Planetary systems based on a quantum-like model

N. Poveda T.*, N. Vera-Villamizar.,† and N. Y. Buitrago C.‡
Universidad Pedagógica y Tecnológica de Colombia, Grupo de Astrofísica y Cosmología

Planetary systems have their origin in the gravitational collapse of a cloud of gas and dust. Through a process of accretion, a massive star and a disk of planetesimals orbiting the star are formed. Using a formalism analogous to quantum mechanics (quantum-like model), the star-planetesimal system is described and the flow quantizing the gravitational field theoretical model parameters are obtained. Goodness of fit (chi-square) of the observed data with model quantum-like, to the solar system, satellites, exoplanets and protoplanetary disk around HL Tau is determined. Shows that the radius, eccentricity, energy, angular momentum and orbital inclination of planetary objects formed take discrete values depending only on the mass star.

I. INTRODUCTION

In the solar system the distance of the planets from the Sun follow a simple geometric progression, known as the Titius-Bode law: 
\[ r = 0.4 + 0.3 \times 2^n \text{ AU with } n = -\infty, 0, 1, \ldots, 7 \] [Lynch 2003, Neslušan 2004]. However, better agreement using the Bohr equation is obtained: 
\[ r = 0.0425 \times n^2 \text{ AU, with } n = 3, 4, 5, 6, 7, 8, 11, 15, 21, 27 \] [Caswell 1929, Penniston 1936, Barnóthy 1946a, Barnóthy 1946b]. Comparisons of the Titius-Bode law with the equation of Bohr raised the issue of the applicability of certain principles of quantum mechanical in systems orbital of the astronomy [Corliss 1986]. To interpret Bohr’s equation within the framework of classical mechanics, the quantization rule of Bohr-Sommerfeld was used [Agnese & Festa 1997]. These basic ideas were extended to the Jovian satellites [Rubčić & Rubčić 1995, 1996] and extrasolar planets [Rubčić & Rubčić 1996, 1999]. Finally, it has attempted to describe planetary systems with the Schrödinger equation [Reinisch 1998, de Oliveira Neto et al 2004, Smarandache & Christianto 2006, Nie 2011].

In quantum mechanics, Planck’s constant is extremely small, \( h \sim 10^{-34} \text{ J.s} \) and the ratio de Broglie wavelength is inversely proportional to momentum. For a massive object length de Broglie wave is very small, and consequently the wave behavior of a macroscopic object is undetectable, making quantum mechanics reduces to classical mechanics. Thus quantum mechanics is a theory developed to explain phenomena only a microscopic scale.

Planetary systems originate from a nebula which collapses to form a star and a disk of planetesimals orbiting the star. The trajectory of the planetesimals is determined by the principle of stationary action, \( \delta S = 0 \). However, planetesimal is disturbed by the presence of the other, oscillating around the classical trajectory; this causes for a closed and stable trajectory, the action is quantized; that is, the action is an integer multiple of a quantum of action, \( S = nh_s \) (see Appendix A). The quantum of action, \( h_s \) (which plays a role equivalent to Planck’s constant \( h \)) is a free parameter to be determined and depends on the physical system in question. Considering the flux quantization of the gravitational field, the value of \( h_s \) is determined. For a massive object: \( h_s \gg h \), this allows to describe a macroscopic system with the formalism of quantum mechanics (quantum-like model). This means that the formalism of quantum mechanics is applicable to any system that quantize the action. One could use the Bohmian interpretation to give meaning to the formalism.

This implies that planetary systems (planets-Sun, satellites-planet, exoplanets-star, exoplanets-pulsar) must quantize not only the orbital radius as demonstrated, but also eccentricity, orbital inclination, angular momentum and energy. Through the goodness of fit (chi-square) shows that there is a very good agreement between the observed data and the quantum-like model, although the formulation is based only on the interaction between two particles.

II. FLUX QUANTIZATION OF THE GRAVITATIONAL FIELD

In a microscopic system, the electric potential energy between a proton and an electron, is given by, \( U_e = -e^2/4\pi\varepsilon_0 \), and an equivalent macroscopic system: the gravitational potential energy between a star and a planetesimal, is given

*Electronic address: nicanor.poveda@uptc.edu.co
†Electronic address: nelson.vera@uptc.edu.co
‡Electronic address: nidia.buitrago@uptc.edu.co
by, \( U_g = -GM_s m / r \). A relationship of proportionality between the electric potential energy and the quantum of action \( \hbar \) (Planck’s constant), is assumed: \( U_g \sim \hbar \), also between the gravitational potential energy and \( \hbar \) parameter: \( U_g \sim \hbar_s \), which plays the same role of Planck’s constant, in a macroscopic system. We can establish the relationship between these two energies, to obtain a dimensionless independent amount of \( r \), and normalize the macroscopic parameter on a unit, the Planck constant: \( U_g / U_e \sim \hbar_s / \hbar \).

The flow of the gravitational field generated by the mass of a star, through a spherical gaussian surface, is given by: \( \Phi_g = 4\pi GM_s \) and the electric field flux generated by a proton is: \( \Phi_e = e / \epsilon_0 \). If we take the relation between the intensity of the gravitational and electromagnetic force in terms of their flows, \( \Phi = m \Phi_g / \epsilon \Phi_e \), we have, \( \Phi \sim \tilde{\Phi} \). It is considered \( \tilde{\Phi} \), as flow of the quantum of action per unit solid angle: \( \Phi = \tilde{\hbar} / \Omega \) (\( \Omega \) acts as the proportionality constant), we obtain

\[
\frac{4\pi GM_s \hbar}{e^2/4\pi \epsilon_0} = \frac{\hbar_s}{m}
\]

the Bohr radius is given by: \( a_s = (\hbar_s / m)^2 / GM_s \), and the energy of the ground state: \( E_s = GM_s / 2a_s \). We can express these quantities in terms of the solar mass: \( \hbar_s = k \hbar \), \( a_s = ka \), and \( E_s = kE \), where \( k = M_s / M \) called the scale factor. The theoretical parameters are given by: \( h / m = 7.62311 \times 10^{-14} \) Js/kg, \( a = 2.92705 \times 10^{-2} \) AU and \( E / m = -15.1538 \) GJ/kg.

### III. ORBITAL DYNAMICS MODEL

In classical mechanics, the trajectory followed by a planetesimal, orbiting a star, is an ellipse with semi-major axis \( a \) and eccentricity \( \epsilon \). The average value of the distance planetesimal-star during a complete orbit, turns out to be \( \langle r \rangle = a (1 + \epsilon^2 / 2) \). The total mechanical energy is \( E / m = -GM_s / 2a \) and the magnitude of the orbital angular momentum is \( L / m = \sqrt{GM_s a (1 - \epsilon^2)} \). By conservation of angular momentum, in the accretion process of planetesimals, planets must be in the same plane \( \theta = 0 \). The equation of motion for the ith planet orbiting a star is: \( \ddot{r} - r = -G (M_s + m_i) / r^3 + P_i \), where \( P_i \) are small magnitudes, which contain the perturbative effects of all other objects on the ith planet, which is not a point particle; this causes the parameters which determine the orbit, vary periodically. The variation of the orbital kinetic energy, due to the variation of the eccentricity is:\( \Delta E_{\epsilon(i)} / m = GM_s / 2a_s \) \( (1 - \epsilon_i^2) / \epsilon_i - (1 - \epsilon_i^2) / \epsilon_i \)

and, due to the variation of the orbital inclination: \( \Delta E_{\phi(i)} / m = (2GM_s / r_i) \sin^2 [(\theta_i - \phi_i) / 2] \).

In the quantum-like model, by substituting the Hamiltonian, \( \tilde{H} = -\hbar^2 \nabla^2 / 2m - GM_s m / r \) (\( \nabla^2 \) is the Laplacian in spherical coordinates) in the Schrödinger-like equation, we obtain: \( \tilde{E}_n / m = -E / n^2 \), where \( E_s = GM_s / 2a_s \) and \( a_s = (\hbar_s / m)^2 / GM_s \). Also, the average distance, \( \langle r_n, \ell \rangle = a_s / 2 [3n^2 - \ell (\ell + 1)] \); the magnitude of the angular momentum, \( L / m = \sqrt{1 + 1} \hbar_s \), and its orientation, \( \theta, \phi \); the probability of finding a planetesimal in space \( \rho \psi_n (r, \theta, \phi) \) is obtained (see Figure 1).

If the planetesimal obeys the quantum-like model, depending on the distribution of matter and the process of accretion of planetesimals, a large object (planet, dwarf planet or asteroid) is formed on the mean value of the distance \( \langle r_i \rangle \equiv a_s n(n + 1) / 2 \), ie, the object must orbit the star formed, following an elliptical trajectory with semi-major axis \( a = a_s n^2 \), eccentricity \( \epsilon_i = 1 / \sqrt{n} \) and inclination \( \theta_i = \arccos \sqrt{1 - 1} \) (because this is the most probable, and stable closed orbit); with a total energy, \( E / m \equiv -E_s / n^2 \) and magnitude of the angular momentum, \( L / m \equiv \sqrt{1} n(n - 1) \hbar_s \). The orbital inclination and eccentricity of the formed objects changes cyclically over time [Lisiecki 2010], so the observed total energy \( E / m = -GM_s / 2a_s \) should be corrected:

\[
E_c = E_o + \Delta E_{\epsilon(t,o)} - \Delta E_{\phi(t,o)} + \Delta \epsilon.
\] (1)

in this equation \( \Delta \epsilon \) corresponds to other effects, which generally lead to a loss of energy, such as planet-asteroids collisions and others.
IV. THE SOLAR SYSTEM

On the origins of the solar system, the formation of objects by the process of accretion of planetesimals can occur anywhere in the protoplanetary disk, but the stable closed orbits give rise to larger objects, according to the quantum-like model, this occurs in \( a_t = k a_\odot n^2 \). Based on the uncertainty principle, \( \Delta p \Delta a_t \simeq \hbar \), we have that \( \Delta a_t = k a_\odot \). For an average radius of \( \sim 300 \) km, the icy moons and rocky asteroids in our solar system passing from a rounded potato to a sphere \[ \text{Lineweaver 2010} \], for this reason, we selected objects with a diameter greater than Vesta \( \geq 525 \) km (NASA JPL Small-Body Database), smaller objects are discarded because they can easily change orbit, due to collisions or internal dynamic processes (eg, Yarkovsky effect). Assuming that the scaling factor is \( k = 1 \), assigning a quantum number to each object and selecting the of greater diameter: (4) Mercury, (5) Venus, (6) Earth, (7) Mars, (9) Vesta, (10) Ceres, (13) Jupiter, (18) Saturn, (26) Uranus, (32) Neptune, (36) Pluto, (38) Haumea, (39) (2010 KZ39), (40) Makemake, (41) (2013 FZ27), (42) 42301 (2001 UR163), (44) 84522 (2002 TC302), (45) (2013 FY27), (46) (2010 RE64), (48) Eris, (50) 229762 (2007 UK126), (56) 145451 (2005 RM43), (58) (2008 ST291), (95) (2012 VP113), (134) Sedna, (Figure 2). The filling of the orbits depends mainly on the mass of the star and the amount of material available in the protoplanetary disk. The Bohr radius value observed is obtained: \( a_\odot = (2.92007 \pm 0.00256) \times 10^{-2} \) AU, \( \chi^2_{DoF} = 0.30574 \), \( R^2 = 0.99997 \), and percentage error \( e\% = 0.24 \) with respect to the theoretical value.

Figure 2 shows the relationship between the corrected energy \( E_c \), equation (1), and the quantum numbers. Once have formed interior objects orbital radius is maintained, while objects in the energy continuum (\( \geq 30 \) AU) can be easily moved (Kuiper belt). Jupiter is at the inflection point. Is obtained \( E_c^2 / m = (14.65188 \pm 0.00160) \) GJ/kg, \( \chi^2_{DoF} = 0.00002 \), \( R^2 = 0.99956 \), and \( e\% = 3.31 \). It is assumed that the percentage error is due mainly to the effect they have had the impact of asteroids on the inner solar system objects. In turn, the degree of fit, is evidence of the quantization of angular momentum, ie, the quantization of the orbital inclination and magnitude of the angular momentum in the period of formation of the solar system.

In Figure 4 shows the observed angular momentum, which depends on the eccentricity, \( L_o / m = \sqrt{G M_s a_o (1 - e_o^2)} \); the objects far away from the Sun have very high orbital inclinations, this generates a variation in the orbital eccentricity (1), which in turn produces an anomalous behavior in the angular momentum. Suppressing effect of orbital inclination on the eccentricity, we can obtain a corrected angular momentum: \( L_c / m = \sqrt{G M_x a_o (1 - e_c^2)} \), is obtained: \( h_c^2 / m = (0.76144 \pm 0.00081) \) PJ/kg, \( \chi^2_{DoF} = 0.03682 \), \( R^2 = 0.99992 \), and \( e\% = 0.11 \).

The Sun corresponds the quantum number \( n = 1 \), consequently, its angular momentum is zero and the probability distribution of matter is a sphere, see Figure 1(a), this could explain why the angular momentum of the Sun corresponds only \( \sim 2\% \) of the entire solar system. The quantum numbers \( n = 2, 3, \ldots \), have a probability distribution of matter in the form of concentric toroids, see Figure 1(b), which have different degrees of inclination and orbital velocity, the absence of planets in numbers quantum \( n = 2, 3 \), indicates that objects migrated outwards or planetesimals were captured by the Sun.
Figure 2: Orbital radius (planets): theoretical, $a_t$ (solid line) and observed, $a_o$ (circles).

Figure 3: Energy (planets): theoretical $E_t$ (solid line), observed $E_o$ (gray circles), and corrected $E_c$ (black circles).

### A. Orbital migration

The parameters obtained correspond to the current solar system, but the planets had a different distribution when they formed: it is assumed that the sequence of the inner planets was, (4) Mercury, (5) Venus, (6) Earth, (7) Mars, (8) Vesta and (9) Ceres. The Jovian planets has the sequence given by [Gomes et al 2005]. In this solar system objects representing at least 26 is considered that at the time of its formation, the same objects were present. Taking the maximum goodness of fit (maximum likelihood) for these conditions we obtain the sequence: (4) Mercury, (5) Venus, (6) Earth, (7) Mars, (8) Vesta, (9) Ceres, (13) Jupiter, (16) Saturn, (19) Neptune (21) Uranus, (34) 78799 (2002 XW93), (35) Pluto, (36) Haumea, (37) Makemake, (38) 55565 (2002 AW197), (39) (2010 RF43), (40) 42301 (2001 UR163), (41) 84522 (2002 TC302), (42) (2004 XR190), (43) (2013 FY27), (44) (2010 RE64), (45) 225088 (2007 OR10), (46) Eris, (48) 229762 (2007 UK126), (53) 145451 (2005 RM43), (55) (2008 ST291), (90) (2012 VP113), (127) Sedna and $a_o = (3.25085 \pm 0.00279) \times 10^{-2}$ AU, $\chi^2_{DOF} = 0.29805$, $R^2 = 0.99997$, and a scale factor $k = 1.11062$, therefore the mass of the sun was $M_s \simeq 1.1M_\odot$, which agrees with [Boothroyd et al 1991], and [Guzik & Cox 1995].
B. Satellites

Is have selected the satellites of the planets of the solar system (Planetary Satellite Physical Parameters, JPL, NASA) with diameters $\geq 350$ km. In order to compare with the solar system, has taken the ratio of the orbital radius and the scaling factor ($a_o/k$). By the same procedure, used for the solar system, is obtained: (10:J) Io, (12:S) Mimas, (13:J) Europe, (14:S) Enceladus, (15:S) Tethys, (16:J) Ganymede, (17:S) Dione, (21:S) Rhea, (21:J) Callisto, (23:N) Proteus, (31:S) Titan, (32:U) Ariel (37:U) Umbriel, (48:U) Titania, (53:S) Iapetus, (55:U) Oberon, (171:E) Moon. Was excluded (40:N) Triton because anomalous behavior with respect to the model, shows that is a captured by Neptune object. The values obtained for the parameters are observed: $a_o^2 = (2.92645 \pm 0.00173) \times 10^{-2}$ AU ($\chi^2_{DoF} = 0.26477$, $R^2 = 0.99999$, and $e\% = 0.02$ (see Figure 5). Is obtained: $\hbar_o^2/m = (0.76581 \pm 0.00185)$ PJ/kg, $\chi^2_{DoF} = 0.14702$, $R^2 = 0.99982$, and $e\% = 0.46$; $E_o^2/m = (15.11796 \pm 0.02153)$ GJ/kg, $\chi^2_{DoF} = 1.268 \times 10^{-7}$, $R^2 = 0.99993$, and $e\% = 0.24$. 

Figure 4: Orbital angular momentum (planets): theoretical $L_t$ (solid line), observed $L_o$ (gray circles), and corrected $L_c$ (black circles).

Figure 5: Orbital radius (satellites): theoretical $a_t$ (solid line) and observed $a_o$ (black circles).
This research has made using the Exoplanet Orbit Database and the Exoplanet Data Explorer at exoplanets.org [Wright et al 2011]. Have been selected systems with a single star with known mass and exoplanets, radio and orbital eccentricity known; assigning a quantum number and selecting the most massive, there are 219 exoplanets. In order to compare with the solar system, has taken the ratio of the orbital radius and the scaling factor \((a_o/k)\). The value obtained for the Bohr radius is: 

\[ a_o = (3.03542 \pm 0.00944) \times 10^{-2} \text{ AU}, \chi^2_{\text{DoF}} = 0.00108, R^2 = 0.99777, \text{ and } e\% = 3.70 \] 

(see Figure 6). 

In classical mechanics, by conservation of angular momentum, the objects must be in the same plane, \(\theta \to 0^\circ\) (in the accretion disk), in the quantum-like model there is a quantization of angular momentum and its orientation, objects have higher orbital inclination between the closer you are to the star, this explains the elevated orbital inclinations found for exoplanets (\(\theta \to 90^\circ\)). It have \( h_o/m = (0.76619 \pm 0.002910 \text{ PJ s/kg}, \chi^2_{\text{DoF}} = 0.00681, R^2 = 0.99978, \text{ and } e\% = 3.70 \) (see Figure 7).
Figure 8: Orbital angular momentum (exoplanets): theoretical $L_t$ (solid line), observed $L_o$ (gray circles), and corrected $L_c$ (black circles).

e\% = 0.51.

The total energy shown large variations when the exoplanet is very close to the star (before continuous), this is due to variations in the eccentricity ($\Delta E_\varepsilon(t,o)$) and the inclination ($\Delta E_\theta(t,o)$) of the orbit, is obtained $E_\odot^2/m = (17,57860 \pm 0.36720) \text{ GJ/kg}$, $\chi^2_{Dof} = 0.24013$, $R^2 = 0.88301$, and $e\% = 16.00$.

A. Protoplanetary disk around HL Tauri

Considering a central protostellar mass for HL Tau given by $k = 0.55$ [Beckwith et al 1990, Sargent & Beckwith 1991], has to be formed planets in $a_t = ka_\odot n^2$. Based on uncalibrated image of the protoplanetary disk HL Tauri taken by ALMA (ESO / NAOJ / NRAO) where planets may be forming, several gaps are distinguished $a_t = (10.06, 24.49, 46.94, 68.02)$ AU which correspond to $n = (25, 39, 54, 65)$, respectively.

VI. CONCLUSIONS

The quantum-like model is applicable to any system that quantize the action, therefore, the formalism of quantum mechanics can be used in macroscopic systems where, the quantum of action, $\hbar_s \gg \hbar$. In this particular case, this occurs because the flow of gravitational field is quantized. The dynamics is determined only by the mass of the central object (as the case, star or planet) independently of the mass of the orbiting object. The probability distribution of planetesimals in space and the physical quantities (radius, eccentricity, energy, orbital angular momentum and its inclination) of the objects in the planetary system, take a certain value (quantized). Therefore, planets and exoplanets tend to be formed at a predetermined distance which depends on the mass of the star and the quantity of matter on the disc, obeying $a_t = ka_\odot n^2$; the orbit may be normal or retrograde. Similarly, the angular momentum and its orientation takes certain value, however, because the planets and exoplanets are not punctual objects, occur periodic variations, in the eccentricity and inclination orbital, causing a change in the value of the observed energy, equation (1). The obtained theoretical model can be used as a tool to parameterize and study the formation of planetary systems.
the Hamiltonian coordinate is given by:

\[ H = p_x^2/2m + U(r), \]

where \( p_r, m \) and \( U(r) \) is an effective potential energy \( U(r) = L_0^2/(2mr^2) - GM_\ast m/r \), in turn \( L_0 \equiv p_\theta = mr^2 \theta \) is the angular momentum. As the force is conservative, \( F = -\nabla U \); planetesimal experience a attractive force, for \( r > r_o \), and repulsive force, for \( r < r_o \), while \( r = r_o \) forces cancel each other out. It can easily be shown that \( U(r) \) has a minimum at \( r = r_o \): \( U_o = U(r_o) \), necessarily \( (\nabla U)_{r=r_o} = 0 \) and \( k_o = (\nabla^2 U)_{r=r_o} > 0 \).

The lower energy state corresponds to \( E = U_o \), planetesimal path is a circumference of radius \( r_o \) centered on the star. The planetesimal on an orbital cycle runs its perimeter \( (x_o = 2\pi r_o) \) at a time \( (t_o) \). As, \( \nabla U_o = 0 \) the planetesimal in this kind of orbit equivalently behaves a free particle, in this case the equation of Hamilton-Jacobi is given by: \( (\nabla S)^2/2m + \partial S/\partial t = 0 \). Is denoted by \( x \) the perimeter distance traveled by the planetesimal. As in the Hamiltonian coordinate \( x \) (cyclic), or the time \( t \) does not appear explicitly, there are two conserved quantities: the linear momentum \( (\nabla S \equiv p_o) \), and the total energy of the system \( (\partial S/\partial t \equiv E_o) \). Consequently the action is separable: \( S(x, t) = S_x(x) + S_t(t) = p_o x - E_o t \), and is periodic: \( S(x, t) = S(x + x_o, t + t_o) \).

Bertrand’s theorem shows that under any initial condition, the only potential producing stable orbits have the functional form \( \sim 1/r \) (Kepler) or \( \sim r^2 \) (harmonic oscillator); additionally closes the orbit if the relationship between the radial and orbital frequency is \( \omega/\omega_o = n/m \), where \( n \) and \( m \) relatively prime. By disturbing the orbit of a planetesimal it can be shown that the planetesimal oscillates around its original path. A perturbed orbit if after \( n \) periods of variation of \( r(t_o = n\tau) \), the complete revolutions planetesimal \( m, n\Delta \theta = m2\pi \) is repeated and a revolution in the planetesimal travels a distance \( \lambda = \Delta \theta r_o \), where Bohr’s rule is obtained for the orbits, \( n\lambda = 2\pi r_o \).

Hamilton’s principal function vanishes for: \( S(0, 0) = S(x_o, t_o) = n(p_o \lambda) - n(E_o \tau) = 0 \). The terms inside the

ACKNOWLEDGMENTS

We thank the Dirección de Investigaciones (DIN) of the Universidad Pedagógica y Tecnológica de Colombia (UPTC).

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Appendix A: QUANTUM-LIKE MODEL

In the protoplanetary disk particles condense into small planetesimals of mass \( m \), which orbit the star. Since the mass of the star, \( M_\ast \), is extremely large compared to the mass of planetesimals, \( M_\ast \gg m \), action star prevails and the interaction between them, only results in weak perturbations to their respective orbits. The Hamiltonian (in polar coordinates) for a planetesimal orbiting a star is:

\[ H = p_x^2/2m + U(r), \]

where \( p_x, m \) and \( U(r) \) is an effective potential energy \( U(r) = L_0^2/(2mr^2) - GM_\ast m/r \), in turn \( L_0 \equiv p_\theta = mr^2 \theta \) is the angular momentum. As the force is conservative, \( F = -\nabla U; \) planetesimal experience a attractive force, for \( r > r_o \), and repulsive force, for \( r < r_o \), while \( r = r_o \) forces cancel each other out. It can easily be shown that \( U(r) \) has a minimum at \( r = r_o \): \( U_o = U(r_o) \), necessarily \( (\nabla U)_{r=r_o} = 0 \) and \( k_o = (\nabla^2 U)_{r=r_o} > 0 \).

The lower energy state corresponds to \( E = U_o \), planetesimal path is a circumference of radius \( r_o \) centered on the star. The planetesimal on an orbital cycle runs its perimeter \( (x_o = 2\pi r_o) \) at a time \( (t_o) \). As, \( \nabla U_o = 0 \) the planetesimal in this kind of orbit equivalently behaves a free particle, in this case the equation of Hamilton-Jacobi is given by: \( (\nabla S)^2/2m + \partial S/\partial t = 0 \). Is denoted by \( x \) the perimeter distance traveled by the planetesimal. As in the Hamiltonian coordinate \( x \) (cyclic), or the time \( t \) does not appear explicitly, there are two conserved quantities: the linear momentum \( (\nabla S \equiv p_o) \), and the total energy of the system \( (\partial S/\partial t \equiv E_o) \). Consequently the action is separable: \( S(x, t) = S_x(x) + S_t(t) = p_o x - E_o t \), and is periodic: \( S(x, t) = S(x + x_o, t + t_o) \).

Bertrand’s theorem shows that under any initial condition, the only potential producing stable orbits have the functional form \( \sim 1/r \) (Kepler) or \( \sim r^2 \) (harmonic oscillator); additionally closes the orbit if the relationship between the radial and orbital frequency is \( \omega/\omega_o = n/m \), where \( n \) and \( m \) relatively prime. By disturbing the orbit of a planetesimal it can be shown that the planetesimal oscillates around its original path. A perturbed orbit if after \( n \) periods of variation of \( r(t_o = n\tau) \), the complete revolutions planetesimal \( m, n\Delta \theta = m2\pi \) is repeated and a revolution in the planetesimal travels a distance \( \lambda = \Delta \theta r_o \), where Bohr’s rule is obtained for the orbits, \( n\lambda = 2\pi r_o \).

Hamilton’s principal function vanishes for: \( S(0, 0) = S(x_o, t_o) = n(p_o \lambda) - n(E_o \tau) = 0 \). The terms inside the
move with a group velocity, \( v \) harmonically oscillate around the non-perturbed trajectory. Consequently, the disturbance makes the planetesimal = \( \psi \) being (the wave with a positive real function, \( \omega = \omega / k \)). As we can see, the boundary conditions have the effect of discretizing the wave number and frequency. The solution of the wave equation takes the form: \( \psi_n(x, t) = C_n \exp[i(k_n x - \omega_n t)] \).

However, substituting relations \( (A1) \) in \( \psi_n(x, t) \) there is no correspondence between the wave speed and the speed of planetesimals. Planetesimals to represent using the wave equation, it is necessary to modulate the amplitude of the wave with a positive real function, \( A(x, t) = A(x - v_o t) \), to limit the extent of the wave and cause the envelope to move with a group velocity, \( v_o \). Obtaining, in general:

\[
\Psi(r, t) = C A(r, t) \exp \left[ \frac{i}{\hbar_s} S(r, t) \right],
\]

the exponential allows taking into account the phenomena related to the superposition of waves, while the coefficient corresponds to a distribution function (or probability density) to find the planetesimal in space: \( \rho = |\Psi(r, t)|^2 \). By integrating over all space the complex constant \( C \) can normalize to unity. The probability current is defined as \( J = rv_o \). Conservation of probability is given by the continuity equation:

\[
\frac{\partial \rho}{\partial t} + \nabla J = 0,
\]

the probability density moves in space with the same velocity \( \nabla S/m = v_o \) and follow the path set of planetesimals. The Hamilton-Jacobi equation we add the term \( U_Q = \frac{1}{2} k_o \xi^2 \) which corresponds to the effect of the disturbance,

\[
H + U_Q + \frac{\partial S}{\partial t} = 0,
\]

this implies that the classical force is being affected by another force, which generates the quantization of the action of the physical system: \( F = -\nabla U - \nabla U_Q \). The eikonal equation: \( \nabla^2 A + (2mU_Q/h_s^2)A = 0 \) allows us to define a relationship between \( U_Q \) and \( A \). Substituting this expression in equation \( (A1) \) and with \( (A3) \) we can construct an analogous equation to Schrödinger, which call Schrödinger-like:

\[
H \Psi(r, t) = i\hbar_s \frac{\partial}{\partial t} \Psi(r, t).
\]

From the Hamilton-Jacobi equation \( (A4) \) we obtain the trajectory followed by the particles and from Schrödinger equation \( (A2) \) we obtain the distribution of particles in the space or the probability density, \( \rho(r, t) \).

In the star-planetesimal system planetesimals that quantized action giving rise to stationary waves (or stable trajectories) which store the energy of the disturbance. These waves interfere constructively giving rise to phenomena of resonance or stationary states (when the system with the resonance frequency is disturbed), are described by the eigenvalue equation \( H \phi(r) = E_n \phi(r) \). Planetesimals not quantized action are represented by traveling waves transporting the disturbance energy, these waves end transferring its energy, so that the planetesimals assume the classical trajectory, which is described by \( (A4) \) where \( U_Q = 0 \), i.e, the formalism of classical mechanics. Therefore, we have a privileged system where states appear when \( U_Q \neq 0 \), which can be described with the formalism of quantum mechanics.