Reinforcement Learning with Analogue Similarity to Guide Schema Induction and Attention

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Research in analogue reasoning suggests that higher-order cognitive functions such as abstract reasoning, far transfer, and creativity are founded on recognizing structural similarities among relational systems. Here we integrate theories of analogy with the computational framework of reinforcement learning (RL). We propose a psychology theory that is a computational synergy between analogy and RL, in which analogue comparison provides the RL learning algorithm with a measure of relational similarity, and RL provides feedback signals that can drive analogue learning. Simulation results support the power of this approach.

KEYWORDS
Reinforcement Learning, Analogy, Schema Induction, Attention, Representation Learning

1 | INTRODUCTION

How do people learn new abstract concepts? Everyone has a repertoire of concepts they use in making sense of the world and filtering the infinite complexity of experience into manageable chunks. Abstract representations that capture reoccurring patterns in the environment are used for generalization (i.e., making predictions or inferences about states of the environment that haven't been experienced before). How are these representations constructed, and how is their usefulness evaluated? For example, how do people discover and apply concepts such as a ‘fork’ in the game of chess? How do scientists create theories such as natural selection?

The goal of the present work is to develop a computational understanding of how people learn abstract concepts. Previous research in analogue reasoning suggests that higher-order cognitive functions such as abstract reasoning,
far transfer, and creativity are founded on recognizing structural similarities among relational systems [Doumas et al., 2008; Gentner, 1983; Hummel and Holyoak, 2003]. However, we argue a critical element is missing from these theories, in that their operation is essentially unsupervised, merely seeking patterns that recur in the environment, rather than focusing on the ones that are predictive of reward or other important outcomes.

Here we integrate theories of analogy with the computational framework of reinforcement learning (RL). RL offers a family of learning algorithms that have been highly successful in machine learning applications (e.g., Bagnell and Schneider, 2001; Tesaro, 1995) and that have neurophysiological support in the brain (e.g., Schultz et al., 1997). A shortcoming of RL is that it only learns efficiently in complex tasks if it starts with a representation (i.e., a means for encoding stimuli or states of the environment) that somehow captures the critical structure inherent in the task. We formalize this notion below in terms of similarity-based generalization [Shepard, 1987] and kernel methods from statistical machine learning [Shawe-Taylor and Cristianini, 2004]. In other words, RL requires a sophisticated sense of similarity to succeed in realistically complex tasks. Psychologically, the question of how such a similarity function is learned can be cast as a question of learning sophisticated, abstract representations.

This paper proposes a computational model of analogical RL, in which analogical comparison provides the RL learning algorithm with a measure of relational similarity, and RL provides feedback signals that can drive analogical learning. Relational similarity enables RL to generalize knowledge from past to current situations more efficiently, leading to faster learning. Conversely, the prediction-error signals from RL can be used to guide induction of new higher-order relational concepts. Thus we propose there exists a computationally powerful synergy between analogy and RL. The simulation experiment reported here supports this claim. Because of the strong empirical evidence for each of these mechanisms taken separately, we conjecture that the brain exploits this synergy as well.

We want to emphasize that this isn’t a theory of specific psychological phenomena. We’re coming from the perspective that human conceptual learning and invention is far beyond current scientific explanation. There are no existing models that can compare to what the brain achieves. Our goal is to explore where this power might come from. The computational framework we propose is grounded in well-established psychological principles that themselves are supported by large bodies of experimental evidence, but the aim of our model is not to explain specific data. It’s to demonstrate the potential power that comes from combining these principles in the way we propose. Thus the scope of this paper is to understand how the proposed mechanism might work in principle. Future work will derive testable predictions and empirical means for testing them.

In the following sections, we will review analogy and RL, then lay out our computational proposal and present a formal model, then present simulation results and discuss implications and limitations of the model.

2 | ANALOGY

Research in human conceptual knowledge representation has shown that concepts are represented not just as distributions of features (cf. Nosofsky, 1986; Rosch and Mervis, 1975) but as relational structures. This relational knowledge includes both internal structure, such as the fact that a robin’s wings allow it to fly [Sloman et al., 1998], as well as external structure, such as the fact that a dog likes to chase cats [Jones and Love, 2007]. Theories of analogical reasoning represent relational knowledge of this type in a predicate calculus that binds objects to the roles of relations, for example \( \text{CHASE(DOG,CAT)} \). According to these theories, an analogy between two complex episodes (each a network of relations and objects) amounts to recognition that they share a common relational structure [Gentner, 1983; Hummel and Holyoak, 2003].

At a more mechanistic level, the dominant theory of analogy is structural alignment [Gentner, 1983]. This process
involves building a mapping between two episodes, mapping objects to objects and relations to relations. The best mapping is one that maps objects to similar objects, maps relations to similar relations, and most importantly, satisfies parallel connectivity. Parallel connectivity means that, whenever two relations are mapped to each other, the objects filling their respective role-fillers are also mapped together. An example is shown in Figure 1. Parallel connectivity is satisfied here because, for each mapped pair of ATTACK relations (red arrows), the objects filling the ATTACKER role are mapped together (knight is mapped to queen), and the objects filling the ATTACKED role are also mapped together (rook to rook and king to king). Thus structural alignment constitutes a (potentially partial or imperfect) isomorphism between two episodes, which respects the relational structure that they have in common. Importantly, if the search for a mapping gives little emphasis to object-level similarity (as opposed to relation-level similarity and parallel connectivity), then structural alignment can find abstract commonalities between episodes having little or no surface similarity (i.e., in terms of perceptual features).

![Figure 1](image)

**Figure 1** An example of structural alignment between two chess positions. Both positions contain instances of the abstract concept of a FORK: black’s piece is simultaneously attacking both of white’s pieces. These attacking relations are represented by the red arrows. Cyan lines indicate the mapping between the two episodes. The mapping satisfies parallel connectivity because it respects the bindings between relations and their role-fillers.

We propose structural alignment is critical to learning of abstract concepts for three reasons. First, perceived similarity of relational stimuli depends on structural alignability [Markman and Gentner, 1993]. Second, structural alignment is important for analogical transfer, which is the ability to apply knowledge from one situation to another, superficially different situation [Gick and Holyok, 1980]. For example, a winning move in one chess position can be used to discover a winning move in a different (but aligned) position, by translating that action through the analogical mapping. Third, a successful analogy can lead to schema induction, which involves extraction of the shared relational structure identified by the analogy [Doumas et al., 2008; Gentner, 1983; Hummel and Holyok, 2003]. In the example of Figure 1 this schema would be a system of relational knowledge on abstract (token) objects, including \text{ATTACK}(\text{PIECE1}, \text{PIECE2}), \text{ATTACK}(\text{PIECE1}, \text{PIECE3}),$ and potentially other shared information such as \text{NOT\_ATTACKED}(\text{PIECE1}) and \text{KING}(\text{PIECE2}).

These three observations suggest that analogy plays an important role in learning and use of abstract relational concepts. The first two observations suggest that analogical transfer is like similarity-based generalization, but it’s also more sophisticated because it takes structure into account, as we elaborate in the next two sections. In brief, structural alignment offers a sophisticated form of similarity that can be used to generalize knowledge between situations that are superficially very different. The third observation suggests that analogy can discover new relational concepts (e.g., the concept of a chess fork, from Figure 1), which can in turn lead to perception of even more abstract similarities among future experiences.

One potential shortcoming of the basic theory of analogy reviewed here is that is it essentially unsupervised. In this
framework, the quality of an analogy depends only on how well the two systems can be structurally aligned, and not on how useful or predictive the shared structure might be. For example, one could list many relational patterns that arise in chess games but that are not especially useful for choosing a move or for predicting the course of the game. In previous work, we have found that implementing structural alignment and schema induction in a rich and structured artificial environment results in discovery of many frequent but mostly useless schemas (Foster et al., 2012). An alternative, potentially more powerful model of analogical learning would involve feedback from the environment, so that the value of an analogy or schema is judged partially by how well it improves predictions of reward or other important environmental variables. For example, the concept of a fork in chess is an important schema not (only) because it is a recurring pattern in chess environments, but because it carries information about significant outcomes (i.e., about sudden changes in each player’s chances of winning). A natural framework for introducing this sort of reward sensitivity into theories of analogy is that of RL, which we review next.

3 | REINFORCEMENT LEARNING

RL is a mathematical and computational theory of learning from reward in dynamic environments. An RL task is characterized by an agent embedded in an environment that exists in some state at any given moment in time. At each time step, the agent senses the state of its environment, takes an action that affects what state occurs next, and receives a continuous-valued reward that depends on the state and its action (Sutton and Barto, 1998). This framework is very general and can encompass nearly any psychological task in which the subject has full knowledge of the state of the world at all times (i.e., there are no relevant hidden variables).

Most RL models work by learning values for different states or actions, which represent the total future reward that can be expected from any given starting point (i.e., from any state or from any action within a state). These values can be learned incrementally, from temporal-difference (TD) error signals calculated from the reward and state following each action (see Model section). There is strong evidence that the brain computes something close to TD error, and thus that RL captures a core principle of biological learning (Schultz et al., 1997).

In principle, this type of simple algorithm could be used to perfectly learn a complex task such as chess, by experiencing enough games to learn the true state values (i.e., probability of winning from every board position) and then playing according to those values. However, a serious shortcoming of this naive approach is that it learns the value of each state independently, which can be hopelessly inefficient for realistic tasks that typically have very large state spaces. Instead, some form of generalization is needed, to allow value estimates for one state to draw on experience in other, similar states.

Many variants of RL have been proposed for implementing generalization among states (e.g., Albus, 1981; Sutton, 1988). Here we pursue a direct and psychologically motivated form of generalization, based on similarity (Jones and Cañas, 2010; Ormoneit and Sen, 2002). We assume the model has a stored collection of exemplar states, each associated with a learned value. This exemplar representation is particularly suited for the analogy model we present below because it allows us to treat schemas as exemplars. The value estimate for any state is obtained by a similarity weighted average over the exemplars’ values; that is, knowledge from each exemplar is used in proportion to how similar it is to the current state. This approach is closely related to exemplar-generalization models in more traditional psychological tasks such as category learning (Nosofsky, 1986). It can also be viewed as a subset of kernel methods from machine learning (Shawe-Taylor and Cristianini, 2004), under the identification of the kernel function with psychological similarity (Jäkel et al., 2008).

A critical consideration for all learning models (including RL models) is how well their pattern of generalization
matches the inherent structure of the task. If generalization is strong only between stimuli or states that have similar values or outcomes, then learning will be efficient. On the other hand, if the model generalizes significantly between stimuli or states with very different outcomes, its estimates or predictions will be biased and learning and performance will be poor. The kernel or exemplar-similarity approach makes this connection explicit, because generalization between two states is directly determined by their similarity. As we propose next, analogy and schema induction offer a sophisticated form of similarity that is potentially quite powerful for learning complex tasks with structured stimuli.

4 | ANALOGICAL RL

The previous two sections suggest a complementary relationship between analogy and RL, which hint at the potential for a computationally powerful, synergistic interaction between these two cognitive processes. We outline here a formal theory of this interaction. The next two sections provide a mathematical specification of a partial implementation of this theory, and then present simulation results offering a proof-in-principle of the computational power of this approach.

The first proposed connection between analogy and RL is that structural alignment yields an abstract form of psychological similarity that can support sophisticated generalization. Incorporating analogical similarity into the RL framework could thus lead to rapid learning in complex, structured environments. For example, an RL model of chess equipped with analogical similarity and a notion of an attack relation should recognize the similarity between the two positions in Figure 1 and hence generalize between them. Consequently the model should learn to create forks and to avoid forks by the opponent much more rapidly than if it had to learn about each possible fork instance individually.

The second proposed connection is that the TD error computed by RL models, for updating value estimates, can potentially drive analogical learning by guiding schema induction and attention. Instead of forming schemas for whatever relational structures are frequently encountered (or are discovered by analogical comparison of any two states), an analogical RL model can be more selective, only inducing schemas from analogies that significantly improve reward prediction. Such analogies indicate that the structure common to the two analogue states may have particular predictive value in the current task, and hence that it might be worth extracting as a standalone concept. For example, if the model found a winning fork move by analogical comparison to a previously seen state involving a fork, the large boost in reward could trigger induction of a schema embodying the abstract concept of a fork. Furthermore, the model can use prediction error to increase attention to concepts that are more useful for predicting reward, and decrease attention to concepts that are less useful. Such an attention-learning mechanism is proposed to bias the model to rely more on concepts that have been consistently useful in the past.

The proposed model thus works as follows (see the next section for technical details). The model maintains a set of exemplars \( E \), each with a learned value, \( v(E) \). To estimate the value of any state \( s \), it compares that state to all exemplars by structural alignment, which yields a measure of analogical similarity for each exemplar \( \text{sim}(s, E) \). The estimated value of the state, \( \hat{V}(s) \), is then obtained as a similarity-weighted average of \( v(E) \). After any action is taken and the immediate reward and next state are observed, a TD error is computed as in standard RL. The exemplar values are then updated in proportion to the TD error and in proportion to how much each contributed to the model’s prediction, that is, in proportion to \( \text{sim}(s, E) \).

The attentions \( u(E) \) implement exemplar-specific attentions weights or learning rates. Attention learning is an additional mechanism of the model whose purpose is to increase the model’s reliance on useful exemplars. Although the model makes sense without this attention learning mechanism, including it improves performance and integrates
analogy, RL, and attention, and has been demonstrated in experiments with humans (Foster and Jones, 2013b; Foster, 2015). A model that increases its repertoire of concepts needs some pruning mechanism to sort through what’s been discovered. This attention learning approach is a reasonable way to handle the pruning problem in an exemplar setting: the model needs to learn which exemplars or schemas to retain, so we attach an attention value to each one.

The \( u(E) \) values can also be thought of as voting weights in the computation of \( \tilde{V}(s) \). Exemplars with higher \( u \) values have greater influence on the similarity-weighted average of exemplar values. The attention weights are updated in proportion to the TD error, in proportion to how much each contributed to the model’s prediction, and in proportion to how much that exemplar’s prediction differed from the overall prediction \( \tilde{V} \). Attention is increased to exemplars that individually made a more accurate prediction that the overall prediction, and attention is decreased to exemplars that individually made a less accurate prediction than the overall prediction.

Attention and RL also mutually facilitate each other through an additional mechanism of schema induction. Whenever the structural alignment between a state and an exemplar produces a sufficient reduction in prediction error (relative to what would be expected if that exemplar were absent), a schema is induced from that analogy. The schema is an abstract representation, defined on token (placeholder) objects, and it contains only the shared information that was successfully mapped by the analogy. The schema is added to the pool of exemplars, where it can acquire value associations directly (just like the exemplars do). Theoretically, we take the position that there is no real psychological difference between schemas and concrete state exemplars, it’s just a continuum of specificity. The advantage conferred by the new schema is that it allows for even faster learning about all states it applies to (i.e., that contain that substructure). For example, rather than learning by generalization among different instances of forks, the model would learn a direct value for the fork concept, which it could immediately apply to any future instances. A consequence of the schema induction mechanism is that the pool of exemplars comes to contain more and more abstract schemas. Thus the model’s representation transitions from initially episodic to more abstract and conceptual. The facilitation between analogy and RL here is that analogy provides new representations for RL to use for its value learning, and RL provides a TD error signal to help analogy decide which schemas to build.

Analogical RL thus integrates three principles from prior research: RL, exemplar generalization, and structural alignment of relational representations. Because each of these principles has strong empirical support as a psychological mechanism, it is plausible that they all interact in a manner similar to what we propose here. Thus it seems fruitful to explore computationally what these mechanisms can achieve when combined.

5 | MODEL

The proposed model applies to Markov Decision Process tasks, where an agent makes decisions based on the current state of the environment. At each time step, the agent chooses from the available actions in the current state and then the environment gives the agent an immediate reward and moves into the next state.

The simulation study presented below uses a variant of RL known as afterstate learning, in which the agent learns values for the possible states it can move into (Sutton and Barto, 1998). This is a reasonable and efficient method for the task we use here—tic-tac-toe, or noughts & crosses—because the agent’s opponent can be treated as part of the environment and is the only source of randomness. Our main proposal regarding the interaction between RL and analogical learning is not limited to this approach.

The operation of the model is illustrated in Figure 2. On each time step, the model identifies all possible actions and their associated afterstates. For each afterstate \( S \), it computes an analogical similarity, \( \text{sim}(S, E) \), to each exemplar, \( E \), by structural alignment. At the theoretical level, the model does not commit to a particular mapping algorithm. Instead,
FIGURE 2  Schema Induction Model operation. Each candidate afterstate is evaluated by analogical comparison to stored exemplars, followed by similarity-weighted averaging among the learned exemplar values. Learning is by TD error applied to the exemplar values. On some trials, especially useful analogies produce new schemas that are added to the exemplar pool. In the example here, S and E both have guaranteed wins for X by threatening a win in two ways. The induced schema embodies this abstract structure. Dots with red arrows indicate ternary "same-rank" relations.

...
systems of relations governed by common higher-order relations).

Analogical similarity is then defined as the value of the best mapping (here the $\theta$ parameter determines specificity of generalization):

$$sim(S, E) = \exp \left( \theta \cdot \max_M \Phi(M) \right).$$

(2)

The activation $a(E)$ of each exemplar is determined by weighting the analogical similarity between that exemplar $E$ and the candidate state $S$ by its attention $u$ and normalizing by the attention-weighted analogical similarity across all exemplars:

$$a(E) = \frac{u(E) \cdot sim(S, E)}{\sum_{E' \in \text{Exemplars}} u(E') \cdot sim(S, E')}.$$  

(3)

The estimated value of $S$, $\tilde{V}(S)$, is computed as a similarity-weighted average of the exemplar values $v(E)$:

$$\tilde{V}(S) = \sum_v v(E) \cdot a(E).$$

(4)

Thus the estimate is based on the learned values of the exemplars most similar to the candidate state and the exemplars with the highest attention weights.

There is a separate pass through the whole network for each candidate state (i.e., an outer loop over candidates). Although our model evaluates all possible afterstates, a more realistic model would be selective. How an agent decides which options to even consider is an important question but is outside the scope of this paper.

Once values $\tilde{V}(S)$ have been estimated for all candidate afterstates, the model uses a softmax (Luce-choice or Gibbs-sampling rule) to select what state to move into (here $\tau$ is an exploration parameter):

$$\Pr[S_t = S] \propto e^{\tilde{V}(S)/\tau}.$$  

(5)

Learning based on the chosen afterstate $S_t$ follows the SARSA rule [Rummery and Niranjan 1994], after the model chooses its action on the next time step. This produces a TD error, which is then used to update the exemplar values and exemplar attention or voting weights by gradient descent. Exemplar values are updated in proportion to the TD error and their activations:

$$\Delta v(E) = \epsilon \cdot TD \cdot a(E)$$

(6)

where $\epsilon$ is a learning rate. Exemplar attentions are updated in proportion to the learning rate $\epsilon$, the TD error, the analogical similarity between the exemplar and candidate state $sim(S, E)$, and the difference between the exemplar value $v(E)$ and the estimated value of the candidate state $\tilde{V}(S)$, normalized by the attention-weighted analogical similarity across all exemplars:

$$\Delta u(E) = \frac{\epsilon \cdot TD \cdot sim(S, E) \cdot (v(E) - \tilde{V}(S))}{\sum_{E' \in \text{Exemplars}} u(E') \cdot sim(S, E')}.$$  

(7)

This exemplar-specific attention weight learning mechanism ($u$ parameter in the equations) allows the model to
learn which exemplars are most useful for reducing prediction error. Over time, the model learns to increase attention to exemplars which are predictive of reward outcome, and decrease attention to exemplars which are not predictive of outcome.

Following learning after each trial, the schema induction mechanism determines how much each exemplar contributed to reducing prediction error, by comparing TD to what it would have been without that exemplar. If the reduction is above some threshold, the analogical mapping found for that exemplar (lower right of Figure 2) produces a schema that is added to the exemplar pool (far right). The schema is given a value of \( v \) initialized at \( \bar{V}(S_t) \). This schema value is updated on future trials just as are the exemplar values. Acquisition of new schemas in this way is predicted to improve the model’s pattern of generalization, tuning it to the most useful relational structures in a task.\(^1\)

### 6 | SIMULATION

The goal of the simulation was to test whether the model could (1) learn more quickly by generalizing between states sharing structure but not surface similarity and (2) bootstrap its learning by discovering composite structures (schemas) that are particularly predictive. If the model succeeded at these two measures, then that would suggest it has potential to exhibit humanlike behavior in more complex settings. For example, the model might offer insight into how humans become experts in complex games like chess and Go, or how they acquire and apply relationally complex sentence structures (see Goldwater et al., 2011).

The analogical RL model was tested on its ability to learn tic-tac-toe (Foster and Jones, 2013a). Tic-tac-toe was chosen as a test domain because it is a simple game with relational structure and a clear task goal. However, we want to be clear that this is not a theory of how people play tic-tac-toe. People are typically given explicit instruction on the rules, goals, and winning states of the game. The model isn't provided any rules of the game, other than implicitly knowing what moves are legal at any point (i.e., you can only move in an empty square) and the model doesn't know what constitutes a win. Instead, the model learns entirely from trial and error. Whereas people are able to look ahead and mentally simulate moves several steps into the future, the model as currently implemented does not look ahead beyond the current move.

In the simulation, each board position was represented by treating the nine squares as objects of types 0 (blank), 1 (focal agent’s), and 2 (opponent’s), and defining 8 ternary “same-rank” relations for the rows, columns, and diagonals. Thus a player wins by filling all squares in any one of these relations (see Figure 3). Object similarity was defined as 1 for matching object types and 0 otherwise. Similarity between relations was always 1 because there was only one type

| Model Variation                | Additional Mechanism Tested              |
|--------------------------------|------------------------------------------|
| Featural                       |                                          |
| Relational                     | Relational similarity                    |
| Unguided Schema Induction, Fixed Attentionss | Schema induction                        |
| Guided Schema Induction, Fixed Attentionss | Schema induction guided by RL            |
| Guided Schema Induction, Learned Attentionss | State and Schema attentions guided by RL |

\(^1\) Schemas can be spawned from the mapping between a candidate state and an exemplar state (schema induction) as well as from the mapping between a candidate state and an exemplar schema (schema refinement) (Doumas et al., 2008).
Reward was given only at the end of a game, as +1 for the winner, -1 for the loser, or 0 for a draw. After the game ended, it moved to a special terminal state with fixed value of 0. For simplicity, all free parameters of the model (\(\beta, \theta, \alpha, \gamma, \tau\)) were set to a default value of 1, except the schema induction threshold which was set to 6 standard deviations above the mean TD error reduction on each turn.

**Figure 3** Relations Defined on the Tic-Tac-Toe board. The figure illustrates the 8 ternary relations that the Relational Model uses to represent a state of the game. There are three relations for the 3 vertical columns (in red), 3 relations for the 3 horizontal rows (in orange), and 2 relations for the diagonals (in yellow). Each relation has 3 roles, which are filled by the objects in their corresponding board locations.

State exemplars (i.e., non-schema exemplars) were added to the model probabilistically, with probability of recruitment inversely proportional to the number of state exemplars already recruited. Recruitment of duplicate exemplars was not allowed.

Five variations of the model were implemented to verify its mechanisms. Each model variation builds on the previous variation by adding an additional mechanism (see Table 1).

The Featural model was restricted to literal mappings between states (upper-left square to upper-left square, etc.). This model still included generalization, but its similarity was restricted to the concrete similarity of standard feature-based models. Featural similarity was defined as the proportion of matching objects in the same absolute board locations.

The Relational model considered all 8 mappings defined by rigid rotation and reflection of the board. This scheme was used in place of searching all 9! possible mappings for every comparison, to reduce computation time (see Figure 4).

The Unguided Schema Schema Induction, Fixed Attentions model extended the Relational model by inducing schemas that capture the relational structure critical to the task. This model variation included schema induction and relational similarity, but the schema induction process was not guided by RL feedback. Instead, this baseline model randomly induced schemas between exemplars and the current state (or between schemas and the current state). The number of schemas induced was determined by yoking to the feedback-guided schema induction model. Thus this yoked baseline model induced the same number of schemas as the feedback-guided schema induction model, but the
FIGURE 4  Featural vs. Relational Similarity in the Tic-Tac-Toe implementation. X indicates the model’s token, O indicates the opponent’s token, blank indicates an open space on the grid. Similarity is inversely related to the number of differences between states and states, or between states and schemas. Asterisks indicate board locations that are not part of the schema, and so do not contribute to the count of differences. (a) The featural and relational model similarity would be the same because all tokens match based on their absolute board locations. (b) The relational model would score these two states as having higher similarity because it allows for the reflection of the board, whereas the featural model would count four differences. (c) The schema on the right contains three X’s in a row, as well as two O’s, two blanks, and two locations that are not part of the schema. (d) The schema on the right has been further refined, and is “clean” in that it only represents the information relevant to a winning board configuration, 3 X’s in a row.

particular schemas learned were induced from comparisons between the current state and randomly chosen exemplars. The hypothesis was that the feedback-guided schema induction model would learn faster than the unguided model, and would also discover more useful representations.

The Guided Schema Induction, Fixed Attentions model extended the Unguided Schema Induction, Fixed Attentions model by using RL to guide schema induction. Whenever an exemplar was particularly useful (meaning it reduced TD error by a thresholded amount), that exemplar was used to induce a schema by comparing it to the candidate state. The threshold used was in terms of standard deviations in reduction of TD error. The reduction in TD error was computed by leaving out each exemplar and computing TD without it. The difference between the TD error with and without an exemplar is the reduction in TD for that exemplar. Exemplars whose reductions were greater than 6 standard deviations above the mean reduction were used to induce schemas. Thus in this model, RL is used both to learn the values of the exemplars and to guide schema induction.

The Guided Schema Induction, Learned Attentions model extended the Guided Schema Induction, Fixed Attentions model by using RL to learn exemplar-specific attention weights (the $\mu$ values). Each exemplar’s attention weight $\mu$ was
initialized to 1. Over time the model uses the TD error signal to update the attention weights so that particularly useful exemplars have their attentions increased and misleading exemplars have their attentions decreased (see Equation 7). Anytime an exemplar’s attention went below 0, it was pruned from the exemplar pool.

Each model variant was trained in blocks of 10 games of self-play. In self-play, at the start of each game the model variant was cloned and played a game against itself. Learning (updates to \( v \) and \( \omega \)) was cached during the game and then applied after the game finished. Learning occurred only during training. Following each block of 10 self-play games, the model was tested in a pair of games against an ideal player (playing first in one game and second in the other). The ideal player was given the correct values for every state and always moved into the highest-valued next state. In testing games, the model was given one point for each non-losing move it made (i.e., moves from which it could still guarantee a draw), for a maximum of 9 points per pair of testing games. In other words, points were awarded for how long the model could play before the ideal opponent could guarantee a win. More rigorously, the ideal player is defined by backward induction, partitioning the state space into states from which ideal play on both sides will lead to a draw, an X win, or an O win. Tic-tac-toe has the property that the initial (blank) state lies in the first of these three subsets. The model’s score is determined by how long it keeps the game in that regime.

**FIGURE 5**  Simulation Results. Performance of the four model variations over 50,000 training games. The featural model (black) is extremely slow to learn. The relational model (red) uses analogical generalization and performs better than the featural model. Adding schema induction (light blue) to the relational model does not improve performance, unless the schema induction process is guided by reinforcement learning (dark blue). Adding the exemplar-specific attention learning mechanism (green) to the guided schema induction model further improves performance. Performance is measured by points, which is the number of moves made before the game is a sure loss against an ideal player, and is averaged across 64 independent copies of the model. Shading around each line indicates standard error bars. Training games are labeled in units of 5000 games.
Averaged learning curves are shown in Figure 5 for 64 independent copies of each model over 5000 blocks (50,000 training games). These results show that the featural model (black) is extremely slow to learn. The relational model (red) uses analogical generalization and performs better than the featural model. Adding schema induction (light blue) to the relational model does not improve performance, unless the schema induction process is itself guided by the prediction error signal (dark blue). There is a further improvement in performance when the model can also learn to adapt its attentions based on prediction error with the exemplar-specific attention learning mechanism (green).

As the model learns from training games, it updates its estimate of the value and attention for each exemplar. Figure 6 shows timelines of the model’s values (v) and attentions (u) as the model learns from more training games. Values and attentions are plotted separately for states recruited as exemplars vs. schemas induced by the model. The values for schemas tend to be more extreme (either highly positive or highly negative) than the values for states, and so are more diagnostic of winning or losing board positions. Additionally, the attentions for schemas tend to be larger than attentions for states. An exemplar with a higher attention value tends to make especially accurate reward predictions. The overall larger attentions for schemas (as compared to states) indicates that schemas are, overall, more useful exemplars for reward prediction. The important conclusion is that the model learns to shift its representation of the environment from states that it has directly experienced to more abstract schematic representations.

Examples of some of the most useful representations learned by the Schema Induction model are shown in Figure 7. The schemas in (a) represent winning states of tic-tac-toe. The top left schema perfectly matches (with a similarity of 1) any tic-tac-toe state with 3 X’s in the bottom row, the left row, the right row, or the top row – regardless of which objects (X, O, or blank) occupy the other grid locations. Similarly, the left schema in (b) perfectly matches any tic-tac-toe state with an O in the center, an O in a corner, and a blank in the opposite corner. This schema represents the state in which the model is about to lose the game by an opponent making 3 O’s along the diagonal. These two schemas are “pure” or “clean” in that they contain no information that’s irrelevant to the winning or about-to-win states. However, the right schema in (a) does contain an irrelevant piece of information - the O in the top center grid location. This schema is
Figure 7  Examples of some of the representations with the highest attentions, as averaged over 64 independent copies of the Guided Schema Induction, Learned Attentions model after 50,000 training games. Because the model is implemented with afterstates, in each of these examples is valued from the perspective of the model (X) just having placed a token, and it being the opponent’s (O) turn. The model associates (a) and (c) with positive reward value, and (b) and (d) with negative reward value, but all have high attentions because they are useful for predicting reward. (a) Examples of schemas where model wins the game. (b) Examples of schemas where the model is about to lose. (c) Examples of states where the model has two ways to win, and the opponent can only block one of them. (d) Examples of states where the opponent may have two ways to win. In the left board, a play by O in the top right corner will create two ways for the opponent to win. In the right board, the opponent is about to win, perhaps because it had produced a fork before X played in the left-center position.

“dirty” in that it contains irrelevant details from the states from which it was induced. With further training, the model would likely refine this schema and learn that abstracting out the O provides a more general, more useful schema. The right schema in (b) is also “dirty” in that it contains the extraneous X in the top left corner. The states in (c) represent something like “forks” in the tic-tac-toe domain, where the model has two ways to win and the opponent can only block one of them. The left state in (d) represents a state the model should avoid, because it provides opportunity for the opponent to create a fork by playing in the top right corner. In the right board in (d), the opponent is about to win, possibly because it had created a fork before X played in the left-center position.

8  |  GENERAL DISCUSSION

The results presented here constitute a proof-of-principle that analogy and schema induction can be productively integrated with a learning framework founded on RL and similarity-based generalization. This integration leads to a model exhibiting sophisticated, abstract generalization derived from analogical similarity, as well as discovery of new relational structures driven by their ability to predict reward.
The basic modeling framework used here applies not just to analogical similarity and schema induction, but to other forms of representational learning as well. Kernel-based RL offers a powerful and general theory of representation learning, because it can be integrated with any form of representation that yields a pairwise similarity function. Its TD error signal can drive changes in representation via the objective of improving generalization. This idea has been applied to learning of selective attention among continuous stimulus dimensions [Jones and Cañas, 2010]. The current model offers a richer form of representation learning, in that it acquires new concepts rather than reweighting existing features.

8.1 | Related Models

The analogical RL model also builds on other models of relational learning. [Tomlinson and Love, 2006] propose a model of analogical category learning (BRIDGES), with essentially the same similarity and exemplar generalization mechanisms adopted in the present model. Our model adds to theirs in that it applies to dynamic tasks and in that it grows its representation through schema induction. In BRIDGES, analogy contributes to RL by providing relational generalization. Our model also has the reverse, in that RL guides schema induction and hence acquisition of abstract concepts. [Van Otterlo, 2012] has developed methods for applying RL to relational representations of the same sort used here, although the approach to learning is quite different. His models are not psychologically motivated and hence learn in batches and form massive conjunctive rules, with elaborate updating schemes to keep track of the possible combinations of predicates. In contrast, the present approach learns iteratively, behaves probabilistically, and grows its representation more gradually and conservatively. This approach is likely to provide a better account of human learning, but a more interesting question may be whether it offers any performance advantages from a pure machine-learning perspective.

In the present model, the activation of each exemplar elicited by a candidate state can be thought of as a feature of that state. The exemplar effectively has a “receptive field” within the state space, defined by the similarity function. This duality between exemplar- and feature-based representations is founded in the kernel framework (see Shawe-Taylor and Cristianini, 2004). The present model takes advantage of this duality, producing a smooth transition from an episodic, similarity-based representation to a more semantic, feature-based representation defined by learned schemas.

The value and attention learning mechanisms have roots in traditional associational learning models. The classic Rescorla-Wagner model introduced joint learning of cue-outcome associations, but had no mechanism for attention learning [Rescorla et al., 1972]. Although the Rescorla-Wagner model allowed for different input cues to have different associabilities, there was no mechanism for learning these attentions. [Mackintosh, 1975] introduced an attention learning mechanism which modifies attention to cues to reduce prediction error and reduce interference between cues. Support for this attention learning mechanism comes from demonstrations of learned inattention in experiments with rats [Mackintosh and Turner, 1971] and humans [Kruschke and Blair, 2000].

Although the analogical RL attention-learning mechanism in the present work is being implemented in an exemplar model, it is compatible with Mackintosh's theory and with extensions by [Kruschke, 2001]. Each of these models would predict that attention would increase to lower-variance cues based on the idea that attention increases to cues that are more predictive (contribute less error), compared to the average individual cue predictiveness (as in [Mackintosh, 1975] and [Kruschke, 2001]'s mixture of experts model) or the overall combined cue predictiveness (as in [Kruschke, 2001]'s EXIT model and the present ARL model). In the exemplar setting, the model needs to learn which exemplars to retain in order to interpolate the reward value of each new stimulus based on learned exemplars. Thus the present ARL model also includes a normalization in the definition of exemplar activation, which makes attention more like a voting weight and less like salience.
The present work is complementary to hierarchical Bayesian models that discover relational structure through probabilistic inference [Tenenbaum et al., 2011]. Whereas our model builds up schemas from simpler representations, the Bayesian approach takes a top-down approach, defining the complete space of possibilities a priori and then selecting among them. The top-down approach applies to any learning model, because any well-defined algorithm can always be circumscribed in terms of its set of reachable knowledge states. This is a useful exercise for identifying inductive biases and absolute limits of learning, but it offers little insight into the constructive processes that actually produce the learning. The present model offers proposals about the mechanisms underlying how the human mind discovers new, abstract concepts. Extensions to this model are discussed in the following sections.

8.2 | Afterstate vs. Forestate Learning

Although the tic-tac-toe domain is well captured with an afterstate representation, most tasks can’t use this simplification. In the forestate version of the model, each state would have a set of available actions, and the model would learn values for state-action pairings rather than learning values just for (after)states.

Furthermore, the afterstate formulation misses an important way in which analogical transfer is more complex than similarity-based generalization (SBG), which doesn’t accommodate the translation provided by the analogical mapping. In the extended forestate version of the present model, generalization is not just via blind similarity, because it takes the mapping (and hence the structure of the stimuli) into account. In a sense, the action is in the schema. A forestate model wherein structure mapping informed only the similarity computation and not the mapping of actions would fail in cases where the extended model succeeds. Consider the fork example from chess in Figure 1. SBG would infer that a capturing action involving the knight in the game on the left would apply to a knight in the game on the right. If instead the action is in the schema, then a capturing action by the knight on the left would be correctly translated through the mapping into a capturing action by the analogous queen on the right.

8.3 | Two-Stage Memory Retrieval

A challenge for the present model is tractability, because it’s not feasible to compute analogical mapping to all stored exemplars in memory. For computational efficiency and simplicity, the current implementation in the tic-tac-toe domain exploits knowledge of the game’s invariance under a predetermined set of simple symmetries. A solution to the tractability problem for the full model that uses structural mapping is to incorporate two-stage memory retrieval, following the MAC/FAC model [Forbus et al., 1995]. The first stage uses fast feature-vector similarity to efficiently retrieve a set of candidate exemplars, and the second stage uses structural alignment to determine the best analogical matches. Such two-stage retrieval enables a more computationally tractable and psychologically plausible form of analogical inference from stored exemplars to novel situations.

The first stage of retrieval (Many Are Called) computes a MAC score which is used to select the set of candidate exemplars that pass on to the second (Few Are Called) stage. The candidate exemplar set is defined by the top N exemplars sorted by descending MAC score. To compute each exemplar’s MAC score, first the model computes the cosine similarity between each exemplar’s feature vector and the candidate state’s feature vector. The choice of cosine similarity is an implementational detail, and the model is not theoretically committed to this particular choice of featural similarity. The present model’s attention-learning mechanism can be incorporated into the MAC score by multiplying the featural similarity measure by the exemplar-specific attention weights u to compute each exemplar’s MAC score:

$$MAC(S, E) = u(E)^P \frac{f(S) \cdot f(E)}{\|f(S)\| \|f(E)\|}.$$  

(8)
These two components (featural similarity and attention) in the MAC score make exemplars with higher featural similarity and higher attentions more likely to be retrieved from memory and included in the set of candidate exemplars. Thus more useful exemplars are more likely to be retrieved and relied upon for generalization.  The relative weighting of the featural similarity and the exemplar attentions is set by an exponent \( p \) on the attention weight. The second stage of retrieval (Few Are Called) computes a FAC score through a more computationally complex structure-mapping process on relational representations. In the present model, the FAC score is the analogous similarity of Equation 2. Although the mapping and similarity scoring process itself is not currently learned, it may be productive to enable the prediction error signal to influence mapping or scoring itself, as in Liang and Forbus (2015).

The MAC/FAC extension also lays groundwork that will be useful for a later model with relational consolidation, which is discussed in detail in the next section. In brief, the exemplars that are learned to be most useful (via the attention-learning mechanism described above) would become new perceptual features which could be leveraged by the MAC retrieval stage to improve the candidate exemplars retrieved for the subsequent structural alignment processing.

### 8.4 Relational Consolidation

Although the present integration of analogy, schema induction, and reinforcement learning proves powerful, there’s a ceiling to what it can learn because all it can do is build configurations of the primitive elements it’s endowed with. It lacks a mechanism to create rich compositional hierarchies of relational concepts. Examples of such compositional hierarchies include computer architecture, mathematical functions, and natural languages, which all exemplify multiple levels of abstraction by chunking systems of relations at one level into building blocks at the next level. In computer architecture, digital logic gates are composed to form adders, which are composed with other digital circuits to form an arithmetic logic unit (ALU), which is a building block in a computer’s CPU. Software design manages complexity by continuing this hierarchy, composing primitive functions into more complex functions, and from there to objects and design patterns. The conceptual progression in mathematics proceeds similarly, composing the counting operation to define adding, which is further composed to form multiplying, and then exponentiation. In traditional views of linguistics, phonemes, morphemes, words, and sentences form another example of a relational hierarchy.

Although we agree with theories of schema induction, we argue it is insufficient to explain human relational learning. Schemas are explicit relational structures, and thus they cannot be bound to roles of yet-higher-order relations in the way unitary objects and relations can. Experiments with chimpanzees suggest that newly learned relations can only fill roles of other relations if they can be represented as atomic entities (Thompson et al., 1997). Therefore, to explain acquisition of relational hierarchies, we put forward the hypothesis that useful schemas are eventually replaced (or supplemented) with unitary representations (Foster et al., 2012). Thus, a concept that was represented as a system of relations (via the schema) can now be represented as an atomic entity, capable of entering into relations itself. We label this process relational consolidation, in a deliberate parallel to theories of episodic memory consolidation (e.g., Squire and Alvarez, 1995).

As summarized in Table 2, consolidation is hypothesized to confer properties to a concept that are not true of (unconsolidated) schemas, because consolidated concepts are recognizable perceptually, without explicit (working-memory dependent) structure mapping (Corral and Jones, 2014, 2017; Foster et al., 2012). A consolidated concept can

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2 In the existing model, exemplar attention \( u \) is standing in for the expected value of retrieval, whereas in the extended model with two-stage retrieval, retrieval probability is linearly related to \( u \). Our theoretical commitment is that an exemplar with a higher \( u \) has higher probability of being retrieved (monotonic relationship).

3 Relational hierarchies are not taxonomic hierarchies. In a taxonomic hierarchy, each concept or category is a union of lower-level categories. In a relational hierarchy, each instance of a concept is a configuration of instances of lower-order concepts.
TABLE 2  Predicted consequences of consolidation.

| Not Consolidated                  | Consolidated                  |
|-----------------------------------|-------------------------------|
| More affected by WM demands       | Less affected by WM demands   |
| Quicker at analogical inference,  | Easier to learn higher-order   |
| because structure mapping is     | structure, because instances  |
| active                            | can be represented by tokens  |
| Serial retrieval                  | Parallel retrieval            |

be recognized automatically, retrieved from memory in parallel, and represented as an element of yet-higher-order relations.

It is important to note that consolidation is not a change in the declarative knowledge embodied by a concept. Rather, it is a proceduralization of the concept that enables future changes in knowledge – similar to the interaction between declarative and procedural knowledge in production systems (Anderson and Lebiere, 1998).

The MAC/FAC model discussed above embodies the assumption that verifying the lower-level elements of an episode (i.e., predefined objects and relations) is fast and automatic, whereas verifying relational structure is slower and requires working-memory resources (Forbus et al., 1995). From this perspective, relational consolidation enables higher-order relational structure to be chunked and treated as a dimension of the feature vector used for memory probing. Prior to consolidation, retrieval of instances of a higher-order relational structure requires something like the FAC stage, in which agents explicitly map between those instances and the schema. Following consolidation, retrieval can rely solely on the MAC stage, thus operating much more rapidly and without requiring working memory. We also propose a similar difference for perceptual recognition of instances of the concept. Before consolidation, episodes must be structurally aligned to a schema. After consolidation, an instance of the concept is explicitly represented and bound to the lower-order relations.

We further propose that analogy, schema induction, and relational consolidation form a cycle that, when iterated, can produce relational hierarchies of arbitrary depth (height). This form of learning leads to a dualist view of objects and relations, in which (nearly) every concept is both a relational structure among its components and an object capable of participating in relations. The conceptual systems built from this hierarchical relational chunking are potentially quite powerful and flexible.

8.5  Conclusions

This paper has proposed a psychological theory that integrates reinforcement learning with relational representations and analogy. The integration produces a computational synergy in which analogy enables abstract generalization, and reinforcement learning drives discovery of useful relational concepts without relying on hand-coded representations. Analogy contributes mechanisms to generalize based on relational similarity, translate rewarding actions between mapped scenarios via analogical transfer, and produce progressively more abstract representations of the environment via schema induction. In return, reinforcement learning contributes mechanisms to learn the long-term value of exemplars, guide schema induction, and learn exemplar-specific attentions based on reward predictiveness. These mutually supportive mechanisms combine in a way that begins to explain how people and machines can learn abstract concepts based on experience with the world.
REFERENCES

Albus, J. S. (1981) *Brains, Behavior and Robotics*. Byte Books.

Anderson, J. J. R. and Lebiere, C. J. (1998) *The Atomic Components of Thought*. Mahwah, NJ: Erlbaum.

Bagnell, J. A. and Schneider, J. C. (2001) Autonomous helicopter control using reinforcement learning policy search methods. *IEEE Int Conf Robo*, 1615–1620.

Corral, D. and Jones, M. (2014) The effects of relational structure on analogical learning. *Cognition, 132*, 280–300.

— (2017) Learning relational concepts through unitary versus compositional representations. *Proceedings of the 39th Annual Meeting of the Cognitive Science Society*.

Doumas, L. A. A., Hummel, J. E. and Sandhofer, C. M. (2008) A theory of the discovery and predication of relational concepts. *Psychological Review, 115*, 1–43.

Falkenhainer, B., Forbus, K. D. and Gentner, D. (1989) The structure-mapping engine: Algorithm and examples. *Artificial Intelligence, 41*, 1–63.

Forbus, K., Gentner, D. and Law, K. (1995) MAC/FAC: A model of similarity-based retrieval. *Cognitive Science, 19*, 141–205.

Forbus, K. D. and Gentner, D. (1989) Structural evaluation of analogies: What counts. *Proceedings of the 11th Annual Conference of the Cognitive Science Society*, 341–348.

Foster, J. M. (2015) Analogical reinforcement learning.

Foster, J. M., Cañas, F. and Jones, M. (2012) Learning conceptual hierarchies by iterated relational consolidation. *Proceedings of the 34th Annual Conference of the Cognitive Science Society*, 324–329.

Foster, J. M. and Jones, M. (2013a) Analogical reinforcement learning. *Proceedings of the 35th Annual Conference of the Cognitive Science Society*.

— (2013b) Are some schemas stronger than others? the reinforcement of relational concepts.

Gentner, D. (1983) Structure-mapping: A theoretical framework for analogy. *Cognitive Science, 7*, 155–170.

Gick, M. L. and Holyoak, K. J. (1980) Analogical problem solving. *Cognitive Psychol, 12*, 306–355.

Goldwater, M. B., Tomlinson, M. T., Echols, C. H. and Love, B. C. (2011) Structural priming as structure-mapping: Children use analogies from previous utterances to guide sentence production. *Cognitive Science, 35*, 156–170.

Hofstadter, D. R., Mitchell, M. et al. (1994) The copycat project: A model of mental fluidity and analogy-making. *Advances in connectionist and neural computation theory*, 2, 29–30.

Hummel, J. E. and Holyoak, K. J. (2003) A symbolic-connectionist theory of relational inference and generalization. *Psychological Review, 110*, 220–264.

Jäkel, F., Schölkopf, B. and Wichmann, F. A. (2008) Generalization and similarity in exemplar models of categorization: Insights from machine learning. *Psychon B Rev, 15*, 256–271.

Jones, M. and Cañas, F. (2010) Integrating reinforcement learning with models of representation learning. *Proceedings of the 32nd Annual Conference of the Cognitive Science Society*, 1258–1263.

Jones, M. and Love, B. C. (2007) Beyond common features: The role of roles in determining similarity. *Cognitive Psychology, 55*, 196–231.
Kruschke, J. K. (2001) Toward a unified model of attention in associative learning. *Journal of Mathematical Psychology, 45*, 812–863.

Kruschke, J. K. and Blair, N. J. (2000) Blocking and backward blocking involve learned inattention. *Psychonomic Bulletin & Review, 7*, 636–645.

Larkey, L. B. and Love, B. C. (2003) CAB: Connectionist Analogy Builder. *Cognitive Science, 27*, 781–794.

Liang, C. and Forbus, K. D. (2015) Learning plausible inferences from semantic web knowledge by combining analogical generalization with structured logistic regression. In *AAAI*, 551–557.

Mackintosh, N. and Turner, C. (1971) Blocking as a function of novelty of cs and predictability of ucs. *The Quarterly Journal of Experimental Psychology, 23*, 359–366.

Mackintosh, N. J. (1975) A theory of attention: Variations in the associability of stimuli with reinforcement. *Psychological Review, 82*, 276.

Markman, A. B. and Gentner, D. (1993) Structural alignment during similarity comparisons. *Cognitive Psychol, 25*, 431–431.

Nosofsky, R. M. (1986) Attention, similarity, and the identification-categorization relationship. *J Exp Psychol Gen, 115*, 39–57.

Ormoneit, D. and Sen, S. (2002) Kernel-based reinforcement learning. *Mach Learn, 49*, 161–178.

Rescorla, R. A., Wagner, A. R. et al. (1972) A theory of pavlovian conditioning: Variations in the effectiveness of reinforcement and nonreinforcement. *Classical conditioning II: Current research and theory, 2*, 64–99.

Rosch, E. and Mervis, C. B. (1975) Family resemblances: Studies in the internal structure of categories. *Cognitive Psychol, 7*, 573–605.

Rumery, G. A. and Niranjan, M. (1994) On-line q-learning using connectionist systems. *Tech. Rep. CUED/F-INFENG/TR 166, Cambridge University.*

Schultz, W., Dayan, P. and Montague, P. R. (1997) A neural substrate of prediction and reward. *Science, 275*, 1593–1599.

Shawe-Taylor, J. and Cristianini, N. (2004) *Kernel Methods for Pattern Analysis*. Cambridge University Press.

Shepard, R. N. (1987) Toward a universal law of generalization for psychological science. *Science, 237*, 1317–1323.

Sloman, S. A., Love, B. C. and Ahn, W. K. (1998) Feature centrality and conceptual coherence. *Cognitive Sci, 22*, 189–228.

Squire, L. R. and Alvarez, P. (1995) Retrograde amnesia and memory consolidation: a neurobiological perspective. *Current Opinion in Neurobiology, 5*, 169–177.

Sutton, R. S. (1988) Learning to predict by the methods of temporal differences. *Mach Learn, 3*, 9–44.

Sutton, R. S. and Barto, A. G. (1998) *Reinforcement Learning: An Introduction*. The MIT Press.

Tenenbaum, J. B., Kemp, C., Griffiths, T. L. and Goodman, N. D. (2011) How to grow a mind: Statistics, structure, and abstraction. *Science, 331*, 1279–1285.

Tesauro, G. (1995) Temporal difference learning and td-gammon. *Commun ACM, 38*, 58–68.

Thompson, R. K., Oden, D. L. and Boysen, S. T. (1997) Language-naive chimpanzees (Pan troglodytes) judge relations between relations in a conceptual matching-to-sample task. *Journal of Experimental Psychology. Animal Behavior Processes, 23*, 31–43.

Tomlinson, M. T. and Love, B. C. (2006) From pigeons to humans: Grounding relational learning in concrete examples. *Proceedings of the 21st National Conference on Artificial Intelligence (AAAI-06)*, 199–204.

Van Otterlo, M. (2012) Solving relational and first-order logical markov decision processes: A survey. *Reinforcement Learning, 12*, 253–292.