Universal properties of the FQH state from the topological entanglement entropy and disorder effects

Na Jiang, Qi Li, Zheng Zhu, Zi-Xiang Hu

Department of Physics, Chongqing University, Chongqing, 401331, P. R. China

Abstract

The topological entanglement entropy (TEE) is a robust measurement of the quantum many-body state with topological order. In fractional quantum Hall (FQH) state, it has a connection to the quantum dimension of the state itself and its quasihole excitations from the conformal field theory (CFT) description. We study the entanglement entropy (EE) in the Moore-Read (MR) and Read-Rezayi (RR) FQH states. The non-Abelian quasihole excitation induces an extra correction of the TEE which is related to its quantum dimension. With considering the effects of the disorder, the ground state TEE is stable before the spectral gap closing and the level statistics seems to have significant change with a stronger disorder, which indicates a many-body localization (MBL) transition.

1. Introduction

Fractional quantum Hall (FQH) liquids are remarkable many-electron systems that occur in two-dimensional electron gas with a perpendicular magnetic field \[1\]. This is the most studied and first experimentally realized system in condensed matter physics that has the topological order \[2\]. It is a typical strong correlated electron system with quenched kinetic energies by magnetic field which fails the application of the perturbation theory. On the other hand, comparing with the Landau theory of the quantum phase transition, there is no order parameter, or symmetry breaking to describe the phase transition between any two FQH states. Therefore, the understanding of the FQH effect has only benefited either from the numerical diagonalizing the Hamiltonian for finite size system, such as exact diagonalization \[3\], density matrix renormalization group \[4, 5, 6\], matrix product state \[7, 8\], et.al., or using of model wavefunctions \[9, 10, 11\]. In a seminal paper of Moore and Read \[11\], it was found that these model wavefunctions can be expressed as correlators of the electron operators in a conformal field theory (CFT), i.e., the so called conformal blocks. Although the model wavefunctions are not the exact ground state wavefunctions of a realistic Hamiltonian, they are supposed to capture the universal properties of the FQH states such as fractional quasiparticle excitations and their fusion relations, statistics, as well as exponents in the edge tunneling and
quantum dimensions for quasiparticles. The most striking theoretically predicted properties of the FQH quasiparticles is the emergence of the Abelian or non-Abelian braiding statistics \cite{11, 12, 13}. The interchange of two Abelian quasiparticles adds a nontrivial phase on the wavefunction. They are named “anyons” since the phase is neither $\pi$ by fermions nor $2\pi$ by bosons. The typical Abelian FQH states are the Laughlin series at $\nu = 1/3, 1/5, 2/3 \cdots$. However, interchange two non-Abelian quasiparticles results in a ground state unitary transformation in the topological degenerate Hilbert space. According to this, the non-Abelian FQH states have received much interests due to their potential applications in the topological quantum computation \cite{14, 15, 16}. Thus far there are two most interesting examples as the candidates for the non-Abelian states which have been realized in experiments, namely the FQH states on the first Landau level at $\nu = 5/2$ \cite{17} and $\nu = 12/5$ \cite{18, 19}. For the even denominator FQH state at $\nu = 5/2$, Moore and Read \cite{11} proposed a $p$-wave paired wavefunction as a candidate ground state. The nature of the 12/5 state is still undetermined. However, the most exciting candidate of the ground state is the $k = 3$ parafermion state proposed by Read and Rezayi \cite{13} which describes a condensate of three-electron clusters.

For the FQH states, it has been established that there is a deep connection between the bipartite EE \cite{20}, or entanglement spectrum \cite{21} and the topological properties embedded in the ground state and its low-lying excitations. This connection is based on the CFT description of the FQH model wavefunctions. For topological states in two dimensional systems, the bipartite EE satisfies the “area law” with a universal order $O(1)$ correction, namely the TEE \cite{22, 23, 24}, i.e., $S = \alpha L - \gamma t$ where the $L$ is the length of the boundary between two subsystems. For example, the bipartite FQH system can be implemented in both the momentum and real space for the two dimensional electron system. The former is called the orbital cut (OC) \cite{20} and the later real space cut (RC) \cite{25}. In this work we mostly use the RC since it has a more accurate definition of the boundary in the “area law”. The EE depends on the way of the partition the system, or $\alpha$ is not universal. However, the TEE $\gamma t$ is a robust measurement of quantum entanglement in a topological phase. It has a connection to the total quantum dimension as $\gamma t = \ln D$ and $D = \sqrt{\sum_i d_i^2}$, where $d_i$s are the quantum dimensions of each sector making up the topological field theory of the corresponding FQH states. In fact, for a general RR state with order-$k$ clustering and at filling fraction $\nu = k \frac{M+2}{kM+2}$, the total quantum dimension is $D_{k,M} = \frac{\sqrt{(k+2)(kM+2)}}{2\sin(\frac{\pi}{k+2})}$. Such as the Laughlin state at $\nu = 1/3$, the MR state at $\nu = 5/2$ and the RR state at $\nu = 12/5$ are corresponding to $M = 1$ and $k = 1, 2, 3$ RR states respectively. The TEE has an additional correction when a topological excitation, or a quasihole is created in the system, i.e., $\gamma_{qh} t = \ln D - \ln d_0$, where $d_0$ is the quantum dimension of the quasihole. In general, Abelian quasihole excitation has quantum dimension $d_0 = 1$ and $d_0 > 1$ for non-Abelian ones. Therefore, the behaviors of the EE, especially the TEE should be very different while a non-Abelian quasihole is created, in other words, we can measure the quantum dimension of the quasihole from the shift of the EE before and after its excitation.

The discussions above are based on a clean system without introducing the effects of the disorder. The topological properties are believed to be robust in the presence of a weak disorder. When the strength of the disorder is comparable to that of the interaction between electrons, the FQH will eventually be destroyed and the system enters into a localized insulating phase. The topological Chern number calculation \cite{26}
Figure 1: The sketch map of the model. A cylinder is bipartite d into two subsystems A and B in real-
space. For non-Abelian FQH state, such as the MR state at $\nu = 5/2$, one $e/4$ quasihole locates on each
edge of the cylinder.

shows that the destruction of the FQHE is related to the continuous collapse of the
mobility gap of the system. On the other hand, because of the strong electron-electron
interaction in the FQH system, the localized phase driven by the disorder can be treated
as a many-body localized phase. The transition between the ergodic and many-body
localized (MBL) phase in the disordered interacting system is a subject of much interest
recently [27, 28, 29, 30]. From the prediction of the random matrix theory, the MBL phase
transition results in the varying of the spectral statistics between the Poisson and the
Wigner-Dyson distribution. Recently Serbyn and Moore [27] found that the existence of
the intermediate statistics between them which can be described by a general distribution
with two parameters, $P(x, \beta, \gamma_p) = \alpha x^\beta e^{-\eta x^\gamma_p}$. In this work, we study the stability of
the Laughlin state and the change of the spectral statistics as varying the strength of the
disorder.

The rest of this paper is arranged as follows. The model and methods are introduced
in Sec.II. In Sec.III, we consider non-Abelian nature for the MR state, RR state and
corresponding quasihole excitations via TEE in the clean systems. The effects of the
disorder and energy statistics are discussed in Sec.IV and Sec.V gives the summary and
discussion.

2. Model and methods

Our model is depicted as in Fig.1. Electrons are put on a cylinder with circumference
$L_y$ in $y$ direction in a magnetic field perpendicular to the surface, the single electron wave
function in the lowest Landau level is

$$\psi_j(r) = \frac{1}{\sqrt{\pi L_y/2 B}} e^{ik_y y} e^{-\frac{1}{2B}(x+k_y B)^2}$$

in which $k_y = \frac{2\pi}{L_y} j$, $j = 0, \pm 1, \pm 2 \cdots$ are the translational momentum in the $y$
direction and the magnetic length is defined as $l_B = \sqrt{\hbar c/eB}$. For a finite size system, the
number of orbits $N_{\text{orb}}$ equals to the number of the magnetic flux quantum. Each orbit
occupies an area $2\pi l_B^2$. Therefore, the length in $x$ direction for a finite system is fixed
with a given aspect ratio $\gamma$, namely $L_x/l_B = \sqrt{N_{\text{orb}} 2\pi/\gamma}$. The advantage of using the
cylinder geometry is that there is no curvatures on the surface and the length \( L_x \) is linearly with aspect to the system size which can be tuned easily. The EE is defined as \( S_A = -\text{Tr} [\rho_A \ln \rho_A] \) where \( \rho_A = \text{Tr}_B \rho \) is the reduced density matrix of the subsystem. If we make a cut in real space along \( y \) direction, saying at the position \( x = l_x \), the electron operators in momentum space can be wrote as a summation of two parts:

\[
c_m = \alpha_m a_m + \beta_m b_m
\]

where \( a_m \) and \( b_m \) are the operators in A/B subblock respectively. \( \alpha_m^2 (\beta_m^2) \) is the probability for an electron at the \( m \)'th orbit locating in block A(B). Therefore,

\[
\alpha_m^2 = \int_0^{L_y} dy \int_{-\infty}^{l_x} dx |\psi_m|^2 = 1 - \frac{1}{2} \text{Erfc}(l_x - \frac{2\pi}{L_y} m)
\]

and \( \alpha_m^2 + \beta_m^2 = 1 \). For the real space partition of the many-body wavefunction, because the total particle number and translational momentum along \( y \) direction are good quantum numbers in the subsystem, it is actually the same as that of the particle partition with the above probabilities. On the other hand, the many-body model wavefunction can be generally obtained from exact diagonalizing a model Hamiltonian with hard-core interaction. For example, the Laughlin, MR and RR wavefunctions are the most compact zero energy eigenstates for the two, three, and four body hard-core Hamiltonian. In the second quantized form, a \( n \)-body hard-core Hamiltonian in an infinite plane is

\[
H_n = \sum_{(m_1, \cdots, m_n)} V(m_1, \cdots, m_n)V(m_{n+1}, \cdots, m_{2n})c_{m_1}^+ c_{m_2}^+ \cdots c_{m_{n+1}}^+ c_{m_{n+2}} \cdots c_{m_{2n}}
\]

where the matrix elements for two, three and four-body interaction are following:

\[
V(m_1, m_2) = \sqrt{\frac{(M-1)!}{2^M m_1! m_2!}} (m_1 - m_2)
\]

\[
V(m_1, m_2, m_3) = \sqrt{\frac{(M-3)!}{4 \times 3^{M-2} m_1! m_2! m_3!}} A(m_1, m_2)
\]

\[
V(m_1, m_2, m_3, m_4) = \frac{1}{3} \sqrt{\frac{(M-6)!}{2^{2M-5} m_1! m_2! m_3! m_4!}} \sum_{\alpha<\beta<\gamma} m_\alpha m_\beta m_\gamma A(m_\alpha - 1, m_\beta - 1, m_\gamma - 1)
\]

in which \( A(m_\alpha, m_\beta, m_\gamma) = A(m_\alpha (m_\alpha - 1) m_\beta) \) is the antisymmetrizer [31]. Another way of producing the model wavefunctions is by the help of the recursive relation of the Jack polynomials (Jacks) [32, 33, 34]. Jacks are homogeneous symmetric polynomials specified by a rational parameter \( \alpha \) and a root configuration. They satisfy a number of differential equations [32] and exhibit clustering properties [33, 34]. For example, Jacks is one of the polynomial solutions for Calogero-Sutherland Hamiltonian:

\[
H_{CS}^\alpha(z_i) = \sum_i (z_i \frac{\partial^2}{\partial z_i^2}) + \frac{1}{\alpha} \sum_{i<j} \frac{z_i + z_j}{z_i - z_j} \left( z_i \frac{\partial}{\partial z_i} - z_j \frac{\partial}{\partial z_j} \right)
\]

It was found [32, 33] that the FQH model wavefunctions for RR \( Z_k \)-parafermion states can be exactly calculated according to Eq.(6) with a negative parameter \( \alpha \) and a root
configuration (or partition). The choice of the root configuration satisfies \((k, r)\) admissibility which means there can be at most \(k\) particles in \(r\) consecutive orbits. The parameter \(\alpha\) is \(-\frac{k+1}{r-1}\) and the corresponding filling factor \(\nu = \frac{k}{r}\) for bosonic system \((\nu = \frac{k}{k+r}\) for fermionic system, the difference between the fermionic and bosonic wavefunction is just a Vandermonde determinant). For example, the Jack with \(k = 2, r = 2\) \((\alpha = -3)\) is the MR wavefunction at \(\nu = 1\), which has root “20202…” in bosonic case and \(\nu = 1/2\), root “1100110011…” in fermionic case. In this paper, we use the Jacks to produce the model wavefunctions on cylinder and bipartite it in real space.

The EE is obtained by a summation of the entropy for all the quantum numbers in the subsystem.

3. TEE in the non-Abelian FQH and quasihole states

The MR state, or its particle-hole conjugate, is the most possible candidate trial wavefunction for the ground state of the FQH at \(\nu = 5/2\). It supports not only the Abelian quasihole with charge \(e/2\) as that in the Laughlin state, but also the non-Abelian quasihole excitation with charge \(e/4\). The origin of the non-Abelian nature of the \(e/4\) quasihole excitation is the majorana zero mode embedded in the quasihole vortex. In the language of the Jacks, the root for the ground state and \(e/2\) quaihole state are “11001100…” and “011001100…” respectively. Because of the majorana nature of the \(e/4\) excitation, it should appears in pairs in the system. For example, the root “101010101…” describes a configuration with one \(e/4\) quasihole on each edge of the cylinder.

Fig. 2(a) shows the EE as a function of the cut position \(l_x\) for the above three states. Obviously, the EE for the ground state and the Abelian quasihole state have negligible discrepancy and the non-Abelian quasihole state has a larger EE in the bulk. In Fig. 2(b) we plot the value of the EE for the three states at \(l_x = 0\) for different system sizes at aspect ratio \(\gamma = 1.0\). Because of the Abelian nature, the difference between the \(S^A_{e/2}\) and \(S^A_{e/4}\) becomes small while increasing the system size and approaches to zero in the thermodynamic limit. The interesting property is the increment of the EE in the \(e/4\)
the length of the cut in the center, we plot EE as a function of \(\nu\) and \(L_y\) for the system with 14 electrons as shown in Fig. 3. The non-Abelian quasihole excitation in this state is called Fibonacci anyon which supports the universal topological quantum computation. Comparing with the MR state, the RR state has smaller energy gap and higher experimental requirements. The exact plateau in the Hall conductance measurement so far has not been observed and the exact nature is still undetermined. Although there are other candidates for this filling, such as hierarchy state, Jain’s composite-fermion state, Bonderson-Slingerland...
Figure 4: (a) \( S_A \) as a function of \( l_x \) for the ground state, \( \epsilon/5 \), \( 2\epsilon/5 \) and \( 3\epsilon/5 \) respectively for 18-electron system. (b) The \( S_A \) of the four states with two equal subsystems for \( N = 12 \sim 21 \) systems. Insert plot: the entropy difference between ground state and quasihole state in the bulk for different system sizes. The horizontal lines are their theoretical predicted values from CFT where \( \Delta S_{1(2)\epsilon/5} - S_{GS} = \ln\left(\frac{1 + \sqrt{5}}{2}\right) \approx 0.48 \) and \( \Delta S_{3\epsilon/5} - S_{GS} = 0 \).

4. The disorder effects

The above calculations are done in a clean system. However, in a realistic sample, the disorder is inevitable existence. It is known that the effect of the disorder is essential in the quantum Hall physics, such as the formation of the Hall conductivity plateaus and the transition between them. Generally, the ground state topological properties are robust in the presence of a weak disorder. However, while the strength of the disorder is enhanced, the ground state can be destroyed and driven into a localized Anderson
insulating phase [47]. In the free case, the system is described by a many-body interaction Hamiltonian, and therefore, the inducing of the disorder is related to the MBL physics [28].

Recently Geraedts et.al. [29] found that the entanglement spectrum (ES) of a spin system shows level repulsion and follows a semi-Poisson distribution in the MBL phase. Ref. [48] found that the MBL behavior is also shown in the ES of the FQH states by introducing the disorder effect. In the following, we want to look at that to what extent, the TEE can survive with increasing the disorder strength, and we also discuss the relation among the disorder effects and the ground state gap closing, and the MBL behavior in the energy spectrum statistics. In the disk geometry, for simplicity, we consider the disorder effects in the model Hamiltonian with hard-core interaction. In principle, the system breaks the translational symmetry after introducing the disorder. However, in the disorder problem, we always need to do the sample average and the translational symmetry recovers when the disorder is randomly distributed and the number of the disorder is large enough. Here, for simplicity, we consider an uncorrelated random potential \( H_D = \sum_m U_m c^+_m c_m \) [49] and assume that the disorder effects are averaged firstly in each Landau orbit and then the Hamiltonian still has rotational symmetry. The \( U_m \) denotes the averaged random potential on the \( m \)th orbit whose value is randomly chosen in the interval of \([-W/2, W/2]\). For a \( N \)-electron Laughlin state, it has total angular momentum \( M_{tot} = 3N(N-1)/2 \). Because of the conserved rotational symmetry assumed above, we work in the same angular momentum subspace as that for the Laughlin state. The ground state phase transition can be understood by energy level crossing between the lowest energy state and the first excited state while increasing the strength of the disorder. We define the spectral gap as their energy difference \( \Delta E = E_1 - E_0 \). Its averaged value as a function of the disorder strength is depicted in Fig. 5(a). The number of the average samples is 1000 for 10 electrons and larger for smaller systems. It shows that the \( \Delta E \) reaches a minimum at around \( W_c \approx 1.3 \) for different systems and the minimal gap at \( W_c \) decreases as increasing the system size. Finite size effect shows that \( \Delta E \) drops to zero in the thermodynamic limit which means the \( W_c \) is the critical disorder strength of the ground state phase transition. Therefore, we expect that the topological properties, such as the TEE, keeps invariant in weak disorder regions \( W < W_c \) which is shown in

Figure 5: (a) The energy gap \( \Delta E = E_1 - E_0 \) in the subspace \( M_{tot} = 3N(N-1)/2 \) as a function of the disorder strength. The minimum reaches at \( W_c \approx 1.3 \) which means the spectral gap closing. (b) The averaged TEE versus the disorder strength. The line depicts the expected value \( \gamma = -\log \sqrt{3} \) in the pure case.
Fig. 6(a). It shows that the averaged TEE stays around the expected value \( \gamma = -\log \sqrt{3} \) in the weak disorder region and dramatically increases to zero while \( W > W_c \) which demonstrates that the topological properties are immune from the weak disorder.

Excepting the ground state phase transition, strong disorder in a many-body Hamiltonian can induce a MBL phase \([50, 51]\) in which all of the states are insulating and their energy levels obey a Poisson distribution. To probe the MBL transition, it is easy to compute the ratio of adjacent energy gaps. For a sorted spectrum \( \{\lambda_n; \lambda_n \leq \lambda_{n+1}\} \), it is defined as

\[
    r_n = \frac{\min(\lambda_n - \lambda_{n-1}, \lambda_{n+1} - \lambda_n)}{\max(\lambda_n - \lambda_{n-1}, \lambda_{n+1} - \lambda_n)}.
\]

The averaged value \( r \) of the adjacent gaps is \( r \simeq 0.53\) \([52]\) for GOE ensembles, \( r \simeq 0.60\) \([52]\) for GUE and \( r \simeq 0.386\) for Poisson ensembles \([28]\). Fig. 6(a) shows the average ratio of the adjacent gaps as a function of the disorder strength. It clearly shows that the energy levels satisfy the GOE distribution for weak disorder and evolve into Poisson for strong disorder limit. As a comparison, the disorder effects on the three-body model Hamiltonian, the ground state of which is the Moore-Read state, are considered in Fig. 6(c) and (d). The results are quit similar to that of the Laughlin state which demonstrates that the universal properties of the topological quantum state in the presence of the disorder.

From Fig. 6(a), we observe that the GOE distribution still persist after the spectral gap closing and there is a large region of \( W \) in which the spectral distribution obeys neither GOE nor Poisson distribution. We perform a full diagonalization and use the 50% central part of the spectrum to investigate the energy spectral statistics. The level
Figure 7: Evolution of the fitted values of $\gamma_p(\beta)$ which from the expression $ax^\beta e^{-\eta x^{\gamma_p}}$ change from about 2.0 1.0(1.0 0.0) as we tune the disorder strength for 5, 6, 7 electrons systems respectively. We observe the transition from GOE to Poisson. The inset plot is for the weak enough strength and the theoretical value can reached in the thermodynamic limit. The fit is to an ES averaged over states in the middle 50% of the full spectrum while the error averaged from the disorder.

The statistics is the distribution of the level spacing

$$S \equiv \frac{\lambda_{n+1} - \lambda_n}{\langle \lambda_{n+1} - \lambda_n \rangle}.$$  \hfill (8)

Recently, Serbyn and Moore \cite{27} proposed that the spectral statistics across the MBL transition can be written as a form of a semi-Poisson distribution:

$$P(x, \beta, \gamma_p) \sim x^\beta e^{-\eta x^{\gamma_p}},$$  \hfill (9)

in which $1 \leq \gamma_p \leq 2$ controls the tails of the statistics and level rigidity, and $0 \leq \beta \leq 1$ determines the level repulsion. The case of $\gamma_p = 2, \beta = 1$ corresponds to the Wigner-Dyson distribution for GOE ensembles and $\gamma_p = 1, \beta = 0$ corresponds to a Poisson distribution in a MBL phase \cite{28}. While $\gamma_p \to 1$, it evolves to a semi-Poisson distribution with generic $\beta$ which describes an intermediate statistics between ergodic Wigner-Dyson and MBL Poisson. Fig. (b) and (d) depict the level distributions with different disorder strengths ($W=2.0, 20.0, 30.0, 100.0$) in the model Hamiltonian. As a comparison, the GOE and Poisson distributions are also plotted. It is obviously that the level distributions evolve from the GOE to Poisson with $W$ increasing. By using the fitting equation in Eq. (9), the two optimal parameters $\gamma_p, \beta$ for different systems are shown in Fig. (c) as a function of the $W$. The inserted plots show the results for much weaker disorder strengths. It is shown that in the limit $W \to 0$, the level distribution satisfies the Wigner-Dyson condition with $\gamma_p \sim 2$ and $\beta \sim 1$ which consistent to the results in Fig. (a). And in the strong disorder limit, the parameters are roughly $\gamma_p \sim 1$ and $\beta \sim 0$ which corresponds to the random Poisson distribution. The behavior of the $\gamma_p$ as varying $W$ is quite similar to the ratio of the adjacent gaps as shown in Fig. (a). It has a dramatic decay from $\gamma_p = 2$ to $\gamma_p = 1$ around $W \sim 10$. At the same time, the $\beta$ drops from 1 to 0 in this region. Therefore, the results of the Fig. (c) are exactly consistent with that of the Fig. (a) and we conclude that the intermediate phase indeed obeys the semi-Poisson distribution.
5. Summary and discussion

In conclusion, we have presented a systematical study of the EE for FQH states and the corrections of their quasihole excitations. The most important feature of the EE for FQH state is its universal correction in the relation of “area law”, namely the TEE which is related to the quantum dimensions of the ground state. Moreover, the non-Abelian quasihole excitation in the subsystem can have an extra correction in the TEE which is related to the quantum dimension of the quasihole itself. According to the behaviors of the EE for the ground state and quasihole states, we extrapolate the quantum dimensions of the Abelian and non-Abelian quasiholes. Our results are consistent to the theoretical prediction from CFT and recent matrix product state (MPS) study [53]. In the MPS work, they used the orbital cut to define the EE. It has a good scaling behavior in the region \( L_y/l_B \in [10, 25] \) which is consistent to our real space EE study in Fig. 3 as varying the aspect ratio in a finite size system. Moreover, our finite size calculation can show the behavior of the EE for \( L_y/l_B \to 0 \) and \( L_y/l_B \to \infty \) which have physical meaning of the Tao-Thouless crystal and the edge-edge coupling respectively. The MPS work did not look into these two regions because of the truncation errors. For the RR state, although the system is limited by the numerical diagonalization, there are still strong evidences that the \( e/5 \) and \( 2e/5 \) quasihole excitations have non-Abelian nature with quantum dimensions larger than one.

With considering the effects of the disorder, we find that the TEE keeps invariant before the spectral gap closing which demonstrates that the robustness of the topological properties. The spectral gap closing near \( W_c \simeq 1.3 \), at which the energy level statistics still satisfies the GOE distribution as shown in Fig. 5 and Fig. 7. As increasing the strength of the disorder, we find the average value of the adjacent gaps evolves from \( r \simeq 0.53 \) to \( r \simeq 0.386 \) which corresponds the GOE and Poisson distributions respectively. The critical strength of the disorder for MBL transition is about \( W \sim 10 \). It is larger than the \( W_c \) since MBL phase need stronger disorder to localize all the eigenstates of the system. The results of the energy level spacing statistics in these two limits are consistent to the distributions. In the intermediate region of this MBL transition, we verify that the energy space follows the distribution \( P(x, \beta, \gamma_p) \sim x^{\beta} e^{-\eta x^{\gamma_p}} \). The behaviors of the parameter \( \gamma_p \) and \( \beta \) are similar to \( r \).

This work was supported by National Natural Science Foundation of China Grants No. 11674041, No. 91630205 and Fundamental Research Funds for the Central Universities Grant No. CQDXWL-2014-Z006.

References

[1] D. C. Tsui, H. L. Stomer, and A. C. Gossard, Phys. Rev. Lett. 48, 1559 (1982).
[2] X.-G. Wen, Adv. Phys. 44, 405 (1995).
[3] R. B. Laughlin, Phys. Rev. Lett. 50, 1395 (1983).
[4] A. E. Feiguin, E. Rezayi, C. Nayak, and S. DasSarma, Phys. Rev. Lett. 100, 166803 (2008).
[5] J. Z. Zhao, D. N. Sheng, and F. D. M. Haldane, Phys. Rev. B 83, 195135 (2011).
[6] Z-X. Hu, Z. Papic, S. Johri, R. N. Bhatt, and P. Schmitteckert, Phys. Lett. A 376, 2157 (2012).
[7] M. P. Zaletel, R. S. K. Mong, and F. Pollmann, Phys. Rev. Lett. 110, 236801 (2013).
[8] M. P. Zaletel, R. S. K. Mong, F. Pollmann, and E. Rezayi, Phys. Rev. B 91, 045115 (2015).
[9] B. I. Halperin, Helv. Rev. Acta 56, 75 (1983).
[10] J. K. Jain, Phys. Rev. Lett. 63, 199 (1991).
[11] G. Moore and N. Read, Nucl. Phys. B 360, 362 (1991).
[12] M. Greiter, X.-G. Wen, and F. Wilczek, Phys. Rev. Lett. 66, 3205 (1991).
[13] N. Read and E. Rezayi, Phys. Rev. B 59, 8084 (1999).
[14] A. Kitaev, Ann. Phys. (N. Y.) 303, 2 (2003).
[15] M. H. Freedman, Proc. Natl. Acad. Sci. U. S. A. 95, 98 (1998).
[16] C. Nayak, S. H. Simon, A. Stern, M. Freedom, and S. DasSarma, Rev. Mod. Phys. 80, 1803 (2008).
[17] R. L. Willett, J. P. Eisenstein, H. L. Stormer, D. C. Tsui, A. C. Gossard, and J. H. English, Phys. Rev. Lett. 59, 1776 (1987).
[18] J. S. Xia, W. Pan, C. L. Vicente, E. D. Adams, N. S. Sullivan, H. L. Stormer, D. C. Tsui, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, Phys. Rev. Lett. 93, 176809 (2004).
[19] A. Kumar, G. A. Csathy, M. J. Manfra, L. N. Pfeiffer, and K. W. West, Phys. Rev. Lett. 105, 246808 (2010).
[20] M. Haque, O. Zozulya, and K. Schoutens, Phys. Rev. Lett. 98, 060401 (2007); O. S. Zozulya et al., Phys. Rev. B 76, 125310 (2007).
[21] H. Li and F. D. M. Haldane, Phys. Rev. Lett. 101, 010504 (2008).
[22] A. Hamma, R. Ionicioiu, and P. Zanardi, Phys. Lett. A 337, 22 (2005).
[23] A. Kitaev and J. Preskill, Phys. Rev. Lett. 96, 110404 (2006).
[24] M. Levin and X.-G. Wen, Phys. Rev. Lett. 96, 110405 (2006).
[25] J. Dubail, N. Read, and E. H. Rezayi, Phys. Rev. B 85, 115321 (2012); A. Sterdyniak, A. Chandran, N. Regnault, B. A. Bernevig, and P. Bonderson, Phys. Rev. B 85, 125308 (2012).
[26] D. N. Sheng, X. Wan, E. H. Rezayi, K. Yang, R. N. Bhatt, and F. D. M. Haldane, Phys. Rev. Lett. 90, 25 (2003).
[27] M. Serbyn and J. E. Moore, Phys. Rev. B 93, 041424(R) (2016).
[28] A. Pal and D. A. Huse, Phys. Rev. B 82, 174411 (2010).
[29] S. D. Geraedts, R. Nandkishore, and N. Regnault, Phys. Rev. B 93, 174202 (2016).
[30] N. Regnault and R. Nandkishore, Phys. Rev. B 93, 104203 (2016).
[31] X. Wan, Z-X. Hu, E. H. Rezayi, and K. Yang, Phys. Rev. B 77, 165316 (2008).
[32] B. A. Bernevig and F. D. M. Haldane, Phys. Rev. Lett. 100, 246802 (2008).
[33] B. A. Bernevig and F. D. M. Haldane, Phys. Rev. Lett. 101, 246806 (2008).
[34] B. A. Bernevig and N. Regnault, Phys. Rev. Lett. 103, 206801 (2009).
[35] B. Feigin, M. Jimbo, T. Miwa, and E. Mukhin, Int. Math. Res. Not. 2002, 1223 (2002); ibid, 2003, 1015 (2006).
[36] Q. Li, N. Jiang, Z. Zhu, and Z-X. Hu, New J. Phys. 17, 095006 (2015).
[37] R. Tao and D. J. Thouless, Phys. Rev. B 28, 1142 (1983).
[38] D. J. Thouless, Surf. Sci. 142, 147 (1984).
[39] F. D. M. Haldane, Phys. Rev. Lett. 51, 605 (1983).
[40] B. I. Halperin, Phys. Rev. Lett. 52, 1583 (1984).
[41] J. K. Jain, Composite Fermions (Cambridge University Press, Cambridge, England, 2007).
[42] P. Bonderson and J. K. Slingerland, Phys. Rev. B 78, 125323 (2008).
[43] P. Bonderson, A. E. Feiguin, G. Moller, and J. K. Slingerland, Phys. Rev. Lett. 108, 036806 (2012).
[44] G. J. Sreejith, C. Toke, A. Wojs, and J. K. Jain, Phys. Rev. Lett. 107, 086806 (2011).
[45] G. J. Sreejith, Y.-H. Wu, A. Wojs, and J. K. Jain, Phys. Rev. B 87, 245125 (2013).
[46] W. Zhu, S. S. Gong, F. D. M. Haldane, and D. N. Sheng, Phys. Rev. Lett. 112, 096803 (2014).
[47] P. W. Anderson, Phys. Rev. 109, 1492 (1958).
[48] Z. Liu and R. B. Bhatt, Phys. Rev. Lett. 117, 206801 (2016).
[49] Z-X. Hu, K-H. Lee, and X. Wan, Int. J. Mod. Phys. Conf. Ser. 11, 70 (2012).
[50] D. Basko, I. Aleiner, and B. Altshuler, Ann. Phys. 321, 1126 (2006).
[51] I. V. Gornyi, A. D. Mirlin, and D. G. Polyakov, Phys. Rev. Lett. 95, 206603 (2005).
[52] L. D’Alessio and M. Rigol, Phys. Rev. X 4, 041048 (2014).
[53] B. Estienne, N. Regnault, and B. A. Bernevig, Phys. Rev. Lett. 114, 186801 (2015).