$S^1 \times S^2$ wormholes and topological charge*

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I investigate solutions to the Euclidean Einstein-matter field equations with topology $S^1 \times S^2 \times R$ in a theory with a massless periodic scalar field and electromagnetism. These solutions carry winding number of the periodic scalar as well as magnetic flux. They induce violations of a quasi-topological conservation law which conserves the product of magnetic flux and winding number on the background spacetime. I extend these solutions to a model with stable loops of superconducting cosmic string, and interpret them as contributing to the decay of such loops.

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1. Introduction

One of the factors driving the development of the wormhole formalism has been the existence of classical wormhole solutions to the Euclidean equations of motion for gravity and matter. A wormhole solution is typically defined as a solution that is asymptotic to two distinct flat spacetimes. A configuration of wormhole topology that is not a solution to the equations of motion is less convincing evidence of topology change than is a classical solution, as the classical solution may give the dominant contribution to some quantum-mechanical amplitude in the semiclassical limit. Since this is why one considers classical solutions, it is reasonable to ask to which quantum-mechanical amplitudes a proposed wormhole solution will give the dominant contribution.

One could propose that the dominant contribution is to an amplitude for the creation or annihilation of a baby universe. Since this process is expected to be unobservable for low-energy processes on a background spacetime, one suspects that this is not a measurable amplitude, and therefore not a physically relevant calculation. One can instead require, however, that the wormhole mediate some process that otherwise could not take place in the underlying field theory. In a field theory with some conserved global charge, charge violation is just such a process.

The first wormhole solutions found by Giddings and Strominger\[1\] carry flux associated with a three-index antisymmetric tensor field, or axion. These wormholes thus violate axion charge conservation. Lee showed\[2\] how to represent this in terms of a massless scalar field dual to the three-index tensor,

\[ H_{\mu\nu\lambda} = \epsilon_{\mu\nu\lambda\sigma} \partial^\sigma a. \]

The effect of wormholes can then be represented by operators that explicitly break the symmetry \( a \to a + c \). In order to find the solutions for the scalar field, or for the more general case of a complex scalar field with a \( U(1) \) global symmetry,\[3\] one must constrain the initial and final states to be states of definite charge. This can result in certain terms in the equations of motion changing signs, the net result being that the solutions for the three-index tensor theory are identical to the solutions for the dual scalar field theory, which would not be the case if the equations of motion were applied naively.
Wormhole solutions have also been found in 3-dimensional electromagnetism by Hosoya and Ogura; these solutions carry magnetic flux down the wormhole throat. These solutions are really the direct 3-dimensional analog of the 4-dimensional Giddings-Strominger axionic wormholes. In both cases, the charge that supports the wormhole throat is topologically conserved. This means that the current conservation equation is an identity when expressed in terms of the gauge potential. In three dimensions, the magnetic flux current is

$$ j^\lambda = \epsilon^{\lambda\mu\nu} F_{\mu\nu}. $$

The flux conservation equation, $\partial_\mu j^\mu = 0$, is an identity when $F$ is written in terms of the gauge potential $A_\mu$.

The effects of these wormholes will be similar to the effects of any finite-action monopole solutions that may exist in the theory. In a three-dimensional theory, a monopole solution can be thought of as an instanton that mediates processes that violate magnetic flux conservation. One can express the effect of such violation in terms of a scalar field dual to $F$, just as in the four-dimensional case one can express the effects of axion charge non-conservation in terms of a scalar field dual to $H$.

In this work, I generalize the magnetic wormhole to four-dimensional electromagnetism. In four dimensions, one still has magnetic flux conservation, in that magnetic flux lines cannot end. (On a spatial slice, $\nabla \cdot B = 0$.) Loops of magnetic flux can, however, shrink to nothing. This can be avoided by giving the wormholes the topology $S^2 \times S^1 \times R$, so that magnetic flux on the two-sphere can wind around the circle. I will also put a topologically conserved charge on $S^1$, the winding number for a periodic scalar field.

I will then discuss the effects of such wormholes. I believe that while insertions of the usual $S^3 \times R$ wormholes induce pointlike operators at low energy, $S^1 \times S^2 \times R$ wormholes induce looplike operators. I will discuss the consequences of this, in particular in a model with stable loops of superconducting cosmic string.

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1 "$n$-dimensional" will refer to $n$-dimensional Euclidean spacetime, with $n-1$ space dimensions and one Euclidean time dimension.

2 Keay and Laflamme have also constructed wormhole solutions of the same topology in the dual of this theory. These wormholes have axion charge rather than winding number.
2. Solutions

I construct wormhole solutions in a theory that includes the electromagnetic field and a massless periodic scalar field (the axion) coupled to gravity. These solutions will have topology $S^1 \times S^2 \times \mathbb{R}$. The periodic scalar has a topological charge associated with it, the winding number on $S^1$. For the electromagnetic field, magnetic flux on the two-sphere is conserved; this conservation law is topological both in the sense that the current conservation law is an identity, and in the sense that the flux is a topological invariant of the two-sphere if charged fields are added to the theory.

The Euclidean action for this theory is given by

$$S = \int d^4 x \sqrt{g} \left[ -\frac{1}{16\pi G} R + \frac{v^2}{2} g^{\mu\nu} \partial_\mu \Theta \partial_\nu \Theta + \frac{1}{4e^2} g^{\mu\nu} g^{\lambda\sigma} F_{\mu\lambda} F_{\nu\sigma} \right].$$

Here $v$ is the axion symmetry-breaking scale; $\Theta$ is a periodic scalar field with period $2\pi$. Now the simplest possible ansatz for a wormhole solution with the desired features will have a Euclidean Kantowski-Sachs geometry: each spatial slice will be homogeneous and characterized by the radius of the two-sphere, the radius of the circle, and the topological charges associated with each. The metric for this is

$$ds^2 = N^2(\tau) d\tau^2 + a^2(\tau) d^2 + b^2(\tau) d\Omega^2,$$

where $l$ is a periodic coordinate with period 1, and $d\Omega^2$ is the solid angle element on $S^2$, $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$. By reparameterizing $\tau$, the lapse function $N^2(\tau)$ can be set to any strictly positive function; here I set it to unity. The quantities $a$ and $b$ are the radii of the circle and the two-sphere, respectively.

The field equation for $\Theta$ is

$$\partial_\mu [\sqrt{g} g^{\mu\nu} \partial_\nu \Theta] = 0.$$

I impose the restriction that $\Theta$ is a function of $l$ and $\tau$ only, and that derivatives of $\Theta$ are functions of $\tau$ only:

$$\Theta = T_l(\tau) l + T_0(\tau).$$

Since $\Theta$ must have an integral winding number on the circle, it must satisfy the boundary conditions

$$\Theta(l, \tau) = \Theta(l + 1, \tau) - 2\pi n,$$
and thus
\[ \Theta = 2\pi nl + T_0(\tau). \]

I am interested in the case \( n \neq 0 \), and in this case, \( T_0(\tau) \) must be constant to avoid off-diagonal terms in the energy-momentum tensor. I eliminate constant \( T_0 \) by shifting \( \Theta \), so
\[ \Theta = 2\pi nl. \]

For the electromagnetic field there is a field equation and a Bianchi identity:
\[ \partial_\mu[\sqrt{g}F^{\mu\nu}] = 0 \quad \text{and} \quad F_{\mu\nu,\rho} + F_{\nu\rho,\mu} + F_{\rho\mu,\nu} = 0. \]

For a purely magnetic solution, \( F^{0i} = 0 \); therefore,
\[ \partial_\mu[\sqrt{g}F^{\mu\nu}] = \partial_0[\sqrt{g}F^{0\nu}] = 0. \]

Thus the field equation is automatically satisfied. For a homogeneous magnetic field on the two-sphere, the correct ansatz (in the coordinate basis) is
\[ F_{\theta\phi} = -F_{\phi\theta} = \frac{\Phi(\tau)}{4\pi}\sin \theta. \]

The Bianchi identity gives \( \partial_\tau F_{\theta\phi} = 0 \), and thus \( \Phi(\tau) = \Phi_0 \), a constant. \( \Phi_0 \) is the conserved magnetic flux on the two-sphere.

The conservation laws allow one to solve the matter field equations quite directly; the only non-trivial equations are Einstein’s equations, which are
\[ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}. \]

For the given field content, the energy-momentum tensor is
\[ T_{\mu\nu} = v^2[\partial_\mu \Theta \partial_\nu \Theta - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta} \partial_\alpha \Theta \partial_\beta \Theta] + \frac{1}{c^2}[g^{\alpha\beta}F_{\alpha\mu}F_{\beta\nu} - \frac{1}{4}g_{\mu\nu}g^{\alpha\beta}g^{\lambda\sigma}F_{\alpha\lambda}F_{\beta\sigma}], \]
where I include the metric explicitly. Substituting the ansatz for the metric and the solutions to the matter equations, Einstein’s equations reduce to the following three equations:
\[ \frac{2\dot{a}\dot{b}}{ab} + b^2 - \frac{1}{b^2} = -\frac{Q_1^2}{a^2} - \frac{Q_2^2}{b^4}, \quad (2.1a) \]
\[ \frac{2\ddot{b}}{b} + \frac{\dot{b}^2}{b^2} - \frac{1}{b^2} = \frac{Q_1^2}{a^2} - \frac{Q_2^2}{b^4}, \quad (2.1b) \]
\[ \frac{\ddot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{a}\dot{b}}{ab} = -\frac{Q_1^2}{a^2} + \frac{Q_2^2}{b^4}, \quad (2.1c) \]
where \( \dot{} \equiv \frac{d}{d\tau} ( ) \), \( Q_1^2 = 8\pi G (2\pi^2 n^2 v^2) \) and \( Q_2^2 = 8\pi G \left( \frac{d_0^2}{32\pi^2 c^2} \right) \). Equation (2.1d) is a constraint equation for the system of second-order differential equations defined by the other two; one can easily verify that it is conserved by equations (b) and (c). One can obtain either of equations (b) and (c) from equation (a) and the remaining equation, thus equations (b) and (c) are redundant and either can be eliminated without loss of generality.

Do wormhole solutions to these equations exist? What exactly do we mean by a wormhole solution? Unlike the case where the topology is \( S^3 \times R \), there will not be any asymptotically Euclidean solutions with topology \( S^2 \times S^1 \times R \). First, the ansatz itself cannot be asymptotic to \( R^4 \), since the topology forbids it. Second, while there are configurations asymptotic to flat \( R^3 \times S^1 \) (i.e., \( a \rightarrow \text{constant}, b \rightarrow \tau + \text{constant} \)), a simple argument shows that these cannot be solutions when \( Q_1 \neq 0 \): Since the circle goes to a constant radius, the energy density that is due to the winding of the scalar goes to a non-zero constant, and thus the curvature cannot go to zero as it must for a flat solution. So what do we mean by a wormhole? In this case, I will define a wormhole solution to be a solution such that a) there exists a small “throat” where both of the radii attain a minimum value, and b) some distance outside the throat both radii become much larger than they are near the throat and are “almost” of the form \( a = \text{constant}, b = \tau + \text{constant} \). I will demonstrate the existence of such solutions.

The first thing to determine is whether a wormhole throat can form. The condition for a wormhole throat (at a particular value of \( \tau \)) is that \( \dot{a} = \dot{b} = 0 \) and \( \ddot{a}, \ddot{b} \geq 0 \). Is this consistent with the equations? Setting \( \dot{a} \) and \( \dot{b} \) to zero, constraint equation (2.1d) gives relation between \( a \) and \( b \) at the throat:

\[
\frac{Q_1^2}{a^2} = \frac{1}{b^2} - \frac{Q_2^2}{b^4}.
\]

(2.2)

Using this relation and equation (2.1d) one has

\[
\frac{\ddot{b}}{b} = \frac{Q_1^2}{a^2}
\]

at the throat, thus \( \ddot{b} \) will always be greater than zero. Finally, using the previous two relations and (2.1d), at the throat,

\[
\frac{\ddot{a}}{a} = \frac{3Q_2^2}{b^4} - \frac{2}{b^2}.
\]
To make $\ddot{a}$ positive at the throat requires $Q_2^2/b^2 > 2/3$. Note also that equation (2.2) implicitly requires that $Q_2^2/b^2 < 1$ at the throat. At the throat, then, $Q_2^2/b^2$ is a free parameter, which must satisfy

$$1 > \frac{Q_2^2}{b^2} > \frac{2}{3},$$

(2.3)

and one can calculate all other quantities from this (and of course, from $Q_1$ and $Q_2$).

First consider the case $Q_1 = 0$. In Euclidean space, the given ansatz is equivalent to static, spherically symmetric spacetime with periodically identified time, where the Euclidean time $\tau$ of the ansatz becomes the radial coordinate of the spherically symmetric spacetime. For $Q_1 = 0$, the solution to these equations is well known: it is the Euclidean magnetic Reissner-Nordstrøm solution. These solutions, of course, seem nothing like wormholes, but there is, in fact, a solution with a “throat”: 

$$b = b_0 \equiv Q_2,$$

$$a = a_0 \cosh(\tau/b_0).$$

The constant radius of the two-sphere is equal to the horizon radius of the extreme Reissner-Nordstrøm black hole. This solution is still nothing like a wormhole; I will therefore ignore it and concentrate on $Q_1 > 0$.

In the general case, $Q_1, Q_2 > 0$, I was unable to obtain analytic solutions. I did obtain some results by numerically integrating the system of ordinary differential equations (2.1). One need use only equations (2.1a) and (b), which give a first-order differential equation for $a(\tau)$ and a second-order differential equation for $b(\tau)$:

\[
\frac{da}{d\tau} = \frac{ab}{2b} \left( \frac{1}{b^2} - \frac{\dot{b}^2}{b^2} - \frac{Q_1^2}{a^2} - \frac{Q_2^2}{b^4} \right),
\]

(2.4a)

\[
\frac{d^2b}{d\tau^2} = \frac{b}{2} \left( \frac{1}{b^2} - \frac{\dot{b}^2}{b^2} + \frac{Q_1^2}{a^2} - \frac{Q_2^2}{b^4} \right).
\]

(2.4b)

I found numerical solutions by performing integrations with initial conditions set to values appropriate for a wormhole throat:

$$b(0) = b_0 \quad \left( \frac{2}{3} < \frac{Q_2^2}{b_0^2} < 1 \right)$$

$$a(0) = Q_1 b_0 \left( 1 - \frac{Q_2^2}{b_0^2} \right)^{-1/2}$$

$$\dot{b}(0) = \epsilon.$$
Note that $\dot{b}(0)$ is set to a small, non-zero value $\epsilon$. This is necessary, because when $\dot{b} = 0$, equation (2.1a) merely imposes the constraint (2.2) without fixing $\dot{a}$. Setting $\dot{b}$ to $\epsilon$ at $\tau = 0$ does set $\dot{a}(0)$ to (nearly) zero with $a(0)$ and $b(0)$ as given. I set $\epsilon$ small enough that computer test runs with the opposite sign for $\epsilon$ showed no significant difference. In general, equation (2.4a) will be undefined whenever $\dot{b}$ goes through zero. The integrator (Mathematica built-in) handled this without difficulty. Graphs of some of the results are displayed in Figures 1–3.

What are some of the general features of the solutions that one can see from the numerical results? Note first that one only need calculate solutions for fixed values of $Q_1$ and $Q_2$, here set to 1. This is possible because if the pair of functions $a(\tau)$ and $b(\tau)$ is a solution to equations (2.1) for any particular values of $Q_1$ and $Q_2$, there is a corresponding solution with $\tilde{Q}_1 = \lambda_1 Q_1$, $\tilde{Q}_2 = \lambda_2 Q_2$ given by

$$\tilde{a}(\tau) = \lambda_1 \lambda_2 \tilde{a}(\tau/\lambda_2)$$
$$\tilde{b}(\tau) = \lambda_2 b(\tau/\lambda_2).$$

(2.5)

The qualitative features of the solutions thus depend only on the parameter $Q_2^2/b_0^2$.

These scaling relations also yield the important relation between the charge carried by the wormhole and its size. For a fixed value of the parameter $Q_2^2/b_0^2$, one can find a solution $A(\tau), B(\tau)$, for $Q_1 = Q_2 = 1$; then the general solution is

$$a(\tau) = Q_1 Q_2 A(\tau/Q_2)$$
$$b(\tau) = Q_2 B(\tau/Q_2).$$

So the overall size of the solution is proportional to $Q_2$, while the length of the $S^1$ loop is also proportional to $Q_1$.

I performed numerical integrations that started at the wormhole throat with various values for the free parameter $Q_2^2/b_0^2$. When $Q_2^2/b_0^2$ is very close to 1, there is a wormhole-like solution, where $a$ starts to rise very rapidly for a time and then levels off, seeming to approach a constant. The $S^2$ radius $b$ starts out fairly flat, then goes to a regime in which $\dot{b}$ is nearly 1. If one continues to integrate to much larger values of $\tau$, $a$ will reach a maximum and start decreasing, eventually collapsing to zero, while $b$ increases rapidly to infinity. When $Q_2^2/b_0^2$ is not close to 1, this “nearly flat” behavior never begins; instead, $a$ just reaches a maximum and then collapses while $b$ diverges — the only difference is that this happens much sooner, never allowing the solution to reach a nearly flat regime.
3. Wormhole Insertions and Topological Charge

What relevance do these wormhole solutions have for a theory of quantum gravity? In particular, what kind of effects does this type of wormhole have on low-energy physics in flat, four-dimensional spacetime? To determine this, one must first understand how these wormhole geometries of $S^1 \times S^2 \times R$ topology can attach to flat $R^4$.

The answer to this question is actually suggested by the solutions themselves. The metric given by $a(\tau) = \text{constant}$ and $b(\tau) = \tau$ is flat; flat $R^3 \times S^1$ has this metric with $\tau \in [0, \infty)$. The subset of this space given by $\tau \in [0, \tau_f]$, is flat $B^3 \times S^1$, where $B^3$ is the three-dimensional ball. There are $B^3 \times S^1$ subsets of $R^4$ that consist of a loop in spacetime with a neighborhood around it. The geometry of these subsets very nearly approximates that of the flat $B^3 \times S^1$ described above, at least in the limit where the loop (the $S^1$) is long and straight on the scale of the ball around it. After excising such a region from the background, there is a boundary left with topology $S^2 \times S^1$, to which one can attach the $S^2 \times S^1$ boundary of flat $B^3 \times S^1$, or any other geometry that approximates this near the boundary. The given wormhole solutions almost match this geometry in the $\dot{a} \approx 0, \dot{b} \approx 1$ regime. It should then be possible to perturb the geometries of the wormhole solution and the background spacetime in such a way that they can be patched together on the $S^1 \times S^2$ boundary. The geometry formed this way is “almost” a classical solution. I conjecture that in theories with appropriate matter content, there exists an exact classical solution which closely approximates this geometry. It is this solution which one should think of as the wormhole under discussion.

What happens in the background of such a solution? From the point of view of the background spacetime, the wormhole end appears as a small neighborhood around a closed curve $C$. As we follow the loop around, we find that the scalar field winds $n$ times. If we look at a three-dimensional slice that intersects the loop, we see magnetic flux coming out; or if our three-dimensional slice contains the loop, it changes the magnetic flux, as follows. Consider the magnetic flux lines in the background “time”-slice before and after the slice in which the loop sits. Since the magnetic flux from the loop changes sign between the “before” and “after” slices, one finds that the insertion of the wormhole end creates a loop of magnetic flux in the background. Thus the effect of inserting the wormhole end will be similar to the effect of insertions of an “t Hooft loop” operator. The ’t Hooft loop, $B(C)$,
is the analog in one extra dimension of the flux creation operator $\phi(x)$ described in [5] and [9]. Its action on a state (i.e., a time slice) is to perform a singular gauge transformation that has a non-zero winding number along a curve that links the loop. In the path integral, an insertion of $B(C)$ means that one integrates over gauge field configurations such that $C$ is the world line of a Dirac monopole singularity of the gauge field. There is no particular conservation law that forbids the formation of loops of magnetic flux. But recall that there is an axion winding number as we follow the loop around: The wormhole insertion creates a loop of flux with axion winding number. This “flux winding number” is topologically conserved, as follows.

Consider a loop of flux that winds in the manner described above. The axion winding number is given by

$$2\pi n = \oint_C \nabla \Theta \cdot dl,$$

where the line integral is along the loop. If one multiplies this by the flux, one has

$$2\pi n \Phi = \int B \cdot da \oint_C \nabla \Theta \cdot dl = \int d^3 x B \cdot \nabla \Theta.$$

This is a conserved charge, because $B \cdot \nabla = J^0$, where

$$J^\lambda = \epsilon^{\lambda\mu\nu\rho} F_{\mu\nu} \partial_\rho \Theta,$$

and $J^\lambda$ is an identically conserved current. So “flux winding” as defined here is a topologically conserved charge.

Unfortunately, it also happens to be zero! Note that because of flux conservation, $B \cdot \nabla = \nabla \cdot (B\Theta)$, so our charge is equal to the integral of $B\Theta$ over the two-sphere at infinity. Since there are no monopoles in the theory, this charge will be zero. In other words, if the field $\Theta$ is continuous everywhere, the winding number around any contractible loop, and therefore any loop in $\mathbb{R}^3$, is zero. This charge can be made non-trivial by allowing singularities in $\Theta$. This is perfectly natural if $\Theta$ is actually the phase of a complex scalar field $\phi$; the singularities of $\Theta$ are simply zeros of $\phi$. 

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4. Superconducting cosmic strings

One can gain a more complete understanding of the low energy effects of these wormholes in a theory with vortex solutions. Consider the simplest theory with bosonic superconducting cosmic strings.[10] This theory has two independent $U(1)$ gauge fields $R_\mu$ and $A_\mu$, and two complex scalar fields $\sigma$ and $\phi$ that are minimally coupled to $R$ and $A$, respectively. The scalar potential is such that $\phi$ has a vacuum expectation value, but $\sigma$ does not. Thus the $A$ gauge symmetry is realized in the Higgs phase, and the $R$ symmetry is realized in the Coulomb phase. The scalar potential also has the property that at the core of a Nielsen-Olesen vortex of the $A$ symmetry, $\sigma$ has a non-zero expectation value. Since $\sigma$ carries the charge of the unbroken gauge field $R_\mu$, the $\sigma$ condensate at the core of the string causes it to be a superconductor. Loops of this string carry persistent currents that are characterized by the winding number of the $\sigma$ field around the loop. There exist static solutions to the field equations of this theory known as “springs”[11] or “vortons,”[12] that consist of a loop of superconducting string prevented from collapsing by a persistent current and the magnetic field that it generates.

If one identifies the $F_{\mu\nu}$ of our wormhole solution with the field strength of the $A_\mu$ gauge field and identifies the $\Theta$ field of our wormhole solution with the phase of $\sigma$, then these vortons are carriers of exactly the topological charge defined. The theories do not match exactly, but in the limit where the radius of the two-sphere boundary of the wormhole is much less than the radius of the core of the string, one might expect that the wormhole solution can successfully patch on to the vorton solution. In the theory with superconducting cosmic strings, of course, the periodic scalar is coupled to a gauge field. Adding this gauge field to the theory explicitly may shed some additional light on the dynamics of the wormhole in this background.

The new theory has the action

$$S = \int d^4x \sqrt{g} \left[ -\frac{1}{16\pi G} R + \frac{\nu^2}{2} g^{\mu\nu}(\partial_\mu \Theta + R_\mu)(\partial_\nu \Theta + R_\nu) 
+ \frac{1}{4\pi^2} g^{\mu\nu} g^{\lambda\sigma} F_{\mu\lambda} F_{\nu\sigma} + \frac{1}{4e^2} g^{\mu\nu} g^{\lambda\sigma} G_{\mu\lambda} G_{\nu\sigma} \right],$$

3 The notation here for $A$ and $R$ is reversed from that of [10].
where $R_\mu$ is the gauge field that couples to $\Theta$, $G_{\mu\nu}$ is the field strength for $R$, and $e'$ is the coupling. In this model, the field equations for $F$ are unchanged. The field equation for $\Theta$ becomes:

$$\partial_\mu [\sqrt{g}g^{\mu\nu}(\partial_\nu \Theta + R_\mu)] = 0.$$  

The $G$ field equation is

$$\partial_\mu [\sqrt{g}G^{\mu\nu}] = (e'v)^2 \sqrt{g}(\partial^\nu \Theta + R^\nu).$$

The energy-momentum tensor (and thus the Einstein equations) will be given by adding a term for $G$, and by replacing $\partial_\mu \Theta$ with $\partial_\mu \Theta + R_\mu$ throughout.

Now consider the wormhole ansatz. The situation can be simplified somewhat by setting $\Theta$ to zero identically via a gauge transformation. Then any non-zero winding number for $\Theta$ around the $S^1$ becomes a non-zero Wilson loop for $R^\mu$. This will not, however, be constrained to be an integer. The ansatz for $R$ will be

$$R_\mu = 2\pi nf(\tau)\delta^l_\mu,$$

that is, only $R_l$ is non-zero and it only depends on $\tau$. With this ansatz, the $\Theta$ field equation is satisfied automatically. The field equation for $G$ becomes

$$\ddot{f} = \left(\frac{\dot{a}}{a} - \frac{2\dot{b}}{b}\right) \dot{f} + m^2 f. \quad (4.1)$$

The Einstein equations become

\begin{align}
\frac{2\dot{a}b}{ab} + \frac{\dot{b}^2}{b^2} - \frac{1}{b^2} &= -Q_1^2 \left(f^2 - \frac{\dot{f}^2}{m^2}\right) - \frac{Q_2^2}{b^4}, \quad (4.2a) \\
\frac{2\ddot{b}}{b} + \frac{\dot{b}^2}{b^2} - \frac{1}{b^2} &= Q_1^2 \left(f^2 + \frac{\dot{f}^2}{m^2}\right) - \frac{Q_2^2}{b^4}, \quad (4.2b) \\
\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab} &= -Q_1^2 \left(f^2 + \frac{\dot{f}^2}{m^2}\right) + \frac{Q_2^2}{b^4}, \quad (4.2c)
\end{align}

where I define $m = e'v$ and define the other quantities as before.

The conditions at the wormhole throat will be mostly unchanged. One still wants $\dot{a} = \dot{b} = 0$ and $\ddot{a} > 0, \ddot{b} > 0$. So that the throat will be time-symmetric, I impose the
additional requirement that $\dot{f} = 0$. Without loss of generality, I set $f_0 = 1$, since one can always rescale $Q_1$ to compensate. In this case, the throat conditions (2.2) and (2.3) will both hold unchanged. The scaling properties given by (2.5) will also hold, provided one scales $m$ as $m \to \lambda_2 m$. In other words, the $Q_1 = Q_2 = 1$ solutions are general, provided one holds $b_0/Q_2$ and $m/Q_2$ fixed as one varies $Q_1$ and $Q_2$.

I can now integrate the equations. I find that when $m$ is small, the solutions are not affected over timescales of order $m^{-1}$. When $m$ is large, however, $f$ grows rapidly and leads to collapse of the wormhole. An intermediate situation is shown in Figure 4.

I also attempt to further approximate the core of a superconducting cosmic string by letting the expectation value of the scalar field go to zero after some distance (i.e. outside the “core”). In this case the solution asymptotically approaches a flat, $a \to$ constant, $b \to \tau +$ constant solution. Indeed, if the expectation value goes to zero in a “nearly flat” regime before the collapse has set in, the relaxation occurs fairly rapidly. In such a region, we can approximate equation (4.1) as

$$\ddot{f} \simeq -\frac{2}{\tau} \dot{f},$$

for which $\dot{f} \sim \tau^{-2}$; thus $f$ relaxes rapidly to a constant, allowing the geometry to be asymptotically flat. This simple calculation shows that putting the solution in a slightly more physical context can improve the matching to the background geometry, and can increase one’s confidence that the hypothesis (that exact solutions of this approximate form exist) is correct.

Insertion of this wormhole solution will induce an operator that creates or destroys a vorton. One can visualize this in the following way: Imagine a loop of superconducting cosmic string propagating forward in time, leaving a world tube. A spacelike hypersurface intersects this tube in a loop, and it is along this loop (in the core of the vorton) that we insert the wormhole end. The vorton core has magnetic flux flowing along it, and has a winding number for the scalar field which is non-zero in the core. As discussed before, the wormhole insertion destroys (or creates) a unit of magnetic flux whenever the scalar field winding number is non-zero. This reduces (or increases) the flux-winding number product

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4 I chose the normalization of $f$ so that in the $m \to 0$ limit, $f(\tau) = 1$ gives solutions equivalent to the ones previously found.
which is the conserved charge of the vorton. Since the vorton is quantum-mechanically
unstable, the charge is not strictly conserved, but if the vorton lifetime is sufficiently long,
the wormhole may be the most important contribution to its decay. (One would expect
this to be the case when the additional action of inserting the wormhole solution is less
than the additional action of inserting an instanton in the world sheet of the string that
allows the winding number of the scalar field along the string to decrease. Note that this
decay, which changes the winding number, is not quite the same as the wormhole-mediated
decay, which changes the flux.) The wormhole contribution to the theory could then be
described in terms of an effective local field that describes the vorton degrees of freedom,
even though this somewhat obscures the non-local nature of the wormhole insertions.

Let me now ask which configurations dominate the path integral. I have found solu-
tions which, for a fixed value of the charges $Q_1$ and $Q_2$, can be patched to the background
spacetime along an arbitrarily long curve $C$ by adjusting the parameter $Q_2^2/b_0^2$ at the worm-
hole throat. In searching for the lowest action classical solution contributing to a given
process, however, one cannot fix the parameters at the throat — one can only fix param-
eters on the background. For example, if one is looking for the leading contribution to
a process which carries away a fixed magnetic flux with a fixed scalar winding number,
one should include only the lowest action solution for fixed $Q_1$ and $Q_2$. I expect that this
will be a circular wormhole solution of some fixed length; this is analogous to the stable
static vorton solution with similar parameters. One could also imagine actually searching
for processes which annihilate a vorton of a given size as well as charge, the dominant
contribution to which will be given by a wormhole of the appropriate size. The correct
wormhole solution always depends on the amplitude under consideration.

5. Conclusions

I have constructed wormhole solutions of topology $S^2 \times S^1 \times R$ in a theory with
electromagnetic fields and periodic scalar fields, as well as in a gauged version of this
theory. While not exactly realistic, this theory is a simple example of a theory with
topologically conserved charges on both the two-sphere and the circle. These wormhole
solutions do not fit the paradigm of having $S^3 \times R$ topology and asymptotic flatness in
both directions.
These wormholes may be sensibly interpreted in terms of effective ’t Hooft loop operators (or monopole loops) on the background spacetime, but this interpretation still leaves a number of loose ends. For example, it is an unproven hypothesis that the asymptotically flat solutions suggested by this work actually exist. Even if they do, they may not contribute to the Euclidean path integral in the simple form suggested.

Another issue concerns the Coleman-Lee solution to the large wormhole problem. Although these wormholes are indeed supported by a conserved charge, it is not obvious how this charge can be “drained away” by smaller wormholes. This might be easiest to see in terms of a field operator that creates or destroys vortons. But in this case, the global, topological aspects of the conserved charge seem to be lost.

Despite these difficulties, it seems quite likely that there are in fact solutions to the Euclidean Einstein equations that connect an asymptotically flat background to an $S^1 \times S^2$ baby universe. These wormhole solutions will give the dominant contributions to violation of vorton charge conservation, and thus must be considered on an equal basis with other wormhole contributions to low-energy physics. Thus any complete description of low-energy physics that includes wormholes must be able to reckon with solutions of the type I have constructed here.

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Fig. 1: $a(\tau)$ (solid) and $b(\tau)$ (dashed) for $Q_1 = Q_2 = 1$ and $Q_2^2/b_0^2 = 0.98$.

Fig. 2: $a(\tau)$ (solid) and $b(\tau)$ (dashed) for $Q_1 = Q_2 = 1$ and $Q_2^2/b_0^2 = 0.8$. The figure does not show it, but $b$ goes to infinity where $a$ goes to zero.

Fig. 3: $a(\tau)$ (top) and $b(\tau)$ (bottom) for $Q_1 = Q_2 = 1$ and $Q_2^2/b_0^2 = 0.999$. The maximum value of $a$ increases dramatically as $Q_2^2/b_0^2 \rightarrow 1$.

Fig. 4: The top figure shows $a(\tau)$ (solid) and $b(\tau)$ (dashed) for $Q_1 = Q_2 = 1$, $Q_2^2/b_0^2 = 0.98$, and $m = 0.1$. The bottom figure shows $f(\tau)$ for these parameters.