Image processing as state reconstruction in optics

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(Dated: November 4, 2018)

The image reconstruction of partially coherent light is interpreted as the quantum state reconstruction. The efficient method based on maximum-likelihood estimation is proposed to acquire information from registered intensity measurements affected by noise. The connection with totally incoherent image restoration is pointed out. The feasibility of the method is demonstrated numerically. Spatial and correlation details significantly smaller than the diffraction limit are revealed in the reconstructed pattern.

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OCIS codes: 030.1640, 100.3010, 100.5070, 100.6640, 100.6950, 110.4980.

1. Introduction

Light conveys considerable part of information on the surrounding world. Information coded into and transmitted by means of light plays the key role in contemporary information technology. That is why any deeper understanding of fundamental limitations imposed by the theory represents a challenging problem. The origin of the diffraction limit restricting the spatial resolution is comprehended since the era of wave optics. This phenomenon, manifested for example by a fuzzy diffraction spot as a consequence of the finite aperture, yields a serious limitation in image processing. But there are still other physical restrictions which must be taken into account. Direct observations are not able to determine the phase of optical fields due to the fast oscillations and due to the effect of the time integration of intensity detectors. The phase must be therefore retrieved adopting sophisticated techniques. This is known phase problem, solution of which is sensitive to noise and requires various regularization treatments.

There are several ways to surpass the mentioned limitations imposed by realistic aspects of optical observations. Sophisticated algorithms of data processing and image reconstruction have been devised, such as analytic continuation of signal and diagonalization of optical device, different regularization and smoothing techniques of direct deconvolution and methods of projections onto convex sets (POCS), solving of the transport-of-intensity equation for phase reconstruction, utilizing of canonical transforms, and statistical methods based on minimum least-squares distance, maximum entropy, maximum Cramer-Rao bound, and maximum likelihood principles. The tomographical synthesis of different intensity observations of an object can considerably improve its reconstruction and allow the phase retrieval. The resolution can also be enhanced by eliminating of out-of-focus light by means of the apodisation technique or by utilizing of confocal arrangement and interaction between light and matter, like in multi-photon fluorescence microscopy and STED technique.

In this article the problem of image processing will be addressed from the viewpoint of statistical reconstruction techniques. The series of tomographical-like intensity measurements will be used up for the reconstruction of a state of partially coherent light. In the particular case of totally incoherent light the proposed approach will be identified with the Richardson algorithm of the image reconstruction.

There is a tight connection between fundamental principles of wave optics and quantum mechanics. Description of the scalar wave in optics is equivalent to description of the de Broglie wave of a mass particle in the framework of quantum mechanics. The pure quantum state in position representation coincides with the complex scalar wave, and similarly, the mixed quantum state in this representation corresponds to the correlation function of the second order. This connection will be systematically exploited and the problem of quantum state reconstruction will be considered in analogy with optical counterpart of image processing.

For the sake of simplicity all the considered problems will be treated as two dimensional problems. The first dimension corresponds to the evolution parameter—time in dynamical problems or longitudinal z-coordinate in the case of image processing. The second dimension corresponds to the observed quantity. This could be position in the former case of dynamical problems or transverse x-coordinate in the later case of image processing. Further generalization to higher dimension can be obtained by a straightforward manner.

2. Wave theory

In this section the analogy between scalar wave optics and quantum mechanics will be highlighted. As will be shown, abstract quantum formulation is advantageous for the purpose of signal reconstruction. In quantum domain, pure quantum state \( |\psi\rangle \) from the Hilbert space represents the complete knowledge about the position and momentum of a particle, of course obey-
Probabilities \( \lambda_k \) are nonnegative and adds to unity, and the \textit{mixed state} \( \hat{\rho} \) satisfies the following relations,

\[
\hat{\rho}^\dagger = \hat{\rho}, \quad \text{Tr}[\hat{\rho}] = 1, \quad \langle \psi|\hat{\rho}|\psi \rangle \geq 0, \quad \forall |\psi\rangle,
\]

where \( \dagger \) means the Hermitian conjugation. Denoting formally the position by a projector \( |x\rangle \langle x| \), complex amplitude \( \psi(x) = \langle x|\psi \rangle \) describes the coherent quasi-monochromatic scalar field. Indeed, it characterizes the amplitude as well as the phase of the propagating wave. In the general case of partially coherent field, the second order correlation function

\[
\Gamma(x,x') = \text{Tr}[\hat{\rho}|x\rangle \langle x'|x\rangle \langle x'|] = \sum_k \lambda_k \langle x|\varphi_k \rangle \langle \varphi_k |x'\rangle = \sum_k \lambda_k \varphi_k(x) \varphi^*_k(x') = \langle \psi^*(x')\psi(x) \rangle_{\text{ens}},
\]

describes the statistical properties of the field. The brackets \( \langle \ldots \rangle_{\text{ens}} \) denote the averaging over complex amplitudes of all modes \( k \). The analogy between the density matrix \( \hat{\rho} \) and the \textit{mutual intensity} \( \Gamma \) is expressed clearly by the relations analogous to \( (2) \).

\[
\Gamma^s(x',x) = \Gamma(x,x'), \quad \int dx I(x) = 1, \quad \Gamma(x,x) \geq 0. \quad (4)
\]

Notice that the function \( I(x) = \Gamma(x,x) \) means the optical intensity of field. The analogy between quantum and wave descriptions can be emphasized in phase space by means of the Wigner quasi-distribution \( \hat{W} \),

\[
W(x,p) = \frac{1}{\pi} \int dx' e^{-i2px'} \Gamma(x + x', x - x'). \quad (5)
\]

This \( (x,p) \) distribution is real bounded function, which is however, not positively defined in general.

Let us proceed further to consider the state transformation. Assuming linearity and causality the equation for evolution of pure state formally reads

\[
|\psi\rangle_{\text{out}} = \hat{T}|\psi\rangle_{\text{in}}.
\]

Here \( \hat{T} \) is linear operator satisfying the equation

\[
\frac{\partial}{\partial z} \hat{T} = \hat{L}\hat{T},
\]

where \( z \) is evolution parameter and the generator \( \hat{L} \) of evolution is considered to be \( z \)-independent. The evolution equation \( (7) \) covers both the Schrödinger equation in quantum mechanics and paraxial Helmholtz equation in Fresnel approximation of scalar wave optics. The \textit{unitary evolution} is governed by the Hamiltonian operator, \( \hat{L} = -\frac{i}{\hbar} \hat{H} \), and the evolution of the mixed state is described by transformation

\[
\hat{\rho}_{\text{out}} = \hat{T}\hat{\rho}_{\text{in}}\hat{T}^+, \quad \hat{T} = \exp \left[ -\frac{i}{\hbar} \hat{H} z \right]. \quad (8)
\]

Evolution \( (\hat{T}) \) of the state \( \hat{\rho} \) in the Schrödinger picture can be equivalently formulated in the Heisenberg picture. This formulation follows the laws of ray optics. Indeed, the relationship between the canonical observables of position and momentum reads

\[
\begin{pmatrix} \hat{x}_{\text{out}} \\ \hat{p}_{\text{out}} \end{pmatrix} = \hat{T}\begin{pmatrix} \hat{x}_{\text{in}} \\ \hat{p}_{\text{in}} \end{pmatrix}, \quad \hat{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \hat{x}_{\text{in}} + s \\ \hat{p}_{\text{in}} + r \end{pmatrix}, \quad (9)
\]

where \( \text{Det}[\hat{T}] = AD - BC = 1 \). This generic \( ABCD \) transformation covers useful cases of wave transformation, for example free evolution, displacement, rotation, phase shift, squeezing, and chirp. Linear transformation of the \( (\hat{x}, \hat{p}) \) operators induces the evolution of the Wigner function by linear transformation of its variables,

\[
W(x,p) = W(Dx - Bp - s, -Cx + Ap - r). \quad (11)
\]

Roughly speaking, all these transformations rotate and rescale the input state. As the consequence, the observable \( A\hat{x} + B\hat{p} \) can be measured offering an important tool for all the tomographical methods.

In the \textit{classical limit} there is a tight connection between evolution of position and momentum operators in the Heisenberg picture \( \hat{T} \), and geometrical paraxial optics represented by the identity between operators \( \hat{x}, \hat{p} \) and its c-values \( (x, p) \). The state vector \( (x, p) \) is used to specify position and angle of the optical ray. Similar description may be adopted for particle in classical mechanics. Evolution operator \( \hat{T} \) is given by the \( ABCD \) matrix \( T \) completed by the transverse shift \( s \) and rotation \( r \) in analogy with the relation \( (10) \). Geometrical optics as well as classical mechanics do not involve interference, what simplifies the input-output relations considerably. This is why the geometrical optics is so suitable for “direct” observations. Indeed, if the positions \( x_1, x_2 \) for the two values \( z_1, z_2 \) are measured, the state vector \( (x_0, p_0) \) for \( z = 0 \) can be completely reconstructed as

\[
\begin{pmatrix} x_0 \\ p_0 \end{pmatrix} = \frac{1}{z_2 - z_1} \begin{pmatrix} x_1 z_2 - x_2 z_1 \\ x_2 - x_1 \end{pmatrix}. \quad (12)
\]

In the case of losses the evolution turns out to be \textit{non-unitary}. Let us imagine the absorbing screen with 2n aperture. The incident state can be decomposed in the
base of eigenstates $|\xi\rangle$ of position $\hat{x}$ on screen. Since only a part of eigenstates spectrum for eigenvalues $\xi \in [-a, a]$ is transmitted, the non-unitary transformation can be described by projection operator

$$\hat{T} = \int_{-a}^{a} d\xi \langle \xi | \xi \rangle$$  \hspace{1cm} (13)$$
corresponding to finite aperture.

Let us conclude the overview by explicit formulation of the state transformation in position representation. The generic evolution (10) of the pure state (coherent wave) takes a well-known form of the superposition integral,

$$\psi_{\text{out}}(x) = \langle x | \psi \rangle_{\text{out}} = \int dx_0 \langle x | \hat{T} | x_0 \rangle \langle x_0 | \psi \rangle_{\text{in}} = \int dx_0 h(x, x_0) \psi_{\text{in}}(x_0).$$  \hspace{1cm} (14)$$
The kernel of the integral transformation (14),

$$h(x, x_0) = \langle x | \hat{T} | x_0 \rangle,$$  \hspace{1cm} (15)$$
is called propagator in the quantum theory and the response function or point-spread function (PSF) in the scalar wave theory. Loosely speaking, it relates the point source in the object (input) plane, $z = 0$, with its image in the image (output) plane with coordinate $x$. This mapping is fuzzy in realistic image processing due to the effect of diffraction caused by non-unitary evolution. In the case of unlimited aperture it may become sharp corresponding to the case of unitary evolution, $\hat{T}^+ = \hat{T}^{-1}$. Analogously, the evolution (9) of the mixed state (mutual intensity) in the position representation reads

$$\Gamma_{\text{out}}(x, x') = \int dq dq' h(x, q) h^*(x', q') \Gamma_{\text{in}}(q, q').$$  \hspace{1cm} (16)$$

3. Detection

According to standard formulation of quantum mechanics the measurement is represented by an observable, a Hermitian operator $\hat{A}$. Eigenvalues of this operator correspond to possible results of elementary measurements. Eigenstates determine the possible states after the measurement and they are complete and orthogonal,

$$\hat{A}|a\rangle = a|a\rangle, \sum_a |a\rangle \langle a | = \hat{1}, \langle a | a' \rangle = \delta_{aa'}.$$  \hspace{1cm} (17)$$
These properties are reflected in the probability $p_a = \text{Tr}[\hat{\rho} |a\rangle \langle a|]$, predicted by quantum theory guaranteeing the normalization of probabilities $\sum_a p_a = 1$ (completeness), and mutual exclusivity of the results $a$ (orthogonality). This description may be further generalized in terms of positive-operator valued measure (POVM) yielding the decomposition of identity operator $\hat{1}$,

$$\hat{\Pi}_b \geq 0, \sum_b \hat{\Pi}_b = \hat{1}.$$  \hspace{1cm} (18)$$

It predicts the probability for registering the output $b$ analogously to the case of orthogonal projectors, $p_b = \text{Tr}[\hat{\rho} \hat{\Pi}_b]$. The notion of POVM plays the crucial role in description a generic quantum measurement in state estimation and discrimination.

Registration of the image intensity $I(x)$ of partially coherent wave in the transverse position $x$ corresponds to the measurement of position operator $\hat{x}$ in the output plane,

$$I(x) = \Gamma_{\text{out}}(x, x) = p(x) = \text{Tr}[\hat{\rho}_{\text{out}} |x\rangle \langle x|].$$  \hspace{1cm} (19)$$

Realistic detector always possesses the finite spatial resolving power. Denoting its pixels by the indices $i$, the simplest representation of detector POVM is given by the operators

$$\hat{O}_i = \int_{\Delta_i} dx |x\rangle \langle x|,$$  \hspace{1cm} (20)$$
where the integration is done along the surface of the $i$-th pixel.

Consider now the generic observation scheme. The input state $\hat{\rho}$ represented by its mutual intensity $\Gamma(x, x')$ in wave description is transformed by optical device $\hat{T} = \hat{T}(A, B, \ldots)$ with the response function $h(x, x_0; A, B, \ldots)$. Resulting output state $\hat{\rho}_{\text{out}}$ is observed by the detector (20) placed in the output plane. The detector counts the elementary clicks in every $i$-th pixel. The numbers $N_i$ of registered clicks represented by relative frequencies $f_i = N_i/N$, $N = \sum_i N_i$, sample the probabilities $p_i$ (intensities $I_i$),

$$p_i = \text{Tr}[\hat{\rho}_{\text{out}} \hat{O}_i] = \text{Tr}[\hat{\rho} \hat{\Pi}_i], \hat{\Pi}_i = \hat{T}^+ \hat{O}_i \hat{T}. \hspace{1cm} (21)$$

Notice however, that this scheme is rather classical and it does not take into account statistics of detection process in accordance with classical image processing, when intensity is considered as measurable quantity. Provided that quantum nature of detection will be considered, the relation (21) should be modified taking into account registration of photons instead.

4. Direct reconstruction

The reconstruction of the signal in wave theory is rather involved and extensive field with many applications. Let us review briefly this topic. Assuming an unknown signal propagating through optical refractive and diffractive elements, the output field may be detected. Provided that properties of the optical device are known, and detection is ideal, the input signal may be predicted from the output one. This is the classical inverse problem of wave optics.

Standard methods use the isoplanatic approximation involving the relation (13) as convolution,

$$\psi_{\text{out}} = \int dx_0 h(x - x_0) \psi_{\text{in}}(x_0) = h * \psi_{\text{in}}.$$  \hspace{1cm} (22)$$
Inversion is given by the Fourier deconvolution
\[
\tilde{\psi}_{\text{in}} = \frac{\tilde{\psi}_{\text{out}} + \tilde{N}}{h}.
\] (23)

Here \(\tilde{\psi}_{\text{in}}, \tilde{\psi}_{\text{out}},\) and \(h\) are Fourier transformations of \(\psi_{\text{in}}, \psi_{\text{out}},\) and \(h\), respectively, and \(\tilde{N}\) represents the spectrum of additive noise \(N\). Typical point-spread function \(h\) has the form of Sinc or BeSinc function and \(h\) corresponds to step-function. Hence, the spatial frequencies of the signal are transmitted only up to certain upper cut-off\(^{44}\). This is why the deconvolution is very sensitive to noise \(N\) and diverges at spatial frequencies for which the transfer function \(h\) turns to be zero. In particular, for frequencies above the cut-off, the transfer function \(h\) vanishes and (23) diverges due to the broad noise spectrum \(\tilde{N}\). Some regularization procedures are necessary in all these cases\(^{10,11,12,13,14,15,16}\). The special attention has been devoted to the more accurate description of the optical device. Detailed analysis needs special choice of eigenfunctions related to finite aperture instead of spatial frequencies\(^{3,4,5}\). Systematic theory of this remarkable basis, so called prolate spheroidal wave functions, was given by Frieden.\(^{6}\) Several further super-resolution techniques as apodisation\(^{54}\) or analytic continuation\(^{28}\) have been suggested. The inverse source problem may be further generalized, taking into account other realistic aspects of detection. For example, optical intensity is detected by real photo-detectors instead of complex amplitude and phase is subject of reconstruction\(^{2,5,9,16,18,19,20,21,22,23,24,25,26}\).

Standard image processing deals with the observation in the image plane revealing the sharpest image. However, the observations in defocused planes are also worth while\(^{45,46,52,62}\). They correspond to the observation of \(A\tilde{\psi} + B\tilde{p}\) in the language of quantum theory\(^{10}\), bringing other piece of information about signal and affording better employment of measured data. Such tomographical technique was suggested by Bertrand, Bertrand\(^{33}\) and Vogel, Risken\(^{42}\) and experimentally verified by group from University of Oregon\(^{14}\). Up to now several other tomographical schemes for observing of various faces of the system have been proposed. This is usually achieved by adjusting of some parameters in the set-up. Particular configuration in dependence on parameter provides the desired group of transformations. Both the classical X-ray tomography (CT) used in medicine\(^{63,64,65}\) and the homodyne tomography\(^{43,44,50,51,53}\) use the group of rotations. Phase-space tomography and chronocyclic tomography are connected with the symplectic group but they are convertible to the classical tomography\(^{45,47,52}\). General non-homogeneous symplectic tomography was introduced by Mancini, Man'ko, and Tombesi\(^{48}\).

All the above mentioned inverse problems are relating measured data \(f_i\) with theoretical probabilities \(p_i\), by means of the equality
\[
\text{Tr} \left[ \hat{\rho} \hat{\Pi}_i \right] = f_i, \tag{24}
\]
where multi-index \(i\) passes over all configurations of optical device and over all pixels of detector. However, the solution of linear equations\(^{24}\) represents an ill-posed problem of the same kind as image reconstruction using deconvolution. This procedure is very sensitive to noise, which is an inevitably involved in any measurement scheme. Ill posed problem implies, that reconstructed “state” need not represent any physically possible object. In the language of quantum theory this means that \(\langle \psi | \hat{\rho} | \psi \rangle < 0\) may hold for some states. Alternatively, the optical intensity \(I(x)\) may drop below zero at some position coordinates in wave-theory language. Linear algorithm is not capable to guarantee such necessary conditions as positive definiteness of density matrix or mutual intensity.

Statistical approaches suggest a remedy to this problem releasing the strict condition\(^{24}\). For example, the equality between \(f_i\) and \(p_i\) could be replaced by requirement of their minimal least-squares distance
\[
\sum_i |f_i - \text{Tr} \left[ \hat{\rho} \hat{\Pi}_i \right]|^2, \tag{25}
\]
an obvious choice in engineering practice. But the other metrics are also eligible. Least-squares method\(^{27,28,65}\), Richardson method\(^{31}\), maximum-likelihood (ML) principle and expectation-maximization (EM) algorithm\(^{35,36,37,38,39,40,41,67,68}\), principle of maximum Cramer-Rao bound\(^{32,33}\), maximum entropy (ME) method\(^{29,30,31,69,70,71}\), and intrinsic correlation function (ICF) model represent several examples of various statistical signal-processing schemes. In the following section the arguments in favor of maximum likelihood estimation will be formulated with the help of quantum treatment.

5. Reconstruction as generalized quantum measurement

Let us assume the generic scheme for the quantum measurement\(^{20,21}\) described above. The conditional probability of detecting \(N_i\) clicks in the \(i\)-th pixel, if the state \(\hat{\rho}\) occurs in the input plane, has the form of multinomial distribution
\[
\mathcal{L}(\hat{\rho}) \approx \prod_i p_i^{N_{fi}}, \tag{26}
\]
where \(f_i\) are the relative frequencies of registered clicks, \(N_{fi} = N_i\). The input state \(\hat{\rho}\) is the subject of estimation procedure. The likelihood functional\(^{20}\) then gives the answer to the question “How is it likely that the given data \(f_i\) were registered provided that the system was in the given quantum state \(\hat{\rho}\)?” For some states the detection of given data is more likely than for others. Using the relation\(^{21}\) the log-likelihood function reads
\[
\ln \mathcal{L}(\hat{\rho}) = \sum_i f_i \ln p_i = \sum_i f_i \ln \text{Tr} \left[ \hat{\rho} \hat{\Pi}_i \right], \tag{27}
\]
Maximum likelihood principle selects such a state $\hat{\rho}_{\text{est}}$ for which the likelihood reaches its maximum,

$$\hat{\rho}_{\text{est}} = \arg \left[ \max_{\hat{\rho}} \ln \mathcal{L}(\hat{\rho}) \right].$$  \hspace{1cm} (28)

The formal necessary condition

$$\frac{\delta \ln \mathcal{L}(\hat{\rho})}{\delta \hat{\rho}} \bigg|_{\hat{\rho}_{\text{est}}} = 0$$  \hspace{1cm} (29)

may be rewritten to the form of extremal operator equation\textsuperscript{72,73,74,75,76}, or alternatively, extremization can be done by means of numerical up-hill simplex method\textsuperscript{77}. Any density matrix may be parameterized in diagonal form\textsuperscript{11} using independent (orthogonal) basis states $|\varphi_k\rangle$ and the variation (29) may be done along these rays. Likelihood function depends on the density matrix through probabilities $p_i$. This yields the system of coupled equations\textsuperscript{72,73,74,75,76}

$$\frac{\delta \ln \mathcal{L}(\hat{\rho})}{\delta |\varphi_k\rangle} = 0 \text{ for any allowed component } k.$$  \hspace{1cm} (30)

and the normalization $\text{Tr} [\hat{\rho}] = 1$, the extremal equation\textsuperscript{72,73,74,75,76} for the density operator $\hat{\rho}$ reads

$$\hat{R} \hat{\rho} = \hat{\rho}.$$  \hspace{1cm} (31)

Here

$$\hat{R} = \sum_i \frac{f_i}{p_i} \hat{\Pi}_i,$$  \hspace{1cm} (32)

and probabilities $p_i$ are state dependent\textsuperscript{71}. Operator equation (31) determines the most likely solution $\hat{\rho}_{\text{est}}$, for which $\hat{R}(\hat{\rho}_{\text{est}}) = \hat{1}$ holds on the Hilbert space of the state $\hat{\rho}_{\text{est}}$\textsuperscript{71,75}. No prior knowledge about the estimated state is needed. Results of the measurement itself are sufficient for analysis.

Let us develop the optical counterpart of this reconstruction problem. In the spatial domain the extremal equation\textsuperscript{71} has the form of integral equation for mutual intensity $\Gamma(x, x')$,

$$\int dx' \mathcal{R}(q, x') \Gamma(x', q') = \Gamma(q, q'),$$  \hspace{1cm} (33)

where the resolution of identity $\hat{1} = \int dx |x\rangle\langle x|$ has been used. Kernel

$$\mathcal{R}(q, x') = \sum_i \frac{f_i}{p_i} \mathcal{P}_i(q, x')$$  \hspace{1cm} (34)

and functions

$$\mathcal{P}_i(q, x') = \int_{\Delta_i} \frac{dx}{\Delta} h^*(x, q) h(x, x')$$  \hspace{1cm} (35)

correspond to the operator $\hat{R}$ and to the POVM operators $\hat{\Pi}_i$, respectively. The probabilities (21) of elementary detection in individual pixels then read

$$p_i = \int dq dq' \Gamma(q, q') \mathcal{P}_i(q', q).$$  \hspace{1cm} (36)

The equation (33) relates measured data $f_i$, properties of optical device, and reconstructed signal $\Gamma(x, x')$. Dependence on the optical apparatus is expressed via point-spread function $h(x, x')$ only. However, this mutual relation is inseparable, since the relation is nonlinear. In comparison to standard treatments in scalar optics, no assumptions about statistical nature of the signal have been done. This seems to be reasonable, since the coherence properties of the light field may change during the propagation (van Cittert–Zernike effect\textsuperscript{22}). The proposed formulation anticipates only the knowledge of the optical apparatus and the measured data without any prior assumptions about the unknown signal.

Special cases of the generic formulation deserve attention. Let us assume an iterative solution of the equation (33) taking the maximally-ignorant initial guess represented by the totally mixed uniform state, $\hat{\rho}^{(0)} = \frac{1}{D} \hat{1}$, where $1/D$ ensures the trace normalization in $D$-dimensional Hilbert space. After evaluating the kernel $\hat{R}^{(0)}$ we are able to write down the first iteration of estimated state, $\hat{\rho}^{(1)} = \hat{R}^{(0)} \hat{\rho}^{(0)} = \sum_i f_i \hat{\Pi}_i / \text{Tr}[\hat{\Pi}_i]$. In the spatial domain this state has the form of partially coherent superposition of response functions\textsuperscript{15} weighted by measured data $f_i$, $\sum_i f_i = 1$,

$$\Gamma^{(1)}(q, q') = \sum_i f_i \frac{\int \int dx h^*(x, q) h(x, q')}{\int \int dx |h(x, \xi)|^2}.$$  \hspace{1cm} (37)

It is clear that the coherence properties of estimated signal are changed during repeated iterations of extremal equation (33). Besides the proposed iterative solution the well-known EM algorithm\textsuperscript{72} completed by unitary step\textsuperscript{75} can also be utilized. This guarantees the convergence and keeps all fundamental properties\textsuperscript{11} of partially coherent signal $\Gamma(x, x')$.

As the second special case the totally incoherent light can be assumed,

$$\Gamma(x, x') = I(x) \delta(x - x'),$$  \hspace{1cm} (38)

where $\delta(x)$ is Dirac distribution. The extremal equation then reduces to

$$\int dq' \mathcal{R}(q, q') I(q') = I(q),$$  \hspace{1cm} (39)

whereas the probabilities (36) read

$$p_i = \int dq \mathcal{P}_i(q, q) I(q).$$  \hspace{1cm} (40)

The relations (39), (41), and (40) provide the extremal equations for unknown optical intensity $I(x)$,

$$\sum_i \int dq \mathcal{P}_i(q, q) I(q) \int dq' \mathcal{P}_i(q', q') I(q') = I(x).$$  \hspace{1cm} (41)
This relationship may be utilized for iterative procedure as was proposed by Richardson in 1972 for incoherent image reconstruction. Notice, however, that in the original derivation the Bayes rule was adopted. The treatment devised here makes it possible to extend the solution to the cases of partially or totally coherent signals.

6. Numerical example

In this section we demonstrate the feasibility and advantages of the presented method by means of the carefully selected example. The partially coherent testing object is chosen below the resolution limit of the simple optical device with finite aperture. Therefore, the observed images do not bear any resemblance with the true object. In spite of this obvious limitations the reconstructed object reveals the original structure. This improvement is achieved by adjusting of the detector position in transverse and longitudinal directions. The background noise is added to simulated intensities. Only these data enter the reconstruction procedure.

Let us consider the optical set-up that consist of free evolution to the distance \( d_{in} \), thin lens with the focal length \( f \), and free evolution to the distance \( d_{out} \). The lens has finite aperture diameter \( 2d \). The whole device can be transversely shifted to a distance \( s \) from axial position. The longitudinal distance \( d_{in} \) from the object and transverse shift \( s \) are parameters of the optical set-up and they may be adjusted during measurement, while the other parameters are kept constant. The point-spread function \( h(x, x_0) = h(x, x_0; d_{in}, s) = \langle x | \hat{T}(d_{in}, s) | x_0 \rangle \) of the device under consideration reads

\[
h(x, x_0) = \langle x | \exp (-i \frac{d_{out}}{2k} \hat{\rho}^2) \int_{-a}^{a} d\xi \xi \rangle \xi | x_0 \rangle \times \exp (-i \frac{k}{2f} x^2) \exp (-i \frac{d_{in}}{2k} \hat{\rho}^2) \exp (-is \hat{\rho}) | x_0 \rangle,
\]

where \( k = 2\pi/\lambda \) is the longitudinal wave number. With the help of the relations \( \hat{1} = \int dx \langle x | x \rangle \hat{1} = \int dp \langle p | p \rangle \hat{1}, \) and \( \langle x | p \rangle = (2\pi)^{-1/2} \exp (i p x) \) the point-spread function \( h(x, x_0; d_{in}, s) = h_{in}(x, x_0) \) can be rewritten to the form

\[
h(x, x_0) = h_{in}(x, x_0) \mathcal{E}(x, x_0).
\]

Here

\[
h_{in}(x, x_0) = \text{const} \int e^{\frac{i}{2} \left[ \left( \frac{x^2}{d_{out}} + \frac{(x_0 + s)^2}{d_{in}} \right) - \frac{x^2}{R} \right]}
\]

is the response function of the ideal refractive focusing device and

\[
\mathcal{E}(x, x_0) = \frac{1}{2} \text{erf} \left[ \frac{1 - i \sqrt{\Delta}}{2} \left( \frac{\theta}{\Delta} + a \right) \right] - \frac{1}{2} \text{erf} \left[ \frac{1 - i \sqrt{\Delta}}{2} \left( \frac{\theta}{\Delta} - a \right) \right]
\]

represents the correction to the finite aperture. This tends to unity for large aperture, \( \lim_{a \to \infty} \mathcal{E}(x, x_0) = 1. \)

The parameter \( \Delta \) characterizes the defocusing from the imaging configuration,

\[
\Delta = \frac{1}{d_{in}} + \frac{1}{d_{out}} - \frac{1}{f},
\]

the parameter \( \theta \) is related to the transverse wave number,

\[
\theta = \theta_{in} + \theta_{out} = \frac{x_0 + s}{d_{in}} + \frac{x}{d_{out}},
\]

and the function \( \text{erf}() \) denotes the common error function,

\[
\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z dt e^{-t^2}.
\]

In the present simulation the parameters have been chosen as \( \lambda = 600 \text{ nm}, \) \( d_{out} = 1.5 \text{ m}, \) \( f = 0.5 \text{ m}, \) and \( a = 0.6 \text{ mm}. \) The image \( (\Delta = 0) \) appears at the distance \( d_{in} = 0.75 \text{ m}. \) However, it is blurred due to the small aperture. In fact, details closer than the diffraction limit

\[
R = C \lambda d_{out} / a
\]

are mapped to two spots with insufficient contrast according to Rayleigh’s criterion. The factor \( C \) depends on aperture shape and coherence properties of the signal. It equals to 0.61 in the case of circular shape and totally incoherent light, or to 0.82 in the case of totally coherent light (Abbe’s resolution limit). We set \( C = 0.5, \) what is equal or smaller than any classical resolution limit for imaging with partially coherent light. The testing object consists from four bright spots separated by dark spaces. The distance 0.15 mm between the edges of the central closest spots is 5-times smaller than resolution limit \( R \). The corresponding optical intensity \( I(q) \) is shown in Fig. 1. The off-diagonal peaks of mutual

![Fig. 1. Optical intensity \( I(q) \) of the true object in the input plane.](image-url)
The object plane is discretized by 100 equidistant points in the interval \([-1.5, 1.5]\) mm, or equivalently \([-2R, 2R]\). The corresponding mutual intensity \(\Gamma(q, q')\) is given on the square mesh of 100 × 100 points \((q_m, q'_n)\) in the process of data generation and subsequent state reconstruction. Similarly, the detection plane \(x\) in the interval \([-4, 4]\) mm is sampled only by 64 pixels \(x_i\) for every longitudinal distance \(d_{inj} = (0.75 - 0.05j)\) m, \(j = 0, \ldots, 5\), and for every transverse shift \(s_l = (-1.2 + 0.3l)\) mm, \(l = 0, \ldots, 8\). Using the relations (35), (36), and (37–39) the intensities \(p_{ijl} = \frac{I_{ijl}(x_i)}{\sum_{j,k,l} I_{ijl}(x_i)}\) in the pixels \(i\) can be evaluated for all the configurations of the optical set-up. For example, in the case of imaging axial configuration \((j = 0, l = 4)\) the intensity reveals the central peak with small side lobes, see Fig. 2. The correspond-

![Fig. 2. Simulated relative frequencies \(f_i\) (points) affected by 20% of background noise sample the optical intensity \(I(x)\) in the detection plane (lines) for imaging axial arrangement, \(d_{in} = 0.75\) m, \(s = 0\).](image)

ings under-sampled data \(f_i\) spoiled by 20% of background noise are shown in the same figure. These simulated relative frequencies \(f_{ijl}\) serve as an input for the reconstruction procedure (36–39). To solve the extremal equation (33) we need to find the mutual intensity on the given mesh as the hermitian matrix of the dimension 100 × 100. As an initial iteration, the uniform incoherent superposition of all pure states on the supposed space is used. It exhibits flat intensity profile. It is interesting to note that the final results seems to be independent of the choice of initial mutual intensity. In the course of repeated iterations of the discretized equation (33) the difference \(\varepsilon = \int dqdq' |\Gamma^{(n+1)}(q, q') - \Gamma^{(n)}(q, q')|^2\) between successive iterations can be used as the criterion for terminating the extremization process. Numerical results show that the difference \(\varepsilon\) reaches the level about \(10^{-10}\) after several tens of iterations and it reaches the level \(10^{-12}\) after approximately 1000 iterations, see Fig. 4. The convergence improves slightly for smaller portion of background noise. Iterated intensity starts to reveal the four-peak structure after relatively small number of steps. The contrast \(V\) of the central part of the estimated optical intensity beyond the diffraction limit \(R\) reaches the value of 0.56 after 1000 iterations, see Fig. 5. The positions and relative intensities of bright spots in estimated object match very well the structure of the true object. This is demonstrated in Fig. 5.

The numerical simulations clearly show that the proposed reconstruction algorithm is feasible and could be implemented. It provides considerable improvement comparing to non-statistical image processing techniques, and significantly, it yields the complete information in the form of correlation function.

### 7. Conclusion

The purpose of the presented paper is twofold. First the tight connection between wave optics and quantum mechanics has been emphasized. The operator language routinely used in quantum theory can simplify the manipulation and description of optical objects, like partially coherent wave and response function of optical device. The second goal of the contribution is the mathematical formulation of the reconstruction algorithm for partially coherent signal proceeded from the maximum-likelihood estimation of mixed quantum states. The solu-

![Fig. 3. The exponentially fast convergence of square distance \(\varepsilon\) during the extremization process.](image)

![Fig. 4. Reconstructed optical intensity \(I(q)\) in the input plane.](image)
tion of extremal equation by means of repeated iterations has been suggested. The first iteration has been explicitly formulated. The proposed method never yields the non-physical results.

The feasibility of the method has been verified by extensive numerical simulations. The realistic experimental data will be considered in the forthcoming publication. The particular numerical example shows the good agreement between the true and estimated states of partially coherent light beyond the diffraction limit, despite of under-sampled data and 20% of background noise.

The method is able to estimate the generic signal without any prior assumptions, utilizing only real noisy data. The potential applications cover wide range of optical inverse problems. The method may be used for the state estimation of the localized mode in photonic band-gap structures (photonic crystals), for the determination of the near-field short-range correlation of the signal transmitted through random media, and for the reconstruction of spatial and coherence properties of light confined and emitted by modern laser-diode sources. Moreover, the general quantum origin of the method allows us to estimate arbitrary continuous (discretized) partially coherent object described by the correlation function or the Wigner function. The reconstruction of de Broglie wave function of a particle and the optical homodyne detection of a quantum state of the light mode are typical examples. In short, the method is applicable to all inverse problems where the precise knowledge of partially coherent signal (mixed state) is essential, providing that the measurement device and real data are known.

Acknowledgments

We would like to thank Z. Bouchal, J. Fiurášek, M. Dušek, and J. Reháček for valuable discussions. This work was supported by the EU grant under QIPC, project IST-1999-13071 (QUICOV) and by Grant LN00A015 of the Czech Ministry of Education.

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Fig. 5. Contour lines (thin) of the reconstructed mutual intensity $I(q, q')$. The positions of the diagonal bright spots as well as the positions of off-diagonal correlations match the true ones (thick lines).
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