SOBOLEV INEQUALITIES AND THE \( \overline{\partial} \)-NEUMANN OPERATOR

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ABSTRACT. We study a complex valued version of the Sobolev inequalities and its relationship to compactness of the \( \overline{\partial} \)-Neumann operator. For this purpose we use an abstract characterization of compactness derived from a general description of precompact subsets in \( L^2 \)-spaces. Finally we remark that the \( \overline{\partial} \)-Neumann operator can be continuously extended provided a subelliptic estimate holds.

1. INTRODUCTION.

Let \( \Omega \) be a bounded open set in \( \mathbb{R}^n \), and \( k \) a nonnegative integer. We denote by \( W^k(\Omega) \) the Sobolev space

\[
W^k(\Omega) = \{ f \in L^2(\Omega) : \partial^\alpha f \in L^2(\Omega), |\alpha| \leq k \},
\]

where the derivatives are taken in the sense of distributions and endow the space with the norm

\[
\| f \|_{k,\Omega} = \left( \sum_{|\alpha| \leq k} \int_{\Omega} |\partial^\alpha f|^2 \, d\lambda \right)^{1/2},
\]

where \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is a multiindex, \( |\alpha| = \sum_{j=1}^n \alpha_j \) and

\[
\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.
\]

\( W^k(\Omega) \) is a Hilbert space. If \( \Omega \subset \mathbb{R}^n, n \geq 2 \), is a bounded domain with a \( C^1 \) boundary, the Rellich-Kondrachov lemma says that for \( n > 2 \) one has

\[
W^1(\Omega) \subset L^r(\Omega), r \in [1, 2n/(n-2))
\]
and that the imbedding is also compact; for \( n = 2 \) one can take \( r \in [1, \infty) \) (see for instance [4]), in particular, there exists a constant \( C_r \) such that

\[
\| f \|_r \leq C_r \| f \|_{1,\Omega}, \tag{1.1}
\]

for each \( f \in W^1(\Omega) \), where

\[
\| f \|_r = \left( \int_{\Omega} |f|^r \, d\lambda \right)^{1/r}.
\]

Now let \( \Omega \subset \mathbb{C}^n (\cong \mathbb{R}^{2n}) \) be a smoothly bounded pseudoconvex domain. We consider the \( \overline{\partial} \)-complex

\[
L^2(\Omega) \xrightarrow{\overline{\partial}} L^2_{(0,1)}(\Omega) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} L^2_{(0,n)}(\Omega) \xrightarrow{\overline{\partial}} 0, \tag{1.2}
\]

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where \( L^2_{(0,q)}(\Omega) \) denotes the space of \((0,q)\)-forms on \( \Omega \) with coefficients in \( L^2(\Omega) \). The \( \overline{\partial} \)-operator on \((0,q)\)-forms is given by

\[
\overline{\partial} \left( \sum_J \alpha_J \, d\zeta_J \right) = \sum_{j=1}^n \sum_J \frac{\partial \alpha_J}{\partial \zeta_j} d\zeta_j \wedge d\bar{\zeta}_j,
\]

where \( \sum_J \) means that the sum is only taken over strictly increasing multi-indices \( J \).

The derivatives are taken in the sense of distributions, and the domain of \( \overline{\partial} \) consists of those \((0,q)\)-forms for which the right hand side belongs to \( L^2_{(0,q+1)}(\Omega) \). So \( \overline{\partial} \) is a densely defined closed operator, and therefore has an adjoint operator from \( L^2_{(0,q+1)}(\Omega) \) into \( L^2_{(0,q)}(\Omega) \) denoted by \( \overline{\partial}^* \).

We consider the \( \overline{\partial} \)-complex

\[
L^2_{(0,q-1)}(\Omega) \xrightarrow{\overline{\partial}} L^2_{(0,q)}(\Omega) \xrightarrow{\overline{\partial}} L^2_{(0,q+1)}(\Omega),
\]

for \( 1 \leq q \leq n-1 \).

We remark that a \((0,q+1)\)-form \( u = \sum_J u_J \, d\zeta_J \) belongs to \( C^\infty_{(0,q+1)}(\Omega) \cap \text{dom}(\overline{\partial}^*) \) if and only if

\[
\sum_{k=1}^n u_{kK} \frac{\partial r}{\partial z_k} = 0
\]

on \( \partial \Omega \) for all \( K \) with \( |K| = q \), where \( r \) is a defining function of \( \Omega \) with \( |\nabla r(z)| = 1 \) on the boundary \( \partial \Omega \).

The complex Laplacian \( \square = \overline{\partial} \overline{\partial}^* + \overline{\partial}^* \overline{\partial} \), defined on the domain

\[
\text{dom}(\square) = \{ u \in L^2_{(0,q)}(\Omega) : u \in \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}^*), \overline{\partial} u \in \text{dom}(\overline{\partial}^*), \overline{\partial}^* u \in \text{dom}(\overline{\partial}) \}
\]

acts as an unbounded, densely defined, closed and self-adjoint operator on \( L^2_{(0,q)}(\Omega) \), for \( 1 \leq q \leq n \), which means that \( \square = \square^* \) and \( \text{dom}(\square) = \text{dom}(\square^*) \).

Note that

\[
(\square u, u) = (\overline{\partial} \overline{\partial}^* u + \overline{\partial}^* \overline{\partial} u, u) = ||\overline{\partial} u||^2 + ||\overline{\partial}^* u||^2,
\]

for \( u \in \text{dom}(\square) \).

If \( \Omega \) is a smoothly bounded pseudoconvex domain in \( \mathbb{C}^n \), the so-called basic estimate says that

\[
||\overline{\partial} u||^2 + ||\overline{\partial}^* u||^2 \geq c \|u\|^2,
\]

for each \( u \in \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}^*), c > 0 \).

This estimate implies that \( \square : \text{dom}(\square) \longrightarrow L^2_{(0,q)}(\Omega) \) is bijective and has a bounded inverse

\[
N : L^2_{(0,q)}(\Omega) \longrightarrow \text{dom}(\square).
\]

\( N \) is called \( \overline{\partial} \)-Neumann operator. In addition

\[
||Nu|| \leq \frac{1}{c} ||u||.
\]

A different approach to the \( \overline{\partial} \)-Neumann operator is related to the quadratic form

\[
Q(u, v) = (\overline{\partial} u, \overline{\partial} v) + (\overline{\partial}^* u, \overline{\partial}^* v).
\]
For this purpose we consider the embedding
\[ j : \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}^*) \to L^2_{(0,q)}(\Omega), \]
where \( \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}^*) \) is endowed with the graph-norm
\[ u \mapsto (\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2)^{1/2}. \]
The graph-norm stems from the inner product \( Q(u,v) \). The basic estimates (1.7) imply that \( j \) is a bounded operator with operator norm
\[ \|j\| \leq \frac{1}{\sqrt{c}}. \]
By (1.7) it follows in addition that \( \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}^*) \) endowed with the graph-norm \( u \mapsto (\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2)^{1/2} \) is a Hilbert space.

The \( \overline{\partial} \)-Neumann operator \( N \) can be written in the form
\[ (1.9) \quad N = j \circ j^*, \]
details may be found in [13].

2. Compactness and Sobolev inequalities.

Here we apply a general characterization of compactness of the \( \overline{\partial} \)-Neumann operator \( N \) using a description of precompact subsets in \( L^2 \)-spaces (see [10]).

**Theorem 2.1.** Let \( \Omega \subset \subset \mathbb{C}^n \) be a smoothly bounded pseudoconvex domain. The \( \overline{\partial} \)-Neumann operator \( N \) is compact if and only if for each \( \epsilon > 0 \) there exists \( \omega \subset \subset \Omega \) such that
\[ \int_{\Omega \setminus \omega} |u(z)|^2 d\lambda(z) \leq \epsilon (\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2) \]
for each \( u \in \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}^*) \).

Now let
\[ W^1_{0,q}(\Omega) := \{ u \in L^2_{(0,q)}(\Omega) : u \in \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}^*) \} \]
edowed with graph norm. As already mentioned above, this “complex” version of a Sobolev space \( W^1_{0,q}(\Omega) \) is a Hilbert space.

It appears to be interesting to compare the standard Sobolev imbedding
\[ W^1(\Omega) \subset L^r(\Omega), \quad r \in [1, 2n/(n - 1)] \]
where the derivatives are taken with respect of the real variables \( x_j = \Re z_j \) and \( y_j = \Im z_j \) for \( j = 1, \ldots, n \), with the imbedding of the space \( W^1_{0,q}(\Omega) \) endowed with graph norm, into \( L^r_{(0,q)}(\Omega) \). We have the following result

**Theorem 2.2.** If \( \Omega \subset \subset \mathbb{C}^n \) is a smoothly bounded pseudoconvex domain and the inequality
\[ (2.1) \quad \|u\|_r \leq C((\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2)^{1/2} \]
for some \( r > 2 \) and for all \( u \in \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}^*) \) holds, then the \( \overline{\partial} \)-Neumann operator
\[ N : L^2_{(0,q)}(\Omega) \to L^2_{(0,q)}(\Omega) \]
is compact.
Proof. To show this we have to check that the unit ball in $W^1_{\epsilon_0}(\Omega)$ is precompact in $L^2_{(0,q)}(\Omega)$. By Proposition 2.1 we have to show that for each $\epsilon > 0$ there exists $\omega \subset \subset \Omega$ such that
\[
\int_{\Omega \setminus \omega} |u(z)|^2 \, d\lambda(z) < \epsilon^2,
\]
for all $u$ in the unit ball of $W^1_{\epsilon_0}(\Omega)$.

By (2.1) and Hölder’s inequality we have
\[
\left( \int_{\Omega \setminus \omega} |u(z)|^2 \, d\lambda(z) \right)^{\frac{1}{2}} \leq \left( \int_{\Omega \setminus \omega} |u(z)|^r \, d\lambda(z) \right)^{\frac{1}{r}} \cdot |\Omega \setminus \omega|^{\frac{1}{2} - \frac{1}{r}} \leq C |\Omega \setminus \omega|^{\frac{1}{2} - \frac{1}{r}}.
\]

Now we can choose $\omega \subset \subset \Omega$ such that the last term is $< \epsilon$. □

In the following Theorem we suppose that a so-called subelliptic estimate holds. Subelliptic estimates are related to the geometric notion of finite type. We remark that the $\overline{\partial}$-Neumann problem for smoothly bounded strictly pseudoconvex domains is subelliptic with a gain of one derivative for $N$ which is considerably stronger than compactness.

Theorem 2.3. Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ with boundary of class $C^\infty$. Suppose that $0 < \epsilon \leq 1/2$ and that
\[
dom(\overline{\partial}) \cap \dom(\overline{\partial}^*) \subseteq W^\epsilon_{(0,q)}(\Omega),
\]
and that there exists a constant $C > 0$ such that
\[
\|u\|_{\epsilon,\Omega} \leq C(\|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2)^{1/2},
\]
for all $u \in \dom(\overline{\partial}) \cap \dom(\overline{\partial}^*)$, where $W^\epsilon_{(0,q)}(\Omega)$ is the standard $\epsilon$-Sobolev space. Then the $\overline{\partial}$-Neumann operator
\[
N : L^2_{(0,q)}(\Omega) \longrightarrow L^2_{(0,q)}(\Omega)
\]
is compact and $N$ can be continuously extended as an operator
\[
\tilde{N} : L^{\frac{2n}{n+\epsilon}}_{(0,q)}(\Omega) \longrightarrow L^{\frac{2n}{n+\epsilon}}_{(0,q)}(\Omega),
\]
which means that there is a constant $C > 0$ such that
\[
\|\tilde{N} u\|_{\frac{2n}{n+\epsilon}} \leq C \|u\|_{\frac{2n}{n+\epsilon}},
\]
for each $u \in L^{\frac{2n}{n+\epsilon}}_{(0,q)}(\Omega)$.

Proof. We use the continuous imbedding for the space $W^\epsilon(\Omega)$:
\[
W^\epsilon(\Omega) \longrightarrow L^r(\Omega),
\]
for $2 \leq r \leq 2n/(n - \epsilon)$, (see [1], Theorem 7.57). Hence we can choose $r_0 > 2$ to get
\[
dom(\overline{\partial}) \cap \dom(\overline{\partial}^*) \subseteq W^\epsilon_{(0,q)}(\Omega) \subseteq L^{r_0}_{(0,q)}(\Omega),
\]
and we can apply Theorem 2.2.

To show that $N$ extends continuously recall that $N = j \circ j^*$, where
\[
j : \dom(\overline{\partial}) \cap \dom(\overline{\partial}^*) \longrightarrow L^2_{(0,q)}(\Omega),
\]
see [14]. In our case \( j \) is a continuous operator into \( L^{2+\epsilon}_{(0,q)}(\Omega) \), hence

\[
j^* : L^{2+\epsilon}_{(0,q)}(\Omega) \to \text{dom}(\partial) \cap \text{dom}(\partial^*) ,
\]

which proves the assertion. □

St. Krantz [12], R. Beals, P.C. Greiner and N.K. Stanton [2], I.Lieb and R. M. Range [13], and A. Bonami and N. Sibony [3] proved \( L^p \) -estimates and Lipschitz estimates for solution operators of the inhomogeneous \( \partial \)-equation and the \( \overline{\partial} \)-Neumann operator using integral representations for the kernel of these operators, but without relationship to compactness and continuous extendability.

**Remark 2.4.** If \( \Omega \) is a bounded strictly pseudoconvex domain in \( \mathbb{C}^n \) with boundary of class \( \mathcal{C}^\infty \), then (2.2) is satisfied for \( \epsilon = 1/2 \) (see [14], Proposition 3.1).

D’Angelo ([8], [9]) and Catlin ([5], [6], [7]) give a characterization of when a subelliptic estimate holds in terms of the geometric notion of finite type, see also [14].

**Corollary 2.5.** Let \( \Omega \) be a smooth bounded pseudoconvex domain in \( \mathbb{C}^n \), \( n \geq 2 \). Let \( P \in b\Omega \) and assume that there is an \( m \)-dimensional complex manifold \( M \subset b\Omega \) through \( P \) (\( m \geq 1 \)), and \( b\Omega \) is strictly pseudoconvex at \( P \) in the directions transverse to \( M \) (this condition is void when \( n = 2 \)). Then (2.1) is not satisfied for \((0,q)\)-forms with \( 1 \leq q \leq m \).

**Proof.** Theorem 4.21 of [14] gives that the \( \overline{\partial} \)-Neumann operator fails to be compact on \((0,q)\)-forms with \( 1 \leq q \leq m \). Hence we can again apply Proposition 2.2 to get the desired result. □

**Remark 2.6.** If the Levi form of the defining function of \( \Omega \) is known to have at most one degenerate eigenvalue at each point (the eigenvalue zero has multiplicity at most 1), a disk in the boundary is an obstruction to compactness of \( N \) for \((0,1)\)-forms. A special case of this is implicit in [11] for domains fibered over a Reinhardt domain in \( \mathbb{C}^2 \).

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