Magnetotransport properties of a magnetically modulated two-dimensional electron gas with the spin–orbit interaction

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Abstract
We study the electrical transport properties of a two-dimensional electron gas (2DEG) with the Rashba spin–orbit interaction in the presence of a constant perpendicular magnetic field ($B_0 \hat{z}$) which is weakly modulated by $B_1 = B_1 \cos(qx) \hat{z}$, where $B_1 \ll B_0$ and $q = 2\pi/a$ with $a$ the modulation period. We obtain the analytical expressions of the diffusive conductivities for spin-up and spin-down electrons. The conductivities for spin-up and spin-down electrons oscillate with different frequencies and produce beating patterns in the amplitude of the Weiss and Shubnikov–de Haas oscillations. We show that the Rashba strength can be determined by analyzing the beating pattern in the Weiss oscillation. We find a simple equation which determines the Rashba spin–orbit interaction strength if the number of Weiss oscillations between any two successive nodes is known from the experiment. We compare our results with the electrically modulated 2DEG with the Rashba interaction. For completeness, we also study the beating pattern formation in the collisional and the Hall conductivities.

1. Introduction

The magnetotransport properties of a two-dimensional electron gas (2DEG) in the presence of a weakly modulated one-dimensional (1D) periodic electric potential have been studied in great detail experimentally and theoretically for a long time [1–5]. In the absence of the modulation and the spin–orbit interaction (SOI), the magnetoconductivity oscillations due to the charged impurities are commonly known as the Shubnikov–de Haas (SdH) oscillations. In the presence of the modulation and at low magnetic fields, it has been observed that the magnetoresistivity tensor oscillates with the inverse of the magnetic field. These oscillations are completely different in periodicity and temperature dependence from the SdH oscillations observed at higher magnetic field. These oscillations are commonly known as the Weiss oscillations. This is due to the effect of the commensurability between the two length scales in the system: the cyclotron diameter at the Fermi energy and the modulation period $a$. The similarity and differences between the Weiss and the SdH oscillations are as follows: (i) both the oscillations are periodic in $1/B_0$; (ii) the period of the Weiss oscillation varies with the electron density ($n_e$) as $\sqrt{n_e}$, whereas that of the SdH ones as $n_e$; (iii) the amplitude of the Weiss oscillations is less dependent on temperature than that of the SdH oscillations; (iv) the Weiss oscillations are visible at weak magnetic field ($B_0 < 0.6$ T) and the SdH oscillations modulated by the Weiss oscillations are visible at higher fields.

In conventional 2DEG systems, magnetotransport properties in the presence of 1D electric and magnetic modulations are continuing to be an active research field. The magnetotransport properties of a 2DEG in the presence of a weakly modulated magnetic field has been studied theoretically [6–13]. It was theoretically observed that the magnetoconductivity oscillates with the inverse of the magnetic field and the amplitude of the oscillation is much higher than in the case of electrical modulation. The magnetothermodynamical properties of electrically or magnetically modulated 2DEG has also been studied where...
the Weiss-type oscillation is also shown [5, 14, 15]. Later, the magnetoresistance oscillation in a magnetically modulated 2DEG system was observed experimentally [16–18]. In these experiments, a 1D spatially varying magnetic modulation was achieved by placing micro-patterned ferromagnet or superconductor on the surface of a 2DEG system.

An internally generated crystal field induces the SOI which is known as the Rashba SOI. The Rashba interaction strength can also be controlled by a strong external electric field acting normal to the 2DEG plane. Using the Rashba interaction it was possible to explain many experimentally observed features like the combined resonances [19, 20] and the beating patterns in the SdH oscillations [21]. The Rashba interaction is responsible for many other novel effects like the spin-FET [22], spin-galvanic effect [23] and the spin Hall effect [24, 25]. The Rashba SOI in the 2DEG influences various properties such as transport [26, 27], magnetotransport [28–31], magnetization [32] etc. In [33–35], the effects of the Rashba interaction on the magnetotransport properties of electrically modulated 2DEG have been studied theoretically.

Generally, beating pattern analysis in the SdH oscillation [21] and weak anti-localization method [36] are used to extract the strength of the Rashba SOI in the 2DEG system. However, in the presence of $B_0 \neq 0$, Zeeman splitting is also accompanied with the Rashba spin splitting. To avoid this problem optical measurement [37] has also been proposed. The Weiss oscillation is due to the modulation induced Landau levels broadening which oscillates with inverse magnetic field. Moreover, this oscillation appears at low magnetic field where the Rashba SOI dominates over the Zeeman splitting. The Weiss oscillation is less influenced by the Zeeman term especially in the periodicity. To determine the Rashba strength, beating pattern analysis in the Weiss oscillation is more reliable than the SdH oscillation.

In this work, our primary goal is to study analytically the Weiss oscillations of the 2DEG with the Rashba interaction and to determine the strength of the Rashba SOI by analyzing the beating pattern of the Weiss oscillations. For completeness, we also study other transport coefficients such as the collisional and the Hall conductivities of the spin-up and spin-down electrons in detail.

This paper is organized as follows. In section 2, we summarize the energy eigenvalues, the corresponding eigenfunction and density of states of the 2DEG with the Rashba interaction in the presence of a uniform magnetic field. Also, the first-order energy corrections due to the magnetic perturbations and group velocities for spin-up and spin-down electrons are evaluated. We calculate and discuss the diffusive, collisional and the Hall conductivities by using the semi-classical Kubo formula in section 3. The summary of our work is presented in section 4.

2. Energy eigenvalue, eigenfunction and density of states in the presence of the Rashba SOI

We consider a 2DEG in the $x$–$y$ plane subjected to a magnetic field $\mathbf{B} = [B_0 + B_1 \cos(qy)]\hat{z}$, where $q = 2\pi/a$ and $a$ is the modulation period. Also, we consider the strength of the magnetic modulation to be very weak, i.e. $B_1 \ll B_0$. In the Landau gauge the corresponding vector potential $\mathbf{A}$ is $\mathbf{A} = \left[B_0x + (B_1/q) \sin(qy)\right]\hat{y}$.

The Hamiltonian of an electron with charge $-e$ in the presence of the perpendicular magnetic field $\textbf{B}$ is

$$H = \frac{(\mathbf{p} + e\mathbf{A})^2}{2m^*} - \frac{\alpha}{\hbar} [\mathbf{\sigma} \times (\mathbf{p} + e\mathbf{A})]_z + \frac{g}{2} \mu_B \mathbf{B} \cdot \mathbf{\sigma}, \quad (1)$$

where $\mathbf{p}$ is the 2D momentum operator, $m^*$ is the effective mass of the electron, $g$ is the Landé $g$-factor, $\mu_B$ is the Bohr magneton, $\mathbf{1}$ is the identity matrix, $\mathbf{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli spin matrices and $\alpha$ is the strength of the Rashba interaction.

Expanding the above Hamiltonian and re-writing as a sum of the various Hamiltonians: $H = H_0 + H_1 + H_2 + H_3 + H_4$, where

$$H_0 = \frac{1}{2m^*} [p_x^2 + (p_y + eB_0x)^2] \mathbf{1} + \frac{\alpha}{\hbar} [\sigma_x(p_y + eB_0x) - \sigma_y p_0],$$

$$H_1 = \frac{V_B}{\hbar q} (p_y + eB_1x) \sin(qy) \mathbf{1},$$

$$H_2 = \frac{V_B}{2\pi} \sigma_z \sin(qy),$$

$$H_3 = \frac{V_B g^* \cos(qy) \sigma_z}{\sigma_z},$$

$$H_4 = \frac{V_B}{(4\pi)^2 e_B} \left[1 - \cos(2qy)\right] \mathbf{1}. \quad (2)$$

Here, $V_B = \hbar cB_1/m^*$ is the strength of the effective magnetic potential determined by the amplitude $B_1$ of the magnetic modulation and $p_0 \ll \omega_0$ with $\omega_0 = eB_0/m^*$ is the cyclotron frequency due to the constant magnetic field $B_0$. The dimensionless parameter $p_0 = a_B k_0$ with $k_0 = a m^*/\hbar^2$ and $g^* = g m^*/(4\omega_0)$. Also, $e_2 = h^2/(m^* a^2)$ is a characteristic energy scale introduced by the modulation period $a$. Here, the term $H_0$ is the Hamiltonian for the electron with the Rashba interaction in the presence of the constant magnetic field $B_0 \hat{z}$ including the Zeeman energy. The terms $H_1$ and $H_2$ are due to the effect of the magnetic modulation on the kinetic energy term. The term $H_2$ is the interaction between the spin–orbit and the magnetic modulation whereas the term $H_3$ is the interaction between the Zeeman energy and the magnetic modulation.

The Hamiltonian $H_0$ can be solved analytically and can be treated as an unperturbed Hamiltonian. The other four Hamiltonians can be treated as a small perturbations since $V_B \ll \hbar c_0$. The eigenfunctions of the unperturbed Hamiltonian $(H_0)$ can be used to find the energy correction due to the small perturbations $H_1, H_2, H_3$ and $H_4$.

Here, we shall briefly summarize the results of the [28] where the analytical solutions of the Hamiltonian $H_0$ have been derived. Using the Landau wavefunctions without the Rashba interaction as the basis, one can write the new wavefunction as

$$\Psi_{h}\mathbf{r} = \frac{e^{ih_y}}{\sqrt{L_y}} \sum_{n=0}^{\infty} \phi_h(x + x_0) \left(\begin{array}{c} C_n^+ \\ C_n \end{array}\right). \quad (7)$$
Here, \( \phi_0(x) = (1/\sqrt{\pi}2^n n!l_0) e^{-x^2/2l_0^2} \) is the normalized harmonic oscillator wavefunction and \( \hbar \) is the Landau level index, \( l_0 \) is the width of the sample in the \( y \) direction, and \( l_0 = \sqrt{\hbar/(eB_0)} \) is the magnetic length and the cyclotron orbit is centered at \(-x_0 \) with \( x_0 = k_y^2 l_0^2 \).

Using these wavefunctions and equation (1), the eigenvalue problem \( H_0 \Psi = E \Psi \) leads to an infinite number of equations that can be solved exactly after decomposing it into independent systems of one or two equations [28]. The resulting eigenstates are labeled by a new quantum number \( s \), instead of \( n \). For \( s = 0 \), there is only one level, the same as the lowest Landau level without SOI, with energy \( E_0^+ = E_0 = (\hbar \omega_0 - g \mu_B B_0)/2 \) and the corresponding wavefunction is

\[
\Psi_{0,k_y}^+ (r) = \frac{e^{i k_y y}}{\sqrt{L_y}} \phi_0(x + x_0) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

For \( s = 1, 2, 3, \ldots \) there are two branches of energy levels, denoted by \( + \) and \( - \) with energies

\[
E_s^\pm = s \hbar \omega_0 \pm \sqrt{E_0^2 + sE_a \hbar \omega_0},
\]

where \( E_a = 2m^* \alpha^2/\hbar^2 \). These energy levels are degenerate in \( k_y \). Therefore, the group velocities, \( v_x \) and \( v_y \), are zero.

The corresponding wavefunction for the + branch is

\[
\Psi_{s,k_y}^+ (r) = \frac{e^{i k_y y}}{\sqrt{L_y A_s}} \begin{pmatrix} D_s \phi_0(x + x_0) \\ \phi_0(x + x_0) \end{pmatrix},
\]

and for the - branch is

\[
\Psi_{s,k_y}^- (r) = \frac{e^{i k_y y}}{\sqrt{L_y A_s}} \begin{pmatrix} \phi_0(x + x_0) \\ -D_s \phi_0(x + x_0) \end{pmatrix},
\]

where \( A_s = 1 + D_s^2 \) and \( D_s = \sqrt{E_a \hbar \omega_0}/|E_0 + \sqrt{E_0^2 + sE_a \hbar \omega_0}| \).

The Landau level quantum numbers \( s^z \) at the Fermi energy are determined from the equation \( E_s = s \hbar \omega_0 \pm \sqrt{E_0^2 + sE_a \hbar \omega_0} \). These quantum numbers \( s^z \) are given by \( s^z \approx k_y^2 l_0^2/2 \pm k_y k_y^2 l_0^2 \), where \( k_F = \sqrt{2 \pi n_c} \) is the Fermi wavevector and \( n_c \) is the electron density.

The approximate density of states (DOS) of the 2DEG with the Rashba SOI in the presence of constant perpendicular magnetic field is given [31] by

\[
D^\pm(E) \approx \frac{m^*}{2 \pi \hbar^2} \left[ 1 + 2 \exp \left\{ -2 \left( \pi \Gamma_0 \right) / \hbar \omega_0 \right\} \right] \times \cos \left\{ 2\pi \left( \frac{E_a}{\hbar \omega_0} + \sqrt{E_0^2 + E_a E} \right) \right\},
\]

where \( \Gamma_0 \) is the Landau level broadening. The detailed derivation of the DOS is given in appendix A. We shall use the above DOS to obtain the analytic expressions of the magnetotransport coefficients in the presence of the modulations.

Using the perturbation theory, we calculate the first-order energy correction for the + branch as well as the - branch. It is to be noted that due to the Hamiltonian \( H_4 \) is of the order of \( \alpha^2 \) which is much smaller than \( \omega_0 \). Therefore, we neglect the contribution of \( H_4 \) to the total energy correction.

The total energy for the + branch and the - branch is given by

\[
E_{s,k_y}^\pm = s \hbar \omega_0 \pm \sqrt{E_0^2 + sE_a \hbar \omega_0} + \cos(qx_0) F_s^\pm(u),
\]

where

\[
F_s^\pm(u) = \frac{V_B}{A_s} \left[ u^{-1/2} g^*(\pm D_s^2 L_s \mp 1/2 \mp 1/2(u)) \mp L_s \mp 1/2 \mp 1/2(u) + D_s^2 G_s \mp 1/2 \mp 1/2(u) \right] \frac{Z}{\sqrt{2s}} e^{-u/2} L_s^1(u) - 1/2(u).
\]

Here, \( u = q^2 l_0^2/2, s = 0, 1, 2, \ldots \) for the + branch and \( s = 1, 2, 3, \ldots \) for the - branch. Also, \( G_s(u) = e^{-u/2} L_s(u/2 + L_s^1(u)) \) and \( Z = 2k_F D_s l_0 \). The upper and lower signs correspond to the + and - branches, respectively. The width of the broadened levels of the two branches is \( |\Delta_s^\pm| \) and it can be written as \( \Delta_s^\pm = 2F_s^\pm(u) \).

For various plots, we use the following parameters: \( \alpha \) in units of \( \alpha_0 = 1.0 \times 10^{-11} \) eV m, modulation strength \( V_B = 0.05 \) meV for \( B_1 = 0.02 \) T, electron density \( n_c = 10^{16} \) m\(^{-2} \), electron effective mass \( m^* = 0.05m_0 \) with \( m_0 \) is the free electron mass, mobility \( \mu = 100 \) m\(^2\) V\(^{-1}\) s\(^{-1} \), \( g = 2 \), modulation period \( \lambda = 800 \) Å and temperature \( T = 1.5 \) K. For these parameters, \( p_0 = 1.05 \) for \( \alpha = 2\alpha_0, p_f = p_k = 20 \), \( E_p = 47.5 \) meV and \( e_a = 0.236 \) meV.

In figure 1, we plot the dimensionless bandwidth \( \Delta^\pm/V_B \) at the Fermi energy as a function of the dimensionless inverse magnetic field \( \lambda = B_y/B_1 \) for two different values of \( \alpha \). Here, \( B_y = \hbar/(e a^2) = 0.102 \) T is the characteristic magnetic field introduced by the modulation period \( \alpha \). We plot \( \Delta^\pm \) instead of \( |\Delta^\pm| \) to show the oscillations more clearly.

The diagonal matrix elements of the velocity operator do not vanish due to the \( k_y \) dependence of the energy levels. It can
be calculated by using \( v_y^\pm = (1/h)\partial E_{k_y}^\pm /\partial k_y \), and their values are given by
\[
v_y^\pm = -\frac{q^2}{h} \sin(\theta_x^0) E_y^\pm(u). \tag{15}\]

These non-zero values give rise to the finite diffusive conductivity whereas in the absence of the magnetic modulation the diffusive conductivity vanishes whether the Rashba interaction is present or not. These velocities for spin-up and spin-down branches do oscillate with slightly different frequencies as a function of the magnetic field \( B_0 \).

3. Magnetotransport coefficients

In the presence of the weak modulation, there are two contributions to the transport properties: diffusive and collisional contributions. The former is due to finite drift velocity gained by the electrons in the presence of the modulation. The latter is the contribution from the hopping of the localized states due to scattering by impurities. The diffusive conductivity decreases with increasing impurity scattering whereas the collisional conductivity increases with the increase of impurity scattering. The collisional scattering is dominant at low temperature.

We follow the formulation of \cite{36} to calculate the magnetotransport coefficients in the presence of the modulation. In the linear response regime and weak scattering potentials, the conductivity tensor, in the one-electron modulation. In the linear response regime and weak scattering the magnetotransport coefficients in the presence of the magnetic field \( B \) are given by
\[
\sigma_{ij}^{\text{diff}} = 2 e^2 \tau \sum_f \frac{f_{i,j}^* f_{j,i}}{Z_0} \int_{E_i}^{E_f} \frac{dE}{h} f(E) \left[ 1 - f(E) \right] (v_y^\pm)^2. \tag{17}\]

In the limit of weak magnetic field \( B_0 \), many Landau levels are filled \((s \gg 1)\), we use the following approximations: \( e^{-n^2/4} \ll 1 \) and \( \sum_s \rightarrow \int_0^\infty dE / (h \omega_0) \). With the use of the above approximations and following 
\cite{5}, Equation (17) reduces to an analytical form which is purely the Weiss contribution, as
\[
\sigma_{\text{Weiss}}^{\pm} = e^2 A_B C_0^\pm \lambda \left[ 1 + \left( \frac{W_0^\pm}{C_0^0} \right) \cos(2 \pi \Omega^\pm \lambda - 2 \pi \Phi_0^\pm) \right], \tag{18}\]

where \( \Omega^\pm = \frac{2}{p_F} (p_F \mp p_a) \) are the frequencies of the conductivity oscillations of spin-up and spin-down electrons, \( A_B = \frac{\hbar^2}{\beta \tau} \), \( T_B = E_F / (2 \pi^2 \hbar p_B \lambda) \) is the characteristic temperature and \( \Phi_0^\pm = \delta_0^\pm + 4 \pi / \beta (p_F \mp p_a) \) are the phase factors. Other parameters are \( H(x) = x / \sin(x) \),
\[
C_0^\pm = \frac{1}{4(p_F \mp p_a) + \pi (p_F \mp p_a)^3} \sin^2 \left( \frac{2 \pi}{p_F \mp p_a} \right),
C_1^\pm = \frac{1}{4(p_F \mp p_a) + \pi (p_F \mp p_a)^3} \sin^2 \left( \frac{2 \pi}{p_F \mp p_a} \right),
\]

and
\[
C_2^\pm = \frac{(p_F \mp p_a)}{4 \pi^2} \sin \left( \frac{2 \pi}{p_F \mp p_a} \right).
\]

The standard semi-classical expression for the diffusive conductivity is given as \cite{36}
\[
\sigma_{\text{diff}}^{\tau} = \frac{e^2}{h} A_B \lambda \left[ C + H \left( \frac{T_B}{2 \pi} \right) W_0^+ \sin(2 \pi \Omega^+ \lambda - 2 \pi \Phi_0^+) \right], \tag{19}\]

where \( C = C_0^+ + C_0^- \). It shows that the total diffusive conductivity exhibits beating patterns in the amplitude of the Weiss oscillations. In figure 2, we compare the analytical result of the Weiss contribution equation (19) to the total diffusive conductivity with the exact numerical results obtained from equation (17). The analytical results are in excellent agreement with the numerical results except at higher magnetic field \( \lambda \leq 0.15 \) where the SdH oscillation dominates over the magnetic Weiss oscillation.
To obtain the analytical expressions of the locations of the beat nodes and the number of oscillations between any two successive nodes, we simplify equation (19) by considering \( p_F \gg p_a \) and \( 1/(p_F \mp p_a) \approx 0 \). Equation (19) reduces to the following form:

\[
\sigma_{\text{Weiss}}^{\text{diff}} \simeq \frac{e^2}{h} \frac{p_F}{2\pi^2} A_B \lambda \left[ 1 + H \left( \frac{T}{T_a} \right) \sin \left( 2\pi \Omega_0 \lambda + \pi \right) \right] \times \cos (2\pi \Omega_a \lambda),
\]

where \( \Omega_0 = (\Omega_+ + \Omega_-)/2 \) and \( \Omega_a = (\Omega_+ - \Omega_-)/2 \).

At the node positions \( B_0 = B_j \), we have the following condition: \( \cos(2\pi \lambda \Omega_a)_{B_0=B_j} = 0 \), which gives us

\[
B_j \approx \frac{8p_a B_a}{(2j + 1)},
\]

where \( j = 0, 1, 2, 3, 4, \ldots \) are the \( j \)th beat node and the corresponding magnetic field is \( B_j \). By using the above equation, the beating nodes appear at \( B_j \). By using the above equation, the beating nodes appear at \( B_j = 0.11, 0.35, 0.59, 0.83, 1.07, \ldots \) which are in excellent agreement with the exact numerical results. The above equation indicates that the magnetic field corresponding to the \( j = 0 \) node is the lower limit of the Weiss oscillation below which the SdH oscillation starts to dominate. In practice, the numbering of the beat nodes is difficult. To remove this problem, the above equation can be re-written for any two successive beat nodes as

\[
\frac{1}{p_a} = 4B_a \left( \frac{1}{B_{j+1}} - \frac{1}{B_j} \right).
\]

So, the strength of the Rashba SOI can be determined from the above equation by knowing the locations of the two successive beating nodes. The number of oscillations between any two successive beat nodes can be obtained from the first sine term of equation (20), which is

\[
N_{\text{osc}} = \frac{2p_F B_a}{(B_{j+1} - B_j)}. \tag{23}
\]

Using equation (22) in the above equation, we get

\[
N_{\text{osc}} = \frac{p_F}{2p_a} = \frac{k_F}{2k_a}. \tag{24}
\]

From the above equation, the following important conclusions can be drawn. (i) The above equation can be re-written as \( \alpha = h^2 k_F/(2m^* N_{\text{osc}}) \). Therefore, the Rashba SOI strength can be easily calculated by just counting the number of oscillations between any two successive nodes. (ii) The number of oscillations between any two successive nodes is constant for given values of \( n_a \) and \( \alpha \), whereas it depends on the magnetic field in the SdH oscillation [31]. (iii) While the frequency of the Weiss oscillation depends on the modulation period, the number of oscillations between any two successive nodes does not depend on it.

The analytical expression of the diffusive conductivity given in equation (19) is not able to explain the origin of the superposition of the SdH oscillation on the Weiss oscillation at higher magnetic field. This can be explained by using the asymptotic expression of the DOS given in equation (12) and replacing the summation [5] over discrete states by integration as \( \sum \rightarrow 2\pi \int_0^\infty D^{\pm}(E) \, dE \), then we get \( \sigma_{\text{Weiss}}^{\pm} = \sigma_{\text{SdH}}^{\pm} \), where \( \sigma_{\text{Weiss}}^{\pm} \) given in equation (18) and the Weiss oscillation modulated by the SdH oscillation \( (\sigma_{\text{SdH}}^{\pm}) \) is given by

\[
\sigma_{\text{SdH}}^{\pm} = \frac{e^2}{h} A_B \lambda \sqrt{\frac{\Omega}{2\pi}} \exp \left[ -2 \left( \frac{\pi \Gamma_0}{\bar{\hbar} \alpha_0} \right)^2 \right] \frac{H \left( T/T_a \right)}{1 + \left( \frac{W^{\pm}}{e^{\pm}} \right)^2 \sin(2\pi \Omega_\lambda \lambda - \Phi^{\pm})} \cos(2\pi f^{\pm}/B_0), \tag{25}
\]

where \( W^{\pm} = \sqrt{(C^{\pm})^2 + (2C^{\pm}_a)^2}, \) \( \Phi^{\pm} = \delta^{\pm} + 4\pi/(p_F \mp p_a) \) with \( \delta^{\pm} = \tan^{-1}(2C^{\pm}_a/C^{\pm}) + \pi \). Also, \( f^{\pm} = (m^*/(\hbar e))(E_F + E_a)/2 \pm \sqrt{E^2_0 + E_0 E_F} \) are the frequencies of the SdH oscillations for spin-up and spin-down electrons in the absence of the modulation [31] and the characteristic temperature for the SdH oscillation is \( T_c = (\hbar \alpha_0)/(2\pi^2 k_B) \). For \( \alpha = 2\alpha_0 \) and \( T = 1.5 \, \text{K} \), the ratio \( \sigma_{\text{SdH}}^{\pm}/\sigma_{\text{Weiss}}^{\pm} \) is 0.3, 0.025, 0.001 for \( B_a/B_0 = 0.3, 0.5, 0.8 \), respectively. Here, \( \sigma_{\text{SdH}}^{\pm} = \sigma_{\text{SdH}}^{\pm} + \sigma_{\text{SdH}}^{\pm} \). Note that the Weiss oscillation frequencies \( (\Omega^{\pm}) \) for spin-up and spin-down electrons are different from the SdH oscillation frequencies \( f^{\pm} \).

To derive equations (18) and (25), we have used the DOS for an unmodulated 2DEG. In the presence of the modulation, the DOS can be written as [39] a sum of the unmodulated \( (D^{\pm}(E)) \) and modulated \( (D^{\pm}_m(E)) \) part as \( D^{\pm}(E) = D^{\pm}(E) + D^{\pm}_m(E) \). Here, \( D^{\pm}(E) \) is given in equation (12) and \( D^{\pm}_m(E) \) is of

Figure 2. Plots of the exact (dashed) and asymptotic (solid) expressions of the diffusive conductivity versus dimensionless inverse magnetic field \( B_a/B_0 \) for electric and magnetic modulations. Here strength of the Rashba SOI \( \alpha = 2\alpha_0 \).
the order of $V_g^2$, which is very small in comparison to $D^\pm(E)$. Moreover, the correction to the Weiss and the SdH contributions due to the $D_m(E)$ will be of the order of $V_g^2$. The effect of the DOS correction due to the modulation on the Weiss and the SdH oscillations is really small and we have neglected it.

Substituting $\alpha = 0$ in equation (18), we get the oscillation period $P = 1/\Omega = 1/(2\pi \nu_f)$ which is the same as obtained in [6]. We compare this theoretical result with the available experimental result [18] for $\alpha = 0$. In this experiment [18], magnetic modulation period $a = 1 \mu m$ and the electron density $n_e = 2.2 \times 10^{15} \text{ m}^{-2}$. Using these parameters, we obtain $P = 0.00425$ which is very close to the experimental result $P \approx 0.00465$.

3.1.1. Comparison with the electrical modulation case.

Here, we would like to compare the above-mentioned results for the magnetically modulated case with the electrically modulated system. The electric modulation potential is described by $V' = V_0 \cos(q_x)$, where $V_0$ is the amplitude of the electric modulation potential. We have used $V_0 = 0.05 \text{ meV}$ in our numerical calculation. In the case of an electrically modulated system, numerical results have been discussed in [33]. Here, we provide an analytical expression of the diffusive conductivity for the electric modulation case. We obtain the Weiss and the SdH contributions to the diffusive conductivity, which are given below:

$$\sigma_{\text{Weiss}}^\pm = \frac{e^2}{\hbar} \frac{A_E \lambda}{\nu_f + \nu_a} \cos^2\left[\pi \left(\nu_f \pm \nu_a\right)\right] \left[1 + H\left(\frac{T}{T_a}\right)\right] \sin[2\pi \Omega^\pm \lambda - 2\pi \nu_f \nu_a] \left[1 + H\left(\frac{T}{T_a}\right)\right] \sin[2\pi \Omega^\pm \lambda - 2\pi \nu_f \nu_a]$$

(26)

and

$$\sigma_{\text{SdH}}^\pm = \frac{e^2}{\hbar} \frac{A_E \lambda}{\nu_f + \nu_a} \cos^2\left[\pi \left(\nu_f \pm \nu_a\right)\right]$$

$$\times \exp\left[-2\left(\frac{\pi \Gamma_0}{\hbar \nu_f}\right)^2 \right] H\left(\frac{T}{T_a}\right) \left[1 + H\left(\frac{T}{T_a}\right)\right] \sin[2\pi \Omega^\pm \lambda - 2\pi \nu_f \nu_a]$$

(27)

where $A_E = V_0^2/(\hbar \varepsilon_a)$. The lower panel of figure 2 shows the comparison between the analytical expression given in equation (26) and the numerical results reproduced from equation (20) of [33]. Our analytical expression compares very well with the numerical result. The Weiss and the SdH oscillation frequencies for spin-up and spin-down electrons are the same for both types of modulations. Therefore, equations (22)–(24) for the magnetic modulation case remain also valid in the electric modulation case.

However, there are a few important differences between the diffusive conductivities in the electric and magnetic modulation cases. There is a definite phase difference between the diffusive conductivities in electric and magnetic modulation cases. The amplitude of the diffusive conductivity in the presence of the magnetic modulation is found to be nearly $\nu_f/(2\pi)^2$ times higher in comparison to the electrical modulation case, which is the same as in the absence of the Rashba SOI [6]. The role played by the Rashba SOI in the case of magnetic modulation is that there is a difference between the amplitudes of conductivity for the spin-up and spin-down electrons by a factor $\nu_f/(2\pi)^2$. In the electric modulation case, amplitudes of the conductivities of spin-up and spin-down branches are nearly the same.

3.2. Collisional conductivity

In electron systems the charge impurities play an important role in magnetotransport properties. Collisional conductivity arises because of the migration of the cyclotron orbit due to scattering from charge impurities. At low temperature, we can assume that electrons are elastically scattered by the charged impurities distributed uniformly. The standard expression for collisional conductivity is given by [38]

$$\sigma_{\mu \mu}^{\text{col}} = \frac{\beta \varepsilon_u^2}{2S_0} \sum_{\xi, \xi'} f^\mu \left(1 - f^\mu\right) W_{\xi, \xi'}(\alpha^\mu_\xi - \alpha^\mu_{\xi'})^2.$$  

(28)

Here $f^\mu = f^\mu$ for elastic scattering, $W_{\xi, \xi'}$ is the transition probability between one-electron states $|\xi\rangle$ and $|\xi'\rangle$. Also, $\alpha^\mu_\xi = (\xi | \nu_\mu | \xi\rangle$ is the expectation value of the $\mu$ component of the position operator for the electron in state $|\xi\rangle$. The scattering rate $W_{\xi, \xi'}$ is given by

$$W_{\xi, \xi'} = \sum_{\Psi_0} \left|U(|\Psi_0\rangle)^2/(|\xi\rangle e^{i|\Psi_0\rangle - R}|\xi'\rangle\right|^2 \delta(E_\xi - E_{\xi'})$$

(29)

where $q_0 = q_0^x + q_0^y \hat{y}$ is a 2D wavevector and $U(|\Psi_0\rangle = 2\pi \varepsilon_u^2/(\sqrt{q_0^x + q_0^y + k_z^2})$ is the Fourier transform of the screened impurity potential $U(r) = (e^2/4\pi \varepsilon\varepsilon_0)(e^{i\hat{y} \hat{r}}) \delta(r - \hat{r})$, where $k_z$ is the inverse screening length and $\varepsilon$ is the dielectric constant of the material. In the limit of small $|q_0| \ll k_s$, $U(|\Psi_0\rangle \approx 2\pi \varepsilon_u^2/(\varepsilon\varepsilon_0 k_s) = U_0$. Here, $r$ and $R$ are the position vector of the electron and impurity, respectively. In this limit, we can use $r \approx \pi \varepsilon_0^2 q_0^2 N_1 U_0^2$, where $N_1$ is the 2D impurity number density.

We follow [5, 6] to calculate the collisional conductivity. We include the correction to the unperturbed eigenstate $|\Psi_{s, k_s}(r)\rangle$ due to the weak perturbative term $\Delta H = H_1 + H_2 + H_3 + H_3$. The first-order correction to the Landau state is obtained by

$$|\Psi_{s, k_s}(r)\rangle = |\Psi_{s, k_s}(r)\rangle + \sum_{s \neq s} \frac{\langle \Psi_{s, k_s}(r) | \Delta H | \Psi_{s', k_s}(r)\rangle}{E_s - E_{s'}} |\Psi_{s', k_s}(r)\rangle,$$

Following [5, 6] and using the perturbed Landau states $|\Psi_{s, k_s}(r)\rangle$, we obtain the collisional conductivity as

$$\sigma_{\text{coll}} = \frac{e^2}{2\pi \alpha_a} \sum_{s} \left[ (I_{s}^{+} M_{s}^{+} + R_{s}^{+} J_{s}^{+}) \right].$$  

(30)

The exact expressions of $I_{s}^{+}$ and $M_{s}^{+}$ are given as

$$I_{s}^{+} = \left[(2s \mp 1)D_{s}^{+} - 2sD_{s}^{+} + 2s \pm 1\right]/\Lambda_{s}^{2}.$$  

(31)
The term $K_F^\pm$ is appearing due to the first-order correction to the Landau wavefunction and it is of the order of $V_B^2$. We neglect this term because of its small contribution. The major contribution to the collisional conductivity is due to the first term proportional to $M^\pm_{s\pm}$. The effect of the magnetic modulation mainly enters through the energy correction due to the modulation in the total energy in the Fermi–Dirac distribution function.

Similar to the diffusive conductivity, the collisional conductivity in the presence of the modulation will have two contributions, namely the SdH and the Weiss contributions: $\sigma_{sx}^\pm = \sigma_{s\mathrm{Dh}}^\pm + \sigma_{\mathrm{Weiss}}^\pm$. It is difficult to get the analytical expression of the Weiss contribution; $(\sigma_{\mathrm{Weiss}}^\pm)$ comes from the energy correction in the Fermi–Dirac distribution function.

The numerical results of the change in the collisional conductivity $\Delta\sigma_{sx}$ versus magnetic field for $\alpha = 2\alpha_0$ are plotted in figure 3. To compare the results of the magnetic modulation case, we present $\Delta\sigma_{sx}$ for the electric modulation case in the lower panel of figure 3.

The oscillatory behavior with beating pattern appears in the changes in the conductivity due to the modulation at low magnetic field range where modulation strength is not much less than the energy scale of the Landau levels. The effect of the modulation diminishes with the increase of the strength of the perpendicular magnetic field. As the magnetic field increases, the SdH oscillation starts to dominate over the modulation induced Weiss oscillation.

The collisional conductivities for spin-up and spin-down electrons oscillate with the same frequencies as the bandwidth $|\Delta^\pm|$. The frequencies of the oscillation of the bandwidths are the same as that of the diffusive conductivities, namely, $\Omega^\pm$. Therefore, the beating condition for the collisional conductivity is the same as given in equation (21) for the diffusive conductivity. Using the beating condition, we get the magnetic fields (in Tesla) corresponding to the beating positions, obtained from the equation (21), to be 0.0506, 0.0574, 0.0662, 0.0782, 0.0957 . . . . These are in good agreement with the exact numerical results shown in figure 3. The phase difference between the oscillations in the diffusive and collisional conductivities is nearly $\pi$.

The modulation gives a very small effect on the SdH part ($\sigma_{s\mathrm{Dh}}^\pm$) of the collisional conductivity. It is also difficult to get an analytical expression of the SdH oscillations superposed on the Weiss oscillations. Ignoring the modulation effect and using the analytic form of the DOS, the asymptotic expression of the SdH oscillation has been studied in [31].

The beating pattern in the SdH oscillation is given by [31]

\[
\frac{\sigma_{s\mathrm{Dh}}}{\sigma_0} \simeq \frac{E_F}{4(\alpha_0)^2} \left[ 1 + 2 \exp \left\{ -2 \left( \frac{\pi \Gamma_0}{B_0} \right)^2 H \left( \frac{T}{T_c} \right) \right\} \right] \times \cos(2\pi f_s/B_0) \cos(2\pi f_d/B_0),
\]

(34)

where $\sigma_0 = n a^2 \tau / m^*$ is the classical Drude conductivity, $E_F = [1 + E_a/(2E_F) \mp (3/2)\sqrt{E_a/E_F}]$ and $f_a = (f^+ + f^-)/2$ and $f_d = (f^+ - f^-)/2$.

3.3. The Hall conductivity

The off-diagonal elements in the conductivity tensor are termed as the Hall conductivity which is given by [38]

\[
\sigma_{xy} = \frac{i e^2 h}{S_0} \sum_{\xi} f_\xi (1 - f_\xi) \langle \xi | v_s | \xi' \rangle \langle \xi' | v_s | \xi \rangle \frac{1 - e^{\beta(E_\xi - E_{\xi'})}}{(E_\xi - E_{\xi'})^2},
\]

(35)

To simplify the above equation, we shall use the following relation

\[
f_\xi (1 - f_\xi') [1 - e^{\beta(E_\xi - E_{\xi'})}] = f_\xi - f_\xi'.
\]

(36)

The matrix elements of the velocity operators are zero except between the nearest Landau levels. Following [5, 6, 28], we calculate the velocity matrix elements and the energy difference between two nearest Landau levels (see appendix B). Finally, we get the expressions of the Hall conductivity as

\[
\sigma_{yx}^\pm = \frac{e^2 B}{h} \sum_{s} \frac{1}{A_s A_{s+1}} \left[ D_{s+1}(D_s \sqrt{s \pm \sqrt{2} k_{sd} B_0}) + \sqrt{s + 1} \right] \frac{f_{s+1,k_s} - f_{s+1,k_s}}{[1 + \gamma_s^\pm \cos qx_0]^2} \langle q | v_s | q \rangle, \tag{37}
\]

In the large $s$ limit, $\gamma_s^\pm$ reduces to

\[
\gamma_s^\pm \simeq \frac{V_B \sqrt{\pi}}{\pi \alpha_a} \left( \frac{p_F + p_a}{\pi^2} \right)^{3/2} \sin^2 \left[ \pi (p_F \mp p_a) \right] \times \cos \left( 2\pi \lambda (p_F \mp p_a) - \frac{\pi}{4} - \frac{\pi}{(p_F \mp p_a)} \right). \tag{38}
\]
When \( T \to 0 \) and \( E_s < E_F < E_{s+1} \), equations (37) and (38) give the asymptotic form of the Hall conductivity as

\[
\sigma_{ys}^\pm \simeq \frac{\sigma_H \alpha}{2} \left[ 1 + \frac{3}{2} \left( \Upsilon_{s}^\pm \right)^2 \right].
\]

(39)

where

\[
\sigma_{H,\alpha} = \sigma_H \left[ 1 - \frac{1}{2} \left( \frac{k_\alpha}{k_F} \right)^2 \right] \simeq \sigma_H \left[ 1 - \frac{k_\alpha}{k_F} \right]^2
\]

(40)

is the total Hall conductivity of the 2DEG with Rashba SOI but without modulation and \( \sigma_H = n_e e^2 / B_0 \) is the classical Hall conductivity. Note that \( n_e \) is the sum of the density of the spin-up and spin-down electrons. By taking the superposition of the analytical expression of the conductivity for the spin-up and spin-down electrons, we get the same beating condition as given in equation (21) for the diffusive conductivity. The numerical results of the change in the Hall conductivity versus magnetic field is shown in figure 4.

Similarly, we obtain the Hall conductivity for the electrical modulation case as \( \sigma_{ys}^\pm \simeq \frac{\sigma_H \alpha}{2} \left[ 1 + (3/2) \left( \Upsilon_{s}^\pm \right)^2 \right] \), where

\[
\Upsilon_{s}^\pm \simeq \frac{V_F}{\pi e a} \sqrt{\lambda} \sin \left[ 2\pi \left( \frac{p_T}{p_T \mp p_\alpha} \right) \right] \times \sin \left[ 2\pi \lambda \left( p_T \mp p_\alpha \right) - \pi / 4 \right].
\]

(41)

In figure 5, we compare the change in the Hall conductivity of the electric and magnetic modulation cases. The beating condition remains the same in the electric modulation case also. The amplitude in the change in conductivity due to the magnetic modulation is nearly \( \left( p_T / (2\pi) \right)^2 \) times higher than the electrical modulation case. There is a \( \pi \) phase difference between the Hall conductivities of the electrical and magnetic modulation cases. In the above analytical expression the modulation effect through the Fermi–Dirac distribution function has been ignored.

4. Conclusion

We have studied magnetotransport properties of the 2DEG in the presence of the Rashba SOI when the perpendicular magnetic field is weakly modulated.

The diffusive conductivity shows a beating pattern due to the interference between the conductivities for spin-up and spin-down electrons. We calculate the asymptotic expression of the Weiss conductivity which matches very well with the exact numerical results. The number of Weiss oscillations between any two successive nodes is fixed for given values of \( n_e \) and \( \alpha \) whereas in the SdH it depends on the magnetic field. We have shown that the Rashba SOI strength can be determined by analyzing the beating pattern in the Weiss oscillation. The strength of the Rashba SOI can be determined by just counting the number of Weiss oscillations between any two successive beat nodes. There is a definite phase difference between the conductivities for magnetic and electric modulation cases. In a magnetically modulated system, there is a difference in amplitude by a factor \( p_\alpha / (2\pi^2) \) between the conductivity due to the spin-up and spin-down electrons whereas in the electrically modulated system, the amplitudes are the same for spin-up and spin-down electrons.

To observe the effect of the modulation, we plot the change in the collisional conductivity due to the modulation at the low range of the magnetic field. It is found that the effect of the magnetic modulation is much higher than the electrical modulation. The major effect of the modulation comes through the energy correction in the total energy in the Fermi–Dirac distribution function. The beating condition in the Weiss contribution to the collisional conductivity is the same as that of the diffusive conductivity.

The modulation effect on the Hall conductivity is shown by plotting the change in the Hall conductivity due to the modulation. The beating pattern appears in the Hall
conductivity and it increases with the increase of the Rashba strength. The oscillations are out of phase between the electric and magnetic modulation cases. The beating condition remains the same as the diffusive conductivity. The amplitude of the fluctuation in the presence of the magnetic modulation is found to be much higher in comparison to the electric modulation case.

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Appendix A

We derive the DOS by taking the imaginary part of the self-energy [40, 41] which is given as

$$\Sigma^-(E) = \frac{\pi}{2} \sum_{s} \frac{1}{E - E_s - \Sigma^-(E)}.$$  \hspace{1cm} (A.1)

The DOS is the imaginary part of the self-energy: $D(E) = \text{Im} \{ \Sigma^-(E) \}$. First we consider the lower branch. To find the summation, we use the residue theorem and neglecting the $(E_0/\hbar\omega_0)^2$ term, we obtain $\Sigma^-(E) = \frac{\pi \Gamma^2_0}{\hbar\omega_0} \text{cot}(\pi s_x)$. Here $s_x \approx \frac{\hbar}{E_0}[E - \Sigma^-(E) + E_{\alpha}/2 + \sqrt{E_0^2 + E_{\alpha}E}]$. We write $\Sigma^-(E) = \Delta + i\Gamma/2$, then the above equation can be re-written as

$$\Delta + i\Gamma/2 = \frac{\pi \Gamma^2_0}{\hbar\omega_0} \text{cot} \left( \frac{(u - iv)}{2} \right)$$

$$= \frac{\pi \Gamma^2_0}{\hbar\omega_0} \frac{\sin u + i \sinh v}{\cosh v - \cos u}.$$  \hspace{1cm} (A.2)

We are using the following standard result to simplify it further:

$$\frac{\sinh v}{\cosh v - \cos u} = 1 + 2 \sum_{k=1}^{\infty} e^{-kv} \cos(ku).$$  \hspace{1cm} (A.3)

Similarly, one can do it for the upper branch. Finally, the DOS for lower and upper branches can be put together as

$$D^{\pm}(E) = \frac{m^*}{2\pi \hbar^2} \left[ 1 + 2 \exp \left( -2 \left( \frac{\pi \Gamma_0}{\hbar\omega_0} \right)^2 \right) \right]$$

$$\times \cos \left( \frac{2\pi}{\hbar\omega_0} \left( E + E_{\alpha}/2 + \sqrt{E_0^2 + E_{\alpha}E} \right) \right).$$  \hspace{1cm} (A.5)

In the absence of the Rashba SOI and the Zeeman term, including the spin degeneracy the above expression can be reduced to the following standard result [40, 41] as

$$D(E) = \frac{m^*}{\pi \hbar^2} \left[ 1 + 2 \exp \left( -2 \left( \frac{\pi \Gamma_0}{\hbar\omega_0} \right)^2 \right) \cos \left( \frac{2\pi E}{\hbar\omega_0 - \pi} \right) \right].$$  \hspace{1cm} (A.6)

Appendix B

The velocity operators are given by

$$v_x = \frac{\partial H_0}{\partial p_x} = \frac{p_x}{m^*} - \frac{\alpha}{\hbar} \bar{s}_y$$

$$v_y = \frac{\partial H_0}{\partial p_y} = \frac{1}{m^*} (p_y + eB_0 x) + \frac{\alpha}{\hbar} s_x$$

$$= \left[ \frac{p_y + eB_0 x}{m^*} + \frac{\alpha}{\hbar} s_x \right].$$  \hspace{1cm} (B.2)

For $\xi = (s+1, \pm, k_x)$, the velocity matrix elements (see [5, 6, 28]) are

$$\langle \Psi^s_{-k_x} (r) | v_x | \Psi^s_{+1,k_x} (r) \rangle = \frac{1}{\sqrt{A_{s+1}}}$$

$$\times \left[ D_{s+1} (\phi^s_{-1} (X)) \frac{p_x}{m^*} | \phi^s_{+1} (X) \rangle + i \frac{\alpha}{\hbar} D_{s+1} \right]$$

$$= \frac{1}{\sqrt{A_{s+1}}} \frac{\hbar\omega_0}{2m^*} [D_{s+1} (D_x \sqrt{s} + \sqrt{2}k_x l_0) + \sqrt{s} + 1].$$  \hspace{1cm} (B.3)

where $X = x + x_0$. Similarly, for the velocity component $v_y$

$$\langle \Psi^s_{-1+k_x} (r) | v_y | \Psi^s_{s+1,k_x} (r) \rangle = \frac{1}{\sqrt{A_{s+1}}} \frac{\hbar\omega_0}{2m^*}$$

$$\times [D_{s+1} (D_x \sqrt{s} + \sqrt{2}k_x l_0) + \sqrt{s} + 1].$$  \hspace{1cm} (B.4)

The energy difference between the two successive Landau levels is

$$E_{s, k_x} - E_{s+1, k_x} \approx -\hbar\omega_0 [1 - \gamma_s^\pm \cos(q x_0)].$$  \hspace{1cm} (B.5)

where $\gamma_s^\pm = (F^s_{s+1} - F^\mp_s)/\hbar\omega_0$. Substituting the above three equations (B.3)-(B.5) in the Hall conductivity expression, we get equation (37). A multiplication factor 2 needs to be used for the contribution coming from the $\xi^1 = \{ s - 1, \pm, k_x \}$ states.
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