MAXIMAL NON-COMMUTING SETS IN CERTAIN UNIPOTENT UPPER-TRIANGULAR LINEAR GROUPS

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ABSTRACT. We find the exact size of a maximal non-commuting set in unipotent upper triangular linear group $UU_4(F_q)$ in terms of a non-commuting geometric structure (Refer Definition 10), where $F_q$ is the finite field with $q$ elements. Then we get bounds on the size of such a set by explicitly finding certain non-commuting sets in the non-commuting structure.

1. INTRODUCTION

We start this section with a few definitions.

**Definition 1.** For any group $G$, we define a subset $N \subset G$ to be a non-commuting set if for any $x \neq y \in N$, $xy \neq yx$.

**Definition 2.** Let $G$ be a group. Let $S \subset G$ be any set. A set is said to be a maximal non-commuting subset of $S$ if it is not a proper subset of a bigger non-commuting subset of $S$ and also has maximum cardinality among all non-extendable non-commuting subsets of $S$. The cardinality of such a set is denoted by $\omega(S)$. This is also known as the clique number of the associated non-commuting subgraph of $S$ of the associated non-commuting graph of $G$.

The clique numbers for various families of groups have been studied by several authors such as R.Brown, A.Abdollahi, C.E.Praeger, A.Azad, H.Liu, Y.L.Wang, A.Y.M. Chin, J.Pakianathan and E.Yalcin, etc. There has been work on the clique number of the non-commuting graph of the symmetric group, see the two papers by R.Brown [7, 8]. On the other hand, there has been work on the commuting graph of finite groups by C.W. Parker, G.L. Morgan and G. Michael (See [15, 16, 19]).

Maximal non-commuting sets in finite groups arise in many contexts in the literature. Among earlier authors who have worked on $\omega(G)$ are B.H.Neumann [17], answering a question of P.Erdős, D.R.Mason [14], giving a bound on $\omega(G)$ by covering the group $G$ by $\left(\frac{|G|}{2} + 1\right)$ abelian groups and L. Pyber [20], relating $\omega(G)$ to the index of the center $Z(G)$ in $G$ as $[G : Z(G)] \leq c\omega(G)$ for some constant $c$.

1.1. Non-commuting sets in groups. In [2], it has been proved that

$$\omega(GL_2(F_q)) = q^2 + q + 1.$$  

The question of maximal non-commuting sets in $GL_3(F_q)$ have been studied by the authors A.Azad and C.E.Praeger using the concept of Singer generators and pseudo Singer generator elements and the exact value of $\omega(GL_3(F_q))$ has been found.

For higher dimension, the following theorem has been proved by the authors A.Azad, M.A.Iranmanesh and C.E.Praeger in [3].

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Theorem 3. Let $G = GL_n(F_q)$. Then for $q \geq 2$,
\[ q^{-n}(1 - q^{-3} - q^{-5} + q^{-6} - q^{-n}) < \frac{\omega(GL_n(F_q))}{|GL_n(F_q)|} \leq q^{-n}l(q), \]
where $l(q) = \prod_{k \geq 1} (1 - q^k)^{-\binom{k+1}{2}-1}$.

For $p \neq 2$, where $p$ is a prime the following has been proved in \[9\] by A.Y.M. Chin
\[ np + 1 \leq \omega(G) \leq \frac{p(p - 1)^n - 2}{p - 1}. \]

In \[13\], upper and lower bounds have been obtained by Y.L.Wang and H.Liu for the size of a maximal non-commuting set in generalized extra-special $p$-groups using an inductive procedure similar to what has been obtained by A.Y.M. Chin in her paper \[9\] for an extra-special $p$-group of order $p^{2n+1}$. For $p = 2$, $\omega(G) = 2n + 1$ has been proved by I.M.Isaacs for any extra-special $2$-group of order $2^{2n+1}$ (see \[4\, p. 40\]).

A homological criterion has been given in \[18\] by J.Pakianathan and E.Yalcin for the existence of maximal non-commuting sets in groups by using non-commuting and commuting simplicial complexes. First a structural result of the simplicial complex as wedge of a base complex and suspension spaces has been observed due to A.Björner et al in \[5\] and then especially in one of the cases of the non-commuting structure, where every centralizer is at least two, a homological criterion has been given.

Solving certain equations over finite fields lead to solutions whose count has polynomial expressions. In this article the authors are interested in the size of a maximal non-commuting set of a non-commuting structure and hence have reduced to a similar question about the non-commuting structure. In this regard about, being a polynomial, a similar result is mentioned just below.

In \[21\], the conjecture of G. Higman has been addressed which says that the number of conjugacy classes of elements in $UU_n(F_q)$ is a polynomial in $q$. An algorithm has been developed which proves that for $n \leq 13$ and the number of conjugacy classes is a polynomial with integer coefficients of degree $\frac{n(n+6)}{12}$ with the number of conjugacy classes, at least $q^{\frac{n(n+6)}{12}}$ for any positive integer $n$.

1.2. Main results and the structure of the paper.

We begin with a few definitions.

**Definition 4.** Let $F_q$ be the finite field with $q = p^r$ elements, where $p$ is a prime. We define for a positive integer $n$
\[ EE_n(F_q) = \{ g = [g_{ij}]_{n \times n} \in GL_n(F_q) \mid g_{ii} = 1, g_{ij} = 0 \text{ for } 1 \leq j < i \leq n \}. \]

**Definition 5.** Let $n > 0$ be a positive integer. Let $C \subset F_q^n$. Let $R$ be any symmetric relation on $C$ which apriori need not be reflexive. We say $x$ commutes with $y$ if $xRy$. A subset $A \subset C$ is said to be abelian or an abelian set if for all $x, y \in A$ with $x \neq y$ we have $xRy$. For the purpose of mentioning about reflexivity we refer Remark \[15\].

A set is said to be a maximal non-commuting subset of $C$ if it is not a proper subset of a bigger non-commuting subset of $C$ and also has maximum cardinality among all non-extendable non-commuting subsets of $C$. The cardinality of such a set is also denoted by $\omega(C)$. We define $Z_{F_q}(x) = \{ y \in F_q^n \mid xRy \}$ to be the centralizer of $x$. 


Let $S \subset \mathbb{F}_q^n$ for some $n > 0$ with symmetric relation or let $S \subset G$, where $(G, \ast)$ is a finite group with the binary operation $\ast$. Let

$$S = \bigsqcup_{i=1}^l C_i, \text{ where } C_i \subset S$$

be a partition of the set $S$. Now we define the following.

**Definition 7** (Abelian decomposition, non-commuting decomposition). We say that the partition (6) is an abelian decomposition or decomposition into abelian sets if each $C_i$ is an abelian set. We immediately observe that

$$0 \leq \omega(S) \leq l, \text{ an upper bound.}$$

We say that the partition (6) is a non-commuting decomposition if for every $i \neq j$ and for each $x \in C_i, y \in C_j$ we have $x \not\sim_R y$. We immediately observe that

$$\omega(S) = \sum_{i=1}^l \omega(C_i) \geq l, \text{ a lower bound.} \quad (8)$$

**Remark 9** (Decomposition into non-commuting sets). We say that the partition 6 is a decomposition into non-commuting sets (Refer Definition 1) if for some $i$, $\#(C_i) \geq 2$ and for each $x, y \in C_i$ with $x \neq y$ we have $x \sim_R y$. We immediately observe that

$$\omega(S) \geq \max_i \#(C_i), \text{ a lower bound.}$$

If in addition we have for every $i \neq j, x \in C_i, y \in C_j, x \sim_R y$ then

$$\omega(S) = \max_i \#(C_i).$$

Here we say a set $S$ is non-abelian if there is a decomposition into non-commuting sets.

To state the main results and for the structure of the paper we need the following two definitions of the non-commuting structures.

**Definition 10.** We define

$$\mathcal{M} = \{(x, y, z) \in \mathbb{F}_q \times \mathbb{F}_q^* \times \mathbb{F}_q\}$$

and with a commuting relation between $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathcal{M}$ given by

$$\text{Det} \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = z_1 - z_2.$$

We define

$$\mathcal{Q} = \{(x, y, z) \in \mathbb{F}_q \times \mathbb{F}_q \times \mathbb{F}_q\}$$

and with a commuting relation between $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathcal{Q}$ given by

$$\text{Det} \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = z_1 - z_2.$$

We note immediately that the relation given by commuting condition is reflexive as well as symmetric. Also refer Definition 14, Remark 15.

Now we are ready to the state the two main results of this article.

**Theorem A.** The size of a maximal non-commuting set in $UU_4(\mathbb{F}_q)$ is given by

$$\omega(UU_4(\mathbb{F}_q)) = q^3 + q + 1 + \omega(\mathcal{M}).$$

**Theorem B.** The following holds for the non-commuting structure $\mathcal{M}$.

1. There exists a decomposition of $\mathcal{M}$ into exactly $q(q - 1)$ disjoint abelian sets each of size $q$. 
(2) $2q \leq \omega(M) \leq q(q-1)$.

We study $\omega(UU_n(F_q))$ especially for $n = 4$, by using the method of “centralizer equivalence relation” on a set $S \subset UU_4(F_q) \setminus Z(UU_4(F_q))$ and deduce that $\omega(S) = \omega(X)$, where $X \subset S$ is a representative set under the equivalence relation. Indeed, we consider a non-commuting decomposition of the complement of $T_q$ (an abelian set) in $UU_4(F_q)$ (Refer Lemma 32) and determine the size of the maximal non-commuting sets in each part (See Sections 4 and 5). One of the parts of the partition corresponds to an extra-special $p$-group and it gives rise to a non-commuting structure $M$ (see, Definition 10, Equation 11). After that, using Theorem 62 we determine the non-commuting size of a subset $S_0 \subset UU_n(F_q)$, which contains all those elements of $UU_n(F_q)$ such that the product of all super-diagonal elements is non-zero. Using the equality in Equation 3 we find the value of $\omega(UU_4(F_q))$ in terms of $\omega(M)$. In fact we prove Theorem A.

In Section 2, we discuss the non-commuting structure $M$ and obtain a non-commuting set of size $2q$ in $M$. We analyze the centralizer of any element of $M$ under the commuting condition of any two elements of $M$. We prove a structure theorem for $M$ by classifying the non-commuting substructure of the centralizer of any element of $M$ and conclude that they are all isomorphic. Later, we use this structure theorem and the method of abelian decompositions to get lower and upper bounds for $\omega(M)$. We also prove that the method of abelian decompositions cannot be used to further improve the upper bound. In fact, we prove Theorem B.

In view of Theorem A and Theorem B, we have

$$q^3 + 3q + 1 \leq \omega(UU_4(F_q)) \leq q^3 + q^2 + 1.$$ 

For $q = 3$, by the above inequality, we get $\omega(UU_4(F_3)) = 37$.

In Sections 7 and 8 we discuss the non-commuting structure $Q$ and obtain a better upper and lower bound for $\omega(Q)$. This betterment plays a key role in the improvement of lower and upper bound for $\omega(UU_4(F_q))$. In Section 9 we consider the possibility of the existence of a non-commuting set which is a union of $m$-distinct lines except a bounded and $o(1)$-set (also refer Remark 53 and the initial part of the Section 9). Here we multi-represent the collection of such sets inside a suitable dimensional affine space over the algebraic closure of the finite field $F_p$ as an algebraic set and also as a quasi-affine algebraic set with a $GL_2(F_p)$ action in $6m$ dimensional affine space over $\overline{F_p}$. In the final Section 10 we ask relevant open questions based on this paper. The methods employed here in this article as we could gather from the survey are not used before.

2. Non-commuting sets in finite groups

Let $G$ be a finite group. In this section, we determine $\omega(S)$ for some $S \subset G \setminus Z(G)$ via a “centralizer relation.” We start with the following few definitions.

**Definition 13** (Centralizer relation). On an arbitrary nonempty subset $S$ of a finite group $G$ define a relation $\sim$ as follows. We say for $x, y \in S, x \sim y$ if $C_G(x) = C_G(y)$, where $C_G(x) = \{ z \in G \mid zx = xz \}$.

It is immediate that $\sim$ is an equivalence relation. Moreover, each equivalence class is an abelian set.

**Definition 14.** Let $T$ be a finite set with a symmetric relation “C”. For any $x, y \in T$ we say “$x$ commutes with $y$” if $xCy$ otherwise we say “$x$ does not commute with $y$” i.e. $x \sim Cy$. 
Let \( Z(T) \) denote the center of \( T \) i.e. \( Z(T) = \{ x \in T \mid xCy \text{ for every } y \in T \} \). Also, let \( Z_T(x) = \{ y \in T \mid xCy \} \). We define the centralizer equivalence relation \( \sim \) on \( T \) as \( x \sim y \) if \( Z_T(x) = Z_T(y) \). We say a set \( S \subset T \) is abelian if for every \( x, y \in S, x \neq y \) we have \( xCy \).

We say a set \( S \subset T \) is non-commuting if for every \( x, y \in S, x \neq y \) we have \( xCy \). A map of a finite set \( \phi : T \rightarrow T \) is a structure-map if \( x_1Cx_2 \Rightarrow \phi(x_1)C\phi(x_2) \) for all \( x_1, x_2 \in T \).

We say it is an isomorphism if in addition it is a bijection.

**Remark 15.** We remark that in definition 14, \( Z_T(x) \) need not contain \( x \). For example consider \( T = \mathbb{Z}_2 \setminus \{(0,0)\} \) with a commuting relation between \( (x_1, y_1), (x_2, y_2) \in T \) given by

\[
\text{Det} \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = \pm 1.
\]

We also remark that if the relation \( C \) is reflexive i.e. \( xCx \) for all \( x \in T \) and if \( x \sim y \), i.e., \( Z_T(x) = Z_T(y) \) then \( xCy \) and \( x, y \in Z_T(x) = Z_T(y) \).

In this article we have for the non-commuting structures \( M \) (Refer Section 6), \( Q \) (Refer Section 7) the commuting conditions are reflexive. Hence we assume that this relation \( \sim \) is “stronger” than relation \( C \) i.e. \( x \sim y \Rightarrow xCy \) for \( x \neq y \).

We also note that the Definition 13, Theorem 16 and Lemma 18 are also valid if we replace the group \( G \) by a finite set \( T \) with the symmetric relation “\( C \)” which need not apriori be reflexive but can be derived as follows for some elements of \( T \). If \( x \neq y, x, y \in T \) and \( Z_T(x) = Z_T(y) \) \( \Rightarrow \) \( x \sim y \Rightarrow xCy \Rightarrow x, y \in Z_T(x) = Z_T(y) \neq \emptyset \) is also non-empty and we have \( xCx \) and \( yCy \).

In the above example \( T \) if we include origin then \( Z_T((0,0)) = \emptyset \). So there is no \( 0 \neq v \in T \) such that \( Z_T((0,0)) = Z_T(v) \) as \( Z_T(v) \neq \emptyset \) if \( v \neq (0,0) \). However in this example reflexivity cannot be derived for any element in \( T \). Here \( Z_T(v) = Z_T(-v) \) but

\[
v \sim (-v) \iff vC(-v).
\]

**Theorem 16.** Let \( G \) be a finite group. Let \( S \subset G \) be an arbitrary nonempty subset of \( G \). Then \( \omega(S) \) is independent of the choice of the representative set i.e \( \omega(S) = \omega(X) \) for any representing set \( X \) of the equivalence classes \( S/\sim \), where the definition of the relation \( \sim \) is given in 13.

**Proof.** Let \( X = \{ x_i \mid i = 1, 2, \ldots, k \} \) and \( Y = \{ y_i \mid i = 1, 2, \ldots, k \} \) be two representing sets for the equivalence classes \( S/\sim = \{ [x_i] = [y_i] \mid i = 1, 2, \ldots, k \} \). Define a bijective map \( \phi : X \rightarrow Y \) such that \( \phi(x_i) = y_i \).

**Claim 17.** The bijection \( \phi \) preserves commutativity.

Suppose \( x_i \sim x_j \). Since \( C_G(x_i) = C_G(y_i), x_j \in C_G(y_i) \). Again, \( C_G(x_j) = C_G(y_j) \Rightarrow y_i \in C_G(y_j) \). Hence Claim 17 follows.

Using Claim 17 we have \( \omega(X) = \omega(Y) \). Now as each equivalence class is an abelian set, \( \omega(S) \leq |X| \). Further, we will show that \( \omega(S) = \omega(X) \). Let \( R \subset S \) be a maximal non-commuting set in \( S \). Then \( R \) is a subset of some representative set, say \( X \), for the equivalence relation \( \sim \). Hence \( |R| \leq \omega(X) \). So we get \( \omega(S) = \omega(X) \).

**Lemma 18.** Let \( G \) be a finite group. Let \( S \subset G \) be an arbitrary nonempty subset such that \( G \setminus S \) is abelian. Then

\[
\omega(G) - 1 \leq \omega(S) \leq \omega(G).
\]

**Proof.** Any maximal non-commuting set in \( G \) can contain at most one element outside \( S \). Hence the inequality follows.
3. Unipotent upper triangular groups

First we start with the matrix multiplication lemma.

**Lemma 19** (Matrix multiplication lemma). Let $A = (a_{ij})_{n \times n}, B = (b_{ij})_{n \times n}$ be two upper triangular unipotent matrices in $UU_n(\mathbb{K})$, where $\mathbb{K}$ is any field. Then $B$ commutes with $A$ if and only if for every $1 \leq i < j \leq n$ we have

$$\sum_{j > k > i} a_{ik} b_{kj} = \sum_{j > k > i} b_{ik} a_{kj} \iff \sum_{j > k > i} \det \begin{pmatrix} a_{ik} & b_{ik} \\ a_{kj} & b_{kj} \end{pmatrix} = 0.$$ 

**Proof.** Directly follows from matrix multiplication. $\square$

**Remark 20.** Here the non-commuting structure condition in the set $UU_n(\mathbb{K})$ is given by determinant sums arising out of matrix multiplication. About the structure we remark the following.

- For $n = 1$, the matrix multiplication is commutative.
- For $n > 1$, if the structure is given as follows. For some $1 \leq i < j \leq n$, $\sum_{j > k > i} \det \begin{pmatrix} a_{ik} & b_{ik} \\ a_{kj} & b_{kj} \end{pmatrix} = \pm 1 \neq 0$.

Then it is symmetric but is not reflexive.

3.1. An involutive anti-isomorphism $\Phi : UU_n(\mathbb{K}) \rightarrow UU_n(\mathbb{K})$.

**Definition 21.** Define a map $\Phi : UU_n(\mathbb{K}) \rightarrow UU_n(\mathbb{K})$ as follows.

For $A = (a_{ij})_{n \times n} \in UU_n(\mathbb{K}), \Phi(A) = \tilde{A} = (\tilde{a}_{ij})$, where $\tilde{a}_{ij} = a_{n-j+1,n-i+1}$.

By the definition of $\Phi$, we have the following lemma.

**Lemma 22.**

1. The map $\Phi$ is an anti-isomorphism. i.e. $\Phi(AB) = \Phi(B)\Phi(A)$.
2. Let $X, Y \subset UU_n(\mathbb{K})$ be two sets such that $\Phi(X) = Y$. Then $\omega(X) = \omega(Y)$.
3. Moreover the map $\Phi$ is an involution i.e. order 2 and moreover $\Phi(A) = wA^t w^{-1}$, where $w$ is the anti-diagonal permutation matrix corresponding to the permutation $(1, n)(2, n-1) \ldots$.

**Proof.** The proof is trivial. $\square$

3.2. Abelian centralizer. Here we compute the value of $\omega(S_0)$, where

$$S_0 = \{ A = (a_{ij}) \in UU_n(\mathbb{F}_q) \mid \prod_{i=1}^{n-1} a_{i,i+1} \neq 0 \}.$$ 

Before we state the following theorem, we mention that in the appendix section we give a proof that the centralizer of an element in $UU_n(\mathbb{K})$ is abelian whenever the super-diagonal entries are all non-zero and $\mathbb{K}$ is any field.

**Theorem 23.** $\omega(S_0) = (q - 1)^{n-2} q^{\binom{n-2}{2}}$.

**Proof.** In view of Theorem 62 we have the following.

- For every $x \in S_0$ we have that $C_G(x)$ is abelian.
- $|C_G(x)| = q^{n-1}$. 

• \(|C_G(x) \cap S_0| = q^{(n-2)(q - 1)}\) (in the proof of Lemma 63 the positions of the free variables are \((1j) : 1 < j \leq n\) and here we should have a non-zero value in the \((12)\) position).

Define a relation \(R_0\) on \(S_0\) as follows. We say \(y R_0 z\) if \(yz = zy\) for \(y, z \in S_0\). Since \(C_G(x)\) is abelian for any \(x \in S_0\), \(R_0\) is an equivalence relation, equivalence classes are abelian and all having same cardinality. The equivalence relation \(R_0\) gives a non-commutative decomposition of \(S_0\). Hence

\[
\omega(S_0) = \frac{|\{[x]_{R_0} \mid x \in S_0\}|}{|S_0|} = \frac{(q - 1)^{(n-1)}q^{\binom{n-1}{2}}}{(q - 1)q^{n-2}} = (q - 1)^{(n-2)}q^{\binom{n-2}{2}}.
\]

This completes the proof. 

\[\square\]

4. \(UU_4(\mathbb{F}_q)\)

In this section, we first divide the group \(UU_4(\mathbb{F}_q) \setminus Z(UU_4(\mathbb{F}_q))\) into various special sets and then determine the cardinality of a maximal non-commuting set in each set. At last we merge all these sets and determine the value of \(\omega(UU_4(\mathbb{F}_q))\).

4.1. Definitions of some special sets in \(UU_4(\mathbb{F}_q)\). We define the following sets in \(G_4 = UU_4(\mathbb{F}_q)\).

1. \(N_0 = \{A = (a_{ij})_{4 \times 4} \in G_4 \mid a_{12}a_{23}a_{34} \neq 0\}\),
2. \(N_1 = \{A = (a_{ij})_{4 \times 4} \in G_4 \mid a_{12}a_{23} \neq 0, a_{34} = 0\}\),
3. \(N_1^{anti} = \Phi(N_1) = \{A = (a_{ij})_{4 \times 4} \in G_4 \mid a_{23}a_{34} \neq 0, a_{12} = 0\}\),
4. \(N_2 = N_2^{anti} = \Phi(N_2) = \{A = (a_{ij})_{4 \times 4} \in G_4 \mid a_{12}a_{34} \neq 0, a_{23} = 0\}\),
5. \(N_3 = \{A = (a_{ij})_{4 \times 4} \in G_4 \mid a_{12} \neq 0, a_{23} = 0, a_{34} = 0\}\),
6. \(N_3^{anti} = \{A = (a_{ij})_{4 \times 4} \in G_4 \mid a_{34} \neq 0, a_{12} = 0, a_{23} = 0\}\),

where \(\Phi\) is given by Definition 21.

4.2. The abelian centralizer case \(N_0\).

Lemma 24. \(\omega(N_0) = q(q - 1)^2\).

Proof. Using Theorem 23 we specialize to the case \(n = 4\) to obtain \(\omega(N_0) = (q - 1)^{4-2}q^{\binom{4-2}{2}} = (q - 1)^2q\).  

\[\square\]

4.3. The sets \(N_1, N_1^{anti}, N_3\) and \(N_3^{anti}\).

Consider the set 
\[
T_1 = \{ \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & 1 & a_{23} & a_{24} \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G_4 \mid a_{34} = 0 \}.
\]
We observe that \( T_1 \) is a group with center

\[
Z(T_1) = \left\{ \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid * \in \mathbb{F}_q \right\}
\]

of size \( q^2 \). By the definition of \( N_1, N_3 \) and \( N_1^{anti}, N_3^{anti} \), we have \( N_1 \bigcup N_3 \subset T_1 \) and \( N_1^{anti} \bigcup N_3^{anti} \subset T_1^{anti} = \Phi(T_1) = \{ A = (a_{ij})_{4 \times 4} \in G_4 \mid a_{12} = 0 \} \), where \( \Phi \) is given by Definition 21.

**Observation 25.** For the sets \( N_1 \) and \( N_3 \), observe the following.

(a) For \( A = (a_{ij})_{4 \times 4} \in N_1, B = (b_{ij})_{4 \times 4} \in G_4 \) if \( AB = BA \), then

\[
b_{34} = 0, b_{24} = \lambda a_{24}, b_{i,i+1} = \lambda a_{i,i+1} \text{ for } i = 1, 2
\]

for some \( \lambda \in \mathbb{F}_q \).

(b) From Observation 25(a), we have for every \( s \in N_1 \subset T_1 \), \( C_{G_4}(s) = C_{T_1}(s) \) and \( |C_{G_4}(s)| = q^3 \). Since \( |Z(T_1)| = q^2 \), \( C_{T_1}(s) \) is an abelian centralizer for every \( s \in N_1 \) follows from a matrix computation.

(c) For \( A = (a_{ij})_{4 \times 4} \in N_3, B = (b_{ij})_{4 \times 4} \in G_4 \) if \( AB = BA \), then

\[
b_{23} = 0, b_{12} a_{24} = a_{12} b_{24} + a_{13} b_{34}.
\]

(d) From Observation 25(c), we have for every \( s \in N_3 \subset T_1 \), \( |C_{T_1}(s)| = q^3 \). Since \( |Z(T_1)| = q^2 \), \( C_{T_1}(s) \) is an abelian centralizer for every \( s \in N_3 \) follows from a matrix computation.

(e) If \( A = (a_{ij})_{4 \times 4} \in T_1 \setminus Z(T_1) \) such that \( a_{12} = 0 \), then

\[
C_{T_1}(A) = \left\{ \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid * \in \mathbb{F}_q \right\}
\]

is an abelian centralizer of size \( q^4 \).

**Definition 26.** A group \( G \) is called an AC-group, if the centralizer of every noncentral element of \( G \) is abelian.

**Lemma 27.** \( T_1 \) is an AC group and \( \omega(T_1) = q^2 + 1 \).

**Proof.** From Observation 25(a),(b),(c),(d) and the structure of the centralizers we get for \( s_1, s_2 \) such that \( s_1 s_2 \neq s_2 s_1 \) or \( C_{T_1}(s_1) \neq C_{T_1}(s_2) \) then \( C_{T_1}(s_1) \cap C_{T_1}(s_2) = Z(T_1) \) and that \( T_1 \) is an AC group. The commuting condition is an equivalence relation and coincides with the centralizer equivalence relation. Let \( X = \{ x_1, x_2, \ldots, x_k \} \) be a maximal non-commuting set in \( T_1 \). Without loss of generality we can assume that \( |C_{T_1}(x_1)| = q^4 \). Suppose for \( x_2 \), \( |C_{T_1}(x_2)| = q^4 \). Then \( q^4 = |T_1/Z(T_1)| = |T_1/(C_{T_1}(x_1) \cap C_{T_1}(x_2))| \leq |T_1/(C_{T_1}(x_1)||T_1/C_{T_1}(x_2))| = q^2 \), which is impossible. Hence \( x_1 \) is a unique element in \( X \) such that \( |C_{T_1}(x_1)| = q^4 \) and therefore, \( C_{T_1}(x_i) = q^3 \) for \( i = 2, \ldots, k \). Moreover we see that \( x_1 \in T_1 \setminus (N_1 \cup N_3) \) which is an abelian set. Now \( T_1 = \bigcup_{i=1}^{k} C_{T_1}(x_i) \). Therefore, we have

\[
|T_1| = \sum_{i=1}^{k} |(C_{G_4}(x_i) \setminus Z(T_1))| + |Z(T_1)|
\]

\[
q^5 = q^4 - q^2 + (k - 1)(q^3 - q^2) + q^2.
\]

This yields that \( k = q^2 + 1 \). \( \square \)
Remark 28. In view of the proof of Lemma [27] and Observation [28](a), all the elements in a maximal non-commuting set $X$ of $T_1$ belongs to $N_1 \cup N_3$ except $x_1$. Therefore, $\omega(N_1 \cup N_3) = q^2$ and so $\omega(N_1 \cup N_3) = q^2$. We observe that the sets $(N_1 \cup N_3)$ and $(N_1 \cup N_3)$ do not commute. Thus $\omega((N_1 \cup N_3) \cup (N_1 \cup N_3)) = 2q^2$ and $\omega(T_1 \cup T_1) = 2q^2 + 1$.

4.4. Set $N_2 = N_2^{anti}$. The set $N_2$ is given by

$$N_2 = \{ \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & 1 & a_{23} & a_{24} \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G_4 \mid a_{23} = 0, a_{12}a_{34} \neq 0 \}.$$

Observation 29. For the set $N_2$, observe the following.

(a) For $A = (a_{ij})_{4 \times 4} \in N_2, B = (b_{ij})_{4 \times 4} \in G_4$. If $AB = BA$, then

$$b_{23} = 0, a_{12}b_{24} + a_{13}b_{34} = b_{12}a_{24} + b_{13}a_{34}.$$  

(b) From Observation [29](a), we have for every $s \in N_2$, $C_{G_4}(s) = C_{T_2}(s)$ and $|C_{G_4}(s)| = q^4$, where

$$T_2 = \{ A = (a_{ij})_{4 \times 4} \in G_4 \mid a_{23} = 0 \}.$$  

Now, define a relation $R$ on $N_2$ as follows. We say $y \sim z$ if $C_{G_4}(y) = C_{G_4}(z)$ for $y, z \in N_2$. It is immediate that $R$ is an equivalence relation. Moreover all equivalence classes are abelian.

Lemma 30. The set

$$X_{N_2} = \{ \begin{pmatrix} 1 & 1 & x_{13} & 0 \\ 0 & 1 & 0 & x_{24} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid x_{34} \in F_q, x_{13}, x_{24} \in F_q \}$$

is a complete representative set for the equivalence classes under the relation $R$ on $N_2$ and size of $X_{N_2}$ is $q^2(q-1)$.

Proof. Let $A, C \in N_2$. From Observation [28](a), we get $C_{T_2}(A) = C_{T_2}(C)$ if and only if

$$\delta(a_{12}, a_{13}, -a_{24}, -a_{34}) = (c_{12}, c_{13}, -c_{24}, -c_{34})$$

for some $\delta \in F^*_q$. So $C$ is of the form

$$C = \begin{pmatrix} 1 & \delta a_{12} & \delta a_{13} & c_{14} \\ 0 & 1 & 0 & \delta a_{24} \\ 0 & 0 & 1 & \delta a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$  

By choosing a suitable $\delta$ in each equivalence class so that $c_{12} = 1$, the Lemma 30 follows. Since cardinality of each equivalence class is $q(q-1)$, the number of equivalence classes is $q^2(q-1)$.

Lemma 31. There exists a bijection $\psi$ between the equivalence classes of $N_2$ under $R$ and the set $M$ of ordered 3-tuples in $F_q \times F_q \times F_q$. Indeed each equivalence has a unique matrix representative in $X_{N_2}$ as in Lemma 30. The bijection $\psi$ sends the matrix

$$\begin{pmatrix} 1 & 1 & x_{13} & 0 \\ 0 & 1 & 0 & x_{24} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
to the 3-tuple \((x_{13}, x_{34}, x_{24}) \in \mathcal{M}\). Moreover, if \(x, y \in X_{N_2}\) representing equivalence classes commute if and only if \(\sigma(x), \sigma(y)\) satisfy the relation \(CC\).

**Proof.** The commutativity condition between \(x = (x_{ij})_{4 \times 4}\) and \(y = (y_{ij})_{4 \times 4}\) in \(X_{N_2}\) gives the following condition
\[y_{24} + x_{13}y_{34} = x_{24} + y_{13}x_{34}.
\]
Now, consider the bijection \(\psi : X_{N_2} \rightarrow \mathcal{M}\) given by
\[\psi(x_{13}, x_{34}, x_{24}) = (x, y, z).
\]
This bijection preserves commutativity. Hence, the Lemma follows. \(\square\)

5. **The size of a maximal non-commuting set in \(UU_4(\mathbb{F}_q)\)**

In this section, we determine \(\omega(UU_4(\mathbb{F}_q))\) in terms of the non-commuting structure \(\omega(\mathcal{M})\). In the coming sections, we give a lower bound of \(\omega(\mathcal{M})\). Here, we start with the following lemma.

**Lemma 32.** Let \(T_4 = \left\{ \begin{pmatrix} 1 & 0 & a_{13} & a_{14} \\ 0 & 1 & a_{23} & a_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid a_{13}, a_{14}, a_{23}, a_{24} \in \mathbb{F}_q \right\}\). Then \(\omega(UU_4(\mathbb{F}_q) \setminus T_4) = q^3 + q + \omega(\mathcal{M})\).

**Proof.** First we observe that \(T_4\) is an abelian subgroup of order \(q^4\). The following is a non-commuting decomposition of \(UU_4(\mathbb{F}_q) \setminus T_4\) given by
\[UU_4(\mathbb{F}_q) \setminus T_4 = N_0 \bigcup (N_1 \cup N_3) \bigcup (N_1^{anti} \cup N_3^{anti}) \bigcup N_2.
\]
Hence \(\omega(UU_4(\mathbb{F}_q) \setminus T_4) = (q - 1)^2 q + 2q^2 + \omega(\mathcal{M}) = q^3 + q + \omega(\mathcal{M})\). \(\square\)

Next, we prove the Theorem by using Lemma 18.

**Proof of Theorem** Any maximal non-commuting set in \(UU_4(\mathbb{F}_q)\) cannot contain more than one element from the set \(T_4\) which is defined in the Lemma. So \(\omega(UU_4(\mathbb{F}_q)) - 1 \leq \omega(UU_4(\mathbb{F}_q) \setminus T_4) \leq \omega(UU_4(\mathbb{F}_q))\). In the non-commuting subset that we have just produced in Remark for \(UU_4(\mathbb{F}_q)\), it contains one additional element coming from the set \(T_4\) other than \(\omega(UU_4(\mathbb{F}_q) \setminus T_4) = q^3 + q + \omega(\mathcal{M})\) non-commuting elements from the set \(UU_4(\mathbb{F}_q) \setminus T_4\). Hence the Theorem follows. \(\square\)

6. **Non-commuting structure \(\mathcal{M}\)**

In this section, we provide a lower and an upper bound of \(\omega(\mathcal{M})\). Let
\[C(x, y, z) = \{(x_1, y_1, z_1) \in \mathcal{M} \mid x_1 y - y_1 x = z_1 - z\}
\]
denote the centralizer of \((x, y, z) \in \mathcal{M}\).

**Lemma 33** (Disjoint decomposition into centralizers).
\[\mathcal{M} = \bigsqcup_{m \in \mathbb{F}_q} C(m, 1, 0)
\]
is a disjoint decomposition into centralizers each of size \(q(q - 1)\).

**Proof.** First we observe that \(C(m, 1, 0) = \{(my + z, y, z) \mid y \in \mathbb{F}_q, z \in \mathbb{F}_q\}\) and \(C(m_1, 1, 0) \cap C(m_2, 1, 0) = \emptyset\) for \(m_1 \neq m_2\). The size of the set \(C(m, 1, 0)\) is \(q(q - 1)\). \(\square\)
The set $C(m, 1, 0)$ can be considered as a $(q - 1) \times q$ matrix $C(my + z, y, z)$ with $(i, j)^{th}$ element entries are given by $(m * i + j, i, j)$, where $i \in F_q^*, j \in F_q$.

**Lemma 34.** For a fixed $m \in F_q$, $\omega(C(m, 1, 0)) = q + 1$.

**Proof.** Fix $y \neq 1 \in F_q^*$. Then, $\{(my + z, y, z) \mid z \in F_q\} \cup \{(m + 1, 1, 1)\}$ is a non-commuting set of $q + 1$ elements. This shows that

$$\omega(C(m, 1, 0)) \geq q + 1$$

i.e. bounded below by $q + 1$.

Next, we prove that $\omega(C(m, 1, 0)) \leq q + 1$. For this purpose, consider the decomposition into abelian sets given by

$$C(m, 1, 0) = \{(m + 1, 1, 1) \mid z \in F_q\} \bigcup \{(rm, r, 0) \mid r \in F_q^*\} \bigcup \{(rz + 1)m + z, (rz + 1), z) \mid - \frac{1}{r} \neq z \in F_q\}.$$ 

The above need not be a completely disjoint decomposition. However we observe that each decomposed part is abelian i.e.

(a) $\{(m + 1, 1, 1) \mid z \in F_q\}$ is abelian.

(b) $\{(rm, r, 0) \mid r \in F_q^*\}$ is abelian.

(c) For any $r \in F_q^*$, $\{(rz + 1)m + z, (rz + 1), z) \mid - \frac{1}{r} \neq z \in F_q\}$ is abelian.

Moreover $C(m, 1, 0)$ is the union of these abelian decomposed parts.

This completes the proof of the Lemma 34. \hfill $\Box$

In the next lemma, we provide a non-commuting set of size $2q$ in the non-commuting structure $M$.

**Lemma 35.** Let $q \neq 2$. For the non-commuting structure $M$, $\omega(M) \geq 2q$.

**Proof.** Fix $y \neq 1 \in F_q^*$. Then the set

$$\{(z, y, z) \mid z \in F_q\} \bigcup \{(m + 1, 1, 1) \mid m \neq y^{-1} - 1\} \bigcup \{(y^{-1} - 1, 1, 0)\}$$

is a non-commuting set of $2q$ elements. Hence we get $\omega(M) \geq 2q$. \hfill $\Box$

**Lemma 36.**

1. For a fixed $a, m, c$ with $am \neq 0$, the set $\{(mx + c, a, amx) \in M \mid x \in F_q\}$ is a non-extendable abelian set in $M$ and its size is $q$.

2. If $(x_1, y_1, z_1) \in M$, then $C(x_1, y_1, z_1) = \{(x, y, z) \in M \mid x_1y - xy_1 = z_1 - z\}$ is of size $q(q - 1)$.

3. If $x_1y_1 \neq 0$, then $\omega(C(x_1, y_1, z_1)) = q + 1$.

4. If $x_1 = 0$, then also we have $\omega(C(0, y_1, z_1)) = \omega(C(0, 1, 0)) = q + 1$.

5. For any $m \in F_q$, the non-commuting substructure $C(m, 1, 0) \subseteq M$ is isomorphic to the non-commuting structure $\mathcal{N}$, where

$$\text{Set } : \mathcal{N} = \{(x, y) \in F_q \times F_q^*\}$$

$$CC : \text{Det} \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = x_1 - x_2.$$

6. $\omega(\mathcal{N}) = q + 1$. 

(6) $\omega(\mathcal{N}) = q + 1$. 

Proof. Suppose \((\alpha, \beta, \gamma)\) commutes with every element of the set \(\{(mx + c, a, amx) \in \mathcal{M} \mid x \in F_q\}\). Then, we get \(\beta = a\) and \(\gamma = a(\alpha - c)\). Hence the Lemma \[36(1)\] follows.

The Lemma \[36(2)\] follows because \(y \neq 0 \neq y_1\) and \(z_1 = z + x_1y - xy_1\).

To prove the Lemma \[36(2)\], we observe the following. Consider the map from \(C(m, 1, 0) \rightarrow C(x_1, y_1, z_1)\) given by 
\[
(x, y, z) \mapsto \left(\frac{x}{y_1}, \frac{my}{x_1}, z_1 + z\right),
\]
where \(m = x_1y_1\).

- Since \(x_1y_1 \neq 0\), we have \((x, y, z) \in C(m, 1, 0)\) if and only if \(\left(\frac{x}{y_1}, \frac{my}{x_1}, z_1 + z\right) \in C(x_1, y_1, z_1)\).
- The map \((x, y, z) \mapsto \left(\frac{x}{y_1}, \frac{my}{x_1}, z_1 + z\right)\) is bijective from \(C(m, 1, 0) \rightarrow C(x_1, y_1, z_1)\).
- Since \(x_1y_1 = m \neq 0\), the map is preserving the commutative property.

Hence the Lemma \[36(3)\] follows.

To prove the Lemma \[36(4)\], we observe the following. Consider the map 
\[C(0, 1, 0) \rightarrow C(0, y_1, z_1)\]
by 
\[
(x, y, x) \mapsto (x, y_1y, xy_1 + z_1).
\]

- We observe that \((x, y, x) \in C(0, 1, 0)\) and \((x, y_1y, xy_1 + z_1) \in C(0, y_1, z_1)\).
- The map \((x, y, x) \mapsto (x, y_1y, xy_1 + z_1)\) is bijective from \(C(0, 1, 0) \rightarrow C(0, y_1, z_1)\).
- The map is a commuting preserving bijection.

Hence the Lemma \[36(4)\] follows.

To prove the Lemma \[36(5)\] we observe that for two elements \((my_1 + z_1, y_1, z_1), (my_2 + z_2, y_2, z_2) \in C(m, 1, 0)\) the commuting condition gives \(z_1y_2 - z_2y_1 = z_1 - z_2\). Consider the bijection 
\[C(m, 1, 0) \rightarrow \mathcal{N}\]
given by 
\[
(my + z, y, z) \mapsto (z, y).
\]
This is a commuting preserving bijection and so the Lemma \[36(5)\] follows.

Using the Lemma \[34\] the Lemma \[36(6)\] follows. \(\square\)

6.1. The geometry of centralizer sets in \(\mathcal{M}\) and the method of abelian decompositions for \(\mathcal{M}\).

The following Theorem \[37\] characterizes the non-commuting structure for the centralizer subsets. They all turn out to be isomorphic. This is a “sort of first structure theorem” for the non-commuting structure \(\mathcal{M}\).

**Theorem 37** (Geometry of centralizer sets in \(\mathcal{M}\)). For the non-commuting structure \(\mathcal{M}\), we have the following.

1. For any \((x, y, z) \in \mathcal{M}\) we have \(\omega(C(x, y, z)) = q + 1\).
2. For any \((x, y, z) \in \mathcal{M}\) the non-commuting substructure \(C(x, y, z) \subset \mathcal{M}\) is isomorphic to \(\mathcal{N}\).

**Proof.** It follows from Lemma \[36\]. \(\square\)

In the next theorem we determine an upper bound of a cover by abelian subsets of \(\mathcal{M}\).

**Theorem 38** (Commuting size for \(\mathcal{M}\)). For the non-commuting structure \(\mathcal{M}\), we have the following.

1. The cardinality of any abelian set in the non-commuting structure \(\mathcal{M}\) is at most \(q\).
2. There does not exist an abelian decomposition into fewer than \(q(q - 1)\) sets for the non-commuting structure \(\mathcal{M}\).
(3) The size of a maximal non-extendable abelian set in the non-commuting structure $\mathcal{M}$ is $q$.

Proof. Any abelian set is contained in a centralizer set. Using Theorem 37, we get that the geometry of the centralizer set is isomorphic to $\mathcal{N}$. Hence it is enough to prove that the cardinality of a maximal abelian set in $\mathcal{N}$ is bounded by $q$. Let us assume without loss generality that the abelian set is contained in $C(m, 1, 0)$ for some $m \in \mathbb{F}_q$. If there exists an element in the abelian set coming from a row other than the first row of the matrix $C(m^*i + j, i, j)$ then the set contains at most one element from each row and there are only $(q - 1)$ rows. Otherwise the set is completely contained in the first abelian row which has $q$ elements. However using the Lemma 36(1) we see that the first row is a non-extendable abelian set. Hence the Theorem 38(1) follows.

Let $\mathcal{M} = \bigcup_{i=1}^n A_i$ be any abelian decomposition into sets. Then by using Theorem 38(1) we have the following.

$$q^2(q - 1) = |\mathcal{M}| \leq \sum_{i=1}^n |A_i| \leq nq.$$  

$$\Rightarrow n \geq q(q - 1).$$

Hence the Theorem 38(2) follows.

Theorem 38(3) is a consequence of the Theorem 38(1).

6.2. Trivial upper and lower bounds for non-commuting sets in the non-commuting structure $\mathcal{M}$.

Proof of Theorem B. Using the Lemma 36(1), consider the decomposition into non-extendable abelian sets as follows.

$$\mathcal{M} = \bigcup_{a \in \mathbb{F}_q^*, c \in \mathbb{F}_q} \{(x + c, a, ax) \in \mathcal{M} | x \in \mathbb{F}_q\}.$$  

We immediately see that since $|\mathcal{M}| = q^2(q - 1)$ and there are $q(q - 1)$ abelian sets indexed by $a \in \mathbb{F}_q^*, c \in \mathbb{F}_q$ each of size $q$, this decomposition is disjoint. Hence

$$\mathcal{M} = \bigcup_{a \in \mathbb{F}_q^*, c \in \mathbb{F}_q} \{(x + c, a, ax) \in \mathcal{M} | x \in \mathbb{F}_q\}.$$  

We could also prove disjointness directly. Now the Theorem B follows by using Lemma 35.

Remark 39 (Method of abelian decompositions for $\mathcal{M}$). In the view of Theorem 38(2), it is clear that, in the case of non-commuting structure $\mathcal{M}$, the method of finding an upper bound for the cardinality of maximal non-commuting set, using abelian decompositions cannot be improved further from $q(q - 1)$.

Observation 40. 

- $\liminf_{q \to \infty} \frac{\omega(\mathcal{M})}{2q} \geq 1$.
- $\limsup_{q \to \infty} \frac{\omega(\mathcal{M})}{q^2} \leq 1$.
- For $q = 3$, $\omega(\mathcal{M}) = 2q = 6$.
- If $\omega(\mathcal{M})$ is a polynomial in $q$, then it is a linear polynomial with leading coefficient $\geq 2$ or a degree 2 polynomial with leading coefficient between 0 and 1.
In the following sections we analyze the non-commuting structure $\mathcal{M}$ via the non-commuting structure $\mathcal{Q}$ and find some geometrically interesting non-commuting subsets in the non-commuting structure $\mathcal{M}$ to improve the lower bound. It is a trivial observation that $\omega(\mathcal{M}) \leq \omega(\mathcal{Q})$.

7. Non-commuting structure $\mathcal{Q}$

First we consider the action of the group $GL_2(\mathbb{F}_q)$ on $\mathcal{Q}$ which does not exist on the non-commuting structure $\mathcal{M}$.

7.1. Action of the group $GL_2(\mathbb{F}_q)$ on $\mathcal{Q}$. The group $GL_2(\mathbb{F}_q)$ acts on the non-commuting structure as follows. Let $(x, y, z) \in \mathbb{F}_q^3$, $A \in GL_2(\mathbb{F}_q)$. Then the action is defined as

$$A.(x, y, z) = ((A(x, y))^t, \det(A)z).$$

This action preserves the commuting condition

$$CC : x_1y_2 - y_2x_1 = z_1 - z_2 \iff \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = z_1 - z_2.$$

We have

$$\det(A \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}) = \det(A)(z_1 - z_2).$$

We say a line $L$ is “commuting line” if for any two points $x \neq y$ on $L$, satisfy $xCCy$. On the other hand if for any two points $x \neq y$ on $L$, satisfy $xCCy$, then we say $L$ is a “non-commuting line”.

Lemma 41. Let $(x_0, y_0, z_0) \in \mathbb{F}_q^3\setminus\{0\}$, $(a, b, c) \in \mathbb{F}_q^3\setminus\{0\}$. In $\mathcal{Q}$, suppose the equations of a line $L$ passing through $(x_0, y_0, z_0)$ and parallel to the vector $<a, b, c>$ is given by

$$L : x = x_0 + at, y = y_0 + bt, z = z_0 + ct, t \in \mathbb{F}_q.$$

Then the line $L$ is a commuting line if and only if $\det \begin{pmatrix} a & b \\ x_0 & y_0 \end{pmatrix} = c$.

Proof. Let $t_1 \neq t_2$ be two elements in $\mathbb{F}_q$. Then

$$\det \begin{pmatrix} x_0 + at_1 & y_0 + bt_1 \\ x_0 + at_2 & y_0 + bt_2 \end{pmatrix} = (z_0 + ct_1) - (z_0 + ct_2) \iff \det \begin{pmatrix} a & b \\ x_0 & y_0 \end{pmatrix} = c.$$

This proves the lemma. \qed

Remark 42. The above lemma gives the fundamental observation about the non-commuting structure $\mathcal{Q}$ that each line is either a commuting line or a non-commuting line.

Lemma 43. For any finite characteristic of the field $\mathbb{F}_q$, there exists a non-extendable non-commuting set of size $2q$ in $\mathcal{Q}$ which is a union of two lines.

Proof. To produce $2q$-size non-commuting set, first consider a union of two non-commuting lines

$$L_1 : x = x_0 + at, y = y_0 + bt, z = z_0 + ct$$
$$L_2 : x = x_1 + at, y = y_1 + bt, z = z_1 + ct$$

i.e. $c \neq \det \begin{pmatrix} a & b \\ x_0 & y_0 \end{pmatrix}$, $\gamma \neq \det \begin{pmatrix} \alpha & \beta \\ x_1 & y_1 \end{pmatrix}$. The commuting condition between these two lines gives rise to the following equation.

$$\det \begin{pmatrix} x_0 & y_0 \\ x_1 & y_1 \end{pmatrix} + \det \begin{pmatrix} a & b \\ x_1 & y_1 \end{pmatrix} t_0 + \det \begin{pmatrix} x_0 & y_0 \\ a & b \end{pmatrix} t_1 + \det \begin{pmatrix} a & b \\ \alpha & \beta \end{pmatrix} t_0 t_1 = z_0 - z_1 + ct_0 - t_1.$$
where \( t_0, t_1 \in \mathbb{F}_q \). If for some \( \lambda \neq 0 \), \( \lambda(a, b) = (\alpha, \beta) \neq 0 \) and

\[
\det \begin{pmatrix} a & b \\ x_1 & y_1 \end{pmatrix} = c, \det \begin{pmatrix} x_0 & y_0 \\ \alpha & \beta \end{pmatrix} = -\gamma \text{ and } \det \begin{pmatrix} a & b \\ x_1 & y_1 \end{pmatrix} \neq \det \begin{pmatrix} a & b \\ x_0 & y_0 \end{pmatrix}
\]

then we get a union of two lines say for example

\[
L_1 : x = t, y = 1 + t, z = 2t \\
L_2 : x = t, y = 2 + t, z = 1 + t.
\]

as non-commuting set of size \( 2q \) which is non-extendable. This proves the lemma. \( \square \)

**Remark 44.**  (1) In the above observation, by choosing \( b = 0, \beta = 0 \) and \( y_0y_1 \neq 0 \) we get a non-commuting set of size \( 2q \) which is a union of two lines in the non-commuting structure \( \mathcal{M} \). For example consider for \( \text{char}(\mathbb{F}_q) \neq 2 \), the lines \( L_1 \) and \( L_2 \)

\[
L_1 : x = 1 + t, y = 2, z = t \\
L_2 : x = 2t, y = 1, z = 4t.
\]

(2) In Lemma 35 we have obtained a non-extendable non-commuting set of size \( 2q \). This set is not a union of two lines. Instead it is a union of a line, a line without a point and another point.

(3) Since we have two geometrically different examples of \( 2q \)-size non-commuting sets in \( \mathcal{Q} \), there is no finite group which acts on the non-commuting structure \( \mathcal{Q} \) preserving linear structure, preserving the commuting condition and acts transitively on the collection of non-extendable non-commuting sets of size \( 2q \) in \( \mathcal{Q} \).

7.2. **Lower bound for the non-commuting structure \( \mathcal{Q} \).** In this subsection, we improve the lower bound for \( \omega(\mathcal{Q}) \).

**Lemma 45.** For the non-commuting structure \( \mathcal{Q} \), there exists a union of 3-lines without a bounded \( o(1) \)- set (i.e of size at least \( 3q - 3 \)), which gives rise to a non-commuting set. Further, for \( q > 3 \), \( \text{char}(\mathbb{F}_q) \neq 2 \), there exists a non-commuting set of size \( 3q - 2 \) which is not contained in a union of three lines.

**Proof.** We again prove this statement by adjusting the determinants. First consider the union of three non-commuting lines

\[
L_1 : x = x_0 + at, y = y_0 + bt, z = z_0 + ct \\
L_2 : x = x_1 + at, y = y_1 + \beta t, z = z_1 + \gamma t \\
L_3 : x = x_2 + pt, y = y_2 + qt, z = z_2 + rt,
\]

where \( c \neq \det \begin{pmatrix} a & b \\ x_0 & y_0 \end{pmatrix} \), \( \gamma \neq \det \begin{pmatrix} \alpha & \beta \\ x_1 & y_1 \end{pmatrix} \), \( r \neq \begin{pmatrix} p & q \\ x_2 & y_2 \end{pmatrix} \). The commuting condition between pairs of lines among \( L_1, L_2, L_3 \) yields the following equations

\[
\det \begin{pmatrix} x_0 & y_0 \\ x_1 & y_1 \end{pmatrix} + \det \begin{pmatrix} a & b \\ x_1 & y_1 \end{pmatrix} t_0 + \det \begin{pmatrix} x_0 & y_0 \\ \alpha & \beta \end{pmatrix} t_1 + \det \begin{pmatrix} a & b \\ \alpha & \beta \end{pmatrix} t_0 t_1 = z_0 - z_1 + ct_0 - \gamma t_1 \\
\det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} + \det \begin{pmatrix} \alpha & \beta \\ x_2 & y_2 \end{pmatrix} t_1 + \det \begin{pmatrix} x_1 & y_1 \\ p & q \end{pmatrix} t_2 + \det \begin{pmatrix} \alpha & \beta \\ p & q \end{pmatrix} t_1 t_2 = z_1 - z_2 + \gamma t_1 - rt_2 \\
\det \begin{pmatrix} x_2 & y_2 \\ x_0 & y_0 \end{pmatrix} + \det \begin{pmatrix} p & q \\ x_0 & y_0 \end{pmatrix} t_2 + \det \begin{pmatrix} x_2 & y_2 \\ a & b \end{pmatrix} t_0 + \det \begin{pmatrix} p & q \\ a & b \end{pmatrix} t_2 t_0 = z_2 - z_0 + rt_2 - ct_0,
\]
where \( t_0, t_1, t_2 \in \mathbb{F}_q \). Suppose
\[
(a, b) = (\alpha, \beta) = (p, q) \neq 0,
\]
\[
det \begin{pmatrix} a & b \\ x_1 & y_1 \end{pmatrix} = c, \ det \begin{pmatrix} a & b \\ x_2 & y_2 \end{pmatrix} = \gamma, \ det \begin{pmatrix} a & b \\ x_0 & y_0 \end{pmatrix} = r \text{ and } c \neq r, \gamma \neq c, r \neq \gamma.
\]

Then the above equations become
\[
det \begin{pmatrix} x_0 & y_0 \\ x_1 & y_1 \end{pmatrix} + (\gamma - r) t_1 = z_0 - z_1,
\]
\[
det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} + (r - c) t_2 = z_1 - z_2,
\]
\[
det \begin{pmatrix} x_2 & y_2 \\ x_0 & y_0 \end{pmatrix} + (c - \gamma) t_0 = z_2 - z_0.
\]

Solve for \( t_0, t_1, t_2 \) and exclude those choices of \( t_i, i = 0, 1, 2 \) on the lines \( L_i, i = 0, 1, 2 \) respectively. This gives a non-commuting set of size \( 3q - 3 \).

For example the lines
\[
L_1 : x = 1 + t_0, y = 1 + t_0, z = t_0
\]
\[
L_2 : x = t_1, y = 1 + t_1, z = -t_1
\]
\[
L_3 : x = 1 + t_2, y = t_2, z = 0
\]
with \( t_1 \neq 1, t_2 \neq -1, t_0 \neq -\frac{1}{2} \)
give rise to a set containing \( 3q - 3 \) non-commuting points. Now consider the commuting line \((0, -1, 0) + s(1, 3, -1)\). From this line we add the point \((2, 5, -2)\) to the above set to get an extension into a bigger non-commuting set of size \( 3q - 2 \) for \( 3 \nmid q \). If \( 3 \mid q \neq 3 \), then we add an element \((s, -1, -s)\) with \( s \notin \mathbb{F}_3 \) to get a bigger non-commuting set. This set is not contained in union of three lines. This completes the proof of the lemma.

7.3. Criteria for the existence of non-commuting sets from the union of \( m \) distinct lines in \( \mathbb{Q} \).

**Lemma 46.** Consider a set which is a union of the following \( m \) distinct lines given by
\[
L_i : x = x_i + a_i t, y = y_i + b_i t, z = z_i + c_i t, i = 0, 1, 2, 3, \ldots, m - 1.
\]

Then, for large \( q \) this gives rise to a non-commutative set except possibly for a bounded \( o(1) \) - subset (also refer Remark [53] and the initial part of the Section [2]) if and only if for every \( 0 \leq i < j \leq m - 1 \) the equation
\[
\left( \det \begin{pmatrix} a_i & b_j \\ x_j & y_j \end{pmatrix} - c_j \right) \left( \det \begin{pmatrix} a_j & b_i \\ x_i & y_i \end{pmatrix} - c_j \right) = \left( z_i - z_j \right) - \det \begin{pmatrix} x_i & y_i \\ x_j & y_j \end{pmatrix} \det \begin{pmatrix} a_i & b_i \\ a_j & b_j \end{pmatrix}
\]
and
\[
\prod_{i=0}^{m-1} \left( c_i - \det \begin{pmatrix} a_i & b_i \\ x_i & y_i \end{pmatrix} \right) \neq 0
\]
holds.

In other words, existence of such a non-commuting set corresponds to an existence of a solution to a collection of equations and an inequation corresponding to the collection of the sets of \( m \) distinct lines.
We observe that in the affine plane \( F \), the equation \( \alpha \det - \lambda = 0 \) factorizes into two linear factors. One necessary and sufficient condition for reducibility is that there exist four such lines in four different horizontal planes.

Hence we will not get a non-commuting set of size \( \approx mq \).

Claim 47. For every \( 0 \leq i < j \leq (m - 1) \), the equation corresponding to the pair \( (i, j) \) factorizes into at most two linear factors.

Suppose not then there exists an equation involving \( i_0, j_0 \) among the above, where we can solve \( t_{j_0} \) in terms of \( t_{i_0} \) and therefore, we get a bijection between lines \( L_{i_0} \) and \( L_{j_0} \) and this bijection is such that, except for one point, it maps a point to another which commutes with it.

Hence we will not get a non-commuting set of size \( \approx mq \).

Claim 48. (1) For \( \delta \neq 0 \), the equation

\[
\alpha + \beta x + \gamma y + \delta xy = 0
\]

has \( 2q - 1 \) solutions over the finite field \( F_q \) if and only if \( \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = 0 \). Otherwise it has \( q - 1 \) solutions. Moreover if it has \( 2q - 1 \) solutions, then the LHS of the equation splits into a product of two linear factors.

(2) The following holds.

\[
\left( \det \begin{pmatrix} a_i & b_i \\ x_j & y_j \end{pmatrix} - c_i \right) \left( \det \begin{pmatrix} a_j & b_j \\ x_i & y_i \end{pmatrix} - c_j \right) = \\
\left( (z_i - z_j) - \det \begin{pmatrix} x_i & y_i \\ x_j & y_j \end{pmatrix} \right) \det \begin{pmatrix} a_i & b_i \\ a_j & b_j \end{pmatrix}.
\]

We observe that in the affine plane \( F_q^2 \), the equation \( xy = c \) has \( q - 1 \) solutions for \( c \neq 0 \) and the equation \( xy = 0 \) has \( 2q - 1 \) solutions. Similarly for \( \delta \neq 0 \), the number of solutions to the equation \( \alpha + \beta x + \gamma y + \delta xy = 0 \) is \( q - 1 \) or \( 2q - 1 \) depending on whether it remains irreducible or factorizes into two linear factors. One necessary and sufficient condition for reducibility into two linear factors is \( \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = 0 \) or equivalently there exists \( \lambda \in F_q \) such that \( (\alpha, \beta) = \lambda (\gamma, \delta) \). To complete the proof of the Claim(2) and Lemma 49 we observe that if \( \det \begin{pmatrix} a_i & b_i \\ a_j & b_j \end{pmatrix} = 0 \), then either \( \det \begin{pmatrix} a_i & b_i \\ x_j & y_j \end{pmatrix} = c_i \) or \( \det \begin{pmatrix} a_j & b_j \\ x_i & y_i \end{pmatrix} = c_j \) and hence, the size of the non-commutative set cannot be \( \approx mq \). This completes the proof of the lemma. \( \square \)

7.4. A non-commuting set of size almost \( 4q \) in \( Q \) for large \( q \) when \( -3 \) is a square.

Lemma 49. Suppose \( \text{char}(F_q) \neq 3 \). There exists non-commuting sets of size more than \( 4q - 12 \) in \( Q \) whenever \( -3 \) is a square in \( F_q \) (i.e. \( q = p^n, p = a^2 + ab + b^2 \) for some \( a, b \in \mathbb{Z} \) or equivalently \( p \equiv 1 \mod 3 \) or when \( q = p^n, n \) even).

Proof. First consider the non-commuting horizontal lines not meeting the \( z \)-axis. Let

\[
L_1 : x = x_1 + a_1 t, y = y_1 + b_1 t, z = z_1 \\
L_2 : x = x_2 + a_2 t, y = y_2 + b_2 t, z = z_2 \\
L_3 : x = x_3 + a_3 t, y = y_3 + b_3 t, z = z_3 \\
L_4 : x = x_4 + a_4 t, y = y_4 + b_4 t, z = z_4
\]

be four such lines in four different horizontal planes \( z = z_1, z = z_2, z = z_3, z = z_4 \).
The factorizing conditions in Lemma 49 among the lines reduces to the following. For 1 \leq i < j \leq 4,
\[ \det \begin{pmatrix} a_i & b_i \\ x_i & y_i \end{pmatrix} \det \begin{pmatrix} a_j & b_j \\ x_j & y_j \end{pmatrix} = (z_i - z_j) \det \begin{pmatrix} a_i & b_i \\ a_j & b_j \end{pmatrix}. \]
Choosing \( y_i = a_i = 1, i = 1, 2, 3, 4, x_1 = \frac{1}{b_1}, x_2 = \frac{1}{b_2}, x_3 = \frac{1}{b_3} \) and \( z_i = x_i \) we get three out of six equations, namely, for 1 \leq i < j \leq 3
\[ \det \begin{pmatrix} a_i & b_i \\ x_i & y_i \end{pmatrix} \det \begin{pmatrix} a_j & b_j \\ x_j & y_j \end{pmatrix} = (z_i - z_j) \det \begin{pmatrix} a_i & b_i \\ a_j & b_j \end{pmatrix} \]
are satisfied. The remaining three equations, we have
\[ (1 - b_4 x_4)(1 - b_i x_i) = (z_4 - x_i)(b_i - b_4) \text{ for } i = 1, 2, 3 \]
for the three unknowns \( b_4, x_4, z_4 \). Now we solve these unknowns.

In order to solve, first we eliminate \( x_4 \) to get
\[ \frac{(z_4 - x_1)(b_1 - b_4)}{1 - b_1 x_1} = \frac{(z_4 - x_2)(b_2 - b_4)}{1 - b_2 x_2} = \frac{(z_4 - x_3)(b_3 - b_4)}{1 - b_3 x_3}. \]
Substituting for \( x_i \) in terms of \( b_j \) and eliminating \( z_4 \) we get the following equation in \( b_4 \) i.e.
\[ \frac{(b_1 - b_2)(b_1 - b_3)}{b_4 - b_1} + \frac{(b_2 - b_1)(b_2 - b_3)}{b_4 - b_2} + \frac{(b_3 - b_1)(b_3 - b_2)}{b_4 - b_3} = 0. \]
This reduces to the following quadratic equation in \( b_4 \) i.e.
\[ (b_1^2 + b_2^2 + b_3^2 - b_1 b_2 - b_2 b_3 - b_3 b_1)b_4^2 - \left( \sum_{1 \leq i \neq j \leq 3} b_i^2 b_j \right) - (6b_1 b_2 b_3) b_4 = 0, \]
or equivalently
\[ \left( (b_2 - b_3)^2 + (b_3 - b_1)^2 + (b_1 - b_2)^2 \right)b_4^2 - 2 \left( b_1 (b_2 - b_3)^2 + b_2 (b_3 - b_1)^2 + b_3 (b_1 - b_2)^2 \right) b_4 \]
\[ + \left( b_1^2 (b_2 - b_3)^2 + b_2^2 (b_3 - b_1)^2 + b_3^2 (b_1 - b_2)^2 \right) = 0 \]
if the coefficient of \( b_4 \) does not vanish.

Solving for \( b_4 \) we get the following as two roots for \( b_4 \) i.e.
\[ b_4 = \frac{b_1 (b_2 - b_3)^2 + b_2 (b_3 - b_1)^2 + b_3 (b_1 - b_2)^2 \pm \sqrt{(b_1 - b_2)(b_2 - b_3)(b_3 - b_1) \sqrt{3}}}{(b_2 - b_3)^2 + (b_3 - b_1)^2 + (b_1 - b_2)^2}. \]

With the help of the value for \( b_4 \), we get \( z_4 \) by using either of the equations
\[ z_4 = \frac{\left( b_3 - b_4 \right)}{b_3 - b_4} \frac{\left( b_2 - b_4 \right)}{b_2 - b_4} \frac{\left( b_1 - b_4 \right)}{b_1 - b_4}, \]
provided the denominators do not vanish. Now \( x_4 \) is given by any of the following equations.
\[ x_4 = \frac{1}{b_4} \left( 1 - \frac{(z_4 - x_1)(b_1 - b_4)}{1 - b_1 x_1} \right) = \frac{1}{b_4} \left( 1 - \frac{(z_4 - x_2)(b_2 - b_4)}{1 - b_2 x_2} \right) = \frac{1}{b_4} \left( 1 - \frac{(z_4 - x_3)(b_3 - b_4)}{1 - b_3 x_3} \right). \]

Suppose \( q = p^n \), where \( n \) is even and \( p \equiv 1 \text{ mod } 3 \). Then consider \( b_1 \neq b_2 \neq b_3 \neq b_4 \) so that \( b_1, b_2, b_3 \in \mathbb{F}_p^* \) and hence give rise to a quadratic equation for \( b_4 \).

**Claim 50.** \( z_4 \in \mathbb{F}_p[\sqrt{-3}] \setminus \mathbb{F}_p \) and the horizontal plane \( z = z_4 \) is different from the other three horizontal planes \( z = z_1, z = z_2, z = z_3 \).
Proof of Claim. First we observe that since $-3$ is not a square in $\mathbb{F}_p$, we have that $b_4 \in \mathbb{F}_p[\sqrt{-3}] \setminus \mathbb{F}_p$. Now consider the following cases.

(1) $z_4 \in \mathbb{F}_p$.

(2) The denominator expression of $z_4$ vanishes.

In both cases we derive $b_i = b_j$ for some $1 \leq i \neq j \leq 3$ (by using linear independence of the basis $\{1, \sqrt{-3}\}$ of $\mathbb{F}_p[\sqrt{-3}]$ over $\mathbb{F}_p$).

Suppose $q = p > 7$, where $p \equiv 1 \mod 3$. Then consider $b_1 \neq b_2 \neq b_3 \neq b_1$ such that $b_1,b_2,b_3 \in \mathbb{F}_p^*$. This give rise to a quadratic equation for $b_4$ (Refer Claim 51). In this case if the denominator expressions for $z_4$ do not vanish, then we immediately conclude the following.

If $z_4 = \frac{1}{b_k}$ for some $1 \leq i \leq 3$, then $b_j = b_k$ for some $1 \leq j \neq k \leq 3$ to get a contradiction. So again we get a horizontal plane $z = z_4$ which is different from $z = z_1, z = z_2, z = z_3$.

Claim 51. There exists a choice of $b_1, b_2, b_3 \in \mathbb{F}_p^*$ such that $b_1 \neq b_2 \neq b_3 \neq b_1$ and $b_1^2 \neq b_j b_k \mod p$ for $\{i, j, k\} = \{1, 2, 3\}$ and $(b_1, b_2, b_3)$ does not satisfy the following equations

$$
\begin{align*}
&b_3(1 + \sqrt{-3}) + b_2(1 \mp \sqrt{-3}) = 2b_1 \\
&b_1(1 + \sqrt{-3}) + b_3(1 \pm \sqrt{-3}) = 2b_2 \\
&b_2(1 + \sqrt{-3}) + b_1(1 \mp \sqrt{-3}) = 2b_3.
\end{align*}
$$

This give rise to a quadratic equation for $b_4$ having two distinct roots other than $b_1, b_2, b_3$ and a choice of one of the roots for $b_4$ such that not all denominator expressions for $z_4$ vanish.

Proof of Claim. Suppose all the denominator expressions for $z_4$ vanish then we get

$$
\begin{align*}
b_3 \left( \frac{b_1 - b_4}{b_3 - b_1} \right) = b_2 \left( \frac{b_3 - b_4}{b_2 - b_3} \right) = b_1 \left( \frac{b_2 - b_4}{b_1 - b_2} \right).
\end{align*}
$$

Solving for $b_4$ we get

$$
b_4 = \left( \frac{b_3 b_4}{b_3 - b_1} - \frac{b_2 b_4}{b_2 - b_3} \right) = \left( \frac{b_2 b_4}{b_2 - b_3} - \frac{b_1 b_4}{b_1 - b_2} \right) = \left( \frac{b_4}{b_3 - b_1} - \frac{b_2}{b_2 - b_3} \right).
$$

The denominators above do not vanish because $b_1^2 \neq b_j b_k \mod p$ for $\{i, j, k\} = \{1, 2, 3\}$. Since we have two possible values for $b_4$ as the discriminant of the quadratic is non-zero because of distinctness of $b_i, i = 1, 2, 3$, we can choose the other value for $b_4$ and hence not all the denominators for $z_4$ vanish.

Since the union of the zero sets of the below equations in the variables $b_1, b_2, b_3$ is not the whole of $\mathbb{F}_p^3$, the distinct choices of $b_1, b_2, b_3$ is possible.

$$
\begin{align*}
b_3(1 + \sqrt{-3}) + b_2(1 \mp \sqrt{-3}) &= 2b_1 \\
b_1(1 + \sqrt{-3}) + b_3(1 \mp \sqrt{-3}) &= 2b_2 \\
b_2(1 + \sqrt{-3}) + b_1(1 \mp \sqrt{-3}) &= 2b_3.
\end{align*}
$$

These equations give rise to two planes passing through the origin in the three dimensional space consisting $(b_1, b_2, b_3)$.

The choice of $b_1, b_2, b_3$ is such that the point $(b_1, b_2, b_3)$ is not in any of the two planes and $b_i^2 \neq b_j b_k$ for all $\{i, j, k\} = \{1, 2, 3\}$. Such a choice can be made if $\mathbb{F}_p^3$ has more than 4 non-squares which it has because $q = p > 7$ as follows.
Choose \(b_1, b_2\) to be two distinct non-zero squares and \(b_3\) to be a non-square hence distinct and if necessary has to avoid four values

- The two square roots of \(b_1 b_2\).
- The two solutions for \(b_3\) given by the two plane equations.

Since \(z_4 \neq \frac{1}{b_i}, b_4 \neq b_i\) for any \(i = 1, 2, 3\) and \(1 - b_4 x_4 \neq 0\), the line \(L_4\) is a non-commuting line.

An example for \(q = p = 7\) is given as follows. Take \(b_1 = 1, b_2 = 2, b_3 = 3\). We get a solution for \(b_4\) as \(b_4 = 5, z_4 = 6\) and \(x_4 = 0\) so the non-commuting lines are given by

\[
L_1 : x = 5 + t, y = 1 + t, z = 5 \\
L_2 : x = 1 + t, y = 1 + 2t, z = 1 \\
L_3 : x = 4 + t, y = 1 + 3t, z = 4 \\
L_4 : x = t, y = 1 + 5t, z = 6.
\]

By substituting values for the variables we can construct non-commuting sets which are almost 4 lines lying in different horizontal planes and the size of the this non-commuting set is almost \(4q\) except possibly at most 12 points over various fields \(\mathbb{F}_q\) whenever \(-3\) is a square.

Hence the Lemma [49] follows.

### 8. Bounds for the sizes of the non-commuting sets for \(UU_4(\mathbb{F}_q)\)

Now, we are ready to give lower and upper bound for \(\omega(UU_4(\mathbb{F}_q))\). Summing up, we have the following lemma.

**Lemma 52.** Let \(G = UU_4(\mathbb{F}_q)\).

- Then there exists a constant \(K\) independent of \(q\) such that for large \(q = p^n\), where \(p\) is an odd prime, \(q^3 + 4q + 1 - K \leq \omega(G) \leq q^3 + q^2 + 1\).
- Suppose \(-3\) is a square in \(\mathbb{F}_q\) with \(\text{char}(\mathbb{F}_q) \neq 3\). Then there exists a constant \(K\) independent of \(q\) such that for large \(q = p^n\), where \(p\) is an odd prime, \(q^3 + 5q + 1 - K \leq \omega(G) \leq q^3 + q^2 + 1\).

**Proof.** This Lemma 52 follows from the lower bounds obtained for the non-commuting structure \(Q\) and restricting the non-commuting sets to the non-commuting structure \(M\) in Lemmas [45] and [49] and using the formula for the size of the non-commuting set of \(UU_4(\mathbb{F}_q)\) in Theorem A.

**Remark 53.** There are total \(q^4\) non-commuting lines and \((q+1)q^2\) commuting lines in the non-commuting structure \(Q\). Let \(m > 0\). In the presence of above discussion, it is natural to ask that “Is it possible to describe a non-commuting set which contains almost \(m\)-distinct lines as an algebraic set”. Here, we want an algebraic set in terms of equations. In fact, we seek solutions to these equations in the algebraic closure \(\overline{\mathbb{F}_p}\) of \(\mathbb{F}_p\) and then descend down to finite algebraic extension to produce a non-commuting set of size \(\approx mq\) for some large \(q\).

### 9. Description of non-commuting sets which contains almost \(m\)-lines as an algebraic set

We have observed that the non-commuting conditions of a geometrical set which is a union of almost \(m\)-distinct lines can be expressed in terms of equations and inequations. The following are the three conditions for the lines.

- The distinctness of \(m\)-lines condition.
- The non-commuting condition for the lines (An inequation).
The factorizability conditions given by the non-commutativity conditions between pairs of lines.

In this section, we show that the above three conditions give rise to an affine set in an affine space of suitable dimension and a quasi-affine set with a linear group action in another affine space of suitable dimension. Moreover the equations which give rise to the affine set/quasi-affine set are such that we can seek solutions to them over any finite field. Hence we can consider the ratio of the number of points in the algebraic set over the finite field $\mathbb{F}_q$ (in the variable $q$ which represents the cardinality of the field) with respect to an appropriate power of $q$ giving rise to the terminology of almost $m$-lines which means union of $m$-distinct lines except a bounded $o(1)$-set (asymptotic in $q$ and not in $m$).

Now we explain the geometric structure of non-commuting sets which contains almost $m$-lines. Indeed, in the following lemma we show that the collection of the non-commuting sets which is the union of almost $m$-lines forms an algebraic set.

**Lemma 54** (Affine algebraic set of non-commuting sets which is the union of almost $m$-lines). Let $m > 0$ be a positive integer and let $p$ be an odd prime. There exists an affine algebraic set $V_m[\mathbb{F}_p]$ defined by some equations in a finite set of variables, which corresponds to non-commuting sets containing almost $m$-lines.

**Proof.** In the view of the proof of Lemma 46 we describe a non-commuting set of size $mq - K$, where $K$ is independent of $q$. First we consider a set containing $m$-distinct non-commuting lines over finite field and then we exclude a finite set $P$ whose cardinality is independent of $q$ and hence obtain a set whose cardinality turns out be $mq - K$ for a certain $K$ where $K$ is bounded by $m(m-1) + m\binom{m}{2} \approx O(m^3)$ and hence bounded and independent of $q$. The set $P$ contains the following points.

- The points corresponding to the solutions $t_i$ in the factorizing conditions which are at most $m(m-1)$ number of points. (see Claim 47, Lemma 46).

We also take care of the following over-count from $mq$ when counting the cardinality of union of $m$-lines. This is done as follows.

- There are at most $\binom{m}{2}$ number of points appearing as intersection points of lines each of which are over-counted at most $m$-times.

So, the number $K$ is independent of $q$ which needs to be subtracted from $mq$ accounts for the above over count and also for the exclusion of points coming out of factorizability condition.

The conditions arising from the non-commuting set of size $mq - K$, where $K \ll O(m^3)$ is a positive integer independent of $q$ are given by

$$\left(\det\left(\begin{array}{cc} a_i & b_i \\ x_j & y_j \end{array}\right) - c_i\right)\left(\det\left(\begin{array}{cc} a_j & b_j \\ x_i & y_i \end{array}\right) - c_j\right) - \left(z_i - z_j\right)\det\left(\begin{array}{cc} x_i & y_i \\ x_j & y_j \end{array}\right)\det\left(\begin{array}{cc} a_i & b_i \\ a_j & b_j \end{array}\right) = 0,$$

where $0 \leq i < j \leq (m-1)$. It is obvious that just the above $\binom{m}{2}$ LHS expressions in the variables $a_i, b_i, x_i, y_i, c_i, z_i, i = 0, 1, \ldots, m - 1$, do not generate a unit ideal. By introducing a new variable $U$ we can rewrite the inequation corresponding to the non-commutativity of the lines condition as

$$\prod_{i=0}^{m-1} \left(c_i - \det\left(\begin{array}{cc} a_i & b_i \\ x_i & y_i \end{array}\right)\right)U = 1.$$

We have another open condition which is the distinctness of lines $L_0, L_1, \ldots, L_{m-1}$.
In order for the line \( L_i, L_j \) to be distinct we need to have that the following matrix has rank at least 3.

\[
\begin{pmatrix}
  x_i & y_i & z_i & 1 \\
  a_i + x_i & b_i + y_i & c_i + z_i & 1 \\
  x_j & y_j & z_j & 1 \\
a_j + x_j & b_j + y_j & c_j + z_j & 1
\end{pmatrix}
\]

However to seek a possibility of a non-commuting set consisting almost \( m \)-distinct lines for \( m < q \) actually, it is enough that the following \( 3 \times 4 \) matrix has full rank 3 for any \( 0 \leq i < j \leq (m-1) \).

\[
\begin{pmatrix}
  x_i & y_i & z_i & 1 \\
  a_i + x_i & b_i + y_i & c_i + z_i & 1 \\
  x_j & y_j & z_j & 1
\end{pmatrix}
\]

This ensures first of all that \((a_i, b_i, c_i) \neq 0\) and given lines \( L_i \) and \( L_j \) for \( i < j \) we get that the point \((x_j, y_j, z_j)\) does not lie on the line joining \((x_i, y_i, z_i)\) and \((a_i + x_i, b_i + y_i, c_i + z_i)\). Hence the lines are all distinct. Conversely given such a non-commuting set with \( m < q \), there exists, for \( 0 \leq i \leq m-1 \), a choice of \((x_i, y_i, z_i) \in \mathbb{F}_q^3\) and a choice of \((a_i, b_i, c_i) \in \mathbb{F}_q^3 \setminus \{(0, 0, 0)\}\) such that each of the above matrices have full rank 3. This choice of finding an \((x_i, y_i, z_i) \in L_i \setminus \bigcup_{j=1,j \neq i}^{m-1} L_j\) is possible because \( m < q \) and we have distinctness of \( L_i : 0, 1, 2, \ldots, (m-1)\)

Equivalently for \( 0 \leq i < j \leq (m-1) \) consider the variables \( u_{ij}, v_{ij}, w_{ij}, t_{ij} \) and the following equations must be satisfied.

\[
\begin{aligned}
(det & \begin{pmatrix}
  x_i & y_i & z_i \\
  a_i + x_i & b_i + y_i & c_i + z_i \\
  x_j & y_j & z_j
\end{pmatrix}) u_{ij} - 1)(det \begin{pmatrix}
  x_i & y_i & 1 \\
  a_i + x_i & b_i + y_i & 1 \\
  x_j & y_j & 1
\end{pmatrix}) v_{ij} - 1) \\
(det & \begin{pmatrix}
  x_i & z_i & 1 \\
  a_i + x_i & c_i + z_i & 1 \\
  x_j & z_j & 1
\end{pmatrix}) w_{ij} - 1)(det \begin{pmatrix}
  y_i & z_i & 1 \\
  b_i + y_i & c_i + z_i & 1 \\
  y_j & z_j & 1
\end{pmatrix}) t_{ij} - 1) = 0
\end{aligned}
\]

(55)

So the affine algebraic set \( V_m[\mathbb{F}_p] \) is given by these three sets of equations corresponding to

- Factorizing condition/configuration of union of \( m \) lines.
- Non-commuting line condition.
- Distinct lines condition.

in the variables \( x_i, y_i, z_i, a_i, b_i, c_i, U, u_{ij}, v_{ij}, w_{ij}, t_{ij} \).

This completes the proof of the Lemma. \( \square \)

**Remark 56.** Two different points in this affine set may represent the same non-commuting set which is a union of \( m \)-lines of almost type.

Now, consider the action of \( GL_2(\mathbb{F}_p) \) on \( \mathbb{F}_p^3 \setminus \{(0, 0, 0)\} \times \mathbb{F}_p^3 = \{(a, b, c, x, y, z) \mid a, b, c, x, y, z \in \mathbb{F}_p\} \) as follows. Let \( A \in GL_2(\mathbb{F}_p) \), then

\[
A(a, b, c, x, y, z) = ((A(a, b)^t)^t, det(A)c, (A(x, y)^t)^t, det(A)z)
\]

In the next lemma, we give a description of such non-commuting sets as a quasi affine algebraic set which is \( GL_2(\mathbb{F}_p) \) invariant.

**Lemma 57.** There exists a quasi affine algebraic set \( O_m[\mathbb{F}_p] \) corresponding to the non-commuting sets of union of \( m \)-lines of almost type on which the above action gives rise to an action of \( GL_2(\mathbb{F}_p) \) on \( O_m[\mathbb{F}_p] \).
In the final section of the paper we raise some interesting questions based on the previous sections.

**Question 58.** Is it possible to show that, given any \( m > 0 \) a positive integer, there exist a non-empty algebraic set/quasi-algebraic set corresponding to collection of non-commuting sets consisting almost \( m \)-distinct lines over \( \mathbb{F}_q \) for large \( q \)?

**Remark 59.** If the answer is in affirmative, then we can improve the lower bound for the size of the non-commuting set in the non-commuting structure \( \mathcal{Q} \) and hence also in \( UU_4(\mathbb{F}_q) \). For the non-commuting structure \( \mathcal{Q} \) the lower bound can be any degree one polynomial in \( q \), for large \( q \), if this phenomenon is true for any positive integer \( m \). If there are congruence conditions on the prime \( p \), where \( q = p^n \) for some \( n \) then the lower bound is applicable with the congruence conditions.

**Question 60.** Is \( \omega(UU_4(\mathbb{F}_q)) \) a polynomial in \( q \) like \( \omega(GL_3(\mathbb{F}_q)) \) for large \( q \)?

We observe that \( \omega(UU_4(\mathbb{F}_q)) \) is a polynomial in \( q \) if and only if \( \omega(\mathcal{M}) \) is a polynomial in \( q \). We have the following open question as well.

**Question 61.** Does there exist a maximal non-commuting set in \( \mathcal{M} \subset \mathbb{F}_q^3 \) which tend to have a “geometric structure” just like the union of almost \( m \)-distinct lines?

In the case of higher dimensional upper triangular unipotent matrix groups over finite fields, because of the existence more non-trivial non-commuting geometric structures than in \( UU_4(\mathbb{F}_q) \) what can be said about the polynomial nature of \( \omega(UU_n(\mathbb{F}_q)) \) for large \( q \)?

We can ask the following relevant questions about the polynomial nature of \( \omega(UU_n(\mathbb{F}_q)) \).

1. Is it the case \( \omega(UU_n(\mathbb{F}_q)) \) a single polynomial for large \( q \) in higher dimensions for \( n \geq 4 \)?
2. Is it the case that we can determine only the highest order of \( q \) in \( \omega(UU_n(\mathbb{F}_q)) \) by \( \omega(S_0) \), where

\[
S_0 = \begin{bmatrix}
1 & a_{12} & a_{13} & \cdots & a_{1,(n-1)} & a_{1n} \\
0 & 1 & a_{23} & \cdots & a_{2,(n-1)} & a_{2n} \\
0 & 0 & 1 & \cdots & a_{3,(n-1)} & a_{3n} \\
0 & 0 & 0 & \cdots & 1 & a_{(n-1),n} \\
0 & 0 & 0 & \cdots & 0 & 1
\end{bmatrix}, \prod_{i=1}^{n} a_{i,i+1} \neq 0}
\]

However lower orders in \( q \) arising out of non-commuting sets in other structures can only suffice to give bounds for large \( q \) but not being exactly the lower part of any polynomial which could possibly describe \( \omega(UU_n(\mathbb{F}_q)) \) as a polynomial. This would be an interesting phenomenon in the case of \( UU_4(\mathbb{F}_q) \) itself because of the non-commuting structure \( \mathcal{Q} \) or \( \mathcal{M} \).

3. Is it the case that \( \omega(UU_n(\mathbb{F}_p)) \) follows Higmann Porc Polynomial Phenomenon for large primes \( p \) depending on congruence classes mod \( N \) for some positive integer \( N > 0 \)? The Lemma 49 suggests this particular question.
In this section we look at $UU_n(\mathbb{F}_q)$ for any $n$ and classify certain abelian centralizers of elements in $UU_n(\mathbb{F}_q)$. Indeed we prove the following theorem for any field $\mathbb{K}$(Refer Remark 63).

**Theorem 62.** Let $G = UU_n(\mathbb{K})$, where $\mathbb{K}$ is any field. Let $x \in G$ be a unipotent upper triangular $n \times n$ matrix such that the product of the super-diagonal entries is non-zero. Then $C_G(x)$ is abelian.

**Lemma 63.** Let $A = (a_{ij})_{n \times n} \in UU_n(\mathbb{K})$ be such that none of the super-diagonal entries in $A$ are zero. Let $B = (b_{ij})_{n \times n} \in UU_n(\mathbb{K})$ commute with $A$ then the first row entries $b_{12}, b_{13}, \ldots, b_{1n}$ of $B$ determine the remaining entries of $B$. If $\mathbb{K} = \mathbb{F}_q$, then the cardinality of the group centralizer $C_{UU_n(\mathbb{F}_q)}(A)$ of $A$ is $q^{n-1}$.

**Proof.** From Lemma 19 it follows that $b_{12}$ determines $b_{i,i+1}$ for $2 \leq i \leq (n-1)$, $b_{13}$ determines $b_{i,i+2}$ for $2 \leq i \leq (n-2)$ and $b_{1,n-1}$ determines $b_{2n}$. $\square$

Now we are ready to prove Theorem 62.

**Proof of Theorem 62** Let

$$x = \begin{pmatrix}
1 & x_{12} & x_{13} & \cdots & x_{1,(n-1)} & x_{1n} \\
0 & 1 & x_{23} & \cdots & x_{2,(n-1)} & x_{2n} \\
0 & 0 & 1 & \cdots & x_{3,(n-1)} & x_{3n} \\
& & & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & x_{(n-1),n} \\
0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix} \in G$$

be a unipotent upper triangular $n \times n$ matrix such that $\prod_{i=1}^{n-1} x_{i,i+1} \neq 0$. In the case when $\mathbb{K} = \mathbb{F}_q$ a finite field, it follows from Lemma 63 the size of the conjugacy class $Cl_G(x)$ of $x$ is $\frac{|G|}{|C_G(x)|} = \frac{q(n)}{q^{n-1}} = q^{n-1}$. We observe the following.

- The super-diagonal entries do not change in any particular conjugacy class over any field $\mathbb{K}$.
- If $\mathbb{K} = \mathbb{F}_q$ the total number of matrices in $UU_n(\mathbb{F}_q)$ with the same super-diagonal entries as that of $x$ is $q^{n-1}$.

In the case of when $\mathbb{K} = \mathbb{F}_q$, by finiteness we can conclude that $x$ is conjugate to $\tilde{x} \in G$, where $\tilde{x}$ has the same super-diagonal entries as that of $x$ and rest of the upper triangular entries of $\tilde{x}$ are zero.

Even otherwise, for any field $\mathbb{K}$, we could actually let $u$ be a variable unipotent upper triangular matrix such that $xu = u\tilde{x}$ and solve a system of linear equations for $u$ using the fact that $\prod_{i=1}^{n-1} x_{i,i+1} \neq 0$.

Now it is enough to prove that $C_G(\tilde{x})$ is abelian. So without loss of generality, let us assume that in $x$ the upper triangular entries $x_{ij} = 0$ for $1 \leq i < i + 1 < j \leq n$ i.e. the non-super-diagonal positions are all zero and $\prod_{i=1}^{n-1} x_{i,i+1} \neq 0$. 

11. Appendix
Let 
\[
y = \begin{pmatrix}
1 & y_{12} & y_{13} & \cdots & y_{1,(n-1)} & y_{1n} \\
0 & 1 & y_{23} & \cdots & y_{2,(n-1)} & y_{2n} \\
0 & 0 & 1 & \cdots & y_{3,(n-1)} & y_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & y_{(n-1),n} \\
0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix} \in C_G(x).
\]

Then by a direct calculation 
\[
y_{ik} = y_{1,k-i+1} \left( \frac{x_{k-1,k}}{x_{i-1,i}} \right) \frac{x_{k-2,k-1}}{x_{i-2,i-1}} \cdots \frac{x_{k-i+2,k-i+3}}{x_{23}} \left( \frac{x_{k-i+1,k-i+2}}{x_{12}} \right)
\]
for \(1 < i < k \leq n\).

Let 
\[
z = \begin{pmatrix}
1 & z_{12} & z_{13} & \cdots & z_{1,(n-1)} & z_{1n} \\
0 & 1 & z_{23} & \cdots & z_{2,(n-1)} & z_{2n} \\
0 & 0 & 1 & \cdots & z_{3,(n-1)} & z_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & z_{(n-1),n} \\
0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix} \in C_G(x).
\]

Then by a direct calculation 
\[
z_{ik} = z_{1,k-i+1} \left( \frac{x_{k-1,k}}{x_{i-1,i}} \right) \frac{x_{k-2,k-1}}{x_{i-2,i-1}} \cdots \frac{x_{k-i+2,k-i+3}}{x_{23}} \left( \frac{x_{k-i+1,k-i+2}}{x_{12}} \right)
\]
for \(1 < i < k \leq n\).

**Claim 64.** \(zy=yz\) i.e. 
\[
\sum_{i<j<k} y_{ij}z_{jk} = \sum_{i<j<k} z_{ij}y_{jk}.
\]

Let \(j = i + t, k = i + r\) for some \(t > 0, r > 0\).

\[
y_{i,i+t}z_{i+t,i+r} = y_{1,t+1}z_{1,t+r+1}P_{i,(i+t),(i+r)}
\]
\[
z_{i,i+t}y_{i+t,i+r} = z_{1,t+1}y_{1,t+r+1}P_{i,(i+t),(i+r)},
\]
where 
\[
P_{i,(i+t),(i+r)} = \left( \frac{x_{i+t-1,i+t}}{x_{i-1,i}} \right) \frac{x_{i+t-2,i+t-1}}{x_{i-2,i-1}} \cdots \frac{x_{i+t+2,t+3}}{x_{23}} \left( \frac{x_{i+t+1,r+t+2}}{x_{12}} \right)
\]
\[
= \frac{x_{i+r-1,i+r}x_{i+r-2,i+r-1} \cdots x_{r-t+2,r-t+3}x_{r-t+1,r-t+2}}{x_{t,t+1}x_{t-1,t} \cdots x_{23}x_{12}}.
\]

Let \(j = i + r - t, k = i + r\) for some \(t > 0, r > 0, r > t\). Now

\[
y_{i,i+r-t}z_{i+r-t,i+r} = y_{1,r-t+1}z_{1,r+t+1}Q_{i,(i+r-t),(i+r)}
\]
\[
z_{i,i+r-t}y_{i+r-t,i+r} = z_{1,r-t+1}y_{1,r+t+1}Q_{i,(i+r-t),(i+r)},
\]
where 
\[
Q_{i,(i+r-t),(i+r)} = \left( \frac{x_{i+r-t-1,i+r-t}}{x_{i-1,i}} \right) \frac{x_{i+r-t-2,i+r-t-1}}{x_{i-2,i-1}} \cdots \frac{x_{i+r-t+2,r-t+3}}{x_{23}} \left( \frac{x_{i+r-t+1,r-t+2}}{x_{12}} \right)
\]
\[
= \frac{x_{i+r-t-1,i+r-t}x_{i+r-t-2,i+r-t-1} \cdots x_{i+r-t+2,r-t+3}x_{r-t+1,r-t+2}}{x_{r-t,t+1}x_{r-t-1,r-t} \cdots x_{23}x_{12}}.
\]
We observe that \( p_{i,(i+t),(i+t)} = q_{i,(i+r-t),(i+t)} \) in both the cases \( r - t \leq t \) and \( r - t \geq t \). Therefore we have for all \( 0 < t < r \)

\[
y_{i,i+t}z_{i,i+t,r} = z_{i,i+r-t}y_{i+r-t,i,t}
y_{i,i+r} = y_{i,i+r-t}z_{i,i+r-t,i,t}.
\]

So

\[
\sum_{i<j<k} y_{ij}z_{jk} = \sum_{i<j<k} z_{ij}y_{jk}
\]

and \( yz = zy \). Claim 63 follows.

Hence the centralizer \( C_G(x) \) is abelian and Theorem 62 follows. \( \square \)

**Remark 65.**

1. (Affineness of the abelian centralizer:) In Theorem 62 when we consider the field \( \mathbb{K} \) as an algebraically closed (actually this condition is not required see next Remark 65(2)), then the space \( C_G(x) \) is a closed algebraic set isomorphic to the affine space \( \mathbb{A}_\mathbb{K}^{n-1} \) under the isomorphism

\[
\phi : \mathbb{A}_\mathbb{K}^{n-1} \rightarrow C_G(x)
\]

\[
\phi(y_{ij} : 1 < j \leq n) := \begin{pmatrix}
1 & y_{12} & y_{13} & \cdots & y_{1,n-1} & y_{1n} \\
0 & 1 & y_{23} & \cdots & y_{2,n-1} & y_{2n} \\
0 & 0 & 1 & \cdots & y_{3,n-1} & y_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & y_{(n-1),n} \\
0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix}.
\]

Using Lemma 63 the other entries \( y_{ij} \) with \( 1 \neq i < j \) are determined in terms of \( y_{ij} : 1 < j < n \) and \( x_{i,i+1} : 1 < i < n \). So the map \( \phi \) is a polynomial isomorphism onto the closed subgroup \( C_G(x) \).

2. Let \( X \) be the set of matrices in \( G \) whose super-diagonal elements are the same as that of \( x \) and the remaining entries can be any elements from the field \( \mathbb{K} \). We note that the proof is general as it goes through over any field \( \mathbb{K} \), need not be finite, need not be algebraically closed and we have the following exact sequence of affine sets (all are isomorphic to affine spaces)

\[
0 \rightarrow \mathbb{A}_\mathbb{K}^{n-1} \rightarrow G \rightarrow X \rightarrow 0
\]

The maps are just polynomial maps. The first one is affine and becomes linear if we replace \( G \) by the set

\[
G - \text{Identity} = \{ g - I \mid g \in G, I \text{ is the identity matrix} \}
\]

and the map \( \phi \) by \( \phi - I \). The second has a linear expression for the super-diagonal and also remains fixed as that of \( x \). The sequence below is an exact sequence of linear maps.

\[
0 \rightarrow \mathbb{A}_\mathbb{K}^{n-1} \rightarrow G - \text{Identity} \rightarrow X - \tilde{x} \rightarrow 0,
\]

where \( \tilde{x} = (\tilde{x}_{ij})_{n \times n} \), where \( \tilde{x}_{ij} = x_{ij}, 1 \leq i + 1 \leq j \leq n \) and 0 otherwise.

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