We prove that any topological group $G$ containing a subspace $X$ of the Sorgenfrey line has spread $s(G) \geq s(X \times X)$. Under OCA, each topological group containing an uncountable subspace of the Sorgenfrey line has uncountable spread. This implies that under OCA a cometrizable topological group $G$ is cosmic if and only if it has countable spread. On the other hand, under CH there exists a cometrizable Abelian topological group that has hereditarily Lindelöf countable power and contains an uncountable subspace of the Sorgenfrey line. This cometrizable topological group has countable spread but is not cosmic.

**Key words:** Sorgenfrey line, topological group, spread, OCA, CH.

### 1. Introduction

The main result of this paper is the following theorem answering the problem [2], posed by the first author on MathOverflow.

**Theorem 1.** Each topological group containing a topological copy of the Sorgenfrey line contains a discrete subspace of cardinality continuum.
We recall that the Sorgenfrey line is the real line endowed with the topology, generated by the half-intervals \([a, b)\) where \(a < b\) are arbitrary real numbers. The Sorgenfrey line endowed with the (continuous) operation of addition of real numbers is a classical example of a paratopological group, which is not a topological group, see \([1, 1.2.1]\). The Sorgenfrey line has countable spread and shows that Theorem 1 cannot be generalized to paratopological groups.

Theorem 1 follows from a more refined theorem evaluating the spread of a topological group that contains a topological copy of an uncountable subspace of the Sorgenfrey line.

We recall that for a topological space \(X\) the cardinal

\[
s(X) = \sup\{|D| : D \subseteq X \text{ is a discrete subspace of } X\}
\]

is called the spread of \(X\).

**Theorem 2.** Assume that a topological group \(G\) contains a subspace \(X\), homeomorphic to an uncountable subspace of the Sorgenfrey line. Then \(s(G) \geq s(X \times X)\).

Theorems 1 and 2 will be proved in Section 2. Theorem 2 has the following corollary holding under OCA (the Open Coloring Axiom, see \([11, \S 8]\)).

**Corollary 1.** Under OCA any topological group \(G\) containing an uncountable subspace \(X\) of the Sorgenfrey line has uncountable spread.

**Proof.** Proposition 8.4(c) of \([11]\) implies that \(X\) contains an uncountable subset \(Z\) admitting a strictly decreasing function \(f : Z \to X\) (with respect to the linear order inherited from the real line). Then \(D = \{(x, f(x)) : z \in Z\}\) is a discrete subspace of \(X \times X\) and hence

\[
s(G) \geq s(X \times X) \geq |D| = |Z| > \omega.
\]

\[\square\]

We shall apply Corollary 1 to detect cosmic topological groups among cometrizable topological groups.

A topological space \(X\)

- is cosmic if it is a continuous image of a separable metrizable space;
- is cometrizable if \(X\) admits a weaker metrizable topology such that each point has a (not necessarily open) neighborhood base consisting of sets which are closed in the metric topology.

Cometrizable spaces were introduced by Gruenhage in \([5]\). The interplay between cometrizable spaces and other generalized metric spaces was studied in \([3, 4]\). It was proved in \([5, 6]\) that the class of cometrizable spaces includes all stratifiable and all sequential \(N_0\)-spaces. On the other hand, there exists a countable (and hence cosmic) space, which is not cometrizable.

In \([5]\) Gruenhage proved that under PFA a regular cometrizable space \(X\) is cosmic if and only if \(X\) has countable spread and contains no uncountable subspace of the Sorgenfrey line. In \([11, 8.5]\) Todorčević observed that this characterization remains true under OCA (which is a weaker assumption than PFA). Unifying Theorem 8.5 \([11]\) of Todorčević with Corollary 1 we obtain the following OCA-characterization of cosmic topological groups.
Corollary 2. Under OCA, a metrizable topological group is cosmic if and only if it has countable spread.

It is interesting that this OCA-characterization of cosmic metrizable groups does not hold under the Continuum Hypothesis (briefly, CH).

Theorem 3. Under CH there exists a metrizable topological group $G$ that contains an uncountable subspace of the Sorgenfrey line (and hence is not cosmic) but has hereditarily Lindelöf countable power $G^\omega$ (and hence $G^\omega$ has countable spread).

Theorem 3 will be proved in Section 3.

Remark 1. By [10], there exists a hereditarily Lindelöf topological group $G$ whose square is not normal. The topological group $G$ has countable spread but is not cosmic. Corollary 2 implies that the space $G$ is not metrizable under OCA.

Remark 2. Using the Continuum Hypothesis, Hajnal and Juhász [7] constructed a hereditarily separable Boolean topological group $G$ with uncountable pseudocharacter. This topological group has countable spread (being hereditarily separable) but is not hereditarily Lindelöf and not cosmic (because it has uncountable pseudocharacter).

2. Proof of Theorem 2

Theorems 1 and 2 will be deduced from the following

Lemma 1. Let $\kappa$ be a cardinal of uncountable cofinality and $X$ be a subspace of the Sorgenfrey line whose square contains a discrete subspace $\Gamma \subset X \times X$ of cardinality $|\Gamma| = \kappa$. If a topological group $G$ contains a subspace homeomorphic to $X$, then $G$ contains a discrete subspace of cardinality $\kappa$.

Proof. We shall identify the subspace $X$ of the Sorgenfrey line with a subspace of the topological group $G$. For every $x \in X$ and a rational number $q > x$ let

$$[x, q) = \{y \in X : x \leq y < q\}$$

be the order half-interval in $X$. Let also

$$\uparrow x = \{y \in X : x \leq y\}.$$ 

By the definition of the Sorgenfrey topology, the countable family $\{[x, q) : x < q \in \mathbb{Q}\}$ is a neighborhood base at $x$ in the space $X$.

Since the subspace $\Gamma \subset X \times X$ is discrete, each point $(x, y) \in \Gamma$ has a neighborhood $O_{(x,y)} \subset X \times X$ such that $\Gamma \cap O_{(x,y)} = \{(x, y)\}$. Find rational numbers $u_{(x,y)}, v_{(x,y)}$ such that

$$(x, y) \in [x, u_{(x,y)}) \times [y, v_{(x,y)}) \subset O_{(x,y)}.$$ 

Since the cardinal $|\Gamma| = \kappa$ has uncountable cofinality, for some rational numbers $u, v$ the set

$$\Gamma' = \{(x, y) \in \Gamma : u_{(x,y)} = u, \ v_{(x,y)} = v\}$$

has cardinality $|\Gamma'| = |\Gamma|$. Replacing the set $\Gamma$ by the set $\Gamma'$, we can assume that $u_{(x,y)} = u$ and $v_{(x,y)} = v$ for all $(x, y) \in \Gamma$.

Let

$$\Gamma_1 := \{x \in X : \exists y \in X \ (x, y) \in \Gamma\} \quad \text{and} \quad \Gamma_2 := \{y \in X : \exists x \in X \ (x, y) \in \Gamma\}$$
be the projections of the set $\Gamma \subset X \times X$ onto the coordinate axes. We claim that $\Gamma$ coincides with the graph of some strictly decreasing function $f : \Gamma_1 \to \Gamma_2$. First observe that for any $x \in \Gamma_1$ there exists a unique $y \in \Gamma$ with $(x, y) \in \Gamma$. Otherwise we could find two real numbers $y_1 < y_2$ with $(x, y_1), (x, y_2) \in \Gamma$ and conclude that

$$(x, y_2) \in [x, u] \times [y_2, v] \subset [x, u] \times [y_1, v] \subset O_{(x, y_1)},$$

which contradicts the choice of the neighborhood $O_{(x, y_1)}$. This contradiction shows that $\Gamma$ coincides with the graph of some function $f : \Gamma_1 \to \Gamma_2$. Let us show that this function is strictly decreasing. Assuming that this is not true, we could find two points $(x_1, y_1), (x_2, y_2) \in \Gamma$ with $x_1 < x_2$ and $y_1 \leq y_2$. Then

$$(x_2, y_2) \in [x_2, u] \times [y_2, v] \subset [x_1, u] \times [y_1, v] \subset O_{(x_1, y_1)},$$

which contradicts the choice of the neighborhood $O_{(x_1, y_1)}$.

Therefore the function $f : \Gamma_1 \to \Gamma_2$ is strictly decreasing, which implies that $|\Gamma_1| = |\Gamma_2| = |\Gamma| = \kappa$.

For any point $x \in X$ choose a neighborhood $V_x \subset G$ of the unit $e$ of $G$ such that

$$X \cap (V_x^{-1}V_x \cup xV_xV_x^{-1}) \subset \uparrow x.$$  

Next, for every point $x \in X$, choose a rational point $r_x > x$ such that $[x, r_x) \subset xV_{f(x)}$ if $x \in \Gamma_1$ and $[x, r_x) \subset V_{f^{-1}(x)}x$ if $x \in \Gamma_2$. Since the cardinal $|\Gamma_1| = \kappa$ has uncountable cofinality, for some $c, d \in \mathbb{Q}$ the set $Z = \{z \in \Gamma_1 : r_z = c, r_{f(z)} = d\}$ has cardinality $\kappa$.

We claim that the subspace $D := \{z \cdot f(z) : z \in Z\}$ has cardinality $\kappa$ and is discrete in $G$. For every $z \in Z$ consider the neighborhood $z(V_z \cap V_{f(z)})f(z)$ of the point $z \cdot f(z)$ in $G$. We claim that $x \cdot f(x) \notin z(V_z \cap V_{f(z)})f(z)$ for any $x \in Z \setminus \{z\}$. To derive a contradiction, assume that $x \cdot f(x) \in zV_{f(z)}f(z)$ for some $x \neq z$ in $Z$.

If $x > z$, then $x \in [z, r_z) \subset zV_{f(z)}$ and

$$f(x) = x^{-1}xf(x) \in x^{-1}zV_{f(z)}f(z) \subset V_{f(z)}^{-1}zV_{f(z)}f(z) = V_{f(z)}^{-1}V_{f(z)}f(z).$$  

Then

$$f(x) \in X \cap V_{f(z)}^{-1}V_{f(z)}f(z) \subset \uparrow f(z)$$

and $f(x) \geq f(z)$, which is not possible as $x > z$ and $f$ is strictly decreasing.

If $z > x$, then $f(x) > f(z)$ and

$$f(x) \in [f(x), r_{f(z)}) = [f(x), d) \subset [f(z), d) = [f(z), r_{f(z)}) \subset V_{f(z)}f(z)$$  

and then

$$x \in zV_{f(z)}f(x)^{-1} \subset zV_{f(z)}f(z)^{-1}V_{f(z)}^{-1} = zV_{f(z)}^{-1} \subset \uparrow z$$

which contradicts $z > x$. \hfill $\Box$

**Proof of Theorem** Assume that a topological group $G$ contains a topological copy of the Sorgenfrey line $\mathbb{S}$. Observe that the square of $\mathbb{S}$ contains a discrete subset $\Gamma = \{(x, -x) : x \in \mathbb{S}\}$ of cardinality continuum $\mathfrak{c}$. By [3, 5.12], the continuum has uncountable cofinality. Applying Lemma 1 we conclude that the topological group $G$ contains a discrete subspace of cardinality $\mathfrak{c}$. \hfill $\Box$
Proof of Theorem 2. Let \( G \) be a topological group containing a subspace \( X \), homeomorphic to an uncountable subspace of the Sorgenfrey line. Assuming that \( s(G) < s(X \times X) \), we conclude that \( s(X \times X) \geq \kappa^+ \) for the cardinal \( \kappa = s(G) \). Then \( X \times X \) contains a discrete subspace \( D \) of cardinality \(|D| = \kappa^+\), which has uncountable cofinality. In this case we can apply Lemma 1 and conclude that \( G \) contains a discrete subspace of cardinality \( \kappa^+ \), which implies that \( \kappa = s(G) \geq \kappa^+ > \kappa \) and this is a desired contradiction.

3. Proof of Theorem 3

In this section we prove Theorem 3. But first we prove that the Sorgenfrey line \( \mathbb{S} \) embeds into a metrizable topological group. In the proof of this embedding result, we use the \( k \)-separability of \( \mathbb{S} \).

A subset \( D \) of a topological space \( X \) is called \( k \)-dense in \( X \) if each compact subset \( K \subset X \) is contained in a compact set \( \tilde{K} \subset X \) such that the intersection \( D \cap \tilde{K} \) is dense in \( \tilde{K} \).

A topological space \( X \) is defined to be \( k \)-separable if it contains a countable \( k \)-dense subset.

Lemma 2. The set \( \mathbb{Q} \) of rational numbers is \( k \)-dense in the Sorgenfrey line \( \mathbb{S} \).

Proof. Given a compact set \( K \subset \mathbb{S} \), observe that \( K \) is metrizable and hence contains a countable dense subset \( \{x_n\}_{n \in \omega} \subset K \). For every \( n, k \in \omega \) fix a rational numbers \( x_{n,k} \) such that \( x_n < x_{n,k} < x_n + \frac{1}{2^n} \). We claim that the subset \( \tilde{K} = K \cup \{x_{n,k}\}_{n,k \in \omega} \) is compact.

Indeed, let \( \mathcal{U} \) be a cover of \( K \) by open subsets of \( \mathbb{S} \). For every \( x \in K \) find a set \( U_x \in \mathcal{U} \) with \( x \in U_x \) and a real number \( b_x \) such that \( [x, b_x) \subset U_x \). By the compactness of \( K \) the open cover \( \{[x, b_x) : x \in K\} \) of \( K \) has a finite subcover \( \{[x, b_x) : x \in F\} \) (here \( F \) is a suitable finite subset of \( K \)). For every \( x \in F \) the set \( [x, b_x) \) is closed in \( \mathbb{S} \) and hence the intersection \( K \cap [x, b_x) \) is compact, which implies that the number \( \varepsilon_x := b_x - \max(K \cap [x, b_x)) \) is strictly positive. Choose \( m \in \mathbb{N} \) such that \( \frac{1}{2^m} < \min_{x \in F} \varepsilon_x \). Then

\[
\tilde{K} \setminus \bigcup_{x \in F} [x, b_x) \subset \{x_{n,k} : n, k \leq m\}
\]

is finite and hence is contained in the union \( \bigcup F \) of some finite subfamily \( F \subset \mathcal{U} \). Then \( \mathcal{F} \cup \{U_x : x \in F\} \subset \mathcal{U} \) is a finite subcover of \( \tilde{K} \), witnessing that the subset \( \tilde{K} \) of \( X \) is compact. By the definition of \( \tilde{K} \), the set \( \tilde{K} \cap \mathbb{Q} = \{x_{n,k}\}_{n,k \in \omega} \) is dense in \( \tilde{K} \).

Lemma 2 implies that the Sorgenfrey line is \( k \)-separable. Now we prove that for any \( k \)-separable space \( X \) and a metrizable space \( Y \) the function space \( C_k(X, Y) \) is metrizable. Here for topological spaces \( X, Y \) by \( C_k(X, Y) \) we denote the space of continuous functions from \( X \) to \( Y \), endowed with the compact-open topology, which is generated by the subbase consisting of the sets

\[
[K, U] := \{f \in C_k(X, Y) : f(K) \subset U\}
\]

where \( K \) is a compact subset of \( X \) and \( U \) is an open subset of \( Y \).

Lemma 3. For any \( k \)-separable space \( X \) and any metrizable space \( Y \) the function space \( C_k(X, Y) \) is metrizable.
Proof. Let $D$ be a countable $k$-dense set in $X$ and $\tau$ be a metrizable topology on $Y$, witnessing that the space $Y$ is cometrizable. By $Y_\tau$ we denote the metrizable topological space $(Y, \tau)$.

The density of the set $D$ in $X$ ensures that the restriction operator

$$r : C_k(X, Y) \to Y_\tau^D, \quad r : f \mapsto f|D,$$

is injective. Let $\sigma$ be the (metrizable) topology on $C_k(X, Y)$ such that the map

$$r : (C_k(X, Y), \sigma) \to Y_\tau^D$$

is a topological embedding. We claim that the topology $\sigma$ witnesses that the space $C_k(X, Y)$ is cometrizable.

Fix any function $f \in C_k(X, Y)$ and an open neighborhood $O_f \subset C_k(X, Y)$. Without loss of generality, $O_f$ is of basic form $O_f = \bigcap_{i=1}^n [K_i, U_i]$ for some non-empty compact sets $K_1, \ldots, K_n \subset X$ and some open sets $U_1, \ldots, U_n \subset Y$. For every $i \leq n$ and point $x \in K_i$, find a neighborhood $V_{f(x)} \subset Y$ of $f(x) \in U_i$ whose $\tau$-closure $\overline{V}_{f(x)}$ is contained in $U_i$. Using the regularity of the cometrizable space $Y$, find two open neighborhoods $N_{f(x)}, W_{f(x)}$ of $f(x)$ such that

$$N_{f(x)} \subset W_{f(x)} \subset \overline{W}_{f(x)} \subset V_{f(x)}.$$

By the compactness of $K_i$, the open cover $\{f^{-1}(N_{f(x)}) : x \in K_i\}$ of $K_i$ has a finite subcover $\{f^{-1}(N_{f(x)}) : x \in F_i\}$ where $F_i \subset K_i$ is a finite subset of $K_i$. By the $k$-density of $D$ in $X$, for every $x \in F_i$ the compact set $K_{i,x} := K_i \cap f^{-1}(N_{f(x)})$ can be enlarged to a compact set $\tilde{K}_{i,x} \subset X$ such that $K_{i,x}$ is contained in the closure of the set $\tilde{K}_{i,x} \cap D$. Replacing the set $K_{i,x}$ by $\tilde{K}_{i,x} \cap f^{-1}(\overline{W}_{f(x)})$, we can assume that $f(\tilde{K}_{i,x}) \subset \overline{W}_{f(x)} \subset V_{f(x)}$.

Consider the open neighborhood

$$V_f = \bigcap_{i=1}^n \bigcap_{x \in F_i} [\tilde{K}_{i,x}, V_{f(x)}]$$

of $f$ in the function space $C_k(X, Y)$. We claim that its $\sigma$-closure $\overline{V}_f$ is contained in $O_f$.

Given any function $g \notin O_f$, we should find a neighborhood $O_g \in \sigma$ of $g$ that does not intersect $V_f$. Since $g \notin O_f$, there exists $i \leq n$ and a point $z \in K_i$ such that $g(z) \notin U_i$. Find a point $x \in F_i$ with $z \in K_{i,x}$. Taking into account that $\overline{V}_{f(x)} \subset U_i \subset Y \setminus \{g(z)\}$, we conclude that $g(z) \notin \overline{V}_{f(x)}$. Since the point $z$ belongs to the closure of the set $K_{i,x} \cap \overline{D}$, the continuity of the function $g : Z \to Y_\tau$ yields a point $d \in \tilde{K}_{i,n} \cap \overline{D}$ such that $g(d) \notin \overline{V}_{f(x)}$. Then $O_g := [(d), Y \setminus \overline{V}_{f(x)}] \in \sigma$ is a required $\sigma$-open neighborhood of $g$ that is disjoint with the neighborhood $V_f$. \hfill $\square$

Lemma 4. The Sorgenfrey line $\mathbb{S}$ admits a topological embedding into the cometrizable locally convex linear vector space $C_k(\mathbb{S})$.

Proof. By Lemma 2 the Sorgenfrey line $\mathbb{S}$ is $k$-separable, and by Lemma 3 the function space $C_k(\mathbb{S})$ is cometrizable. It remains to observe that the map $\chi : \mathbb{S} \to C_k(\mathbb{S})$ assigning to each point $x \in \mathbb{S}$ the function $\chi_x : \mathbb{S} \to \{0, 1\}$ defined by $\chi_x^{-1}(1) = [-x, \infty)$ is a
topological embedding of $S$ into the function space $C_k(S)$, which has the structure of a locally convex topological vector space.

\[ \square \]

**Proof of Theorem 3.** By Lemma 4, the Sorgenfrey line $S$ can be identified with a subspace of some cometrizable Abelian topological group $H$. According to Michael [9], under CH the Sorgenfrey line contains an uncountable subspace $X$ whose countable power $X^\omega$ is hereditarily Lindelöf. Observe that the topological sum $X^{<\omega} = \bigoplus_{n\in\omega} X^n$ of finite powers of $X$ admits a topological embedding into $X^\omega$, which implies that $X^{<\omega}$ is hereditarily Lindelöf as well as its countable power $(X^{<\omega})^\omega$.

Observing that the group hull $G$ of $X$ in the group $H \supset S \supset X$ is a continuous image of $X^{<\omega}$, we conclude that the space $G$ is hereditarily Lindelöf. Moreover, the countable power $G^\omega$ is hereditarily Lindelöf, being a continuous image of the hereditarily Lindelöf space $(X^{<\omega})^\omega$.

\[ \square \]

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ПРО СПРЕД ТОПОЛОГІЧНИХ ГРУП, ЩО МІСТЯТЬ ПІДМНОЖИНИ СТРІЛКИ ЗОРГЕНФРЕЯ

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Доведено, що топологічна група $G$, яка містить підпростір $X$ стрілки Зоргенфрея, має спред $s(G) \geq s(X \times X)$. В припущені ОСА, довільна топологічна група, що містить несівічений підпростір стрілки Зоргенфрея має несівічений спред. Звідси випливає, що при ОСА кометризована топологічна група має зівічну сітку тоді і лише тоді, коли вона має зливачений спред. З іншого боку, при ЧН існує кометризована абелева топологічна група, що має спадкову лінійну сітку зливачну степінь і містить деякий несівічний підпростір стрілки. Ця топологічна група має зливачений спред, проте не має зливачної сітки.

Ключові слова: стрілка Зоргенфрея, топологічна група, спред, ОСА, ЧН.