Field theory of higher-order topological crystalline response, generalized global symmetries and elasticity tetrads

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Abstract
I discuss aspects of higher-order topological field theory of crystalline insulators with no other symmetries. I show how the topology and geometry of the crystalline lattice is organized in terms of so-called elasticity tetrads which are ground state degrees of freedom labelling translational lattice topological charges, higher-form conservation laws and responses on sub-dimensional manifolds of the bulk insulator. The quasi-topological responses obtained in this way depend on the lattice and its embedding in space, as expected for weak topology. In a topological crystalline insulator, they classify higher-order responses and global symmetries in a transparent fashion in generic dimensions. This hierarchy coincides with the dimensional hierarchy of topological terms, the multipole expansion, and anomaly inflow, related to a mixed number of elasticity tetrads and electromagnetic gauge fields. In the continuum limit of the elasticity tetrads, the semi-classical expansion in momentum space can be used to derive the higher-order or subdimensional topological responses to local U(1) symmetries, such as electromagnetic gauge fields, with explicit formulas for the higher-order quasi-topological invariants in terms of the elasticity tetrads and Green's functions. The topological responses in arbitrary dimensions are readily generalized to parameter space to allow for e.g. multipole pumping. The simple results further bridge the recently appreciated connections between topological field theory, higher-form symmetries and gauge fields and their relation to fractonic excitations and topological defects with restricted mobility in the elasticity of crystalline insulators.

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1. Introduction

Higher order topological insulators [1–3] generalize symmetry protected topological phases and are protected e.g. by spatial symmetries of crystalline origin (see, however, e.g. [4–6] for higher-order topology with non-crystalline symmetries). Compared to traditional topological insulators, they harbour protected edge excitations of higher codimensionality: on the $d - 1$-dimensional boundary, $d - 2$- or lower-dimensional modes robust to symmetry preserving perturbations occur in $d$ spatial dimensions. In particular, multipole insulators with localized boundary moments, i.e. localized dipoles, quadrupoles etc. are a prominent example [1,2].

The close relation of higher-order topology to subsystem [6,7] and higher-order (gauge) symmetries in field theories has been recently noted and explored in various directions [8–10]. This connection is powered by the notion generalized global symmetries that constitute a convenient organizing framework for the phase structure and the spectrum of allowed operators in (gauge) field theories [11]. In fact, the essential physics following from this organizing principle were implicitly contained in the early, seminal Refs. [12] on confinement in gauge theories. More generally one expects that, also in gauge theories, global symmetries are more useful than local ones in analysing the phase structure, since the latter are really redundancies in the degrees of freedom. The generalized global symmetries couple to (topological) line, surface, hypersurface and volume operators instead of the conventional global symmetries that couple to point operators. In such theories, conserved charges are, respectively, measured on spatial hypersurfaces of suitable codimension. Systems with such symmetries in topological phases with long-range entanglement have been shown to have subextensive ground state degeneracy and excitations with restricted mobility. These phases and excitations are now referred to as fractonic, see e.g. [13–18] for reviews.

On the other hand, the geometric responses of (ordinary) topological crystalline systems involving translational lattice gauge fields with direct relation to elasticity were identified recently [24–26]. Given the lattice, these gauge fields are of tautological and composite nature themselves (see also [27]) but can lead to non-trivial global symmetries and quasitopological responses in crystalline topological systems. In this approach, the weakly protected anomalous quantum Hall (QH) response and generalized electric polarization and their associated anomaly inflow structures follow in a general framework in $d > 2$ and $d > 1$, respectively [25,26,28]. These anomalies are, respectively, the mixed axial-gravitational and Luttinger-type filling anomaly of the boundary theory. In both cases, lattice dislocations carry topological charges, and, in three spatial dimensions, these two responses are dual to each other in terms of the lattice geometry. The former singles out one-dimensional lattice directions (1-cycles) with polarization invariant $P_a$ and the latter two-dimensional (2-cycles) with QH invariants $N_{ab}$ with $a, b$ along lattice directions. Such a formulation is reminiscent of the higher-form symmetries and currents where, in the electromagnetic response, all fields couple explicitly to the electromagnetic fields with the ordinary gauge and global charge conservation symmetry. Similar lattice and higher form fields involving generalized global symmetries were considered also earlier in the context of magnetohydrodynamics [29] and elasticity [30,31].

Here we discuss their role in topological crystalline responses. Motivated by the close analogy of the lattice fields to generalized global symmetries, we consider classes of mixed lattice-electromagnetic (EM) responses in relation to weak subdimensional and higher-order topology in crystalline systems. We will look at the possible forms of insulators with additional higher-form crystalline generalized global conservation laws. Utilizing these lattice fields, the expectations is that...
higher-form responses can be derived from the semi-classical expansion in momentum space, much like the momentum space invariants of the electromagnetic response (EM) of ordinary topological insulators [32]. Concerning the crystalline lattice, we will phrase our results in the continuum formulation of elasticity and cohomology, expecting that the end results with topological character do not qualitatively change upon the incorporation of the discrete lattice theory, space or point group symmetries and discrete (co)homology. For simplicity, we focus exclusively on spatial lattices although our results can straightforwardly extended to driven time-periodic (Floquet) systems.

The rest of this paper is organized as follows. In Section 2, we review higher-order symmetries in U(1) Maxwell gauge theory which is sufficient for our purposes and connects directly with the responses to EM fields in the presence of higher-order global crystalline symmetries. Then we discuss the main application, (continuum) lattice field theory with higher-form crystalline generalized global symmetries and currents. The connection to the EM response in higher-order topological crystalline insulators is made in Section 4. The associated momentum space invariants are discussed in 5 and the Appendices. We end with Conclusions and Outlook.

Notations and conventions. For ease of notation, throughout we employ differential forms. For example, the electromagnetic 2-form \( F = dA = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \) with \( A = A_\mu dx^\mu \) the 1-form EM gauge potential and \( \wedge \) the graded antisymmetric wedge product. \( \star \) is the spacetime Hodge dual, mapping \( n \)-forms to \( d+n-n \)-forms. The \((d+1)\)-dimensional metric is set to \( g_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(-1, 1, 1, \ldots, 1) \), leading to \( \star^2 = -(-1)^{n(d+1-n)} \) for \( n \)-forms. See e.g. [33] for more details. Although the construction is general, we mostly specialize to 3+1 dimensions and work in units where \( e^2 = \hbar = 1 \). The electric and magnetic fields are denoted as \( E_i = F_{0i} \) and \( B_i = \frac{1}{2} \epsilon^{ijk} F_{jk} \) and the lattice one-forms \( E^a = E^a_{\mu} dx^\mu \) are the (continuum) elasticity tetrads.

2. Review of higher form symmetries

We start by reviewing higher-form symmetries from the perspective of generalized global symmetries [11,30,34], focusing to a relevant example for our applications: ordinary U(1) Maxwell gauge theory in 3+1d. In the following we shall eventually consider higher-form global symmetries in insulating crystalline systems that are inherited from the U(1) number, i.e. the charge conservation of the lattice constituents.

In \( d+1 \)-dimensions, \( p \)-form global symmetries are an extension of the usual global symmetries with point like, i.e. 0-form, charges. The existence of a \( p \)-form symmetry leads to the existence of conserved \( p+1 \)-form currents. The charged objects are \( p \)-dimensional with charges measured on \( d-p \)-dimensional surfaces. Similar to ordinary global symmetries, \( p \)-form can be spontaneously broken in \( d-p > 2 \) dimensions, can be anomalous on their own or when gauged in quantum theory [11]. For example, in 0-form global U(1) theory with field \( \phi \to \phi + \lambda \), \( q \)-charged point-operators are transformed as \( Q_q(x) \to e^{iq\lambda} Q_q(x) \). In 1-form symmetric theory, \( Q \)-charged operators are loops \( C \) transforming as \( W_Q[C] \to e^{iQ} \int_C \Lambda W_Q[C] \) under global transformations \( \Phi \to \Phi + \Lambda \), where \( \Phi \) and \( \Lambda \) are the elementary abelian 1-form charged field and a closed shift parameter, respectively. Generalizing this to the general abelian U(1) case, the hypersurface operators are \( H_Q[S] \to e^{iQ} \int_S \Lambda H_Q[S] \), for a general \( p \)-form \( \Phi \to \Phi + \Lambda \) and charge \( Q = \int_{S'} \star_j \), where \( \star_j \) is a conserved \( d-p \)-form current evaluated on a suitable codimension \( p+1 \) dual surface \( S' \) at constant time [11]. Such operators for lattice fields will be discussed in Section 3, often in terms of the conserved currents \( \star_j \). We focus on closed hypersurfaces \( S \), so that additional boundary conditions are not needed.

Let us consider (3+1)d U(1) 0-form Maxwell gauge theory as an example. The gauge action is

\[
S_{\text{EM}}[A,j] = \int \sqrt{|g|} dx^4 \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + A_\mu j^\mu
\]

\[
= \int F \wedge \star F + A \wedge \star j
\]

(1)

where \( F \equiv dA \) and \( \star \) is the spacetime Hodge dual. The 1-form \( j \equiv j_\mu dx^\mu \) represents electically charged matter. The equations of motion are

\[
d \star F = \star j, \quad dF = 0
\]

(2)
where the latter is a topological Bianchi identity from \( d^2 A = 0 \), tantamount to 0-form gauge symmetry of \( A \to A + d\lambda \). The corresponding current is conserved

\[
d \star j = \partial_\mu j^\mu = 0.
\]  

This statement is equivalent to the (global and local) 0-form charge conservation

\[
\frac{dq}{dt} = \frac{d}{dt} \int_{\text{space}} \star j = 0.
\]  

In gauge theories, the Wilson loop operators are \( \langle W_A[C] \rangle = \int DA \exp \left[i \oint_C A \right] e^{-iS[A]} \) and their asymptotic behaviour is diagnostic for the phase and charged spectrum [12]. However, and not unrelated to this, since local gauge symmetries represent redundancies in the physical degrees of freedom, the properties of the theory are better understood from 1-form global symmetries. These are given by 2-form currents

\[
J_{\mu\nu}^e = F_{\mu\nu}, \quad J_{\mu\nu}^m = \frac{1}{4} \epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho},
\]  

leading to, over two-dimensional spatial surfaces \( \Sigma^{(2)} \),

\[
Q_{m}^{(2)} = \frac{1}{2\pi} \int_{\Sigma^{(2)}} \star J_m = \frac{1}{2\pi} \int_{\Sigma^{(2)}} F = \mathbb{Z},
\]

\[
Q_{e}^{(2)} = \frac{1}{2\pi} \int_{\Sigma^{(2)}} \star J_e = \frac{1}{2\pi} \int_{\Sigma^{(2)}} \star F = \frac{1}{2\pi} \int_{\Sigma^{(2)}} \mathcal{E} \cdot dS = \int_{\Sigma^{(3)}} j^0.
\]  

Physically \( Q_{m}^{(2)} \) is just the magnetic flux, which is conserved due to the Bianchi identity (magnetic field lines are closed; absence of magnetic monopoles). On the other hand \( Q_{e}^{(2)} \) is the electric flux, which is not conserved (electric field lines end on charges). On the last equality of (8), we used \( \partial \Sigma^{(3)} = \Sigma^{(2)} \) and (6). It is important to consider compact (or closed) spatial manifolds for the charges, since they are well-defined without additional boundary conditions [11]. For example, if \( \Sigma^{(3)} \) is the total spatial volume, the charge is of course conserved by (4). Similarly, for topologically non-trivial spaces, we can transform e.g. a line operator \( \exp[i \frac{1}{2\pi} \oint_C A] = \exp[\oint_C \frac{\alpha}{2\pi}] W_A[C] \) on a closed homology loop with a large gauge transformation \( \alpha \) and \( Q_{m}^{(2)} \) is still conserved.

We see that if there are no free, light charges in the vacuum, \( j = 0 \), both electric and magnetic fields become 1-form symmetric. The 2-form currents and conservation laws are \( \star \text{dual} \) and \( \star F = d\tilde{A} \) for a magnetic photon \( \tilde{A} \). The usual Coulomb phase of free electric charges corresponds to the phase with magnetic 1-form symmetry, whereas confinement adds the electric one (as relevant e.g. in [12] or non-abelian gauge theory). Likewise, proliferation of (dual) magnetic/electric vortex operators can break (restore) the symmetry to (from) a subgroup. With magnetic monopoles \( \tilde{j} \), the theory features exactly dual versions of the above conservation laws (by electromagnetic duality with magnetic monopoles).  

3. Crystalline topology of elasticity tetrads: almost conserved higher order gauge symmetries

Now we introduce the lattice gauge fields from elasticity, dubbed “elasticity tetrads” (more precisely “vielbein” in general dimensions) in Refs. [25,35] in connection to topological responses. These fields were first considered in [36,37], whereas closely related works concerning topology are [16,38,39]. The same fields have been utilized in gapless superconductors and semimetals e.g. in [40–44].

We can now ask the following question: at low energies, what phases of matter can have approximate higher-form symmetries related to the U(1) gauge group of electromagnetism and what consequences this implies for the EM response? Essentially, we look for all the different ways the light (in a suitable sense) electric charges can be excluded from the vacuum in insulators in
addition to \( j = 0 \). Next, we address this question by identifying higher-form global symmetries in crystalline systems. This viewpoint allows to connect the higher-form elastic symmetries to topological responses in crystalline insulators when free charge currents are excluded at low enough energies below the mobility gap in Section 4.

3.1. Elasticity tetrads and lattice geometry in the continuum limit

In \( d \)-dimensions, a lattice \( L \) can be embedded in space as a system of \( d \) crystallographic coordinate planes of constant phase \( X^a(x) \), i.e. \( e^{iX^a(x)} = e^{2\pi n^a \cdot \mathbf{r}} \), \( n^a \in \mathbb{Z} \) with \( a = 1, \ldots, d \) [36,37]. In \( d = 3 \), the intersections of the surfaces

\[
X^1(\mathbf{r}, t) = 2\pi n^1, \quad X^2(\mathbf{r}, t) = 2\pi n^2, \quad X^3(\mathbf{r}, t) = 2\pi n^3,
\]

then define the (possibly deformed) crystal lattice

\[
L = \{ \mathbf{r} = \mathbf{R}(n_1, n_2, n_3) | \mathbf{r} \in \mathbb{R}^3, n^a \in \mathbb{Z}^3 \}. \tag{10}
\]

Instead of the periodic scalars \( X^a \), the elasticity tetrads \( E^a_{\mu}(x) = \partial_{\nu}X^a \) represent the conventional hydrodynamic variables of elasticity theory [25,37]. The depend slowly on the spacetime coordinates and encode the geometry and topology of the lattice embedded in spacetime. Note that this lattice has not necessarily anything to do with the underlying crystal lattice, but instead is a coarse-grained ground state property [45]. The lattice \( X^a \rightarrow X^a + 2\pi \) symmetries are tantamount to the existence of the translational gauge fields, the elasticity tetrads.

In \( d = 3 \), the elasticity tetrads are gradients of the three U(1) phase fields \( X^a, a = 1, 2, 3 \),

\[
E^a_{\mu}(x) = \partial_\mu X^a(x) \tag{11}
\]

and have units of crystal momentum. In the simplest undeformed case, \( X^a(\mathbf{r}, t) = \mathbf{K}^a \cdot \mathbf{r} \), where \( E^a_0 = \mathbf{K}^a \) are the (primitive) reciprocal lattice vectors \( \mathbf{K}^a \). In the general case, they depend on space (and time) but are still quantized in terms of the lattice \( L \) in Eq. (10).

The basis \( E^a_{\mu} \) can be made non-degenerate by adding \( E^{00}_{\mu} = 1/\delta_{\mu}^0 \), where \( T \) is time-periodicity and letting \( T \to \infty \) at the end. While the index \( a \) is understood to be spatial, the \( \mu = t, x, y, z \) for time dependent deformations, with velocities \( V^a = \partial_\mu X^a \). Moreover, in order to couple to EM, we have to assume the continuity equation of charge conservation. For small deformations \( X^a = \frac{2\pi}{\delta^a}(\delta^a + \mu_\mu u^\mu) \), leading to \( E^0_{\mu} = \frac{2\pi}{\delta^a}(\delta^a_{\mu} + \partial_{\mu}u^\mu) \). The currents from time-deformations are

\[
\partial_\mu \rho + \nabla \cdot j = \rho_{\text{u.c.}}(-\partial_\mu \partial_\tau + \partial_\tau \partial_\mu) u^\tau = 0. \tag{12}
\]

where “u.c.” denotes unit-cell averaged (charge) density. This conservation law holds in the crystal in the absence of interstitial/vacancies, being equivalent to the dislocation glide constraint (no motion parallel to the Burgers vector) and is an additional symmetry constraint [46]. In the absence of dislocations, the \( X^a(x) \) are globally well-defined, meaning that the tetrads \( E^a_{\mu}(x) \) are “pure gauge” and satisfy the general integrability (flatness) condition:

\[
T^a := dE^a = \frac{1}{2}(\partial_\mu E^a_{\nu} - \partial_\nu E^a_{\mu}) dx^\mu \wedge dx^\nu = 0. \tag{13}
\]

In the rest of the paper, for simplicity we assume \( V^a = 0 \), focusing on the electronic couplings proportional to \( \mathbf{E} \) in a (dielectric) crystalline insulator with localized charges. Effects of non-zero velocity would also induce magnetic terms, proportional to \( \mathbf{V} \times \mathbf{B} \) corresponding to terms \( V^a \wedge dA \) below. This extension has to be made bearing in mind that the dual form coupling to \( p \)-form currents and charges change their dimensionality when time-like components in \( d + 1 \)-dimensions are included, in addition to the spatial lattice.

By the inverse function theorem, we can define the inverse vectors,

\[
E^a_{\mu}(x)E^\nu_a(x) = \delta^\nu_{\mu}, \tag{14}
\]

and the lattice metric \( G \) associated with these tetrads,

\[
G_{ab} = E^a_{\mu}E^b_{\nu} \eta_{\mu \nu}, \quad G^{ab} = E^a_{\mu}E^b_{\nu} \eta^{\mu \nu}. \tag{15}
\]
where $\eta$ is the metric associated to the background spacetime, say the spatial Euclidean (or the four-dimensional Minkowski metric). Note that $ds^2 = G_{ab} N^a N^b$ counts distances in terms of the number of lattice points $N^a$ of $L$ [36]. We assume a simple cubic or orthorhombic lattice. Loosely speaking, the elasticity tetrads are the trivial gauge fields corresponding to the $U(1)^d$ translational symmetries along these directions.

3.2. Topology

Now we describe the differential topology of this lattice and embedding in detail. Upon making the identifications $\mathbb{T}^d \simeq \mathbb{R}^d / L$, we can compute the holonomies

$$\int_{C_b} E^a = 2\pi n^a \delta_{ab} = \langle C_b, E^a \rangle, \quad a, b = 1, \ldots, d$$

(16)

where we fix the origin at $X^a = 0$ and there are $n^a = n^a(L)$ lattice points along the direction $C_a = \{X^b = \text{const.} | a \neq b \}$. The lattice $L$ in (10) is then defined by the trivial holonomy sections $e^{i\langle C_b(x), E^a \rangle} = 1$, where $C_b(x)$ is the open loop from the origin to the point $x$. When discussing the higher-form symmetries, we restrict to closed loops (16) and surfaces.

Equivalently, the flat gauge fields $T^a = dE^a = 0$ on the torus $\mathbb{T}^d$ take the form

$$E^a = K^a(x) \cdot dx + E^a(x)$$

(17)

where $K^a(X^a) = \frac{2\pi}{\eta^a}$ are the lattice directions of periodicity; the $E^a$ are $n^a$-fold generators of the 1-cohomology group of $\mathbb{T}^d$ in the coordinates $x$ if there are $n^a = L^a / \eta^a$ lattice points along $a$. The smooth gauge part $dE^a(x) = 0$ is arbitrary and does not depend on the lattice topology, $E^a(x) = \partial_\mu \tilde{X}^a$, where $\tilde{X}^a$ is a smoothly connected to the zero. We conclude that the $\{E^a\}, \{C_b\}$ are (after normalization) simply the basis of $H^1(\mathbb{T}^d, \mathbb{Z})$ and $H_1(\mathbb{T}^d, \mathbb{Z})$ pulled back via $X^a(x)$ in the spacetime. Similarly, the wedge $q$-fold products $E^a \wedge \cdots \wedge E^b$ generate (a pullback of) the (co)homology $H^q(\mathbb{T}^d, \mathbb{Z})$, see [11,33] for the definitions and context in terms of generalized global symmetries with the flat gauge fields $E^a$.

In the presence of dislocations, $T^a \neq 0$, and the embeddings $X^a(x)$ are multivalued [19]. Then we can define the embedding by also including the operators on surfaces $C^{(2)}$,

$$\int_{C^{(2)}} T^a = 2\pi B^a$$

(18)

where $B^a$ is the total Burgers vector integrated over $C^{(2)}$ (e.g. over lattice plaquettes where $n^a, n^b$ increase by specific amounts) [19]. In the presence of a dislocation, the lattice lines defined by $E^a$ are no longer conserved, corresponding to the removal/addition of lattice points compared to the dislocation free lattice. Similarly, it can be shown that the lattice with dislocations is connected by a set of singular gauge transformations to the defect-free lattice if we allow gauge transformations that add/remove lattice points (corresponding to multivalued $X^a$). The translational symmetry, spontaneously broken by the lattice, can be restored when such vortex line-operatros proliferate or condense [21]. We will only consider dislocation defects and translational fields $E^a$, since the rotational disclination defects have considerably higher energies and are confined [20]. As such they only occur as bound dipoles, equivalent to dislocations with Burgers vector along dipole moment. This allows us to set curvature to zero in the coarse-grained continuum limit, while dislocations can break the torsion-free constraint of $E^a$ [19].

3.3. Higher form crystalline symmetries

The connection of elasticity, dislocations and disclinations to higher rank gauge theories and symmetries has recently attracted attention [21,23,47]. In particular, the restricted motion of dislocations, as disclination dipoles, is elegantly formulated in terms of fractonic higher-rank (dual) gauge fields. From this perspective, the dislocation glide constraint (motion only transverse to the Burgers vector), originating from the number conservation of lattice constituents, manifests itself as the higher-rank conservation law in the (dual) gauge theory.
Motivated by this and elastic fields in the topological responses in insulators [25,26,32], we now consider the localized (electronic) lattice charges in a crystalline background and their relation to the global U(1) charge conservation. Specifically, we want to consider the effects of the crystalline symmetries in the insulator in combination with the local U(1) gauge symmetry of electromagnetism. This is supported by the intuitive picture of weak crystalline topological insulators, both as subdimensional insulators and higher-form topological insulators. For the latter especially, we already anticipate the non-trivial higher-order boundary responses and charges with restricted motion to be related to higher-form symmetries.

We imagine that the crystalline insulator is composed of unit cells with integer number of charges, say with a simple cubic or orthorhombic symmetry. Since the system is an insulator, the (electronic) charges are localized in the unit cell below a mobility gap, even in the presence of external fields. The simplest occurrence of insulation is when the unit-cell (u.c.) (or “voxel” [2]) coarse-grained charge density \( \rho_{\text{u.c.}} \) is (locally) conserved

\[
\frac{d}{dt} \int_{\text{space}} \rho_{\text{u.c.}} = \int_{\text{space}} \nabla \cdot j_{\text{u.c.}} = 0
\]

the last equality over total volume is non-trivial with open boundary conditions only. In addition, and less constraining, it might be that

\[
\frac{d}{dt} \int_{\text{surfaces}} \rho_{s} = 0, \quad \frac{d}{dt} \int_{\text{lines}} \rho_{l} = 0,
\]

etc. in higher dimensions. Here \( \rho_{s} \) and \( \rho_{l} \) are really charge densities per unit plaquettes and links on the lattice. Above we have specialized to three dimensions, although the conservation of hypersurface charges is an easy generalization. In terms of the elasticity tetrads, the conservation laws are written respectively as, \( \det(E^a) \sim \rho_{\text{u.c.}} \),

\[
d(E^1 \wedge E^2 \wedge E^3) = 0 \quad [0\text{-form symmetry}]
\]

and, for \( a, b, c = 1, 2, 3 \) in three dimensions,

\[
d\left( \frac{1}{2} \epsilon_{abc} E^b \wedge E^c \right) = 0, \quad [1\text{-form symmetry}]
\]

\[
dE^a = 0, \quad [2\text{-form symmetry}],
\]

where we have indicated the associated global p-form symmetry for clarity. Although the breaking of these conservation laws is easily understood in terms of dislocation defects (i.e. missing lattice points), here we emphasize this as the breaking of the conservation of lattice charges, along lattice lines, planes and volumes. These equations are of the form (compare to Section 2)

\[
d(\ast j^{(p)}) = 0
\]

where now the duality \( \ast \) is the Hodge star on elements of \( \wedge^k E^a \), incorporating the lattice metric (15). Although we can couple \( A \) to ordinary conserved 1-form current, the higher-order versions will lead to different responses when combined with topological terms. We note the dual nature of the currents

\[
\ast 1 = E^1 \wedge E^2 \wedge E^3, \quad \ast E^a = \frac{1}{2} \epsilon_{abc} E^b \wedge E^c.
\]

we see that the “most trivial” conservation law in the crystalline insulator is actually the 3-form volume conservation law

\[
d1 = d(\ast E^1 \wedge E^2 \wedge E^3) = 0 \quad [3\text{-form symmetry}].
\]

The elastic currents are reminiscent of the non-trivial 2-form electric conservation law in U(1) Maxwell theory (8) for \( j = 0 \), essentially dictated by constrained dynamics. In general \( d \)-spatial dimensions, such \( p \)-form symmetry current conservation laws would look like

\[
d(\ast j^{(p)}) = d\left( \frac{1}{n!} \epsilon_{a_1...a_p} E^{a_1+1} \wedge \cdots \wedge E^{a_d} \right) = 0
\]
and we can write down a hierarchy of $p$-form currents and symmetries along specified lattice hypersurfaces.

We will discuss below the EM responses corresponding to these conservations laws. Before that, let us summarize the salient points of the lattice currents and higher-form symmetries: First, we note that the conserved elastic currents are tautological since the $E^a$ are by construction flat in the absence of dislocations, by the existence of the crystalline order (and $U(1)$ number conservation). Dislocations represent the vortex-like (dual)disorder operators of the translational order. The topological $p$-form conservation laws remain valid under arbitrary smooth deformations as long the order persist. On the other hand, also the global $U(1)$ number conservation law is tautological to EM gauge symmetry. Nevertheless, constrained dynamics can lead to emergent higher-form conservation laws [20], as was discussed with the $3+1d$ Maxwell theory. Second, the construction of a very similar lattice higher-form theory was described in Ref. [11] and a $\mathbb{Z}_n$ generalization in [34]. In our context, we utilize the ambiguity of lattice flat gauge fields $E^a$ on homologically non-trivial cycles. Concretely, this amounts to the lattice as the network of defect operators $\exp(i \oint E^a)$, such that their presence is consistent with allowed translational charges of the theory [11]: the defect operators of [11] implement large gauge transformations on the lattice points along transverse hypersurfaces, their combinations to $\exp(i \oint E^a \wedge E^b)$ on surfaces etc, and so on. This defines the lattice $L$ via intersections and open sets in the embedding. More abstractly, the construction applies to a topological space with (triangulated) Cech (co)homology of open covers, or a space admitting a CW-complex up to homotopy equivalence [11,33].

Finally, since the higher-form $U(1)$ theory we consider is constructed with the Wilson lines of $\exp(i \oint E^a)$ with holonomy fluxes, or equivalently, operators implementing the $T^n$ large gauge transformations along transverse sections, a natural question is what happens with surface operators replacing the role of the elementary line operators. This is trivially realized if we restrict $E^a \wedge E^b \rightarrow E^{ab}$ for some 2-form without lower-dimensional resolution in terms of 1-forms, or construct higher-rank gauge fields from $E^{ab}_\mu$ [16,21,47]. Below, such a two-form symmetry without a resolution to 1-form symmetries will correspond to quadrupolar HOTIs in three dimensions. Likely new classes of responses of higher-form theories can be obtained in this way which are different to those with symmetries (21), (22), (23) or (26). Next we classify responses related to topological states with $p$-form symmetries corresponding to each of these.

4. Higher form topological crystalline responses

We have discussed the conservation laws related to higher-form symmetries of $E^a$ above without specifying their coupling to EM fields. The response to a $p$-form background gauge field works if we pair it with a conserved $p$-form current $d \star f^{[p]} = 0$ [11], compare to (1),

$$S^{[p]}[A,J] = \int A^{[p]} \wedge \star f^{[p]} \Rightarrow S^{[p]}[B,J] = \int B^{(p+1)} \wedge \star f^{(p)}, \quad (28)$$

where we specialized to the spatial lattice currents (21)–(23), (26) in the second line. We now explore such a $p$-form current coupling in the case of the translational crystalline order coupled to the local EM $U(1)$ gauge symmetry. This sets

$$B^{(p+1)} = B^{(p+1)}[A,F, \partial F, \ldots] \quad (29)$$

An equivalent problem is the coupling of higher $p$-form background gauge fields and currents. However, we focus on the global lattice symmetries, especially since we currently do not now any (simple) incarnations of higher-form gauge fields in Nature, except those from dual (magnetic) gauge fields. In other words, for the crystalline systems we study, we are interested in the response to the EM $U(1)$ gauge fields in the presence of the (approximate) $p$-form currents $\star f^{(p)}$ in Eqs. (21), (22), (23). An important things is that the spatial lattice (and metric) enters (28) through the dual lattice $p$-form coupling $\star \rightarrow \star$ Eq. (24) and the elastic fields carry only spatial indices $a$, while the dual currents change for spacetime fields.

The heuristic rule for the relevant higher-form local EM $U(1)$ couplings and responses are suitable $B^{(p+1)}[A,F, \partial F]$ in (29), the latter allowed to be quadratic in $A$. We note that (28) is understood to apply for classical background fields, the elastic $E^a$ and electromagnetic $A$. Equivalently, (28) is a term
in the background effective action that is induced after integrating out the quantum constituents on the lattice, e.g. electrons. Quantum elasticity with $E^a$ is another matter, see e.g. [16,21,38,47]. We now discuss these responses by stating the results, in terms of the dual, higher-form current pairs. The semi-classical expansion identifying the couplings, effective actions and associated invariants in momentum space is discussed briefly for each case in Section 5.

4.1. Bulk theta term and charge

In 3+1d, the simplest crystalline responses corresponds to the 3-form symmetry (26) of lattice volume conservation,

$$S_\theta[A] = \int \frac{\theta}{8\pi^2} (\ast E^1 \wedge E^2 \wedge E^3) \wedge F = \int \frac{\theta}{8\pi^2} F \wedge F,$$

(30)

where $\theta$ is quantized in the presence of time-reversal symmetry to $0, \pi$ (see also below). Although the crystalline 3-form symmetry superficially trivial/tautological in bulk, it is familiar that the theta term implies protected boundary modes with anomalous $T$-symmetry [32].

The dual 0-form symmetry response is

$$S_\omega[E, A] = \int N_\omega(\ast 1) \wedge A = \int \frac{N_\omega}{(2\pi)^3} E^1 \wedge E^2 \wedge E^3 \wedge A$$

(31)

corresponding to just the global U(1) charge conservation, where the invariant $N_\omega$ counts the occupied bands in the BZ. Naturally to first order in deformations, $\det(E^a_i) = 1 + \partial_i u^i + O(\partial_i u^2)$, representing lattice volume.

These dual responses were (briefly) mentioned in Ref. [25,26,28] in the present context of elasticity tetrads.

4.2. Polarization and the quantum hall effect

Now we describe responses corresponding to 1-form and 2-form lattice symmetries. The electric polarization [48–55] is defined along one-dimensional spatial submanifolds and the quantum Hall response on two-dimensional spatial sections. Both couple to the external electromagnetic field gradient $dA = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$, the latter through the Chern–Simons (CS) 3-form $A \wedge dA$.

4.2.1. d + 1-d. polarization

The 1-form symmetric response is obtained by $B^{(2)} \rightarrow dA$,

$$S_{pol}[E, A] = \int P^a (\epsilon_{abc} \cdots E^b \wedge \cdots \wedge E^c) \wedge dA,$$

(32)

where the dimensionless coefficient $P^a$ is defined on lattice 1-cycles. In other words, we find that the $P^a$ describes polarization in the direction of the omitted $E^a$ from the response, allowing e.g. open boundary conditions where a-periodicity is violated. It is quantized (mod $2\pi$ ) in terms of protecting (mirror, inversion) symmetries, which are realized anomalously on the boundaries, similar to time-reversal invariant 0-form insulators with quantized theta. Under the large gauge transformations of $E^b \rightarrow E^b + \frac{2\pi}{\ell_b} dx^b$,

$$\delta P^a(\epsilon_{abc} \cdots E^b \wedge \cdots \wedge E^d) = 2\pi P^a(\epsilon_{abc} \cdots dn^b E^c \wedge \cdots \wedge E^d), \ a \neq b,$$

where $dn^b = dx^b/\ell_b$ is the density of lattice points in the $b$-direction. This constraint is tantamount to the integer periodicity of polarization density $P^a(\epsilon_{abc} \cdots E^b \wedge \cdots \wedge E^c)$ and was also noted in two-dimensions in [56].

The polarization response was derived using the crystalline 1-form symmetry but we actually see that it implies also emergent 1-form electric gauge symmetry in the sense that we can shift the local U(1) EM field $A \rightarrow A + A$ where $A$ is a 1-form (not necessarily exact). This viewpoint was studied in Ref. [56]. We see that here it just a restatement of the crystalline 1-form symmetry $d(\ast f^{(1)}) = 0$, inherited from the 0-form U(1) charge conservation in the insulator. Viewing the polarization as a
dipole charge, this is the generalization of the 1-form electric symmetry \( j = 0 \) to the crystalline insulator. We note that \( S_{\text{pol}}[E, A] \) is a pure boundary term, since

\[
S_{\text{pol}}[E, A] = \frac{\epsilon_{abc}}{(d-1)(2\pi)^{d-1}} \int d(P^b E^c - \cdots - E^c \wedge A)
\]

(33)

\[
S_{\text{pol}}[E, A] = \frac{\epsilon_{abc}}{(d-1)(2\pi)^{d-1}} \Delta P^a \int_{\partial_a} E^b \wedge \cdots \wedge E^c \wedge A.
\]

where \( \Delta P^a \) is the integrated boundary jump between the insulator and vacuum (for simplicity assumed constant along the transverse directions). We see that the boundary \((d-1)\)-volume-form symmetry term is generated. This implies protected boundary modes and the possibility of anomalous Luttinger’s theorem on the boundary via anomaly inflow from the bulk to the boundary [26].

4.2.2. \( d + 1 \)-d. quantum Hall

There is a dual response to the polarization which is more familiar. This is simply the (weak) crystalline quantum Hall phase. The response follows from the crystalline 2-form symmetry and the replacement \( B^{(3)} \to A \wedge dA \), the CS 3-form. It is given as

\[
S_{\text{QH}}[E, A] = \frac{N^{ab}}{2!(d-2)(2\pi)^{d-1}} \int \epsilon_{abc...d} E^c \wedge \cdots \wedge E^d \wedge \cdots \wedge A \wedge dA.
\]

(34)

The coefficient \( N^{ab} \) is defined on lattice 2-cycles. Note that Refs. [25,28] used the dual convention \( \tilde{N}^a = \frac{1}{2} \epsilon_{abc} N^{bc} \) for the topological charges for the QHE in 3+1d.

4.2.3. Discussion

The expressions (32), (34) are both defined without reference to any spacetime metric. They depend on (the metric of) the elasticity tetrads \( E^a = E_a^\mu dx^\mu \) and the local deformations. In other words, the responses are not universally quantized. It still follows that the responses are topological in the sections defined by the \( d \)-cycles of appropriate “missing coordinates” and depend only on the ground state properties which determine the elasticity tetrads up to elastic deformations.

The bulk polarization is a total derivative and leads to an emergent electric 1-form gauge symmetry. On the other hand, it is well-known that \( S_{\text{QHE}} \) is not a total derivative and moreover is not gauge invariant in the presence of boundaries. If we perform 0-form gauge transformation, \( A \to A + d\lambda \), the result is the 1+1d (consistent) chiral anomaly from the bulk to the boundary:

\[
\delta_{\lambda} S_{\text{QH}}[E, A] = \int \frac{1}{2!(d-2)(2\pi)^{d-1}} d(N^{ab} \epsilon_{abc...d} E^c \wedge \cdots \wedge E^d \wedge \cdots \wedge A \wedge dA) \]

(35)

which, as perhaps expected, now describes a possibly anomalous surface polarization response, where the quantity in brackets is \( \lambda d \ast j \neq 0 \) at the boundary in the absence of dislocations, so that gauge invariance holds overall in the bulk and boundary.

4.3. Multipole insulators

Continuing in 3+1d, we now seem to have exhausted the possible responses, since we cannot straightforwardly construct a \( p \leq 4 \) form out of \( A \). On the other hand, the topological responses relevant to multipolar subsystem insulators and electric multipole HOTIs [2] were not yet obtained, except for the polarization arising from \( B^{(2)} \to dA \).

The resolution to the missing ingredient to Eq. (28) for such systems is the fact that we expect additional derivatives of EM fields for multipole responses, as well as from recent results for HOTIs and higher-rank gauge theories related to crystalline systems [8]. The bulk multipoles themselves satisfy cycle-like conditions with surface multipoles when overlapping at a corner, e.g. for quadrupole moment \( q_{ij} \) and octopole moment \( o_{ijk} \),

\[
q_{xy} = p_x + p_y - q_c
\]

\[
o_{xyz} = q_{xy} + q_{yz} + q_{xz} - p_x - p_y - p_z + q_c.
\]
etc. [2], where \( q_{ij} \) denotes bulk or surface quadrupole, \( p_i \) edge polarizations and \( q_c \) is the corner charge. Related to these, there are the higher-rank conservation laws of multipole currents [18]: the current of \( p \)-multipoles is a conserved \( p + 1 \) current. Importantly, for multipole electronic HOTIs, we now require that all lower order bulk multipoles must vanish. In this way, the simple and tautological conserved currents from elasticity tetrads can allow for the correct non-trivial multipole transport. This leads to the form \( B^{(p)} \sim \partial^{p−1} F \) for \( p \)-multipoles that enter as follows in the response (28).

Let us now discuss this in two spatial dimensions for a quadrupole response. We can construct the following symmetric 2-rank EM field that couples to \( f^{(1)} \) via derivatives,

\[
\begin{align*}
\mathcal{L}_{\text{Weinberg}} &= \frac{1}{2} \partial_{\mu} A_{\nu} \partial^{\mu} A^{\nu} - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} \sim \frac{1}{2} \partial_{\mu} A_{\nu} \partial^{\mu} A^{\nu} - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} \\
&= \frac{1}{2} ( \partial_\mu A_\nu + \partial_\nu A_\mu ) - \frac{1}{2} \partial_i A_j \partial_j A_i - \frac{1}{2} \partial_i A_j \partial_j A_i = \frac{1}{2} ( \partial_\mu A_\nu + \partial_\nu A_\mu ).
\end{align*}
\]

The last line identifies basically that \( f^{(2)} \sim F \), the EM field strength (up to finite lattice rotations; we assume simple cubic). The spatial components are obtained from a symmetric form \( A_{ij} \), as appropriate for multipole moments. From the last identity we see that for multipole response, we couple only to spatial derivatives of \( A_0 \) like for polarization, and the time derivative of the relevant multipole gauge field \( A_{ij} \).

Plugging this in (28), we obtain a quadrupolar HOTI from crystalline 1-form symmetry in two dimensions as

\[
\theta \int \frac{\partial f^{(2)}_{ij}}{2\pi} \wedge \hat{E}_c = \theta \int d^2 \mathbf{x} t \partial_i \mathbf{E}_c + \partial_j \mathbf{E}_c
\]

where now \( f^{(2)} = F \) and the lattice derivative \( E_\mu \sim \hat{E}_c \partial_c \sim \partial_\mu \) is equivalent to the elasticity tetrads. The matrix \( e^{ai}_b \) and angle \( \theta \) form the quadrupole response which we identify as \( q_{ab} \) for chiral 3+1d HOTIs in Section 5.3.2. As for 1-form polarization response, the response is a total derivative. Emergent 2-form gauge symmetry results from \( F \rightarrow F + d\Lambda^{(2)} \) in combination with the conserved current \( d \neq f^{(2)} = dE^c = 0 \). We note in passing that the quantity \( \partial_i d\Lambda^{(2)} \) is the one which enters, and seems to realize an additional dipole shift symmetry \( d\Lambda^{(2)} \sim \partial_x \hat{d}\Lambda^{(2)} \) [57].

The boundary response is, choosing a simple cubic lattice and speciaing to the \( xy \)-plane for simplicity,

\[
S_{\partial q_{xy}} = \int \frac{\theta}{2\pi} \left( f^{(2)}_{xy}(F) \wedge \hat{E}_c \big|_{y=\text{const.}} + f^{(2)}_{q_{xy}}(F) \wedge \hat{E}_c \big|_{x=\text{const.}} \right)
\]

which is the appropriate for anomalous boundary polarization response, say, along:

\[
S_{\partial q_{xy}} = \int d x d t d z \hat{E}_c
\]

which naturally leads to polarization \( P^a = \theta/(2\pi) = 1/2 \), in our units, and can been shown to harbour corner charges of \( 1/4 \) if \( \theta \) is quantized [8].

In 4+1d we also have the octopole insulator which corresponds to 3-form symmetry without time reversal but instead crystalline symmetries, as distinct from the bulk theta term. In this case \( B^{(3)} \rightarrow f^{(3)}_{ij}(\partial^2 F) \) and the response is given as

\[
\begin{align*}
S_0 &= \int \frac{\theta_{ab}}{2\pi} f^{(3)}_{ij}(\partial_\mu A_\nu) \wedge \hat{E}_{ab} \\
f^{(3)}_{ij} &= \frac{1}{6} ( \partial_\mu A_\nu + \text{symm.}) \sim A_{ij}, \quad f^{(3)}_{q,ij} = \frac{1}{2} ( \partial_\mu A_\nu + \text{symm.}) \sim A_{ij}.
\end{align*}
\]

This response was analysed in [8] without the lattice fields. In this case, since the action is a total derivative and the two-form \( \hat{E}_{ab} \neq \hat{E}_a \wedge \hat{E}_b \) by symmetry in \( a, b \), there seems to be no a priori quantization in three-dimensions nor emergent higher-form gauge symmetry with a multipole Chern–Simons [8]. Of course, one could consider octopole insulator to descend from 4+1d chiral octopole insulator with the prescription \( E_{ab} \rightarrow E_a dE_b \), see Section 5.3.2, which in 3+1d explicitly depends on the boundary data.
4.4. Remarks on higher dimensions

The above construction is naturally generalized to arbitrary dimensions by identifying the possible higher-form lattice currents and substituting lower-dimensional topological states in terms of $B^{(p)} \rightarrow f^{(p)}(A, \partial A, \ldots, \partial^{p-1} F)$. For example, a subsystem theta term becomes possible in 4+1d. The 3+1d responses we considered in detail were just the ones we obtained from gluing the usual Chern, $F, F \wedge F, \ldots$ and Chern–Simons terms $A, A \wedge dA, \ldots$ up to that dimension along lattice cycles. For multipoleis, we utilized spatially “multipolized” $E^a$, as obtained in multipolar Chern–Simons terms and gauge theories [8,16]. The higher-dimensional construction is directly physically relevant when considering multipole pumping, essentially by relaxing periodic lattice directions and adding adiabatic parameters, possibly with non-trivial topology. Natural questions concern the detailed form and realization of anomaly inflow, as well as the possibility of realizing states that have non-anomalous response only when considered as boundaries higher-dimensional theories.

Related to the above, the multipole responses likely generalize to multipole Chern–Simons theories [8] by replacing elasticity tetrads with the proper components of higher-rank gauge fields. Such a replacement should potentially arise when integrating out elastic deformations quadratically. It would be interesting to consider such multipole Chern–Simons theories in full generality and their dependence on the lattice deformations and geometry. On the other hand, we can construct the odd spatial-dimensional axion extension of multipole CS, simply by adding elasticity tetrads.

5. The semi-classical expansion and momentum space invariants

We shall now (briefly) discuss the higher-order symmetric responses and momentum space invariants in 3+1d. In this case, the relevant crystalline topological responses are bulk theta-term, polarization, quantum Hall and quadrupole multipole responses. For generality, we still write formulas in $d+1$ spacetime dimensions.

The responses are not in general quadratic neither in the elasticity tetrads and the EM gauge field $A$, and the response formulas should be strictly understood to apply for semi-classical, infinitesimal background fields. We note that even in this limit, the translational lattice fields $E^a$ are quantized in terms of large gauge transformations. Accordingly, we compute the invariants in the semi-classical expansion in momentum space for weakly varying coordinate dependence. The assumptions are gauge invariance and the validity gradient expansion. Naturally, this expansion can be utilized only in the continuum limit of the elasticity tetrads but we expect that the results are robust to finite lattices and space group symmetries.

Following e.g. Refs. [32,58–60], we imagine that the fermionic degrees of freedom $\Psi$ have been integrated out and focus on the (low-energy) ground state effective action, defined as

$$S_{\text{eff}}[E, A] = -i \log \int \mathcal{D}\Psi \ e^{iS[E, A, \Psi]}$$  \hspace{1cm} (43)

Assuming that the fermion interactions have been decoupled in terms of mean-field Hubbard–Stratonovich fields, the effective action is expanded around saddle-point solutions

$$S_{\text{eff}}[E, A] = i \text{Tr} \log G[E, A] = -i \text{Tr} \int_0^1 du \ G \partial_u G^{-1}$$  \hspace{1cm} (44)

where $G[E, A]$ is the Green’s function and $E, A$ are background fields. We imagine that the electromagnetic field $A$ is turned on adiabatically as a function of the parameter or extra coordinate $u$ as $A_{u=0} = 0$ and $A_{u=1} = A$. By gauge invariance, $A_\mu$ minimally couples to $p_\mu$. Similarily, we can define in terms of the adiabatic coordinate $u$,

$$E^a_{u=1} = E^a_\mu(\bar{x})d\bar{\mu},$$  \hspace{1cm} (45)

$$E^a_{u=0} = E^a_\mu(0)d\bar{\mu} = \frac{2\pi}{\epsilon^a} \delta_\mu^a d\bar{\mu}.$$  \hspace{1cm} (46)

where $E^a = \frac{2\pi}{\epsilon^a} \delta_\mu^a d\bar{\mu}$, with the convention that $E^a_\mu = \frac{1}{\epsilon^a} \delta_\mu^a \rightarrow 0$ in the case without any periodicity $T$ in the time direction. For the elasticity tetrads, we simply assume “geometric” minimal coupling:
the lattice momenta \( p^\mu \) in the BZ and spacetime coordinate momenta \( p^\mu \) are related by the matrices \( \hat{E}_\mu^a(x) \), note that former are dimensionless, and \( a = 1, \ldots, d \),

\[
  p^\mu = \hat{E}_\mu^a(x)p^a, \quad p_a = \hat{E}_\mu^a(x)p_\mu, \quad p^a \in \text{BZ}.
\]

In this way \( E^a \sim K^a \), where \( K^a = \oint dp^a \) is a reciprocal lattice vector. As a result, the background fields enter as

\[
  G^{-1}[E, A] = G^{-1}[p_a - A_a] = G^{-1}[\hat{E}_\mu^a p^a - A_a] = \ii \omega + A_0 - H[\hat{E}_\mu^a(x)p_\mu - A_0(x)].
\]

We evaluate spacetime derivatives as

\[
  \partial_\mu G^{-1}[A] = \partial_k G^{-1}[u_0 = 0] A_\mu + \partial_\mu E^{-1}[u_0 = 0] \hat{E}_\mu^a \partial_a \hat{E}_\mu^b
\]

where we used \( \partial_{\mu} G^{-1} = \partial_{k\mu} G^{-1}[u_0 = 0] \) and \( \partial_{E \mu} G^{-1} = -\partial_{\mu} G^{-1}[u_0 = 0] \). We also neglected the second order couplings \( A_0(x) \approx E^{(0)}[A] A_\mu = \delta_0^a A_\mu \) in \( G^{-1}[E, A] \), i.e. when \( u = 0 \), we make no distinction between the lattice and spacetime coordinates, momenta or gauge fields \( E^{(0)}[A] A_\mu = \delta_0^a \).

To obtain the responses, we expand the Green’s function \( G[E, A] \) in the semi-classical gradient expansion, where \( A \) and \( \delta E^a = E^a - E^{(0)}[A] \) are both small, slowly varying adiabatic background fields.

### 5.1. Bulk theta term and charge

These dual invariants are familiar \([32,58]\). The response first order in \( A \) zeroth order term in gradients \( \partial A \) is

\[
  S^{(1,0)}_{\text{eff}}[A, E] = \ii \int_0^1 du \int_{\text{BZ}} \frac{d^dp d\omega}{(2\pi)^{d+1}} \int d^dxdtr[G\partial_\mu G^{-1}]|_{u = 0} \partial_{\mu} A_\mu
\]

Since we assume no periodicity in time, only the integral over lattice spatial directions is well defined. This is just the lattice volume, i.e. the total charge density coupling to \( A \),

\[
  S^{(1,0)}[A, E] = \frac{N_1}{d!(2\pi)^d} \ii \int \epsilon_{a\cdots c} E^a \wedge E^b \wedge \cdots \wedge E^c \wedge A
\]

where the invariant corresponding to the 0-form bulk lattice charge is just

\[
  N_\omega = N_{\omega}(p) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \, \text{tr} G(p_\mu) \partial_\omega G^{-1}(p_\mu).
\]

\( N_\omega \) counts the number of occupied states and can only change at BZ momenta \( p \) where the gap closes \([25,26,58,61]\). In the actual system, the imaginary frequency is cutoff from above by the charge gap, as well as from below. As discussed, the same response arises on the boundary of the system with non-trivial electric polarization \([26]\) and is related to the boundary Luttinger anomaly \([25,26]\).

For the 3-form crystalline symmetry, the second order in \( O(\partial A^2) \) term in the expansion, \( S^{(0,2)}_{\text{eff}}[A] \) with no elasticity tetrads, produces the bulk theta term \([30]\) with \([32]\)

\[
  \theta = \frac{1}{48\pi^2} \ii \int_0^{2\pi} du \int_{\text{BZ}} d^d p e^{\ii u_{\mu}^{\lambda}} \text{tr} \left[ (G\partial_\mu G^{-1})(G\partial_\nu G^{-1})(G\partial_\nu G^{-1})(G\partial_\mu G^{-1}) \right]
\]

this invariant can be reduced to a pure momentum space winding number, through its relation to the 2nd Chern class of the BZ,

\[
  \theta = \frac{1}{24\pi^2} \int_{\text{BZ}} \text{tr} (GdG^{-1})^3,
\]

making the dual nature to \([51]\) more explicit. We also note the relation of these invariants via dimensional reduction \([58,62]\).
5.2. Quantum Hall response

The semi-classical expansion for the quantum Hall response in terms of elasticity tetrads was discussed in [25]. This term is [63–66]

\[
S^{(1,1)}_{\text{eff}}[E, A] = -\frac{1}{4} \int d^2xdt \int_{BZ} \frac{d^2p}{(2\pi)^2} \tr[(G\partial_{k_x}G^{-1}G\partial_{k_x}G^{-1} - G\partial_{k_y}G^{-1}G\partial_{k_y}G^{-1})G\partial_{p_x}G^{-1}]|_{\omega=0} \partial_{\mu}A_{\mu}A_{\nu}.
\]

which is first order in gradients \( \partial A \) and second order in \( A \). Again the integral splits in momentum-space and the result is Eq. (34) with the familiar invariant given as

\[
N_\alpha(k^a) = \frac{1}{8\pi^2} \epsilon_{ijk} \int_{-\infty}^{\infty} d\omega \int_{BZ} dS^3 \tr[(G\partial_{\omega}G^{-1})(G\partial_{p_j}G^{-1})(G\partial_{p_k}G^{-1})].
\]

This invariant is quantized to integers as an element of \( \pi_3(\mathbb{T}^3, U(N)) \) [64], where \( N \) is the number of bands, where \( \mathbb{T}^3 \) is the doubly pinched 3-torus formed by identifying \( \omega \to \pm \infty \) in the sections of 2D BZ transverse to \( a \). Under these constraints, also \( N_\alpha(k^a) \) is quantized and can change as a function of \( k^a \) only when it closes in the BZ.

5.3. Polarization and quadrupolarization

5.3.1. Polarization

We now give an elementary argument for the quantized momentum space polarization invariant \( P^a \) inspired by the results [58,60,62,67]. Consider a system which is half-open to the \( x \)-direction with a boundary region at \( x = \pm L_x \) with bulk polarization. The three-dimensional polarization response is

\[
S_{pol} = \frac{1}{2(4\pi^2)} \int d^3x P^a \epsilon_{abc} E^b \wedge E^c \wedge dA,
\]

we focus on the polarization along \( P^1 = P^x \) with \( E^1_0 = \delta_{lx} \). It is given by

\[
S_{pol} = \frac{1}{4\pi^2} \int dydz \int dx dt P^1 E^y_0 E^z_0 \epsilon^{xyz} \partial_x A_0 = n_y n_z \int_{-L_x}^{L_x} dx dt (\partial_x P^1) A_0
\]

where for simplicity \( A_0 \) is constant along \( y, z, \int E^2 = n_y, \int E^3 = n_z \), and the \( x \)-integral is over the boundary “soliton”, where the bulk polarization changes to its vacuum value. For such a 1d theta term, in the paper [60] the following invariant was given (BZ denotes the suitably restricted Brillouin zone depending on the symmetries of the Wigner transformed open system)

\[
N_{\text{soliton}} = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} d\omega \int_{-L_x}^{L_x} dx \int_{BZ} dk_x \times \tr(G\partial_k G^{-1}G\partial_\omega G^{-1}G\partial_k G^{-1})
\]

\[
= P^1|_{L_x} - P^1|_{-L_x} = \frac{1}{2\pi} \int_{BZ} dk_x \tr[G\partial_k G^{-1}]
\]

where, on the last line, we have used the dimensional reduction formula [62,67]. Extending this result to the open-system with two macroscopic boundaries at \( x = -L, L \), where \( L \to \infty \), and comparing the opposite jumps at both ends, we obtain the result (assuming that \( P^a \) is constant in bulk and \( P^a = 0 \) mod integers outside)

\[
P^a = \frac{1}{2\pi} \int_{BZ} dk^a \tr(S G\partial_k G^{-1})
\]

where \( \Sigma^2 = 1 \) is an operator that (anti)commutes with the Hamiltonian for e.g. chiral crystalline systems [68]. Remarkably the invariant is well-defined in the bulk BZ mod integers and non-trivial if the bulk crystalline symmetry implies \( P^a \to -P^a \). In the recent paper [28], this invariant and boundary response was explicitly studied in PT-invariant chiral inversion-time symmetric insulator.
In particular, when evaluated for the effective Wigner-transformed Hamiltonian $H(k_0) = G^{-1}(\omega = 0, k_0)$ singular at the edge due to gapless flat band excitations. This response is calculated directly from the semi-classical expansion in the Appendix.

5.3.2. Quadrupole response

Let us now discuss the quadrupole response. In contrast to the QH or polarization responses, in this case the response is not obtained from combinations of topological terms $A$, $dA, AdA$ but from the “multipole” electric field form $\partial aF$. This term appears in the semi-classical expansion in chiral HOTIs protected by the $C_4T$ symmetry if we couple the fermions to momentum charged symmetric tensor resembling elasticity tetrads.

The quadrupole response was discussed in Ref. [3] for chiral HOTI in terms of the time-reversal protected theta term, although a constant $\theta$ is not quantized by $C_4T$. To make contact with the 4-component chiral model in [3], we switch to a massive Dirac fermion description in 3+1d:

$$L_{\text{Dirac}} = \overline{\psi} \left[ \gamma^\mu (i\partial_\mu + QA_\mu - \delta B_\mu \cdot \tau) - m(1 + i\gamma^5 \theta \cdot \tau) \right] \psi,$$  \hspace{1cm} (60)

where $A_\mu, \delta B_{\mu a}$ and $\theta_a$ are slowly varying perturbations, $\gamma^\mu, \gamma^5$ Dirac gamma matrices and $\tau^a$ are isospin Pauli matrices; the angle $\theta_\alpha$ interpolates between $\pm m$ for $0, \pi$. The $Q = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right)$ projects the 8-component fermion to the 4-component Dirac fermion corresponding to the lattice model [3]. The $T$ and $C_4$-breaking terms $\delta B_{\mu a} \cdot \tau = \delta E_{\mu a}^z p_a \tau^2$ is a lattice momentum dependent isospin field, generalizing the elasticity tetrads. The relevant quadrupolar term in the gradient expansion of the effective action (60) is (see [69] for related 2+1d model)

$$S_{\text{eff}}^{(3)} = i \text{Tr} \left[ Q \frac{1}{p - m} \frac{1}{p - m} \frac{1}{\gamma^5} \left( \gamma^\mu \partial_\mu + QA_\mu - \delta B_\mu \cdot \tau \right) \right] = i^4 \text{Tr} \left[ \frac{m^2}{(p^2 - m^2)^3} \partial_\mu \partial_\nu \partial_\alpha (\theta_1 \delta E_\mu^a ) \gamma^5 \right]$$ \hspace{1cm} (61)

where $\hat{A} = \gamma^\mu A_\mu$, etc. and we utilized the semiclassical relation $[p_\mu, f(x)] = i\partial_\mu f(x)$, in particular for obtaining the $\partial_a F$ term. The related quadratic in $A_\mu$ term produces the theta term (30) from the mass interpolating angle $\theta_\alpha$. In terms of the semi-classical greens function, the momentum space invariant multiplying the response is simply proportional to (52).

For non-zero $\tau^2$ term proportional to $O(p^2)$, the insulator is in the quadrupole phase. Instead of assuming the $\delta E_{\mu a}^z p_a$ term $\delta B_{\mu a}$ in (60), we could consider a $O(p^2)$ term in a $k \cdot p$ expansion with non-trivial tetrads $\delta E_{\mu i}^a$:

$$\frac{1}{2m} k_i g^{ij} k_j \Rightarrow \frac{1}{2m} \gamma^a \delta E_{\mu i}^a \gamma^b \delta E_{\mu j}^b k_j = \frac{g^{ij}}{2m} k_i k_j - \frac{1}{4m} i\gamma^j \gamma^i T_{\mu i}^a k_a.$$ \hspace{1cm} (62)

where $T_{\mu i}^a = \partial_\mu \delta E_{\nu i}^a - (\mu \leftrightarrow \nu)$ is the (effective) torsion. After discarding $O(k^2)$ terms, this represents a non-minimal, non-universal dipole-like coupling in terms of the isospin-massive Dirac fermions.

Expanding the matrix trace and performing the momentum space integral (61), we are left with

$$S_{\text{eff}}^{(3)}[E, A] = \frac{1}{16\pi^2} \int d^4 x \epsilon^{\mu \nu \lambda \rho} \partial_\mu (\theta_1 \delta E_{\nu i}^a) \partial_\lambda F_{\rho i}.$$ \hspace{1cm} (63)

Interestingly, this is a total derivative giving a non-trivial 2+1d response when $d(\theta_1 \delta E^a) \neq 0$ at a boundary,

$$S_{\text{q-pole}}[E, A] = \frac{1}{8\pi^2} \int d^3 x \epsilon^{ij} (\theta_1 \delta E_{\nu i}^a) \partial_\nu \epsilon_{ij}.$$ \hspace{1cm} (64)

This coincides with the 2+1d quadrupole response [8] provided we identify, $\theta_1 = \pi \mod \mathbb{Z}$ and

$$\epsilon_{ai} \delta E_{\mu i}^a \equiv \epsilon_{ij} = 2\pi q_{ij}.$$ \hspace{1cm} (65)
Although the terms $\delta B_2 \tau^2$ and $\theta_1 \tau^1$ are trivial constants in the bulk, the quadrupole term arises on the boundary as the bulk mass interpolates between $\pm m$. We note that the quantization of the response $q_{ij} = \frac{\theta_{ij}}{2\pi}$ follows by $\theta_1 \rightarrow \theta_1 + 2\pi$ and $\delta \hat{E}^a \rightarrow 2\pi \delta_{xj}$. The same quantization of the response was discussed in [8] although from a different viewpoint.

While the relation (65) looks strange, we note the following salient features: (i) The coefficient $q_{ij}$ in the response is $T$-even but $C_4 T$-odd, which can quantize the response to be $1/2 \mod \pi$. In particular we do not need invoke solely $T$-symmetry for the quantization of the theta-response with non-trivial spatial dependence, in contrast to [3]. As discussed, a bulk EM theta term (30) arises from the mass parameter $\theta_3$; this is detailed in the Appendix. (ii) The same relation (65) was discovered for chiral topological elasticity for an asymmetric traceless gauge field $q_{ij}$ [16], when generalizing fractions to curved space. Although the $C_4$-boundary of the chiral 3+1d quadrupole insulator is sensitive to the bulk symmetries it preserves, in the presence of a mass and coupling to elasticity, we can generate the same torsional chiral Chern–Simons action in 2+1d, with the property that the equations of motion give zero torsion or $\frac{d q_{ij}}{d \tau} = 0$ and $\nabla \times p_j = 0$ which are the defining properties of the quadrupole insulator with vanishing bulk polarization and non-trivial boundary charges from $p_j = \partial_i q_{ij}$, $q = \partial_i \partial_j q_{ij}$ at the boundaries.

Finally we remark that the octupole response can derived from a $k^3$-dispersion and the term

$$\sim e^{ikm} e^{iT h p_a p_b F_{ab}}. \tag{66}$$

This is not well-defined 3+1d since the induced boundary term is trivial since $E^a E^b \partial_a \partial_b$ is symmetric in $a, b$. A quantized response follows in 4+1d, however, with the mixed Chern–Simons term (66). We leave the detailed study of chiral elasticity, torsion and multipoles in Dirac models for the future.

6. Conclusions

We have shown how the topological response of crystalline insulators, both subdimensional weak topological insulators and HOTIs, arises from coupling EM fields to conserved generalized global U(1) lattice symmetries from elasticity. The periodic lattice directions are first embedded to the underlying spacetime using tautological gauge fields, the elasticity tetrads, encoding UV lattice momenta (even in the continuum). More formally, we can consider the lattice theory and symmetries as a network of monopole translation operators and allowed translations in Čech cohomology [11,33].

The semiclassical field theory expansion incorporating the (almost) conserved higher-form currents in terms of elasticity tetrads is related to (weak) momentum space invariants of the gapped insulating lattice on transverse, submanifold cycles. The overall response is not exactly quantized, however, due to the remaining lattice fields. Essentially, this refinement of different topological insulating responses was achieved by coupling them to elastic deformations and then considering the possible higher-order global symmetries from $E^a$. When these lattice fields are coupled to EM gauge fields representing topological responses, emergent higher-form gauge symmetries emerge, see also [56]. Both of these can appear in the response if the insulator retains the conservation of the charged constituents along some lattice directions and/or associated charge multipoles, respectively.

The form of the quasitopological response with lattice dependence is particularly clear for the polarization and the quantum Hall invariants, which both descend from lower dimensional topological classes, the 1+1d theta term and 2+1d CS term. The latter appears if there is a two-form conservation law along the QH planes, meaning that layers are independent in topological charge transport, while the former can appear if there are lattice directions along which the charge and topological polarizations are conserved. The almost conserved symmetry is defined with respected to the charge gap and therefore is not fundamental but emerges at low energy in the topological insulator. The higher-order topological multipole response is similarly well-defined if the almost conserved symmetry localizes to a boundary of the same order and the appropriate multipole charge is quantized by symmetries to be non-zero mod integers with lower multipoles vanishing. This geometric interpretation of higher-rank multipole fields in terms of elasticity tetrads corresponds to the proposal in [16].
In short, the proposed framework is a higher-order extension of the topological field theory of traditional insulators, in the sense that topological terms are induced with quantized momentum space invariants multiplying quasitopological terms with mixed field content. The responses are not quantized overall, as can be deduced by simple dimensional analysis, and change under deformations of $E^a$. A hierarchy of lower-dimensional momentum space invariants can be quantized on the other hand, and leads to expect that several topological invariants are required to specify generic crystalline responses involving higher-order topology. This requires more work on generalizing and extending the results in [67], along with clear identification of the gauge fields corresponding to non-trivial multipoles. Nevertheless, our framework naturally explains how higher-form gauge symmetries can arise in crystals and is equivalent to the reduction of corresponding higher-rank gauge theories to global U(1) symmetry [8,56]. The framework requires the crystalline symmetries and connects the higher-order gauge symmetry to charge transport via the almost conserved global symmetries encoded by the elasticity tetrads. The lattice defects carry topological charges and have restricted mobility, whose connection to fractonic phases with higher rank gauge fields has already been emphasized [47]. In general, the dimensional hierarchy of topological response, subdimensional conservation laws and anomalies is realized in a transparent way by utilizing lattice gauge fields.

In the classification of topological crystalline states with symmetry $G$ and ground states $\Omega$, the cohomology $H^d(BG, \Omega)$ appears [38,70]. Here we have focused only on the finite lattice translation symmetries $\mathbb{Z}^d$ in $d+1$-dimensions. The classifying space $BG = EG/G$ is extremely simple and follows as $B\mathbb{Z}^d = \mathbb{T}^d$, when we choose $EG = \mathbb{R}^d$ and fix the origin of the lattice at $X^a = 0$. The elasticity tetrads are flat gauge fields on $\mathbb{T}^d$, i.e. elements of $H^d(\mathbb{Z}, \mathbb{T})$. Here we have outlined the higher-order responses of HOTIs and topological insulators coupled to a conserved U(1) charge and translations. This approach pertaining to unit cell translational lattice symmetries should be developed to include the space group symmetries and discrete lattices in any real crystalline material. Until now the complete topological classification of crystalline matter has been hindered by the lack of generalization of $K$-theory to include finite group symmetries. It is possible that the higher-form symmetries and elasticity tetrads could lead to a simpler description of associated vector bundles with appropriate higher rank gauge fields, since the higher-form response is stabilized by the additional crystalline symmetries and would not be necessary to add to the bundle structure group. Finally, here we assumed integer filled bands, while the connection of the results to Lieb-Schultz-Mattis-type theorems and responses in semimetals should be explored [43,45,56,70].

Note added: While the preprint version arXiv:2009.14184 of this paper was prepared, the preprint [56] appeared which discusses HOTIs from the perspective of higher-order gauge symmetries. Our results are similar where they overlap. Since then, many works have proposed similar responses and lattice gauge fields in closely related context, e.g. Refs. [38,39,42,69]. The lattice fields $E^a$, with units of lattice momenta, are also closely related to foliations and subsystem symmetries [71,72]; here we focused on generalized symmetries defined with flat gauge fields on cycles of the lattice torus $\mathbb{T}^d$, following [11].

CRediT authorship contribution statement

Jaakko Nissinen: Conceptualization, Methodology, Formal analysis.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.
Appendix A. Polarization and quadrupole in the semi-classical Green’s function expansion

Now we consider the $\partial_A E_3^a$ terms in the semiclassical expansion, which heuristically are momentum insertions. These are terms from

$$S'_{\text{eff}} = -i \text{Tr} \int_0^1 du \ G\partial_{E_3^a} G^{-1}|_{u=0} \partial_u \hat{E}_3^a = i \text{Tr} \int_0^1 du \ G\partial_{p_3} G^{-1}|_{u=0} p_3 \partial_u \hat{E}_3^a$$  \hspace{1cm} (A.1)$$

Now we expand the semiclassical $G$ in the expansion, see e.g. [62]. The first $\partial A$ term is

$$S^{(1')} = -i \text{Tr} \int_0^1 du \ G\partial_{\hat{E}_3^a} G^{-1}|_{u=0} \partial_u \hat{E}_3^a = i \text{Tr} \int_0^1 du \ G\partial_{p_3} G^{-1}|_{u=0} p_3 \partial_u \hat{E}_3^a$$  \hspace{1cm} (A.2)$$

$$= \int \frac{dtd^dxdod^dp}{(2\pi)^{d+2}} \text{tr} \left[ \left( G\partial_{p_3} G^{-1} G\partial_{p_3} G^{-1} - G\partial_{p_3} G^{-1} G\partial_{p_3} G^{-1} \right) G\partial_{p_3} G^{-1} \right] u=0 \hat{E}_3^a p_3 \partial_u A_\nu$$

$$= \int \frac{dtd^dxdod^dp}{(2\pi)^{d+2}} \frac{p^a}{(d-1)(2\pi)^d-1} \epsilon_{abc} E^b \wedge \cdots \wedge E^c \wedge F$$  \hspace{1cm} (A.3)$$

To first order in $\delta E^3$, the integral is the flux of the (multivalued) 3d invariant in four dimensions, with $\int d\rho A_\rho^a = E_3^a$ (no sum),

$$P(p) = \int \frac{d\rho d\rho}{(2\pi)^{d}} \int \frac{d^d}{d} \text{tr} \left[ GdG^{-1} \right] = \left[ \omega = A \right] \epsilon_{abc} E^b \wedge \cdots \wedge E^c \iint \frac{1}{2\pi} \text{tr} GdG^{-1}$$  \hspace{1cm} (A.4)$$

over suitable sections of the BZ, which is reduced by the results of [67] to the one-dimensional invariant (A.4) in Eqs. (32), (59). In particular, the invariant can change only at a band-inversion at imaginary frequency $\omega = 0$, implying the existence of boundary modes (when $P^a = 1/2$ mod Z; see also [28]).

Appendix B. Chiral quadrupole insulator and theta term

The quadrupole response was discussed in the main text based on general elasticity tetrads (38) and the Dirac model (60).

From the semiclassical expansion, second order terms $\partial^2 A$ are from $\partial_\mu \partial_\nu G^{-1} = \partial_\nu \partial_\mu G^{-1} \partial_\lambda A_\lambda \partial_\nu A_\rho + \partial_\nu G^{-1} \partial_\mu \partial_\lambda A_\lambda$ and for us only the second insertion is important. We get

$$S^{(2')} = i \text{Tr} \int_0^1 du \ G\partial_{p_3} G^{-1}|_{u=0} p_3 \partial_u \hat{E}_3^a$$  \hspace{1cm} (B.1)$$

$$= \frac{i}{4} \int \frac{dtd^dxdod^dp}{(2\pi)^{d+2}} \text{tr} \left[ \left( G\partial_{p_3} G^{-1} G\partial_{p_3} G^{-1} + G\partial_{p_3} G^{-1} G\partial_{p_3} G^{-1} \right) G\partial_{p_3} G^{-1} \right] u=0 \hat{E}_3^a p_3 \partial_u A_\nu$$

$$- G\partial_{p_3} G^{-1} \partial_\mu \partial_\nu G - \partial_\nu \partial_\mu G^{-1} G\partial_{p_3} G^{-1} G\partial_{p_3} G^{-1} \right) G\partial_{p_3} G^{-1} \right] u=0 \hat{E}_3^a p_3 \partial_u A_\nu$$  \hspace{1cm} (B.2)$$
This vanishes upon taking the proper gauge invariant, antisymmetric combinations of \( \hat{E}_\alpha^a \partial_\mu \partial_\lambda A_\lambda \), since the 2nd order term in the semiclassical gradient expansion is symmetric. On the other hand, we can utilize the relation \([g_{\alpha \beta}, \partial_\mu A_\lambda] = i \partial_\mu g_{\alpha \beta} A_\lambda \) in (A.4). The result is then proportional to the 2+1d QH invariant, which, however, vanishes on the boundary with non-zero quadrupole of a chiral HOTI [8]. A model where the higher-dimensional winding number, similar to the 3+1d theta term, produces a 2+1d quadrupole was given in Eqs. (60), (64).

Let us now discuss and compare these somewhat contradictory findings to the \( C_4T \)-invariant model of a chiral HOTI with the theta term [3,8]. The response (38) with symmetries

\[
C_4 : (t, x, y, z) \rightarrow (t, y, -x, z), \quad (A_0, A_x, A_y, A_z) \rightarrow (A_0, -A_y, A_x, -A_z)
\]

\[
T : t \rightarrow -t, \quad (A_0, A_i) \rightarrow (A_0, -A_i).
\]

fixes \( O(\hat{A}^2) \) response to \( \partial_\mu \partial_\lambda A_\lambda = \partial_\lambda \varepsilon_{\lambda xy} + \partial_\lambda \varepsilon_{\lambda x} \). In the paper [3], this response was related to a non-constant theta term, now quantized with respect the \( C_4T \) symmetry as \( \theta(x, y, z, t) \rightarrow \theta(y, -x, z, t) \), similar to the TRI invariant \( \mathbb{Z}_2 \) insulator with theta term quantized in the bulk and \( T \)-anomalous Dirac fermions at the boundary. We now want to connect this response to our results. For a spacetime dependent \( \theta(x) \), the action making is sense is really the one corresponding to the 3+1d QH,

\[
S'_{q_{xy}}[A, \theta] = \frac{1}{16\pi^2} \int d^4x \, \epsilon^{\alpha\beta\gamma\delta} A_\alpha F_{\beta\gamma} \partial_\delta \theta.
\]

This produces the QH-like chiral response in terms of the vortices of \( \theta \) in the \( xy \)-plane, instead of the topological invariant (30). Due to the \( C_4T \)-symmetry, \( \theta \) winds by \( 2\pi \)-jumps in the four quadrants, which can be pushed to the corners as \( \pm 2\pi \) vortices. The singularities \( (\partial_\alpha \partial_\beta - \partial_\beta \partial_\alpha) \theta \neq 0 \) are localized on the corners in the \( xy \)-plane for a \( q_{xy} \)-quadrupole. The QH-like currents, polarizations and magnetizations are

\[
\begin{align*}
J^\alpha &= \frac{\delta S'_{q_{xy}}}{\delta A_\alpha} = \frac{1}{8\pi^2} \epsilon^{\alpha\beta\gamma\delta} F_{\beta\gamma} \partial_\delta \theta, \\
p^i &= \frac{\delta S'_{q_{xy}}}{\delta \varepsilon_i} = \frac{1}{8\pi^2} \epsilon^{ijk} A_j \partial_k \theta, \quad M^i = \frac{\delta S'_{q_{xy}}}{\delta B_i} = \frac{1}{8\pi^2} A_0 \partial_i \theta.
\end{align*}
\]

and imply charges on the vortices of \( \theta \). In Ref. [60] the following solitonic formula, e.g. at the \( z \)-boundary of the system, was considered,

\[
\partial_\mu \theta = \frac{1}{2\pi^2 i} \int_{BZ} d^3k d\omega \, e^{0lmn} \text{tr} \left( G \partial_\mu G^{-1} G \partial_\omega G^{-1} G \partial_k G^{-1} G \partial_k G^{-1} \right)
\]

\[
= \frac{1}{10\pi^2 i} \int_{BZ} d\omega d^2k \partial_\mu n_{5}^\mu,
\]

where \( n_{5}^\mu \) is defined by the above formula. For the chiral HOTI theta term, this is replaced with the 7-form winding formula, where the Green’s function depends on the coordinates \( (\omega, k; x, y, z) \) in the presence of “solitonic” boundaries in all directions. In the bulk where \( \theta \) is constant, the 7-dimensional invariant can be shown to coincide with the 3-dimensional invariant (53) of Ref. [3] over momentum space, see e.g. [32].

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