On the role of the far fields and of cosmology for the theory of gravitation and for the dark matter problem

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Abstract

We give an estimate of the gravitational field of force exerted on a test particle by the far galaxies, in the frame of the weak field approximation. In virtue of Hubble’s law, the action of the far matter turns out to be non negligible, and even the dominant one. An extremely simplified cosmological model is considered. A nonvanishing contribution is obtained only if the discrete and fractal nature of the matter distribution is taken into account. The force per unit mass acting on a test particle is found to be of the order of $0.1cH_0$, where $c$ is the speed of light and $H_0$ the present value of Hubble’s constant.

1 Introduction

The present paper had a rather occasional origin. During a conference in honour of Claude Froeschlè, held in Spoleto at the end of June 2007, one of us had the opportunity to illustrate some recent results concerning the classical microscopic theory of matter–radiation interaction for a system of dipoles located at the sites of an infinite lattice [1] [2]. Particular emphasis was put on two features, namely, the retarded character of the forces and the globality of the interactions of the infinitely many dipoles composing the system. Such features had indeed played an essential role in the deduction of the main result, i.e., a proof of the Wheeler–Feynman identity. It then naturally occurred that, in a meeting mostly devoted to celestial mechanics, the question would be raised whether such features of retardation (with the
corresponding appearing of the far fields) and of globality may perhaps play some role also in the theory of gravitation. Many discussions in this connection took place with George Contopoulos and Christos Efthymiopoulos.

Motivated in such a way, rather soon we were led to understand that the occurring of far fields in gravitation (as enforced by general relativity), combined with simple considerations of a global cosmological character (namely, Hubble’s law), implies that the dominant contribution to the gravitational field of force on a test particle is provided by the far matter.

On the other hand it became immediately evident that such a fact might have a strong impact on the problem of dark matter, at least in the prototype example in which its existence was first conceived, namely in connection with the virial theorem applied to clusters of galaxies. Indeed, according to that theorem, the virial of the forces (per unit mass and per galaxy) should be equal to the variance $\sigma_v^2$ of the velocity distribution of the galaxies of the cluster, while on the other hand, considering the case of Coma, Zwicky had pointed out [3] [4] that the contribution to the virial due to the visible mass of the cluster is by far smaller than the amount required by the observed velocity variance. This led him to conceive of the presence of dark matter, whose contribution to the virial would restore the balance. Now, in Zwicky’s argument no mention at all is made of the external forces. On the other hand, we had just happened to understand how huge the contribution of far matter to gravitational forces might be. So very naturally we came to conceive that in order to restore the balance in the virial theorem, what was lacking is the taking into account of the gravitational forces of the external matter, rather than of some dark matter inside the cluster.

One has thus the problem of estimating the gravitational force (per unit mass) due to the external galaxies. The first relevant result one obtains in this connection is a negative one, inasmuch as it can be established that the sum of the gravitational forces due to all the external galaxies exactly vanishes, in the approximation in which the external matter is described as a continuous medium with a spherically symmetric density.

So we started looking for some less naive approach, in which the discrete character of the distribution of the far matter as a source of gravitation would be taken into account. The simplest approach appeared to be the statistical one that had been introduced by Chandrasekhar and von Neumann (see the review [5]) in the study of the Newtonian forces acting on a star, due to the surrounding ones. In such an approach, the discreteness of the source matter is trivially taken into account because the sources are introduced as individual point particles, while the statistical character is introduced through a probabilistic description of the positions of such point particles. So we took an analogous approach in estimating the contribution of the far matter to the gravitational field of force on a test particle.

However, the present situation proved to be much more complicated than that considered by Chandrasekhar and von Neumann. Indeed they were
working in the assumption that the positions of the sources were independent identically distributed random variables, while it turns out that such an assumption leads to an estimate of the force which doesn’t qualitatively fit the observed velocity variance. It turns out instead that a qualitative fit with the observations, and even a quantitative one, is obtained if the distribution of matter is assumed to have a fractal character \cite{6} \cite{7} \cite{8} \cite{9}.

This is the way in which we came to conceive of the thesis that general relativity, combined with some features of a cosmological character (Hubble’s law, and the fractal nature of the Universe), entails the existence of a relevant contribution to gravitation due to the far matter, which might even eliminate the need of introducing any dark matter at all.

We hope that it will be possible in the future to provide for such a thesis some rigorous proof in the frame of a cosmological model, with the fractal character of the Universe taken into account. Nevertheless it seems to us that in the meantime it may be worth to illustrate in the simplest possible way, mostly of a heuristic character, the main ideas which are at the basis of such a thesis, by adding some more details to the considerations just sketched. This is indeed the aim of the present paper.

2 The virial theorem, and the role of the external forces

Let us recall what the virial theorem in its simplest form is, comparing the case of Clausius to that of Zwicky. One considers a system $S$ composed of $n$ points. For Clausius $S$ is a gas in a box, for Zwicky it is the Coma cluster, whose “points” are galaxies, immersed in the Universe. One considers Newton’s equations of motion $\ddot{x}_i = F_i/m_i = f_i$, $i = 1, \ldots, n$ (the dot denoting time derivative) where $x_i$ is the position vector of the $i$–th particle with respect to the center of mass of the system $S$, while $F_i$ is the force acting on the $i$–th particle (of mass $m_i$) and $f_i = F_i/m_i$ is the corresponding force per unit mass, i.e., the corresponding acceleration. Then one takes the dot product with $x_i$, and adds over $i$. Performing a time average, under the hypothesis that the system remains confined one immediately obtains

$$\overline{\sigma_v^2} = -\overline{V}/n, \quad (1)$$

where $\sigma_v^2 = (1/n) \sum_i v_i^2$ is the variance of the velocity distribution of the galaxies of the cluster, whereas $V = \sum_i f_i \cdot x_i$ is called the virial of the forces (per unit mass), and overline denotes time–average. This is the form of the theorem suited to the case of gravitation (involving forces per unit mass), whereas the analogous theorem for the case of general forces relates twice the kinetic energy to the virial of the forces.

Notice that one has the decomposition $F_i = F_i^{int} + F_i^{ext}$ of the force acting on the $i$–th particle as the sum of an internal force and an external one (that
exerted by the walls of the box confining the gas in the case of Clausius, and by the galaxies external to Coma in the case of Zwicky). In the case of Clausius one has \( F_{int} = 0 \) for a perfect gas, and in any case the virial of the internal forces is considered to be negligible with respect to the external one. In turn, the virial of the external forces is related to the pressure, and so from the virial theorem one obtains the thermodynamic interpretation of the translational kinetic energy as proportional to temperature. In the case of Coma, instead, Zwicky doesn’t make any mention of the external forces at all, and considers only the internal ones, given by Newton’s gravitational law. He thus finds, analogously to the case of gases, that \( \langle V_{int} \rangle / n \) is just a negligible fraction of \( \sigma^2 \), and so he is led to the conjecture that some non visible mass exists, whose contribution to the virial may restore the balance with the observed velocity variance.

Now, why should the external virial \( V_{ext} \) be neglected at all? The main idea underlying the present paper is that in astrophysics, just as in the case of gases, the virial of the external forces may actually be what is needed to restore the balance in the virial theorem.

3 Relevance of the far matter if Hubble’s law is taken into account. Definition of the model (first part)

So we have the problem of estimating the gravitational field of force due to the external galaxies. From the point of view of general relativity, in the weak–field approximation this amounts to writing down the equations for the geodesic motion when the metric tensor \( g_{\mu\nu} \) is a solution of the Einstein equation with the external galaxies as sources. It is well known that, for small velocities of the test particle, the equation for the geodesic motion is the same as for a point particle with a Lagrangian

\[
L = g_{ik}\dot{x}^i \dot{x}^k + c g_{0k} \dot{x}^k + c^2 g_{00},
\]

where \( c \) is the speed of light (with the summations over the spatial indices \( i \) and \( k \) extending from 1 to 3). So at first sight, neglecting the corrections due to the kinetic energy, the forces per unit mass appear to be the same as if the test particle were in presence of an electromagnetic field having \( g_{00} \) as scalar potential and \( g_{0k} \) as vector potential (although relevant differences exist between the two cases, as particularly emphasized by Zeldovich and Novikov [10]). On the other hand, it is very well known that in the weak–field approximation, writing the metric tensor as a perturbation of the Lorentzian background \( \eta_{\mu\nu} \), namely, as \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \), the perturbation \( h_{\mu\nu} \) turns out to be a solution of the wave equation, so that its components are the familiar
retarded potentials of electrodynamics. In fact one finds

\[ h_{\mu\nu} = \frac{-2\pi G}{c^4} \sum_j M_j \frac{1}{\gamma_j} \left| \frac{2\dot{q}_j^{(j)} \cdot \dot{q}_j^{(j)} - c^2 \eta_{\mu\nu}}{|x - q_j^{(j)}|} \right|_{t=t_{ret}}, \]  

(3)

where \( G \) is the gravitational constant, while \( M_j, q_j^{(j)} \) and \( \gamma_j, j = 1, \ldots, N \), are the mass, the position vector and the Lorentz factor of the \( j \)-th source galaxy, dealt with as a point particle, and the dot denotes derivative with respect to proper time along the worldline of the source. The notation \( q_j^{(j)} \) in place of \( q_j \) was introduced just in this formula, in order to avoid confusion with the tensorial indices.

Formula (3) implies first of all that, according to general relativity in the weak–field approximation, the gravitational field of force presents (in analogy with electrodynamics) both a “classical” Newtonian near–field term, decaying as \( 1/r^2 \), and a far–field one, decaying as \( 1/r \). This in turn implies that the contribution to the gravitational field of force due to the far matter is in principle dominant with respect to that due to the near matter. In other terms, the problem of estimating the external gravitational field of force acting on a localized system such as the Coma cluster, immediately presents itself as a problem of a cosmological character, and this compels us to introduce a cosmological model.

To this end we introduce an oversimplified model. We make reference to a local chart, with Lorentzian coordinates, having as origin the center of mass of the considered localized system (the Coma cluster). As \( h_{\mu\nu} \) depends on each source not only through its position \( q_j \), but also through its velocity, the latter has to be assigned in order that the model be defined. In our simple model, this is obtained by introducing Hubble’s law as a phenomenological prescription, namely, by requiring that for the external galaxies one has

\[ \dot{q}_j = \gamma_j^{-1} H_0 q_j, \quad j = 1, \ldots, N \]  

(4)

(the dot denoting now derivative with respect to the background Lorentzian time). Here, \( H_0 \) is the Hubble constant which, in our extremely simplified model, we take fixed to its present value.

Notice that the Hubble assumption (4) has an essential impact on the size of the gravitational field of force. Indeed, in the field of force one has a term (decreasing as \( 1/r^2 \)) proportional to the velocity of the source, and a term (decreasing as \( 1/r \)) proportional to the acceleration of the source. Thus Hubble’s law (4), which implies that also the acceleration has a contribution proportional to the distance, has the consequence that the term proportional to the acceleration actually doesn’t depend on distance at all. This is the main reason why the far matter gives the dominant contribution to the gravitational field of force.

The considerations of the present section seem to just extend to cosmology the classical arguments that had been put forward in connection
with the Mach principle (see [11], page 102). The main difference being that in such a case, lacking Hubble’s law, the velocities of the sources were neglected. Consequently, only the Newtonian, fast decaying, potential was considered, so that only the near matter, and not the far one, did play a role.

4 The mean metric. Estimate of an effective density which takes the far matter into account

We take now a point of view analogous to that of Chandrasekhar and von Neumann [5], introducing a probabilistic description of the positions of the external galaxies. The present section is devoted to an estimate of the corresponding mean metric of our model, in the assumption that the distribution of the galaxies be spherically symmetric.

By taking the mean, which we denote by $\langle \cdot \rangle$, it is immediately seen that the off–diagonal coefficients of the metric vanish, and that the spatial coefficients are all equal. In fact one finds

$$\begin{align*}
    ds^2 &= \langle g_{\mu\nu} \rangle \ dx^\mu dx^\nu = (1 - \alpha - 3\beta) c^2 dt^2 - (1 + \alpha + \beta) dl^2 \quad (5)
\end{align*}$$

where $dl^2 = dx^2 + dy^2 + dz^2$ and

$$\begin{align*}
    \alpha &= \frac{2G}{c^2} \langle \sum_j M_j |q_j| \rangle \quad (6) \\
    \beta &= \frac{4GH_0^2}{3c^4} \langle \sum_j M_j |q_j| \rangle . \quad (7)
\end{align*}$$

Notice that this expression for $\beta$ was obtained in the approximation of small velocities, for which one may assume $\gamma_j \simeq 1 + (1/2)|\dot{q}_j|^2/c^2$.

In agreement with the assumption of spherical symmetry for the distribution of the galaxies, the means entering the coefficients $\alpha$ and $\beta$ can be estimated as

$$\begin{align*}
    \langle \sum_j M_j |q_j| \rangle &\simeq 4\pi \int_0^{R_0} \rho r dr \\
    \langle \sum_j M_j |q_j| \rangle &< 4\pi \int_0^{R_0} \rho r^3 dr \quad (8, 9)
\end{align*}$$

with a suitable density $\rho(r)$, where $R_0$ is the radius of the present horizon.

If for $\rho(r)$ we take the constant value $\rho_0$ usually considered, namely,

$$\rho_0 = \Omega_0 \frac{3H_0^2}{8\pi G} \quad (10)$$

with $\Omega_0 \simeq 0.05$, one finds that, at the present time $t = 0$, the quantities $\alpha$ and $\beta$ are small of the same order as $\Omega_0$. 6
On the other hand, however, the density $\rho$ has to depend on $r$, because on receding from the center one meets with values corresponding to previous times, and one would even meet with a divergence at $r = R_0$ (the present horizon). Any cosmological model should provide a special choice for the function $\rho(r)$. In order to keep the discussion at a rather simple level, we now introduce a constant density $\rho_{\text{eff}}$ playing the role of an “effective density”, having the properties

$$
\int_0^{R_0} \rho r^2 dr \simeq \rho_{\text{eff}} \frac{R_0^2}{2}, \quad \int_0^{R_0} \rho r^3 dr \simeq \rho_{\text{eff}} \frac{R_0^4}{4}.
$$

For the coefficients $\alpha$ and $\beta$, this leads to the estimates

$$
\alpha \simeq \frac{8\pi G}{c^2} \rho_{\text{eff}} \frac{R_0^2}{2}, \quad (11)
\beta < \frac{16\pi G H_0^2}{3c^4} \rho_{\text{eff}} \frac{R_0^4}{4}, \quad (12)
$$

i.e., with $R_0 H_0 = c$,

$$
\beta < \frac{2}{3} \alpha. \quad (13)
$$

Notice that the quantities $\alpha$ and $\beta$ depend on time not through $H_0$, which was assumed to be constant, but because of the fact that, as time increases, new matter enters by crossing the horizon, which recedes with speed $c$.

One clearly has $\rho_{\text{eff}} > \rho_0$, and it is immediately checked that the condition $1 - \alpha + 3\beta > 0$ leads for $\rho_{\text{eff}}$ to the bound $\rho_{\text{eff}} < \frac{H_0^2}{8\pi G}$. We show now that our model provides a stronger consistency condition, which gives

$$
\rho_{\text{eff}} \simeq \frac{1}{4} 3H_0^2, \quad (14)
$$

namely,

$$
\rho_{\text{eff}} \simeq 5\rho_0. \quad (15)
$$

The consistency condition is the requirement that the expansion rate calculated with the metric (5) does actually coincide with the rate that was introduced into the definition of the model. This is expressed in the form

$$
\frac{d}{dt} \log \frac{R}{1 + \alpha + \beta} = H_0. \quad (16)
$$

From (11) and (12), with $\dot{R}_0 = c$, one has

$$
\dot{\alpha} \simeq \frac{8\pi G}{c^2} \rho_{\text{eff}} R_0 c, \quad (17)
\dot{\beta} \simeq \frac{2}{3} \dot{\alpha}. \quad (18)
$$
Using these expressions for $\dot{\alpha}$ and $\dot{\beta}$, the consistency condition (16) then becomes an algebraic one, which gives for $\rho_{\text{eff}}$ a value which we have rounded off to (14).

In conclusion it has been shown that, in the hypothesis of a spherically symmetric distribution for the external galaxies, the mean metric is a Friedmann one, in which however there appears, in place of the present density $\rho_0$, an effective density $\rho_{\text{eff}}$. The latter takes into account the contribution of the far matter, and is given by $\rho_{\text{eff}} \simeq 5\rho_0$. This is the point where the far matter is actually seen to play a role which substitutes that of dark matter, inasmuch as it provides for the density an increment over the presently measured value $\rho_0$.

5 Estimate of the gravitational forces. The role of the discreteness of the sources, and of the fractal nature of the Universe. Definition of the model (second part)

It is immediately seen that at any point $x$ the gravitational field of force due to the external matter, exactly vanishes if it is computed through the mean metric (5), i.e., if the external matter is described as a continuous medium with a constant effective density. This should be expected, in agreement with Birkhoff’s theorem (we thank Rudolf Thun for kindly pointing this out to us).

We come now to an estimate of the variance of the field of force. It will be seen that the result depends on the further assumptions one introduces concerning the spatial distribution of the external galaxies.

Assume first that the positions $q_j$ of the $N$ galaxies are independent random variables, uniformly distributed with respect to the Lebesgue measure; then the sum defining the perturbation $h_{\mu\nu}$ to the metric is found to grow as $\sqrt{N}$. This indeed is just a consequence of the central limit theorem, because in such a case $h_{\mu\nu}$ is the sum of $N$ independent identically distributed random variables having zero mean and a nonvanishing variance. By the way, such a result is the analogue of that obtained by Chandrasekhar and von Neumann for the case of Newtonian forces, the only difference being that in their case the variance is infinite (due to the divergence of Newton’s force at zero distance, a property which plays no role in our case). For what concerns the corresponding estimate of the virial, one easily sees that with the present assumption it is by far too small to account for the observations, just because the considered sum behaves as $\sqrt{N}$ rather than as $N$ (see later).

So we modify the previous assumption, and consider the case in which the distribution of mass is fractal [6]. This means first of all that the positions of the galaxies are no more independently distributed, and this has the relevant
consequence that $g_{\mu\nu}$ is no more constrained to grow as $\sqrt{N}$, and can instead have a faster growth, as required by the observations. We mention in passing that the fractal nature of the Universe was particularly advocated in recent years [7] [8] [9] in connection with the need to eliminate some inconsistencies that are met in analyzing the galaxy catalogues if uniformity with respect to Lebesgue measure is assumed.

Notice that the removal of the assumption of independence of the positions of the galaxies entails that the analytical computation of the probability distribution of the field of force becomes now a quite nontrivial task, with respect to the much simpler case considered by Chandrasekhar and von Neumann. So we are forced, at least provisionally, to investigate the problem by numerical methods.

We proceeded as follows. We consider altogether the total force per unit mass on a test particle at the origin of the coordinates, due to a system of $N$ galaxies, whose positions are extracted (with the method described in [6]) in such a way that the mass distribution has a fractal dimension, precisely the fractal dimension 2. Such a total force per unit mass is thus a random variable, and its probability distribution can be estimated by extracting a certain number of samples for the collection of the positions of the $N$ galaxies.

Actually, among the terms entering the total force per unit mass, we addressed our attention just to the one which is proportional to the acceleration of the source and thus, as previously mentioned, is the dominating one at large distances. This term, which we denote by $f$, has the form

$$f = \frac{4GH_0^2 M}{c^2} \mathbf{u}$$

where we have introduced the vector $\mathbf{u}$ defined by

$$\mathbf{u}(N) = \sum_{j=1}^{N} \frac{\mathbf{q}_j}{|\mathbf{q}_j|};$$

moreover, the masses of the sources were all put equal to a common value $M$, and the Lorentz factors $\gamma_j$ were put equal to 1 for the reasons to be mentioned in a moment. Notice the extremely simple nature of this force per unit mass (or acceleration). Apart from a multiplicative factor, such a force is just the sum of all the unit vectors pointing to each of the external galaxies. Finally, our attention was addressed to the component of such a force $f$ (or of the corresponding vector $\mathbf{u}$) along a given direction. Such a component of the force $f$ will be simply denoted by $f$; analogously, the corresponding component of $\mathbf{u}$ will be denoted by $u$. The quantity $u$ is the one whose probability distribution was actually determined. The corresponding histogram, obtained through 10,000 samples of configurations of the galaxies, is shown in Fig. [1] for $N = 512,000$. 

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Figure 1: Probability density of the random variable $u$, which is proportional to the force per unit mass due to $N$ external galaxies. The histogram was computed with 10,000 samples of $N = 512,000$ galaxies, by counting the fraction of times the value $u$ belongs to a given interval of width $\sqrt{N}$.

We come now to the dependence of $u$ on the number $N$ of external galaxies, which was made to vary in the range $1000 \leq N \leq 512,000$, with the density kept constant. This means that the positions of the $N$ points were taken to lie inside a cutoff sphere whose volume was made to increase as $N$. For the values of $N$ investigated, the corresponding radius turns out to be so small with respect to the present horizon, that the Lorentz factors $\gamma$ could altogether be put equal to 1, and more in general the special relativistic character of our model was actually justified.

The mean of $u$ turns out to practically vanish for all $N$, while its variance $\sigma_u^2$ is found to grow as $N^2$ (actually, as $0.2N^2$), rather than as $N$, as occurs in the uniform case. This is shown in Fig. 2. We thus conclude that the standard deviation $\sigma_f$ of the component of the force per unit mass along a
direction is proportional to $N$, being given by

$$\sigma_f \simeq 0.2 \frac{4GH_0^2}{c^2} MN = 0.2 \frac{4G}{R_0^2} MN .$$

(21)

Figure 2: The variance $\sigma_u^2$ of $u$ versus the number of galaxies in log-log scale. The dashed line is the curve $\sigma_u^2 = 0.2 N^2$.

We now take such a result, which was obtained for extremely small values of $N$, and extrapolate it up to the present horizon $R_0 = c/H_0$, i.e., we put in formula (21) the actual value of $N$, so that the quantity $MN$ can be identified with the total visible mass of the Universe. In terms of the previously determined effective density $\rho_{\text{eff}}$, we thus set

$$MN = \frac{4}{3}\pi \rho_{\text{eff}} R_0^3 ,$$

i.e., with $R_0 = c/H_0$,

$$MN = \frac{1}{8} \frac{c^3}{GH_0} .$$

(22)
This gives
\[ \sigma_f \simeq 0.1 \, cH_0 \, . \] (23)

On the other hand, if a random variable \( f \) has zero mean and a finite variance \( \sigma_f^2 \), with great probability it will take on values very near to its standard deviation \( \sigma_f \). In such a sense we may say to have found
\[ f \simeq 0.1 \, cH_0 \, , \] (24)

which perhaps constitutes the main result of the present work.

This shows why it is so important that the variance of the force grows as \( N^2 \) and not as \( N \) (which is the growth obtained in the assumption of a uniform, rather than fractal, distribution). Indeed, with a growth as \( N \) one would obtain an estimate of the type \( f \simeq cH_0/\sqrt{N} \) where \( N \) is the number of external galaxies, i.e., essentially \( f \simeq 0 \). In other words, without the fractal hypothesis, the Zwicky procedure of neglecting at all the gravitational contribution of the external matter, would be justified. Instead, in our oversimplified model within the fractal hypothesis, the force per unit mass, i.e., the acceleration, exerted by the far matter on a test particle, is found to have a value of the order of \( cH_0 \), which is the one that is needed in most cases in which the presence of a dark matter is advocated.

6 Application to the virial theorem for the Coma cluster

We can now apply our estimate to the case of the virial theorem for a cluster of galaxies. To this end, first of all one has to assume that locally, in the region of interest, the field of force has somehow a central character, because otherwise the cluster itself could not exist at the considered place. In other words, we are assuming that, for a given realization of the positions of the external galaxies, on the average (over the positions of the internal galaxies) locally the field of force is directed towards a center. However, apart from such a correlation, the intensity of the field of force should not be thought of as a smooth function, being for the rest uncorrelated with respect to the position.

We have to estimate the quantity
\[ \sum_{i=1}^{n} \overline{f_i} \cdot \overline{x_i} \, . \] (25)

In the conditions we have assumed, all the terms of such a sum can be taken to be equal, so that we just have to estimate one of them. We can take
\[ \overline{f_i} \cdot \overline{x_i} \simeq -f \frac{1}{|x_i|} \]
where $f$ is a typical value, given by (24), of the modulus of the projection of $f_i$ on a direction, while one can take $|x_i| \simeq L/4$, where $L$ is the diameter of the cluster. So one finds

$$|V_{\text{ext}}| \simeq \frac{n f L}{4}.$$  \hfill (26)

Inserting for $f$ the expression (24), we thus obtain the result that, according to the virial theorem, the velocity variance of a cluster should obey the law

$$\frac{\sigma_v^2}{c^2} \simeq 0.1 \frac{c H_0 L}{4},$$  \hfill (27)

where $L$ is the linear dimension of the cluster. In the case of Coma, one thus finds a value $\simeq 3 \cdot 10^5 \text{Km}^2/\text{sec}^2$, which is very near to the value $5 \cdot 10^5 \text{Km}^2/\text{sec}^2$ reported by Zwicky.

Notice the linear dependence on $L$ in the formula (27). In this connection one may point out that, if the external force were smooth, by a Taylor expansion about the origin one would have $f_i$ proportional to $x_i$, and this would lead to a virial (and thus also a velocity variance) proportional to $L^2$ rather than to $L$. Instead, the observations seem to require a proportionality to $L$. Apparently, this was first pointed out by Kazanas and Mannheim [12], in a paper in which some data were reported in a range of $L$ covering five orders of magnitude (see [12], Fig. 2, page 539). This property is also confirmed by a dimensional analysis. Indeed, with the parameters entering the problem, the square of a velocity can be formed only as $c^2$, or as $c H_0 L$ or as $(H_0 L)^2$. But the first term is by far too large, the last term by far too small, while the term linear in $L$ is indeed about of the correct order of magnitude.

7 Conclusion

A rather rough estimate has been given of the gravitational acceleration on a test particle due to the far matter, when Hubble’s law and the possible fractal nature of the distribution of galaxies are taken into account. The estimated value is of the order of $c H_0$, which is the one that shows up in several cases in which the presence of dark matter is advocated, and in particular leads, for the velocity variance of the Coma cluster, to a value in agreement with the observations. Thus, the presence of a gravitational field of force per unit mass (or an acceleration) due to far matter should perhaps be considered as a concrete physical effect. This naturally suggests that, concerning the cosmological models, the point of view might be taken that they should be required, as a constraint, to predict an acceleration on test particles which fits the observations.
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