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Abstract

Let $G_\infty$ be a semisimple real Lie group with unitary dual $\hat{G}_\infty$. We produce new upper bounds for the multiplicities with which representations $\pi \in \hat{G}_\infty$ of cohomological type appear in certain spaces of cusp forms on $G_\infty$. The main new idea is to apply noncommutative Iwasawa theory to certain $p$-adic completions of the cohomology of locally symmetric spaces.

1. Introduction

Let $G_\infty$ be a semisimple real Lie group with unitary dual $\hat{G}_\infty$. The goal of this note is to produce new upper bounds for the multiplicities with which representations $\pi \in \hat{G}_\infty$ of cohomological type appear in certain spaces of cusp forms on $G_\infty$.

More precisely, we suppose that $G_\infty := G(\mathbb{R} \otimes_{\mathbb{Q}} F)$ for some connected semisimple linear algebraic group $G$ over a number field $F$. Let $K_\infty$ be a maximal compact subgroup of $G_\infty$. We fix an embedding $G \hookrightarrow \text{GL}_N$ for some $N$, and for any ideal $q$ of $\mathcal{O}_F$, we let $G(q)$ denote the intersection of $G_\infty$ with the congruence subgroup of $\text{GL}_N(\mathcal{O}_F)$ of full level $q$. We also fix an arithmetic lattice $\Gamma$ in $G_\infty$ (i.e., a subgroup commensurable with the congruence subgroups $G(q)$) and write $\Gamma(q) := \Gamma \cap G(q)$. For any $\pi \in \hat{G}_\infty$, let $m(\pi, \Gamma(q))$ denote the multiplicity with which $\pi$ occurs in the decomposition of the regular representation of $G_\infty$ on $L^2_{\text{cusp}}(\Gamma(q) \backslash G_\infty)$. Let $V(q)$ denote the volume of the arithmetic quotient $\Gamma(q) \backslash G_\infty$.

In terms of this notation, we may state our main results.

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THEOREM 1.1. Let \( p \) be a prime ideal in \( \mathcal{O}_F \). Let \( \pi \in \hat{G}_\infty \) be of cohomological type. Suppose either that \( G_\infty \) does not admit discrete series, or, if \( G_\infty \) admits discrete series, that \( \pi \) contributes to cohomology in degrees other than \( \frac{1}{2} \dim(G_\infty / K_\infty) \). Then

\[
m(\pi, \Gamma(p^k)) \ll V(p^k) \Gamma^{1/2} \dim(G_\infty) \text{ as } k \to \infty,
\]

with the implied constant depending on \( \epsilon \) and \( p \).

THEOREM 1.2. Let \( p \) be a prime ideal in \( \mathcal{O}_F \). Suppose \( W \) is a finite-dimensional representation of \( G_\infty \), and let \( \mathcal{V}_{W,k} \) denote the local system on \( \Gamma(p^k) \backslash G_\infty \) induced by \( W \) (assuming that \( k \) is taken large enough for \( \Gamma(p^k) \) to be torsion free). Let \( n \geq 0 \), and if \( G_\infty \) admits discrete series, then suppose furthermore that \( n \neq \frac{1}{2} \dim(G_\infty / K_\infty) \). Then

\[
\dim H^n(\Gamma(p^k) \backslash G_\infty, \mathcal{V}_{W,k}) \ll V(p^k) \Gamma^{1/2} \dim(G_\infty) \text{ as } k \to \infty,
\]

with the implied constant depending on \( \Gamma \) and \( p \).

These two theorems are evidently closely related, in light of the results of [Fra98], which show that \( H^n(\Gamma(p^k) \backslash G_\infty, \mathcal{V}_{W,k}) \) may be computed in terms of automorphic forms.

In the rest of this introduction, we discuss the relation of Theorem 1.1 to prior results in this direction before briefly describing the main ingredients in the proof of the two theorems.

De George and Wallach established general upper bounds for \( m(\pi, \Gamma) \) in the case where \( \Gamma \) is cocompact (see [dGW78]). In particular (ibid., Corollary 3.2), they showed that

\[
m(\pi, \Gamma) \leq \left( \int_B |\phi(g)|^2 \, dg \right)^{-1} \text{vol}(\Gamma \backslash G_\infty),
\]

where \( \phi(g) = \langle \pi(g) v, v \rangle \) is a matrix coefficient, and \( B \) is the preimage in \( \Gamma \backslash G_\infty \) of a ball in \( G_\infty / K_\infty \) of radius equal to the injectivity radius of \( \Gamma \backslash G_\infty / K_\infty \). Suppose, however, that \( \pi \) is not a discrete series. In particular, the corresponding matrix coefficients of \( \pi \) are then not square integrable. If \( \Gamma(q) \) denotes the mod \( q \) congruence subgroup of \( \Gamma \), then \( \text{inj.rad}(\Gamma(q) \backslash G_\infty) \to \infty \) as \( N_{F/Q}(q) \to \infty \), and thus the formula of de George and Wallach implies that

\[
\lim_{N_{F/Q}(q) \to \infty} V(q)^{-1} \cdot m(\pi, \Gamma(q)) = 0.
\]

For noncocompact \( \Gamma \), an analogous result was established by Savin [Sav89].

It is natural to try to improve this result so as to obtain an estimate on the rate of decay in (1.2) as \( N_{F/Q}(q) \to \infty \). If \( \pi \) is nontempered, then (1.1) itself implies
an estimate of the form

\[ m(\pi, \Gamma(q)) \ll V(q)^{1-\mu} \quad \text{for some } \mu > 0. \]  

(See [SX91, Lemma 1 and displayed equation (6)].) In fact Sarnak and Xue in [SX91] have conjectured an inequality of the following form (in the case of cocompact \( \Gamma \)):

**Conjecture 1.3 (Sarnak and Xue).** For \( \pi \in \hat{G}_\infty \) fixed,

\[ m(\pi, \Gamma(q)) \ll V(q)\left(\frac{1}{2}/p(\pi)\right)^{+\varepsilon} \quad \text{for all } \varepsilon > 0, \]

where \( p(\pi) \) is the infimum over \( p \geq 2 \) such that the \( K \)-finite matrix coefficients of \( \pi \) are in \( L^p(G_\infty) \).

Sarnak and Xue proved their conjecture for arithmetic lattices in \( \text{SL}_2(\mathbb{R}) \) and \( \text{SL}_2(\mathbb{C}) \), obtaining partial results in the direction of this conjecture for \( \text{SU}(2, 1) \). Note, however, that their conjecture is nontrivial only for non-tempered representations, since, for tempered representations, \( p(\pi) = 2 \). In particular, in the tempered but nondiscrete series case, Conjecture 1.3 is weaker than the known result (1.2).

In Theorem 1.1, we restrict our attention to congruence covers of the form \( \Gamma(p^k) \) for the fixed prime \( p \). For such covers we obtain a quantitative improvement of (1.2) even in the case of tempered representations (at least for those of cohomological type; note that nondiscrete series tempered representations of cohomological type exist precisely when \( G_\infty \) admits no discrete series — see [BW80, Thm. 5.1, p. 101]). For such representations, our result provides the first general bound of the form (1.3) for any \( \mu > 0. \)

As we already noted, our two main theorems are closely related. Indeed, Theorem 1.1 is an easy corollary of Theorem 1.2 (see the end of Section 3 below), and most of our efforts will be concentrated on establishing the latter.

When studying the Betti numbers of arithmetic quotients of symmetric spaces, it is natural to try to use tools such as Euler characteristics and the Lefschetz trace formula. When applied to analyzing contributions from the discrete series, such methods tend to be very powerful; for example, the \((g, \ell)\)-cohomology of a discrete series representation is concentrated in a single dimension [BW80], and so no cancellations occur when taking alternating sums. However, in other situations, these methods can be useless. For example, if \( \pi \) is tempered but not a discrete series, then the Euler characteristic of its \((g, \ell)\)-cohomology vanishes [BW80]. Similarly, in situations where the symmetric space is a real manifold of odd dimension \( n \), Poincaré duality leads to cancellations in the natural sum \( \sum_{k=0}^{n} (-1)^k \dim(H^k) \). One is thus forced to find different techniques. The proof of Theorem 1.2 takes as input the inequality (1.2) of [dGW78] and [Sav89] and a
spectral sequence from [Eme06], proceeding via a bootstrapping argument relying on noncommutative Iwasawa theory.

2. Iwasawa theory

Let $G \subseteq \text{GL}_N(\mathbb{Z}_p)$ be an analytic pro-$p$ group. Let

$$G_k = G \cap (1 + p^k M_N(\mathbb{Z}_p)) \subseteq \text{GL}_N(\mathbb{Z}_p).$$

The subgroups $G_k$ form a fundamental set of open neighborhoods of the identity in $G$; moreover, for large $k$, there exists a constant $c$ such that $[G : G_k] = c \cdot p^{dk}$, where $d = \dim(G)$.

Fix a finite extension $E$ of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_E$. Write $\Lambda = \mathcal{O}_E[[G]]$ and $\Lambda_E = E \otimes_{\mathcal{O}_E} \Lambda$. The module theory of $\Lambda$ falls under the rubric of Iwasawa theory. A fundamental result of Lazard [Laz54] states that $\Lambda$ is Noetherian; the same is thus true of the ring $\Lambda_E$. The rings $\Lambda$ and $\Lambda_E$ are noncommutative domains admitting a common field of fractions, which we will denote by $\mathcal{L}$. Thus, $\mathcal{L}$ is a division ring which contains $\Lambda$ and $\Lambda_E$ and is flat over each of them (on both sides). If $M$ is a finitely generated left $\Lambda$-module (resp. $\Lambda_E$-module), then $\mathcal{L} \otimes_{\Lambda} M$ (resp. $\mathcal{L} \otimes_{\Lambda_E} M$) is a finite-dimensional left $\mathcal{L}$-vector space; we define the rank of $M$ to be the $\mathcal{L}$-dimension of this vector space. Note that rank is additive in short exact sequences of finitely generated $\Lambda$-modules (resp. $\Lambda_E$-modules), by virtue of the flatness of $\mathcal{L}$ over $\Lambda$ and $\Lambda_E$.

Recall that a continuous representation of $G$ on an $E$-Banach space $V$ is called admissible if its topological $E$-dual $V'$ (which is naturally a $\Lambda_E$-module) is finitely generated over $\Lambda_E$. (See [ST02]; a key point is that since $\Lambda_E$ is Noetherian, the category of admissible continuous $G$-representations is abelian. Indeed, passing to topological duals yields an anti-equivalence with the abelian category of finitely generated $\Lambda_E$-modules.) We define the corank of an admissible $G$-representation to be the rank of the finitely generated $\Lambda_E$-module $V'$.

A coadmissible $G$-representation $V$ is not determined by the collection of subspaces of invariants $V^{G_k}$ ($r \geq 1$). However, the following result of Harris shows that its corank is so determined.

**Theorem 2.1** ([Har00, Thm. 1.10]). Let $V$ be an $E$-Banach space equipped with an admissible continuous $G$-representation, and let $d = \dim(G)$. Then as $k \to \infty$,

$$\dim E V^{G_k} = r \cdot [G : G_k] + O(p^{(d-1)k}) = r \cdot c \cdot p^{dk} + O(p^{(d-1)k}),$$

where $r$ is the corank of $V$ and $c$ depends only on $G$.

Using this result, we may obtain bounds on the dimensions of the continuous cohomology groups $H^i(G_k, V)$ in terms of $k$ for admissible continuous
Let us remark that the continuous $G_k$-cohomology on the category of admissible continuous $G$-representations may also be computed as the right-derived functors of the functor of $G_k$-invariants; see [Eme06, Prop. 1.1.3].)

**Lemma 2.2.** Let $V$ be an admissible continuous $G$-representation. For each $i \geq 1$,
\[
\dim_E H^i(G_k, V) \ll p^{(d-1)k} \quad \text{as} \quad k \to \infty.
\]

**Proof.** Let $\mathcal{C} := \mathcal{C}(G, E)$ denote the Banach space of continuous $E$-valued functions on $G$, equipped with the right regular $G$-action. The module $\mathcal{C}$ has corank one (indeed, it is cofree — its topological dual is free of rank one over $\Lambda_E$). Moreover, $\mathcal{C}$ is injective in the abelian category of admissible $G$-representations and is therefore acyclic. If $V$ is an admissible continuous $G$-representation, then there exists an exact sequence
\[
0 \to V \to \mathcal{C}^n \to W \to 0
\]
of admissible continuous $G$-representations for some integer $n \geq 0$. Since $\mathcal{C}$ is acyclic, from the long exact sequence of cohomology we obtain

(2.1) \quad $0 \to V^{G_k} \to (\mathcal{C}^{G_k})^n \to W^{G_k} \to H^1(G_k, V) \to 0$,

(2.2) \quad $H^i(G_k, V) \cong H^{i-1}(G_k, W)$ for $i \geq 2$.

The lemma for $i = 1$ follows from a consideration of (2.1), taking into account Theorem 2.1 and the fact that corank of $W$ is equal to $n$ minus the corank of $V$ (since corank is additive in short exact sequences). We now proceed by induction on $i$. Assume the result for $i \leq m$ and all admissible continuous representations, in particular for $W$. The result for $i = m + 1$ then follows directly from the isomorphism (2.2). This completes the proof. \hfill $\square$

**3. Cohomology of arithmetic quotients of symmetric spaces**

We now return to the situation considered in the introduction and use the notation introduced there. In particular, we fix a connected semisimple linear group $G$ over $F$, an embedding $G \hookrightarrow \GL_N$ over $F$, an arithmetic lattice $\Gamma$ of the associated real group $G_{\infty}$, and a prime $p$ of $F$.

If we write
\[
G := \varprojlim_k \Gamma / \Gamma(p^k),
\]
then $G$ is a compact open subgroup of the $p$-adic Lie group $G(F_p)$ (where $F_p$ denotes the completion of $F$ at $p$); alternatively, we may define $G$ to be the closure of $\Gamma$ in $G(F_p)$. If we replace $\Gamma$ by $\Gamma(p^k)$ for some sufficiently large value of $k$ (i.e., discarding finitely many initial terms in the descending sequence of lattices $\Gamma(p^k)$), then $G$ will be pro-$p$ and, hence, will be an analytic pro-$p$-group. Note
that $\Gamma$ is a dense subgroup of $G$. Let $e$ and $f$ denote respectively the ramification and inertial indices of $p$ in $F$ (so that $[F_p : \mathbb{Q}_p] = ef$). For each $k \geq 0$, write $G_k$ to denote the closure of $\Gamma(p^{ek})$ in $G$. Alternatively, if we consider the embedding
\[ G(F_p) \hookrightarrow GL_N(F_p) \hookrightarrow GL_{efN}(\mathbb{Q}_p), \]
then $G_k = G \cap (1 + p^k M_{efN}(\mathbb{Z}_p))$; thus, our notation is compatible with that of the preceding section. We let $d$ denote the dimension of $G$ and note that $d = (ef/[F : \mathbb{Q}]) \cdot \dim(G_{\infty})$.

For each $k \geq 0$, we write
\[ Y_k := \Gamma(p^{ek}) \backslash G_{\infty} / K_{\infty}. \]
There is a natural action of $G$ on $Y_k$ through its quotient $G/G_k \ (\cong \Gamma / \Gamma(p^{ek}))$, which is compatible with that of the preceding section. We let $d$ denote the dimension of $G$ and note that $d = (ef/[F : \mathbb{Q}]) \cdot \dim(G_{\infty})$.

Fix a finite-dimensional representation $W$ of $G$ over $E$, and let $W_0$ denote a $G$-invariant $\mathcal{O}_E$-lattice in $W$. Let $\mathcal{V}_k$ denote the local system of free finite rank $\mathcal{O}_E$-modules on $Y_k$ associated to $W_0$. If $0 \leq k' \leq k$, then the sheaf $\mathcal{V}_k$ on $Y_k$ is naturally isomorphic to the pull-back of the sheaf $\mathcal{V}_{k'}$ on $Y_{k'}$ under the projection $Y_k \to Y_{k'}$. In particular the sheaf $\mathcal{V}_k$ is $G/G_k$-equivariant.

Recall the definitions
\[ \tilde{H}^n(\mathcal{V}) := \lim_{\leftarrow k} \lim_{\to l} H^n(Y_k, \mathcal{V}_k / p^l) \quad \text{and} \quad \tilde{H}^n(\mathcal{V})_E := E \otimes_{\mathcal{O}_E} \tilde{H}^n(\mathcal{V}) \]
from [Eme06, p. 21]. Each $\tilde{H}^n(\mathcal{V})$ is a $p$-adically complete $\mathcal{O}_E$-module, equipped with a left $G$-action in a natural way, and hence each $\tilde{H}^n(\mathcal{V})_E$ has a natural structure of $E$-Banach space and is equipped with a continuous left $G$-action. In fact, they are admissible continuous representations of $G$ [Eme06, Thm. 2.1.5(i)], and in particular, Theorem 2.1 and Lemma 2.2 apply to them. (Note that the results of [Eme06] are stated in the adèlic language. We leave it to the reader to make the easy translation to the more classical language we are using in this paper.)

The following result, which is [Eme06, Thm. 2.1.5(ii), p. 22], is a “control theorem” relating $G_k$ invariants in $\tilde{H}^n(\mathcal{V})_E^{G_k}$ to the classical cohomology classes $H^n(Y_k, \mathcal{V}_k) \otimes E$.

**THEOREM 3.1.** Fix an integer $k$. There is a spectral sequence
\[ E_2^{i,j}(Y_k) = H^i(G_k, \tilde{H}^j(\mathcal{V})_E) \Longrightarrow H^{i+j}(Y_k, \mathcal{V}_k)_E. \]
One should view this spectral sequence as a version of the Hochschild-Serre spectral sequence “compatible in the $G$-tower.”

**THEOREM 3.2.** For any $n \geq 0$, if $r_n$ denotes the corank of $\tilde{H}^n(\mathcal{V})$, then
\[ \dim_E H^n(Y_k, \mathcal{V}_k)_E = r_n \cdot c \cdot p^{dk} + O(p^{(d-1)k}) \quad \text{as} \quad k \to \infty. \]
(Here $c$ is the constant appearing in Theorem 2.1; it depends only on $G$.)
Proof. For each \(i, j \geq 0\) and \(l \geq 2\), let \(E^{i,j}_l(Y_k)\) denote the terms in the spectral sequence of Theorem 3.1. Because \(H^j(\mathcal{V})\) is admissible, Lemma 2.2 implies that \(\dim_E H^i(G_k, H^j(\mathcal{V})_E) \ll p^{(d-1)k}\) if \(i > 0\), and therefore that \(\dim_E E^{i,j}_l(Y_k) \ll p^{(d-1)k}\) as \(k \to \infty\) if \(i > 0\) (since \(E^{i,1}_l(Y_k)\) is a subquotient of \(E^{i,j}_l(Y_k) := H^i(G_k, H^j(\mathcal{V})_E)\) for \(l \geq 2\)). Theorem 2.1 shows that

\[
\dim_E E^{0,n}_E = \dim_E H^n(\mathcal{V})_E = r_n \cdot c \cdot p^{d^k} + O(p^{(d-1)k}).
\]

On the other hand, since the spectral sequence of Theorem 3.1 is an upper right quadrant exact sequence, \(E^{0,n}_E\) is obtained by taking finitely many successive kernels of differentials \(d_l\) to \(E^{i,j}_l\), which all have dimension \(\ll p^{(d-1)k}\) by the first part of our argument. Thus,

\[
\dim_E E^{0,n}_E = r_n \cdot c \cdot p^{d^k} + O(p^{(d-1)k}).
\]

Since \(H^n(Y_k, \mathcal{V}_k)_E\) admits a finite length filtration whose associated graded pieces are isomorphic to \(E^{i,j}_\infty\) for \(i + j = n\), we conclude that

\[
\dim_E E^{0,n}_E = r_n \cdot c \cdot p^{d^k} + O(p^{(d-1)k}). \quad \square
\]

The following lemma quantifies the precise relationship between multiplicities and the dimensions of cohomology groups that we will require to deduce Theorem 1.1 from Theorem 1.2.

**Lemma 3.3.** Fix a cohomological degree \(n\), and let \(S\) denote the set of isomorphism classes \([\pi]\) of \(\pi \in \hat{G}\) that contribute to cohomology with coefficients in \(\mathcal{V}\) in degree \(n\). Then

\[
\sum_{[\pi] \in S} m(\pi, Y(p^{ek})) \geq \dim_E H^n_{\text{cusp}}(Y_k, \mathcal{V}_k)_E.
\]

**Proof.** Since the set \(S\) is finite, there is an integer \(d \geq 1\) such that

\[
1 \leq \dim H^n(\mathfrak{g}, \mathfrak{k}; \pi \otimes W) \leq d
\]

for each isomorphism class \([\pi] \in S\). This implies that

\[
\sum_{[\pi] \in S} m(\pi, Y(p^{ek})) \leq \dim_E H^n_{\text{cusp}}(Y_k, \mathcal{V}_k)_E \leq d \left( \sum_{[\pi] \in S} m(\pi, Y(p^{ek})) \right)
\]

for each \(k \geq 0\). \(\square\)

We can now prove our main result.

**Theorem 3.4.** Let \(n \geq 0\), and suppose either that \(G_\infty\) does not admit discrete series, or that \(n \neq \frac{1}{2} \dim(G_\infty / K_\infty)\). Then, for all \(n \geq 0\),

\[
\dim_E H^n(Y_k, \mathcal{V}_k)_E \ll p^{(d-1)k} \quad \text{as} \; k \to \infty.
\]
Proof. In the case when \( G_\infty \) admits discrete series, recall from [BW80, Thm. 5.1, p. 101] that these contribute to cohomology only in the dimension \( \frac{1}{2} \dim(G_\infty/K_\infty) \). Thus, under the assumptions of the theorem, there is no contribution from the discrete series to \( H^n_{\text{cusp}}(Y_k, \mathcal{V}_k) \). The inequality (1.2) of [dGW78] and [Sav89], together with Lemma 3.3 and the main result of [RS87] (which states that \( \dim_E \left( H^n(Y_k, \mathcal{V}_k) / H^n_{\text{cusp}}(Y_k, \mathcal{V}_k) \right) = o(p^{d_k}) \)), thus shows that

\[
\dim_E H^n(Y_k, \mathcal{V}_k)_E = o(p^{d_k})
\]

as \( k \to \infty \) for all \( n \geq 0 \). From Theorem 3.2, we then infer that each \( \bar{H}^n(\mathcal{V}) \) has corank 0. Another application of the same theorem now gives our result. \( \square \)

Note that \( V(p^{e_k}) \sim [G : G_k] \sim e \cdot p^{d_k} \). Thus Theorem 3.4 implies Theorem 1.2 since

\[
(3.1) \quad \dim(G) \leq \dim(G_\infty).
\]

Theorem 1.2 and Lemma 3.3 together imply Theorem 1.1.

Remark 3.5. We have equality in (3.1) precisely when \( p \) is the unique prime lying over \( p \) in \( \mathcal{O}_F \). If there is more than one prime lying over \( p \), then \( \dim(G) \) is strictly less than \( \dim(G_\infty) \), and we obtain a corresponding improvement in the bounds of Theorems 1.1 and 1.2, namely (in the notation of their statements), that

\[
m(\pi, \Gamma(p^k)) \quad \text{and} \quad \dim E H^i(Y(p^k), \mathcal{V}_{W,k})_E \ll V(p^k)^{1-1/\dim(G)}
\]

(where, as we noted above, \( \dim(G) = (ef/[F : \mathbb{Q}]) \cdot \dim(G_\infty) \), with \( e \) and \( f \) being the ramification and inertial index of \( p \), respectively).

Example/Question 3.6. Let \( F/\mathbb{Q} \) be an imaginary quadratic field; let \( G = \text{SL}_2/F \). The corresponding symmetric space \( G_\infty/K_\infty = \text{SL}_2(\mathbb{C})/\text{SU}(2) \) is real hyperbolic three-space \( \mathcal{H} \), and the quotients \( Y \) are commensurable with the Bianchi manifolds \( \mathcal{H}/\text{PGL}_2(\mathcal{O}_K) \). Choose a local system \( \mathcal{W}_0 \) associated to some finite-dimensional representation \( W \) of \( G_\infty = \text{GL}_2(\mathbb{C}) \) and a congruence subgroup \( \Gamma \). Assume that \( p = p\mathfrak{p}_p \) splits in \( \mathcal{O}_F \), and apply Theorem 3.4 to the \( p \)-power tower. We obtain the inequality

\[
H^1_{\text{cusp}}(Y_k, \mathcal{V}_k) \ll p^{2k} \quad \text{as} \quad k \to \infty.
\]

It is natural to ask how tight this inequality is.

The main result of Calegari and Dunfield [CD06] shows that there exists at least one \( (F, \Gamma, p) \) for which

\[
H^1_{\text{cusp}}(Y_k, \mathbb{C}) = 0 \quad \text{for all} \quad k.
\]
On the other hand, if there exists at least one newform on $\mathcal{C}_k$ for some $k$, then a consideration of the associated oldforms shows that

$$H^1_{\text{cusp}}(Y_k, \mathcal{V}_k) \gg p^k \quad \text{as } k \to \infty.$$  

Are there situations in which this lower bound gives the true rate of growth?

**Remark 3.7.** Our results are most interesting in the case when $G_\infty$ does not admit any discrete series, since, as we noted in the introduction, in this case (and only in this case) $G_\infty$ admits (nondiscrete series) tempered representations of cohomological type.

On the other hand, Theorem 3.2 does have a consequence in the case when $G_\infty$ admits discrete series, which may be of some interest. Recall the following result from [Sav89] (established in [dGW78] in the cocompact case): if $\pi \in \hat{G}_\infty$ lies in the discrete series, then

\begin{equation}
(3.2) \quad m(\pi, \Gamma(p^k)) = d(\pi)V(p^k) + o(V(p^k)) \quad \text{as } k \to \infty.
\end{equation}

Fix a finite-dimensional representation $W$ of $G_\infty$, and let $\hat{G}_\infty(W)_d$ denote the subset of $\hat{G}_\infty$ consisting of discrete series representations that contribute to cohomology with coefficients in $W$. Summing over all $\pi \in \hat{G}_\infty(W)_d$, we obtain the formula

\begin{equation}
(3.3) \quad \frac{1}{|\hat{G}_\infty(W)_d|} \sum_{\pi \in \hat{G}_\infty(W)_d} m(\pi, \Gamma(p^k)) = d(\pi)V(p^k) + o(V(p^k))
\end{equation}

(a result first proved in [RS87]). The following result provides an improvement in the error term of (3.3).

**Theorem 3.8.** There exists a $\mu > 0$ such that

$$\frac{1}{|\hat{G}_\infty(W)_d|} \sum_{\pi \in \hat{G}_\infty(W)_d} m(\pi, \Gamma(p^k)) = d(\pi)V(p^k) + O(V(p^k)^{1-\mu}).$$

**Proof.** Let $n = \frac{1}{2} \dim(G_\infty/K_\infty)$. As we have already noted, it follows from [BW80, Thm. 5.1, p. 101] that all nondiscrete series contributions to $H^n_{\text{cusp}}(Y_k, \mathcal{V}_k)$ are nontempered. The same result shows that each discrete series has one-dimensional ($g, \mathfrak{g}$)-cohomology in dimension $n$. As we recalled in the introduction, the multiplicity of any nontempered representations is bounded by $V(p^k)^{1-\mu}$ for some $\mu > 0$ [SX91], and thus, Theorem 3.2 and (the proof of) Lemma 3.3 show that

$$\frac{1}{|\hat{G}_\infty(W)_d|} \sum_{\pi \in \hat{G}_\infty(W)_d} m(\pi, \Gamma(p^k)) = C \cdot V(p^k) + O(V(p^k)^{1-\mu}).$$

Comparing this formula with (3.3) yields the theorem. \[\square\]
QUESTION 3.9. Does the result of Theorem 3.8 hold term-by-term? That is, does (3.2) admit an improvement of the form
\[ m(\pi, \Gamma(p^k)) = d(\pi) V(p^k) + O(V(p^k)^{1-\mu}) \quad \text{for some } \mu > 0? \]

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References

[BW80] A. Borel and N. R. Wallach, Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups, Ann. of Math. Studies 94, Princeton Univ. Press, Princeton, NJ, 1980. MR 83c:22018 Zbl 0443.22010

[CD06] F. Calegari and N. M. Dunfield, Automorphic forms and rational homology 3-spheres, Geom. Topol. 10 (2006), 295–329. MR 2007h:57013 Zbl 1103.57007

[dGW78] D. L. de George and N. R. Wallach, Limit formulas for multiplicities in \( L^2(\Gamma\backslash G) \), Ann. of Math. 107 (1978), 133–150. MR 58 #11231 Zbl 0397.22007

[Eme06] M. Emerton, On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms, Invent. Math. 164 (2006), 1–84. MR 2007k:22018 Zbl 1090.22008

[Fra98] J. Franke, Harmonic analysis in weighted \( L_2 \)-spaces, Ann. Sci. École Norm. Sup. 31 (1998), 181–279. MR 2000f:11065 Zbl 0938.11026

[Har79] M. Harris, \( p \)-adic representations arising from descent on abelian varieties, Compositio Math. 39 (1979), 177–245. MR 80j:14035 Zbl 0417.14034

[Har00] ______, Correction to the article [Har79], Compositio Math. 121 (2000), 105–108. MR 2001b:11050 Zbl 1060.14524

[Laz54] M. A. Lazard, Sur les groupes nilpotents et les anneaux de Lie, Ann. Sci. École Norm. Sup. 71 (1954), 101–190. MR 19,529b Zbl 0055.25103

[RS87] J. Rohlf and B. Speh, On limit multiplicities of representations with cohomology in the cuspidal spectrum, Duke Math. J. 55 (1987), 199–211. MR 88k:22010 Zbl 0626.22008

[SX91] P. Sarnak and X. Xue, Bounds for multiplicities of automorphic representations, Duke Math. J. 64 (1991), 207–227. MR 92h:22026 Zbl 0741.22008

[Sav89] G. Savin, Limit multiplicities of cusp forms, Invent. Math. 95 (1989), 149–159. MR 90c:22035 Zbl 0673.22003

[ST02] P. Schneider and J. Teitelbaum, Banach space representations and Iwasawa theory, Israel J. Math. 127 (2002), 359–380. MR 2003c:22026 Zbl 1006.46053

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