CRITICAL BEHAVIOUR IN THE DENSE PLANAR NJL MODEL

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We present results of a Monte Carlo simulation of a 2+1 dimensional Nambu – Jona-Lasinio model including diquark source terms. A diquark condensate \((qq)\) is measured as a function of source strength \(j\). In the vacuum phase \((qq)\) vanishes linearly with \(j\) as expected, but simulations in a region with non-zero baryon density suggest a power-law scaling and hence a critical system for all \(\mu > \mu_c\). There is no diquark condensation signalling superfluidity. Comparisons are drawn with known results in two dimensional theories, and with the pseudogap phase in cuprate superconductors.

We also measure the dispersion relation \(E(k)\) for fermionic excitations, and find results consistent with a sharp Fermi surface. Any superfluid gap \(\Delta\) is constrained to be much less than the constituent quark mass scale \(\Sigma_0\).

In the absence of a reliable non-perturbative method for calculating the properties of QCD at high densities, models of strongly-interacting matter continue to rely on results derived from effective theories such as the Nambu – Jona-Lasinio (NJL) model \([1]\). Recently such approaches have suggested the intriguing possibility of superconducting behaviour in quark matter at high density as a result of BCS condensation between diquark pairs at the Fermi surface \([2,3]\). Even in simplified models, however, systematic methods of calculation are hard to find, and up to now mostly self-consistent methods have been applied \([4,5]\). Scenarios such as color-flavor locking \([6]\), however, imply that in the high density phase the global U(1)\(_{B}\) symmetry in the QCD Lagrangian corresponding to conservation of baryon number is spontaneously broken, leading to superfluid behaviour and a massless scalar excitation via Goldstone’s theorem. A description of the fluctuations of these modes and the resulting long-range interactions may need a more systematic non-perturbative approach; in this Letter we will use numerical results from lattice field theory to argue that the situation is more complex than hitherto thought.

The Lagrangian for the NJL model, incorporating a baryon chemical potential \(\mu\), is

\[
\mathcal{L}_{NJL} = \bar{\psi}(\partial + m + \mu \gamma_0)\psi - \frac{g^2}{2} \left[ (\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\vec{\tau}\psi)^2 \right],
\]

(1)

where \(\psi, \bar{\psi}\) carry a global isospin index acted on by Pauli matrices \(\vec{\tau}\). The model in 2+1 dimensions has been studied with \(\mu \neq 0\) using staggered lattice fermions \(\chi, \bar{\chi}\) \([7]\); the reader is referred to this paper for further details of the formulation and symmetries of the lattice model. Apart from the obvious numerical advantages of working in a reduced dimensionality, the model in this case also has an interacting continuum limit for a critical value of the coupling \(1/g_c^2 \simeq 1\) \([8]\), making controlled predictions possible in principle. In the simulations of this Letter, as in \([9]\), \(1/g^2 = 0.5\), rather far from the continuum limit. For \(\mu = 0\), this results in a ground state with broken chiral symmetry and a dynamically generated fermion mass \(\Sigma_0 \simeq 0.71\), together with a triplet of Goldstone pions. The “constituent quark mass” \(\Sigma_0\) defines the physical scale in cutoff units. As \(\mu\) is raised from zero, the system remains essentially unaltered until \(\mu = \mu_c \simeq 0.65\), whereupon the constituent quark mass \(\Sigma\) falls to zero as chiral symmetry is restored in a strong first-order transition, and baryon density \(n = \langle \bar{\psi}\gamma_0\psi \rangle\) jumps discontinuously from zero to a non-zero value; by \(\mu = 0.8\), the density \(n \simeq 0.25\) quarks of each isospin component per lattice site.

The tendency for diquark pairing at high density was investigated via measurements of the diquark propagator 

\[
G(t) = \sum_\chi \langle q\bar{q}(0, \mathbf{0})q\bar{q}(t, \mathbf{x}) \rangle.
\]

In particular it was found that for \(\mu > \mu_c\) and the choice of isoscalar diquark operator \(q\bar{q} = \chi^{\tau r}(x)\tau_2\chi(x)\) (the “spectral scalar” channel of \([10]\)), \(G(t)\) plateaus at a non-zero value as \(t \to \infty\), implying \(\langle q\bar{q}\rangle\langle q\bar{q} \rangle \neq 0\) by the cluster property. Studies on varying spatial volumes, however, reveal that the plateau height is not an extensive quantity as naively expected, and that diquark condensation signalling a superfluid state is not unambiguously indicated.

The situation can be clarified by the introduction of source terms for diquark and anti-diquark pairs:

\[
\mathcal{L}[j, \bar{j}] = \mathcal{L}_{NJL} + j\chi^{\tau r}(x)\tau_2\chi(x) + \bar{j}\bar{\chi}(x)\tau_2\bar{\chi}^{\tau r}(x).
\]

(2)

The functional integral may now be written using the Gor’kov representation \([11]\) as

\[
Z[j, \bar{j}] = \langle \text{Pf}(\mathcal{A}[j, \bar{j}]) \rangle,
\]

(3)

where \(\langle \ldots \rangle\) denotes averaging with respect to bosonic auxiliary fields \(\sigma\) and \(\bar{\sigma}\) introduced to make \(\mathcal{L}_{NJL}\) bilinear in \(\chi\) and \(\bar{\chi}\), and the antisymmetric matrix \(\mathcal{A}\) is

\[
\mathcal{A} = \left( \begin{array}{cc} -\frac{1}{2}M^{\tau r} & \frac{1}{2}M \\ -\frac{1}{2}M & j_2 \end{array} \right),
\]

(4)


where $M = M(\mu; \sigma, \bar{\sigma})$ is the conventional kinetic operator of $\mathcal{L}_{NJL}$. For convenience we now define diquark operators $qq_\pm$ via

$$qq_\pm(x) = \frac{1}{2} \left[ \chi^{\tau}(x)\tau_2 \chi(x) \pm \bar{\chi}(x)\tau_2 \chi^{\tau}(x) \right],$$

with corresponding sources $j_\pm = j \pm \bar{j}$. The diquark condensate is readily calculated to be

$$\langle qq_+ \rangle = \frac{1}{V} \frac{\partial \ln Z}{\partial j_+} = \frac{1}{4V} \langle \tr_2 A^{-1} \rangle.$$  

Under U(1)$_B$ transformations $\chi \mapsto e^{i\beta} \chi$, $\bar{\chi} \mapsto e^{-i\beta} \bar{\chi}$, the operators $qq_\pm$ rotate into each other; condensation of $\langle qq_+ \rangle \neq 0$ implies a massless Goldstone mode in the $qq_-$ channel as $j_+ \to 0$, illustrated by a Ward identity for the susceptibility $\chi_-$:

$$\chi_- \equiv \sum_x \langle qq_- (0)qq_-(x) \rangle \bigg|_{j_-=0} = \frac{\langle qq_+ \rangle}{j_+}. \tag{7}$$

Numerical simulation using the functional measure requires us to address the issue of calculating the Pfaffian $[10]$. We begin by observing that $PfA = \pm \det A$, the sign depending on the details of the ordering of the Grassmann integration. Now, using the property of a square block matrix

$$\det \begin{pmatrix} X & Y \\ W & Z \end{pmatrix} = \det X \det(Z - WX^{-1}Y), \tag{8}$$

together with the property $\tau_2 M \tau_2 = M^*$, we find $\det A = \det(jj + M^t M/4)$, and hence is real and positive if $jj$ is chosen real and positive. It follows that $PfA$ is real. To avoid the sign problem, however, we choose to simulate with measure $\det^2(A^t A) = Pf^2A$, corresponding to a doubling in the number of fermion species. This is usual practice in simulations of four-fermi models; “crosstalk” between the two species, which might conceivably lead to problems for $\mu \neq 0$ due to light unphysical baryonic states in the spectrum, is subleading in this case and causes no problems $[11]$. It is straightforward to implement the simulation using a hybrid molecular dynamics “R” algorithm $[12]$, the resulting dynamics are those of 2 flavors of staggered lattice fermion, corresponding to $N_f = 4$ flavors of four-component physical fermion.

We have performed simulations on lattice sizes $16^3, 16^2 \times 24, 24^3, 32^3$ and $48^3$, with chemical potentials $\mu = 0.2$ corresponding to the low density chiral-broken phase, and $\mu = 0.8$ in the high density chirally-restored phase $[13]$. Diquark source values ranging from $j = 0.3$ down to as low as $j = 0.025$ were used; throughout we took $\bar{j} = j$ and kept a bare Dirac mass $m = 0.01$ to assist identification of chirally broken and restored phases. We plot $\ln \langle qq_+ \rangle$ vs. $\ln j$ in Fig. 1. At low density the results for $\mu = 0$ and $\mu = 0.2$ are very similar and support a linear relation $\langle qq_+ \rangle \propto j^\alpha$, suggesting that $U(1)_B$ is unbroken in the zero-source limit. For $\mu = 0.8$, when chiral symmetry is restored, the situation is different: whilst there is still no evidence for a non-zero condensate as $j \to 0$, the log-log plot now displays a marked curvature, together with evidence for significant finite volume effects. Empirically we find that on a $L_\nu^3 \times L_\tau$ system the dominant correction scales as $1/L_\tau$ — this enables an extrapolation of the data with $L_\nu \geq 24$ to the thermodynamic limit, shown in Fig. 2.

\[ \langle qq_+ \rangle \propto j^\alpha, \tag{9} \]

with $\alpha \simeq 0.31$. There is thus no superfluid condensate

\[ FIG. 1. \ln \langle qq_+ \rangle \text{ vs. } \ln j \]

\[ FIG. 2. \text{Extrapolation of } \langle qq_+ \rangle \text{ at } \mu = 0.8 \text{ to the infinite volume limit.} \]

The extrapolated data, denoted by filled squares in Fig. 2, suggest the best description in the high-density phase is a power law:
in this region of parameter space, and no consequent Goldstone pole. The Ward identity (5), however, implies that the transverse susceptibility \( \chi_- \) diverges as \( j^{d-1} \), thus long-wavelength fluctuations remain important, and there are strongly-interacting massless states present. This is the first main result of this Letter.

Now, the value \( \mu = 0.8 \) was originally chosen merely as characteristic of the high-density phase; moreover data from partially quenched simulations (4) suggest that the power-law scaling (3) may be generic for \( \mu > \mu_c \). This is analogous to the behaviour of the 2d X-Y model (13), which in accordance with well-known theorems (14) does not have spontaneous magnetisation, but instead exhibits a line of critical points for all \( T < T_c \) (13). In this low-temperature phase the modulus of the spin variables is constant, but the phase fluctuates; the divergence of \( \chi_- \) signals correlation of the phase over macroscopic distance scales without long-range phase coherence. This behaviour without an order parameter can be characterised by two exponents \( \delta \) and \( \eta \), defined in our notation by \( \langle qq_+ \rangle \propto j^{\frac{\delta}{2}} \), and \( \lim_{j \to 0} \langle qq_-(0)qq_-(x) \rangle \propto 1/x^{d-2+\eta} \). For the X-Y model (13) \( \delta \geq 15 \) and \( \eta \leq \frac{1}{4} \) (the inequalities being saturated up to logarithmic corrections at \( T = T_c \)), and satisfy the hyperscaling relation

\[
\delta = \frac{d+2-\eta}{d-2+\eta}. \tag{10}
\]

For the dense planar NJL model, by contrast, our results suggest \( \delta \approx 3 \).

The phenomenon of long-range phase coherence being wiped out by soft transverse fluctuations is particularly interesting for a composite order parameter field, suggesting as it does that some dynamical pairing mechanism is in play even in the absence of symmetry breaking. It has been suggested that such a state may describe a pseudogap phase, characterised by a suppression of spectral weight in the vicinity of the Fermi surface, and observed as a precursor to the superconducting state for underdoped cuprate superconductors (16). It is intriguing in the current context that cuprate superconductivity is a planar phenomenon (16).

A field theoretic example of possible relevance is studied in (17), where results from an exactly soluble 2d fermionic model are generalised to the Gross-Neveu model with U(1) chiral symmetry (18). Once again due to the low dimensionality no long-range ordering via \( \langle \bar{\psi}\psi \rangle \neq 0 \) is possible; however for sufficiently large number of flavors \( N_f \) the model has a phase analogous to the low temperature phase of the X-Y model, with \( \eta \sim O(1/N_f) \). The physical fermion is a superposition of positive and negative chirality states and has zero net chirality; despite the absence of chiral symmetry breaking it propagates as \( C(x) \propto x^{-1/4+\eta}e^{-\Sigma x} \), implying massive propagation as \( x \to \infty \), but also a non-trivial spectral function, signalled by the departure from canonical scaling of the pre-exponential factor.

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*An important distinction is that in cuprates the pairing operator is d-wave, whereas for the NJL model it is s-wave.
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