BASINS OF ATTRACTION OF NONLINEAR SYSTEMS’ EQUILIBRIUM POINTS: STABILITY, BRANCHING AND BLOW-UP *

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Abstract  The nonlinear dynamical model consisting the system of differential and operator equations is studied. Here differential equation contains a nonlinear operator acting in Banach space, a nonlinear operator equation with respect to two elements from different Banach spaces. This system is assumed to enjoy the stationary state (rest points or equilibrium). The Cauchy problem with the initial condition with respect to one of the desired functions is formulated. The second function controls the corresponding nonlinear dynamic process, the initial conditions are not set. The sufficient conditions of the global classical solution’s existence and stabilization at infinity to the rest point are formulated. It is demonstrated that a solution can be constructed by the method of successive approximations under the suitable sufficient conditions. If the conditions of the main theorem are not satisfied, then several solutions may exist. Some of solutions can blow-up in a finite time, while others stabilize to a rest point. The special case of considered dynamical models are nonlinear differential-algebraic equation (DAE) have successfully modeled various phenomena in circuit analysis, power systems, chemical process simulations and many other nonlinear processes. Three examples illustrate the constructed theory and the main theorem. Generalization on the non-autonomous dynamical systems concludes the article.

Key words  nonlinear dynamics, stability, rest point, branching solution, Cauchy problem, DAE, bifurcation, equilibrium, blow-up.

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In decompositions (1.3), (1.5) there are derivatives and Frechet differentials calculated in point \(x, u\) of real Banach spaces \(X, U\); \(G\) is linear operator \(X \rightarrow U\) and \(F\) is nonlinear operator \(X + U \rightarrow U\) are continuous in the neighborhoods \(\|x - x_0\|_X \leq r_1, \|u - u_0\|_U \leq r_2\) of real Banach spaces \(X, U\); \(E\) is linear real normed space. It is assumed that the following operators decompositions are valid

\[
F(x, u) = F(x_0, u_0) + A_1(x - x_0) + A_2(u - u_0) + R(x, u); \\
G(x, u) = G(x_0, u_0) + \sum_{k=1}^{n} d^k(G(x_0, u_0); (x - x_0, u - u_0)) + r(x, u);
\]

In decompositions (1.3), (1.5) there are derivatives and Frechet differentials calculated in point \((x_0, u_0)\) as follows

\[
A_1 := \left. \frac{\partial F(x, u)}{\partial x} \right|_{x=x_0, u=u_0} \in \mathcal{L}(X \rightarrow E); \\
A_2 := \left. \frac{\partial F(x, u)}{\partial u} \right|_{x=x_0, u=u_0} \in \mathcal{L}(U \rightarrow E); \\
A_3 := \left. \frac{\partial G(x, u)}{\partial x} \right|_{x=x_0, u=u_0} \in \mathcal{L}(X \rightarrow U); \\
A_4 := \left. \frac{\partial G(x, u)}{\partial u} \right|_{x=x_0, u=u_0} \in \mathcal{L}(U \rightarrow U); \\
d^k(G(x_0, u_0); (x - x_0, u - u_0)) = \\
\sum_{i+j=k} C^k_i \left. \frac{\partial^k G(x, u)}{\partial x^i \partial u^j} \right|_{x=x_0, u=u_0} (x - x_0)^i(u - u_0)^j; \\
\frac{\partial^k G(x, u)}{\partial x^i \partial u^j} : X + \cdots + X + U + \cdots + U \rightarrow U.
\]

Obviously, the nonlinear operator equation (1.2) should enjoy the real solution in order for existence of solution of system (1.1) – (1.2).

Therefore, in this work it is assumed that elements \(x_0, u_0\) are from real Banach spaces \(X, U\) satisfy operator equations \(F(x, u) = 0, G(x, u) = 0\). Therefore, \(x_0, u_0\) is stationary solution to system (1.1) – (1.2) (steady state, equilibrium point or rest point).

In various power engineering problems (here readers may refer e.g. to [1–4]), there the special cases of system (1.1) – (1.2) were considered, when \(X = E = \mathbb{R}^n, U = \mathbb{R}^m\) are finite spaces. In works [1–3] such systems are known as algebraic-differential systems and considered with initial Cauchy conditions

\[
x(0) = \Delta,
\]
where $\Delta$ is element from neighborhood of equilibrium point $x_0$. Solutions $x(t), u(t)$ on semiaxis $[0, +\infty)$ are constructed such as 

$$
\lim_{t \to +\infty} (\|x(t)\| + \|u(t)\|) = 0.
$$

Functions $u(t)$ are selected such as solution $x(t), u(t)$ is stabilized to the rest point $x_0, u_0$ as $t \to +\infty$.

Such problem is important for solution of various automatic control problem. In classic works of Russian mathematicians (Anatoliy Lure, Evgenii Barbashin [5] et al.) the fundamentals of the modern theory for automatic control are considered. In number of works (see e.g. [1–3, 6]) in this field the interesting results of numerical analysis of the electric engineering models are considered. The complexity of this problem is demonstrated, which is caused by solutions’ bifurcation and blow-up, ref. [1, 3]. Therefore, it is important to consider the solvability of the Cauchy problem (1.1), (1.2), (1.7) as well as numerical methods to attack this challenging problem both from theoretical and applied sides.

This problem is also important for nonlinear dynamic systems’ mathematical modeling using “input” – “output” approach [7, 8], when $u$ is “input”, and $y = h(x, u)$ is “output”. In case of closed loop the output must satisfy the given criteria in the form of equation $G(y) = 0$. (see Fig. 1)

![Fig. 1. Scheme of dynamic system with closed loop](image)

In works [1, 2], [3] only models (1.1), (1.2) with ordinary differential equations have been considered. Calculations were performed to demonstrate the problem complexity caused by stability analysis and possible blow-up of the solution.

Results of the present work were announced in [9] in Russian language and concentrates on the nonlinear differential-operator systems in general settings. Constructed theory enables the unified consideration of “input-output” models involving both differential and integral-differential equations.

Let $(0, 0)$ be equilibrium point of system (1.1), (1.2) i.e. in (1.3) – (1.6) and below $x_0 = 0, u_0 = 0, F(0, 0) = 0, G(0, 0) = 0$.

**Definition 1.1** We call the ball $\|x\| \leq r$ as basin of attraction of equilibrium point $(0, 0)$ of system (1.1), (1.2) such as for arbitrary $\Delta$ from this ball there exist solutions $x : [0, +\infty) \to X, u : [0, +\infty) \to U$ with initial condition $x(0) = \Delta$ stabilizing to zero on the positive semiaxis.

**Definition 1.2** If basin of attraction of the rest point is nonempty ($r > 0$), then stationary solution $(0, 0)$ of system (1.1), (1.2) is called asymptotically stable.

The objective of this work is to construct the sufficient condition of non-emptiness of the basin of attraction of equilibrium points, proof of the existence and uniqueness theorem for the solution, the construction of sufficient conditions for the nonempty basin of attraction.
of the equilibrium point, the proof of the existence and uniqueness theorem for the solution \((1.1), (1.2)\) with the initial condition \(x(0) = \Delta\); the development of the method of successive approximations of the solution of the Cauchy problem on semiaxis \([0, +\infty)\). In addition, for the first time sufficient conditions are formulated for the Cauchy problem’s solution branching for the system \((1.1), (1.2)\) with stability analysis of individual branches of this solution.

The article is organized as follows. The main part of this work (sec. 1-4) concentrated on reducing a non-linear system in the neighborhood of an equilibrium point to a single differential equation.

2 Reduction of a non-linear system in the neighborhood of an equilibrium point to a single differential equation

Let \(G(x, u) = A_3 x + A_4 u + r(x, u)\), where \(||r(x, u)|| = o(||x|| + ||u||)\).

**Lemma 2.1** Let operator \(A_4\) has bounded inverse, here \(A_4 = \frac{\partial G(x, u)}{\partial u}|_{x=u=0}\) is Frechet derivative. Then for arbitrary ball \(S_1 : ||x|| \leq r_1\) exists ball \(S_2 : ||u|| \leq r_2\) such as for any \(x \in S_1\) equation \((1.2)\) has unique continuous solution \(u(x)\) in ball \(S_2\). In this case we have an asymptotic representation of the solution as \(||x|| \to 0\):

\[
u = -A_3^{-1}A_3 x + o(||x||), \tag{2.1}\]

where \(A_3 = \frac{\partial G(x, u)}{\partial x}|_{x=u=0}\) is Frechet derivative on \(x\).

**Proof** Equation \((1.2)\) due to invertability of operator \(A_4\) and equality \(G(0, 0) = 0\) can be reduced to \(u = \Phi(u, x)\). Here \(\Phi(u, x) = -A_3^{-1}A_3 x - A_1^{-1}r(x, u)\). Hence for \(\forall q \in (0, 1)\) exists neighborhood \(||u|| \leq r_2\), in which operator \(\Phi\) will be uniformly contracting on \(x\) from the ball \(||x|| \leq r_1\): \(||\Phi(u_1, x) - \Phi(u_2, x)|| \leq q||u_1 - u_2||\). Moreover, in this neighborhood the following inequality is valid \(||\Phi(u, x)|| \leq q r_2 + ||\Phi(0, x)||\). Since \(\Phi(0, 0) = 0\), then due to continuity \(\Phi(0, x)\) there exists \(r\) in the interval \((0, r_1]\) such as \(||\Phi(0, x)|| \leq (1 - q) r_2\) for \(||x|| \leq r\). Hereby, for any \(x\) from ball \(||x|| \leq r\) and \(\forall u\) from \(||u|| \leq r_2\) operator \(\Phi\) will be contracting and transfers ball \(S(0, r_2)\) into itself. Hence, using the principle of contracting mappings, the sequence \(\{u_n\}\), where \(u_n = \Phi(u_{n-1}, x) \in S(0, r)\), \(u_0 = 0\) converges uniformly on \(x\) to unique solution \(u(x)\) in the ball \(||u|| \leq r_2\). Since \(u_n(x) \sim -A_3^{-1}A_3 x\) as \(||x|| \to 0\), then we have asymptotics

\[
u(x) \sim -A_3^{-1}A_3 x. \tag{2.2}\]
as \(||x|| \to 0\).

From Lemma 2.1 it follows

**Lemma 2.2** Exist neighborhood \(||x|| \leq r, ||u|| \leq r_2\), such as system (1.1), (1.2) can be uniquely reduced to the following differential equation

\[
A \frac{dx}{dt} = f(x).
\]  

(2.3)

Here mapping \(f : S(0, r) \subset X \to E_2 := \text{F}(x, u(x))\) is as follows

\[
f(x) = A_1 x - A_2 A_{\lambda}^{-1} A_3 x + L(x).
\]  

(2.4)

Nonlinear mapping \(L : X \to E\) satisfies the estimate \(||L(x)|| = o(||x||)\).

**Proof** For proof it is sufficient to substitute the estimate \(||L(x)|| = o(||x||)\).

Remark 2.3 After determination of \(x(t)\) one may find the approximate \(u(t)\). Indeed, let \(x(t)\) is solution to differential equation \((2.3)\) constructed in Lemma 2. Then function \(u(t)\) using Lemma 1 is constructed by following asymptotic \(u(t) \sim -A_{\lambda}^{-1} A_3 x(t)\) as \(||x|| \to 0\).

3 **An a priori estimate of the Cauchy problem’s solution**

Let us consider the system (1.1), (1.2) with initial Cauchy condition \(x(0) = \Delta\).

**Lemma 3.1** Let \(x : [0, +\infty) \to X\) be solution to Cauchy problem for system (1.1), (1.2), where \(||\Delta||\) in the initial condition is sufficiently small. Let \(Re \lambda \leq -l < 0\) for all \(\lambda \in \sigma(M)\). Then there are \(C \geq 1\) and \(\varepsilon \in (0, l)\), such as \(||\exp M t||_{\mathcal{L}(X \to X)} \leq Ce^{-lt}\) and \(||x(t)||_{X} \leq C||\Delta||e^{(\varepsilon-l)t} for \(t \in [0, +\infty)\).

**Proof** Using Lemma 2.1 let us find function \(u(t)\) using successive approximations. Substitution of this function in the right hand side of eq. (1.1) leads to eq. (2.3). Using identity \(A^{-1} f(x) = M x + A^{-1} L(x)\) we get

\[
\frac{dx}{dt} = M x + A^{-1} L(x), \quad x(0) = \Delta.
\]

(3.1)

Therefore

\[
x(t) = \exp M t \Delta + \int_{0}^{t} \exp M (t - s) A^{-1} L(x(s)) \, ds =: \Phi(x, t)
\]

(3.2)

is Volterra integral equation for determination of the desired solution \(x(t)\) of Cauchy problem. It is known (see e.g., [17]), that due to the conditions of Lemma 3.1 the operator exponential on the spectrum \(\sigma(M)\) satisfies the following estimate

\[||\exp M t||_{\mathcal{L}(X \to X)} \leq C e^{-lt}, \quad C \geq 1.\]

Therefore, we have the inequality

\[||x(t)|| \leq C e^{-lt} ||\Delta|| + C ||A^{-1}|| \int_{0}^{t} e^{-l(t-s)} ||L(x(s))|| \, ds.\]

For \(\forall \varepsilon > 0\) due to the estimate \(||L(x)|| = o(||x||)\) there exists \(\delta = \delta(\varepsilon) > 0\) such as \(||A^{-1}|| ||L(x)|| \leq \varepsilon ||x||\) for \(||x|| \leq \delta\). Therefore, until \(||x(s)|| \leq \delta\) there will be inequality \(e^{lt} ||x(t)|| \leq C ||\Delta|| + C \int_{0}^{t} e^{\lambda s} \varepsilon ||x(s)|| \, ds.\) Hence, in view of the Gronwall-Bellman inequality, we have

\[e^{lt} ||x(t)|| \leq C ||\Delta|| e^{l t} \int_{0}^{t} e^{\lambda s} \varepsilon \, ds = C ||\Delta|| e^{rt}.
\]
Therefore $\|x(t)\| \leq C\|\Delta\|e^{(e-\ell)t}$ for $t > 0$. Let us select $\varepsilon \in (0, 1)$. Then $\|x(t)\| \leq C\|\Delta\|e^{-(l-\varepsilon)t} < C\|\Delta\|$ for $t \in (0, +\infty)$. Let in initial condition $x(0) = \Delta$ sufficiently small $\Delta$ is selected, i.e. $\|\Delta\| < \frac{\delta}{C\|\Delta\|}$. Since $C \geq 1$ then $\|x(0)\| < \delta$. Then $\|x(t)\| < \delta$ for $\forall t \in [0, +\infty)$ and condition $\|x\| \leq \delta$ is satisfied. Moreover, $\sup_{0 \leq t < \infty} \|x(t)\| < \delta$. For $\forall t^* \in [0, +\infty)$ $\|x(t^*)\| < \delta$. Thus, Lemma 3.1 is valid, a continuous solution exists and it is unique on semiaxis $[0, +\infty)$.

A priori assessment of the solution justifies the continuation of a local continuous solution of the Cauchy problem to the entire interval $[0, +\infty)$. In the monograph [11] the proof of the possibility of continuous continuation of the solution was also based on the presence of an a priori estimate of the solution. A priori estimate of the solution, as a rule, is used in proving nonlocal existence theorems by the method of continuation with respect to a parameter, see, for example, sec. 14 in [11].

4 Existence, uniqueness, and asymptotic stability

**Theorem 4.1** Let $(0, 0)$ be equilibrium point of system (1.1), (1.2). Let $Re\lambda \leq -l < 0$ for all $\lambda \in \sigma(M)$, $\|\Delta\|$ is sufficiently small. Then system (1.1), (1.2) with condition $x(0) = \Delta$ enjoys unique solution $x : [0, +\infty) \to X$, $u : [0, +\infty) \to U$. Moreover, $\lim_{t \to +\infty} (\|x(t)\| + \|u(t)\|) = 0$.

**Proof**

By virtue of Lemmas 1 and 2 and the obvious validity of Picard’s theorem for the equation (3.1), the Cauchy problem (1.1), (1.2), $x(t_0) = x_0$ has a unique local solution for $\forall t_0 \in [0, \infty)$. Therefore, the set of values of the arguments $t$, for which the local solution continuously extends, is open in any interval $[t_0, +\infty)$.

Since the solution of the Cauchy problem for sufficiently small $\|x(0)\|$ on the basis of Lemma 3, satisfies the a priori estimate $x(t) < \delta$ for $\forall t \in (0, +\infty)$ and does not reach $\delta$, then the set of values of the arguments $t$, on which the solution can be continued, will be closed. Therefore, on the basis of known facts about the method of continuation with respect to a parameter (see, for example, [11], sec.14) the solution of the equation (3.1) continuously extends to the entire interval $[0, +\infty)$. In view of Lemmas 1, 2 and 3, the desired functions $x(t), u(t)$ stabilize as $t \to +\infty$ to equilibrium point $(0, 0)$.

Additional information on operator $L(x)$ in equation (3.2) enables lower estimation of $\|x\|$, formulated in Theorem 1. Indeed, let positive continuous decreasing function $q(r)$ is found $q(0) = 0$ such as

$$\|L(x_1) - L(x_2)\| \leq q(r)\|x_1 - x_2\|$$

for any $x_1, x_2$ from ball $\|x\| \leq r$. Then for operator $\Phi$ in equation (3.2) we have an estimate

$$\sup_{0 \leq t < \infty} \|\Phi(x_1, t) - \Phi(x_2, t)\| \leq C\|A^{-1}\| \int_0^t e^{-l(t-s)}\|L(x_1(s)) - L(x_2(s))\| ds \leq \frac{c}{l} \|A^{-1}\| q(r) \sup_{0 \leq s < \infty} \|x_1(s) - x_2(s)\|.$$

Let us select $r^* > 0$ such as for $\forall r \leq r^*$ the following inequality

$$\frac{C}{l} \|A^{-1}\| q(r) \leq q^* < 1$$
takes place. Then operator $\Phi$ will be contracting in the ball $S(0,r^*)$ and for $\|x\| \leq r^*$ and $\forall t \in [0, +\infty)$

$$
\|\Phi(x,t)\| \leq \|\Phi(x,t) - \Phi(0,t)\| + \|\Phi(0,t)\| \leq q^* r^* +
+ \exp M|\Delta| \leq q^* r^* + C|||\Delta||.
$$

Let us now select $||\Delta|| \leq (\frac{1-q^*}{C})r^*$.

For such a sufficiently small initial value $\Delta$, the contraction operator $\Phi$ maps the ball $S(0,r^*)$ into itself and hence the Cauchy problem (5.1) has a unique global solution.

**Example 4.2**

$$
\begin{cases}
\frac{\partial x(t,z)}{\partial t} = -x(t,z) + x^3(t,z) + u^2(t,z), \\
x(0, z) = \Delta(z), \quad t \in [0, +\infty), \quad z \in [0,1], \\
u(t, z) + \int_0^1 zs u(t, s) \, ds + x^2(t,z) + u^2(t,z) = 0,
\end{cases}
$$

$|\Delta(z)| \leq \varepsilon$, $\varepsilon$ is sufficiently small. Here $A = A_1 = I$, $A_3 = 0$, $X = E = U = C_{0,1}$. Operator $A_4 = I + \int_0^1 zs[\cdot] \, ds$ has bounded inverse

$$
A_4^{-1} = I - \frac{2}{3} \int_0^1 zs[\cdot] \, ds,
$$

$M = -I$. If $|x(t,z)|$ is sufficiently small then sequence $\{u_n(t,z)\}$, where

$$
u_n(t,z) = -x^2(t,z) - u_{n-1}^2(t,z) + \frac{2}{3} \int_0^1 zs \{x^2(t,s) + u_{n-1}^2(t,s)\} \, ds,
$$

$u_0(t,z) = 0$ converges and enables the algorithm for construction of solution $u(t,z)$ of integral equation as function of $x(t,z)$. Substituting this solution into a differential equation, we reduce the problem to the following differential equation:

$$
\frac{\partial x(t,z)}{\partial t} = -x(t,z) + O(|x|^2), \quad x(0,z) = \Delta(z).
$$

Here $|x| = O(|x|^2)$.

Therefore, this model, consisting of differential and integral equations satisfies conditions of the Theorem 1 and on the semiaxis $[0, \infty)$ enjoys unique continuous solution $x(t,z)$, $u(t,z)$, stabilizing as $t \to +\infty$ to the rest point $(0,0)$ if $\max_{0 \leq z \leq 1} |\Delta(z)|$ is sufficiently small.

**5 On the construction of a solution of a nonlinear system by the method of successive approximations**

Under the conditions of Theorem 1, the desired solution $x(t), u(t)$ of the system (1.1), (1.2) with the condition $x(0) = \Delta$ can be constructed without prior system’s reduction to one differential equation. Indeed, we introduce two sequences $\{x_n(t)\}$, $\{u_n(t)\}$ with conditions $x_n(0) = \Delta$, $n = 0, 1, \ldots$, where $|||\Delta||$ is sufficiently small. Let $u_0 = 0$, and $|||\Delta||$ is sufficiently small, $x_n(t)$ is solution to Cauchy problem $A \frac{dx_n}{dt} = F(x_n, u_{n-1})$, $x_n(0) = \Delta$, $n = 1, 2, \ldots$. Obviously solution $x_n(t)$ exists and unique for $t \geq 0$ due to the Theorem 1.

Next, let us construct functions $u_n$ using the iterations $u_n = u_{n-1} + w_n$, where $u_0 = 0$. Due to invertibility of the operator $A_4$, functions $w_n$ can be found from the linear equation $A_4 w_n + G(x_n, u_{n-1}) = 0, n = 1, 2, \ldots$. Then, under Theorem 1 assumptions $\lim_{n \to \infty} x_n(t) = x(t)$, $\lim_{n \to \infty} u_n(t) = u(t)$, $\lim_{t \to \infty} (|||x(t)||| + |||u(t)|||) = 0$. It is essential to require small $|||\Delta||$ otherwise solution to nonlinear differential equation may blow-up in the point $t^*$ (refer to examples below).
Example 5.1 Let us consider the system

$$\begin{cases}
\frac{dx(t)}{dt} = -\frac{x(t)}{2} - u(t) + x^2(t), \\
0 = 2u(t) - x(t) + 2u(t)\sin u(t) - x(t)\sin u(t).
\end{cases}$$

with initial condition $x(0) = \Delta$, $0 \leq t < +\infty$. The replacement $u(t) = \frac{x(t)}{2}$ will reduce this system to Cauchy problem

$$\begin{cases}
\dot{x}(t) = -x(t) + x^2(t), \\
x(0) = \Delta.
\end{cases}$$

It is easy to verify that the latter model has an exact solution

$$x(t) = \frac{\Delta}{e^t(1 - \Delta) + \Delta}.$$ 

Let us demonstrate that point $t^* = \ln \frac{\Delta}{\Delta - 1}$ may appear to be blow-up of the constructed solution. We consider the following 4 cases.

Case 1. If $\Delta \in (0, 1)$, then blow-up point $t^*$ is complex, solution is continuous for $t \in (0, +\infty)$ and stabilizing to the rest point $x = 0$ as $t \to +\infty$ (ref. Fig. 2).

![Fig. 2. $\Delta \in (0, 1)$](image)

Case 2. If $\Delta \in (-\infty, 0)$, then blow-up point is negative, and on semiaxis $[0, +\infty)$ solution is continuous and stabilizing to the rest point $x = 0$ as $t \to +\infty$ (ref. Fig. 3).
Case 3. If $1 < \Delta < \infty$, then solution blows-up for $t^* = \ln \frac{\Delta}{\Delta^2 - 1}$, where $\frac{\Delta}{\Delta^2 - 1} > 0$. If $t > t^*$ then solution is continuous and also stabilizing to the rest point $x = 0$ as $t \to +\infty$ (ref. Fig. 4).

Case 4. For $\Delta = 0$ and $\Delta = 1$ we get the stationary solutions.

Based on the above, the following conclusion can be drawn. In example 2 for $\Delta \in (-\infty, 1)$ exists the unique solution to the Cauchy problem as $t \geq 0$, this solution stabilizing to the rest point as $t \to +\infty$. It is to be mentioned that for $\Delta > 1$ the Cauchy problem’s solution will blow-up on the finite time $\ln \frac{\Delta}{\Delta^2 - 1}$.

Remark 5.2 The absence of real rest points can generate solutions with a countable set of blow-up points.
6 Branching of the Cauchy problem’s solution

Let in (1.1), (1.2) \( E = \mathbb{R}^n, U = \mathbb{R}^m, A \) is unit \( n \times n \) matrix. The broad class of the systems (6.1), (6.2) appear in various applications

\[
\begin{aligned}
\frac{dx_i}{dt} &= \sum_{k=1}^{n} a_{ik} x_k + \sum_{k=1}^{m} b_{ik} u_k + R_i(x; u), \\
 x_i(0) &= \Delta_i, i = 1, \ldots, n,
\end{aligned}
\]

\( 0 = q_k(x; u), k = 1, \ldots, m, \) (6.2)

where

\[
\begin{aligned}
x &:=(x_1, \ldots, x_n)^T, \ u := (u_1, \ldots, u_m)^T, \ m \leq n.
\end{aligned}
\]

Let \( R_i(x, u) = o(||x|| + ||u||), \)

\[
\begin{aligned}
q_k(x, u) &= A_k(x_k, u_k) + x_k^{N_k+1} r_k(x, u), k = 1, \ldots, m,
\end{aligned}
\]

\( A_k(x_k, u_k) := \sum_{s=0}^{N_k} m_{ks} x_k^{N_k-s} u_k^s + o(||x|| + ||u||)^{N_k}, \)

\( r_k(x, u) = o(1) \ ||x|| + ||u|| \to 0, \ N_k \geq 2. \)

We will search for small solutions of (6.2) \( u = u(x) \to 0 \) as \( ||x|| \to 0 \) in the form of products \( u_i = x_i w_i(x), i = 1, \ldots, m \) under condition

\[
\sum_{i=1}^{m} |w_i(0)| \neq 0.
\]

Then functions \( w_i \) must satisfies the following system

\[
\begin{aligned}
x_k^{N_k} \sum_{s=0}^{N_k} m_{ks} w_k^s(x_k) + x_k^{N_k+1} r_k(x, x_1 w_1, \ldots, x_m w_m) &= 0, \\
k = 1, \ldots, m.
\end{aligned}
\]

After reduction, we come to the system

\[
\begin{aligned}
\sum_{s=0}^{N_k} m_{ks} w_k^s(x_k) + x_k r_k(x, x_1 w_1, \ldots, x_m w_m) &= 0, k = 1, \ldots, m.
\end{aligned}
\]

Therefore, the vector \( w(0) = (w_1(0), \ldots, w_m(0))^T \) consists of the polynomials roots

\[
\sum_{s=0}^{N_k} m_{ks} w_k^s(0) = 0, k = 1, \ldots, m.
\]

Let \( w_k^*, k = 1, \ldots, m \) be simple roots of the corresponding polynomials. On the basis of the implicit function theorem, these roots will have a small solution of systems (6.2) of the form

\[
u_k(x) = x_k w_k^* + r_k(x), k = 1, \ldots, m,
\]

where \( |r_k(x)| = o(||x||) \). Functions \( r_k(x) \) for small \( x \) can be approximated using successive approximations. Substitution of \( u_k(x) \) into differential equations (6.1) yields (similar with proof of the Lemma 2) the differential system with respect to vector-function \( x(t) : \)

\[
\frac{dx_i}{dt} = \sum_{k=0}^{n} c_{ik} x_k + o(||x||)
\]
with conditions $x_i(0) = \Delta_i$, $i = 1, \ldots, n$. Here

$$c_{ik} = \begin{cases} a_{ik}, & i = 1, \ldots, n, \ k = m + 1, \ldots, n, \\ a_{ik} + b_{ik} u^*_k, & i = 1, \ldots, n, \ k = 1, \ldots, m. \end{cases}$$

Let us introduce the matrix $M = \{c_{ik}\}_{i,k=1}^n$. Finally, we make

**Remark.** Let $\sum_{k=1}^n |\Delta_k|$ be sufficiently small. If all the eigenvalues of the matrix $M$ has negative real parts, then exists solution to problem (6.1), (6.2), this solution is unique and stabilizing to rest point $(0, 0)$ as $t \to +\infty$. Since polynomials $\sum_{k=0}^{N_k} m_k w_k^k$, $k = 1, \ldots, m$ can have several simple roots such as the corresponding matrix $M$ has only eigenvalues with negative real parts, then solution to problem (6.1), (6.2) in general case may have several stable solutions $x(t)$. The simple roots of these polynomials, under which the matrix $M$ has an eigenvalue with positive real part will correspond to an unstable solution $x(t)$.

**Example 6.1**

$$\begin{cases} \frac{dx}{dt} = \alpha x + \beta u + u^2 + x^3, \\ x(0) = \Delta, \\ ax^2 + 2bxu + u^2 = 0, 0 \leq t < \infty. \end{cases}$$

Here $(0, 0)$ is the rest point. Under assumption $u = cx$, where $c$ is const we get the following quadratic equation $c^2 + 2bc + a = 0$. Then $u$ is double-valued

$$u_{1,2} = x(t)(-b \pm \sqrt{b^2 - a}).$$

(6.3)

Let $a < b^2$.

We substitute the found values of the function $u$ into the differential equation. Then the problem of determining the function $x(t)$ is reduced to the solution of two Cauchy problems

$$\begin{cases} \frac{dx}{dt} = (\alpha + \beta(-b \pm \sqrt{b^2 - a})x_\pm + (-b \pm \sqrt{b^2 - a})^2 x_{\pm}^2 + x_{\pm}^3, \\ x_{\pm}(0) = \Delta. \end{cases}$$

Let $\alpha + \beta(-b - \sqrt{b^2 - a}) < 0$. Then exists branch $x_-(t)$ for small $|\Delta|$ for $t \geq 0$ and stabilizing to zero as $t \to +\infty$.

### 7 Possible generalizations

In this work until now only autonomous systems were considered. This can be relaxed. For example, if we have system

$$\begin{cases} A \frac{dx}{dt} = (A_1 + \tilde{A}_1(t))x(t) + (A_2 + \tilde{A}_2(t))u(t) + R(x, u, t), \\ 0 = A_3 x + A_4 u + r(x, u, t), \end{cases}$$

where $\tilde{A}_1(t) \to 0, \tilde{A}_2(t) \to 0$ as $t \to +\infty$, $||R(x, u, t)|| = o(||x|| + ||u||)$ and $||r(x, u, t)|| = o(||x|| + ||u||)$ for $||x|| + ||u|| \to 0$ coversges uniformly $t \geq 0$, then results of Theorem 1 remains correct.

In the theory of systems (1.1), (1.2), the most difficult case was when the Frechet derivative $\frac{\partial}{\partial u}G(x, u)$ is not invertible at the rest point, and therefore the implicit operator theorem is not fulfilled for the map $G(x, u) = 0$. 

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In sec. 5 only one case was addressed when this condition is not satisfied and solution is branching. Other more complex cases of branching solutions can be investigated using the results of modern analytical branching theory solutions of non-linear equations obtained in the works of V.A. Trenogin, B.V. Loginov, N.A. Sidorov, A.D. Bruno, M.G. Krein, J. Toland et al. Equally interesting is the problem of analyzing systems (1.1), (1.2) with a discontinuity in a neighborhood of the rest points, when the stability condition in the first approximation is not satisfied, and more advanced methods must be used, for example, methods related to the construction Lyapunov functions, to evaluate the location of potential blow-up points using method of convex majorants of L.V. Kantorovich used in works [13–19].

In this case, when developing algorithms for analyzing stability and constructing estimates of the regions of attraction of the rest points of the power systems of input-output type, it is expedient to use methods based on the theory of the Lyapunov vector-function.

Finally, it is interesting to consider the system (1.1), (1.2) with rest points for an irreversible operator \( A \). In this case, the standard Cauchy problem can has no classical solutions and it is advisable to introduce other initial conditions. If the irreversible operator \( A \) admits a finite-length skeleton decomposition, then new correct initial conditions for the problem (1.1), (1.2) can be formulated using the results of the works [13, 21].

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