The renormalization structure and quantum equivalence of 2D dilaton gravities

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Abstract

The one-loop effective action corresponding the general model of dilaton gravity given by the Lagrangian
\[ L = -\sqrt{g} \left[ \frac{1}{2} Z(\Phi) g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \Phi + C(\Phi) R + V(\Phi) \right], \]
where \( Z(\Phi), C(\Phi) \) and \( V(\Phi) \) are arbitrary functions of the dilaton field, is found. The question of the quantum equivalence of classically equivalent dilaton gravities is studied. By specific calculation of explicit examples it is shown that classically equivalent quantum gravities are also perturbatively equivalent at the quantum level, but only on-shell. The renormalization group equations for the generalized effective couplings \( Z(\Phi), C(\Phi) \) and \( V(\Phi) \) are written. An analysis of the equations shows, in particular, that the Callan-Giddings-Harvey-Strominger model is not a fixed point of these equations.

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1 Introduction

2D dilaton gravity constitutes a very nice example of a toy model for 4D quantum gravity, a theory that has not been formulated yet in a consistent way. The study of 2D dilaton gravity can throw new light into some of the general properties of 4D gravity and, of course, it may actually predict some new unexpected phenomena.

In particular, one of the topics which is currently under discussion in 2D dilaton gravity (motivated by the recent identification of black holes in string theories [1]) concerns the quantum structure of 2D dilaton gravity and of 2D black holes [1-9] (for a review and extended list of references see [6]). There is the hope that the longstanding mystery concerning the Hawking evaporation of black holes [11] will be solved in a theory of 2D gravity with matter (see [2-4,6] for a discussion of recent progress on this point).

Actually, two different models of 2D dilaton gravity have been discussed in the literature. A very general model of such theory, which is multiplicatively renormalizable (first ref. of [7],[8] and [15]) is given by the following action

\[ S = - \int d^2 x \sqrt{g} \left[ \frac{1}{2} Z(\Phi) g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi + C(\Phi) R + V(\Phi) \right] , \]  

where \( Z(\Phi), C(\Phi) \) and \( V(\Phi) \) are some functions of the dilaton field \( \Phi \). This theory can be considered as a kind of \( \sigma \)-model or one-dimensional string (see, for example [12]). A popular choice of these functions —which corresponds to the model of Callan, Giddings, Harvey and Strominger (CGHS) [2] is the following

\[ Z(\Phi) = 8e^{-2\Phi}, \quad C(\Phi) = e^{-2\Phi}, \quad V(\Phi) = 4\lambda^2 e^{-2\Phi}. \]

One can also consider other choices for the functions in (1). Some of them lead to nonsingular theories [8].

Let us now discuss an important problem which appears in the different 2D quantum gravities based on the action (1). The idea is the following. One can start from some particular model of the family (1), with specified functions \( Z(\Phi) = Z_1(\Phi), C(\Phi) = C_1(\Phi) \) and \( V(\Phi) = V_1(\Phi) \) (for instance, a model motivated by string theory), or from a different one (let us say coming from a \( \sigma \) model as the one above), with corresponding functions \( Z_2(\Phi), C_2(\Phi) \) and \( V_2(\Phi) \). Before going on, we shall perform the transformation

\[ g_{\mu\nu} \rightarrow e^{-2\rho(\varphi)} \bar{g}_{\mu\nu}, \quad \Phi \rightarrow f(\varphi), \]  

where \( \rho(\varphi) \) and \( f(\varphi) \) are functions to be determined.
and work with the new variables, what renders some expressions more simple. So the
formal setting is that we have two theories of the class (1) characterized by two different
sets of functions: \( \{ Z_1(\Phi), C_1(\Phi), V_1(\Phi) \} \) and \( \{ Z_2(\Phi), C_2(\Phi), V_2(\Phi) \} \). Of course, at the
classical level these two theories are equivalent and lead to the same classical physics.
Now, the natural question is if this equivalence will be maintained at the quantum level.
Are any two classically equivalent theories of the family (1) also quantum equivalent?
If the answer is no, then the physics of these two theories will be different. (This is
certainly an important question which also showed up in the early days of string theories
and concerned the classical, semiclassical and quantum equivalence of the different string
models.) In fact there are some indications [4] that classical equivalence does not carry
over to the quantum level.

The present work is devoted to the study of the effective action and to the question of
quantum equivalence in 2D gravity within the covariant perturbative approach. We will
calculate the one-loop effective action and shall show that, in general, classically equivalent
2D dilatonic gravities are not quantum equivalent off-shell. The paper is organized as
follows. In the next section we discuss a popular model of dilatonic gravity. We calculate
the one-loop effective action in two minimal gauges. After that, the one-loop effective
action in a classically equivalent version of the same dilaton gravity in one of the two
gauges considered (conveniently transformed) is obtained. Comparison of the results will
show, in fact, that these two models of dilaton gravity that are classically equivalent are
also quantum equivalent on-shell. In sect. 3 the one-loop effective action for the general
model (1) of dilaton gravity interacting with a Maxwell field is found. It is shown that this
effective action is given by a total derivative term on shell (finiteness property). Sect. 4 is
devoted to the renormalization group analysis of the generalized effective couplings \( Z(\Phi),
C(\Phi) \) and \( V(\Phi) \). Some variants of fixed points for renormalization group \( \beta \)-functions are
presented. Sect. 5 is devoted to conclusions. We also make there some final remarks on
our results. In an appendix we show that the results of sect. 3 provide also the one-loop
counterterms corresponding to 2D \( R^2 \)-gravity.
2 One-loop effective action and quantum equivalence of dilatonic gravities: an example

In this section we will study the popular version of 2D dilatonic gravity which is given by the action

\[ S_1 = - \int d^2 x \sqrt{g} \left[ \frac{1}{2} g^{\mu \nu} \partial_\mu \Phi \partial_\nu \Phi + CR \Phi + V(\Phi) \right], \]  

where \( C \) is a positive constant and \( V(\Phi) \) an arbitrary function. The one-loop renormalization of the theory (3) has been performed in refs. [7] in different covariant gauges.

Let us here briefly summarize the results of refs. [7,14] concerning the one-loop effective action. We use the background field method

\[ g_{\mu \nu} \rightarrow \bar{g}_{\mu \nu} = g_{\mu \nu} + h_{\mu \nu}, \quad \Phi \rightarrow \bar{\Phi} = \Phi + \phi, \]  

where \( h_{\mu \nu} \) and \( \phi \) are the quantum fields. The simplest minimal covariant gauge is given by

\[ S_{GF} = -\frac{1}{2} \int d^2 x c_{\mu \nu} \chi^\mu \chi^\nu, \]  

where

\[ c_{\mu \nu} = -C \Phi \sqrt{g} g_{\mu \nu}, \quad \chi^\mu = -\nabla^\nu \bar{h}_\nu + \frac{1}{\Phi} \nabla^\mu \phi, \]  

and \( \bar{h}_{\mu \nu} = h_{\mu \nu} - \frac{1}{2} g_{\mu \nu} h \). The divergences of the one-loop effective action (including all surface terms) have been calculated in ref. [14]

\[ \Gamma_{\text{div}} = -\frac{1}{2\epsilon} \int d^2 x \sqrt{g} \left[ 4R + \frac{2}{C \Phi} V + \frac{2}{C} V' + \left( \frac{1}{\Phi} - \frac{1}{C} \right) (\Delta \Phi) - \frac{3}{\Phi^2} (\nabla^\lambda \Phi)(\nabla_\lambda \Phi) \right]. \]  

Notice that in the first and third references of [7] such calculation has been done without taking into account the surface terms and also, that the theory (3) is not invariant under the change \( \Phi \rightarrow -\Phi \) (no \( Z_2 \) symmetry). Hence, the classical restriction \( \Phi \geq 0 \) seems reasonable here (see the paper by Hamada and Tsuchiya in [4]). This renders the discussion of the quantum dynamics of black holes quite difficult (usually, however, this restriction has been ignored).

Let us now make in the theory (3) the field transformation

\[ \Psi^2 = \frac{C}{\gamma} \Phi^2, \quad g_{\mu \nu} \rightarrow e^{-2\rho} \bar{g}_{\mu \nu}, \]  

where

\[ \gamma > 0, \quad \rho = \frac{\gamma \Psi^2}{4C^2} - \frac{1}{8\gamma} \ln \Psi. \]
Then, action (3) becomes

\[ S_2 = - \int d^2 x \sqrt{g} \left[ \frac{1}{2} \bar{g}^{\mu\nu} \partial_\mu \Psi \partial_\nu \Psi + \gamma R \Psi^2 + U(\Psi) \right], \tag{9} \]

where we have defined \( U(\Psi) \equiv e^{-2\rho V(\Phi(\Psi))} \) and dropped off a total derivative term. The actions \( S_1 \) and \( S_2 \) (eqs. (3) and (9), respectively) belong to the same class (1) but are parametrized through different triplets of functions \( \{Z, C, V\} \). They are classically equivalent and lead to the same classical physics.

We shall now investigate the one-loop effective action for the theory (9). The calculation will be done in the same gauge (5)-(6), that is also to be transformed in accordance with (8). So, the natural prescription is

1. For the background fields we shall make the transformation (8), where \( \Psi \) and \( \bar{g}_{\mu\nu} \) will be now the background fields of the theory (9).

2. The quantum fields will be transformed according to the first order Taylor expansion of eq. (8), that is

\[ \varphi \rightarrow \frac{2\gamma}{C} \Psi \eta, \quad h_{\mu\nu} \rightarrow e^{-2\rho(\Psi)} \left[ \bar{h}_{\mu\nu} + \left( \frac{1}{4\gamma \Psi} - \frac{\gamma}{C^2 \Psi} \right) \bar{g}_{\mu\nu} \eta \right]. \tag{10} \]

We should recall now that in the background field method for the theory (9),

\[ \bar{g}_{\mu\nu} \rightarrow \bar{g}_{\mu\nu} + \bar{h}_{\mu\nu}, \quad \Psi \rightarrow \Psi + \eta, \tag{11} \]

where \( \bar{h}_{\mu\nu} \) and \( \eta \) are the quantum fields. Notice also that from the functional integral point of view the change of variables (10) is local.

Taking into account all the remarks above, we get the following covariant gauge for the theory (9):

\[ S_{GF} = - \frac{1}{2} \int d^2 x \, c_{\mu\nu} \chi^\mu \chi^\nu, \tag{12} \]

where

\[ c_{\mu\nu} = -\gamma \Psi^2 \sqrt{g} g_{\mu\nu}, \]
\[ \chi^\mu = -\nabla^\nu \bar{h}_{\mu\nu} + \left( \frac{\gamma}{C^2 \Psi} - \frac{1}{4\gamma \Psi} \right) (\nabla^\nu \Psi) \bar{h}_{\mu\nu} + \frac{2}{\Psi} \nabla^\mu \eta + \frac{2}{\Psi^2} (\nabla^\mu \Psi) \eta. \tag{13} \]

In order to simplify notation, in what follows we shall suppress tildas over \( g_{\mu\nu} \) and \( h_{\mu\nu} \). Since we are working with theory (9) only, this should not lead to any confusion. On the
other hand, it is no surprise at all that the gauge (12) is minimal again (namely a minimal gauge is mapped into another minimal gauge).

For the calculation of the one-loop effective action we will use the standard technique

\[
\Gamma_{\text{div}} = \frac{i}{2} \text{Tr} \ln \hat{H} = \left. \frac{i}{2} \text{Tr} \ln (\hat{1} \Delta + 2 \hat{E}^\lambda \nabla_\lambda + \hat{\Pi}) \right|_{\text{div}}
\]

\[
= \frac{1}{2\epsilon} \int d^2x \sqrt{g} \text{Tr} \left[ \hat{\Pi} + \frac{R}{6} - \hat{E}_\lambda \hat{E}_\lambda - \nabla_\lambda \nabla_\lambda \right],
\]

(14)

where \( \epsilon = 2\pi(n - 2) \) and dimensional regularization has been used.

The total quadratic expansion of the action (9) with gauge-fixing term (12) can be written as follows

\[
- \frac{1}{2} \varphi^i \hat{H} \varphi^j \equiv - \frac{1}{2} \varphi^i \left[ - \hat{K}_{ij} \Delta + \hat{L}_{\lambda,ij} \nabla^\lambda + \hat{P}_{ij} \right] \varphi^j,
\]

(15)

where \( \varphi^i \equiv \{ \eta, h, \bar{h}_{\mu\nu} \} \) and

\[
\hat{K}_{ij} = \begin{pmatrix}
1 - 4\gamma & \gamma \Psi & 0 \\
\gamma \Psi & 0 & 0 \\
0 & 0 & -\frac{1}{2} \gamma \Psi^2 P_{\mu\nu,\alpha\beta}
\end{pmatrix},
\]

(16)

where \( P_{\mu\nu,\alpha\beta} = \delta_{\mu\nu,\alpha\beta} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \) and

\[
\hat{L}_i^\lambda = -\frac{8\gamma}{\Psi} (\nabla^\lambda \Psi), \quad \hat{L}_i^{\lambda,21} = -\hat{L}_i^{\lambda} = \gamma (\nabla^\lambda \Psi),
\]

\[
\hat{L}_i^{\lambda,31} = \hat{L}_i^{\lambda,31} = \left( \frac{1}{2} + \frac{2\gamma^2 \Psi^2}{C^2} \right) (\nabla_\omega \Psi) P^{\alpha\beta,\lambda\omega},
\]

\[
\hat{L}_2^2 = -\gamma \Psi (\nabla^\lambda \Psi), \quad \hat{L}_2^{\lambda,23} = -\hat{L}_2^{\lambda} = \gamma \Psi (\nabla_\omega \Psi) P^{\alpha\beta,\lambda\omega},
\]

\[
\hat{L}_3^\lambda = \left( \frac{\gamma^2 \Psi^3}{C^2} + 2\gamma \Psi - \frac{\Psi}{4} \right) (\nabla^\omega \Psi) (P^{\mu\nu,\alpha\beta} - P^{\mu\nu,\alpha\beta} P_{\omega\kappa} P_{\alpha\beta,\lambda\kappa}) - 3\gamma \Psi (\nabla^\lambda \Psi) P^{\mu\nu,\alpha\beta},
\]

\[
\hat{P}_{12} = \hat{P}_{21} = \frac{1}{2} U', \quad \hat{P}_{22} = 0,
\]

\[
\hat{P}_{33} = -\frac{1}{2} \left[ \gamma \Psi^2 R + U + \frac{1}{2} (\nabla_\lambda \Psi) (\nabla^\lambda \Psi) \right] P^{\mu\nu,\alpha\beta}
\]

\[
+ \left[ \frac{5}{4} - 4\gamma - \frac{1}{16} \gamma - \frac{3\gamma^2 \Psi^2}{C^2} + \frac{\gamma \Psi^2}{2 C^2} - \frac{\gamma^3 \Psi^4}{C^4} \right] (\nabla^\omega \Psi) (\nabla_\omega \Psi) P^{\mu\nu,\alpha\beta} P_{\omega\kappa} P_{\alpha\beta,\lambda\kappa},
\]

\[
+ \left[ \frac{\Psi}{4} - 4\gamma \Psi - \frac{\gamma^2 \Psi^3}{C^2} \right] (\nabla^\omega \nabla_\lambda \Psi) P^{\mu\nu,\alpha\beta} P_{\omega\kappa} P_{\alpha\beta,\lambda\kappa}.
\]

(17)

It is easy to see that the operator \( \hat{H} \) is not uniquely defined, since arbitrary integrations by parts can be performed. In order to eliminate this possibility and end up with a uniquely
defined hermitean operator, the doubling trick of ’t Hooft and Veltman [13] is very useful. Applying it amounts to doing the following redefinitions in \( \hat{H} \) (15):

\[
\hat{H} \rightarrow \hat{H}' = -\hat{K} \Delta + \hat{L}_\lambda \nabla^\lambda + \hat{P}',
\]

\[
\hat{L}_\lambda = \frac{1}{2}(\hat{L}_\lambda - \hat{L}_\lambda^T) - \nabla^\lambda \hat{K},
\]

\[
\hat{P}' = \frac{1}{2}(\hat{P} + \hat{P}^T) - \frac{1}{2} \nabla^\lambda \hat{L}_\lambda^T - \frac{1}{2} \Delta \hat{K},
\]

(18)

Introducing the notations \( \hat{E}^\lambda = -(1/2)\hat{K}^{-1}\hat{L}^\lambda \) and \( \hat{\Pi} = -\hat{K}^{-1}\hat{P}' \), the operator \( \hat{H}' \) can be put in the form

\[
\hat{H}' = -\hat{K}(\hat{1}\Delta + 2\hat{E}^\lambda \nabla_\lambda + \hat{\Pi}).
\]

(19)

where

\[
(\hat{E}^\lambda)^1 = \frac{1}{2}(\nabla^\lambda \Psi), \quad (\hat{E}^\lambda)^2 = 0, \quad (\hat{E}^\lambda)^3 = \frac{1}{2}(\nabla^\lambda \Psi)P^{\alpha\beta,\lambda\omega},
\]

\[
(\hat{E}^\lambda)^2 = \frac{4\gamma - 1}{\gamma \Psi^2}(\nabla^\lambda \Psi), \quad (\hat{E}^\lambda)^2 = 0, \quad (\hat{E}^\lambda)^3 = \left( \frac{1}{4\gamma \Psi} - \frac{2}{\Psi} - \frac{\gamma \Psi}{C^2} \right)(\nabla^\lambda \Psi)P^{\alpha\beta,\lambda\omega},
\]

\[
\hat{\Pi}^1 = \frac{1}{2\gamma \Psi} \nabla - \frac{1}{2\gamma \Psi} \nabla U', \quad \hat{\Pi}^2 = -\frac{1}{2\gamma \Psi} \nabla U + \frac{1}{2\gamma \Psi} \nabla \Delta \Psi + \frac{1}{2\gamma \Psi} \nabla (\nabla^\lambda \Psi)(\nabla_\lambda \Psi),
\]

\[
\hat{\Pi}^3 = \left[ \frac{8\gamma - 1}{2\gamma \Psi^2}(\nabla^\lambda \Psi)(\nabla_\lambda \Psi) + \frac{4}{\Psi} \nabla \Delta \Psi - \frac{1}{\gamma \Psi^2} \nabla U \right]P^{\alpha\beta},
\]

\[
+ \left[ \frac{1}{2\gamma \Psi} \nabla - \frac{8}{\Psi} - \frac{2\gamma}{C^2} \right](\nabla^\lambda \nabla_\lambda \Psi)P^{\alpha\beta,\lambda\omega} + \frac{20\gamma - 64\gamma^2 - 1}{8\gamma^2 \Psi^2} + \frac{1}{C^2} - \frac{2\gamma P^{\alpha\beta}}{\Psi \Psi^2} \right](\nabla^\lambda \Psi)(\nabla_\lambda \Psi)
\]

(20)

Notice that we do not need the off-diagonal components of \( \hat{\Pi} \).

Evaluating the functional traces in (14), we get

\[
\Gamma_{2-div} = -\frac{1}{2\epsilon} \int d^4x \sqrt{g} \left[ \frac{4}{3} R + \frac{2}{\gamma \Psi^2} U + \frac{1}{\gamma \Psi} U' + \left( \frac{4\gamma - 1}{\gamma \Psi} + \frac{2\gamma \Psi}{C^2} \right)(\Delta \Psi) \right.
\]

\[
+ \left. \left( \frac{2\gamma}{C^2} - \frac{2}{\Psi^2} \right)(\nabla^\lambda \Psi)(\nabla_\lambda \Psi) \right].
\]

(21)

The ghost operator turns out to be

\[
\mathcal{M}_{\nu} = g_{\nu} \Delta + \left( \frac{1}{4\gamma \Psi} - \frac{\gamma \Psi}{C^2} \right)(\nabla^\nu \Psi)\nabla_\lambda \left( g_{\nu} g^\lambda \lambda + g_{\nu} g^\lambda - g_{\nu} g^\lambda \right)
\]

\[
- \frac{2}{\Psi} (\nabla_\nu \Psi)\nabla^\nu - \frac{1}{\Psi^2} (\nabla^\nu \nabla_\nu \Psi) - \frac{2}{\Psi^2} (\nabla_\nu \Psi)(\nabla^\nu \Psi) + R_{\nu},
\]

(22)
and the divergent ghost contribution can be written as

\[ \Gamma_{\text{ghost-div}} = -\frac{1}{2\epsilon} \int d^2 x \sqrt{g} \left[ \frac{8}{3} R + \left( \frac{2\gamma \Psi}{C^2} - \frac{1 + 4\gamma}{2\gamma \Psi} \right) (\Delta \Psi) \right. \]

\[ + \left. \left( \frac{2\gamma}{C^2} + \frac{1 - 16\gamma}{2\gamma \Psi^2} \right) (\nabla^\lambda \Psi)(\nabla_\lambda \Psi) \right]. \tag{23} \]

The total one-loop divergence, \( \Gamma_{\text{div}} = \Gamma_{2-\text{div}} + \Gamma_{\text{ghost-div}} \), is given by

\[ \Gamma_{\text{div}} = -\frac{1}{2\epsilon} \int d^2 x \sqrt{g} \left[ 4R + \frac{2}{\gamma \Psi^2} U + \frac{1}{\gamma \Psi} U' + \left( \frac{4\gamma - 3}{2\gamma \Psi} + \frac{4\gamma \Psi}{C^2} \right) (\Delta \Psi) \right. \]

\[ + \left. \left( \frac{4\gamma}{C^2} + \frac{1 - 20\gamma}{2\gamma \Psi^2} \right) (\nabla^\lambda \Psi)(\nabla_\lambda \Psi) \right]. \tag{24} \]

We must observe that in this final formula all surface terms have been kept. A few remarks are in order. First of all, let us do an integration by parts in (24) and drop the surface terms. We get

\[ \Gamma_{\text{div}} = -\frac{1}{2\epsilon} \int d^2 x \sqrt{g} \left[ \frac{2}{\gamma \Psi^2} U + \frac{1}{\gamma \Psi} U' - \frac{1 + 8\gamma}{\gamma \Psi^2} g^{\mu\nu}(\nabla_\mu \Psi)(\nabla_\nu \Psi) \right]. \tag{25} \]

The one-loop renormalized action is given by

\[ S_R = S - \Gamma_{\text{div}}. \tag{26} \]

Making use of the renormalization transformation

\[ g^R_{\mu\nu} = \exp \left( \frac{1 + 8\gamma}{8\epsilon \gamma^2 \Psi^2} \right) g^R_{\mu\nu}, \tag{27} \]

where \( g^R_{\mu\nu} \) is the renormalized metric, we get the one-loop renormalized action

\[ S_R = -\int d^2 x \sqrt{g_R} \left[ \frac{1}{2} g^R_{\mu\nu} \partial_\mu \Psi \partial_\nu \Psi + \gamma \Psi^2 R_R + U + \frac{U'}{8\gamma^2 \epsilon \Psi^2} - \frac{U'}{2\epsilon \gamma \Psi} \right]. \tag{28} \]

It follows from (28) that the theory under discussion is one-loop off-shell renormalizable in the usual sense for the following choice of \( U \):

\[ U = \exp \left( \frac{1}{4\gamma} \ln \Psi - a_1 \gamma \Psi^2 + a_2 \right), \tag{29} \]

where \( a_1 \) and \( a_2 \) are arbitrary constants.

Let us now discuss the on-shell limit of \( \Gamma_{\text{div}} \), eq. (24). Keeping all the surface terms and using the classical field equations resulting from the action (9), namely

\[ -\Delta \Psi + 2\gamma R \Psi + U' = 0, \quad -\gamma \Delta \Psi^2 + U = 0, \tag{30} \]
we obtain
\[ \Gamma_{\text{off-shell}} = -\frac{1}{2\epsilon} \int d^2x \sqrt{g} \left[ 2R + \Delta \left( \frac{12\gamma - 1}{2\gamma} \ln \Psi + \frac{2\gamma}{C^2} \Psi^2 \right) \right]. \] (31)

Notice that this is a total derivative.

As a second example of the theory (9) we will consider the gauge fixing action of the following form
\[ S_{GF} = -\frac{1}{2} \int d^2x c_{\mu\nu} \chi^\mu \chi^\nu, \] (32)
where \( c_{\mu\nu} = -\gamma \Psi^2 \sqrt{g} g_{\mu\nu} \) and \( \chi^\mu = -\nabla^\nu h^\mu_\nu + \frac{2}{\Psi} \nabla^\mu \eta \). Repeating the calculation above with this new gauge choice, one finds the following final result for the one-loop effective action:
\[ \Gamma_{\text{div}} = -\frac{1}{2\epsilon} \int d^2x \sqrt{g} \left[ 4R + \frac{2}{\gamma \Psi^2} U + \frac{1}{\gamma \Psi} U' + \frac{4\gamma - 1}{2\gamma \Psi} \Delta \Psi - \frac{1 + 20\gamma}{2\gamma \Psi^2} \left( \nabla^\lambda \Psi \right) \left( \nabla_\lambda \Psi \right) \right]. \] (33)
As we can see, \( \Gamma_{\text{div}} \), eq. (33), differs from \( \Gamma_{\text{div}} \), eq. (24), in surface terms (a total derivative). After integration by parts, eq. (33) becomes eq. (25). Hence, the off-shell one-loop renormalization is the same in both gauges (32) and (5). However, on shell \( \Gamma_{\text{div}} \) in eq. (33), differs from eq. (31) in some total derivative terms. Summing up, we see that the on-shell effective action in both gauges is given by surface divergences only (finiteness of the \( S \) matrix), but these terms depend yet on the choice of gauge condition.

We are now going to investigate the theory (3) in the variables (8). Transforming \( \Gamma_{\text{div}} \), eq. (7), to the new variables (8), we get
\[ \Gamma_{\text{div}} = -\frac{1}{2\epsilon} \int d^2x \sqrt{g} \left[ 4R + \frac{2}{\gamma \Psi^2} U + e^{-2\rho(\Psi)} \frac{2}{\gamma} \frac{\partial V(\Psi)}{\partial \Psi} \right. \\
+ \left. \left( \frac{4\gamma - 2}{2\gamma \Psi} + \frac{2\gamma \Psi}{C^2} \right) \Delta \Psi + \left( \frac{2\gamma}{C^2} + \frac{2 - 20\gamma}{2\gamma \Psi^2} \right) \left( \nabla^\lambda \Psi \right) \left( \nabla_\lambda \Psi \right) \right], \] (34)
where all surface terms have been kept. As we see, there is no perturbative quantum equivalence between the two classically equivalent dilaton gravities (3) and (9). This result is a clear confirmation of the preliminary conclusions in ref. [4]. In fact, the one-loop effective action (34) which comes from action (3) in the gauge (5) does not coincide with \( \Gamma_{\text{div}} \), eq. (24), which is obtained when starting from the classically equivalent theory (9) in the gauge (12) —that can be made correspond with the gauge (5). There are, however, some similarities between the two results.
In particular, the theory (3) in the gauge (5) is one-loop off-shell multiplicatively renormalizable in the usual sense for the following potential [7]:

\[ V(\Phi) = e^{\alpha \Phi} + \Lambda, \]  

(35)

where \( \alpha \) and \( \Lambda \) are arbitrary constants. Making use of the transformation (8) we obtain

\[ U(\Psi) = e^{-2\rho(\Psi)}V(\Psi) = \exp \left( \frac{1}{4\gamma} \ln \Psi - \frac{\gamma \Psi^2}{4C^2} \right) \left( e^{\ln \Lambda} + e^{\alpha \gamma \Psi^2/C} \right). \]  

(36)

This function \( U \) belongs to the same class as the \( U \) of eq. (29). We thus see from our example that two classically equivalent dilaton gravities lead to the same class of multiplicatively renormalizable potentials (which are of Liouville form).

Let us now consider the on-shell effective action of the theory (3). Using classical field equations and keeping all the total derivative terms in (7), we get

\[ \Gamma_{\text{on-shell}}^{\text{div}} = -\frac{1}{2\epsilon} \int d^2 x \sqrt{g} \left[ 2R + \Delta \left( \frac{1}{C} \Phi + 3 \ln \Phi \right) \right]. \]  

(37)

Transforming the variables in (37) according to the change (8), we obtain

\[ \Gamma_{\text{on-shell}}^{\text{div}} = -\frac{1}{2\epsilon} \int d^2 x \sqrt{g} \left[ 2R + \Delta \left( \frac{2\gamma \Psi^2}{C^2} + \frac{12\gamma - 1}{2\gamma} \ln \Psi \right) \right]. \]  

(38)

This expression coincides completely with eq. (31).

From the discussion above, we conclude that there is on-shell perturbative quantum equivalence for the classically equivalent dilaton gravities (3) and (9). This statement is not true off-shell, generally speaking, as follows from the preceding analysis.

3 The one-loop effective action in a general model of 2D dilaton-Maxwell gravity

In this section we will present an analysis of the one-loop effective action in a general model of 2D dilaton-Maxwell gravity. It is defined by the following action

\[ S = -\int d^2 x \sqrt{g} \left[ \frac{1}{2} Z(\Phi) g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + C(\Phi) R + V(\Phi) + \frac{1}{4} f(\Phi) F^2_{\mu\nu} \right]. \]  

(39)

We shall use again the background field method. Fields will be split according to (4) with the additional expression \( A_\mu \rightarrow A_\mu + Q_\mu \). In this section we will continue denoting
C \equiv C(\Phi) \text{ (now } C \text{ is a function, not a constant as in the previous section). The model with the action (39) is connected (via some compactification) with the four-dimensional Einstein-Maxwell theory, which admits charged black hole solutions [10]. Particular cases of (39) describe the bosonic string and the heterotic string effective actions, respectively.}

The simplest minimal gauge fixing is

\[ S_{GF} = -\frac{1}{2} \int d^2x \chi^A c_{AB} \chi^B, \]  

where \( A \equiv \{\mu, *\} \), \( c_{\mu\nu} = -C\sqrt{g} g_{\mu\nu}, \chi^\mu = -\nabla_\nu \tilde{h}^{\mu\nu} + \frac{C'}{C} \nabla^\mu \varphi, \ C_* = \sqrt{gf}, \chi_* = -\nabla^\nu Q_\nu. \)

Let us introduce \( \Phi^i = \{Q_\mu, \varphi, h, \bar{h}_{\mu\nu}\}. \)

Now the quadratic contribution takes the usual form

\[ S_{(2)}^{\text{tot}} = \int d^2x \sqrt{g} \Phi^i \left( -\hat{K} \Delta + \hat{L}^\lambda \nabla_\lambda + \hat{P}_{ij} \Phi^j, \right), \]  

where

\[ \hat{K}_{ij} = \begin{pmatrix} f g^{\mu\alpha} & 0 & 0 & 0 \\ 0 & Z - \frac{C^2}{C'} & C'/2 & 0 \\ 0 & C'/2 & 0 & 0 \\ 0 & 0 & 0 & -\frac{C}{2} P_{\mu\nu,\alpha\beta} \end{pmatrix}, \]

so that

\[ \hat{K}^{-1}_{ij} = \begin{pmatrix} \frac{1}{2} g_{\rho\mu} & 0 & 0 & 0 \\ 0 & 0 & 2/C' & 0 \\ 0 & 2/C' \left( \frac{4}{C'} - \frac{4Z}{C^2} \right) & 0 \\ 0 & 0 & 0 & -\frac{2}{C} P_{\rho\sigma,\mu\nu} \end{pmatrix}, \]

and the other essential matrix elements are:

\[ \hat{L}_{11}^\lambda = f'(\nabla^\alpha \Phi) g^{\mu\lambda} - f'(\nabla^\mu \Phi) g^{\alpha\lambda} - f'(\nabla^\lambda \Phi) g^{\mu\alpha}, \]

\[ \hat{L}_{12}^\lambda = -\hat{L}_{21}^\lambda = f' F^{\mu\lambda}, \]

\[ \hat{L}_{13}^\lambda = -\hat{L}_{31}^\lambda = -\frac{1}{2} f F^{\mu\lambda}, \]

\[ \hat{L}_{14}^\lambda = -\hat{L}_{41}^\lambda = f F^\lambda_\omega P^{\alpha\beta,\mu\omega} - f F^\mu_\omega P^{\alpha\beta,\lambda\omega}, \]

\[ \hat{L}_{22}^\lambda = \left( 2 \frac{C'^2}{C} - \frac{C'^3}{C^2} + Z' \right) (\nabla^\lambda \Phi), \]
\[ \hat{L}_{23} = -\hat{L}_{32} = \frac{1}{2} C''(\nabla^\lambda \Phi), \]

\[ \hat{L}_{24} = -\hat{L}_{42} = (Z - C'')(\nabla_\omega \Phi) P^{\alpha\beta,\lambda\omega}, \]

\[ \hat{L}_{33} = -\frac{1}{2} C'(\nabla^\lambda \Phi), \]

\[ \hat{L}_{34} = -\hat{L}_{43} = \frac{1}{2} C'(\nabla_\omega \Phi) P^{\alpha\beta,\lambda\omega}, \]

\[ \hat{L}_{44} = C'(\nabla^\omega \Phi) \left[ P_{\omega k}^{\mu\nu} p^{\alpha\beta,\lambda\kappa} - P_{\mu\nu,\lambda\kappa}^{\mu\nu,\alpha\beta} \right] - \frac{3}{2} C'(\nabla^\lambda \Phi) P^{\mu\nu,\alpha\beta}, \]

\[ \hat{P}_{11} = f R^{\mu\alpha}, \]

\[ \hat{P}_{23} = \hat{P}_{32} = \frac{1}{2} V' - \frac{1}{8} f' F^2, \]

\[ \hat{P}_{33} = \frac{1}{8} f F^2, \]

\[ \hat{P}_{44} = \left[ (Z - 2C'')(\nabla_\omega \Phi)(\nabla^\lambda \Phi) - 2C'(\nabla_\omega \nabla^\lambda \Phi) + f F_{\omega \rho} P^{\lambda\rho} \right] P^{\mu\nu,\omega k} P^{\alpha\beta}_{\lambda\kappa} \]

\[ - \frac{1}{2} \left[ C R + V + \frac{1}{4} f F^2 + \frac{1}{2} Z(\nabla^\lambda \Phi)(\nabla_\lambda \Phi) \right] P^{\mu\nu,\alpha\beta} \]

\[ + \frac{1}{2} f F_{\omega k} F^{\lambda\rho} P^{\mu\nu}_{\omega\lambda} P^{\alpha\beta}_{\rho\rho}. \quad (44) \]

The divergent part may be expressed in terms of the matrices \( \hat{E}^\lambda \) and \( \hat{\Pi} \), in accordance with (19)

\[ (\hat{E}^\lambda)^1 = \frac{f'}{2f} \left[ (\nabla^\lambda \Phi) g_\rho^\alpha - (\nabla^\alpha \Phi) g^\lambda_\rho + (\nabla_\rho \Phi) g^{\alpha\lambda} \right], \]

\[ (\hat{E}^\lambda)^2 = -\frac{f'}{2f} F_\rho^\lambda, \]

\[ (\hat{E}^\lambda)^3 = \frac{1}{4} F_\rho^\lambda, \]

\[ (\hat{E}^\lambda)^4 = \frac{1}{2} F_{\rho\omega} P^{\alpha\beta,\lambda\omega} - \frac{1}{2} F^{\lambda\omega} P^{\alpha\beta}_{\rho\omega}. \]
\[(\hat{E}^\lambda_1)_1 = -\frac{f}{2C'} F^{\alpha\lambda},\]
\[(\hat{E}^\lambda_1)_2 = \frac{C''}{C'} (\nabla^\lambda \Phi),\]
\[(\hat{E}^\lambda_2)_3 = 0,\]
\[(\hat{E}^\lambda_3)_4 = -\frac{1}{2} (\nabla_\omega \Phi) P^{\alpha\beta, \lambda\omega},\]
\[(\hat{E}^\lambda_4)_1 = \left( \frac{f Z'}{C^2} - \frac{f}{C} + \frac{f'}{C'} \right) F^{\alpha\lambda},\]
\[(\hat{E}^\lambda_2)_2 = \left( \frac{Z'}{C'} - 2 \frac{C'' Z}{C'^2} + \frac{C'^2}{C^2} \right) (\nabla^\lambda \Phi),\]
\[(\hat{E}^\lambda_3)_3 = 0,\]
\[(\hat{E}^\lambda_4)_2 = \left( \frac{C''}{C'} - \frac{Z'}{C'} \right) (\nabla_\omega \Phi) P^{\lambda\omega},\]
\[(\hat{E}^\lambda_3)_4 = -\frac{C''}{2C} (\nabla^\lambda \Phi) P^{\lambda\omega},\]
\[(\hat{E}^\lambda_4)_3 = \frac{C' (\nabla^\omega \Phi) \left[ P_{\rho\sigma, \omega\kappa} P^{\alpha\beta, \lambda\kappa} - P_{\omega\kappa} P^{\alpha\beta, \lambda\kappa} \right] + \frac{C''}{2C} (\nabla^\lambda \Phi) P^{\alpha\beta}_{\rho\sigma},\]
\[\hat{\Pi}_1 = -R^\alpha_\rho,\]
\[\hat{\Pi}_2 = \frac{C''}{C'} (\nabla^\lambda \Phi)(\nabla^\lambda \Phi) + \frac{C''}{C'} (\Delta \Phi) - \frac{1}{C'} V'' + \frac{f'}{4C'} F^2,\]
\[\hat{\Pi}_3 = \left( \frac{C'' Z}{C'^2} - \frac{C''}{C} \right) (\nabla^\lambda \Phi)(\nabla^\lambda \Phi) + \left( \frac{Z}{C'} - \frac{C''}{C} \right) (\Delta \Phi) - \frac{1}{C'} V'' + \left( \frac{f Z}{2C'^2} - \frac{f}{2C} + \frac{f'}{4C'} \right) F^2,\]
\[ \hat{\Pi} = \left[ \left( \frac{2Z}{C} - 4\frac{C''}{C} \right)(\nabla_{\omega}\Phi)(\nabla^{\lambda}\Phi) - 4\frac{C''}{C}(\nabla_{\omega}\nabla^{\lambda}\Phi) + \frac{2f}{C}F_{\omega\nu}F^{\lambda\nu} \right] P_{\rho\sigma}F_{\lambda\kappa}^{\alpha\beta} \\
+ \left[ \left( \frac{2C''}{C} - \frac{Z}{2C} \right)(\nabla^{\lambda}\Phi)(\nabla_{\lambda}\Phi) + 2\frac{C''}{C}(\Delta\Phi) - R - \frac{1}{C}V - \frac{f}{4C}F^{2} \right] P_{\rho\sigma}^{\alpha\beta} \\
+ \frac{f}{C}F_{\omega\kappa}F^{\lambda\nu}P_{\rho\sigma,\omega\lambda}P^{\alpha\beta}_{\kappa\nu}. \] (45)

To obtain the divergent part, \( \Gamma_{2-\text{div}} \), we have to evaluate the functional traces of the matrices above as in eq. (14). Thus, we get,

\[ \Gamma_{2-\text{div}} = -\frac{1}{\epsilon} \int d^2x \sqrt{g} \left\{ 3R - \frac{C''}{C}(\Delta\Phi) + \left( \frac{C''}{C} - \frac{3C'^{2}}{2C^2} \right) (\nabla^{\lambda}\Phi)(\nabla_{\lambda}\Phi) \right\} . \] (46)

What is left to do is to calculate the divergent structure of the both ghost operators: that corresponding to diffeomorphisms,

\[ \hat{\mathcal{M}}_{\mu} = g_{\nu}^{\mu} \Delta - \frac{C'}{C}(\nabla_{\nu}\Phi)\nabla^{\mu} - \frac{C'}{C}(\nabla^{\mu}\nabla_{\nu}\Phi) + R_{\nu}^{\mu}, \] (47)

and the one corresponding to the Maxwell gauge transformations,

\[ \hat{\mathcal{M}} = \Delta. \] (48)

Hence,

\[ \Gamma_{\text{gh-div}} = -\frac{1}{\epsilon} \int d^2x \sqrt{g} \left\{ 5R + \frac{2}{C}V + \frac{2}{C'}V' + \left( \frac{f'}{2C'} - \frac{f}{2C} \right) F^{2} \right. \\
+ \left. \left( \frac{f'}{f} + \frac{C'}{C} - \frac{Z}{C'} \right) (\Delta\Phi) \right\} . \] (49)

The total divergent contribution is

\[ \Gamma_{\text{div}} = -\frac{1}{\epsilon} \int d^2x \sqrt{g} \left\{ 5R + \frac{2}{C}V + \frac{2}{C'}V' + \left( \frac{f'}{2C'} - \frac{f}{2C} \right) F^{2} \right. \\
+ \left. \left( \frac{f'}{f} + \frac{C'}{C} - \frac{Z}{C'} \right) (\Delta\Phi) \right\} . \] (50)
Notice that from eq. (50) one can also get the one-loop effective action for pure dilaton gravity with the action (1). The result is

\[
\Gamma_{\text{div}} = -\frac{1}{2\epsilon} \int d^2 x \sqrt{g} \left\{ 4R + \frac{2}{C} V + \frac{2}{C'} V' + \left( \frac{C''}{C} - \frac{Z}{C'} \right) \Delta \Phi \\
+ \left( \frac{C'''}{C} - \frac{3 C'^2}{C^2} - \frac{C'' Z}{C'^2} \right) (\nabla^\lambda \Phi)(\nabla_\lambda \Phi) \right\}. \tag{51}
\]

In the expressions (50) and (51) for the effective action all surface divergent terms have been kept.

Let us now discuss the on-shell limit of the effective action (50). The classical field equations that we need are

\[
\frac{\delta S}{\delta \Phi} = -\nabla_\nu (Z g^{\mu \nu} \partial_\mu \Phi) + \frac{1}{2} Z' (\nabla^\mu \Phi)(\nabla_\mu \Phi) + C' R + V' + \frac{1}{4} f F^2 = 0,
\]

\[
g^{\mu \nu} \frac{\delta S}{\delta g^{\mu \nu}} = -\Delta C + V - \frac{1}{4} f F^2 = 0. \tag{52}
\]

Substituting eqs. (52) into the effective action (50) and keeping all the surface counterterms, one obtains

\[
\Gamma_{\text{on-shell}}^{\text{div}} = -\frac{1}{2\epsilon} \int d^2 x \sqrt{g} \left\{ 3R + \Delta \left[ \ln(f C^3) \right] + \nabla^\lambda \left[ \frac{Z}{C'} \nabla_\lambda \Phi \right] \right\}. \tag{53}
\]

The theory with the action (39) is finite on-shell as well as the dilaton gravities discussed in sect. 2. Hence, we can propose the very plausible conjecture that dilaton gravities of the family (1), if they are classically equivalent, are also quantum equivalent on shell.

4 Renormalization and renormalization group equations

In this section we will study the renormalization structure and renormalization group equations for dilaton gravity with the action (1). One can discuss the renormalization of the metric as in sect. 2 and find the restrictions imposed by off-shell multiplicative renormalizability (in the usual sense) on the form of the functions \( Z(\Phi), C(\Phi) \) and \( V(\Phi) \). Instead of doing this, we will not renormalize the fields but rather will consider the functions \( Z, C \) and \( V \) as generalized effective couplings (generalized renormalizability).
Then the generalized $\beta$-functions will be found and generalized renormalization group equations will be generated.

Let us start from the theory defined by action (1). (For simplicity, we shall not discuss the case of dilaton-Maxwell gravity.) The general structure of renormalization for general couplings is given by

$$Z_0 = (\mu^2)^{\epsilon'} \left[ Z + \sum_{k=1}^{\infty} \frac{a_k Z(Z, C, V)}{\epsilon^k} \right],$$

where $\epsilon' = n - 2$, and similar expressions for $C$ and $V$. As it follows from one-loop renormalization, eq. (51),

$$a_1 Z = -Z' C + 2 \frac{C'^2}{C} + 2 \frac{C'' Z}{C'^2}, \quad a_1 C = 0, \quad a_1 V = -\frac{V}{C} - \frac{V'}{C'}.$$

Now, the generalized one-loop $\beta$-functions are given by the standard relations:

$$\beta_T = -a_1 T + Z \frac{\delta a_1 T}{\delta Z} + C \frac{\delta a_1 T}{\delta C} + V \frac{\delta a_1 T}{\delta V},$$

where $T \equiv \{Z, C, V\}$. Using (55) and (56) we obtain

$$\beta_C = 0,$$

$$\beta_V = V \frac{C'}{C'} - \frac{C'}{C'^2} - \frac{C C''}{C'^2} + 2 \frac{C' V'}{C'^2},$$

$$\beta_Z = \frac{Z'}{C'} + \frac{2 C'^2}{C'^2} - \frac{4 C''}{C} - \frac{Z C''}{C'^2} + 3 \frac{C Z''}{C'^2} - 2 \frac{C' Z}{C'^3}.$$

Of course, in the case that the theory includes a Maxwell sector, similar $\beta$-functions for $f$ can be easily obtained too.

The renormalization group equations have the following form:

$$\frac{\partial C}{\partial t} = \beta_C, \quad \frac{\partial V}{\partial t} = \beta_V, \quad \frac{\partial Z}{\partial t} = \beta_Z,$$

with $t = \ln \mu^2$. The system of equations (58) is very difficult to solve. It depends not only on the scaling parameter $t$, as in usual field theory, but also on some unknown functions of the field variables. Moreover, nobody has any idea about the proper boundary conditions (initial data) that the partial differential equation system (58) should satisfy.

Notwithstanding that, we can get some useful information yet from the renormalization group equations (58). In particular, we can look for fixed points of this system (what does not at all involve the knowledge of initial data). The equations they must satisfy are

$$\beta_C = 0, \quad \beta_V = 0, \quad \beta_Z = 0.$$
The system of differential equations (59) is still very complicated. Nevertheless, some basic, particular solutions of the same can be discovered. For example, motivated by the CGHS action [2], we can look for fixed points of the following type:

\[ C = e^{a_1 \Phi}, \quad V = e^{a_2 \Phi}, \quad Z = e^{a_3 \Phi}, \]  

where \( a_1, a_2 \) and \( a_3 \) are some constants. Substituting (60) into eq. (59) we obtain the following solutions

\[
\begin{align*}
  a_1 &= 0, \quad a_2 = 1, \quad a_3 = -\frac{1}{3}; \\
  a_1 &= \frac{2}{3}, \quad a_2 = 2, \quad a_3 = \frac{1}{6} \left(1 \pm \sqrt{19/3}\right); \quad \text{etc.}
\end{align*}
\]

In the same way, different particular cases of fixed points can be considered. For instance, for

\[ C = \Phi^{\alpha_1}, \quad V = \Phi^{\alpha_2}, \quad Z = \Phi^{\alpha_3}, \]  

we get

\[
\begin{align*}
  \alpha_2 (\alpha_2 - 4\alpha_1 + 1) &= 0, \\
  3\alpha_3^2 - \alpha_3\alpha_1 - \alpha_2^2 - \alpha_3 + \alpha_1 &= 0.
\end{align*}
\]

Particular solutions are

\[
\begin{align*}
  \alpha_1 &= 2, \quad \alpha_2 = 0, \quad \alpha_3 = \frac{1}{2} \left(3 \pm \sqrt{33}\right); \\
  \alpha_1 &= 2, \quad \alpha_2 = 7, \quad \alpha_3 = \frac{1}{2} \left(3 \pm \sqrt{33}\right); \quad \text{etc.}
\end{align*}
\]

Thus, we have shown that (at least in principle) one can find ultraviolet stable fixed points for the generalized couplings \( Z(\Phi), C(\Phi) \) and \( V(\Phi) \).

\section{Conclusions}

In summary, we have investigated in this paper the one-loop renormalization structure of the general model of 2D dilaton gravity (1). The divergences of the one-loop effective action have been found.

The calculation of the one-loop effective action for two different but classically equivalent dilaton gravities, eqs. (3) and (9), respectively, in the same gauge (5) and (12), has
shown that these theories are quantum equivalent on shell. The one-loop on-shell effective action is just given by surface terms, i.e., it is finite. Since the on-shell effective action for the general model of dilaton gravity (namely, including a Maxwell field) has the same property, we have been led to conjecture that all classically equivalent dilaton gravities are in fact quantum equivalent on-shell.

Generalized renormalizability of the general model of dilaton gravity has been discussed, and the corresponding generalized $\beta$ functions have been found. The analysis of the renormalization group equations yields some set of generalized couplings \{Z, C, V\} which are fixed points of such equations. It is interesting to notice that the CGHS model does not belong to this set. A further remark is the fact that if one requires the renormalizability of the theory in the usual sense, then one can renormalize the metric in the theory (39) through the following transformation

$$g_{\mu\nu} = \exp \left[ \frac{1}{\epsilon} \left( \frac{1}{C} + \frac{Z}{2C'^2} \right) \right] g_{\mu\nu}^R.$$  \(\text{(65)}\)

Then, the 2D dilaton-Maxwell theory (39) is multiplicatively renormalizable off-shell in the usual sense for the following choice of potentials:

$$V = \exp \left[ aC + \int \frac{Z}{2C'} d\Phi \right], \quad f = \exp \left[ -bC - \int \frac{Z}{2C'} d\Phi \right].$$  \(\text{(66)}\)

where $a$ and $b$ are arbitrary constants. For the theory (3) the above potentials are of Liouville type, what is in full agreement with refs. [7,14].

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A Appendix

In this appendix we will show that the results of sect. 3 actually give also the one-loop counterterms in 2D $R^2$-gravity (for a discussion of different models of such theory see [16-18]).
Let us consider 2D $R^2$-gravity as defined by the action

$$S = - \int d^2x \sqrt{g} \left( \Lambda - \frac{a}{4} R^2 \right),$$

(67)

where $a$ and $\Lambda$ are dimensional parameters. One can rewrite (67) by introducing the auxiliary scalar field (dilaton), as in [18],

$$S = - \int d^2x \sqrt{g} \left( R \Phi + \frac{1}{a} \Phi^2 + \Lambda \right).$$

(68)

Theories with the actions (67) and (68) are classically equivalent. The theory given by (68) belongs to class (1), with

$$Z = 0, \quad C = \Phi, \quad V = \Lambda + \frac{1}{a} \Phi^2.$$  

(69)

The one-loop effective action for this theory has been actually calculated in sect. 3. The result is

$$\Gamma_{\text{div}} = - \frac{1}{2\epsilon} \int d^2x \sqrt{g} \left[ 4R + \frac{2}{\Phi} \left( \Lambda + \frac{1}{a} \Phi^2 \right) + \frac{4}{a} \Phi + \frac{1}{\Phi} \Delta \Phi - \frac{3}{\Phi^2} (\nabla^\lambda \Phi)(\nabla_\lambda \Phi) \right].$$

(70)

Hence, we see that, with our procedure, we are able to calculate the one-loop effective action explicitly in some version of 2D $R^2$-gravity. It is also interesting to notice that $\Gamma_{\text{div}}$ is on-shell finite too.
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