Density-dependent nn-potential from subleading chiral three-neutron forces: Long-range terms

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Abstract

The long-range terms of the subleading chiral three-nucleon force [published in Phys. Rev. C77, 064004 (2008)] are specified to the case of three neutrons. From these 3n-interactions an effective density-dependent neutron-neutron potential \( V_{\text{med}} \) in pure neutron matter is derived. Following the division of the pertinent 3n-diagrams into two-pion exchange, two-pion-one-pion exchange and ring topology, all self-closings and concatenations of two neutron-lines to an in-medium loop are evaluated. The momentum and \( k_n \)-dependent potentials associated with the spin-operators 
\[
1, \vec{\sigma}_1 \cdot \vec{q} \vec{\sigma}_2, \vec{\sigma}_1 \cdot \vec{p} \vec{\sigma}_2 + \vec{\sigma}_1 \cdot \vec{\sigma}_2 \vec{\sigma}_2 \cdot \vec{p'}, \vec{\sigma}_1 \cdot \vec{\sigma}_2 \cdot (\vec{q} \times \vec{p}) \\
\]
are expressed in terms of functions, which are either given in closed analytical form or require at most one numerical integration. The subsubleading chiral 3N-force is treated in the same way. The obtained results for \( V_{\text{med}} \) are helpful to implement the long-range chiral three-body forces into advanced neutron matter calculations.

1 Introduction

Three-nucleon forces are an indispensable ingredient in accurate few-nucleon and nuclear structure calculations. Nowadays, chiral effective field theory is the appropriate tool to construct systematically the nuclear interactions in harmony with the symmetries of QCD. Three-nucleon forces appear first at \( N^2 \)LO, where they consist of a zero-range contact-term \( (\sim c_E) \), a mid-range 1\( \pi \)-exchange component \( (\sim c_D) \) and a long-range 2\( \pi \)-exchange component \( (\sim c_{1,3,4}) \). The complete calculation of the chiral 3N-forces to subleading order \( N^3 \)LO \[1\, 2\] and even to subsubleading order \( N^4 \)LO \[3\, 4\] has been achieved during the past decade by the Bochum-Bonn group. At present the focus lies on constructing 3N-forces in chiral effective field theory with explicit \( \Delta(1232) \)-isobars, for which the long-range 2\( \pi \)-exchange component has been derived recently in ref. \[5\] at order \( N^3 \)LO.

However, for the variety of existing many-body methods, that are commonly employed in calculations of nuclear matter or medium mass and heavy nuclei, it is technically very challenging to include the chiral three-nucleon forces directly. An alternative and approximate approach is to use instead a density-dependent two-nucleon interaction \( V_{\text{med}} \) that originates from the underlying 3N-force. When restricting to on-shell scattering of two nucleons in isospin-symmetric spin-saturated nuclear matter, \( N_1(\vec{p}) + N_2(\vec{p}) \rightarrow N_1(\vec{p'}) + N_2(\vec{p'}) \), the resulting in-medium NN-potential \( V_{\text{med}} \) has the same isospin- and spin-structure as the free NN-potential. The analytical expressions for \( V_{\text{med}} \) from the leading chiral 3N-force at \( N^2 \)LO (involving the parameters \( c_{1,3,4} \), \( c_D \) and \( c_E \)) have been presented in ref. \[6\] and these have found many applications in recent years. But in order to perform nuclear many-body calculations that are consistent with their input at the two-body level, one needs also \( V_{\text{med}} \) derived from the subleading chiral 3N-forces at order \( N^3 \)LO. In two recent works this task has been completed for the short-range terms and relativistic \( 1/M \)-corrections in ref. \[7\], and for the long-range terms in ref. \[8\]. The continuation of this construction from the intermediate-range terms of the subsubleading chiral 3N-force at order \( N^3 \)LO has been reported recently in ref. \[9\]. The normal ordering of chiral 3N-forces to density-dependent NN-interactions has also been performed by the

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Darmstadt group using a decomposition in a $J j$-coupled 3N partial-wave momentum basis\cite{10,11}. An advantage of this purely numerical approach is that the restriction to on-shell kinematics in the center-of-mass frame can be avoided and a regulator function $f_R(p,q) = \exp[-(p^2 + 3q^2/4)^4/\Lambda^8]$ may be included.

In this work the specification to the case of three neutrons is considered and an effective density-dependent neutron-neutron potential $V_{\text{med}}$ in pure neutron matter is derived. The 3n-interaction is obtained from a given expression for the 3N-force by the simple substitution of isospin-operators: $\vec{\tau}_1 \cdot \vec{\tau}_2 \rightarrow 1$, $\vec{\tau}_1 \cdot \vec{\tau}_3 \rightarrow 1$, $\vec{\tau}_2 \cdot \vec{\tau}_3 \rightarrow 1$ and $\vec{\tau}_1 \cdot (\vec{\tau}_2 \times \vec{\tau}_3) \rightarrow 0$. In the next step three self-closings and six concatenations of two neutron-lines to an in-medium loop need to be evaluated. This introduces a set of loop-functions that depend on the momentum $p = |\vec{p}|$, the momentum-transfer $q = |\vec{p}' - \vec{p}|$, and the neutron Fermi-momentum $k_n$. Since the calculational procedure follows closely that in previous works\cite{7,8,9}, it is sufficient to list the results for the nn-potential $V_{\text{med}}$ in pure neutron matter without further explanation. The analytical results for the in-medium nn-potential $V_{\text{med}}$ derived from the subleading short-range terms and relativistic $1/M$-corrections have been reported in ref.\cite{12}.

Figure 1: $2\pi$-exchange topology, $2\pi 1\pi$-exchange topology and ring topology which comprise the long- and intermediate-range chiral 3n-forces.

2 Two-pion exchange topology

One starts with the longest-range component of the subleading chiral 3N-interaction. It arises from $2\pi$-exchange with the corresponding symbolic diagram shown on the left of Fig. 1. Using for simplification the relation $q_2^2 = q_1^2 + q_3^2 + 2\vec{q}_1 \cdot \vec{q}_3$, the expression in eq.(2.9) of ref.\cite{1} specified to three neutrons takes the form:

$$2V_{3n} = \frac{g^4_A}{128\pi f^6_{\pi}} \frac{\vec{\sigma}_1 \cdot \vec{q}_1 \vec{\sigma}_3 \cdot \vec{q}_3}{(m^2_\pi + q_1^2)(m^2_\pi + q_3^2)} \left\{ m_\pi (m^2_\pi + q_1^2 + q_3^2 + 2q_2^2) \ight. + \left. (2m^2_\pi + q_2^2)(3m^2_\pi + q_1^2 + q_3^2 + 2q_2^2) A(q_2) \right\},$$

with the pion-loop function $A(s) = (1/2s) \arctan(s/2m_\pi)$.

2.1 Contributions to in-medium nn-potential

The self-closing of neutron-line 2 gives (after relabeling 3 $\rightarrow$ 2) the contribution

$$V_{\text{med}}^{(0)} = -\frac{g_A^4 m_\pi k^3_n}{2(4\pi f^2_{\pi})^3} \frac{\vec{\sigma}_1 \cdot \vec{q} \vec{\sigma}_2 \cdot \vec{q}}{(m^2_\pi + q^2)^2} \left( q^2 + \frac{5m^2_\pi}{6} \right).$$

2
From pionic vertex corrections on either neutron-line one obtains the (total) contribution:

\[
V_{\text{med}}^{(1)} = \frac{2g_A^4}{(8\pi f_\pi^2)^3} \frac{\bar{\sigma}_1 \cdot \bar{q} \bar{\sigma}_2 \cdot \bar{q}}{m_\pi^4 + q^2} \left\{ m_\pi \left[ 2k_n^3 - 8\Gamma_2 - 2m_\pi^2 (\Gamma_0 + \Gamma_1) + q^2 (\Gamma_0 - \Gamma_1 - 2\Gamma_3) \right] + S(p) + J(p, q) \right\},
\]

(3)

where the functions \( \Gamma_\nu(p, k_n) \) are defined in the appendix of ref. [7]. The part linear in \( m_\pi \) comes obviously from the first line in eq.(1) and the decomposition \( S(p) + J(p, q) \) is obtained by canceling momentum factors against a pion-propagator. The two functions appearing at the end of eq.(3) read:

\[
S(p) = \left[ \frac{k_n^5}{15p^3} \left( \frac{m_\pi^4}{14} + \frac{k_n^2}{3p} \frac{m_\pi^4}{2} + \frac{k_n^4}{3} \frac{m_\pi^4}{4} + 2m_\pi^2 \right) p + \frac{p^2}{5} \left( \frac{k_n^2}{2} - \frac{4m_\pi^2}{3} \right) + \frac{p^4}{84} \right] \arctan \frac{p + k_n}{2m_\pi}
+ \left[ \frac{k_n^5}{15p^3} \left( \frac{m_\pi^4}{14} + \frac{k_n^2}{3p} \frac{m_\pi^4}{2} + \frac{k_n^4}{3} \frac{m_\pi^4}{4} + 2m_\pi^2 \right) p + \frac{p^2}{5} \left( \frac{4m_\pi^2}{3} - \frac{k_n^2}{2} \right) + \frac{p^4}{84} \right] \arctan \frac{p - k_n}{2m_\pi}
+ m_\pi^3 \left[ \frac{7p}{24} + \frac{1}{p} \left( \frac{m_\pi^4}{3} - \frac{k_n^2}{4} \right) + \frac{1}{p^3} \left( \frac{6m_\pi^4}{35} + \frac{k_n^2 m_\pi^4}{15} - \frac{k_n^4}{24} \right) \right] \ln \frac{4m_\pi^2 + (p + k_n)^2}{4m_\pi^2 + (p - k_n)^2}
+ \frac{k_n m_\pi}{7} \left[ \frac{p^2}{3} - \frac{61m_\pi^4}{30} - \frac{37k_n^2 m_\pi^4}{15} - \frac{1}{p^2} \left( \frac{2k_n^4}{15} + \frac{k_n^2 m_\pi^4}{6} + \frac{6m_\pi^4}{5} \right) \right],
\]

(4)

\[
J(p, q) = \frac{1}{4q^2} \int_{p-k_n}^{p+k_n} ds \left( 2m_\pi^2 + s^2 \right) \left( 2m_\pi^2 + q^2 + 2s^2 \right) \arctan \frac{s}{2m_\pi}
\times \left\{ \frac{k_n^2 - (p - s)^2}{p} + \frac{q^2 - m_\pi^2 - s^2}{q} \ln \frac{qX + 2\sqrt{W}}{(2p + q)\left( m_\pi^2 + (q - s)^2 \right)} \right\},
\]

(5)

with the auxiliary polynomials:

\[
X = m_\pi^2 + 2(k_n^2 - p^2) + q^2 - s^2,
\]

\[
W = k_n^2 q^4 + p^2 (m_\pi^2 + s^2)^2 + q^2 \left[ (k_n^2 - p^2)^2 + m_\pi^2 (k_n^2 + p^2) - s^2 (k_n^2 + p^2 + m_\pi^2) \right].
\]

(6)

Finally, the two diagrams related to double exchange lead to the expression:

\[
V_{\text{med}}^{(3)} = \frac{g_A^4}{(8\pi f_\pi^2)^3} \left\{ m_\pi + (2m_\pi^2 + q^2)A(q) \right\} \left[ \frac{8k_n^3}{3} + 2q^2 (\Gamma_0 - \Gamma_1) ight] - 2m_\pi^2 \Gamma_0 \left[ 3m_\pi + (2m_\pi^2 + q^2)A(q) \right] + (2m_\pi^2 + q^2)A(q) \left[ m_\pi^3 - 2m_\pi q^2 + (3m_\pi^2 + 5m_\pi^2 q^2 + 2q^4)A(q) \right] G_0
+ i(\bar{\sigma}_1 + \bar{\sigma}_2) \cdot (\bar{q} \times \bar{p}) \left[ 2 \left[ m_\pi + (2m_\pi^2 + q^2)A(q) \right] (\Gamma_0 - \Gamma_1) + \Gamma_1 \right]
+ \left[ 2m_\pi q^2 - m_\pi^2 + (2m_\pi^2 + 5m_\pi^2 q^2 + 2q^4)A(q) \right] (G_0 - 2G_1) \right\}.
\]

(7)

### 3 Two-pion-one-pion exchange topology

Next, one considers the \( 2\pi \) exchange three-neutron interaction represented by the middle diagram in Fig. 1. According to eqs.(2.16)-(2.20) in ref.[1] this chiral 3n-interaction can be written in the form:

\[
V_{3n} = \frac{g_A^4}{256\pi f_\pi^6 m_\pi^2 + q_3^2} \left\{ \bar{\sigma}_3 \cdot \bar{q}_3 \left[ \bar{\sigma}_2 \cdot \bar{q}_1 \bar{q}_1 \cdot \bar{q}_3 f_1(q_1) + \bar{\sigma}_2 \cdot \bar{q}_1 f_2(q_1) \right]
+ \bar{\sigma}_2 \cdot \bar{q}_3 f_3(q_1) + \bar{\sigma}_1 \cdot \bar{q}_1 \bar{q}_1 \cdot \bar{q}_3 f_4(q_1) + \bar{\sigma}_1 \cdot \bar{q}_3 f_5(q_1) + \bar{\sigma}_2 \cdot \bar{q}_3 \bar{q}_1 \cdot \bar{q}_3 f_8(q_1) \right\},
\]

(8)
where a common factor $g_A^4/(256\pi f_\pi^6)$ has been pulled out, and thus the reduced functions $f_j(s)$ read:

$$f_1(s) = \frac{5g_A^2}{2}(1 - 2g_A^2) - \frac{g_A^2m_\pi}{4m_\pi^2 + s^2} + \left[1 + g_A^2 + \frac{4m_\pi^2}{s^2}(2g_A^2 - 1)A(s)\right], \quad (9)$$

$$f_2(s) = 6m_\pi + 6(2m_\pi^2 + s^2)A(s) \quad (10)$$

$$f_3(s) = (1 - g_A^2)\left[3m_\pi + (8m_\pi^2 + 3s^2)A(s)\right], \quad (11)$$

$$f_5(s) = -s^2 f_4(s) = 2g_A^3s^2A(s), \quad (12)$$

with $A(s)$ defined after eq.(1). In perspective one notes that at subleading order $\mathcal{Q}$ the chiral $2\pi 1\pi$-exchange $3n$-interaction has a richer spin- and momentum-dependence and also the last term involving $f_5(q_1)$ is needed to represent all diagrams belonging to this topology at $N^4$LO. In the three-neutron case the other relevant functions are combined as: $f_1(s) \equiv f_1(s) + f_6(s)$, $f_2(s) \equiv f_2(s) + f_7(s)$, $f_3(s) \equiv f_3(s) + f_8(s)$ with $f_{1,2,3,6,7,9}(s)$ on the right hand sides given in eqs.(2-12) of ref.[9].

### 3.1 Contributions to in-medium NN-potential

Again, one lists the four contributions from $V_{3n}$ in eq.(8) to the in-medium nn-potential $V_{\text{med}}$. The self-closing of the neutron lines 1 and 2 lead to nonvanishing contributions, and with the relations $f_3(0) = 5(1 - g_A^2)m_\pi$ and $f_5(q) + q^2 f_4(q) = 0$ one obtains:

$$V_{\text{med}}^{(0)} = \frac{5g_A^4(1 - g_A^2)m_\pi k_0^3}{384\pi^3 f_\pi^6} \frac{\bar{\sigma}_1 \cdot \bar{\sigma}_2 \cdot \bar{q}}{m_\pi^2 + q^2}, \quad (13)$$

which is of the form: $1\pi^0$-exchange nn-interaction times a factor linear in the neutron density $\rho_n = k_0^3/3\pi^2$. On the other hand the vertex corrections by $1\pi^0$-exchange, incorporated in eq.(8) through the second factor $\bar{\sigma}_3 \cdot \bar{\sigma}_4/(m_\pi^2 + q^2_3)$, produce the contribution:

$$V_{\text{med}}^{(1)} = \frac{g_A^4}{(8\pi f_\pi^2)^3} \left\{ \left(2m_\pi^2 \Gamma_0 - \frac{4k_0^3}{3}\right) f_3(q) - \left(2\Gamma_2 + \frac{q^2}{2} \Gamma_3\right) q^2 f_1(q) + \Gamma_1 q^2 f_2(q) - 2\bar{\sigma}_1 \cdot \bar{\sigma}_2 \Gamma_2 f_5(q) \right. \right.
\nonumber - \left(\bar{\sigma}_1 \cdot \bar{p}\bar{\sigma}_2 \cdot \bar{\rho} + \bar{\sigma}_1 \cdot \bar{\rho}' \bar{\sigma}_2 \cdot \bar{\rho}'\right) \bar{\Gamma}_3 f_5(q) - \bar{\sigma}_1 \cdot \bar{q} \bar{\sigma}_2 \cdot \bar{q} \left(2\Gamma_2 + \frac{q^2}{2} \Gamma_3\right) f_4(q) \nonumber + i(\bar{\sigma}_1 + \bar{\sigma}_2) \cdot (\bar{q} \times \bar{p}) \left[ \frac{2k_0^3}{3} \bar{\Gamma}_3 f_1(q) - \bar{\Gamma}_1 f_2(q) \right] + \left. \left. \left(2k_0^3 \bar{\sigma}_3 \cdot \bar{\sigma}_4 \right) \right) q^2 f_8(q) \right\} \quad (14)$$

Here, the frequently occurring combinations $\bar{\Gamma}_1(p) = \Gamma_1(p) + \Gamma_1(p)$ and $\bar{\Gamma}_3(p) = \Gamma_0(p) + 2\Gamma_1(p) + \Gamma_3(p)$ have been introduced. Moreover, the vertex corrections by $2\pi\pi$-exchange (compiled in the expression in curly brackets of eq.(8)) can be summarized as the $1\pi^0$-exchange nn-interaction times a $\{p, q, k_f\}$-dependent factor:

$$V_{\text{med}}^{(2)} = \frac{g_A^4}{(8\pi f_\pi^2)^3} \frac{\bar{\sigma}_1 \cdot \bar{q} \bar{\sigma}_2 \cdot \bar{q}}{m_\pi^2 + q^2} \left[ S_1(p) + q^2 S_2(p) \right], \quad (15)$$

with the two functions $S_{1,2}(p, k_n)$ defined by:

$$S_1(p) = \frac{1}{8p^3} \int_{p-k_n}^{p+k_n} ds s \left[ k_n^2 - (p - s)^2 \right] \left\{ \left[ (p + s)^2 - k_n^2 \right] f_2(s) - 8p^2 \left[ f_3(s) + f_5(s) \right] \right\} + \frac{1}{3} \left[ (k_n^2 - (p - s)^2) (k_n^2 - p^2 - 4ps - s^2) \left[ f_1(s) + f_4(s) \right] \right], \quad (16)$$
\[ S_2(p) = \frac{1}{32p^2} \int_{p-k_n}^{p+k_n} ds \left[ k_n^2 - (p-s)^2 \right] \left[ k_n^2 - (p+s)^2 \right] \left\{ (p^2 + s^2 - k_n^2) \left[ f_1(s) + f_4(s) \right] - 4p^2 f_8(s) \right\}. \] (17)

Note that the integrals for \( S_{1,2}(p) \) with input functions \( f_{1,2,3,4,5}(s) \) as in eqs.(9-12) could be solved in terms of the functions \( \arctan \left[ (p \pm k_n)/2m_\pi \right] \) and \( \ln [4m_\pi^2 + (p \pm k_n)^2] \).

Finally, the more complicated contribution from double exchange reads:

\[ V_{\text{med}}^{(3)} = \frac{g_A^4}{(8\pi f^2_\pi)} \left\{ \bar{\sigma}_1 \cdot \bar{\sigma}_2 \left( 2l_{1,2} - 2l_{3,2} - H_{1,2} - \bar{l}_{1,2} + H_{8,2} + \bar{l}_{8,2} \right) + \bar{\sigma}_1 \cdot \bar{q} \bar{\sigma}_2 \cdot \bar{q} \left( \frac{H_{1,1} + \bar{l}_{1,4} + \bar{l}_{8,5}}{2} \right. \right. \]
\[- \left. I_{2,4} - I_{3,5} + H_{8,0} - H_{8,1} \right) + \bar{\sigma}_1 \cdot \bar{p} \bar{\sigma}_2 \cdot \bar{p} + \bar{\sigma}_1 \cdot \bar{p} \bar{\sigma}_2 \cdot \bar{p} \right) \left( I_{2,3} - I_{3,3} - \frac{H_{1,3} + \bar{l}_{1,3}}{2} \right) \]
\[ + \frac{H_{8,3} + \bar{l}_{8,3}}{2} \left] + 2m^2 I_{5,0} - 2H_{5,0} - 3H_{4,2} - 3\bar{l}_{4,2} + \frac{q^2}{2} (H_{4,1} + \bar{l}_{4,4}) - p^2 (H_{4,3} + \bar{l}_{4,3}) \right) \left. \right. \]
\[- \frac{i}{2} \left( \bar{\sigma}_1 + \bar{\sigma}_2 \right) \cdot (\bar{q} \times \bar{p}) (H_{4,1} + \bar{l}_{4,1}) \right\} . \] (18)

The definitions of the double-indexed functions \( H_{j,\nu}(p, k_n), I_{j,\nu}(p, q, k_n) \) and \( \bar{l}_{j,\nu}(p, q, k_n) \) can be found in eqs.(25-34) of ref.\[8\].

One should point to a typing error in the third line of eq.(24) in ref.\[8\]. There the spin-orbit operator should read \( (\bar{\sigma}_1 + \bar{\sigma}_2) \cdot (\bar{q} \times \bar{p}) \) and not \( i(\bar{\sigma}_1 + \bar{\sigma}_2) \cdot (\bar{p} \times \bar{q}) \). The same typing error has occurred in the second line of eq.(19) in ref.\[9\]. Further typing errors that need to be corrected in ref.\[7\] are: \( (q^2/4 - 2p^2) \) at the beginning of the last line of eq.(29), \( [m_\pi^2(\gamma_0 + \gamma_1) - \Gamma_0 - \Gamma_1] \) at the end of the fourth line of eq.(30), and \( (\cdots + G_{0*} + 2G_{1*}) \) in the fifth line of eq.(45).

### 4 Ring topology

The three-neutron ring-interaction, represented by the right diagram in Fig.1, possesses a more complicated structure, because any factorization property in the three momentum transfers \( \vec{q}_{1,2,3} \) is lost. One starts with the basic expression for \( V_{3n} \) in the form of a three-dimensional loop-integral over (static) pion-propagators and momentum-factors \[1\]:

\[ V_{3n} = \frac{g_A^4}{16f^2_\pi} \int \frac{d^3l_2}{(2\pi)^3 (m_\pi^2 + l_2^2)(m_\pi^2 + l_3^2)(m_\pi^2 + l_4^2)} \left\{ \bar{l}_1 \cdot \bar{l}_2 \bar{l}_3 \cdot \bar{l}_4 - \bar{\sigma}_1 \cdot (\bar{l}_2 \times \bar{l}_3) \bar{\sigma}_2 \cdot (\bar{l}_1 \times \bar{l}_2) \right\} \] \[- \frac{g_A^2}{m_\pi^2 + l_2^2} \left[ \bar{l}_1 \cdot \bar{l}_2 \bar{l}_3 \cdot \bar{l}_4 \left( 2\bar{\sigma}_2 \cdot (\bar{l}_1 \times \bar{l}_3) \bar{\sigma}_3 \cdot (\bar{l}_1 \times \bar{l}_2) \right) \left( \bar{l}_2 \cdot \bar{l}_3 \right) - \frac{3}{2} \bar{\sigma}_1 \cdot (\bar{l}_2 \times \bar{l}_3) \bar{\sigma}_3 \cdot (\bar{l}_1 \times \bar{l}_2) \bar{l}_1 \cdot \bar{l}_3 \right\}. \] (19)

where one has to set \( \bar{l}_1 = \bar{l}_2 - \bar{q}_3 \) and \( \bar{l}_3 = \bar{l}_2 + \bar{q}_1 \).

#### 4.1 Self-closings of neutron-lines

For the self-closings of neutron-lines the Fermi-sphere integral gives just a factor density \( \rho_n = k_n^3/3\pi^2 \). Sorted according to powers of \( g_A^2 \), the self-closing contributions to \( V_{\text{med}} \) from the 3n-ring interaction \( V_{3n} \) in eq.(19) read:

\[ V_{\text{med}}^{(0)} = \frac{g_A^4 k_n^3}{192\pi^3 f^6_\pi} \left\{ - 7m_\pi - \frac{m_\pi^3}{4m_\pi^2 + q^2} - \frac{8m_\pi^2 + 3q^2}{q} \arctan \frac{q}{2m_\pi} \right\}, \] (20)
Using the assignments of momenta \( \vec{l} \), the subtraction of the asymptotic constant

\[ V_{\text{med}}^{(0)} = \frac{g_A^4 k_n^3}{192 \pi^3 f_\pi^6} \left\{ \frac{19 m_\pi^2}{2} - \frac{5 m_\pi^2}{2(4 m_\pi^2 + q^2)} + \frac{4 m_\pi^5}{(4 m_\pi^2 + q^2)^2} + \frac{8 m_\pi^2 + 3 q^2}{q} \arctan \frac{q}{2 m_\pi} \right\}
\]

\[ + \frac{1}{2 q^2} \left( \vec{\sigma}_1 \cdot \vec{q}_2 q^2 - \vec{\sigma}_1 \cdot \vec{q} \vec{\sigma}_2 \cdot \vec{q} \right) \left[ m_\pi - \frac{5 m_\pi^2}{4 m_\pi^2 + q^2} + \frac{m_\pi^2 - 7 q^2}{2 q} \arctan \frac{q}{2 m_\pi} \right] \} ,
\]

where the linear divergences in the central parts have been treated by dimensional regularization.

### 4.2 Concatenations of neutron-lines for ring interaction \( \sim g_A^4 \)

Treating the concatenations of two neutron-lines is somewhat simpler for the \( g_A^4 \)-part of the 3n-ring interaction \( V_{3n} \) in eq.(19), since for this component the three pion-propagators are on an equal footing. Using the assignments of momenta \( \vec{l}_1, \vec{l}_2, \vec{l}_3 \) for the six possible concatenations in table 1 of ref.[8], one obtains the following contributions to \( V_{\text{med}} \), which are listed individually by specifying first their type.

- **Central potential** (three pieces added):

\[
V_{\text{med}}^{(cc)} \equiv \frac{g_A^4}{64 \pi^4 f_\pi^6} \int_0^\infty dl \left\{ \mu \tilde{l}_1(l) \left[l \left( \frac{q^2}{2 p^2} - 2 \right) + \left( \frac{q^2}{2} + 4 m_\pi^2 - 2 l^2 + 2 p^2 - \frac{q^2}{2 p^2} (m_\pi^2 + l^2) \right) \Lambda(l) \right] + (l^2 - m_\pi^2 - p^2) (2 m_\pi^2 + q^2) \Omega(l) \right\} + l \left[ \frac{2 k_3^3}{3} - m_\pi^2 \Gamma_0(l) \left[ (2 m_\pi^2 + q^2) \Omega(l) - 2 \Lambda(l) \right] + 4 k_3^3 \right\} ,
\]

with the following auxiliary functions arising from the solid-angle integration:

\[
\Lambda(l) = \frac{1}{4 p} \ln \frac{m_\pi^2 + (l + p)^2}{m_\pi^2 + (l - p)^2} ,
\]

\[
\Omega(l) = \frac{1}{q \sqrt{B + q^2 l^2}} \ln q l + \sqrt{B + q^2 l^2} ,
\]

and the abbreviation \( B = [m_\pi^2 + (l + p)^2][m_\pi^2 + (l - p)^2] \). In accordance with dimensional regularization the subtraction of the asymptotic constant \(-4 k_3^3 \) ensures the convergence of the radial integral \( \int_0^\infty dl \) in eq.(22).

- **Spin-spin and tensor potentials** (two pieces added):

\[
V_{\text{med}}^{(cc)} = \frac{g_A^4}{32 \pi^4 f_\pi^6} \tilde{\vec{\sigma}}_1 \cdot \vec{q}_2 q^2 - \vec{\sigma}_1 \cdot \vec{q} \vec{\sigma}_2 \cdot \vec{q} \int_0^\infty dl \frac{\tilde{l}_1(l)}{4 p^2 - q^2} \left[ (B + q^2 l^2) \Omega(l) - (m_\pi^2 + l^2 + p^2) \Lambda(l) \right] ,
\]

- **Quadratic spin-orbit potential** (two pieces added):

\[
V_{\text{med}}^{(cc)} = \frac{g_A^4}{16 \pi^4 f_\pi^6} \vec{\sigma}_1 \cdot (\vec{q} \times \vec{p}) \vec{\sigma}_2 \cdot (\vec{q} \times \vec{p}) \int_0^\infty dl \frac{l \tilde{l}_1(l)}{4 p^2 - q^2} \left\{ - \frac{l}{2 p^2} + \frac{m_\pi^2 + l^2 - p^2}{2 p^2} + \frac{4 (m_\pi^2 + l^2 + p^2)}{4 p^2 - q^2} \right\} \Omega(l) + \left[ m_\pi^2 + 3 l^2 + p^2 - \frac{4 (m_\pi^2 + l^2 + p^2)}{4 p^2 - q^2} \right] \Omega(l) ,
\]

- **Spin-orbit term** (one piece):

\[
V_{\text{med}}^{(cc)} = \frac{g_A^4}{64 \pi^4 f_\pi^6} i (\vec{\sigma}_1 + \vec{\sigma}_2) \cdot (\vec{q} \times \vec{p}) \int_0^\infty dl \frac{l}{4 p^2 - q^2} \left\{ \Gamma_2(l) \left[ (2 m_\pi^2 + 2 l^2 - 2 p^2 + q^2) \Omega(l) - 2 \Lambda(l) \right] + \tilde{\Gamma}_3(l) \left[ (B + q^2 l^2) \Omega(l) - (m_\pi^2 + l^2 + p^2) \Lambda(l) \right] \right\} .
\]
4.3 Concatenations of neutron-lines for ring interaction $\sim g_A^6$

When treating the concatenations of two neutron-lines for the $g_A^6$-part of the 3n-ring interaction $V_{3n}$, one has to distinguish the cases with $\vec{l}_2 = \pm(\vec{l}_4 + \vec{l})$, because this momentum enters a squared pion-propagator in eq.(19). The Fermi-sphere integral over the squared pion-propagator (yielding the functions $\gamma_\nu(l, k_n)$) combined with the solution of the angular integral leads to the following contributions to $V_{\text{med}}$ from the concatenations $n_1$ on $n_3$ and $n_3$ on $n_1$.

- **Central potential:**

\[
V^{(2)}_{\text{med}} = \frac{g_A^6}{64\pi^4 f_0^6} \int_0^\infty dl \left\{ 2l \gamma_2(l) \left[ l \left( 4 - \frac{q^2}{p^2} \right) + \left( m_\pi^2 + l^2 \right) \frac{q^2}{p^2} - 8m_\pi^2 - 3q^2 \right] \Lambda(l) + \left( 2m_\pi^2 + q^2 \right) \Omega(l) \right\} \\
+ \frac{l}{8} \gamma_3(l) \left[ 2l(2m_\pi^2 + 9l^2 + p^2) - \frac{lq^2}{p} (m_\pi^2 + 5l^2) + \left( 2m_\pi^2 + q^2 \right) [B + 2l^2 (4m_\pi^2 + 3q^2)] \Omega(l) \right] \\
+ \left[ \frac{q^2}{p^2} (m_\pi^2 + l^2) (m_\pi^2 + 5l^2) - q^2 (16l^2 + p^2) - 2B - 4m_\pi^2 (m_\pi^2 + 9l^2 + p^2) \right] \Lambda(l) \right\} - \frac{4k_n^3}{3},
\]  

(28)

- **Spin-orbit potential:**

\[
V^{(2)}_{\text{med}} = \frac{3g_A^6}{(4\pi)^1 f_0^6} i(\vec{\sigma}_1 + \vec{\sigma}_2) \cdot (\vec{q} \times \vec{p}) \int_0^\infty dl \left\{ \gamma_2(l) \left[ 4(p^2 - l^2 - 2m_\pi^2) - 3q^2 + \frac{q^2}{p^2} (m_\pi^2 + l^2) \right] \Lambda(l) \\
+ l \left( 4 - \frac{q^2}{p^2} \right) + (2m_\pi^2 + q^2) (2m_\pi^2 + 2l^2 - 2p^2 + q^2) \Omega(l) \right\} \\
+ \frac{\gamma_3(l)}{8} \left[ l \left( 1 - \frac{q^2}{4p^2} \right) (m_\pi^2 + l^2 + p^2) + \left[ (l^2 + p^2 - m_\pi^2)^2 + \frac{q^2}{4} (2m_\pi^2 - 2l^2 + p^2) \right. \\
- 4m_\pi^2 - 4m_\pi^2 p^2 - 4p^4 + \frac{q^2}{4p^2} (m_\pi^2 + l^2)^2 \right] \Lambda(l) + \left\{ 2(m_\pi^2 - l^2 + p^2) (B + q^2l^2) \right\} \Omega(l) \right\},
\]  

(29)

- **Spin-spin and tensor potentials:**

\[
V^{(2)}_{\text{med}} = \frac{g_A^6}{16\pi^4 f_0^6} (\vec{\sigma}_1 \cdot \vec{\sigma}_2 q^2 - \vec{\sigma}_1 \cdot \vec{q} \vec{\sigma}_2 \cdot \vec{q}) \int_0^\infty dl \left\{ \gamma_2(l) \left[ (m_\pi^2 + l^2 + p^2) \Lambda(l) - (B + q^2l^2) \Omega(l) \right] \\
+ \gamma_3(l) \left[ l \left( 1 - \frac{q^2}{4p^2} \right) (m_\pi^2 + l^2 + p^2) + \left[ (l^2 + p^2 - m_\pi^2)^2 + \frac{q^2}{4} (2m_\pi^2 - 2l^2 + p^2) \right. \\
- 4m_\pi^2 - 4m_\pi^2 p^2 - 4p^4 + \frac{q^2}{4p^2} (m_\pi^2 + l^2)^2 \right] \Lambda(l) + \left\{ 2(m_\pi^2 - l^2 + p^2) (B + q^2l^2) \right\} \Omega(l) \right\},
\]  

(30)

- **Quadratic spin-orbit potential:**

\[
V^{(2)}_{\text{med}} = \frac{g_A^6}{16\pi^4 f_0^6} \vec{\sigma}_1 \cdot (\vec{q} \times \vec{p}) \vec{\sigma}_2 \cdot (\vec{q} \times \vec{p}) \int_0^\infty dl \left\{ \gamma_2(l) \left[ \frac{l}{p^2} + \left[ 3 - \frac{m_\pi^2 + l^2}{p^2} \right] \Lambda(l) + \left[ \frac{8(m_\pi^2 + l^2 + p^2)^2}{4p^2 - q^2} - 4m_\pi^2 - 8l^2 - q^2 \right] \Omega(l) \right] \\
+ \gamma_3(l) [m_\pi^2 + p^2 - l^2] \left[ \frac{-l}{4p^2} + \left[ \frac{m_\pi^2 + l^2 - p^2}{4p^2} + \frac{2(m_\pi^2 + l^2 + p^2)}{4p^2 - q^2} \right] \Lambda(l) \right. \\
+ \left[ \frac{l}{2} (m_\pi^2 + 3l^2 + p^2) - \frac{2(m_\pi^2 + l^2 + p^2)^2}{4p^2 - q^2} \right] \Omega(l) \right\}. 
\]  

(31)
For the other four concatenations one has $\tilde{l}_{1,3} = \pm (\tilde{l}_4 + \tilde{l}_5)$ and the Fermi-sphere integral goes over an ordinary pion-propagator (yielding the functions $\gamma_{\nu}(l, k_n)$). When adding pieces of the same type, one obtains the following additional contributions to $V_{\text{med}}$ from concatenations.

- **Central potential (two pieces added):**

$$V_{\text{med}}^{(cc)} = \frac{g_A^6}{128 \pi^4 f_\pi^6} \int_0^\infty dl \left\{ \frac{2k_n^3}{3} - m^2_\pi \Gamma_0(l) \right\} \left[ (8 - \frac{q^2}{p^2}) \Lambda(l) + \frac{l}{B} \left( \frac{q^2}{p^2} (m^2_\pi + l^2) - 8m^2_\pi - 3q^2 + \frac{(2m^2_\pi + q^2)^2}{B + q^2 l^2} (m^2_\pi + l^2 + p^2) \right) + (2m^2_\pi + q^2) \left( \frac{2m^2_\pi + q^2}{B + q^2 l^2} (m^2_\pi + l^2 + p^2) - 4 \right) \Omega(l) \right\} - \frac{16k_n^3}{3} \right\}, \quad (32)$$

- **Spin-orbit potential (one piece):**

$$V_{\text{med}}^{(cc)} = \frac{g_A^6}{128 \pi^4 f_\pi^6} \int_0^\infty dl \left\{ \frac{2k_n^3}{3} - m^2_\pi \Gamma_0(l) \right\} \left[ \frac{1}{p^2} \left( 1 - \frac{m^2_\pi + l^2}{p^2} \right) + \frac{2m^2_\pi + q^2}{B + q^2 l^2} (l^2 - m^2_\pi - p^2) \right) + \left[ - \frac{4(m^2_\pi + l^2 + p^2)}{4p^2 - q^2} + \frac{2m^2_\pi + q^2}{B + q^2 l^2} (l^2 - m^2_\pi - p^2) \right] \Omega(l) \right\} - \frac{l}{B} \left( 1 - \frac{m^2_\pi + l^2}{p^2} \right) + \frac{2m^2_\pi + q^2}{B + q^2 l^2} (l^2 - m^2_\pi - p^2) \right) \Lambda(l) \right\} + \frac{\tilde{\Gamma}_3(l)}{4} \left\{ 3l \left( \frac{q^2}{2p^2} - 2 \right) + \left[ 5(3m^2_\pi + l^2 + 3p^2) - \frac{3q^2}{2p^2} (m^2_\pi + l^2 + p^2) \right] \Lambda(l) + \left[ 4l^2(2p^2 - 2m^2_\pi - q^2) + l^4 - 9(m^2_\pi + p^2)^2 \right] \Omega(l) \right\} \right\}, \quad (33)$$

- **Spin-spin and tensor potentials (two pieces added):**

$$V_{\text{med}}^{(cc)} = \frac{g_A^6}{64 \pi^4 f_\pi^6} \int_0^\infty dl \frac{l}{4p^2 - q^2} \left\{ 5\tilde{\Gamma}_2(l) \left[ (m^2_\pi + l^2 + p^2) \Omega(l) - \Lambda(l) \right] + \frac{\tilde{\Gamma}_3(l)}{4} \left\{ 3l \left( \frac{q^2}{2p^2} - 2 \right) + \left[ 5(3m^2_\pi + l^2 + 3p^2) - \frac{3q^2}{2p^2} (m^2_\pi + l^2 + p^2) \right] \Lambda(l) + \left[ 4l^2(2p^2 - 2m^2_\pi - q^2) + l^4 - 9(m^2_\pi + p^2)^2 \right] \Omega(l) \right\} \right\}, \quad (34)$$

- **Quadratic spin-orbit potential (two pieces added):**

$$V_{\text{med}}^{(cc)} = \frac{g_A^6}{64 \pi^4 f_\pi^6} \int_0^\infty dl \frac{l}{4p^2 - q^2} \left\{ 5\tilde{\Gamma}_2(l) \left[ \left( \frac{1}{p^2} + \frac{8}{4p^2 - q^2} \right) \Lambda(l) - \frac{l}{B} \left[ \frac{m^2_\pi + l^2}{p^2} \right] \right\} + \frac{4m^2_\pi + q^2}{B + q^2 l^2} (m^2_\pi + l^2 + p^2 - 3) + \left[ 4 - (m^2_\pi + l^2 + p^2) \right] \left( \frac{8}{4p^2 - q^2} + \frac{4m^2_\pi + q^2}{B + q^2 l^2} \right) \Omega(l) \right\} + \frac{\tilde{\Gamma}_3(l)}{4} \left\{ 2 \left( \frac{l^2 - 9m^2_\pi - 9p^2}{4p^2 - q^2} - \frac{m^2_\pi}{p^2} \right) \Lambda(l) + \frac{l}{B} \left[ \frac{m^2_\pi - (l^2 - p^2)^2}{q^2} + \frac{2m^2_\pi}{p^2} (m^2_\pi + l^2) \right] \right\} + \frac{11m^2_\pi}{2} + \frac{5}{2} \left( l^2 + p^2 - q^2 \right) + \frac{5(4m^2_\pi + q^2)}{2(B + q^2 l^2)} ((m^2_\pi + p^2)^2 + l^2(m^2_\pi - 3p^2 + q^2)) \right\} + \left[ \frac{(l^2 - p^2)^2 - m^4_\pi}{q^2} + \frac{2(m^2_\pi + l^2 + p^2)^2}{4p^2 - q^2} (9m^2_\pi + 9p^2 - l^2) - \frac{l^2}{2} \right] + \frac{5}{2} \left( (m^2_\pi + p^2)^2 + l^2(m^2_\pi - 3p^2 + q^2) \right) \Omega(l) \right\} \right\}. \quad (35)$$
5 \(2\pi^0\)-exchange three-neutron force at N\(^4\)LO

In this section the longest-range \(2\pi^0\)-exchange 3n-interaction is treated, following the work of ref. \[3\]. Modulo terms of shorter range it can be written according to eq.(3.1) in ref. \[3\] in the general form:

\[
2V_{3n} = \frac{g_A^2}{4f_\pi^2} \frac{\vec{\sigma}_1 \cdot \vec{q}_1 \vec{\sigma}_2 \cdot \vec{q}_2}{(m_\pi^2 + q_1^2)(m_\pi^2 + q_2^2)} \tilde{g}_+(q_2),
\]

(36)

with both sides multiplied by a factor 2 due to the \(1 \leftrightarrow 3\) symmetry. The structure function \(\tilde{g}_+(q_2)\) is \(f_\pi^2\) times the isoscalar non-spin-flip \(\pi N\)-scattering amplitude at zero pion-energy \(\omega = 0\) and squared momentum-transfer \(t = -q_2^2\). The corresponding expression up to N\(^4\)LO is given in eq.(59) of ref. \[8\].

5.1 Contributions to in-medium nn-potential

The self-closing of neutron-line 2 for \(V_{3n}\) in eq.(36) gives (after relabeling 3 \(\rightarrow\) 2) the contribution:

\[
V^{(0)}_{\text{med}} = -\frac{g_A^2 k_n^3}{16\pi^2 f_\pi^4} \frac{\vec{\sigma}_1 \cdot \vec{q}_1 \vec{\sigma}_2 \cdot \vec{q}_2}{(m_\pi^2 + q_2^2)^2} \tilde{g}_+(0).
\]

(37)

From pionic vertex corrections on either neutron-line one obtains the (total) contribution:

\[
V^{(1)}_{\text{med}} = \frac{g_A^2}{16\pi^2 f_\pi^4} \frac{\vec{\sigma}_1 \cdot \vec{q}_1 \vec{\sigma}_2 \cdot \vec{q}_2}{(m_\pi^2 + q_2^2)^2} \int_{p-k_n}^{p+k_n} ds \tilde{g}_+(s) \left[ \frac{k_n^2 - (p - s)^2}{p} \right]
+ \left( q - \frac{m_\pi^2 + s^2}{q} \right) \ln \left( \frac{qX + 2\sqrt{W}}{2(p + q)(m_\pi^2 + (q - s)^2)} \right),
\]

(38)

with the auxiliary polynomials \(X\) and \(W\) defined in eq.(6). Finally, the two diagrams related to double exchange lead to the expression:

\[
V^{(3)}_{\text{med}} = \frac{g_A^2 \tilde{g}_+(q)}{16\pi^2 f_\pi^4} \left[ 2\Gamma_0 - (2m_\pi^2 + q^2)G_0 + i(\vec{\sigma}_1 + \vec{\sigma}_2) \cdot (\vec{q} \times \vec{p}) (G_0 + 2G_1) \right],
\]

(39)

where one reminds that the functions \(\Gamma_0\) depends on \((p, k_n)\), and the functions \(G_{0,1}\) depend on \((p, q, k_n)\).

6 Subleading ring diagrams

6.1 3n-ring interaction \(~ g_A^0 \)

According to eq.(31) in ref. \[3\] the subleading 3n-ring interaction proportional to \(g_A^0\) reads:

\[
V_{3n} = \frac{1}{f_\pi^6} \int_0^\infty dl_1 \int_0^\infty dl_2 \int_1^{l_1} d\bar{l}_1 (2\pi)^4 (\bar{m}^2 + l_1^2)(\bar{m}^2 + l_2^2)(\bar{m}^2 + \bar{l}_1^2) \left[ 2c_1 m_\pi^2 + (c_2 + c_3)l_0^2 + c_3 \bar{l}_2 \cdot \bar{l}_3 \right],
\]

(40)

with \(\bar{m} = \sqrt{m_\pi^2 + l_0^2}\) and one has to set \(l_1 = \bar{l}_2 - \bar{q}_3\) and \(l_3 = \bar{l}_2 + \bar{q}_1\). The sum of the three self-closings of a neutron-line rise up to the central potential:

\[
V^{(0)}_{\text{med}} = \frac{k_n^3}{32\pi^4 f_\pi^6} \left\{ \left( (2c_1 - \frac{3c_2}{2} - 3c_3) m_\pi^2 - \left( \frac{c_2}{4} + \frac{5c_3}{9} \right) q^2 \right) \ln \frac{m_\pi}{\lambda} + \left( \frac{5c_2}{8} + \frac{37c_3}{27} \right) \frac{q^2}{4}
+ \left( \frac{3c_2}{2} - 2c_1 + \frac{23c_3}{9} \right) \frac{m_\pi^2}{4} + \left( (2c_1 - c_2 - \frac{17c_3}{9}) m_\pi^2 - \left( \frac{c_2}{4} + \frac{5c_3}{9} \right) q^2 \right) L(q) \right\},
\]

(41)
with logarithmic loop-function

\[ L(q) = \frac{\sqrt{4m_\pi^2 + q^2}}{q} \ln \frac{q + \sqrt{4m_\pi^2 + q^2}}{2m_\pi}. \]  

(42)

Note that the four-dimensional euclidean loop-integral in eq.(40) has been regularized with a cutoff \( \lambda \) and the \( \lambda^2 \)-divergence has been dropped in the expression in eq.(41). The total contribution from the six possible concatenations of two neutron-lines is a central potential with the following double-integral represenation:

\[ V_{\text{med}}^{(cc)} = \frac{1}{4\pi^5 f_\pi^6} \int_0^\lambda \int_0^{\pi/2} drr \int_0^\infty dv \psi \left\{ l_0^2 \left[ \bar{\Gamma}_0(l) + \left( 6c_1 m_\pi^2 + 3(c_2 + c_3)l_0^2 - \frac{c_3}{2}(2\bar{m}^2 + q^2) \right) \Omega(l) \right] + c_3 \bar{\Gamma}_1(l) \left[ \bar{\Lambda}(l) + (l^2 - \bar{m}^2 - p^2)\bar{\Omega}(l) \right] \right\} + \frac{k_\pi^3}{2} \left( c_3 + c_2 \cos^2 \psi \right) \sin^2 2\psi, \]  

(43)

where one has to set \( l_0 = r \cos \psi \) and \( l = r \sin \psi \). Note that all barred functions are to be evaluated with \( \bar{m} = \sqrt{m_\pi^2 + l_0^2} \) instead of \( m_\pi \). The behavior of the subtracted double-integral for large \( \lambda \) is:

\[ \frac{\pi k_\pi^3}{16} \left[ c_4 \left( 3m_\pi^2 + \frac{3k_\pi^2}{10} + \frac{p^2}{2} + \frac{q^2}{4} \right) - 4c_1 m_\pi^2 + c_3 \left( 6m_\pi^2 + \frac{2k_\pi^2}{3} + \frac{5}{9}(2p^2 + q^2) \right) \right] \ln \frac{m_\pi}{\lambda}. \]  

(44)

A good check is given by the fact that the \( \lambda^2 \)-divergences behind \( V_{\text{med}}^{(0)} \) and \( V_{\text{med}}^{(cc)} \) cancel each other. This feature must hold, because there is no 3n-contact coupling that could absorb this divergence.

### 6.2 3n-ring interaction \( \sim g_\Lambda^2 \)

According to eq.(44) in ref.\[9\] the subleading 3n-ring interaction proportional to \( g_\Lambda^2 \) reads:

\[ V_{3n} = - \frac{g_\Lambda^2}{f_\pi^6} \int_0^\infty dl_0 \int_0^{\infty} \frac{d^3l_2}{(2\pi)^4} \frac{1}{(m_\pi^2 + l_0^2)(m_\pi^2 + l_2^2)(m_\pi^2 + l_3^2)} \left\{ 2c_1 m_\pi^2 + (c_2 + c_3)l_0^2 + c_3 \bar{l}_2 \cdot \bar{l}_3 \right\} l_1 \cdot (\bar{l}_2 + \bar{l}_3) + c_4 \left[ \bar{m}^2 (\bar{\sigma}_2 \times \bar{l}_3) \cdot (\bar{\sigma}_3 \times \bar{l}_2) + \bar{l}_2 \cdot \bar{l}_3 \bar{\sigma}_2 \cdot \bar{l}_1 \bar{\sigma}_3 \cdot \bar{l}_1 + \bar{l}_3 \cdot \bar{l}_2 \bar{\sigma}_3 \cdot \bar{l}_1 \bar{\sigma}_2 \cdot \bar{l}_2 \bar{\sigma}_3 \cdot \bar{l}_1 \right]. \]  

(45)

The sum of the three self-closings of a neutron-line gives rise to the following contributions to \( V_{\text{med}} \).

- **Central potential:**

\[ V_{\text{med}}^{(0)} = \frac{g_\Lambda^2 k_\pi^3}{96 \pi^4 f_\pi^6} \left\{ 9 m_\pi^2 (4c_1 - c_2 - 6c_3) - \left( \frac{11c_2}{6} + \frac{32c_3}{3} \right) q^2 \right\} \ln \frac{m_\pi}{\lambda} + \left( \frac{19c_2}{12} - 5c_1 + \frac{25c_3}{6} \right) m_\pi^2 + \left( \frac{137c_2}{16} + 58c_3 \right) \frac{q^2}{9} + \left[ 2m_\pi^2 \left( 18c_1 - \frac{8c_2}{3} - \frac{43c_3}{3} \right) - \left( \frac{11c_2}{6} + \frac{32c_3}{3} \right) q^2 + \frac{8(c_3 - 2c_1)m_\pi^4}{4m_\pi^2 + q^2} \right] L(q), \]  

(46)

- **Spin-spin potential:**

\[ V_{\text{med}}^{(0)} = \frac{g_\Lambda^2 c_4 k_\pi^3}{96 \pi^4 f_\pi^6} \bar{\sigma}_1 \cdot \bar{\sigma}_2 \left\{ - (4m_\pi^2 + q^2) \ln \frac{m_\pi}{\lambda} + \frac{q^2}{4} - m_\pi^2 - q^2 L(q) \right\}, \]  

(47)

- **Tensor potential:**

\[ V_{\text{med}}^{(0)} = \frac{g_\Lambda^2 c_4 k_\pi^3}{96 \pi^4 f_\pi^6} \bar{q} \bar{\sigma}_1 \cdot \bar{\sigma}_2 \cdot \bar{q} \left\{ \ln \frac{m_\pi}{\lambda - \frac{1}{4}} + L(q) \right\}. \]  

(48)
Next, one evaluates the concatenations $n_3$ on $n_2$ and $n_2$ on $n_3$ and obtains the following contributions to $V_{\text{med}}$.

**Central potential:**
\[
V_{\text{med}}^{(1)} = \frac{g^2_A}{4\pi^5 f_\pi^2} \int_0^\Lambda \int_0^{\pi/2} d\psi \left\{ l \tilde{\Gamma}_1(l) \left\{ c_3 l \left( 1 - \frac{q^2}{4p^2} \right) + \left[ 2c_1 m^2_\pi + c_2 l_0^2 + c_3 \left( r^2 - 2\bar{m}^2 - p^2 \right) + \frac{g^2}{4p^2} \left( l^2 + \bar{m}^2 - p^2 \right) \right] \Lambda(l) + \left[ 2c_1 m^2_\pi + c_2 l_0^2 - c_3 \left( m^2_\pi + \frac{q^2}{2} \right) \right] \left( l^2 - \bar{m}^2 - p^2 \right) \Omega(l) \right\} + c_4 l \left\{ \left[ \bar{m}^2 \tilde{\Gamma}_0(l) + 2\tilde{\Gamma}_2(l) \right] \left[ 2\Lambda(l) - (2\bar{m}^2 + q^2)\Omega(l) \right] + \frac{1}{4} \tilde{\Gamma}_3(l) \left[ 2(3l^2 - \bar{m}^2 - p^2)\Lambda(l) + l + (\bar{B} - 2l^2(4\bar{m}^2 + q^2))\Omega(l) \right] \right\} - \frac{4k^3}{3} (c_3 + c_4 + c_2 \cos^2 \psi) \sin^4 \psi \right\},
\]
\[ \tag{49} \]

where the (subtracted) double-integral has the large-$\lambda$ behavior:
\[
\pi k^3 \left[ \frac{c_2}{4} \left( 6m^2_\pi + k^2_\eta + \frac{5p^2}{3} + \frac{q^2}{6} \right) - 6c_1 m^2_\pi + c_3 \left( 9m^2_\pi + k^2_\eta + \frac{5p^2}{3} + q^2 \right) + c_4 \left( 6m^2_\pi + \frac{3k^2_\eta}{5} + p^2 + \frac{7q^2}{6} \right) \right] \ln \frac{m_\pi}{\lambda}.
\]
\[ \tag{50} \]

**Spin-orbit potential:**
\[
V_{\text{med}}^{(1)} = \frac{c_4 g^2_A}{8\pi^5 f_\pi^2} i(\hat{\sigma}_1 + \hat{\sigma}_2) \cdot (\hat{q} \times \hat{p}) \int_0^\Lambda \int_0^{\pi/2} d\psi \left\{ l \int_0^{\pi/2} d\psi \frac{l}{4p^2 - q^2} \left\{ \left[ \bar{m}^2 \tilde{\Gamma}_0(l) + 2\tilde{\Gamma}_2(l) \right] \left[ 2\Lambda(l) \right] + (2p^2 - 2l^2 - 2\bar{m}^2 - q^2)\Omega(l) \right\} + \tilde{\Gamma}_3(l)(\bar{m}^2 + p^2 - l^2) \left\{ (\bar{m}^2 + l^2 + p^2)\Omega(l) - \Lambda(l) \right\} \right\},
\]
\[ \tag{51} \]

with a large-$\lambda$ behavior of the double-integral: $(\pi k^3_{\text{n}}/24) \ln(m_\pi/\lambda)$.

The other four concatenations ($n_3$ on $n_1$, $n_1$ on $n_3$, $n_1$ on $n_2$, $n_2$ on $n_1$) lead to the following contributions to $V_{\text{med}}$.

**Central potential:**
\[
V_{\text{med}}^{(cc)} = \frac{g^2_A}{4\pi^5 f_\pi^2} \int_0^\Lambda \int_0^{\pi/2} d\psi \left\{ l \left[ 2c_1 m^2_\pi + (c_2 + c_3)l_0^2 \right] \left\{ \tilde{\Gamma}_0(l) \left[ 2\Lambda(l) - (2\bar{m}^2 + q^2)\Omega(l) \right] + \tilde{\Gamma}_1(l) \left[ \Lambda(l) + (l^2 - p^2 - \bar{m}^2)\Omega(l) \right] \right\} + c_3 l \left\{ \tilde{\Gamma}_2(l) \left[ 2\Lambda(l) - (2\bar{m}^2 + q^2)\Omega(l) \right] + \tilde{\Gamma}_3(l) \left[ \frac{l}{2} + (l^2 - p^2 - \bar{m}^2)\Lambda(l) + \frac{l}{2} (l^2 - p^2 - \bar{m}^2)^2\Omega(l) \right] + \tilde{\Gamma}_1(l) \left[ l \left( 1 - \frac{q^2}{4p^2} \right) \right] + l \left( l^2 - p^2 - 2m^2 - \frac{q^2}{4} + \frac{q^2}{4p^2} (l^2 + \bar{m}^2) \right) \Lambda(l) + \left( \bar{m}^2 + \frac{q^2}{2} \right) (m^2 + p^2 - l^2)\Omega(l) \right\} \right\}
\]
\[ - \frac{8k^3}{3} (c_3 + c_2 \cos^2 \psi) \sin^4 \psi \right\},
\]
\[ \tag{52} \]

with a large-$\lambda$ behavior of the (subtracted) double-integral:
\[
\frac{\pi k^3_{\text{n}}}{4} \left[ c_2 \left( m^2_\pi + \frac{k^2_\eta}{10} + \frac{p^2}{6} + \frac{5q^2}{36} \right) - 4c_1 m^2_\pi + c_3 \left( 6m^2_\pi + \frac{11k^2_\eta}{15} + \frac{11p^2 + 5q^2}{9} \right) \right] \ln \frac{m_\pi}{\lambda}.
\]
\[ \tag{53} \]
\[ V^{(cc)}_{\text{med}} = \frac{c_4 g^2_A}{16 \pi^2 f^6_\pi} \tilde{\sigma}_1 \cdot \tilde{\sigma}_2 \int_0^\Lambda dr \int_0^{\pi/2} d\psi \left\{ \frac{l \tilde{1}_1(l)}{p^2} (3p^2 - l^2 - \bar{m}^2 - q^2) \\
+ \left( 2l^2 - 2\bar{m}^2 - 3p^2 + (\bar{m}^2 + l^2 + q^2) \bar{m}^2 + l^2 \right) + \frac{q^2 (4\bar{m}^2 + 4l^2 + q^2)}{4p^2 - q^2} \right\} \tilde{\Lambda}(l) \\
+ \frac{2q^2 (\bar{m}^2 + l^2 + p^2)}{4p^2 - q^2} (2p^2 - 2\bar{m}^2 - q^2) \tilde{\Omega}(l) \right\}, \quad (54) \]
with a large-\( \lambda \) behavior of the (subtracted) double-integral: \((2\pi k^3/9) [\bar{6} m^2 + k^2 + 2p^2 + 3q^2/4] \ln(m_\pi/\lambda).\)

- Ordinary tensor potential:

\[ V^{(cc)}_{\text{med}} = \frac{c_4 g^2_A}{4 \pi^2 f^6_\pi} \tilde{\sigma}_1 \cdot \tilde{\sigma}_2 \cdot \tilde{q} \int_0^\Lambda dr \int_0^{\pi/2} d\psi \left\{ \frac{l \tilde{1}_1(l)}{4p^2 - q^2} \right\} \tilde{\Lambda}(l) \\
+ \frac{q^2}{4p^2 - q^2} \left( 2\bar{m}^2 + 2l^2 + \frac{q^2}{2} \right) \tilde{\Omega}(l) \\
+ \frac{3\tilde{B} + 2l^2 (5p^2 + q^2) + (\bar{m}^2 + p^2) (q^2 - 2p^2)}{4p^2 - q^2} (\bar{m}^2 + l^2 + p^2)^2 \tilde{\Omega}(l) \right\}, \quad (55) \]

with a large-\( \lambda \) behavior of the double-integral: \(-(\pi k^3/24) \ln(m_\pi/\lambda)\).

- Tensor-type potential:

\[ V^{(cc)}_{\text{med}} = \frac{c_4 g^2_A}{4 \pi^2 f^6_\pi} (\tilde{\sigma}_1 \cdot \tilde{p} \tilde{\sigma}_2 \cdot \tilde{p} + \tilde{\sigma}_1 \cdot \tilde{p}' \tilde{\sigma}_2 \cdot \tilde{p}') \int_0^\Lambda dr \int_0^{\pi/2} d\psi \left\{ \frac{l \tilde{1}_1(l)}{4p^2 - q^2} \right\} \tilde{\Lambda}(l) \\
+ \frac{3q^2}{4p^2} (\bar{m}^2 + l^2) + \frac{p^2}{2} + l^2 - \bar{m}^2 - \frac{5q^2}{8} - \frac{6(\bar{m}^2 + l^2)^2 + q^2 (\bar{m}^2 + l^2)}{4p^2} + \frac{3q^2}{8p^4} (\bar{m}^2 + l^2)^2 \\
- \frac{q^2 (4\bar{m}^2 + 4l^2 + q^2)}{4p^2 - q^2} \tilde{\Omega}(l) + \left[ \frac{4(\bar{m}^2 + l^2 + p^2)^2}{4p^2 - q^2} - \bar{m}^2 - 3l^2 - p^2 \right] q^2 \tilde{\Omega}(l) \right\}, \quad (56) \]

with a large-\( \lambda \) behavior of the double-integral: \(-(\pi k^3/36) \ln(m_\pi/\lambda)\). The total \( \lambda^2 \)-divergence behind the calculated central and spin-spin potentials adds up to \( c_4 g^2_A k^3_\pi \lambda^2 / (96 \pi^4 f^6_\pi) [3 + \tilde{\sigma}_1 \cdot \tilde{\sigma}_2] \). As required by the renormalizability condition this is equivalent to zero, since \( \tilde{\sigma}_1 \cdot \tilde{\sigma}_2 \) has the eigenvalue \(-3\) in the \( ^1S_0 \)-state of two neutrons.

### 6.3 3n-ring interaction \( \sim g^4_A \)

In the case of the subleading 3n-ring interaction terms proportional to \( g^4_A \), as obtained by setting \( \tilde{\tau}_i \cdot \tilde{\tau}_j \rightarrow 1 \) and \( \tilde{\tau}_i \cdot (\tilde{\tau}_2 \times \tilde{\tau}_3) \rightarrow 0 \) in the very lengthy expression in eq.(58) of ref.[9], only the results for the self-closing contributions are given here.

- Central potential:

\[ V^{(0)}_{\text{med}} = \frac{g^4_A k^3_\pi}{96 \pi^4 f^6_\pi} \left\{ \frac{45m^2_\pi}{4} \left( \frac{11c_3 - 3c_2}{4} - 2c_1 \right) + \left( 177c_3 - 53c_2 \right) q^2 \right\} \ln \frac{m_\pi}{\lambda} - \frac{4m^4_\pi (2c_1 + 3c_3)}{4m^2_\pi + q^2} \\
+ \left( 705c_3 - 73c_2 - 528c_1 \right) m^2_\pi \left( \frac{7q^2}{64} \frac{53c_2}{3} - 35c_3 \right) + \left[ m^2_\pi \frac{81c_3 - 35c_2}{2} - 82c_1 \right) \\
+ \left( 177c_3 - 53c_2 \right) q^2 + \frac{2m^4_\pi}{4m^2_\pi + q^2} \left( 40c_1 + 3c_2 + 21c_3 \right) - \frac{16m^6_\pi (2c_1 + 3c_3)}{(4m^2_\pi + q^2)^2} \right\} L(q) \right\}, \quad (57) \]
• Spin-spin potential:

\[
V_{\text{med}}^{(0)} = \frac{g_4 k_3^3}{96 \pi^4 f_0^6} \bar{\sigma}_1 \cdot \bar{\sigma}_2 \left\{ 6 m_\pi^2 \left( 6 c_1 + c_2 - \frac{13 c_3}{2} + 5 c_4 \right) + \left( c_2 - \frac{9 c_3}{2} + \frac{11 c_4}{3} \right) q^2 \right\} \ln \frac{m_\pi}{\lambda} \\
+ \left( 9 c_1 + \frac{c_2}{2} - \frac{33 c_4}{4} + \frac{35 c_4}{6} \right) m_\pi^2 - \left( \frac{c_2}{6} + \frac{c_4}{9} \right) q^2 + \left[ \left( 24 c_1 + 2 c_2 - 11 c_3 + \frac{26 c_4}{3} \right) m_\pi^2 \right. \\
+ \left. \left( \frac{c_2}{2} + \frac{11 c_4}{3} \right) q^2 + \frac{4 m_\pi^4}{4 m_\pi^2 + q^2} (3 c_3 - 6 c_1 - 2 c_4) \right] L(q) \right\},
\] (58)

\[
\frac{V_{\text{med}}}{\rho_n} = 2 (E_1 + E_2) \left[ \frac{6 k_n^2}{5} + 2 p^2 - q^2 \right] + 2 (E_3 + E_4) (\bar{\sigma}_1 \cdot \bar{\sigma}_2 + 3) \left( \frac{3 k_n^2}{5} + p^2 \right) \\
+ 3 (E_5 + E_6) \left[ \bar{\sigma}_1 \cdot \bar{p} \bar{\sigma}_2 \cdot \bar{p} + \bar{\sigma}_1 \cdot \bar{p}^\prime \bar{\sigma}_2 \cdot \bar{p}^\prime - \frac{2 p^2}{3} \bar{\sigma}_1 \cdot \bar{\sigma}_2 \right] + (E_7 + E_8) i (\bar{\sigma}_1 + \bar{\sigma}_2) \cdot (\bar{q} \times \bar{p}) \\
+ (E_9 + E_{10}) \left[ q^2 - 2 p^2 - \frac{6 k_n^2}{5} - i (\bar{\sigma}_1 + \bar{\sigma}_2) \cdot (\bar{q} \times \bar{p}) \right] \\
+ (E_{11} + E_{12} + E_{13}) \left[ q^2 - 2 p^2 - \frac{6 k_n^2}{5} + i (\bar{\sigma}_1 + \bar{\sigma}_2) \cdot (\bar{q} \times \bar{p}) \right],
\] (60)

7 Subleading three-nucleon contact potential

The subleading three-nucleon contact potential (appearing at N^4LO in the chiral counting) has been reexamined recently in ref. [13]. Its corrected expression, given in eq. (15) of ref. [13], depends quadratically on momenta and it involves 13 parameters, called \( E_1, \ldots, E_{13} \). Specifying to the case of three neutrons and closing two neutron-lines in all possible ways to a loop, one obtains the following contribution to the in-medium nn-potential:

\[
\langle S_0 | V_{\text{med}} | S_0 \rangle = \left. \frac{6 \rho_n k_n^2}{5} (2 E_1 + 2 E_2 - E_9 - E_{10} - E_{11} - E_{12} - E_{13}) \frac{9 \rho_n k_n^2}{5} E_c, \right.
\]

\[
\langle 3 P_0 | V_{\text{med}} | 3 P_0 \rangle = \frac{2}{3} \rho_n p^2 (2 E_1 + 2 E_2 + 2 E_7 + 2 E_8 - 3 E_9 - 3 E_{10} + E_{11} + E_{12} + E_{13}) = \rho_n p^2 (E_c - 2 E_o),
\]

\[
\langle 3 P_1 | V_{\text{med}} | 3 P_1 \rangle = \frac{2}{3} \rho_n p^2 (2 E_1 + 2 E_2 + E_7 + E_8 - 2 E_9 - 2 E_{10}) = \rho_n p^2 (E_c - E_o),
\]

\[
\langle 3 P_2 | V_{\text{med}} | 3 P_2 \rangle = \frac{2}{3} \rho_n p^2 (2 E_1 + 2 E_2 - E_7 - E_8 - 2 E_{11} - 2 E_{12} - 2 E_{13}) = \rho_n p^2 (E_c + E_o),
\] (61)
where the parts $\sim (E_3 + E_4)$ and $\sim (E_5 + E_6)$ in eq.(60) are effectively zero and only a central parameter $E_c$ and a spin-orbit parameter $E_s$ are independent. As a remarkable feature one observes that there is no term proportional to $p_n^2$ in the S-wave matrix element $\langle 1S_0|V_{med}|1S_0 \rangle$. In order to establish that this restriction is consistent with the renormalizability condition, one exploits $\bar{\sigma}_1 = -\bar{\sigma}_2$ in the spin-singlet state and uses the angular average $\langle q^2 \rangle = 2p^2$ of the squared momentum-transfer. By following in subsections 6.1 and 6.2 all terms that contribute proportional to $c_{2,3}k^3_\pi p^2/(64\pi^4 f^6_\pi) \ln(m/\lambda)$ or $c_{2,3,4}g^2_\pi k^3_\pi p^2/(96\pi^4 f^6_\pi) \ln(m/\lambda)$ to the S-wave matrix element, one verifies that these indeed sum up to zero. The same exact cancelation occurs also for the terms proportional to $c_{1,2,3}m^2_\pi/(64\pi^4 f^6_\pi) \ln(m/\lambda)$ or $c_{1,2,3}g^2_\pi m^2_\pi/(96\pi^4 f^6_\pi) \ln(m/\lambda)$.

From the corrected version of the subleading 3N-contact potential given in eq.(15) of ref.[13] the following in-medium NN-potential in isospin-symmetric nuclear matter of density $\rho = 2k^3/3\pi^2$ is derived:

$$V_{med}(k^2, p^2) = E_1 \left[ \frac{6}{5} k_j^2 + 2p^2 - 3q^2 \right] + E_3 \left[ (\bar{\sigma}_1 \cdot \bar{\sigma}_2 + 3) \left( \frac{3}{5} k_j^2 + p^2 \right) - \bar{\sigma}_1 \cdot \bar{q} q^2 \right]
$$

$$+ E_3 \left[ (\bar{\sigma}_1 \cdot \bar{\sigma}_2 + 9) \left( \frac{3}{5} k_j^2 + p^2 \right) - \bar{\sigma}_1 \cdot \bar{\sigma}_2 q^2 \right]
$$

$$+ E_3 \left[ (\bar{\sigma}_1 \cdot \bar{\sigma}_2 q^2 - p^2) - 3\bar{\sigma}_1 \cdot \bar{q} \bar{\sigma}_2 \cdot q + \frac{3}{2} \bar{\sigma}_1 \cdot \bar{p} \bar{\sigma}_2 \cdot \bar{p} + \bar{\sigma}_1 \cdot \bar{p}' \bar{\sigma}_2 \cdot \bar{p}' \right]
$$

$$+ E_5 \left[ \frac{7E_7 - 9E_8}{4}(\bar{\sigma}_1 + \bar{\sigma}_2) \cdot (\bar{q} \times \bar{p}) \right]
$$

$$+ E_7 \left[ \bar{\sigma}_1 \cdot \bar{q} \bar{\bar{\sigma}}_2 \cdot \bar{q} + \frac{3}{2} q^2 - 3p^2 - \frac{9}{5} k_j^2 - \frac{3i}{2} (\bar{\sigma}_1 + \bar{\sigma}_2) \cdot (\bar{q} \times \bar{p}) \right]
$$

$$+ E_9 \left[ \bar{\sigma}_1 \cdot \bar{q} \bar{\bar{\sigma}}_2 \cdot \bar{q} + \frac{3}{2} q^2 - 3p^2 - \frac{9}{5} k_j^2 - \frac{3i}{2} (\bar{\sigma}_1 + \bar{\sigma}_2) \cdot (\bar{q} \times \bar{p}) \right]
$$

$$+ E_{10} \left[ \bar{\sigma}_1 \cdot \bar{q} \bar{\bar{\sigma}}_2 \cdot \bar{q} + \frac{3}{2} q^2 - 3p^2 - \frac{9}{5} k_j^2 - \frac{3i}{2} (\bar{\sigma}_1 + \bar{\sigma}_2) \cdot (\bar{q} \times \bar{p}) \right]
$$

which should replace the (incomplete) expression written in eq.(49) of ref.[7].

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