How Does Risk Hedging Impact Operations? Insights from a Price-Setting Newsvendor Model

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Financial asset price movement impacts product demand and thus influences operational decisions of a firm. We develop and solve a general model that integrates financial risk hedging into a price-setting newsvendor. The optimal hedging strategy is found analytically, which leads to an explicit objective function for optimization of pricing and service levels. We find that, in general, the presence of hedging reduces the optimal price. It also reduces the optimal service level when the asset price trend positively impacts product demand ("asset price benefits demand"), while it may increase the optimal service level by a small margin when the impact is negative ("asset price hurts demand"). We construct the mean-variance efficient frontier that characterizes the risk-return trade-off, and we quantify the risk reduction achieved by the hedging strategy. Our numerical case study using real data of Ford Motor Company shows that the markdown in price and decrease in service level are small under our model, and the hedging strategy substantially reduces risk without materially reducing operational profit.

Key words: newsvendor; pricing; risk hedging; mean-variance framework

1. Introduction
Effective risk management for firms facing substantial volatility in product demand needs to account for various factors that impact demand. One important exogenous factor is the price of some tradable financial asset (e.g., stock market index, commodities, foreign currency); that is, product demand fluctuates in response to financial asset price movement. For example, the “Big Three” in automotive industry—General Motors, Fiat Chrysler Automobiles, and Ford Motor Company—all recognize, in their annual reports, that oil price is a major risk factor that impacts their sales volumes (General Motors, 2020 10-K Filing; Fiat Chrysler Automobiles, 2020 10-K Filing; Ford Motor Company, 2020 10-K Filing). This is supported by both economic theory and empirical evidence. Theoretically, it is known that the demand for a product is affected by the price of a complementary good (Mankiw 2014). This is because, in the automaker’s case, a higher oil price leads to a higher gasoline price, which directly increases the expense of using a car, thereby leading
customers to switch from gas guzzlers to more fuel-efficient vehicles. Empirical studies that support this theory include those of Busse et al. (2009), Klier and Linn (2010), Busse et al. (2013), and Langer and Miller (2013). Examples of financial asset prices impacting product demand can be found in other industries too. For instance, John Deere, the largest farming equipment manufacturer in the United States, discloses that prices of agricultural commodities, such as corn, have a major impact on their sales, as these prices directly affect the revenues of their customers (e.g., farmers). Another example is Caterpillar Inc., the largest industry equipment producer, who recognizes metal prices as demand risk factors. The analysis in this paper is not limited to any specific industry.

A major endogenous factor that affects demand is, clearly, product price. Basic economic theory suggests that a higher price leads to a lower demand, and the slope of the demand curve describes how demand changes with product price. While the financial asset price is uncontrollable, product price is controllable. Observing the former, the producer may set the latter proactively to mitigate the asset price’s impact on product demand. For example, it is observed that automakers respond to the fuel price’s impact on vehicle demand by setting prices strategically (Busse et al. 2009; Langer and Miller 2013; Allcott and Wozny 2014). In the scenario of rising oil prices, demand for fuel-inefficient cars decreases while demand for fuel-efficient cars increases. Car producers reduce the price of fuel-inefficient cars to offset the impact of the rising energy costs, while they raise the price of fuel-efficient cars in response to the increased demand.

We will demonstrate our results using data of two popular car models produced by Ford Motor Company: the Explorer and the Focus. The Explorer is categorized as fuel-inefficient (low miles per gallon [MPG]), and the Focus is fuel-efficient (high MPG). The relevant financial asset here is the crude oil, West Texas Intermediate (WTI). For each car model, we regress, respectively, the monthly sales volume and the selling price on the monthly average WTI price. The regression results are summarized in Table 1. It can be observed that the selling price of the Explorer (resp., the Focus)

| Car   | Estimate | Regression Equation                          |
|-------|----------|---------------------------------------------|
|       |          | Price $\sim$ WTI                           |
| Explorer | slope    | -40.16 (3.60)                               |
|        | p-value  | $6.6 \times 10^{-20}$                      |
| Focus  | slope    | 9.32 (1.90)                                |
|        | p-value  | $4.2 \times 10^{-6}$                       |
|        |          | Sales Volume $\sim$ WTI                     |
| Explorer |          | -168.55 (23.53)                             |
|        |          | $1.7 \times 10^{-11}$                      |
| Focus  |          | 132.99 (25.20)                              |
|        |          | $2.2 \times 10^{-7}$                       |

*Note.* The numbers in parentheses represent the standard errors.

is significantly negatively (resp., positively) correlated with oil price. The same pattern holds for
sales volume. These are consistent with our discussions above and existing empirical findings (Klier and Linn 2010; Busse et al. 2013; Allcott and Wozny 2014). Two important ideas are conveyed by the discussion and data analysis above. First, financial asset price movement may significantly impact product demand. Second, in practice, there exist interactions between the pricing decision and the asset price: to respond to the latter’s impact on demand, the firm can adjust the former (as in the practice of automakers discussed above). Here we also point out that since the financial asset is tradable, another way to manage risk is to adopt a risk-hedging strategy via trading this asset. Then, a more effective risk-management strategy is to jointly optimize pricing and hedging (and also production) decisions, which simultaneously account for both the endogenous and exogenous factors of demand. There are established bodies of literature on, separately, pricing strategies and risk management using financial assets (refer to §1.2 for details), but the literature on how to jointly optimize pricing and risk hedging is scarce. To our best knowledge, we are the first to study the interaction of pricing and risk hedging.

We propose the following research questions:

(i) How can a risk-management strategy be developed by jointly optimizing pricing, production, and hedging decisions using the relevant financial assets?

(ii) How does risk hedging affect the pricing decision, compared with the no-hedging case?

To answer these questions, we start by building a demand model that incorporates impacts from both the product pricing decision and the financial asset price. The model allows general relationship between demand and asset price. Based on this demand model, we set up the risk management problem under the mean-variance criterion. With production and pricing decisions given, we analytically solve for the optimal risk-hedging strategy. This gives an explicit risk objective function, which we minimize to find the optimal production and pricing decisions. All this amounts to a complete characterization of the mean-variance efficient frontier, upon which we quantify the improvement relative to the no-hedging model. Then, we apply the risk-management model to real data of the Explorer and the Focus.

1.1. Main Results and Contributions

Our main results are Theorems 1, 2, and 3. Given pricing and service levels (refer to (3) for the definition of service level), Theorem 1 calculates explicitly the optimal hedging strategy and its associated variance. The optimal hedging strategy is a combination of a risk-mitigation position and an investment position. The variance associated with this hedging strategy is the sum of investment risk and unhedgeable risk (i.e., the kind of risk that is irrelevant to and hence cannot be hedged by financial asset), with the latter being increasing in both the pricing and service levels. This
variance, as a function of the pricing and service levels, provides an explicit objective to be further minimized.

Assuming the asset price trend positively impacting ("benefiting") demand (e.g., rising oil price boosts demand of fuel-efficient cars), Theorem 2 characterizes the optimal pricing and service levels in the presence of hedging, with the target mean set as the newsvendor’s maximum profit. We find that both the optimal pricing and service levels are lower in the presence of hedging than those without hedging. Hedging cancels the positive effect of the asset price trend on demand, so the optimization of the operational policy (i.e., the pricing and service levels) essentially assumes a hypothetically smaller market size compared to that without hedging. Adapting to this smaller market size, the pricing and service levels also decrease while reducing unhedgeable risk.

Theorem 3 assumes the asset price trend negatively impacting ("hurting") demand (e.g., rising oil price decreases demand of fuel-inefficient cars), the scenario opposites the one considered in Theorem 2. The results are analogous: the optimal price is lower with hedging than without hedging, and the optimal service level in the presence of hedging is lower than—or at most exceeds by a small margin—the optimal service level without hedging. We interpret the result as follows. Because the hedging strategy cancels the negative effect of the asset price trend on demand, optimization of operational policy assumes a hypothetically larger market size. A negative payoff from the investment position of hedging never induces optimal return-risk trade-off. Therefore, with the enlarged market size, to leave leeway in the target return for hedging to fill, the pricing level needs to be adjusted down, which also reduces unhedgeable risk. For the optimal service level, we identify a condition—the detrimental effect of asset price is sufficiently strong—under which the service level with hedging does not exceed the service level without hedging (Theorem 3 (ii)), or the amount of excess is small (Theorem 3 (iii)).

This paper makes both technical and managerial contributions. Technically, we develop and solve a general risk-management model that integrates pricing, production, and hedging (using financial assets) decisions. The model does not assume any specific functional relationship between demand and asset price; thus it may incorporate application-specific data analytics. In particular, we explicitly solve for the pricing, production, and hedging decisions if the asset price follows the exponential Ornstein–Uhlenbeck process, a standard model for oil price.

Concerning managerial insights, the leading message of our paper is that hedging adjusts the pricing level down. To our best knowledge, we are the first to study how risk hedging impacts pricing. This points to the insight that hedging not only reduces risk but also enhances a firm’s competitiveness in the market. In addition, we show that the service level in the presence of hedging is either lower than or exceeds by a small margin the service level without hedging. To apply our results, we conduct a comprehensive numerical case study using real-world data sets of Ford Motor
Company. The analysis shows that the hedging performs well: risk can be reduced as much as 40%. In particular, markdowns for both the pricing and service levels are small—not exceeding 1.03% for price reduction and 1.70% for service level reduction. This is desirable, as firms usually do not want to excessively reduce the price (e.g., concern for brand name) or service level (e.g., concern for maintaining market share). This indicates that, while substantially reducing risk, hedging will not materially reduce a firm’s operational levels.

1.2. Literature Review

This study falls within the scope of integrated operations and financial risk management. Two main streams of literature in this realm are related to our work. One is joint pricing and production/inventory management, and the other is incorporating financial hedging in operations management.

In the stream of literature on joint pricing and production/inventory management, for the single-period setting, Whitin (1955) is the first to study the fundamental connection between pricing and inventory control theory. Petruzzi and Dada (1999) consider a risk-neutral firm facing a price-dependent random demand and examine how the stocking quantity decision interacts with the selling price decision. Agrawal and Seshadri (2000) study how a risk-aversion retailer facing a price-dependent random demand makes order quantity and pricing decisions to maximize an expected utility. Chen et al. (2009) consider a risk-averse decision maker similar to that analyzed by Agrawal and Seshadri (2000), with a different risk objective. Extensions include multi-period settings (Federgruen and Heching 1999; Chen and Simchi-Levi 2004a,b) and multi-product settings (Aydin and Porteus 2008; Zhu and Thonemann 2009; Song et al. 2021).

The other stream of literature concerns incorporating financial hedging in operational risk management. Gaur and Seshadri (2005) study the construction of an optimal trading strategy to hedge inventory risk using financial market instruments. A work closely related to ours is the study of Caldentey and Haugh (2006), in which the authors formulate a general modeling framework that incorporates risk hedging into operations with a quadratic utility function. Due to its generality, the interaction of risk hedging and any specific operational policy is not studied. Caldentey and Haugh (2009) consider a supply chain contracting problem with a supplier and a retailer engaging in a Stackelberg game, and in their setting the product market size depends on the price of some financial asset. Ding et al. (2007) consider an international firm who sells to both domestic and foreign markets and uses operational and financial hedging to manage currency exchange risk. The paper most relevant to ours is Wang and Yao (2017). The authors study a newsvendor model in which demand dynamics is partially driven by a financial asset price change. Our paper differs from theirs by including price as a decision variable and adopting a more general demand and asset price
model. Kouvelis and Li (2019) study a newsvendor problem with correlated operational and financial risks under value-at-risk constraints. Guiotto and Roncoroni (2021) develop a general framework of combined custom hedging for a risk-averse firm exposed to claimable and non-claimable risks.

None of the papers on integrated operational and financial risk management reviewed above include product price as a decision variable. An exception appears to be Chen et al. (2007). This paper studies joint dynamic inventory control and pricing strategies for a risk averse decision maker in a multi-period setting. In the extension part, they assume that the model parameters are correlated with some financial asset price and then include financial hedging. Our paper differs from this paper by considering a mean-variance optimization criterion as opposed to an expected utility objective. In contrast to deriving the structure of the dynamic programming problem involved in pricing and inventory control, our focus is to examine how financial risk hedging affects pricing and service levels.

In a broader scope, our work is also related to a relatively new research area on the interface of finance, operations and risk management such as Chod et al. (2010), Secomandi et al. (2015), and Iancu et al. (2017). More references can be found in a recent review by Babich and Kouvelis (2018) and a tutorial by Babich and Birge (2020).

The rest of the paper is organized as follows. In §2, we discuss the price-setting newsvendor model. In §3, we develop the demand-asset model and formulate the risk-management problem. The hedging problem is solved in §4, and optimal production/pricing in the presence of hedging is discussed in §5. In §6, the analytical model is applied to real-world data sets of automakers. Concluding remarks are provided in §7.

2. Base Model: Price-Setting Newsvendor

The price-setting newsvendor is a base model considered in this paper, and it is extensively studied in the literature (see, for example, Petruzzi and Dada (1999) and Agrawal and Seshadri (2000); also refer to DeYong (2020) for a more recent survey). In this section, we describe this model in details. For a selling period $[0,T]$, a newsvendor faces a stochastic demand, $D_T$, which is realized at time $T$. At time 0, the newsvendor needs to decide a unit selling price $P$ and a production quantity $Q$. With unit production cost $c$ and unit salvage value $s$, the newsvendor’s payoff function is

$$H_T(P, Q) = (P - c)Q - (P - s)(Q - D_T(P))^+, \tag{1}$$

where $(x)^+ = \max\{x, 0\}$. The demand function is modeled as

$$D_T(P) = A_T - bP,$$

where $A_T$ is the market potential (i.e., market size) independent from $P$, and $b$ is a positive parameter capturing demand’s sensitivity to price. We assume that $A_T$ has a continuous distribution,
with $f(\cdot)$, $F(\cdot)$, and $r(\cdot) = f(\cdot)/(1 - F(\cdot))$ denoting its density function, distribution function, and hazard rate function, respectively.

The price-setting newsvendor problem finds an optimal $(P, Q)$ to maximize the expected production payoff:

\[
(P^{NV}, Q^{NV}) := \arg \max_{P,Q} \mathbb{E}[H_T(P, Q)].
\]

To facilitate the analysis, we adopt the following change of variable (Petruzzi and Dada 1999; Agrawal and Seshadri 2000):

\[
R := Q + bP.
\]

Note that $P\{D_T \leq Q\} = P\{A_T \leq R\}$. It is clear that $R$, as a quantity combining production and pricing policies, essentially determines the service level, which measures the proportion of demand that is served. Then,

\[
\mathbb{E}[H_T(P, R)] = (P - c)(R - bP) - (P - s)\mathbb{E}[(R - A_T)^+]\] 

For fixed $P$ (resp., $R$), $\mathbb{E}[H_T(P, R)]$ is concave in $R$ (resp., $P$). In addition, $\mathbb{E}[H_T(P, R)]$ is supermodular in $(P, R)$. Clearly, (2) is equivalent to

\[
(P^{NV}, R^{NV}) := \arg \max_{P,R} \mathbb{E}[H_T(P, R)] \quad \text{s.t.} \quad P \geq c, \quad R - bP \geq 0.
\]

We make the following assumptions throughout the paper.

**Assumption 1.** $2r(a)^2 + r'(a) > 0$ for all $a$. In addition, $[1 - F(a)]^2/r(a) \to 0$ as $a \to \infty$ and $F^2(a)/r(a) \to 0$ as $a \to -\infty$. Further, the three conditions also hold for distribution of $A_T$ under the probability measure $P^M$ defined in (14) below.

**Assumption 2.** $P^{NV} > c$, $Q^{NV} = R^{NV} - bP^{NV} > 0$ and $\mathbb{E}[H_T(P^{NV}, R^{NV})] > 0$.

**Assumption 3.** $\mathbb{E}[(bc - A_T)^+] \leq [2b(c - s)] \wedge \sqrt{4b\mathbb{E}[H_T(P^{NV}, R^{NV})]}$.

Assumption 1 is standard in the literature (Petruzzi and Dada 1999). It ensures a unique solution to the problem in (4). Assumption 2 excludes trivial cases. Assumption 3 indicates that the market potential $A_T$ should not fall too far below $bc$, which is reasonable because it should be substantially larger than this value.

The result regarding the optimal solution to (4) is summarized in the following lemma, with proof detailed in the Appendix. Analogous results are known in literature (Petruzzi and Dada 1999).

**Lemma 1.** Under Assumptions 1 and 2, the profit-maximization problem in (4) has a unique solution characterized by the following optimality equations:

\[
2bP^{NV} - bc = \mathbb{E}(R^{NV} \wedge A_T), \quad R^{NV} = F^{-1}\left(\frac{P^{NV} - c}{P^{NV} - s}\right),
\]

where $x \wedge y = \min\{x, y\}$.
The solution to the base model specified in Lemma 1 represents a risk-neutral decision: only expected payoff is maximized, and risk is not considered. In this study, our ultimate goal is to develop an effective risk-management strategy, which calls for first quantifying risk. We follow the common practice of quantifying risk by variance:

$$\text{Var}(H_T(P,R)) = (P - s)^2\text{Var}((R - A)^+).$$  \hspace{1cm} (6)

It is clearly determined that $\text{Var}(H_T(P,R))$ increases in both $P$ and $R$. Higher $R$ represents a higher service level and thus increases the exposure to demand volatility, which in turn increases the payoff risk. A higher pricing level also leads to a higher payoff risk. This is because a higher pricing level increases the positive (resp., negative) impact on payoff from each sold (resp., unsold) product, leading to a higher volatility of the payoff. Of course, it does not make economic sense to separate risk from return, and the focal issue here is to consider trade-off between risk and return, which we discuss below.

Of particular interest are two optimization problems that examine the risk-return trade-off induced by pricing decision $P$ or service level decision $R$. For the former, given $P \geq c$, we define $R^{NV}(P) := \arg \max_R \mathbb{E}[H_T(P,R)]$, $m(P) := \mathbb{E}[H_T(P,R^{NV}(P))]$, $v(P) := \text{Var}(H_T(P,R^{NV}(P)))$.

$$\hspace{1cm} (7)$$

In parallel, for a given $R \geq bc$, we define $P^{NV}(R)$, $m(R)$, and $v(R)$:

$$P^{NV}(R) := \arg \max_P \mathbb{E}[H_T(P,R)], \quad m(R) := \mathbb{E}[H_T(P^{NV}(R),R)], \quad v(R) := \text{Var}(H_T(R^{NV}(R),R)).$$   \hspace{1cm} (8)

It turns out that a larger $P$ (resp., $R$) induces both larger return and greater risk, which is detailed in Proposition 1 below.

To explicitly express risk-return trade-off, which is the focus of our study, we formulate the following mean-variance risk-management problem:

$$(P^{NV}_m, R^{NV}_m) := \arg \min_{P,R} \text{Var}(H_T(P,R)) \quad \text{s.t.} \quad \mathbb{E}[H_T(P,R)] = m,$$ \hspace{1cm} (9)

where $m \in [0, \mathbb{E}[H_T(P^{NV}, R^{NV})]]$ is the target mean payoff. Here we note that, with $P = c$ or $R = bP$, the maximum level that can be achieved by the production payoff is $-(c - s)\mathbb{E}[(bc - A)^+] \leq 0$. Thus, with $m \geq 0$, neither $P \geq c$ nor $R \geq bP$ can be binding, so we omit them from the formulation in (9). The same applies to all relevant settings throughout the paper.

The results of the problems in (7), (8), and (9) are summarized in the following proposition, with proof detailed in the Appendix.

**Proposition 1.** Suppose Assumptions 1–3 hold.
(i) For the problem in (7), \( R^{NV}(P) \) increases with \( P \). Both \( m(P) \) and \( v(P) \) increase in \( P \) for \( P \in [P, P^{NV}] \), where \( P \) is the smallest root of \( m(P) = 0 \). Hence, \((m(P), v(P))\) constitutes an efficient frontier.

(ii) For the problem in (8), \( P^{NV}(R) \) increases with \( R \). Both \( m(R) \) and \( v(R) \) increase in \( R \) for \( R \in [R, P^{NV}] \), where \( R \) is the smallest root of \( m(R) = 0 \). Hence, \((m(R), v(R))\) constitutes an efficient frontier.

(iii) The minimal variance of the problem in (9)—that is \( \text{Var}(H_{T}(P^{NV}_m, R^{NV}_m)) \)—increases in \( m \). Hence, \((m, \text{Var}(H_{T}(P^{NV}_m, R^{NV}_m)))\) constitutes an efficient frontier for \( m \in [0, E[H_{T}(P^{NV}, R^{NV})]] \). In particular, \( \frac{d}{dm}\text{Var}(H_{T}(P^{NV}_m, R^{NV}_m)) = \infty \) at \( m = E[H_{T}(P^{NV}, R^{NV})] \).

For part (i) (resp., (ii)), we expect \( P \) (resp., \( R \)) to be close to \( c \) (resp., \( bc \))—or at least substantially smaller than \( P^{NV} \) (resp., \( R^{NV} \))—and this is reconfirmed in the data analysis in §6.

Proposition 1 offers fundamental insights into how operational levels induce risk-return trade-offs. Part (i) indicates that higher pricing level \( P \) induces higher return while increasing risk. Part (ii) is analogous to part (i) but considers the perspective of service level \( R \). Part (iii) mirrors the previous two parts: higher return induces higher risk (after \( P \) and \( R \) are optimized). Moreover, Part (iii) indicates that the profit-maximizing solution to the base model is just one point on the efficient frontier, and it induces both maximum risk and maximum incremental risk: as return approaches the newsvendor’s maximum profit in the base model, the additional risk effected by a slight increase in return is enormous. In other words, the solution to the base model induces an extremely risky payoff. That the newsvendor’s maximum profit is very risky is also noted by Wang and Yao (2019), but their production model involves only production decisions while treating the pricing level as a given. We propose a risk-hedging model that improves the efficient frontier via substantial risk reductions from the base model.

Parts (i) and (ii) of Proposition 1 are somewhat symmetrical: higher \( P \) (resp., \( R \)) induces higher \( R \) (resp., \( P \)). This is analytically confirmed by the supermodularity of \( E[H_{T}(P, R)] \) in \((P, R)\), because

\[
\frac{\partial E[H_{T}(P, R)]}{\partial P} = -2bP + bc + R - E[(R - A)_{T}^{+}]
\]

which is increasing in \( R \). Economically, this means that a higher service level (i.e., higher \( R \)) increases the marginal value of price. To understand this, note that increasing the service level has two effects on the marginal value of price. On one hand, revenue is increased by capitalizing on the increased price from the increased service level, which corresponds to \( R \) in the derivative above. On the other hand, a higher service level also induces greater loss due to overproduction, which corresponds to \( E[(R - A)_{T}^{+}] \) in the marginal value of price. The positive effect outweighs the negative effect, as there is always a positive probability for a unit to be sold. In summary, a higher service level induces a higher pricing level due to its positive impact on the marginal value of price. This insight will be crucial in proving our main results, Theorems 2 and 3, in §5.
3. Price-Setting Newsvendor with Risk Hedging

3.1. Financial Asset Price and Demand

Let \( \{\Omega, \mathcal{F}_t, \mathbb{P}\} \) be a filtered probability space upon which all processes below are defined. \( \mathcal{F}_t \) is generated by two independent standard Brownian motions, \( B_t \) and \( \tilde{B}_t \). The asset price, \( X_t > 0 \), is a general diffusion process driven by \( B_t \):

\[
dX_t = X_t(\mu_t dt + \sigma_t dB_t),
\]

where \( \mu_t \) and \( \sigma_t > 0 \) are continuous processes adapted to \( \mathcal{G}_t \), the filtration generated by \( B_t \) (note: \( \mathcal{G}_t \subset \mathcal{F}_t \)). In addition, we assume that \( (X_t; \mu_t; \sigma_t) \) is Markovian. Then, the associated market price of risk process is

\[
\eta_t := \frac{\mu_t}{\sigma_t}.
\]

We make the following integrability assumption:

**Assumption 4.** \( \int_0^T \mathbb{E}[(X_t \sigma_t^2)] dt < \infty \).

Let \( D_t \) be the cumulative demand up to time \( t \):

\[
D_t = C_t + \tilde{\sigma} \tilde{B}_t - bP,
\]

where \( C_t \) is a continuous process adapted to \( \mathcal{G}_t \) (recall, \( \mathcal{G}_t \) is the information generated by asset price up to time \( t \)). In (12), demand is modeled by three factors: \( C_t \) incorporates impact from asset price fluctuation, \( \tilde{\sigma} \tilde{B}_t \) captures demand’s intrinsic volatility, and \( -bP \) incorporates demand’s sensitivity to selling price. Note that, in relation to (1), (12) models the market potential, \( A_T \), as

\[
A_T = C_T + \tilde{\sigma} \tilde{B}_T.
\]

The demand model above is general, as we do not assume any specific functional form of \( C_t \) in \( \{X_t, 0 \leq s \leq t\} \). This allows the model to be adapted to various specific application scenarios. A simple and natural model for the market potential is \( C_T = g(X_T) \) for some function \( g(\cdot) \), and this is adopted in literature (e.g., Gaur and Seshadri (2005)). This model can be extended to allow dependence on the path of asset prices: \( C_T = \int_0^T \tilde{\mu}(X_t) dt \). Here \( \tilde{\mu}(\cdot) \) models the demand rate, which is a function of asset price, and we will use this model for the numerical case study in §6.

The risk-adjusted trend of the asset price is captured by \( \eta_t \) in (11), which may benefit or hurt demand. For example, with rising oil prices, the demand for fuel-inefficient cars decreases while the demand for fuel-efficient cars (usually sedans or small SUVs) increases. To clarify this argument, we first introduce the Radon-Nikodym derivative, its density process, and the associated risk-neutral measure for the asset price process:

\[
Z_T := \frac{d\mathbb{P}^M}{d\mathbb{P}} = e^{-\int_0^T m dB_t - \frac{1}{2} \int_0^T \eta_t^2 dt}, \quad Z_t := \mathbb{E}(Z_T | \mathcal{F}_t) = e^{-\int_0^t \eta_s dB_s - \frac{1}{2} \int_0^t \eta_s^2 ds}.
\]
Under the probability measure $P^M$, $X_t$ is *de-trended* (i.e., it becomes a local martingale). The non-financial noise, $\tilde{B}_t$, is not affected. Let $C^M_T$ be the version of $C_T$ under $P^M$—that is, $P(C^M_T \leq x) = P^M(C_T \leq x)$ for all $x$. In other words, $C^M_T$ is the version of the financial component of the demand that incorporates no asset price trend. Now we can formalize how the asset price trend benefits or hurts demand, and thus also the production payoff ($\succeq$ and $\preceq$ below refer to stochastic order):

- If $C_T \succeq C^M_T$, we say that the asset price trend benefits demand. Also, $E[H_T(P, R)] \geq E^M[H_T(P, R)]$ for all $(P, R)$ in this case.
- If $C_T \preceq C^M_T$, we say that the asset price trend hurts demand. Also, $E[H_T(P, R)] \leq E^M[H_T(P, R)]$ for all $(P, R)$ in this case.

As we will see in the subsequent sections, $Z_t$ in (14) is a key object in determining the optimal hedging strategy. Below, we impose a mild technical assumption on $Z_t$ to enable application of the quadratic hedging technique to later solve the hedging problem.

**Assumption 5.** $Z_t$ is a square-integrable martingale over $[0, T]$ under $P$.

Now that asset price is known to impact product demand, analyzing the financial asset price provides information about future demand. If the asset price trend benefits demand, we expect the pricing to be higher when maximizing the expected payoff. This is formalized in the following (more general) result.

**Lemma 2.** $(P^{NV}, R^{NV})$ is the expected production payoff maximizer defined in (4). If the asset-dependent market potential, $C_T$ in (13), increases stochastically, then both $P^{NV}$ and $R^{NV}$ increase.

To understand what is indicated by Lemma 2, we examine the marginal benefits effected by $P$ and $R$ on the expected production payoff:

\[
\frac{\partial E(H_T)}{\partial P} = R - 2bP + bc - E[(R - A_T)^+], \quad \frac{\partial E(H_T)}{\partial R} = (P - c) - (P - s)P(A_T \leq R).
\]  

Both derivatives are increasing in $A_T$, indicating that a larger market size induces greater marginal benefits from increases in $P$ and $R$. As a result, by stochastically increasing $A_T$, optimal $P$ and $R$ both increase. The economic reasoning is as follows. For $P$, a larger market size leads to a higher demand for the same pricing level, buffering the negative impact of the higher price. In other words, with a larger market size, the producer has more room to increase price without excessively hurting demand, thereby increasing profit. As for $R$, to benefit from increasing the market size, the producer can increase production (to capture the larger demand) or set a higher price (to induce greater unit sales revenue from the greater market potential). $R$ integrates the combined effect of production and pricing. So, to benefit from the larger market size, the optimal $R$ is also increased. Both analyses are consistent with the observation made in (15). In summary, when asset price induces a larger market size, the profit-maximizing $P$ and $R$ will also become larger.
3.2. Mean-Variance Risk-Hedging Model

With the impact of asset price built into the demand model, a real-time hedging strategy using asset price is designed to mitigate production payoff risk. Specifically, a risk-hedging strategy is added as another decision variable to the mean-variance optimization problem in (9), in addition to \( P \) and \( R \):

\[
\min_{P, R, \vartheta = \{\theta_t, t \in [0, T]\}} \mathbb{Var}(H_T(P, R) + \chi_T(\vartheta))
\]

s.t. \( \chi_t(\vartheta) := \int_0^t \theta_s dX_s, \quad \mathbb{E}[H_T(P, R) + \chi_T(\vartheta)] = m. \) \hspace{1cm} (16)

In the problem formulation above, \( \theta_t \) is the number of shares to be held in the asset at time \( t \), and \( \chi_t \) is the cumulative profit/loss from the hedging strategy until time \( t \). \( A_X \) is the set of all admissible trading strategies that satisfy certain conditions, including \( \theta_t \in \mathcal{F}_t \) and \( \chi_T(\vartheta) \) has a finite second moment (the former condition ensures that the strategy does not look into future and the latter the variance exists); others technical conditions are described in the Appendix.

To solve the problem in (16), we first fix \( (P, R) \) and solve the following hedging problem:

\[
B(m, P, R) := \min_{\vartheta = \{\theta_t, t \in [0, T]\}} \mathbb{Var}(H_T(P, R) + \chi_T(\vartheta))
\]

s.t. \( \chi_t(\vartheta) := \int_0^t \theta_s dX_s, \quad \mathbb{E}[H_T(P, R) + \chi_T(\vartheta)] = m. \) \hspace{1cm} (17)

After solving the hedging problem in (17), the operational policies \( P \) and \( R \) are further optimized in the presence of hedging to further reduce risk:

\[
(P^h_m, R^h_m) := \arg \min_{P, R} B(m, P, R).
\hspace{1cm} (18)
\]

In subsequent sections, we will (i) derive the optimal hedging strategy to solve the hedging problem in (17) (§4), and (ii) examine how hedging affects optimal operational policy compared to the no-hedging case, and quantify the risk reduction involved in the no-hedging model (§5).

4. Solution to the Hedging Problem

We apply the quadratic hedging technique (Gourieroux et al. 1998) to solve the hedging problem in (17). To prepare, we define the following martingale under \( P^M \):

\[
Z_t^M := \mathbb{E}^M(Z_T | \mathcal{F}_t) = \frac{\mathbb{E}(Z_T^2 | \mathcal{F}_t)}{Z_t},
\hspace{1cm} (19)
\]

and, using the martingale representation theorem, we define the associated dynamics

\[
dZ_t^M = \zeta_t dX_t,
\hspace{1cm} (20)
\]
where $\zeta_t$ is an adapted stochastic process. The solution approach essentially relies on the projected production payoff process:

$$V_t(P, R) := \mathbb{E}^M[H_T(P, R) | \mathcal{F}_t].$$

Clearly, $V_t$ is a martingale under $\mathbb{P}^M$. The martingale representation of $V_t$ plays a key role in determining the optimal hedging strategy, so we summarize it and some related quantities in the following lemma, with proof detailed in the Appendix.

**Lemma 3.** $V_t(P, R)$ defined in (21) has the following representation:

$$V_t(P, R) = V_0(P, R) + \int_0^t \xi_s(P, R) dX_s + \int_0^t \delta_s(P, R) d\tilde{B}_s,$$  

with $V_0(P, R) = \mathbb{E}^M[H_T(P, R)]$. $\xi_t$ and $\delta_t$ are processes adapted to $\mathcal{F}_t$, and

$$\xi_t(P, R) = -(P - s) f_x(t, X_t, A_t, \mu_t, \sigma_t), \quad \delta_t(P, R) = -\tilde{\sigma}(P - s) f_x(t, X_t, A_t, \mu_t, \sigma_t),$$

where $f(t, x, a, \mu, \sigma) := \mathbb{E}^M[(R - A_T^+) | X_t = x, A_t = a, \mu_t = \mu, \sigma_t = \sigma]$. In particular,

$$\delta_t(P, R) = \tilde{\sigma}(P - s) \mathbb{P}^M(A_T \leq R | \mathcal{F}_t),$$

which increases in both $P$ and $R$, where $A_T$ is defined in (13).

### 4.1. Optimal Hedging Strategy and Minimal Variance

Here we present the solution to the problem in (17), with proof detailed in the Appendix.

**Theorem 1.** Suppose Assumptions 4–5 hold.

(i) The optimal solution to the problem in (17) is

$$\theta^*_t(P, R) = -\underbrace{\xi_t(P, R)}_{\text{risk-mitigation position}} - \frac{\zeta_t}{Z_t^M} \underbrace{[\lambda_m - V_t(P, R) - \chi_t^*]}_{\text{investment position}},$$

where $\xi_t$ is defined in (22), $Z_t^M$ in (19), and $\zeta_t$ in (20). $\chi_t^* = \int_0^t \theta^*_s(P, R) dX_s$. $\lambda_m$ is defined as

$$\lambda_m = \frac{m Z_0^M - V_0(P, R)}{Z_0^M - 1}.$$  

(Note: $Z_0^M = \mathbb{E}(Z_T^2) > 1$.)

(ii) The optimal objective function value in (17) has the following expression:

$$B(m, P, R) = \left[\frac{m - V_0(P, R)}{Z_0^M - 1}\right]^2 + \int_0^T \mathbb{E}\left[\frac{Z_t}{Z_t^M} \delta_t^2(P, R)\right] dt,$$

where $\delta_t(P, R)$ is expressed in (24). In particular, $Z_0^M = \mathbb{E}(Z_T^2) > 1$, and $0 \leq Z_t/Z_t^M \leq 1$. 

Part (i) of Theorem 1 is intuitively appealing. The structure in (25) reveals two aspects of the optimal hedging strategy. The term $\xi_t(P, R)$, which appears in (22), captures the impact of asset price movement (i.e., $dX_t$) on production payoff. By taking the opposite position, $-\xi_t(P, R)$, the hedging strategy offsets such impact to mitigate risk. The second term in (25) involves the difference between $\lambda_m$ (a proxy of the target mean $m$) and $V_t(P, R) + \chi^*_t$ (the projected total wealth at time $t$). This term functions as an investment to close this gap between the target and the total wealth by taking position in the asset. These two aspects are also noted by Wang and Yao (2017), but here, the asset price model is general, in contrast to the geometric Brownian motion (GBM) in that earlier paper.

With regard to part (ii) of Theorem 1, the variance function, $B(m, P, R)$ in (27), is closely related to the structure of the optimal hedging strategy in (25)—and quite revealing. It has two terms: the first term is proportional to the square of $m - V_0$, and the second term does not involve $m$. To understand the first term, note that after combining the contribution from the first position (i.e., $-\xi_t$) of the hedging strategy, the expected net production payoff is

$$E[H_T + \int_0^T (-\xi_t) dX_t] = E[V_0 + \int_0^T \xi_t dX_t + \int_0^T \delta_t d\tilde{B}_t + \int_0^T (-\xi_t) dX_t] = V_0.$$  

This indicates that $V_0$ is the expected production payoff net profit/loss resulting from the risk-mitigation position of the hedging strategy, which we refer to as the “hedged production payoff.” Then, the gap left between the target $m$ and this hedged production payoff, $m - V_0$, needs to be closed by the investment position of the hedging strategy in (25). The larger this gap is, the larger the investment position size taken in the asset; thus, its risk contribution is greater, leading to a higher value of the first term in the variance function $B(m, P, R)$. The second term of $B(m, P, R)$, involving only the operational policy $(P, R)$, is increasing in demand noise, $\tilde{\sigma}$ (via $\delta_t$; see (24)), which cannot be hedged by trading financial assets; thus, this term reflects unhedgeable risk. This structure of the variance function is similar to that noted by Wang (2021), but it incorporates a more general asset price model and involves the pricing decision in addition to the production policy.

Moreover, in the presence of hedging, the optimal operational policy (i.e., $(P, R)$) that minimizes $B(m, P, R)$ essentially assumes a market size of $A^M_T$—the version of $A_T$ under the risk-neutral probability measure $P^M$. This is both intuitively and technically appealing. Hedging, specifically the $-\xi_t$ term of the hedging strategy, offsets any impact of asset price movement (and thus also its trend) on market size, thereby effectively replacing the actual market size (i.e., $A_T$) faced by the decision maker in the real world with a market size under $P^M$ (i.e., $A^M_T$) in the risk-neutral world where asset price has no trend. Technically, all quantities in the expression of $B(m, P, R)$
involve the distributions under $P^M$ only (note: $E\left[ \frac{Z^2_t}{\delta_t^2} \right] = E^M\left[ \frac{\delta_t^2}{Z^2_t} \right]$, where both $Z^M_t$ and $\delta_t$ are defined under $P^M$ (see (19) and (24)). This is consistent with the discussion above. The insight that hedging replaces the actual market size with a market size in the risk-neutral world, will be crucial in analyzing the behaviour of optimal pricing and service levels in §5.

4.2. Efficient Frontiers

For a given pricing and production policy, we construct the efficient frontier and quantify the variance reduction from the base model.

**Proposition 2.** Given $(P,R)$, $B(m,P,R)$ decreases in $m$ for $m \leq V_0(P,R)$ and increases in $m$ for $m \geq V_0(P,R)$. Therefore, it is not efficient to set $m \leq V_0(P,R)$, while $(m,B(m,P,R))$ constitutes an efficient frontier for $m \geq V_0(P,R)$.

**Proof.** From (27), it is clear that, given $(P,R)$, $B(m,P,R)$ is a convex quadratic function in $m$, with the global minimizer being $V_0(P,R)$; thus, this variance function decreases (resp., increases) for $m \leq V_0$ (resp., $m \geq V_0$). According to the definition of mean-variance efficiency, the stated result immediately follows. □

Proposition 2 indicates that, given a production and pricing policy, which determines the operating payoff $H_T$, it is not optimal, in terms of risk-return trade-off, to demand a mean return that is less than $V_0$. This is consistent with Theorem 1. As the discussion following this theorem indicates, $V_0$ is the hedged production payoff. Thus, demanding less than $V_0$ means requiring the investment position of hedging to generate a negative mean payoff while enduring additional risk effected by this position. Clearly, it can never be optimal to do so.

4.3. Example: Asset Price Following the Exponential Ornstein–Uhlenbeck Process

Theorem 1 assumes a general diffusion process for the asset price $X_t$ (see (10)). In any specific application context, $X_t$ will be further specified. For example, when the asset is a stock, $X_t$ can be modeled using GBM, which is considered by Wang and Yao (2017). To be commensurate with the automakers’ example discussed in §1 and to be studied in §6, in this section, we let $X_t$ follow the exponential Ornstein–Uhlenbeck (EOU) process, which is commonly used to model the prices of commodities such as oil (Schwartz 1997).

The EOU process $X_t$ is specified by

$$X_t = e^{Y_t}, \quad \text{with} \quad dY_t = \kappa(\alpha - Y_t)dt + \sigma dB_t,$$

where $\kappa$, $\alpha$, and $\sigma$ are all positive constants. In particular, $\alpha$ represents the long-term mean of $Y_t$, and $\kappa$ is the mean-reversion coefficient. Applying Itô’s Lemma,

$$dX_t = \kappa\left(\alpha + \frac{\sigma^2}{2\kappa} - \log X_t\right)X_t dt + \sigma X_t dB_t,$$
it is clear that Assumption 4 holds. In order to ensure Assumption 5 holds, we need the following parametric constraint:

$$\kappa T < \frac{\pi}{4}$$  \hspace{1cm} (30)

and the details are collected in the Appendix. For this asset price model, the market price of risk $\eta_t$ in (11) is

$$\eta_t = \frac{\kappa}{\sigma} \left( \alpha + \frac{\sigma^2}{2\kappa} - \log X_t \right).$$  \hspace{1cm} (31)

When the mean-reversion coefficient $\kappa$ is eliminated, $Y_t$ in (28) reduces to $\sigma B_t$, then $X_t$ in (28) reduces to GBM, and the market price of risk $\eta_t$ in (31) becomes a constant, $\sigma/2$. For this case, all the results in this section still hold and are adapted to the GBM model; in particular, (30) holds.

Applying Theorem 1, we have the following proposition.

**Proposition 3.** Suppose $X_t$ follows (28) and Assumptions 4–5 hold.

(i) The optimal solution to the problem in (17) is

$$\theta_t^*(P,R) = -\xi_t(P,R) - a(T-t) + \frac{\kappa}{\sigma^2} [\lambda_t - V_t(P,R) - \chi^*_t],$$

where $\chi_t^* = \int_0^t \theta_s(P,R) dX_s$, $\lambda_t$ is defined in (26), and $\xi_t(P,R)$ is defined in (22). The functions $a(\tau)$ and $b(\tau)$ for all $\tau \in [0,T]$ are defined as follows:

$$a(\tau) = \frac{1}{2} + \frac{1}{\cos \kappa \tau - \sin \kappa \tau}, \quad b(\tau) = \frac{\cos \kappa \tau + \sin \kappa \tau}{\cos \kappa \tau - \sin \kappa \tau}. \hspace{1cm} (32)$$

(ii) The optimal objective function value of the problem in (17) is

$$B_m(P,R) = \frac{[m - V_0(P,R)]^2}{Z_m^M - 1} + \int_0^T E[e^{-f_0(T-t)-f_1(T-t)Y_t-f_2(T-t)Y_t^2} \delta_t(P,R)] dt,$$

where $\delta_t(P,R)$ is defined in (22), and the functions $f_0$, $f_1$, and $f_2$ for $\tau \in [0,T]$ are

$$f_0(\tau) = -\alpha - \frac{1}{2} \kappa + \frac{1}{4} \sigma^2 \tau - \frac{1}{2} \log[\cos \kappa \tau - \sin \kappa \tau] + \frac{\alpha + \left( \frac{\sigma^2}{\sigma^2} + \frac{\sigma^2}{2 \kappa} \right) \sin \kappa \tau}{\cos \kappa \tau - \sin \kappa \tau},$$

$$f_1(\tau) = \frac{-1 + \cos \kappa \tau - (\frac{2 \alpha}{\sigma^2} + 1) \sin \kappa \tau}{\cos \kappa \tau - \sin \kappa \tau},$$

$$f_2(\tau) = \frac{\kappa}{\sigma^2} \frac{\cos \kappa \tau - \sin \kappa \tau}{\cos \kappa \tau - \sin \kappa \tau}.$$  \hspace{1cm} (34)

Note that $a(\tau)$ and $b(\tau)$ in (32) are both well defined under the condition (30). With further specification of the market size $A_T$ in (13), both $\xi_t(P,R)$ and $\delta_t(P,R)$ can be derived explicitly (see (40) and (41) in §6.2).
5. Optimal Pricing and Service Levels in the Presence of Hedging

Now we consider the operational task: finding the jointly optimal pricing and service levels in the presence of hedging—that is, solving (18). We begin by presenting some basic properties of the optimal solution (with proof in the Appendix).

**Proposition 4.** (i) Define the profit-maximizing \((P, R)\) under the risk-neutral measure as

\[
(P^{NV(M)}, R^{NV(M)}) := \arg \max_{P, R} E^M[H_T(P, R)].
\]

Then, for any \(m \geq 0\), \(P^h_m \leq P^{NV(M)}_m\) and \(R^h_m \leq R^{NV(M)}_m\). In particular, \(V_0(P^h_m, R^h_m) \leq m\).

(ii) If \(C^M_T \preceq C_T\) (i.e., the asset price trend benefits demand), \(P^{NV} \geq P^{NV(M)}\) and \(R^{NV} \geq R^{NV(M)}\); conversely, if \(C^M_T \succeq C_T\) (i.e., the asset price trend hurts demand), \(P^{NV} \leq P^{NV(M)}\) and \(R^{NV} \leq R^{NV(M)}\).

Part (i) of Proposition 4 is both intuitively and technically appealing. First, \(V_0(P^h_m, R^h_m) \leq m\) is consistent with what is indicated by Proposition 2: targeting a return less than the hedged production payoff never induces optimal risk-return trade-off. As a result, increasing the pricing and service levels beyond those maximizing \(V_0\) can never be optimal, which leads to the stated bounds of \(P^h_m\) and \(R^h_m\). These bounds are also crucial in applying numerical global optimization methods to find \((P^h_m, R^h_m)\) by bounding the feasible region to a compact set. The intuition of part (ii) of Proposition 4 is as follows: when the asset price trend benefits (resp., hurts) demand, the producer faces a smaller (resp., larger) market size and will adjust the pricing level down (resp., upwards) and decrease (resp., increase) the service level.

It turns out that more aspects of the variance function, \(B(m, P, R)\) in (27), can be explored to enhance efficiency of the numerical procedure. This is summarized in the following lemma, with proof in the Appendix.

**Lemma 4.** For a given \(R\), \(\bar{P}(R)\) is the smaller root of \(V_0(P, R) = m\) if \(m \leq \max_P V_0(P, R)\); otherwise, let \(\bar{P}(R) = \arg \max_P V_0(P, R)\). Then,

\[
\arg \min_P B(m, P, R) \leq \bar{P}(R),
\]

and \(B(m, P, R)\) is convex in \(P\) over \([c, \bar{P}(R)]\).

Based on Lemma 4, for each given \(R\), the corresponding optimal \(P\) can be found efficiently. Then, finding the optimal \(R\) amounts to a line search over \([0, R^{NV(M)}]\) (see Proposition 4).

In the subsequent sections, we present the main results of this paper regarding the properties of optimal \(P\) and \(R\) in the presence of hedging, depending on whether the asset price trend benefits (§5.1) or hurts (§5.2) demand. Then, we fully characterize the efficient frontier for the hedging model and quantify the risk reduction from the base model (§5.3).
5.1. Optimal $P$ and $R$: Asset Price Trend Benefiting Demand

We consider the case of the asset price trend benefiting demand (i.e., $C^M_T \preceq C_T$). A concrete example is from automakers: when the price of oil exhibits a downward trend, demand for fuel-inefficient vehicles (e.g., SUVs or pickup trucks) will increase. The following theorem follows immediately from Proposition 4.

**Theorem 2.** Suppose Assumptions 1–5 hold and $C^M_T \preceq C_T$ (i.e., the asset price trend benefits demand). Then, for $m = E[H_T(P^{NV}, R^{NV})]$,

$$P^h_m \leq P^{NV}, \quad R^h_m \leq R^{NV}.$$  

Theorem 2 sheds light on the properties of the optimal operational policy in the presence of risk hedging in the case of the asset price trend benefiting demand. As the discussion regarding Theorem 1 reveals, with risk hedging, the operational policy essentially assumes a market size in a risk-neutral world—that is, $A^M_T$. When the asset price trend benefits demand, $A^M_T \preceq A_T$ (which follows from $C^M_T \preceq C_T$ and the independence of $\tilde{B}_T$ from the asset price process). In other words, in the presence of hedging, the operational policy faces a decreased market size. Considering that an optimal operational policy should strike a balance between contributing to the hedged production payoff (i.e., $V_0$) and controlling for unhedgeable risk (i.e., the second term of the variance function $B(m, P, R)$), the pricing and service levels should not exceed those of the base model, because with a smaller market size (relative to that without hedging), the pricing and service levels should both decrease (see Lemma 2 and the discussion there). Pricing and service levels beyond those of the base model (which correspond to a larger market size) only decrease $V_0$ (due to overstocking and overpricing) and thus also increase both investment risk (i.e., the first term of $B(m, P, R)$ in (27)) and unhedgeable risk (i.e., the second term of $B(m, P, R)$) by increasing exposure to the unhedgeable volatility of the demand.

5.2. Optimal $P$ and $R$: Asset Price Trend Hurting Demand

Now we consider the case of the asset price trend hurting demand (i.e., $C^M_T \succeq C_T$). For example, an increasing oil price hurts the demand for fuel-inefficient cars.

To prepare for the main results presented below, we define

$$P^{NV(M)}(R) := \arg \max_P V_0(P, R), \quad R^{NV(M)}(P) := \arg \max_R V_0(P, R).$$  \hspace{1cm} (35)

Based on (35), we define a critical point, $(P^*, R^*)$:

$$R^* \text{ is the smallest root of } V_0(P^{NV(M)}(R^*), R) = E[H_T(P^{NV}, R^{NV})], \quad P^* := P^{NV(M)}(R^*).$$  \hspace{1cm} (36)

The properties of $P^*$ and $R^*$ are presented in the following lemma, with proof in the Appendix.
Lemma 5. Suppose Assumptions 1–3 hold and $C_M^T \succeq C_T$. Then, $(P^*, R^*)$ defined in (36) satisfies

\[ P^* \leq P^{NV} \quad \text{and} \quad R^* \leq R^{NV}. \]

Lemma 5 offers interesting economic insights. When the asset price trend hurts demand, in the presence of hedging, the production and pricing decisions essentially assume an enlarged market size, $A_M^T \succeq A_T$. Then, to attain the newsvendor’s maximum profit in the base model (i.e., $E[H_T(P^{NV}, R^{NV})]$) that corresponds to a smaller market size, the service level needs to be decreased after the pricing decision has been optimized. Otherwise, $V_0$, the profit that assumes a larger market size, will exceed $E[H_T(P^{NV}, R^{NV})]$. With Lemma 5, we can prove the following result with details provided in the Appendix.

Theorem 3. Suppose Assumptions 1–5 hold and $C_M^T \succeq C_T$ (i.e., the asset price trend hurts demand). Let $m = E[H_T(P^{NV}, R^{NV})]$. Then,

(i) $P^h_m \leq P^* \leq P^{NV}$.

(ii) Let $\tilde{P} = P - s$ for any $P$. Define

\[ P^\circ := P^{NV(M)}(R^{NV}), \quad r^\circ := \frac{\tilde{P}^\circ}{P^{NV}}. \]

(Note: $r^\circ \geq 1$. If

\[ \frac{\tilde{P}^\circ}{r^\circ + \sqrt{(r^\circ)^2 - 1}}, \quad P^M(A_T \geq R^{NV}) \leq c - s, \]

then $R^h_m \leq R^{NV}$.

(iii) If (38) does not hold, then $R^h_m \leq R^\circ$, where $R^\circ$ (which exists uniquely) is determined by

\[ \frac{E^M[A_T 1\{A_T \leq R^\circ\}]}{P^M(A_T \geq R^\circ)} = \frac{\tilde{P}^{NV}(2b\tilde{P}^\circ + 2bs - bc)}{(c - s)(r^\circ + \sqrt{(r^\circ)^2 - 1})} - R^{NV}. \]

In particular, $R^{NV} \leq R^\circ \leq R^{NV(M)}$.

Part (i) of Theorem 3 generates the same indication as that of Theorem 2—in the presence of hedging, the optimal pricing level is lower—but the economic insight behind it is fundamentally different. Under the conditions of Theorem 2, the asset price trend benefits demand, so the hedging effectively causes the operational policy to face a smaller market size compared to that without hedging (i.e., $A_M^T \preceq A_T$), which induces a lower pricing level. The scenario associated with Theorem 3 is the opposite: in the presence of hedging, the operational policy faces a larger market size. To understand the finding that the pricing level is still adjusted down, it is critical to note that operational policy needs to leave leeway in the expected payoff for hedging to fill (i.e., $V_0 \leq m$), otherwise it cannot induce an optimal risk-return trade-off (see part (i) of Proposition 4). According to the definitions in (36), $P^*$ is the optimal (in terms of maximizing $V_0$, which corresponds to a
larger market size, $A_T^M \geq A_T$) pricing level associated with a service level that makes $V_0$ attain the target $m$. So, to make $V_0 \leq m$ (and also to decrease unhedgeable risk), the optimal pricing level does not exceed $P^\ast$, which in turn is smaller than $P^{NV}$ (see Lemma 5).

Parts (ii) and (iii) of Theorem 3 characterize the optimal service level, $R_{m}^{h}$, in the presence of hedging, according to the magnitude of the detrimental effect of the asset price trend on demand. Consider the condition in (38) of part (ii) first. Suppose the detrimental effect of the asset price trend on demand strengthens. For example, fixing the initial oil price ($X_0$) below the long-term average oil price, any increment in $\alpha$ of (28) will hurt the demand for fuel-inefficient cars more due to the upward oil price trend. In such a case, $A_T^M$ is not affected (since it does not involve the oil price trend) while $A_T$ is reduced, which further induces a decreased $R^{NV}$. As a result, both $P^{NV}$ and $P^{o}$ decrease, since

$$P^{NV} = \frac{1}{2b} \left[ E(R^{NV} \wedge A_T) + bc \right], \quad P^{o} = \frac{1}{2b} \left[ E(R^{NV} \wedge A_T^M) + bc \right].$$

Since $A_T$ is reduced while $A_T^M$ is not affected, we expect $P^{NV}$ to decrease more than $P^{o}$ does (so long as both $P^{NV}$ and $P^{o}$ are bounded sufficiently away from $c$), so $r^o$, defined in (37), will increase as the detrimental effect strengthens. Thus, the term in the bracket on the left hand side of (38), which is the main factor controlling the magnitude of the left side of that inequality, decreases. Further, the probability term, $P^{M}(A_T \geq R^{NV})$, suggests that $R^{NV}$ should not be too low—that is, $A_T$ should not be reduced too much. This is intuitive. Consider the extreme case in which $A_T$ approaches an unrealistically low level such that both $P^{NV}$ and $P^{o}$ are suppressed close to $c$, which will lead to a near-zero profit. In such a scenario, both $r^o$ and the probability term are close to 1 while $P^{NV}$ is close to $c$, and whether the condition in (38) holds becomes ambiguous. Of course, this kind of extreme scenario is not likely to occur in reality. In summary, the economic meaning of the condition in (38) is that the detrimental effect of the asset price trend on demand is strong (but not unrealistically so), and part (ii) indicates that in such a scenario, the service level with hedging is lower than the service level without hedging.

In contrast to part (ii), part (iii) implies that, although the detrimental effect is not sufficiently strong (thus, (38) does not hold), $R_{m}^{h}$ can be further bounded by a level that is smaller than $R^{NV(M)}$. In this case, we expect $P^{M}$ to be close to $P$; thus, so are $R^{NV}$ and $R^{NV(M)}$. Therefore, even if $R_{m}^{h}$ exceeds $R^{NV}$, we expect it to still be close to $R^{NV}$. In summary, parts (ii) and (iii) of Theorem 3 indicate that, in the presence of hedging, the service level is either smaller than $R^{NV}$ (when the detrimental effect is moderately high), or it is larger than but close to $R^{NV}$. This aligns with what Theorem 2 indicates: in the presence of hedging, the service level is adjusted down, or it at most exceeds by a small amount the service level without hedging.
5.3. Efficient Frontiers

With the operational and hedging policies jointly optimized, we characterize the efficient frontier for this optimal model. Clearly, this efficient frontier lies lower than that specified in Proposition 1: for any target mean \( m \geq 0 \), \( B(m, P^h_m, R^h_m) \) is smaller than \( \text{Var}(H_T(P^{NV}_m, R^{NV}_m)) \). The gap between these two frontiers represents the risk reduction, which we quantify by its lower bound. The results are summarized in the following proposition (with proof in the Appendix).

**Proposition 5.** \( B(m, P^h_m, R^h_m) \) increases in \( m \); thus, \( (m, B(m, P^h_m, R^h_m)) \) constitutes an efficient frontier. For any \( m \geq 0 \), the risk reduction achieved by hedging is lower bounded as follows:

\[
\text{Var}(H_T(P^{NV}_m, R^{NV}_m)) - B(m, P^h_m, R^h_m) \geq \int_0^T \mathbb{E}\left[ \frac{\sigma^2 X^2 Z^2_t}{Z^M_t} y_t(P^{NV}_m, R^{NV}_m) \right] dt,
\]

where \( y_t(P^{NV}_m, R^{NV}_m) = \xi_t(P^{NV}_m, R^{NV}_m) + \frac{\xi_t}{Z^M_t}(\lambda_m - V_t(P^{NV}_m, R^{NV}_m)), \) with \( \lambda_m = \frac{m Z^M_0 - V_0(P^{NV}_m, R^{NV}_m)}{Z^M_0 - 1} \).

Being the sum of two terms, the structure of \( y_t \) above reveals the sources of risk reduction: offsetting the impact of asset price on demand (the first term), and pulling the payoff process toward the target by investing (the second term).

6. Numerical Case Study with Data of Ford Motor Company

In this section, using real-world financial and automotive sales/price data sets, we implement the hedging model developed in this paper. We first apply calibration methods to estimate the asset price and demand models. Then, with the estimated models, we conduct a comprehensive numerical study to illustrate various aspects of the analytical results derived in the previous sections.

6.1. Data Description

Two sets of data are used: one is financial data, while the other U.S. automakers’ operational data. The financial asset in our context is WTI crude oil, a major global oil benchmark. The data source is the Federal Reserve Bank of St. Louis database. The data set includes daily spot prices between 2010 and 2019.

The U.S. automakers’ operational data, including monthly sales volumes and manufacturer-suggested retail prices (MSRPs), was purchased from a commercial vendor specializing in automotive business data. The data set includes brands, models, versions, MSRPs, and combined MPG. We focus on two car models—the Explorer and the Focus—which are two popular models manufactured by Ford Motor Company. Recall, the Explorer is categorized as fuel-inefficient (low MPG) and the Focus is fuel-efficient (high MPG). The Explorer data is dated from January 2011 to December 2019, and the Focus data ranges from January 2010 to May 2018. The sales data is deseasonalized. In addition, monthly Consumer Price Index (CPI) data ranging from January 2010 to December 2019 for all urban consumers was collected from the Federal Reserve Bank of St. Louis database,
which is used to adjust the MSRP for inflation. The sales volume for each model is aggregated from sales across various versions, and the price is the weighted (by sales volume) average price of different versions. Summary statistics for the two models are presented in Table 2.

| Car  | Sample Size | Price ($) | Sales Volume | MPG  |
|------|-------------|-----------|--------------|------|
| Explorer | 108    | 40,615    | 15,933       | 16.89 |
|       |         | (1,239)   | (3,516)      | (1.96) |
| Focus | 101       | 22,300    | 16,190       | 24.92 |
|       |         | (444)     | (3,378)      | (3.34) |

*Note. The numbers in parentheses are standard deviations.*

### 6.2. Parameter Estimation

In the automaker’s case under consideration, a selling period is one month, i.e., \( T = 1/12 \). The relevant asset is WTI, with price dynamics following the EOU process, introduced in §4.3. Regarding the market size in (13), we follow the specification of Wang and Yao (2017):

\[
A_T = \int_0^T \tilde{\mu}(X_t) dt + \tilde{\sigma} \tilde{B}_T. \tag{39}
\]

Here, \( \tilde{\mu}(x) \) is the demand rate function, which is to be determined from the data. With (28) and (39), we can further specify \( \xi_t \) and \( \delta_t \) as follows:

\[
\delta_t(P, R) = \tilde{\sigma}(P - s) \mathbb{E} \left[ \mathcal{N} \left( \frac{R - a - \int_t^T \tilde{\mu}(x X_{u-t}^M) du}{\tilde{\sigma} \sqrt{T - t}} \right) \bigg| X_t = x, A_t = a \right], \tag{40}
\]

and

\[
\xi_t(P, R) = (P - s) \mathbb{E} \left[ \mathcal{N} \left( \frac{R - a - \int_t^T \tilde{\mu}(x X_{u-t}^M) du}{\tilde{\sigma} \sqrt{T - t}} \right) \int_t^T \tilde{\mu}'(x X_{u-t}^M) X_{u-t}^M du \bigg| X_t = x, A_t = a \right]. \tag{41}
\]

Here, \( X_{u-t}^M := e^{\sigma(B_u - B_t) - \frac{1}{2} \sigma^2(u-t)} \), and \( \mathcal{N} \) is an independent standard normal random variable.

#### 6.2.1. WTI Price

We calibrate the EOU process to the WTI price data by applying ordinary least squares (OLS) to estimate the parameters involved. Here, \( X_t \) stands for the WTI price. Recall, \( Y_t = \log X_t \) is modeled in (28). First, discretize \( Y_t \):

\[
Y_{t+\nu} = (1 - \kappa \nu) Y_t + \kappa \alpha \nu + \sigma \sqrt{\nu} \cdot \epsilon_t, \tag{42}
\]

where \( \nu = 1/252 \) is the time step size (i.e., one trading day), and \( \epsilon_t \) is a standard normal random variable independent from \( Y_t \). Then, (42) leads to the following linear regression model:

\[
Y_{t+1} = a Y_t + b + \epsilon,
\]
where $a = 1 - \kappa \nu$, $b = \kappa \alpha \nu$, and $E[\epsilon^2] = \sigma^2 \nu$. Applying OLS produces estimators of $a$, $b$, and standard error of residuals, which are denoted by $\hat{a}$, $\hat{b}$, and $\hat{\sigma}^2$, respectively. Then, estimators (denoted by corresponding symbols with hats) for $\kappa$, $\alpha$, and $\sigma^2$ are

$$\hat{\kappa} = \frac{1 - \hat{a}}{\nu}, \quad \hat{\alpha} = \frac{\hat{b}}{1 - \hat{a}} \quad \text{and} \quad \hat{\sigma}^2 = \frac{1}{N \nu} \sum_{n=1}^{N} (Y_{n+1} - (\hat{a} Y_n + \hat{b}))^2,$$

where $N = 2600$ is the number of observations. The estimated parameters are as follows:

$$\hat{\kappa} = 0.5356, \quad \hat{\alpha} = 4.1847, \quad \text{and} \quad \hat{\sigma} = 0.3327.$$

The estimated parameters suggests that the long-run average of WTI is

$$E[X_\infty] = e^{\alpha + \frac{\sigma^2}{\kappa}} = 72.82.$$

### 6.2.2. Demand Model Calibration

Recall, market size is assumed to take the form shown in (39). Further, we assume a linear form for $\tilde{\mu}(x)$:

$$\tilde{\mu}(x) = \mu_0 + \mu_1 x,$$

where $\mu_0$ and $\mu_1$ are the two parameters to be estimated from data. To evaluate the integral involved in (39) in each period (month), we apply the discretized numerical integration:

$$C_T = \int_0^T \tilde{\mu}(X_t) dt \approx T (\mu_0 + \mu_1 \bar{X}) \equiv A + B \bar{X},$$

where $\bar{X}$ is the average asset price within one month: $\bar{X} = (\sum_{j=0}^{J} X_j) / (J + 1)$, and $J = 20$—that is, there are 21 trading days in a month. Thus, $\mu_0 = A / T$, and $\mu_1 = B / T$, with $T = 1 / 12$.

Plugging (43) in (39), we have the following linear regression model:

$$D = A + B \bar{X} - b P + \tilde{\sigma} \sqrt{T} \epsilon_1,$$

where $\epsilon_1$ is an independent standard normal error.

The pricing level $P$ in (44) and the production quantity $Q$ are determined by the manufacturer, which we assume to accord with the newsvendor’s profit-maximization problem:

$$(P^*, Q^*) = \arg \max_{P,Q} E[(P - c)Q - P(Q - D)^+],$$

where $c$ is the average direct production cost, including raw material and labor/assembly costs. Here, for both vehicle models, we set the salvage value per car, $s$, to be zero. This does not mean that the excess inventory for the current month will be worthless; this only reflects that the profit/loss accounting is restricted to the current month. That is, the inventory may be carried over into the next period, but the payoff generated is not counted toward the current month.
The set of model parameters to be calibrated is \( \{A, B, b, c, \tilde{\sigma}\} \). Given the oil price \( X_0 \) on the first trading day of month \( i \), the model-implied price and production quantity for the same month are

\[
(P_i^*, Q_i^*) = \arg \max_{P, Q} \mathbb{E}[(P - c)Q - P(Q - D_i)^+],
\]

where \( D_i \) is simulated given \( \{A, B, b, c, \tilde{\sigma}\} \) and \( X_0 \). In principle, we need to minimize the distance between the model-implied \( P \) and \( Q \) and those observed in data. While \( P \) is observable, \( Q \) is not available in our data set. Thus, instead of fitting \( Q \), we turn to minimizing the error of the demand model in (44). Here, we use sales volume to proxy the demand, which is appropriate because the inventory-to-sales ratio in the automotive industry in the U.S. market has been around 2.5 for decades (Dunn and Vine 2016). Therefore, we minimize the distance between the model-implied expected demand \( A + B\bar{X}_i - bP_i^* \) and observed sales volume \( S_i \). In summary, our calibration model is

\[
\min_{A, B, b, c, \tilde{\sigma}} \sum_{i=1}^{n} [(P_i^* - P_i)^2 + (A + B\bar{X}_i - bP_i^* - S_i)^2].
\]

The calibrated parameters are summarized in Table 3. We have validated that Assumptions 1–3 are satisfied based on the calibrated demand parameters, estimated oil price model parameters, and initial oil prices we set in \( \S \)6.3. Also, we numerically check that \( P \) (resp., \( R \)) is close to \( c \) (resp., \( bc \)) for both car models.

| Car     | \( A (= \mu_0 T) \) | \( B (= \mu_1 T) \) | \( b \) | \( c \) (cost) | \( \tilde{\sigma} \) |
|---------|---------------------|---------------------|--------|---------------|------------------|
| Explorer| 111,155.66          | -185.42             | 2.02   | 34,543.91     | 11,577.37        |
| Focus   | 151,887.67          | 157.41              | 6.59   | 20,467.10     | 8,619.46         |

Note. \( D_T = \int_0^T (\mu_0 + \mu_1 X_t) dt + \tilde{\sigma} \bar{B}_T - bP \), and \( T = 1/12 \).

Note that the sign of the parameter \( B \) determines how asset price impacts demand. Specifically, the plus (resp., minus) sign of \( B \) represents the positive (resp., negative) impact of oil price on demand. That is, the demand of the Explorer (resp., the Focus), the fuel-inefficient (resp., fuel-efficient) model, is negatively (resp., positively) impacted by oil price. This is consistent with the economic intuition and empirical evidence discussed in \( \S \)1. Another point worth noting is that the estimated production costs \( c \) for both cars are around 85% of the average selling price, which matches the profit margin observed in the automobile industry. Moreover, the estimated profit margin is 17.6% for the Explorer and 9.0% for the Focus, consistent with the fact that larger cars are more profitable than smaller ones (e.g., Ford Motor Company, 2020 10-K Filing).
6.3. Numerical Implementation of the Hedging Model

In this part, we conduct numerical experiments with the parameters estimated in §6.2 to illustrate various analytical results (especially Theorems 2 and 3) developed in §4 and §5, focusing on pricing and production decisions, service level, hedging performance, and efficient frontiers.

Numerical Procedures. To simulate sample paths, we set the discretized time step size at $\nu = 1/252$ and assume that each month has 21 trading days. To evaluate the variance function $B(m, P, R)$ in (27), the first term is easily obtained via Monte Carlo simulation with simulated paths of $X_t$ and $D_t$ (the simulation procedure is described below). Evaluating the second term involves the following steps:

(i) Use Monte Carlo simulation to generate $N_1$ sample paths of $(Y_t, X_t, A_t)$ according to (28), (29), (13), and (43).

(ii) Given each path $(Y_t = y, X_t = x, A_t = a)$, generate $N_2$ paths of $X_u^M$ for $u \in [t, T]$ with initial value $X_0^M = 1$ and evaluate the integral $\int_t^T [\mu_0 + \mu_1 \cdot (x X_{u-1}^M)] du$ along each path of $X_u^M$.

(iii) Evaluate $\delta_t(P, R)$ in (40) via simulation using the paths generated in step (ii) for each path $(X_t = x, A_t = a, Y_t = y)$ generated in step (i).

(iv) Compute the functions in (34) and use $\delta_t(P, R)$ to evaluate, for $t \in [0, T]$,

$$E[e^{-f_0(T-t)-f_1(T-t)Y_t-f_2(T-t)Y_t^2} \delta_t^2(P, R)].$$

(v) Evaluate the second term in $B(m, P, R)$ via the trapezoidal rule.

To minimize $B(m, P, R)$ over $(P, R)$, we follow two steps. First, given $R$, we perform a line search for the corresponding optimal $P$ over $[c, P^{NV(M)}]$. This is fairly efficient, as $B(m, P, R)$ is convex in $P$ given $m$ and $R$ (see Lemma 4). Then, we perform a line search for the optimal $R$ over $[bc, R^{NV(M)}]$.

In all our numerical experiments below, we compare two models: (i) the price-setting newsrvendor model without risk hedging (i.e., the base model) and (ii) the price-setting newsrvendor model with risk hedging.

Optimal Solutions and Hedging Performance. We focus on the target return at the newsrvendor’s maximum profit in the base model (i.e., $m = E[H_T(P^{NV}, R^{NV})]$). Then, we consider three initial oil prices $(X_0)$: 40, 70, and 100. Recall, the long-run average oil price is around $70. Thus, $X_0 = 40$ represents an upward asset price trend; in this case, the trend benefits the demand of the Focus but hurts the demand of the Explorer. By contrast, $X_0 = 100$ represents a downward asset price trend; in this case, the effect of the trend on the demands of these two models is reversed. The case of $X_0 = 70$ represents a negligible trend scenario as it is very close to the long-run average. With these three initial oil prices, we compute—for both the Explorer and the Focus—the optimal prices, service levels, production quantities, contributions from production to the total return in the hedging model, and risk reduction. The results are summarized in Table 4.
Table 4  Production, Price, Service Level, Return, and Risk.

| Car   | \(X_0\) | Model | \(Q\) | \(P\) | \(R\)   | Return \((\times10^7)\) | Risk \((\times10^7)\) |
|-------|---------|-------|-------|-------|--------|--------------------------|--------------------------|
|       | 40      | NV    | 15,547| 42,079| 100,546| 10.34                     | 4.02                     |
|       |         | Hedging | 14,713| (5.37\%) | 41,645| (1.03\%) | 98,835| (1.70\%) | (96.35\%) | 2.50 | (37.85\%) |
| Explorer | 70      | NV    | 12,555| 40,628| 94,624 | 6.56                      | 3.50                     |
|       |         | Hedging | 12,350| (1.63\%) | 40,502| (0.31\%) | 94,164| (0.49\%) | (99.60\%) | 3.08 | (12.16\%) |
|       | 100     | NV    | 9,513 | 39,153| 88,602 | 3.60                      | 2.93                     |
|       |         | Hedging | 8,907| (6.37\%) | 38,843| (0.79\%) | 87,369| (1.39\%) | (96.38\%) | 2.20 | (25.12\%) |
| Focus  | 40      | NV    | 9,826 | 21,946| 154,510| 1.29                      | 0.82                     |
|       |         | Hedging | 9,113| (7.26\%) | 21,837| (0.50\%) | 153,077| (0.93\%) | (93.78\%) | 0.48 | (41.60\%) |
|       | 70      | NV    | 12,265| 22,313| 159,367| 2.05                      | 0.97                     |
|       |         | Hedging | 12,090| (1.43\%) | 22,285| (0.13\%) | 159,005| (0.23\%) | (99.62\%) | 0.84 | (13.90\%) |
|       | 100     | NV    | 14,647| 22,671| 164,108| 2.96                      | 1.12                     |
|       |         | Hedging | 14,169| (3.26\%) | 22,595| (0.33\%) | 163130| (0.60\%) | (98.07\%) | 0.75 | (33.37\%) |

Note. The percentages in parentheses represent the decrease relative to the base model in columns \(Q\), \(P\), and \(R\); represent the contribution from the production payoff in the column labeled “Return”; and represent the reduction relative to the base model in the column labeled “Risk.” The target return of both the base model and the hedging model are set at the newsvendor’s maximum profit in the base model.

A couple of observations can be made from Table 4. First, when \(X_0\) (the initial oil price for a particular month) deviates from the long-run average (i.e., \(X_0 = 40\) or \(100\)), the risk reductions achieved by the hedging model are prominent (compared to \(X_0 = 70\), which is close to the long-run average). This observation holds for both the fuel-efficient model (the Focus) and the fuel-inefficient model (the Explorer). Note that for the Explorer (resp., the Focus), a low (resp., high) initial price represents an upward (resp., downward) oil price trend, inducing detrimental effects on demand, and vice versa when the initial oil price is high (resp., low) for the Explorer (resp., the Focus). This is economically intuitive: when asset price trend significantly affects demand, hedging is especially effective as it offsets such impacts. This demonstrates that when asset price trend imposes a prominent beneficial or detrimental effect on demand, the hedging model achieves substantial risk reductions. The next observation concerns the optimal operational policy in the presence of hedging. We can see that for both car models, pricing level \((P)\), production \((Q)\), and service level \((R)\) are adjusted down relative to those without hedging, but the decrements are small. As a result, the operational payoff still contributes most of the total wealth. This is reassuring: hedging does not excessively decrease the operational level. A particularly desirable trait is that the price markdown is small: while price reduction enhances market competitiveness, automakers are usually reluctant to cut prices too much in order to protect their brands’ value. Another observation is that when the impact of the asset price trend on demand is strong \((X_0 = 40\) or \(100\)), hedging adjusts the
operational policy more, relative to the weaker impact case ($X_0 = 70$). This is consistent with the finding that hedging is more effective in the former case.

**Efficient Frontiers.** In Figure 1, we plot the efficient frontiers for both car models with an initial oil price $X_0 = 40$. The target mean ranges from 90% to 100% of $\mathbb{E}[H_T(P^NV, R^NV)]$, the newsvendor’s maximum profit in the base model. Risk is measured by the standard deviation of the terminal wealth (the right ends of the curves correspond to the results in Table 4). All curves are upward sloping, which is the hallmark of efficient frontiers: after all decisions are optimized, an increase in return is always accompanied by increased risk. As the analyses confirmed, the hedging model’s curves (blue) lie lower than those of the base model (red), and the gap between these two represents the risk reduction due to hedging. We can observe that this gap increases as return (thus, also risk) increases. Also, as return increases, the slope of the hedging model’s frontiers grows substantially more slowly than that of the base model’s frontiers, implying that the hedging model bears significantly lower increments from increasing returns, and this phenomenon is most prominent when the return reaches the newsvendor’s maximum profit in the base model. Both points indicate that hedging is more effective in high return (and risk) cases.

![Efficient Frontiers](image)

*Figure 1 Efficient Frontiers.*

*Note.* Initial WTI price is $X_0 = 40$.

Figure 2 illustrates the contribution of production payoff to total wealth over the presented range of target returns. The numerical results show that production payoff accounts for at least 94.5% of total wealth in all instances, indicating that operations is the primary source of profit for the automakers. We can also observe that, as the target return increases, the contribution from hedging increases (i.e., the contribution from production decreases). This is consistent with how hedging reduces risk. As return increases, for the base model, risk also increases due to higher pricing and service levels; the increment in risk is most prominent when the return approaches the newsvendor’s maximum profit in the base model (see Proposition 1). By contributing more to the return as
it increases, hedging suppresses the growth of production payoff (induced by higher pricing and service levels), which in turn suppresses the growth of unhedgeable risk in order to control the total risk. This is reflected by the observed higher percentage contribution from hedging when the target return is higher.

Figure 2 Production Payoff Contribution with Risk Hedging.

Note. Initial WTI price is $X_0 = 40$.

Optimal $P$, $R$, and $Q$. Figure 3 plots the optimal price ($P$), service levels ($R$), and production quantities ($Q$) for models with and without hedging (with an initial oil price $X_0 = 40$) over a range of target returns.

Theorems 2 and 3 indicate that the optimal price with hedging is lower than that without hedging when the target mean is set at the newsvendor’s maximum profit in the base model. Although we do not have analytical results for other values of the target return, we can observe from Figure 3 that the optimal price with hedging never exceeds the optimal price without hedging over the given range of target returns. We also observe that the price markdown is small for all return levels in the range. It is also worth noting that as return increases, the optimal price grows more slowly with hedging than without hedging. This indicates that as the producer demands higher return, in the presence of hedging the price increment does not have to be as high as in the absence of hedging. This helps the manufacturer to stay competitive in the market.

Similar patterns are observed for optimal $R$ and $Q$. Both the service and production levels of the hedging model are lower than those without hedging, and the gap widens as return increases, reaching the maximum value when $m$ approaches the newsvendor’s maximum profit in the base model. The decreases in both $R$ and $Q$ are small.

In summary, the operational level (represented by $P$, $R$, and $Q$) decreases in the presence of hedging, but the reduction is small, which is desirable.
Risk Decomposition. In Figure 4, we examine the contribution of unhedgeable risk (i.e., the second term of $B(m, P, R)$ in (27)) to total risk (i.e., $B(m, P, R)$). Unhedgeable risk contributes to most of the total risk for all instances—at least 77% for the Explorer and 64% for the Focus—indicating that the main risk factor (after being hedged) is the intrinsic demand volatility. In other words, investment risk (i.e., the first term of $B(m, P, R)$) constitutes only a moderate part of the total risk. This is a desirable trait; a manufacturer, which is non-financial in nature, does not want to bear too much risk originating from financial investment.

The other observation is that this percentage drops as $m$ increases. This is consistent with Figure 2: the contribution from hedging to the total return increases with the target return, leading to a higher investment risk that has to be borne (note that production payoff is bounded from above, so as $m$ increases, the additional payoff has to be contributed by hedging, which leads to a higher investment risk).

Illustration of Parts (ii) and (iii) of Theorem 3. Part (ii) of Theorem 3 indicates that when the detrimental effect of the asset price trend on market size is moderately strong, $R^{h_m}$ will never exceed $R^{NV}$. By contrast, part (iii) of the same theorem predicts that even if the detrimental effect is not sufficiently strong, $R^{h_m}$ will not too much exceed $R^{NV}$.

To demonstrate these results, we create hypothetical demand-asset models in order to magnify detrimental effects, based on the parameters estimated from §6.2.1 and §6.2.2. Specifically, to make
the asset price trend more prominent and to increase its volatility, we increase the values of both \( \kappa \) and \( \sigma \) by a factor of 10. In addition, we increase \( B \) in (43) by a factor of 9 (resp., 10) for the Explorer (resp., the Focus) and adjust \( A \) in (43) accordingly to keep \( \mathbb{E}(A_T) \) in the same place as in the estimated model in §6.2.2. Other parameters remain unchanged. It is numerically checked that \( P \) (resp., \( R \)) is close to \( c \) (resp., \( bc \)) for the Focus and substantially smaller than \( P^{\text{NV}} \) (resp., \( R^{\text{NV}} \)) for the Explorer (see Proposition 1). To vary magnitude of detrimental effect, for the Explorer, fixing \( X_0 \) at 30, we increase \( \alpha \) in (28) (i.e., long-run average of the logarithm of oil price). For the Explorer, increasing \( \alpha \) induces a stronger detrimental effect. For the Focus, we set \( X_0 \) at 250 to create a prominent downward oil price trend, and then we vary \( \alpha \). For this car model, decreasing \( \alpha \) induces a stronger detrimental effect. Based on this setup, we demonstrate parts (ii) and (iii) of Theorem 3 as follows. We quantify the detrimental effect, \( \Delta \), by the difference between the right and left sides of (38):

\[
\Delta := c - s - \left[ \frac{\bar{P}^{\text{NV}}}{r^\circ + \sqrt{(r^\circ)^2 - 1}} \right] \cdot P^M(A_T \geq R^{\text{NV}}).
\]

(Note: \( \Delta \geq 0 \) is equivalent to (38).) For part (ii), we examine \( \Delta \) and \( r^\circ \) (defined in (37)) as \( \alpha \) varies. For part (iii) (i.e., \( \Delta < 0 \)), we numerically examine \( R^\circ, R^{\text{NV}}, \) and \( R^{\text{NV}(M)} \) as \( \alpha \) varies.

Figure 5 illustrates part (ii) of Theorem 3. As \( \alpha \) increases (resp., decreases), the upward (resp., downward) sloping WTI price trend, reflected by \( E(X_\infty - X_0) \) (resp., \( E(X_0 - X_\infty) \)), is more prominent, and the detrimental effect of WTI price on the demand of the Explorer (resp., the Focus) strengthens. This is clearly observed from the two graphs in the first column of the panel: \( \Delta \) increases as the detrimental effect strengthens. We can see from the second column of the graphs that the difference (in mean) between \( A_T^M \) and \( A_T \) increases as the detrimental factor (\( \Delta \)) increases, as the strengthening asset price trend reduces \( A_T \) but does not affect \( A_T^M \). Then, as discussed
following Theorem 3, $r^o$ (defined in (37)) increases with the detrimental effect for both car models (third column).

Part (iii) of Theorem 3 is illustrated in Figure 6. As expected, when (38) does not hold (i.e., $\Delta < 0$), $R^o$ (recall, $R^h_m \leq R^o$) is close to $R^{NV}$ (compared to its distance to $R^{NV(M)}$), reconfirming that in this case (i.e., when the detrimental effect is not sufficiently strong), $R^h_m$ can possibly exceed $R^{NV}$, but only by a small margin.
7. Concluding Remarks

In this study, we develop and solve a general model that integrates pricing, production, and risk hedging using financial assets. We completely characterize the return-risk efficient frontier and quantify the risk reduction from the no-hedging model. We find that the pricing level is lower with hedging than without hedging, both when the asset price trend benefits demand and when it hurts demand. This is desirable for firms that operate in a competitive market. In addition, the service level is lower with hedging than without hedging when the asset price trend benefits demand; when the asset price trend hurts demand, the service level with hedging may exceed the service level without hedging, but only by a small margin. Our case study using data sets of Ford Motor Company shows that the hedging model performs substantially better than a price-setting newsvendor without hedging. The markdowns in pricing and service levels are small, which are appealing because hedging does not materially decrease operational profit, and it substantially reduces risk.

We conclude by pointing out a couple of potential extensions of this work. The model could be extended to a multi-period model that allows dynamic pricing and inventory decisions to be integrated with risk hedging. This is an important problem, as many industry sectors dynamically adjust pricing and inventory over time. The significant analytical challenge lies in determining how to align the dynamic programming involved in the pricing/inventory decision (which occurs in discrete time) with the martingale method of the solution approach to the hedging problem (which occurs in real time). Another extension is to consider a portfolio of products simultaneously. For example, WTI price impacts the demand of sedans and SUVs at the same time. It would be interesting to investigate how risk hedging adjusts the prices of multiple products simultaneously. The solution approach to the hedging problem in this paper can be immediately extended to this setup, but the analysis of pricing for multiple products can be analytically challenging.

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Supplemental Material

EC.1. Proof of Lemma 1

The expected profit, as a function of \((P, R)\), is

\[
E[H_T(P, R)] = (P - c)(R - bP) - (P - s)E[(R - A_T)^+].
\]

Given \(P\), differentiate \(E[H_T(P, R)]\) with respect to \(R\):

\[
\frac{\partial E[H_T(P, R)]}{\partial R} = (P - c) - (P - s)F(R). \tag{EC.1.1}
\]

Then, taking second-order derivative to \(R\):

\[
\frac{\partial^2 E[H_T(P, R)]}{\partial R^2} = -(P - s)f(R) < 0 \tag{EC.1.2}
\]

Thus, for a given \(P\), the expected profit is a concave function in \(R\), and the optimal solution, denoted \(R^{NV}(P)\), is solved from setting \(\partial E[H_T(P, R)]/\partial R\) to zero. This leads to

\[
R^{NV}(P) = F^{-1}\left(\frac{P - c}{P - s}\right).
\]

Clearly, \(R^{NV}(P)\) increases in \(P\). Define:

\[
m(P) := E[H_T(P, R^{NV}(P))] = (P - c)(R^{NV}(P) - bP) - (P - s)E[(R^{NV}(P) - A_T)^+].
\]

As \(P \to c\), \(R^{NV}(c) \to -\infty\), \((P - s)E[(R^{NV}(P) - A_T)^+] \to 0\), then,

\[
\lim_{P \to c} m(P) = \lim_{P \to c} (P - c)(R^{NV}(P) - bP) = \lim_{P \to c} \frac{R^{NV}(P) - bP}{1 - \frac{1}{(P - c)^2}} - b
\]

\[
= -\lim_{P \to c} \frac{1}{f(R)}(c - s)(\frac{P - s}{P - c})^2 + \lim_{P \to c} b(P - c)^2 = -(c - s)\lim_{R \to -\infty} \frac{F^2(R)}{f(R)}
\]

\[
= -(c - s) \lim_{R \to -\infty} \frac{F^2(R)}{r(R)}(1 - F(R)).
\]

Under Assumption 1, \(F^2(a)/r(a) \to 0\) as \(a \to -\infty\), so the limit above is 0.

Next, differentiate \(m(P)\) with respect to \(P\):

\[
m'(P) = R^{NV}(P) - 2bP + bc - E[(R^{NV}(P) - A_T)^+] \tag{EC.1.3}
\]

As \(P \to c\), \(R^{NV}(c) \to -\infty\) and \(m'(c) \to -\infty\).

Differentiating \(m'(P)\) with respect to \(P\):

\[
m''(P) = -2b + (1 - F(R^{NV}(P)))\frac{1}{dR^{NV}(P)} = -2b + \frac{(1 - F(R^{NV}(P)))^3}{(c - s)f(R^{NV}(P))} = -2b + \frac{(1 - F(R^{NV}(P)))^2}{(c - s)r(R^{NV}(P))} \tag{EC.1.4}
\]
where we use the definition of the hazard rate for $A_T$, i.e., $r(a) = \frac{f(a)}{1-F(a)}$. Further differentiating $m''(P)$ with respect to $P$:

$$\frac{dm''(P)}{dP} = - \frac{(2r(R_{NV}(P))^2 + r'(R_{NV}(P))(1-F(R_{NV}(P)))^2}{(c-s)r^2(R_{NV}(P))} \frac{dR_{NV}(P)}{dP}. $$

By Assumption 1, we have $\frac{dm''(P)}{dP} < 0$, so $m''(P)$ is decreasing in $P$. Thus, as $P$ increases from $c$ to $\infty$, $m''(P)$ decreases from $\infty$ to $-2b$. (As $P \to c$, $R_{NV}(P) \to -\infty$, $r(R_{NV}(P)) \to 0$, leading to $m''(P) \to \infty$; As $P \to \infty$, $\frac{(1-F(R_{NV}(P)))^3}{f(R_{NV}(P))} \to 0$ by Assumption 1). Combining the above, $m'(P)$ increases first and then decreases with $m'(P) \to -\infty$ as $P \to c$ and $m'(P) \to -\infty$ as $P \to \infty$.

Combining the analysis above and Assumption 2, there must exist some $P^0 > c$ such that $m'(P^0) > 0$, otherwise, the optimal expected profit is negative. Therefore, $m'(P) = 0$ has two zeros for $P > c$. The smaller one is the minimizer, the larger one is the maximizer. As $P$ increases over $c$, $m(P)$ first decreases from 0 to a negative value, which is the minimum level of $m(P)$. Then, it increases to a positive value, which is the maximum level of $m(P)$ and the associated optimal price is denoted as $P_{NV}$, which follows the following optimality equation:

$$m'(P_{NV}) = R_{NV}(P) - 2bP_{NV} + bc - E[(R_{NV}(P) - A_T)^+] = 0. \quad (EC.1.5)$$

The equation above is equivalent to:

$$2bP_{NV} - bc = E(R_{NV}(P_{NV}) \land A_T), \quad (EC.1.6)$$

and $m(P)$ decreases when $P > P_{NV}$. Let $P$ be the smaller zero of $m(P) = 0$ and then $P_{NV} > P > c$. Let $R_{NV} = R_{NV}(P_{NV})$, which satisfies the other optimality equation:

$$P_{NV} - c - (P_{NV} - s)F(R_{NV}) = 0.$$

Combining the optimality equations of $(P_{NV}, R_{NV})$ leads to (5), and this completes the proof. \(\square\)

**EC.2. Proof of Proposition 1**

We prove the three results stated in this proposition one by one.

For part (i), we have already proved that $m(P)$ is increasing in $P \in [P, P_{NV}]$ in §EC.1. The variance of production payoff is:

$$\text{Var}(H_T(P, R)) = (P - s)^2 \{E[(R - A_T)^+]^2 - E^2[(R - A_T)^+]\}. $$

Differentiate $\text{Var}(H_T(P, R))$ with respect to $P$:

$$\frac{\text{Var}(H_T(P, R))}{\partial P} = 2(P - s)\left[\left(E[(R - A_T)^+]^2 - E^2[(R - A_T)^+]\right] > 0. $$

$$
We claim that

\[ \text{Var}(H_T(P, R)) \text{ with respect to } R: \]

\[ \frac{\partial \text{Var}(H_T(P, R))}{\partial R} = 2(P - s)^2 \{E[(R - A_T)^+](1 - F(R)) > 0. \]

Next, differentiating \( v(P) = \text{Var}(H_T(P, R^{\text{NV}}(P))) \) with respect to \( P \), we have

\[ \frac{dv(P)}{dP} = \frac{\partial \text{Var}(H_T(P, R^{\text{NV}}(P)))}{\partial P} + \frac{\partial \text{Var}(H_T(P, R^{\text{NV}}(P)))}{\partial R} \frac{dR^{\text{NV}}(P)}{dP} > 0 \quad (\text{EC.2.1}) \]

Then \( \text{Var}[H_T(P, R^{\text{NV}}(P))] \) is increasing in \( P \) for \( P \geq c \).

For part (ii), fixing \( R \), define

\[ m(R) = E[H_T(P^{\text{NV}}(R), R)] = (P^{\text{NV}}(R) - c)(R - bP^{\text{NV}}(R)) - (P^{\text{NV}}(R) - s)E[(R - A_T)^+]. \]

Recall, \( P^{\text{NV}}(R) = (E[R \wedge A_T] + bc)/(2b) \). It is straightforward to verify that \( P^{\text{NV}}(bc) < c \). Further, from Assumption 3, we have:

\[ E[(bc - A_T)^+] \leq 2b(c - s). \]

Then,

\[ P^{\text{NV}}(bc) = \frac{1}{2b} (E[bc \wedge A_T] + bc) = \frac{1}{2b} (bc - E[(bc - A_T)^+] + bc) > s, \]

so

\[ m(bc) = b(P^{\text{NV}}(bc) - c - P^{\text{NV}}(bc)) - (P^{\text{NV}}(bc) - s)E[(bc - A_T)^+] < 0. \]

Next, differentiate \( m(R) \) with respect to \( R \) and taking into account \( \partial E[H_T(P^{\text{NV}}(R), R)]/\partial P = 0 \):

\[ \frac{dm(R)}{dR} = \frac{dE[H_T(P^{\text{NV}}(R), R)]}{dR} = \frac{\partial E[H_T(P^{\text{NV}}(R), R)]}{\partial R} = (P^{\text{NV}}(R) - c) - (P^{\text{NV}}(R) - s)F(R). \]

Let

\[ J(R) = \frac{dm(R)}{dR} = (P^{\text{NV}}(R) - c) - (P^{\text{NV}}(R) - s)F(R), \]

and differentiate \( J(R) \) with respect to \( R \):

\[ \frac{dJ(R)}{dR} = f(R) \left[ \frac{1 - F(R)}{2br(R)} - (P^{\text{NV}}(R) - s) \right]. \]

Then, differentiate \( \frac{dJ(R)}{dR} \) with respect to \( R \):

\[ \frac{d^2 J(R)}{dR^2} = \frac{df(R)}{dR} \frac{dJ(R)}{dR} - \frac{f(R)(1 - F(R))}{2br^2(R)} (2r^2(R) + r'(R)). \]

We claim that \( J(R) \) is either monotone or unimodal in \( R \). First note for any \( R, \frac{f(R)(1 - F(R))}{2br^2(R)} (2r^2(R) + r'(R)) > 0 \) by Assumption 1. Then, there are two cases. The first one is that for all \( R, \frac{dJ(R)}{dR} > 0 \) then \( J(R) \) is monotone in \( R \). The other case is that there exists \( R \) such that \( \frac{dJ(R)}{dR} = 0 \). Then, at this \( R, \frac{d^2 J(R)}{dR^2} < 0 \), which indicates that for any stationary point \( R \), it must be a local
maximizer. Thus, $J(R)$ is unimodal for this case. Note, $J(bc) < 0$ and $J(R) \to -(c-s) < 0$ as $R \to \infty$. Then, by Assumption 2, there must exist some $R > bc$ such that $J(R) > 0$. Otherwise $J(R) < 0$ for all $R \geq bc$ which implies that $m(R)$ is decreasing in $R$ and $m(R) \leq m(bc) < 0$, and the optimal service level is $R = bc$, which contradicts this assumption. By the analysis above, we conclude that $J(R)$ is unimodal in $R \geq bc$. Furthermore, there are two roots of $J(R) = 0$ and let $R_1^0$ (resp., $R_2^0$) be the smaller (resp., larger) one. Clearly, $m(R)$ is decreasing in $R < R_1^0$, increasing in $R_1^0 < R < R_2^0$ and decreasing in $R > R_2^0$. Since $m(bc) < 0$, there are two roots for $m(R) = 0$ for $R \geq bc$ and let $R$ denote the smaller one. By Lemma 1, $R^{NV} = R_2^0$. Thus, we conclude that $m(R)$ is increasing in $R < R < R^{NV}$ (and decreasing in $R > R^{NV}$).

Analogous to Proposition 1 (i), it is straightforward to prove that $\text{Var}(H_T(P^{NV}(R), R))$ increase in $R$.

For part (iii), let $\bar{m} = \mathbb{E}[H_T(P^{NV}, R^{NV})]$. For any $m \in (0, \bar{m})$, we claim that both $\partial \mathbb{E}[H_T(P^{NV}, R^{NV})]/\partial P$ and $\partial \mathbb{E}[H_T(P^{NV}, R^{NV})]/\partial R$ are positive. Otherwise, suppose $\partial \mathbb{E}[H_T(P^{NV}, R^{NV})]/\partial P \leq 0$. As $\mathbb{E}[H_T(P, R^{NV})]$ is concave in $P$ and $\mathbb{E}[H_T(c, R^{NV})] = m$, we know that $\mathbb{E}[H_T(P, R^{NV})]$ is increasing in $P \in (c, P^{NV}(R^{NV}))$ and decreasing in $P > P^{NV}(R^{NV})$. Then $P^{m^*}_m \geq P^{NV}(R^{NV})$ as we have assumed $\partial \mathbb{E}[H_T(P^{NV}(m, R^{NV}))]/\partial P \leq 0$. So there exists $P'_m$ such that $P'_m \leq P^{NV}(R^{NV}) \leq P^{NV}_m$ and we have:

$$
\mathbb{E}[H_T(P^{NV}_m, R^{NV}_m)] = \mathbb{E}[H_T(P'_m, R^{NV}_m)] = m, \quad \text{Var}(H_T((P'_m, R^{NV}_m))) \leq \text{Var}(H_T(P^{NV}_m, R^{NV}_m)).
$$

In other words, keeping the target $m$, $(P'_m, R^{NV}_m)$ has a smaller variance than $(P^{NV}_m, R^{NV}_m)$ does, and thus $(P^{NV}_m, R^{NV}_m)$ cannot be the solution to the problem in (9). By contradiction, $\partial \mathbb{E}[H_T(P^{NV}, R^{NV})]/\partial P$ must be positive. Similarly, $\partial \mathbb{E}[H_T((P^{NV}_m, R^{NV}_m))] / \partial R > 0$ always holds.

Next, introducing the Lagrange multiplier $\lambda$, the Lagrangian function of the problem in (9) is:

$$
\mathcal{L} = \text{Var}(H_T(P, R)) - \lambda (\mathbb{E}[H_T(P, R)] - m).
$$

Then, $(P^{NV}_m, R^{NV}_m, \lambda^*_m)$ satisfies the Karush–Kuhn–Tucker (KKT) equations:

$$
\frac{\partial \text{Var}(H_T(P, R))}{\partial P} - \lambda \frac{\partial \mathbb{E}[H_T(P, R)]}{\partial P} = 0; \quad (\text{EC.2.3})
$$

$$
\frac{\partial \text{Var}(H_T(P, R))}{\partial R} - \lambda \frac{\partial \mathbb{E}[H_T(P, R)]}{\partial R} = 0; \quad (\text{EC.2.4})
$$

$$
\mathbb{E}[H_T(P, R)] - m = 0. \quad (\text{EC.2.5})
$$

we have $\lambda^*_m > 0$ as we have proved that $\partial \mathbb{E}[H_T(P^{NV}_m, R^{NV}_m)] / \partial P > 0$ and $\partial \mathbb{E}[H_T(P^{NV}_m, R^{NV}_m)] / \partial R > 0$ hold. By Envelop Theorem, we have:

$$
\frac{d}{dm} \text{Var}(H_T(P^{NV}_m, R^{NV}_m)) = \lambda^* > 0
$$
this proves that $\text{Var}(H_T(P_{m}^{NW}, R_{m}^{NW}))$ is increasing in $m$ for $m \in (0, E[H_T(P^{NW}, R^{NW})])$.

When $(P, R) \rightarrow (P^{NW}, R^{NW})$, $m \rightarrow E[H_T(P^{NW}, R^{NW})]$, $\partial E[H_T(P, R)]/\partial P \rightarrow 0$ and $\partial E[H_T(P, R)]/\partial R \rightarrow 0$. Therefore, KKT condition (EC.2.3) or condition (EC.2.4) implies that $\lambda^* \rightarrow \infty$, this proves that the incremental risk approaches infinity when the expected payoff approaches the newsvendor’s maximum proft in the base model. □

**EC.3. Proof of Lemma 2**

Suppose $C_T^{(1)}$ is stochastically larger than $C_T^{(2)}$, i.e. $C_T^{(1)} > C_T^{(2)}$, then it is straightforward to verify that the associated market potentials satisfy $A_T^{(1)} > A_T^{(2)}$. Let the distribution functions of $A_T^{(i)}$ be $F_i$.

Let $(P_i^{NW}, R_i^{NW})$ be the maximizer of the expected newsvendor’s profit given the market potential is $A_T^{(i)}$, $i = 1, 2$. For a given $P$, let $R_i^{NW}(P)$ be the corresponding profit-maximizing decision, i.e.,

$$R_i^{NW}(P) := \arg \max_R E[H_T(P, R)]$$

$$= \arg \max_R (P - c)(R - bP) - (P - s)E[(R - A_T^{(i)})^+] = F_i^{-1}\left(\frac{P - c}{P - s}\right).$$

Clearly, $R_1^{NW}(P) > R_2^{NW}(P)$.

Define

$$H_i(P) = (P - c)(R - bP) - (P - s)E[(R_i^{NW}(P) - A_T^{(i)})^+], \quad i = 1, 2.$$ 

Then, the first-order derivative with respect to $P$ is:

$$H'_i(P) = R_i^{NW}(P) - 2bP + bc - E[(R_i^{NW}(P) - A_T^{(i)})^+], \quad i = 1, 2$$

For any given $P$, we show $H'_1(P) > H'_2(P)$:

$$H'_1(P) = R_1^{NW}(P) - 2bP + bc - E[(R_1^{NW}(P) - A_T^{(1)})^+]$$

$$> R_1^{NW}(P) - 2bP + bc - E[(R_1^{NW}(P) - A_T^{(2)})^+]$$

$$> R_2^{NW}(P) - 2bP + bc - E[(R_2^{NW}(P) - A_T^{(2)})^+] = H'_2(P).$$

The first inequality is due to $A_T^{(1)} > A_T^{(2)}$. The second inequality follows from the fact that $R - E[(R - A_T^{(2)})^+] = E(R \wedge A_T^{(2)})$ is increasing in $R$ and $R_1^{NW}(P) > R_2^{NW}(P)$. Suppose $P_i^{NW}$ is the one of the larger zeros to $H'_i(P) = 0$. (Recall, $H'_i(P) = 0$ has two solutions, the larger one is the newsvendor’s solution; refer to §EC.1.) Then,

$$0 = H'_1(P_i^{NW}) > H'_2(P_i^{NW}),$$

thus $H'_2(P)$ becomes negative before $P$ exceeds $P_i^{NW}$, indicating $P_2^{NW} < P_1^{NW}$. Furthermore,

$$R_1^{NW}(P_1^{NW}) = F_1^{-1}\left(\frac{P_1^{NW} - c}{P_1^{NW} - s}\right) > F_2^{-1}\left(\frac{P_2^{NW} - c}{P_2^{NW} - s}\right) > F_2^{-1}\left(\frac{P_2^{NW} - c}{P_2^{NW} - s}\right) = R_2^{NW}(P_2^{NW}) = R_2^{NW}.$$

This completes the proof. □
EC.4. Proof of Lemma 3

We reiterate the projected production payoff process in (21):

\[ V_t(P, R) = \mathbb{E}^M[H_T(P, R) | \mathcal{F}_t] \]
\[ = (R - bP)(P - c) - (P - s)\mathbb{E}^M[(R - A_T)^+ | \mathcal{F}_t] \]
\[ = (R - bP)(P - c) - (P - s)\mathbb{E}^M[(R - A_T)^+ | X_t = x, A_t = a, \mu_t = \mu, \sigma_t = \sigma] \]
\[ = (R - bP)(P - c) - (P - s)f(t, x, a, \mu, \sigma), \]

where the second equality follows from that \((X_t, A_t, \mu_t, \sigma_t)\) is Markovian and the last equality is based on the definition of \(f(t, x, a, \mu, \sigma)\):

\[ f(t, x, a, \mu, \sigma) := \mathbb{E}^M[(R - A_T)^+ | X_t = x, A_t = a, \mu_t = \mu, \sigma_t = \sigma]. \]

By definition, \(V_t\) is a martingale under \(\mathbb{P}^M\). Applying Martingale Representation Theorem, \(V_t(P, R)\) has the following representation, with \(V_0(P, R) = \mathbb{E}^M[H_T(P, R)]\):

\[ V_t(P, R) = V_0(P, R) + \int_0^t \xi_s(P, R) dX_s + \int_0^t \delta_s(P, R) dB_s. \]

\(\xi_t\) and \(\delta_t\) are processes adapted to \(\mathcal{F}_t\) and can be directly derived as follows, by applying Itô’s Lemma:

\[ \xi_t(P, R) = -(P - s)f_x(t, X_t, A_t, \mu_t, \sigma_t), \quad \delta_t(P, R) = -\tilde{\sigma}(P - s)f_a(t, X_t, A_t, \mu_t, \sigma_t), \]

Furthermore,

\[ A_T = \int_0^T \tilde{\mu}(X_s) ds + \tilde{\sigma} \tilde{B}_T = \int_0^T \tilde{\mu}(X_s) ds + \int_0^T \tilde{\mu}(X_s) ds + \tilde{\sigma} \tilde{B}_s + \tilde{\sigma}(\tilde{B}_T - \tilde{B}_t). \]

Then, given \(A_t = a\),

\[ A_T = a + \int_t^T \tilde{\mu}(X_s) ds + \tilde{\sigma}(\tilde{B}_T - \tilde{B}_t), \]

and thus

\[ f(t, x, a, \mu, \sigma) = \mathbb{E}^M[(R - A_T)^+ | X_t = x, A_t = a, \mu_t = \mu, \sigma_t = \sigma] \]
\[ = \mathbb{E}^M[(R - (a + \int_t^T \tilde{\mu}(X_s) ds + \tilde{\sigma}(\tilde{B}_T - \tilde{B}_t)))^+ | X_t = x, A_t = a, \mu_t = \mu, \sigma_t = \sigma] \]
\[ = \mathbb{E}^M[(R - (a + \int_t^T \tilde{\mu}(X_s) ds + \tilde{\sigma}(\sqrt{T-t}Z))|^+ | X_t = x, A_t = a, \mu_t = \mu, \sigma_t = \sigma], \]

where \(Z = (\tilde{B}_T - \tilde{B}_t)/\sqrt{T-t}\) follows standard normal distribution that is independent of \(\{X_s, t \leq s \leq T\}\). Taking first derivative of \(f(t, x, a, \mu, \sigma)\) with to \(a\):

\[ f_a(t, x, a, \mu, \sigma) = -\mathbb{E}^M\left[ 1\{a + \int_t^T \tilde{\mu}(X_s) ds + \tilde{\sigma}\sqrt{T-t}Z \leq R\} | X_t = x, A_t = a, \mu_t = \mu, \sigma_t = \sigma \right] \]
\[- \mathbb{P}^M(A_T \leq R \mid X_t = x, A_t = a, \mu_t = \mu, \sigma_t = \sigma) = -\mathbb{P}^M(A_T \leq R \mid \mathcal{F}_t) \].

Therefore, we have:
\[
\delta_t(P, R) = \tilde{\sigma}(P - s)\mathbb{P}^M(A_T \leq R \mid \mathcal{F}_t).
\]

Clearly, \( \delta_t(P, R) \) increases in both \( P \) and \( R \). \( \square \)

**EC.5. Proof of Theorem 1**

In this part, we apply the quadratic hedging technique in Gourieroux et al. (1998) to solve our hedging problem. The setup in Gourieroux et al. (1998) is semimartingale-based and abstract, hence the solution is not as explicit as ours since our setup is based on Brownian motions. In §EC.5.1, we lay out technical preparations that are needed in the proof of Theorem 1 in §EC.5.2.

**EC.5.1. Technical Preparation**

Recall, the risk-neutral measure \( \mathbb{P}^M \) is defined via the associated Radon-Nikodym (R-N) derivative:
\[
Z_T = \frac{d\mathbb{P}^M}{d\mathbb{P}} := e^{-\int_0^T \eta_t dB_t - \frac{1}{2} \int_0^T \eta_t^2 dt},
\]
and \( Z_t^M := \mathbb{E}^M(Z_T \mid \mathcal{F}_t) = \mathbb{E}(Z_T^2 \mid \mathcal{F}_t)/Z_t \) in (19), and thus \( Z_0^M = \mathbb{E}(Z_T^2) \). We introduce a process \( N_t \) and another probability \( \mathbb{P}^R \). With Assumption 5, \( \mathbb{P}^R \) below is well-defined:
\[
\frac{d\mathbb{P}^R}{d\mathbb{P}} = \left( \frac{Z_T^M}{Z_0^M} \right)^2, \quad \text{thus} \quad \frac{d\mathbb{P}^R}{d\mathbb{P}^M} = \frac{Z_T^M}{Z_0^M}. \quad (EC.5.1)
\]

Note that \( Z_t^M \) is a \( \mathbb{P}^M \)-martingale and below we denote its martingale representation as:
\[
dZ_t^M = \psi_t dB_t^M, \quad (EC.5.2)
\]
where \( \psi_t \) is a adapted process to \( \mathcal{G}_t \) (recall, \( \mathcal{G}_t \) is the filtration generated by \( B_t \), hence independent from \( \tilde{B}_t \)); and \( B_t^M = \eta_t dt + dB_t \), which is a Brownian motion under \( \mathbb{P}^M \). Matching (EC.5.2) to the alternative representation for \( Z_t^M \) in (20), \( dZ_t^M = \zeta_t dX_t = \zeta_t \sigma_t X_t dB_t^M \), we have
\[
\zeta_t = \frac{\psi_t}{X_t \sigma_t}. \quad (EC.5.3)
\]

Now, from (EC.5.1), \( Z_t^M / Z_0^M \) is the density process associated with \( d\mathbb{P}^R / d\mathbb{P}^M \). Then, applying Girsanov’s Theorem and accounting for (EC.5.2), the market price of risk process associated with \( d\mathbb{P}^R / d\mathbb{P}^M \) is
\[
\eta_t^M := -\frac{\psi_t}{Z_t^M}. \quad (EC.5.4)
\]

In addition, \( B_t^R \) defined below is a Brownian motion under \( \mathbb{P}^R \):
\[
dB_t^R := dB_t^M + \eta_t^M dt = dB_t^M - \frac{\psi_t}{Z_t^M} dt. \quad (EC.5.5)
\]
Next, we introduce \( N_t = (N^0_t, N^1_t) \) as follows:

\[
N^0_t := \frac{1}{Z^M_t}, \quad N^1_t := \frac{X_t}{Z^M_t}.
\]

(EC.5.6)

Recall, \( N^0_t \) and \( N^1_t \) are interpreted as the original assets, dollar (i.e. 1) and financial asset \( (X_t) \), denominated in \( Z^M_t \). By change of measure and Jensen’s inequality,

\[
Z^M_t = \frac{1}{Z_t} E(Z^2_T | G_t) \geq \frac{1}{Z_t} E^2(Z_T | G_t) = Z_t > 0,
\]

with \( Z_t \) defined in (14). By Assumption (4), \( X_t > 0 \). Therefore, \( N^0_t \) and \( N^1_t \) are well-defined, and both are strictly positive.

Applying Itô’s Lemma and accounting for (EC.5.5):

\[
dN^0_t = -(N^0_t)^2 \psi_t dB^M_t - \psi_t N^0_t dt = -(N^0_t)^2 \psi_t dB^R_t,
\]

\[
dN^1_t = -N^0_t [N^1_t \psi_t - \sigma_t X_t] [dB^M_t - \psi_t N^0_t dt] = -N^0_t [N^1_t \psi_t - \sigma_t X_t] dB^R_t.
\]

(Ec.5.7)

Clearly, both \( N^0_t \) and \( N^1_t \) are local martingales under \( P^R \); being nonnegative, they are also super-martingales. It is easy to verify that they are indeed \( P^R \)-martingales by having constant means (and being supermartingales):

\[
E^R(N^0_t) = \frac{1}{Z^M_t} E^M \left[ \frac{Z^M_T}{Z^M_t} \right] = \frac{1}{Z^M_0} E^M \left[ \frac{1}{Z^M_t} E^M (Z^M_T | G_t) \right] = \frac{1}{Z^M_0};
\]

the first equality is change of measure using (EC.5.1), the second one uses iterated conditioning on \( G_t \) and the fact that \( Z^M_t \) is adapted to \( G_t \). The last equality uses the definition of \( Z^M_t \) in (20), noting \( Z^M_T = Z^M_t \). Analogous verification can be applied to \( N^1_t \):

\[
E^R(N^1_t) = E^M \left[ \frac{Z^M_T}{Z^M_0} \frac{X_t}{Z^M_t} \right] = \frac{X_0}{Z^M_0} = N^1_0;
\]

the first equality uses the fact that \( Z^M_t / Z^M_0 \) is the density process for \( dP^R / dP^M \) and \( N^1_t \) is adapted to \( G_t \), and the second equality is by martingale property of \( X_t \). We summarize the analysis above into the following lemma.

**Lemma EC.1.** \( N^0_t \) and \( N^1_t \) in (EC.5.6) are both martingales under \( P^R \).

Now, we are ready to define \( \mathcal{M}_X, \mathcal{M}_N, \mathcal{A}_X \) and \( \mathcal{A}_N \), all of which are technically crucial in defining admissible class of hedging strategies. (\( \sim \) stands for the equivalence between probability measures.)

\[
\mathcal{M}_X := \left\{ P^M : \frac{dP^M}{dP} \in L_2(P), X_t \text{ is a } P^M\text{-martingale} \right\}.
\]

(5.8)
\( \mathcal{M}_X \) contains the equivalent martingale measures that have square-integrable R-N derivatives. By Assumption 4, \( \mathcal{P}^M \subset \mathcal{M}_X \), hence \( \mathcal{M}_X \neq \emptyset \). Similarly for \( N_t \), we define

\[
\mathcal{M}_N := \left\{ \mathcal{P}^\mathcal{R} \sim \mathcal{P} : \frac{1}{Z_M^T} \frac{d\mathcal{P}^\mathcal{R}}{d\mathcal{P}} \in L_2(\mathcal{P}), N^0_t \text{ and } N^1_t \text{ are } \mathcal{P}^\mathcal{R}\text{-martingales} \right\}.
\]  

(EC.5.9)

It is straightforward to verify that for \( \mathcal{P}^R \) defined in (EC.5.1),

\[
\frac{1}{Z_M^T} \frac{d\mathcal{P}^\mathcal{R}}{d\mathcal{P}} \in L_2(\mathcal{P}), \text{ and together with Lemma EC.1 this implies } \mathcal{P}^R \in \mathcal{M}_N, \text{ hence } \mathcal{M}_N \neq \emptyset.
\]

Based on \( \mathcal{M}_X \) and \( \mathcal{M}_N \), we define admissible classes of trading strategies. We start with \( \mathcal{A}_X \), the admissible class of hedging strategies in (17). A \( \mathcal{G}_t\)-predictable process \( \vartheta = \{\theta_t, t \in [0,T]\} \) is admissible by belonging to the following set:

\[
\mathcal{A}_X := \{ \vartheta : \vartheta \text{ is } X_t\text{-integrable}; \chi_T(\vartheta) \in L_2(\mathcal{P}); \forall \mathcal{P}^M \in \mathcal{M}_X, \{\chi_t(\vartheta), t \in [0,T]\} \text{ is a } \mathcal{P}^M\text{-martingale} \}.
\]  

(EC.5.10)

(Recall, \( \vartheta = \{\theta_t, t \in [0,T]\} \) and \( \chi_t(\vartheta) = \int_0^t \theta_s dX_s \).) Next, we define the set of all terminal wealth attainable by admissible trading strategies:

\[
\chi_T(\mathcal{A}_X) := \{ \chi_T(\vartheta) | \vartheta \in \mathcal{A}_X \}.
\]  

(EC.5.11)

We remark that \( \chi_T(\mathcal{A}_X) \) is closed in \( L^2(\mathcal{P}) \); refer to Lemma 2.6 and Theorem 2.8 of Černý and Kallsen (2008); and for a brief review on this, refer to Theorem A.1 of Wang and Wissel (2013). This property of \( \chi_T(\mathcal{A}_X) \) allows us to establish the following technical result, with proof collected in §EC.5.1.1.

**Lemma EC.2.** Let \( Z^M_t \) be defined in (19), with dynamics specified in (20) which is reiterated below:

\[
dZ^M_t = \zeta_t dX_t.
\]

Under Assumptions 4 and 5, \( \zeta_t \in \mathcal{A}_X \); in other words, \( \zeta_t \) is an admissible hedging strategy with respect to \( X_t \). Hence, by definition of \( \mathcal{M}_X \) in (EC.5.10), \( Z^M_t \) is a \( \mathcal{P}^\mathcal{M}\text{-martingale for each } \mathcal{P}^\mathcal{M} \) in \( \mathcal{M}_X \).

It will become clear later that Lemma EC.2 is crucial in establishing connection between \( \mathcal{M}_X \) and \( \mathcal{M}_N \), which plays a key role in solving the quadratic hedging problem.

Next, recall that \( N_t \) in (EC.5.6) can be viewed as asset prices *denominated* in \( Z^M_t \), hence we can also define admissible trading strategies with respect to \( N_t \). A \( \mathcal{G}_t\)-predictable process \( \varphi = \{\phi_t = (\phi^0_t, \phi^1_t), t \in [0,T]\} \) is admissible if it belongs to the following set:

\[
\mathcal{A}_N := \{ \varphi : \varphi \text{ is } N_t\text{-integrable and } Z^M_T \pi_T(\varphi) \in L_2(\mathcal{P}), \forall \mathcal{P}^R \in \mathcal{M}_N, \{\pi_t(\varphi), t \in [0,T]\} \text{ is a } \mathcal{P}^R\text{-martingale} \},
\]  

(EC.5.12)
where the notation parallels those for $\mathcal{A}_X$: $\varphi = \{\phi_t, t \in [0, T]\}$, with $\phi_t = (\phi^0_t, \phi^1_t)$ being a two-dimensional process adapted to $\mathcal{G}_t$. And

$$\pi_t(\varphi) := \int_0^t \phi_s \cdot dN_s = \int_0^t \phi^0_s dN^0_s + \int_0^t \phi^1_s dN^1_s. \quad \text{(EC.5.13)}$$

Similar to (EC.5.11), we define $\pi_T(\mathcal{A}_N) := \{\pi_T(\varphi) | \varphi \in \mathcal{A}_N\}$ to be the attainable terminal wealth by admissible strategies in $\mathcal{A}_N$.

Now, we establish bijection between $\mathcal{M}_X$ in (EC.5.8) and $\mathcal{M}_N$ in (EC.5.9), which will be used later in proving the key lemma of this section. The following lemma is a special case of Proposition 3.1 in Gourieroux et al. (1998), and our proof here will make explicit uses of Baye's formula based on Doob's martingale.

**Lemma EC.3.** Recall $P^M$ and $P^R$ are defined in (14) and (EC.5.1), respectively.

(i) For any given $\mathcal{M}_X$, the probability measure defined below is in $\mathcal{M}_N$.

$$\frac{dP^R}{dP} := \frac{dP^\mathcal{M} \cdot Z^M_T}{dP^\mathcal{M}} = \frac{dP^\mathcal{M}}{dP} \cdot \frac{dP^R}{dP^\mathcal{M}}.$$

(ii) For any $\mathcal{M}_N$, the probability measure defined below is in $\mathcal{M}_X$.

$$\frac{dP^M}{dP} := \frac{dP^\mathcal{R} \cdot Z^0_0}{dP^\mathcal{R}} = \frac{dP^R}{dP} \cdot \frac{1}{dP^\mathcal{R}}.$$

The proof is to check the conditions specified in (19) and (EC.5.8) for each case respectively, and we collect the details in §EC.5.1.1.

To this point, we are ready to present the key lemma of this section, which spells out an one-to-one relationship between the two admissible classes $\mathcal{A}_X$ in (EC.5.10) and $\mathcal{A}_N$ in (EC.5.12).

**Lemma EC.4.** (i) For any given $X_t$-admissible trading strategy $\vartheta = \{\theta_t, t \in [0, T]\} \in \mathcal{A}_X$, there exists an $N_t$-admissible strategy $\varphi = \{\phi_t = (\phi^0_t, \phi^1_t), t \in [0, T]\} \in \mathcal{A}_N$, such that $\forall t \in [0, T]$,

$$\frac{\chi^1_t(\vartheta)}{Z^M_t} = \pi_t(\varphi), \quad \text{and} \quad \phi_t = (\chi^1_t(\vartheta) - \theta_t X_t, \theta_t).$$

(ii) Conversely, given any $N_t$-admissible strategy $\varphi = \{\phi_t = (\phi^0_t, \phi^1_t), t \in [0, T]\} \in \mathcal{A}_N$, there exists an $X_t$-admissible trading strategy $\vartheta = \{\theta_t, t \in [0, T]\} \in \mathcal{A}_X$, such that $\forall t \in [0, T]$,

$$\frac{\chi^1_t(\vartheta)}{Z^0_0} = \pi_t(\varphi), \quad \text{and} \quad \theta_t = \zeta_t(\pi_t(\varphi) - \phi_t \cdot N_t) + \phi^1_t \quad \text{with } \zeta_t \text{ defined in (19)}.$$

(iii) Combining (i) and (ii), we have:

$$\pi_T(\mathcal{A}_N) = \frac{\chi_T(\mathcal{A}_X)}{Z^M_T} := \left\{ \frac{\chi_T(\vartheta)}{Z^M_T} : \vartheta \in \mathcal{A}_X \right\};$$

recall, $\chi_T(\mathcal{A}_X)$ is the set of attainable wealth defined in (EC.5.11) and $\pi_T(\mathcal{A}_N)$ is similarly defined.
Proof. We first show part (i) of this lemma. Given \( \vartheta = \{ \vartheta, t \in [0, T] \} \), write \( \chi_t = \chi_t(\vartheta) = \int_0^t \vartheta_s dX_s \) for lighter notation. Apply Itô’s Lemma on \( \frac{\chi_t}{Z_t^M} \), accounting for the dynamics of \( Z_t^M \), \( N_t^0 \) and \( N_t^1 \) in, respectively, (EC.5.2) and (EC.5.7):

\[
d\chi_t N_t^0 = \chi_t dN_t^0 + N_t^0 d\chi_t + d\chi_t dN_t^0
\]

\[
= \chi_t dN_t^0 + \theta_t N_t^0 \sigma_t X_t dB_t^M - \theta_t \sigma_t X_t N_t^0 (N_t^0 \psi_t) dt
\]

\[
= (\chi_t - \theta_t X_t) dN_t^0 + \left[ \theta_t X_t dN_t^0 + \theta_t N_t^0 \sigma_t X_t dB_t^M - \theta_t \sigma_t X_t N_t^0 (N_t^0 \psi_t) dt \right]
\]

\[
= (\chi_t - \theta_t X_t) dN_t^0 + \left[ (\theta_t X_t) [ -N_t^0 (N_t^0 \psi_t) dB_t^R] + \theta_t \sigma_t X_t N_t^0 [dB_t^M - (N_t^0 \psi_t) dt] \right]
\]

\[
= (\chi_t - \theta_t X_t) dN_t^0 + \theta_t ( - (N_t^0 \psi_t) N_t^1 + N_t^0 \sigma_t X_t ) dB_t^R
\]

\[
= (\chi_t - \theta_t X_t) dN_t^0 + \theta_t dN_t^1
\]

and this leads to the equality stated in (i).

The rest is to show \( \varphi \in A_N \). First, \( Z_t^M \int_0^T \phi_t \cdot dN_t = \chi_T(\vartheta) \in L^2(P) \), since \( \vartheta \in A_X \). Next, \( \forall P^R \in \mathcal{M}_N \), we have \( E^R[|\chi_T(\vartheta)|] = E[|\chi_T(\varphi)(dP^R / Z_t^M)|] < \infty \), by Cauchy-Schwarz inequality, the fact \( dP^R / Z_t^M \in L^2(P) \) (since \( P^R \in \mathcal{M}_N \)) and \( \chi_T(\vartheta) \in L^2(P) \) (by \( \vartheta \in A_X \)). Then \( \pi_T(\varphi) \in L^1(P^R) \) and we can take conditional expectation:

\[
E^R\left[ \frac{\chi_T(\vartheta)}{Z_T^M} \left| G_t \right. \right] = E\left[ \frac{\chi_T(\vartheta) dP^R}{(Z_T^M/Z_0^M)} \left| G_t \right. \right] \frac{1}{E\left[ \frac{dP^R}{dP} \left| G_t \right. \right] Z_0^M}.
\]

Applying change of measure formula:

\[
E\left[ \frac{dP^R}{dP} \left| Z_T^M / Z_0^M \right. \right] \left| G_t \right. \right] = Z_0^M E\left[ \frac{dP^R}{dP} N_0^0 \left| G_t \right. \right] \frac{1}{E\left[ \frac{dP^R}{dP} \left| G_t \right. \right]} E\left[ \frac{dP^R}{dP} \left| G_t \right. \right] 
\]

\[
= Z_0^M E\left[ \frac{dP^R}{dP} \left| G_t \right. \right] E\left[ \frac{dP^R}{dP} \left| G_t \right. \right]
\]

\[
= Z_0^M N_t^0 E\left[ \frac{dP^R}{dP} \left| G_t \right. \right] Z_0^M
\]

the above accounts for the fact that \( N_t^0 \) is a \( P^R \)-martingale.

Combining the above, we derive:

\[
E^R\left[ \frac{\chi_T(\vartheta)}{Z_T^M} \left| G_t \right. \right] = \left[ E\left[ \frac{\chi_T(\vartheta) dP^R}{(Z_T^M/Z_0^M)} \left| G_t \right. \right] \frac{1}{E\left[ \frac{dP^R}{dP} \left| G_t \right. \right] Z_0^M} \right] N_t^0
\]

\[
= E^M[\chi_T(\vartheta)|G_t] N_t^0
\]

\[
= \chi_t(\vartheta) / Z_t^M,
\]
where $\mathbb{P}^M$ is defined by:

$$
\frac{d\mathbb{P}^M}{d\mathbb{P}} := \frac{d\mathbb{P}_0^M}{d\mathbb{P}} \frac{Z_t^M}{Z_0^M},
$$

and by Lemma EC.3, $\frac{d\mathbb{P}^M}{d\mathbb{P}} \in \mathcal{M}_X$. Then, since $\vartheta \in \mathcal{A}_X$, $\chi_t(\vartheta)$ must be a martingale under $\mathbb{P}^M$, which gives the last equality in the derivation above. Then, the above implies that $\pi_t(\varphi) = \chi_t(\vartheta)/Z_t^M$ is a $\mathbb{P}^R$-martingale $\forall \mathbb{P}^R \in \mathcal{M}_N$, hence $\varphi \in \mathcal{A}_N$ and (i) is proved.

Now we prove part (ii). Given $\varphi = \{\phi_t = (\phi^0_t, \phi^1_t), t \in [0, T]\} \in \mathcal{A}_N$, we apply Itô’s Lemma on $Z_t^M \int_0^t \phi_s \cdot dN_s$:

$$
dZ_t^M \int_0^t \phi_s \cdot dN_s = Z_t^M (\phi_t^0 dN_t^0 + \phi_t^1 dN_t^1) + \left( \int_0^t \phi_s \cdot dN_s \right) dB_t^M + \phi_t^0 dZ_t^M dN_t^0 + \phi_t^1 dZ_t^M dN_t^1
$$

$$
= Z_t^M \phi_t^0 dN_t^0 + Z_t^M \phi_t^1 dN_t^1 + \left( \int_0^t \phi_s \cdot dN_s \right) dB_t^M - \phi_t^0 (\psi_t N_t^0)^2 dt
$$

$$
- \phi_t^1 (\psi_t N_t^1) [\psi_t N_t^1 - \sigma_t X_t] dt. \tag{EC.5.15}
$$

Now, use (EC.5.7) to express $dN_t^0$ and $dN_t^1$ (and choose the representation involving $dB_t^M$), then it is straightforward to verify that $dt$-term vanishes and the equation above reduces to:

$$
dZ_t^M \int_0^t \phi_s \cdot dN_s = \left[ - \phi_t^0 N_t^0 \psi_t - \phi_t^1 (N_t^0 X_t \psi_t - \sigma_t) + \psi_t \int_0^t \phi_s \cdot dN_s \right] dB_t^M
$$

$$
= \left[ - \phi_t^0 N_t^0 \zeta_t - \phi_t^1 (N_t^0 X_t \zeta_t - 1) + \zeta_t \int_0^t \phi_s \cdot dN_s \right] \sigma_t dB_t^M
$$

$$
= \left[ \zeta_t \left( \int_0^t \phi_s \cdot dN_s - \phi_t^0 N_t^0 - \phi_t^1 N_t^1 \right) + \phi_t^1 \right] dX_t, \tag{EC.5.16}
$$

where the second equality uses $N_t^1 = X_t N_t^0$, as well as the relation between $\zeta_t$ and $\psi_t$ in (EC.5.3); the third equality uses the $\mathbb{P}^M$-dynamics of $X_t$: $dX_t = \sigma_t X_t dB_t^M$. The integrand with respect to $dX_t$ in the last line above gives the expression for $\theta_t$ specified in (ii).

What remains is to show $\vartheta$ stated in (ii) is in $\mathcal{A}_X$. First, note $\chi_T(\vartheta) = Z_T^M \pi_T(\varphi) \in L^2(\mathbb{P})$ follows from $\varphi \in \mathcal{A}_N$. Next, $\forall \mathbb{P}^M \in \mathcal{M}_X$, define $\mathbb{P}^R$ by

$$
\frac{d\mathbb{P}^R}{d\mathbb{P}} := \frac{d\mathbb{P}^M}{d\mathbb{P}} \frac{Z_T^M}{Z_0^M}.
$$

By Lemma EC.3, $\mathbb{P}^R \in \mathcal{M}_X$. Note $E^M[\chi_T(\vartheta)] = E \left[ \frac{d\mathbb{P}^M}{d\mathbb{P}} \pi_T(\varphi) Z_t^M \right] < \infty$ is easily verified by using Cauchy-Schwarz inequality, $\frac{d\mathbb{P}^M}{d\mathbb{P}} \in L^2(\mathbb{P})$ and $\pi_T(\varphi) Z_t^M \in L^2(\mathbb{P})$ (since $\varphi \in \mathcal{A}_N$). Hence we compute the following conditional expectation under $\mathbb{P}^M$ and apply change of measure:

$$
E^M[Z_T^M \pi_T(\varphi)|G_t] = E \left[ (Z_T^M \pi_T(\varphi) \frac{d\mathbb{P}^Q}{d\mathbb{P}}) | F_t \right] \frac{1}{E[\frac{d\mathbb{P}^M}{d\mathbb{P}} | F_t]} \frac{d\mathbb{P}^Q}{d\mathbb{P}} | G_t \right].
$$

By Baye’s formula,

$$
E \left[ \frac{d\mathbb{P}^R}{d\mathbb{P}} | G_t \right] = E \left[ \frac{d\mathbb{P}^M}{d\mathbb{P}} \frac{Z_T^M}{Z_0^M} | G_t \right] = E^M \left[ \frac{Z_T^M}{Z_0^M} | G_t \right] E \left[ \frac{d\mathbb{P}^M}{d\mathbb{P}} | G_t \right] = \frac{Z_T^M}{Z_0^M} E \left[ \frac{d\mathbb{P}^M}{d\mathbb{P}} | G_t \right];
$$
where the first quality uses definition of \( \frac{db^R}{d\mathbb{P}} \) above; the second equality switches measure between \( \mathbb{P}^M \) and \( \mathbb{P} \); and the last equality is by the fact that \( Z_t^M \) is a martingale under \( \mathbb{P}^M \) as implied by Lemma EC.3.

Combining all above,

\[
E^M[Z_t^M \pi_T(\varphi) \mid \mathcal{G}_t] = E \left[ \frac{Z_t^M \pi_T(\varphi) \frac{d\mathbb{P}^M}{d\mathbb{P}} \mid \mathcal{G}_t}{\frac{d\mathbb{P}^M}{d\mathbb{P}} \mid \mathcal{G}_t} \right] = E \left[ \frac{Z_t^M \pi_T(\varphi) \frac{d\mathbb{P}^M}{d\mathbb{P}} \mid \mathcal{G}_t}{\frac{d\mathbb{P}^M}{d\mathbb{P}} \mid \mathcal{G}_t} \right] Z_t^M = \pi_t(\varphi) Z_t^M
\]

where the last equality is due to \( \pi_t(\varphi) \) is a \( \mathbb{P}^R \)-martingale implied by \( \varphi \in \mathcal{A}_N \) (recall, \( \mathbb{P}^R \in \mathcal{M}_N \) by Lemma EC.3). This concludes that \( \chi_t(\vartheta) = \pi_t(\varphi) Z_t^M \) is a \( \mathbb{P}^M \)-martingale \( \forall \mathbb{P}^M \in \mathcal{M}_X \); hence \( \vartheta \) defined in (ii) is in \( \mathcal{A}_X \). This completes the proof of (ii). \( \square \)

**EC.5.1.1. Proof of Lemma EC.2 and and EC.3.** We first prove Lemma EC.2 and here is an outline of the proof. Recall, \( \chi_T(\mathcal{A}_X) \), the set of wealth attainable by admissible strategies with \( X_t \) is introduced in (EC.5.11), which is a nonempty set closed in \( L^2(\mathbb{P}) \). Our approach is to first show \( Z_T^M - Z_0^M \in \chi_T(\mathcal{A}_X) \). Once this is established, then \( \exists \{\theta, t \in [0, T]\} \in \mathcal{A}_X \) such that \( Z_T^M - Z_0^M \) is in (ii). The second equality uses the first equality follows the definition of \( \mathcal{A}_X \); the last equality uses the admissibility of \( \theta_t \); an admissible strategy attaining \( Z_T^M - Z_0^M \); the last equality uses the admissibility of \( \theta_t \); the induced wealth process is an martingale under any measure from \( \mathcal{M}_X \) (defined in (EC.5.8)), and in particular, recall that \( \mathbb{P}^M \in \mathcal{M}_X \). In this way, we establish that \( \int_0^t \zeta_s dX_s = \int_0^t \theta_s dX_s \); note both integrals are continuous martingales under \( \mathbb{P}^M \), hence \( \zeta_t = \theta_t \). Thus, we can establish \( \zeta_t \in \mathcal{A}_X \), which in turn implies \( Z_t^M = Z_0^M + \int_0^t \zeta_s dX_s \) is an martingale under any measure from \( \mathcal{M}_X \), and the lemma is proved.

Now we proceed with showing \( Z_T^M - Z_0^M \in \chi_T(\mathcal{A}_X) \). Recall, \( \chi_T(\mathcal{A}_X) \) is closed in \( L^2(\mathbb{P}) \), hence it is sufficient to find a sequence of elements in this set, with limit (in \( L^2(\mathbb{P}) \)) as \( Z_T^M - Z_0^M \), then the desired result will follow from the closedness. To do this, we follow the approach similar to that in the proof of Theorem 3.5 in Wang and Wissel (2013).
Define the sequence of $\mathcal{G}_t$-stopping times:

$$
\tau_k := \inf\{t \geq 0 : |\eta_t| \geq k\} \wedge T; \ k \in \mathbb{N};
$$

(EC.5.17)

recall $\eta_t$ is the market price of risk process defined in (20). Since $\eta_t$ is continuous, we have $\tau_k \uparrow T$ as $k \to \infty$. Clearly,

$$
Z_{\tau_k} \to Z_T \quad \text{a.s.}
$$

(EC.5.18)

Recall $Z_t$ is assumed to be a continuous square-integrable martingale under $\mathbb{P}$ (by Assumption 5), hence Doob’s $L^p$ inequality implies:

$$
\mathbb{E} \left[ \sup_{t \in [0,T]} Z_t^2 \right] \leq 2 \mathbb{E}(Z_T^2) < \infty;
$$

(EC.5.19)

Clearly, $\sup_{k \in \mathbb{N}} Z_{\tau_k}^2 \leq \sup_{t \in [0,T]} Z_t^2$ (note $Z_t$ is a positive process), hence (EC.5.19) implies

$$
\mathbb{E} \left[ \sup_{k \in \mathbb{N}} Z_{\tau_k}^2 \right] < \infty.
$$

The above invokes dominated convergence in (EC.5.18), and we establish

$$
Z_{\tau_k} \to Z_T \quad \text{in } L^2(\mathbb{P}).
$$

(EC.5.20)

Clearly, $Z_{\tau_k} \in L^2(\mathbb{P})$, hence also in $L^1(\mathbb{P}^M)$. So, for each $k \in \mathbb{N},$ define

$$
M_t^{(k)} := \mathbb{E}^M(Z_{\tau_k}|\mathcal{G}_t) = \frac{1}{Z_t} \mathbb{E}(Z_T Z_{\tau_k}, \mathcal{G}_t) = M_0^{(k)} + \int_0^t \theta_s^{(k)} dX_s.
$$

(EC.5.21)

the first equality is the change of measure formula, and the second equality is martingale representation, with $\theta_t^{(k)}$ being a process predictable to $\mathcal{G}_t$. In particular, note $M_T^{(k)} = Z_{\tau_k}$, and also $M_0^{(k)} = \mathbb{E}(Z_T Z_{\tau_k}) \to \mathbb{E}(Z_T^2) = Z_T^M$ by (EC.5.20); so we have

$$
M_T^{(k)} - M_0^{(k)} \to Z_T - Z_T^M = Z_T^M - Z_0^M \quad \text{in } L^2(\mathbb{P});
$$

Now we have found a sequence of elements, $M_t^{(k)} - M_0^{(k)}$, converging to $Z_T^M - Z_0^M$ in $L^2(\mathbb{P})$. As outlined, the next step is to show $M_t^{(k)} - M_0^{(k)} \in \chi_T(A_X)$ for each $k$. To this end, fix $k$ and $\mathbb{P}^M \in \mathcal{M}_X$, we will show $M_t^{(k)} - M_0^{(k)} = \int_0^t \theta_s^{(k)} dX_s$ defined in (EC.5.21) is a martingale in $\mathbb{P}^M$, as follows. Clearly, $\int_0^t \theta_s^{(k)} dX_s$ is a $\mathbb{P}^M$-local-martingale, since $X_t$ is a martingale under this probability measure. To proceed, the crux is to examine the following. Recall $Z_t$ has the exponential form as defined in (14), hence

$$
Z_{t \wedge \tau_k} = \exp \left\{ - \int_0^{t \wedge \tau_k} \eta_s dB_s - \frac{1}{2} \int_0^{t \wedge \tau_k} \eta_s^2 ds \right\}
$$

$$
= \exp \left\{ - \int_0^t \eta_s 1\{s \leq \tau_k\} dB_s - \frac{1}{2} \int_0^t \eta_s^2 1\{s \leq \tau_k\} ds \right\}
$$
The above immediately implies \( W_t \) is a \( \mathbb{P}^M \)-Brownian-motion \( (B_t^M = \eta_t dt + dB_t) \), and \( \hat{\eta}_t \) is defined as \( \hat{\eta}_t = \eta_t \mathbf{1}\{s \leq \tau_k\} \) (note \( \hat{\eta}_s \eta_s = \hat{\eta}_s^2 \)); also note \( \hat{\eta}_t \leq k \) by definition of \( \tau_k \) in (EC.5.17). Denote

\[
W_t = \exp \left\{ - \int_0^t \hat{\eta}_s dB^M_s - \frac{1}{2} \int_0^t \hat{\eta}_s^2 ds \right\} \quad \text{and} \quad C_t = \exp \left\{ \int_0^t \hat{\eta}_s^2 ds \right\};
\]

then the expression above for \( Z_{t \wedge \tau_k} \) becomes

\[
Z_{t \wedge \tau_k} = W_t C_t.
\]

For \( C_t \), since each \( \hat{\eta}_s \) is bounded by \( k \), we have

\[
1 \leq C_t \leq e^{k^2 T}, \forall t \in [0, T].
\]

The above immediately implies \( W_t \leq Z_{t \wedge \tau_k} \).

Next, clearly \( W_t \) is a local martingale under \( \mathbb{P}^M \), and since \( \hat{\eta}_t \leq k \), we have

\[
\mathbb{E}^M \left[ \frac{1}{2} \exp \left\{ \int_0^T \hat{\eta}_t^2 dt \right\} \right] \leq e^{\frac{1}{2} k^2 T} < \infty;
\]

in other words, Novikov’s condition holds, indicating that \( W_t \) is a \( \mathbb{P}^M \)-martingale.

Combining the above, we have

\[
0 \leq M^{(k)}_0 + \int_0^t \theta^{(k)}_s dX_s = M^{(k)}_t \\
= \mathbb{E}^M (M^{(k)}_T | \mathcal{G}_t) = \mathbb{E}^M (Z_{\tau_k} | \mathcal{G}_t) \\
= \mathbb{E}^M (Z_{t \wedge \tau_k} | \mathcal{G}_t) = \mathbb{E}^M (W_T C_T | \mathcal{G}_t) \\
\leq e^{k^2 T} W_t \leq e^{k^2 T} Z_{t \wedge \tau_k} \\
\leq e^{k^2 T} \sup_{t \in [0,T]} Z_t \tag{EC.5.22}
\]

the first line is just the definition in (EC.5.21), and the second line uses \( M^{(k)}_T = Z_{\tau_k} \); the first equality on the third line uses the obvious fact \( \tau_k \wedge T = \tau_k \), and the second equality makes use of the representation of \( Z_{t \wedge \tau_k} \) established above; the fourth line is based on the bound on \( C_T \) and the martingale property of \( W_t \) established above, as well as \( W_t \leq Z_{t \wedge \tau_k} \) as shown above.

Now, combining (EC.5.22) and (EC.5.19) implies \( \sup_{t \in [0,T]} \int_0^t \theta^{(k)}_s dX_s \in L^2(\mathbb{P}) \). Finally, by Cauchy-Schwarz inequality and \( d\mathbb{P}^M / d\mathbb{P} \in L^2(\mathbb{P}) \)

\[
\mathbb{E}^{\mathbb{P}^M} \left[ \sup_{t \in [0,T]} \int_0^t \theta^{(k)}_s dX_s \right] = \mathbb{E} \left[ \frac{d\mathbb{P}^M}{d\mathbb{P}} \right] \sup_{t \in [0,T]} \int_0^t \theta^{(k)}_s dX_s \]
What remains is to show that
\[ N_t^k - M_t^k \]
\[ \text{is a } \mathbb{P}^\mathcal{M} \text{-martingale.} \]
This establishes the desired result, and proves Lemma EC.2.

Based on Lemma EC.2, we are now ready to prove Lemma EC.3.

For part (i), suppose \( \mathbb{P}^\mathcal{M} \in \mathcal{M}_X \) is given and \( \mathbb{P}^\mathcal{R} \) follows the stated definition. First note that clearly \( \frac{d\mathbb{P}^\mathcal{R}}{d\mathbb{P}} > 0 \) almost surely since \( \frac{d\mathbb{P}^\mathcal{M}}{d\mathbb{P}} > 0 \) and \( Z_T^\mathcal{M} = Z_T \) takes exponential form (see (14)); hence \( \frac{d\mathbb{P}^\mathcal{R}}{d\mathbb{P}} \sim \mathbb{P} \). Next, we have
\[
\mathbb{E}\left[ \left( \frac{d\mathbb{P}^\mathcal{R}}{d\mathbb{P}} \right)^2 \right] = \mathbb{E}\left[ \left( \frac{d\mathbb{P}^\mathcal{M} Z_T^\mathcal{M}}{d\mathbb{P} Z_0^\mathcal{M}} \right)^2 \right] = \mathbb{E}^\mathcal{M}\left[ \frac{Z_T^\mathcal{M}}{Z_0^\mathcal{M}} \right] = 1.
\]
The last equality follows from Lemma EC.2, which indicates that \( Z_t^\mathcal{M} \) is a \( \mathbb{P}^\mathcal{M} \)-martingale since \( \mathbb{P}^\mathcal{M} \in \mathcal{M}_X \). Next, derive
\[
\mathbb{E}\left[ \left( \frac{1}{Z_T^\mathcal{M}} \frac{d\mathbb{P}^\mathcal{R}}{d\mathbb{P}} \right)^2 \right] = \mathbb{E}\left[ \left( \frac{1}{Z_T^\mathcal{M}} \frac{d\mathbb{P}^\mathcal{M} Z_T^\mathcal{M}}{d\mathbb{P} Z_0^\mathcal{M}} \right)^2 \right] = \left( \frac{1}{Z_T^\mathcal{M}} \right)^2 \mathbb{E}\left[ \left( \frac{d\mathbb{P}^\mathcal{M}}{d\mathbb{P}} \right)^2 \right] < \infty.
\]
The \( < \) follows from the fact that \( \mathbb{P}^\mathcal{M} \in \mathcal{M}_X \); hence the above implies \( \frac{1}{Z_T^\mathcal{M}} \frac{d\mathbb{P}^\mathcal{R}}{d\mathbb{P}} \in L^2(\mathbb{P}) \).
What remains is to show that \( N_t^0 \) and \( N_t^1 \) are \( \mathbb{P}^\mathcal{R} \)-martingales. First note
\[
\mathbb{E}^\mathcal{R}(N_t^0) = \mathbb{E}\left( \frac{d\mathbb{P}^\mathcal{M} Z_T^\mathcal{M}}{d\mathbb{P} Z_0^\mathcal{M}} \frac{1}{Z_T^\mathcal{M}} \right) = \frac{1}{Z_T^\mathcal{M}} < \infty;
\]
hence we can apply conditional expectation and compute
\[
\mathbb{E}^\mathcal{R}(N_t^0 | \mathcal{G}_t) = \mathbb{E}\left[ \frac{d\mathbb{P}^\mathcal{M} Z_T^\mathcal{M}}{d\mathbb{P} Z_0^\mathcal{M}} \frac{1}{Z_T^\mathcal{M}} \right] \mathbb{E}\left( \frac{d\mathbb{P}^\mathcal{M} Z_T^\mathcal{M}}{d\mathbb{P} Z_0^\mathcal{M}} \left| \mathcal{G}_t \right. \right) = \frac{1}{Z_T^\mathcal{M}} = N_t^0;
\]
the first equality applies change of measure from \( \mathbb{P}^\mathcal{R} \) to \( \mathbb{P} \), and the second equality follows from changing measure from \( \mathbb{P} \) to \( \mathbb{P}^\mathcal{M} \) on the term \( \mathbb{E}\left( \frac{d\mathbb{P}^\mathcal{M} Z_T^\mathcal{M}}{d\mathbb{P} Z_0^\mathcal{M}} \left| \mathcal{G}_t \right. \right) \); the third equality again uses the fact that \( Z_t^\mathcal{M} \) is \( \mathbb{P}^\mathcal{M} \)-martingale based on Lemma EC.2. From above, we can conclude that \( N_t^0 \) is a martingale under \( \mathbb{P}^\mathcal{R} \). Similar derivation applies to \( N_t^1 \) as follows. First note that
\[
\mathbb{E}^\mathcal{R}(N_t^1) = \mathbb{E}\left( \frac{d\mathbb{P}^\mathcal{M} Z_T^\mathcal{M} X_T}{d\mathbb{P} Z_0^\mathcal{M} Z_T^\mathcal{M}} \right) = \frac{1}{Z_T^\mathcal{M}} \mathbb{E}^\mathcal{M}(X_T) = \frac{X_0}{Z_0^\mathcal{M}} < \infty;
\]
the last equality accounts for the fact that \( \mathbb{P}^\mathcal{M} \) is a martingale measure with respect to \( X_t \). Now we can compute the conditional expectation:
\[
\mathbb{E}^\mathcal{R}(N_t^1 | \mathcal{G}_t) = \mathbb{E}\left( \frac{d\mathbb{P}^\mathcal{M} Z_T^\mathcal{M} X_T}{d\mathbb{P} Z_0^\mathcal{M} Z_T^\mathcal{M}} \left| \mathcal{G}_t \right. \right) \mathbb{E}\left( \frac{d\mathbb{P}^\mathcal{M} Z_T^\mathcal{M}}{d\mathbb{P} Z_0^\mathcal{M}} \left| \mathcal{G}_t \right. \right) = \frac{X_0}{Z_0^\mathcal{M}} < \infty.
\]
\[= \mathbb{E}^\bar{M}(X_T | \mathcal{G}_t) \mathbb{E}\left(\frac{d\mathbb{P}^\bar{M}}{d\mathbb{P}} \bigg| \mathcal{G}_t\right) \frac{1}{\mathbb{E}^M(Z_t^M | \mathcal{G}_t) \mathbb{E}\left(\frac{d\mathbb{P}^\bar{M}}{d\mathbb{P}} \bigg| \mathcal{G}_t\right)}\]
\[= \frac{X_t}{Z_t^M} = N_t^1; \quad \text{(EC.5.24)}\]

the first equality applies change of measure formula on \(\mathbb{E}^\bar{R}(N_t^1 | \mathcal{G}_t)\); the second equality does the same for \(\mathbb{E}^\bar{M}(X_T | \mathcal{G}_t)\) and \(\mathbb{E}^\bar{M}(Z_t^M | \mathcal{G}_t)\), respectively; the third equality uses the fact that \(Z_t^M\) is a \(\mathbb{P}^\bar{M}\)-martingale. Hence, \(N_t^1\) is also a martingale under \(\mathbb{P}^\bar{M}\) by the derivation above. To this point, we have checked that \(\mathbb{P}^\bar{R}\) satisfies conditions specified in \(\mathcal{M}_N\), hence belongs to this set; this proves (i).

For part (ii), the proof is analogous. Suppose \(\mathbb{P}^R \in \mathcal{M}_N\) is given, and define \(\mathbb{P}^\bar{M}\) as stated. First note by the same argument as that for part (i), \(\mathbb{P}^\bar{M} > 0\) almost surely, hence equivalent to \(\mathbb{P}\). And,

\[\mathbb{E}\left[\frac{d\mathbb{P}^\bar{M}}{d\mathbb{P}}\right] = \mathbb{E}\left[\frac{d\mathbb{P}^R Z_0^M}{d\mathbb{P} Z_T^M}\right] = Z_0^M \mathbb{E}^\bar{R}(N_T^0) = Z_0^M N_0^1 = 1;\]

the third equality uses the fact that \(\mathbb{P}^\bar{R}\) is a martingale measure for \(N_t^0\) and \(N_t^1\). Next, check

\[\mathbb{E}\left[\left(\frac{d\mathbb{P}^\bar{M}}{d\mathbb{P}}\right)^2\right] = (Z_0^M)^2 \mathbb{E}\left[\left(\frac{d\mathbb{P}^R}{d\mathbb{P}} \cdot \frac{1}{Z_T^M}\right)^2\right] < \infty;\]

< follows from \(\frac{1}{Z_T^M} \frac{d\mathbb{P}^R}{d\mathbb{P}} \in L^2(\mathbb{P})\); hence the above implies \(\frac{d\mathbb{P}^\bar{M}}{d\mathbb{P}} \in L^1(\mathbb{P})\).

Then, the rest is to show \(X_t\) is a martingale under \(\mathbb{P}^\bar{M}\). We start with checking the integrability condition:

\[\mathbb{E}^\bar{M}(X_T) = \mathbb{E}\left(\frac{d\mathbb{P}^R Z_0^M}{d\mathbb{P} Z_T^M} X_T\right) = Z_0^M \mathbb{E}^\bar{R}(N_T^1) = Z_0^M N_0^1 = X_0 < \infty;\]

the third equality follows from that \(\mathbb{P}^\bar{R} \in \mathcal{M}_N\) is a martingale measure for \(N_t^0\) and \(N_t^1\). Next, compute the conditional expectation

\[\mathbb{E}^\bar{M}(X_T | \mathcal{G}_t) = \mathbb{E}\left[\frac{d\mathbb{P}^R Z_0^M}{d\mathbb{P} Z_T^M} X_T | \mathcal{G}_t\right] \frac{1}{\mathbb{E}\left[\frac{d\mathbb{P}^\bar{M}}{d\mathbb{P}} \frac{Z_0^M}{Z_T^M} | \mathcal{G}_t\right]};\]
\[= \mathbb{E}\left[\frac{d\mathbb{P}^R}{d\mathbb{P}} N_t^1 | \mathcal{G}_t\right] \frac{1}{\mathbb{E}\left[\frac{d\mathbb{P}^R}{d\mathbb{P}} N_T^0 | \mathcal{G}_t\right]};\]
\[= \mathbb{E}^\bar{R}(N_T^1 | \mathcal{G}_t) \mathbb{E}\left[\frac{d\mathbb{P}^R}{d\mathbb{P}} | \mathcal{G}_t\right] \frac{1}{\mathbb{E}\left[\frac{d\mathbb{P}^\bar{M}}{d\mathbb{P}} | \mathcal{G}_t\right] \mathbb{E}^\bar{R}(N_T^0 | \mathcal{G}_t)}\]
\[= \frac{N_t^1}{N_t^0} = X_t; \quad \text{(EC.5.25)}\]

the first equality applies change of measure formula on \(\mathbb{E}^\bar{M}(X_T | \mathcal{G}_t)\); the second equality does the same, respectively, for \(\mathbb{E}^\bar{R}(N_T^1 | \mathcal{G}_t)\) and \(\mathbb{E}^\bar{R}(N_T^0 | \mathcal{G}_t)\); the third equality recognizes that \(N_t^0\) and \(N_t^1\) are martingales under \(\mathbb{P}^\bar{R}\). This concludes that \(X_t\) is a \(\mathbb{P}^\bar{M}\)-martingale and proves part (ii). \(\Box\)
EC.5.2. Proof of Theorem 1

In this section, we derive the optimal hedging strategy, $\theta^*_t$, of (25) as well as the associated minimum variance function $B(m, P, R)$, and thereby provide proofs to Theorem 1; all derivations are based on results established in §EC.5.1.

We will first transform the equality-constrained problem in (17) to an equivalent unconstrained quadratic hedging problem, and then solve the latter by applying the numeraire-based technique. Define

$$A(\lambda) := \inf_{\vartheta \in \mathcal{A}_X} \mathbb{E}\left[\left(\lambda - H_T(P, R) - \chi_T(\vartheta)\right)^2\right],$$

(EC.5.26)

where $A(\lambda)$ relates to $B(m, P, R)$ by (refer to Proposition 6.6.5 in Pham (2009)):

$$B(m, P, R) = \max_{\lambda} \left[A(\lambda) - (m - \lambda)^2\right].$$

(EC.5.27)

In addition, the optimal hedging strategy induced by the problem in (EC.5.26), with $\lambda$ being set as the optimal solution to the maximization problem (EC.5.27), is also optimal to the problem (17).

We will show $A(\lambda)$ takes the following expression:

$$A(\lambda) = \frac{[\lambda - V_0(P, R)]^2}{Z_0^M} + \int_0^T \mathbb{E}\left[\frac{Z_t}{Z_t^M} \delta_t^2(P, R)\right] dt,$$

(EC.5.28)

where $V_0(P, R)$ and $\delta_t(P, R)$ are terms involved in the martingale representation of $V_t(P, R)$ in (21); in particular, $V_0(P, R) = \mathbb{E}^M[H_T(P, R)]$, and $\delta_t(P, R)$ is defined in (24). $Z_t$ and $Z_t^M$ follow (14) and (19), respectively.

In (EC.5.28), $\lambda$ only enters the first component as a quadratic term; the second component is independent from $\lambda$. Thus, the minimization problem in (EC.5.27) has a quadratic objective function, since both $A(\lambda)$ expressed in (EC.5.28) and $(m - \lambda)^2$ are quadratic functions in $\lambda$, and it is straightforward to verify that the $\lambda$ specified in (26) solves the right hand side of (EC.5.27) and gives the expression of $B(m, P, R)$ in (27).

Starting from here, we begin to prove (EC.5.28) by deriving solution to the hedging problem in (EC.5.26). Write $\hat{H}_T(\lambda) := \lambda - H_T$, and start with definition of $A(\lambda)$ in (EC.5.26):

$$A(\lambda) = \inf_{\vartheta \in \mathcal{A}_X} \mathbb{E}\left[\left(\hat{H}_T(\lambda) - \chi_T(\vartheta)\right)^2\right]$$

$$= \inf_{\vartheta \in \mathcal{A}_X} Z_0^M \mathbb{E}\left[\left(\frac{Z_t^M}{Z_0^M} \left(\hat{H}_T(\lambda) - \frac{\chi_T(\vartheta)}{Z_t^M}\right)^2\right)\right]$$

$$= Z_0^M \inf_{\vartheta \in \mathcal{A}_X} \mathbb{E}^R\left[\left(\frac{\hat{H}_T(\lambda)}{Z_t^M} - \frac{\chi_T(\vartheta)}{Z_t^M}\right)^2\right];$$

(EC.5.29)

recall the probability measure $P$ is defined in (EC.5.1).
Continue with (EC.5.29), by Lemma EC.4 part (iii),

\[ A(\lambda) = Z_0^M \inf_{\varphi \in \mathcal{A}_N} E^R \left[ \left( \frac{\hat{H}_T(\lambda)}{Z_T^M} - \int_0^T \phi_t \cdot dN_t \right)^2 \right]. \]  

(EC.5.30)

A natural next step is to express \( \frac{\hat{H}_T(\lambda)}{Z_T^M} \) as a stochastic integral with respect to \( N_t \), so that we can choose the optimal \( \phi_t \) based on this representation. By the fact that \( \hat{H}_T(\lambda) \) is bounded hence has finite second moment in \( P \), it is easy to check that \( \frac{\hat{H}_T(\lambda)}{Z_T^M} \) has finite second moment in \( P^R \).

Hence, we are able to define the Doob’s martingale:

\[ \hat{M}_t := E^R \left[ \frac{\hat{H}_T(\lambda)}{Z_T^M} \right | \mathcal{F}_t], \]  

(EC.5.31)

note \( \frac{\hat{H}_T(\lambda)}{Z_T^M} = \hat{M}_T \). Furthermore, \( \hat{M}_t \) is a square-integrable martingale under \( P^R \).

Now, Since \( N_0^0 \) and \( N_1^1 \) are both \( P^R \)-martingales, applying martingale representation on \( \hat{M}_t \) get:

\[ \hat{M}_t = E^R \left[ \frac{\hat{H}_T(\lambda)}{Z_T^M} \right] + \int_0^t \phi_s^H \cdot dN_s + \int_0^t \gamma_s \cdot d\tilde{B}_s, \]  

(EC.5.32)

where \( \phi_s^H \) and \( \gamma_t \) are some adapted processes to \( \mathcal{F}_t \). Note the representation in (EC.5.32) has used the fact that \( \tilde{B}_t \) is a Brownian motion under \( P^R \) since the market risk of price process associated with it is 0.

Substitute (EC.5.32) to (EC.5.30) and take into the consideration that \( N_t \) and \( \tilde{B}_t \) are independent, we can further expand \( A(\lambda) \):

\[ A(\lambda) = Z_0^M \inf_{\varphi \in \mathcal{A}_N} \left[ E^R \left[ \left( \frac{\hat{H}_T(\lambda)}{Z_T^M} \right) + \int_0^T \gamma_t d\tilde{B}_t + \int_0^T \phi_t^H \cdot dN_t - \int_0^T \phi_t \cdot dN_t \right)^2 \right] \]

\[ = Z_0^M \inf_{\varphi \in \mathcal{A}_N} \left[ \left( \int_0^T \phi_t^H \cdot dN_t \right)^2 + \left( \int_0^T \phi_t^H \cdot dN_t - \int_0^T \phi_t \cdot dN_t \right)^2 \right]. \]  

(EC.5.33)

To reach the second line of above, note the cross term is zero, following the fact that \( \int_0^t \gamma_s \cdot d\tilde{B}_s \), \( \int_0^t \phi_s^H \cdot dN_s \) and \( \int_0^t \phi_s \cdot dN_s \) are all square-integrable martingales under \( P^R \). Specifically, for \( \int_0^t \gamma_s \cdot d\tilde{B}_s \) and \( \int_0^t \phi_s^H \cdot dN_s \), this follows from (EC.5.32) and that \( \hat{M}_t \) is a \( P^R \)-square-integrable martingale.

For \( \int_0^t \phi_s \cdot dN_s \), this follows from the \( N_t \)-admissibility of \( \phi_t \); \( \int_0^t \phi_s \cdot dN_s \) is \( P^R \)-martingale and \( \int_0^T \phi_t \cdot dN_t \in L^2(P) \) (since \( Z_T^M \int_0^T \phi_t \cdot dN_t \in L^2(P) \)). Hence, \( \int_0^t \gamma_s \cdot d\tilde{B}_s \) and \( \int_0^t (\phi_s^H - \phi_s) \cdot dN_s \) are two square-integrable martingales under \( P^R \), and they are independent since \( N_t \) is adapted to \( \sigma(B_t) \), which is independent from \( \tilde{B}_t \). This makes the cross term vanish.

Two results follow (EC.5.33). First, it is obvious that the optimal \( \phi_t \) of this problem, denoted by \( \hat{\phi}_t = (\phi_t^0, \phi_t^1) \), should be set as:

\[ \hat{\phi}_t = \phi_t^H; \]  

(EC.5.34)
substituting this to (EC.5.33), apply Itô’s isometry and switch measure from $P^R$ to $P$, we have:

$$A(\lambda) = Z_0^M E^R \left[ \left( E^R[\frac{\hat{H}_T(\lambda)}{Z_T^M}] + \int_0^T \gamma_t d\hat{B}_t \right)^2 \right]$$

$$= Z_0^M \left( E^M[\frac{\hat{H}_T(\lambda)}{Z_T^M}] \right)^2 + Z_0^M \int_0^T E^M \left[ \gamma_t^2 \cdot \frac{Z_T^M}{Z_0^M} \right] dt$$

$$= \frac{(\lambda - V_0)^2}{Z_0^M} + \int_0^T E \left( \gamma_t^2 Z_t Z_t^M \right) dt. \quad (EC.5.35)$$

The expression above for $A(\lambda)$ does not coincide with (EC.5.28) yet (the second term takes a different form); we will come back to this later.

The other result from (EC.5.33) is the expression for the optimal hedging strategy $\theta_t^*$. With (EC.5.34), invoking part (ii) of Lemma EC.4, we have the expression for $\theta_t^*$:

$$\theta_t^* = \zeta_t (\int_0^t \phi_s^* \cdot dN_s - \phi_t^* \cdot N_t) + \phi_t^1, \quad (EC.5.36)$$

and recall, $\zeta_t$ is defined in (20).

To this end, we have obtained the expressions for both $A(\lambda)$ and $\theta_t^*$, in (EC.5.35) and (EC.5.36) respectively, and both expressions involve terms related to $N_t$. Next, we will replace such terms by terms associated with $X_t$ and $V_t$ only. To do so, the crux is to compare (21) and (EC.5.32) and match integrands for $dt$, $dB_t^M$ and $d\hat{B}_t$. Start with

$$\hat{M}_t = \frac{\lambda - V_0}{Z_0^M} + \int_0^t \phi_s^* \cdot dN_s + \int_0^t \gamma_s d\hat{B}_s,$$

and this equation comes directly from (EC.5.32), changing measure from $P^R$ to $P^M$ for the first term, and accounting for $\phi_t^* = \phi^H$. Alternatively, $\hat{M}_t$ can also be represented as the following by applying change of measure formula:

$$\hat{M}_t = E^M \left[ \frac{\lambda - H_T}{Z_T^M} \cdot \frac{Z_T^M}{Z_0^M} \left| F_t \right. \right] \cdot \frac{1}{E^M \left[ \frac{Z_T^M}{Z_0^M} \left| F_t \right. \right]} = (\lambda - V_t) N_t^0$$

$$= \left[ \lambda - V_0 - \int_0^t \xi_s dX_s - \int_0^t \delta_s d\hat{B}_s \right] N_t^0, \quad (EC.5.37)$$

where the first equality switches the measure from $P^R$ to $P^M$; the second equality accounts for the fact $Z_T = Z_T^M$ and definition of $N_t^0$ in (EC.5.7); the third equality makes use of the martingale representation of $V_t$ in (21). Now, apply Itô’s Lemma on both (EC.5.37) and (EC.5.37), use the dynamics for $N_t$ in (EC.5.7)), and match the $d\hat{B}_t$ term, we have:

$$\gamma_t = -\frac{\delta_t}{Z_t^M}. \quad (EC.5.38)$$

Substituting (EC.5.38) in (EC.5.35) gives (EC.5.28), as desired.
Next, match $dB_t^M$ (matching for $dt$ term gives the same result) and obtain:

$$
\xi_t + (\lambda - V_t)N_t^0 \zeta_t = \xi_t(\phi_t^1 \cdot N_t) - \phi_t^1.
$$

(EC.5.39)

Recall, $\zeta_t$ and $\psi_t$ are defined in (20) and (EC.5.2) respectively, and to reach (EC.5.39) the relation in (EC.5.3) is used. Substituting (EC.5.39) to (EC.5.36) gives (25), taking into the following fact implied by Lemma EC.4 part (ii):

$$
\int_0^t \theta \ast_s dX_s = \int_0^t \phi \ast_s \cdot dN_s.
$$

What remains is to show that $\theta^*_t$ above is an admissible trading strategy. By part (ii) of Lemma EC.4, it is sufficient to show $\phi^*_t$ in (EC.5.34) is in $A_N^T$. Note $\phi^*_t = \phi^H_t$. Denote $\hat{M}^\circ_t = \int_0^t \phi^H_s \cdot dN_s$; we already showed that (see the arguments below (EC.5.33)) $\hat{M}^\circ_t$ is a square-integrable martingale under $P^R$, and this implies it has finite expected quadratic variation under $P^R$:

$$
E^R([\hat{M}^\circ, \hat{M}^\circ]_t) < \infty.
$$

(EC.5.40)

Then, for any $P^R \in \mathcal{M}_N$, we have

$$
E^R(\sqrt{[\hat{M}^\circ, \hat{M}^\circ]_t}) = E^R(\frac{dP^R}{dP} \sqrt{[\hat{M}^\circ, \hat{M}^\circ]_t}) \leq E^R\left(\left(\frac{dP^R}{dP}\right)^2\right) E^R([\hat{M}^\circ, \hat{M}^\circ]_t) < \infty; \quad (EC.5.41)
$$

where the $\leq$ follows Cauchy–Schwarz inequality; and the $<$ follows (EC.5.40) and the following

$$
E^R\left(\left(\frac{dP^R}{dP}\right)^2\right) = E\left[\left(\frac{dP^R}{dP}\right)^2 \frac{(Z^M_t)^2}{Z^0_M}\right] = E\left[\left(\frac{dP^R}{dP}\right)^2 \left(\frac{Z^M_0}{Z^0_M}\right)^2 \left(\frac{Z^M_T}{Z^0_M}\right)^2\right] = Z^0_M E\left[\left(\frac{1}{Z^0_M} \frac{dP^R}{dP}\right)^2\right] < \infty;
$$

(EC.5.42)

where the $<$ follows from the definition of $\mathcal{M}_N$; see (EC.5.9). Now by Burkholder-Davis-Gundy inequality, $\hat{M}^\circ_t$ is an martingale under $P^R$. Last, $Z^M_T \hat{M}^\circ_T \in L^2(P)$ easily follows from that $\hat{M}^\circ_t$ is square-integrable under $P^R$. Combining the arguments above, we have $\phi^*_t = \phi^H_t \in A_N$, hence $\theta^*_t$ is admissible as argued. This completes the proof for Theorem 1. □

EC.6. Proof of Proposition 3

EC.6.1. Proof of Lemma EC.5

First, we approve that under the parametric condition in (30), Assumption 5 holds.

**Lemma EC.5.** For $X_t$ specified in (28), let $Z_t$ follow the definition in (14) (then $\eta_t$ involved in $Z_t$ is defined in (31)). Under the parametric condition in (30) holds, i.e.,

$$
\kappa T < \frac{\pi}{4},
$$

Assumption 5 holds—that is, $Z_t$ is a square-integrable martingale under $P$. 
Proof. By definition of $Z_t$ in (14), we have

$$dZ_t = -\eta_t Z_t dB_t = -Z_t \frac{\kappa}{\sigma} \left( \alpha + \frac{\sigma^2}{2\kappa} - Y_t \right) dB_t; \tag{EC.6.1}$$

the second equality uses the expression of $\eta_t$ in (31).

Note that (EC.6.1) implies that $Z_t$ is a local martingale under $P$. We will prove that it is a martingale as follows. Introduce the following process, which is a candidate for $E(Z^2_t \mid G_t)$.

$$J_t := Z^2_t f(t, Y_t) = Z_t^2 \exp\{f_0(T-t) + f_1(T-t)Y_t + f_2(T-t)Y^2_t\}; \tag{EC.6.2}$$

where $f_i$, $i = 0, 1, 2$ are functions defined in (34); and

$$f(t, y) := \exp\{f_0(T-t) + f_1(T-t)y + f_2(T-t)y^2\}. \tag{EC.6.3}$$

Since $f_i(0) = 0$ for all $i = 0, 1, 2$, we have $f(T, Y_t) = 0$ and thus $J_T = Z^2_T$. Below we show that properties of $J_t$ guarantee $Z_t$ to be a square-integrable martingale.

Applying Itô’s Lemma, it is straightforward to obtain the following dynamics:

$$d\left(\frac{J_t}{Z_t}\right) = dZ_t f(t, Y_t) = f(t, Y_t)dZ_t + Z_t df(t, Y_t) + dZ_t df(t, Y_t) = Z_t \left[ f_t - \frac{1}{2} \sigma^2 f_y + \frac{1}{2} \sigma^2 f_{yy} \right] dt + Z_t (\sigma f_y - \eta_t f(t, Y_t)) dB_t = Z_t \left[ f_t - \kappa (\alpha + \frac{\sigma^2}{\kappa} - Y_t) f_y + \frac{1}{2} \sigma^2 f_{yy} + \eta_t^2 f(t, Y_t) \right] dt - Z_t \left[ \eta_t f(t, Y_t) - \sigma f_y \right] d\left( dB_t + \eta_t dt \right); \tag{EC.6.4}$$

$f_t$, $f_y$ and $f_{yy}$ are usual notations for partial derivatives; note they all depend on $(t, Y_t)$, but for simplicity the arguments are dropped. The second line uses the expression of $\eta_t$ in (31).

Next, applying Itô’s Lemma on $J_t$, it is straightforward to obtain:

$$dJ_t = Z_t d\left(\frac{J_t}{Z_t}\right) + \frac{J_t}{Z_t} dZ_t + d\left(\frac{J_t}{Z_t}\right) dZ_t = Z_t^2 \left[ f_t - \kappa (\alpha + \frac{\sigma^2}{\kappa} - Y_t) f_y + \frac{1}{2} \sigma^2 f_{yy} + \eta_t^2 f(t, Y_t) \right] dt + \left( \sigma f_y - \eta_t f(t, Y_t) - \eta_t^2 Z_t^2 f(t, Y_t) \right) dB_t \tag{EC.6.5}$$

It can be verified that the function $f(t, y)$ defined in (EC.6.3) solves the following partial differential equation:

$$f_t - \kappa (\alpha + \frac{\sigma^2}{\kappa} - y) f_y + \frac{1}{2} \sigma^2 f_{yy} + \left( \frac{\kappa}{\sigma} \right)^2 (\alpha + \frac{\sigma^2}{2\kappa} - y)^2 f(t, y) = 0$$

s.t. $f(T, y) = 1, \quad \forall y \in \mathbb{R}. \tag{EC.6.6}$
To check this, take derivatives of $f(t,y)$ using (EC.6.3). Then, collect the coefficients for $y^2$, $y$ and the term independent of $y$, and set these coefficients to 0, then (EC.6.6) reduces to an ordinary differential equation system:

$$
-f'_2 + (2\kappa)f_2 + (2\sigma^2)f_2^2 + \left(\frac{\kappa}{\sigma}\right)^2 = 0 \\
-f'_1 + \kappa[f_1 - 2(\alpha + \frac{\sigma^2}{\kappa})f_2] + (2\sigma^2 f_2)f_1 - 2\left(\frac{\kappa}{\sigma}\right)^2(\alpha + \frac{\sigma^2}{2\kappa}) = 0 \\
-f'_0 - \kappa(\alpha + \frac{\sigma^2}{\kappa})f_1 + \frac{1}{2}\sigma^2(f_1^2 + 2f_2) + \left(\frac{\kappa}{\sigma}\right)^2(\alpha + \frac{\sigma^2}{2\kappa})^2 = 0
$$

s.t. $f_i(0) = 0, i = 0, 1, 2$. (EC.6.7)

It is straightforward to verify that under the parameter condition in (30), $f_i, i = 0, 1, 2$ specified in (34), solve this ODE system. So, $f(t,y)$ solves the PDE above, and this makes the $dt$-term of $dJ_t$ in (EC.6.5) vanish, reducing (EC.6.5) to:

$$
dJ_t = \left[\sigma f_y - \eta_t f(t,Y_t) - \eta_t Z_t^2 f(t,Y_t)\right] dB_t. \tag{EC.6.8}
$$

And, (EC.6.4) reduces to (note the PDE expression is also involved in the $dt$-term on the second line of (EC.6.4)):

$$
d\left(\frac{J_t}{Z_t}\right) = -Z_t \left[\eta_t f(t,Y_t) - \sigma f_y\right] d\left( dB_t + \eta_t dt \right). \tag{EC.6.9}
$$

Clearly, by (EC.6.8), $J_t$ is a local martingale under $P$.

Now, observe that the term inside the exponential of $f(t,Y_t)$ is a quadratic function in $Y_t$ at each time $t$. Under (30), $f_2(T-t) \geq 0$ for all $t \in [0,T]$, hence

$$
f(t,Y_t) \geq \exp\left\{f_0(T-t) - \frac{f_1^2(T-t)}{4f_2(T-t)}\right\}, \quad t \in [0,T];
$$

It is easy to verify that $f_1^2(T-t)/f_2(T-t) \to 0$ as $t \to T$. Clearly, the function $f_0(T-t) - \frac{f_1^2(T-t)}{4f_2(T-t)}$ is continuous, hence admits a minimum on $[0,T]$; so, there exists a positive number, $c > 0$, such that $f(t,Y_t) \geq c > 0$ for all $t \in [0,T]$. Therefore,

$$
J_t = Z_t^2 f(t,Y_t) \geq cZ_t^2 \Rightarrow Z_t \leq \sqrt{\frac{1}{c}J_t}. \tag{EC.6.10}
$$

Next, we will make use of (EC.6.10) to prove $Z_t$ is a square-integrable martingale under $P$. Define the following sequence of $\mathcal{G}_t$-stopping times:

$$
\tau_k := \inf\{t \in [0,T] \mid J_t \geq k\} \wedge T, \quad k \in \mathbb{N}. \tag{EC.6.11}
$$
Clearly, $J_t$ is a continuous process, hence $\tau_k \uparrow T$ as $k \to \infty$. Since each $\tau_k$ bounds the stopped version of $J$, hence also bounds the stopped version of $Z_t$ via (EC.6.10). So, both $Z_t \wedge \tau_k$ and $J_t \wedge \tau_k$ are bounded $\mathbb{P}$-martingales, and we can apply Doob’s inequality:

$$
E\left[ \sup_{t \in [0,T]} Z_t^2 \right] \leq 2E(Z_T^2) \leq \frac{2}{c}E(J_{\tau_k}) = \frac{2}{c}J_0.
$$

(EC.6.12)

the first $\leq$ is application of Doob’s inequality, and the following equality uses the obvious fact $\tau_k \leq T$. The second $\leq$ uses (EC.6.10) and the $= \text{is application of optional stopping theorem on the bounded martingale } J_t \wedge \tau_k$. So, (EC.6.12) implies

$$
E\left[ \sup_{t \in [0,T]} Z_t^2 \right] \leq \frac{2}{c}J_0;
$$

clearly, because $\tau_k$ increases in $k$, so does $\sup_{t \in [0,T]} Z_t^2$ (since the sup is taken on a longer time interval for larger $\tau_k$), so we can let $k \to \infty$ and apply monotone convergence to above to reach the following, accounting for $\tau_k \uparrow T$:

$$
E\left[ \sup_{t \in [0,T]} Z_t^2 \right] \leq \frac{2}{c}J_0;
$$

this is sufficient to establish that $Z_t$ is a square-integrable martingale under $\mathbb{P}$, and completes the proof of Lemma EC.5. □

EC.6.2. Proof of Proposition 3

Now that we know $Z_t$ is a $\mathbb{P}$-martingale, $\mathbb{P}^M$ specified in (14) is well-defined. Then, Girsanov Theorem applies and we have the $\mathbb{P}^M$-Brownian-motion:

$$
 dB_t^M = dB_t - \eta_t dt.
$$

Then, (EC.6.9) becomes

$$
 d\left( \frac{J_t}{Z_t} \right) = -Z_t \left[ \eta_t f(t,Y_t) - \sigma f_y \right] dB_t^M;
$$

(EC.6.13)

using the expression of $f(t,y)$ in (EC.6.3), it is straightforward to derive an explicit expression for $f_y$. Then, arrange the terms to explicitly write (EC.6.13) as:

$$
 d\left( \frac{J_t}{Z_t} \right) = -\sigma \left( \frac{J_t}{Z_t} \right) \left[ \frac{\kappa}{\sigma^2} b(T-t)(\alpha - Y_t) + a(T-t) \right] dB_t^M;
$$

(EC.6.14)

with the two deterministic functions $a(\cdot)$ and $b(\cdot)$ specified in (32); note $b(T-t) > 0$ for all $t \in [0,T]$. Next, observe that $V_t := \frac{\kappa}{\sigma^2} b(T-t)(\alpha - Y_t) + a(T-t)$ follows a linear stochastic differential equation of the following form:

$$
 dV_t = (x_t + y_t V_t) dt + z_t dB_t^M;
$$
where \( x_t, y_t \) and \( z_t \) above are deterministic functions; the linearity comes from \( Y_t \), which also follows a linear SDE (see (28)), as well as from \( \eta_t \) in (31), which is linear in \( Y_t \). Now, Lemma A.4 in Wang and Wissel (2013) immediately applies, and we conclude that \( J_t/Z_t \) is a martingale under \( P^M \). Next, \[
Z^M_t := E^M(Z_T \mid G_t) = E^M\left( \frac{J_T}{Z_T} \mid G_t \right) = \frac{J_t}{Z_t} = Z_t f(t, Y_t). \tag{EC.6.15}\]
The first = follows from \( J_T = Z_T^2 \), and the second one uses the established fact that \( J_t/Z_t \) is a martingale under \( P^M \). The above can be written as:
\[
\frac{Z_t}{Z_t^M} = \frac{1}{f(t, Y_t)} = \exp\{-f_0(T - t) - f_1(T - t)Y_t - f_2(T - t)Y_t^2\}. \tag{EC.6.16}\]
Now, (EC.6.14) can be written as:
\[
dZ^M_t = -Z^M_t \left[ \frac{K}{\sigma^2} b(T - t)(\alpha - Y_t) + a(T - t) \right] \frac{\sigma dB_t^M}{dt} = -Z^M_t \frac{\sigma}{X_t} \left[ \frac{K}{\sigma^2} b(T - t)(\alpha - Y_t) + a(T - t) \right] dX_t; \tag{EC.6.17}\]
where the second line uses \( dX_t = \sigma X_t dB_t \). Hence, here the quantity \( \zeta_t \) defined in (20) has the expression
\[
\zeta_t = -\frac{Z^M_t}{X_t} \left[ \frac{K}{\sigma^2} b(T - t)(\alpha - Y_t) + a(T - t) \right]. \tag{EC.6.18}\]

Now we can apply Theorem 1 to establish Proposition 3. In particular, \( Z_t/Z_t^M \) involved in \( B(m, P, R) \) expressed in (27) follows (EC.6.16), and \( \zeta_t \) involved in \( \theta_t \) specified in (25) follows (EC.6.18). This completes the proof. \( \square \)

**EC.7. Proof of Proposition 4**

Denote the second term of \( B(m, P, R) \) as \( \Psi(P, R) \). Clearly, \( \Psi(P, R) \) strictly increases in both \( P \) and \( R \). Then, write
\[
B(m, P, R) = C[(m - V_0(P, R))]^2 + \Psi(P, R),
\]
where \( C = 1/(Z^M_0 - 1) > 0 \). (Below, we refer to \( C[(m - V_0(P, R))]^2 \) (resp., \( \Psi(P, R) \)) as the “first term” (resp., “second term”) of \( B(m, P, R) \).) Note that \( (P^h_m, R^h_m) \) satisfy the optimality equations
\[
2C(V_0 - m) \frac{\partial V_0(P^h_m, R^h_m)}{\partial P} + \frac{\partial \Psi(P^h_m, R^h_m)}{\partial P} = 0, \quad 2C(V_0 - m) \frac{\partial V_0(P^h_m, R^h_m)}{\partial R} + \frac{\partial \Psi(P^h_m, R^h_m)}{\partial R} = 0.
\]
Since \( \Psi(P, R) \) is strictly increasing in both \( P \) and \( R \), we must have
\[
(V_0 - m) \frac{\partial V_0(P^h_m, R^h_m)}{\partial P} < 0, \quad (V_0 - m) \frac{\partial V_0(P^h_m, R^h_m)}{\partial R} < 0. \tag{EC.7.1}\]

Recall, \( V_0(P, R) \) is concave in \( P \) (resp., \( R \)) for any given \( R \) (resp., \( P \)). We use the notation in (35), and it is straightforward to derive:
\[
P^{NV(M)}(R) = \frac{bc + E^M(R \wedge A_T)}{2b}, \quad R^{NV(M)}(P) = F^{-1}_M\left( \frac{P - c}{P - s} \right); \tag{EC.7.2}\]
where $F_M$ is the distribution function of $A_T$ under $P^M$. Clearly, $P^{NV(M)}(R)$ (resp., $R^{NV(M)}(P)$) increases in $R$ (resp., $P$). Moreover, by its partial concavity in $P$ (resp., $R$), for given $R$ (resp., $P$), $V_0(P,R)$ increases in $P$ (resp., $R$) for $P \leq P^{NV(M)}(R)$ (resp., $R \leq R^{NV(M)}(P)$) and then decreases. Thus, for any given $R$, the corresponding $P$ that minimizes $B(m,P,R)$ cannot exceed $P^M(R)$.

Otherwise, a $P$ smaller than $P^{NV(M)}(R)$, $P'$, can be found to satisfy $V_0(P,R) = V_0(P',R)$. Then, $B(m,P',R) < B(m,P,R)$, contradicting with $P$ minimizing $B(m,P,R)$. Completely analogous argument can be applied to $R$. Applying the above to optimality of $(P^h_m, R^h_m)$, we must have:

**Lemma EC.6.** $P^h_m \leq P^{NV(M)}(R^h_m)$, $R^h_m \leq R^{NV(M)}(P^h_m)$.

Clearly, Lemma EC.6 implies:

$$\frac{\partial V_0(P^h_m, R^h_m)}{\partial P} \geq 0, \quad \frac{\partial V_0(P^h_m, R^h_m)}{\partial R} \geq 0.$$  

(EC.7.3)

Combining (EC.7.3) with (EC.7.1), we conclude $V_0(P^h_m, R^h_m) \leq m$.

Now, we proceed with showing the other result of part (i) of this proposition, $P^h_m \leq P^{NV(M)}$ and $R^h_m \leq R^{NV(M)}$. First, we consider the case of $m \geq V_0(P^{NV(M)}, R^{NV(M)})$. For this case, $P^h_m$ and $R^h_m$ cannot both exceed $P^{NV(M)}$ and $R^{NV(M)}$, respectively; otherwise both terms of $B(m, P^h_m, R^h_m)$ will exceed those of $B(m, P^{NV(M)}, R^{NV(M)})$, contradicting optimality of $(P^h_m, R^h_m)$. Then, if $P^h_m \leq P^{NV(M)}$, by Lemma EC.6,

$$R^h_m \leq R^{NV(M)}(P^h_m) \leq R^{NV(M)}(P^{NV(M)}) = R^{NV(M)}.$$

If $R^h_m \leq R^{NV(M)}$, again by Lemma EC.6,

$$P^h_m \leq P^{NV(M)}(P^h_m) \leq P^{NV(M)}(R^{NV(M)}) = P^{NV(M)}.$$

Hence, for both cases, we must have $P^h_m \leq P^{NV(M)}$ and $R^h_m \leq R^{NV(M)}$.

Now, we consider the other case, $m < V_0(P^{NV(M)}, R^{NV(M)})$. Let $v(P) := V_0(P,R^{NV(M)}(P))$. Clearly, $v(c) \leq 0 \leq m$. On the other hand,

$$v(P^{NV(M)}) = V_0(P^{NV(M)}, R^M(P^{NV(M)})) = V_0(P^{NV(M)}, R^{NV(M)}) > m.$$

Thus, $\exists P_1 \in [c, P^{NV(M)}]$ such that $v(P_1) = m$; let $R_1 = R^{NV(M)}(P_1)$. Note, $R_1 = R^{NV(M)}(P_1) \leq R^{NV(M)}(P^{NV(M)}) = R^{NV(M)}$. Then, $P^h_m$ and $R^h_m$ cannot both exceed, respectively, $P_1$ and $R_1$, otherwise both terms of $B(m, P^h_m, R^h_m)$ are larger than those of $B(m, P_1, R_1)$ (the first term of which is zero). Then, if $P^h_m \leq P_1(\leq P^{NV(M)})$, applying Lemma EC.6,

$$R^h_m \leq R^{NV(M)}(P^h_m) \leq R^{NV(M)}(P_1) \leq R^{NV(M)}(P^{NV(M)}) = R^{NV(M)}.$$

If $R^h_m \leq R_1(\leq R^{NV(M)})$, then again by Lemma EC.6,

$$P^h_m \leq P^{NV(M)}(R^h_m) \leq P^{NV(M)}(R_1) \leq P^{NV(M)}(R^{NV(M)}) = P^{NV(M)}.$$
Then, for both cases, we must also have \( P^h_m \leq P^{NV(M)} \) and \( R^h_m \leq R^{NV(M)} \). Summarizing all above leads to \( P^h_m \leq P^{NV(M)} \) and \( R^h_m \leq R^{NV(M)} \). This concludes the proof of part (i).

Proof of part (ii) is completely analogous to that of Lemma 2, which has been presented in EC.3.

\[\Box\]

EC.8. Proof of Lemma 4

For any given \( R \), \( V_0(P,R) \) is concave in \( P \); in particular, it increases in \( P \) up to \( P^{NV(M)}(R) \) and then decreases. First, suppose \( m > \max_P V_0(P,R) \). Then, increasing \( P \) beyond \( P^{NV(M)}(R) \) increases both terms of \( B(m,P,R) \), thus is not optimal. Therefore, for this case the optimal \( P \) is bounded by \( P^{NV(M)}(R) \).

In summary, for a given \( R \), the optimal \( P \) satisfies \( V_0(P,R) \leq m \) over \( P \leq \bar{P}(R) \) (i.e. the smaller root of \( V_0(P,R) = m \)) is not optimal as it also increases both terms of \( B(m,P,R) \).

EC.9. Proof of Lemma 5

Let \( f(R) := V_0(P^{NV(M)}(R),R) \). For notational simplicity, let \( P(R) = P^{NV(M)}(R) \) and \( m = E[H_T(P^{NV},R^{NV})] \). Given \( R \), \( V_0(P,R) \) is concave in \( P \), hence by setting \( \partial V_0/\partial P \) to zero, \( P(R) \) satisfies the following optimality equation:

\[
2bP(R) = E^M[R \wedge A_T] + bc. \tag{EC.9.1}
\]

Then, it is straightforward to verify:

\[
f(R) = b(P(R))^2 - 2bP(R) - (c - s)R + bcs.
\]

Next, we will show \( f(R^{NV}) \geq m \) and \( f(bc) \leq m \), which in turn indicates that \( f(R) \) has at least one root within \([bc,R^{NV}]\) and thus proves the result. First, examine \( f(R^{NV}) \). With \( C^M_T \geq C_T \), by independence of \( \tilde{B}_T \) from \( \{B_t, 0 \leq t \leq T\} \), it is straightforward to verify that \( A^M_T \geq A_T \) (\( A^M_T \) is the version of \( A_T \) under \( P^M \)). Then,

\[
2bP(R^{NV}) = E^M[R^{NV} \wedge A_T] + bc \geq E[R^{NV} \wedge A_T] + bc = 2bP^{NV},
\]

where the last equality follows from (5), leading to \( P(R^{NV}) \geq P^{NV} \). Then, by the first optimality equation in (5), it is straightforward to verify the following:

\[
m = b(P^{NV})^2 - 2bsP^{NV} - (c - s)R^{NV} + bcs.
\]
Analogously, it can be verified that
\[ f(R^{NV}) = b(P(R^{NV}))^2 - 2bsP(R^{NV}) - (c - s)R^{NV} + bcs. \]
Since \( P(R^{NV}) \geq P^{NV} \geq s \), we have \( f(R^{NV}) \geq m \).

Next, we check \( f(bc) \). Let \( \epsilon = E[M((bc - A_T)^+)] \), then clearly \( \epsilon \leq [E[(bc - A_T)^+]] \). From Assumption 3, we have
\[ \frac{[E[(bc - A_T)^+]]^2}{4bm} \leq 1. \]
Note \( P(bc) = [E^M(bc \wedge A_T) + bc]/2b = c - \epsilon/2b \), then it is straightforward to verify that:
\[ f(bc) < \frac{\epsilon^2}{4b}. \]
Thus,
\[ \frac{f(bc)}{m} < \frac{\epsilon^2}{4bm} \leq \frac{[E[(bc - A_T)^+]]^2}{4bm} \leq 1 \implies f(bc) < m. \]
Combining the above, \( f(R) \) must have root(s) within \([bc, R^{NV}]\), hence by definition of \( R^*, R^* \leq R^{NV} \).

Next, we proceed with showing \( P^* \leq P^{NV} \). Using (EC.9.1), it is easy to verify that
\[ (m =) f(R^*) = b(P^*)^2 - 2bsP^* - (c - s)R^* + bcs. \]
Comparing with the expression of \( m \) above and taking into account \( R^* \leq R^{NV} \), clearly, \( P^* \leq P^{NV} \) must hold. This completes the proof. \( \square \)

**EC.10. Proof of Theorem 3**

We layout key definitions and technical preparations in §EC.10.1 and then prove Theorem 3 in §EC.10.2.

**EC.10.1. Definitions and Technical Preparations**

Throughout the proof, \( m \) denotes the newsvendor’s maximum expected profit, i.e.,
\[ m = E[H_T(P^{NV}, R^{NV})] > 0. \]
(The > 0 is due to Assumption 2.) Also recall,
\[ P^{NV(M)}(R) := \arg \max_P V_0(P, R) = \frac{E^M(R \wedge A_T) + bc}{2b}, \]
\[ R^{NV(M)}(P) := \arg \max_R V_0(P, R) = F_{R}^{-1}(\frac{P - c}{P - s}). \]

Related to \( m \), below we define three critical values.

**Definition EC.1.** (i) \( P_1 \) is the smallest solution to \( E^M[H_T(P_1, R^{NV(M)}(P_1))] = m \) and \( R_1 =: R^{NV(M)}(P_1) \). (ii) \( P_2 \) is the solution to \( R^{NV(M)}(P_2) = R^{NV} \). (iii) \( P_3 \) is the smallest solution to \( V_0(P_3, R^{NV}) = m \).
The lemma below collects properties of \( P_i, i = 1, 2, 3. \)

**Lemma EC.7.** All \( P_1, P_2 \) and \( P_3 \) exist, hence by definition they are unique. Furthermore,

\[
P_2 = \frac{c - s P^M(A_T \leq R^{NV})}{1 - P^M(A_T \leq R^{NV})} = s + \frac{c - s}{P^M(A_T \geq R^{NV})}. \\
(\text{EC.10.1})
\]

And

\[
P_3 = P^{NV(M)}(R^{NV}) - \sqrt{(P^{NV(M)}(R^{NV}) - P^{NV})(P^{NV(M)}(R^{NV}) + P^{NV} - 2s)}. \\\n(\text{EC.10.2})
\]

Moreover,

\[
P_i \leq P^{NV}, \quad i = 1, 2, 3. \\\n(\text{EC.10.3})
\]

**Proof.** We first show \( P_1 \) exists. (Existence of \( P_2 \) and \( P_3 \) will be clear after (EC.10.1) and (EC.10.2) are proved.) Recall,

\[
V_0(P, R) = (P - c)(R - bP) - (P - s)E^M[(R - A_T)^+].
\]

As \( P \to c, R^{NV(M)}(P) = F^{-1}_M((P - c)/(P - s)) \to -\infty \). Thus, as \( P \to c, (P - c)(R - bP) \) is eventually nonpositive (hence \( < m \)) and so is \( V_0(P, R^{NV(M)}(P)) \). Now, we check \( V_0(P^{NV}, R^{NV(M)}(P^{NV})) \):

\[
V_0(P^{NV}, R^{NV(M)}(P^{NV})) = \max_R V_0(P^{NV}, R) \\
\geq V_0(P^{NV}, R^{NV}) \\
= (P^{NV} - c)(R^{NV} - bP^{NV}) - (P^{NV} - s)E^M[(R^{NV} - A_T)^+] \\
\geq (P^{NV} - c)(R^{NV} - bP^{NV}) - (P^{NV} - s)E[(R^{NV} - A_T)^+] \\
= E[H_T(P^{NV}, R^{NV})] = m.
\]

The second \( \leq \) is due to \( A_T \leq A^M_T \). Summarizing the above, there exists value(s) of \( P \) in \([c, P^{NV}]\) such that \( V_0(P, R^{NV(M)}(P)) = m \), hence \( P_1 \) uniquely exists and in particular, \( P_1 \leq P^{NV} \).

For \( P_2 \), by definition:

\[
R^{NV(M)}(P_2) = F^{-1}_M\left(\frac{P_2 - c}{P_2 - s}\right) = R^{NV},
\]

which immediately leads to the expression in (EC.10.1). Furthermore,

\[
P_2 = s + \frac{c - s}{P^M(A_T \geq R^{NV})} \leq s + \frac{c - s}{P(A_T \geq R^{NV})} = P^{NV}(R^{NV}) = P^{NV}.
\]

The \( \leq \) is due to \( A_T \leq A^M_T \).

For \( P_3 \), note that given \( R = R^{NV} \), \( V_0(P, R^{NV}) = m \) is a quadratic equation in \( P \). Rearranging the terms and accounting for the following:

\[
P^{NV(M)}(R^{NV}) = \frac{E^M(R^{NV} \land A_T) + bc}{2b},
\]

\[\text{EC.10.4}\]
the equation becomes
\[-bP^2 + 2bP^{NV(M)}(R^{NV}) P - (c - s)R^{NV} - 2bsP^{NV(M)}(R^{NV}) + bcs = m.\] (EC.10.4)

Using the optimality equations specified in Lemma 1, it is straightforward to derive the following:
\[m = b(P^{NV})^2 - (c - s)R^{NV} - 2bsP^{NV} + bcs.\]

Substituting this expression of \(m\) to (EC.10.4), the equation becomes:
\[-P^2 + 2P^{NV(M)}(R^{NV}) P - [(P^{NV})^2 + 2sP^{NV(M)}(R^{NV}) - 2sP^{NV}] = 0.\] (EC.10.5)

Then, it is straightforward to verify that (EC.10.2) is the smallest solution to (EC.10.5). In particular, by \(A_T \preceq A^M_T\),
\[P^{NV} = \frac{1}{2b}[E(R^{NV} \land A_T) + bc] \leq \frac{1}{2b}[E^M(R^{NV} \land A_T) + bc] \leq P^{NV(M)}(R^{NV}).\]

In addition, by Assumption 2, \(P^{NV} > c > s\), hence \(P^{NV(M)}(R^{NV}) > c > s\). Therefore, \(P_3\) in (EC.10.2) is well-defined (i.e., real-valued). Next, using the fact that the function \(x - \sqrt{(x-y)(x+y-2s)}\) decreases in \(x\) for \(x \geq y \geq s\), \(P_3 \leq P^{NV}\) immediately follows from \(P^{NV(M)}(R^{NV}) \geq P^{NV}\). \(\square\)

The next result indicates that \(R_1\) upper bounds \(R^h_m\).

**Lemma EC.8.** \(R^h_m \leq R_1\).

**Proof.** If \(R^h_m > R_1\) and \(P^h_m > P_1\), then \(B(m, P_1, R_1) < B(m, P^h_m, R^h_m)\) since the both the first term (which is zero) and the second term of the former are smaller than the latter. This contradicts with optimality of \((P^h_m, R^h_m)\). Therefore \(R^h_m > R_1\) and \(P^h_m > P_1\) cannot hold at the same time. If \(R^h_m > R_1\), we must have \(P^h_m \leq P_1\), but this introduces contradiction because by Lemma EC.6, we must also have
\[R^h_m \leq P^{NV(M)}(P^h_m) \leq P^{NV(M)}(P_1) = R_1.\]
Therefore, \(R^h_m \leq R_1\) must hold. \(\square\)

**EC.10.2. Proof of Theorem 3**

**Proof of Part (i):** By Lemma 5, \(P^* \leq P^{NV}\), so it is sufficient to prove \(P^h_m \leq P^*\). To this end, we consider two cases, \(R^h_m \leq R^*\) or \(R^h_m \geq R^*\). For the first case, apply Lemma EC.6, we have
\[P^h_m \leq P^{NV(M)}(P^h_m) \leq P^{NV(M)}(R^*) = P^*.\]

Now, consider the other case, \(R^h_m \geq R^*\). Note that \(P^h_m\) and \(R^h_m\) cannot exceed \(P^*\) and \(R^*\) at the same time: the first term of \(B(m, P^*, R^*)\) is zero (since \(V_0(P^*, R^*) = m\) and thus increasing both
$P$ and $R$ beyond $P^*$ and $R^*$ increases both terms $B(m, P, R)$. Therefore, $R^h_m \geq R^*$ must lead to $P^h_m \leq P^*$.

**Proof of Part (ii):** Applying Lemma EC.7, it is straightforward to verify that the stated inequality in (38) is equivalent to $P_3 \leq P_2$. By Lemma EC.8, it is sufficient to prove $R_1 \leq R^{NV}$ under this condition. To this end, first note that

$$P^{NV(M)}(R^{NV}) = \frac{1}{2b}[E^M(R^{NV} \land A_T) + bc] \geq \frac{1}{2b}[E(R^{NV} \land A_T) + bc] = P^{NV}.$$  

The $\geq$ is due to $A^M_T \succeq A_T$. Combining this with (EC.10.3) and accounting for the concavity of $V_0(P, R)$ in $P$ with a given $R$, $P_3 \leq P_2$ leads to

$$V_0(P_1, R_1) = m = V_0(P_3, R^{NV}) \leq V_0(P_2, R^{NV}).$$

By $V_0(P_1, R_1) = \max_R V_0(P_1, R) \geq V_0(P_1, R^{NV})$, we have:

$$V_0(P_1, R^{NV}) \leq V_0(P_2, R^{NV}).$$

Applying $P_1, P_2 \leq P^{NV(M)}(R^{NV})$ and concavity of $V_0(P, R)$ in $P$ with given $R$ again, we have

$$P_1 \leq P_2.$$ 

Therefore,

$$R_1 = R^{NV(M)}(P_1) \leq R^{NV(M)}(P_2) = R^{NV},$$

which completes the proof of part (ii).

**Proof of Part (iii):** Violation of (38) is equivalent to $P_2 < P_3$. Reversing the argument in proof of part (ii) above, it is easy to obtain $P_1 > P_2$, hence $R_1 = R^{NV(M)}(P_1) > R^{NV(M)}(P_2) = R^{NV}$. Next,

$$V_0(P_1, R_1) = \max_R V_0(P_1, R) = m = V_0(P_3, R^{NV}) \Rightarrow V_0(P_1, R^{NV}) \leq V_0(P_3, R^{NV}).$$

Again, by concavity of $V_0(P, R^{NV})$ in $P$ and $P_1, P_3 \leq P^{NV} \leq P^{NV(M)}(R^{NV})$, we have $P_1 \leq P_3$. In summary, we have

$$c < P_2 < P_1 \leq P_3, \quad R_1 > R^{NV}. \quad (EC.10.6)$$

($P_2 > c$ is obvious from (EC.10.1.)

Next, rearranging the terms of $V_0(P, R)$ leads to:

$$V_0(P, R) = (-bP^2 + bcP) + (P - s)E^M(R \land A_T) - (c - s)R.$$ 

Then, $m = V_0(P_1, R_1) = V_0(P_3, R^{NV})$ leads to:

$$(-bP_1^2 + bcP_1) - (-bP_3^2 + bcP_3) + (P_1 - s)E^M[R_1 \land A_T] - (c - s)R_1$$
\[(P_3 - s)E^M[R^{NV} \land A_T] - (c - s)R^{NV}.
\]

Since the function \(-bP^2 + bcP\) decreases in \(P \geq c/2\) and \(c < P_1 \leq P_3\), we have \((-bP_1^2 + bcP_1) - (-bP_3^2 + bcP_3) \geq 0\). Then, the equality above implies:

\[(P_1 - s)E^M[R_1 \land A_T] - (c - s)R_1 \leq (P_3 - s)E^M[R^{NV} \land A_T] - (c - s)R^{NV}. \quad \text{(EC.10.7)}
\]

The LHS of (EC.10.7) is:

\[E^M(R_1 \land A_T) = R_1P^M(A_T \geq R_1) + E^M[A_T1\{A_T \leq R_1\}] = R_1\frac{c - s}{P_1 - s} + E^M[A_T1\{A_T \leq R_1\}],\]

where the equality is due to definition of \(R_1\): \(P^M(A_T \leq R_1) = (P_1 - c)/(P_1 - s)\), substituting the above expression of \(E^M[R_1 \land A_T]\) to the LHS of (EC.10.7), we have:

\[
\text{LHS of (EC.10.7)} = R_1(c - s) + (P_1 - s)E^M[A_T1\{A_T \leq R_1\}] - (c - s)R_1
\]

\[= (P_1 - s)E^M[A_T1\{A_T \leq R_1\}]
\]

\[= E^M[A_T1\{A_T \leq R_1\}] \frac{c - s}{P_1 - s},\]

where the last equality is due to: \(P^M(A_T \geq R_1) = (c - s)/(P_1 - s)\).

Now, we focus on the RHS of (EC.10.7). We first show that it is positive. Using \(V_0(P_3, R^{NV}) = m > 0\) and rearranging terms of \(V_0(P_3, R^{NV})\), we have:

\[0 < m = V_0(P_3, R^{NV}) = (-bP_3)(P_3 - c) + (P_3 - s)E^M(R^{NV} \land A_T) - (c - s)R^{NV}.
\]

Note \(P_3 > P_2 > c\) and thus \((-bP_3)(P_3 - c) < 0\), which implies

\[
\text{RHS of (EC.10.7)} = (P_3 - s)E^M(R^{NV} \land A_T) - (c - s)R^{NV} > 0.
\]

Next, we have:

\[E^M[R^{NV} \land A_T] = 2bP^{NV(M)}(R^{NV}) - bc = 2b\bar{P}^{NV(M)}(R^{NV}) + 2bs - bc = 2b\bar{P}^o + 2bs - bc,
\]

where the last equality follows from the definition of \(\bar{P}^o\) in (37). By (EC.10.2) and definition of \(r^o\) in (37), it is straightforward to verify the following:

\[\bar{P}_3 \equiv P_3 - s = \frac{\bar{P}^{NV}}{r^o + \sqrt{(r^o)^2 - 1}}.
\]

Therefore, the RHS of (EC.10.7) is rewritten as:

\[
\text{RHS of (EC.10.7)} = \frac{\bar{P}^{NV}(2b\bar{P}^o + 2bs - bc)}{r^o + \sqrt{(r^o)^2 - 1}} - (c - s)R^{NV} (> 0).
\]
Combining the above and dividing both sides by $c - s$, the inequality (EC.10.7) is equivalent to:

$$
\frac{E^M[A_T 1\{A_T \leq R_1\}]}{P^M(A_T \geq R_1)} \leq \frac{P^{NV}(2b\bar{P}^o + 2bs - bc)}{(c - s)(r^o + (r^o)^2 - 1)} - R^{NV}.
$$

(EC.10.8)

In particular, the RHS of (EC.10.8) is positive.

Next, we show that (EC.10.8) produces an upper bound on $R_1$, $R^o$, hence by Lemma EC.8 also bounds $R_m^b$. Define the following function in $R$:

$$
T(R) = \frac{E^M[A_T 1\{A_T \leq R\}]}{P^M(A_T \geq R)}.
$$

Taking derivative of $T(R)$ leads to:

$$
T'(R) = \frac{f^M(R)}{[P^M(A_T \geq R)]^2} E^M(R \land A_T),
$$

where $f^M(r)$ is the probability density function of $A_T$ under $P^M$. Note, by $A_T^M \preceq A_T$ and Assumption 2,

$$
E^M(R^{NV} \land A_T) \geq E(R^{NV} \land A_T) = 2bP^{NV} - bc > 2bc - bc = bc > 0.
$$

So, clearly $T'(R) > 0$ for $R \geq R^{NV}$. Thus, $T(R)$ strictly increases in $R$ for $R \in [R^{NV}, \infty)$.

Next, we show $R(R^{NV})$ is smaller than RHS of (EC.10.8). There are two cases: $E^M[A_T 1\{A_T \leq R^{NV}\}] < 0$ or $E^M[A_T 1\{A_T \leq R^{NV}\}] \geq 0$. For the first case, clearly $T(R^{NV}) < 0$ and thus smaller than RHS of (EC.10.8). Now, suppose the other case holds, i.e., $E^M[A_T 1\{A_T \leq R^{NV}\}] \geq 0$, then we have:

$$
T(R^{NV}) = \frac{E^M[A_T 1\{A_T \leq R^{NV}\}]}{P^M(A_T \geq R^{NV})} \leq \frac{P_3 - s}{c - s} E^M[A_T 1\{A_T \leq R^{NV}\}]
$$

$$
= \frac{P_3 - s}{c - s} \left( E^M[A_T 1\{A_T \leq R^{NV}\}] + R^{NV} \frac{c - s}{P_3 - s} \right) - R^{NV}
$$

$$
\leq \frac{P_3 - s}{c - s} \left( E^M[A_T 1\{A_T \leq R^{NV}\}] + P^M(A_T \geq R^{NV})R^{NV} \right) - R^{NV}
$$

$$
= \frac{P_3 - s}{c - s} E^M[R^{NV} \land A_T] - R^{NV} = \frac{P^{NV}(2b\bar{P}^o + 2bs - bc)}{(c - s)(r^o + (r^o)^2 - 1)} - R^{NV}
$$

$$
= \text{RHS of (EC.10.8)}.
$$

The two $\leq$ involved in the derivations above uses the fact $P^M(A_T \leq R^{NV}) \leq (P_3 - c)/(P_3 - s)$, which follows from (based on the proved fact that $R_1 \geq R^{NV}$ and $P_3 \geq P_1$):

$$
P^M(A_T \leq R^{NV}) \leq P^M(A_T \leq R_1) = \frac{P_1 - c}{P_1 - s} \leq \frac{P_3 - c}{P_3 - s}, \quad \text{and thus} \quad P^M(A_T \geq R^{NV}) \geq \frac{c - s}{P_3 - s}.
$$

Combining both cases, we have:

$$
T(R^{NV}) \leq \text{RHS of (EC.10.8)}.
$$
Furthermore, since $E^M(A_T) \geq E^M(R^{NV} \land A_T) \geq E(R^{NV} \land A_T) = 2bP^{NV} - bc > 0$, clearly $T(R) \to \infty$ as $R \to \infty$. Recall, $T(R)$ strictly increases in $R$ for $R \geq R^{NV}$, the analysis above indicates that there exist a unique $R^* \geq R^{NV}$ such that:

$$T(R^*) = \frac{E^M[A_T 1\{A_T \leq R^*\}]}{P^M(A_T \geq R^*)} = \text{RHS of (EC.10.8)} = \frac{P^{NV}(2b\overline{P} + 2bs - bc)}{(c - s)(r^0 + \sqrt{(r^0)^2 - 1})} - R^{NV}.$$ 

Combining the proved fact that $R_1 \geq R^{NV}$ with $T(R_1) \leq \text{RHS of (EC.10.8)}$, $R_1 \leq R^*$. Applying Lemma EC.8, $R^*_m \leq R_1 \leq R^*$ immediately follows.

What remains is to show $R^* \leq R^{NV(M)}$. Let $R = R^{NV(M)}$ and use $P^M(AT \geq R^{NV(M)}) = (c - s)/(P^{NV(M)} - s)$:

$$T(R^{NV(M)}) = \frac{E^M[A_T 1\{A_T \leq R^{NV(M)}\}]}{P^M(A_T \geq R^{NV(M)})} = \frac{P^{NV(M)} - s}{c - s}E^M[A_T 1\{A_T \leq R^{NV(M)}\}] = \text{(EC.10.9)}.$$ 

Here we show $E^M[A_T 1\{A_T \leq R^{NV(M)}\}] > 0$ (hence $T(R^{NV(M)}) > 0$) as follows. First note that

$$0 < m = \mathbb{E}[H_T(P^{NV}, R^{NV})] \leq \mathbb{E}[H_T(P^{NV}, R^{NV})] \leq \max_{P,R} V_0(P, R) = V_0(P^{NV}, R^{NV})$$

(the first $\leq$ is due to $A_T^M \leq A_T$). Next,

$$V_0(P^{NV(M)}, R^{NV(M)}) = (P^{NV(M)} - c)(P^{NV(M)} - bP^{NV(M)}) - (P^{NV(M)} - s)E^M[(P^{NV(M)} - A_T)^+] = (P^{NV(M)} - c)(P^{NV(M)} - bP^{NV(M)}) - (P^{NV(M)} - s)R^{NV(M)}P^M(A_T \leq R^{NV(M)}) + (P^{NV(M)} - s)E^M[A_T 1\{A_T \leq R^{NV(M)}\}] = (-bP^{NV(M)})(P^{NV(M)} - c) + (P^{NV(M)} - s)E^M[A_T 1\{A_T \leq R^{NV(M)}\}] > 0.$$ 

The third equality uses the optimality equation $P^M(A_T \leq R^{NV(M)}) = (P^{NV(M)} - c)/(P^{NV(M)} - s)$.

Since $(-bP^{NV(M)})(P^{NV(M)} - c) < 0$ (by $P^{NV(M)} > c > s$), we must have $E^M[A_T 1\{A_T \leq R^{NV(M)}\}] > 0$.

Let $R_3 := F^{-1}_M((P_3 - c)/(P_3 - s))$, i.e., $R_3 = R^{NV(M)}(P_3)$. As $P_2 < P_3$,

$$R^{NV} = R^{NV(M)}(P_2) \leq R^{NV(M)}(P_3) = R_3.$$ 

By $P_3 \leq P^{NV}$,

$$R_3 = R^{NV(M)}(P_3) \leq R^{NV(M)}(P^{NV}) \leq R^{NV(M)}.$$ 

Combining the above, we have $R^{NV} \leq R_3 \leq R^{NV(M)}(P^{NV}) \leq R^{NV(M)}$. Let

$$g(R) := \frac{P_3 - s}{c - s}E^M[R \land A_T] - R.$$ 

Clearly, $g(R^{NV}) = \text{RHS of (EC.10.8)}$. Taking derivative of $g(R)$, we have

$$g'(R) = \frac{P_3 - s}{c - s}P^M(A_T \geq R) - 1 \geq 0, \quad \text{for any } R \leq R_3,$$
since $P^M(A_T \geq R) \geq (c - s)/(P_3 - s)$ for $R \leq R_3$. Thus, $g(R^{NV}) \leq g(R_3)$. Now we check $g(R_3)$:

$$g(R_3) = \frac{P_3 - s}{c - s} E^M[R_3 \wedge A_T] - R_3$$

$$= \frac{P_3 - s}{c - s} E^M[A_T \{A_T \leq R_3\}] + \frac{P_3 - s}{c - s} R_3 P^M(R_3 \leq A_T) - R_3$$

$$= \frac{P_3 - s}{c - s} E^M[A_T \{A_T \leq R_3\}]$$

$$\leq \frac{P_{NV(M)} - s}{c - s} E^M[A_T \{A_T \leq R^{NV(M)}\}]$$

$$= T(R^{NV(M)}).$$

The third $=$ is due to the definition of $R_3$: $P^M(R_3 \leq A_T) = (c - s)/(P_3 - s)$ and the last $=$ is due to (EC.10.9). The $\leq$ is due to $P^{NV(M)} \geq P^{NV} \geq P_3 > c > s$, $E^M[A_T \{A_T \leq R^{NV(M)}\}] > 0$ and $R^{NV(M)} \geq R_3$. Therefore,

$$T(R^0) = \text{RHS of (EC.10.8)} = g(R^{NV}) \leq g(R_3) \leq T(R^{NV(M)}),$$

which leads to $R^0 \leq R^{NV(M)}$. This completes the proof. \hfill \Box

**EC.11. Proof of Proposition 5**

**EC.11.1. Technical Preparation**

The following lemma is crucial in proving Proposition 5.

**Lemma EC.9.** Given $(P, R)$ and $m = E[H_T(P, R)]$ (i.e. $E(\chi_T^*) = 0$, where $\chi_T^*$ is the terminal wealth attained by the associated optimal hedging strategy specified in Theorem 1), the risk reduction from the base model can be expressed as:

$$\text{Var}(H_T(P, R)) - B(m, P, R) = \int_0^T E\left[\frac{(\sigma_t X_t)^2 Z_t}{Z_t^M} y_t^2(P, R)\right] dt,$$

(EC.11.1)

where $y_t(P, R) = \xi_t(P, R) + \frac{\xi_t}{Z_t^M} (\lambda - V_t(P, R))$ with

$$\lambda = \frac{m Z_0^M - V_0(P, R)}{Z_0^M - 1}.$$

**Proof.** Throughout the proof, the argument $(P, R)$ is dropped whenever possible. We start with the identity:

$$\text{Var}(H_T) = E[(\lambda - H_T)^2] - (\lambda - m)^2.$$

(EC.11.2)

This holds for any constant $\lambda$, and here we choose $\lambda = \lambda_m$ as defined in (26). Next, we derive $E[(\lambda - H_T)^2]$. Note

$$E[(\lambda - H_T)^2] = Z_0^M E^R \left[\left(\frac{\lambda - H_T}{Z_T}\right)^2\right],$$
where the probability measure $P^R$ is defined in (EC.5.1). Use the decomposition representation of $M_T := (\lambda_m - H_T)/Z_T$ (recall $Z_T = Z_T^M$) defined in (EC.5.31, EC.5.32), the above can be represented as:

$$
E[(\lambda_m - H_T)^2] = Z_0^M \left( \sigma^2 \right) + \int_0^T \gamma_t d\tilde{B}_t + \int_0^T \phi_t^H \cdot dN_t^R.
$$

where the second equality uses Itô’s isometry and independence between $N_t$ and $\tilde{B}_t$, as well as (EC.5.38) to express $\gamma_t$ using $\delta_t$, and the change of measure from $P^R$ to $P^M$ following (EC.5.1).

Next, substitute the above into (EC.11.2), we reach:

$$
\text{Var}(H_T) = \frac{1}{Z_0^M} (\lambda_m - V_0)^2 - (\lambda_m - m)^2 + \int_0^T \sigma^2 \left( \frac{\sigma^2}{Z_t^M} \right) dt + Z_0^M \left( \int_0^T \phi_t^H \cdot dN_t^R \right)^2,
$$

where the second equality uses expression of $\lambda_m$ in (26).

The rest is to explicitly express the second term above so as to reach (EC.11.1)). For the derivations below, dynamics of $N_t^0$ and $N_t^1$ in (EC.5.7), the representation of $dZ_t^M$ in (20) and the relationship in (EC.5.3) are used.

$$
Z_0^M \left( \int_0^T \phi_t^H \cdot dN_t^R \right)^2 = Z_0^M \left( \int_0^T \left( - \phi_t^0 (N_t^0)^2 \psi_t - \phi_t^1 N_t^0 (N_t^1 \psi_t - \sigma_t X_t) \right) dB_t^R \right)^2
$$

$$
= Z_0^M \int_0^T \left( \sigma_t X_t N_t^0 (\xi_t + (\lambda_m - V_t) N_t^0 \xi_t) \right)^2 dt
$$

$$
= \int_0^T \left( \sigma_t X_t N_t^0 (\xi_t + (\lambda_m - V_t) N_t^0 \xi_t) \right)^2 dt
$$

Now note $Z_t^M = E^M (Z_T | F_t) = \text{E}(Z_T^2 | F_t)/E(Z_T | F_t)$, which is equivalent to $Z_t Z_t^M = E(Z_T^2 | F_t)$. Then, the expression above reduces to:

$$
Z_0^M \left( \int_0^T \phi_t^H \cdot dN_t^R \right)^2 = \int_0^T \left[ \int_0^T \left( \sigma_t X_t N_t^0 (\xi_t + (\lambda_m - V_t) N_t^0 \xi_t) \right)^2 dt
$$

where the second equality uses $N_t^0 = 1/Z_t^M$. Substituting the above to (EC.11.3) leads to the expression in (EC.11.1).
EC.11.2. Proof of Proposition 5

To establish the efficient frontier, direct differentiation using the expression of $B(m, P, R)$ in (27) yields (with $C = 1/(Z_0^M - 1) > 0$):

$$\frac{dB(m, P^h, R^h)}{dm} = 2C[m - V_0(P^h, R^h)] + \frac{\partial B(m, P^h, R^h)}{\partial P} \frac{dP^h}{dm} + \frac{\partial B(m, P^h, R^h)}{\partial R} \frac{dR^h}{dm}$$

$$= 2C[m - V_0(P^h, R^h)] \geq 0,$$

where the second equality follows from optimality of $P^h$ and $R^h$ (which makes the partial derivatives vanish) and the $\geq 0$ follows from Proposition 4.

To prove the lower bound of risk reduction, note that $B(m, P^{NV}, R^{NV})$ is the minimum variance when $(P, R) = (P^{NV}, R^{NV})$. As $B(m, P^h, R^h) \leq B(m, P^{NV}, R^{NV})$, we have

$$\text{Var}(H_T(P^{NV}, R^{NV})) - B(m, P^h, R^h) \geq \text{Var}(H_T(P^{NV}, R^{NV})) - B(m, P^{NV}, R^{NV}).$$

Then, accounting for $m = E[H_T(P^{NV}, R^{NV})]$ and applying Lemma EC.9 leads to the stated expression of the risk reduction. This completes the proof. \(\square\)