Star–Matrix Models

E.Vinteler
International School for Advanced Studies (SISSA/ISAS)
Via Beirut 2, 34014 Trieste, Italy
e-mail:vinteler@gandalf.sissa.it

Abstract

The star-matrix models are difficult to solve due to the multiple powers of the Vandermonde determinants in the partition function. We apply to these models a modified Q-matrix approach and we get results consistent with those obtained by other methods. As examples we study the inhomogenous gaussian model on Bethe tree and matrix $q$-Potts-like model. For the last model in the special cases $q = 2$ and $q = 3$, we write down explicit formulas which determinate the critical behaviour of the system. For $q = 2$ we argue that the critical behaviour is indeed that of the Ising model on the $\phi^3$ lattice.
1 Introduction

The star-matrix models were considered first in [1] in connection with the q-Potts model and percolation problem. Another field of interest is the "induced" QCD which in the large $N$ limit is equivalent with the star-model on Bethe tree. The star-models are also a direct generalization of the $c = 1$ matrix model, which is the particular case for $q = 1$.

The difficulties in solving exactly these models are related with the multiple powers of the Vandermonde determinants in the partition function. In the Q-matrix approach, another problem related with the first, is the choosing of a proper basis in which to define consistently the Q-matrices.

In this work we show that it is possible to apply a modified Q-matrix approach and to define consistently the Q-matrices. The method gives consistent results for the cases where other methods can be applied: the saddle-point method, Schwinger-Dyson approach, gaussian integration etc. The gaussian models and matrix model on Bethe tree give similar results with the previous ones. However, our method can be applied in all genera, and permits a more precise study of particular parts of the Bethe tree.

In the last section we consider the matrix formulation of the q-Potts-like model and study it as a polymer on a random surface. It is important that we still get consistent results for this model. It is a non-trivial result, because the coupling conditions are overdetermined: the Q-matrices depend on $3q + 3$ variables and we have $4q + 3$ equations. More generally when the central potential $V_0$ is gaussian and the lateral potentials $V_\alpha$ are of order $n$ we have $3q + n$ variables and $(n + 1)q + 3$ equations always with compatible solutions.

2 Q-matrix approach

The Q-matrix approach was introduced for the 1-matrix model [1] and further developed for the two-matrix and the chain multi-matrix models in [2]. Here we are applying this approach to another class of multi-matrix models – the star-matrix models.

The partition function of the $q$-star model is given by:

$$Z = \int \prod_{\alpha=1}^{q} dM_\alpha dM_0 \exp(V_0 + \sum_{\alpha} V_\alpha + M_0 \sum_{\alpha=1}^{q} c_\alpha M_\alpha)$$  \hspace{1cm} (2.1)

with the potentials $V_\alpha = \sum_{r=1}^{p_\alpha} t_{\alpha,r} M_\alpha^r$, $\alpha = 0, 1 \ldots q$. It is possible to integrate over the angular degrees of freedom and to remain with integrals only over the eigenvalues:

$$Z = \int \prod_{i=1}^{N} \prod_{\alpha=1}^{q} \Delta(\lambda_{0,i}) \Delta(\lambda_{\alpha,i})^{2-q} \prod_{\alpha=1}^{q} \Delta(\lambda_\alpha) e^V$$  \hspace{1cm} (2.2)

with

$$V = \sum_{i=1}^{N} (V_0(\lambda_{0,i}) + \sum_{\alpha=1}^{q} V_\alpha(\lambda_{\alpha,i}) + \lambda_{0,i} \sum_{\alpha=1}^{q} c_\alpha \lambda_{\alpha,i})$$
We define the orthogonal functions basis as $\xi_n$ and $q + 1$-conjugate functions $\eta_{\alpha,m}$:

$$\int d\lambda_0 \prod_{\alpha=1}^{q} d\lambda_\alpha \xi_{q,n}^\alpha(\lambda_0) e^{V_\alpha} \prod_{\alpha=1}^{q} \eta_{\alpha,m_\alpha}(\lambda_\alpha) = h_n \delta_{nm} \prod_{\alpha=1}^{q} \delta_{m,m_\alpha} \tag{2.3}$$

The basic functions have the property:

$$\frac{h_{\alpha,m}}{h_{\alpha,m}} \eta_{0,m}(\lambda_0) = \int d\lambda_\alpha e^{V_\alpha+c_\alpha \lambda_0 \lambda_\alpha} \eta_{\alpha,m}(\lambda_\alpha) \tag{2.4}$$

Introducing this relation (2.4) in the orthogonality condition (2.3) we get:

$$\int d\lambda_0 \xi_{q,0,n}^q(\lambda_0) e^{V_0} \prod_{\alpha=1}^{q} \eta_{0,m_\alpha}(\lambda_0) = h_n \prod_{\alpha=1}^{q} \delta_{n,m_\alpha} \tag{2.5}$$

Because the basic functions $\xi_{0,n}(\lambda_0), \eta_{0,n}(\lambda_0)$ at power $q$ are linear combinations of basic functions:

$$\xi_{q,0,n}(\lambda_0) = \xi_{0,nq}(\lambda_0) + \sum_{k=1}^{nq} a_k \xi_{0,nq}(\lambda_0), \quad \eta_{q,0,n}(\lambda_0) = \eta_{0,nq}(\lambda_0) + \sum_{k=1}^{nq} \pi_k \eta_{0,nq}(\lambda_0) \tag{2.6}$$

we can show that the integral (2.7) follows from the orthogonal condition:

$$\int d\lambda_0 \xi_{0,n}(\lambda_0) e^{V_0} \eta_{0,m}(\lambda_0) = h_n \delta_{n,m} \tag{2.7}$$

and:

$$h_n = n_{0,nq} + \sum_{k=1}^{nq} |a_k|^2 h_{0,nq-k}$$

Inserting the expression of $\eta_{0,m}$ (2.4) in the relation (2.3) we get the orthogonal condition:

$$\int d\lambda_0 d\lambda_\alpha \xi_{q,0,n}(\lambda_0) e^{V_0+V_\alpha+c_\alpha \lambda_0 \lambda_\alpha} \eta_{q,m}(\lambda_\alpha) = h_{\alpha,n} \delta_{n,m} \tag{2.8}$$

The property (2.4) gives the possibility to integrate over Vandermonde determinants:

$$\prod_{i=0}^{N-1} \frac{h_{\alpha,i}}{h_{0,i}} \det[\eta_{0,i}(\lambda_{0,j})] = \int d\lambda_\alpha e^{V_\alpha+c_\alpha \lambda_0 \lambda_\alpha} \det[\eta_{\alpha,i}(\lambda_{\alpha,j})]$$

and permits to calculate the partition function:

$$Z = \text{const.} N! \prod_{i=0}^{N-1} h_{0,i}^{-q} \left( \prod_{\alpha=1}^{q} h_{\alpha,i} \right) \tag{2.9}$$

We introduce the $Q$-matrices as:

$$\int d\lambda_0 \prod_{\alpha=1}^{q} d\lambda_\alpha \xi_{q,n}^\alpha(\lambda_0) e^{V_\alpha} \prod_{\alpha=1}^{q} \eta_{\alpha,m_\alpha}(\lambda_\alpha) = h_n Q_{\alpha,nm} \prod_{\alpha=1}^{q} \delta_{m,m_\alpha} \tag{2.10}$$
and the $P$-matrices as:

\[
\int d\lambda_0 \prod_{\alpha=1}^{q} d\lambda_\alpha \xi^n_{\alpha}(\lambda_0) \frac{\partial}{\partial \lambda_0} \left( \xi_\alpha(\lambda_0) e^{V_0 + \lambda_0 \sum_{\alpha=1}^{q}} \right) e^{\sum_{\alpha=1}^{q} V_\alpha \prod_{\alpha=1}^{q} \eta_{\alpha,m_\alpha}(\lambda_\alpha) = (2.11)}
\]

\[
= P_{0,nm} h_m \prod_{\alpha=1}^{q} \delta_{m,m_\alpha}
\]

\[
\int d\lambda_0 \prod_{\alpha=1}^{q} d\lambda_\alpha \xi^n_{\alpha}(\lambda_0) e^{V_0 - c_\alpha \lambda_\alpha \lambda_\alpha} \prod_{\alpha=1}^{q} \frac{\partial}{\partial \lambda_\beta} \left( \eta_{\beta,m_\beta}(\lambda_\beta) e^{V_\alpha + c_\alpha \lambda_0 \lambda_\alpha} \right) \prod_{\alpha \neq 0, \beta} \eta_{\alpha,m_\alpha}(\lambda_\alpha) =
\]

\[
= h_n P_{\beta,nm} \prod_{\alpha=1}^{q} \delta_{m,m_\alpha}
\]

We can now derive the coupling conditions:

\[
q P_0 + V_0'(Q_0) + \sum_{\alpha=1}^{q} c_\alpha Q_\alpha = 0 \tag{2.12}
\]

\[
\mathcal{P}_\alpha + V_\alpha'(Q_\alpha) + c_\alpha Q_0 = 0, \quad \alpha = 1, \ldots q
\]

We consider only the symmetric case when the order of potentials is $p_\alpha = p$. The calculation of degree for matrices gives:

\[
Q_\alpha \in [-m,n], \quad \text{for all } \alpha
\]

\[
Q_0 \in [-m_0,n_0], \tag{2.13}
\]

where $m = 1, n = p_0 - 1; m_0 = p - 1, n_0 = 1$.

The notation shows that we have $Q_\alpha$-matrices with finite band with $m$ lower and $n$ higher diagonals.

We consider only 1-point correlation functions, hence we do not need the flow equations. The 1-point correlation functions can be calculated in the usual way as:

\[
< \text{Tr} M_1^{k_1} \cdots M_q^{k_q} >= \text{Tr}(Q_1^{k_1} \cdots Q_q^{k_q}) \tag{2.14}
\]

These 1-point correlation functions can be calculated in every genus $h$ and in principle for arbitrary potentials, not only gaussian.

## 3 The gaussian model

After the exact solution of $c = 1$ matrix model (or chain q-multimatrix model) with gaussian potential, the star q-multimatrix model is an obvious target for study. Even with a trivial gaussian potential, it could give an interesting string description. For example, due to the additional permutation symmetry $S_q$, the tachyonic field structure can be quite different from the original chain model. The 3-star matrix model is also the first one in the class of matrix models having the target space- the $D_n$ Dynkin diagramm \[\text{[ Diagram]\[3].}\]
The gaussian star-matrix model has the partition function:

\[ Z = \int \prod_{\alpha=1}^{q} dM_{\alpha} dM_{0} \exp[t_{0}M_{0}^{2} + u_{0}M_{0} + M_{0} \sum_{\alpha=1}^{q} c_{\alpha}M_{\alpha} + \sum_{\alpha=1}^{q} (t_{\alpha}M_{\alpha}^{2} + u_{\alpha}M_{\alpha})] \]  (3.1)

The coupling conditions are:

\[ qP_{0} + 2t_{0}Q_{0} + u_{0} + \sum_{\alpha=1}^{q} c_{\alpha}Q_{\alpha} = 0 \]

\[ P_{\alpha} + 2t_{\alpha}Q_{\alpha} + u_{\alpha} + c_{\alpha}Q_{0} = 0, \quad \alpha = 1, \ldots q \]  (3.2)

With the following parametrization of \( Q \)-matrices:

\[ Q_{0} = I_{+} + a_{0}I_{0} + a_{1}\epsilon_{-} \]

\[ Q_{\alpha} = b_{\alpha}/R_{\alpha}I_{+} + d_{\alpha}I_{0} + R_{\alpha}\epsilon_{-}, \quad \alpha = 1, \ldots q \]  (3.3)

we arrive at following equations:

\[ 2t_{\alpha}R_{\alpha} + c_{\alpha}a_{1} = 0 \]
\[ 2t_{\alpha}b_{\alpha} + n + c_{\alpha}R_{\alpha} = 0 \]
\[ 2t_{\alpha}d_{\alpha} + u_{\alpha} + c_{\alpha}a_{0} = 0 \]
\[ 2t_{0} + \sum_{\alpha} c_{\alpha}b_{\alpha}R_{\alpha} = 0 \]  (3.4)
\[ 2t_{0}a_{0} + u_{0} + \sum_{\alpha} c_{\alpha}d_{\alpha} = 0 \]
\[ 2t_{0}a_{1} + qn + \sum_{\alpha} c_{\alpha}R_{\alpha} = 0 \]

Solving the coupling conditions we get:

\[ a_{1} = \frac{2q}{A}, a_{0} = \frac{1}{A} \left( \sum_{\alpha} c_{\alpha}u_{\alpha}/t_{\alpha} - 2u_{0} \right) \]
\[ b_{\alpha} = -\frac{1}{2t_{\alpha}^{2}}(c_{\alpha}q + t_{\alpha}), R_{\alpha} = c_{\alpha}q/t_{\alpha}A \]
\[ d_{\alpha} = \frac{1}{A} \left( c_{\alpha}u_{0} - 2t_{0}u_{\alpha} + u_{\alpha} \sum_{\beta} \frac{c_{\beta}^{2}}{2t_{\beta}} - c_{\alpha} \sum_{\beta} \frac{c_{\beta}u_{\beta}}{2t_{\beta}} \right) \]

where \( A = 4t_{0} - \sum c_{\alpha}^{2}/t_{\alpha} \).

For quadratic potentials \( V_{\alpha} \), the basic functions are Hermite polynomials:

\[ \xi_{n}(\lambda) = \eta_{n}(\lambda) = H_{n}(\frac{\lambda_{0} - a_{0}}{\sqrt{2a_{1}}}), \]  (3.5)

\[ \eta_{\alpha,n}(\lambda_{\alpha}) = H_{n}(\frac{\lambda_{\alpha} - d_{\alpha}}{\sqrt{2b_{\alpha}}}, \alpha = 1 \ldots q \]

To calculate the partition function we observe that:

\[ h_{0,n} = \frac{1}{A}, h_{\alpha,n} = R_{\alpha}^{n} \]  (3.6)
giving:

\[ Z = \text{const} \left( \frac{1}{A} \prod_{\alpha=1}^{q} \frac{c_{\alpha}}{t_{\alpha}} \right)^{N^2} \]  

(3.7)

\text{const} \text{ in our case is the exponent } \exp[-1/4(Aa_{0}^2 + \sum_{\alpha} u_{\alpha}^2/t_{\alpha})] \text{ obtained by shifting the matrices } M_{\alpha} \text{ so that the linear terms in the potential vanish.}

The partition function and the first simplest 1-point correlation functions as \( \text{Tr} Q_{\alpha}^{2} \), \( \text{Tr} Q_{\alpha} \) can be calculated by direct integration of the original integral (3.1). This is not the case for more complicated 1-point correlation functions as those given by (2.14). Instead with the Q-matrix approach this is easy, using the explicit form of Q-matrices. In the Dyson-Schwinger approach, this calculation is also possible but only on the sphere.

As an example we give the result for the 1-point correlation function (when \( u_{0} = u_{\alpha} = 0 \) and \( a_{0} = d_{\alpha} = 0 \)):

\[ \text{Tr} Q_{\alpha}^{n} Q_{\beta}^{m} = \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{n!m!(-1)^{k}(l+k)}{i!j!(n-i)!(m-j)!} b_{\alpha}^{n-i} b_{\beta}^{m-j} R_{\alpha}^{2i} R_{\beta}^{2j} N \frac{n+m+1}{2} \]  

(3.8)

\[ b_{\alpha}^{n-i}(i+j+l) b_{\beta}^{m-j}(j+l) R_{\alpha}^{2i} R_{\beta}^{2j} N \frac{n+m+1}{2} \]  

(3.9)

In the large \( N \) limit we have \( l + k = 0 \) or \( l = 0, k = 0, i + j = \frac{n+m}{2} \) and the previous formula simplifies to:

\[ \text{Tr} Q_{\alpha}^{n} Q_{\beta}^{m} = \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{n!m!}{i!j!(n-i)!(m-j)!} b_{\alpha}^{n-i} b_{\beta}^{m-j} R_{\alpha}^{2i} R_{\beta}^{2j} N \frac{n+m+1}{2} \]  

(3.9)

For \( m = 0 \) and \( n = 2r \) we obtain the correlation function in genus 0:

\[ \text{Tr} Q_{\alpha}^{2r} = \frac{(2r)!}{r!(r+1)!} N^{r+1} (b_{\alpha})^{r} \]  

(3.10)

We have sum over \( i = r, j = 0 \).

4 The gaussian model on Bethe tree

Kazakov and Migdal [7] have obtained the so-called induced "induced QCD" - a matrix model embedded in the regular D-dimensional lattice. For gaussian potential, the model was solved by Gross [8]. In the limit \( N \rightarrow \infty \) the Kazakov-Migdal model with generic potential is equivalent to the matrix model with a Bethe tree target space [9].

Because the model was studied only with the saddle-point method, it is interesting to study it in a different framework, that of Q-matrices approach. It gives higher accuracy in studying different regions of Bethe tree and also permits computations in higher genera.

Our model is the inhomogenous version of matrix model on Bethe tree, at every level of branch we assigna a specific partition function and propagator.
The gaussian matrix model on Bethe tree is \((i = 2j)\):

\[
Z = \int dM_i \exp \text{Tr} \left[ \sum_i (t_i M_i^2 + u_i M_i) + \sum_{<ij>} c_{ij} M_i M_j \right]
\]

(4.1)

where \(<ij>\) denotes the permitted links of Bethe tree, and \(c_i\) are the coupling constants of the \(i\)-th level branch.

The coupling conditions are:

\[
q P_i + 2t_i Q_i + u_i + c_i Q_{i-1} + (q - 1)c_{i+1} Q_{i+1} = 0
\]

\[
q P_{i+1} + 2t_{i+1} Q_{i+1} + u_{i+1} + c_{i+1} Q_i + (q - 1)c_{i+2} Q_{i+2} = 0
\]

(4.2)

Introducing \(Q\)-matrices with the form:

\[
Q_i = S_i I_+ + a_i I_0 + f_i \epsilon_-
\]

\[
Q_{i+1} = b_i / R_i I_+ + d_i I_0 + R_i \epsilon_-
\]

(4.3)

we get from the coupling conditions the following equations for coefficients:

\[
2t_i S_i + c_i b_{i-1} / R_{i-1} + (q - 1)c_{i+1} b_i / R_i = 0
\]

\[
2t_i a_i + u_i + c_i d_{i-1} + (q - 1)c_{i+1} d_i = 0
\]

\[
2t_i f_i + \frac{q n}{S_i} + c_i R_{i-1} + (q - 1)c_{i+1} R_i = 0
\]

(4.4)

\[
2t_{i+1} R_i + c_{i+1} f_i + (q - 1)c_{i+2} f_{i+1} = 0
\]

\[
2t_{i+1} d_i + u_{i+1} + c_{i+1} a_i + (q - 1)c_{i+2} a_{i+1} = 0
\]

\[
2t_{i+1} b_i / R_i + q n / R_i + c_{i+1} S_i + (q - 1)c_{i+2} S_{i+1} = 0
\]

Let suppose that all coefficients for various \(i\) are equal \(c_i = c, t_i = t, u_i = u\). In this case the equations reduce to the following set of equations:

\[
f(S_i) + g(R_{i-1}) = 0, \quad f(R_i) - g(S_i) = 0
\]

\[
f(a_i) + K = 0, \quad f(d_i) + K = 0
\]

\[
f\left(\frac{b_i}{R_i}\right) - \frac{2qnt}{c^2} \frac{1}{R_i} = 0, \quad f(f_i) - \frac{2qnt}{c^2} \frac{1}{S_i} = 0
\]

(4.5)

where the constant \(K = (q - \frac{2q}{c})u\) and the functions \(f, g\) are:

\[
f(x_i) = x_{i-1} + \left[ 2(q - 1) - 4 \left( \frac{t}{c} \right)^2 \right] x_i + (q - 1)^2 x_{i+1}
\]

\[
g(x_i) = \frac{q n}{c} \left( \frac{1}{x_i} + \frac{q - 1}{x_{i+1}} \right)
\]

(4.6)

### 4.1 Fractal regime

In the scaling limit \(R_i \rightarrow R, S_i \rightarrow S, b_i \rightarrow b, f_i \rightarrow f, d_i \rightarrow d, a_i \rightarrow a\) and (we take \(n = 1\)):

\[
RS = \frac{q^2 c}{4t^2 - (qc)^2},
\]
\[ b = fS = -\frac{qt}{4t^2 - (qc)^2} \]  
\[ a = d = \frac{qc - 2t}{4t^2 - (qc)^2}u \]  

This case represents the limiting case of a matrix model on the fractal curve of Bethe tree. We can define the free energy on the unit length:

\[ F_{frac} = \log RS = \log \frac{q^2c}{4t^2 - (qc)^2} \]  

We observe a singularity of the free energy at \((2t/c)^2 = 1\). This critical point is the analog of critical point for \(c = 1\) matrix model (or \(q\)-multimatrix model) at the self-dual radius \(R^2 = 1\). We must consider the physical domain when \(2t < qc\) \((t < 0\) to have a well-defined path-integral). We choose \(c < 0\). We see that the other region \(2t > qc, c > 0\) is not reached.

We can calculate the 1-point correlation function (for simplicity we take \(u = 0\) and genus 0)

\[ \langle \text{Tr}M_i^{2k}\rangle_0 = \langle \text{Tr}M_{i+1}^{2k}\rangle_0 = \text{Tr}(Q^{2k}) = b^k \text{Tr}(I_+ + \epsilon_-)^{2k} = \frac{(2k)!N^{k+1}}{k!(k + 1)!} \left(\frac{-qt}{4t^2 - (qc)^2}\right)^k \]  

\[ 4.9 \]

### 4.2 Asymptotic regime

We have considered the fractal curve or a tiny strip of surface which is filling densely the extremity of the Bethe tree. We can study a larger strip which tends asymptotically to the fractal curve.

In the large \(N\) limit we can scale the coefficients \(R_i \rightarrow \sqrt{NR_i(x)}, S_i \rightarrow \sqrt{NS_i(x)}, b_i \rightarrow Nb_i(x), f_i \rightarrow \sqrt{N}f_i(x), d_i \rightarrow d_i(x), a_i \rightarrow a_i(x)\), where \(x = n/N\). We can see now that all second terms in the equations (4.6) are proportional with \(x, 0 \leq x < 1\) and can be considered as perturbations. We neglect the function \(g\). In this case we can solve the recursion relations (4.6) with the result:

\[ R_i = r^i, S_i = s^i, \]
\[ a_i = r^i - \frac{2t}{q^2c}K, d_i = s^i - \frac{2t}{q^2c}K, K = \frac{qc u}{2t} - u \]  

\[ 4.10 \]

where \(r = r_+, s = r_-\) or viceversa with \(r_\pm\) being the solution of second order eq.:

\[ (q - 1)^2r_\pm^2 + (2(q - 1) - \frac{4t^2}{c^2})r_\pm + 1 = 0 \]  

\[ 4.11 \]

and \(b = b_+, f = b_-\) or viceversa with \(b_\pm\) being the solution of second order eq.:

\[ (q - 1)^2b_\pm^2 + (2(q - 1) + \frac{2tq}{c^2} - \frac{4t^2}{c^2})b_\pm + 1 = 0 \]  

\[ 4.12 \]

We see that the alternating \(Q_i, Q_{i+1}\) in the asymptotic regime can be interchanged; hence it does not matter if \(i\) is odd or even.
We can define also in this regime the free energy on the unit length:

\[ F_{i,i+2} = \log(q - 1)^{2i} R_i S_i = 0 \]  

(4.13)

(q must be bigger than 2 to have at least one branch). The free energy is 0 and is different from \( F_{frac} \). This is understandable because \( F_{frac} \) is proportional with \( x \), but we have considered the case when \( x = 0 \), hence \( F_{i,i+2} = 0 \). To see if \( F_{i,i+2} \) really tends to \( F_{frac} \) we must include the perturbation in \( x \).

As we said before, our model differs from the one studied by Gross and Boulatov [8][9]. Their model is homogeneous; for the Bethe lattice with coordination number \( 2D \) their partition function is:

\[
Z = \int dM \exp \left[ -\sum_i \frac{m^2}{2} M_i^2 + \sum_{<ij>} M_i M_j \right] = \int dM Z(M)^{2D} \exp \left( -\frac{m^2}{2} \text{Tr} M^2 \right)
\]

where the partition function of a branch \( Z(M) \) satisfies the equation:

\[
Z(M) = \int dM' Z(M')^{2D-1} \exp \left( -\frac{m^2}{2} M'^2 + M'M \right)
\]

Our model is inhomogeneous. Every branch of different level \( i \) has a different partition function \( Z_i(M) \). Hence the total partition function is:

\[
Z = \int dM Z(M)^q \exp \left( -\frac{t}{c} \text{Tr} M^2 \right)
\]

and \( Z_i(M) \) satisfies the equation:

\[
Z_i(M) = \int dM' Z_{i+1}(M')^{q-1} Z_{i-1}(M') \exp \left( -\frac{t}{c} M'^2 + M'M \right)
\]  

(4.14)

We see that the partition function for \( i \)-th level branch is expressed not only in terms of higher level branches \( (i + 1) \), but we have also the back-reaction on the lower level branches \( (i - 1) \). We also observe that the eq. (4.14) is different for \( i \) odd or even.

Solving the equation is equivalent with the first two equations (4.6). Solving them as recursion equations we get the power-like solution (4.11)

\[
Z_j(M) = r^j \text{Tr} M^2,
\]

where \( r \) satisfies the second order equation (4.11). If we solve (4.6) as differential equations (after proper scaling when \( |(i+1) - i| \ll i \)) we get the exponential solution

\[
Z_j(M) = \exp( j r \text{Tr} M^2 )
\]

where \( r \) is (from equation (4.11)):

\[
r_{\pm} = \frac{2 \left( \frac{t}{c} \right)^2 - (q - 1) \pm 2 \left( \frac{t}{c} \right) \sqrt{ (\frac{t}{c})^2 - (q - 1) } }{ (q - 1)^2 }
\]

The signs alternate for \( i \) odd and even. This result can be compared with that of homogenous model if we identify \( t/c = m^2/2, q = 2D \). The partition function per branch for homogenous model behaves as \( Z(M) = \exp(-\alpha \text{Tr} M^2) \), where \( \alpha \) is:

\[
\alpha_{\pm} = \frac{m^2(D - 1) \pm D \sqrt{m^4 - 4(2D - 1) } }{2D - 1}
\]

We have an interesting property in the asymptotic regime: the coefficient \( R_k \) near the point \( \frac{1}{q-1} (t/c)^2 = 1/2 \) has a slow oscillation with the period \( T_{\Delta k} \).
The coefficient $R_k$ is directly related with the free energy of branches of level between $k$ and $k+1$:

$$F_{k,k+1} = \log(q-1)^{2k}R_k$$

$S_k$ has a complementary oscillation such that $F_{k,k+1}(R_k)+F_{k+1,k+2}(S_k) = F_{k,k+2} = 0$. This behaviour is typical to an antiferromagnet: $R_k, S_k$ are like spin-up and spin-down configurations which group pair-wise having a total energy zero.

We take for convenience $R_0 = 1, R_1 = 0$. The coefficient $R_k$ is:

$$R_k = \frac{r_+^{k+1} - r_+^{k+1}}{r_+ - r_-}$$

where $r_\pm$ are the solutions of the equation (5.11). We introduce the notation $\beta = \frac{1}{q-1}(t/c)^2$ and consider the region $0 \leq \beta \leq 1$. In this case:

$$r_\pm = \frac{1}{(q-1)^2}e^{\pm \omega}, \quad \arctan \omega = \frac{2\sqrt{\beta(1-\beta)}}{2\beta - 1}$$

(4.15)

In this region $R_k = (q-1)^{-2k} \frac{\sinh k\omega}{\sinh \omega}$ has a fast decaying in amplitude (for $q > 2$) and an oscillatory character with pulsation $\omega$. For $\beta \sim \frac{1}{2}$ the expression (5.16) is singular and the pulsation is $\omega \sim \frac{\pi}{2}$. If we take $\beta = \frac{1}{2} + \epsilon$ then $\omega = \frac{\pi}{2} + \Delta \omega$ and $\Delta \omega \sim \epsilon$. This property induces a modulation of the oscillation with the period:

$$T_{\Delta k} \sim \left(1 - \frac{2}{q-1} \left(\frac{t}{c}\right)^2\right)^{-1}$$

(4.16)

For $\beta \to 1/2$ the modulation disappears because the period $T_{\Delta k} \to \infty$.

5 The q-Potts-like model

q-state Potts spins are an interesting generalization of the Ising model ($q = 2$). On planar random lattice they were studied first time by Kazakov [1] [2]. The cases $q \to 0$ and $q \to 1$ represent the models of tree-polymers, respectively that of percolation. The $q = 2$ case, that of Ising model on random lattice, can be expressed in terms of the 2-matrix model.

The partition function of q-state Potts model on a random lattice is:

$$Z(g, \beta, H) = \sum_n g^n \sum_{\{G^{(n)}\}} \sum_{\{\sigma\}} \exp[-\beta \sum_{k,j} G_{k,j}^{(n)}(\delta_{\sigma_k \sigma_j} - 1) + H \sum_k (\delta_{1, \sigma_k} - 1)]$$

(5.1)

where the summ run over all triangulations with $n$ triangles $\{G^{(n)}\}$ and all spin configurations $\{\sigma\}$. $G_{k,j}^{(n)}$ is the adjacency matrix of planar lattice with $n$ vertices, $\beta$- inverse temperature, $H$- magnetic field.

For zero magnetic field, the partition function can be expressed in terms of the matrix model:

$$Z = \int \prod_{\alpha=1}^q dM_\alpha \exp[2c \sum_{\alpha > \beta} M_\alpha M_\beta - \sum_{\alpha=1}^q (M_\alpha^2 + g/3M_\alpha^3)]$$

(5.2)
The partition functions (5.2) and (5.1) are equal due to the equivalence of the Feynman graphs of matrix model on dual lattice with inverse temperature $\beta_{dual} = \log(1 + q(e^\beta - 1)^{-1})$ and the Boltzmann weights of Potts model on the original lattice.

Introducing a new matrix $M_0$ the previous integral can be rewritten as:

$$Z = \int \prod_{\alpha=1}^{q} dM_{\alpha} dM_0 \exp[\tilde{c}M_0 \sum_{\alpha=1}^{q} M_{\alpha} - M_0^2/2 - \sum_{\alpha=1}^{q} (M_{\alpha}^2/2 - \tilde{g}/3M_{\alpha}^3)]$$  (5.3)

with $\tilde{c}^2 = c/(1 + c), \tilde{g}^2 = g^2/(2(1 + c))^3$. The coupling constant $c$ is connected with the inverse temperature $\beta$ by the formula

$$c^2 = (e^\beta + q - 1)^{-1}$$

We consider the star-matrix model with partition function:

$$Z = \int \prod_{\alpha=1}^{q} dM_{\alpha} dM_0 \exp[t_0 M_0^2 + u_0 M_0 + M_0 \sum_{\alpha=1}^{q} c_\alpha M_{\alpha} +]$$

$$+ \sum_{\alpha=1}^{q} (s_\alpha M_{\alpha}^3 + t_\alpha M_{\alpha}^2 + u_\alpha M_{\alpha})]$$  (5.4)

The coupling conditions are:

$$qP_0 + 2t_0 Q_0 + u_0 + \sum_{\alpha=1}^{q} c_\alpha Q_\alpha = 0$$

$$P_\alpha + 3s_\alpha Q_\alpha^2 + 2t_\alpha Q_\alpha + u_\alpha + c_\alpha Q_0 = 0, \alpha = 1, \ldots q$$

With the following parametrization of $Q$-matrices:

$$Q_0 = I_+ + a_0 I_0 + a_1 I_- + a_2 I_{-2}$$

$$Q_\alpha = b_\alpha / R_\alpha I_+ + d_\alpha I_0 + R_\alpha I_{-\alpha}, \alpha = 1, \ldots q$$

we arrive at following equations:

$$\begin{align*}
3s_\alpha R_{\alpha}^2 + c_\alpha a_2 &= 0 \\
6s_\alpha R_\alpha d_\alpha + 2t_\alpha R_\alpha + c_\alpha a_1 &= 0 \\
6s_\alpha b_\alpha d_\alpha + 2t_\alpha b_\alpha + n + c_\alpha R_\alpha &= 0 \\
3s_\alpha (d_\alpha^2 + 2b_\alpha) + 2t_\alpha d_\alpha + u_\alpha + c_\alpha a_0 &= 0
\end{align*}$$

$$\begin{align*}
2t_0 + \sum_{\alpha=1}^{q} \frac{c_\alpha b_\alpha}{R_{\alpha}} &= 0 \\
2t_0 a_0 + u_0 + \sum_{\alpha=1}^{q} c_\alpha d_\alpha &= 0 \\
2t_0 a_1 + qn + \sum_{\alpha=1}^{q} c_\alpha R_\alpha &= 0
\end{align*}$$  (5.5)

5.1 Symmetric case

We solve the special symmetric case when $s_\alpha = s, t_\alpha = t, u_\alpha = u, c_\alpha = c$. In this case $R_\alpha = R, d_\alpha = d, b_\alpha = b$.

We can express all the coefficients in terms of only two of them $R$ and $d$:
\[ a_2 = -\frac{3s}{c} R^2, a_1 = -\frac{q(n + cR)}{2t_0}, a_0 = -\frac{u_0 + qcd}{2t_0}, b = -\frac{2t_0R}{qc} \] 

(5.7)

\[ R \text{ and } d \text{ satisfy a system of 2 non-linear equations:} \]

\[ d = -\frac{t}{3s} + \frac{cq}{12st_0}(c + \frac{n}{R}) \]

\[ R = \frac{qc}{4t_0} d^2 + \left( \frac{qct}{6st_0} - \frac{(qc)^2c}{24st_0^2} \right) d + \frac{qcud}{12st_0} - \frac{(qc)cu_0}{24st_0^2} \] 

(5.8)

Hence all coefficients of the \( Q \)-matrices can be expressed in terms of the \( R_\alpha = R \) coefficient which satisfies a third-order equation:

\[ R^3 + R^2 \frac{qc}{(12st_0)^2} \left( \frac{(4t_0 - qc)^2}{4t_0} + 6s(cu_0 - 2t_0u) \right) - \frac{(qc)^3n^2}{4t_0(12st_0)^2} = 0 \] 

(5.9)

For simplicity we choose \( 6s = 2t_0 = 2t = -1, u_0 = 0, c^2 = 1/q \). If we denote \( z = R/qc \) we have instead (5.9) the equation:

\[ z^3 - z^2u + 1/2 = 0 \] 

(5.10)

Two roots coincide \( z_1 = z_2 = z_* = 1 \) at the critical point when the "cosmological constant" \( u_* = 3/2 \). Near critical point, the variable \( R \) related with the free energy will scale as \( R - R_* \sim (u - u_*)^{2/(p+q-1)} \) for the \( (p, q) \) matter models coupled with the 2d gravity.

Expanding \( u \) and \( z \) near critical point:

\[ u = u_* + \mu \delta^2, z = z_* + Z \delta^2 \]

we get in the lowest order of \( \delta \) the relation \( \mu = 3Z^2 \). This means that the variable \( R \) scales as \( R - R_* \sim (u - u_*)^{1/2} \). This critical point (the continuum limit) corresponds to the pure gravity model or \( \phi^3 \) 1-matrix model. The results remain true for arbitrary \( s, t_0, t, u_0, c \) in the symmetric case.

### 5.2 Non-symmetric case

We can write the system of equations (5.5), (5.6) in a different way which will permit to remain with a single type of variables \( X_\alpha \):

\[ X_\alpha = 3s_\alpha d_\alpha + t'_\alpha \] 

(5.11)

We denote by:

\[ u'_\alpha = u_\alpha - \frac{c_\alpha u_0}{2t_0}, c_{\alpha\beta} = -\frac{c_\alpha c_\beta}{4t_0}, t'_\alpha = t_\alpha - \frac{c_\alpha^2}{4t_0} \]
Then we can express the variables $R_\alpha$ in terms of $X_\alpha$ from the system:

$$X_\alpha R_\alpha + \sum_{\beta \neq \alpha} c_{\alpha \beta} R_\beta = \frac{qn}{4t_0}, \quad \alpha = 1 \ldots q$$ (5.12)

and also the variables $b_\alpha$ in terms of $X_\alpha$:

$$\frac{2b_\alpha X_\alpha + n}{R_\alpha} + \sum_{\beta \neq \alpha} c_{\alpha \beta} \frac{2b_\beta}{R_\beta} = 0, \quad \alpha = 1 \ldots q$$ (5.13)

We remain with the system:

$$\frac{X_\alpha^2}{3s_\alpha} + 3s_\alpha (2b_\alpha) + u'_\alpha - \frac{t'_\alpha^2}{3s_\alpha} + \sum_{\beta \neq \alpha} \frac{2c_{\alpha \beta}}{3s_\alpha} (X_\beta - t'_\beta) = 0, \quad \alpha = 1 \ldots q$$ (5.14)

We have the supplementary constraint which can be imposed on the variables $X_\alpha$:

$$\frac{3s_\alpha}{c_\alpha} R_\alpha^2 = \text{const}, \quad \alpha = 1 \ldots q$$ (5.15)

When we tend to the symmetric case with $c_{\alpha \beta} = c, t'_\alpha = 0, 6s_\alpha = 2t_0 = -1, u'_\alpha = u + c^2(q - 1)^2$ we get from the system (5.13) the following equation:

$$2(X + c(q - 1))^2 - \frac{1}{2(X + c(q - 1))} - u = 0$$

With the notation $z = R/q = -1/(2X + 2c(q - 1))$ we obtain the equation (5.10).

$q=2$ case

We argue that the critical behaviour of the case $q = 2$ coincides with the Ising model on random $\phi^3$ lattice.

Solving the system (5.13) we get:

$$2b_1 = -\frac{n(X_2 - cr)}{X_1 X_2 - c^2}, \quad 2b_2 = -\frac{n(X_1 - c/r)}{X_1 X_2 - c^2}$$

where:

$$r = \frac{R_1}{R_2} = \left(\frac{c_1 s_2}{c_2 s_1}\right)^{1/2}$$ (5.16)

Because $R_1, R_2$ are:

$$R_1 = -\frac{(X_2 - c)}{X_1 X_2 - c^2} \frac{2n}{4t_0}, \quad R_2 = -\frac{(X_1 - c)}{X_1 X_2 - c^2} \frac{2n}{4t_0}$$

the relation (5.16) can be rewritten as:

$$r = \frac{X_2 - c}{X_1 - c}$$

We remain with the system:

$$\frac{X_1^2}{3s_1} + \frac{2c}{3s_2} X_2 - 3s_1 n \frac{X_2 - cr}{X_1 X_2 - c^2} + u'_1 = \frac{t'_1}{3s_1} + 2c \frac{t'_2}{3s_2}, \quad (5.17)$$

$$\frac{X_2^2}{3s_2} + \frac{2c}{3s_1} X_1 - 3s_2 n \frac{X_1 - c/r}{X_1 X_2 - c^2} + u'_2 = \frac{t'_2}{3s_2} + 2c \frac{t'_1}{3s_1}$$
We point out the great similarity between this system and that of the Ising model on $\phi^3$ lattice [10] (which corresponds to the case $q = 2$ Potts model). To show this, we integrate the intermediate matrix $M_0$ in the relation (5.4). We get the two-matrix model:

$$Z = \int dM_1 dM_2 \exp \left[ \sum_{\alpha=1}^{2} (s_{\alpha} M_1^{\alpha} + t_{\alpha}' M_2^{\alpha} + u_{\alpha}' M_\alpha) + 2c M_1 M_2 \right]$$

with the previous notations for $t_{\alpha}', u_{\alpha}', c = c_{12}$.

Solving the coupling conditions we remain with three equations:

$$X_1 X_2 = c^2 + \frac{nc}{2R},$$

$$\frac{X_1^2}{3s_1} + \left( \frac{2c}{3s_2} - \frac{3s_1 4R}{2c} \right) X_2 + u_1' = \frac{t_1'^2}{3s_1} + 2c \frac{t_1'}{3s_2},$$

$$\frac{X_2^2}{3s_2} + \left( \frac{2c}{3s_2} - \frac{3s_2 4R}{2c} \right) X_1 + u_2' = \frac{t_2'^2}{3s_2} + 2c \frac{t_2'}{3s_1}.$$  (5.18)

Introducing the expression of $2R = nc/(X_1 X_2 - c^2)$ in the last two equations, we see that they differ from the equations (5.18) only by the terms containing the $r$ quantity. We expect that these terms are only an artefact of the different basis of orthogonal polynomials we have chosen and that they do not change the critical behaviour of the free energy. The variable $R$ scales at the critical point as $R - R_\ast \sim (u - u_\ast)^{1/3}$ (we take $u_1' = u, u_2' = 0$).

$q = 3$ case

This is the first non-trivial member of the $q$-Potts-type set of models. It can not be derived from the two-matrix model or other more complicated chain models. It represents the matrix model on the Dynkin diagram $D_3$.

We have not managed to derive the critical behaviour. However, for further developments, we write down the system of equations which gives the critical scaling.

We introduce the function:

$$Y_{123}(1, r_1, r_2) = X_2 X_3 - c_{23}^2 + r_1 (c_{13} c_{23} - c_{12} X_3) + r_2 (c_{12} c_{23} - c_{13} X_2)$$  (5.19)

The indices 123 of the function $Y_{123}$ are related with the indices of the variables $X_{\alpha}, c_{\alpha \beta}$. The function is symmetric only in the last two indices: $Y_{123}(1, r_1, r_2) = Y_{132}(1, r_1, r_2)$.

We also introduce:

$$Y = X_1 X_2 X_3 + 2c_{12} c_{13} c_{23} - (c_{13}^2 X_2 + c_{23}^2 X_1 + c_{12}^2 X_3)$$

Then the coupling conditions (5.14) in this case are (for simplicity we consider $t_{\alpha}' = 0$):

$$\frac{X_1^2}{3s_1} + \frac{2c_{12}}{3s_2} X_2 + \frac{2c_{13}}{3s_3} X_3 - 3s_1 n \frac{Y_{123}(1, r_1, r_2)}{Y} + u_1' = 0,$$

$$\frac{X_2^2}{3s_2} + \frac{2c_{12}}{3s_1} X_1 + \frac{2c_{23}}{3s_3} X_3 - 3s_2 n \frac{Y_{231}(1, r_2/r_1, 1/r_1)}{Y} + u_2' = 0,$$

$$\frac{X_3^2}{3s_3} + \frac{2c_{13}}{3s_2} X_1 + \frac{2c_{23}}{3s_1} X_2 - 3s_3 n \frac{Y_{312}(1, r_1/r_2, 1/r_2)}{Y} + u_3' = 0.$$  (5.20)
The variables \( r_1, r_2 \) can be written in terms of \( X_\alpha \):

\[
r_1 = \frac{Y_{123}(1,1,1)}{Y_{231}(1,1,1)}, \quad r_2 = \frac{Y_{123}(1,1,1)}{Y_{312}(1,1,1)}
\]

(5.21)

6 Conclusions

We have studied the inhomogenous matrix model on the Bethe tree and have obtained similar results with the homogenous model in the saddle-point method. We have two regimes: the fractal and the asymptotic. In the asymptotic regime, we get the exponential behaviour for the partial partition function \( Z_j(M) = \exp(jr \text{Tr} M^2) \), where \( r \) satisfies a second order equation like in the homogenous case. For large \( j \)- the level of the branch- we expect that the properties of the model become independent of \( j \) and is a transition from the asymptotic regime to the fractal regime. Also in the asymptotic regime when \( \beta = \left( \frac{1}{q-1} \right) \left( \frac{t}{c} \right)^2 = 1/2 \) we have a slow oscillation of the free energy with the period \( T \sim (1/2 - \beta)^{-1} \).

For the q-Potts-like model with arbitrary \( q \) we write down the general coupling conditions. For the special cases \( q = 2 \) and \( q = 3 \) we solve the coupling conditions in terms of only one type of variables \( X_\alpha \). For \( q = 2 \) we have argued that the system has the critical behaviour of the Ising model on \( \phi^3 \) lattice.

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