EXPONENTIAL MAPS, COMMUTING NILPOTENT VARIETIES, AND SATURATION

PAUL SOBAJE

Abstract. Let $G$ be a connected reductive algebraic group over an algebraically closed field of characteristic $p > 0$. We prove that when $p$ is good for $G$ there exists a uniquely defined exponential map on $\mathcal{N}_1(\mathfrak{g})$, the restricted nullcone of $\text{Lie}(G)$. We use this map to relate the cohomological variety of the $r$-th Frobenius kernel of $G$ with the variety of $r$-tuples of commuting elements in $\mathcal{N}_1(\mathfrak{g})$. A second application is given as we show that the exponential provides a solution to the saturation problem, which involves finding a way to canonically embed each $p$-unipotent element of $G$ in a one-parameter subgroup. This reproves a result due to Seitz.

Let $G$ be a connected reductive group over an algebraically closed field $k$ of characteristic $p > 0$. Its Lie algebra $\mathfrak{g}$ has a $p$-th power operation $x \mapsto x^{[p]}$, and we denote by $\mathcal{N}_1(\mathfrak{g})$ the restricted nullcone of $\mathfrak{g}$, which is the set of those $x$ such that $x^{[p]} = 0$. Equivalently, this is the set of all $x \in \mathfrak{g}$ which act $p$-nilpotently on every rational $G$-module. In this paper we prove that in good characteristic there exists a unique “exponential” map on $\mathcal{N}_1(\mathfrak{g})$. This map can be thought of as the characteristic $p$ analogue of the exponential map in characteristic 0 (defined on the entire nullcone in that case). When $G = GL_n$, this map is just given by the $p$-truncated exponential series.

The exponential map has several applications, particularly within representation theory, and we provide a few in this paper. First we note that it extends (by holding for all $p$ good) and strengthens (by establishing uniqueness) a similar result proved by Carlson, Lin, and Nakano [1, Theorem 3]. That result was the foundation upon which the authors constructed a map from the cohomological support variety of a rational $G$-module over a finite subgroup of Lie type to its support variety over $\mathfrak{g}$ (see [1, §4] for precise statements). We do not revisit their work here, but instead give another important application of the exponential map to the theory of cohomological support varieties.

For a finite group scheme $\mathcal{G}$ over $k$, Friedlander and Suslin have shown [4, Theorem 1.1] that the even cohomology ring $H^\bullet(\mathcal{G}, k)$ is a finitely generated commutative $k$-algebra, hence one can consider the affine variety $X$ whose coordinate algebra is isomorphic to $H^\bullet(\mathcal{G}, k)_{\text{red}}$. Suslin, Friedlander, and Bendel further showed in [13] that if $\mathcal{G}$ is an infinitesimal group scheme of height $r$, then $X$ is homeomorphic to the variety $\text{Hom}_{\mathfrak{g}/k}(\mathbb{G}_{a(r)}, \mathcal{G})$, where $\mathbb{G}_{a(r)}$ is the $r$-th Frobenius kernel of the additive group $\mathbb{G}_a$. In particular, this holds when $\mathcal{G}$ is the $r$-th Frobenius kernel of some linear algebraic group over $k$.

In [13] it was shown that there is a bijection between $\text{Hom}_{\mathfrak{g}/k}(\mathbb{G}_{a(r)}, GL_n)$ (which is the same as $\text{Hom}_{\mathfrak{g}/k}(\mathbb{G}_{a(r)}, GL_{n(r)})$) and the variety of $r$-tuples of pairwise commuting elements in $\mathcal{N}_1(\mathfrak{gl}_n)$, which we denote by $C_r(\mathcal{N}_1(\mathfrak{gl}_n))$. A natural question to ask in view of this result is whether or not such an identification holds
for an arbitrary connected reductive group. This question was implicitly raised in 
[13], where the authors gave an answer in the affirmative for the classical groups
$SL_n, SO_n,$ and $Sp_{2n}$. In [7], McNinch effectively showed that this description also
holds for any connected reductive group $G$ provided that $p > 2h - 2$, where $h$ is the
Coexeter number of $G$. In [12] we extended this by showing that the result is true
when $p \geq h$ and $p$ is very good for $G$ (in particular, it is always true if $p > h$). The
proof employed in [12] differed from those in [13] and [7] in that the description was
given intrinsically, without relying on an embedding of $G$ into some general linear
group. This latter feature then answered an explicit question asked in [13] Remark
1.9]. Here we make use of the exponential map to prove that the bijection between
$C_r(N_1(g))$ and $\text{Hom}_{g/k}(G_{0(r)}, G)$ holds for all $G$ reductive provided only that $p$
is very good for $G$. In particular if the derived subgroup $G'$ is simply connected then
this holds if $p$ is good for $G$.

The results just mentioned also find an application in recent work by Friedlander
[3], in which a theory of support varieties for rational $H$-modules is introduced, $H$
being a linear algebraic group. In order for this theory to be developed, the
author restricts to the case where $H$ can be given a structure of exponential type
[3] Definition 1.6]. Theorems 2.4 and 3.2 below show that any connected reductive
group in very good characteristic admits such a structure.

The second application of the exponential map is that it solves the saturation
problem for $G$. This notion is due to Serre, and involves assigning to each $p$-
unipotent element $g \in G$ a one-parameter subgroup of $G$ whose image contains
$g$, and which can be specified in some canonical way. In [10], saturation for $p$-
unipotent elements in $GL_n$ was used by the author to prove the semisimplicity
tensor products of certain group representations in characteristic $p$. Seitz then
explored saturation more generally in [9], proving that it can be achieved for any
connected reductive group in good characteristic. We observe that the exponential
map provides an immediate manner in which to embed $p$-unipotent elements into
canonical one-parameter subgroup of $G$, and that these subgroups are in fact the
same as those specified by Seitz.

1. Preliminaries and Notation

For the rest of this paper, $G$ will denote a connected reductive algebraic group
over a field $k$ of characteristic $p > 0$. We fix a maximal torus $T$ of $G$, and denote
by $\Phi$ the root system of $G$ with respect to $T$. We choose a set of simple roots
$\Pi \subseteq \Phi$ which determines the set of positive roots $\Phi^+$. Let $B$ denote the Borel
subgroup of $G$ containing the root subgroups corresponding to every positive root.
For any $J \subseteq \Pi$, let $P_J$ be the corresponding parabolic subgroup of $G$. We define
$\Phi_J = \Phi \cap Z_J$, and $\Phi^+_J = \Phi^+ \cap \Phi_J$.

For $g, h \in G$, $X \in \mathfrak{g}$, we write $g \cdot h = ghg^{-1}$, while $g \cdot X$ denotes the adjoint
action. The centralizer of $g$ is $C_G(g)$, $C_G(X)$ is the stabilizer of $X$ in $G$, and $C_\mathfrak{g}(X)$
is the centralizer of $X$ in $\mathfrak{g}$.

The prime $p$ is good for $G$ if it is good for $\Phi$. Specifically, this means that $p > 2$ if
$\Phi$ has a component of type $B, C,$ or $D$; $p > 3$ if $\Phi$ has a component of type $E_6, E_7, F_4$
or $G_2$; and $p > 5$ if $E_8$ is a component of $\Phi$. The derived group $G' = (G, G)$ is
semisimple, and we say that $p$ is a very good prime for $G$ if the surjective map
$G'_{\text{sc}} \to G'$ is separable, where $G'_{\text{sc}}$ is the simply-connected semisimple group with
the same root system as $G$. 


If $H$ is an affine algebraic group over $k$, then it is also an affine group scheme over $k$, and by abuse of notation we will use $H$ to denote both the scheme and the group of $k$-points of the scheme (as will be clear by the context).

For any affine group scheme $H$ (i.e., not necessarily coming from an algebraic group), we write $k[H]$ for its coordinate ring. We denote by $\text{Dist}(H)$ its algebra of distributions (see [6, I.7]), and by $\mathfrak{h}$ its Lie algebra. If $\varphi$ is a homomorphism of affine group schemes from $H_1$ to $H_2$, then $d\varphi$ is the induced map from $\text{Dist}(H_1)$ to $\text{Dist}(H_2)$. We also use $d\varphi$ to denote the differential of $\varphi$, i.e., the induced map from $\mathfrak{h}_1$ to $\mathfrak{h}_2$ (in fact, this can simply be viewed as a restriction of the map on distribution algebras).

Let $H_{(r)}$ denote the $r$-th Frobenius kernel of $H$ (see [4, 1.9] regarding how to define such a kernel without specifying an $F_p$-structure on $H$). We then have that $\text{Dist}(H_{(r)}) \subseteq \text{Dist}(H_{(r+1)})$, and $\text{Dist}(H) = \bigcup_{r \geq 1} \text{Dist}(H_{(r)})$.

The additive group $\mathbb{G}_a$ has coordinate algebra $k[\mathbb{G}_a] \cong k[t]$, and $\text{Dist}(\mathbb{G}_a)$ is spanned by the elements $d^{(j)}$, where

$$\frac{d}{dt} \left(t^j\right) = \delta_{ij}$$

If we set $u_j = \frac{d}{dt} \left(t^j\right)$, and if $m$ is an integer with $p$-adic expansion $m = m_0 + m_1 p + \cdots + m_q p^q$, then

$$\frac{d}{dt} \left(t^m\right) = \frac{u_0^{m_0} \cdots u_q^{m_q}}{m_0! \cdots m_q!}$$

Therefore $\text{Dist}(\mathbb{G}_a)$ is generated as an algebra over $k$ by the set $\{u_j\}_{j \geq 0}$, while $\text{Dist}(\mathbb{G}_{a(r)})$ is generated by the subset where $j < r$. Also, for any affine group scheme $H$, a homomorphism from $\mathbb{G}_{a(r)}$ to $H$ is equivalent to a Hopf algebra homomorphism from $\text{Dist}(\mathbb{G}_{a(r)})$ to $\text{Dist}(H)$, this latter homomorphism being determined by the images of the elements $u_j$.

For each $\alpha \in \Phi$, fix a root homomorphism $\varphi_\alpha : \mathbb{G}_a \to G$. We then set $e_\alpha = d\varphi_\alpha \left(\frac{d}{dt} \left(t^{(1)}\right)\right) \in \mathfrak{g}$. Let $U_i(G)$ denote the closed subset of $p$-unipotent elements in the algebraic group $G$, which is the set of all $x \in G$ such that $x^p = 1$.

2. **Exponential Maps**

When $p$ is good for $G$, it was proved in [8] that there exists a parabolic subgroup $P_J \leq G$ with unipotent radical $U_J$ for which

$$G \cdot U_J = \mathcal{N}_1(\mathfrak{g}).$$

It was further shown in [11, Theorem 2] that one may choose $P_J$ so that the nilpotence class of $U_J$ is less than $p$, and

$$G \cdot U_J = U_1(G).$$

When the nilpotence class of $U_J$ is less than $p$, we know by a result due to Serre that there exists a unique exponential map on $U_J$ which arises from base-changing the exponential isomorphism in characteristic 0 (see [7, §8] for a nice account of this). Before stating this result precisely, we recall that if $U_J$ has nilpotence class less than $p$ then it can be made into a group via the Baker-Campbell-Hausdorff
formula. Henceforth, any reference to a Lie algebra (of nilpotence class less than $p$) as an algebraic group will be assuming this group structure.

**Proposition 2.1.** Let $U_J$ has nilpotence class less than $p$, then there is a unique $P_J$-equivariant isomorphism of algebraic groups

$$\varepsilon_J : u_J \sim U_J$$

such that $\varepsilon_J(se_\alpha) = \varphi_\alpha(s)$ for all $s \in k$ and $\alpha \in (\Phi + \Phi_J^+)$. 

**Remark 2.2.** This is variation of the result as stated in [3], but is easily seen to be equivalent when looking at its proof.

It was the idea of Carlson-Lin-Nakano in [1] to use these results above to obtain a type of exponential map on the restricted nullcone by extending $\varepsilon_J$. Indeed this was achieved in Theorem 3 of loc. cit. under the assumption that $N_1(g)$ is a normal variety. This assumption is known to hold in particular cases, for example if $p \geq h$ (in which case the entire nullcone is restricted), and also for simple groups of type $A$.

Our first result shows that $\varepsilon_J$ always extends to a well-defined map on $N_1(g)$, regardless of whether or not the latter variety is normal. Furthermore, we show that this extended map is independent of the choice of $J \subseteq \Pi$. As noted in [2], the choice of $J$ for which $G \cdot u_J = N_1(g)$ is not necessarily unique, and the exponential map in [1] Theorem 3 is seemingly dependent this choice (of course in the case that $p \geq h$, this is not an issue).

**Proposition 2.3.** Let $I, J \subseteq \Pi$ be such that both $U_I$ and $U_J$ are of nilpotence class less than $p$.

1. If $g \in G$ and $x \in u_J$ are such that $g \cdot x \in u_I$, then $\varepsilon_J(g \cdot x) = g \cdot \varepsilon_J(x)$.
2. If $g \in G$, $x \in u_I$, and $y \in u_J$ are such that $y = g \cdot x$, then $\varepsilon_I(y) = g \cdot \varepsilon_J(x)$.

**Proof.**

1. Suppose first that $g \in N_G(T)$. Conjugation by $g$ sends $P_J$ to some other parabolic subgroup $P$ with unipotent radical $U$. This map defines group isomorphisms $U_J \sim U$ and $u_J \sim u$, and it is not hard to see then that the Serre exponential $\varepsilon : u \rightarrow U$ is given by $\varepsilon(g \cdot x) = g \cdot \varepsilon_J(x)$. Indeed, this is true on the root subgroups which generate $U$, and hence on all $g \cdot x \in u$. Therefore to prove the statement above we must prove that $\varepsilon_J$ and $\varepsilon$ agree on $u_I \cap u$. We have that $u_I \cap u$ is generated by those $e_\alpha$ where both $\alpha$ and $g \cdot \alpha$ are in $(\Phi + \Phi_J^+)$. Both $\varepsilon$ and $\varepsilon_J$ then restrict to group isomorphisms between $u_J \cap u$ and $U_J \cap U$. We also have that $\varepsilon(e_\alpha) = \varepsilon_J(e_\alpha)$ for all $e_\alpha \in u_J \cap u$. This then says that $\varepsilon_J$ and $\varepsilon$ are group isomorphisms which agree on a set of group generators for $u_J \cap u$, hence they agree on all of $u_I \cap u$. Thus $\varepsilon_J(g \cdot x) = \varepsilon_J(g \cdot x) = g \cdot \varepsilon_J(x)$, so the result is true for all $g \in N_G(T)$.

Now let $g$ and $x$ be any pair such that both $x$ and $g \cdot x$ are in $u_I$. By the Bruhat decomposition of $G$, we know that $g = b_1b_2$, where $b_1, b_2 \in B$ and $n \in N_G(T)$. We have then that $b_1n_2 \cdot x \in u_I$ if and only if $nb_2 \cdot x \in u_J$. We also know that $b_2 \cdot x \in u_J$. By the $P_J$-equivariance of $\varepsilon_J$ we have that $\varepsilon_J(b_2 \cdot x) = b_2 \cdot \varepsilon_J(x)$ and $\varepsilon_J(b_1 \cdot (nb_1 \cdot x)) = b_1 \cdot \varepsilon_J(nb_2 \cdot x)$. Combining these with the above result for $n$ then establishes the first claim.

The proof of (2) is similar. Given $g \cdot x = y \in u_J$ for some $x \in u_J$, we again can write $g = b_1n_2$ as above. Replacing $x$ with $b_2 \cdot x \in u_J$ and $y$ with $b_1^{-1} \cdot y \in u_J$, the
Theorem 2.4. Let \( G \) be a connected reductive group over \( k \), and assume \( p \) is good for \( G \). Then there is a unique \( G \)-equivariant bijection
\[
\exp : \mathcal{N}_1(\mathfrak{g}) \xrightarrow{\sim} \mathcal{U}_1(G)
\]
with the following properties:

1. If \( U \) is the unipotent radical of a parabolic subgroup \( P \leq G \) such that \( U \) has nilpotence class less than \( p \), then \( \exp \) restricts to an isomorphism \( \mathfrak{u} \xrightarrow{\sim} U \) of algebraic groups having tangent map equal to the identity on \( \mathfrak{u} \).

2. For each \( X \in \mathcal{N}_1(\mathfrak{g}) \) we obtain a one-parameter additive subgroup of \( G \) \( \exp_X \) defined by sending \( s \in \mathbb{G}_a \) to \( \exp(sX) \).

3. For \( X, Y \in \mathcal{N}_1(\mathfrak{g}) \), \( [X, Y] = 0 \iff (\exp(X), \exp(Y)) = 1 \).

Proof. Since \( G \) is a connected reductive group, the \( p \)-nilpotent elements in \( \mathfrak{g} \) and the \( p \)-unipotent elements in \( G \) all come from \( G' \), so we may assume that \( G \) is semisimple. Suppose first that \( G \) is also simply-connected. In the case that \( G \) is simple and simply-connected. For all \( X \in \mathcal{N}_1(\mathfrak{g}) \), we can write \( X = g \cdot Y \) for some \( g \in G, Y \in \mathfrak{u}_J \). We then define \( \exp(X) = g \cdot \varepsilon_J(Y) \). By Proposition 2.3 \( \exp \) is well-defined since if \( X = g_1 \cdot Y_1 = g_2 \cdot Y_2 \), with \( Y_1, Y_2 \in \mathfrak{u}_J \), then we have
\[
\varepsilon_J(Y_1) = \varepsilon_J(g_1^{-1}g_2 \cdot Y_2) = g_1^{-1}g_2 \cdot \varepsilon_J(Y_2),
\]
so that \( g_1 \cdot \varepsilon_J(Y_1) = g_2 \cdot \varepsilon_J(Y_2) \). The uniqueness of \( \exp \), as well as properties (1)-(2), follow from Propositions 2.1 and 2.3. To see that \( \exp \) is a bijection, we note that the argument used in Proposition 2.3 applies equally well to the inverse maps \( \varepsilon_J^{-1} \), thus it follows that there is a well-defined map \( \log : \mathcal{U}_1(G) \to \mathcal{N}_1(\mathfrak{g}) \) which is inverse to \( \exp \). As this bijection is \( G \)-equivariant, \( C_G(X) = C_G(\exp(X)) \) for all \( X \in \mathcal{N}_1(\mathfrak{g}) \).

Suppose now that \( X, Y \in \mathcal{N}_1(\mathfrak{g}) \) and \( [X, Y] = 0 \). The assumptions on \( G \) imply by [5] \( \S \)2.5 and \( \S \)2.6 that \( \text{Lie}(C_G(X)) = \mathfrak{g}_p(X) \) (as the nilpotent and unipotent varieties of \( GL_n \) and \( SL_n \) coincide, we may work with \( GL_n \) for type \( A \)). Thus,
\[
Y \in \mathfrak{g}_p(X) = \text{Lie}(C_G(X)) = \text{Lie}(C_G(\exp(X)))
\]
and so the adjoint action of \( \exp(X) \) on \( Y \) is trivial. This implies then that \( \exp(X) \in C_G(Y) = C_G(\exp(Y)) \), so that \( (\exp(X), \exp(Y)) = 1 \).

Conversely, if \( \exp(X) \) and \( \exp(Y) \) commute, then the adjoint action of \( \exp(X) \) fixes \( Y \), hence fixes \( sY \) for all \( s \in k \). But this means that \( \exp(X) \) commutes with \( \exp(sY) \) for all \( s \), so that \( \exp_X \) defines a one-parameter subgroup of \( C_G(\exp(X)) \), and thus \( Y \in \text{Lie}(C_G(\exp(X))) = \text{Lie}(C_G(X)) \), hence \( [X, Y] = 0 \).

Suppose now that \( G \) is any semisimple group. Then there is a central isogeny \( \psi : G_{sc} \to G \), and this map induces homeomorphisms (and in particular bijections) between both the \( p \)-nilpotent and \( p \)-unipotent varieties of the two groups, which are variety isomorphisms in very good characteristic. Further, by Lemmas 4.3 and 9.7 of [7] we see that for any \( X, Y \in \mathcal{N}_1(\mathfrak{g}_{sc}) \), \( [X, Y] = 0 \) if and only if \( [d\psi(X), d\psi(Y)] = 0 \).
Remark 2.5. If \( p \) is a very good prime for \( G \), then \( G : X \cong G/C_G(X) \) for every \( X \in \mathcal{N}(\mathfrak{g}) \) (see sections 2.1, 2.2, and 2.9 of [X]). Since \( C_G(X) = C_G(\exp(X)) \) for all \( X \in \mathcal{N}_1(\mathfrak{g}) \), \( \exp \) restricts to variety isomorphisms between the \( G \)-orbits of \( \mathcal{N}_1(\mathfrak{g}) \) and \( \mathcal{U}_1(G) \), and if \( \mathcal{N}_1(\mathfrak{g}) \) is normal, to an isomorphism from \( \mathcal{N}_1(\mathfrak{g}) \) to \( \mathcal{U}_1(G) \). Such an isomorphism is given, under this normality hypothesis, in [1, Theorem 3]. The main distinction between that result and Theorem [2.4] is that we show the map \( \exp \) to exist regardless of normality, and moreover prove it to be independent of the choice of \( J \).

3. Infinitesimal One-parameter Subgroups

Let \( F \) denote the standard Frobenius morphism on \( \mathbb{G}_a \), defined by \( F(s) = sp \). For any \( X \in \mathcal{N}_1(\mathfrak{g}) \) and any \( i \geq 0 \), we denote by \( \exp_{X}^{(i)} \) the one parameter subgroup of \( G \) given by \( \exp_{X} \circ F^{i} \). It follows from Theorem 2.4 that given any set \((X_0, X_1, \ldots, X_i)\) of pairwise commuting elements in \( \mathcal{N}_1(\mathfrak{g}) \) we obtain a one-parameter subgroup of \( G \)

\[
\exp_{X_0} \exp_{X_1} \cdots \exp_{X_i}^{(0)}(s) = \exp(sX_0)\exp(sX_1) \cdots \exp(s^{i}X_i)
\]

We will show below that every infinitesimal one-parameter subgroup of \( G \) comes from restricting a one-parameter subgroup of the form above. We start by establishing a useful lemma which is valid for any algebraic group over \( k \).

Lemma 3.1. Let \( H \) be an affine group scheme such that \( k[H] \) is a finitely generated \( k \)-algebra, and let \( \varphi_1, \varphi_2 \in \text{Hom}_{k/k}(\mathbb{G}_{a(r)}, H) \). Let \( 0 < m < r \), and suppose that \( d\varphi_1(u_i) = d\varphi_2(u_i) \), for all \( i < m \). Then \( d\varphi_1(u_m) - d\varphi_2(u_m) \) is \( \mathfrak{h} \).

Proof. Let \( \Delta_H \) and \( \Delta_{G_a} \) denote the comultiplication maps on \( \text{Dist}(H) \) and \( \text{Dist}(\mathbb{G}_a) \) respectively. We have then that

\[
\Delta_H(d\varphi_1(u_m) - d\varphi_2(u_m)) = d\varphi_1 \otimes d\varphi_1(\Delta_{G_a}(u_m)) - d\varphi_2 \otimes d\varphi_2(\Delta_{G_a}(u_m)),
\]

and

\[
\Delta_{G_a}(u_m) = u_m \otimes 1 + 1 \otimes u_m + \sum y_1 \otimes y_2,
\]

with each \( y_i \) contained in \( \text{Dist}(\mathbb{G}_{a(r-1)}) \subseteq \text{Dist}(\mathbb{G}_{a(r)}) \). By assumption, \( d\varphi_1 \) agrees with \( d\varphi_2 \) on \( \text{Dist}(\mathbb{G}_{a(r-1)}) \). From this it follows that

\[
\Delta_H(d\varphi_1(u_m) - d\varphi_2(u_m)) = (d\varphi_1(u_m) - d\varphi_2(u_m)) \otimes 1 + 1 \otimes (d\varphi_1(u_m) - d\varphi_2(u_m)).
\]

Thus, the comultiplication of \( d\varphi_1(u_m) - d\varphi_2(u_m) \) is primitive. By [13, §3.18], the primitive elements in \( \text{Dist}(H) \) all lie in \( \mathfrak{h} \), proving the claim. \( \square \)

Theorem 3.2. Let \( G \) be a connected reductive group over \( k \), and assume that \( p \) is very good for \( G \). Then there is an identification between \( \text{Hom}_{k/k}(\mathbb{G}_{a(r)}, G) \) and

\[
C_r(\mathcal{N}_1(\mathfrak{g})) := \{(X_0, X_1, \ldots, X_{r-1}) \mid X_i \in \mathcal{N}_1(\mathfrak{g}), [X_i, X_j] = 0 \}.
\]

This map sends the \( r \)-tuple \((X_0, X_1, \ldots, X_{r-1})\) to the homomorphism
Proof. As before, we may assume that $G$ is semisimple as any homomorphism from a unipotent group scheme to $G$ must factor through its derived subgroup (since $G/G'$ is diagonalizable). Suppose also that $G$ is simply-connected. We may then assume that $G$ is simple and simply-connected. In the case of $SL_n$ we can again work instead with $GL_n$.

1) First, any commuting $r$-tuple $(X_0, X_1, \ldots, X_{r-1})$ defines a one-parameter subgroup of $G$, and thus by restriction an element in $\text{Hom}_{a/k}(G(a(r)), G)$. To see that this assignment is injective, we can follow the proof of [7, Theorem 9.6]. Suppose that $\exp X_0 \exp^{(1)} \cdots \exp^{(r-1)} = \exp Y_0 \exp^{(1)} \cdots \exp^{(r-1)}$. Because these homomorphisms agree when restricted to $G(a(1))$, we must have that $X_0 = Y_0$. We can therefore multiply each homomorphism by $\exp -X_0$, and since $\exp X_0 \exp -X_0 = Id$, it follows that $\exp^{(1)} X_1 \cdots \exp^{(r-1)} X_{r-1} = \exp^{(1)} Y_1 \cdots \exp^{(r-1)} Y_{r-1}$. With $F$ denoting the Frobenius morphism on $G(a)$, this says that

$$(\exp X_1 \cdots \exp^{(r-2)} X_{r-1}) \circ F = (\exp Y_1 \cdots \exp^{(r-2)} Y_{r-1}) \circ F$$

when restricted to $G(a(r))$. As $F(G(a(r)) = G(a(r-1))$, we can proceed by induction to see that $X_i = Y_i$ for all $i$.

To prove the map is surjective, suppose that $\phi : G(a(r)) \rightarrow G$ is an infinitesimal one-parameter subgroup of $G$. We define recursively the following sequence of elements in $\text{Dist}(G)$:

$$X_0 = d\phi(u_0); \quad X_i = d\phi(u_i) - d(\exp X_0 \exp^{(1)} \cdots \exp^{(i-1)})(u_i)$$

We will prove by induction that this defines a sequence of commuting elements in $N_1(g)$, and further that for any $i \leq r$, the homomorphism $\exp X_0 \exp^{(1)} \cdots \exp^{(i-1)}$ is the same as $\phi$ when restricted to $G(a(i))$.

Because $u_i^p = 0$, we have that $X_0 \in N_1(g)$, and that $\exp X_0$ and $\phi$ are equal on $G(a(1))$. It follows from Lemma 3.3 that $X_1 = d\phi(u_1) - d\exp X_0(u_1) \in g$. To see that $X_i^p = 0$, we note first that both $d\phi(u_1)$ and $d\exp X_0(u_1)$ commute with $X_0$, as $u_0$ and $u_1$ commute in $\text{Dist}(G_a)$ and $X_0 = d\phi(u_0) = d\exp X_0(u_0)$. Thus $X_1 \in C_g(X_0) = \text{Lie}(C_G(X_0)) = \text{Lie}(C_G(\exp(sX_0)))$ for all $s \in k$. Thus the image of the one parameter subgroup of $G$ given by $\exp X_0$ acts trivially in the adjoint action on $X_1$, so that $d\exp X_0(u_i)$ commutes with $X_1$ for all $i$. But $[d\exp X_0(u_0), X_1] = 0$ if and only if $[d\exp X_0(u_1), d\phi(u_1)] = 0$, therefore

$$X_i^p = (d\phi(u_1) - d\exp X_0(u_1))^p = d\phi(u_1)^p - d\exp X_0(u_1)^p = 0$$

so that $X_i \in N_1(g)$. Suppose now that $X_0, \ldots, X_{i-1}$ have been chosen as above, so that $\exp X_0 \cdots \exp^{(i-1)}$ is the same as $\phi$ when restricted to $G(a(i))$, and that $X_i = d\exp X_0 \cdots \exp^{(i-1)}(u_i) - d\phi(u_i)$ is $p$-nilpotent. Then to complete the inductive step we must show that $\exp X_0 \cdots \exp^{(i)} X_i$ is equal to $\phi$ on $G(a(i+1))$ and that $d\exp X_0 \cdots \exp^{(i)}(u_{i+1})$ commutes with $d\phi(u_{i+1})$. 


Following the general argument presented in [11 Section 2], we first note that we can factor \( \exp_{X_0} \cdots \exp_{X_i}^{(i)} \) as

\[
\mathbb{G}_a \xrightarrow{\delta} \mathbb{G}_a \times \mathbb{G}_a \xrightarrow{\exp_{X_0} \cdots \exp_{X_{i-1}}^{(i-1)} \times \exp_{X_i}^{(i)}} G \times G \xrightarrow{m} G
\]

where \( m \) is the multiplication map on \( G \) and \( \delta \) is the diagonal map \( s \mapsto (s, s) \). By the reasoning given in the proof of [11, Proposition 2.3], it then follows that

\[
d(\exp_{X_0} \cdots \exp_{X_i}^{(i)})(u_i) = d(\exp_{X_0} \cdots \exp_{X_{i-1}}^{(i-1)})(u_i) + \exp_{X_i}(u_0)
\]

\[
= d(\exp_{X_0} \cdots \exp_{X_{i-1}}^{(i-1)})(u_i) + X_i
\]

\[
= d\phi(u_i)
\]

Therefore these maps agree on \( Dist(\mathbb{G}_a(i+1)) \). We further see that \( X_{i+1} \) commutes with \( X_0 \) for the same reason that \( X_1 \) did above. Hence \( X_{i+1} \) commutes with \( \exp_{X_0}(u_1) \), and so commutes with \( d\phi(u_1) - \exp_{X_0}(u_1) = X_1 \). Proceeding in this way, we see that \( X_{i+1} \) commutes with \( X_j \) for \( j \leq i \), hence with \( \exp_{X_i}(u_\ell) \) for all \( \ell \), and therefore \( X_i \) commutes with \( \exp_{X_0} \cdots \exp_{X_i}^{(i)}(u_{i+1}) \). This implies that \( X_{i+1} \) is \( p \)-nilpotent, completing the inductive step.

\[\square\]

4. Saturation

In this section we recall the “saturation problem” for \( p \)-unipotent elements in \( G \), which was introduced by Serre in [10], and then investigated successfully by Seitz in [9]. We show that the exponential map provides a solution to this problem, and that this is the same answer as that already given in [9].

The problem is to associate to all \( g \in \mathcal{U}_1(G) \) a canonical one-parameter subgroup \( \phi_g \) of \( G \) with the property that \( \phi_g(1) = g \). In [10 §4] it was shown that for \( GL_n \) the saturation problem is solved by assigning to each \( g \in \mathcal{U}_1(GL_n) \) the one-parameter subgroup \( \phi_g \) defined by \( \phi_g(s) = \exp_p(s(\log_p(g))) \), where

\[
\log_p(g) = (g - 1) - \frac{(g-1)^2}{2} + \frac{(g-1)^3}{3} + \cdots + \frac{(-1)^p(g-1)^{p-1}}{p-1}!
\]

and for \( X \in \mathcal{N}_1(\mathfrak{g}_l) \)

\[
\exp_p(X) = 1 + X + \frac{X^2}{2} + \cdots + \frac{X^{p-1}}{(p-1)!}
\]

For \( GL_n \), the map \( \exp \) in Theorem 2.3 corresponds precisely to the \( p \)-truncated exponential series. This is not surprising, and can be shown by Proposition 4.3 (below) together with the remarks following [9 Theorem 1.3].

We immediately have the following:

**Proposition 4.1.** If \( G \) is a connected reductive group and \( p \) is good for \( G \), then every \( p \)-unipotent element \( g \in G \) lies in a canonical one-parameter subgroup \( \phi_g \), where

\[
\phi_g(s) = \exp(s(\exp^{-1}(g))), \text{ for all } s \in k.
\]
In order to compare this solution to that given in [9], we must first recall the definition of a good $A_1$ subgroup. A closed subgroup $A \leq G$ is of type $A_1$ if $A$ is isomorphic to $SL_2$ or $PSL_2$. Let $T_A$ be a maximal torus of $A$. We say that $A$ is good if $g$, as a $T_A$-module, has weights which are $\leq 2p - 2$. Good $A_1$ subgroups were used by Seitz to specify the canonical one-parameter subgroups which contain $p$-unipotent elements. Specifically, he proved the following:

**Theorem 4.2.** [9] There is a unique monomorphism $\psi_g : G_a \to G$ with image contained in a good $A_1$ and satisfying $\psi_g(1) = g$.

We are now in position to prove:

**Proposition 4.3.** The one-parameter subgroups $\phi_g$ and $\psi_g$ agree for every $g \in U_1(G)$.

**Proof.** First, by the definition of $\psi_g$ given in [9] we may assume that it comes from a homomorphism $\psi : SL_2 \to G$ such that

$$\psi_g(a) = \psi\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\right).$$

Let $T_A$ be the image of the diagonal subgroup of $SL_2$, and for each $c \in k^\times$ we will use the notation

$$\psi_T(c) = \psi\left(\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}\right).$$

Let $X_0$ be the image of $u_0 \in Dist(G_a)$ under $d\psi_g$, the differential of the one-parameter subgroup $\psi_g$. We see that in the adjoint action of $G$, $X_0$ is a weight vector for $T_A \leq G$ with weight 2. Moreover, $d\psi_g(u_i)$ is a weight vector of weight $2p^i$ for $T_A \leq G$ acting on $Dist(G)$.

Since $\exp$ is $G$-equivariant, we have for each $c \in k^\times$ and $a \in k$ that

$$\psi_T(c)\exp_{X_0}(a)\psi_T(c^{-1}) = \exp_{X_0}(c^2a).$$

It follows that each element $d\exp_{X_0}(u_1) \in Dist(G)$ is also a weight vector for $T_A$ of weight $2p^i$. By Lemma [3.1] $d\psi_g(u_1) - d\exp_{X_0}(u_1)$ is an element of $g$, and by preceding remarks is a weight vector of $T_A$ of weight $2p$. But all non-zero weight vectors of $T_A$ on $g$ are $\leq 2p - 2$, thus $d\psi_g(u_1) = d\exp_{X_0}(u_1)$. Continuing in this way we have that $d\psi_g(u_i) = d\exp_{X_0}(u_i)$ for all $i$, from which is follows that $\psi_g = \exp_{X_0}$. As $\exp_{X_0}(1) = g$, we have that $\phi_g = \exp_{X_0}$, completing the proof. □

**Acknowledgements:** This research was partially supported by grants from the Australian Research Council (DP1095831, DP0996774 and DP120101942).

**References**

[1] J. Carlson, Z. Lin, and D. Nakano, *Support Varieties for modules over Chevalley groups and classical Lie algebras*, Trans. A.M.S. 360 (2008), 1870-1906.

[2] J. Carlson, Z. Lin, D. Nakano, and B. Parshall, *The restricted nullcone*, Contemp. Math., 325 (2003), 51-75.

[3] E. Friedlander, *Support varieties for rational $G$-modules*, preprint.

[4] E. Friedlander, A. Suslin, *Cohomology of a finite group scheme over a field*, Invent. Math. 127 (1997), 209-270.

[5] J.C. Jantzen, *Nilpotent Orbits in Representation Theory*, Progr. Math. 228, Birkhäuser, Boston, 2004.
[6] J.C. Jantzen, *Representations of Algebraic Groups*, 2nd ed. Mathematical Surveys and Monographs, 107, American Mathematical Society 2003.

[7] G. McNinch, *Abelian unipotent subgroups of reductive groups*, J. Pure Appl. Algebra, 167 (2002), no. 2-3, 269-300.

[8] D.K. Nakano, B.J. Parshall, and D.C. Vella, *Support varieties for algebraic groups*, J. Reine Angew. Math. 547 (2002), 15-49.

[9] G. Seitz, *Unipotent elements, tilting modules, and saturation*, Invent. Math. 141 (2000) 3, 467-502.

[10] J.-P. Serre, *Sur la semi-simplicite des produits tensoriels de representations de groupes*, Invent. Math. 116 (1994)

[11] P. Sobaje, *Support varieties for Frobenius kernels of classical groups*, J. Pure and Appl. Algebra, 216 (2012), pp. 2657-2664.

[12] P. Sobaje, *On exponentiation and infinitesimal one-parameter subgroups of reductive groups*, J. Algebra 385 (2013), pp. 14-26.

[13] A. Suslin, E. Friedlander, and C. Bendel, *Infinitesimal 1-parameter subgroups and cohomology*, Journal of the A.M.S. 10 (1997), 693-728.

[14] M. Takeuchi, *Tangent coalgebras and hyperalgebras I.*, Japan. J. Math. 42 (1974), 1-143.