REMARKS ON PLANAR BLASCHKE-SANTALÓ INEQUALITY

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Abstract. We prove the Blaschke-Santaló inequality restricted to \( n \)-gons: the extremal polygons are the affine regular \( n \)-gons. If either the John or the Löwner ellipse of a planar \( o \)-symmetric convex body \( K \) is the unit circle about \( o \), then a sharpening of the Blaschke-Santaló inequality holds: even the arithmetic mean \((V(K) + V(K^*)) / 2\) is at least \( \pi \). We give stability variants of the Blaschke-Santaló inequality for the plane. If for some \( n \geq 3 \) the planar convex body \( K \) is \( n \)-fold rotationally symmetric about \( o \), then we give the exact maximum of \( V(K^*) \), as a function of \( V(K) \) and the area of either the John or the Löwner ellipse.

We introduce some notation that will be used in the paper. For \( x, y \in \mathbb{R}^2, x \neq y \) we write \( \text{aff} \{x, y\} \) for the line passing through \( x \) and \( y \). For compact sets or points \( X_1, \ldots, X_k \) in \( \mathbb{R}^2 \), let \( [X_1, \ldots, X_k] \) denote the convex hull of their union, where the \( X_i \)'s which are points are replaced by \( \{X_i\} \). For \( x, y \in \mathbb{R}^2 \) we write \( |xy| \) for the length of the segment \( [x, y] \). For any point \( p \neq o \), let \( p^* \) be the polar line with equation \( \langle x, p \rangle = 1 \). For \( l = p^* \), let \( p = l^* \). By the intersection point of two parallel lines we mean their common point at infinity (in the projective plane). For two non-negative quantities \( f, g \) we write \( g = \Theta(f) \) (\( g \) is of exact order \( f \)), if \( g = O(f) \) and \( f = O(g) \).

Optimality of regular \( n \)-gons in the \( o \)-symmetric case

First we treat \( o \)-symmetric polygons because the argument is much easier in this case.

Theorem A. For even \( n \geq 6 \), if \( K \) is an \( o \)-symmetric convex polygon of at most \( n \) vertices, then

\[
V(K)V(K^*) \leq n^2 \sin^2 \frac{\pi}{n},
\]

with equality if and only if \( K \) is a regular \( n \)-gon.

Proof. 1. Let \( K \) maximize the area product among \( o \)-symmetric convex polygons of at most \( n \) vertices. Since \( n^2 \sin^2(\pi/n) \) is strictly monotonically increasing with \( n \), by induction we may suppose that \( K \) has exactly \( n \) sides. By compactness of

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affine equivalence classes of convex bodies such a $K$ exists. In this case, $K^*$ is also a maximal body. The vertices of $K$ will be denoted by $x_1, x_2, \ldots, x_n$, in positive sense.

First we observe that $K$ has exactly $n$ vertices. In fact, if $k < n$, consider the vertex $x_2$ of $K$. Let $l$ be a line such that $K \cap l = \{x_2\}$. Let $\varepsilon > 0$ be sufficiently small. Push $l$ parallelly towards $K$ through a distance $\varepsilon$. Then the vertex $x_2$ is cut off. (Without further mentioning we will make always the same changes at the opposite vertex $-x_2$ so that $K$ remains $o$-symmetric.) We write $K_{\text{new}}$ for the so obtained new $o$-symmetric polygon, that still has at most $n$ vertices by hypothesis. Then $l$ cuts off an area $\Theta(\varepsilon^2)$ from $K$, so

$$V(K_{\text{new}}) = V(K) - \Theta(\varepsilon^2).$$

Then $K^*$ has neighbouring vertices $\text{aff} \{x_1, x_2\}^*$ and $\text{aff} \{x_2, x_3\}^*$ but $K^*_{\text{new}}$ will have a new vertex $l^*$ between these two old vertices. The distance of $l^*$ and the side $x_2^2$ of $K$ is $\Theta(\varepsilon)$. Hence $K^*_{\text{new}}$ is obtained from $K^*$ by adding to it a small triangle of height, and also of area $\Theta(\varepsilon)$. Thus

$$V(K^*_{\text{new}}) = V(K) + \Theta(\varepsilon).$$

Therefore for the volume products we have

$$V(K_{\text{new}})V(K^*_{\text{new}}) = V(K)V(K^*) + \Theta(\varepsilon) > V(K)V(K^*),$$

a contradiction.

Next we prove two basic properties of $K$:

(i) If $x_1, x_2, x_3$ are consecutive vertices of $K$, then $o, (x_1 + x_3)/2$ and $x_2$ are collinear.

(ii) If $y_1, y_2, y_3, y_4$ are consecutive vertices of $K$, and $m$ is the intersection point of $\text{aff} \{y_1, y_2\}$ and $\text{aff} \{y_3, y_4\}$ (observe that $m$ is a finite point separated from $K$ by $\text{aff} \{y_2, y_3\}$), then $o, (y_2 + y_3)/2$ and $m$ are collinear.

Observe that both (i) and (ii) are affine invariant. Then we may suppose that in (i) $x_1, x_2, x_3 \in S^1$, and in (ii) that $\text{aff} \{y_1, y_2\}$, $\text{aff} \{y_2, y_3\}$, $\text{aff} \{y_3, y_4\}$ are tangents to $S^1$. Then (i) expresses that $\{x_1, x_2, x_3\}$ is symmetrical to the axis $\mathbb{R} \cdot (x_1 + x_3)/2$, while (ii) expresses that $\{\text{aff} \{y_1, y_2\}, \text{aff} \{y_2, y_3\}, \text{aff} \{y_3, y_4\}\}$ is symmetrical to $\mathbb{R} \cdot m$. Otherwise said, in the metric of the $o$-symmetric ellipse containing $x_1, x_2, x_3$, the rotation carrying $x_1$ to $x_2$ carries $x_2$ to $x_3$, and in the metric of the $o$-symmetric ellipse touched by $\text{aff} \{y_1, y_2\}$, $\text{aff} \{y_2, y_3\}$, $\text{aff} \{y_3, y_4\}$, the rotation carrying $\text{aff} \{y_1, y_2\}$ to $\text{aff} \{y_2, y_3\}$ carries $\text{aff} \{y_2, y_3\}$ to $\text{aff} \{y_3, y_4\}$. Then these two symmetry properties, one taken for $K$, the other taken for $K^*$, clearly imply each other.

Since (i) for $K^*$ is equivalent to (ii) for $K$, therefore it is sufficient to prove (i).

We suppose that (i) does not hold, and seek a contradiction with a method going back to Mahler. (Recall that we assumed $x_1, x_2, x_3 \in S^1$.) We may assume that $x_1$ and $x_2$ lie on the same open side of $\text{aff} \{o, (x_1 + x_3)/2\}$. For $i = 1, 5, 2.5$, we write $y_i$
for the intersection of $x_1^*$ and $x_2^*$, and hence $y_{1.5}$ and $y_{2.5}$ are consecutive vertices of $K^*$. Let $l$ be the line through $x_2$ parallel to $\text{aff}\{x_1, x_3\}$. We move $x_2$ into a position $\tilde{x}_2$ along $l$ towards $\text{aff}\{o, (x_1 + x_3)/2\}$. In other words, $\tilde{x}_2 - x_2 = \varepsilon(x_3 - x_1)$ for small $\varepsilon > 0$, where $\varepsilon$ is small enough to ensure that $x_1$ and $\tilde{x}_2$ lie on the same open side of both $\text{aff}\{o, (x_1 + x_3)/2\}$ and the line containing the other side of $K$ with endpoint $x_3$ (say, $\text{aff}\{x_3, x_4\}$). (That is, we move toward the symmetric situation.) Therefore there exists an $o$-symmetric convex polygon $\tilde{K}$ obtained from $K$ by removing $x_2$ and $-x_2$ from the set of vertices, and adding $\tilde{x}_2$ and $-\tilde{x}_2$. Then $V(\tilde{K}) = V(K)$.

For $i = 1.5$, or $i = 2.5$, let $\tilde{y}_i$ be the intersection of $x_1^*$ and $\tilde{x}_2^*$, or $x_3^*$ and $\tilde{x}_2^*$, respectively, and hence $\tilde{y}_{1.5}$ and $\tilde{y}_{2.5}$ are the two new vertices of $K^*$ replacing the vertices $y_{1.5}$ and $y_{2.5}$ of $K^*$. In addition $\tilde{y}_{1.5} \notin K^*$ and $\tilde{y}_{2.5} \in K^*$, and $l^* = [y_{1.5}, y_{2.5}] \cap [\tilde{y}_{1.5}, \tilde{y}_{2.5}]$, moreover

$$||l^* - y_{1.5}|| > ||l^* - y_{2.5}|| \quad \text{and hence} \quad ||l^* - \tilde{y}_{1.5}|| > ||l^* - \tilde{y}_{2.5}||,$$

for $\varepsilon > 0$ sufficiently small. Since the triangles $[l^*, y_{1.5}, \tilde{y}_{1.5}]$ and $[l^*, y_{2.5}, \tilde{y}_{2.5}]$ have the same angle at $l^*$, we have

$$V(\tilde{K}^*) - V(K^*) = 2(V([l^*, y_{1.5}, \tilde{y}_{1.5}]) - V([l^*, y_{2.5}, \tilde{y}_{2.5}])) > 0$$

(observe that because of $o$-symmetry, an analogous change has to be made at the side opposite to $y_{1.5}y_{2.5}$, that gives the factor 2). This contradiction proves (i).

2. Now we prove Theorem A based on (i) and (ii). Applying a linear transformation, we may assume that $x_1, x_2, x_3 \in S^1$, and hence $x_2$ lies on the perpendicular bisector of $[x_1, x_3]$ by (i).

Now (ii) yields that $\text{aff}\{x_1, x_2\}$ and $\text{aff}\{x_3, x_4\}$ are symmetric with respect to the perpendicular bisector of $[x_2, x_3]$ (that contains $o$). It follows that $\text{aff}\{x_2, x_3\}$ and $\text{aff}\{x_3, x_4\}$ are symmetric with respect to $\text{aff}\{o, x_3\}$. Together with (i), applied to $x_2, x_3, x_4$, this yields that $x_4$ is uniquely determined, and we have $x_4 \in S^1$, and the rotation about $o$ taking $x_1$ to $x_2$ takes $x_2$ to $x_3$ and $x_3$ to $x_4$. Continuing like this, we conclude that $K$ is a regular $n$-gon inscribed into $B^2$.

**AREA SUM FOR NORMALIZED $o$-SYMMETRIC PLANAR CONVEX BODIES**

Recall that in [BMMR] the main tool was the lower estimate for polar sectors of convex bodies in $\mathbb{R}^2$. There we had a sector $xoy$, with vertex at $o$, and of angle in $(0, \pi)$, with $x, y \in \partial K$, and we had two supporting lines of $K$ at $x$ and $y$, respectively, intersecting in the half-plane bounded by $\text{aff}\{x, y\}$ not containing $K$. Thus the sector contained $[x, o, y]$ and was contained in the convex quadrangle with two sides $[o, x]$ and $[o, y]$, and other two sides lying on the given supporting lines. This quadrangle could be an arbitrary convex quadrangle, up to affinities. (Observe that the only affine invariants of a convex quadrangle are the two ratios in which the two diagonals bisect each other.) Below we will use a convex deltoid, that is up to affinities characterized by the fact that one diagonal bisects the other one in its midpoint. However, this will be sufficient for our theorems. In fact we cannot solve the general case about the maximum polar area, this remains an open
question.

Here we proceed analogously. First we maximize the area product for an angular domain of angle $2\alpha \in (0, \pi)$, say $\angle (\cos \alpha, -\sin \alpha)o(\cos \alpha, \sin \alpha)$, where we suppose that the convex sector contains the triangle $T := [(\cos \alpha, -\sin \alpha), o, (\cos \alpha, \sin \alpha)]$, and is contained in the deltoid $Q := [(\cos \alpha, -\sin \alpha), o, (\cos \alpha, \sin \alpha), (1/ \cos \alpha, 0)]$. Observe that the tangents to $S^1$ at $(\cos \alpha, \pm \sin \alpha)$ contain the sides $[(\cos \alpha, \pm \sin \alpha), (1/ \cos \alpha, 0)]$ of our deltoid. Of course, our arguments will be linearly invariant, as is the volume product, due to the formula $(TK)^* = (T^*)^{-1}K^*$ for a non-singular linear transformation $T$.

For this we need Steiner symmetrization. For an $o$-symmetric planar convex body $K$, and a line $l$ through $o$, translate each chord of $K$ orthogonal to $l$ (including the ones that degenerate to points) orthogonally to $l$ in such a way that the midpoint of the translated chord should lie on $l$. The union of these translates is the Steiner symmetrical $K'$ of $K$ with respect to $l$. Clearly $K'$ is an $o$-symmetric planar convex body with $V(K') = V(K)$. According to K.M. Ball’s PhD thesis [...] (see also M. Meyer, A. Pajor [...] ), we have

\begin{equation}
V ((K')^*) \geq V(K^*),
\end{equation}

with equality if and only if $K'$ is a linear image of $K$, by a linear map preserving all straight lines orthogonal to $l$. (A similar statement holds, without $o$-symmetry, for $V ((C - s(C))^*)$, with “linear” replaced by “affine”, cf. ??). For the following proposition we need some notations.

Let $\alpha \in (0, \pi/2)$, and let $a := (\cos \alpha, -\sin \alpha), b := (1/ \cos \alpha, 0), c := (\cos \alpha, \sin \alpha)$. Let $T := [o, a, c]$ and $Q := [o, a, b, c]$. Then $a, c \in S^1$ and $\text{aff} \{a, b\}$ and $\text{aff} \{b, c\}$ are tangents to $S^1$. Let $K$ be an $o$-symmetric convex body with $a, c \in \partial K$, and let the counterclockwise arcs $I := c(-a)$ and $-I = (-c)a$ of $S^1$ be contained in $\partial K$. Then

$$[I, -I] \subset K \subset B^2 \cup [a, b, c] \cup [-a, -b, -c].$$

Then we have also

$$[I, -I] \subset K^* \subset B^2 \cup [a, b, c] \cup [-a, -b, -c].$$

Let $C := K \cap Q$ and $C^* := K^* \cap Q$. Observe that $\partial C^*$ also contains $I \cup (-I)$, and that $T \subset C \subset Q$. Further, we have that $K$ or $K^*$ is the union of $C \cup (-C)$ or $C^* \cup (-C^*)$, and the two sectors $[\pm I, o]$ of $B^2$. Observe that we have $V(C) \in [\cos \alpha \sin \alpha, \tan \alpha]$. Both the minimal and maximal values of $V(C)$ are attained for a unique $K$: namely for $K = [I, -I]$ and for $K = B^2 \cup [a, b, c] \cup [-a, -b, -c]$. Still observe

\begin{equation}
V(K) = 2V([o, I]) + 2V(C) \quad \text{and} \quad V(K^*) = 2V([o, I]) + 2V(C^*).
\end{equation}

Hence maximization of $V(C^*)$ is equivalent to maximization of $V(K^*)$.

**Proposition.** With the above notations, let $V(C) \in (\cos \alpha \sin \alpha, \tan \alpha)$ be fixed.
Then the maximum of $V(C^*)$ occurs, e.g., for the following cases.
(i) for $V(C) < \alpha$ e.g., for $\partial C$ being the union of $[o, a] \cup [o, c]$ and an ellipsoidal arc $J$ in $B^2$ joining $a$ and $c$ (in the positive sense), the ellipse having as centre $o$;
(ii) for $V(C) = \alpha$ e.g., for $C = B^2 \cap Q$;
(iii) for $V(C) > \alpha$ e.g., for $\partial C$ being the union of $[o, a] \cup [o, c]$ and two segments of equal length $[a, a'] \subset \text{aff} \{a, b\}$ and $[c, c'] \subset \text{aff} \{c, b\}$, and an ellipsoidal arc $J$ joining $a'$ and $c'$ (in the positive sense), the ellipse having as centre $o$, and having $\text{aff} \{a, b\}$ and $\text{aff} \{b, c\}$ as supporting lines.

Observe that for $C$ of the form (i) or (iii) we have that $C^*$ is of the form in (iii) or (i), respectively (for suitable areas).

Proof. 1. We begin with Steiner’s symmetrization of $K$ with respect to the $x$ axis, obtaining $K'$. Observe that then $K \cap [I, -I]$ remains invariant, and $K \cap [a, b, c]$ will be replaced by $K' \cap [a, b, c]$, which is just the Steiner symmetrization of $K \cap [a, b, c]$ with respect to the $x$ axis. We introduce the notation

$$C' := [o, (K' \cap [a, b, c])].$$

Then

$$(P3) \quad \begin{cases} V(K) = 2V([o, I]) + 2V(C) = V(K') = 2V([o, I]) + 2V(C') \text{ and} \nonumber \\ V(K^*) = 2V([o, I]) + 2V(C^*) = V((K')^*) = 2V([o, I]) + 2V((C')^*). \end{cases} \nonumber$$

Hence maximization of $V(C^*)$ is equivalent to maximization of $V(K^*)$. Still observe that by the equality case of $(P1) V(K^*)$ can attain its maximum only in the case when $K' = K$, since the only linear map preserving vertical lines and taking $[I, -I]$ to itself is the identity.

Therefore we may suppose that $K$ and thus also $C$ is symmetric with respect to the $x$ axis. In particular,

$$(P4) \quad \begin{cases} [a, b] \cap \partial K \text{ and } [b, c] \cap \partial K \text{ will be symmetric images of each} \\
\text{other with respect to the }x\text{ axis, thus they have the same length.} \end{cases} \nonumber$$

This length can be 0 or positive, but in the second case by the hypothesis $V(C) \in (\cos \alpha \sin \alpha, \tan \alpha)$ of the proposition its length is smaller than the length of $[a, b]$.

2. Now let us consider a convex body $K$ satisfying the hypotheses of the Proposition for which $V(K^*)$ is maximal, and the body $C$ corresponding to this $K$. Then by 1 both $K$ and $C$ are necessarily symmetric with respect to the $x$-axis. We have two cases.

(i) $[a, b] \cap \partial K$ (and then also $[b, c] \cap \partial K$) has a positive length.
(ii) $[a, b] \cap \partial K$ (and then also $[b, c] \cap \partial K$) is a point.

For this $K$ and $C$ let us inscribe to $[(\partial C) \setminus T] \cup \{a, c\}$ a convex polygonal arc $x_1 \ldots x_n$, in positive orientation, in the following way. We have $x_1 = a$ and $x_n = c$.

In case (i) the segment $[a, b] \cap \partial K$ (and $[b, c] \cap \partial K$) should be $[x_1, x_2]$ (or $[x_{n-1} x_n]$, respectively). The further points $x_3, \ldots, x_{n-2}$ in case (i) or $x_2, \ldots, x_{n-1}$ in case (ii) are placed so that they divide the counterclockwise arc $x_2 \ldots x_{n-1}$ or $x_1 \ldots x_n$.
of $\partial C$ to subarcs of equal length. This guarantees symmetry of our polygonal arc with respect to the $x$ axis. Since the length of the arc $[\partial C \setminus T] \cup \{a, c\}$ is at most the length of the arc $\bar{a}bc$, therefore the length of the subarcs $\bar{x}_i \xrightarrow{i+1}$, excepting $\bar{x}_1 \bar{x}_2$ and $\bar{x}_{n-1} \bar{x}_n$ in case (i), is $O(1/n)$.

We write

$$K_n := [I, -I] \cup \text{conv}\{x_1, \ldots , x_n\} \cup \text{conv}\{-x_1, \ldots , -x_n\}$$

and $C_n := [o, K_n \cap [a, b, c]]$.

Then $K_n \to K$ and $C_n \to C$ in the Hausdorff metric in both cases (i) and (ii). In fact, we have on one hand $K_n \subset K$ and $C_n \subset C$. On the other hand, if some $p \in C$ is separated from $C_n$ by a side $x_i \xrightarrow{i+1}$ (in case (i) this cannot be $\bar{x}_1 \bar{x}_2$ or $\bar{x}_{n-1} \bar{x}_n$), then

$$\min\{|x_i p|, |x_{i+1} p|\} \leq (|x_i p| + |x_{i+1} p|)/2 \leq |\bar{x}_i \xrightarrow{i+1}|/2 = O(1/n).$$

Hence the Hausdorff distance of $K$ and $K_n$, as well as that of $C$ and $C_n$ is $O(1/n)$.

Now we define $\bar{K}_n := [I, -I, J, -J]$, where $J$ is a strictly convex polygonal arc $J := \bar{x}_1, \ldots , \bar{x}_k$ which has the following properties. We have $k \leq n$. Further, $J$ lies in $[a, b, c]$ and satisfies $x_1 = a$ and $x_k = c$, and in case (i) still $[x_1, x_2] \supset [a, b] \cap \partial K$ and $[x_{k-1}, x_k] \supset [b, c] \cap \partial K$, and $V([\bar{x}_1, \ldots , \bar{x}_k]) = V([x_1, \ldots , x_n])$. Lastly, $J$ is such a one among all polygonal arcs satisfying all above listed properties, for which $V((\bar{K}_n)^*)$ attains its maximal value. (By a (strictly) convex polygonal arc we mean one for which the sides following each other turn (strictly) counterclockwise.)

Observe that till now we could not exclude $[\bar{x}_1, \bar{x}_2] \neq [a, b] \cap \partial K$ and $[\bar{x}_{k-1}, \bar{x}_k] \neq [b, c] \cap \partial K$ in either case (i) or case (ii), and also we do not yet know $k = n$.

Then $V((K_n)^*) \leq V((\bar{K}_n)^*)$. Then for some subsequence $K_{n(i)}$ of the $K_n$'s there exists a limit convex body, and also the equal lengths of the intersections $K_{n(i)} \cap [a, b]$ and $\bar{K}_{n(i)} \cap [b, c]$ are convergent. Then also $V((\lim K_{n(i)})^*)$ will have a maximal value among the considered convex bodies $K$.

3. Now we want to apply the method of proof of Theorem A. Its part 1 consisted of three main steps. First we showed $k = n$. Then we showed (i) and (ii) from the proof of Theorem A.

Now, rather than (i) and (ii) from 2 we have to distinguish between the following cases.

(i') $\bar{x}_2 \in (a, b)$ and $\bar{x}_{k-1} \in (b, c)$, and

(ii') $\bar{x}_2 \notin (a, b)$ and $\bar{x}_{k-1} \notin (b, c)$. Observe that by strict convexity, in case (i') we have

$$\bar{x}_3, \ldots , \bar{x}_{n-2} \in \text{int} [a, b, c],$$

while in case (ii') we have

$$\bar{x}_2, \ldots , \bar{x}_{n-1} \in \text{int} [a, b, c].$$

As soon as some vertex $\bar{x}_i$ lies in $[a, b, c]$, it is freely movable till some small distance. Therefore the statements in the proof of Theorem A, 1, which used small movements of such $x_i$'s, remain valid also here.
In particular, (i) of Theorem A, 1 remains true for any three consecutive vertices $\tilde{x}_i, \tilde{x}_{i+1}, \tilde{x}_{i+2}$, for $2 \leq i \leq k - 3$ in case (i'), and for $1 \leq i \leq k - 2$ for case (ii').

However, for the proof of $k = n$ and (ii) in the proof of Theorem A, we have to take more care in the proof here.

4. We begin with the proof of $k = n$. In the proof of Theorem A, we cut off from $K$ a part of area $\Theta(\varepsilon^2)$, so for keeping the area constant, we have to give this area back at some other place.

Namely, supposing $k < n$ consider a straight line $l$ such that $K_n \cap l = \{\tilde{x}_{i+1}\}$. Then push parallelly $l$ towards $\varepsilon$ through a small distance $\varepsilon = \varepsilon(K_n) > 0$, obtaining $l'$. Then consider the $\varepsilon$-symmetric convex body $(\tilde{K}_n)' \subset (\tilde{K}_n)$ obtained from $\tilde{K}_n$ by cutting $\tilde{x}_{i+1}$ and $-\tilde{x}_{i+1}$ from $\tilde{K}_n$ by $l'$ and by $-l'$. Then $(\tilde{K}_n)'$ has in the closed angular domain $\angle\text{aoc} k + 1 \leq n$ vertices, and $V(\tilde{K}_n) - V((\tilde{K}_n)') = \Theta(\varepsilon^2)$.

Moreover, $(\tilde{K}_n)'$ has vertices $\tilde{x}_i, \tilde{x}_{i+1}', \tilde{x}_{i+2}'$, $\tilde{x}_{i+2}'$ in positive sense. Now fix $\tilde{x}_i$ and aff $\{\tilde{x}_{i+1}', \tilde{x}_{i+2}'\}$, and rotate the side line aff $\{\tilde{x}_i, \tilde{x}_{i+1}'\}$ about $\tilde{x}_i$ outwards through an angle $\Theta(\varepsilon^2)$ (and making the analogous change also at $-\tilde{x}_i$) so that for the new $\varepsilon$-symmetric convex body $(\tilde{K}_n)'''$, that has also $k + 1$ vertices in the closed angular domain $\angle\text{aoc}$ we should have $(\tilde{K}_n)'' \supset (\tilde{K}_n)'$ and $V((\tilde{K}_n)') = V(\tilde{K}_n)$. Now we turn to the polar bodies. By $\tilde{K}_n \supset (\tilde{K}_n)' \subset (\tilde{K}_n)''$ we have $(\tilde{K}_n)'' \supset (\tilde{K}_n)' \supset ((\tilde{K}_n)')^* \supset ((\tilde{K}_n)'')^*$. Then $(\tilde{K}_n)^*$ has neighboring vertices $(\tilde{x}_i \tilde{x}_{i+1})^*$ and $(\tilde{x}_{i+1} \tilde{x}_{i+2})^*$, connected by the side on the line $\tilde{x}_{i+1}^*$. Then $((\tilde{K}_n)')^*$ will have a new vertex $(l')^*$, at a distance $\Theta(\varepsilon)$ from $\tilde{x}_{i+1}^*$, hence $V(((\tilde{K}_n)')^*) - V((\tilde{K}_n)^*) = \Theta(\varepsilon)$. The rotation of the side line aff $\{\tilde{x}_i, \tilde{x}_{i+1}\}$ about $\tilde{x}_i$ through an angle $O(\varepsilon^2)$ implies motion of (aff $\{\tilde{x}_i, \tilde{x}_{i+1}\}$)* on the line $\tilde{x}_i^*$ through a distance $O(\varepsilon^2)$, hence $V(((\tilde{K}_n)')^*) - V(((\tilde{K}_n)'')^*) = O(\varepsilon^2)$. Therefore,

$$V(((\tilde{K}_n)'')^*) = V((\tilde{K}_n)^*) + \Theta(\varepsilon) - O(\varepsilon^2) > V((\tilde{K}_n)^*)$$

for $\varepsilon > 0$ sufficiently small, a contradiction. This contradiction proves our claim $k = n$.

5. For the proof of (ii) in the proof of Theorem A, 1 (by dualizing (i) in the proof of Theorem A), the area of $K$ was not preserved, even just conversely, the area of the polar was preserved. Therefore we have to give a new proof, which preserves the area of $K$.

So we consider four consecutive vertices $\tilde{x}_i, \tilde{x}_{i+1}, \tilde{x}_{i+2}, \tilde{x}_{i+3}$ of $\tilde{K}_n$, and $m$ is the intersection point of aff $\{\tilde{x}_i, \tilde{x}_{i+1}\}$ and aff $\{\tilde{x}_{i+2}, \tilde{x}_{i+3}\}$. Then $m$ is a finite point, separated from $\tilde{K}_n$ by aff $\{\tilde{x}_{i+1}, \tilde{x}_{i+2}\}$.

We may suppose that the lines aff $\{\tilde{x}_i, \tilde{x}_{i+1}\}$, aff $\{\tilde{x}_{i+1}, \tilde{x}_{i+2}\}$ and aff $\{\tilde{x}_{i+2}, \tilde{x}_{i+3}\}$ are tangent to $S^1$, and also that $m$ lies on the positive $x$ axis. So aff $\{o, m\}$ is the $x$ axis. Then aff $\{\tilde{x}_i, \tilde{x}_{i+1}\}$ and aff $\{\tilde{x}_{i+2}, \tilde{x}_{i+3}\}$ are symmetric with respect to
the $x$ axis. Then of course, also their polars \((\text{aff} \{\tilde{x}_i, \tilde{x}_{i+1}\})^*\) and \((\text{aff} \{\tilde{x}_{i+2}, \tilde{x}_{i+3}\})^*\)
are symmetric with respect to the $x$ axis, thus
\[
\text{(P5) } \begin{cases} 
\text{the line connecting } (\text{aff} \{\tilde{x}_i, \tilde{x}_{i+1}\})^* \\
\text{and } (\text{aff} \{\tilde{x}_{i+2}, \tilde{x}_{i+3}\})^* \text{ is vertical.}
\end{cases}
\]

We may assume, for contradiction, that \((\tilde{x}_{i+1} + \tilde{x}_{i+2})/2\) lies below the $x$ axis.

We begin with rotating the side line \(\text{aff} \{\tilde{x}_{i+1}, \tilde{x}_{i+2}\}\) about the midpoint \((\tilde{x}_{i+1} + \tilde{x}_{i+2})/2\) of the side \([\tilde{x}_{i+1}, \tilde{x}_{i+2}]\), through some small angle \(\varepsilon > 0\) in positive sense. (That is, we move toward the symmetric position, like in the proof of (i) in 1 of the proof of Theorem A.) The new positions of \(\tilde{x}_{i+1}\) and \(\tilde{x}_{i+2}\) are denoted by \((\tilde{x}_{i+1})'\) and \((\tilde{x}_{i+2})'\). The body obtained by this change is denoted by \((\tilde{K}_n)'\). Then we have
\[
V \left( (\tilde{K}_n)' \right) - V(\tilde{K}_n) = O(\varepsilon^2).
\]

Of course we have to still to restore the original area. This is done by a parallel replacement of the already rotated side line through a distance \(O(\varepsilon^2)\). The new positions of \((\tilde{x}_{i+1})'\) and \((\tilde{x}_{i+2})'\) will be denoted by \((\tilde{x}_{i+1})''\) and \((\tilde{x}_{i+2})''\), and the thus obtained body will be denoted by \((\tilde{K}_n)''\).

Observe that the centre of rotation \((\tilde{x}_{i+1} + \tilde{x}_{i+2})/2\) of our first motion lies in \([m, s, t] \setminus B^2 \subset [m, s, t]\), where s and t are the points of tangency of \(S^1\) with the side lines \(\text{aff} \{\tilde{x}_i, \tilde{x}_{i+1}\}\) and \(\text{aff} \{\tilde{x}_{i+2}, \tilde{x}_{i+3}\}\). Therefore \((\tilde{x}_{i+1} + \tilde{x}_{i+2})/2\) lies in the open right halfplane given by \(x > 0\). By hypothesis, it also lies in the open lower halfplane given by \(y < 0\). So it lies in the open fourth coordinate quadrant. Therefore
\[
\text{(P6) } \text{the slope of the polar line } [([\tilde{x}_{i+1} + \tilde{x}_{i+2}]/2]^* \text{ lies in } (0, \infty) .
\]

Now we consider the polars. We begin with the first part of the motion, i.e., with the rotation of the side line about the side midpoint. Rotation of the side line \(\text{aff} \{\tilde{x}_{i+1}, \tilde{x}_{i+2}\}\) about the midpoint \((\tilde{x}_{i+1} + \tilde{x}_{i+2})/2\) of the side \([\tilde{x}_{i+1}, \tilde{x}_{i+2}]\) implies moving the polar point \((\text{aff} \{\tilde{x}_{i+1}, \tilde{x}_{i+2}\})^*\) on the polar line \([([\tilde{x}_{i+1} + \tilde{x}_{i+2}]/2]^*\), so that this point moves counterclockwise, when looked upon from \(O\). Therefore by (P5) its \(x\) coordinate increases, by a quantity \(\Theta(\varepsilon)\).

The polar body \((\tilde{K}_n)'\) has consecutive vertices \((\text{aff} \{\tilde{x}_i, \tilde{x}_{i+1}\})^*, (\text{aff} \{(\tilde{x}_{i+1})', (\tilde{x}_{i+2})'\})^*\) and \((\text{aff} \{\tilde{x}_{i+2}, \tilde{x}_{i+3}\})^*\). The first and third of these vertices are fixed, and their connecting line is vertical, by (P5). The second vertex lies on the right hand side of this vertical line, and in its new position \((\text{aff} \{\tilde{x}_{i+1}, \tilde{x}_{i+2}\})^*\) its \(x\) coordinate is greater than in its original position \((\text{aff} \{\tilde{x}_{i+1}, \tilde{x}_{i+2}\})^*\), namely the difference is \(\Theta(\varepsilon)\). Therefore
\[
\text{(P7) } V(\tilde{K}'^*) - V(\tilde{K}^*) = \Theta(\varepsilon).
\]

Now we consider the second part of this motion, i.e., the parallel displacement of the already rotated side, through a distance \(O(\varepsilon^2)\). Then the vertices \((\text{aff} \{\tilde{x}_i, \tilde{x}_{i+1}\})^*,\) and \((\text{aff} \{\tilde{x}_{i+2}, \tilde{x}_{i+3}\})^*\) remain fixed, but the vertex \((\text{aff} \{(\tilde{x}_{i+1})',
(\tilde{x}_{i+2})^* moves to its new position (aff \{(\tilde{x}_{i+1})^\prime, (\tilde{x}_{i+2})^\prime\})^*, and the distance of these last two points is \(O(\varepsilon^2)\). Hence

(P8) \[ V((K^\prime)^*) - V((K)^*) = O(\varepsilon^2) \]

Then (P7) and (P8) imply

(P9) \[ V((K^{\prime\prime})^*) = V(K^*) + \Theta(\varepsilon) + O(\varepsilon^2) = V(K^*) + \Theta(\varepsilon) > V(K^*) , \]

that is a contradiction. This proves (ii) in the proof of Theorem A, 1.

8. By 5, 6 and 7 we have that the extremal polygonal line \(x_1 \ldots x_n\) has exactly \(n\) vertices, and both (i) and (ii) from the proof of Theorem A, 1 hold.

Recall the cases (i') and (ii') from 3. From 3, in case (i') we have

\[ \tilde{x}_3, \ldots, \tilde{x}_{n-2} \in \text{int } [a, b, c], \]

while in case (ii') we have

\[ \tilde{x}_2, \ldots, \tilde{x}_{n-1} \in \text{int } [a, b, c]. \]

Then, like in the proof of Theorem A, we obtain that in case (i') \(\tilde{x}_2 \ldots \tilde{x}_{n-1}\), while in case (ii) \(\tilde{x}_1 \ldots \tilde{x}_n\) is inscribed to an elliptical arc, where the respective ellipse \(E_n\) has centre \(o\), and, in the metric of \(E_n\), the rotation carrying \(x_2\) to \(x_3\) in case (i) or carrying \(x_1\) to \(x_2\) in case (ii) also carries all \(x_i\) to \(x_{i+1}\), for \(3 \leq i \leq n - 2\), or \(2 \leq i \leq n - 2\), respectively. Since any two metrics on \(\mathbb{R}^2\) are equivalent, therefore changing the original metric of \(\mathbb{R}^2\) to the metric determined by \(E_n\) preserves lengths and areas, up to an at most constant positive factor. Therefore we may use henceforth for fixed \(n\) the metric of \(E_n\).

Now let us replace the polygonal arc \(\tilde{x}_3 \ldots \tilde{x}_{n-2}\) in case (i'), and \(\tilde{x}_2 \ldots \tilde{x}_{n-1}\) in case (ii') by the respective arc of \(E_n\) (and similarly for their mirror images with respect to \(o\)). Then in both cases we obtain a convex body \(L_n\), since the chords \([\tilde{x}_2, \tilde{x}_3]\) and \([\tilde{x}_{n-2}, \tilde{x}_{n-1}]\) in case (i'), and \([\tilde{x}_1, \tilde{x}_2]\) and \([\tilde{x}_{n-1}, \tilde{x}_n]\) in case (ii') span lines intersecting in \([a, b, c]\).

Then we have \(K_n \subset L_n\), and the Hausdorff distance of \(K_n\) and \(L_n\) is \(O(1/n^2) = o(1)\). For a suitable subsequence \(n(i)\) (cf. 2) we have that

(a) \(K_n\) is convergent, and then necessarily \(L_{n(i)}\) has the same limit, and we have case (i') for all \(n(i)\) or case (ii') for all \(n(i)\), and

(b) in case (i') the points \(\tilde{x}_2 \in (a, b)\) and \(\tilde{x}_{n-1} \in (b, c)\) are convergent, and also

(c) \(E_{n(i)}\) is convergent.

Therefore we claim that we may use the metric of any \(E_n\), or the metric of \(E := \lim_{i \to \infty} E_{n(i)}\), the changes in lengths and areas are still bounded by an at most constant factor.

For this we have to show that all \(E_n\)'s lie in a compact family. All of them pass through all four vertices \((\pm \cos \alpha, \pm \sin \alpha)\) of an axis-parallel rectangle, hence have axis-parallel axes themselves. The positive horizontal semiaxes have endpoints in \(\text{int } [a, b, c]\), hence are bounded from below and from above. The positive vertical
semiaxes are bounded from below by the \( y \)-coordinate of the vertex \( \tilde{x}_1 \) in case (i’) and of the vertex \( \tilde{x}_2 \) in case (ii’). In case (i’) this is fixed. Also in case (ii’) these cannot be arbitrarily small, since then the areas of \( [\tilde{x}_1, \ldots, \tilde{x}_n] \) would tend to \( V([a, b, c]) \), which was excluded. Also they cannot be arbitrarily large. Namely then the chords of \( E_n \) from the endpoints of their vertical semiaxes with positive (negative) \( y \)-coordinates to the endpoints with positive \( x \)-coordinates and positive (negative) \( y \)-coordinates of the above considered elliptical arcs span lines on whose left hand side lies \( [\tilde{x}_1, \ldots, \tilde{x}_n] \). Then the areas of \( [\tilde{x}_1, \ldots, \tilde{x}_n] \) would tend to 0, which was also excluded.

Therefore the limit of the \( \tilde{L}_{n(i)} \)'s exists. It is \( o \)-symmetric. In the closed angular domains \([\alpha, \pi - \alpha]\) and its mirror image with respect to \( o \) it is bounded by the respective arcs of \( S^1 \). In the closed angular domains \([\alpha, \pi - \alpha]\) it is bounded in case (i’) by the segment \([\lim \tilde{x}_1, \lim \tilde{x}_2]\), its mirror image w.r.t. the \( x \)-axis, and an elliptical arc symmetric w.r.t. the \( x \)-axis, while in case (ii’) only with the hyperbola arc symmetric w.r.t. the \( x \)-axis. In case (i’) possibly \( \lim \tilde{x}_1 = \lim \tilde{x}_2 \) — this case we count to case (ii’).

In case (ii’) the area of the convex hull of the arc of ellipse (that is equal to \( V([\tilde{x}_1, \ldots, \tilde{x}_n]) \) for each \( n \)) uniquely determines the arc of ellipse. Namely the boundary of the ellipse passes through the four points \((\pm \cos \alpha, \pm \sin \alpha)\). Therefore any two of them does not have any further intersection points, so their parts in the open angular domain \((\cos \alpha, \pm \sin \alpha)\) are disjoint, hence they are linearly ordered by inclusion. The arc giving the minimal area is \([a, c]\), which however cannot be attained by the restriction on the area. We assert that the maximal of them is the arc of \( S^1 \) in this angular domain. Namely any two such ellipses have a transversal intersection point at \( a \) (and \( c \)). Else they would have eight common points with multiplicities, hence would be equal. In particular, the arc of \( S^1 \) in this angular domain and any elliptical arc passing through the four points \((\pm \cos \alpha, \pm \sin \alpha)\) have different tangents at \( a \). In case of strict inclusion of the sector of \( B^2 \) and the respective sector of the elliptical arc the tangent of the elliptical arc at \( a \) would point outside from the quadrangle \( Q \), that is impossible, since it lies in \( Q \).

In case (i’) the situation is more complicated. Then we have on the boundary of \( \tilde{L} \) segments of equal positive lengths \([\tilde{x}_1, \tilde{x}_2]\) and \([\tilde{x}_{n-1}, \tilde{x}_n]\). On \( \partial L \), at \( \tilde{x}_2 \) and its mirror image w.r.t. the \( x \)-axis there joins to these segments an arc of an ellipse, also symmetric w.r.t. the \( x \)-axis. Now we have two parameters: the positive length of the segment \([\tilde{x}_1, \tilde{x}_2]\), and one more parameter that distinguishes the elliptical arc among all those elliptical arcs that pass through the four points \( \pm \tilde{x}_2 \) and their symmetric images w.r.t. the \( x \)-axis.

However, here is still one restriction. The body \( L \) cannot have a non-smooth point at \( \tilde{x}_2 \). This can be proved similarly as formerly already several times. Namely, we put a line \( l \) that is a supporting line of \( L \) at \( \tilde{x}_2 \), but is not equal to any of the half-tangents at \( \tilde{x}_2 \). Now let \( \varepsilon > 0 \) be sufficiently small, and push \( l \) inward to \( L \) (and similarly at all three symmetric images of \( \tilde{x}_2 \)). The loss of area is \( \Theta(\varepsilon^2) \). Since the area of \( K \) must remain constant, we have to give this loss of area somewhere back. For this consider the elliptical arc in question, and close to the midpoint of this elliptical arc we choose a point \( p \) on the positive \( x \)-axis outside of \( L \), having a distance \( \delta \) from \( L \). We consider the convex hull \([L, p - p]\). This has area
V(L) + Θ(δ^{3/2}). Similarly, [L, p−p]∗ has an area V(L∗) + Θ(δ^{3/2}). Then choosing δ suitably, the added area will be exactly equal to the lost area, so the area of K will not change by performing these two operations. At the same time, δ^{3/2} = Θ(ε^2). By cutting off K with l (and its three symmetric images) K∗ will change to the convex hull of the original K∗ and l∗ and its three symmetric images. However, this means an increase in the area by Θ(ε^2). Then cutting the new K∗ by the line p∗ (and its three symmetric images) causes the area to decrease with Θ(δ^{3/2}) = Θ(ε^2). That is, the original V(K∗) changed to V(K∗) + Θ(ε) + Θ(ε^2) = V(K∗) + Θ(ε) > V(K∗) a contradiction.

This contradiction implies that the extremal body L is smooth at x̂2 (and its three symmetric images). This means that it is already uniquely determined (by passing through the point x̂2 and having given tangents there — namely two such ellipses would have eight common points with multiplicities, which is a contradiction). Observe that the polars of such arcs, consisting of two symmetrically placed segments and an arc of a hyperbola, are just elliptical arcs of the form in case (ii'), hence the corresponding (polar) sectors are totally ordered by inclusion. Therefore, for a given area of the sector of K in the closed angular domain aob the part of the boundary in this angular domain is uniquely determined. By the upper inequality on the area of the sector of K in this angular domain the degenerate position, when the sector of K would coincide with the deltoid [o, a, b, c], cannot be attained.

We obtain case (ii) of the Proposition when both (i') and (ii') hold. ■

The John ellipse E_1(K), or the Löwner ellipse E_0(K) of an o-symmetric convex body K ⊂ R^d is a the o-symmetric ellipsoid of maximal volume contained in K, or of minimal volume containing K, respectively. Both of them are unique, and the polar of the John ellipsoid of K is the Löwner ellipsoid of K∗ (using duality). By [Be], B^2 is the John ellipse or the Löwner ellipse of K ⊂ R^2 if and only if it is contained in K, or contains K, and (∂K) ∩ S^1 contains the vertices of a square with vertices p_1, ..., p_4 in positive cyclic order, inscribed to S^1, or the vertices of an o-symmetric convex hexagon with vertices p_1, ..., p_6, in positive cyclic order, where ∠p_iop_{i+1} < π/2 (Behrend, [Behrend]).

**Theorem B.** If B^2 is the John ellipse or the Löwner ellipse of a planar o-symmetric convex body K, then

V(K) + V(K∗) ≤ 2π.

**Remark.** This inequality is specific to o-symmetric planar convex bodies. If either K is a regular triangle inscribed into S^1, or a regular cross-polytope circumscribed about S^{d−1} (for the John ellipsoid) or a cube inscribed to S^{d−1} (for the Löwner ellipsoid) of dimension d ≥ 3, the analogous statement (i.e., when π is replaced by the volume of the unit ball B^d in R^d) does not hold.

In fact, for K a regular triangle inscribed to S^1 we have V(K) + V(K∗) = 15√3/4 = 6.4951... > 2π. Further, observe that the John ellipsoid of a regular cross-polytope D^d, circumscribed about B^d, is B^d (by the criterion of John [John]). Then,
by polarity, the Löwner ellipsoid of a cube $C^d$, inscribed to $B^d$, is $B^d$. For $d = 3$, we have $V(C^3) + V(D^3) = 44\sqrt{3}/9 = 8.4678... > 2V(B^3) = 8.3775...$. For $d = 4$, already $V(D^4) = 10.6666... > 9.8696... = 2V(B^4)$ and $V(D^5) = 14.9071... > 10.5275... = 2V(B^5)$. Further, using the recursion formulas, one easily sees that $V(D^d)/(2V(B^d))$ is increasing, separately for even, and for odd $d \geq 4$, that proves our claim. (For comparing the cases of dimensions $d$ and $d + 2$, the increasing property is equivalent to $(1 + 2/d)^{d/2} \cdot [(d + 2)/(d + 1)] \cdot (2/\pi) > 1$, which holds even omitting the second factor.)

Proof of Theorem B. We may assume that $B^2$ is the Löwner ellipse of $K$. It is sufficient to prove that if $p := p_i, q := p_{i+1} \in S^1 \cap \partial K$, and the angle of the vectors $p$ and $q$ is $2\alpha \in (0, \pi/2]$, then

\[(*) \quad V(K \cap S) + V(K^* \cap S) \leq 2\alpha\]

for the convex cone $S := \{ tp + sq : t, s \geq 0 \}$. Namely, summing all four, or all six such inequalities, we obtain the statement of the theorem.

We may suppose $p = (1, 0)$. Let $r := (0, 1) \in S^1$, and let $\tilde{S} = \{ tr + sq : t, s \geq 0 \}$.

For

$$\tilde{K}_{pq} = [\pm(S \cap K), \pm(\tilde{S} \cap B^2)],$$

we have

$$\tilde{K}^*_{pq} = [\pm(S \cap K^*), \pm(\tilde{S} \cap B^2), \pm\sqrt{2}(r - p)].$$

Therefore (*) is equivalent to

$$V(\tilde{K}_{pq}) + V(\tilde{K}^*_{pq}) \leq 3 + \pi.$$ 

Let $E$ be an $\alpha$-symmetric ellipse such that $p, r \in \partial E$, and for the part $M$ between the chords $[-r, p]$ and $[-p, r]$, we have $V(M) = V(\tilde{K}_{pq}) \leq 1 + \pi/2$. By $\pm p, \pm r \in \partial E$ we have $V(E) \geq \pi$. It follows from the Proposition that $V(\tilde{K}^*_{pq}) \leq V(M^*)$, and hence (*) follows from

\[(**) \quad V(M \cap S_0) + V(M^* \cap S_0) \leq \pi/2,\]

where $S_0 := S \cup \tilde{S} = \{ tp + sr : t, s \geq 0 \}$ and $V(A \cap S_0) \leq \pi/4$. Thus it suffices to show (**), which we are going to do.

Let $a \leq 1$ and $b \geq 1$ be the half axes of $E$. For $E = B^2$ (**) is fulfilled, therefore we may assume $a < 1 < b$. We choose a new orthonormal system of coordinates with basic vectors $(1/\sqrt{2}, 1/\sqrt{2}), (-1/\sqrt{2}, 1/\sqrt{2})$, in this order. Then $E = \Phi B^2$, for a diagonal matrix

$$\Phi = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$ 

In particular, $E \cap S_0 = \Phi \sigma$, where $\sigma \subset S_0$ is a sector of $B^2$, having an acute angle $2\alpha$.
at 0, and being symmetric with respect to the perpendicular bisector of \([p, r]\).

(For \(2\alpha < \pi/2\) observe \(V(E) > \pi\), and hence \(V(E \cap S_0) \leq \pi/4 < V(E)/4\).) It follows that \(\Phi (\cos \alpha, \sin \alpha) = (a \cos \alpha, b \sin \alpha) = (1/\sqrt{2}, 1/\sqrt{2})\), hence \(a = 1/(\sqrt{2} \cos \alpha)\) and \(b = 1/(\sqrt{2} \sin \alpha)\), hence \(\tan(\alpha) = a/b\) and

\[
M^* \cap S_0 = [p, r, \tau],
\]

where \(\tau\) is the sector of the polar ellipse \(E^*\) corresponding to the sector \(\sigma\) of \(E\) by polarity. Hence

\[
V(M \cap S_0) + V(M^* \cap S_0) = ab\alpha + \frac{\alpha}{ab} + \frac{1 - \tan^2 \alpha}{1 + \tan^2 \alpha}.
\]

In particular, \(V(M \cap S_0) + V(M^* \cap S_0) = f(\alpha)\) for \(\alpha \in (0, \pi/4]\), where \(f(\pi/4) = \pi/2\), and

\[
f(\alpha) = \alpha (\sin(2\alpha) + 1/\sin(2\alpha)) + \cos(2\alpha).
\]

We write \(\beta := 2\alpha\) and \(g(\beta) := f(\beta/2)\), where \(\beta \in (0, \pi/2]\). The inequality \(g'(\beta) > 0\) becomes, rearranging, \(\beta < \tan \beta\), for \(\beta \in (0, \pi/2]\), hence it holds. Therefore

\[
V(M \cap S_0) + V(M^* \cap S_0) = g(\beta) \leq g(\pi/2) = \pi/2,
\]

completing the proof of (**), and in turn the proof of Theorem B. ■

**Stability of the Blaschke-Santaló Inequality**

**Theorem D.** Let \(K\) be a planar convex body, and \(s(K)\) the Santaló point of \(K\). If the Banach-Mazur distance of \(K\) from the ellipses is at least \(1 + \varepsilon\), where \(\varepsilon > 0\), then

\[
V(K)V(K - s(K))^* \leq \pi^2 (1 - c\varepsilon^6),
\]

where \(c\) is a positive absolute constant. If additionally \(K\) is 0-symmetric, we have

\[
V(K)V(K - s(K))^* = V(K)V(K^*) \leq \pi^2 \left(1 - (8 + o(1))\varepsilon^3\right),
\]

for \(\varepsilon \to 0\).

**Proof.** It follows by [Bö], Theorem 1.4, together with [MR06], Theorem 1 (and Theorem 13), that there exists an \(o\)-symmetric planar convex body \(\tilde{K}\) with \(V(\tilde{K})V(\tilde{K}^*) \geq V(K)V((K - s(K))^*)\), whose Banach-Mazur distance from the ellipses is at least \(1 + c_0\varepsilon^2\), where \(c_0\) is a positive absolute constant \(\blacksquare\). We may assume for the John ellipse of \(\tilde{K}\) that \(E_i(\tilde{K}) = B^2\). Then there is a point \(p\) of \(\tilde{K}\) of distance at least \(1 + c_0\varepsilon^2\) from \(o\). We deduce that \(V(\tilde{K}) \geq \pi + c_0^{3/2} (2\sqrt{2}/3 + o(1))\varepsilon^3\), for \(\varepsilon \to 0\). Similarly, \((\tilde{K})^*\) has a supporting line \(p^*\) at distance at most \(1 - (c_0 + o(1))\varepsilon^2\) from \(o\), hence \(V(\tilde{K}^*) \leq \pi - c_0^{3/2} (4\sqrt{2}/3 + o(1))\varepsilon^3\). Therefore Theorem B yields
\[
\begin{array}{l}
4V(\tilde{K})V((\tilde{K})^*) = (V(\tilde{K}) + V((\tilde{K})^*))^2 - (V(\tilde{K}) - V((\tilde{K})^*))^2 \\
\leq 4\pi^2 - \left( \left( \sqrt{8\varepsilon_0^{3/2} + o(1)} \right) \varepsilon^3 \right)^2.
\end{array}
\]

The \(o\)-symmetric case follows similarly.

Remark. A somewhat weaker result is given by K. M. Ball, K. J. Böröczky [BB], Theorem 2.2, case of \(\mathbb{R}^2\) (observe that there \(n \geq 3\) is written, but the arguments are valid for \(n \geq 2\) as well). By a slight rewriting it says the following. (Observe that in the end of the proof of the cited theorem the exponent of \(|\log \varepsilon|\) can in fact be doubled at the application of (3) and (4) there, for the \(o\)-symmetric case.) Under the conditions of Theorem D, except that \(K \subset \mathbb{R}^n\), we have

\[
V(K)V(K^*) \leq \kappa_n^2(1 - \text{const}_n \varepsilon^{3n+3}|\log \varepsilon|^{-4}),
\]

and for the \(o\)-symmetric case

\[
V(K)V(K^*) \leq \kappa_n^2(1 - \text{const}_n \varepsilon^{(3n+3)/2}|\log \varepsilon|^{-4}).
\]

For \(n = 2\) these give

\[
V(K)V(K^*) \leq \pi^2(1 - \text{const}_\varepsilon^9|\log \varepsilon|^{-4}),
\]

and

\[
V(K)V(K^*) \leq \pi^2(1 - \text{const}_\varepsilon^{9/2}|\log \varepsilon|^{-4}).
\]

As stated in [BB], we cannot have a better upper estimate, even in the \(o\)-symmetric case, than \(\kappa_n^2(1 - \text{const}_n \varepsilon^{(n+1)/2})\).

For the planar \(o\)-symmetric case we can let \(K_0 := \{(x, y) \in B^2 \mid |x| \leq 1 - \varepsilon\}.\) Then the John ellipse \(E_i(K)\) of \(K\) has, by Behrends' characterization, the equation

\[
(x/(1 - \varepsilon))^2 + y^2 = 1.
\]

Possibly the Banach-Mazur distance is attained for \(E_i(K)\) and that inflated copy \(\lambda E_i(K)\) of it that passes through the four nonsmooth points \((\pm(1 - \varepsilon, \pm\sqrt{2\varepsilon - \varepsilon^2})\) of \(K\). If this were true, we would have \(\lambda^2 = ((1 - \varepsilon)/(1 - \varepsilon))^2 + (\sqrt{2\varepsilon - \varepsilon^2})^2\), hence \(\lambda - 1 \sim \varepsilon\). Also we have \(V(K)V(K^*) = \pi^2 - (\pi \cdot 4\sqrt{2} + o(1)) \varepsilon^{3/2}/3\). Observe that the same Banach-Mazur distance is attained if we consider the intersection \(K'\) of \(B^2\) with an \(o\)-symmetric square \(Q\) of sidelengths \(1 - \varepsilon\), that follows from the fact that the John and Löwner ellipses of \(K'\) are the incircle of \(Q\) and \(B^2\), respectively. But \(V(K')V(K'^*) = \pi^2 - (\pi \cdot 8\sqrt{2} + o(1)) \varepsilon^{3/2}/3\), so \(K'\) is worse than \(K\). Similarly, for the general case the intersection of \(B^2\) with a regular triangle of sides at a distance \(1 - \varepsilon\) from \(o\) is probably worse than the body \(K := \{(x, y) \in B^2 \mid x \leq 1 - \varepsilon\}.\) Whether \(K_0\) and \(K'\) could be conjectured as optimal, is not clear: possibly some truncation of \(B^2\) with some other curve(s) can be better.
Let us review some basic formulas about polar bodies, due to Santaló and Meyer-Pajor. Let $M$ be a convex body in $R^d$. The support function of $M$ is

$$h_M(u) = \max_{x \in M} \langle u, x \rangle.$$ 

In particular, if $z \in \text{int} M$, then the support function of $M - z$ is $h_M(u) - \langle u, z \rangle$, and the radial function of $(M - z)^*$ at $u \in S^{d-1}$ is $(h_M(u) - \langle u, z \rangle)^{-1}$. It follows that (with $V(\cdot)$ denoting volume)

$$V((M - z)^*) = d^{-1} \int_{S^{d-1}} (h_M(u) - \langle z, u \rangle)^{-d} du.$$ 

Here $V((M - z)^*)$ is a strictly convex analytic function of $z \in \text{int} M$, that tends to infinity as $\text{dist} (z, \partial M) \to 0$. In fact, its second differential is a positive definite quadratic form. E.g.,

$$\left( \frac{\partial}{\partial x_1} \right) V((M - z)^*) = (d + 1) \int_{S^{d-1}} u_1^2 (h_M(u) - \langle z, u \rangle)^{-d} du > 0,$$

and the respective formula holds for the second derivative along any direction. Hence it has a unique minimum at some $s(M) \in \text{int} M$, which is called the Santaló point $s(M)$ of $M$. Differentiation yields that

$$\int_{S^{d-1}} \frac{u}{(h_M(u) - \langle s(M), u \rangle)^{d+1}} du = o.$$ 

Thus $\int_{(M - s(M))^*} y dy = o$, and hence $o$ is the center of mass of $(M - s(M))^*$. This implies that

$$-(M - s(M))^* \subset d(M - s(M))^*,$$ 

hence

$$-(M - s(M)) \subset d(M - s(M)).$$

(0)

There are known several statements about stability of the Santaló point, or behaviour of $V(K - z)^*$ for $z$ close to the Santaló point $s(K)$ of $K$. See e.g., Santaló [San], 2, pp. 156-157, Kim-Reisner [KR], Propositions 1 and 2, and the first part of this paper [BMMR], Lemma 11, first statement (about $c_1(K_0)$ — the second statement, about $c_2(K_0)$, will not be needed here).

Now we cite [BMMR], Lemma 11, first part. [BMMR], Lemma 11 becomes correct.) This is a more explicit version of Kim-Reisner [KR], Proposition 1, inasmuch the constants are explicitly given. This will render it possible to give estimates with absolute constants (for fixed dimension).
Lemma E'. ([BMMR], Lemma 11, first part) Let \( d \geq 2 \) be an integer, \( K_0 \subset \mathbb{R}^d \) be a convex body, and let \( 0 < \varepsilon_1 \leq \varepsilon_1(K_0) := \min \{ 1/2, 2^{-2d-1}(\kappa_{d-1}/d\kappa_d^2) \} \cdot |K_0|/(\text{diam } K_0)^d \}. \) Let \( K \subset \mathbb{R}^d \) be a convex body, and let

\[
(1 - \varepsilon_1)K_0 + a \subset K \subset (1 + \varepsilon_1)K_0 + b, \text{ where } a, b \in \mathbb{R}^d.
\]

Additionally, let (E2) from above hold. Then

\[
\|s(K) - s(K_0)\| \leq c(K_0) \cdot \varepsilon_1,
\]

where

\[
c(K_0) := (\text{diam } K_0)^{d+1} |K_0|^{-d-2} \cdot d(d\kappa_d/\kappa_{d-1})^{d+2}.
\]

Theorem E. Let \( d \geq 2 \) be an integer. Then there exists an \( \varepsilon(d) > 0 \) such that for all \( \varepsilon \in [0, \varepsilon(d)) \) the following holds. Let \( K_0, K \subset \mathbb{R}^d \) be convex bodies, and let

\[
(1 - \varepsilon)K_0 + a \subset K \subset (1 + \varepsilon)K_0 + b,
\]

where \( a, b \in \mathbb{R}^d \) and

\[
[((1 - \varepsilon)K_0 + a) + ((1 + \varepsilon)K_0 + b)]/2 = K_0.
\]

Then we have \( s(K) \in \text{int } K_0 \), and

\[
\begin{align*}
V((K_0 - s(K))^*) - V((K_0 - s(K_0))^*) & \leq \varepsilon, \\
2^{2d^2+3d+1}d^{2d^2+6d+9} \kappa_{d-1}^{-2d-4} \kappa_d(d + 1)^{d+2} \cdot \varepsilon^2.
\end{align*}
\]

Proof. We proceed on the lines of Santaló [San] and Kim-Reisner [KR], Proposition 2 and [BMMR], Lemma 11, second part.

Observe that the statement of Theorem E is affine invariant. Therefore we may suppose, by John’s theorem, that \( B^d \subset K_0 \subset dB^d \). Then (E3) remains valid if in the expression of \( c(K_0) \) in Lemma E’ we replace \( |K_0| \) with its lower bound \( \kappa_d \) and \( \text{diam } K_0 \) by its upper bound \( 2d \), obtaining

\[
\|s(K) - s(K_0)\| \leq c \cdot \varepsilon,
\]

where

\[
c = 2^{(d+1)^2}d^{d^2+3d+4} \kappa_{d-1}^{-d-2}.
\]

The minimum of \( f(z) := V((K_0 - x)^*) \) for \( z \in \text{int } K_0 \) is attained for \( z = s(K_0) \), the Santaló point of \( K_0 \). The function \( f(z) \) is analytic, hence has a power series
expansion at \( s(K_0) \), convergent in any open ball of centre \( s(K_0) \) and contained in \( \text{int} \, K_0 \). Let us suppose that \( s(K) - s(K_0) \) is non-zero (else we have nothing to prove), and it points, say, to the direction of the positive \( x_d \)-axis. The first derivatives of \( f(z) \) vanish at \( z = s(K_0) \), hence

\[
\begin{align*}
V((K_0 - s(K))^*) - V((K_0 - s(K_0))^*) = \\
&= f(s(K)) - f(s(K_0)) = (1/2)\|s(K) - s(K_0)\|^2 \cdot (\partial/\partial x_d)^2 f(x) = \\
&= (1/2)\|s(K) - s(K_0)\|^2 \cdot (d+1) \int_{S_{d-1}} u_0^2 (h_{K_0} - \langle u, x \rangle)^{-d-2} du,
\end{align*}
\]

where \( x \) lies in the relative interior of the segment \([s(K_0), s(K)]\), and hence \( \|x - s(K_0)\| < \|s(K) - s(K_0)\| \). We estimate \((\partial/\partial x_d)^2 f(x) \) in (E6). We estimate \( u_0^2 \) from above by 1. Then still we have to estimate \( h_{K_0} - \langle u, x \rangle \) from below by some positive number. We have

\[
h_{K_0}(u) - \langle u, x \rangle = (h_{K_0}(u) - \langle u, s(K_0) \rangle) + (\langle u, s(K_0) \rangle - \langle u, x \rangle).
\]

So we have to estimate from below \((h_{K_0}(u) - \langle u, s(K_0) \rangle)\) and to estimate from above \(\langle u, s(K_0) \rangle - \langle u, x \rangle\).

For the first estimate we recall (0):

\[
K_0 - s(K_0) \subset -d(K_0 - s(K_0)),
\]

or, equivalently,

\[
h_{K_0 - s(K_0)} \leq dh_{s(K_0) - K_0}.
\]

This last inequality can be interpreted geometrically as follows. Take any supporting parallel strip of \( K_0 \). Then for the (positive) distances of \( s(K_0) \) from the two boundary hyperplanes of the parallel strip we have that their quotient is at most \( d \), or, otherwise said, the distance of \( s(K_0) \) to any of these boundary hyperplanes cannot be less than \( 1/(d+1) \) times the width of the parallel strip. We apply this for \( K_0 \). Thus the distance of \( s(K_0) \) to any supporting hyperplane of \( K_0 \) cannot be less than \( 1/(d+1) \) times the minimal width of \( K_0 \), which width is by \( K_0 \supset B^d \) at least 2. Returning to the original formulation, we have that

\[
h_{K_0} - \langle s(K_0), u \rangle \geq 2/(d+1).
\]

For the second estimate we have, by \( \|u\| = 1 \),

\[
\begin{align*}
|\langle u, x \rangle - \langle u, s(K_0) \rangle| &= \\
|\langle u, x - s(K_0) \rangle| &\leq \|x - s(K_0)\| < \|s(K) - s(K_0)\| \leq c_1 \varepsilon < c_1 \varepsilon(d).
\end{align*}
\]
Suppose that
\[ (E11) \quad c_1 \varepsilon(d) \leq 1/(d + 1). \]

Then, using (E7), (E9) and (E10), we get
\[ (E12) \quad \begin{cases} h_{K_0}(u) - \langle u, x \rangle = (h_{K_0}(u) - \langle u, s(K_0) \rangle) + \\ (\langle u, s(K_0) \rangle - \langle u, x \rangle) \geq 2/(d + 1) - 1/(d + 1) = 1/(d + 1). \end{cases} \]

This implies, by (E6), (E12) and (E4), (E5), that
\[ (E13) \quad \begin{cases} V[(K - s(K))^\ast] - V[(K - s(K_0))^\ast] = (1/2)\|s(K) - s(K_0)\|^2. \\ \int_{S^{d-1}} u_d^2 (h_{K_0}(u) - \langle u, x \rangle)^{-d-2} du \leq \\ (1/2)c^2 \varepsilon^2 \cdot d\kappa_d[1/(d + 1)]^{-d-2}, \end{cases} \]

and this is equivalent to the inequality of the Theorem.

**Optimality of regular n-gons**

We introduce some notation that will be used in the next section, as well. For \( x \neq y \in \mathbb{R}^2 \), we write \( \text{aff} \{ x, y \} \) for the line passing through \( x \) and \( y \). For compact sets \( X_1, \ldots, X_k \) in \( \mathbb{R}^2 \), let \([X_1, \ldots, X_k]\) denote the convex hull of their union. For any point \( p \neq o \), let \( p^\ast \) be the polar line with equation \( \langle x, p \rangle = 1 \). For \( 0 \neq l = p^\ast \), let \( p = l^\ast \).

**Theorem F.** Among convex polygons \( K \) of at most \( n \) vertices whose Santaló point is \( o \), the ones maximizing the area product \( V(K)V(K^\ast) \) are the non-singular linear images of regular \( n \)-gons with centre at \( o \).

**Proof.** As all triangles are affine images of each other, we may assume \( n \geq 4 \). By Theorem 7 we may assume even \( n \geq 5 \). Observe that the area product of a regular \( n \)-gon, i.e., \( n^2 \sin^2(\pi/n) \) is strictly increasing with \( n \). Therefore, using induction w.r.t. \( n \), with base of induction \( n = 4 \), we may restrict our attention to convex polygons with exactly \( n \) sides. By the intersection point of two distinct parallel lines, we mean their common point of infinity (in the projective plane).

Let \( K \) maximize the area product among convex polygons of at most \( n \) vertices, whose Santaló point is \( o \). We prove two basic properties of \( K \):

(i) If \( x_1, x_2, x_3 \) are consecutive vertices of \( K \), then \( o, \frac{1}{2}(x_1 + x_3) \) and \( x_2 \) are collinear.

(ii) If \( x_1, x_2, x_3, x_4 \) are consecutive vertices of \( K \), and \( m \) is the intersection point of \( \text{aff} \{ x_1, x_2 \} \) and \( \text{aff} \{ x_3, x_4 \} \), then \( o, (x_2 + x_3)/2 \) and \( m \) are collinear. Here \( m \) is meant as a point of the projective plane.

To prove these claims, we show with a method going back to Mahler that if either (i) or (ii) does not hold, then \( K \) can be deformed into a convex polygon with the same number of vertices but with a larger area product. We write \( \gamma_1, \gamma_2, \ldots \).
for positive constants that depend on \(K\), but not on its deformations.

First we suppose that (i) does not hold, and seek a contradiction. For \(i = 1, 3\), we write \(y_{1+i/2}\) for the intersection of \(x_{i}^*\) and \(x_{2}^*\), and hence \(y_{1.5}\) and \(y_{2.5}\) are consecutive vertices of \(K^*\). Let \(l\) be the line through \(x_{2}\) parallel to aff \(\{x_{1}, x_{3}\}\), and hence \(l^* \in [y_{1.5}, y_{2.5}]\). We consider two cases depending on the position of \(o\) with respect to the triangle \([x_{1}, x_{2}, x_{3}]\).

**Case 1.** \(o \not\in \text{int} [x_{1}, x_{2}, x_{3}]\)

We may assume that \(x_{1}\) and \(x_{2}\) lie on the same side of aff \(\{o, (x_{1} + x_{3})/2\}\) if \(o \not\in [x_{1}, x_{2}, x_{3}]\) (and hence \(o \not\in [x_{1} x_{2} x_{3}]\)), or \(\|x_{1}\| < \|x_{3}\|\) if \(o \in [x_{1}, x_{3}]\). In particular,

\[\|l^* - y_{1.5}\| > \|l^* - y_{2.5}\|.

We move \(x_{2}\) into a position \(\tilde{x}_{2}\) where \(\tilde{x}_{2} - x_{2} = \varepsilon(x_{3} - x_{1})\) for small \(\varepsilon > 0\), where \(\varepsilon\) is small enough to ensure that \(x_{1}\) and \(\tilde{x}_{2}\) lie on the same side of the line containing the other side of \(K\) with endpoint \(x_{3}\), and if \(o \not\in [x_{1}, x_{2}, x_{3}]\), then still \(x_{1}\) and \(\tilde{x}_{2}\) lie on the same side of aff \(\{o, (x_{1} + x_{3})/2\}\). Therefore there exists a convex polygon \(\tilde{K}\) obtained from \(K\) by removing \(x_{2}\) from the set of vertices, and adding \(\tilde{x}_{2}\). Clearly \(V(\tilde{K}) = V(K)\).

For \(i = 1, 3\), let \(\tilde{y}_{(i+2)/2}\) be the intersection of \(x_{i}^*\) and \(\tilde{x}_{2}^*\), and hence \(\tilde{y}_{1.5}\) and \(\tilde{y}_{2.5}\) are the two new vertices of \(\tilde{K}^*\) replacing the vertices \(y_{1.5}\) and \(y_{2.5}\) of \(K^*\). In addition, \(\tilde{y}_{1.5} \not\in K^*\) and \(\tilde{y}_{2.5} \in K^*\), because \(o \not\in \text{int} [x_{1}, x_{2}, x_{3}]\). Writing \(\alpha := \angle x_{2}o \tilde{x}_{2}\) for the angle of \(x_{2}\) and \(\tilde{x}_{2}\), we have \(\gamma_{1}\varepsilon < \alpha < \gamma_{2}\varepsilon\). Since

\[V(\tilde{K}^*) - V(K^*) = V([l^*, y_{1.5}, \tilde{y}_{1.5}]) - V([l^*, y_{2.5}, \tilde{y}_{2.5}])\]

and \(\alpha\) is the angle of both triangles \([l^*, y_{1.5}, \tilde{y}_{1.5}]\) and \([l^*, y_{3}, \tilde{y}_{3}]\) at \(l^{*}\), we have

\[V(\tilde{K}^*) - V(K^*) > \frac{(||l^* - y_{1.5}||^2 - ||l^* - y_{2.5}||^2) \sin \alpha}{2} - \gamma_{3}\alpha^2 > \gamma_{4}\varepsilon.

It follows by Theorem E that \(V((\tilde{K} - s(\tilde{K}^*)) > V(\tilde{K}^*) - \gamma_{5}\varepsilon^2\). Therefore if \(\varepsilon > 0\) is small enough, then \(V((\tilde{K} - s(\tilde{K}))^*) > V(K^*) + \gamma_{6}\varepsilon > V(K^*),\) which is a contradiction, and thus we have proved (i) in Case 1.

**Case 2.** \(o \in \text{int} [x_{1}, x_{2}, x_{3}]\)

We may assume that \(x_{1}\) and \(x_{2}\) lie on the same side of aff \(\{o, (x_{1} + x_{3})/2\}\), and hence

\[\|l^* - y_{1}\| < \|l^* - y_{3}\|.

We move \(x_{2}\) into a position \(\tilde{x}_{2}\) where \(\tilde{x}_{2} - x_{2} = \varepsilon(x_{1} - x_{3})\) for small \(\varepsilon > 0\), and \(\varepsilon\) is small enough to ensure that there exists a convex polygon \(\tilde{K}\) obtained from \(K\) by removing \(x_{2}\) from the set of vertices, and adding \(\tilde{x}_{2}\). Clearly \(V(\tilde{K}) = V(K)\).

For \(i = 1, 3\), let again \(\tilde{y}_{(i+2)/2}\) be the intersection of \(x_{i}^*\) and \(\tilde{x}_{2}^*\), and hence \(\tilde{y}_{1.5}\) and \(\tilde{y}_{2.5}\) are the two new vertices of \(\tilde{K}^*\) replacing the vertices \(y_{1.5}\) and \(y_{2.5}\) of \(K^*\). In this case \(\tilde{y}_{1.5} \in K^*\) and \(\tilde{y}_{2.5} \not\in K^*\), thus
\[ V(\tilde{K}^*) - V(K^*) = V([l^*, y_{2.5}, \tilde{y}_{2.5}]) - V([l^*, y_{1.5}, \tilde{y}_{1.5}]). \]

Now the argument can be finished as in Case 1, completing the proof of (i).

Next we suppose that (ii) does not hold. We may assume that

\[ (*) \quad \begin{cases} [x_1, x_2] \text{ and } p = (x_2 + x_3)/2 \text{ lie on the same side} \\ \text{of the line connecting } o \text{ and } m. \end{cases} \]

There are three cases. The point \( m \) can be a finite point, lying on the other side of \( \text{aff} \{x_2, x_3\} \) as \( K \), or can be a point at infinity, or can be a finite point lying on the same side of \( \text{aff} \{x_2, x_3\} \) as \( K \). Let \( l_{1.5} = \text{aff} \{x_1, x_2\} \), \( l_{3.5} = \text{aff} \{x_3, x_4\} \) and \( l = \text{aff} \{x_2, x_3\} \). We observe that if \( m \in \mathbb{R}^2 \), then \( \langle m, l_{1.5}^* \rangle = \langle m, l_{3.5}^* \rangle = 1 \), and also \( \langle 0, l_{1.5}^* \rangle = \langle 0, l_{3.5}^* \rangle = 0 \). Therefore the assumption on \( p \) yields that

\[ (**): \quad \langle p, l_{1.5}^* \rangle > \langle p, l_{3.5}^* \rangle \]

holds independently whether \( m \in \mathbb{R}^2 \), or \( m \) is a point at infinity.

We rotate \( l \) about \( p \) into a new position \( \tilde{l} \) through a small angle \( \varepsilon \in (0, \pi/2) \), in a way such that \( \tilde{l} \) intersects \( [x_3, x_4] \setminus \{x_4\} \) in a point \( \tilde{x}_3 \), but intersects \( l_{1.5} \) in a point \( \tilde{x}_2 \) lying outside of \( K \). In particular,

\[ ||\tilde{l}^* - l^*|| > \gamma_6 \varepsilon. \]

Let \( \tilde{K} \) be the convex polygon obtained from \( K \) by removing \( x_2 \) and \( x_3 \) from the set of vertices, and adding \( \tilde{x}_2 \) and \( \tilde{x}_3 \). Then

\[ V(\tilde{K}) = V(K) + V([p, x_2, \tilde{x}_2]) - V([p, x_3, \tilde{x}_3]) > V(K) - \gamma_7 \varepsilon^2. \]

Now \( \tilde{K}^* \) is obtained from \( K^* \) by removing \( l^* \) from the set of vertices, and adding \( \tilde{l}^* \). Here \( l^*, \tilde{l}^* \in p^* \), in all the three cases for \( m \), i.e., if it is a point at infinity, or it is a finite point, lying on either side of \( \text{aff} \{x_2, x_3\} = l \). We have three different figures, but the following calculations are valid for each of these three figures. Supposing that the vertices of \( K \) follow each other in the positive sense, then assumption \((*)\) on \( p \) yields \( \langle (l_{3.5}^* - l_{1.5}^*) \times (\tilde{l}^* - l^*), (0, 0, 1) \rangle > 0 \), hence by \((**)\) actually

\[ \langle (l_{3.5}^* - l_{1.5}^*) \times (\tilde{l}^* - l^*), (0, 0, 1) \rangle > \gamma_8 \varepsilon \]

(observe that the angle of \( m^* \) and \( p^* \) only depends on \( K \), but not on \( \varepsilon \)). Therefore
\[ V(\tilde{K}^*) - V(K^*) = \langle (l_{3,5}^* - l_{1,5}^*) \times (\tilde{l}^* - l^*), (0, 0, 1) \rangle / 2 > \gamma_9 \varepsilon. \]

Furthermore, \( V((\tilde{K} - s(\tilde{K}))^*) > V(\tilde{K}^*) - \gamma_{10} \varepsilon^2 \) by Theorem E. We conclude that

\[
\begin{align*}
& V(\tilde{K}) V((\tilde{K} - s(\tilde{K}))^*) > (V(K) - \gamma_7 \varepsilon^2) (V(K^*) - \gamma_{10} \varepsilon^2) \\
& > (V(K) - \gamma_7 \varepsilon^2) (V(K^*) + \gamma_9 \varepsilon - \gamma_{10} \varepsilon^2) \\
& > V(K) V(K^*) + \gamma_{11} \varepsilon > V(K) V(K^*),
\end{align*}
\]

provided \( \varepsilon \in (0, \pi/2) \) is small enough. This contradiction proves (ii).

Now we prove Theorem F based on (i) and (ii). Let \( x_1, \ldots, x_k \), with \( 3 \leq k \leq n \) be the vertices of \( K \) in this order. By the beginning of the proof of this theorem we have \( k = n \geq 5 \). By \( n \geq 5 \) we have that the average value of \( \angle x_{i-1} ox_i + \angle x_i ox_{i+1} \) is \( 4\pi/n < \pi \) (indices meant cyclically). Let, e.g., \( \angle x_1 ox_2 + \angle x_2 ox_3 < \pi \). (Observe that this property is invariant under linear maps.) Applying a linear transformation, we may assume that \( x_1, x_2, x_3 \in S^1 \) and hence \( x_2 \) lies on the perpendicular bisector of \([x_1, x_3]\) by (i). Applying another linear transformation, we may also assume that \( x_1, x_2, x_3 \in S^1 \) — actually, they lie in an open half-circle.

Now (ii) yields that \( \text{aff} \{x_1, x_2\} \) and \( \text{aff} \{x_3, x_4\} \) are symmetric with respect to the perpendicular bisector of \([x_2, x_3]\), and \( m \) is a finite point separated from \( K \) by \( l \).

It follows that \( \text{aff} \{x_2, x_3\} \) and \( \text{aff} \{x_3, x_4\} \) are symmetric with respect to \( \text{aff} \{o, x_3\} \).

Therefore (i) yields that \( x_4 \in S^1 \), with \( \|x_4 - x_3\| = \|x_3 - x_2\| = \|x_2 - x_1\| \).

Continuing like this, we conclude that \( K \) is a regular \( k \)-gon inscribed into \( B^2 \).

From the first paragraph of the proof we have \( k = n \). ■

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