A NOTE ON $q$-BERNOULLI NUMBERS AND $q$-BERNSTEIN POLYNOMIALS

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Abstract. The purpose of this paper is to investigate some properties of several $q$-Bernstein type polynomials to express the bosonic $p$-adic $q$-integral of those polynomials on $\mathbb{Z}_p$.

1. INTRODUCTION

Let $p$ be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of $p$-adic integers, the field of $p$-adic numbers and the field of $p$-adic completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$. Let $q$ be regarded as either a complex number $q \in \mathbb{C}$ or a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, then we always assume $|q| < 1$. If $q \in \mathbb{C}_p$, we usually assume that $|1 - q|_p < 1$. In this paper we use the notation of $q$-number as $[x]_q = 1 - q^x$.

Let $UD(\mathbb{Z}_p)$ be the set of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the bosonic $p$-adic $q$-integral on $\mathbb{Z}_p$ is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)(q^x), \quad (\text{see } [2-5]). \tag{1}$$

In [2], the Carlitz’s $q$-Bernoulli numbers are inductively defined by

$$\beta_0,q = 1, \quad \text{and} \quad q(q\beta + 1)^k - \beta_{k,q} = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases} \tag{2}$$

with the usual convention of replacing $\beta^i$ by $\beta_{i,q}$.

The Carlitz’s $q$-Bernoulli polynomials are also defined by

$$\beta_{n,q}(x) = (qx + [x]_q)^k = \sum_{i=0}^{k} \binom{k}{i} q^i x^i \beta_{i,q}[x]_q^{k-i}. \tag{3}$$

In [2], Kim proved that the Carlitz $q$-Bernoulli numbers and polynomials are represented by $p$-adic $q$-integral as follows: for $n \in \mathbb{Z}_+$,

$$\beta_{n,q} = \int_{\mathbb{Z}_p} [x]_q^n d\mu_q(x), \quad \text{and} \quad \beta_{n,q}(x) = \int_{\mathbb{Z}_p} [x + y]_q^n d\mu_q(y). \tag{4}$$

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The Kim’s $q$-Bernstein polynomials are defined by

$$B_{k,n}(x, q) = \binom{n}{k} [x]_q^k [1 - x]_{q^{-1}}^{n-k}, \quad \text{(see [1-8])},$$

where $n, k \in \mathbb{Z}_+$, and $x \in [0, 1]$.

Let $f$ be continuous functions on $[0, 1]$. Then the linear Kim’s $q$-Bernstein operator of order $n$ for $f$ are defined by

$$B_{n,q}(f \mid x) = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) B_{k,n}(x, q), \quad \text{(see [5])}.$$

In this paper, we consider the $p$-adic analog of the extended Kim’s $q$-Bernstein polynomials on $\mathbb{Z}_p$ and investigate some properties of several extended Kim’s $q$-Bernstein polynomials to express the bosonic $p$-adic q-integral of those polynomials.

2. Extended $q$-Bernstein Polynomials

In this section we assume that $q \in \mathbb{R}$ with $0 < q < 1$. Let $C[0, 1]$ be the set of continuous function on $[0, 1]$.

For $f \in C[0, 1]$, we consider the extended Kim’s $q$-Bernstein operator of order $n$ as follows:

$$B_{n,q}(f \mid x_1, x_2) = \sum_{k=0}^{n} \frac{n}{k} \binom{n}{k} [x_1]_q^k [1 - x_2]_{q^{-1}}^{n-k}$$

$$= \sum_{k=0}^{n} f \left( \frac{k}{n} \right) B_{k,n}(x_1, x_2 \mid q).$$

For $n, k \in \mathbb{Z}_+$, and $x_1, x_2 \in [0, 1]$, the extended Kim’s $q$-Bernstein polynomials of degree $n$ are defined by

$$B_{k,n}(x_1, x_2 \mid q) = \binom{n}{k} [x_1]_q^k [1 - x_2]_{q^{-1}}^{n-k}. \quad \text{(7)}$$

In the special case $x_1 = x_2 = x$, then $B_{k,n}(x_1, x_2 \mid q) = B_{k,n}(x, q)$.

From (6) and (7) we can derive the generating function for $B_{k,n}(x_1, x_2 \mid q)$ as follows:

$$F^{(k)}_q(x_1, x_2 \mid t) = \frac{(t[x_1]_q^k e_x(t[1 - x_2]_{q^{-1}})}{k!}, \quad \text{(8)}$$

where $k \in \mathbb{Z}_+$ and $x_1, x_2 \in [0, 1]$.

By (8), we get

$$F^{(k)}_q(x_1, x_2 \mid t) = \sum_{n=k}^{\infty} \frac{[x_1]_q^k [1 - x_2]_{q^{-1}}^{n-k} t^n}{k! n!}$$

$$= \sum_{n=k}^{\infty} \binom{n}{k} [x_1]_q^k [1 - x_2]_{q^{-1}}^{n-k} \frac{t^n}{n!}$$

$$= \sum_{n=k}^{\infty} B_{k,n}(x_1, x_2 \mid q) \frac{t^n}{n!}$$

Thus, we have

$$B_{k,n}(x_1, x_2 \mid q) = \begin{cases} \binom{n}{k} [x_1]_q^k [1 - x_2]_{q^{-1}}^{n-k}, & \text{if } n \geq k \\ 0, & \text{if } n < k, \end{cases}$$

for $n, k \in \mathbb{Z}_+$. 
It is easy to check that
\[ B_{n-k,n}(1 - x_2, 1 - x_1 \mid q^{-1}) = B_{k,n}(x_1, x_2 \mid q). \] (10)

For \( 0 \leq k \leq n \), we have
\[
1 - x_2]q^{-1}B_{k,n-1}(x_1, x_2 \mid q) + [x_1]_q B_{k-1,n-1}(x_1, x_2 \mid q) \\
= [1 - x_2]q^{-1}(\binom{n-1}{k}[x_1]_q^{k}[1 - x_2q^{-k}] + [x_1]_q \binom{n-1}{k-1}[x_1]_q^{k-1}[1 - x_2q^{-k}] + [x_1]_q^{k}1 - x_2]q^{-k} \] (11)
\[
= \binom{n}{k}[x_1]_q^{k}[1 - x_2]q^{-k} = B_{k,n}(x_1, x_2 \mid q).
\]

Therefore, we obtain the following theorem.

**Theorem 1.** For \( x_1, x_2 \in [0, 1] \) and \( n, k \in \mathbb{Z}_+ \), we have
\[
1 - x_2]q^{-1}B_{k,n}(x_1, x_2 \mid q) + [x_1]_q B_{k-1,n}(x_1, x_2 \mid q) = B_{k,n+1}(x_1, x_2 \mid q).
\]

The partial derivative of \( B_{k,n}(x_1, x_2 \mid q) \) are also \( q \)-polynomials of degree \( n - 1 \):
\[
\frac{\partial}{\partial x_1} B_{k,n}(x_1, x_2 \mid q) = \frac{\log q}{q - 1} q^{x_1} n B_{k-1,n-1}(x_1, x_2 \mid q),
\]
and
\[
\frac{\partial}{\partial x_2} B_{k,n}(x_1, x_2 \mid q) = \frac{\log q}{q - 1} q^{x_2} n B_{k,n-1}(x_1, x_2 \mid q).
\]

Therefore, we obtain the following lemma.

**Lemma 2.** For \( k \in \mathbb{Z}_+ \) and \( n \in \mathbb{N}, x_1, x_2 \in [0, 1] \), we have
\[
\frac{\partial}{\partial x_1} B_{k,n}(x_1, x_2 \mid q) = \frac{\log q}{q - 1} n \{ (q-1)[x_1]_q B_{k-1,n-1}(x_1, x_2 \mid q) + B_{k-1,n-1}(x_1, x_2 \mid q) \},
\]
and
\[
\frac{\partial}{\partial x_2} B_{k,n}(x_1, x_2 \mid q) = \frac{\log q}{q - 1} n \{ (q-1)[x_2]_q B_{k,n-1}(x_1, x_2 \mid q) + B_{k,n-1}(x_1, x_2 \mid q) \}.
\]

For \( f = 1 \), by (6), we have
\[
B_{n,q}(1 \mid x_1, x_2) = \sum_{k=0}^{n} B_{k,n}(x_1, x_2 \mid q) = \sum_{k=0}^{n} \binom{n}{k}[x_1]_q^{k}[1 - x_2]q^{-k}
\]
\[
= (1 + [x_1]_q - [x_2]_q)^n.
\] (12)

By (12), we see that
\[
\frac{1}{(1 + [x_1]_q - [x_2]_q)^n} B_{n,q}(1 \mid x_1, x_2) = 1.
\]

For \( f(t) = t \), by (6), we get
\[
B_{n,q}(t \mid x_1, x_2) = \sum_{k=0}^{n} \binom{k}{n} [x_1]_q^{k}[1 - x_2]q^{-k} \binom{n}{k}
\]
\[
= \sum_{k=1}^{n} [x_1]_q^{k}[1 - x_2]q^{-k} \binom{n-1}{k-1}
\]
\[
= [x_1]_q \sum_{k=0}^{n-1} \binom{n-1}{k} [x_1]_q^{k}[1 - x_2]q^{-k-1},
\]
where \( n \in \mathbb{N} \) and \( x_1, x_2 \in [0, 1] \).
Thus, we have
\[
\frac{1}{(1 + [x_1]_q - [x_2]_q)^{n+1}} B_{n,q}(f \mid x_1, x_2) = [x_1]_q.
\]

For \( f(t) = t^2 \), by (6), we have
\[
B_{n,q}(t^2 \mid x_1, x_2)
= \frac{n-1}{n} [x_1]^2 q (1 + [x_1] - [x_2])^{n-2} + \frac{n}{n} (1 + [x_1] - [x_2])^{n-1}.
\]

In the special case, \( x_1 = x_2 = x \),
\[
B_{n,q}(t^2 \mid x_1, x_2) = \frac{n-1}{n} [x_1]^2 q + \frac{n}{n} [x_1]^2 q, \tag{13}
\]

From (13), we note that
\[
\lim_{n \to \infty} B_{n,q}(t^2 \mid x, x) = [x_1]^2 q.
\]

By (6), we see that
\[
B_{n,q}(f \mid x_1, x_2) = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) B_{k,n}(x_1, x_2 \mid q)
= \sum_{k=0}^{n} f \left( \frac{k}{n} \right) \left( \frac{n}{k} \right) [x_1]^k \sum_{j=0}^{n-k} \left( \frac{n-k}{j} \right) (-1)^j [x_2]^j
= \sum_{t=0}^{n} \left( \frac{n}{t} \right) [x_2]^t \sum_{k=0}^{t} \left( \frac{t}{k} \right) (-1)^{t-k} f \left( \frac{k}{n} \right) \left( \frac{[x_1]^k}{[x_2]^t} \right).
\]

From the definition of \( B_{k,n}(x_1, x_2 \mid q) \), we have
\[
\frac{n-k}{n} B_{k,n}(x_1, x_2 \mid q) + \frac{k+1}{n} B_{k+1,n}(x_1, x_2 \mid q)
= \frac{(n-1)!}{k!(n-k-1)!} [x_1]^k [1 - x_2]^{n-k} + \frac{(n-1)!}{k!(n-k-1)!} [x_1]^{k+1} [1 - x_2]^{n-k-1} \tag{14}
= ([x_1]^k + [1 - x_2]^{n-k}) B_{k,n}(x_1, x_2 \mid q)
= ([x_1] q + [1 - x_2] q) B_{k,n}(x_1, x_2 \mid q),
\]

where \( n \in \mathbb{N} \) and \( k \in \mathbb{Z}_+, x_1, x_2 \in [0, 1] \).

By the binomial theorem, we get
\[
B_{k,n}(x_1, x_2 \mid q) = \left( \frac{[x_1]^k}{[x_2]^t} \right) \sum_{l=k}^{n} \left( \frac{l}{k} \right) \left( \frac{n}{l} \right) (-1)^{l-k} [x_2]^l.
\]

It is possible to write \( [x_1]^k \) as a linear combination of \( B_{k,n}(x_1, x_2 \mid q) \) by using the degree evaluation formulae and mathematical induction:
\[
\frac{1}{(1 + [x_1] q - [x_2] q)^{n-1}} \sum_{k=1}^{n} \frac{k}{n} B_{k,n}(x_1, x_2 \mid q) = [x_1]^q.
\]

By the same method, we get
\[
\frac{1}{(1 + [x_1] q - [x_2] q)^{n-2}} \sum_{k=2}^{n} \frac{k}{n} B_{k,n}(x_1, x_2 \mid q) = [x_1]^{2 q}.
\]

Continuing this process, we obtain the following theorem.
Thus, we obtain the following theorem.

**Theorem 3.** For \( j \in \mathbb{Z}_+ \) and \( x_1, x_2 \in [0, 1] \), we have
\[
\frac{1}{(1 + [x_1]_q - [x_2]_q)^n} \sum_{k=j}^{n-1} \binom{n}{k} B_{k,n}(x_1, x_2 \mid q) = [x_1]_q^n.
\]

From Theorem 3, we have
\[
\frac{1}{(1 + [x_1]_q - [x_2]_q)^n} \sum_{k=j}^{n-1} \binom{n}{k} B_{k,n}(x_1, x_2 \mid q) = \sum_{k=0}^{j} q^\left(\begin{array}{c} j \\ k \end{array}\right) [k]_q S_q(k, j - k),
\]
where \([k]_q! = [k]_q [k - 1]_q \cdots [2]_q [1]_q\) and \(S_q(k, j - k)\) is the \(q\)-Stirling numbers of the second kind.

3. \(q\)-Bernoulli Polynomials associated with the bosonic \(p\)-adic \(q\)-Integral on \(\mathbb{Z}_p\).

In this section we assume that \(q \in \mathbb{C}_p\) with \(|1 - q|_p < 1\). For \(n \in \mathbb{Z}_+\), by (1), we get
\[
\int_{\mathbb{Z}_p} [1 - x + x_1]_{q^{-1}}^n d\mu_{q^{-1}}(x_1) = (-1)^n q^n \int_{\mathbb{Z}_p} [x + x_1]_q^n d\mu_q(x_1). \tag{15}
\]
From (4) and (15), we have
\[
\beta_{n,q^{-1}}(1 - x) = (-1)^n q^n \beta_{n,q}(x) \quad \text{for} \quad n \in \mathbb{Z}_+. \tag{16}
\]
By (2), (3) and (16), we get
\[
q^2 \beta_{n,q}(2) - (n + 1)q^2 + q = q(q + 1)^n = \beta_{n,q} \quad \text{if} \quad n > 1.
\]
Thus, we have
\[
\beta_{n,q}(2) = (n + 1) - \frac{1}{q} + \frac{1}{q^2} \beta_{n,q}. \tag{17}
\]
By simple calculation, we see that
\[
\int_{\mathbb{Z}_p} [1 - x]_{q^{-1}}^n d\mu_q(x) = (-1)^n q^n \beta_{n,q}(-1) = \beta_{n,q^{-1}}(2).
\]
From (15), (16) and (17), we can derive the following equation (18).
\[
\int_{\mathbb{Z}_p} [1 - x]_{q^{-1}}^n d\mu_q(x) = q^2 \beta_{n,q^{-1}} + (n + 1) - q \quad \text{if} \quad n > 1. \tag{18}
\]
Taking double bosonic \(p\)-adic \(q\)-integral on \(\mathbb{Z}_p\), by (18), we set
\[
\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} B_{k,n}(x_1, x_2 \mid q) d\mu_q(x_1) d\mu_q(x_2)
\]
\[
= \binom{n}{k} \int_{\mathbb{Z}_p} [x_1]_q^n d\mu_q(x_1) \int_{\mathbb{Z}_p} [1 - x_2]_{q^{-1}}^{n-k} d\mu_q(x_2). \tag{19}
\]
Thus, we obtain the following theorem.

**Theorem 4.** For \(x_1, x_2 \in [0, 1]\) and \(n, k \in \mathbb{Z}_+\), we have
\[
\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} B_{k,n}(x_1, x_2 \mid q) d\mu_q(x_1) d\mu_q(x_2)
\]
\[
= \begin{cases} 
\binom{n}{k} \beta_{k,q}(q^2 \beta_{n-k,q^{-1}} + (n - k + 1) - q), & \text{if} \ n > k + 1 \\
0, & \text{if} \ n < k \\
\beta_{k,q}, & \text{if} \ n = k \\
1, & \text{if} \ n = k = 0
\end{cases}
\]
From the $q$-symmetric properties (see Eq. (10)) of the $q$-Bernstein polynomials, we have

\[
\int_{\mathbb{Z}} \int_{\mathbb{Z}} B_{k,n}(x_1, x_2 \mid q) d\mu_q(x_1) d\mu_q(x_2) = \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \int_{\mathbb{Z}} \int_{\mathbb{Z}} [1 - x_1]^{k-l} [1 - x_2]^{n-k} d\mu_q(x_1) d\mu_q(x_2)
\]

\[
= \int_{\mathbb{Z}} [1 - x_2]^{n-k} d\mu_q(x_2) \{1 - k \int_{\mathbb{Z}} [1 - x_1]^{k-l} d\mu_q(x_1) + \sum_{l=0}^{k-2} \binom{k}{l} (-1)^{k+l} \int_{\mathbb{Z}} [1 - x_1]^{k-l} d\mu_q(x_1) \}. \tag{20}
\]

For $n, k \in \mathbb{Z}_+$, by (20), we get

\[
\int_{\mathbb{Z}} \int_{\mathbb{Z}} B_{k,n}(x_1, x_2 \mid q) d\mu_q(x_1) d\mu_q(x_2) = \int_{\mathbb{Z}} [1 - x_2]^{n-k} d\mu_q(x_2) \{1 - k \frac{2}{q} \beta_{k-1,q-1} + k - l + 1 - q \}.
\]

By (19) and (21), we obtain the following theorem.

**Theorem 5.** For $n, k \in \mathbb{Z}_+$, we have

\[
\binom{n}{k} \beta_{k,q} = (1 - k - \frac{2}{q}) + \sum_{l=0}^{k-2} \binom{k}{l} (-1)^{k+l} (q^2 \beta_{k-l,q-1} + k - l + 1 - q).
\]

Let $m, n, k \in \mathbb{Z}_+$. Then we have

\[
\int_{\mathbb{Z}} \int_{\mathbb{Z}} B_{k,n}(x_1, x_2 \mid q) B_{k,m}(x_1, x_2 \mid q) d\mu_q(x_1) d\mu_q(x_2) = \binom{n}{k} \binom{m}{k} \int_{\mathbb{Z}} [x_1]^{2k} d\mu_q(x_1) \int_{\mathbb{Z}} [1 - x_2]^{n+m-2k} d\mu_q(x_2)
\]

\[
= \binom{n}{k} \binom{m}{k} \beta_{2k,q} \int_{\mathbb{Z}} [1 - x_2]^{n+m-2k} d\mu_q(x_2). \tag{22}
\]

By the $q$-symmetric properties of $q$-Bernstein polynomials, we get

\[
\int_{\mathbb{Z}} \int_{\mathbb{Z}} B_{k,n}(x_1, x_2 \mid q) d\mu_q(x_1) d\mu_q(x_2) = \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} \int_{\mathbb{Z}} [1 - x_1]^{2k-l} d\mu_q(x_1) \int_{\mathbb{Z}} [1 - x_2]^{n+m-2k} d\mu_q(x_2)
\]

\[
= \int_{\mathbb{Z}} [1 - x_2]^{n+m-2k} d\mu_q(x_2) \{1 - 2k \int_{\mathbb{Z}} [1 - x_1]^{2k-l} d\mu_q(x_1) + \sum_{l=0}^{2k-2} \binom{2k}{l} (-1)^{2k+l} \int_{\mathbb{Z}} [1 - x_1]^{2k-l} d\mu_q(x_1) \}. \tag{23}
\]
By (22) and (23), we obtain the following theorem.

**Theorem 6.** For \( m, n, k \in \mathbb{Z}_+ \), we have

\[
\binom{n}{k} \binom{m}{k} \beta_{k,q} = 1 - 2k - \frac{2k}{[2]_q} + \sum_{l=0}^{2k-2} \binom{2k}{l} (-1)^{2k+l} (q^2 \beta_{2k-l,q-1} + 2k - l + 1 - q).
\]

Let \( n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+ \), and \( s \in \mathbb{N} \). Then

\[
\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \prod_{i=1}^{s} B_{k,n_i}(x_1, x_2 \mid q) d\mu_q(x_1) d\mu_q(x_2)
\]

\[
= \left( \prod_{i=1}^{s} \binom{n_i}{k} \right) \int_{\mathbb{Z}_p} [1 - x_1]_{q-1}^{n_k} d\mu_q(x_1) \int_{\mathbb{Z}_p} [1 - x_2]_{q-1}^{n_1+\cdots+n_s-sk} d\mu_q(x_2)
\]

\[
= \prod_{i=1}^{s} \left( \frac{n_i}{k} \right) \beta_{sk,q} \int_{\mathbb{Z}_p} [1 - x_1]_{q-1}^{n_k} d\mu_q(x_1) \int_{\mathbb{Z}_p} [1 - x_2]_{q-1}^{n_1+\cdots+n_s-sk} d\mu_q(x_2).
\]

By the binomial theorem, we get

\[
\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \prod_{i=1}^{s} B_{k,n_i}(x_1, x_2 \mid q) d\mu_q(x_1) d\mu_q(x_2)
\]

\[
= \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk+l} \int_{\mathbb{Z}_p} [1 - x_1]_{q-1}^{sk-l} d\mu_q(x_1) \int_{\mathbb{Z}_p} [1 - x_2]_{q-1}^{n_1+\cdots+n_s-sk} d\mu_q(x_2).
\]

From (24) and (25), we note that

\[
\left( \prod_{i=1}^{s} \binom{n_i}{k} \right) \beta_{sk,q}
\]

\[
= \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk+l} \int_{\mathbb{Z}_p} [1 - x_1]_{q-1}^{sk-l} d\mu_q(x_1)
\]

\[
= 1 - sk \int_{\mathbb{Z}_p} [1 - x_1]_{q-1} d\mu_q(x_1) + \sum_{l=0}^{sk-2} \binom{sk}{l} (-1)^{sk+l} \int_{\mathbb{Z}_p} [1 - x_1]_{q-1}^{sk-l} d\mu_q(x_1).
\]

By (26), we obtain the following theorem.

**Theorem 7.** Let \( s \in \mathbb{N}, n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+ \). Then we have

\[
\left( \prod_{i=1}^{s} \binom{n_i}{k} \right) \beta_{sk,q} = 1 - sk - \frac{sk}{[2]_q} + \sum_{l=0}^{sk-2} \binom{sk}{l} (-1)^{sk+l} (q^2 \beta_{sk-l,q-1} + sk - l + 1 - q).
\]

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