proven $v_\theta$ is uniformly bounded in all $D_m$.

**Lemma 3.3.** Let $v$ be a solution to Problem [2.7] with $m \geq 2$. There exists a positive constant $\bar{C}_1$, depending only on $\|v_\theta(\cdot, 0)\|_{L^{\infty}(D_1)}$ and $\|v(\cdot, 0)\|_{L^{2}(D_m)}$ such that

$$
\|v_\theta(\cdot, t)\|_{L^{\infty}(D_1 \cap [3/4r < 1])} \leq \bar{C}_1, \forall t > 0.
$$

(3.4)

**Proof.** We will use Moser’s iteration to prove the $L^\infty$ bound for $\Gamma = rv_\theta$ in $D_1 \times [0, \infty)$. The Navier boundary condition for $v_\theta$ on the right vertical boundary of $D_m$ and the lack of good bounds on $b = v_r e_r + v_3 e_3$ will present a small obstacle. In order to proceed, we will use a trace inequality and adopt the method in [41], which can prove local boundedness of solutions under weaker than usual condition on the drift term $b$. Alternatively one can also work with (2.7), the equation for $h = \frac{v_\theta}{r}$, which satisfies the more friendly Neumann boundary condition. But one still needs to deal with the drift term and an additional potential term $-v_r/r$, using dimension reduction and energy bound (3.1) (c).

Let us start with the equation for $\Gamma$:

$$
\begin{aligned}
\Delta \Gamma - b \nabla \Gamma - \frac{2}{r} \partial_r \Gamma - \partial_t \Gamma &= 0, \quad \text{in } D_m \times (0, \infty) \\
\Gamma(x, 0) = rv_\theta(x, 0), x \in D_m.
\end{aligned}
$$

(3.5)

For any rational number $p > 1$ in the form of $2k/l$ where $k, l$ are positive integers, we know that $\Gamma^p$ is a sub-solution, namely

$$
\begin{aligned}
\Delta \Gamma^p - b \nabla \Gamma^p - \frac{2}{r} \partial_r \Gamma^p - \partial_t \Gamma^p &\geq 0, \quad \text{in } D_m \times (0, \infty) \\
\Gamma^p(x, 0) = (rv_\theta(x, 0))^p, x \in D_m.
\end{aligned}
$$

(3.6)
Let $\phi = \phi(r)$ be a cut off function defined on the interval $[0, 1]$ on the $r$ axis such that $0 \leq \phi \leq 1$, $\phi(r) = 1$, $r \in [3/4, 1]$; $\phi(r) = 0$, $r \in [0, 1/2]$ and that $||\phi'/\phi^{0.99}||_{L^\infty} < \infty$. For $T \geq 1$, let $\eta = \eta(t)$ be a cut off function in time, supported in $[T-1, T]$ which will be specified later. Since a nonuniform $L^\infty$ bound for $v_0$ in Section 2 (Step 2 in the proof of Proposition 2.2) is already proved, we can use $\Gamma^p(\phi \eta)^2$ as a test function on (3.6) to deduce (3.7)

$$LS \equiv \int_{T-1}^T \int_{D_m} |\nabla (\Gamma^p \phi \eta)|^2 \, dx \, dt + \frac{1}{2} \int_{D_m} \Gamma^{2p}(\phi \eta)^2(x, T) \, dx$$

$$\leq \int_0^T \int_{\partial D_m} \partial_n \Gamma^p \Gamma^p(\phi \eta)^2 \, dS \, dt - \int_{T-1}^T \int_{D_m} b \nabla \Gamma^p \Gamma^p(\phi \eta)^2 \, dx \, dt - \int_{T-1}^T \int_{D_m} \frac{2}{r} \partial_r \Gamma^p \Gamma^p(\phi \eta)^2 \, dx \, dt$$

$$+ \int_{T-1}^T \int_{D_m} \Gamma^{2p}(|\nabla (\phi \eta)|^2 + \eta' \eta) \, dx \, dt + \frac{1}{2} \int_{D_m} \Gamma^{2p}(\phi \eta)^2(x, T-1) \, dx,$$

where $n$ is the exterior normal of $\partial D_m$. Next we will find bounds for $R_1$, $R_2$ and $R_3$.

To bound $R_1$, we use the boundary conditions $\partial_{x_3} \Gamma = r \partial_{x_3} v_\theta = 0$ on $\partial^H D_m$. Therefore

$$R_1 = \int_{T-1}^T \int_{\partial^V D_m} \partial_n \Gamma^p \Gamma^p(\phi \eta)^2 \, dS \, dt$$

$$= - \sum_{j = 1}^m \int_{T-1}^T \int_{L_j} \partial_j \Gamma^p \Gamma^p(\phi \eta)^2 \, dx \, dt + \int_{T-1}^T \int_{L_{m+1}} \partial_r \Gamma^p \Gamma^p(\phi \eta)^2 \, dx \, dt$$

$$= \int_{T-1}^T \int_{L_{m+1}} \partial_r \Gamma^p \Gamma^p(\phi \eta)^2 \, dx \, dt. \quad \text{(because $L_1, \ldots, L_m$ are cut off by $\phi$)}$$

Here $\partial^V D_m = \cup_{j=1}^m L_j \cup L_{m+1}$, with $L_1, \ldots, L_m$ being the vertical boundary segments to the left of $D_m$ and $L_{m+1}$ being the only vertical boundary segment to the right of $D_m$. Notice that $\partial_n = -\partial_r$ on $L_1, \ldots, L_m$ and $\partial_n = \partial_r$ on $L_{m+1}$. On $L_{m+1}$ and $L_j$, $j = 1, 2, \ldots, m$, the Navier boundary condition reads

$$\partial_r \Gamma^p = \partial_r(r pv_\theta^p) = pr^{p-1}v_\theta^p + r^p pv_\theta^{p-1} \partial_r v_\theta = 2pr^{p-1}v_\theta = \frac{2p}{r} \Gamma^p.$$ 

Consequently

$$R_1 = 2p \int_{T-1}^T \int_{L_{m+1}} \Gamma^{2p} \eta^2 \, dx \, dt.$$

Next, via integration by parts, we see that

$$R_3 = \int_{T-1}^T \int_{L_{m+1}} \Gamma^{2p} \eta^2 \, dx \, dt - \int_{T-1}^T \int_{D_m} \Gamma^{2p} \partial_r(\phi \eta)^2 \frac{1}{r} \, dx \, dt.$$

Using integration by parts again together with the boundary condition $v_r = 0$ on $\partial^V D_m$, $v_3 = 0$ on $\partial^H D_m$, we noticed that

$$R_2 = - \int_{T-1}^T \int_{D_m} v_r \Gamma^{2p} \phi \partial_r(\phi \eta)^2 \, dx \, dt.$$
The combination of (3.9), (3.11), (3.10) and (3.7) yields
(3.12)
$$
LS \leq (2p - 1) \int_{T-1}^{T} \int_{L_{m+1}} \Gamma^{2p} \eta^2 dx dt + \int_{T-1}^{T} \int_{D_m} v_r \Gamma^{2p} \phi \partial_r \phi \eta^2 dx dt \\
+ \int_{T-1}^{T} \int_{D_m} \Gamma^{2p} \left( |\nabla(\phi \eta)|^2 + \eta' \eta + \partial_r(\phi \eta)^2 \frac{1}{r} \right) dx dt + \frac{1}{2} \int_{D_m} \Gamma^{2p}(\phi \eta)^2(x, T-1) dx
$$

We need to absorb $I_1$ and $I_2$ by $LS$.

Since $\phi = 1$ on $L_{m+1}$ and $\phi = 0$ when $r = 1/2$, we can compute
$$
\int_{L_{m+1}} \Gamma^{2p} dx_3 = \int_{L_{m+1}} \Gamma^{2p} \phi^2 dx_3 - \int_{D_m \cap \{r = 1/2\}} \Gamma^{2p} \phi^2 dx_3 \\
= \int_{0}^{1} \int_{1/2}^{0} \partial_r \left( \Gamma^{2p} \phi^2 \right) dr dx_3 = 2 \int_{0}^{1} \int_{1/2} \left( \Gamma^{p} \phi \right) \partial_r (\Gamma^{p} \phi) dr dx_3.
$$

Therefore
$$
I_1 \leq 4(2p - 1) \int_{T-1}^{T} \int_{1/2} \left( \Gamma^{p} \phi \eta \right) \sqrt{r} \sqrt{r} |\partial_r (\Gamma^{p} \phi \eta)| dr dx_3 dt
$$
$$
\leq 0.5 \int_{T-1}^{T} \int_{D_m} |\nabla(\Gamma^{p} \phi \eta)|^2 dx dt + 8(2p - 1)^2 \int_{T-1}^{T} \int_{D_m} (\Gamma^{p} \phi \eta)^2 dx dt
$$
(3.13)

Next we use the argument on p254-p255 in [41] with $m = 4/3$, $b = v_r e_r + v_3 e_3$ and $\omega = \Gamma^p$ and Corollary 2 there to deduce
(3.14)
$$
I_2 \leq 0.25 \int_{T-1}^{T} \int_{D_m} |\nabla(\Gamma^{p} \phi \eta)|^2 dx dt + C \|v_r\|^4\|L^{p/2}(D_1)\|\partial_r \phi/\phi^{0.99}\|\|_{\infty} \int_{T-1}^{T} \int_{D_1} |\Gamma^{p} \phi|^2 dx dt \\
\leq 0.25 \int_{T-1}^{T} \int_{D_m} |\nabla(\Gamma^{p} \phi \eta)|^2 dx dt + C \|v_3\|^4\|L^{p/2}(D_m)\|\partial_r \phi/\phi^{0.99}\|\|_{\infty} \int_{T-1}^{T} \int_{D_1} |\Gamma^{p} \phi|^2 dx dt.
$$

In the last step, part (a) of the energy inequality (3.1) has been used. Also note that $v_3$ drops out since $\phi$ depends only on $r$. Now we can plug (3.13) and (3.14) into (3.12) to reach, since $\phi$ is supported in $D_1$,
(3.15)
$$
\int_{T-1}^{T} \int_{D_1} |\nabla(\Gamma^{p} \phi \eta)|^2 dx dt + 2 \int_{D_1} \Gamma^{2p}(\phi \eta)^2(x, T) dx
$$
$$
\leq 32(2p - 1)^2 \int_{T-1}^{T} \int_{D_1} (\Gamma^{p} \phi \eta)^2 dx dt + 4C \|v_r\|^4\|L^{p/2}(D_m)\|\partial_r \phi/\phi^{0.99}\|\|_{\infty} \int_{T-1}^{T} \int_{D_1} |\Gamma^{p} \phi|^2 dx dt \\
+ 4 \int_{T-1}^{T} \int_{D_1} \Gamma^{2p} \left( |\nabla(\phi \eta)|^2 + \eta' \eta + \partial_r(\phi \eta)^2 \frac{1}{r} \right) dx dt + 2 \int_{D_1} \Gamma^{2p}(\phi \eta)^2(x, T-1) dx
$$

Take $T = 1$ and $\eta = 1$ first. Then the last term in (3.15) is bounded by $2 \|v_r(\cdot, 0)\|^2\|L^{p/2}(D_1)\|$. By standard Moser’s iteration from (3.15), it is clear that (3.4) holds for $t \in [0, 1]$. Note that only the standard $L^2$ Sobolev inequality is needed because the spatial domain is $D_1$. For $T > 1$,
we take suitable sequences of \( \phi \) and \( \eta \) such that \( \eta(T - 1) = 0 \). Then the last term in (3.15) drops out. Moser’s iteration again tells us, for a positive number \( q > 1 \),
\[
||\Gamma(\cdot, T)||_{L^q(D_m \cap \{3/4 < r < 1\})} \leq C[1 + ||v(\cdot, 0)||_{L^q(D_m)}]^q \int_{T-1}^T \int_{D_m} \Gamma^2 dxdt.
\]
Since \( \Gamma = rv_\theta \), this and the energy inequality imply (3.4) for all \( T > 1 \), completing the proof of the lemma. \( \square \)

**Lemma 3.4.** Let \( v \) be a solution to Problem 2.1 and \( \Gamma = rv_\theta \). Then there is a positive constant \( C_1 \), depending only on \( \|v_\theta(\cdot, 0)\|_{L^{\infty}(D_1)} \) and \( \|v(\cdot, 0)\|_{L^2(D_m)} \) such that
\[
||\Gamma(\cdot, t)||_{L^{\infty}(D_m)} \leq ||\Gamma(\cdot, 0)||_{L^{\infty}(D_m)} + C_1, \quad \forall t > 0.
\]

**Proof.** Recall from (3.6) that for any positive even integer \( p \), \( \Gamma^p \) is a subsolution, namely
\[
\begin{align*}
\Delta \Gamma^p - b
\nabla \Gamma^p - \frac{2}{p} \partial_r \Gamma^p - \partial_t \Gamma^p & \geq 0, \quad \text{in} \ D_m \times (0, \infty) \\
\Gamma^p(x, 0) & = (rv_\theta(x, 0))^p, \ x \in D_m.
\end{align*}
\]
This implies
\[
\partial_t \int_{D_m} \Gamma^p(x,t) dx \leq \int_{D_m} \Delta \Gamma^p(x,t) dx - \int_{D_m} b \nabla \Gamma^p(x,t) dx - \int_{D_m} \frac{2}{p} \partial_r \Gamma^p(x,t) dx \\
\equiv T_1 + T_2 + T_3.
\]
Since we already proved a nonuniform \( L^{\infty} \) bound for \( v_\theta \) in Section 2 (Step 2 in the proof of Proposition 2.2), the above integrals are well defined. We will find uniform bounds for \( T_2 \), \( T_1 \) and \( T_3 \) respectively.

Using the boundary condition \( v_r = 0 \) on \( \partial^V D_m \), \( v_3 = 0 \) on \( \partial^H D_m \), we see that
\[
T_2 = \int_{D_m} \text{div} b \Gamma^p(x,t) dx = 0.
\]
To bound \( T_1 \), we use the boundary conditions \( \partial x_3 \Gamma = r \partial x_3 v_\theta = 0 \) on \( \partial^H D_m \). Therefore
\[
T_1 = \int_{\Gamma^p D_m} \partial_n \Gamma^p dS = - m \int_{L_j} \partial_\tau \Gamma^p rdx_3 + \int_{L_{m+1}} \partial_t \Gamma^p rdx_3.
\]
Here \( n \) is the outward normal of \( \partial D_m \) and \( \partial^V D_m \) \( = \bigcup_{j=1}^m L_j \cup L_{m+1} \), with \( L_1, \ldots, L_m \) being the vertical boundary segments to the left of \( D_m \) and \( L_{m+1} \) being the only vertical boundary segment to the right of \( D_m \). Notice that \( \partial_n = -\partial_\tau \) on \( L_1, \ldots, L_m \) and \( \partial_n = \partial_t \) on \( L_{m+1} \), which accounts for the sign in (3.19). By (3.8), on \( L_j, j = 1, 2, ..., m + 1 \), the Navier boundary condition reads \( \partial_\tau \Gamma^p = \frac{2p}{r} \Gamma^p \). Substituting this to (3.19) gives
\[
T_1 = -2p \sum_{j=1}^m \int_{L_j} \Gamma^p dx_3 + 2p \int_{L_{m+1}} \Gamma^p dx_3.
\]
Similarly,
\[
T_3 = - \int_{D_m} \frac{2}{r} \partial_t \Gamma^p(x,t) dx = 2 \sum_{j=1}^m \int_{L_j} \Gamma^p dx_3 - 2 \int_{L_{m+1}} \Gamma^p dx_3.
\]
Substituting (3.18), (3.20) and (3.21) into (3.17), we find, since \( p \) is an even integer, that
\[
\partial_t \int_{D_m} \Gamma^p(x,t)dx \leq -2(p-1) \sum_{j=1}^{m} \int_{L_j} \Gamma^p dx_3 + 2(p-1) \int_{L_{m+1}} \Gamma^p dx_3 \\
\leq 2(p-1) \int_{L_{m+1}} \Gamma^p dx_3.
\]
Therefore
\[
\| \Gamma(\cdot,t) \|_{L^p(D_m)} \leq \left[ 2(p-1) \int_0^t \int_{L_{m+1}} \Gamma^p dx_3 ds + \int_{D_m} \Gamma^p(x,0)dx \right]^{1/p} \\
\leq (2(p-1))^{1/p} \left[ \int_0^t \int_{L_{m+1}} \Gamma^p dx_3 ds \right]^{1/p} + \| \Gamma(\cdot,0) \|_{L^p(D_m)}.
\]
This shows, after letting \( p \to \infty \), that
\[
\| \Gamma(\cdot,t) \|_{L^\infty(D_m)} \leq \| v_0 \|_{L^\infty(D_m \cap [3/4<r<1])} + \| \Gamma(\cdot,0) \|_{L^\infty(D_m)},
\]
which, together with Lemma 3.3, completes the proof of the lemma.

**Lemma 3.5.** Let \( v \) be a solution to Problem 2.1 coming from Proposition 2.2. Then for a.e. \( t \), the following hold
\[(3.22) \quad \| \nabla \frac{v_r}{r} \|_{L^2(D_m)} \leq \| \frac{\partial_\theta \omega_0}{r} \|_{L^2(D_m)}, \quad \| \nabla \frac{\partial_{x_3} v_r}{r} \|_{L^2(D_m)} \leq \| \partial_{x_3} \frac{\partial_\theta \omega_0}{r} \|_{L^2(D_m)}.
\]

**Proof.** Again we choose those \( t \) so that \( v_r(\cdot,t) \in W^{2,2}(D_m) \). According to Proposition 2.2, those \( t \) form a set of full measure on the time axis. For simplicity, we write \( \Omega = \omega_0/r \) and suppress the \( t \) variable. It is known that the following relation is true:
\[(3.23) \quad (\Delta + \frac{2}{r} \partial_r) \frac{v_r}{r} = \partial_{x_3} \Omega.
\]
For example it can be derived from (2.16) easily.

Using \( \frac{\partial_{x_3} v_r}{r} \) as a test function on (3.23) and using the boundary condition \( v_r = 0 \) on \( \partial^V D_m \), \( \partial_{x_3} v_r = 0 \) on \( \partial^H D_m \) and \( \Omega = 0 \) on \( \partial D_m \), we deduce
\[
\int_{D_m} \left| \nabla \frac{v_r}{r} \right|^2 dx = \int_{D_m} \Omega \partial_{x_3} \frac{v_r}{r} dx
\]
which yields the first inequality in (3.22) immediately.

To prove the 2nd inequality, we use \( \frac{\partial^2_{x_3} v_r}{r} \) as a test function on equation (3.23). Using the notation \( f = \frac{v_r}{r} \) for simplicity, we see that
\[
\int_{D_m} \partial_{x_3} \Omega \partial^2_{x_3} f dx = \int_{D_m} \Delta f \partial^2_{x_3} f dx + \int_{D_m} \frac{2}{r} \partial_r f \partial^2_{x_3} f dx \equiv T_1 + T_2.
\]
Using integration by parts and the boundary condition \( \partial_{x_3} f = 0 \) on \( \partial^V D_m \), we find
\[
T_2 = -\int_{D_m} \frac{2}{r} \partial_r \partial_{x_3} f \partial_{x_3} f dx = -\int_{D_m} \frac{1}{r} \partial_r (\partial_{x_3} f)^2 dx = 0.
\]
Next we work on $T_1$. Note that $\partial x_3 f \in W^{1,2}_0(D_m)$ due to the Navier boundary condition. Hence there exists a sequence of $C^2_0(D_m)$ functions, say $\{h_j\}$, such that $h_j$ converges to $\partial x_3 f$ in $W^{1,2}_0(D_m)$ norm as $j \to \infty$. Thus, since $\nabla f_j$ vanishes on $\partial D_m$,

$$T_1 = \lim_{j \to \infty} \int_{D_m} \Delta f \partial x_3 h_j \, dx = - \lim_{j \to \infty} \int_{D_m} \nabla f \partial x_3 \nabla h_j \, dx = \lim_{j \to \infty} \int_{D_m} \nabla \partial x_3 f \nabla h_j \, dx = \int_{D_m} |\nabla \partial x_3 f|^2 \, dx$$

The last three identities tell us

$$\int_{D_m} |\nabla \partial x_3 f|^2 \, dx = \int_{D_m} \partial x_3 \Omega \partial x_3 f \, dx$$

which proves the 2nd inequality in (3.22).

We need a lemma which states that a uniform Sobolev inequality holds for a subclass of $W^{1,2}(D_m)$ functions on the domain $D_m$. Due to the thinness of $D_m$ near the $x_3$ axis, some extra condition is needed.

**Lemma 3.6.** Let $D_m$ be the domain in (1.9) and $f \in W^{1,2}(D_m)$. Suppose either $f = 0$ on $\partial^2 D_m$, the horizontal boundary of $D_m$, or $\int_{D_m \cap \{r = \text{constant}\}} f(r, x_3) \, dx_3 = 0$. Then there is a uniform constant $s_0$, independent of $m$ or $f$, such that

$$\|f\|_{L^6(D_m)} \leq s_0 \|\nabla f\|_{L^2(D_m)}.$$

**Proof.** By definition $D_m = \cup_{i=1}^m S_i$ and we can write

$$S_i = \cup_{k=1}^{m_i} S_{ik}$$

where $m_i$ is some positive integer and $S_{ik}$ are rectangles in the $r x_3$ plane with the following properties: the above union is non-overlapping and the ratio of the height and width of $S_{ik}$ is between 1 and 2. Note that $S_{ik}$ is not exactly an open rectangle, which may contain a vertical edge. Denote the height of $S_{ik}$ by $h_i$. Since $h_i$ is between 1 and 2 times the width of $S_{ik}$, the standard Sobolev inequality holds on $S_{ik}$. Namely, there exists a uniform positive constant $C_0$ such that

$$\left( \int_{S_{ik}} f^6 \, dx \right)^{1/3} \leq C_0 \int_{S_{ik}} |\nabla f|^2 \, dx + \frac{C_0}{h_i^2} \int_{S_{ik}} f^2 \, dx.$$

Due to the extra assumptions on $f$, we can apply the Poincaré inequality in the $x_3$ direction, which says, for a positive constant $C_1$, 

$$\int_{S_{ik}} f^2 \, dx \leq C_1 h_i^2 \int_{S_{ik}} |\partial x_3 f|^2 \, dx.$$

A combination of the preceding two inequalities yields:

$$\left( \int_{S_{ik}} f^6 \, dx \right)^{1/3} \leq C_0 (1 + C_1) \int_{S_{ik}} |\nabla f|^2 \, dx.$$

Taking $s_0 = \sqrt{C_0(1 + C_1)}$ and using the elementary inequality

$$\left( \sum_{i=1}^j a_i \right)^{1/3} \leq \sum_{i=1}^j a_i^{1/3}, \quad a_i \geq 0, \quad j = 1, 2, 3, \ldots,$$
we find
\[
\left( \int_{D_m} f^6 \, dx \right)^{1/3} = \left( \sum_{i=1}^{m} \sum_{k=1}^{m_i} \int_{S_{ik}} f^6 \, dx \right)^{1/3} \leq \sum_{i=1}^{m} \sum_{k=1}^{m_i} \left( \int_{S_{ik}} f^6 \, dx \right)^{1/3}
\]
\[
\leq s_0^6 \sum_{i=1}^{m} \sum_{k=1}^{m_i} \int_{S_{ik}} |\nabla f|^2 \, dx = s_0^2 \int_{D_m} |\nabla f|^2 \, dx,
\]
which proves the lemma.

Now we turn to the key estimate on the vorticity. Let \( \omega = \nabla \times v = \omega_1 e_r + \omega_2 e_\theta + \omega_3 e_3 \) be the vorticity. Define
\[
J = \frac{\omega_r}{r}, \quad \Omega = \frac{\omega_\theta}{r}.
\]
Then the triple \( J, \Omega, \omega_3 \) satisfy the system: for \( b = v_r e_r + v_3 e_3 \),
\[
\begin{align*}
\Delta J - (b \cdot \nabla) J + \frac{2}{r} \partial_r J + (\omega_1 \partial_r + \omega_2 \partial_\theta + \omega_3 \partial_3) \frac{v_r}{r} - \partial_t J &= 0, \\
\Delta \Omega - (b \cdot \nabla) \Omega + \frac{2}{r} \partial_r \Omega - \frac{2v_\theta}{r} J - \partial_t \Omega &= 0, \\
\Delta \omega_3 - (b \cdot \nabla) \omega_3 + \omega_1 \partial_r \omega_3 + \omega_2 \partial_\theta \omega_3 - \partial_t \omega_3 &= 0.
\end{align*}
\]
These follow from direct computation based on the vorticity equation
\[
\begin{align*}
\left( (\Delta - \frac{1}{r^2}) \omega_r - (b \cdot \nabla) \omega_r + \omega_1 \partial_r \omega_r + \omega_2 \partial_\theta \omega_r - \partial_t \omega_r \right) &= 0, \\
\left( (\Delta - \frac{1}{r^2}) \omega_\theta - (b \cdot \nabla) \omega_\theta + 2 \frac{v_\theta}{r} \partial_r \omega_\theta + \omega_\theta \omega_\theta - \partial_t \omega_\theta \right) &= 0, \\
\left( (\Delta - \frac{1}{r^2}) \omega_3 - (b \cdot \nabla) \omega_3 + \omega_1 \partial_r \omega_3 + \omega_2 \partial_\theta \omega_3 - \partial_t \omega_3 \right) &= 0.
\end{align*}
\]
and the relations
\[
\omega_r = -\partial_3 v_\theta, \quad \omega_\theta = \partial_3 v_r - \partial_r v_3, \quad \omega_3 = \partial_r v_\theta + \frac{v_\theta}{r}.
\]

**Lemma 3.7.** Let \( v \) be a solution to Problem 2.2 coming from Proposition 2.2. There exists a constant \( \lambda_0 = \lambda_0(\|v_\theta(\cdot, 0)\|_{L^\infty(D_m)}), \|v_r(\cdot, 0)\|_{L^\infty(D_m)}, \|v(\cdot, 0)\|_{L^2(D_m)}, \beta) \), depending only on the initial value in terms of the stated quantities and independent of \( m \) such that
\[
\|\Omega(\cdot, t)\|_{L^2(D_m)} + \|J(\cdot, t)\|_{L^2(D_m)} \leq e^{\lambda_0 t} \left[ \|\Omega(\cdot, 0)\|_{L^2(D_m)} + \|J(\cdot, 0)\|_{L^2(D_m)} \right].
\]
Moreover
\[
\left\| \frac{v_r(\cdot, t)}{r} \right\|_{L^2(D_m)} \leq s_0 e^{\lambda_0 t} \left[ \|\Omega(\cdot, 0)\|_{L^2(D_m)} + \|J(\cdot, 0)\|_{L^2(D_m)} \right],
\]
where \( s_0 \) is the Sobolev constant in Lemma 3.6.

**Proof.** The proof is based on inequalities (2.4) for \( \Omega \) and (2.5) for \( J \) in Proposition 2.2. Using integration by parts on (2.4) and the boundary condition that \( \Omega = 0 \) on \( \partial D_m \), we find
\[
\int_0^T \int_{D_m} |\nabla \Omega|^2 \, dx \, dt + \frac{1}{2} \int_0^T \int_{D_m} |\Omega(x, T)|^2 \, dx = \frac{1}{2} \int_0^T \int_{D_m} |\Omega(x, 0)|^2 \, dx
\]
\[
= -\int_0^T \int_{D_m} \frac{v_\theta}{r} \partial_3 \Omega \, dx \, dt = -2 \int_0^T \int_{D_m} \frac{v_\theta}{r} \Omega J \, dx \, dt,
\]
where we have used \( J = -\partial_3 v_\theta/r \).
To control the function \( J \), we need to do integration by parts on the first integral on the right hand side of (2.5).

\[
\int_0^T \int_{D_m} J \frac{2}{r} \partial_r J dx dt = \int_0^T \int \chi_{D_m} \partial_r J^2 dr dx dt \\
= - \sum_{j=1}^m \int_0^T \int_{L_j} J^2 dx dt + \int_0^T \int_{L_{m+1}} J^2 dx dt.
\]

Here as before the vertical part of the boundary \( \partial^V D_m = \bigcup_{j=1}^m L_j \cup L_{m+1} \), with \( L_1, \ldots, L_m \) being the vertical boundary segments to the left of \( D_m \) and \( L_{m+1} \) being the only vertical boundary segment to the right of \( D_m \). Since \( r = 1 \) on \( L_{m+1} \), the above identity implies

\[
\int_0^T \int_{D_m} J \frac{2}{r} \partial_r J dx dt \leq 4 \int_0^T \int_{L_{m+1}} J^2 \left( r - \frac{1}{2} \right) dx dt = 4 \int_0^T \int_0^1 \int_{1/2}^1 \partial_r \left[ J^2 \left( r - \frac{1}{2} \right)^2 \right] dr dx dt \\
= 4 \int_0^T \int_0^1 \int_{1/2}^1 \left[ 2J \partial_r J \left( r - \frac{1}{2} \right)^2 + 2J^2 \left( r - \frac{1}{2} \right) \right] dr dx dt \\
\leq \frac{1}{2} \int_0^T \int_{D_m} |\nabla J|^2 dx dt + 10 \int_0^T \int_{D_m} J^2 dx dt.
\]

Substituting this to the right hand side of (2.5), we find

\[
\int_0^T \int_{D_m} |\nabla J|^2 dx dt + \int_0^T \int_{D_m} |J(x, T)|^2 dx - \int_0^T \int_{D_m} |J(x, 0)|^2 dx \\
\leq 2 \int_0^T \int_{D_m} \left\{ v_\theta \partial_\theta v_\rho \partial_\rho J - v_\theta \partial_\theta v_\rho \partial_\rho J \right\} dx dt + 1000 \int_0^T \int_{D_m} J^2 dx dt.
\]

This infers, after applying Cauchy-Schwarz, that

\[
\int_0^T \int_{D_m} |\nabla J|^2 dx dt + 2 \int_0^T \int_{D_m} |J(x, T)|^2 dx - 2 \int_0^T \int_{D_m} |J(x, 0)|^2 dx \\
\leq 4 \int_0^T \int_{D_m} v^2 \left( \partial_\theta v_\rho^2 + \partial_\rho v_\rho^2 \right) dx dt + 2000 \int_0^T \int_{D_m} J^2 dx dt.
\]

A combination of (3.28) and (3.29) yields

\[
\int_{D_m} \left( J^2 + \Omega^2 \right) dx \bigg|_0^T + \frac{1}{2} \int_0^T \int_{D_m} \left( |\nabla J|^2 + |\nabla \Omega|^2 \right) dx dt \\
\leq - \int_0^T \int_{D_m} \frac{4v_\theta^2}{r} \Omega J dx dt + 2 \int_0^T \int_{D_m} v_\theta^2 \left( \left| \partial_\theta v_\rho \right|^2 + \left| \partial_\rho v_\rho \right|^2 \right) dx dt + 1000 \int_0^T \int_{D_m} J^2 dx dt.
\]

We will show that the first two terms on the right hand side of (3.30) can be absorbed by the left hand side, modulo lower order terms. According to Lemma 3.4, there is a positive constant \( \tilde{C}_1 = \tilde{C}_1(||v(\cdot, 0)||_{L^\infty(D_m)}, ||v(\cdot, 0)||_{L^2(D_m)}) \),

\[
||\Gamma(\cdot, t)||_{L^\infty(D_m)} \leq ||\Gamma(\cdot, 0)||_{L^\infty(D_m)} + \tilde{C}_1, \quad \forall t > 0.
\]
Since $\Gamma = rv_\theta$, it follows that
\begin{equation}
|v_0(r, x_3, t)| \leq \frac{||rv_\theta(\cdot, 0)||_{L^\infty(D_m)} + \bar{C}_1}{r}, \quad \forall t > 0, (r, x_3) \in D_m.
\end{equation}

Plugging (3.31) into (3.30), we deduce
\begin{align}
&\int_{D_m} \left( J^2 + \Omega^2 \right) dx + \frac{1}{2} \int_0^T \int_{D_m} \left( |\nabla J|^2 + |\nabla \Omega|^2 \right) dx dt \\
&\leq 2C_s(v_0) \int_0^T \int_{D_m} \left( \frac{1}{r^2} \Omega^2 dx dt \right) + 2C_s(v_0) \int_0^T \int_{D_m} \left( \frac{1}{r^2} J^2 dx dt \right) \\
&+ 2C_s^2(v_0) \int_0^T \int_{D_m} \left( \frac{1}{r^2} \left( \left| \partial_r v_r \right|^2 + \left| \partial_x v_r \right|^2 \right) dx dt \right) + 1000 \int_0^T \int_{D_m} J^2 dx dt.
\end{align}

Let us estimate $T_1, T_2$ and $T_3$ respectively.

Recall from (1.9) that $D_m = \bigcup_{j=1}^m S_j$, the union of rectangles
\[ S_j = \{(r, x_3) | 2^{-j} \leq r < 2^{-(j-1)}, 0 < x_3 < 2^{-\theta(j-1)} \} \]
in the $rx_3$ plane. Denote by $h_j$ the height of $S_j$ (maximum value of $x_3$) and $r_j$ the width of $S_j$. According to the boundary condition, the functions $\Omega = \omega_0/r$, $J = -\partial_{x_3} v_\theta/r$ and $\partial_{x_3} v_r/r$ all vanishes on $\partial^\mu D_m$ in point-wise or trace sense a.e. $t$. Recall from Lemma 3.2
\[ \int_{D_m \cap \{r=\text{const.}\}} v_r(r, x_3, t) dx_3 = 0, \]
which infers
\[ \int_{D_m \cap \{r=\text{const.}\}} \partial_r [v_r(r, x_3, t)/r] dx_3 = 0. \]

Hence the one dimensional Poincaré inequality
\begin{equation}
\int_{S_j \cap \{r=\text{const.}\}} f^2(x_3) dx_3 \leq h_j^2 \int_{S_j \cap \{r=\text{const.}\}} |\partial_{x_3} f|^2(x_3) dx_3
\end{equation}
holds for all four of these functions
\[ f = f(x_3) = \Omega(r, x_3, t), J(r, x_3, t), \partial_{x_3} v_r(r, x_3, t)/r, \text{or } \partial_t [v_r(r, x_3, t)/r]. \]

Let $j_0$ be a positive integer to be determined later. We can estimate, using (3.33),
\begin{align}
T_1 &= \sum_{j=1}^m \int_{S_j} \frac{1}{r^2} \Omega^2 dx = \sum_{j=j_0}^m \int_{S_j} \frac{1}{r^2} \Omega^2 dx + \sum_{j=1}^{j_0-1} \int_{S_j} \frac{1}{r^2} \Omega^2 dx \\
&\leq \sup_{j_0 \leq j \leq m} \frac{h_j^2}{r_j^2} \int_{D_m} |\partial_{x_3} \Omega|^2 dx + 4j_0 \int_{D_m} \Omega^2 dx.
\end{align}
Analogously

\[
T_2 \leq \sup_{j_0 \leq j \leq m} \frac{h^2_j}{r_j^2} \int_{D_m} |\partial_{x_i} J|^2 dx + 4^{j_0} \int_{D_m} J^2 dx;
\]

\[
T_3 \leq \sup_{j_0 \leq j \leq m} \frac{h^2_j}{r_j^2} \int_{D_m} \left| \nabla \partial_{x_i} \frac{v_r}{r} \right|^2 dx + 4^{j_0} \int_{D_m} \left| \nabla \frac{v_r}{r} \right|^2 dx
\]

\[
\leq \sup_{j_0 \leq j \leq m} \frac{h^2_j}{r_j^2} \int_{D_m} \left| \partial_{x_i} \Omega \right|^2 dx + 4^{j_0} \int_{D_m} \Omega^2 dx
\]

(3.36)

where we have used Lemma 3.5. Substituting (3.34), (3.35) and (3.36) into (3.32) we find

\[
\int_{D_m} \left( J^2 + \Omega^2 \right) dx \bigg|_0^T + \frac{1}{2} \int_0^T \int_{D_m} \left( |\nabla J|^2 + |\nabla \Omega|^2 \right) dx dt
\]

\[
\leq [2C_\ast(v_0) + 2C^2_\ast(v_0)] \sup_{j_0 \leq j \leq m} \frac{h^2_j}{r_j^2} \int_0^T \int_{D_m} \left| \partial_{x_i} \Omega \right|^2 dx dt + 2C_\ast(v_0) \sup_{j_0 \leq j \leq m} \frac{h^2_j}{r_j^2} \int_0^T \int_{D_m} \left| \partial_{x_i} J \right|^2 dx dt
\]

\[
+ [2C_\ast(v_0) + 2C^2_\ast(v_0)] 4^{j_0} \int_0^T \int_{D_m} \Omega^2 + [2C_\ast(v_0) 4^{j_0} + 1000] \int_0^T \int_{D_m} J^2 dx dt.
\]

By our choice of \( \beta \in (1, 1.1) \) in the construction of the domain \( D_m \), we can pick the smallest integer \( j_0 \) such that

\[
j_0 \geq (\beta - 1)^{-1} \ln(4^{2-1}[2C_\ast(v_0) + 2C^2_\ast(v_0)]) / \ln 4.
\]

Then

\[
[2C_\ast(v_0) + 2C^2_\ast(v_0)] \sup_{j_0 \leq j \leq m} \frac{h^2_j}{r_j^2} \leq 1/4.
\]

Consequently

\[
\int_{D_m} \left( J^2 + \Omega^2 \right) dx \bigg|_0^T + \frac{1}{4} \int_0^T \int_{D_m} \left( |\nabla J|^2 + |\nabla \Omega|^2 \right) dx dt
\]

\[
\leq [2C_\ast(v_0) + 2C^2_\ast(v_0)] 4^{j_0} \int_0^T \int_{D_m} \Omega^2 + [2C_\ast(v_0) 4^{j_0} + 1000] \int_0^T \int_{D_m} J^2 dx dt.
\]

Taking

\[
\lambda_0 = [2C_\ast(v_0) + 2C^2_\ast(v_0)] 4^{j_0} + 1000.
\]

and applying Gronwall’s inequality, we have verified the first inequality in the statement of the lemma. From the definition of \( C_\ast(v_0) \) in (3.31), we see that \( \lambda_0 \) depends only on the 3 stated quantities about the initial value and the parameter \( \beta \) for \( D_m \). The second one follows from the first one, Lemma 3.5 and Lemma 3.6, which is applicable since the line integral in the \( x_3 \) direction for \( v_r/r \) is 0 by Lemma 3.2. \( \square \)
3.2. **A priori bounds for** \( \|v_0\|_{L^5_\infty} \), \( \|v_t\|_{L^5_\infty} \), \( \|v_3\|_{L^5_\infty} \). Based on the a priori bounds from the previous subsection, we will prove an a priori bound for \( \|v\|_{L^5_\infty} \). Let us start with \( v_0 \). We will apply Moser’s iteration on the equation for \( v_0 \) in (1.3):

\[
(\Delta - \frac{1}{r})v_\theta - (v_r \partial_r + v_3 \partial_3)v_\theta - \frac{v_\theta}{r} - \partial_1 v_\theta = 0, \\
v_\theta(x, 0) = (v_0)_0(x).
\]

Intuitively, we regard \( v_r / r \) as a potential function for \( v_\theta \). According to Lemma 3.7, \( v_r / r \in L^\infty_\mu L^6_\lambda \), which makes it a subcritical potential. Hence there is a chance to prove \( L^\infty \) bound for \( v_\theta \). There is an extra hurdle to face in though, due to the lack of uniform Sobolev inequality for functions like \( v_\theta \) on \( D_m \). Instead, we will use a Sobolev inequality with a smaller power and extra weight \( 1/r^2 \), which is weaker than the standard Sobolev inequality but is sufficient to carry out Moser’s iteration. The constant in this weaker inequality is uniform for all \( D_m \). The exponent \( 13/5 \) may be improved, which will result in an improvement in the range of \( \beta \) in the definition of the domain in the main theorem.

**Lemma 3.8.** Let \( f \in W^{1,2}(D_m) \), \( m = 1, 2, 3, \ldots \). There exists a uniform constant \( C_0 \) such that

\[
\left( \int_{D_m} |f|^{13/5} \, dx \right)^{10/13} \leq C_0 \int_{D_m} \left( |\nabla f|^2 + \frac{1}{r^2} f^2 \right) \, dx.
\]

**Proof.** By definition again \( D_m = \bigcup_{j=1}^m S_j \) where \( S_j \) can be regarded as a rectangle in the \( r x_3 \) plane with width \( r_j = 2^{-j} \) and height \( h_j = 2^{-\beta(j-1)} \) with \( \beta \in (1, 1.1] \). By the standard Sobolev inequality on \( S_j \), there exists a uniform positive constant \( C_0 \) such that

\[
\left( \int_{S_j} f^6 \, dx \right)^{1/3} \leq C_0 \int_{S_j} |\nabla f|^2 \, dx + \frac{C_0}{h_j^2} \int_{S_j} f^2 \, dx.
\]

Observe that \( h_j \) is much smaller than \( r_j \) for large \( j \) since \( \beta \in (1, 1.1] \). This and Hölder inequality imply

\[
\left( \int_{S_j} |f|^{13/5} \, dx \right)^{10/13} \leq \left( \int_{S_j} f^{(13/5)(30/13)} \, dx \right)^{1/3} \left( \int_{S_j} \, dx \right)^{17/39} \leq C \left( \int_{S_j} f^6 \, dx \right)^{1/3} (r_j^2 h_j)^{17/39}.
\]

By the choice that \( \beta \in (1, 1.1] \), we have \( r_j^{103/61} \leq h_j = o(r_j) \) for \( j \geq 100 \). Therefore

\[
(r_j^2 h_j)^{17/39} / h_j^2 = r_j^{34/39} / h_j^{61/39} \leq r_j^{-69/39} \leq r_j^{-2}.
\]

Substituting this to (3.38), we find

\[
\left( \int_{S_j} |f|^{13/5} \, dx \right)^{10/13} \leq C \left[ C_0 \int_{S_j} |\nabla f|^2 \, dx + \frac{C_0}{r_j^2} \int_{S_j} f^2 \, dx \right]
\]

Summing up, we deduce

\[
\left( \int_{D_m} |f|^{13/5} \, dx \right)^{10/13} \leq \sum_{j=1}^m \left( \int_{S_j} |f|^{13/5} \, dx \right)^{10/13} \leq CC_0 \left[ \int_{D_m} |\nabla f|^2 \, dx + \int_{D_m} \frac{1}{r^2} f^2 \, dx \right]
\]
The lemma is proven by taking $C_s = C C_0$. □

The next lemma provides a $L^\infty$ bound for $v_\theta$.

**Lemma 3.9.** Let $v$ be a solution to Problem 2.1 coming from Proposition 2.2. Then, there is an absolute constant $C$ such that

$$
\|v_\theta(\cdot, T)\|_{L^\infty(D_m)} \leq \begin{cases}
C \left[ 1 + C_s^{13/5} \|v_\theta\|_{L^{18/5}_{t=0} L^6(D_m)} \right]^{8/3} |D_m|^{1/2} \|v_\theta(t)\|_{L^\infty(D_m)}, & T \leq 1, \\
C \left[ 1 + C_s^{13/5} \|v_\theta\|_{L^{18/5}_{t=0} L^6(D_m)} \right]^{8/3} \left( \int_{D_m} |v_\theta(t)|^2 dx \right)^{1/2}, & T \geq 1.
\end{cases}
$$

Here $C_s$ is the Sobolev constant in Lemma 3.8.

**Proof.** We will work on the equation (3.37). Except for the term $-v_r v_\theta/r$, the treatment of other terms are similar to those in Lemma 3.3.

For any rational number $p > 1$ in the form of $2k/l$ where $k, l$ are positive integers, we know that $v_\theta^p$ is a sub-solution, namely

$$
\begin{align*}
&\Delta v_\theta^p - \frac{1}{r} v_\theta^p - b \nabla v_\theta^p - p \frac{v_r v_\theta^p}{r} - \partial_r v_\theta^p \geq 0, \quad \text{in} \quad D_m \times (0, \infty) \\
&v_\theta^p(x, 0) = [(v_\theta(t)]^p, x \in D_m.
\end{align*}
$$

For $T \geq 1$, let $\eta = \eta(t)$ be a cut off function in time, supported in $[T - 1, T]$ which will be specified later. Since a nonuniform $L^\infty$ bound for $v_\theta$ is already proven (Step 2 in the proof of Proposition 2.2), we can use $v_\theta^p \eta^2$ as a test function on (3.40) to deduce

$$
\begin{align*}
&\frac{L_S}{2} \equiv \int_{T-1}^T \int_{D_m} \|
abla(v_\theta^p \eta)\|^2 dx \, dt + \int_{T-1}^T \int_{D_m} \frac{1}{r^2} v_\theta^{2p} \eta^2 dx \, dt + \frac{1}{2} \int_{D_m} v_\theta^{2p} \eta^2(x, T) dx \\
&\leq \int_{T-1}^T \int_{D_m} \partial_n v_\theta^p v_\theta \eta^2 dS \, dt - \int_{T-1}^T \int_{D_m} b \nabla v_\theta^p v_\theta \eta^2 dx \, dt \\
&\quad - \int_{T-1}^T \int_{D_m} \frac{v_r}{r} v_\theta^{2p} \eta^2 dx \, dt + \int_{T-1}^T \int_{D_m} v_\theta^{2p} \eta^2(x) \, dx \, dt + \frac{1}{2} \int_{D_m} v_\theta^{2p} \eta^2(x, T - 1) dx,
\end{align*}
$$

where $n$ is the exterior normal of $\partial D_m$. Next we will find bounds for $R_1, R_2$ and $R_3$.

First, using integration by parts together with the boundary condition $v_r = 0$ on $\partial^V D_m$, $v_3 = 0$ on $\partial^H D_m$, we see that

$$
R_3 = 0.
$$

To bound $R_1$, we use the boundary conditions $\partial_{x_3} v_\theta = 0$ on $\partial^H D_m$. Therefore

$$
R_1 = \int_{T-1}^T \int_{\partial^V D_m} \partial_n v_\theta^p v_\theta \eta^2 dS \, dt \\
= - \sum_{j=1}^m \int_{T-1}^T \int_{L_j} \partial_{x_3} v_\theta^p v_\theta \eta^2 dx_3 dt + \int_{T-1}^T \int_{L_{m+1}} \partial_{x_3} v_\theta^p v_\theta \eta^2 dx_3 dt.
$$

Here again $\partial^V D_m = \bigcup_{j=1}^m L_j \cup L_{m+1}$, with $L_1, \ldots, L_m$ being the vertical boundary segments to the left of $D_m$ and $L_{m+1}$ being the only vertical boundary segment to the right of $D_m$. Notice
that \( \partial_n = -\partial_r \) on \( L_1, \ldots, L_m \) and \( \partial_n = \partial_r \) on \( L_{m+1} \). On \( L_{m+1} \) and \( L_j, i = 1, 2, \ldots, m \), the Navier boundary condition reads

\[
\partial_r y_\theta^p = pv_\theta^{p-1} \partial_r y_\theta = pr^{-1} y_\theta^p.
\]

Consequently

\[
R_1 = p \int_{T-1}^{T} \int_{L_{m+1}} v_\theta^{2p} \eta^2 dx_3 dt - p \sum_{j=1}^{m} \int_{T-1}^{T} \int_{L_j} v_\theta^{2p} \eta^2 dx_3 dt
\]

\[
\leq p \int_{T-1}^{T} \int_{L_{m+1}} v_\theta^{2p} \eta^2 dx_3 dt.
\]

For simplicity we write \( f \equiv v_\theta^p \). Since \( r = 1 \) on \( L_{m+1} \), we see that

\[
R_1 \leq 4p \int_{T-1}^{T} \int_{L_{m+1}} f^2 \left( r - \frac{1}{2} \right)^2 \eta^2 dx_3 dt = 4p \int_{T-1}^{T} \int_{0}^{1} \int_{1/2}^{1} \partial_r \left[ f^2 \left( r - \frac{1}{2} \right)^2 \right] \eta^2 dr dx_3 dt.
\]

This implies

\[
R_1 \leq \frac{1}{2} \int_{T-1}^{T} \int_{D_m} |\nabla f|^2 \eta^2 dx dt + 10p^2 \int_{T-1}^{T} \int_{D_m} f^2 \eta^2 dx dt.
\]

To estimate \( R_3 \), let us notice that

\[
\int_{D_m} \frac{|y_\theta|}{r} v_\theta^{2p} dx = \int_{D_m} \frac{|y_\theta|}{r} f^2 dx = \int_{D_m} \frac{|y_\theta|}{r} f^{13/9} f^{5/9} dx
\]

\[
\leq \left( \int_{D_m} \frac{|y_\theta|^6}{r^6} dx \right)^{1/6} \left( \int_{D_m} f^{13/5} dx \right)^{5/9} \left( \int_{D_m} f^2 dx \right)^{5/18} \quad \text{(by Hölder inequality)}
\]

\[
\leq \epsilon^{18/13} \left( \int_{D_m} f^{13/5} dx \right)^{10/13} + \epsilon^{-18/5} \| \frac{y_\theta}{r} \|_{L^5(D_m)}^{18/5} \int_{D_m} f^2 dx
\]

\[
\leq C_3 \epsilon^{18/13} \int_{D_m} \left( |\nabla f|^2 + \frac{1}{r^2} f^2 \right) dx + \epsilon^{-18/5} \| \frac{y_\theta}{r} \|_{L^5(D_m)}^{18/5} \int_{D_m} f^2 dx \quad \text{(by Lemma 3.8)}.
\]

Thus

\[
|R_3| \leq pC_3 \epsilon^{18/13} \int_{T-1}^{T} \int_{D_m} \left( |\nabla f|^2 + \frac{1}{r^2} f^2 \right) dx + p \epsilon^{-18/5} \| \frac{y_\theta}{r} \|_{L^5(D_m)}^{18/5} \int_{T-1}^{T} \int_{D_m} f^2 dx.
\]

From this inequality, we choose \( \epsilon \) so that \( pC_3 \epsilon^{18/13} = 1/8 \) or \( p \epsilon^{-18/5} = (8C_3)^{13/5} p^{18/5} \) to obtain

\[
|R_3| \leq \frac{1}{8} \int_{T-1}^{T} \int_{D_m} \left( |\nabla f|^2 + \frac{1}{r^2} f^2 \right) \eta^2 dx dt + (8C_3)^{13/5} p^{18/5} \| \frac{y_\theta}{r} \|_{L^5(D_m)}^{18/5} \int_{T-1}^{T} \int_{D_m} f^2 \eta^2 dx dt.
\]
The combination of (3.46), (3.45), (3.42), and (3.41) yields
\[
\frac{1}{4} \int_{T-1}^T \int_{D_m} |\nabla (f \eta)|^2 dx dt + \frac{1}{4} \int_{T-1}^T \int_{D_m} \frac{1}{r^2} f^2 \eta^2(x, t) dx dt + \frac{1}{2} \int_{D_m} f^2 \eta^2(x, T) dx
\]
\[
\leq \left[ 10p^2 + (8C_s)^{13/5} p^{18/5} \| \frac{V_f}{R} \|^{|18/5|}_{13/5}(D_m) \right] \int_{T-1}^T \int_{D_m} f^2 \eta^2 dx dt +
\]
\[
+ \int_{T-1}^T \int_{D_m} f^2 \eta' dx dt + \frac{1}{2} \int_{D_m} f^2 \eta^2(x, T - 1) dx.
\]
(3.47)

Next
\[
\int_{T-1}^T \int_{D_m} (f \eta)^{32/13} dx dt = \int_{T-1}^T \int_{D_m} (f \eta)^2 (f \eta)^{6/13} dx dt
\]
\[
\leq \int_{T-1}^T \left( \int_{D_m} |f \eta|^{13/5} dx \right)^{10/13} \left( \int_{D_m} |f \eta|^{(6/13)(13/3)} dx \right)^{3/13} dt
\]
\[
\leq \sup_{t \in [T-1, T]} \left( \int_{D_m} |f \eta|^{2} (x, t) dx \right)^{3/13} \int_{T-1}^T \left( \int_{D_m} |f \eta|^{13/5} dx \right)^{10/13} dt
\]
\[
\leq C_s \sup_{t \in [T-1, T]} \left( \int_{D_m} |f \eta|^{2} (x, t) dx \right)^{3/13} \int_{T-1}^T \int_{D_m} \left[ \nabla (f \eta)^2 + \frac{1}{r^2} f^2 \eta^2 \right] dx dt
\]
where we have used Lemma 3.8. Plugging (3.47) to the right hand side in the preceding inequality gives, since \( f = v_p' \),
\[
\left( \int_{T-1}^T \int_{D_m} (v_p')^{32/13} dx dt \right)^{13/16} \leq 4 \left[ 10p^2 + (8C_s)^{13/5} p^{18/5} \| \frac{V_f}{R} \|^{|18/5|}_{13/5}(D_m) + \sup |\eta'| \right] \times \left( \int_{T-1}^T \int_{D_m} v_p^2 \eta dx dt + \int_{D_m} v_p^2 \eta^2(x, T - 1) dx \right)
\]
(3.48)

Take \( T = 1 \) and \( \eta = 1 \) first. Then the integrand in the last term in the above is bounded by \( 2 \| (v_0) \|_{10}^2 \|_D(D_m) \). Writing \( \alpha_0 = \| (v_0) \|_{10}^2 \|_D(D_m) \), then (3.48) implies
\[
\left( \int_{0}^{1} \int_{D_m} [(v_0) \vee \alpha_0]^{2p} \right)^{16/13} dx dt \leq C \left[ 10p^2 + (8C_s)^{13/5} p^{18/5} \| \frac{V_f}{R} \|^{|18/5|}_{13/5}(D_m) \right] \times \left( \int_{0}^{1} \int_{D_m} (v_0) \vee \alpha_0 \right)^{2p} dx dt.
\]
(3.49)

Here \( |v_0| \vee \alpha_0 = \max(|v_0|(x, t), \alpha_0) \) and \( C \) is an absolute constant. By Moser’s iteration from (3.49) with \( p = (16/13)^j, j = 0, 1, 2, \ldots \), we deduce
\[
\| v_0 \|_{10}^2 \|_D(D_m \times [0, 1]) \leq C \left[ 1 + C_s^{13/5} \| \frac{V_f}{R} \|^{|18/5|}_{10}(D_m) \right]^{8/3} \left( \int_{0}^{1} \int_{D_m} (|v_0| + \alpha_0) \right)^{1/2} dx dt.
\]
(3.50)
where $C$ is another absolute constant. For $T > 1$, we take suitable sequences of $\eta$ such that $\eta(T - 1) = 0$. Then the last term in (3.48) drops out. Moser’s iteration again tells us,

$$
\|v_0\|_{L^\infty(D_m \times [T-0.5,T])} \leq C \left[ 1 + C_s^{13/5} \left( \frac{V_r}{r} \right)_{L^\infty(D_m)}^{18/5} \right]^{8/3} \left( \int_{T-0.75}^{T} \int_{D_m} |v_\theta|^2 \, dx \, dt \right)^{1/2},
$$

(3.51)

$$
\leq C \left[ 1 + C_s^{13/5} \left( \frac{V_r}{r} \right)_{L^\infty(D_m)}^{18/5} \right]^{8/3} \left( \int_{D_m} |v_\theta|^2 \, dx \right)^{1/2}
$$

where $C$ is an absolute constant and the energy inequality is used. The lemma follows from (3.50) and (3.51).

\[ \square \]

The next lemma provides a $L^\infty$ bound for $\omega_\theta$, which is needed to bound $v_r$ and $v_3$.

**Lemma 3.10.** Let $v$ be a solution to Problem 2.1 coming from Proposition 2.2. Then, there is an absolute constant $C$ such that

$$
\|\omega_\theta(\cdot, T)\|_{L^\infty(D_m)} \leq \begin{cases} 
C \Lambda_0^{5/4} \left( \int_{D_m} (|v_0| + \|\omega_\theta\|_{L^\infty(D_m)}) + 1 \right)^2 \, dx \right)^{1/2}, & T \leq 1, \\
C (\Lambda + 1)_0^{5/4} \left( \int_{D_m} (|v_0| + 1)^2 \, dx \right)^{1/2}, & T \geq 1,
\end{cases}
$$

(3.52)

where

$$
\Lambda_0 \equiv C_s^{13/5} \left( \frac{V_r}{r} \right)_{L^\infty(D_m)}^{18/5} + \|v_\theta\|_{L^\infty(D_m \times [T-1,T])}^2 \left( \frac{\partial_x v_\theta}{r} \right)_{L^\infty(D_m \times [T-1,T])}^2 + 1
$$

Here $C_s$ is the Sobolev constant in Lemma 3.8.

**Proof.** The proof is based on the equation for $\omega_\theta$ which was already stated as in (2.44) e.g.

$$
\begin{cases} 
(\Delta - \frac{1}{r^2}) \omega_\theta - (b\nabla) \omega_\theta + 2 \frac{w_\theta}{r} \partial_x \omega_\theta + \omega_\theta \frac{\partial_x v_\theta}{r} - \partial_t \omega_\theta = 0, & D_m \times (0, T) \\
\omega_\theta = 0, & \partial D_m \times (0, T); \\
\omega_\theta(x, 0) = (\omega_0)(x), & x \in D_m.
\end{cases}
$$

(3.53)

The procedure is similar to that of Lemma 3.9 with two differences. One is that the boundary value of $\omega_\theta$ is 0 which makes the proof shorter in this aspect. The other, however, is the presence of the extra nonhomogeneous term $2 \frac{w_\theta}{r} \partial_x v_\theta$ which needs to be dealt with.

For any rational number $p > 1$ in the form of $2k/l$ where $k, l$ are positive integers, we know that $f = \omega_\theta^p$ is a sub-solution to the above equation, namely

$$
\begin{cases} 
(\Delta - \frac{1}{r^2}) f - (b\nabla) f + p \frac{w_\theta}{r} f + 2p \frac{w_\theta}{r} \partial_x v_\theta \omega_\theta^{p-1} - \partial_t f \geq 0, & D_m \times (0, T) \\
f = 0, & \partial D_m \times (0, T); \\
f(x, 0) = (\omega_0)^p(x), & x \in D_m.
\end{cases}
$$

(3.54)

As in the previous lemma, for $T \geq 1$, let $\eta = \eta(t)$ be a cut off function in time, supported in $[T - 1, T]$ which will be specified later. Since a nonuniform $L^q$ bound with any $q > 1$ for $\omega_\theta$ (c.f. (2.15)) and actually $L^\infty$ bound is already proved in Section 2, we can use $f \eta^2$ as a test
function on \(3.54\) to deduce (3.55)

\[
LS \equiv \int_{T-1}^{T} \int_{D_m} |\nabla (f \eta)|^2 dx dt + \int_{T-1}^{T} \int_{D_m} \frac{1}{r^2} f^2 \eta^2 (x, t) dx dt + \frac{1}{2} \int_{D_m} f^2 \eta^2 (x, T) dx
\]

\[
\leq - \int_{T-1}^{T} \int_{D_m} b \nabla f \cdot f \eta^2 dx dt + p \int_{T-1}^{T} \int_{D_m} \frac{V}{r^2} f^2 \eta^2 dx dt
\]

\[
+ 2p \int_{T-1}^{T} \int_{D_m} \frac{V}{r^2} \partial_{x_3} \omega \frac{p-1}{p} f \eta^2 dx dt + \frac{8}{p} \int_{T-1}^{T} \int_{D_m} f^2 \eta^2 dx dt + \frac{1}{2} \int_{D_m} f^2 \eta^2 (x, T - 1) dx,
\]

As in the previous lemma, \(T_1 = 0\) by integration by parts. The term \(T_2\) can be estimated exactly as the term \(R_3\) in the previous lemma, although the function \(f\) is different. So, like (3.46), we have (3.56)

\[
|T_2| \leq \frac{1}{8} \int_{T-1}^{T} \int_{D_m} \left( |\nabla f|^2 + \frac{1}{r^2} f^2 \right) \eta^2 dx dt + (8C_s)^{13/5} p^{18/5} \frac{V}{r^2} \frac{\eta^2}{L_{\infty} (D_m)} \int_{T-1}^{T} \int_{D_m} f^2 \eta^2 dx dt.
\]

We remark that the power \(18/5\) can be improved since the standard Sobolev inequality holds for \(f\) due to the 0 boundary condition. Since a lower power is not needed we will not pursue it here.

So we are left to bound \(T_3\) which requires the \(L_{r}^{\infty}L_{\dot{\chi}}^{2}\) bound for \(J = -\partial_{x_3} \theta / r\) in Lemma \(3.7\) and the \(L^{m}\) bound for \(\theta\) in the \(L^{m}\) bound for \(\theta\) in Lemma \(3.9\). Since \(|\omega \frac{p-1}{p} f| \leq (|\omega| + 1) f = (f + 1) f\), we see that

(3.57)

\[
|T_3| \leq 2p \|v_o\|_{L^\infty (Q_1)} \int_{T-1}^{T} \int_{D_m} |J| f (f + 1) dx \eta^2 dt,
\]

where \(Q_1 = D_m \times [T - 1, T]\). For the spatial integral on the right hand side, we can proceed as follows. By Hölder inequality, for \(f = f(\cdot, t), J = J(\cdot, t)\) and \(\epsilon \in (0, 1),\)

\[
\int_{D_m} |J| (f + 1)^2 dx = \int_{D_m} |J| (f + 1)^{3/2} (f + 1)^{1/2} dx
\]

\[
\leq \left( \int_{D_m} |J|^2 dx \right)^{1/2} \left( \epsilon \int_{D_m} (f + 1)^6 dx \right)^{1/4} \left( \epsilon^{-1} \int_{D_m} (f + 1)^2 dx \right)^{1/4}
\]

\[
\leq \epsilon^{1/3} \left( \int_{D_m} (f + 1)^6 dx \right)^{1/3} + \epsilon^{-1} \|J(\cdot, t)\|_{L^2} \int_{D_m} (f + 1)^2 dx
\]

\[
\leq 2 \epsilon^{1/3} \left( \int_{D_m} f^6 dx \right)^{1/3} + [\epsilon^{-1} \|J(\cdot, t)\|_{L^2}^2 + 2] \int_{D_m} (f + 1)^2 dx.
\]
Here we have used $|D_m| > 1/2$. Substituting this to (3.57) and using the standard 3 dimensional Sobolev inequality for functions with 0 boundary, we deduce
\[
|T_3| \leq 4e^{1/3}p\|v_0\|_{L^\infty(Q_t)\mathcal{S}_0^2}^2 \int_{T-1}^T \int_{D_m} |\nabla f|^2 \eta^2 dx dt \\
+ 2p\|v_\theta\|_{L^\infty(Q_t)}[e^{-1}\|J\|_{L^1_t L^4_x}^4 + 2] \int_{T-1}^T \int_{D_m} (f + 1)^2 \eta^2 dx dt.
\]
Here $\mathcal{S}_0$ is the $L^2$ Sobolev constant in $\mathbb{R}^3$. This infers, after taking $\epsilon$ such that $4e^{1/3}p\|v_0\|_{L^\infty(Q_t)\mathcal{S}_0} = 1/4$, that
\[
|T_3| \leq \frac{1}{4} \int_{T-1}^T \int_{D_m} |\nabla f|^2 \eta^2 dx dt \\
+ 4(16\mathcal{S}_0^2)^3\|v_\theta\|_{L^\infty(Q_t)}^4\| J \|_{L^4_t L^2_x}^4 + 1) p^4 \int_{T-1}^T \int_{D_m} (f + 1)^2 \eta^2 dx dt.
\]
After substituting (3.56) and (3.58) into (3.55), we arrive at
\[
\frac{1}{2} \int_{T-1}^T \int_{D_m} \left[ |\nabla (f\eta)|^2 dx dt \right] + \frac{1}{r^2} f^2 \eta^2 dx dt + \frac{1}{2} \int_{D_m} f^2 \eta^2(x,T) dx \\
\leq \Lambda p^4 \int_{T-1}^T \int_{D_m} (f + 1)^2 \eta^2 dx dt + \int_{T-1}^T \int_{D_m} f^2 \eta' \eta dx dt + \frac{1}{2} \int_{D_m} f^2 \eta^2(x,T-1) dx,
\]
where
\[
\Lambda \equiv \left( 8C_s^{13/5} F_\infty \right)^{18/5} + 4(16\mathcal{S}_0^2)^3\|v_\theta\|_{L^\infty(D_m)}^4(\| J \|_{L^4_t L^2_x}^4 + 1).
\]
We treat the case $T = 1$ first. Taking $\eta = 1$ in (3.59), we obtain
\[
\int_0^1 \int_{D_m} \left[ |\nabla f|^2 dx dt \right] + \frac{1}{r^2} f^2 dx dt + \int_{D_m} f^2(x,1) dx \\
\leq 2\Lambda p^4 \int_0^1 \int_{D_m} (f + 1)^2 dx dt + \int_{D_m} f^2(x,0) dx,
\]
which implies, via the standard Sobolev inequality and interpolation that
\[
\left( \int_0^1 \int_{D_m} \omega_\theta^{10/3} dx dt \right)^{3/5} \leq \mathcal{S}_0^{6/5} 2\Lambda p^4 \int_0^1 \int_{D_m} (|\omega_\theta|^p + 1 + \beta_0)^2 dx dt,
\]
where $\beta_0 \equiv \sup_{D_m} (|\omega_\theta|_\theta(x))$. Since $|D_m| > 1/2$, this implies, for some absolute constant $C$,
\[
\left( \int_0^1 \int_{D_m} (|\omega_\theta| \vee (1 + \beta_0))^{2p(5/3)} dx dt \right)^{3/5} \leq C\Lambda p^4 \int_0^1 \int_{D_m} (|\omega_\theta| \vee (1 + \beta_0))^{2p} dx dt.
\]
Here $|\omega_0|\vee \beta_0 = \max\{|\omega_0|(x, t), \beta_0\}$. From this inequality, by Moser’s iteration with $p = (5/3)^j$, $j = 0, 1, 2, \ldots$, we deduce

$$
(3.61) \quad \|\omega_0\|_{L^p(D_m \times [0, 1])} \leq CA^{5/4} \left( \int_0^1 \int_{D_m} (|\omega_0| + \beta_0 + 1)^2 \, dx \, dt \right)^{1/2},
$$

where $C$ is another absolute constant. For $T > 1$, we take suitable sequences of $\eta$ such that $\eta(T - 1) = 0$. Then in (3.59) implies, since the last term drops out,

$$
\left( \int_{T-1}^T \int_{D_m} (|\omega_0| \vee 1)^{2p(5/3)} \eta^2 \, dx \, dt \right)^{3/5} \leq C(\Lambda + \sup |\eta'|) p^4 \int_{T-1}^T \int_{D_m} (|\omega_0| \vee 1)^{2p} \eta \, dx \, dt,
$$

where is an absolute constant depending only on $\mathcal{S}_0$. Moser’s iteration again tells us,

$$
(3.62) \quad \|\omega_0\|_{L^p(D_m \times [T-0.75, T])} \leq C(\Lambda + 1)^{5/4} \left( \int_{T-0.75}^T \int_{D_m} (|\omega_0| + 1)^2 \, dx \, dt \right)^{1/2},
$$

where $C$ may have changed in value but is still an absolute constant. Since $\Lambda$ is given by (3.60), the lemma follows from (3.62) and (3.61) together with the energy inequality in Lemma 3.1(c).

Now we are in a position to prove $L^\infty$ bounds for $v_r$ and $v_3$.

**Lemma 3.11.** Let $v$ be a solution to Problem 2.1 coming from Proposition 2.2. Then, there is an absolute constant $C$ such that

$$
(3.63) \quad \|v_r(\cdot, t)\|_{L^\infty(D_m)} \leq CC_s^{13/6} \|\omega_0(\cdot, t)\|_{L^\infty(D_m)}^{13/3} \left( \int_{D_m} (|v| + 1)^2 (\cdot, 0) \, dx \right)^{1/2},
$$

$$
(3.64) \quad \|v_3(\cdot, t)\|_{L^\infty(D_m)} \leq Cs_0^{13/2} \|\omega_0(\cdot, t)\|_{L^\infty(D_m)}^{31/2} \left( \int_{D_m} (|v| + 1)^2 (\cdot, 0) \, dx \right)^{1/2}.
$$

Here $C_s$ and $s_0$ are the Sobolev constants in Lemma 3.8 and Lemma 3.6 respectively.

**Proof.** According to the Biot-Savart law and the Navier boundary condition, the following equations hold for each $t > 0$.

$$
(3.65) \quad \begin{cases} 
(\Delta - \frac{1}{r^2}) v_r = \partial_{x_3} \omega_0, & \text{in } D_m, \\
\partial_{x_3} v_r = 0, & \text{on } \partial^H D_m; \quad v_r = 0, & \text{on } \partial^V D_m,
\end{cases}
$$

$$
(3.66) \quad \begin{cases} 
\Delta v_3 = -\left( \partial_t \omega_0 + \frac{\omega_0}{r} \right), & \text{in } D_m, \\
v_3 = 0, & \text{on } \partial^H D_m; \quad \partial_r v_3 = 0, & \text{on } \partial^V D_m,
\end{cases}
$$

where the $t$ variable is omitted for simplicity.

First let us bound $v_r$. By the a priori estimate in Proposition 2.2, for any $p \geq 0$, the function $v_r^{p+1}$ can be used as a test function of (3.65), which infers, after using the boundary condition,

$$
\int_{D_m} |\nabla v_r^{p+1}|^2 \, dx + \frac{(p + 1)^2}{2p + 1} \int_{D_m} \frac{1}{r^2} v_r^{2(p+1)} \, dx = (p + 1) \int_{D_m} \omega_0 v_r^p \partial_{x_3} v_r^{p+1} \, dx.
$$
By Cauchy Schwarz, we deduce
\[
\int_{D^m} |\nabla v_r|^{p+1} + \frac{1}{r^2} v_r^{2(p+1)} \, dx \leq 2(p + 1)^2 \int_{D^m} \omega_0 r v_r^{2p} \, dx.
\]
From the Sobolev inequality in Lemma 3.8, this shows,
\[
\left( \int_{D^m} |v_r|^{(p+1)13/5} \, dx \right)^{10/13} \leq 2(p + 1)^2 C_3 \|\omega_0\|_{L^\infty(D^m)}^{12/5} \int_{D^m} |v_r|^{2p} \, dx.
\]
Hence, for some absolute constant \( C \), we have
\[
\left( \int_{D^m} (|v_r| \lor 1)^{(p+1)13/5} \, dx \right)^{10/13} \leq C(p + 1)^2 C_3 \|\omega_0\|_{L^\infty(D^m)} \int_{D^m} (|v_r| \lor 1)^{(p+1)} \, dx.
\]
Taking \( p + 1 = (13/10)^j \), \( j = 0, 1, 2, \ldots \) in Moser’s iteration, we conclude, after inserting the \( t \) variable and using the energy inequality, that
\[
\|v_r(\cdot, t)\|_{L^\infty(D^m)} \leq C C_3^{13/6} \|\omega_0(\cdot, t)\|_{L^\infty(D^m)}^{13/3} \left( \int_{D^m} (|v_r| + 1)^2 (\cdot, t) \, dx \right)^{1/2}
\leq C C_3^{13/6} \|\omega_0(\cdot, t)\|_{L^\infty(D^m)}^{13/3} \left( \int_{D^m} (|v| + 1)^2 (\cdot, 0) \, dx \right)^{1/2},
\]
which is (3.63).

To bound \( v_3 \), we use \( v_3^{2p+1} \) as a test function on equation (3.66) to obtain
\[
\int_{D^m} |\nabla v_3|^{p+1} \, dx = (p + 1) \int_{D^m} \omega_0 v_3^p \partial_r v_3^{p+1} \, dx.
\]
In the above we have used the boundary condition for \( v_r \), so that boundary integrals vanish.
By Cauchy Schwarz, we deduce
\[
\int_{D^m} |\nabla v_3|^{p+1} \, dx \leq 2(p + 1)^2 \int_{D^m} \omega_0^2 v_3^{2p} \, dx.
\]
Since \( v_3 = 0 \) on the horizontal boundary \( \partial^H D^m \), the Sobolev inequality in Lemma 3.6 can be used for \( v_3^{p+1} \) in the above identity. Therefore
\[
\left( \int_{D^m} |v_3|^{6(p+1)} \, dx \right)^{1/3} \leq 2s_0^2 (p + 1)^2 \|\omega_0\|_{L^\infty(D^m)}^2 \int_{D^m} |v_3|^{2p} \, dx.
\]
Hence, for some absolute constant \( C \), we have
\[
\left( \int_{D^m} (|v_3| \lor 1)^{6(p+1)} \, dx \right)^{1/3} \leq C s_0^2 (p + 1)^2 \|\omega_0\|_{L^\infty(D^m)}^2 \int_{D^m} (|v_3| \lor 1)^{2(p+1)} \, dx.
\]
Taking \( p + 1 = 3^j \), \( j = 0, 1, 2, \ldots \) in Moser’s iteration, we conclude, after inserting the \( t \) variable, that
\[
\|v_3(\cdot, t)\|_{L^\infty(D^m)} \leq C s_0^{3/2} \|\omega_0(\cdot, t)\|_{L^\infty(D^m)}^{3/2} \left( \int_{D^m} (|v_3| + 1)^2 (\cdot, t) \, dx \right)^{1/2},
\]
which implies (3.64) by the energy inequality. \( \Box \)
3.3. **Completion of the proof.** Pick any \( v_0 \in C^2(D_m) \). By definition, there exists a sequence of \( v_0^{(m)} \in C^2(D_m) \) satisfying the Navier boundary condition such that \( v_0^{(m)} \) converge to \( v_0 \) in \( C^2 \) norm. According to Proposition 2.2, Lemma 3.9 and Lemma 3.11, equation (1.3) with initial value \( v_0^{(m)} \) and Navier slip boundary condition on \( D_m \) has a finite energy solution \( v^{(m)} \) such that \( \| v^{(m)}(\cdot, t) \|_{L^\infty(D_m)} \) are uniformly bounded for each \( t \geq 0 \) and all \( m \). Also one can see from Lemma 3.1 that the energy norm of \( v^{(m)} \) is uniformly bounded for any finite time. This assertion is clear for \( \nabla v^{(m)}_r, \partial_x v^{(m)}_\theta \) and \( \nabla v^3 \) from statement (c) in that lemma. To see that \( \partial_r v^{(m)}_\theta \) is in \( L^2_{L^2} \) uniformly, we argue as follows. From statement (b) in Lemma 3.1, we know that

\[
\int_0^T \int_{D_m} \left( |\partial_r v^{(m)}_\theta|^2 + \frac{1}{r^2} |v^{(m)}_\theta|^2 \right) dx dt - 2 \int_0^T \int_{D_m} \partial_r v^{(m)}_\theta \frac{v^{(m)}_\theta}{r} dx dt \leq \frac{1}{2} \int_{D_m} |v^{(m)}(0, 0)|^2 dx.
\]

After integration by parts in the polar coordinates, we infer

\[
\int_0^T \int_{D_m} \left( |\partial_r v^{(m)}_\theta|^2 + \frac{1}{r^2} |v^{(m)}_\theta|^2 \right) dx dt + \sum_{j=2}^{m+1} \int_0^T \int_{L_j} |v^{(m)}_\theta|^2 d x_3 dt
\]

\[
\leq \frac{1}{2} \int_{D_m} |v^{(m)}(0, 0)|^2 dx + \int_0^T \int_{L_1} |v^{(m)}_\theta|^2 d x_3 dt.
\]

Again, \( L_1 \) is the right most vertical boundary of \( D_m \) and \( L_2, ..., L_{m+1} \) are the connected segments of the left boundary of \( D_m \). Since, for each fixed \( T \), the right hand side of the above inequality is uniformly bounded due to Lemma 3.3, we see that \( \partial_r v^{(m)}_\theta \) is uniformly bounded in \( L^2_{L^2} \) norm for each fixed \( T > 0 \). This proves the assertion. Therefore we can extract a subsequence, still denoted by \( \{v^{(m)}\} \) which converges weakly in the energy norm to a solution \( v \) in \( E \cap L^\infty \). Due to standard regularity theory, the convergence can also be made point wise except at the corners of the boundary of \( D_m \). This concludes the proof of the theorem. \( \Box \)

4. **Uniqueness results for elliptic equations in polygons with low integrability**

In this section we state and prove the uniqueness results for elliptic equations on domains with non-convex corners, which were used in Section 2. The study of elliptic equations in rough domains has been very active over the years. Comparing with standard theory for elliptic equations on Lipschitz domains, the integrability assumption is lower than usual, in view of the non-convex corners with angle \( 3\pi/2 \). We feel that these results may be known, but are unable to find them in the literature.

**Proposition 4.1.** Let an axially symmetric function \( u \in W^{1, s}_0(D_m), s = 3^-/2 \), be a solution to the Dirichlet problem

\[
\begin{cases}
\Delta u(x) - V(x)u(x) = 0, & x \in D_m, \\
u(x) = 0, & x \in \partial D_m;
\end{cases}
\]

where \( V \) is a given axially symmetric function such that \( V \geq 1/(4r^2) \) and \( V \in L^\infty(D_m) \). Then \( u = 0 \).
Notice that we are not making any assumption on the non-tangential maximal function of \(|\nabla u|\). Therefore the established theory of elliptic equations in Lipschitz domains does not seem to apply directly. The idea of the proof is to regard the equation as a two dimensional one in the \(r, x_3\) variable first. Next the problem is converted via conformal mapping to the case with smooth boundary but weaker integrability conditions on the solution and its gradient. Then a duality argument will infer uniqueness. One can also use the regularity result for very weak solutions in [35] after the conversion, even replacing the condition \(V \geq 1/(4r^2)\) by \(V \geq 0\). See the proof of Proposition 4.4 below.

Before proving the proposition, we need the following lemma on uniqueness of \(L^2\) solutions on \(C^3\) domains in \(\mathbb{R}^2\), which may be of independent interest.

**Lemma 4.2.** Let \(U \subset \mathbb{R}^2\) be a bounded \(C^3\) domain and \(Q = \{x_1, \ldots, x_k\}\) be finitely many points on the boundary \(\partial U\). Given any fixed \(p > 2\), let \(u \in L^p(U) \cap C^3(\bar{U} \setminus Q)\) be a solution to

\[
\begin{cases}
\Delta u(x) - V(x)u(x) = 0, & x \in U, \\
u(x) = 0, & x \in \partial U \setminus Q,
\end{cases}
\]

where \(0 \leq V(x) \leq \sum_{i=1}^k \frac{C}{|x_i - x|}\) for all \(x \in U\) and a constant \(C\). Then \(u = 0\).

**Remark 4.3.** Comparing with standard results, the solution \(u\) is not assumed to be bounded near \(x_i, i = 1, 2, \ldots, k\), nor is there any assumption on \(|\nabla u|\). So the result is more akin to removable singularity theorems, but with a boundary condition.

**Proof.** (of Lemma 4.2). Let \(u\) be a solution to (4.2). We first find a solution \(\phi \in W^{1,2}_0(U)\) to the standard nonhomogeneous Dirichlet problem

\[
\begin{cases}
\Delta \phi(x) - V(x)\phi(x) = u(x), & x \in U, \\
\phi(x) = 0, & x \in \partial U.
\end{cases}
\]

Since \(V \geq 0\), \(V \in L^{2^*}(U)\) and \(u \in L^p(U)\) with \(p > 2 > 1\), which is 1/2 of the dimension, the potential function \(V\) and nonhomogeneous term \(u\) are in the regularity class. Therefore the problem admits a unique solution in \(W^{1,2}_0(U) \cap C^\alpha(U)\) by standard theory. Here \(\alpha \in (0, 1)\).

Now we show that

\[
|\nabla \phi| \in L^{\infty}(U).
\]

It is clear that we just need to prove this in small balls around the potential singular points \(x_1, \ldots, x_k, y_1, \ldots, y_l\).

Let \(\Gamma = \Gamma(x, y)\) be the Dirichlet Green’s function on \(U\). Then

\[
\phi(x) = -\int_U \Gamma(x, y)u(y)dy - \int_U \Gamma(x, y)V(y)\phi(y)dy.
\]

By standard theory, there exists a constant \(\bar{C}_0\) depending only on \(U\) such that

\[
|\Gamma(x, y)| \leq \bar{C}_0[|\ln|x - y|| + 1], \quad |\nabla_x \Gamma(x, y)| \leq \bar{C}_0/|x - y|, \quad x, y \in U.
\]
From these bounds, \( \phi \in W^{1,2}_0(U) \cap C^0(U) \subset L^q(U), \forall q > 1 \), the bound on \( V \), and \( u \in L^p(U), \)
\( p > 2 \), it is easy to see by (4.5) that
\[
|\nabla \phi(x)| \leq \tilde{C}_0 \int_U \frac{|u(y)|}{|x-y|}dy + C\tilde{C}_0 \int_U \frac{1}{|x-y|} \sum_{i=1}^k |\phi(y) - \phi(x_i)| dy
\]
\[
\leq \tilde{C}_0 \int_U \frac{|u(y)|}{|x-y|}dy + C\tilde{C}_0 \int_U \frac{1}{|x-y|} \sum_{i=1}^k \frac{1}{|y-x_i|} dy \|\phi\|_{C^0(U)}
\]
\[
\leq C\tilde{C}_0 (|u|_{L^p(U)} + \|\phi\|_{W^{1,2}(U)}),
\]
proving (4.4). In the last step, we have used the standard Hölder estimate for \( \phi \) using its \( W^{1,2} \)
and \( L^p \) norm of \( V \).

Pick a small number \( \epsilon > 0 \), denote \( U_\epsilon = U \setminus \bigcup_{i=1}^k B(x_i, \epsilon) \). Since all possible singularities
of \( u \) are outside of \( U_\epsilon \), we can compute:
\[
\int_{U_\epsilon} u^2 dx = \int_{U_\epsilon} u(x) (\Delta u(x) - V(x) \phi(x)) dx
\]
\[
= \int_{\partial U_\epsilon} u \partial_n \phi dS - \int_{\partial U_\epsilon} \phi \partial_n u dS + \int_{U_\epsilon} (\Delta u(x) - V(x) u(x)) \phi(x) dx,
\]
where \( n \) is the outward normal of \( \partial U_\epsilon \). Due to the boundary condition, the above identity becomes
\[
(4.6) \quad \int_{U_\epsilon} u^2 dx = - \sum_{i=1}^k \int_{\partial B(x_i, \epsilon) \cap U} u \partial_n \phi dS + \sum_{i=1}^k \int_{\partial B(x_i, \epsilon) \cap U} \phi \partial_n u dS.
\]
where \( n \) is the outward normal of \( \partial B(x_i, \epsilon) \) or \( \partial B(y_j, \epsilon) \). Next we argue that the right hand
side of the preceding identity goes to 0 as \( \epsilon \to 0 \).

Pick any \( x \in \partial B(x_i, \epsilon) \). Then either \( B(x, \epsilon/2) \) intersects with \( U^c \), then we make 0 extension
for \( u \) across \( B(x, \epsilon/2) \cap \partial U \), or \( B(x, \epsilon/2) \subset U \). We can choose \( \epsilon \) sufficiently small so that
\( B(x, \epsilon/2) \cap \{x_1, x_2, ..., x_k\} \) is empty. So in either case we can regard \( u^2 \) as a subsolution, namely
\[
\Delta u^2 \geq 2Vu^2 \geq 0, \text{ in } B(x, \epsilon/2).
\]
Hence the standard mean value inequality states, for some absolute constant \( \tilde{C}_1 \), that
\[
|u(x)| \leq \tilde{C}_1 \left( |B(x, \epsilon/4)|^{-1} \int_{B(x, \epsilon/4)} u^2(y) dy \right)^{1/2}.
\]
By the assumption that \( u \in L^p(U), p > 2 \), this implies that
\[
(4.7) \quad u(x) = o(1) / \epsilon \quad \text{as} \quad \epsilon \to 0.
\]

Further more, when \( \epsilon \) is sufficiently small, the region \( [B(x_i, 1.5\epsilon) - B(x_i, 0.5\epsilon)] \cap U \) is free
of potential singular points in \( Q \). Since \( \partial U \) is \( C^3 \) and \( V \) has mild singularity near \( x_i \), we can obtain a gradient estimate for \( u \) on \([B(x_i, 1.2\epsilon) - B(x_i, 0.8\epsilon)] \cap U \), by straighten out the boundary. Since \( V \) may be unbounded near \( y_j \), we give a proof here. Pick \( x \in \partial B(x_i, \epsilon) \). For any small \( \epsilon \), we choose a smooth cut off function \( \eta = \eta(x) \) supported in \( B(x, \epsilon/4) \) such that
\( \eta = 1 \) on \( B(x, \epsilon/8) \) and that \( |\nabla \eta| \leq C/\epsilon \) and \( |\Delta \eta| \leq C/\epsilon^2 \). If \( B(x, \epsilon/4) \) straddles the boundary
of $U$, we can, as usual, straighten out the boundary and make odd reflection for $u$ and even reflection for $V$ afterwards. So we can just assume $u$ is a classical solution in the full ball $B(x, \epsilon/4)$. By (4.2), $u\eta$ satisfies
\[
\Delta(u\eta) - Vu\eta = 2\nabla u\eta + u\Delta\eta.
\]
Let $\Gamma_0$ be the Green’s function of the Laplacian in $\mathbb{R}^2$, we have, $\forall z \in \mathbb{R}^2$,
\[
u(z) = -\int \Gamma_0(z, y)V u\eta(y)dy - 2\int \Gamma_0(z, y)\nabla u\eta(y)dy - \int \Gamma_0(z, y)\Delta u\eta(y)dy
\]
\[
= -\int \Gamma_0(z, y)V u\eta(y)dy + 2\int \nabla_y \Gamma_0(z, y)\nabla u\eta(y)dy + \int \Gamma_0(z, y)\Delta u\eta(y)dy.
\]
By our assumptions, $0 \leq V \leq C/\epsilon$ in $B(x, \epsilon/4)$ and $\eta = 1$ in $B(x, \epsilon/8)$. Thus we deduce, after using the usual bounds for $|\nabla \Gamma_0|$ and $|\nabla^2 \Gamma_0|$, that
\[
|\nabla u(x)| \leq \frac{C}{\epsilon} \int_{B(x, \epsilon/4)} \frac{|u(y)|}{|x - y|} dy + \frac{C}{\epsilon} \int_{B(x, \epsilon/4)} \frac{|u(y)|}{|x - y|^2} dy + \frac{C}{\epsilon^2} \int_{B(x, \epsilon/4)} \frac{|u(y)|}{|x - y|} dy
\]
\[
\leq \frac{C}{\epsilon} \int_{B(x, \epsilon/4)} \frac{|u(y)|}{|x - y|} dy + \frac{C}{\epsilon^3} \int_{B(x, \epsilon/4)} |u(y)| dy
\]
This shows, since $u \in L^p$, $p > 2$, that for each $x \in \partial B(x_i, \epsilon)$, we have
\[
|\nabla u(x)| \leq C_1 \epsilon^{-1} \sup_{B(x, \epsilon/4)} |u(y)| = o(1)/\epsilon^2.
\]
For the same $x$, by the gradient bound (4.4) and the boundary condition that $\phi = 0$ on $\partial U$, we know that
\[
|\phi(x)| = O(1)\epsilon.
\]
Substituting (4.7), (4.8) and (4.9) to the right side of (4.6), we conclude that $u = 0$ after letting $\epsilon \to 0$. This proves the lemma.

Now we are in a position to give a

**Proof of Proposition 4.1**

The idea of the proof is to regard the equation in (4.1) as a two dimensional elliptic equation in the $r \times 3$ plane, and the domain $D_m$ as a polygon. Then we use the Schwarz-Christoffel mapping to conformally transform $D_m$ into the upper half plane and eventually to the unit disk. Problem 4.1 is thus converted to an elliptic problem on the unit disk. The condition that the solution $u \in W^{1,2}_{0}(D_m)$ will imply that the transformed solution, say $f$, is in $L^{2+}$ on the unit disk. Here and later $2^+$ is a number close to and greater than 2. Then we can conclude from Lemma 4.2 that $f = 0$. Thus the original solution is also 0.

Since there are only finitely many corners on $D_m$ and the Schwarz-Christoffel mapping is smooth except at the corners, we can just prove that the transformed solution $f$ is in $L^{2+}$ near the images of these corners, which are points on the boundary of the upper half plane. The worst corners are those with $3\pi/2$ angles. Since the corners are isolated, without loss of generality, we can just carry out the proof of $L^{2+}$ integrability of $f$ near the image of one of the corners. After a change of variable, the Schwarz-Christoffel mapping is essentially a power function of the complex variable and we can carry out the calculation explicitly as follows.
In \( r x_3 \) coordinate, the equation for \( u \) in (4.1) is
\[
\partial^2_r u + \frac{1}{r} \partial_r u + \partial^2_{x_3} u - V u = 0.
\]
In the identity,
\[
\partial^2_r (r^{\beta} u) = r^{\beta} \partial^2_r u + 2 \beta r^{\beta-1} \partial_r u + \beta(\beta - 1) r^{\beta-2} u,
\]
we take the constant \( \beta = 1/2 \). Then the above equation for \( u \) is transformed into
\[
(4.10) \quad \partial^2_r (\sqrt{r} u) + \partial^2_{x_3} (\sqrt{r} u) - (V - \frac{1}{4r^2})(\sqrt{r} u) = 0.
\]
Recall that in \( D_m \), the variable \( r \) is bounded away from 0. So there is no singularity in the coefficients of this equation.

Case 1. We work near one of the nonconvex corners, located say at \((r^0, x_3^0)\).

Let us make a translation \( r - r^0, x_3 - x_3^0 \) and rotation of angle \( \pi \) clockwise, namely
\[
(4.11) \quad \tilde{r} = -(r - r^0), \quad \tilde{x}_3 = -(x_3 - x_3^0).
\]
Then, in the \( \tilde{r} \tilde{x}_3 \) plane, we can assume that the corner is located at the origin and the boundary of \( D_m \) near the corner coincides with the positive horizontal (\( \tilde{r} \)) axis and negative vertical (\( \tilde{x}_3 \)) axis respectively. Consider the functions
\[
(4.12) \quad f = f(\tilde{r}, \tilde{x}_3) = \sqrt{r} u(r, x_3) = \sqrt{r^0 - \tilde{r}} u(\tilde{r} - r^0, x_3^0 - x_3), \quad \tilde{V} = \tilde{V}(\tilde{r}, \tilde{x}_3) = V(r, x_3) - \frac{1}{4r^2}.
\]
Then in the \( \tilde{r} \tilde{x}_3 \) variables, \( f \) satisfies
\[
(4.13) \quad \partial^2_{\tilde{r}} f + \partial^2_{x_3} f - \tilde{V} f = 0, \quad f = 0, \quad \text{on} \quad \partial \tilde{D}_m.
\]
Here \( \tilde{D}_m \) is the region \( D_m \) in the \( \tilde{r} \tilde{x}_3 \) plane, which is still a rectangular polygon. Let us introduce the complex variable
\[
(4.14) \quad z = \tilde{r} + i \tilde{x}_3.
\]
In terms of \( z \), the equation for \( f \) becomes, since \( \frac{\partial}{\partial \tilde{r}} = \frac{1}{2} \left( \frac{\partial}{\partial r} - i \frac{\partial}{\partial x_3} \right) \),
\[
(4.15) \quad \frac{\partial^2 f}{\partial z \partial \bar{z}} - \frac{\tilde{V}}{4} f = 0.
\]
Now let \( w = \mathcal{M}(z) \) be a Schwarz-Christoffel mapping that maps \( \tilde{D}_m \) to a unit circle \( \mathbb{D}_1 \) in the \( w = w_1 + i w_2 \) plane, whose center is at \((w_1, w_2) = (0, 1) \) and that the corner point \( z = 0 \) is mapped to the point \( w = 0 \). Near the corner \( z = 0 \), say in the region \( B(0, \epsilon) \cap D_m \), the mapping is given by, for a constant \( \sigma \),
\[
(4.16) \quad w = \sigma z^{2/3}(1 + \text{lower order terms}), \quad \frac{\partial w}{\partial z} = \frac{2\sigma}{3} z^{-1/3}(1 + \text{lower order terms}), \quad \frac{\partial z}{\partial w} = \frac{3\sigma^{-3/2}}{2} w^{1/2}(1 + \text{lower order terms}) = \frac{3\sigma^{-3/2}}{2} w^{1/2}(1 + a_1 w + a_2 w^2 + ...).
\]
Note \( |\frac{\partial z}{\partial w}| \) is bounded when \( z \) is away from the corners since \( z \) is an analytic function of \( w \) and vice versa.
Let
\begin{equation}
U = \mathbb{M}(B(0, \epsilon) \cap \bar{D}_m)
\end{equation}
where the ball $B(0, \epsilon)$ in the $z$ plane does not contain any other corners with $\epsilon$ sufficiently small. So $U$ is the image under $\mathbb{M}$ of a small corner in $\bar{D}_m$. Consider the function
\[ g = g(w) = f(z), \quad w \in \mathbb{D}_1, \quad w = \mathbb{M}(z). \]

Then, for any $p \in (2, 4)$, we have, from \eqref{4.16}
\begin{align*}
\int_U |g|^p dwd\bar{w} & = \frac{4}{9} \sigma^2 \int_{B(0, \epsilon) \cap \bar{D}_m} |f|^p |z|^{-2/3} (1 + o(1)) dz d\bar{z} \\
& = \frac{4}{9} \sigma^2 \left( \int_{B(0, \epsilon) \cap \bar{D}_m} |f|^6 dz d\bar{z} \right)^{2/3} \left( \int_{B(0, \epsilon) \cap \bar{D}_m} |z|^{-2} (1 + o(1)) dz d\bar{z} \right)^{1/3} \\
& \leq C \|f\|_{W^{1,1}(D_m)}^p = C \|u\|_{W^{1,1}(D_m)}^p.
\end{align*}

Here we just used the 2 dimensional Sobolev imbedding and the assumption that $s = 3^~/2$. Since there are only finitely many corners, this tells us that
\[ \int_{\mathbb{N}_\epsilon \cap \bar{D}_1} |g|^p dwd\bar{w} \leq C \|u\|_{W^{1,1}(D_m)}^p < \infty, \]
where $\mathbb{N}_\epsilon$ is the $\epsilon$ neighborhood of images of nonconvex corners in the $w$ plane. From the equation for $f$ \eqref{4.15} and the conformal nature of $w = \mathbb{M}(z)$, we know that $g$ satisfies the equation
\[ \begin{cases}
\frac{\partial^2 g}{\partial w \partial \bar{w}} - \left| \frac{\partial g}{\partial w} \right|^2 \overline{\bar{\nu}} g = 0, & w \in \overline{\mathbb{D}_1}\setminus\text{images of corners}, \\
g(w) = 0, & w \in \partial\mathbb{D}_1\setminus\text{images of corners}.
\end{cases} \]

By \eqref{4.16} the potential function in front of $g$ is bounded and nonnegative near the images of the nonconvex corners.

**Case 2.** We work near one of the convex corners, located say at $(r^0, x^0_\pi)$.

After similar translation, we have the following formula for the Schwarz-Christoffel mapping near these corners.
\begin{align*}
w &= \sigma z^2 (1 + \text{lower order terms}), \\
\frac{\partial w}{\partial z} &= 2z (1 + \text{lower order terms}), \\
\frac{\partial z}{\partial w} &= \frac{1}{2\sigma} w^{-1/2} (1 + \text{lower order terms}) = \frac{1}{2\sigma} w^{-1/2} (1 + a_1 w + a_2 w^2 + \ldots).
\end{align*}

Again, near the images of the convex corners in the $w$ plane, it is easy to check that that $g \in L^p$ with some $p > 2$ and that
\[ \begin{cases}
\frac{\partial^2 g}{\partial w \partial \bar{w}} - \left| \frac{\partial g}{\partial w} \right|^2 \overline{\bar{\nu}} g = 0, & w \in \overline{\mathbb{D}_1}\setminus\text{images of corners}, \\
g(w) = 0, & w \in \partial\mathbb{D}_1\setminus\text{images of corners}.
\end{cases} \]

In fact, by standard theory, $u$ is smooth near the convex (i.e. $\pi/2$) corners of $D_m$, which infers that $g$ is Hölder continuous near the image of the convex corners in the $w$ plane. Recall that $w = 0$ is actually the image of one of the corners in the $w$ plane. By \eqref{4.18} for images
of convex corners and Case 1 for those of nonconvex corners, we know that the potential function $\left| \frac{\partial u}{\partial \nu} \right|^2 \geq \frac{\hat{V}}{4}$ satisfies

$$0 \leq \left| \frac{\partial \xi}{\partial w} \right|^2 \hat{V}(y) \leq C \sum_{i=1}^{k} \frac{1}{|y - x_i|}, \quad \forall y \in \mathbb{D}_1.$$  

(4.19)

Here $x_i$ are the images of corners in the $w$ plane. Recall that near the images of nonconvex corners, by Case 1, we know that the potential function is bounded. Since both $u$ and the Schwarz-Christoffel mapping are smooth except at the corners, we see that

$$g \in L^p(\mathbb{D}_1) \cap C^3(\mathbb{D}_1 \setminus \text{images of corners}).$$

Now we can use Lemma 4.2 on $g$ to conclude that $g = 0$ and hence $f = 0$ and $u = 0$, completing the proof of the proposition. In that lemma, we take $x_i$ as the images of the corners and the potential function as $\left| \frac{\partial u}{\partial \nu} \right|^2 \frac{\hat{V}}{4}$ which satisfies (4.19).

Similarly we have the following uniqueness result for a mixed Dirichlet-Neumann problem, which was also used in Section 2.

**Proposition 4.4.** Let an axially symmetric function $u \in W^{1,s}(D_m)$, $s = 3^-/2$, be a solution to the mixed Dirichlet-Neumann problem

$$\begin{aligned}
\Delta u(x) - V(x)u(x) &= 0, \quad x \in D_m, \\
u(x) &= 0, \quad x \in \partial^H D_m, \quad \partial_n u(x) = 0, \quad x \in \partial^V D_m.
\end{aligned}$$

(4.20)

Here $V \geq 0$ is a $L^\infty(\bar{D}_m)$ axially symmetric function. Then $u = 0$.

**Proof.** The main task is to prove that $u \in C(\bar{D}_m)$. Taking this statement for granted, one can quickly prove the uniqueness as follows. If $u \neq 0$, we can assume without loss of generality that $\sup u > 0$. Since the potential $V \geq 0$, by the maximum principle, $\sup u$ must occur at $\partial D_m$. By the boundary condition $u = 0$ on $\partial^H D_m$, $\sup u$ must occur at a point, say $x_0 \in \partial^V D_m$. This point $x_0$ can not be a corner point due to the continuity of $u$ in $\bar{D}_m$. So $\partial_n u(x_0) = 0$. But this contradicts with the Hopf maximum principle.

Now let us prove $u \in C(\bar{D}_m)$. It is clear we only need to prove $u$ is continuous in a neighborhood of non-convex corners with angle $3\pi/2$. In other places, standard reflection method can show that $u \in C^1$. The proof is similar to that of Proposition 4.1. We use a Schwarz-Christoffel mapping to convert the problem to a mixed Dirichlet-Neumann problem on a smooth domain: the unit disk, which was studied long time ago. The transformed solution will inherit enough integrability property to allow us to use an earlier result to finish the proof.

As in the proof of the previous proposition, we may just work near one of the corners located, say, at $(r^{1/2}, x^0)$. Using exactly the same notations as before, we see the function $g = g(w) = f(z) = r^{1/2} u$ satisfies

$$\begin{aligned}
\frac{\partial^2 g}{\partial \nu \partial w} - \left| \frac{\partial g}{\partial w} \right|^2 \frac{\hat{V}}{4} g &= 0, \quad w \in \mathbb{D}_1, \\
g(w) &= 0, \quad \text{or} \quad \partial_n g(w) = 0, \quad w \in \partial \mathbb{D}_1 \setminus \text{images of corners}.
\end{aligned}$$

(4.21)

Due to the boundary condition $u = 0$ on $\partial^H D_m$ and the assumption that $u \in W^{1,s}$, $s = 3^-/2$, we still can apply the 2 dimensional Sobolev inequality to obtain $u \in L^6(D_m)$. Hence the
same proof as before shows that \( g \in L^4(D_1) \). Next we prove that \( |\nabla g| \in L^{4/3}(\mathbb{N}_e) \), where \( \mathbb{N}_e \) is again the \( \epsilon \) neighborhood of images of nonconvex corners in the \( w \) plane..

Let \( U \) be the domain in (4.17). Then, for \( q = 4^{-}/3 \), we compute, from (4.16),

\[
\int_U \left| \frac{\partial g}{\partial w} \right|^q dwd\bar{w} = \frac{4}{9} \sigma^{4-q} \int_{B(0,\epsilon) \cap \bar{D}_m} \left| \frac{\partial f}{\partial z} \right|^q \left( \frac{3}{2} \right) \left| \sigma^{4/3} |z|^{-2/3} (1 + o(1))dzd\bar{z} \right.
\]

\[
= (2/3)^{2-q} \sigma^{-q} \left( \int_{B(0,\epsilon) \cap \bar{D}_m} \left| \frac{\partial f}{\partial z} \right|^{3/2} dzd\bar{z} \right)^{2q/3} \left( \int_{B(0,\epsilon) \cap \bar{D}_m} \left| \sigma^{3/2} |z|^{1/2} (1 + o(1))dzd\bar{z} \right) \right)^{(3^{-}-2q)/3^{-}}
\]

\[
\leq C||f||^q_{W^{1,1}(\bar{D}_m)} \leq C||u||^q_{W^{1,1}(D_m)},
\]

where we have used \( \frac{3^{-}(q-2)}{3^{-}-2q} > -2 \) when \( 3^{-} \) is sufficiently close to 3. Likewise the same bound holds for \( |\nabla g| \). Hence \( g|_{\bar{D}_m} \in W^{1,4^{-}/3}(\mathbb{N}_e) \subset L^{4^{-}}(\mathbb{N}_e) \), where the inclusion is due to the 2 dimensional Sobolev inequality. Now we can apply the main result in Shamir [35] on (4.21) to conclude that \( g \in C^{1/2}(\bar{D}_1) \). The detail is as follows. Notice that in small neighborhoods around the images of the nonconvex corners, we have \( |\frac{\partial g}{\partial \sigma}|^2 = C|w|(1 + \text{lower order terms}) \) by (4.16). So the coefficient for \( g \) in (4.21) is Lipschitz in these neighborhoods. By bootstrapping from \( g|_{\bar{D}_m} \in W^{1,4^{-}/3}(\mathbb{N}_e) \subset L^{4^{-}}(\mathbb{N}_e) \), we can treat the term \( \left| \frac{\partial \sigma}{\partial w} \right|^2 \frac{3^{-}}{4} g|_{\bar{D}_m} \in L^{4^{-}}(\mathbb{N}_e) \) as an inhomogeneous term and apply local versions of Lemma 5.1 or Theorem 5.3 there with large \( p \) to deduce \( g|_{\bar{D}_m} \in W^{1,4}(\mathbb{N}_e) \) and so on, giving us \( g|_{\bar{D}_m} \in C^{1/2}(\mathbb{N}_e) \). As mentioned, outside of \( \mathbb{N}_e \), we already know \( u \) is in \( C^1 \). Therefore \( u \in C(\bar{D}_m) \) which completes the proof of the proposition.

\[ \square \]

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