INSTANTONS AND SINGULARITIES IN THE YANG-MILLS FLOW

ALEX WALDRON

Abstract. Several results on singularities and convergence of the Yang-Mills flow in dimension four are given. We show that a singularity of pure + or − charge cannot form within finite time, in contrast to the analogous situation of harmonic maps between Riemann surfaces. We deduce long-time existence given low initial self-dual energy, and in this case we study convergence of the flow at infinite time. If a global weak Uhlenbeck limit is anti-self-dual and has vanishing self-dual second cohomology, then the limit exists smoothly and exponential convergence holds. We also recover the classical grafting theorem, and derive asymptotic stability of this class of instantons in the appropriate sense.

The Yang-Mills flow

\[
\frac{\partial A}{\partial t} = -D_A^* F_A
\]

evolves a connection \( A \) on a vector or principal bundle by the \( L^2 \) gradient of the Yang-Mills functional

\[
YM(A) = \frac{1}{2} \int |F_A|^2 dV.
\]

Over compact base manifolds of dimension two or three, it was shown by G. Daskalopoulos [4] and Rade [17] that the Yang-Mills flow exists for all time and converges. Finite-time blowup is known to occur in dimension five or higher [16], and explicit examples of Type-I shrinking solitons were produced on \( \mathbb{R}^n, 5 \leq n \leq 9 \), by Weinkove [30]. Hong and Tian [13] showed that the singular set has codimension at least four, and gave a complex-analytic description in the compact Kahler case (where an application of the maximum principle shows that singularities can only form at infinite time, see [22], Ch. 1). In complex dimension two, Donaldson’s early results [8] for the flow on stable holomorphic bundles have recently been generalized by Daskalopoulos and Wentworth ([5], [6]).

The behavior of the Yang-Mills flow on Riemannian manifolds of dimension four, however, has not been understood well. The foundational work of Struwe [23] gives a global weak solution with finitely many point singularities, by analogy with harmonic map flow in dimension two [24]. To date, outside of the Kahler setting, long-time existence and convergence have only been fully established in specific cases, by appealing to energy restrictions on blowup limits [19] or by imposing a symmetric Ansatz [20]. Moreover, finite-time singularities have long been known as a characteristic feature of critical harmonic map flow [2].

This paper provides several theorems concerning long-time existence and smooth convergence of the Yang-Mills flow in dimension four. As with the classical results of Taubes [25] and Donaldson [7], ours will rely on the splitting of two-forms into self-dual and anti-self-dual parts, as well as a number of useful observations in the parabolic setting.
Outline and discussion of results. In Section 1, we briefly review the Yang-Mills formalism and derive the relevant identities, in particular the split Bochner-Weitzenbock formula.

In Section 2 (p. 6), we give a simple yet generic criterion for long-time existence, namely, that either of $F^+$ or $F^-$ does not concentrate in $L^2$. The proof relies on a borderline Moser iteration (Proposition 2.1), together with a manipulation of the local energy inequality with a logarithmic cutoff (Theorem 2.4). We note that this criterion is not sufficient to rule out singularity formation at infinite time. Moreover, the two results hold simultaneously only in dimension four (see Remark 2.6).

We draw several conclusions: first, that a singularity of pure positive or negative charge, hence modeled on an instanton, cannot occur at finite time. This suggests that finite-time singularities are very unlikely to form on low-rank bundles, and should be unstable if they do. Second, if the global self-dual energy is less than $\delta$, a computable constant, then the flow exists for all time and blows up at most exponentially. Third, a proof of long-time existence in the $SO(4)$-equivariant case studied by [20] follows from Theorem 2.4 (see Example 2.7).

We also note that finite-time blowup of equivariant harmonic map flow $S^2 \to S^2$ occurs even with low holomorphic energy [2], hence lacks this additional level of “energy quantization.” In this sense, Theorem 2.4 draws a geometric contrast between the dynamics of the two flows (see [11] for a comparative study of the respective scalar equations). The coupling between $F^+$ and $F^-$ also invites a comparison with Topping’s repulsion estimates [27].

In Section 3 (p. 10), assuming low initial self-dual energy, we give a characterization of infinite-time singularities along classical gauge-theoretic lines. If the self-dual second cohomology $H^{2+}$ of an anti-self-dual Uhlenbeck limit is zero, e. g. if it is irreducible of charge one, then a Poincaré inequality holds on self-dual two-forms. The estimate is inherited by connections along the flow, implying the exponential decay of $||F^+||^2$. This results in smooth convergence, once one is sufficiently close to the limit modulo gauge on an open set (Theorem 3.8). The set of bubbling points is therefore empty and the limit unique, in this case.

We conclude that an anti-self-dual limit must have $H^{2+} \neq 0$, if bubbling occurs at infinite time. Since this need not be the case either for a general weakly convergent sequence of instantons, or a priori within Taubes’s framework [20]. Theorem 3.8 may yield additional information about the topology of the instanton moduli spaces.

In the final section (p. 16), we deduce further properties of the flow at low self-dual energy. We recover the grafting theorem for pointlike instantons [25], which requires a brief new gauge-fixing argument at short time. We also obtain the following (Corollary 4.3).

Assume the bundle $E$ has structure group $SU(2)$ with $c_2(E) = 1$, and the base manifold $M$ is simply-connected with $H^{2+}(M) = 0$. If $||F^+||_{L^2} < \delta_1$ initially, then the flow exists for all time and has a smooth subsequential limit. If the limit is anti-self-dual and irreducible then it is unique, and the flow converges exponentially.

Note that on certain manifolds with $H^{2+}(M) \neq 0$, e. g. $\mathbb{CP}^2$, $SU(2)$-instantons of charge one do not exist, and therefore the flow cannot have a smooth limit. This is also the simplest
demonstration that Atiyah-Bott’s description of Morse theory \[1\] does not generalize naively to dimension four.

In the case that the ground state of a certain physical system is not locally unique, the natural question is that of “asymptotic” stability under small perturbations. This has been studied chiefly in the hyperbolic setting, but also by Gustafson, Nakanishi, and Tsai \[12\] for the Landau-Lifshitz equations, which include harmonic map flow as the purely parabolic case. In the Yang-Mills context we observe Theorem 4.4 which gives an \(H^1\) asymptotic stability result in the parabolic sense for the instantons with \(H^{2+} = 0\).

**Note on dependence of constants.** Several of our estimates will have constants, e.g. \(C_1\) with a particular dependence which we state in the corresponding proposition. The letter \(C\) itself denotes a numerical constant which can be taken to be increasing throughout the paper, although it will be used similarly within individual proofs. The constant \(C_M\) also depends on the geometry of the fixed base manifold \(M\). In Section 3.1 we will also define a Poincaré constant \(C_A\), labeled by the corresponding connection.

1. **Preliminaries**

Let \((M, g)\) be a compact Riemannian manifold of dimension four, \(\pi : E \to M\) a vector bundle with fiber \(\mathbb{R}^n\), fiberwise inner-product \(\langle \cdot , \cdot \rangle\) and smooth reference connection \(A_{ref}\). We recall the relevant definitions (see \[9\], \[10\], \[14\]) and make several basic derivations. The proofs of our main results begin in Section 2 (p. 6).

1.1. **Yang-Mills functional and instantons.** Writing \(|F_A|^2\) for the pointwise norm of the curvature form in the fixed metric \(g\), the Yang-Mills energy is defined as above. We may compute its gradient using the formula

\[
F_{A+a} = F_A + da + A \wedge a + a \wedge A + a \wedge a
\]

in order to obtain

\[
\frac{d}{dt} YM(A + ta) = \frac{1}{2} \frac{d}{dt} \left( \int (|F_A|^2 + 2t \langle F_A, D_A a \rangle) \, dV + O(t^2) \right)
\]

\[
= \int \langle a, D^*_A F_A \rangle dV.
\]

We conclude that a critical point, or Yang-Mills connection, satisfies

\[
D^*_A F_A = 0.
\]

Moreover the Yang-Mills flow is given in local components

\[
\frac{\partial}{\partial t} A^\alpha_{ij} = \nabla^i F^\alpha_{ij \beta}.
\]

By definition, we have the energy inequality

\[
YM(A(0)) - YM(A(T)) = \int_0^T \|D^*_A F_A \|^2 dt
\]
as long as the connection is sufficiently smooth. Therefore, if the flow exists for all time, we expect a weak limit which, if not an absolute minimum of YM, is at least a Yang-Mills connection. Note that we will often abbreviate

$$|| \cdot ||_{L^2(M)} = || \cdot ||.$$

We will write

$$F^\pm = \frac{1}{2} (F \pm *F)$$

for the self-dual and anti-self-dual parts of the curvature form, respectively. In normal coordinates, these satisfy the relations

$$F^\pm_{12} = \pm F^\pm_{34} \quad F^\pm_{13} = \mp F^\pm_{24} \quad F^\pm_{14} = \pm F^\pm_{23}.$$  

From the second Bianchi identity, remark that

$$2D_A^* F^\pm = - * (D * F \pm D *^2 F) = D_A^* F.$$  

Therefore, if a connection is anti-self-dual ($F^+ = 0$) or self-dual ($F^- = 0$), it is a critical point of YM. These special critical points are called instantons.

Recall from Chern-Weil theory that the integer

$$\kappa(E) = \frac{1}{8\pi^2} \int \operatorname{Tr} F_A \wedge F_A$$

is a topological invariant which does not depend on the connection $A$ (for complex bundles, this coincides with the second Chern character). From the definition of the Hodge star operator, we compute

$$\int \operatorname{Tr} F_A \wedge F_A = - \int \langle F^+ + F^-, F^+ - F^- \rangle dV$$

$$= ||F^-||^2 - ||F^+||^2$$

but by orthogonality, also

$$||F||^2 = ||F^+||^2 + ||F^-||^2.$$  

Changing the orientation of $M$ if necessary, we may assume that $\kappa$ is nonnegative. We obtain the formula

$$||F||^2 = 8\pi^2 \kappa + 2||F^+||^2.$$  

Thus a connection is anti-self-dual if and only if it attains the energy $8\pi^2 \kappa$, which then must be the absolute minimum for connections on $E$. 
1.2. **Evolution of curvature and Weitzenbock formulae.** In what follows, we will always assume that the initial connection is smooth (unless otherwise stated). Although the flow is not strictly parabolic, short-time existence of a solution is guaranteed for smooth data by a De Turck-type trick (see [9], Ch. 6). A solution $A(t)$ was constructed by Struwe so as to achieve the following characterization of long-time existence. We say, for a certain $\epsilon_0 > 0$, that the curvature $F(t) = F_A(t)$ concentrates in $L^2$ at $x \in M$ if

$$\inf_{R>0} \limsup_{t \to T} \int_{B_R(x)} |F(t)|^2 dV \geq \epsilon_0.$$  

**Theorem 1.1.** (Struwe [23], Theorem 2.3) *The maximal smooth existence time $T$ of $A(t)$ is characterized by concentration of the curvature $F(t)$ at some $x \in M$ as $t \to T$.*

The primary remaining task is to study concentration of the curvature along the Yang-Mills flow. From (1.1), we compute the evolution

$$\frac{\partial}{\partial t} F_A = D_A(-D_A^* F_A).$$

In view of the second Bianchi identity $D_A F_A = 0$, we may rewrite this as the tensorial heat equation

$$\left(\frac{\partial}{\partial t} + \Delta_A\right) F_A = 0$$

where $\Delta_A = DD^* + D^* D$ is the Hodge Laplacian with respect to the evolving connection.

We compute, for $\omega \in \Omega^k(\mathfrak{g}_E)$

$$(D^* D + DD^*)\omega_{i_1 \ldots i_k} = -\nabla^j (\nabla j \omega_{i_1 \ldots i_k} - \nabla^j \omega_{i_1 \omega_{j i_2 \ldots i_k}} - \cdots - \nabla^j \omega_{i_1 \ldots i_k \ldots 1j})$$

$$- \nabla_i \nabla^j \omega_{i j i_2 \ldots i_k} + \nabla^j \omega_{i i_1 j i_2 \ldots i_k} + \cdots + \nabla_i \nabla^j \omega_{i j i_2 \ldots i_k i_1}.$$  

Permuting $j$ and $i_1$ in the positive terms of the second line, we may group all but the very first term into commutators. We obtain the Weitzenbock formula

$$(D^* D + DD^*)\omega_{i_1 \ldots i_k} = \nabla^* \nabla \omega_{i_1 \ldots i_k} + \nabla \omega_{i_1 \ldots i_k} Rm \# \omega - [F_{i_1}^j, \omega_{j i_1 \ldots i_k}] - \cdots - [F_{i_k}^j, \omega_{i_1 \ldots i_k}]$$

In particular, for a two-form, we have

$$-\Delta_A \omega_{ij} = \nabla^k \nabla_k \omega_{ij} + [F_j^k, \omega_{ki}] - [F_j^k, \omega_{ki}]$$

$$- R_{i j k \ell} \omega_{k \ell j} - R_{i j k \ell} \omega_{k \ell j} + R_{i j k \ell} \omega_{k \ell i} + R_{i j k \ell} \omega_{k \ell i}$$

(1.5)

We now make a simple observation about the zeroth-order terms (see [14], appendix). Assume we are in geodesic coordinates at a point, so (anti)-self-duality is defined as in (1.2). For $\omega \in \Omega^2^+ \epsilon_0 d$ and $\eta \in \Omega^2^- \epsilon_0 d$, we may write

$$\omega_{k \ell} \eta_{k \ell} - \omega_{k \ell} \eta_{k \ell} = \omega_{13} \eta_{32} - \omega_{23} \eta_{31} + \omega_{14} \eta_{42} - \omega_{24} \eta_{41}$$

$$= (-\omega_{24}) (-\eta_{41}) - \omega_{14} \eta_{42} + \omega_{14} \eta_{42} - \omega_{24} \eta_{41}$$

$$= 0$$

and similarly for any choice of indices. A similar calculation shows that for $\omega, \omega'$ self-dual, $\omega_{k \ell} \omega'_{k \ell} - \omega_{k \ell} \omega'_{k \ell}$ is again self-dual. These facts amount to the splitting of Lie algebras

$$so(4) = so(3) \oplus so(3).$$
For the $Rm$ terms, one notes that the first and third are skew in $i, j$, as are the second and fourth, and that these are each self-dual if the same is true of $\omega$ (as explained in [10], appendix). We conclude that the extra terms of the Weitzenbock formula (1.5) in fact split into self-dual and anti-self-dual parts. Note also that $\Delta A^* = * \Delta A$, and the trace Laplacian clearly preserves the identities (1.2) in an orthonormal frame.

We obtain, finally, for $\omega$ self-dual
\begin{equation}
- \Delta A \omega_{ij} = \nabla^k \nabla_k \omega_{ij} + [F_{i+k}^+, \omega_{kj}] - [F_{j+k}^+, \omega_{ki}] + Rm \# \omega
\end{equation}
as well as a similar formula for anti-self-dual forms. Applied to the self-dual curvature $F^+$, this yields the key evolution equation
\begin{equation}
\frac{\partial}{\partial t} F_{ij}^+ = \nabla^k \nabla_k F_{ij}^+ + 2 [F_{i+k}^+, F_{j+k}^+] + Rm \# F^+.
\end{equation}

1.3. Sobolev spaces. Any connection can now be uniquely written $A_{ref} + A$, with $A \in \Omega^1(g_E)$, and any norms applied to a connection will be applied to the global one-form $A$.

We define the Sobolev norms
\[ ||\omega||_{H^k} = \left( \sum_{\ell=0}^{k} ||\nabla_{ref}^\ell \omega||_{L^2}^2 \right)^{\frac{1}{2}} \]
as well as the corresponding spaces of forms and connections over any open set $\Omega \subset M$. A different reference connection defines uniformly equivalent norms. Our proofs will not deal directly with Sobolev spaces of gauge transformations and connections, as we are able to cite the highly developed regularity theory.

For any $\Omega' \subset \subset \Omega$, there is a local Sobolev inequality
\[ ||\omega||_{L^4(\Omega')}^2 < C_{\Omega', \Omega} ||\omega||_{H^1(\Omega)}^2 \]
for the norms defined with respect to $A_{ref}$. The difficulty with Yang-Mills in dimension four and above is that due to the zeroth-order terms of the Weitzenbock formula, the Sobolev constant for $D_A \oplus D_A^*$ blows up as the curvature of $A$ concentrates.

2. (Anti)-self-dual singularities

In order to obtain separate control of the self-dual curvature, we apply the inner product with $F^+$ to (1.7). Letting $u = |F^+|^2$, we obtain the differential inequality
\begin{equation}
\left( \frac{\partial}{\partial t} + \Delta \right) u \leq -2 |\nabla F^+|^2 + A u^{3/2} + Bu
\end{equation}
where $B$ is a multiple of $||Rm||_{L^\infty(M)}$.

**Proposition 2.1.** Let $u(x, t) \geq 0$ be a smooth function satisfying
\[ \left( \frac{\partial}{\partial t} + \Delta \right) u \leq A u^{3/2} + Bu \]
on $M \times [0, T)$, with $M$ compact of dimension four. There exist $R_0 > 0$ (depending on the geometry of $M$) and $\delta > 0$ (depending on $A, B, R_0$) as follows:
Assume $R < R_0$ is such that $\int_{B_R(x_0)} u(x,t)dx < \delta^2$ for all $x_0 \in M, t \in [0,T)$. Then for $t \in [\tau, T)$, we have
\[ ||u(t)||_{L^\infty(M)} < C_{2.1} ||u||_{L^1(M \times [\tau, T))}. \]
The constant depends on $||u(0)||_{L^2}, R,$ and $\tau$. If $u$ is defined for all time, then
\[ \limsup_{t \to \infty} ||u(t)||_{L^\infty(M)} < C_M/R^4. \]

**Proof.** Let $\varphi \in C_0^\infty(B_R(x))$. Multiplying by $\varphi^2 u$ and integrating by parts, we obtain
\[ \frac{1}{2} \frac{d}{dt} \left( \int \varphi^2 u^2 \right) + \int \nabla(\varphi^2 u) \cdot \nabla u \leq A \int \varphi^2 u^{5/2} + B \int \varphi^2 u^2 \]
\[ \frac{1}{2} \frac{d}{dt} \left( \int \varphi^2 u^2 \right) + \int |\nabla(\varphi u)|^2 \leq \int |\nabla \varphi|^2 u^2 + A \int \varphi^2 u^{5/2} + B \int \varphi^2 u^2. \]
Applying the Sobolev and Hölder inequalities on $B_R$,
\[ \frac{1}{2} \frac{d}{dt} \int \varphi^2 u^2 + \frac{1}{C_S} \left( \int (\varphi u)^4 \right)^{1/2} \leq ||\nabla \varphi||_{L^\infty}^2 \int_{B_R} u^2 + B \int \varphi^2 u^2. \]
Assuming $R < R_0$, depending on the geometry of $M$, we have $Vol(B_R(x)) \leq c^2 R^4$ for all $x \in X$ as well as a uniform Sobolev constant $C_S$. We may also choose a cover of $M$ by geodesic balls $B_{R/2}(x_i)$ in such a way that no more than $N$ of the balls $B_i = B_{R}(x_i)$ intersect a fixed ball, with $N$ universal in dimension four. For each $i$, let $\tilde{\varphi}_i$ be a standard cutoff for $B_{R/2}(x_i) \subset B_R(x_i)$ with $||\nabla \tilde{\varphi}_i||_{L^\infty} < 4/R$. Define $\varphi_i = \tilde{\varphi}_i/\sqrt{\sum_j \tilde{\varphi}_j^2}$, so that $\{\varphi_i^2\}$ is a partition of unity with $||\nabla \varphi_i||_{L^\infty} < C/R$.

We now apply the above differential inequality to $\varphi_i$ and sum
\[ \sum_i \left( \frac{1}{2} \frac{d}{dt} \int \varphi_i^2 u^2 + (C_S^{-1} - A\delta) \left( \int (\varphi_i u)^4 \right)^{1/2} \right) \leq \sum_i \left( CR^{-2} \int_{B_i} (\sum_j \varphi_j^2) u^2 + B \int \varphi_i^2 u^2 \right) \]
\[ \leq \left( \frac{CN}{R^2} + B \right) \sum_i \int \varphi_i^2 u^2. \]
Note that for $\theta > 0$, we have by Hölder’s and Young’s inequalities
\[ \int (\varphi_i u)^2 \leq \delta \left( \int_{B_R} (\varphi_i u)^3 \right)^{1/2} \leq \delta \left( \int_{B_R} \left( \theta^3 + \frac{(\varphi_i u)^4}{\theta} \right) \right)^{1/2} \]
\[ \leq \delta \left( CR^4 \theta^3 + \theta^{-1/2} \left( \int (\varphi_i u)^4 \right)^{1/2} \right). \]
Taking $\theta = R^{-4}$, we obtain
\[ \sum_i \left( \frac{1}{2} \frac{d}{dt} \int \varphi_i^2 u^2 + (C_S^{-1} - A\delta) \left( \int (\varphi_i u)^4 \right)^{1/2} \right) \leq \delta \left( \frac{C}{R^2} + B \right) \sum_i \left( \frac{C}{R^4} + R^2 \left( \int (\varphi_i u)^4 \right)^{1/2} \right) \]
and subtracting the last term
\[ \sum_i \left( \frac{d}{dt} \int \varphi_i^2 u^2 + \epsilon \left( \int (\varphi_i u)^4 \right)^{1/2} \right) \leq \frac{C\delta (1 + BR^2)}{R^6} \left( \text{# of balls} \right) , \]
where we now choose $\delta$ so that

$$\epsilon = 2 \left( C_s^{-1} - \delta \left( A + (C + BR_0^2) \right) \right) > 0.$$  

We may finally apply Hölder’s inequality to the left-hand side and absorb the partition of unity

$$\sum_i \left( \frac{d}{dt} \int \varphi_i^2 u^2 + \frac{\epsilon}{cR^2} \int \varphi_i^2 u^2 \right) \leq \frac{C\delta (1 + BR^2)}{R^6} \left( Vol(M) \right) \left( V ol(M) \right)$$

$$\frac{d}{dt} \int u^2 + \frac{\epsilon}{cR^2} \int u^2 \leq \frac{C\delta (1 + BR^2) Vol(M)}{R^{10}}.$$  

Integrating, we obtain the estimate

$$\int u(t)^2 \leq e^{-\epsilon cR^2 t} \int u(0)^2 + \frac{C\delta (1 + BR^2) Vol(M)}{\epsilon R^8} \left( 1 - e^{-\epsilon cR^2 t} \right).$$  

This gives a uniform $L^2$ bound on $u(t)$ for $t > 0$, hence a uniform $L^4$ bound on $Au^{1/2} + B$. Standard Moser iteration (see [13] Lemma 19.1) on cylinders of radius $R_0$ and height $\tau$ then implies the stated $L^\infty$ bounds.  

**Lemma 2.2.** (C. f. [14], 7.2.10) There is a constant $L$ and for any $N > 1, R > 0$ a smooth function $\beta = \beta_{N,R}$ on $\mathbb{R}^4$ with $0 \leq \beta(x) \leq 1$ and

$$\beta(x) = \begin{cases} 1 & |x| \leq R/N \\ 0 & |x| \geq R \end{cases}$$

and

$$||\nabla \beta||_{L^4}, ||\nabla^2 \beta||_{L^2} < \frac{L}{\sqrt{\log N}}.$$  

Assuming $R < R_0$, the same holds for $\beta(x - x_0)$ on any geodesic ball $B_R(x_0) \subset M$.  

**Proof.** We take

$$\beta(x) = \tilde{\phi} \left( \frac{\log N}{R} x \right)$$

where

$$\tilde{\phi}(s) = \begin{cases} 1 & s \leq 0 \\ 0 & s \geq 1 \end{cases}$$

is a standard cutoff function (with respect to the cylindrical coordinate $s$).  

**Remark 2.3.** The construction of Lemma 2.2 is possible in dimension four and above due to the scaling of the $L^4$ norm on 1-forms ($L^2$ norm on 2-tensors), together with the failure of these norms to control the supremum. Proposition 2.1 holds only in dimension less than or equal to four.
**Theorem 2.4.** Let $A(t)$ satisfy the Yang-Mills flow equation on $M \times [0,T)$. For $R < R_0$ and $N > 1$, we have the local bound

$$
\|F(T)\|_{L^2(B_{R/N})}^2 \leq \|F(0)\|_{L^2(B_R)}^2 + \int_0^T \frac{\|F^+(t)\|_{L^\infty(B_R)}}{\sqrt{\log(N)}} \left(C + \|F(t)\|_{L^2(B_R)}^2\right) dt
$$

on concentric geodesic balls in $M$. Therefore if $\|F^+\|_{L^\infty(M)} \in L^1([0,T))$, or in particular if $F^+$ does not concentrate in $L^2$, then the flow extends smoothly past time $T$.

**Proof.** Recall the evolution of the curvature tensor

$$
\frac{\partial}{\partial t} F_A = -DD^* F.
$$

Multiplying by $\varphi^2 F$ and integrating by parts, we obtain

$$
\frac{1}{2} \frac{d}{dt} \|\varphi F\|^2 + \|\varphi D^* F\|^2 = 2(\varphi D \varphi \cdot F, D^* F)
$$

On the right-hand side we switch $D^* F = 2D^* F^+$ \([13]\), and integrate by parts again to obtain

$$
\frac{1}{2} \frac{d}{dt} \|\varphi F\|^2 + \|\varphi D^* F\|^2 = 4 \int_M \left( \langle \nabla_i \varphi \nabla^k \varphi + \varphi \nabla_i \nabla^k \varphi \rangle F_{kj} + \varphi \nabla^k \varphi \nabla_i F_{kj} , (F^+)_{ij} \right) dV
$$

In the inner product with the self-dual 2-form $F^+$, we may replace the term $\varphi \nabla^k \varphi \nabla_i F_{kj}$ via the identity

$$
(\nabla^k \varphi \nabla_i F_{kj} - \nabla_j F_{ki})^+ = (\nabla^k \varphi \langle (-\nabla_j F_{ik} - \nabla_k F_{ji}) - \nabla_j F_{ki} \rangle)^+
$$

$$
= (\nabla^k \varphi \nabla_k F_{ij})^+
$$

$$
= \nabla^k \varphi \nabla_k F_{ij}^+.
$$

We then write

$$
\langle \nabla_k F_{ij}^+ , (F^+)_{ij} \rangle = \frac{1}{2} \nabla_k |F^+|^2
$$

and integrate by parts once more to obtain

$$
\frac{1}{2} \frac{d}{dt} \|\varphi F\|^2 + \|\varphi D^* F\|^2 = 4 \int_M \left( \langle \nabla_i \varphi \nabla_k \varphi + \varphi \nabla_i \nabla_k \varphi \rangle \left( \langle F^+_{kj} , (F^+)_{ij} \rangle + g^{ik} |F^+|^2 \right) + 4 \|F^+\|_{L^\infty(B_r)} \left( \epsilon^{-1} \|F\|^2_{L^2} + \epsilon \left( \|\nabla \varphi\|^4_{L^4} + \|\varphi \nabla^2 \varphi\|^2_{L^2} \right) \right) \right) dV
$$

Choose $\epsilon = 8\sqrt{\log(N)}$ and $\varphi = \beta_{N,r}$ to obtain the desired estimate.

By Theorem \([13]\) (the work of Struwe \([23]\)), to prove the second claim it suffices to show that the full curvature does not concentrate in $L^2$ at time $T$. Note that $\|F(t)\|^2$ is decreasing. Therefore if the curvature on $B_r$ is initially less than $\delta/2$, then for $N$ sufficiently large, the estimate implies that the full curvature on $B_{r/N}$ remains less than $\delta$ until time $T$.

Moreover by Proposition \([27]\) non-concentration of $F^+$ implies a uniform $L^\infty$ bound, and hence the required $L^1(L^\infty)$ bound at finite time. \qed
Corollary 2.5. If the maximal existence time is finite, then both $F^+$ and $F^-$ must concentrate.\(^2\)

Remark 2.6. In view of the Corollary, one can modify the standard rescaling argument \([18]\) at a finite-time singularity to obtain a weak limit which has either nonzero $F^+$ or nonzero $F^-$ (it follows from Proposition 2.1 that this energy cannot be lost in the limit). Thus one cannot have a finite-time singularity for which every weak blowup limit is strictly self-dual, or anti-self-dual. Since any stable Yang-Mills connection on an $SU(2)$ or $SU(3)$-bundle over $S^4$ is either self-dual, anti-self-dual, or reducible, Theorem 2.4 in this case should imply that finite-time singularities are unstable.

Example 2.7. It is straightforward, from the maximum principle applied directly to its evolution, to show that $F^-$ is bounded in the $SO(4)$-symmetric setting of \([20]\). Hence Theorem 2.4 gives a geometric proof of long-time existence in that case. This clarifies the contrast with harmonic map flow \([2]\) (attributed by \([11]\) to a coefficient in the equivariant ansatz corresponding to the “rotation number”). For, finite-time blowup of equivariant maps readily occurs even with low holomorphic energy (see \([27]\) for definitions).

Corollary 2.8. For $\delta$ as in Proposition 2.1, if an initial $H^1$ connection has self-dual curvature $\|F^+_{A(0)}\|_{L^2(M)} < \delta$ then the Yang-Mills flow exists for all time and blows up at most exponentially, with asymptotic rate bounded uniformly for $M$. On any $SU(2)$-bundle, there exists a nonempty $H^1$-open set of initial connections for which the Yang-Mills flow exists for all time, and converges exponentially if $H^{2+}(M) = 0$.

Proof. The connection is smooth after a short time, modulo gauge. Proposition 2.1 then implies a uniform bound on $F^+$ for all future time, and long-time existence follows from Theorem 2.4.

Following Freed and Uhlenbeck \([10]\), for any $\delta_1$ one can construct smooth pointlike $SU(2)$-connections with $\|F^+\|_{L^2} < \delta_1$ and $\|F^+\|_{L^\infty} < C$ (p. 124). Provided $H^{2+}(M) = 0$, Theorem 4.1 (below) yields convergence at infinite time, which holds in an $H^1$-open neighborhood of the resulting instantons (Theorem 4.4). \(\square\)

3. CONVERGENCE AT INFINITE TIME

In this section we assume that all connections have globally small self-dual energy

$$\|F^+_A\|_{L^2(M)} < \delta.$$  

By (1.4), this condition is preserved by the flow, which exists for all time by Corollary 2.8. It is also attained for a nonempty set of connections on bundles with $c_2(E) \geq 0$ and structure group $SU(2)$, and in this case should represent the generic end-behavior of the flow.

We first recall and adapt several standard pieces of Yang-Mills theory. For an open set $\Omega \subset M$, we will write

$$\Omega_r = \{ x \in \Omega \mid d(x, \Omega^c) > r \} \subset \subset \Omega.$$  

\(^2\)Since the singularities are isolated, $F^\pm$ clearly must concentrate at the same point. This is easily shown by adding a boundary term to Proposion 2.1.
Lemma 3.1. There exists $\epsilon_0 > 0$ as follows. For $R < R_0$, if the energy $||F||^2_{L^2(B_R)} < \epsilon_0$ for $t \in [-R^2, 0]$, then for $k \geq 0$

$$||\nabla^k_A F||_{L^\infty(B_{R_k} \times [-R^2, 0])} < \frac{C_k}{R^{2+k}}$$

where $R_k = R/2^k$.

Proof. See [3], [13] for standard proofs of the $k = 0$ estimate via monotonicity formulae. For $k \geq 1$, this is the result of the Bernstein-Hamilton-type derivative estimates of [30]. □

Lemma 3.2. Assume $||F^+(t)||_{L^\infty(\Omega)} < K^+$ for $0 \leq t \leq \tau$. Let $\epsilon_0$ be as above, and assume that for some $r_0 < R_0$ there holds

$$||F(t)||^2_{L^2(B_{r_0}(x))} < \frac{\epsilon_0}{3}$$

for all $x \in \Omega_{r_0}$, with $0 < r_0^2 < \tau$. If

$$||F(0)||^2_{L^2(M)} - ||F(t)||^2_{L^2(M)} \leq \epsilon_0/3$$

then we have

$$||F(\tau)||_{L^\infty(\Omega_{r_0})} < \frac{C_{3.2}}{r_0^2}.$$

The constant depends on $K^+$ and $||F(0)||^2$.

Proof. Assume the contrary. Then by $\epsilon_0$-regularity (Lemma 3.1), for any $N$, at some time $0 < t < \tau$ and $x \in \Omega_r$ we have

$$||F(t)||^2_{L^2(B_{r_0/N}(x))} \geq \epsilon_0.$$

Letting $\varphi$ be the cutoff of Lemma 2.2 for $B_{r_0/N} \subset B_{r_0}$, we apply the proof of Theorem 2.4 using $\overline{\varphi} = 1 - \varphi$. This gives

$$||F(\tau)||^2_{L^2(M \setminus B_{r_0})} - ||F(t)||^2_{L^2(M \setminus B_{r_0/N})} < \frac{\epsilon_0}{3}$$

for $N$ large enough based on $||F||^2$ and $K^+$. But we also have

$$||F(t)||^2_{L^2(B_{r_0/N})} - ||F(\tau)||^2_{L^2(B_{r_0})} > 2\epsilon_0/3$$

by assumption. Subtracting, we obtain

$$||F(t)||^2 - ||F(\tau)||^2 > \epsilon_0/3$$

which is a contradiction. □

Definition 3.3. For a sequence $t_j \to \infty$, we say that $(A_\infty, E_\infty)$ is an Uhlenbeck limit for the flow if the following holds. There exists a subsequence of times $t_{j_k}$ and smooth bundle isometries $u_k : E \to E_\infty$ defined on an exhaustion of open sets

$$U_1 \subset \cdots \subset U_k \subset \cdots \subset M_0 = M \setminus \{x_1, \ldots, x_n\}$$

such that on any open set $\Omega \subset M_0$, we have $u_k^*(A_{t_k}) \to A_\infty$ smoothly.

\[\text{In fact the } k = 0 \text{ bound also follows from the derivative estimate, for smooth solutions in dimension four.}\]
Theorem 3.4. Assuming $|F^+| < \delta$, any sequence $t_j \to \infty$ necessarily contains an Uhlenbeck limit which is a Yang-Mills connection on $E_\infty$.

Proof. This is a standard improvement of the detailed arguments found in [18], by analogy with the Kahler case (see [9], Ch. 6).

The existence of weak $H^1$ limits on a countable family of balls in $M_0$ is the result of compactness theory for connections with bounded $L^2$ curvature ([21], [28]). By Lemma 3.2 we in fact have $L^\infty$ bounds on the curvature of $A(t_{j_k})$ and all its derivatives on each ball, for $k$ large enough. By [9], Lemma 2.3.11, upon taking further subsequences, the weak limit can be taken to be a smooth limit over each ball.

By the argument of [9], §4.4.2, these gauge transformations can be patched together over the open sets $U_i$. The fact that the limiting connection is Yang-Mills away from the bubbling points, and therefore extends to a smooth Yang-Mills connection on $E_\infty$, follows from the energy inequality, [29], and the next estimate. \hfill \qedsymbol

Lemma 3.5. Assume $|F(t)||_{L^\infty(B_R(x_0))} < K$ on $[0, T)$. Then for $R < R_0$ we have the estimate

$$
||D^*F(t)||^2_{L^\infty(B_{R/2})} \leq C \|D^*F\|^2_{L^2(B_R \times [t - \tau, t])}
$$

for $t \geq \tau > 0$. The constant depends on $K, R,$ and $\tau$.

Proof. One computes the evolution

$$
\left( \frac{\partial}{\partial t} + \nabla^* \nabla \right) D^*F_i = 2 \{F_{ij}, D^*F_j\} + Rm \# D^*F_j.
$$

The estimate then follows again from Moser iteration ([15], 19.1). \hfill \qedsymbol

3.1. Sobolev and Poincaré inequality. Assuming $|F^+_A| < \delta$, Hölder’s inequality applied to the Weitzenbock formula ([16]) implies the Sobolev inequality

$$
||\omega||^2_{L^4} + ||\nabla_A \omega||^2_{L^2} \leq C_M (||D_A \omega||^2_{L^2} + ||D^*_A \omega||^2_{L^2} + ||\omega||^2_{L^2}).
$$

for any $\omega \in \Omega^2_+ (\text{End}E)$. Recall the basic instanton complex

$$
0 \to g_E \xrightarrow{D_A} \Omega^1(g_E) \xrightarrow{\pi_1D_A^*} \Omega^2_+(g_E).
$$

Under the assumption $H^2_A = 0$, there are no $L^2$ self-dual two-forms $\omega$ with $D^*_A \omega = D_A \omega = 0$ in the distributional sense. Therefore, by the standard compactness argument, we have

$$
||\omega||^2_{L^4} \leq C_A (||D_A \omega||^2_{L^2} + ||D^*_A \omega||^2_{L^2}).
$$

Hence this term can be dropped from the RHS of (3.1), and we obtain

$$
||\omega||^2_{L^4} + ||\omega||^2_{L^2} + ||\nabla_A \omega||^2_{L^2} \leq C_A (||D_A \omega||^2_{L^2} + ||D^*_A \omega||^2_{L^2})
$$

for $\omega \in \Omega^2_+(\text{End}E)$. We will always take $C_A \geq C_M$.\footnote{Note that Theorem 1.3(ii) of Schlatter does not include any patching, because this may not be possible with $H^{2,2}$ gauge transformations.}
Lemma 3.6. Let $A_0$ be a connection on a bundle $E_0$ over $M$ which satisfies the Poincaré inequality
\begin{equation}
||\omega||^2_{L^4} + ||\omega||^2_{L^2} < C_{A_0} (||D_A\omega||^2_{L^2} + ||D^*_A\omega||^2_{L^2}).
\end{equation}
Assume $A$ is a connection on $E$ for which there exists a smooth bundle isometry $u : E_0 \to E$ defined over $M_r = M - \overline{B}_r(x_1) \cup \cdots \cup \overline{B}_r(x_n)$ with $||u^*(A) - A_0||_{L^4} \leq \epsilon$. Then if $r, \epsilon$ are sufficiently small, $A$ satisfies \((3.3)\) with constant $4C_{A_0}$.

Proof. Assume first that Supp$(\omega) \subset M_r$. Write $\tilde{A} = u^*(A), \tilde{\omega} = u^*(\omega), a = A_0 - \tilde{A}$. We then have
\begin{equation*}
||D_A\omega||^2_{L^2} + ||D^*_A\omega||^2_{L^2} = ||D\tilde{A}\tilde{\omega}||^2_{L^2} + ||D^*\tilde{A}\tilde{\omega}||^2_{L^2}.
\end{equation*}

On the other hand, if Supp$(\omega) \subset B_r(x_1) \cup \cdots \cup B_r(x_n)$, then $||\omega||^2_{L^2} \leq cnr^2 ||\omega||^2_{L^4}$.

Now let $\varphi = \sum \beta_{N,r}(x - x_i)$ be a sum of the logarithmic cutoffs of Lemma 2.2, $\varphi = 1 - \varphi$. Choose $\epsilon, r, N$ such that
\begin{equation*}
4\epsilon^2 + cnr^2 + 2L^2/\log(N) < (4C_{A_0})^{-1}.
\end{equation*}

Then
\begin{equation*}
||\omega||^2_{L^4} + ||\omega||^2_{L^2} \leq 2 \left(||\varphi\omega||^2_{L^4} + ||\varphi\omega||^2_{L^2} + ||v\omega||^2_{L^4} + ||v\omega||^2_{L^2}\right)
\end{equation*}
\begin{equation*}
\leq 2C_M \left(||D_A(\varphi\omega)||^2 + ||D^*_A(\varphi\omega)||^2 + ||\varphi\omega||^2\right)
+ 2C_{A_0} \left(||D_{A_0}(v\omega)||^2 + ||D^*_{A_0}(v\omega)||^2\right)
\leq 2C_{A_0} \left(||\varphi D_A\omega||^2 + ||\varphi D^*_A\omega||^2 + ||v D_A\omega||^2 + ||v D^*_A\omega||^2\right)
+ 2||D\varphi\#\omega||^2 + 4||a||^2_{L^4}|\omega||^2_{L^4} + cnr^2 \left(||\omega||^2_{L^4}\right),
\end{equation*}
which upon rearranging yields the claim (we replace $r/N$ by $r$ in the statement). \qed

3.2. Convergence. We now proceed to the proofs of our main convergence results.

Proposition 3.7. Assume $||F(0)||^2 - ||F(T)||^2 < 1$ and $||F(t)||_{L^\infty(\Omega)} < K$ for $t \in [0, T)$. Then we have the $L^\infty$ bound
\begin{equation*}
||A(T) - A(\tau)||_{L^\infty(\Omega)} < C_{3.7}^0 \left(||F(0)||^2 - ||F(T)||^2\right) (T - \tau).
\end{equation*}
We also have the Sobolev bounds
\begin{equation*}
||A(T) - A(\tau)||^2_{H^k(\Omega)} < C_{3.7}^k \left(||F(0)||^2 - ||F(T)||^2\right) (T - \tau) (1 + ||A||^2_{L^\infty}).
\end{equation*}
The constants depend on $K, r, \tau$, and $\Omega \subset M$.

Proof. Let $\epsilon = ||F(0)||^2 - ||F(T)||^2$. In this proof the constant $C$ depends on $K, \tau, r$ and $\Omega$, but not $\epsilon$ or $T$.

From the energy inequality we have $||D^*F||^2_{L^2(M \times [0, T])} = \epsilon$, and
\begin{equation*}
||D^*F(t)||^2_{L^\infty(\Omega)} < C_{3.5} \epsilon \text{ for } t > \tau.
\end{equation*}
Now let \( \varphi \) be a cutoff for \( \Omega_{2r} \subset \Omega_r \). Multiplying the evolution of the full curvature by \( \varphi \) and squaring, we obtain

\[
||\varphi \frac{\partial}{\partial t} F||^2 + ||\varphi DD^* F||^2 = -2(DD^* F, \varphi^2 \frac{\partial}{\partial t} F)
\]

\[
= -2 \left( D^* F, -2\varphi D\varphi \cdot \frac{\partial}{\partial t} F + \varphi^2 D^* \left( \frac{\partial}{\partial t} F \right) \right)
\]

\[
\leq \frac{1}{2} ||\varphi \frac{\partial}{\partial t} F||^2 + 8 ||D^* F||_{L^4(\Omega_r)}^2 ||D\varphi||_{L^4}^2
\]

\[
+ 2 \left( \varphi^2 D^* F, D^* F \right)
\]

\[
\leq \frac{1}{2} ||\varphi \frac{\partial}{\partial t} F||^2 - \frac{d}{dt} ||\varphi D^* F||^2 + C \left( K + ||D^* F||_{L^\infty(\Omega_r)}^2 \right) ||D^* F||^2.
\]

Rearranging and integrating in time yields

\[
||\frac{\partial}{\partial t} F||_{L^2(\Omega_{2r} \times [\tau, T])}^2 + 2 ||DD^* F||_{L^2(\Omega_{2r} \times [\tau, T])}^2 + 2 ||D^* F(T)||_{L^2(\Omega_{2r})}^2
\]

\[
\leq 2 ||D^* F(\tau)||_{L^2(\Omega_r)}^2 + C ||D^* F||_{L^2(M \times [\tau, T])}^2
\]

\[
< C\epsilon.
\]

Also remark that from Lemma \ref{lemma3.5} applied on cylinders \( B_r \times [t - \tau, t] \), we have

\[
\int_\tau^T ||D^* F(t)||_{L^\infty(\Omega_r)}^2 dt < C\epsilon.
\]

Therefore

\[
||A(T) - A(\tau)||_{L^\infty(\Omega_{2r})} \leq C \int_\tau^T ||D^* F||_{L^\infty} dt \leq C\epsilon^{1/2} (T - t)^{1/2}
\]

as claimed.

For the \( H^1 \) norm, we write \( F_A = F_{A,ref} + D_{ref} A + A \wedge A \), and

\[
\frac{\partial}{\partial t} D_{ref} A = \frac{\partial}{\partial t} F_A + D^* F \# A
\]

which implies

\[
||D_{ref} A(T) - D_{ref} A(\tau)||_{L^2} \leq \int_\tau^T \left( ||\frac{\partial F}{\partial t}||_{L^2(\Omega_{2r})} + 2 ||A||_{L^\infty} ||D^* F||_{L^2} \right)
\]

\[
\leq C\epsilon^{1/2} (T - \tau)^{1/2} (1 + ||A||_{L^\infty})
\]

Finally, notice that

\[
\frac{\partial}{\partial t} \left( D_{ref}^* A \right) = D_{ref}^* D^* F = D^* D^* F + A \# D^* F = A \# D^* F
\]

which implies a similar bound. This suffices to control the \( H^1 \) distance, and higher Sobolev norms are controlled similarly. \( \square \)
Theorem 3.8. Assume $||F^+(0)|| < \delta$, and there exists an Uhlenbeck limit $A_\infty$ on $(M, E_\infty)$ which is an instanton with $H_{A_\infty}^2 = 0$. Then $E = E_\infty$, and the flow converges smoothly to a connection which is gauge-equivalent to $A_\infty$.

More precisely, if $A_\infty$ is a connection satisfying (3.3), then for any $\tau_1 \geq \tau_0 > 0$ there exist $\delta_1, \epsilon_1$, and $r_1 > 0$ as follows. If for some $\tau \geq \tau_1$, $||F^+(\tau - \tau_0)|| < \delta_1$ and $A(\tau)$ is within $\epsilon_1$ of $A_\infty$ in $H^1(M_{r_1})$ modulo gauge, then for $t \geq \tau$ the flow converges exponentially (in the sense below). The constants $\delta_1, \epsilon_1$ also depend on $||F^+(0)||_{L^1}$, but are independent of it for $\tau_1$ sufficiently large.

Proof. Let $M_0 = M \setminus \{x_1, \ldots, x_n\}$ be as in Definition 3.3. Let $r_1 = r/3$ (where $r$ is as in Proposition 3.6), and choose $r_0 < \min(r_1, R_0, \sqrt{\tau_0})$ such that for every $x \in M_{2r_1}$, we have

$$||F_{A_\infty}||_{L^2(B_r(x))}^2 < \epsilon_0/3.$$  

Now, let $\tau \geq \tau_1$ be such that

$$||F^+(\tau - \tau_0)||^2 < \delta_1^2$$

and there exists a smooth isometry $u$ such that

$$(3.4) \quad ||u^*(A(\tau)) - A_\infty||_{H^1(M_{r_1})} < \epsilon_1.$$  

By the local Sobolev inequality, we have

$$||u^*(A(\tau)) - A_\infty||_{L^4(M_{2r_1})} \leq C\epsilon_1.$$  

Choosing $\epsilon_1$ such that $C\epsilon_1 < \epsilon/2$ (where $\epsilon$ is as in Proposition 3.9), the Poincaré inequality holds for $A(t)$ with constant $C_{\infty} = C_{A_\infty}$ on some maximal interval $[\tau, T)$. We will argue that if $\delta_1 > 0$ is small enough, then $T = \infty$ and the flow converges.

Note that $|DF^+| = | - *D * F^+ | = |D^*F^+|$. Applied to the global energy inequality for $F^+$, the Poincaré inequality

$$||F^+||^2 \leq C_{\infty}||D^*F^+||^2$$

yields

$$\partial_t||F^+||^2 + C_{\infty}^{-1}||F^+||^2 \leq \partial_t||F^+||^2 + ||D^*F^+||^2 = 0.$$  

This implies the exponential decay for $t \geq \tau$

$$(3.5) \quad ||D^*F||_{L^2(M \times [\tau, T])}^2 \leq ||F^+(t)||^2 \leq \delta^2_1 e^{-(t-\tau)/C_{\infty}}.$$  

By Proposition 2.11 we have the global $L^\infty$ bound

$$(3.6) \quad ||F^+(t)||_{L^\infty(M)} \leq K^+(t)^2 := C_{2.11}^+ \epsilon^2_1 e^{-(t-\tau)/C_{\infty}}$$

for $t \geq \tau$. Therefore, if $\delta_1$ is sufficiently small we have

$$(3.7) \quad \left(C + ||F(t)||^2\right) \int^T_\tau K^+(t)dt < \epsilon_0/3.$$  

By Theorem 2.4, the full curvature cannot concentrate on $M_{2r_1}$ before time $T$, and we have a uniform bound

$$(3.8) \quad ||F(t)||_{L^\infty(M_{2r_1})} < K$$

applied with respect to a smooth reference connection for $E_\infty.$
for \( \tau + r_0^2 < t < T \).

In order to apply Proposition 3.7 we need this curvature bound on \( M_{2r_1} \) also from time \( \tau - r_0^2 \). Note that

\[
\delta_1^2 > \||F^+(\tau - r_0^2)||^2 \geq \frac{1}{2} \left( ||F(\tau - r_0^2)||^2 - ||F(T)||^2 \right).
\]

By Lemma 3.2 provided \( \delta_1^2 < \epsilon_0/6 \), we in fact have a larger uniform bound (3.8) on \( M_{2r_1} \) for \( \tau - r_0^2 < t \leq \tau + r_0^2 \).

With this curvature bound, we may now apply Proposition 3.7 and (3.5) at each time \( \tau + \delta \), to conclude

\[
||A(\tau + \delta) - A(\tau)||_{L^4(M)_{2r_1}} \leq C \sum_i K^+(\tau + \delta) \leq C K^+(\tau) = C \delta_1.
\]

By the triangle inequality and geometric series, we have

\[
||A(T) - A(\tau)||_{L^4(M)_{2r_1}} \leq C \sum_i K^+(\tau + \delta) \leq C K^+(\tau) = C \delta_1.
\]

If \( \delta_1 \) is small enough that \( C \delta_1 < \epsilon/2 \), we conclude

\[
||u^*(A(T)) - A_\infty||_{L^4(M)_{2r_1}} \leq ||u^*(A(T)) - u^*(A(\tau))||_{L^4(M)_{2r_1}} + ||u^*(A(\tau)) - A_\infty||_{L^4(M)_{2r_1}} \\
\leq C \delta_1 + C \epsilon_1 < \epsilon.
\]

Therefore \( T = \infty \), and the above estimates continue as \( t \to \infty \).

Note that Theorem 2.3 and (3.7) imply that the curvature does not concentrate anywhere on \( M \) as \( t \to \infty \). Therefore the flow converges globally and strongly in \( H^1 \) (and by Proposition 3.7 and (3.5) applied on \( M \), at least exponentially). This proves the second statement.

In the case that \( F^+_{A_\infty} = 0 \), by taking \( r_1 \) and \( \epsilon_1 \) smaller in the second statement, we can clearly satisfy the assumption \( ||F^+(\tau - \tau_0)|| < \delta \). Hence the second statement implies the first.

\[\square\]

4. Further results

**Theorem 4.1.** (Taubes’s grafting theorem, parabolic version.) Let \((E_0, A_0)\) be a flat bundle on \( M \) with \( H^2_{A_0} = 0 \). For any \( K^+ \), and points \( x_1, \ldots, x_n \in M \), there exist \( \delta_1, \epsilon_1, r_1 > 0 \) such that if \( A \) is a connection on \( E \) with \( ||F^+_{A_0}|| < \delta_1 \), \( ||F^+_A||_{L^\infty(M)} < K^+ \), and

\[
||A - A_0||_{H^1(M_{r_1})} < \epsilon_1
\]

then the flow with initial data \( A(0) = A \) converges and remains \( L^4 \)-close to \( A_0 \) modulo gauge on \( M_{r_1} = M \setminus \{x_1, \ldots, x_n\} \).

**Proof.** By assumption, a Poincaré estimate (3.3) holds, and we choose \( \epsilon_1 \leq \epsilon/2 \), \( r_1 = r/2 \) according to Lemma 3.6.

By (4.1), we have \( ||F(0)||_{L^2} < C \epsilon_1 \). Applying the maximum principle to the evolution of \( ||F^+||^2 \), equation (2.1), we have \( ||F^+(t)||_{L^\infty(M)} < 2 K^+ \) for \( 0 \leq t < \tau < 1 \). Therefore, taking \( \delta_1 \) sufficiently small, Proposition 2.1 and Theorem 2.3 imply

\[
||F(t)||_{L^2(M_{r_1})} < 2C \epsilon_1
\]
for $0 \leq t \leq \tau$. Assume first that $M$ is simply-connected, so we may take $A_0 = 0$. According to [9], Prop. 4.4.10, there exists a gauge transformation $u$ on $M_{2r_1}$ (also simply-connected) with

$$ ||u^*A(\tau)||_{L^4(M_{2r_1})} < C\epsilon_1 $$

for $\delta_1$ sufficiently small. The claim now follows from the precise statement of Theorem 3.8.

If $M$ is not simply-connected, we argue as follows. Let $\pi : \tilde{M} \to M$ be the universal cover, and choose a simply-connected domain $\Omega \subset \tilde{M}$ covering $M_{2r_1}$, which is a finite union of preimages of $B_i \subset M_{r_1}$, with $B_i \cap B_j$ connected. Now choose a gauge $u$ on $\Omega$ such that $\tilde{A} = \pi^*A(\tau)$ has

$$ ||u^*\tilde{A}||_{L^4(\Omega)} < C\epsilon_1. $$

If this is done using Coulomb gauges on the $B_i$, then $u^{-1}du$ is well-defined on $M$.

Note that we also have

$$ ||A(\tau) - A_0||_{L^2(M_{r_1})} \leq ||A(\tau) - A(0)||_{L^2(M_{r_1})} + C||A(0) - A_0||_{L^4(M_{r_1})} $$

$$ \leq \tau^{1/2} \left( \int ||D^*F||_{L^2(M_{r_1})}^2 dt \right)^{1/2} + C\epsilon_1 $$

$$ \leq \delta_1 + C\epsilon_1. $$

Hence over $\Omega$, we have

$$ ||du|| = ||u^{-1}du|| \leq ||u^*\tilde{A}|| + ||u\tilde{A}u^{-1}|| < C\epsilon_1. $$

By the Poincaré inequality, in each ball

$$ ||u - \bar{u}||_{L^2(B_i)} < C\epsilon_1. $$

We may therefore choose points $p_i \in B_i$ such that $d(p_i, p_j) \geq c > 0$ and

$$ |u(\bar{p}_i) - u(\bar{p}_j)| < C\epsilon_1 $$

for each $\bar{p}_i, \bar{p}_j \in \Omega$ such that $\pi(\bar{p}_i) = p_i$ and $\pi(\bar{p}_j) = p_j$.

It is clearly possible to construct a frame $v$ over $\Omega$ such that $v(\bar{p}_i) = u(\bar{p}_i) \forall i$, $||dv||_{L^\infty} < C\epsilon_1$, (depending on $\Omega$) and $v^{-1}dv$ is well-defined on $M_{r_1}$. The frame $w = v^{-1}u$ then satisfies $w(\bar{p}_i) = 1$ for all $\bar{p}_i$, and descends to a frame on $E$ over $M_{r_1}$. Note that

$$ ||w^*\tilde{A}||_{L^4(\Omega)} \leq ||v^{-1}dv + v^{-1}(u^*\tilde{A})v||_{L^4(\Omega)} \leq 2C\epsilon_1 $$

and so downstairs

$$ ||w^*A(\tau) - A_0||_{L^4(M_{r_1})} \leq C\epsilon_1. $$

Convergence follows for $\epsilon_1$ and $\delta_1$ sufficiently small as before. ~\hfill\Box

**Remark 4.2.** A similar argument can be used to recover the gluing theorem for connected sums with long necks of small volume, i.e. [9], Theorem 7.2.24.

---

6 Here the $\delta_1$-dependent bounds of Proposition 3.7 take the place of the anti-self-dual condition used to obtain the curvature-dependent bounds of [9] 2.3.11.

7 This can be done for instance by lifting the geodesic balls $B_i$ to $\tilde{M}$ using a set of based paths which form a spanning tree for their incidence graph.
Corollary 4.3. Assume \( \pi_1(M) \) is abelian or has no nontrivial representations in \( SU(2) \), and \( H^{2+}(M) = 0 \). For any initial connection on the bundle \( E \) with structure group \( SU(2) \) and \( c_2(E) = 1 \), assuming \( ||F^+(0)|| < \delta_1 \), no bubbling occurs and the flow has a smooth subsequential limit as \( t \to \infty \). If this limit is an irreducible instanton, then it is unique and the flow converges exponentially.

\[ \text{Proof.} \] Assume by way of contradiction that bubbling occurs as \( t \to \infty \). The blowup limits of \([18]\) at a presumed singularity, as well as the Uhlenbeck limit, preserve the structure group. Due to the \( L^\infty \) bound on \( F^+ \), the blowup limit at a bubble must be anti-self-dual, and therefore contains all but \( 2\delta_1 \) of the energy. If the Uhlenbeck limit obtained from Theorem 3.4 on the same sequence of times is also anti-self-dual, it must be flat by integralty of \( \kappa \). By the assumptions on \( \pi_1(M) \), it acts trivially on the adjoint bundle. But then its cohomology is exactly \( H^{2+}(M) = 0 \), and by the Theorem the flow converges, which is a contradiction. If the Uhlenbeck limit is not anti-self-dual, it nonetheless must be \( L^4 \)-close to a flat connection by the argument of the previous Theorem, which is still a contradiction.

Therefore a smooth Uhlenbeck limit exists. If it is irreducible then \( H^{2+} = 0 \), and again by Theorem 3.8 we have exponential convergence. \( \Box \)

Theorem 4.4. The instantons with \( H^{2+} = 0 \) are asymptotically stable in the \( H^1 \) topology. In other words, given an \( H^1 \) neighborhood \( U \) of \( A \), there exists a neighborhood \( U' \subset U \) of initial connections for which the limit under the flow will again be an instanton with \( H^{2+} = 0 \), lying in \( U \) modulo smooth gauge transformations.

Moreover, there exists an \( H^1 \)-open neighborhood \( N \) for which the flow gives a deformation retraction from \( N \cap H^k \), \( k >> 1 \), onto the moduli space of instantons with \( H^{2+} = 0 \).

\[ \text{Proof.} \] By Struwe’s construction \([23]\), §4.2-4.3, choosing the instanton \( A \) itself as reference connection, the gauge-equivalent flow remains in \( U \) for a time \( \tau \), long enough for \( \epsilon \)-regularity to take effect. This gives a uniform bound on the curvature at time \( \tau \), including on \( ||F^+||_{L^\infty} \). Choosing \( U' \) small enough, we also obtain \( ||F^+|| < \delta_1 \). We are then in the situation of Theorem 3.8 which can be applied with \( \{x_i\} = \emptyset \).

The latter refinement follows from standard parabolic theory. For, two connections in \( N \) which are initially \( H^k \)-close remain so under the gauge-equivalent flow for a large time \( T \); but then both are close to their respective limits under the Yang-Mills flow. \( \Box \)

Acknowledgements. These results will form part of the author’s forthcoming PhD thesis at Columbia University, and he thanks his advisor, Panagiota Daskalopoulos, for vital direction and support. Thanks also to Richard Hamilton for noticing a simplification, to Michael Struwe for an encouraging discussion, and to D. H. Phong for initially suggesting the problem.

References

[1] M. F. Atiyah and R. Bott. The Yang-Mills equations over Riemann surfaces. Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences (1983), 523-615.

[2] K. C. Chang, W. Y. Ding, and R. Ye. Finite-time blowup of harmonic maps from surfaces. J. Diff. Geom. 36 (1992), 507-515.
3. Y. Chen. and C.-L. Shen. Monotonicity formula and small action regularity for Yang-Mills flows in higher dimensions. Calc. Var. 2 (1994), 389-403.
4. G. Daskalopoulos. The topology of the space of stable bundles on a compact Riemann surface. J. Differential Geom. 36 (1992), no. 3, 699-746.
5. G. Daskalopoulos and R. Wentworth. Convergence properties of the Yang-Mills flow on Kahler surfaces. J. Reine Angew. Math. 575 (2004), 69-99.
6. —. On the blow-up set of the Yang-Mills flow on Kahler surfaces. Math. Z. 256 (2007), no. 2, 301-310.
7. S. K. Donaldson. An application of gauge theory to four dimensional topology. J. Differential Geom. 18 (1983), 279-315.
8. —. Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles. Proceedings of the London Mathematical Society 50.1 (1985), 1-26.
9. S. K. Donaldson and P. B. Kronheimer. The Geometry of Four-Manifolds. Oxford University Press (1990).
10. D. Freed and K. Uhlenbeck. Instantons and Four-Manifolds. MSRI Research Publications I (1984).
11. J. Grotowski and J. Shatah. Geometric evolution equations in critical dimensions. Calc. Var. 30 (2007), 499-512.
12. S. Gustafson, K. Nakanishi, and T. P. Tsai. Asymptotic stability, concentration, and oscillation in harmonic map flow, Landau-Lifshitz, and Schrodinger maps on $\mathbb{R}^2$. Comm. Math. Phys. 300 (2010), no. 1, 205-242.
13. M. C. Hong and G. Tian. Asymptotic behavior of the Yang-Mills flow and singular Yang-Mills connections. Math. Ann. 330 (2004), 441-472.
14. H. B. Lawson. The Theory of Gauge Fields in Four Dimensions. Regional Conference Series in Mathematics, Number 58, AMS (1985).
15. P. Li. Geometric Analysis. Cambridge University Press (2012).
16. H. Naito. Finite time blowing-up for Yang-Mills gradient flow in higher dimensions. Hokkaido Math. J. 23 (1994), no. 3, 451-464.
17. J. Rade. On the Yang-Mills heat equation in two and three dimensions. J. Reine Angew. Math. 120 (1998), 117-128.
18. A. E. Schlatter. Long-time behavior of the Yang-Mills flow in four dimensions. Ann. Global Anal. Geom. 15 (1997), no. 1, 1-25.
19. —. Global existence of the Yang-Mills flow in four dimensions. J. Reine Angew. Math. 479 (1996), 133-148.
20. A. E. Schlatter, M. Struwe and A. S. Tahvildar-Zadeh. Global existence of the equivariant Yang-Mills heat flow in four space dimensions. Am. J. Math. 120 (1998), 117-128.
21. S. Sedlacek. The Yang-Mills functional over four-manifolds. Comm. Math. Phys. 86 (1982), 515-527.
22. Y. T. Siu. Lectures on Hermitian-Einstein Metrics for Stable Bundles and Kahler-Einstein Metrics. Birkhauser, 1986.
23. M. Struwe. The Yang-Mills flow in four dimensions. Calc. Var. 2 (1994), 123-150.
24. —. On the evolution of harmonic mappings of Riemann surfaces. Comment. Math. Helv. 60 (1985), no. 4, 558-581.
25. C. H. Taubes. Self-dual Yang-Mills connections on non-self-dual 4-manifolds. Journal of Differential Geometry 17.1 (1982), 139-170.
26. —. A framework for Morse theory for the Yang-Mills functional. Inventiones mathematicae 94.2 (1988), 327-402.
27. P. Topping. Repulsion and quantization in almost-harmonic maps. Ann. of Math. (2) 159 (2004), no. 2, 465-534.
28. K. Uhlenbeck. Connections with $L^p$ bounds on curvature, Comm. Math. Phys. 83 (1982), 31-42.
29. K. Uhlenbeck. Removable singularities in Yang-Mills fields, Comm. Math. Phys. 83 (1982), 11-30.
30. B. Weinkove. Singularity formation in the Yang-Mills flow. Calc. Var. 19 (2004), 221-220.