Review of “Knots” by Alexei Sossinsky, Harvard University Press, 2002, ISBN 0-674-00944-4

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1 Introduction

This is a brilliant and sharply written little book about knots and theories of knots. Listen to the author’s preface.

“Butterfly knot, clove hitch knot, Gordian knot, hangman’s knot, vipers’ tangle - knots are familiar objects, symbols of complexity, occasionally metaphors for evil. For reasons I do not entirely understand, they were long ignored by mathematicians. A tentative effort by Alexandre-Th[e]ophile Vandermonde at the end of the eighteenth century was short-lived, and a preliminary study by the young Karl Friedrich Gauss was no more successful. Only in the twentieth century did mathematicians apply themselves seriously to the study of knots. But until the mid 1980s, knot theory was regarded as just one of the branches of topology: important, of course, but not very interesting to anyone outside a small circle of specialists (particularly Germans and Americans).

Today, all that has changed. Knots - or more accurately, mathematical theories of knots - concern biologists, chemists, and physicists. Knots are trendy. The French “nouveaux philosophes” (not so new anymore) and post-modernists even talk about knots on television, with their typical nerve and
incompetence. The expressions “quantum group” and “knot polynomial” are used indiscriminately by people with little scientific expertise. Why the interest? Is it a passing fancy or the provocative beginning of a theory as important as relativity or quantum physics?"

Sossinsky continues for 119 pages in that quick, vivid, irreverent way, telling the story of knots and knot theory from practical knot tying to speculations about the relationship of knots and physics. I will first give a personal sketch of some history and ideas of knot theory. Then we will return to a discussion of Sossinsky’s remarkable book.

2 A Sketch of Knot Theory

Knot theory had its most recent beginnings in the nineteenth century due to the curiosity of Karl Friedrich Gauss, James Clerk Maxwell and Peter Guthrie Tait, and the energy of Lord Kelvin (Sir William Thomson). The latter had the popular physical theory of the day with the hypothesis that atoms were three dimensional knotted vortices in the all pervading ether of space. Thomson enlisted the aid of mathematicians Tait, Little and Kirkman to produce the first tables of knots, with the hopes that these tables would shed light on the structure of the chemical elements. Eventually, this theory foundered, first on the vast prolixity of knotted forms and later in the demise of the etheric point of view about the nature of space. But at the same time, mathematical concepts of manifolds and topology were coming forth in the hands of Gauss, Riemann and later Poincare. With these tools it became possible to analyze topological phenomena such as knotting and the properties of three dimensional manifolds. It was Poincare’s fundamental group (of a topological space) that became the first significant tool in knot theory. From the properties of the fundamental group of the complement of the knot, Max Dehn (in the early 1900’s) was able to prove the knottedness of the trefoil knot and its inequivalence to its mirror image. In this way the deep question of detecting the chirality of knots was born. In the 1920’s James W. Alexander of Princeton University discovered a polynomial invariant of knots and links [1] that enabled many extensive computations. Alexander’s polynomial could not distinguish knots from their mirror images, but it was remarkably effective in other ways, and it was based upon ideas
from the fundamental group and from the structure of covering spaces of the knot complement. In fact, Alexander based the theory of his polynomial invariant of knots on the newly discovered Reidemeister moves, expressing the topological equivalence relation for isotopy of knots and links in terms of a language of graphical diagrams. (Reidemeister’s exposition of the moves can be found in his book [25].) Alexander’s version of his polynomial was expressed by the determinant of a matrix that one can read directly from the knot diagram. Invariance of the polynomial is proved by examining how this determinant behaved under the Reidemeister moves. Reidemeister wrote the first book on knot theory and based it upon his moves.

It would take another sixty years to realize the power inherent in Reidemeister’s approach to knot theory. Topology evolved from the 1920’s onward with the seminal work of Seifert [27] on knots and three manifolds and the rapid evolution of algebraic topology. In the early 1950’s the precise role of the fundamental group in knot theory was made clear by the work of Ralph Fox [9] and his students. Fox showed how one could extract the analogues of Alexander polynomials directly from the presentations of the groups by a remarkable algebraic technique (derived from the theory of covering spaces) called the free differential calculus. It was a non-commutative and discrete version of Newtonian calculus, adapted to the needs of algebraic topology and combinatorial group theory.

Then in the late 1960’s John Horton Conway published a startling paper [7] in which he showed how to compute Alexander polynomials without any matrices, free calculi or determinants. His method relied on a recursive formula that expressed the Conway version of the Alexander polynomial in terms of simpler knots and links. The Conway skein theory was met by puzzlement on the part of topologists. It took about ten years for knot theorists to start thinking about the Conway approach, and the first thoughts became proofs of various sorts that the Conway method was valid and that it resulted in a normalized version of the Alexander polynomial. The author of this review was one of those captured souls, who puzzled about the Conway magic. He found two approaches to it. The first [12] went back to techniques of Seifert. The second [13] went back to Alexander’s original paper. The second approach rewrote and normalized Alexander’s determinant, converting it to a sum over combinatorial states of the link diagram. This sum over
states made it easy to prove that the resulting polynomials satisfy Conway’s identities and that they are invariant under the Reidemeister moves. The state sum produced a fully combinatorial (graph theoretic) way to understand Alexander’s original determinant. The state summation is analogous to certain sums in graph theory and to partition functions in statistical mechanics. The full significance of this analogy was not apparent in 1980/81 when these relations were discovered.

At this point it is worth making a digression about the Reidemeister moves. In the 1920’s Kurt Reidemeister proved an elementary and important theorem that translated the problem of determining the topological type of a knot or link to a problem in combinatorics. Reidemeister observed that any knot or link could be represented by a diagram where a diagram is a graph in the plane with four edges locally incident to each node, and with extra structure at each node that indicates an over-crossing of one local arc (consisting in two local edges in the graph) with another. See Figure 1. The diagram of a classical knot or link has the appearance of a sketch of the knot, but it is a rigorous and exact notation that represents the topological type of the knot. Reidemeister showed that two diagrams represent the same topological type (of knottedness or linkedness) if and only if one diagram can be obtained from another by planar homeomorphisms coupled with a finite sequence of the Reidemeister moves illustrated in Figure 2. Each of the Reidemeister moves is a local change in the diagram that is applied as shown in this Figure.

\[\text{Figure 1 - A Knot Diagram}\]
The first move is special for a number of reasons. One can permute the
performance of the first move with the other moves. So one can save up
the doings of the first Reidemeister move until the very end of a process, if
one so desires. And from a physical point of view, there are good reasons to
not use the first Reidemeister move. The move is designed to remove a curl
in the line, but a curl in a rope does not go away, it just gets hidden as a
twist when you pull on the rope. The reader should try this with a bit of
string or a rubber band. Form a curl as in Figure 3 and then pull on the
ends of the string or band. You will find that the curl is transmuted into a
twist, and if you relax the string or band, the curl can reappear. For this
reason, it is useful to consider just the equivalence relation generated by the
second and third Reidemeister moves. This relation is called *regular isotopy*.
Regular isotopy was first defined by Bruce Trace in [32] and has turned out
to be a useful companion to the full equivalence relation defined by all three
Reidemeister moves. One way of thinking about regular isotopy is that one is
talking about a *framed knot or link* where, by framing one means that there
is an embedding of a band (i.e. the cross product of a circle with the unit

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**Figure 2 - Reidemeister Moves**

I

II

III

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interval) for each component of the link, so that the bands do not intersect one another. The bands can twist and this twisting models the twisting of a physical rope or a rubber band. In order to model embedded bands with curly diagrams we actually do not just remove the first Reidemeister move, but we replace it by the move illustrated in Figure 3. Note that there are two ways, shown in this Figure, to make a curl out of a full twist in a band. These two curls are equated with one another and this becomes the replacement for the first Reidemeister move. With this replacement, one refers to the equivalence classes as framed links in the blackboard framing (since these diagrams can be drawn on a blackboard). We shall speak about “measuring the framing” when speaking about the curls in a diagram taken up to regular isotopy because of this connection with framed links.

![Figure 3 - Framing Equivalence](image)
In 1984 Vaughan Jones dropped a bombshell [10] from which knot theory and indeed modern mathematics has yet to recover. By following an analogy between the structure of Artin Braid Group and certain identities in a class of von Neumann algebras, Jones discovered new representations of the braid group and used these representations to produce an entirely new Laurent polynomial invariant of knots and links. On top of this, Jones showed that his new (one variable) invariant satisfied a skein relation that was almost the same as the relation for the Conway polynomial. Only the coefficients were changed. This was shocking. On top of that, the Jones polynomial could distinguish many knots from their mirror images, leaving the Alexander polynomial in the dust.

Jones’ invariant was quickly generalized to a two-variable polynomial invariant of knots and links that goes by the acronym “Homflypt” polynomial after the people who discovered it: Hoste, Ocneanu, Millett, Freyd, Lickorish, Yetter, Przytycki and Traczyk. There were collaborations with Millett and Lickorish working together, Freyd and Yetter working together and Przytycki and Traczyk working together. This generalization was proved to have its properties in a number of different ways: direct induction on the properties of knot and link diagrams, algebras related to skeins of knots and links, representations of Hecke algebras (generalizing the von Neumann algebras used by Jones), some category theory in the bargain. A few months went by, and then Brandt, Millett and Ho discovered a new one-variable invariant of unoriented knots and links with a different skein relation. This invariant did not detect the difference between knots and their mirror images, but there was a new idea in it, namely that one could smooth an unoriented crossing in two different ways (as shown in Figure 4). When the author of this review received a copy of the announcement of this invariant, he was inspired and astonished to realize that their invariant could be generalized to a two variable polynomial invariant $L_K(z, a)$ of knots and links that did detect the difference between many links and their mirror images. The key idea for this generalization is to regard the original invariant as an invariant of framed links, adding an extra variable that measures the framing. The reviewer had earlier found an analogous way to understand the Homflypt polynomial. This was not published until later in [14, 15].
The invariant $L_{K}$ satisfies the following formulas

\[ L_{\chi} + L_{\overline{\chi}} = z(L_{\bowtie} + L_{\gamma}) \]
\[ L_{\gamma} = aL_{\bowtie} \]
and
\[ L_{\overline{\gamma}} = a^{-1}L_{\bowtie} \]

where the small diagrams represent parts of larger diagrams that are identical except at the site indicated in the diagram. We take the convention that the letter chi, $\chi$, denotes a crossing where the curved line is crossing over the straight segment. The barred letter denotes the switch of this crossing, where the curved line is undercrossing the straight segment. In this formula we have used the notations $\bowtie$ and $\gamma$ to indicate the two new diagrams created by the two smoothings of a single crossing in the diagram $K$. That is, the four diagrams differ at the site of one crossing in the diagram $K$. Here $\gamma$ denotes a curl of positive type as indicated in Figure 5, and $\overline{\gamma}$ indicates a curl of negative type, as also seen in this figure. The type of a curl is the sign of the crossing when we orient it locally. Our convention of signs is also given in Figure 5. Note that the type of a curl does not depend on the orientation we choose. The small arcs on the right hand side of these formulas indicate the removal of the curl from the corresponding diagram.
The polynomial $L_K(z, a)$ is invariant under regular isotopy and can be normalized to an invariant of ambient isotopy by the definition

$$F_K(z, a) = a^{-w(K)}L_K(z, a),$$

where we chose an orientation for $K$, and where $w(K)$ is the sum of the crossing signs of the oriented link $K$. $w(K)$ is called the *writhe* of $K$. The convention for crossing signs is shown in Figure 5.

This use of regular isotopy is a key ingredient in defining the polynomial $L_K$, and it turns out to be important in defining many other knot polynomials as well. In the case of the reviewer’s research, discovering $L_K$ opened a doorway to finding a remarkably simple model of the original Jones polynomial the *bracket state sum model* [16, 17]. The bracket state sum is also an invariant of regular isotopy, and can be normalized just like $L_K$ by using the writhe. The idea behind the bracket state sum is the notion that one might form an invariant of knots and links by summing over “states” of the link diagram in analogy with summations over states of physical systems (called partition functions) that occur in statistical mechanics. The reviewer had earlier discovered a model for the Alexander- Conwy polynomial in this form [13] and was convinced that such models should exist for the new knot polynomials. The first case of such a model occurs with the bracket state summation.
The \textit{bracket polynomial}, \(< K > = < K > (A)\), assigns to each unoriented link diagram \(K\) a Laurent polynomial in the variable \(A\), such that

1. If \(K\) and \(K'\) are regularly isotopic diagrams, then \(< K > = < K' >\).

2. If \(K \uplus O\) denotes the disjoint union of \(K\) with an extra unknotted and unlinked component \(O\), then

\[< K \uplus O > = \delta < K >,\]

where

\[\delta = -A^2 - A^{-2}.\]

3. \(< K >\) satisfies the following formulas

\[< \chi > = A < \bowtie > + A^{-1} < >\]
\[< \overline{\chi} > = A^{-1} < \bowtie > + A < >,\]

where the small diagrams represent parts of larger diagrams that are identical except at the site indicated in the bracket. We take the same conventions as described for the \(L-\) polynomial. Note, in fact that it follows from the bracket formulas that

\[< \chi > + < \overline{\chi} > = (A + A^{-1})(< \bowtie > + < >)(>).\]

Making the bracket polynomial a special case of the \(L-\) polynomial.

It is easy to see that Properties 2 and 3 define the calculation of the bracket on arbitrary link diagrams. The choices of coefficients \((A \text{ and } A^{-1})\) and the value of \(\delta\) make the bracket invariant under the Reidemeister moves II and III (see [16]). Thus Property 1 is a consequence of the other two properties.

In order to obtain a closed formula for the bracket, we now describe it as a state summation. Let \(K\) be any unoriented link diagram. Define a \textit{state}, \(S\), of \(K\) to be a choice of smoothing for each crossing of \(K\). There are two choices for smoothing a given crossing, and thus there are \(2^N\) states of a diagram with \(N\) crossings. In a state we label each smoothing with \(A\) or \(A^{-1}\) according to the left-right convention discussed in Property 3 (see Figure 4).
The label is called a vertex weight of the state. There are two evaluations related to a state. The first one is the product of the vertex weights, denoted

\[ < K|S > . \]

The second evaluation is the number of loops in the state \( S \), denoted

\[ ||S||. \]

Define the state summation, \( < K > \), by the formula

\[ < K > = \sum_S < K|S > \delta^{||S||-1}. \]

It follows from this definition that \( < K > \) satisfies the equations

\[ < \chi > = A < \bowtie > + A^{-1} < >, \]
\[ < K \bowtie O > = \delta < K >, \]
\[ < O > = 1. \]

The first equation expresses the fact that the entire set of states of a given diagram is the union, with respect to a given crossing, of those states with an \( A \)-type smoothing and those with an \( A^{-1} \)-type smoothing at that crossing. The second and the third equation are clear from the formula defining the state summation. Hence this state summation produces the bracket polynomial as we have described it at the beginning of the section.

In computing the bracket, one finds the following behaviour under Reidemeister move I:

\[ < \gamma > = - A^3 < \bowtie > \]

and

\[ < \overline{\gamma} > = - A^{-3} < \bowtie > \]

where \( \gamma \) denotes a curl of positive type as indicated in Figure 5, and \( \overline{\gamma} \) indicates a curl of negative type, as also seen in this figure. The type of a curl is the sign of the crossing when we orient it locally. Our convention of signs is also given in Figure 5. Note that the type of a curl does not depend on the orientation we choose. The small arcs on the right hand side of these formulas indicate the removal of the curl from the corresponding diagram.
The bracket is invariant under regular isotopy and can be normalized to an invariant of ambient isotopy by the definition

\[ f_K(A) = (-A^3)^{-w(K)} \langle K \rangle (A), \]

where we chose an orientation for \( K \), and where \( w(K) \) is the writhe of \( K \). By a change of variables one obtains the original Jones polynomial, \( V_K(t) \), for oriented knots and links from the normalized bracket:

\[ V_K(t) = f_K(t^{-\frac{1}{4}}). \]

The bracket model for the Jones polynomial is quite useful both theoretically and in terms of practical computations. One of the neatest applications is to simply compute \( f_K(A) \) for the trefoil knot \( T \) and determine that \( f_T(A) \) is not equal to \( f_T(A^{-1}) = f_{-T}(A) \). This shows that the trefoil is not ambient isotopic to its mirror image, a fact that is quite tricky to prove by classical methods. To this day, it is still an open problem whether there are any non-trivial classical knots with unit Jones polynomial.

After the bracket polynomial model demonstrated the idea of a state summation directly related to the knot diagram there came lots of other examples of state summations, using algebraic machinery from statistical mechanics and Hopf algebras (See [18] for an exposition of some of this development). The subject grew rapidly and then underwent another change in the late 1980’s when Edward Witten showed how to think about such invariants in terms of quantum field theory [34]. Witten’s methods indicated that one should be able to construct invariants of three-manifolds, and in fact David Yetter [35, 36] had shown a similar pattern earlier using categories and Reshetikhin and Turaev [26] showed explicitly how to accomplish this goal using the algebra of quantum groups (Hopf algebras). Witten’s approach brought gauge field theory into the subject of knot invariants. In Witten’s approach there is a Lie algebra valued connection (a differential one-form) defined on three dimensional space. One can measure the trace of the holonomy of this connection around a knotted loop \( K \). This measurement is called the Wilson loop \( W_K(A) \). The Wilson loop is not itself a topological invariant of \( K \), but a suitable average of \( W_K(A) \) over many connections \( A \) should be a (framed) invariant. Witten suggested the specific form of this averaging process, and indeed that average works just as advertised at a formal level.
The formalism leads to the idea of a topological quantum field theory due jointly to Witten and Atiyah [2], a concept that has been highly influential since that time. Witten’s techniques are highly suggestive of fully rigorous combinatorial approaches. His work continues to act as a catalyst for new approaches to the structure of the invariants, and as a connection of the subject to mathematical physics. One of the most exciting aspects of the Wilson loop formulation is that it leads to connections between knot theory and theories of quantum gravity [28, 29] and string theory [22, 21].

Other approaches to link invariants grew out of Witten’s work, most notably the theory of Vassiliev invariants [33]. Vassiliev invariants were first seen as coming from considerations of the topology of the space of all knots and singular knots, but were then quickly connected to combinatorial topology [6] on the one hand, and to Witten’s work [3, 4] on the other. In [4] there is an excellent account of the approach to Vassiliev invariants via the Kontsevich integrals and in [18, 19] the reader will find a heuristic account telling how the Kontsevich integrals arise in the Feynman diagram expansion of Witten’s functional integral. The result of this evolution has been a very clear view of just how it is that Lie algebraic structures are related to invariants of knots and links. The reader should recall that a Lie algebra is a linear algebra with a non-associative multiplication (here denoted \(ab\)) such that \(ab = -ba\) for all \(a\) and \(b\) in the algebra, and such that \(a(bc) = (ab)c + b(ac)\) (the Jacobi identity) for all elements \(a, b, c\) in the algebra. A diagrammatic version of the Jacobi identity can be seen just beneath the surface of the Reidemeister moves, when one takes the Vassiliev point of view, and it is this transition through diagrammatic (or categorical) algebra that makes a deep connection between knot theory, algebra and mathematical physics.

Three recent developments are worth mentioning. First is the discovery of a generalization (categorification) of the bracket polynomial by Misha Khovanov [11, 5] that writes the original Jones polynomial as an Euler characteristic of a complex whose graded cohomology leads to new invariants and to invariants of surfaces in four-dimensional space. One extraordinary outgrowth of the Khovanov work is the deep work of Ozwath and Szabo [23] finding a categorification of the Alexander polynomial that is related to the state sum in [13]. The second is the discovery by Morwen Thistlethwaite
of examples of non-trivial links that have the same Jones polynomial as trivial links, and the extension of this result by Eliahou, Kauffman and Thistlethwaite [8] to infinite families of links with this property. The third is the discovery of virtual knot theory [20], a generalization of classical knot theory to knots in abstract surfaces where there exist infinitely many non-trivial virtual knots with unit Jones polynomial. We do not yet know if any of these virtual counterexamples will yield classical knots with unit Jones polynomial. The fundamental problem remains open.

3 Sossinsky’s “Knots”

The book “Knots” [30] by Alexei Sossinsky is for general readers. It is a companion piece for the book “Knots, Links, Braids and 3-Manifolds - An Introduction to the New Invariants in Low-Dimensional Topology” [24] by Prasolov and Sossinsky, published in translation by the American Mathematical Society. The latter book is not under review here, but is recommended to all readers who enjoyed reading the book that is under review. Sossinsky’s “Knots” begins with a very nicely illustrated tour of practical knots and decorative knotwork. It is recalled how Lord Kelvin (Sir William Thomson) in the 1860’s had a theory of vortex atoms in which the atomic constituents of matter were to be modeled on three dimensional vortices, knots in the ether. This theory inspired Kelvin to enlist the aid of mathematicians Tait, Kirkman and Little to construct the first tables of knots. The vortex theory eventually came to a stop with the rejection of the ether in favor of relativity, but the idea of knots related to physics lives on to this day. Sossinsky’s book describes this beginning for knot theory and then continues with a discussion of topological equivalence, wild knots, braids and the braid group and a proof of the basic theorem of Alexander that any knot can be represented as a closed braid. Sossinsky then goes on to describe a modern algorithm due to Pierre Vogel for transforming a knot into a braid, and a new algorithm of Dehornoy for finding a minimal representation of a braid. Chapter 3 discusses planar diagrams of knots and the Reidemeister moves. Chapter 4 discusses the arithmetic of knots under connected sum, and gives a sketch of one proof of the theorem that you cannot cancel knots. That is, if $K$ and $K'$ have an unknotted connected sum, then $K$ and $K'$ are individually unknotted. This is a fact that people often find surprising at first and the proof
given by Sossinsky uses a technique known to topologists as “the method of infinite repetition”. This method of proof is probably more astonishing than the Theorem! The chapter ends with a discussion of the factorization of knots into prime knots. Here is the beginning of a fruitful analogy of knots and numbers that is really just beginning to be explored.

Chapter 5 is packed with a number of things. There is a discussion of Conway’s approach and reformulation of the Alexander polynomial. Conway’s approach uses oriented diagrams and involves two operations that can be performed at a diagrammatic crossing. One can switch the two strands, or one can smooth the crossing by reattaching the strands, preserving orientation and removing the crossover in the process. These two operations are reminiscent of the operations of topoisomerase enzymes and of combinatorics of DNA recombination (as was observed some years after Conway devised his methods). Sossinsky takes the opportunity to make a quick digression into the subject of DNA topology. He then discusses how one calculates with Conway’s polynomial, and how natural it is to generalize (with 20/20 hindsight!) this construction of Conway to the 2-variable Homflypt polynomial. Chapter 6 returns the discussion to the original Jones polynomial and constructs the bracket polynomial state summation in some detail, including its relationship with the Potts model in statistical mechanics. The reviewer is pleased with this exposition of the bracket. The reader will find some speculation by Sossinsky on how this construction was discovered. In this review the author of the review has given his version of this story in section 2. Chapter 7 goes on to discuss the Vassiliev invariants. The discussion of this advanced topic is accomplished neatly, with an emphasis on the ideas and the diagrammatic context. In fact, Sossinsky discusses the so-called four term relation for the Vassiliev invariants, and in this way comes right up to the fundamental relationship between knot invariants and Lie algebras. This is an example of the daring nature of this popular exposition. Chapter 8 treats relationships of knot theory with physics, including a bit about statistical mechanics, quantum field theory and quantum gravity.

This book is a tour de force. It is a great read and, I believe that it is a book that can actually be understood by a very wide readership. When large scale topics are discussed, Sossinsky deftly, sometimes ironically, places key ideas before the reader. A myriad of details will have to be dealt with
by one who wants to master these topics but here the ideas are laid bare. I
would recommend this book as a first book on knot theory to anyone who
asks.

There are a few slips and misprints. I will tell them here to the best
of my knowledge. This is a book that deserves many editions and it is
to be assumed that the author will correct these few stumbling blocks in
the very next edition of the book. Page 43, Figure 3.4 shows a diagram
that actually can be reduced to the unknot by Reidemeister moves without
making the number of crossings increase (contrary to the claim in the book).
Diagrams with that property are not hard to construct, but this is not one
of them. On page 70 in Figure 5.7, $A$ is a figure eight knot, contrary to
the book’s assertion, and $B$ is a trefoil knot. On page 71 it is claimed that
the Conway polynomial of the trefoil knot and the figure eight knot are the
same. This is not the case. In the book’s notation we have $1 + x^2$ for the
Conway polynomial of the trefoil knot and $1 - x^2$ for the Conway polynomial
of the figure eight knot. It is important for the next edition to correct this
error, since a beginner in calculating these polynomials could become mighty
confused by such a claim. On page 73 it is said that Louis Kauffman is at the
University of Chicago. In fact, he is at the University of Illinois at Chicago.

I cannot resist ending this review with one more quote from the book.
Sossinsky expresses throughout his amazing enthusiasm for mathematics. In
this quote he is speaking of the last touches, completing the construction and
proof of invariance of the bracket polynomial. He says

“God knows I do not like exclamation points. I generally prefer Anglo-
Saxon understatement to the exalted declarations of the Slavic soul. Yet I
had to restrain myself from putting two exclamation points instead of just
one at the end of the previous section. Why? Lovers of mathematics will
understand. For everyone else: the emotion a mathematician experiences
when he encounters (or discovers) something similar is close to what the
art lover feels when he first looks at Michelangelo’s *Creation* in the Sistine
Chapel. Or better yet (in the case of a discovery), the euphoria that the
conductor must experience when all the musicians and the choir, in the same
breath that he instills and controls, repeat the “Ode to Joy” at the end of
the fourth movement of Beethoven’s *Ninth.*”

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I could not agree more. Read the book!

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