Compositeness Effects in the Bose-Einstein Condensation

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Abstract

Small deviations from purely bosonic behavior of trapped atomic Bose-Einstein condensates are investigated with the help of the quon algebra, which interpolates between bosonic and fermionic statistics. A previously developed formalism is employed to obtain a generalized version of the Gross-Pitaevskii equation. Two extreme situations are considered, the collapse of the condensate for attractive forces and the depletion of the amount of condensed atoms with repulsive forces. Experimental discrepancies observed in the parameters governing the collapse and the depletion of the condensates can be accounted for by universal fittings of the deformation parameter for each case.

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I. INTRODUCTION

In many physical problems one has to deal with a large number of identical particles that are not of a fundamental character but which are known to be composed by a bound system of several fermions. Examples include systems of identical atoms, molecules or nuclei. If the number of “fundamental” fermions contained in the composite particle is odd, it is a fermion, otherwise it is a boson. In many situations the internal structure of the composite particle can be ignored and the system as a whole treated as a collection of interacting or noninteracting point-like particles. This is the case for instance, in the theories of Bose-Einstein condensation (BEC) of trapped bosonic gases \[1\]. Another example is provided by electron-hole bosonic states in semiconductors, called excitons \[2\]. In such systems, the bosons have internal structure and finite size, i.e., they are composite bosonic particles. The rational for neglecting the internal structure of the atoms in atomic BEC is that one is dealing with a very dilute system in the trap. The low density regime makes it very improbable that the internal structures of the atoms overlap in the trap, since the average distance between atoms in typical condensates is several times the size of an atom. For excitons, the situation is not as favorable as in atomic BEC and effects of the internal structure of the bosons might play an important role \[2\].

A departure from the purely bosonic behavior of the atoms in a trap might occur in situations that the central density of the condensate grows beyond a critical value. This happens, for example, when interatomic attractive forces tend to push the atoms to the center of the trap and the zero-point kinetic energy is no longer able to stabilize the system. The collapse is expected to occur when the number \(N\) of atoms exceeds a critical value, \(N_{cr}\), leading to an interaction energy larger than the kinetic energy.

The aim of the present paper is to set up a framework to evaluate the departure from the purely bosonic behavior of a BEC of composite particles. A complete theory aimed at such a task should include all the possible degrees of freedom for the constituent particles, which is in general highly prohibitive from the computational point of view. In the present paper, we use a phenomenological approach, making use of the concept of quons \[3\]. Quons are particles that are neither bosons, nor fermions, and the quon creation and annihilation operators obey a particular algebra that interpolates between Fermi and Bose algebras. The quon algebra is in fact a deformation of the Fermi and Bose algebras, and is such that when
a parameter \((q)\) runs from \(-1\) and \(+1\), it interpolates between the Fermi and Bose algebras.

Recently, a systematic way to build a many-body quon state has been discussed \([3, 4]\) and a general formula for a normalized many-quon symmetric state \([4]\) has been found. The developed formalism allows to apply the quon algebra to describe physical systems presenting deviations from an idealized situation, in the same fashion of the more usual Quantum Algebras \([3]\). An example is \([4]\) the three dimensional quonic harmonic oscillator which describes in a effective way anharmonic effects. The quonic deformation is also effective for incorporating correlations in many-body systems. Applications involving the antisymmetric subspace (fermion-like particles) can be also considered and are now under way.

The use of the symmetric subspace, and \(q\) close enough to \(+1\), allows to describe in a very natural way the departure from a purely bosonic behavior of systems of composite bosons. In order to apply such ideas to the BEC in trapped gases, we derive in Section III a quonic version of the Gross-Pitaevskii (GP) equation, which we denote qGP. In the limit of \(q = 1\), the qGP equation reduces to the usual GP equation, widely used in the literature \([1]\). Initially, in Section II, we present a brief discussion on the description of a composite boson built from two non-identical fermions and provide a connection with the quon algebra. This will allow a first estimate of the deformation parameter \(q\) in terms of parameters of the system \([2, 6]\). In Section IV we present numerical results, and compare with observables related to trapped bosonic atoms. Conclusions and future perspectives are presented in Section V.

II. RELATION BETWEEN COMPOSITE PARTICLES AND QUONS

A. Composite particles

Let us consider a composite boson state with quantum number \(\alpha\) as a bound state of two distinct fermions

\[
A_\alpha^\dagger |0\rangle = \sum_{\mu\nu} \Phi_{\alpha}^{\mu\nu} a_\mu^\dagger b_\nu^\dagger |0\rangle ,
\]

where \(\Phi_{\alpha}^{\mu\nu}\) is the Fock-space bound-state amplitude, \(a_\mu^\dagger\) and \(b_\nu^\dagger\) are the fermion creation operators, and \(|0\rangle\) is the no-particle state (vacuum). The quantum number \(\alpha\) stands for the center of mass momentum, the internal energy, the spin, and other internal degrees of freedom of the composite boson. For instance, the composite boson could be the hydrogen...
atom, where $a_\mu^\dagger$ and $b_\mu^\dagger$ create an electron and a proton respectively. The $\mu$ and $\nu$ stand for the space and internal quantum numbers of the constituent fermions. The sum over $\mu$ and $\nu$ is to be understood as a sum over discrete quantum numbers and an integral over continuous variables.

The fermion creation and annihilation operators satisfy canonical anti-commutation relations:

$$\{a_\mu, a_\nu\} = \{a_\mu^\dagger, a_\nu^\dagger\} = 0 \quad \{b_\mu, b_\nu\} = \{b_\mu^\dagger, b_\nu^\dagger\} = 0,$$

$$(a_\mu, a_\nu^\dagger) = \delta_{\mu,\nu} \quad (b_\mu, b_\nu^\dagger) = \delta_{\mu,\nu}.$$  \hspace{1cm} (2)

It is convenient to work with normalized amplitudes $\Phi^{\mu,\nu}_\alpha$, such that

$$\langle \alpha | \beta \rangle = \delta_{\alpha,\beta},$$

and therefore

$$\sum_{\mu\nu} \Phi^{\mu,\nu}_\alpha \Phi^{\mu,\nu}_\beta = \delta_{\alpha,\beta}. \hspace{1cm} (3)$$

Using the fermion anticommutation relations of Eq. (2) and the Fock-space amplitude normalization Eq. (4), one can easily show that the composite boson operators satisfy the following commutation relations:

$$[A_\alpha, A_\beta] = [A_\alpha^\dagger, A_\beta^\dagger] = 0,$$

$$[A_\alpha, A_\beta^\dagger] = \delta_{\alpha,\beta} - \Delta_{\alpha\beta},$$

where $\Delta_{\alpha\beta}$ is given by

$$\Delta_{\alpha\beta} = \sum_{\mu\nu} \Phi^{\mu,\nu}_\alpha \left( \sum_{\mu'} \Phi^{\mu',\nu}_\beta a_{\mu'} a_{\mu} + \sum_{\nu'} \Phi^{\mu,\nu'}_\beta b_{\nu'} b_{\nu} \right). \hspace{1cm} (6)$$

One can also easily show the following commutation relations:

$$[a_\mu, A_\alpha^\dagger] = \sum_{\mu'\nu'} \delta_{\mu,\mu'} \Phi^{\mu',\nu'}_\alpha b_{\nu'}, \quad [b_\nu, A_\alpha^\dagger] = -\sum_{\mu'\nu'} \delta_{\nu',\nu} \Phi^{\mu',\nu'}_\alpha a_{\mu'}. \hspace{1cm} (8)$$

The composite nature of the bosons is evident from the presence of $\Delta_{\alpha\beta}$, which is a sort of “deformation” of the canonical boson algebra. The effect of this term becomes unimportant in the infinite tight binding limit, i.e. in the limit of point-like bosons. Eq. (8) shows the lack of kinematical independence of the microscopic operators $a_\mu$ and $b_\nu$ from the macroscopic ones $A_\alpha^\dagger$s.
B. The quon algebra

The quon algebra is defined by the deformed commutation relation

\[ A_\alpha A^\dagger_\beta - qA^\dagger_\beta A_\alpha = \delta_{\alpha,\beta}, \]  

(9)

where \( q \) is the deformation parameter of the algebra; \( A_\alpha \) annihilates the vacuum

\[ A_\alpha |0\rangle = 0. \]  

(10)

The quon algebra interpolates between the Fermi and Bose algebras as the parameter \( q \) varies from \(-1\) to \(+1\). Polynomials in the creation operators acting on the vacuum form a Fock-like space of vectors, i.e., the quonic Fock space. In Ref. [8] it was shown that the squared norm of all vectors in the quonic Fock space remains positive definite, provided that \( q \) ranges from \(-1\) to \(+1\). Note that no commutation relation can be imposed on \( A^\dagger_\alpha, A^\dagger_\beta \) and \( A_\alpha A_\beta \). However, as remarked by Greenberg [3], similarly to the case of normal Bose commutation relations, no such rule is needed for practical evaluation of expectation values of polynomials in \( A_\alpha \) and \( A^\dagger_\alpha \) when Eq. (10) holds. Such matrix elements can be evaluated with the repeated use of Eq. (9) solely; annihilation operators are moved to the right using Eq. (9) until they annihilate the vacuum or creation operators are moved to the left using the adjoint of Eq. (9) until they annihilate the vacuum.

Since there is no commutation relations between \( A^\dagger_\alpha, A^\dagger_\beta \) and \( A_\alpha A_\beta \) all \( n! \) states given by the order permutations of the creation operators acting on the vacuum (and carrying different quantum numbers) are linearly independent. This is in sharp contrast to the usual bosonic or fermionic algebras and important differences between the quonic and usual Fock space appears due to this property. As an example, we consider a two quons system occupying the arbitrary states 1 and 2

\[ |1\rangle = A^\dagger_1 A^\dagger_2 |0\rangle \quad |2\rangle = A^\dagger_2 A^\dagger_1 |0\rangle, \]  

(11)

which are not independent states and the overlap matrix is

\[ \begin{pmatrix} \langle 1|1 \rangle & \langle 1|2 \rangle \\ \langle 2|1 \rangle & \langle 2|2 \rangle \end{pmatrix} = \begin{pmatrix} 1 \\ q \\ q \end{pmatrix}. \]  

(12)
One can show that the basis that diagonalizes the overlap matrix is

\[ |S\rangle = \frac{1}{\sqrt{2\sqrt{1 - q}}} (A_1^\dagger A_2^\dagger + A_2^\dagger A_1^\dagger) |\ 0\rangle, \] (13)

\[ |A\rangle = \frac{1}{\sqrt{2\sqrt{1 + q}}} (A_1^\dagger A_2^\dagger - A_2^\dagger A_1^\dagger) |\ 0\rangle, \] (14)

where \(|S\rangle\) is a symmetric state and \(|A\rangle\) is an antisymmetric state. Analogously for a three quons system occupying three distinct states 1, 2, 3, the non-orthogonal basis will contain the six independent states \(A_i^\dagger A_j^\dagger A_k^\dagger |\ 0\rangle\), where the indexes \(i, j, k\) are determined by the 3! permutations of 1, 2, 3. The orthonormal basis may be obtained classifying these states according to the irreducible representations of the permutation group of three elements, \(S_3\).

Besides the well known symmetric and antisymmetric states, there are four more exotic mixed symmetric states. To shorten the corresponding expressions we adopt the convention

\[ A_i^\dagger A_j^\dagger A_k^\dagger |\ 0\rangle \equiv |ijk\rangle \]

and can write: states:

\[ |S\rangle = \frac{1}{\sqrt{1 + 2q^2 + 2q^3}} \frac{1}{\sqrt{6}} [\ |ijk\rangle + |jik\rangle + |ikj\rangle + |kij\rangle + |kji\rangle ] , \] (15)

\[ |A\rangle = \frac{1}{\sqrt{1 + 2q^2 - 2q^3}} \frac{1}{\sqrt{6}} [\ |ijk\rangle - |jik\rangle - |ikj\rangle - |kij\rangle - |kji\rangle ] , \] (16)

\[ |MS1\rangle = \frac{1}{\sqrt{1 - q^2 + q - q^3}} \frac{1}{\sqrt{12}} [\ |ijk\rangle - |jik\rangle + 2|ikj\rangle + |kij\rangle - 2|kji\rangle - |kji\rangle ] , \] (17)

\[ |MS2\rangle = \frac{1}{\sqrt{1 - q^2 + q - q^3}} \frac{1}{2} [\ -|ijk\rangle - |jik\rangle + |ikj\rangle + |kji\rangle ] , \] (18)

\[ |MS3\rangle = \frac{1}{\sqrt{1 - q^2 - q + q^3}} \frac{1}{2} [\ |ijk\rangle - |jik\rangle - |ikj\rangle + |kji\rangle ] , \] (19)

\[ |MS4\rangle = \frac{1}{\sqrt{1 - q^2 - q + q^3}} \frac{1}{\sqrt{12}} [\ |ijk\rangle + |jik\rangle - 2|ikj\rangle + |kji\rangle - 2|kij\rangle + |kji\rangle ] , \] (20)

where \(i, j, k = 1, 2, 3\). Also, the cases \(i = j\), \(i = k\), \(j = k\) and \(i = j = k\) are automatically included in the expressions of Eqs. (13) to (20) up to a normalization factor which is \(q\)-independent. Evidently, the above basis states can be built from the well known procedure based on the Young tableaux method \[4\], or through the diagonalization of the overlap matrix obtained from all possible order permutations from the state \(A_i^\dagger A_j^\dagger A_k^\dagger |0\rangle\). In fact, the \(q\)-polynomials which appear in the square roots in Eqs. (15) to (20), correspond to the eigenvalues of the overlap matrix and measure the degree of violation of statistics in the system.

Finally, we remark that for the quon algebra one may also define the transition operator
\( N_{ij} \), obeying the commutation relations\[3, 10\]

\[
\begin{align*}
[N_{ij}, A_k^\dagger] &= \delta_{jk} A_i^\dagger, \\
[N_{ji}, A_k] &= -\delta_{jk} A_i.
\end{align*}
\] (21)

The \( N_{ij} \) operator is given by an infinite series expansion in terms of the quonic annihilation and creation operators, and its first few terms are given by

\[
N_{ij} = A_i^\dagger A_j + (1 - q^2)^{-1} \sum_\gamma (A_\alpha^\dagger A_\gamma^\dagger - q A_i^\dagger A_\gamma^\dagger)(A_j A_\gamma - q A_\gamma A_i) + \cdots .
\] (22)

The number operator \( N_i \) is obtained when \( i = j \).

C. The quon algebra and composite particles

If one writes \( q = 1 - x \), the deformed commutator, Eq. (6), can be written as

\[
[A_\alpha, A_{\beta}^\dagger] = \delta_{\alpha\beta} - x A_\alpha^\dagger A_\beta .
\] (23)

The similarity of this relation with Eq. (3) is evident. In some sense, the product of fermion operators \( a_\mu^\dagger a_\mu \) and \( b_\nu^\dagger b_\nu \) weighted by the \( \Phi \)'s in Eq. (3) is effectively modelled by the term \( x A_\alpha^\dagger A_\beta \) in Eq. (23).

We note that the commutation relations of Eq. (3) between \( A_\alpha, A_\beta \) and \( A_\alpha^\dagger, A_\beta^\dagger \) are not algebraic relations valid for quons. Since we wish to describe a system of “identical” composite bosons, we impose that the state vectors of a many-body composite particles system are invariant by the permutation of the particle indexes. So we assume that the physical subspace which is adequate for the description of a bosonic composite system is composed only of totally symmetric states. To build the physical subspace we have to project out from the basis of states only totally symmetric states.

It is possible to show that the most general symmetric state for a system of \( N \) quons can be written as \[4\]:

\[
|n_\alpha n_\beta n_\gamma \cdots ; S\rangle = \sqrt{\frac{n_\alpha! n_\beta! n_\gamma! \cdots}{N! [N]!}} \hat{S}_N (A_\alpha^\dagger)^{n_\alpha} (A_\beta^\dagger)^{n_\beta} (A_\gamma^\dagger)^{n_\gamma} \cdots |0\rangle,
\] (24)

where \( \hat{S}_N \) is an operator that generates all possible combinations that are symmetric under the permutation of any of the creation operators, \( n_\alpha + n_\beta + n_\gamma + \cdots = N \) and \[3, 11\]

\[
[N] = \frac{1 - q^N}{1 - q},
\] (25)
with \([N]! = [N][N-1]...[2][1]\) and \([0]!=1\). Another important result that we are going to use next is the following [4]:

\[
A_\alpha |n_\alpha n_\beta n_\gamma ...;S\rangle = \sqrt{\frac{[N]}{N}} \sqrt{n_\alpha} |n_\alpha - 1, n_\beta n_\gamma ...;S\rangle.
\] (26)

This last expression allows one to calculate matrix elements between symmetric quonic states with any number of quons. Using Eq. (26) one can easily show that, within the physical subspace, i.e., the one built only from totally symmetric states, \(|\psi, S\rangle\), the commutation relations of Eq. (5) are now valid also for the quons

\[
\langle \psi, S | (A^\dagger_\alpha A^\dagger_\beta - A^\dagger_\beta A^\dagger_\alpha) | \psi, S \rangle = 0 ,
\] (27)

\[
\langle \psi, S | (A_\alpha A_\beta - A_\beta A_\alpha) | \psi, S \rangle = 0 .
\] (28)

In this way, the analogy between a composite particle and a quon is complete.

**D. Free composite gas**

Now let us consider a system of \(N\) composite bosons in a large box of volume \(V\) at zero temperature. If the bosons were ideal point-like particles, the ground state of the system would be the one where all bosons condense in the zero momentum state. In the case of composite bosons, the closest analog of the ideal gas ground state is

\[
|N\rangle = \frac{1}{\sqrt{N!}} (A^\dagger_0)^N |0\rangle ,
\] (29)

where \(A^\dagger_0\) is the creation operator of a composite boson in its ground state (ground state \(\Phi\)) and with zero center of mass momentum. Due to the composite nature of the bosons, this state incorporates kinematical correlations implied by the Pauli exclusion principle which operates on the constituent fermions. Among other effects, the Pauli principle forbids the macroscopic occupation of the zero momentum state. The closest analog to the boson occupation number in the state of Eq. (29) is

\[
N_0 = \frac{\langle N|A^\dagger_0 A_0|N\rangle}{\langle N|N\rangle} .
\] (30)
In order to evaluate Eq. (30), we consider a spin zero boson and use for the spatial part of $\Phi$ a simple Gaussian form such that the r.m.s radius of the boson is $r_0$. To lowest order in the density of the system $n = N/V$, $N_0$ is given by

$$N_0 = N \left(1 - \gamma n r_0^3\right),$$

where $\gamma$ is a geometrical factor. The exact value of the geometrical factor $\gamma$ depends on the form of the internal wave-function of the composite boson. Depending on the exact form used for the internal boson wave-function this factor runs from 50 to 250 [2]. Note that Eq. (31) is valid only for $\gamma n r_0^3 \ll 1$, i.e. for extremely low-density systems. For higher-density systems, a nonlinear $N$ dependence of $N_0/N$ is obtained [2].

It is apparent from Eq. (31) that in the limit of infinite tight binding, $r_0 \to 0$, one has the familiar Bose-Einstein condensation. For finite values of the size of the bosons, the effects of the Pauli principle become important and one has a depletion on the amount of condensed bosons. Moreover, from Eq. (31) one has that if the size of the bound state is of the order of the mean separation of the bosons in the medium, $d \sim n^{-1/3}$, the depletion is almost total. The depletion of the condensation is a direct consequence of the deformation of the boson algebra by the term $\Delta_{\alpha\beta}$.

In the same spirit as in the previous case, we take as the closest analog of the ideal boson gas ground state the $N$ quon state

$$|N\rangle = \frac{1}{\sqrt{[N]!}}(A_0^\dagger)^N|0\rangle.$$

(32)

Note that we are using the same symbols for both the composite and quon algebra annihilation/creation operators.

As before, the operator $A_0^\dagger A_0$ is the number operator in the zero deformation limit only. The effect of the deformation can be evaluated taking the expectation value of $A_0^\dagger A_0$ in the state $|N\rangle$, Eq. (32). To evaluate the expectation value, we make use of the result given by equation Eq. (23)

$$N_0 = \langle N|A_0^\dagger A_0|N\rangle = [N] = \frac{1 - q^N}{1 - q}.$$

(33)

Using the above result one obtains to lowest order in $x$,

$$N_0 = \frac{1}{x}[1 - (1 - x)^N] \simeq N \left(1 - \frac{1}{2}Nx\right).$$

(34)
Comparing this result with the one of Eq. (31) it is clear that the effect of the deformation, for sufficiently low densities, is such that

\[ Nx \sim 2\gamma \frac{N r_0^3}{V}. \] (35)

That is, the effect of the deformation parameter is proportional to the ratio of the volume occupied by the bosons to the volume of the system. Note that for densities not extremely low, a nonlinear \( N \) dependence is expected for the deformation parameter, as remarked above.

**III. THE GROSS-PITAEVSKII EQUATION FOR COMPOSITE BOSONS**

We now use the quon algebra formalism in the BEC. We consider a system of \( N \) composite bosons interacting in a spherical harmonic oscillator trap. We assume that the effective Hamiltonian describing such system is given by

\[
H = T + V + V_{\text{trap}} = \sum_{\alpha,\beta} \langle \alpha | T | \beta \rangle A^\dagger_{\alpha} A_{\beta} + \frac{1}{2} \sum_{\alpha,\beta,\gamma,\delta} \langle \alpha \beta | V | \gamma \delta \rangle A^\dagger_{\alpha} A^\dagger_{\beta} A_{\gamma} A_{\delta} + \sum_{\alpha,\beta} \langle \alpha | V_{\text{trap}} | \beta \rangle N_{\alpha\beta}, \tag{36}
\]

where \( T, V_{\text{trap}} \) and \( V \) correspond to the kinetic energy, trap harmonic oscillator potential and the interaction among the composite bosons respectively. We take as usual

\[
T(\vec{x}) = -\frac{\hbar^2}{2m} \Delta \vec{x}, \tag{37}
\]

and

\[
V(\vec{x}, \vec{y}) = g \delta(\vec{x} - \vec{y}). \tag{38}
\]

To obtain the equation describing the condensate of composite bosons we assume that the lowest energy state is given by

\[
|\psi\rangle = \frac{(A^\dagger_0)^N}{\sqrt{N!}} |0\rangle. \tag{39}
\]

So we impose the variational principle

\[
\langle \psi, S | H | \delta \psi, S \rangle = 0 , \tag{40}
\]

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where the arbitrary variational symmetrized state has the form

$$|\delta \psi, S \rangle = \hat{S} A^\dagger_\mu A_0 |\psi, S \rangle, \ (\mu \neq 0) \quad (41)$$

and the symbol \( \hat{S} \) is a symmetrizer operator. Eq. (40) is equivalent to the expression

$$\langle \psi, S | \sum_{\alpha\beta} (\langle \alpha | T | \beta \rangle A^\dagger_\alpha A^\dagger_\beta + \frac{1}{2} \sum_{\gamma\delta} (\langle \alpha | V | \gamma \delta \rangle A^\dagger_\alpha A_\gamma A^\dagger_\delta A_\delta + \langle \alpha | V^{\text{trap}} | \beta \rangle) N_{\alpha\beta}) |\delta \psi, S \rangle = 0 \quad (42)$$

From Eq. (26) one obtains the matrix elements

$$\langle \psi, S | A^\dagger_\alpha A^\dagger_\beta |\delta \psi, S \rangle = \frac{[N]}{N} \sqrt{N} \delta_{\alpha 0} \delta_{\beta \mu}, \quad (43)$$

$$\langle \psi, S | N_{\alpha\beta} |\delta \psi, S \rangle = \sqrt{N} \delta_{\alpha 0} \delta_{\beta \mu} \langle \psi, S | A^\dagger_\alpha A^\dagger_\beta A\gamma A_\delta |\delta \psi, S \rangle = \frac{[N]}{N} \frac{[N-1]}{N-1} \sqrt{N} (N-1) (\delta_{\alpha 0} \delta_{\beta 0} \delta_{\gamma \mu} \delta_{\delta 0} + \delta_{\alpha 0} \delta_{\beta 0} \delta_{\gamma 0} \delta_{\delta \mu}). \quad (44)$$

Using these in Eq. (42), one obtains

$$\frac{[N]}{N} T_{0\mu} + V_{0\mu}^{\text{trap}} + \frac{[N][N-1]}{N} V_{000\mu} = 0. \quad (45)$$

Therefore, the variational principle leads in coordinate space to the equation

$$\left[ -\frac{\hbar^2}{2m} \Delta x + \frac{N}{[N]} V^{\text{trap}}(x) + g[N-1] |\phi(x)|^2 \right] \phi(x) = \epsilon \phi(x). \quad (46)$$

This equation may be interpreted as a generalization of the Gross-Pitaevskii equation to quons and, as mentioned previously will be denoted as qGP.

**IV. THE SOLUTION OF THE QUON GROSS-PITAЕVSKII EQUATION AND APPLICATIONS**

Many authors have recently presented different methods to solve the usual GP equation. Here, we follow a variational approach, in which we expand the ground state wave-function in a three-dimensional harmonic oscillator basis with angular momentum \( l = 0 \). Good convergence has been achieved within the method and we were able to reproduce to very good accuracy results obtained in the literature with other methods when we go to the limit \( q = 1 \).

Initially we consider the case of an attractive interaction, represented by a negative value of the scattering length \( [1] \). Of particular interest is the study of the well known collapse,
which occurs when the number of atoms in the trap reaches a critical number for which the system becomes unstable. In that case, the behavior of the GP solution is characterized by the dimensionless constant

$$k = \frac{N_0 |a|}{b_t},$$

where $b_t$ is the trap oscillator length and $a$ the s-wave scattering length. Several authors (see for instance Ref. [14] and references therein) have calculated the critical value ($k_c$) for this constant using different methods. The value, for an isotropic trap, was always very close to $k_c = 0.575$. Recent measurements [14] indicate a value for $k_c$ that is between $\sim 10\%$ and $25\%$ smaller than the theoretical predictions.

Let us now consider now the qGP equation. According to our discussion in Section (II), one may interpret the number of condensed atoms as $N_0 = [N]$. Eq. (46) can be rewritten in terms of dimensionless variables in the same fashion as done in the normal GP equation [1]:

$$[\triangle \tilde{x} + \tilde{r}^2 - 8\pi k(q)|\phi(\tilde{x})|^2] \phi(\tilde{x}) = 2 \tilde{\epsilon} \phi(\tilde{x}).$$

(48)

Here we have used explicitly the oscillator form for the potential and have written the interaction constant in terms of the scattering length $\frac{g}{4\pi \hbar^2 a/m}$. The tilde means energy in units of the oscillator energy, $\tilde{\epsilon} = \epsilon/\hbar\omega_t$, and length is expressed in units of the modified oscillator constant, $\tilde{x} = x/b_t(q)$, with

$$b_t(q) = b_t \left( \frac{[N]}{N} \right)^{1/4}.$$

(49)

In addition,

$$k(q) = \frac{|a|[N][N-1]}{b_t \left[ \frac{[N]}{N} \right]} \left( \frac{N}{[N]} \right)^{1/4} \simeq k \left( \frac{N}{[N]} \right)^{1/4}.$$

(50)

Obviously the qGP equation of Eq. (48) leads to a critical value $k_c(q) = 0.575$. However, in the present case the critical constant to be compared with the experimental value is given by

$$k_c = 0.575 \left( \frac{[N]}{N} \right)^{1/4}.$$

(51)

For $N_0 = [N] = 2000$ and using $x$ independent of $N$ and as small as $x = 3 \times 10^{-4}$ (or $q = 1 - x = 0.9999$), one finds that it is possible to fit the experimental result $k_c = 0.522$ [14]. This value of $x$ should be compared to the expression coming from our estimate of the
deformation parameter in Eq. (35). Using for $r_0$ the scattering length $|a| \sim 15\AA$, and using the linear dimension of the trap as given by $b \sim 100\AA$ [16] one obtains precisely
\[ x = 2\gamma \frac{|a|^3}{b^3} \sim 10^{-4}, \] (52)
where for the geometrical factor $\gamma$ we used the value $\gamma = 100$, as estimated in Ref. [2]. This result is of the correct order of magnitude to bring the theoretical value for the critical constant within the range of experimental values.

As a second application, we consider the case of a repulsive interaction. In this case the number of atoms in the trap can reach large numbers ($N$ up to $\sim 10^7$), which can raise the question of validity for the use of the GP equation for such high densities. In a recent calculation [15], effects that go beyond the usual GP (or mean-field) solution were also considered and shown to lead to a systematic, small increase in the chemical potential of the condensate. In our formalism we may obtain the same behavior by preserving the original GP dynamics but relaxing the condition that the particles are true bosons. To see how this happens, we now rewrite Eq. (46) in a slightly different form:
\[
\left[ H_{osc} + \left( \frac{N}{[N]} - 1 \right) V^{trap}(\vec{x}) + g[N-1]|\phi(\vec{x})|^2 \right] \phi(\vec{x}) = \epsilon \phi(\vec{x}), \] (53)
where $H_{osc}$ is the usual three-dimensional harmonic oscillator Hamiltonian. Following our previous interpretation that the number of condensed atoms is $[N]$, we conclude that only the term proportional to $N/[N] - 1$ differs from the usual GP equation. However, for $q$ close enough to 1, this term is small ($\ll 1$) even for large values of $N$, and therefore one may treat this term as a perturbation to the non-deformed solution. At this point it is worthwhile to note that this term will always increase the energy of the condensate by a small amount. Of course, the increase will depend on the value of $q$ (or $x$) used. In order to make a phenomenological estimation of the effect of this perturbation we relate again our results of section II.D for the depletion of the condensate to the one presented in Section III of Ref. [1]. There, an estimation for the depletion is given by:
\[
\frac{N-N_0}{N_0} = \frac{5\sqrt{\pi}}{8} \sqrt{a^3 n(0)}, \] (54)
where $n(0)$ is the central ($r = 0$) density of the condensate. Equating our result, Eq. (34), with the above and taking in to account that $x \ll 1$, we obtain:
\[ x \approx \frac{10\sqrt{\pi}}{8N} \sqrt{a^3 n(0)}. \] (55)
Note that a dependence of the deformation parameter on the number of atoms (or on the density) appears very naturally here. As we have discussed in Section II D, an $N$-independent deformation parameter $x$ is physically reasonable only for low density systems. In the present situation of a repulsive interaction, the number of atoms in the trap can be many orders of magnitude larger than in the attractive case and it is gratifying to see the internal consistency of our formalism, in that it is capable of describing both extremes in a very natural way.

In order to perform numerical calculations, one needs $n(0)$, which can be taken from the $x = 0$ solution of the GP equation and the value for $N$, which in turns depends on $x$. However, for our purposes here we may evaluate the order of magnitude of $x$ making $N = N_0$ in Eq. (55). We also use $a/b_t = 5 \times 10^{-2}$. This last value might seem quite big when compared to the one used for example in Refs. [1] and [14]. However, recent experimental developments [16] allow to change the scattering length by orders of magnitude thanks to a technique known as Feshbach resonance. The value chosen here is consistent with, for example, the experiment reported in Ref. [16]. In Table I we present our numerical results for the chemical potential $\epsilon$, for different $N_0$ values. Note that, although our initial conditions are different from the ones chosen for the calculations in Ref. [15], the results are qualitatively equivalent, in the sense that we get always a bigger value for the chemical potential, as compared to the usual GP solution.

V. CONCLUSIONS

We have considered in this work the quon algebra to describe in an effective and phenomenological way the departure from a purely bosonic behavior of a system of composite bosons. The formalism was developed previously and relies on the projection of the whole quonic space onto the symmetric subspace. The main idea is to preserve the bosonic behavior and leave to the deformation parameter the description of possible deviations. As a specific application we have considered the derivation of the Gross-Pitaevskii equation within the quon algebra, which can be done in a straightforward way using our formalism. The interpretation of the modified Gross-Pitaevskii equation obtained is consistent with our initial qualitative considerations.

As for numerical calculations we have in first place solved our deformed GP equation
for the collapse in the case of an attractive force in a trapped bosonic gas. Reinterpreting
the value of the critical constant for the collapse, we obtained that the deformation of the
bosonic algebra changes the value of this constant, and the change is in the right direction
to explain experimental results. Of course, the modification depends on the chosen value for
the deformation parameter $q$, which is not easy to estimate at the conditions of the experi-
ment. We have found that a decrease of order $10^{-4}$ compared to the bosonic limit $q = 1$, and
independent of the number of atoms in the trap, brings the critical value within the recent
experimental measurements. Secondly, we have calculated the chemical potential for the
case of a repulsive force between the trapped atoms. This is a more favorable situation to
test our model, since recently it became experimentally feasible to create condensates with
a much larger number of atoms as compared to the attractive case. Using a phenomenolog-
ical estimation for the deformation parameter, for which a density dependence arises very
naturally, we are able to calculate the correction in the chemical potential.

Natural extensions of the present work would include the investigation of the effect of
the $q$-deformation on other observables of the BE condensates. Also, there are many other
interesting systems for which the quon algebra could be useful to describe deviations from
a purely bosonic behavior of many-body systems of composite bosons. One example, which
seems particularly interesting in the present context, is the case of excitons [2], which are
boson-like particles formed by a electron-hole pair in a semiconductor. Work in these direc-
tions is underway.

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TABLE I: Chemical potential $\epsilon$ - in units of $\hbar \omega_t$ - of the condensate for four values of the number of atoms $N_0$, using the usual GP formalism and the qGP equation. The values of $x$ used in each case are also shown and were obtained as explained in the text.

| $N_0$  | $\epsilon_{GP}$ | $\epsilon_{qGP}$ | $x$       |
|-------|-----------------|-----------------|-----------|
| 1000  | 7.31            | 7.47            | $1.0 \times 10^{-3}$ |
| 10000 | 17.84           | 18.32           | $1.3 \times 10^{-4}$ |
| 100000| 44.59           | 46.69           | $2.0 \times 10^{-5}$ |
| 1000000| 111.94          | 131.16          | $5.1 \times 10^{-6}$ |