If all geodesics are closed on the projective plane

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Abstract

Given a $C^\infty$ Riemannian metric $g$ on $\mathbb{R}P^2$ we prove that $(\mathbb{R}P^2, g)$ has constant curvature iff all geodesics are closed. Therefore $\mathbb{R}P^2$ is the first non trivial example of a manifold such that the smooth Riemannian metrics which involve that all geodesics are closed are unique up to isometries and scaling. This remarkable phenomenon is not true on the 2-sphere, since there is a large set of $C^\infty$ metrics whose geodesics are all closed and have the same period $2\pi$ (called Zoll metrics), but no metric of this set can be obtained from another metric of this set via an isometry and scaling. As a corollary we conclude that all two dimensional P-manifolds are SC-manifolds.

0 Introduction

Gromoll and Grove proved in [G] that if $(S^2, g)$ is a P-manifold then it is a SC-manifold (see definition below). We prove that this even holds if $\mathbb{R}P^2$ is a P-manifold. Moreover we show that $\mathbb{R}P^2$ has constant curvature iff all geodesics are closed. The first result in this direction was proven by Green in [Gr]. He proved that $S^2$ has constant curvature iff it is a Blaschke manifold. From this theorem it follows that $(\mathbb{R}P^2, g)$ has constant curvature iff all geodesics are closed, have the same period and are without selfintersections, since the orientable double cover of $(\mathbb{R}P^2, g)$ is then a Blaschke manifold. In the complete paper all geodesics are parametrized by arc length and the geodesic flow is complete. The manifolds and the Riemannian metrics are $C^\infty$. $\pi : TM \to M$ denotes the canonical projection. $\gamma_v$ denotes the geodesic with $\dot{\gamma}_v(0) = v$. For interested readers who want to know more about P-manifolds we refer to the book [Bs].

1 Preliminaries

Definition 1.1. $(M, g)$ is called a P-manifold if all geodesics are closed. $(M, g)$ is called a C-manifold if all geodesics are closed and they have all the same period. $(M, g)$ is called a SC-manifold if $(M, g)$ is a C-manifold and all geodesics are simple closed. The associated metrics are called P-metrics (resp. C-metrics and SC-metrics).

Definition 1.2. Let $(X, d)$ be a metric space. A dynamical system $\Phi : \mathbb{R} \times X \to X$ is called
i) equicontinuous (regular) if for all \( \epsilon > 0 \) there exists a \( \delta(\epsilon) > 0 \) such that for all \( x, y \) with \( d(x, y) < \delta(\epsilon) \) we have \( d(\Phi_t(x), \Phi_t(y)) < \epsilon \) for all \( t \in \mathbb{R} \).

ii) uniformly almost periodic if there exists for every \( \epsilon > 0 \) a \( \tau > 0 \) such that in every interval \( I \) of length \( \tau \) there exists a \( t \in I \) such that for all \( x \) we have \( d(\Phi_t(x), x) < \epsilon \).

iii) periodic if there exists an \( L > 0 \) such that \( \Phi_L = Id \), i.e. every point is periodic and all points have a common period.

Note that the terms above are independent from the metric which defines the topology if the space \( X \) is compact.

**Theorem 1.3.** An equicontinuous flow \( \Phi : X \times \mathbb{R} \to X \) on a compact metric space is uniformly almost periodic.

Proof: See theorem 2.2 in [A]. \( \square \)

**Theorem 1.4** (Wadsley). Given a P-manifold \( (M, g) \) and let denote \( \Phi : \mathbb{R} \times SM \to SM \) its geodesic flow. Define for \( v \in SM \) \( P(v) \) to be the least period of \( \Phi_t(v) \). Then \( M \) is compact, the geodesic flow is periodic and \( P \) is a discrete, bounded function. Moreover, \( P(v) \) is a multiple of the smallest period, i.e. \( P(SM) \subseteq NP(v_0) \) for some \( v_0 \in SM \).

Proof: The compactness of \( M \) is clear. Wadsley showed in [W] a characterization of periodic flows on manifolds. One can conclude from this article that the geodesic flow is periodic. A proof of this fact can be found in [Bs] (see lemma 7.11).

Since the geodesic flow is periodic (\( M \) is a P-manifold), all geodesics have a common period and \( P \) is a discrete, bounded function. It is known that in this case \( P(v) \) is a multiple of the smallest period, i.e. \( P(SM) \subseteq NP(v_0) \) for some \( v_0 \in SM \) (see page 490 in [Be]). \( \square \)

**Lemma 1.5.** If \( (M, g) \) is a P-manifold then the geodesic flow \( \Phi : \mathbb{R} \times SM \to SM \) is equicontinuous with respect to the Sasaki-metric.

Proof: We know from theorem [1.4] that the flow is periodic and \( M \) is compact. Let \( L \) denote the period of the geodesic flow. Choose for \( \epsilon > 0 \) a \( \delta > 0 \) such that \( d(v, w) < \delta \) for \( v, w \in SM \) implies

\[
d(\Phi_t(v), \Phi_t(w)) < \epsilon
\]

for all \( |t| < 2L \), hence it holds for all \( t \). \( \square \)

Now let \( M \) always be a closed surface such that all geodesics are closed (unless otherwise stated).

The set \( S := \{ v \in SM \mid \gamma_v \text{ is simple closed} \} \) is a closed subset of \( SM \) since the dimension of \( M \) is 2. Let \( S^c \) denotes the complement of \( S \) in \( SM \).

The next lemma will be useful in the next section:
Lemma 1.6. If $v_i \in S^c \rightarrow v \in S$ then $\liminf_{i \rightarrow \infty} P(v_i) \geq 2P(v)$.

Proof: Assume given $0 \leq t_{1,i} < t_{2,i} < P(v_i)$ such that $\gamma_{v_i}(t_{1,i}) = \gamma_{v_i}(t_{2,i})$. We have $|t_{1,i} - t_{2,i}| \geq \text{inj}(M)$, and $P(v_i) - |t_{1,i} - t_{2,i}| \geq \text{inj}(M)$.

Let $L = \liminf_{i \rightarrow \infty} P(v_i)$ and choose a subsequence (still denoted by $v_i$) such that $P(v_i) \rightarrow L$ and $t_{1,i} \rightarrow s_1$ and $t_{2,i} \rightarrow s_2$. Note that since $P$ is a discrete function we have $P(v_i) = L$ for large $i$. Since $\gamma_{v_i} \rightarrow \gamma_v$ we have $\gamma_{v_i}(s_1) = \gamma_{v_i}(s_2)$. The assumption that $\gamma_v$ is simple closed implies $|s_1 - s_2| = nP(v)$ where $n \in \mathbb{N}$. Furthermore $\gamma_v(0) = \gamma_v(L)$ and therefore $L = mP(v)$ where $m \in \mathbb{N}$. This implies:

$$L = \lim P(v_i) \geq |s_1 - s_2| + \text{inj}(M) \geq P(v) + \text{inj}(M)$$

and therefore $L \geq 2P(v)$. \qed

The existence of simple closed geodesics on compact Riemannian manifolds $(M,g)$ of dimension 2 will play a crucial role for proving our main result. The following theorem guarantees us the existence of a sufficient large number of simple closed geodesics.

Theorem 1.7 (Ballmann). Every compact Riemannian manifold $(M,g)$ of dimension 2 has at least three simple closed geodesics.

Proof: See [B]. \qed

Lemma 1.8. If $\gamma$ is a closed geodesic on a surface $(M,g)$ and $(M,g)$ is a P-manifold then every geodesic (except $\gamma$) intersects $\gamma$.

Proof: Take $x \notin \gamma$ and set $O = \{w \in S_\gamma M \mid \gamma_w$ intersects $\gamma\}$ and $W = SM - T\gamma$ (the set of all unit tangent vectors based on $\gamma$ which are not element of $T\gamma$). It is clear that $O$ is open, but it is also closed. Indeed, choose a sequence $v_i \in O \rightarrow v$ with $0 < t_i \leq \max P$ and $\Phi_{t_i}(v_i) \in W$. Since $t_i$ is bounded, there exists a converging subsequence (still denoted by $t_i$). If $\Phi_{t_i}(v_i) \rightarrow \Phi_s(v) \notin W$, then $\Phi_s(v) \in T\gamma$ hence $x \in \gamma$. \qed

In the next section the Euclidean isometries of $\mathbb{R}^3$ are used to characterize the geodesic flows. The following theorem is important.

Theorem 1.9. If $F$ is a periodic homeomorphism (i.e. $F^n = Id$ for some $n > 0$) of $S^2$ then $F$ is topologically conjugate to the restriction of an Euclidean isometry of the ambient 3-space.

Proof: See [C] or [K]. \qed

Corollary 1.10. Let $F$ be a periodic homeomorphism of $S^2$. If $F$ is orientation preserving and has at least 3 fixed points then $F = Id$. If $F$ is orientation reversing and has at least one fixed point, then the fixed point set is an embedded circle.

Proof: Theorem [1.9] implies that $F$ is conjugate to an Euclidean isometry. If $F$ is orientation preserving then $F$ is a rotation whose normal form is given
by

\[
A = \begin{pmatrix}
\cos(\phi) & \sin(\phi) & 0 \\
-sin(\phi) & \cos(\phi) & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

If this map has 3 fixed points then the map is the identity, hence \( F = Id \).
If \( F \) is orientation reversing then \( F \) is conjugate to a reflection whose normal form is given by

\[
A = \begin{pmatrix}
\cos(\phi) & \sin(\phi) & 0 \\
-sin(\phi) & \cos(\phi) & 0 \\
0 & 0 & -1
\end{pmatrix}.
\]

If this map has one fixed point then the normal form is given by

\[
A = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

The fixed point set is an embedded circle and therefore the fixed point set of \( F \) is an embedded circle as well. \( \Box \)

2 \( \mathbb{R}P^2 \) as a \( \text{P-manifold} \)

In this section we prove our main result, so let us assume that \( M = \mathbb{R}P^2 \) and \((M,g)\) is a \( \text{P-manifold} \). In this case the geodesic flow \( \Phi \) is periodic (see theorem [1.4]) and equicontinuous (see lemma [1.5]). A well-known technic to study a geodesic flow on a surface is to study the geodesic return map. Let \( A = S^1 \times (0,1) \) denote the open annulus. Let \( \gamma \) be a simple closed geodesic. Denote by \( W = SM|_{\gamma-T\gamma} \) (the set of all unit tangent vectors based on \( \gamma \) which are not element of \( T\gamma \)). We want to prove by using the geodesic return map that \((M,g)\) has infinitely many distinct simple closed geodesics. It is possible to define the geodesic return map on \( M \), but we must separate two cases.

Let us assume that \( \gamma \) is a simple closed geodesic that preserves the orientation. Note that in this case \( W \) is homeomorphic to an union of two open annulus \( A_0 \) and \( A_1 \). We can identify \( v \in A_0 \) with \((x,\theta)\) where \( \pi(v) = x \) and \( \theta \in (0,1) \) is the angle of \( v \) and \( \dot{\gamma}(t) \) divided by \( \pi \).

Since every orbit is periodic we define for our flow \( \Phi \) the map \( F : A_0 \to A_0 \) by

\[
F(x,\theta) = (x_0,\theta_0)
\]

where \( x_0 = \pi(\Phi_{t_0}(x,\theta)) \) is the next intersection point of \( \{\pi(\Phi_t(x,\theta))|t > 0\} \) with \( \gamma \) such that \( \Phi_{t_0}(x,\theta) = (x_0,\theta_0) \in A_0 \). We just simple write \( F : A \to A \).

If \( \gamma \) reverses the orientation, then we find a neighbourhood of \( \gamma \) looking like a Möbius strip. Denote \( W = SM|_{\gamma-T\gamma} \) (the set of all unit tangent vectors based on \( \gamma \) which are not element of \( T\gamma \)). The set \( W \) is homeomorphic to an open annulus, since \( W \) is connected and orientable. Since every orbit is periodic (recurrent) we define for our flow \( \Phi \) the map \( F : W \to W \) by

\[
F(v) = v_0
\]
where $\Phi_{t_0}(v) = v_0 \in W$ and $x_0 = \pi(\Phi_{t_0}(v))$ is the next intersection point of \{ $\pi(\Phi_t(v))$ $|$ $t > 0$ \}. Since $W$ is homeomorphic to $A$, we simple write $F: A \to A$.

**Lemma 2.1.** Let $S^2$ be the two-point-compactification of $A$, i.e. $S^1 \times \{0\} \sim -\infty$ and $S^1 \times \{1\} \sim \infty$. Then $F$ can be extended to an homeomorphism $F: S^2 \to S^2$ such that $\{-\infty, \infty\}$ are fixed points.

Proof: It is clear that $F: A \to A$ is an homeomorphism. If for $v = (x, \theta)$ we have $\theta$ near zero, the geodesic $\Phi_t(v)$ stays near $\dot{\gamma}$ (equicontinuity) hence the second coordinate of $F(x, \theta)$ is near zero, thus $\{\infty\}$ and $\{-\infty\}$ are fixed points and moreover $F$ is continuous on $S^2$. \hfill \square

We call in this paper the extension of $F$ to $S^2$ the geodesic return map and denote it with $F: S^2 \to S^2$. Note that $S$ (see chapter 2) induces a set $S_A := \{ v \in A \mid \gamma_v \text{ is simple closed} \} \subset A$ in $A \subset S^2$, due to lemma [1.8] and theorem [1.7]. $F$ is a pointwise periodic homeomorphism on $S^2$ and therefore $F$ is periodic, i.e. $F^n = Id$ for some $n > 0$, since the following theorem holds:

**Theorem 2.2** (Montgomery). Let $F: X \to X$ be a pointwise periodic homeomorphism on a compact manifold $X$ then $F$ is periodic.

Proof: See [M]. \hfill \square

**Lemma 2.3.** Given a simple closed geodesic $\gamma$ with length $L$ and the geodesic return map $F: S^2 \to S^2$ constructed from this geodesic. Set $\text{Per}(x) := \inf\{ n > 0 \mid F^n(x) = x \}$ for $x \in A$. Then the following holds:

i) If $\gamma$ preserves the orientation then

$$2\text{Per}(v) = \sharp\{ \gamma_v([0, P(v)]) \cap \gamma([0, L]) \}.$$  

ii) If $\gamma$ reverses the orientation then

$$\text{Per}(v) = \sharp\{ \gamma_v([0, P(v)]) \cap \gamma([0, L]) \}.$$  

Proof: ii) is clear by construction and therefore we only prove i). We take the model of $M = \mathbb{RP}^2$ as a closed disk where we identify the boundary points. Set the boundary curve to be $\beta$. Let $V$ be a vectorfield along $\gamma$ with $g(V, V) = 1$ and $g(V(t), \dot{\gamma}(t)) = 0$. Therefore $V(t) = V(t + P_0)$, where $P_0$ is the period of $\gamma$.

We define what lies above and below near $\gamma$. Given a small neighbourhood $U$ of $\gamma$, so small that $U$ is diffeomorphic to $S^1 \times (0, 1)$. $U - \gamma$ is an union of two open annulus $U_0$ and $U_1$. Let $U_0$ denote the set where the vectors of $A$ are pointing in. We say that a point in $U_0$ lies above and a point in $U_1$ lies below. Fix a closed geodesic $c$ that cuts $\gamma$ transversally. We show that $c$ intersects $\gamma$ mutually from above and below. If this holds for a small perturbation of $\gamma$ in the
space of the $C^\infty$ simple closed curves then it holds for $\gamma$. Thus we can suppose that $\gamma$ intersects $\beta$ and $c$ transversally. It is easy to see that $\gamma$ intersects $\beta$ for an even times, otherwise it would reverse orientation. Thus we can lift $\gamma$ to a simple closed curve on $S^2$. We lift the vectorfield $V$ up and conclude that $c$ intersects $\gamma$ as expected (regard a lift of $c$ and use the Jordan curve theorem).

□

Lemma 2.4. Given the geodesic return map $F : S^2 \to S^2$ induced from $\gamma$. Set $m_0 = \min \{ \per(x) | x \in A \}$. Then the following holds:

i) $F^{m_0}$ has three fixed points

ii) $\per(A) = \{ m_0, 2m_0 \}$

iii) If $F^{m_0}$ is orientation preserving then $F^{m_0} = \text{Id}$. Furthermore all geodesics are simple closed.

iv) If $F^{m_0}$ is orientation reversing the fixed point set is an embedded circle.

v) If for some $v \in S_A$ we have $\per(v) = 2m_0$ then $v$ is an interior point of $S_A$, i.e. $S_A$ contains an open set.

Proof: i) is clear, since $\{ \pm \infty \}$ are fixed points.

ii) If $m_0 < \per(y) = p < 2m_0$ then $F^{2m_0-p}(y) = y$ and $0 < 2m_0 - p < m_0$.

iii) Since $F$ has three fixed points, we conclude from corallary [1.10] that $F = \text{Id}$. We show that all geodesics are simple closed. We first prove the case when $\gamma$ preserves orientation.

Lemma [2.3, i)] shows that all geodesics (except $\gamma$) are intersecting $\gamma$ for $2m_0$ times, i.e. $m_0 \equiv \per$. We show that $S_A$ is an open set. Assume there is a sequence $v_i \to v \in S_A$ such that $\gamma_{v_i}$ has selfintersections. It follows from lemma [1.6] that we can suppose $P(v_i) \geq nP(v)$ where $n > 1$. If $i$ grows it follows from transversality that $\gamma_{v_i}$ intersects $\gamma$ more than $2m_0$ times. Take a look at the picture.
Regard a local coordinate system around $\gamma$. Choose a sequence 
$0 \leq t_{1,i} < t_{2,i} < P(v_i)$ such that $\gamma_{v_i}(t_{1,i}) = \gamma_{v_i}(t_{2,i})$. Set $x_i = \gamma_{v_i}(t_{1,i})$. Note 
that $\dot{\gamma}_{v_i}(t_{1,i})$ and $\dot{\gamma}_{v_i}(t_{2,i})$ converge to tangent vectors of $\gamma_v$.

W.l.o.g. $x_i \to x \notin \gamma$. Thus if $i$ grows, we have for some $\delta > 0$ that $t_{1,i} + 
P(v_i) - \delta < t_{2,i} + \delta$. Moreover, we can assume that $\gamma_{v_i}([t_{1,i} + \delta, t_{1,i} + P(v_i) - \delta])$ and $\gamma_{v_i}([t_{2,i} + \delta, t_{2,i} + P(v_i) - \delta])$ intersect $\gamma$ for at least 4$m_0$ times, thus $\gamma_{v_i}(0, P(v_i))$ intersects $\gamma$ for at least 4$m_0$ times. This gives a contradiction. Hence $S_A$ is 
onempty, but it is closed in $A$. If $\gamma$ is orientation reversing the proof is analogue. One must apply lemma [2.3, ii)] instead of lemma [2.3, i]a).

iv) If $F^{m_0}$ is orientation reversing one can deduce as in the proof of lemma [2.3, iii)] from corollary [1.10] that its fixed point set is an embedded circle.

v) We only prove the case where $\gamma$ is orientation preserving, since the other case is analog. All geodesics (except $\gamma$) are intersecting $\gamma$ at most 4$m_0$ times!

Assume there is a sequence $v_i \in S_A^c \to v \in S_A$ where $Per(v) = 2m_0$. Since $P(v_i) \to nP(v)$ where $n > 1$ we conclude as in the proof of lemma [2.4, iii)] that $\gamma_{v_i}$ intersects $\gamma$ more than 4$m_0$ times for $i$ sufficiently large. \hfill \square

**Proposition 2.5.** Given a simple closed geodesic $\gamma$ and the associated geodesic return map $F : S^2 \to S^2$. Then $|S_A| = \infty$, hence there are infinitely many distinct simple closed geodesics.

Proof: Let us set $G := F^{m_0}$. W.l.o.g. $G$ is orientation reversing, otherwise we apply lemma [2.4, iii)]. By corollary [1.10] we have $F^{2m_0} = G^2 = Id$, since $G^2$ has at least three fixed points and is orientation preserving.

We first regard the case where $\gamma$ preserves the orientation (i.e. we can find a neighbourhood of $\gamma$ homeomorphic to a cylinder). Assume $|S_A|$ is finite thus $Per(S_A) = m_0$ by lemma [2.4, v)] and $G := F^{m_0}$ has at least 2 + 2 fixed points. Indeed, we have

$$Fix(G) = \{v \in A \mid Per(v) = m_0\} \cup \{\pm \infty\},$$
and the remaining two simple closed geodesics given by theorem [1.7] and $\{\pm \infty\}$ gives us at least 4 fixed points by the previous results. $G$ is orientation reversing and therefore by corollary [1.10] its fixed point set is an embedded circle. We conclude from the method of lemma [2.4, iii)] that $S_A$ contains an arc connecting $\infty$ and $-\infty$. Indeed, $Fix(G) - \{\infty, -\infty\}$ is the union of two arcs that contains at least two fixed points (that induce simple closed geodesics). There is an arc $\nu \subset Fix(G) - \{\infty, -\infty\}$ that contains at least one point from $S_A$ and is connecting $\infty$ and $-\infty$. We show that $S_A \cup \nu$ is open in the space $\nu$!

Take $v_i \in S_A^c \cap \nu \to v \in S_A \cap \nu$, therefore we have $Per(v) = m_0 = Per(v_i)$. Again $P(v_i) \to nP(v)$ where $n > 1$ and we conclude as in lemma [2.4, iii)] that $Per(v_i) > m_0$ if $i$ grows, which gives a contradiction. Therefore $S_A \cap \nu$ is an open set in $\nu$.

If $\gamma$ reverses the orientation then the proof is analagous. \hfill \square

Now we are able to prove our theorem:

**Theorem 2.6.** Let $g$ be a metric on $\mathbb{R}P^2$. Then $(\mathbb{R}P^2, g)$ has constant curvature iff all geodesics are closed.
Proof: We show that \((\mathbb{R}P^2, g)\) is a SC-manifold, since this implies that \((\mathbb{R}P^2, g)\) has constant curvature (see 10.10.3 theorem 258 in [Be]). Let us denote \(M = \mathbb{R}P^2\) and \(SM\) its unit tangent bundle. We know that the geodesic flow \(\Phi\) on \(SM\) is a smooth semi-free (sometimes called locally-free) \(S^1\)-action. Suppose that \(u \in SM\) is a point such that the orbit of \(\Phi\) does not have maximal period, i.e. the isotropy group \(S^1_u\) at \(u\) is a non-trivial cyclic group generated by say \(f_k = e^{2\pi i} \in S^1\). Note that \(SM\) as a contact manifold is orientable and the finite order diffeomorphism \(f_k : SM \rightarrow SM\) is orientation preserving. Thus its fixed point set is one dimensional and hence exactly the orbit through \(u\). Any orbit of this type (that has no maximal period) is isolated and therefore there are at most finitely many such exceptional orbits. Let us prove this in detail: Given \(SM\) and a smooth circle action of \(SM\), then we can find a smooth Riemannian metric \(g_0\) invariant under the circle action by averaging of a smooth Riemannian metric on \(SM\). Thus any element of our action becomes an isometry of \((SM, g_0)\). Given now \(f_k\) and \(u \in Fix(f_k)\). Note that the orbit of \(u\) under the geodesic flow \(\Phi\) lies in \(Fix(f_k)\). Suppose that \(Fix(f_k)\) is not the union of finitely many such exceptional orbits, therefore for some point \(x \in Fix(f_k)\) we find for \(Df(x)\) 2 independent eigenvectors with eigenvalue 1. Take a look at the picture that shows how \(Fix(f_k)\) looks likewise if it is not a manifold of dimension 1.

![Diagram](image)

Note that \((Df_k(x))^k = Id\) (thus \(Df_k(x)\) has no nilpotent part) and therefore we conclude \(f_k = Id\), since \(f_k : SM \rightarrow SM\) is orientation preserving and an isometry of \((SM, g_0)\) such that \(Df_k(x) = Id\). This implies that \(Fix(f_k)\) is near \(u\) a segment of the unique orbit through \(u\) and therefore the finite union of closed orbits. The same arguments can be applied to \(f_k^l\) where \(l \in \{2, \cdots, k - 1\}\).

Note that \(S\) is the set of all \(v \in SM\) that induces simple closed geodesics. Let us denote by \(E\) the dense, open and connected set in \(SM\) containing only the no-exceptional orbits (\(SM\) has dimension three). Note that \(S^c \cap E\), as a subset of \(E\), is open and there must exist an orbit \(v \in E\) that is simple closed by proposition [2.5], since \(E^c\) is the union of finitely many orbits. The set \(S \cap E\) is open in \(E\). Otherwise we approximate \(v \in S \cap E\) with \(v_i \in S^c \cap E\) and conclude that \(v\) has selfintersections by lemma [1.6]. Thus \(E = E \cap S \cap S\) and since \(S\) is closed in \(SM\), we conclude \(S = SM\). This argument was also used in [G].

Finally we show that the period for all geodesics are the same. Take a geodesic \(\gamma\) that has maximal period and construct the map \(F : A \rightarrow A\). All exceptional points \(E^c\) induce a set \(E^c_A = E^c \cap A\), since lemma [1.8] holds. We prove only the case where \(\gamma\) preserves orientation, since the other case is analogue. Note \(|E^c_A| < \infty\), since \(E^c\) is the union of finitely many orbits. Given any \(w \in E^c_A\) that has no maximal period. Let \(P_0\) denote the maximal period. Take a sequence \(w_i = E_A = E \cap A \rightarrow w\) such that \(Per(w_i) = Per(w)\) and
$bP_0 = bP(w_i) = aP(w)$, where $a, b$ are integers. This is possible since $Per$ is constant on $A$ or $Fix(F^{m_0})$ is an embedded circle and the set of those points $v$ such that $Per(v) \neq m_0$ is open. The closed curves $\gamma_{w_i} : [0, bP_0] \to M$ converge to $\gamma_w : [0, aP(w)] \to M$, thus, if $i$ grows, the intersection number of these closed curves with $\gamma$ is the same as that of $\gamma_w$ with $\gamma$. Therefore $2bPer(w_i) = 2aPer(w)$ and this shows $a = b$ and so $P(w) = P_0$. \hfill \square$

As a corollary we conclude:

**Theorem 2.7.** All two dimensional $P$-manifolds are SC-manifolds.

Proof: The only $P$-manifolds in dimension 2 are $S^2$ and $\mathbb{R}P^2$ (see theorem 7.37 in [Bs]). The first case follows from [G] and the second case follows from theorem [2.6]. \hfill \square

### 3 Zoll surfaces

In this small section we want to point out the remarkable difference between $P$-metrics on $S^2$ and $\mathbb{R}P^2$. We state the results of Zoll and Darboux that easily shows the existence of many different $P$-metrics on $S^2$. The following classes of Riemannian manifolds $(S^2, g)$ are important in the theory of 2-dimensional $P$-manifolds:

**Definition 3.1.** A metric $g$ on $S^2$ is called a Zoll metric if $g$ is continuous and $(S^2, g)$ is a $C^{2\pi}$-surface (i.e. all geodesics are periodic and have the period $2\pi$). In this case $(S^2, g)$ is called a Zoll surface.

**Definition 3.2.** A metric $g$ on $S^2$ is called a metric of revolution if $(S^2, g)$ has $S^1$ as an effective isometry subgroup. The surface $(S^2, g)$ is called a surface of revolution.

If $(S^2, g)$ is a surface of revolution then the action $S^1$ has exactly two fixed points, say $N$ and $S$ (the so-called North and South poles). Since $U := S^2 - \{S, N\}$ is diffeomorphic to the punctured disc

$$\{z \in \mathbb{C} \mid |z| < 1, z \neq 0\},$$

we can find a coordinate system such that $g$ can be written on $U$ as

$$g = du^2 + a^2(u)\theta^2$$

such that $a^2$ is bounded by 1. From this we can find a coordinate system $[U, (r, \theta)]$ such that $g$ can be written on $U$ as

$$g = f(cos r)^2 dr^2 + sin^2 r d\theta^2,$$

where $f$ is a function from $(-1, 1)$ to $\mathbb{R}^+$ (compare proposition 4.10 in [Bs]). The following theorem holds:
**Theorem 3.3** (Darboux). Let \((S^2, g)\) be a surface of revolution and \(g\) be written in the parametrization \([U, (r, \theta)]\) as

\[ g = f(\cos r)^2 dr^2 + \sin^2 r d\theta^2. \]

A necessary and sufficient condition in order that all geodesics are closed is that for every \(t\) one has

\[ \int_t^{\pi-t} \frac{f(\cos r) \sin t}{\sin r \sqrt{\sin^2 r - \sin^2 t}} dr = \frac{p \pi}{q} \]

where \(p\) and \(q\) are integers.

Proof: See [D] or theorem 4.11 in [Bs]. □

If we assume that \((S^2, g)\) is a Zoll surface of revolution then the following theorem holds:

**Theorem 3.4.** A metric \(g\) is a Zoll metric of revolution on \(S^2\) iff \(g\) can be written in the parametrization \([U, (r, \theta)]\) as

\[ g = (1 + h(\cos r))^2 dr^2 + \sin^2 r d\theta^2 \]

where \(h\) is an odd function from \([-1, 1]\) to \((-1, 1)\), whose value at 1 is 0. Furthermore, \((S^2, g)\) is a \(SC_{2\pi}\)-manifold (i.e. a \(C_{2\pi}\)-manifold whose geodesics are all simple closed). Moreover, \(g\) is \(C^k\) at \(N\) (resp. \(S\)) iff \(h\) extends to a \(C^{k-1}\) function in a neighbourhood of 1 (resp. \(-1\)).

Proof: See theorem 4.13 and corollary 4.16 in [Bs]. □

It is remarkable that the choice of the function \(h\) does not involve any condition on the derivative. Since the sectional curvature can be expressed in the form

\[ \sigma(r) = \frac{1}{(1 + h(\cos r))^3} (1 + h(\cos r) - \cos r \cdot h'(\cos r)), \]

we see that there is a large class of \(C^\infty\) Zoll metrics whose sectional curvature is not constant. One can choose for example

\(h(\cos r) = \cos r \cdot \sin(2k + 1)r\) or \(h(\cos r) = \cos r \cdot \frac{\sin^2 r}{2}\). In particular, one can prove that there is a large set of different \(P\)-metrics.

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