Consequence of Modified Boundary Condition On Natural Convection in a Porous Medium Saturated by Nanofluid – A Computational Approach

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Abstract: In the present numerical simulation, steady, laminar, two-dimensional flow in a porous medium saturated by nanofluid [1] along an isothermal vertical plate is presented. Here we have studied a more realistic situation where the nanoparticle volume fraction at the plate (boundary condition) is passively controlled by assuming that its flux there is zero. We employ Boungiorno model [2] that treats the nanofluid as a two-component mixture, incorporating the effects of Brownian motion and thermophoresis. Darcy model is utilized for the presence of porous medium. With the help of appropriate similarity variables, the governing nonlinear partial differential equations of flow are changed to a bunch of nonlinear ordinary differential equations. Afterwards, they are reduced to a first order system and integrated using Newton Raphson and adaptive Runge-Kutta methods. The computer codes are developed for this numerical analysis in Matlab environment. Dimensionless stream function (s), longitudinal velocity (s’), temperature (θ) and nanoparticle volume fraction (f) are figured and outlined graphically for various values of four dimensionless parameters, namely, Lewis number (Le), buoyancy-ratio parameter (Nr), Brownian motion Parameter (Nb), and thermophoresis parameter (Np). The dependence of the reduced Nusselt number on these parameters is illustrated.

Keywords: Brownian Motion, Isothermal Vertical Plate, Natural Convection, NanoFluid, Porous Medium, Thermophoresis.

1. Introduction

Nanofluids are a comparatively new class of liquids which comprise of a base liquid with nano-sized particles (1 – 100 nm) suspended. It is introduced by Choi in 1995. Nanofluids have novel properties that make them potentially useful in many applications in heat transfer: transportation (engine cooling/vehicle thermal management), nuclear system cooling, electronics cooling, solar water heating, heat exchanger, biomedicine, fuel cells etc. Nanofluid in porous media is an emerging topic, as heat transfer efficiency is optimized by porous media. Cheng and Minkowycz [3] presented similarity solutions for free convective heat transfer from a vertical plate in a fluid-saturated porous medium. Gorla and Tornabene [4] and Gorla and Zinolabedini [5] solved problems of free convective heat transfer from a vertical plate embedded in a saturated porous medium with constant, arbitrarily varying surface temperature or heat flux. The case of combined convection from vertical plates in porous media was studied by Minkowycz et al. [6] and Ranganathan and Viskanta [7]. All of these studies were concerned with Newtonian fluid flows. The boundary layer flows in nanofluids have been analyzed recently by [8, 9].

In those and other related papers, a constant volume fraction of nanoparticles at the surface of the plate was assumed by the authors, but no information was given as how to achieve this. In the present numerical study, the boundary
condition on nanoparticle volume fraction at the surface of the plate was reformulated such that its mass flux is zero there.

2. Mathematical Model

We suppose the natural convection flow to be steady, laminar, two dimensional. The direction along the vertical plate is \( x \), and the direction normal to surface is \( y \) (Fig 1).

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 0 \quad 1 \]

\[ \frac{\partial \phi}{\partial x} = -\frac{\mu}{\kappa} u + [(1 - \phi)\rho_f \beta \beta(T - T_\infty) - (\rho - \rho_f) \beta(\phi - \phi_0)] \quad 2 \]

\[ \frac{\partial \phi}{\partial y} = 0 \quad 3 \]

\[ u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha_m \nabla^2 T + \tau \left[ D_B \frac{\partial \phi}{\partial y} \frac{\partial T}{\partial y} + \left( \frac{\nabla^2 T}{\tau_m} \right) \left( \frac{\partial T}{\partial y} \right)^2 \right] \quad 4 \]

\[ \frac{1}{\varepsilon} \left( \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \right) = \frac{D_B}{\nabla^2 T} \left( \frac{\partial \phi}{\partial y} + \left( \frac{\nabla^2 T}{\tau_m} \right) \left( \frac{\partial T}{\partial y} \right)^2 \right) \quad 5 \]

\[ \alpha_m = \frac{k_m}{\rho_c f}, \quad \tau = \frac{(\rho_c)_{np}}{\rho_c f} \quad 6 \]

where, Darcy velocity components along \( x \) and \( y \) are represented by \( u \) and \( v \), respectively, \( T_x \) and \( \phi_0 \) are temperature and nanoparticle volume fraction far away from the plate respectively, \( \rho, \mu \) and \( \beta \) are the density, viscosity and volumetric thermal expansion coefficient of the fluid, \( \rho_f \) is the density of the nanoparticles, \( k_m \) and \( (\rho_c)_m \) are
effective thermal conductivity and effective heat capacity of the porous medium respectively. \( g \) is the acceleration due to gravity, \((\rho c)_f\) is the heat capacity of the fluid, \((\rho c)_p\) is the effective heat capacity of the nanoparticle material, \(D_B\) and \(D_T\) are the Brownian diffusion coefficient and thermophoretic diffusion coefficient, respectively. \( \varepsilon \) and \( K \) are porosity and permeability of the porous medium. The boundary conditions on the solution are:

\[
\begin{align*}
  & \text{At } y = 0: \quad v = 0, \quad T = T_w, \quad \frac{D_B}{e} \frac{\partial \phi}{\partial y} = 0, \\
  & \text{For large } y: \quad u = v = 0, \quad T = T_\infty, \quad \phi = \phi_\infty.
\end{align*}
\]

The last statement of (7c) indicates the modified boundary condition on nanoparticle volume fraction at the surface of the plate. It is reformulated such that its mass flux at the plate is zero. The continuity equation (1) is naturally fulfilled through presentation of the stream function:

\[
\begin{align*}
  u &= \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \\
  \phi &= \phi_\infty, \quad T = T_\infty
\end{align*}
\]

And, we can eliminate pressure \( p \) from Eqs (2) and (3) by cross-differentiation, and get the following equations from eqs (2) to (5):

\[
\begin{align*}
  \frac{\partial^2 \psi}{\partial y^2} &= \frac{(1 - \phi_\infty)\rho_f\beta g K \frac{\partial T}{\partial y} - (\rho_p - \rho_f)gK \frac{\partial \phi}{\partial y}}{\mu}, \quad 10 \\
  \frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial y} &= a_{in} V^2 T + \tau [D_B \frac{\partial \phi}{\partial y} + \frac{D_T}{\eta_a} \frac{\partial^2 T}{\partial y^2}] \\
  \frac{1}{\varepsilon} \left( \frac{\partial \psi}{\partial y} \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \phi}{\partial y} \right) &= D_B \frac{\partial^2 \psi}{\partial y^2} + D_T \frac{\partial^2 T}{\partial x^2}, \quad 12
\end{align*}
\]

The following similarity variables are introduced:

\[
\begin{align*}
  \eta &= \frac{y}{R_0 \sqrt{x}} \quad 13 \\
  \sigma &= \frac{\psi}{a_{in} R_0 \sqrt{x}} \quad 14 \\
  \theta &= \frac{T - T_\infty}{T_w - T_\infty} \quad 15 \\
  f &= \frac{\phi - \phi_\infty}{\phi_\infty} \quad 16
\end{align*}
\]

As a result, equations (10), (11) and (12) may be modified as (with a prime signifying differentiation with respect to \( \eta \)).
Then the appropriate boundary conditions are:
\begin{align}
  s & = 0, \theta = 1, Nbf' + Nt\theta' = 0 \quad \text{at } \eta = 0 \quad \text{(20a, b, c)} \\
  s' & = 0, \theta = 1, f = 0 \quad \text{at } \eta = \infty \quad \text{(21a, b, c)}
\end{align}

where Local Rayleigh number $Ra_x$, the buoyancy-ratio parameter ($Nr$), Brownian motion parameter ($Nb$), thermophoresis parameter ($Nt$) and Lewis number ($Le$) are defined as follows:

\begin{align}
  Ra_x & = \frac{(1 - \varphi_{\infty})\rho_{\infty} \beta \gamma Kx}{\mu \alpha_m} \\
  Nr & = \frac{(\rho_p - \rho_{\infty})\varphi_{\infty}}{\rho_{\infty} \beta (T_w - T_{\infty}) (1 - \varphi_{\infty})} \\
  Nb & = \frac{\nu (\rho c) \gamma D_B \varphi_{\infty}}{(\rho c) \gamma \alpha_m} \\
  Nt & = \frac{\nu (\rho c) \gamma D_T (T_w - T_{\infty})}{(\rho c) \gamma \alpha_m T_w} \\
  Le & = \frac{\alpha_m}{\epsilon D_B}
\end{align}

Eq (17) after integration once with boundary conditions (21) gives
\[ s' - \theta + Nr f = 0 \quad \text{(27)} \]

Associated boundary conditions are
\[ \text{At } \eta \to \infty, \quad \theta = 0, f = 0 \quad \text{(28a, b)} \]

Local Nusselt number is defined as
\[ Nu = \frac{q''x}{h_m (T_w - T_{\infty})} \quad \text{(29)} \]

where $q''$ is the wall heat flux. The reduced local Nusselt number is
\[ Nu_{\text{reduced}} = \frac{Nu}{Ra_x^{1/3}} = -\theta' (0) \quad \text{(30)} \]

3. Solution Procedure
Eqs (274), (18) and (19) are coupled nonlinear higher order ordinary differential equations. There are three unknown initial values at the wall, \( \theta(0), f(0) \) and \( f'(0) \).

3.1 Reduction of Equations to First-order System

We reduce the order easily by introducing new variables:

\[
\begin{align*}
    z_1 &= s \\
    z_2 &= \theta \\
    z_3 &= z_2' = \theta' \\
    z_4 &= f \\
    z_5 &= z_4' = f'
\end{align*}
\]

Therefore, from eqs (27), (18) and (19), we get the following set of differential equations

\[
\begin{align*}
    z_1' &= s' = \theta - N_r f = z_2 - N_r z_4 \\
    z_2' &= z_3 \\
    z_3' &= z_2'' = \theta'' = -0.5s\theta' - N_b f'\theta' - N_t \theta'^2 = -0.5z_1z_3 - N_b z_5 z_3 - N_t z_3^2 \\
    z_4' &= z_5 \\
    z_5' &= z_4'' = f'' = -0.5L_s f' - (N_t/N_b)\theta'' = -0.5L_s z_1 z_5 - (N_t/N_b)(-0.5z_1z_3 - N_b z_5 z_3 - N_t z_3^2) 
\end{align*}
\]

Eq (27) is first-order and is replaced by a first-order equation (32a), whereas eqs (18) and (19) are second-order each and are replaced with two first-order equations each 32(b, c) and 32(d, e, f) respectively. The boundary conditions (28) and (20) then become:

\[
\begin{align*}
    z_1(0) &= s(0) = 0 \\
    z_2(0) &= \theta(0) = 1 \\
    N_b z_5(0) + N_t z_3(0) &= N_b f'(0) + N_t \theta'(0) = 0 \\
    z_2(\infty) &= \theta(\infty) = 0 \\
    z_4(\infty) &= f(\infty) = 0 
\end{align*}
\]

3.2 Conversion to Initial Value Problems

To solve eqs (27, 18, 19), we denote the two unknown initial values \( z_3(0) \) and \( z_4(0) \) by \( a_1 \) and \( a_2 \) respectively, the set of initial conditions is then:

\[
\begin{align*}
    z_1(0) &= s(0) = 0 \\
    z_2(0) &= \theta(0) = 1 \\
    z_3(0) &= \theta'(0) = a_1 \\
    z_4(0) &= f(0) = a_2 \\
    z_5(0) &= f'(0) = -(N_t/N_b) a_1 
\end{align*}
\]

To solve eqs (32), we utilize adaptive Runge-Kutta method along with the initial conditions in eq (34). The calculated boundary values at \( \eta = \infty \) depend on the selection of \( a_1 \) and \( a_2 \) respectively. We express this dependence as

\[
\begin{align*}
    z_2(0) &= \theta(0) = f_1(a_1) \\
    z_4(0) &= f(0) = f_2(a_2) 
\end{align*}
\]

The correct choice of \( a_1, a_2 \) and \( a_3 \) yields the given boundary conditions at \( \eta = \infty \); that is, it satisfies the equations

\[
\begin{align*}
    z_2(0) &= \theta(0) = f_1(a_1) = 0 
\end{align*}
\]
z_0 = f(0) = f_z(a_0) = 0

These nonlinear equations can be solved by the Newton-Raphson method. A value of 10 is all right for infinity, with increasing values of \( \eta_{\text{max}} \), no significant changes are observed in the results. Pr, Le, Nr, Nb and Nt are parameters.

3.3 Program Details

Using Newton Raphson and adaptive Runge-Kutta methods a set of Matlab routines for the solution of eqs (32) along with the boundary conditions (34) are developed. The final output of these codes give the tabulated values of \( s, s', 0, \theta', f, f' \) as functions of \( \eta \) with Le, Nr, Nb and Nt are parameters. The set of discrete values of the parameters Le, Nr, Nb and Nt of the present study used in the codes are given in the Table 1.

| Input parameters | values              |
|------------------|---------------------|
| Le               | 1, 10, 100, 1000    |
| Nr               | 0.1, 0.2, 0.3, 0.4, 0.5 |
| Nb               | 0.1, 0.2, 0.3, 0.4, 0.5 |
| Nt               | 0.1, 0.2, 0.3, 0.4, 0.5 |

4. Results and Discussion

4.1 Boundary layer profiles.

Distributions of dimensionless stream function \( s \), longitudinal velocity \( s' \), temperature \( \theta \) and nanoparticle volume fraction \( f \) for four different values of Le with Nr = Nb = Nt = 0.5 obtained from the codes are shown in Figs. 2(a) – (d).
Fig. 2b.

Fig. 2c.

Fig. 2d.

Fig. 2a – Fig. 2d. Plots of dimensionless similarity functions $s(\eta)$, $s'(\eta)$, $\theta(\eta)$, $f(\eta)$ for $Nr = Nb = Nt = 0.5$. 
It is clear that the profiles for the temperature function \( \theta (\eta) \) and the stream function \( s(\eta) \) possess similar form to the case of a regular fluid. For regular fluid, the profile for longitudinal component of velocity, \( s'(\eta) \) is identical with that for temperature, \( \theta (\eta) \). It is noteworthy from the above plots that these two profiles diverge when \( Le \) is relatively small but tends to coincide when \( Le \) is large.

### 4.2 Correlation

The codes developed in the present study, have been run for 125 sets of values of \( Nr, Nb, Nt \) in the range \([0.1, 0.2, 0.3, 0.4, 0.5]\) with \( Le = 10 \), and the reduced Nusselt number (eq. 30) from the solutions are shown in Table 2. The linear regression, performed on the results, yielded the correlation

\[
N_{\text{urest}} = 0.4433 - 0.0003Nr + 0.0007Nb - 0.1045Nt
\]

with a maximum error about 1%. This may be put side by side with the correlation where the nanoparticle volume fraction at the plate is actively controlled [8], which was

\[
N_{\text{urest}} = 0.444 - 0.111Nr - 0.245Nb - 0.150Nt
\]

Table 2 Values of \( -\theta'(0) \) with \( Le = 10 \).

| \( Nb \) | \( Nt \) | \( Nr = 0.1 \) | \( Nr = 0.2 \) | \( Nr = 0.3 \) | \( Nr = 0.4 \) | \( Nr = 0.5 \) |
|---|---|---|---|---|---|---|
| 0.1 | 0.1 | 0.4311 | 0.4332 | 0.4333 | 0.4334 | 0.4335 |
| 0.2 | 0.4224 | 0.4226 | 0.4226 | 0.4226 | 0.4224 |
| 0.3 | 0.4120 | 0.4121 | 0.4119 | 0.4115 | 0.4109 |
| 0.4 | 0.4017 | 0.4017 | 0.4012 | 0.4004 | 0.3992 |
| 0.5 | 0.3917 | 0.3915 | 0.3907 | 0.3891 | 0.3875 |
| 0.2 | 0.1 | 0.4330 | 0.4331 | 0.4331 | 0.4332 | 0.4333 |
| 0.2 | 0.4223 | 0.4224 | 0.4225 | 0.4226 | 0.4226 |
| 0.3 | 0.4118 | 0.4120 | 0.4121 | 0.4121 | 0.4120 |
| 0.4 | 0.4016 | 0.4017 | 0.4018 | 0.4017 | 0.4015 |
| 0.5 | 0.3916 | 0.3917 | 0.3917 | 0.3915 | 0.3912 |
| 0.3 | 0.1 | 0.4329 | 0.4330 | 0.4331 | 0.4331 | 0.4332 |
| 0.2 | 0.4222 | 0.4223 | 0.4224 | 0.4225 | 0.4226 |
| 0.3 | 0.4117 | 0.4119 | 0.4120 | 0.4120 | 0.4121 |
| 0.4 | 0.4015 | 0.4017 | 0.4018 | 0.4018 | 0.4018 |
| 0.5 | 0.3916 | 0.3917 | 0.3917 | 0.3917 | 0.3916 |
| 0.4 | 0.1 | 0.4329 | 0.4330 | 0.4331 | 0.4331 | 0.4331 |
| 0.2 | 0.4222 | 0.4223 | 0.4223 | 0.4224 | 0.4224 |
| 0.3 | 0.4117 | 0.4119 | 0.4120 | 0.4120 | 0.4120 |
| 0.4 | 0.4015 | 0.4016 | 0.4017 | 0.4017 | 0.4018 |
| 0.5 | 0.3915 | 0.3916 | 0.3917 | 0.3917 | 0.3917 |
| 0.5 | 0.1 | 0.4329 | 0.4330 | 0.4331 | 0.4331 | 0.4331 |
| 0.2 | 0.4222 | 0.4223 | 0.4223 | 0.4224 | 0.4224 |
| 0.3 | 0.4117 | 0.4119 | 0.4119 | 0.4119 | 0.4120 |
| 0.4 | 0.4015 | 0.4016 | 0.4017 | 0.4017 | 0.4017 |
| 0.5 | 0.3915 | 0.3916 | 0.3917 | 0.3917 | 0.3917 |

and had a maximum error of about 12%. With the modified boundary condition on nanoparticle volume fraction at the plate, \( N_{\text{urest}} \) turns out to be almost independent of the Brownian motion parameter \( Nb \) and the buoyancy-ratio parameter \( (Nr) \) [eq. (37)], whereas these parameters have significant effects in case actively controlled nanoparticle
volume fraction at the plate [eq (38)]. Eq (37) demonstrates the reduced Nusselt number decreases as the parameter Nt (thermophoresis parameter) increases, which increases thermal boundary layer.

5. Conclusion and Outlook

The present numerical study deals with the effect of modified boundary condition on natural convection in a porous medium saturated by nanofluid. Detailed numerical procure is outlined. The dependence of the reduced Nusselt number (Nur) on the pertinent four dimensionless parameters, namely, Lewis number (Le), buoyancy-ratio parameter (Nr), Brownian motion Parameter (Nb), and thermophoresis parameter (Nt) is given in tabulated form. A linear regression correlation between them is also developed. The reduced Nusselt number (Nur) is found to be a decreasing function of Nt (Thermophoresis parameter), and practically independent of Nb (Brownian motion parameter) and Nr (buoyancy-ratio parameter), whereas, in the case of actively controlled nanoparticle volume fraction at the plate, Nb and Nr have significant effects

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