Anisotropic Compact Star Model on Finch–Skea Space-Time

Ankita Jangid1*, B. S. Ratanpal2**, and K. K. Venkataratnam1***

1Department of Physics, Malaviya National Institute of Technology, Jaipur, 302017 India
2Department of Applied Mathematics, Faculty of Technology and Engineering, The Maharaja Sayajirao University of Baroda, Vadodara, 390001 India

Received September 20, 2022; revised November 3, 2022; accepted December 5, 2022

Abstract—We demonstrate a new anisotropic solution to the Einstein field equations in Finch–Skea space-time. The physical features of a stellar configuration have been studied in previous investigations. We create a model that meets all physical plausibility conditions for a variety of stars and plot graphs for 4U 1820-30.

DOI: 10.1134/S0202289323020068

1. INTRODUCTION

In astrophysics, compact stars are usually the endpoint of stellar evolution. These compact objects have incredibly high densities compared to other atomic matter. In astronomy, researchers attracted much attention to studying white dwarfs, neutron stars, and black holes, commonly referred to as compact stars, because they have similar structures and characteristics. In compact stars, the matter could be anisotropic. The nature of anisotropy can change the evolution, and subsequently the physical properties of stellar objects. Understanding of the nonnegligible effects of anisotropy on the stellar object parameters such as mass, pressure, composition, etc., is essential for a viable physical model of a compact star.

In a stellar configuration, anisotropy can happen for a variety of reasons, including the presence of a solid core, type $P$ superfluid, phase transitions, mixing of two fluids, the presence of an external field, etc. The authors of [1–3] have looked at models of a compact star as uncharged spheres with isotropic pressures. Neutral anisotropic matter was examined in [4–7]. Among the authors who highlight charged isotropic compact models are those of [8–12]. The general model with charge and anisotropy was analyzed in [13–15]. Ref. [16] is a pioneering work on compact objects. The authors described a hydrostatic equilibrium after studying static anisotropic spherically symmetric configurations. They conclude that the anisotropy may not have been disregarded when calculating the mass and surface redshift of compact stars. Ref. [17] argued that matter might be anisotropic when it comes to objects with densities significantly higher than the nuclear density. According to [18], superfluid development inside the star may also cause anisotropy in pressure to increase. Ref. [19] proposed a number of models with spherical anisotropic distributions at constant densities. The identical task was considered in [20], but with variable densities. Ref. [21] explored how an anisotropy in pressure affected the mass, structure, and physical characteristics of compact objects and obtained an equation of state relating radial and tangential pressure in various versions of the precise solution. For spherically symmetric static anisotropic stellar configuration, [22–24] all provide a class of exact solutions to the Einstein field equations. An anisotropy was suggested to be a crucial prerequisite for a dense matter regime in [25, 26]. Ref. [27] purposefully introduced a local anisotropy in the self–gravitating system. In modern research, as proposed by [28], in relativistic compact objects, a pressure anisotropy cannot be ignored. The investigations in [29, 30] highlighted a number of intriguing characteristics of exact solutions to the Einstein–Maxwell system for charged anisotropic quark stars. The existence of charge and anisotropy in a stellar interior has been a subject of numerous recent studies [31]. Generalized isothermal models were studied in [32] and superdense models were examined in [33]. The study of [34] contains further new precise solutions for charged anisotropic stars. Ref. [35] discovered exact models for charged anisotropic materials with a quadratic equation of state. In [36, 37], charged stellar models with Van der Waals and modified Van der Waals equations of state, respectively, were considered.

In this study, we will produce a solution to Ein-
stein’s field equations with the use of the Finch–Skea ansatz [39], for anisotropic fluid distributions that are static and spherically symmetric. The metric potential of the Finch–Skea ansatz assumed a form used by numerous investigators [40–42], to create solutions to the Einstein field equation in various astrophysical contexts. We assume that matter inside the fluid sphere is uncharged, and the anisotropy plays a significant role. In this model, we use the compact object 4U 1820-30 to compare our model to the observational data.

The work is organized as follows: Section 2 contains the Einstein field equations and their solutions. The integration constants are obtained in Section 3. Section 4 contains a physical analysis, and we conclude the work in Section 5.

2. EINSTEIN’S FIELD EQUATIONS AND THEIR SOLUTIONS

Consider a static spherically symmetric space-time metric as
\[ ds^2 = e^\nu(r) dt^2 - e^{\lambda(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \]  
(1)
where \( \lambda \) and \( \nu \) are unknown functions of the radial coordinate \( r \). The stress-energy-momentum tensor for an anisotropic matter distribution is of the form
\[ T_{ij} = (\rho + p)u_iu_j - pg_{ij} + \pi_{ij}, \]  
(2)
where \( \rho \) and \( p \) are the energy density and isotropic pressure, respectively, \( u^i \) is the radial 4-velocity vector, and \( \pi_{ij} \) is the anisotropic stress tensor
\[ \pi_{ij} = \sqrt{3} S \left[ C_i C_j - \frac{1}{3} (u_i u_j - g_{ij}) \right]. \]  
(3)
The nonvanishing components of the energy-momentum tensor are
\[ T^{00} = \rho, \quad T^{11} = -(p + 2S/\sqrt{3}), \]  
\[ T^{22} = T^{33} = p - S/\sqrt{3}. \]  
(4)
The relations between the radial pressure, tangential pressure, and anisotropic pressure are
\[ p_r = p + \frac{2S}{\sqrt{3}}, \]  
(5)
\[ p_t = p - \frac{S}{\sqrt{3}}, \]  
(6)
\[ p_r - p_t = \sqrt{3} S. \]  
(7)
The Einstein field equations corresponding to the metric (1) and the energy-momentum tensor (2) are obtained as (\( G = c = 1 \))
\[ 8\pi \rho = \frac{1}{r^2} - e^{-\lambda} \left( \frac{1}{r^2} - \frac{\lambda'}{r} \right), \]  
(8)
\[ 8\pi p_r = e^{-\lambda} \left( \frac{1}{r^2} + \frac{\nu'}{r} \right) - \frac{1}{r^2}, \]  
(9)
\[ 8\pi p_t = e^{-\lambda} \frac{1}{4} \left[ 2\nu'' + (\nu' - \lambda') \left( \nu' + \frac{2}{r} \right) \right], \]  
(10)
\[ 8\pi \sqrt{3} S = 8\pi p_r - 8\pi p_t. \]  
(11)
We have a system of five unknown functions, namely, \( e^\nu, e^\lambda, \rho, p_r, p_t \). To obtain the space-time metric for a stellar configuration to solve the model, we use the well-known Finch–Skea ansatz [42] for the metric potential \( g_{rr} \), to find closed-form solutions for a static anisotropic matter distribution with spherically symmetry, which has a clear geometric interpretation for the associated background space-time.

We assume the Finch–Skea ansatz for the metric as
\[ e^\lambda = 1 + \frac{r^2}{R^2}, \]  
(12)
\[ e^\nu = F^2, \]  
(13)
then Eq. (11) is written as
\[ F'' - \left( \frac{1 + 2r^2/R^2}{r(1 + r^2/R^2)} \right) F' - \frac{1 + 2r^2/R^2}{r^2(1 + r^2/R^2)} \]  
\[ + \left( 8\pi \sqrt{3} S + \frac{1}{r} \right) \left( 1 + \frac{1}{R^2} \right) F = 0. \]  
(14)
We assume \( 8\pi \sqrt{3} S \) as
\[ 8\pi \sqrt{3} S = \frac{r^2/R^2 - 4r^4/R^4}{4r^2(1 + r^2/R^2)^3}, \]  
(15)
this choice is physically viable since \( 8\pi \sqrt{3} S = 0 \) at \( r = 0 \).

The solution of Eq. (14) is given by
\[ F = C \left( 1 + \frac{r^2}{R^2} \right)^{5/2} + D \left( 1 + \frac{r^2}{R^2} \right)^{1/2}, \]  
(16)
where \( C \) and \( D \) are integration constants. This leads to
\[ e^\nu = \left[ C \left( 1 + \frac{r^2}{R^2} \right)^{5/4} + D \left( 1 + \frac{r^2}{R^2} \right)^{1/4} \right]^2, \]  
(17)
and the density, radial pressure, and tangential pressure then take the form
\[ 8\pi \rho = \frac{(r^2 + 3R^2)}{(r^2 + R^2)^2}, \]  
(18)
\[ 8\pi p_r = \frac{-Dr^2R^2 + C(-r^4 + 3r^2R^2 + 4R^4)}{(r^2 + R^2)^2(DR^2 + C(r^2 + R^2))}, \]  
(19)
\[ 8\pi p_t = \frac{-5Dr^2R^4 + C(11r^4R^2 + 27r^2R^4 + 16R^6)}{4(r^3 + R^3)^3(DR^2 + C(r^2 + R^2))}. \]  
(20)
The space–time metric (1) now takes the form
\[
ds^2 = \left[ C \left( 1 + \frac{r^2}{R^2} \right)^{5/4} + D \left( 1 + \frac{r^2}{R^2} \right)^{1/4} \right]^2 dt^2
- \left( 1 + \frac{r^2}{R^2} \right) dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2).
\] (21)

The integration constants will be calculated in the next section.

3. INTEGRATION CONSTANTS

The space–time metric (21) should continuously match with the Schwarzschild exterior metric
\[
ds^2 = \left( 1 - \frac{2M}{r} \right) dt^2 - \left( 1 - \frac{2M}{r} \right)^{-1} dr^2
- r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\] (22)
at the boundary of the star \( r = a \), and \( p_r|_{r=a} = 0 \). These two conditions give
\[
M = \frac{a^3}{2(R^2 + a^2)},
\] (23)
\[
C = a^2 \left\{ \left( 1 + \frac{a^2}{R^2} \right)^{3/4} \left[ a^2 \left( 1 + \frac{a^2}{R^2} \right) + R^2 \left( -\frac{a^4}{R^4} + \frac{3a^2}{R^2} + 4 \right) \right] \right\}^{-1},
\] (24)
\[
D = \frac{4R^2 - a^2}{4R^2 (1 + a^2/R^2)^{3/4}}.
\] (25)

Substituting the values of \( C \) and \( D \) into Eqs. (18)–(20), we get
\[
8\pi \rho = \frac{(r^2 + 3R^2)}{(r^2 + R^2)^2},
\] (26)
\[
8\pi p_r = \frac{a^4 r^2 - 4a^2 R^4 - a^2 (r^4 - 4R^4)}{(r^2 + R^2)^2 (-a^4 + 4R^4 + a^2 (r^2 + 4R^2))},
\] (27)
\[
8\pi p_t = R^2 \left[ 5a^4 r^2 - 20a^2 R^4 + a^2 (11r^2 + 12r^2 R^2 + 16R^4) \right]
\times \left\{ (r^2 + R^2)^3 \left[ -a^4 + 4R^4 + a^2 (r^2 + 4R^2) \right] \right\}^{-1}.
\] (28)

4. PHYSICAL ANALYSIS

The distribution of anisotropic material described in the background of spherically symmetric space-time contains two geometric parameters \( a \) and \( R \). The radius of the star is given by the parameter \( a \). The following are the physical plausibility requirements to be used in determining the bounds on the other parameter \( R \):

\[
(i) \quad \rho(r), p_r(r), p_t(r) \geq 0 \text{ for } 0 \leq r \leq a.
\]
\[
(ii) \quad \frac{dp}{dr}, \frac{dp_r}{dr}, \frac{dp_t}{dr} < 0 \text{ for } 0 \leq r \leq a.
\]
\[
(iii) \quad \rho + p_r > 0 \text{ for } 0 \leq r \leq a.
\]
\[
(iv) \quad \rho + p_r + 2p_t \geq 0 \text{ for } 0 \leq r \leq a.
\]
\[
(v) \quad \rho - p_r - 2p_t \geq 0 \text{ for } 0 \leq r \leq a.
\]
\[
(vi) \quad 0 < \frac{dp_r}{dp} < 1, \quad 0 < \frac{dp_t}{dp} < 1 \text{ for } 0 \leq r \leq a.
\]

For any given values of \( R \) and \( a \), the condition \( \rho(r) \geq 0 \) is always satisfied throughout the distribution. Similarly, considering all the relevant inequalities, we found lower and upper admissible bound for \( R \) given by the following conditions.

The condition \( p_t(r) \geq 0 \) imposes a restriction on the value of \( R \) as \( 0 < R < 2a \).

In order to investigate the trace energy condition, we assess the expression \( \rho - p_r - 2p_t > 0 \) at \( r = 0 \).
Table 1. The estimated and observed values for compact stars

| S. no. | Star         | $a$ (km) | $M \odot$ | $R(a/\sqrt{2} < R < 2a)$ (km) | $\rho_o$ (MeV/Fm$^3$) | $\rho_R$ (MeV/Fm$^3$) |
|--------|--------------|----------|-----------|-------------------------------|-----------------------|----------------------|
| 01     | 4U 1820-30   | 9.1      | 1.58      | 6.5                           | 2138.24               | 403.491              |
| 02     | Her-X 1     | 8.1      | 0.85      | 5.785                         | 2698.79               | 509.268              |
| 03     | SMC X-1     | 8.831    | 1.29      | 6.307                         | 2270.49               | 428.447              |
| 04     | 4U 1538-52  | 7.866    | 0.87      | 5.618                         | 2861.75               | 540.019              |
| 05     | PSR J1903+327 | 9.43    | 1.667     | 6.735                         | 1991.21               | 375.745              |
| 06     | Vela X-1    | 9.56     | 1.77      | 6.828                         | 1937.42               | 365.595              |
| 07     | LMC X-4     | 8.301    | 1.04      | 5.929                         | 2569.68               | 484.904              |
| 08     | PSR J1614-2230 | 9.69   | 1.97      | 6.921                         | 1885.78               | 355.852              |

and at $r = a$. That leads to a lower bound of $R$, namely, $R > a/\sqrt{2}$.

Thus to represent a relativistic star in the model, we should require $a/\sqrt{2} < R < 2a$.

A star with radius $a$ and curvature in the valid range of $a/\sqrt{2} < R < 2a$, would satisfy all the physical plausibility conditions. Just for plotting the graphs and Table 1, we chose a particular choice of $R = 0.71a$, in the valid range of $R$, for which the model meets all the physical plausibility criteria. We have further confirmed that our model agrees well with the estimated $R$ of several compact stars like 4U 1820-30, Her-X 1, SMC X-1, 4U 1538-52, PSR J1903+327, Vela X-1, LMC X-4, PSR J1614-2230,
and demonstrated plots for the star 4U 1820-30. The radial pressure, tangential pressure, and density should be positive throughout the mass distribution. Figs. 1–3 show that these conditions are satisfied, and the relevant quantities decrease with mass. The gradient of density and pressures, all of these being negative throughout the distribution, are plotted in Figs. 4–6. The condition $\rho \geq 0$ is related to the weak energy condition, which is also satisfied. The third condition is the null energy condition, which is $\rho + p_r > 0$, it is fulfilled. Further, the strong energy condition, $\rho + p_r + 2p_t \geq 0$ and the trace energy condition $\rho - p_r - 2p_t \geq 0$ are also satisfied.
throughout the distribution, as shown in Figs. 7, 8. The fifth condition is called the causality condition, followed by the star 4U 1820–30, as shown in Figs. 9, 10.

It is clear from Fig. 11 that the adiabatic index $\Gamma > 4/3$. The variation of the anisotropy factor is shown in Fig. 12, and the computed values of the central and surface density for the particular choice of $R = 0.71a$ are shown in Table 1.

5. CONCLUSIONS

In this work, we have obtained an exact non-singular solution of Einstein’s field equations using the Finch–Skea [39] metric potential. The salient features of the model are that all the physical plausibility conditions are satisfied for the stars 4U 1820–30, Her–X 1, SMC X–1, 4U 1538–52, PSR J1903+327, Vela X–1, LMC X–4, PSR J1614–2230. We have calculated the mass $M$, the central and surface density for these stars, and this has been shown in Table 1, and we have plotted the physical parameters’ behavior for the star 4U 1820–30. According to [38], the adiabatic index is $\Gamma > 4/3$, the model is potentially stable. The data demonstrate that the model satisfies all physical criteria for the star 4U 1820–30.

ACKNOWLEDGMENTS

AJ and BSR are grateful to IUCAA, Pune, for their hospitality and the workspace, they were given while working on this project.

CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

REFERENCES

1. M. H. Murad and N. Pant, “A class of exact isotropic solutions of Einstein’s equations and relativistic stellar models in general relativity,” Astroph. Space Sci. 350, 349–359 (2014).
2. M. K. Mak and T. Harko, “Relativistic compact objects in isotropic coordinates,” Pramana 65, 185–192 (2005).
3. R. Sharma, S. Karmakar, and S. Mukherjee, “Maximun mass of a class of cold compact stars,” Int. J. Mod. Phys. D 15, 405–418 (2006).
4. B. C. Paul, P. K. Chattopadhyay, S. Karmakar, and R. Tiikekar, “Relativistic strange stars with anisotropy,” Mod. Phys. Lett. A 26, 575–587 (2011).
5. T. Harko and M. K. Mak, “Anisotropic relativistic stellar models,” Ann. der Physik 11, 3–13 (2002).
6. F. Rahaman et al., “Strange stars in Krori–Barua space-time,” Eur. Phys. J. C 72, 1–9 (2012).
7. M. Kalam, A. A. Usmani, et al., “A relativistic model for strange quark star,” Int. J. Theor. Phys. 52, 3319–3328 (2013).
8. S. K. Maurya and Y. K. Gupta, “A family of well-behaved charge analogues of a well-behaved neutral solution in general relativity,” Astroph. Space Sci. 332, 481–490 (2011).
9. S. K. Maurya and Y. K. Gupta, “A class of charged analogues of Durgapal and Fuloria superdense star,” Astroph. Space Sci. 331, 135–144 (2011).
10. R. P. Negreiros and F. Weber, “Electrically charged strange quark stars,” Phys. Rev. D 80, 083006 (2009).
11. M. H. Murad and S. Fatema, “A family of well-behaved charge analogues of Durgapal’s perfect fluid exact solution in general relativity II,” Astroph. Space Sci. 344, 69–78 (2013).
12. N. Bijalwan, “Charged analogues of Schwarzschild interior solution in terms of pressure,” Astroph. Space Sci. 336, 413–418 (2011).
13. M. Esculpi and E. Aloma, “Conformal anisotropic relativistic charged fluid spheres with a linear equation of state,” Eur. Phys. J. C 67, 521–532 (2010).
14. P. Maia Takisa and S. D. Maharaj, “Compact models with regular charge distributions,” Astroph. Space Sci. 343, 569–577 (2013).
15. F. Rahaman et al., “Anisotropic strange star with de Sitter spacetime,” Eur. Phys. J. C 72, 1–7 (2012).
16. R. L. Bowers and E. P. T. Liang, “Anisotropic spheres in general relativity,” Astroph. J. 188, 657 (1974).
17. M. Ruderman, “Pulsars: Structure and dynamics,” Ann. Rev. Astron. Astroph. 10, 427–476 (1972).
18. R. Kippenhahn and A. Weigert, “Stellar structure and evolution,” Astron. Astroph. 192, 1027 (1990).
19. S. D. Maharaj and R. Maartens, “Anisotropic spheres with uniform energy density in general relativity,” Gen. Rel. Grav. 21, 899–905 (1989).
20. M. K. Gokhroo and A. L. Mehra, “Anisotropic spheres with variable energy density in general relativity,” Gen. Rel. Grav. 26, 75–84 (1994).
21. K. Dev and M. Gleiser, “Anisotropic stars: Exact solutions,” Gen. Rel. Grav. 34, 1793–1818 (2002).
22. K. N. Singh and N. Pradhan, “A New charged anisotropic compact star model in general relativity,” Int. J. Theor. Phys. 54, 3408 (2015).
23. K. N. Singh and N. Pant, “Charged anisotropic superdense stars with constant stability factor,” Astroph. Space Sci. 358, 1–13 (2015).
24. K. N. Singh, and N. Pant, “Singularity free charged anisotropic solutions of Einstein–Maxwell field equations in general relativity,” Indian J. Phys. 90, 843–851 (2016).
25. M. K. Mak and T. Harko, “Anisotropic stars in general relativity,” Proc. Roy. Soc. London, Ser. A: Math., Phys., Engineering Sciences 459, 393–408 (2003).
26. R. Sharma, S. Mukherjee, and S.D. Maharaj, “General solution for a class of static charged spheres,” Gen. Rel. Grav. 33, 999–1009 (2001).
27. L. Herrera and N. O. Santos, “Local anisotropy in self-gravitating systems,” Phys. Rep. 286, 53–130 (1997).
28. L. Herrera, “Stability of the isotropic pressure condition,” Phys. Rev. D 101, 104024 (2020).
29. S. D. Maharaj, J. M. Sunzu, and S. Ray, “Some simple models for quark stars,” Eur. Phys. J. Plus 129, 1–10 (2014).
30. J. M. Sunzu, S. D. Maharaj, and S. Ray, “Charged anisotropic models for quark stars,” Astroph. Space Sci. 352, 719–727 (2014).
31. S. D. Maharaj and P. Mafa Takisa, “Regular models with quadratic equation of state,” Gen. Rel. Grav. 44, 1419–1432 (2012).
32. S. D. Maharaj and S. Thirukkanesh, “Generalized isothermal models with strange equation of state,” Pramana 72, 481–494 (2009).
33. S. K. Maurya and Y. K. Gupta, “A family of anisotropic super-dense star models using a space-time describing charged perfect fluid distributions,” Phys. Scripta 86, 025009 (2012).
34. P. Takisa and S. D. Maharaj, “Some charged polytropic models,” Gen. Rel. Grav. 45, 1951–1969 (2013).
35. T. Feroze and A. A. Siddiqui, “Charged anisotropic matter with a quadratic equation of state,” Gen. Rel. Grav. 43, 1025–1035 (2011).
36. M. Malaver, “Regular model for a quark star with Van der Waals modified equation of state,” World Applied Programming 3, 309–313 (2013).
37. M. Malaver, “Analytical model for charged polytropic stars with Van der Waals modified equation of state,” Am. J. Astron. Astroph. 1, 41–46 (2013).
38. B. S. Ratanpal, “Cracking and stability of non-rotating relativistic spheres with anisotropic internal stresses,” IOP Science 1, 025207 (2020).
39. M. R. Finch and J. E. F. Skea, “A realistic stellar model based on an ansatz of Duorah and Ray,” Class. Quantum Grav. 6, 467 (1989).
40. B. S. Ratanpal and R. Sharma, “A realistic stellar model admitting a quadratic equation of state,” Int. J. Mod. Phys. D 22, 1350074 (2013).
41. R. Tikekar and V. O. Thomas, “A relativistic core-envelope model on pseudospheroidal space-time,” Pramana J. Phys. 64, 5–15 (2005).
42. D. M. Pandya, V. O. Thomas, and R. Sharma, “Modified Finch and Skea stellar model compatible with observational data,” Astrophys. Space Sci. 356, 285 (2015).