BRACKETS BY ANY OTHER NAME

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In Memory of Kirill Mackenzie (1951-2020)

Abstract. Brackets by another name - Whitehead or Samelson products - have a history parallel to that in Kosmann-Schwarzbach’s “From Schouten to Mackenzie: notes on brackets”. Here I sketch the development of these and some of the other brackets and products and braces within homotopy theory and homological algebra and with applications to mathematical physics.

In contrast to the brackets of Schouten, Nijenhuis and of Gerstenhaber, which involve a relation to another graded product, in homotopy theory many of the brackets are free standing binary operations. My path takes me through many twists and turns; unless particularized, bracket will be the generic term including product and brace. The path leads beyond binary to multi-linear n-ary operations, either for a single n or for whole coherent congeries of such assembled into what is known now as an ∞-algebra, such as in homotopy Gerstenhaber algebras. It also leads to more subtle invariants. Along the way, attention will be called to interaction with ‘physics’; indeed, it has been a two-way street.

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1. **Introduction.** Here is a complement to [35] in this volume, emphasizing a parallel development of ‘brackets’ (and products and braces, etc.) from a homotopy/homological point of view (cf. also [?]). I will include higher structures\(^1\) in two versions: \(n\)-ary operations for \(n > 2\) and in the sense of classic secondary operations, defined only when a primary operation vanishes\(^2\).

Brackets by another name - Whitehead or Samelson products - have a history parallel to that in Kosmann-Schwarzbach’s “From Schouten to Mackenzie: notes on brackets”. Here I sketch their development as well as that of some of the other brackets and products and braces within homotopy theory and homological algebra and applications to mathematical physics.

In contrast to the brackets of Schouten, Nijenhuis and of Gerstenhaber, which involve a relation to another graded product, in homotopy theory many of the brackets are free standing binary operations. My path takes me through many twists and turns; unless particularized, bracket will be the generic terms including products and braces. The path leads beyond binary to multi-linear \(n\)-ary operations, either for a single \(n\) or for whole coherent congeries of such assembled into what is known now as an \(\infty\)-algebra, such as in homotopy Gerstenhaber algebras, as well as more subtle invariants; for example, secondary operations which are defined only when a primary operation vanishes. It reveals missed (or seriously delayed) opportunities for collaboration within mathematics and between mathematics and physics; some leads have ‘withered on the vine’. As Dyson remarked about missed opportunities [18]:

> occasions on which mathematicians and physicists lost chances of making discoveries by neglecting to talk to each other.

The situation has improved since Dyson’s lecture, but not completely; a common vocabulary does not necessarily imply a common viewpoint.

2. **Brackets known as products.** In 1941, J.H.C. Whitehead constructed his eponymous Whitehead product\(^3\) [76]. It might better be called the Whitehead bracket; indeed, it is denoted by \([ , ]\) and later was shown to satisfy a shifted graded version of the Jacobi identity.

For a space \(X\) and base point \(x\), the homotopy classes of maps \(f : S^p \to X\) (respecting base points) form a group \(\pi_p(X)\). For classes \(\alpha \in \pi_p(X), \beta \in \pi_q(X)\), the Whitehead product \([\alpha, \beta] \in \pi_{p+q-1}\) is represented as follows: Let \(f : S^p \to X\) and \(g : S^q \to X\) represent \(\alpha, \beta\) respectively. The product \(S^p \times S^q\) has a cellular decomposition as \(S^p \vee S^q \sim e^{p+q}\), where the cell \(e^{p+q}\) is attached by a map \(h : S^{p+q-1} \to S^p \vee S^q\). The composition \((f \vee g) \circ h\) represents the homotopy class \([\alpha, \beta] \in \pi_{p+q-1}\).

However, it was not until the mid-1950s that several authors\(^4\), independently, showed that the Whitehead product satisfied a graded version of the Jacobi

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\(^1\) There is now the Journal of Higher Structures.
\(^2\) When iterated operations are used, that will be made explicit.
\(^3\) The name appears to have been used for the first time in 1953 by Hilton and Whitehead [31].
\(^4\) S. C. Chang, Hilton, Massey-Uehara, Nakaoka-Toda, G.W. Whitehead and others not published.
identity. It was around the same time that Schouten introduced his bracket of multivector fields on a manifold, followed by the contributions of his student Nijenhuis. These two visions of graded Lie algebra were contemporaneous, but no-one noticed, at least no-one remarked, until decades later. Here in this section is a sketch of the ‘other’ development of the theory of graded Lie brackets in early homotopy theory, based primarily on historical research in 1995 by Kristen Haring: On the Events Leading to the Formulation of the Gerstenhaber Algebra:1945-19665 [30].

Nijenhuis’ title Jacobi-type identities for bilinear differential concomitants of certain tensor fields [53] indicates the pangs of birthing the new concept of graded Lie bracket, as did the many attempts to recognize it in homotopy theory.

The graded Jacobi identity There was a long delay from Whitehead’s 1941 paper until 1954 when many independent proofs of the Jacobi identity for Whitehead products were published or acknowledged elsewhere6. The difficulty (beyond the issue of grading dependent signs) was that \((f \vee g) \circ h\) defines the homotopy class \([\alpha, \beta] \in \pi_{p+q-1}\), but the Jacobi identity at the map level will hold only up to homotopy.

As Massey remarked to Haring: this question was “in the air” among homotopy theorists in the early 1950’s, I don’t believe you can point to any one person and say he or she raised this question.

One of the issues that may have delayed recognition/formulation was the fact that all three brackets (Nijenhuis-Schouten, Gerstenhaber, Whitehead) were graded Lie only after a shift. Samelson’s development [63] of what is now called the Samelson product produced the shift topologically by passing to the based loop space \(\Omega X\) of a pointed space \(X\) where \(\pi_p(\Omega X)\) is isomorphic to \(\pi_{p+1}(X)\). Samelson’s product

\[\pi_p \Omega X \otimes \pi_q \Omega X \rightarrow \pi_{p+q} \Omega X\]

was realized in terms of the commutator with respect to loop multiplication, though his emphasis was on the corresponding Pontryagin product after applying the Hurewicz morphism. As a graded commutator at the level of homotopy classes, the Jacobi identity follows as for a graded associative algebra.

3. The Gerstenhaber bracket revisited. Gerstenhaber’s bracket was originally defined on the Hochschild cochain complex \(C^*(A, A)\) of an associative algebra \(A\) with \(C^p(A, A) := \text{Hom}(A^p, A)\) and coboundary \(\delta : C^p(A, A) \rightarrow C^{p+1}(A, A)\) given, for \(f \in C^p(A, A)\), by

\[(\delta f)(a_1 \cdots a_{p+1}) = a_1 f(a_2 \cdots a_{p+1}) + \sum_{i=1}^{p} \pm f(a_1 \cdots a_i a_{i+1} \cdots a_{p+1}) \pm f(a_1 \cdots a_p) a_{p+1}\]

This cochain complex is itself an associative algebra with respect to the cup product \(f \smile g := f(\cdots)g(\cdots)\). Notice, as in algebraic topology, \(\smile\) is not commutative, even

5 Including contributions from many of the main researchers involved: Gerstenhaber, Hochschild, Lian, Massey, Nijenhuis, Samelson, Serre, Spencer, Sternberg, G.W. Whitehead and Zuckerman.

6 See Haring [30] for a chronological account of the submission of proofs and the recognition within those articles of the existence of still others.
up to sign. However, Gerstenhaber introduced his bracket $[\ ,\ ]$ with a description that related it to the cup product by\textsuperscript{7}

$$\delta [f, g] = [\delta f, g] \pm [f, \delta g] \pm f \smile g$$

(2)

On the Hochschild cochain complex above, Gerstenhaber defined his $\circ$ operation:

$$f \circ g \in C^{p+q-1}$$

for $f \in C^p, g \in C^q$ by summing operations $\circ_i$ inserting (with signs) $g$ into $f$ in the $i$-th place. On the cohomology, what is now known as the Gerstenhaber bracket $[f, g]$ is represented by the commutator $f \circ g = g \circ f$. The combination of the cup product $f \smile g$ with the Gerstenhaber bracket $[f, g]$ and the Hochschild differential satisfies many of the relations of a differential graded Poisson algebra, but not all. For example, the operation $[\cdot, \cdot]$ is up to sign a right inner derivation of degree 1 of $f \smile g$ of the graded algebra $(C^\ast(A, A), \smile)$, but generally not a left derivation. Instead, the deviation from being a left derivation of degree 1 is given by a coboundary.

**Remark 1.** The nomenclature evolved from ‘graded or odd Poisson algebra’ to G-algebra around 1987 and then to Gerstenhaber algebra. According to Google Scholar, the first published appearance of “Gerstenhaber algebra” was in 1992\textsuperscript{42,55}.

This brings us back to one of the subtleties of Gerstenhaber’s construction: what is now called a Gerstenhaber algebra exists on the cohomology of the complex of Hochschild cochains. Gerstenhaber originally apologized\textsuperscript{8} for not presenting his results on Hochschild cohomology in terms of a more modern $Ext$-functor (at the cohomology level). I beg to differ. He derived his structures from operations at the cochain level where some of the defining relations held only up to homotopy. The cochain homotopy structure as he developed it bore fruit in ways the cohomology level $Ext$-functor might not have suggested.

The Fröhlicher-Nijenhuis bracket of vector-valued forms is the graded commutator of derivations of the algebra of differential forms\textsuperscript{35}. Gerstenhaber’s is also a commutator, in fact, from two different points of view: first, in Gerstenhaber, as a commutator with respect to the $\circ$-product, later as the commutator of coderivations of $\bigoplus A^p$ as a coalgebra with the de-concatenation diagonal. Indeed, the Hochschild chain complex can be described in terms of the bar construction on $A$ in one of its incarnations. This is how I came to see Gerstenhaber’s structures, which led me to the interpretation of $C(A, A) \equiv Hom(TA, A)$ as $Coder(TA)$, the space of coderivations of $TA$, the tensor coalgebra on $A$. There I saw, many years later, the Gerstenhaber bracket as (up to sign) the commutator bracket of coderivations\textsuperscript{68}.

**Remark 1.** The bracket notation $[a, b]$ was often used to indicate a commutator in an associative algebra and only gradually became standard for an abstract Lie algebra. Even the appellation “Lie algebra”, due to Weyl, did not occur until the 1930’s! (Compare the alternate notations discussed in\textsuperscript{35}.)

**Homotopy Gerstenhaber algebras** In a letter to several of us\textsuperscript{15}, P. Deligne pointed out that since the structure of what is now called a Gerstenhaber algebra can be described as the structure of an algebra over the homology of the little discs operad\textsuperscript{50}, for this relationship which exists at the homology level there must

\textsuperscript{7} The resemblance to Steenrod’s relation between $\sim$ and $\sim_1$ was first observed (as far as I know) by Kadeishvili in 1988\textsuperscript{33}.

\textsuperscript{8} “at the risk of seeming old fashioned”\textsuperscript{23}. 
be more between the little disks operad and the Hochschild cochains. Later this became quoted as (some variant of) the following conjecture.

**Deligne’s Conjecture** now a Theorem: The structure of an algebra over the homology of the little disks operad on the Hochschild cohomology may be naturally lifted to the cochain level.

This gave rise to a ‘cottage industry’ producing a variety of proofs, first by by Tamarkin [71] though unpublished, then A. Voronov [74], then McClure-Smith [51] and several others. The essential point is that homotopy relations in Gerstenhaber’s original construction are but a small part of a whole conger of higher homotopy relations, similar to those appearing in the algebraic topology of based loop spaces about the same time, beginning in 1963 [66], the same year as [23]. It was a very good year! Like the Whitehead product, these homotopy relations did not involve another basic operation such as the cup product or exterior product. Still more higher homotopies were needed for iterated loop spaces, giving rise to May’s introduction of **operads** [50].

Intuition similar to Deligne’s is a recurring theme: classical structure on the homology of a chain complex indicates the possible presence of a corresponding ∞-structure on the original chain complex. Conversely, there is the process of homotopy transfer of a strict graded algebra on the chain level (e.g. a differential graded associative algebra) which will often give rise to an ∞-structure on the homology so as to be suitably equivalent to the original on the chain complex. A pioneering example is due to Kadeshvili, giving rise in the associative setting to an A∞-structure [67]:

**Theorem 3.1.** [34] Given a differential graded algebra (A, d), there is an A∞-algebra structure on H(A, d) which is equivalent as A∞-algebra to (A, d).

**L∞-algebras, deformation theory and physics** Since the Schouten-Nijenhuis bracket developed out of differential geometry, it is not surprising it had physical relevance; what has been called **cohomological physics** can be traced at least as far back as to Gauss [22]. More surprising, perhaps, is that Gerstenhaber’s bracket from homological algebra appeared in physics in the development of deformation quantization [8, 9]. Here the Gerstenhaber bracket gave the first obstruction to the existence of a *star product*. If that obstruction vanishes, then there are higher order obstructions as in general deformation theory, including the original example for deformations of complex structure [17]. Kosmann-Schwarzbach comments that in 1964, Nijenhuis and Richardson remarked on the striking similarity of Gerstenhaber’s treatment of deformations and the existing treatments of deformations of complex structure. Indeed, Gerstenhaber confirms that deformations of complex structure inspired his own work.

For the Whitehead product or for the Gerstenhaber bracket, the Jacobi identity holds only up to homotopy. A choice of homotopy can be taken as part of the structure on a chain complex (L, d) with a bracket [x, y] and a ternary operation
\[ [x, y, z] \text{ such that} \]
\[ [x, [y, z]] \pm [[x, y], z] \pm [y, [x, z]] = d[x, y, z] \pm [dx, y, z] \pm [x, dy, z] \pm [x, y, dz] \]
\[ (3) \]
\[ \text{or for cycles } x, y, z: \]
\[ [x, [y, z]] \pm [[x, y], z] \pm [y, [x, z]] = d[x, y, z] \]
\[ (4) \]

Asking if \([x, y, z]\) satisfies an appropriate relation leads to the concept of an \(L_\infty\)-algebra. These algebras are a generalization of differential graded Lie algebras in which the Jacobi identity again holds only up to homotopy, but now included is a choice of such a homotopy and a relation involving four elements, to be satisfied up to homotopy and on and on. \(L_\infty\)-algebras were first introduced in the context of deformations of rational homotopy types, in preliminary versions of [64], but other later basic references are listed in the n-lab entry for \(L\)-infinity-algebra.

**Definition 3.1.** An \(L_\infty\)-algebra consists of a differential graded vector space \((L, d)\) with differential \(d = \ell_1\) of degree 1 and graded skew-symmetric \(n\)-ary brackets \(\ell_n : L^{\otimes n} \to L\) of degree \(2 - n\) satisfying a coherent set of differential relations, called \(\text{generalized Jacobi relations}$. An \(L_\infty\)-algebra may arise from an \(A_\infty\)-algebra by suitable skew-symmetrization; in return, there is an \(A_\infty\)-algebra given by a universal enveloping functor.

In 1989, the \(L_\infty\) structure of CSFT (Closed String Field Theory) was first recognized as such when Zwiebach fortuitously gave a talk in Chapel Hill at the last GUT (Grand Unification Theory) Workshop [77, 78]. This was the proper ‘birth certificate’ for \(L_\infty\)-structures in physics, which gradually became a part of the ‘standard tool kit’ in gauge field theory (cf. [42]). Closed String Field Theory is a field theory in the physics sense, starting with an algebra of functions on the space of loops (closed strings) on a manifold. A convolution product on this algebra produces a binary bracket and a full panoply of higher homotopies.

A little later, Lian and Zuckerman [45] were working on conformal field theory (CFT) in the BRST formalism, which includes a generalization of a Chevalley-Eilenberg complex. The theories studied by Lian and Zuckerman led to a BV-algebra (Batalin-Vilkovisky) on cohomology, which extends a Gerstenhaber algebra. Just as Gerstenhaber presented some first level homotopies which ultimately gave rise to homotopy Gerstenhaber algebras, Lian and Zuckerman presented some first level homotopies which ultimately gave rise to homotopy BV-algebras. (See the next section for Nambu Hamiltonian dynamics.)

4. \textbf{n-variations on a theme.} So far, the emphasis has been on either binary operations or systems of compatible multilinear operations: \(\infty\)-algebras. There is another tradition of \(n\)-ary algebras: vector spaces (without grading) with only one multilinear \(n\)-ary operation \(V^{\otimes n} \to V, n \geq 3\). According to an excellent survey of many \(n\)-ary operations by de Azcarraga and Izquierdo [13]:

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9 According to folklore, “there exists a set of signs” which are spelled out in many of the relevant references.

10 There are alternate descriptions with different notation, e.g. \([\cdot, \cdots, \cdot]_n\), with shifted grading and with \(d\) of degree \(-1\).
ternary operations appeared for the first time associated with the cubic matrices that had been introduced by A. Cayley in the middle of the XIXth century and that were also considered by J.J. Sylvester some forty years later.

**WARNING:** There is a confusion of nomenclature, especially for variations on the definition of Lie algebra, including both $n$-Lie algebras and Lie $n$-algebras, as well as $\text{Lie}(n)$ and $\text{Lie}_n$. These are *not* all the same, not at all (see the Appendix for clarification).

**n-ary Lie algebras**

Let us begin with $n$-ary Lie algebra, which has two major variants, corresponding to two versions of a (generalized Jacobi) characteristic identity\(^{11}\). In a “Lie” context, appropriate graded symmetries are always assumed.

One version is the defining relation for an $L_\infty$-algebra restricted to a vector space with just one $n$-ary bracket,

$$\sum \pm l_n(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \otimes v_{\sigma(n+1)} \otimes \cdots \otimes v_{\sigma(n+1)}) = 0; \quad (5)$$

no grading is involved.

Such algebras have been studied quite independently of $\infty$-structures and of each other by Hanlon and Wachs\[^{28}\] (combinatorial algebraists) and by de Azcarraga and Bueno\[^{14}\] (physicists).

**n-ary Filippov algebra**

A major alternative *generalized Jacobi relation* was introduced by Filippov.

**Definition 4.1.** An $n$-ary Filippov algebra (or $n$-Filippov algebra)\[^{19}\] consists of a vector space $V$ with an $n$-ary bracket $[\, , \cdots , \, ] : V^{\otimes n} \to V$ such that $[X_1,X_2,\ldots,X_{n-1},\, ]$ acts as a left derivation\(^{12}\):

$$[X_1,X_2,\ldots,X_{n-1},[Y_1,Y_2,\ldots,Y_n]] =$$

$$[[X_1,X_2,\ldots,X_{n-1},Y_1],Y_2,\ldots,Y_n] + \cdots + [Y_1,\ldots,Y_{n-1},[X_1,\ldots,X_{n-1},Y_n]].$$

Unfortunately, algebras with either characteristic identity are often called $n$-Lie algebras; I urge use of *n-ary Filippov* to lessen the confusion of terms (see the Appendix for clarification of these and other terms).

As I recall, I first learned of the Filippov identity from Alexander Vinogradov when we met at the Conference on Secondary Calculus and Cohomological Physics, Moscow, August 1997. (This identity arose independently by yet another name in\[^{20}\]) See A. and M. Vinogradov’s\[^{73}\] for a comparison of these two distinct generalizations of the ordinary Jacobi identity to $n$-ary brackets.

**Nambu $n$-Hamiltonian mechanics**

\(^{11}\) In older literature, these two different generalized Jacobi relations are distinguished by respective *fundamental identities*, but, as suggested in\[^{13}\], a better name is *characteristic identities*.

\(^{12}\) That identity was known also to Sahoo and Valsakumar\[^{62}\].
Nambu mechanics \cite{52, 70} is a generalization of Hamiltonian mechanics proposed by Yoichiro Nambu in 1973\textsuperscript{13}. In his formulation, a triple (or, more generally, n-tuple) of “canonical variables” replaces a canonically conjugated pair in the Hamiltonian formalism and a ternary (or, more generally, n-ary) operation the Nambu bracket, which generalizes the Poisson bracket to n-variables, \( n \geq 3 \). Nambu dynamics is described by the flow given by Nambu-Hamilton equations of motion—a system of ODE’s which involves \( n - 1 \) “Hamiltonians”. Nambu’s original work \cite{52} was a generalization of the binary Poisson bracket of Hamiltonian mechanics to a ternary bracket \( (6) \). In \cite{7}, Bayen and Flato formalized the notion of a Nambu algebra, including the n-ary version. The defining relation was the characteristic identity for the Filippov bracket above, though apparently not remarked at the time. Similarly, in \cite{70} there is no mention of Filippov; apparently the Siberian journal was not well read in Moscow or Leningrad. Takhtajan emphasizes the role ternary and higher order algebraic operations and mathematical structures related to them play in passing from Hamilton’s to Nambu’s dynamical picture. He writes:

We start by formulating the fundamental identity \( (\text{FI}) \) for the Nambu bracket as a consistency condition for Nambu’s dynamics. It yields the analog of the Poisson theorem that the Poisson bracket of integrals of motion is again an integral of motion. A ‘canonical’ Nambu bracket is defined for a triple of classical observables on the three-dimensional phase space \( \mathbb{R}^3 \) with coordinates \( x, y, z \) by the following beautiful formula

\[
\{f_1, f_2, f_3\} = \frac{\partial (f_1, f_2, f_3)}{\partial (x, y, z)},
\]

where the right-hand side stands for the Jacobian of the mapping \( f = (f_1, f_2, f_3) : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \). This formula naturally generalizes the usual Poisson bracket from binary to ternary operation on classical observables\textsuperscript{14}.

The Vinogradovs \cite{73} provide a thorough survey, both historically and cross-culturally (analysis, combinatorics, homological algebra, deformation theory) including mathematical physics. My understanding (though I’ve been unable to verify it) is that Filippov had noticed that, for \( n = 3 \), Nambu’s bracket was a specific example \cite{52, 70, 69}.

As can be seen, there has been a rather spotty interaction among the various points of view in both physics and mathematics, delaying opportunities for interaction.

All these algebras are important in geometry and in physics where the corresponding structures are on vector bundles over a smooth manifold (see, for example, \cite{75} and references therein). For these, there are a variety of terms, e.g. \( n \)-(ary) Lie algebroids. Mackenzie played a leading role in clarifying the basic mathematical structure here \cite{48}.

\textsuperscript{13} Gaetano Vilasi has alerted me to a possible predecessor in 1887, Albeggiani’s \( n \)-Poisson bracket \cite{4}.

\textsuperscript{14} M. Flato informed Takhtajan that, apparently, Nambu introduced this bracket in order to develop a “toy model” for quarks considered as triples and that, together with Fronsdal, Flato independently introduced such a relation.
5. **Brace algebras.** Products and brackets and braces -oh my!\(^{15}\) Recall that Gerstenhaber’s bracket on the Hochschild cochain complex arose from summing operations \(\circ_i\) inserting (with signs) \(g\) into \(f\) in the \(i\)-th place. To handle more and more significantly coherent multi-linear operations, there are more elaborate structures, **brace algebras** and **symmetric brace algebras**\(^{16}\) in which the insertions occur simultaneously in multiple slots, not necessarily consecutive.

Such braces were originally introduced in 1988 (without the name and with different notation) by Kadeishvili \([33]\); he recognized later that his \(\sim_1\) on the Hochschild complex is exactly Gerstenhaber’s circle product. For uniqueness and functorality of the minimal \(A_\infty\)-model, he needed the notion of \(A_\infty\)-morphisms of \(A_\infty\)-algebras, for which he constructed the higher braces. Later in 1993, Getzler \([25]\) used essentially the above description, again without the name ‘brace’.

Not until 1995 in work of Gerstenhaber and A. Voronov \([24]\) was it shown that the braces on the Hochschild cochain complex satisfy certain identities; they called the resulting algebraic structure a **brace algebra** with a homotopy Gerstenhaber algebra structure as an application.

**Definition 5.1.** A (non-symmetric) **brace algebra** is a graded vector space \(B\) together with a collection of degree 0 multilinear braces \(x, x_1, \ldots, x_n \mapsto x\{x_1, \ldots, x_n\}\) that satisfy the identities

\[
x\{x_1, \ldots, x_m\}\{y_1, \ldots, y_n\} = \\
\sum \epsilon x\{y_1, \ldots, y_{i_1}, x_1\{y_{i_1+1}, \ldots, y_{j_1}\}, y_{j_1+1}, \ldots, y_{i_m}\}, \\
x_m\{y_{i_m+1}, \ldots, y_{j_m}\}, y_{j_m+1}, \ldots, y_n\},
\]

where the sum is taken over all sequences \(0 \leq i_1 \leq j_1 \leq \cdots \leq i_m \leq j_m \leq n\) where \(x\{\} = x\) and \(\epsilon\) is an appropriate sign.

There is a corresponding symmetric version of the brace algebra where the sum is taken over unshuffle sequences with appropriate signs \([41]\). They provide an important machinery for handling panoplies of higher operations, for example, in the definition of homotopy Gerstenhaber algebras and homotopy BV-algebras and applications to physics.

A still more elaborate notion of **multi-braces** is due to and applied by Akman \([3, 2]\), again in physics.

6. **Secondary and higher products and brackets.** There is an ancient and honorable world view of **secondary or conditional** invariants, ranging from classical algebra to algebraic topology and even mathematical physics. The essential idea:

> When a known invariant vanishes, it is often possible to define a secondary invariant, but only on objects where the primary invariant vanishes.

\(^{15}\) Lions and tigers, and bears, oh my! - Dorothy in Wizard of Oz (1939)

\(^{16}\) The name *brace* refers to the symbol \(\{, \}\), not to be confused with its use for Poisson brackets. Parentheses, brackets, and braces are sometimes referred to as "round," "square," and "curly."
More precisely [58], given a set of transformations $T$, a *conditional invariant modulo $R$* means invariance for an object under only those transformation $T$ for which $R$ holds\footnote{Physicists might say “on shell”}.

In cohomology, there are *Massey products*, defined on cohomology classes $u, v, w$ if the products $uv = 0 = vw$ [49]. Then suitable vanishing for Massey products of four classes leads to a ternary invariant and so on. This works for general dg associative algebras and for $A_\infty$-algebras with a little more effort.

For a dg Lie algebra, the analogous operations are named *Massey-Lie brackets*, developed by Retakh [60]. After transferring the structure of a dg Lie algebra $(L, d)$ to an $L_\infty$-algebra with respect to the differential (the 1-bracket), there are again multi-linear operations on the cohomology $H(L)$ which are all primary in the sense of $L_\infty$-algebras.

### Toda (secondary) brackets

The *Toda bracket* is an operation on homotopy classes of maps, in particular on homotopy groups of spheres, defined by Hiroshi Toda [72]. Here the algebra in question concerns the composition of maps up to homotopy, in particular maps from spheres to spheres and hence on the algebra of homotopy groups of spheres.

For spaces $A$ and $B$, denote by $[A, B]$ the set of homotopy classes of maps $A \to B$. Suppose that

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$$

is a sequence of maps between spaces such that the compositions $g \circ f$ and $h \circ g$ are both nullhomotopic. Given a space $A$, let $CA$ denote the cone on $A$. Then we get a (non-unique) map

$$F : CW \to Y$$

induced by a homotopy from $g \circ f$ to a trivial map. Similarly we get a non-unique map

$$G : CX \to Z$$

induced by a homotopy from $h \circ g$ to a trivial map. Appropriate compositions give two maps $CW \to Z$ which agree on $W$. Joining them together on the suspension $SW$, the union of these two cones, we get a map

$$< f, g, h > : SW \to Z.$$
7. **Leibniz “brackets”**. In 1993, Loday [47], for use in algebraic K-theory, formalized the notion of a (right) Leibniz algebra\(^{18}\), using the \([\ , \ ]\)-notation to display the defining relation as visibly equivalent to the Jacobi relation:

\[
[x, y], z = [x, [y, z]] + [[x, z], y].
\]

Such algebras with the left handed convention had been described earlier by Bloh [11] with the name (in Russian) left \(\Delta\)-algebra, where \(\Delta\) refers to distributive, although it could just as well stand for derivation.

In [47], there is a hint of relation to (Hamiltonian) physics. Earlier, in the late 1980’s, Dirac’s theory of constraints led to work of Irene Dorfman in the context of Dirac structures in field theory [16]. She developed a bracket that bears her name, that is a special case of what is now known as a Leibniz bracket (or product).

In the context of Courant algebroids, Liu, Weinstein and Xu [46] introduced a non-skewsymmetric bracket they called ‘a twisted\(^{19}\) bracket’. Later Ševera and Weinstein wrote [65]:

> It was observed in 1998 by Kosmann-Schwarzbach, Xu, and Ševera (all unpublished !!) that the non-skewsymmetric version of the bracket satisfied the Jacobi identity written in Leibniz form.

Roytenberg in his thesis [61] pointed out that their formula agreed with Dorfman’s. He also showed that this bracket can be expressed as the derived bracket of the bracket of a differential graded Lie algebra introduced in [37]): In any graded differential Lie algebra, \((A, [, ], D)\), with bracket of degree \(\pm 1\), one can define a bilinear map, called the derived bracket of \([\ , \ ]\) by \(D\), as

\[
(a, b) \in A \times A \mapsto (-1)^{|a|}[Da, b] \in A,
\]

where \(|a|\) is the degree of \(a\) (see [36]).

In 1994, Loday invited Shavkat Ayupov for two months to IRMA (the Institute in Strasbourg of which Loday was director). Ayupov found discussions with Loday very stimulating. Although Loday studied these algebras from a cohomological point of view, upon return to Tashkent, Ayupov and his colleagues began to develop the structure theory of Leibniz algebras as algebras in their own rite; indeed, they are are principal developers of this theory. For a quite complete picture of these algebraic developments, see their book [5].

In physics, Leibniz algebras have recently found increased use, in particular for gauging procedures in supergravity, replacing the more classical Lie algebras of symmetries, while using \(\circ\) in place of the bracket notation \([\ , \ ]\).

**Definition 7.1.** A (left) Leibniz algebra \((V, \circ)\) consists of a vector space \(V\) with a bi-linear operation \(\circ: V \otimes V \to V\) such that

\[
x \circ (y \circ z) = ((x \circ y) \circ z) + y \circ (x \circ z).
\]

The need for a unified perspective on gauging procedures in supergravity, as well as in Double and Exceptional field theories, has been salient in theoretical physics for some years now. The first author to notice that Leibniz algebras could

\(^{18}\) Some authors have called these Loday algebras, but Loday himself strongly urged calling them Leibniz algebras.

\(^{19}\) A term much overworked, even in this intersection of math and physics.
be a crucial element in gauging procedures in supergravity may have been Strobl [39].

Recent mathematical interpretations of such ‘physical’ structures have led to the development of tensor hierarchies [39, 44, 43] which are differential graded Lie algebras (differential Lie crossed modules are a particular case). Crucial to the development of such tensor hierarchies is an embedding tensor: for a Lie algebra \( \mathfrak{g} \) and a Leibniz algebra \((V, \circ)\) for which \( V \) is a \( \mathfrak{g} \)-module, an embedding tensor is a map \( \Theta : V \to \mathfrak{g} \) satisfying some compatibility conditions. From that data, a tensor hierarchy can be constructed, much as is done for Sullivan models or Postnikov towers.

Many physical gauge ‘field’ theories, especially of Lagrangian type, involve fields which can be recognized as differential forms with the infinitesimal symmetries of the theory being differential forms with values in a Lie algebra. Certain field theories where the gauge structure of the ‘free’ (think non-interacting particles) theory is given in terms of a strict Lie algebra often require an \( L_\infty \) algebra for the interacting theory, the algebra of gauge symmetries being field dependent. The latter is an idea going back to [10] (compare [21]). Recently there has been further progress using differential forms with values in an \( L_\infty \)-algebra. In fact, Hohm and Zwiebach [32] have described general gauge invariant perturbative field theories in terms of an \( L_\infty \)-algebra \( L = \{ L_n \} \) in which the fields are elements of \( L_{-1} \), gauge parameters encoding the gauge symmetries are elements of \( L_0 \) while \( L_{-2} \) contains the equation of motion and \( L_{-3} \) contains the Noether identities of her second variational theorem [38, 54]. A promising alternative [12] is to start with an algebra of differential forms with values in a Leibniz algebra and a Lagrangian that ‘misbehaves’ due to a lack of ‘covariance’. To achieve covariance, the Leibniz algebra of values is extended step by step to kill the obstructions to covariance, arriving ultimately at an \( L_\infty \)-algebra.

### 8. Evolution of notation.

Another historical accident that may have hindered recognition of the these two parallel developments of the Jacobi relation is the notation. First, in the differential geometry of Schouten, ‘tensors’ (in fact tensor fields) were objects represented by symbols with indices with elaborate rules for combining them, as Ricci, Levi-Civita and Einstein would have used the word [21]. Nijenhuis grew up in that tradition, but during the 1950’s, differential geometers gradually moved away from heavy dependence on indices and Nijenhuis updated his notation. As he wrote to Haring:

> I saw how to state theorems with minimal use of indices, but not how to prove them. Once I caught on, the transformation went quickly.

20 Their convention is that the differential \( d = \ell_1 \) is of degree \(-1\)

21 In 1923, Rainich [57] wrote:

> As to the method of the study it seemed to me better to avoid, as far as possible, the introduction of things which have no intrinsic meaning, such as coordinates, the \( g \)'s, the three-indices symbols,...

Rainich tried again in 1950 to re-emphasize ‘the idea of the tensor itself and to consider the components as something secondary’ [59]. However, heavy use of indices persists to this day in the physics literature.
9. **For Kirill.** I was able to meet Kirill at a conference in Paris in January 2007 including an evening hosted for a few of us by Yvette and Bertram. Plans for him to meet and work with me in the US were delayed by bureaucratic restrictions in the United Kingdom. I had hoped to develop the higher structure version of his major work \[48\], which he with his collaborators had begun to do \[27\]. Then, all too soon, sadly we lost him quite unexpectedly.

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The items that have been mentioned in the text are meant only to whet the reader’s appetite in the hope they will follow up on one or more of these tidbits. In particular, even at the binary level, there are other brackets attributed to e.g. Vinogradov, Balavoine and others.

11. **Appendix.** There are two major distinct meanings of \(n\)-algebra:

- \(n\) indicating an algebra of \(n\)-ary operations
- \(n\) indicating an algebra structure on a graded vector space \(V = \{V_i\}\) for \(0 \leq i \leq n - 1\).

Unfortunately, algebras satisfying one or the other characteristic identity are called \(n\)-Lie algebras by many authors; I urge use of \(n\)-ary Filippov to lessen the confusion of terms.

To further confusion, in the existing literature, in addition to Lie \(n\)-algebra in the above sense, there are terms: \(\text{Lie}(k), \text{Lie}_n\) and \(L(m)\).

\(\text{Lie}(k)\) in Hanlon and Wachs \[28\] has maps \(V^{k+1} \to V\) satisfy an ungraded version of the \(L_\infty\)-relations. These are also considered by de Azcarraga and Bueno \[14\] (physicists), but defined in terms of structure constants for use in physics.

There is a totally different use of \(\text{Lie}(n)\); it is a representation of the symmetric group on \(n\) letters called \(\text{Lie}(n)\) which has dimension \((n-1)!\). It can be realized as a certain sub-algebra of the free Lie algebra on \(n\) ‘letters’.

In contrast to truncating an \(L_\infty\)-algebra definition by limiting grading to \(0 \leq i \leq n - 1\), Lada and Markl \[40\] define \(L(m)\) via bounds on the \(l_k\): \(\{l_k\mid 1 \leq k \leq m, k < \infty\}\).

Gnedbaye \[26\] treats, from an operadic point of view, \(k\)-ary algebras satisfying various possible generalizations of associativity, commutativity and Lie structure.

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