A note on the longest common Abelian factor problem

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Abstract. Abelian string matching problems are becoming an object of considerable interest in last years. Very recently, Alatabbi et al. [1] presented the first solution for the longest common Abelian factor problem for a pair of strings, reaching $O(\sigma n^2)$ time with $O(\sigma n \log n)$ bits of space, where $n$ is the length of the strings and $\sigma$ is the alphabet size. In this note we show how the time complexity can be preserved while the space is reduced by a factor of $\sigma$, and then how the time complexity can be improved, if the alphabet is not too small, when superlinear space is allowed.

1 Introduction

The longest common Abelian factor (LCAF) problem, posed at the String Masters 2013 meeting by Thierry Lecroq and Arnaud Lefebvre, can be stated like that: Given two strings $A$ and $B$, both of length $n$, over the alphabet $\Sigma$, compute the maximal length of a factor in $A$ such that there exists a factor in $B$ being its permutation (i.e., being an Abelian match). Moreover, it is desirable to return some (or all) occurrences of such factors in $A$ and $B$.

To our knowledge, the only work on this problem was presented very recently by Alatabbi et al. [1], in which they obtained $O(\sigma n^2)$ worst-case time with $O(\sigma n \log n)$ bits of space, where $n$ is the length of the strings and $\sigma$ is the alphabet size. Further on, we will express the space in words, and the cited space becomes $O(\sigma n)$ words.

While the Alatabbi et al. algorithm is simple, let us note that the same result can be immediately obtained by a reduction from a well-known problem, the (standard) longest common factor (LCF) [1]. Hui [3] showed that using a generalized suffix tree it is possible to find the LCF for a pair of strings of length $n$ in $O(n)$ time. We use this algorithm $n$ times, for each factor length $\ell$, replacing each $\ell$ symbol long factor by its Parikh vector followed with a unique terminator (e.g., for the factors taken from $A$ the subsequent terminators can be $-1$, $-2$).

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1 Also known as the longest common substring (LCS) problem. We prefer the word “factor” in the problem name, to avoid confusion with the abbreviation for the longest common subsequence.
..., while for the factors taken from $B$ they can be $-n-1$, $-n-2$, ...). The terminators disallow to have matches longer than $\sigma$. If the found LCF is of length exactly $\sigma$, it must correspond to a pair of factors, one from $A$ and one from $B$, of length $\ell$. This is obtained in $O(\sigma n)$ time for one value of $\ell$, using $O(\sigma n)$ space, hence the total time, for all possible factor lengths, becomes $O(\sigma n^2)$ with $O(\sigma n)$ space (we build and discard the generalized suffix trees one by one). In this way, we obtained the same time and space as Alatabbi et al. did.

2 Preliminaries

Let $S$ be a string of length $n$ over an alphabet $\Sigma$ of size $\sigma = |\Sigma|$. It can also be written as $S[1 \ldots n]$, where $S[i], 1 \leq i \leq n,$ denotes its $i$-th symbol. An analogous notation will be used for arrays.

Throughout the note we assume that $\sigma = O(n)$ and $\Sigma = \{1, 2, \ldots, \sigma\}$. (If this is not the case, we can remap the alphabet for both input strings at the start with standard means, in $O(n \log n)$ time and $O(n)$ extra space.)

The Parikh vector for string $S$, denoted as $P(S)[1 \ldots \sigma]$, is defined as a vector (array) of size $\sigma$ storing the number of occurrences of each alphabet symbol in $S$. Formally, $P(S)[c] = k$ if $|\{i : S[i] = c\}| = k$, for any alphabet symbol $c$. For two strings $S$ and $T$ of equal length and over a common alphabet, we say that the Parikh vector $P(S)$ is (lexicographically) smaller than the Parikh vector $P(T)$, denoted as $P(S) < P(T)$, iff there exists an alphabet symbol $c'$, $1 \leq c' \leq \sigma$, such that $P(S)[c] = P(T)[c]$ for all $c < c'$ and $P(S)[c'] > P(T)[c']$. The two Parikh vectors are equal, i.e., $P(S) = P(T)$, when $P(S)[c] = P(T)[c]$ for all symbols $c$.

3 Reducing the space

First, let us note that recently Kociumaka et al. showed that for any tradeoff parameter $1 \leq \tau \leq n$, the LCF problem can be solved in $O(\tau)$ space and $O(n^2/\tau)$ time. Applying this to the LCAF problem, we obtain $O(\tau \sigma n^2)$ time using $O(\sigma n/\tau)$ space, for any $1 \leq \tau \leq \sigma n$.

Yet, the specifics of LCAF allow for a better result. We consider each factor length $\ell$ separately. For a given $\ell$, we sort all $n - \ell + 1$ factors of $A$ according to their Parikh vectors, using the LSD radix sort. Each factor is represented as its start position in $A$. There are $\sigma$ passes of the radix sort and accessing the keys’ “digits” seems to be the soft spot of this variant. Yet, before each pass of the radix sort we scan $A$ and for each $\ell$-sized window collect the count of the corresponding symbol in it. More precisely, just before the $i$-th pass of the radix sort, in which the keys will be distributed according to $P(\cdot)[\sigma - i + 1]$, we compute and store $P(A[j \ldots j + \ell - 1])[\sigma - i + 1]$ for each factor $A[j \ldots j + \ell - 1]$, using $O(n)$ time and $O(n)$ extra space. Thanks to it, we can access a digit in the radix sort in constant time. After the $i$-th pass, the $P(\cdot)[\sigma - i + 1]$ statistics are discarded. In this way, sorting of the $\ell$-long factors of $A$ takes $O(\sigma n)$ time and its output (and working area) requires $O(n)$ words of space.
We sort the factors of $B$ in the same way. Additionally, for every $\sigma$-th evenly sampled $\ell$-long factor of $A$ and $B$, we store explicitly its Parikh vector using $O(\sigma)$ space. More precisely, we compute and store the Parikh vectors for the factors $A[1 \ldots \ell], A[\sigma+1 \ldots \sigma+\ell], A[2\sigma+1 \ldots 2\sigma+\ell], \ldots$, and similarly for $B[1 \ldots \ell], B[\sigma+1 \ldots \sigma+\ell], B[2\sigma+1 \ldots 2\sigma+\ell], \ldots$. As we scan the strings from left to right and compute the successive Parikh vectors incrementally (first making a copy of the previous vector), this phase takes $O(n + (n/\sigma)\sigma) = O(n)$ time and $O(n)$ space.

The computed Parikh vectors serve to speed up factor comparisons during the last phase, which is to intersect the lists of factors from $A$ and $B$, similarly as in a binary merge operation. Thanks to the Parikh vectors kept in regular intervals of $A$ and $B$, each factor comparison takes $O(\sigma)$ time, therefore the intersection takes $O(\sigma n)$ time.

The total cost of the described procedure, over all relevant factor lengths, becomes $O(\sigma n^2)$ and the required space is $O(n)$. This matches the time complexity of the Alatabbi et al. solution, yet the space usage is decreased by a factor of $\sigma$.

4 Reducing the time

4.1 The general variant

In this section we present a variant which achieves $o(\sigma n^2)$ time for the price of superlinear space. The key idea is to sort together factors of varying (yet close) lengths.

The whole sorting phase runs in $\Theta(n/k)$ steps, $k < \sigma$, where in the $i$-th step the factors of both $A$ and $B$ of all lengths from $ik+1$ to $(i+1)k$ are considered (yet, each group of factors, defined by their length, is sorted separately). The required space grows to $O(kn)$. To improve the time complexity, it is crucial to perform one step in $o(k\sigma n)$ time. To this end, we make use of a data-oblivious sorting algorithm. An algorithm is called data-oblivious if its sequence of possible memory accesses is independent of its input values. There exist such sort algorithms working in $O(n \log n)$ worst-case time (assuming that keys can be accessed in constant time), see [2] and references therein.

In our scenario, we compare the Parikh vectors of two factors of length $ik+1$ in $O(\sigma)$ time and also collect all the positions $i, 1 \leq i \leq \sigma$, at which the respective Parikh vectors have different values. These positions are inserted in bulk into a balanced binary search tree $T$, in $O(\sigma)$ time. Let the two factors be $A[u \ldots u+ik]$ and $A[v \ldots v+ik]$. The next comparison concerns the factors of length $ik+2$: $A[u \ldots u+ik+1]$ and $A[v \ldots v+ik+1]$. Their Parikh vectors can be obtained with updating only one counter in the previous vectors, which can also affect $T$, as up to two elements should now be added to $T$ and up to two elements should be removed $T$. The operations on $T$, including finding its minimum (or finding out that $T$ is empty), which immediately serves to resolve the factor comparison, take $O(\log |T|) = O(\log \sigma)$ time. Similarly we handle the next pairs of factors, up to length $(i+1)k$. Each time when equal (in the Abelian sense) factors are found and one of them is from $A$ and the other from $B$, we record their starting
positions (in $A$ or $B$) and length. In this way, we cannot miss the longest Abelian matching factors. Note that in a comparison based sort, and in particular in a deterministic data-oblivious sort, it is impossible not to compare equal items at some moment, if such exist. To see this, imagine that we associate a real number with each item according to the sorted order; that is, the smallest number will have the smallest number and the largest item the largest number, and equal items will have equal associated numbers. Now, if two items, $x$ and $y$, are equal and no other item in the collection is equal to $x$, not comparing $x$ to $y$ in the sorting process would mean that $x$ and $y$ are indistinguishable. If, say, after the sorting $x$ stands (just) before $y$ and imagine $x$ is modified in such a way that its associated value gets greater by $\varepsilon/2$, where $\varepsilon$ is the minimum absolute difference between the associated values for any non-equal items in the collection, the hypothetical sort algorithm not comparing $x$ to $y$ would produce the same output as before, which of course means that the algorithm is incorrect.

One step of the presented sort algorithm takes $O((\sigma + k \log \sigma)n \log n)$ time, which sums up to $O((\sigma/k + \log \sigma)n^2 \log n)$ time over all steps, and the space usage is $O(kn)$. Note that a space-time tradeoff is obtained with $k$ between 2 and $\sigma/\log \sigma$. For example, we can set $k = \sqrt{\sigma}$, which gives $O(\sqrt{\sigma}n^2 \log n)$ time and $(\sqrt{\sigma}n)$ space. This time complexity is $o(\sigma n^2)$ when $\sigma = \omega(\log^2 n)$.

4.2 Faster, sometimes

In the algorithm above, the Parikh vectors of factors of length $ik + 1$ were compared in $O(\sigma)$ time. Let us try to reduce this time, trying to obtain a better overall space-time tradeoff.

To this end, for each length $ik + 1$ we compute and store the Parikh vectors for factors of $A$ and $B$ sampled every $d$-th position, where $d < \sigma$ will be chosen later. Additionally, we compute the positions of the differences between each of the $\Theta(n^2/d^2)$ pairs of Parikh vectors, storing them in a balanced binary search tree, as described in the previous subsection. This requires overall $O(\sigma n^3/(d^2k))$ extra time and $O(\sigma n^2/d^2)$ extra space. However, the “main” time component gets reduced to $O((d/k + \log \sigma)n^2 \log n)$. As we are interested in improving the space-time tradeoff, we need to check if $d$ can be set to such value that the space complexity is not compromised, yet the time complexity improves, at least for some $k$ and $\sigma$. Clearly, it requires that $\sigma n^2/d^2 = O(kn)$, i.e., $d = \Omega(\sqrt{\sigma n}/k)$. As only $d = o(\sigma)$ may improve the time complexity, we need to have $n/k = o(\sigma)$ (and of course $\sigma = \omega(1)$). An extra requirement is $k = o(\sigma/\log \sigma)$. Finally, improving the time complexity means that $(d/k + \log \sigma)n^2 \log n + \sigma n^3/(d^2k) = o((\sigma/k + \log \sigma)n^2 \log n)$, which does not introduce an extra constant since $\sigma n^3/(d^2k) = O(n^3)$, given the aforementioned lower bound on $d$.

We set $d = \Theta(\sqrt{\sigma n}/k)$. This implies $d = \omega(\sqrt{n \log \sigma})$ and thus also $\sigma = \omega(\sqrt{n \log n})$, which eventually gives $d = \omega(\sqrt{n \log n})$.

To sum up, if $\sigma = \omega(\sqrt{n \log n})$ and $k = \omega(n/\sigma)$ but also $k = o(\sigma/\log \sigma)$, by choosing $d = \Theta(\sqrt{\sigma n}/k)$ we preserve the $O(kn)$ space and improve the time to $O((\sqrt{\sigma n}/k^3 + \log \sigma)n^2 \log n)$. In most cases the improvement is not large: for example, if $\sigma = n^{0.8}$ and $k = n^{0.4}$, the time complexity is slashed by a factor
of $n^{0.1}$. On the other hand, if e.g. $\sigma = \Theta(n \log n)$ and $k = \Theta(n^{2/3} \log n)$, then the time complexity becomes $O((\log \sigma)n^2 \log n)$, an improvement by a factor of $n^{1/3}$.

5 Conclusions

Finding the longest common Abelian factor is a recently posed problem, with a solution given in [1], achieving $O(\sigma n^2)$ worst-case time and needing $O(\sigma n)$ words of space. A significant weakness of that result is its space requirement, which may be unacceptable with a larger alphabet. In this work we improve this result in two ways.

One algorithm keeps the time complexity of the previous result, while it reduces its space to $O(n)$. This is obtained with very simple means (the key component is the LSD radix sort). The other algorithm of ours increases the space to $O(kn)$ and achieves the time complexity of $O((\sigma/k + \log \sigma)n^2 \log n)$, where $k \leq \sigma/\log \sigma$ is a freely chosen parameter. When $\sigma = \omega(\log n \log \log n)$ it is always possible to choose such $k$ that this algorithm beats the result from [1] in both time and space complexity. This variant is also simple conceptually, yet it makes use of a deterministic data-oblivious sort algorithm of optimal complexity in the comparison based model. There are several such algorithms known, but none of them is really simple. A more practical choice could be the textbook Shell sort algorithm with the sequence of gaps of the form $2^p3^q$, proposed by Pratt in 1972 [5]. Applying this Shell sort variant would deteriorate our time complexity by a factor of $\log n$. The latter of the two algorithms is also improved slightly for convenient values of $\sigma$ and $k$.

We are convinced that better algorithms for the LCAF problem are possible. One obvious line of attack is using word-level parallelism (in the word-RAM model) for Parikh vector comparisons. The anticipated speed-up factor is however only about $w/\log(n/\sigma)$, where $w$ is the machine word size. A more interesting question is whether sharing computations for different factor lengths could be exploited with a stronger effect than presented here.

References

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