The stress-strain state of the spherical body under uniform compression taking into account the radial non-monotonic distribution of the material elastic characteristics

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Abstract. The mathematical model describing the centrally symmetric stress-strain state of an inhomogeneous elastic spherical body under the action of uniform all-sided compression is constructed. Approximate analytical solutions are found. The inhomogeneity of the elastic characteristics of the material is modeled by a radial non-monotonic elastic modulus at a constant Poisson's ratio. Based on experimental data, the components of the stress-strain state are numerically calculated for the problem under consideration, and characteristic effects are revealed.

1. Introduction
The strength calculation of a large number of structural elements used in such industries as aircraft construction, rocket science, and construction industry is reduced to solving the problems of determining the stress-strain state (SSS) of inhomogeneous bodies under the effect of elastic deformations, that is, to the problems of the heterogeneous elasticity theory.

Inhomogeneous elastic bodies are divided into three categories according to the type of functions that model the distribution of elastic characteristics of materials: bodies with continuous inhomogeneity; piecewise homogeneous bodies; randomly inhomogeneous bodies. The functional dependencies describing changes in the mechanical properties of the material (hereinafter, heterogeneity functions) are continuous, piecewise constant [1], and random, respectively.

In the present paper, we will consider elastic bodies with continuous heterogeneity. This type of heterogeneity of mechanical properties of the material occurs, for example, in vessels and flowlines of the nuclear power station (reactor shells, main circulation pipe systems, and others) working for a long time under high internal pressure at high temperature, corrosion and radiation conditions. This leads to the heterogeneity of the material thickness of these structures [2–5].

Note that the problems of mechanics of continuously inhomogeneous bodies lead to differential equations with variable coefficients [6]. Such differential equations differ significantly according to the form of functional dependencies modeling changes in mechanical characteristics along the coordinates. Therefore, fairly simple dependencies should be used as heterogeneity functions to obtain a simpler equation. Besides, these functions should most adequately describe the experimental data,
since even small differences in the choice of approximating functions can lead to significantly different results.

Linear, exponential, or power functions were chosen in [5, 7–10] as the distribution model of the elastic modulus when searching for an analytical solution. However, the specified functional dependencies used for approximation of variable mechanical characteristics do not always satisfy quantitative and qualitative assessment of the approximation results in comparison with experimental data. This applies to structures where the distribution of elastic characteristics of materials is significantly non-monotonous. For example, the functional dependence with extremes of the mechanical characteristics of materials is observed near the inner surface when creating mine workings by drilling and blasting or creating underground cavities using camouflage explosions [11].

A numerical and analytical study of the stress-strain state of an elastic massif near a spherical cavity was carried out in [12–14], taking into account the technological heterogeneity of mechanical properties of the massif. The problem of the distribution of stress and displacement fields in a two-layer spherical structure, taking into account the porous structure of the inner layer and the heterogeneity of the outer layer, was considered in [15]. The problem of stability of the equilibrium state of a porous cylindrical body subject to the inelastic behavior of the completely compressed matrix was investigated in [16]. In the present paper, the authors solve the problem of determining the stress-strain state of a hollow ball within the framework of a centrally symmetric formulation for inhomogeneous elastic materials, taking into account the non-monotonic nature of the thickness distribution of mechanical characteristics.

2. Symmetric deformation of an elastic compressible spherical body taking into account the non-monotonic thickness distribution of the material elastic characteristics

In this section, the problem of deforming of a spherical body with an outer radius \( b \) and an inner radius \( a \) is considered within the framework of a model of deforming of an elastically compressible material with a continuous dependence of the Lame parameter \( \mu = \mu(r) \) on the radial coordinate with a constant Poisson's ratio \( \nu = \text{const} \neq 0.5 \) (figure 1). The outer and inner surfaces are under uniformly distributed radial loads with the intensities \( q_b \) and \( q_a \), respectively.

![Figure 1](image_url)

*Figure 1.* Calculation scheme for a radially inhomogeneous spherical body under uniformly compressive loads: (a) the spherical body under uniform radial compression; (b) non-monotonic distribution of the Lame parameter \( \mu = \mu(r) \) of material on the radial coordinate.
As noted above, problems of calculating the reactor shells of nuclear power plants, support constructions of underground spherical cavities created using camouflage explosions [17], etc. are reduced to such problems from the point of view of mathematical modeling.

In the present paper, the inhomogeneity of elastic properties of a material is proposed to be modeled by a non-monotone thickness distribution $\mu = \mu(r)$ of the spherical body under consideration in the form of the following dependence

$$\mu(r) = \mu_0 \left(1 + k e^{-\alpha r} \left(\cos(\beta r + \omega) + \sin(\beta r + \omega)\right)\right), \quad (1)$$

where $k, \alpha, \beta, \omega$ are the approximation parameters ($\alpha > 0$), $\mu_0$ is the Lame parameter for the corresponding homogeneous material.

Note that heterogeneity function (1) differs from the well-known monotone ones [2, 6, 7, 9].

The distribution graphs (1) for different values of the approximating coefficients are shown in figure 2. Unless otherwise specified, the values of the parameters of the approximating model (1) are as follows: $\mu_0 = 1$, $\alpha = 1$, $\beta = 2$, $\omega = 0$, $k = 0.1$.

From the analysis of the presented graphs, it follows that the dependence $\mu(r)$ describes the non-monotonic character of the distribution of this characteristic with extremes near the inner surface. The parameter $\alpha$ affects the rate of fluctuations’ attenuation of inhomogeneous properties of the material, so the greater the $\alpha$, the faster the approximating curve gets close to $\mu_0$. The parameters $k, \beta, \omega$ are responsible for the amplitude, the number of waves along the radial coordinate, and the initial phase of the non-monotonic distribution of inhomogeneous elastic properties of the material, respectively. The case $k = 0$ corresponds to a homogeneous material.

![Figure 2](image-url)

**Figure 2.** Distribution of the Lame parameter $\mu = \mu(r)$ of an inhomogeneous material along the radial coordinate $r$ (a) at varying values of the approximating coefficient $\beta$ and $k = 0.2$ (curve 1: $\beta = 2$; curve 2: $\beta = 4$; curve 3: $\beta = 6$); (b) at varying values of the approximating coefficient $\omega$ and $k = 0.2$ (curve 1: $\omega = -1$; curve 2: $\omega = 0$; curve 3: $\omega = 1$).

Note that the approximating coefficients of the model (1) are determined based on experimental data in such a way that the resulting curve is closest to the original data, for example, in the sense of "least squares". This can be done in various ways that are not the subject of this study.

Based on the problem statement described above, a Central (polar) symmetry takes place. Then the SSS of the considered hollow ball in the spherical coordinate system $r, \theta, \varphi$ will be described only by the main non-zero components of the stress and strain tensors, as well as the radial component of
the displacement vector. All these components will be functions of the radial coordinate and, due to the Central symmetry, satisfy the following conditions

\[
\begin{align*}
\sigma_r &= \sigma_r(r), \quad \sigma_\theta = \sigma_\theta(r) = \sigma_\phi = \sigma_\phi(r), \\
\varepsilon_r &= \varepsilon_r(r), \quad \varepsilon_\theta = \varepsilon_\theta(r) = \varepsilon_\phi = \varepsilon_\phi(r), \\
u &= u(r).
\end{align*}
\]

(2)

Taking into account (2), three equilibrium equations are converted to one

\[
2r \frac{d\sigma_r}{dr} + 2(\sigma_r - \sigma_\theta) = 0;
\]

(3)

the Cauchy relations take the form

\[
\varepsilon_r = \frac{du}{dr}, \quad \varepsilon_\theta = \varepsilon_\phi = \frac{u}{r};
\]

(4)

the formulas of Hooke’s law taking into account (1) in terms of the Lamé parameters \(\lambda\) and \(\mu\) are written in the form

\[
\sigma_r = \left(\lambda(r) + 2\mu(r)\right)\varepsilon_r + 2\lambda(r)\varepsilon_\theta, \quad \sigma_\theta = \sigma_\phi = \lambda(r)\varepsilon_r + 2\left(\lambda(r) + \mu(r)\right)\varepsilon_\phi,
\]

(5)

or through the parameters \(\mu\) and \(\nu\), are converted to the form

\[
\sigma_r = 2\mu(r)\left(\frac{\nu}{1-2\nu} \cdot \Theta + \varepsilon_r\right), \quad \sigma_\theta = \sigma_\phi = 2\mu(r)\left(\frac{\nu}{1-2\nu} \cdot \Theta + \varepsilon_\phi\right),
\]

(6)

where the volumetric deformation \(\Theta\) is determined by the formula

\[
\Theta = \varepsilon_r + 2\varepsilon_\theta;
\]

(7)

the stress boundary conditions are written in the form

\[
\sigma_r|_{r=a} = -q_a, \quad \sigma_r|_{r=b} = -q_b.
\]

(8)

We rewrite the equilibrium equation (2) in displacements taking into account (5) and (4)

\[
\frac{d}{dr}\left[(\lambda(r) + 2\mu(r))\frac{du}{dr} + 2\lambda(r)\frac{u}{r}\right] + 4\left(\frac{d}{dr}\left[\mu(r)\frac{u}{r}\right] - \frac{u}{r} \mu'(r)\right) = 0.
\]

Note that the elastic characteristics are related by the formula \(\lambda(r) = \frac{2\nu}{1-2\nu}\mu(r)\), then the last relation is rewritten as

\[
\frac{d}{dr}\left[(\frac{du}{dr} + 2\frac{u}{r})\mu(r)\right] = 2\frac{1-2\nu}{1-\nu} \mu'(r)\frac{u}{r}.
\]

(9)

Then, following the paper [9], we organize the iterative process according to the formula

\[
\frac{d}{dr}\left[(\frac{du_{k+1}}{dr} + 2\frac{u_{k+1}}{r})\mu(r)\right] = 2\frac{1-2\nu}{1-\nu} \mu'(r)\frac{u_{k+1}}{r}.
\]

(10)

The solution of the inhomogeneous equation (10) will be found as the sum of the general solution of the corresponding homogeneous equation and an arbitrary partial solution of the inhomogeneous equation. As a result, the iterative formula for determining the radial component of the displacement
vector will take the form

\[ u_{k+1} = \frac{1}{r^2} \left( A_{k+1} \int_a^r \frac{x^2}{\mu(x)} \, dx + B_{k+1} \right) + G_k(r), \tag{11} \]

where \( A_{k+1} \), \( B_{k+1} \) are integration constants,

\[ G_k(r) = \frac{2 - 4v}{1 - v} \frac{1}{r^2} \int_a^r Q_k(y) \, dy, \quad Q_k(y) = \frac{y^2}{\mu(y)} \int_a^y \frac{\mu'(x)}{x} \, dx. \tag{12} \]

Using the Cauchy relations (4) and the formula (7), taking into account (11), we obtain iterative relations for the components of deformations and volumetric deformation \( \Theta \)

\[ (\varepsilon_r)_{k+1} = A_{k+1} \left( \frac{1}{\mu(r)} \int_a^r \frac{x^2}{\mu(x)} \, dx \right) - \frac{2B_{k+1}}{r^3} + \frac{dG_k(r)}{dr}, \]

\[ (\varepsilon_\theta)_{k+1} = (\varepsilon_\phi)_{k+1} = \frac{1}{r^2} \left( A_{k+1} \int_a^r \frac{x^2}{\mu(x)} \, dx + B_{k+1} \right) + \frac{G_k(r)}{r}, \tag{13} \]

Iterative formulas for determining the stress components are obtained from Hooke's law (6) taking into account (13) in the form

\[ (\sigma_r)_{k+1} = 4\mu(r) \left( A_{k+1} F(r) - \frac{B_{k+1}}{r^3} + H_k(r) \right), \tag{14} \]

\[ (\sigma_\theta)_{k+1} = (\sigma_\phi)_{k+1} = 2\mu(r) \left( M(r) A_{k+1} + \frac{B_{k+1}}{r^3} + T_k(r) \right), \]

where the following notation is entered

\[ F(r) = \frac{1 - v}{2(1 - 2v)} \frac{1}{\mu(r)} - \frac{1}{r^3} \int_a^r \frac{x^2}{\mu(x)} \, dx, \quad M(r) = \frac{\nu + 1}{2(1 - 2v)} \frac{1}{\mu(r)} - F(r), \]

\[ H_k(r) = \frac{Q_k(r)}{r^2} + \frac{4v - 2}{1 - v} \frac{1}{r^3} \int_a^r Q_k(x) \, dx, \quad T_k(r) = \frac{\nu + 1}{1 - v} \frac{Q_k(r)}{r^2} - H_k(r). \tag{15} \]

The integration constants \( A_{k+1} \), \( B_{k+1} \) are determined from the boundary conditions (8), taking into account (14), (15) in the form

\[ A_{k+1} = \frac{a^3 \left( \frac{q_a}{4\mu(a)} + H_k(a) \right) - b^3 \left( \frac{q_b}{4\mu(b)} + H_k(b) \right)}{b^3 F(b) - a^3 F(a)}, \quad B_{k+1} = a^3 \left( A_{k+1} F(a) + H_k(a) + \frac{q_a}{4\mu(a)} \right), \tag{16} \]

where according to (15) and (12)

\[ F(a) = \frac{1 - v}{2(1 - 2v)} \frac{1}{\mu(a)} - \frac{1}{b^3} \int_a^b \frac{r^2}{\mu(r)} \, dr, \]

\[ H_k(a) = 0, \quad H_k(b) = \frac{Q_k(b)}{b^2} + \frac{4v - 2}{1 - v} \frac{1}{b^3} \int_a^b Q_k(r) \, dr. \tag{17} \]
Iterative relations (11), (13), (14), (16) taking into account the entered notations (12), (15), (17) describe the distribution of fields of displacements, deformations, and stresses in a spherical body made of an elastically compressible inhomogeneous material with a continuous dependence of the Lamé parameter on the radial coordinate \( \mu = \mu(r) \) at a constant Poisson's ratio \( v = \text{const} \neq 0.5 \).

The results of the numerical experiment for the first and second approximations based on the obtained relations are shown in figures 3–6.

We choose \( u_0 = 0 \) for the initial approximation to determine consistently the integration constants and the SSS-components of the problem under consideration. A non-monotonic function of the form (1) is used as a dependence approximating radially inhomogeneous elastic properties of the material. Unless otherwise specified, the values of the approximating parameters are as follows: \( \alpha = 1.5, \beta = 6, \omega = 0, k = 2.5 \).

Calculations are performed in relative terms. All values of the stress dimension are referred to the displacement modulus \( \mu_0 \) of a homogeneous material. All values of the length dimension are referred to the inner radius \( a \) of the hollow ball under consideration. The values of relative initial physical, mechanical, and geometric parameters are as follows: \( \mu_0 = 1, q_a = 0.003, q_b = 0.08, a = 1, b = 3a \).

**Figure 3.** Graph of the radial component of the displacement vector \( u(r) \) in the second approximation against the radial coordinate \( r \) at varying values of the Poisson's ratio \( v \) (curve 1: \( v = 0.2 \); curve 2: \( v = 0.3 \); curve 3: \( v = 0.4 \)).

**Figure 4.** Graph of the radial component of the displacement vector \( u(r) \) against the radial coordinate \( r \) in the first (curve 1) and second (curve 2) approximations at \( v = 0.34 \).
Figure 5. Graph of the (a) radial stress component $\sigma_r$ in the second approximation; (b) circumferential stress component $\sigma_{\theta}$ in the second approximation against the radial coordinate $r$ at varying values of the Poisson's ratio $\nu$ (curve 1: $\nu=0.2$; curve 2: $\nu=0.3$; curve 3: $\nu=0.4$; dotted line: $\nu=0.5$ – the case of incompressible material).

Figure 6. Graph of the (a) radial stress component $\sigma_r$; (b) circumferential stress component $\sigma_{\theta}$ against the radial coordinate $r$ in the first (curve 1) and second (curve 2) approximations at $\nu=0.34$.

Note that the relations (11), (13), (14) and (16), which describe the SSS for the model of compressible material, cannot be generalized to the case of incompressible material ($\nu=0.5$), since in this case the equation (9) turns into an identity when the incompressibility condition is met.

3. Mathematical model of the SSS of an elastic incompressible spherical body taking into account the radial inhomogeneity of its mechanical properties

We present a solution to the problem stated in the previous section for the case of an elastic incompressible radially inhomogeneous material $\mu=\mu(r), \nu=0.5$.

In this case, the relations of Hooke's law for incompressible material in the Central symmetric formulation under consideration are written in the form

$$S_r = 2\mu(r)\varepsilon_r, \quad S_{\theta} = S_{\phi} = 2\mu(r)\varepsilon_{\theta},$$

(18)

where the components of the stress deviator tensor $S_r, S_{\theta}, S_{\phi}$ are defined by the formulas...
\[
S_r = \sigma_r - \frac{1}{3} (\sigma_r + 2\sigma_\theta), \quad S_\theta = \sigma_\theta - \frac{1}{3} (\sigma_r + 2\sigma_\theta).
\] (19)

The condition \( \Theta = 0 \) (the incompressibility condition of the material), taking into account (7) and (4), is written as
\[
\frac{du}{dr} + \frac{2u}{r} = 0.
\] (20)

Thus, to determine the SSS of an elastic incompressible spherical body taking into account a non-monotonic distribution of the parameter \( \mu = \mu(r) \) is necessary to solve a closed boundary value problem (3), (4), (18)–(20),(8).

From the incompressibility condition (20), which is a linear homogeneous equation with respect to the radial component of the displacement vector, we get
\[
u = \frac{C_1}{r^3},
\] (21)
where \( C_1 \) is the integration constant.

From (18), taking into account (19) and (4), it follows that
\[
\sigma_r - \sigma_\theta = 2\mu(r) \left( \frac{du}{dr} - \frac{u}{r} \right),
\]
or taking into account (21)
\[
\sigma_r - \sigma_\theta = -6C_1 \frac{\mu(r)}{r^3}.
\] (22)

Substituting (22) into the equilibrium equation (3), we obtain an equation for determining the radial component of stresses
\[
\frac{d\sigma_r}{dr} = 12C_1 \frac{\mu(r)}{r^3},
\]
integrating it we get
\[
\sigma_r = 12C_1 \int_a^r \frac{\mu(x)}{x^4} dx + \sigma_r \bigg|_{r=a}.
\]

Taking into account the boundary conditions (8), the last relation is rewritten in the form
\[
\sigma_r = \frac{q_a - q_b}{\int_a^b \frac{\mu(r)}{r^3} dr} \int_a^r \frac{\mu(x)}{x^4} dx - q_a.
\] (23)

The remaining components of the stress tensor are determined from (22), taking into account (23) and (2)
\[
\sigma_\theta = \sigma_\phi = \frac{q_a - q_b}{2\int_a^b \frac{\mu(r)}{r^3} dr} \left( 2 \int_a^r \frac{\mu(x)}{x^4} dx + \frac{\mu(r)}{r^3} \right) - q_a.
\] (24)

Finally, the displacement and strain components are defined in the form
Thus, the SSS of a spherical body made of an elastic radially inhomogeneous non-compressible material is described by the relations (23)–(26).

As in section 2, we use a function of the form (1) as the heterogeneity function.

The results of the numerical experiment are shown in figures 7–8.

**Figure 7.** Graph of the (a) relative radial stress component \( \sigma_r \); (b) relative circumferential stress component \( \sigma_\theta \) against the radial coordinate \( r \) at varying values of the approximating parameters \( \alpha, \omega, k \) for the case of incompressible material: \( \nu = 0.5 \) (curve 1: \( \alpha = 1.8, \omega = 0, k = 13 \); curve 2: \( \alpha = 1.5, \omega = 1, k = 6.5 \); curve 3: \( \alpha = 1.2, \omega = -1, k = 3.9 \); dotted line: \( k = 0 \) – the case of homogeneous material).

**Figure 8.** Distribution of the radial coordinate of the displacement vector \( u(r) \) over the thickness of the hollow ball at varying values of the approximating parameters \( \alpha, \omega, k \) for the case of incompressible material: \( \nu = 0.5 \) (curve 1: \( \alpha = 1.8, \omega = 0, k = 13 \); curve 2: \( \alpha = 1.5, \omega = 1, k = 6.5 \); curve 3: \( \alpha = 1.2, \omega = -1, k = 3.9 \); dotted line: \( k = 0 \) – the case of homogeneous material).

In each of figures 9–11, it is taken \( \beta = 6 \). The values of relative initial physical, mechanical, and geometric parameters are as follows: \( \mu_0 = 1, q_a = 0.01, q_b = 0.1, a = 1, b = 3a \).
4. Conclusions
We have carried out mathematical modeling of the SSS of a spherical body (hollow ball) under uniform compression, taking into account the non-monotonic distribution of elastic characteristics of the material. The non-monotonic distribution was modeled by the functional dependence (1) of the Lame parameter $\mu(r)$ along the radial coordinate on condition $\nu = \text{const}$. For the model of a compressible body ($\nu \neq 0.5$), the fields of displacements, deformations, and stresses are described by iterative relations (11)–(17), and in the case of an incompressible body ($\nu = 0.5$) the SSS of a hollow ball is determined by the formulas (23)–(26).

Based on the solutions found, a numerical experiment was performed. The results of a numerical experiment were analyzed to identify the influence of approximation parameters on the distribution of stress and displacement fields. From the analysis of the presented dependencies, the following conclusions can be made:
- there is a dependence of the SSS-components on the approximation parameters; changes in the approximation parameters can affect both the quantitative value and the qualitative behavior of the SSS-components;
- for a compressible material model, a change of the Poisson's ratio leads to a significant change of the radial component of the displacement vector; in lesser degree, the change of the Poisson's ratio affects the behavior of the radial and circumferential components of the stress tensor;
- for the case of a compressible material, there is a non-monotonous distribution of the SSS-components along the radial coordinate; if the incompressibility condition is accepted, the radial displacement component has a monotonically decreasing distribution along the thickness, while the stress plots (especially the circumferential component) have a significantly non-monotonous distribution along the radial coordinate;
- for the approximate relations (11)–(17), the similarity of the corresponding dependencies in the first and second approximations is observed. This fact confirms the convergence of the obtained iterative formulas.

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