TEST CONFIGURATIONS FOR K-STABILITY AND GEODESIC RAYS

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Abstract

Let $X$ be a compact complex manifold, $L \to X$ an ample line bundle over $X$, and $H$ the space of all positively curved metrics on $L$. We show that a pair $(h_0, T)$ consisting of a point $h_0 \in H$ and a test configuration $T = (L \to X \to \mathbb{C})$, canonically determines a weak geodesic ray $R(h_0, T)$ in $H$ which emanates from $h_0$. Thus a test configuration behaves like a vector field on the space of Kähler potentials $H$. We prove that $R$ is non-trivial if the $\mathbb{C}^* \times$ action on $X_0$, the central fiber of $X$, is non-trivial. The ray $R$ is obtained as limit of smooth geodesic rays $R_k \subseteq H_k$, where $H_k \subseteq H$ is the subspace of Bergman metrics.

1 Introduction

Let $X$ be a compact complex manifold. According to a basic conjecture of Yau [33], the existence of canonical metrics on $X$ should be equivalent to a stability condition in the sense of geometric invariant theory. A version of this conjecture, due to Tian [31] and Donaldson [14], says that if $L \to X$ is an ample line bundle, then $X$ has a metric of constant scalar curvature in $c_1(L)$ if and only if the pair $(X, L)$ is K-stable, that is, if and only if the Futaki invariant $F(T)$ is negative for each non-trivial test configuration $T$. In particular, $F(T) < 0$ for all such $T$ should imply that the K-energy $\nu : H \to \mathbb{R}$ is bounded below, where $H$ is the space of all positively curved metrics on $L$.

Now it is well known that the K-energy is convex along geodesics of $H$ (Donaldson [12]). Thus, if $h_0 \in H$ and if $R : (-\infty, 0] \to H$ is a smooth geodesic ray emanating from $h_0$, then the restriction of $\nu$ to $R$ is a smooth convex function $\nu_R : (-\infty, 0] \to \mathbb{R}$ and hence $\lim_{t \to -\infty} \dot{\nu}_R = a(R)$ is well defined (here $\dot{\nu}_R$ is the time derivative of the K-energy). In particular, if $a(R) < 0$, then $\nu$ is bounded below on the ray $R$.

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We are thus led to the following plan for relating K-stability to lower bounds for the K-energy: Given a non-trivial test configuration $T = (\mathcal{L} \to X \to \mathbb{C})$ and a point $h_0 \in \mathcal{H}$,

A) Associate to $(h_0, T)$ a canonical non-trivial geodesic ray $R(T, h_0)$ emanating from $h_0$.

B) Prove that $\lim_{t \to -\infty} \dot{\nu}_R = F(T) + d(T)$.

where $d(T) \geq 0$ has the property: $d(T) = 0$ if $X_0$, the central fiber of $X$, has no multiplicity and $F(T) < 0$ implies $F(T) + d(T) < 0$. If this plan could be implemented, then $F(T) < 0$ would imply that $\nu$ is bounded below on all the rays $R(T, h_0)$ emanating from $h_0$.

In this paper, we take a step in the direction of the plan outlined above.: For step A), we start with an arbitrary test configuration $T$ and an arbitrary point $h_0 \in \mathcal{H}$. We associate to this data a weak geodesic $R(h_0, T)$ which is upper semi-continuous (but may not be smooth). If the $\mathbb{C} \times$ action on $X_0$ is non-trivial (in particular, if $F(T) \neq 0$), then we show that $R(h_0, T)$ is a non-trivial geodesic.

We also provide evidence for step B): The ray $R(h_0, T)$ is constructed as a limit of Bergman geodesic rays $h(t; k)$. Under certain geometric conditions (which are necessary for our proofs, but we expect can be removed) we observe that the limit of the K-energy time derivative along $h(t; k) = h_0 e^{-\phi(t; k)}$ converges to the Futaki invariant $F(T)$ as $k \to \infty$ if $X_0$ is multiplicity free.

After raising $\mathcal{L}$ and $L$ to sufficiently high powers, we may assume that $L$ is very ample, that $H^0(X, L)$ generates $\bigoplus_{k=0}^{\infty} H^0(X, L^k)$, and that $\mathcal{L}$ has exponent one (note that raising the power of the line bundle will just amount to a reparametrization of the geodesic). These assumptions will be made throughout this paper.

Our main results are Theorem 1 and Theorem 2 below (with relevant notation provided in §4):

**Theorem 1** Let $L \to X$ be a very ample line bundle, $h_0$ a positively curved metric on $L$, and $T$ a test configuration for $(X, L)$. Let

$$\phi_t = \lim_{k \to \infty} (\sup_{l \geq k} [\phi(t; l)])^*$$

Then $h(t) = h_0 e^{-\phi_t}$ is a weak geodesic ray emanating from $h_0$. Here we make use of the notation $u^*(\zeta_0) = \lim_{t \to 0} \sup_{|\zeta - \zeta_0| < \epsilon} u(\zeta)$ for any locally bounded $u : X \times (-\infty, 0] \to \mathbb{R}$.

**Theorem 2** Assume that the action of $\mathbb{C}^\times$ on $X_0$ is non-trivial. Then the weak geodesic defined by $\phi_t$ in Theorem 1 is non-trivial.

We note that the $\mathbb{C}^\times$ action on $X_0$ is non-trivial if the Futaki invariant $F(T)$ of the test configuration $T$ does not vanish.
We also note the following result:

**Theorem 3** Assume that the test configuration can be equivariantly imbedded in a proper family $\mathcal{X} \to B$, where $\mathcal{X}$ and $B$ are smooth compact manifolds with the property that the Chern class map $\text{Pic}(B) \to H^2(B, \mathbb{Z})$ is injective and $X_0$ is multiplicity free. Then, for each $k > 0$,\[ \lim_{t \to -\infty} \dot{\nu}_k = F(T) \] (1.2)

Here $\nu_k$ is the restriction of $\nu$ to the Bergman geodesic $h(t; k)$.

**Remark:** Theorem 1 holds in a wider context than that stated above - our proofs show that one can associate a weak geodesic ray to an arbitrary traceless hermitian matrix $A \in \text{gl}(H^0(X, L))$ (thus the eigenvalues of $A$ are real numbers and not restricted to lie in $\mathbb{Z}$).

To define what is meant by a weak geodesic, we start by recalling that $\mathcal{H}$ is an infinite dimensional symmetric space with respect to its natural Riemannian structure (see Mabuchi [19], Semmes [29] and Donaldson [12]). Furthermore, the geodesic equation for $h_0 e^{-\phi t}$ is equivalent to the degenerate Monge-Ampère equation\[ \Omega^{n+1} = 0 \quad \text{on} \quad X \times A \] (1.3)

where $A \subseteq \mathbb{C}$ is an annulus (in the case of a geodesic segment) or a punctured disk (in the case of a geodesic ray). Here $\Omega = \Omega_\phi$ is the smooth $(1, 1)$-form on $X \times A$ determined by: $\Omega = \Omega_0 + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \Phi$ where $\Omega_0 = \phi^{-1} \omega_0$, $\omega_0$ is the curvature of $h_0$, $p_1(x, w) = x$, $\Phi(x, w) = \phi_t(x)$, and $t = \log |w|$. A weak geodesic $\phi_t$ is one for which $\Omega_\phi$ is a plurisubharmonic solution to (1.3) in the sense of pluripotential theory [2].

The problem of constructing geodesic rays from test configurations has been considered previously by Arezzo-Tian [1]. They show that, if the central fiber of the test configuration $T$ is smooth, then one can use the Cauchy-Kowalevskya theorem to find a local analytic solution near infinity to the geodesic equation, and in this way, they construct a geodesic ray $R(T)$ in $\mathcal{H}$. In fact, they construct a family of rays $R_j(T)$ where $j$ ranges over certain free parameters which determine the power series coefficients. These rays have the advantage of being real-analytic, but it doesn’t appear that their origins can be prescribed by this method. Moreover, the relation of $R_j(T)$ to $F(T)$ is unclear.

We now provide an outline of the paper. The starting point is the approximation theorem for Kähler metrics by Bergman metrics: For $k \geq 1$, the space $\mathcal{H}_k \subseteq \mathcal{H}$ of Bergman metrics associated to $L^k$ is a finite dimensional symmetric Riemannian sub-manifold. If $h \in \mathcal{H}$ and $h(k) \in \mathcal{H}_k$ is the associated Bergman metric, then the theorem of Tian-Yau-Zelditch [34],[31],[35] implies $h(k) \to h$ in the $C^\infty$ topology.

Now fix $h_0, h_1 \in \mathcal{H}$, a pair of distinct elements, and let $h(t; k)$ be the unique smooth geodesic segment in $\mathcal{H}_k$ defined by the conditions $h(0; k) = h_0(k)$ and $h(1; k) = h_1(k)$.
It was proved in [27] that the sequence \( h(t; k) \) converges uniformly, in the weak \( C^0 \) sense of Theorem 1, to a weak geodesic segment \( h(t) \) in \( \mathcal{H} \) with the property: \( h(0) = h_0 \) and \( h(1) = h_1 \). Moreover, \( h(t) \) equals the \( C^{1,1} \) geodesic joining \( h_0 \) to \( h_1 \), whose existence was established by Chen [9]. We note that another approximation of the \( C^{1,1} \) geodesic by potentials \( \tilde{h}(t; k) \) in \( c_1(L) + \frac{1}{k}c_1(K_X) \) has been very recently constructed by Berndtsson [5].

The proof of Theorem 1 follows the method of [27]. First, we construct a geodesic ray \( h(t; k) = h_0 e^{-\phi(t; k)} \) with \( h(0; k) = h_0(k) \) that “points in the direction of \( T \)”. Then we prove that

\[
\int_{X \times A} \Omega_k^{n+1} = O(k^{-1})
\]

where \( \Omega_k \) is associated to \( \phi(t; k) \). This step relies on the ideas developed in the recent work of Donaldson [16]. It also requires some estimates on test configurations, which include the following very simple, but basic estimate for the endomorphisms \( A_k \) on \( H^0(X_0, L_0^k) \) determined by a test configuration,

\[
\|A_k\|_{op} = O(k).
\]

Next, we use the methods of pluripotential theory to establish the convergence of the \( \phi(t; k) \). In the case of geodesic rays, the annulus \( A \) is actually a punctured disk, and the boundary behavior at the puncture has to be treated carefully, by controlling the asymptotics for the \( \phi(t; k) \) at the puncture.

For Theorem 2, we show that, when the test configuration is non-trivial, the sup norm of \( \phi_t \) goes to \( \infty \) near the puncture. This implies that the geodesic is non-trivial. A key ingredient is Donaldson’s formula [16] for the leading coefficient of \( \text{Tr}(A_k^2) \).

Theorem 3 is a direct consequence of the work of Tian [31] and Paul-Tian [22]: We apply the formula in [31] which relates the metric of the CM line bundle \( L_{CM} \) to the K-energy. We then use [22] which relates the line bundle \( \lambda_{CM} \) on the Hilbert scheme to \( L_{CM} \).

We would like to add some references that have come to our attention since the posting of the first version of this paper. In a paper [23] which appeared shortly after ours, Paul and Tian present several results which include in particular Theorem 3. In fact, they actually prove a stronger result, in which the assumption on the injectivity of the Chern map is removed. As should be clear from its proof and as we already noted above, Theorem 3 was in any case an immediate consequence of their earlier work. In the recent paper [11], X.X. Chen shows that geodesic rays parallel to a given geodesic ray can be constructed under a certain assumption of tame ambient geometry. We would also like to note that constructions involving upper envelopes appear frequently in pluripotential theory, notably in the work of Kolodziej [17].

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2 Test Configurations: preliminaries

2.1 Definition

Let $L \to X$ be an ample line bundle over a compact complex manifold. A test configuration, as defined by Donaldson [14], consists of the following data:

(1) A scheme $\mathcal{X}$ with a $\mathbb{C}^\times$ action $\rho$.

(2) A $\mathbb{C}^\times$ equivariant line bundle $\mathcal{L} \to \mathcal{X}$ which is ample on all fibers.

(3) A flat $\mathbb{C}^\times$ equivariant map $\pi : \mathcal{X} \to \mathcal{C}$ where $\mathbb{C}^\times$ acts on $\mathcal{C}$ by multiplication

satisfying the following: The fiber $X_1$ is isomorphic to $X$ and the pair $(X, L^r)$ is isomorphic to $(X_1, L_1)$ where, for $w \in \mathcal{C}$, $X_w = \pi^{-1}(w)$ and $L_w = \mathcal{L}|_{X_w}$. After raising $\mathcal{L}$ and $L$ to sufficiently high powers, we may assume that $L$ is very ample, that $H^0(X, L)$ generates $\oplus_{k=0}^{\infty} H^0(X, L^k)$, and that $\mathcal{L}$ has exponent one. Thus we set $r = 1$.

If $\tau \in \mathbb{C}^\times$ and $w \in \mathcal{C}$, let $\rho_k(\tau, w) : H^0(X_w, L^k_w) \to H^0(X_{\tau w}, L^k_{\tau w})$ be the isomorphism induced by $\rho$. If $w = 0$ we write $\rho_k(\tau, 0) = \rho_k(\tau)$. We also let $B_k \in \text{End}(V_k)$ be defined by

$$\rho_k(e^t) = e^{tB_k} \quad (2.1)$$

for $t \in \mathbb{R}$, and $A_k$ the traceless part of $B_k$. The eigenvalues of $A_k$ are denoted by $\lambda_0^{(k)} \leq \lambda_1^{(k)} \leq \cdots \leq \lambda_{N_k}^{(k)}$, and the eigenvalues of $B_k$ are denoted by $\eta_0^{(k)} \leq \eta_1^{(k)} \leq \cdots \leq \eta_{N_k}^{(k)}$. Thus $\rho_k : \mathbb{C}^\times \to GL(V_k)$ where $V_k = H^0(X_0, L^k_0)$. Let $d_k = \dim V_k$ and $w(k) = \text{Tr}(B_k)$, the weight of the induced action on $\text{det}(V_k)$. Then, as was observed in [14], there is an asymptotic expansion

$$\frac{w(k)}{kd_k} = F_0 + F_1 k^{-1} + F_2 k^{-2} + \cdots \quad \text{as} \quad k \to \infty \quad (2.2)$$

The Donaldson-Futaki invariant $F(T)$, or simply Futaki invariant, of $T$ is defined by the formula: $F(T) = F_1$.

2.2 Equivariant imbeddings of test configurations

The construction of the Bergman geodesics associated to a test configuration $T$ relies on the existence of an equivariant, unitary imbedding of $T$ into projective space, whose existence was first established by Donaldson [16]. In this section, we begin by recalling the statement of Donaldson’s result.

Let $T$ be a test configuration of exponent $r = 1$ for the pair $(X, L)$. For $k$ large, since $L$ is very ample, we have canonical compatible imbeddings $i_k : X \subseteq \mathcal{P}(H^0(X, L^k_1)^*)$ and $i_k : L^k_1 \to O(1)$ where $\mathcal{O}_w(1) \to \mathcal{P}(H^0(X_w, L^k_w)^*)$ is the hyperplane line bundle, where $H^0(X_w, L^k_w)^*$ is the dual of $H^0(X_w, L^k_w)$. 

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One can show that the bundle $\pi_*L^k \to C$ has an equivariant trivialization and thus the test configuration has an equivariant imbedding into projective space. To be precise: Let $\Theta$ be an arbitrary vector space isomorphism $\Theta : H^0(X_0, L^k_0) \to H^0(X_1, L^k_1)$; Let $\mathcal{X}^* = \pi^{-1}(C^*)$ and let $\mathcal{L}^* = \mathcal{L}|_{\mathcal{X}^*}$. Define an imbedding $I_\Theta : (\mathcal{L}^*)^k \hookrightarrow O_0(1) \times C^*$ by the formula

$$I_\Theta(\rho(\tau)l) = [(\rho_k(\tau)\Theta^*(\iota_k(l)), \tau) \quad (2.3)$$

where $\tau \in C^*$, $l \in L^k_1$ and $\Theta^* : O_1(1) \to O_0(1)$ is the isomorphism induced by the dual vector space isomorphism $\Theta^* : H^0(X_1, L^k_1)^* \to H^1(X_0, L^k_0)^*$. We similarly define the imbedding $I_\Theta : \mathcal{X}^* \hookrightarrow \mathcal{P}(H^0(X_0, L^k_0)) \times C^*$. Then we say $\Theta : H^0(X_0, L^k_0) \to H^0(X_1, L^k_1)$ is a “regular generator of $T$” if $I_\Theta$ extends to an imbedding $\mathcal{L}^k \to O_0(1) \times C$ which restricts, over the central fiber, to the canonical embedding $L^k_0 \hookrightarrow O_0(1)$.

Next let $h$ be a fixed metric on $L$. It is shown in [16] that there exists an regular generator $\Theta$ which respects $h$ structure in the following sense: The metric $h$ defines a hermitian metric $H_k$ on $H^0(X, L^k)$ by the formula $\langle s, s' \rangle = f_X(s, s')h_k \omega^2$ where $\omega$ is the curvature of $h$. If $\Theta$ is a regular generator of $T$, then we can use the isomorphism $\Theta : V_k \to H^0(X, L^k)$ to define a metric on $V_k$, which we call $H_k(\Theta)$. Let $B_k$ be the endomorphism of $V_k$ defined by: $\rho_k(e^t) = e^{tb_k}$ for $t \in R$. We say $\Theta$ is a regular hermitian generator if $B_k$ is hermitian with respect to $H_k(\Theta)$. In other words, $\Theta$ is regular hermitian if $\rho_k(\tau) : V_k \to V_k$ is an isometry for $|\tau| = 1$.

In [16] the following is proved:

**Lemma 1** Let $T$ be a test configuration for $(X, L)$ and $h$ a positively curved metric on $L$. Then there exists $\Theta$, a regular hermitian generator for $T$. The metric $H_k = H_k(\Theta)$ is independent of the choice of such a $\Theta$. Moreover, the map $\Theta : V_k \to H^0(X_1, L^k_1)$ is unique up to an isometry of $V_k$ which commutes with $B_k$.

Our formulation of Lemma 1 is somewhat different than that given in [16] and in order to make the relationship between the two precise, we shall provide a complete proof (which is of course essentially the one which appears in [16]):

Let $E \to C$ be an algebraic vector bundle of rank $r$. Then $E(C)$, the space of global sections of $E$, is a free $C[t]$ module of rank $N + 1$. A “trivialization of $E$” is just a choice of ordered basis $S_0, ..., S_N$ of the $C[t]$ module $E(C)$.

If $S_0, ..., S_N$ is a trivialization of $E$, and if $t \in C$, then $S_0(t), ..., S_N(t)$ is a basis of the fiber $E_t$ so, we have a well defined isomorphism $\phi_{t_2,t_1} : E_{t_1} \cong E_{t_2}$ for any pair $t_1, t_2 \in C$, which takes the basis $S_j(t_1)$ to the basis $S_j(t_2)$. The collection $\{\phi_{t_2,t_1}\}$ defines a regular cocycle, that is: $\phi_{t_3,t_2}\phi_{t_2,t_1} = \phi_{t_3,t_1}$ and for every $e \in E_{t_1}$, the map $t \mapsto \phi_{t_1,t}(e)$ is a global section of $E$. Conversely, a regular cocycle $\phi_{t_2,t_1}$ defines a trivialization of $E$.

Now suppose $E \to C$ is a vector bundle with a $C^*$ action, covering the usual action of $C^*$ on $C$. This means that we are given an algebraic map $\rho : C^* \to Aut(E \to C)$. Thus,
if \( \tau \in \mathbb{C}^\times \) then \( \rho(\tau) : E \to E \) is a function with the following properties:

1. The function \( \rho(\tau) \) maps the fiber \( E_t \) into the fiber \( E_{rt} \), that is: \( \pi(\rho(\tau)e) = \rho(\tau)\pi(e) \).
2. The function \( \rho(\tau) : E_t \to E_{rt} \) is an isomorphism of vector spaces.
3. If \( \tau_1, \tau_2 \in \mathbb{C}^\times \), then \( \rho(\tau_1\tau_2) = \rho(\tau_1)\rho(\tau_2) \).
4. The map \( \mathbb{C}^\times \times E \to E \) given by \( (\tau, e) \to \rho(\tau)e \) is algebraic.

Let \( S_0, \ldots, S_N \) be a basis of global sections for \( E \). If \( S : \mathbb{C} \to E \) is an arbitrary global section, and if \( \tau \in \mathbb{C}^\times \), then \( S^{\rho(\tau)}(t) = \rho(\tau)^{-1}S(\tau t) \) is also a global section. Hence, there is a matrix \( A(\tau, t) \in GL(N+1, \mathbb{C}[\tau, \tau^{-1}, t]) \) with the property:

\[
\begin{align*}
\begin{bmatrix}
S^{\rho(\tau)}
\end{bmatrix} & = A(\tau, t)\begin{bmatrix}
S
\end{bmatrix}
\end{align*}
\tag{2.4}
\]

where \( S \) is the column vector whose components are the \( S_j \). Note that

\[
\begin{align*}
\begin{bmatrix}
S^{\rho(\tau_1\tau_2)}
\end{bmatrix} & = \rho(\tau_2\tau_1)^{-1}\begin{bmatrix}
S_{\tau_2\tau_1}
\end{bmatrix} = \rho(\tau_1)^{-1}A(\tau_2, \tau_1 t)\begin{bmatrix}
S_{\tau_1}
\end{bmatrix} = A(\tau_2, \tau_1 t)A(\tau_1, t)\begin{bmatrix}
S
\end{bmatrix}
\end{align*}
\]

where, in the last equality, we are using the fact that \( \rho(\tau_1)^{-1} \) is linear on the fibers. Hence:

\[
A(\tau_2\tau_1, t) = A(\tau_2, \tau_1 t)A(\tau_1, t)
\tag{2.5}
\]

In particular, if \( A(\tau) = A(\tau, 0) \), then \( A(\tau) : \mathbb{C}^\times \to GL(N + 1, \mathbb{C}) \) is a one parameter subgroup.

With these preliminaries in place, we now show that if \( E \to \mathbb{C} \) is an vector bundle with \( \mathbb{C}^\times \) action, then \( E \) has a \( \mathbb{C}^\times \) equivariant trivialization:

**Lemma 2** Let \( E \to \mathbb{C} \) be a vector bundle of rank \( r = N + 1 \) with a \( \mathbb{C}^\times \) action. Then there exists a basis of global sections \( S_0, ..., S_N \) such that \( \mathbb{A}(\tau, t) \) is independent of \( t \), that is, \( A(\tau, t) = A(\tau, 0) \equiv A(\tau) \). In other words, there exists a regular cocycle \( \{\phi_{t_2, t_1}\} \) satisfying

\[
\rho(\tau)\phi_{t_2, t_1}\rho(\tau)^{-1} = \phi_{t_2, \tau t_1}
\tag{2.6}
\]

The basis \( S_0, ..., S_N \) is unique up to change of basis matrices \( M(t) \in GL(N + 1, \mathbb{C}[t]) \) with the property: \( M(\tau t) = A(\tau)M(t)A(\tau)^{-1} \).

**Proof.** Choose any \( \mathbb{C}[t] \) basis \( S_0, ..., S_N \in E(\mathbb{C}) \) and define \( A(\tau, t) \in GL(N + 1, \mathbb{C}[\tau, \tau^{-1}, t]) \) as in equation (2.4). Thus \( \det(A(\tau, t)) = a\tau^p = \det(A(\tau)) \) for some integer \( p \) and some \( a \in \mathbb{C}^\times \). Now consider the set

\[
\mathcal{S} = \{ S_j^{\rho(\tau)} : \tau \in \mathbb{C}^\times, 0 \leq j \leq N \}
\]
Let $V \subseteq E(\mathbb{C})$ be the complex vector space generated by $S$. We claim that $V$ is finite dimensional and invariant under the action of $\mathbb{C}^\times$. In fact, since $S^{\rho(\tau)} = A(\tau,t)S$ we see that the $S_j^{\rho(\tau)}$ are all linear combinations, with $\mathbb{C}$ coefficients, of elements in the set\( \{ t^mS_j : 0 \leq j \leq N, 0 \leq m \leq M \} \), where $M$ is chosen so that the entries of $A(\tau,t)$, which are polynomials in $t$ with coefficients in $A[\tau,\tau^{-1}]$, all have degree at most $M$.

Choose a basis $\{ T_\mu ; 0 \leq \mu \leq K \}$ of $V$ with the property $T^{\rho(\tau)}_\mu = \tau^{l_\mu}T_\mu$ for some integers $l_\mu$. Choose $\mu_j, 0 \leq j \leq N$, such that $T_{\mu_j}(0)$ are linearly independent. This can certainly be done since the $T_\mu$ span $V$, and $V$ contains the $S_j$. Let $T$ be the column vector consisting of the $T_{\mu_j}$. Then $T(t) = C(t)S(t)$ for some $(N+1) \times (N+1)$ matrix $C(t)$ with coefficients in $\mathbb{C}[t]$, for which $C(0)$ is invertible. The existence of such a matrix is guaranteed by the fact that the $S_j$ form a $\mathbb{C}[t]$ basis of $E(\mathbb{C})$. Replacing $S$ by $C(0)S$ doesn’t change $V$ and allows us to assume $C(0) = I$. Now

\[
T^{\rho(\tau)}(t) = \rho(\tau)^{-1}T(\tau) = \rho(\tau)^{-1}C(\tau t)S(\tau t) = C(\tau t)A(\tau,t)S(t)
\]

On the other hand, $T^{\rho(\tau)}(t) = U(\tau)T(t)$ where $U(\tau)$ is diagonal with diagonal entries of the form $\tau^l$. Hence

\[
U(\tau)S(0) = U(\tau)T(0) = T^{\rho(\tau)}(0) = S^{\rho(\tau)}(0) = A(\tau)S(0)
\]

so $U(\tau) = A(\tau)$. Thus

\[
C(\tau t)A(\tau,t)S(t) = T^{\rho(\tau)}(t) = A(\tau)T(t) = A(\tau)C(t)S(t)
\]

which implies: $A(\tau)C(t) = C(\tau t)A(\tau,t)$. Since $\det(A(\tau)) = \det(A(\tau,t))$ for all $t$, we have $\det(C(\tau t)) = \det(C(t))$ which means that $\det(C(t))$ is independent of $t$. Since $C(0) = I$, we conclude $\det(C(t)) = 1$ and this implies that $T$ is a $\mathbb{C}[t]$ basis of $E(\mathbb{C})$. This now establishes Lemma 2.

At this point we can prove the existence of a regular generator for $T$: Let $E = \pi_*(\mathcal{L}_k)^*$ so that $E_t = H(X_t, L_k^t)^*$. Then we define $\Theta^* : E_1 \to E_0$ by the formula: $\Theta^* = \phi_{0,1}$ where $\phi_{t_2,t_1}$ satisfies (2.6), with $\rho(\tau)$ replaced by $\rho^*(\tau) = \rho(\tau^{-1})^*$. One easily checks that $\Theta$ is a regular generator of $T$.

**Lemma 3** Let $H_1$ be a hermitian metric on $E_1$. Then there is a unique equivariant trivialization $\phi_{t_2,t_1}$ such that $\rho(\tau)^{-1}\phi_{r,1} : E_1 \to E_1$ is an isometry for all $\tau \in \mathbb{C}^\times$ with $|\tau| = 1$.

**Proof.** Let $\{ \phi_{t_2,t_1} \}$ be any equivariant trivialization. Consider the decomposition $E_0 = \oplus V_i$ into eigenspaces for the action of $\mathbb{C}^\times$. Let $\tau^{w_j}$ be the restriction of $\rho(\tau)$ to the subspace $V_j$. We may assume that $w_1 < w_2 < \cdots < w_l$. Thus $\sum_{j=1}^l w_j \dim(V_j) = N + 1 = \dim(E_0)$. Let $e_0, ..., e_N$ of $E_0$ be given by the union of the bases of the $V_j$ and define $S_j(t) = \phi_{t,0}(e_j)$ and let $W_i = \phi_{t,0}(V_i) \subseteq E_1$. Then $S_0, ..., S_N$ is a trivialization of $E \to \mathbb{C}$. 

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Let $A(\tau)$ be the diagonal matrix which represents the automorphism $\rho(\tau) : E_0 \to E_0$ with respect to the basis $e_j$. Then $A(\tau)$ also represents the automorphism $\rho(\tau)^{-1} \phi_{r,1} : E_1 \to E_1$ with respect to the basis $S_j(1)$. We want to modify the equivariant trivialization $\phi_{t_2, t_1}$ in such a way that this automorphism is an isometry. To do this, we must find a matrix $M(t) \in GL(N + 1, \mathbb{C}[t])$ satisfying:

1. $M(\tau t)A(\tau)M(t)^{-1} = A(\tau)$ for all $t, \tau$.

2. $M(1)S_j(1)$ is orthonormal with respect to $H$.

The first condition says that $M(t)$ is a block matrix with blocks $t^{w_i-w_j} \alpha_{ij}$ where $\alpha_{ij}$ is independent of $t$. Since $M(t) \in GL(N + 1, \mathbb{C}[t])$, this implies that $\alpha_{ij} = 0$ if $i < j$. Thus $M(t)$ is lower block triangular. On the other hand, the usual Gram-Schmidt process allows us to choose an $M(1)$ of this form which satisfies condition 2: First choose an orthonormal basis of $W_0$. Then choose an orthonormal basis of $W_0^\perp \subseteq W_0 \oplus W_1$, etc.

Finally we prove uniqueness. Let $M(t) \in GL(N + 1, \mathbb{C}[t])$ satisfy 1. and 2. and assume furthermore that the $e_j$ are orthonormal and that $M(0) = I$. Then we must show that $M(t) = I$ for all $t$. Since the $e_j$ are orthonormal, the matrix $M(1)$ is unitary. On the other hand, it is lower block triangular. This implies it is block diagonal. Since the $i, j$ block is of the form $t^{w_i-w_j} \alpha_{ij}$, and since $\alpha_{ij} = 0$ for $i \neq j$, we see that $M(t)$ is independent of $t$ so $M(t) = M(0) = I$. The lemma is proved.

Note that if $\phi$ is any equivariant trivialization, then $\rho(\tau)^{-1} \phi_{r,1} : \mathbb{C}^\times \to GL(E_1)$ is a homomorphism:

$$\rho(\tau_1)^{-1} \phi_{r_1,1} \rho(\tau_2)^{-1} \phi_{r_2,1} = \rho(\tau_1 \tau_2)^{-1} \phi_{r_1 \tau_2, \tau_2} \phi_{r_2,1} = \rho(\tau_1 \tau_2)^{-1} \phi_{r_1, \tau_2,1},$$

where the first equality makes use of the equivariance property of $\phi$, and the second follows from the cocycle property of $\phi$. Thus the theorem can be restated as follows: There exists an equivariant trivialization $\phi$ such that $\rho(\tau)^{-1} \phi_{r,1} : S^1 \to GL(E_1)$ is a unitary representation.

To deduce Lemma 1 from Lemma 3, we again define $\Theta^* = \phi_{0,1}$. Let $\tau \in \mathbb{C}^\times$ be of unit length. Then to show $\rho_k(\tau)^* : V_k^* \to V_k^*$ is an isometry is equivalent, by definition of the metric on $V_k$, to showing $(\Theta^*)^{-1} \rho_k(\tau)^* \Theta^* : H(X_1, L_k)^* \to H(X_1, L_k)^*$ is an isometry. Thus we must show $\phi_{1,0} \rho_k(\tau)^* \phi_{0,1} = \phi_{1,0} \phi_k^* (\tau^{-1}) \phi_{0,1}$ is an isometry. But (2.6) implies $\phi_{1,0} \rho_k^* (\tau^{-1}) \phi_{0,1} = \rho^* (\tau^{-1}) \phi_{r,0} \phi_{0,1} = \rho^* (\tau)^{-1} \phi_{r,1}$ which is an isometry by the result of Lemma 3. This proves Lemma 1.
3 Estimates for test configurations

3.1 Bounds for $A_k$

Let $T$ be a test configuration and define the endomorphisms $A_k$ and $B_k$ and their eigenvalues $\lambda_{\alpha}^{(k)}$ and $\eta_{\alpha}^{(k)}$ as in Section §2.1. The following simple estimate for the operator norm $\|A_k\|_{op}$ of the endomorphisms $A_k$ plays an important role in the subsequent bounds for the total masses of the Monge-Ampère currents:

Lemma 4 There is a constant $C > 0$ which is independent of $k$ such that $|\lambda_{\alpha}^{(k)}| \leq Ck$ for all $k > 0$ and all $\alpha$ such that $0 \leq \alpha \leq N_k$.

Proof of Lemma 4. After applying Lemma 1 with $k = 1$, we may assume that $X \subseteq \mathbb{P}^m \times \mathbb{C}$, $m = N_1 + 1$, and that $\rho(\tau)$ is a diagonal matrix in $GL(m+1)$ whose entries are $\tau_0^{\eta_0}, \ldots, \tau_m^{\eta_m}$ where $\eta_0 \leq \cdots \leq \eta_m$ are integers. The scheme $X_0 \subseteq \mathbb{P}^m$ is defined by a homogenous ideal $I \subseteq \mathbb{C}[X_0, \ldots, X_m]$ and we write

$$\mathbb{C}[X_0, \ldots, X_m]/I = \bigoplus_{k \geq 0} S_k/I_k$$

where $S_k \subseteq \mathbb{C}[X_0, \ldots, X_m]$ is the space of polynomials which are homogeneous of degree $k$ and $I_k = S_k \cap I$. Then, for $k >> 0$, we have $H^0(X_0, L_0^k) = S_k/I_k$. The matrix $\rho(\tau)$ defines an automorphism of $\mathbb{C}[X_0, \ldots, X_m]$, determined by the formula: $X_j \mapsto \tau_j^{\eta_j} X_j$. This automorphism leaves $S_k$ and $I_k$ invariant, and thus it induces an automorphism of $S_k/I_k$ which is, by definition, the map $\rho_k(\tau)$.

The monomials of degree $k$ form a basis of $S_k$ which are eigenfunctions of $\rho(\tau)$: More precisely, if $X^p$ is a monomial, with $p = (p_0, \ldots, p_m)$ and $p_0 + \cdots + p_m = k$, then we have $\rho(\tau) \cdot X^p = \tau^{p_\alpha} X^\alpha$. Since the monomials of degree $k$ span $S_k/I_k$, some subset form a basis of eigenvectors for that space. Thus the eigenvalues of the $B_k$ form a subset of $\{p \cdot \eta : p_0 + \cdots + p_m = k\}$. On the other hand, for such an $p$, we clearly have $|p \cdot \eta| \leq \sup |\eta_j| \cdot k$ and this proves that

$$|\eta_{\alpha}^{(k)}| \leq Ck,$$

with $C = \sup_{0 \leq j \leq m} |\eta_j|$. On the other hand,

$$\lambda_{\alpha}^{(k)} = \eta_{\alpha}^{(k)} - \frac{\text{Tr}(B_k)}{N_k + 1} = \eta_{\alpha}^{(k)} + O(k).$$

This proves Lemma 4.
3.2 An alternative characterization of the Futaki invariant

3.2.1 The $F^0_\omega$ functional

Let $X$ be a compact complex manifold of dimension $n$ and $\omega = \omega_0$ a Kähler metric on $X$. Let $H = H_\omega$ be the space of Kähler potentials:

$$H_\omega = \{ \phi \in C^\infty(X) : \omega_\phi = \omega + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \phi > 0 \}$$ (3.3)

The functionals $F^0_\omega, \nu_\omega : H \to \mathbb{R}$ play an important role in Kähler geometry and are defined as follows:

$$F^0_\omega(\phi) = -\frac{1}{n+1} \left( \int_X \omega^n \right)^{-1} \sum_{j=0}^{n} \int_X \phi \omega_j^\phi \omega^{n-j} = -\frac{1}{n+1} \left( \int_X \omega^n \right)^{-1} E_\omega(\phi)$$

$$\nu_\omega(\phi) = -\left( \int_X \omega^n \right)^{-1} \int_0^1 \int_X \phi(s - \hat{s}) \omega^n_t \, dt,$$ (3.4)

Here $\phi_t, 0 \leq t \leq 1$, is a smooth path in $H_\omega$ joining the potential $\phi_0$ for $\omega_0$ to $\phi = \phi_1$. Then a simple calculation shows

$$\dot{E}_\omega(\phi_t) = (n+1) \int_X \dot{\phi}_t \omega^{n}_{\phi_t} \quad \text{and} \quad \ddot{E}_\omega(\phi_t) = (n+1) \int_X (\ddot{\phi}_t - |\partial \dot{\phi}_t|^2) \omega^n_{\phi_t},$$ (3.5)

Thus $E$ satisfies the cocycle property: $E_\omega(\phi) + E_{\omega_\psi}(\psi) = E_\omega(\phi + \psi).$ Note as well that if $f : Y \to X$ is a biholomorphic map, then

$$E_{f^* \omega}(\phi \circ f) = E_\omega(\phi)$$ (3.6)

3.2.2 The Chow weight and the Futaki invariant

Let $V$ be a finite dimensional vector space, $Z \subseteq \mathbb{P}(V)$ a smooth subvariety, and $B \in gl(V)$. Then we wish to define the generalized Chow weight $\mu(Z, B) \in \mathbb{R}$. We start by assuming the $V = \mathbb{C}^{N+1}$ so that $B$ is a $(N+1) \times (N+1)$ matrix. Let $\omega_{FS}$ be the Fubini-Study metric on $\mathbb{P}^N$. We shall also denote by $\omega_{FS}$ the restriction of the Fubini-Study metric to $Z$. For $t \in \mathbb{R}$ let $\sigma_t \in GL(N+1, \mathbb{C})$ be the matrix $\sigma_t = e^{tB}$ and let $\psi_t : \mathbb{P}^N \to \mathbb{R}$ be the function

$$\psi_t(z) = \log \frac{|\sigma_t z|^2}{|z|^2}$$ (3.7)

Here we view $z$ as an element in $\mathbb{P}^N$ and, when there is no fear of confusion, a column vector in $\mathbb{C}^{N+1}$.

Then $\psi_t$ is a smooth path in $H$: In fact, $\sigma_t^* \omega_{FS} = \omega_{FS} + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \psi_t$. Define
\[ \mu(Z, B) = - \lim_{t \to -\infty} \dot{E}_{\omega_{FS}}(\psi_t) = -\dot{E}(-\infty) \]  

Equation (3.8)

Note that the function \( E(t) = E_{\omega_{FS}}(\psi_t) : \mathbb{R} \to \mathbb{R} \) is convex (see [24, 25]), so the limit in (3.8) exists.

Next we compute the derivative of \( E(t) \):

\[ \frac{d}{dt} E_{\omega_{FS}}(\psi_t) = (n + 1) \int_{Z} z^* \sigma_t^* \cdot (B + B^*) \cdot \sigma_t z \, \omega_{FS}^n = (n + 1) \int_{\sigma_t(Z)} z^* \cdot (B + B^*) \cdot z \, \omega_{FS}^n \]  

Equation (3.9)

where, for \( C \) a matrix with complex entries, we write \( C^* = \overline{t} C \). In particular,

\[ \dot{E}_{\omega_{FS}}(\psi_t) \big|_{t=0} = \dot{E}(0) = (n + 1) \text{Tr}((B + B^*) \cdot M) \]  

Equation (3.10)

where

\[ M_{\alpha\beta} = M_{\alpha\beta}(Z) = \int_{Z} z_{\alpha} \overline{z}_{\beta} \omega_{FS}^n \]  

Equation (3.11)

**Lemma 5** Let \( V \) be a finite dimensional complex vector space, \( B \in \text{gl}(V) \) and \( Z \subseteq \mathbb{P}(V) \) a smooth subvariety. Let \( \theta : V \to \mathbb{C}^{N+1} \) be an isomorphism. Then \( \mu(\theta(Z), \theta B \theta^{-1}) \) is independent of \( \theta \).

**Proof.** We make use of the formula of Zhang [36] and Paul [21] (see also [24]): If \( Z \subseteq \mathbb{P}^N(\mathbb{C}) \) is a subvariety of dimension \( n \) and degree \( d \), let \( \text{Chow}(Z) \in \mathbb{P}(H^0(Gr(N-n, \mathbb{C}^{N+1}, O(d)))) \) be the Chow point of \( Z \subseteq \mathbb{P}^N \). If \( B \in \text{gl}(N+1, \mathbb{C}) \), \( \sigma_t = e^{tB} \), and \( \psi_{\sigma_t} = \log \frac{|\sigma_t(z)|^2}{|z|^2} \) then

\[ E_{\omega_{FS}}|_Z(\psi_{\sigma_t}) = \log \frac{||\sigma_t \cdot \text{Chow}(Z)||^2}{||\text{Chow}(Z)||^2} = \log \frac{||\text{Chow}(\sigma_t Z)||^2}{||\text{Chow}(Z)||^2} \]  

Equation (3.12)

where \( || \cdot || \) is the Chow norm defined on \( H^0(Gr(N-n, \mathbb{C}^{N+1}, O(d))) \).

Suppose \( M \in \text{GL}(N+1, \mathbb{C}) \). Then

\[ E_{\omega_{FS}|_M}(\psi_{M\sigma_t M^{-1}}) = \log \frac{||M\sigma_t M^{-1} \cdot \text{Chow}(MZ)||^2}{||\text{Chow}(MZ)||^2} \]  

Equation (3.13)

Subtracting (3.12) from (3.13) we get

\[ E_{\omega_{FS}|_M}(\psi_{M\sigma_t M^{-1}}) - E_{\omega_{FS}|_Z}(\psi_{\sigma_t}) = \log \frac{||M\sigma_t \cdot \text{Chow}(Z)||^2}{||\sigma_t \cdot \text{Chow}(Z)||^2} - \log \frac{||M \cdot \text{Chow}(Z)||^2}{||\text{Chow}(Z)||^2} \]

which is a bounded function of \( t \), and hence the limit of its first derivative is zero. This proves Lemma 5.
Now let $Z \subseteq \mathbb{P}(V)$ and $B \in gl(V)$. Let $\theta : V \to \mathbb{C}^{N+1}$ be an isomorphism and define $\mu(Z, B) = \mu(\theta(Z), \theta B \theta^{-1})$. The lemma guarantees that this definition is unambiguous. Note that (3.12) shows that $\mu(Z, B)$ is just the usual Chow weight. (The Chow weight is normally defined only when $B$ is a traceless diagonalizable matrix with integer eigenvalues, but we find it convenient to work with this somewhat more general notion).

If $\tau \in GL(V)$ then $\mu(\tau(Z), B) = \mu(\theta(\tau(Z)), \theta B \theta^{-1}) = \mu((\theta \tau)(Z), (\theta \tau)^{-1} B \tau (\theta \tau)^{-1})$. We conclude

$$\mu(\tau(Z), B) = \mu(Z, \tau^{-1} B \tau)$$

(3.14)

In particular, if $\tau$ commutes with $B$, then $\mu(Z, B) = \mu(\tau(Z), B)$.

If we replace the functional $E$ by $\nu$, the $K$ energy functional, we may define a corresponding invariant $\tilde{\mu}(Z, B)$ for $Z \subseteq \mathbb{P}^N(C)$ and $B \in gl(N + 1, C)$:

$$\tilde{\mu}(Z, B) = \lim_{t \to -\infty} \nu_{\omega_{FS}}(\psi_t)$$

(3.15)

It will be convenient for us to introduce an alternative characterization of the Futaki invariant: Fix, once and for all, an isomorphism $\kappa : (X, L) \to (X_1, L_1)$. We continue to assume that $r = 1$ (the case $r > 1$ can be treated in a similar fashion). Then we have an induced isomorphism $H^0(X, L^k) = H^0(X_1, L_1^k)$.

Let $\Theta$ be an equivariant trivialization of $\pi_*\mathcal{L}^k$. Then $I_{\Theta}|X_1 : X_1 \hookrightarrow \mathbb{P}(H^0(X_0, L_0^k))$. Let $Z_k \subseteq \mathbb{P}(H^0(X_0, L_0^k)^r)$ be the image of $I_{\Theta}|X_1$ and $Z_{k}^{(0)}$ the image of the canonical imbedding $X_0 \subseteq \mathbb{P}(H^0(X_0, L_0^k)^r)$. Note that $Z_k$ depends on the choice of $\Theta$, but that if $\Theta'$ is another choice, then $\Theta' = U\Theta$ where $UA_k = A_k U$, and thus the value $\mu(Z_k, A_k)$ is independent of the choice of equivariant $\Theta$.

**Lemma 6** We have

$$F(T) = -c(X, \omega) \cdot \lim_{k \to \infty} \frac{\mu(Z_k, A_k)}{k^n}$$

(3.16)

where $c(X, \omega) = \frac{1}{n!(n+1)!} \int_X \omega^n$.

**Proof.** Since this argument is implicit in Donaldson [14], we only briefly sketch the proof (see as well Ross-Thomas [28]): If $Z \subseteq \mathbb{P}^N$ and $\lambda : \mathbb{C}^r \to SL(N+1, \mathbb{C})$ is a one parameter subgroup, let $A \in sl(N + 1)$ be such that $\lambda(e^t) = e^{tA}$ and $Z(0) = \lim_{\tau \to 0} \lambda(\tau)(Z)$ (the flat limit) so $Z(0) \subseteq \mathbb{P}^N$ is a subscheme of $\mathbb{P}^N$ with the same Hilbert polynomial as $Z$. Let $M_0 = O(1)|_{Z(0)}$. Then $\lambda(\tau)$ defines an automorphism of $H^0(Z(0), M_0^p)$ and we let $\tilde{w}(Z, A, p)$ be the weight of this action on $det(H^0(X_0, M_0^p))$. It is known that $\tilde{w}(p)$ is a polynomial in $p$ for $p$ large such that

$$\tilde{w}(Z, A, p) = \frac{\mu(Z, A)}{(n+1)!} p^{n+1} + O(p^n)$$

and $\tilde{w}(Z(0), 1) = 0$

(3.17)
(see for example, Mumford [20]). Now let $T$ be a test configuration, let $r > 0$ and consider
$Z_r \subseteq \mathbf{P}(H^0(X_0, L_0^*)$. Applying (3.17) to $Z = Z_r$, $A = r N_r A_r$ and $M_0 = L_0^*$, we get
\[
\tilde{w}(Z_r, r N_r A_r, p) = \frac{\mu(Z_r, r N_r A_r)}{(n + 1)!} \cdot p^{n+1} + O(p^n)
\] (3.18)
On the other hand, since $M_0^p = L_0^p$, we get, with $k = rp$:
\[
\tilde{w}(Z_r, r N_r A_r, p) = w(k) r N_r - w(r) k N_k = e_T(r) k^{n+1} + O(k^n)
\] (3.19)
where $e_T$ is a polynomial in $r$ of degree at most $n$. If follows from the definition of $F(T)$
that $-F(T)$ is the leading coefficient of $e_T(r)$. Comparing with (3.18) we get
\[
\lim_{r \to \infty} \frac{\mu(Z_r, r N_r A_r)}{r^n r^{n+1} (n + 1)!} = -F(T)
\]
Since $r^{-n} N_r = \frac{1}{n!} \int \omega^n + O(r^{-1})$, Lemma 6 follows.

4 Completion of the Proof of Theorem 1

4.1 The Tian-Yau-Zelditch expansion

Let $L \to X$ be an ample line bundle over a compact complex manifold $X$. If $h$ is a smooth
hermitian metric on $L$ then the curvature of $h$ is given by $\omega = R(h) = -\frac{i}{2} \partial \bar{\partial} \log h$. Let $\mathcal{H}$
be the space of positively curved hermitian metrics on $L$. Then $\mathcal{H}$ contains a canonical
family of finite-dimensional negatively curved symmetric spaces $\mathcal{H}_k$, the space of Bergman
metrics, which are defined as follows: For $k >> 0$ and for $s = (s_0, ..., s_{N_k})$ an ordered
basis of $H^0(X, L^k)$, let $t_s : X \hookrightarrow \mathbf{P}^{N_k}$
be the Kodaira imbedding given by $x \mapsto (s_0(x), ..., s_{N_k}(x))$. Then we have a canonical
isomorphism $t_s : L^k \to t_s^* O(1)$ given by
\[
t_s(l) = \left[ \left( \frac{s_0}{s}, \frac{s_1}{s}, ..., \frac{s_{N_k}}{s} \right) \mapsto \frac{l}{s} \right] \]
where $l \in L^k$ and $s$ is any locally trivializing section of $L^k$.

Fix $h_0 \in \mathcal{H}$. Let $h_{FS}$ be the Fubini-Study metric on $O(1) \to \mathbf{P}^{N_k}$ and let
\[
h_s = (t_s^* h_{FS})^{1/k} = \frac{h_0}{\left( \sum_{\alpha = 0}^{N_k} |s_\alpha|^2 h_0^k \right)^{1/k}}
\] (4.2)
Note that the right side of (4.2) is independent of the choice of $h_0 \in \mathcal{H}$. In particular
\[
\sum_{\alpha = 0}^{N_k} |s_\alpha|^2 h_s^k = 1
\] (4.3)
Let
\[ \mathcal{H}_k = \{ h_\alpha : \_ a basis of H^0(X, L^k) \} \subseteq \mathcal{H} \]
Then \( \mathcal{H}_k = GL(N_k + 1)/U(N_k + 1) \) is a finite-dimensional negatively curved symmetric space sitting inside of \( \mathcal{H} \). It is well known that the \( \mathcal{H}_k \) are topologically dense in \( \mathcal{H} \): If \( h \in \mathcal{H} \) then there exists \( h(k) \in \mathcal{H}_k \) such that \( h(k) \to h \) in the \( C^\infty \) topology. This follows from the Tian-Yau-Zelditch theorem on the density of states (Yau [33], Tian [30] and Zelditch [35]; see also Catlin [7] for corresponding results for the Bergman kernel). In fact, if \( h \in \mathcal{H} \), then there is a canonical choice of the approximating sequence \( h(k) \): Let \( \_ \) be a basis of \( H^0(X, L^k) \) which is orthonormal with respect to the metrics \( h \). In other words,
\[
\langle s_\alpha, s_\beta \rangle_h = \int_X (s_\alpha, s_\beta)_h \omega^n = \delta_{\alpha\beta} \text{ where } \omega = R(h) .
\] (4.4)
The basis \( \_ \) is unique up to an element of \( U(N_k + 1) \). Define \( \rho_k(h) = \rho_k(\omega) = \sum_\alpha |s_\alpha|_{h^k}^2 \). Then Theorem 1 of [35], which is the \( C^\infty \) version of the \( C^2 \) approximation result first established in [30], says that for \( h \) fixed, we have a \( C^\infty \) asymptotic expansion as \( k \to \infty \):
\[
\rho_k(\omega) \sim k^n + A_1(\omega)k^{n-1} + A_2(\omega)k^{n-1} + \cdots
\] (4.5)
Here the \( A_j(\omega) \) are smooth functions on \( X \) defined locally by \( \omega \) which can be computed in terms of the curvature of \( \omega \) by the work of Lu [18]. In particular, it is shown there that
\[
A_1(\omega) = \frac{s(\omega)}{2\pi}
\] (4.6)
where \( s(\omega) \) is the scalar curvature of \( \omega \).
Let \( \_ = k^{-n/2} \_ \) and \( h(k) = h_\_ \). Then (4.2) and (4.5) imply that
\[
\frac{h(k)}{h} = 1 - \frac{s(\omega)}{2\pi} \cdot \frac{1}{k^2} + O \left( \frac{1}{k^3} \right) , \quad \omega(k) = \omega + O \left( \frac{1}{k^2} \right) , \quad \phi(k) = \phi + O \left( \frac{1}{k^2} \right)
\] (4.7)
Here, as before, \( \omega = R(h) \), \( \omega(k) = R(h(k)) \), \( h = h_0e^{-\phi} \) and \( h(k) = h_0e^{-\phi(k)} \). In particular,
\[
\omega_0 + \sqrt{-1} \partial\bar{\partial} \phi(k) = \omega(k) = \frac{1}{k} \_^* \omega_{FS}.
\]
Lemma 1 can now be conveniently reformulated as follows:

**Lemma 7** Let \( \rho : C^\infty \to Aut(\mathcal{L} \to X \to C) \) be a test configuration \( T \) of exponent one for the pair \( (X, L) \), where \( L \to X \) is ample. Let \( h_0 \) be a positively curved metric on \( L \to X \). Let \( k \) be an integer such that \( L^k \) is very ample. Then there is

1. An orthonormal basis \( \_ = (s_0, ..., s_{N_k}) \) of \( H^0(X, L^k) = H^0(X_1, L_1^k) \),
2. An imbedding \( I_\_ : (L^k \to X \to C) \hookrightarrow (O(1) \times C \to P^{N_k} \times C \to C) \)

satisfying the following property: the imbedding \( I_\_ \) restricts to \( i_\_ \) on the fiber \( L_1^k \) and \( I_\_ \) intertwines \( \rho(\tau) \) and \( \tau^{B_k} \). More precisely: for every \( \tau \in C^\infty \) and every \( l_w \in L^k_w \),
\[ I_\Delta(\rho(\tau)l_w) = \left( \tau^{B_k} \cdot I_\Delta(l_w) , \tau w \right) \]  

(4.8)

where \( \tau^{B_k} \) is a diagonal matrix whose eigenvalues are the eigenvalues of \( \rho_k(\tau) : V_k \to V_k \).

The matrix \( B_k \) is uniquely determined, up to a permutation of the diagonal entries, by \( k \) and the test configuration \( T \). Moreover, the basis \( \mathcal{B} \) is uniquely determined by \( h_0 \) and \( T \), up to an element of \( U(N_k + 1) \) which commutes with \( B_k \). The image of \( X_1 \) is \( Z_k \subseteq \mathbb{P}^{N_k} \).

4.2 Growth bounds for the Bergman geodesic rays

We make precise the notation which appears in Theorem 1: Let \( L \to X \) be an ample line bundle over a compact complex manifold, and \( \mathcal{H} \) the space of positively curved metrics on \( L \). Let \( h_0 \in \mathcal{H} \) and let \( T \) be a test configuration for the pair \( (X, L) \) of exponent \( r \). We wish to associate to the pair \( (h_0, T) \) an infinite geodesic ray in \( \mathcal{H} \) whose initial point is \( h_0 \).

After replacing \( L \) by \( L^r \) we may assume, without loss of generality, that \( r = 1 \) and that \( L \) is very ample.

Let \( k \) be a large positive integer and choose \( \mathcal{B} \), an orthonormal basis of \( \mathcal{H}^0(X, L_k) \) as in Lemma 7. Define \( A_k \) to be the traceless part of \( B_k \) and let \( \lambda_0^{(k)}, \lambda_1^{(k)}, \ldots, \lambda_{N_k}^{(k)} \) be the diagonal entries of \( A_k \). Set \( \hat{s} = k^{-n/2} s \) so that \( h_\hat{s} = h_0(k) \), where \( h_\hat{s} \) is defined as in (4.2).

Now let \( \hat{s}(t; k) = (e^{t\lambda_0} \hat{s}_0, e^{t\lambda_1} \hat{s}_1, \ldots, e^{t\lambda_{N_k}} \hat{s}_{N_k}) \), and define

\[ h(t; k) = h_\hat{s}(t; k) = h_0 e^{-\phi(t; k)} = h_0(k) e^{-\phi(t; k) - \phi(k)} \]  

(4.9)

so that \( h(t; k) : (-\infty, 0] \to \mathcal{H}_k \) is a geodesic ray in \( \mathcal{H}_k \) and \( h(0; k) = h_0(k) \). In particular we have

\[ \phi(t; k) = \frac{1}{k} \log \left( k^{-n} \cdot \sum_{\alpha=0}^{N_k} e^{2t\lambda_\alpha} |s_\alpha| h_0^2 \right) = \frac{1}{k} \log \left( k^{-n} \cdot \sum_{\alpha=0}^{N_k} e^{2t\lambda_\alpha} |s_\alpha| h_0(k)^2 \right) + \phi(k) \]  

(4.10)

Let

\[ f(k) = \frac{w(k)}{kd_k} - F_0 = \frac{F(T)}{k} + O\left( \frac{1}{k^2} \right) \]  

(4.11)

where \( w(k), d_k \) and \( F_0 \) are defined as in (2.2). In particular, \( f(k) = O\left( \frac{1}{k} \right) \).

**Lemma 8** Let \( k, l \) be positive integers with \( k < l \). Then there exists \( C_{k,l} > 0 \) with the following property:

\[ -C_{k,l} < \left[ \phi(t; l) + 2t \cdot f(l) \right] - \left[ \phi(t; k) + 2t \cdot f(k) \right] < C_{k,l} \]  

(4.12)
Proof. If suffices to prove (4.12) in the case \( k = 1 \). Then, replacing \( l \) by \( k \), we have

\[
\tilde{\phi}(t; k) - \tilde{\phi}(t; 1) = [\phi(t; k) + 2t \cdot f(k)] - [\phi(t; 1) + 2t \cdot f(1)] = \log \left( \frac{(k^{-n} \cdot \sum_{\alpha=0}^{N_k} e^{2\tau_{\alpha}^{(k)}} |\tilde{s}_\alpha|_{h_0}^2}^{1/k}}{= \sum_{\beta=0}^{N} e^{2\tau_{\beta}^{(1)}} |s_\beta|_{h_0}^2} \right)
\]

where \( \eta_0^{(k)} \leq \eta_1^{(k)} \cdots \leq \eta_{N_k}^{(k)} \) are the eigenvalues of the diagonal matrix \( B_k \), \( N = N_1 \) and \( s_\beta = s_\beta^{(1)} \).

We now have

\[
\log(\sum_{\beta=0}^{N} e^{2\tau_{\beta}^{(1)}} |s_\beta|_{h_0}^2) = \frac{1}{k} \log(\sum_{\alpha=0}^{N_k} e^{2\tau_{\alpha}^{(k)}} |\tilde{s}_\alpha|_{h_0}^2) + O(1)
\]

where the \( O(1) \) term is independent of \( t \), and, for \( \eta \in \mathbb{Z} \),

\[
\{ \tilde{s}_\alpha^{(k)} : \eta_\alpha^{(k)} = \eta \} \subset \{ s_0^{p_0} \otimes \cdots \otimes s_N^{p_N} \in H^0(X, L^k) : \sum_\beta p_\beta = k \text{ and } \sum_\beta p_\beta \eta_\beta^{(1)} = \eta \},
\]

is a maximally linearly independent subset. On the other hand, \( (s_0^{(k)}, ..., s_{N_k}^{(k)}) \) and \( (\tilde{s}_0^{(k)}, ..., \tilde{s}_{N_k}^{(k)}) \) are two bases of the same vector space which differ by a lower block triangular matrix. This proves Lemma 8.

### 4.3 The volume formula

Let \( \phi_t : [a, b] \to \mathcal{H}_\omega \) be a smooth path and let \( U_{a,b} = \{ w \in \mathbb{C}^\times : e^a \leq |w| \leq e^b \} \). Let \( M_{a,b} = X \times U_{a,b} \) and \( \Omega_0 \) be the \((1,1)\) form on \( M_{a,b} \) defined by pulling back \( \omega_0 \). Define \( \Phi(z, w) : M_{a,b} \to \mathbb{R} \) by

\[
\Phi(z, w) = \phi_t(z) \quad \text{where} \quad t = \log |w|
\]

Let \( \Omega_\Phi \) be the \((1,1)\) form on \( M_{a,b} \) defined by \( \Omega_\Phi = \Omega_0 + \overline{\partial} \partial \Phi \). Then

\[
\Omega_{\Phi}^{n+1} = \frac{1}{4} (\tilde{\phi}_t - |\partial \phi_t|^2) \omega_{\phi_t}^n \wedge (\frac{\sqrt{-1}}{2} dw \wedge d\overline{w})
\]

In particular, we have the key observation of [19], [29], [12]:

\[
\Omega_{\Phi}^{n+1} = 0 \iff \tilde{\phi}_t - |\partial \phi_t|^2 \omega_{\phi_t} = 0 \iff \phi_t \text{ is a smooth geodesic in } \mathcal{H}
\]

We say that a function \( \phi_t(x) \) on \([a, b] \times X\) is a weak geodesic if \( \Phi \) is bounded, plurisubharmonic with respect to \( \Omega_0 \), and if \( \Omega_{\Phi}^{n+1} = 0 \).
Finally, we obtain, using (3.5), the following useful volume formula [27]:

\[(n + 1) \int_{X \times U,a,b} \Omega_{\Phi}^{n+1} = \dot{E}_\omega(b) - \dot{E}_\omega(a)\]  

(4.19)

where \(E_\omega(t) = E_\omega(\phi_t)\).

### 4.4 Volume estimates for the Monge-Ampère measure

We first need a few lemmas. Let \(D^\ast = \{w \in \mathbb{C} : 0 < |w| < 1\}\). We associate to \(\phi(t; k)\) the function \(\Phi(k)\) on \(X \times D^\ast\) as in (4.16):

\[\Phi(k)(z, w) = \phi(t; k)(z) \quad \text{where} \quad t = \log |w|\]  

(4.20)

and we let \(\Omega_0\) be the pullback of \(\omega_0\) to \(X \times D^\ast\) and we let \(\Omega_{\Phi(k)} = \Omega_0 + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \Phi(k)\).

**Lemma 9** We have \(\lim_{k \to \infty} \int_{X \times D^\ast} \Omega_{\Phi(k)}^{n+1} = 0\). In fact,

\[\int_{X \times D^\ast} \Omega_{\Phi(k)}^{n+1} = O(k^{-1})\]  

(4.21)

**Proof.** According to (4.19),

\[\int_{X \times D^\ast} \Omega_{\Phi(k)}^{n+1} = \int_X \dot{\phi}(0; k) \omega^n_{\phi(0; k)} - \lim_{t \to \infty} \int_X \dot{\phi}(t; k) \omega^n_{\phi(t; k)}\]  

(4.22)

Hence it suffices to show that each of the two terms in (4.21) is \(O(\frac{1}{k})\).

Let \(\psi(t; k) = k[\phi(t; k) - \phi(k)] + n \log k\). Then \(\psi(t; k) = \log \frac{|z|}{|z|}^2\) where \(\sigma_t = e^{tA_k}\). Recall that \(\omega_0 + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \phi(k) = \omega_0(k)\) and \(\omega_0(k) = \frac{1}{k^2} \omega_{FS}\). Thus

\[\omega_{\phi(t; k)} = \omega_0 + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \phi(t; k) = \omega_0(k) + \frac{1}{k} \frac{\sqrt{-1}}{2} \partial \bar{\partial} \psi_t = \frac{1}{k} \frac{1}{2} \left(\omega_{FS} + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \psi_t\right)\]

Since we also have \(\dot{\phi}(t; k) = \frac{1}{k} \dot{\psi}(t; k)\) we conclude:

\[(n + 1) \int_X \dot{\phi}(t; k) \omega^n_{\phi(t; k)} = (n + 1) \cdot \frac{1}{k} \frac{1}{k^n} \int_{Z_k} \dot{\psi}_t \left(\omega_{FS} + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \psi_t\right)^n = \frac{1}{k} \frac{1}{k^n} \dot{E}_{FS}(\psi_t)\]  

(4.23)

Thus

\[-\lim_{t \to \infty} \int_X \dot{\phi}(t; k) \omega^n_{\phi(t; k)} = \frac{1}{k} \frac{1}{k^n} \mu(Z_k, A_k)\]  

(4.24)

Now, according to Lemma 6, \(\frac{\mu(Z_k, A_k)}{k^n}\) has a finite limit as \(k\) tends to infinity. In fact, the limit is equal to \(F(T)\), the Futaki invariant. Thus the second term in (4.22) is \(O(\frac{1}{k})\).

In order to treat the first term, we require the following result from [16]:

\[\text{\textsuperscript{18}}\]
Lemma 10 Let $L \to X$ be an ample line bundle over a compact complex manifold $X$ and $h$ a metric on $L$ with positive curvature $\omega$. Let $\mathbf{e}$ be an orthonormal basis of $H^0(X, L^k)$ and let $\iota_{\mathbf{e}} : X \to \mathbb{P}^{N_k}$ be the associated Kodaira imbedding. Let $Z_k$ be the image of $\iota_{\mathbf{e}}$. Define $M^{(k)}_{\alpha\beta} = M_{\alpha\beta}(Z_k)$ as in (3.11). Then

$$M^{(k)}_{\alpha\beta} = \int_{Z_k} \frac{z_{\alpha} \bar{z}_{\beta}}{|z|^2} \omega^n_{FS} = \int_X (s_{\alpha}, s_{\beta}) h^k \cdot \left(\frac{h(k)}{h}\right)^k \cdot \frac{\omega(n)}{\omega^n} \cdot \omega^n$$

$$= \delta_{\alpha\beta} - k^{-1} \int_X (s_{\alpha}, s_{\beta}) h^k \cdot s(\omega) \omega^n + O(k^{-2})$$

$$= \left(\frac{1}{n!} \int_X \omega^n\right) \frac{k^n}{N_k} \cdot \delta_{\alpha\beta} - k^{-1} \int_X (s_{\alpha}, s_{\beta}) h^k \cdot [s(\omega) - \hat{s}] \omega^n + O(k^{-2})$$

$$= \left(\frac{1}{n!} \int_X \omega^n\right) \frac{k^n}{N_k} \cdot \delta_{\alpha\beta} - k^{-1} \int_X \frac{x_{\alpha} \bar{x}_{\beta}}{|x|^2} \cdot [s(\omega) - \hat{s}] \omega^n_{FS} + O(k^{-2})$$

(4.25)

where $\hat{s} \in \mathbb{R}$ is defined by: $\int_X [s(\omega) - \hat{s}] \omega^n = 0$. The proof of Lemma 10 follows from (4.7).

We return to the proof of Lemma 9: Applying (4.25) to (3.10) we obtain:

$$\frac{1}{n+1} \int_X \phi(0; k) \omega^n_{\phi(0,k)} = \frac{1}{k \cdot \frac{1}{k^n}} \cdot \bar{E}_{\omega_{FS}}(\psi_1) = \frac{1}{k \cdot \frac{1}{k^n}} \cdot \tr(A_k M^{(k)})$$

$$= - \frac{1}{k \cdot \frac{1}{k^n}} \int \sum (s_{\alpha}, s_{\alpha}) h^k \cdot \frac{\lambda(k)}{k} \cdot [s(\omega) - \hat{s}] \omega^n + O(k^{-2}) \frac{1}{k \cdot \frac{1}{k^n}} \sum |\lambda_{\alpha}|$$

(4.26)

where in the first equality we use the fact that $A_k = A_k^*$ and in the last equality we have made use of the fact that $A_k$ is traceless.

If we apply Lemma 4 to equation (4.26) we obtain

$$\left| \int_X \phi(0; k) \omega^n_{\phi(0,k)} \right| \leq C' \frac{1}{k \cdot \frac{1}{k^n}} \sum \int |s_{\alpha}|^2 h^k \omega^n + O(k^{-2}) \frac{1}{k \cdot \frac{1}{k^n}} C k N_k$$

(4.27)

where $C' = C \sup_X |s(\omega) - \hat{s}|$. The first term equals $C' \frac{1}{k \cdot \frac{1}{k^n}} N_{k+1}$. Since $\frac{N_{k+1}}{k^n}$ is bounded as a function of $k$, the first term is $O(k^{-1})$. Similarly, the second term is $O(k^{-2})$.

4.5 The Monge-Ampère equation on a punctured disk

We now complete the proof of Theorem 1. As in the proof of Theorem 3, [27], we can choose a sequence of positive real numbers $c_k \searrow 0$ in such a way that

$$\phi(0; k) + c_k > \phi(0; k + 1) + c_{k+1}$$

(4.28)
Indeed, by the Tian-Yau-Zelditch theorem, \(\sup X|\phi(0; k) - \phi| \leq Ck^{-2}\), so that the sequence \(c_k = 2C \sum_{j \geq k} j^{-2}\) is such a choice. Choose also \(\epsilon_k = k^{-1/2}\) and make the replacement

\[
\phi(t; k) \longrightarrow \phi(t; k) + c_k - \epsilon_k t. \tag{4.29}
\]

Then it is still true that \(\phi(0; k) \to \phi\), and that \(\int \Omega_{\Phi(k)}^{n+1} = O(1/k)\). Moreover, the value of \(\phi_t\), as defined in (1.1) does not change under this replacement.

Next, we show that \(\phi_t\) is continuous at \(t = 0\) and has the desired initial value. As in [27], the essential ingredient is a uniform bound for \(|Y|\phi(t; k)| near the boundary \(S^1 \times X\), where \(Y = \partial_t\). In fact, differentiating the expression for \(\phi(t; k)\) gives

\[
|\dot{\phi}(t; k)| \leq \frac{2}{k} \sum_{\alpha=0}^N |\lambda_\alpha| e^{2\lambda_\alpha} s_\alpha^2 + \epsilon_k
\leq \frac{2}{k} \sup_\alpha |\lambda_\alpha| + \epsilon_k \leq C, \tag{4.30}
\]

where in the last step, we made use of the bound \(\|A_k\|_{op} \leq Ck\) provided by Lemma 4. On the boundary \(X \times S^1\), the monotonicity of \(\phi(t; k)\) guarantees that, for any pair \(k, l\) with \(k < l\),

\[
\phi(t; k) - \phi(t; l) > \phi(t; k) - \phi(t; k + 1) > \delta_k \text{ on } X \times S^1, \tag{4.31}
\]

where \(\delta_k\) is a strictly positive constant independent of \(l\). Since \(|\dot{\phi}(t; m)|\) is uniformly bounded in \(m\), it follows that \(\phi(t; k) - \phi(t; l) > \frac{1}{2}\delta_k\) in a neighborhood \(U_k\) of \(X \times S^1\) independent of \(l\). Thus, we have for any \(k\),

\[
[\sup_{l \geq k}\phi(t; l)]^* = \phi(t; k) \tag{4.32}
\]

in an open neighborhood \(U_k\) of \(X \times S^1\). Extend now the original potential \(\phi\) on \(X\) as a function in a neighborhood of \(X \times S^1\), by making it constant along the flow lines of \(Y\). For any \(\epsilon > 0\), choose \(k\) large enough so that \(\sup_X|\phi(0; k) - \phi| < \epsilon\). Then the above estimate for \(\dot{\phi}(t; k)\) shows that we have

\[
\sup_U|\phi(t; k) - \phi| < 2\epsilon \tag{4.33}
\]

for some neighborhood \(U\) of \(X \times S^1\) in \(X \times D^\times\), independent of \(k\). This implies that \(\phi_t = \lim_{k \to \infty}[\sup_{l \geq k}\phi(t; k)]^*\) is continuous at \(X \times S^1\), and that \(\phi_t = \phi\) at \(t = 0\).

On the other hand, for fixed \(l \geq j > k > 0\), Lemma 8 implies that

\[
\phi(t; j) - \phi(t; k) \leq C_{k,j} + c_j - c_k + 2tf(k) - 2tf(j) - (\epsilon_k - \epsilon_j)|t|
\leq \sup_{k \leq m \leq l} C_{k,m} + 1 - \frac{1}{2}(\epsilon_k - \epsilon_{k+1})|t|. \tag{4.34}
\]
for $k$ sufficiently large. Thus, we can make sure that

$$\phi(t; k) > 1 + \phi(t; j)$$

(4.35)

for all $j$ such that $k < j \leq l$ and all $t$ such that

$$|t| > 2 \frac{2 + \sup_{k \leq m \leq l} C_{k,m}}{\epsilon_k - \epsilon_{k+1}} \equiv -\log r_{k,l}.$$  

(4.36)

Clearly, we have $r_{k,l+1} < r_{k,l}$ and, by choosing $C_{k,l}$ in Lemma 8 large enough, we can make sure that

$$\lim_{l \to \infty} r_{k,l} = \lim_{l \to \infty} r_{l,l+1} = 0$$

(4.37)

for each $k > 0$. Thus, if we set

$$\phi(t; k, l) = \sup_{k \leq j \leq l} [\phi(t; j)],$$

(4.38)

and let $\Omega_{k,l}$ be the $(1,1)$ form on $X \times D^\times$ corresponding to $\phi(t; k, l)$ via (4.16), we have, for $l > k$,

$$\int_{X \times D^r_{k,k+1}} \Omega_{k,l}^{n+1} \leq \int_{X \times D^r_{k,l}} \Omega_{k,l}^{n+1} = \int_{X \times D^r_{k,k,l}} \Omega_{k,l}^{n+1} \leq \int_{X \times D^\times} \Omega_{k,l}^{n+1} \leq \frac{C}{k}$$

(4.39)

where $C$ is independent of $k, l$ and

$$D_r = \{ w \in \mathbb{C} : 1 > |w| > r \}.$$

In the middle equality above, we made use of the fact that the volume integrals depend only on the values of the currents in a neighborhood of the boundary of $X \times D^r_{k,l}$ (see Lemma 2 in [27]).

Now $\phi(t; k, l)$ is an increasing sequence in the index $l$ which converges pointwise, almost everywhere, to $\xi(t; k) = \sup_{k \leq j \leq l} [\phi(t; j)]^*$. Let $\Xi_k$ be the $(1,1)$ form on $D^\times \times M$ corresponding to $\xi(t; k)$. Then, by the Bedford-Taylor monotonicity theorem [3] applied to the increasing sequence $\phi(t; k, l)$ (see also Blocki [6] and Cegrell [8]), we have

$$\int_{X \times D^r_{k,k+1}} \Xi_k^{n+1} = \lim_{l \to \infty} \int_{X \times D^r_{k,k,l}} \Omega_{k,l}^{n+1} \leq \frac{C}{k}$$

(4.40)

Moreover, if $l \geq k$,

$$\int_{X \times D^r_{l,l+1}} \Xi_l^{n+1} \leq \int_{X \times D^r_{l,l+1}} \Xi_l^{n+1} \leq \frac{C}{l}$$

(4.41)
Finally, since \( \xi(t; l) \) is monotonically decreasing to \( \phi(t) \) (by definition of \( \phi(t) \)) we have, using the Bedford-Taylor monotonicity theorem again (but this time for decreasing sequences):

\[
\int_{X \times D_{r,k,k+1}} \Omega_{\Phi}^{n+1} = \lim_{l \to \infty} \int_{X \times D_{r,k,k+1}} \Xi_l^{n+1} = 0 \quad (4.42)
\]

Since this is true for all \( k \), we obtain

\[
\int_{X \times D} \Omega_{\Phi}^{n+1} = 0. \quad (4.43)
\]

Thus \( \Omega_{\Phi}^{n+1} = 0 \), and this proves the theorem. Q.E.D.

5 Proof of Theorem 2

In this section we show that if the expression \( N_2(T)^2 \) defined below by (5.3) is strictly positive, then \( \phi_t \) is a non-trivial geodesic, i.e., \( \phi_t \) is not a constant function of \( t \).

Let \( N_1 + 1 = \dim(X, L) \) and set \( N = N_1 \). Let \( \lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_N \) be the diagonal entries of \( A_1 \) and let \( N_1 + 1 = \dim H^0(X, L) \). By Lemma 1 we may assume that \( T \) is imbedded in \( P^n \) and that the action \( \rho(\tau) \) is given by the diagonal matrix whose diagonal entries are given by \( \tau^{\lambda_0}, \ldots, \tau^{\lambda_N} \). As usual, we denote by \( X_0 \subseteq P^n \) the central fiber of \( T \).

Define \( h : P^n \to \mathbb{R} \) by

\[
h(z) = \frac{\sum_{\alpha=0}^{N} \lambda_\alpha |z_\alpha|^2}{\sum_{\alpha=0}^{N} |z_\alpha|^2}. \quad (5.1)
\]

We next recall the formula in Donaldson [16]:

\[
\text{Tr}(A_k^2) = N_2(T)^2 \cdot k^{n+2} + O(k^{n+1}), \quad (5.2)
\]

where the coefficient \( N_2(T) \) is given by

\[
N_2(T)^2 = \int_{X_0} (h - \hat{h})^2 \omega_{FS}^n, \quad (5.3)
\]

and \( \hat{h} \) is determined by \( \int_{X_0} (h - \hat{h}) \omega_{FS}^n = 0 \). The test configuration \( T \) is trivial if and only if \( N_2(T) = 0 \).

Now let \( \lambda = \lambda_N \) so that \( \lambda \leq \lambda_\alpha \) for all \( \alpha \). Denote by \( \lambda_0^{(k)} \geq \lambda_1^{(k)} \geq \cdots \geq \lambda_N^{(k)} \) the eigenvalues of the endomorphism \( A_k \) (for convenience, the eigenvalues are ordered here in the opposite order than previously). Set \( \lambda^{(k)} = \lambda_N^{(k)} \). Then \( \lambda^{(k)} = k\lambda - \frac{\text{Tr} B_k}{N_k} \), and \( \lambda^{(k)} k^{-1} \) has a limit. Set

\[
\Lambda = \lim_{k \to \infty} \lambda^{(k)} / k. \quad (5.4)
\]
If $N_2(T) > 0$ then (5.3) implies that the average absolute eigenvalue $|\lambda_j^{(k)}|$ has size at least $k$. On the other hand, since $\text{Tr} A_k = 0$, one easily sees that $|\lambda^{(k)}|$ has size at least $k$ and thus $\Lambda > 0$.

Next, recall that

$$\phi(t; k) = \frac{1}{k} \log \left( k^{-n} \cdot \sum_{\alpha=0}^{N_k} e^{2t \lambda_\alpha } |s_\alpha| h_0^2 \right)$$

Observe that

$$\int \sum_{\alpha=0}^{N_k} e^{2t \lambda_\alpha } |s_\alpha| h_0^2 \omega_0^n \geq e^{2t |\lambda^{(k)}|}$$

so

$$2|t| \frac{|\lambda^{(k)}|}{k} + O\left( \frac{1}{k^2} \right) \geq \sup_{X_t} \phi(t; k) \geq 2|t| \frac{|\lambda^{(k)}|}{k} - n \frac{\log k}{k} - \frac{1}{k} \log \left( \int \omega_0^n \right)$$

so, letting $k \to \infty$,

$$\sup_{X_t} \phi_t = 2|t| \cdot |\Lambda|$$

(5.6)

Since $|\Lambda| > 0$, this already shows that $\phi_t$ is non-trivial if $N_2(T) > 0$. This establishes Theorem 2. Q.E.D.

Under the additional technical assumption (which we expect can be removed) that for $k_0$ large enough, $[\sup_{k \geq k_0} \phi(t; k)]^* = \sup_{k \geq k_0} \phi(t; k)$ for $|t| > t_0 >> 1$, then the geodesic $\phi_t$ can be shown to be non-trivial in the stronger sense that it defines a non-trivial ray in $H/\mathbb{R}$.

To show strong non-triviality, we observe that $N_2(T) > 0$ implies (and is in fact, equivalent to) the following: There exist $p \in X$ such that $s_\alpha(p) = 0$ for all $\alpha$ such that $\lambda_\alpha = \lambda$. Fix such a $p$. Let $\gamma = \inf \{ \lambda_\alpha : \lambda_\alpha > \lambda \}$ and $\gamma^{(k)} = \inf \{ \lambda_\alpha^{(k)} : \lambda_\alpha^{(k)} > \lambda^{(k)} \}$. Note that $|\gamma| < |\Lambda|$. Again, $\gamma^{(k)} = k \gamma - \frac{\text{Tr} B_k}{N_k}$, and $\gamma^{(k)}$ has a limit as $k \to \infty$,

$$\Gamma = \lim_{k \to \infty} \frac{\gamma^{(k)}}{k}$$

(5.7)

satisfying $|\Gamma| < |\Lambda|$. Now, at the point $p$, we have for all $k$,

$$\phi(t; k)(p) \leq 2|t| \frac{|\gamma^{(k)}|}{k} + O\left( \frac{1}{k} \right).$$

(5.8)

Fix $\epsilon > 0$ so small that $|\Gamma| + 2\epsilon < |\Lambda|$. Then there exists $k_0$ so that

$$\phi(t; k)(p) \leq 2(|\Gamma| + \epsilon)|t|$$

(5.9)

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for all $|t| > 1$ and all $k \geq k_0$. We have then, for $|t|$ sufficiently large,

$$\phi_t(p) \leq [\sup_{k \geq k_0} \phi(t; k)]^*(p) = \sup_{k \geq k_0} \phi(t; k)(p) \leq 2(|\Gamma| + \epsilon)|t|. \quad (5.10)$$

In view of (5.6), this shows $\lim_{t \to -\infty} \text{osc}_{X_t} \phi_t = \infty$ where $\text{osc}_{X_t} \phi_t = \sup_{X_t} \phi_t - \inf_{X_t} \phi_t$. Thus $\phi_t$ is strongly non-trivial.

## 6 Proof of Theorem 3

The formula in Lemma 8.8 of Tian [31] implies that

$$\lim_{t \to -\infty} \dot{\nu}_k = F_{CM}(T) \quad (6.1)$$

where $F_{CM}(T)$ is the CM-Futaki invariant (see [31] for the precise definition).

On the other hand, the recent work of Paul-Tian [22] shows that $F_{CM}(T) = F(T)$ under the hypothesis of Theorem 3.
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