On the Consistency of Top-$k$ Surrogate Losses

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Abstract
The top-$k$ error is often employed to evaluate performance for challenging classification tasks in computer vision as it is designed to compensate for ambiguity in ground truth labels. This practical success motivates our theoretical analysis of consistent top-$k$ classification. To this end, we define top-$k$ calibration as a necessary and sufficient condition for consistency, for bounded below loss functions. Unlike prior work, our analysis of top-$k$ calibration handles non-uniqueness of the predictor scores, and extends calibration to consistency – providing a theoretically sound basis for analysis of this topic. Based on the top-$k$ calibration analysis, we propose a rich class of top-$k$ calibrated Bregman divergence surrogates. Our analysis continues by showing previously proposed hinge-like top-$k$ surrogate losses are not top-$k$ calibrated and thus inconsistent. On the other hand, we propose two new hinge-like losses, one which is similarly inconsistent, and one which is consistent. Our empirical results highlight theoretical claims, confirming our analysis of the consistency of these losses.

1. Introduction
Consider a multiclass classifier which is granted $k$ guesses, so its prediction is declared error-free only if any one of the guesses is correct. This conceptually defines the top-$k$ error (Akata et al., 2012) (equiv. top-$k$ accuracy). Top-$k$ error is popular in computer vision, natural language processing, and other applied problems where there are a large number of possible classes, along with potential ambiguity regarding the label of a sample and/or when a sample may correspond to multiple labels e.g. an image of a park containing a pond may be correctly labeled as either a park or a pond Russakovsky et al. (2015); Xiao et al. (2010); Zhou et al. (2018).

Like the zero-one loss for binary classification, the top-$k$ error is typically not minimized directly because it is discontinuous and only has zero gradients. Instead, practical algorithms depend on minimizing a surrogate loss, often a convex upper bound (Lapin et al., 2015; 2016).

Most commonly, the predictive model is trained to output a continuous-valued score vector, and the classes corresponding to the top $k$ entries of the score vector constitute the classification prediction (Lapin et al., 2018). While popular in practice, there is limited work on the theoretical properties of top-$k$ error and its surrogate losses. We are particularly interested in the consistency of surrogate losses, which says whether the learned classifier converges to the Bayes optimal in the infinite sample limit.

We begin by characterizing the Bayes optimal classifier for the top-$k$ error. Our careful analysis reveals precise conditions required for a prediction to be top-$k$ calibrated. Importantly, our analysis does not use implicit uniqueness assumptions in prior work. Our consistency analysis gives rise to the notion of the top-$k$ calibration of a surrogate loss function, which, in the case where a minimizer exists, informally states that any minimizer of the loss also minimizes the top-$k$ error. We further show that this condition, clearly necessary for consistency, is also sufficient assuming the loss function is bounded below.

Following the calibration analysis, we formulate a class of top-$k$ consistent surrogate functions based on Bregman divergences, motivated by Ravikumar et al. (2011). At this point we generalize our framework to the weighted top-$k$ evaluation metric, where each class has a different misclassification penalty. Rounding up our analysis, we investigate several hinge-like top-$k$ surrogates proposed in Lapin et al. (2015), while proposing two of our own. We find all but one of the 5 hinge losses to be inconsistent, and one we propose to be consistent.

Main Contributions. In summary, our main contributions are outlined as follows:

- We carefully analyze (weighted) top-$k$ Bayes-optimal classifiers. This results in the formulation of a property fundamental to top-$k$ consistency (top-$k$ preserving) and a notion of calibration which is necessary and sufficient to construct (weighted) consistent top-$k$ sur-
rogate losses.

- We propose a family of consistent (weighted) top-k surrogate losses based on Bregman divergences. We show the inconsistency of previously proposed top-k hinge-like surrogate losses and propose new ones, one of which is (weighted) top-k consistent.

In addition to our theoretical analysis, empirical results are provided to highlight our claims. In particular, we are able to empirically observe our results on the consistency of the hinge-like losses in action via a synthetic experiment.

1.1. Notation

For any $N \in \mathbb{Z}^+$, we use $[N] = \{1, \ldots, N\}$. We assume there are $M$ classes and denote the input space as $\mathcal{X}$. In addition to associating the label with an index $l \in [M]$, we represent the label as a vector $y \in \{0, 1\}^M =: \mathcal{Y}$ where $y_l = 1$ and $y_i = 0$ for all $i \in [M] \setminus \{l\}$. We slightly abuse this notation for conciseness so that when $y$ appears in a subscript or as the index in a sum, it refers to the index/label in $[M]$. The data is assumed to be generated i.i.d. from some distribution $\mathbb{P}$ over $\mathcal{X} \times \mathcal{Y}$.

Define the probability simplex $\Delta_M := \{v \in \mathbb{R}^M \mid \forall m \in [M], v_m \geq 0, \sum_{m=1}^M v_m = 1\}$, and let $\eta(x) \in \Delta_M$ be the conditional distribution of $y \in \mathcal{Y}$ given $x \in \mathcal{X}$, i.e., $\eta(x)_m = P(l = m \mid X = x) = P(y_m = 1 \mid X = x)$. Furthermore, given a vector $v \in \mathbb{R}^m$, let $v[j]$ denote the $j$th greatest entry of $v$. For example, if $v = (1, 4, 4, 2)$, then $v[1] = 4, v[2] = 4, v[3] = 2, v[4] = 1$.

1.2. Related Work

The statistical properties of surrogates for binary classification are well-studied (Zhang, 2004b; Bartlett et al., 2003). Furthermore, many of these results have been extended to multiclass classification with the accuracy metric (Zhang, 2004a; Tewari & Bartlett, 2005). Usually, $y \in \{1, \ldots, M\}$, $s \in \mathbb{R}^M$ is a vector-valued score, and the prediction is the index of the entry of $s$ with the highest value. There have also been recent studies on a general framework for consistent classification with more general concave and fractional linear multiclass metrics (Narasimhan et al., 2015).

In the realm of multilabel classification, there is work on extending multiclass algorithms to multilabel classification (Lapin et al., 2018), characterizing consistency for multilabel classification (Gao & Zhou, 2013), and constructing a general framework consistent for classification with multilabel metrics (Koyejo et al., 2015).

On the other hand, statistical properties such as consistency of surrogate loss functions for the top-k error are not so thoroughly characterized. It is known that softmax loss $-\log \left( \frac{e^{s[l]}}{\sum_{m=1}^M e^{s[m]}} \right)$ is top-k consistent and that the multiclass hinge loss $\max_{m \in [M]} \{ \mathbb{1}[m \neq y] + s_m - s_y \}$ proposed by Cramer & Singer (2001) is top-k inconsistent (Zhang, 2004a). However, the consistency of recently proposed improved top-k surrogates such as proposals in Berrada et al. (2018); Lapin et al. (2015; 2016; 2018) has so far remained unresolved. Our work resolves some of these open questions by showing their inconsistency, in addition to providing a more robust framework for top-k consistency.

2. Top-k Consistency

We begin by formally defining the top-k error.

**Definition 2.1** (Top-k error). Given label vector $y \in \mathcal{Y}$ with $y_l = 1$ and prediction $s \in \mathbb{R}^M$, the top-k error is defined as

$$\text{err}_k(s, y) = \mathbb{I}[l \notin r_k(s)],$$

where $r_k : \mathbb{R}^M \rightarrow \mathcal{P}([M])$ is the top-k thresholding operator which selects the $k$ indices of the greatest entries of the input, breaking ties arbitrarily.

In general, $s$ is the output of some predictor $\theta$ given a sample $x \in \mathcal{X}$. The goal of a classification algorithm under the top-k metric is to learn a predictor $\theta : \mathcal{X} \rightarrow \mathbb{R}^M$ that minimizes the risk

$$L_{\text{err}_k}(\theta) := \mathbb{E}_{(x,y) \sim \mathbb{P}}[\text{err}_k(\theta(x), y)].$$

Given $s \in \mathbb{R}^M$ and $\eta \in \Delta_M$, we may define the conditional risk

$$L_{\text{err}_k}(s, \eta) := \mathbb{E}_{y \sim \eta}[\text{err}_k(s, y)].$$

Furthermore, we define optimal risk and conditional risk

$$L_{\text{err}_k}^*(\eta) := \inf_{s \in \mathbb{R}^M} L_{\text{err}_k}(s, \eta).$$

Analogous population statistics for arbitrary loss functions $\psi : \mathbb{R}^M \times \mathcal{Y} \rightarrow \mathbb{R}$ are denoted using the standard notation (i.e. swapping out the metrics) e.g. $\psi$ risk is defined as $L_\psi(\theta) := \mathbb{E}_{(x,y) \sim \mathbb{P}}[\psi(\theta(x), y)]$.

2.1. Bayes Optimality

Here we define and characterize Bayes optimal predictors for the top-k error.

**Definition 2.2** (Top-k Bayes optimal). The predictor $\theta^* : \mathcal{X} \rightarrow \mathbb{R}^M$ is top-k Bayes optimal if

$$L_{\text{err}_k}(\theta^*) = L_{\text{err}_k}^*.$$

We refer to $r_k \circ \theta^*$, the classifier obtained from $\theta^*$, as the top-k Bayes decision rule.

**Remark 2.1.** Definition 2.2 does not imply that the optimal score $s \in \mathbb{R}^M$ for a conditional distribution $\eta \in \Delta_M$ is achieved when the top-$k$ indices of $s$ are a subset of the top-$k$ indices of $\eta$. 

Remark 2.1 shows that nuances of the top-k error can lead to seemingly natural definitions being incorrect. For instance, Lapin et al. (2016; 2018) write top-k optimality as:

\[ \{ y \mid s_y \geq s[k] \} \subseteq \{ y \mid \eta_y \geq \eta[k] \}. \]

Consider the following counter-example. Let \( s = (0,1,1), \eta = (1,0,0) \) and \( k = 2 \). Note \( s[k] = 1, \eta[k] = 0 \). Denote the left set \( T_k(s) = \{ y \mid s_y \geq s[k] \} \) and the right set \( T_k(\eta) = \{ y \mid \eta_y \geq \eta[k] \} \). Then, \( T_k(s) = \{2,3\} \subseteq T_k(\eta) = \{1,2,3\} \). By the above definition, such an \( s \) is considered optimal. Yet, it achieves 100% top-k error since it results in a prediction \( r_k(s) = \{2,3\} \) even though according to \( \eta \), the classes \( \{2,3\} \) have 0 probability of occurring.

We define top-k preserving, a necessary and sufficient property for top-k optimality. This property will be fundamental to our theoretical analysis of top-k consistency.

**Definition 2.3 (Top-k preserving).** Given \( x \in \mathbb{R}^M \) and \( y \in \mathbb{R}^M \), we say that \( y \) is top-k preserving with respect to \( x \), denoted \( P_k(y,x) \), if for all \( m \in [M], \)

\[ x_m > x[k+1] \implies y_m > y[k+1] \]

\[ x_m < x[k] \implies y_m < y[k]. \]

The negation of this statement is \( \neg P_k(y,x) \).

This is not a symmetric condition. For example, although \( y = (4,3,2,1) \) is top-2 preserving with respect to \( x = (4,2,2,1) \), \( x \) is not top-2 preserving with respect to \( y \).

**Proposition 2.2.** \( \theta : \mathcal{X} \rightarrow \mathbb{R}^M \) is top-k Bayes optimal for any top-k thresholding operator \( r_k \) if and only if \( \theta(X) \) is top-k preserving with respect to \( \eta(X) \) almost surely.

**Proof.** Fix \( x \in \mathcal{X} \) and \( s \in \mathbb{R}^M \), with \( \eta = \eta(x) \). We have

\[ L_{err_k}(s,\eta) = \mathbb{E}_{y \sim \eta} [err_k(s,y)] = \sum_{m \in [M]/r_k(s)} \eta_m \]

\[ = 1 - \sum_{m \in r_k(s)} \eta_m \geq 1 - \sum_{m=1}^k \eta_{[m]}. \]

The last inequality holds because \( |r_k(s)| = k \), so \( \sum_{m \in r_k(s)} \eta_m \leq \sum_{m=1}^k \eta_{[m]} \). Equality occurs if and only if \( \sum_{m \in r_k(s)} \eta_m = \sum_{m=1}^k \eta_{[m]} \). If equality does not hold, there exists \( i \in r_k(s), j \in [M] \setminus r_k(s) \) such that \( \eta_j > \eta_i \). If \( \eta_j > \eta_{[k+1]} \), then since \( s_j \not\in r_k(s), s_j \not\in s[k+1]. \) If \( \eta_j \leq \eta_{[k+1]} \), then \( \eta_j < \eta_{[k+1]} \leq \eta[k] \). However, \( s_i \not\in s[k] \), because \( i \in r_k(s) \). Either way, \( \neg \neg P_k(s,\eta) \).

If \( \neg P_k(s,\eta) \), then there exists \( i \in [M] \) such that \( \eta_i > \eta_{[k+1]} \) but \( s_i \not\in s[k+1], \) or \( \eta_i < \eta[k] \) but \( s_i \not\in s[k] \). In the first case, there is an \( r_k \) such that \( i \not\in r_k(s), \) because there are at least \( k \) indices \( j \in [M], j \neq i \) such that \( s_j \geq s_i \). In the second case, there is an \( r_k \) such that \( i \in r_k(s), \) because \( s_i \) is one of the top \( k \) values of \( s \). In either case, there is an \( r_k \) such that \( \sum_{m \in r_k(s)} \eta_m < \sum_{m=1}^k \eta_{[m]} \). Thus, \( L_{err_k}(s,\eta) \) is optimal for any thresholding operator \( r_k \) if and only if \( P_k(s,\eta) \), i.e. \( s \) is top-k preserving with respect to \( \eta \).

Finally, we note that

\[ L_{err_k}(\theta) = \mathbb{E}_{X \sim \mu} [L_{err_k}(\theta(X), \eta(X))], \]

where \( \mu \) is the conditional distribution of \( X \). It follows that \( \theta \) minimizes \( L_{err_k}(\theta) \) if and only if \( \theta(X) \) minimizes \( L_{err_k}(\theta(X), \eta(X)) \) almost surely. In other words, \( \theta \) is a Bayes optimal predictor for any \( r_k \) if and only if \( P_k(\theta(X), \eta(X)) \) almost surely.

**2.2. Top-k calibration**

We define top-k calibration, which is intended to capture when the minimizer of a loss function leads to the Bayes decision rule. Analogous notions of binary classification calibration can be found in Bartlett et al. (2003); Lin (2004). For multiclass classification (i.e. top-k classification), Zhang (2004a) calls the notion infinite sample consistent.

**Definition 2.4 (Top-k calibration).** A loss function \( \psi : \mathbb{R}^M \times \mathcal{Y} \rightarrow \mathbb{R} \) is top-k calibrated if for all \( \eta \in \Delta_M \),

\[ \inf_{s \in \mathbb{R}^M : \neg P_k(s,\eta)} L_{\psi}(s,\eta) > \inf_{s \in \mathbb{R}^M} L_{\psi}(s,\eta) = L_{\psi}^*(\eta). \]

If a minimizer \( s^* \) of \( L_{\psi}(s,\eta) \) exists, this implies that \( s^* \) must be top-k preserving with respect to \( \eta \).

More generally, if \( \{s^{(n)}\} \) is a sequence such that \( L_{\psi}(s^{(n)},\eta) \rightarrow \inf_{s} L_{\psi}(s,\eta), \) then for all \( n \) greater than some \( N, P_k(s^{(n)},\eta) \).

If this condition does not hold, then the classifier learned from minimizing \( \psi \) does not return the top-k Bayes rule.

**2.3. Obtaining consistency**

We can convert top-k calibration into top-k consistency for all lower bounded loss functions, i.e. \( \psi \) such that \( \psi(s,y) \geq B \) for all \( s \in \mathbb{R}^M, y \in \mathcal{Y} \) and some \( B \in \mathbb{R} \). WLOG, we assume \( \psi \) is nonnegative, i.e. \( B = 0 \), because a constant shift does not change the minimization of the loss. We begin with the lemma that \( L_{\psi}^* \) is continuous.

**Lemma 2.3. Let \( \psi : \mathbb{R}^M \times \mathcal{Y} \rightarrow [0, \infty) \) be a nonnegative loss function. \( L_{\psi}^* : \Delta_M \rightarrow \mathbb{R} \) defined by \( L_{\psi}^*(\eta) = \inf_{s \in \mathbb{R}^M} \sum_{i=1}^M \eta_i \psi(s,i) \) is continuous.**

**Proof.** See Appendix for full proof. In summary, we first argue that \( L_{\psi}^* \) is lower semicontinuous by Theorem 10.2 from Rockafellar (1970), since it is concave and its domain \( \Delta_M \) is locally simplicial. Then, we directly show that it is upper semicontinuous, completing the proof.
Now we obtain that any nonnegative \( \psi \) that is top-\( k \) calibrated is also top-\( k \) consistent.

**Theorem 2.4.** Suppose \( \psi \) is a nonnegative top-\( k \) calibrated loss function. Then \( \psi \) is top-\( k \) consistent in the sense that for any sequence of measurable functions \( f^{(n)} : \mathcal{X} \rightarrow \mathbb{R}^M \), we have

\[
L_{\psi}(f^{(n)}) \rightarrow L_{\psi}^* \iff L_{\text{err}_k}(f^{(n)}) \rightarrow L_{\text{err}_k}^*.
\]

**Proof.** (Sketch.) For full proof, see Appendix. Let \( \Delta L_f := L_f - L_f^* \) for a loss \( f \). Then, by Corollary 26 of Zhang (2004a) we are done if we show that the quantity

\[
\Delta H(\epsilon) = \inf \{ \Delta L_{\psi}(s, \eta) \mid \Delta L_{\text{err}_k}(s, \eta) \geq \epsilon \}
\]

is greater than 0 whenever \( \epsilon > 0 \). I.e., if we do not have 0 top-\( k \) error, then there must be a positive constant which lower bounds the loss. We do so by contradiction; if for some \( \epsilon > 0 \), \( \Delta H(\epsilon) = 0 \), there is a sequence \( \{s^{(n)}, \eta^{(n)}\} \) such that \( \Delta L_{\text{err}_k}(s^{(n)}, \eta^{(n)}) \geq \epsilon \) for all \( n \in \mathbb{N} \) and yet \( \Delta L_{\psi}(s^{(n)}, \eta^{(n)}) \rightarrow 0 \). We argue that this implies \( L_{\psi}(s^{(n)}, \eta) \rightarrow L_{\psi}^*(\eta) \), where \( \eta = \lim_{n \rightarrow \infty} \eta^{(n)} \). We apply top-\( k \) calibration and continuity of \( L_{\text{err}_k}^* \) to obtain that \( \Delta L_{\text{err}_k}(s^{(n)}, \eta^{(n)}) = L_{\text{err}_k}(s^{(n)}, \eta^{(n)}) - L_{\text{err}_k}^*(\eta^{(n)}) < \epsilon \) eventually, a contradiction. \( \blacksquare \)

### 3. Bregman Divergence Top-\( k \) Consistent Surrogates

Next, we outline top-\( k \) consistent surrogates based on Bregman divergences. Given a convex, differentiable function \( \phi : \mathbb{R}^M \times \mathbb{R}^M \rightarrow \mathbb{R} \), define the Bregman divergence \( D_\phi \) by

\[
D_\phi(s, \cdot) = \phi(y) - \phi(s) - \nabla \phi(s) ^ {\top} (y - s). \tag{2}
\]

\( D_\phi(s, \cdot) \) can be interpreted as the error when approximating \( \phi(\cdot) \) by the first order Taylor expansion of \( \phi \) centered at \( s \). Bregman divergences include squared loss and KL divergence as special cases.

Here, we present the result that any Bregman divergence composed with a reverse top-\( k \) preserving function is top-\( k \) calibrated. First we define top-\( k \) and reverse top-\( k \) preserving functions.

**Definition 3.1** (Top-\( k \) preserving function.). Given \( A \subseteq \mathbb{R}^M \) and \( B \subseteq \mathbb{R}^M \), \( f : A \rightarrow B \) is top-\( k \) preserving if \( \forall x \in A, \text{P}_k(f(x), x) \).

We say that \( f \) is reverse top-\( k \) preserving if \( \forall x \in A, \text{P}_k(x, f(x)) \).

Now we give the following top-\( k \) calibrated Bregman divergence formulation as a theorem.

**Theorem 3.1.** Suppose \( \phi : \mathbb{R}^M \rightarrow \mathbb{R}^M \) is strictly convex and differentiable. If \( g : \mathbb{R}^M \rightarrow \mathbb{R}^M \) is reverse top-\( k \) preserving, continuous, and \( \Delta M \subseteq \text{range}(g) \), then \( \psi : \mathbb{R}^M \times \mathcal{Y} \rightarrow \mathbb{R} \) defined by

\[
\psi(s, y) = D_\phi(g(s), y)
\]

is top-\( k \) calibrated.

**Proof.** See Appendix. \( \blacksquare \)

### 3.1. Examples of top-\( k \) calibrated losses

We can use Theorem 3.1 to verify the top-\( k \) calibration of loss functions. For example, the commonly used softmax with cross-entropy loss is top-\( k \) calibrated:

\[
\psi(s, y) = -\ln \left( \frac{e^{s_i}}{\sum_{m=1}^M e^{s_m}} \right) = \sum_{m=1}^M y_m \ln \left( \frac{y_m}{e^{s_m}/(\sum_{i=1}^M e^{s_i})} \right)
\]

can be rewritten as \( \psi(s, y) = D_\phi(g(s), y) \) with \( \phi(x) = \sum_{m=1}^M x_m \ln x_m \) and \( g(s)_m = \frac{e^{s_m}}{\sum_{i=1}^M e^{s_i}} \). \( \phi \) is strictly convex and differentiable, and \( g \) satisfies the assumptions of 3.1. In fact, \( g \) satisfies the stronger rank preserving condition,

\[
\forall i, j \in [M], s_i > s_j \iff g(s)_i > g(s)_j.
\]

As a result, \( \psi(s, y) \) is top-\( k \) calibrated for every \( k \), i.e. rank consistent. An interesting question is whether there is a viable loss which does not satisfy such a strong property, and is top-\( k \) calibrated for just a specific \( k \). We eventually answer this by proposing the \( \psi_k \) hinge loss, which is calibrated for particular \( k \).

Another top-\( k \) loss which is rank consistent is the squared loss:

\[
\psi(s, y) = (s - y)^2 = D_\phi(g(s), y)
\]

with \( \phi(x) = \|x\|^2 \) and \( g(s) = s \).

### 3.2. Generalization to cost-sensitive top-\( k \) error

In some contexts, it may make sense to penalize not recognizing certain classes more than others. For example, it could be more important for a robot to correctly classify people as people than to correctly classify various inanimate objects. Taking this into account, the cost sensitive top-\( k \) error is

\[
\text{err}_k(s, y) = c_y \mathbb{1}[y \not\in \text{r}_k(s)].
\]

It is straightforward to show in the same way as our earlier arguments that given a distribution \( \eta \in \Delta_M \), the Bayes
optimal weighted top-\( k \) error is

\[
1 - \sum_{m=1}^{K} \eta'_m,
\]

where \( \eta'_m = c_m \eta_m \) for every \( m \in [M] \). The conditions under which \( s \) is optimal or \( \psi \) is top-\( k \) calibrated can be modified for this setting by replacing \( \eta \) with \( \eta' \).

To show that we can discuss the case where \( c_m = 1 \) for all \( m \in [M] \) as we have been doing without loss of generality, we present the following proposition. It states that a top-\( k \) calibrated loss function can be modified by a simple weighting to obtain a weighted top-\( k \) calibrated loss function.

**Proposition 3.2.** Suppose \( \psi \) is top-\( k \) calibrated for the unweighted top-\( k \) error. Then, \( \psi' \) defined by

\[
\psi'(s, y) = c_y \psi(s, y),
\]

is top-\( k \) calibrated for the cost-sensitive error weighted by \( c_y \). The converse holds as well.

**Proof.** See Appendix. \( \square \)

### 4. Top-\( k \) hinge-like losses

Hinge-like losses for top-\( k \) classification have been proposed by Lapin et al. (2015), one of which is a modification of the general class of ranking losses in Usunier et al. (2009). We begin with a generalization of the multiclass loss proposed in Crammer & Singer (2001):

\[
\psi_1(s, y) = \max\{1 + (s_{iy})_k - s_y, 0\}. \tag{3}
\]

This loss is first discussed in Lapin et al. (2015) as a direct extension of the Crammer-Singer multiclass loss. Berrada et al. (2018) describe the main problem with \( \psi_1 \) as the sparsity of its gradients, which leads to poor results in practice. Thus, they smooth \( \psi_1 \) by rewriting it as a difference in maximums over subsets of \( [M] \) of size \( k \) then apply the \( \logsumexp \) ≜ max trick. On the other hand, Lapin et al. (2015) raise the issue of \( \psi_1(s, y) \) being non-convex in \( s \). They propose the following alternative convex loss and motivate it by pointing out that it is a relatively tight upper bound on \( \psi_1(s, y) \):

\[
\psi_2(s, y) = \max\left\{\frac{1}{k} \sum_{i=1}^{k} (s + \bar{1}(y))_i - s_y, 0\right\}, \tag{4}
\]

where \( c = \bar{1}(y) \in \{0, 1\}^M \) is defined by \( c_m = 1 \) if \( m \neq y \) and \( c_y = 0 \).

Inspired by the general family of ranking losses proposed in Usunier et al. (2009), Lapin et al. (2015) also propose the loss

\[
\psi_3(s, y) = \frac{1}{k} \sum_{i=1}^{k} \max\{(s + \bar{1}(y))_i - s_y, 0\}.
\]

They note that \( \psi_2 \) is a tighter upper bound on \( \psi_1 \) than \( \psi_3 \). In fact, we propose a loss which is convex and a tighter upper bound on \( \psi_2 \) than \( \psi_1 \) when \( s_{iy} \) is \( s \) with its \( y \)th entry removed:

\[
\psi_4(s, y) = \max\left\{\frac{1}{k} \sum_{i=1}^{k} (1 + (s_{i \setminus y})_i) - s_y, 0\right\}.
\]

Finally, we propose loss that is similar to \( \psi_1 \), but is a tighter upper bound on \( \text{err}_k(s, y) \), and turns out to be the only top-\( k \) calibrated hinge loss we have discovered so far:

\[
\psi_5(s, y) = \max\{1 + s_{[k+1]} - s_y, 0\}.
\]

To see how this upper bounds \( \text{err}_k(s, y) \), notice that \( \text{err}_k(s, y) = 1 \) implies that \( s_y \leq s_{[k+1]} \), otherwise, \( s_y > s_{[k+1]} \) and \( y \) would have to be selected by \( r_k \). Thus, \( 1 - s_y + s_{[k+1]} \geq 1 \). And, \( \psi_5 \geq 0 \) always, establishing the upper bound. Note that \( 1[s_y \leq s_{[k+1]}] \) is extremely close to the definition of \( \text{err}_k \) — we have seen that if \( \text{err}_k(s, y) = 1 \), then \( s_y \leq s_{[k+1]} \). Conversely, if \( s_y \leq s_{[k+1]} \), then there is an \( r_k \) such that \( y \not\in r_k \), giving \( \text{err}_k(s, y) = 1 \).

In Lapin et al. (2016), the authors leave the top-\( k \) calibration of \( \psi_2 \) and \( \psi_3 \) as an open question. Here, we resolve these open questions. Furthermore, the top-\( k \) calibration of \( \psi_1 \) has not been discussed in the literature until now. We show that \( \psi_1, \psi_2, \psi_3, \) and \( \psi_4 \) are not top-\( k \) calibrated, and that \( \psi_5 \) is top-\( k \) calibrated. Moreover, we derive the explicit solution to \( \arg \min_x L_{\psi_j}(s, \eta) \). Although these losses are not top-\( k \) calibrated in general, they may be calibrated under low-noise type restrictions on the set of possible conditional distributions \( \eta \). However, the precise conditions may differ between losses.

#### 4.1. Characterization of hinge-like losses

We precisely characterize the minimizers of the expected loss \( L_{\psi_j}(s, \eta) = \mathbb{E}_{y \sim \eta} [\psi_j(s, y)] \) given a conditional distribution \( \eta \in \Delta_M \). Though we arrive at inconsistency, our
results also indicate that if $\eta$ is from the restricted probability simplex $\{ \eta \in \Delta_M \mid \eta[k] > \sum_{i=k+1}^M \eta[i] \}$, $\psi_1$ is top-$k$ calibrated.

**Theorem 4.1.** Let $\Pi_M$ denote the set of permutations from $[M]$ to $[M]$. Say $\pi \in S_M$ sorts a vector $v \in \mathbb{R}^M$ if $v_{\pi_1} \geq v_{\pi_2} \geq \ldots \geq v_{\pi_M}$.

Let $\eta \in \Delta_M$, and suppose it has no zero entries. Then,

1. If $\eta[k] > \sum_{i=k+1}^M \eta[i]$, then $s^* \in \arg\min_{s} L_{\psi}(s, \eta)$ if and only for some $c \in \mathbb{R}$ and $\pi \in \Pi_M$ which sorts $\eta$,

$$s_{\pi_{k+1}} = \ldots = s_{\pi_{M}} = c, \quad s_{[k]} = c + 1, \quad \forall i \in \{1, \ldots, k-1\}, s_{[i]} \in [c+1, \infty).$$

Furthermore, $L_{\psi}(\eta) = 2 \sum_{i=k+1}^M \eta[i].$

2. If $\eta[k] < \sum_{i=k+1}^M \eta[i]$, then $s^* \in \arg\min_{s} L_{\psi}(s, \eta)$ if and only for some $c$,

$$s_{\pi_k} = \ldots = s_{\pi_{M}} = c, \quad \forall i \in \{1, \ldots, k-1\}, s_{[i]} \in [c + 1, \infty).$$

Furthermore, $L_{\psi}(\eta) = \sum_{i=k}^M \eta[i].$

3. If $\eta[k] = \sum_{i=k+1}^M \eta[i]$, then $s^*$ is not top-$k$ preserving if and only for some $c$,

$$s_{\pi_{k+1}} = \ldots = s_{\pi_{M}} = c, \quad s_{\pi_k} \in (c, c+1), \quad \forall i \in \{1, \ldots, k-1\}, s_{[i]} \in [c + 1, \infty).$$

Furthermore, $L_{\psi}(\eta) = \sum_{i=k}^M \eta[i].$

**Proof.** See Appendix.

This implies that $\psi_1$ is not top-$k$ calibrated: if $\eta$ is such that $\eta_1 > \ldots > \eta_M$ and $\eta_k < \sum_{m=k+1}^M \eta_m$, then $s^* \in \arg\min_{s} L_{\psi}(s, \eta)$ where $s_m^* = 1$ for all $m \in [k-1]$ and $s_m^* = 0$ for all $m \in \{k, k+1, \ldots, M\}$.

The following proposition implies that $\{\psi_2, \psi_3, \psi_4\}$ are not top-$k$ calibrated, and are thus inconsistent.

**Proposition 4.2.** For any $\psi \in \{\psi_2, \psi_3, \psi_4\}$, if $\sum_{m=k+1}^M \eta[m] > \frac{k}{k+1}$, we have $0 \in \arg\min_{s} L_{\psi}(s, \eta)$, and hence $L_{\psi}(\eta) = \min_{s} L_{\psi}(s, \eta) = L_{\psi}(0, \eta) = 1$.

**Proof.** See Appendix.

To show this leads to inconsistency, take $\eta = (1/8, 1/8, 1/12, 1/12, \ldots, 1/12) \in \Delta_5$ with $k = 2$. $\eta$ satisfies $\sum_{i=k+1}^M \eta[i] = \frac{3}{4} > \frac{2}{3} = \frac{k}{k+1}$, so the optimal is $s^* = 0$. But, $s^*$ is not top-$k$ preserving wrt $\eta$. This implies that $\psi \in \{\psi_2, \psi_3, \psi_4\}$ is not top-$k$ calibrated. Nonetheless, these loss functions may be effective in practice for well behaved $\eta$.

**Proposition 4.3.** $\psi_5 : \mathbb{R}^M \times \mathcal{Y}$ is top-$k$ calibrated.

**Proof.** See Appendix.

Since $\psi_5$ is bounded below, by 2.4, it is top-$k$ consistent. It is the only calibrated top-$k$ hinge loss we encounter.

5. Experiments

Here we describe experiments comparing an assortment of top-$k$ surrogate loss functions on synthetic and real data. Our goal here is to obtain a basic picture of how the different losses compare with each other, especially in the context of the theory discussed. One synthetic experiment empirically showcases our theoretical results on the inconsistency of $\psi_2, \psi_3, \psi_4$ and consistency of $\psi_5$. A second synthetic experiment and experiments on the real data empirically show that the newly proposed top-$k$ hinge loss functions, $\psi_4$ and $\psi_5$, are tighter bounds on the top-$k$ error. In addition to the proposed top-$k$ hinge-like losses, we use the multiclass loss $\psi_{CS}$ from Crammer & Singer (2001), classic softmax with cross entropy denoted Ent, and the following truncated cross entropy losses:

$$\text{Ent}_{\text{Tr}_1}(s, y) = \ln g(s)_y$$

$$\text{Ent}_{\text{Tr}_2}(s, y) = \ln g(s)_y + 1 - \sum_{i=1}^M g(s)_i,$$

with $g(s)_j = \ln \left(1 + \sum_{i=k}^{M-1} \exp((s_j)_i - s_j)\right)$.

**Proof.** See Appendix.

This implies that $\psi_1$ is not top-$k$ calibrated: if $\eta$ is such that $\eta_1 > \ldots > \eta_M$ and $\eta_k < \sum_{m=k+1}^M \eta_m$, then $s^* \in \arg\min_{s} L_{\psi}(s, \eta)$ where $s_m^* = 1$ for all $m \in [k-1]$ and $s_m^* = 0$ for all $m \in \{k, k+1, \ldots, M\}$.

The neural nets were implemented in Python and Keras (Chollet et al., 2015) and trained on an Intel Core i7 8th-gen CPU with 16GB of RAM.

5.1. Synthetic Data

The first synthetic experiment we conduct highlights the consistency/inconsistency of the top-$k$ hinge losses. By Proposition 4.2, if the $k+1$ least likely classes altogether have a probability of occurring greater than $\frac{k}{k+1}$, the predictions made by $\psi_2, \psi_3, \psi_4$ equal a constant vector, and by Theorem 4.1, $\psi_1$ will assign a value of $c + 1$ to the $k-1$ most probable classes and $c$ to the rest. This behavior is inconsistent. On the other hand, $\psi_5$, which is top-$k$ consistent, will assign values of $c + 1$ to the $k$ most probable classes, and $c$ to the rest.

We construct training data which matches the above setting. The data contains 68 data points with each input
data point equal to the zero vector in $\mathbb{R}^2$. Each class in \{1, 2\} is assigned to 10 data points, and each class in \{3, 4, 5, 6, 7, 8\} is assigned to 8 data points. We set $k = 2$ so that $\sum_{i=4}^M \eta_i = \frac{48}{68} > \frac{2}{3}$, as described in Proposition 4.2. We train our neural architecture on the data using batch gradient descent, setting the loss of the last layer to be each of $\{\psi_1, \ldots, \psi_5\}$ with $k = 2$. For each classifier obtained, we evaluate the top-2 error on the training set. This is repeated for 100 trials to ensure the robustness of our results.

One may surmise that even if the theoretical minimizers for a loss are not top-$k$ Bayes optimal, they may be effective in practice due to the optimization process. For example, the learned classifier for $\psi_2$ could output a vector close to 0, but with the first two entries minutely greater than the rest. Interestingly, this is not the case: the returned classifiers for $\psi_2, \psi_3, \psi_4$ essentially pick randomly amongst the 8 possible classes. The classifier returned by $\psi_1$ chooses one of \{0, 1\}, and randomly picks from the rest of the classes. Finally, the classifier returned by $\psi_5$ returns the Bayes decision rule, \{0, 1\}. These results closely align with the theoretical optima of these losses.

We report average top-2 accuracy over the 100 trials in Table 2. For reference, predicting \{0, 1\} yields a top-2 accuracy of $\frac{20}{68} = 0.294$, predicting one of them gives $\frac{18}{68} = 0.265$, and predicting none of them gives $\frac{16}{68} = 0.235$. Examples of score vectors returned by each loss are in Table 6 in the Appendix. We note that the neural net trained with $\psi_5$ predicts \{0, 1\} every trial.

| $\psi_1$ | $\psi_2$ | $\psi_3$ | $\psi_4$ | $\psi_5$ |
|---------|---------|---------|---------|---------|
| Top-2:  | 0.2671  | 0.2515  | 0.2500  | 0.2468  | 0.2941  |

We briefly summarize how the datasets were obtained and featured in the following. ALOI and Letter were downloaded from the LibSVM website (Chang & Lin, 2011). Caltech 101 was obtained from Benjamin Marlin’s website\(^2\). For each of these three datasets, the original features were used without modification. The Flower 102 category dataset was downloaded from the Oxford vision website\(^3\) and the CUB 200 dataset (Welinder et al., 2010) from the Caltech vision website\(^4\). CIFAR-100 was downloaded using Keras (Chollet et al., 2015). The images from Flower 102 and CUB 200 corresponding to the train and test splits were converted to $150 \times 150 \times 3$ tensors using Keras and divided by 255. We used pre-trained features obtained from last max-pooling layer of VGGnet-16 (Simonyan & Zisserman, 2014) trained on Imagenet, obtained from Keras. The Indoor 67 dataset was downloaded from the website of Antonio Torralba\(^5\). Pre-trained features were extracted from the VGGnet-16 architecture, but trained on Places 365 (Kalliatakis, 2017).

Results averaged over the datasets for each loss are given in Table 5. The individual results for each dataset are given in Table 7 in the Appendix.

Looking at the average top-5 and accuracy values in 5, we notice that the entropy based losses $\text{Ent}, \text{Ent}_{\text{Tr}1}, \text{Ent}_{\text{Tr}2}$ perform the best, with $\text{Ent}$ performing the best overall. This may be because the hinge losses minimized by a neu-

\(^2\)https://people.cs.umass.edu/~marlin/data.shtml
\(^3\)http://www.robots.ox.ac.uk/~vgg/data/flowers/
\(^4\)http://www.vision.caltech.edu/visipedia/CUB-200.html
\(^5\)http://web.mit.edu/torralba/www/indoor.html
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Table 3. Second set of synthetic experiments, each value averaged over 10 trials. $N$ is the number of Gaussian centers. Superscript on top-$k$ losses indicates the value of $k$ for that loss. Top-5 is top-5 accuracy=$1 - err_5$, Acc. is accuracy, $\Delta_1 =$ test loss - test top-5 error, $\Delta_2 =$ training loss - test top-5 error.

| Dataset | $n_{tr}$ | $n_{test}$ | $d$ | $M$ |
|--------|---------|-----------|-----|-----|
| ALOI   | 54k     | 54k       | 128 | 1000|
| Caltech 101 Sil | 6364 | 2307 | 784 | 101 |
| CIFAR-100 | 50k  | 10k     | 512 | 100 |
| CUB 200 | 2976   | 3008     | 8192| 200 |
| FLOWER | 5248   | 1920     | 8192| 102 |
| INDOOR 67 | 5312  | 1312    | 8192| 67  |
| LETTER | 15k    | 5k       | 16  | 26  |

Table 4. Description of datasets. $n_{tr}$, $n_{test}$, $d$, $M$ are number of training samples, testing samples, features, and classes, respectively.

Table 5. Averaged results on real datasets. See text for details.

| Dataset | Top-5 | Acc. | $\Delta_1$ | $\Delta_2$ |
|--------|-------|------|------------|------------|
| Ent    | 0.615 | 0.450| 7.018      | 3.667      |
| $\psi_{CS}$ | 0.342 | 0.280| 0.228      | 0.072      |
| $\psi_1$ | 0.415 | 0.243| 0.197      | 0.117      |
| $\psi_2$ | 0.352 | 0.270| 0.223      | 0.062      |
| $\psi_3$ | 0.348 | 0.243| 0.198      | 0.066      |
| $\psi_4$ | 0.349 | 0.259| 0.195      | 0.060      |
| $\psi_5$ | 0.388 | 0.187| 0.126      | 0.049      |
| Ent$^5_{Tr_1}$ | 0.610 | 0.265| 5.538      | 2.311      |
| Ent$^5_{Tr_2}$ | 0.573 | 0.444| 3.085      | 2.512      |

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On the other hand, the hinge losses better approximate the top-$k$ error as reflected by lower $\Delta_1$, $|\Delta_2|$. $\psi_4$ and $\psi_5$ possess the lowest average values of $\Delta_1$ and $|\Delta_2|$, with $\psi_5$ in particular noticeably outperforming the other losses in this respect, achieving $\hat{\Delta}_1 = 0.126$ and $\hat{\Delta}_2 = 0.049$. This is in line with them being the tightest bounds on the top-$k$ error: $err_k(s, y) \leq \psi_4^k(s, y) \leq \psi_5^k(s, y)$ and $err_k(s, y) \leq \psi_4^k(s, y) \leq \psi_5^k(s, y) \leq \psi_5^k(s, y)$.

The performance bottleneck of the hinge losses seems to the difficulty of optimizing them with neural nets. Due to the fidelity of $\psi_4$ and $\psi_5$ to the top-$k$ error, one expects their minima to best minimize the top-$k$ error. Combined with the success of a smoothed $\psi_1$ neural net loss in Berrada et al. (2018), this suggests that smoothing $\psi_4, \psi_5$ is a promising direction for obtaining even lower top-$k$ error.

6. Conclusion

We have derived a rigorous theoretical framework for top-$k$ classification, introducing and making analytic use of concepts such as top-$k$ preserving and top-$k$ calibration to establish results on the consistency of surrogate losses. We then turned our attention to hinge-like top-$k$ losses, showing that previously proposed ones are not top-$k$ calibrated and thus inconsistent. At the same time, we propose two new hinge-like losses, one which we also show is not calibrated and one which is calibrated. In a synthetic experiment, these losses perform just as predicted by our consistency analysis. In terms of accuracy, the hinge losses perform similarly on real and synthetic data. However, the new hinge losses we propose achieve significantly smaller differences in loss and top-$k$ error. This reflects that they are tighter bounds on the top-$k$ error.

Future directions include providing explicit bounds on the risk in terms of the expected loss, and bounds on the estimation error. Also, we would like to come up with "low noise" conditions on the distribution of the data for top-$k$ classification. If defined properly, we may obtain consistency for a general class of loss functions that are otherwise inconsis-
tent, such as the hinge-like losses.

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A. Proofs

In addition to providing the proofs, we restate what is being proved for convenience.

A.1. Proof of Lemma 2.3

Lemma A.1. Let \( \psi : \mathbb{R}^M \times \mathcal{Y} \to [0, \infty) \) be a nonnegative loss function. \( L_\psi^* : \Delta_M \to \mathbb{R} \) defined by \( L_\psi^*(\eta) = \inf_{s \in \mathbb{R}^M} \sum_{i=1}^M \eta_i \psi(s, i) \) is continuous.

Proof. First, note that \( L_\psi^* \) is concave, because it is a pointwise infimum of affine functions of \( \eta \). Also, it is finite valued, because \( \psi \) is lower bounded (thus \( L_\psi^*(\eta) > -\infty \)) and clearly \( L_\psi^*(\eta) < \infty \).

By Theorem 10.2 of Rockafellar (1970), any concave function taking finite real values on a locally simplicial subset \( S \subseteq \mathbb{R}^M \) is lower semicontinuous. That is, for all \( x \in S \) and sequences \( \{x(n)\} \) converging to \( x \), \( f(x) \leq \lim_{n \to \infty} f(x(n)) \) if the limit on the right exists.

\( \Delta_M \) is locally simplicial (it is the probability simplex) and \( L_\psi^* \) satisfies the assumptions, so \( L_\psi^* \) is lower semicontinuous.

Now we just need to show upper semicontinuity, which can be stated as: for any \( \epsilon > 0 \), \( \eta \in \Delta_M \), there exists \( \delta > 0 \) where for all \( \eta' \in \Delta_M \), \( \|\eta' - \eta\|_2 \leq \delta \) implies \( L_\psi^*(\eta') \leq L_\psi^*(\eta) + \epsilon \).

Let \( \eta \in \Delta_M \), \( \epsilon > 0 \). Choose \( s \) so that \( L_\psi(s, \eta) \leq L_\psi^*(\eta) + \epsilon/2 \), which is possible by definition of \( L_\psi^* \). Now set \( \delta = \epsilon \left( 2 \max \left\{ \sqrt{\sum_{i=1}^M \psi(s, i)^2}, 1 \right\} \right)^{-1} \) (taking the max with 1 to avoid a zero in the denominator), and suppose \( \eta' \in \Delta, \|\eta - \eta'\|_2 \leq \delta \). We have,

\[
L_\psi^*(\eta') \leq L_\psi(s, \eta') = \sum_{i=1}^M \eta_i^* \psi(s, i) = \sum_{i=1}^M \eta_i \psi(s, i) + \sum_{i=1}^M (\eta_i' - \eta_i) \psi(s, i) \\
\leq L_\psi^*(\eta) + \epsilon/2 + \|\eta' - \eta\|_2 \sum_{i=1}^M \psi(s, i)^2 \\
\leq L_\psi^*(\eta) + \epsilon/2 + \epsilon/2 = L_\psi^*(\eta) + \epsilon.
\]

The first inequality is by definition of \( L^* \), and the second inequality uses the Cauchy-Schwartz inequality. Therefore, \( L^* \) is upper semicontinuous. Since it is also lower semicontinuous, it is continuous.

A.2. Proof of Theorem 2.4

Theorem A.2. Suppose \( \psi \) is a nonnegative top-k calibrated loss function. Then \( \psi \) is top-k consistent in the sense that for any sequence of measurable functions \( f^{(n)} : \mathcal{X} \to \mathbb{R}^M \), we have

\[
L_\psi(f^{(n)}) \to L_\psi^* \implies L_{err^k}(f^{(n)}) \to L_{err^k}^*.
\]

Proof. We place top-k classification in the abstract decision model in Appendix A. of Zhang (2004a) with output-model space \( Q = \Delta_M \), decision space \( D \) equal to the set of subsets of \( [M] \) of size \( k \), and estimation-model space \( \Omega = \mathbb{R}^M \). The risk function is the top-k error and the decision rule is equal to \( r_k \), the top-k thresholding operator.

By Corollary 26 of Zhang (2004a) we just need to show that for any \( \epsilon > 0 \),

\[
\Delta(\epsilon) = \inf \{ \Delta L_\psi(s, \eta) | L_{err^k}^*(s(n), \eta(n)) \geq \epsilon \} > 0,
\]

where \( \Delta L(s, \eta) := L(s, \eta) - L^*(\eta) \). In other words, we need to show that given any \( \epsilon > 0 \), there is a \( \delta > 0 \) such that \( \Delta L_{err^k}(s, \eta) \geq \epsilon \) implies \( \Delta L_\psi(s, \eta) \geq \delta \).

Proof by contradiction. Given \( \epsilon > 0 \), assume there does not exist \( \delta > 0 \) such that the above holds. Then, there is a sequence \( \{s(n), \eta(n)\} \) such that \( \Delta L_{err^k}(s(n), \eta(n)) \geq \epsilon \) for all \( n \in \mathbb{N} \) and yet \( \Delta L_\psi(s(n), \eta(n)) \to 0 \). Since \( \eta(n) \) comes from a compact set \( \Delta_M \), we may assume that \( \eta(n) \to \eta \) without loss of generality, since otherwise we could take a convergent subsequence.

We will show that \( \Delta L_\psi(s(n), \eta) \to 0 \), which provides a contradiction in the following. Because \( \psi \) is top-k calibrated, \( s(n) \) is top-\( k \) preserving with respect to \( \eta \) for all \( n \) greater than some \( N \). This means there exists \( N \) where \( \Delta L_{err^k}(s(n), \eta) = 0 \) for all \( n > N \), i.e. \( L_{err^k}(s(n), \eta) = L_{err^k}^*(\eta) \). By continuity of \( L_{err^k}^* \), there exists \( N' \) such that \( |L_{err^k}^*(\eta(n)) - L_{err^k}^*(\eta)| < \frac{\epsilon}{2} \) for all \( n > N' \). But this means \( \Delta L_{err^k}^*(s(n), \eta(n)) < \frac{\epsilon}{2} \) for \( n > \max\{N, N'\} \), a contradiction.

Since \( \Delta L_\psi(s(n), \eta(n)) \to 0 \), for any \( \epsilon' > 0 \), there exists \( N > 0 \) such that for all \( n > N \), we have

\[
|L_\psi(s(n), \eta(n)) - L_\psi^*(\eta(n))| \leq \epsilon'/2.
\]
Moreover, since $L_{\psi}^*$ is continuous by Lemma 2.3 and $\eta^{(n)} \to \eta$, there exists $N' > 0$ such that for all $n > N'$, we have
\[
|L_{\psi}^*(\eta^{(n)}) - L_{\psi}^*(\eta)| \leq \epsilon'/2.
\]
Then, for all $n > \max\{N, N'\}$,
\[
|L_{\psi}(s^{(n)}, \eta^{(n)}) - L_{\psi}^*(\eta)| \leq \left|L_{\psi}(s^{(n)}, \eta^{(n)}) - L_{\psi}^*(\eta^{(n)})\right| + |L_{\psi}^*(\eta^{(n)}) - L_{\psi}^*(\eta)| \leq \epsilon'.
\]
Since $\epsilon'$ was arbitrary, we have $L_{\psi}(s^{(n)}, \eta^{(n)}) \to L_{\psi}^*(\eta)$. Now we extend to $L_{\psi}(s^{(n)}, \eta) \to L_{\psi}^*(\eta)$ by showing that $L_{\psi}(s^{(n)}, \eta^{(n)})$ is close to $L_{\psi}(s^{(n)}, \eta)$. Given any $\epsilon' > 0$, let $N$ be such that for all $n > N$, $L_{\psi}(s^{(n)}, \eta^{(n)}) - L_{\psi}(s^{(n)}, \eta) \leq \epsilon$.
Then we have for all $n > N$
\[
|L_{\psi}(s^{(n)}, \eta^{(n)}) - L_{\psi}(s^{(n)}, \eta)| \leq L_{\psi}(s^{(n)}, \eta^{(n)}) - L_{\psi}^*(\eta) \leq \epsilon'.
\]
Let $I$ be the support of $\eta$. For every $i \in I$, $\{\psi(s^{(n)}, i), i\}$ is bounded, since $\psi \geq 0$ and if it were unbounded above then
\[
L_{\psi}(s^{(n)}, \eta^{(n)}) \geq \frac{1}{M} \psi(s^{(n)}, i) \to \infty > L_{\psi}^*(\eta) \text{ eventually.}
\]
Now suppose $C > 0$ upper bounds $\{\psi(s^{(n)})\}$ for every $i \in I$. Since $\eta^{(n)} \to \eta$, there exists $N'$ such that $n > N'$ implies $\eta^{(n)}_i \geq \eta_i - \epsilon'/MC$ for every $i \in [M]$. Then,
\[
L_{\psi}(s^{(n)}, \eta^{(n)}) - L_{\psi}(s^{(n)}, \eta) = \sum_{i=1}^{M}(\eta_i^{(n)} - \eta_i)\psi(s^{(n)}, i) \\
\geq \sum_{i \in I}(\eta_i^{(n)} - \eta_i)\psi(s^{(n)}, i) \\
\geq M \left( -\frac{\epsilon'}{MC}C \right) = -\epsilon'.
\]
Therefore, for all $n > \max\{N, N'\}$, we have
\[
|L_{\psi}(s^{(n)}, \eta^{(n)}) - L_{\psi}(s^{(n)}, \eta)| \leq \epsilon'.
\]
Since $\epsilon' > 0$ was arbitrary, this implies that $\{L_{\psi}(s^{(n)}, \eta)\}$ converges to the same limit as $\{L_{\psi}^*(s^{(n)}, \eta^{(n)})\}$. Thus, $L_{\psi}(s^{(n)}, \eta) \to L_{\psi}^*(\eta)$. We have thus reached the contradiction laid out earlier.

A.3. Proof of Theorem 3.1

To prove Theorem 3.1, we use two lemmas which are interesting in their own right. The first establishes the openness of the set $\{s \in \mathbb{R}^M \mid P_k(s, \eta)\}$ for any $\eta \in \mathbb{R}^M$. The second says that a convex function with a unique minimizer has bounded sublevel sets.

Lemma A.3. $P_k(\eta) := \{s \in \mathbb{R}^M \mid P_k(s, \eta)\}$ is open for any $\eta \in \mathbb{R}^M$, $k \in \mathbb{Z}^+$.

Proof. Let $\eta \in \mathbb{R}^M$ and $s \in P_k(\eta)$. Define
\[
\delta_1 = \min_{i \in [M]} \{s_i - s_{[k+1]} \mid s_i > s_{[k+1]}\} \\
\delta_2 = \min_{i \in [M]} \{s_{[k]} - s_i \mid s_i < s_{[k]}\}.
\]
Take $\delta = \min\{\delta_1, \delta_2\}$, and notice $\delta > 0$. Then, take $s' \in \mathbb{R}^M$ with $|s'_i - s_i| < \delta/2$ for all $i \in [M]$. If $s_i > s_{[k+1]}$, then
\[
s'_i > s_i - \delta/2 > s_{[k+1]} + \delta/2 > s'_{[k+1]},
\]
and similarly if $s_i < s_{[k]}$ then $s'_i < s'_{[k]}$. Therefore, $P_k(s', \eta)$. This holds for every $s'$ in the neighborhood – thus $P_k(\eta)$ is open.

Lemma A.4. If $f : \mathbb{R}^M \to \mathbb{R}$ is convex and has a unique minimizer, the sublevel sets $\{x \in \mathbb{R}^M \mid f(x) \leq \alpha\}$ are bounded for every $\alpha \in \mathbb{R}$.

Proof. Suppose $x_0 \in \mathbb{R}^M$ is the unique minimizer. We can assume $x_0 = 0$ by taking $f(x + x_0)$, which has the same sublevel sets just shifted by $x_0$, and a unique minimizer at $x = 0$.

Then, $f(x) > f(0)$ for all $x \in \mathbb{R}^M$. Consider the set $B = \{x \in \mathbb{R}^M \mid \|x\|_2 = 1\}$. $B$ is compact. Therefore, the image of $B$ under $f$, $f(B) \subseteq \mathbb{R}$, is compact and has a minimum. Since $f(x) > f(0)$ for all $x \in B$, we have
\[
\delta := \min(f(B)) - f(0) > 0.
\]
Now, suppose $x \in \mathbb{R}^M$ such that $\|x\|_2 = D \geq 1$. Since $D \geq 1$, we have $0 < 1/D \leq 1$. Note $\|x/D\|_2 = 1$. Now we apply convexity:
\[
f \left( \frac{x}{D} \right) \leq \frac{1}{D} f(x) + \left(1 - \frac{1}{D}\right) f(0).
\]
Rearranging,
\[
f(x) \geq Df \left( \frac{x}{D} \right) + (1 - D)f(0) \\
= D(f(x/D) - f(0)) + f(0) \\
\geq D\delta + f(0).
\]
Thus, if $D \geq 1$, we have $\|x\|_2 \geq D$ implies $f(x) > D\delta/2 + f(0)$. The contrapositive is, $f(x) \leq D\delta/2 + f(0)$ implies $\|x\|_2 < D$ for $D \geq 1$. Therefore, for all $x \in \mathbb{R}^M$
\[
f(x) \leq \alpha \implies \|x\|_2 \leq \max \left\{ \frac{2(\alpha - f(0))}{\delta}, 1 \right\}.
\]
This says that the sublevel sets are bounded.

Now we prove the theorem.

Theorem A.5. Suppose $\phi : \mathbb{R}^M \to \mathbb{R}$ is strictly convex and differentiable. If $g : \mathbb{R}^M \to \mathbb{R}$ is reverse top-$k$ surrogate preserving, continuous, and $\Delta_M \subseteq \text{range}(g)$, then $\psi : \mathbb{R}^M \times \mathcal{Y} \to \mathbb{R}$ defined by
\[
\psi(s, y) = D\phi(g(s), y)
\]
is top-$k$ calibrated.
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Proof. Let \( \eta \in \Delta_M \). By Theorem 1 from Banerjee et al. (2005),
\[
\arg\min_{\eta \in \Delta_M} E_{y \sim \eta} D_\phi(\tilde{y}, Y) = E[Y] = \eta.
\]
Therefore,
\[
\arg\min_{s \in \mathbb{R}^M} L_\psi(s, \eta) = \arg\min_{s \in \mathbb{R}^M} E_{y \sim \eta} D_\phi(g(s), Y) = \{ s \in \mathbb{R}^M \mid g(s) = \eta \},
\]
and since \( \Delta_M \subseteq \text{range}(g) \) the last set is nonempty. Let \( s^* \) be such that \( g(s^*) = \eta \).

Since \( g \) is reverse top-k preserving, \( P_k(s^*, \eta) \). This holds for any \( s^* \) in \( O := \{ s \in \mathbb{R}^M \mid g(s) = \eta \} \). Given any \( s \) for which \( \neg P_k(s, \eta), s \not\in O \), and thus \( g(s) \not= \eta \), \( L_\psi(s, \eta) = E_{y \sim \eta} D_\phi(g(s), Y) > E_{y \sim \eta} D_\phi(\eta, Y) \). Therefore,
\[
\inf_{s \in \mathbb{R}^M : \neg P_k(s, \eta)} L_\psi(s, \eta) > \inf_{s \in \mathbb{R}^M} L_\psi(s', \eta).
\]

To see this, first note \( E_{y \sim \eta} D_\phi(g, y) \) is convex in \( g \) while attaining a unique minimum by Banerjee et al. (2005). Therefore, by Lemma A.4 the sublevel sets \( \{ g \mid E_{y \sim \eta} D_\phi(g, y) \leq \alpha \} \) are bounded for any \( \alpha \in \mathbb{R} \). Then
\[
\inf_{g \in \mathbb{R}^M : \neg P_k(g, \eta)} E_{y \sim \eta} D_\phi(g, y) > \inf_{g \in \mathbb{R}^M : \neg P_k(g, \eta)} E_{y \sim \eta} D_\phi(g, y) > \inf_{s \in \mathbb{R}^M} L_\psi(s, \eta),
\]
as \( \{ g \in \mathbb{R}^M : \neg P_k(g, \eta) \} \) is closed by A.3, and for the infimum we only have to consider its intersection with some bounded closed (i.e. compact) set, due to the boundedness of the sublevel sets. Then since continuous functions map compact sets to compact sets, we can switch the infimum to a minimum.

Because \( g \) is reverse top-k preserving, \( P_k(s, g(s)) \). Then, if \( P_k(g(s), \eta) \), we see by transitivity of \( P_k \) that \( P_k(s, \eta) \). Therefore, \( \neg P_k(s, \eta) \implies \neg P_k(g(s), \eta) \). So, \( A := \{ L_\psi(s, \eta) \mid \neg P_k(g(s), \eta) \} \subseteq \mathbb{R}^M : \neg P_k(g, \eta) \} := B \), and
\[
\inf A \geq \min B > \inf_{s \in \mathbb{R}^M} L_\psi(s, \eta).
\]
Thus, \( \psi \) is top-k calibrated. \( \square \)

A.4. Proof of Proposition 3.2

Proposition A.6. Suppose \( \psi \) is top-k calibrated for the unweighted top-k error. Then, \( \psi' \) defined by
\[
\psi'(s, y) = c_y \psi(y, y),
\]
is top-k calibrated for the cost-sensitive error weighted by \( c_y \). The converse holds as well.

A.5. Proof of Theorem 4.1

Theorem A.7. Let \( \Pi_M \) denote the set of permutations from \([M] \) to \([M] \). Say \( \pi \in \mathbb{S}_M \) sorts a vector \( v \in \mathbb{R} \) if \( v_{\pi_1} \geq v_{\pi_2} \geq \ldots \geq v_{\pi_M} \).

Let \( \eta \in \Delta_M \), and suppose it has no zero entries. Then,

1. If \( \eta[k] > \sum_{i=k+1}^{M} \eta[i] \), then \( s^* = \arg\min_{s \in \mathbb{R}^M} L_\psi(1, \eta) \) iff for some \( c \in \mathbb{R} \) and \( \pi \in \Pi_M \) which sorts \( \eta \).
2. \( s_{\pi_k} = \ldots = s_{\pi_M} = c, \quad s_{[k]} = c + 1, \quad \forall i \in \{1, \ldots, k-1\}, \quad s[i] \in \{c+1, \infty\} \).

Furthermore, \( L_\psi(1, \eta) = 2 \sum_{i=k+1}^{M} \eta[i] \).

3. If \( \eta[k] < \sum_{i=k+1}^{M} \eta[i] \), then \( \) iff for some \( c \),
2. \( s_{\pi_k} = \ldots = s_{\pi_M} = c, \quad \forall i \in \{1, \ldots, k-1\}, \quad s[i] \in \{c+1, \infty\} \).

Furthermore, \( L_\psi(1, \eta) = 2 \sum_{i=k+1}^{M} \eta[i] \).

Proof. Suppose \( \tau \in \Pi_M \) sorts \( s \). Define \( \delta := s_{\tau_k} - s_{\tau_{k+1}} = s_{[k]} - s_{[k+1]} \) \geq 0. Since
\[
\max\{1 + s_{\tau_{k+1}} - s_{\tau_k}, 0\} = \max\{1 - \delta, 0\}
\]
and \( \max\{1 + s_{\tau_k} - s_{\tau_{k+1}}, 0\} = 1 + \delta, \forall i \in \{k + 1, \ldots, M\} \),
Thus, the equality holds if and only if in addition to the common requirements, regardless of case. Since \( \eta \) has no zero entries, the first line is an equality if and only if \( s_{\tau_k} \geq s_{\tau_{k+1}} + 1 = c + 1 \) for all \( i \in [k-1] \), and \( s_{\tau_{k+1}} = s_{\tau_{k+2}} = \ldots = s_{\tau_M} = c \). And in any case where the second line is an equality, the sums on the right of both lines equal, which happens if and only if \( \{\tau_{k+1}, \ldots, \tau_M\} = \{\pi_{k+1}, \ldots, \pi_M\} \) for some \( \pi \in \Pi_M \) which sorts \( \eta \).

Case 1: If \( \eta[k] > \sum_{i=k+1}^M \eta[i] \), then \( F(\delta) \) is minimized uniquely at \( \delta = 1 \) in the interval \([0, 1]\); by our assumption that \( \eta \) does not have 0 entries and \( k < M \), \( \delta > 1 \) is suboptimal. Thus, \( L_\psi(\eta) \geq 2 \sum_{i=k+1}^M \eta[i] \) (achieved by a described below).

The equality is achieved if and only if the common requirements hold and \( \delta = 1 \), giving \( s_{\tau_k} = c + 1 \).

Case 2: If \( \eta[k] < \sum_{i=k+1}^M \eta[i] \), then \( F(\delta) \) is minimized by \( \delta = 1 \) and \( L_\psi(\eta) = \sum_{i=k+1}^M \eta[i] \). Therefore, the equality holds if and only if \( s_{\tau_k} = s_{\tau_{k+1}} = c \) and \( \tau_k = \pi_k \) for some \( \pi \in S_M \) which sorts \( \eta \), along with the common requirements.

Case 3: If \( \eta[k] = \sum_{i=k+1}^M \eta[i] \), then \( L_\psi(\eta) = 2 \sum_{i=k+1}^M \eta[i] \). Thus \( F(\delta) \) is minimized by \( \delta \in [0, 1] \).

If \( \delta \in (0, 1) \), the inequality in (5) requires

\[
\sum_{i=k}^M \eta[i] = \sum_{i=k}^M \eta[i] = 2 \sum_{i=k+1}^M \eta[i] = 2 \sum_{i=k+1}^M \eta[i].
\]

Thus, the equality holds if and only if in addition to the common requirements, \( s_{\tau_k} \in (c, c+1) \), and for some \( \pi \in S_M \) which sorts \( \eta, \pi_k = s_{\tau_k} \).

If \( \delta = 1 \) or \( \delta = 0 \), we have the same iff conditions for the equality as in case 1 and case 2.

A.6. Proof of Proposition 4.2

Proposition A.8. For any \( \psi \in \{\psi_2, \psi_3, \psi_4\} \), if \( \sum_{m=k+1}^M \eta[m] > \frac{k}{k+1} \), we have \( \delta \in \{\arg \min \}_{\delta \in I} L_\psi(s, \eta) \), and thus \( L_\psi(\eta) = \min_{\eta} L_\psi(s, \eta) = L_\psi(0, \eta) = 1 \).

Proof. We will show that \( L_\psi(\eta) = 1 \). WLOG, we can assume that \( \eta_1 \geq \ldots \geq \eta_M, s_1 \geq s_2 \geq \ldots \geq s_M \), and \( s_{k-1} = s_k = \ldots = s_M = 0 \).

Suppose \( s_i \geq 1 \) for some \( i \in [M] \). Then, for each \( \psi \in \{\psi_2, \psi_3, \psi_4\} \), \( \psi(s, i) \geq 1 + \frac{1}{k} \) for all \( i = k+1, \ldots, M \), and so \( L_\psi(s, \eta) \geq (1 + \frac{1}{k}) (\eta_1 + \ldots + \eta_M) > \frac{k+1}{k} = \frac{k}{k+1} \), which gives \( \delta = 1 \). This implies that \( s \) is suboptimal, since \( L_\psi(0, \eta) = 1 \).

Thus, at optimum \( 0 \leq s_i < 1 \) for every \( i \), under which \( \psi_2(s, i) = \psi_3(s, i) = \psi_4(s, i) \) for every \( i \). This is because in this regime, \( \max \{1 + s_j - s_i, 0\} = 1 + s_j - s_i \), and the \( k \)th highest value of \( s(i) + \) coincides with the \( k \)th highest value of \( 1 + s \) excluding the \( i \)th index. Now for all \( i \in [k] \), we have \( s_i \in (0, 1) \) and thus

\[
\frac{\partial L_\psi(s, \eta)}{\partial s_i} = \frac{1}{k} \sum_{m \neq i} \eta_m - \eta_i = \frac{1}{k} (1 - \eta_i) - \eta_i \geq \frac{1}{k} \cdot \frac{1}{k+1} - \frac{1}{k+1} = 0.
\]

The derivative is positive (and constant) in \((0, 1)\), so the minimum value of \( s_i \) is achieved at 0, for every \( i \). Therefore, \( L_\psi(\eta) = 1 \), achieved by a score vector of 0. This proves the desired statement.

A.7. Proof of Proposition 4.3

Proposition A.9. \( \psi_5 : \mathbb{R}^M \times \mathcal{Y} \) defined by \( \psi_5(s, y) = \max \{1 + s_{[k+1]} - s_i, 0\} \) is top-k calibrated.

Proof. Let \( \eta \in \Delta_M \). For any \( s \in \mathbb{R}^M \), we have

\[
L_\psi(s, \eta) = \sum_{i=1}^M \eta_i \psi_5(s, i) = \sum_{i=1}^M \eta_i \max\{1 + s_{[k+1]} - s_i, 0\}.
\]

We may assume \( \eta_1 \geq \eta_2 \geq \ldots \geq \eta_M \) WLOG. By inspection, setting \( s_1 = \ldots = s_k = 1 \) and \( s_{k+1} = \ldots = s_M = 0 \) gives \( L_\psi(s, \eta) = \sum_{i=k+1}^M \eta[i] \geq C \).

We will show that any \( s \in \mathbb{R}^M \) such that \( \neg P_k(s, \eta) \) has \( L_\psi(s, \eta) - L_\psi(\eta) \geq L_\psi(s, \eta) - C \geq \delta \) for some constant \( \delta > 0 \), which implies top-k calibration.

Suppose \( \neg P_k(s, \eta) \). Define \( \delta_1 = \min\{\eta_i - \eta_{[k+1]} \mid i \in [M], \eta_i > \eta_{[k+1]} \} \) and \( \delta_2 = \min\{\eta_i \mid i \in [M], \eta_i < \eta_{[k+1]} \} \). If either set is empty, define its minimum to be \( \infty \).

Furthermore, define the set \( I := \{i \in [M] \mid s_i \leq s_{[k+1]}\} \). Note by definition of \( s_{[k+1]} \), \( |I| \geq M - k \). We have \( L_\psi(s, \eta) \geq \sum_{i \in I} \eta_i \). There are two cases.

If there exists \( i \in [M] \) such that \( \eta_i \geq \eta_{[k+1]} \) and \( s_i \leq s_{[k+1]} \), then \( i \in I \). But then \( \sum_{j \in I} \eta_j \geq \sum_{j=k+1}^M \eta[j] + \delta_1 \).

If there exists \( i \in [M] \) such that \( \eta_i < \eta_{[k+1]} \), then \( s_i \geq s_{[k+1]} \), then consider if \( s_i > s_{[k+1]} \). Then, \( i \notin I \). That is, \( \eta_i \) does not appear in the sum \( \sum_{j \in I} \eta_j \). Since \( |I| \geq M - k \), \( \eta_i \)
must be replaced with a term \( \eta_i' \geq \eta_i[k] \). Thus, \( \sum_{j \in I} \eta_j \geq \sum_{j=k+1}^{M} \eta_{[j]} + \delta_2 \). If \( s_i = s_{[k+1]} \), then since \( s_i \geq s_{[k]} \geq s_{[k+1]} \), we have \( s_i = s_{[k]} \). This implies \( |I| > M - k \), and \( \sum_{j \in I} \eta_j \geq \sum_{j=k+1}^{M} \eta_{[j]} + \delta_2 \). Thus, for any \( s \) such that \( \neg P_k(s, \eta) \), we have \( L_\psi(s, \eta) \geq L_\psi^*(\eta) + \delta \) where \( \delta = \min\{\delta_1, \delta_2\} > 0 \). Therefore,

\[
\inf_{s: \neg P_k(s, \eta)} L_\psi(s, \eta) \geq \inf_s L_\psi(s, \eta) + \delta > \inf_s L_\psi(s, \eta),
\]

so \( \psi = \psi_5 \) is top-\( k \) calibrated.

\[ \Box \]

**B. Discussion of general hinge-like losses**

Recall that the hinge loss for binary classification is defined by \( \phi(x) = \max\{1 - x, 0\} \). There are several extensions of the binary hinge loss to the setting of multiclass classification (often with multiclass error i.e. top-1 loss). We list them here because they serve as inspiration for designing hinge-like top-\( k \) losses, and the analysis of their consistency in the literature also informs the analysis of the top-\( k \) case.

The method of Crammer & Singer (2001) uses as its loss function \( \psi : \mathbb{R}^M \times Y \to \mathbb{R} \) where

\[
\psi(s, y) = \max\{1 + (s \setminus y)[1] - s_y, 0\} = \phi(s_y - \max_{y' \neq y} s_{y'}).
\]

(6)

When \( y \in Y \) appears in a subscript it refers to the label as an index in \( \{1, \ldots, M\} \). Furthermore, the notation \( s \setminus y = (s_1, \ldots, s_{y-1}, s_{y+1}, \ldots, s_M) \in \mathbb{R}^{M-1} \) denotes the vector \( s \) with the \( y \)th entry removed.

The method of Weston & Watkins (1999) solves a multiclass SVM problem for which the corresponding loss function is

\[
\psi(s, y) = \sum_{y' \neq y} \phi(s_y - s_{y'}),
\]

where \( \phi \) is still the binary hinge loss. Furthermore, the one vs. all method Rifkin & Klautau (2004) solves \( M \) binary classification problems using the hinge loss for each class, using the instances of the class as positive examples and the rest of the instances as negative examples. The \( M \) scores returned by the \( M \) resulting classifiers are compiled into an \( M \) length vector, and the method proceeds like all the above methods by taking the argmax of the vector. Similarly, the method of Lee et al. (2004) minimizes the expectation of the loss function

\[
\psi(s, y) = \sum_{y' \neq y} \phi(-s_{y'})
\]

under the constraint that \( \sum_{m=1}^{M} s_m = 0 \). Interestingly, Zhang (2004a) showed the first three Crammer & Singer (2001); Weston & Watkins (1999); Rifkin & Klautau (2004) to be inconsistent, i.e. not top-1 calibrated, and the constrained Lee et al. (2004) to be consistent. These results were also found by Tewari & Bartlett (2005).
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Table 6. Examples of predicted score vector \( s = f(0) \) with the zero vector as input, where \( f \) is a neural net trained with the losses below.

| \( s_1 \) | \( s_2 \) | \( s_3 \) | \( s_4 \) | \( s_5 \) | \( s_6 \) | \( s_7 \) | \( s_8 \) |
|---|---|---|---|---|---|---|---|
| \( \psi_1 \) | 0.87793601 | -0.12823531 | -0.12382337 | -0.12676451 | -0.12382337 | -0.12235278 | -0.12529394 | -0.12764691 |
| \( \psi_2 \) | 0.995 0.921 0.241 0.246 | 0.852 0.526 0.578 0.588 | 0.135 0.051 4.715 4.298 | 0.030 0.007 0.032 0.032 | 0.024 0.007 0.010 0.016 |
| \( \psi_3 \) | 0.75734961 | 0.75734961 | -0.25529474 | -0.24823636 | -0.2523534 | -0.24823636 | -0.25529483 | -0.25529486 |

Table 7. Results on all real datasets.

| ALOI | CALTECH 101 | CIFAR-100 | CUB-200 |
|---|---|---|---|
| Ent | Top-5 Acc. | \( \Delta_1 \) | \( \Delta_2 \) | Top-5 Acc. | \( \Delta_1 \) | \( \Delta_2 \) | Top-5 Acc. | \( \Delta_1 \) | \( \Delta_2 \) |
| \( \psi_{CS} \) | 0.827 0.681 3.65 0.112 | 0.861 0.654 2.88 3.30 | 0.549 0.273 3.65 3.73 | 0.192 0.056 11.8 10.8 |
| \( \psi_1 \) | 0.824 0.674 1.13 0.016 | 0.377 0.285 0.122 0.133 | 0.058 0.0158 0.588 0.078 | 0.023 0.008 0.022 0.028 |
| \( \psi_{s} \) | 0.809 0.519 0.717 -0.137 | 0.852 0.526 0.578 0.588 | 0.081 0.019 0.21 0.02 | 0.024 0.007 0.10 0.016 |
| \( \psi_2 \) | 0.829 0.628 1.072 -0.085 | 0.403 0.349 0.092 0.10 | 0.093 0.029 0.072 0.086 | 0.030 0.007 0.032 0.032 |
| \( \psi_3 \) | 0.822 0.611 1.051 -0.095 | 0.405 0.192 0.089 0.14 | 0.064 0.020 0.063 0.067 | 0.019 0.004 0.021 0.021 |
| \( \psi_4 \) | 0.822 0.601 1.061 -0.115 | 0.383 0.343 0.050 0.053 | 0.089 0.024 0.073 0.08 | 0.028 0.004 0.029 0.029 |
| \( \psi_{w} \) | 0.803 0.508 0.709 -0.144 | 0.702 0.263 0.127 0.10 | 0.058 0.017 0.010 0.016 | 0.019 0.002 0.006 0.007 |
| \( \psi_{s} \) | 0.802 0.500 2.728 -0.053 | 0.858 0.008 1.119 1.11 | 0.562 0.26 3.687 3.87 | 0.212 0.060 9.231 8.294 |
| \( \psi_{Tr} \) | 0.778 0.700 3.578 1.134 | 0.789 0.648 2.244 2.266 | 0.524 0.266 3.059 3.104 | 0.135 0.051 4.715 4.298 |

| FLOWER | INDOOR 67 | LETTER |
|---|---|---|
| Ent | Top-5 Acc. | \( \Delta_1 \) | \( \Delta_2 \) | Top-5 Acc. | \( \Delta_1 \) | \( \Delta_2 \) | Top-5 Acc. | \( \Delta_1 \) | \( \Delta_2 \) |
| \( \psi_{CS} \) | 0.042 0.017 0.037 | 0.025 | 0.077 0.016 | 0.080 0.080 | 0.996 0.946 | 0.137 0.146 |
| \( \psi_1 \) | 0.070 0.013 0.026 | -0.014 | 0.076 0.016 | 0.016 0.027 | 0.993 0.604 | 0.013 0.016 |
| \( \psi_2 \) | 0.037 0.006 0.038 | 0.038 | 0.077 0.016 | 0.221 0.056 | 0.997 0.853 | 0.038 0.035 |
| \( \psi_3 \) | 0.067 0.012 0.061 | 0.050 | 0.064 0.014 | 0.069 0.061 | 0.997 | 0.850 | 0.033 | 0.028 |
| \( \psi_4 \) | 0.046 0.010 0.042 | 0.032 | 0.077 0.016 | 0.078 0.078 | 0.997 | 0.815 | 0.033 | 0.030 |
| \( \psi_{w} \) | 0.072 0.023 0.004 | 0.023 | 0.074 0.014 | 0.009 -0.031 | 0.987 | 0.482 | 0.020 | 0.018 |
| \( \psi_{s} \) | 0.783 0.465 1.166 | 1.123 | 0.061 0.009 | 20.796 1.707 | 0.989 | 0.551 | 0.036 | 0.023 |
| \( \psi_{Tr} \) | 0.713 0.502 2.645 | 2.564 | 0.077 0.015 | 3.833 2.694 | 0.992 | 0.924 | 1.523 | 1.525 |