We derive the generalized Gauss-Codazzi-Mainardi (GCM) equation for a general affine connection with torsion and non-metricity. Moreover, we show that the metric compatibility and torsionless condition of a connection on a manifold are inherited to the connection of its hypersurface. As a physical application to these results, we derive the (3+1)-Einstein Field Equation (EFE) for a special case of Metric-Affine $f(\mathcal{R})$-gravity when $f(\mathcal{R}) = \mathcal{R}$, the Metric-Affine General Relativity (MAGR). Motivated by the concept of geometrodynamics, we introduce additional variables on the hypersurface as a consequence of non-vanishing torsion and non-metricity. With these additional variables, we show that for MAGR, the energy, momentum, and the stress-energy part of the EFE are dynamical, i.e., all of them contain the derivative of a quantity with respect to the time coordinate. For the Levi-Civita connection, one could recover the Hamiltonian and the momentum (diffeomorphism) constraint, and obtain the standard dynamics of GR.

I. INTRODUCTION

With the recent discoveries of gravitational waves and black holes, General Relativity (GR) has been one of the most successful theories in physics. However, several problems have not been covered by GR, these include dark matter, dark energy, and quantum gravity. Attempts to solve these problems have led to some motivation in modifying the original General Relativity, proposed by Einstein in 1915. Different types of modification had flourished with rapid progression; for a complete review on theories of modified gravity, one could consult [1]. A specific way to achieve a modified theory is by adding higher-order terms on curvature to the Einstein-Hilbert action. In $f(\mathcal{R})$-gravity, the higher-order terms are restricted to the power series of the Ricci scalar curvature [1,2]. Another way to modify GR is by considering a general affine connection with torsion, and even non-metricity, hence the connection is not necessarily Levi-Civita as in the standard GR.

Different theories of gravity do not use curvature to describe the effect of gravity, they instead use torsion in Teleparallel Gravity [3], or non-metricity in Symmetric Teleparallel Gravity [4,5]. There exists even a more general theory that uses both torsion and non-metricity (but zero curvature) in General Teleparallel Quadratic Gravity [6]. On the other hand, the Metric-Affine Gravity (MAG) is a curvature theory of gravity that includes both non-metricity and torsion for the connection [7,8]. The choice of action for MAG could vary greatly, for example, the Ricci scalar $\mathcal{R}$, the power series of Ricci scalar $f(\mathcal{R})$, and other exotic actions as discussed in [8]. In this article, we only consider Metric-Affine General Relativity (MAGR), a class of MAG with Ricci scalar $\mathcal{R}$ as the curvature part of action. It could also be considered as a special case of the most general version of the $f(\mathcal{R})$-gravity, the Metric-Affine $f(\mathcal{R})$-gravity, that is a modified theory of gravity with $f(\mathcal{R})$-action and general affine connection [2]. The existence of torsion in the connection is important for the coupling of gravity with fermions [9], while the non-metricity part is important if one considers certain kinds of matter fields that contain a non-vanishing symmetric part of the hypermomentum [2,10–12].

To obtain the dynamics of a covariant gravitational theory, one needs to decompose the covariant fields into their temporal and spatial parts via the (3+1) decomposition. It was first considered by Darmois in [7], Lichnerowicz in [14–16], then by Choquet-Bruhat in [17,18]. However, it gained a lot of attention after the work by Arnowitt, Deser, and Misner in [19]. The (3+1) ADM decomposition is a formulation of GR which split the spacetime and the covariant fields into the temporal and spatial components, thus allowing one to obtain the dynamics of the variables on the 3D hypersurface via the equation of motions. At the core of the ADM formulation, lies the Gauss-Codazzi-Mainardi equations, which relates the intrinsic (Riemann) curvatures of a manifold with the extrinsic and intrinsic curvature of its submanifold. The result of the formulation is the (3+1) Einstein Field Equations (EFE): 4 constraint equations and 6 dynamical ones.

Attempts to apply the (3+1) formulation to $f(\mathcal{R})$-gravity had been done for special cases; in $f(\mathcal{R})$-gravity with Levi-Civita connection [20,21] and in teleparallel-gravity with torsion [22]. Even more, the Hamiltonian analysis and possible quantization procedure of $f(\mathcal{R})$-gravity had been done in [23–25]. Little attention had been given to the (3+1) formulation of gravity with non-metricity, with the latest work being the derivation of the (1+3)
Raychaudhuri equation [8, 26]. However, for various reasons, it is interesting to perform the (3+1) decomposition for MAG and \(f(R)\)-gravity which includes non-metricity, particularly, if one considers matter fields with a general hypermomentum. With this motivation in mind, in this article, we apply the (3+1) ADM formulation for Metric-Affine General Relativity (MAGR) with general affine connection that includes torsion and non-metricity, as a special case to MAG and \(f(R)\)-gravity.

The article is organized as follows: In Section II, we derive the generalized Gauss-Codazzi-Mainardi (GCM) equations for a general affine connection with torsion and non-metricity. A similar treatment is also applied to torsion, resulting in a torsion (3+1) decomposition. These results are valid for manifold with any dimension. For the derivation, we use 2 different definitions of extrinsic curvature that exist in the literature; both differ by the quantity \(\nabla X g\) (with \((\nabla, g)\) are the connection and the metric on the manifold, and \(X\) is a vector on the hypersurface) if the connection is non-metric. The generalized GCM equations are written in terms of these quantities, together with the definition of the acceleration tensor.

In Section III, we show that the metric compatibility and the torsionless condition of a connection \(\nabla\) on a manifold are inherited to the connection \(3\nabla\) on the hypersurface. This was already a well-known result and was used to derive the original GCM equation for the Levi-Civita connection. Since the generalized Riemann curvature tensor of an affine connection is lacking the symmetries possessed by the Levi-Civita connection, we need to define three different types of independent contraction of the Riemann tensor, following the definition introduced in [27–29]. These contractions are important for the derivation of the Einstein tensor in the next section.

Section IV consists of the physical application of the generalized GCM and the torsion decomposition. Minimization of the \(f(R)\)-action with respect to the variation of metric, connection, and a Lagrange multiplier results in 2 Euler-Lagrange equations and 1 constraint equation. However, considering the scope of this article, our focus will be on the generalized stress-energy-momentum equation (EFE); the one that comes from the variation of the metric. We perform the ADM formulation to the generalized EFE using the generalized GCM equation we derive in Section II, resulting in 3 parts of the equations: the energy, momentum, and stress-energy equation. With the introduction of some additional variables on the hypersurface, we could write the energy, momentum, and stress-energy part of the EFE in terms of tensor fields defined only on the hypersurface, motivated by the concept of geometrodynamics in [31]. An interesting result here is, unlike the standard GR case, there is no constraint originating from the generalized EFE; all the 3 equations contain dynamics, i.e., contain the derivative of a quantity with respect to the time coordinate. However, with the Levi-Civita condition, one could recover the Hamiltonian and momentum (or diffeomorphism) constraint, together with the standard dynamics of GR.

We discuss some subtleties on the results in Subsection VA, and for the completeness of the analysis in this article, we present a special case of (3+1) decomposition of the 2 remaining equations of motion: the hypermomentum and the projective constraint equation. This is done in Subsection VB. The detailed derivation and general treatment of these equations will be presented in our companion article. Finally, we conclude our works in Subsection VC.

II. THE GENERALIZED GAUSS-CODAZZI-MAINARDI EQUATIONS

The Metric and Connection on the Hypersurface

Let \(\mathcal{M}\) be an \(n\)-dimensional manifold, equipped with a (pseudo)-Riemannian metric \(g\) and an affine connection \(\nabla\). In the most general case, \(g\) and \(\nabla\) are independent of one another. Let \(\Sigma\) be the hypersurface, namely, an \((n-1)\)-dimensional submanifold embedded on \(\mathcal{M}\). One could define the unit vector normal to \(\Sigma\) at each point \(p \in \Sigma\), namely, \(\hat{n}_p\), satisfying \(g_p (\hat{n}_p, \hat{n}_p) = \pm 1\), where the − sign (the one on the upper side) and the + sign (the one on the lower side) depends on whether \((\mathcal{M}, g)\) is Riemannian or Lorentzian, respectively. From this point up to the rest of the article, the signature on the upper side of the equation is the one valid for the Riemannian case, while the lower side is for the Lorentzian’s. The existence of metric \(g\) on \(\mathcal{M}\) induces a metric on the hypersurface \(\Sigma\) as follows:

\[
3q = g \mp \hat{n}^* \otimes \hat{n}^*,
\]

(1)

with \(\hat{n}^* \in T_p^* \Sigma\) is the covariant vector to \(\hat{n}\), satisfying \(\hat{n}^* = g (\hat{n}, \cdot)\) (here, we omit the label \(p\) for simplicity).

Let \(V, X \in T_p \Sigma\), then one could construct a new vector \(\nabla_V X \in T_p \mathcal{M}\), not necessarily an element of \(T_p \Sigma\), since \(\nabla\) is defined on \(\mathcal{M}\). The decomposition of \(\nabla_V X\) into the components parallel and perpendicular to \(\Sigma\) is the following:

\[
\nabla_V X = \underbrace{\nabla_V X \mp g (\nabla_V X, \hat{n}) \hat{n}}_{(\nabla_V X)_\parallel} \pm \underbrace{g (\nabla_V X, \hat{n}) \hat{n}}_{(\nabla_V X)_\perp}.
\]

(2)

The labels \(\parallel\) and \(\perp\) denote, respectively, the parallel part of \(\nabla_V X\) which lies on \(\Sigma\) and the part normal to \(\Sigma\). The covariant derivative \(\nabla\) on the quantity \((\nabla_V X)_\parallel\) associates \(V, X \in T_p \Sigma\) to a transformed vector \((\nabla_V X)_\parallel \in T_p \Sigma\), hence,
one defines:

\[
(\nabla_V X)_\parallel := ^3\nabla_V X = \nabla_V X \mp g(\nabla_V X, \hat{n}) \hat{n},
\]

where \(^3\nabla\) is the covariant derivative on \(\Sigma\). It could be easily shown that \(^3\nabla\) is an affine connection. One could define an entirely different affine connection \(^3\nabla\) on \(\Sigma\), which in general, is not induced from \(\nabla\). However, since the objective of the work is to obtain the generalized Gauss-Codazzi relation, we only consider the case where \(^3\nabla\) is induced by \(\nabla\) as in (2).

**Extrinsic Curvature of the First Kind**

The component of the perpendicular part of (2) is defined as the extrinsic curvature of \(\Sigma\) in the direction of \((V,X)\) as follows [32]:

\[
(\nabla_V X)_\perp := K(V,X) = \pm g(\nabla_V X, \hat{n}).
\]

Written in terms of components, one could show that the extrinsic curvature of the first kind satisfies:

\[
K(V,X) = \pm V^\mu X^\alpha \omega_\alpha^\beta n^\mu,
\]

using the fact that \(n_\alpha V(X^\alpha) = 0\), since \(X, V \in T_p \Sigma\) are perpendicular to \(\hat{n}\). The quantity \(\omega_\alpha^\beta \partial_\alpha = \nabla_\alpha \partial_\beta\) is the vector potential of \(\nabla\). Notice that in general, \(\omega_\alpha^\beta\) of an affine connection does not possess any symmetry in the indices \((\mu, \alpha, \beta)\), therefore, one needs to be careful not to switch the position of these indices. With definitions (3) and (4), (2) could be written as:

\[
\nabla_V X = ^3\nabla_V X + K(V,X) \hat{n}.
\]

There exists another definition of extrinsic curvature that will be explored in the next sections.

**The Torsion Tensor**

The torsion tensor on \(\mathcal{M}\) is defined as:

\[
T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \quad X,Y \in T_p \mathcal{M}.
\]

For the case where \(X,Y \in T_p \Sigma\), using the parallel and perpendicular decomposition of \(\nabla_V X\) as in (6), one could obtain:

\[
T(X,Y) = ^3T(X,Y) + (K(X,Y) - K(Y,X)) \hat{n},
\]

with:

\[
^3T(X,Y) = ^3\nabla_X Y - ^3\nabla_Y X - [X,Y],
\]

is the torsion on the hypersurface \(\Sigma\). Moreover, using (5), one could obtain:

\[
K(X,Y) - K(Y,X) = \pm g(T(X,Y), \hat{n}),
\]

and hence (8) could be regarded as the torsion decomposition into the parallel and normal part of the hypersurface. The torsion tensor is antisymmetric in the first and second argument:

\[
T(X,Y) = -T(Y,X), \quad \forall X,Y \in T_p \mathcal{M}.
\]

**The Generalized (Riemann) Intrinsic Curvature**

The original Riemann curvature is defined only for the curvature of a Levi-Civita connection in the context of Riemannian geometry. In this article, we use a similar definition to describe the intrinsic curvature of a general affine connection in non-Riemannian geometry:

\[
R(U,V)X = \nabla_U \nabla_V X - \nabla_V \nabla_U X - \nabla_{[U,V]} X, \quad U,V,X \in T_p \mathcal{M}.
\]
To obtain the intrinsic curvature on $\Sigma$, let $U, V, X \in T_p \Sigma$, then with the definitions in (3) and (4), one could obtain the double derivative of $X$:

$$\nabla_U \nabla_V X = 3\nabla_U \nabla_V X + K (V, X) \nabla_U \hat{n} + \left(U [K (V, X)] + K (U, 3\nabla_V X)\right) \hat{n},$$

(10)

from a direct calculation. Notice here that $U [f (x)]$ denotes a vector $U$ acting on a scalar function $f (x)$ as $U^\mu \partial_\mu f (x)$. Moreover one could have the following quantity:

$$\nabla_V \nabla_U X = 3\nabla_V \nabla_U X + K (U, X) \nabla_V \hat{n} + \left(V [K (U, X)] + K (V, 3\nabla_U X)\right) \hat{n},$$

(11)

$$\nabla_{[U, V]} X = 3\nabla_{[U, V]} X + K ([U, V], X) \hat{n},$$

(12)

using the fact that $[U, V] \in T_p \Sigma$ for $U, V \in T_p \Sigma$. By a little tensor algebra, one could show that the following relation is valid:

$$K (U, 3\nabla_V X) - K (V, 3\nabla_U X) - K ([U, V], X) + U [K (V, X)] - V [K (U, X)] = \left(3\nabla_U K\right) (V, X) - \left(3\nabla_V K\right) (U, X) + K \left(3T (U, V), X\right),$$

(13)

with $3T (U, V)$ is the torsion tensor on $\Sigma$. Inserting (10), (11), and (12) to (9) and then using relation (13), one obtains the decomposition of intrinsic curvature on $\mathcal{M}$ into the intrinsic and extrinsic curvature of $\Sigma$:

$$R (U, V, X) = 3\ n = 3\ n = 3\ n,$$

(14)

$$+ \left((3\nabla_U K) (V, X) - (3\nabla_V K) (U, X) + K \left(3T (U, V), X\right)\right) \hat{n}.$$

$3\ R$ is the generalized Riemann curvature of the connection $3\nabla$ on the hypersurface $\Sigma$. The only symmetry possessed by the generalized Riemann curvature (9) is:

$$R (U, V) X = -R (V, U) X.$$

As a comment concerning the terminology, the definition in (9) is defined originally for the Levi-Civita connection, and so does the term Riemann curvature tensor. However, in this article, we use a similar term and definition (i.e., (9)) for the intrinsic curvature of a generalized affine connection. From the decomposition of generalized Riemann tensor (14), the Gauss-Codazzi-Mainardi equations could be obtained.

**The Acceleration Tensor**

In the Eulerian frame, one could regard the normal $\hat{n}$ as the 4-velocity of an event $p \in \mathcal{M}$ [33]. Let us choose the spatial part $x^i$ of $x^\mu = (x^0, x^i)$ as an 'adapted' local $(n-1)$ coordinate on $\Sigma$ (notice that in general, the spatial part $x^i$ of $x^\mu = (x^0, x^i)$ is not necessarily a local coordinate on $\Sigma$). The vector basis related to this coordinate is $\partial_i$. Together with $\hat{n}$, they define a complete basis on $T_p \mathcal{M}$, namely $(\hat{n}, \partial_i)$. Let us define the acceleration tensor $a$ in the coordinate basis $(\hat{n}, \partial_i)$ as follows:

$$a = a_\nu = (a_{\hat{n}}, a_i) := (\nabla_{\hat{n}} \hat{n}, \nabla_i \hat{n}).$$

(15)

where:

$$a_\nu = a_{\nu}^\mu \partial_\mu \in T_p \mathcal{M}, \quad \nu = (\hat{n}, i), \quad i = 1, 2, \ldots, n - 1.$$

Hence, for an arbitrary vector $U \in T_p \Sigma$, one has:

$$\nabla_U \hat{n} = U^i \nabla_i \hat{n} = U^i a_i = a (U).$$

The quantity $a_{\hat{n}} := \alpha$ is known as the 4-acceleration.

It is a well-known fact that in the original General Relativity with Levi-Civita connection, the 4-acceleration is orthogonal to the 4-velocity, namely $g (\alpha, \hat{n}) = 0$. This is a consequence of the metric compatibility $\nabla_X g = 0$, $\forall X \in T_p \mathcal{M}$. With the same condition, one could show that the acceleration tensor is orthogonal to the 4-velocity as well:

$$g (a_\nu, \hat{n}) = g (\nabla_\nu \hat{n}, \hat{n}) = 0, \quad \nu = (\hat{n}, i), \quad i = 1, 2, \ldots, n - 1,$$
using the fact that:
\[
\nabla_\nu g (\hat{n}, \hat{n}) = (\nabla_\nu g) (\hat{n}, \hat{n}) + g (\nabla_\nu \hat{n}, \hat{n}) + g (\hat{n}, \nabla_\nu \hat{n}),
\]
and the symmetricity of \( g \). Notice that if the metricity condition is violated, one has:
\[
g (a_\nu, \hat{n}) = -\frac{1}{2} (\nabla_\nu g) (\hat{n}, \hat{n}),
\]
which defines the 'angle' between \( a_\nu \) and \( \hat{n} \). Let us label the angle as:
\[
\Theta_\nu = \pm g (a_\nu, \hat{n}).
\]
(16)

For an arbitrary vector \( U \in T_p \Sigma \), one could write \( \Theta (U) = \pm g (a (U), \hat{n}) \), that is, the angle between the acceleration in the spatial direction \( U \), with the normal \( \hat{n} \). The definition of the angle \( \Theta \) will be useful for the next sections.

**Extrinsic Curvature of the Second Kind**

As we had mentioned in the previous sections, besides the extrinsic curvature defined in (4), there exists another definition of the extrinsic curvature [33]:
\[
K (U, V) = g (\nabla_U \hat{n}, V), \quad U, V \in T_p \Sigma.
\]
(17)

For the original General Relativity where the connection is Levi-Civita, the first and second extrinsic curvature coincide (by the factor of \( \mp 1 \) for Riemannian/ Lorentzian case):
\[
\nabla_U g (V, \hat{n}) = (\nabla_U g) (V, \hat{n}) + g (\nabla_U V, \hat{n}) + g (V, \nabla_U \hat{n}).
\]
Notice that without the metric compatibility, they differ by \( \nabla_U g (V, \hat{n}) \).

Written in terms of components, one could show that the extrinsic curvature of the second kind satisfies:
\[
K (U, V) = g (\nabla_U \hat{n}, V) = U^\mu V^\nu g (\nabla_\mu \hat{n}, \partial_\nu) = V_\alpha U^\mu \partial_\mu n^\alpha + U^\mu V_\alpha \omega^\alpha_\beta,
\]
(18)

where the first term is zero since \( U^\mu \partial_\mu V, n^\alpha \) and \( n^\alpha U^\mu \partial_\mu V_\alpha \) are zero for \( U, V \in T_p \Sigma \). The geometrical interpretation of \( K (\partial_\mu, \partial_\nu) \) differs from \( K (\partial_\mu, \partial_\nu) \): it describes the \( \nu ^{th} \)-component of the change of the normal in the direction of \( \partial_\mu \).

Before we proceed to the main calculation, it is convenient to fix the notations and conventions we use concerning the metric and the inner-product. First, the covector \( V^* \in T^*_p \mathcal{M} \) is defined as follows:
\[
g (V, W) = \langle V^*, W \rangle, \quad \forall \ W \in T_p \mathcal{M},
\]
(19)

with \( g = g_{\mu \nu} dx^\mu \otimes dx^\nu \) is the metric on \( \mathcal{M} \), \( V \in T_p \mathcal{M} \), and \( \langle \cdot, \cdot \rangle \) is an inner product on \( T_p \mathcal{M} \). With this definition, then (17) could be written as \( K (U, V) = \langle V^*, \nabla_U \hat{n} \rangle \), for example. Second, as we have mentioned at the beginning of this chapter, one could write (19) as \( g (V, \cdot) = g (\cdot, V) = V^* \). Moreover, given \( V^*, W^* \in T^*_p \mathcal{M} \), (19) could be written as:
\[
g (V, W) = g^* (V^*, W^*),
\]
(20)

with \( g^* = g^{\mu \nu} \partial_\mu \otimes \partial_\nu \) is the dual of metric \( g \), such that \( g (g^*) = 1 \) (or in indices: \( g^{\mu \nu} g_{\sigma \tau} = \delta^\mu_\sigma \)). However, these relations should be used carefully, for example \( g (\partial_\mu, \partial_\nu) = g_{\mu \nu} \), is not equal to either \( (dx^\mu, \partial_\nu) \) or \( g^* (dx^\mu, dx^\nu) = g^{\mu \nu} \). As another example, even if \( g (V, \cdot) = g (\cdot, V) = V^* \), this is not valid for basis vectors and covectors, as \( g (\partial_\mu) \neq dx^\mu \). The correct relation is:
\[
dx^\mu = g^{\mu \nu} g (\partial_\nu).
\]
(21)
The Torsion Decomposition, Gauss, and Codazzi-Mainardi Equations

The results (8) and (14) will be contracted with vectors \( \hat{n} \) and \( Y \in T_p \Sigma \) to obtain the temporal and spatial parts of the equations. As a first step, we need to decompose the quantity \( \nabla_U \hat{n} \) into the normal and parallel parts. Let us consider the following quantity:

\[
\nabla_U (g (X, \hat{n})) = (\nabla_U g) (X, \hat{n}) + g (\nabla_U X, \hat{n}) + g (X, \nabla_U \hat{n}) , \quad X \in T_p \Sigma .
\]  

(22)

The LHS of (22) could be written as \( \nabla_U (g (X, \hat{n})) = U [g (X, \hat{n})] = 0 \), since \( X \in T_p \Sigma \) is perpendicular to \( \hat{n} \). One could do the following decomposition on the quantity \( \nabla_U \hat{n} \):

\[
\nabla_U \hat{n} = \pm (\hat{n}^* , \nabla_U \hat{n}) \hat{n} + \left( (3 \, dx^i , \nabla_U \hat{n}) \right) \partial_i .
\]  

(23)

with \( \partial_i \) and \( 3 \, dx^i = 3 q^{ij} q (\partial_j) \) are the vector and covector basis corresponding to \( x^i \), the adapted coordinate on \( \Sigma \). With the definitions in (16) and (17), one could obtain the following relations:

\[
(\nabla_U \hat{n})_{\perp \Sigma} = \pm (\hat{n}^* , \nabla_U \hat{n}) \hat{n} = \pm g (\nabla_U \hat{n}, \hat{n}) = \Theta (U) \hat{n},
\]

\[
(\nabla_U \hat{n})_{\parallel \Sigma} = \left( (3 \, dx^i , \nabla_U \hat{n}) \right) \partial_i = 3 q^{ij} g (\partial_j, \nabla_U \hat{n}) \partial_i = 3 q^{ij} K (U, \partial_j) \partial_i,
\]

using equation (1), (17), and (21). Inserting (23) to (14) gives:

\[
R (U, V) X = 3 R (U, V) X + 3 q^{ij} (K (V, X) K (U, \partial_j) - K (U, X) K (V, \partial_j)) \partial_i + \left( (3 \, \nabla_U K) (V, X) - (3 \, \nabla_V K) (U, X) + K (3 T (U, V), X) + K (\Theta (U) V - \Theta (V) U, X) \right) \hat{n} .
\]

(24)

For the next step, by contracting (24) with \( \hat{n} \), we obtain the generalized Codazzi-Mainardi equation:

\[
g (\hat{n}, R (U, V) X) = \pm \left( (3 \, \nabla_U K) (V, X) - (3 \, \nabla_V K) (U, X) + K (3 T (U, V), X) + K (\Theta (U) V - \Theta (V) U, X) \right) \hat{n} .
\]

(25)

Moreover, contracting (24) by \( Y \in T_p \Sigma \) gives the generalized Gauss equation:

\[
g (Y, R (U, V) X) = g (Y, 3 R (U, V) X) + K (V, X) K (U, Y) - K (U, X) K (V, Y) ,
\]

(26)

where \( Y \) could be written in the adapted coordinate as \( Y = Y^i \partial_i \), with \( g (Y^i \partial_j , \partial_i) = 3 Y^i \), and \( Y^* = 3 Y_i \, dx^i \in T_p \Sigma \). Equation (8) could be equally contracted with \( \hat{n} \) and \( Z \in T_p \Sigma \) to give the torsion decomposition relations:

\[
g (\hat{n}, T (X, Y)) = (K (X, Y) - K (Y, X)) ,
\]

(27)

\[
g (Z, T (X, Y)) = g (Z, 3 T (X, Y)) .
\]

(28)

These four equations are the main results in this article and will be heavily used in the next sections. Notice that if the connection is metric and torsionless, then \( K (U, V) = \mp K (U, V) \), \( \Theta (U) = 0 \), and \( T (U, V) = 0 \), hence the generalized GCM returns to the standard original GCM.

III. METRIC, TORSIONLESS, AND LEVI-CIVITA CASES

In this section, we will discuss some special cases of the affine connection, in particular, metric, torsionless, and Levi-Civita connection. We will show that the conditions applied on the connection \( \nabla \) on \( \mathcal{M} \) are inherited to the connection \( 3 \nabla \) on the hypersurface \( \Sigma \).

To prevent any confusion, we need to clarify the conventions we use to write the notations indices of some geometrical objects. This is important because of the different conventions used in the literature. First, we write, using indices, the (vector potential of the) connection as \( \nabla_{\mu} \partial_{\beta} = \omega_{\mu}^{\alpha} \partial_{\alpha} \), the torsion tensor as \( T (\partial_{\mu} , \partial_{\beta}) = T_{\mu}^{\alpha} \partial_{\alpha} \), and the generalized Riemann tensor as \( R (\partial_{\mu} , \partial_{\nu}) \partial_{\beta} = R_{\mu \nu}^{\alpha} \partial_{\alpha} \). These conventions are different from the ones used in, for examples [2, 34].
A metric connection is a connection that satisfies the metric compatibility condition:
\[ \nabla_X g = X^\mu \nabla_\mu \left( g_{\alpha\beta} dx^\alpha dx^\beta \right) = 0, \quad \forall X \in T_p \mathcal{M}. \] (29)

The condition could be written in components, with abuse of notation as follows:
\[ \nabla_\mu g_{\alpha\beta} = \partial_\mu g_{\alpha\beta} - \omega_{\mu\beta} - \omega_{\mu\alpha} = 0. \] (30)

It describes the failure of the last two indices of \( \omega \) (borrowing the term from [35], the 'rotation' bivector indices) to be antisymmetric, differing with the quantity \( \partial_\mu g_{\alpha\beta} \). In the next paragraph, we will show that the metric compatibility of a connection in \( \mathcal{M} \) will induce also a metric connection in \( \Sigma \).

Let us remember that the metric \( g \) in \( \mathcal{M} \) could be decomposed into the perpendicular and parallel parts with respect to the embedded hypersurface \( \Sigma \), as in (1):
\[ g = g_{\parallel} \pm \frac{g_{\perp}}{3_q} n^* \otimes n^*. \] (31)

With this, we could obtain the following relation:
\[ \nabla_X g = \nabla_X g_{\parallel} \pm \nabla_X g_{\perp} = 0, \quad X \in T_p \Sigma, \] (32)

from the metric compatibility. On the other hand, we could directly decompose \( \nabla_X g \) as:
\[ \nabla_X g = (\nabla_X g)_{\parallel} \pm (\nabla_X g)_{\perp} = 0, \quad \implies (\nabla_X g)_{\parallel} = 0, \quad (\nabla_X g)_{\perp} = 0. \]

Notice that in general, \( (\nabla_X g)_{\parallel} \neq (\nabla_X g)_{\parallel} \) and \( (\nabla_X g)_{\perp} \neq (\nabla_X g)_{\perp} \). Now let us consider the following quantity:
\[ \nabla_X 3_q = (\nabla_X 3_q)_{\parallel} \pm (\nabla_X 3_q)_{\perp}. \] (33)

Since \( X \in T_p \Sigma \) and \( (\nabla_X 3_q)_{\parallel} \) live in the hypersurface \( \Sigma \), we could define:
\[ (\nabla_X 3_q)_{\parallel} := 3 \nabla_X 3_q, \] (34)

with \( 3 \nabla \) is the induced connection on hypersurface \( \Sigma \). With the definition (34) and (32), we obtain:
\[ 3 \nabla_X 3_q = \mp \nabla_X g_{\perp} \pm (\nabla_X g_{\perp})_{\perp}. \]

Decomposing \( \nabla_X g_{\perp} \) in its parallel and perpendicular part (and also using (32)), we have:
\[ 3 \nabla_X 3_q = \mp (\nabla_X g_{\parallel}). \]

Using (31), then one obtains:
\[ 3 \nabla_X 3_q = \mp ((\nabla_X n^* \otimes n^*)_{\parallel} = \mp ((\nabla_X n^*) \otimes n^* + n^* \otimes \nabla_X n^*). \] (35)

Notice that the RHS of (35) could be written as a projection of such quantity to \( \Sigma \) using a projection operator \( p_\Sigma = g^{\mu\lambda} 3_q \lambda \mu \partial_\mu \otimes dx^\nu \), namely:
\[ ((\nabla_X n^*) \otimes n^* + n^* \otimes \nabla_X n^*)_{\parallel} = p_\Sigma \cdot ((\nabla_X n^*) \otimes n^* + n^* \otimes \nabla_X n^*) \cdot p_\Sigma, \]

with \( \cdot \) denotes the inner product (matrix multiplication) between matrices. One could write:
\[ p_\Sigma \cdot (n^* \otimes \nabla_X n^*) = p_\Sigma (n^*) \otimes \nabla_X n^*, \]
\[ ((\nabla_X n^*) \otimes n^*) \cdot p_\Sigma = (\nabla_X n^*) \otimes p_\Sigma (n^*). \]

Since \( p_\Sigma (n^*) = 0 \) (\( 3_q \) lives in \( \Sigma \) and \( n \) is perpendicular to \( \Sigma \) ), we have:
\[ 3 \nabla_X 3_q = 0. \]

Therefore, we have shown that the metric compatibility of a connection \( \nabla \) on (\( \mathcal{M}, g \)), induces a connection \( 3 \nabla \) on the hypersurface (\( \Sigma, 3_q \)), which also satisfies the metric compatibility with respect to \( 3_q \).
Torsionless Condition

Let us perform a similar derivation for a special condition on the torsion tensor. The torsionless condition is a case where the connection has zero torsion:

\[ T(X,Y) = 0, \quad \forall X,Y \in T_pM. \]  

(36)

With the definition of torsion in (7), a direct consequence is:

\[ T(\partial_\alpha, \partial_\beta) = \left( \omega_\sigma^\sigma - \omega_\beta^\sigma \right) \partial_\sigma = 0, \quad \omega_\alpha^\sigma = \omega_\beta^\sigma, \]  

(37)

which results in the symmetry of the first and third index of the connection. Notice here that we do not use factor 2 on the definition of torsion, as in [2, 34]. Using the torsion decomposition (8), the torsionless condition becomes:

\[ 3T(X,Y) + (K(X,Y) - K(Y,X)) \hat{n} = 0. \]

From the definition of the first extrinsic curvature (5), and then using (37), we have:

\[ 3T(X,Y) = 0. \]

Hence, the torsionless connection \( \nabla \) on \( M \) induced a torsionless connection \( 3\nabla \) on \( \Sigma \).

Levi-Civita Connection

Since the metric compatibility and torsionless condition are independent of one another, one could define a connection that satisfies both of the conditions. This connection is unique and known as the Levi-Civita connection. A metric compatible and torsionless connection \( \nabla \) on \((M,g)\) induces a connection \( 3\nabla \) on \((\Sigma, 3q)\) which is also metric (with respect to \( 3q \)) and torsionless.

The Levi-Civita connection could be derived as follows. With the metric compatibility condition (30), one considers a certain combination of the vector potential satisfies equality as follows:

\[ \omega_{\mu\alpha\beta} + \omega_{\mu\beta\alpha} - \omega_{\alpha\beta\mu} + \omega_{\beta\alpha\mu} = \partial_\mu g_{\alpha\beta} - \partial_\alpha g_{\beta\mu} + \partial_\beta g_{\mu\alpha}. \]  

(38)

Using the torsionless condition (37) on the LHS, one could obtain, and then define the (vector potential of the) Levi-Civita connection:

\[ \omega_{\mu\alpha\beta} := \Gamma_{\mu\alpha\beta} = \frac{1}{2} \left( \partial_\mu g_{\alpha\beta} - \partial_\alpha g_{\beta\mu} + \partial_\beta g_{\mu\alpha} \right), \]  

(39)

also known as the Christoffel symbol. It is easy to show that the Levi-Civita connection is symmetric on the first and third indices, namely \( \Gamma^\mu_{\alpha\beta} = \Gamma^\beta_{\alpha\mu} \).

The Symmetric and Antisymmetric Parts of Connections, Torsion, and Curvatures

A general affine connection could be decomposed into its Levi-Civita, torsion, and non-metricity part. The derivation could be found in classic literature, for example, in [36]. Let us consider the previous combinations of \( \omega \) as in (38), but now without the metric compatibility. With (29), one has:

\[ \omega_{\mu\alpha\beta} + \omega_{\mu\beta\alpha} - \omega_{\alpha\beta\mu} + \omega_{\beta\alpha\mu} = \partial_\mu g_{\alpha\beta} - \nabla_\mu g_{\alpha\beta} - \partial_\alpha g_{\beta\mu} + \nabla_\alpha g_{\beta\mu} + \partial_\beta g_{\mu\alpha} - \nabla_\beta g_{\mu\alpha}. \]

Notice here that we do not use any requirement for the equation. Using (7) to rewrite the LHS, one obtains:

\[ 2\omega_{\mu\alpha\beta} + T_{\mu\alpha\beta} + T_{\beta\alpha\mu} = \partial_\mu g_{\alpha\beta} - \partial_\alpha g_{\beta\mu} + \partial_\beta g_{\mu\alpha} - (\nabla_\mu g_{\alpha\beta} - \nabla_\alpha g_{\beta\mu} + \nabla_\beta g_{\mu\alpha}), \]

that could be written as:

\[ \omega_{\mu\alpha\beta} = \Gamma_{\mu\alpha\beta} + C_{\mu\alpha\beta} + Q_{\mu\alpha\beta}, \]  

(40)
with the Levi-Civita connection $\Gamma$ satisfies (39), and:

$$Q_{\mu\alpha\beta} = -\frac{1}{2} (\nabla_\mu g_{\alpha\beta} - \nabla_\alpha g_{\beta\mu} + \nabla_\beta g_{\mu\alpha}),$$

(41)

$$C_{\mu\alpha\beta} = \frac{1}{2} (T_{\mu\alpha\beta} - T_{\alpha\beta\mu} + T_{\beta\mu\alpha}).$$

(42)

$Q$ and $C$ are the non-metricity and the torsion parts, usually known respectively, as the disformation/deflection and the contorsion tensor. If the disformation $Q$ is zero, the connection is metric, and if the contorsion $C$ is zero, the connection is torsionless. If both of them are zero, the connection is Levi-Civita.

Let $(\alpha \ldots |\beta)$ and $[\alpha \ldots |\beta]$ denote, respectively, the symmetric and antisymmetric part of an arbitrary tensor with indices $\alpha$ and $\beta$. With these notations, one could write the symmetries of $\Gamma$, $Q$ and $C$ as follows:

$$\Gamma_{\mu[\alpha|\beta]} = \frac{1}{2} \partial_{[\alpha} g_{\beta]|\mu],$$

$$Q_{[\mu|\alpha|\beta]} = -\frac{1}{2} \nabla_{[\mu} g_{\alpha|\beta]},$$

$$C_{[\mu|\alpha|\beta]} = \frac{1}{2} (T_{\mu|\alpha|\beta} - T_{\alpha|\beta|\mu}),$$

$$Q_{\mu[\alpha|\beta]} = 0,$$

$$C_{\mu[\alpha|\beta]} = 0,$$

$$\Gamma_{\mu[\alpha|\beta]} = -\partial_{\mu} g_{\alpha|\beta],}$$

$$Q_{\mu[\alpha|\beta]} = -\frac{1}{2} \nabla_{\mu} g_{\alpha|\beta]},$$

$$C_{\mu[\alpha|\beta]} = 0.$$  

From these relations, it is clear that $\Gamma$ and $Q$ have similar symmetries, in particular, they are symmetric in the first and second indices. On the other hand, the contorsion $C$ is antisymmetric in the second and third indices. One should keep in mind that other authors use different conventions for the order of the indices, in particular [2, 34], but these do not alter the physical symmetries of the connection.

With the decomposition of connection (40), one could obtain directly the decomposition of extrinsic curvature of the first kind:

$$K(X,Y) = \pm X^\mu Y^\beta \frac{\Gamma_{\mu|\alpha\beta n}}{K_\Gamma(X,Y)} + \pm X^\mu Y^\beta Q_{\mu|\alpha\beta n} + \pm X^\mu Y^\beta C_{\mu|\alpha\beta n},$$

$$K_\Gamma(X,Y),$$

$$K_Q(X,Y),$$

$$K_C(X,Y),$$

where $K_\Gamma$ and $K_Q$ are symmetric on the first and second argument, hence, the torsionless condition $C = 0$, gives a symmetric extrinsic curvature. On the other hand, the second extrinsic curvature satisfies the following decomposition:

$$K(X,Y) = X^\mu Y^\alpha n^\beta \frac{\Gamma_{\mu|\alpha\beta}}{K_\Gamma(X,Y)} + X^\mu Y^\alpha n^\beta Q_{\mu|\alpha\beta} + X^\mu Y^\alpha n^\beta C_{\mu|\alpha\beta},$$

$$K_\Gamma(X,Y),$$

$$K_Q(X,Y),$$

$$K_C(X,Y),$$

Notice that now $K_\Gamma$ and $K_Q$ are not symmetric on the first and second argument. However, for a metric connection, the extrinsic curvatures could be written as:

$$K(X,Y) = \mp \left( (\nabla_X g)(Y,n) + g(Y,\nabla_X n) \right) = \mp K(X,Y),$$

which comes from relation (22). Hence, the extrinsic curvature of the first kind coincides with the second kind.

The torsion could be written as:

$$T(X,Y) = T_\Gamma(X,Y) + T_C(X,Y) + T_Q(X,Y) = X^\mu Y^\beta \left( C_{\mu|\alpha\beta} - C_{\beta|\mu\alpha} \right) \partial_\alpha,$$

(43)

from the symmetries of the parts of the affine connection.

The Riemann curvature (9) could be rewritten in indices as follows:

$$R(\partial_\mu, \partial_\nu) \partial_\beta = \nabla_\mu \nabla_\nu \partial_\beta - \nabla_\nu \nabla_\mu \partial_\beta = \left( \partial_\mu \omega_\nu^{\alpha\beta} - \partial_\nu \omega_\mu^{\alpha\beta} + \omega_\mu^{\alpha\gamma} \omega_\nu^{\gamma\beta} - \omega_\nu^{\alpha\gamma} \omega_\mu^{\gamma\beta} \right) \partial_\alpha.$$  

(44)

One could insert (40) to (44), to obtain the decomposition of the Riemann tensor in terms of $\Gamma$, $Q$, and $C$, but it is not necessary for the derivation in this article. Unlike the first extrinsic curvature and the torsion tensor, the Riemann tensor could not be cleanly decomposed because of the existence of the non-linear terms in (44).
It is convenient to lower all the indices of the Riemann tensor to obtain clearer symmetries:

\[ R_{\mu\nu\alpha\beta} = \partial_\mu \omega_{\nu\alpha\beta} - \partial_\nu \omega_{\mu\alpha\beta} + \omega_\mu^\lambda \omega_{\nu\beta\lambda\alpha} - \omega_\nu^\lambda \omega_{\mu\beta\lambda\alpha} + \omega_\mu^\lambda \nabla_\nu g_{\lambda\alpha} - \omega_\nu^\lambda \nabla_\mu g_{\lambda\alpha}. \]  

(45)

Notice that the generalized Riemann tensor (44) only satisfies the antisymmetricity on the first and second index, namely, on the loop orientation indices [35]: \( R_{\mu\nu\alpha\beta} = -R_{\mu\nu\beta\alpha} \). Only if the connection is metric; \( Q = 0 \), one could show that the curvature satisfies an additional symmetry, namely \( R_{\mu\nu\alpha\beta} = -R_{\mu\nu\beta\alpha} \). Moreover, if the metric connection is also torsionless, it is symmetric by the switching between the loop orientation indices \((\mu, \nu)\) and the ‘rotation’ bivector indices \((\alpha, \beta)\), thus recovering the usual symmetries of the Riemannian tensor of the Levi-Civita connection: \( R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu} \), where the loop orientation and the rotation bivector indices are symmetric.

**Bianchi Identities**

The generalized Riemann tensor satisfies the second Bianchi identity: For any vector \( X, Y, Z \in T_p M \), the second Bianchi identity is:

\[(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0.\]  

(46)

It is a consequence of the orthogonality of \( \omega \) and \( R \), namely, \( \delta_X R = d\omega \wedge \omega = 0 \), where \( \delta_X \) is the exterior covariant derivative of the connection \( \nabla \), see [32]. The Bianchi identity (46) could be written in terms of torsion as follows:

\[((\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y)) = \nabla_{[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]]} W \]

\[ + (R(X,T(Y,Z)) + R(Y,T(Z,X)) + R(Z,T(X,Y))) W.\]

The first term contains the Jacobi identity, which is zero: \([X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0\]. Hence, we have the second Bianchi identity in terms of torsion:

\[R(X,T(Y,Z)) + R(Y,T(Z,X)) + R(Z,T(X,Y)) = 0.\]

A torsionless connection satisfies another similar identity, the first Bianchi identity. One could show that the following relation is true:

\[R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = \nabla_X T(Y,Z) + \nabla_Y T(Z,X) + \nabla_Z T(X,Y) + T(X,[Y,Z]) \]

\[+ T(Y,[Z,X]) + T(Z,[X,Y]) - [X,[Y,Z]] - [Y,[Z,X]] - [Z,[X,Y]].\]

With the Jacobi identity and the torsionless condition, one obtains the first Bianchi identity:

\[R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.\]

**Contractions of the Generalized Riemann Tensor**

To understand the significance of Gauss-Codazzi equations for General Relativity, it is convenient to introduce the contractions of Riemann tensors, the Ricci tensor \( \text{Ric} \) and Ricci scalar \( \mathcal{R} \) as follows:

\[\text{Ric} = \langle dx^\mu, R(\partial_\mu, \partial_\nu) \partial_\beta \rangle dx^\nu \otimes dx^\beta = \frac{\delta^\alpha\beta}{R_{\alpha\beta}} R_{\mu\nu} \alpha dx^\nu \otimes dx^\beta,\]

(47)

\[\mathcal{R} = \text{trRic} = g^{\alpha\beta} R_{\alpha\beta}.\]

(48)

The lack of symmetries of the generalized Riemann tensor results in the ambiguities in defining the Ricci tensor. Contracting the index \( \nu \) with \( \alpha \) of the Riemann tensor \( R_{\mu\nu\alpha\beta} \) will give the minus of (47), but contracting either \( \mu \) or \( \nu \) with \( \beta \) instead of \( \alpha \) will give a different quantity due to the lack of symmetry in the indices \((\alpha, \beta)\). This quantity is defined as the co-Ricci tensor [28]:

\[\bar{\text{Ric}} = \frac{g^{\alpha\beta} R_{\mu\nu\alpha\beta} dx^\nu \otimes dx^\alpha}{R_{\alpha\beta}}.\]

(49)
Another type of contraction also exists, known as the homothetic curvature [27–29]:

\[ \text{tr} R(\partial_\mu, \partial_\nu) = \delta^\beta_\alpha R^\alpha_{\mu\nu} \beta, \]

this, again, is not zero as in the Levi-Civita case, due to the non-metricity nature of the general affine connection.

Moreover, the Ricci tensor is no longer symmetric in the indices \((\nu, \beta)\). This will cause problem in constructing the Einstein tensor \(G\), that needs to be symmetric since it is proportional to the stress-energy tensor \(T\) via the Einstein-Field Equation \(G = \kappa T\). However, in Metric-Affine and \(f(R)\)-gravity, only the symmetric part of the Ricci tensor enters the equation of motion [2].

**IV. APPLICATION TO GRAVITY**

**A. The Modified Theories of Gravity**

*Second Order Formulation: The Standard GR and \(f(R)\)-Gravity*

The original derivation of Einstein Field Equation (EFE) from a variational principle was first considered by Einstein and Hilbert, where the action is \(S_{EH}[g] = R \text{vol} + S_{\text{matter}}[g]\), as a functional of the metric \(g\). The connection is assumed *a priori* to be Levi-Civita and therefore is a function of the metric. Minimizing the action with respect to the variation of \(g\) gives the standard EFE. This is known as the second-order formulation of gravity.

\(f(R)\)-gravity is one of the modification of General Relativity via a more general choice of action:

\[ S[g] = \int_M f(R) \text{vol} + S_{\text{matter}}[g], \quad (50) \]

where \(f(R)\) is the power series function of Ricci scalar with the form as follows [1, 2]:

\[ f(R) = \ldots + \frac{c_2}{R^2} + \frac{c_1}{R} - 2\Lambda + R + \frac{R^2}{k_2} + \frac{R^3}{k_3} + \ldots \]

Minimizing the action (50) with respect to the metric gives the \(f(R)\)-EFE as follows:

\[ f'(R) R_{(\alpha\beta)} - \frac{1}{2} f(R) g_{\alpha\beta} = \kappa T_{\alpha\beta}. \quad (51) \]

\(T_{\alpha\beta}\) is the stress-energy-momentum tensor, obtained from the variation of \(g\) on \(S_{\text{matter}}\):

\[ \delta g S_{\text{matter}}[g] = - \int_M \kappa T_{\alpha\beta} \delta g^{\alpha\beta} \text{vol}. \quad (52) \]

Standard GR is a special case of \(f(R)\)-gravity, where \(f(R) = R\).

*First Order Formulation: The Palatini Formalism*

In a more general approach known as the Palatini formalism (or the first-order formulation of gravity), the connection is not assumed to be Levi-Civita, hence it is independent of the metric on \(M\) [34]. The Einstein-Hilbert action is now a functional over the metric \(g\) and connection \(\omega\):

\[ S_{EH}[g, \omega] = \int_M R \text{vol} + S_{\text{matter}}[g]. \quad (53) \]

\(\omega\) is a general affine connection defined in (40). Minimizing the action with respect to the variation of \(g\) gives an equation of motion similar to the EFE:

\[ R_{(\alpha\beta)} - \frac{1}{2} R g_{\alpha\beta} = \kappa T_{\alpha\beta}. \quad (54) \]
but with the torsional and non-metricity contributions inside $R_{[\alpha \beta]}$ and $\mathcal{R}$. Minimizing the action (53) with respect to the connection $\omega$ results in another equation of motion:

$$
\left( T^\lambda_\mu - \frac{1}{2} g^{\sigma \lambda} \nabla_\sigma g_{\kappa \lambda} - \nabla_\mu \right)_{\lambda} \left( 2 \delta_\mu^\nu g^{\alpha \beta} \right) + T^\alpha_\mu \omega^\beta = 0. \tag{55}
$$

$G_{\alpha \beta}$ and $P^\alpha_\mu$ are known, respectively, as the Einstein and Palatini tensor [29]. Although (55) is different from the result found in the literature [2, 29, 34], it could be shown that they are equal, up to the factor 2 in the third term. This is due to the difference in the definition of torsion in (7) which does not include the factor $\frac{1}{2}$, as defined in literatures, in particular [2, 29, 34].

As clearly derived in [29], the solution to (55) contains unspecified vector degrees of freedom on the connection. However, since the Einstein-Hilbert action (53) is invariant under a projective transformation, these vector degrees of freedom could be eliminated, which results in a condition that the connection must be metric compatible and torsionless, hence Levi-Civita [2, 34]. This will be discussed in the next subsection. With the Levi-Civita connection, (54) returns to the original EFE. Therefore, by considering the projective transformation, the Palatini formalism gives a similar dynamics with the second-order formulation [29].

**The Projective Invariance Problem**

The projective transformation is defined as follows [2, 8, 29]:

$$
\omega^\alpha_\mu \rightarrow \omega^\alpha_\mu + \delta^\alpha_\beta \xi_\mu. \tag{56}
$$

By a direct calculation, one could show that the Ricci scalar $\mathcal{R}$ is invariant under (56). This property could be utilized to obtain the Levi-Civita condition from (55) as follows. First, one could show that the Palatini tensor is traceless: $P^\alpha_\mu = 0$ [8, 29]. Therefore, in 4-dimension, one could only obtain 60 independent equations from (55). These equations are not enough to determine completely the connection, as it will need 64 independent equations. Instead, the maximal condition on the connection one could obtain from (55) is:

$$
\omega^\alpha_\mu = \Gamma^\alpha_{\mu \beta} - \frac{1}{(n-1)} T^\lambda_\mu \delta^\alpha_\beta = \Gamma^\alpha_{\mu \beta} + \frac{1}{n} \left( -\frac{1}{2} g^{\sigma \lambda} \nabla_\mu g_{\sigma \lambda} \right) \delta^\beta_\mu, \tag{57}
$$

with $n$ is the dimension of $\mathcal{M}$, as shown in a detailed derivation carried in [8, 29]. $\Gamma^\alpha_{\mu \beta}$ is the Levi-Civita connection, while $T_\mu$ and $Q_\mu$, are respectively, the torsion and disformation vectorial degrees of freedom ($Q_\mu$ is also known as the Weyl vector). Note that there exist differences in the factors in front of $T_\mu$ and $Q_\mu$ with [8, 29]; these are due to the different definition on torsion tensor (7).

However, since the action (53) is invariant under (56), one could eliminate the vectorial degrees of freedom using the transformation (56), by setting an appropriate value of $\xi_\mu$. In this article, we only consider the case where $\xi_\mu = \frac{1}{n} T_\mu$, with $n = 4$. For the case where $\xi_\mu = -\frac{1}{n} Q_\mu$, or moreover, a linear combination of $T_\mu$ and $Q_\mu$, one could consult [29, 30]. In the end, by performing the projective transformation (56) on (57), one could fix the 4 vectorial degrees of freedom on $\omega^\alpha_\mu$ such that (57) becomes Levi-Civita:

$$
\omega^\alpha_\mu = \Gamma^\alpha_{\mu \beta},
$$

with the condition:

$$
T_\mu = T^\lambda_\mu - C^\lambda_\mu = 0. \tag{58}
$$

(58) is known as the traceless torsion constraint. It needs to be kept in mind that the tracelessness of the torsion (58) is introduced at the kinematical level; it does not result from the dynamics, i.e. the equation of motion.

To obtain the Levi-Civita connection from the solution of the equation of motion, one needs to add a new term corresponding to the constraint into the action [2]:

$$
S_{EH} [g, \omega, \chi] = \int_\mathcal{M} \mathcal{R} [\omega] \text{vol} [g] + S_{\text{matter}} [g] + S_{LM} [\chi], \tag{59}
$$
where \( S_{LM} = \int_{\mathcal{M}} \chi^{\mu} T_{\mu} \text{vol} \) and \( \chi^{\mu} \) is the Lagrange multiplier corresponding to \( T_{\mu} \). Minimizing (59) with respect to \( g, \omega, \) and \( \chi \) gives:

\[
R_{(\alpha\beta)} - \frac{1}{2} \left( \mathcal{R} + \chi^{\mu} T_{\mu}^{\lambda} \right) g_{\alpha\beta} = \kappa T_{\alpha\beta},
\]

(60)

\[
(T_{\lambda}^{\lambda} + Q_{\lambda}^{\lambda} - \nabla_{\nu}) \left( 2\delta^{[\nu}_{\mu} g^{\alpha]\beta] \right) + T_{\mu}^{\alpha\beta} = \chi^{\alpha\delta}_{\mu} - \chi^{\beta\delta}_{\mu},
\]

(61)

\[
T_{\lambda}^{\lambda} = 0,
\]

(62)

that could be solved to obtain \( \chi^{\alpha} = 0 \), hence, giving the Levi-Ciita condition and the standard EFE. One could conclude that the Palatini formalism of gravity with the traceless torsion constraint is equivalent with GR \([8, 29]\).

**Metric-Affine Gravity, Metric-Affine f (R)-Gravity, and Metric-Affine GR**

Metric-Affine Gravity (MAG) is a large class of theories based on the first-order formalism with a general affine connection that includes torsion and non-metricity. The choice of action for MAG could vary greatly: the Ricci scalar \( \mathcal{R} \) for the Metric-Affine General Relativity (MAGR) or Generalized Palatini theory could be obtained from \( f (R, R_{\mu\nu} R^{\mu\nu}) \) and \( \mathcal{L} (g_{\mu\nu}, R_{\mu\nu}^{\alpha\beta}) \) \([29]\). In this article, we only consider Metric-Affine \( f (R) \) Gravity and Metric-Affine General Relativity.

The action of Metric-Affine \( f (R) \)-gravity is defined as:

\[
S [g, \omega, \chi] = \int_{\mathcal{M}} f (R [\omega]) \text{vol} + S_{\text{matter}} [g, \omega] + S_{LM} [\chi].
\]

(63)

Notice that now the matter action \( S_{\text{matter}} \) is also a functional of the affine connection \( \omega \). The variation of \( S_{\text{matter}} \) with respect to \( \omega \) gives the hypermomentum tensor \( \mathcal{H} \):

\[
\delta \omega S_{\text{matter}} [g, \omega] = -\kappa \mathcal{H}^{\alpha\beta}_{\mu} \delta \omega^{\mu} \text{vol},
\]

(64)

and therefore, minimizing the action (63) with respect to the \( g, \omega, \) and \( \chi \), results in three equations of motion:

\[
f' (R) R_{(\alpha\beta)} - \frac{1}{2} \left( f (R) + \chi^{\mu} T_{\mu}^{\lambda} \right) g_{\alpha\beta} = \kappa T_{\alpha\beta},
\]

(65)

\[
\left( T_{\lambda}^{\lambda} + Q_{\lambda}^{\lambda} - \nabla_{\nu} \right) \left( 2\delta^{[\nu}_{\mu} g^{\alpha]\beta] \right) + T_{\mu}^{\alpha\beta} = \kappa \mathcal{H}^{\alpha\beta}_{\mu} + 2\chi^{[\alpha}_{\nu} \delta^{\beta]}_{\mu},
\]

(66)

\[
T_{\lambda}^{\lambda} = 0.
\]

(67)

The last equation is the constraint equation. Notice that the hypermomentum only exists if the matter term \( S_{\text{matter}} \) is a functional of the connection, hence in the Palatini formalism, \( \mathcal{H} = 0 \). One could show that the torsion enters the dynamics through the antisymmetric part of the second and third indices \( \mathcal{H}^{[\alpha\mu\beta]} \), while the non-metricity enters through the symmetric part of the first and third indices \( \mathcal{H}^{(\alpha)\mu\beta} \), and if \( \mathcal{H}^{\alpha\beta}_{\mu} = 0 \), (66) and (67) give the requirements of Levi-Civita connection, hence the theory coincides with the original GR, for \( f (R) = R \). The equations of motions (66) and (67) are the crucial results in the \( f (R) \)-theory of gravity. However, due to the scope of this article, our focus will be on the stress-energy-momentum equation (65).

The theory of Metric-Affine General Relativity (MAGR) or Generalized Palatini theory could be obtained from Metric-Affine \( f (R) \)-gravity by setting \( f (R) = R \). With this requirement, (65)-(67) becomes:

\[
R_{(\alpha\beta)} - \frac{1}{2} \left( \mathcal{R} + \chi^{\mu} T_{\mu}^{\lambda} \right) g_{\alpha\beta} = \kappa T_{\alpha\beta},
\]

(68)

\[
\left( T_{\lambda}^{\lambda} + Q_{\lambda}^{\lambda} - \nabla_{\nu} \right) \left( 2\delta^{[\nu}_{\mu} g^{\alpha]\beta] \right) + T_{\mu}^{\alpha\beta} = \kappa \mathcal{H}^{\alpha\beta}_{\mu} + 2\chi^{[\alpha}_{\nu} \delta^{\beta]}_{\mu},
\]

(69)

\[
T_{\lambda}^{\lambda} = 0.
\]

(70)

which could be simplified to obtain:

\[
R_{(\alpha\beta)} - \frac{1}{2} \mathcal{R} g_{\alpha\beta} = \kappa T_{\alpha\beta},
\]

(71)

\[
\left( Q_{\lambda}^{\lambda} - \nabla_{\nu} \right) \left( 2\delta^{[\nu}_{\mu} g^{\alpha]\beta] \right) + T_{\mu}^{\alpha\beta} = \kappa \left( \mathcal{H}^{\alpha\beta}_{\mu} - \frac{2}{3} \mathcal{H}^{(\alpha}_{\nu} \delta^{\beta]}_{\mu} \right),
\]

(72)

\[
T_{\lambda}^{\lambda} = 0,
\]

(73)
by inserting (70) to (68)-(69) and solving $\chi$ from (69). Let us first focus only on the stress-energy-momentum equation (71). The quantity in the LHS of (71) is known as the generalized Einstein tensor:

$$G_{\alpha\beta} = R_{(\alpha\beta)} - \frac{1}{2}Rg_{\alpha\beta},$$

which is symmetric on the $(\alpha, \beta)$-indices. The stress-energy-momentum equation (71) could be written in an equivalent form as follows $[33]$:

$$R_{(\alpha\beta)} = \kappa T_{\alpha\beta} - \frac{1}{2}Tg_{\alpha\beta},$$

with $T = g^{\alpha\beta}T_{\alpha\beta}$ is the trace of the stress-energy momentum tensor (52). In the next part of this section, (71) will be decomposed into its temporal and spatial part using the GCM equations derived in (24).

\[ \text{B. (3+1) Decomposition for MAGR} \]

The Adapted Coordinate, Lapse Function, and Shift Vector

Let $M$ be a globally hyperbolic Lorentzian manifold, and let $x^\mu = \{x^0, x^i\}$ be a general local coordinate on $M$. The coordinate vector basis on $T_pM$ is $\partial_\mu = \{\partial_0, \partial_i\}$. Let $\partial_0$ be the temporal component of $\partial_\mu$, that could be decomposed according to the hypersurface $\Sigma$ as follows:

$$\partial_0 = \partial_0 + g(\partial_0, \hat{n}) \hat{n} + g(\partial_0, \hat{n}) \hat{n},$$

(75)

(and hence we take the lower part of the $\pm$ signature in (75)). The scalar $N$ is the lapse function describing the normal component of $\partial_0$, while the vector $N = N^i \partial_i$ is the shift vector describing the parallel part of $\partial_0$.

Let the metric $g$ be written in terms of the component in the local coordinate $\partial_\mu$, using (75):

$$g(\partial_0, \partial_0) = g_{00} = N^u N_\mu - N^2,$$

$$g(\partial_0, \partial_i) = g_{0i} = N_i + N n_i,$$

$$g(\partial_i, \partial_j) = g_{ij} = \delta_{ij} - n_i n_j,$$

(78)

Comparing with (1), one could obtain $g_{00} = N_0 = N^u N_\mu$, $g_{0i} = N_i$, and $n_0 = -N$.

Let us consider the adapted coordinate on $M$ (it had been mentioned on the previous sections) where $\{x^i\}$, $i = 1, 2, 3$, is a local coordinate on $\Sigma$. Notice that in this special coordinate, $n \perp \partial_i$, so that $n_i = g(n, \partial_i) = 0$. Moreover, in this coordinate, the shift $N \in T_p \Sigma$ does not have a temporal component, namely $N^0 = 0$. Using the adapted coordinate, the spatial and the temporal part of $\Sigma$ could be cleanly separated.

The components of metric $g$ could be written in the adapted coordinate as follows; (76)-(78) becomes:

$$g(\partial_0, \partial_0) = g_{00} = N^i N_i - N^2,$$

$$g(\partial_0, \partial_i) = g_{0i} = N_i,$$

$$g(\partial_i, \partial_j) = g_{ij} = \delta_{ij} - q_{ij},$$

while the components of the inverse metric $g^{-1}$, using the convention in (20), are:

$$g^* (dx^0, dx^0) = g^{00} = -N^{-2},$$

$$g^* (dx^0, dx^i) = g^{0i} = \delta_{ij} - (N^i N_j) N^{-2},$$

with the coordinate basis vector on $T_p^* M$ satisfies $dx^0 = -\hat{n}^* N^{-1}$ and $dx^i = \hat{n}^* N^i N^{-1} + \delta q dx^i$. Notice that $dx^i$ is not necessarily equal to $\delta q dx^i = \delta q^j \delta q (\partial_j)$ in (23).

In the adapted coordinate, one could check that the following relations are true:

$$n = n^0 \partial_0 + n^i \partial_i = N^{-1} (\partial_0 - N^i \partial_i),$$

$$n^* = n_0 dx^0 + n_i dx^i = -N dx^0,$$

$$N = N^0 \partial_0 + N^i \partial_i = -N n^i \partial_i,$$

$$N^* = N_0 dx^0 + N_i dx^i = N^i N_i dx^0 + \delta q_{ij} N^j dx^i.$$
These relations will be useful for the following derivation.

Applying the adapted coordinate to the Riemann curvature and torsion decomposition, namely (24) and (27)-(28), one obtains:

\[
R_{ij}^0{}^k = (3\nabla_i K_{jk} - 3\nabla_j K_{ik} + 3T_{ij}^l K_{lk} + \Theta_i K_{jk} - \Theta_j K_{ik}) n^0, \tag{79}
\]
\[
R_{ij}^k = 3R_{ij}^k + K_{jl}K_{ij}^k - K_{il}K_{ij}^k + (3\nabla_i K_{jl} - 3\nabla_j K_{il} + 3T_{ij}^m K_{ml} + \Theta_i K_{jl} - \Theta_j K_{il}) n^k, \tag{80}
\]
which are, respectively, the components of Codazzi-Mainardi and Gauss equations, and:

\[
\hat{n}_\mu T_{ij}^\mu = K_{ij} - K_{ji}, \tag{81}
\]
\[
g_{\mu\nu} T_{ij}^\mu = 3q_{kl}^3 T_{ij}^l, \tag{82}
\]
which are the components of torsion decomposition. Here, the indices of the extrinsic curvatures are raised with the 3-metric, for example, \( K_{ij}^\beta = 3q^{ij} 3 K_{ik} \).

**The Energy Part**

Let us consider the purely-time part of the Einstein tensor, where the \( \alpha, \beta \) indices of \( G_{\alpha\beta} \) is contracted with the normal \( \hat{n} \) to give the following scalar quantity:

\[
G_{\alpha\beta} n^\alpha n^\beta = G(\hat{n}, \hat{n}) = \text{Ric}(\hat{n}, \hat{n}) - \frac{1}{2} g(\hat{n}, \hat{n}) \mathcal{R}. \tag{83}
\]

By a direct calculation, one could show that the generalized Ricci scalar could be decomposed into:

\[
\mathcal{R} = 3\mathcal{R} - \text{tr} (KK) + (K) (trK) - \text{Ric}(\hat{n}, \hat{n}) + \text{Ric}(\hat{n}, \hat{n}), \tag{84}
\]
where \( \text{Ric} \) is the co-Ricci tensor satisfying (49). Inserting (84) to (83) gives:

\[
G(\hat{n}, \hat{n}) = \frac{1}{2} \left( \text{Ric}(\hat{n}, \hat{n}) + \text{Ric}(\hat{n}, \hat{n}) \right) + \frac{1}{2} \left( 3\mathcal{R} - \text{tr} (KK) + (K) (trK) \right). \]

The first term is the additional part due to the non-metricity and torsion, this will be clear in the next subsections.

The normal part of the stress-energy-momentum tensor (52) is the energy density, namely \( \mathcal{T}(\hat{n}, \hat{n}) = E \), and the energy equation in MAG is:

\[
\frac{1}{2} \left( \text{Ric}(\hat{n}, \hat{n}) + \text{Ric}(\hat{n}, \hat{n}) \right) + \frac{1}{2} \left( 3\mathcal{R} - \text{tr} (KK) + (K) (trK) \right) = \kappa E. \tag{85}
\]

For metric connections, \( \text{Ric} = -\text{Ric} \) and \( K = K \), hence, for Levi-Civita connection, (85) returns to the original form, the Hamiltonian constraint. Notice that (85) is also valid for a metric connection with torsion, since the antisymmetric part of \( 3\mathcal{R} \) and \( K \), resulting from a non-vanishing torsion, do not contribute to the energy equation.

**The Momentum Part**

The momentum part of the Einstein tensor is a mixture between the time and spatial parts. In the adapted coordinate, it could be written as follows:

\[
G_{\mu\nu} n^\mu = G(\partial_i, \hat{n}) = G(\hat{n}, \partial_i) = \frac{1}{2} \left( \text{Ric}(\partial_i, \hat{n}) + \text{Ric}(\hat{n}, \partial_i) \right). \tag{86}
\]

where the term containing the Ricci scalar is zero due to the fact that in the adapted coordinate, \( g(\partial_i, \hat{n}) = n_i = 0 \). From a direct calculation, one could show that the terms containing the 3-covariant derivative of the extrinsic curvature in (79) comes from the co-Ricci tensor, instead of the Ricci tensor:

\[
\text{Ric}(\partial_i, \hat{n}) = q^{jk} (3\nabla_i K_{jk} - 3\nabla_j K_{ik} + 3T_{ij}^l K_{lk} + \Theta_i K_{jk} - \Theta_j K_{ik}) - g(\hat{n}, R(\hat{n}, \partial_i) \hat{n}). \tag{87}
\]
For a metric connection, \( \text{Ric} = -\overline{\text{Ric}} \), so one could immediately insert (87) to (86). This is not the case for a general connection. Hence, in MAGR, there is no direct way to write the momentum equation in terms of the intrinsic curvature of the first kind \( K \). Let us postponed this problem for the moment and write the momentum equation as follows:

\[
\frac{1}{2} (\text{Ric} (\partial_i, \hat{n}) + \text{Ric} (\hat{n}, \partial_i)) = \kappa p_i, \tag{88}
\]

with \( p_i \) is the 'mixed' part of the stress-energy-momentum tensor (52), namely, the (relativistic) momentum \( T (\partial_i, \hat{n}) = T (\hat{n}, \partial_i) = p_i \).

For a metric connection with torsion, the momentum equation becomes:

\[
3 \nabla_j K^j_i - 3 \nabla_i K^j_j - 3 T^k_i K^{kj} + \frac{1}{2} \eta^\nu (\nabla_\nu T^\mu_\mu) = \kappa p_i, \tag{89}
\]

where in general, the extrinsic curvature \( K \) contains an antisymmetric part. For the Levi-Civita case, the torsion vanishes, and (88) returns to the original momentum (or diffeomorphism) constraint, with symmetric \( K \).

**The Stress-Energy Part**

The purely-spatial part of the stress-energy-momentum equation are the following set of equations:

\[
G_{ij} = G (\partial_i, \partial_j) = \frac{1}{2} (\text{Ric} (\partial_i, \partial_j) + \text{Ric} (\partial_j, \partial_i)) - \frac{1}{2} g (\partial_i, \partial_j) \mathcal{R}. \tag{90}
\]

The first term, namely, the spatial part of the Ricci tensor, could be decomposed as follows:

\[
\text{Ric} (\partial_i, \partial_j) = 3 \text{Ric} (\partial_i, \partial_j) + K_{ij} \text{tr} \mathcal{K} - K^{k}_{(i} K^{k}_{j)} - g (\hat{n}, \textbf{R} (\hat{n}, \partial_i) \partial_j). \tag{91}
\]

Inserting (91) and the Ricci scalar (84) to (90) gives immediately the spatial part of \( G \):

\[
G (\partial_i, \partial_j) = G (\partial_i, \partial_j) - g (\hat{n}, \textbf{R} (\hat{n}, \partial_i) \partial_j) + K_{(ij)} \text{tr} \mathcal{K} - K^{k}_{(i} K^{k}_{j)}
- \frac{1}{2} q_{ij} ((\text{tr} \mathcal{K}) (\text{tr} \mathcal{K}) - \text{tr} (K \mathcal{K}) - \text{Ric} (\hat{n}, \hat{n}) + \overline{\text{Ric}} (\hat{n}, \hat{n})),
\]

with:

\[
3 G (\partial_i, \partial_j) = \frac{1}{2} (3 \text{Ric} (\partial_i, \partial_j) + 3 \text{Ric} (\partial_j, \partial_i) - 3 q_{ij} 3 \mathcal{R})
\]

is the Einstein tensor on \( \Sigma \).

The spatial part of the stress-energy-momentum tensor (52) is the stress tensor \( T (\partial_i, \partial_j) = S (\partial_i, \partial_j) = S (\partial_j, \partial_i) \), hence the stress-energy equations of EFE is:

\[
3 G (\partial_i, \partial_j) - g (\hat{n}, \textbf{R} (\hat{n}, \partial_i) \partial_j) + K_{(ij)} \text{tr} \mathcal{K} - K^{k}_{(i} K^{k}_{j)}
- \frac{1}{2} q_{ij} ((\text{tr} \mathcal{K}) (\text{tr} \mathcal{K}) - \text{tr} (K \mathcal{K}) - \text{Ric} (\hat{n}, \hat{n}) + \overline{\text{Ric}} (\hat{n}, \hat{n})) = \kappa S (\partial_i, \partial_j). \tag{92}
\]

One could write (92) in a more convenient way as follows. Projecting (74) to the hypersurface \( \Sigma \) gives:

\[
\frac{1}{2} (\text{Ric} (\partial_i, \partial_j) + \text{Ric} (\partial_j, \partial_i)) = \kappa S (\partial_i, \partial_j) - \frac{1}{2} g (\partial_i, \partial_j) \kappa T.
\]

Using (91), one could obtain:

\[
3 R_{(ij)} + K_{(ij)} \text{tr} \mathcal{K} - K^{k}_{(i} K^{k}_{j)} - g (\hat{n}, \textbf{R} (\hat{n}, \partial_i) \partial_j) = \kappa S_{ij} - \frac{1}{2} (3 q_{ij} \kappa T). \tag{93}
\]

In the original GR, the purely-spatial part of the EFE is the only part that contains the dynamics of the system, i.e., the equations which contain the time derivative of the 3-metric. In (93), the time derivative is hidden such that it is contained implicitly in the covariant term \( g (\hat{n}, \textbf{R} (\hat{n}, \partial_i) \partial_j) \). We will show that this is indeed the case in the following subsections.
The Additional Variables on the Hypersurface

In the geometrodynamics concept introduced by Wheeler [31], the system of GR could be equivalently described using only fields in $\Sigma$ which evolve in time. In this perspective, one does not need to refer to the 4-manifold $\mathcal{M}$ explicitly. However, the EFE (85), (88), and (93) contain some terms that are still covariant, therefore, these terms need to be decomposed into the temporal and spatial parts.

Let us regard $\hat{n}$, the unit normal to $\Sigma$, as a 4-velocity in a Eulerian frame. The 4-acceleration, defined as $\nabla_{\hat{n}}\hat{n} = a = \alpha^\mu \partial_\mu$ in (15), has time and spatial components as follows:

$$\begin{align*}
\langle \hat{n}^*, \nabla_{\hat{n}}\hat{n} \rangle &= g\left( \nabla_{\hat{n}}\hat{n}, \hat{n} \right) = g\left( \alpha, \hat{n} \right) := -\Theta \left( \hat{n} \right), \\
\langle 3dx^i, \nabla_{\hat{n}}\hat{n} \rangle &= \langle 3dx^i, \alpha \rangle := \alpha^i.
\end{align*}$$

Equation (94), already defined in (16), is the angle between the 4-velocity with the 4-acceleration which is zero for a metric connection. (95) is the components of the (relativistic) 3-acceleration, $\Theta$ defined in (16), is the angle between the 4-velocity with the 4-acceleration which is zero for a metric connection. (95) is the components of the (relativistic) 3-acceleration, $\Theta$ (94), already defined in (16), is the angle between the 4-velocity with the 4-acceleration which is zero for a metric connection. (95) is the components of the (relativistic) 3-acceleration, $\Theta$ defined in (16), is the angle between the 4-velocity with the 4-acceleration which is zero for a metric connection. (95) is the components of the (relativistic) 3-acceleration, $\Theta$ defined in (16), is the angle between the 4-velocity with the 4-acceleration which is zero for a metric connection.

The acceleration $\alpha$ is a part of the acceleration tensor $a$ defined in (15), the other part of the tensor are the following:

$$\begin{align*}
\langle \hat{n}^*, \nabla_{\hat{n}}\hat{n} \rangle &= g\left( \nabla_{\hat{n}}\hat{n}, \hat{n} \right) = g\left( a_i, \hat{n} \right) := -\Theta_i,
\end{align*}$$

Together with $a_i = \alpha$, define the acceleration tensor $a = (a_i, a_j)$, $a_i$ is the rate of change of the 4-velocity in the spatial direction $\partial_i$; therefore $\Theta_i$ is the temporal components of $a_i$ or the angle between $a_i$ and $\hat{n}$. One could similarly obtain the spatial components of $a_i$, but one could prove using equation (1), (17), and (21), that this is only the extrinsic curvature of the second kind:

$$\begin{align*}
\langle 3dx^j, \nabla_i\hat{n} \rangle &= 3q^{jk}g\left( \nabla_i\hat{n}, \partial_k \right) = 3q^{jk}g\left( a_i, \partial_k \right) = 3q^{jk}\hat{K}_{ik} = \hat{K}_{i}^j,
\end{align*}$$

which had been defined in (17).

Having defined all the covariant derivative of $\hat{n}$ in all directions and its components, now we could proceed to define the covariant derivative of the spatial direction. The first two are the components of $\nabla_i\partial_j$, the rate of change of $\partial_j$ in the direction $\partial_i$:

$$\begin{align*}
\langle \hat{n}^*, \nabla_i\partial_j \rangle &= g\left( \nabla_i\partial_j, \hat{n} \right) = g\left( \omega_i^\mu \partial_\mu, \hat{n} \right) = -K_{ij}, \\
\langle 3dx^k, \nabla_i\partial_j \rangle &= \langle 3dx^k, \omega_i^\mu \partial_\mu \rangle = \omega_i^k.
\end{align*}$$

(98) is the time component of $\nabla_i\partial_j$ which is exactly the extrinsic curvature of the first kind defined in (4). The spatial component of $\nabla_i\partial_j$ is exactly the induced 3-connection (99) on $\Sigma$.

The last quantities are the components of $\nabla_i\partial_i$, which define the rate of change of $\partial_i$ in the direction of $\hat{n}$, i.e., the evolution of $\partial_i$ in time:

$$\begin{align*}
\langle \hat{n}^*, \nabla_i\partial_i \rangle &= g\left( \nabla_i\partial_i, \hat{n} \right) = g\left( n^\alpha \omega_\alpha^\mu \partial_\mu, \hat{n} \right) = \omega \left( \hat{n} \right)^\mu n_\mu := -\Delta_i, \\
\langle 3dx^i, \nabla_i\partial_i \rangle &= \langle 3dx^i, n^\alpha \omega_\alpha^\mu \partial_\mu \rangle = \omega \left( \hat{n} \right)^i := \Delta_i.
\end{align*}$$

One could think of $\Delta_i$ and $\Delta_i^j$ as the generator of the evolution of $\partial_i$. (100) is the temporal component of the evolution, which drags $\partial_i$ along the normal direction, while (101) are the spatial components of the evolution, which moves $\partial_i$ along $\Sigma$. Since $\nabla$ is not only a differentiation $\partial$, but it also rotates and shears vectors by $\omega$, the existence of the spatial components (101) is understandable.

Notice that in the original GR, the EFE could be written as functions of the following variables: the extrinsic curvature, the 3-connection (which is a function of metric) in terms of 3-Ricci scalar and 3-Einstein tensor, the 3-acceleration in terms of the lapse $N$, and the evolution part $\Delta^j_i$, which contains the shift $N$. These variables are not independent of one another. For MAGR (and hence MAG), we have 8 different (with 4 additional) variables: $K, \hat{K}, \hat{\dot{\theta}}$, $\Theta (\hat{n})$, $\Theta_i$, $\Delta_i$, and $\Delta_i^j$.

With the full variables on $\Sigma$, one could write the decompositions of the derivatives of $\hat{n}$ and $\partial_i$:

$$\begin{align*}
\nabla_i\hat{n} &= \Theta \left( \hat{n} \right) \hat{n} + \alpha^j \partial_j, \\
\nabla_i\partial_j &= \Theta_i \partial_j + \hat{K}^j_i \partial_j, \\
\nabla_i \partial_i &= \Delta_i \partial_i + \Delta_i^j \partial_j,
\end{align*}$$

\(102, 103, 104, 105\)
to rederive the torsion and the non-metricity factor in terms of the additional variables as follows. The decomposition of the torsion tensor is:

\[ T(n,n) = 0, \]
\[ T(n,\partial_i) = -T(\partial_i,n) = (\Delta_i - \Theta_i - g(\partial_i\hat{n}, \hat{n})) \hat{n} + \left( \Delta_i - K_i^j + \langle 3dx^j, \partial_i\hat{n} \rangle \right) \partial_j, \]
\[ T(\partial_i,\partial_j) = 3T(\partial_i,\partial_j) + (K(\partial_i,\partial_j) - K(\partial_j,\partial_i)) \hat{n}, \]

while the decomposition of the non-metricity factor is:

\[ \nabla_n g^* = -2\Theta(n) \hat{n} \otimes \hat{n} + (\Delta^i - \alpha^i)(\hat{n} \otimes \partial_i + \partial_i \otimes \hat{n}) + 3\nabla_n 3q^* \]
\[ \nabla_i g^* = -2\Theta_i \hat{n} \otimes \hat{n} + (K_i^j - K_i^j)(\hat{n} \otimes \partial_j + \partial_j \otimes \hat{n}) + 3\nabla_i 3q^* \]

where:

\[ 3\nabla_n 3q^* = \langle 3\nabla_n 3q^j \rangle \partial_i \otimes \partial_j = \left( n [3q^j] + \Delta^i 3q^{jk} + \Delta^j 3q^i \right) \partial_i \otimes \partial_j, \]
\[ 3\nabla_i 3q^* = \langle 3\nabla_i 3q^j \rangle \partial_i \otimes \partial_k = \left( \partial_i 3q^{jk} + \omega_i^{jk} \right) \partial_i \otimes \partial_k. \]

Therefore, the torsionless condition \( T = 0 \) is equivalent to:

\[ 3\omega_i^{jk} = 3\omega_i^{jk}, \quad K_{ij} = K_{ij}, \]
\[ \Delta_i = g(\partial_i\hat{n}, \hat{n}) + \Theta_i, \quad \Delta_j = K_i^j - \langle 3dx^j, \partial_i\hat{n} \rangle, \]

with:

\[ g(\hat{n}, \partial_i\hat{n}) = \partial_i \ln N, \]
\[ \langle dx^j, \partial_i\hat{n} \rangle = \frac{1}{N} \left( N^j \partial_i \ln N - \partial_i N^j \right), \]
\[ \langle 3dx^j, \partial_i\hat{n} \rangle = -\frac{1}{N} \partial_i N^j, \]

while the metric compatibility \( \nabla_\mu g^* = 0 \) is equivalent to:

\[ \Theta(\hat{n}) = 0, \]
\[ \Theta_i = 0, \quad 3\alpha_i = \Delta_i, \quad 3\alpha_j = \Delta_j, \quad K_{ij} = K_{ij}, \]

together with:

\[ 3\nabla_n 3q^* = 0, \]
\[ 3\nabla_i 3q^* = 0. \]

The relation (106)-(110) could be used to confirm the theorems we proved in Section III: The metricity condition \( \nabla_\mu g = 0 \) will cause \( 3\nabla_i \hat{q}^* = 0 \), and \( T(\partial_i, \partial_i) = 0 \) will cause \( 3T(\partial_i,\partial_j) \), but these relations are not valid vice-versa. The Levi-Civita connection must satisfy the metric compatibility and torsionless condition, and hence, the 8 additional variables are constrained by (113)-(117). Applying these constraints, the variables in the stress-energy-momentum equation for the Levi-Civita connection reduce to 4: \( K = K, \quad 3\omega, \quad 3\alpha_i = \Delta_i, \quad 3\alpha_j = \Delta_j, \) and \( \nabla_\mu 3q^* = 0 \).

The Results

Relations (102)-(105) are used to split the covariant parts in (85), (88), and (93) into (3+1) forms, in particular:

\[ \text{Ric}(\hat{n}, \hat{n}) = \Theta(\hat{n}) \text{tr}K - \Theta_i \alpha^i - \hat{n} \left[ \text{tr}K \right] - K_i^j \Delta^j_i + 3\nabla_i \alpha^i + \alpha^i g(\hat{n}, \partial_i\hat{n}) - K_j^i \langle 3dx^j, \partial_i\hat{n} \rangle, \]
\[ \text{Ric}(\hat{n}, \partial_i) = 3q^j \Theta K_{ij} - \Theta_i \Delta_j + \hat{n} [K_{ij}] - K_{ik} \Delta^k_j - 3\nabla_i \Delta_j - \Delta_j g(\hat{n}, \partial_i\hat{n}) + K_{ij} \langle 3dx^k, \partial_i\hat{n} \rangle, \]
\[ \text{Ric}(\partial_i, \hat{n}) = 3\nabla_j K_{ij} - 3\nabla_i K_{ij} + 3\nabla_j K_{ij} + 3T_j^i K_{ij} + \Theta_j K_{ij} - \Theta_j K_{ij} + K_{ij} \Delta_j - K_{ij} \alpha_j - \Theta_i (\hat{n}) + \hat{n} [\Theta_i] - \Theta(\hat{n}) g(\hat{n}, \partial_i\hat{n}) + \Theta_j \langle 3dx^i, \partial_i\hat{n} \rangle, \]
and:

\[ g(\hat{n}, \mathbf{R}(\hat{n}, \partial t) \partial t_j) = 3\nabla_i \Delta_j + \Theta_i \Delta_j - \Theta(\hat{n}) K_{ij} + K_{ik} \Delta_j^k - \hat{n} [K_{ij}] + \Delta_j g(\hat{n}, \partial_t \hat{n}) - K_{kj} \left( 3dx^k, \partial_t \hat{n} \right). \]

One could observe that the Ricci tensor \( \text{Ric}(\partial_t, \hat{n}) \) contains the 3-covariant derivative of \( K \) instead of \( K \) as in (79). Together with (114) and (115), the (3+1) field equations for MAG could be written in partial differential equations containing only 3-dimensional variables on the hypersurface \( \Sigma \):

\[
\frac{1}{2} \left( 3\mathcal{R} - \text{tr} \left( \text{tr} \mathcal{K} \right) \mathcal{K} \right) + \left( \text{tr} \mathcal{K} \right) \left( \text{tr} \mathcal{K} \right) - \left( K_{ij} + K_{ij} \right) \Delta_j - \hat{n} \left[ \text{tr} \mathcal{K} \right] - \Theta_i \left( \alpha^i + 3 q^{ij} \Delta_j \right) + \Theta(\hat{n}) + \left( \hat{n} - \Theta(\hat{n}) \right) \partial_i \ln N + \frac{1}{N} \left( K_{ij} - K_{ij} \right) \partial_i N^j = \kappa E.
\]

\[
\frac{1}{2} \left( 3\nabla^i K^j_i - 3\nabla^i K^j_i + 3T_{ij}^k K^j_k - \alpha^j (K_{ij} + K_{ji}) + 3\nabla^j \Delta_i - \hat{n} \left[ 3\omega_{ji}^j \right] + \hat{n} \left[ \Theta_i \right] - \partial_i \Theta(\hat{n}) + (\Delta_i + \Theta_i) K^j_i + (\Delta_j - \Theta_j) K^j_i + \Delta_j \partial_i \ln N - \Theta(\hat{n}) \partial_i \ln N + \frac{1}{N} \left( 3\omega_{ji}^j \partial_j N^k - \Theta_j \partial_j N^j \right) \right) = \kappa p_i,
\]

\[
\hat{n} [K_{ij}] - \frac{1}{N} K_{k(ij}, \partial_j N^k - (\partial_{(i}) \ln N + \Theta_{(i)} - 3\nabla_{(ij)} \Delta_{ij})
\]

\[
+ 3R_{(ij)} + \left( \text{tr} \mathcal{K} + \Theta(\hat{n}) \right) K_{ij} - K_{k(i} K_{k(j)} - K_{k(i \partial j)^k} = \kappa S_{ij} - \frac{1}{2} 3 q_{ij} \kappa (S - E),
\]

where we use the fact that \( T = S - E \), with \( S = g^{ij} S_{ij} \). (119)-(121), are respectively, the energy, momentum, and stress-energy equations for MAGR. Notice the existence of the additional variables. One could show that by inserting the Levi-Civita condition (113) and (117)-(118), they return to the original (3+1) Einstein field equation.

V. DISCUSSIONS AND CONCLUSIONS

A. The Stress-Energy-Momentum Equation

Equation (119)-(121) are the (3+1) decomposition of the first Euler-Lagrange equation (71); it comes from the variation of action (63) (for \( f(\mathcal{R}) = \mathcal{R} \)) with respect to metric \( g \). One could see that they provide 1+3+6=10 differential equations. For simplicity, let us take the time gauge (or the Gauss normal coordinate [33]), namely \( N = 1 \), and \( N = 0 \). Hence, equation (119)-(121) becomes:

\[
\frac{1}{2} \left( 3\mathcal{R} - \text{tr} \left( \text{tr} \mathcal{K} \right) \mathcal{K} \right) + \left( \text{tr} \mathcal{K} \right) \left( \text{tr} \mathcal{K} \right) - \left( K_{ij} + K_{ij} \right) \Delta_j - \hat{n} \left[ \text{tr} \mathcal{K} \right] + 3q_{ij} \hat{n} [K_{ij}] + 3\nabla_i \alpha^i - 3q^{ij} 3\nabla_i \Delta_j - \Theta_i \left( \alpha^i + 3 q^{ij} \Delta_j \right) + \Theta(\hat{n}) \left( \text{tr} \mathcal{K} + \text{tr} \mathcal{K} \right) = \kappa E,
\]

\[
\frac{1}{2} \left( 3\nabla^i K^j_i - 3\nabla^i K^j_i + 3T_{ij}^k K^j_k - \alpha^j (K_{ij} + K_{ji}) + 3\nabla^j \Delta_i - \hat{n} \left[ 3\omega_{ji}^j \right] + \hat{n} \left[ \Theta_i \right] - \partial_i \Theta(\hat{n}) + (\Delta_i + \Theta_i) K^j_i + (\Delta_j - \Theta_j) K^j_i + \Delta_j \partial_i \ln N - \Theta(\hat{n}) \partial_i \ln N + \frac{1}{N} \left( 3\omega_{ji}^j \partial_j N^k - \Theta_j \partial_j N^j \right) \right) = \kappa p_i,
\]

\[
\hat{n} [K_{ij}] - \Theta_i \Delta_j - 3\nabla_i \Delta_j + 3R_{(ij)} + (\text{tr} \mathcal{K} + \Theta(\hat{n}) \right) K_{ij} - K_{k(i} K_{k(j)} - K_{k(i \partial j)^k} = \kappa S_{ij} - \frac{1}{2} 3 q_{ij} \kappa (S - E),
\]

but the physical interpretation is invariant under the change of coordinate.

The energy equation (122), contains time evolution (and hence the dynamics) from the terms \( \hat{n} [\text{tr} \mathcal{K}] \) and \( \hat{n} [K_{ij}] \). These originate from the term \( \text{Ric}(\hat{n}, \hat{n}) + \hat{\text{Ric}}(\hat{n}, \hat{n}) \), which is not zero due to the non-metricity and torsion. The momentum equation (123) also contains dynamics from \( \hat{n} \left[ 3\omega_{ji}^j \right] \) and \( \hat{n} \left[ \Theta_i \right] \). The first term contains the change of 3-connection in time, while the second is the change of the angle between \( \alpha_i \) and \( \hat{n} \). The second term will vanish for the Levi-Civita connection. For the first term, it enters the equation of motion because the connection is treated as
On the other hand, equation (124), in the standard GR, becomes:

\[ \text{antisymmetric part of the extrinsic curvatures} K \]

The last equation, the stress-energy part, contains dynamics via \( \hat{n}[K_{ij}] \) as in the energy equation. The torsion only enters the momentum equation explicitly, but it is contained implicitly in all the equations, for example, in the antisymmetric part of the extrinsic curvatures \( K \) and \( \mathcal{K} \). All these three equations are dynamical.

One might ask where are the terms containing the time derivatives of the 3-metric in equation (122)-(124). In the original EFE, equation (122) and (123) become, respectively, the Hamiltonian and momentum (or diffeomorphism) constraint with Levi-Civita condition (113) and (117)-(118), there is no term containing the derivative with respect to time in such equations because the additional parts cancel with each other:

\[
\frac{1}{2} \left( 3R - \text{tr} (K^2) + (\text{tr}K)^2 \right) = \kappa E, \tag{125}
\]

\[
3\nabla_j K^j_i - 3\nabla_i K^j_j = \kappa p_i, \tag{126}
\]

On the other hand, equation (124), in the standard GR, becomes:

\[
\mathcal{L}_\hat{n} K_{ij} + 3R_{ij} + (\text{tr}K) K_{ij} - 2K^k_i K_{kj} = \kappa S_{ij} - \frac{1}{2} 3q_{ij}\kappa (S - E), \tag{127}
\]

with \( \mathcal{L}_\hat{n} K_{ij} \) is the Lie derivative of \( K \) in the direction \( \hat{n} \):

\[
\mathcal{L}_\hat{n} K_{ij} = \hat{n}[K_{ij}] + K_{ik}\partial_j n^k + K_{kj}\partial_i n^k,
\]

where the last two terms in the RHS are zero in the normal coordinate. The dynamics in (124) is contained in the term \( \hat{n}[K_{ij}] \), however, the 3-metric and \( K \) are related by the following equation:

\[
\hat{n} \left[ 3q_{ij} \right] = (\nabla_n g)(\partial_i, \partial_j) + K_{ij} + K_{ji}, \tag{128}
\]

where for the Levi-Civita case becomes:

\[
\hat{n} \left[ 3q_{ij} \right] = 2K_{ij}, \tag{129}
\]

hence, one could insert (129) to (127) to obtain the terms containing the double derivative of the 3-metric with respect to the time coordinate \( \partial_0^2 3q_{ij} \), giving the standard dynamics of GR. However, for the general affine connection, \( \nabla_n g \neq 0 \), but:

\[
(\nabla_n g)(\partial_i, \partial_j) = \hat{n} \left[ 3q_{ij} \right] - \Delta_{ij} - \Delta_{ji},
\]

caus​ing the \( \hat{n} \left[ 3q_{ij} \right] \)'s in (128) to cancel each other, while leaving \( K_{ij} + K_{ji} = \Delta_{ij} + \Delta_{ji} \), free from \( 3q_{ij} \). Therefore, the momentum and stress-energy equation (123)-(124) are free from the time derivative of \( 3q_{ij} \). However, the dynamics of the 3-metric enters the energy equation (122) from the term \( \hat{n}[\text{tr}K] = \hat{n} \left[ 3q_{ij}K^{ij} \right] \).

One needs to keep in mind that at this stage, we are only working with the first equation of motion (71); there still exists another equation of motion, namely, the one obtained from the variation of the action with respect to the connection (72). The first equation of motion (71) provides only 10 differential equations, whereas the unknown variables are 74 (10 from the metric, 64 from the connections, assuming the theory does not have constraint). With the additional variables on the hypersurface, we have introduced 64 unknown variables (1 for \( \Theta (\hat{n}) \), 3 for each \( \Theta_i, \Delta_i, 3\alpha_i \), 9 for \( K, \mathcal{K}, \Delta^i_i \), and 27 for \( 3\omega \)), and 10 more unknowns should come from the metric \( g_{\mu\nu} \), in terms of the 3-metric \( 3q_{ij} \), the lapse \( N \), and the shift \( \mathcal{N} \). The second Euler-Lagrange equation (72) will provide 60 more differential equations, leaving 4 vectorial degrees of freedom on the connection. The last 4 equations come from the traceless torsion constraint (73), by taking the projective invariance transformation (56) into account. Without the \( (3+1) \) decomposition of these equations of motions as well, it is impossible to do a complete analysis of the \( (3+1) \) MAGR theory.

### B. The Hypermomentum Equation and Traceless Torsion Constraint

For the completeness of the discussion in this article, we add some of the results from our companion paper. Here, we only present the \( (3+1) \) decomposition of the hypermomentum equation in the normal (Gauss) coordinate. The general treatment and the detailed derivation of the result will be discussed in our companion paper.
(3+1) Decomposition of Hypermomentum

The hypermomentum (64) could be written with the spacetime index α as hidden as $\mathcal{H}(\partial_{\mu}, dx^\beta) = \mathcal{H}^\alpha_\mu \partial_{\alpha}$. Using this notation, one could decompose the hypermomentum by its 'internal' indices into the normal and parallel parts, with respect to the hypersurface $\Sigma$:

$$\mathcal{H}(\hat{n}, \hat{n}^*) = Kn \beta \mathcal{H}^\alpha_\mu \partial_{\alpha} = - \langle \hat{n}^*, \mathcal{H}(\hat{n}, \hat{n}^*) \rangle \hat{n} + \langle 3 dx^i, \mathcal{H}(\hat{n}, \hat{n}^*) \rangle \partial_i,$$

$$\mathcal{H}(\hat{n}, 3 dx^i) = K_\mu^\beta \mathcal{H}^\alpha_\mu \partial_{\alpha} = - \langle \hat{n}^*, \mathcal{H}(\hat{n}, 3 dx^i) \rangle \hat{n} + \langle 3 dx^i, \mathcal{H}(\hat{n}, 3 dx^i) \rangle \partial_j,$$

$$\mathcal{H}(\partial_i, \hat{n}^*) = K_\mu^\beta \mathcal{H}^\alpha_\mu \partial_{\alpha} = - \langle \hat{n}^*, \mathcal{H}(\partial_i, \hat{n}^*) \rangle \hat{n} + \langle 3 dx^i, \mathcal{H}(\partial_i, \hat{n}^*) \rangle \partial_j,$$

$$\mathcal{H}(\partial_i, 3 dx^i) = K_\mu^\beta \mathcal{H}^\alpha_\mu \partial_{\alpha} = - \langle \hat{n}^*, \mathcal{H}(\partial_i, 3 dx^i) \rangle \hat{n} + \langle 3 dx^i, \mathcal{H}(\partial_i, 3 dx^i) \rangle \partial_k.$$

(130)

Following the decomposition of $\mathcal{T}$ into its 3 components, i.e., the energy $E$, the momentum $p_i$, and the stress $S_{ij}$, one could apply the same procedure to the hypermomentum $\mathcal{H}$, where the decomposition is based on the split of the 'spacetime' into space and time. This should not be confused with the split of $\mathcal{H}$ based on the symmetricity of the indices which gives the spin, shear, and dilatation parts as in [7].

The (3+1) Hypermomentum Equation in Normal Coordinate

With the (3+1) decomposition in (130), the hypermomentum equation (72) could be split into 4 equations as follows:

$$n^\mu n^\lambda T_\mu^\alpha \lambda \partial_{\alpha} - n_\beta (\nabla \alpha g^{\beta \alpha}) \partial_{\alpha} + n_\beta (\nabla \beta g^{\alpha \beta}) \n = \kappa n^\mu n_\beta \mathcal{H}^\alpha_\mu \partial_{\alpha} - \frac{2}{3} n^\mu n_\beta \kappa \mathcal{H}^\alpha_\mu \sigma_\delta^\beta \partial_{\alpha},$$

(131)

$$n_\beta (\nabla \alpha g^{\beta \alpha}) \partial_{\alpha} + \frac{1}{2} \left( g^{\alpha \beta} \nabla \gamma g^{\alpha \lambda} \right) (n^\gamma g^{\beta i} \partial_{\alpha} - g^{\nu \beta} \n) - (\nabla \alpha g^{\beta i}) \partial_{\alpha} + (\nabla g^{\alpha i}) \n = \kappa n^\alpha \mathcal{H}^\alpha_\mu \partial_{\alpha} - n^\alpha \delta^i_\beta \frac{2}{3} \kappa \mathcal{H}^\alpha_\mu \sigma_\delta^\beta \partial_{\alpha},$$

$$n^\alpha \mathcal{H}^\alpha_\mu \partial_{\alpha} + \frac{1}{2} \left( g^{\alpha \beta} \nabla \gamma g^{\alpha \lambda} \right) (\delta^i_\nu \n - n^\nu \partial_i) - (\nabla \gamma g^{\beta i}) \partial_{\alpha} + (\nabla i g^{\beta i}) \partial_i = \kappa \mathcal{H}^\alpha_\mu \partial_{\alpha} - \delta^i_\beta \frac{2}{3} \kappa \mathcal{H}^\alpha_\mu \sigma_\delta^\beta \partial_{\alpha},$$

$$g^{\alpha \beta} \nabla \gamma g^{\alpha \lambda} \left( \delta^i_\nu \n - n^\nu \partial_i \right) - (\nabla i g^{\beta i}) \partial_{\alpha} + (\nabla i g^{\beta i}) \partial_i = \kappa \mathcal{H}^\alpha_\mu \partial_{\alpha} - \delta^i_\beta \frac{2}{3} \kappa \mathcal{H}^\alpha_\mu \sigma_\delta^\beta \partial_{\alpha}. $$

Using equation (130) together with torsion and non-metricity decomposition in (106)-(110), one could rewrite the four equations (131) in terms of the additional variables as follows:

$$\left( K^i_j - K^i_j \right) \hat{n} + \left( \Delta^i - \alpha^i \right) \partial_i = \kappa \left( \mathcal{H}(\hat{n}, \hat{n}^*) + \frac{1}{3} \left( \text{tr} \mathcal{H} + \langle \hat{n}, \text{tr} \mathcal{H} \rangle \hat{n} \right) \right),$$

(132)

$$\left( 3 q^{ij} \left( \Theta(\hat{n}) + \frac{1}{2} q_{kl} \left( n \left[ 3 q^{kl} \right] + \Delta^{kl} + \Delta^{jk} \right) \right) - K^{ij} - n \left[ 3 q^{ij} \right] - \Delta^{ij} \right) \partial_j$$

$$+ \left( \Delta^i - 3 \Theta^i + 3 \nabla_j g^{qij} - \frac{1}{2} q_{jk} \nabla^i q^{ijk} \right) \n = \kappa \left( \mathcal{H}(\hat{n}, dx^i) + \frac{1}{3} \langle dx^i, \text{tr} \mathcal{H} \rangle \hat{n} \right),$$

(133)

$$\left( K^j_j - K^j_j \right) + \Theta(\hat{n}) - \frac{1}{2} q_{kj} \left( n \left[ 3 q^{kj} \right] + \Delta^{kj} + \Delta^{jk} \right) \partial_k$$

$$+ \left( K^j_j - \Delta^j \right) \partial_j + \left( \frac{1}{2} q_{jk} \nabla^i q^{ijk} - \Delta^i \right) \hat{n} = \kappa \left( \mathcal{H}(\partial_i, \hat{n}^*) + \frac{1}{3} \langle \hat{n}, \text{tr} \mathcal{H} \rangle \partial_i \right),$$

(134)

$$\left( K^i_j - K^i_j \right) \hat{n} + \left( \Delta^j - \alpha^j - \Theta^j + 3 \nabla_k q^{jik} - \frac{1}{2} q_{kl} \nabla^i q^{jkl} \right) \partial_i$$

$$+ \left( \Theta_i + \frac{1}{2} q_{lm} \nabla^i q^{kl} \right) q^{ijk} \left( \nabla^i q^{jkl} - \nabla_i q^{jkl} \right) \partial_k + q^{ijk} \left( \nabla^i q^{jkl} + \nabla^i q^{jkl} T(\partial_i, \partial_k) = \kappa \left( \mathcal{H}(\partial_i, dx^i) - \frac{1}{3} \left( \delta^i_\beta \text{tr} \mathcal{H} - \langle dx^i, \text{tr} \mathcal{H} \rangle \partial_k \right) \right),$$

(135)

with $\text{tr} \mathcal{H} = \mathcal{H}(\partial_{\mu}, dx^\mu) = \mathcal{H}^\alpha_\mu \partial_{\alpha}$. These are the (3+1) hypermomentum equations in normal coordinates. Notice that the quantity $\mathcal{H}(U, V^*) \in T_{\mu} \mathcal{M}$ is a vector and each equation still has the normal and parallel parts with respect to $\Sigma$. Moreover, one could contract (132)-(135) with $\hat{n}^*$ and $dx^i$ to obtain 8 equations.
As explained in the previous sections, the hypermomentum equation (72) only provides 60 equations; one needs
4 more equations to reduce the vectorial degrees of freedom in the connection. These are provided by the traceless
torsion constraint (73), which could be decomposed (in normal coordinate) as follows. Notice that
\[ T^\nu = T^\mu_\nu = g(dx^\mu, T(\partial_\mu, \partial_\nu)) = 0. \]

By contracting \( T \) with the normal \( \hat{n} \) and \( \partial_i \), then using torsion decomposition (106)-(108), one obtains (in normal
coordinate):
\[ \langle T, \hat{n} \rangle = n^\nu T_\nu = n^\nu T^\mu_\nu = \Delta^i_i - K^i_i = 0, \]
(136)
\[ \langle T, \partial_i \rangle = T_i = T^\mu_i = \Delta_i - \Theta_i + \frac{3}{2}T^j_j i = 0. \]
(137)

The Zero Hypermomentum Case

When the hypermomentum is zero, the hypermomentum equation (72), together with the traceless torsion constraint
(73) must give the condition for the Levi-Civita connection. Let us check if the (3+1) equations (132)-(135) agree
with this fact. Setting \( \mathcal{H} = 0 \) and then contracting (132)-(135) with \( \hat{n}^* \) and \( dx^i \), we obtain 8 equations, which include
1 scalar equation:
\[ K^i_j - K^j_i = 0, \]
(138)
3 vector equations:
\[ \Delta^i - \alpha^i = 0, \]
(139)
\[ \Delta^i - 2\Theta^i + \frac{3}{2} \nabla^j q^{ji} - \frac{1}{2} \nabla^i q^{jk} = 0, \]
(140)
\[ \frac{1}{2} \nabla^i q^{jk} - \Delta^i = 0, \]
(141)
3 matrix equations:
\[ q^{ij} \left( \Theta (\hat{n}) + \frac{1}{2} \nabla^k q^{[ik]} \right) - K^{ij} - n \left[ q^{ij} \right] - \Delta^{ij} = 0, \]
(142)
\[ K^i_j - K^j_i + \left( K^k_i - K^i_k + \Theta (\hat{n}) - \frac{1}{2} \nabla^k q^{[ik]} \right) \delta^j_i = 0, \]
(143)
\[ K^i_j - K^j_i = 0, \]
(144)
and 1 tensor equation of order \( \frac{3}{2} \):
\[ \left( \Delta^j - \alpha^j - \Theta^j + \frac{3}{2} \nabla^l q^{ji} - \frac{1}{2} \nabla^i q^{km} \right) q^k_j + \frac{3}{2} q^{jl} q^{ki} = 0. \]
(145)

The traceless torsion constraint (73) provides 1 scalar equation (136) and 1 vector equation (137).
These equations could be simplified as follows. Solving the vector equations (140)-(141) for \( \Theta^i \) and \( \Delta^i \) gives:
\[ \Theta^i = \frac{1}{2} \nabla^j q^{ji}, \]
(146)
\[ \Delta^i = \frac{1}{2} q^{jk} \nabla^i q^{jk}, \]
(147)
while solving the matrix equations (142)-(143) gives:
\[ \Theta (\hat{n}) = \frac{1}{6} q_{kl} \nabla^i q^{ki}, \]
(148)
\[ K^{ij} - K^{ij} = \nabla^i q^{ij} - \frac{1}{3} q^{ij} q_{kl} \nabla^i q^{kl}, \]
(149)
with the help of the scalar equation (138). At last, using the vector equations (139), (146), and (147) to (145), then decomposing this tensor equation into the symmetric and antisymmetric parts of the \((i,j)\)-indices, gives:

\[
3\nabla (q^3 j^k) - 3q^k (3\nabla q^3 j)^l = 0, \\
3T^{ikj} = 3q^{kl} 3q_{lm} 3\nabla j 3q^m + 3\nabla (q^3 j^k).
\]

Let us solve these simplified equations. The easiest way is to start from (150), which is satisfied if:

\[
3\nabla (q^3 j^k) = 0.
\]

Inserting (152) to (151) gives:

\[
3T^{ikj} = 0,
\]

while inserting (152) to vector equation (146) gives:

\[
\Theta^i = 0.
\]

Inserting (153) to (137) gives:

\[
\Delta_i - \Theta_i = 0.
\]

Now let us focus on the matrix equation (144) and (149). Applying the symmetries decomposition to the indices \((i,j)\) gives the following conditions:

\[
K^{(ij)} - K^{(ji)} = 0, \\
K^{[ij]} + K^{[ji]} = 0, \\
K^{(ij)} - K^{(ji)} = \nabla_n 3q^{ij} - \frac{1}{3} 3q^{lj} 3q_{kl} \nabla_n 3q^{kl}, \\
K^{[ij]} - K^{[ji]} = 0,
\]

where (156)-(157) comes from (144), while (158)-(159) comes from (149). Comparing (157) and (159), then \(K^{[ij]} = K^{[ji]} = 0\), which means that both the extrinsic curvatures do not possess antisymmetric parts. On the other hand, from (156) and (158), we obtain \(K^{(ij)} = K^{(ji)}\), that are equivalent with the following conditions:

\[
K^{ij} = K^{ji}, \\
K^{ij} = K^{ji},
\]

and:

\[
\nabla_n 3q^{ij} - \frac{1}{3} 3q^{lj} 3q_{kl} \nabla_n 3q^{kl} = 0,
\]

which is satisfied if:

\[
\nabla_n 3q^{ij} = 0.
\]

Inserting (162) to (148) gives:

\[
\Theta (\hat{n}) = 0.
\]

Finally, using (111) and inserting (138), (162), and (163) to (143) gives:

\[
K^{j}_{i} - \Delta^{j}_{i} = 0.
\]

The conditions obtained from solving the trivial hypermomentum equation could be classified as follows: \{(153),(155), (161), (164)\} and \{(139), (152), (154), (160), (162), (163)\} are, respectively, the torsionless and metric condition (113) and (117)-(118) in the normal (Gauss) coordinate where \(N = 1\) and \(\mathcal{N} = 0\). Hence, for zero hypermomentum, the affine connection becomes Levi-Civita, and the \((3+1)\) stress-energy-momentum equations for MAGR (122)-(124) return to the original EFE (125)-(127). Notice that without the traceless torsion constraint (136)-(137), it is not possible to retrieve the metric compatibility and torsionless condition, in the absence of hypermomentum. A more detailed analysis of the hypermomentum equation will be discussed in our companion article.
C. Conclusions and Further Remarks

In this subsection, we summarize the results achieved from our work. First, we have derived the generalized Gauss-Codazzi-Mainardi equations and torsion (3+1) decomposition, which are valid for manifold with any dimension. This is done in Section II with equation (24) and (27)-(28) (or equation (79)-(80) and (81)-(82), written in terms of components) as the results. Second, we have shown that a metric connection \( \nabla \) on \((M, g)\), induce a connection \( ^q\nabla \) on the hypersurface \((\Sigma, ^qg)\), which also satisfies the metric compatibility with respect to \(^qg\). A similar statement is also valid for a torsionless connection. These had been proven in Section III. Third, having the previous results in hand, we have performed the ADM formulation to the generalized EFE, \( \text{Ric} - \frac{1}{2}gR = \kappa T \) for Metric-Affine General Relativity, resulting in 3 parts of the equation: the energy equation (119), momentum equation (120), and stress-energy equation (121), with the help of the additional variables on the hypersurface introduced in Section IV. Unlike the standard GR case, the energy and momentum part of the Einstein field equations are not constraint equations, since all of the 3 equations contain the derivative of quantities with respect to the time coordinate. We have also shown in the discussion that by applying metric compatibility (117) and torsionless condition (113) to (119)-(121), one will recover the Hamiltonian and momentum (or diffeomorphism) constraint, together with the standard dynamics of GR.

For the completeness of the discussion in this article, we present some results of our companion paper, which is the (3+1) decomposition (in normal coordinates) of the hypermomentum equation (132)-(135) and the traceless torsion constraint (136)-(137). We have shown that in the (3+1) perspective, setting hypermomentum to zero, one will recover the metric compatibility and torsionless condition, hence, the connection must be Levi-Civita. These conditions, in MAGR (and MAG) is not \( a \text{ priori} \) given but as a consequence of the absence of hypermomentum.

However, the tracelessness of torsion in (70) does not result from the equation of motion, it is introduced in the kinematical level, via the projective transformation. If one expects to derive the tracelessness of torsion from the dynamics, the action must be added with new terms at least quadratic in torsion [37]. This will be an interesting subject to pursue.

Another remark is concerning the way to break the projective invariance. As we had mentioned at the beginning of Section IV, fixing the torsion to be traceless is only one of the possibilities to break the invariance. It is also possible to choose the non-metricity factor, or moreover, a linear combination of the non-metricity factor and torsion to be traceless, as discussed in [29, 30]. These choices might result in different (3+1) decompositions, in particular, the form of equation (136)-(137), and will also be an interesting subject for further works.

As we had already mentioned in the discussion, in this article, the focus is on the first equation of motion (71); we are writing another article that focuses on the hypermomentum equation and the projective invariant constraint, as a companion to this article. With a complete ADM decomposition to these equations of motions as well, it will be possible to do a complete analysis on the (3+1) formulation of MAGR, in particular, the Cauchy problem and/or the Hamiltonian analysis of the theory. As further works, it will be interesting to perform the (3+1) ADM formulation to more general theories such as Metric-Affine \( f(R) \)-Gravity. We expect our preliminary work in this article could motivate more research in this direction.

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