POSITIVE LYAPUNOV EXPONENTS AND A LARGE DEVIATION THEOREM FOR CONTINUUM ANDERSON MODELS, BRIEFLY

VALMIR BUCAJ, DAVID DAMANIK, JAKE FILLMAN, VITALY GERBUZ, TOM VANDENBOOM, FENGPENG WANG, AND ZHENGHE ZHANG

Abstract. In this short note, we prove positivity of the Lyapunov exponent for 1D continuum Anderson models by leveraging some classical tools from inverse spectral theory. The argument is much simpler than the existing proof due to Damanik–Sims–Stolz, and it covers a wider variety of random models. Along the way we note that a Large Deviation Theorem holds uniformly on compacts.

1. Introduction

It is well understood that random Schrödinger operators in one space dimension exhibit Anderson localization!

While this introductory statement is correct in many ways, it is nevertheless important to clarify what is actually meant. Does one talk about spectral localization or dynamical localization? Does one consider the discrete setting or the continuum setting? Even if one considers the (easier) discrete setting, is the assertion made for the standard model, or for more general models such as the ones considered in [7]? What is assumed about the single-site distribution?

It is true that no matter how one answers these questions, localization is indeed known. However, the difficulty of the known proofs depends heavily on the answers. For example, the proofs are short and elegant in the discrete setting with an absolutely continuous single-site distribution, but they can be quite difficult once the continuum setting and/or singular single-site distributions are considered.

Some of the landmark papers are Kunz-Souillard [12] (standard discrete model with an absolutely continuous single-site distribution), Carmona-Klein-Martinelli [4] (standard discrete model with a general single-site distribution), and Damanik-Sims-Stolz [6] (standard continuum model with a general single-site distribution).

In the case of a general single-site distribution, the localization proof typically consists of two steps. First, one proves that the Lyapunov exponent is positive for a sufficiently large set of energies by an application of Fürstenberg’s theorem, and second, one parleys this positivity statement into the exponential decay of generalized eigenfunctions, showing in effect that the spectrum is pure point and the eigenfunctions decay exponentially. A second look at the structure of the eigenfunctions then allows one to control their semi-uniform localization properties, which in turn yields dynamical localization. Traditionally, this second step was performed via multi-scale analysis.

The first step is very easy to implement in the discrete case and the Lyapunov exponent turns out to be positive for every energy via a straightforward verification of the assumptions of Fürstenberg’s

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Theorem. In the continuum case, on the other hand, verifying these assumptions is less straightforward; it is was accomplished in [6] away from a discrete set of energies via inverse scattering theory.

In scenarios where the initial proofs were involved, it is of interest to find simplifications of the arguments. For the standard discrete model with a general single-site distribution studied in [4], there have been several recent papers proposing such simplifications [3, 9, 11]. These new proofs simplify the second step in the two-step procedure described above (since the first step cannot be simplified, as indicated above).

In this paper we take a new look at the first step in this procedure for continuum models. Rather than using inverse scattering theory we will use inverse spectral theory, and the resulting proof of positive Lyapunov exponents for energies outside a discrete set turns out to be significantly simpler. Our setting is also more general than that of [6], so that technically speaking, we generalize the scope of the approach.

We also discuss a Large Deviation Theorem (LDT), which demonstrates that the second step in the localization proof for continuum models can then be carried out in complete analogy to the treatment of the discrete case developed in [3]. Since this is entirely straightforward, we do not carry this out explicitly, but merely note that with the present work and [3], both steps in the two-step procedure to prove localization for 1D continuum Anderson models have been simplified.

2. Main Result

2.1. Setting and Notation. Fix two parameters $0 < \delta \leq m$, and define

$$W = \bigcup_{\delta \leq s \leq m} L^2[0,s].$$

To distinguish the fibers, let us denote the length of the domain by $s = \ell(f)$ whenever $f \in L^2[0,s)$. We specify a continuum Anderson model by choosing a probability measure $\tilde{\mu}$ on $W$ such that

$$(\tilde{\mu} \text{Bd}) \quad \tilde{\mu} \text{-ess sup } \|f\|_{L^2} < \infty.$$

We naturally obtain the full shift

$$\Omega = W^\mathbb{Z}, \quad \mu = \tilde{\mu}^\mathbb{Z}, \quad [T \omega]_n = \omega_{n+1}.$$ Then, for each $\omega \in \Omega$, we obtain a potential $V_\omega$ by concatenating $\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots$, and an associated Schrödinger operator $H_\omega = -\partial_x^2 + V_\omega$. More specifically, define

$$s_n = s_n(\omega) := \begin{cases} \sum_{j=0}^{n-1} \ell(\omega_j) & n \geq 1 \\ 0 & n = 0 \\ -\sum_{j=n}^{-1} \ell(\omega_j) & n \leq -1, \end{cases}$$

(2.1) denote $I_n = [s_n, s_{n+1})$, and define

$$V_\omega(x) = \omega_n(x - s_n), \quad \text{for each } x \in I_n.$$ Let us note that this setting, which is related to those considered in [5, 7], includes that of [6] as a special case.

2.2. Lyapunov Exponent. For each $w \in W$, $E \in \mathbb{C}$, $A^E(w)$ is the unique $\text{SL}(2, \mathbb{C})$ matrix with

$$\begin{bmatrix} \psi(s_1) \\ \psi'(s_1) \end{bmatrix} = A^E(w) \begin{bmatrix} \psi(0) \\ \psi'(0) \end{bmatrix}$$

whenever $H_\omega \psi = E \psi$ with $\omega_0 = w$. For each $E$, the Lyapunov exponent is given by

$$L(E) = \lim_{n \to \infty} \frac{1}{n} \int_\Omega \log \|A^E_n(\omega)\| \, d\mu(\omega), \quad \text{where } A^E_n(\omega) = A^E(\omega_{n-1}) \cdots A^E(\omega_0), \ n \geq 1.$$
One obvious obstruction to localization is if all elements of the support of $\tilde{\mu}$ commute in the free product sense, so that one cannot distinguish permutations of elements of the support after concatenation. When this is the case, all realizations $V_\omega$ are periodic, and localization clearly fails. This is the only obstruction to localization; we formulate the negation of this as our nontriviality condition. For $f_j \in L^2[0,a_j)$, $j = 1, 2$, we write

$$(f_1 \ast f_2)(x) = \begin{cases} f_1(x) & 0 \leq x < a_1 \\ f_2(x-a_1) & a_1 \leq x < a_1 + a_2. \end{cases}$$

The nontriviality condition is then the following:

(NC) There exist $f_j \in \text{supp } \tilde{\mu}$ such that $f_1 \ast f_2 \neq f_2 \ast f_1$.

Let us note that the equality that fails in (NC) is in $L^2$, so we really mean that $f_1 \ast f_2$ and $f_2 \ast f_1$ differ on a set of positive Lebesgue measure.

**Theorem 2.1.** If $\tilde{\mu}$ satisfies (NC) and $(\tilde{\mu} \text{ Bd})$, then there is a discrete set $D \subset \mathbb{R}$ such that the Lyapunov exponent is positive away from $D$:

$$L(E) > 0 \quad \text{for all } E \in \mathbb{R} \setminus D.$$ 

Furthermore, for any compact set $K \subset \mathbb{R} \setminus D$ and any $\varepsilon > 0$, there exist $C = C(\varepsilon, K) > 0$ and $\eta = \eta(\varepsilon, K) > 0$ such that

$$\mu \left\{ \omega : \left| \frac{1}{n} \log \| A_n^E(\omega) \| - L(E) \right| \geq \varepsilon \right\} \leq C e^{-\eta m}$$

for all $n \in \mathbb{Z}_+$ and all $E \in K$.

In particular, the nontriviality condition (NC) implies the critical assumptions of Fürstenberg’s theorem and a contractivity property crucial to proving the Large Deviation result. The boundedness assumption $(\tilde{\mu} \text{ Bd})$ then provides a sufficient regularity property of the cocycle to complete the proof.

### 3. Proof of Theorem

Let $(\Omega, \mu, T)$ denote the ergodic dynamical system generated by the full shift on a probability space $(W, \tilde{\mu})$. A special case of a classical theorem of Fürstenberg will yield positive Lyapunov exponent in our setting:

**Theorem 3.1.** For $E \in \mathbb{R}$, define $A^E : W \to \text{SL}(2, \mathbb{R})$ as above, let $\nu_E := A_x^E \tilde{\mu}$ be the pushforward of $\tilde{\mu}$ under $A^E$, and suppose that $\int \log \| M \| \, d\nu_E(M) < \infty$. If

(FI) there exist $M, M' \in \text{supp } \nu_E$ which have no common eigenvectors,

then the Lyapunov exponent $L(E) > 0$ is positive.

Theorem 3.1 was originally proved by Fürstenberg under the assumption that $G_{\nu_E}$ is noncompact and strongly irreducible [8]. The sufficient criterion stated here implies strong irreducibility and noncompactness and is due to Ishii [10, Theorem 4.1].

Under regularity assumptions on the cocycle one can conclude a uniform Large Deviation Theorem:

**Theorem 3.2** (Theorem 3.1, [3]). Let $K \subset \mathbb{R}$ be a compact set, and consider a map $A : K \times W \to \text{SL}(2, \mathbb{R})$ such that, for every $E \in K$, $A^E := A(E, \cdot)$ satisfies the assumptions of Theorem 3.1. Suppose that $A$ also satisfies the following properties:

(UnifEq) \quad \{ E \mapsto A^E(w) : w \in W \} \text{ is uniformly equicontinuous};

(UnifBd) \quad \exists C > 0 \text{ such that } \sup_{E \in K, w \in W} \| A^E(w) \| \leq C \text{ for } \tilde{\mu} \text{-a.e. } w;
and, for all $E \in K$,

$$\exists \{g_n\}^\infty_{n=1} \subset G_{\nu_E} \text{ for which } \|g_n\|^{-1} g_n \text{ converges to a rank one operator.}$$

Then, for any $\varepsilon > 0$, there exists $C = C(\varepsilon, K) > 0$, $\eta = \eta(\varepsilon, K) > 0$ such that

$$\mu \left\{ \omega : \left| \frac{1}{n} \log \| A^E_n(\omega) \| - L(E) \right| \geq \varepsilon \right\} \leq Ce^{-\eta n}$$

for all $n \in \mathbb{Z}_+$ and all $E \in K$.

For the 1D continuum Anderson model described above, (NC) implies that conditions (FI) and (Ctrct) hold true away from a discrete set of energies:

**Theorem 3.3.** With notation as above, if $\mu$ satisfies (NC), then there is a discrete set $D \subset \mathbb{R}$ such that, for any $E \in \mathbb{R} \setminus D$, conditions (FI) and (Ctrct) hold true.

Theorem 3.3 resolves the foremost obstructions to proving Theorem 2.1. Indeed, the boundedness condition ($\mu$ Bd) and the cocycle structure (2.3) of $A^E$ imply (UnifEq) and (UnifBd) (cf. [3, Lemma 3.3]). Thus, Theorem 2.1 follows from Theorems 3.1, 3.2, and 3.3.

To prove Theorem 3.3 we will follow the following Lemma, which essentially follows from classical inverse spectral theory – namely the Borg–Marchenko theorem; compare [1, 2, 13].

**Lemma 3.4.** If $V_1, V_2 \in L^2[0, T]$ and

$$(3.1) \quad A^E(V_1) = A^E(V_2) \text{ for every } E \in \mathbb{C},$$

then $V_1 = V_2$ Lebesgue almost everywhere on $[0, T]$.

**Proof.** Denote the $m$-function associated with $V_j$ by $m_j$. That is, taking $\beta$ large enough, then, for every $E \in \mathbb{C} \setminus [-\beta, \infty)$, there is a unique (modulo an overall multiplicative constant) solution $u_j = u_j(\cdot, E)$ of $-u''_j + V_j u_j = E u_j$ that satisfies a Dirichlet boundary condition at $T$. One then defines the $m$-functions by

$$m_j(E) = \frac{u'_j(0, E)}{u_j(0, E)}.$$

However, by equality of the cocycles (3.1), it is easy to see $m_1 \equiv m_2$, whence $V_1 \equiv V_2$ (a.e.) by [14, Theorem 1.1]. \[\square\]

We are now ready to prove Theorem 3.3, and consequently Theorem 2.1.

**Proof of Theorem 3.3.** Let $f_j$ be as in assumption (NC), denote by $M_j(E) = A^E(f_j)$ the associated monodromies, and let $Q(E) = M_1(E)M_2(E) - M_2(E)M_1(E)$ be their commutator. To show (FI) holds at $E$, it suffices to show $\det Q(E)$ is nonzero.

The commutator $Q(E)$ and determinant $\det Q(E)$ are analytic. Lemma 3.4 and (NC) imply that $Q(E)$ does not vanish identically, hence only vanishes on a discrete set by analyticity. Suppose for the sake of contradiction that $\det Q(E) = 0$ identically. Fix an interval $I \subset \mathbb{R}$ such that $M_1(E)$ is elliptic for all $E \in I$; such an interval exists since $M_1$ is a monodromy for a periodic Schrödinger operator. Since $\det Q(E) = 0$ for $E \in I$, $M_1$ and $M_2$ have a common eigenvector. Furthermore, since $M_1$ is elliptic, the eigenvector may be chosen of the form $v = (1, w)^T$, $w \in \mathbb{C} \setminus \mathbb{R}$. Then, since $M_1$ and $M_2$ are real entries, one deduces that they both have $(1, \bar{w})^T$ as an eigenvector, whence $Q(E) = 0$ for all $E \in I$. But $Q(E)$ only vanishes on a discrete set. Thus, $\det Q(E)$ is a nonzero analytic function, and there is a discrete set $D \subset \mathbb{R}$ such that (FI) holds for $E \in \mathbb{R} \setminus D$.

It remains to show that (Ctrct) holds for any $E \in \mathbb{R} \setminus D$. Under (NC) we have seen above that, for any $E \in \mathbb{R} \setminus D$, the group $G_{\nu_E}$ contains noncommuting elements $M_j(E)$. But any nonabelian subgroup $G$ of $\text{SL}(2, \mathbb{R})$ must contain a non-elliptic element $M$ (cf. [15, Theorem 10.4.14]); taking $g_n := M^n$ yields the property (Ctrct). \[\square\]
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Department of Mathematics, United States Military Academy, West Point, NY 10996, USA

E-mail address: valmir.bucaj@westpoint.edu

Department of Mathematics, Rice University, Houston, TX 77005, USA

E-mail address: damanik@rice.edu

Department of Mathematics, Virginia Tech, 225 Stanger Street, Blacksburg, VA 24061, USA

E-mail address: fillman@vt.edu

Department of Mathematics, Rice University, Houston, TX 77005, USA

E-mail address: vitaly.gerbuz@rice.edu

Department of Mathematics, Yale University, New Haven, CT 06511, USA

E-mail address: thomas.vandenboom@yale.edu

School of Mathematics, Ocean University of China, Qingdao, China 266100

E-mail address: wfpouc0163.com

Department of Mathematics, UC Riverside, Riverside, CA 92521, USA

E-mail address: zhenghe.zhang@ucr.edu