On string field theory for $C \leq 1$ \footnote{to appear in Proceedings of 16th Johns Hopkins Workshop on Current Problems in Particle Theory: Pathways to Fundamental Theories}

A. Marshakov

Theory Department
P.N. Lebedev Physics Institute
Leninsky prospect, 53, Moscow, 117 924, Russia, \footnote{E-mail address: tdparticle@glas.apc.org}

ABSTRACT

I present a short review of our results with S. Kharchev, A. Mironov, A. Morozov and A. Zabrodin on Generalized Kontsevich model which in a sense can be interpreted as unifying “string field theory” for $c < 1$ minimal series coupled to 2d gravity. The problem of interpolation between different models is discussed. It is found that this problem is closely connected with “deformations” within the set of solutions to KP hierarchy, described by a sort of reparameterization of a spectral curve and change of asymptotics of the basis in the Grassmannian. The $c \to 1$ limit is considered along this line.
Recently [1,2,3] a new model was proposed which describes a non-perturbative solution to the $c < 1$ 2d conformal matter coupled to the two-dimensional gravity. It seems now that this model could be considered as the first successful attempt of constructing a *string field theory* or effective theory of string models. It is necessary to point out from the beginning that by string field theory we would mean more than a conventional definition as a field theory of functionals defined on string loops - it must rather mean a sort of effective theory which gives all the solutions to classical string equations of motion (2d conformal field theories coupled to 2d gravity) as its vacua and allows us to consider all of them on equal footing (within the same Lagrangian framework) and maybe even describe the flows between different string vacua. Of course, it should reproduce perturbation expansion around any of these vacua. In this sense, the conventional string field theory was not true effective model, because it contained a *fixed* set of variables which correspond to a concrete vacuum (say, 26 free scalar fields). Therefore, it doesn't have even *a priori* a possibility to make a flow to another classical solution (maybe only except for some simple change of a background), *i.e.* conventional string field theory might describe only some small perturbation around given classical solution in terms of the coordinates equivalent to the matter variables in the Polyakov path integral.

The other side of the problem is connected with the non-perturbative definition of string theory. In conventional approach (critical string) even the perturbative expansion was ill-defined due to the presence of tachyon in the spectrum or in other words due to the instability of the classical solution. The only chance to get a sensible effective theory appears after we make sense to the non-perturbative description. Such description appeared (and exists now only in the case of non-critical and moreover “non-tachyonic” strings) in the language of ordinary matrix models after the double scaling limit has been found [4-7]. The deep connection between matrix models and integrable theories [8] made possible “axiomatic” description of string field theory which was actually formulated in [9] by three following statements:

(i)  
\[ F(T) = \log \tau(T) \]  
\[(1)\]

*i.e.* the partition function (or better generating function) for string correlators is given by logarithm of *tau-function* of the Kadomtsev-Petviashvili (KP) hierarchy
or the particular choice of the KP-solution is determined by so-called string equation.

The notation $L^{-1}$ means that the string equation is the first from the tower of (extended) Virasoro constraints which vanish the $\tau$-function reproducing the partition function (generating function for all the correlators) in the non-perturbative string theory acting as differential operators with respect to the infinite set of KP times $\{T\} = T_1, T_2, \ldots$

(iii) Depending on particular reduction of the KP hierarchy and the particular form of string equation and/or (extended) Virasoro algebra one can describe the series of conformal matter plus 2d gravity where the matter central charges are given by

$$c_{\text{matter}} = c_{p,q} = 1 - 6\frac{(p - q)^2}{pq}$$

with one of the coprime numbers, say $q$ is fixed and determines the particular $q$-reduction of the KP hierarchy (e.g. $q = 2$ corresponds to the Korteweg de Vries (KdV) hierarchy, $q = 3$ - to the Boussinesq hierarchy, and in general fixed finite $q$-reduction is called $q$-th KdV) and $p$ is arbitrary.

Of course, the central charges (3) are known as the central charges of minimal conformal models [10] which can be described (before coupling to gravity) as highly reduced $c = 1 \, 2d$ free-field theories [11,12]. That means that what we know up to now is the only case of “highly-noncritical” string models where the total matter-gravity central charge

$$c_{\text{matter}} + c_{\text{gravity}} = 26$$

is “dominated” by the contribution of 2d gravity. This is far from the case of critical string ($c_{\text{matter}} = 26$) and close to the case of pure two-dimensional gravity. Unfortunately, up to now this is the only region where it is possible to formulate string theory consistently and at least put the question what is the internal principle which might allow one to choose dynamically a string vacuum.

The idea to play with highly reduced theories leads automatically to a conclusion that the effective string field theory should not necessarily be a field theory in a literal sense, and this is at least naively consistent with the result, being presented in a form of (not really functional) matrix integral. Moreover, one can actually consider this matrix integral
as a finite one, because the result in fact is independent of the size \( N \) of matrices. Indeed, the only place where this parameter appears is the definition of coupling constants in terms of \( N \) Miwa “spectral parameters”; see below). The fact of \( N \)-independence in the theory could be interpreted as a cutoff independence of topological models, though the field theoretical sense of Miwa transform is still not clear.

Another crucial role of the Miwa coordinates is connected with the problem of interpolation between different string vacua. We know, that in terms of the KP times themselves there is no good flow between various critical points of usual matrix models (see for example [13]). In a sense this means that in terms of KP time-variables certain limits of them to zero values could be singular\(^3\). Introducing Miwa’s variables allows us in principle to go around this problem, considering flows as certain reparametrizations of spectral curve, which in the language of hierarchies of integrable equations is connected with so-called equivalent hierarchies [14,15].

2. The Lagrangian description of the effective \( c < 1 \) string theory is based on the equivalence of the topological and quantum 2d gravity [16,17,1,2]. In other words the solution to the topological 2d gravity found by Kontsevich [18] is equivalent to the particular solution of the KdV hierarchy obeying Virasoro constraints or string equation (\( \mathcal{L}_{-1} \) – constraint) (2) with

\[
\mathcal{L}_n = \mathcal{L}_n^{(2)} = \frac{1}{2} \sum_{\text{odd}} kT_k \frac{\partial}{\partial T_{k+2n}} + \frac{1}{4} \sum_{a+b=2n} \frac{\partial^2}{\partial T_a \partial T_b} + \\
+ \frac{1}{4} \sum_{a+b=-2n} aT_a bT_b + \frac{1}{16} \delta_{n,0} - \frac{\partial}{\partial T_{3+2n}}
\]

(5)

being second order differential Virasoro operators acting to the KdV-hierarchy \( \tau \)-function, depending only (in this particular case of \( q = 2 \) reduction) upon odd times

\[
\tau_{KdV}(T) = \tau_{KdV}(T_1, T_3, \ldots)
\]

(6)

This is the case of “pure gravity”, i.e. \( c \) is given by (3) with \( q = 2, p = 2k + 1 (k \geq 0) \). The idea is that the solution to the eqs.(5), (6) can be represented in the form of matrix integral

\[^3\]These “non-vanishing” parameters were also found in continuum formulation (see, for example, N.Seiberg’s contribution to this volume.)
\[ Z[M] = \frac{\int DX \exp -Tr(MX^2 + X^3/3)}{\int DX \exp -TrMX^2} = \]
\[ = C[M] \int DX \exp Tr(M^2X - X^3/3) \tag{7} \]

where \( X \equiv \|X_{ij}\| \) and \( M = \|M_{ij}\| - N \times N \) hermitean matrices (the last one can be taken to be diagonal because \( Z[M] \) depends only on its eigenvalues) and the normalization \( C[M] \) is given by

\[ C[M] = \exp(-\frac{2}{3}TrM^3) \text{det}(M^T \otimes I + I \otimes M)^{1/2} \tag{8} \]

The partition function \( Z[M] \) can be decomposed in series over

\[ T_n = \frac{1}{n} TrM^{-n} = \frac{1}{n} \sum_{j=1}^{N} \mu_j^{-n} \tag{9} \]

(\( n \) odd) and the coefficients reproduce the intersection indices on module spaces of Riemann surfaces with punctures [18] or the (1,2) (topological) gravity theory with \( \epsilon_{\text{matter}} = c_{1,2} = -2 \). In terms of times (9) the partition function (7) gives a representation for \( \tau \)-function of KdV hierarchy obeying Virasoro constraints (5). This can be proven directly using the properties of matrix integral (7) [17]. Kontsevich’s potential \( V(X) = X^3/3 \) is the simplest one and describes the \( q = 2 \) series of “pure gravity”. However, the generalization to higher series is straightforward [1,2], and is given by (7) with an arbitrary polynomial potentials (non-polynomial potentials would rather correspond to less trivial cases both from “stringy” and “integrable” point of view, see below)

\[ Z_{GKM}[M, V] = \frac{\int DX \exp -TrU(M, X)}{\int DX \exp -TrU_2(M, X)} = \]
\[ = C[M|V] \int DX \exp Tr[V'(M)X - V(X)] \tag{10} \]

with

\[ C[V|M] = \exp Tr[V'(M) - MV'(M)] \text{det}[V''(M)]^{1/2} \frac{\Delta(V'(M))}{\Delta(M)} \]
\[ U(M, X) = V(M + X) - V(M) - XV'(M) \]
\[ U_2(M, X) = \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} U(M, \epsilon X) \tag{11} \]

which is nothing but a sort of “effective potential” for matrix theory. For the case \( V(X) = X^3/3 \) eqs. (10), (11) reduce to (7), (8). Two general statements (i), (ii) are easily derived from the matrix-integral representation, first is that

\[ Z[T(M)|V] \equiv \exp \mathcal{F}[T(M)|V] = \]
\[
\frac{\det_{ij} \Phi_i^{(V)}(\mu_j)}{\Delta(\mu)} = \tau[T_n = \frac{1}{n} Tr M^{-n}|V] \tag{12}
\]  

with 

\[
\Phi_i^{(V)}(\mu) = \exp[V(\mu) - \mu V'(\mu)]V''(\mu)^{1/2} \times 
\int dx \ x^{i-1} \exp Tr[V'(\mu)x - V(x)] \tag{13}
\]

where the determinant formula (12) means that the partition function satisfies the Hirota difference bilinear relation in Miwa coordinates (9) [19] and thus is a \( \tau \)-function of KP hierarchy while the second reads 

\[
\frac{\mathcal{L}_1^{(V)} \mathcal{Z}[T(M)|V]}{\mathcal{Z}[T(M)|V]} = 0 \tag{14}
\]

where 

\[
\mathcal{L}_1^{(V)} = \sum_{n \geq 1} Tr[V''(M)M^{n+1}]^{-1} \partial/\partial T_n + 
\frac{1}{2} \sum_{i,j} \frac{1}{V''(\mu_i)V''(\mu_j)} \frac{V''(\mu_i) - V''(\mu_j)}{\mu_i - \mu_j} - \partial/\partial T_1 \tag{15}
\]

(see [2] for details). Equations (14), (15) mean that for any (at least polynomial) potential \( V(X) \) we get from GKM a \( \tau \)-function of KP hierarchy which satisfies string equation, so, at least naively we can preserve both these properties (integrability and string equation) varying the potential \( V(X) \) smoothly between, say, two monomials corresponding to particular \( (p = \text{fixed}, q) \) series (monomial \( X^{p+1}/(p+1) \) gives \( (p, q) \) solution with fixed \( p \) in terms of \( p \)-th KdV reduction of KP hierarchy). In this sense we immediately obtain a string field theory (in the sense of sect.1) description of all discrete series with \( c < 1 \) coupled to two-dimensional gravity.

However this is not true exactly due to two important things which are now in order. First one is connected with the choice of a particular “critical point” within one series with \( \text{fixed} \ p \), and the second one concerns flows between two different \( p \)'s or two different classes of potentials.

Naively Generalized Kontsevich model gives us a “topological” solutions within \( c_{p,q} \) series - \( i.e. \) only points where \( (p, q) = (p, 1) \ c = c_{p,1} = 1 - \frac{(p-1)^2}{p} \). This is determined

\[\text{Of course, these two are actually just the same problem due to } p - q \text{ duality, which is however not manifest in the language of matrix models (see, for example [20])}. \]

\[\text{Again, even at the level of 2d conformal field theory these are singled points (with negative integer central charges, integer dimensions primary fields etc); see also P.West’s contribution to this volume.} \]
by a simple fact that $\tau$-functions are defined as formal series in times \(\{T\}\) or in other words for small \(T\)'s and this corresponds to the limit \(M \to \infty\) in terms of spectral parameter (\(\mu_j \to \infty\) altogether). The particular critical point is determined by the following constraints [9]

\[
\hat{T}_2 = \hat{T}_3 = \ldots = \hat{T}_{p+q-1} = 0
\]

\[
\hat{T}_{p+q} = \text{const} \neq 0
\]

(16)

where for GKM we defined \((V(X) \equiv \sum v_k X^k)\)

\[
\hat{T}_n = \hat{T}^{(V)}_n = T_n - (n-1)v_n
\]

(17) (in particular for monomial potential \(v_n = \delta_{n,p+1}\), \(\hat{T}_n = T_n - \frac{n-1}{n}\delta_{n,p+1}\)) and the shift (17) is determined by the requirement of absorption of linear (like \(\partial/\partial T_1\)) terms in the expressions for Virasoro generators, i.e. by the exact coincidence with the constraints for ordinary matrix models [9,21]. Thus, we see that for any monomial potential in GKM one gets

\[
\hat{T}_n = -\frac{n-1}{n}\delta_{n,p+1} + O(1/M^n)
\]

(18)

i.e. \(\hat{T}_{p+q} \neq 0\) only for \(q = 1\), and the first correction in \(1/M\) will be given by \(T_1\).

For “pure” Kontsevich’s case \(V(X) = X^3/3\) we obtain in such a way the theory with \(c = c_{2,1} = -2\) [22,23] which doesn’t correspond to any critical behaviour of the Hermitian 1-matrix model which starts from \(q = 3\), \(\hat{T}_5 \neq 0\), \(c_{2,3} = 0\) (pure gravity), \(q = 5\), \(\hat{T}_7 \neq 0\), \(c_{2,5} = -22/5\) (Yang-Lee model) etc.

In order to get higher critical points one should try to consider more complicated choices for matrix \(M\), or in other words to consider expansion near different from infinity points on the surface of spectral parameter. For example, instead of expansion over \(Z = 1/M\) near \(Z = 0\) we can take for the simplest case \(V(X) = X^3/3\) the following choice

\[
Z = \frac{1}{3} \begin{bmatrix}
  z\omega_1 + \epsilon \\
  z\omega_2 + \epsilon \\
  z\omega_3 + \epsilon
\end{bmatrix}
\]

(19)

i.e. take matrix \(Z\) in block form where any block \(z\) is multiplied by corresponding root of unity \(\omega_k = \exp(\frac{2\pi ik}{3})\). Then obviously

\[
\hat{T}_1 = Tr\epsilon
\]
\[ \hat{T}_3 = Trz^3 - \frac{2}{3} + Tr\epsilon^3 \]
\[ \hat{T}_5 = 10Trz^3\epsilon^2 + \ldots \]
\[ \hat{T}_7 = 7Trz^6\epsilon + \ldots \]
\[ \hat{T}_9 = Trz^9 + O(\epsilon^3) \] (20)

and if we adjust \( Trz^3 = \frac{2}{3} \), then the first non-zero time will be \( \hat{T}_9 \) (we remind that the case of cubic potential corresponds to the KdV reduction of KP hierarchy and the partition function is independent of all even times \( T_{2n} \), in particular of \( T_6 \)). There will be other non-zero times (say, \( \hat{T}_{27} \)) but they can be switched off by tuning proper behavior in the limit \( N \to \infty \) and the important one is \( \hat{T}_9 \), moreover the critical behaviour is determined by the lower degree term on sphere

\[ \partial/\partial T_1 \frac{L^{(2)}_{KdV}}{\tau_{KdV}} = \sum_{k \geq 1} (2k + 1)\hat{T}_{2k+1}u^k - \frac{1}{8}\hat{T}_1 + O(\partial u, \ldots) = 0 \] (21)

so that

\[ u \sim T_1^{1/k} \] (22)

where \( \hat{T}_{2k+1} \) is the first non-vanishing term. The behaviour is determined by \( 2k + 1 = p + q \), or

\[ \frac{1}{k} = -\frac{2}{p + q - 1} = \gamma_{str} \] (23)

The other problem is that now \( \hat{T}_7 \) is of the same order as \( \hat{T}_1 \), but this should be changed by usual renormalization (dependent upon a particular critical point) in string equation.

Of course, the above example only touches the whole problem and is not satisfactory, because, for example, it doesn’t lead to the most interesting point of pure gravity. However, it demonstrates the main idea – already this flow in “p-direction” is connected with a specific reparametrization of spectral curve and change the asymptotics of basis vector in the Grassmannian (now \( z_j = 1/\mu_j \) have different limits for different \( j \), and expansion should be taken in different points). Below, we’ll see that that actually the same phenomenon appears when we consider other problems of formulating string field theory.
3. Now let us pass to interpolation between various series with different \( p \)'s (flows in "\( p \)-direction"). The example can be given by the potential

\[
V(X) = \alpha \frac{X^{k+1}}{k+1} + \beta \frac{X^k}{k}
\]

(24)

with \( \alpha + \beta = 1 \). This should describe the flow between points with \( \alpha = 0 \) and \( \alpha = 1 \) which is nonsingular everywhere except for the place of choosing of integration contour. In principle any matrix integral of the type (10) is determined by analytical continuation (or contour deformation) from a conventional definition of corresponding (generalized) Airy function. For various monomials in the exponent the definitions of such contours are different and the addition even of a small piece like that in (24) can change the contours drastically. However, this is nothing but the same sort difficulty as in the ordinary non-perturbative definition of field theory path integral (with the same potential (24)) and it is not too serious problem. The deformation (24) around monomial potential by lower order terms is described by so called Landau-Ginsburg flows and it uses the fact that the derivatives with respect to the first \( k \leq p \) times are related to the insertion of corresponding monomials (see [24] for details)

\[
\frac{\partial Z}{\partial T_k} = \langle TrM^k - TrX^k \rangle, \quad k = 1, \ldots, p
\]

(25)

(\( Z \equiv \langle 1 \rangle \)). Moreover, the flow around a given “critical point” (determined by the higher-degree term in potential; \( k + 1 \) - in (24)) can be at least partially absorbed into the definition of times via the change of spectral parameter

\[
\tilde{M} = [V'(M)]^{1/k}
\]

(26)

(a sort of “positive” Virasoro reparameterization – not moving the point of expansion, in contrast to the case considered in sect.2 above) and the redefined partition function

\[
\tilde{Z}[\tilde{T}|\tilde{V}] = \frac{C[\tilde{M}|\tilde{V}]}{C[M|V]} Z[T|V]
\]

(27)

is still a \( \tau \)-function of \( k \) – reduced KP hierarchy in terms of new \( \tilde{T}_n = \frac{1}{n} Tr\tilde{M}^{-n} \) variables.

However, the limit \( \alpha \to 0 \) will be a singular one. It is exactly the case when the contour jumps, or in other words one has to change the region of definition of spectral parameter. The other thing is that the string equation is deformed smoothly in terms
of Miwa coordinates but has two absolutely different expansions in times $T_n$. Another immediate consequence of (26) is different role of times $T_n$ for $n \leq p$ and $n > p$. In the language of topological theories the first ones correspond to so called primary fields while the second to their descendants (see, for example, [25] and references therein).

The “Landau-Ginsburg” deformation (27) doesn’t even change the “critical” point of concrete solution, because in new $\tilde{T}$-variables the solution has the same reduction. It corresponds rather to an infinitesimal deformation of a GKM in the vicinity of a critical point. The corresponding change of spectral parameter or spectral curve reparameterization (26) is also infinitesimal in the sense that it doesn’t move (in contrast to the previous section) the point of expansion (and even the asymptotics of basis vectors). In the language of integrable hierarchies this corresponds to so called equivalent hierarchies [14,15], where new and old times are connected by a triangular linear transformation, and the potentials of one solution are functionals of the potentials of another one. The main feature of equivalent hierarchies is that not any transformation of times but only those, induced by spectral reparametrizations like (26). We demonstrated above that in more general situation we have a more complicated case of spectral reparametrization (as well as when considering $c \to 1$ limit below) but this is not yet formulated in terms of the properties of the hierarchy of integrable equation. We are going to return to this problem elsewhere [26].

4. Now, let us discuss the $c \to 1$ limit of Generalized Kontsevich model. The exact way to do this is to take $p \to \infty$ limit keeping $q = p + 1$ (or at least difference $p - q$ fixed and finite, then $c_{p,p+1} = 1 - 6/(p+1)$ $p \to \infty$ 1). Unfortunately, this is hard to perform exactly (due to all mentioned above problems) let us instead try to analyze the naive limit $p \to \infty$ in the model with monomial potential $V(X) = X^{p+1}/(p+1)$. Changing the variables

$$Y = -\frac{1}{p+1} \Lambda^{p+1} X^{p+1}$$

in the integral

$$\int DX \exp Tr[\Lambda X - X^{p+1}/(p+1)]$$

one gets

$$\int DY \exp\{Tr[-(p+1)Y^{1/(p+1)} + \tilde{\Lambda} Y] + \log \frac{\partial(X)}{\partial(Y)}\}$$
\[
\tilde{\Lambda} = (-p - 1)p\Lambda^{-(p+1)}
\]

\[
T_n = \frac{1}{n} Tr \Lambda^{-n/p} \sim \frac{1}{n} Tr \tilde{\Lambda}^{n/p(p+1)}
\]  

(30)

where the last term stands for Jacobian of the transformation (28). This Jacobian can be determined from

\[
DY \sim (\text{det}(\Lambda^{p+1}))^N D(X^{p+1})
\]  

(31)

and

\[
D(X^{p+1}) = (p + 1)^N (\text{det}(X)^p \left( \prod_{i<j} \sum_{a+b=p} x_i^a x_j^b \right)^2 DX
\]  

(32)

where \( N \) – the size of the matrix \( X \), and \( \{x_i\} \) - its eigenvalues. (Eq. (32) easily follows from decomposition \( X = \Omega^t x \Omega \), \( X^{p+1} = \Omega^t x^{p+1} \Omega \), where \( x = diag(x_1, ..., x_N) \), and

\[
DX = D\Omega^t D\Omega \prod_i dx_i \Delta^2(x)
\]  

(33)

with \( \Delta(x) = \prod_{i<j} (x_i - x_j) \) – Vandermonde determinant). It means that

\[
DX \sim DY \left( (\text{det}(X)^p \prod_{i\neq j} \sum_{a+b=p} x_i^a x_j^b \right)^{-1} \sim DY \exp \left[ -\frac{p}{p+1} \text{Tr} \log Y - \sum_{i\neq j} \log \sum_{a+b=p} \lambda_i^{-a} y_i^{a/(p+1)} \lambda_j^{-b} y_j^{b/(p+1)} \right] \sim 
\]

\[
D Y \exp \left[ -\frac{p}{p+1} \text{Tr} \log Y - \sum_{i\neq j} \log \sum_{a+b=p} (\tilde{\lambda}_i y_i)^{a/(p+1)} (\tilde{\lambda}_j y_j)^{b/(p+1)} \right]
\]  

(34)

Finally, in the limit \( q \to \infty \) the integral (30) turns to be

\[
\int DY \exp[Tr \tilde{\Lambda} Y - 2Tr \log Y - \sum_{i\neq j} \log \frac{\tilde{\lambda}_i y_i - \tilde{\lambda}_j y_j}{\log(\tilde{\lambda}_i y_i / \tilde{\lambda}_j y_j)}] \sim 
\]

\[
\sim \int D\Xi \exp[Tr \Xi - 2Tr \log \Xi - \sum_{i\neq j} \log \frac{\xi_i - \xi_j}{\log(\xi_i / \xi_j)}]
\]  

(35)

So, we see that in such limit \( p \to \infty \) one gets a construction similar to Penner model [27,28] and all the nontrivial “Miwa” times decouple from (35) (this is consistent with (30) which stands that all nontrivial times are tend to zero in such limit). Thus, in naive limit we can get nothing in addition to effective theory for puncture operator (tachyon with zero momenta) which decouples from rest of the theory in such limit (see for example [29] and references therein).
However, the determinant form of Penner model partition function implies already that for fixed values of times it is a Toda lattice tau-function in the sense of [3] and allows us to apply to this case the main idea of [3] – the Toda theory representation for Generalized Kontsevich models. Indeed, the solution to the Penner model

\[ \mathcal{Z} \sim \det \mathcal{H}^{(\alpha)}_{ij} \tag{36} \]

with

\[ \mathcal{H}^{(\alpha)}_{ij} = \Gamma(\alpha + i + j - 1) \tag{37} \]

is nothing but a specific case of GKM.

The solution to generic Toda lattice hierarchy looks like

\[ \tau_n[T] = \det H_{i+n,j+n}[T] \tag{38} \]

where matrix elements \( H_{ij} \) satisfy the following equations

\[ \frac{\partial H_{ij}}{\partial T_p} = H_{i,j-p} \text{ for “positive times” } T_p \]
\[ \frac{\partial H_{ij}}{\partial T_{-p}} = H_{i-p,j} \text{ for “negative times” } T_{-p} \tag{39} \]

(\( n \) being integer-valued “zero-time”). In particular, for generalized Kontsevich model the solution of (38), (39) has the form

\[ H_{ij}[T_{-p}, T_p] = \sum_{k \leq i, l \geq -j} P_{i-k}[T_{-p}] h_{kl} P_{l+j}[T_p] \tag{40} \]

with

\[ h_{kl} = \oint \Phi^{\{V\}}_k(z) z^l \tag{41} \]

where \( \Phi^{\{V\}}_k(z) \) are defined in (11) \(^6\).

Now one can easily introduce positive- and negative-times dependence in (37) according to (40) and then reconstruct \( \Phi^{\{V\}}_k(z) \) from (41) \(^7\). Indeed,

\[ h_{ij}^{(\alpha)} = \mathcal{H}^{(\alpha)}_{ij} = \Gamma(\alpha - 1 + i + j) = \]

\(^6\)The most interesting example of non-trivial Toda generalization of GKM is given by a discrete Hermitean matrix model [3,30] which corresponds to \( Tr(\Lambda X - X^2/2 + n \log X) \) in the exponent, (\( n \) being Toda zero-time), when for \( n = 0 \) the GKM integral is Gaussian and thus trivial.

\(^7\)C.Vafa told me that negative Toda times in order to describe \( c = 1 \) theory by means of GKM were also used by R.Dijkgraaf and G.Moore.
\[
\int_0^\infty \frac{dy}{y} e^{-y} y^{\alpha - 1 + i} = \int_0^\infty \phi_{\alpha}(z) z^j \tag{42}
\]
immediately gives
\[
\phi_{\alpha}(z) = \int_0^\infty \frac{dy}{y} e^{z y - y} y^{\alpha - 1 + i} \tag{43}
\]
which is a sort of GKM-like representation. The difference with more common situation for \(c < 1\) is in the definition of the contour in (43) and also in the fact that \(z\)-dependence is trivial in (43), because integral is easily taken with the result
\[
\phi_{\alpha}(z) = \frac{\Gamma(\alpha + i)}{(z - 1)^{\alpha + i}} \equiv \phi_{\alpha + i}(z) \tag{44}
\]
and
\[
\left( \frac{\partial}{\partial z} \right)^j \phi_{\alpha}(z) = (-)^j \phi_{\alpha + j}(z) = (-)^j \frac{\Gamma(\alpha + i + j)}{(z - 1)^{\alpha + i + j}} \tag{45}
\]
Introducing negative times according to [3], one gets
\[
\phi_{\alpha}(z|T_{-p}) = z^{-\alpha} \exp \left( -\sum_{p > 0} T_{-p} z^{-p} \right) \sum_k P_k[T_{-p}] \phi_{i-k}(z) \tag{46}
\]
where \(P_k[T_p]\) are Schur polynomials (\(\exp \sum T_p z^p = \sum z^n P_n[T_p]\)), or simply
\[
Z_{c=1} \sim \int DY \exp TrZY + \alpha Tr\log Y + \sum_{k > 0} T_{-k} TrY^{-k} \tag{47}
\]
with
\[
T_{+k} = \frac{1}{k} TrZ^k \tag{48}
\]
Now, we see again that the formulas (44) looks like basis vector of a trivial element of Grassmannian (sphere) but in a shifted point of spectral curve (expansion near \(z = 1\)). Thus, we conclude that the same phenomena which corresponds to flows between various \(pq\)-solutions is also valid when taking \(c \to 1\) limit of GKM. This seems to be a generic feature of \(c \leq 1\) effective string theory.

I tried to demonstrate above that matrix models in general and especially the Generalized Kontsevich model give at the moment the most advanced understanding of \(c \leq 1\) non-perturbative string theory. This is mostly connected with the appearance of a non-trivial dynamics over coupling constants in this formulation, described by integrable equations of KP (or Toda lattice) hierarchy. The particular stringy solutions to KP can be described in a form of matrix integrals, having the sense of effective string field theory. The
central point is introducing specific coupling by a matrix of Miwa spectral parameters, thus making the conventional flows in the space of coupling constants to be a sort of “spectral flows”. From the point of view of integrable models themselves these spectral flows correspond to equivalent hierarchies, which have not yet been investigated a lot. We are going to return to all these problems elsewhere.

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